A blow-up criterion for the compressible liquid crystals system

Xiangao LIU∗ Lanming LIU∗

Abstract In this paper, we establish a blow-up criterion for the compressible liquid crystals equations in terms of the gradient of the velocity only, similar to the Beale-Kato-Majda criterion [1] for ideal incompressible flows and the criterion obtained by Huang and Xin [8] for the compressible Navier-Stokes equations.

Key words Blow-up criterion; Strong solutions; Liquid crystals equations; Compressible Navier-Stokes equations

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1 Introduction

In this paper we consider the following simplified model of the Ericksen-Leslie theory for nematic liquid crystals and study a blow-up criterion for it.

\begin{align*}
  \rho_t + \text{div}(\rho u) &= 0, \quad \text{(1.1)} \\
  (\rho u)_t + \text{div}(\rho u \otimes u) + \nabla p &= \mu \Delta u - \lambda \text{div}(\nabla d \otimes \nabla d) - \frac{1}{2}(|\nabla d|^2 + F(d))I, \quad \text{(1.2)} \\
  d_t + u \cdot \nabla d &= \nu(\Delta d - f(d)) \quad \text{(1.3)}
\end{align*}

in $\Omega \times (0, T)$, for a bounded smooth domain $\Omega$ in $\mathbb{R}^3$.

In the above system, the velocity field $u(x, t)$ of the flow, the direction field $d(x, t)$ representing the orientation parameter of the liquid crystal are vectors in $\mathbb{R}^3$. The density $\rho(x, t)$ is a scalar and $p$ is the pressure dependent on the density $\rho$. $\mu, \lambda, \nu$ are positive

∗Department of Mathematical Sciences, Fudan University, Shanghai, 200433, China.

E-mail: xgliu@fudan.edu.cn 09110180017@fudan.edu.cn
The unusual term $\nabla d \otimes \nabla d$ denotes the $3 \times 3$ matrix whose $(i,j)$-th element is given by $\sum_{k=1}^{3} \partial_x d_k \partial_x d_k$ and $I$ is the unite matrix. $f(d)$ is a polynomial of $d$ which satisfies $f(d) = \frac{\partial}{\partial d} F(d)$ where $F(d)$ is the bulk part of the elastic energy. Usually we choose $F(d)$ to be the Ginzburg-Landau penalization, that is, $F(d) = \frac{1}{4\pi^2}(|d|^2 - 1)^2$ and $f(d) = \frac{1}{4\pi^2}(|d|^2 - 1)d$, where $\sigma$ is a positive constant.

As the paper [11], we assume the pressure $p$ satisfies

$$p = p(\cdot) \in C^1[0, \infty), \quad p(0) = 0.$$ (1.4)

The authors of the paper [11] have proved the following local existence of strong solutions to (1.1)-(1.3) with initial data:

$$\rho(0, x) = \rho_0 \geq 0, \quad u(0, x) = u_0, \quad d(0, x) = d_0(x),$$ (1.5)

boundary conditions: $\forall (t, x) \in (0, T) \times \partial \Omega$,

$$u(t, x) = 0, \quad d(t, x) = d_0(x), \quad |d_0(x)| = 1,$$ (1.6)

and some compatibility condition on the initial data:

$$\mu \triangle u_0 - \lambda \text{div}(\nabla d_0 \otimes \nabla d_0) - \frac{1}{2}(|\nabla d_0|^2 + F(d_0))I - \nabla p_0 = \frac{\rho_0^\frac{4}{3}}{\rho_0^2} g \quad \text{for some} \quad g \in L^2.$$ (1.7)

Throughout this paper, we adopt the following simplified notations for Sobolev spaces

$$L^q = L^q(\Omega), \quad W^{k,q} = W^{k,q}(\Omega), \quad H^k = H^k(\Omega), \quad H^1_0 = H^1_0(\Omega).$$

**Proposition 1.** If $(\rho_0, u_0, d_0)$ satisfies the following regularity condition

$$\rho_0 \in W^{1,6}, \quad u_0 \in H^1_0 \cap H^2 \quad \text{and} \quad d_0 \in H^3,$$ (1.8)

and the compatibility condition (1.7), then there exists a small $T^* \in (0, T)$, and a unique strong solution $(\rho, u, d)$ to (1.1)-(1.3) with initial-boundary data (1.5)-(1.6) such that

$$\rho \in C([0, T^*); W^{1,6}), \quad \rho_t \in C([0, T^*); L^6),$$
$$u \in C([0, T^*); H^1_0 \cap H^2) \cap L^2(0, T^*; W^{2,6}), \quad u_t \in L^2(0, T^*; H^1_0),$$
$$d \in C([0, T^*); H^3), \quad d_t \in C([0, T^*); H^1_0) \cap L^2(0, T^*; H^2),$$
$$d_{tt} \in L^2(0, T^*; L^2), \quad \sqrt{\rho} u_t \in C([0, T^*); L^2).$$ (1.9)
It is an interesting and natural question whether there is a global strong solution. The paper [12] has proved there is a global weak solution to the compressible liquid crystals system (1.1)-(1.3) where vacuum is allowed initially. And recently the authors of paper [11] have proved the system (1.1)-(1.3) has a global strong solution with small initial data. Since the compressible liquid crystals system (1.1)-(1.3) is coupled by Navier-Stokes equations and liquid crystals equation, it is expected to non-existence of global strong solutions when vacuum regions are present initially. In order to establish a blow-up criterion for the system (1.1)-(1.3), we turn to the Navier-Stokes equations. There are many results concerning blow-up criteria of the incompressible or compressible flow. It is well known that Beal-Kato-Majda established the following blow-up criterion for the incompressible Euler equation in the paper [1]:

\[
\lim_{T \to T^*} \int_0^{T^*} \| \nabla \times u \|_{L^\infty} \, dt = +\infty.
\]

