A PRIORI ESTIMATES FOR RELATIVISTIC LIQUID BODIES

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ABSTRACT. We demonstrate that a sufficiently smooth solution of the relativistic Euler equations that represents a dynamical compact liquid body, when expressed in Lagrangian coordinates, determines a solution to a system of non-linear wave equations with acoustic boundary conditions. Using this wave formulation, we prove that these solutions satisfy energy estimates without loss of derivatives. Importantly, our wave formulation does not require the liquid to be irrotational, and the energy estimates do not rely on divergence and curl type estimates employed in previous works.

1. INTRODUCTION

1.1. Relativistic Euler equations. On a 4-dimensional spacetime, the relativistic Euler equations are given by

\[ \nabla_\mu T^{\mu \nu} = 0 \]  

(1.1)

where \( T^{\mu \nu} = (\rho + p)v^\mu v^\nu + pg^{\mu \nu} \) is the stress energy tensor, \( g = g_{\mu \nu}dx^\mu dx^\nu \) is a Lorentzian metric of signature \((-+,+,+,+)\), \( \nabla_\mu \) is the Levi-Civita connection of \( g_{\mu \nu} \), \( v^\mu \) is the fluid four-velocity normalized by \( g_{\mu \nu}v^\mu v^\nu = -1 \), \( \rho \) is the proper energy density of the fluid, and \( p \) is the pressure. Projecting (1.1) into the subspaces parallel and orthogonal to \( v^\mu \) yields the following well known form of the relativistic Euler equations

\[ v^\mu \nabla_\mu \rho + (\rho + p)\nabla_\mu v^\mu = 0, \]

(1.2)

\[ (\rho + p)v^\mu \nabla_\mu v^\nu + h^{\mu \nu}\nabla_\mu p = 0, \]

(1.3)

where \( h_{\mu \nu} = g_{\mu \nu} + v_\mu v_\nu \) (1.4) is the induced positive definite metric on the subspace orthogonal to \( v^\mu \). In this article, we will be concerned with fluids with a barotropic equation of state of the form

\[ \rho = \rho(p) \]

where \( \rho \) satisfies

\[ \rho \in C^\infty([0, \infty), [\rho_0, \rho_1]), \quad \rho(0) = \rho_0, \]  

(1.5)

and

\[ \rho'(p) > 0, \quad 0 \leq p < \infty, \]  

(1.6)

for some constants \( 0 < \rho_0 < \rho_1 \).

For fluid bodies with compact support, the timelike matter-vacuum boundary is defined by the vanishing of the pressure. Due to the above restrictions on the equation of state, the type of fluids considered in this article are liquids, which are characterized by having a jump discontinuity in the proper energy density at the matter-vacuum boundary. The main aim of this article is to derive a priori estimates for sufficiently smooth solutions of the relativistic Euler equations that represent dynamical compact liquid bodies. The precise form of the a priori estimates can be found in Theorem 8.1, which represent the main

1With the exception of Section 7, we use lower case Greek indices, i.e. \( \mu, \nu, \gamma \), to label spacetime coordinate indices which run from 0 to 3.

2Following standard conventions, we lower and raise spacetime coordinate indices, i.e. \( \mu, \nu, \gamma \), using the metric \( g_{\mu \nu} \) and inverse metric \( g^{\mu \nu} \), respectively.
result of this article. The key analytic difficulties in establishing the a priori estimates are due to the presence matter-vacuum boundary, which is free.

Our approach to establishing a priori estimates begin with showing that sufficiently smooth solutions, which represent dynamical liquid bodies, of the relativistic Euler equations satisfy, when expressed in Lagrangian coordinates, a system of non-linear wave equations with acoustic boundary conditions; see (6.1)-(6.9) for the complete initial boundary value problem (IBVP). Although it is well known that the Euler equations can be reduced to a non-linear scalar wave equation in the special case of irrotational fluids, see [6, 28] for details, our formulation is different and able to handle the general case where rotation is present. The importance of our wave formulation is that it is well suited for deriving energy estimates without derivative loss in the presence of a free matter-vacuum boundary. This is due, in part, to the wave structure of the equations, and in part, to the nature of the acoustic boundary conditions. Indeed, in Section 2 we establish a local existence and uniqueness theory for linear systems of wave equations with acoustic boundary conditions; see, in particular, Theorem 7.12. This linear theory provides the key technical result needed to establish our a priori estimates. We anticipate that the linear theory developed in Section 2 may be of independent interest as it can be applied more generally to other systems of wave equations having acoustic boundary conditions.

1.2. Comparison with existing results. The only existing work that contains a mathematical analysis of compact, relativistic liquid bodies is [27]. There, the local existence and uniqueness of solutions is established using the theory of symmetric hyperbolic systems. However, the energy estimates derived from the symmetric hyperbolic theory involve a derivative loss that is repaired using a Nash-Moser iteration scheme. This leads to a rather high requirement on the regularity of the initial data in order to close the scheme. In contrast, the energy estimates established in this work do not involve derivative loss and require less regularity on the initial data.

In the non-relativistic limit, the relativistic Euler equations reduce to the compressible Euler equations. In this setting, there are more results available with the first local existence and uniqueness result for compressible liquids due to Lindblad [18]. The estimates derived in [18] also required the use of a Nash-Moser scheme due to derivative loss. We note that the work of Trakhinin [27] applies in the non-relativistic setting and provides an alternate approach.

More recently, a local existence and uniqueness theory without derivative loss for non-relativistic, compact, compressible liquid bodies has been developed in [7]. The regularity requirements, as measured by the amount of regularity assumed on the initial data for the map that defines the Lagrangian coordinates, for the energy estimates derived in this article are comparable with those of [7] with both needing 5.5 derivatives bounded in an $L^2$ sense. One key technical difference between the approach taken here compared to that taken in [7] and also [18] is the energy estimates derived in this article do not rely on the divergence and curl estimates developed in [7, 18].

It is worth noting that there are a number of other related results for the non-relativistic Euler equations that involve either incompressible, or gaseous fluid bodies with compact support. For example, see [9, 8, 10, 15, 19, 20, 24].

1.3. Future directions. In work that is currently in preparation, we use the techniques developed in this article to establish the local existence and uniqueness of solutions to the relativistic Euler equations that represent dynamical, compact liquids bodies. The key technical step in going from the a priori estimates presented here to existence and uniqueness is to view the relations (3.11)-(3.13) as constraints, and show that solutions of IBVP (6.1)-(6.9) that satisfy these constraints initially continue to satisfy these constraints to the future; that is, we show that the constraints propagate. Since it is relatively straightforward to establish the existence and uniqueness of solutions for (6.1)-(6.9) using the linear theory developed in Section 7 the key difficulties are reduced to showing that the IBVP satisfied by the constraints (3.11)-(3.13) has unique solutions. This uniqueness problem is then solved by applying standard results from hyperbolic theory. We also are able to show that once the local existence and uniqueness problem is settled for a fixed metric, this theory can be used in conjunction with the techniques developed in [2, 3], suitably adapted, to establish, in a relatively straightforward manner, the local existence of solutions to the Einstein-Euler equations.

1.4. Overview of this paper. In Section 2 we review the Frauendiener-Walton formulation of the relativistic Euler equations, which is the starting point for the derivation of our wave formulation. We set out there the class of solutions for which we establish a priori estimates. The derivation of our wave formulation, in the Eulerian picture, is carried out in Section 3 with the key equation being (3.35).
Lagrangian coordinates adapted to our problem are introduced in Section 4 and the wave equation (3.35) is transformed into these coordinates there with the resulting wave equation given by (4.39). A time differentiative version of the wave equations (4.39) is also derived in this section, see (4.65). In the following section, Section 5, we show that the liquid boundary condition, which corresponds to the vanishing of the pressure at the matter-vacuum interface, implies acoustic type boundary conditions for both wave equations (4.39) and (4.65). The complete IBVP given by (6.1)-(6.9) is then presented in Section 6. A linear existence and uniqueness theory, which includes energy estimates, for this type of IBVP is developed in Section 7. These energy estimates are then used in the final section, Section 8, to obtain the desired a priori estimates that are presented in Theorem 8.1, which represent the main result of this article. Finally, a number of useful calculus inequalities, elliptic estimates, and determinant formulas are listed in the Appendices A, B and C, respectively.

2. THE FRAUendiener-Walton FORMULATION OF THE RELATIVISTIC EULER EQUATIONS

The derivation of our wave formulation of the Euler equations is based on the Frauendiener-Walton formulation of the Euler equations [12, 29], which we quickly review. In the Frauendiener-Walton formulation of the Euler equations, the proper energy density $\rho$ and the normalized fluid 4-velocity $v^\mu$ are combined into a single timelike vector field $w^\mu$ satisfying the symmetric hyperbolic equation

$$A_{\mu\nu}\gamma w^\nu = 0$$

(2.1)

where

$$A_{\mu\nu} = \left(3 + \frac{1}{s^2}\right)\frac{w_\mu w_\nu}{w^2} - \delta_\mu^\gamma w_\nu + \delta_\nu^\gamma w_\mu + w_\gamma g_{\mu\nu},$$

and the square of the sound speed $s^2$ is a function of $\zeta$ defined by

$$\zeta = \frac{1}{\sqrt{w^2}} \quad (w^2 := -w_\nu w^\nu > 0).$$

(2.2)

An explicit formula for $s^2$ can be calculated as follows: first, the pressure $p = p(\zeta)$ is determined by solving the initial value problem

$$\frac{dp}{d\zeta} = \frac{1}{\zeta} \left(\rho(p) + p\right),$$

(2.3)

$$p(\zeta_0) = p_0,$$

(2.4)

for any particular choice of $\zeta_0 > 0$ and $p_0 \geq 0$. To be definite, we set $p_0 = 0$ and $\zeta = 1$.

(2.5)

Solving (2.3)-(2.4) then yields, by standard ODE theory and (1.5), a solution

$$p \in C^\infty((0, \infty)).$$

(2.6)

With $p(\zeta)$ determined, $s^2$ is then given by the formula

$$\frac{1}{s^2} = \left(\frac{\zeta f'(\zeta)}{f(\zeta)} - 3\right)$$

where

$$f(\zeta) = \zeta^3 p'(\zeta).$$

An easy consequence of (2.6) and the assumption (1.6) on the equation of state is that $s^2$ satisfies

$$s^2 \in C^\infty((0, \infty)) \quad \text{and} \quad s^2(\zeta) > 0, \quad 0 < \zeta < \infty.$$

The standard parameterization of a barotropic perfect fluid in terms of the proper energy density $\rho$ and the fluid velocity $v^\mu$ is recovered using the relations

$$\rho = \frac{f(\zeta)}{\zeta} - p(\zeta)$$

(2.7)

and

$$v^\mu = \zeta w^\mu.$$  

(2.8)

As shown in [12, 29], $\rho$ and $v^\mu$ calculated this way satisfy the relativistic Euler equations (1.1).

Before proceeding, we fix the class of relativistic solutions that will be of interest to us. As noted in the introduction, our assumptions on the equation of state imply that this class of solutions represent dynamical, compact liquid bodies.
Assumptions: We assume the following:

(A.1) The \((x^\mu), \mu = 0, 1, 2, 3\) are (global) Cartesian coordinates on \(\mathbb{R}^4\), and \(g = g_{\mu\nu}dx^\nu dx^\mu\) is a smooth Lorentzian metric on \(\mathbb{R}^4\).

(A.2) \(U\) is an open, bounded set in \(\mathbb{R}^4\) that is diffeomorphic to a timelike cylinder with base \(\Omega_0 = \{0\} \times \Omega\) where \(\Omega\) is a bounded, open set in \(\mathbb{R}^3\) with smooth boundary.

(A.3) The vector field \(w = w^\mu \partial_\mu\) is timelike, has components \(w^\mu \in C^k(U)\) for \(k \geq 5\), and satisfies the Frauendiener-Walton-Euler equations (2.1) on \(U\). Here, and in the following, we employ the notation \(\partial_\mu = \frac{\partial}{\partial x^\mu}\) for partial derivatives with respect to the coordinates \((x^\mu)\).

(A.4) The set \(U\) is invariant under the flow of \(w\). Letting \(F_\tau(x^\mu) = (F^\mu_\tau(x^\nu))\) denote the flow map of the vector field \(w\), so that \(F^\mu_\tau(x^\nu)\) is the unique solution to the initial value problem

\[
\frac{d}{d\tau}F^\mu_\tau(x^\nu) = w^\mu(x^\nu), \quad F^\mu_0(x^\nu) = x^\mu,
\]

it is clear from this assumption that there exists a \(T > 0\) such that

\[
U_T := \bigcup_{0 \leq \tau \leq T} F_\tau(\{0\} \times \Omega) \subset U.
\]

(A.5) The vector field \(w\) is tangent to the timelike boundary \(B_T := \bigcup_{0 \leq \tau \leq T} F_\tau(\{0\} \times \partial \Omega)\) of \(U_T\), i.e.

\[w|_{B_T} \in TB_T.\] (2.10)

This not actually an independent assumption since it is a consequence of the previous assumption. However, we state it here separately to emphasize the condition (2.10).

(A.6) There exists constants \(0 < c^- < c^+ < 1\) such that

\[0 < c^- \leq s^2 \leq c^+ < 1 \quad \text{in} \ U.\]

(A.7) The pressure vanishes on the timelike boundary \(B_T\), i.e.

\[p|_{B_T} = 0, \] (2.11)

and satisfies, for some constant \(c_p > 0\), the Taylor sign condition

\[0 < c_p \leq -\nabla_n p \quad \text{in} \ B_T\] (2.12)

where \(n = n^\nu \partial_\nu\) is the outward pointing unit normal to \(B_T\).

Remark 2.1:

(i) Since the vector field \(w^\mu\) is timelike by assumption (A.3), there exists a constant \(0 < c_w\) such that

\[0 < c_w \leq w^2 \quad \text{in} \ U.\]

(ii) Assumption (A.5) implies via (2.8) that

\[v|_{B_T} \in TB_T.\] (2.13)

Together, (2.11) and (2.13) make up the standard representation of the free boundary conditions for a fluid body.

(iii) Using (2.3), we have that

\[-\nabla_n p = \frac{\zeta^2}{2}(\rho(p) + p)\nabla_n w^2.\]

Evaluating this on the boundary yields

\[-\nabla_n p|_{B_T} = \frac{\rho_0}{2} \nabla_n w^2\]

by (1.5), (2.5) and (2.11). From this it is then clear that

\[0 < \frac{2c_p}{\rho_0} \leq \nabla_n w^2 \quad \text{in} \ B_T\] (2.14)
3. An Eulerian wave formulation of the Euler equations

The starting point for the derivation of our wave formulation is the frame formulation of the relativistic Euler equations from [22]. Following [22], we introduce a frame

\[ e_i = e_i^\mu \partial_\mu \]

where

\[ e_0 := w, \]

and the remaining frame fields \( \{e_I\}^3_{I=1} \) are determined by solving the Lie transport equations

\[ [e_0, e_I] = 0. \]

Remark 3.1. Writing (3.2) as

\[ e_0^\mu \partial_\mu e_I^\nu - (\partial_\nu e_0^\mu)e_I^\nu = 0, \]

it is clear, since \( U \) is invariant under the flow of \( e_0 = w \) by assumption, that we can solve (3.3) for given initial data

\[ e_I^\mu|_{\Omega_0} = f_I^\mu \in C^k(\Omega), \]

using the method of characteristics to get a solution

\[ e_I^\mu \in C^{k-1}(U_T). \]

We let

\[ \theta^i = \theta^i_\mu dx^\mu \quad (\theta^i_\mu := (e_\mu^I)^{-1}) \]

denote the co-frame, and we recall that the connection coefficients \( \omega^k_{ij} \) are defined via the relation

\[ \nabla_{e_i} e_j = \omega^k_{ij} e_k. \]

We define the associated connection 1-forms \( \omega^k_\cdot \) in standard fashion by

\[ \omega^k_j = \omega^k_{ij} \theta^i, \]

and we set

\[ \omega_{kj} = \gamma_{kl} \omega^l_j = \omega_{ikj} \theta^i, \]

where

\[ \gamma_{ij} := g(e_i, e_j) = g_{\mu\nu} e_i^\mu e_j^\nu, \]

is the frame metric and

\[ \omega_{ikj} := \gamma_{kl} \omega^l_j = g(\nabla_{e_i} e_j, e_k). \]

We also let \( \gamma^{ij} \) denote the inverse frame metric, i.e.

\[ (\gamma^{ij})^{-1} = (\gamma_{ij}), \]

and note that

\[ w^2 = -\gamma_{00} \quad \text{and} \quad \zeta = \left( \frac{1}{-\gamma_{00}} \right)^{\frac{1}{2}} \]

by (2.2) and (3.1). Due to the choice (2.5), the boundary condition (2.11) is equivalent to

\[ \gamma_{00}|_{B_T} = -1. \]

Next, we let

\[ F \in C^\infty((0, \infty), (0, \infty)) \]

denote the unique solution to the initial value problem

\[ F'(\zeta) = -\frac{F(\zeta)}{\zeta s^2(\zeta)}, \]

\[ F(1) = 1, \]

and we choose initial data (3.4) so that

\[ \gamma_{0j}|_{\Omega_0} = 0 \quad \text{and} \quad \det(\gamma_{ij}|_{\Omega_0}) = F\left((-\gamma_{00}|_{\Omega_0})^{-\frac{1}{2}}\right)^2. \]
As we shall see in Section 4.2, this is always possible.

With this choice of initial data, it follows from Proposition IV.3 of [22] that the frame coefficient $e^\mu_i$ satisfy the relations

$$[e_0, e_J] = 0, \quad \gamma_{0J} = 0, \quad F((-\gamma_{00})^{-1/2})^2 = \det(\gamma_{IJ}),$$

and

$$e_0(\sigma_i^J k) = 0$$

in $\mathcal{U}_T$ where

$$\sigma_i^J k := \theta^i([e_l, e_k]) = \theta^i(e_l^\sigma e_k^\lambda - e_k^\sigma e_l^\lambda).$$

From (3.11) and (3.15), we observe that $\sigma_i^J k$ is anti-symmetric in the $i, k$ indices and satisfies

$$\sigma_0^J k = \sigma_k^J 0 = 0.$$

Using the Cartan structure equations

$$d\theta^i = -\omega^i J \wedge \theta^j,$$

and

$$d\gamma_{ij} = \omega_{ij} + \omega_{ji},$$

it is not difficult, see the proof of Proposition IV.1 in [22] for details, to show that (3.11)-(3.14) imply that connection coefficients $\omega_i^J k$ satisfy

$$\frac{1}{s^2\gamma_{00}}(-\gamma_{00}^J \omega_{0J}) = 0$$

and

$$\omega_{00} + \omega_{00} = 0$$

in $\mathcal{U}_T$. Writing the Cartan Structure equations (3.17) in the alternative form

$$\theta^i([e_l, e_k]) = \omega^i_k - \omega^k_i,$$

we see from (3.15) that $\sigma_i^J k$ may be expressed as

$$\sigma_i^J k = \omega_i^k J - \omega_j^k.$$

We also note that, due to (3.12), the inverse frame metric $\gamma^0 J$ satisfies

$$\gamma^0 J = 0$$

and

$$\gamma^0 J = \frac{1}{\gamma_{00}},$$

which, in turn, allow us to express (3.13) as

$$F(\sqrt{\gamma_{00}})^2 = \frac{1}{\det(\gamma_{IJ})}.$$  

Appealing to the definition of the Hodge star operator and the linear independence of the co-frame $\theta^i$, we know that the 1-form $\ast(\theta^1 \wedge \theta^2 \wedge \theta^3)$ is non-vanishing and orthogonal to the $\theta^i$, and consequently,

$$\theta^0 = \ast(\theta^1 \wedge \theta^2 \wedge \theta^3)$$

for some non-vanishing function $\ast$ by (3.20). To determine $\ast$, we apply the Hodge star operator to (3.24) to get

$$\frac{1}{\ast} \theta^0 = \theta^1 \wedge \theta^2 \wedge \theta^3.$$

\footnote{Recall that $\ast \ast \lambda = (-1)^{p(q-p)+1} \lambda$ for $p$-forms $\lambda$.}
Wedging this with $\theta^0$ gives
\[
\frac{1}{f} \theta^0 \wedge *\theta^0 = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3,
\]
which we can write as
\[
\frac{1}{f} \gamma_{00} \sqrt{-\text{det}(\gamma^{ij})} \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3.
\]
From this, we conclude that
\[
f = -\gamma_{00} \sqrt{-\text{det}(\gamma^{ij})},
\]
which, with the help of (3.20) and (3.23), implies
\[
f = f(\zeta) := -\zeta F(\zeta) \quad (3.27)
\]
where $\zeta = \sqrt{-\gamma_{00}}$.

Applying the operator $*d$ to (3.25) yields
\[
*\left( \frac{1}{f} * \theta^0 \right) = * \left( d\theta^1 \wedge \theta^2 \wedge \theta^3 - \theta^1 \wedge d\theta^2 \wedge \theta^3 + \theta^1 \wedge \theta^2 \wedge d\theta^3 \right).
\]
Noting that the right hand side of this vanishes, since
\[
d\theta^i = -\frac{1}{2} \sigma_M^i \theta^M \wedge \theta^L
\]
by (3.10) and (3.17), and
\[
\theta^I \wedge \theta^J \wedge \theta^K \wedge \theta^L = 0
\]
for any choice of $I, J, K, L \in \{1, 2, 3\}$, we see that
\[
*\left( \frac{1}{f} * \theta^0 \right) = 0,
\]
or equivalently, in terms of components,
\[
\nabla^\mu \left( \frac{1}{f} \theta^0_\mu \right) = 0.
\]
Applying the covariant derivative $\nabla_\nu$ to this expression, yields, after commuting the covariant derivatives, the relation
\[
\nabla^\mu \left( \nabla_\nu \left( \frac{1}{f} \theta^0_\mu \right) \right) = \frac{1}{f} R^\lambda_\nu \theta^0_\lambda.
\]
Continuing on, we compute
\[
\nabla_\nu \zeta = -\frac{g(\theta^0_\nu, \nabla_\nu \theta^0)}{\sqrt{-\gamma_{00}}} \quad (\zeta = \sqrt{-\gamma_{00}}),
\]
and using (3.27) and (3.38),
\[
\frac{d}{d\zeta} \ln(-f(\zeta)) = \frac{1}{\zeta} + \frac{1}{F(\zeta)} \frac{dF}{d\zeta} = \frac{1}{\zeta} - \frac{1}{\zeta s^2(\zeta)}.
\]
From these two expressions, we find that
\[
\nabla_\nu \left( \frac{1}{f} \theta^0_\mu \right) = \frac{1}{f} \left[ \nabla_\nu \theta^0_\mu - \frac{d}{d\zeta} \ln(-f(\zeta)) \nabla_\nu \zeta \theta^0_\mu \right] = \frac{1}{f} \left[ \nabla_\nu \theta^0_\mu - \left( 1 - \frac{1}{s^2} \right) \frac{g(\theta^0_\nu, \nabla_\nu \theta^0)}{\gamma_{00}} \theta^0_\mu \right].
\]
Using (2.8) and (3.1), it is clear from the definition (1.4) that the projection tensor $h_{\mu\nu}$ can be written as
\[
h_{\mu\nu} = g_{\mu\nu} - \epsilon_{\mu\nu\lambda}\epsilon_{00}. \quad (3.31)
\]
We also note that it follows directly from (3.12) and the definition of the co-frame that
\[
\theta^0_\mu = \frac{g_{\mu\nu}}{\gamma_{00}} \epsilon^\nu_0,
\]
\[
\text{We are using the well known identity } \lambda \wedge * \beta = g(\lambda, \beta)_\mu, \text{ which holds for all one forms } \alpha \text{ and } \beta \text{ with } \mu \text{ the volume form.}
\]
\[
\text{In terms of the co-frame } \theta^0, \mu \text{ is given by } \mu = (-\det(\gamma^{ij}))^{-1/2} \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3.
\]
\[
\text{Here, we use } \nabla_\nu \nabla_\sigma \gamma - \nabla_\nu \nabla_\sigma \gamma = R_{\mu\nu\sigma} \gamma^0 \lambda_\sigma \text{ and } R_{\mu\nu} = R_{\mu\nu\gamma}.\]
which, we can use to write (3.31) as
\[ h_{\mu\nu} = g_{\mu\nu} - \frac{1}{\gamma_{00}} \theta_{0}^{\mu} \theta_{0}^{\nu} = g_{\mu\nu} - \gamma_{00} \theta_{0}^{\mu} \theta_{0}^{\nu}. \]  
(3.33)

Setting \( i = 0 \) in (3.28) and expressing the result in terms of covariant derivatives\( ^{4} \) we get that
\[ \nabla_{\nu} \theta_{0}^{\mu} = \nabla_{\mu} \theta_{0}^{\nu} + \sigma_{M}^{0} L_{\mu}^{\lambda} \theta_{0}^{L}. \]  
(3.34)

Using (3.33) and (3.34), we can write (3.30) as
\[ \nabla_{\nu} \left( \frac{1}{f} a_{\alpha\beta} \right) = \frac{1}{f} \left( \frac{1}{s^{2}} \gamma_{00} \right) \left[ \nabla_{\alpha} \theta_{0}^{\beta} + \sigma_{M}^{0} L_{\alpha}^{\lambda} \theta_{0}^{L} \right]. \]  
(3.35)

Remark 3.2.

(i) From (3.32), it is clear that we can write \( a_{\alpha\beta} \) as
\[ a_{\alpha\beta} = h_{\alpha\beta} + \frac{1}{s^{2}} \frac{\theta_{0}^{\alpha} \theta_{0}^{\beta}}{\gamma_{00}} - \left( 1 - \frac{1}{s^{2}} \right) \frac{\theta_{0}^{\alpha} \theta_{0}^{\beta}}{\gamma_{00}}, \]  
(3.36)

or equivalently, using (2.8) and (3.1),
\[ a_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{s^{2}} \psi_{\mu} \psi_{\nu}, \]
which we recognize as the inverse of the acoustic metric
\[ a_{\alpha\beta} = h_{\alpha\beta} - s^{2} \psi_{\mu} \psi_{\nu}. \]

(ii) Although \( \theta_{0}^{\alpha} \) is equivalent by (3.32) to the vector field \( e_{0} = w \), which we know from the discussion in Section 2 is completely equivalent to the usual parameterization of a barotropic perfect fluid in terms of \( \rho \) and \( v^{\nu} \), the wave equation (3.35) does not, in general, represent a complete evolution equation for the fluid. This is due to the presence of the co-frame fields \( \theta_{0}^{M} \), which satisfy their own evolution equations that can be derived from (3.11). The one exception to this is when the fluid is irrotational, which can be shown to be equivalent to the condition \( \sigma_{i}^{k} j = 0 \). In this case, it is clear that (3.35) reduces to a wave equation involving only \( \theta_{0}^{\alpha} \), and hence provides a complete evolution equation for the fluid.

Although (3.35) does not, in general, provide a complete evolution equation for the perfect fluid, we show in the next section that this defect can be remedied by transforming to Lagrangian coordinates. The introduction of the Lagrangian coordinates is also essential to fix the free boundary and allow us to work on a fixed domain. To prepare for the change to Lagrangian coordinates, we express the covariant derivatives in (3.35) in terms of the partial derivatives and Christoffel symbols to get
\[ \partial_{\alpha} Y_{\nu}^{\alpha} = -\Gamma_{\alpha\gamma}^{\alpha} Y_{\nu}^{\gamma} + \Gamma_{\alpha\nu}^{\alpha} Y_{\gamma}^{\alpha} + \frac{1}{f} R_{\lambda}^{\alpha} \theta_{0}^{\lambda}. \]  
(3.37)

where
\[ Y_{\nu}^{\alpha} = \frac{1}{f} a^{\alpha\beta} \left[ \partial_{\beta} \theta_{0}^{\mu} - \Gamma_{\beta\nu}^{\alpha} \theta_{0}^{\mu} + \sigma_{M}^{0} L_{\beta}^{\lambda} \theta_{0}^{L} \right]. \]  
(3.38)

\( ^{9} \)Note that \( d \theta^{i} = -\frac{1}{f} \sigma_{M}^{j} L_{\theta}^{M} \wedge L^{L} \iff \nabla_{\mu} \theta_{0}^{i} = \sigma_{M}^{j} L_{\theta}^{M} \theta_{0}^{L}. \)
4. The Lagrangian wave formulation of the Euler equations

4.1. Lagrangian coordinates. We introduce Lagrangian coordinates \((\bar{x}^\mu)\) adapted to the vector field \(e_0 = w\) via the formula

\[ x^\mu = \phi^\mu(\bar{x}^\lambda) := F^\mu_{\nu \lambda}(0, \bar{x}^\lambda) \]  

(4.1)

where \(F_{\nu \lambda}\) is the flow of \(w\) defined by (2.9). From the regularity assumption (A.3) for the components of the vector field \(w^\mu\) and the standard properties of flows, it is not difficult to verify that \(\phi\) defines a \(C^k\) diffeomorphism

\[ \phi : \Omega_T := [0, T] \times \Omega \rightarrow U_T : (\bar{x}^\lambda) \mapsto (\phi^\mu(\bar{x}^\lambda)), \]

and that

\[ \phi|_{\Omega_0} = \text{id}_{\Omega_0}. \]  

(4.2)

Moreover, since \(\phi\) is generated by the flow of \(e_0\), the pullback \(\bar{e}_0 = \phi^*e_0\) satisfies

\[ \bar{e}_0 = \bar{\partial}_0 \iff \bar{e}_0^\mu = \delta_0^\mu \]  

(4.3)

where here and below, we use

\[ \bar{\partial}_\mu = \frac{\partial}{\partial \bar{x}^\mu} \]

to denote the partial derivative with respect to the Lagrangian coordinates \((\bar{x}^\mu)\).

For use below, we let

\[ J = (J^\mu_{\nu}) := (\bar{\partial}_\nu \phi^\mu) \]  

(4.4)

denote the Jacobian matrix of the coordinate transformation (4.1) and

\[ \bar{J} = (\bar{J}^\mu_{\nu}) := J^{-1} \]  

(4.5)

its inverse. Using this notation, the components of the pullback \(\bar{e}_0^\mu\) can be computed via the formula

\[ \bar{e}_0^\mu = \bar{J}^\mu_{\nu}e_0^\nu \circ \phi. \]

From this, it follows immediately that

\[ \bar{\partial}_0 \phi^\mu = e_0^\mu \circ \phi \]  

(4.6)

is equivalent to (4.3).

