SEMI-ORTHOGONAL DECOMPOSITION OF SYMMETRIC PRODUCTS OF CURVES AND CANONICAL SYSTEM

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ABSTRACT. Let $C$ be a smooth complex projective curve of genus $g \geq 2$ and $C_d$ its $d$-fold symmetric product. In this paper, we study the question of semi-orthogonal decomposition of the derived category of $C_d$. This entails investigations of the canonical system on $C_d$, in particular its base locus.

1. Introduction

Let $C$ be a smooth complex projective curve of genus $g \geq 2$. Let $C^d = C \times \cdots \times C$ be its Cartesian product, and let $C_d$ be the $d$-fold symmetric product, meaning the quotient of $C^d$ by the action of the symmetric group. We address the question whether the bounded derived category of coherent sheaves $D(C_d) := D^b_{\text{coh}}(C_d)$ admits a non-trivial semi-orthogonal decomposition (SOD for short).

Let us briefly recall the notion of semi-orthogonal decomposition. A triangulated category $T$ admits a nontrivial SOD if there are two full non-trivial triangulated subcategories $A, B$ of $T$ such that (1) $\text{Hom}_T(b, a) = 0$ for every $b \in B$, $a \in A$ and (2) $A, B$ generate $T$. Semi-orthogonal decomposition is one of the basic notions in the theory of derived categories of coherent sheaves on algebraic varieties.

It is well-known that a semi-orthogonal decomposition of the derived category of an algebraic variety is closely related to the base locus of the canonical bundle of the variety (cf. [KO, Ok]). This motivated us to study base locus of the canonical line bundle of $C_d$.

Studying base locus of canonical bundles and semi-orthogonal decompositions of derived categories of $C_d$ was inspired by a conjecture of M. S. Narasimhan. Recently, Narasimhan proved in [Na1, Na2] that the derived category of $C$ can be embedded into the derived category of the moduli space $SU_C(2, L)$ of stable bundles of rank 2 and fixed determinant $L$ of degree 1. A similar result was obtained by Fonarev and Kuznetsov for general curves via different method (cf. [FK]). Narasimhan conjectured that the derived category of the moduli space admits a semi-orthogonal decomposition as follows:

**Conjecture 1.1.** The derived category of $SU_C(2, L)$ has the following semi-orthogonal decomposition

$$D(SU_C(2, L)) = \langle D(pt), D(pt), D(C), D(C), \cdots, D(C_{g-2}), D(C_{g-2}), D(C_{g-1}) \rangle.$$

(two copies of $D(C_i)$ for $i < g-1$ and one copy for $i = g-1$).

It turns out that there is a motivic decomposition of $SU_C(2, L)$ which is compatible with the above conjecture (cf. [Lee]). From the point of view of this conjecture

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it is of interest whether the derived categories of symmetric powers of curves can be further decomposed. Okawa proved that the derived category of a curve of genus \( g \geq 2 \) cannot have a non-trivial semi-orthogonal decomposition [Ok].

The gonality \( \text{gon}(C) \) of a curve \( C \) is the lowest degree among all nonconstant morphisms from \( C \) to the projective line \( \mathbb{P}^1 \). Equivalently, it is the lowest degree of a line bundle \( L \) on \( C \) with \( h^0(L) \geq 2 \). In this paper, we prove the following theorem.

**Theorem 1.2** (Theorem 4.3). Let \( C \) be a smooth complex projective curve of genus \( g \geq 3 \), and let \( d \) be a positive integer with \( d < \text{gon}(C) \). Then there is no non-trivial semi-orthogonal decomposition of \( D(C_d) \).

We note that, for a generic curve \( C \) of genus \( g \), the gonality satisfies \( \text{gon}(C) \leq \left\lfloor \frac{g + 3}{2} \right\rfloor \).

A stronger version when \( d = 2 \):

**Theorem 1.3** (Theorem 4.8). Let \( C \) be a smooth projective curve of genus \( g \geq 3 \). Then there is no non-trivial semi-orthogonal decomposition on \( D(C_2) \).