Similarly, Huang and Xin [8] also give a blow-up criterion for the isentropic compressible Navier-Stokes equations as follows:

\[
\lim_{T \to T^*} \int_0^{T^*} \| \nabla u \|_{L^\infty} \, dt = +\infty.
\]

Inspired by these ideas, we establish the following criterion for the compressible liquid crystals system (1.1)-(1.3):

**Theorem 1.** (Blow-up Criterion) Assume that the initial data satisfies the regularity (1.8) and the compatibility condition (1.7). Let \((\rho, u, d)\) be the unique strong solution to the problem (1.1)-(1.3) with the initial boundary conditions (1.5)-(1.6). If \(T^*\) is the maximal time of the existence and \(T^*\) is finite, then

\[
\lim_{T \to T^*} \int_0^{T^*} \| \nabla u \|_{L^\alpha}^{\beta} + \| u \|_{W^{1,\infty}} \, dt = +\infty \tag{1.10}
\]

where \(\alpha, \beta\) satisfy

\[
\frac{3}{\alpha} + \frac{2}{\beta} < 2 \quad \text{and} \quad \beta \geq 4. \tag{1.11}
\]

**Remark 1.** This criterion given by theorem 1 only involves the velocity \(u\) because thanks to the constraint (1.11), the first part of (1.10) plays a role as the direction \(d\).

As usual, we will prove theorem 1 by contradiction in the next section.
2 Proof of Theorem

Let \((ρ, u, d)\) be the unique strong solution to the problem (1.1)-(1.6). We assume the opposite to (1.10) holds, i.e.
\[
\lim_{T \to T^*} \int_0^T \| \nabla u \|_{L^α}^β + \| u \|_{W^{1,∞}} \, dt \leq C < +∞.
\]
Hence for all \(T < T^*\)
\[
\int_0^T \| \nabla u \|_{L^α}^β + \| u \|_{W^{1,∞}} \, dt \leq C,
\]
from which we will get the same regularity at time \(T^*\) as the initial data, a contraction to the maximality of \(T^*\). Thanks to the assumption (1.11) on \((α, β)\), we have by interpolation
\[
\int_0^T \| u \|_{L^∞}^2 \, dt, \quad \int_0^T \| \nabla u \|_{L^2}^4 \, dt, \quad \int_0^T \| \nabla u \|_{L^∞}^2 \, dt \leq C.
\]
In the following proof, we will employ energy law and higher order energy law.

2.1 Estimate for \(ρ\)

It is easy to see that the continuity equation (1.1) on the characteristic curve \(\frac{d}{dt} \chi(t) = u(t, \chi(t))\) can be written as
\[
\frac{d}{dt} ρ(t, \chi(t)) = -ρ(t, \chi(t)) \text{div}(u(t, \chi(t))).
\]
So
\[
ρ(t, \chi(t)) = ρ(0, \chi(0)) \exp\left(-\int_0^t \text{div}(u(\tau, \chi(\tau))) \, d\tau\right)
\]
Thus
\[
0 \leq ρ(t, x) \leq \|ρ_0\|_{L^∞} \exp(\int_0^T \|\text{div}u\|_{L^∞} \, dt) \leq C \quad \forall (t, x) \in [0, T] \times \overline{Ω}.
\]
According to the assumption (1.4) on the pressure \(p\) and the above estimate (2.3),
\[
\sup_{0 \leq t \leq T} \{\|p(ρ)\|_{L^∞}, \|p'(ρ)\|_{L^∞}\} \leq C.
\]
As the final section of the paper [3], we construct sequences \( \{ \rho_0^k \} \) and \( \{ u^k \} \) of smooth scalar and vector fields such that

\[
\rho_0^k \in H^2 \cap C^2(\Omega), \quad u^k \in L^2(0,T; H^1_0 \cap H^3) \cap C^2([0,T] \times \Omega) \quad \text{and} \quad \| \rho_0^k - \rho_0 \|_{W^{1,6}} + \int_0^T \| \nabla (u^k - u)(t) \|_{W^{1,6}}^2 \, dt \to 0 \quad \text{as} \quad k \to \infty. \tag{2.6}
\]

Then it follows from the classical linear hyperbolic theory that there is a unique solution \( \rho^k \in C^2([0,T] \times \Omega) \) to the following problem:

\[
\begin{align*}
\rho_t + \text{div}(\rho \rho^k) & = 0 \quad \text{in} \ (0, T) \times \Omega, \\
\rho(0) & = \rho_0^k \quad \text{in} \ \Omega. 
\end{align*} \tag{2.7}
\]

The final section of the paper [3] proves that for each fixed \( t \in [0, T) \),

\[
\rho^k(t) \to \rho(t) \quad \text{weakly in} \ W^{1,6}. \tag{2.8}
\]

Applying the operator \( \nabla \) to the equation (2.7), then multiplying by \( \nabla \rho^k \) and integrating over \( \Omega \) give us we get

\[
\frac{d}{dt} \int_\Omega \| \nabla \rho^k \|_{L^2}^2 \, dx \leq - \int_\Omega \| \nabla \rho^k \|_{L^2} \| \nabla u^k \|_{L^\infty} + C \| \nabla \rho^k \|_{L^2} \| \nabla \text{div} u^k \|_{L^2},
\]

that is,

\[
\frac{d}{dt} \| \nabla \rho^k \|_{L^2} \leq C \| \nabla \rho^k \|_{L^2} \| \nabla u^k \|_{L^\infty} + C \| \nabla \text{div} u^k \|_{L^2} \tag{2.9}
\]

