ABSTRACT. In this paper we prove a Lukács type characterization theorem of the Wishart distribution on Euclidean simple Jordan algebras under weak and natural regularity assumptions on the densities.

1. INTRODUCTION AND PRELIMINARIES

The main purpose of this paper is to prove a generalization of the most celebrated result in the field of characterization problems concerning probability distribution, the theorem of Lukács. This is contained in the following, see also Lukács [15].

Theorem 1.1 (Lukács). If $X$ and $Y$ are non-degenerate, independent random variables, then the random variables

$$V = X + Y \quad \text{and} \quad U = \frac{X}{X + Y}$$

are independently distributed if and only if both $X$ and $Y$ have gamma distributions with the same scale parameter.

Since its 1955 appearance this theorem has been extended and generalized in several ways. For example, in Olkin–Rubin [19] and also in Casalis–Letac [4] the authors extended the above result to matrix and symmetric cone variable distributions, respectively. The main problem here is that in such situations there is no unique way of defining the quotient. Therefore, the authors quoted above initiated the notion of a so-called division algorithm and under some strong assumption they proved an analogue of Lukács’s result. To avoid this assumption, in Bobecka–Wesołowski [3] it was assumed that the densities are strictly positive and they are twice differentiable. This regularity assumption was weakened in Kołodziejek [12]. In that paper only continuity was assumed.

Therefore the main aim of this paper is to give a unified proof of analogue of Lukács’s theorem in the symmetric cone setting, assuming only the existence of densities.

1.1. Euclidean Jordan algebras. In this subsection we collected some definitions and statements from the theory of Euclidean Jordan algebras that will be used subsequently. For further results we refer to the monograph of Faraut and Korányi, see [7] Section III. 1.].

Definition 1. We say that $(\mathbb{E}; \langle \cdot, \cdot \rangle; \cdot)$ is a Euclidean Jordan algebra, if

(i) $(\mathbb{E}; \langle \cdot, \cdot \rangle)$ is an inner product space

(ii) there exists $e \in \mathbb{E}$ such that $xe = ex = x$

is fulfilled for any $x \in V$

(iii) for all $x, y, z \in \mathbb{E}$
(a) \( xy = yx \)
(b) \( x(x^2y) = x^2(xy) \)
(c) \( \langle x, yz \rangle = \langle xy, z \rangle \)
holds.

**Definition 2.** Let \((\mathbb{E}; \langle \cdot, \cdot \rangle; \cdot)\) be a Euclidean Jordan algebra, fix an \( x \in \mathbb{E} \) and let us consider the maps \( P(x) \) and \( L(x) \) defined by

\[
L(x)y = xy
\]

and

\[
P(x) = 2L^2(x) - L(x^2).
\]

Then the mapping

\[
\mathbb{E} \ni x \mapsto P(x) \in \text{End}(\mathbb{E})
\]

is called the quadratic representation of \( \mathbb{E} \).

An element \( z \in \mathbb{E} \) is said to be invertible if there exists \( w \in \mathbb{E} \) such that

\[
L(z)w = e.
\]

Then \( w \) is called the inverse of \( z \) and it will be denoted by \( z^{-1} \).

**Definition 3.** The Euclidean Jordan algebra \((\mathbb{E}; \langle \cdot, \cdot \rangle; \cdot)\) is said to be simple if it is not a Cartesian product of two nontrivial Euclidean Jordan algebras.

**Example 1.** Let \( \mathbb{K} = \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\} \) and let denote \( S_r(\mathbb{K}) \) the set of all \( r \times r \) Hermitian matrices with entries in \( \mathbb{K} \). If \( x, y \in S_r(\mathbb{K}) \) let

\[
\langle x, y \rangle = \text{Tr}(x^*y)
\]

and

\[
L(x)y = \frac{xy + yx}{2}.
\]

Then \( (S_r(\mathbb{K}); \langle \cdot, \cdot \rangle; \cdot) \) is a Euclidean Jordan algebra. Furthermore, in this case

\[
P(y)x = yxy \quad (x, y \in \mathbb{K}).
\]

**Example 2.** Let \( n \geq 2 \) and let us consider \( \mathbb{R}^{n+1} \) with the usual inner product and with the Jordan product

\[
(x_0, x_1, \ldots, x_n) \cdot (y_0, y_1, \ldots, y_n) = \left( \sum_{i=0}^{n} x_iy_j, x_0y_1 + x_1y_0, \ldots, x_0y_n + x_ny_0 \right).
\]

Then \( \mathbb{R}^{n+1} \) is a Euclidean Jordan algebra.

