Limiting values of rational functions at the points of discontinuity

Yaacov Tzeitlin

16.02.2000

AMS Subject Classification: 26C15, 41A20, 46J10.

1 Introduction and conventions

In the present article we describe a class of algebraic curves on which rational functions of two arguments may reach all their possible limiting values. We also solve a similar question for functions that can be represented as a uniform limit of a sequence of bounded rational functions.

Some standard notation. Below we denote by \( \mathbb{N}, \mathbb{Z}^+, \mathbb{Q}, \mathbb{R} \) the sets of all natural, nonnegative integer, rational and real numbers, respectively. Further, all functions are real valued.

**Definition 1** Let \( f(x, y) \) be a real valued function of two real arguments. We say that a point \((x_0, y_0)\) is a point of bounded discontinuity if \( f \) is defined and bounded in a certain nbd of a point \((x_0, y_0)\) and \((x_0, y_0)\) is a point of the discontinuity.

Let \( f(x, y) = \frac{p(x, y)}{q(x, y)} \) be a rational function when \( p(x, y) \) and \( q(x, y) \) are polynomials. For every \((x_0, y_0)\) there exists the limit

\[
\lim_{x \to x_0} \lim_{y \to y_0} f(x, y).
\]

If \( \lim_{x \to x_0} \lim_{y \to y_0} f(x, y) \neq \infty \), then we will suppose that \( f(x_0, y_0) = \lim_{x \to x_0} \lim_{y \to y_0} f(x, y) \).

That is, we assume that such rational functions are defined at the points of bounded discontinuity. Considering the rational function \( \frac{p(x, y)}{q(x, y)} \) we will assume below that \( p(x, y) \) and \( q(x, y) \) are mutually prime. Let us introduce some additional notation:

\( \mathcal{R} = \{ \text{the set of all rational functions with two arguments} \} \)

\( \mathcal{R}(x, y) = \{ f(x, y) \in \mathcal{R} | f(x, y) \text{ is bounded in some nbd of } (x, y) \} \)

\( \mathcal{R}_G = \bigcap \{ \mathcal{R}(x, y) | (x, y) \in G \} \) for a certain \( G \subset \mathbb{R}^2 \).

Note that \( \mathcal{R}(x, y) \) and \( \mathcal{R}_G \) are commutative real algebras with respect to pointwise operations.

**Lemma 1** If \( f(x, y) \in \mathcal{R} \) then the set of all points of bounded discontinuity is finite.

**Proof** Indeed if \( f(x, y) = \frac{p(x, y)}{q(x, y)} \), then the set of all bounded discontinuity is a subset of all roots of the following system

\[
p(x, y) = 0, \quad q(x, y) = 0.
\]

On the other hand the number of such solutions is at most \( \deg(p) \cdot \deg(q) \), where \( p \) and \( q \) are mutually prime. \( \square \)
It is important to know what is the behaviour of the function \( f(x, y) \) near the point \((x_0, y_0)\). This can be described by the set \( D_f(x_0, y_0) \) of all limiting values of \( f \) at \((x_0, y_0)\). By the definition,
\[
D_f(x_0, y_0) = \bigcap \{ \overline{f(V(x_0, y_0))} \mid V \in N(x_0, y_0) \},
\]
where \( N(x_0, y_0) \) denotes the set of all nbd’s of the point \((x, y)\). Let \( \underline{f} = \inf D_f(x_0, y_0) \), \( \overline{f} = \sup D_f(x_0, y_0) \).

**Lemma 2** If \( f(x, y) \in \mathcal{R}_{(x_0, y_0)} \), then \( D_f^{(x_0, y_0)} = [\underline{f}, \overline{f}] \).

**Proof** Observe that
\[
\underline{f} \in D_f^{(x_0, y_0)}, \quad \overline{f} \in D_f^{(x_0, y_0)}
\]
Let \( d \in (\underline{f}, \overline{f}) \). Then by Lemma 1, \( f \) is continuous at every point \( \neq (x_0, y_0) \) of a sufficiently small nbd of \((x_0, y_0)\). Therefore, by the intermediate value theorem there exists a point \((x_d, y_d)\) such that \( f(x_d, y_d) = d \). \( \square \)

**Lemma 3** Let \( G \) be a domain in \( \mathbb{R}^2 \) and \( f(x, y) = \frac{p(x, y)}{q(x, y)} \in \mathcal{R}_G \). Then \( q \) has a constant sign on \( G \).

**Proof** Since \( G \) is an open path-connected subset of \( \mathbb{R}^2 \), the same is true for the subset \( \{(x, y) \in G \mid q(x, y) \neq 0 \} \) and arbitrary pair \((x_1, y_1), (x_2, y_2)\) of points from this subset can be joined in this open subspace by a path. This implies that \( q(x_1, y_1), q(x_2, y_2) \) have the same sign. \( \square \)

Further we will deal with the characters on the algebra \( \mathcal{R}_{(x_0, y_0)} \). Recall that a (nontrivial) real **character** on an algebra \( A \) is a (nontrivial) homomorphism from \( A \) into \( \mathbb{R} \) (see 2). Denote by \( X^A \) the set of all nontrivial characters. Endow it with the weak topology \( \sigma(X^A, A) \), that is, topology generated by all maps \( \tilde{a} : X^A \to \mathbb{R}, \tilde{a}(\chi) = \chi(a) \).

We need the following

**Definition 2** 1. Let \( \{(x_\gamma, y_\gamma)\}_{\gamma \in \Gamma} \) be a net \( \mathbb{R}^2 \) that converges to the point \((x_0, y_0)\) in \( \mathbb{R}^2 \) such that
\[
\forall f \in \mathcal{R}_{(x_0, y_0)}, \exists \lim_{\gamma \in \Gamma} f(x_\gamma, y_\gamma) \neq \infty.
\]

The formula \( \chi(t) = \lim_{t \in \Gamma} f(x_\gamma, y_\gamma) \) defines a character \( \chi \) of the algebra \( \mathcal{R}_{(x_0, y_0)} \). We will say that the net \( \{(x_\gamma, y_\gamma)\}_{\gamma \in \Gamma} \) determines the character \( \chi \).

