O-minimal cohomology and definably compact definable groups

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Abstract

Let \( \mathcal{N} \) an o-minimal expansion of a real closed field. We develop the cohomology theory for the category of \( \mathcal{N} \)-definable manifolds with continuous \( \mathcal{N} \)-definable maps and use this to solve the Peterzil-Steinhorn problem [ps] on the existence of torsion points on \( \mathcal{N} \)-definably compact \( \mathcal{N} \)-definable abelian groups. Namely we prove the following result: Let \( G \) be an \( \mathcal{N} \)-definably compact \( \mathcal{N} \)-definably connected \( \mathcal{N} \)-definable group of dimension \( n \). Then the o-minimal Euler characteristic of \( G \) is zero. Moreover, if \( G \) is abelian then \( \pi_1(G) = \mathbb{Z}^n \) and for each \( k > 1 \), the subgroup \( G[k] \) of \( k \)-torsion points of \( G \) is isomorphic to \( (\mathbb{Z}/k\mathbb{Z})^n \). We also compute the cohomology rings of \( \mathcal{N} \)-definably compact \( \mathcal{N} \)-definable groups.

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1 Introduction

We work over an o-minimal expansion $\mathcal{N} = (\mathbb{N}, 0, 1, <, +, \cdot, \ldots)$ of a real closed field. Definable means $\mathcal{N}$-definable (possibly with parameters). The results of this paper are an extension to the work of A. Woerheide in [Wo] where o-minimal homology was introduced. We develop the cohomology theory for the category of definable manifolds with continuous definable maps and use this to solve the problem of existence of torsion points on definably compact definable abelian groups (see [ps]). Namely we prove the following result (see theorem 7.8 and theorem 11.6).

**Fact 1.1** Let $G$ be a definably compact definably connected definable group of dimension $n$. Then the o-minimal Euler characteristic of $G$ is zero. Moreover, if $G$ is abelian then $\pi_1(G) \cong \mathbb{Z}^n$ and for each $k > 1$, the subgroup $G[k]$ of $k$-torsion points of $G$ is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^n$.

We also compute the cohomology rings of definable groups (see theorem 11.6). Other main results of this paper are: Poincaré duality (theorem 9.2), Alexander duality (theorem 9.4), Lefschetz duality (theorem 11.11), the Lefschetz fixed point theorem (theorem 10.10) in the category of Hausdorff definable manifolds with continuous definable maps. From this we obtain the usual corollaries, in particular the generalised Jordan-Brouwer separation theorem (corollary 9.9) for Hausdorff definable manifolds. Note that the Lefschetz fixed point theorem and the generalised Jordan-Brouwer separation theorem are also proved in [bo1] for the category of affine $C^p$ definable manifolds with (definable) $C^p$ definable maps with $p \geq 3$. The method used in [bo1] is based on intersection theory for such definable manifolds adapted from classical differential topology.

The main motivation for the work presented here is the classification problem for groups definable in o-minimal structures. The results from [e1] together with those from [pps1] reduce this problem to the classification problem for definably compact definable abelian groups in o-minimal structures. In [e2] we try to classify definably compact definable abelian groups $G$ under the assumption that $\pi_1(G) \cong \mathbb{Z}^{\dim G}$. As we mentioned above this assumption is true when we are working in an o-minimal expansion of a real closed field, we believe that this remains true in general, although to prove it one would
need to develop possibly an o-minimal analogue of “Grothendieck theory of duality”.

Most of the results of this paper are modifications of classical results in algebraic topology (see for example [d], [f], [g] or [ro]). For the convenience of model theorists and since the setting is different we will be careful enough to include all the details. In particular, in order to introduce terminology and notation we include a brief description of Woerheide work. We will follow L. van den Dries book [vdd] for results on o-minimal structures and for the non model theorists we recall in section 2 some basic notions of o-minimality.

2 O-minimal structures

Definition 2.1 A structure \( \mathcal{N} \) consists of a nonempty set \( N \) together with:

(i) a set of constants \((c^N_k)_{k \in K}\), where \( c^N_k \in N \);

(ii) a family of maps \((f^N_j)_{j \in J}\), where \( f^N_j \) is an \( n_j \)-ary map, \( f^N_j : N^{n_j} \rightarrow N \) and

(iii) a family \((R^N_i)_{i \in I}\) of relations that is, for each \( i \), \( R^N_i \) is a subset of \( N^{n_i} \) for some \( n_i \geq 1 \).

The language \( \mathcal{L} \) associated to a structure \( \mathcal{N} \) consists of:

(i) for each constant \( c^N_k \), a constant symbol \( c^k \); (ii) for each map \( f^N_j \) a function symbol, \( f^j \) of arity \( n_j \), and (iii) for each relation \( R^N_i \), a relation symbol, \( R^i \) of arity \( n_i \).

We often use the following notation \( \mathcal{N} = (N, (c^N_k)_{k \in K}, (f^N_j)_{j \in J}, (R^N_i)_{i \in I}) \) and sometimes we omit the superscripts. Similarly we use the notation \( \mathcal{L} = \{(c_k)_{k \in K}, (f_j)_{j \in J}, (R_i)_{i \in I}\} \). If \( \mathcal{L} \) is the language associated with the structure \( \mathcal{N} \) we say that \( \mathcal{N} \) is an \( \mathcal{L} \)-structure. If \( \mathcal{L}' \subseteq \mathcal{L} \) are two languages and \( \mathcal{N}' \) and \( \mathcal{N} \) are respectively an \( \mathcal{L}' \)-structure and an \( \mathcal{L} \)-structure, such that \( \mathcal{N}' = \mathcal{N} \) then we say that \( \mathcal{N} \) is an expansion of \( \mathcal{N}' \) or that \( \mathcal{N}' \) is a reduct of \( \mathcal{N} \).

Let \( \mathcal{L} \) be a language and \( \mathcal{N} \) an \( \mathcal{L} \)-structure. We are going to define inductively the set of \( \mathcal{L} \)-formulas and satisfaction of an \( \mathcal{L} \)-formula \( \phi \) in \( \mathcal{N} \), in order to define the \( \mathcal{N} \)-definable sets.

Definition 2.2 The set of \( \mathcal{L} \)-terms is generated inductively by the following rules: (i) every variable \( x_i \) from a countable set of variable \((x_q)_{q \in Q}\) is an \( \mathcal{L} \)-term, (ii) every constant of \( \mathcal{L} \) is an \( \mathcal{L} \)-term and (iii) if \( f \) is in \( \mathcal{L} \) is an \( n \)-ary function, and \( t_1, \ldots, t_n \) are \( \mathcal{L} \)-terms, then \( f(t_1, \ldots, t_n) \) is an \( \mathcal{L} \)-term. An atomic \( \mathcal{L} \)-formula is an expression of the form: \( t_1 = t_2 \) or \( R(t_1, \ldots, t_n) \)
where $R$ is an $n$-ary relation in $\mathcal{L}$ and $t_1, \ldots, t_n$ are $\mathcal{L}$-terms. We sometimes write $R(t_1(x_1, \ldots, x_k), \ldots, t_n(x_1, \ldots, x_k))$ if we want to explicitly show the variables occurring in the atomic $\mathcal{L}$-formula. Given a tuple $a \in N^k$, we say that $a$ satisfies $R(t_1(x), \ldots, t_n(x))$ in $\mathcal{N}$ where $x = (x_1, \ldots, x_k)$ if $R^N(t_1(a), \ldots, t_n(a))$ holds. We denote this by $\mathcal{N} \models R(t_1(a), \ldots, t_n(a))$.

The set of $\mathcal{L}$-formulas is generated inductively by the following rules: (i) all atomic $\mathcal{L}$-formulas are $\mathcal{L}$-formulas; (ii) if $\phi_1(x)$ and $\phi_2(x)$ are $\mathcal{L}$-formulas, then $(\phi_1 \land \phi_2)(x)$ and $(\phi_1 \lor \phi_2)(x)$ are $\mathcal{L}$-formulas, and for $a \in N^k$, $\mathcal{N} \models (\phi_1 \land \phi_2)(a)$ iff $\mathcal{N} \models \phi_1(a)$ and $\mathcal{N} \models \phi_2(a)$, we also have the obvious clause for $\forall$; (iii) if $\phi(x)$ is an $\mathcal{L}$-formula, $\neg \phi(x)$ is an $\mathcal{L}$-formula, and for $a \in N^k$, $\mathcal{N} \models \neg \phi(a)$ iff $\phi(a)$ does not hold in $\mathcal{N}$ (this is denote by $\mathcal{N} \not\models \phi(a)$); (iv) if $\phi(x, x_{k+1})$ is an $\mathcal{L}$-formula, then $\exists x_{k+1} \phi(x, x_{k+1})$ is an $\mathcal{L}$-formula, and for $a \in N^k$, $\mathcal{N} \models \exists x_{k+1} \phi(a, x_k)$ iff there exists $b \in N$ such that $\mathcal{N} \models \phi(a, b)$; (v) the obvious clauses for $\forall$.

We are now ready to define the important notion of $\mathcal{N}$-definable sets and $\mathcal{N}$-definable maps.

**Definition 2.3** A subset $D \subseteq N^k$ is an $\mathcal{N}$-definable subset (defined over $A \subseteq N$) if there is a $\mathcal{L}$-formula $\phi(x, y)$ with $x = (x_1, \ldots, x_k)$ and $y = (x_{k+1}, \ldots, x_{k+m})$ and some $b \in A^m$ such that $D = \{a \in N^k : \mathcal{N} \models \phi(a, b)\}$. If $X \subseteq N^k$ and $Y \subseteq N^m$ are $\mathcal{N}$-definable sets (over $A \subseteq N$), a function $f : X \rightarrow Y$ is $\mathcal{N}$-definable (over $A$) if its graph is an $\mathcal{N}$-definable set (over $A$). More generally, a structure $\mathcal{M} = (M, (c^M_k)_{k \in K}, (f^M_j)_{j \in J}, (R^M_i)_{i \in I})$ is $\mathcal{N}$-definable (over $A$) if: (i) $M \subseteq N^l$ is $\mathcal{N}$-definable (over $A$); (ii) for each $k \in K$ there is a point $m_k \in M$ corresponding to $c^M_k$; (iii) for each $j \in J$ the function $f^M_j : M^n_{\jmath} \rightarrow M$ is $\mathcal{N}$-definable (over $A$) and (iv) for each $i \in I$ the relation $R^M_i \subseteq M^m_i$ is $\mathcal{N}$-definable (over $A$). Note that, in this case every $\mathcal{M}$-definable set is also an $\mathcal{N}$-definable set.

Finally we include here one more definition from basic model theory which palys a crucial role in our paper.

**Definition 2.4** Given two $\mathcal{L}$-structures $\mathcal{N}$ and $\mathcal{M}$, we say that $\mathcal{N}$ is an $\mathcal{L}$-substructure of $\mathcal{M}$, denoted by $\mathcal{N} \subseteq \mathcal{M}$, if $N \subseteq M$ and: (i) for every constant symbol $c$ in $\mathcal{L}$, $c^N = c^M$; (ii) for every $n$-ary function symbol $f$ in $\mathcal{L}$, for every $a \in N^n$, $f^N(a) = f^M(a)$ and (iii) for every $n$-ary relation symbol $R$ in $\mathcal{L}$
and for every $a \in N^n$, $R^N(a)$ iff $R^M(a)$. Let $\mathcal{N} \subseteq \mathcal{M}$. We say that $\mathcal{M}$ is an elementary extension of $\mathcal{N}$ (or that $\mathcal{N}$ is an elementary substructure of $\mathcal{M}$), denoted by $\mathcal{N} \preceq \mathcal{M}$, if for every $\mathcal{L}$-formula $\phi(x)$, for all $a \in N^k$, we have $\mathcal{N} \models \phi(a)$ iff $\mathcal{M} \models \phi(a)$.

Note that (by Tarski-Vaught test) $\mathcal{N} \preceq \mathcal{M}$ iff for every non empty $\mathcal{M}$-definable set $E \subseteq M^l$, defined with parameters from $N$ and $\mathcal{N} \preceq \mathcal{M}$, then the $\mathcal{L}$-formula which determines $S$ determines an $\mathcal{M}$-definable set $S(M) \subseteq M^l$ ("the $\mathcal{M}$-points of $S$").

The theory $Th(\mathcal{N})$ of an $\mathcal{L}$-structure $\mathcal{N}$ is the collection of all $\mathcal{L}$-sentences (i.e., $\mathcal{L}$-formulas without free variables) $\sigma$ such that $\mathcal{N} \models \sigma$. $\mathcal{N}$ is elementarily equivalent to $\mathcal{M}$, denoted $\mathcal{N} \equiv \mathcal{M}$ iff $Th(\mathcal{N}) = Th(\mathcal{M})$. Clearly, if $\mathcal{N} \preceq \mathcal{M}$ then $\mathcal{N} \equiv \mathcal{M}$. The following two facts (the L"owenheim-Skolem theorems) are fundamental theorems of basic model theory: (1) if $\mathcal{N}$ an $\mathcal{L}$-structure with $X \subseteq N$, then for every cardinal $\kappa$ such that $|X| + |\mathcal{L}| \leq \kappa \leq |N|$, $\mathcal{N}$ has an elementary substructure $\mathcal{M}$ such that $X \subseteq M$ and $|M| = \kappa$; and (2) if $\mathcal{N}$ be an infinite $\mathcal{L}$-structure, then for any cardinal $\kappa > |N|$, $\mathcal{N}$ has an elementary extension of cardinality $\kappa$.

**Definition 2.5** An o-minimal structure is an expansion $\mathcal{N} = (N, <, \ldots)$ of a linearly ordered nonempty set $(N, <)$, such that every $\mathcal{N}$-definable subset of $N$ is a finite union of points and intervals with endpoints in $N \cup \{-\infty, +\infty\}$.

Note the following important results: let $\mathcal{N}$ be an o-minimal structures and $\mathcal{L}$ its language. Then: (1) every $\mathcal{N}$-definable structure $\mathcal{M}$ which is an expansion of a linearly ordered nonempty set $(M, <_M)$ is also o-minimal; (2) [KPS] if $\mathcal{M}$ is a structure (in the language of $\mathcal{N}$) such that $\mathcal{N} \equiv \mathcal{M}$ then $\mathcal{M}$ is also o-minimal; (3) [PiS1] for every $A \subseteq N$ there is a prime model of $Th(\mathcal{N})$ over $A$ i.e., there is an o-minimal structure $\mathcal{P}$ such that $A \subseteq P$ and for all $\mathcal{M}$ with $A \subseteq M$ and either $\mathcal{M} \preceq \mathcal{N}$ or $\mathcal{N} \preceq \mathcal{M}$ we have $\mathcal{P} \preceq \mathcal{M}$; and (4) for every $\kappa > \max\{\aleph_0, |\mathcal{L}|\}$ there up to isomorphism $2^\kappa$ o-minimal structures $\mathcal{M}$ such that $|M| = \kappa$ and $\mathcal{M} \equiv \mathcal{N}$ (see [Sh]), and if $\mathcal{L}$ is countable then up to isomorphism there are either $2^{\aleph_0}$ or $6^{\aleph_0}3^{m}$ countable o-minimal structures $\mathcal{M}$ such that $\mathcal{M} \equiv \mathcal{N}$ (see [M]).
There are many geometric properties of $\mathcal{N}$-definable sets and $\mathcal{N}$-definable maps in an o-minimal structures $\mathcal{N}$. For example, two of the most powerful results are the monotonicity theorem for definable one variable functions and the $C^p$-cell decomposition theorem for definable sets and definable maps. We will now explain the $C^p$-cell decomposition theorem (here $p = 0$ if $\mathcal{N}$ is not an expansion of a (real closed) field) in order to introduce the notion of o-minimal dimension and Euler characteristic.

**Definition 2.6** $C^p$-cells and o-minimal dimension are defined inductively as follows: (i) the unique non empty $\mathcal{N}$-definable subset of $\mathcal{N}^0$ is a $C^p$-cell of dimension zero, a point in $\mathcal{N}^1$ is a $C^p$-cell of dimension zero and an open interval in $\mathcal{N}^1$ is a $C^p$-cell of dimension one; (ii) a $C^p$-cell in $\mathcal{N}^{l+1}$ of dimension $k$ (resp., $k+1$) is an $\mathcal{N}$-definable set of the form $\Gamma(f)$ (the graph of $f$) where $f : C \rightarrow \mathcal{N}$ is a $C^p$-definable function and $C$ is a $C^p$-cell in $\mathcal{N}^l$ of dimension $k$ (resp., of the form $(f,g)_C := \{(x,y) \in C \times \mathcal{N} : f(x) < y < g(x)\}$ where $f,g : C \rightarrow \mathcal{N}$ are $C^p$-definable function with $-\infty \leq f < g \leq +\infty$ and $C$ is a $C^p$-cell in $\mathcal{N}^l$ of dimension $k$. The Euler characteristic $E(C)$ of a $C^p$-cell $C$ of dimension $k$ is defined to be $(-1)^k$.

**Definition 2.7** A $C^p$-cell decomposition of $\mathcal{N}^m$ is a special kind of partition of $\mathcal{N}^m$ into finitely many $C^p$-cells: a partition of $\mathcal{N}^1$ into finitely many disjoint $C^p$-cells of dimension zero and one is a $C^p$-cell decomposition of $\mathcal{N}^m$ and, a partition of $\mathcal{N}^{k+1}$ into finitely many disjoint $C^p$-cells $C_1, \ldots, C_m$ is a $C^p$-cell decomposition of $\mathcal{N}^{k+1}$ if $\pi(C_1), \ldots, \pi(C_m)$ is a $C^p$-cell decomposition of $\mathcal{N}^k$ (where $\pi : N^{k+1} \rightarrow N^k$ is the projection map onto the first $k$ coordinates). Let $A_1, \ldots A_k \subseteq A \subseteq \mathcal{N}^m$ be $\mathcal{N}$-definable sets. A $C^p$-cell decomposition of $A$ compatible with $A_1, \ldots A_k$ is a finite collection $C_1, \ldots, C_l$ of $C^p$ partitioning $A$ obtained from a $C^p$-cell decomposition of $\mathcal{N}^m$ such that for every $(i,j) \in \{1, \ldots, k\} \times \{1, \ldots, l\}$ if $C_j \cap A_i \neq \emptyset$ then $C_j \subseteq A_i$.

**Fact 2.8** ($C^p$-cell decomposition theorem) Given $\mathcal{N}$-definable sets $A_1, \ldots, A_k \subseteq A \subseteq \mathcal{N}^m$ there is a $C^p$-cell decomposition of $A$ compatible with $A_1, \ldots A_k$ and, for every $\mathcal{N}$-definable function $f : A \rightarrow \mathcal{N}$, $A \subseteq \mathcal{N}^m$, there is a $C^p$-cell decomposition of $A$, such that each restriction $f|_C : C \rightarrow \mathcal{N}$ is $C^p$ for each cell $C \subseteq A$ of the $C^p$-cell decomposition.
The o-minimal dimension \( \dim(X) \) and Euler characteristic \( E(X) \) of a \( \mathcal{N} \)-definable set \( X \) are defined by \( \dim(X) = \max\{ \dim(C) : C \in \mathcal{C} \} \) and \( E(X) = \sum_{C \in \mathcal{C}} E(C) \) where \( \mathcal{C} \) is some (equivalently any) \( C^p \)-cell decomposition of \( X \). These notions are well behaved under the usual set theoretic operations on \( \mathcal{N} \)-definable sets, are invariant under \( \mathcal{N} \)-definable bijections and given an \( \mathcal{N} \)-definable family of \( \mathcal{N} \)-definable sets, the set of parameters whose fibre in the family has a fixed dimension (resp., Euler characteristic) is also an \( \mathcal{N} \)-definable set. The cell decomposition theorem is also used to show that every \( \mathcal{N} \)-definable set has only finitely many \( \mathcal{N} \)-definably connected components, and given an \( \mathcal{N} \)-definable family of \( \mathcal{N} \)-definable sets there is a uniform bound on the number of \( \mathcal{N} \)-definably connected components of the fibres in the family.

Since in this paper we are concerned only with o-minimal expansions of ordered fields (necessarily real closed fields) we will from now on assume that \( \mathcal{N} \) is an o-minimal expansion of a real closed field and definable means \( \mathcal{N} \)-definable. Below we list the some of properties of definable sets and definable maps that we will be using through this paper.

**Definition 2.9** Let \( S_1, \ldots, S_k \subseteq S \subseteq N^m \) be definable sets. A **definable triangulation** in \( N^m \) of \( S \) compactible with \( S_1, \ldots, S_k \) is a pair \((\Phi, K)\) consisting of a complex \( K \) in \( N^m \) and a definable homeomorphism \( \Phi : S \rightarrow |K| \) such that each \( S_i \) is a union of elements of \( \Phi^{-1}(K) \). We say that \((\Phi, K)\) is a **stratified definable triangulation** of \( S \) compactible with \( S_1, \ldots, S_k \) if: \( m = 0 \) or \( m > 0 \) and there is a stratified definable triangulation \((\Psi, L)\) of \( \pi(S) \) compactible with \( \pi(S_1), \ldots, \pi(S_k) \) (where \( \pi : N^m \rightarrow N^{m-1} \) is the projection onto the first \( m-1 \) coordinates) such that \( \pi|_{Vert(K)} : K \rightarrow L \) is a simplicial map and the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\Phi} & |K| \\
\downarrow^\pi & & \downarrow^\pi \\
\pi(S) & \xrightarrow{\Psi} & |L|.
\end{array}
\]

We say that \((\Phi, K)\) is a **quasi-stratified definable triangulation** of \( S \) compatible with \( S_1, \ldots, S_k \) if there is a linear bijection \( \alpha : N^m \rightarrow N^m \) such that \((\alpha \Phi \alpha^{-1}, \alpha K)\) is a stratified definable triangulation of \( \alpha(S) \) compatible with \( \alpha(S_1), \ldots, \alpha(S_k) \).
Fact 2.10 (Definable triangulation theorem). Let $S_1, \ldots, S_k \subseteq S \subseteq N^m$ be definable sets. Then, there is a definable triangulation of $S$ compatible with $S_1, \ldots, S_k$. Moreover, if $S$ is bounded then, there is a quasi-stratified definable triangulation of $S$ compatible with $S_1, \ldots, S_k$.

Several other geometric properties from semialgebraic and subanalytic geometry also hold for definable sets and definable maps: we have a definable curve selection theorem, a definable trivialization theorem and finally (see [DM2]) a definable analogue of the uniform bounds on growths theorem, the $C^p$-multiplier theorem, the generalised Lojasiewicz inequality, the $C^p$ zero set theorem, the $C^p$ Whitney stratification theorem, etc.

3 Definable manifolds

3.1 Definable manifolds

Definition 3.1 A definable manifold (of dimension $m$ defined over $A$) is a triple $X := (X, (X_i, \phi_i)_{i \in I})$ where $\{X_i : i \in I\}$ is a finite cover of the set $X$ and for each $i \in I$, (1) we have injective maps $\phi_i : X_i \rightarrow N^m$ such that $\phi_i(X_i)$ is an open definably connected definable set (defined over $A$); (2) each $\phi_i(X_i \cap X_j)$ is an open definable subset of $\phi_i(X_i)$ (defined over $A$) and (3) the map $\phi_{ij} : \phi_i(X_i \cap X_j) \rightarrow \phi_j(X_i \cap X_j)$ given by $\phi_{ij} := \phi_j \circ \phi_i^{-1}$ is a definable homeomorphism (defined over $A$) for all $j \in I$ such that $X_i \cap X_j \neq \emptyset$.

Definition 3.2 Let $X = (X, (X_i, \phi_i)_{i \in I})$ and $Y = (Y, (Y_j, \psi_j)_{j \in J})$ be definable manifolds. A definable subset of $X$ is a set $Z \subseteq X$ such that the sets $\phi_i(Z \cap X_i)$ are definable. If all of $\phi_i(Z \cap X_i)$ are definable over $B$ we say that $Z$ is definable over $B$. Let $Z$ be a definable subset of $X$. A map $f : Z \subseteq X \rightarrow Y$ is a definable map if $\psi_j \circ f \circ \phi_i^{-1} : \phi_i(X_i \cap Z) \rightarrow \psi_j(Y_j)$ is a definable map whenever it is defined.

A definable manifold $X$ has a natural topology generated by the definable open subsets of $X$ i.e., those definable subsets $U$ such that for all $i \in I$, $\phi_i(U \cap X_i)$ is an open definable subset of $\phi_i(X_i)$. We say that a definable manifold $X$ is definably connected if there are no two disjoint definable open subsets of $X$ whose union is $X$. By [2], $X$ is definably connected iff it is
definably path connected i.e., for all \( x, y \in X \) there a definable continuous map \( \alpha : [0, 1] \to X \) such that \( \alpha(0) = x \) and \( \alpha(1) = y \).

**Definition 3.3** A definable submanifold of \( X \) is a definable manifold \( Y \) such that \( Y \subseteq X \) and each \( \psi_j : Y_j \to \psi_j(Y_j) \) are definable and whose topology is the induced topology from \( X \). The definable submanifolds of \( N^m \) are called affine definable manifolds.

