Strategy for implementing stabilizer-based codes on solid-state qubits

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We present a method for implementing stabilizer-based codes with encoding schemes of the operator quantum error correction paradigm, e.g., the “standard” five-qubit and CSS codes, on solid-state qubits with Ising or XY-type interactions. Using pulse sequences, we show how to dynamically generate the effective dynamics of the stabilizer Hamiltonian, the sum of an appropriate set of stabilizer operators for a given code. Within this approach, the encoded states (ground states of the stabilizer Hamiltonian) can be prepared without measurements and preserved against both the time evolution governed by the original qubit Hamiltonian, and errors caused by local sources.

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I. INTRODUCTION

A variety of quantum error-correcting codes (QECCs) have been widely investigated aiming at a robust computing system similar to the classical digital computer [1–15]. In particular, codes based on the stabilizer formalism constitute an important class of QECCs. This formalism has proven useful not only for the standard codes [1,2], but also for the subsystem code [5–7], topological [5,8,16], and Majorana codes [9]. On the experimental side, Knill et al. demonstrated its usefulness in the NMR domain [11,12]. Stabilizer-based QECCs in systems with always-on coupling have recently attracted a great deal of interest [17,18].

Stabilizer operators $G_j$ ($j = 1, \ldots, l$) are mutually commuting operators given by products of multiple Pauli matrices $X_i$, $Y_i$, and $Z_i$ ($i = 1, \ldots, n$) [2]. Conventionally, logical qubit states are encoded through measurements into a joint, 2$^l$-dimensional, eigenspace $\mathcal{H}_S$ of these operators. For $l$ stabilizer operators and $n$ physical qubits, a maximum number of $k = n - l$ logical qubits can be encoded into $\mathcal{H}_S$, while $k < n - l$ in case of subsystem encoding. Since the ground states of the stabilizer Hamiltonian $H_{stab} := -\sum_{j=1}^l G_j$ are joint eigenstates of all stabilizer operators, its ground-state manifold can play the role of $\mathcal{H}_S$.

It is important to note that stabilizer operators of many error-correction codes, e.g., the surface code [3] or color code [8], are given by products of more than two Pauli matrices. Therefore the corresponding stabilizer Hamiltonians cannot be directly implemented in natural solid-state qubit systems, where the interactions between qubits are of two-body type [18,20].

In this work, we demonstrate how to prepare ground states of $H_{stab}$, as encoded states and preserve them by inducing the effective dynamics of this Hamiltonian using sequences of pulses in the form of single-qubit rotations. Being based on single-qubit rotations only, our method works for an always-on physical (qubit) Hamiltonian with two-qubit interactions, i.e., it does not require switching on and off any of its parts (single-qubit or interaction). That local manipulation schemes are in general sufficient to induce arbitrary Hamiltonian dynamics was shown by Bennet et al. [21]. Benjamin and Bose [22] proposed a particular implementation based on single-qubit rotations to perform quantum computations on a one-dimensional system of bare qubits with always-on Heisenberg interactions.

The distinguishing feature of our method is that it allows the preparation of QECC encoded states without measurements, thus avoiding measurement-induced decoherence. The method can be used not only for standard codes (i.e., five-qubit and CSS codes) but also for the extended class of codes with encoding schemes within the general operator quantum error correction framework [5–6]. Even in the presence of inevitable pulse (rotation angle) errors the ground-state fidelity scales favorably with the system size.

Our scheme provides an essential ingredient for the implementation of stable solid-state quantum memories. This, in turn, facilitates the realization of quantum gates [18] within the limitations imposed by the size and coherence time of a system. In fact, our approach even allows us to directly realize arbitrary single- and multiqubit gates on error-correcting code-words using only single-qubit rotations. Although the scheme requires a rather large number of pulses (rotations), its feasibility can be anticipated based on the recent progress in qubit-manipulation techniques [23].

