Instanton Action for Type II Hypermultiplets

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We analyze the hypermultiplet moduli space describing the universal sector of type IIA theory compactified on a Calabi-Yau threefold. The classical moduli space is described in terms of the coset $SU(2,1)/U(2)$. The flux quantization condition of the antisymmetric tensor field of M-theory implies discrete identifications for the scalar fields describing this four-dimensional quaternionic geometry. Non-perturbative corrections of the classical theory are described in terms of a membrane-fivebrane instanton action which we construct herein.
1. Introduction

Compactifications of type IIA superstring theory on a Calabi-Yau threefold results in a four-dimensional theory with $N = 2$ supersymmetry. Supersymmetry requires that the moduli space is a product space $\mathcal{M} = \mathcal{M}_V \times \mathcal{M}_H$, where $\mathcal{M}_V$ corresponds to the moduli space of vector multiplets and $\mathcal{M}_H$ corresponds to the moduli space for hypermultiplets. The vector multiplet moduli space is described in terms of a special Kähler manifold that has been well understood for some time. The hypermultiplet moduli space is described in terms of a quaternionic geometry [1] whose quantum corrections turn out to be more difficult to understand, in part because of the complicated structure of these manifolds 1. Quantum corrections to the classical hypermultiplet geometry have been studied in some cases in the limit where gravity decouples [3] [4]. In this case the geometry is formulated in terms of hyper-Kähler geometry which turns out to be easier to understand.

Some time ago it was shown that non-perturbative corrections to the hypermultiplet geometry of type II theories can be obtained from membrane and fivebrane instantons [5]. The explicit evaluation of some of these corrections was performed in the hyper-Kähler limit in [6] and [7]. In this paper we would like to consider the quantum moduli space of hypermultiplets without decoupling gravity i.e. the full quaternionic geometry. This question is of interest since in some cases quantum corrections to the classical geometry are essentially gravitational in nature so that the hyper-Kähler limit becomes trivial.

We shall be interested in the quantum moduli space for the so-called universal sector [8] which appears in every Calabi-Yau compactification of type II theories. In [9] it was shown that the classical moduli space for the universal hypermultiplet is the coset $SU(2,1)/U(2)$. A first step to understand the quantum moduli space for hypermultiplets was done in [10] [11]. These papers evaluated perturbative corrections to the classical hypermultiplet moduli space and showed that the moduli space receives a one-loop correction proportional to the Euler number of the internal Calabi-Yau. This correction originates from the $R^4$-term of the M-theory action of [12] [13]. Furthermore in [10] there appeared a proposal for the correction to the classical metric to all orders in perturbation theory. The corrected

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1 A nice mathematical description can be found e.g. in [2].
metric is related to the classical metric by a field redefinition.

In this paper we shall be interested in the explicit form of membrane and fivebrane instanton corrections to the classical geometry. These non-perturbative corrections are encoded in a membrane-fivebrane instanton action whose explicit form we would like to compute. Such an action was calculated in [3] for the special case where the R-R background vanishes. However, in order to describe the universal sector we are interested in keeping the R-R fields coming from the eleven-dimensional three-form field strength, so that a generalization of the results appearing in [3] is required.

Some mathematical aspects of our discussion overlap with a recent paper by O. Ganor [14]. In section two we discuss the classical moduli space for the universal sector in some detail. The coset space $SU(2,1)/U(2)$ describing hypermultiplets has eight isometries which leave the classical action invariant. The explicit form of these symmetry transformations was found in [15] and [16]. We will be particularly interested in three Peccei-Quinn symmetries corresponding to shifts in the NS-NS axion and the two R-R three-form potentials. Charge quantization implies discrete identifications for these fields. These transformations form a discrete subgroup $Z$ so that the moduli space is $\mathcal{M} = Z\backslash SU(2,1)/U(2)$. This discrete subgroup of the Peccei-Quinn symmetries is preserved in the quantum theory. Associated with the above isometries are a number of conserved Noether charges. The instanton action is expressed in terms of an invariant combination of three Noether charges. In section three we construct the membrane-fivebrane instanton action explicitly.

2. Classical Moduli Space and Discrete Identifications

Let us start by considering the classical moduli space for the universal sector.