4.2. Initial conditions. Since the vector field \(e_0^\mu = w^\mu\) completely determines the proper energy density \(\rho\) and the fluid four-velocity \(v^\nu\),

\[ f^\mu_{\nu} := e_0^\mu |_{\Omega_0} \]

contains all of the initial data for the fluid. We are then free to choose the other initial data \(f^\mu_{\nu}\), see (3.4), as we like as long as the constraints (3.10) are satisfied. For our purposes, we need to fix the \(f^\mu_{\nu}\) in a specific fashion beyond just satisfying the constraints (3.10). However, before we discuss this, we first make some observations starting with the pullback frame

\[ \bar{e}_i = \phi^* \bar{e}_i \iff \bar{e}_i^\mu = \bar{J}^\mu_{\nu} \bar{e}_i^\nu \]  

(4.7)

where

\[ \bar{e}_i^\nu = e_i^\nu \circ \phi. \]  

(4.8)

or equivalently

\[ \bar{e}_i^\mu = \bar{e}_i^\lambda \bar{\partial}_\lambda \phi^\mu. \]  

(4.9)

Since the \(e_i\) satisfy (3.11) and the Lie bracket is natural with respect to pullbacks by diffeomorphisms, the \(\bar{e}_i\) must also satisfy \([\bar{e}_0, \bar{e}_j] = 0\), or equivalently, by (4.3),

\[ \bar{\partial}_0 \bar{e}_j^\mu = 0. \]  

(4.10)

In particular, this shows that the frame fields \(\bar{e}_i^\mu\) are independent of \(\bar{x}^0\). We also note that

\[ \bar{e}_j^\mu |_{\{0\} \times \Omega} = f^\mu_{\nu}, \]  

(4.11)

and

\[ (J^\mu_{\nu}) |_{\{0\} \times \Omega} = \left( f_{\nu}^\mu, \delta_\nu^\mu \right) \]  

(4.12)

\[ \text{That is, the Lie bracket } [\cdot, \cdot] \text{ satisfies } \psi^*[X, Y] = [\psi^*X, \psi^*Y] \text{ for all diffeomorphisms } \psi \text{ and vector fields } X, Y. \]
by (4.2) and (4.6). Using (4.12), we compute
\[
(J^\mu)_{\{0\} \times \Omega} = \begin{pmatrix} 1 & f_0^A \\ -\frac{f_0^T}{f_0} f_0^A & \delta^A_2 \end{pmatrix}.
\]
From this, (4.11) and the fact that the \( \tilde{e}_\mu \) are \( \bar{x}^0 \)-independent, we find that
\[
(\tilde{e}_\mu) = \begin{pmatrix} 1 & f_0^T \\ 0 & -\frac{f_0^T}{f_0} f_0^A + f_0^A \end{pmatrix}.
\] (4.13)

By assumption, \( \bar{e}_0 = \bar{\partial}_0 \) is tangent to the matter-vacuum boundary, which is given in Lagrangian coordinates by
\[
\Gamma_T := [0, T] \times \partial \Omega.
\]
Using the Gram-Schmidt algorithm, we can complete \( \bar{e}_0 \) to a basis
\[
\{ \bar{e}_0, \bar{z}_1 = \bar{z}^\nu \bar{J}_\nu, \bar{z}_2 = \bar{z}^\nu \partial_\nu, \bar{z}_3 = \bar{x}^0 \partial_\nu \}
\] (4.14)
of \( \Gamma_T \) such that the frame is orthogonal at \( \bar{x}^0 = 0 \), and the components \( \bar{z}_3^\nu, \bar{A} = 1, 2 \), are \( \bar{x}^0 \)-independent. Here, the orthogonality at \( \bar{x}^0 = 0 \) is determined with respect to the pull-back metric
\[
\bar{g} := \phi^* \bar{g} = \bar{g}_{\alpha \beta} dx^\alpha dx^\beta
\]
with components given by
\[
\bar{g}_{\alpha \beta} = J^\mu_\alpha J^\nu_\beta g_{\mu \nu} \circ \phi.
\] (4.15)
We can then extend the basis (4.13) to \( \Omega_T \), while keeping the components \( \bar{z}_3^\nu, \bar{A} = 1, 2 \), \( \bar{x}^0 \)-independent, and complete it to a full frame
\[
\{ \bar{e}_0, \bar{z}_1 = \bar{z}^\nu \partial_\nu, \bar{z}_2 = \bar{z}_2^\nu \partial_\nu, \bar{z}_3 = \bar{z}_3^\nu \partial_\nu \}
\]
such that: \( \bar{z}_3 \) is outward pointing at the boundary \( \Gamma_T \), the components \( \bar{z}_3^\nu \) are \( \bar{x}^0 \)-independent, and the frame is orthogonal at \( \bar{x}^0 = 0 \). Thus, in particular,
\[
\text{Span}\{ \{ \bar{e}_0, \bar{z}_3 \} | \Gamma_T, | \bar{A} = 1, 2 \} = T \Gamma_T
\] (4.16)
and
\[
\bar{g}(\bar{e}_0, \bar{z}_3)|_{\Gamma_T} = 0.
\]
We also observe via (4.2), (4.12) and (4.15) that
\[
\bar{g}(\bar{z}_I, \bar{z}_J)|_{\Gamma_T} = (\bar{z}_I^\nu f_I^\nu + \bar{z}_J^\nu \delta^\nu_2) (\bar{z}_I^\nu f_I^\nu + \bar{z}_J^\nu \delta^\nu_2) g_{\mu \nu}|_{\Gamma_T}.
\]
Next, we set
\[
f_I^0 = \det(\bar{g}(\bar{z}_I, \bar{z}_J))^{-\frac{1}{2}} F((-g_{00})^{\frac{1}{2}}) \bar{z}_I^0|_{\Gamma_T}
\] (4.17)
and
\[
f_I^A = \det(\bar{g}(\bar{z}_I, \bar{z}_J))^{-\frac{1}{2}} F((-g_{00})^{\frac{1}{2}}) \left( \bar{z}_I^A + f_I^A \bar{z}_I^0 \right)|_{\Gamma_T}.
\] (4.18)
From the above two expressions and (4.13), we obtain
\[
\tilde{e}_\mu = \det(\bar{g}(\bar{z}_I, \bar{z}_J))^{-\frac{1}{2}} F((-g_{00})^{\frac{1}{2}}) \bar{z}_I^\mu,
\]
which, in turn, implies that
\[
g(\bar{e}_0, \bar{e}_J)|_{\Gamma_T} = \bar{g}(\bar{e}_0, \bar{e}_J)|_{\Gamma_T} = 0
\]
and
\[
\det(g(\bar{e}_I, \bar{e}_J)|_{\Gamma_T} = \det(\bar{g}(\bar{e}_I, \bar{e}_J))|_{\Gamma_T} = F((-g_{00})^{\frac{1}{2}})^2|_{\Gamma_T}.
\]
This shows the constraints (3.10) are satisfied for the choice of initial data \( f_I^\mu \) given by (4.17) and (4.18), and that
\[
\text{Span}\{ \bar{e}_a | \Gamma_T, | a = 0, 1, 2 \} = T \Gamma_T
\] (4.19)
by virtue of (4.16). It is also not difficult to see from the above construction that we can always ensure that
\[ c_f > \det(\bar{e}) > \frac{1}{c_f} > 0 \quad \text{in } \Gamma_T \]
holds for some positive constant \( c_f \).

In terms of the co-frame, we have that
\[ \bar{\theta}^i = \phi^* \theta^i \quad \iff \quad \bar{\theta}^i_\mu = J^\nu_\mu \bar{\theta}^\nu_i \]
where
\[ (\bar{\theta}^i_\nu) := (\theta^i_\nu \circ \phi) = (\bar{e}^i_\nu)^{-1} \quad \text{and} \quad (\bar{\theta}^i_\nu) = (\bar{e}^i_\nu)^{-1}. \]
By definition, \( \bar{\theta}^i_\mu \bar{e}^\mu_j = \delta_j^i \), and so, by (4.13) and (4.19), \( \bar{\theta}^i_\mu \) satisfies \( \theta^0_0 = 0 \), and, consequently,
\[ \nu_\mu := \bar{\theta}^1_\mu = \delta^0_\mu \theta^3_\Sigma \]
defines a \( \bar{x}^0 \)-independent, outward pointing (non-normalized) co-normal to the boundary \( \Gamma_T \), while \( \nu_\Sigma \) defines an outward pointing co-normal to the boundary \( \partial\Omega \). Finally, letting
\[ (f^j_\mu) = (f^j_\mu)^{-1} \]
denote the inverse of \( f^j_\mu \), a short calculation shows that
\[ (\bar{\theta}^i_\mu) = (\delta^0_0 \ f^i_\lambda). \]

4.3. The wave formulation in Lagrangian coordinates. The key to transforming the wave equation (5.37) into Lagrangian coordinates is the following well known transformation formula for the divergence of a vector field:
\[ |\bar{g}|^{-\frac{1}{2}} \bar{\partial}_\mu (|\bar{g}|^{\frac{1}{2}} \bar{X}^\mu) = \phi^* (|g|^{-\frac{1}{2}} \partial_\mu (|g|^{\frac{1}{2}} X^\mu)) \]
where
\[ \bar{X}^\mu = J^\nu_\mu X^\nu \circ \phi, \]
\[ |\bar{g}| = -\det(\bar{g}_{\mu\nu}), \] and \[ |g| = -\det(g_{\mu\nu}). \] Setting
\[ Y^\alpha = \frac{1}{|\bar{g}|} \bar{X}^\alpha \]
where \( Y^\alpha \) is as defined previously by (3.33), a short calculation, using the chain rule, shows that
\[ |\bar{g}|^{-\frac{1}{2}} \bar{X}^\alpha = \left( \frac{|\bar{g}|}{|g|} \phi \right)^{-\frac{1}{2}} \left[ \bar{a}^{\alpha\beta} \bar{\theta}_\beta \bar{g}^0_0 + \frac{1}{4} \bar{a}^{\alpha\beta} \left( -J^\beta_\gamma \bar{\Gamma}_{\gamma\sigma} \right) \bar{g}^0_\sigma \right] \]
where
\[ \bar{a}^{\alpha\beta} = (\phi^* a)^{\alpha\gamma} = J^\mu_\alpha J^\beta_\mu a^{\mu\nu} \circ \phi \]
denotes the pull-back of the acoustic metric \( a^{\mu\nu} \) by \( \phi \), and we have introduced the definitions
\[ \bar{\theta}^0_\nu = \theta^0_\nu \circ \phi, \]
\[ \bar{\sigma}^M_0_L = \sigma^M_0_L \circ \phi, \]
\[ \bar{\Gamma}^\mu_\nu_\sigma = \Gamma^\mu_\nu_\sigma \circ \phi, \]
\[ \bar{f} = f \circ \phi \]
and
\[ \bar{a}^{\mu\nu} = a^{\mu\nu} \circ \phi. \]

Next, we define
\[ \bar{g}_{\mu\nu} = g_{\mu\nu} \circ \phi \quad \text{and} \quad \bar{g}^{\mu\nu} = g^{\mu\nu} \circ \phi, \]
and note that
\[ |\bar{g}| = -\det(J^\alpha_\beta \bar{g}^\alpha_\beta J^\alpha_\nu) = \det(J)^2 |\bar{g}| = \det(J)^2 |g| \circ \phi, \]
or equivalently
\[ \left( \frac{|\bar{g}|}{|g|} \circ \phi \right)^{\frac{1}{2}} = \det(J). \]
We also define
\[ \bar{\gamma}_{ij} = \gamma_{ij} \circ \phi \quad \text{and} \quad \bar{\gamma}^{ij} = \gamma^{ij} \circ \phi, \]
and observe that
\[ \bar{f} = f((-\bar{\gamma}_{00})^{-1/2}) \]
where
\[
\tilde{\gamma}_{00} = \tilde{g}_{\alpha\beta}\tilde{e}_0^\alpha\tilde{e}_0^\beta = \frac{1}{\tilde{g}^{\alpha\beta}\tilde{\theta}_{\alpha}^\beta},
\] (4.36)
and that
\[
\tilde{a}^{\mu\nu} = \tilde{g}^{\mu\nu} - \frac{1}{\tilde{\gamma}_{00}} \left(1 - \frac{1}{s^2}\right) \tilde{e}_0^\mu \tilde{e}_0^\nu = \tilde{g}^{\mu\nu} - \tilde{\gamma}_{00} \left(1 - \frac{1}{s^2}\right) \tilde{g}^{\alpha\beta} \tilde{\theta}_{\alpha}^\gamma \tilde{\theta}_{\beta}^\gamma \] (4.37)
where
\[
\tilde{s}^2 = s^2 \left(-\tilde{\gamma}_{00}\right)^{-1/2}.
\] (4.38)

Using (4.24), (4.25), and (4.33), it follows that the wave equation (3.37) when expressed in Lagrangian coordinates becomes
\[
\tilde{\partial}_\alpha \left(\tilde{A}^{\alpha\beta}\tilde{\partial}_\beta\tilde{\theta}_{\nu}^\gamma + \tilde{L}^\alpha_{\nu} \right) = \tilde{F}_{\nu}
\] (4.39)
where
\[
\tilde{A}^{\alpha\beta} = -\text{det}(J)\tilde{a}^{\alpha\beta},
\] (4.40)
\[
\tilde{L}^\nu_{\nu} = -\text{det}(J)\tilde{a}^{\alpha\beta} \left(-J_\beta^\gamma \tilde{\Gamma}_{\nu\gamma} \tilde{\theta}_{\beta}^\nu + \tilde{\sigma}_{M}^0 L\tilde{\theta}_{M}^L \tilde{\theta}_{\nu}^L\right),
\] (4.41)
\[
\tilde{F}_{\nu} = -\text{det}(J) \left(-\tilde{\Gamma}_{\alpha\gamma} \tilde{V}_{\nu}^\gamma + \tilde{\Gamma}_{\nu\gamma} \tilde{V}_{\alpha}^\gamma + \frac{1}{f} \tilde{R}_{\nu}^{\lambda} \tilde{\theta}_{\lambda}^0\right),
\] (4.42)
\[
\tilde{Y}_{\nu}^\gamma = \frac{1}{f} \tilde{a}^{\alpha\beta} \left[\tilde{J}_{\beta}^\gamma \tilde{\theta}_{\nu}^\beta - \tilde{R}_{\beta}^{\gamma} \tilde{\theta}_{\nu}^\beta + \tilde{\sigma}_{M}^0 L\tilde{\theta}_{M}^L \tilde{\theta}_{\nu}^L\right],
\] (4.43)
and
\[
\tilde{R}_{\nu}^{\lambda} = R_{\nu}^{\lambda} \circ \phi.
\] (4.44)

We note that by using the chain rule, (3.14) and (4.6) it is straightforward to verify that \(\tilde{\sigma}_{i}^{k} j\) satisfies the evolution equation
\[
\tilde{\partial}_0 \tilde{\sigma}_{i}^{k} j = 0.
\] From this and (4.2), it follows immediately that
\[
\tilde{\sigma}_{i}^{k} j(x^0, x^\Sigma) = \sigma_{i}^{k} j(0, x^\Sigma).
\] (4.45)
We also observe that (3.32) and (4.6) imply that
\[
\tilde{\partial}_0 \tilde{\theta}_{\nu}^\mu = \tilde{\gamma}_{00} \tilde{g}^{\mu\nu} \tilde{\theta}_{\nu}^0,
\] (4.46)
while
\[
\tilde{\theta}_{\nu}^I = \tilde{J}_{\nu}^{\nu} \tilde{\theta}_{\nu}^I,
\] (4.47)
is a consequence of (1.21).

The two evolution equations (4.39) and (4.46) along with the definitions (4.4), (4.5), (4.29), (4.32), (4.35)-(4.38), (4.40)-(4.45), and (4.47) constitute our Lagrangian wave formulation of the Euler equations.

Remark 4.1. In the simpler setting where the pressure never vanishes and there is no free boundary, it already follows from known results that our Lagrangian wave formulation is complete in the sense that the evolution equations (4.39) and (4.46) for the pair \(\tilde{\phi}_{\nu}^L, \tilde{\theta}_{\nu}^0\) are well-posed, and in particular, solutions of (4.39) and (4.46) satisfy energy estimates. Indeed, the evolution equations (4.39) and (4.46) fit within the class of wave equations considered by [17]. Although boundary conditions, which do not include the boundary conditions considered in this article, were imposed in [17], it is clear that results of [17] continue to hold in the absence of a boundary, and hence apply to the system (4.39) and (4.46) immediately when no boundary is present.
4.4. The time differentiated wave equation. Due to the free boundary, it is not enough to consider the evolution equations (4.39) and (4.40) alone. Instead, we must supplement them with what amounts to a $\bar{x}^0$-differentiated version of the wave equation (4.39). With this in mind, we define

$$\psi_{\nu} = (\nabla_{\epsilon_{\nu}} \theta_{\nu}^0) \circ \phi,$$

and note using (4.3) and the chain rule that this can be written as

$$\psi_{\nu} = \tilde{\partial}_{\nu} \theta_{\nu}^0 + \beta^\nu \theta_{\nu}^0,$$

where we have set

$$\beta^\nu := -\dot{\gamma}_0 \tilde{\Gamma}_{\gamma}^\nu \overset{\text{(4.32)}}{=} -\dot{\gamma}_{00} \tilde{g}^{\gamma\sigma} \tilde{\theta}_{\sigma}^0 \tilde{\Gamma}_{\gamma}^\nu.$$

Differentiating (4.39) with respect to $\bar{x}^0$, a short calculation shows that $\psi_{\nu}$ satisfies the wave equation

$$\tilde{\partial}_\alpha (\bar{A}_{\alpha\beta} \tilde{\partial}_\beta \psi_{\nu} + L^\nu_{\nu}) = F_{\nu},$$

where

$$L^\nu_{\nu} = \tilde{\partial}_\alpha \bar{A}_{\alpha\beta} \tilde{\partial}_\beta \theta_{\nu}^0 + \tilde{\partial}_\alpha \bar{L}^\alpha_{\nu} + \beta_{\nu} \bar{L}^\alpha_{\nu} - \bar{A}_{\alpha\beta} \tilde{\partial}_\beta \theta_{\nu}^0,$$

and

$$F_{\nu} = \tilde{\partial}_\nu \bar{F}_{\nu} + \beta_{\nu} \bar{F}_{\nu} + \tilde{\partial}_\alpha \beta_{\nu} (\bar{A}_{\alpha\beta} \tilde{\partial}_\beta \theta_{\nu}^0 + \bar{L}^\alpha_{\nu}).$$

Next, we define a positive definite, symmetric 2-tensor by

$$m^{\alpha\beta} := g^{\alpha\beta} - \frac{1}{\gamma_{00}} \epsilon_{\alpha} \epsilon_{\beta} \overset{\text{eq. (4.32)}}{=} g^{\alpha\beta} - \gamma_{00} g^{\mu\nu} g_{\mu\nu} \theta_{\nu}^0 \theta_{\nu}^0,$$

and let

$$(m_{\alpha\beta}) = (m^{\alpha\beta})^{-1}$$

denote its inverse. A short calculation then shows that

$$m_{\alpha\beta} = g_{\mu\nu} - \frac{1}{\gamma_{00}} \theta_{\mu}^0 \theta_{\nu}^0.$$
Differentiating $\hat{\psi}_\nu$, we find that
\[ \partial_\mu \hat{\psi}_\nu = \frac{1}{\mu} \pi^\lambda_\mu \partial_\mu \psi_\lambda - \frac{1}{2} \partial_\mu \check{m}^{\alpha\beta} \hat{\psi}_\alpha \hat{\psi}_\beta \hat{\psi}_\nu \] (4.57)
where
\[ \pi^\lambda_\mu = \delta^\lambda_\mu - \check{\psi}^\lambda \hat{\psi}_\nu \]
(4.58)
is the projection operator uniquely defined by the properties
\[ \pi^\lambda_\nu \hat{\psi}_\lambda = 0 \quad \text{and} \quad \pi^\lambda_\nu \pi^\mu_\lambda = \pi^\mu_\nu, \]
(4.59)
while differentiating $\mu$ yields
\[ \bar{\partial}_\nu \mu = \check{\psi}^\alpha \bar{\partial}_\nu \psi_\alpha + \frac{\mu}{2} \bar{\partial}_\nu \check{m}^{\alpha\beta} \hat{\psi}_\alpha \hat{\psi}_\beta. \]
(4.60)
For use below, we define the following variants of the projection operator $\pi^\mu_\nu$:
\[ \pi^{\mu\nu} = \check{m}^{\mu\lambda} \pi^\nu_\lambda \quad \text{and} \quad \mu^{\mu\nu} = \check{m}\pi^\lambda_\nu. \]
(4.61)
From (4.57), we see that
\[ \check{A}^{\alpha\beta} \partial_\beta \hat{\psi}_\nu = \frac{1}{\mu} \pi^\omega_\nu \check{A}^{\alpha\beta} \partial_\beta \psi_\omega - \frac{1}{2} \check{A}^{\alpha\beta} \partial_\beta \check{m}^{\gamma\nu} \hat{\psi}_\sigma \hat{\psi}_\gamma \hat{\psi}_\nu, \]
and hence that
\[ \bar{\partial}_\alpha \left( \mu \check{A}^{\alpha\beta} \partial_\beta \hat{\psi}_\nu \right) = \bar{\partial}_\alpha \left( \pi^\omega_\nu \check{A}^{\alpha\beta} \partial_\beta \psi_\omega - \frac{\mu}{2} \check{A}^{\alpha\beta} \partial_\beta \check{m}^{\gamma\nu} \hat{\psi}_\sigma \hat{\psi}_\gamma \hat{\psi}_\nu \right). \]
Using this and (4.49), we see that $\hat{\psi}_\nu$ satisfies the wave equation
\[ \bar{\partial}_\alpha \left( \mu \hat{\psi}^\gamma_\mu \check{A}^{\alpha\beta} \partial_\beta \hat{\psi}_\nu + \frac{\mu}{2} \hat{\psi}^\gamma_\nu \check{A}^{\alpha\beta} \partial_\beta \check{m}^{\gamma\nu} \hat{\psi}_\sigma \hat{\psi}_\gamma \hat{\psi}_\nu + \mu \pi^{\mu\nu} \check{L}_\omega \right) = \pi^\nu_\nu \check{F}_\omega + \bar{\partial}_\alpha \pi^\nu_\nu \left( \hat{A}^{\alpha\beta} \partial_\beta \hat{\psi}_\nu + \check{L}_\gamma \right). \]
(4.62)
A similar computation starting from the identity
\[ \check{A}^{\alpha\beta} \partial_\beta \mu \hat{\psi}_\nu = \frac{\psi^\gamma}{\mu} \check{A}^{\alpha\beta} \partial_\beta \psi_\gamma + \frac{\mu}{2} \check{A}^{\alpha\beta} \partial_\beta \check{m}^{\gamma\nu} \hat{\psi}_\sigma \hat{\psi}_\gamma, \]
which holds by (4.60), shows that $\mu$ satisfies the wave equation
\[ \bar{\partial}_\alpha \left( \check{A}^{\alpha\beta} \partial_\beta \mu + \frac{\mu}{2} \check{A}^{\alpha\beta} \partial_\beta \check{m}^{\gamma\nu} \hat{\psi}_\sigma \hat{\psi}_\gamma + \check{\psi}^\gamma \check{L}_\omega \right) = \check{\psi}^\gamma \check{F}_\gamma + \bar{\partial}_\alpha \check{\psi}^\gamma \left( \check{A}^{\alpha\beta} \partial_\beta \check{\psi}_\gamma + \check{L}_\gamma \right). \]
(4.63)
Setting
\[ \Psi = \begin{pmatrix} \check{\psi}_\nu \\ \mu \end{pmatrix}, \]
(4.64)
we combine (4.62) and (4.63) into the single equation
\[ \bar{\partial}_\alpha \left( \check{A}^{\alpha\beta} \partial_\beta \Psi + \check{L}_\alpha \right) = \check{F} \]
(4.65)
where
\[ \check{A}^{\alpha\beta} = \begin{pmatrix} \mu^2 \check{A}^{\alpha\beta} \check{m}^{\mu\nu} & 0 \\ 0 & \check{A}^{\alpha\beta} \end{pmatrix}, \]
(4.66)
\[ \check{L}_\alpha = \begin{pmatrix} \mu \check{m}^{\mu\nu} \left( \frac{\mu}{2} \check{A}^{\alpha\beta} \partial_\beta \check{m}^{\gamma\nu} \hat{\psi}_\sigma \hat{\psi}_\gamma + \pi^\nu_\nu \check{L}_\omega \right) & 0 \\ 0 & -\frac{\mu}{2} \check{A}^{\alpha\beta} \partial_\beta \check{m}^{\gamma\nu} \hat{\psi}_\sigma \hat{\psi}_\gamma + \check{\psi}^\gamma \check{L}_\omega \end{pmatrix}, \]
(4.67)
and
\[ \check{F} = \begin{pmatrix} f_\nu \\ f \end{pmatrix}, \]
(4.68)
with
\[ f_\nu = \mu \check{m}^{\mu\nu} \left( \pi^\nu_\nu \check{F}_\omega + \bar{\partial}_\alpha \pi^\nu_\nu \left( \hat{A}^{\alpha\beta} \partial_\beta \hat{\psi}_\nu + \check{L}_\omega \right) \right) \]
\[ + \bar{\partial}_\alpha \left( \mu \check{m}^{\mu\nu} \left( \check{A}^{\alpha\beta} \partial_\beta \check{\psi}_\nu + \frac{\mu}{2} \check{A}^{\alpha\beta} \partial_\beta \check{m}^{\gamma\nu} \hat{\psi}_\sigma \hat{\psi}_\gamma \hat{\psi}_\nu + \pi^\nu_\nu \check{L}_\omega \right) \right), \]
\[ f = \check{\psi}^\gamma \check{F}_\gamma + \bar{\partial}_\alpha \check{\psi}^\gamma \left( \hat{A}^{\alpha\beta} \partial_\beta \check{\psi}_\gamma + \check{L}_\gamma \right). \]
5. Boundary conditions

By virtue of (4.3), the boundary condition (2.11) is automatically incorporated into the definition of the Lagrangian coordinates (4.1). Therefore, from this perspective the only non-trivial boundary condition is (2.11), or equivalently (5.7), which in Lagrangian coordinates, is given by

\[ \tilde{\gamma}_{00}[\Gamma_T] = -1. \]  

(5.1)

Written this way, there does not seem to be enough boundary conditions to derive energy estimates using our wave formulation. However, as we show below this single boundary condition does, in fact, imply a sufficient set of boundary conditions. For use below, we note that (3.7) implies, by (3.9), (3.27) and (3.35), that

\[ \tilde{f}[\Gamma_T] = -1. \]  

(5.2)

5.1. Neumann boundary conditions. The first set of boundary conditions that we derive from (5.1), or equivalently (3.7) in the Eulerian picture, are of Neumann type, albeit degenerate. The derivation of these boundary conditions begins with the identity

\[ \tilde{\theta}_\alpha^3 g^{\alpha\beta} \nabla_\beta \theta_\nu^0 = \tilde{\theta}_\alpha^3 g^{\alpha\beta} \nabla_\beta \theta_\nu^0 - \sigma_{M_0}^3 L_{\alpha}^3 g^{\alpha\beta} \bar{L}_\beta^3 \theta_\nu^I \]  

(by 3.34)

\[ = g(\theta^3, \nabla_\nu \theta^0) - \sigma_{M_0}^3 L_{\alpha}^3 g(\theta^3, \theta^M) \bar{L}_\beta^3 \theta_\nu^I \]  

\[ = -g(\nabla_\nu \theta^3, \theta^0) - \sigma_{M_0}^3 L_{\alpha}^3 g(\theta^3, \theta^M) \bar{L}_\beta^3 \theta_\nu^I \]  

(by 3.20)

\[ = g^{\alpha\beta} \bar{L}_\alpha^3 \theta_\beta^3 - \sigma_{M_0}^3 L_{\alpha}^3 g(\theta^3, \theta^M) \bar{L}_\beta^3 \theta_\nu^I \]  

(5.3)

Noting that

\[ \tilde{\theta}_\alpha^3 g^{\alpha\beta} = \tilde{\theta}_\alpha^3 g^{\alpha\beta} \]

follows from (3.36), we can write (5.3) as

\[ \tilde{\theta}_\alpha^3 (a^{\alpha\beta}[\nabla_\beta \theta_\nu^0 + \sigma_{M_0}^3 L_{\beta}^3 \theta_\nu^I]) = -\frac{1}{\gamma_{00}} \tilde{\theta}_\alpha^3 \nabla_\nu \theta_\alpha^3. \]  

(5.4)

Transforming this expression into Lagrangian coordinates gives

\[ \nu_\alpha (A^\alpha_{\beta} \tilde{\theta}_{\beta} \tilde{\theta}_\nu^0 + \tilde{L}_\nu^\alpha) \big|_{\Gamma_T} = \frac{\det(J)}{\bar{\gamma}_{00}} \left( \tilde{\theta}_0 \tilde{\theta}_\nu^3 + \beta_\nu^3 \tilde{\theta}_\gamma^3 \right). \]  

(5.5)

where here, we are employing the notation (4.26).

Next, we calculate

\[ |g| \circ \phi = -\det(\tilde{g}_{\mu\nu}) \]

\[ = \det(\tilde{\theta}_0^\nu \tilde{\gamma}_{ij} \theta_\nu^i) \]

\[ = \det(\tilde{\theta}^2 \det(\tilde{\gamma}_{ij}) \]

\[ = \det(\tilde{\theta} J)^2 \left[ \tilde{\gamma}_{00} f((-\tilde{\gamma}_{00})^{-1/2}) \right]^2 \]  

(by 5.26 and 4.21)

\[ = \left( \frac{\det(\tilde{\theta})}{\det(J)} \right)^2 \left[ \tilde{\gamma}_{00} f((-\tilde{\gamma}_{00})^{-1/2}) \right]^2, \]

which after taking the square root, gives

\[ \det(J) = \frac{\det(\tilde{\theta})}{(|g| \circ \phi)^2 \tilde{\gamma}_{00}}, \]  

(5.6)

and allows us to write (5.5) as

\[ \tilde{\theta}_\alpha^3 (A^\alpha_{\beta} \tilde{\theta}_{\beta} \tilde{\theta}_\nu^0 + \tilde{L}_\nu^\alpha) \big|_{\Gamma_T} = \frac{\det(\tilde{\theta})}{|g|^2 \tilde{\gamma}_{00}} \left( \tilde{\theta}_0 \tilde{\theta}_\nu^3 + \beta_\nu^3 \tilde{\theta}_\gamma^3 \right). \]  

(5.7)
We now examine the structure of the righthand side of (5.7). By definition, \((\tilde{\theta}_\mu^\nu)\) is the inverse of \((\tilde{e}_\mu^\nu)\), and so, we must have that
\[
\tilde{\theta}_\mu^\nu = \frac{\text{cof}(\tilde{e})^\nu_\mu}{\det(\tilde{e})}.
\]
But,
\[
\det(\tilde{e}) = \det(J\tilde{e}) = \det(J) \det(\tilde{e}) = \frac{\det(\tilde{\theta}) \det(\tilde{e}) \tilde{\gamma}_{00}\bar{f}}{\sqrt{|\bar{g}|}} \quad \text{(by (5.7))}
\]
and so
\[
\tilde{\theta}_\mu^\nu = \frac{|\bar{g}|^{1/2}}{\tilde{\gamma}_{00}} \text{cof}(\tilde{e})^\nu_\mu.
\]
By definition of the cofactor matrix\(^\footnote{Recall for \(a = (a_{\mu}^\nu) \in \mathbb{M}_{n \times n}\) that}
\[
\text{cof}(\tilde{e})^\nu_\mu = -\epsilon_{\mu\alpha\beta\gamma}\tilde{e}_0^\alpha \tilde{e}_1^\beta \tilde{e}_2^\gamma,
\]
which gives
\[
\tilde{\theta}_\mu^\nu = -\frac{|\bar{g}|^{1}}{2} \epsilon_{\mu\alpha\beta\gamma} \tilde{e}_0^\alpha \tilde{e}_1^\beta \tilde{e}_2^\gamma.
\]
Evaluating this at the boundary, we see, with the help of (5.2), that
\[
\tilde{\theta}_\mu^\nu \big|_{\Gamma_T} = -|\bar{g}|^{1/2} \epsilon_{\mu\alpha\beta\gamma} \tilde{e}_0^\alpha \tilde{e}_1^\beta \tilde{e}_2^\gamma.
\]
Differentiating (5.8) with respect to \(\bar{x}^0\), we find, using (4.9) and (4.10), that
\[
-\partial_0 \tilde{\theta}_\mu^\nu \big|_{\Gamma_T} = \bar{g}^{1/2} (\epsilon_{\mu\alpha\gamma} \tilde{e}_0^\alpha \tilde{e}_2^\gamma + \epsilon_{\alpha\mu\beta} \tilde{e}_0^\alpha \tilde{e}_1^\beta + \epsilon_{\alpha\beta\mu} \tilde{e}_0^\alpha \tilde{e}_1^\beta \tilde{e}_2^\gamma) \partial_\nu \tilde{e}_0^\nu + \frac{1}{2} \frac{\partial |\bar{g}|}{\partial \bar{x}^\nu} \tilde{e}_0^\mu \tilde{e}_0^\nu \tilde{e}_1^\beta \tilde{e}_2^\gamma,
\]
while a short calculation, using (3.32), (4.19) and (5.1), shows that
\[
\bar{e}_a(e_0^\nu) \big|_{\Gamma_T} = -\bar{g}^{\nu\omega} e_a(\tilde{\theta}_0^\nu) - \bar{e}_a(\tilde{e}^\nu) \tilde{\theta}_0^\nu = -\bar{m}^{\nu\omega} \bar{e}_a(\tilde{\theta}_0^\nu) - \bar{e}_a(\tilde{e}^\nu) \tilde{\theta}_0^\nu.
\]
Taken together, (5.7), (5.9), (5.10) and (5.11) imply that
\[
\nu_\alpha (\lambda^{\alpha\beta}\partial_\beta \tilde{\theta}_0^\nu + \tilde{L}_\nu^\alpha) \big|_{\Gamma_T} = \tilde{S}_{\nu\mu} \bar{m}^{\nu\gamma} \tilde{\partial}_\gamma \tilde{\theta}_0^\nu + \tilde{Z}_\nu
\]
where
\[
\tilde{S}_{\nu\mu} \bar{m}^{\nu\gamma} \tilde{\partial}_\gamma \tilde{\theta}_0^\nu = \partial(\tilde{\theta}) (\epsilon_{\mu\alpha\gamma} \tilde{e}_0^\alpha \tilde{e}_1^\beta \tilde{e}_2^\gamma + \epsilon_{\nu\alpha\beta} \tilde{e}_0^\alpha \tilde{e}_1^\beta \tilde{e}_2^\gamma + \epsilon_{\nu\beta\alpha} \tilde{e}_0^\alpha \tilde{e}_1^\beta \tilde{e}_2^\gamma),
\]
and
\[
\tilde{Z}_\nu = \tilde{S}_{\nu\mu} \bar{m}^{\nu\gamma} \tilde{\partial}_\gamma \tilde{\theta}_0^\nu - \partial(\tilde{\theta}) \left( \frac{1}{2} \frac{\partial |\bar{g}|}{\partial \bar{x}^\nu} \tilde{e}_0^\mu \tilde{e}_0^\nu \tilde{e}_1^\beta \tilde{e}_2^\gamma + \beta_{\nu}^{\nu\gamma} \epsilon_{\lambda\alpha\beta\gamma} \tilde{e}_0^\alpha \tilde{e}_1^\beta \tilde{e}_2^\gamma \right).
\]
Setting
\[
\tilde{S}^{\mu\nu} := \tilde{m}^{\nu\alpha} \tilde{S}_{\gamma\alpha} \bar{m}^{\gamma\beta} \tilde{\theta}_0^\nu,
\]
we observe from that \(\tilde{S}^{\mu\nu}\) satisfies
\[
\tilde{S}^{\mu\nu} = -\tilde{S}^{\nu\mu} \quad \text{and} \quad \nu_\nu \tilde{S}^{\mu\nu} = 0.
\]