The paper is organized as follows. After the introduction of the scheme to induce the dynamics of a stabilizer operator starting from a simple initial Hamiltonian in Sec. II, we explain the extraction of suitable initial Hamiltonians and two-qubit gates from typical solid-state qubit Hamiltonians using only single-qubit rotations in Sec. III. The whole procedure is applied to the examples of the five-qubit, Steane, and Kitaev’s surface code in Sec. IV illustrating the versatility and generality of our method. This is followed by an illustration of how to use pulses both to prepare codewords without measurements and apply gate operations on logical qubits in Sec. V. A discussion of the robustness of the scheme against pulse errors is provided in Sec. VI. Finally, in Sec. VII we present our conclusions.

II. DYNAMICAL GENERATION OF STABILIZER OPERATORS

As a first step, we show how a stabilizer operator $G_j$ can be dynamically generated from a simple initial Hamiltonian...
TABLE I: Stabilizer operators for the five-qubit code [1] as realized with $XY$-type interactions. Starting from $H_{\text{ini}}$, the generating sequence will realize the desired stabilizer operator. Each appearance of $H_{XY}^{i+1}$ indicates an application of the transformation $H \rightarrow e^{-i\tau_{\text{top}} H_{XY}^i} e^{i\tau_{\text{top}} H_{XY}^0}$, while each instance of $(\pi/2)^b_i$ denotes a $\pi/2$ rotation about the $b$ axis. The rightmost column shows the time required to generate each stabilizer operator; $\tau_{\text{rot}}$ is the time needed to perform a single-$\theta$ rotation.

| $H_{\text{ini}}$ generation sequence | time for generation |
|---------------------------------------|---------------------|
| $G_1$ X Z Z X I $\Omega_2 X_2$ $H_{XY}^{23} \rightarrow (\pi/2)^2_5 \rightarrow H_{XY}^{12} + H_{XY}^{34}$ | $\tau_{\text{ini}} + 24\tau_{\text{rot}} + 4\tau_{\text{top}}$ |
| $G_2$ I X Z Z X $\Omega_3 X_3$ $H_{XY}^{34} \rightarrow (\pi/2)^3_5 \rightarrow H_{XY}^{12} + H_{XY}^{25}$ | $\tau_{\text{ini}} + 24\tau_{\text{rot}} + 4\tau_{\text{top}}$ |
| $G_3$ X I X Z Z $\Omega_4 X_4$ $G_1 \rightarrow H_{XY}^{25} \rightarrow (\pi/2)^5_5 \rightarrow H_{XY}^{23}$ | $\tau_{\text{ini}} + 43\tau_{\text{rot}} + 8\tau_{\text{top}}$ |
| $G_4$ Z X I X Z $\Omega_2 X_2$ $G_1 \rightarrow H_{XY}^{34} \rightarrow (\pi/2)^3_5 \rightarrow H_{XY}^{12} \rightarrow (\pi/2)^3_5 \rightarrow H_{XY}^{23}$ | $\tau_{\text{ini}} + 45\tau_{\text{rot}} + 8\tau_{\text{top}}$ |

$H_{\text{ini}} \propto X_i, Y_i, \text{or } Z_i$. The time evolution corresponding to the generation process is illustrated with the schematic notation $\rho(0) \overset{H_{XY}}{\xrightarrow{t}} \rho(t)$, where $\rho(t) = \exp(-iHt)\rho(0)\exp(iHt)$ is the density matrix for a time-independent Hamiltonian $H$, or for an effective $H$ in the sense of average-Hamiltonian theory [23]. After the application of mutually inverse, unitary operations according to $\rho(0) \overset{\tau_{\text{top}}}{\xrightarrow{H_{XY}}} \rho(\tau_{\text{ini}}) \overset{\tau_{\text{rot}}}{\xrightarrow{H_{XY}}} \rho(0)$, the system has evolved as if propagated by the effective Hamiltonian $\exp(-i\tau_{\text{top}} H_{XY}) H_{\text{ini}} \exp(i\tau_{\text{top}} H_{XY})$ for a time $\tau_{\text{ini}}$ [23]. To build the stabilizer operator $G_1$ from $H_{\text{ini}}$, we need two elementary transformations: one that rotates arbitrary single-qubit terms about an angle of $\pi/2$ and another one that increases the order of Pauli-matrix terms by 1. If $H_{XY}$ is the generator of a single-qubit rotation, say $-JX_i$, such a sequence dynamically generates the time evolution of $H_{\text{ini}}$ rotated about the $x$ axis through angle $2J\tau_{\text{top}}$. Higher-order products of Pauli matrices can be generated using the following transformations [25]:

$$e^{-itH_{XY}^{i+1}} X_i e^{itH_{XY}^{i+1}} = c_0 X_i - s_0 Z_i Y_{i+1},$$

$$e^{-itH_{XY}^{i+1}} Y_i e^{itH_{XY}^{i+1}} = c_0 Y_i + s_0 Z_i X_{i+1},$$

$$e^{-itH_{XY}^{i+1}} Z_i e^{itH_{XY}^{i+1}} = c_0^2 Z_i + s_0^2 Y_{i+1} + c_0 s_0 Y_i X_{i+1},$$

where $H_{XY}^{i+1} = \sum_j H_{XY}^{i+1}_j$ is the (two-body) $\tilde{X} \tilde{Y}$ interaction and $c_0 \equiv \cos(2\theta)$ and $s_0 \equiv \sin(2\theta)$. For $\theta = Jt = \pi/4$, these transformations increase the order of the Pauli-matrix terms as $X_i \rightarrow -Z_i Y_{i+1}$ and $Y_i \rightarrow Z_i X_{i+1}$. Similarly, one obtains $Z_i \rightarrow Z_{i+1}$. Analogous relations hold for the Ising interaction given by $H_{\text{Ising}} = J\sum_i Z_i Z_{i+1}$.

With $\tau_{\text{rot}} = \pi/(4J)$ and a properly constructed, nested sequence of operations $H_{XY}$, framing a period of propagation with $H_{\text{ini}}$ and duration $\tau_{\text{ini}}$, we can therefore induce the dynamics of arbitrary stabilizer operators $G_j$. Table I shows such sequences for the case of the five-qubit code. The time $\tau_{\text{ini}}$ has to be chosen such that the entire process can be carried out in a time interval sufficiently shorter than the coherence time.

### III. EXTRACTING $H_{\text{ini}}$ AND $H_{XY}$ FROM A QUBIT HAMILTONIAN

The key step in dynamically generating the stabilizer operators is extracting a single-qubit part or a pure two-body interaction part from a qubit system with Hamiltonian $H = H_0 + H_{XY}$, where $H_0 = \sum_i \Omega_i X_i + \varepsilon_i Z_i$ is a single-qubit part. Note that instead of the $XY$ Hamiltonian we could also use the Ising Hamiltonian.

This process is carried out using the Baker-Campbell-Hausdorff (BCH) formula [24]. For simplicity, we explain this procedure for $\varepsilon_i = 0$ in $H_0$, where only rotations about the $z$ axis will be needed, and set $\Omega_i = \Omega$. In the general case, the procedure requires a slightly more complex pulse sequence.

A part $H_{XY}$ can be extracted from $H_0$ by applying a single appropriate $\pi$ pulse, if that pulse transforms $H_0$ to $H_0 - H_{XY}$, where $H_0 - H_{XY}$ consists of the unwanted terms. For 2n alternating periods of propagation with $A = i\tau (H_0 + H_{XY})$ and $B = i\tau (H_0 - H_{XY})$, the BCH formula yields

$$(e^{A} e^{B})^n \approx \exp(i2n\tau H_0 + n\tau^2 [H_0, H_0])$$

where the duration of the pulse sequence is $2n\tau$. Thus, as long as $\tau \|H_0\| \ll 1$, where $\|A\| = |\text{Tr}(A^\dagger A)/d|^{1/2}$ is the standard operator norm in a Hilbert space of dimension $d$, we can neglect the second term. As the number $n$ of repetitions increases, this approximation becomes progressively better.