2.1. Eleven Dimensions

The bosonic part of the eleven-dimensional supergravity action is $^2$ [17]:

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-\hat{g}} \hat{R} - \frac{1}{4\kappa_{11}^2} \int \left[ \hat{F}_4 \wedge \ast \hat{F}_4 - \frac{1}{3} \hat{A}_3 \wedge \hat{F}_4 \wedge \hat{F}_4 \right],$$

(2.1)

$^2$ Our signature is $(-, +, +, \ldots, +)$. 
where $\hat{A}_3$ is the three-form potential, $\hat{F}_4 = d\hat{A}_3$ is the four-form field strength, $\kappa_{11}$ is the eleven-dimensional gravitational coupling constant and the ‘hat’ denotes eleven-dimensional quantities. The field strength obeys the Bianchi identity

$$d\hat{F}_4 = 0,$$  \hspace{1cm} (2.2)

and the field equation

$$d \ast \hat{F}_4 + \frac{1}{2} \hat{F}_4^2 = d \left( \ast \hat{F}_4 + \frac{1}{2} \hat{A}_3 \wedge \hat{F}_4 \right) = 0.$$  \hspace{1cm} (2.3)

Equations (2.2) and (2.3) give rise to two classically conserved charges (see e.g. [18] and [19]). The electric charge or ‘Page charge’

$$q_e = \frac{1}{\sqrt{2\kappa_{11}}} \int_{S^7} \left( \ast \hat{F}_4 + \frac{1}{2} \hat{A}_3 \wedge \hat{F}_4 \right),$$  \hspace{1cm} (2.4)

follows from the equation of motion for $\hat{F}_4$ and is the charge associated to the membrane. Here $S^7$ is a seven-sphere surrounding the membrane. The magnetic charge or topological charge

$$q_m = \frac{1}{\sqrt{2\kappa_{11}}} \int_{S^4} \hat{F}_4,$$  \hspace{1cm} (2.5)

is the charge associated to the fivebrane. Here $S^4$ is an asymptotic four-sphere surrounding the fivebrane. The electric and magnetic charges obey the Dirac quantization condition (see [20] and references therein)

$$q_e q_m = 2\pi \mathbb{Z}.$$  \hspace{1cm} (2.6)

These charges can be expressed in terms of the membrane and fivebrane tensions, which are functions of $\kappa_{11}$ [21] [22]. To see this we need the form of the worldvolume action for the membrane in eleven dimensions. It is given by

$$S_m = T_2 \int d^3 \xi \left[ -\frac{1}{2} \sqrt{-\gamma} \gamma^{ij} \partial_i X^M \partial_j X^N \hat{g}_{MN} + \frac{1}{2} \sqrt{-\gamma} \right.$$  \hspace{1cm} (2.7)

$$+ \frac{1}{3!} \epsilon^{ijk} \partial_i X^M \partial_j X^N \partial_k X^P \hat{A}_{MNP}(X) \right],$$

\footnote{We will be following the conventions of [20].}
where $\xi^i$ (with $i = 1, 2, 3$) are the worldvolume coordinates, $\gamma^{ij}$ is the worldvolume metric, $X^M(\xi^i)$ with $M = 0, \ldots, 10$ describes the bosonic part of the membrane configuration and $T_2$ is the membrane tension.

Demanding the membrane to have a well defined quantum theory, it follows from (2.7) that $\hat{A}_3$ has period $2\pi/T_2$. Therefore, $\hat{F}_4$ is quantized according to:

$$\int_{S^4} \hat{F}_4 = \frac{2\pi}{T_2} \mathbb{Z}. \quad (2.8)$$

The consistency of the $\hat{A}_3$ periods with the supergravity action (2.1) gives a relation between $\kappa_{11}$ and $T_2$

$$\frac{2\pi^2}{\kappa_{11} T_2^3} = \mathbb{Z}. \quad (2.9)$$

This relation has been derived in the appendix of [23].

The dual seven-form $\ast \hat{F}_4$ couples to the worldvolume of the eleven-dimensional fivebrane. Demanding the fivebrane action to describe a well defined quantum theory gives a quantization conditions for the electric charge

$$\int_{S^7} (\ast \hat{F}_4 + \frac{1}{2} \hat{A}_3 \wedge \hat{F}_4) = \frac{2\pi}{T_5} \mathbb{Z}. \quad (2.10)$$

Inserting (2.8) and (2.10) in the Dirac quantization condition (2.6) one gets:

$$2\kappa_{11}^2 T_2 T_5 = 2\pi \mathbb{Z}. \quad (2.11)$$

Using (2.11) and (2.9) one obtains a relation between membrane and fivebrane tensions which was first derived by Schwarz [24]

$$T_5 = \frac{1}{2\pi} T_2^3, \quad (2.12)$$

using duality arguments. The expressions (2.9) and (2.12) imply that membrane and fivebrane tensions can both be expressed in terms of the eleven-dimensional gravitational coupling constant $\kappa_{11}$.