**Remark 3.4** It's easy to see that definable manifolds are T1, but one can construct easily an example of a definable manifold \( X \) which is not Hausdorff: Let \( X_1 := N \times \{0\} \), \( X_2 := N \times \{1\} \) with the subsets \( X^{-}_1 := N^{<0} \times \{0\} \) and \( X^{-}_2 := N^{<0} \times \{1\} \) identified. But affine definable manifolds are definably normal \([vdd]\) and by lemma 3.5 below every Hausdorff definable manifold \( X \) is definably regular. By \([vdd]\) every definably regular definable manifold is definably homeomorphic to an affine definable manifold.

**Lemma 3.5** Every Hausdorff definable manifold \( X \) is definably regular.

**Proof.** This is contained in the proof of lemma 10.4 \([bo1]\). For each \( i \in I \) and \( x, y \in X_i \), let \( d_i(x, y) := |\phi_i(x) - \phi_i(y)| \). Let \( K \) be a closed definable subset of \( X \) and \( a_0 \in X \setminus K \). Let \( I^K := \{ i \in I : K \cap X_i \neq \emptyset \} \), \( I_{a_0} := \{ j \in I : a_0 \in X_j \} \) and for each \( i \in I^K \) let \( K_i := K \cap X_i \). For \( \epsilon \in N \), \( \epsilon > 0 \) and \( i \in I^K \) define \( K^{\epsilon}_i \) to be the set of points \( y \in \bigcup_{j \in I} X_j \) such that there is a point \( x \) in \( K_i \cap \bigcup_{j \in I} X_j \) with \( d_j(x, y) < \epsilon \). Since \( I \) is finite, \( K^{\epsilon}_i \) is open (and contains \( K_i \)). Let \( K^\epsilon := \bigcup_{i \in I^K} K^{\epsilon}_i \). Then \( K^\epsilon \) is an open definable subset containing \( K \). Similarly we define \( L^\epsilon \) containing \( a_0 \) to be the open definable subset of all \( y \in \bigcup_{j \in I} X_j \) such that there is \( x \in \bigcup_{j \in I} X_j \) with \( d_j(x, y) < \epsilon \).

If for some \( \epsilon > 0 \) \( K^{\epsilon} \cap L^\epsilon = \emptyset \) we are done. Otherwise, there is a finite subset \( J \) of \( I^K \) such that \( K^{\epsilon}_j \cap L^\epsilon \neq \emptyset \) for all sufficiently small \( \epsilon > 0 \), where \( K^{\epsilon}_j := \bigcup_{i \in J} K^{\epsilon}_i \). Now by definable choice (chapter 6, proposition 1.2 \([vdd]\)) and lemma 10.3 \([bo1]\), there is a definable continuous map \( a : (0, \epsilon) \to X \) such that \( a(\epsilon) \in K^{\epsilon}_j \cap L^\epsilon \). Since \( X \) is Hausdorff \( a_0 = \lim_{\epsilon \to 0} a(\epsilon) \). We reach a contradiction by showing that \( a_0 \in K \). Choose \( i \) such that \( a_0 \in X_i \). Then for all sufficiently small \( \epsilon > 0 \) we have \( a(\epsilon) \in X_i \) so \( d_i(a(\epsilon), K \cap X_i) \) is
well defined and must be less than $\epsilon$ since $a(\epsilon)$ belongs to $K_j$. Therefore, 
$\lim_{\epsilon \to 0} d_i(a(\epsilon), K \cap X_i) = 0$ i.e., $d_i(a_0, K \cap X_i) = 0$ and $a_0 \in K$. \hfill \Box

Note that if $X$ (resp., $\mathcal{X}$) is a definable manifold over $A$ (resp., a collection 
of definable manifold over $A$ and definable maps over $A$ between definable 
manifolds in $\mathcal{X}$) then $X$ (resp., $\mathcal{X}$) can be identified with a definable structure 
over $A$. Moreover, if $\mathcal{M}$ is a structure in the language of $\mathcal{N}$ such that $A \subseteq M$ 
and either $\mathcal{N} \preceq \mathcal{M}$ or $\mathcal{M} \preceq \mathcal{N}$ then $X$ (resp., $\mathcal{X}$) can also be seen as an $\mathcal{M}$- 
definable manifold over $A$ (resp., a collection of $\mathcal{M}$-definable manifold over $A$ and $\mathcal{M}$-definable maps over $A$ between $\mathcal{M}$-definable manifolds in $\mathcal{X}$).

Although many topological notions (such has Hausdorff, regular, definably connected, etc.), of definable manifolds and their definable maps are 
invariant under the process mentioned above, other notions such as compact 
and locally compact are not good notions in our context since, for example 
if $N$ is non standard then no nontrivial definable manifold will be locally 
compact or compact. For this reason a weaker notion of compactness for 
definable manifold was introduced in [ps]:

**Definition 3.6** A definable manifold $X$ is *definably compact* if it is Hausdorff 
and for every definable curve $\sigma : (a, b) \subseteq N \to X$, where $-\infty \leq a < b \leq +\infty$ 
there are $c, d \in X$ such that $\lim_{x \to a^+} \sigma(x) = c$ and $\lim_{x \to b^-} \sigma(x) = d$.

By [ps], an affine definable manifold $X$ is definably compact iff $X$ is a 
closed and bounded definable subset of some $N^m$. Therefore, this notion of 
definable compactness is invariant under elementary extension/substructure.

**Example 3.7** Let $H := (H, \cdot)$ be a definable a group. Results from [p] and 
[ps] show that: (1) $H$ has a unique structure of a definable manifold such 
that the group operations are definable continuous maps. Note that, since 
definable groups are Hausdorff (as a definable manifolds), every definable 
group is isomorphic to an affine definable group. Moreover, there is in a 
uniformly definable family of subsets of $H$ containing the identity element $e$, 
$\{V_a : a \in S\}$ such that $\{V_a : a \in S\}$ is a basis for the open neighbourhoods of 
e; (2) the topology of a definable subgroup $G$ of $H$ agrees with the topology 
induced on $G$ by $H$, $G$ is closed in $H$ and if $\dim G = \dim H$ then $G$ is open 
in $H$; and (3) a definable homomorphism of definable groups $\alpha : H \to K$ is 
a continuous (in fact $C^p$ for every $p$) definable map.
3.2 Definable manifolds with boundary

**Definition 3.8** A **definable manifold with boundary** of dimension $m$ is a triple $X := (X, (X_i, \phi_i)_{i \in I})$ where $\{X_i : i \in I\}$ is a finite cover of the set $X$ and for each $i \in I$, (1) we have maps $\phi_i : X_i \rightarrow \{(x_1, \ldots, x_m) \in N^m : x_m \geq 0\}$ such that $\phi_i(X_i)$ is a definable open definably connected set; (2) each $\phi_i(X_i \cap X_j)$ is definable and open in $\phi_i(X_i)$; (3) the map $\phi_{ij} : \phi_i(X_i \cap X_j) \rightarrow \phi_j(X_i \cap X_j)$ given by $\phi_{ij} := \phi_j \circ \phi_i^{-1}$ is a definable homeomorphism for all $j \in I$ such that $X_i \cap X_j \neq \emptyset$ and (4), there is $i \in I$ such that $\{x \in X_i : \phi_i(x) \in N^{m-1} \times \{0\}\} \neq \emptyset$.

Just like for definable manifolds, we have the notions of definable subset, open definable subset, definable submanifold and definable maps etc., for definable manifolds with boundary.

**Definition 3.9** We have the following definable submanifolds of $X$ of dimension $m$ and $m-1$ respectively: $\hat{X} := (\hat{X}, (\hat{X}_i, \phi_{ii})_{i \in I})$ where $\hat{X} := \bigcup_{i \in I} \hat{X}_i$ and $\hat{X}_i := \{x \in X_i : \phi_i(x) \in \{(x_1, \ldots, x_m) \in N^m : x_m > 0\}\}$; and $\partial X := (\partial X, (\partial X_i, \phi_{ii})_{i \in I})$ where $\partial X := \bigcup_{i \in I} \partial X_i$ and $\partial X_i := \{x \in X_i : \phi_i(x) \in N^{m-1} \times \{0\}\}$. $\partial X$ is called the **boundary** of $X$.

We include here the following remark which will be use in subsection $$.

**Remark 3.10** Then $X$ can be definably embedded in two different ways in a definable manifold $2X$ of dimension $m$ such that $X$ is a closed definable submanifold, and there are open definable submanifolds $Y_1$ and $Y_2$ of $Y$ each containing a copies $X_1$ and $X_2$ of $X$ and such that $X_i$ is a definable deformation retract of $X_i$ and $Y_1 \cap Y_2$ is definably homotopically equivalent to $\partial X$.

In fact let $2X = (Y, (Y_i, \psi_i)_{i \in I})$ be given by: $Y_i = \hat{X}_i^1 \cup \partial X_i \times [0, -1] \cup \hat{X}_i^2$ where $\hat{X}_i^1$ and $\hat{X}_i^2$ are copies of $\hat{X}_i$ and $\psi_i$ is given by $\psi_i = \phi_i$ on $\hat{X}_i^1$, $\psi_i = \phi_i \times 1_{[0, -1]}$ on $\partial X_i \times [0, -1]$ and for $x \in \hat{X}_i^2$ define $\psi_i(x) = (-1 - \phi_i^1(x), \ldots, -1 - \phi_i^m(x))$ where $\phi_i = (\phi_i^1, \ldots, \phi_i^m)$. Let $Y_1 = \hat{X}^1 \cup \partial X \times [0, -1]$ and $Y_2 = \partial X \times (0, 1] \cup \hat{X}^2$ where for $l = 1, 2$, $\hat{X}^l = \bigcup_{i \in I} \hat{X}_i^l$.

*We will from now on assume that, all definable manifolds are Hausdorff definable manifolds, hence affine.*
3.3 The fundamental group

Let $X = (X, (X_i, \phi_i)_{i \in I})$ and $Y = (Y, (Y_j, \psi_j)_{j \in J})$ be definably connected definable manifolds. Given $A_1, \ldots, A_l \subseteq X'$ and $B_1, \ldots, B_l \subseteq Y'$ definable subsets of $X$ and $Y$ respectively and a continuous definable map $f : X' \to Y'$ such that $f(A_i) \subseteq B_i$, we write $f : (X', A_1, \ldots, A_l) \to (Y', B_1, \ldots, B_l)$ (if $l = 1$, $A_1 = \{x\}$ and $B_1 = \{y\}$ we write $f : (X', x) \to (Y', y)$).

**Definition 3.11** Two continuous definable maps $f, g : (X', A_1, \ldots, A_l) \to (Y', B_1, \ldots, B_l)$ are *definably homotopic* if there is a continuous definable map $H : (X \times [0, 1], A_1 \times [0, 1], \ldots, A_l \times [0, 1]) \to (Y, B_1, \ldots, B_l)$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X'$. $H$ is called a *definable homotopy*. This is an equivalence relation compatible with composition in the set of all such continuous definable maps. We denote by $[f]$ the equivalence class of $f$ and by $[(X', A_1, \ldots, A_l), (Y', B_1, \ldots, B_l)]$ the set of all such classes. $(X', A_1, \ldots, A_l)$ and $(Y', B_1, \ldots, B_l)$ are said to have the same *definable homotopy type* if there are continuous definable maps $f : (X', A_1, \ldots, A_l) \to (Y', B_1, \ldots, B_l)$ and $g : (Y', B_1, \ldots, B_l) \to (X', A_1, \ldots, A_l)$ such that $[g \circ f] = [1_{X'}]$ and $[f \circ g] = [1_{Y'}]$. This is an equivalence relation and we denote by $[(X', A_1, \ldots, A_l)]$ the equivalence class of $(X', A_1, \ldots, A_l)$.

The following result is proved just like in the classical case, with the word definable add where necessary (for details see chapter 12 [f]):

**Fact 3.12** There is a covariant functor $\pi_1$ from the category of pointed definable manifolds into the category of groups. $\pi_1(X, x) := [(0, 1), \{0, 1\}], (X, x)$ is a group called the definable fundamental group of $X$ at $x$ with the product defined by $[\sigma][\gamma] := [\sigma \cdot \gamma]$ and given definable maps $f, g : (X, x) \to (Y, y)$, $\pi_1(f)$ (which is denoted $f_*$) is defined by $f_*([\sigma]) = [f \circ \sigma]$. Moreover, if $[f] = [g]$ then $f_* = g_*;$ if there is a definable path in $X$ from $x_0$ to $x_1$ then $\pi_1(X, x_0) \simeq \pi_1(X, x_1);$ and finally $\pi_1(X \times Y, (x, y)) \simeq \pi_1(X, x) \times \pi_1(Y, y)$.

**Definition 3.13** A definably connected definable manifold $X$ is called *definably simply connected* if $\pi_1(X, x) = 0$ for some (equivalently for all) $x \in X$.

We finish this subsection, with the following result from [bo2] (for a different proof see [e2]) on how to compute the definable fundamental group
of definable manifolds. Note however that the main results of this paper on definable groups were proved (in an old version of this paper) without using fact 3.15 below. In fact, although we need to know that the fundamental group of a definable group is a finitely generated abelian group, this will be proved directly in fact 5.18 and lemma 11.4.

**Definition 3.14** Let $K$ be a simplicial complex, $v$ a vertex of $K$ and $T$ a maximal tree in $K$ ($T$ contains all the vertices of $K$). An edge path in $K$ is a sequence $\sigma = u_1, u_2, \ldots, u_l$ of vertices of $K$ such that for all $i = 1, \ldots, l - 1$, $u_i, u_{i+1}$ are vertices of an edge in $K$. If $\gamma = w_1, w_2, \ldots, w_k$ is another edge path in $K$ and $u_l, w_1$ are vertices of an edge in $K$ then the concatenation $\sigma \cdot \gamma$ is also an edge path in $K$. $\sigma$ is an edge loop in $K$ at $v$ if $v = u_1 = u_l$. $E(K, v)$ is the group under the operation $[\sigma][\gamma] := [\sigma \cdot \gamma]$ of classes $[\sigma]$ of edge loops $\sigma$ in $K$ at $v$ under the equivalence relation: $v, u, a, b, c, \ldots, w, v$ $v, u, a, c, \ldots, w, v$ if $abc$ spans a $k$-simplex of $K$ where $k = 0, 1, 2$. $E(K, v)$ is isomorphic to the group $G(K, T)$ generated by the 1-simplexes of $K$, denoted $g_{a,b}$ for each edge $a, b$ and with relations: (1) $g_{a,b} = 1$ if $a, b$ spans a simplex of $L$ and (2) $g_{a,b}g_{b,c} = g_{a,c}$ if $a, b, c$ spans a simplex of $K$.

**Fact 3.15** (Tietze theorem, [bo2]). Suppose that $X$ is a definably compact, definable manifold. If $(\Phi, K)$ is a definable triangulation of $X$ and $T$ is a maximal tree in $K$ then, $\pi_1(X, x) \simeq G(K, T)$. In particular, $\pi_1(X, x)$ is invariant under taking elementary extensions, elementary substructures of $\cal N$ (containing the parameters over which $X$ is defined) and under taking expansions of $\cal N$ and reducts of $\cal N$ on which $X$ is definable.

### 4 Basic homological algebra

The category of chain complexes, denoted $\textbf{Comp}$ is the category whose objects are chain complexes $(E_*, \partial_*)$ - which are sequences $(E_n)_{n \in \mathbb{Z}}$ of abelian groups with morphisms $\partial_n : E_n \to E_{n-1}$ such that $\partial_{n-1} \circ \partial_n = 0$ for all $n \in \mathbb{Z}$ - and whose morphisms are chain maps $f : E_* \to F_*$ between chain complexes $(E_*, \partial_*)$ and $(F_*, \delta_*)$, i.e., $f$ is a sequence $(f_n)_{n \in \mathbb{Z}}$ of group homomorphisms $f_n : E_n \to F_n$ such that $f_{n-1} \circ \partial_n = \delta_n \circ f_n$ for all $n \in \mathbb{Z}$. A chain complex $E_*$ is nonnegative if $E_n = 0$ for $n < 0$.
The category of augmented chain complexes $\tilde{\text{Comp}}$, is the subcategory whose objects are those chain complexes $(E_*, \partial_*)$ with $E_{-1} = \mathbb{Z}$ and $E_n = 0$ for all $n < -1$, and whose morphisms are all those chain maps $f$ between such chain complexes with $f_{-1} = 1_{\mathbb{Z}}$. An augmented chain complex $(\tilde{E}_*, \tilde{\partial}_*)$ is an augmentation of the nonnegative chain complex $(E_*, \partial_*)$ if $E_n = \tilde{E}_n$ for $n \geq 0$ and $\partial_n = \tilde{\partial}_n$ for $n > 0$.

**Definition 4.1** The covariant functors $H_n : \text{Comp} \rightarrow \text{Ab}$ for $n \in \mathbb{Z}$ are defined by: $H_n(E_*) = \ker \partial_n/\im \partial_{n+1}$ and for $f : E_* \rightarrow F_*$, $H_n(f) : H_n(E_*) \rightarrow H_n(F_*)$ is given by $\cls_{E_*, z} \rightarrow \cls_{F_*, f_n(z)}$ where $\cls_{E_*, z}$ denotes the equivalence class of $z \in \ker \partial_n$ in $H_n(E_*)$. $H_n(E_*)$ is the $n$-th homology group of $E_*$ and $f_*$ is usually used to denote the sequence $(H_n(f))_{n \in \mathbb{Z}}$.

The homology groups $H_n(E_*)$ measure how far from being exact the chain complex $(E_*, \partial_*)$ is. We say that $E_*$ is exact (or acyclic) if $H_n(E_*) = 0$ for all $n$. Under some conditions we may have $H_n(E_*) \simeq H_n(F_*)$ for all $n$, for example if $F_*$ is an adequate subcomplex of $E_*$ - the theorem on removing cells, tells us that under certain conditions, we can remove an $n$-cell and an $n + 1$-cell (i.e., elements from the set of generators of $E_n$ and $E_{n+1}$ respectively) in order to obtain an adequate subcomplex $F_*$ of $E_*$. Also, if a chain map $f : E_* \rightarrow F_*$ is a chain equivalence i.e., there is a chain map $g : F_* \rightarrow E_*$ such that $g \circ f \simeq 1_{E_*}$ and $f \circ g \simeq 1_{F_*}$, then $H_n(f) : H_n(E_*) \rightarrow H_n(F_*)$ is an isomorphism for every $n$. Here, $f \simeq h$ means that the chain maps $f : E_* \rightarrow F_*$ and $h : F_* \rightarrow E_*$ are chain homotopic i.e., there is a sequence of maps $s_n : E_n \rightarrow F_{n+1}$ such that $f_n - h_n = \delta_{n+1} \circ s_n + s_{n-1} \circ \partial_n$ for all $n$.

**Fact 4.2** (Long exact sequence). Let $0 \rightarrow E'_* \xrightarrow{i} E_* \xrightarrow{p} E''_* \rightarrow 0$ be a short exact sequence of chain maps. There exists a sequence $d = (d_n : H_n(E''_*) \rightarrow H_{n-1}(E'_*))$ of (connecting) homomorphisms such that the following sequence is exact.

$$
\cdots \rightarrow H_n(E'_*) \xrightarrow{i} H_n(E_*) \xrightarrow{p} H_n(E''_*) \xrightarrow{d_n} H_{n-1}(E'_*) \rightarrow \cdots
$$
Fact 4.3 (Naturality of the connecting homomorphism). Assume that there is a commutative diagram of chain complexes with exact rows:

\[
\begin{array}{ccc}
0 & \rightarrow & E_0' \\
\uparrow f' & & \uparrow f \\
0 & \rightarrow & E_0''
\end{array}
\]

Then there is a commutative diagram of abelian groups with exact rows:

\[
\begin{array}{ccc}
\cdots & \rightarrow & H_n(E_0') \\
\downarrow f'_n & & \downarrow f_n \\
\cdots & \rightarrow & H_n(E_0'')
\end{array}
\]

Definition 4.4 Let \( F : C \rightarrow Ab \) be a functor, and let \( M \) be a set of models for \( C \) (i.e., a subset of \( \text{Obj} C \)). We say that \( F \) is free with base in \( M \) if:

1. \( FC \) is a free abelian group for every \( C \in \text{Obj} C \);
2. There is an indexed family \( (M_j)_{j \in J} \) of models in \( M \) and an element \( x = (x_j) \in \prod_{j \in J} FM_j \) such that, for every \( C \in \text{Obj} C \), the family \( ((F\sigma)(x_j))_{j \in J, \sigma : M_j \rightarrow C} \) is a basis of \( FC \).

In this situation, we call \( x \) a basis of \( F \) in \( M \).

Let \( E : C \rightarrow \text{Comp} \) be a functor. We say that \( E \) is nonnegative if \( EC \) is a nonnegative chain complex for all \( C \in \text{Obj} C \); \( E \) is acyclic on a set of models \( M \) for \( C \) if \( EM \) is an acyclic chain complex for every \( M \in M \).

A natural transformation between functors \( E, F : C \rightarrow \text{Comp} \) is called a natural chain map.

Fact 4.5 (Acyclic Models). Let \( C \) be a category with set of models \( M \) and let \( E, F : C \rightarrow \text{Comp} \) be nonnegative functors. Assume that \( F_n \) is free with basis in \( M \) for all \( n > 0 \) and that \( E \) is acyclic on \( M \). Then:

1. every natural transformation \( \tau_0 : F_0 \rightarrow E_0 \) lifts to a natural chain map \( \tau : F \rightarrow E \);
2. any two liftings \( \tau, \tau' : F \rightarrow E \) are naturally chain homotopic i.e., there are natural transformations \( s_n : F_n \rightarrow E_{n+1} \) such that \( \tau_n - \tau'_n = \delta_{n+1} \circ s_n + s_{n-1} \circ \partial_n \) for all \( n \). Furthermore, we can choose \( s_0 = 0 \).

An augmented natural chain map \( \tau : E \rightarrow F \) is by definition a natural transformation \( \tau : E \rightarrow F \) between two functors \( E, F : C \rightarrow \text{Comp} \). Note that in this case, for every \( C \in \text{Obj} C \), the map \( (\tau C)_{-1} : E_{-1} C = \mathbb{Z} \rightarrow F_{-1} C = \mathbb{Z} \) is \( 1_{\mathbb{Z}} \).
Fact 4.6 (Acyclic Models in $\tilde{\text{Comp}}$). Let $C$ be a category with set of models $M$ and let $E, F : C \to \tilde{\text{Comp}}$ be functors. Then: (1) if $F_n$ is free with basis in $M$ for all $n > 0$ and if $E$ is acyclic on $M$ then there exist an augmented natural chain map $\tau : F \to E$ and any two are naturally chain homotopic; (2) if both $E$ and $F_n$ are free with basis in $M$ for all $n > 0$ and if both $E$ and $F$ are acyclic on $M$ then every augmented natural chain map $\tau : F \to E$ is a natural chain equivalence.

5 Homology

We denote by $D\text{TOP}(\mathcal{N})$ (resp., $D\text{CTOP}(\mathcal{N})$) the category of definable manifolds (resp., the subcategory of definably compact manifolds) with continuous definable maps.

5.1 Homology

Definition 5.1 The category of pairs over $\mathcal{N}$, $D\text{TOPP}(\mathcal{N})$ is the category whose objects are the pairs $(X, A)$ where $X$ is a definable manifold and $A$ a definable subset of $X$, and whose morphisms $f : (X, A) \to (Y, B) \in \text{Mor} D\text{TOPP}(\mathcal{N})$, are the continuous definable maps $f : X \to Y$, such that $f(A) \subseteq B$.

The lattice of $(X, A) \in \text{Obj} D\text{TOPP}(\mathcal{N})$ consists of the pairs $(\emptyset, \emptyset)$, $(A, \emptyset)$, $(A, A)$, $(X, \emptyset)$, $(X, A)$ and $(X, X)$ all their identity maps, the inclusion maps and all their compositions. $f : (X, A) \to (Y, B) \in \text{Mor} D\text{TOPP}(\mathcal{N})$, induces a map of every pair of the lattice of $(X, A)$ into the corresponding pair of the lattice of $(Y, B)$.

Definition 5.2 A subcategory $C$ of $D\text{TOPP}(\mathcal{N})$ is called admissible if the following conditions are satisfied.

1. If $(X, A) \in \text{Obj} C$, then the lattice of $(X, A)$ is in $C$.

2. If $f : (X, A) \to (Y, B) \in \text{Mor} C$, then the induced maps on every pair of the lattice of $(X, A)$ into the corresponding pair of the lattice of $(Y, B)$ is in $\text{Mor} C$. 

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(3) If \((X, A) \in \text{Obj}\ C\), then \((X \times [0, 1], A \times [0, 1]) \in \text{Obj}\ C\), and for \(i \in \{0, 1\}\),
\[g_i : (X, A) \rightarrow (X \times [0, 1], A \times [0, 1]) : x \mapsto (x, i) \in \text{Mor}\ C.\]

(4) Let \(P = \{p\}\) be a one point set. Then \((P, \emptyset) \in \text{Obj}\ C\), and for each
\((X, \emptyset) \in \text{Obj}\ C\) and each point \(x \in X\), then \((P, \emptyset) \rightarrow (X, \emptyset) : p \mapsto x \in \text{Mor}\ C.\)

(5) If \((X, A), (Y, B) \in \text{Obj}\ C\), \(X \subseteq Y\) and \(A \subseteq B\) then the inclusion

\((X, A) \rightarrow (Y, B) \in \text{Mor}\ C.\)

The subcategory of \(DTOPP(\mathcal{N})\) of all definably compact pairs over \(\mathcal{N}\)

is denoted by \(DCTOPP(\mathcal{N})\). For example \(DTOPP(\mathcal{N})\) and \(DCTOPP(\mathcal{N})\)

are admissible subcategories of \(DTOPP(\mathcal{N})\).

