An Analysis on Non-Adaptive Group Testing based on Sparse Pooling Graphs

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Abstract—In this paper, an information theoretic analysis on non-adaptive group testing schemes based on sparse pooling graphs is presented. The binary status of objects to be tested are modeled by i.i.d. Bernoulli random variables with probability \( p \). An \((l, r, n)\)-regular pooling graph is a bipartite graph with left node degree \( l \) and right node degree \( r \), where \( n \) is the number of left nodes. Two scenarios are considered; one is the noiseless setting and another is the noisy setting. The main contributions of this paper are direct part theorems which give conditions for existence of an estimator achieving arbitrary small estimation error probability. The direct part theorems are proved by averaging an upper bound on estimation error probability of the typical set estimator over an \((l, r, n)\)-regular pooling graph ensemble. The numerical results obtained here indicate sharp threshold behaviors in the asymptotic regime.

1. INTRODUCTION

The paper by Dorfman [8] introduced the idea of group testing and also presented a simple analysis which indicates advantages of the idea. His main motivation was to devise an economical way to detect infected persons among a population by using blood tests. It is assumed that an outcome of a blood test tells whether the blood used in the test contains certain target viruses (or bacteria) or not.

Of course, blood tests for each person in the population clearly distinguishes infected individuals from the persons who are not infected. The Dorfman’s idea for reducing the number of tests is the following. We first divide the population into several disjoint groups and then make pools in such a way that a pool is a mixture of the bloods of persons in a group. The whole test process consists of two-stages. In the first stage, blood tests are carried out for each pool and the pools containing infected bloods are detected. In the second stage, all the individuals in the groups giving positive results are tested. Numerical examples showing certain reductions of the number of tests without losing the detection capability are shown in [8].

The Dorfman’s invention triggered emergence of subsequent theoretical works on group testing and varieties of practical applications such as DNA clone library screening, detection of faulty parts of machines [9] [10]. Recent advancements of theories of compressed sensing [8] stimulate research activities on theoretical aspects of group testing as well.

The group testing scheme due to Dorfman can be classified into the class called adaptive group testing such that a part of test are designed based on the partial results of preceding test results. Another class of the group testing is called non-adaptive group testing, in which a pool design is completely given in advance of the execution of tests. Intuitively, the adaptive group testing is natural and advantageous over the non-adaptive group testing because the number of required tests is fewer than that of non-adaptive group tests. However, the non-adaptive group testing also provides its own benefits such that all the tests can be executed in parallel. Note that the adaptive group testing requires sequential tests which leads to certain restrictions on parallel execution of the tests.

In order to develop a non-adaptive group testing scheme, pool design is crucial for achieving reasonably good detection performance. In the field of combinatorial group testing, a pooling matrix which defines the set of pools to be tested is constructed by using tools and theories from combinatorial design and combinatorics. Deterministic constructions for \( K \)-disjunct matrix are one of the central themes in the combinatorial group testing [9] [10]. For a \( K \)-disjunct matrix, there are simple and efficient reconstruction algorithms that realize correct estimation for \( K \)-sparse instances.

Another class of construction for generating a pooling matrix is random construction; i.e., \((0,1)\)-elements of a pooling matrix is probabilistically determined. Several reconstruction algorithms have been proposed for such probabilistically constructed pooling matrices. For example, Sedjino and Johnson [18], Kanamori et al. [13] recently proposed belief propagation-based reconstruction algorithms. Maloufat and Malaytov [16], Chan et al. [5] studied linear programming(LP)-based reconstruction.

From theoretical point of view, clarifying the scaling behavior of the number of required tests for correct reconstruction has been one of the most important topics in this field. Berger and Levenshtein [2] studied a two-stage group testing scheme and unveiled the scaling law for the number of required tests based on an information theoretic framework. Mézard and Toninelli [17] provided a novel analysis for two-stage schemes based on theoretical techniques from statistical mechanics. Recently, Atia and Saligrama [11] presents an information theoretic analysis for non-adaptive group testing with/without noises. They showed a direct part theorem which gives a condition for existence of an estimator achieving arbitrary small estimation error probability and a converse part theorem which gives a condition for non-existence of good estimators. Their argument for the proof of these theorems are based on the
proof of channel coding theorems for multiple access channels and it can be applied to both noiseless and noisy observations. For example, in the noiseless case, it is shown that a \( K \)-sparse instance of \( n \)-objects can be perfectly recovered from the test results if the number of tests is asymptotically \( O(K \log n) \).