Due to the charge quantization conditions (2.8) and (2.10) the moduli space of the four-dimensional theory will have a set of discrete identifications or periodicities in the dual
scalar fields as we will see later on. To finish this section we would like to remark that equation (2.3) can be interpreted as the Bianchi identity of the eleven-dimensional fivebrane. As was argued in [21] this equation will, in general, receive gravitational Chern-Simons corrections associated with the sigma-model anomaly on the six-dimensional fivebrane worldvolume. The corrected fivebrane Bianchi identity contains a term proportional to $X_8$, which is an eight-form polynomial quartic in the gravitational curvature. This will give a gravitational contribution to the quantization condition of the electric charge (2.4). We will not take this effect into account in the following. The existence of gravitational contributions to the flux quantization law for the antisymmetric tensor field of M-theory was first pointed out in [23].

2.2. Ten Dimensions

To reduce the action (2.1) to ten dimensions we make the following ansatz for the metric (see [26] and references therein)

$$ds^2_{11} = e^{-2\phi/3} g_{mn} dx^m dx^n + e^{4\phi/3} (dx^{11} - A_m dx^m)^2,$$

(2.13)

where $m, n = 1, \ldots, 10$ are ten-dimensional indices and $\phi$ is the ten-dimensional dilaton. In the following we will not take the dependence on the type IIA gauge field $A_m$ into account as this field contributes to the four-dimensional vectormultiplet moduli space. We will be using conventions where the range of the eleventh dimension is $x_{11} \rightarrow x_{11} + 2\pi \sqrt{\alpha'}$ and $2\kappa_{10}^2 = \kappa_{11}^2 / \pi \sqrt{\alpha'} = (2\pi)^7 \alpha'^4$. In order to obtain a canonical Einstein term in ten dimensions it is convenient to use the Weyl rescaling formula (valid in any dimension $d$). Under a rescaling of the metric of the form

$$\tilde{g}_{ac} = \Omega^2 g_{ac},$$

(2.14)

the scalar curvature term transforms as

$$\Omega^{2-d} \sqrt{g} \tilde{R} = \sqrt{g} \left[ R - (d-2)(d-1)g^{ac} (\partial_a \log \Omega) \partial_c \log \Omega \right].$$

(2.15)

Here and in the following we will set $g_s = 1$ for simplicity.
After reducing to $d = 10$ and Weyl rescaling the metric with a factor $\Omega^2 = e^{\phi/2}$ we obtain:

$$\sqrt{-\tilde{g}} \tilde{R} \rightarrow \sqrt{-g} \left[ R - \frac{1}{2} (\partial_m \phi)^2 \right]. \quad (2.16)$$

From the eleven-dimensional three-form we can define ten-dimensional gauge field potentials

$$B_{mn} = \hat{A}_{mn11},$$
$$A_{mnp} = \hat{A}_{mnp}, \quad (2.17)$$

and ten-dimensional field strengths $H_3 = dB_2$ and $F_4 = dA_3$. After compactification and Weyl rescaling we get

$$\hat{F}_4 \wedge \ast \hat{F}_4 \rightarrow e^{\phi/2} F_4 \wedge F_4 + e^{-\phi} H_3 \wedge \ast H_3. \quad (2.18)$$

The topological term is invariant under this rescaling and has the form

$$\hat{A}_3 \wedge \hat{F}_4 \wedge \hat{F}_4 \rightarrow B_2 \wedge F_4 \wedge F_4. \quad (2.19)$$

To summarize, the ten-dimensional supergravity action takes the form

$$S_{10} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left[ R - \frac{1}{2} (\partial_m \phi)^2 \right] - \frac{1}{4\kappa_{10}^2} \int \left[ e^{\phi/2} F_4 \wedge \ast F_4 + e^{-\phi} H_3 \wedge \ast H_3 \right] + \frac{1}{4\kappa_{10}^2} \int B_2 \wedge F_4 \wedge F_4. \quad (2.20)$$

In ten dimensions a string is dual to a fivebrane and the membrane is dual to a fourbrane. The string and the fourbrane are obtained by double dimensional reduction of the eleven-dimensional membrane and fivebrane respectively [27]. The quantization condition for the fivebrane charge follows from (2.8)

$$\int_{S^3} H_3 = \frac{2\pi}{T_1} \mathbb{Z}, \quad (2.21)$$

where $T_1 = 1/2\pi\alpha'$ is the string tension which is related to the eleven-dimensional membrane tension as $2\pi\sqrt{\alpha'} T_2 = T_1$. The charge associated with the dual string follows from (2.10) and is quantized as

$$\int_{S^7} \left( e^{-\phi} \ast H_3 + \frac{1}{2} A_3 \wedge F_4 \right) = \frac{2\pi}{T_5} \mathbb{Z}, \quad (2.22)$$
where $S^7$ is a seven-sphere surrounding the string. Using (2.12) we can express the fivebrane tension in terms of $\alpha'$ as $T_5 = (2\pi)^{-5}\alpha'^{-3}$. The fourbrane charge is quantized as
\[
\int_{S^4} F_4 = \frac{2\pi}{T_2} \mathbb{Z},
\]
where the membrane tension is $T_2 = (2\pi)^{-2}\alpha'^{-3/2}$. Finally, the dual membrane charge follows from (2.11)
\[
\int_{S^6} \left( e^{\phi/2} * F_4 + A_3 \wedge H_3 \right) = \frac{2\pi}{T_4} \mathbb{Z}.
\]

