ON THE MEAN-FIELD LIMIT FOR THE CONSENSUS-BASED OPTIMIZATION

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Abstract. This paper is concerned with the large particle limit for the consensus-based optimization (CBO), which was postulated in the pioneering works [6, 28]. In order to solve this open problem, we adapt a compactness argument by first proving the tightness of the empirical measures \( \{\mu_N\}_{N \geq 2} \) associated to the particle system and then verifying that the limit measure \( \mu \) is the unique weak solution to the mean-field CBO equation. Such results are extended to the model of particle swarm optimization (PSO).

Keywords: Consensus-based optimization, particle swarm optimization, propagation of chaos, tightness, weak convergence.

1. Introduction

The global optimization of a potentially nonconvex nonsmooth cost function is of great interests in various areas such as economics, physics, and artificial intelligence. In the sequel, we consider the following optimization problem

\[
 x^* \in \arg\min_{x \in \mathbb{R}^d} \mathcal{E}(x),
\]

where \( \mathcal{E}(x) : \mathbb{R}^d \to \mathbb{R} \) is a given continuous cost function, which one wishes to minimize. Many methods have been designed to tackle this kind of problems. The present paper is in particular concerned with the methods of so-called metaheuristics [1, 2, 5, 18] which provide empirically robust solutions to tackle hard optimization problems with fast algorithms. Metaheuristics are methods that orchestrate an interaction between local improvement procedures and global/high level strategies and combine random and deterministic decisions, to create a process capable of escaping from local optima and performing a robust search of a solution space. Noble examples of metaheuristics include Simplex Heuristics [27], Evolutionary Programming [12], Genetic Algorithms [20], Particle Swarm Optimization [21], Ant Colony Optimization [10], and Simulated Annealing [1]. Recently a new type of metaheuristics was proposed in [6, 28], which is referred to as consensus-based optimization (CBO) method.

The consensus-based optimization takes advantage of an interacting \( N \)-particle system \( \{(X^{i,N}_t)_{t \geq 0}\}_{i=1}^N \), which is described by a system of stochastic differential equations (SDEs)

\[
 dX^{i,N}_t = -\lambda (X^{i,N}_t - X_\alpha(\mu^N_t))dt + \sigma D(X^{i,N}_t)dB^{i}_t,
\]

where \( \lambda, \sigma > 0 \),

\[
 X_\alpha(\mu^N_t) = \frac{\int_{\mathbb{R}^d} x e^{-\alpha \mathcal{E}(x)} \mu^N_t(dx)}{\int_{\mathbb{R}^d} e^{-\alpha \mathcal{E}(x)} \mu^N_t(dx)} \quad \text{with} \quad \mu^N_t = \frac{1}{N} \sum_{i=1}^N \delta_{X^{i,N}_t},
\]

and \( \{(B^{i}_t)_{t \geq 0}\}_{i=1}^N \) are \( N \) independent \( d \)-dimensional Wiener processes. We also use the following notation for the diagonal matrix

\[
 D(X_t) := \text{diag}\{(X_t)_1, \ldots, (X_t)_d\} \in \mathbb{R}^{d \times d},
\]

H. H. is partially supported by the Pacific Institute for the Mathematical Sciences (PIMS) postdoc fellowship. J. Q. is partially supported by the National Science and Engineering Research Council of Canada (NSERC) and by the start-up funds from the University of Calgary.
where \((X_i)_k\) is the \(k\)-th component of \(X_t\). The choice of the weight function
\[
\omega^\alpha_t(x) := \exp(-\alpha \mathcal{E}(x)),
\]
comes from the well-known Laplace’s principle \([9, 26]\), a classical asymptotic method for integrals, which states that for any probability measure \(\mu \in \mathcal{P}(\mathbb{R}^d)\), there holds
\[
\lim_{\alpha \to \infty} \left( -\frac{1}{\alpha} \log \left( \int_{\mathbb{R}^d} \omega^\alpha_t(x) \mu(dx) \right) \right) = \inf_{x \in \text{supp}(\mu)} \mathcal{E}(x) .
\]
Thus for \(\alpha\) large enough, one expects that
\[
X^\alpha_\infty(\mu^N) \approx \text{argmin} \ \{ \mathcal{E}(X^1_t), \ldots, \mathcal{E}(X^N_t) \} ,
\]
which means that \(X^\alpha_\infty(\mu^N_t)\) is a global best location at time \(t\). It has been proved that CBO can guarantee global convergence under suitable assumptions \([10]\) and it is a powerful and robust method to solve many interesting non-convex high-dimensional optimization problems in machine learning \([7,14]\). By now, CBO methods have also been generalized to optimization over manifolds \([13,15,22]\) and several variants have been explored, which use additionally, for instance, personal best information \([30]\), binary interaction dynamics \([3]\) or connect CBO with Particle Swarm Optimization \([8,19]\). The readers are referred to \([31]\) for a comprehensive review on the recent developments of the CBO methods.

Because of the nonlinear and nonlocal term \(X^\alpha_\infty(\mu^N_t)\), the conventional method (see e.g. \([11,29]\)) for the mean-field limit does not work here and the pioneering CBO works \([6,28]\) postulated the large particle limit (as \(N \to \infty\)) of the system \((1.2)\) towards the Mckean process
\[
dX_t = -\lambda(X_t - X^\alpha_\infty(\mu_t))dt + \sigma D(X_t - X^\alpha_\infty(\mu_t))dB_t ,
\]
where
\[
X^\alpha_\infty(\mu_t) = \frac{\int_{\mathbb{R}^d} xe^{-\alpha \mathcal{E}(x)} \mu_t(dx)}{\int_{\mathbb{R}^d} e^{-\alpha \mathcal{E}(x)} \mu_t(dx)} \quad \text{with} \quad \mu_t = \text{Law}(X_t) .
\]
Throughout this paper, we denote by \(\mathcal{L}(X)\) the the law of random variable \(X\). Applying the Itô-Doeblin formula, one can see that \(\mu\) is a weak solution to the following mean-field partial differential equation (PDE):
\[
\frac{\partial}{\partial t} \mu_t = \frac{\sigma^2}{2} \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2}((x - X^\alpha_\infty(\mu_t))^2 \mu_t) + \lambda \nabla \cdot ((x - X^\alpha_\infty(\mu_t)) \mu_t) ,
\]
in the sense of Definition \([31]\). We refer to \([6]\) Theorem 2.1 for the well-posedness of the particle system \((1.2)\) and \([6]\) Theorem 3.2 for the nonlinear SDE \((1.5)\). While the existence of the weak solution \(\mu\) to PDE \((1.7)\) follows straightforwardly from an application of the Itô-Doeblin formula, the uniqueness may be obtained without much effort on the basis of the well-posedness of Mckean process \((1.5)\); see Lemma \([32]\) and the appendix for a sketched proof.

This paper is devoted to solving the open problem suggested in \([6,28,31]\) by providing a rigorous proof of the mean-field limit for the CBO method \((1.2)\) through a tightness argument. We first prove that the sequence of empirical measures \(\{\mu^N\}_{N \geq 2}\) (\(\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{X^i,N}\) are \(\mathcal{P}((0,T]; \mathbb{R}^d)\)-valued random variables) is tight. Prokhorov’s theorem indicates that there exists a subsequence of \(\{\mu^N\}_{N \geq 2}\) converging in law to a random measure \(\mu\). Then, to identify the limit, we verify that the limit measure \(\mu\) is a weak solution to the mean-field PDE \((1.7)\) underlying the process \((1.3)\) almost surely, while the uniqueness of the weak solution to PDE \((1.7)\) yields that \(\mu\) is actually deterministic. The approach mixes certain probabilistic and stochastic arguments and some analysis on PDEs. For such a probabilistic method with tightness arguments, we refer to \([29]\) for an introduction and the interested readers are also referred to \([17,23,24]\) for the application to the study of the propagation of chaos for the large Brownian particle system with particular Coulomb type interaction forces.
Throughout this paper the cost function $\mathcal{E}$ satisfies the following assumption.

