Big pure mapping class groups are never perfect

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Appendix with Ryan Dickmann

July 30, 2020

Abstract

We show that the closure of the compactly supported mapping class group of an infinite type surface is not perfect and that its abelianization contains a direct summand isomorphic to \( \bigoplus_{\aleph_0} \mathbb{Q} \). We also extend this to the Torelli group and show that in the case of surfaces with infinite genus the abelianization of the Torelli group contains an indivisible copy of \( \bigoplus_{\aleph_0} \mathbb{Z} \) as well. Finally we give an application to the question of automatic continuity by exhibiting discontinuous homomorphisms to \( \mathbb{Q} \).

1 Introduction

Let \( S \) be a connected, orientable, second-countable, surface. The mapping class group, \( \text{MCG}(S) \), is the group of orientation preserving homeomorphisms of \( S \) up to homotopy, where all homeomorphisms and homotopies fix the (possibly empty) boundary of \( S \) point-wise. When \( S \) is infinite type; that is, when \( \pi_1(S) \) is not finitely generated, we will often call \( \text{MCG}(S) \) a big mapping class group. The pure mapping class group, \( \text{PMCG}(S) \), is the subgroup of \( \text{MCG}(S) \) consisting of elements which fix the ends of \( S \). In the finite type setting it is a classic result of Powell \[\text{Pow78}\] that \( \text{PMCG}(S) \) is perfect, has a trivial abelianization, whenever \( S \) has genus at least 3. We will see that this is not the case in the infinite type setting.

Let \( \text{PMCG}_c(S) \) denote the subgroup of \( \text{PMCG}(S) \) consisting of compactly supported mapping classes. We prove the following when \( S \) has more than one end. The one ended case is proved with Ryan Dickmann in the attached appendix by applying the Birman Exact Sequence.

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*Partially supported by NSF DMS-1607236, NSF DMS-1840190, and NSF DMS-1246989.
**Theorem A.** \( \text{PMCG}_c(S) \) is not perfect if \( S \) is an infinite type surface.

This disproves Conjecture 5 in [APV17]. In [APV17] the authors show that once \( S \) has at least two ends accumulated by genus there exist nontrivial homomorphisms from \( \text{PMCG}(S) \) to \( \mathbb{Z} \) so that \( \text{PMCG}(S) \) cannot be perfect. The maps they build come from handleshifts and for genus 2 and greater they prove that the integral cohomology of the closure of the compactly supported mapping classes is trivial. The authors in [DP19] prove the same for genus 1 surfaces. Note that we get nontrivial homomorphisms to \( \mathbb{Z} \) from \( \text{PMCG}(S) \) when \( S \) has genus 0 for free by first taking a forgetful map to a sphere with finitely many punctures (see [DP19] for a discussion on this). Thus \( \text{PMCG}(S) \) also cannot be perfect when \( S \) has genus 0.

In [PV18] the authors prove that \( \text{PMCG}(S) = \text{PMCG}_c(S) \) if and only if \( S \) has at most one end accumulated by genus. Combining the previous work in [APV17] with our main theorem we see that big pure mapping class groups are never perfect.

**Theorem B.** \( \text{PMCG}(S) \) is not perfect if \( S \) is an infinite type surface.

Now that we know that \( \text{PMCG}_c(S) \) is not perfect we can ask: What is \( H_1(\text{PMCG}_c(S); \mathbb{Z}) \)? Throughout this paper when we refer to the homology of a group we refer to its homology as a discrete group. We make use of the tools involved in the proof of Theorem A to find an uncountable direct sum of \( \mathbb{Q} \)'s inside the abelianization. We can then apply tools from abelian group theory to conclude the following.

**Theorem C.** Let \( S \) be an infinite type surface. \( H_1(\text{PMCG}_c(S); \mathbb{Z}) = \bigoplus_{2^\aleph_0} \mathbb{Q} \oplus B \) where all divisible subgroups of \( B \) are torsion.

We can similarly find such a direct summand in the abelianization of the Torelli group. However, in the Torelli group we can make use of the Johnson homomorphism to see that the abelianization also contains many indivisible copies of \( \mathbb{Z} \) whenever \( S \) has infinite genus.

**Theorem D.** Let \( S \) be an infinite type surface. \( H_1(\mathcal{I}(S); \mathbb{Z}) = \bigoplus_{2^\aleph_0} \mathbb{Q} \oplus B \) where all divisible subgroups of \( B \) are torsion. If \( S \) also has infinite genus, then \( B \) contains a copy of \( \bigoplus_{2^\aleph_0} \mathbb{Z} \).

Finally we provide an application of Theorem C to the question of automatic continuity of big mapping class groups.

**Theorem E.** Let \( S \) be an infinite type surface. There exists \( 2^\alpha \) discontinuous homomorphism from \( \text{PMCG}_c(S) \) to \( \mathbb{Q} \).
This gives some progress towards Questions 2.4 and 2.6 in [Man20] which ask for which infinite-type surfaces do the mapping class groups or pure mapping class groups have automatic continuity.

Our proof of the main result uses the projection complex machinery of [BBF15] and [BBFS17]. Projection complexes have proven very useful in the setting of finite type mapping class groups (see [BBF16], [BBF19], and the original two papers mentioned previously). Recently the authors in [HQR20] make use of the projection complex machinery to study the question of which big mapping class groups admit nonelementary continuous actions on hyperbolic spaces.

Acknowledgements: The author thanks Mladen Bestvina for numerous helpful conversations and suggestions and for patiently reading through many drafts of this paper. Thanks also to Ryan Dickmann for pointing out the case of the Loch Ness Monster and the application to automatic continuity and to Jesús Hernández Hernández, Paul Plummer, and Priyam Patel for many helpful discussions about big mapping class groups.

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Here we give an outline of the proof of Theorem A. While Powell in [Pow78] shows that in the finite type case $\mathrm{PMCG}(S)$ is perfect once $S$ has genus at least 3, it is also known in the finite type setting that pure mapping class groups are not uniformly perfect. In fact, the number of commutators needed to write a power of a Dehn twist grows linearly in the power [EK01]. This idea gives some intuition as to the proof of the main theorem. Consider the mapping class $f$ on the surface $S$ with two ends accumulated by genus given by an infinite product of increasing powers of Dehn twists about disjoint separating curves. That is, $f = \prod_{i\in\mathbb{Z}} T_{\gamma_i}^{[i]}$ where $\gamma_i$ is a bi-infinite sequence of disjoint separating curves. See Figure 1 for an example of curves we twist about.

Now, if we approximate $f$ on bigger and bigger finite type subsurfaces the number of commutators needed to write these approximations of $f$ grows;
however, if $\text{PMCG}_c(S)$ were perfect then we would be able to write $f$ as a finite product of commutators. The challenge is how do we actually build a contradiction using this intuition.

We will build a quasimorphism on $\text{PMCG}_c(S)$ for each curve $\gamma_i$ that “measures” the twisting of $f$ about $\gamma_i$. However, quasimorphisms are always bounded on commutators, so we get a uniform upper bound on the value of $f$ for each of these quasimorphisms. This will yield a contradiction as $f$ twists more and more about each $\gamma_i$ as $i$ increases.

This transforms the problem into building quasimorphisms with these properties. To do this we will use the projection complex machinery of [BBF15] and [BBFS17]. We will use $\gamma_i$ and the orbit of $\gamma_i$ under $\text{PMCG}_c(S)$ to build a projection complex for each $i$. This will give an action of $\text{PMCG}_c(S)$ on a quasi-tree with $T_{\gamma_i}$ acting as a WWPD element. We then use a generalization of the Brooks construction from [BBF16] to build our desired quasimorphisms.

Section 3 establishes the definitions and previous results we use. Sections 4 and 5 pertain to building the projection complexes and quasimorphisms we use. Section 6 proves that certain types of infinite products of Dehn twists about separating curves are nontrivial in $H_1(\text{PMCG}_c(S); \mathbb{Z})$. As a corollary we obtain Theorem A. Section 7 shows that we can similarly replace the infinite product of Dehn twists with infinite products of partial pseudo-Anosovs supported on disjoint subsurfaces and obtain the same result. This is important for proving Theorem C in the low genus case. Section 8 contains the prove of Theorem C. Section 9 contains a discussion on the Torelli group and the proof of Theorem D. Section 10 discusses an application to the question of automatic continuity by building discontinuous homomorphisms and contains the proof of Theorem E. Section 11 contains a conversation and poses some questions on other possible elements in $H_1(\text{PMCG}_c(S); \mathbb{Z})$.

3 Background

We will always assume that our surfaces are connected, orientable, second-countable, and possibly with finitely many compact boundary components.

3.1 Ends, Classification of, and Exhaustions of Infinite Type Surfaces

The space of ends of a surface $S$ is given by $\text{Ends}(S) = \lim_{-} (S \setminus K)$ where $K$ ranges over the compact subsets of $S$. It can be given a topology
which is totally disconnected, separable, and compact so that it is always homeomorphic to a subset of the Cantor set. We say an end is **accumulated by genus** if every open set in $S$ containing that end has infinite genus. We denote the set of ends accumulated by genus as $\text{Ends}_\infty(S)$.

**Theorem 3.1** (Classification of Surfaces, [Ker23] [Ric63]). A surface, $S$, with finitely many compact boundary components is determined up to homeomorphism by the quadruple $(g, b, \text{Ends}(S), \text{Ends}_\infty(S))$, where $g \in \mathbb{N} \cup \{\infty\}$ is the genus of $S$, $b \in \mathbb{N}$ is the number of boundary components, and the pair $(\text{Ends}(S), \text{Ends}_\infty(S))$ is considered up to topological type.

Note that this classification subsumes the classical classification of finite type surfaces.