2. Let \( L \) be a curve in \( \mathbb{R}^2 \) and \((x_0, y_0) \in L \). then if
\[
\forall f \in \mathcal{R}_{x_0, y_0}, \exists \lim_{(x, y) \to (x_0, y_0)} f(x, y)
\]
then the formula \( \chi(f) = \lim_{(x, y) \to (x_0, y_0)} f(x, y) \) defines a character of the algebra \( \mathcal{R}_{x_0, y_0} \). We will say that the curve \( L \) determines the character \( \chi \).

3. Let \( \{L_N\}_{N \in \mathbb{N}} \) be a sequence of curves such that each of them determines the character \( \chi_N \). Then if \( \{L_N\}_{N \in \mathbb{N}} \) converges in \( \sigma(\mathcal{R}_{(x_0, y_0)}, \mathcal{R}_{(x_0, y_0)}) \) to the character \( \chi \), then we say that \( \{L_N\}_{N \in \mathbb{N}} \) determines \( \chi \).

In the following definition we introduce functions and series that allow us to describe the characters of the algebra \( \mathcal{R}_{(x_0, y_0)} \).
Definition 3 (i) For every $N \in \mathbb{Z}^+$ define the function $\Phi_N(t)$ by

$$\forall N \in \mathbb{N} \quad \Phi_N(t) = \sum_{k=1}^{N} \phi_k |t|^e_k$$

where $\phi_k \in \mathbb{R}$, $\phi_k \neq 0$, $e_k \in \mathbb{Q}$ and $1 \leq e_1 < \cdots < e_N$. If $N = 0$ then by definition $\Phi_0(t) = 0$.

All such functions will be called functions of type (i).

(ii) We say that the series $\sum_{k=1}^{\infty} \phi_k |t|^e_k$ is of type (ii), if all partial sums are of type (i).

Denote by $I$ the set of all functions of the type (i) and by $II$ the set of all series of the type (ii). Finally, denote by $\mathbb{II}_{\infty}$ the set of all series $\{ \sum_{k} \phi_k |t|^e_k \in II | \lim_{k \to \infty} e_k = \infty \}$.

We naturally can define on the sets $I, \mathbb{II}_{\infty}$ the binary operations: addition and multiplication. Namely, let $\sum_i \alpha_i |t|^{m_i}$ and $\sum_j \beta_j |t|^{r_j}$ belong to $I \cup \mathbb{II}_{\infty}$. Suppose that $\sum_k \phi_k |t|^e_k$, where $\{e_k\}$ is a sequence which is obtained by writing elements of $m_i, r_j$ in their natural order and $\phi_k = \begin{cases} \alpha_i & \text{if } e_k = m_i \notin \{r_j\} \\ \beta_j & \text{if } e_k = r_j \notin \{m_i\} \\ \alpha_i + \beta_j & \text{if } e_k = m_i = r_j \end{cases}$.

The multiplication is defined by $(\sum_i \alpha_i |t|^{m_i}) \cdot (\sum_j \beta_j |t|^{r_j}) = \sum_k \phi_k |t|^k$. Here $\phi_k = \sum_{m_i + r_j = e_k} \alpha_i \beta_j$ and $\{e_k\}$ is obtained by rewriting the elements of $\{m_i + r_j\}_{i,j}$ as an increasing sequence. Thus $I \cup \mathbb{II}_{\infty}$ is an algebra over reals. It is impossible to extend these operations on $I \cup \mathbb{II}_{\infty}$. However such extension exists for natural degrees.

Lemma 4 For every $\sum_{k} \phi_k |t|^e_k \in I \cup \mathbb{II}_{\infty}$ and $n \in N$ is defined the following equality $\left( \sum_{k} \phi_k |t|^e_k \right)^n = \sum_{m} g_m |t|^{s_m}$. Here, $s_m$ is obtained by rewriting the elements of $\{e_{k_1} + \cdots + e_{k_n} | k_1 \leq k_2 \cdots \leq k_n \}$ as an increasing sequence and $g_m = \sum_{k_{1} + k_{2} + \cdots + k_{n} = s_m} \phi_{k_1} \cdots \phi_{k_n}$.

Proof For $\sum_{k} \phi_k |t|^e_k \in I \cup \mathbb{II}_{\infty}$ the assertion is true because $I \cup \mathbb{II}_{\infty}$ is an algebra. It suffices to prove for the series $\sum_k \phi_k |t|^e_k$ of the type $I$ such that $\lim_{k \to \infty} e_k = e \neq \infty$.

We have to show that $\sum_{e_{k_1} + \cdots + e_{k_n} = s_m} \phi_{k_1} \cdots \phi_{k_n}$ is well-defined for every $s \in \{e_{k_1} + \cdots + e_{k_n} | k_1 \leq \cdots \leq k_n \}$ and that $\{e_{k_1} + \cdots + e_{k_n} | k_1 \leq \cdots \leq k_n \}$ can be represented as an increasing sequence $\{s_m\}_{m \in N}$ for each $n \in N$.

In order to show this it is sufficient to establish that the number of the roots of the equation $e_{k_1} + \cdots + e_{k_n} = s$ is finite for every $s \in \{e_{k_1} + \cdots + e_{k_n} | k_1 \leq \cdots \leq k_n \}$.

We use the induction on $n$. For $n = 1$ the assertion is trivial. We have to check the case of $n + 1$ assuming that the assertion is true for $n$. Consider the equation $e_{k_1} + \cdots + e_{k_{n+1}} = s$. Then $(n+1)e_{k_1} < e_{k_1} + \cdots + e_{k_{n+1}} < (n+1)e_{k_{n+1}} < (n+1)e$ and $e_{k_1} < \frac{e}{n+1} < e$. However, for sufficiently big $N$ and for $k \geq N$ holds $e_k > \frac{e}{n+1}$. Hence we obtain that $k_1 < N$. By inductive assumption, the equation $e_{k_2} + \cdots + e_{k_{n+1}} = s - e_{k_1}$ has finitely many roots for every $k_1 < N$. Therefore, the equation $s = e_{k_1} + \cdots + e_{k_n} + e_{k_{n+1}}$ also has finite number of the roots. This proves the lemma.