Applying Gronwall’s inequality to it, we obtain

\[
\| \nabla \rho^k \|_{L^2} \leq (\| \rho_0^k \|_{H^1} + C \int_0^t \| \nabla \text{div} u^k \|_{L^2} \, d\tau) \exp(C \int_0^t \| \nabla u^k \|_{L^\infty} \, d\tau), \quad \forall t \in [0, T].
\]

Hence because of the assumption (2.6) and the convergence (2.8), we can get

\[
\| \nabla \rho \|_{L^2} \leq (\| \rho_0 \|_{H^1} + C \int_0^t \| \nabla \text{div} u \|_{L^2} \, d\tau) \exp(C \int_0^t \| \nabla u \|_{L^\infty} \, d\tau) \quad \forall t \in [0, T]. \tag{2.9}
\]

As the above similar process, we obtain

\[
\| \nabla \rho \|_{L^6} \leq (\| \rho_0 \|_{W^{1,6}} + C \int_0^t \| \nabla \text{div} u \|_{L^6} \, d\tau) \exp(C \int_0^t \| \nabla u \|_{L^\infty} \, d\tau) \quad \forall t \in [0, T]. \tag{2.10}
\]
2.2 Energy law

Multiplying the momentum equation (1.2) by \( u \) and then integrating over \( \Omega \), we can obtain

\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho |u|^2 dx + \int_{\Omega} u \cdot \nabla p dx = -\mu \int_{\Omega} |\nabla u|^2 dx - \lambda \int_{\Omega} (u \cdot \nabla) d \cdot (\Delta d - f(d)) dx. \tag{2.11}
\]

Because of the estimate (2.5), we have

\[
|\int_{\Omega} u \cdot \nabla p dx| = |\int_{\Omega} p \text{div} u dx| \leq \epsilon \int_{\Omega} |\nabla u|^2 dx + C \epsilon^{-1}. \tag{2.12}
\]

By liquid crystals equation (1.3), we can get

\[
\int_{\Omega} (u \cdot \nabla) d \cdot (\Delta d - f(d)) dx = \frac{d}{dt} \int_{\Omega} \frac{1}{2} |\nabla d|^2 + F(d) dx + \nu \int_{\Omega} |\Delta d - f(d)|^2 dx. \tag{2.13}
\]

So substituting (2.12) and (2.13) into the corresponding terms of (2.11) and taking \( \epsilon \) small enough give us

\[
\frac{dE}{dt} + \int_{\Omega} |\Delta d - f(d)|^2 dx + \int_{\Omega} |\nabla u|^2 dx \leq C \tag{2.14}
\]

where

\[ E = \int_{\Omega} \rho |u|^2 + |\nabla d|^2 + F(d) dx. \]

Applying Gronwall’s inequality to (2.14), we can obtain the desire energy law of the liquid crystals system

\[
\sup_{0 \leq t \leq T} \int_{\Omega} \rho |u|^2 + |\nabla d|^2 + F(d) dx + \int_{0}^{T} \int_{\Omega} |\Delta d - f(d)|^2 dx dt + \int_{0}^{T} \int_{\Omega} |\nabla u|^2 dx dt \leq C. \tag{2.15}
\]

2.3 Estimate for \( d \)

Multiply the liquid equation (1.3) by \( d \), we know that \( |d| \leq 1 \) by the maximal principle of parabolic equation. So \( f(d) \) and \( F(d) \) are bounded.

**Lemma 1.**

\[
\sup_{0 \leq t \leq T} \|d\|_{H^2}^2 + \int_{0}^{T} \|\nabla d\|_{L^2}^2 dt \leq C. \tag{2.16}
\]
Proof. Multiplying (1.3) by $\Delta d_t$, we have
\[\frac{d}{dt} \int_{\Omega} |\Delta d|^2 dx + \int_{\Omega} |\nabla d_t|^2 dx \]
\[\leq C(\int_{\Omega} u \cdot \nabla d_\Delta d_t dx + \int_{|d|^2 - 1} d \Delta d_t dx) \]
\[\leq C(\int_{\Omega} |\nabla u||\nabla d||\nabla d_t|dx + \int_{\Omega} |u||\nabla^2 d||\nabla d_t|dx + \int_{\Omega} |\nabla d||\nabla d_t|dx) \]
\[\leq \epsilon \|\nabla d_t\|_{L^2}^2 + C \epsilon^{-1} \|\nabla u\|_{L^3} \|\nabla d_t\|_{L^6} + C \epsilon^{-1} \|\nabla^2 d\|_{L^2}^2 \|u\|_{L^\infty}^2 + C \epsilon^{-1} \|\nabla d_t\|_{L^2}^2 \]
\[\leq \epsilon \|\nabla d_t\|_{L^2}^2 + C \epsilon^{-1} \|\nabla u\|_{L^3} (\|\Delta d_t\|_{L^2}^2 + ||d_0||_{H^2}^2 + C) \]
\[+ C \epsilon^{-1} (\|\Delta d_t\|_{L^2}^2 + ||d_0||_{H^2}) \|u\|_{L^\infty}^2 + C \epsilon^{-1} \]
\[\leq C(1 + \int_0^T |u|_{L^\infty}^2 + \|\nabla u\|_{L^3}^2 dt) \exp(\int_0^T |u|_{L^\infty}^2 + \|\nabla u\|_{L^3}^2 dt) \]
\[\leq C \]  (2.17)
\[\] where in the last inequality we employ the elliptic regularity result \(\|\nabla^2 d\|_{L^2} \leq C(\|\Delta d\|_{L^2} + ||d_0||_{H^2})\) and the energy inequality (2.15).