**Definition 3.2.** We say that an essential, simple closed curve $\gamma$ in a surface $S$ is **end-separating** if it separates the space of ends of $S$. Likewise we say that a finite-type subsurface $B \subset S$ is **end-separating** if $\partial B \setminus \partial S$ is a collection of essential, end-separating, simple closed curves.

We will make use of a modification of the notion of a principal exhaustion as defined in [HMV19]. First we recall that the **topological complexity** of a finite type surface $S$ is $3g - 3 + b + n$ where $g$ is the genus of $S$, $b$ is the number of boundary components of $S$, and $n$ is the number of punctures of $S$.

**Definition 3.3.** Let $\{S_i\}$ be an increasing sequence of subsurfaces of $S$ and infinite type surface. We say that $\{S_i\}$ is a **separating principal exhaustion** if $S = \bigcup_{i=1}^{\infty} S_i$ and for all $i$ it satisfies the following:

(i) each $S_i$ is an end-separating surface,

(ii) $S_i$ is contained in the interior of $S_{i+1}$,

(iii) and each component of $S_{i+1} \setminus S_i$ has topological complexity at least 6.

A separating principal exhaustion always exists for any infinite type surface with at least two ends.

### 3.2 Big Mapping Class Groups

For $S$ a surface, possibly with boundary, let $\text{Homeo}_0^+(S)$ be the group of orientation preserving homeomorphisms which fix the boundary pointwise. The **mapping class group**, $\text{MCG}(S)$ is defined to be

$$\text{MCG}(S) = \text{Homeo}_0^+(S) / \sim$$
where two homeomorphisms are equivalent if they are isotopic relative to the boundary of $S$. When $S$ is of finite type, $\text{MCG}(S)$ is discrete. In the infinite type setting we equip $\text{Homeo}_0^+(S)$ with the compact open topology, which induces the quotient topology on $\text{MCG}(S)$. The pure mapping class group, $\text{PMCG}(S)$, is the kernel of the action of $\text{MCG}(S)$ on the space of ends of $S$ equipped with the subspace topology.

We say $f \in \text{MCG}(S)$ is compactly supported if $f$ has a representative that is the identity outside of a compact subset of $S$. The subgroup consisting of compactly supported mapping classes is denoted $\text{PMCG}_c(S) \subset \text{MCG}(S)$. Note that any compactly supported mapping class is in the subgroup $\text{PMCG}(S)$.

In [APV17] the authors decompose $\text{PMCG}(S)$ as a semi-direct product of $\text{PMCG}_c(S)$ and a group generated by handle shifts.

**Theorem 3.4** ([APV17], Corollary 4). $\text{PMCG}(S) = \text{PMCG}_c(S) \rtimes H$ where $H \cong \prod_{n \geq 1} \mathbb{Z}$ with $n \in \mathbb{N} \cup \{\infty\}$ the number of ends of $S$ accumulated by genus and $H$ trivial if $n \leq 1$. Furthermore, $H$ is generated by pairwise commuting handle shifts.

See [PV18] and [APV17] for the definition of a handle shift and a more thorough introduction to big mapping class groups. We note that if $S$ has only one end accumulated by genus, then $\text{PMCG}(S) = \text{PMCG}_c(S)$.

### 3.3 Projection Complexes

In this section we will review the projection complex machinery of [BBF15] and [BBFS17] which will be used to build an action of $\text{PMCG}_c(S)$ on a quasi-tree.

Let $Y$ be a set and for each $Y \in Y$ let $\mathcal{C}(Y)$ be a geodesic metric space. For $X, Z \in Y$ with $X \neq Z$ let $\pi_Z(X) \subset \mathcal{C}(Z)$ be the projection from $X$ to $Z$. Define $d_Y(X, Z) = \text{diam}(\pi_Y(X) \cup \pi_Y(Z))$ for $X, Y, Z \in Y$.

**Definition 3.5.** The collection $\{(\mathcal{C}(Y), \pi_Y)\}_{Y \in Y}$ satisfies the projection axioms for a projection constant $\theta \geq 0$ if

- (P0) $\text{diam}(\pi_Y(X)) \leq \theta$ when $X \neq Y$,
- (P1) if $X, Y, Z$ are distinct and $d_Y(X, Z) > \theta$ then $d_X(Y, Z) \leq \theta$,
- (P2) if $X \neq Z$, the set $\{Y \in Y | d_Y(X, Z) > \theta\}$ is finite.
Given such a collection one can define the projection complex \( \mathcal{P}_K(Y) \) to be the graph with vertex set \( Y \) and edges joining \( X, Z \in Y \) whenever \( d_Y(X, Z) < K \) for all \( Y \in Y \setminus \{X, Z\} \). We then get the blown up projection complex \( \mathcal{C}_K(Y) \) by replacing each vertex \( Y \in Y \) with \( \mathcal{C}(Y) \) and joining points in \( \pi_X(Z) \) with points in \( \pi_Z(X) \) by an edge of length \( L = L(K) \) whenever \( X \) and \( Z \) have an edge between them in \( \mathcal{P}_K(Y) \). Technically \( \mathcal{C}_K(Y) \) depends on a choice of \( L \) and \( K \) but we will fix \( L \) as a function of \( K \). Each \( \mathcal{C}(Y) \) will be isometrically embedded in \( \mathcal{C}_K(Y) \).

We say that a group \( G \) acting on \( Y \) preserves the projection structure if for every \( Y \in Y \) and \( g \in G \) there are isometries \( F_g^Y : \mathcal{C}(Y) \to \mathcal{C}(g(Y)) \) so that

(i) \( F_{g'}^Y F_g^Y = F_{g'g}^Y \) for all \( g, g' \in G, Y \in Y \) and

(ii) \( g(\pi_Y(X)) = \pi_{g(Y)}(g(X)) \) for all \( g \in G \) and \( X, Y \in Y \).

If \( G \) acts in this way it preserves the projection distances and acts naturally on \( \mathcal{P}_K(Y) \) and \( \mathcal{C}_K(Y) \) by isometries. For our uses we only need that \( K \) is sufficiently larger than the projection constant and so we will often drop the \( K \) and simply write \( \mathcal{P}(Y) \) and \( \mathcal{C}(Y) \) for the projection complex and blown up projection complex, respectively.

Provided that \( K \) is large enough this construction gives a group action on a quasi-tree.

Theorem 3.6 ([BBF15], Theorem 3.16, Theorem 4.14, and Theorem 4.17). If \( \{ (\mathcal{C}(Y), \pi_Y) \}_{Y \in Y} \) satisfies the projection axioms with projection constant \( \theta \) and \( K > 3\theta \) then

(i) \( \mathcal{P}_K(Y) \) is a quasi-tree.

(ii) If all \( \mathcal{C}(Y) \) are quasi-trees with uniform bottleneck constants for all \( Y \in Y \), \( \mathcal{C}_K(Y) \) is a quasi-tree. Furthermore, the bottleneck constant of \( \mathcal{C}_K(Y) \) depends only on the bottleneck constants of the \( \mathcal{C}(Y) \) and the projection constant.

(iii) If all \( \mathcal{C}(Y) \) are \( \delta \)-hyperbolic with the same \( \delta \) then \( \mathcal{C}_K(Y) \) is hyperbolic with hyperbolicity constant depending only on \( \delta \) and the projection constant.

We will primarily be utilizing the blown up projection complex.
3.4 Curve Graphs and Projections

We will be using the curve graphs and subsurface projections as defined in [MM99] and [MM00] to build our projection complex. Recall that the curve graph of an orientable surface with boundary, $S$, is the graph $\mathcal{C}(S)$ with vertices homotopy classes of simple closed curves and edges between any two classes which can be realized disjointly on $S$. We can then define projections between curve graphs of essential subsurfaces of $S$. If $Y$ and $Z$ are essential subsurfaces with $\partial Z$ intersecting $Y$, then $\partial Z \cap Y$ is a collection of curves and arcs in $Y$. For each of these arcs one can perform surgery, in potentially two different ways, with $\partial Y$ to close it up to a curve in $Y$. Then we define $\pi_Y(Z) \subset \mathcal{C}(Y)$ to be the union of all curves and closed up arcs coming from $\partial Z \cap Y$. This gives a definition for whenever our subsurfaces have negative Euler characteristic; however, we will also be concerned with annular subsurfaces and projections between them.

We now define the curve graph for a simple closed curve in $S$, or equivalently an annular subsurface of $S$. Fix a hyperbolic metric on the interior of $S$. If $\gamma$ is an essential non-peripheral simple closed curve let $X_\gamma$ be the annular cover of $S$ corresponding to $\gamma$. Now let $\mathcal{C}(\gamma)$ be the graph with vertices complete geodesics in $X_\gamma$ which cross the core curve and an edge between any pair of geodesics which are disjoint. In [MM00] it is shown that $\mathcal{C}(\gamma)$ is quasi-isometric to $Z$. In fact, there is always a $(1, 2)$-quasi-isometry, regardless of the topological type of the underlying surface or curve.

For $\beta$ another essential non-peripheral simple closed curve intersecting $\gamma$ we define the projection $\pi_\gamma(\beta)$ to be the components of the pre-image of the geodesic representative of $\beta$ in $X_\gamma$ which intersects the core curve.

We say that two subsurfaces $Y$ and $Z$ overlap if $\partial Y \cap \partial Z \neq 0$ where if $\gamma$ is an essential non-peripheral simple closed curve we say that the boundary of the corresponding annular subsurface is simply $\gamma$. Note that projections between subsurfaces are only defined when they overlap.