The following result can also be found in Faraut–Korányi, see [7, Theorem V.3.7].

**Theorem 1.2.** Up to linear isomorphism there exist only the following Euclidean simple Jordan algebras

(i) \( S_r(\mathbb{R}) \), where \( r \geq 1 \) is arbitrary;
(ii) \( S_r(\mathbb{C}) \), where \( r \geq 2 \) is arbitrary;
(iii) \( S_r(\mathbb{H}) \), where \( r \geq 2 \) is arbitrary;
(iv) \( S_3(\mathbb{O}) \);
(v) \( \mathbb{R}^{n+1} \) appearing in the previous example.

**Theorem 1.3** ([7, Section III.2]). Let \( \mathbb{E} \) be a Euclidean Jordan algebra. Then the set

\[
\mathcal{V} = \{ x^2 \mid x \text{ is invertible} \}
\]

is an open convex cone that has the following properties.
(i) The closure of $\mathcal{V}$ is the set
$$\overline{\mathcal{V}} = \{ x^2 \mid x \in \mathbb{E} \}.$$  

(ii) If $y \in \mathbb{E}$ and for all $x \in \overline{\mathcal{V}} \setminus \{0\}$ holds, then $y \in \mathcal{V}.$

(iii) Let $y \in \mathcal{V}$, then for all $x \in \overline{\mathcal{V}} \setminus \{0\}$ we have
$$\langle y, x \rangle > 0.$$  

(iv) If $x \in \mathbb{E}$ is invertible, then
$$\mathbb{P}(x)(\mathcal{V}) = \mathcal{V}.$$  

(v) For all $x \in \mathcal{V}$ $\mathbb{P}(x) > 0.$

(vi) The cone $\mathcal{V}$ is the connected component of the identity in the set of invertible elements. The cone $\mathcal{V}$ is the set of all elements $x \in \mathbb{E}$ for which $L(x)$ is positive definite.

Since each simple Jordan algebra corresponds to a symmetric cone, up to a linear isomorphism there exist also only five types of symmetric cones. We remark that the cone corresponding to the Euclidean Jordan algebra $\mathbb{R}^{n+1}$ is called the Lorentz cone. This case was investigated in [11]. Therefore, in what follows we will implicitly assume that the symmetric cone $\mathcal{V}$ is not the Lorentz cone.

As we wrote in at the beginning of this paper, our main aim is to present a characterization theorem for the Wishart distribution. Therefore, we also have to review some notations concerning it. For further details, we refer to Eaton [6, Chapter VIII., 302–333].

Let $a \in \mathcal{V}$ and $p \in \left\{ 0, \frac{d}{2}, \ldots, \frac{d(r-1)}{2} \right\} \cup \left[ d(r-1), +\infty \right[.$ We say that the random variable $X$ has Wishart distribution with scalar parameter $a$ and shape parameter $p$, if the Laplace transform of its distribution is
$$\int_{\mathcal{V}} \exp \left( -\langle t, y \rangle \right) d(\gamma_{p,a}y) = \frac{1}{(\det (1 + ta^{-1}))^{p}},$$
where $\Gamma_{\mathcal{V}}$ denotes the multivariate Gamma function.

If $p \leq \frac{\dim(\mathcal{V})}{2} - 1$ then the Wishart no longer has a density, but it represents a singular distribution.

If $p > \frac{\dim(\mathcal{V})}{r} - 1$ then the Wishart distribution is absolutely continuous with density function
$$\gamma_{p,a}(dy) = \frac{\det(a)^p}{\Gamma_{\mathcal{V}}(p)} (\det(y))^{p-\frac{r}{2}} \cdot \exp \left( -\langle a, y \rangle \right) I_{\mathcal{V}}(y) dy.$$  

1.2. Prerequisites from the theory of functional equations. This subsection contains some basic definitions and results from the theory of functional equations. Concerning this topic we refer to the two basic monographs Aczél [1] and Kuczma [13].

Subsequently we will work on the cone $\mathcal{V}$. Therefore, when reviewing some definition (e.g. the notion of additive and logarithmic mappings, resp.) we will always restrict our considerations only to a Euclidean Jordan algebra $\mathbb{E}$.