The fourbrane tension can be obtained from the eleven-dimensional fivebrane tension as $T_4 = 2\pi \sqrt{\alpha'} T_5 = (2\pi)^{-4}\alpha'^{-5/2}$.

Non-perturbative corrections to the four-dimensional universal hypermultiplet appear when the ten-dimensional type IIA membrane wraps a supersymmetric three-cycle and when the fivebrane wraps the six-dimensional Calabi-Yau manifold. The fivebrane charge (2.21) and the membrane charge (2.24) give rise to four-dimensional Noether charges that are associated to certain isometries in the four-dimensional hypermultiplet moduli space. Due to the above charge quantization conditions a discrete subgroup of the four-dimensional isometries will be preserved once instanton effects are taken into account. We will see this in more detail in the following section.

2.3. Four Dimensions

The complete action resulting from compactification of the type IIA theory on a Calabi-Yau threefold was computed in [28] [29] [30]. Here we will follow the $SU(3)$-invariant reduction of [31], which was used by Ferrara and Sabharwal [9] to construct the manifold describing the universal sector of any Calabi-Yau compactification of type II theories. The number of hypermultiplets in a four-dimensional Calabi-Yau compactification of the type IIA theory is $h_{21} + 1$, while the number of vector multiplets is $h_{11}$. In the general case, with arbitrary $h_{11}$ and $h_{21}$ the moduli space is given by a sigma model with target manifold described by the product of a Kähler manifold of complex dimension $h_{11}$ and a dual quaternionic manifold of complex dimension $2(h_{21} + 1)$. Compactifying the type IIA theory on a manifold with $h_{21} = 0$ yields only one hypermultiplet which is the so called
‘universal’ hypermultiplet \[8\]. The four bosonic fields in this multiplet are the dilaton, the
NS-NS axion and two additional scalars of R-R type. To perform the dimensional reduction
to four dimensions one should keep only the SU(3) singlets in the internal indices. We will
make the following ansatz for the metric
\[
ds_{10}^2 = e^{-\phi/2} g_{IJ} dx^I dx^J + e^{3\phi/2} g_{\mu\nu} dx^\mu dx^\nu, \quad (2.25)
\]
where \(\mu, \nu = 1, \ldots, 4\) are four-dimensional indices, \(I, J = 5, \ldots, 10\) are six-dimensional in-
dices and \(\phi = \phi(x_\mu)\). We shall be using complex coordinates to describe the six-dimensional
internal space. We will make an ansatz \(g_{i\bar{j}} = g_{\bar{j}i} = \delta_{i\bar{j}}\), with \(i, \bar{i} = 1, 2, 3\) for the internal
metric.

The expectation value of the three-form potential is parametrized in terms of a com-
plex R-R scalar \(C\) as
\[
A_{ijk} = \sqrt{2} C \epsilon_{ijk}. \quad (2.26)
\]
The remaining fields contributing to the ‘universal’ hypermultiplet are the NS-NS tensor
field \(B_{\mu\nu}\) with field strength \(H_3 = dB_2\) and the dilaton \(\phi\). In order to obtain the scalar
manifold describing the four-dimensional quaternionic geometry one has to integrate \(H_3\)
out. This can be done by adding to the four-dimensional action resulting from compacti-
fication of (2.20) a Lagrange multiplier
\[
S_4 = \frac{1}{2\kappa_4^2} \int d^4 x \sqrt{-g} \left[ R - 2(\partial_\mu \phi)^2 - 2e^{2\phi} |\partial_\mu C|^2 \right] \\
- \frac{1}{4\kappa_4^2} \int \left[ e^{-4\phi} H_3 \wedge *H_3 - 2i H_3 \wedge \tilde{C} dC - 4H_3 \wedge dD \right]. \quad (2.27)
\]
Here \(\kappa_4^2 = \kappa_{10}^2/(2\pi \sqrt{\alpha'})^6 = \pi \alpha'\) is the four-dimensional gravitational constant. We can
dualize \(H_3\) in terms of the pseudoscalar \(D\) and the complex field \(C\) as
\[
H_3 = e^{4\phi} \star \left( 2dD + i\tilde{C} dC \right). \quad (2.28)
\]
After integrating \(H_3\) out the dual action describing the scalar fields takes the form
\[
S_4 = -\frac{1}{\kappa_4^2} \int d^4 x \sqrt{-g} \left[ (\partial_\mu \phi)^2 + e^{2\phi} |\partial_\mu C|^2 + e^{4\phi} \left( \partial_\mu D + \frac{i}{2} \tilde{C} \partial_\mu C \right)^2 \right]. \quad (2.29)
\]
We can introduce a new complex field $S$