**Assumption 1.** For the given cost function $\mathcal{E} : \mathbb{R}^d \to \mathbb{R}$, it holds that:

1. There exists some constant $L > 0$ such that $|\mathcal{E}(x) - \mathcal{E}(y)| \leq L(|x| + |y|)|x - y|$ for all $x, y \in \mathbb{R}^d$;
2. $\mathcal{E}$ is bounded from below with $-\infty < \underline{\mathcal{E}} := \inf \mathcal{E}$ and there exists some constant $C_u > 0$ such that $\mathcal{E}(x) - \underline{\mathcal{E}} \leq C_u (1 + |x|^2)$ for all $x \in \mathbb{R}^d$;
3. $\mathcal{E}$ has quadratic growth at infinity. Namely, there exist constants $C_l, M > 0$ such that $\mathcal{E}(x) - \underline{\mathcal{E}} \geq C_l |x|^2$ for all $|x| \geq M$.

The rest of the paper is organized as follows: In Section 2 we prove the tightness of the empirical measures $\{\mu^N\}_{N \geq 2}$ associated to the CBO particle system (1.2) through the Aldous criteria; see Theorem 2.1. Then in Section 3 we verify that the limit measure $\mu$ of a subsequence of $\{\mu^N\}_{N \geq 2}$ is the unique weak solution to the mean-field CBO equation (1.7); see Theorem 3.3. In Section 4, the result is extended to the model of particle swarm optimization. Finally, the existence and uniqueness of the weak solution is proved for a class of linear PDEs in Appendix.

## 2. Tightness of the empirical measures

First, let us recall the following lemma on a uniform moment estimate for the particle system (1.2) from [6, Lemma 3.4]

**Lemma 2.1.** Let $\mathcal{E}$ satisfy Assumption 1 and $\mu_0 \in \mathcal{P}_d(\mathbb{R}^d)$. For any $N \geq 2$, assume that $\{(X_{i,t}^N)_{t \in [0,T]}\}_{i=1}^N$ is the unique solution to the particle system (1.2) with $\mu_0$-distributed initial data $\{X_{0,i}^N\}_{i=1}^N$. Then there exists a constant $K > 0$ independent of $N$ such that

$$\sup_{i=1,\ldots,N} \left\{ \sup_{t \in [0,T]} \mathbb{E}\left[|X_{i,t}^N|^2 + |X_{i,t}^N|^4\right] + \sup_{t \in [0,T]} \mathbb{E}\left[|X_{\alpha}(\mu^N_t)|^2 + |X_{\alpha}(\mu^N_t)|^4\right] \right\} \leq K. \quad (2.1)$$

We treat $X_{i,t}^N : \Omega \to \mathcal{C}([0,T];\mathbb{R}^d)$. Then $\mu^N = \sum_{i=1}^N \delta_{X_{i,t}^N} : \Omega \to \mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d))$ is a random measure. Let us denote $\mathcal{L}(\mu^N) := \text{Law}(\mu^N) \in \mathcal{P}(\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d)))$. We can prove that $\{\mathcal{L}(\mu^N)\}_{N \geq 2}$ is tight, or we say $\{\mu^N\}_{N \geq 1}$ is tight.

**Theorem 2.1.** Under the same assumption as in Lemma 2.1, recall the empirical measure $\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_{i,t}^N}$. Then the sequence $\{\mathcal{L}(\mu^N)\}_{N \geq 2}$ is tight in $\mathcal{P}(\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d)))$.

**Proof.** According to [29] Proposition 2.2 (ii), we only need to prove that $\{\mathcal{L}(X_{1,N}^t)\}_{N \geq 2}$ is tight in $\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d))$ because of the exchangeability of the particle system. We shall do this by verifying the Aldous criteria below.

**Lemma 2.2.** Let $\{(X^n)_{n \in \mathbb{N}}\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and valued in $\mathcal{C}([0,T];\mathbb{R}^d)$. The sequence of probability distributions $\{\mu_{X^n}\}_{n \in \mathbb{N}}$ of $\{X^n\}_{n \in \mathbb{N}}$ is tight on $\mathcal{C}([0,T];\mathbb{R}^d)$ if the following two conditions hold.

1. (Con1) For all $t \in [0,T]$, the set of distributions of $X_{i,t}^N$, denoted by $\{\mu_{X_{i,t}^N}\}_{n \in \mathbb{N}}$, is tight as a sequence of probability measures on $\mathbb{R}^d$.
2. (Con2) For all $\varepsilon > 0$, $\eta > 0$, there exists $\delta_0 > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all discrete-valued $\sigma(X_{i,s}^N; s \in [0,T])$-stopping times $\beta$ with $0 \leq \beta + \delta_0 \leq T$, it holds that

$$\sup_{\delta \in [0,\delta_0]} \mathbb{P}\left(|X_{\beta+\delta}^N - X_{\beta}^N| \geq \eta\right) \leq \varepsilon. \quad (2.2)$$
It is now sufficient to justify conditions (Con1) and (Con2):

- **Step 1: Checking (Con1).** For any $\varepsilon > 0$, there exists a compact subset $U_\varepsilon := \{x : |x|^2 \leq \frac{K}{\varepsilon}\}$ such that by Markov's inequality

$$\mathcal{L}(X_1^{1,N})((U_\varepsilon)') = \mathbb{P}\left(|X_1^{1,N}|^2 > \frac{K}{\varepsilon}\right) \leq \frac{\varepsilon \mathbb{E}[|X_1^{1,N}|^2]}{K} \leq \varepsilon, \quad \forall N \geq 2,$$

where we have used Lemma 2.1 in the last inequality. This means that for each $t \in [0,T]$, the sequence $\{\mathcal{L}(X_t^{1,N})\}_{N \geq 2}$ is tight, which verifies condition (Con1) in Lemma 2.2.

- **Step 2: Checking (Con2).** Let $\beta$ be a $\sigma(X_t^{1,N};s \in [0,T])$-stopping time with discrete values such that $\beta + \delta_0 \leq T$. Recalling (2.2), we have

$$X_{\beta+\delta}^{1,N} - X_{\beta}^{1,N} = - \int_{\beta}^{\beta+\delta} \lambda(X_s^{1,N} - X_\alpha(\mu^N_s))ds + \sigma \int_{\beta}^{\beta+\delta} D(X_s^{1,N} - X_\alpha(\mu^N_s))dB_s^1.$$

Notice that

$$\mathbb{E}\left[\int_\beta^{\beta+\delta} \lambda(X_s^{1,N} - X_\alpha(\mu^N_s))ds\right]^2 \leq \lambda^2 \delta \int_0^T \mathbb{E}[|X_t^{1,N} - X_\alpha(\mu^N_s)|^2] ds$$

$$\leq 2\lambda^2 \delta T \left(\sup_{t \in [0,T]} \mathbb{E}[|X_t^{1,N}|^2] + \sup_{t \in [0,T]} \mathbb{E}[|X_\alpha(\mu^N_s)|^2]\right) \leq 2TK\lambda^2\delta, \quad (2.3)$$

where we have used Lemma 2.1 in the last inequality. Further we apply Itô’s isometry

$$\mathbb{E}\left[\sigma \int_{\beta}^{\beta+\delta} D(X_s^{1,N} - X_\alpha(\mu^N_s))dB_s^1\right]^2 = \sigma^2 \mathbb{E}\left[\int_{\beta}^{\beta+\delta} |X_s^{1,N} - X_\alpha(\mu^N_s)|^2 ds\right]$$

$$\leq \sigma^2 \delta \mathbb{E}\left[\left(\int_{0}^{T} |X_s^{1,N} - X_\alpha(\mu^N_s)|^4 ds\right)^{\frac{1}{2}}\right] \leq \sigma^2 \delta \left(\int_{0}^{T} \mathbb{E}[|X_s^{1,N} - X_\alpha(\mu^N_s)|^4] ds\right)^{\frac{1}{2}}$$

$$\leq \sigma^2 \delta T^{\frac{1}{2}} (8K)^{\frac{1}{2}}. \quad (2.4)$$

Combining estimates 2.3 and 2.4 one has

$$\mathbb{E}[|X_{\beta+\delta}^{1,N} - X_{\beta}^{1,N}|^2] \leq C(\lambda, \sigma, T, K) \left(\delta^\frac{1}{2} + \delta\right). \quad (2.5)$$