Taking $\epsilon$ small, integrating it over $[0,T]$ and using Gronwall’s inequality, we can deduce
\[\sup_{0 \leq t \leq T} \int_{\Omega} |\Delta d_t|^2 dx + \int_0^T |\nabla d_t|^2 dx dt \]
\[\leq C \]  (2.18)
\[\] where the last inequality uses the estimate (2.14).

Using the elliptic estimate, (2.14) yields (2.16). \[\Box\]

Differentiating (1.3) with respect to space gives us
\[\nu \Delta (\nabla d) = \nabla d_t + \nabla (u \cdot \nabla d) + \frac{\nu}{\sigma^2} \nabla [(|d|^2 - 1)d]. \quad (2.18)\]

Applying elliptic regularity result to (2.18), from the estimate (2.16), one can estimate the term $|\nabla d|_{H^2}$ as follows
\[\|\nabla d\|_{H^2} \leq C(\|\nabla d_t\|_{L^2} + \|\nabla (u \cdot \nabla d)\|_{L^2} + \|\frac{\nu}{\sigma^2} \nabla [(|d|^2 - 1)d]\|_{L^2} + ||d_0||_{H^3} + \|\nabla d_t\|_{L^2}) \]
\[\leq C(\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^3} \|\nabla d\|_{L^6} + ||u||_{L^\infty} \|\nabla^2 d\|_{L^2} + \|\nabla d\|_{L^2} \|d_0\|_{L^\infty} \]
\[\|\nabla d_t\|_{L^2} + ||d_0||_{H^3}) \]
\[\leq C(\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^3} + ||u||_{L^\infty} + C). \quad (2.19)\]
So
\[
\int_0^T \| \nabla d \|^2_{H^2} dt \leq C \int_0^T (\| \nabla d_t \|^2_{L^2} + \| \nabla u \|^2_{L^4} + \| u \|^2_{L^8} + C) dt \leq C
\] (2.20)
where the second inequality can be obtained by the estimates (2.2) and (2.16).

2.4 Estimate for \( u \)

At the beginning, we prove a key lemma

**Lemma 2.**
\[
\sup_{0 \leq t \leq T} \int_\Omega |u|^{3+\delta} dx \leq C
\] (2.21)
where \( \delta(< 1) \) is a small nonegative constant.

**Proof.** Multiplying (1.2) by \( q|u|^{q-2}u \) and using the estimate (2.5), we can deduce
\[
\frac{d}{dt} \int_\Omega \rho|u|^q dx + \int_\Omega q|u|^{q-2}(\mu |\nabla u|^2 + \nu \lambda \Delta d - f(d))^2 + \mu(q-2)|\nabla |u|^2| dx
\]
\[
= q \int_\Omega \text{div}(|u|^{q-2}u)dx + \lambda q \int_\Omega |u|^{q-2}d_t(\Delta d - f(d)) dx
\]
\[
\leq C \int_\Omega |u|^{q-2}|\nabla u|^2 dx + \epsilon \int_\Omega |u|^{q-2} \Delta d - f(d)^2 dx + C\epsilon^{-1} \int_\Omega |u|^{q-2}d_t^2 dx
\]
\[
\leq \epsilon \int_\Omega |u|^{q-2}|\nabla u|^2 dx + C\epsilon^{-1} \int_\Omega |u|^{q-2}dx + \epsilon \int_\Omega |u|^{q-2} \Delta d - f(d)^2 dx
\]
\[
+ C\epsilon^{-1} \int_\Omega |u|^{q-2}d_t^2 dx.
\]

Hence
\[
\frac{d}{dt} \int_\Omega \rho|u|^q dx + \int_\Omega |u|^{q-2}(\| \nabla u \|^2 + |\Delta d - f(d)|^2 + |\nabla |u|^2|) dx
\]
\[
\leq C \int_\Omega |u|^{q-2}dx + C \int_\Omega |u|^{q-2}d_t^2 dx.
\] (2.22)

Let \( q = 3 + \delta \) and integrate (2.22) over \([0, T]\). Using (2.2), (2.4) and (2.15), we can obtain
\[
\sup_{0 \leq t \leq T} \int_\Omega \rho|u|^{3+\delta} dx + \int_0^T \int_\Omega |u|^{1+\delta}(\| \nabla u \|^2 + |\Delta d - f(d)|^2 + |\nabla |u|^2|) dx dt
\]
\[
\leq C \int_0^T \int_\Omega |u|^{1+\delta} dx dt + C \int_0^T \int_\Omega |u|^{1+\delta}d_t^2 dx dt
\]
\[
\leq C + C \int_0^T \| u \|_{L^6}^4 + \| d_t \|_{L^{12}}^4 dt
\]
\[
\leq C + C \int_0^T \| d_t \|_{L^4}^4 dt.
\] (2.23)
From the liquid crystal equation (1.3) and using (2.2), (2.16) and (2.20), we get
\[
\int_0^T \| d_t \|_{L^4}^4 dt \leq \int_0^T \| \triangle d \|_{L^2}^2 dt + \int_0^T \| u \cdot \nabla d \|_{L^2}^2 dt + C
\]
\[
\leq \int_0^T \| \triangle d \|_{L^2}^2 \| \triangle d \|_{L^2}^2 dt + \int_0^T \| \nabla u \|_{L^2}^4 \| \nabla d \|_{L^2}^4 dt + C
\]
\[
\leq C.
\]  
(2.24)
Taking (2.24) into (2.23), we obtain the conclusion (2.21).

**Lemma 3.**
\[
\sup_{0 \leq t \leq T} (\| \nabla u \|_{L^2}^2 + \| \nabla \rho \|_{L^2}^2) + \int_0^T \| \sqrt{\rho} u_t \|_{L^2}^2 dt \leq C(1 + \eta^{-1}) + \eta \int_0^T \| u_t \|_{L^2}^2 dt
\]  
(2.25)
where \( \sigma \) is a small positive constant and will be determined later.