**Remark 5.1.** Equation (5.12) is the fundamental Neumann boundary condition satisfied by our system. It is important to realize that this Neumann boundary condition is degenerate in the sense that it does not yield coercive elliptic estimates of the type (7.18), and as such, is not directly useful for deriving energy estimates. However, by considering the time differentiated version of these boundary conditions, we show below that this degeneracy can be removed, although it should be noted that the resulting boundary conditions are of acoustic type.
To proceed, we differentiate (5.12) with respect to \( \bar{x}^0 \) to find, after a short calculation, that \( \psi_\nu \) satisfies

\[
\nu_\alpha \left( \hat{A}^{\alpha \beta} \hat{\partial}_\beta \psi_\nu + L^\alpha_\nu \right) |_{\Gamma_T} = \hat{S}_{\nu \mu} \bar{\hat{m}}^\mu \gamma \hat{\partial}_\gamma \psi_\nu + Z_\nu
\]

(5.16)

where

\[
Z_\nu = -\hat{S}_{\nu \mu} \bar{\hat{m}}^\mu \gamma \hat{\partial}_\gamma \lambda_0 \beta_0 + \hat{\partial}_0 \left( \hat{S}_{\nu \mu} \bar{\hat{m}}^\mu \gamma \right) \hat{\partial}_\gamma \beta_0 + \hat{\partial}_0 \hat{Z}_\nu.
\]

From (4.57), (4.59), (4.60), and (5.15), we see also that

\[
\pi^\mu_\nu \hat{S}_{\nu \mu} \bar{\hat{m}}^\mu \gamma \hat{\partial}_\gamma \psi_\nu = \mu \pi_{\nu \beta} \bar{\hat{S}}^{\beta \mu \omega} \pi^\tau_\nu \hat{\partial}_\tau \psi_\nu + \pi_{\nu \beta} \bar{\hat{S}}^{\beta \mu \omega} \hat{\partial}_\omega \psi_\nu \hat{\partial}_\omega \mu - \frac{\mu^2}{2} \pi_{\nu \beta} \bar{\hat{S}}^{\beta \mu \omega} \hat{\partial}_\omega \mu \bar{\hat{m}}^\sigma \lambda \hat{\partial}_\sigma \psi_\lambda + \mu \pi_{\nu \beta} Z_\omega,
\]

(5.17)

and

\[
\hat{\psi}_\nu \bar{\hat{S}}^{\nu \mu \omega} \hat{\partial}_\omega \psi_\gamma = \mu \hat{\psi}_\nu \bar{\hat{S}}^{\nu \mu \omega} \pi^\tau_\nu \hat{\partial}_\tau \psi_\tau.
\]

(5.18)

A straightforward calculation using (5.16)-(5.18) then shows that

\[
\nu_\alpha \left( \mu^2 \hat{A}^{\alpha \beta} \hat{\partial}_\beta \hat{\psi_\nu} + \frac{\mu^2}{2} \hat{A}^{\alpha \beta} \hat{\partial}_\beta \bar{\hat{m}}^\mu \gamma \hat{\partial}_\gamma \psi_\nu + \mu \pi_{\nu \beta} L^\alpha_\nu \right) |_{\Gamma_T} = \mu^2 \pi_{\nu \beta} \bar{\hat{S}}^{\beta \mu \omega} \pi^\tau_\nu \hat{\partial}_\tau \psi_\tau
\]

\[
+ \mu \pi_{\nu \beta} \bar{\hat{S}}^{\beta \mu \omega} \hat{\partial}_\omega \psi_\nu \hat{\partial}_\omega \mu - \frac{\mu^2}{2} \pi_{\nu \beta} \bar{\hat{S}}^{\beta \mu \omega} \hat{\partial}_\omega \mu \bar{\hat{m}}^\sigma \lambda \hat{\partial}_\sigma \psi_\lambda + \mu \pi_{\nu \beta} Z_\omega,
\]

(5.19)

and

\[
\nu_\alpha \left( \hat{A}^{\alpha \beta} \hat{\partial}_\beta \mu \bar{\hat{S}}^{\nu \mu \omega} \hat{\partial}_\omega \psi_\nu \hat{\partial}_\omega \psi_\gamma + \hat{\psi}_\nu L^\alpha_\nu \right) |_{\Gamma_T} = \mu \hat{\psi}_\nu \bar{\hat{S}}^{\nu \mu \omega} \pi^\tau_\nu \hat{\partial}_\tau \psi_\tau + \hat{\psi}_\nu Z_\omega.
\]

(5.20)

Employing the definition (4.64), we collect (5.19)-(5.20) into the single boundary condition

\[
\nu_\alpha \left( A^{\alpha \beta} \hat{\partial}_\beta \Psi + L^\alpha \right) |_{\Gamma_T} = S^\omega \hat{\partial}_\omega \Psi + G
\]

(5.21)

where

\[
S^\omega = \begin{pmatrix}
\mu^2 \pi_{\nu \beta} \bar{\hat{S}}^{\beta \mu \omega} \pi^\tau_\nu \hat{\partial}_\tau \psi_\nu \\
\mu \pi_{\nu \beta} \bar{\hat{S}}^{\beta \mu \omega} \pi^\tau_\nu \hat{\partial}_\tau \psi_\nu
\end{pmatrix}
\]

(5.22)

and

\[
G = \begin{pmatrix}
\mu^2 \pi_{\nu \beta} \bar{\hat{S}}^{\beta \mu \omega} \hat{\partial}_\omega \psi_\nu \hat{\partial}_\omega \mu \\
\mu \pi_{\nu \beta} \bar{\hat{S}}^{\beta \mu \omega} \hat{\partial}_\omega \mu \bar{\hat{m}}^\sigma \lambda \hat{\partial}_\sigma \psi_\lambda
\end{pmatrix}.
\]

(5.23)

Additionally, we note that

\[
(S^\omega)^* = -S^\omega \quad \text{and} \quad \nu_\alpha S^\omega = 0
\]

(5.24)

by (5.16).

5.2. **Acoustic boundary conditions.** The boundary conditions (5.21) for the time differentiated system are still degenerate in the sense of Remark 5.1. The first step in removing this degeneracy is to observe that at the boundary

\[
\omega^0_i |_{B_T} = -\omega^0_{0i} \quad \text{(by (3.7) and (3.12))}
\]

\[
= -\omega^0_{i0} \quad \text{(by (5.16) and (5.19))}
\]

\[
= -\frac{1}{2} \epsilon_i (\gamma_{00}) \quad \text{(by (3.18))}
\]

\[
= -\frac{1}{2} \epsilon_i (\gamma_{00}) \delta^3_i \quad \text{(by (3.7) and (4.19))}.
\]

From this and the connection formula

\[
\nabla e_0 \theta^0_\mu = -\omega^0_0 \theta^i_\mu,
\]

we obtain

\[
\theta^3_1 |_{B_T} = \frac{2}{e_3 (\gamma_{00})} \nabla e_0 \theta^0_\mu,
\]

(5.25)

which is well defined since

\[
e_3 (\gamma_{00}) |_{B_T} \leq c < 0
\]

(5.26)

for some constant \( c \) by (2.14). Clearly, (5.25) implies that

\[
\theta^3_1 |_{B_T} = -\theta^3_1 \frac{\nabla e_0 \theta^0_\mu}{|\nabla e_0 \theta^0_\mu|}_g,
\]
or equivalently, since \( |\nabla_e \theta^0|_g \big|_{B_T} = |\nabla_e \theta^0|_m \) by (3.7) and (4.50),

\[
\theta^\mu_1 \big|_{B_T} = - |\theta^\alpha_0 \big|_g \frac{\nabla_e \theta^0_\mu}{|\nabla_e \theta^0|_m}.
\]  

Using this, we can write (5.4) as

\[
\theta^\alpha_0 \left( \alpha^{\alpha \beta} \left[ \nabla_\beta \theta^0_\nu + \sigma M^0 \theta^M_\beta \theta^L \nu \right] \right) \big|_{B_T} = \frac{1}{\gamma_{00}} \nabla_{e0} \left( |\theta^\alpha_0 \big|_g \frac{\nabla_e \theta^0_\mu}{|\nabla \theta^0|_m} \right).
\]  

We also observe that

\[
e_0 (|\nabla_e \theta^0|_m^2) \big|_{B_T} = \nabla_{e0} \left( g^{\mu \nu} \nabla_e \theta^0_\mu \nabla_e \theta^0_\nu - \frac{\left( \mu^{\mu \nu} \nabla_e \theta^0_\mu \right)^2}{\gamma_{00}} \right) \tag{by (4.50)}
\]

\[
= \nabla_{e0} \left( g^{\mu \nu} \nabla_e \theta^0_\mu \nabla_e \theta^0_\nu - \frac{\left( e_0 \gamma_{00} \right)^2}{4 \gamma_{00}} \right)
\]

\[
= 2 g^{\mu \nu} \nabla_e \theta^0_\mu \nabla_e \theta^0_\nu \frac{\gamma_{00}}{\gamma_{00}} \tag{by (3.7)}
\]

\[
= 2 \mu^{\mu \nu} \nabla_e \theta^0_\mu \nabla_e \theta^0_\nu + \frac{2}{\gamma_{00}} \mu^{\mu \nu} \nabla_e \theta^0_\mu \nabla_e \theta^0_\nu \mu^{\alpha \beta} \theta^0_\alpha \nabla_e \theta^0_\beta
\]

\[
= 2 \mu^{\mu \nu} \nabla_e \theta^0_\mu \nabla_e \theta^0_\nu + \frac{1}{\gamma_{00}} e_0 (\gamma_{00}) \mu^{\alpha \beta} \theta^0_\alpha \nabla_e \theta^0_\beta
\]

\[
= 2 \mu^{\mu \nu} \nabla_e \theta^0_\mu \nabla_e \theta^0_\nu \tag{by (3.7)}
\]  

When expressed in Lagrangian coordinates, (5.28) and (5.29) become

\[
\bar{\theta}^\alpha_0 \left( \bar{A}^{\alpha \beta} \bar{\partial}_\beta \bar{v}_\nu + \bar{L}_\nu^0 \right) \bigg|_{B_T} = - \frac{\det (J)}{\gamma_{00}} D_0 \left( \bar{\theta}^{3}_\beta \bar{v}_\beta \right) - \frac{\det (\bar{\theta})}{|\bar{g}|^{1/2}} \left( \bar{\theta}^{3}_\beta \bar{v}_\beta D_0 \bar{v}_\nu + \bar{\partial}_\beta \bar{\theta}^{3}_\beta \bar{v}_\nu \right),
\]

and

\[
\bar{\partial}_\beta \mu^2 \left|_{B_T} \right. = 2 \bar{\mu}^{\mu \nu} \bar{v}_\mu D_0 \bar{v}_\nu,
\]

respectively, where \( D_0 \) is defined by

\[
D_0 \zeta_\nu = \bar{\partial}_\beta \zeta_\nu + \beta^{3}_\nu \zeta_\lambda.
\]

From (1.68) and (5.31), we then obtain

\[
D_0 \bar{v}_\nu \bigg|_{B_T} = D_0 \left( \frac{1}{\nu} \bar{v}_\nu \right) \bigg|_{B_T} = \frac{1}{\nu} \mu_\nu D_0 \bar{v}_\nu,
\]

which in turn, implies that

\[
\bar{v}_\nu D_0 \bar{v}_\nu \bigg|_{B_T} = 0.
\]  

Next, differentiating (5.30) with respect to \( \bar{v}_\nu \), we see that

\[
\nu_\alpha \left( \bar{A}^{\alpha \beta} \bar{\partial}_\beta \bar{v}_\nu + \bar{L}_\nu^0 \right) \bigg|_{B_T} = D_0 (\alpha D_0 \bar{v}_\nu + \lambda \bar{v}_\nu)
\]

where

\[
\alpha = - \frac{\det (\bar{\theta}) |\bar{\theta}^3|_{\bar{g}}}{|\bar{g}|^{1/2}}, \quad \lambda = - \frac{\det (\bar{\theta}) \bar{\partial}_3 |\bar{\theta}^3|_{\bar{g}}}{|\bar{g}|^{1/2}},
\]

and it is understood that \( \bar{\theta}^3 |_{\bar{g}} \) is calculated using the right hand side of (5.9). From this and (4.57), we then get that

\[
\nu_\alpha \left( \mu^2 \bar{A}^{\alpha \beta} \bar{\partial}_\beta \bar{v}_\nu + \frac{\mu^2}{2} \bar{A}^{\alpha \beta} \bar{\partial}_\beta \bar{m}^{\gamma \nu} \bar{v}_\sigma \bar{v}_\gamma \bar{v}_\nu + \mu \bar{\pi}_{\alpha}^\nu \bar{L}_\nu^0 \right) \bigg|_{B_T} = \mu \bar{\pi}_{\alpha}^\nu D_0 (\alpha D_0 \bar{v}_\nu + \lambda \bar{v}_\nu).
\]  

But for any \( \kappa \in \mathbb{R} \),

\[
\bar{\pi}_{\alpha}^\nu D_0 (\alpha D_0 \bar{v}_\nu + \lambda \bar{v}_\nu) = \alpha \bar{\pi}_{\alpha}^\nu D_0 D_0 \bar{v}_\nu + (\alpha \bar{\partial}_3 + \lambda) \bar{\pi}_{\alpha}^\nu D_0 \bar{v}_\nu
\]

\[
= \alpha \bar{\pi}_{\alpha}^\nu D_0 D_0 \bar{v}_\nu + \left( (\bar{\partial}_3 \alpha + \lambda) \bar{\pi}_{\alpha}^\nu + \kappa \bar{\pi}_{\alpha}^\nu \right) D_0 \bar{v}_\nu \tag{by (5.32)},
\]

and so, we see that

\[
\nu_\alpha \left( \mu^2 \bar{A}^{\alpha \beta} \bar{\partial}_\beta \bar{v}_\nu + \frac{\mu^2}{2} \bar{A}^{\alpha \beta} \bar{\partial}_\beta \bar{m}^{\gamma \nu} \bar{v}_\sigma \bar{v}_\nu + \mu \bar{\pi}_{\alpha}^\nu \bar{L}_\nu^0 \right) \bigg|_{B_T} = \mu \alpha \bar{\pi}_{\alpha}^\nu D_0 D_0 \bar{v}_\nu + (\mu (\bar{\partial}_3 + \lambda) \bar{\pi}_{\alpha}^\nu + \kappa \bar{\pi}_{\alpha}^\nu) D_0 \bar{v}_\nu \tag{by (5.32)},
\]

follows from (5.34).
To proceed, we calculate

\[
D_0 D_0 \psi_\omega = \partial_0^2 \psi_\omega + 2 \beta_0^2 D_0 \psi_\lambda + \partial_0 \beta_0^2 \psi_\lambda - \beta_0^2 \beta_0^2 \psi_\gamma \\
= \partial_0^2 \psi \omega + 2 \beta_0^2 \left( \partial_0^2 + \psi^2 \right) D_0 \psi_\lambda + \partial_0 \beta_0^2 \psi_\lambda - \beta_0^2 \beta_0^2 \psi_\gamma \\
= \partial_0^2 \psi \omega + 2 \beta_0^2 \partial_0^2 D_0 \psi_\beta + \partial_0 \beta_0^2 \psi_\lambda - \beta_0^2 \beta_0^2 \psi_\gamma
\]

(by 5.32).

Using this, we can express (5.35) as

\[
\nu_\alpha \left( \mu^2 \tilde{m}^{\mu \nu} \Lambda^{\alpha \beta} \partial_\beta \psi_\nu + \frac{\mu^2}{2} \Lambda^{\alpha \beta} \partial_\beta \tilde{m}^{\gamma \sigma} \psi_\sigma \psi_\nu + \mu \pi^{\mu \nu} L_\omega^\alpha \right) \bigg|_{\Gamma_T} = q^{\mu \nu} \partial_0^2 \psi_\nu + p^{\mu \nu} \partial_0 \psi_\nu + r^\mu
\]

where

\begin{align*}
q^{\mu \nu} &= \mu \alpha \pi^{\mu \nu}, \\
p^{\mu \nu} &= 2 \mu \alpha \pi^{\mu \nu} \beta_0^2 \pi_{,\lambda}^\nu + \mu (\tilde{\partial} \alpha + \lambda) \tilde{m}^{\mu \nu} + \kappa \psi_\mu \psi_\nu,
\end{align*}

and

\[
r^\mu = \tilde{m}^{\mu \nu} \left[ \mu \alpha \pi_{,\nu} \partial_0 \beta_0^2 \psi_\lambda - \mu \alpha \pi_{,\nu} \beta_0^2 \psi_\lambda + (\mu (\tilde{\partial} \alpha + \lambda) \psi_\nu + \kappa \psi_\mu \psi_\nu) \beta_0^2 \psi_\sigma + 2 \mu \alpha \pi_{,\nu} \beta_0^2 \beta_0^2 \psi_\sigma \right].
\]

Remark 5.2. The boundary conditions (5.36) for the wave equation (4.62) are of a (generalized) acoustic type\footnote{Acoustic boundary conditions were first defined in \cite{5, 21} and further analyzed in \cite{4}. See also \cite{14} for more recent work and relations to Wentzell boundary conditions.}. Although it is not obvious at the moment, these boundary conditions can be used to remove the degeneracy from the Neumann boundary conditions (5.21), and provide effective boundary conditions for the wave equation (4.65). For details, see Lemmas 8.2 and 8.4.

6. THE COMPLETE INITIAL BOUNDARY VALUE PROBLEM

From the above calculations and results, in particular, (5.32), (4.30), (4.51), (4.53), (4.63), (4.65), (5.12), (5.21), and (5.36), it follows from a straightforward calculation that, for any choice of $\kappa, \epsilon, \delta \in \mathbb{R}$, the triple $\{\phi = (\psi^{(\mu)}, \theta^0 = (\bar{\theta}^0)^\mu, \Psi = (\bar{\psi}^{(\mu)}, \mu)^{1/2}\}$ derived from a solution of the Frauenneiner-Walton-Euler equations satisfying the assumptions (A.1)-(A.7) from Section 2 and (1.56) solves the (overdetermined) IBVP:

\begin{align*}
\partial_\alpha \left( B^{\alpha \beta} \partial_\beta \bar{\theta}^0 + M^\alpha \right) &= H \quad \text{in } \Omega_T, \\
\nu_\alpha \left( B^{\alpha \beta} \partial_\beta \bar{\theta}^0 + M^\alpha \right) &\bigg|_{\Gamma_T} = K \quad \text{in } \Gamma_T, \\
\partial_\alpha \left( B^{\alpha \beta} \partial_\beta \Psi + M^\alpha \right) &= H \quad \text{in } \Omega_T, \\
\nu_\alpha \left( B^{\alpha \beta} \partial_\beta \Psi + M^\alpha \right) &\bigg|_{\Gamma_T} = Q \partial_0^2 \Psi + P \partial_0 \Psi + K \quad \text{in } \Gamma_T, \\
\partial_0 \phi^{(\mu)} - \gamma_{0 \alpha} \tilde{\theta}^{(\nu)} \bar{\theta}^0 &= 0 \quad \text{in } \Omega_T, \\
\partial_0 \bar{\theta}^0 + \beta_0 \bar{\theta}^0 &= \bar{\psi}_\mu \quad \text{in } \Omega_T, \\
(\phi^{(0), \phi^{(\gamma)}}) &= (0, \text{id}_{\partial_0}) \quad \text{in } \Omega_0, \\
\bar{\theta}^0 &= \theta^0 \bigg|_{\partial_0} \quad \text{in } \Omega_0, \\
(\Psi, \bar{\theta} \Psi) &= (\Psi_0, \Psi_1) \quad \text{in } \Omega_0.
\end{align*}
where

\[
\Psi_0 = \left( \frac{\nabla \nu_0 \theta_0 |_{\Omega_0}}{|\nabla \nu_0 \theta_0|^2_{\Omega_0}} \right) \right|_{\Omega_0},
\]

\[
\Psi_1 = \left( \frac{\nu_0 (\nabla \nu_0 \theta_0 |_{\Omega_0}) - \nu_0 \theta_0 |_{\Omega_0}}{|\nabla \nu_0 \theta_0|^2_{\Omega_0}} \right) \right|_{\Omega_0},
\]

\[
B^\alpha = \left( \nu_0 (\nabla \nu_0 \theta_0 |_{\Omega_0}) - \nu_0 \theta_0 |_{\Omega_0} \right) (\tilde{\theta}_0 |_{\Omega_0}), \quad \nu_0 := \delta^{\alpha \beta} \nu \beta,
\]

\[
M^\alpha = \left( \nu_0 (\tilde{\theta}_0 |_{\Omega_0}) \tilde{L}_0 \right),
\]

\[
H = 2 \tilde{\theta}_0 (\tilde{S} \nu_0 |_{\Omega_0}) \tilde{\theta}_0 + (\tilde{m}_0 \nu_0 + \epsilon \pi_0) \tilde{F}_0 + \tilde{\theta}_0 (\tilde{m}_0 \nu_0 + \epsilon \pi_0) (\tilde{L}_0 \tilde{\theta}_0 + \tilde{L}_0),
\]

\[
K = \left( \nu_0 \pi_0 (\tilde{\theta}_0 \tilde{\psi}_0 + \beta \tilde{\psi}_0 \tilde{\psi}_0) + \tilde{m}_0 \tilde{Z}_0 \right),
\]

\[
P = \left( \begin{array}{cc}
\delta^\mu_0 & 0 \\
0 & 0 
\end{array} \right),
\]

\[
B^\alpha = (\mathbb{I} + \delta \mathbb{P}) A^\alpha + 2 \nu_0 \tilde{S}^\alpha,
\]

\[
M^\alpha = (\mathbb{I} + \delta \mathbb{P}) \mathbb{L}^\alpha,
\]

\[
H = (\mathbb{I} + \delta \mathbb{P}) \mathbb{F} + 2 \tilde{\theta}_0 (\nu_0 \tilde{S}^\alpha) \tilde{\theta}_0, \quad \nu_0 \tilde{S}^\alpha \tilde{\theta}_0 = 0
\]

\[
Q = \left( \begin{array}{cc}
\delta q_0 & 0 \\
0 & 0 
\end{array} \right),
\]

\[
P = \left( \begin{array}{cc}
\delta p_0 & 0 \\
0 & 0 
\end{array} \right),
\]

\[
K = \mathcal{G} + (\delta \pi_0) \left( \begin{array}{cc}
\delta^\pi_0 & 0 \\
0 & 0 
\end{array} \right),
\]

and all other variables are as previously defined.

It is worthwhile remarking at this point that, for the purposes of deriving energy estimates, we view (6.3)-(6.6) from the above systems as the primary evolution equations, while (6.1)-(6.2) will be treated as an elliptic constraint equation for \( \hat{\theta}_0 \) by using (6.6) to express the time derivatives of \( \hat{\theta}_0 \) in terms of the variables \( \{ \hat{\theta}_0, \phi, \hat{\psi} \} \). We further remark that in the following, it turns out to be convenient to “forget” that that \( \hat{\psi}_0 \) satisfies \( |\hat{\psi}_0|^2 = 1 \). This necessitates redefining \( \pi^\mu_0 \) as

\[
\pi^\mu_0 = \delta^\mu_0 - \frac{\hat{\psi}_0 \hat{\psi}_0}{|\hat{\psi}_0|^2}
\]

so that it remains a projection operator that agrees with the previous definition (4.58) for \( \hat{\psi}_0 \) satisfying \( |\hat{\psi}_0|^2 = 1 \). We also redefine \( p^{\mu \nu} \) by

\[
p^{\mu \nu} = 2 \mu \alpha \pi^{\mu \nu} \beta \hat{\psi}_0 \hat{\psi}_0 + (\tilde{\theta}_0 \alpha + \lambda) \tilde{m}_0 \tilde{q}_0 + \kappa \frac{\hat{\psi}_0 \hat{\psi}_0}{|\hat{\psi}_0|^2}
\]

in order to agree with the previous definition (5.38).

We make the following simple observations:

(i) From the definitions (4.40) and (4.66), see also (4.26), (4.37) and (4.52), it is not difficult to verify that the maps

\[
\hat{A}^\alpha = \hat{A}^\alpha_0(\phi, J), \quad \text{and} \quad A^\alpha = A^\alpha(\phi, J, \mu)
\]

are smooth for \((\phi, J, \mu) \in \mathcal{U} \times \mathbb{R}_{>0}\) where

\[
\mathcal{U} = \{ (\phi, J) \in \mathbb{R}^4 \times M_4 \times 4 \mid \det(J) > 0, \ -g_{\mu \nu}(\phi)J_0^{\mu} J_0^{\nu} > 0, \ s^2 \left( (\nu_0(\phi)J_0^{\mu} J_0^{\nu})^{-1/2} \right) > 0 \}
\]

and moreover, that

\[
\hat{A}^\alpha = \hat{A}^\alpha_0 \quad \text{and} \quad A^\alpha = A^\alpha_0 \text{ tr},
\]
respectively, and for any bounded open subsets $\tilde{U} \subset U$ and $I \subset \mathbb{R}_{>0}$, there exists a constants $c^0_A$, $c^1_A > 0$ such that
\[
\begin{align*}
(\Psi, A^{00}(\phi, J, \mu)\Psi) &\leq -c^0_A |\Psi|^2 \quad \forall \, ((\phi, J), \mu, \Psi) \in \tilde{U} \times I \times \mathbb{R}^5, \\
\tilde{A}^{00}(\phi, J) &\leq -c^1_A \quad \forall \,(\phi, J) \in \tilde{U}.
\end{align*}
\]

Here and below, we use $(\cdot, \cdot)$ and $|\cdot|$ to denote the Euclidean inner product and norm, respectively.

(ii) From (6.25), the antisymmetry conditions (5.15) and (5.24), and the obvious symmetry $\pi^{\mu\nu} = \pi^{\nu\mu}$, it is clear that $B^{\alpha\beta}$ and $B^{\alpha\beta}$ satisfy the symmetry conditions
\[ (B^{\alpha\beta})^{\text{tr}} = B^{\beta\alpha} \quad \text{and} \quad (B^{\beta\alpha})^{\text{tr}} = (B^{\beta\alpha})^{\text{tr}}, \]

respectively.

(iii) Setting
\[ \tilde{\sigma}^0 = (\tilde{\sigma}_1^0(\bar{x}^\Lambda)), \quad f = (f_1^0(\bar{x}^\Lambda)), \quad J_0 = (J_0^\mu), \]

and using
\[ \tilde{D}(\cdot) = (\tilde{D}_1(\cdot), \tilde{D}_2(\cdot), \tilde{D}_3(\cdot)) \quad \text{and} \quad \tilde{\sigma}(\cdot) = (\tilde{\sigma}_0(\cdot), \tilde{D}(\cdot)) \]

to denote the spatial and spacetime gradients, respectively, we see from the definitions (6.13), (6.14), (6.18), (6.19) and the results of the previous sections that the maps
\[ M^\alpha = M^\alpha(\tilde{\sigma}^0, f, \phi, J, \Psi), \]
\[ M^\alpha = \tilde{M}^\alpha(\bar{D}, f, \phi, J, \tilde{\sigma}_0, \Psi), \]
\[ H = \tilde{H}(\tilde{\sigma}^0, f, \bar{D}, f, \phi, J, \tilde{\sigma}_0, \Psi), \]

are well defined and smooth in all variables provided that $f \in \text{GL}(4, \mathbb{R}) \times U \times (\mathbb{R} \times \mathbb{R}^4)$.

(iv) Finally, letting
\[ \tilde{\mathcal{D}}(\cdot) = (\tilde{\mathcal{D}}_1(\cdot), \tilde{\mathcal{D}}_2(\cdot), \tilde{\mathcal{D}}_3(\cdot)) \]

denote the collection of derivatives tangent to the boundary $\Gamma_T$, it is also not difficult to see from the definitions (6.14), (6.22), (6.15) and (6.20)-(6.22), and the results of the previous sections that
\[ \tilde{S}^\alpha = \tilde{S}^{\alpha\mu}(f, \phi, \tilde{\mathcal{D}} \phi), \]
\[ S^\alpha = S^\alpha(f, \phi, \tilde{\mathcal{D}} \phi, \Psi), \]
\[ K = K(f, \phi, \tilde{\mathcal{D}} \phi, \Psi, \tilde{\mathcal{D}} J_0, \Psi), \]
\[ \mathcal{K} = \mathcal{K}(f, \phi, \mathcal{D} \phi, \mathcal{D} J_0, \Psi), \]
\[ P = P(f, \phi, \mathcal{D} \phi, \mathcal{D} J_0, \Psi) \]

and
\[ Q = Q(f, \phi, \mathcal{D} \phi, \Psi) \]

where the maps $K, K, \mathcal{K}, S^\alpha, P$ and $Q$ are smooth in all the variables provided that $f, \Psi \in \text{GL}(4, \mathbb{R}) \times (\mathbb{R} \times \mathbb{R}^4)$.

7. Linear Wave Equations

With our wave formulation complete, we now turn in this section to the problem of establishing the existence, uniqueness, and energy estimates for solutions to linear equations that include equations of the form (6.3)-(6.4).

7.1. Preliminaries. Before proceeding, we first introduce some notation and fix our conventions that will be used throughout this section. Unlike the previous sections, we work here in arbitrary dimensions.

\footnote{Here, we are using the notation $\mathbb{R}^4_+ = \mathbb{R}^4 \setminus \{0\}$.}
7.1.1. Notation. We use \((x^\mu)_{\mu=0}^n\) to denote Cartesian coordinates on \(\mathbb{R}^{n+1}\), and we use \(x^0\) and \(t\), interchangeability, to denote the time coordinate, and \((x^i)_{i=1}^n\) to denote spatial coordinates. We also use \(x = (x^1, \ldots, x^n)\) and \(x = (x^0, \ldots, x^n)\) to denote spatial and spacetime points, respectively.

As before, partial derivatives are denoted by
\[
\partial_\mu = \frac{\partial}{\partial x^\mu},
\]
and we use \(Du(x) = (\partial_1 u(x), \ldots, \partial_n u(x))\) and \(\partial u(x) = (\partial_0 u(x), Du(x))\) to denote the spatial and spacetime gradients, respectively. For time derivatives, we often employ the notation
\[
\gamma_{\ell} = \partial_t^\ell u, \tag{7.1}
\]
and use
\[
\mathbf{u}_r = ((u_0, u_1, \ldots, u_r)^T \tag{7.2}
\]
to denote the collection of partial derivatives of \(u\) with respect to \(t\). We also use the notation \((7.1)\) more generally for vectors with components \(\gamma_{\ell}, 0 \leq \ell \leq r\) that are not necessarily of the type \((7.1)\).

It will be clear from the context whether \(\gamma_{\ell}\) represents some general vector component, or is of the type \((7.1)\).

7.1.2. Function spaces.

Spatial function spaces: In the following, we let \(\Omega\) denote a bounded, open set in \(\mathbb{R}^n\) with a \(C^\infty\) boundary, and employ similar notation when \(\Omega\) is replaced by a smooth, closed \(n\)-dimensional manifold. Given a finite dimensional vector space \(V\), we let \(W^{s,p}(\Omega, V), s \in \mathbb{R}\) \((s \in \mathbb{Z}_{\geq 0}), 1 < p < \infty \left(1 \leq p \leq \infty\right)\), denote the space of \(V\)-valued maps on \(\Omega\) with fractional (integral) Sobolev regularity \(W^{s,p}\). Particular cases of interest for \(V\) will be \(V = \mathbb{R}^N\) and \(V = \mathbb{M}_{N \times N}\), where, here, we use \(\mathbb{M}_{N \times N}\) to denote the set of \(N\) by \(N\) matrices. In the special case of \(V = \mathbb{R}\), we employ the more compact notation \(W^{s,p}(\Omega) = W^{s,p}(\Omega, \mathbb{R})\).

When \(p = 2\), we employ the notation \(H^s(\Omega, V) = W^{s,2}(\Omega, V)\) and on \(L^2(\Omega, \mathbb{R}^N)\), we denote the inner product by
\[
\langle u|v\rangle_\Omega = \int_\Omega (u(x)|v(x)) \, dx, \quad u, v \in L^2(\Omega, \mathbb{R}^N)
\]
where, as previously,
\[
(\xi|\zeta) = \xi^s \zeta, \quad \xi, \zeta \in \mathbb{R}^N,
\]
is the Euclidean inner product on \(\mathbb{R}^N\).