Hence, for any $\varepsilon > 0$, $\eta > 0$, there exists some $\delta_0 > 0$ such that for all $N \geq 2$ it holds that

$$\sup_{\delta \in [0,\delta_0]} \mathbb{P}\left(|X_{\beta+\delta}^{1,N} - X_{\beta}^{1,N}|^2 \geq \eta\right) \leq \sup_{\delta \in [0,\delta_0]} \frac{\mathbb{E}[|X_{\beta+\delta}^{1,N} - X_{\beta}^{1,N}|^2]}{\eta} \leq \varepsilon. \quad (2.6)$$

This justifies condition Con2 in Lemma 2.2 and completes the proof of Theorem 2.1 \(\square\)

As a consequence of the tightness in Theorem 2.1 we obtain the following results.

**Lemma 2.3.**  
(1) There exist a subsequence of $\left\{\mu^N\right\}_{N \geq 2}$ (denoted w.l.o.g. by itself) and a random measure $\mu : \Omega \to \mathcal{P}(C([0,T];\mathbb{R}^d))$ such that

$$\mu^N \to \mu \text{ in law as } N \to \infty, \quad (2.7)$$

which is equivalent to say $\mathcal{L}(\mu^N)$ converges weakly to $\mathcal{L}(\mu)$ in $\mathcal{P}(\mathcal{P}(C([0,T];\mathbb{R}^d)))$

(2) For the subsequence in (1), the time marginal $\mu^N_t$ of $\mu^N$, as $\mathcal{P}(\mathbb{R}^d)$ valued random measure converges in law to $\mu_t \in \mathcal{P}(\mathbb{R}^d)$, the time marginal of $\mu$. Namely $\mathcal{L}(\mu^N_t)$ converges weakly to $\mathcal{L}(\mu_t)$ in $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$
Proof. By Prokhorov’s theorem, assertion (1) follows from the tightness of \( \{ \mathcal{L}(\mu^N) \}_{N \geq 2} \) in \( \mathcal{P}(\mathcal{C}([0,T]; \mathbb{R}^d)) \) as shown in Theorem 2.1.

As for assertion (2), we first notice that a sequence \( \nu^n \in \mathcal{P}(\mathcal{C}([0,T]; \mathbb{R}^d)) \) that converges weakly to \( \nu \in \mathcal{P}(\mathcal{C}([0,T]; \mathbb{R}^d)) \) will imply that \( \nu^n \in \mathcal{P}(\mathbb{R}^d) \) converges weakly to \( \nu \in \mathcal{P}(\mathbb{R}^d) \) for each time \( t \). Indeed, for each \( \phi \in C_b(\mathbb{R}^d) \), we have \( \int_{\mathbb{R}^d} \phi(x)\nu^n_t(dx) = \int_{\mathcal{C}([0,T]; \mathbb{R}^d)} \phi(x_t)\nu^n(dx) \) (see [23] Lemma 2.8). Note that for all \( x \in \mathcal{C}([0,T]; \mathbb{R}^d) \), \( x \mapsto \phi(x_t) \) is a bounded continuous functional on \( \mathcal{C}([0,T]; \mathbb{R}^d) \), which leads to

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} \phi(x)\nu^n_t(dx) = \lim_{n \to \infty} \int_{\mathcal{C}([0,T]; \mathbb{R}^d)} \phi(x_t)\nu^n(dx) = \int_{\mathcal{C}([0,T]; \mathbb{R}^d)} \phi(x_t)\nu(dx) = \int_{\mathbb{R}^d} \phi(x)\nu_t(dx).
\]

Now we consider a bounded continuous functional \( \Gamma : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R} \), then one defines \( \Gamma_1 : \mathcal{P}(\mathcal{C}([0,T]; \mathbb{R}^d)) \to \mathbb{R} \) as

\[
\Gamma_1(\nu) := \Gamma(\nu_t) \quad \text{for any} \quad \nu \in \mathcal{P}(\mathcal{C}([0,T]; \mathbb{R}^d)).
\]

This means that \( \Gamma_1 \) is a bounded continuous functional on \( \mathcal{P}(\mathcal{C}([0,T]; \mathbb{R}^d)) \) according to what has been justified. Consequently,

\[
\mathbb{E}[\Gamma_1(\mu^N)] \to \mathbb{E}[\Gamma_1(\mu)] \Rightarrow \mathbb{E}[\Gamma(\mu^N)] \to \mathbb{E}[\Gamma(\mu)],
\]

which implies assertion (2). \( \square \)

3. Identification of the Limit Measure via PDE

**Definition 3.1.** We say \( \mu_\theta \in \mathcal{C}([0,T]; \mathcal{P}_2(\mathbb{R}^d)) \) is a weak solution to PDE (1.7) if

(i) The continuity in time is in \( C^0_b \) topology, namely it holds

\[
\int_{\mathbb{R}^d} \phi(x)\mu_t(dx) \to \int_{\mathbb{R}^d} \phi(x)\mu_t(dx)
\]

for all \( \phi \in C_b(\mathbb{R}^d) \) and \( t_n \to t \);

(ii) The following holds

\[
\langle \varphi(x), \mu_t(dx) \rangle - \langle \varphi(x), \mu_0(dx) \rangle + \lambda \int_0^t \langle (x - X_\alpha(\mu_s)) \cdot \nabla \varphi(x), \mu_s(dx) \rangle ds
\]

\[
-\int_0^t \sum_{k=1}^d \left( x - X_\alpha(\mu_s) \right)_k^2 \frac{\partial^2}{\partial x_k^2} \varphi(x), \mu_s(dx) \right) ds = 0,
\]

for all \( \varphi \in C^2_b(\mathbb{R}^d) \).

First, for each \( \varphi \in C^2_b(\mathbb{R}^d) \), we define a functional on \( \mathcal{P}(\mathcal{C}([0,T]; \mathbb{R}^d)) \) as following

\[
F_\varphi(\nu) := \langle \varphi(x_t), \nu(dx) \rangle - \langle \varphi(x_0), \nu(dx) \rangle + \lambda \int_0^t \langle (x_s - X_\alpha(\nu_s)) \cdot \nabla \varphi(x_s), \nu(dx) \rangle ds
\]

\[
-\int_0^t \sum_{k=1}^d \left( x_s - X_\alpha(\nu_s) \right)_k^2 \frac{\partial^2}{\partial x_k^2} \varphi(x_s), \nu_s(dx) \right) ds
\]

\[
= \langle \varphi(x), \nu_t(dx) \rangle - \langle \varphi(x), \nu_0(dx) \rangle + \lambda \int_0^t \langle (x_s - X_\alpha(\nu_s)) \cdot \nabla \varphi(x_s), \nu_s(dx) \rangle ds
\]

\[
-\int_0^t \sum_{k=1}^d \left( x_s - X_\alpha(\nu_s) \right)_k^2 \frac{\partial^2}{\partial x_k^2} \varphi(x_s), \nu_s(dx) \right) ds,
\]

for all \( \nu \in \mathcal{P}(\mathcal{C}([0,T]; \mathbb{R}^d)) \) and \( x \in \mathcal{C}([0,T]; \mathbb{R}^d) \). Recall that here

\[
X_\alpha(\nu_s) = \frac{\int_{\mathbb{R}^d} xe^{-\alpha \xi(x)}\nu_s(dx)}{\int_{\mathbb{R}^d} e^{-\alpha \xi(x)}\nu_s(dx)} = \frac{\langle xe^{-\alpha \xi(x)}, \nu_s(dx) \rangle}{\langle e^{-\alpha \xi(x)}, \nu_s(dx) \rangle}.
\]

Then we have the following estimate.
Proposition 3.2. Let $\mathcal{E}$ satisfy Assumption 1 and $\mu_0 \in \mathcal{P}_4(\mathbb{R}^d)$. For any $N \geq 2$, assume that $\{(X_{i,N}^t)_{t \in [0,T]}\}_{i=1}^N$ is the unique solution to the particle system (1.2) with $\mu_0$-distributed initial data $\{X_{i,N}^0\}_{i=1}^N$. There exists a constant $C > 0$ depending only on $\sigma, K, T,$ and $\|\nabla \varphi\|_\infty$ such that
\[
E[|F_\varphi(\mu^N)|^2] \leq \frac{C}{N},
\]
where $\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_{i,N}}$ is the empirical measure.