**Definition 4.** Let $A \subset \mathbb{E}$ be an arbitrary nonempty set and
$$\mathcal{A} = \left\{ (x, y) \in \mathbb{E}^2 \mid x, y, x + y \in A \right\}.$$  

A function $a : A \rightarrow \mathbb{R}$ is called additive on $A$, if for all $(x, y) \in \mathcal{A}$
$$a(x + y) = a(x) + a(y).$$  

(1.1)
If \( A = \mathbb{E} \), then the function \( a \) will be called simply additive.

It is well-known that the solutions of the equation above, under some mild regularity condition, are of the form

\[
a(x) = \langle \lambda, x \rangle \quad (x \in A),
\]

with a certain constant \( \lambda \in \mathbb{E} \). It is also known, however, that there are additive functions that are nowhere continuous, see Kuczma [13].

In the sequel we will use the following extension theorem concerning the so-called Pexider equation, this result is a special case of [5, Theorem 3] if we choose the normed space \( X \) to be \( \mathbb{E} \) and the open and connected set \( D \) to be \( \mathbb{V} \times \mathbb{V} \).

**Theorem 1.4.** Assume that for the functions \( k, l, n : \mathbb{V} \to \mathbb{R} \)

\[
k(x + y) = l(x) + n(y)
\]

is fulfilled for any \( x, y \in \mathbb{V} \). If the function \( k \) is nonconstant, then these functions can be uniquely extended to functions \( \overline{k}, \overline{l}, \overline{n} : \mathbb{E} \to \mathbb{R} \) so that

\[
\overline{k}(x + y) = \overline{l}(x) + \overline{n}(y) \quad (x, y \in \mathbb{E}).
\]

Especially, if a function \( a : \mathbb{V} \to \mathbb{R} \) is additive on \( \mathbb{V} \), then it can always be uniquely extended to an additive function \( \overline{a} : \mathbb{E} \to \mathbb{R} \). We remark this follows also from Theorem 4 of Páles [20].

In what follows we will present the general solution of the Pexider equation on a restricted domain. This theorem follows immediately from [5, Theorem 1], with exactly the same choice as above.

**Theorem 1.5.** Assume that for the functions \( k, l, n : \mathbb{V} \to \mathbb{R} \)

\[
k(x + y) = l(x) + n(y)
\]

is fulfilled for any \( x, y \in \mathbb{V} \) and that the function \( k \) is nonconstant. Then and only then there exists a uniquely determined additive function \( \overline{a} : \mathbb{E} \to \mathbb{R} \) and real constants \( b \) and \( c \) so that

\[
\begin{align*}
k(x) &= a(x) + b + c \\
l(x) &= a(x) + b \\
n(x) &= a(x) + c
\end{align*}
\]

(\( x \in \mathbb{V} \)).

Finally, the following statement concerns the constant solutions of the Pexider equation.

**Corollary 1.1.** Let \( k \in \mathbb{R} \) be fixed and \( l, n : \mathbb{V} \to \mathbb{R} \) be functions so that

\[
k(x) = l(x) + n(y)
\]

is fulfilled for all \( x, y \in \mathbb{V} \). Then there exists \( c \in \mathbb{R} \) such that

\[
n(x) = c \quad \text{and} \quad l(x) = k - c \quad (x \in \mathbb{V}).
\]

The following lemma will also be used frequently in the sequel.

**Lemma 1.1.** Let \( a : \mathbb{V} \to \mathbb{R} \) be a additive function and assume that at least one of the following statements is valid.

(i) The function \( a \) is continuous at a point \( x_0 \in \mathbb{V} \).
(ii) There exists a set \( A \subset \mathbb{V} \) of positive Lebesgue measure such that the function \( a \) is bounded above or below on \( A \);
(iii) There exists a set \( A \subset \mathbb{V} \) of positive Lebesgue measure such that the restriction of \( a \) to the set \( A \) is measurable (in the sense of Lebesgue).

Then there exists \( \lambda \in \mathbb{E} \) such that

\[
a(x) = \langle \lambda, x \rangle \quad (x \in \mathbb{V}).
\]
Theorem 1.6 (Járai). Let $Z$ be a regular topological space, $Z_i (i = 1, \ldots, n)$ be topological spaces and $T$ be a first countable topological space. Let $Y$ be an open subset of $\mathbb{R}^k$, $X_i$ an open subset of $\mathbb{R}^n$ ($i = 1, \ldots, n$) and $D$ an open subset of $T \times Y$. Let further $T' \subset T$ be a dense subset, $F : T' \to Z$, $g_i : D \to X_i$ and $h : D \times Z_1 \times \cdots \times Z_n \to Z$. Suppose that the function $f_i$ is defined almost everywhere on $X_i$ (with respect to the $r_i$-dimensional Lebesgue measure) with values in $Z_i$ and the following conditions satisfied.