Note that \( D(C_2) \) admits a semi-orthogonal decomposition when \( g = 2 \). Indeed, it is easy to see that when \( d \geq g \), there is a nontrivial semi-orthogonal decomposition on \( D(C_d) \) obtained from the Albanese map. The geometry of \( C_d \) is very different when \( d < g \). We conjecture the following.

**Conjecture 1.4.** Let \( C \) be a projective smooth curve of genus \( g \geq 2 \). Then there is no non-trivial semi-orthogonal decomposition on \( D(C_d) \) for \( 1 \leq d \leq g - 1 \).

In this direction, we prove some results on the base locus of the canonical divisor \( K_{C_d} \) of \( C_d \).

**Proposition 1.5** (Proposition 3.4). Let \( 1 \leq d \leq g - 1 \). The base locus of the canonical divisor \( K_{C_d} \) is the set of points \( (x_1, \cdots, x_d) \) in \( C_d \) such that \( h^0(\mathcal{O}_C(x_1 + \cdots + x_d)) > 1 \).

Equivalently, the base locus is the set of points in \( C_d \) where the Albanese map is not injective.

We came to know of the following conjecture from the experts.

**Conjecture 1.6.** Let \( X \) be a smooth projective variety. If the canonical bundle \( K_X \) is nef and \( h^0(K_X) > 0 \), then \( X \) admits no non-trivial SOD.

Assuming Conjecture 1.6, we show that for any curve \( C \) with \( g \geq 3 \) and \( 1 < d < g \), the symmetric product \( C_d \) admits no non-trivial SOD (cf. Lemma 2.2).

### 2. Nefness of the Canonical Divisor of \( C_d \)

Let \( \Theta \) be the theta divisor on the Jacobian \( J(C) \). Fixing a point \( p \in C \), the Albanese map of the symmetric product \( C_d \) is constructed as follows

\[
u : C_d \to J(C), \quad D \mapsto \mathcal{O}_C(D - dp);
\]

we also define

\[
i : C_{d-1} \to C_d, \quad D \mapsto D + p.
\]

Let \( \theta := \nu^*\Theta \). The class of the divisor \( i(C_{d-1}) \) of \( C_d \) will be denoted by \( x \).
Lemma 2.1. The canonical class of the symmetric product $C_d$ is given by the formula

$$K_{C_d} = (g - d - 1)x + \theta.$$  \hfill (2.1)

Proof. Let $\Delta \subset C^d$ be the big diagonal where at least two points coincide. The image of $\Delta$ under the quotient map $\pi : C^d \to C_d$, for the action of the symmetric group, will be denoted by $\Delta'$, so we have the diagram

$$\Delta \quad \pi|_{\Delta} \quad C^d \quad \pi \quad \Delta' \quad \pi \quad C_d$$

Note that $\pi^*(\Delta') = 2\Delta$, and hence $\Delta$ is the ramification divisor. The divisor $\Delta'$ is divisible by 2; in fact,

$$K_{C_d} = (2g - 2)x - \Delta/2$$ \hfill [K2] Proposition 2.6]. On the other hand,

$$-\Delta/2 = \theta - (d + g + 1)x$$ \hfill [K1] Lemma 7]. The lemma follows from these two facts. \hfill $\square$

Lemma 2.2. If Conjecture 1.6 holds, then for any curve $C$ with $g \geq 3$ and $1 < d < g$, the symmetric product $C_d$ admits no non-trivial SOD.

Proof. The class $\theta$ is nef, being the pullback of an ample class, while the class $x$ is ample, hence $K_{C_d} = (g - d - 1)x + \theta$ is nef under the given conditions on $d$. Furthermore $H^0(K_{C_d}) = \bigwedge^d H^0(K_C) \neq 0$ [Ma], so all the conditions in Conjecture 1.6 are satisfied. \hfill $\square$

3. Base locus of canonical divisor of $C_d$

Let $C$ be a smooth projective curve of genus $g$. Take any positive integer $d \leq g - 1$. In this section we identify the base locus of the canonical line bundle $K_{C_d}$ of the symmetric product $C_d$. When we write a point of $C_d$ as $z = (z_1, \ldots, z_d)$, the points $z_i$ of $C$ need not be distinct. We also denote by $z$ the subscheme of $C$ defined by $\sum z_i$.

