Accelerating Asynchronous Algorithms for Convex Optimization by Momentum Compensation

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Abstract
Asynchronous algorithms have attracted much attention recently due to the crucial demands on solving large-scale optimization problems. However, the accelerated versions of asynchronous algorithms are rarely studied. In this paper, we propose the “momentum compensation” technique to accelerate asynchronous algorithms for convex problems. Specifically, we first accelerate the plain Asynchronous Gradient Descent, which achieves a faster $O(1/\sqrt{\epsilon})$ (v.s. $O(1/\epsilon)$) convergence rate for non-strongly convex functions, and $O(\sqrt{\kappa} \log(1/\epsilon))$ (v.s. $O(\kappa \log(1/\epsilon))$) for strongly convex functions to reach an $\epsilon$-approximate minimizer with the condition number $\kappa$. We further apply the technique to accelerate modern stochastic asynchronous algorithms such as Asynchronous Stochastic Coordinate Descent and Asynchronous Stochastic Gradient Descent. Both of the resultant practical algorithms are faster than existing ones by order. To the best of our knowledge, we are the first to consider accelerated algorithms that allow updating by delayed gradients and are the first to propose truly accelerated asynchronous algorithms. Finally, the experimental results on a shared memory system show acceleration leads to significant performance gains on ill-conditioned problems.

1 Introduction
With the popularity of multi-core computers and the crucial demands for handling large-scale data in machine learning, designing parallel algorithms have attracted lots of interests in recent years. A straightforward way to implement parallelization is through synchronous update. Since each thread has to wait the precedent one to finish computing, a limited speed up caused by serious overhead can be observed from synchronous algorithms, especially when the computation costs for each thread are different, or a large load imbalance exists. To avoid the frequent usage of synchronization operation, asynchronous algorithms are designed as a more sophisticated way for parallelization.

The main difference between asynchronous and synchronous algorithms lies in the state of the parameters for computing the gradient. For synchronous algorithms, their results are essentially identical to the serial one with variants only on implementation. Asynchronous algorithms are different, because when one thread is computing the gradient, other threads might have updated the parameters. Take Asynchronous Gradient Decent as an example, if we assign a global counter $k$ to indicate each update from any thread, the iteration can be formulated as:

$$x^{k+1} = x^k - \gamma \nabla f(x^{j(k)}),$$

(1)
Table 1: Convergence rates of asynchronous algorithms and their corresponding serial algorithms for convex optimization. (‘P’ is short for optimization problem. ‘T’ is short for type, ‘S’ is short for serial, ‘A’ is short for asynchronous, and ‘B’ is short for bounded delay assumption. \( \mu \) is the strong convexity modulus, \( L \) and \( L_c \) are the Lipschitz and coordinate Lipschitz constants in Eq. (3) and Eq. (4), respectively.)

| P | T | Algorithm             | Convergence Rate for NC | Convergence Rate for SC | Assumptions                              |
|---|---|-----------------------|-------------------------|-------------------------|------------------------------------------|
| 1 | S | AGD (Nesterov 1983)   | \( \sqrt{L/\epsilon} \) | \( \frac{L}{\sqrt{\epsilon}} \log(1/\epsilon) \) |                                          |
|   | A | AAGD (ours)           | \( \tau^2 \sqrt{L/\epsilon} \) | \( \frac{L^2}{\tau^2 \epsilon} \log(1/\epsilon) \) | B                                        |
| 5 | S | APCG (Lin et al. 2014) | \( (n\sqrt{L} + n\sqrt{L_c})/\epsilon \) | \( n \frac{\sqrt{L} + \sqrt{L_c}}{\epsilon} \log(1/\epsilon) \) | B, \( \tau \leq \frac{L}{\sqrt{L_c}} \), NC: \( |\mathbf{x}^j| \leq R \) |
|   | A | ASCD (Liu et al. 2015b) | \( nL_c/\epsilon \) | \( n \frac{L_c}{\epsilon} \log(1/\epsilon) \) | B                                        |
|   | A | AROCK (Peng et al. 2016) | \( \tau nL_c/\epsilon \) | \( \tau n \frac{L_c}{\epsilon} \log(1/\epsilon) \) | B                                        |
|   | A | AASCD (ours)          | \( (\sqrt{L} + n\tau\sqrt{L_c})/\epsilon \) | \( n\tau \frac{\sqrt{L} + \sqrt{L_c}}{\epsilon} \log(1/\epsilon) \) | B                                        |
| 12| S | Katyusha (Allen-Zhu 2017) | \( n + (n + \sqrt{n})\sqrt{L/\epsilon} \) | \( n + \frac{n\sqrt{L} + \sqrt{n}}{\epsilon} \log(1/\epsilon) \) | B, Sparse, Smooth |
|   | A | Hogwild (Niu et al. 2011) | Not analysis | \( \frac{O(L)}{\tau^2 \epsilon} \) | B, Sparse, Smooth |
|   | A | ASGD (Agarwal & Duchi 2011) | \( \frac{n}{\tau^2 \epsilon} + \frac{\sqrt{n}}{\epsilon} \) | Not analysis | B, Smooth |
|   | A | ASVRG (Reddi et al. 2015) | Not analysis | \( n + (1 + \triangle \tau)^2 \frac{n}{\epsilon} \log(1/\epsilon) \) | B, Sparse, Smooth |
|   | A | AASGD (Meng et al. 2016b) | Not analysis | \( n + \frac{\tau^2}{Lc} \frac{n}{\epsilon} \log(1/\epsilon) \) | B, Sparsity, Smooth |
|   | A | ASVRG (ours) | Not analysis | \( n + \tau^2 Ln/\epsilon \) | B |
|   | A | AASVRG (ours)          | \( n + (n + \tau \sqrt{n})\sqrt{L/\epsilon} \) | \( n + \tau \frac{n\sqrt{L} + \sqrt{n}}{\epsilon} \log(1/\epsilon) \) | B |

where \( \gamma \) is the step size and \( x^{j(k)} \) is the state of \( x \) at the reading time. Typically, \( x^{j(k)} \) can be any of \( \{x^1, \ldots, x^K\} \) when the parameters are updated with locks (see Section 2). So for asynchronous algorithms, the gradient might be delayed.

Up to now, lots of plain asynchronous algorithms are designed. For example, Niu et al. (2011) and Agarwal & Duchi (2011) propose Asynchronous Stochastic Gradient (ASGD), which achieves \( O(1/\epsilon) \) convergence rate for strongly convex (SC) functions, where \( \epsilon \) is the approximate error satisfying \( F(x) - F(x^*) \leq \epsilon \). Some Variance Reduction (VR) based asynchronous algorithms (Reddi et al. 2015, Cong & Lin 2017, Huo & Huang 2016) are also designed later. For Asynchronous Stochastic Coordinate Descent (ASCD) (Liu et al. 2015b, Peng et al. 2016), the provable convergence rate is \( O(1/\epsilon) \) for non-strongly convex (NC) functions and \( \kappa \log(1/\epsilon) \) for SC, where \( \kappa \) is the condition number. A more detailed comparison for convergence results of asynchronous algorithms for convex problems is shown in Table 1.

On the other hand, Nesterov (1983, 1988) has proposed a well-known accelerated version of gradient descent (AGD) for \( L \)-smooth convex functions. AGD achieves \( O(1/\sqrt{\epsilon}) \) rate for NC and \( O(\sqrt{\kappa} \log(1/\epsilon)) \) for SC, which provably meets the lower bound (ignoring the constant) and is also observed to be faster than existing ones. After that, many accelerated algorithms have been designed to achieve faster convergence rates. For example, FISTA (Beck & Teboulle 2009) is a proximal version of AGD. APCG (Percoco & Richtárik 2015, Lin et al. 2014) is a proximal and accelerated version of Stochastic Coordinate Descent (SCD). AccSDCA (Shalev-Shwartz & Zhang 2014) uses the black-box technique to accelerate the Stochastic Dual Coordinate Ascent. Katyusha (Allen-Zhu 2017) is an accelerated version of VR methods.

Comparing those plain asynchronous algorithms with serial ones, there is a gap in convergence rate. It is an open problem to fill in the gap by proposing accelerated asynchronous algorithms. We find that Meng et al. (2016b) integrates momentum, VR, tricks, coordinate sampling to accelerate ASGD, named AASGD. But the convergence rate is still \( O(\kappa \log(1/\epsilon)) \) for SC functions. There is no improvement in convergence rate comparing with ASVRG. Designing an asynchronous accelerated algorithms is not easy. The reason are two-folded:

- In serial accelerated schemes, the extrapolation point are subtly and strictly connected with \( x^k \) and

1 To the best of our knowledge, there is still no analysis on asynchronous VR algorithms for NC. As a byproduct of our analysis, the convergence rate is \( O(n + \tau^2 Ln/\epsilon) \), as shown in Table 1. The proof is shown in Supplementary Material.
x^{k-1}, i.e. y^{k} = x^{k} + \frac{\delta(1-\delta^{k-1})}{p_{\delta}}(x^{k} - x^{k-1}). However, such information might not be available for asynchronous algorithms because there are unknown delays in updating the parameters.

- Since x^{k+1} is updated based on y^{k}, i.e. x^{k+1} = y^{k} - \frac{1}{\tau} \nabla f(y^{k}), x^{k+1} is related to past updates (to generate y^{k}). This is different from unaccelerated algorithms. For example, in gradient descent, x^{k+1} = x^{k} - \frac{1}{\tau} \nabla f(x^{k}), so x^{k+1} only depends on x^{k}.

In this paper, we attempt to fill in the gap to some degree. We propose a technique called “momentum compensation” to accelerate asynchronous algorithms for convex problems. We first consider accelerating plain Asynchronous Gradient Descent. We demonstrate that doing only one original step of momentum prevents us from bounding the distance between delayed gradient and the latest one. Instead, by “momentum compensation” we design Accelerated Asynchronous Stochastic Coordinate Descent (AASCD) and Asynchronous Stochastic Gradient Descent (AASVRG). Both of the resultant algorithms are faster than existing ones by order and even with less order of \tau comparing with AAGD. We also show that for sparse datasets, the delay will be largely reduced and linear speed up is achievable for our algorithms under certain conditions. Finally, we conduct lots of experiments on a shared memory system to demonstrate the fast convergence of our algorithms. To summarize, we list the contributions of our work as follows:

1. We propose the “momentum compensation” technique to accelerate asynchronous algorithms for convex problems. To the best of our knowledge, we are the first to consider accelerated algorithms for delayed gradients. Our results are strong (improve the rate), general (includes analysis for proximal version and NC), and easy to combine with other techniques (see 2).

2. We show that our technique can be applied to modern stochastic asynchronous algorithms. The resultant algorithms, i.e. AASCD and AASVRG, are also faster than existing ones by order.

3. We perform lots of numerical experiments on a shared memory system to demonstrate that acceleration can lead to significant performance improvements. We will put our C++ implementation with POSIX threads on website once our paper is accepted.

2 Preliminaries and Notations

In most asynchronous parallelism, there are typically two schemes:

- Atom (consistent read) scheme: The parameter x is updated as an atom. When x is read or updated in the central node, it will be locked. So x^{(k)} \in \{x^{0}, x^{1}, \cdots, x^{k}\}.

Table 2: Notations for different algorithms in this paper

| Algorithm | Objective Function | \( f(x) + h(x) \) | \( f(x) \): L-Lipschitz continuous gradient |
|-----------|--------------------|------------------|---------------------------------------------|
| AGD       | Objective function | \( x^{k}, x^{j(k)} \) | \( x \) in k-th, j(k)-th iteration, respectively |
| AASCD     | Objective function | \( f(x) + h(x) \) | \( f(x) \): Lipschitz coordinate continuous gradient; \( h(x) = \sum_{i=1}^{n} h_i(x_i), x \in \mathbb{R}^n \). |
|           | Subscript | \( x^{s}, x^{j(k)} \) | \( x \) in k-th, j(k)-th iteration, respectively. |
|           | Subscript | \( x_i, \nabla f(x) \) | The i-th coordinate of x and \( \nabla f(x) \) respectively. |
| AASVRG    | Objective function | \( f(x) + h(x) \) | \( f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x), x \) is updated as an atom. |
|           | Subscript | \( x^{s}, x^{j(k)} \) | \( x \) in k-th, j(k)-th iteration, at s-th epoch respectively. |
| x, \nabla | Snapshot vector and VR gradient followed by {Johnson & Zhang [2013]}.
Algorithm 1 AGD (Nesterov, 1983)

Input $\theta^k$, step size $\gamma$, $x^0 = 0$, and $z^0 = 0$.

for $k = 0$ to $K$ do
    1. $y^k = (1 - \theta^k)x^k + \theta^k z^k$.
    2. $\delta^k = \text{argmin}_\delta h(z^k + \delta) + \langle \nabla f(y^k), \delta \rangle + \frac{\theta^k}{2\gamma}\|\delta\|^2$.
    3. $z^{k+1} = z^k + \delta^k$.
    4. $x^{k+1} = \theta^k z^{k+1} + (1 - \theta^k)x^k$.
end for

Output $x^{K+1}$.

3 AAGD

We first illustrate our momentum compensation technique for plain AGD algorithms. The objective function is:

$$\min_x f(x) + h(x),$$

where $f(x)$ has $L$-Lipschitz continuous gradient and both $f(x)$ and $h(x)$ are convex.
We find that \( x \) which is known as extrapolation. Set \( y \) we have

\[
\text{Algorithm 2 AAGD}
\]

Input \( \theta^k \), step size \( \gamma \). \( x^0 = 0 \) and \( z^0 = 0 \).

for \( k = 0 \) to \( K \) do

1. \( w^{j(k)} = x^{j(k)} + \left( \sum_{i=j(k)}^{k} b(j(k), i) \right) (x^{j(k)} - x^{j(k)-1}) \).

2. \( \delta^k = \arg\min_{\delta} h(x^k + \delta) + \langle \nabla f(w^{j(k)}), \delta \rangle + \frac{\delta^k}{2\gamma} \| \delta \|^2 \).

3. \( z^{k+1} = x^k + \delta^k \).

4. \( x^{k+1} = \theta^k z^{k+1} + (1 - \theta^k) x^k \).

end for

Output \( x^{K+1} \).

3.1 Momentum Compensation

Recall the serial Accelerated Gradient Descent [Nesterov, 1983], shown in Algorithm 1. If we directly implement AGD (Nesterov, 1983) asynchronously, we can only get the gradient \( \nabla f(y^{j(k)}) \) at Step 1 due to the delay. Now we need to measure the distance between \( y^{j(k)} \) and \( x^k \). With some algebraic transformation, we have

\[
y^{k+1} = x^k + \frac{\theta^k(1 - \theta^k)}{\theta^k} (x^k - x^{k-1}),
\]

which is known as extrapolation. Set \( \alpha^k = \frac{\theta^k(1 - \theta^k)}{\theta^k} \), and \( b(l, k) = \prod_{i=l}^{k} a^i \), where \( l \leq k \). Then by applying Eq. (38) recursively, for \( k \geq j(k) \geq 0 \), we have,

\[
y^k = y^{k-1} + \sum_{i=j(k)+1}^{k} (1 + b(i, k)(x^i - y^{i-1})) + b(j(k), k)(x^{j(k)} - x^{j(k)-1}).
\]

Summing Eq. (7) with superscript from \( j(k) \) to \( k \), we obtain the relation between \( y^{j(k)} \) and \( y^k \):

\[
y^k = y^{j(k)} + \sum_{i=j(k)+1}^{k} \left( 1 + \sum_{l=i}^{k} b(i, l) \right) (x^i - y^{i-1}) + \left( \sum_{i=j(k)+1}^{k} b(j(k), i) \right) (x^{j(k)} - x^{j(k)-1}).
\]

We find that \( x^{j(k)} - x^{j(k)-1} \) is related to all the past updates before \( j(k) \). If we directly implement AGD asynchronously like most asynchronous algorithms, then \( j(k) < k \) (due to delay), so \( \sum_{i=j(k)+1}^{k} b(j(k), i) > 0 \). Since \( x^{j(k)} - x^{j(k)-1} \) is hard to bound, it causes difficulty to obtain the accelerated convergence rate.

Instead, we compensate the momentum term and introduce a new extrapolation point \( w^{j(k)} \), such that

\[
w^{j(k)} = x^{j(k)} + \left( \sum_{i=j(k)}^{k} b(j(k), i) \right) (x^{j(k)} - x^{j(k)-1}).
\]

One can find these are actually several steps of momentum. Then the difference between \( y^k \) and \( w^{j(k)} \) can be directly bounded by the norm of several latest updates, namely \( \| \sum_{i=j(k)+1}^{k} \left( 1 + \sum_{l=i}^{k} b(i, l) \right) (x^i - y^{i-1}) \|^2 \). So we are able to obtain the accelerated rate. The Algorithm is shown in Algorithm 2.
Algorithm 3 AAGD-implementation

\begin{algorithm}
\begin{algorithmic}
\State **Input** $\theta^0$, step size $\gamma$, $u^0 = 0$, $v^0 = 0$, and $d^0 = 1$.
\For{$k = 0$ to $K$}
\State $d^{k+1} = d^k (1 - \theta^k)$.
\State $w^{j(k)} = u^{j(k)} + d^{k+1} v^{j(k)}$.
\State $\delta^k = \arg\min_{\delta} h(z^k + \delta) + \langle \nabla f(w^{j(k)}), \delta \rangle + \frac{\theta^k}{2} \|\delta\|^2$.
\State $u^{k+1} = u^k + \delta^k$.
\State $v^{k+1} = v^k - \frac{\delta^k}{\gamma}$.
\EndFor
\State **Output** $x^{K+1} = u^{K+1} + d^{K+1} v^{K+1}$.
\end{algorithmic}
\end{algorithm}

3.2 Convergence Results

After introducing $w^{j(k)}$, we separately analysis $f(x^{k+1}) - f(x^*)$ and $\|z^{k+1} - z^*\|^2$ like the Lyapunov technique \cite{Reddi et al. 2015, Cong & Lin 2017}, and bound them through the existing terms in serial AGD \cite{Nesterov 1983} and additional $\|w^{j(k)} - y^k\|^2$. Then we choose a proper step size to obtain a faster convergence rate. We directly give the convergence results of AAGD. All the proofs can be found in Supplementary Material.

**Theorem 1** Under Assumption 1, for Algorithm 2 for non-strongly convex case, if the step size satisfies $2\gamma L + 3\gamma L (\tau^2 + 3\tau)^2 \leq 1$, $\theta^k = \frac{2}{\tau + 2}$, and the first $\tau$ iterations are updated in serial, we have

$$F(x^{K+1}) - F(x^*) \leq (\theta^k)^2 \left( \frac{1}{2\gamma} \|z^0 - x^*\|^2 \right).$$

When $h(x)$ is strongly convex with modulus $\mu \leq L$, the step size satisfies $\frac{2}{\tau + 2} L + \gamma L (\tau^2 + 3\tau)^2 \leq 1$, and $\theta^k = \frac{-\gamma + \sqrt{\gamma^2 + 4\gamma \mu}}{2}$ is denoted as $\theta$ instead, we have

$$F(x^{K+1}) - F(x^*) \leq (1 - \theta)^{K+1} \left( F(x^0) - F(x^*) \right) + (1 - \theta)^K \left( \frac{\theta^2}{2\gamma} + \frac{\mu \theta}{2} \right) \|z^0 - x^*\|^2.$$

**Corollary 1** For Algorithm 2 under the assumption of Theorem 7, the Iteration First-Order (IFO) calls are $O(\tau^2 \sqrt{L/\epsilon})$ for NC and $O(\tau^2 \sqrt{L/\mu \log(1/\epsilon)})$ for SC.

The order of $\tau$ is large for AAGD, we will show that for stochastic asynchronous algorithms, the order of $\tau$ will be largely reduced.

3.3 AAGD in Implementation

In Eq. (9), we need to compute $\sum_{i=j(k)}^k b(j(k), i)$, which is a little complicated. To make our algorithm clearer, inspired by \cite{Fercoq & Richtárik 2015, Lin et al. 2014}, we can change variable as follows: $z^k = u^k$, $x^k = u^k + d^k v^k$, and $y^k = u^k + d^k + 1 v^k$. The algorithm is shown in Algorithm 3. The equivalent of Algorithm 2 and 3 is shown Supplementary Material. Another advantage for Algorithm 3 is the ability to sparse update for the sparse dataset.

4 Practical Asynchronous Algorithms

\begin{footnotesize}\textsuperscript{2}We use this assumption only for simplicity. This assumption is removed in the analysis of AASCD and AASVRG.\end{footnotesize}
We have the following theorem:

\[ f \]

Algorithm 4 AASCD

Input \( \theta^k \), step size \( \gamma \), \( x^0 = 0 \) and \( z^0 = 0 \).

Define \( a^k = \theta^k(1-\theta^k), \ b(l,k) = \prod_{i=1}^{k} a^i \).

for \( k = 0 \) to \( K \) do
1. \( w^{(k)} = y^{(k)} + \sum_{i=(k)+1}^{k} b(i,k)(y^{(k)} - x^{i,(k)-1}) \).
2. Randomly choose an index \( i_k \) form \([1, 2, \ldots, n]\).
3. \( \delta^k = \arg\min_{\delta} h_{i_k}(z^k + \delta) + \langle \nabla_{i_k} f(w^{(k)}), \delta \rangle + \frac{\theta^k}{2\gamma} \| \delta \|^2 \).
4. \( z^k_{i_k} = z^k_{i_k} + \delta^k \) with other coordinates unchanged.
5. \( y^k = (1 - \theta^k)x^k + \theta^k z^k \).
6. \( x^{k+1} = (1 - \theta^k)x^k + \theta^k z^{k+1} - (n - 1)\theta^k z^k \).
end for
Output \( x^{K+1} \).

To meet the large-scale of machine learning, most asynchronous algorithms are designed in a stochastic fashion. We are now to demonstrate that our technique can further be applied to accelerate modern state-of-the-art stochastic asynchronous algorithms, such as ASCD (Liu et al., 2015b) and ASVRG (Reddi et al., 2015) [Meng et al., 2016a]. The proofs of our AASCD and AASVRG are similar to that of AAGD, but are much involved. It needs to further fuse other techniques, such as Estimate Sequence technique in (Fercoq & Richtárik, 2015) for AASCD and the negative momentum technique (Allen-Zhu, 2017) for AASVRG. Like AAGD, the two algorithms also be changed variables to be clearer and able to sparse update. We directly demonstrate the algorithms and the convergence results. All the proofs can also be found in Supplementary Material.

4.1 AASCD

(Asynchronous) Stochastic Coordinate Descent algorithms mainly solves the following problem:

\[
\min_{x \in \mathbb{R}^n} f(x) + h(x),
\]

where \( f(x) \) has \( L_c \)-Lipschitz coordinate continuous gradient, \( h(x) \) has coordinate separable structure, i.e.

\[
h(x) = \sum_{i=1}^{n} h_i(x_i),
\]

\( f(x) \) and \( h(x) \) are convex. At each iteration, the algorithms choose one coordinate \( x_i \), to sufficiently reduce the objective value while keeping other coordinates fixed which reduces the per-iteration cost. In more detail, in each iteration the following types of proximal subproblem is solved:

\[
\delta^k = \arg\min_{\delta} h_{i_k}(x^k + \delta) + \langle \nabla_{i_k} f(x^k), \delta \rangle + \frac{\theta^k}{2\gamma} \| \delta \|^2,
\]

where \( \nabla_{i_k} f(x^k) \) denotes the partial gradient of \( f \) with respect to \( x_i \).