(i) for all $t \in T'$ and for almost all $y \in D_t = \{y \in Y \mid (t, y) \in D\}$

(*) \quad $F(t) = f_1(g_1(t, y)), \ldots, f_n(g_n(t, y));$

(ii) for all fixed $y \in Y$, the function $h$ is continuous in the other variables;

(iii) $f_i$ is Lebesgue measurable on $X_i$ ($i = 1, \ldots, n$);

(iv) $g_i$ and also the partial derivative $\frac{\partial g_i}{\partial y}$ are continuous on $D$ for all $i = 1, \ldots, n$;

(v) for each $t \in T$ there exists $y$ such that $(t, y) \in D$ and the partial derivative $\frac{\partial g_i}{\partial y}$ has rank $r_i$ at $(t, y) \in D$ for all $i = 1, \ldots, n$.

Then there exists a unique continuous function $\tilde{F}$ such that $F = \tilde{F}$ almost everywhere on $T$ and if $F$ is replaced by $\tilde{F}$ in (*), then equation (*) is satisfied everywhere.

2. The Olkin–Baker functional equation

In what follows we will investigate the so-called Olkin–Baker equation on the cone $\mathcal{V}$, that is, functional equation

$$a(x) + b(y) = c(x + y) + d\left(\mathbb{P}\left((x + y)^{\frac{1}{2}}\right)x\right)$$

(2.1) \quad $(x, y) \in \mathcal{V}$,

where $a, b, c : \mathcal{V} \to \mathbb{R}$ and $d : \mathcal{D} \to \mathbb{R}$ are unknown functions and

$$\mathcal{D} = \{z \in \mathcal{V} \mid 1 - z \in \mathcal{V}\}.$$
Firstly, we will determine the general solutions. After that the a description of the regular solutions will follow.

In 1975, during the Twelfth International Symposium on Functional Equations I. Olkin (see Olkin [18]) posed the problem of solving the function equation will follow.

\[ f(x)g(y) = p(x + y)q \left( \frac{x}{y} \right) \quad (x, y \in ]0, +\infty[), \]

where the unknown functions \( f, g, p, q : ]0, +\infty[ \to \mathbb{R} \) are assumed to be positive. The general solution of this equation was described in Baker [2]. After that, this functional equation was investigated by several authors. For example, in Lajkó–Mészáros [14] it is assumed the the functional equation is satisfied for almost all pairs \((x, y) \in ]0, +\infty[^2\) and the unknown functions are measurable. Furthermore, in Ger–Misiewicz–Wesołowski [8] it is supposed only that the above equation is fulfilled for almost all pairs \((x, y) \in ]0, +\infty[^2\). However, no regularity assumption was imposed on the unknown positive functions \( f, g, p, q \).

2.1. The general solution. In this subsection we begin with the description of the general solution of equation (2.1). Since we assumed that the cone \( \mathcal{V} \) is not the Lorentz cone, equation (2.1) has the form

\[ a(x) + b(y) = c(x + y) + d \left( (x + y)^{-\frac{1}{2}} x(x + y)^{-\frac{1}{2}} \right). \]

\textbf{Lemma 2.1.} Let us assume that the functions \( a, b, c : \mathcal{V} \to \mathbb{R} \) and \( d : \mathcal{D} \to \mathbb{R} \) fulfill equation (2.2) for all \( x, y \in \mathcal{V} \). Then there exists an additive function \( A : \mathcal{V} \to \mathbb{R} \), logarithmic functions \( \ell_1, \ell_2 : ]0, +\infty[ \to \mathbb{R} \) and \( \kappa_1, \kappa_2 \in \mathbb{R} \) such that

\[
\begin{align*}
a(x) &= A(x) + \ell_1 (\text{det}(x)) + e(x) \\
b(x) &= A(x) + \ell_2 (\text{det}(x)) + f(x) \\
c(x) &= A(x) + (\ell_1 + \ell_2) (\text{det}(x)) + g(x) \\
d(u) &= \ell_1 (\text{det}(u)) + \ell_2 (\text{det}(1 - u)) + h(u)
\end{align*}
\]