We can now define distances between subsurface projections. For $X,Y,Z$ three overlapping subsurfaces (potentially annuli) define

$$d_Y(X, Z) = \text{diam}_{\mathcal{C}(Y)}(\pi_Y(X), \pi_Y(Z)).$$

For $\beta$ any curve intersecting $\gamma$ transversely we have that $d_\gamma(T^n_\gamma(\beta), \beta) = 2 + |n|$ for all $n \neq 0$. The additive factor of two comes from the fact that the Dehn twist in $S$ will affect every lift of $\gamma$ to $X_\gamma$, so that the lifts of $T^n_\gamma(\beta)$ are twisted an extra amount, causing it to pick up two additional intersections. Now we have the following lemma which follows from work in [MM00] and the Behrstock inequality, [Beh06]. The explicit bound of 10...
for the Behrstock inequality follows from a proof of Leininger as recorded in [Man10].

**Lemma 3.7 ([BBF15], Section 5.1).** Let $Y$ be a collection of pairwise overlapping subsurfaces in a compact orientable finite-type surface $S$ such that $\chi(S) < 0$, possibly with finitely many punctures (compact after the punctures are filled in). Then $\{(C(Y), \pi_Y)\}_{Y \in Y}$, where $C(Y)$ denotes the curve graph of $Y$, satisfies the projection axioms (P0)-(P2) with projection constant $\theta = 10$.

This lemma also holds in the infinite type setting.

**Lemma 3.8.** Let $Y$ be a collection of pairwise overlapping finite-type subsurfaces in an orientable infinite-type surface $S$. Then $\{(C(Y), \pi_Y)\}_{Y \in Y}$, where $C(Y)$ denotes the curve graph of $Y$, satisfies the projection axioms (P0)-(P2) with projection constant $\theta = 10$.

**Proof.** (P0) and (P1) follow exactly as in the finite-type setting. For (P2), if $X, Y \in Y$ we let $A$ be the smallest finite-type subsurface of $S$ that contains both $X$ and $Y$. Note that if $Z$ is a third subsurface and $Z$ is not contained within $A$ then there is some curve $\gamma$ in $Z$ disjoint from $\partial X \cap Z$ and $\partial Y \cap Z$, so it suffices to only consider subsurfaces $Z$ contained in $A$. Then (P2) holds due to the fact that it holds for $X$ and $Y$ as subsurfaces of $A$. \qed

### 3.5 WWPD Elements

The construction above gives an action of a group $G$ on a $\delta$-hyperbolic space, $C_K(Y)$. This action will not be proper; however, we will still have some control on how certain elements of $G$ act.

**Definition 3.9.** Let $G$ act on a $\delta$-hyperbolic graph $X$. We say that $g \in G$ is a **WWPD** element if

(i) $g$ acts as a hyperbolic isometry on $X$,

(ii) $C = C(g) < G$ is a subgroup that fixes the points $g^{\pm \infty}$ at infinity fixed by $g$; equivalently, for every virtual quasi-axis $\gamma$ the orbit $C \gamma$ is contained in a Hausdorff neighborhood of $\gamma$ and no element of $C$ flips the ends, and

(iii) there is a $\xi = \xi_g > 0$ and quasi-axis, $\ell$, for $g$ such that for every $h \in G \setminus C$ we have that the projection of $h \cdot \ell$ to $\ell$ has diameter $\leq \xi$.

We will say that $(G, X, g, C)$ satisfy WWPD with constant $\xi_g$.

WWPD elements will be important in our construction of quasimorphisms.
3.6 Quasimorphisms

Definition 3.10. A quasimorphism of a group $G$ is a function $F : G \to \mathbb{R}$ such that

$$D(F) := \sup_{g,h \in G} |F(gh) - F(g) - F(h)| < \infty.$$  

We say $D(F)$ is the defect of $F$. We say that $F$ is antisymmetric if $F(g^{-1}) = -F(g)$ for all $g \in G$.

Note that any antisymmetric quasimorphism $F : G \to \mathbb{R}$ is bounded on commutators; that is,

$$|F([g,h])| \leq 3D(F)$$

for all $g, h \in G$.

In [BBF16] the authors generalize the classical Brooks construction to the setting of groups acting on quasi-trees with WWPD elements.

Proposition 3.11 ([BBF16], Proposition 3.1). For every $\Delta > 0$ there is $M = M(\Delta)$, a fixed multiple of $\Delta + 1$, such that the following holds. Let $(G, Q, g, C)$ satisfy WWPD with constant $\xi_g$ where $Q$ is a quasi-tree with bottleneck constant $\Delta$ and assume that $\tau_g \geq \xi_g + M$ where $\tau_g$ is the translation length of $g$. Then there is an antisymmetric quasimorphism $F : G \to \mathbb{R}$ such that

(i) $D(F) \leq 12,$ and

(ii) $F$ is unbounded on the powers of $g$. In fact, $F(g^n) \geq \frac{n}{2} - 1$.

The proposition as stated in [BBF16] has more consequences but we’ve only listed the two that we will take advantage of.

Sketch of Proof. We will give a sketch of the construction, which is a modification of the classical Brooks construction for free groups acting on trees [Bro81]. First fix a $(4, A)$-quasi-isometry $\phi : Q \to T$ where $T$ is a tree and an $\epsilon > 0$ so that the image of a $(2, 10\delta + 10)$-quasi-geodesic $[a, b]$ is in the $\epsilon$-neighborhood of $[\phi(a), \phi(b)]$. $A$ and $\epsilon$ depend only on $\Delta$ and can be chosen to be fixed multiples of $\Delta + 1$. Pick $x_0$ a vertex in $Q$ so that $d(x_0, g(x_0)) = D$ is minimal and let $w = [x_0, g(x_0)]$, where $w$ is oriented. Taking $M$ sufficiently large gives $D >> \delta, A, \epsilon$ so that the union of the $\langle g \rangle$-translates of $w$ forms a quasi-axis $\ell$ of $g$.  

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We say that a copy of \( w \) is a translate \( \gamma w \). We say that a copy \( \gamma w \) is contained in a segment \([q, q'] \subset Q\) if, possibly after translating both the copy and segment by an element of \( G \), the \( \phi \)-image of \( \gamma w \) is contained in the \( \epsilon \)-neighborhood of the \( \phi \)-image of \([q, q']\) and the orientations agree.

Two copies \( \gamma w \) and \( \gamma' w \) are non-overlapping if for some \( \beta \in G \) the images \( \phi(\beta \gamma w) \) and \( \phi(\beta \gamma' w) \) are disjoint. Now the non-overlapping count \( N_w(q, q') \) is the maximal number of pairwise non-overlapping copies of \( w \) contained in the segment \([q, q']\). One can check that \( N_w(q, q') \) is finite and if \( r \in Q \) is \( 2\delta \)-close to a geodesic between \( q \) and \( q' \) then

\[
|N_w(q, q') - N_w(q, r) - N_w(r, q')| \leq 2. \tag{1}
\]

Define \( F : G \to \mathbb{R} \) by

\[
F(\alpha) = N_w(x_0, \alpha(x_0)) - N_{w^{-1}}(x_0, \alpha(x_0)) \\
= N_w(x_0, \alpha(x_0)) - N_w(\alpha(x_0), x_0).
\]

Note that for any \( \alpha \in G \), \( N_w(q, q') = N_w(\alpha(q), \alpha(q')) \) so that \( F \) is antisymmetric. Indeed we can check

\[
F(\alpha^{-1}) = N_w(x_0, \alpha^{-1}(x_0)) - N_w(\alpha^{-1}(x_0), x_0) \\
= N_w(\alpha(x_0), \alpha^{-1}(x_0)) - N_w(\alpha(x_0), \alpha^{-1}(x_0), \alpha(x_0)) \\
= N_w(\alpha(x_0), x_0) - N_w(x_0, \alpha(x_0)) = -F(\alpha).
\]

We can now check (i). Let \( \alpha, \beta \in G \) and pick \( r \in Q \) that is within \( 2\delta \) of the three sides of a geodesic triangle with vertices \( x_0, \alpha(x_0) \), and \( \alpha \beta(x_0) \). Now one can apply the inequality (1) above 6 times to get that \( D(F) \leq 12 \).

For (ii) we first note that the set of copies of \( w \) \( \{g^{2k}(w)\}_{k \in \mathbb{Z}} \) are non-overlapping. Therefore, \( N_w(x_0, g^{2k}(x_0)) \geq k \) for all \( k > 0 \). We also have that \( N_w(g^k(x_0), x_0) = 0 \) since \( g \) is a WWPD element. Indeed, \( \gamma \in C \) cannot flip the orientation of \( w \), and if \( \gamma \in G \setminus C \) any possible overlap of \( \gamma w \) and \([g^k(x_0), x_0]\) is small relative to the length of \( w \). Therefore we must have that \( F(g^n) \geq \frac{n}{2} - 1 \) as desired.

\[\square\]

4 Building Projection Complexes

Let \( S \) be an infinite type surface with more than one end. We will build various projection complexes out of the \( \text{PMCG}_c(S) \)-orbit of either an end-separating curve in \( S \) or an end-separating subsurface \( B \) of \( S \).
Lemma 4.1. Let $g, h \in \overline{\text{PMCG}}_c(S)$. Then

(i) for any end-separating simple closed curve $\gamma$ in $S$ the translates $g(\gamma)$ and $h(\gamma)$ overlap. That is, $h(\gamma) \cap g(\gamma) \neq \emptyset$.

(ii) for any end-separating subsurface $B \subset S$ the translates $g(B)$ and $h(B)$ overlap. That is, $\partial(h(B)) \cap \partial(g(B)) \neq \emptyset$.

Proof. We start by proving (i). Without loss of generality we will show that $\gamma \cap g(\gamma) \neq \emptyset$ for all $g \in \overline{\text{PMCG}}_c(S)$. Since $g \in \overline{\text{PMCG}}_c(S)$ there is some $g' \in \text{PMCG}_c(S)$ such that $g'(\gamma) = g(\gamma)$. There is a finite type subsurface $K \subset S$ such that $\gamma, g(\gamma), \text{supp}(g')$ are contained in $K$. Thus we can realize $g'$ as a pure mapping class of the surface $K$. Now since $\gamma$ separates the boundary curves and/or the punctures of $K$ we must have that $\gamma \cap g'(\gamma) \neq \emptyset$, or equivalently, $\gamma \cap g(\gamma) \neq \emptyset$.