$$S = e^{-2\phi} + 2iD + C\bar{C}. \quad (2.30)$$

In terms of the complex fields $C$ and $S$ the classical Lagrangian $\mathcal{L}^{(2.29)}$ can be written as

$$\mathcal{L} = -\frac{1}{\kappa^2} \left( \mathcal{K}_{SS} \partial_\mu S \partial^\mu S + \mathcal{K}_{SC} \partial_\mu S \partial^\mu C + \mathcal{K}_{CS} \partial_\mu C \partial^\mu S + \mathcal{K}_{CC} \partial_\mu C \partial^\mu \bar{C} \right). \quad (2.31)$$

Here $\mathcal{K}$ is the Kähler potential, which has the Fubini-Study form

$$\mathcal{K} = -\log(S + \bar{S} - 2C\bar{C}). \quad (2.32)$$

Notice that $e^{2\phi} = 2e^\mathcal{K}$. The line element is explicitly

$$ds^2 = e^{2\mathcal{K}} \left( dS d\bar{S} - 2C dS d\bar{C} - 2\bar{C} d\bar{S} dC + 2(S + \bar{S}) dC d\bar{C} \right). \quad (2.33)$$

This is a quaternionic manifold of real dimension four corresponding to the coset space $SU(2,1)/U(2)$ \cite{9}. The explicit form of the eight symmetry transformations which leave the classical Lagrangian invariant was found in \cite{15} and \cite{16}. Four classical symmetries have the form

$$S \to S - \epsilon_0 S - \frac{i}{4} \epsilon_1 S^2 - \frac{1}{2} (\epsilon_3 + i\epsilon_4) CS, \quad C \to C - \frac{\epsilon_0}{2} C - \frac{i}{4} \epsilon_1 CS - \frac{1}{2} \epsilon_3 (C^2 - \frac{S}{2}) - \frac{i}{2} \epsilon_4 (C^2 + \frac{S}{2}). \quad (2.34)$$

Here the epsilons are real parameters. In particular, $\epsilon_0$ parametrizes a scale transformation. We expect that these symmetries are not preserved in the quantum theory.

The situation is rather different for the remaining four symmetries. First, a duality symmetry exchanges $ReC$ and $ImC$. We will later see that this symmetry is present when instantons are taken into account. Furthermore, there are three isometries associated with constant shifts of the NS-NS axion $D$ and the two R-R scalars $C$ and $\bar{C}$ \cite{28}

$$S \to S + i\alpha + 2(\gamma + i\beta)C + \gamma^2 + \beta^2, \quad C \to C + \gamma - i\beta, \quad (2.35)$$
under which the classical Lagrangian is invariant. Here \( \alpha, \beta \) and \( \gamma \) are real parameters. The generators of these transformations form a continuous non-Abelian group \( G \) known as the Heisenberg group. The generators of \( G \) obey the commutation relations

\[
[T_\alpha, T_\beta] = [T_\alpha, T_\gamma] = 0 \quad \text{and} \quad [T_\beta, T_\gamma] = T_\alpha.
\] (2.36)

It is expected that a discrete subgroup of these three symmetry transformations remains in the quantum theory. The reason for this is as follows. Associated with the three isometries (2.35) are three classically conserved Noether currents. These are

\[
J_\alpha = \frac{i}{\kappa_4^2} e^{2\kappa} \left( dS - d\bar{S} + 2C \bar{dC} \right),
\]

\[
J_\beta = -\frac{2i}{\kappa_4^2} e^{\kappa} \left( dC - d\bar{C} \right) + 2(C + \bar{C}) J_\alpha,
\]

\[
J_\gamma = -\frac{2}{\kappa_4^2} e^{\kappa} \left( dC + d\bar{C} \right) - 2i(C - \bar{C}) J_\alpha.
\] (2.37)

There are three corresponding conserved charges

\[
Q_{\alpha,\beta,\gamma} = \int_{\Sigma_3} * J_{\alpha,\beta,\gamma},
\] (2.38)

where \( \Sigma_3 \) is a three cycle. The charge \( Q_\alpha \) corresponds to the fivebrane charge (2.21) while \( Q_\beta \) and \( Q_\gamma \) correspond to membrane charges coming from (2.24). There are two membrane charges because there are two homology classes for three cycles.