The main motivation of this work is to provide another information theoretic analysis for non-adaptive group testing based on sparse pooling graphs. In this paper, we assume that status (0 or 1) of \( n \)-objects (persons) are modeled by i.i.d. Bernoulli random variables with probability \( p \). In other words, we here consider the scenario where the sparsity parameter \( K \) scales as \( K \approx np \) asymptotically. In most of conventional analysis such as [11], \( K \) is assumed to be independent of \( n \). Such assumption is reasonable to clarify the dependency of the required number of tests to the sparsity parameter and the number of objects. Although our assumption is different from the conventional one, it may be also natural in information theoretic sense and suitable for observing sharp threshold behaviors in asymptotic regime.

Another new ingredient of this work is that the analysis is carried out under the assumption of an \((l, r, n)\)-regular pooling graph ensemble which is a bipartite graph ensemble with left node degree \( l \) and right node degree \( r \), where \( n \) is the number of left nodes. This model is suitable for handling a very sparse pooling matrix and amenable for an ensemble analysis. We will present direct and converse theorems which predict the asymptotic behavior of a group testing scheme with an \((l, r, n)\)-pooling graph. The asymptotic conditions appeared in the direct and converse theorems are parameterized by \( p, l \) and \( r \). Therefore, for a given pair \((l, r)\), an achievable region for \( p \), where arbitrary accurate estimation is possible, can be clarified. Our analysis is inspired from the analysis of low-density parity-check (LDPC) codes due to Gallager and others [11] [14] [13].

II. PRELIMINARIES

In this section, we will introduce two scenarios for group testing to be discussed in this paper. The first one is the noiseless system where test results can be seen as a function of an input vector. The second one is the noisy system where an test result are disturbed by additive noises.

A. Problem setting for noiseless system

The random variable \( X \) \(\triangleq (X_1, \ldots, X_n)\) represents the status of \( n \)-objects. We assume that \( X_i (i \in [1, n]) \) is an i.i.d. Bernoulli random variable with the probability distribution \( Pr(X_i = 0) = 1 - p, Pr(X_i = 1) = p \) for \( 0 \leq p \leq 1 \). The notation \([a, b]\) represents the set of consecutive integers from \( a \) to \( b \). It might be abuse of notation but the notation \([a, b]\) is also used for representing closed interval over \( \mathbb{R} \) if there are no fear of confusion. A realization of \( X \) is denoted by \( x \) \(\triangleq (x_1, \ldots, x_n)\). The test function \( OR(z_1, \ldots, z_r) : \{0, 1\}^r \rightarrow \{0, 1\} \) is the logical OR (disjunctive) function with \( r \)-arguments \((r \text{ is a positive integer)} \) defined by

\[
OR(z_1, \ldots, z_r) \triangleq \begin{cases} 
0, & z_1 = z_2 = \cdots = z_r = 0 \\
1, & \text{otherwise}
\end{cases}
\]  

The results of pooling tests which is abbreviated as test results are represented by \( Y \) \(\triangleq (Y_1, \ldots, Y_m)\). A realization of \( Y \) is denoted by \( y \) \(\triangleq (y_1, \ldots, y_m)\).

Let \( G \) \(\triangleq (V_L, V_R, E) \) be a bipartite graph, called a pooling graph, with the following properties. The \( n \)-nodes in \( V_L \) are called left nodes and other \( m \)-nodes in \( V_R \) are called right nodes. The set \( E \) represents the set of edges. For convince, we assume that left nodes are labeled from 1 to \( n \). The left node with label \( i \in [1, n] \) (for simplicity, we call it as the left node \( i \) hereafter) corresponds to \( X_i \). In a similar manner, right nodes are labeled from 1 to \( m \). In this paper, \( G \) is assumed to be an \((l, r, n)\)-regular bipartite graph, which means that any left and right nodes have its degree \( l \) and \( r \) and that the number of the left nodes is \( n \).

For the right node \( j \in [1, m] \), the neighbor set of the node \( j \) is defined by \( M(j) \triangleq \{ i \in [1, n] \mid (i, j) \in E \} \). We are now ready to describe the relationship between \( X \) and \( Y \). For a given pooling graph \( G \), \( Y_j (j \in [1, m]) \) is related to \( X_i (i \in [1, n]) \) by \( Y_j = OR(X_i)_{i \in M(j)} \). The notation \((X_i)_{i \in M(j)}\) represents \((X_{i_1}, \ldots, X_{i_r})\) when \( M(j) = \{X_{i_1}, \ldots, X_{i_r}\} \). Namely, a pooling graph \( G \) defines a function from \( X \) to \( Y \). We will denote this relationship as \( Y = F_G(X) \) for short.