For (ii) we apply (i) to each of the curves in $\partial B$.

We can now apply Lemma 3.8 to obtain an action of $\overline{\text{PMCG}}_c(S)$ on a projection complex.

Proposition 4.2. Let $S$ be an infinite type surface and let $Y = \{g(A) | g \in \overline{\text{PMCG}}_c(S)\}$ where $A$ is either an end-separating curve on $S$ or an end-separating subsurface of $S$. Then $\overline{\text{PMCG}}_c(S)$ acts on a quasi-tree, the projection complex $P_A(Y)$ corresponding to $\{C(Y), \pi_Y\}_{Y \in Y}$, where the bottleneck constant for $P_A(Y)$ is independent of the surface $S$ or $A$. Furthermore, $\overline{\text{PMCG}}_c(S)$ also acts on the blown up projection complex $C_A(Y)$. When $A$ is an end-separating curve $C_A(Y)$ is again a quasi-tree and when $A$ is an end-separating subsurface $C_A(Y)$ is $\delta$-hyperbolic. In either case the respective bottleneck constant or $\delta$ is independent of $S$ and $A$.

Remark 4.3. The independence of all of the constants follows from the fact that they only depend on two quantities:

- The projection constant, which in all cases is 10,
- The bottleneck constant of the curve graph in the case that $A$ is a curve, which is shown to be constant in [MM00], and the hyperbolicity constant of the curve graph when $A$ is a surface which is shown to be independent of topological type independently in [Aou13, Bow14, CRS15, HPW15].
5 Constructing Quasimorphisms

In this section we construct quasimorphisms on $\text{PMCG}_c(S)$ that will “see” elements which are nontrivial in $H_1(\text{PMCG}_c(S); \mathbb{Z})$. We will do this by showing that Dehn twists about end-separating curves are WWPD elements when acting on the projection complexes arising from Proposition 4.2 and then applying Proposition 3.11. Let $\gamma$ be an end-separating simple closed curve on $S$.

**Lemma 5.1.** $(\text{PMCG}_c(S), C_\gamma(Y), T_\gamma, \text{Stab}(\gamma))$ satisfies WWPD with constant $\xi$ depending only on the projection constant of $C_\gamma(Y)$ and with translation length 1. Furthermore, for any power $n > 0$, $(\text{PMCG}_c(S), C_\gamma(Y), T_n^\gamma, \text{Stab}(\gamma))$ also satisfies WWPD with the same constant.

To prove this we need to make use of the fact that nearest point projections in the projection complex are uniformly close to the given projections.

**Proposition 5.2** ([BBF15], Corollary 4.10). For every $Z \in Y$ the nearest point projection $C(Y) \rightarrow C(Z)$ is coarsely Lipschitz and the image of $C(Y)$ for $Y \neq Z$ is in a uniform neighborhood of the bounded set $\pi_Z(Y)$. The uniform bound is a function of the projection constant.

**Proof of Lemma 5.1.** We first note that $T_\gamma$ acts hyperbolically with translation length 1. Indeed, within $C(\gamma)$ we have that the projection distances satisfy $d_\gamma(T_n^\gamma(\alpha), \alpha) = 2 + |n|$ where $\alpha$ is some curve in $S$ that intersects $\gamma$ transversely and for all $n \neq 0$. $C(\gamma)$ is then isometrically embedded within $C(Y)$.

Fix a quasi-axis $\ell \subset C(\gamma)$ for $T_\gamma$. For any $h \in \text{PMCG}_c(K) \setminus \text{Stab}(\gamma)$ we have that $h$ must move $C(\gamma)$ to some other $C(h(\gamma))$. Thus by Proposition 5.2 the diameter of the nearest point projection of $h \cdot \ell$ to $\ell$ is bounded by the nearest point projection of a uniform neighborhood of $\pi_{C(\gamma)}(C(h \cdot \gamma))$ to $\ell$. This in turn is uniformly bounded by a function of the projection constant.

We can now apply Proposition 3.11 to this construction. By making an appropriate choice of basepoint in the construction of our quasimorphism we can gain control over the value of the quasimorphism on group elements that are sufficiently “independent” of our Dehn twist.

**Lemma 5.3.** Suppose $(\text{PMCG}_c(S), C_\gamma(Y), T_\gamma^n, \text{Stab}(\gamma))$ is as in Lemma 5.1. If $h \in \text{PMCG}_c(S)$ fixes $C(\gamma)$ and $C(\gamma')$ for some $\gamma' \in Y$ with $\gamma' \neq \gamma$ then
the quasimorphism $F$ obtained via Proposition 3.11 (when $n$ is sufficiently large) can be chosen to be bounded on $h$. Furthermore, if $n$ is greater than the projection constant, 10, of $C(Y)$, then $F(h) = 0$.

Proof. If $h$ preserves $C(\gamma)$ and $C(\gamma')$ then it must preserve the projections $\pi_\gamma(\gamma')$ and $\pi_{\gamma'}(\gamma)$. These projections are bounded in diameter by the projection constant, 10. Now we can pick our basepoint, $x_0$, in the construction of $F$ to be in the set $\pi_\gamma(\gamma')$ so that $d_{C_\gamma}(x_0, h(x_0)) \leq 10$. We conclude that $|F(h)| \leq \frac{10}{n}$ since the translation length of $T^n_\gamma$ is $n$. The furthermore statement follows from the fact that $F$ is integer valued. □

6 Proof of Theorem A

We will actually prove a stronger theorem than stated in the introduction.

Theorem 6.1. Let $S$ be an infinite type surface with more than one end, $\Gamma = \{\gamma_i\}_{i \in \mathbb{N}}$ and $\Gamma' = \{\gamma'_i\}_{i \in \mathbb{N}}$ be two collections of disjoint end-separating curves so that $\gamma'_i \neq \gamma_i$ is a translate of $\gamma_i$ by a compactly supported mapping class and $\gamma'_i \cap \gamma_j = \emptyset$ for all $i \neq j$, and $A = \{a_i\}_{i \in \mathbb{N}}$ be an unbounded sequence of natural numbers. Then the mapping class

$$f_{\Gamma,A} := \prod_{i=1}^{\infty} T^{a_i}_{\gamma_i} \in \overline{\text{PMCG}}_c(S)$$

cannot be written as a product of commutators in $\overline{\text{PMCG}}_c(S)$. The same also holds for products $\phi f_{\Gamma,A}$ where $\phi \in \text{PMCG}_c(S)$ is a mapping class that fixes $\gamma_i$ and $\gamma'_i$ for infinitely many $i$.

We can first note that since the $\gamma_i$ are disjoint, maps of the form $f_{\Gamma,A}$ are indeed defined and contained in $\overline{\text{PMCG}}_c(S)$.

Proof of Theorem 6.1. The proof is a direct application of the following lemma.

Lemma 6.2. For all $C > 0$, there exists an $N_0$ so that all $g \in \overline{\text{PMCG}}_c(S)$ that can be written as $g = hT^N_\gamma$, where $N > N_0$, $\gamma$ is an end-separating curve, and $h \in \overline{\text{PMCG}}_c(S)$ is a mapping class that fixes $\gamma$ and some $\overline{\text{PMCG}}_c(S)$-translate of $\gamma$, cannot be written as a product of $C$ commutators.
**Proof.** Let \( \xi \) be the WWPD constant coming from Lemma 5.1 and \( M \) be the constant coming from Proposition 3.11 when we apply it to the quasi-tree arising from a projection complex as in Proposition 4.2. Note that \( \xi \) and \( M \) depend only on the projection constant of 10 coming from Lemma 3.8. In particular, they do not depend on \( S \) or \( \gamma \).

Next we let \( N_1 = \max\{M + \xi, 11\} \), \( N_2 = 2(12C + 25) \), and \( N_0 = N_1N_2 \). Suppose \( g = hT_N^\gamma \) as in the statement of the lemma. Apply Proposition 4.2 to \( S \) and the curve \( \gamma \) to get an action of \( \text{PMCG}_c(\mathcal{S}) \) on the quasi-tree \( C_\gamma(\mathcal{Y}) \). Denote the length metric on \( C_\gamma(\mathcal{Y}) \) by \( d_{C_\gamma} \). By Lemma 5.1, \( (\text{PMCG}_c(\mathcal{S}), C_\gamma(\mathcal{Y}), T_N^{\gamma}, \text{Stab}(\gamma)) \) satisfies WWPD with constant \( \xi \) and \( T_N^{\gamma} \) has translation length \( N_1 > 10 \) and \( N_1 \geq M + \xi \). By our choice of \( N_1 \) we can apply Proposition 3.11 and Lemma 5.3 to \( (\text{PMCG}_c(\mathcal{S}), C_\gamma(\mathcal{Y}), T_N^{\gamma}, \text{Stab}(\gamma)) \) in order to build a quasimorphism \( F : \text{PMCG}_c(\mathcal{S}) \rightarrow \mathbb{R} \) with basepoint \( x_0 \) so that \( F(h) = 0 \). Thus we see that

\[
|F(hT_N^\gamma) - F(T_N^\gamma)| < 12.
\]

Now write \( N = AN_1 + B \) for \( A \geq N_2 \) and \( B < N_1 \). Note that \( B < N_1 \) so that \( d_{C_\gamma}(x_0, T_B^\gamma x_0) < d_{C_\gamma}(x_0, T_N^{\gamma_1} x_0) \) and hence \( F(T_B^\gamma) = 0 \). Then we have

\[
|F(T_N^{AN_1+B}) - F((T_N^{\gamma_1})^A) - F(T_B^\gamma)| = |F(T_N^{AN_1+B}) - F((T_N^{\gamma_1})^A)| < 12.
\]

We chose \( N_2 \) so that \( F(T_N^{\gamma_1})^A > 12C + 24 \) by Proposition 3.11(ii). Thus we see that

\[
F(T_N^{\gamma_1}) > 12C + 12,
\]

and

\[
F(g) = F(hT_N^\gamma) > 12C.
\]

If \( g \) could be written as a product of \( C \) commutators than we would have \( F(g) \leq 12C \), which would contradict the lower bound found above.