**Definition 5.3** Let \(C\) be an admissible subcategory of \(DTOPP(\mathcal{N})\). A

homology \((H_\ast, d_\ast) = (H_n, d_n)_{n \geq 0}\) on \(C\) is a sequence of covariant functors

\(H_n : C \rightarrow \text{Ab}\) for \(n \geq 0\) such that the following axioms hold.

**Homotopy Axiom.** If \(f_0, f_1 : (X, A) \rightarrow (Y, B) \in \text{Mor}\ C\) and there is a

definable homotopy in \(C\) from \(f_0\) to \(f_1\) (i.e., there is \(F : (X \times [0, 1], A \times [0, 1]) \rightarrow (Y, B) \in \text{Mor}\ C\), with \(F \circ g_0 = f_0\) and \(F \circ g_1 = f_1\) where for

\(i \in \{0, 1\}\), \(g_i : (X, A) \rightarrow (X \times [0, 1], A \times [0, 1]) : x \mapsto (x, i) \in \text{Mor}\ C)\), then

\[H_n(f_0) = H_n(f_1) : H_n(X, A) \rightarrow H_n(Y, B)\]

for all \(n \geq 0\).

**Exactness Axiom.** There is a sequence of natural transformations \(d_n : H_n \rightarrow H_{n-1} \circ G\), where \(G : C \rightarrow C\) is the functor \((X, A) \rightarrow (A, \emptyset)\),

such that for each pair \((X, A) \in C\) with inclusions \(i : (A, \emptyset) \rightarrow (X, \emptyset)\) and

\(j : (X, \emptyset) \rightarrow (X, A)\), the following sequence is exact.

\[
\cdots \rightarrow H_n(A, \emptyset) \xrightarrow{H_n(i)} H_n(X, \emptyset) \xrightarrow{H_n(j)} H_n(X, A) \xrightarrow{d_n} H_{n-1}(A, \emptyset) \rightarrow \cdots
\]

**Excision Axiom.** For every \((X, A) \in \text{Obj}\ C\) and every definable open

subset \(U\) of \(X\) such that \(cl_X(U) \subseteq int_X(A)\) and \((X - U, A - U) \in \text{Obj}\ C\), the

inclusion \((X - U, A - U) \rightarrow (X, A)\) induces isomorphisms

\[H_n(X - U, A - U) \rightarrow H_n(X, A)\]

for all \(n \geq 0\).

**Dimension Axiom.** If \(X\) is a one point set, then \(H_n(X, \emptyset) = 0\) for all \(n > 0\). (The group \(H_0(X, \emptyset)\), is called the coefficient group).
**Notation:** We will write $X \in \text{ObjC}$ for $(X, \emptyset) \in \text{ObjC}$, $f : X \rightarrow Y \in \text{MorC}$ for $f(X, \emptyset) \rightarrow (Y, \emptyset) \in \text{MorC}$ and $H_n(X)$ for $H_n(X, \emptyset)$.

**Definition 5.4** Two homologies $(H_*, d_*)$ and $(H'_*, d'_*)$ on $C$ are **isomorphic** if there is a sequence of natural equivalences $\tau_n : H_n \rightarrow H'_n$ for all $n \geq 0$ such that

$$
\begin{align*}
H_{n+1}(X, A) & \xrightarrow{d_{n+1}} H_n(A) \\
\downarrow{\tau_{n+1}} & \quad \downarrow{\tau_n} \\
H'_{n+1}(X, A) & \xrightarrow{d'_{n+1}} H'_n(A)
\end{align*}
$$

for all $(X, A) \in \text{ObjC}$ and for all $n \geq 0$.

**Fact 5.5** (*Eilenberg-Steenrod Theorem*) Homology functors on $\text{DTOPP}(\mathbb{N})$ with isomorphic coefficient group are isomorphic.

**Proof.** The proof in [h] can be adapted to show that the Eilenberg-Steenrod axioms characterise homology on $\text{DTOPP}(\mathbb{N})$. □

### 5.2 O-minimal simplicial homology

A. Woerheide [Wo] defines *simplicial homology* on $\text{DCTOPP}(\mathbb{N})$. Like in the case for simplicial homology over $\mathbb{R}$, the main complication is defining the induced homomorphisms between the homology groups. The standard method, using repeated barycentric subdivision and the Lebesgue number property fails because $(\mathbb{N}, 0, 1, <, +, \cdot)$ may be non-archemedian. But the use of the method of acyclic models and the o-minimal triangulation theorem makes the construction possible.

We give here a brief description of Woerheide’s construction. First we need some basic terminology.

**Definition 5.6** Let $K$ be a closed simplicial complex in $\mathbb{N}^n$. The *simplicial chain complex* $(C_*(K), \partial_*)$ is defined by: $C_l(K) := A_l(K)/A'_l(K)$ where $A_l(K)$ is the free abelian group generated by the set of $(l + 1)$-tuples $(v_0, \ldots, v_l)$ and $A'_l(K)$ is the subgroup generated by the elements of
the form \((v_0, \ldots, v_l) - sgn(\alpha)(v_{\alpha(0)}, \ldots, v_{\alpha(l)})\) where \(\alpha\) is a permutation of \(\{0, \ldots, l\}\) and \(v_0, \ldots, v_l\) spans an \(l\)-simplex in \(K\); and for \(l > 0\), \(\partial_l\) is induced by the homomorphism \((v_0, \ldots, v_l) \mapsto \sum_{i=0}^l (-1)^i(v_0, \ldots, \hat{v}_i, \ldots, v_l)\). We define \(\tilde{\partial}_0: C_0(K) \to \mathbb{Z}\) by \(\sum \alpha_i v_i \mapsto \sum \alpha_i\), in this way we obtain an augmentation \(\tilde{C}_*(K)\) of \(C_*(K)\). Note that each \(\tilde{C}_*(K)\) is a free abelian group, and given any total order on the vertices of \(K\), the set of all classes \(v_0 < \ldots < v_l\) provides a basis for \(\tilde{C}_*(K)\). If \((K, K')\) is a closed simplicial pair, then \(\tilde{C}_*(K')\) is a subcomplex of \(\tilde{C}_*(K)\) and we define the relative simplicial chain complex \(C_*(K, K') := \tilde{C}_*(K)/\tilde{C}_*(K')\). Note that \(C_*(K) \simeq C_*(K, \emptyset)\). We define \(H_*(K, K') := H_*(C_*(K, K'))\).

For a closed simplicial complex \(K\), let \(A(K)\) denote the category whose objects are closed simplicial subcomplexes of \(K\) and whose morphisms are the inclusion maps. A map \(f: |K| \to |L|\) between the geometric realizations of closed simplicial complexes \(K\) and \(L\) is said to compatible if, for each simplex \(t \in K\), there is a simplex \(s \in L\) such that \(f(t) \subseteq s\). For each compatible map \(f: |K| \to |L|\), we define the functor \(A(f): A(L) \to A(K)\) by \(A(f)(L') = \{t \in K : f(t) \subseteq |L'|\}\). Let \(\tilde{C}_*^{L'}: A(L) \to \tilde{\text{Comp}}\) be the functor which sends a closed subcomplex \(L'\) of \(L\) to \(\tilde{C}_*(L')\) and sends an inclusion \(L'' \to L'\) to the inclusion \(\tilde{C}_*(L'') \to \tilde{C}_*(L')\).

An easy application of the theorem on acyclic models for \(\tilde{\text{Comp}}\) (see [Wo]) shows that: (1) there is an augmented natural chain map between the functors \(\tilde{C}_*^{K} \circ A(f)\) and \(\tilde{C}_*^{L'}\) and any two are naturally chain homotopic; (2) if \(K\) is a closed simplicial complex and \((\Psi, L)\) is a quasi-stratified triangulation of \(|K|\) (i.e., after a linear change of coordinates its a stratified triangulation) such that \(\Psi^{-1}: |L| \to |K|\) is compactible then every augmented natural chain map \(\tilde{C}_*^{L'} \circ A(\Psi^{-1}) \to \tilde{C}_*^{K}\) is a natural chain equivalence. The proof of (2) also involves the method of acyclic models to establish that if \(|K|\) is convex then \(\tilde{C}_*(L)\) is an acyclic chain complex (see [Wo]).

We are now recall Woerheide’s definition of the o-minimal simplicial homology.

**Definition 5.7** [Wo] Let \(f: (K, K') \to (L, L')\). If \(f\) is compatible then there is an augmented natural chain map \(\tau : \tilde{C}_*^{K} \circ A(f) \to \tilde{C}_*^{L}\), then chain map \(\tau_L: \tilde{C}_*(K) \to \tilde{C}_*(L)\) induces a chain map \(\tau_L: C_*(K, K') \to C_*(L, L')\). Define \(H_*(f)\) to be \(H_*(\tau_L)\). If \(f: (K, K') \to (L, L')\) is not
necessarily compactible, by the definable triangulation theorem, there is a
quasi-stratified triangulation \((\Phi, M)\) of \(|K|\) such that \(\Phi^{-1} : |M| \to |K|\) and
\(f \circ \Phi^{-1} : |M| \to |L|\) are compactible, so that \(H_*(\Phi^{-1})\) and \(H_*(f \circ \Phi^{-1})\) are
defined. On the other hand, by the above, every augmented natural chain
map \(\tilde{C}^M_\ast \circ A(\Phi^{-1}) \to \tilde{C}^K_\ast\) is a natural chain equivalence. So \(H_*(\Phi^{-1})\) is an
isomorphism. Define \(H_*(f)\) to be the composition
\(H_*(f \circ \Phi^{-1}) \circ (H_*(\Phi^{-1}))^{-1}\).

Its easy to see that \(H_\ast\) is a well defined functor.

**Definition 5.8 [Wo]** For \((X, A) \in \text{ObjDCTOPP}(\mathcal{N})\), let \(T(X, A)\) denote
the set of all definable triangulations \((\Phi, K)\) of \(X\) which respects \(A\), and
for each \((\Phi, K) \in T(X, A)\), \(K'\) denotes the closed subcomplex of \(K\) such
that \(\Phi(A) = |K'|\). The \(n\)-th homology group \(H_\ast(X, A)\) is defined to be the
subgroup of \(\Pi_{(\Phi, K) \in T(X, A)} H_\ast(K, K')\) consisting of all elements \(\alpha\) such that, for
all \((\Phi_1, K_1), (\Phi_2, K_2) \in T(X, A)\), we have \(\alpha_{(\Phi_2, K_2)} = H_\ast(\Phi_2 \circ \Phi_1^{-1})(\alpha_{(\Phi_1, K_1)}).

Given \(f : (X, A) \to (Y, B) \in \text{MorDCTOPP}(\mathcal{N})\) we define \(n\)-th in-
duced homomorphism \(H_\ast(f) : H_\ast(X, A) \to H_\ast(Y, B) : \alpha \to \beta\) where, for
all \((\Phi, K) \in T(X, A)\) and \((\Psi, L) \in T(Y, B)\), we have \(\beta_{(\Psi, L)} = H_\ast(\Psi \circ f \circ \Phi^{-1})(\alpha_{(\Phi, K)}).

The verification of the Eilenberg-Steenrod axioms is now easy and shows
that:

**Fact 5.9 [Wo]** There is a homology \((H_\ast, d_\ast)\) on \(\text{DCTOPP}(\mathcal{N})\) such that if
\((X, A) \in \text{ObjDCTOPP}(\mathcal{N}), (\Phi, K)\) is a definable triangulation of \(X\) which
respects \(A\) and \(K'\) is the subcomplex of \(K\) such that \(|K'| = \Phi(A)\), then we
have isomorphisms \(\pi_n(\Phi, K') : H_n(X, A) \to H_n(K, K')\) for all \(n \geq 0\) and such
that if \(f : (X, A) \to (Y, B) \in \text{MorDCTOPP}(\mathcal{N})\) and \((\Psi, L)\) is a definable
triangulation of \(Y\) respecting \(B\) and such that \(|L'| = \Psi(B)\), then
\[
\begin{array}{ccc}
H_n(X, A) & \xrightarrow{H_n(f)} & H_n(Y, B) \\
\downarrow_{\pi_n(\Phi, K)} & & \downarrow_{\pi_n(\Psi, L)} \\
H_n(K, K') & \xrightarrow{H_n(\Psi \circ f \circ \Phi^{-1})} & H_n(L, L')
\end{array}
\]
5.3 O-minimal singular homology

A. Woerheide [Wo] also defines singular homology on $DTOPP(\mathcal{N})$. In this case the construction is essentially the same as for the standard singular homology, only with the word “definable” added here and there. But, for the same reasons as above the standard proof of the excision axiom fails, and the difficulty is avoided by the use of the o-minimal triangulation theorem and the results obtained while constructing the simplicial homology.

**Definition 5.10** The standard $n$-simplex over $N$, $\Delta^n$ is the convex hull of the standard basis vectors $e_0, \ldots, e_n$ in $N^{n+1}$. Let the standard $(-1)$-simplex $\Delta^{-1}$ be the empty set. Let $X \in ObjDTOP(N)$ be a definable set. For $n \geq -1$, we define $\tilde{S}_n(X)$ to be the free abelian group on the set of definable continuous maps $\sigma: \Delta^n \rightarrow X$ (the (singular) $n$-simplexes in $X$) and for $n < -1$, we define $\tilde{S}_n(X) = 0$. Note that $\tilde{S}_{-1}(X) = \mathbb{Z}$. The elements of $\tilde{S}_n(X)$ are called the $n$-chains.

For $n > 0$ and $0 \leq i \leq n$ let $\epsilon_i^n: \Delta^{n-1} \rightarrow \Delta^n$ be the continuous definable map given by $\epsilon_i^n(\sum_{i=0}^{n-1} a_je_j) := \sum_{j \neq i} a_je_j$. Let $\epsilon_0^0: \Delta^{-1} \rightarrow \Delta^0$ be the unique map. We define the boundary homomorphism $\partial_n: \tilde{S}_n(X) \rightarrow \tilde{S}_{n-1}(X)$ to be the trivial homomorphism for $n < 0$ and for $n \geq 0$, $\partial_n$ is given on basis elements by

$$\partial_n(\sigma) := \sum_{i=0}^{n} (-1)^i \sigma \circ \epsilon_i^n.$$  

One verifies that $\partial^2 = 0$ and so $(\tilde{S}_n(X), \partial)$ is a chain complex, the augmented singular chain complex.

Given $(X, A) \in ObjDTOPP(\mathcal{N})$, then the relative singular chain complex $(S_*(X, A), \partial)$ is the quotient chain complex $(\tilde{S}_*(X)/\tilde{S}_*(A), \partial)$. We define the singular chain complex $S_*(X)$ to be $S_*(X, \emptyset)$.

For $f: (X, A) \rightarrow (Y, B) \in MorDTOPP(\mathcal{N})$, we have an induced chain map $f_*: S_*(X, A) \rightarrow S_*(Y, B)$ given on the basis elements of $\tilde{S}_*(X)$ by $f_*(\sigma) = f \circ \sigma$. We get a sequence of functors $H_n: DTOPP(\mathcal{N}) \rightarrow Ab$ by setting $H_n(X, A) := H_n(S_*(X, A))$ and $H_n(f) := H_n(f_*)$. We set $\tilde{H}_n(X) := H_n(\tilde{S}_*(X))$ and $H_n(X) := H_n(X, \emptyset)$. 

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**Definition 5.11** Let $X \subseteq \mathbb{N}^m$ be a convex definable set and let $p \in X$. The *cone construction over $p$ in $X$* is a sequence of homomorphisms $\tilde{S}_s(X) \rightarrow \tilde{S}_{s+1}(X)$: $z \mapsto p.z$ defined as follows: For $n < -1$, $p.$ is defined as the trivial homomorphism and for $n \geq -1$ and a basis element $\sigma$, $p.(\sum_{i=0}^{n+1} t_i e_i) = p$ if $t_0 = 1$ or $t_0 + (1 - t_0)\sigma(\sum_{i=1}^{n+1} \frac{t_i}{1-t_0} e_i)$ if $t_0 \neq 1$.

Given definable sets $X,Y$ and $Z$, let $F(X,Y)$ denote the free abelian group on the set of all definable continuous maps from $X$ into $Y$. Given $\alpha = \sum_i s_i \alpha_i \in F(X,Y)$ and $\beta = \sum_j t_j \beta_j \in F(Y,Z)$, we define the *sharp operator* by

$$ \beta \sharp \alpha := \sum_{i,j} s_i t_j (\beta_j \circ \alpha_i) \in F(X,Z). $$

Note that $\sharp$ is associative, and since $\tilde{S}_s(X) = F(\Delta^n, X)$, every $z \in \tilde{S}_s(X)$ yields a chain map $z_\sharp : \tilde{S}_s(\Delta^n) \rightarrow \tilde{S}_s(X)$.

**Definition 5.12** Let $X$ be a definable set. The *barycentric subdivision* $Sd_n : \tilde{S}_n(X) \rightarrow \tilde{S}_n$ is defined as follows: for $n \leq -1$, $Sd_n : \tilde{S}_s(X) \rightarrow \tilde{S}_s(X)$ is the trivial homomorphism, $Sd_{-1} : \tilde{S}_{-1}(X) \rightarrow \tilde{S}_{-1}(X)$ is the identity and for $n \geq 0$, $Sd_n(z) := z_\sharp(b_n.Sd_{n-1}(1_{\Delta^n}))$, where $b_n$ is the barycenter of $\Delta^n$.

Note that $Sd_0$ is the identity and $Sd_n$ is natural i.e., it commutes with $f_\sharp : \tilde{S}_n(X) \rightarrow \tilde{S}_n(Y)$. Some lemmas on the cone construction and the sharp operator show that $Sd = (Sd_n)_{n \in \mathbb{Z}}$ is a chain map $\tilde{S}_s(X) \rightarrow \tilde{S}_s(X)$.

**Definition 5.13** Let $K$ be a closed simplicial complex in $\mathbb{N}^m$. We define the chain map $\tau_K : \tilde{C}_s(K) \rightarrow \tilde{S}_s(|K|)$ by setting $\tau_K = 1_Z : \tilde{C}_{-1}(K) \rightarrow \tilde{S}_{-1}(|K|)$ and for $n \geq 0$, $\tau_K(<v_0,\ldots,v_n>) := \sigma$ where $\sigma : \Delta^n \rightarrow |K|$ is such that $\sigma(\sum_{i=0}^{n} t_i e_i) = \sum_{i=0}^{n} t_i v_i$.

**Definition 5.14** Let $(\Phi,K)$ be a definable triangulation of $\Delta^n$ compatible with the standard simplicial complex of $E^n$. Let $X$ be a definable set. We define the subdivision operator $Sd_i^K : \tilde{S}_i(X) \rightarrow \tilde{S}_i(X)$ where $i \leq n$ by: $Sd_{-1} = 1_Z$ and for $0 \leq i \leq n$,

$$ Sd_i^K(z) := (Sdz_\sharp(\gamma_i^n)_\sharp(\Phi^{-1})_\sharp \tau_K F_n < e_{n-i}, \ldots, e_n > $$
where $F_n : \tilde{C}_* (E^n) \to \tilde{C}_* (K)$ is the chain map induced the unique (up to natural chain homotopy) augmented natural chain map $\tilde{C}_*^{E^n} \to \tilde{C}_*^K \circ A (\Phi^{-1})$, and $\gamma^n_i : \Delta^n \to \Delta^i$ is defined by

$$\gamma^n_i (\sum_{j=0}^n a_j e_j) = \sum_{j=0}^i (a_{n-i+j} + \sum_{k=0}^{n-i-1} \frac{a_k}{i+1}) e_j.$$

It’s clear from the definition that $Sd^K_n$ is natural. A simple computation shows that $Sd^K := (Sd^K_i)_{i \leq n}$ is a natural partial chain map $\tilde{S}_* \to \tilde{S}_*$ of order $n$. The proof of the theorem on acyclic models shows that there is a natural partial chain homotopy between the subdivision operator $Sd^K$ and the restriction $1_{\leq n}$ of the identity natural chain map $1_* : \tilde{S}_* \to \tilde{S}_*$. This latter fact together with the result that says that given a definable set $X$ with definable open sets $U$ and $V$ such that $X = U \cup V$, if $z \in \tilde{S}_n (X)$ then there is a definable triangulation $(\Phi, K)$ of $\Delta^n$ compatible with $E^n$ such that $Sd^n_K (z) \in \tilde{S}_n (U) + \tilde{S}_n (V)$ implies the excision axiom for the singular o-minimal homology, and therefore:

**Fact 5.15** [Wo] The sequence $H = (H_n)_{n \in \mathbb{Z}}$ of functors defined above is a homology for $DTOPP (\mathcal{N})$, called singular homology.

### 5.4 Some properties of homology

Like in the classical case, we have the following results which are consequence of the axioms for homology.

**Remark 5.16** Consider the diagram in $DTOPP (\mathcal{N})$:

$$
\begin{array}{ccc}
A & \xrightarrow{i} & X \\
\downarrow{1_A} & \searrow{r} & \nearrow{1_X} \\
A & \xrightarrow{1} & X.
\end{array}
$$

We have a **definable weak retract** if $[r \circ i] = [1_A]$ (i.e., the map $r \circ i$ is definably homotopic to the map $1_A$); a **definable retract** if $r \circ i = 1_A$; a **definable weak deformation retract** if $[r \circ i] = [1_A]$ and $[i \circ r] = [1_X]$; and a **definable deformation retract** if $r \circ i = 1_A$ and $[i \circ r] = [1_X]$. 

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If \( i : A \rightarrow X \) is a definable (weak) retract, then the exactness axiom implies for all \( n \) that \( H_n(X) \cong H_n(A) \oplus H_n(X, A) \) (we have \( ker_r = H_n(X, A) \)).

If \( i : A \rightarrow X \) is a definable weak deformation retract then for all \( n \geq 0 \), \( H_n(X, A) = 0 \). In particular, \( H_n(X, X) = 0 \) for all \( n \geq 0 \).

**Theorem 5.17** Let \( X \in \text{Obj}_{\text{DTOP}}(\mathcal{N}) \), then \( H_q(X) \) is finitely generated for every \( q \geq 0 \), \( H_q(X) = 0 \) for all \( q > \text{dim}X =: m \), \( H_m(X) \) is a free abelian group (possibly trivial) and \( H_0(X) = \mathbb{Z}^k \) where \( k \) is the number of definable connected components of \( X \). Moreover, \( H_*(X) \) is invariant under taking elementary extensions or substructures of \( \mathcal{N} \) that contain the parameters over which \( X \) is defined and under taking expansions and reducts of \( \mathcal{N} \) on which \( X \) is definable.

**Proof.** Note that by remark 5.16 and since by proposition 3.3 in chapter 8 of [vdd], \( X \) is a definable deformation retract of some \( Y \in \text{DCTOP}(\mathcal{N}) \), we may assume that \( X \in \text{DCTOP}(\mathcal{N}) \). Let \( (\Phi, K) \) be a definable triangulation of \( X \). Then \( H_q(X) = H_q(K) \) which by theorem 7.14 [ro] is finitely generated for every \( q \geq 0 \), \( H_q(X) = 0 \) for all \( q > \text{dim}X =: m \), \( H_m(X) \) is a free abelian group (possibly trivial). The fact that \( H_0(X) = \mathbb{Z}^k \) where \( k \) is the number of definable connected components of \( X \) is proved in theorem 4.14 [ro]. \( \square \)

**Fact 5.18** (Hurewicz Theorem). Suppose that \( X \) is a definable manifold. Then \( \pi_1(X, x)/\pi_1(X, x)_{(1)} \cong H_1(X) \). In particular, \( \pi_1(X, x)/\pi_1(X, x)_{(1)} \) is finitely generated with finitely many relations.

**Proof.** See theorem 4.29 [ro] (on lemma 4.26 [ro] instead of the function \( u : [0, 1] \rightarrow S^1 \) given by \( u(t) = e^{2\pi it} \), use any definable continuous \( v : [0, 1] \rightarrow S^1 \) such that \( v(0) = v(1) \) and such that \( v|[0,1) \) is a bijection of \( [0, 1) \) and \( S^1 \)). \( \square \)

Note that the excision axiom is equivalent to the following: for all \( X \in \text{DTOP}(\mathcal{N}) \) and all \( X_1, X_2 \) subsets of \( X \) such that \( X = \text{int}_X(X_1) \cup \text{int}_X(X_2) \), the inclusion \((X_1, X_1 \cap X_2) \hookrightarrow (X, X_2)\) induces isomorphisms

\[ H_n(X_1, X_1 \cap X_2) \rightarrow H_n(X, X_2) \]

for all \( n \geq 0 \).
Fact 5.19 (Mayer-Vietoris) Consider the commutative diagram of inclusions in $\text{DTOPP}(\mathcal{N})$

\[
\begin{array}{ccc}
(X_1 \cap X_2, Z) & \xrightarrow{i_1} & (X_1, Z) \xrightarrow{p} (X_1, X_1 \cap X_2) \\
\downarrow{i_2} & & \downarrow{g} \\
(X_2, Z) & \xrightarrow{j} & (X, Z) \xrightarrow{q} (X, X_2),
\end{array}
\]

where $X_1, X_2, Z \subseteq X$ with $X = \text{int}_X(X_1) \cup \text{int}_X(X_2)$ and $Z \subseteq X_1 \cap X_2$. Then there is an exact sequence for all $n \in \mathbb{N}$

\[
H_n(X_1 \cap X_2, Z) \xrightarrow{(i_1)_*} H_n(X_1, Z) \oplus H_n(X_2, Z) \xrightarrow{\partial} H_n(X, Z) \xrightarrow{D} H_{n-1}(X_1 \cap X_2, Z)
\]

with $D = l_*dh^{-1}_*q_*$, where $d$ is the connecting homomorphism of the pair $(X_1, X_1 \cap X_2)$ and $l : (X_1 \cap X_2, \emptyset) \rightarrow (X_1 \cap X_2, Z)$ is the inclusion.