Making use Lemma 4, we can obtain an asymptotic characterization of the behaviour of a polynomial on partial sums of the series of type $\mathbb{II}_{\infty}$.
Lemma 5 Let \( P(x, y) \) be a polynomial, \( \sum_{k=1}^{\infty} \phi_k |t|^k \in I I \) and let \( \{ \Phi_N(t) \}_{N \in \mathbb{N}} \) be the sequence of all partial sums of the above series. Then there exists \( N_0 \in \mathbb{N} \) such that for every \( N > N_0 \) we have \( p(x, y + \Phi_N(x - x_0)) = p(x_0, y_0) + c(x - x_0)^n + o((x - x_0)^n) \) for \( x > x_0 \) and \( p(x, y) = p(x_0, y_0) + d(x_0 - x)^\nu + o((x_0 - x)^\nu) \) for \( x < x_0 \). Here \( u, v \) are rational and \( c, d, \) real numbers.

2 Characters of the algebra \( R_{(x_0, y_0)} \)

Now we are in the position to describe curves and sequences of curves that determine characters of the algebra \( R_{(x_0, y_0)} \).

Theorem 1 Let \( \Phi_N(x - x_0) \) be an arbitrary function of type I. Then the curves and sequences of curves with the following equations determine characters of the algebra \( R_{(x_0, y_0)} \).

1. \( y = y_0 + \Phi_N(x - x_0) \quad N \in \mathbb{Z}^+ \).
2. \( y = y_0 + \Phi_N(x - x_0) = |x - x_0|^e \), where \( N \in \mathbb{Z}^+ \) and \( e \notin \mathbb{Q} \). We assume in addition that \( 1 \leq e \) if \( N = 0 \) and \( e_N \leq e \) if \( N \in \mathbb{N} \).
3. \( y = \begin{cases} y_0 + \Phi_N(x - x_0) + |x - x_0|^\nu \ln |x - x_0| & \text{if } x \neq x_0 \\ y_0 & \text{if } x = x_0 \end{cases} \)

Here and in (4) we assume that \( N \in \mathbb{Z}^+ \), \( e \in \mathbb{Q} \), \( 1 \leq e \) if \( N = 0 \) and \( e_N \leq e \) for \( N \in \mathbb{N} \).

4. \( y = \begin{cases} y_0 + \Phi_N(x - x_0) + |x - x_0|^\nu \ln |x - x_0| & \text{if } x \neq x_0 \\ y_0 & \text{if } x = x_0 \end{cases} \)

5. The sequence \( \{ y_0 + \Phi_N(x - x_0) \}_{N \in \mathbb{N}} \), where \( \{ \Phi_N(x - x_0) \}_{N \in \mathbb{N}} \) is the sequence of partial sums of series from \( I I \).

Each of above equations from (1,2,3,4) and the series from (5) define, in general, different characters for the cases \( x_0 \leq x \) and \( x \leq x_0 \).

Proof In the cases (1,2,3,4) it is straightforward to show that for every \( f(x, y) \in R_{(x_0, y_0)} \), there exists a limit which determines the corresponding character. In the case (5), we use also Lemma 3. Analogously can be established.

Theorem 2 Let \( \Phi_N(y - y_0) \) be an arbitrary function of type I. Then the curves and sequences of curves with the following equations determine characters of the algebra \( R_{(x_0, y_0)} \).

1. \( x = x_0 + \Phi_N(y - y_0) \quad N \in \mathbb{Z}^+ \).
2. \( x = x_0 + \Phi_N(y - y_0) = |y - y_0|^e \), where \( N \in \mathbb{Z}^+ \) and \( e \) be irrational. We assume in addition that \( 1 \leq e \) if \( N = 0 \), and \( e_N \leq e \) if \( N \in \mathbb{N} \).
3. \( x = \begin{cases} x_0 + \Phi_N(y - y_0) + |y - y_0|^\nu \ln |y - y_0| & \text{if } y \neq y_0 \\ x_0 & \text{if } y = y_0 \end{cases} \)

Here and in (4) we assume that \( N \in \mathbb{Z}^+ \), \( e \in \mathbb{Q} \). Moreover \( 1 \leq e \) if \( N = 0 \) and \( e_N \leq e \) for \( N \in \mathbb{N} \).
4. \[ x = \begin{cases} 
\frac{y_0 + \Phi_N(x - x_0) + \ln|x - x_0|}{y_0} & \text{if } x \neq x_0 \\
y_0 & \text{if } x = x_0 
\end{cases} \]

5. The sequence \( \{x_0 + \Phi_N(y - y_0)\}_{N \in \mathbb{N}}, \) where \( \{\Phi_N(y - y_0)\}_{N \in \mathbb{N}} \) is the sequence of partial sums of series from \( \mathcal{I} \).

Each of above equations from (1, 2, 3, 4) and the series from (5) define, in general, different characters for the cases \( y_0 \leq y \) and \( y \leq y_0 \).

Now we describe the nets that determine the characters of the algebra \( \mathcal{R}(x_0,y_0) \). First of all some useful remarks about such nets.

Every such net necessarily tends to \( (x_0,y_0) \) in \( \mathbb{R}^2 \). We can ignore the trivial case of (finally) stationary nets. If \( \{(x_\gamma,y_\gamma)\}_{\gamma \in \Gamma} \) is not a finally stationary net, then there exists a cofinal subnet \( \{(x_\gamma,y_\gamma)\}_{\Gamma_0 \subset \Gamma} \) such that one of the following conditions is fulfilled:

a) \( \lim_{\gamma \in \Gamma_0} \frac{y_\gamma - y_0}{x_\gamma - x_0} = \varphi \neq \infty \)

b) \( \lim_{\gamma \in \Gamma_0} \frac{y_\gamma - x_0}{y_\gamma - y_0} = \varphi \neq \infty \)

Indeed, otherwise \( \lim_{\gamma \in \Gamma} f(x_\gamma,y_\gamma) \) does not exist for the function

\[ f(x,y) = \frac{(x - x_0)^2 - (y - y_0)^2 - (x - x_0)(y - y_0)}{(x - x_0)^2 + (y - y_0)^2} \in \mathcal{R}(x_0,y_0). \]

Moreover we can assume that in both cases of (a), (b) the denominators preserve the sign.