Proof. Using the definition of $F_\varphi$, one has
\[
F_\varphi(\mu^N) = \frac{1}{N} \sum_{i=1}^N \varphi(X_{i,N}^t) - \frac{1}{N} \sum_{i=1}^N \varphi(X_{0,N}^t) + \lambda \int_0^t \frac{1}{N} \sum_{i=1}^N (X_{s,N}^{i,N} - X_\alpha(\mu^N)_s) \cdot \nabla \varphi(X_{s,N}^{i,N}) ds
- \frac{\sigma^2}{2} \int_0^t \frac{1}{N} \sum_{i=1}^N \sum_{k=1}^d (X_{s,N}^{i,N} - X_\alpha(\mu^N)_s)_k \frac{\partial^2}{\partial x_k^2} \varphi(X_{s,N}^{i,N}) ds
\]
\[= \frac{1}{N} \sum_{i=1}^N \left( \varphi(X_{i,N}^t) - \varphi(X_{0,N}^t) + \lambda \int_0^t (X_{s,N}^{i,N} - X_\alpha(\mu^N)_s) \cdot \nabla \varphi(X_{s,N}^{i,N}) ds
- \frac{\sigma^2}{2} \int_0^t \sum_{k=1}^d (X_{s,N}^{i,N} - X_\alpha(\mu^N)_s)_k \frac{\partial^2}{\partial x_k^2} \varphi(X_{s,N}^{i,N}) ds \right).
\]
(3.6)

For each $i = 1, \cdots, N$ and $\varphi \in C^2_c(\mathbb{R}^d)$, applying Itô-Doeblin formula gives
\[
\varphi(X_{i,N}^t) = \varphi(X_{0,N}^t) - \int_0^t \lambda \nabla \varphi(X_{i,N}^s) \cdot (X_{s,N}^{i,N} - X_\alpha(\mu^N)_s) ds + \sigma \int_0^t D(X_{s,N}^{i,N} - X_\alpha(\mu^N)_s) \nabla \varphi(X_{s,N}^{i,N}) \cdot dB_s^i + \frac{\sigma^2}{2} \int_0^t \sum_{k=1}^d (X_{s,N}^{i,N} - X_\alpha(\mu^N)_s)_k \frac{\partial^2}{\partial x_k^2} \varphi(X_{s,N}^{i,N}) ds.
\]
(3.7)

This implies that
\[
F_\varphi(\mu^N) = \frac{\sigma}{N} \sum_{i=1}^N \int_0^t D(X_{s,N}^{i,N} - X_\alpha(\mu^N)_s) \nabla \varphi(X_{s,N}^{i,N}) \cdot dB_s^i.
\]
(3.8)

Then it holds that
\[
E[|F_\varphi(\mu^N)|^2] = \frac{\sigma^2}{N^2} E \left[ \sum_{i=1}^N \int_0^t D(X_{s,N}^{i,N} - X_\alpha(\mu^N)_s) \nabla \varphi(X_{s,N}^{i,N}) \cdot dB_s^i \right]^2
\]
\[= \frac{\sigma^2}{N^2} \sum_{i=1}^N E \left[ \int_0^t D(X_{s,N}^{i,N} - X_\alpha(\mu^N)_s) \nabla \varphi(X_{s,N}^{i,N}) \cdot dB_s^i \right]^2
\]
\[= \frac{\sigma^2}{N^2} \sum_{i=1}^N E \left[ \int_0^t |X_{s,N}^{i,N} - X_\alpha(\mu^N)_s|^2 |\nabla \varphi(X_{s,N}^{i,N})|^2 ds \right]
\]
\[\leq C(\sigma, K, T, \|\nabla \varphi\|_\infty) \frac{1}{N},
\]
(3.9)
where we have used Lemma 2.1 in the last inequality. This completes the proof. \hfill \Box

By Skorokhod’s lemma (see [1] Theorem 6.7 on page 70), using Lemma 2.3 we may find a common probability space $(\Omega, \mathcal{F}, P)$ on which the processes $\{\mu^N\}_{N \in \mathbb{N}}$ converge to some process $\mu$ as a random variable valued in $\mathcal{P}(\mathcal{C}([0,T]; \mathbb{R}^d))$ almost surely. In particular, we have that for all $t \in [0,T]$ and $\phi \in C_b(\mathbb{R}^d),$
\[
\lim_{N \to \infty} |\langle \phi, \mu^N_t - \mu_t \rangle + |X_\alpha(\mu^N_t) - X_\alpha(\mu_t)|| = 0, \text{ a.s.}
\]
(3.10)
Indeed, according to Assumption 1 one has $xe^{-\alpha x(t)}$, $e^{-\alpha x(t)} \in C_b(\mathbb{R}^d)$, which gives

$$\lim_{N \to \infty} X_\alpha(\mu^N) = \lim_{N \to \infty} \frac{\langle xe^{-\alpha x(t)}, \mu^N \rangle}{\langle e^{-\alpha x(t)}, \mu^N \rangle} = \frac{\langle xe^{-\alpha x(t)}, \mu(t) \rangle}{\langle e^{-\alpha x(t)}, \mu(t) \rangle} = X_\alpha(\mu) \quad \text{a.s.}$$ (3.11)

**Lemma 3.1.** [Lemma 3.3] Let $\mathcal{E}$ satisfy Assumption 2 and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. Then it holds that

$$|X_\alpha(\mu)|^2 \leq b_1 + b_2 \int_{\mathbb{R}^d} |x|^2 \mu(dx),$$ (3.12)

where $b_1$ and $b_2$ depends only on $M$, $C_u$, and $C_t$.

For each $A > 0$, let us take $\phi = | \cdot |^4 \wedge A \in C_b(\mathbb{R}^d)$. It follows from (3.10) that

$$E \left[ \int_{\mathbb{R}^d} (|x|^4 \wedge A) \mu_t(x) \right] = E \left[ \lim_{N \to \infty} \int_{\mathbb{R}^d} (|x|^4 \wedge A) \mu^N_t(x) \right] \leq \lim_{N \to \infty} \frac{\sum_{i=1}^N E[|X_{i,N}^4|]}{N} \leq K,$$ (3.13)

where we have used Lemma 2.1. Letting $A \to \infty$, we have

$$\sup_{t \in [0,T]} E \left[ \int_{\mathbb{R}^d} |x|^4 \mu_t(x) \right] \leq K.$$ (3.14)

Then Lemma 3.1 implies that

$$E[|X_\alpha(\mu)|^4] < \infty,$$ (3.15)

for all $t \in [0,T]$.