We note that

$$H^0(C_d, K_{C_d}) = \bigwedge^d H^0(C, K_C)$$ \hfill (3.1)

[Ma], and there is a canonical isomorphism for the fiber of $K_{C_d}$ over a point $z = (z_1, \ldots, z_d) \in C_d$

$$K_{C_d}|_z = \bigwedge^d H^0(K_C|_z)$$ \hfill (3.2)

For divisors $D$ and $D'$ on $C$, if

$$D' = D + D'',$$  

where $D''$ is an effective divisor, then we say that $D' - D$ is effective.

Proposition 3.1. Let $z = (z_1, \ldots, z_d) \in C_d$ be a point of the base locus of the complete linear system $|K_{C_d}| = P(H^0(C_d, K_{C_d}))$. Then the dimension of

$$H^0(C, \mathcal{O}_C(z)) = H^0(C, \mathcal{O}_C(\sum_{i=1}^d z_i))$$

is at least two.
Proof. We will first describe a subset of $P(H^0(C, K_C))$ whose linear span is entire $P(H^0(C_d, K_{C_d}))$.

Let $S \subset H^0(C, K_C)$ be a linear subspace of dimension $d$. Now define

$$D_S := \{ (y_1, \ldots, y_d) \in C_d \mid \text{div}(\omega) - \sum_{i=1}^d y_i \text{ is effective for some } \omega \in S \setminus \{0\} \}.$$  

Note that $\text{div}(\omega) - \sum_{i=1}^d y_i$ is effective if and only if $\omega$ vanishes on the subscheme of $C$ defined by $\sum_{i=1}^d y_i$.

We claim that $D_S$ is a divisor on $C_d$ linearly equivalent to $K_{C_d}$ and moreover the collection $\{D_S\}_{S \in \text{Gr}(d, H^0(C, K_C))}$ spans $H^0(C_d, K_{C_d})$.

To prove this, using (3.1) the above divisor $D_S$ corresponds to the line

$$\bigwedge^d S \subset \bigwedge^d H^0(C, K_C) = H^0(C_d, K_{C_d}).$$

The collection of all such lines with $S$ running over $\text{Gr}(d, H^0(C, K_C))$ evidently spans $\bigwedge^d H^0(C, K_C)$, proving the claim.

As in the statement of the proposition, take a point $z = (z_1, \ldots, z_d) \in C_d$ of the base locus of $P(H^0(C_d, K_{C_d}))$. Note that this means that for every linear subspace $S \subset H^0(C, K_C)$ of dimension $d$, there is a nonzero $\omega \in S$ such that $\text{div}(\omega) - \sum_{i=1}^d z_i$ is effective. We will now interpret this condition in order to be able to use it. Consider the short exact sequence of sheaves

$$0 \to K_C \otimes \mathcal{O}_C(-z) = K_C \otimes \mathcal{O}_C(-\sum_{i=1}^d z_i) \to K_C \to K_C|_z \to 0 \quad (3.3)$$

on $C$. Let

$$0 \to H^0(C, K_C \otimes \mathcal{O}_C(-z)) \xrightarrow{\beta} H^0(C, K_C) \xrightarrow{\gamma} H^0(K_C|_z) \quad (3.4)$$

be the long exact sequence of cohomologies associated to (3.3). This implies that

$$\dim \beta(H^0(C, K_C \otimes \mathcal{O}_C(-z))) \geq g - d,$$

because $\dim H^0(K_C|_z) = d$. Note that the proposition is vacuously true if $g \leq 1$.

We will show that

$$\dim \beta(H^0(C, K_C \otimes \mathcal{O}_C(-z))) \geq g - d + 1. \quad (3.5)$$

To prove this, if $\dim \beta(H^0(C, K_C \otimes \mathcal{O}_C(-z))) = g - d$, then take a subspace of dimension $d$

$$S \subset H^0(C, K_C)$$

which is complementary to the subspace $\beta(H^0(C, K_C \otimes \mathcal{O}_C(-z)))$ of $H^0(C, K_C)$. Then the restriction $\gamma|_S$, where $\gamma$ is the homomorphism in (3.4), is injective. Therefore, there is no nonzero $\omega \in S$ such that $\text{div}(\omega) - \sum_{i=1}^d z_i$ is effective, because such an element $\omega$ has to be in the kernel of $\gamma|_S$. This proves (3.5).