for all \( x \in \mathcal{V} \) where the functions \( e, f, g : \mathcal{V} \to \mathbb{R} \) and \( h : \mathcal{D} \to \mathbb{R} \) satisfy the Olkin–Baker equation

\[ e(x) + f(y) = g(x + y) + h \left( (x + y)^{-\frac{1}{2}} x(x + y)^{-\frac{1}{2}} \right). \]

and \( e, f \) and \( g \) are homogeneous of zero order, that is

\[ e(sx) = e(x), \quad f(sx) = f(x), \quad \text{and} \quad g(sx) = g(x) \]

is fulfilled for arbitrary \( s \in ]0, +\infty[ \) and \( x \in \mathcal{V} \).

\textbf{Proof.} Let us assume that for the unknown functions \( a, b, c \) and \( d \) equation (2.2) is valid. Let \( s \in ]0, +\infty[ \) be arbitrary and substitute \( sx \) and \( sy \) in place of \( x \) and \( y \), respectively to get

\[ a(sx) + b(sy) = c(s(x + y)) + d \left( (s(x + y))^{-\frac{1}{2}} x(x + y)^{-\frac{1}{2}} \right). \]

This yields that

\[ a_s(x) + b_s(y) = c_s(x + y) \quad (s \in ]0, +\infty[, x, y \in \mathcal{V}), \]

where the functions \( a_s, b_s, c_s : \mathcal{V} \to \mathbb{R} \) are defined by

\[
\begin{align*}
a_s(x) &= a(sx) - a(x) \\
b_s(x) &= b(sx) - b(x) \\
c_s(x) &= c(sx) - c(x)
\end{align*}
\]
Thus there exists an additive function $A_s : \mathcal{V} \to \mathbb{R}$ and functions $\alpha, \beta : ]0, +\infty[ \to \mathbb{R}$ such that

\[
\begin{align*}
    a_s(x) &= A_s(x) + \alpha(s) \\
    b_s(x) &= A_s(x) + \beta(s) \quad (s \in ]0, +\infty[, x \in \mathcal{V}) \\
    c_s(x) &= A_s(x) + \alpha(s) + \beta(s)
\end{align*}
\]

Let $s, t \in ]0, +\infty[$ and $z \in \mathcal{V}$, in view of the definition of the function $a_s$,

\[
a_{st}(z) = a((st)z) - a(z) = a((st)z) - a(sz) + a(sz) - a(z) = a_s(sz) + a_s(z).
\]

Therefore,

\[
A_{st}(z) + \alpha(st) = A_s(sz) + \alpha(t) + A_s(z) + \alpha(s) \quad (s, t \in ]0, +\infty[, z \in \mathcal{V})
\]

holds, that is

\[
A_{st}(z) - A_s(sz) - A_s(z) = \alpha(t) + \alpha(s) - \alpha(st) \quad (s, t \in ]0, +\infty[, z \in \mathcal{V})
\]

Since the right hand side of this identity does not depend on $z \in \mathcal{V}$, however, on the left hand side $z \in \mathcal{V}$ can be arbitrary, we obtain that

\[
\alpha(st) = \alpha(s) + \alpha(t)
\]

holds for all $s, t \in ]0, +\infty[$. This means that the function $\alpha : ]0, +\infty[$ is logarithmic. A similar computation shows that the function $\beta : ]0, +\infty[ \to \mathbb{R}$ is also logarithmic on $]0, +\infty[$.

Thus there exist logarithmic functions $\ell_1, \ell_2 : ]0, +\infty[ \to \mathbb{R}$ such that

\[
\alpha(s) = \ell_1(s) \quad \text{and} \quad \beta(s) = \ell_2(s) \quad (s \in ]0, +\infty[)
\]

is fulfilled. Let again $s, t \in ]0, +\infty[$ and $z \in \mathcal{V}$ be arbitrary. Then

\[
a_{st}(z) = a_s(sz) + a_s(z)
\]

holds. Since the left hand side is symmetric in $s$ and $t$, we also have

\[
a_{st}(z) = a_s(tz) + a_s(z).
\]

Hence,

\[
a_s(sz) + a_s(z) = a_s(tz) + a_s(z) \quad (s, t \in ]0, +\infty[, z \in \mathcal{V})
\]