The goal of an examine to infer the realization of a hidden random variable \( X \) from the test observation \( y \) as correct as possible. Assume that the examiner uses an estimator (i.e., estimation function) \( \Phi : \{0, 1\}^m \rightarrow \{0, 1\}^n \) for the inference. The estimator gives an estimate of \( x, \hat{x} = \Phi(y) \), from the test observation \( y \). The estimator \( \Phi \) should be chosen to reduce the estimation error probability \( P_e \triangleq Pr(\Phi(F_G(X)) \neq X) \) as small as possible.

B. Problem setting for noisy system

The problem setting for the noisy system is almost same as the setting for the noiseless system described in the previous subsection. The crucial difference between the noiseless system and the noisy system is the assumption on the observation noises. In the case of the noisy system, the examiner observes a realization of random variable \( Z \) defined by

\[
Z = Y + E = F_G(X) + E,
\]

where \( E \) \(\triangleq (E_1, \ldots, E_m) \) represents an observation noise. We here simply assume that \( E_i (i \in [1, m]) \) is an i.i.d. Bernoulli random variable with the probability distribution \( Pr(E_i = 0) = 1 - q, Pr(E_i = 1) = q \) \((0 \leq q \leq 1)\).

III. CONVERSE PART ANALYSIS

In this section, lower bounds on estimation error probability for the noiseless/noisy systems will be shown. The key of the proofs are Fano’s inequality that ties estimation error probability to the conditional entropy.

A. Lower bound for noiseless system

Fano’s inequality is an inequality that relates the conditional entropy to the estimation error probability and it has been often used as the main tool for the converse part of a channel
coding theorem [7]. This inequality plays also a crucial role in
the following analysis in order to clarify the limit of accurate estima-
tion for the noiseless and noisy systems.

Lemma 1 (Fano’s inequality): Assume that random variables
A, B are given. The cardinality of the domains (alphabets) of
A and B are assumed to be finite. For any estimator \(\phi\) for
estimating the hidden value of A from the observation of B,
the inequality \(1 + P( A \neq \phi(B)) \log_2 |A| \geq H(A|B)\) holds
where the domain of A is denoted by \(A\).

We here use the Fano’s inequality for deriving a lower
bound on the error probability of an estimation for the noise-
less system. Note that this lower bound does not depend on
the choice of a pooling graph and an estimator. The proof of
the theorem is resemble to the proof of the upper bound on
code rate for LDPC codes [11] [4]. Similar argument can be
found in [11], [8] as well.

Theorem 1 (Noiseless system): Assume the noiseless system.
For any pair of an \((l, r, n)\)-pooling graph and an estimator, the
error probability \(P_e\) is bounded from below by

\[
h(p) - \frac{l}{r} h((1-p)^r) - \frac{1}{n} \leq P_e. \tag{3}
\]

(Proof) For any estimator having the error probability \(P_e\), we have

\[
H(X) = I(X; Y) + H(X|Y) \leq I(X; Y) + 1 + P_e \log_2 |X| \tag{4}
\]

\[
= H(Y) - H(Y|X) + 1 + P_e n \tag{5}
\]

\[
= H(Y) + 1 + P_e n. \tag{6}
\]

The inequality (4) holds since \(X = \{0,1\}^n\). Note that, in the noiseless system,
the random variable Y is a function of X, namely \(Y = F_G(X)\)
and that it implies \(H(Y|X) = 0\). The last equality (6) is the
consequence of \(H(Y|X) = 0\).