Furthermore, it holds that

$$\lim_{N \to \infty} E \left[ (\phi, \mu^N_t - \mu_t)^2 + |X_\alpha(\mu^N_t) - X_\alpha(\mu_t)|^2 \right] = 0,$$ (3.16)

which follows directly from the pointwise convergences of $\langle \phi, \mu^N_t - \mu_t \rangle$ and $X_\alpha(\mu^N_t) - X_\alpha(\mu_t)$, and the uniform estimate (2.1) in Lemma 2.1 and (3.15). To see this, let us consider a sequence of random variables $\{X_n\}_{n \geq 1}$, which satisfies that $X_n \to 0 \ (n \to \infty)$ pointwisely and $\sup_{n \geq 1} E[|X_n|^4] \leq C$ uniformly in $n$. For all $A > 0$, we compute

$$E[|X_n|^2] = E[|X_n|^2 I_{|X_n| \leq A} + E[|X_n|^2 I_{|X_n| > A}] \leq E[|X_n|^2 I_{|X_n| \leq A} + (E[|X_n|^4])^{\frac{1}{2}}(E[|X_n|^4])^{\frac{1}{2}}].$$ (3.17)

It is obvious that $E[|X_n|^2 I_{|X_n| \leq A}] \to 0 \ (n \to \infty)$ holds by the dominated convergence theorem. One also notices that

$$(E[|X_n|^4])^{\frac{1}{2}}(E[|X_n|^4])^{\frac{1}{2}} \leq (E[|X_n|^4])^{\frac{1}{2}}(E[|X_n|^4])^{\frac{1}{2}} \leq \frac{C}{A^2} \to 0 \quad \text{as } A \to \infty,$$ (3.18)

which leads to $E[|X_n|^2] \to 0$ as $n \to \infty$.

**Theorem 3.3.** Let $\mathcal{E}$ satisfy Assumption 2 and $\mu_0 \in \mathcal{P}_d(\mathbb{R}^d)$. For any $N \geq 2$, assume that $\{(X_{i,N}^t)_{t \in [0,T]}\}_{i=1}^N$ is the unique solution to the particle system (1.2) with $\mu_0^{\otimes N}$-distributed initial data $(X_{0,N}^i)_{i=1}^N$. Then the limit (denoted by $\mu$) of the sequence of the empirical measure $\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_{i,N}}$ exists. Moreover, $\mu$ is deterministic and it is the unique weak solution to PDE (1.7).

**Proof.** Suppose the $\mathcal{P}(C([0,T];\mathbb{R}^d))$-valued random variable $\mu$ is the limit of a subsequence of the empirical measure $\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_{i,N}}$. W.l.o.g., Denote the subsequence by itself. We may continue to work on the above common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by Skorokhod’s lemma where the convergence is holding almost surely (see 2.10 for instance). We may first check that $\mu_t$ is a.s. continuous in time in the sense of 2.11. Indeed for any $\phi \in C_b(\mathbb{R}^d)$ and $t_n \to t$ we may apply dominated convergence theorem

$$\int_{C([0,T];\mathbb{R}^d)} \phi(x_{i,N}) \mu(dx) \to \int_{C([0,T];\mathbb{R}^d)} \phi(x_t) \mu(dx) \quad \text{a.s.},$$
which gives 
\[ \int_{\mathbb{R}^d} \phi(x) \mu_t(dx) \to \int_{\mathbb{R}^d} \phi(x) \mu_t(dx) \quad \text{a.s.} \]

For \( \varphi \in C^2_c(\mathbb{R}^d) \), using the convergence result in (3.16) one has
\[ \lim_{N \to \infty} \mathbb{E} \left[ \left| \langle \varphi(x), \mu^N_t(dx) \rangle - \langle \varphi(x), \mu^0_t(dx) \rangle \right| ight] = 0. \tag{3.19} \]

Further we notice that
\[ \left| \int_0^t \langle (x - X_\alpha(\mu^N_s)), \nabla \varphi(x), \mu^N_s(dx) \rangle ds - \int_0^t \langle (x - X_\alpha(\mu_s)), \nabla \varphi(x), \mu_s(dx) \rangle ds \right| \]
\[ \leq \int_0^t \left| \langle (x - X_\alpha(\mu^N_s)), \nabla \varphi(x), \mu^N_s(dx) - \mu_s(dx) \rangle \right| ds + \int_0^t \left| \langle (X_\alpha(\mu_s) - X_\alpha(\mu^N_s)), \nabla \varphi(x), \mu_s(dx) \rangle \right| ds \]
\[ =: \int_0^t |I^N_t(s)| ds + \int_0^t |I^N_t(s)| ds. \tag{3.20} \]

One computes
\[ \mathbb{E}[|I^N_t(s)|] \leq \mathbb{E}[|x \cdot \nabla \varphi(x) - \mu^N_s(dx) - \mu_s(dx)\rangle] + \mathbb{E}[|X_\alpha(\mu^N_s) \cdot (\nabla \varphi(x), \mu^N_s(dx) - \mu_s(dx))\rangle] \]
\[ \leq \mathbb{E}[|x \cdot \nabla \varphi(x) - \mu^N_s(dx) - \mu_s(dx)\rangle] + K^2 \left( \mathbb{E}[|\nabla \varphi(x), \mu^N_s(dx) - \mu_s(dx)\rangle|^2] \right)^{\frac{1}{2}}, \tag{3.21} \]

where we have used Lemma 2.1 in the second inequality. Since \( \varphi \) has a compact support, applying (3.16) leads to
\[ \lim_{N \to \infty} \mathbb{E}[|I^N_t(s)|] = 0. \tag{3.22} \]

Moreover, the uniform boundedness of \( \mathbb{E}[|I^N_t(s)|] \) follows directly from (3.14), (3.15), and the estimates in Lemma 2.1 which by the dominated convergence theorem implies
\[ \lim_{N \to \infty} \int_0^t \mathbb{E}[|I^N_t(s)|] ds = 0. \tag{3.23} \]

As for \( I^N_2 \), we know that
\[ \left| \langle X_\alpha(\mu_s) - X_\alpha(\mu^N_s), \nabla \varphi(x), \mu_s(dx) \rangle \right| \leq \| \nabla \varphi \|_{\infty} |X_\alpha(\mu_s) - X_\alpha(\mu^N_s)|. \tag{3.24} \]

Hence by (3.16) it yields that
\[ \lim_{N \to \infty} \mathbb{E}[|I^N_2(s)|] = 0. \tag{3.25} \]

Again by the dominated convergence theorem, we have
\[ \lim_{N \to \infty} \int_0^t \mathbb{E}[|I^N_2(s)|] ds = 0. \tag{3.26} \]

This combined with (3.22) leads to
\[ \lim_{N \to \infty} \mathbb{E}\left[ \left| \int_0^t \langle (x - X_\alpha(\mu^N_s)), \nabla \varphi(x), \mu^N_s(dx) \rangle ds - \int_0^t \langle (x - X_\alpha(\mu_s)), \nabla \varphi(x), \mu_s(dx) \rangle ds \right| \right] = 0. \tag{3.27} \]

Similarly we split the error
\[ \left| \int_0^t \langle (x - X_\alpha(\mu^N_s)), \nabla \varphi(x), \mu^N_s(dx) \rangle ds - \int_0^t \langle (x - X_\alpha(\mu_s)), \nabla \varphi(x), \mu_s(dx) \rangle ds \right| \]
\[ \leq \int_0^t \left| \langle (x - X_\alpha(\mu^N_s)), \nabla \varphi(x), \mu^N_s(dx) - \mu_s(dx) \rangle \right| ds + \int_0^t \left| \langle (x - X_\alpha(\mu^N_s)), \nabla \varphi(x), \mu_s(dx) \rangle \right| ds \]
\[ =: \int_0^t |I^N_2(s)| ds + \int_0^t |I^N_4(s)| ds. \tag{3.28} \]