Fact 5.20 (Exactness Axiom for triples). If we have inclusions

\[
(A, \emptyset) \xrightarrow{a} (A, B) \xrightarrow{b} (X, A)
\]

in $\text{DTOPP}(\mathcal{N})$, then there is an exact sequence for all $n \in \mathbb{N}$

\[
\cdots \rightarrow H_n(A, B) \xrightarrow{a_*} H_n(X, B) \xrightarrow{b_*} H_n(X, A) \xrightarrow{c_* \circ d_*} H_{n-1}(A, B) \rightarrow \cdots
\]

Definition 5.21 Let $(X, A) \in \text{DTOPP}(\mathcal{N})$ and let $G$ be an abelian group. The singular chain complex $(S_*(X, A; G), \partial^2)$ with coefficients in $G$ is defined as $S_n(X, A; G) := S_n(X, A) \otimes G$ and $\partial^2_n := \partial_n \otimes 1$. For $f : (X, A) \rightarrow (Y, B) \in \text{Mor}\text{DTOPP}(\mathcal{N})$, we have an induced chain map $f_* : S_*(X, A; G) \rightarrow S_*(Y, B; G)$ given on the basis elements by $f_*(\sigma \otimes g) = (f \circ \sigma) \otimes g$. We have in this way a sequence of functors $H_n(\_; G) : \text{DTOPP}(\mathcal{N}) \rightarrow \text{Ab}$ by setting $H_n(X, A; G) := H_n(S_*(X, A; G))$ and $H_n(f) := H_n(f_*)$.

Below, $\text{Tor} = \text{Tor}^R$ where for a ring $R$, $\text{Tor}^R$ is the torsion functor, and for $R$-modules $A$ and $B$, $\text{Tor}^R(\_; B)$ measures the failure of functor $\otimes_R B$ to be exact. Thus, if $0 \rightarrow C \rightarrow F \rightarrow A \rightarrow 0$ is a free resolution of $A$, then

\[
\text{Tor}^R(A, B) = \ker(C \otimes_R B \rightarrow F \otimes_R B).
\]

We have the following result. For a proof see theorem 9.32 \[\text{[ro]}\].
Fact 5.22 (Universal Coefficients Theorem for Homology). For every $X \in \text{Obj} DTOP(N)$ and for all $n \geq 0$, there are canonical exact sequences

$$0 \to H_n(X) \otimes G \overset{\alpha}{\to} H_n(X;G) \to \text{Tor}(H_{n-1}(X),G) \to 0.$$  

These sequences split, that is there are canonical isomorphisms $H_n(X;G) \simeq H_n(X) \otimes G \oplus \text{Tor}(H_{n-1}(X),G)$. In particular, if $K_p$ is a field of characteristic $p$ (a prime or zero) then we have an isomorphism $H_n(X;K_p) \simeq H_n(X) \otimes K_p$ iff $H_*(X)$ is $p$-torsion free.

Theorem 5.23 (Künneth Formula for Homology). Let $R$ be a principal ideal domain. For every $X,Y \in \text{Obj} DTOP(N)$ and for all $n \geq 0$, there are canonical exact sequences of $R$-modules

$$0 \to \sum_{i+j=n} H_i(X;R) \otimes_R H_j(Y;R) \overset{\alpha''}{\to} H_n(X \times Y;R) \to$$

$$\to \sum_{p+q=n-1} \text{Tor}^R(H_p(X;R), H_q(Y;R)) \to 0.$$  

These sequences split, that is there are canonical isomorphisms

$$H_n(X \times Y;R) \simeq \sum_{i+j=n} H_i(X;R) \otimes_R H_j(Y;R) \oplus$$

$$\oplus \sum_{p+q=n-1} \text{Tor}^R(H_p(X;R), H_q(Y;R)).$$  

In particular, if $H_*(X;R)$ or $H_*(Y;R)$ is a finitely generated free $R$-module, then the homology (external) cross product $\alpha''$ is an isomorphism of graded $R$-modules.

Proof. We include here a proof for the case $R = \mathbb{Z}$. For the general case see proposition 2.6 [4]. By the Eilenberg-Zilber theorem (see theorem 9.33 [7]) there is a natural chain equivalence $\zeta : S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$ which induces isomorphisms $H_n(X \times Y) \simeq H_n(S_*(X) \otimes S_*(Y))$ for all $n \geq 0$. Here, $S_*(X) \otimes S_*(Y)$ is the chain complex with

$$(S_*(X) \otimes S_*(Y))_n = \sum_{i+j=n} S_i(X) \otimes S_j(Y)$$
and whose differentiation \( D_n : (S_*(X) \otimes S_*(Y))_n \rightarrow (S_*(X) \otimes S_*(Y))_{n-1} \) is defined on the generators by

\[
D_n(a_i \otimes b_j) = \partial a_i \otimes b_j + (-1)^i a_i \otimes \partial b_j, \quad i + j = n.
\]

The Künneth theorem (see theorem 9.36 [4]), gives a split exact sequence with middle term \( H_n(S_*(X) \otimes S_*(Y)) \) and the Eilenberg-Zilber theorem identifies this term with \( H_n(X \times Y) \). Here \( \alpha'' \) is given by

\[
\alpha''((cls a_i \otimes cls b_j)) := cls(\zeta'(a_i \otimes b_j)) =: cls a_i \times cls b_j
\]

where \( \zeta' \) is the inverse of an Eilenberg-Zilber chain equivalence \( \zeta \).

By theorem 12.25 [5], the Eilenberg-Zilber chain equivalence \( \zeta : S_*(X \times Y) \rightarrow S_*(X) \otimes S_*(Y) \) can be defined as the Alexander-Whitney map

\[
\zeta_n(\sigma) := \sum_{i+j=n} \sigma' \lambda_i^n \otimes \sigma'' \mu_j^n,
\]

where \( \sigma : \Delta^n \rightarrow X \times Y \) and \( \sigma' = \pi' \sigma, \sigma'' = \pi'' \sigma \) (and where \( \pi', \pi'' \) are the projections of \( X \times Y \) onto \( X, Y \), respectively). And for \( 0 \leq i \leq n \), the (affine) maps \( \lambda_i^n, \mu_i^n : \Delta^i \rightarrow \Delta^n \) are given by

\[
\lambda_i^n(\sum_{j=0}^i a_j e_j) := \sum_{j=0}^i a_j e_j + \sum_{j=i+1}^n 0 e_j
\]

and

\[
\mu_i^n(\sum_{j=0}^i a_j e_j) := \sum_{j=0}^{n-i} 0 e_j + \sum_{j=0}^{n-(i+1)} a_j e_{j+(i+1)}.
\]

\( \square \)

For the proof of the next fact see proposition 2.6 [5].

**Fact 5.24** The homology cross product satisfies the following properties: (1) \((f \times g)_*(\alpha \times \beta) = (f_* \alpha) \times (g_* \beta)\) (naturality); (2) \(t_*(\alpha \times \beta) = (-1)^{deg \beta \deg \alpha} \beta \times \alpha\) (skew-commutativity) where \( t : X \times Y \rightarrow Y \times X \) commutes factors; (3) \((\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)\) (associativity); (4) \(1 \times \alpha = \alpha \times 1 = \alpha\) (unit element).
Remark 5.25 Note that all the results presented in this subsection can be generalised to relative homology. More precisely, we have a relative version of the universal coefficient theorem, the Künneth formula with corresponding cross product. For details see [4].

6 Cohomology

6.1 Cohomology

Definition 6.1 Let $G$ be an abelian group. Suppose that $C$ is an admissible subcategory of $\text{DTOPP}(\mathcal{N})$. A cohomology $(H^*, d) = (H^n, d^n)_{n \geq 0}$ on $C$ with coefficients in $G$ is a sequence of contravariant functors $H^n(\ , G) : C \rightarrow Ab$ for $n \geq 0$ satisfying the axioms dual to those satisfied by a homology functor. We denote by $H^n(X, A; G)$ the image of $(X, A) \in \text{Obj}C$ under $H^n$ and by $H^n(X; G)$ the image of $(X, \emptyset) \in \text{Obj}C$ under $H^n(\ , G)$.

A. Woerheide constructions together with classical arguments (see chapter 12 of [ro]) show that there exist simplicial cohomology functor for the category $\text{DCTOPP}(\mathcal{N})$ and singular cohomology functors for the category $\text{DTOPP}(\mathcal{N})$.

Definition 6.2 For a fixed abelian group $G$, recall that $\text{Hom}(\ , G) : Ab \rightarrow Ab$ is a contravariant functor. For $(X, A) \in \text{Obj} \text{DTOPP}(\mathcal{N})$, we have the augmented singular cochain complex $(\tilde{S}^*(X; G), \delta)$ with coefficients in $G$, where $\tilde{S}^n(X; G) := \text{Hom}(\tilde{S}_n(X), G)$ and $\delta^n := \text{Hom}(\partial_{n+1}, G)$. We also have the relative singular cochain complex $(S^*(X, A; G), \delta)$ with coefficients in $G$ given by $S^n(X, A; G) := \text{Hom}(S_n(X, A), G)$ and $\delta^n := \text{Hom}(\partial_{n+1}, G)$. Given $f : (X, A) \rightarrow (Y, B) \in \text{Mor} \text{DTOPP}(\mathcal{N})$ we have a chain homomorphism $f^2 : S^*(Y, B; G) \rightarrow S^*(X, A; G)$ given by $f^2 := \text{Hom}(f_2, G)$.

The singular cohomology functors $H^n(\ , G) : \text{DTOPP}(\mathcal{N}) \rightarrow Ab$ with coefficients in $G$ are given by $H^n(X, A; G) := H^n(S^n(X, A; G))$ and $H^n(f) := H^n(f^2)$. We set $\tilde{H}^n(X; G) := H^n(\tilde{S}^*(X; G))$ and $H^n(X; G) := H^n(X, \emptyset; G)$. As usual, $H^n(X) := H^n(X, \mathbb{Z})$.

We now list some properties of the (singular) cohomology groups. There are analogue results for homology groups, and the proofs are obtained by
taking their “dual”. For fact 6.3 see theorem 12.11 [ro] and for fact 6.4 see theorem 12.15 [ro].

Below, \( \text{Ext}(\cdot, \cdot) \) is the extension functor defined as follows. Let \( A \) and \( B \) be abelian groups and consider a free resolution \( 0 \rightarrow R \rightarrow F \rightarrow A \rightarrow 0 \) of \( A \) (i.e., \( F \) is a free abelian group). Then

\[
\text{Ext}(A, B) = \text{coker}(\text{Hom}(F, B) \rightarrow \text{Hom}(R, B)),
\]

so that \( \text{Ext}(\cdot, B) \) measures the failure of \( \text{Hom}(\cdot, B) \) to be exact.

**Fact 6.3 (Dual Universal Coefficients).** For every \( X \in \text{ObjDTOP}(\mathcal{N}) \) and for all \( n \geq 0 \), there are canonical exact sequences

\[
0 \rightarrow \text{Ext}(H_{n-1}(X), G) \rightarrow H^n(X; G) \xrightarrow{\partial} \text{Hom}(H_n(X), G) \rightarrow 0.
\]

These sequences split, that is there are canonical isomorphisms \( H^n(X; G) \cong \text{Hom}(H_n(X), G) \oplus \text{Ext}(H_{n-1}(X), G) \). In particular, if \( K_p \) is a field of characteristic \( p \), then for all \( n \geq 0 \), \( H^n(X; K_p) \cong \text{Hom}(H_n(X), K_p) \) iff \( H_*(X) \) is \( p \)-torsion free.

**Fact 6.4 (Universal Coefficients Theorem for Cohomology).** For every \( X \in \text{ObjDTOP}(\mathcal{N}) \) and for all \( n \geq 0 \), there are canonical exact sequences

\[
0 \rightarrow H^n(X) \otimes G \xrightarrow{\alpha} H^n(X; G) \rightarrow \text{Tor}(H^{n+1}(X), G) \rightarrow 0.
\]

These sequences split, that is there are canonical isomorphisms \( H^n(X; G) \cong H^n(X) \otimes G \oplus \text{Tor}(H^{n+1}(X), G) \). In particular, if \( K_p \) is a field of characteristic \( p \), then for all \( n \geq 0 \), \( H^n(X; K_p) \cong H^n(X) \otimes K_p \) iff \( H_*(X) \) is \( p \)-torsion free (equivalently iff \( H^*(X) \) is \( p \)-torsion free).

Recall that a ring \( R \) (resp., an algebra over a ring \( S \)) is a graded ring (resp., graded \( S \)-algebra) if there are additive subgroups (resp., \( S \)-submodules) \( R^n \), \( n \geq 0 \) such that \( R = \sum_{n \geq 0} R^n \) (direct sum of additive subgroups (resp., \( S \)-submodules)) with a homomorphism \( R \otimes R \rightarrow R \) (called multiplication) of graded additive subgroup (resp., \( S \)-modules) such that \( R^n R^m \subseteq R^{n+m} \). A unit is an element \( 1 \in R^0 \) such that \( 1r = r1 = r \) for all \( r \in R \). The multiplication is associative if \( a(bc) = (ab)c \) for all \( a, b, c \in R \). If \( x \in R^n \) we say that \( x \) is homogeneous of degree \( n \), we denote this by \( \text{deg}(x) = n \). If \( x = x_1 + \cdots + x_k \), with \( x_i \in R^{n_i} \) we define the degree of \( x \) to be \( \text{deg}(x) := \sum_{i=1}^k \text{deg}(x_i) \). The multiplication is skew-commutative if \( ab = (-1)^{\text{deg}(a)\text{deg}(b)}ba \) for all \( a, b \in R \).
Definition 6.5 The product of \( c \in S_n(X,A;G) \) and \( \sigma \in S^n(X,A;G) \) defined by \((\sigma,c) := \sigma(c)\) satisfies \((\delta \sigma,c) = (\sigma,\partial c)\) and hence induces the Kronecker product
\[
(\ ,\ ) : H^n(X,A;G) \otimes H_n(X,A;G) \rightarrow G.
\]

Theorem 6.6 Let \( R \) be a commutative ring. For every \( X \in \text{ObjDTOP}(\mathcal{N}) \), \( H^*(X;R) = \sum_{n \geq 0} H^n(X;R) \) is a graded \( R \)-algebra with a canonical multiplication (called cup product) \( \cup : H^*(X;R) \otimes H^*(X;R) \rightarrow H^*(X;R) \) satisfying:
\[
\theta \cup \psi = (-1)^{\text{deg}(\theta)\text{deg}(\psi)} \psi \cup \theta.
\]
Moreover, \( H^*(\ ,R) \) is a functor from \( \text{DTOP}(\mathcal{N}) \) into the category of graded skew-commutative (associative) \( R \)-algebra with unit element.

Proof. The cup product \( \cup : S^n(X,R) \times S^m(X,R) \rightarrow S^{n+m}(X,R) \) is defined by
\[
(c,\phi \cup \theta) := (c\Lambda^{n+m}_n,\phi)(p_m^{n+m},\theta).
\]
Here and through \((c,\phi) := \phi(c)\). Lemma 12.19 [ro] shows that \( S^*(X,R) \) is a graded \( R \)-algebra under cup product, corollary 12.21 [ro] shows that \( S^*(\ ,R) \) is a contravariant functor from \( \text{DTOP}(\mathcal{N}) \) into the category of graded \( R \)-algebras. Lemma 12.22 [ro] and theorem 12.23 [ro] shows that \( \cup \) can be defined on \( H^n(X,R) \times H^m(X,R) \rightarrow H^{n+m}(X,R) \) by \( \text{cls}(\phi \cup \theta) := \text{cls}(\phi \cup \theta) \) and \( H^*(\ ,R) \) is a contravariant functor from \( \text{DTOP}(\mathcal{N}) \) into the category of graded \( R \)-algebras.

Theorem 12.26 [ro] shows that the cup product is the composite \( \Delta \gamma^* \pi \), where \( \Delta : X \rightarrow X \times X \) is the diagonal and
\[
\pi : S^*(X,R) \otimes S^*(Y,R) \rightarrow \text{Hom}(S_*(X) \otimes S_*(Y),R)
\]
is defined as follows: If \( \phi \in S^n(X,R) \) and \( \theta \in S^m(Y,R) \), then there is a function \( S^n(X,R) \otimes S^m(Y,R) \rightarrow \text{Hom}(S_n(X) \otimes S_m(Y),R) \) by \( (\phi \otimes \theta) \rightarrow (\ ,\phi \otimes \theta) \), where \((\sigma \otimes \tau,\phi \otimes \theta) = (\sigma,\phi)(\tau,\theta)\). Since this function is bilinear, it extends to the homomorphism \( \pi \). This is used in theorem 12.29 [ro] to show the skew-commutativity. \(\square\)
Remark 6.7 The same proof also shows that we have the following relative cup products:

\[ \cup : H^*(X, A; R) \otimes_R H^*(X; R) \rightarrow H^*(X, A; R); \]
\[ \cup : H^*(X; R) \otimes_R H^*(X, A; R) \rightarrow H^*(X, A; R); \]
\[ \cup : H^*(X, A; R) \otimes_R H^*(X; R) \rightarrow H^*(X, A; R). \]

satisfying the above properties. They are interrelated via the homomorphism \( j^* : H(X, A; R) \rightarrow H^*(X; R) \). The relation with \( d^* : H^*(A; R) \rightarrow H^{*+1}(X, A; R) \) is given by \( d^*(\alpha \cup i^*\beta) = d^*\alpha \cup \beta \), for \( \alpha \in H^p(A; R) \), \( \beta \in H^q(X; R) \), where \( i^* : H^*(X; R) \rightarrow H^*(A; R) \).

Theorem 6.8 (K"unneth Formula for Cohomology). Let \( R \) be any principal ideal domain. For every \( X, Y \in \text{Obj}_{\text{DTOP}}(\mathcal{N}) \) and for all \( n \geq 0 \), there are canonical exact sequences of skew-commutative (associative) \( R \)-algebras with unit

\[ 0 \rightarrow \sum_{i+j=n} H^i(X; R) \otimes_R H^j(Y; R) \xrightarrow{\alpha'} H^n(X \times Y; R) \rightarrow \]
\[ \rightarrow \sum_{p+q=n+1} \text{Tor}^R(H^p(X; R), H^q(Y; R)) \rightarrow 0. \]

These sequences split, that is there are canonical isomorphisms

\[ H^n(X \times Y; R) \simeq \sum_{i+j=n} H^i(X; R) \otimes_R H^j(Y; R) \oplus \sum_{p+q=n+1} \text{Tor}^R(H^p(X; R), H^q(Y; R)). \]

In particular, if \( H^*(X; R) \) or \( H^*(Y; R) \) is a finitely generated free \( R \)-module, then the cohomology (external) cross product \( \alpha' \) is an isomorphism of graded skew-commutative (associative) \( R \)-algebra with unity element.

Proof. See theorem 12.16 \( \square \) and theorem 12.31 \( \square \). The cross product homomorphism \( \alpha' \) is determined by the (external) cross product homomorphism

\[ \zeta^\pi : S^*(X, R) \otimes S^*(Y, R) \rightarrow S^*(X \times Y, R) \]

(it is common to write \( \phi \times \theta \) for \( \zeta^\pi(\phi \otimes \theta) \) and also \( \phi \times \theta \) for \( \alpha'(\phi \otimes \theta) \)). \( \square \)
Remark 6.9 The cohomology cross product satisfies the dual properties of those satisfied by the homology cross product. Moreover, it also satisfies the following: \((\alpha \times \beta, \sigma \times \tau) = (-1)^{\deg \sigma \deg \alpha} (\alpha, \beta)(\beta, \tau)\) (duality). For details see chapter VII section 7 in [d].

Remark 6.10 Note that, the cohomology cross product is related to the cup product by 
\[\alpha \times \beta = p_{X}^{*}\alpha \cup p_{Y}^{*}\beta\]
where \(p_{X} : X \times Y \to X\) and \(p_{Y} : X \times Y \to Y\). This can be used to prove relative versions of the Künneth formula for cohomology.

The following two results are proved just like in the classical case.

**Theorem 6.11** Let \(R\) be a commutative ring. For all \(X \in \text{Obj} \mathcal{D} \text{TOP}(\mathcal{N})\), \(H_{\ast}(X; R)\) is graded (right) \(H^{\ast}(X; R)\)-module under the bilinear pairing
\[\cap : H_{p+q}(X; R) \times H^{p}(X; R) \to H_{q}(X; R)\]
called the cap product. Furthermore, the cap product satisfies the following properties: (1) \((\alpha \cap \beta, \gamma) = (\alpha, \beta \cup \gamma)\) (duality); (2) for \(f : X \to Y \in \text{Mor} \mathcal{D} \text{TOP}(\mathcal{N})\), we have \(f_{*}(\alpha \cap f^{*}\beta) = f_{*}\alpha \cap \beta\) (naturality).

**Proof.** The cap product is induced by \(\cap : S_{p+q}(X, R) \times S^{p}(X, R) \to S_{q}(X, R)\) give by
\[(\sigma \otimes r) \cap c := \sigma \lambda_{q}^{p+q} \otimes r(\sigma \mu_{q}^{p+q}, c)\]
For details see theorem 66.2 [m] and also corollary 24.22 [g].

**Theorem 6.12** There is a bilinear pairing
\[\mathcal{L} : H^{p+q}(X \times Y, R) \times H^{p}(X, R) \to H^{q}(Y, R)\]
called the slant product which satisfies: (1) \((\beta, \gamma/\alpha) = (\alpha \times \beta, \gamma)\) (duality); (2) \(1/\alpha = (\alpha, 1)1\) (units); (3) for \(f : X \to X' \in \text{Mor} \mathcal{D} \text{TOP}(\mathcal{N})\) \(f : Y \to Y' \in \text{Mor} \mathcal{D} \text{TOP}(\mathcal{N})\) we have \((f \times g)^{*}(\gamma)/\alpha = g^{*}(\gamma/f_{s}\alpha)\) (naturality) and (4) \([\zeta \times \beta] \cup \gamma]/\alpha = (-1)^{g(p+q+r-s)} \beta \cup [\gamma/\alpha \cap \zeta].\]
Proof. For details see (29.19), (29.20), (29.21), (29.22) and (29.23) in [g]. □

Remark 6.13 Note that as for the homology case, all the results presented in this subsection can be generalised to relative cohomology. More precisely, we have a relative version of the Mayer-Vietoris sequence, the universal coefficient theorem, the Künneth formula with corresponding cross product. We also have cup products, cap products and slant products for the relative case. For details see [d] or [g]. In subsequent sections we will need the following versions of cap products:

\[ \cap : H_{p+q}(X, A; R) \times H^p(X, A; R) \to H_q(X; R) \]

and

\[ \cap : H_{p+q}(X, A; R) \times H^p(X; R) \to H_q(X, A; R). \]

7 The Euler characteristic

7.1 Hopf algebras

A graded skew-commutative associative \( R \)-algebra \( H = \sum_{k \geq 0} H^k \) with unity element is called a quasi Hopf \( R \)-algebra if each \( H^k \) is a finite dimensional \( R \)-module and there is a degree preserving \( R \)-algebra homomorphism \( \mu : H \to H \otimes_R H \) called comultiplication, for which \( \mu(H^k) \subseteq \sum_{i+j=k} H^i \otimes_R H^j \).

A quasi Hopf \( R \)-algebra \( H = \sum_{k \geq 0} H^k \) is connected if \( H^0 \) is an \( R \)-algebra of dimension one with generator \( e \) and the map \( \epsilon : H \to R \), defined by \( \epsilon(e) = 1 \) and \( \epsilon(h) = 0 \) for all \( h \in H^k \) with \( k \geq 1 \) is a co-unit i.e., for all \( h \in H \),

\( (\epsilon \otimes_R 1)\mu(h) = h \otimes_R 1 \) and \( (1 \otimes_R \epsilon)\mu(h) = 1 \otimes_R h \). A quasi-Hopf algebra \( H \) is called an Hopf algebra if \( \mu \) is associative i.e., \( (\mu \otimes_R 1)\mu = 1 \otimes_R \mu \). \( \mu \) is called commutative if \( T \circ \mu = \mu \) where \( T(x \otimes_R y) = (-1)^{\deg(x)\deg(y)} y \otimes_R x \).