To finish the proof of our theorem we simply note that since \( A \) is unbounded, for any \( C > 0 \) we can always write \( f_{\Gamma,A} = hT_N^\gamma \) where \( a_i > N_0(C) \) coming from Lemma 6.2. Here \( h \) will be the product of all Dehn twists appearing in \( f_{\Gamma,A} \) other than the twists about \( \gamma_i \) and so satisfies the conditions of Lemma 6.2. For the final claim we simply include \( \phi \) into the expression for \( h \).  

\[\Box\]
Finally, we can conclude Theorem A from Theorem 6.1 provided that families of curves as in the statement of Theorem 6.1 always exist. To find such a family of curves we can fix a separating principal exhaustion of a given infinite type surface $S$ (with at least two ends) and take $\Gamma = \{\gamma_i\}_{i \in \mathbb{N}}$ to be a choice of one boundary curve of each $S_i$ in the exhaustion. Now for each $i$ pick a curve $\alpha_i$ in $S_{i+1} \setminus S_{i-1}$ that intersects $\gamma_i$. Set $\gamma'_i = T_{\alpha_i}(\gamma_i)$. These collections of curves $\Gamma$ and $\Gamma'$ satisfy the hypotheses of Theorem 6.1. This proves Theorem A provided that $S$ has at least two ends. The one-ended case is proved in the appendix by applying the Birman exact sequence.

7 Pseudo-Anosovs on Disjoint Subsurfaces

Now we see that the proof of Theorem 6.1 also works when we replace the Dehn twists by pseudo-Anosovs on homeomorphic disjoint subsurfaces. This version will be used in the following section to prove that $H_1(\text{PMCG}_c(S); \mathbb{Z})$ contains an uncountable sum of $\mathbb{Q}$'s when $S$ has genus less than 3.

**Theorem 7.1.** Let $S$ be an infinite type surface with at least two ends, $\mathcal{B} = \{B_i\}_{i \in \mathbb{N}}$ be a collection of disjoint subsurfaces of $S$, each of which is end-separating and is homeomorphic to some fixed finite type surface $B$ and $A = \{a_i\}_{i \in \mathbb{N}}$ be an unbounded sequence of natural numbers. Suppose that $f \in \text{PMCG}(B)$ is a pseudo-Anosov and let $f_i \in \text{PMCG}_c(S)$ be the mapping class which is equal to $f$ on $B_i$ and the identity outside of $B_i$. Then the mapping class

$$f_{B,A} := \prod_{i=1}^{\infty} f_i^{a_i} \in \text{PMCG}_c(S)$$

cannot be written as a product of commutators in $\text{PMCG}_c(S)$. The same also holds for any mapping class of the form $\phi f_{B,A}$ where $\phi \in \text{PMCG}_c(S)$ fixes $B_i$ for infinitely many $i$.

To prove this we want to follow the same steps used in the proof above. Proposition 4.2 gives an action of $\text{PMCG}_c(S)$ on a blown up projection complex built out of the curve graphs of the $\text{PMCG}_c(S)$-orbit of $B_i$ for each $i$. Just as above we will build quasimorphisms on $\text{PMCG}_c(S)$ for each $B_i$. For now we assume that $i$ is fixed and abuse notation to write $B = B_i$.

Next we have to do two things. We have an action of $\text{PMCG}_c(S)$ on a $\delta$-hyperbolic space, but we really need an action on a quasi-tree. Also we need to see that a pseudo-Anosov, $f$, supported on $B$ is a WWPD element for this action. This will all follow from work in [BF02], [BBF15], and [BBF16].
The first step is to see that $f$ acts as a WWPD element on $C_B(Y)$. The following proposition informs us that $f$ is a WPD element for the action of PMCG($B$) on the curve graph of $B$.

**Proposition 7.2 ([BF02], Proposition 11).** Let $A$ be a nonsporadic finite type surface. The action of MCG($A$) on the curve graph of $A$ is WPD.

Now Proposition 4.20 in [BBF15] tells us that this WPD element for the action on a single curve graph gives a WWPD element for the action on the entire blown up projection complex with WWPD constant depending only on the projection constant. Finally we can use the following proposition to upgrade our WWPD action on a $\delta$-hyperbolic graph to a WWPD action on a quasi-tree.

**Proposition 7.3 ([BBF16], Proposition 2.9).** Let $X$ be a $\delta$-hyperbolic graph and assume $(G,X,g,C)$ satisfies WWPD with constant $\xi = \xi^X_g$. Then there is an action of $G$ on a quasi-tree $Q$ such that:

1. The bottleneck constant, $\Delta$, for $Q$ depends only on $\delta$ and $\xi$ and is bounded by a multiple of $\delta + \xi + 1$,
2. $(G,Q,g,C)$ satisfies WWPD with $\xi^Q_g$ bounded by a multiple of $\delta + \xi + 1$.

**Sketch of Proof.** We apply the projection complex construction again. Say two conjugates of $g$ are equivalent if they have parallel quasi-axes. Now for each equivalence class we take the union of the quasi-axes of its members. This is a quasi-line with the subspace metric. Let the collection $Y$ be all of these quasi-lines. This collection will satisfy the projection axioms and so when we construct the projection complex we get a quasi-tree, $Q$. We get (i) by realizing that the projection constant used in the construction only depends on $\delta$ and $\xi$. (ii) follows again from Proposition 4.20 in [BBF15].

We collect these facts in the following lemma.

**Lemma 7.4.** Let $S$ be an infinite type surface and suppose that $B \subset S$ is an end-separating subsurface. Given $f \in$ PMCG($B$) a pseudo-Anosov, there exists a subgroup $N < \text{PMCG}_c(S)$ and quasi-tree $Q$ so that $(\text{PMCG}_c(S),Q,f,N)$ satisfies WWPD such that $Q$ is a quasi-tree with bottleneck constant which does not depend on how $B$ embeds as a subsurface of $S$.

This lemma allows us to apply Proposition 3.11. Now we get an analogous result as in Lemma 5.3.
Lemma 7.5. Let \((\text{PMCG}_c(S), Q, f^n, N)\) be as above. If \(h \in \text{PMCG}_c(S)\) acts as the identity on \(B\) then the quasimorphism \(F\) obtained via Proposition 3.11 (when \(n\) is sufficiently large) can be chosen to be trivial on \(h\).

Proof. \(h\) acts as the identity on \(B\) and so it fixes \(C(B)\) pointwise in \(C_B(Y)\). Thus \(h\) must fix pointwise a quasi-axis of \(f\) in \(C(B)\). This quasi-axis is one of the objects used to construct the projection complex \(Q\). Now when we build \(F\) we can take the basepoint to be on this quasi-axis so that \(F(h) = 0\). \(\square\)

We can also follow the exact same proof for Lemma 6.2 to get a version in this setting.

Lemma 7.6. Let \(S\) be an infinite type surface and \(f \in \text{PMCG}_c(S)\) a partial pseudo-Anosov supported on an end-separating subsurface \(B\) of \(S\). For all \(C > 0\), there exists an \(N_0\), dependent on \(f\), so that all \(g \in \text{PMCG}_c(S)\) that can be written as \(g = h f^N\), where \(N > N_0\) and \(h \in \text{PMCG}_c(S)\) a mapping class that fixes \(B\), cannot be written as a product of \(C\) commutators.

With all of these pieces the proof of Theorem 7.1 follows exactly as the proof of Theorem 6.1. Note that the constant \(N_0\) is dependent on \(f\) as an element of \(\text{PMCG}(B)\). This is the reason that we take higher and higher powers of the same pseudo-Anosov in the statement of Theorem 7.1. The result should also work for higher and higher powers of different pseudo-Anosovs provided that their translation lengths fell in a bounded range.

8 Divisible Subgroup of \(H_1(\text{PMCG}_c(S))\)

We have seen that for any infinite type surface with more than one end, \(S\), \(H_1(\text{PMCG}_c(S); \mathbb{Z})\) is nontrivial. Next we will use our main theorem to find a subgroup isomorphic to \(\oplus_{2^{\aleph_0}} \mathbb{Q}\) within the abelianization.

Definition 8.1. An element \(g\) of a group \(G\) is said to be divisible by \(n\) if the equation \(g = x^n\) has a solution in \(G\). We say that \(g\) is divisible if it is divisible by \(n\) for all \(n \in \mathbb{N}\). An abelian group is called divisible if every element is divisible.

We first find a divisible element in the abelianization, then we construct uncountably many independent elements, and finally we combine these two constructions to prove Theorem C. We adopt the notation that an overbar represents the image of a mapping class in \(H_1(\text{PMCG}_c(S); \mathbb{Z})\).
8.1 Constructing Divisible Elements

We will follow the construction of Bogopolski and Zastrow for infinitely divisible elements in the first homology of the Hawaiian Earring and Griffiths’ space as seen in [BZ12]. We will need a slight modification when $S$ has genus less than 3.

8.1.1 $S$ has Genus at Least 3

We first consider the case that $S$ is an infinite type surface of genus at least 3 and with more than one end.

Let $\{\gamma_i\}_{i \in \mathbb{N}}$ and $\{\gamma'_i\}_{i \in \mathbb{N}}$ be as in the statement of Theorem 6.1. Let

$$f = \prod_{j=1}^{\infty} T_{\gamma_j}^{\gamma_j}.$$ 

Note that $f$ satisfies the hypotheses of Theorem 6.1 so that $\bar{f}$ is nontrivial in $H_1(\text{PMCG}_c(S); \mathbb{Z})$.