For asynchronous algorithms, the partial gradient will be delayed, and at iteration \( k \) we could only obtain \( \nabla_{i_k} f(x^{(k)}) \) instead of \( \nabla_{i_k} f(x^k) \).

Now we propose our accelerated algorithm. For simplicity, we assume that each coordinate Lipschitz constant \( L_i \) are the same, then \( L_c = L_i, i = 1,2,\cdots,i \). By judging the distance between the delayed extrapolation points and the newest noes and compensating the “lost” momentum term, we obtain Algorithm 4.

We have the following theorem:

\footnote{This is the case that the data are normalized. When \( L_i \) are different, \( n^2 L_i \) can be extended to \( (n \sum_{i=1}^{n} L_i)^2 \), also we can fuse the non-uniform sampling (Allen-Zhu et al., 2016) technique and replace it with smaller \((\sum_{i=1}^{n} L_i)^2 \) in convergence rate.}
When we consider the following composite finite-sum convex optimization problem:

\[ \min_{x} \left\{ f(x) + h(x) : \|x\| \leq \frac{1}{\varepsilon} \right\} \]

We can find that the order of \( \tau \) is strongly convex with modulus \( \gamma L \).

Algorithm 5 AASVRG

Input \( \theta_1 \), step size \( \gamma \), \( x_0 = 0 \), \( \tilde{x} = 0 \), and \( z_0^0 = 0 \), \( \theta_2 = \frac{1}{2} \), \( m = n \), and \( a^* = 1 - \theta_2 - \theta_1 \).

for \( s = 0 \) to \( S \) do

\( \text{for } k = 0 \) to \( m - 1 \) do

1. \( w_j^{(k)} = x_j^{(k)} + a^* \frac{1}{\tau} ((x_j^{(k)}) - x_j^{(k-1)}) \)
2. \( \nabla_k^s = \nabla f_i^k(w_j^{(k)}) - \nabla f_i^k(\tilde{x}) \)
3. \( \delta_k^s = \arg \min_{\delta} \mathcal{H}(x_k^s + \delta) + \langle \nabla_k^s, \delta \rangle + \frac{\gamma}{2}\|\delta\|^2 \)
4. \( x_k^{s+1} = x_k^s + \theta_2 \tilde{x} + a^*x_k^s \)

\( \text{end for } k \)

\( x_0^{s+1} = x_0^s, x_0^{s+1} = z_m^s \)

For NC: \( \tilde{x}^{s+1} = \frac{1}{m} \sum_{k=0}^{m-1} x_k^s \)

For SC: \( \tilde{x}^{s+1} = \left( \sum_{k=0}^{m-1} (1 + \theta_1^s)^i \right)^{-1} \sum_{k=0}^{m-1} (1 + \theta_1^s)x_k^s \)

end for \( s \).

Output \( x_0^{S+1} \).

Theorem 2 Under Assumption 1, and \( \tau \leq \sqrt{n} \), for Algorithm 4 if the step size satisfies \( 2\gamma L_c + (1 + \frac{1}{n})\gamma L_c (\frac{2C}{n} 2\tau)^2 \leq 1 \), and \( \theta^k = \frac{2}{2n+k} \), we have

\[ \frac{E[F(X^{K+1})] - F(x^*)}{(\theta K)^2} + \frac{n^2}{2\gamma} \| z^{K+1} - x^* \|^2 \]

\[ \leq \frac{F(x^0) - F(x^*)}{(\theta K)^2} + \frac{n^2}{2\gamma} \| z^0 - x^* \|^2. \]  \hspace{1cm} (14)

When \( h(x) \) is strongly convex with modulus \( \mu \leq L_c \), the step size satisfies \( 2\gamma L_c + (\frac{3}{4} + \frac{3}{8h})\gamma L_c (\tau^2 + \tau) / n + 2\tau)^2 \leq 1 \), and \( \theta^k = \frac{\mu}{2n} \) is denoted as \( \theta \) instead, we have

\[ E[F(X^{K+1})] - F(x^*) \leq (1 - \theta)^K + 1 \left( F(x^0) - F(x^*) \right) \]

\[ + (1 - \theta)^K \frac{n^2(\theta^2) + n\theta^2\mu}{2\gamma} \| z^0 - x^* \|^2. \] \hspace{1cm} (15)

Corollary 2 For Algorithm 4 under the assumption of Theorem 2, the IFO calls are \( O \left( (n\sqrt{L} + n\tau\sqrt{L_c})\sqrt{1/\varepsilon} \right) \) for NC and \( O \left( n\tau \sqrt{\frac{L}{n}} \log(1/\varepsilon) \right) \) for SC.

We can find that the order of \( \tau \) are reduced comparing with AAGD due to the stochastic effect.

4.2 AASVRG

We consider the following composite finite-sum convex optimization problem:

\[ F(x) = h(x) + \frac{1}{n} \sum_{i=1}^n f_i(x). \] \hspace{1cm} (16)

where \( f_i(x) \)'s, \( i = 1, 2, \cdots, n \), are convex and have Lipschitz continuous gradients, and \( h(x) \) is also convex. We denote \( f(x) = \frac{1}{n} \sum_{i=1}^n f_i(x) \). To solve Eq. (16), stochastic methods computes a gradient estimator from
When we focus on solving Empirical Risk Minimization problems: We can find that the order of $\lambda > 0$ where $\lambda$ is lower than AAGD. The algorithm is shown in Algorithm 5. We have the following theorem:

**Theorem 3** Under the Assumption 1, for Algorithm 4 if the step size satisfies $5\gamma L + 10\gamma L\tau^2 \leq 1$, and $\theta_1^* = \frac{2}{s + 1}$, we have

\[
\mathbb{E}\left(F(x^S) - F(x^*)\right) + (\theta_2 + \theta_1^*) \sum_{k=1}^{n-1} \mathbb{E}\left(F(x^E_k) - F(x^*)\right) \\
\leq 2n(\theta_1^*)^2(F(x^0) - F(x^*)) + \frac{(\theta_1^*)^2}{2\gamma} ||z_0^0 - x^*||^2.
\]

(17)

When $h(x)$ is strongly convex with modulus $\mu \leq \frac{L\tau^2}{\alpha n}$, the step size satisfies $5\gamma L + \frac{\alpha\gamma}{8}\tau^2 \gamma L \leq 1$, $\theta_1^* = \frac{1}{\tau} \sqrt{\frac{\mu}{\gamma L}}$, and $\theta_3 = 1 + \frac{\alpha\gamma}{\theta_1^*}$, we have

\[
(F(x^{S+1}) - F(x^*)) \leq (\theta_3)^{-S_n}\left(\frac{7}{4n} ||z_0^0 - x^*||^2\right) \\
+ (\theta_3)^{-S_n}\left(1 + \frac{1}{n} \left(F(x_0^0) - F(x^*)\right)\right)
\]

(18)

**Corollary 3** For Algorithm 4 under the assumption of Theorem 3 the IFO calls are $O\left(n + (n + \tau\sqrt{n})\sqrt{\frac{L}{\epsilon}}\right)$ for NC and $O\left(n + \tau\sqrt{\frac{nL}{\mu}} \log(1/\epsilon)\right)$ for SC.

We can find that the order of $\tau$ is also lower than AAGD.

### 5 Applications

We focus on solving Empirical Risk Minimization problems:

\[
\min_{x \in \mathbb{R}^n} P(x) = \frac{1}{n} \sum_{i=1}^{n} \phi_i(A_i^T x) + \lambda g(x),
\]

(19)

where $\lambda > 0$, $g(x)$ is typical a regular terms, and $\sum_{i=1}^{n} \phi_i(A_i^T x)$ are loss functions over training samples. Lots of machine learning problem can be formulated into Eq (19), such as linear SVM, Ridge Regression, and Logistic Regression. For ASVRG, solving Eq (19) is equivalent to Eq (16). For AASCD, we consider solve Eq (19) in dual. When $g(x) = ||x||^2$. The dual formulation of Eq (19) is:

\[
\min_{a \in \mathbb{R}^n} D(a) = \frac{1}{n} \sum_{i=1}^{n} \phi_i^*(-a_i) + \frac{\lambda}{2} \frac{1}{\lambda n} \|Aa\|^2.
\]

(20)

Through the technique of [Lin et al., 2014], for SC, we can obtain the convergence rate for AASCD on primal:

**Theorem 4** Assume that each function $\phi_i$ is $L_2$-smooth, $g(\cdot)$ has a unit convexity modulus 1, and $\|A_i\| \leq R$ for all $i = 1, \cdots, n$. Then the IFO calls to reach both the dual optimality gap $(\mathbb{E}[D^* - D(a^*)] \leq \epsilon)$ and the primal one $(\mathbb{E}[P(x) - P^*] \leq \epsilon)$ through Algorithm 4 are $O\left(n + \tau\sqrt{\frac{nR^2 L \log(1/\epsilon)}{\lambda}}\right)$. 

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Table 3: Details of sparse datasets.

| Datasets   | #samples | #features | #nonzeros |
|------------|----------|-----------|-----------|
| real-sim   | 72,309   | 20,958    | 3,709,083 |
| news20     | 19,996   | 1,355,191 | 9,097,916 |
| rcv1       | 20,242   | 47,236    | 49,556,258|
| url        | 2,396,130| 3,231,961 | 277,058,644|

5.1 Sparse Dataset

One crucial application for asynchronous algorithms in shared memory systems is to solve sparse data. The main reasons are two folded: (1) as the non-zeros coordinates for samples are varying, the computation cost is different for each thread. In this case, asynchronous algorithms are more practical than synchronous ones because threads do not need to wait for synchronization; (2) the data matrix are sparse and “disjoint”, so the delay effect will be largely reduced. We formulate this fact in the following proposition:

**Proposition 1** For a given dataset, if each example is generated i.i.d and has non-zero component $i$ with probability $\beta_i$, then $\mathbb{E}(m_i(k, j(k))) = \beta_i(k - j(k))$, where $m_i(k, j)$ is the total number of nonzero updates in component $i$ from iteration $k_j$ to iteration $k_i$.

Since $\beta \ll 1$, the delay effect is largely reduced. Reddi et al. (2015) proposes the $\triangle$-assumption to judge the sparsity (see Supplementary Material). Under this assumption, our algorithms are able to achieve linear speedup. For example, for AASVRG we have the following property:

**Proposition 2** Under the $\triangle$-assumption ($\triangle \ll 1$) proposed by Reddi et al. (2015), for Algorithm 5, the IFO calls is $O\left(n + (n + (1 + \triangle \tau)\sqrt{n}) \sqrt{L/\epsilon}\right)$ for NC and $O\left(n + (1 + \triangle \tau)\sqrt{nL/\mu \log(1/\epsilon)}\right)$ for SC, respectively. Thus linear speedup is achievable.

6 Experiments

We have conducted extensive experiments to demonstrate the effectiveness of our method. We study the problem of Linear SVM for AASCD and Ridge Regression for AASVRG. We have performed experiments on lots of datasets. For sparse datasets, we choose four benchmark sparse datasets rcv1, real-sim, news20, and url. The details of the datasets are shown in Table 4. Similar to Reddi et al. (2015), we have a careful implementation for sparse gradient and computation. We mainly focus on ill-condition problems, so we set the regularizer weight to be $1/(100n)$ in all experiments, and we tune the step size to give the best convergence results. All experiments are done on an Intel multi-core 4-socket machine with each one contains 8 cores.

These datasets can be downloaded from [https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/](https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/)
Figure 2: Residuals vs CPU training time (s) for solving the Linear SVM problem on four test datasets.

Figure 3: Residuals vs Iterations and CPU training time (s) for solving the Ridge Regression problem on four test datasets.

6.1 Experiments for AASCD

We compare AASCD with the following methods: 1) Pegasos (Shalev-Shwartz et al., 2011), which can be considered as one of the best single thread implementation to solve Linear SVM; 2) ASCD (Liu et al., 2015a); we also compare some accelerated algorithms, though they have no theoretic guarantees. We compare with RMPE (Scieur et al., 2016), which a regularized nonlinear acceleration algorithm, we also implement AASGD (Meng et al., 2016b). However, we find it hard to converge on the sparse data.

6.2 Experiments for AASVRG

We compare AASVRG with the following methods: 1) HOGWILD, a lock-free asynchronous variant of SGD and D-HOGWILD, which chooses decaying step size as $\eta_0 \sqrt{\sigma_0/(t + \sigma_0)}$; 2) ASVRG, the lock-free asynchronous variant of SVRG; and 3) RMPE (Scieur et al., 2016), a regularized nonlinear acceleration algorithm.

6.3 Experiments for Dense Dataset

We also do experiments on Dense Dataset for AASVRG which performs on USPS, SENSIT, MNIST, and EPSILON with the details shown in Supplementary Material.
6.4 Results

We first measure the speedup achieved by our algorithms on the sparse dataset. The time speedup is defined as the ratio of the runtime to achieve a given precision with the serial implementation to the runtime with the asynchronous implementation with $P$ threads, and the iteration speedup is defined as

$$\text{iteration speedup} = \frac{\text{# of iters for seri. algorithms}}{\text{# of iters for asyn. algorithms}} \times P.$$ 

The results are shown in Fig. 1. There is a linear speedup for iteration, and a nearly linear speedup for time, which verifies our theorem on the sparse dataset. Asynchronous algorithm achieves higher speedup than synchronous one.

To compare these algorithms, we consider the training loss residual versus CPU time. For AASCD, the results is shown in Fig. 2. The experiments are conducted on 10 cores. It is clear that our algorithm converges fastest in all four datasets among other the algorithms.

The experiment results for AASVRG is shown in Fig. 4. To demonstrate that our algorithm has a faster speed, we also reports the training loss residual versus iteration. It is also clear that our algorithm are much faster.

Due to space limit, implementation details and more experimental results, e.g. variant regularizer weight terms are shown in Supplementary Material.

7 Supplementary Materials

The Supplementary Material is structured as follows: in Section 7.1, we give the proof for AAGD; in Section 7.2, we give the proof for AASCD; in Section 7.3, we give the proof for AASVRG; Also an outline of the proof is at the beginning of each Section. In Section 7.4, we give the proof for ASVRG. In Section 7.5, we show some implementation details and more experimental results.

7.1 AAGD

We set

$$y^k = (1 - \theta^k)x^k + \theta^k z^k. \quad (21)$$

The through the step 4 in Algorithm 2 in the paper, we have

$$x^{k+1} = y^k + \theta^k \delta^k. \quad (22)$$

Outline of the Proof:

Step 1: Through the update rule, we have that

$$y^k - w^{j(k)} = \sum_{i=j(k)+1}^{k} \left(1 + \sum_{l=i}^{k} b(i, l)\right)(x^i - y^{i-1}). \quad (23)$$

Step 2: By analyzing the function value, we have

$$f(x^{k+1}) \leq f(y^k) - \gamma (1 - \gamma L/2) \left\|x^{k+1} - y^k\right\|^2 \gamma - \langle \xi^k, x^{k+1} - y^k \rangle + \gamma L^2/2C 1 \left\|w^{j(k)} - y^k\right\|^2 + \gamma C 1 \left\|x^{k+1} - y^k\right\|^2/\gamma. \quad (24)$$
Step 3: By analyzing the \( \|z^{k+1} - x^*\|^2 \), we have

\[
\frac{1}{2\gamma} \|\theta^k z^{k+1} - \theta^k x^*\|^2 = \frac{1}{2\gamma} \|\theta^k z - \theta^k x^*\|^2 + \frac{1}{2\gamma} \|x^{k+1} - y^k\|^2 - \langle z^k, \theta^k z - \theta^k x^* \rangle + (1 - \theta^k) f(x^k) + \theta^k f(x^*) - f(y^k) + (\nabla f(y^k) - \nabla f(w^{j(k)})) (y^k - w^{j(k)}).
\]

Step 4: By adding Eq. (24) and Eq. (25), we have

\[
F(x^{k+1}) \leq (1 - \theta^k) F(x^k) + \theta^k F(x^*) - \gamma \left( \frac{1}{2} \gamma L \right) \|x^{k+1} - y^k\|^2 + \gamma \left( \frac{\gamma^2 L^2}{2C_1} + \gamma L \right) \|x^{k+1} - y^k\|^2 + \frac{1}{2\gamma} \|\theta^k z^k - \theta^k x^*\|^2 - \left( \frac{1}{2\gamma} + \frac{\mu}{2\theta^k} \right) \|\theta^k z^{k+1} - \theta^k x^*\|^2.
\]

Step 5: We choose proper step size and obtain Theorem 1 in the paper.

**Proof of step 1:**

Through Eq. (21), we have

\[
\theta^k z = y^k - (1 - \theta^k)x^k, \quad k \geq 0.
\]

and through the Step 4 in Algorithm 2 in the paper,

\[
\theta^k z^{k+1} = x^{k+1} - (1 - \theta^k)x^k, \quad k \geq 0.
\]

Eliminating \( z^k \), we have

\[
\frac{y^k - (1 - \theta^k)x^k}{\theta^k} = \frac{x^k - (1 - \theta^k)x^{k-1}}{\theta^{k-1}}, \quad k \geq 1.
\]

Thus

\[
y^k = x^k + \frac{\theta^k (1 - \theta^k)}{\theta^{k-1}} (x^k - x^{k-1}), \quad k \geq 1.
\]

Set \( a^k = \frac{\theta^k (1 - \theta^k)}{\theta^{k-1}} \), we have \( a^k \leq 1 \). We have

\[
y^k = x^k + a^k (x^k - y^{k-1}) + a^k (y^{k-1} - x^{k-1}) = y^{k-1} + (a^k + 1)(x^k - y^{k-1}) + a^k a^{k-1} (x^{k-1} - x^{k-2}), \quad k \geq 2.
\]

For \( x^{k-1} - x^{k-2} \), and \( k \geq j(k) + 2 \geq 2 \), we have

\[
x^{k-1} - x^{k-2} = x^{k-1} - y^{k-2} + y^{k-2} - x^{k-2} = x^{k-1} - y^{k-2} + a^{k-2} (x^{k-2} - x^{k-3}) = x^{k-1} - y^{k-2} + a^{k-2} (x^{k-2} - y^{k-3}) + a^{k-2} a^{k-3} (x^{k-3} - x^{k-4}) = x^{k-1} - y^{k-2} + \sum_{i=k(j)+1}^{k-2} \left( \prod_{l=i}^{k-2} a^l \right) (x^i - y^{i-1}) + \left( \prod_{l=j(k)}^{k-2} a^l \right) (x^{j(k)} - x^{j(k)-1}).
\]
Set \( b(l, k) = \prod_{i=l}^{k} a^i \), where \( l \leq k \). Substituting Eq. (32) into Eq. (31), we have

\[
y^k = y^{k-1} + (b(k, k) + 1)(x^k - y^{k-1}) + b(k-1, k)(x^{k-1} - y^{k-2})
+ \sum_{i=j(k)+1}^{k-2} (b(i, k)(x^i - y^{i-1}) + b(j(k), k)(x^{j(k)} - x^{j(k)-1})
\]

\[
= y^{k-1} + (x^k - y^{k-1}) + \sum_{i=j(k)+1}^{k} (b(i, k)(x^i - y^{i-1}) + b(j(k), k)(x^{j(k)} - x^{j(k)-1}).
\]

By checking, when \( k = j(k) \) and \( k = j(k) + 1 \), Eq. (33) is right. So Eq. (33) holds for any \( k \geq j(k) \geq 0 \). Summing Eq. (33) with \( k = j(k) + 1 \) to \( k \), we have

\[
y^k = y^{j(k)} + \sum_{i=j(k)+1}^{k} (x^i - y^{i-1}) + \sum_{l=j(k)+1}^{k} \sum_{i=j(k)+1}^{l} b(i, l)(x^i - y^{i-1})
+ \left( \sum_{i=j(k)+1}^{k} b(j(k), i) \right) (x^{j(k)} - x^{j(k)-1})
\]

\[
= y^{j(k)} + \sum_{i=j(k)+1}^{k} (x^i - y^{i-1}) + \sum_{l=i}^{k} \left( \sum_{i=j(k)+1}^{k} b(i, l) \right) (x^i - y^{i-1})
+ \left( \sum_{i=j(k)+1}^{k} b(j(k), i) \right) (x^{j(k)} - x^{j(k)-1})
\]

Eq. (\ref{eq34})

\[
x^{j(k)} + \sum_{i=j(k)+1}^{k} (x^i - y^{i-1}) + \sum_{l=i}^{k} \left( \sum_{i=j(k)+1}^{k} b(i, l) \right) (x^i - y^{i-1})
+ \left( \sum_{i=j(k)}^{k} b(j(k), i) \right) (x^{j(k)} - x^{j(k)-1}),
\]

where \( \overset{a}{=} \) is obtained by rearrange terms. Then by comparing the results, we obtain Step 1.

**Proof of step 2:**

Through the optimal solution of \( z^{k+1} \) in Step 2 of Algorithm 2 in the paper, we have that

\[
\theta^k(z^{k+1} - z^k) + \gamma \nabla f(w^{j(k)}) + \gamma \xi^k = 0,
\]

where \( \xi^k \in \partial h(z^{k+1}) \). And through Eq. (22), we have

\[
(x^{k+1} - y^k) + \gamma \nabla f(w^{j(k)}) + \gamma \xi^k = 0.
\]
For $f$ has Lipschitz continues gradient, we obtain

$$
f(x^{k+1}) \leq f(y^k) + \langle \nabla f(y^k), x^{k+1} - y^k \rangle + \frac{L}{2} \|x^{k+1} - y^k\|^2
$$

Substituting Eq. (38) into Eq. (37), we obtain the results of Step 2.

Proof of step 3:

Substituting Eq. (38) into Eq. (37), we obtain the results of Step 2. 

Proof of step 3:

For the last term of Eq. (37), applying Cauchy-Schwarz inequality, we have

$$
\langle \nabla f(w^{(j)}), -\nabla f(y^k) \rangle \leq \frac{\gamma L^2}{2C_1} \|w^{(j)} - y^k\|^2 + \frac{\gamma C_1}{2} \left\| \frac{x^{k+1} - y^k}{\gamma} \right\|^2.
$$

where in $\overset{a}{\sim}$, we use Eq. (36); in $\overset{b}{\sim}$, we use $-\nabla f(y^k) = -\nabla f(w^{(j)}) - \xi^k + \xi^k - \nabla f(y^k) + \nabla f(w^{(j)})$; in $\overset{c}{\sim}$, we reuse Eq. (36).

For the last term of Eq. (37), applying Cauchy-Schwarz inequality, we have

$$
\langle \nabla f(w^{(j)}) - \nabla f(y^k), x^{k+1} - y^k \rangle
\leq \frac{\gamma}{2C_1} \left\| \nabla f(w^{(j)}) - \nabla f(y^k) \right\|^2 + \frac{\gamma}{2} \left\| \frac{x^{k+1} - y^k}{\gamma} \right\|^2
\leq \frac{\gamma L^2}{2C_1} \|w^{(j)} - y^k\|^2 + \frac{\gamma C_1}{2} \left\| \frac{x^{k+1} - y^k}{\gamma} \right\|^2.
$$

(38)

Substituting Eq. (38) into Eq. (37), we obtain the results of Step 2.