**Theorem 3** Let the net \( \{(x_\gamma,y_\gamma)\}_{\gamma \in \Gamma} \) tends to \( (x_0,y_0) \) in \( \mathbb{R}^2 \) and one of the following conditions hold:

a) \( \lim_{\gamma \in \Gamma_0} \frac{y_\gamma - y_0}{x_\gamma - x_0} = \varphi \neq \infty \), where \( x_0 < x_\gamma \) for every \( \gamma \in \Gamma \). Or, \( x_0 > x_\gamma \) for every \( \gamma \in \Gamma \);

b) \( \lim_{\gamma \in \Gamma_0} \frac{y_\gamma - x_0}{y_\gamma - y_0} = \varphi \neq \infty \), where \( y_\gamma < y_0 \) for every \( \gamma \in \Gamma \). Or, \( y_0 > y_\gamma \) for every \( \gamma \in \Gamma \).

Then this net determines a character of the algebra \( \mathcal{R}(x_0,y_0) \) iff one of the following conditions hold:

1. a) \( \exists \Phi_N(t) \in \mathcal{I} \) \( \forall e \in \mathbb{R} \) \( \lim_{\gamma \in \Gamma} \frac{y_\gamma - y_0 - \Phi_N(x_\gamma - x_0)}{|x_\gamma - x_0|^r} = 0 \).

   b) \( \exists \Phi_N(t) \in \mathcal{I} \) \( \forall e \in \mathbb{R} \) \( \lim_{\gamma \in \Gamma} \frac{x_\gamma - x_0 - \Phi_N(y_\gamma - y_0)}{|y_\gamma - y_0|^r} = 0 \).

2. a) \( \exists \Phi_N(t) \in \mathcal{I} \) \( \exists e \notin \mathbb{Q} \)
   \[ \lim_{\gamma \in \Gamma} \frac{y_\gamma - y_0 - \Phi_N(x_\gamma - x_0)}{|x_\gamma - x_0|^r} = 0 \text{ for } r < e \text{ and } \]
   \[ \lim_{\gamma \in \Gamma} \frac{y_\gamma - y_0 - \Phi_N(x_\gamma - x_0)}{|x_\gamma - x_0|^r} = \infty \text{ for } r > e. \]

   b) \( \exists \Phi_N(t) \in \mathcal{I} \) \( \exists e \notin \mathbb{Q} \)
   \[ \lim_{\gamma \in \Gamma} \frac{x_\gamma - x_0 - \Phi_N(y_\gamma - y_0)}{|y_\gamma - y_0|^r} = 0 \text{ for } r < e \text{ and } \]
   \[ \lim_{\gamma \in \Gamma} \frac{x_\gamma - x_0 - \Phi_N(y_\gamma - y_0)}{|y_\gamma - y_0|^r} = \infty \text{ for } r > e. \]

3. a) \( \exists \Phi_N(t) \in \mathcal{I} \) \( \exists e \notin \mathbb{Q} \)
   \[ \lim_{\gamma \in \Gamma} \frac{y_\gamma - y_0 - \Phi_N(x_\gamma - x_0)}{|x_\gamma - x_0|^r} = 0 \text{ for } r \leq e \text{ and } \]
   \[ \lim_{\gamma \in \Gamma} \frac{y_\gamma - y_0 - \Phi_N(x_\gamma - x_0)}{|x_\gamma - x_0|^r} = \infty \text{ for } r > e. \]
b) $\exists \Phi_N(t) \in I \quad \exists \gamma \in \mathbb{Q}$

\[
\lim_{\gamma \to \Gamma} \frac{x_{\gamma} - x_0 - \Phi_N(y_0 - y_0)}{|y_0 - y_0|} = 0 \quad \text{for } r \leq e \quad \text{and} \\
\lim_{\gamma \to \Gamma} \frac{x_{\gamma} - x_0 - \Phi_N(y_0 - y_0)}{|y_0 - y_0|} = \infty \quad \text{for } r > e.
\]

4. a) $\exists \Phi_N(t) \in I \quad \exists \gamma \in \mathbb{Q}$

\[
\lim_{\gamma \to \Gamma} \frac{y_{\gamma} - y_0 - \Phi_N(x_{\gamma} - x_0)}{|x_{\gamma} - x_0|} = 0 \quad \text{for } r < e \quad \text{and} \\
\lim_{\gamma \to \Gamma} \frac{y_{\gamma} - y_0 - \Phi_N(x_{\gamma} - x_0)}{|x_{\gamma} - x_0|} = \infty \quad \text{for } e \leq r.
\]

b) $\exists \Phi_N(t) \in I \quad \exists \gamma \in \mathbb{Q}$

\[
\lim_{\gamma \to \Gamma} \frac{x_{\gamma} - x_0 - \Phi_N(y_0 - y_0)}{|y_0 - y_0|} = 0 \quad \text{for } r < e \quad \text{and} \\
\lim_{\gamma \to \Gamma} \frac{x_{\gamma} - x_0 - \Phi_N(y_0 - y_0)}{|y_0 - y_0|} = \infty \quad \text{for } e \leq r.
\]

5. a) $\exists \sum_{k=1}^{\infty} \phi_k |t|^{e_k} \in \mathbb{Z} \quad \forall \gamma \in \mathbb{Z}$

\[
\lim_{\gamma \to \Gamma} \frac{y_{\gamma} - y_0 - \Phi_N(x_{\gamma} - x_0)}{|x_{\gamma} - x_0|} = \phi_{N+1}.
\]