An example of a connected Hopf \( R \)-algebra, is the free, skew-commutative graded Hopf \( R \)-algebra

\[ R[x_1, \ldots, x_k, \ldots] \otimes_R \wedge[y_1, \ldots, y_l, \ldots]_R. \]
where the $x_i$'s are of even degrees and the $y_j$'s are of odd degrees, and we have the relations:

$$y_j^2 = -y_j^2 = 0, \ y_iy_j = -y_jy_i, \ y_jx_i = x_iy_j, \ x_ix_j = x_jx_i$$

and the number of $x_i$ and $y_j$ of each degree is finite. In view of the freeness of this algebra, comultiplication is determined by its values on the generators $x_i$ and $y_j$:

$$\mu(x_i) = x_i \otimes_R 1 + 1 \otimes_R x_i + \sum_{\deg(u_p) + \deg(u_q) = \deg(x_i)} u_p \otimes_R u_q$$

and

$$\mu(y_j) = y_j \otimes_R 1 + 1 \otimes_R y_j + \sum_{\deg(u_p) + \deg(u_q) = \deg(y_j)} u_p \otimes_R u_q.$$ 

**Fact 7.1** (Theorem 7.6 [dfn]) If $R$ is a field of characteristic zero, then a connected Hopf $R$-algebra is a free, skew-commutative graded Hopf $R$-algebra.

We also have the following result:

**Fact 7.2** ([mt] chapter VII, corollary 1.4). Let $H$ be a quasi Hopf algebra over a perfect field $K_p$ of characteristic $p$. Then we have the following ring isomorphisms:

1. For $p = 0$; $H \simeq (\otimes_\alpha [x_\alpha]_{K_p}) \otimes (\otimes_\beta K_0[x_\beta])$, where $\deg x_\alpha$ is odd and $\deg x_\beta$ is even.
2. For $p = 2$; $H \simeq (\otimes_\alpha K_2[x_\alpha]/(x_\alpha^{h_\alpha})) \otimes (\otimes_\beta K_2[x_\beta])$, where $h_\alpha$ is a power of 2.
3. For $p \neq 0,2$; $H \simeq (\otimes_\alpha [x_\alpha]_{K_p}) \otimes (\bigwedge_\beta K_p[x_\beta]) \otimes (\otimes_\gamma K_p[x_\gamma]/(x_\gamma^{h_\gamma}))$, where $\deg x_\alpha$ is odd, $\deg x_\beta$ and $\deg x_\gamma$ are even, and $h_\gamma$ is a power of $p$.

Here, if $\dim H < \infty$, then there is no term of $K_p[x_\beta]$.

**Definition 7.3** Let $(X, e, m)$ be a definable $H$-manifold i.e., $X$ is a definable manifold with a continuous definable $H$-multiplication $m : (X \times X, (e, e)) \rightarrow (X, e)$ and a $H$-unit $e \in X$ such that $[m \circ i_1] = [1_X] = [m \circ i_2]$ where $i_1, i_2 : X \rightarrow X \times X$ are the inclusions $i_1(x) := (x, e)$ and $i_2(x) := (e, x)$. We
say that \( m \) is definably \( H \)-commutative if \([m] = [m \circ t]\) (where \( t : X \times X \to X \times X \) commutes factors); \( m \) is definably \( H \)-associative if \([m \circ (m \times 1_X)] = [m \circ (1_X \times m)]\). A definable \( H \)-manifold \((X, e, m)\) is a definable \( H \)-group if \( m \) is \( H \)-associative and has an \( H \)-inverse i.e., a definable continuous map \( \iota : X \to X \times X \) such that \([m \circ (\iota \times 1_X) \circ \Delta_X] = [e] = [m \circ (1_X \times \iota) \circ \Delta_X]\) where \( \Delta_X : X \to X \times X \) is the diagonal map.

A definable continuous map \( f : (X, e) \to (X', e') \) between definable \( H \)-manifolds (resp., definable \( H \)-groups) \((X, e, m)\) and \((X', e', m')\) is called a definable \( H \)-map (resp., \( H \)-homomorphism) if \([h \circ m] = [m' \circ (h \times h)]\) (resp., also \([h \circ \iota] = [\iota' \circ h]\)).

Fact [7.3] together with the Künneth formula gives the following result.

**Theorem 7.4** Let \( X \in \text{ObjDTOP}(\mathcal{N}) \) and let \( R \) be either a field or \( \mathbb{Z} \) in which case we assume that \( H_n(X) \) are all free abelian groups. If \( X \) is a definably connected definable \( H \)-manifold (in particular, if \( X \) is a definable group) then \( H^*(X; R) \) is a connected quasi Hopf \( R \)-algebra. Moreover, if \( R \) is a field of characteristic zero, then \( H^*(X; R) \) is isomorphic to the finitely generated free exterior \( R \)-algebra \( \bigwedge[y_1, \ldots, y_r]_R \) and therefore comultiplication is given by \( \mu(x) = x \otimes_R 1 + 1 \otimes_R x \). If \( K_p \) is a field of characteristic \( p \) then graded \( K_p \)-algebra \( H^*(X; K_p) \) is given in fact 7.4.

**Proof.** For the fact that \( H^*(X; R) \) is a connected quasi Hopf \( R \)-algebra see the proof of theorem 12.42 [4] and the remark that follows it. Note that comultiplication \( \mu \) is given by \( \mu := (\alpha')^{-1} \circ m^* \).

By fact [7.3] if \( R \) is a field of characteristic zero then \( H^*(X; R) \) is isomorphic to a free, skew-commutative graded Hopf \( R \)-algebra. If there were any free generators \( x_i \) of positive even degree, then there would be elements of arbitrary high degree and this is impossible since for \( n > \dim X \) we have \( H^n(X; R) = 0 \). Therefore, \( H^*(X; R) \) is isomorphic to the finitely generated free exterior \( R \)-algebra \( \bigwedge[y_1, \ldots, y_r]_R \) and comultiplication is given by \( \mu(x) = x \otimes_R 1 + 1 \otimes_R x \).

Let \( X \) be as above. We call \( r \) the rank of \( X \) and denote this by \( r = \text{rank}(X) \). Also for each \( i = 1, \ldots, r \), \( g_i := \deg y_i \) and we write \( y_{ij} \) instead of \( y_i \) if \( g_i = j \).

We will now prove the dual of theorem 7.4.
Theorem 7.5 Let $R$ be a commutative ring. Then $\mathbb{H}_*(\ ; R)$ is a covariant functor from the category of definable (resp., definable $\mathbb{H}$-commutative, $\mathbb{H}$-associative) definable $\mathbb{H}$-manifold into the category of graded (resp., skew-commutative, associative) $R$-algebras with unit element.

If $(X, e, m)$ be a definably connected definable $\mathbb{H}$-commutative and $\mathbb{H}$-associative definable $\mathbb{H}$-manifold of finite type, and $R$ is either a field or $\mathbb{Z}$ in which case we assume that $\mathbb{H}_n(X)$ are all free abelian groups. Then $\mathbb{H}_*(X; R)$ is a connected quasi Hopf $R$-algebra. Moreover, if $R$ is a field of characteristic zero, then $\mathbb{H}_*(X; R)$ is isomorphic to the finitely generated free exterior $R$-algebra $\bigwedge[y_1, \ldots, y_r]_R$ and therefore comultiplication is given by $\mu(x) = x \otimes_R 1 + 1 \otimes_R x$.

Proof. Let $(X, e, m)$ be a definable $\mathbb{H}$-manifold. The multiplication in $\mathbb{H}_*(X; R)$ is called the Pontrijagin product and is defined by $\alpha'' \circ m^*$. For details see chapter VII section 3 in [d]. Co-multiplication in the other case is given by $(\alpha'')^{-1} \circ \Delta_X^*$ (for details see chapter VII section 10 in [d]).

7.2 The Euler characteristic

Definition 7.6 Let $X \in \text{Obj}_{DTOP}(\mathcal{N})$, $f : X \rightarrow X \in \text{Mor}_{DTOP}(\mathcal{N})$ and let $R$ be a commutative ring. Suppose that $\text{dim}X = m$. The Lefschetz number of $f$ over $R$ is defined by $\lambda(f; R) := \sum_{i=0}^{m} (-1)^i \text{tr} f^*_i$, where $\text{tr} f^*_i$ is the trace of the $R$-module homomorphism $f^* : \mathbb{H}^*(X; R) \rightarrow \mathbb{H}^*(X; R)$. We denote $\lambda(f; \mathbb{Q})$ by $\lambda(f)$.

Fact 7.7 (Hopf Trace Theorem)

$$\sum_{i=0}^{m} (-1)^i \text{tr} f_{i*} = \sum_{i=0}^{m} (-1)^i \text{tr} f_{i*} = \sum_{i=0}^{m} (-1)^i \text{tr} f_{i}^2 = \sum_{i=0}^{m} (-1)^i \text{tr} f_{i*}.$$

Proof. For the first equality see lemma 9.18 [e], for the last equality see theorem I.D.2 [f] and for the second equality see corollary I.D.3 [f]. This shows that $\lambda(f; R)$ is an integer and is independent of $R$.

Recall that the o-minimal Euler characteristic $E(X)$ of a definable set $X$ of dimension $m$ is defined by $E(X) := \sum_{i=0}^{m} (-1)^i a_i$ where for each $i$, $a_i$ is
the number of cells of dimension \( i \) in a cell decomposition of \( X \). This number does not depend on the chosen cell decomposition.

**Theorem 7.8** If \( X \in Obj\,DTOP(\mathcal{N}) \) then \( E(X) = \lambda(1_X) = \sum_{p=0}^{m}(-1)^p b_p \)
where, \( b_p := \text{dim}_Q H^p(X; \mathbb{Q}) \) are the Betti numbers.

**Proof.** By the triangulation theorem, we can assume that \( X = |K| \) for a simplicial complex \( K \). Its clear that \( E(X) = \sum_{p=0}^{m}(-1)^p a_p \) where for each \( p, a_p \) is the number \( p \)-simplexes in \( X \). There is a canonical chain equivalence \( j_z : C_*(K) \rightarrow S_*(|K|) \) and for each \( p \) the rank of the \( \mathbb{Z} \)-module \( C_p(K) \) is exactly \( a_p \) and so \( \sum_{p=0}^{m}(-1)^p a_p = \lambda(1_X) \). The other equality is clear. \( \square \)

We now combine theorem 7.8 with theorem 7.4 to prove the following result.

**Theorem 7.9** Let \( G \) be a definably compact definable group and for each \( k \geq 1 \) let \( p_k : G \rightarrow G \) be given by \( p_k(x) = x^k, p_0(x) = 1 \) and \( p_{-k}(x) = (x^{-1})^k \). Then there is \( r > 0 \) such that for all \( l \in \mathbb{Z} \), \( \lambda(p_l) = (1 - l)^r \). In particular, \( E(G) = 0 \).

**Proof.** By theorem 7.4, \( H^*(G; \mathbb{Q}) \) is isomorphic to the finitely generated free exterior \( \mathbb{Q} \)-algebra \( \bigwedge[y_1, \ldots, y_r]_\mathbb{Q} \) and therefore comultiplication is given by \( \mu(x) = x \otimes_\mathbb{Q} 1 + 1 \otimes_\mathbb{Q} x \). An induction on \( k \) shows that \( (p_k)^*(x) = kx \) for \( x \in \{y_1, \ldots, y_r\} \) (use the fact that if \( \Delta : G \rightarrow G \times G \) is the diagonal map, then \( (p_{k+1})^*(x) = \Delta^*((p_k)^* \otimes_\mathbb{Q} 1_G^*)\mu(x) \), for details see theorem F.1 chapter III in [3] and therefore, if \( x_i = y_{i_1} \cdots y_{i_r} \) is a generating monomial for \( H^*(G; \mathbb{Q}) \) then \( (p_k)^*(x_i) = k^{\text{len}(x_i)} x_i \) where \( \text{len}(x_i) = u \). This implies that \( \lambda(p_k) = \sum (-1)^{\text{deg}(x_i)} k^{\text{len}(x_i)} + 1 \) where the sum is taken over all monomials \( x_i \) generating \( H^*(G; \mathbb{Q}) \). Since \( y_j \)'s have odd degrees, we have \( (-1)^{\text{deg}(x_i)} = (-1)^{\text{len}(x_i)} \). Using this, a simple calculation shows that \( \lambda(p_k) = (1 - k)^n \). For negative \( k \) use the same argument together with the fact that \( x^*(z) = (-1)^p z \) for all \( z \in H^p(G; \mathbb{Q}) \) where \( x \) is the inverse in \( G \). The case \( k = 0 \) is trivial.

It will follow from remark 8.10 that \( r > 0 \). \( \square \)

Combining theorem 7.9, the main theorem of [2] together with results from [8] we get.

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Corollary 7.10 If $G$ is a definable abelian group such that $E(G) = 0$ then for each $k > 1$ the subgroup $\{x \in G : kx = 0\}$ of $k$-torsion points is finite and non trivial.

Proof. The fact that the subgroup $\{x \in G : kx = 0\}$ of $k$-torsion points is finite and non trivial follows from proposition 6.1 and lemma 5.11.

Corollary 7.11 (1) A definable solvable group has Euler characteristic zero if and only if it has a definably compact part. (2) If $G$ is a definable group such that $E(G) = 0$ then it has a definable abelian subgroup $H \neq 1$ such that $E(H) = 0$. (3) Maximal definably compact definably connected definable abelian subgroups of a definable group are conjugate.

Proof. (1) follows from the main theorem of [3] and theorem 7.3. (2) is lemma 2.9 and (3) is corollary 5.17 since by (1) and corollary 5.17 “maximal tori” (in the sense of [3]) are exactly maximal definably compact definably connected definable abelian subgroups.

8 Orientable definable manifolds

Through this section, $X = (X, (X_i, \phi_i)_{i \in I})$ is a definable manifold of dimension $n$.

8.1 Lemma on definable triangulation

Since $X$ if affine, $X \subseteq N^k$ for some $k$ and so we can definably triangulate the definable set $X$. But below we will be interested in a special definable triangulation of $X$ compatible with the definable charts $(X_i, \phi_i)_{i \in I}$. We make this notion precise in the following definition:

Definition 8.1 Let $A_i, \ldots, A_n, B, Z$ be definable subsets of $X$. Let $I = \{1, 2, \ldots, k\}$ be a numbering of $I$. Define inductively $(X'_i, (\Psi_i, M_i), (\Phi_i, N_i))$ for $i \in I$ by: $X'_1 = X_1$, $(\Psi_1, M_1)$ is a definable triangulation of $\phi_1(X_1)$ compatible with the definable subsets $\phi_1(X_1 \cap X_j), \phi_1(X_1 \cap A_i)$ and $\phi_1(X_1 \cap$
$X_j \cap A_l$) for all $l \in \{1, \ldots, n\}$ and all $j \in I_1$, and $(\Phi_1, N_1) = (\Psi_1, M_1)$; let $X'_{i+1} := \bigcup\{C : \phi_r(C) \in N_i, \ r = 1, \ldots, i\}$ and be $(\Psi_{i+1}, M_{i+1})$ be a definable triangulation of $\phi_{i+1}(X'_{i+1})$ compatible with the definable sets $\phi_{i+1}(X'_{i+1} \cap X_j)$, $\phi_{i+1}(X'_{i+1} \cap A_l)$ and $\phi_{i+1}(X'_{i+1} \cap X_j \cap A_l)$ for all $l \in \{1, \ldots, n\}$ and all $j \in I_{i+1}$ and such that $(\Phi_{i+1}, N_{i+1})$ which is equal to $(\Psi_{i+1}, M_{i+1})$ together with all the $(\Phi_j, N_j)$ for $j \in I_{i+1} \cap \{1, \ldots, i\}$ is a definable triangulation of $\phi_{i+1}(X_{i+1})$.

By a definable triangulation of $X$ compatible compatible with the definable charts and with $A_1, \ldots, A_n$ we mean a sequence $(\Phi, K) = \{(\Phi_i, N_i) : i \in I\}$ some $\{(\Phi, N_i) : i \in I\}$ like above. By a definable triangulation of $B$ compatible with the definable charts and with $A_1, \ldots, A_n$ we mean a sequence $(\Phi_B, K_B) = \{(\Lambda, L) \in (\Phi, K) : |L| \subseteq B\}$ for some definable triangulation $(\Phi, K)$ of $X$ compatible with the definable charts and with $A_1, \ldots, A_n, B$. Note that for each $C \in K$ such that $\phi_i(C) \subseteq K_i$ for some $i \in I$, $C$ is definably homeomorphic to a $k$-simplex. If $C \subseteq Z$ we say that $C$ is a $k$-simplex of $(\Phi, K)$ in $Z$.

**Lemma 8.2** There is a finite cover of $X$ by open definable subsets definably homeomorphic to open balls in $\mathbb{N}^n$. Moreover, if $A \subseteq X$ is a definable closed subset then, there is a finite family $A_1, \ldots, A_l$ closed under intersection of closed definable subsets $A_i$ and a finite cover of $X$ by open definable subsets $U_j$ definably homeomorphic to open balls in $\mathbb{N}^n$, such that $A = \bigcup A_i$ and for each $i$ there is a $j_i$ such that $A_i \subseteq U_{j_i}$.

**Proof.** Let $(\Phi, K)$ be a definable triangulation of $X$ compatible with the definable charts. Clearly each $\phi_i(X_i)$ has a finite cover by open definable subsets $V$ definably homeomorphic to open balls in $\mathbb{N}^n$: for each 0-simplex $v$ of the induced definable triangulation of $\phi_i(X_i)$ let $V$ be the star centred at $v$ (i.e, the union of $\{v\}$ together with all open simplexes of the triangulation which contain $v$ as a point in the closure. Now its also clear that the definable open sets $\phi_i^{-1}(V)$ satisfy the lemma.

Now suppose that $A \subseteq X$ is a definable closed subset. Take a definable triangulation of $X$ compatible with the definable charts and with $A$. For each 0-simplex $v$ of such triangulation let $U$ be the star centred at $v$ and let $A_i$ be the closed simplexes obtained from the barycentric subdivision of the closed simplexes contained in $A$. Then clearly each $A_i$ is contained in the a star of a 0-simplex of the triangulation contained in $A$. \[\square\]
8.2 The orientation sheaf of a definable manifold

Note that the standard proof of the fact that $\tilde{H}_q(S^n) = \mathbb{Z}$ iff $q = n$ and zero otherwise, remains valid in our case.

**Lemma 8.3** For any point $x \in X$, $H_n(X, X - x; R) = R$.

**Proof.** Suppose that $x \in X_i$. Let $U$ be an open definable subset of $X_i$ such that $x \in U$ and $\phi_i(U)$ is an open ball in $\phi_i(X_i)$. Then

$$H_n(X, X - x; R) \cong H_n(U, U - x; R) \cong \tilde{H}_{n-1}(U - x; R) \cong \tilde{H}_{n-1}(S^{n-1})$$

and $\tilde{H}_{n-1}(S^{n-1}) \cong R$.

The first equality is obtained by excising the closed definable subset $X - U$ of the definable open set $X - x$, the second follows from the exact sequence of the pair $(U, U - x)$ since $U$ is definably contractible, the third equality is obtained from the fact that $U - x$ is definably homotopically equivalent to $S^{n-1}$ and the last equality from remark above. \(\square\)

**Lemma 8.4** Given an element $\alpha_x \in H_n(X, X - x; R)$ there is an open definable neighbourhood $U$ of $x$ and $\alpha \in H_n(X, X - U; R)$ such that $\alpha_x = j_x^U(\alpha)$, where

$$j_x^U : H_n(X, X - U; R) \longrightarrow H_n(X, X - x; R)$$

is the canonical homomorphism induced by inclusion.

**Proof.** Let $a$ be the relative cycle representing $\alpha_x$. Then the support $|\partial a|$ of $\partial a$ is a definably compact subset of $X$ contained in $X - x$, so that $U := X - |\partial a|$ is an open definable neighbourhood of $x$. Let $\alpha \in H_n(X, X - U; R)$ be the homology class of $a$ relative to $X - U$. \(\square\)

We call $\alpha$ of lemma 8.4 the **continuation of $\alpha_x$ in $U$**.

**Lemma 8.5** There is a finite cover of $X$ by open definable subsets $U_i$ definably homeomorphic to open balls in $\mathbb{N}^n$ such that if $x \in U_i$ then for every $y \in U_i$, $j_y^{U_i}$ is an isomorphism (hence $\alpha_x$ has a unique continuation in $U_i$).
Proof. The existence of a finite cover of $X$ by open definable subsets $U_i$ definably homeomorphic to open balls in $\mathbb{N}^n$ is proved in lemma 8.2.

Let $V$ be one such open definably contractible definable subset of $X$, let $x \in V$ and let $H : V \times [0,1] \to V$ be a definable contraction of $V$ to a point in $V$. Let $U := V_t := H(V,t)$ be such that $x \in V_t$ and $0 < t < 1$. Then we have the following commutative diagram for any $y \in U$:

$$
\begin{array}{ccc}
H_n(X, X - U; R) & \cong & H_n(V, V - U; R) \\
\downarrow j_U & & \downarrow \\
H_n(X, X - y; R) & \cong & H_n(V, V - y; R) \cong \tilde{H}_{n-1}(V - y; R)
\end{array}
$$

in which the left horizontal isomorphism are excisions and the right ones are connecting homomorphisms ($V$ is definably contractible). The right vertical arrow is an isomorphism because the inclusion $V - U \to V - y$ is a definable homotopy equivalence (move out radially from $y$). Therefore, $j_U$ is an isomorphism. Since $V = \bigcup_{0<t<1} V_t$ the lemma follows.  

Definition 8.6 The $R$-orientation presheaf $O^X$ on $X$ is the presheaf of $R$-modules given by $O^X(U) := H_n(X, X - U; R)$ i.e., $O^X$ is a contravariant functor from the category of open definable subsets of $X$ with the inclusion maps into the category of $R$-modules. If $V \subseteq U$ are open definable subsets of $X$, $j_U^*: H_n(X, X - U; R) \to H_n(X, X - V; R)$ denotes the homomorphism induced by inclusion. $j_U^*$ is called a restriction map.

Note that by lemma 8.4 the stalk $O^X_x$ of $O^X$ at $x \in X$, which is by definition, the direct limit $O^X_x := \lim_{x \in U} O^X(U)$ with respect to the restriction maps is $H_n(X, X - x; R)$. And $j_U^*: H_n(X, X - U; R) \to H_n(X, X - x; R)$ is exactly the natural map induced by the direct limit.

Definition 8.7 The étalé space $\tilde{O}^X$ associated to the $R$-orientation presheaf $O^X$ is the topological space

$$
\tilde{O}^X := \{(x, \alpha_x) : x \in X, \alpha_x \in H_n(X, X - x; R)\}
$$

where the basis for the topology on $\tilde{O}^X$ is given by $(U, \alpha_U) := \{(x, \alpha_x) : x \in U, \alpha_x = j_U^*(\alpha_U)\}$ for open definable subsets $U$ of $X$ together with the étalé map $p : \tilde{O}^X \to X$ given by $p(x, \alpha_x) = x$ (i.e., $p$ is locally a homeomorphism).
For any definable subset $A \subseteq X$, a continuous map $s : A \rightarrow \tilde{O}^X$ such that $p \circ s = 1_A$ is called a \textit{section over} $A$. A section over $X$ is called a \textit{global section}. The set $\Gamma(A; R)$ of all sections over $A$ is in a natural way an $R$-module. We denote by $\Gamma_c(A; R)$ the $R$-submodule of $\Gamma(A; R)$ of all sections $s \in \Gamma(A; R)$ which agree with the zero section outside some definably compact definable subset of $A$.

We have a canonical homomorphism

$$j_A : H_n(X, X - A; R) \rightarrow \Gamma(A; R)$$

defined by $j_A(\alpha)(x) := (x, j^A_x(\alpha))$ for $x \in A$ (see remark 22.23). If $B \subseteq A$, we have the commutative diagram

$$
\begin{array}{ccc}
H_n(X, X - A; R) & \xrightarrow{j_A} & \Gamma(A; R) \\
\downarrow j^A_B & & \downarrow r \\
H_n(X, X - B; R) & \xrightarrow{j_B} & \Gamma(B; R)
\end{array}
$$

where $r$ is the restriction map.

\textbf{Definition 8.8} $\tilde{O}^X$ determines the presheaf of sections $U \rightarrow \tilde{O}^X(U) := \Gamma(U; R)$. And we have a morphism of presheafs $j : O^X \rightarrow \tilde{O}^X$ given by $j_U : O^X(U) \rightarrow \tilde{O}^X(U)$. The presheaf $\tilde{O}^X$ is actually a \textit{sheaf} i.e., for every collection of open definable subsets $U_i$ of $X$ with $U = \bigcup U_i$ then $\tilde{O}^X$ satisfies (1) if $\alpha, \beta \in \tilde{O}^X(U)$ and $j^U_i(\alpha) = j^U_i(\beta)$ for all $i$, then $\alpha = \beta$ and (2) if $\alpha_i \in \tilde{O}^X(U_i)$ and if for $U_i \cap U_j \neq \emptyset$ we have $j^U_{i \cap j}(\alpha_i) = j^U_{i \cap j}(\alpha_j)$ for all $i$, then there exists an $\alpha \in \tilde{O}^X(U)$ such that $j^U_i(\alpha) = \alpha_i$ for all $i$.