A general membrane-fivebrane instanton in four dimensions is described by the Euclidean continuation of the three charges \((Q_\alpha, Q_\beta, Q_\gamma)\). The charge quantization conditions will imply that only a discrete subgroup of the symmetry transformations (2.35) will remain once instanton effects are taken into account. The discrete identification for the pseudoscalar \( D \) can be derived from the quantization condition for the electric charge which follows from (2.22)

\[
D \rightarrow D + \frac{n_\alpha}{2} + \ldots,
\] (2.39)

where \( n_\alpha \) is an integer. The field \( S \) has then the periodicity

\[
S \rightarrow S + in_\alpha + \ldots
\] (2.40)
The periodic identification for the R-R scalar $C$ follows from the quantization condition of the magnetic charge (2.23)

$$C \rightarrow C + n_\gamma - in_\beta,$$

where $n_\beta$ and $n_\gamma$ are both integers.

To summarize, periodicity in the fields following from R-R charge quantization conditions implies that we must identify under the action of a discrete non-abelian subgroup $Z$ of $G$

$$S \rightarrow S + in_\alpha + 2(n_\gamma + in_\beta)C + n_\gamma^2 + n_\beta^2,$$

$$C \rightarrow C + n_\gamma - in_\beta.$$

The resulting moduli space is then $\mathcal{M} = Z \backslash SU(2,1)/U(2)$.

The charges (2.38) are not single valued on the moduli space. Under the discrete identifications (2.41) and (2.40) they transform as

$$Q_\alpha \rightarrow Q_\alpha \quad Q_\beta \rightarrow Q_\beta + 4n_\gamma Q_\alpha \quad \text{and} \quad Q_\gamma \rightarrow Q_\gamma - 4n_\beta Q_\alpha.$$

This interesting non-abelian structure is due to the presence of the $\int \hat{A}_3 \wedge \hat{F}_4$ term in the eleven-dimensional electric charge (2.4). This structure will enable us to evaluate instanton effects coming from eleven-dimensional membranes and fivebranes using a rather simple argument, as we shall later see.

The existence of multi-valued charges may sound strange at first. This type of behavior was first discussed by Witten [32] for CP-violating field theories containing axions and monopoles. According to Witten a magnetic monopole in a theta vacuum becomes a dyon with an electric charge proportional to theta.\footnote{A nice discussion on nonabelian vortices can be found in [33].} In the context of string theories an analog situation was discussed by Greene, Shapere, Vafa and Yau in relation to the “stringy cosmic string” [34]. Here it was noticed that if one followed certain string states adiabatically around closed loops they would not come back as the same state. This is precisely what happens in our context. An element of $Z$, characterized by three integers $(n_\alpha, n_\beta, n_\gamma)$ can be associated to every closed loop in spacetime. These integers are not invariant under $Z$ transformations because the generators of $Z$ obey the non-abelian Heisenberg algebra.
Since a string in four dimensions is surrounded by a closed loop, an element of $Z$ is associated to every string. A fundamental string has $(n_\alpha, n_\beta, n_\gamma) = (\pm 1, 0, 0)$. This particular element of $Z$ is invariant under all $Z$ transformations so the notion of a fundamental string is globally defined. The other elements of $Z$ are carried by strings which may be described as fourbranes wrapping supersymmetric three-cycles in one of the two homology classes. When these strings are dragged around one another they pick up fundamental string charge.

Invariant charges can be defined as
\[
\hat{Q}_\alpha = Q_\alpha \quad \hat{Q}_\beta = Q_\beta - 4\zeta_0 Q_\alpha \quad \text{and} \quad \hat{Q}_\gamma = Q_\gamma - 4\tilde{\zeta}_0 Q_\alpha.
\] (2.44)

Here we have defined $C = \zeta + i\tilde{\zeta}$ and the subindex indicates the value of the field at infinity.

The classical action for the universal sector could in principle receive both perturbative and non-perturbative corrections in the quantum theory. Perturbative corrections have been discussed in [10] and [11]. These papers showed that the $R^4$-term of the M-theory action of [12] and [13] gives a one-loop correction to the hypermultiplet metric which is proportional to the Euler number of the internal Calabi-Yau
\[
\chi = 2(h_{11} - h_{21}).
\] (2.45)

Furthermore in [10] there appeared a proposal for a perturbative all-orders corrected metric which is related to the classical metric by a field redefinition. In the following we will discuss non-perturbative corrections to the classical moduli space of the universal hypermultiplet.