From (3.5) and (3.4) it follows immediately that the homomorphism $\gamma$ in (3.4) is not surjective.

Now consider the short exact sequence of sheaves

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(z) = \mathcal{O}_C(\sum_{i=1}^d z_i) \to \mathcal{O}_C(z)|_z \to 0$$
Semi-orthogonal decomposition of symmetric products of curves

Let

\[ 0 \to H^0(C, \mathcal{O}_C) \to H^0(C, \mathcal{O}_C(z)) \xrightarrow{\eta} H^0(\mathcal{O}_C(z)_x) \xrightarrow{\phi} H^1(C, \mathcal{O}_C) \]

be the corresponding long exact sequence of cohomologies. By Serre duality,

\[ H^1(C, \mathcal{O}_C) = H^0(C, \mathcal{K}_C^*) \]

Using this duality, the homomorphism \( \phi \) in (3.6) is the dual of the homomorphism \( \gamma \) in (3.4). We proved earlier that \( \gamma \) is not surjective. Consequently, \( \phi \) is not injective. Hence from (3.6) it follows that \( \dim H^0(C, \mathcal{O}_C(z)) \geq 2 \). This completes the proof.

□

In view of the above proof, the following converse of Proposition 3.1 is now rather straightforward.

Lemma 3.2. Let \( z = (z_1, \ldots, z_d) \in C_d \) be a point such that the dimension of

\[ H^0(C, \mathcal{O}_C(z)) = H^0(C, \mathcal{O}_C(\sum_{i=1}^{d} z_i)) \]

is at least two. Then \( z \) lies on the base locus of the complete linear system \( |K_{C_d}| = P(H^0(C_d, K_{C_d})) \).

Proof. Since \( \dim H^0(C, \mathcal{O}_C(z)) \geq 2 \), the homomorphism \( \eta \) in (3.6) is nonzero. Hence \( \phi \) in (3.6) is not injective. Consequently, the dual homomorphism \( \gamma \) in (3.4) is not surjective. Therefore,

\[ \dim \gamma(H^0(C, \mathcal{O}_C(z))) = \dim H^0(K_{C_d}) = d. \]

This implies that for any linear subspace \( S \subset H^0(C, K_C) \) of dimension \( d \), the restriction \( \gamma|_S \) is not injective. Now for any nonzero \( \omega \in \ker(\gamma|_S) \) the divisor \( \text{div}(\omega) - \sum_{i=1}^{d} y_i \) is effective. Consequently, \( z \) lies on the base locus of the complete linear system \( |K_{C_d}| \).

□

Let \( f : C \to \mathbb{P}^1 \) be a surjective map of degree \( d \). For any \( b \in \mathbb{P}^1 \), we have \( f^{-1}(b) \subset C_d \), where \( f^{-1}(b) \) is the scheme theoretic inverse image. Therefore, we have morphism

\[ \widehat{f} : \mathbb{P}^1 \to C_d, \quad b \mapsto f^{-1}(b). \]

Corollary 3.3. The image of the above map \( \widehat{f} \) is contained in the base locus of the complete linear system \( |K_{C_d}| \).

Proof. For any \( b \in \mathbb{P}^1 \), we have

\[ \dim H^0(C, \mathcal{O}_C(f^{-1}(b))) \geq \dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b)) = 2. \]

So Lemma 3.2 completes the proof.

□

Proposition 3.4. Let \( 1 \leq d \leq g - 1 \). The base locus of the canonical divisor \( K_{C_d} \) is the set of points \( (x_1, \ldots, x_d) \) in \( C_d \) such that \( h^0(\mathcal{O}_C(x_1 + \cdots + x_d)) > 1 \).

Equivalently, the base locus is the set of points in \( C_d \) where the Albanese map is not injective.