Here $\Phi_N(x_{\gamma} - x_0) = \sum_{k=1}^{\infty} \phi_k |x_{\gamma} - x_0|^{e_k}$ for $N \in \mathbb{N}$ and $\Phi_0(x_{\gamma} - x_0) = 0$.

b) $\exists \sum_{k=1}^{\infty} \phi_k |t|^{e_k} \in \mathbb{Z} \quad \forall \gamma \in \mathbb{Z}$

\[
\lim_{\gamma \to \Gamma} \frac{x_{\gamma} - x_0 - \Phi_N(y_0 - y_0)}{|y_0 - y_0|} = \phi_{N+1}.
\]

Analogously, here $\Phi_N(y_{\gamma} - y_0) = \sum_{k=1}^{\infty} \phi_k |y_{\gamma} - y_0|^{e_k}$ for $N \in \mathbb{N}$ and $\Phi_0(y_{\gamma} - y_0) = 0$.

**Proof**

For simplicity assume $(x_0, y_0) = (0, 0)$. We consider only the case of (a) with the assumption $x_0 \leq x_\gamma$. Other cases are very similar.

We will present polynomials in the form

\[
p(x, y) = \sum_{k=0}^{n} (y - \Phi_N(x))^k x^{r_k} p_k(x),
\]

where $x^{r_k} p_k(x) = \frac{\partial^n}{\partial x^n} x^{y_0} |y = \Phi_N(x)| = x^{r_k} p_k(0) + 0(x^{r_k})$. Therefore $f(x, y) = \frac{p(x, y)}{q(x, y)} \in \mathbb{R}(x_0, y_0)$ we can represent in the form

\[
f(x, y) = \frac{\sum_{k=0}^{n} (y - \Phi_N(x))^k x^{r_k} p_k(x)}{\sum_{k=0}^{n} (y - \Phi_N(x))^k x^{s_k} q_k(x)}.
\]

Now we consider our conditions (1-5). If (1) holds, that is, if $y_\gamma = 0(x_{\gamma})$ for $e \in \mathbb{R}$, then

\[
\lim_{\gamma \to \Gamma} f(x_{\gamma}, y_{\gamma}) = \lim_{\gamma \to \Gamma} \frac{\sum_{k=0}^{n} (y_{\gamma} - \Phi_N(x_{\gamma}))^k x^{r_k} p_k(x_{\gamma})}{\sum_{k=0}^{n} (y_{\gamma} - \Phi_N(x_{\gamma}))^k x^{s_k} q_k(x_{\gamma})} = \lim_{\gamma \to \Gamma} \frac{(y_{\gamma} - \Phi_N(x_{\gamma}))^n x^{r_n} p_n(x_{\gamma})}{(y_{\gamma} - \Phi_N(x_{\gamma}))^m x^{s_m} q_m(x_{\gamma})}
\]

where $n = \min\{k | \frac{\partial^n}{\partial x^n} x^{y_0} |y = \Phi_N(x) \neq 0 \}$ and $m = \min\{k | \frac{\partial^n}{\partial x^n} x^{y_0} |y = \Phi_N(x) \neq 0 \}$. Clearly, $\lim_{\gamma \to \Gamma} f(x_{\gamma}, y_{\gamma}) \neq \infty$ exists for every $f(x, y) \in \mathbb{R}(0, 0)$. In order to examine the conditions (2), (3), (4), write $f(x, y)$ in the form

\[
f(x, y) = \frac{\sum_{k=0}^{n} (y - \Phi_N(x))^k x^{r_k + ke} p_k(x)}{\sum_{k=0}^{n} (y - \Phi_N(x))^k x^{s_k + ke} q_k(x)}
\]

Then

\[
\lim_{\gamma \to \Gamma} f(x_{\gamma}, y_{\gamma}) = \lim_{\gamma \to \Gamma} \frac{(y_{\gamma} - \Phi_N(x_{\gamma}))^n x^{r_n + ne} p_n(x_{\gamma})}{(y_{\gamma} - \Phi_N(x_{\gamma}))^m x^{s_m + me}} = \begin{cases} \frac{\partial^n}{\partial x^n} p_n(0) \quad &\text{if } m = n, r_n = s_n \\ 0 \quad &\text{in other cases of (2)} \end{cases}
\]
Now we give formulas like (**) and methods for finding suitable $n, m$ for other cases.

For (2). Then $n$ and $m$ are determined by the conditions:

$$r_n + ne = \min \{ r_k + ke \big| \frac{\partial^k p}{\partial y^k} \big|_{y = \Phi_N(x)} \neq 0 \}$$

For (3). Then we set: $n = \min \{ k| r_k + ke = u \}$, where $u = \min \{ r_k + ke \big| \frac{\partial^k p}{\partial y^k} \big|_{y = \Phi_N(x)} \neq 0 \}$ and $m = \min \{ k| s_k + ke = u \}$, where $u = \min \{ s_k + ke \big| \frac{\partial^k q}{\partial y^k} \big|_{y = \Phi_N(x)} \}$.

For (4). Set: $n = \max \{ k| r_k + ke = u \}$, where $u = \min \{ r_k + ke \big| \frac{\partial^k p}{\partial y^k} \big|_{y = \Phi_N(x)} \neq 0 \}$, $m = \max \{ k| s_k + ke = u \}$, where $u = \min \{ s_k + ke \big| \frac{\partial^k q}{\partial y^k} \big|_{y = \Phi_N(x)} \} \neq 0$.