Following the same argument as for \( I^N_1 \) and \( I^N_2 \), one has
\[ \lim_{N \to \infty} \int_0^t \mathbb{E}[|I^N_3(s)|] ds = 0 \quad \text{and} \quad \lim_{N \to \infty} \int_0^t \mathbb{E}[|I^N_4(s)|] ds = 0. \tag{3.29} \]
This implies that
\[
\lim_{N \to \infty} \mathbb{E} \left[ \int_0^t \sum_{k=1}^d (x - X_\alpha(\mu^N_k))^2 \frac{\partial^2}{\partial x_k^2} \varphi(x, \mu^N_k(\text{d}x)) \text{d}s - \int_0^t \sum_{k=1}^d (x - X_\alpha(\mu_k))^2 \frac{\partial^2}{\partial x_k^2} \varphi(x, \mu_k(\text{d}x)) \text{d}s \right] = 0. \quad (3.30)
\]

Collecting estimates (3.19), (3.27) and (3.30) we have
\[
\lim_{N \to \infty} \mathbb{E}[|F_\varphi(\mu^N) - F_\varphi(\mu)|] = 0. \quad (3.31)
\]
Then we have
\[
\mathbb{E}[|F_\varphi(\mu)|] \leq \mathbb{E}[|F_\varphi(\mu^N) - F_\varphi(\mu)|] + \mathbb{E}[|F_\varphi(\mu^N)|] \leq \mathbb{E}[|F_\varphi(\mu^N) - F_\varphi(\mu)|] + \frac{C}{\sqrt{N}} \to 0 \quad \text{as } N \to \infty, \quad (3.32)
\]
where we have used Proposition 3.2 in the last inequality. This implies that
\[
F_\varphi(\mu) = 0 \quad \text{a.s.} \quad (3.33)
\]
In other words, it holds that
\[
\langle \varphi(x), \mu_t(\text{d}x) \rangle - \langle \varphi(x), \mu_0(\text{d}x) \rangle + \lambda \int_0^t \langle (x - X_\alpha(\mu_s)), \nabla \varphi(x, \mu_s(\text{d}x)) \rangle \text{d}s - \frac{\sigma^2}{2} \int_0^t \sum_{k=1}^d \langle (x - X_\alpha(\mu_s))^2 \frac{\partial^2}{\partial x_k^2} \varphi(x, \mu_s(\text{d}x)) \rangle \text{d}s = 0 \quad \text{a.s.}, \quad (3.34)
\]
for any \( \varphi \in C^2_0(\mathbb{R}^d) \).

Until now we have proved that \( \mu \) a.s. is a weak solution to PDE (1.7). Finally combining the uniqueness of weak solution to (1.7) (see in Lemma 3.2 below) and the arbitrariness of the subsequence of \( \{\mu^N\}_{N \geq 1} \), the (deterministic) weak solution \( \mu \) to PDE (1.7) must be the limit of the whole sequence \( \{\mu^N\}_{N \geq 1} \). We complete the proof.

**Lemma 3.2.** Assume that \( \mu^1, \mu^2 \in C([0,T];\mathcal{P}_2(\mathbb{R}^d)) \) are two weak solutions to PDE (1.7) in the sense of Definition 3.1 with the same initial data \( \mu_0 \). Then it holds that
\[
\sup_{t \in [0,T]} W_2(\mu^1_t, \mu^2_t) = 0,
\]
where \( W_2 \) is the 2-Wasserstein distance.

**Proof.** We construct two linear processes \((\hat{X}^i_t)_{t \in [0,T]} \, i = 1, 2\) satisfying
\[
d\hat{X}^i_t = -\lambda(\hat{X}^i_t - X_\alpha(\mu^i_t))dt + \sigma D(\hat{X}^i_t - X_\alpha(\mu^i_t))dB_t, \quad (3.35)
\]
with the common initial data \( \hat{X}_0 \) distributed according to \( \mu_0 \). Above processes are linear because that \( \mu^i \) are prescribed. Let us denote \( \text{law}(\hat{X}^i_t) = \hat{\mu}^i_t \, (i = 1, 2) \), which are weak solutions to the following linear PDE
\[
\partial_t \hat{\mu}^i_t = \frac{\sigma^2}{2} \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2} \left( (x - X_\alpha(\mu^i_t))^2 \right) \hat{\mu}^i_t - \lambda \nabla \cdot ((x - X_\alpha(\mu^i_t))\hat{\mu}^i_t),
\]
where \( X_\alpha(\mu^i_t) \in C([0,T];\mathbb{R}^d) \) for given \( \mu^i \in C([0,T],\mathcal{P}_2(\mathbb{R}^d)) \). By the uniqueness of weak solution to the above linear PDE (see Theorem 3.3 in Appendix) and the fact that \( \mu^i \) is also a solution to the above PDE, it follows that \( \hat{\mu}^i_t = \mu^i_t \, (i = 1, 2) \). Consequently, the process \((\hat{X}^i_t)_{t \in [0,T]} = (\overline{X}^i_t)_{t \in [0,T]} \) are solutions to the nonlinear SDE (1.3), for which the uniqueness has been obtained in [3, Theorem 3.2]. In particular, it holds that
\[
\sup_{t \in [0,T]} \mathbb{E} \left[ |\overline{X}^i_t - \overline{X}^j_t|^2 \right] = 0, \quad (3.36)
\]
which by the definition of Wasserstein distance implies
\[
\sup_{t \in [0,T]} W_2(\mu^i_t, \mu^j_t) = \sup_{t \in [0,T]} W_2(\hat{\mu}^i_t, \hat{\mu}^j_t) \leq \sup_{t \in [0,T]} \mathbb{E}[|\hat{X}^i_t - \hat{X}^j_t|^2] = \sup_{t \in [0,T]} \mathbb{E}[|\hat{X}^i_t - \hat{X}^j_t|^2] = 0.
\]

Thus the uniqueness is obtained.

4. MEAN-FIELD LIMIT FOR PARTICLE SWARM OPTIMIZATION

In this section we extend our discussions to the model of particle swarm optimization (PSO) proposed recently by Grassi and Pareschi [13], where they only numerically verified the mean-limit result. We consider PSO based on a continuous description in the form of a system of stochastic differential equations:

\[
\begin{cases}
    dX^i_t = V^i_t dt, \\
    dV^i_t = -\frac{\gamma}{m} V^i_t dt + \frac{\alpha}{m} (X^\alpha_t - X^i_t) dt + \frac{\sigma}{m} dB^i_t, \\
\end{cases}
\]

(4.1)

where the \( \mathbb{R}^d \)-valued functions \( X^i_t \) and \( V^i_t \) denote the position and velocity of the \( i \)-th particle at time \( t \), \( m > 0 \) is the inertia weight, \( \gamma = 1 - m \geq 0 \) is the friction coefficient, \( \lambda > 0 \) is the acceleration coefficient, \( \sigma > 0 \) is the diffusion coefficient, and \( \{(B^i_t)_{t \geq 0}\}_{i=1}^N \) are \( N \) independent \( d \)-dimensional Brownian motions.

Here the weighted average is given by

\[
X^\alpha_t = \frac{1}{N} \sum_{i=1}^N \delta_{X^i_t} = \frac{1}{N} \sum_{i=1}^N \delta_{X^i_t, V^i_t},
\]

(4.2)

with the empirical measure \( \rho^N := \frac{1}{N} \sum_{i=1}^N \delta_{X^i_t, V^i_t} \), which is the spacial marginal of

\[
f^N = \frac{1}{N} \sum_{i=1}^N \delta_{(X^i_t, V^i_t)} : \Omega \rightarrow \mathcal{P}(\mathcal{C}([0,T]; \mathbb{R}^d) \times \mathcal{C}([0,T]; \mathbb{R}^d)).
\]

Parallel to Theorem 2.1, we can prove the tightness of the empirical measures \( \{f^N\}_{N \geq 2} \) by verifying the Aldous criteria presented in Lemma 2.2.