The sheaf $\tilde{O}^X$ is called the $R$-\textit{orientation sheaf} of $X$.

\textbf{8.3 Orientable definable manifolds}

\textbf{Definition 8.9} If $x \in X$, an $R$-orientation of $X$ at $x$ is a generator $\alpha_x$ of the $R$-module $H_n(X, X - x; R)$. Given a definable subset $A \subseteq X$, an $R$-orientation of $X$ along $A$ is a section $s \in \Gamma(A, R)$ such that for each $a \in A$, $s(a)$ is an $R$-orientation of $X$ at $a$. An $R$-orientation of $X$ is an $R$-orientation of $X$ along $X$. $X$ is $R$-orientable along $A$ if such $s$ exists, $X$ is $R$-orientable if $X$ is $R$-orientable along $X$. We also say that $X$ is orientable if it is $\mathbb{Z}$-orientable.
Note that, if there is an element $\alpha \in H_n(X, X - A; R)$ such that $j_y^A(\alpha)$ generates $H_n(X, X - y; R)$ for each $y \in A$ then $s(a) := (a, j_a^A(\alpha))$ is an $R$-orientation of $X$ along $A$. We call such $\alpha$ an $R$-orientation of $X$ along $A$. If there is a family $(U_i, \alpha_i)$ with $U_i$'s open definable neighbourhoods which cover $X$ and each $\alpha_i$ is an $R$-orientation of $X$ along $U_i$ which satisfies the compatibility conditions: for any $x \in X$, if $x \in U_i \cap U_j$, then $j_{x_i}^{U_i}(\alpha_i) = j_{x_j}^{U_j}(\alpha_j)$ then, $s \in \Gamma(X, R)$ given by $s(x) := (x, j_x^{U_i}(\alpha_i))$ if $x \in U_i$ is an $R$-orientation of $X$. We call such family $(U_i, \alpha_i)$ is called an $R$-orientation system on $X$.

By the universal coefficient theorem, if $X$ is orientable then it is $R$-orientable for all coefficient rings $R$.

**Remark 8.10** Let $G$ be an definably connected, definably compact definable group of dimension $n$. Then $G$ is $R$-orientable: let $\alpha_1$ be an $R$-orientation in an open definable neighbourhood $U_1$ of 1 (the identity element of $G$). For $a \in G$ let $l_a : G \rightarrow G$ be $l_a(x) := ax$, $U_a := l_a(U_1)$ and $\alpha_a := l_{a*}(\alpha_1)$. Then $(U_a, \alpha_a)_{a \in G}$ is an $R$-orientation system for $G$. This follows from the fact that for $x \in U_1$ and $a \in G$ we have $l_{a*} \circ j_{x_1}^{U_1} = j_{x}^{U_a} \circ l_{a*}$.

Below, we talk of *definable covering maps*, for details see [2].

**Theorem 8.11** (1) If $X$ is definably connected and non-orientable then there is a 2-fold definable covering map $p : E \rightarrow X$ such that $E$ is a definable orientable definably connected manifold. In particular, every definably connected manifold whose fundamental group contains no subgroup of index 2 is orientable. (2) An open definable submanifold of an $R$-orientable $X$ is $R$-orientable. In particular, $X$ is $R$-orientable if and only if all its definably connected components are $R$-orientable. (3) If $X$ is definably connected then any two $R$-orientations on $X$ that agree at one point are equal. In particular, if $X$ is orientable then it has exactly two distinct orientations. (4) Every definable manifold has a unique $\mathbb{Z}/2\mathbb{Z}$-orientation.

**Proof.** (1) Let $E$ be the set of pairs $(x, \alpha_x)$, where $x \in X$ and $\alpha_x$ is one of the generators of $H_n(X, X - x; \mathbb{Z})$. $E$ is a definable manifold with a basis of open definable neighbourhoods given by $(U, \alpha_U)$, where $U$ is an open definable neighbourhood in $X$ and $\alpha_U$ is a local orientation of $X$ along $U$. To see this use lemma [8.4]. The definable map $p : E \rightarrow X$ given by $p(x, \alpha_x) = x$ is a 2-fold definable covering map, and $E$ is oriented by the system whose opens...
are the \((U,\alpha_U)\) and whose orientations are the elements \(H_n(p_{|(U,\alpha_U)})^{-1}(\alpha_U)\) in \(H_n(E,E-(U,\alpha_U);\mathbb{Z})\). Moreover, \(E\) is definably connected for otherwise, \(p\) induces a homeomorphism between \(X\) and a component of \(E\) contradicting the orientability of \(E\) and its components. The rest of (1) follows from the fact that \(p_*\pi_1(E,e_0)\) has index 2 in \(\pi_1(X,x_0)\).

(2) Follows from the fact that if \(V \subseteq X\) is an open definable subset and \(x \in V\) then \(H_n(V,V-x;R) \simeq H_n(X,X-x;R)\) and definably connected components of \(X\) are open definable subsets.

(3) Follows from lemma 8.5 since the set on which two \(R\)-orientations agree is a clopen definable subset of \(X\).

(4) Follows also from lemma 8.3. \(\square\)

**Remark 8.12** Let \(X\) and \(Y\) be \(R\)-orientable definable manifolds. Then an \(R\)-orientation of \(X\) and of \(Y\) determine in a canonical way an \(R\)-orientation of \(X \times Y\). If fact we have by the relative K"unneth formula for homology a continuous map \(\mu : \tilde{\mathcal{O}}^X \times \tilde{\mathcal{O}}^Y \to \tilde{\mathcal{O}}^{X \times Y}\) commuting with the natural maps \(p^{X \times Y} : \tilde{\mathcal{O}}^{X \times Y} \to X \times Y\) and \(p^X \times p^Y : \tilde{\mathcal{O}}^X \times \tilde{\mathcal{O}}^Y \to X \times Y\). This map determines a map \(\Gamma(A;R) \times \Gamma(B;R) \to \Gamma(A \times B;R)\) for every definable subsets \(A \subseteq X\) and \(B \subseteq Y\). For details see VII.2.13 [d].

**Remark 8.13** Suppose now that \(X\) is a definable manifold with boundary. Then just like in 28.7[g] and 28.8[g], we have that if \(V\) is an open definable subset of \(X\) and \(\hat{V} = V \cap \hat{X}\) and \(\partial V = V \cap \partial X\) then there are unique homomorphisms \(\partial_V : \Gamma(V) \to \Gamma(\partial V)\) which are compatible with restriction to smaller \(V\) and which take local \(R\)-orientations of \(\hat{X}\) along \(\hat{V}\) into a local \(R\)-orientation of \(\partial X\) along \(\partial V\). In particular, if \(\hat{X}\) is \(R\)-orientable then so is \(\partial X\).

Note that by 28.12 [g], if \(\hat{X}\) is \(R\)-orientable, then there is a unique \(R\)-orientation of \(2\hat{X}\) (see remark 3.11) inducing the given \(R\)-orientation on \(\hat{X}^1\) and \(\hat{X}^2\).

### 8.4 The fundamental class

We start this subsection with the following easy remark.
Remark 8.14 For any definable subset $A \subseteq X$, $X$ is $R$-orientation along $A$ iff the covering map $p : p^{-1}(A) \to A$ is trivial. In which case, $\Gamma(A; R)$ is isomorphic to the $R$-module of all continuous maps $A \to R$. In particular, if $A$ has $k$ connected components then $\Gamma(A; R) \simeq R^k$.

Proof. Suppose that $s \in \Gamma(A; R)$ is an $R$-orientation of $X$ along $A$. Then for each $a \in A$, $s(a) = (a, s'(a))$ and $s'(a)$ is an $R$-orientation at $a$. If $(x, \alpha_x) \in p^{-1}(A)$ then there is a unique $\lambda_x \in R$ such that $\alpha_x = \lambda_x s'(x)$. The map $\phi : p^{-1}(A) \to A \times R$ given by $\phi(x, \alpha_x) = (x, \lambda_x)$ is a homeomorphism by lemma 8.4. Conversely, given $\phi$ we can recover $s$ by $s(x) = \phi^{-1}(x, 1)$ for $x \in A$. \qed

The main result of this subsection is the following theorem.

Theorem 8.15 Suppose that $A \subseteq X$ is a closed definable subset. Then, for all $q > n$ $H_q(X, X - A; R) = 0$ and

$$j_A : H_n(X, X - A; R) \to \Gamma_c(A; R)$$

is an isomorphism.

Proof. By lemma 8.2, there is a finite family $A_1, \ldots, A_l$ closed under intersection of closed definable subsets $A_i$ and finitely many open definable subsets $U_j$ definably homeomorphic to open balls such that $A = \cup A_i$ and for each $i$ there is a $j_i$ such that $A_i \subseteq U_{j_i}$.

Claim (1): If the result holds for closed definable subsets $A$, $B$ and $A \cap B$ then it holds for $C := A \cup B$.

Proof of Claim (1): Using the relative Mayer-Vietoris sequence for the triad $(X, X - A, X - B)$ we get $H_q(X, X - C; R) = 0$ for $q > n$ and we have the commutative diagram

$$\begin{array}{cccccc}
0 & \to & G_n(C) & \to & G_n(A) \oplus G_n(B) & \to & G_n(A \cap B) \\
& & \downarrow j_C & & \downarrow j_{A \oplus B} & & \downarrow j_{A \cap B} \\
0 & \to & \Gamma_c(C; R) & \xrightarrow{(r_A \oplus r_B)} & \Gamma_c(A; R) \oplus \Gamma_c(B; R) & \xrightarrow{r_A \oplus r_B} & \Gamma_c(A \cap B; R)
\end{array}$$

where $G_n(Z) := H_n(X, X - Z; R)$, and chasing the diagram shows that $j_C$ is an isomorphism. \qed
Claim (2): The result holds for $A$ definably compact.

Proof of Claim (2): Note that since $A$ is definably compact, it is a closed complex i.e., it contains all its faces, and also $A_i$'s are closed simplexes (all its faces are in $X$). By claim (1), the theorem follows by induction on the number of $A_i$'s such that $A = \bigcup A_i$. So suppose that $A = A_1$ and let $A \subseteq U_{j_1} =: U$. Since $U$ is definably homeomorphic to an open ball of dimension $n$ the result follows from $H_q(U, U - A; R) \simeq \tilde{H}_{q-1}(U - A; R) \simeq H_{q-1}(S^{n-1}; R)$, $\tilde{H}_{n-1}(S^{n-1}; R) = R$ and remark 8.14. □

Claim (3): If $A \subseteq U$, where $U$ is an open definable subset of $X$ with definably compact closure $U$ then the result holds for $U$ and $A$.

Proof of Claim (3): We use the exact homology sequence for the triple $(X, U \cup (X - \overline{U}), (U - A) \cup (X - \overline{U}))$. Note that by excision, $H_q(U, U - A; R) \simeq H_q(U \cup (X - \overline{U}), (U - A) \cup (X - \overline{U}); R)$. For $q > n$ we have $H_{q+1}(X, U \cup (X - \overline{U}); R) \rightarrow H_q(U, U - A; R) \rightarrow H_q(X, (U - A) \cup (X - \overline{U}); R)$.

For $q = n$, we have

$$
\begin{array}{c}
0 \rightarrow H_n(U, U - A; R) \rightarrow H_n(X, W; R) \rightarrow H_n(X, V; R) \\
\downarrow j_A^U \quad \downarrow \quad \downarrow \\
0 \rightarrow \Gamma_c(A; R) \rightarrow \Gamma_c(U \cup (X - \overline{U}); R) \rightarrow \Gamma_c(U - U; R)
\end{array}
$$

where $W := (U - A) \cup (X - \overline{U})$, $V := U \cup (X - \overline{U})$, $\Gamma_c(A; R)$ and $j_A$ are computed in the definable manifold $U$, and the monomorphism $i$ is defined as follows: Let $s \in \Gamma_c(A; R)$ be zero outside an definably compact $K \subseteq A$. Then $i(s)|_A = s$ and $i(s) = 0$ outside $K$.

Applying the result for the definably compacts $\overline{U} - U$ and $\overline{A} \cup (\overline{U} - U)$ we see that $H_q(U, U - A; R) = 0$ for $q > n$ and $j_A^U$ is an isomorphism. □

Claim (4): The result holds for any closed definable subset $A$.

Proof of Claim (4): Given $s \in \Gamma_c(A; R)$ zero outside the definably compact $K \subseteq A$. There is an open definable subset $K \subseteq U$ such that $\overline{U}$ is definably compact: As we have seen, by the triangulation theorem, $X$ is covered by finitely many open definable subsets $U_i$'s definably homeomorphic to open balls. We get $U$ by taking it to be $\bigcup_i V_i$ where $V_i$ is obtained from $U_i$ after a small definable contraction.
Consider $A' = A \cap U$, $s' = s|_A$. By Claim (3) applied to $U$ and $A'$, and the commutative diagram

$$
\begin{align*}
H_n(U, U - A'; R) &\longrightarrow H_n(X, X - A; R) \\
\downarrow^{j_{A'}} &\downarrow^{j_A} \\
0 &\longrightarrow s' \in \Gamma_c(A'; R) \quad \mapsto \quad s \in \Gamma_c(A; R)
\end{align*}
$$

we see that $j_A$ is surjective.

Now let $\alpha \in H_q(X, X - A; R)$. If $q = n$ suppose that $j_A(\alpha) = 0$. Let $z$ be the relative cycle representing $\alpha$. Applying the above argument to $|z|$, there is an open definable subset $|z| \subseteq U$ such that $U$ is definably compact. Let $A' = A \cap U$. By the same commutative diagram, we have $\alpha = 0$. For $q > n$, we know that the class of $z$ in $H_q(U, U - A'; R)$ is zero by Claim (3), so $\alpha = 0$. $\blacksquare$

**Definition 8.16** We see that, an $R$-orientation of $X$ along a definably compact $A$ determines a generator $\zeta_{X,A}$ of $H_n(X, X - A; R)$ called the fundamental class of the $R$-orientation of $X$ along $A$. If $A = X$ we let $\zeta_X := \zeta_{X,X}$.

The orientation class of $X$ is the element $\omega_X \in H^n(X; R)$ such that $(\zeta_X, \omega_X) = 1$.

**Corollary 8.17** If $A$ is a closed definably connected and not definably compact definable subset of $X$ then $H_n(X, X - A; R) = 0$. In particular, if $X$ is definably connected and not definably compact then $H_n(X; R) = 0$.

**Proof.** $j_A$ is locally constant and zero outside an definably compact subset of $A$. By definably connectedness of $A$, if $\alpha \in H_n(X, X - A; R)$, then $j_A(\alpha)$ is zero and so $\alpha = 0$. $\blacksquare$

**Corollary 8.18** Suppose that $X$ is definably compact and definably connected. Assume that for any $a \neq 0$, $a \in R$ and any unity $u \in R$, $ua = a$ implies $u = 1$. Then $H_n(X; R) = R$ if $X$ is $R$-orientable and $H_n(X; R) = 0$ otherwise.
Proof. If $X$ is $R$-orientable, apply remark 8.14. Suppose that there is a global section $s \in \Gamma(X; R)$, $s \neq 0$. Then by lemma 8.4 there is $a \in R$, $a \neq 0$, such that $s'(x)$ is $a$ times a generator of $H_n(X, X-x; R)$ for all $x \in X$, where $s(x) = (x, s'(x))$. The hypothesis on $R$ implies that $s'(x)/a$ is a well defined generator, and so $s/a \in \Gamma(X; R)$ given by $(s/a)(x) := (x, s'(x)/a)$ is an $R$-orientation of $X$. 

\[\blacksquare\]

8.5 Degrees

Definition 8.19 Let $f : X \longrightarrow Y$ be a continuous definable map between orientable definable manifolds of dimension $n$, and let $K \subseteq Y$ be a definably compact definably connected non-empty definable subset such that $f^{-1}(K)$ is definably compact. The degree of $f$ over $K$ is the integer $\deg_K f$ such that $f^*(\zeta_{X,K}) = (\deg_K f)\zeta_{Y,K}$.

We say that $f$ is a definably proper map if for all definably compact $K \subseteq Y$, $f^{-1}(K)$ is definably compact. For example $f$ is a definably proper map if $X$ is definably compact.

Lemma 8.20 Let $f : X \longrightarrow Y$ and $K \subseteq Y$ be as in definition 8.13. Then we have: (1) if $K' \subseteq f^{-1}(K)$ is definably compact then $f_*(\zeta_{X,K'}) = (\deg_K f)\zeta_{Y,K}$; (2) if $L$ is a definably compact definable subset of $K$ then we have $f_*(\zeta_{X,f^{-1}(L)}) = (\deg_K f)\zeta_{Y,K}$ in particular $\deg_L f = \deg_K f$; (3) if $X$ is a finite union of open definable subsets $X_1, \ldots, X_r$ such that the sets $K_i = f^{-1}(K) \cap X_i$ are mutually disjoint then $\deg_K f = \sum_{i=1}^r \deg_K f_{|K_i}$; (4) if $g : Z \longrightarrow X$ is a continuous definable map between orientable definable manifolds of dimension $n$, and $g^{-1}(f^{-1}(K))$ is definably compact then $\deg_K(f \circ g) = (\deg_{f^{-1}(K)}g)\deg_K f$; (5) if $f$ is a definably proper map and if $Y$ is definably connected then $\deg_K f = \deg f$ and is independent from $K$.

Proof. (1) The inclusion homomorphism

$$H_n(X, X - K'; R) \longrightarrow H_n(X, X - f^{-1}(K); R)$$

takes $\zeta_{X,K'}$ into $\zeta_{X,f^{-1}(K)}$. Therefore, the composition with

$$f_* : H_n(X, X - f^{-1}(K); R) \longrightarrow H_n(X, X - K; R)$$
takes $\zeta_{X,K}$ into $f_*(\zeta_{X,f^{-1}(K)}) = (\deg_K)\zeta_{X,K}$.

(2) Consider the commutative diagram

$$
\begin{align*}
H_n(X, X - f^{-1}(K); R) &\xrightarrow{i^*} H_n(X, X - K; R) \\
\downarrow i_* & \quad \downarrow i_* \\
H_n(X, X - f^{-1}(L); R) &\xrightarrow{i^*} H_n(X, X - L; R)
\end{align*}
$$

Chasing $\zeta_{X,f^{-1}(K)}$ through the diagram and using (1) gives the result.

(3) Consider the maps

$$
\bigoplus_{j=1}^r i_*^j \bigoplus_{j=1}^r i_*^j (\zeta_{X,K_j}) = \zeta_{X,Q}
$$

for every $Q \in f^{-1}(K)$, hence $\bigoplus_{j=1}^r i_*^j (\zeta_{X,K_j}) = \zeta_{X,f^{-1}(K)}$ by (1). Now, we have

$$
(\deg_K f)\zeta_{X,K} = f_*(\zeta_{X,f^{-1}(K)}) = f_*(\bigoplus_{j=1}^r i_*^j (\zeta_{X,K_j})) = \\
\bigoplus_{j=1}^r f_{|K_j}^* (\zeta_{X,K_j}) = (\sum_{j=1}^r \deg_K f_{|K_j}) \zeta_{X,K}.$$

(4) Is obvious. (5) Let $K, L \subseteq Y$ be definably compact and let $k \in K$ and $l \in L$ then by (2) $\deg_K f = \deg_K f$ and $\deg_L f = \deg_L f$. Since $Y$ is definably connected there is a definable path $\alpha : [0, 1] \to Y$ from $k$ to $l$ (see [2]). $\alpha([0, 1])$ is definably compact and by (2) again $\deg_K f = \deg_{\alpha([0,1])} f = \deg_L f$.

\[\square\]

9 Duality on definable manifolds

Through this section, $X = (X, (X_i, \phi_i)_{i \in I})$ is an $R$-orientable definable manifold of dimension $n$ with $R$-orientation $s \in \Gamma(X, R)$. Otherwise we take $R = \mathbb{Z}/2\mathbb{Z}$ and take the unique $\mathbb{Z}/2\mathbb{Z}$-orientation.

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9.1 Poincaré duality

Definition 9.1 The singular cohomology with compact supports is defined to be the direct limit

\[ H^q_c(X; R) := \lim_{K} H^q(X, X - K; R) \]

where \( K \) are definably compact. Note that if \( X \) is definably compact then 
\( H^q_c(X; R) = H^q(X; R) \). Definably proper definable maps \( f : X \to Y \) induce homomorphisms 
\( H^q_c(f) : H^q_c(Y; R) \to H^q_c(X; R) \).

Now for any definably compact \( K \subseteq X \), consider the homomorphism 
\( \zeta_{X,K} : H^q(X, X - K; R) \to H_{n-q}(X; R) \) given by cap product. Passing to the limit, these homomorphisms give a homomorphism 

\[ D : H^q_c(X; R) \to H_{n-q}(X; R). \]

Theorem 9.2 (Poincaré Duality Theorem). \( D : H^q_c(X; R) \to H_{n-q}(X; R) \) is an isomorphism (for all \( q \)).

Proof. First note that, by lemma 8.2, \( X \) is the union of a finite family \( U_1, \ldots, U_k \) closed under intersection of open definable subsets definably homeomorphic to open balls. We prove the result by induction on \( k \).

Case \( k = 1 \): Let \( U_1 = U \). Since \( U \) is definably homeomorphic to an open ball \( B \) of dimension \( n \) centred at the origin and of radius 1, in computing the inductive limit \( \lim_K H^q(B, B - K; R) \) it suffices to let \( K \) run through the final system of closed balls of radius < 1 centred at the origin. But for such \( K \), the modules in question are zero unless \( q = n \), and \( \zeta_K : H^n(B, B - K; R) \to H_0(B; R) \simeq R \) is an isomorphism and so the limiting homomorphism is also an isomorphism.

Claim: If the result holds for open definable subsets \( U, V \) and \( W = U \cap V \), then the result holds for \( Y = U \cup V \).

Proof of Claim: Let \( K \) (resp., \( L \)) be an definably compact subset of \( U \) (resp., \( V \)). We use the Mayer-Vietoris sequence for the triad \((Y, Y - K, Y - L)\). The diagram (where we put \( W' := W - K \cap L, Y' := Y - K \cup L, U' := U - K \),
$V' := V - L$; and for $Z = W, Y, U, V$ we set $T^m(Z) := H^m(Z, Z'; R)$, and $k := n - q + 1$ and $l := n - q$,

$$
T^{q-1}(W) \longrightarrow T^q(Y) \longrightarrow T^q(U) \oplus T^q(V) \longrightarrow T^q(W)
$$

$H_k(W; R) \longrightarrow H_l(Y; R) \longrightarrow H_l(U; R) \oplus H_l(V; R) \longrightarrow H_l(W; R)
$

is commutative except possibly a $+$ or $-$ sign. Moreover, every definably compact in $Y$ has the form $K \cup L$. Passing to the limit gives a sign commutative diagram (where $k = n - q + 1$ and $l = n - q$)

$$
H^q_c(W; R) \longrightarrow H^q_c(Y; R) \longrightarrow H^q_c(U; R) \oplus H^q_c(V; R) \longrightarrow H^q_c(W; R)
$$

$D \downarrow$

$H_k(W; R) \longrightarrow H_l(Y; R) \longrightarrow H_l(U; R) \oplus H_l(V; R) \longrightarrow H_l(W; R)$

in which all the rows are exact and all the vertical arrows except those involving $Y$ are isomorphisms. By the 5-lemma (19.12 [g]), the result holds for $Y$. $\square$

We get from Poincaré duality theorem the usual corollaries. In particular, suppose that $X$ is definably compact orientable. If $\dim(X)$ is odd then $E(X) = 0$ and if $\dim(X)$ is even and not divisible by 4 then $E(X)$ is even (see 26.10 and 26.11 [g]). Also if $X$ is definably compact with boundary and $\dot{X}$ is orientable, then $E(\partial X)$ is even (see 28.13 [g]).

9.2 Alexander duality

Through this subsection, $A \subseteq X$ is a closed definable subset and $U := X - A$. Let

$$\overline{H}^q(A; R) = \varprojlim V H^q(V; R)$$

be the direct limit where $V$ are open definable neighbourhoods of $A$ directed by reverse inclusion.

**Lemma 9.3** Assume that $X$ is definably compact. Then there is an exact sequence

$$\cdots \rightarrow H^q_c(U; R) \xrightarrow{i} H^q(X; R) \xrightarrow{j} \overline{H}^q(A; R) \xrightarrow{\delta} H^q_{c+1}(U; R) \rightarrow \cdots$$

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Proof. The homomorphism $H^q_c(U;R) \rightarrow H^q(X;R)$ is the unique homomorphism making the diagram

$$
\begin{array}{ccc}
H^q_c(U;R) & \rightarrow & H^q(X;R) \\
\downarrow & & \downarrow \\
H^q(U,U-K;R) & \rightarrow & H^q(X,X-K;R)
\end{array}
$$

commutative for all $K \subseteq U$ definably compact, where

$$H^q(U,U-K;R) \rightarrow H^q(X,X-K;R)$$

is the inverse to the excision isomorphism.

$\tilde{H}^q(A;R) \rightarrow H^{q+1}_c(U;R)$ is induced by the homomorphisms

$$H^q(V;R) \rightarrow H^{q+1}_c(X,V;R) \simeq H^{q+1}(U,U-K;R)$$

where $V$ is an open definable neighbourhood of $A$ and $K := X - V$ is a definably compact contained in $U$.