**Proof.** Multiplying the momentum equation (1.2) by \( u_t \), integrating over \( \Omega \) and then using Young’s inequality, we have
\[
\frac{\mu}{2} \frac{d}{dt} \int_\Omega | \nabla u |^2 dx + \frac{1}{2} \int_\Omega \rho | u_t |^2 dx \leq 2 \int_\Omega \rho | u_t | \cdot \nabla u dx + \int_\Omega p \text{div} u dx - \int_\Omega (u_t \cdot \nabla) d (\Delta d - f(d)) dx.
\]  
(2.26)
Using the continuity equation (1.1) gives us
\[
\int_\Omega p \text{div} u dx = \frac{d}{dt} \int_\Omega p \text{div} u dx - \int_\Omega p \text{div} u dx + \int_\Omega p' (\nabla \rho \cdot u + \rho \text{div} u) \text{div} u dx.
\]  
(2.27)
Using the liquid crystal equation (1.3), we can get
\[
\int_\Omega (u_t \cdot \nabla) d (\Delta d - f(d)) dx = \frac{1}{\nu} \int_\Omega (u_t \cdot \nabla) d (d_t + u \cdot \nabla d) dx.
\]  
(2.28)
Substituting the above equations (2.27) and (2.28) into (2.26), integrating over \( (0, t) \) and using Young’s inequality, we obtain
\[
\int_\Omega | \nabla u |^2 dx + \int_0^t \int_\Omega \rho | u_t |^2 dx dt \leq C + C \int_0^t \int_\Omega \rho | u_t | \cdot \nabla u dx dt + C \int_\Omega p^2 (\rho) dx + C \int_0^t \int_\Omega p' (\nabla \rho \cdot u
\]
\[+ \rho \text{div} u) \text{div} u dx dt + C \int_0^t \int_\Omega | u_t | \cdot \nabla d \| \Delta d \| + | u_t | \cdot \nabla d \| f(d) dx dt.
\]  
(2.29)
In order to estimate the second term of the right side of (2.29), we need to control $\|u\|_{H^2}$. Thanks to the estimate (2.21), we obtain

$$
\int_{\Omega} \rho|u|^2|\nabla u|^2 \leq C \int_{\Omega} \rho \frac{|u|^2}{1+\delta} |\nabla u|^2 \leq C \frac{\|\rho\|^2_{L^\infty} \|\nabla u\|^2_{L^2}}{1+\delta} \leq \epsilon^2 \|\nabla u\|^2_{H^1} + C\epsilon^{-2} \|\nabla u\|^2_{L^2} (2.30)
$$

where in the last inequality we use the inequality (2.21), the interpolation inequality and Young’s inequality.

Rewriting the momentum equation (1.2),

$$
\mu \Delta u = \rho u_t + \rho u \cdot \nabla u + \nabla p + \lambda(\nabla d)^T (\Delta d - f(d)).
$$

Using elliptic estimate and the inequality (2.30), we can get

$$
\|u\|_{H^2} \leq C(\|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} + \|\nabla p\|_{L^2} + \|\nabla d|\Delta d - f(d)|\|_{L^2})
$$

Taking $\epsilon$ small enough and using the estimate (2.16), we can get from the above inequality.

$$
\|u\|_{H^2} \leq C(\|\rho u_t\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla p\|_{L^2} + \|\nabla d_t\|_{L^2}). (2.31)
$$

We continue our proof.

Thanks to (2.31),

$$
\int_{\Omega} \rho|u|^2 |\nabla u|^2 \leq C \int_{\Omega} (\epsilon^2 \|\nabla u\|^2_{H^1} + C\epsilon^{-2} \|\nabla u\|^2_{L^2}) d\tau (2.32)
$$

$$
\leq \int_{0}^{t} \epsilon^2 \|\nabla u\|^2_{H^1} + C\epsilon^{-2} \|\nabla u\|^2_{L^2} d\tau (2.33)
$$

$$
\leq C\epsilon^2 \int_{0}^{t} \|\rho u_t\|^2_{L^2} + \|\nabla u\|^2_{L^2} + \|\nabla p\|^2_{L^2} + \|\nabla d_t\|^2_{L^2} d\tau + C\epsilon^{-2} \int_{0}^{t} \|\nabla u\|^2_{L^2} d\tau
$$

$$
\leq C\epsilon^2 \int_{0}^{t} \|\rho u_t\|^2_{L^2} + C(\epsilon^2 + \epsilon^{-2}) \int_{0}^{t} \|\nabla u\|^2_{L^2} d\tau + C\epsilon^2 \int_{0}^{t} \|\nabla \rho\|^2_{L^2} d\tau + C\epsilon^2. (2.34)
$$
where the last inequality utilizes the estimate (2.20).

Using the above estimates (2.2), (2.4), (2.5), (2.16) and (2.20), we can obtain

\[ \int_0^t \int_{\Omega} p'(\rho)(\nabla \rho \cdot u) \text{div} u \, dx \, d\tau \leq C \int_0^t \| \nabla \rho \|_{L^2}^2 \| u \|_{L^\infty} \, d\tau + C \int_0^t \| \nabla u \|_{L^2}^2 \| u \|_{L^\infty} \, d\tau \]  
\[ (2.35) \]

\[ \int_0^t \int_{\Omega} \rho |u| \| \nabla d \|_{L^2} \, dx \, d\tau \leq \int_0^t \| u \|_{L^2} \| \nabla d \|_{L^\infty} \, d\tau \leq \eta \int_0^t \| u \|_{L^2}^2 \, d\tau + C \eta^{-1} \int_0^t \| d \|_{H^3} \, d\tau \]
\[ \leq \eta \int_0^t \| u \|_{L^2}^2 \, d\tau + C \eta^{-1} \]  
\[ (2.36) \]

and

\[ \int_0^t \int_{\Omega} |u| \| \nabla d \|_{L^2} \| f(d) \|_{L^2} \, dx \, d\tau \leq C \int_0^t \| u \|_{L^2} \| \nabla d \|_{L^2} (\| d \|_{L^\infty}^2 + 1) \| d \|_{L^\infty} \, d\tau \]
\[ \leq \eta \int_0^t \| u \|_{L^2}^2 \, d\tau + C \eta^{-1} \]  
\[ (2.37) \]