3. Non-Perturbative Corrections

3.1. Four-Fermi Coupling and Quaternionic Geometry

In the following we would like to compute instanton corrections to the low energy effective action described by (2.32). Recall that in the context of $N = 2, D = 4$ supergravity it was shown in [4] that the $4n$ scalars of $n$ hypermultiplets describe a quaternionic geometry with holonomy group $Sp(n)Sp(1)$ and a non-vanishing $Sp(1)$ connection. The Riemann tensor of this geometry can be written in the form
\[
R_{ijkl} = \epsilon_{CB} R_{i j k l}^{c B} + \epsilon_{B J} R_{i j C B},
\] (3.1)
where \( C, B = 1, 2 \) and \( i, j = 1, \ldots, 4n \); \( R_{ijAB} \) and \( R_{ijIJ} \) are the \( Sp(1) \) and \( Sp(n) \) curvatures respectively and \( \gamma^i_{A}, \gamma^j_{B} \) are covariantly constant functions of the \( 4n \) scalars that satisfy identities similar to those of Dirac gamma matrices. The \( Sp(1) \) connection can be written in the form

\[
R_{ijAB} = \kappa^2 \left( \gamma^i_{A} \gamma^j_{B} - \gamma^j_{A} \gamma^i_{B} \right),
\]

and the \( Sp(n) \) connection is given by

\[
R_{ijIJ} = \kappa^2 \left( \gamma^i_{A} \gamma^j_{B} - \gamma^j_{A} \gamma^i_{B} \right) + \gamma^K_{A} \gamma^I_{B} R_{IJKL},
\]

where \( R_{IJKL} \) is totally symmetric in its indices and \( \kappa^2 \) is proportional to the four-dimensional gravitational constant. In the following we will be computing non-perturbative corrections to the four-fermi coupling

\[
\int d^4 x \sqrt{g} (\bar{\chi}^I \chi^J) (\bar{\chi}^K \chi^L) R_{IJKL},
\]

where \( \chi^I \) is the fermionic component of the universal hypermultiplet. As noticed in \cite{5} this correction implies a non-perturbative correction to the classical metric on the moduli space by \( N = 2 \) supersymmetry.

### 3.2. Membrane-Fivebrane Instanton Action

There are two types of non-perturbative (instanton) corrections to the \( SU(2,1)/U(2) \) geometry described by \cite{2.32}. These corrections originate from the ten-dimensional Euclidean fivebrane wrapping the Calabi-Yau space and from the ten-dimensional Euclidean membrane wrapping a non-trivial Calabi-Yau three-cycle. The type IIA fivebrane can be described in terms of an exact conformal field theory \cite{35}. The instanton action for the fivebrane was computed in \cite{5} for the special case when the R-R background \( C \) vanishes. To describe a correction to the universal hypermultiplet we are interested in keeping the R-R scalar \( C \), so that a generalization of the result \cite{5} is in order. Let us recapitulate the main points of this calculation. Closely related are computations appearing in \cite{36} and the D-instanton calculations of \cite{37}. The basic idea used in \cite{5} to compute the fivebrane instanton action is that the type IIA fivebrane can be described as a soliton \cite{35}. Since
the internal six-dimensional geometry is unaffected by the fivebrane instanton, we used standard four-dimensional instanton methods to compute the instanton action.

The fivebrane soliton is a solution to the ten-dimensional field equations which is asymptotically flat in the four transverse directions. We will make the ansatz (2.25) for the ten-dimensional metric. The relevant terms of the Euclidean four-dimensional action are:

\[ S_4 = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{g} \left[ -R + 2(\partial_\mu \phi)^2 \right] + \frac{1}{4\kappa_4^2} \int e^{-4\phi} H_3 \wedge * H_3, \quad (3.5) \]

or in terms of the dual field \( D \) (2.28):

\[ S_4 = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{g} \left[ -R + 2(\partial_\mu \phi)^2 + 2e^{4\phi}(\partial_\mu D)^2 \right]. \quad (3.6) \]

If one considers variations of (3.5) which vanish on the boundary one obtains the Euclidean equations of motion

\[ R - 2(\partial_\mu \phi)^2 + \frac{1}{12} e^{-4\phi} H_3^2 = 0, \]
\[ \partial^2_\mu \phi + \frac{1}{12} e^{-4\phi} H_3^2 = 0. \quad (3.7) \]

The ‘neutral’ solution describing the fivebrane soliton is

\[ g_{\mu\nu} = \delta_{\mu\nu} e^{-2\phi}, \]
\[ e^{2\phi} = e^{2\phi_0} + \frac{\alpha' n_\alpha}{y^2}, \quad (3.8) \]
\[ H_{\mu\nu\rho} = 2\epsilon_{\mu\nu\rho} \chi \partial_\lambda \phi. \]