Given \(s = k/2\) for \(k \in \mathbb{Z}_{\geq 0}\), we define the spaces
\[
X^{s,r}(\Omega, V) = \prod_{\ell=0}^r H^{s-\frac{\ell}{2}}(\Omega, V) \tag{7.3}
\]
for \(0 \leq r \leq 2s\), and set
\[
X^s(\Omega, V) = X^{s,2s}(\Omega, V). \tag{7.4}
\]
Using the vector notation \((7.2)\), we can write the norms for the spaces \((7.3)\) and \((7.4)\) as
\[
\|\mathbf{u}_r\|_{X^{s,r}}^2 = \sum_{\ell=0}^r \|\gamma_{\ell}\|_{H^{s-\frac{\ell}{2}}}^2 \quad \text{and} \quad \|\mathbf{u}_{2s}\|_{X^{s,2s}}^2 = \|\mathbf{u}_{2s}\|_{X^{s,2s}}^2,
\]
respectively.

Spacetime function spaces: Given \(T > 0\), and \(s = k/2\) for \(k \in \mathbb{Z}_{\geq 0}\), we define the spaces
\[
X_T^{s,r}(\Omega, V) = \bigcap_{\ell=0}^r W^{\ell,\infty}([0, T], H^{s-\frac{\ell}{2}}(\Omega, V)), \tag{7.5}
\]
for \(0 \leq r \leq 2s\),
\[
X_T^s(\Omega, V) = X_T^{s,2s}(\Omega, V), \tag{7.6}
\]
\[
X_T^{2s-3}(\Omega, V) = \bigcap_{\ell=0}^{2s-3} W^{\ell,\infty}([0, T], H^{s-\frac{m_\ell}{2}}(\Omega, V)), \tag{7.7}
\]
and
\[
X_T^s(\Omega, V) = \bigcap_{\ell=0}^{2s-2} W^{\ell,\infty}([0, T], H^{s-\frac{m_\ell}{2}}(\Omega, V)) \tag{7.8}
\]
where

\[ m_\ell = \begin{cases} 
\ell & \text{if } 0 \leq \ell \leq 2s - 4 \\
2s - 2 & \text{if } \ell = 2s - 3 \\
2s & \text{if } \ell = 2s - 2 
\end{cases} \]

We also define the following energy norms:

\[ \|u\|_{E^{r,s}}^2 = \sum_{\ell=0}^r \|\partial^\ell_t u\|_{H^{s-\frac{m_\ell}{2}}(\Omega)}^2, \quad \|u\|_{E^s}^2 = \|u\|_{E^{r,s}}^2, \]

\[ \|u\|_{E^{2s-3}}^2 = \sum_{\ell=0}^{2s-3} \|\partial^\ell_t u\|_{H^{s-\frac{m_\ell}{2}}(\Omega)}^2 \quad \text{and} \quad \|u\|_{E^{2s}}^2 = \sum_{\ell=0}^{2s-2} \|\partial^\ell_t u\|_{H^{s-\frac{m_\ell}{2}}(\Omega)}^2. \]

In terms of these energy norms, we can write the norms of the spaces (7.5)-(7.8) as

\[ \|u\|_{X^{r,s}} = \sup_{0 \leq t \leq T} \|u(t)\|_{E^{r,s}}, \quad \|u\|_{X^s} = \sup_{0 \leq t \leq T} \|u(t)\|_{E^s}, \]

\[ \|u\|_{X^{2s-3}} = \sup_{0 \leq t \leq T} \|u(t)\|_{E^{2s-3}}, \quad \|u\|_{X^2} = \sup_{0 \leq t \leq T} \|u(t)\|_{E^2}, \]

respectively. Finally, we define the subspace

\[ C^r_T(\Omega, V) = \bigcap_{\ell=0}^{2s-2} C^\ell([0,T], H^{s-\frac{m_\ell}{2}}(\Omega, V)). \]

7.1.3. Estimates and constants. We employ that standard notation

\[ a \lesssim b \]

for inequalities of the form

\[ a \leq Cb \]

in situations where the precise value or dependence on other quantities of the constant \( C \) is not required. On the other hand, when the dependence of the constant on other inequalities needs to be specified, for example if the constant depends on the norms \( \|u\|_{L^\infty(\mathbb{T}^n)} \) and \( \|v\|_{L^\infty(\Omega)} \), we use the notation

\[ C = C(\|u\|_{L^\infty(\mathbb{T}^n)}, \|v\|_{L^\infty(\Omega)}). \]

Constants of this type will always be non-negative, non-decreasing, continuous functions of their arguments.

7.2. A model class of linear wave equations. Rather than directly considering linear wave equations that include equations of the form (6.3)-(6.4), we instead consider a related model class of equations for which it is easier to establish an existence and uniqueness result. The desired existence and uniqueness result will then follow from this one.

The model class that we consider are wave equations of the form:

\[ \partial_\alpha (b^{\alpha\beta} \partial_\beta v + L^\alpha) + \lambda cv = F \quad \text{in } \Omega_T, \]

\[ \nu_\alpha (b^{\alpha\beta} \partial_\beta v + L^\alpha) = g \partial_t^2 v + P \partial_t v + G \quad \text{in } \Gamma_T, \]

\[ (v, \partial_t v) = (\tilde{v}_0, \tilde{v}_1) \quad \text{in } \Omega_0, \]

\[ \partial_t v = \tilde{w}_1 \quad \text{in } \Gamma_0 \]

where

(i) \( \lambda \in \mathbb{R} \),

(ii) \( \Omega \subset \mathbb{R}^n \) is open and bounded with smooth boundary,

(iii) \( \nu_\alpha = \delta_\alpha^i \nu_i \) where \( \nu_i \) is the outward pointing unit normal to \( \partial \Omega \),

(iv) \( v = v(t,x) \) is a \( \mathbb{R}^N \)-valued map,

(v) \( L = (L^\alpha) \in W^{1,2}([0,T], L^2(\Omega, \mathbb{R}^N)) \), \( F \in L^2(\Omega_T, \mathbb{R}^N) \) and \( G \in L^2(\Gamma_T, \mathbb{R}^N) \),

(vi) the matrix valued maps \( c \in L^\infty([0,T], L^p(\Omega, M_{N \times N})) \), \( q \in W^{1,\infty}(\Omega_T, M_{N \times N}) \), and \( P \in W^{1,\infty}(\Omega_T, M_{N \times N}) \)

satisfy

\[ \partial_t q = q, \quad q \leq 0, \]

\[ c \leq -\sigma \]

\[ q^{\sigma} = q, \quad q \leq 0, \]

\[ c \leq -\sigma \]

\[ q^{\sigma} = q, \quad q \leq 0, \]

\[ c \leq -\sigma \]
and
\[ \text{rank}(q) = N_q \]  
(7.15)

for some \( \sigma > 0 \) and \( N_q \in \{0, 1, \ldots, N\} \).

(vii) the matrix valued maps \( b^{\alpha \beta} \in W^{1, \infty}(0, T; L^\infty(\Omega, \mathbb{M}_{N \times N})) \) satisfy
\[ (b^{\alpha \beta})^\tau = b^{\beta \alpha} \]  
(7.16)

and
\[ b^{00} \leq -\kappa_0, \]  
(7.17)

(viii) and there exists constants \( \kappa_1 > 0 \) and \( \mu \geq 0 \) such that
\[ \langle \partial_t v | b^{ij}(t) \partial_j v \rangle_{L^2} \geq \kappa_1 \|v\|_{H^1}^2 - \mu \|v\|_{L^2}^2 \]  
(7.18)

for all \( v \in H^1(\Omega) \) and \( t \in [0, T] \).

Remark 7.1. The coercive condition (7.18) is known to be equivalent to the matrix \( b^{ij} \) being strongly elliptic at each point \( \bar{\Omega} \) and satisfying the strong complementing condition at each point on the boundary \( \partial \Omega \). For a proof of this equivalence, see Theorem 3 in Section 6 of [24].

7.3. Weak solutions.

Definition 7.2. A pair \((v, w) \in H^1(\Omega, \mathbb{R}^N) \times L^2(\Gamma_T, \mathbb{R}^N)\) is called a weak solution of (7.9)–(7.12) if \((v, w)\) define maps \( v : [0, T] \to H^1(\Omega, \mathbb{R}^N) \), \( \partial_t v : [0, T] \to L^2(\Omega, \mathbb{R}^N) \) and \( w : [0, T] \to L^2(\Gamma, \mathbb{R}^N) \) that satisfy
\[ (u(t), \partial_t v(t)) \to (\bar{v}_0, \bar{v}_1) \text{ in } H^1(\Omega, \mathbb{R}^N) \times L^2(\Omega, \mathbb{R}^N), \text{ and } w(t) \to \bar{w} \text{ in } L^2(\Gamma, \mathbb{R}^N) \]

as \( t \to 0 \),
\[ w \in \text{ran}(q) \text{ in } \Gamma_T, \]
\[ \langle qw | \psi \rangle_{\Gamma_T} = -\langle \partial_t qv | \psi \rangle_{\Gamma_T} - \langle qv | \partial_t \psi \rangle_{\Gamma_T} \]

for all \( \psi \in C^1_0([0, T], C^0(\partial \Omega, \mathbb{R}^N)) \), and
\[ \langle b^{\alpha \beta} \partial_{\alpha} v + L^\alpha | \partial_{\alpha} \phi \rangle_{\Omega_T} + \langle (\partial_t q - P) \partial_{\alpha} v | \phi \rangle_{\Gamma_T} - \langle G | \phi \rangle_{\Gamma_T} + \langle qw | \partial_t \phi \rangle_{\Gamma_T} = \langle \lambda c v - F | \phi \rangle_{\Omega_T} \]

for all \( \phi \in C^1_0([0, T], C^1(\bar{\Omega}, \mathbb{R}^N)) \).

Remark 7.3. In the above definition, the condition \( \langle qw | \psi \rangle_{\Gamma_T} = -\langle \partial_t qv | \psi \rangle_{\Gamma_T} - \langle qv | \partial_t \psi \rangle_{\Gamma_T} \) for all \( \psi \in C^1_0([0, T], C^0(\partial \Omega, \mathbb{R}^N)) \) implies that \( v \) weakly satisfies
\[ qw = q \partial_t v \text{ in } \Gamma_T. \]  
(7.19)

Also, as in [17], the boundary term \( \langle (\partial_t q - P) \partial_{\alpha} v | \phi \rangle_{\Gamma_T} \) is defined via the expression
\[ \langle (\partial_t q - P) \partial_{\alpha} v | \phi \rangle_{\Gamma_T} = (\nu(\partial_t q - P) \partial_{\alpha} v - \partial_t (\partial_t q - P) \nu(v) + \partial_{\alpha} \nu^\alpha (\partial_t q - P) \partial_t v | \phi \rangle_{\Omega_T} + \langle (\partial_t q - P) \partial_t v | \nu(\phi) \rangle_{\Omega_T} - \langle (\partial_t q - P) \nu(v) \partial_t | \phi \rangle_{\Omega_T} \]  
(7.20)

where \( \nu(\cdot) = \nu^\alpha \partial_{\alpha}(\cdot) \) and \( \nu^\alpha = \delta^{\alpha \beta} \nu_\beta \). Using (7.19) and (7.20) together with a suitable approximation scheme, it is not difficult to see that a weak solution satisfies
\[ \langle b^{\alpha \beta} \partial_{\beta} v + L^\alpha | \partial_{\alpha} v \rangle_{\Omega_T} + \langle (\partial_t q - P) \partial_{\alpha} v | \phi \rangle_{\Gamma_T} - \langle G | \phi \rangle_{\Gamma_T} + \langle qw | \phi \rangle_{\Gamma_T} = \langle \lambda c v - F | \phi \rangle_{\Omega_T}. \]

When \( q = 0 \) and \( P \leq 0 \), the existence and uniqueness of weak solutions to the IBVP (7.9)–(7.11), in this case \( w = 0 \) and (7.12) is redundant, is a consequence of Theorem 2.2 from [17]. Using similar arguments, we establish the following generalization.

---

10 For sufficiently differentiable vector valued and matrix valued maps \( \{v, \phi\} \) and \( S \), respectively, the identity
\[ 0 = \int_{\Omega_T} \partial_\beta \left[ b^{\beta \alpha}(S v)(\phi) \right] d^\alpha x = \langle \partial_t S v | \phi \rangle_{\Omega_T} + \langle S v | \partial_t \phi \rangle_{\Omega_T} \]
follows from the divergence theorem. This together with one more application of the divergence theorem then yields the identity
\[ \langle S \partial_t v | \phi \rangle_{\Gamma_T} = \int_{\Omega_T} \partial_\beta \left[ b^{\beta \alpha}(S \partial_t v)(\phi) \right] d^\alpha x = \langle \nu(S) \partial_t v - \partial_t S v + \partial_{\alpha} \nu^\alpha S \partial_{\alpha} v | \phi \rangle_{\Omega_T} + \langle S \partial_t v | \nu(\phi) \rangle_{\Omega_T} - \langle S v | \partial_t \phi \rangle_{\Omega_T}. \]
Theorem 7.4. Suppose \( \tilde{v}_0 \in H^1(\Omega, \mathbb{R}^N), \tilde{v}_1 \in L^2(\Omega, \mathbb{R}^N), \tilde{w}_1 \in L^2(\partial\Omega, \mathbb{R}^N) \) with \( \tilde{w}_1(x) \in \text{ran}(q(0,x)) \) for all \( x \in \partial\Omega \), the assumptions (i)-(viii) from Section 7.2 are fulfilled, and
\[
P - \frac{1}{2} \partial_t q - rq \leq 0 \quad \text{in } \Gamma_T
\]
for some \( r \in L^\infty(\Gamma_T) \). Then there exists a unique weak solution \((v, w)\) to the IBVP (7.9)-(7.12), and this solution satisfies the energy estimate
\[
E(t)^{\frac{1}{2}} \leq E(0)^{\frac{1}{2}} + C(\kappa_0, \kappa_1, \mu) \int_0^t \left( 1 + \|\partial_t b(\tau)\|_{L^\infty(\Omega)} + \|r(\tau)\|_{L^\infty(\partial\Omega)} \right)\]
\[
+ \|c(\tau)\|_{L^\infty(\Omega)} \right) E(\tau)^{\frac{1}{2}} + \|\partial_t L(\tau)\|_{L^2(\Omega)} + \|F(\tau)\|_{L^2(\Omega)} + \|G(\tau)\|_{L^2(\partial\Omega)} \, d\tau
\]
for \( 0 \leq t \leq T \) where
\[
E(t) = \frac{1}{2} (\partial_t v(t) - b^{ij}(t) \partial_i v(t))_\Omega - \frac{1}{2} (\partial_t v(t) - b^{ij}(t) \partial_i v(t))_\Omega + \langle \partial_t v(t) | L^i(t) \rangle_\Omega
\]
\[
- \frac{1}{2} (w(t) - q(t)) w(t)_\Omega + \frac{\mu}{2} w(t)^2_\Omega + \frac{1}{4\kappa_0} \|\tilde{L}(t)\|^2_{L^2(\Omega)},
\]
\( \tilde{L} = (L^i), \ b = (b^{ij}), \) and \( E(t) \) satisfies
\[
E(t) \geq \min \left\{ \frac{\kappa_0}{4}, \frac{\kappa_1}{2}, \frac{1}{2} \right\} \|(v(t), w(t))\|^2_E
\]
with \( \|(v(t), w(t))\|^2_E \) given by
\[
\|(v(t), w(t))\|^2_E = \|(v(t))\|_{H^1(\Omega)}^2 + \|\partial_v v\|_{L^2(\Omega)}^2 + \langle w(t) | (-q(t)) w(t) \rangle_{\partial\Omega}.
\]
Moreover, \((v, w) \in \bigcap_{t=0}^{T} C^2([0, T], H^1(\Omega, \mathbb{R}^N)) \times C^0([0, T], L^2(\partial\Omega, \mathbb{R}^N)) \).

Proof. The existence of weak solutions to the IBVP (7.9)-(7.12) can be established using a variation of the Galerkin approximation method employed in [17]; see the proof Theorem 2.2 in [17]. We will not reproduce these details, but instead, we focus on the key result, which is the energy estimate, which we derive for classical solutions. The validity of the energy estimate for weak solutions can be established by adapting the arguments used in proof of Theorem 2.2 from [17]. As usual, uniqueness follows immediately from the energy estimate by applying it to the difference between two weak solutions. The regularity statement \((v, w) \in \bigcap_{t=0}^{T} C^2([0, T], H^1(\Omega, \mathbb{R}^N)) \times C^0([0, T], L^2(\partial\Omega, \mathbb{R}^N)) \) can also be established using a variation of the method employed in the proof of Theorem 2.2 from [17].

We begin the proof of the energy estimate by assuming that \( v \in C^2(\Omega_T) \) is a classical solution of the IBVP (7.9)-(7.12), which in particular, implies that the pair \((v, w := \mathbb{P}_q \partial_\Omega v |_{\Gamma_T})\) is a weak solution where \( \mathbb{P}_q(y), y \in \Gamma_T, \) is the projection onto the range of \( q(y) \). Testing (7.9) with \( \partial_\Omega u \), we obtain, after integrating by parts and applying the divergence theorem,
\[
- \int_{\Omega_t} (\partial_\Omega \partial_t v) b^{ij} \partial_j v + L^\alpha d^{n+1}x + \int_{\partial\Omega_t} n_\alpha (\partial_\Omega v) b^{ij} \partial_j v + L^\alpha dS = \int_{\Omega_t} (\partial_\Omega u) (f) d^{n+1}x
\]
(7.21)
where \( t \in (0, T), \ n_\alpha \) is the outward pointing unit conormal to \( \partial\Omega_t \), \( dS \) is the induced volume form on \( \partial\Omega_t \), and
\[
f = F - \lambda cv.
\]
(7.22)
Using the boundary conditions (7.10) and the symmetry condition (7.10), we can write (7.21) as
\[
\langle \partial_\Omega v(\tau) | b^{ij} \partial_j v(\tau) + L^\alpha(\tau) \rangle_{\Omega_t} = \int_{\tau=0}^{\tau=t} \int_{\Omega_t} \partial_\Omega \partial_\Omega v(\tau) (b^{ij} \partial_j v(\tau))_{\Omega_t} \partial_\Omega u(\tau) (f(\tau))_{\Omega_t} \, d\tau
\]
\[
- \int_{\tau=0}^{\tau=t} \int_{\Omega_t} (\partial_\Omega v(\tau) q(\tau) \partial_\Omega v(\tau) + g(\tau))_{\partial\Omega_t} \, d\tau = \int_{\tau=0}^{\tau=t} \int_{\Omega_t} (\partial_\Omega u(\tau) (f(\tau))_{\Omega_t} \, d\tau
\]
(7.23)
where
\[
g = P \partial_\Omega v + G.
\]
(7.24)
Performing an integration by parts with respect to time in the term \( \int_{\tau=0}^{\tau=t} \langle \partial_\Omega v(\tau) | L^\alpha(\tau) \rangle_{\Omega_t} d\tau \), we obtain
\[
\int_{\tau=0}^{\tau=t} \langle \partial_\Omega v(\tau) | L^\alpha(\tau) \rangle_{\Omega_t} d\tau = \langle \partial_\Omega v(\tau) | L^\alpha(\tau) \rangle_{\Omega_t} \bigg|_{\tau=0}^{\tau=t} - \int_{\tau=0}^{\tau=t} \langle \partial_\Omega v(\tau) | \partial_\Omega L^\alpha \rangle_{\Omega_t} d\tau.
\]
Using this, we can express (7.24) as

\[ J(\tau) |_{\tau=t}^{T} = \int_0^T \frac{1}{2} (\partial_\alpha v(\tau) | \delta b^{\alpha \beta}(t) \partial_\beta v(\tau)) \Omega - (\partial_\alpha v(\tau) | f(\tau)) \Omega + (\partial_\alpha v(\tau) | \partial_\beta L^\alpha) \Omega \, d\tau \]

(7.25)

where

\[ J(t) = (\partial_\alpha v(t) | \frac{1}{2} (b^{\alpha \beta}(t) - \delta^{\alpha \beta} b^{\alpha \beta}(t) - b^{\alpha \beta}(t) \delta^{\alpha \beta}) \partial_\beta v(t)) \Omega + (\partial_\alpha v(t) | L^i(t)) \Omega - \frac{1}{2} (\partial_\alpha v(t) | q(t) \partial_\alpha v(t)) \Omega. \]

Integrating by parts in time, we see that the term \( \int_0^T (\partial_\alpha v(\tau) | \partial_\beta^2 v(\tau) + g(\tau)) \Omega \, d\tau \) can be expressed as

\[ \int_0^T (\partial_\alpha v(\tau) | \partial_\beta^2 v(\tau) + g(\tau)) \Omega \, d\tau = \frac{1}{2} (\partial_\alpha v(t) | q(\tau) \partial_\alpha v(\tau)) |_{\tau=t}^{T} - \frac{1}{2} (\partial_\alpha q(\tau) \partial_\alpha v(\tau) \Omega \, d\tau, \]

which in turn, allows us to write (7.25) as

\[ J(\tau) |_{\tau=t}^{T} = \int_0^T (\partial_\alpha v(\tau) | q(\tau) \partial_\alpha v(\tau) | \Omega - (\partial_\alpha v(\tau) | f(\tau)) \Omega + (\partial_\alpha v(\tau) | \partial_\beta L^\alpha) \Omega \, d\tau. \]

Differentiating this in time then gives

\[ J'(t) = (\partial_\alpha v(t) | g(t) - \frac{1}{2} \partial_\alpha q(t) \partial_\alpha v(t)) \Omega + \frac{1}{2} (\partial_\alpha v(t) | \delta b^{\alpha \beta}(t) \partial_\beta v(t)) \Omega - (\partial_\alpha v(t) | f(t)) \Omega + (\partial_\alpha v(t) | \partial_\beta L^\alpha(t)) \Omega. \]

(7.26)

Next, we observe that

\[ J(t) = \frac{1}{2} (\partial_\alpha v(t) | b^{\alpha \beta}(t) \partial_\beta v(t)) \Omega - \frac{1}{2} (\partial_\alpha v(t) | b^{\alpha \beta}(t) \partial_\beta v(t)) \Omega + (\partial_\alpha v(t) | L(t)) \Omega - \frac{1}{2} (\partial_\alpha v(t) | q(t) \partial_\alpha u) \Omega \]

\[ \geq \frac{\kappa_1}{2} ||v(t)||^2_{H^1(\Omega)} - \frac{\mu}{2} ||v(t)||^2_{L^2(\Omega)} + \frac{\kappa_0}{2} ||\partial_\alpha v(t)||_{L^2(\Omega)} - \kappa_1 ||Dv(t)||^2_{L^2(\Omega)} - \frac{1}{\kappa_1} ||\tilde{L}(t)||^2_{L^2(\Omega)} - \frac{1}{2} (\partial_\alpha v(t) | q(t) \partial_\alpha v(t)) \]

where in deriving the last inequality we used (7.17), (7.18), the Cauchy-Schwartz inequality, and Young’s inequality, and we have set \( \tilde{L} = (L^i). \)

Defining

\[ E(t) = J(t) + \frac{\kappa_0}{2} ||v(t)||^2_{L^2(\Omega)} + \frac{1}{\kappa_1} ||\tilde{L}(t)||^2_{L^2(\Omega)}, \]

it is clear that \( E(t) \) satisfies

\[ E(t) \geq \min \left\{ \frac{\kappa_0}{2}, \frac{\kappa_1}{4}, \frac{1}{2} \right\} ||v(t)||^2_E \]

(7.27)

where

\[ ||v||^2_E = ||v||^2_{H^1(\Omega)} + ||\partial_\alpha v||^2_{L^2(\Omega)} + (\partial_\alpha v | (-q) \partial_\alpha v) \Omega. \]

Differentiating \( E(t) \), we find using (7.26) that

\[ E'(t) = \mu (v(t) | \partial_\alpha v(t))_{L^2(\Omega)} + \frac{2}{\kappa_1} (\tilde{L}(t) | \partial_\alpha \tilde{L}(t))_{L^2(\Omega)} + (\partial_\alpha v(t) | g(t) - \frac{1}{2} \partial_\alpha q(t) \partial_\alpha v(t)) \Omega \]

\[ + \frac{1}{2} (\partial_\alpha v(t) | \delta b^{\alpha \beta}(t) \partial_\beta v(t)) \Omega - (\partial_\alpha v(t) | f(t)) \Omega + (\partial_\alpha v(t) | \partial_\beta L^\alpha(t)) \Omega, \]

and hence, by (7.22), (7.24), (7.27), the Hölder and Sobolev inequalities, see Theorems A.1 and A.2 and the assumption \( P - \frac{1}{2} \partial_\alpha q - r q \leq 0 \) in \( \Gamma_T \), that

\[ E'(t) \leq C(\mu, \kappa_0, \kappa_1) \left[ (1 + ||\partial_\alpha b(t)||_{L^\infty(\Omega)} + ||v(t)||_{L^\infty(\Omega)}) \right. \]

\[ + \left. ||\partial_\alpha L(t)||_{L^2(\Omega)} + ||f(t)||_{L^2(\Omega)} + ||G(t)||_{L^2(\Omega)} \right] E(t)^{\frac{1}{2}}. \]

But, this is equivalent to

\[ \left[ E(t)^{\frac{1}{2}} \right]' \leq C(\mu, \kappa_0, \kappa_1) \left[ (1 + ||\partial_\alpha b(t)||_{L^\infty(\Omega)} + ||v(t)||_{L^\infty(\Omega)}) \right. \]

\[ + \left. ||\partial_\alpha L(t)||_{L^2(\Omega)} + ||f(t)||_{L^2(\Omega)} + ||G(t)||_{L^2(\Omega)} \right] E(t)^{\frac{1}{2}} + \left( ||\partial_\alpha L(t)||_{L^2(\Omega)} + ||f(t)||_{L^2(\Omega)} + ||G(t)||_{L^2(\Omega)} \right) \]}
and so, integrating in time, we obtain the desired energy estimate

\[
E(t)^{\frac{1}{2}} \leq E(0)^{\frac{1}{2}} + C(\mu, \kappa_0, \kappa_1) \int_0^t \left( 1 + \|\partial_b b(\tau)\|_{L^\infty(\Omega)} + \|r(\tau)\|_{L^\infty(\partial \Omega)} \\
+ |\lambda| \|c(\tau)\|_{L^\infty(\Omega)} \right) E(\tau)^{\frac{1}{2}} + \|\partial_b L(\tau)\|_{L^2(\Omega)} + \|F(\tau)\|_{L^2(\Omega)} + \|G(\tau)\|_{L^2(\partial \Omega)} \, d\tau,
\]

which holds for \(0 \leq t \leq T\).

\[\square\]

**Remark 7.5.** Using similar arguments, it is not difficult to show that existence statements and energy estimates from Theorem [7.4] continue to hold for

\[F = f + \tilde{f}v + \tilde{\ell}^{\alpha} \partial_\alpha v, \quad L^\alpha = \ell^\alpha + \tilde{\ell}^\alpha v \quad \text{and} \quad G = g + \tilde{g}v\]

where

\[f \in L^2(\Omega_T, \mathbb{R}^N), \quad \ell^\alpha \in W^{1,\infty}(\Omega_T), \quad g \in L^2(\Delta_0, \mathbb{R}^N)\]

\[\tilde{f} \in L^\infty(\Omega_T), \quad \tilde{\ell} \in W^{1,\infty}(\Omega_T), \quad \tilde{g} \in L^\infty(\Omega_T).\]

More generally, the existence statement and energy estimates from Theorem [7.4] will continue to hold for

\[F = f + \tilde{f}v, \quad L^\alpha = \ell^\alpha + \tilde{\ell}^\alpha v \quad \text{and} \quad G = g + \tilde{g}v\]

where \(f, \ell^\alpha\) and \(g\) are as given above\(^{17}\)

\[\tilde{f} \in L^\infty(\Omega_T), \quad \tilde{\ell} \in W^{1,\infty}(\Omega_T), \quad \tilde{g} \in L^\infty(\Omega_T).\]

### 7.4. The rescaled system.

We rescale in time by letting \(t \rightarrow t/\epsilon, \epsilon > 0\), so that we can introduce a small parameter into our IBVP. Under this rescaling, the IBVP \([7.9] - [7.12]\) transforms according to

\[
\partial_\alpha \left( B^{\alpha\beta} \partial_\beta v + M^\alpha \right) + \lambda \epsilon \epsilon v = F \quad \text{in} \quad \Omega_T,
\]

\[
\nu_\alpha \left( B^{\alpha\beta} \partial_\beta v + M^\alpha \right) = \epsilon^2 \epsilon \partial_\beta^2 v + \epsilon \epsilon P \partial_\beta v + G \quad \text{in} \quad \Gamma_T,
\]

\[
(v, \partial_\nu v) = (\tilde{v}, \partial_\nu \tilde{v}) \quad \text{in} \quad \Omega_0,
\]

\[
\partial_\nu v = \tilde{w}_1 \quad \text{in} \quad \Gamma_0
\]

where

\[
B^{\alpha\beta} = \epsilon^2 \delta_0^0 \delta_0^0 \delta^0 b_{i0} + \epsilon \delta^0 \delta^0 \delta^0 b_{i0} + \epsilon \delta^0 \delta^0 \delta^0 b_{i0} + \delta^0 \delta^0 \delta^0 b_{i0},
\]

\[
M^\alpha = \epsilon \delta^0 \delta^0 \delta^0,
\]

and we have rescaled the final time \(T\) accordingly.