Substituting (2.35) and (2.36) into (2.29) and taking \( \epsilon \) small, we can obtain

\[ \int \| \nabla u \|_{L^2}^2 \, dx + \int \rho |u| \| \nabla d \|_{L^2}^2 \, dx \]
\[ \leq C(1 + \eta^{-1}) + C \int_0^t \| \nabla u \|_{L^2}^2 + \| \nabla \rho \|_{L^2}^2 \| u \|_{L^\infty} + \eta \int_0^t \| u \|_{L^2}^2 \, d\tau. \]
\[ (2.39) \]

Taking (2.31) into (2.9), we obtain

\[ \zeta \| \nabla \rho \|_{L^2}^2 \leq C \zeta(1 + \int_0^t \| \nabla u \|_{L^2}^2 + \| \nabla \rho \|_{L^2}^2 + \| \nabla d \|_{L^2}^2) \]  
\[ (2.40) \]

Taking \( \zeta \) small, combing (2.39) with (2.40) and using the estimates (2.2) and (2.10), we have

\[ \int \| \nabla u \|_{L^2}^2 + \| \nabla \rho \|_{L^2}^2 \, dx + \int \rho |u| \| \nabla d \|_{L^2}^2 \, dx \]
\[ \leq C(1 + \eta^{-1}) + C \int_0^t \| \nabla u \|_{L^2}^2 + \| \nabla \rho \|_{L^2}^2 \| u \|_{L^\infty} + \eta \int_0^t \| u \|_{L^2}^2 \, d\tau. \]
\[ (2.41) \]

Applying generalized Gronwall’s inequality to (2.41), we deduce

\[ \sup_{0 \leq t \leq T} \int \| \nabla u \|_{L^2}^2 + \| \nabla \rho \|_{L^2}^2 \, dx + \int_0^T \rho |u| \| \nabla d \|_{L^2}^2 \, dx \]
\[ \leq C(1 + \eta^{-1}) + \eta \int_0^T \| u \|_{L^2}^2 \, d\tau. \]
\[ \blacksquare \]
2.5 Higher order energy inequality

We will use higher order energy inequality to deduce the following lemma:

Lemma 4.

\[
\sup_{0 \leq t \leq T} (\| \nabla u \|^2_{L^2} + \| \nabla \rho \|^2_{L^2} + \| \nabla \rho u_t \|^2_{L^2} + \| \nabla d_t \|^2_{L^2}) + \int_0^T (\| \nabla u_t \|^2_{L^2} + \| \nabla u_t \|_{L^2}^2 + \| (\Delta d - f(d))_t \|_{L^2}^2) dt \leq C.
\] (2.42)

Proof. Rewrite the momentum equation (1.2) in a non conservative form as

\[
\rho u_t + \rho u \cdot \nabla u + \nabla p_t = \mu \Delta u - \lambda (\nabla d)^T (\Delta d - f(d)).
\] (2.43)

Then differentiate the above equation (2.43) with respect to time, multiply the resulting equation by \(u_t\) and integrate it over \(\Omega\) to get

\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho |u_t|^2 dx + \int_{\Omega} \mu |\nabla u_t|^2 dx - \int_{\Omega} p_t \text{div} u_t dx = -\int_{\Omega} \rho u \cdot \nabla (\frac{1}{2}|u_t|^2 + (u \cdot \nabla u) \cdot u_t) + \rho (u_t \cdot \nabla u) \cdot u_t dx -\lambda \int_{\Omega} (u_t \cdot \nabla d_t) \cdot (\Delta d - f(d)) + (u_t \cdot \nabla) d_t \cdot (\Delta d - f(d))_t dx.
\] (2.44)

Differentiating liquid crystals equation (1.3) with respect to time derives

\[
u (\Delta d - f(d))_t - \Delta d_t - u \cdot \nabla d_t.
\]

Then

\[
\int_{\Omega} (u_t \cdot \nabla) d_t \cdot (\Delta d - f(d))_t dx
\]

\[
\int_{\Omega} |\nu (\Delta d - f(d))_t|^2 dx - \Delta d_t + \Delta d_t f(d)_t - (u \cdot \nabla) d_t (\Delta d - f(d))_t dx
\]

\[
\int_{\Omega} -(u_t \cdot \nabla d + u \cdot \nabla d_t) f(d)_t + (\nu (f(d))_t - u \cdot \nabla d_t) (\Delta d - f(d))_t dx
\]

\[
+ \int_{\Omega} |\nu (\Delta d - f(d))_t|^2 dx + \frac{d}{dt} \int_{\Omega} |\nabla d_t|^2 dx.
\] (2.45)