Since we have assumed that \(X = (X_1, \ldots, X_n)\) is an n-
tuple of i.i.d. Bernoulli random variables, the entropy of X
is given by \(H(X) = n h(p)\) where \(h(p)\) is the binary entropy
function defined by \(h(p) = -p \log_2 p - (1-p) \log_2 (1-p)\).
We thus have

\[
h(p) \leq H(Y) + 1 + P_e n. \tag{7}
\]

For further evaluation, we require to evaluate \(H(Y) =
H(Y_1, \ldots, Y_m)\). It should be remarked that the random vari-
bles \(Y_1, Y_2, \ldots, Y_n\) are binary random variables and they
are correlated in general. A simple upper bound on \(H(Y)\) can
be obtained as \(H(Y_1, Y_2, \ldots, Y_m) \leq \sum_{i=1}^m H(Y_i)\). This is
simply due to the chain rule and property of the conditional
probabilities (i.e., conditioning reduces entropy [17]). From our
assumptions that \(Y_i = OR(X_1)_{i \in [1,m]}\) and that
\(|\mathcal{M}(j)| = r(j \in [1, m])\), we have \(H(Y_j) = h((1-p)^r)\)
because \(Pr[Y_j = 0] = (1-p)^r\). Combining these results,
we obtain an inequality \(nh(p) \leq mh((1-p)^r) + 1 + P_e n.\)
From this inequality, we immediately obtain the claim of this
theorem.

B. Lower bound for noisy system

Let us recall the problem setup for the noisy system. The
random variable \(Z \triangleq (Z_1, \ldots, Z_m)\) representing a noisy
observation is defined by \(Z = Y + E = F_G(X) + E\). As
in the case of the noiseless system, a lower bound on the
error probability for the noisy system can be derived based
on Fano’s inequality.

Theorem 2 (Noisy system): Assume the noiseless system.
For any pair of an \((l, r, n)\)-pooling graph and an estimator,
the error probability \(P_e\) is bounded from below by

\[
h(p) + \frac{l}{r} h(q) - \frac{l}{r} h((1-p)^r(1-q) + (1-(1-p)^r)q) - \frac{1}{n} \leq P_e. \tag{8}
\]

IV. DIRECT PART ANALYSIS

In the previous section, we have discussed the limitation of
accurate estimation of any estimator; i.e., a lower bound on
error probability. This result is similar to the converse part
of a coding theorem. In this section, we shall discuss a direct part;
i.e., existence a sequence of estimators achieving arbitrary
small error probability. As in the case of coding theorems, we
here rely on the standard bin coding argument [7] to prove the
main theorems. In order to apply such an information theoretic
argument, we here introduce a novel class of estimators which
is called a typical set estimator.

A. Pooling graph ensemble

In the following analysis, we will take ensemble average
of the error probability of the typical set estimator over an
ensemble of pooling graphs. The pooling graph ensemble
introduced below is resembled to the bipartite graph ensemble
for regular LDPC codes. The following definition gives the
details of the pooling graph ensemble [14].

Definition 1 (Pooling graph ensemble): Let \(G_{l,r,n}\) be the
set of all \((l, r, n)\)-regular bipartite graphs with \(n\)-left and \(m =
(l/r)n\)-right nodes. The cardinality of \(G_{l,r,n}\) is \((nl)!\). Assume
that equal probability \(P(G) = 1/(nl)!\) is assigned for each
graph \(G \in G_{l,r,n}\). The probability space defined based on
the pair \((G_{l,r,n}, P)\) is referred to as the \((l, r, n)\)-pooling graph
ensemble.

In order to prove the direct part theorems, we need to
evaluate the expectation of the number of typical sequences
\(x\) satisfying \(y = F_G(x)\) over the \((l, r, n)\)-pooling graph
ensemble. The next lemma is required for deriving the main
theorems.

Lemma 2: Assume that \(s \in [0, n]\) and \(w \in [0, n]\) are given.
Let \(y_s \in \{0, 1\}^m\) be a binary \(m\)-tuple with weight \(s\)
and \(x_w \in \{0, 1\}^n\) be a binary \(n\)-tuple with weight \(w\). The probability of
the event \(y_s = F_G(x_w)\) is given by

\[
E [\|y_s = F_G(x_w)\|] = \frac{1}{\binom{m}{w}} \text{Coeff}([(1+z) - 1^w, z^w], \tag{9}
\]

where \(\text{Coeff}([g(z), z^w])\) represents the coefficient of \(z^w\) in the
polynomial \(g(z)\). The function \(\|\text{cond}\|\) is the indicator function
taking value 1 if \(\text{cond}\) is true; otherwise it gives value 0.
The combinatorial argument presented in the proof of Lemma 2 is closely related to the derivation of an average input-output weight distribution of LDPC codes over a regular bipartite graph ensemble due to Hsu and Anastasopoulos [12].

B. Analysis on error probability for noiseless system

In this subsection, we will define the typical set estimator for the noiseless system and give an error performance analysis. Before describing the typical set estimator, we introduce the definition of the typical set [7] as follows.