### 7.5. Elliptic estimates.

Formally differentiating \([7.25] - [7.29]\) \(\ell\)-times with respect to \(t\) for \(\ell = 0, 1, \ldots, 2s-2\), we see, employing the notation \([7.8] - [7.11]\), that the \(v, 0 \leq \ell \leq 2s-2\), satisfy

\[
\partial_\alpha \left( b_{ij} \partial_\beta v + \epsilon d_{ij}^\alpha v + L_{ij}^\alpha \right) = \epsilon \epsilon d_{ij}^\alpha v + \lambda \epsilon \epsilon v = F^\alpha \quad \text{in} \quad \Omega_T
\]

\[
\nu_\alpha \left( b_{ij} \partial_\beta v + \epsilon d_{ij}^\alpha v + L_{ij}^\alpha \right) = \epsilon \epsilon d_{ij}^\alpha v + G \quad \text{in} \quad \Gamma_T
\]
where
\[ d_i^\ell = \ell b_i^{00}, \]
\[ a_i^\ell = (1 + \ell) b_i^{0j}, \]
\[ c_\ell = c + \frac{\ell(\ell + 1)}{2\lambda} b_2^{00}, \]
\[ \mathcal{L}_\ell^t = \epsilon b_0^{00} v_{\ell+1} + L_\ell^t + \epsilon b_0^{0i} v_t + \epsilon \sum_{r=1}^{\ell-2} \left( \frac{\ell}{r} \right) b_0^{0r} \partial_r v_{\ell+1} + \sum_{r=1}^{\ell-1} \left( \frac{\ell}{r} \right) b_0^{ij} \partial_j v_r + \sum_{r=1}^{\ell-1} \left( \frac{\ell}{r} \right) b_0^{ij} \partial_j v_r, \]
\[ \mathcal{F}_\ell = \epsilon b_0^{0j} \partial_j v_{\ell+1} + (\ell + 1) b_0^{00} v_{\ell+1} + \epsilon b_0^{00} v_{\ell+2} + F_\ell - \lambda \left( c_\epsilon v_0 + \sum_{r=1}^{\ell-1} \left( \frac{\ell}{r} \right) c_\epsilon v_r \right) - L_\ell^0 - \epsilon \left( b_0^{0j} \partial_j v_0 + \sum_{r=1}^{\ell-1} \left( \frac{\ell}{r} \right) b_0^{ij} \partial_j v_r + \sum_{r=1}^{\ell-2} \left( \frac{\ell}{r} \right) b_0^{ij} \partial_j v_{r+1} \right) - \epsilon^2 \left( b_0^{00} v_{\ell+1} + b_0^{00} v_2 + \sum_{r=1}^{\ell-2} \left( \frac{\ell}{r} \right) b_0^{00} \partial_r v_{\ell+1} + \sum_{r=1}^{\ell-3} \left( \frac{\ell}{r} \right) b_0^{ij} \partial_j v_{r+1} \right), \]
\[ h_\ell = \frac{\ell(\ell + 1)}{2} q_2 + \ell P_1, \]
\[ g_\ell = \epsilon^2 q_0 v_{\ell+2} + \epsilon (\ell q_1 + P_0) v_{\ell+1} + G_\ell + \epsilon^2 q_0 v_2 + \epsilon^2 \sum_{r=1}^{\ell-3} \left( \frac{\ell}{r} \right) q_\ell v_r + \epsilon \sum_{r=1}^{\ell-2} \left( \frac{\ell}{r} \right) P_\ell v_r + \epsilon \sum_{r=1}^{\ell-2} \left( \frac{\ell}{r} \right) P_\ell v_{r+1}. \]

For the purpose of establishing existence and uniqueness results for the IBVP (7.28)-(7.31), it turns out to be useful to interpret (7.32)-(7.33) as a system of elliptic equations, because this will allow us to use elliptic estimates to bound the \( v_\ell \). To prepare for the elliptic estimates, we first estimate the coefficients and sources terms that appear in (7.32)-(7.33) and collect the estimates in the following lemma:

**Lemma 7.6.** Suppose \( s > n/2, \) \( \bar{s} \in [0, s], \) \( 0 \leq \epsilon \leq 1, \) and \( (s, \bar{s}) = (k/2, \tilde{k}/2) \) for \( k, \tilde{k} \in \mathbb{Z}. \) Then the following estimates hold:

(i)
\[ \|G_\ell\|_{H^{s-\frac{3}{2}}(\Omega)} \lesssim \epsilon \left( \|q_0\|_{H^s(\Omega)} \|v_{\ell+2}\|_{H^{s-1-\frac{3}{2}}(\Omega)} + \left( \|q_1\|_{H^s(\Omega)} + \|P_0\|_{H^s(\Omega)} \right) \|v_{\ell+1}\|_{H^{s-1-\frac{3}{2}}(\Omega)} \right), \]

for \( 0 \leq \ell \leq 2\bar{s} - 4, \)
\[ \|G_{2\bar{s}\pm 3}\|_{H^{s-\frac{1}{2}}(\Omega)} \lesssim \epsilon \left( \|q_0\|_{H^s(\Omega)} \|v_{2\bar{s}\pm 1}\|_{H^{s+1}(\Omega)} + \left( \|q_1\|_{H^s(\Omega)} + \|P_0\|_{H^s(\Omega)} \right) \|v_{2\bar{s}\pm 2}\|_{H^{s+1-\frac{3}{2}}(\Omega)} \right), \]

and
\[ \|G_{2\bar{s}\pm 2}\|_{H^{s-\frac{1}{2}}(\Omega)} \lesssim \epsilon \left( \|q_0\|_{L^2(\Omega)} + \left( \|q_1\|_{H^s(\Omega)} + \|P_0\|_{H^s(\Omega)} \right) \|v_{2\bar{s}\pm 1}\|_{H^{s+1}(\Omega)} \right), \]

(ii)
\[ \|L_\ell^t\|_{H^{s-\frac{3}{2}}(\Omega)} \lesssim \|L_\ell^t\|_{H^{s-\frac{3}{2}}(\Omega)} + \|b_\ell\|_{X^{s,\ell}} \|v_{\ell-1}\|_{X^{s+1,\ell-1}}, \]

for \( 0 \leq \ell \leq 2\bar{s} - 4, \) and
\[ \|L_{2\bar{s}\pm 2}\|_{H^{s-\frac{3}{2}}(\Omega)} \lesssim \epsilon \left( \|b_0\|_{H^s(\Omega)} \|v_{2\bar{s}\pm 1}\|_{H^{s+1}(\Omega)} + \|L_{2\bar{s}\pm 2}\|_{H^{s-1-\frac{3}{2}}(\Omega)} + \|b_{2\bar{s}\pm 2}\|_{X^{s,\ell}} \|v_{2\bar{s}\pm 3}\|_{X^{s+1,\ell-1}}, \right) \]
(iii) and
\[ \|F_\ell\|_{H^{s-\frac{1}{2}}(\Omega)} \lesssim \epsilon (\|b_0\|_{H^s(\Omega)} + \|b_1\|_{H^{s-\frac{1}{2}}(\Omega)})(\|v_{\ell+1}\|_{H^{s+\frac{3}{2}}(\Omega)} + \|v_{\ell+2}\|_{H^{s+\frac{5}{2}}(\Omega)}), \]
\[ + \|F_\ell\|_{H^{s-\frac{1}{2}}(\Omega)} + \|L_\ell\|_{H^{s-\frac{1}{2}}(\Omega)} + (\|b_\ell\|_{X^{s,\epsilon}} + \|c_\ell\|_{X^{s,\epsilon}})\|v_{\ell-1}\|_{X^{s+\frac{1}{2},\epsilon-1}} \]
for 0 \leq \ell \leq 2\tilde{s} - 4,
\[ \|F_{2\ell-3}\|_{H^{s-\frac{2}{3}}(\Omega)} \lesssim \epsilon (\|b_0\|_{H^s(\Omega)} + \|b_1\|_{H^{s-\frac{2}{3}}(\Omega)})(\|v_{2\ell-3}\|_{H^{s+\frac{3}{2}}(\Omega)} + \|v_{2\ell-2}\|_{H^{s+\frac{5}{2}}(\Omega)}), \]
\[ + \|F_{2\ell-3}\|_{H^{s-\frac{2}{3}}(\Omega)} + \|L_{2\ell-3}\|_{H^{s-\frac{2}{3}}(\Omega)} + (\|b_{2\ell-3}\|_{X^{s,\epsilon}} + \|c_{2\ell-3}\|_{X^{s,\epsilon}})\|v_{2\ell-2}\|_{X^{s+\frac{1}{2}},2\ell-4} \]
and
\[ \|F_{2\ell-2}\|_{H^{s-\frac{2}{2}}(\Omega)} \lesssim \epsilon (\|b_0\|_{H^s(\Omega)} + \|b_1\|_{H^{s-\frac{2}{2}}(\Omega)})(\|v_{2\ell-2}\|_{H^{s+\frac{3}{2}}(\Omega)} + \|v_{2\ell-1}\|_{L^2(\Omega)} + \|F_{2\ell-2}\|_{H^{s-\frac{2}{2}}(\Omega)} + \|L_{2\ell-2}\|_{H^{s-\frac{2}{2}}(\Omega)} + (\|b_{2\ell-2}\|_{X^{s,\epsilon}} + \|b_{2\ell-2}\|_{L^2(\Omega)} + \|c_{2\ell-2}\|_{X^{s,\epsilon}})\|v_{2\ell-2}\|_{X^{s+\frac{1}{2},2\ell-3}}, \]

Proof. From the assumptions \( s = k/2 > n/2 \) and \( \tilde{s} = \tilde{k}/2 \in [0, s] \), it follows directly from the fractional multiplication inequality, Theorem A3.7, that the estimates
\[ \|q_0v_{\ell+2}\|_{H^{s-\frac{1}{2}}(\Omega)} \lesssim \|q_0\|_{H^s(\Omega)}\|v_{\ell+2}\|_{H^{s+\frac{1}{2}}(\Omega)}, \]
\[ ||(\ell q_1 + P_0)v_{\ell+1}\|_{H^{s-\frac{1}{2}}(\Omega)} \lesssim (\|q_1\|_{H^s(\Omega)} + \|P_0\|_{H^s(\Omega)})\|v_{\ell+1}\|_{H^{s+\frac{1}{2}}(\Omega)}, \]
\[ ||q_0v_{2\ell}\|_{H^{s-\frac{1}{2}}(\Omega)} \lesssim \|q_0\|_{H^s(\Omega)}\|v_{2\ell}\|_{H^{s+\frac{1}{2}}(\Omega)}, \]
\[ ||q_{\ell-r}v_{r+2}\|_{H^{s-\frac{1}{2}}(\Omega)} \lesssim \|q_{\ell-r}\|_{H^{s-\frac{1}{2}}(\Omega)}\|v_{r+2}\|_{H^{s+\frac{1}{2}}(\Omega)}, \]
\[ ||P_{\ell}v_{\ell}\|_{H^{s-\frac{1}{2}}(\Omega)} \lesssim \|P_{\ell}\|_{H^{s-\frac{1}{2}}(\Omega)}\|v_{\ell}\|_{H^{s+\frac{1}{2}}(\Omega)}, \]

and
\[ ||P_{\ell-1}v_{r+1}\|_{H^{s-\frac{1}{2}}(\Omega)} \lesssim \|P_{\ell-1}\|_{H^{s-\frac{1}{2}}(\Omega)}\|v_{r+1}\|_{H^{s+\frac{1}{2}}(\Omega)}, \]
for 0 \leq \ell \leq 2\tilde{s} - 4. Using these estimates, it is clear from (7.40) that \( G_\ell \) can be estimated by
\[ \|G_\ell\|_{H^{s-\frac{1}{2}}(\Omega)} \lesssim \epsilon (\|q_0\|_{H^s(\Omega)}\|v_{\ell+2}\|_{H^{s+\frac{1}{2}}(\Omega)} + (\|q_1\|_{H^s(\Omega)} + \|P_0\|_{H^s(\Omega)})\|v_{\ell+1}\|_{H^{s+\frac{1}{2}}(\Omega)}), \]
\[ + \|G_\ell\|_{H^{s-\frac{1}{2}}(\Omega)} + (\|P_{\ell}\|_{X^{s,\epsilon}} + \|q_{\ell}\|_{X^{s,\epsilon}} + \|q_1\|_{H^s(\Omega)})\|v_{\ell-1}\|_{X^{s+1,\epsilon-1}}, \]
for 0 \leq \ell \leq 2\tilde{s} - 4, which proves the first estimate. The remaining estimates can be established in a similar fashion.

□

Proposition 7.7. Suppose \( s > n/2 + 1, s = k/2 \) for \( k \in \mathbb{Z}_{\geq 0}, m \in \{0, 1, 2, \ldots, 2s - 2\} \), and \((v_{m+1}, v_{m+2})\) satisfy
\[ (v_{m+1}, v_{m+2}) \in \begin{cases} H^{s+1-\frac{m+1}{2}}(\Omega, \mathbb{R}^N) \times H^{s+1-\frac{m+2}{2}}(\Omega, \mathbb{R}^N) \quad &\text{if } 0 \leq m \leq 2s - 4 \\ H^2(\Omega, \mathbb{R}^N) \times H^1(\Omega, \mathbb{R}^N) \quad &\text{if } m = 2s - 3 \end{cases} \]
and
\[ (v_{m+1}, q_0v_{m+2}) \in H^1(\Omega, \mathbb{R}^N) \times L^2(\partial\Omega, \mathbb{R}^N) \quad \text{if } m = 2s - 2. \]

Then there exist constants
\[ \lambda^* = \lambda^*(\sigma, \mu) \geq 1 \quad \text{and} \quad \delta^* = \delta^*(k_1, \|b_0\|_{H^{s-\frac{1}{2}}(\Omega)}, \|q_2\|_{H^{s-\frac{1}{2}}(\Omega)}, \|P_1\|_{H^{s-\frac{1}{2}}(\Omega)}) \geq 1, \]
such that for each \((\lambda, \epsilon) \in [\lambda^*, \infty) \times [0, 1/\delta^*] \) there exists a unique solution
\[ v_m = (v_0, v_1, \ldots, v_m) \in X^{s+1,m}(\Omega, \mathbb{R}^N) \]
to the sequence of equations (7.32)-(7.33) for 0 \leq \ell \leq m. Moreover, this solution satisfies the estimate
\[ \|v_m\|_{X^{s+1,m}} \leq C(\|F_m\|_{X^{s-1,m}} + \|L^0_{m+1}\|_{X^{s-1,m}} + \|G_m\|_{X^{s,m}} + \|L^1_m\|_{X^{s,m}} + \epsilon V_m)) \]
where
\[
V_m = \begin{cases} 
\|v_{m+1}\|_{H^{s+1-\frac{m+1}{2}}(\Omega)} + \|v_{m+2}\|_{H^{s+1-\frac{m}{2}}(\Omega)} & \text{if } 0 \leq m \leq 2s - 4 \\
\|v_{m+1}\|_{H^2(\Omega)} + \|v_{m+2}\|_{H^1(\Omega)} & \text{if } m = 2s - 3 \\
\|v_{m+1}\|_{H^1(\Omega)} + \|v_{m+2}\|_{L^2(\Omega)} + \|q_0v_{m+2}\|_{L^2(\partial\Omega)} & \text{if } m = 2s - 2
\end{cases}
\]

and
\[
C = C(\kappa_1, \mu, \sigma, \lambda, \|b_{2s-2}\|_{X^{s,2s-2}}, \|b_{2s-1}\|_{L^2(\Omega)}, \|c_{2s-2}\|_{X^{s,2s-2}}, \|c_{2s-2}\|_{X^{s,2s-2}}, \|q_{2s-2}\|_{X^{s,2s-2}}, \|q_1\|_{H^s(\Omega)}).
\]

Proof. We use proof by induction.

Base case: From Theorems [3.3] and [3.4], Lemma [7.6] and the Sobolev inequalities, Theorems [A,3] and [A,3] we see that there exists constant\(11\)

\[
\delta^* = \delta^*\left(\kappa_1, \|b_0\|_{H^{s-\frac{1}{2}}(\Omega)}, \|q_2\|_{H^{s-1}(\Omega)}, \|P_1\|_{H^{s-\frac{1}{2}}(\Omega)} \right) \geq 1 \quad \text{and} \quad \lambda^* = \lambda^*(\kappa_1, \mu, \sigma) \geq 1
\]

such that (7.32)-(7.33) has a unique solution \(v_0 = v_0 \in X^{s+1,0}(\Omega, \mathbb{R}) = H^{s+1}(\Omega, \mathbb{R})\) for \(0 \leq \ell + 2s - 3\) and \((\lambda, \epsilon) \in [\lambda^*, \infty) \times [0, \frac{1}{r}]\) that satisfies the bound

\[
\|v_0\|_{X^{s+1,0}} \leq C_0(\|F_0\|_{X^{s-1,0}} + \|P_0\|_{X^{s-1,0}} + \|G_0\|_{X^{s,0}} + \|L_0\|_{X^{s,0}} + \epsilon V_0)
\]

where
\[
C_0 = C_0(\kappa_1, \mu, \sigma, \lambda, \|b_{2s-2}\|_{X^{s,2s-2}}, \|b_{2s-1}\|_{L^2(\Omega)}, \|c_{2s-2}\|_{X^{s,2s-2}}, \|P_{2s-2}\|_{X^{s,2s-2}}, \|q_{2s-2}\|_{X^{s,2s-2}}, \|q_1\|_{H^s(\Omega)}).
\]

Induction hypothesis: With the base case covered, we fix \(m \in \{0, 1, \ldots, 2s - 3\}\) and assume that the system (7.32)-(7.33), \(0 \leq \ell \leq m\), has for \((\lambda, \epsilon) \in [\lambda^*, \infty) \times [0, \frac{1}{r}]\) with \(\delta \geq \delta^*\) a unique solution \(v_m \in X^{s+1,m}(\Omega, \mathbb{R})\) satisfying the bound

\[
\|v_m\|_{X^{s+1,m}} \leq C_m(\|F_m\|_{X^{s-1,m}} + \|P_m\|_{X^{s-1,m}} + \|G_m\|_{X^{s,m}} + \|L_m\|_{X^{s,m}} + \epsilon V_m)
\]

for some constant \(C_m\) of the form (7.41).

Induction step: Appealing again to Theorems [3.3] and [3.4], we see using Lemma [7.6] and the Sobolev inequalities that, for \((\lambda, \epsilon) \in [\lambda^*, \infty) \times [0, \frac{1}{r}]\), the BVP (7.32)-(7.33) with \(\ell = m + 1\) has a unique solution \(v_{m+1} \in H^{s+1-\frac{m+1}{2}}(\Omega, \mathbb{R})\) that satisfies the bound

\[
\|v_{m+1}\|_{H^{s+1-\frac{m+1}{2}}(\Omega)} \leq c_{m+1}(\|F_{m+1}\|_{X^{s-1,m+1}} + \|P_{m+2}\|_{X^{s-1,m+1}} + \|G_{m+1}\|_{X^{s,m+1}} + \|L_{m+1}\|_{X^{s,m+1}} + \epsilon V_{m+1})
\]

where \(c_{m+1}\) is a constant of the form (7.41). Fixing \(\delta \geq 2C_m c_{m+1}\), (7.32) and (7.43) imply that the estimate

\[
\|v_{m+1}\|_{H^{s+1-\frac{m+1}{2}}(\Omega)} \leq C_{m+1}(\|F_{m+1}\|_{X^{s-1,m+1}} + \|P_{m+2}\|_{X^{s-1,m+1}} + \|G_{m+1}\|_{X^{s,m+1}} + \|L_{m+1}\|_{X^{s,m+1}} + \epsilon V_{m+1})
\]

holds for all \((\lambda, \epsilon) \in [\lambda^*, \infty) \times [0, \frac{1}{r}]\) where \(C_{m+1}\) is again a constant of the form (7.41). Combining this estimate with (7.42) yields the desired estimate

\[
\|v_{m+1}\|_{X^{s+1,m+1}} \leq C_{m+1}(\|F_{m+1}\|_{X^{s-1,m+1}} + \|P_{m+2}\|_{X^{s-1,m+1}} + \|G_{m+1}\|_{X^{s,m+1}} + \|L_{m+1}\|_{X^{s,m+1}} + \epsilon V_{m+1}).
\]

\[\square\]

---

\[\text{We fix } \delta^* \text{ to be the maximum of the constants computed by setting } a^i = a^i_1, d^i = d^i_1, \text{ and } h = h_{\ell=1,0} \text{ in Theorem } [3.3] \text{ for } \ell = 0, 1, \ldots, 2s - 2.\]
7.6. Existence and uniqueness for the model problem. The first step of our existence and uniqueness proof for the model problem is to consider the following elliptic-hyperbolic IBVP, which is obtained from the model problem through formally differentiating in time multiple times:

\[
\begin{align*}
\partial_t (b^\ell v^\ell_t + \epsilon c^\ell v^\ell_t + \mathcal{L}^\ell_t) + \epsilon c^\ell \partial_t v^\ell_t + \lambda c_t v^\ell_t &= \mathcal{F}_t & \text{in } \Omega, \\
\nu_t (b^\ell \partial_t v^\ell_t + \epsilon c^\ell v^\ell_t + \mathcal{L}^\ell_t) &= \epsilon \delta_t v_t + \mathcal{G}_t & \text{in } \partial \Omega, \\
\partial_t (B^{\alpha \beta} \partial_\beta v^{2s-1} + \mathcal{M}_{2s-1}^\alpha - \lambda c v^{2s-1}) &= \mathcal{F}_{2s-1} & \text{in } \Omega_T, \\
\nu_t (B^{\alpha \beta} \partial_\beta v^{2s-1} + \mathcal{M}_{2s-1}^\alpha - \lambda c v^{2s-1}) &= \epsilon^2 q \partial_t^2 v^{2s-1} - \epsilon \mathcal{P} \partial_t v^{2s-1} = \mathcal{G}_{2s-1} & \text{in } \Gamma_T, \\
(\nu^{2s-1}_t, \partial_t v^{2s-1}) &= (\tilde{v}_t^{2s-1}, \tilde{v}_t^{2s}) & \text{in } \Omega_0, \\
\mathcal{P}_q \partial_t v^{2s-1} &= \tilde{w}_t^{2s} & \text{in } \Gamma_0.
\end{align*}
\]

where \(0 \leq \ell \leq 2s - 2\), \(\mathbb{P}_q(y), y \in \Gamma_T\), is the projection onto the range of \(q(y)\),

\[
\mathcal{M}_{2s-1}^\alpha = (2s - 1) \partial_t B^{\alpha 0} v^{2s-1} + \sum_{r=0}^{2s-1} \left(\frac{2s - 1}{r}\right) B^{\alpha 0}_{2s-1-r} v^{r+1} + \sum_{r=0}^{2s-2} \left(\frac{2s - 1}{r}\right) B^{\alpha i}_{2s-1-r} \partial_i v_t + M_{2s-1},
\]

\[
\mathcal{P} = P + \epsilon (2s - 1) \partial_t q,
\]

\[
\mathcal{F}_{2s-1} = F_{2s-1} - \lambda \sum_{r=0}^{2s-2} \left(\frac{2s - 1}{r}\right) c_{2s-1-r} v_{r},
\]

\[
\mathcal{G}_{2s-1} = G_{2s-1} + \epsilon \sum_{r=0}^{2s-3} \left(\frac{2s - 1}{r}\right) P_{2s-1-r} v_t + (2s - 1) \partial_t P \partial_t v^{2s-1}
\]

\[
+ \epsilon^2 \sum_{r=0}^{2s-4} \left(\frac{2s - 1}{r}\right) q_{2s-1-r} v_{r+2} + \frac{(2s - 1)(2s - 2)}{2} \partial_t^2 q_{v^{2s-1}},
\]

all other quantities are as previously defined, and we are using the notation \(v^{2s} = \partial_t v^{2s-1}\) and \(\mathbb{P}_q v^{2s} |_{\Gamma_T} = w_{2s}\) while, otherwise, treating the \(v^\ell\), \(0 \leq \ell \leq 2s - 2\), as independent variables.

**Lemma 7.8.** Suppose \(s > n/2, \tilde{s} \in [0, s]\), \(0 \leq \epsilon \leq 1\), and \((s, \tilde{s}) = (k/2, \tilde{k}/2)\) for \(k, \tilde{k} \in \mathbb{Z}_{\geq 0}\). Then the following estimates hold:

\[
\|\mathcal{M}_{2s-1}^\alpha \|_{L^2(\Omega)} \lesssim \|\partial_t^{2s-1} L^1 \|_{L^2(\Omega)} + \|b\|_{E^{s+2s-1}} \left(\|v^{2s-2}\|_{X^{s+1,2s-2}} \|v^{2s-1}\|_{H^s(\Omega)}\right),
\]

\[
\|\partial_t \mathcal{M}_{2s-1}^\alpha \|_{L^2(\Omega)} \lesssim \|\partial_t^{2s} L \|_{L^2(\Omega)} + \left(\|b\|_{E^{s+2s}} \right.
\]

\[
+ \|\partial_t b\|_{H^s(\Omega)} \left(\|v^{2s-2}\|_{X^{s+1,2s-2}} + \|v^{2s-1}\|_{H^s(\Omega)} + \|\partial_t v^{2s-1}\|_{L^2(\Omega)}\right),
\]

\[
\|\mathcal{F}_{2s-1} \|_{L^2(\Omega)} \lesssim \|\partial_t^{2s-1} F \|_{L^2(\Omega)} + \lambda \|c\|_{E^{s+2s}} \|v^{2s-2}\|_{X^{s+1,2s-2}}
\]

and

\[
\|\mathcal{G}_{2s-1} \|_{L^2(\Omega)} \lesssim \|\partial_t^{2s-1} G \|_{L^2(\Omega)} + \epsilon \left(\|P\|_{E^{s+1,2s-1}} \right.
\]

\[
+ \epsilon \|q\|_{E^{s+1,2s-1}} + \|\partial_t^2 q\|_{H^s(\Omega)} \left(\|v^{2s-2}\|_{X^{s+1,2s-2}} + \|v^{2s-1}\|_{H^s(\Omega)}\right).
\]

**Proof.** The proof of this lemma follows from similar arguments used to prove the estimates from Lemma 7.6. We omit the details. \(\square\)

With the preliminary estimates out of the way, we are now ready to prove the existence and uniqueness of solutions to the elliptic-hyperbolic IBVP [7.11]-[7.39].

**Theorem 7.9.** Suppose \(s > n/2 + 1\), \(s = k/2\) for \(k \in \mathbb{Z}\), \(\epsilon_0 > 0\),

\[
\begin{align*}
b^{\alpha \beta} &\in X_T^s(\Omega, \mathbb{M}_{\mathbb{N} \times \mathbb{N}}), \quad c \in X_T^{s,2s-1}(\Omega, \mathbb{M}_{\mathbb{N} \times \mathbb{N}}), \quad P, q \in X_T^{s+1,2s-1}(\Omega, \mathbb{M}_{\mathbb{N} \times \mathbb{N}}), \\
F, \partial_t F, L^0, \partial_t L^0, \partial_t^2 L^0 &\in X_T^{s+1,2s-2}(\Omega, \mathbb{R}^N), \quad L^1, \partial_t L^1, G, \partial_t G \in X_T^{s+2s-2}(\Omega, \mathbb{R}^N), \\
\partial_t b^{\alpha \beta}, \partial_t q, \partial_t^2 q &\in L^\infty([0, T], H^s(\Omega, \mathbb{M}_{\mathbb{N} \times \mathbb{N}})), \quad r \in L^\infty([0, T], H^s(\Omega)), \\
\partial_t^{2s-1} L^i, \partial_t^{2s+1} L^i &\in L^\infty([0, T], L^2(\Omega, \mathbb{R}^N)), \quad \partial_t^{2s-1} G |_{\partial \Omega} \in L^\infty([0, T], L^2(\partial \Omega, \mathbb{R}^N)),
\end{align*}
\]
the coefficients \(\{b^{\alpha}, c, q\}\) satisfy (7.13) for constants \(\kappa_0, \kappa_1, \sigma > 0\) and \(\mu \geq 0\), \(\epsilon P + \epsilon^2 (2s - \frac{3}{2})\partial q - r q \leq 0\) in \(G_T\) for \(0 < \epsilon \leq \epsilon_0\), \(q^2 + \gamma q \leq 0\) in \(G_T\) for some \(\gamma > 0\), \(\tilde{v}_{2s-1} \in H^1(\Omega, \mathbb{R}^N), \tilde{v}_{2s} \in L^2(\Omega, \mathbb{R}^N)\), and \(\tilde{w}_{2s} \in L^2(\partial \Omega, \mathbb{R}^N)\) satisfies \(P \tilde{w}_{2s} = \tilde{w}_{2s}\). Then there exist constants \(\lambda^* = \lambda^*(\sigma, \mu) \geq 1\) and

\[
\delta^* = \delta^*(\kappa_1, \sup_{0 \leq t \leq T} \|\partial_t b(t)\|_{H^{-\frac{1}{2}}(\Omega)}, \sup_{0 \leq t \leq T} \|\partial_t^2 q(t)\|_{H^{-1}(\Omega)}, \sup_{0 \leq t \leq T} \|\partial_t P(t)\|_{H^{-\frac{1}{2}}(\Omega)}) \geq 1,
\]

such that for each \((\lambda, \epsilon) \in [\lambda^*, \infty) \times (0, \frac{1}{\delta^*}]\) there exists a unique solution

\[
(v_{2s-2}, v_{2s-1}, w_{2s}) \in C^0([0, T], X^{s+1,2s-2}(\Omega, \mathbb{R}^N)) \times \prod_{j=0}^1 C^j([0, T], H^{1-j}(\Omega, \mathbb{R}^N)) \times C([0, T], L^2(\partial \Omega, \mathbb{R}^N)),
\]

to (7.44)-(7.45). Moreover, \((v_{2s-2}, v_{2s-1}, w_{2s})\) satisfies the energy estimate

\[
\|(v_{2s-2}(t), v_{2s-1}(t), w_{2s}(t))\|_{s+1} \leq C \left(\|(v_{2s-2}(0), v_{2s-1}(0), w_{2s}(0))\|_{s+1} + \alpha_0\right)
+
\int_0^t \alpha_1(\tau) \|(v_{2s-2}(\tau), v_{2s-1}(\tau), w_{2s}(\tau))\|_{s+1} + \alpha_2(\tau) d\tau
\]

where

\[
\|(v_{2s-2}(t), v_{2s-1}(t), w_{2s}(t))\|_{s+1} = \|v_{2s-2}(t)\|_{X^{s+1,2s-2}} + \|v_{2s-1}(t)\|_{X^{s+1,2s-2}} + \|w_{2s}(t)\|_E,
\]

\[
\alpha_0 = \|L'(0)||_{E^{s,2s-2}} + \|F(0)|_{E^{s,2s-2}} + \|\partial_t L'(0)|_{E^{s,2s-2}} + \|G(0)|_{E^{s,2s-2}},
\]

\[
\alpha_1(t) = 1 + \|b(t)|_{E^{s,2s-2}} + \|\partial_t b(t)|_{H^{1}(\Omega)} + \|c(t)|_{E^{s,2s-1}} + \|P(t)|_{E^{s+1,2s-1}} + \|q(t)|_{E^{s+1,2s-1}} + \|\partial_t^2 q(t)|_{H^{0}(\Omega)}
\]

\[
\alpha_2(t) = \|\partial_t F(t)|_{E^{s,2s-2}} + \|\partial_t L'(t)|_{E^{s,2s-2}} + \|\partial_t^2 L'(t)|_{E^{s,1,2s}} + \|\partial_t^2 L(t)|_{E^{s+1,2s-3}} + \|\partial_t^2 L(t)|_{L^2(\Omega)} + \|\partial_t G(t)|_{E^{s,2s-2}} + \|\partial_t^2 G(t)|_{E^{s+1,2s}} \cdot \|\partial_t^2 G(t)|_{L^2(\partial \Omega)} \cdot \|\partial_t^2 G(t)|_{L^2(\partial \Omega)},
\]

and

\[
C = C(\kappa_0, \kappa_1, \mu, \sigma, \gamma, \lambda, \rho)
\]

with

\[
\rho = \|b|_{X^{s,2s-2}} + \|c|_{X^{s,2s-2}} + \|P|_{X^{s,2s-2}} + \|q|_{X^{s,2s-2}} + \sup_{0 \leq t \leq T} \left(\|\partial_t^2 b(t)|_{L^2(\Omega)} + \|\partial_t q(t)|_{H^0(\Omega)}\right).
\]

Proof. Given \(v_{2s-1}(t) \in H^1(\Omega, \mathbb{R}^N), \partial_t v_{2s-1}(t) \in L^2(\Omega, \mathbb{R}^N)\), and \(w_{2s}(t) \in L^2(\partial \Omega, \mathbb{R}^N)\) with \(\text{ran}(w_{2s}(t)) \subseteq \text{ran}(q(t))\), it follows from Proposition (7.7) that there exists constants \(\lambda^* = \lambda^*(\sigma, \mu) \geq 1\) and

\[
\delta^* = \delta^*(\kappa_1, \sup_{0 \leq t \leq T} \|\partial_t b(t)\|_{H^{-\frac{1}{2}}(\Omega)}, \sup_{0 \leq t \leq T} \|\partial_t^2 q(t)\|_{H^{-1}(\Omega)}, \sup_{0 \leq t \leq T} \|\partial_t P(t)\|_{H^{-\frac{1}{2}}(\Omega)}) \geq 1,
\]

such that for each \((\lambda, \epsilon) \in [\lambda^*, \infty) \times (0, \frac{1}{\delta^*}]\), there exists a unique solution \(v_{2s-2}(t) \in X^{s+1,2s-2}(\Omega, \mathbb{R}^N)\)

\[
\|(v_{2s-2}(t))_{X^{s+1,2s-2}} \leq C(\kappa_0, \mu, \sigma, \gamma, \lambda, \rho) \left(\|F(t)|_{E^{s,1,2s-2}} + \|\partial_t L'(t)|_{E^{s,1,2s-2}} + \|L'(t)|_{E^{s,1,2s-2}} + \|\partial_t^2 L(t)|_{E^{s,1,2s-2}} + \epsilon \|(v_{2s-1}(t), w_{2s}(t))|_E\right)
\]

(7.50)

where

\[
\rho = \|b|_{X^{s,2s-2}} + \|c|_{X^{s,2s-2}} + \|P|_{X^{s,2s-2}} + \|q|_{X^{s,2s-2}} + \sup_{0 \leq t \leq T} \left(\|\partial_t^2 b(t)|_{L^2(\Omega)} + \|\partial_t q(t)|_{H^0(\Omega)}\right).
\]

By virtue of the above estimate, Lemma (7.8) and Theorem (7.4) it follows that there exists a unique weak solution

\[
(v_{2s-1}, w_{2s}) \in \prod_{j=0}^1 C^j([0, T], H^{1-j}(\Omega, \mathbb{R}^N)) \times C([0, T], L^2(\partial \Omega, \mathbb{R}^N))
\]

(7.51)

of the IBVP defined by (7.47)-(7.48), and furthermore, that this solution satisfies the energy estimate

\[
\|(v_{2s-1}(t), w_{2s}(t))|_E \leq C \left(\|(v_{2s-1}(0), w_{2s}(0))|_E + \int_0^t \alpha_1(\tau) \|(v_{2s-2}(\tau), v_{2s-1}(\tau), w_{2s}(\tau))|_{s+1} + \beta_1(\tau)\right)
\]

(7.52)

\[\text{Note that } q^2 + \gamma q \leq 0 \text{ and } q^{12} = q \text{ imply that } \|qw_{2s}\|_{L^2(\partial \Omega)} \leq \gamma^{\frac{1}{2}}(w_{2s}(q^2))^{\frac{1}{2}}\]
where \( C = C(\kappa_0, \kappa_1, \sigma, \gamma, \mu, \lambda, \rho) \),
\[
\beta_1(t) = \| \partial_t^{2s-1}F(t) \|_{L^2(\Omega)} + \| \partial_t^{2s-1}L^i(t) \|_{L^2(\Omega)} + \| \partial_t^{2s-1}L(t) \|_{L^2(\Omega)} + \| \partial_t^{2s-1}G|_{\partial \Omega}(t) \|_{L^2(\partial \Omega)}
\]
and
\[
\| (v_{2s-2}(t), v_{2s-1}(t), w_{2s}(t)) \|_{s+1} = \| (v_{2s-2}(t))_{X^{s+1,2s-2}} + \| (v_{2s-1}(t), w_{2s}(t)) \|_{E}.
\]
Since the linear map \( H^1(\Omega, \mathbb{R}^N) \times L^2(\partial \Omega, \mathbb{R}^N) \times L^2(\partial \Omega, \mathbb{R}^N) \in (v_{2s-1}, \partial_t v_{2s-1}, w_{2s}) \rightarrow v_{2s-2} \in X^{s,2s-2}(\Omega, \mathbb{R}^N) \) is bounded, it is also clear from (7.51) that \( v_{2s-2} \in C^0([0, T], X^{s,2s-2}(\Omega, \mathbb{R}^N)) \).