Finally consider the case (5). Note that rewriting polynomials in the form

$$p(x, y) = \sum_{k=0}^{n_p} (y - \Phi_N(x))^k x^r y^k,$$

by Lemma 3 we may state that starting from certain $N$ the value of $r_k$ are unchanged for $k = 0, 1, \ldots, n_p$. We will show next that for every polynomial $p(x, y)$ there exists a $k(p) \in N$ such that for sufficiently big $N$ holds

$$p(x, y) = (\frac{y - \Phi_N(x)}{x^{e_N+1}})^{k(p)} x^{r_k(p)} y^{k(p)} e_{N+1} + 0(x^{r_k(p)} y^{k(p)} e_{N+1} + 1) \quad (***)$$

Indeed, if $\lim_{N \to \infty} e_N = \infty$ then clearly (***') holds for sufficiently big $N$ and for $k(p) = \min \{ k| \frac{\partial^k p}{\partial y^k} \neq 0 \}$. Alternatively if $\lim_{N \to \infty} e_N = e < \infty$ and $r = \min \{ r_k + ke \big| \frac{\partial^k p}{\partial y^k} \neq 0 \}$. Then if $k(p) = \max \{ k| r_k + ke = r \}$ and $N$ is so big that $r < r_k + ke$ implies that $r_k(p) + e_N k(p) < r_k + e_N k$ whenever $k \neq k(p)$ and hence (***') holds. In order to finish the subcase of (5) observe that applying (***') to $p(x, y), q(x, y)$ for $f(x, y) = \frac{p(x, y)}{q(x, y)} \in \mathcal{R}_{(0,0)}$ the limit $\lim_{x,y \to \Gamma} f(x, y)$ exists. Therefore we have proved that each net which satisfies any of the conditions (1-5) determines a character of the algebra $\mathcal{R}_{(x_0, y_0)}$. The present proof concerns all possibilities except the cases when $e \in \mathbb{Q}$ and the limit (finite or infinite) $\lim_{x, y \to \Gamma} \frac{p(x, y)}{q(x, y)} \in \mathcal{R}_{(0,0)}$ does not exist. But for these exceptional cases easily can be constructed appropriate examples of $f(x, y) \in \mathcal{R}_{(x_0, y_0)}$ such that $\lim_{x, y \to \Gamma} f(x, y)$ does not exist. In particular, if $\lim_{x, y \to \Gamma} \frac{p(x, y)}{q(x, y)} \in \mathcal{R}_{(0,0)}$ exists for $r \neq e$ and if $\lim_{x, y \to \Gamma} \frac{p(x, y)}{q(x, y)} \in \mathcal{R}_{(0,0)}$ exists for $e = \frac{n_0}{m_0}$, then $\lim_{x, y \to \Gamma} f(x, y)$ does not exist for $f(x, y) = \frac{y^{n_0} x^{m_0}}{x^{e_N+1} y^{e_N} + x^{e_N} y^{e_N}}$. The proof is finished.

**Corollary 1** If a net satisfies one the conditions i.(a) or i.(b) ($i = 1, 2, 3, 4$) of Theorem 1 then it determines the same character as the curves described in i. of Theorem 1 or in i. of Theorem 3. If a net satisfies (3), then the character determined by this net and the character determined by the sequence of the curves from Theorem 1 or in Theorem 3 are the same.

**Corollary 2** If a character is determined by some net then such character can be determined also by a sequence.

**Corollary 3** 1. Let $\{(x_n, y_n)\}_{n \in N}$ tends to $(x_0, y_0)$ then there exists a subsequence which determines a character of the algebra $\mathcal{R}_{(x_0, y_0)}$. 

7
2. \( \forall f(x, y) \in \mathcal{R}_{(x_0, y_0)} \quad \forall d \in D^f_{(x_0, y_0)} \exists \chi \in X^\mathcal{R}_{(x_0, y_0)} \chi(t) = d. \)

We will denote by \( X_{(x_0, y_0)} \) the set of all characters in the algebra \( \mathcal{R}_{(x_0, y_0)} \) determined by nets. Analogously, we denote by \( X^i_{(x_0, y_0)} \) the characters determined by the corresponding condition \( i \) (where, \( i = 1, 2, 3, 4, 5 \)) of Theorem 3.

**Theorem 4** \( \forall f(x, y) \in \mathcal{R}_{(x_0, y_0)} \quad \forall d \in D^f_{(x_0, y_0)} \exists \chi \in X^1_{(x_0, y_0)} \chi(t) = d. \)

**Proof** If \( f(x, y) \in \mathcal{R}_{(x_0, y_0)} \) and \( d \in D^f_{(x_0, y_0)} \) then by Corollary 3 of Theorem 3 we can conclude that there exists \( \chi \in X_{(x_0, y_0)} \) such that \( \chi(t) = d. \)

If \( \chi \notin X^1_{(x_0, y_0)} \) then we show that for \( i = 2, 3, 4, 5 \) by the description of \( \chi \) in terms of Theorem 1 (or Theorem 2), we can choose \( \chi f \in X^i_{(x_0, y_0)} \) and \( \chi f(f) = \chi(f) = d. \)

It suffices to consider the case where the character is described by conditions of Theorem 1 for \( x_0 \leq x \). Let \( f(x, y) = \frac{p(x, y)}{q(x, y)} \). Suppose that \( \chi \in X^2_{(x_0, y_0)} \) is determined by the curve with the equation \( y = y_0 + \Phi_N(x - x_0) + (x - x_0)^e \) where \( e \notin \mathbb{Q} \), then

\[
p(x, y_0 + \Phi_N(x - x_0) + (x - x_0)^e) = a(x - x_0)^{u+ne} + 0((x - x_0)^{u+me})
\]

and

\[
q(x, y_0 + \Phi_N(x - x_0) + (x - x_0)^e) = b(x - x_0)^{v+me} + 0((x - x_0)^{v+me}).
\]

Now choose \( e_{N+1} \in \mathbb{Q} \) sufficiently close to \( e \) such that: \( e_{N} < e_{N+1} \),

\[
p(x, y_0 + \Phi_N(x - x_0) + (x - x_0)^{e_{N+1}}) = p(x, y_0) + a(x - x_0)^{u+ne_{N+1}} + 0((x - x_0)^{u+ne_{N+1}})
\]

\[
q(x, y_0 + \Phi_N(x - x_0) + (x - x_0)^{e_{N+1}}) = q(x, y_0) + b(x - x_0)^{v+me_{N+1}} + 0((x - x_0)^{v+me_{N+1}}).
\]