Proof of step 3:

$$
\frac{1}{2\gamma} \| \theta^k z^{k+1} - \theta^k x^* \|^2
$$

(39)

where in $\overset{a}{\sim}$, we use Eq. (35). Then for the last term, we have that

$$
-\langle \nabla f(w^{(j)}), \theta^k z^k - \theta^k x^* \rangle
$$

(40)

where in $\overset{a}{\sim}$, we insert $w^{(j)}$; in $\overset{b}{\sim}$, we use the convexity of $f$, namely applying

$$
f(w^{(j)}) + \langle \nabla f(w^{(j)}), a - w^{(j)} \rangle \leq f(a),
$$

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on $a = x^*$, and $a = x^k$, respectively; in $\le$, we use that

$$-f(w^{(k)}) \le -f(y^k) + \langle \nabla f(y^k), y^k - w^{(k)} \rangle.$$  \hspace{1cm} (41)

Substituting Eq. (40) into Eq. (39), after simplifying, we obtain the result of Step 3.

**Proof of step 4:** Adding Eq. (24) and Eq. (25), we have that

$$f(x^{k+1}) \le (1 - \theta^k)f(x^k) + \theta^k f(x^*) - \gamma(\frac{1}{2} - \frac{\gamma L}{2}) \|x^{k+1} - y^k\|^2 - \langle \xi^k, x^{k+1} - y^k \rangle$$

$$+ \frac{\gamma L^2}{2C_1} \|w^{(k)} - y^k\|^2 + \frac{\gamma C_1}{2} \|x^{k+1} - y^k\|^2 - \langle \xi^k, \theta^k z^k - \theta^k x^* \rangle$$

$$+ \langle \nabla f(y^k) - \nabla f(w^{(k)}), y^k - w^{(k)} \rangle + \frac{1}{2\gamma} \|\theta^k z^k - \theta^k x^*\|^2 - \frac{1}{2\gamma} \|\theta^k z^{k+1} - \theta^k x^*\|^2$$

$$\le (1 - \theta^k)f(x^k) + \theta^k f(x^*) - \gamma(\frac{1}{2} - \frac{\gamma L}{2}) \|x^{k+1} - y^k\|^2 - \langle \xi^k, x^{k+1} - y^k \rangle$$

$$+ \left(\frac{\gamma L^2}{2C_1} + L\right) \|w^{(k)} - y^k\|^2 + \frac{\gamma C_1}{2} \|x^{k+1} - y^k\|^2 - \langle \xi^k, \theta^k z^k - \theta^k x^* \rangle$$

$$+ \frac{1}{2\gamma} \|\theta^k z^k - \theta^k x^*\|^2 - \frac{1}{2\gamma} \|\theta^k z^{k+1} - \theta^k x^*\|^2,$$ \hspace{1cm} (42)

where in $\le$, we use $\langle \nabla f(y^k) - \nabla f(w^{(k)}), y^k - w^{(k)} \rangle \le \|y^k - w^{(k)}\|^2$. Since $\xi \in \partial h(z^{k+1})$, we have that

$$-\langle \xi^k, x^{k+1} - y^k \rangle - \langle \xi^k, \theta^k z^k - \theta^k x^* \rangle \le \theta^k \langle \xi^k, x^* - z^{k+1} \rangle$$

$$\le \theta^k h(x^*) - \theta^k h(z^{k+1}) - \frac{\mu\theta^k}{2} \|z^{k+1} - x^*\|^2.$$ \hspace{1cm} (43)

For the convexity of $h(z^{k+1})$, and the step 4 in Algorithm 2 in the paper, we have

$$\theta^k h(z^{k+1}) + (1 - \theta^k) h(x^k) \ge h(x^{k+1}).$$ \hspace{1cm} (44)

Substituting Eq. (43) into Eq. (42), and using Eq. (44), we have

$$F(x^{k+1}) \le (1 - \theta^k)F(x^k) + \theta^k F(x^*) - \gamma(\frac{1}{2} - \frac{\gamma L}{2}) \|x^{k+1} - y^k\|^2$$

$$+ \left(\frac{\gamma L^2}{2C_1} + L\right) \|w^{(k)} - y^k\|^2 + \frac{\gamma C_1}{2} \|x^{k+1} - y^k\|^2$$

$$+ \frac{1}{2\gamma} \|\theta^k z^k - \theta^k x^*\|^2 - \left(\frac{1}{2\gamma} + \frac{\mu}{2\theta^k}\right) \|\theta^k z^{k+1} - \theta^k x^*\|^2.$$ \hspace{1cm} (45)

**Proof of step 5:**
We first consider the not-strongly convex case. Through Eq. (23), we have

\[
\| w^{(k)} - y^k \|^2
\]

\[
= \left\| \sum_{i=j(k)+1}^{k} \left( 1 + \sum_{l=1}^{k} b(i, l) \right) (x^i - y^{i-1}) \right\|^2
\]

\[\leq a \left( \sum_{i=j(k)+1}^{k} \left( 1 + \sum_{l=1}^{k} b(i, l) \right) \right) \left( 1 + \sum_{l=1}^{k} b(i, l) \right) \| x^i - y^{i-1} \|^2
\]

\[\leq b \left( \sum_{i=j(k)+1}^{k} \left( 1 + \sum_{l=1}^{k} b(i, l) \right) \right) \left( 1 + \sum_{l=1}^{k} b(i, l) \right) \| x^i - y^{i-1} \|^2
\]

\[\leq c \left( \sum_{i=1}^{k-j(k)} \left( 1 + \sum_{l=1}^{i} b(i, l) \right) \right) \left( 1 + \sum_{l=1}^{i} b(i, l) \right) \| x^{k-i+1} - y^{k-i} \|^2
\]

\[\leq d \left( \sum_{i=1}^{\min(\tau, k)} \left( 1 + \sum_{l=1}^{i} b(i, l) \right) \right) \left( 1 + \sum_{l=1}^{i} b(i, l) \right) \| x^{k-i+1} - y^{k-i} \|^2
\]

\[\leq \frac{\tau^2 + 3\tau}{2} \min(\tau, k) \| x^{k-i+1} - y^{k-i} \|^2,
\]

where in \( a \), we use the fact that for \( c_i \geq 0, \ 0 \leq i \leq n, \)

\[\| c_1 a_1 + c_2 a_2 + \cdots + c_n a_n \|^2 \leq (c_1 + c_2 + \cdots + c_n)(c_1 \| a_1 \|^2 + c_2 \| a_2 \|^2 + \cdots + c_n \| a_n \|^2), \quad (46)
\]

since the function \( f(x) = \| x \|^2 \) is convex, and so

\[
\left\| \frac{c_1}{\sum_{i=0}^{n} c_i} a_1 + \frac{c_2}{\sum_{i=0}^{n} c_i} a_2 + \cdots + \frac{c_n}{\sum_{i=0}^{n} c_i} a_n \right\|^2
\]

\[\leq \frac{c_1}{\sum_{i=0}^{n} c_i} \| a_1 \|^2 + \frac{c_2}{\sum_{i=0}^{n} c_i} \| a_2 \|^2 + \cdots + \frac{c_n}{\sum_{i=0}^{n} c_i} \| a_n \|^2 ;
\]

in \( b \), we use \( b(i, l) \leq 1 \); in \( c \), we change variable \( ii = k - i + 1 \); and in \( d \), we use \( k - j(k) \leq \tau \).

As we are more interested the limited case, namely \( k \) is large. We suppose at the first \( \tau \) step, we run our
algorithm in serial. Diving \((\theta^k)^2\) on Eq. (45) and summing the results with \(k = 0\) to \(K\), we have

\[\sum_{k=0}^{K} \frac{1}{(\theta^k)^2} \left\| w^{(k)} - y^k \right\|^2 = \sum_{k=\tau}^{K} \frac{1}{(\theta^k)^2} \left\| w^{(k)} - y^k \right\|^2 \]

\[\leq \frac{\tau^2 + 3\tau}{2} \sum_{k=\tau}^{K} \sum_{i=1}^{\min(k - \tau, \tau - i)} \frac{1}{(\theta^k)^2} \left\| x^{k-i} - y^{k-i} \right\|^2 \]

\[\leq \frac{\tau^2 + 3\tau}{2} \sum_{k=\tau}^{K} \sum_{i=1}^{\min(k - \tau, \tau - i)} \frac{4(i + 1)}{(\theta^{k-i})^2} \left\| x^{k-i} - y^{k-i} \right\|^2 \]

\[\leq (\tau^2 + 3\tau)^2 \sum_{k=\tau}^{K} \frac{1}{(\theta^k)^2} \left\| x^{k+1} - y^k \right\|^2 , \]

where in \(\leq\), we use that \((k + j)^2 \leq 4k^2\), since \(k \geq \tau \geq j\), so \(\frac{1}{(\theta^k)^2} \leq \frac{4}{(\theta^{k+j})^2}\) with \(k \geq \tau\) and \(i \leq \min(\tau, \tau - i)\); \(\leq\) is because that for each \(\frac{1}{(\theta^k)^2} \left\| x^k - y^{k-1} \right\|^2 \) (1 \(\leq k \leq K\)) there are most \(\tau\) terms with coefficient from 8 to 4(\(\tau + 1\)).

Diving \((\theta^k)^2\) on both sides of Eq. (26), and use \(\mu = 0\), we have

\[\frac{F(x^{k+1}) - F(x^*)}{(\theta^k)^2} \leq \frac{1 - \theta^k}{(\theta^k)^2} \left( F(x^k) - F(x^*) \right) - \frac{\gamma}{(\theta^k)^2} \left( \frac{1}{2} - \frac{\gamma L}{2} \right) \left\| x^{k+1} - y^k \right\|^2 \]

\[+ \frac{\gamma}{(\theta^k)^2} \left( \frac{\gamma^2 L^2}{2C_1} + \gamma L \right) \left\| w^{(k)} - y^k \right\|^2 + \frac{\gamma C_1}{2(\theta^k)^2} \left\| x^{k+1} - y^k \right\|^2 \]

\[+ \frac{1}{2\gamma} \left\| z^k - x^* \right\|^2 - \frac{1}{2\gamma} \left\| z^{k+1} - x^* \right\|^2 \]

\[\leq \frac{F(x^k) - F(x^*)}{(\theta^{k-1})^2} - \gamma \frac{1}{(\theta^k)^2} \left( \frac{1}{2} - \frac{\gamma L}{2} \right) \left\| x^{k+1} - y^k \right\|^2 \]

\[+ \gamma \frac{\gamma^2 L^2}{2C_1} + \gamma L \right) \left\| w^{(k)} - y^k \right\|^2 + \frac{\gamma C_1}{2(\theta^k)^2} \left\| x^{k+1} - y^k \right\|^2 \]

\[+ \frac{1}{2\gamma} \left\| z^k - x^* \right\|^2 - \frac{1}{2\gamma} \left\| z^{k+1} - x^* \right\|^2 , \]

where in \(\leq\), we use that \(\frac{1 - \theta^k}{(\theta^k)^2} \leq \frac{1}{(\theta^{k-1})^2}\) for \(k \geq 1\). When \(k = 0\), we have \(1 - \theta^0 = 0\).
Summing Eq. (48) with $k$ from 0 to $K$, and applying Eq. (47), we have that

$$
\frac{F(x^{K+1}) - F(x^*)}{(\theta^K)^2} \leq - \sum_{k=0}^{K} \frac{\gamma}{(\theta^k)^2} \left( \frac{1}{2} - \frac{\gamma L}{2} \right) \| x^{k+1} - y^k \|^2 \\
+ \sum_{k=0}^{K} \frac{\gamma}{(\theta^k)^2} \left( \frac{\gamma^2 L^2}{2C_1} + \gamma L \right) \| w_j(k) - y^k \|^2 + \sum_{k=0}^{K} \frac{\gamma C_1}{2(\theta^k)^2} \| x^{k+1} - y^k \|^2 \\
+ \frac{1}{2\gamma} \| z^0 - x^* \|^2 - \frac{1}{2\gamma} \| z^{K+1} - x^* \|^2 \\
\leq \frac{1}{2\gamma} \| z^0 - x^* \|^2 - \frac{1}{2\gamma} \| z^{K+1} - x^* \|^2 \\
+ \left( \frac{1}{2} - \frac{\gamma L}{2} - C_1 \left( \frac{\gamma^2 L^2}{2C_1} + \gamma L \right) (\tau^2 + 3\tau)^2 \right) \sum_{k=0}^{K} \frac{\gamma}{(\theta^k)^2} \| x^{k+1} - y^k \|^2.
$$

Set $C_1 = \gamma L$, we have that

$$
2\gamma L + 3\gamma L(\tau^2 + 3\tau)^2 \leq 1,
$$

So

$$
\frac{F(x^{K+1}) - F(x^*)}{(\theta^K)^2} + \frac{1}{2\gamma} \| z^{K+1} - x^* \|^2 \leq \frac{1}{2\gamma} \| z^0 - x^* \|^2.
$$

Now we consider the strongly convex case. In the following, we set $\theta = \theta^k$, and use $\theta^a$ to denote the $a$’s power of $\theta$, instead. Multiply Eq. (45) with $(1 - \theta)^{K-K'}$, and summing the results with $k$ from 0 to $K$, we have

$$
\sum_{k=0}^{K} (1 - \theta)^{K-k} \| w_j(k) - y^k \|^2
$$

$$
\leq \frac{\tau^2 + 3\tau}{2} \sum_{k=0}^{K} \min_{i=1}^{\min(\tau,k)} (i+1)(1 - \theta)^{K-k} \| x^{k-i+1} - y^{k-i} \|^2,
$$

$$
\leq \frac{\tau^2 + 3\tau}{2} \sum_{k=0}^{K} \min_{i=1}^{\min(\tau,k)} (1 - \theta)^{-i}(i+1)(1 - \theta)^{K-(k-i)} \| x^{k-i+1} - y^{k-i} \|^2,
$$

$$
\leq \frac{\tau^2 + 3\tau}{2(1 - \theta)^\tau} \sum_{k=0}^{K} \min_{i=1}^{\min(\tau,k)} (i+1)(1 - \theta)^{K-(k-i)} \| x^{k-i+1} - y^{k-i} \|^2,
$$

$$
\leq \frac{(\tau^2 + 3\tau)^2}{4(1 - \theta)^\tau} \sum_{k=0}^{K-1} (1 - \theta)^{K-i} \| x^{i+1} - y^i \|^2,
$$

$$
\leq \frac{(\tau^2 + 3\tau)^2}{4(1 - \theta)^\tau} \sum_{k=0}^{K} (1 - \theta)^{K-i} \| x^{i+1} - y^i \|^2,
$$

where $\leq$ is because that for each $(1 - \theta)^{K-i} \| x^{i+1} - y^i \|^2 (1 \leq k \leq K)$ there are most $\tau$ terms with coefficient from 2 to $\tau + 1$, like Eq. (47).
By arrange term on Eq. (26), we have that
\[
F(x^{k+1}) - F(x^*) + \left( \frac{\theta^2}{2\gamma} + \frac{\mu \theta}{2} \right) \| z^{k+1} - x^* \|^2
\leq (1 - \theta) \left( F(x^0) - F(x^*) + \left( \frac{\theta^2}{2\gamma} + \frac{\mu \theta}{2} \right) \| z^0 - x^* \|^2 \right)
- \gamma \left( \frac{1}{2} - \frac{\gamma L}{2} \right) \| x^{k+1} - y^k \|^2 + \gamma \left( \frac{\gamma^2 L^2}{2C_1} + \gamma L \right) \| \frac{w^{j(k)} - y^k}{\gamma} \|^2,
\] (52)
since we have set \( \theta = \frac{\gamma L}{1 - \gamma L} \), which satisfies
\[
\left( \frac{\theta^2}{2\gamma} + \frac{\mu \theta}{2} \right) (1 - \theta) = \frac{\theta^2}{2\gamma},
\]
solving it, we will have to solve \( g(x) = x^2 + \mu \gamma x - \gamma L = 0 \), we will have \( \sqrt{\gamma L} \leq \theta \leq \sqrt{\gamma L} \), since \( \gamma \mu \leq 1 \).
For the assumption of \( \gamma \), we have
\[
9\gamma L \tau^2 \leq \frac{5}{2} \gamma L + \gamma L (\tau^2 + 3\tau)^2 \leq 1,
\] (53)
We then consider that \( \frac{1}{(1 - \theta)^\gamma} \), without loss of generality, we assume that \( \tau \geq 2 \), we have that
\[
\frac{1}{(1 - \theta)^\gamma} \leq \frac{1}{(1 - \sqrt{\gamma L})^\gamma} \leq \frac{1}{(1 - \frac{1}{3\tau})^\gamma} \leq \frac{1}{(1 - \frac{1}{4})} \leq \frac{3}{2},
\] (54)
where in \( a \), we use \( \theta \leq \sqrt{\gamma L} \); in \( b \), we use Eq. (53); \( c \), we use \( \frac{\gamma L}{\mu} \leq 1 \), and \( d \), we use the fact that function \( g(x) = (1 - \frac{2}{\gamma})^x \) is monotonous increasing when \( x \in (0, 1] \).
Multiply Eq. (52) with \( \theta^{K-k} \), and summing the result with \( k \) from 0 to \( K \), we have that
\[
F(x^{K+1}) - F(x^*) + \left( \frac{\theta^2}{2\gamma} + \frac{\mu \theta}{2} \right) \| z^{K+1} - x^* \|^2
\leq (1 - \theta)^{K+1} \left( F(x^0) - F(x^*) + \left( \frac{\theta^2}{2\gamma} + \frac{\mu \theta}{2} \right) \| z^0 - x^* \|^2 \right)
- \gamma \left( \frac{1}{2} - \frac{\gamma L}{2} \right) \sum_{i=0}^{K} (1 - \theta)^{K-k} \| x^{i+1} - y^k \|^2 + \gamma \left( \frac{\gamma^2 L^2}{2C_1} + \gamma L \right) \sum_{k=0}^{K} (1 - \theta)^{K-k} \| \frac{w^{j(k)} - y^k}{\gamma} \|^2
\leq (1 - \theta)^{K+1} \left( F(x^0) - F(x^*) + \left( \frac{\theta^2}{2\gamma} + \frac{\mu \theta}{2} \right) \| z^0 - x^* \|^2 \right)
- \gamma \left( \frac{1}{2} - \frac{\gamma L}{2} \right) \left( \frac{\gamma^2 L^2}{2C_1} + \gamma L \right) \left( \frac{(\tau^2 + 3\tau)^2}{4(1 - \theta)^\gamma} \right) \sum_{i=0}^{K} (1 - \theta)^{K-k} \| x^{i+1} - y^k \|^2
\leq (1 - \theta)^{K+1} \left( F(x^0) - F(x^*) + \left( \frac{\theta^2}{2\gamma} + \frac{\mu \theta}{2} \right) \| z^0 - x^* \|^2 \right)
- \gamma \left( \frac{1}{2} - \frac{\gamma L}{2} \right) \left( \frac{\gamma^2 L^2}{2C_1} + \gamma L \right) \left( \frac{3(\tau^2 + 3\tau)^2}{8} \right) \sum_{i=0}^{K} (1 - \theta)^{K-k} \| x^{i+1} - y^k \|^2.
\]
where in \( a \), we use Eq. (51), and in \( b \), we use Eq. (54). Setting \( C_1 \) to be \( \frac{3}{2} \gamma L \), we have that
\[
1 - \frac{5}{2} \gamma L - \left( \frac{2}{3} + 2 \right) \gamma L \left( \frac{3(\tau^2 + 3\tau)^2}{8} \right) \geq 0,
\] (56)
this is the result.
7.2 AASCD

Lemma 1 Each $x^k$ is a convex combination of $z^0, \ldots, z^k$, suppose $x^k \sum_{i=0}^k c_{k,i}z^i$, we have $\epsilon_{0,0} = 1, \epsilon_{1,0} = 1 - n\theta^0, \epsilon_{1,1} = n\theta$. And for $k > 1$, we have

$$e_{k+1,i} = \begin{cases} n(1 - \theta^k)\theta^{k-1} + \theta^k - n\theta^k, & i = k \leq k - 1 \\ n\theta^k, & i = k + 1. \end{cases}$$

(57)

Supposing $\hat{h}^{k+1} = \sum_{i=0}^k a_{k,i}h(z^i)$, we have

$$\mathbb{E}_{ik}(\hat{h}^{k+1}) = (1 - \theta^k)\hat{h}^k + \theta^k \sum_{i_k=1}^n h_{ik}(z_{ik}^{k+1}),$$

(58)

where $\mathbb{E}_{ik}$ denote the random expectation is only taken on $i_k$ under the condition that $x^k, z^k$ is known.

The proof of Lemma 1 is directly taken from Lin et al. [2014] Fercoq & Richtárik [2015]. For completeness, we provide a proof in the end of the section.

Because all proof only uses the Lipschitz coordinate constant, we use $L$ instead of $L_c$ to represent it for simply.