Next, using the integral representation \( F(t) = F(0) + \int_0^t F(\tau) \, d\tau \), we can estimate \( F(t) \) by
\[
\| F(t) \|_{E^{s-1,2s-1}} \leq \| F(0) \|_{E^{s-1,2s-1}} + \int_0^t \| \partial_t F(\tau) \|_{E^{s-1,2s-1}} \, d\tau.
\]
From this and similar estimates for \( G(t) \) and \( L(t) \), it then follows from (7.50) that
\[
\| v_{2s-2}(t) \|_{X^{s+1,2s-2}} \leq C(\kappa_1, \mu, \sigma, \gamma, \lambda, \rho) \left( \alpha_0 + \int_0^t \| \partial_t F(\tau) \|_{E^{s-1,2s-1}} + \| \partial_t^2 L^0(\tau) \|_{E^{s-1,2s-2}} + \| \partial_t G(\tau) \|_{E^{s-1,2s-2}} + \| \partial_t L^i(\tau) \|_{E^{s,2s-2}} d\tau + \epsilon \| (v_{2s-1}(t), w_{2s}(t)) \|_E \right)
\]
where
\[
\alpha_0 = \| L^i(0) \|_{E^{s,2s-2}} + \| F(0) \|_{E^{s-1,2s-1}} + \| \partial_t L^0(0) \|_{E^{s-1,2s-2}} + \| G(0) \|_{E^{s,2s-2}}.
\]
Finally, combining the two estimates (7.52) and (7.53) yields the energy estimate
\[
\| (v_{2s-2}(t), v_{2s-1}(t), w_{2s}(t)) \|_{s+1} \leq C \left( \| (v_{2s-2}(0), v_{2s-1}(0), w_{2s}(0)) \|_{s+1} + \alpha_0 \right)
+ \int_0^t \alpha_1(\tau) \| (v_{2s-2}(\tau), v_{2s-1}(\tau), w_{2s}(\tau)) \|_{s+1} + \alpha_2(\tau) \, d\tau)
\]
where
\[
\alpha_2(\tau) = \| \partial_t F(\tau) \|_{E^{s-1,2s-2}} + \| \partial_t L^i(\tau) \|_{E^{s,2s-2}} + \| \partial_t^2 L^0(\tau) \|_{E^{s-1,2s-2}}
+ \| \partial_t^2 L(t) \|_{L^2(\Omega)} + \| \partial_t G(\tau) \|_{E^{s-1,2s-2}} + \| \partial_t^{2s-1}G|_{\partial \Omega}(t) \|_{L^2(\partial \Omega)}.
\]
and the proof is complete.

In order to go from solutions of (7.44)-(7.49) to solutions of (7.28)-(7.31), we need to ensure that the initial data
\[
(v, \partial_t v)_{t=0} = (\tilde{v}_0, \tilde{v}_1) \in H^{s+1}(\Omega, \mathbb{R}^N) \times H^{s+\frac{1}{2}}(\Omega, \mathbb{R}^N)
\]
satisfies the compatibility conditions given by
\[
\tilde{v}_\ell := \partial_t^\ell v|_{t=0} \in H^{s+1-\frac{\ell}{2}}(\Omega, \mathbb{R}^N), \quad 2 \leq \ell \leq 2s,
\]
and
\[
w_{2s} := \mathbb{P}_q \partial_t^{2s} v|_{t=0} \in L^2(\partial \Omega, \mathbb{R}^N).
\]
Here, the time derivatives \( \partial_t^\ell v|_{t=0} \) \( \ell \geq 2 \) are generated from the initial data (7.54) by formally differentiating (7.28) with respect to \( t \) and evaluating at \( t = 0 \).

**Corollary 7.10.** Suppose \( s > n/2 + 1, \) \( s = k/2 \) for \( k \in \mathbb{Z}, \) \( \epsilon_0 > 0, \)
\[
\begin{align*}
&b^\alpha \in X^s_T(\Omega, \mathbb{M}_{N \times N}), \quad c \in X^{s,2s-1}_T(\Omega, \mathbb{M}_{N \times N}), \quad P, q \in X^{s+\frac{1}{2},2s-1}_T(\Omega, \mathbb{M}_{N \times N}), \\
&F, \partial_t F, L^0, \partial_t L^0, \partial_t^2 L^0 \in X^{s-1,2s-2}_T(\Omega, \mathbb{R}^N), \quad L^i, \partial_t L^i, G, \partial_t G \in \mathcal{C} X^{s,2s-2}_T(\Omega, \mathbb{R}^N), \\
&\partial_t b^\alpha, \partial_t q, \partial_t^2 q \in L^\infty([0, T], H^s(\Omega, \mathbb{M}_{N \times N})), \quad \tau \in L^\infty([0, T], H^s(\Omega)), \\
&\partial_t^{2s-1} L^i, \partial_t^{2s-1} L^i \in L^\infty([0, T], L^2(\mathbb{R}^N)), \quad \partial_t^{2s-1} G|_{\partial \Omega} \in L^\infty([0, T], L^2(\partial \Omega, \mathbb{R}^N)),
\end{align*}
\]
the coefficients \( \{ b^\alpha, c, q \} \) satisfy (7.13), (7.18) for constants \( \kappa_0, \kappa_1, \sigma > 0 \) and \( \mu \geq 0, \) \( \epsilon P + \epsilon^2(2s - \frac{3}{2})\partial_t q - r q \leq 0 \) in \( \Gamma_T \) for \( 0 < \epsilon \leq \epsilon_0, \) \( \epsilon^2 q + \gamma q \leq 0 \) in \( \Gamma_T \) for some \( \gamma > 0, \) and the initial data
\((\tilde{v}_0, \tilde{v}_1) \in H^{s+1}(\Omega, \mathbb{R}^N) \times H^{s+\frac{1}{2}}(\Omega, \mathbb{R}^N)\) satisfies the compatibility conditions \((7.55)-(7.56)\) for \(0 < \epsilon \leq \epsilon_0\). Then there exist constants \(\lambda^* = \lambda^*(\sigma, \mu) \geq 1\) and

\[
\delta^* = \delta^* \left( \kappa_1, \sup_{0 \leq t \leq T} \| \partial_t^\epsilon \tilde{v}_0(t) \|_{H^{s+\frac{1}{2}}(\Omega)} + \sup_{0 \leq t \leq T} \| \partial_t^\epsilon \tilde{g}(t) \|_{H^{s+\frac{1}{2}}(\Omega)} + \sup_{0 \leq t \leq T} \| \partial_t \tilde{P}(t) \|_{H^{s+\frac{1}{2}}(\Omega)} \right) \geq 1,
\]

such that for each \((\lambda, \epsilon) \in [\lambda^*, \infty) \times (0, \frac{1}{\delta^*})\) there exists a unique solution \(v \in C^\lambda(\Omega, \mathbb{R}^N)\) to the IBVP \((7.28)-(7.31)\). Moreover, there exists a map \(w_{2s} \in C([0, T], L^2(\Omega, \mathbb{R}^N))\) such that \((\partial_t^{2s-1} v, w_{2s})\) is a weak solution of the linear wave equations obtained by differentiating \((7.28)-(7.29)\) \((2s-1)\)-times with respect to \(t\), and the pair \((v, w_{2s})\) satisfy the energy estimate

\[
\|(v(t), w_{2s}(t))\|_{s+1} \leq C \left( \|(v(0), w_{2s}(0))\|_{s+1} + \alpha_0 \right)
\]

\[
+ \int_0^t \alpha_1(\tau) \|(v(\tau), w_{2s}(\tau))\|_{s+1} + \alpha_2(\tau) \, d\tau
\]

where \(C = C(\kappa_0, \kappa_1, \sigma, \gamma, \mu, \lambda, \rho, \rho\) and the \(\alpha_i\) are as defined above in Theorem 7.9 and

\[
\|(v(t), w_{2s}(t))\|_{s+1} = \|(v(t))\|_{E^{s+1,2s-2}} + \|w_{2s}(t))\|_{E}.
\]

Proof. Given initial data \((\tilde{v}_0, \tilde{v}_1) \in H^{s+1}(\Omega, \mathbb{R}^N) \times H^{s+\frac{1}{2}}(\Omega, \mathbb{R}^N)\) satisfying the compatibility conditions \((7.55)-(7.56)\), we let

\[
(\mathbf{v}_{2s-2}, \mathbf{v}_{2s-1}, \mathbf{w}_{2s}) \in C^0([0, T], X^{s+1,2s-2}(\Omega, \mathbb{R}^N)) \times \bigcap_{j=0}^1 C^j([0, T], H^{1-j}(\Omega, \mathbb{R}^N)) \times C([0, T], L^2(\partial \Omega, \mathbb{R}^N)),
\]

denote the unique solution to \((7.44)-(7.49)\), which we know exists for \((\lambda, \epsilon) \in [\lambda^*, \infty) \times (0, \frac{1}{\delta^*})\) by Theorem 7.9.

To proceed, we assume that the \(v_\ell(t), \ell = 0, \ldots, 2s-2\), are differentiable in time and satisfy \(\partial_\ell v_\ell(t) \in H^{s+1-\frac{2s-1}{2}}(\Omega, \mathbb{R})\). This assumption can be justified by replacing the derivative \(\partial_\ell v_\ell\) by the difference quotient \(\Delta_\ell v_\ell(t, x) = h^{-1} \left( \Delta_\ell v_\ell(t+h,x) - \Delta_\ell v_\ell(t,x) \right)\) and sending \(h \rightarrow 0\) at the end of the computation. Under the differentiability assumption, a straightforward calculation shows that the differences \(v_\ell := \partial_\ell v_\ell - v_\ell, \ell = 1, \ldots, 2s-1\), define a weak solution to a collection of elliptic equations of the form

\[
\partial_\ell (\partial^{\ell+1}_1 \partial_1 v_\ell + \epsilon_1 \partial^{\ell+1}_2 v_\ell + \lambda_\ell v_\ell = \mathbf{f}_\ell) \quad \text{in} \Omega, \quad (7.57)
\]

\[
\nu_\ell (\partial^{\ell+1}_1 \partial_1 v_\ell + \epsilon_1 \partial^{\ell+1}_2 v_\ell + \lambda_\ell v_\ell = \mathbf{g}_\ell) \quad \text{in} \partial \Omega, \quad (7.58)
\]

where the source terms \(\mathbf{f}_\ell, \mathbf{g}_\ell, \mathbf{f}_\ell, \mathbf{g}_\ell, \mathbf{f}_\ell, \mathbf{g}_\ell\), and \(\mathbf{f}_\ell, \mathbf{g}_\ell\) are homogenous in the variables \(v_\ell\) and satisfy homogeneous versions of the estimates from Lemma 7.6 that is, estimates that arise from making the replacements:

\[
(G_\ell, L_\ell, F_\ell) \mapsto (\mathbf{f}_\ell, \mathbf{g}_\ell, \mathbf{f}_\ell) \quad \text{and} \quad (G_\ell, L_\ell, F_\ell) \mapsto (0, 0, 0) \quad \text{for} \ 1 \leq \ell \leq 2s-2,
\]

\[
\|\mathbf{f}_\ell\|_{L^2(\Omega)} + \|\mathbf{g}_\ell\|_{L^2(\Omega)} + \|\mathbf{f}_\ell\|_{L^2(\Omega)} + \|\mathbf{g}_\ell\|_{L^2(\Omega)} \leq \|v_\ell\|_{X^{s+1,2s-2}} + \|v_{2s-1}\|_{H^1(\Omega)}
\]

for \(\ell = 2s-1\). By Proposition 7.7 and Theorem 5.9, we know that solutions to \((7.57)-(7.58)\) are unique for \((\lambda, \epsilon) \in [\lambda^*, \infty) \times (0, \frac{1}{\delta^*})\), and so, we conclude that the trivial solution, given by \(v_\ell = 0\) for \(1 \leq \ell \leq 2s-1\), is the unique solution. From this, we see that \(\partial_\ell v_\ell = v_{\ell+1}, 0 \leq \ell \leq 2s-2\), and hence, that

\[
v_\ell = \partial_\ell^\ell v, \quad 0 \leq \ell \leq 2s-1. \quad (7.59)
\]

The proof now follows since it is clear from the properties of the solution \((\mathbf{v}_{2s-2}, \mathbf{v}_{2s-1}, \mathbf{w}_{2s})\), see Theorem 7.10 and \((7.28)-(7.31)\) that \(v \in C^\lambda(\Omega, \mathbb{R}^N)\), \(v\) solves \((7.28)-(7.31)\), \((\partial_t^{2s-1} v, w_{2s})\) is a weak solution of the linear wave equation obtained by differentiating \((7.28)-(7.29)\) \((2s-1)\)-times with respect to \(t\), and \((v, w_{2s})\) satisfies the desired energy estimate.

\[\square\]

**Remark 7.11.** There is a straightforward generalization of Corollary 4.10 that allows for source terms \(F\), and \(G\) that are of the form

\[
F = f + f^\alpha \partial_\alpha v \quad \text{and} \quad G = g + g^\alpha v
\]

where

\[
f, \partial_\ell f \in X^{s-1,2s-2}(\Omega, \mathbb{R}), \quad f^\alpha, \partial_\ell f^\alpha \in X^{s-1,2s-2}(\Omega, M_{N \times N}), \quad g, \partial_\ell g \in X^{s,2s-2}(\Omega, \mathbb{R}), \\
\partial_\ell^{2s-1} g|_{\partial \Omega} \in L^\infty([0, T], L^2(\partial \Omega, \mathbb{R}))), \quad \text{and} \quad \tilde{g}, \partial_\ell \tilde{g} \in X^{s,2s-2}(\Omega, M_{N \times N}).
\]

With this change, the energy estimate continues to hold provided that we replace \(F\) and \(G\) by \(f\) and \(g\), respectively, in the \(\alpha_i\) followed by adding \((\|f(0)\|_{E^{s-1,2s-2}} + \|g(0)\|_{E^{s,2s-2}})\) and \((\|f(t)\|_{E^{s-1,2s-2}} + \|g(t)\|_{E^{s,2s-2}})\).
\[ \| \tilde{g}(t) \|_{E^{s, 2r} - 2} + \| \partial_t \tilde{f}(t) \|_{E^{s - 1, 2r} - 2} + \| \partial_t \tilde{g}(t) \|_{E^{s, 2r} - 2} \] to \( \alpha_0 \) and \( \alpha_1(t) \), respectively. The proof of this generalization is an easy consequence of Corollary 7.10 and a contraction argument.

7.7. Local existence and uniqueness. We are now prepared to use the existence and uniqueness results for the model problem to establish an existence and uniqueness result for linear wave equations that include equations of the form (6.3) - (6.4). The precise class of linear wave equations that we consider are:

\[
\begin{align*}
\partial_t (b^{\alpha \beta} \partial_\beta u + \ell^\alpha) &= f & \text{in } \Omega_T, \\
\nu (b^{\alpha \beta} \partial_\beta u + \ell^\alpha) &= g \partial_t^2 u + p \partial_t u + g & \text{in } \Gamma_T, \\
(u, \partial_t u) &= (\tilde{u}_0, \tilde{u}_1) & \text{in } \Omega_0
\end{align*}
\]

where the initial data

\[
(u, \partial_t u)\vert_{\Omega_0} = (\tilde{u}_0, \tilde{u}_1) \in H^{s+1}(\Omega, \mathbb{R}^N) \times H^{s+\frac{1}{2}}(\Omega, \mathbb{R}^N)
\]
satisfies compatibility conditions given by

\[
\tilde{u}_\ell := \partial_\ell^\ell u \vert_{\Omega_0} \in H^{s+1-\frac{ma}{2s}}(\Omega, \mathbb{R}^N), \quad 2 \leq \ell \leq 2s,
\]
and

\[
\tilde{w}_{2s} := \mathbb{P}_t \partial_t^{2s} u \vert_{\Omega_0} \in L^2(\partial \Omega, \mathbb{R}^N).
\]

As before, the higher time derivatives \( \partial_\ell^\ell v \vert_{\Omega_0} \), \( \ell \geq 2 \), are generated from the initial data (7.63) by formally differentiating (7.60) with respect to \( t \) at \( t = 0 \).

**Theorem 7.12.** Suppose \( s > n/2 + 1, s = k/2 \) for \( k \in \mathbb{Z} \),

\[
b^{\alpha \beta} \in X_T^s(\Omega, M_{N \times N}), \quad p, q \in X_T^{s + \frac{1}{2}, 2s - 1}(\Omega, M_{N \times N}),
\]

\[
f, \partial_t f, \ell^\alpha, \partial_t \ell^\alpha, \partial_t^2 \ell^\alpha \in X_T^{s - 1, 2s - 2}(\Omega, \mathbb{R}^N), \quad \ell^\alpha, \partial_t \ell^\alpha, g, \partial_t g \in X_T^{s, 2s - 2}(\Omega, \mathbb{R}^N),
\]

\[
\partial_t b^{\alpha \beta}, \partial_t q, \partial_t^2 q \in L^\infty([0, T], H^s(\Omega, M_{N \times N})), \quad r \in C([0, T], H^s(\Omega)),
\]

\[
\partial_t^{2s - 1} \ell^\alpha, \partial_t^2 \ell^\alpha \in L^\infty([0, T], L^2(\Omega, \mathbb{R}^N)), \quad \partial_t^{2s - 1} q \in L^\infty([0, T], L^2(\partial \Omega, \mathbb{R}^N)),
\]

the coefficients \( \{ b^{\alpha \beta}, q \} \) satisfy (7.13), (7.15) - (7.18) for constants \( \kappa_0, \kappa_1 > 0 \) and \( \mu \geq 0, q^2 + q \leq 0 \) in \( \Gamma_T \) for some \( \gamma > 0 \), \( p + (2s - \frac{1}{2}) \partial_t q - rq \leq 0 \) in \( \Gamma_T \), and the initial data \( (\tilde{v}_0, \tilde{v}_1) \) satisfies the compatibility conditions (7.64) - (7.65). Then there exists a unique solution \( u \in C \mathcal{X}_T^{s+1}(\Omega, \mathbb{R}^N) \) to the IBVP (7.60) - (7.62). Moreover, there exists a map \( w_{2s} \in C([0, T], L^2(\Omega, \mathbb{R}^N)) \) such that \( (\partial_\ell^{2s - 1} u, w_{2s}) \) is a weak solution of the linear wave equations obtained by differentiating (7.60) - (7.61) \((2s - 1)\)-times with respect to \( t \), and the pair \( (u, w_{2s}) \) satisfy the energy estimate

\[
\| (u(t), w_{2s}(t)) \|_{s+1} \leq C\left( \| (u(0), w_{2s}(0)) \|_{s+1} + a_0 + \int_0^t \alpha_1(\tau) \| (u(\tau), w_{2s}(\tau)) \|_{s+1} \alpha_2(\tau) d\tau \right)
\]

where \( C = C(\kappa_0, \kappa_1, \mu, \gamma, \rho) \),

\[
\rho = \| b \|_{X_T^{s, 2s - 2}} + \| p \|_{X_T^{s, 2s - 2}} + \| q \|_{X_T^{s, 2s - 2}} + \sup_{0 \leq t \leq T} \| \partial_t^{2s - 1} b(t) \|_{L^2(\Omega)} + \| \partial_t q(t) \|_{H^s(\Omega)},
\]

\[
\alpha_0 = \| \ell^\alpha \|_{E^{s, 2s - 2}} + \| \partial_t \ell^\alpha \|_{L^2(\Omega)} + \| f(t) \|_{E^{s, 1, 2s - 2}} + \| \partial_t f(t) \|_{E^{s, 1, 2s - 2}} + \| \partial_t \ell^\alpha(0) \|_{E^{s - 1, 2s - 2}} + \| g(0) \|_{E^{s, 2s - 2}},
\]

\[
\alpha_1(t) = \| b(t) \|_{E^{s, 2s}} + \| \partial_t b(t) \|_{H^s(\Omega)} + \| \tau(t) \|_{H^s(\Omega)} + \| \partial_t^2 q(t) \|_{H^s(\Omega)},
\]

\[
\alpha_2(t) = \| f(t) \|_{E^{s - 1, 2s - 2}} + \| \partial_t f(t) \|_{E^{s - 1, 2s - 2}} + \| \partial_t^2 \ell^\alpha(t) \|_{E^{s - 1, 2s - 2}} + \| \partial_t \ell^\alpha(t) \|_{E^{s, 2s - 2}} + \| \partial_t \ell^\alpha(0) \|_{E^{s - 1, 2s - 2}} + \| g(t) \|_{E^{s, 2s - 2}} + \| \partial_t g(t) \|_{E^{s, 2s - 2}} + \| \partial_t^2 \ell^\alpha(t) \|_{L^2(\Omega)} + \| g(t) \|_{E^{s, 2s - 2}} + \| \partial_t g(t) \|_{E^{s, 2s - 2}} + \| \partial_t^2 \ell^\alpha(t) \|_{L^2(\Omega)} + \| g(t) \|_{E^{s, 2s - 2}} + \| \partial_t g(t) \|_{E^{s, 2s - 2}} + \| \partial_t^2 \ell^\alpha(t) \|_{L^2(\Omega)}
\]

\[
\| (u(t), w_{2s}(t)) \|_{s+1} = \| u(t) \|_{E^{s+1, 2s - 2}} + \| (u(t), w_{2s}(t)) \|_E.
\]
Proof. First, a short computation shows that the IBVP (7.60)-(7.62) transform as
\[ \partial_\alpha (b^{\alpha \beta} \partial_\beta v + L^{\alpha}) + cv = F \quad \text{in } \Omega_T, \]
\[ \nu_\alpha (b^{\alpha \beta} \partial_\beta v + L^{\alpha}) = q \partial_t^2 v + P \partial_t v + G \quad \text{in } \Gamma_T, \]
\[ (v, \partial_t v) = (\tilde{v}_0, \tilde{v}_1) := (e^{-\omega} \tilde{u}_0, e^{-\omega} (\tilde{u}_1 - \partial_t (\omega \tilde{u}_0))) \quad \text{in } \Omega_0, \]
under change of variables
\[ u = e^\omega v \]
where
\[ c = b^{\alpha \beta} \partial_\alpha \omega \partial_\beta \omega, \]
\[ F = e^{-\omega} f - e^{-\omega} \ell^\alpha \partial_\alpha \omega - \partial_\alpha \omega b^{\alpha \beta} \partial_\beta v, \]
\[ L^{\alpha} = e^{-\omega} \ell^\alpha + b^{\alpha \beta} \partial_\beta \omega, \]
\[ P = p + 2 \partial_t \omega q, \]
and
\[ G = e^{-\omega} g + [(\partial_t \omega)^2 + \partial_t^2 \omega] q v + \partial_t \omega pv. \]
Setting
\[ \omega = \sqrt{\lambda t}, \quad \lambda > 0, \]
then gives
\[ c = \lambda b_{00}, \]
while
\[ P - (2s - \frac{3}{2}) \partial_t q - (2\sqrt{\lambda} + r) q \leq 0 \]
follows from the assumption \( p - (2s - \frac{3}{2}) \partial_t q - r q \leq 0 \).

Using the localization technique\(^{20}\) that was employed in the proof of Theorem 3.4 from [2], it is enough, in order to establish the existence and uniqueness of solutions to (7.66)-(7.68), to assume that
(i) the coefficients \( \{b^{\alpha \beta}, q, p\} \) are of the form
\[ b^{\alpha \beta}(x) = b^{\alpha \beta}_c + \epsilon^2 b^{\alpha \beta}(x), \]
\[ q(x) = q_c + \epsilon^2 q^{\alpha \beta}(x), \]
\[ p(x) = p_c + \epsilon^2 p^{\alpha \beta}(x), \]
on the domain
\[ \Omega := \{ (x^1, \ldots, x^n) \in \mathbb{R}^n | -1 < x^1, x^2, \ldots, x^{n-1} < 1, \quad 0 < x^n < 1 \} / \sim \subset \mathbb{T}^n \]
where \( b^{\alpha \beta}_c, q_c \) and \( p_c \) are constant matrices, the coefficients \( b^{\alpha \beta}_c, q \) and \( p \) are bounded in the same spaces at \( b^{\alpha \beta}, q \) and \( p \), respectively, and otherwise \( b^{\alpha \beta}, q \) and \( p \) satisfy the hypotheses of the theorem for some constant\(^{21}\) \( \mu \geq 0 \) and \( \kappa_0, \kappa_1, \gamma > 0 \) independent of \( \epsilon \in (0, \epsilon_0] \) for some \( \epsilon_0 > 0 \).
(ii) the source terms \( \{f, g, \ell^\alpha\} \) satisfy the hypotheses of the theorem,
(iii) the initial data satisfies the compatibility conditions.

Rescaling the time \( t \mapsto t/\epsilon \) and multiplying through by \( \epsilon \), the system (7.66)-(7.68) transforms according to
\[ \partial_\alpha (b^{\alpha \beta}_e \partial_\beta V_e + \epsilon L^{\alpha}_e) + \lambda^{00}_e V_e = \epsilon F_e \quad \text{in } \Omega_e T, \]
\[ \nu_\alpha (b^{\alpha \beta}_e \partial_\beta V_e + \epsilon L^{\alpha}_e) = \epsilon^2 q e \partial_t^2 V_e + \epsilon P e \partial_t V_e + \epsilon G_e \quad \text{in } \Gamma_e T, \]
\[ (V_e, \partial_t V_e) = (\epsilon \tilde{v}_0, \epsilon \tilde{v}_1) \quad \text{in } \Omega_0, \]
where we have set
\[ V_e = \epsilon v_e, \quad P_e = p_c + 2 \sqrt{\lambda} q_c, \quad b^{\alpha \beta}_e = 2 \delta_0^\alpha \delta_0^\beta b^{00}_c + \epsilon \delta_0^\alpha \delta_0^\beta b^{00}_c + \epsilon \delta_1^\alpha \delta_1^\beta b^{00}_c + \epsilon \delta_1^\alpha \delta_0^\beta b^{00}_c + \delta_1^\alpha \delta_0^\beta b^{00}_c, \]

\(^{20}\)This amounts to localizing about a point on the boundary following by a rescaling to get estimates in a neighborhood of the boundary. Away from the boundary, estimates follow by standard techniques. Interior and boundary estimates are then patched together to get a global estimate.

\(^{21}\)This statement is obvious except for the constants \( \mu \) and \( \kappa_1 \) involved in the coercive estimate. That appropriate constants exist for the localized problem is due to the fact, see Remark 7.1, that the coercivity estimate is equivalent to pointwise algebraic conditions on the \( b^{ij} \) that are easily verified to continue to hold for the localized problem.
and we employ the notation \( f_\epsilon(t, x) := f(t/\epsilon, x) \) for all other variables, e.g. \( b_\epsilon, p_\epsilon, q_\epsilon, F_\epsilon, L_\epsilon, \) and \( v_\epsilon. \) Importantly, we have that
\[
\partial_t b_\epsilon^\mu(t) = \epsilon(\partial_t \phi^\mu) \left( \frac{t}{\epsilon} \right), \quad \partial_t q_\epsilon(t) = (\partial_t q) \left( \frac{t}{\epsilon} \right), \quad \partial_t P_\epsilon = \epsilon \left( \frac{t}{\epsilon} \right) + 2\sqrt{\Lambda}(\partial_t q) \left( \frac{t}{\epsilon} \right)
\]
and hence, by assumption (ii) above, that
\[
\text{Theorem 8.1.}
\]
that the triplet (8.1) satisfies the IBVP (6.1)-(6.9) for any of the freely specifiable constants \( \delta, \epsilon, \kappa \in \mathbb{R}. \) In order to apply the energy estimates from Theorem 7.12 and the elliptic estimates from Theorem 11.4 to the solution (5.3), we first need to show the free parameters \( \delta, \epsilon, \kappa \) can be chosen so that \( B_{\Sigma A} \) and \( \Sigma A \) satisfy coercive estimates of the form (7.18), and \( P \) and \( Q \) satisfy the conditions \( P + (2s - \frac{3}{2})\partial_0 Q - rQ \)
and \( Q^2 + c_Q Q \leq 0 \) for some constants \( r \) and \( c_Q > 0 \). That this is possible is the content of the following three lemmas.

**Lemma 8.2.** There exists constants \( \delta, c_Q > 0, \) independent of \( \epsilon, \kappa \in \mathbb{R}, \) such that
\[
\left( \partial_0 \Sigma |B_{\Sigma A}(\vec{x})\partial_0 \Sigma \right)_{\Omega} \geq c_B \| \Sigma \|^2_{L_2(\Omega)}
\]
for all \( \vec{x} \in [0, T] \) and \( \Sigma \in C^4(\mathbb{T}, \mathbb{R}^5). \)

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22 From the assumption \( e_0 = (e_0^\mu) \in C^5(\mathbb{T}, \mathbb{R}^4), \) which, we note implies that \( \phi = (\phi^\mu) \in C^5(\mathbb{T}, \mathbb{R}^4) \) by solving the ODE (4.10), it follows from the definition (11.9) that \( \tilde{e}_0^\mu \in C^4(\mathbb{T}, \mathbb{R}^4). \) Similarly, from the fact that \( e_\epsilon = (e_\epsilon^\mu) \in C^4(\mathbb{T}, \mathbb{R}^4), \) we also have that \( \tilde{e}_\epsilon = (\tilde{e}_\epsilon^\mu) \in C^4(\mathbb{T}, \mathbb{R}^4). \) At first glance, these two statements seem to imply via definition (11.22) that \( \theta^0 = (\theta^0_\epsilon) \in C^4(\mathbb{T}, \mathbb{R}^4). \) However, due to the relation (3.32), the definition \( \theta^0_\epsilon = \theta^0_\epsilon \circ \phi, \) and the smoothness of the metric \( g_{\mu\nu}, \) we, in fact, have that \( \theta^0 \in C^4(\mathbb{T}, \mathbb{R}^4). \)
Proof. From the assumptions (A.1)-(A.7) from Section 2 and formulas (4.10), (4.18), (4.26), and (4.52), it is clear that there exists constants $c_A > 0$ and $c_{\mathcal{A},0} > 0$ such that
\begin{equation}
\zeta \Sigma \zeta \delta^\Sigma (\bar{\lambda}^\lambda) \geq c_A |\zeta|^2 \quad \text{and} \quad \omega_\mu \omega_\nu \tilde{m}^\mu (\bar{\lambda}^\lambda) \geq c_{\mathcal{A}} |\omega|^2
\end{equation}
for all $(\bar{\lambda}^\lambda, \zeta, \omega_\mu) \in \mathbb{T} \times \mathbb{R} \times \mathbb{R}$. Fixing $\Xi = (\zeta, \tilde{\zeta}, \xi) \in C^1(\mathbb{T} \times \mathbb{R} \times \mathbb{R})$, it then follows immediately from the bounds (8.2), the definitions (4.66), (4.82), and (4.22), and H"older’s inequality that there exists constants $c_A > 0$, $c_S > 0$, independent of $\Xi$, such that
\begin{equation}
\langle \partial_\Sigma \Xi \delta^\Sigma A^\Sigma (\bar{x}^\lambda) \partial_\lambda \Xi \rangle \geq c_A \left( \|D \xi\|_{L^2(\Omega)}^2 + (1 + \delta) \|D \tilde{\xi}\|_{L^2(\Omega)}^2 \right)
\end{equation}
and
\begin{equation}
|\langle \partial_\Sigma \Xi \delta^\Sigma 2\nu |^\Sigma N(\bar{x}^\lambda) \partial_\lambda \Xi \rangle| \leq c_S \left( \|D \xi\|_{L^2(\Omega)}^2 + \|D \tilde{\xi}\|_{L^2(\Omega)}^2 + \|D \hat{\xi}\|_{L^2(\Omega)}^2 \right)
\end{equation}
for all $\bar{x} \in [0, T]$ and $\delta \geq 0$. Applying Young’s inequality to the right hand side of (8.3) gives
\begin{equation}
|\langle \partial_\Sigma \Xi \delta^\Sigma 2\nu |^\Sigma N(\bar{x}^\lambda) \partial_\lambda \Xi \rangle| \leq c_S \left( 1 + \frac{c_S}{2c_A} \right) \|D \xi\|_{L^2(\Omega)}^2 + \frac{c_A}{2} \|D \xi\|_{L^2(\Omega)}^2.
\end{equation}
Setting
$$
\delta = \frac{c_S}{c_A} \left( 1 + \frac{c_S}{2c_A} \right) + \frac{1}{2},
$$
the coercive estimate
\begin{equation}
\langle \partial_\Sigma \Xi \delta^\Sigma 2\nu |^\Sigma N(\bar{x}^\lambda) \partial_\lambda \Xi \rangle \geq \frac{c_A}{2} \|D \Xi\|_{L^2(\Omega)}^2, \quad 0 \leq x^0 \leq T,
\end{equation}
then follows directly from (8.17), and the estimates (8.3) and (8.5).

Lemma 8.3. Suppose $\epsilon > 0$ and let $\delta > 0$ be as in Lemma 8.2. Then there exist constants $c_B > 0$, independent of $\kappa \in \mathbb{R}$, such that
\begin{equation}
\langle \partial_\Sigma \xi \delta^\Sigma B^\Sigma A (\bar{x}^\lambda) \partial_\lambda \xi \rangle \geq c_B \|D \xi\|_{L^2(\Omega)}^2 - c_B \|\xi\|_{L^2(\Omega)}^2
\end{equation}
for all $\bar{x} \in [0, T]$ and $\xi \in C^1(\mathbb{T} \times \mathbb{R})$.