In order to extract a single-qubit (local) part of the system Hamiltonian, relation [2] has to be applied twice, leading to (case $n = 1$)

$$e^{A} e^{B} e^{A'} \approx \exp(2h_a \pm [h_a, h_a]) e^{2h_a'} - [h_a', h_a'] \approx \exp(2h_a - [h_a, h_a] - [h_a', h_a'] + 4[h_a, h_a'])$$

where $h_a^{(r)} := i\tau H_0^{(r)} A' = h_a' + h_b' + B' = h_a' - h_a'$. Consider extracting a single-qubit Hamiltonian $X_2$ for a one-dimensional five-qubit array. With $h_i = i\tau \Omega X_i$ (single-qubit Hamiltonian with $\varepsilon_i = 0$) and $h_{ij} = i\tau H_{XY}^{ij}$, we set

$$h_a = h_2 + h_34 + h_{45},$$

$$h_b = h_1 + h_3 + h_4 - h_5 + h_{12} + h_{23},$$

$$h_a' = h_2 - h_34 - h_{45},$$

$$h_b' = h_1 - h_3 + h_4 - h_5 + h_{12} - h_{23}.$$
an unwanted perturbation term. This term scales like $J_{ij} \tau \ll 1$ and hence can be reduced by shortening the duration of the pulse sequence.

Similarly, the operator $e^{-i\tau H_{XY}^{12}}$ is obtained by extracting $H_{XY}^{12}$ from the system Hamiltonian using

$$
h_a = h_{23} + h_1 + h_4,
\quad h_b = h_{12} + h_{34} + h_{45} + h_2 + h_3 + h_5,
\quad h'_a = h_{23} - h_1 - h_4,
\quad h'_b = -h_{12} - h_{34} - h_{45} + h_2 + h_3 + h_5.
$$

The perturbation terms can be neglected for $J/\Omega \ll 1$.

In the following, we apply our scheme to the five-qubit code and Steane’s seven-qubit code (the smallest single-error correcting CSS code) [1], as well as the surface code [3].

IV. REALIZATION OF FIVE-QUBIT, STEANE, AND KITAEV’S SURFACE CODES

The generation processes of the four stabilizer operators $G_j (j = 1, \ldots, 4)$ of the five-qubit code [1] are shown in Table I. For example, starting from the initial Hamiltonian $H_{ini} = \Omega_2 X_2$, the stabilizer operator $G_1$ of the five-qubit code is realized through the sequence

$$
e^{-i\tau_{op} H_{X}^{12}} X_2 e^{-i\tau_{op} H_{XY}^{12}} \rightarrow -Z_2 Y_3
\quad -e^{-i(\pi/4) X_2} Z_3 e^{i(\pi/4) X_2} \rightarrow Y_2 Y_3
\quad e^{-i\tau_{op} (H_{X}^{12} + H_{XY}^{12})} Y_2 Z_3 e^{i\tau_{op} (H_{X}^{12} + H_{XY}^{12})} \rightarrow X_1 Z_2 Z_3 X_4.
$$

The minimal time required for this process is $\tau_{ini} + 24 \tau_{rot} + 4 \tau_{op}$. The effective dynamics of $H_{stab} = -\sum_{i=1}^{4} G_i$ is induced by subsequent generation of the four stabilizer operators.