Definition 2 (Typical set): Assume that an i.i.d. random variables $A_i$ ($i \in [1, n]$), a positive constant $\epsilon$ and a positive integer $n$ are given. The typical set $T_{n, \epsilon}$ is defined by

$$T_{n, \epsilon} \triangleq \{ (a_1, \ldots, a_n) \in A^n : |H(A) - \kappa(a_1, \ldots, a_n)| \leq \epsilon \},$$

where $A$ is the finite alphabet of $A_i$ and $H(A) = H(A_i)$ holds for $i \in [1, n]$. The function $\kappa$ is defined by $\kappa(a_1, \ldots, a_n) \triangleq (-1/n) \log_2 Pr(a_1, \ldots, a_n)$.

The typical set estimator defined below is almost same as the typical set decoder assumed in a proof of several coding theorems such as [13]. It is exploited for a simpler proof and it is, in general, computationally infeasible for carrying out. Despite of its computational complexity, the performance the typical set estimator can be used as a benchmark of other estimation algorithms.

Definition 3 (Typical set estimator): Assume the noiseless system. Suppose that an $(l, r, n)$-pooling graph $G \in G_{l, r, n}$ and a positive real value $\epsilon$ are given. The typical set estimator $\Phi: \{0, 1\}^m \rightarrow \{0, 1\}^n \cup \{E\}$ is defined by

$$\Phi(y) \triangleq \begin{cases} x \in D(y), & \text{if } |D(y)| = 1, \\ E, & \text{otherwise}, \end{cases}$$

where $D(y)$ ($y \in \{0, 1\}^m$) is the decision set defined by $D(y) \triangleq \{ x \in T_{n, \epsilon} : y = F_G(x) \}$. The symbol $E$ represents failure of estimation.

The typical set estimator $\Phi$ depends on the bins defined on the typical set $T_{n, \epsilon}$. A bin $D(y)$ consists of the inverse image of $y$ in the typical set. For an observed vector $y$, if the cardinality of the bin $D(y)$ is 1, the estimator declare that $x \in D(y)$ has occurred. The failure of estimation occurs when the cardinality of $D(y)$ is greater than 1. For evaluating the error probability of the typical set estimator, an analysis for this event is indispensable and it will be the main topic of the following analysis.

The next lemma provides the existence of a pair $(G, \Psi)$ achieving a given upper bound on the error probability, which can be regarded as a counter part of the coding theorem.

Lemma 3: Assume the noiseless system. For any given $\gamma > 0$, if

$$-(l-1)h(p) + \max_{\sigma \in [0,l/r]} \left[ \log_2 \inf_{z>0} \frac{(1+z)^e-1}{z^p} \right] + \gamma < 0,$$

holds, then there exists a pair $(G \in G_{l, n, \epsilon}, \Phi)$ with the error probability smaller than $\gamma$.

(Proof) The proof is based on the bin-coding argument. Assume that a positive real number $\epsilon$ is given (later, we will see that $\epsilon$ is determined according to $\gamma$ but, for a while, we consider that $\epsilon$ is a given parameter). Note that there are two events that the typical set estimator fails to estimate correctly. The event in which a realization of $X, x$, is not a typical sequence is denoted by Event I. Another event, Event II, corresponds to the case where a realization $x$ is a typical sequence but $|D(F_G(x))| > 1$ holds.

We therefore have

$$P_e = P_{II}(X) \Phi(F_G(X)) = P_I + P_{II}(G),$$

(13)

where $P_I, P_{II}(G)$ are probabilities corresponding to Event I and II, respectively. Note that the probability $P_I$ depends only on the parameters $n$ and $\epsilon$.

We first consider the probability $P_{II}(G)$, which can be upper bounded as follows:

$$P_{II}(G) \leq \sum_{x \in T_{n, \epsilon}} \sum_{x' \in T_{n, \epsilon}, x' \neq x} Pr(x) [D(F_G(x')) = D(F_G(x))].$$

(14)

By taking the expectation of (14) over the $(l, r, n)$-pooling graph ensemble, we obtain

$$E[P_{II}(G)] \leq \sum_{x \in T_{n, \epsilon}} \sum_{x' \in T_{n, \epsilon}, x' \neq x} E[|D(F_G(x')) = D(F_G(x))|].$$

(15)

where $w_{min}$ and $w_{max}$ are defined by

$$w_{max} = \max_{x \in T_{n, \epsilon}} wt(x), \quad w_{min} = \min_{x \in T_{n, \epsilon}} wt(x),$$

(16)

where $wt(x)$ represents the Hamming weight of $x$. The vector $y_s$ is an arbitrary binary $m$-tuple with weight $s$ and $x_w$ is an arbitrary binary $n$-tuple with weight $w$.