This yields

\[
A_s(sz) + \alpha(t) + A_s(z) + \alpha(s) = A_s(tz) + \alpha(s) + A_s(z) + \alpha(t) \quad (s, t \in ]0, +\infty[, z \in \mathcal{V})
\]

With the substitution $s = 2$ we get

\[
A_s(z) = A_2(tz) - A_2(z) \quad (t \in ]0, +\infty[) \,.
\]

Thus

\[
a_s(z) = A_2(sz) - A_2(z) + \ell_1(s) \quad (s \in ]0, +\infty[, z \in \mathcal{V})
\]

Define the function $e : \mathcal{V} \to \mathbb{R}$ by

\[
e(x) = a(x) - A_2(x) - \frac{1}{r} \ell_1(\det(x)) \quad (x \in \mathcal{V}) ,
\]

where $r$ denotes the rank of the cone $\mathcal{V}$. The above identities imply that the function $e$ is homogeneous of order zero. A similar computation shows that for the function $b_s$

\[
b_s(x) = A_2(sz) - A_2(z) + \ell_2(s) \quad (s \in ]0, +\infty[, z \in \mathcal{V})
\]

holds. Therefore let us define the functions $f, g, h : \mathcal{V} \to \mathbb{R}$ through

\[
f(x) = b(x) - A_2(x) - \frac{1}{r} \ell_2(\det(x)) \quad (x \in \mathcal{V}) ,
\]
In lemma 2.2, we consider the functions $a$, $b$ and $c$ can be assumed, otherwise the functions satisfy equation (2.2).

In what follows, we will investigate the functions $e$, $f$, $g$ and $h$ appearing in the previous lemma.
On the other hand, for all \( u, v \in \mathcal{V} \)
\[-2c(u + v) + c(u) + c(v) = d\left((u + v)^{-\frac{1}{2}}u(u + v)^{-\frac{1}{2}}\right) + d\left((u + v)^{-\frac{1}{2}}v(u + v)^{-\frac{1}{2}}\right) - 2d\left(\frac{1}{2}\right).\]

Thus,
\[-2c(1) + c\left((u + v)^{-\frac{1}{2}}u(u + v)^{-\frac{1}{2}}\right) + c\left((u + v)^{-\frac{1}{2}}v(u + v)^{-\frac{1}{2}}\right) = -2c(u + v) + c(u) + c(v) \quad (u, v \in \mathcal{V}).\]

Let us define the functions \( C \) and \( D \) on \( \mathcal{V}^2 \) through
\[ C(x, y) = -c(x + y) + c(x) + c(y) \quad (x, y \in \mathcal{V}) \]
and
\[ D(x, y) = -2c(1) + c(x + y) + c\left((x + y)^{-\frac{1}{2}}x(x + y)^{-\frac{1}{2}}\right) + c\left((x + y)^{-\frac{1}{2}}y(x + y)^{-\frac{1}{2}}\right) \quad (x, y \in \mathcal{V}).\]

In this case the previous equation yields that
\[ C(x, y) = D(x, y) \]
is fulfilled for arbitrary \( x, y \in \mathcal{V} \). Since the function \( C \) is a Cauchy difference, it satisfies the cocycle equation, i.e.,
\[ C(x + y, z) + C(x, y + z) + C(y, z) \quad (x, y \in \mathcal{V}), \]
furthermore, the definition of the function \( C \) immediately implies that \( C \) is a symmetric function. Therefore, the function \( D \) is also symmetric and
\[ D(x + y, z) + D(x, y + z) = D(x, y + z) + D(y, z) \]
holds for all \( x, y, z \in \mathcal{V} \). This equation with \( y = z \) yields that
\[ D(x + y, z) + D(x, z) = D(x, 2z) + D(z, z) \]
or
\[ D(x, z) = D(x, 2z) + D(z, z) - D(x + z, z) \quad (x, z \in \mathcal{V}). \]