Here \( y^2 = \delta_{\mu\nu} y^\mu y^\nu \) is the distance in the four-dimensional transverse space. If we consider variations of \( D \) that do not vanish on the boundary, we have to add to the action the boundary term:

\[ -2i \oint_{\partial M} d\Sigma^\mu J^\alpha_\mu D, \quad (3.9) \]

in order to make it stationary. Here \( J^\alpha_\mu \) represents the Euclidean continuation of the current appearing in (2.37) with \( C = 0 \). This boundary term plays a similar role as the \( \theta \)-term appearing in conventional Yang-Mills theories. We have restored the \( g_s \) dependence of the instanton action by comparing with the \( C = 0 \) result of [5] where a careful analysis of the \( g_s \) dependence was performed.
Taking into account the field configuration (3.8) as well as the form of the four-dimensional action (3.5) plus the boundary term, we obtain the instanton action

\[ S_{\text{inst}} = -Q_\alpha \left( \frac{1}{g_s^2} + 2iD_0 \right) = -2\pi n_\alpha S_0, \quad (3.10) \]

where \( g_s = e^{\phi_0} \) and \( D_0 \) are the asymptotic values of the dilaton and the axion respectively. Consequently, the field \( S_0 \) is the value of the field (2.30) at infinity (again for \( C = 0 \)).

This instanton couples to the charge \( Q_\alpha \) as this charge is related to \( H_3 \) and the fivebrane is a source for \( H_3 \). From the counting of fermionic zero modes it was shown in [3] that the above instanton action gives a non-perturbative correction to the coupling of four dilatinos. By supersymmetry such a correction is related to a non-perturbative correction of the \( S - \bar{S} \) component of the classical metric on the moduli space. The \( e^{-1/g_s^2} \) dependence is typical for ordinary Yang-Mills instantons.

As we had already mentioned, a general membrane-fivebrane instanton is characterized by the three charges (2.38). The charges \( Q_\beta \) and \( Q_\gamma \) correspond to membrane charges. The exponential of the instanton action should be invariant under the discrete identifications (2.41) and (2.40). This constraint together with the result (3.10) for \( C = 0 \) uniquely fixes the action for an instanton with charges \((Q_\alpha, Q_\beta, Q_\gamma)\) as

\[ S_{\text{inst}} = -\frac{1}{g_s^2} \hat{Q}_\alpha - \frac{1}{g_s} \left( \hat{Q}_\beta + \hat{Q}_\gamma \right) - i \left( 2D_0 Q_\alpha + \frac{\zeta_0}{2} Q_\beta + \frac{\bar{\zeta}_0}{2} Q_\gamma \right), \quad (3.11) \]

As is by now well known, membrane instantons, as opposed to fivebrane instantons, give \( e^{-1/g_s} \) corrections to the classical action. This is the origin for the different \( g_s \) dependence for the bulk terms of the action which contain the membrane charges. Notice that this action is invariant under the duality symmetry which exchanges \( \zeta \) and \( \bar{\zeta} \). As argued in [3] the counting of Goldstino zero modes in the instanton background tells us that the above action contributes to a correction to the four-fermi interaction which by supersymmetry is related to a correction of the metric on the moduli space appearing in (2.33). However, we have not worked out the explicit form of the metric on the moduli space once these instanton effects have been taken into account. We hope to report on more details elsewhere.
Taking this computation as a guiding principle other more complicated quaternionic geometries could be understood along the same lines. A more complicated example of quaternionic geometry was discussed by Bodner and Cadavid [38] in the context of type IIB compactifications. Compactifying the type IIB theory on a Calabi-Yau manifold with \( h_{11} = 1 \) and \( h_{21} = 0 \) yields a four dimensional theory described in terms of two hypermultiplets. The classical moduli space of these hypermultiplets is a quaternionic manifold of real dimension eight which is the coset space \( G_{2,2}/SO(4) \). Equivalently, by mirror symmetry this manifold can be obtained by compactifying the type IIA theory on a Calabi-Yau manifold with \( h_{21} = 1 \). The instanton action describing non-perturbative corrections of this geometry can be computed with similar methods as we have done in this paper. Work in this direction is in progress [39].

4. Conclusion

The universal hypermultiplet is the first quaternionic manifold appearing in a four-dimensional Calabi-Yau compactification whose perturbative and non-perturbative corrections have been understood in some detail. These corrections cannot be evaluated in the simpler hyper-Kähler limit, as is usually done in the literature. This simple but important example may teach us how to explicitly compute the quantum moduli space of a general quaternionic geometry. Our understanding of vector multiplet and hypermultiplet moduli spaces in the context of type II compactifications would then finally be on equal footing.

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Note Added

Some mathematical aspects of this paper overlap with a recent paper by O. Ganor [14].
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