Outline of the Proof:
Step 1: Set $y^k = \theta^k z^k + (1 - \theta^k)x^k$. Through the update rule, we have that

$$y^k = y^{j(k)} + \sum_{i=1}^k (1 + c^i)(x^i - y^{i-1}) + \sum_{i=j(k)+1}^k \left( \sum_{l=i}^{k-1} c^l b(i + 1, l) \right)(x^i - y^{i-1})$$

$$+ \left( \sum_{i=j(k)+1}^k b(j(k), i) \right)(y^{j(k)} - x^{j(k)-1}).$$

Step 2: By analyzing the function value, we have

$$f(x^{k+1}) \leq f(y^k) - \gamma(1 - \frac{\gamma L}{2})\left\| x_{ik}^{k+1} - y_{ik}^k \right\|_2^2 - \langle \xi_{ik}^k, x_{ik}^{k+1} - y_{ik}^k \rangle$$

$$+ \frac{\gamma L^2}{2C^2} \left\| w_{ik}^{j(k)} - y_{ik}^k \right\|_2^2 + \frac{\gamma C^2}{2} \left\| x_{ik}^{k+1} - y_{ik}^k \right\|_2^2,$$

(60)

Step 3: By analyzing the $\|z^{k+1} - x^*\|^2$, we have

$$\frac{n^2}{2\gamma} \mathbb{E}_{ik} \| \theta^k z^{k+1} - \theta^k x^* \|^2$$

$$= \frac{n^2}{2\gamma} \| \theta^k z^k - \theta^k x^* \|^2 + \frac{1}{2\gamma n} \sum_{i_k=1}^n \| x_{ik}^{k+1} - y_{ik}^k \|^2 - \sum_{i_k=1}^n \langle \xi_{ik}^k, \theta^k z_{ik}^k - \theta^k x_{ik}^k \rangle$$

$$+ (1 - \theta^k)f(x^k) + \theta^k f(x^k) - f(y^k) + (\nabla f(y^k) - \nabla f(w^{j(k)}), y^k - w^{j(k)}).$$

Step 4: Taking expectation on Eq. (60), and adding and Eq. (61), and simplifying, we have

$$\mathbb{E}_{ik} f(x^{k+1}) + \mathbb{E}_{ik} [\hat{h}^{k+1}] - F(x^*) + \frac{n^2 (\theta^k)^2 + n\theta^k \mu \gamma}{2\gamma} \mathbb{E}_{ik} \| z^{k+1} - x^* \|^2$$

$$\leq (1 - \theta^k) f(x^k) + \hat{h}^k - F(x^*)$$

$$- \gamma(1 - \frac{\gamma L}{2}) - C^2 \mathbb{E}_{ik} \| x^{k+1} - y^k \|^2$$

$$+ \left( \frac{\gamma L^2}{2nC^2} + L \right) \left\| w^{j(k)} - y^k \right\|_2^2 + \frac{n^2 (\theta^k)^2 + (n - 1)\theta^k \mu \gamma}{2\gamma} \| z^k - x^* \|^2.$$
We have
\[
\theta^k z^k = y^k - (1 - \theta^k)x^k,
\] (63)
and
\[
n\theta^k z^{k+1} = x^{k+1} - (1 - \theta^k)x^k + (n - 1)\theta^k z^k.
\] (64)

We have
\[
x^{k+1} = y^k + n\theta^k(z^{k+1} - z^k).
\] (65)

Multiplying Eq. (63) with \((n - 1)\), and adding with Eq. (64), we have
\[
n\theta^k z^{k+1} = x^{k+1} - (1 - \theta^k)x^k + (n - 1)y^k - (n - 1)(1 - \theta^k)x^k.
\] (66)

Eliminating \(z^k\) using Eq. (66) and Eq. (63), for \(k \geq 1\), we have
\[
\frac{y^k - (1 - \theta^k)x^k}{\theta^k} = \frac{x^k - (1 - \theta^{k-1})x^{k-1} + (n - 1)y^{k-1} - (n - 1)(1 - \theta^{k-1})x^{k-1}}{n\theta^{k-1}}.
\] (67)

Computing out \(y^k\) through Eq. (67), we have
\[
y^k = x^k - \theta^k x^k + \frac{\theta^k x^k}{n\theta^{k-1}} - \frac{\theta^k(1 - \theta^{k-1})x^{k-1}}{n\theta^{k-1}} - \frac{(n - 1)\theta^k(1 - \theta^{k-1})x^{k-1}}{n\theta^{k-1}} + \frac{(n - 1)\theta^k y^{k-1}}{n\theta^{k-1}}
\] (68)
\[
= x^k + \frac{\theta^k}{\theta^{k-1}}(1 - \theta^{k-1})(x^k - y^{k-1}) + \frac{\theta^k(1 - \theta^{k-1})}{\theta^{k-1}}(y^{k-1} - x^{k-1}).
\]

Still, we set \(a^k = \frac{\theta^k(1 - \theta^{k-1})}{\theta^{k-1}}(y^{k-1} - x^{k-1})\) and \(b(l, k) = \prod_{i=l}^{k} a^i\), where \(l \leq k\). Then by setting \(c^k = \frac{\theta^k}{\theta^{k-1}}(\frac{1}{n} - \theta^{k-1})\), we have
\[
y^k = x^k + c^k(x^k - y^{k-1}) + a^k(y^{k-1} - x^{k-1})
\] (69)
\[
= y^{k-1} + (1 + c^k)(x^k - y^{k-1}) + a^k(y^{k-1} - x^{k-1})
\]
\[
= y^{k-1} + (1 + c^k)(x^k - y^{k-1}) + a^k c^k - 1(x^{k-1} - y^{k-2}) + a^k a^{k-1}(y^{k-2} - x^{k-2})
\]
\[
= y^{k-1} + (1 + c^k)(x^k - y^{k-1}) + \sum_{i=j(k)+1}^{k-1} b(i+1, k)c^i(x^i - y^{i-1})
\]
\[
+ b(j(k) + 1, k)(y^{j(k)} - x^{j(k)-1}), \quad k \geq j(k) + 1 \geq 1.
\]
Like Eq. (34), summing Eq. (69) with \( k = j(k) + 1 \) to \( k \), we have

\[
y^k = y^{j(k)} + \sum_{i=j(k)+1}^{k} (1 + c^i)(x^i - y^{i-1}) + \sum_{i=j(k)+1}^{k} \sum_{l=i+1}^{k-1} c^i b(i, l)(x^i - y^{i-1}) \]
\[
+ \left( \sum_{i=j(k)+1}^{k} b(j(k) + 1, i) \right) (y^{j(k)} - x^{j(k)-1})
\]
\[
= y^{j(k)} + \sum_{i=j(k)+1}^{k} (1 + c^i)(x^i - y^{i-1}) + \sum_{i=j(k)+1}^{k} \left( \sum_{l=i}^{k-1} c^i b(i, l) \right) (x^i - y^{i-1})
\]
\[
+ \left( \sum_{i=j(k)+1}^{k} b(j(k), i) \right) (y^{j(k)} - x^{j(k)-1}).
\]

Comparing the result, we obtain step 1.

**Proof of step 2:**

Through the optimal solution of \( z_{ik}^{k+1} \) in step 4, we have that

\[
n\theta^k (z_{ik}^{k+1} - z_{ik}^k) + \gamma \nabla_{ik} f(w^{j(k)}) + \gamma \xi_{ik}^k = 0,
\]

where we denote \( \xi_{ik}^k \) as a subgradient of \( h_{ik} \), i.e. \( \xi_{ik}^k \in h_{ik}(x_{ik}^{k+1}) \). Through Eq. (65),

\[
x_{ik}^{k+1} - y_{ik}^k + \gamma \nabla_{ik} f(w^{j(k)}) + \gamma \xi_{ik}^k = 0.
\]

Since \( f \) has Lipschitz continue gradient on coordinate \( i_k \), we have

\[
f(x^{k+1}) \leq f(y^k) + \langle \nabla_{ik} f(y^k), x_{ik}^{k+1} - y_{ik}^k \rangle + \frac{L}{2} \left\| x_{ik}^{k+1} - y_{ik}^k \right\|^2
\]
\[
= f(y^k) - \gamma (\nabla_{ik} f(y^k), \nabla_{ik} f(w^{j(k)}) + \xi_{ik}^k) + \frac{L}{2} \left\| x_{ik}^{k+1} - y_{ik}^k \right\|^2
\]
\[
= f(y^k) - \gamma (\nabla_{ik} f(w^{j(k)}) + \xi_{ik}^k, \nabla_{ik} f(w^{j(k)}) + \xi_{ik}^k) + \frac{L}{2} \left\| x_{ik}^{k+1} - y_{ik}^k \right\|^2
\]
\[
+ \gamma (\xi_{ik}^k, \nabla_{ik} f(w^{j(k)}) + \xi_{ik}^k) + \gamma (\nabla_{ik} f(w^{j(k)}) - \nabla_{ik} f(y^k), \nabla_{ik} f(w^{j(k)}) + \xi_{ik}^k)
\]
\[
= f(y^k) - \gamma (1 - \frac{\gamma L}{2}) \left\| x_{ik}^{k+1} - y_{ik}^k \right\|^2 - \langle \xi_{ik}^k, x_{ik}^{k+1} - y_{ik}^k \rangle
\]
\[
- (\nabla_{ik} f(w^{j(k)}) - \nabla_{ik} f(y^k), x_{ik}^{k+1} - y_{ik}^k)
\]
\[
\leq f(y^k) - \gamma (1 - \frac{\gamma L}{2}) \left\| x_{ik}^{k+1} - y_{ik}^k \right\|^2 - \langle \xi_{ik}^k, x_{ik}^{k+1} - y_{ik}^k \rangle
\]
\[
+ \frac{\gamma L^2}{2C_2} \left\| w_{ik}^{j(k)} - y_{ik}^k \right\|^2 + \frac{\gamma C_2}{2} \left\| x_{ik}^{k+1} - y_{ik}^k \right\|^2,
\]

where in \( \leq \), we use Eq. (72); in \( \leq \), we insert \( \nabla_{ik} f(y^k) + \xi_{ik}^k \); in \( \leq \), we use Cauchy-Schwarz inequality.
Taking expectation on Eq. (60), we have

Proof of step 3:

By the same technology of Eq. (40), for the last second term of Eq. (75), we have that

Proof of step 4:

Substituting Eq. (76) into Eq. (75), we have the results of Step 3.

So taking expectation on Eq. (74), we have

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Substituting Eq. (76) into Eq. (75), we have the results of Step 3.

Proof of step 4:

Taking expectation on Eq. (60), we have

Taking expectation on Eq. (60), we have

Taking expectation on Eq. (60), we have

Taking expectation on Eq. (60), we have
Adding Eq. (77) and Eq. (61) we have

\[
\mathbb{E}_{i_k} f(x^{k+1}) \leq (1 - \theta^k) f(x^k) + \theta^k f(x^*) - \frac{\gamma}{n} \sum_{i_k=1}^{n} \left( \frac{1}{2} - \frac{\gamma L}{2} - \frac{C_2}{2} \right) \left\| x_{i_k}^{k+1} - y_{i_k}^k \right\|^2 \\
+ \left( \frac{\gamma L^2}{2nC_2} + L \right) \left\| w^{(k)}(x^k) - y^k \right\|^2 - \sum_{i_k=1}^{n} \langle \xi_{i_k}, \theta^k z_{i_k}^k - \theta^k x_i^* + \frac{1}{n} (x_{i_k}^{k+1} - y_{i_k}^k) \rangle \\
+ \frac{n^2}{2 \gamma} \left\| \theta^k z^k - \theta^k x^* \right\|^2 - \frac{n^2}{2 \gamma} \mathbb{E}_{i_k} \left\| \theta^k z_{i_k}^{k+1} - \theta^k x_i^* \right\|^2 \\
\equiv (1 - \theta^k) f(x^k) + \theta^k f(x^*) - \frac{\gamma}{n} \sum_{i_k=1}^{n} \left( \frac{1}{2} - \frac{\gamma L}{2} - \frac{C_2}{2} \right) \left\| x_{i_k}^{k+1} - y_{i_k}^k \right\|^2 \\
+ \left( \frac{\gamma L^2}{2nC_2} + L \right) \left\| w^{(k)}(x^k) - y^k \right\|^2 - \sum_{i_k=1}^{n} \langle \xi_{i_k}, \theta^k z_{i_k}^{k+1} - \theta^k x_i^* \rangle \\
+ \frac{n^2}{2 \gamma} \left\| \theta^k z^k - \theta^k x^* \right\|^2 - \frac{n^2}{2 \gamma} \mathbb{E}_{i_k} \left\| \theta^k z_{i_k}^{k+1} - \theta^k x_i^* \right\|^2 \tag{78}
\]

where in \( \equiv \), we use Eq. (65).

The same as Eq. (43), since \( h_{i_k} \) is convex, we have

\[
\theta^k \langle \xi_{i_k}, x_i^* - z_{i_k}^{k+1} \rangle \leq \theta^k h_{i_k}(x^*) - \theta^k h_{i_k}(z_{i_k}^{k+1}) - \frac{\mu \theta^k}{2} \left\| z_{i_k}^{k+1} - x_i^* \right\|^2. \tag{79}
\]

Analyzing the expectation, we have

\[
\mathbb{E}_{i_k} \left\| z_{i_k}^{k+1} - x^* \right\|^2 = \frac{1}{n} \sum_{i_k=1}^{n} \left[ \left\| z_{i_k}^{k+1} - x_i^* \right\|^2 + \sum_{j \neq i_k} \left\| z_j^k - x_i^* \right\|^2 \right] \\
= \frac{1}{n} \sum_{i_k=1}^{n} \left\| z_{i_k}^{k+1} - x_i^* \right\|^2 + \frac{n-1}{n} \left\| z^k - x^* \right\|^2, \tag{80}
\]

Since as Eq. (80), we can find that

\[
\mathbb{E}_{i_k} \left\| x_{i_k}^{k+1} - y_i^k \right\|^2 = \frac{1}{n} \sum_{i_k=1}^{n} \left[ \left\| x_{i_k}^{k+1} - y_i^k \right\|^2 + \sum_{j \neq i_k} \left\| x_j^k - y_i^k \right\|^2 \right] \\
\geq \frac{1}{n} \sum_{i_k=1}^{n} \left\| x_{i_k}^{k+1} - y_i^k \right\|^2. \tag{81}
\]
We have that

\[ E_{i_k} f(x^{k+1}) + \sum_{i_k=1}^{n} h(z_{i_k}^{k+1}) \]

\[ \leq (1 - \theta^k) f(x^k) + \theta^k F(x^*) - \frac{\gamma}{n} \sum_{i_k=1}^{n} \left( \frac{1}{2} - \frac{\gamma L}{2} - \frac{C^2}{2} \right) \left\| x_{i_k}^{k+1} - y_{i_k}^k \right\|^2 + \left( \frac{\gamma L^2}{2nC^2} + L \right) \left\| w^{j(k)} - y^k \right\|^2 - \sum_{i_k=1}^{n} \frac{n\theta^k}{2} \left\| z_{i_k}^{k+1} - x_{i_k}^k \right\|^2 + \frac{n^2}{2\gamma} \left\| \theta^k z_{i_k}^{k+1} - \theta^k x^k \right\|^2 \]

\[ = (1 - \theta^k) f(x^k) + \theta^k F(x^*) - \frac{\gamma}{n} \sum_{i_k=1}^{n} \left( \frac{1}{2} - \frac{\gamma L}{2} - \frac{C^2}{2} \right) \left\| x_{i_k}^{k+1} - y_{i_k}^k \right\|^2 + \left( \frac{\gamma L^2}{2nC^2} + L \right) \left\| w^{j(k)} - y^k \right\|^2 - \sum_{i_k=1}^{n} \frac{n^2(\theta^k)^2 + (n - 1)\theta \mu \gamma}{2\gamma} \left\| z^k - x^* \right\|^2 \]

where in \( \leq \), we use Eq. (80). We obtain the results of Step 4.

**Proof of step 5:**

Through Eq. (59), using the same technique of Eq. (45), we have

\[ \left\| w^{j(k)} - y^k \right\|^2 \]

\[ \leq \left\| \sum_{i=j(k)+1}^{k} \left( 1 + c^e + c^i \sum_{l=i}^{k-1} b(i, l) \right) (x^l - y^{i-1}) \right\|^2 \]

\[ \leq a \left( \sum_{i=j(k)+1}^{k} \left( 1 + c^e + c^i \sum_{l=i}^{k-1} b(i, l) \right) \right) \left( \sum_{i=1}^{k} \left( 1 + c^e + c^i \sum_{l=i}^{k-1} b(i, l) \right) \right) \left\| x^l - y^{i-1} \right\|^2 \]

\[ \leq b \left( \sum_{i=j(k)+1}^{k} \left( 1 + c^e \sum_{l=1}^{k-i} 1 \right) \right) \left( \sum_{i=1}^{k} \left( 1 + c^e \sum_{l=1}^{k-1} 1 \right) \right) \left\| x^l - y^{i-1} \right\|^2 \]

\[ \leq c \left( \min(\tau, k) \sum_{i=1}^{k-j(k)} \left( 1 + \frac{1}{n} \sum_{l=1}^{i} 1 \right) \right) \left( \sum_{i=1}^{\tau} \left( 1 + \frac{1}{n} \sum_{l=1}^{i} 1 \right) \right) \left\| x^{k-i+1} - y^{k-i} \right\|^2 \]

\[ \leq d \left( \sum_{i=1}^{\min(\tau, k)} \left( 1 + \frac{1}{n} \sum_{l=1}^{i} 1 \right) \right) \left( \sum_{i=1}^{\tau} \left( 1 + \frac{1}{n} \sum_{l=1}^{i} 1 \right) \right) \left\| x^{k-i+1} - y^{k-i} \right\|^2 \]

\[ \leq \left( \frac{\tau^2 + \tau}{2n} + \tau \right) \sum_{i=1}^{\min(\tau, k)} \left( i + 1 \right) \left\| x^{k-i+1} - y^{k-i} \right\|^2 , \]

where in \( \leq \), we use Eq. (46); in \( \leq \), we use \( b(i, l) \leq 1 \); in \( \leq \), we change variable \( ii = k - i + 1 \), and use \( c^e \leq \frac{1}{n} \); and in \( \leq \), we use \( k - j(k) \leq \tau \). Since \( \tau \leq \sqrt{n} \leq 2n \). Diving \((\theta^k)^2\) on Eq. (45) and summing the
results with $k = 0$ to $K$, we have
\[
\sum_{k=0}^{K} \frac{1}{(\theta^k)^2} \|w^{(k)} - y^k\|^2
\]
\[\leq \left( \frac{\tau^2 + \gamma + \gamma}{2n} \right) \sum_{k=0}^{K} \sum_{i=1}^{\min(k, \tau)} \frac{4(\frac{i}{n} + 1)}{(\theta^k)^2} \|x^{k-i+1} - y^{k-i}\|^2
\]
\[\leq \left( \frac{\tau^2 + \gamma + 2\tau}{n} \right) \sum_{k=0}^{K-1} \frac{1}{(\theta^k)^2} \|x^{k+1} - y^k\|^2,
\]
\[\leq \left( \frac{\tau^2 + \gamma + 2\tau}{n} \right) \sum_{k=0}^{K} \frac{1}{(\theta^k)^2} \|x^{k+1} - y^k\|^2,
\]
where in $\leq$, we use that $(k + j)^2 \leq 4k^2$, when $k \geq 2n \geq \tau$ and $j \leq \tau$, and $\frac{1}{\theta^k} = \frac{(2n+k)}{2}$ for $k \geq 0$; $\leq$ is because that for each $\left( \frac{1}{\theta^k} \right)^2 \|x^k - y^{k-1}\|^2$ (1 $\leq k \leq K$) there are most $\tau$ terms.

Diving $(\theta^k)^2$ on both sides of Eq. (62), and use $\mu = 0$, we have
\[
\frac{E_i f(x^{k+1}) + E_i \hat{h}^{k+1} - F(x^*)}{(\theta^k)^2} + \frac{n^2}{2\gamma} E_i \|z^{k+1} - x^*\|^2
\]
\[\leq \frac{1 - \theta^k}{(\theta^k)^2} \left( f(x^k) + \hat{h}^k - F(x^*) \right) - \frac{\gamma}{(\theta^k)^2} \left( \frac{1}{2} - \frac{\gamma L}{2} - \frac{C_1}{2} \right) E_i \|x^{k+1} - y^k\|^2
\]
\[+ \left( \frac{\gamma L^2}{2nC_2} + L \right) \|w^{(k)} - y^k\|^2 + \frac{n^2}{2\gamma} \|z^k - x^*\|^2
\]
\[\leq \frac{1}{(\theta^k)^2} \left( f(x^k) + \hat{h}^k - F(x^*) \right) - \frac{\gamma}{(\theta^k)^2} \left( 1 - \frac{\gamma L}{2} - \frac{C_1}{2} \right) E_i \|x^{k+1} - y^k\|^2
\]
\[+ \left( \frac{\gamma L^2}{2nC_2} + L \right) \|w^{(k)} - y^k\|^2 + \frac{n^2}{2\gamma} \|z^k - x^*\|^2
\]
where in $\leq$, we use that $\frac{1 - \theta^k}{(\theta^k)^2} \leq \frac{1}{(\theta^k)^2}$. Taking expectation on the first $k$ iteration for Eq. (85), and summing it with $k$ from 0 to $K$, we have that
\[
\frac{E f(x^{K+1}) + E \hat{h}^{K+1} - F(x^*)}{(\theta^k)^2} + \frac{n^2}{2\gamma} E \|z^{K+1} - x^*\|^2
\]
\[\leq \frac{1 - \theta^k}{(\theta^k)^2} \left( f(x^0) + \hat{h}^0 - F(x^*) \right) - \frac{\gamma}{(\theta^k)^2} \left( \frac{1}{2} - \frac{\gamma L}{2} - \frac{C_1}{2} \right) \sum_{k=0}^{K} E \|x^{k+1} - y^k\|^2
\]
\[+ \left( \frac{\gamma L^2}{2nC_2} + L \right) \sum_{k=0}^{K} E \|w^{(k)} - y^k\|^2 + \frac{n^2}{2\gamma} \|z^0 - x^*\|^2
\]
\[\leq \frac{1}{(\theta^k)^2} \left( f(x^0) + \hat{h}^0 - F(x^*) \right) + \frac{n^2}{2\gamma} \|z^0 - x^*\|^2
\]
\[+ \frac{1}{(\theta^k)^2} \left( \frac{\gamma L^2}{2nC_2} + L \right) \sum_{k=0}^{K} \frac{\gamma}{(\theta^k)^2} E \|x^{k+1} - y^k\|^2
\]
Set $C_2 = \gamma L$, we have that
\[2\gamma L + (1 + \frac{1}{n}) \gamma L \left( \frac{\tau^2 + \gamma}{2n} + 2\tau \right) \leq 1,
\]
So
\[
\frac{\mathbb{E}f(x^{K+1}) - F(x^*)}{(\theta g)^2} + \frac{n^2}{2\gamma} \mathbb{E} \|z^{K+1} - x^*\|^2 \leq \frac{\mathbb{E}f(x^{K+1}) + \mathbb{E}[\hat{h}^{K+1}] - F(x^*)}{(\theta g)^2} + \frac{n^2}{2\gamma} \mathbb{E} \|z^{K+1} - x^*\|^2 \leq \frac{\mathbb{E}f(x^0) + \hat{h}^0 - F(x^*)}{(\theta - 1)^2} + \frac{n^2}{2\gamma} \|z^0 - x^*\|^2 \leq \frac{F(x^0) - F(x^*)}{(\theta - 1)^2} + \frac{n^2}{2\gamma} \|z^0 - x^*\|^2,
\]
where \( \leq \) use \( h(x^{K+1}) = h(\sum_{i=0}^{K+1} e_{k+1, i}z^i) \leq \sum_{i=0}^{K+1} e_{k+1, i}h(z^i) = \hat{h}^{K+1} \) and in \( \leq \) use \( h(x^0) = \hat{h}^0 \).

Now we consider the strongly convex case. In the following, again we set \( \theta = \theta^k \) and use \( \theta^a \) to denote the \( a \)'s power of \( \theta \). Multiply Eq. (83) with \( (1 - \theta)^{K-k} \), and summing the results with \( k \) from 0 to \( K \), we have
\[
\sum_{k=0}^{K} (1 - \theta)^{K-k} \|w^{j(k)} - y^k\|^2 \leq \frac{(\tau^2 + \tau + \tau)}{2n} \sum_{k=0}^{K} \sum_{i=1}^{\min(\tau, k)} \left( \frac{i}{n} + 1 \right)(1 - \theta)^{K-k} \|x^{k-i+1} - y^{k-i}\|^2,
\]
where \( \leq \) is because that for each \( (1 - \theta)^{K-k} \|x^{k-i+1} - y^i\|^2 (1 \leq k \leq K) \) there are most \( \tau \) terms, like Eq. (47).