Now using the definition of the function \( D \), we get that
\[ c(x + z) + c\left((x + z)^{-\frac{1}{2}}x(x + z)^{-\frac{1}{2}}\right) + c\left((x + z)^{-\frac{1}{2}}x(x + z)^{-\frac{1}{2}}\right) = c(x + 2z) + c\left((x + 2z)^{-\frac{1}{2}}x(x + 2z)^{-\frac{1}{2}}\right) + c\left((x + 2z)^{-\frac{1}{2}}2z(x + 2z)^{-\frac{1}{2}}\right) + c\left((x + 2z)^{-\frac{1}{2}}2z(x + 2z)^{-\frac{1}{2}}\right) + c\left((x + 2z)^{-\frac{1}{2}}z(x + 2z)^{-\frac{1}{2}}\right), \]
or after some rearrangement,
\[ c(x + z) + c\left((x + z)^{-\frac{1}{2}}x(x + z)^{-\frac{1}{2}}\right) + c\left((x + z)^{-\frac{1}{2}}x(x + z)^{-\frac{1}{2}}\right) = c\left((x + 2z)^{-\frac{1}{2}}x(x + 2z)^{-\frac{1}{2}}\right) + c(z) - c\left((x + 2z)^{-\frac{1}{2}}z(x + z)(x + 2z)^{-\frac{1}{2}}\right), \]
\[ (x, z \in \mathcal{V}). \]

On the other hand, \( D(x, z) = C(x, z) \), that is,
\[-c(x + z) + c(x) + c(z) = c(x + z) + c\left((x + z)^{-\frac{1}{2}}x(x + z)^{-\frac{1}{2}}\right) + c\left((x + z)^{-\frac{1}{2}}x(x + z)^{-\frac{1}{2}}\right) + c\left((x + z)^{-\frac{1}{2}}x(x + z)^{-\frac{1}{2}}\right). \]
Thus, for all $x, z \in \mathcal{V}$ we have

$$-c(x + z) + c(x) + c(z) = c \left( (x + 2z)^{-\frac{1}{2}} x(x + 2z)^{\frac{1}{2}} \right) + c(z) - c \left( (x + 2z)^{-\frac{1}{2}} (x + z)(x + 2z)^{\frac{1}{2}} \right),$$

that is,

$$-c(x + z) + c(x) = c \left( (x + 2z)^{-\frac{1}{2}} x(x + 2z)^{\frac{1}{2}} \right) - c \left( (x + 2z)^{-\frac{1}{2}} (x + z)(x + 2z)^{\frac{1}{2}} \right) \quad (x, z \in \mathcal{V}).$$

Furthermore, for all $x, z \in \mathcal{V}$

$$-2c(x + 2z) + c(x) + c(x + z) = c \left( (x + 2z)^{-\frac{1}{2}} x(x + 2z)^{\frac{1}{2}} \right) + c \left( (x + 2z)^{-\frac{1}{2}} (x + z)(x + 2z)^{\frac{1}{2}} \right)$$

Thus,

$$c(x) - c(x + 2z) = c \left( (x + 2z)^{-\frac{1}{2}} x(x + 2z)^{\frac{1}{2}} \right) \quad (x, z \in \mathcal{V})$$

holds. Let us observe that this latter identity yields that

$$c(x) = c \left( y^{-\frac{1}{2}} xy^{-\frac{1}{2}} \right) + c(y) \quad (x, y \in \mathcal{V}).$$

\[ \square \]

**Remark 2.1.** The case $\mathcal{V} = [0, +\infty[$ is trivial, since if a function $f : [0, +\infty[ \to \mathbb{R}$ is homogeneous of order zero, i.e.,

$$f(sx) = f(x) \quad (s, x \in [0, +\infty[),$$

then with the substitution $s = \frac{1}{x}$ we obtain that

$$f(x) = f(1) \quad (x \in [0, +\infty[),$$

which means that $f$ is a constant function. Therefore, in case $\mathcal{V} = [0, +\infty[$, the general solution of equation (2.2) is

$$a(x) = A(x) + \ell_1 (\det(x)) - \kappa$$
$$b(x) = A(x) + \ell_2 (\det(x)) + \kappa$$
$$c(x) = A(x) + (\ell_1 + \ell_2) (\det(x))$$
$$d(u) = \ell_1 (\det(u)) + \ell_2 (\det(1 - u))$$

where $A : \mathcal{V} \to \mathbb{R}$ is an additive function, $\ell_1, \ell_2 : [0, +\infty[ \to \mathbb{R}$ are logarithmic functions on $[0, +\infty[$ and $\kappa \in \mathbb{R}$ is a certain constant.

### 2.2. Regular solutions

Using the results of the previous subsection, we are able to determine the regular solutions of equation (2.2). This result is contained in the following statement.