To prove that this sequence is exact is a simple diagram chasing, for details see theorem 27.3 [g].

Now let $\zeta_{X,A}$ be the fundamental class determined by the $R$-orientation of $X$ along $A$. For any open definable neighbourhood $V$ of $A$, we have $H_n(V,V-A;R) \simeq H_n(X,X-A;R)$ by excision; the pre-image of $\zeta_{X,A}$ under this isomorphism is still denoted by $\zeta_{X,A}$. Taking the cap product with $\zeta_{X,A}$ gives a homomorphism

$$\zeta_{X,A} \cap : H^q(V;R) \rightarrow H_{n-q}(V,V-A;R) \simeq H_{n-q}(X,X-A;R)$$

which induces a homomorphism

$$D_A : \tilde{H}^q(A;R) \rightarrow H_{n-q}(X,X-A;R).$$

Theorem 9.4 (Alexander Duality Theorem). Assume that $X$ is definably compact. Then $D_A$ is an isomorphism for all $q$. 

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Proof. The diagram
\[
\begin{array}{c}
\rightarrow H^q_c(U; R) \rightarrow H^q(X; R) \rightarrow \check{H}^q(A; R) \rightarrow \\
\downarrow_{D_U} \downarrow_{D_X} \downarrow_{D_A} \\
\rightarrow H_{n-q}(U; R) \rightarrow H_{n-q}(X; R) \rightarrow H_{n-q}(X, X - A; R) \rightarrow
\end{array}
\]
is sign-commutative (where $D_U$ and $D_X$ are the isomorphisms of the Poincaré Duality Theorem). Now apply the 5-lemma (19.15 [g]) which still works when the diagrams are only sign commutative.

\[\square\]

9.3 General separation theorem

The following lemma is an adaptation of results from 26.17 [g] on absolute neighbourhood retracts (namely, lemma 26.17.6 [g]).

Lemma 9.5 Let $Y$ be a definable manifold, $B$ a closed definable subset of $Y$ and $f : B \rightarrow X$ a continuous definable map. Then $f$ extends to a continuous definable map from an open definable neighbourhood of $B$ in $Y$.

Proof. By lemma [52], $X$ can be covered by a finite family $U_1, \ldots, U_l$ closed under intersection of open definable subsets definably homeomorphic to open balls. We prove the result by induction on $l$. The case $l = 1$ is corollary 3.10 [vdd].

Now let $X_1 = U_1 \cup \cdots \cup U_{l-1}$ and let $X_2 = U_l$ and assume that the result also holds for $X_1$. Let $A_i = B - f^{-1}(X_i)$ for $i = 1, 2$. Then $A_1 \cap A_2$ is empty, and $A_1, A_2$ are closed. Since $Y$ is affine, by lemma 3.5 [vdd], it is normal i.e., we can separate $A_1$ and $A_2$ by open definable sets $Y_1$ and $Y_2$.

Let $Y_0 = Y - (Y_1 \cup Y_2)$, closed in $Y$, hence also normal. Let $B_i = Y_i \cap B$, $i = 0, 1, 2$. Then $f(B_i) \subseteq X_i$ for $i = 1, 2$ and $f(B_0) \subseteq X_1 \cap X_2$. Since the result holds for $X_1 \cap X_2$, $f|_{B_0}$ can be extended to a continuous definable map $g_0$ on an open definable neighbourhood $U_0$ of $B_0$ in $Y_0$. Then $U_0 \cap B = B_0$, so $f$ together with $g_0$ define a continuous definable map $g : U_0 \cup B \rightarrow X$.

Again by lemma 3.5 [vdd], there are disjoint (relative) open definable subsets $V, W \subseteq Y_0$ such that $B_0 \subseteq V$ and $Y_0 - U_0 \subseteq W$. Then $U'_0 = Y_0 - W$ is closed and $U'_0 \subseteq U_0$. Now for $i = 1, 2$, $g(U'_0 \cup B_i) \subseteq X_i$ and $U'_0 \cup B_i$ is closed in $Y$. Since the result holds for $X_i$, $g|_{U'_0 \cup B_i}$ extends to a continuous definable map $G_i : U_i \rightarrow X_i$ on an open definable neighbourhood $U_i$, $i = 1, 2$. Now
\[ U'_i = U_i \cap (U'_0 \cup Y_i) \text{ is closed in } U := U'_1 \cup U'_2, \ i = 1, 2 \text{ and } U'_0 = U'_1 \cap U'_2. \]

Hence we can define a well defined continuous definable map \( F : U \to X \) by \( F|_{U'_i} = G_i, \ i = 1, 2. \) Moreover, \( U \) contains the open definable neighbourhood \((U_1 \cap (V \cup Y_1)) \cup (U_2 \cap (Y \cup Y_2))\) of \( B. \)

**Proposition 9.6** There is an isomorphism \( \kappa : \tilde{H}^q(A; R) \to H^q(A; R). \)

**Proof.** Passing the inclusion homomorphisms \( H^q(V; R) \to H^q(A; R) \) to the limit we get a canonical homomorphism \( \kappa : \tilde{H}^q(A; R) \to H^q(A; R). \) By proposition 3.3 \[vdd\] there is a definable retraction \( r : V \to A \) of an open definable neighbourhood \( V \) of \( A. \) Let \( i : A \to V \) be the inclusion. Then \( H^q(ri) = \text{identity}, \) and so \( H^q(i) \) is an epimorphism, hence \( \kappa \) is also an epimorphism.

Claim: There is an open definable neighbourhood \( W \) of \( A \) contained in \( V \) such that if \( j : W \to V \) is the inclusion, then there is a definable homotopy between \( i \circ r_{|W} \) and \( j. \)

**Proof of Claim:** On the closed subset \((V \times 0) \cup (A \times [0, 1]) \cup (V \times 1)\) of \( V \times [0, 1], \) set \( F(x, t) = x \) if \((x, t) \in V \times 0, \) \( F(x, t) = r(x) \) if \((x, t) \in V \times 1 \) and \( F(x, t) = x \) if \((x, t) \in A \times [0, 1]. \) Since \( V \times [0, 1] \) is normal, by lemma 9.3 \( F \) extends to a definable map of an open definable neighbourhood of this set into \( V. \) That open definable neighbourhood contains a definable set of the form \( W \times [0, 1], \) where \( W \) is an open definable neighbourhood of \( A. \) This gives the desired definable homotopy.

By the Claim, \( H^q(j) = H^q(r_{|W})H^q(i) \) and we have a factorisation

\[
\begin{align*}
H^q(U'; R) & \to H^q(A; R) \\
\downarrow & \uparrow \\
H^q(V; R) & \to H^q(A; R) \to H^q(W'; R)
\end{align*}
\]

where \( U' \) is any open definable neighbourhood of \( A \) containing \( V. \) It follows that any class in \( H^q(U'; R) \) going to zero in \( H^q(A; R) \) goes to zero in \( H^q(W'; R) \) and thus \( \kappa \) is a monomorphism.

**Corollary 9.7** There is an isomorphism \( H^q_c(X - A; R) \to H^q(X, A; R). \)
Proof. The same as the proof of corollary 27.4 [3].

Remark 9.8 (For details see 27.6 [3] and 27.7 [3]). Let \((K, L)\) be a pair of definably compact subsets of \(X\). Then there is a relative Alexander duality

\[
\hat{H}^q(K, L; R) \longrightarrow H_{n-q}(X - L, X - K; R)
\]

which is an isomorphism in the case \(K = X\). Here, \(\hat{H}^q(K, L; R)\) is the direct limit \(\lim_{(U, V)} H^q(U, V; R)\) where \((U, V)\) runs through the directed set of open definable sets containing \((K, L)\).

We can similarly define a canonical homomorphism

\[
H^q(X - L, X - K; R) \longrightarrow \hat{H}_{n-q}(K, L; R)
\]

which is an isomorphism if \(R\) is a field. Here, \(\hat{H}_{n-q}(K, L; R)\) is the inverse limit \(\lim_{(U, V)} H^n(U, V; R)\).

Corollary 9.9 (Separation Theorem). If \(A\) is a definably compact submanifold of \(N^m\) of dimension \(m - 1\) and having \(k\) definably connected components, then the complement of \(A\) has \(k + 1\) definably connected components.

Proof. By proposition 9.6 we have \(\hat{H}^q(A; R) = H^q(A; R)\). Regarding \(N^m\) as \(S^m\)-point, we have the isomorphisms \(H^q(A; R) \simeq H_{m-q}(S^m, S^m - A; R) \leftarrow H_{m-q}(N^m, N^m - A; R) \simeq \hat{H}_{m-q-1}(N^m - A; R)\).
9.4 Lefschetz duality theorem

Lemma 9.10 Let \( X \) be a definably compact definable manifold with boundary and let \( s \in \Gamma(\tilde{X}; R) \) an \( R \)-orientation of \( \tilde{X} \). Then there is a unique homology class \( \zeta \in H_n(X, \partial X; R) \) such that for any \( x \in \tilde{X} \), \( s(x) = j_x^\ast(\zeta) \). Moreover, \( \partial \zeta \in H_{n-1}(\partial X; R) \) is the fundamental class for the induced \( R \)-orientation of \( \partial X \).

Proof. The same as proposition 28.15 and corollary 28.16.

\( \blacksquare \)

Theorem 9.11 (Lefschetz Duality Theorem). Suppose that \( X \) is a definably compact definable manifold with boundary, such that let \( \tilde{X} \) be \( R \)-orientable. Let \( \partial \zeta \in H_{n-1}(\partial X; R) \) be the fundamental class. The the diagram

\[
\begin{array}{cccccc}
\rightarrow H^{q-1}(X; R) & \rightarrow H^{q-1}(\partial X; R) & \rightarrow H^q(X, \partial X; R) & \rightarrow H^q(X; R) \\
\downarrow \zeta \cap & \downarrow \partial \zeta \cap & \downarrow \zeta \cap & \downarrow \\
H_{n-q+1}(X, \partial X; R) & \rightarrow H_{n-q}(\partial X; R) & \rightarrow H_{n-q}(X; R) & \rightarrow H_{n-q}(X, \partial X; R)
\end{array}
\]

is sign-commutative and the vertical arrows are isomorphisms.

Proof. Similar to the proof of 28.18.

\( \blacksquare \)

10 Lefschetz fixed point theorem

Through this section, \( X = (X, (X_i, \phi_i)_{i \in I}) \) is an \( R \)-orientable manifold of dimension \( n \) with \( R \)-orientation \( s \in \Gamma(X, R) \).

10.1 Thom class

Consider the dual sheaf \( U \rightarrow \Gamma^*(U, R) \) of the \( R \)-orientation sheaf \( \tilde{O}^X \). \( s \) determines a global section \( s^* \in \Gamma^*(X, R) \) such that \( (s(x), s^*(x)) = 1 \) for all \( x \in X \).

Let \( \Delta_X \) be the diagonal of \( X \times X \) and \( i^U_x : (X, X - x) \rightarrow (X \times U, X \times U - \Delta_X) \) be the map given by \( i^U_x(z) := (z, x) \) for \( z \in X \).
Theorem 10.1 (Thom isomorphism theorem). Let \( U \subseteq X \) be an open definable subset. Then for all \( q < n \), \( H^q(X \times U, X \times U - \Delta_X; R) = 0 \) and there is a unique isomorphism \( \phi : H^n(X \times U, X \times U - \Delta_X; R) \to \Gamma^*(U; R) \) such that \( \phi(\beta)(x) = H^n(i_U^!(\beta)) \) for all \( \beta \in H^n(X \times U, X \times U - \Delta_X; R) \), \( x \in U \).

Proof. As we saw before, by lemma 8.2, \( X \) is the union of a finite family \( U_1, \ldots, U_l \) closed under intersection of open definable subsets definably homeomorphic to open balls.

For any \( \beta \in H^n(X \times U, X \times U - \Delta_X; R) \) define a set-theoretic section \( \Phi(\beta) : U \to (\tilde{O}_X)^* \) by \( \Phi(\beta)(x) = H^n(i_U^!(\beta)) \) for all \( x \in U \), where \( (\tilde{O}_X)^* \) is the étale space dual to \( \tilde{O}_X \) i.e., whose fibre at \( x \) is the local cohomology \( R \)-module \( H^n(X, X - x; R) \).

If \( \Gamma'(U; R) \) denotes the \( R \)-module of set-theoretic sections \( U \to (\tilde{O}_X)^* \), then for \( V \subseteq U \) we have the commutative diagram

\[
\begin{array}{ccc}
H^n(X \times U, X \times U - \Delta_X; R) & \to & H^n(X \times V, X \times V - \Delta_V; R) \\
\downarrow \Phi & & \downarrow \Phi \\
\Gamma'(U; R) & \to & \Gamma'(V; R)
\end{array}
\]

Thus to verify that the homomorphism \( \Phi \) takes its values in \( \Gamma^*(U; R) \), it suffices to consider the following cases:

Case (1): \( U \) is definably homeomorphic to an open ball and is contained in an open definable set \( V \) which is definably homeomorphic to an open ball.

Proof of Case (1): For each \( x \in U \) we have a commutative diagram

\[
\begin{array}{ccc}
H^q(X \times U, X \times U - \Delta_X; R) & \to & H^q(V \times U, V \times U - \Delta_V; R) \\
\downarrow i_U^! & & \downarrow i_U^! \\
H^q(X, X - x; R) & \to & H^q(V, V - x; R)
\end{array}
\]

where the horizontal isomorphisms are excisions. Thus we may assume that \( X = V = N^n \). In this case we have a homeomorphism

\[
f : (N^n \times U, (N^n - 0) \times U) \to (N^n \times U, N^n \times U - \Delta_{N^n})
\]

given by \( f(y, x) := (y + x, x) \) and for each \( x \in U \), a commutative diagram

\[
\begin{array}{ccc}
(N^n \times 0, N^n \times 0) & \to & (N^n \times U, N^n \times U) \\
\downarrow f & & \downarrow f \\
(N^n \times 0, N^n \times 0) & \to & (N^n \times N^n)
\end{array}
\]
where $N^n_z = N^n - z$, $f_x(y) := x + y$, and $j_x$ is the map of the point 0 onto $x$.
We may assume that $0 \in U$.

Claim (1.1): The map $s' \mapsto s'(0)$ is an isomorphism of $\Gamma^*(U; R)$ onto 
$H_n(N^n, N^n - 0; R)$.

Proof of Claim (1.1): This follows from the fact that $j_U^U_0$ is an isomorphism (by lemma 8.5).

Claim (1.2): If $s' \in \Gamma^*(U; R)$ and $x \in U$ then $s'(x) = H^n(f_x)(s'(0))$.

Proof of Claim (1.2): Let $\alpha \in H^n(N^n, N^n - U; R)$ be the unique class such that $\alpha = j_U^U_x(s'(x))$ for all $x \in U$. Now the maps $l_U^U_0, f_x l_U^0_0 : (N^n, N^n - U) \to (N^n, N^n - x)$ (where $l_U^0_x$ is the natural inclusion, and $H^n(l_U^0_x) = j_U^U_x$) are definably homotopic (at time $t$ the map is $f_x t x l_U^0_0$), whence $s'(x) = H^n(l_U^0_x)^{-1}(\alpha) = H^n(f_x)^{-1} H^n(l_U^0_0)^{-1} H^n(l_U^0_0)(s'(0)) = H^n(f_{-x})(s'(0))$.

Claim (1.3): $H^q(i_U^U_0)$ is an isomorphism for all $q$.

Proof of Claim (1.3): Using the commutative diagram above, we must show that 
$H^q(1_{N^n} \times j_0) : H^q(N^n \times U, (N^n - 0) \times U; R) \to H^q(N^n \times 0, (N^n - 0) \times 0; R)$
is an isomorphism, which follows from the fact that $U$ is definably contractible.

Now the theorem for Case (1) follow from Claims (1.1) and (1.3), since by Claim (1.2) and the fact that $i_U^U_x$ is definably homotopic to $i_U^U_0 \circ f_{-x}$ we have $\Phi(\beta)(x) := H^n(i_U^U_x)(\beta) = H^n(f_{-x}) H^n(i_U^0_0)(\beta)$.

Case (2): If the theorem holds for open definable subsets $U, V$ and $W = U \cap V$, then it holds for the open $Y = U \cup V$.

Proof of Case (2): Let $U' = X \times U - \Delta_X$, $V' = X \times V - \Delta_X$, $W' = X \times W - \Delta_X$ and $Y' = X \times Y - \Delta_X$.  

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Claim (2.1): There is an exact sequence

\[ \rightarrow H^q(X \times Y, Y'; R) \xrightarrow{i} H^q(X \times U, U'; R) \oplus H^q(X \times V, V'; R) \]
\[ \xrightarrow{j} H^q(X \times W, W'; R) \xrightarrow{k} H^{q+1}(X \times Y', R) \rightarrow \]

where \( i \) is induced by the chain homomorphism \( z \mapsto (z, z) \), \( j \) by the chain homomorphism \( (z, w) \mapsto z - w \), and \( k \) is the connecting homomorphism.

Proof of Claim (2.1): By the Universal Coefficient Theorem, it's enough to prove Claim (2.1) for \( R = \mathbb{Z} \). Consider the monomorphism of chain complexes

\[ i : S_\ast(X \times W)/S_\ast(W') \rightarrow S_\ast(X \times U)/S_\ast(U') \oplus S_\ast(X \times V)/S_\ast(V') \]

given by \( i(\overline{z}) = (\overline{z}, \overline{z}) \), and the chain epimorphism

\[ j : S_\ast(X \times U)/S_\ast(U') \oplus S_\ast(X \times V)/S_\ast(V') \rightarrow \]
\[ \rightarrow (S_\ast(X \times U) + S_\ast(X \times V))/(S_\ast(U') + S_\ast(V')) \]

given by \( j(\overline{z}, \overline{w}) = \overline{z} - \overline{w} \). Clearly \( ji = 0 \). Suppose that \( j(\overline{z}, \overline{w}) = 0 \). If \( z = \sum \mu_i \sigma_i \), let \( v \) be the sum of those \( \mu_i \sigma_i \) such that \( |\sigma_i| \) meets \( \Delta_X \); then \( v \) is equal to the chain defined in the same way using \( w \) instead of \( z \), since \( z - w \in S_\ast(U') + S_\ast(V') \), so that \( i(\overline{v}) = (\overline{z}, \overline{w}) \). Therefore, we have an exact sequence of chain complexes. Since these complexes are free, dualizing gives an exact sequence of cochain complexes, hence an infinite exact cohomology sequence

\[ \rightarrow H^q(C/C') \xrightarrow{i} H^q(X \times U, U') \oplus H^q(X \times V, V') \]
\[ \xrightarrow{j} H^q(X \times W, W') \xrightarrow{k} H^{q+1}(C/C') \rightarrow \]

where \( C = S_\ast(X \times U) + S_\ast(X \times V) \) and \( C' = S_\ast(U') + S_\ast(V') \). Since \( X \times Y = X \times U \cup X \times V \), the inclusion \( C \rightarrow S_\ast(X \times Y) \) is a chain homotopy (see the proof of the excision theorem in [Wo]). This holds similarly for the inclusion \( C' \rightarrow S_\ast(Y') \), and by passage to the quotient, for the inclusion \( C/C' \rightarrow S_\ast(X \times Y)/S_\ast(Y') \). Thus we can replace \( H^q(C/C') \) by \( H^q(X \times Y, Y') \) in the above exact sequence. \( \square \)

We now prove Case (2). By the assumption we see that

\[ H^q(X \times Y, Y'; R) = 0 \]

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for $q < n$ and the commutative diagram

\[
\begin{array}{ccc}
H^n(X \times Y, Y'; R) & \xrightarrow{i} & H^n(X \times U, U') \oplus H^n(X \times V, V') \\
\downarrow\Phi & & \downarrow\Phi \oplus \Phi \\
0 & \xrightarrow{\Phi} & \Gamma^*(Y; R) \oplus \Gamma^*(V; R) \xrightarrow{j} \Gamma^*(W; R)
\end{array}
\]

together with the 5-lemma imply that $\Phi$ is an isomorphism for $Y$. □ □

**Definition 10.2** This means that there is a unique cohomology class $\mu_X$ in $H^n(X \times X, X \times X - \Delta_X; R)$ such that $s^*(x) = H^n(i^X_*)(\mu_X)$. $\mu_X$ is called the *Thom class* of the given $R$-orientation. The *Lefschetz class* of $X$ is the image $\Lambda_X = H^n(j)(\mu_X)$ of the Thom class $\mu_X$ under the homomorphism $H^n(j) : H^n(X \times X, X \times X - \Delta_X; R) \to H^n(X \times X; R)$ induced by the inclusion $j : X \times X \to (X \times X, X \times X - \Delta_X)$.

**10.2 The Lefschettez isomorphism**

Before we proceed we need the following lemma

**Lemma 10.3** Suppose that $X$ is definably compact. If $\tau \in H^p(X \times X, X \times X - \Delta_X; R)$ and $\sigma \in H^q(X; R)$, then

\[
H^p(j)(\tau) \cup (\sigma \times 1) = H^p(j)(1 \times \sigma).
\]

**Proof.** Claim: There is an open definable neighbourhood $V$ of $\Delta_X$ in $X \times X$ and a definable retraction $r : V \to \Delta_X$ such that $i \circ r$ is definably homotopic to $k$, where $i : \Delta_X \to X \times X$ and $k : V \to X \times X$ are the inclusions.

**Proof of Claim:** Suppose that $X \subseteq N^m$. Then by proposition 3.3 [vdd] there is an open definable neighbourhood $U$ of $X$ having a definable retraction $s : U \to X$. Let $\epsilon = \text{distance from } X \to N^m - U$, and let $V$ be the $\epsilon$-neighbourhood of $\Delta_X$ in $X \times X$. Define $F : X \times X \times [0, 1] \to N^m$ by $F(x, y, t) := (1-t)x + ty$. Then $F$ maps $V \times [0, 1]$ into $U$. Let $G := r \circ F|_{V \times [0, 1]} : V \times [0, 1] \to X$ so that $G(x, y, 0) = s(x) = x$, $G(x, y, 1) = s(y) = y$. 

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The required definable homotopy \( H : V \times [0, 1] \to X \times X \) is defined by 
\( H(x, y, t) := (x, G(x, y, t)) \).

Let \( k' : (V, V - \Delta_X) \to (X \times X, X \times X - \Delta_X) \) denote the inclusion. Note that \( k' \) is an excision. We have 
\[
\begin{align*}
\lambda(\beta) &= \tau \cup H^q(p_i)(\sigma) = H^{p+q}(k')(\tau) \cup H^q(p_i; \sigma) \\
\rho(\alpha) &= H^p(j)(\tau) \cup H^q(p_i)(\sigma) = H^{p+q}(k') \cup H^q(p_i; \sigma)
\end{align*}
\]
which proves the lemma. \( \square \)

**Lemma 10.4** Suppose that \( X \) is definably connected and definably compact. Then for any \( p \leq n \), the inverse to the Poincaré duality isomorphism 
\( D_X : H^p(X; R) \to H_{n-p}(X; R), \quad \sigma \to \zeta_X \cap \sigma \) 
is given by 
\( D_X^{-1} : H_{n-p}(X; R) \to H^p(X; R), \quad \alpha \to (-1)^p \Lambda_X / \alpha. \)

**Proof.** We first show that \( \Lambda_X / \zeta_X = 1 \). For \( x \in X \), consider the commutative diagram
\[
\begin{array}{ccc}
(X, X - x) & \xrightarrow{i_x} & (X \times X, X \times X - \Delta_X) \\
X & \xrightarrow{i} & X \times X
\end{array}
\]
where \( i_x = i^X_x \) and the vertical arrows are inclusions. Note that, if \( \overline{x} \) is the homology class of \( x \) (\( x \) is a 0-cycle), then \( H_n(i_x)\zeta_X = \zeta_X \times \overline{x} \) (since \( X \simeq X \times x \)). We have, \( 1 = (s(x), H^n(i_x)(\mu_X)) = (H_n(j_x)\zeta_X, H^n(i_x)\mu_X) = (\zeta_X, H^n(i_xj_x)\mu_X) = (\zeta_X, H^n(j_x)\mu_X) = (H_n(i_x)\zeta_X, \Lambda_X) = (\zeta_X \times \overline{x}, \Lambda_X) = (\overline{x}, \Lambda_X/\zeta_X) \) (by theorem 6.12). 

Consider \( \sigma \in H^p(Y; R) \), then we have \( \Lambda_X/\zeta_X \cap \sigma = 1 \cup (\Lambda_X/\zeta_X \cap \sigma) = (-1)^{p(n+p+0-n)}[(\sigma \times 1) \cup \Lambda_X]/\zeta_X \) (by theorem 6.12) \( = (-1)^p(1 \times \sigma) \cup \Lambda_X \) (by lemma 10.3). \( \) \( = (-1)^p(1 \times \sigma) \cup (\Lambda_X/\zeta_X) = (-1)^p\sigma \cup 1 = (-1)^p\sigma \). \( \square \)

Let \( f : X \to Y \) be a continuous definable map, where \( Y \) is another definably compact, \( R \)-oriented definable manifold of dimension \( m \). We define the cohomology class \( \mu_f \) of the graph of \( f \) by

\[
\mu_f = H^m(f \times 1_Y)(\Lambda_Y) \in H^m(X \times Y; R).
\]

If \( X = Y \), the Lefschetz class \( L_f \) of \( f \) is defined by \( L_f := H^n(\Delta_X)(\mu_f) \in H^n(X; R) \).