**Theorem 4.1.** Let \( E \) satisfy Assumption 1 and \( f_0 \in \mathcal{P}_4(\mathbb{R}^d \times \mathbb{R}^d) \). For any \( N \geq 2 \), we assume that \( \{(X^i_t, V^i_t)_{t \in [0,T]}\}_{i=1}^N \) is the unique solution to the particle system (4.1) with \( f_0^N \)-distributed initial data \( \{X^0_t, V^0_t\}_{i=1}^N \). Then the sequence \( \{\mathcal{L}(f^N)\}_{N \geq 2} \) is tight in \( \mathcal{P}(\mathcal{P}(\mathcal{C}([0,T]; \mathbb{R}^d) \times \mathcal{C}([0,T]; \mathbb{R}^d))). \)

**Proof.** It is sufficient to justify conditions (Con1) and (Con2) in Lemma 2.2

- **Step 1: Checking (Con1)**. It is obvious that

\[
\mathbb{E}[|X^i_t|^4] \leq 2^3 \mathbb{E}[|X^0_t|^4] + 2^3 T^3 \int_0^T \mathbb{E}[|V^i_s|^4] ds,
\]

(4.3)

holds for each \( i = 1, \cdots, N \). Applying Doob’s martingale inequality, we further obtain

\[
\begin{align*}
\mathbb{E}[|V^i_t|^4] &\leq 4^3 \mathbb{E}[|V^0_t|^4] + 4^3 \frac{\lambda^4}{m^3} T^3 \int_0^T \mathbb{E}[|V^i_s|^4] ds + 4^3 \frac{\lambda^4}{m^3} T^3 \int_0^T \mathbb{E}[|X^\alpha_s - X^i_s|^4] ds \\
&\quad + \frac{4^7 \sigma^4}{3^4 m^4} T \int_0^T \mathbb{E}[|X^\alpha_s - X^i_s|^4] ds \\
&\leq C \mathbb{E}[|V^0_t|^4] + C \int_0^T \mathbb{E}[|V^i_s|^4 + |X^i_s|^4] ds + C \int_0^T \mathbb{E}[|X^\alpha_s|^4] ds,
\end{align*}
\]

(4.4)

where \( C \) is independent of \( N \). Thus it holds that

\[
\mathbb{E}[|X^i_t|^4 + |V^i_t|^4] \leq C \mathbb{E}[|X^0_t|^4 + |V^0_t|^4] + C \int_0^T \mathbb{E}[|V^i_s|^4 + |X^i_s|^4] ds + C \int_0^T \mathbb{E}[|X^\alpha_s|^4] ds.
\]

(4.5)
Summing the above estimate over \( i = 1, \ldots, N \), dividing by \( N \) and using the linearity of the expectation, we have

\[
\mathbb{E} \int (|x|^4 + |v|^4) f^N_t (dx, dv) \leq C \mathbb{E} \int (|x|^4 + |v|^4) f^N_0 (dx, dv) + C \int_0^t \mathbb{E} \int (|x|^4 + |v|^4) f^N_s (dx, dv) ds + C \int_0^t \mathbb{E} ||X^\alpha (\rho^N_s)||^4 ds .
\]

(4.6)

It follows from Lemma 3.1 that

\[
||X^\alpha (\rho^N_s)||^4 \leq (b_1 + b_2 \int |x|^2 \rho^N_s (dx))^2 \leq 2(b_1^2 + b_2^2 \int |x|^4 \rho^N_s (dx)) \leq 2(b_1^2 + b_2^2 \int (|x|^4 + |v|^4) f^N_s (dx, dv)) .
\]

(4.7)

Inserting this into (4.6) and applying Gronwall’s inequality yield that

\[
\sup_{t \in [0, T]} \mathbb{E} \int (|x|^4 + |v|^4) f^N_t (dx, dv) \leq K ,
\]

(4.8)

where \( K \) is independent of \( N \). This implies \( \sup_{t \in [0, T]} \mathbb{E} ||X^\alpha (\rho^N_s)||^4 \leq 2(b_1^2 + b_2^2 K) \). Then applying Gronwall’s inequality on (4.5) we have

\[
\sup_{t \in [0, T]} \sup_{i=1, \ldots, N} \mathbb{E} ||X^i_t||^4 + |V^{i,N}_t|^4 \leq K' ,
\]

(4.9)

where \( K' > 0 \) is independent of \( N \). Then (Con1) may be verified in a similar way to Theorem 2.1.

* Step 1: Checking (Con2). Let \( \beta \) be a \( \sigma((X^1_s, V^{s,1}_s); s \in [0, T]) \)-stopping time with discrete values such that \( \beta + \delta_0 \leq T \). It is easy to see that

\[
\mathbb{E} [||X^{1,N}_{\beta+\delta} - X^{1,N}_\beta||^2] \leq \delta \int_0^T \mathbb{E} [||V^{1,N}_s||^2] ds \leq C \delta ,
\]

(4.10)

where \( C > 0 \) is independent of \( N \) by (4.9). Furthermore, following similar arguments as in (2.3) – (2.4), one obtains

\[
\mathbb{E} [||V^{1,N}_{\beta+\delta} - V^{1,N}_\beta||^2] \leq C(\delta + \delta^{1/2}) .
\]

(4.11)

Hence (Con2) is verified.

For any \( \varphi \in C^2_c(\mathbb{R}^d \times \mathbb{R}^d) \), define a functional on \( \mathcal{P}(\mathcal{C}([0, T]; \mathbb{R}^d) \times \mathcal{C}([0, T]; \mathbb{R}^d)) \) as following

\[
F_\varphi (f) := \langle \varphi (x_t, v_t), f (dx, dv) \rangle - \langle \varphi (x_0, v_0), f (dx, dv) \rangle + \int_0^t \langle v_s \cdot \nabla_x \varphi, f (dx, dv) \rangle ds - \frac{\gamma}{m} \int_0^t \langle v_s \cdot \nabla_x \varphi, f (dx, dv) \rangle ds + \frac{\lambda}{m} \int_0^t \langle (x_t - X_{\alpha} (\rho_s)) \cdot \nabla_x \varphi, f (dx, dv) \rangle ds
\]

\[
- \frac{\sigma^2}{2m^2} \int_0^t \sum_{k=1}^d \langle (x_t - X_{\alpha} (\rho_s))^2 \frac{\partial^2 \varphi}{\partial v_k^2}, f (dx, dv) \rangle ds
\]

\[
= \langle \varphi (x, v), f_s (dx, dv) \rangle - \langle \varphi (x, v), f_0 (dx, dv) \rangle + \int_0^t \langle v \cdot \nabla_x \varphi, f_s (dx, dv) \rangle ds - \frac{\gamma}{m} \int_0^t \langle v \cdot \nabla_x \varphi, f_s (dx, dv) \rangle ds + \frac{\lambda}{m} \int_0^t \langle (x - X_{\alpha} (\rho_s)) \cdot \nabla_x \varphi, f_s (dx, dv) \rangle ds
\]

\[
- \frac{\sigma^2}{2m^2} \int_0^t \sum_{k=1}^d \langle (x - X_{\alpha} (\rho_s))^2 \frac{\partial^2 \varphi}{\partial v_k^2}, f_s (dx, dv) \rangle ds ,
\]

(4.12)

for all \( f \in \mathcal{P}(\mathcal{C}([0, T]; \mathbb{R}^d) \times \mathcal{C}([0, T]; \mathbb{R}^d)) \) and \( x, v \in \mathcal{C}([0, T]; \mathbb{R}^d) \), where \( \rho_s (x) = \int_{\mathbb{R}^d} f_s (x, dv) \). Then similar to Proposition 3.2, one can easily prove that

\[
\mathbb{E} [||F_\varphi (f^N) ||^2] \leq \frac{C}{N} ,
\]