**Theorem 2.1.** Let $a, b, c : \mathcal{V} \to \mathbb{R}$ and $d : \mathcal{D} \to \mathbb{R}$ be functions and let us assume that (2.2) holds for all $x, y \in \mathcal{V}$. If the functions $a, b, c$ and $d$ are continuous, then

$$a(x) = A(x) + C_1 \ln (\det(x)) - \kappa$$
$$b(x) = A(x) + C_2 \ln (\det(x)) + \kappa$$
$$c(x) = A(x) + (C_1 + C_2) \ln (\det(x))$$
$$d(u) = C_1 \ln (\det(u)) + C_2 \ln (\det(1 - u))$$

holds for all $x \in \mathcal{V}$ and $u \in \mathcal{D}$, where the function $A : \mathcal{V} \to \mathbb{R}$ is a continuous additive function.
3. The main result

In view of the results of the previous section, we are able to prove our main theorem.

**Theorem 3.1.** Let \( X \) and \( Y \) be independent random variables valued in a symmetric cone \( V \) of rank \( r > 2 \), with strictly positive and Lebesgue measurable densities. Let further
\[
V = X + Y \quad \text{and} \quad U = (X + Y)^{-\frac{2}{r}} X (X + Y)^{-\frac{2}{r}}.
\]
If \( U \) and \( V \) are independent then there exist \( a \in V \) and \( p_1, p_2 > \frac{\dim(V)}{r} - 1 \) such that
\[
X \sim \gamma_{p_1, a} \quad \text{and} \quad Y \sim \gamma_{p_2, a}
\]
holds.

**Proof.** Let \( f_X, f_Y, f_U \) and \( f_V \) denote the densities of \( X, Y, U \) and \( V \), respectively. Furthermore, let us consider the functions \( a, b, c : V \to \mathbb{R} \) and \( d : \mathcal{D} \to \mathbb{R} \) defined by
\[
\begin{align*}
a(x) &= \ln(f_X(x)), \\
b(x) &= \ln(f_Y(x)), \\
c(x) &= \ln(f_V(x)) - \frac{\dim(V)}{r} \ln(\det(x)), \\
d(u) &= \ln(f_V(u))
\end{align*}
\]
In this case
\[
a(x) + b(y) = c(x + y) + d\big(\mathbb{E}\big((x + y)^{-\frac{2}{r}}\big)\big)x
\]
holds for almost all \( x, y \in V \), where the functions \( a, b, c \) and \( d \) are measurable. Now, applying the theorem of Járai successively, in a similar way as in Lemma 1 of Mészáros [16] we get the following. There exist uniquely determined, continuous functions \( \tilde{a}, \tilde{b}, \tilde{c} : V \to \mathbb{R} \) and \( \tilde{d} : \mathcal{D} \to \mathbb{R} \) such that
\[
a = \tilde{a}, \quad b = \tilde{b}, \quad c = \tilde{c}, \quad \text{and} \quad d = \tilde{d}
\]
holds almost everywhere and
\[
a(x) + b(y) = c(x + y) + d\big(\mathbb{E}\big((x + y)^{-\frac{2}{r}}\big)\big)x
\]
is valid for all \( x, y \in V \). In view of Theorem [2, 11] this means that
\[
\begin{align*}
a(x) &= \langle \lambda, x \rangle + \ell_1(\det(x)) - \kappa \\
b(x) &= \langle \lambda, x \rangle + \ell_2(\det(x)) + \kappa
\end{align*}
\]
\((x \in V)\).

Thus
\[
\begin{align*}
f_X(x) &= \exp(a(x)) = \exp(C_1) \exp(\langle \lambda, x \rangle) \det(x)^{k_1} \\
f_Y(x) &= \exp(b(x)) = \exp(C_2) \exp(\langle \lambda, x \rangle) \det(x)^{k_2}
\end{align*}
\]
\((x \in V)\).

Since, \( f_X \) and \( f_Y \) are densities, we obtain that \( a = \lambda \in V \),
\[
k_1 = p_1 - \frac{\dim(V)}{r} > -1 \quad k_2 = p_2 - \frac{\dim(V)}{r} > -1
\]
and
\[
\exp(C_1) = \frac{\det(a)^{p_1}}{\Gamma_r(p_1)} \quad \exp(C_2) = \frac{\det(a)^{p_2}}{\Gamma_r(p_2)}.
\]

All in all,
\[
X \sim \gamma_{p_1, a} \quad \text{and} \quad Y \sim \gamma_{p_2, a}
\]
can be concluded. \( \square \)
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