Since $S$ has genus at least 3, each individual $T_{\gamma_i}$ can be written as a product of commutators. Therefore, if we delete finitely many of the $T_{\gamma_i}$ from $f$, the resulting equivalence class in $H_1(\text{PMCG}_c(S); \mathbb{Z})$ will be unchanged. By deleting all of the occurrences of $T_{\gamma_1}$ in $\bar{f}$ we see that we can write $\bar{f}$ as a square in $H_1(\text{PMCG}_c(S); \mathbb{Z})$. Indeed, $\bar{f} = \bar{f}' \bar{f}'$ where $\bar{f}'$ is given by

$$\bar{f}' = \prod_{k=2}^{\infty} T_{\gamma_k}^{k!.}$$

We also verify that $\bar{f}'$ is nontrivial in $H_1(\text{PMCG}_c(S); \mathbb{Z})$ by Theorem 6.1.

Similarly, for all $n \in \mathbb{N}$, by deleting all occurrences of $T_{\gamma_1}, \ldots, T_{\gamma_n}$ from $\bar{f}$ we see that $\bar{f}$ is an $(n+1)$-th power in $H_1(\text{PMCG}_c(S); \mathbb{Z})$. Thus we see that $\bar{f}$ is divisible in $H_1(\text{PMCG}_c(S); \mathbb{Z})$.

8.1.2 $S$ has Genus Less than 3

Suppose that $S$ has genus less than 3. We can no longer simply use Dehn twists because we no longer get for free that they can be written as a product of commutators. Instead we will run the same construction using a pseudo-Anosov on a punctured sphere that we can write as a commutator. Here we will need to make use of Theorem 7.1.

Suppose that $h$ is a pseudo-Anosov on a six-times punctured sphere (or similarly a sphere with six boundary components) that can be written as
a product of commutators. Now since $S$ is infinite type we can follow the same steps as at the end of Section 6 to find a collection $B = \{B_i\}_{i \in \mathbb{N}}$ where each $B_i$ is end-separating and homeomorphic to a sphere with six boundary components. Let $h_i \in \text{PMCG}_c(S)$ be the mapping class that is $h$ on $B_i$ and the identity elsewhere. We can now apply the same exact construction as in the previous section with $h_i$ instead of $T_{\gamma_i}$ to get a divisible element. We apply Theorem 7.1 to see that it is nontrivial in $H_1(\text{PMCG}_c(S); \mathbb{Z})$. Note that we used a six-times punctured sphere since Proposition 7.2 required a surface with sufficiently large complexity.

Now we just need to find a pseudo-Anosov on a six-times punctured sphere that can be written as a commutator. We will obtain such a pseudo-Anosov from the following lemma which is an application of Thurston’s construction from [T+88] as stated in [FM11].

**Lemma 8.2.** Suppose $\alpha$ and $\beta$ are curves which fill a finite type surface $S$. Then the element $T_\alpha^2 T_\beta^2 T_\alpha^{-2} T_\beta^{-2}$ is a pseudo-Anosov in $\text{MCG}(S)$.

**Proof.** Thurston’s construction gives that there is a representation $\rho : \langle T_\alpha, T_\beta \rangle \to \text{PSL}(2, \mathbb{R})$ given by

$$T_\alpha \to \begin{pmatrix} 1 & -i(\alpha, \beta) \\ 0 & 1 \end{pmatrix}, \quad T_\beta \to \begin{pmatrix} 1 & 0 \\ i(\alpha, \beta) & 1 \end{pmatrix}.$$ 

Furthermore, this representation has the property that $f \in \langle T_\alpha, T_\beta \rangle$ is periodic, reducible, or pseudo-Anosov if and only if $\rho(f)$ is elliptic, parabolic, or hyperbolic, respectively. Finally, we note that two filling curves intersect at least once so that the element $\rho(T_\alpha^2 T_\beta^2 T_\alpha^{-2} T_\beta^{-2})$ has trace in absolute value greater than 2.

This lemma allows us to obtain our desired pseudo-Anosov by taking two curves which fill the six-times punctured sphere.

### 8.2 Uncountably Many Independent Elements

Let $S$ be any infinite type surface with more than one end (not necessarily of genus at least 3). We will apply a trick used in [RS07] and [Man20]. For each $a \in \mathbb{R}$ let $\Lambda_a$ be an infinite subset of $\mathbb{N}$ such that $\Lambda_a \cap \Lambda_b$ is finite for all $a \neq b$. We can obtain $\Lambda_a$ by putting $\mathbb{N}$ in bijection with $\mathbb{Q}$ and then letting $\Lambda_a$ be a sequence of rational numbers approximating $a$. 

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Once again let \( \{\gamma_i\}_{i \in \mathbb{N}} \) and \( \{\gamma'_i\}_{i \in \mathbb{N}} \) be as in the statement of Theorem 6.1. For \( a \in \mathbb{R} \) enumerate elements of \( \Lambda_a \) as \( \{a_i\}_{i \in \mathbb{N}} \) and let

\[
f_a := \prod_{i=1}^{\infty} T_{\gamma_{a_i}}^{i} \in \text{PMCG}_c(S).
\]

By Theorem 6.1 \( \bar{f}_a \) is nontrivial in \( H_1(\text{PMCG}_c(S); \mathbb{Z}) \) for all \( a \in \mathbb{R} \). Note also that since \( \Lambda_a \cap \Lambda_b \) is finite for any \( a \neq b \), any finite product of such \( f_a \) also satisfies the hypotheses of Theorem 6.1 so that any finite product is also nontrivial in \( H_1(\text{PMCG}_c(S); \mathbb{Z}) \). Again, this did not rely on \( S \) having genus at least 3. Thus we have the following proposition.

**Proposition 8.3.** Let \( S \) by any infinite type surface with more than one end. Then \( H_1(\text{PMCG}_c(S); \mathbb{Z}) \) contains an uncountable collection of independent elements.

Note that we could have applied this same technique to products of powers of pseudo-Anosovs on subsurfaces as in Theorem 7.1.

### 8.3 Proof of Theorem C

We can now modify the construction of a divisible element to find uncountably many divisible elements. For \( a \in \mathbb{R} \), let \( \Lambda_a \) be as above. In the genus greater than two case, for each \( a \in \mathbb{R} \) we construct \( f_a \) as in Section 8.1.1 except by using only twists in \( \Lambda_a \). That is, \( f_a \) is defined as:

\[
f_a = \prod_{j=1}^{\infty} T_{\gamma_{a_j}}^{j}.
\]

In the genus less than three case we do the same construction but using pseudo-Anosovs as in Section 8.1.2.

In both cases this gives an uncountable collection of independent divisible elements \( \{f_a\}_{a \in \mathbb{R}} \) in \( H_1(\text{PMCG}_c(S); \mathbb{Z}) \). Let \( A \) be the minimal divisible subgroup containing \( \{f_a\}_{a \in \mathbb{R}} \). We can now apply the Structure Theorem of Divisible Groups. First we recall that a **quasicyclic group** is a group isomorphic to the group of \( p^n \)th complex roots of unity for all \( n \) and for some prime \( p \). Note that these groups are all torsion.

**Theorem 8.4** ([Fuc70], Theorem 23.1). Any divisible group \( D \) is a direct sum of quasicyclic and full rational groups. The cardinal numbers of the sets of quasicyclic components and \( \mathbb{Q}'s \) form a complete and independent system of invariants for \( D \).
Every element in the collection \( \{ f_a \}_{a \in \mathbb{R}} \) is torsion free since any power is non-trivial in the abelianization by Theorem 6.1. Therefore we see that \( A \) has uncountably many torsion-free elements and so must contain a subgroup isomorphic to \( \oplus_{2^{\aleph_0}} \mathbb{Q} \). Next we use the following.

**Theorem 8.5** ([Fuc70], Theorem 21.3). *Every abelian group \( A \) is the direct sum \( A = D \oplus C \) where \( D \) is divisible and \( C \) has no divisible subgroups other than the identity.*

**Proof of Theorem 8.5.** Write \( H_1(\text{PMCG}_{\text{c}}(S); \mathbb{Z}) = D \oplus C \) where \( D \) is divisible and \( C \) as in Theorem 8.5. Next we can further decompose \( D = Q \oplus T \) where \( Q \) is a direct sum of \( \mathbb{Q} \)'s and \( T \) is torsion. By the above discussion and Theorem 8.4 we have \( Q = \oplus_{2^{\aleph_0}} \mathbb{Q} \). Letting \( B = T \oplus C \) finishes the proof. \( \square \)

## 9 Torelli Group

The **Torelli Group**, \( \mathcal{I}(S) \), is the kernel of the natural homomorphism \( \text{MCG}(S) \to \text{Aut}(H_1(S; \mathbb{Z})) \). The Torelli group has been widely studied in the finite type case. In particular, Johnson in [Joh85] explicitly computed the abelianization of the Torelli group when \( S \) is a finite type surface of genus at least 3 and with 1 boundary component. We will see that all of our arguments in the previous sections can be carried out in the Torelli group to obtain the same results.

**Theorem 9.1.** Let \( S \) be an infinite type surface. \( H_1(\mathcal{I}(S); \mathbb{Z}) = \oplus_{2^{\aleph_0}} \mathbb{Q} \oplus B \) where all divisible subgroups of \( B \) are torsion.

The case of the infinite type surface with one end is also handled via a Birman Exact Sequence argument in the appendix.

In [AGK+18] the authors found a topological generating set for \( \mathcal{I}(S) \) when \( S \) is infinite type.