Proof. From the bounds (8.2), the definition (6.12) and the fact that $\pi^{\mu \nu}$ is non-negative, it is clear that there exists a constant $c_B > 0$, independent of $\epsilon > 0$, such that
\begin{equation}
\zeta \Sigma \zeta \delta^\Sigma |B^\Sigma A (\bar{x}^\lambda)| \geq c_B |\zeta|^2 \langle \omega \rangle^2
\end{equation}
for all $(\bar{x}^\lambda, \zeta, \omega_\mu) \in \Omega_T \times \mathbb{R} \times \mathbb{R}$. By definition, this establishes the strong ellipticity of $B^\Sigma A$. From Theorem 3 in Section 6 of [23], we see that the proof follows if we can verify that the BVP
\begin{align}
\partial_\Sigma (B^\Sigma A (\bar{x}^\lambda) \partial_\lambda \xi) &= 0 \quad \text{in } \Omega, \\
\nu_\Sigma B^\Sigma A (\bar{x}^\lambda) \partial_\Sigma \xi &= 0 \quad \text{in } \partial \Omega,
\end{align}
(8.6)
(8.7)
satisfies the strong complementing condition; see [23, §4] for a precise definition, for each $\bar{x} \in [0, T]$.

To verify that (8.6)-(8.7) satisfies the complementing condition, we “freeze” the coefficients at a point $(\bar{x}^0, \bar{\lambda}^\lambda) \in [0, T] \times \partial \Omega$, and consider the following BVP on the half-plane:
\begin{align}
\partial_\Sigma (B^\mu \nu (\bar{x}^\lambda) \partial_\lambda \xi^\nu) &= \alpha^2 \hat{m}^\mu \xi^\nu \quad \text{in } \mathbb{R} \times \mathbb{R}_{>0}, \\
\nu_\Sigma B^\mu \nu (\bar{x}^\lambda) \partial_\Sigma \xi^\nu &= 0 \quad \text{in } \mathbb{R},
\end{align}
(8.8)
(8.9)
where $\alpha \in \mathbb{R}$, $\nu_\Sigma = -\delta_3^\Sigma$ is an outward pointing co-normal, $\nu^\Sigma = -\delta_3^\Sigma$, $B^\Sigma A = (\hat{m}^\mu \nu + \epsilon \pi^\nu) \hat{A}^\Sigma A + 2 \hat{S}^\mu [\Sigma, \nu^\lambda]$, and for notational simplicity, we use the same notation for any of the previously defined geometric objects and their frozen versions, e.g. we denote $\pi^{\mu \nu} (\bar{x}^0, \bar{\lambda}^\lambda)$ by $\pi^{\mu \nu}$. In the following, upper case calligraphic and Fraktur letters, e.g. $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{A}, \mathfrak{B}$, will run from 1, 2 and index the boundary coordinate and frame indices, respectively.

We proceed by making the following definitions:
\begin{align}
S_{ij}^\lambda &= \hat{S}^\mu_{ij} \xi^\nu \xi^\nu, \\
m_{ij} &= \hat{m}^\mu c^\mu_{ij}, \\
\pi_{ij} &= \pi^\mu c^\mu_{ij},
\end{align}
for all $\bar{\lambda}^\lambda$.
and \[ \xi^i = \bar{\theta}^i \xi^c. \]

From these definitions, it is not difficult to verify that (8.8)-(8.9) is equivalent to
\begin{equation}
(m_{ij} + \epsilon\pi_{ij}) A^\gamma A \partial_\gamma \partial_\lambda \xi^j = \alpha^2 m_{ij} \xi^j \quad \text{in } \mathbb{R}^2 \times \mathbb{R}_{>0},
\end{equation}
\begin{equation}
-(m_{ij} + \epsilon\pi_{ij}) \tilde{A}^\gamma A \partial_\gamma \xi^j = S_{ij} A \partial_\lambda \xi^j \quad \text{in } \mathbb{R}^2.
\end{equation}

Recalling that \( \tilde{\gamma}_{0j} = 0 \) by (8.12), \( \tilde{\gamma}_{00}^{\Gamma_T} = -1 \), and \( \tilde{\psi}_\mu^{\Gamma_T} = -|\tilde{\gamma}^{33}|^{-1/2} \tilde{\mu}^3 \) by (5.27), we see from (8.6), (8.34), (8.51), (8.58), (8.61), and (6.13) that
\begin{equation}
(m_{ij}) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \tilde{\gamma}_{33} & \tilde{\gamma}_{33} \\
0 & \tilde{\gamma}_{33} & \tilde{\gamma}_{33}
\end{pmatrix},
\end{equation}
\begin{equation}
(\pi_{ij}) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \tilde{\gamma}_{33} & \tilde{\gamma}_{33} \\
0 & \tilde{\gamma}_{33} & \tilde{\gamma}_{33} - \frac{1}{\tilde{\gamma}_{33}}
\end{pmatrix},
\end{equation}
and
\begin{equation}
(S_{ij} A \omega_A) = \det(\bar{\theta}) \epsilon_{0123} \begin{pmatrix}
0 & 0 & \bar{e}_1^A \omega_A \\
0 & 0 & \bar{e}_2^A \omega_A \\
-\bar{e}_1^A \omega_A & -\bar{e}_2^A \omega_A & 0
\end{pmatrix},
\end{equation}
where \( \epsilon_{0123} = \epsilon_{\mu\nu\lambda} \bar{e}_0^\mu \bar{e}_1^\nu \bar{e}_2^\lambda \bar{e}_3^\lambda = \det(\bar{\theta}) \).

The fact that \( \nu_\mu = -\delta^3_\mu \) is an outward pointing co-normal to \( \Gamma_T \) implies via (4.19) that the frame coefficients \( \bar{e}_A^\mu \) satisfy \( \bar{e}_A^3 = 0 \). Using this together with the boundary condition (5.2) and the identity
\[
det(J) = \det(J) \det(\bar{\theta}) = \det(J \bar{c}) \det(\bar{\theta}) = \det(\bar{c}) \det(\bar{\theta})
\]
allows us to express (8.14) as
\begin{equation}
(S_{ij} A \omega_A) = \ell \begin{pmatrix}
0 & 0 & \bar{e}_1^A \omega_A \\
0 & 0 & \bar{e}_2^A \omega_A \\
-\bar{e}_1^A \omega_A & -\bar{e}_2^A \omega_A & 0
\end{pmatrix},
\end{equation}
where \( \ell = -\frac{\det(J)}{\bar{f}} > 0 \).

We note also that, by making a linear change of coordinates if necessary, we can always arrange that
\begin{equation}
\bar{g}^\gamma A = \bar{g}^\gamma A = \delta^\gamma A \quad \text{and} \quad \bar{g}^{33} = \bar{g}^{33} = 1.
\end{equation}

Next, we look for bounded exponential solutions to (8.10)-(8.11) that are of the form
\[
\xi^i = z^i(\bar{x}^3) \exp(i \omega_A \bar{x}^A), \quad (\omega_A) \in \mathbb{R}^2.
\]
Since \( \pi_{ij} \) arises from lowering the index of the projection operator \( \pi^i_j \) using the positive definite metric \( m_{ij} \), it follows that there exists an invertible matrix \( U_j^i \) such that
\[
U_j^k U_j^l m_{kl} = \delta_{ij} \quad \text{and} \quad U_j^k U_j^l \pi_{kl} = \delta_{ij} - \delta^3_i \delta^3_j.
\]

Letting \( \tilde{U}_j^i \) denote the inverse of \( U_j^i \), we can write (8.18) as
\[
\tilde{z}^j = U_j^i z^i(\bar{x}^3) \exp(i \omega_A \bar{x}^A), \quad \text{where} \quad \tilde{z} = \tilde{U}_j^i z^j.
\]

Substituting this into (8.10), while noting that \( \bar{A}^\gamma A = \ell \delta^\gamma A \) by (4.40), (8.16) and (8.17), we find, after a short calculation, that \( \tilde{z}^j \) satisfies the differential equation
\[
\tilde{z}^j''(\bar{x}^3) - (|\tilde{\omega}|^2 + \tilde{\alpha}^2) \tilde{z}^j(\bar{x}^3) = 0, \quad \bar{x}^3 > 0,
\]
where \( \tilde{\omega} = (\omega_A) \) and \( \tilde{\alpha} \in \left\{ \frac{\alpha}{\sqrt{\lambda}}, \frac{\alpha}{\sqrt{(1+\epsilon)\lambda}} \right\} \).
From standard ODE theory, $\ddot{z}(\hat{x}^3) = (\dot{z}^j(\hat{x}^3))$ must be expressible as a linear combination of exponential solutions of the form

$$\ddot{z}(\hat{x}^3) = \exp(i\hat{x}^3\omega_3)\hat{Y}, \quad \omega_3 \in \mathbb{C}, \quad \hat{Y} = (\hat{Y}^j) \in \mathbb{R}^4.$$ 

Substituting this into (8.20) gives

$$(\omega_3^2 + |\omega|^2 + \hat{\alpha}^2)\hat{Y} = 0.$$ 

Assuming that $\hat{Y} \neq 0$, we see that

$$\omega_3 = \pm i\sqrt{|\omega|^2 + \hat{\alpha}^2}.$$ 

Of these two solutions, only

$$\omega_3 = i\sqrt{|\omega|^2 + \hat{\alpha}^2}$$  \hspace{1cm} (8.21)

is compatible with $\ddot{z}(\hat{x}^3)$ being bounded as $\hat{x}^3 \to \infty$, and consequently, every bounded solution of (8.10) must be a linear combination of terms of the form

$$\xi = (\xi^j) = \exp(i\hat{x}^3\omega_3)\exp(i\omega_A \hat{x}^A)\hat{Y}$$ 

with $\omega_3$ given by (8.21) and $\hat{Y} := (U^j_k \hat{Y}^k) \in \mathbb{R}_+^4$.

Substituting (8.22) into (8.11), we see, using (8.12)-(8.13) and (8.17), that

$$(1 + \epsilon)\sqrt{|\omega|^2 + \hat{\alpha}^2}\hat{Y}^0 = 0$$ 

and

$$M\hat{Y} = 0,$$  \hspace{1cm} (8.24)

where

$$M = \sqrt{|\omega|^2 + \hat{\alpha}^2}\begin{pmatrix} (1 + \epsilon)\gamma_{A3} & (1 + \epsilon)\gamma_{3B} \\ (1 + \epsilon)\gamma_{3A} & (1 + \epsilon)\gamma_{33} - \epsilon \end{pmatrix} - i\lambda\begin{pmatrix} 0 & e^{\epsilon\omega_3} \\ -e^{\epsilon\omega_3} & 0 \end{pmatrix}, \quad \text{and} \quad \hat{Y} = \begin{pmatrix} Y^B \\ Y^3 \end{pmatrix}.$$ 

With the help of Lemma C.1, we compute

$$\det(M) = (|\omega|^2 + \hat{\alpha}^2)^{3/2}\begin{pmatrix} (1 + \epsilon)\gamma_{A3} & (1 + \epsilon)\gamma_{3B} \\ (1 + \epsilon)\gamma_{3A} & (1 + \epsilon)\gamma_{33} - \epsilon \end{pmatrix} - (1 + \epsilon)(|\omega|^2 + \hat{\alpha}^2)^{1/2}\det(\gamma_{AB})\gamma_{C3} e^C e_B P\omega_C \omega_D,$$ 

where

$$\gamma_{AB} := (\gamma_{AB})^{-1}.$$ 

Next, we decompose $\gamma_{IJ}$ as

$$\gamma_{IJ} = \begin{pmatrix} \gamma_{A3} & \gamma_{3B} \\ \gamma_{3A} & \gamma_{33} \end{pmatrix} = \begin{pmatrix} \gamma_{A3} & \beta_{3B} \\ \beta_{3A} & n^2 + \beta_{\epsilon}\beta_{\epsilon} \end{pmatrix},$$

where we have defined

$$\beta_{\epsilon} := \gamma_{A3} \beta_{3B}.$$ 

Inverting $\gamma_{IJ}$, we find that

$$\gamma^{IJ} = \begin{pmatrix} \gamma_{A3}^{1/2} + \beta_{A3} \beta_{3B} & \beta_{3A}^{1/2} \\ \beta_{3A}^{1/2} & n^2 \end{pmatrix}.$$ 

But, $\gamma^{33} = 1$, and so we have that

$$n = 1$$ 

and

$$\gamma^{IJ} = \begin{pmatrix} \gamma_{A3}^{1/2} + \beta_{A3} \beta_{3B} & \beta_{3A}^{1/2} \\ \beta_{3A}^{1/2} & 1 \end{pmatrix}.$$  \hspace{1cm} (8.26)

Using Lemma C.2, we compute

$$\det\left(\begin{pmatrix} (1 + \epsilon)\gamma_{A3} & (1 + \epsilon)\gamma_{3B} \\ (1 + \epsilon)\gamma_{3A} & (1 + \epsilon)\gamma_{33} - \epsilon \end{pmatrix}\right) = (1 + \epsilon)^2 \det(\gamma_{A3})(1 + \epsilon)\gamma_{33} - \epsilon - (1 + \epsilon)\beta_{3A}\beta_{3B}$$

$$= (1 + \epsilon)^2 \det(\gamma_{A3}),$$

where in deriving the last equality, we used $\gamma_{33} = \beta_{3A}\beta_{3B} = n^2 = 1$. Substituting the above expression into (8.25) yields

$$\det(M) = (1 + \epsilon)(|\omega|^2 + \hat{\alpha}^2)^{1/2}\det(\gamma_{A3})(1 + \epsilon)(|\omega|^2 + \hat{\alpha}^2) - \gamma_{A3} e^C e_B P\omega_C \omega_D.$$
However,
\[
\gamma_{AB} e_A e_B e_B^D \omega_{(2)} = (\gamma_{AB} - \beta_{AB}) \tilde{e}_A e_B e_B^D \omega_{(2)} = \gamma_{AB} e_A e_B e_B^D \omega_{(2)} - (\beta_{AB})^2 (\tilde{e}_A e_B e_B^D \omega_{(2)} - (\tilde{e}_A e_B e_B^D \omega_{(2)})^2
\]
(by \(S.2b\)),
\[
= \tilde{\gamma}_{AB} e_A e_B e_B^D \omega_{(2)} - 2 \gamma_{AB} e_A e_B e_B^D \omega_{(2)} - \gamma_{AB} (\tilde{e}_A e_B e_B^D \omega_{(2)})^2 - (\beta_{AB})^2 (\tilde{e}_A e_B e_B^D \omega_{(2)})^2
\]
(by \(S.2b\)),
\[
= [\tilde{\omega}]^2 - \left( [\beta_{AB} e_A e_B + \beta_{AB} e_B e_B^D \omega_{(2)}] \right)^2
\]
(by \(S.17\)),
and so, we see that
\[
\det(M) = (1 + \epsilon)([\tilde{\omega}]^2 + \tilde{a}^2)^{1/2} \det(\tilde{\gamma}_{AB}) \left( \epsilon [\tilde{\omega}]^2 + (1 + \epsilon) \tilde{a}^2 + \left( [\beta_{AB} e_A e_B + \beta_{AB} e_B e_B^D \omega_{(2)}] \right)^2 \right).
\]
Since \(\epsilon > 0\) and \(\det(\tilde{\gamma}_{AB}) > 0\), we conclude that
\[
\det(M) = 0 \iff \tilde{\omega} = 0 \quad \text{and} \quad \alpha = 0,
\]
and hence, that there are no non-trivial solutions \(\Upsilon = (\hat{\Upsilon}, \bar{\Upsilon}) \neq 0\) to \((S.2b)-(S.2c)\) with \(\bar{\Upsilon} \neq 0\). This, in turn, implies that only bounded solutions of the type \((S.13)\) to the frozen BVP \((S.5)-(S.7)\) are those for which the \(\alpha_j\) are constant (zero if \(\alpha \neq 0\)) and \(\omega_A = 0\). By definition, this verifies that the BVP \((S.5)-(S.7)\) satisfies the strong complementing condition for each \((\hat{x}_0) \in \Gamma_T\), and the proof is complete. \(\Box\)

**Lemma 8.4.** Suppose \(\tau \geq 0\) and let \(\delta > 0\) be as in Lemma 8.2. Then
\[
Q(\hat{x}_0)^{\tau} = Q(\hat{x}_0), \quad Q(\hat{x}_0) \leq 0, \quad \text{rank} Q(\hat{x}_0) = 3,
\]
and there exists a constants \(c_\omega, \tau > 0\) and \(\kappa < 0\) such that
\[
P(\hat{x}_0) + \tau \partial_0 Q(\hat{x}_0) + \kappa Q(\hat{x}_0) \leq 0 \quad \text{and} \quad Q(\hat{x}_0)^2 + c_\omega Q(\hat{x}_0) \leq 0
\]
for all \((\hat{x}_0) \in \Omega_T\).

**Proof.** From the assumptions \((A.1)-(A.7)\) from Section 2 and \((4.56)\), the choice of initial data from Section 4.2, see in particular \((4.40)\), and the formulas \((4.42)\), \((4.48)\), \((4.33)\), and \((5.34)\), it is not difficult to verify that there exists constants \(c_2 > c_1 > 0\) such that
\[
c_1 |\xi|^2 \leq \xi_{\mu} \bar{m}_{\mu \nu}(\hat{x}_0) \xi_{\nu} \leq c_2 |\xi|^2,
\]
\[
c_1 - \alpha(\hat{x}_0) \mu(\hat{x}_0) \leq c_2,
\]
\[
|\mu(\hat{x}_0) + \beta(\hat{x}_0) (\hat{x}_0) + \lambda(\hat{x}_0)) | \leq c_2,
\]
and \(E^{23}\)
\[
\|\beta(\hat{x}_0)\|_{op} + \|\bar{m}(\hat{x}_0)\|_{op} + \|\partial_0 q(\hat{x}_0)\|_{op} \leq c_2
\]
for all \((\hat{x}_0) \in \Omega_T\) and \(\tau = (\xi_{\mu}) \in \mathbb{R}^4\). Fixing \(\xi = (\xi_{\mu}) \in \mathbb{R}^4\), we then observe that
\[
|\pi \xi|^2 \leq \frac{c_2}{c_1} \xi_{\mu} \bar{m}_{\mu \nu}(\hat{x}_0) \xi_{\nu} = \frac{c_2}{c_1} \alpha(\hat{x}_0) \xi_{\mu} \xi_{\nu} = \frac{c_2}{c_1} \alpha(\hat{x}_0)
\]
by \((S.24)\), \((S.38)\) and \((5.37)\).

Next, we estimate
\[
|\partial_0 q^{\mu \nu}(\hat{x}_0) \xi_{\mu} \xi_{\nu}| = \left| \frac{\pi_{\mu} \xi_{\lambda} + \hat{\psi}_\mu \hat{\psi}_\lambda \xi_{\lambda}}{\|\psi\|^2 m} \right| \partial_0 q^{\mu \nu}(\hat{x}_0) \xi_{\mu} \xi_{\nu} \left| \frac{\hat{\psi}_\mu \hat{\psi}_\nu \xi_{\nu}}{\|\psi\|^2 m} \right| \left| \frac{\hat{\psi}_\mu \hat{\psi}_\nu \xi_{\nu}}{\|\psi\|^2 m} \right|
\]
\[
\leq \left| \pi_{\mu} \xi_{\lambda} \partial_0 q^{\mu \nu}(\hat{x}_0) \xi_{\nu} \right| + 2 \xi_{\mu} \pi_{\mu} \partial_0 q^{\mu \nu}(\hat{x}_0) \xi_{\nu} \left| \frac{\hat{\psi}_\mu \hat{\psi}_\nu \xi_{\nu}}{\|\psi\|^2 m} \right| \left| \frac{\hat{\psi}_\mu \hat{\psi}_\nu \xi_{\nu}}{\|\psi\|^2 m} \right|^2
\]
\[
\leq \left| \partial_0 q\|_{op} |\pi \xi|^2 + 2 \left| \partial_0 q\|_{op} \left| \frac{\hat{\psi}_\mu \hat{\psi}_\nu \xi_{\nu}}{\|\psi\|^2 m} \right| \left| \frac{\hat{\psi}_\mu \hat{\psi}_\nu \xi_{\nu}}{\|\psi\|^2 m} \right|^2 \right| \left| \partial_0 q\|_{op} |\pi \xi|^2 + 2 \left| \partial_0 q\|_{op} \left| \frac{\hat{\psi}_\mu \hat{\psi}_\nu \xi_{\nu}}{\|\psi\|^2 m} \right| \left| \frac{\hat{\psi}_\mu \hat{\psi}_\nu \xi_{\nu}}{\|\psi\|^2 m} \right|^2 \right| |
\]
\[
\leq 2 \left| \partial_0 q\|_{op} |\pi \xi|^2 + 2 \left| \partial_0 q\|_{op} \left| \frac{\hat{\psi}_\mu \hat{\psi}_\nu \xi_{\nu}}{\|\psi\|^2 m} \right| \left| \frac{\hat{\psi}_\mu \hat{\psi}_\nu \xi_{\nu}}{\|\psi\|^2 m} \right|^2 \right| ,
\]
(8.32)

\(\text{Given } A \in M_{n \times n}, \text{ the operator norm of } A \text{ is defined, as usual, by } \|A\|_{op} = \sup_{|\xi|=1} |A\xi|\).
where in obtaining the last inequality, we used (8.27) and (8.31). Additionally, we estimate
\[ |2\mu \alpha \pi^{\mu \nu} \beta_0^{\lambda} \pi^\lambda_{\kappa} \epsilon_{\mu \nu}| = |2\mu \alpha \pi^{\mu \nu} \beta_0^{\lambda} \pi^\lambda_{\kappa} \epsilon_{\mu \nu}| \leq 2c_2^2 |\xi|^2 \] (by (8.27), (8.28) & (8.30))
\[ \leq \frac{2c_2^3}{c_1} \frac{1}{\mu \alpha} (|\xi| q \xi) \] (by (8.28) & (8.31))
(8.33)
and
\[ \left( \mu (\partial_0 \alpha + \lambda) \tilde{m}^{\mu \nu} + \kappa \frac{\hat{\psi} \mu \hat{\psi} \nu}{|\psi|^2_m} \right) \epsilon_{\mu \nu} \leq \left( \mu (\partial_0 \alpha + \lambda) + \kappa + \kappa \right) \frac{\hat{\psi} \mu \hat{\psi} \nu}{|\psi|^2_m} \epsilon_{\mu \nu} \leq \frac{c_2}{\mu \alpha} (|\xi| q \xi) \] (8.34)
where in deriving the last inequality, we have used (8.29). Fixing \( \tau \geq 0 \), it then follows from the definitions (5.37) and (6.21), and the inequalities (8.29)-(8.34) that
\[ (p^{\mu \nu} + \tau \partial_0 \pi^{\mu \nu}) \epsilon_{\mu \nu} \leq \left( \kappa + c_2 \frac{2c_2 \tau}{c_1} \right) \frac{\hat{\psi} \omega}{|\psi|^2_m} \epsilon_{\omega} \leq \left( \kappa + c_2 \frac{2c_2 \tau}{c_1} + \frac{2c_2 \tau}{c_1} \right) \frac{1}{\mu \alpha} (|\xi| - q \xi).
\]
Setting \( \kappa = -c_2 - \frac{2c_2 \tau}{c_1} < 0 \) then yields, with the help of (8.28), the inequality
\[ (|\xi| q \xi) + \tau (|\xi| q \xi) + \left( \frac{c_2}{c_1} + \frac{2c_2 \tau}{c_1} + \frac{2c_2 \tau}{c_1} \right) (|\xi| q \xi) \leq 0, \]
or equivalently, see (6.20) and (6.21),
\[ P + \tau \partial_0 Q + \left( \frac{c_2}{c_1} + \frac{2c_2 \tau}{c_1} + \frac{2c_2 \tau}{c_1} \right) Q \leq 0. \]
To conclude, we note that the statements
\[ Q^{\tau r} = Q, \quad Q \leq 0, \quad \text{and} \quad \text{rank } Q = 3 \]
are a direct consequence of the definitions (5.37) and (6.20), the fact that \( \pi^{\mu}_\nu \) is a projection operator with a 1-dimensional kernel, and the positivity of \( \delta, \mu, \) and \( -\alpha \). Moreover, from (8.27) and (5.53), we see that
\[ \omega^\mu \omega^\nu \tilde{m}_{\mu \nu} = \frac{1}{c_2} |\omega|^2, \quad \forall \omega = (\omega^\mu) \in \mathbb{R}^4. \]
From this, (4.59) and (4.61), we obtain
\[ \omega^\mu \omega^\nu \pi^{\mu \nu} = \omega^\mu \omega^\nu \pi^{\mu \nu} \tilde{m}_{\sigma \omega} \pi^{\rho \nu} \geq \frac{1}{c_2} \omega^\mu \omega^\nu \pi^{\mu \nu} \delta_{\sigma \omega} \pi^{\rho \nu}, \quad \forall \omega = (\omega^\mu) \in \mathbb{R}^4, \]
or equivalently
\[ \pi^2 - c_2 \pi \leq 0, \]
where \( \pi = (\pi^{\mu \nu}) \). We see immediately from this result and (8.28) that \( Q \) satisfies
\[ Q^2 + c_2^2 Q \leq 0. \]
□

Next, we estimate the various coefficients and source terms from the equations (6.11)-(6.4), which we collect in the following three lemmas.

**Lemma 8.5.** Let
\[ \mathcal{R} = ||\phi||_{E^{+}+\frac{1}{2},2^{-1}} + ||\hat{\theta}||_{E^{+}+\frac{1}{2},2^{-1}} + ||\phi||_{E^{+}+1,2^{-2}} + ||\hat{\theta}||_{E^{+}+1,2^{-2}} + ||\Psi||_{E^{+}+1}. \]
Then
\[ ||B||_{E^{+}+2} + ||\tilde{B}||_{E^{+}+2} + ||H||_{E^{+}+2} + ||\hat{\theta}||_{E^{+}+1,2^{-2}} + ||\hat{\partial}_0 H||_{E^{+}+1,2^{-2}} \]
\[ + ||K||_{E^{+}+2} + ||\tilde{K}||_{E^{+}+2} + ||\hat{\partial}_0 K||_{H^1(\Omega)} + ||M||_{E^{+}+2} + ||\hat{\partial}_0 M||_{E^{+}+2} \]
\[ + ||\tilde{M}||_{H^1(\Omega)} + ||\tilde{M}||_{L^2(\Omega)} + ||P||_{E^{+}+\frac{1}{2},2^{-1}} + ||Q||_{E^{+}+\frac{1}{2},2^{-1}} + ||\hat{\partial}_0 Q||_{H^1(\Omega)} \leq C(\mathcal{R}). \]
Proof. We only prove one of the estimates with the rest following from similar arguments. To begin, we recall from Section [C] that \( M \) depends smoothly on its arguments and is, after suppressing the consequential dependence of the time-independent quantities \( (\tilde{\partial}^0, f) \), of the form
\[
M = M(\phi, J, \tilde{\partial} J_0, \Psi).
\]

From this, we see that
\[
\|\tilde{\partial} J_0\|_{E^{s, \frac{s}{2} - 2}} = \|D M(\phi, J, \tilde{\partial} J_0, \Psi) \cdot (\tilde{\partial} \phi, \tilde{\partial} J, \tilde{\partial} \tilde{\partial} J_0, \tilde{\partial} \Psi)\|_{E^{s, \frac{s}{2} - 2}} \leq C\left(\|\phi\|_{E^{s+1, \frac{s}{2} - 1}}, \|J\|_{E^{s, \frac{s}{2} - 2}}, \|\tilde{\partial} J_0\|_{E^{s, \frac{s}{2} - 2}}, \|\tilde{\partial} \tilde{\partial} J_0\|_{E^{s, \frac{s}{2} - 2}}, \|\tilde{\partial} \Psi\|_{E^{s+1}}\right), \tag{8.35}
\]
where in deriving the inequality we have used Proposition [A.9] the identity
\[
\tilde{\partial} J_0 = \tilde{\partial} J_0,
\]
and the obvious inequalities:
\[
\|\phi\|_{E^{s, \frac{s}{2} - 2}} + \|\tilde{\partial} \phi\|_{E^{s, \frac{s}{2} - 2}} \lesssim \|\phi\|_{E^{s+1, \frac{s}{2} - 1}} \quad \text{and} \quad \|\Psi\|_{E^{s, \frac{s}{2} - 2}} + \|\tilde{\partial} \Psi\|_{E^{s, \frac{s}{2} - 2}} \lesssim \|\Psi\|_{E^{s+1}}.
\]

Since \( J = (\tilde{\partial} \mu \phi^0) \), we see also that
\[
\|J\|_{E^{s, \frac{s}{2} - 2}} \lesssim \|\tilde{\partial} \phi\|_{E^{s, \frac{s}{2} - 2}} + \|\tilde{\partial} \phi\|_{E^{s, \frac{s}{2} - 2}} \lesssim \|\tilde{\partial} \phi\|_{E^{s+1, \frac{s}{2} - 1}} + \|\phi\|_{E^{s+1, \frac{s}{2} - 2}}. \tag{8.36}
\]

Next, by (8.35), we have that
\[
J_0 = \mathcal{J}(\phi, \tilde{\partial}^0), \tag{8.37}
\]
for some smooth map \( \mathcal{J} \). Differentiating this with respect to \( \tilde{x}^0 \) then shows with the help of (8.6) that
\[
\tilde{\partial} J_0 = \tilde{\mathcal{J}}(\phi, \tilde{\partial}^0, \Psi) \tag{8.38}
\]
for some smooth map \( \tilde{\mathcal{J}} \), and hence that
\[
\tilde{\partial}^2 J_0 = D \tilde{\mathcal{J}}(\phi, \tilde{\partial}^0, \Psi) \cdot (\tilde{\partial} \phi, \tilde{\partial} \tilde{\partial} \phi^0, \tilde{\partial} \Psi). \tag{8.39}
\]

From repeated applications of Proposition [A.9] we then obtain from (8.37)-(8.39) the following estimates:
\[
\|D J_0\|_{E^{s, \frac{s}{2} - 2}} \leq \|J_0\|_{E^{s+1, \frac{s}{2} - 2}} \leq C\left(\|\phi\|_{E^{s+1, \frac{s}{2} - 2}}, \|\tilde{\partial}^0\|_{E^{s+1, \frac{s}{2} - 2}}\right), \tag{8.40}
\]
\[
\|\tilde{\partial} J_0\|_{E^{s, \frac{s}{2} - 2}} \leq C\left(\|\phi\|_{E^{s, \frac{s}{2} - 2}}, \|\tilde{\partial}^0\|_{E^{s, \frac{s}{2} - 2}}, \|\Psi\|_{E^{s, \frac{s}{2} - 2}}\right), \tag{8.41}
\]
\[
\|D \tilde{\partial} J_0\|_{E^{s, \frac{s}{2} - 2}} \leq \|\tilde{\partial} J_0\|_{E^{s+1, \frac{s}{2} - 2}} \leq C\left(\|\phi\|_{E^{s+1, \frac{s}{2} - 2}}, \|\tilde{\partial}^0\|_{E^{s+1, \frac{s}{2} - 2}}, \|\Psi\|_{E^{s+1, \frac{s}{2} - 2}}\right), \tag{8.42}
\]
and
\[
\|\tilde{\partial}^2 J_0\|_{E^{s, \frac{s}{2} - 2}} \leq C\left(\|\phi\|_{E^{s+1, \frac{s}{2} - 2}}, \|\tilde{\partial} \phi\|_{E^{s, \frac{s}{2} - 2}}, \|\tilde{\partial} \tilde{\partial} \phi^0\|_{E^{s, \frac{s}{2} - 2}}, \|\tilde{\partial} \tilde{\partial} \tilde{\partial} \phi^0\|_{E^{s, \frac{s}{2} - 2}}, \|\tilde{\partial} \tilde{\partial} \tilde{\partial} \Psi\|_{E^{s, \frac{s}{2} - 2}}, \|\tilde{\partial} \tilde{\partial} \tilde{\partial} \Psi\|_{E^{s, \frac{s}{2} - 2}}\right). \tag{8.43}
\]

Letting \( \tilde{\mathcal{R}} \) be as defined in the statement of the lemma, the desired estimate
\[
\|\tilde{\partial} M\|_{E^{s, \frac{s}{2} - 2}} \leq C(\tilde{\mathcal{R}})
\]
follows directly from (8.35)-(8.36), and (8.40)-(8.43).

\[\square\]

**Lemma 8.6.** Let
\[
\tilde{\mathcal{R}} = \|\phi\|_{E^{s+1, \frac{s}{2} - 2}} + \|\tilde{\partial}^0\|_{E^{s+1, \frac{s}{2} - 2}} + \|\Psi\|_{E^{s+1}}.
\]

Then
\[
\|\tilde{\partial}^0\|_{E^{s+1, \frac{s}{2} - 1}} \lesssim \|\tilde{\partial}^0\|_{H^{s+\frac{s}{2}}(\Omega)} + C(\tilde{\mathcal{R}}) \quad \text{and} \quad \|\tilde{\partial}^0\|_{E^{s, \frac{s}{2} - 2}} \leq C(\tilde{\mathcal{R}}).
\]

\[\square\]

**Proof.** The proof follows from the same arguments used to establish Lemma 8.5.

**Lemma 8.7.** Let
\[
\tilde{\mathcal{R}} = \|\phi\|_{E^{s+1, \frac{s}{2} - 1}(\Omega)} + \|\tilde{\partial}^0\|_{E^{s+1, \frac{s}{2} - 2}(\Omega)} + \|\Psi\|_{E^{s+1}}.
\]

Then
\[
\|\tilde{\partial}_0(B^{0\alpha} \tilde{\partial} \tilde{\partial} \tilde{\partial}^0)\|_{H^{s+\frac{s}{2}}(\Omega)} + \|\tilde{\partial}_0 M^0\|_{H^{s+\frac{s}{2}}(\Omega)} + \|H\|_{H^{s+\frac{s}{2}}(\Omega)}
\]
\[
+ \|B^{\Sigma} \tilde{\partial} \tilde{\partial} \tilde{\partial}^0\|_{H^{s+\frac{s}{2}}(\Omega)} + \|B^{\Sigma} \tilde{\partial} \tilde{\partial} \tilde{\partial}^0\|_{H^{s+\frac{s}{2}}(\Omega)} + \|M^{\Sigma}\|_{H^{s+\frac{s}{2}}(\Omega)} \leq C(\tilde{\mathcal{R}}).
\]

\[\square\]

**Proof.** The proof follows from the same arguments used to establish Lemma 8.5.
Viewing equation (6.1) as an elliptic equation for $\hat{\theta}^0_\mu$, we find that
\[
\|\hat{\theta}^0_\mu\|_{H^{s+\frac{3}{2}}(\Omega)} \leq C(\tilde{R}).
\] (8.44)
by Theorem 14.4 and Lemma 8.7. Integrating (6.3) in time, we obtain
\[
\phi^\mu(t) = \phi^\mu(0) + \int_0^t \tilde{\gamma}_00(\tau)\hat{g}^\mu(\tau)\hat{\theta}^0_\mu(\tau) \, d\tau.
\] (8.45)
Setting
\[
\tilde{R} = \|\phi\|_{E^{s+\frac{3}{2}}(\Omega)} + \|\phi\|_{E^{s+1.2,2-\frac{3}{2}}} + \|\tilde{\theta}\|_{E^{s+1,2-\frac{3}{2}}} + \|\Psi\|_{E^{s+1}}
\]
it follows from the estimate (8.44), Lemmas 8.6 and 8.7, Proposition A.9 and the integral representation (8.45) that we can bound $\phi(t)$ by
\[
\|\phi(t)\|_{E^{s+\frac{3}{2}}(\Omega)} \leq \|\phi(0)\|_{E^{s+\frac{3}{2}}(\Omega)} + \int_0^t C(\tilde{R}(\tau)) \, d\tau.
\] (8.46)
Via similar arguments, we find that
\[
\|\phi(t)\|_{E^{s+1.2,2}} \leq \|\phi(0)\|_{E^{s+1.2,2}} + \int_0^t C(\tilde{R}(\tau)) \, d\tau
\] (8.47)
also holds.