If \( \chi f \) is defined by \( y - y_0 + \Phi_N(x - x_0) + (x - x_0)^{e_{N+1}} \) then \( \chi f \in X^1_{(x_0, y_0)} \) and \( \chi f(f) = d. \)

In the cases of \( \chi f \in X^2_{(x_0, y_0)} \) and \( \chi f \in X^4_{(x_0, y_0)} \) the proof is as in the case of \( \chi f \in X^1_{(x_0, y_0)}. \)

Finally, if \( \chi f \in X^5_{(x_0, y_0)} \) then \( \chi(f) = \lim_{N \to \infty} \chi_N(f) \), where \( \chi_N \in X_{(x_0, y_0)} \). It follows from Lemma 5 that for sufficiently big \( N = N_f \) holds \( \chi(f) = \chi_{N_f}(f) \). \( \square \)

**Corollary** Let \( f(x, y) \) be a rational function. Then \( \lim_{(x, y) \to (x_0, y_0)} f(x, y) = a \in \mathcal{R} \) exists iff there exist the limits

\[
\lim_{x \to x_0} f(x, y_0 + \Phi_N(x - x_0)) = a, \quad \lim_{y \to y_0} f(x_0 + \Phi_N(y - y_0), y) = a
\]

for arbitrary \( \Phi_N(t) \in (\mathcal{I}). \)

**Proof** Indeed, if \( f \in \mathcal{R}_{(x_0, y_0)} \) then our assertion is a particular case of Theorem 4. If \( f(x, y) \) is unbounded in every nbd of \( (x_0, y_0) \) then repeating the arguments in the proof of Theorem 3 for \( d = \infty \), it is easy to check that there exists \( \Phi_N(t) \) such that one of the following facts hold

\[
\lim_{x \to x_0, x < x} f(x, y_0 + \Phi_N(x - x_0)) = \infty
\]

\[
\lim_{x \to x_0, x < x} f(x, y_0 + \Phi_N(x - x_0)) = \infty
\]

\[
\lim_{y \to y_0, y < y} f(x_0 + \Phi_N(y - y_0), y) = \infty
\]

\[
\lim_{y \to y_0, y < y} f(x_0 + \Phi_N(y - y_0), y) = \infty
\]

\( \square \)
3 Algebras $\tilde{R}_G$ and $\tilde{R}_G$

Now we apply our results for description of characters to the case of certain Banach algebras which contain dense subalgebras of rational functions. For these purposes we recall some known facts from the theory of Banach algebras. Let $G$ be a connected open subset of $\mathbb{R}^2$ having compact closure $\bar{G}$. Denote by $B(\bar{G})$ the Banach algebra of all real bounded functions on $\bar{G}$ endowed with $sup$-norm.

By Gelfand-Naimark Theorem every closed subalgebra $A$ of $B(\bar{G})$ is isomorphic with the Banach algebra $C(X^A)$ of all continuous functions on the compactum $X^A$, where $X^A$ is the set of all nontrivial characters of $A$ endowed with the weak topology $\sigma(X^A, A)$.

The desired isomorphism between $A$ and $C(X^A)$ is just the Gelfand Transform $\tilde{f}(\chi) = \chi(f)$, where $f \in A, f \in C(X^A)$ and $\chi \in X^A$.

If a closed subalgebra $A$ separates the points of $B(\bar{G})$ then identifying the character $\chi_{(x_0, y_0)} \in X^A$ (defined by $\chi_{(x_0, y_0)}(t) = f(x_0, y_0)$) with the point $(x_0, y_0)$, we may suppose that $\bar{G}$ is a dense subset of $X^A$. For instance, $\bar{G}$ is dense in $X^A$ if $C(\bar{G}) \subset A$.

If $C(\bar{G}) \subset A$ then $X^A = \cup_{(x, y) \in \bar{G}} \chi_{(x, y)}^A$, where

$$\chi_{(x, y)}^A = \{ \chi \in X^A \mid \forall f \in C(\bar{G}) \chi(f) = f(x, y) \}.$$

Clearly $\chi_{(x, y)}^A$ is compact for every $(x, y) \in \bar{G}$. Since $\bar{G}$ is dense in $X^A$ then

$$\forall \chi \in X^A \exists \{(x_\gamma, y_\gamma)_{\gamma \in F} \mid \lim_{\gamma \in F}(x_\gamma, y_\gamma) = (x, y) \}.$$

Moreover, if $\chi \in X_{(x, y)}^A$ then $\lim_{\gamma \in F}(x_\gamma, y_\gamma) = (x, y)$ in $\mathbb{R}^2$.

Note also that $D_{f(x, y)} = \tilde{f}(X_{(x, y)}^A)$ for every $f \in A$.

In the sequel we consider in $B(\bar{G})$ the following two subalgebras: $\tilde{R}_G$ - the closure of a subalgebra consisting of all rational functions that are bounded on $\bar{G}$; and also $\bar{R}_G$ - the closure of the subalgebra consisting of all rational functions that are bounded on $G$ and continuous on the boundary of $\bar{G}$. Obviously, $C(\bar{G}) \subset \bar{R}_G \subset \tilde{R}_G$.

Let $X^G$ ($X^G$) be the set of all characters of the algebra $\tilde{R}_G$ (respectively, $\bar{R}_G$) both endowed with the weak topology.

**Proposition 1** The spaces $X^G$ and $X^G$ are connected. Moreover, the sets $X^G_{(x, y)}$ and $X^G_{(x, y)}$ are connected for every $(x, y) \in \bar{G}$.

This proposition is a consequence of Lemma 3.

**Proposition 2** $\tilde{R}_G$ is not separable and $X^G$ is not metrizable.

**Proof** Assuming the contrary, let $\tilde{R}_G$ be separable. Since the set of all rational functions is dense in $\tilde{R}_G$, by we obtain that there are countably many points at which a function from $\tilde{R}_G$ can be discontinuous. On the other hand, clearly every point may be a point of discontinuity for suitable function from $\tilde{R}_G$. This contradiction proves the first assertion. Now, the Banach algebra $C(X^G)$ is not separable being topologically isomorphic to the Banach algebra $\tilde{R}_G$. Therefore $X^G$ is not metrizable.