By arrange term on Eq. (82), we have that
\[
\mathbb{E}_{i_k} f(x^{k+1}) + \mathbb{E}_{i_k} [\hat{h}^{k+1}] - F(x^*) + \frac{n^2(\theta)^2 + n\theta\mu\gamma}{2\gamma} \mathbb{E}_{i_k} \|z^{k+1} - x^*\|^2 \leq (1 - \theta) \left( f(x^k) + \hat{h}^k - F(x^*) + \frac{n^2(\theta)^2 + n\theta\mu\gamma}{2\gamma} \|z^k - x^*\|^2 \right)
- \gamma \left( 1 - \frac{\gamma L}{2} \right) \mathbb{E}_{i_k} \|x^{k+1} - y^k\|^2 + \left( \frac{\gamma L^2}{2nC_2} + L \right) \|w^{j(k)} - y^k\|^2.
\]
\[
\text{since we have set } \theta = \frac{-\mu + \sqrt{\mu^2 + 4\gamma \mu}}{2n}, \text{ which satisfies that}
\left( \frac{\theta^2 n^2}{2\gamma} + \frac{n\mu\theta}{2} \right) (1 - \theta) = \frac{\theta^2}{2\gamma} + \frac{(n-1)\mu\theta}{2},
\]
solving it, we will have to solve \( g(x) = n^2x^2 + n\gamma x - \mu \gamma = 0 \), we assume \( \mu/L \leq n^2 \), and we will have \( \sqrt{\gamma\mu}/2 \leq n\theta \leq \sqrt{\gamma\mu} \). For the assumption of \( \gamma \), we have
\[
3\gamma L \tau^2 \leq 2\gamma L + \left( \frac{3}{4} + \frac{3}{8n} \right) \gamma L \left( (\tau^2 + \tau)/n + 2\tau \right) \leq 1,
\]
which gives \( \gamma \leq \frac{2\gamma}{(\tau^2 + \tau)/n + 2\tau} \).
We then consider that $\frac{1}{(1-\theta)^2}$, without loss of generality, we assume that $n \geq 2$, we have that

$$\frac{1}{(1-\theta)^2} \leq \frac{1}{(1-\sqrt{\gamma} \mu/n)^2} \leq \frac{1}{(1-\frac{1}{\tau})^2} \leq \frac{1}{(1-\frac{1}{2\sqrt{3}})^2} \leq \frac{3}{2},$$

(91)

where in $\leq$, we use $\theta \leq \sqrt{\gamma} \mu$; in $\leq$, we use Eq. (53); $\leq$, we use $\sqrt{\frac{\tau}{n}} \leq \frac{1}{2}$.

Taking expectation Eq. (89), and Multiply Eq. (89) with $\theta^{K-k}$, then and summing the result with $k$ from 0 to $K$, we have that

$$\mathbb{E} f(x^{K+1}) + \mathbb{E}[\tilde{h}^{K+1}] - F(x^*) + \frac{n^2(\theta)^2 + n\theta \mu \gamma}{2\gamma} \mathbb{E}\|z^{K+1} - x^*\|^2$$

(92)

$$\leq (1-\theta)^{K+1} \left( f(x^0) + \tilde{h}^k - F(x^*) + \frac{n^2(\theta)^2 + n\theta \mu \gamma}{2\gamma} \|z^0 - x^*\|^2 \right)$$

$$- \gamma \left( \frac{1}{2} - \frac{\gamma L}{2} - \frac{C_2}{2} \right) \sum_{i=0}^K (1-\theta)^{K-k} \mathbb{E}\|x^{k+1} - y^k\|^2 + \left( \frac{\gamma L^2}{2nC_2} + L \right) \sum_{i=0}^K (1-\theta)^{K-k} \mathbb{E}\|w^{i(k)} - y^k\|^2$$

$$\leq (1-\theta)^{K+1} \left( f(x^0) + \tilde{h}^k - F(x^*) + \frac{n^2(\theta)^2 + n\theta \mu \gamma}{2\gamma} \|z^0 - x^*\|^2 \right)$$

$$- \gamma \left( \frac{1}{2} - \frac{\gamma L}{2} - \frac{C_2}{2} - \left( \frac{\gamma L^2}{2nC_2} + \gamma L \right) \left( \frac{(\tau^2 + \tau)/n + 2\tau}{4(1-\theta)^2} \right) \right) \sum_{i=0}^K (1-\theta)^{K-k} \mathbb{E}\|x^{k+1} - y^k\|^2.$$

Set $C_2 = \gamma L$, we have

$$2\gamma L + \left( \frac{\gamma L}{n} + 2\gamma L \right) \left( \frac{(\tau^2 + \tau)/n + 2\tau}{4(1-\theta)^2} \right) \leq 2\gamma L + \left( \frac{\gamma L}{n} + 2\gamma L \right) \left( \frac{3((\tau^2 + \tau)/n + 2\tau)^2}{8} \right) \leq 1,$$

(93)

Then using $h(x^{K+1}) \leq \tilde{h}^{K+1}$ and $h(x^0) = \tilde{h}^0$, we obtain the results.

$$\mathbb{E}[F(x^{K+1})] - F(x^*) + \frac{n^2(\theta)^2 + n\theta \mu \gamma}{2\gamma} \mathbb{E}\|z^{K+1} - x^*\|^2$$

(94)

$$\leq (1-\theta)^{K+1} \left( F(x^0) - F(x^*) + \frac{n^2(\theta)^2 + n\theta \mu \gamma}{2\gamma} \|z^0 - x^*\|^2 \right).$$

Proof of Lemma 1 taken from [Lin et al., 2014; Fercoq & Richtarik, 2015].

We proof $c_{k+1,i}$ first. When $k = 0$ and 1, it is right. We then proof Eq. (57). Since

$$x^{k+1} = (1-\theta^k)x^k + \theta^k z^k + n\theta^k(z^{k+1} - z^k)$$

(95)

$$= (1-\theta^k) \sum_{i=0}^k c_{k,i} z^i + \theta^k z^k + n\theta^k(z^{k+1} - z^k)$$

$$= (1-\theta^k) \sum_{i=0}^{k-1} c_{k,i} z^i + ((1-\theta^k)c_{k,k} + \theta^k - n\theta^k) z^k + n\theta^k z^{k+1}.$$
Comparing the results, we obtain Eq. \((57)\). For Eq. \((58)\), we have

\[
\mathbb{E}_{\delta_k} [\hat{h}^{k+1}] = \sum_{i=0}^{k} \hat{h}(\mathbf{z}^{i}) + \mathbb{E}_{\delta_k} n \theta^k \hat{h}(\mathbf{z}^{k+1})
\]

\[
= \sum_{i=0}^{k} c_{k+1,i} \hat{h}(\mathbf{z}^{i}) + \frac{1}{n} \sum_{i_k} n \theta^k \left( h_{i_k}(\mathbf{z}^{k+1}_{i_k}) + \sum_{j \neq i_k} h_j(\mathbf{z}^{k}) \right)
\]

\[
= \sum_{i=0}^{k} c_{k+1,i} \hat{h}(\mathbf{z}^{i}) + \theta^k \sum_{i_k} h_{i_k}(\mathbf{z}^{k+1}_{i_k}) + (n-1) \theta^k \hat{h}(\mathbf{z}^{k})
\]

\[
= \sum_{i=0}^{k-1} c_{k+1,i} \hat{h}(\mathbf{z}^{i}) + n(1-\theta^k) \theta^{k-1} h(\mathbf{z}^{k}) + \theta^k \sum_{i_k} h_{i_k}(\mathbf{z}^{k+1}_{i_k})
\]

\[
= \sum_{i=0}^{k-1} c_{k,i} (1-\theta^k) \hat{h}(\mathbf{z}^{i}) + (1-\theta^k) c_{k,k} \hat{h}(\mathbf{z}^{k}) + \theta^k \sum_{i_k} h_{i_k}(\mathbf{z}^{k+1}_{i_k})
\]

\[
= \sum_{i=0}^{k-1} c_{k,i} (1-\theta^k) \hat{h}(\mathbf{z}^{i}) + \theta^k \sum_{i_k} h_{i_k}(\mathbf{z}^{k+1}_{i_k}) = \hat{h}^k + \theta^k \sum_{i_k} h_{i_k}(\mathbf{z}^{k+1}_{i_k}),
\]

where in \(a\), we use \(c_{k+1,k+1} = n \theta^k\); in \(b\), we use \(c_{k+1,k} = n(1-\theta^k) \theta^{k-1} + \theta^k - n \theta^k\); and in \(c\), we use \(c_{k+1,i} = (1-\theta^k) c_{k,i} \) for \(i \leq k-1\), and \(c_{k,k} = n \theta^{k-1}\).

### 7.3 AASVRG

**Lemma 2** Define \(f(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^{n} f_i(\mathbf{x})\), if \(f_i(\mathbf{x})\)'s, \(i = 1, 2, \cdots, n\) have Lipschitz continuous gradients, for any \(\mathbf{u}\) and \(\tilde{\mathbf{x}}\), defining

\[
\hat{\nabla} f(\mathbf{u}) = \nabla f_k(\mathbf{u}) - \nabla f_k(\tilde{\mathbf{x}}) + \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\tilde{\mathbf{x}}),
\]

we have

\[
\mathbb{E} \left( \left\| \hat{\nabla} f(\mathbf{u}) - \nabla f(\mathbf{u}) \right\|^2 \right) \leq 2L(f(\tilde{\mathbf{x}}) - f(\mathbf{u}) + \langle \nabla f(\mathbf{u}), \mathbf{u} - \tilde{\mathbf{x}} \rangle),
\]

where the expectation is taken on the random number of \(k\) under the condition that \(\mathbf{u}\) and \(\tilde{\mathbf{x}}\) are known.

The lemma is directly taken from (Allen-Zhu 2017) and (Johnson & Zhang 2013). For completeness, we provide a proof of Lemma 2 in the end of the section. We define

\[
\mathbf{y}^k = (\theta^*_1) \mathbf{z}^k + \theta^*_2 \tilde{\mathbf{x}} + a^s \mathbf{x}^s_k,
\]

Through the step 6 in Algorithm 5 in the paper, we have

\[
\mathbf{x}^{s}_{k+1} = \mathbf{y}^k + \theta^*_1 (\mathbf{z}^{k+1} - \mathbf{z}^k).
\]

**Outline of the Proof:**

**Step 1:** and set \(b^s(l, k) = (a^s)^{k-l+1}\). Through the update rule, we have that

\[
\mathbf{y}^k - w^s_{j(k)} = \sum_{i=j(k)+1}^{k} (\mathbf{x}^s_i - \mathbf{y}^s_{i-1}) + \sum_{i=j(k)+1}^{k} \left( \sum_{l=i}^{k} b^s(i, l) \right) (\mathbf{x}^s_i - \mathbf{y}^s_{i-1}).
\]
Thus we obtain

\[ E_k f(x^*_{k+1}) \leq f(y^*_{k}) - \gamma(1 - \frac{\gamma L}{2} - C_3) - C_4)E \left\| x^*_{k+1} - y^*_{k} \right\|^2 - \frac{\gamma L^2}{C_3} \left\| w^*_{j(k)} - y^*_{k} \right\|^2 \]

Step 2: By analyzing the function value, we have

\[ f(y^*_{k}) - \gamma(1 - \frac{\gamma L}{2} - C_3 - C_4)E \left\| x^*_{k+1} - y^*_{k} \right\|^2 - \frac{\gamma L^2}{2C_4} \left\| w^*_{j(k)} - y^*_{k} \right\|^2 \]

Step 3: By analyzing the \( x^*_{k+1} - x^* \), we have

\[ \frac{1}{2\gamma}E_k \left\| \theta^*_1 z^*_{k+1} - \theta^*_1 x^* \right\|^2 \leq \frac{1}{2\gamma} \left\| \theta^*_1 z^*_{k+1} - \theta^*_1 x^* \right\|^2 - E_k \left( \xi^*_{k+1} + \gamma L \left( f(\tilde{x}^*) - f(w^*_{j(k)}) + (\nabla f(w^*_{j(k)})) \right) \right) \]

Step 4: By adding Eq. (101) and Eq. (102), and simplifying, we have

\[ E_k F(x^*_{k+1}) + \frac{1 + \frac{\eta L}{2\gamma}}{2\gamma} \left\| \theta^*_1 z^*_{k+1} - \theta^*_1 x^* \right\|^2 \leq a^k F(x^*_k) + \theta^*_1 F(x^*) + \theta^*_2 F(\tilde{x}^*) + \frac{1}{2\gamma} \left\| \theta^*_1 z^*_{k+1} - \theta^*_1 x^* \right\|^2 - \gamma \left( \frac{L}{2} - \frac{3\gamma L}{2} - \frac{C_3}{2} \right) \left\| x^*_{k+1} - x^*_k \right\|^2 - \left( \frac{\gamma L^2}{2C_3} + L \right) \left\| w^*_{j(k)} - y^*_{k} \right\|^2 \]

Step 5: We choose proper step size and obtain Theorem 3 in the paper.

**Proof of step 1:**

Proof: Through Eq. (98), we have that

\[ \theta^*_1 z^*_{k+1} + \theta^*_2 \tilde{x}^* = y^*_{k} - (1 - a^k)x^*_k, \quad k \geq 0 \]  

and

\[ \theta^*_1 z^*_{k+1} + \theta^*_2 \tilde{x}^* = x^*_{k+1} - (1 - a^k)x^*_k, \quad k \geq 0 \]  

Thus we obtain

\[ y^*_{k} = x^*_k + a^k(x^*_k - x^*_{k-1}), \quad k \geq 1. \]  

Eq. (106) is the same with Eq. (30), thus by the same proof, we can obtain that:

\[ y^*_{k} = x^*_{j(k)} + \sum_{i=j(k)+1}^{k} (x^*_i - y^*_{i-1}) + \sum_{i=j(k)+1}^{k} \left( \sum_{l=i}^{k} b^*(i, l) \right) (x^*_i - y^*_{i-1}) \]

\[ + \left( \sum_{i=j(k)}^{k} b^*(j(k), i) \right) (x^*_j(k) - x^*_j(k-1)), \quad k \geq 1. \]
Comparing Eq. (107) with the definition of \( w^{a}_{j(k)} \), we obtain the results.

**Proof of step 2:**

Through the optimal solution of \( z_{k+1}^{*} \) in Step 4 of Algorithm 5 in the paper, there exists \( \xi_{k+1}^{*} \in \partial h(z_{k+1}^{*}) \), satisfying

\[
\theta_{1}^{*}(z_{k+1}^{*} - z_{k}^{*}) + \gamma \nabla_{k}^{*} + \gamma \xi_{k+1}^{*} = 0. \tag{108}
\]

Through Eq. (104) and Eq. (105), eliminating \( x_{k}^{*} \), we have

\[
\theta_{1}^{*}(z_{k+1}^{*} - z_{k}^{*}) + x_{k+1}^{*} - y_{k}^{*} = 0.
\]

So we have

\[
x_{k+1}^{*} - y_{k}^{*} + \gamma \nabla_{k}^{*} + \gamma \xi_{k+1}^{*} = 0. \tag{109}
\]

For \( f(\cdot) \) has Lipschitz continuous gradients, we have

\[
f(x_{k+1}^{*}) \leq f(y_{k}^{*}) + \langle \nabla f(y_{k}^{*}), x_{k+1}^{*} - y_{k}^{*} \rangle + \frac{L}{2} \| x_{k+1}^{*} - y_{k}^{*} \|^2 \tag{110}
\]

\[
= f(y_{k}^{*}) - \gamma \langle \nabla f(y_{k}^{*}), \nabla_{k}^{*} + \xi_{k+1}^{*} \rangle + \frac{L}{2} \| x_{k+1}^{*} - y_{k}^{*} \|^2
\]

\[
\leq f(y_{k}^{*}) - \gamma \langle \nabla f(y_{k}^{*}), x_{k+1}^{*} - y_{k}^{*} \rangle - (\xi_{k+1}^{*}, x_{k+1}^{*} - y_{k}^{*})
\]

\[
\leq f(y_{k}^{*}) - \gamma \langle \nabla f(w^{a}_{j(k)}), x_{k+1}^{*} - y_{k}^{*} \rangle - (\xi_{k+1}^{*}, x_{k+1}^{*} - y_{k}^{*}),
\]

where in equality \( a \), we add and subtract the term \( \langle \nabla_{k}^{*} + \xi_{k}^{*}, \nabla_{k}^{*} + \xi_{k+1}^{*} \rangle \); equality \( b \) uses the equality Eq. (109); equality \( \leq \), we add and subtract \( \langle \nabla f(w^{a}_{j(k)}), x_{k+1}^{*} - y_{k}^{*} \rangle \).

For the third last term of Eq. (110), we have

\[
\mathbb{E}_{k} \langle \nabla_{k}^{*} - \nabla f(w^{a}_{j(k)}), y_{k}^{*} - x_{k+1}^{*} \rangle \tag{111}
\]

\[
\leq \frac{\gamma}{2C_{3}} \mathbb{E}_{k} \left\| \nabla_{k}^{*} - \nabla f(w^{a}_{j(k)}) \right\|^2 + \frac{\gamma C_{3}}{2} \mathbb{E}_{k} \left\| x_{k+1}^{*} - y_{k}^{*} \right\|^2
\]

\[
\leq \frac{\gamma L}{C_{3}} \left( f(x^{*}) - f(w^{a}_{j(k)}) + \langle \nabla f(w^{a}_{j(k)}), w^{a}_{j(k)} - x^{*} \rangle \right) + \frac{\gamma C_{3}}{2} \mathbb{E}_{k} \left\| x_{k+1}^{*} - y_{k}^{*} \right\|^2,
\]

where we use \( \mathbb{E}_{k} \) to denote that expectation is taken on the random number of \( i_{k}^{*} \) (step \( k \) and epoch \( s \)) under the condition that \( y_{k}^{*} \) and \( w^{a}_{j(k)} \) are known; in \( \leq \), we use the Cauchy-Schwarz inequality; \( \leq \) uses Eq. (97).

For the second last term of Eq. (110), we have

\[
\langle \nabla f(w^{a}_{j(k)}), y_{k}^{*} - x_{k+1}^{*} \rangle
\]

\[
\leq \frac{\gamma}{2C_{4}} \mathbb{E}_{k} \left\| \nabla f(w^{a}_{j(k)}) - \nabla f(y_{k}^{*}) \right\|^2 + \frac{\gamma C_{4}}{2} \left\| x_{k+1}^{*} - y_{k}^{*} \right\|^2
\]

\[
\leq \frac{\gamma L}{2C_{4}} \left\| w^{a}_{j(k)} - y_{k}^{*} \right\|^2 + \frac{\gamma C_{4}}{2} \left\| x_{k+1}^{*} - y_{k}^{*} \right\|^2,
\]

(112)
where in inequality \( \leq \), we use Cauchy-Schwarz inequality; in inequality \( \leq \), we use the fact that \( f(\cdot) \) has Lipschitz continuous gradients.

Taking expectation for Eq. (110) and Eq. (112) on the random number \( i_k^* \), and adding Eq. (111), we obtain the results.

**Proof of step 3:**

We have that

\[
\begin{align*}
\| \theta_1^* z_{k+1}^* - \theta_1^* x^* \|^2 \\
= & \| x_{k+1}^* - a^* x_k^* - \theta_2 \hat{x}^* - \theta_1^* x^* \|^2 \\
= & \| y_k^* - a^* x_k^* - \theta_2 \hat{x}^* - \theta_1^* x^* - (y_{k+1}^* - x_{k+1}^*) \|^2 \\
= & \| y_k^* - a^* x_k^* - \theta_2 \hat{x}^* - \theta_1^* x^* \|^2 + \| y_k^* - x_{k+1}^* \|^2 - 2\gamma \langle \xi_{k+1}^* + \tilde{\nabla}_k^*, y_k^* - a^* x_k^* - \theta_2 \hat{x}^* - \theta_1^* x^* \rangle \\
= & \| \theta_1^* z_k^* - \theta_1^* x^* \|^2 + \| y_k^* - x_{k+1}^* \|^2 - 2\gamma \langle \xi_{k+1}^* + \tilde{\nabla}_k^*, y_k^* - a^* x_k^* - \theta_2 \hat{x}^* - \theta_1^* x^* \rangle.
\end{align*}
\]

For the last term of Eq. (113), we have

\[
\begin{align*}
\mathbb{E}_k \langle \hat{\nabla}_k^*, a^* x_k^* + \theta_2 \hat{x}^* + \theta_1^* x^* - y_k^* \rangle \\
= & \mathbb{E}_k \langle \hat{\nabla}_k^*, a^* x_k^* + (\theta_2 - \gamma L C_4) \hat{x}^* + \theta_1^* x^* - (1 - \gamma L C_4) w_{j(k)}^* \rangle \\
+ & \mathbb{E}_k \langle \hat{\nabla}_k^*, w_{j(k)}^* - y_k^* + \gamma L C_4 (\hat{x}^* - w_{j(k)}^*) \rangle \\
= & \langle \nabla f(w_{j(k)}^*), a^* x_k^* + (\theta_2 - \gamma L C_4) \hat{x}^* + \theta_1^* x^* - (1 - \gamma L C_4) w_{j(k)}^* \rangle \\
+ & \langle \nabla f(w_{j(k)}^*), w_{j(k)}^* - y_k^* + \gamma L C_4 (\hat{x}^* - w_{j(k)}^*) \rangle \\
\leq & a^* f(x_k^*) + \theta_1^* f(x^*) - (1 - \theta_2) f(w_{j(k)}^*) \\
+ & \langle \nabla f(w_{j(k)}^*), w_{j(k)}^* - y_k^* \rangle + \langle \nabla f(w_{j(k)}^*), \theta_2 (\hat{x}^* - w_{j(k)}^*) \rangle.
\end{align*}
\]

where in inequality \( \leq \), we use the convexity of \( f(\cdot) \) and so for any vector \( u \),

\[
\langle \nabla f(w_{j(k)}^*), u - w_{j(k)}^* \rangle \leq f(u) - f(w_{j(k)}^*),
\]

and set that \( C_4 = \frac{\gamma L}{\theta_2} \). For \( f(w_{j(k)}^*) \), through the convexity of \( f(\cdot) \), we have

\[
-f(w_{j(k)}^*) \leq -f(y_k^*) + \langle \nabla f(y_k^*), y_k^* - w_{j(k)}^* \rangle,
\]

Adding Eq. (113) with Eq. (114), we have

\[
\begin{align*}
\mathbb{E}_k \langle \hat{\nabla}_k^*, (a^* x_k^* + \theta_2 \hat{x}^* + \theta_1^* x^* - y_k^*) \rangle \\
\leq & (1 - \theta_2 - \theta_1^*) f(x_k^*) + \theta_1^* f(x^*) - f(y_k^*) + \theta_2 f(w_{j(k)}^*) \\
+ & \langle \nabla f(w_{j(k)}^*), w_{j(k)}^* - y_k^* \rangle + \langle \nabla f(w_{j(k)}^*), \theta_2 (\hat{x}^* - w_{j(k)}^*) \rangle \\
\leq & (1 - \theta_2 - \theta_1^*) f(x_k^*) + \theta_1^* f(x^*) - f(y_k^*) + \theta_2 f(w_{j(k)}^*) \\
+ & L \| w_{j(k)}^* - y_k^* \|^2 + \langle \nabla f(w_{j(k)}^*), \theta_2 (\hat{x}^* - w_{j(k)}^*) \rangle,
\end{align*}
\]
where in $\leq$, we use the $\langle \nabla_k - \nabla f(y^*_k), w^*_{j(k)} - y^*_k \rangle \leq L \| w^*_{j(k)} - y^*_k \|^2$; Thus dividing Eq. (113) by $2\gamma$, and taking expectation on the random number of $i^*_k$, we have

$$
\frac{1}{2\gamma} E_k \| \theta^*_i z^*_{k+1} - \theta^*_i x^* \|^2
$$

(117)

$$
\leq \frac{1}{2\gamma} \| \theta^*_i z^*_{k} - \theta^*_i x^* \|^2 + \frac{\gamma}{2} E_k \| \frac{y^*_k - x^*_{k+1}}{\gamma} \|^2 - E_k \left( \xi^*_{k+1} + \tilde{\nabla} z^*, y^*_k - a^* x^*_k - \theta z^* - \theta^*_i x^* \right)
$$

$$
\leq \frac{1}{2\gamma} \| \theta^*_i z^*_{k} - \theta^*_i x^* \|^2 + \frac{\gamma}{2} E_k \| \frac{y^*_k - x^*_{k+1}}{\gamma} \|^2 - E_k \left( \xi^*_{k+1}, y^*_k - a^* x^*_k - \theta z^* - \theta^*_i x^* \right) + (1 - \theta - \theta^*_i) f(x^*_k) + \theta^*_i f(x^*) - f(y^*_k) + \theta f(w^*_{j(k)}) + L \| w^*_{j(k)} - y^*_k \|^2 + \langle \nabla f(w^*_{j(k)}), \theta (\tilde{x} - w^*_{j(k)}) \rangle,
$$

where $\leq$ uses Eq. (116). This is the result.