We would now like to address the feasibility of this scheme in a typical superconducting qubit system. For two superconducting qubits in a circuit-QED setup the resulting effective interqubit interaction is also of XY type [26,27]. For instance, for $g/\Delta = 0.1$, $g/(2\pi) = 200$ MHz, $\Delta/(2\pi) = 2$ GHz, where $g$ is the Jaynes-Cummings coupling constant and $\Delta$ is the detuning between the resonator frequency and the qubit splitting, we have $J/(2\pi) = 20$ MHz. Assuming $\tau_{rot} \sim 1$ ns [28], we obtain a minimal total time of $\tau_{coding} = 24 \tau_{rot} + 136 \tau_{rot} \approx 300$ ns, which is significantly shorter than $T_2 \sim 20 \mu$s reported in [29].

Table II shows how to generate the Steane code. The stabilizer operators $G_4, G_5, G_6$ are obtained by $e^{-i \sum_j Y_j/4} (G_1 + G_2 + G_3) e^{i \sum_j Y_j/4}$. Thus, the minimal total time is $\tau_{Steane} = 44 \tau_{rot} + 246 \tau_{rot} \approx 600$ ns, i.e., again much shorter than the $T_2$ given in Ref. [29].

For the realization of Kitaev’s surface code, we need to generate four types of stabilizer qubits. Qubits are placed at the edges of the square lattice; see Fig. [1] Stabilizer operators $H_s = \Pi_{j \in \text{star}(s)} X_j$ are assigned to each vertex $s$, and $H_p = \Pi_{j \in \text{boundary}(p)} Z_j$ to each face $p$. Using the relations $X_i \rightarrow -Z_i Y_{i+1}$ and $Y_i \rightarrow Z_i X_{i+1}$, we can form products of nearest-neighbor operators such as $Y_1 \rightarrow Z_1 X_2 \rightarrow -Z_1 Z_2 Y_3 \rightarrow Z_1 Z_2 Z_3 X_4$. In generating adjacent stabilizer operators, care should be taken to avoid mixing them. This can be achieved by decompositions like $H_s = H_{s_1} + H_{s_2}$ [see Figs. [1]a and [1]b)] and $H_p = H_{p_1} + H_{p_2}$ [Figs. [1]c and [1]d)]. The four elements can then be combined into the total surface-code Hamiltonian.

V. PREPARATION OF ENCODED STATES AND GATE OPERATIONS

Our approach also allows us to prepare encoded states (or codewords) of general stabilizer-based codes without performing measurements on the system and to implement arbitrary single- and multiqubit gate operations.

We show this in detail for the standard codes, which encode $k$ logical qubits into a subspace of dimension $2^k$. However, this procedure also works for subsystem encoding provided suitable stabilizer operators are added. For any given code, only those $G_j$ with $1 \leq j \leq m$ and $m \leq n - k$ that contain $X$ or $Y$ operators are needed for the preparation:

$$
|c_1 \cdots c_k\rangle = (1 + G_1) \cdots (1 + G_m) \tilde{X}^{c_1} \cdots \tilde{X}^{c_k} |0\cdots 0\rangle
= \prod_{i=1}^{k} \tilde{X}^{c_i} \prod_{j=1}^{m} \exp\left(\frac{i\pi}{4} G_j^{a_j}\right) |0\cdots 0\rangle,
$$

where $c_i = 0, 1$ and operators $\tilde{X}_i$ act in the logical state space $\{|\{0\}_i\rangle, |\{1\}_i\rangle\}$. Here, $G_j^{a_j}$ denotes a modified stabilizer operator obtained from $G_j$ by replacing the $X$ operator acting on qubit $a_j$ by a $Y$ operator, or vice versa. This is done in order to match the effect of an individual factor $\exp[i(\pi/4) G_j^{a_j}]$ with the action of the projector $(1 + G_j)$ when qubit $a_j$ is in state $|0\rangle$. To fulfill Eq. (7) for all $1 \leq j \leq m$ simultaneously, all the $a_j$ have to be different and the modified stabilizers have to be generated in an order such that prior to $G_j^{a_j}$ none of the $G_k^{a_k}$ with $k < j$ have acted on qubit $a_j$ with an $X$ or $Y$.