By the continuity (1.1), the term of \(p\) in (2.44) becomes

\[
\int_{\Omega} p_t \text{div} u_t dx = -\int_{\Omega} p'(\rho) (\nabla \rho \cdot u + \rho \text{div} u) \text{div} u_t dx.
\] (2.46)
Substituting (2.45) and (2.46) into (2.44), we get the first order energy inequality

\[
\frac{d}{dt} \int |\frac{1}{2} \rho |u_t|^2 + \lambda | \nabla d_t|^2 \rangle dx + \int \mu | \nabla u_t|^2 + \lambda u^2 |(\Delta d - f(d))c|^2 \rangle dx \\
\leq C(\int |\rho|u_t|| u||_{L^\infty} + \rho|u||_{L^6} || \nabla u||_{L^2} + \rho|u|^2 | \nabla^2 u| + \rho|u|^2 || u||_{L^\infty} || \nabla u_t||_{L^2} dx \\
+ \int \rho|u_t|^2 || u||_{L^2} dx + \int |(u_t \cdot \nabla d)f(d)_t| + |(u \cdot \nabla d_t)f(d)_t| dx \\
+ \int |(\Delta d - f(d))_t f(d)_t| + |(u \cdot \nabla d_t)(\Delta d - f(d))_t| dx \\
+ \int |(u_t \cdot \nabla d_t)(\Delta d - f(d))| dx + \int |p'(\rho)|| \nabla \rho| || u||_{L^6} || \text{div} u_t||_{L^2} dx \\
+ \int \rho|p'(\rho)|| \text{div} u|| \text{div} u_t||_{L^2} dx \\
= C \sum_{i=1}^{12} I_i. \tag{2.47}
\]

Now we estimate each term $I_i$. In the following calculations, we will make full use of Sobolev inequality, Hölder inequality and estimate (2.4), (2.16), (2.5) and (2.31).
where

\[ I_6 \leq \|u_t\|_{L^2} \|\nabla d\|_{L^2}(\|d\|_{L^2}^2 + 1)\|d_t\|_{L^6} \]

\[ \leq \epsilon\|u_t\|^2_{L^2} + C\epsilon^{-1}\|\nabla d_t\|^2_{L^2}, \]

\[ I_7 \leq C\|u\|_{L^\infty} \|\nabla d_t\|_{L^2}\|d_t\|_{L^2}(\|d\|_{L^2}^2 + 1) \]

\[ \leq C\|u\|_{L^\infty} \|\nabla d_t\|^2_{L^2}, \]

\[ I_8 \leq \epsilon\|(\triangle d - f(d))t\|_{L^2}^2 + C\epsilon^{-1}\|f(d)t\|^2_{L^2} \]

\[ \leq \epsilon\|(\triangle d - f(d))t\|_{L^2}^2 + C\epsilon^{-1}\|d_t\|^2_{L^2}, \]

\[ I_9 \leq \epsilon\|(\triangle d - f(d))t\|_{L^2}^2 + C\epsilon^{-1}\|u\|^2_{L^2} \|\nabla d_t\|^2_{L^2}, \]

\[ I_{10} \leq \|u_t\|_{L^2} \|\nabla d_t\|_{L^2} \|\triangle d\|_{L^1} + \|u_t\|_{L^2} \|\nabla d_t\|_{L^2}(\|d\|_{L^2}^2 + 1)\|d\|_{L^\infty} \]

\[ \leq \epsilon\|\nabla u_t\|^2_{L^2} + C\epsilon^{-1}\|\nabla d_t\|^2_{L^2}(1 + \|\triangle d\|_{L^2}^2), \]

\[ I_{11} \leq \epsilon\|\nabla u_t\|^2_{L^2} + C\epsilon^{-1}\|\nabla \rho\|^2_{L^2} \|u\|^2_{L^\infty} \]

and

\[ I_{12} \leq \epsilon\|\nabla u_t\|^2_{L^2} + C\epsilon^{-1}\|\nabla u\|^2_{L^2}. \]

Substituting all the estimates into (2.47) and taking \(\epsilon\) small, we obtain

\[
\frac{d}{dt} \int \rho|u_t|^2 + |\nabla d_t|^2 \, dx + \int \Omega |\nabla u|^2 + |(\triangle d - f(d))t|^2 \, dx
\]

\[
\leq C(\|\sqrt{\rho}u_t\|_{L^2} A(t) + \|\nabla u\|^2_{L^2} B(t) + \|\nabla \rho\|^2_{L^2} C(t) + \|\nabla d_t\|^2_{L^2} D(t)) \quad (2.48)
\]

where

\[ A(t) = \|u\|^2_{L^\infty} + \|u\|^2_{H^1} + \|\nabla u\|_{L^\infty}, \]

\[ B(t) = \|u\|^2_{H^1} + 1, \]

\[ C(t) = \|u\|^2_{H^1} + \|u\|^2_{L^\infty}, \]

\[ D(t) = \|u\|^2_{H^1} + \|u\|_{L^\infty} + \|u\|^2_{L^\infty} + \|\triangle d\|^2_{L^\infty} + 1. \]

The estimates (2.42) and (2.43) yield

\[
\int_0^T (A(t) + B(t) + C(t) + D(t)) \, dt \leq C. \quad (2.49)
\]

Applying the Gronwall’s inequality to the inequality (2.48), we deduce

\[
\sup_{0 \leq t \leq T} (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|^2_{L^2} + \int_0^T |\nabla u|^2 \, dx + |(\triangle d - f(d))t|^2 \, dx \, dt \leq C \int_0^T (\|\nabla u\|^2_{L^2} B(t) + \|\nabla \rho\|^2_{L^2} C(t)) \, dt \exp \int_0^T (A(t) + D(t)) \, dt \]

\[ \leq C \int_0^T (\|\nabla u\|^2_{L^2} B(t) + \|\nabla \rho\|^2_{L^2} C(t)) \, dt. \quad (2.50)
\]
Combing with the estimate (2.25) and taking \( \eta \) small, we get the desired finial inequality

\[
\sup_{0 \leq t \leq T} (\| \nabla u \|_{L^2}^2 + \| \nabla \rho \|_{L^2}^2 + \| \sqrt{\rho u_t} \|_{L^2}^2 + \| \nabla d_t \|_{L^2}^2) \\
+ \int_0^T (\| \sqrt{\rho u_t} \|_{L^2}^2 + \| \nabla u_t \|_{L^2}^2 + \| (\Delta d - f(d))_t \|_{L^2}^2) dt \\
\leq C + C \int_0^T (\| \nabla u \|_{L^2}^2 B(t) + \| \nabla \rho \|_{L^2}^2 C(t)) dt.
\] (2.51)