**Theorem 9.2** ([AGK+18], Corollary 2). *Let \( S \) be a connected oriented surface of infinite type. Then \( \mathcal{I}(S) \) is topologically generated by separating twists and bounding-pair maps.*

We thus immediately see that mapping classes of the form used in Theorem 6.1 are contained within \( \mathcal{I}(S) \). This gives us nontrivial elements in \( H_1(\mathcal{I}(S); \mathbb{Z}) \) when \( S \) has more than one end. We have to be a little bit more careful when it comes to finding divisible elements in the abelianization. Just as in the low genus case we no longer can be sure that individual Dehn twists are contained in \([\mathcal{I}(S); \mathcal{I}(S)]\). However, we can use the exact
same arguments as in Section 8.1.2 to apply Theorem 7.1 to obtain divisible elements in $H_1(I(S); \mathbb{Z})$. To do this we simply apply Lemma 8.2 to a pair of separating, filling curves. Theorem 9.1 now follows by the same argument as in Section 8.3.

Remark 9.3. This theorem did not depend on the fact that we were considering the Torelli group. The conclusion of Theorem 9.1 holds for any subgroups of $\text{PMCG}_c(S)$ that contain sufficiently many elements of the form found in Theorems 6.1 or 7.1.

9.1 Johnson Homomorphism

Johnson makes use of the Johnson and Birman-Craggs-Johnson homomorphisms to explicitly compute $H_1(I(S); \mathbb{Z})$ in the finite type setting. We can extend the Johnson homomorphism to the infinite type setting and use it to find an indivisible copy of $\bigoplus_{2^{\aleph_0}} \mathbb{Z}$ in $H_1(I(S); \mathbb{Z})$ provided that $S$ has infinite genus.

Let $S$ be an infinite type surface, $H = H_1(S; \mathbb{Z})$, and $a \in H$. Represent $a$ by an oriented multicurve $\mu$ on the surface $S$. Given $f \in I(S)$ and a representative homeomorphism $\phi$ of $f$ let $M_\phi$ be the mapping torus of $\phi$. The cylinder $C = \mu \times [0, 1]$ maps into $M_\phi$. Since $f \in I(S)$ we have that $\phi(\mu)$ is homologous to $\mu$ so that there is an immersed surface in $S \times \{0\} \subset M_\phi$ that closes up the cylinder $C$ to a surface $S_a$ in $M_\phi$. Note that since $S$ has at least one end the choice of this surface is unique. $S_a$ gives rise to a homology class $[S_a] \in H_2(M_\phi; \mathbb{Z})$. By Poincaré duality this gives a class $[S_a] \in H^1_c(M_\phi; \mathbb{Z})$, the first cohomology with compact support of $M_\phi$.

Given a triple $a \wedge b \wedge c \in \wedge^3 H$, the third exterior power of $H$, we get an element $[S_a] \sim [S_b] \sim [S_c] \in H^3_c(M_\phi; \mathbb{Z})$. Finally, we can pair this homology class with the fundamental class of $M_\phi$ in locally finite homology to obtain an element of $\mathbb{Z}$. This gives a homomorphism, the Johnson homomorphism

$$\tau : I(S) \to \text{Hom}(\wedge^3 H, \mathbb{Z}).$$

Alternatively, $\tau(f)(a \wedge b \wedge c)$, can be thought of as the triple algebraic intersection $S_a \cap S_b \cap S_c$. Also, just as in the finite type case the Johnson homomorphism satisfies a naturality property: for $f \in I(S)$, $h \in \text{MCG}(S)$, and $a \wedge b \wedge c \in \wedge^3 H$ we have

$$\tau(hfh^{-1})(a \wedge b \wedge c) = \tau(f)(h_*^{-1}(a) \wedge h_*^{-1}(b) \wedge h_*^{-1}(c)).$$
Remark 9.4. Just as in the finite type case it can be shown that \( \tau \) is trivial on any separating twist. In fact, \( \tau \) is trivial on any infinite product of separating twists, provided the infinite product actually defines a mapping class. This triviality follows by building and applying the resulting homomorphism to a geometric homology basis as defined in [FHV19]. The Johnson homomorphism can thus be seen as capturing some new information about \( H_1(\mathcal{I}(S); \mathbb{Z}) \) not coming from the previous constructions in this paper.

Lemma 9.5. Let \( \{\alpha, \beta\} \) be a bounding pair.

(i) \( \tau(T_\alpha T_\beta^{-1}) \) is non-zero.

(ii) Let \( B \) be the subsurface of \( S \) with boundary \( \alpha \cup \beta \). If \( a', b', c' \) is a triple of curves that do not intersect \( B \), then \( \tau(T_\alpha T_\beta^{-1})(a' \land b' \land c') = 0 \).

Proof. The change of coordinates principle and naturality property allows us to consider a standard bounding pair \( \{\alpha, \beta\} \) as in Figure 2. Consider the triple of curves \( a \land b \land c \) in Figure 2. Here \( c \) and \( T_\alpha T_\beta^{-1}(c) \) cobound the surface, \( A \), on the left hand side of the figure. Also, \( T_\alpha T_\beta^{-1} \) fixes \( a \) and \( b \) pointwise. Thus we see that \( \tau(T_\alpha T_\beta^{-1})(a \land b \land c) = 1 \), proving (i).
For (ii) we simply note that $T_\alpha T_{\beta}^{-1}$ fixes each of $a', b', c'$ so that in the mapping torus each corresponding cylinder closes up. Thus the three corresponding subsurfaces have trivial triple intersection.

This allows us to build uncountably many linearly independent elements in the image of $\tau$ provided that $S$ has infinite genus.

**Proposition 9.6.** Let $S$ be a surface with infinite genus. Then the image of $\tau : \mathcal{I}(S) \to \text{Hom}(\wedge^3 H; \mathbb{Z})$ has uncountably many linearly independent elements.

**Proof.** If $f$ is a bounding-pair map we let $S_f$ be the finite type surface that the corresponding bounding pair cobound. Let $\{f_i\}_{i \in \mathbb{N}}$ be a sequence of bounding-pair maps such that $\{S_{f_i}\}_{i \in \mathbb{N}}$ is a pairwise disjoint sequence of subsurfaces of $S$. Now we apply the same trick as in Section 8.2 to obtain for each $a \in \mathbb{R}$ an infinite subset $\Lambda_a$ of $\mathbb{N}$ such that $\Lambda_a \cap \Lambda_b$ is finite for all $a \neq b$. For each $a \in \mathbb{R}$ let

$$f_a = \prod_{i \in \Lambda_a} f_i.$$

We claim that the collection $\{\tau(f_a)\}_{a \in \mathbb{R}}$ is linearly independent. We first check that $\tau(f_a) \neq \tau(f_b)$ for all $a \neq b$. Let $i_a \in \Lambda_a \setminus \Lambda_b$ and $i_b \in \Lambda_b \setminus \Lambda_a$. Consider two triples of curves $x_{ia} \land y_{ia} \land z_{ia}$ and $x_{ib} \land y_{ib} \land z_{ib}$ where $x_{ia}$ and $y_{ia}$ are two curves contained in $S_{ia}$ intersecting once and $z_{ia}$ is a curve intersecting each of the bounding-pair curves making up $f_{ia}$ that is disjoint from $S_{ib}$, likewise for the other triple of curves.

Then by Lemma 9.5 we have the following.

$$\tau(f_a)(x_{ia} \land y_{ia} \land z_{ia}) = 1,$$

$$\tau(f_b)(x_{ia} \land y_{ia} \land z_{ia}) = 0,$$

$$\tau(f_a)(x_{ib} \land y_{ib} \land z_{ib}) = 0,$$

$$\tau(f_b)(x_{ib} \land y_{ib} \land z_{ib}) = 1.$$

Finally, given any finite linear combination of such maps we will always be able to find such a triple which evaluates to something non-zero since any collection of the $\Lambda_a$ has finite intersection.

Since $\tau$ must factor through $H_1(\mathcal{I}(S); \mathbb{Z})$ this proves that the abelianization of $\mathcal{I}(S)$, unlike the abelianization of $\text{PMCG}_c(S)$, also contains many indivisible copies of $\mathbb{Z}$ when $S$ has infinite genus.
Proof of Theorem \[\text{D}.\] The first statement is the content of Theorem \[9.1.\] Assume that \(S\) has infinite genus. We first note that \(\text{Hom}(\wedge^3 H; \mathbb{Z}) \cong \mathbb{Z}^3\) does not contain any divisible elements. We must then have that all divisible elements in \(H_1(I(S); \mathbb{Z})\) are contained in the kernel of \(\tau\). Therefore, given the splitting \(H_1(I(S); \mathbb{Z}) = \oplus_{2^{\aleph_0}} \mathbb{Q} \oplus B\) as in Theorem \[9.1\] we have that \(\tau(H_1(I(S); \mathbb{Z})) = \tau(B)\). By Proposition \[9.6\] we see that the image of \(\tau\) contains a copy of \(\oplus_{2^{\aleph_0}} \mathbb{Z}\). Finally, this is a free abelian group and so lifts to a copy of \(\oplus_{2^{\aleph_0}} \mathbb{Z}\) in \(B\).

\[\square\]

10 A Discontinuous Homomorphism

In this section we give counterexamples to automatic continuity in the setting of the closure of the compactly supported mapping class group using Theorem \[C\]. This application was pointed out to the author by Ryan Dickmann and in conversations with Paul Plummer, Jesús Hernández Hernández, and Ryan Dickmann. A shorter, broader proof was pointed out to the author by Mladen Bestvina. In this section we write \(\mathfrak{c} = 2^{\aleph_0}\) for the cardinality of the continuum.

A topological group is said to be \textbf{Polish} if it is separable and completely metrizable. In [APVI17] the authors show that for an infinite type surface, \(S\), \(\text{MCG}(S)\) is Polish and hence so are all closed subgroups including \(\text{PMCG}(S)\) and \(\overline{\text{PMCG}}_c(S)\).