By implementing the second row of Eq. (7), the quantum information is encoded into the logical qubit after the basis state $|\{0\}\rangle$ is generated by applying appropriate logical gate operations (see below). It is also possible, however, to start from
TABLE II: Stabilizer operators for the Steane code as realized with XY-type interactions. The operators $G_4$, $G_5$, and $G_6$ are obtained by replacing $X$ with $Z$ in $G_1$, $G_2$, and $G_3$, respectively. The rightmost column shows the time required to generate each stabilizer operator; $\tau_{\mathrm{rot}}$ is the time needed to perform a single-qubit rotation.

| $G_j$  | $H_\text{ini}$ generation sequence | $\tau_{\text{rot}}$ for generation |
|--------|----------------------------------|----------------------------------|
| $G_1$  | $X\;X\;X\;I\;I\;I$ $\Omega_2X_2$ $H^{XY}_{34} \rightarrow (\pi/2)^j_2 \rightarrow H^{XY}_{13} + H^{XY}_{34} \rightarrow (\pi/2)^j_2 (\pi/2)^j_3$ | $\tau_{\text{ini}} + 26\tau_{\text{rot}} + 4\tau_{\text{op}}$ |
| $G_2$  | $X\;X\;I\;I\;X\;X$ $-\Omega_3X_3$ $H^{XY}_{34} \rightarrow (\pi/2)^j_3 \rightarrow H^{XY}_{34} + H^{XY}_{15} \rightarrow H^{XY}_{34} + H^{XY}_{15}$ | $\tau_{\text{ini}} + 45\tau_{\text{rot}} + 8\tau_{\text{op}}$ |
| $G_3$  | $X\;I\;X\;I\;X\;X$ $-\Omega_3X_3$ $H^{XY}_{34} \rightarrow (\pi/2)^j_3 \rightarrow H^{XY}_{34} + H^{XY}_{15} \rightarrow H^{XY}_{34} + H^{XY}_{15}$ | $\tau_{\text{ini}} + 51\tau_{\text{rot}} + 10\tau_{\text{op}}$ |

Since the codeword states are encoded in the twofold-degenerate ground-state manifold $|0\rangle$ and $|1\rangle$ of $H_{\text{stab}}$, the robustness of this method is limited by the rate of leakage out of this manifold. In principle, precise estimates of the leakage due to the thermal environment could be obtained by studying the stability of the ground state to various perturbations as in Ref. [30]. However, energy nonconserving single-qubit errors—often a prevalent kind of error created by a thermal bath—are exponentially suppressed for temperatures that are small compared to the Zeeman-splitting $\Omega$. Hence, besides local imperfections and noise sources, unavoidable pulse errors are likely to be the predominant cause of leakage, at low temperatures.

To estimate this effect, we consider pulse errors that can be modeled by randomly distributed, unbiased, and uncorrelated deviations $\delta\theta$ with $\sigma_\theta = \sqrt{\langle \delta\theta^2 \rangle}$ from the ideal angle of $\pi/2$. The leakage can then be estimated by looking at the average of this manifold. In principle, precise estimates of the leakage due to the thermal environment could be obtained by studying the stability of the ground state to various perturbations as in Ref. [30]. However, energy nonconserving single-qubit errors—often a prevalent kind of error created by a thermal bath—are exponentially suppressed for temperatures that are small compared to the Zeeman-splitting $\Omega$. Hence, besides local imperfections and noise sources, unavoidable pulse errors are likely to be the predominant cause of leakage, at low temperatures.

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tion of QECC Hamiltonian dynamics also provides protection against certain classes of local errors such as impurities.

Further steps towards a stable quantum memory would require to protect the code against thermal fluctuations, which could be achieved, e.g., by a coupling to appropriate, non-local external fields \[31\]-\[33\]. Once implemented experimentally, our scheme will therefore pave the way for robust quantum information processing.

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