Applying the upper bound on the size of the typical set and Lemma 2 to (15), we have

$$E[P_{II}(G)] \leq 2^{n(h(p) + \epsilon)} \max_{s \in [0, m]} \max_{w \in [w_{min}, w_{max}]} T(n, l, r, w, s),$$

(17)

where $T(n, l, r, w, s) \triangleq \frac{1}{(n/w)} \text{Coeff}[(1+z)^e-1]$, $z^w$. By letting $\omega \triangleq w/n$ and $\sigma \triangleq s/n$, the above inequality can be rewritten as $E[P_{II}(G)] \leq 2^{n(h(p) + \epsilon + Q)}$ where

$$Q \triangleq \frac{1}{n} \log_2 \max_{s \in [0, m]} \max_{w \in [w_{min}, w_{max}]} T(n, l, r, w, s).$$

(18)

For evaluating the coefficient of the generating function, the theorem by Burshtein and Miller [11] can be exploited and $Q$ can be expressed as

$$Q = -h(p) + \max_{\sigma \in [0, l/r]} \left[ \log_2 \inf_{z>0} \frac{(1+z)^e-1}{z^p} \right] + \delta(n) + \xi(\epsilon),$$

(18)
where $\zeta(\epsilon)$ is a function of $\epsilon$ such that $\zeta(\epsilon) \to 0$ as $\epsilon \to 0$ and the function $\delta(n)$ is a function such that $\delta(n) \to 0$ as $n \to \infty$. Assume that a positive real number $\gamma$ is given and

$$-(l-1)h(p) + \max_{\sigma \in \{0,1\}} \left[ \log_2 \inf_{z>0} \frac{((1+z)^r-1)^{\sigma}}{2^p}\right] + \gamma < 0$$

holds. For sufficiently large $n$ and sufficiently small $\epsilon$, there exists a pair $(n, \epsilon)$ satisfying $\epsilon + \delta(n) + \zeta(\epsilon) < \gamma$ and the following two conditions. The first condition is that $E[P_{11}(G)] < \frac{\gamma}{2}$. Note that, due to the assumption (19), the exponential growth rate of the upper bound on $E[P_{11}(G)]$ is negative and thus the upper bound on $E[P_{11}(G)]$ can be arbitrary small as $n \to \infty$. The second condition is that $P_1 < \gamma/2$ which is guaranteed by the asymptotic equipartition property (AEP) for the typical set $\mathcal{T}$. As a result, we have $E[P_{1}^*] = P_1 + E[P_{11}(G)] < \gamma$ and this implies the existence of a pair $(G \in G_{l,n,r}, \Phi)$ with the error probability smaller than $\gamma$. 

From this lemma, we can immediately derive the following direct part result.

**Theorem 3 (Achievability for noiseless system):** Assume the noiseless system. For any given $\gamma > 0$, if

$$-(l-1)h(p) - lp \log_2(2^{1/r} - 1) + \gamma < 0$$

holds, then there exists a pair $(G \in G_{l,n,r}, \Phi)$ with the error probability smaller than $\gamma$. 

From Theorem 2 and Theorem 3, it is natural to conjecture the existence of the threshold value $p^*(l,r)$ partitioning the range of $p$ into two regions. Namely, if $p < p^*(l,r)$, arbitrary accurate estimation is possible. Otherwise, i.e., $p > p^*(l,r)$, no estimator achieving arbitrary small error probability exists in the asymptotic limit $n \to \infty$. An upper bound on the threshold can be obtained from Theorem 2. The upper bound $p^*_L(l,r)$ is given by $p^*_L(l,r) \geq \inf \left\{ p \mid p \text{ satisfies } h(p) - (l-1)h((1-p)^r) > 0 \right\}$. On the other hand, a lower bound on the threshold is defined by $p^*_L(l,r) \geq \sup \left\{ p \mid p \text{ satisfies } -(l-1)h(p) - lp \log_2(2^{1/r} - 1) < 0 \right\}$, which is a direct consequence of Theorem 3. Table 1 presents the values of the lower and upper bounds on the threshold for the two cases $1/r = 1/2$ and $1/r = 1/4$.