Next, integrating (6.6) in time yields
\[
\hat{\theta}^0_\mu(t) = \hat{\theta}^0_\mu(0) + \int_0^t \tilde{\psi}_\mu(t) - \beta^\mu_\lambda(t)\hat{\varphi}^\lambda_\mu(t) \, d\tau.
\]
Using this, we find with the help of Proposition A.9 that
\[
\|\hat{\theta}(t)\|_{E^{s+1,2-\frac{3}{2}}} \lesssim \|\hat{\theta}(0)\|_{E^{s+1,2-\frac{3}{2}}} + \int_0^t C\left(\|\phi(t)\|_{E^{s+1,2-\frac{3}{2}}}, \|\hat{\theta}(t)\|_{E^{s+1,2-\frac{3}{2}}}, \|\Psi(t)\|_{E^{s+1,2-\frac{3}{2}}}, \|\hat{\theta}(t)\|_{E^{s+1,2-\frac{3}{2}}} \right) \, d\tau.
\] (8.48)
Applying the energy estimates from Theorem 14.12, which is possible in view of Lemma 8.2 and Lemma 8.7 with $\tau = 2t = \frac{3}{2}$, we obtain, with the help of the elliptic estimate (8.44) and the estimates from Lemmas 8.3 and 8.6, the energy estimate
\[
\|\Psi(t)\|_{E^{s+1}} + \langle \bar{\tilde{\theta}}_0^0, \Psi(t) \rangle_{(-Q)^{1/\alpha}} \leq C(\rho_{T_\ast}) \left( \|\hat{\theta}(0)\|_{E^{s+1,2-\frac{3}{2}}} + \|\Psi(0)\|_{E^{s+1,2-\frac{3}{2}}} \right) + \int_0^t C(\tilde{R}(\tau)) \left(\|\Psi(t)\|_{E^{s+1}} + \|\bar{\tilde{\theta}}_0^0, \Psi(t) \rangle_{(-Q)^{1/\alpha}} \right) + C(\tilde{R}(\tau)) \, d\tau,
\] (8.49)
for $0 \leq t \leq T_\ast$, where $T_\ast \in (0, T]$ and
\[
\rho_{T_\ast} = \|\tilde{B}\|_{X^{s+1,2-\frac{3}{2}}} + \|Q\|_{X^{s+1,2-\frac{3}{2}}} + \|P\|_{X^{s+1,2-\frac{3}{2}}} + \sup_{0 \leq t \leq T_\ast} \|\bar{\tilde{\theta}}_0^0, \tilde{B}(t) \|_{L^2(\Omega)} + \|\tilde{\theta}(t)\|_{L^2(\Omega)}.
\] (8.50)
Collectively, the estimates (8.46)-(8.49) imply that
\[
\tilde{R}(t) \leq C(\rho_{T_\ast}) \left( \tilde{R}(0) + \int_0^t C(\tilde{R}(\tau)) \, d\tau \right),
\]
where
\[
\tilde{R} = \tilde{R} + \langle \bar{\tilde{\theta}}_0^0, \Psi(-Q)^{1/\alpha} \rangle_{E^{s+1,2-\frac{3}{2}}},
\]
and hence that
\[
\|\tilde{R}\|_{L^\infty([0,T_\ast])} \leq C(\rho_{T_\ast}) \left( \tilde{R}(0) + \int_0^{T_\ast} C\left(\|\tilde{R}\|_{L^\infty([0,T_\ast])}\right) \, d\tau \right).
\] (8.51)
Using the integral representation $B^{\alpha\beta}(t) = B^{\alpha\beta}(0) + \int_0^t B^{\alpha\beta}(\tau)\,d\tau$, we estimate
\[
\|B(t)\|_{L^{1,2}-2} \leq \|B(0)\|_{L^{1,2}-2} + \int_0^t \|B(\tau)\|_{L^{1,2}-2}\,d\tau.
\]
By Lemma 8.5 it follows immediately that
\[
\|B\|_{L^{1,2}-2} \leq C(\mathcal{R}(0)) + T_* C\left(\|\mathcal{R}\|_{L^{\infty}(0,T_*)}\right).
\]
Estimating the other terms in (8.50) in a similar fashion, we obtain the bound
\[
\rho_{T_*} \leq C(\mathcal{R}(0)) + T_* C\left(\|\mathcal{R}\|_{L^{\infty}(0,T_*)}\right).
\]
(8.52)
Taken together, (8.51) and (8.52) imply that $\mathcal{R}(t)$ satisfies an estimate of the form
\[
\|\mathcal{R}\|_{L^{\infty}(0,T_*)} \leq c_1 \left(\mathcal{R}(0) + T_* c_2\left(\|\mathcal{R}\|_{L^{\infty}(0,T_*)}\right)\right)
\]
(8.53)
for some non-decreasing, continuous functions $c_i : [0,\infty) \to [1,\infty)$, $i = 1, 2$.

Setting
\[
r(t) = t - \frac{T}{1 + c_2\left(\|\mathcal{R}\|_{L^{\infty}(0,T_*)}\right)},
\]
we see that $r : [0,T] \to \mathbb{R}$ is continuous and satisfies $r(0) < 0$ and $r(T) > 0$. The Intermediate Value Theorem then guarantees the existence of a $T_* \in (0,T)$ such that $r(T_*) = 0$, or equivalently
\[
T_* = \frac{1}{1 + c_2\left(\|\mathcal{R}\|_{L^{\infty}(0,T_*)}\right)}.
\]
Substituting this into (8.53) yields the estimate
\[
\|\mathcal{R}\|_{L^{\infty}(0,T_*)} \leq c_1 (\mathcal{R}(0) + 1),
\]
which, in turn, implies that
\[
T_* \geq \frac{T}{1 + c_2\left(\|\mathcal{R}\|_{L^{\infty}(0,T_*)}\right)}.
\]
\[\square\]

Remark 8.8. From the proof of Theorem 8.1 it is clear that the same result also holds in the non-physical dimensions $n \neq 3$ for $s$ satisfying $s > n/2 + 1$ and $s = k/2$ with $k \in \mathbb{Z}$.

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Appendix A. Calculus inequalities

In this appendix, we state, for the convenience of the reader, some well known calculus inequalities. As above, $\Omega$ is a bounded, open subset of $\mathbb{R}^n$, $n \geq 2$, with smooth boundary.

A.1. Spatial inequalities. The proof of the following calculus inequalities are well known and may be found, for example, in the references [1, 13, 23, 20]. In the following, $M$ will denote either $\Omega$, or a closed $n$-manifold.

Theorem A.1. [Hölder’s inequality] If $0 < p, q, r \leq \infty$ satisfy $1/p + 1/q = 1/r$, then
\[
\|uv\|_{L^r(M)} \leq \|u\|_{L^p(M)}\|v\|_{L^q(M)}
\]
for all $u \in L^p(M)$ and $v \in L^q(M)$.

Theorem A.2. [Integral Sobolev inequalities] Suppose $s \in \mathbb{Z}_{\geq 1}$ and $1 \leq p < \infty$.

(i) If $sp < n$, then
\[
\|u\|_{L^q(M)} \lesssim \|u\|_{W^{s,p}(M)} \quad p \leq q \leq \frac{np}{n - sp}
\]
for all $u \in W^{s,p}(M)$.

(ii) If $s \in \mathbb{Z}_{\geq 1}$, $1 \leq p < \infty$ and $sp > n$, then
\[
\|u\|_{L^q(M)} \lesssim \|u\|_{W^{s,p}(M)}
\]
for all $u \in W^{s,p}(M)$.

Theorem A.3. [Fractional Sobolev inequalities] Suppose $s \in \mathbb{R}$ and $1 < p < \infty$. 
(i) If \( sp < n \), then
\[
\| u \|_{L^p(M)} \lesssim \| u \|_{W^{s,p}(M)} \quad p \leq \frac{np}{n-sp}
\]
for all \( u \in W^{s,p}(M) \).

(ii) If \( sp > n \), then
\[
\| u \|_{L^\infty(M)} \lesssim \| u \|_{W^{s,p}(M)}
\]
for all \( u \in W^{s,p}(M) \).

**Theorem A.4.** [Trace theorem] If \( s > 1/2 \), then the trace operator
\[
H^s(\Omega) \ni u \mapsto u|_{\partial \Omega} \in H^{s-\frac{1}{2}}(\partial \Omega)
\]
is well-defined, continuous (i.e. bounded), and surjective.

**Lemma A.5.** [Ehrlich’s lemma] Suppose \( 1 < p < \infty \), \( s_0 < s < s_1 \). Then for any \( \delta > 0 \) there exists a constant \( C = C(\delta) \) such that
\[
\| u \|_{W^{s,p}(M)} \leq \delta \| u \|_{W^{s_1,p}(M)} + C \| u \|_{W^{s_0,p}(M)}
\]
for all \( u \in W^{s_1,p}(M) \).

**Theorem A.6.** [Integral multiplication inequality] Suppose \( s_1, s_2, s_3 \in \mathbb{Z}_{\geq 0}, s_1, s_2 \geq s_3 \geq 0, 1 \leq p \leq \infty \), and \( s_1 + s_2 - s_3 > n/p \). Then
\[
\| uv \|_{W^{s_3,p}(M)} \lesssim \| u \|_{W^{s_1,p}(M)} \| v \|_{W^{s_2,p}(M)}
\]
for all \( u \in W^{s_1,p}(M) \) and \( v \in W^{s_2,p}(M) \).

**Theorem A.7.** [Fractional multiplication inequality] Suppose \( 1 \leq p < \infty \), \( s_1, s_2, s_3 \in \mathbb{R}, s_1 + s_2 > 0, s_1, s_2 \geq s_3 \), and \( s_1 + s_2 - s_3 > n/p \). Then
\[
\| uv \|_{W^{s_3,p}(M)} \lesssim \| u \|_{W^{s_1,p}(M)} \| v \|_{W^{s_2,p}(M)}
\]
for all \( u \in W^{s_1,p}(M) \) and \( v \in W^{s_2,p}(M) \).

**A.2. Spacetime inequalities.** The following spacetime calculus inequalities are fractional versions of the spacetime inequalities from Appendix A.2 of [2]. The proofs of the two propositions below follow from a straightforward adaptation of the proof of Proposition A.8 from [2] that involves replacing the integral multiplication inequalities used there with their fractional versions, i.e. Theorem A.7.

**Proposition A.8.** Suppose \( s_1 = k_1/2 \) and \( s_1 = k_1/2 \) for \( k_1, k_2 \in \mathbb{Z}_{\geq 0}, s_3 \in \mathbb{R}, s_1, s_2 \geq s_3, s_1 + s_2 - s_3 > n/2, r \in \mathbb{Z}, \) and \( 0 \leq r \leq 2s_3 \). Then
\[
\| \partial^r_t (u(t) \varphi(t)) \|_{H^{s_3-r}(\Omega)} \lesssim \| u(t) \|_{E^{s_1-r}} \| \varphi(t) \|_{E^{s_2, r}}
\]
for \( 0 \leq t \leq T, 0 \leq \ell \leq r, \) and all \( u \in X^{s_1-r}_T(\Omega) \) and \( \varphi \in X^{s_2, r}_T(\Omega) \).

**Proposition A.9.** Suppose \( s = k/2, s > n/2, r \in \mathbb{Z}, 0 \leq r \leq 2s, f \in C^r(\mathbb{R}), \) and \( f(0) = 0 \). Then
\[
\| \partial^r_t (u(t)) \|_{H^{s-r}(\Omega)} \lesssim C \| u(t) \|_{E^{s-r}} \| u(t) \|_{E^{s-r}}
\]
for \( 0 \leq t \leq T, 0 \leq \ell \leq r, \) and all \( u \in X^{s-r}_T(\Omega) \).

**Appendix B. Elliptic systems.**

In this appendix, we recall some well known existence and regularity results for elliptic systems of the form
\[
\partial_i (b^{ij} \partial_j v + \epsilon \partial_i v + L^i) + \epsilon \partial_i \partial_j v = F \quad \text{in } \Omega,
\]
\[
\nu_i (b^{ij} \partial_j v + \epsilon \partial_i v + L^i) = \epsilon v + G \quad \text{in } \partial \Omega,
\]
where
(i) as above, \( \Omega \) is open, bounded in \( \mathbb{R}^n, n \geq 2 \), with smooth boundary,
(ii) \( \nu \) is the outward pointing unit co-normal to \( \partial \Omega \),
(iii) \( L^i \in L^2(\Omega, \mathbb{R}^N), F \in L^2(\Omega, \mathbb{R}^N) \) and \( G \in H^{-1/2}(\partial \Omega, \mathbb{R}^N) \),
(iv) \( a^i, \epsilon \in L^p(\Omega, \mathbb{M}_{N \times N}) \) and \( h \in L^\infty(\Omega, \mathbb{M}_{N \times N}) \cap W^{1,n}(\Omega, \mathbb{M}_{N \times N}) \),
(v) \( c \in L^p(\Omega, \mathbb{M}_{N \times N}) \) and satisfies
\[
c \geq \sigma > 0 \quad \text{in } \Omega
\]
for some positive constant \( \sigma \),
(vi) \( b^{ij} \in L^\infty(\Omega, \mathbb{M}_{N\times N}) \) and satisfies
\[
(b^{ij})^\text{tr} = b^{ji},
\]  
(B.3)

(vii) and there exists a \( \kappa_1 > 0 \) and \( \mu \geq 0 \) such that
\[
\langle \partial_i v \rangle b^{ij} \partial_j v \rangle_{L^2(\Omega)} \geq \kappa_1 \|v\|_{H^1(\Omega)}^2 - \mu \|v\|_{L^2(\Omega)}^2
\]  
(B.4)

for all \( v \in H^1(\Omega) \).

**Definition B.1.** Under the assumptions (i)-(vii) above, \( v \in H^1(\Omega, \mathbb{R}^N) \) is called a weak solution of (B.1)-(B.2) if

\[
\langle b^{ij} \partial_i v | \partial_j \phi \rangle_{\Omega} + \epsilon \langle d^j v | \partial_j v \rangle_{\Omega} - \epsilon \langle a^i \partial_i v | \phi \rangle_{\Omega} + \lambda \langle c v | \phi \rangle_{\Omega}
\]
\[
- \epsilon \langle (hv)_{|\partial\Omega} | \phi_{|\partial\Omega} \rangle_{\partial\Omega} = -\langle F \rangle_{\Omega} - \langle L^j \partial_j v \rangle_{\Omega} + \langle G \rangle_{\partial\Omega}
\]  
(B.5)

for all \( \phi \in H^1(\Omega, \mathbb{R}^N) \).

**Remark B.2.** That the above definition makes sense follows from repeated use of Hölder’s inequality, the Trace theorem, the Sobolev inequalities, Erhling’s lemma and the duality relation \((H^\infty(\partial\Omega))^* \cong H^{-\infty}(\partial\Omega)\). To see this, we observe from Theorems [A.1] and [A.2] that
\[
\|uv\|_{L^2(\Omega)} \leq \|u\|_{L^\infty(\Omega)} \|v\|_L^\infty(\Omega) \lesssim \|u\|_{L^\infty(\Omega)} \|v\|_{H^1(\Omega)},
\]  
(B.6)

which in turn, implies via the Cauchy-Schwartz inequality that
\[
\langle uv \rangle_{\Omega} \lesssim \|u\|_{L^\infty(\Omega)} \|v\|_{H^1(\Omega)} \|w\|_{L^2(\Omega)}.
\]  
(B.7)

Using Theorems [A.1] [A.2] and [A.6] we observe also that
\[
\|uv\|_{H^1(\Omega)} \lesssim \|u\|_{L^\infty(\Omega)} \|v\|_{H^1(\Omega)} + \|u\|_{W^{1,\infty}(\Omega)} \|v\|_{L^\infty(\Omega)} \lesssim \|u\|_{L^\infty(\Omega)} \|v\|_{H^1(\Omega)}.
\]  
(B.8)

This inequality, together with Theorem [A.4] the Cauchy-Schwartz inequality, and Lemma [A.5] implies that
\[
\langle (uv)_{|\partial\Omega} \rangle_{\partial\Omega} \lesssim \|uv\|_{H^1(\Omega)} \|w\|_{L^2(\Omega)}
\]
\[
\lesssim \|u\|_{L^\infty(\Omega)} \|v\|_{H^1(\Omega)} \left( \gamma \|w\|_{H^1(\Omega)} + C(\gamma) \|w\|_{L^2(\Omega)} \right)
\]  
(B.9)

for any \( \gamma > 0 \). Finally,
\[
\langle u | v_{|\partial\Omega} \rangle_{\partial\Omega} \lesssim \|u\|_{H^{1/2}(\partial\Omega)} \|v\|_{H^1(\Omega)}
\]  
(B.10)

follows by the duality relation \((H^\infty(\partial\Omega))^* \cong H^{-\infty}(\partial\Omega)\) and Theorem [A.4].

The following existence result is a slight modification of Theorem B.2 from [10], and is proved using similar arguments.

**Theorem B.3.** Suppose assumptions (i)-(vii) above are satisfied. Then
(i) there exists an \( \delta^* = \delta^*(\kappa_1, \|a\|_{L^\infty(\Omega)}, \|d\|_{L^\infty(\Omega)}, \|h\|_{L^\infty(\Omega)} \cap W^{1,\infty}(\Omega) \geq 1 \) such that every weak solution \( v \) of (B.1)-(B.2) with \( \epsilon \in [0, \frac{1}{\delta^*}] \) satisfies the estimate
\[
\|v\|_{H^1(\Omega)} \leq C \left( \|v\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)} + \|G\|_{H^{-1/2}(\partial\Omega)} \right),
\]
where
\[
C = C(\kappa_1, \mu, \lambda, \|a\|_{L^\infty(\Omega)}, \|d\|_{L^\infty(\Omega)}, \|h\|_{L^\infty(\Omega)} \cap W^{1,\infty}(\Omega), \|c\|_{L^\infty(\Omega)}),
\]
and
(ii) there exists a \( \lambda^* = \lambda^*(\sigma, \mu) > 0 \) such that for \( \lambda \geq \lambda^* \) and \( \epsilon \in [0, \epsilon^*] \) there exists a unique weak solution of (B.1)-(B.2).

**Proof.** (i): Given a weak solution \( v \) of (B.1)-(B.2), we see, after setting \( v = \phi \) in (B.5), and using (B.4) and (B.6)-(B.10), that
\[
\kappa_1 \|v\|_{H^1(\Omega)} \leq K \left( \epsilon \|a\|_{L^\infty(\Omega)} + \epsilon \|d\|_{L^\infty(\Omega)} + \epsilon \|h\|_{L^\infty(\Omega)} \cap W^{1,\infty}(\Omega) \right) \|v\|_{H^1(\Omega)} + K \left( \|\lambda\|_{L^\infty(\Omega)} + \epsilon \|h\|_{L^\infty(\Omega)} \cap W^{1,\infty}(\Omega) \right) \|v\|_{L^2(\Omega)} + \epsilon \|v\|_{L^2(\Omega)} + \epsilon \|F\|_{L^2(\Omega)} + \epsilon \|L\|_{L^2(\Omega)} + \epsilon \|G\|_{H^{-1/2}(\partial\Omega)} \|v\|_{H^1(\Omega)}
\]
for some constant $K > 0$ independent of $\epsilon > 0$. Setting

$$\delta^* = \max \left\{ 1, \frac{2K}{\kappa_1} \left( \|a\|_{L^\infty(\Omega)} + \|d\|_{L^\infty(\Omega)} + \|h\|_{L^\infty(\Omega) \cap W^{1,n}(\Omega)} \right) \right\},$$

and choosing $\epsilon \in (0, \frac{1}{\delta^*}]$, it follows immediately from the above estimate that

$$\|v\|_{H^1(\Omega)} \leq C(\|v\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)} + \|L\|_{L^2(\Omega)} + \|G\|_{H^{\frac{1}{2}}(\partial\Omega)}),$$

where

$$C = C(\kappa_1, \mu, \lambda, \|a\|_{L^\infty(\Omega)}, \|d\|_{L^\infty(\Omega)}, \|h\|_{L^\infty(\Omega) \cap W^{1,n}(\Omega)}, \|c\|_{L^\infty(\Omega)}).$$

(ii) Setting

$$B(v, \phi) = \langle b^{ij} \partial_i v \partial_j \phi \rangle_{\Omega} + \langle d^{ij} \partial_i v \partial_j \phi \rangle_{\Omega} + \langle \alpha^i \partial_i v \phi \rangle_{\Omega} + \lambda(\alpha^i v \phi)_{\Omega} - \epsilon(\langle f v \rangle_{\partial\Omega} \phi |_{\partial\Omega})_{\partial\Omega},$$

and

$$\Lambda(\phi) = -\langle F | \phi \rangle_{\Omega} - \langle L^j | \partial_j \phi \rangle_{\Omega} - \langle G | \phi \rangle_{\partial\Omega},$$

it follows from the estimates \((\text{B.4})\) and \((\text{B.6})-(\text{B.10})\) that $B$ and $\Lambda$ define bounded forms on $H^1(\Omega)$, and, moreover, that $B$ satisfies the estimate

$$B(\phi, \phi) \geq (\kappa_1 - K \epsilon) \|\phi\|_{L^\infty(\Omega)}^2 + \epsilon \|\phi\|_{L^\infty(\Omega)} \|\phi\|_{W^{1,n}(\Omega)}^2,$$

wherever $\epsilon \geq \lambda^*$ and $0 \leq \epsilon \leq \frac{1}{\delta^*}$. Fixing $\lambda \in [\lambda^*, \infty)$ and $\epsilon \in (0, \frac{1}{\delta^*}]$, we can apply the Lax-Milgram theorem, see Theorem 1 from Section 6.2.1 of [11], to obtain the existence of a unique $v \in H^1(\Omega)$ satisfying $B(v, \phi) = \Lambda(\phi)$ for all $\phi \in H^1(\Omega)$. By definition of $B$ and $\Lambda$, it is clear that there exists a constant $\lambda^* = \lambda^*(\sigma, \mu) > 0$ such that

$$B(\phi, \phi) \geq \frac{\kappa_1}{2} \|\phi\|_{H^1(\Omega)}^2$$

whenever $\lambda \geq \lambda^*$ and $0 \leq \epsilon \leq \frac{1}{\delta^*}$. Fixing $\lambda \in [\lambda^*, \infty)$ and $\epsilon \in (0, \frac{1}{\delta^*}]$, we can apply the Lax-Milgram theorem, see Theorem 1 from Section 6.2.1 of [11], to obtain the existence of a unique $v \in H^1(\Omega)$ satisfying $B(v, \phi) = \Lambda(\phi)$ for all $\phi \in H^1(\Omega)$. By definition of $B$ and $\Lambda$, it is clear that there exists a constant $\lambda^* = \lambda^*(\sigma, \mu) > 0$ such that

$$B(\phi, \phi) \geq \frac{\kappa_1}{2} \|\phi\|_{H^1(\Omega)}^2$$

whenever $\lambda \geq \lambda^*$ and $0 \leq \epsilon \leq \frac{1}{\delta^*}$.

In addition to the above existence result, we will also require the following version of elliptic regularity.

**Theorem B.4.** Suppose $r, s \in \mathbb{R}$, $s > n/2$, $1 \leq r \leq s$, $b^{ij} \in H^s(\Omega, \mathbb{R}^{N \times N})$, $L^i \in H^r(\Omega, \mathbb{R}^N)$, $F \in H^{r-1}(\Omega, \mathbb{R}^N)$, $G \in H^{r-\frac{1}{2}}(\partial\Omega, \mathbb{R}^N)$, the $b^{ij}$ satisfy \((\text{B.3})\) and \((\text{B.4})\), $a^i = c = d = h = 0$, and $v$ is a weak solution of \((\text{B.1})-(\text{B.2})\). Then $v \in H^{r+1}(\Omega, \mathbb{R}^N)$ and satisfies

$$\|v\|_{H^{r+1}(\Omega)} \leq C \left( \|v\|_{H^r(\Omega)} + \|F\|_{H^{r-1}(\Omega)} + \|L\|_{H^r(\Omega)} + \|G\|_{H^{r-\frac{1}{2}}(\partial\Omega)} \right),$$

where $C = C(\kappa_1, \mu, \|b\|_{H^r(\Omega)})$.

**APPENDIX C. DETERMINANT FORMULAS**

**Lemma C.1.** Suppose that $X, Y \in \mathbb{C}^2$, $N \in \mathbb{C}$, and $L \in \text{Gl}(2, \mathbb{C})$ satisfies $L^t = L$. Then

$$\det \begin{pmatrix} L & X + Y^t N \\ -X^t Y + Y^t & N \end{pmatrix} = \det \begin{pmatrix} L & Y^t \\ Y^t & N \end{pmatrix} + \det(L)X^tL^{-1}X.$$

**Proof.** Direct computation. \(\square\)

**Lemma C.2.** Suppose that $X \in \mathbb{C}^2$, $N \in \mathbb{C}$, and $L \in \text{Gl}(2, \mathbb{C})$ satisfies $L^t = L$. Then

$$\det \begin{pmatrix} L & X \\ X^t N + X^t L^{-1}X & X \end{pmatrix} = N \det(L).$$

**Proof.** Noting that

$$\det \begin{pmatrix} L & X \\ X^t N + X^t L^{-1}X & X \end{pmatrix} = \det \left( \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} L & X \\ -X^t L^{-1}X & X \end{pmatrix} \right) = -\det \begin{pmatrix} L & X \\ -X^t L^{-1}X & X \end{pmatrix},$$
the proof follows since
\[
\det \left( \frac{L}{-X^{tr}} - N - X^{tr}L^{-1}X \right) = -N \det(L)
\]
by Lemma C.1.

 REFERENCES

1. R.A. Adams and J. Fournier, *Sobolev spaces*, 2nd ed., Academic Press, 2003.
2. L. Andersson and T.A. Oliynyk, *A transmission problem for quasi-linear wave equations*, J. Differential Equations 256 (2014), 2023–2078.
3. L. Andersson, B.G. Schmidt, and T.A. Oliynyk, *Dynamical compact elastic bodies in general relativity*, submitted, preprint [arXiv:1410.4894], 2014.
4. J.T. Beale, *Spectral properties of an acoustic boundary condition*, Indiana Univ. Math. J. 25 (1976), 895–917.
5. J.T. Beale and S.I. Rosencrans, *Acoustic boundary conditions*, Bull. Amer. Math. Soc. 80 (1974), 1276–1278.
6. D. Christodoulou, *The formation of shocks in 3-dimensional fluids*, EMS, 2007.
7. D. Coutand, J. Hole, and S. Shkoller, *Well-posedness of the free-boundary compressible 3-d Euler equations with surface tension and the zero surface tension limit*, SIAM Journal on Mathematical Analysis 45 (2013), 3690–3767.
8. D. Coutand, H. Lindblad, and S. Shkoller, *A priori estimates for the free-boundary 3d compressible euler equations in physical vacuum*, Commun. Math. Phys. 296 (2010), no. 2, 559–587 (English).
9. D. Coutand and S. Shkoller, *Well-posedness of the free-surface incompressible Euler equations with or without surface tension*, J. Amer. Math. Soc. 20 (2007), 829–930.
10. , *Well-posedness in smooth function spaces for the moving-boundary three-dimensional compressible euler equations in physical vacuum*, Arch. Rational Mech. Anal. 206 (2012), 515–616 (English).
11. L.C. Evans, *Partial differential equations*, 2nd ed., AMS, 2010.
12. J. Frauendiener, *A note on the relativistic Euler equations*, Class. Quantum Grav. 20 (2003), L193–L196.
13. A. Friedman, *Partial differential equations*, Krieger Publishing Company, 1976.
14. C.G. Gal, G.R. Goldstein, and J.A. Goldstein, *Oscillatory boundary conditions for acoustic wave equations*, J. Evol. Equ. 3 (2003), 623–635.
15. J. Jang and N. Masmoudi, *Well-posedness of compressible Euler equations in a physical vacuum*, to appear Comm. Pure Appl. Math, preprint [arXiv:1005.4441], 2010.
16. H. Koch, *Hyperbolic equations of second order*, Ph.D. thesis, Ruprecht-Karls-Universität, Heidelberg, 1990.
17. , *Mixed problems for fully nonlinear hyperbolic equations*, Math. Z. 214 (1993), 9–42.
18. H. Lindblad, *Well posedness for the motion a compressible liquid with free surface boundary*, Commun. Math. Phys. 260 (2005), 319–392.
19. , *Well-posedness for the motion of an incompressible liquid with free surface boundary*, Ann. of Math. 162 (2005), 109194.
20. H. Lindblad and K.H. Nordgren, *A priori estimates for the motion of a selfgravitating incompressible liquid with free surface boundary*, JHDE 6 (2009), 407–432.
21. P.M. Morse and K.U. Ingard, *Theoretical acoustics*, McGraw-Hill, 1968.
22. T.A. Oliynyk, *Lagrange coordinates for the Einstein-Euler equations*, Phys. Rev. D 85 (2012), 044019.
23. T. Runst and W. Sickel, *Sobolev spaces of fractional order, nemytskij operators, and nonlinear partial differential equations*, Walter de Gruhter, 1996.
24. J. Shatah and C. Zeng, *Geometry and a priori estimates for free boundary problems of the euler’s equation*, Comm. Pure Appl. Math. 61 (2008), no. 5, 698–744.
25. H.C. Simpson and S.J. Spector, *On the positivity of the second variation in finite elasticity*, Archive for Rational Mechanics and Analysis 98 (1987), 1–30.
26. M.E. Taylor, *Partial differential equations iii: nonlinear equations*, Springer, 1996.
27. Y. Trakhinin, *Local existence for the free boundary problem for nonrelativistic and relativistic compressible Euler equations with a vacuum boundary condition*, Comm. Pure Appl. Math. 62 (2009), 1151–1594.
28. M. Visser and C. Molina-Parí as, *Acoustic geometry for general relativistic barotropic irrotational fluid flow*, New Journal of Physics 12 (2010), 095014.
29. R.A. Walton, *A symmetric hyperbolic structure for isentropic relativistic perfect fluids*, Houston J. Math. 31 (2005), 145–160.