**Proof of step 4:**
Adding Eq. (117) and Eq. (101), we obtain the that:

$$
E_k f(x^*_k) + \frac{1}{2\gamma} \| \theta^*_i z^*_{k+1} - \theta^*_i x^* \|^2
$$

(118)

$$
\leq \ a^* f(x^*_k) + \theta^*_i f(x^*) + \theta f(w^*_{j(k)}) + \langle \nabla f(w^*_{j(k)}), \theta (\tilde{x} - w^*_{j(k)}) \rangle
$$

$$
- \gamma \left( \frac{1}{2} - \frac{L}{2} - \frac{C_3}{2} - \frac{C_4}{2} \right) E \left( \frac{x^*_{k+1} - y^*_k}{\gamma} \right)^2 - \left( \frac{\gamma L^2}{2 C_3} + L \right) \| w^*_{j(k)} - y^*_k \|^2
$$

$$
- E_k \left( \xi^*_{k+1}, x^*_{k+1} - y^*_k \right) + \theta f(x^*_{k+1}) + \langle \nabla f(w^*_{j(k)}), w^*_{j(k)} - \tilde{x}^* \rangle
$$

$$
- E_k \left( \xi^*_{k+1}, y^*_k - a^* x^*_k - \theta z^* - \theta^*_i x^* \right) + \frac{1}{2\gamma} \| \theta^*_i z^*_{k} - \theta^*_i x^* \|^2
$$

$$
\leq \ a^* f(x^*_k) + \theta^*_i f(x^*) + \theta f(w^*_{j(k)}) + \frac{1}{2\gamma} \| \theta^*_i z^*_{k} - \theta^*_i x^* \|^2
$$

$$
- \gamma \left( \frac{1}{2} - \frac{3L}{2} - \frac{C_3}{2} \right) E \left( \frac{x^*_{k+1} - y^*_k}{\gamma} \right)^2 - \left( \frac{\gamma L^2}{2 C_3} + L \right) \| w^*_{j(k)} - y^*_k \|^2
$$

$$
- E_k \left( \xi^*_{k+1}, x^*_{k+1} - a^* x^*_k - \theta z^* - \theta^*_i x^* \right),
$$

where in $\leq$, we use $C_4 = \frac{\gamma L}{2\gamma}$. For the last term of Eq. (118), we have

$$
- \left( \xi^*_{k+1}, \theta^*_i z^*_{k+1} - \theta^*_i x^* \right)
$$

(119)

$$
= - \left( \xi^*_{k+1}, \theta^*_i z^*_{k+1} - \theta^*_i x^* \right)
$$

$$
\leq \ \theta^*_i \ h(x^*) - \theta^*_i h(z^*_{k+1}) - \frac{\mu}{2} \| z^{k+1} - x^* \|^2
$$

$$
\leq \ \theta^*_i \ h(x^*) - h(x^*_k) + \theta^*_i h(\tilde{x}^*) + a^* h(x^*_k) - \frac{\mu}{2} \| z^{k+1} - x^* \|^2,
$$

where in $\leq$, we use $x^*_{k+1} = a^* x^*_k + \theta z^* + \theta^*_i x^*_{k+1}$, and the convexity of $h(\cdot)$. Substituting Eq. (119) into Eq. (118), we obtain the result.

**Proof of step 5:**
\[ \left\| w^{(k)} - y^k \right\|^2 \leq \left( \sum_{i=j(k)+1}^{\min(\tau, k)} \left( 1 + \sum_{l=1}^{\frac{k-1+i}{2}} \right) \right) \sum_{i=j(k)+1}^{\min(\tau, k)} \left( 1 + \sum_{l=1}^{\frac{k-1+i}{2}} \right) \left\| x^i - y^{i-1} \right\|^2 \]

Taking expectation on Eq. (103) for the first and summing it with \( k \) technique of Eq. (45), summing Eq. (120) with \( k \) uses Eq. (121). By setting \( C = a \), we use \( \tau \) when \( \tau \leq a \). We first consider the not-strongly convex case. Using the same technique of Eq. (120), summing Eq. (120) with \( k \) to \( m - 1 \), we have

\[ \sum_{k=0}^{m-1} \left\| w^{(k)} - y^k \right\|^2 \leq 4\tau \sum_{k=0}^{m-1} \sum_{i=1}^{\min(\tau, k-\tau)} \left\| x^{k-i+1} - y^{k-i} \right\|^2 \]

where in \( \leq \), we use \( b(i, l) \leq \frac{1}{2} \) when \( l \geq i \). We first consider the not-strongly convex case. Using the same technique of Eq. (45), summing Eq. (120) with \( k \) to \( m - 1 \), we have

\[ \sum_{k=0}^{m-1} \left\| w^{(k)} - y^k \right\|^2 \leq 4\tau \sum_{k=0}^{m-1} \sum_{i=1}^{\min(\tau, k-\tau)} \left\| x^{k-i+1} - y^{k-i} \right\|^2 \]

Taking expectation on Eq. (103) for the first \( k - 1 \) iteration (all the random numbers coming from epoch \( s \)), and summing it with \( k = 0 \) to \( m - 1 \), we have

\[ \sum_{k=0}^{m-1} \mathbb{E} \left( F(x^{s}_{k+1}) - F(x^*) \right) + \frac{1}{2\gamma} \mathbb{E} \left\| \theta^{s}_{1}z^{s}_{m} - \theta^{s}_{1}x^{*} \right\|^2 \]

\[ \leq \sum_{k=0}^{m-1} a^s \mathbb{E} \left( F(x^{s}_{k}) - F(x^*) \right) + m\theta_2 (F(x^s) - F(x^*)) + \frac{1}{2\gamma} \mathbb{E} \left\| \theta^{s}_{1}z^{s}_{m} - \theta^{s}_{1}x^{*} \right\|^2 \]

\[ -\gamma \left( \frac{1}{2} - \frac{3\gamma L}{2} - C_3 \right) \sum_{k=0}^{m-1} \mathbb{E} \left\| x_{k+1}^{s} - y_k^{s} \right\|^2 + \left( \frac{\gamma L^2}{2C_3} + L \right) \sum_{k=0}^{m-1} \left\| w_k^{(k)} - y_k^{s} \right\|^2 \]

\[ \leq \sum_{k=0}^{m-1} a^s \mathbb{E} \left( F(x^{s}_{k}) - F(x^*) \right) + m\theta_2 (F(x^s) - F(x^*)) + \frac{1}{2\gamma} \mathbb{E} \left\| \theta^{s}_{1}z^{s}_{m} - \theta^{s}_{1}x^{*} \right\|^2 \]

\[ -\gamma \left( \frac{1}{2} - \frac{3\gamma L}{2} - C_3 \right) - 4\tau^2 \left( \frac{\gamma L^2}{2C_3} \right) \sum_{k=0}^{m-1} \mathbb{E} \left\| x_{k+1}^{s} - y_k^{s} \right\|^2 \],

where \( \leq \) uses Eq. (121). By setting \( C_3 = 2\gamma L \), we obtain the that

\[ 1 - \frac{3\gamma L}{2} - C_3 - 4\tau^2 \left( \frac{\gamma L^2}{2C_3} \right) \leq 0 \]
The rest proof is similar to [Allen-Zhu 2017]. Diving \((\theta_1^s)^2\) on both side of Eq. \((123)\) and arranging terms, we have

\[
\frac{1}{(\theta_1^s)^2} \mathbb{E} (F(x_m^s) - F(x^*)) + \frac{\theta_2 + \theta_1^s}{(\theta_1^s)^2} \sum_{k=1}^{m-1} \mathbb{E} (F(x_k^s) - F(x^*))
\]

\[
\leq \frac{1 - \theta_1^s - \theta_2}{(\theta_1^s)^2} (F(x_0^s) - F(x^*)) + \frac{\theta_2}{(\theta_1^s)^2} (F(x^s) - F(x^*))
\]

\[
+ \frac{1}{2\gamma} \|z_0^s - x^*\|^2 - \frac{1}{2\gamma} \mathbb{E} \|z_m^s - x^*\|^2.
\]

When \(s > 0\), through the definition of \(\tilde{x}^s\), we have

\[
F(\tilde{x}^s) = F\left(\frac{1}{m} \sum_{k=0}^{m-1} x_k^{s-1}\right) \leq \frac{1}{m} \sum_{k=0}^{m-1} F(x_k^{s-1}) = \frac{1}{m} F(x_0^s) + \frac{1}{m} \sum_{k=1}^{m-1} F(x_k^{s-1}).
\]

Through the definition of \(z_0^s\), we have

\[
z_0^s = z_m^{s-1}.
\]

Substituting Eq. \((126)\) and Eq. \((127)\) into Eq. \((125)\), we have

\[
\frac{1}{(\theta_1^s)^2} \mathbb{E} (F(x_m^s) - F(x^*)) + \frac{\theta_2 + \theta_1^s}{(\theta_1^s)^2} \sum_{k=1}^{m-1} \mathbb{E} (F(x_k^s) - F(x^*))
\]

\[
\leq \frac{1 - \theta_1^s - \theta_2}{(\theta_1^s)^2} (F(x_0^s) - F(x^*)) + \frac{\theta_2}{(\theta_1^s)^2} \sum_{k=1}^{m-1} (F(x_k^{s-1}) - F(x^*))
\]

\[
+ \frac{1}{2\gamma} \|z_0^s - x^*\|^2 - \frac{1}{2\gamma} \mathbb{E} \|z_0^{s+1} - x^*\|^2, \quad s > 0.
\]

Since \(\theta_1^s = \frac{2}{s+4} \leq \frac{1}{2}\), we have

\[
\frac{1}{(\theta_1^s)^2} \geq \frac{1 - \theta_1^{s+1}}{(\theta_1^{s+1})^2}, \quad s \geq 0,
\]

and

\[
\frac{\theta_2 + \theta_1^s}{(\theta_1^s)^2} \geq \frac{\theta_2}{(\theta_1^{s+1})^2}, \quad s \geq 0.
\]

So

\[
\frac{1}{(\theta_1^s)^2} \mathbb{E} (F(x_m^s) - F(x^*)) + \frac{\theta_2 + \theta_1^s}{(\theta_1^s)^2} \sum_{k=1}^{m-1} \mathbb{E} (F(x_k^s) - F(x^*))
\]

\[
\leq \frac{1}{(\theta_1^{s+1})^2} (F(x_m^{s-1}) - F(x^*)) + \frac{\theta_2 + \theta_1^{s-1}}{(\theta_1^{s-1})^2} \sum_{k=1}^{m-1} (F(x_k^{s-1}) - F(x^*))
\]

\[
+ \frac{1}{2\gamma} \|z_0^s - x^*\|^2 - \frac{1}{2\gamma} \mathbb{E} \|z_0^{s+1} - x^*\|^2, \quad s > 0.
\]
When \( s = 0 \), through Eq. (125), use \( \tilde{x}^0 = x^0_0 \), we have
\[
\frac{1}{(\theta_1^2)} \mathbb{E} (F(x^0_m) - F(x^*)) + \frac{\theta_2 + \theta_1^0}{(\theta_1^2)} \sum_{k=1}^{m-1} \mathbb{E} (F(x^k) - F(x^*)) \leq 1 - \theta_1 + (m - 1) \theta_2 (F(x^0) - F(x^*))
\]
\[
+ \frac{1}{2\gamma} \|z_0^* - x^*\|^2 - \frac{1}{2\gamma} \mathbb{E} \|z_{s+1}^* - x^*\|^2.
\]
Taking expectation for Eq. (131) with \( s \) from 1 to \( S \) (random numbers coming from the 0 to \( s - 1 \) epochs) and summing the result with \( S \) from 1 to \( S - 1 \), and adding Eq. (132), we obtain
\[
\frac{1}{(\theta_1^2)} \mathbb{E} (F(x^S_m) - F(x^*)) + \frac{\theta_2 + \theta_1^0}{(\theta_1^2)} \sum_{k=1}^{m-1} \mathbb{E} (F(x^k) - F(x^*)) \leq 1 - \theta_1 + (m - 1) \theta_2 (F(x^0) - F(x^*))
\]
\[
+ \frac{1}{2\gamma} \|z_0^* - x^*\|^2 - \frac{1}{2\gamma} \mathbb{E} \|z_{s+1}^* - x^*\|^2
\]
\[
\leq 2m (F(x^0) - F(x^*)) + \frac{1}{2\gamma} \|z_0^* - x^*\|^2,
\]
where in \( \leq \), we use \( \theta_1^0 = \theta_2 = \frac{1}{2} \).

Now we consider the strongly convex case. Through the definition of \( \gamma \), we have
\[
8\gamma L r^2 \leq 5\gamma L + \frac{95}{8} \gamma L r^2 \leq 1,
\]
and \( \theta^*_1 = \frac{1}{\gamma} \sqrt{\frac{\beta}{F}} \). Set \( \theta_3 = \frac{\theta^*_1 \gamma}{2} + 1 \leq \frac{1}{8\gamma} \sqrt{\frac{\beta}{F}} + 1 \). Multiply Eq. (120) with \( \theta_3 \), and summing the results with \( k \) from 0 to \( m - 1 \), we have
\[
\sum_{k=0}^{K} \theta_3^k \|w^{(k)} - y^k\|^2 \leq \sum_{k=0}^{K} \frac{4\tau \theta_3}{4} \|x^{k-i} - y^{k-i}\|^2
\]
\[
\leq \sum_{k=0}^{K} \frac{4\tau}{4} \|x^{k-i} - y^{k-i}\|^2
\]
\[
\leq \sum_{k=0}^{K} \frac{4\tau}{4} \|x^{k-i} - y^{k-i}\|^2
\]
\[
\leq \frac{a}{b} \sum_{k=0}^{K} \frac{4\tau}{4} \|x^{k-i} - y^{k-i}\|^2
\]
\[
\leq \frac{19\gamma^2}{4} \sum_{k=0}^{K} \theta_3^k \|x^{k+1} - y^k\|^2 \leq \frac{19\gamma^2}{4} \sum_{k=0}^{K} \theta_3^k \|x^{k+1} - y^k\|^2,
\]
where \( \leq \) is because that for each \( \theta_3^k \|x^{k+1} - y^k\|^2 \) \((1 \leq k \leq K)\) there are most \( \tau \) terms, like Eq. (47): in \( \leq \), we use the fact that for the function \( g(x) = (1 + x)^a \leq 1 + \frac{3}{2} ax \), when \( a \leq 1 \), and \( x \leq \frac{1}{a} \). To prove it, we can use Taylor expansion at point \( x = 0 \) to obtain
\[
(1 + x)^a = 1 + ax + \frac{a(a - 1)}{2} x^2 \leq 1 + ax + \frac{a(a - 1)}{2} \frac{1}{a} x \leq 1 + \frac{3}{2} ax,
\]
\[
\text{(136)}
\]
where \( \xi \in [0, x] \), and

\[
\theta_3^* \leq (1 + \frac{1}{8\tau} \sqrt{\frac{\mu}{nL}})^7 \leq (1 + \frac{1}{8\tau} \frac{\tau}{n})^7 \leq 1 + \frac{3\tau}{16n} \leq \frac{19}{16},
\]

where in \( a \), we use the assumption that \( n\mu \leq \tau^2 L \); and in \( b \), we use \( \tau \leq n \). Taking expectation on Eq. (137) for the first \( k - 1 \) iterations, and then multiply it with \( \theta_3^* \), and summing the results with \( k \) from 0 to \( m \), we have

\[
\sum_{k=1}^{m} \theta_3^{k-1} (F(x_k^*) - F(x^*)) - (a^*) \sum_{k=0}^{m-1} \theta_3^k (F(x_k^*) - F(x^*)) - \theta_2 \sum_{k=0}^{m-1} \theta_3^k (F(x_k^*) - F(x^*)) \\
+ \frac{1}{2\gamma} ||\theta_3^* z_0^* - \theta_3^* x^*||^2 - \frac{\theta_3^m}{2\gamma} ||\theta_3^* z_m^* - \theta_3^* x^*||^2 z_0^*
\]

\[
\leq -\gamma (1 - \frac{3\gamma L}{2} - \frac{C_3}{2}) \sum_{k=0}^{m-1} \theta_3^k \|x_{k+1} - y_k^*\|^2 \gamma - \sum_{k=0}^{m-1} \left( \frac{\gamma \sqrt{L^2 + L}}{2C_3} \right) \|w_{j(k)} - y_k^*\|^2
\]

\[
\leq -\gamma \left( \frac{1}{2} - \frac{3\gamma L}{2} - \frac{C_3}{2} - \left( \frac{\gamma \sqrt{L^2 + L}}{2C_3} \right) \frac{19\tau^2}{4} \right) \sum_{k=0}^{m-1} \theta_3^k \|x_{k+1} - y_k^*\|^2. \tag{137}
\]

Set \( C_3 = 2\gamma L \), we have we have

\[
\frac{1}{2} - \frac{5\gamma L}{2} - \frac{95\tau^2\gamma^2 L^2}{16} \geq 0.
\]

The rest proof is similar to \([Allen-Zhu 2017]\). By arranging the terms of Eq. (137), we have

\[
(\theta_3^* + \theta_2 - (1 - 1/\theta_3)) \sum_{i=1}^{m} \theta_3^i (F(x_k^*) - F(x^*)) + \theta_3^m a^* (F(x_k^*) - F(x^*))
\]

\[
\leq \theta_2 \sum_{k=0}^{m-1} \theta_3^k (F(x_k^*) - F(x^*)) + a^* (F(x_k^*) - F(x^*))
\]

\[
+ \frac{1}{2\gamma} ||\theta_3^* z_0^* - \theta_3^* x^*||^2 - \frac{\theta_3^m}{2\gamma} ||\theta_3^* z_m^* - \theta_3^* x^*||^2. \tag{138}
\]

Through the definition of \( \bar{x}^{s+1} = (\sum_{j=1}^{m=1} \theta_3^j)^{-1} \sum_{j=0}^{m-1} x_j^s \theta_3^j \), we have

\[
(\theta_3^* + \theta_2 - (1 - 1/\theta_3)) \theta_3 \sum_{i=1}^{m-1} \theta_3^i (F(\bar{x}^{s+1}) - F(x^*)) + \theta_3^m a^* (F(x_k^*) - F(x^*))
\]

\[
\leq \theta_2 \sum_{k=0}^{m-1} \theta_3^k (F(\bar{x}^s) - F(x^*)) + a^* (F(x_k^*) - F(x^*))
\]

\[
+ \frac{1}{2\gamma} ||\theta_3^* z_0^* - \theta_3^* x^*||^2 - \frac{\theta_3^m}{2\gamma} ||\theta_3^* z_m^* - \theta_3^* x^*||^2. \tag{139}
\]

Since

\[
\theta_2(\theta_3^{m-1} - 1) + (1 - 1/\theta_3)
\]

\[
\leq \frac{1}{2} \left( \left( 1 + \frac{1}{8\tau} \sqrt{\frac{\mu}{nL}} \right)^{m-1} - 1 \right) + \frac{1}{8\tau} \sqrt{\frac{L}{\theta_3^m}}
\]

\[
\leq \frac{3}{4} \frac{\sqrt{\tau n\mu}}{L} + \frac{1}{8\tau} \sqrt{\frac{\tau n\mu}{\theta_3}}
\]

\[
\leq \frac{1}{\tau} \sqrt{\frac{\tau n\mu}{L}} = \theta_3^*, \tag{140}
\]

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where in \( \leq \), we use \( \mu n \leq Lr^2 \), and Eq. (136). Eq. (140) indicates that \( (\theta_1^* + \theta_2 - (1 - 1/\theta_3)) \leq \theta_2 \theta_3^{-m-1} \), so

\[
\theta_2 \theta_3^{-m} \sum_{k=0}^{m-1} \theta_3^k (F(\hat{x}^{s+1}) - F(x^*)) + \theta_3^m a^s (F(x_m^*) - F(x^*))
\]

\[
\leq \theta_2 \sum_{k=0}^{m-1} \theta_3^k (F(\hat{x}^s) - F(x^*)) + a^s (F(x_m^*) - F(x^*))
\]

\[
+ \frac{\gamma}{2} \|\theta_1^* z_0^* - \theta_1^* x^*\|^2 \leq \frac{\theta_3^m \gamma}{2} \|\theta_1^* z_m^* - \theta_1^* x^*\|^2 .
\]

By telescope the above inequality from \( s = 0, \cdots, S \), we have that

\[
\theta_2 \sum_{k=0}^{m-1} \theta_3^k (F(\hat{x}^{s+1}) - F(x^*)) + (1 - \theta_1^* - \theta_2) (F(x_m^*) - F(x^*))
\]

\[
\leq (\theta_3)^{-Sn} \left( \theta_2 \sum_{k=0}^{m-1} \theta_3^k + (1 - \theta_1^* - \theta_2) \right) (F(x^0_0) - F(x^*)) + \frac{(\theta_1^*)^2 \gamma}{2} \|z_0^0 - x^*\|^2 \right).
\]

Since \( \theta_3^k \geq 1 \), and \( \theta_2 = \frac{1}{2} \), and so \( \sum_{k=0}^{m-1} \theta_3^k \geq n, \theta_1^* \leq \frac{1}{2} \), we have

\[
(F(\hat{x}^{S+1}) - F(x^*)) \leq (\theta_3)^{-Sn} \left( (1 + \frac{1}{n}) (F(x^0_0) - F(x^*)) + \frac{\gamma}{4n} \|z_0^0 - x^*\|^2 \right).
\]

This ends proof.