### C. Analysis on error probability for noisy system

As in the case of the noiseless system, we can derive a direct part theorem for the noisy system shown below.

**Theorem 4 (Achievability for noisy system):** Assume the noisy system. For any given $\gamma > 0$, if $-(l-1)h(p) + \frac{1}{p}h(q) - lp \log_2(2^{1/r} - 1) + \gamma < 0$ holds, then there exists a pair $(G \in G_{l,n,r}, \Phi)$ with the error probability smaller than $\gamma$.

### V. Conclusion

There are strong similarity between group testing schemes and linear error correction schemes for binary symmetric channels. The analysis presented in the paper is inspired from the theoretical works on LDPC codes [13] [4]. From numerical evaluation, it is shown that the gap between the upper bound $p^*_L(l,r)$ and lower bound $p^*_L(l,r)$ is usually quite small. It suggests the existence of the sharp threshold which is similar to the Shannon limit for a channel coding problem.

### REFERENCES

[1] G. Atia and V. Saligrama, “Boolean compressed sensing and noisy group testing,” IEEE Trans. on Information Theory, vol.58 (3), pp. 1800 – 1901, 2012.

[2] T. Berger and V.I. Levenshtein, “Asymptotic efficiency of two-stage decision testing,” IEEE Trans. on Information Theory, vol.48 (7), pp. 1741 – 1749, 2002.

[3] D. Burshtein and G. Miller, “Asymptotic enumeration methods for analyzing LDPC codes,” IEEE Trans. on Information Theory, vol.50 (6), pp. 1115 – 1133, 2004.

[4] D. Burshtein, M. Krivelevich, S. Litsyn and G. Miller, “Upper bounds on the rate of LDPC codes,” IEEE Transactions on Information Theory, vol.48 (9), pp. 2437-2449, 2002.

[5] C.L. Chan, S. Jaggi, V. Saligrama, and A. Aghnihat, “Non-adaptive group testing: explicit bounds and novel algorithms,” arXiv:1202.0206v4 [cs.IT], 2012.

[6] E. Candes, J. Romberg and T. Tao, “Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information,” IEEE Trans. on Information Theory, vol.52 (2), pp. 489 – 509, 2006.

[7] T.M. Cover and J.A. Thomas, “Elements of Information Theory,” 2nd ed. Wiley-Interscience 2006.

[8] R. Dorfman, “The detection of defective members of large populations,” Ann. Statist., vol. 14, pp.436-440, 1943.

[9] D.-Z Du and F.K. Hwang, “Combinaitorial Group Testing and Its Applications,” 2nd ed. World Scientific Publishing Company, 2000.

[10] D.-Z Du and F.K. Hwang, “Pooling Designs and Non-Adaptive Group Testing: Important Tools for DNA Sequencing,” Singapore: World Scientific, 2002.

[11] R.G. Gallager, “Low Density Parity Check Codes,” Cambridge, MA-MIT Press 1963.

[12] J.H. Hsu and A. Anastasopoulos, “Capacity-achieving codes with bounded graphical complexity and maximum likelihood decoding,” IEEE Trans. Inform. Theory, pp.992-1006, vol.56, no. 3, Mar. 2010.

[13] T. Kanamori, H. Uehara, M. Jimbo, “Pooling design and bias correction in DNA library screening,” Journal of Statistical Theory and Practice, vol. 6, issue 1, pp. 220-238, 2012.

[14] S. Litsyn and V. Shevelev, “On ensembles of low-density parity-check codes: asymptotic distance distributions,” IEEE Trans. Inform. Theory, vol. IT-48, pp.887–908, Apr. 2002.

[15] D.J.C. MacKay, “Good error-correcting codes based on very sparse matrices,” IEEE Trans. Inform. Theory, vol. IT-45 (2), pp.399-431, 1999.

[16] D. Malioutov and M. Malyutov, “Boolean compressed sensing: LP relaxation for group testing,” IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), 2012.

[17] M. Mézard and C. Toninelli, “Group teasing with random pools: optimal two-stage algorithms,” IEEE Trans. Inform. Theory, vol. IT-57 (3), pp.1736–1745, 2011.

[18] D. Sejdinovic and O. Johnson, “Note on noisy group testing: asymptotic bounds and belief propagation reconstruction,” arXiv:1010.2441v1 [cs.IT], 2010.