Note that, for \( \sigma \in H^p(Y) \) we have \( \mu_f/\zeta_Y \cap \sigma = H^p(f)(\mu'_Y/\zeta_Y \cap \sigma) \) (by theorem 6.12 (3)) \( = (-1)^pH^p(f)(\sigma) \) (by lemma 10.4).

**Definition 10.5** Let \( L^p(X; R) := Hom_R(H^p(X; R), H^p(X; R)) \) and let

\[
L^*(X; R) := \sum_{p=0}^n L^p(X; R).
\]

For each \( p \) we have a canonical isomorphism

\[
k^p : H^p(X; R) \otimes_R H_p(X; R) \to L^p(X; R)
\]

which induces a canonical isomorphism

\[
k : \sum_{p=0}^n H^p(X; R) \otimes_R H_p(X; R) \to L^*(X; R)
\]

given by \( k := \sum_{p=0}^n (-1)^p k^p \). The Lefschetz isomorphism for \( X \) is the isomorphism of \( R \)-modules

\[
\lambda_X : L^*(X; R) \to H^n(X \times X; R)
\]

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given by \( \lambda_X := \alpha' \circ (1_X \otimes_R D_X^{-1}) \circ k^{-1} \) where, \( \alpha' \) is the Künneth isomorphism and \( D_X^{-1} \) is the inverse of the Poincaré duality isomorphism. Note that, \( \Lambda_X = \lambda_X(1_X^*) \).

**Lemma 10.6** Let \( Tr : L^*(X; R) \to R \) be the linear map given by \( Tr \sigma := \sum_{p=0}^{n} (-1)^p tr\sigma^p \) where \( \sigma = \sum_{p=0}^{n} \sigma^p, \sigma^p \in L^p(X; R) \). Then

\[
Tr \sigma = \left( \zeta, \Delta_X^* \lambda_X(\sigma) \right).
\]

**Proof.** Its enough to consider \( \sigma = k(\beta \otimes_R D_X \gamma) \) with \( \beta \in H^p(X; R) \) and \( \gamma \in H^{n-p}(X; R) \). Then, by ordinary linear algebra \( Tr \sigma = (-1)^{np+p}(D_X \gamma, \beta) = (-1)^{p(n-p)}(\zeta_X \cap \gamma, \beta) = (-1)^{p(n-p)}(\zeta_X, \gamma \cup \beta) = (\zeta_X, \beta \cup \gamma) = (\zeta_X, \Delta_X^* \alpha' (\beta \otimes_R \gamma)) = (\zeta_X, \Delta_X^* \lambda_X(\sigma)) \). □

**Definition 10.7** The Poincaré adjoint of \( f^* \) where \( f : X \to Y \) is a continuous definable map, is the unique linear map \( \tilde{f} := \sum_{p=0}^{n} \tilde{f}^p \) where \( \tilde{f}^n : H^n(X; R) \to H^{m-n}(Y; R) \) is determined by \( (D_Y \tilde{f}^n \alpha, \beta) = (D_X \alpha, f^* \beta) \) for all \( \alpha \in H^n(X; R) \) and \( \beta \in H^p(Y; R) \).

It’s easy to see that the Poincaré adjoint of a composition of continuous definable maps is the composition of the Poincaré adjoints, and if \( dimX = dimY \) then \( f \circ f^* = (degf)^1_X \). In particular, if \( X = Y \) then \( f^* \) is a linear isomorphism iff \( degf \neq 0 \), in which case \( (f^*)^{-1} = \frac{1}{degf} \tilde{f}^n \).

**Lemma 10.8** Let \( \sigma \in L^*(Y; R), f, g : X \to Y \) be continuous definable maps and suppose that \( dimX = dimY \). Then

\[
(f \times g)^*(\lambda_Y(\sigma)) = \lambda_X(f^* \circ \sigma \circ g).
\]

In particular, if \( X = Y \) then \( \mu_f = \lambda_X(f^*) \).

**Proof.** It’s enough to take \( \sigma = k(\alpha \otimes_R D_Y \beta) \) with \( \alpha \in H^p(Y; R) \) and \( \beta \in H^{n-p}(Y; R) \). We have \( (f^* \circ \sigma \circ g)(\gamma) = (-1)^{np}(D_Y \beta, g(\gamma)) f^*(\alpha) = (-1)^{np}(D_X g^*(\beta), \gamma) f^*(\alpha) = [k(f^* \alpha \otimes_R D_X g^* \beta)](\gamma) \) for all \( \gamma \in H^p(X; R) \). Therefore, \( \lambda_X(f^* \circ \sigma \circ g) = (f \times g)^* \circ \alpha' (\alpha \otimes_R \beta) = (f \times g)^*(\lambda_Y(\sigma)) \). □
10.3 The Lefschetz fixed point theorem

**Definition 10.9** Let $f, g : X \rightarrow Y$ be continuous definable maps and suppose that $\dim X = \dim Y$. The coincidence number of $f$ and $g$ is defined by

$$\lambda(f, g; R) := \sum_{p=0}^{n} (-1)^p \text{tr}(f^* p \circ \tilde{g}^p).$$

Note that if $X = Y$ then $\lambda(f, 1_X; R) = \lambda(f; R)$.

We have the following (see volume I chapter X [g1v]): $\lambda(f, g; R) = (-1)^n \lambda(g, f; R)$ and if $h : Z \rightarrow X$ is a third continuous definable map from a definably connected, definably compact, $R$-orientable definable manifold then $\lambda(f \circ h, g \circ h; R) = \lambda((degh)f, g; R)$.

**Theorem 10.10** Let $X$ and $Y$ be a $R$-orientable, definably compact definable manifolds of dimension $n$, where $R$ is a field. If $f, g : X \rightarrow Y$ are continuous definable maps then

$$\lambda(f, g; R) = (\zeta_X, \Delta_X^* \circ (f \times g)^*(\Lambda_Y)).$$

If $\lambda(f, g; R) \neq 0$, then there is $x \in X$ such that $f(x) = g(x)$.

**Proof.** We have $\lambda(f, g; R) = Tr(f^* \tilde{g}) = (\zeta_X, \Delta_X^* (\lambda_X(f^* \tilde{g}))) = (\zeta_X, \Delta_X^* (f \times g)^*(\Lambda_Y)).$

If there is no $x \in X$ such that $f(x) = g(x)$, then we have a factorisation

$$\begin{array}{ccc}
X & \xrightarrow{f \times g} & Y \times Y \\
\downarrow \Delta_X & & \uparrow i \\
X \times X & \xrightarrow{f \times g} & Y \times Y - \Delta_Y
\end{array}$$

where $i$ is the inclusion. Since $H^n(i)H^n(j) = 0$ and $\Lambda_Y = H^n(j)(\mu_Y)$, we have $0 = \Delta_X^* \circ (f \times g)^* \circ i^*(\Lambda_Y) = (f \times g)^*(\Lambda_Y)$ and therefore $\lambda(f, g; R) = 0$.

**Corollary 10.11** Let $X$ be a definably connected, definably compact definable manifold. If $X$ admits a continuous definable map $f : X \rightarrow X$ definably homotopic to the identity and without fixed points, then $E(X) = 0$.

Note that, like in the classical case, all the results of this section generalise to definable manifolds with boundary (see remark in 30.14 [g]).
11 Cohomology rings of definable groups

Below $G$ will be an definably connected, definably compact definable group of dimension $n$. We will use $^{-1}$ and $\iota$ for inverse in $G$ and $\cdot$ and $m$ for multiplication in $G$. $X$ is like in the previous section and $R$ will be a field of characteristic zero.

**Lemma 11.1** If $f, g : X \to G$ are two continuous definable maps then

$$\lambda(f,g; R) = \deg(f^{-1} \cdot g).$$

**Proof.** Consider the definable map $q : G \times G \to G$ given by $q := m \circ (\iota \times 1_G)$. A simple calculation using co-multiplication in $H^*(G; R)$, the fact that $\iota^*(\alpha) = (-1)^p \alpha$ for $\alpha \in H^p(G; R)$ and the definition of the Lefschetz isomorphism $\lambda_G$ shows that $q^*\omega_G = (-1)^n \omega_G \times 1 + 1 \times \omega_G = \lambda_G(1_G) = \Lambda_G$.

The result follows from the fact that $f^{-1} \cdot g = q \circ (f \times g) \circ \Delta_X$. \hfill $\square$

Before we prove our main theorem, computing the cohomology rings of definably compact, definably connected definable groups we recall some examples of $\mathbb{R}$-semialgebraic groups and the notion of Lie algebra cohomology.

**Remark 11.2** (see [mt] and [ghv]). For $F = \mathbb{R}$, $\mathbb{C}$ or $H$ the division ring of quaternions over $\mathbb{R}$, let

$$GL(n,F) := \{A \in M(n,F) : \exists A^{-1} \in M(n,F), AA^{-1} = I_n\}$$

(the general linear group over $F$), $U(n,F) := \{A \in GL(n,F) : AA^* = I_n\}$ (the unitary group over $F$). For $F = \mathbb{R}$ or $\mathbb{C}$ let $SL(n,F) := \{A \in GL(n,F) : det A = 1\}$ (the special linear group over $F$), $O(n,F) := \{A \in GL(n,F) : A^tA = I_n\}$, (the $F$ orthogonal groups) $Sp(n,F) := \{A \in M(2n,F) : AJ_nA = J_n\}$ (the $F$ symplectic groups) where $J_n := (J_{i,j})$ with $J_{1,1}^n = J_{2,2}^n = 0_n$, $J_{1,2}^n = -I_n$ and $J_{2,1}^n = I_n$.

The orthogonal groups are $O(n) := O(n,\mathbb{R}) = U(n,\mathbb{R})$, the special orthogonal groups are $SO(n) := U(n,\mathbb{R}) \cap SL(n,\mathbb{R})$, the unitary groups are $U(n) := U(n,\mathbb{C})$, the special unitary groups are defined as $SU(n) := U(n,\mathbb{C}) \cap SL(n,\mathbb{C})$ and the symplectic groups are $Sp(n) := U(n,\mathbb{H})$. We have a definable extension $1 \to SO(n) \to O(n) \to \mathbb{Z}_2 \to 1$ and the other groups of this list are $\mathbb{R}$-compact and $\mathbb{R}$-connected except for
GL(n, F) := U(n, F) × \mathbb{R}^{dn(n-1)/2+n} (for \ F = \mathbb{R}, \mathbb{C}, \text{or } \mathbb{H} \text{ and } d = 1, 2 \text{ or } 4 \text{ respectively}), \ SL(n, F) := SU(n, F) × \mathbb{R}^{dn(n-1)/2+n} (for \ F = \mathbb{R} \text{ or } \mathbb{C} \text{ and } d = 1 \text{ or } 2 \text{ respectively}), \ SP(n, \mathbb{R}) = U(n) × \mathbb{R}^{n(n+1)}, \ SP(n, \mathbb{C}) = Sp(n) × \mathbb{R}^{n(2n+1)} \text{ and } O(n, \mathbb{C}) = O(n) × \mathbb{R}^{n(n-1)/2}.

Remark 11.3 Let \( g \) be a Lie algebra over \( \mathbb{R} \). For each \( k \geq 0 \) let \( C^k(g, \mathbb{R}) := \mathcal{L}(\wedge^k g, \mathbb{R}) \) and define a differential by the Cartan formula

\[
dc(g_1, \ldots, g_{k+1}) = \sum_{1 \leq j < l \leq k+1} (-1)^{j+l+1}c([g_j, g_l], g_1, \ldots, \widehat{g_j}, \ldots, \widehat{g_l}, \ldots, g_{k+1}) + \sum_{j=1}^{k+1} (-1)^j g_j c(g_1, \ldots, \widehat{g_j}, \ldots, g_{k+1}).
\]

The cohomology of this complex is denoted by \( H^*(g, \mathbb{R}) \) and is called the cohomology of the Lie algebra. If \( G \) is a Lie group with Lie algebra \( g \), and \( Ad : G \to GL(g) \) the adjoint representation, \( g_I \) denotes the Lie subalgebra of \( g \) invariant under \( Ad \). If \( G \) is a connected, compact Lie group, then \( H^*(G, \mathbb{R}) = H^*(g_I, \mathbb{R}) \) (see [hvh]).

Lemma 11.4 If \((X,e,m)\) is a definable \( H \)-manifold, then \( \pi_1(X,e) \) is an abelian group. In particular, there are non negative integers \( s, m_1^l, \ldots, m_u^l \), such that \( \pi_1(X) = \mathbb{Z}^s \oplus \mathbb{Z}_{m_1^l} \oplus \cdots \oplus \mathbb{Z}_{m_u^l} \).

Proof. For definable paths \( \alpha \) and \( \beta \) in \( X \) let \( \alpha \beta \) be defined by \( \alpha \beta(t) := m(\alpha(t), \beta(t)) \). It follows from the definition of definable \( H \)-manifold that \([\epsilon_e \alpha] = [\alpha \epsilon_e] = [\alpha] \); if \([\alpha] = [\alpha'] \) and \([\beta] = [\beta'] \) then \([\alpha \beta] = [\alpha' \beta'] \). Further, we have the equality \((\alpha \cdot \beta)(\alpha' \cdot \beta') = (\alpha \alpha')(\beta \beta') \). Now the lemma follows from the definable homotopies: \([\alpha \cdot \beta] = [(\alpha \epsilon_e) \cdot (\epsilon_e \beta)] = [(\alpha \cdot \epsilon_e)(\epsilon_e \cdot \beta)] = [\alpha \beta], [\beta \cdot \alpha] = [(\epsilon_e \beta) \cdot (\alpha \epsilon_e)] = [(\epsilon_e \cdot \alpha)(\beta \cdot \epsilon_e)] = [\alpha \beta] \). \( \Box \)

The next lemma follows from results from [32] on cover of definable groups together with lemma 11.4.

Lemma 11.5 If \( G \) is a definably connected, definable abelian group then there are \( s, m_1^l, \ldots, m_u^l \in \mathbb{N} \cup \{0\} \) such that \( \pi_1(G) = \mathbb{Z}^s \oplus \mathbb{Z}_{m_1^l} \oplus \cdots \oplus \mathbb{Z}_{m_u^l} \) and \( \text{card}(\{x \in G : kx = 0\}) = k^s \cdot (k, m_1^l) \cdots (k, m_u^l) \).
Proof. The definable map \( p_k : (G, 0) \rightarrow (G, 0) \) is a surjective (since \( G \) is divisible) homomorphism with finite kernel. Therefore, by \([e2]\) \( p_k : (G, 0) \rightarrow (G, 0) \) is a definable covering map and since \( p_k* : \pi_1(G) \rightarrow \pi_1(G) \) is given by \( p_k*([\alpha]) = k[\alpha] \), we have that for all \( x, y \in G, |p_k^{-1}(x)| = |p_k^{-1}(y)| = |\pi_1(G, 0) : p_k* (\pi_1(H, 0))| = k^s \cdot (k, m_1) \cdots (k, m_u). \)

Finally we prove our main theorem.

Theorem 11.6 Let \( G \) be an definably connected, definably compact definable group. Then the rank of \( G \) equals the dimension of a maximal definably connected definably compact definable abelian subgroup, \( \sum_{i=1}^{\text{rank}(G)} g_i = \text{dim}G \) and \( \text{dim}G \equiv \text{rank}(G) \text{(mod}2) \). We have,

\[
H^*(G; R) = H^*(Z(G)^0; R) \otimes_R H^*(G/Z(G)^0; R),
\]

\[
\pi_1(G) = \pi_1(Z(G)^0) \oplus \pi_1(G/Z(G)^0).
\]

Moreover, we have (1) \( H^*(Z(G)^0; R) = \bigwedge[y_1, \ldots, y_{\text{dim}Z(G)^0}]_R \), \( \pi_1(Z(G)^0) = \mathbb{Z}^{\text{dim}Z(G)^0} \), and for each \( k > 1 \), \( \text{card}\{x \in Z(G)^0 : kx = 0\} = k^{\text{dim}Z(G)^0} \), (2) \( \pi_1(G/Z(G)^0) \) is finite and \( H^*(G/Z(G)^0; R) \) is obtained by taking tensor products over \( R \) of the following types of free, skew-commutative graded Hopf \( R \)-algebras:

- Type \( A_l \) (\( l \geq 1 \)) : \( \bigwedge[y_{13}, \ldots, y_{2l+1}]_R \);
- Type \( B_l \) (\( l \geq 2 \)) : \( \bigwedge[y_{13}, \ldots, y_{4l-1}]_R \);
- Type \( C_l \) (\( l \geq 3 \)) : \( \bigwedge[y_{13}, \ldots, y_{4l-1}]_R \);
- Type \( D_l \) (\( l \geq 4 \)) : \( \bigwedge[y_{13}, \ldots, y_{4l-5}]_R \);
- Type \( E_6 \) : \( \bigwedge[y_{13}, y_{19}, y_{111}, y_{115}, y_{117}, y_{123}]_R \);
- Type \( E_7 \) : \( \bigwedge[y_{13}, y_{111}, y_{115}, y_{119}, y_{123}, y_{127}, y_{135}]_R \);
- Type \( E_8 \) : \( \bigwedge[y_{13}, y_{115}, y_{123}, y_{135}, y_{139}, y_{147}, y_{159}]_R \);
- Type \( F_4 \) : \( \bigwedge[y_{13}, y_{111}, y_{115}, y_{123}]_R \);
- Type \( G_2 \) : \( \bigwedge[y_{13}, y_{111}]_R \).

Proof. We have a definable extension

\[
1 \rightarrow Z(G)^0 \rightarrow G \rightarrow G/Z(G)^0 \rightarrow 1
\]

and therefore, \( H^*(G; R) = H^*(Z(G)^0; R) \otimes_R H^*(G/Z(G)^0; R) \) and its enough to show the theorem separately for \( Z(G)^0 \) and \( G/Z(G)^0 \).
We first prove the result for $Z(G)^0$. By lemma \[1.1\] we have that for each $k \in \mathbb{Z}$, $(1-k)^r = \lambda(p_k) = deg_{p_{1-k}}$ where $r = rank(Z(G)^0)$. Therefore for each $k > 1$, card$(\{x \in Z(G)^0 : kx = 0\}) = k^r$ (since $p_k : Z(G)^0 \rightarrow Z(G)^0$ does not change local $R$-orientation, recall that $(p_k)^*(x) = k^{\text{len}(x)}x$). But by lemma \[1.3\] there are $s, m_1^l, \ldots, m_u^l \in \mathbb{N}\{0\}$ such that $\pi_1(Z(G)^0) = \mathbb{Z}^s \oplus \mathbb{Z}_{m_1^l} \oplus \cdots \oplus \mathbb{Z}_{m_u^l}$ and card$(\{x \in Z(G)^0 : kx = 0\}) = k^* \cdot (k, m_1^l) \cdots (k, m_u^l)$. It follows that $s = r$, $\pi_1(Z(G)^0) = \mathbb{Z}^r$ and $H^*(Z(G)^0; R) = \bigwedge[y_1, \ldots, y_r]_R$. But since $H^n(Z(G)^0; R)$ is non trivial ($Z(G)^0$ is $R$-orientable), we have $dim Z(G)^0 = r$.

Let $d := dim(G/Z(G)^0)$ and $t := rank(G/Z(G)^0)$. By \[1.2\] $G/Z(G)^0$ is definably semisimple definable group and by \[pps1\] and \[pps3\], $G/Z(G)^0$ is definably isomorphic to an $N$-semialgebraic subgroup (definable with parameters from $\mathbb{Q}$) of some $GL(m, N)$ and therefore since the cohomology groups depend only on the triangulation, $G/Z(G)^0$ and $(G/Z(G)^0)(\mathbb{R})$ have the same cohomology rings. But $(G/Z(G)^0)(\mathbb{R})$ is a Lie group and it follows from theorem IV, volume II chapter IV in \[ghv\] that $rank(G/Z(G)^0)$ is the dimension of a maximal definably connected definably compact definable abelian subgroup, $\sum_{i=1}^t g_i = d$ and $d \equiv t(\text{mod} 2)$.

Also by results from \[ghv\] it follows together with what we have proved so far that $H^*(G; R) = H^*(g; R)$ where $g$ is the Lie algebra of $G$. On the other hand, by \[gps\] we have a definable extension

$$1 \rightarrow Z(G/Z(G)^0) \rightarrow G/Z(G)^0 \rightarrow G_1 \times \cdots \times G_k \rightarrow 1$$

where each $G_i$ is a definably simple centerless definably connected definable group (definable with parameters from $\mathbb{Q}$) and each one of $G_i(\mathbb{R})$ is a simple, compact, connected Lie group of one of the types $A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4$ or $G_2$. Now the result follows from a similar result for Lie groups (see \[int\] or \[ghv\]): For example, the classical groups $SU(l+1)$, $SO(2l+1)$, $Sp(l)$ and $SO(2l)$ are of type $A_l, B_l, C_l$ and $D_l$ respectively, $H^*(SU(n); R) = \bigwedge[y_{13}, \ldots, y_{12n-1}]_R$, $H^*(SO(2m+1); R) = \bigwedge[y_{13}, \ldots, y_{4m-1}]_R$, and $H^*(SO(2m); R) = \bigwedge[y_{13}, \ldots, y_{4m-3}]_R$.

Since $H^1(G/Z(G)^0; R) = 0$ we have that $\pi_1(G/Z(G)^0)$ is finite. \hfill \Box

Note that $H^*(G; R) = \bigotimes_{R}^{G[G]} H^*(G^0; R)$. In particular, $H^*(O(n); R) = \bigotimes_{R}^2 H^*(SO(n); R)$ The cohomology ring of $U(n)$ is given by $H^*(U(n); R) = \bigwedge[y_1, \ldots, y_{2n-1}]_R$, since $Z(U(n)) = U(1) = SO(2)$ and the projective uni-
primary group $PU(n) := U(n)/Z(U(n)) = SU(n)/\mathbb{Z}_n$.

**Corollary 11.7** Let $G$ be a definably compact, definably connected definable group. Then $G$ is definably semisimple iff $Z(G)$ is finite iff $Ad : G \rightarrow Ad(G)$ is a definable covering map iff $\pi_1(G)$ is finite iff $H^1(G; \mathbb{Z}) = 0$ iff the universal covering group $\tilde{G}$ of $G$ is a definably compact (definably semisimple) definable group.

**Proof.** This follows from theorem 11.6 and results from [e2].

**Remark 11.8** The cohomology ring $H^*(G; \mathbb{Z}_p)$ of a definably compact definably, definably simply connected, definably simple definable group $G$ is equal the cohomology ring of the corresponding simply connected, compact simple Lie group $G(\mathbb{R})$ of the same type as $G$. Explicit computation of these cohomology rings is given in [mt]. We will not include here the full description of these rings, but we note that the (co)homology ring of definably simply connected, definably compact, definably simple definable group $G$ are $p$-torsion free in the following cases:

- $p \geq 2$, $G$ of type $A_l$, $C_l$;
- $p \geq 3$, $G$ of type $B_l$, $D_l$, $G_2$;
- $p \geq 5$, $G$ of type $F_4$, $E_6$, $E_7$;
- $p \geq 7$, $G$ of type $E_8$.

Finally note that if $G$ is a definably connected, definably compact definable abelian group of dimension $n$ then, since $H_1(G) = \pi_1(G) = \mathbb{Z}^n$, $H^*(G)$ is $p$-torsion free for all $p$ and so $H^*(G; K_p) = \bigwedge[x_1, \ldots, x_n]_{K_p}$.

**Remark 11.9** The following results are obtained by transferring similar results for simple Lie groups (see [mt]) and from result from [2]. Let $G$ be a definably connected, definably compact, definably semisimple definable group with universal covering group $\tilde{G}$. Then

$$\pi_1(G) \leq Z(\tilde{G}) = Z(\tilde{G}_1) \times \cdots \times Z(\tilde{G}_k)$$

where $\tilde{G}_i$’s are definably simple, simply connected definably compact definable groups such that $\tilde{G}/Z(\tilde{G}) = \tilde{G}_1 \times \cdots \times \tilde{G}_k$. 69
Moreover, $Z(\tilde{G}_i)$ is $\mathbb{Z}_{d+1}, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_2$, 0, 0 or 0 if $\tilde{G}_i(\mathbb{R})$ is of type $A_l$, $B_l$, $C_l$, $D_{2l}$, $D_{2l+1}$, $E_6$, $E_7$, $E_8$, $F_4$ or $G_2$ respectively.

For example, $\pi_1(SU(n)) = 0$ and $Z(SU(n+1)) = \mathbb{Z}_{n+1}$; for $n > 1$ we have $\pi_1(SO(n)) = \mathbb{Z}_2$, $Z(SO(2n+1)) = 0$, $Z(SO(2n)) = \mathbb{Z}_2$ the universal covering group of $SO(n)$ is $Spin(n)$ (the spinor groups) and $Z(Spin(2n+1)) = \mathbb{Z}_2$; $\pi_1(Sp(n)) = 0$ and $Z(Sp(n)) = \mathbb{Z}_2$; $Z(Spin(4n)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $Z(Spin(4n+2)) = \mathbb{Z}_4$.

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