**Lemma 3** Suppose \( f(x) \) has Lipschitz continuous gradients, for any \( x \) and \( y \), we have

\[
\|\nabla f(x) - \nabla f(y)\|^2 \leq 2L (f(x) - f(y) + \langle \nabla f(y), y - x \rangle).
\]

Lemma [3] is Theorem 2.1.5 of the textbook of Nesterov (Nesterov 2013).

**Proof of Lemma 3**

\[
E \left( \|\nabla f(u) - \nabla f(u)\|^2 \right) = E \left( \|\nabla f_k(u) - \nabla f_k(\hat{x}) + \nabla f(\hat{x}) - \nabla f(u)\|^2 \right)
\]

\[
= E \left( \|\nabla f_k(u) - \nabla f_k(\hat{x}) - (\nabla f(u) - \nabla f(\hat{x}))\|^2 \right)
\]

\[
\leq E \left( \|\nabla f_k(u) - \nabla f_k(\hat{x})\|^2 \right) \]

where in inequality \( \leq \), we use that

\[
E (\nabla f_k(u) - \nabla f_k(\hat{x})) = \nabla f(u) - \nabla f(\hat{x}),
\]

and \( E (\|x - E(x)\|^2) = E \|x\|^2 - \|E(x)\|^2 \leq E \|x\|^2 \). Then by directly applying Lemma [3] to Eq. (145), we obtain Eq. (97).

### 7.4 ASVRG

The algorithm of ASVRG is shown in Algorithm [5]. We analyze ASVRG in the wild scheme. For the wild scheme, there is no locks in implementation. So different coordinates of \( x \) read by any child node may at
We can find that atom scheme also satisfies Eq. (146). Now we begin our proof. The proof can be also
represented as an extension of (Cong & Lin, 2017). Lemma 4 is similar to Lemma 1 in (Cong & Lin, 2017). The
proof is as follows:

\[ I_{k(l)}^{s_j} = \mathbf{1} \iff v_{j(l)}^s \text{ has been returned}, \]
\[ v_{j(l)}^s(p), \quad \text{otherwise.} \quad (147) \]

We can find that atom scheme also satisfies Eq. (146). Now we begin our proof. The proof can be also
considered as an extension of (Cong & Lin, 2017). Lemma 4 is similar to Lemma 1 in (Cong & Lin, 2017). The
variant that we adopt is to consider the term \( E(\| \nabla f(x_k^*) \|^2) + 4LE (f(x_k^*) + f(x^*) - 2f(x^*)) \). For simply,
we assume \( h(x) = 0 \). We first prove Lemma 4.

**Lemma 4** Suppose \( f(x) \) has Lipschitz continuous gradients, for ASVRG, if the step size \( \gamma \) satisfies

\[ \gamma \leq \min \left\{ \frac{1}{L} \left( \frac{\rho_1 - 1}{10\rho_1 \sqrt{\rho_2}} \right) \frac{\rho_2 - 1}{10\rho_1^2 \rho_2 \rho_2^2 - \rho_1 - 1} \right\}, \quad (148) \]

for some \( \rho_1 > 1 \) and \( \rho_2 > 1 \), then for any \( s \geq 0 \) and \( k \geq 0 \), we have

\[ E(\| \nabla f(x_k^*) \|^2) + 4LE (f(x_k^*) + f(x^*) - 2f(x^*)) \]
\[ \leq \rho_1 \left[ E(\| \nabla f(x_{k+1}^*) \|^2) + 4LE (f(x_{k+1}^*) + f(x^*) - 2f(x^*)) \right], \quad (149) \]

and

\[ E(\| \nabla f(x_{j(k)}^*) \|^2) + 4LE (f(x_{j(k)}^*) + f(x^*) - 2f(x^*)) \]
\[ \leq \rho_2 E(\| \nabla f(x_k^*) \|^2) + 4LE (f(x_k^*) + f(x^*) - 2f(x^*)). \quad (150) \]
Proof of Lemma 4
We analyze \( \| \nabla f(x_k^*) \|^2 \) and \( f(x_k^*) - f(x^*) + f(\tilde{x}^*) - f(x^*) \), respectively. For \( \| \nabla f(x_k^*) \|^2 \), we have

\[
\mathbb{E} \left( \| \nabla f(x_k^*) \|^2 - \| \nabla f(x_{k+1}^*) \|^2 \right) \\
\leq^a 2 \mathbb{E} \left( \| \nabla f(x_k^*) \| \| \nabla f(x_k^*) - \nabla f(x_{k+1}^*) \| \right) \\
\leq^b 2 L \mathbb{E} \left( \| \nabla f(x_k^*) \| \| x_k^* - x_{k+1}^* \| \right) \\
\leq^c L \gamma \mathbb{E} \left( \frac{1}{C_3^4} \| \nabla f(x_k^*) \|^2 + C_3 \| \nabla f(x_j(k)) \|^2 \right) , \quad (C_3 \geq 0). (151)
\]

where in inequality \( \leq^a \), we use the equality that \( \| a \|^2 - \| b \|^2 \leq 2 \| a \| | a - b | \); inequality \( \leq^b \) uses the fact that \( f(\cdot) \) is \( L \)-smooth; inequality \( \leq^c \) uses the Cauchy-Schwarz inequality. For \( f(x_k^*) - f(x^*) + f(\tilde{x}^*) - f(x^*) \), we have

\[
\mathbb{E} \left( f(x_k^*) - f(x^*) + f(\tilde{x}^*) - f(x^*) \right) - \mathbb{E} \left( f(x_{k+1}^*) - f(x^*) + f(\tilde{x}^*) - f(x^*) \right) \\
= \mathbb{E} \left( f(x_k^*) - f(x_{k+1}^*) \right) \\
\leq^a \mathbb{E} \left( \langle \nabla f(x_k^*), x_k^* - x_{k+1}^* \rangle \right) \\
\leq^b \frac{1}{C_3^4} \mathbb{E} \| \nabla f(x_k^*) \|^2 + \gamma C_3 \mathbb{E} \| \nabla f(x_j(k)) \|^2 , \quad (152)
\]

where in inequality \( \leq^a \) we use the convexity of \( f(\cdot) \); inequality \( \leq^b \) uses the Cauchy-Schwarz inequality. Then similarly, we have

\[
\mathbb{E} \left( \| \nabla f(x_j^*(k+1)) \|^2 - \| \nabla f(x_{j+1}^*(k+1)) \|^2 \right) \leq 2 \mathbb{E} \left( \| \nabla f(x_j^*(k+1)) \| \| \nabla f(x_j^*(k+1)) - \nabla f(x_{j+1}^*(k+1)) \| \right) \\
\leq \frac{L \gamma}{C_4} \mathbb{E} \left( \| \nabla f(x_j^*(k+1)) \|^2 \right) + \frac{L C_4}{\gamma} \mathbb{E} \left( \| x_j^*(k+1) - x_{j+1}^*(k+1) \|^2 \right) \quad (C_4 > 0), \quad (153)
\]

and

\[
\mathbb{E} \left( f(x_j^*(k)) - f(x^*) + f(\tilde{x}^*) - f(x^*) \right) - \mathbb{E} \left( f(x_{k+1}^*) - f(x^*) + f(\tilde{x}^*) - f(x^*) \right) \\
= \mathbb{E} \left( f(x_j^*(k)) - f(x_{k+1}^*) \right) \\
\leq^a \mathbb{E} \left( \langle \nabla f(x_j^*(k)), x_j^*(k) - x_{k+1}^* \rangle \right) \\
\leq^b \frac{1}{C_4} \mathbb{E} \| \nabla f(x_j^*(k)) \|^2 + \gamma C_4 \| x_{k+1}^* - x_j^*(k+1) \|^2 . \quad (154)
\]

For convenience, we set

\[
B_k = \mathbb{E} \| \nabla f(x_{k-1}^*) \|^2 + 4 LE \left( f(x_{k-1}^*) - f(x^*) + f(\tilde{x}^*) - f(x^*) \right) , \quad (155)
\]

which has omitted the superscript \( s \). For the fact that \( \mathbb{E} \| x \|^2 = (\mathbb{E} \| x \|)^2 + D(x) \), we have

\[
\mathbb{E} \left( \| \nabla f(x_j^*(k)) \|^2 \right) = \| \nabla f(x_j^*(k)) \|^2 + \mathbb{E} \left( \| \nabla f(x_j^*(k)) - \nabla f(x_j^*(k)) \|^2 \right). \quad (156)
\]

Then from Lemma 2 we have

\[
\mathbb{E} \left( \| \nabla f(x_k^*) \|^2 \right) \leq B_k. \quad (157)
\]
Multiplying Eq. (152) by $4L$ and adding Eq. (202), we obtain

$$B_k - B_{k+1} \leq 5L\gamma\mathbb{E}\left(\frac{1}{C_3} \left\| \nabla f(x_k^*) \right\|^2 + C_3 \left\| \nabla f(x_{j(k)}) \right\|^2 \right) \leq 5L\gamma \frac{1}{C_3} B_k + 5L\gamma C_3 B_{j(k)}. \quad (158)$$

Now we use induction to prove $B_{k-1} \leq \rho_1 B_k$, and $B_{j(k)} \leq \rho_2 B_k$. Suppose $k = 1$, $B_{j(k)} = B_0$, we have

$$B_0 - B_1 \leq 10L\gamma B_0, \quad (159)$$

where we set $C_3 = 1$. Simplifying Eq. (159), we have

$$B_0 \leq \frac{1}{1 - 10L\gamma} B_1. \quad (160)$$

Recalling the $\gamma$, we have

$$L\gamma \leq \frac{\rho_1 - 1}{10\rho_1 \sqrt{\rho_2}} \leq \frac{\rho_1 - 1}{10\rho_1} = \frac{1}{10} \left(1 - \frac{1}{\rho_1}\right). \quad (161)$$

so

$$B_0 \leq \frac{1}{1 - 10L\gamma} B_1 \leq \rho_1 B_1. \quad (162)$$

On the other hand, multiplying Eq. (154) by $4L$ and then adding Eq. (153), we have

$$B_{j(k+1)} - B_{k+1} \leq \frac{5L\gamma}{C_4} B_{j(k+1)} + \frac{5LC_4}{\gamma} \mathbb{E}\left(\left\| x_{k+1}^s - x_{j(k+1)}^s \right\|^2 \right). \quad (163)$$

When $k = 1$,

$$\mathbb{E}\left(\left\| x_1^* - x_{j(1)}^* \right\|^2 \right) \leq \gamma^2 \mathbb{E}\left(\left\| I_{0(0)}(\nabla f(x_0^*)) \right\|^2 \right) \leq \gamma^2 \mathbb{E}\left(\left\| \nabla f(x_0^*) \right\|^2 \right) \leq \gamma^2 \rho_1 B_1, \quad (164)$$

where in inequality $\leq$ we use the definition of $x_{j(k)}^s$ in Eq. (146) and Eq. (147). Substituting Eq. (164) into Eq. (163), we have

$$B_{j(1)} - B_1 \leq \frac{5L\gamma}{C_4} B_{j(1)} + 5\gamma LC_4 \rho_1 B_1. \quad (165)$$

Setting $C_4$ to be $\frac{1}{\sqrt{\rho_1}}$,

$$B_{j(1)} - B_1 \leq 5\sqrt{\rho_1} L\gamma B_{j(1)} + 5\sqrt{\rho_1} L\gamma B_1. \quad (166)$$

Then

$$B_{j(1)} \leq \frac{1 + 5\sqrt{\rho_1} L\gamma}{1 - 5\sqrt{\rho_1} L\gamma} B_1. \quad (167)$$
Recalling the assumption on $\gamma$, we have
\[
L\gamma \leq \frac{\rho_2 - 1}{10\sqrt{\rho_1\rho_2}} \leq \frac{\rho_2 - 1}{10\sqrt{\rho_1\rho_2}},
\]
(168)
So we have
\[
10\sqrt{\rho_1}L\gamma \leq 1 - \frac{1}{\rho_2} < 1.
\]
(169)
Then
\[
B_{J(1)} \leq 1 + 5\sqrt{\rho_1}L\gamma B_1
\]
\[
\leq 1 - 10\sqrt{\rho_1}L\gamma B_1
\]
\[
\leq \rho_2 B_1
\]
(170)
where we use the fact that $\frac{1+x}{1+2x} \leq \frac{1}{1+2x}$ when $2x < 1$ in the second inequality.

When $B_k$ satisfies $B_{k-1} \leq \rho_1 B_k$, and $B_{J(k)} \leq \rho_2 B_k$, we consider $B_{k+1}$. From Eq. (158),
\[
B_k - B_{k+1} \leq 5L\gamma B_k + 5LC_3\gamma\rho_2 B_k.
\]
(171)
Setting $C_3 = \frac{1}{\sqrt{\rho_2}}$, we have
\[
B_k - B_{k+1} \leq 10\sqrt{\rho_2}L\gamma B_k.
\]
(172)
Then
\[
B_k \leq \frac{1}{1 - 10\sqrt{\rho_2}L\gamma} B_{k+1}.
\]
(173)
From the assumption on $\gamma$, we have $B_k \leq \rho_1 B_{k+1}$. The same as Eq. (28) and Eq. (29) in [Cong & Lin 2017], we have
\[
E\left(\left\|x_{k+1}^s - x_{J(k+1)}^s\right\|^2\right)
= \gamma^2 E\left(\left\|\sum_{l=k-\tau+1}^{k} I_{k(l)} \left(\nabla f(x_{J(l)}^s)\right)\right\|^2\right)
\leq \gamma^2 E\left(\sum_{p=1}^{d} \left(\sum_{l=k-\tau+1}^{k} |\nabla f(x_{J(l)}^s)(p)|\right)^2\right).
\]
(174)
where $\nabla f(x_k^s)(p)$ is the $p$-th coordinate of vector $\nabla f(x_k^s)$, the first inequality uses the inequality that $(a_1 + a_2 + \cdots + a_r)^2 \leq (|a_1| + |a_2| + \cdots + |a_r|)^2$ on each dimension. For any $i = 0, 1, \ldots, \tau - 1$ and
\[ z = 0, 1, \ldots, \tau - 1, \] we have

\[
E \left( \sum_{\rho = 1}^{d} \left( 2|\nabla f(x_{j(k-i)}^{\rho})(p)| \times |\nabla f(x_{j(k-z)}^{\rho})(p)| \right) \right)
\leq E \left( \sum_{\rho = 1}^{d} \left( \rho_{1}^{(z-i)/2} |\nabla f(x_{j(k-i)}^{\rho})(p)|^{2} + \rho_{1}^{(i-z)/2} |\nabla f(x_{j(k-z)}^{\rho})(p)|^{2} \right) \right)
\]

\[ \leq \rho_{1}^{(z-i)/2} \rho_{1} B_{j(k-i)} + \rho_{1}^{(i-z)/2} \rho_{1} B_{j(k-z)} \]

\[ \leq \rho_{2} \rho_{1}^{(z-i)/2} \rho_{1} B_{k} + \rho_{2} \rho_{1}^{(i-z)/2} \rho_{1} B_{k} \]

\[ \leq 2 \rho_{2} \rho_{1}^{(i+z)/2} B_{k}, \]

where in inequality \( \leq \), we use Cauchy-Schwarz. So

\[
E \left( \left\| x_{k+1}^{\gamma} - x_{j(k+1)}^{\gamma} \right\|^{2} \right)
\leq \gamma^{2} \rho_{2} \sum_{i=0}^{\tau-1} \sum_{z=0}^{\tau-1} \rho_{1}^{(i+z)/2} B_{k}
\]

\[ \leq \gamma^{2} \rho_{2} \left( \sum_{i=0}^{\tau-1} \rho_{1}^{i/2} \right) B_{k}
\]

\[ \leq \gamma^{2} \rho_{2} \frac{(\rho_{1}^{\tau/2} - 1)^{2}}{(\sqrt{\rho_{1}} - 1)^{2}} B_{k}. \]

Substituting Eq. (176) into Eq. (163), we have

\[ B_{j(k+1)} - B_{k+1} \]

\[ \leq \frac{5L\gamma}{C_{4}} B_{j(k+1)} + 5LC_{4}\gamma \rho_{2} \frac{(\rho_{1}^{\tau/2} - 1)^{2}}{(\sqrt{\rho_{1}} - 1)^{2}} B_{k}
\]

\[ \leq \frac{5L\gamma}{C_{4}} B_{j(k+1)} + 5LC_{4}\gamma \rho_{2} \rho_{1} \frac{(\rho_{1}^{\tau/2} - 1)^{2}}{(\sqrt{\rho_{1}} - 1)^{2}} B_{k+1}. \]

Setting \( C_{4} = \frac{1}{\sqrt{\rho_{1}^{\tau/2} - 1}} \), we have

\[ B_{j(k+1)} - B_{k+1} \leq 5L\gamma \sqrt{\rho_{1}^{\rho_{2}} \rho_{1}^{\tau/2} - 1} (B_{j(k+1)} + B_{k+1}). \]

Considering the assumption on \( \gamma \), like Eq. (168), we have

\[ 10L\gamma \sqrt{\rho_{1}^{\rho_{2}} \rho_{1}^{\tau/2} - 1} \leq 1 - \frac{1}{\rho_{2}} < 1, \]

(179)
then like Eq. (170), we have

\[
B_{j(k+1)} \leq \frac{1 + 5L\sqrt{\rho_1\rho_2}\gamma_j/2 - 1}{1 - 5L\sqrt{\rho_1\rho_2}\gamma_j/2 - 1} B_{k+1}
\]
\[
\leq \frac{1}{1 - 10L\sqrt{\rho_1\rho_2}\gamma_j/2 - 1} B_{k+1}
\]
\[
\leq \rho_2 B_{k+1}.
\]

So Lemma 4 is proved.

**Proof of the convergence results:**

**Theorem 5** Suppose the step size \( \gamma \) in ASVRG satisfies \( \gamma \leq \left\{ \frac{(\sqrt{3} - \sqrt{2})\sqrt{2}}{20\sqrt{\gamma(\sqrt{3} - 1)}/2} : \frac{1}{12\sqrt{5\gamma(\gamma - 1)}/2} \right\} \), we have

\[
\mathbb{E} \left( F(x_{s+1}) - F(x^*) \right) \leq \left( F(x_0) - F(x^*) \right) + \frac{9}{16\gamma m} \| x_0 - x^* \|^2.
\]  

Recalling Eq. (176), we have

\[
\mathbb{E} \left\| x^*_k - x^*_j(k) \right\|^2 \leq \gamma^2 \rho_2 \frac{(\rho_j/2 - 1)^2}{(\sqrt{\rho_1} - 1)^2} B_{k-1}
\]
\[
\leq \gamma^2 \rho_2 \rho_1 \frac{(\rho_j/2 - 1)^2}{(\sqrt{\rho_1} - 1)^2} B_k.
\]  

We first consider \( f(x) \). For \( f(x) \) has Lipschitz continuous gradients, we have

\[
\mathbb{E}_k f(x_{k+1}) \leq f(x^*_k) + \mathbb{E}_k \langle \nabla f(x^*_k), x_{k+1} - x^*_k \rangle + \frac{L}{2} \mathbb{E}_k \left\| x_{k+1} - x^*_k \right\|^2
\]
\[
= f(x^*_k) - \gamma \mathbb{E}_k \langle \nabla f(x^*_k), \nabla f(x^*_j(k)) \rangle + \mathbb{E}_k \frac{\gamma^2 L}{2} \left\| \nabla f(x^*_j(k)) \right\|^2
\]
\[
\leq f(x^*_k) - \gamma \langle \nabla f(x^*_k), \nabla f(x^*_k) \rangle + \mathbb{E}_k \frac{\gamma^2 L}{2} \left\| \nabla f(x^*_j(k)) \right\|^2 + \gamma \mathbb{E}_k \langle \nabla f(x^*_k), \nabla f(x^*_k) - \nabla f(x^*_j(k)) \rangle
\]
\[
\leq f(x^*_k) - \gamma \langle \nabla f(x^*_k), \nabla f(x^*_k) \rangle + \mathbb{E}_k \frac{\gamma^2 L}{2} \left\| \nabla f(x^*_j(k)) \right\|^2 + \gamma \mathbb{E}_k \langle \nabla f(x^*_k), \nabla f(x^*_k) - \nabla f(x^*_j(k)) \rangle,
\]

where the expectation \( \mathbb{E}_k \) is taken over the random numbers of \( i_{k,a} \) under the condition that \( x^*_k \) is known; in equality \( \triangleq \), we use \( x^*_{k+1} = x^*_k - \gamma \nabla f(x^*_j(k)) \); in equality \( \triangleq \), we replace \( \langle \nabla f(x^*_k), \nabla f(x^*_k) \rangle \) with \( \langle \nabla f(x^*_k), \nabla f(x^*_k) - \nabla f(x^*_k) + \nabla f(x^*_k) \rangle \); in equality \( \triangleq \), we use \( \mathbb{E}_k \langle \nabla f(x^*_k), \nabla f(x^*_k) \rangle = \langle \nabla f(x^*_k), \nabla f(x^*_k) \rangle \). Taking
expectation on all the random numbers on Eq. (183), we have

\[
\mathbb{E} f(x^*_{k+1})
\]

\[\leq \mathbb{E} f(x^*_k) - \gamma \mathbb{E} \langle \nabla f(x^*_k), \nabla f(x^*_k) \rangle + \frac{\gamma^2 L}{2} \mathbb{E} \left\| \nabla f(x^*_k) \right\|^2 + \gamma \mathbb{E} \langle \nabla f(x^*_k), \nabla f(x^*_k) - \tilde{\nabla} f(x^*_j(k)) \rangle
\]

\[\leq a \mathbb{E} f(x^*_k) - \gamma \mathbb{E} \left\| \nabla f(x^*_k) \right\|^2 + \frac{\gamma^2 L}{2} \mathbb{E} \left\| \nabla f(x^*_k) \right\|^2 + \frac{C_5}{2} \mathbb{E} \left\| \nabla f(x^*_k) \right\|^2 + \frac{\gamma^2 L}{2C_5} \mathbb{E} \left\| x^*_j(k) - x^*_k \right\|^2
\]

\[\leq b \mathbb{E} f(x^*_k) - \gamma \mathbb{E} \left\| \nabla f(x^*_k) \right\|^2 + \frac{\gamma^2 L_2 \rho_2}{2} B_k + \gamma \frac{C_5}{2} \mathbb{E} \left\| \nabla f(x^*_k) \right\|^2 + \frac{\gamma^2 L_2 \rho_2 \rho_1}{2C_5} \left( \frac{\rho_1/2 - 1}{\sqrt{\rho_1 - 1}} \right)^2 B_k
\]

\[\leq c \mathbb{E} f(x^*_k) - \gamma \left( 1 - \frac{\rho_2 \gamma L}{2} - \frac{C_5}{2} - \frac{\gamma^2 L_2 \rho_2 \rho_1 (\rho_1/2 - 1)}{2C_5} \left( \frac{\rho_1/2 - 1}{\sqrt{\rho_1 - 1}} \right)^2 \right) \mathbb{E} \left\| \nabla f(x^*_k) \right\|^2
\]

\[+ 4 L^2 \gamma^2 \left( \frac{\rho_2}{2} + \frac{\gamma L_2 \rho_1 (\rho_1/2 - 1)}{2C_5} \left( \frac{\rho_1/2 - 1}{\sqrt{\rho_1 - 1}} \right)^2 \right) \mathbb{E} \left\| f(x^*_k) - f(x^*) + f(\hat{x}^*) - f(x^*) \right\|^2
\]

\[\leq d \mathbb{E} f(x^*_k) - \gamma \left( 1 - \frac{\rho_2 \gamma L}{2} - \frac{\gamma L_2 \rho_2 \rho_1 (\rho_1/2 - 1)}{2C_5} \left( \frac{\rho_1/2 - 1}{\sqrt{\rho_1 - 1}} \right)^2 \right) \mathbb{E} \left\| \nabla f(x^*_k) \right\|^2
\]

\[+ 4 L^2 \gamma^2 \left( \frac{\rho_2}{2} + \frac{\rho_2 \rho_1 (\rho_1/2 - 1)}{2C_5} \left( \frac{\rho_1/2 - 1}{\sqrt{\rho_1 - 1}} \right)^2 \right) \mathbb{E} \left\| f(x^*_k) - f(x^*) + f(\hat{x}^*) - f(x^*) \right\|^2
\]

where in \(a\), we use Cauchy-Schwarz inequality and the smoothness of \(f(\cdot)\), i.e. \(\left\| \nabla f(x^*_k) - \tilde{\nabla} f(x^*_j(k)) \right\|^2 \leq L^2 \left\| x^*_j(k) - x^*_k \right\|^2\), in \(b\), we substitute \(\mathbb{E} \left\| \nabla f(x^*_j(k)) \right\|^2 \leq \rho_2 B_k\) and Eq. (182); in \(c\), we use the definition of \(B_k\) in Eq. (155); in \(d\), we set \(C_5 = \gamma L\).