Applying Gronwall’s inequality to the inequality (2.51) again, we deduce

\[
\sup_{0 \leq t \leq T} (\| \nabla u \|_{L^2}^2 + \| \nabla \rho \|_{L^2}^2 + \| \sqrt{\rho u_t} \|_{L^2}^2 + \| \nabla d_t \|_{L^2}^2) \\
+ \int_0^T (\| \sqrt{\rho u_t} \|_{L^2}^2 + \| \nabla u_t \|_{L^2}^2 + \| (\Delta d - f(d))_t \|_{L^2}^2) dt \\
\leq C.
\]

\[\square\]

### 2.6 Estimate for \( \| u \|_{H^2} \), \( \| d \|_{H^3} \) and \( \| \rho \|_{W^{1,6}} \)

From the estimate (2.52), (2.31) yields

\[
\| u \|_{H^2} \leq C(\| \sqrt{\rho u_t} \|_{L^2} + \| \nabla u \|_{L^2} + \| \nabla \rho \|_{L^2} + \| \nabla d_t \|_{L^2}) \leq C.
\] (2.52)

From the estimate (2.52) and the above inequality (2.52), (2.19) yields

\[
\| \nabla d \|_{H^2} \leq C(\| \nabla d_t \|_{L^2} + \| \nabla u \|_{L^2} + \| \nabla \rho \|_{L^\infty} + C) \\
\leq C(\| \nabla d_t \|_{L^2} + \| \nabla u \|_{L^2}^2 \| u \|_{H^2} + \| \nabla \rho \|_{L^2} + C) \\
\leq C.
\] (2.53)

**Lemma 5.**

\[
\sup_{0 \leq t \leq T} \| \nabla \rho \|_{L^6} + \int_0^T \| \nabla^2 u \|_{L^6}^2 dt \leq C.
\] (2.54)

**Proof.** Using the elliptic regularity result \( \| \nabla^2 u \|_{L^6} \leq C(\| \Delta u \|_{L^6}) \) and the above estimates (2.52) and (2.53) give

\[
\| \nabla^2 u \|_{L^6} \leq C(\| \rho u_t \|_{L^6} + \| \rho u \cdot \nabla u \|_{L^6} + \| \nabla \rho \|_{L^6} + \| \nabla d \|_{L^6} + \| \nabla^2 u \|_{L^6}) \\
\leq C(\| \nabla u_t \|_{L^2} + \| \nabla u \|_{L^\infty} + \| \nabla \rho \|_{L^6} + \| \nabla d \|_{L^\infty} + \| \Delta d \|_{L^6} \\
+ \| \nabla d \|_{L^6} \| f(d) \|_{L^\infty}) \\
\leq C(\| \nabla u_t \|_{L^2} + \| u \|_{H^6}^2 + \| \nabla \rho \|_{L^6} + \| d \|_{L^2}^2 + \| d \|_{H^2}^2) \\
\leq C(\| \nabla u_t \|_{L^2} + \| \nabla \rho \|_{L^6} + 1).
\] (2.55)
Taking the above inequality (2.55) into (2.10), we get
\[ \| \nabla \rho \|_{L^6} \leq (\| \rho_0 \|_{W^{1,6}} + C \int_0^t (\| \nabla u_t \|_{L^2} + \| \nabla \rho \|_{L^6} + 1) \, dt) \exp(C \int_0^t \| \nabla u \|_{L^\infty} \, dt) \] (2.56)

Using the assumption (2.1) and the estimate (2.42), and then applying Gronwall’s inequality to (2.56), we obtain
\[ \sup_{0 \leq t \leq T} \| \nabla \rho \|_{L^6} \leq C. \] (2.57)

Moreover from (2.55) and (2.57) we have
\[ \int_0^T \| \nabla^2 u \|_{L^6} \, dt \leq C. \]

From the Proposition 1, \( \| u \|_{H^2(t)}, \| \rho \|_{W^{1,6}(t)}, \| d \|_{H^3(t)}, \| \sqrt{\rho} u_t \|_{L^2(t)} \) are all continuous on time \([0, T^*]\). From the above estimates (2.4), (2.16), (2.42) and (2.52)-(2.54), we see that
\[ \left( \| \rho \|_{W^{1,6}}, \| u \|_{H^2}, \| d \|_{H^3}, \| \sqrt{\rho} u_t \|_{L^2} \right) \bigg|_{t = T^*} = \lim_{t \to T^*} \left( \| \rho \|_{W^{1,6}}, \| u \|_{H^2}, \| d \|_{H^3}, \| \sqrt{\rho} u_t \|_{L^2} \right) \]
\[ \leq C < \infty. \] (2.58)

The finite of \( \| \sqrt{\rho} u_t \|_{L^2} \big|_{t = T^*} \) means there is a compatibility condition at time \( T^* \). Hence we can take \( (\rho, u, d) \big|_{t = T^*} \) as the initial data and apply the Proposition 1 to extend our local solution beyond \( T^* \) in time which contradicts with the maximality of \( T^* \). Therefore the assumption (2.1) doesn’t hold, that is, (1.10) holds if \( T^* \) is the maximal time of the existence and \( T^* \) is finite.

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