Proof. The first part follows from the combination of Proposition 3.1 and Lemma 3.2. The second part follows from the geometric interpretation given below. □
We will now give a geometric interpretation of the proof of Proposition 3.1. Consider the Albanese map

\[ u : C_d \rightarrow J(C) \]

Let \( z = (z_1, \ldots, z_d) \in C_d \) be a point, which can also be thought as a subscheme in \( C \). The tangent space \( T_z C_d \) of \( C_d \) at \( z \) is

\[ \text{Hom}(\mathcal{O}_C(-z), \mathcal{O}_z) = H^0(\mathcal{O}_z(z)) \]

Therefore, the differential of the Albanese map gives a linear map

\[ \phi : H^0(\mathcal{O}_z(z)) \rightarrow T_{u(z)} J(C) = H^1(C, \mathcal{O}_C) \]

It can be shown that this map \( \phi \) is identified with the homomorphism in the long exact sequence of cohomologies associated to the short exact sequence of sheaves

\[ 0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_C(z) \rightarrow \mathcal{O}_z(z) \rightarrow 0 \]

on \( C \). The dual to the map \( \phi \) is the homomorphism

\[ \gamma : H^0(K_C) \rightarrow H^0(K_C|z) \]

in the long exact sequence of cohomologies associated to the short exact sequence

\[ 0 \rightarrow K_C(-z) \rightarrow K_C \rightarrow K_C|z \rightarrow 0 \]

obtained by taking dual and tensorization with \( K_C \) of the previous short exact sequence.

Using the identifications (3.1), and (3.2) and taking the \( d \)-fold exterior product, we get that

\[ e_z : H^0(C_d, K_{C_d}) = \bigwedge^d H^0(C, K_C) \xrightarrow{d_z} \bigwedge^d H^0(K_C|z) = K_{C_d}|z. \]

The above map \( e_z \) is the evaluation map at \( z \). We note that a point \( z \in C_d \) is in the base locus of \( K_{C_d} \) if and only if \( e_z \) is zero. This is equivalent to the map \( \gamma \) being non-surjective, which in turn is equivalent to the assertion that the map \( \phi \) is not injective. We have identified the map \( \phi \) with the differential \( du_z \) of the Albanese map at \( z \). Therefore, a point \( z \in C_d \) is in the base locus of \( K_{C_d} \) if and only the Albanese map is not injective at \( z \).

4. Semi-orthogonal decompositions of \( D(C_d) \)

In this section we use the previous results on base locus of canonical bundle and the following results of Kawatani and Okawa:

**Theorem 4.1** ([KO, Corollary 1.3]). If the base locus of the canonical divisor of a smooth proper variety \( X \) is a finite set, then \( D(X) \) has no non-trivial SOD.

**Theorem 4.2** ([KO, Theorem 1.8]). Let \( S \) be a minimal smooth projective surface of general type with \( h^0(K_S) > 1 \) and satisfying the condition that for any one-dimensional connected component \( Z \subset Bs |K_S| \), its intersection matrix is negative definite. Then \( D(S) \) admits no non-trivial SOD.

**Theorem 4.3.** Let \( C \) be a smooth complex projective curve of genus \( g \geq 3 \) and let \( d \) be an integer with \( d < \text{gon}(C) \). Then there is no non-trivial semi-orthogonal decomposition of \( D(C_d) \).
Proof. Note that $h^0(\sum z_i) = 1$ for any $z = (z_1, \cdots, z_d)$, because $d < \text{gon}(C)$. So Proposition 3.1 implies that $z$ is not a base point of $K_{C_d}$, and hence the canonical divisor $K_{C_d}$ is base-point free. Now from Theorem 4.1 it follows that $D(C_d)$ has no non-trivial SOD. □

When $d = 2$ we are able to prove a stronger result which disposes of condition on the gonality of $C$. For this we start with some results about the geometry of $C_2$.

**Lemma 4.4.** The surface $C_2$ is minimal. It has an embedded rational curve if and only if $C$ is hyperelliptic, and in this case

- the rational curve is $\Gamma = \{x + \sigma(x)\}$, where $\sigma$ is the hyperelliptic involution,
- $\Gamma^2 = 1 - g$, i.e., $\Gamma$ is a $(1 - g)$-curve.