On the other hand, for \(\left\| x^*_{k+1} - x^* \right\|^2\), we have

\[
\left\| x^*_{k+1} - x^* \right\|^2
\]

\[= \left\| x^*_k - x^* - \gamma \tilde{\nabla} f(x^*_j(k)) \right\|^2
\]

\[= \left\| x^*_k - x^* \right\|^2 - 2 \gamma \left\langle \tilde{\nabla} f(x^*_j(k)), x^*_k - x^* \right\rangle + \gamma^2 \left\| \tilde{\nabla} f(x^*_j(k)) \right\|^2
\]

\[= a \left\| x^*_k - x^* \right\|^2 - 2 \gamma \left\langle \tilde{\nabla} f(x^*_j(k)), x^*_j(k) - x^* \right\rangle + \gamma^2 \left\| \tilde{\nabla} f(x^*_j(k)) \right\|^2
\]

\[\leq b \left\| x^*_k - x^* \right\|^2 + 2 \gamma \left( f(x^*) - f(x^*_j(k)) \right) + \gamma^2 \left\| \tilde{\nabla} f(x^*_j(k)) \right\|^2
\]

\[\leq c \left\| x^*_k - x^* \right\|^2 + 2 \gamma \left( f(x^*) - f(x^*_j(k)) \right) - 2 \gamma \left\langle \tilde{\nabla} f(x^*_j(k)), x^*_k - x^*_j(k) \right\rangle
\]

\[+ \gamma^2 \left\| \tilde{\nabla} f(x^*_j(k)) \right\|^2
\]

\[\leq d \left\| x^*_k - x^* \right\|^2 + 2 \gamma \left( f(x^*) - f(x^*_k) \right)
\]

\[+ \gamma^2 \left\| \tilde{\nabla} f(x^*_j(k)) \right\|^2
\]

where in equality \(a\), we replace \(\langle \tilde{\nabla} f(x^*_j(k)), x^*_k \rangle\) with \(\langle \tilde{\nabla} f(x^*_j(k)), x^*_j(k) - x^*_k + x^*_k \rangle\); in inequality \(b\), we use the convexity of \(f(\cdot)\):

\[f(x^*) - f(x^*_k) \geq \langle \nabla f(x^*_j(k)), x^* - x^*_k \rangle;
\]
in inequality $\leq$, we add and subtract the term $\langle \nabla f(x_k^*), x_k^* - x_j^*(k) \rangle$; in inequality $\leq$, we use the fact that

$$f(x_j^*(k)) - f(x_k^*) \geq -\langle \nabla f(x_k^*), x_k^* - x_j^*(k) \rangle.$$  

Taking expectation only on the random number $i_{k,s}$ on Eq. (185), and use the fact that

$$\mathbb{E}_k \langle \nabla f(x_j(k)) - \nabla f(x_j(k)), x_k - x_j(k) \rangle = 0,$$

we have

$$\mathbb{E}_k \| x_{k+1}^* - x^* \|^2$$

(187)

$$\leq \mathbb{E}_k \| x_k^* - x^* \|^2 + 2\gamma \mathbb{E}_k (f(x^*) - f(x_k^*))$$

$$+ \gamma^2 \mathbb{E}_k \| \nabla f(x_j(k)) \|^2 - 2\gamma \mathbb{E}_k \langle \nabla f(x_j(k)), x_k^* - x_j^*(k) \rangle,$$

Taking expectation on all the random numbers on Eq. (187), we have

$$\mathbb{E} \| x_{k+1}^* - x^* \|^2$$

(188)

$$\leq \mathbb{E} \| x_k^* - x^* \|^2 + 2\gamma \mathbb{E} (f(x^*) - f(x_k^*))$$

$$+ \gamma^2 \mathbb{E} \| \nabla f(x_j(k)) \|^2 - 2\gamma \mathbb{E} \langle \nabla f(x_j(k)), x_k^* - x_j^*(k) \rangle,$$

where in $\leq$, we use Eq. (182) and the smoothness of $f(\cdot)$. Diving Eq. (188) by $2\gamma$ on both size, and using the definition of $B_k$ in Eq. (155), we have

$$\mathbb{E} f(x_k^*) - f(x^*) + \frac{1}{2\gamma} \mathbb{E} \| x_{k+1}^* - x^* \|^2$$

(189)

$$\leq \frac{1}{2\gamma} \mathbb{E} \| x_k^* - x^* \|^2 + \frac{\gamma}{2} \mathbb{E} \| \nabla f(x_j(k)) \|^2 + \gamma^2 L\rho_2 \rho_1 \frac{(\rho_1^{1/2} - 1)^2}{(\sqrt{s} - 1)^2} B_k$$

$$\leq \frac{1}{2\gamma} \mathbb{E} \| x_k^* - x^* \|^2 + \gamma \left( \frac{\rho_2}{2} + \gamma L\rho_2 \rho_1 \frac{(\rho_1^{1/2} - 1)^2}{(\sqrt{s} - 1)^2} \right) \mathbb{E} \| \nabla f(x_k^*) \|^2$$

$$+ 4\gamma L \left( \frac{\rho_2}{2} + \gamma L\rho_2 \rho_1 \frac{(\rho_1^{1/2} - 1)^2}{(\sqrt{s} - 1)^2} \right) \mathbb{E} (f(x_k^*) - f(x^*) + f(\hat{x}^*) - f(x^*)).$$

Multiply Eq. (189) by $\frac{9}{8}$ and add it to Eq. (184), we have

$$\mathbb{E} f(x_{k+1}^*) - f(x^*) + \frac{1}{8} \mathbb{E} \| x_{k+1}^* - x^* \|^2$$

$$\leq C_0 \mathbb{E} (f(\hat{x}^*) - f(x^*)) + \frac{9}{16\gamma} \mathbb{E} \| x_k^* - x^* \|^2 - \gamma C_7 \mathbb{E} \| \nabla f(x_k^*) \|^2.$$  

(190)

where

$$C_0 = \left( \frac{9}{4} \gamma L + 2\gamma^2 L^2 \right) \rho_2 + \frac{13}{2} \gamma^2 L^2 \rho_2 \rho_1 \frac{(\rho_1^{1/2} - 1)^2}{(\sqrt{s} - 1)^2},$$

(191)
\[ C_7 = 1 - \frac{\rho_2 \gamma L}{2} - \frac{\gamma L}{2} - \frac{9 \rho_2}{16} - \frac{13}{8} \gamma L \rho_2 \rho_1 \left( \frac{\rho_1^{1/2} - 1}{\sqrt{\rho_1 - 1}} \right)^2. \]

We first verify that \( \gamma \leq \left\{ \frac{\sqrt{5} - \sqrt{2}}{20 \cdot \frac{5}{4} \sqrt{e(\sqrt{e} - 1) \tau L}} \right\}, \rho_1 = e \) and \( \rho_2 = \sqrt{\varphi} \) satisfies the condition of Lemma 4.

\[
\frac{\rho_1 - 1}{10 \rho_1 \rho_2} \geq \frac{e^{\frac{\rho_1 - 1}{10 \rho_1 \rho_2}} \left( \frac{\sqrt{\varphi}}{10} \right)}{10 \cdot e^{\frac{1}{10 \rho_1 \rho_2}} \left( \frac{\sqrt{\varphi}}{10} \right)} \geq \frac{\sqrt{\varphi}}{10 \cdot \frac{e^{\frac{1}{10 \rho_1 \rho_2}} \left( \frac{\sqrt{\varphi}}{10} \right)}} \geq L \gamma, \tag{192}
\]

where we use the fact that \( e^x - 1 \geq x \) for \( x \geq 0 \) and \( \frac{1}{e^x} \geq \frac{1}{e} \) in inequality. In addition, we have

\[
\frac{\rho_2 - 1}{10 \rho_1 \rho_2} \geq \frac{e^{\frac{\rho_2 - 1}{10 \rho_1 \rho_2}} \left( \frac{\sqrt{\varphi}}{10} \right)}{10 \cdot e^{\frac{1}{10 \rho_1 \rho_2}} \left( \frac{\sqrt{\varphi}}{10} \right)} \geq \frac{\sqrt{\varphi}}{10 \cdot \frac{e^{\frac{1}{10 \rho_1 \rho_2}} \left( \frac{\sqrt{\varphi}}{10} \right)}} \geq L \gamma, \tag{193}
\]

where in \( \geq \), we use \( \rho_1^{\frac{1}{2}} - 1 \geq \frac{1}{2} \). Since

\[
L \gamma \rho_2 \rho_1 \left( \frac{\rho_1^{1/2} - 1}{\sqrt{\rho_1 - 1}} \right)^2 \leq \frac{\sqrt{5}}{2} \cdot e(e - 1) \frac{1}{(e^{\frac{1}{2} - 1})^2} \leq 2 \sqrt{e} e(e - 1)^2, \tag{194}
\]

from the assumption of \( \gamma \leq \frac{1}{12 \sqrt{e} (e - 1)^2 L} \), we can also get

\[
L \gamma \rho_2 \rho_1 \left( \frac{\rho_1^{1/2} - 1}{\sqrt{\rho_1 - 1}} \right)^2 \leq \frac{1}{12 \sqrt{e} (e - 1)^2} \cdot 2 \sqrt{e} e(e - 1)^2 \leq \frac{1}{6}, \tag{195}
\]

From the assumption of \( \gamma \leq \frac{\sqrt{5} - \sqrt{2} \sqrt{2}}{20 \cdot \frac{5}{4} \sqrt{e(\sqrt{e} - 1) \tau L}} \), we have \( L \gamma \leq \frac{1}{100} \), so

\[
C_6 \leq \frac{\sqrt{5}}{2} - \frac{1}{100} \cdot \frac{9}{4} + 2 \cdot \frac{1}{100} \cdot \frac{1}{100} + \frac{13}{2} \cdot \frac{1}{100} \cdot \frac{1}{6} \leq \frac{1}{8}, \tag{196}
\]

and

\[
C_7 \geq 1 - \frac{\sqrt{5}}{400} - \frac{1}{200} - \frac{9 \sqrt{5}}{32} - \frac{13}{8} \cdot \frac{1}{6} \geq 0. \tag{196}
\]
We obtain
\[ Ef(x_{k+1}^*) - f(x^*) + \frac{9}{16\gamma} E \|x_{k+1}^* - x^*\|^2 \]
\[ \leq C_6 E (f(x^*) - f(x^*)) + \left( C_6 - \frac{1}{8} \right) E (f(x_k^*) - f(x^*)) + \frac{9}{16\gamma} E \|x_k^* - x^*\|^2 \]
\[ \leq C_6 E (f(x^*) - f(x^*)) + \frac{9}{16\gamma} E \|x_k^* - x^*\|^2 \]
\[ \leq E (f(x^*) - f(x^*)) + \frac{9}{16\gamma} E \|x_k^* - x^*\|^2. \]  
(197)

Summing \( k \) from 0 to \( m - 1 \) and using the fact that
\[ f(x^{s+1}) \leq \sum_{k=1}^{m} f(x_k^s), \]  
(198)
we have
\[ mE (f(x^{s+1}) - f(x^*)) + \frac{9}{16\gamma} E \|x_m^s - x^*\|^2 \]
\[ \leq mE (f(x^s) - f(x^*)) + \frac{9}{16\gamma} E \|x_0^s - x^*\|^2. \]  
(199)

Summing Eq. (199) with \( s \) from 0 to \( s \), and using \( x_0^s = x_{m-1}^s \), we have the results that
\[ mE (f(x^{s+1}) - f(x^*)) + \frac{9}{16\gamma} E \|x_m^s - x^*\|^2 \]
\[ \leq mE (f(x_0^s) - f(x^*)) + \frac{9}{16\gamma} E \|x_0^s - x^*\|^2. \]  
(200)
So
\[ E (f(x^{s+1}) - f(x^*)) \]
\[ \leq (f(x_0^s) - f(x^*)) + \frac{9}{16\gamma m} \|x_0^s - x^*\|^2. \]  
(201)

### 7.5 Other Material

#### 7.5.1 Sparse Update

Proof of the algorithm 1 and Algorithm 2 are equivalent. We use \( w_1^{j(k)} \) and \( \delta_1^k \) to denote \( w_1^{j(k)} \) and \( \delta_1^k \) generated by Algorithm 1, and use \( w_2^{j(k)} \) and \( \delta_2^k \) to denote \( w_2^{j(k)} \) and \( \delta_2^k \) generated by Algorithm 2. To prove the results, we use induction to show that \( z^k = u^k, x^k = u^k + d^k v^k \). When \( k = 0 \), we have \( z^k = u^k = 0, x^k = u^k + d^k v^k = 0 \). For \( j(0) = 0 \), then \( w_1^{j(0)} = y^0 = w_2^{j(0)} = 0 = u^0 + d^1 v^0 \). So we have \( \delta_1^0 = \delta_2^0 \). Then we have that \( z^1 = u^1 \). So
\[ x^1 = y^0 + \theta^0 \delta_1^0 = u^0 + d^1 v^0 + \theta^0 \delta_2 = u^1 + d^1 v^1 - \delta_2 + \frac{d^1 \delta_2^0}{\theta^0} + \theta^0 \delta_2^0 = u^1 + d^1 v^1, \]  
(202)
where in the third equality, we use \( d^{k+1} = d^k (1 - \theta^k) \).

When \( k > 0 \), suppose we have \( z^k = u^k \), and \( x^k = u^k + d^k v^k \), then
\[ y^k = (1 - \theta^k) x^k + \theta^k z^k = (1 - \theta^k) d^k v^k + u^k = d^{k+1} v^k + u^k. \]  
(203)
If we obtain $w_1^{(k)} = w_2^{(k)}$, then $\delta_1^k = \delta_2^k$ and $z^{k+1} = u^{k+1}$. For $x^{k+1}$, we have

$$
x^{k+1} = y^k + \theta^k \delta_1^k = u^k + d^{k+1} v^k + \theta^k \delta_2
$$

\hspace{1cm}

\begin{equation}
= u^{k+1} + d^{k+1} v^{k+1} - \delta_2^k \frac{d^{k+1} \delta_2^{k+1}}{d^k} + \theta^k \delta_2^k = u^{k+1} + d^{k+1} v^{k+1}.
\end{equation}

(204)

Now we are to prove $w_1^{(j)} = w_2^{(j)}$. We introduce an auxiliary algorithm, shown in Algorithm 3. The algorithm is the serial AGD by setting $\delta_2^k = 0$. The result of Eq. (23) can be directly used by setting $x_j - y_i^{j-1} = 0$ when $i > j(k)$. So we obtain that $y_1^k = w_1^{j(k)}$. Now we are to prove that $y_1^k = w_2^{j(k)}$, that is to prove that

$$
y_1^k = u^{j(k)} + d^{k+1} v^{j(k)}.
$$

(205)

To proof this, we show that Algorithm 3 is equivalent to Algorithm 4. By the induction same as Eq. (202).

Algorithm 7 AAGD-auxiliary

\begin{algorithm}
\caption{AAGD-auxiliary}
\label{alg:auxiliary}
\begin{algorithmic}
\State \textbf{Input} $x_1^{(i)} = x^{(i(k))}$ and $z_1^{(i)} = z^{(i(k))}$.
\For {$k = j(k)$ to $K-1$}
\State $y_1^k = (1 - \theta^k)x^k + \theta^k z^k$
\State $z_1^{k+1} = z_1^k$
\State $x_1^{k+1} = y_1^k + \theta^k(z_1^{k+1} - z_1^k)$.
\EndFor
\end{algorithmic}
\end{algorithm}

(203), (204), we can obtain that $x_1^k = u_1^k + d^k v_1^k$, $z_1^k = u_1^k$, and $y_1^k = u_1^k + d^{k+1} v_1^k$. As $u_1^k = u_1^{j(k)}$ and $v_1^k = v_1^{j(k)}$, we obtain $y_1^k = u_1^{j(k)} + d^{k+1} v_1^{j(k)}$. This ends proof.

7.5.2 Pre-define Update Order

Our technique need to predefine the update order to obtain $w^{j(k)}$. Once such an order has been set, each thread may update the gradient estimator accordingly. If one thread returns the gradient early, the gradient can be stored and it will go on for the next iteration. The master thread will use the gradient to update parameters after receiving all the required gradient.

However, though the threads will never be hanged up, the large inconsistency of real order will amplify the delay effect. We found that for dense datasets, the computation costs are roughly the same for each child node, so one may directly set $k = j(k) + \zeta - 1$, where $\zeta$ is the number of cores. This works well in practice. While for sparse datasets, simply setting the predefined order is not advised. We introduce way to avoid predefining the order.

Through our algorithm, we can find that when smooth-part of the objective function is quadratic, such as $f(x) = \|Ax\|^2$, then

$$
\nabla f(w^{j(k)}) = \nabla f(u^{j(k)}) + d^{k+1} \nabla f(v^{j(k)}). 
$$

(206)

\clearpage
We have also verified the convergence speed for AASVRG on another three datasets, namely the sparse dataset Vector (Pearlmutter, 1994) product to approximate the gradient. Set AASVRG as an example, applying p

\[ \nabla f_{v^p}(w^s_{j(k)}) \approx \nabla f_{v^p}(p^s_{j(k)}) - \alpha^s_k H_{i,k}(p^s_{j(k)})(x^s_{j(k)} - x^s_{j(k)-1}), \]

(207)

where \( p^s_{j(k)} = x^s_j + \frac{a(1-a^{s+1})}{1-a^s}(x^s_{j(k)} - x^s_{j(k)-1}), \) and

\[ \alpha^s_k = \frac{(a^s)^{k-j(k)+2} (1 - (a^s)^{(k-j(k)})}{1 - a^s}. \]

\( H_{i,k}(w^s_{j(k)}) \) denotes the Hessian Matrix of \( f_{i,k} \) at point \( w^s_{j(k)} \). In this way, \( k - j(k) \) does not need to be known before computing the gradient estimator.

We can find that Eq. (207) has the following property: 1) \( \alpha^s_k \) decreases exponentially with respect to the growth of delay \( k - j(k) \). For severely delayed system (lots of cores are running), we can assume that \( k - j(k) \) is large, so \( \alpha \) is small; 2) when \( f \) is quadratic, Eq. (207) holds strictly, so E-ASVRG also achieves the accelerated convergence rate. For lots of machine learning problems, the Hessian-Vector product can be efficiently computed through Hessian Free techniques (Pearlmutter, 1994; Martens, 2010), which is in \( O(d) \) time, the same as computing the gradient, where \( d \) is the dimension of the parameter.

7.5.3 Implementation Details

Deadlock Avoidance We can avoid deadlock, we associate an ordering for all the locks such that each thread follows the same ordering to acquire the locks.

Sparse Update We can find that by changing variable, it is able to spare date on the sparse dataset. For ASCiA, like Lin et al. (2014), we can introduce \( x_1 = Au \) and \( x_2 = A_v v \) to fast obtain the gradient. When \( \theta \) is fixed, such as for SC and AASVRG, the update of \( v^k \) will cause numerical problems because \( d^k \to 0 \), we can store \( d^k v^k \) as \( v^k_1 \) and \( v^k_2 \), with the first one store the value, and the second store the power.

Spin Locks Also observed by Hsieh et al. (2015), when there are no locks, due to the memory conflict, \( x_1 \neq Au \) and \( x_2 \neq Av \), this is harmful and will lead the algorithm solving a deflected problem. To tackle it, we create \( n \)'s spin lock, and add lock when the corresponding coordinate of \( u_1 \) and \( u_2 \) are updated.

7.5.4 Sparsity \( \Delta \)-assumption (Reddi et al., 2015)

The sparsity \( \Delta \)-assumption in (Reddi et al., 2015) is as follows: for problem of composite finite-sum problem, i.e. Eq. (16) in the paper, suppose \( f_i \) only depends on \( x_{e_i} \), where \( e_i \subseteq [d] \), i.e., \( f_i \) acts only on the components of \( x \) indexed by the set \( e_i \). Let \( \|x\|_2^2 \) denote \( \sum_{j=e_{\forall}} \|x_j\|^2 \); then the convergence depends on \( \Delta \), the smallest constant such that \( \text{E}_d[\|x\|_2^2] = \Delta \|x\|^2 \), and \( \Delta \ll 1 \).

One can find by the assumption that changes of each update are small, so through the proof, step 1, e.g. Eq. (100), \( \|y^k - w^{j(k)}\|^2 \) will \( \Delta \) times smaller. So Proposition 2 in the paper is obtained.

7.6 More Experimental Results

We have also verified the convergence speed for AASVRG on another three datasets, namely the sparse dataset new20 and dense datasets asps, combined. The results are shown in Fig. 4. It turns out that our algorithm has competitive results on all of these datasets.

Our algorithm has big advantages for ill-condition problem, i.e., when the regularization term \( \lambda \) in Ridge Regression is small. As we can see from Fig. 5, when \( \lambda \) is large, our algorithm has similar performance as other state-of-the-art algorithms. However, when \( \lambda \) is small, we gains the huge advantages in terms of the convergence.
Table 4: Details of the dense datasets. (Dim., is short for dimensionality)

| Datasets | #training | Dim. | Class | #mini-batch |
|----------|-----------|------|-------|-------------|
| USPS     | 7291      | 256  | 10    | 50          |
| MNIST    | 60000     | 784  | 10    | 50          |
| SENSIT   | 78823     | 100  | 3     | 50          |
| EPSILON  | 400000    | 2000 | 2     | 200         |

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Figure 4: Residuals vs CPU training time (s) and iterations for solving Ridge Regression problem. “news20” is a sparse dataset, while “usps” and “combined” are dense.

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Figure 5: Residuals vs CPU training time and iterations for solving Ridge Regression problem with different $\lambda$ on rcv1 datasets.