\textbf{Definition 10.1.} We say that a Polish group \(G\) has \textbf{automatic continuity} if every homomorphism from \(G\) to a separable topological group is necessarily continuous.

\textbf{Theorem 10.2.} Let \(S\) be an infinite type surface. There exists \(2^\mathfrak{c}\) discontinuous homomorphisms from \(\overline{\text{PMCG}}_c(S)\) to \(\mathbb{Q}\).

\textbf{Proof.} Since \(H_1(\overline{\text{PMCG}}_c(S); \mathbb{Z})\) has a direct summand isomorphic to \(\oplus_{\mathfrak{c}} \mathbb{Q}\) we have \(2^\mathfrak{c}\) nontrivial homomorphisms from \(H_1(\overline{\text{PMCG}}_c(S); \mathbb{Z})\) to \(\mathbb{Q}\).

By pre-composing each of these with the quotient homomorphism \(\overline{\text{PMCG}}_c(S) \to H_1(\text{PMCG}_c(S); \mathbb{Z})\) we have \(2^\mathfrak{c}\) nontrivial homomorphisms from \(\overline{\text{PMCG}}_c(S)\) to \(\mathbb{Q}\). However, since \(\overline{\text{PMCG}}_c(S)\) is separable only \(\mathfrak{c}\) of these can be continuous.

\[\square\]

Note that when \(S\) has at most one end accumulated by genus we have \(\overline{\text{PMCG}}_c(S) = \text{PMCG}(S)\). Thus this theorem gives discontinuous homomorphisms with domain the full pure mapping class group in this setting. When
$S$ is the Loch Ness monster (one end accumulated by genus) we get a discontinuous homomorphism with domain the full mapping class group. This is in contrast to the sphere minus a Cantor set for which it is known that the full mapping class group has automatic continuity \[Man20\] and is uniformly perfect [Calegari Blog].

11 Elements not in $H_1(\overline{\text{PMCG}}_c(S) ; \mathbb{Z})$ and Other Possible Nontrivial Elements

In this section we give examples of elements in $\overline{\text{PMCG}}_c(S) \setminus \text{PMCG}_c S$ that are trivial in the abelianization and pose some questions about other possible nontrivial elements.

**Proposition 11.1.** Let $S$ be an infinite type surface. Suppose $f \in \overline{\text{PMCG}}_c(S)$ can be written as $f = \prod_{i=1}^{\infty} f_i$ where each $f_i \in \text{PMCG}_c(S)$ with $\text{supp}(f_i) \cap \text{supp}(f_j) = \emptyset$ for all $i \neq j$. Furthermore, suppose that each $f_i$ can be written as a product of commutators in $\text{PMCG}(\text{supp}(f_i))$ and that their commutator lengths are uniformly bounded by $N > 0$. Then $f$ can be written as a product of $N$ commutators.

**Proof.** For each $i$ write $f_i = [g_{i_1}, g_{i_2}] \cdots [g_{i_{2N-1}}, g_{i_{2N}}]$ where $g_{i_j} \in \text{PMCG}(\text{supp}(f_i))$. We allow for some of the $g_{i_j}$ to be the identity if $f_i$ has commutator length less than $N$. Thus we have

$$f = \prod_{i=1}^{\infty} \prod_{j=1}^{2N-1} [g_{i_j}, g_{i_{j+1}}].$$

Since we have $\text{supp}(f_i) \cap \text{supp}(f_j) = \emptyset$ we can rearrange this product to get

$$f = \prod_{j=1}^{2N-1} \left[ \prod_{i=1}^{\infty} g_{i_j}, \prod_{i=1}^{\infty} g_{i_{j+1}} \right].$$

$\square$

An example of this is an infinite product of uniformly bounded powers of commuting Dehn twists. We now ask whether the converse holds.

**Question 11.2.** Let $S$ be an infinite type surface. Suppose $f \in \overline{\text{PMCG}}_c(S)$ can be written as $f = \prod_{i=1}^{\infty} f_i$ where each $f_i \in \text{PMCG}_c(S)$ with $\text{supp}(f_i) \cap \text{supp}(f_j) = \emptyset$ for all $i \neq j$. Furthermore, suppose that each $f_i$ can be written as a product of commutators in $\text{PMCG}(\text{supp}(f_i))$ with unbounded commutator lengths. Then is $f$ nontrivial in $H_1(\overline{\text{PMCG}}_c(S) ; \mathbb{Z})$?
Each component of $f$ having unbounded commutator length was the inspiration for Theorem 6.1 and each element we construct satisfies the hypotheses of the question. However, our techniques relied heavily on the fact that each component is a power of the same mapping class on homeomorphic subsurfaces. So far we do not know how to get a large lower bound on a quasimorphism purely from the fact that the commutator lengths of the components grow.

Our technique was also only able to detect torsion free elements. This begs a second question.

**Question 11.3.** Let $S$ be an infinite type surface. Are there torsion elements in $H_1(\text{PMCG}_c(S);\mathbb{Z})$?

## A Appendix: The Loch Ness Monster

Ryan Dickmann and George Domat

In this appendix we prove that the mapping class group of the Loch Ness Monster surface is also not perfect. Note that for this surface and its once-punctured variant $\text{MCG}(S)$, $\text{PMCG}(S)$, and $\text{PMCG}_c(S)$ are all the same.

**Theorem A.1.** Let $L$ be the surface with one end and infinite genus. Then $\text{MCG}(L)$ is not perfect. In fact, $H_1(\text{MCG}(L);\mathbb{Z}) = \bigoplus_{2^{\aleph_0}} \mathbb{Q} \oplus B$ where all divisible subgroups of $B$ are torsion.

To prove this we make use of the Birman Exact Sequence for infinite type surfaces. We could not find a discussion of the infinite type case in the literature so we present one here.

The proof is identical to the standard proof in [FM11]. One only needs to check that $\pi_1(\text{Homeo}^+(S))$ is trivial in the infinite type case. The result then follows from the long exact sequence of homotopy groups given by the fiber bundle

$$\text{Homeo}^+(S, x) \to \text{Homeo}^+(S) \to S.$$ 

Here $\text{Homeo}^+(S)$ is equipped with the compact-open topology. One can verify from the standard proof that this is indeed a fiber bundle in the infinite type case as well. It was shown in [Yag00] that the connected component of the identity in $\text{Homeo}^+(S)$ is homotopy equivalent to a point for general non compact 2-manifolds minus some degenerate finite type cases.
**Theorem A.2** (Birman Exact Sequence). Let $S$ be a surface of negative Euler characteristic or infinite type. Let $(S, x)$ be the surface obtained from $S$ by adding a marked point $x$ in the interior of $S$. Then there is an exact sequence:

$$1 \to \pi_1(S, x) \to \text{MCG}(S, x) \to \text{MCG}(S) \to 1$$

**Proof of Theorem A.1.** We first note that by applying the same abelian group theory argument as in Section 8.3 it suffices to show that $H_1(\text{MCG}(L); \mathbb{Z})$ contains a copy of $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$. The general proof fails in the case of the Loch Ness Monster because we do not have end-separating curves. For the Loch Ness Monster with a puncture we do now have a separating principle exhaustion and the methods in the paper show that $H_1(\text{MCG}(L, x); \mathbb{Z})$ contains a copy of $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$.

The fundamental group of any infinite type surface is a free group with countably many generators [Sti93]. Therefore we have the following exact sequence:

$$1 \to F_{\infty} \to \text{MCG}(L, x) \to \text{MCG}(L) \to 1$$

Abelianization is right-exact so we get the following exact sequence of abelianizations:

$$\mathbb{Z}_{\infty} \to H_1(\text{MCG}(L, x); \mathbb{Z}) \to H_1(\text{MCG}(L); \mathbb{Z}) \to 1$$

Here $\mathbb{Z}_{\infty}$ is the free abelian group with countably many generators. It follows that $H_1(\text{MCG}(L); \mathbb{Z})$ is the quotient of an uncountable group by some countable subgroup and is therefore uncountable itself.

In fact we can do better and find a copy of $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$ inside $H_1(\text{MCG}(L); \mathbb{Z})$. Since $\mathbb{Z}_{\infty}$ is countable we must have that

$$\bigoplus_{2^{\aleph_0}} \mathbb{Q} \cap \ker(H_1(\text{MCG}(L, x); \mathbb{Z}) \to H_1(\text{MCG}(L); \mathbb{Z}))$$

is countable where $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$ refers to the copy found in Section 8. Thus the image of $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$ is a divisible group with uncountably many non-torsion elements. Then by the Structure Theorem of Divisible Groups this image must again contain a copy of $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$. $\square$

We can also apply this method of proof to the Torelli group for the Loch Ness Monster.

**Theorem A.3.** Let $L$ be the surface with one end and infinite genus. Then $H_1(I(L); \mathbb{Z}) = \bigoplus_{2^{\aleph_0}} \mathbb{Q} \oplus B$ where all divisible subgroups of $B$ are torsion.
Proof. Once again, it suffices to show that $H_1(\mathcal{I}(L);\mathbb{Z})$ contains a copy of $\oplus_{2^{\aleph_0}}\mathbb{Q}$. As in Section 9 we can find elements in $\mathcal{I}(L,x)$ that give rise to a copy of $\oplus_{2^{\aleph_0}}\mathbb{Q}$ in the abelianization. We can pick these elements so that they remain in $\mathcal{I}(L)$ after applying the forgetful map. Indeed, we can ensure that the curves we twist about remain separating after forgetting the marked point. Now the same counting argument as in the previous theorem gives a copy of $\oplus_{2^{\aleph_0}}\mathbb{Q}$ in $H_1(\mathcal{I}(L);\mathbb{Z})$. \hfill \Box

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