**Corollary** $\tilde{R}_G$ is not separable and $X^G$ is not metrizable.
Note that Theorem 3 provides a description of $X^G$ and $X^\tilde{G}$. Note also that $X(x, y) \subset X^G(x, y) = \tilde{X}(x, y)$ for any $(x, y) \in G$. If $(x, y) \in \partial G$ ($\partial G$ means the boundary of $G$), then $\tilde{X}(x, y) \subset X^G(x, y)$, where $\tilde{X}(x, y)$ is the set of all characters in $X(x, y)$, that can be determined by nets all members of which are in $G$.

**Proposition 3** $X^G = \cup_{(x,y) \in G}X(x, y) \cup \partial G$ and $X^\tilde{G} = \cup_{(x,y) \in G}X(x, y) \cup (\cup_{(x,y) \in \partial G} \tilde{X}(x, y))$.

**Proof** As we already mentioned, $X^G = \cup_{(x,y) \in G}X^G(x, y)$. Consider $(x_0, y_0) \in G$ and $\chi \in X^G(x_0, y_0)$. There exists a net $\{(x_\gamma, y_\gamma)\}_{\gamma \in \Gamma}$ in $G$ which converges to $\chi$ in weak topology. Then this net determines a character from $X(x_0, y_0)$. Indeed, if not, as in the proof of Theorem 3, there exist natural $n_0, m_0$ such that the limit $\lim_{\gamma \in \Gamma} f(x_\gamma, y_\gamma)$ does not exist, where

$$f(x, y) = \frac{(x - x_0)^{n_0}(y - y_0)^{m_0}}{(x - x_0)^{2n_0} + (y - y_0)^{2m_0}}$$

This is a contradiction because $f(x, y) \in \mathcal{R}_G$. Therefore, $\chi \in X(x_0, y_0)$ and $X^G(x_0, y_0) = X(x_0, y_0)$, as desired. Analogous proof is valid for the second statement.

**Corollary 1** $X(x_0, y_0) = X^\mathcal{R}(x_0, y_0)$ for every $(x_0, y_0) \in \mathbb{R}^2$.

**Corollary 2** $\tilde{G}$ is sequentially dense in $X^G$, that is, for every $\chi \in X^G$ there exists a sequence which converges to $\chi$ and consist of points from $\tilde{G}$.

It is actually a reformulation of Corollary 2 of Theorem 4.

**Corollary 3** $X^1(x,y)$ is dense in $X^G(x,y) = X(x,y)$ for all $(x,y) \in G$ and $\tilde{X}^1(x,y)$ is dense in $X^\tilde{G}(x,y) = \tilde{X}(x,y)$ for all $(x,y) \in \partial G$.

This assertion follows from Theorem 4.

**Corollary 4** Let $f(x, y) \in \mathcal{R}_{\tilde{G}}$ and $(x_0, y_0) \in \tilde{G}$. Then $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = a$ exists iff $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = a$ exists for every curve $L \subset G$ that are described in the assertion 1 of Theorem 4 and the assertion 1 of Theorem 3.

The verification is easy by the previous corollary and Corollary 1 of Theorem 3.

Let $f(x, y) \in B(\tilde{G})$ satisfies the following conditions:

1. $f(x, y)$ is continuous on the boundary of $G$;
2. a) For every $(x_0, y_0) \in G$ and a curve $L$ that is described in one of the assertions (1),(2),(3),(4) of Theorem 4 or Theorem 4 there exists the limit

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y);$$

b) For every $(x_0, y_0) \in G$ and a sequence of the curves $\{L_N\}_{N \in \mathbb{N}}$ that are described in Theorem 4 or in Theorem 4 there exists

$$\lim_{N \rightarrow \infty} \lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y).$$
The set of all such functions clearly is a closed subalgebra of the Banach algebra $B(\hat{G})$. Define this subalgebra by $L(\hat{G})$. Obviously, $\mathcal{R}_G \subset L(\hat{G})$. However, $L(\hat{G}) \neq \mathcal{R}_G$ as it follows from the following known counterexample

$$f(x, y) = \frac{e^{-\frac{1}{(x-x_0)^2}}(y-y_0)}{e^{-\frac{x}{(x-x_0)^2}} + (y-y_0)^2}$$

(see [1]).

**Corollary 5** If $f(x, y) \in \mathcal{R}_G$ then:

1. The set of discontinuity points of $f(x, y)$ is at most countable;
2. $f(x, y) \in L(\hat{G})$;
3. For every $(x_0, y_0) \in G$ and $d \in D^f(x_0, y_0)$ there exists a character $\chi \in X^G_{(x_0, y_0)}$ such that $\chi(f) = d$.

We state here two natural questions inspired by Corollaries 4 and 5 of Proposition 3.

1. How can be generalized Theorem 3 for $\mathcal{R}_G$ or $\mathcal{R}\hat{G}$?
2. Is it true that the conditions of Corollary 5 are also sufficient in order to ensure that $f(x, y) \in \mathcal{R}\hat{G}$?

Note that the main results of the present paper can be generalized for functions of $n$ arguments with $n > 2$ and for arbitrary points of discontinuity of rational functions.

Finally the author thanks to E. Shustin (Tel-Aviv University) for several stimulating conversations and important suggestions that have significant influence on results of the present paper. We thank also to M. Megrelishvili (Bar-Ilan University) for his support.
References

[1] Gelbaum B., Olmsted J., Counterexamples in Analysis, Moscow, 1967.
[2] Khelemskii A., Banach and multilinear algebras, Moscow, 1989.
[3] Engelking R., General Topology, PWN, Warsaw, 1985.

address: Neot Golda str. 606/14, Netanya 42345, Israel
tel: (972) 09-8356839
email:megereli@macs.biu.ac.il