**Proof.** A rational curve in $C_2$ has to be in the fiber of the Albanese map $u : C_2 \rightarrow J(C)$, but the only positive dimensional fiber of this map is the fiber over the hyperelliptic divisor, when $C$ is hyperelliptic.

We denote the above mentioned fiber of $u$ by $\Gamma$, so $\Gamma$ is isomorphic to $\mathbb{P}^1$. We now calculate its self-intersection. The self-intersection of the diagonal $\Delta \subset C \times C$ is

$$\Delta^2 = 2 - 2g$$

(Poincaré–Hopf theorem). The automorphism of $C \times C$, which is identity on the first factor and the hyperelliptic involution on the second, sends the diagonal $\Delta$ to the graph $\tilde{\Gamma} = \{x, \sigma(x)\}$ of the hyperelliptic involution, hence also $\tilde{\Gamma}^2 = 2 - 2g$.

Consider the diagram

$$\begin{array}{ccc}
\tilde{\Gamma} & \xrightarrow{\pi} & C \times C \\
\downarrow & & \downarrow \pi \\
\Gamma & \xrightarrow{f} & C_2
\end{array}$$

The vertical arrow on the left is a 2-to-1 cover. In fact, it is isomorphic to the quotient of the curve $C$ by the hyperelliptic involution. Therefore we have $f_\ast \tilde{\Gamma} = 2\Gamma$ and $f^\ast \Gamma = \tilde{\Gamma}$ as cycles, and the projection formula for intersection gives $\Gamma^2 = 1 - g$. □

Now we prove that $C_2$ is a surface of general type.

**Lemma 4.5.** If $g \geq 3$, then the symmetric product $C_2$ is of general type.

**Proof.** In view of [Be, Proposition X.1] it suffices to show that

- the self-intersection of the canonical divisor of $C_2$ is positive, and $C_2$ is irrational surface.

From [Be, Proposition I.8], we can compute the self-intersection on $C^2$. We know that the pull back of the canonical divisor on $C_2$ to $C^2$ is $K_C \boxtimes K_C(-\Delta)$. The self-intersection of $K_C \boxtimes K_C(-\Delta)$ is

$$2(2g - 2)^2 - (2 - 2g) - 4(2g - 2) = (2g - 2)(4g - 9)$$

and it is positive when $g \geq 3$.

To prove that $C_2$ is not rational by contradiction, assume that $C_2$ is rational. Then $C_2$ can be covered by rational curves, which implies that there cannot be a nonconstant map from $C_2$ to $J(C)$. But we have Albanese map. Hence $C_2$ is not rational.
Therefore $C_2$ is of general type when $g \geq 3$.

**Remark 4.6.** When $g = 2$, we know that $C_2$ is the blow-up of the $J(C)$ at a point and hence in that case $C_2$ is not a surface of general type.

Finally we check that $p_g > 1$ for $C_2$.

**Lemma 4.7.** The canonical bundle $K_{C_2}$ has $h^0(K_{C_2}) = \binom{g}{2} > 1$ (recall $g \geq 3$).

**Proof.** Macdonald, [Ma], proves that $H^0(K_{C_2}) = \bigwedge^d H^0(K_C)$.

**Theorem 4.8.** Let $C$ be a smooth projective curve of genus $g \geq 3$. Then there is no non-trivial semi-orthogonal decomposition on $D(C_2)$.

**Proof.** If $C$ is not hyperelliptic, then the Albanese map is injective. Now Proposition 3.4 implies that $K_{C_2}$ has no base locus, and hence there is no non-trivial SOD by Theorem 4.1.

On the other hand, if $C$ is hyperelliptic, the Albanese map fails to be injective exactly on $\Gamma \subset J(C)$. Therefore, Proposition 3.4 implies that

$$\text{Bs} |K_{C_2}| = \Gamma$$

and the only connected component of the base locus is $\Gamma$, which is irreducible. Hence the intersection matrix is just $\Gamma^2 = 1 - g < 0$, so it is negative definite.

In view of Lemmas 4.4, 4.5, and 4.7, the hypothesis of Theorem 4.2 are satisfied, so there is no non-trivial SOD.

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