(4.13)
Finally, following similar arguments as in Theorem 4.2, there exists a subsequence of \( \{f^N\}_{N \geq 2} \) converging in law to a deterministic measure \( f \in \mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d) \times \mathcal{C}([0,T];\mathbb{R}^d)) \), which is the unique weak solution to the following PDE

\[
\partial_t f_t + v \cdot \nabla_x f_t = \nabla_x \left( \frac{\gamma}{m} f_t - \frac{\lambda}{m} (x - X^\alpha(\rho_t)) f_t + \frac{\sigma^2}{2m^2} D (x - X^\alpha(\rho_t))^2 \nabla_v f_t \right),
\]

(4.14)

where \( \rho_t(x) = \int_{\mathbb{R}^d} f_t(x, dv) \). This can be summarized in the following theorem

**Theorem 4.2.** Let \( E \) satisfy Assumption 4 and \( f_0 \in \mathcal{P}_d(\mathbb{R}^d \times \mathbb{R}^d) \). For any \( N \geq 2 \), we assume that \( \{(X_t^{i,N}, V_t^{i,N})_{t \in [0,T]}\}_{i=1}^N \) is the unique solution to the particle system (4.1) with \( f_0^N \)-distributed initial data \( \{X_0^{i,N}, V_0^{i,N}\}_{i=1}^N \). Then the limit (denoted by \( f \)) of the sequence of the empirical measure \( f^N = \frac{1}{N} \sum_{i=1}^N \delta_{(X_t^{i,N}, V_t^{i,N})} \) exists. Moreover, \( f \) is deterministic and it is the unique weak solution to PDE PDE (4.14).

**Appendix**

**Theorem 4.3.** For any \( T > 0 \), let \( b \in \mathcal{C}([0,T];\mathbb{R}^d) \) and \( \mu_0 \in \mathcal{P}_2(\mathbb{R}^d) \). Then the following linear PDE

\[
\partial_t \mu_t = \frac{\sigma^2}{2} \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2} (x - b_i)^2 \mu_t - \lambda \nabla \cdot ((x - b_i) \mu_t),
\]

(4.15)

has a unique weak solution \( \mu \in \mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d)) \).

**Sketch of the proof.** The existence is obvious, which can be obtained as the law of the solution to the associated linear SDE. To show the uniqueness we can follow a duality argument.

For each \( t_0 \in (0,T] \) and compactly supported smooth function \( \psi \) (i.e., \( \psi \in \mathcal{C}_c^\infty(\mathbb{R}^d) \)), we consider the following backward PDE

\[
\partial_t h_t = -\frac{\sigma^2}{2} \sum_{k=1}^d (x - b_i)^2 \frac{\partial^2}{\partial x_k^2} h_t - \lambda (x - b_i) \cdot \nabla h_t, \quad (t, x) \in [0, t_0] \times \mathbb{R}^d; \quad h_{t_0} = \psi.
\]

(4.16)

which admits a classical solution \( h \in C^1([0, t_0], C^2(\mathbb{R}^d)) \). Indeed, we can explicitly construct a solution

\[
h_t(x) = \mathbb{E}[\psi(X_{t_0}^{t,x})] \quad t \in [0, t_0],
\]

(4.17)

where \( (X_{s}^{t,x})_{0 \leq t \leq s \leq t_0} \) is the strong solution to the following linear SDE

\[
dX_t^{t,x} = \lambda(X_s^{t,x} - b_s)ds + \sigma D(X_s^{t,x} - b_s)dB_s, \quad X_t^{t,x} = x,
\]

(4.18)

with \( D(y) = \text{diag}(y_1, \ldots, y_d) \) for \( y \in \mathbb{R}^d \) and \( B = (B^1, \ldots, B^d) \) being a d-dimensional Wiener process. We can first check the regularity. For each \( (t, x) \in [0, t_0] \times \mathbb{R}^d \), the chain rule gives

\[
\nabla_k h_t(x) = \mathbb{E}\left[\nabla_k \psi(X_{t_0}^{t,x}) \nabla_k (X_{t_0}^{t,x})^k\right], \quad k = 1, \ldots, d.
\]

Note that \( \nabla_k (X_{s}^{t,x})^k = 0 \) when \( k' \neq k \) and that \( \nabla_k (X_{t}^{t,x}) \) is a Geometric Brownian motion satisfying SDE (c.f. [25], Theorem 4.2)

\[
d\nabla_k (X_{s}^{t,x})^k = \lambda_k (X_{s}^{t,x})^k ds + \sigma_k D (X_{s}^{t,x})^k dB_s^k, \quad \nabla_k (X_{t}^{t,x})^k = 1.
\]

This gives \( \nabla_k (X_{s}^{t,x})^k = \exp\left(\lambda (s - t) - \frac{\sigma^2(s-t)}{2} - \sigma (B_s^k - B_t^k)\right) \). Accordingly, we may obtain the time-space continuity of \( \nabla_k h_t(x) \) and in particular, there holds the following uniform boundness

\[
\sup_{(t, x) \in [0, t_0] \times \mathbb{R}^d} |\nabla_k h_t(x)| \leq C \mathbb{E}\left[|\nabla_k (X_{t_0}^{t,x})^k|\right] \leq Ce^{\lambda T} < \infty, \quad k = 1, \ldots, d,
\]
where \( C > 0 \) is a constant depending on \( \psi \). Analogously, we may derive the uniform boundness of \( \nabla^2 h \) and even of \( \nabla^3 h \) together with associated time-space continuity. On the other hand, for \( 0 \leq t < t + \delta < t_0 \), the flow property of solution to SDE (4.18) implies \( X_s^{t,x} = X_s^{t+\delta,x} \) for \( t + \delta < s \leq t_0 \) and thus,

\[
\frac{h_{t+\delta}(x) - h_t(x)}{\delta} = \frac{1}{\delta} \mathbb{E} \left[ \psi(X^{t+\delta,x}_{t_0}) - \psi(X^{t,x}_{t_0}) \right] = \frac{1}{\delta} \mathbb{E} \left[ \psi(X^{t,x}_{t_0}) - \psi(X^{t+\delta,x}_{t_0}) \right] = \frac{1}{\delta} \mathbb{E} \left[ h_{t+\delta}(x) - h_{t+\delta}(X^{t,x}_{t_0}) \right]
\]

Through a simple limiting procedure we may get the time-differentiability of \( h_t(x) \) and further verify that the defined \( h_t(x) \) is a classical solution of PDE (4.10).

Suppose that \( \mu^1 \) and \( \mu^2 \) are two weak solutions of (4.15) with the same initial condition \( \mu^1_0 = \mu^2_0 \). Put \( \delta \mu = \mu^1 - \mu^2 \). Using the above defined solution \( h \) to the backward PDE (4.10) as a test function, we have

\[
\langle h_{t_0}(x), \delta \mu_{t_0}(dx) \rangle = \int_0^{t_0} \langle \partial_t h_s(x), \delta \mu_s(dx) \rangle ds + \int_0^{t_0} \frac{\sigma^2}{2} \sum_{k=1}^{d} \langle (x - b_s)^2 \partial^2_{x_k^2} h_s(x), \delta \mu_s(dx) \rangle ds + \int_0^{t_0} \langle \lambda(x - b_s) \cdot \nabla h_s(x), \delta \mu_s(dx) \rangle ds
\]

\[
= \int_0^{t_0} \langle \partial_x h_s(x), \delta \mu_s(dx) \rangle ds + \int_0^{t_0} \langle -\partial_s h_s(x), \delta \mu_s(dx) \rangle ds = 0,
\]

which gives \( \int_{\mathbb{R}^d} \psi(x) \delta \mu_{t_0}(dx) = 0 \) for arbitrary \( \psi \in C^\infty_c(\mathbb{R}^d) \). This implies that \( \delta \mu_{t_0} = 0 \), which yields the uniqueness by the arbitrariness of \( t_0 \).

\[\square\]

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