Strong subgraph $k$-connectivity bounds

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Abstract

Let $D = (V, A)$ be a digraph of order $n$, $S$ a subset of $V$ of size $k$ and $2 \leq k \leq n$. Strong subgraphs $D_1, \ldots, D_p$ containing $S$ are said to be internally disjoint if $V(D_i) \cap V(D_j) = S$ and $A(D_i) \cap A(D_j) = \emptyset$ for all $1 \leq i < j \leq p$. Let $\kappa_S(D)$ be the maximum number of internally disjoint strong digraphs containing $S$ in $D$. The strong subgraph $k$-connectivity is defined as

$$\kappa_k(D) = \min \{ \kappa_S(D) \mid S \subseteq V, |S| = k \}.$$ 

A digraph $D = (V, A)$ is called minimally strong subgraph $(k, \ell)$-connected if $\kappa_k(D) \geq \ell$ but for any arc $e \in A$, $\kappa_k(D - e) \leq \ell - 1$. In this paper, we first give a sharp upper bound for the parameter $\kappa_k(D)$ and then study the minimally strong subgraph $(k, \ell)$-connected digraphs.

1 Introduction

The generalized $k$-connectivity $\kappa_k(G)$ of a graph $G = (V, E)$ was introduced by Hager [3] in 1985 ($2 \leq k \leq |V|$). For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, an $S$-Steiner tree or, simply, an $S$-tree is a subgraph $T$ of $G$ which is a tree with $S \subseteq V(T)$. Two $S$-trees $T_1$ and $T_2$ are said to be internally disjoint if $E(T_1) \cap E(T_2) = \emptyset$ and $V(T_1) \cap V(T_2) = S$. The generalized local connectivity $\kappa_S(G)$ is the maximum number of internally disjoint $S$-trees in $G$. For an integer $k$ with $2 \leq k \leq n$, the generalized $k$-connectivity is defined as

$$\kappa_k(G) = \min \{ \kappa_S(G) \mid S \subseteq V(G), |S| = k \}.$$ 

Observe that $\kappa_2(G) = \kappa(G)$. If $G$ is disconnected and vertices of $S$ are placed in different connectivity components, we have $\kappa_S(G) = 0$. Thus, $\kappa_k(G) = 0$ for a disconnected graph $G$. Generalized connectivity of graphs has become an established area in graph theory, see a recent monograph [7] by Li and Mao on generalized connectivity of undirected graphs.

To extend generalized $k$-connectivity to directed graphs, Sun, Gutin, Yeo and Zhang [8] observed that in the definition of $\kappa_S(G)$, one can replace “an
Let $F$ hold only for $k$ every edge. The reason is given in the next paragraph. For a digraph $D$ of order $n$, let $\kappa_k(D)$ denote the digraph obtained from $G$ by replacing every edge $xy$ with two arcs $xy$ and $yx$. While minimal connected spanning subgraphs of undirected graphs are all trees, even a simple digraph $C_n$ has two types of such strong subgraphs: a directed cycle and a path $P_n$. A less trivial reason is given in the next paragraph.

The main aim of [8] was to study complexity of computing $\kappa_k(D)$ for an arbitrary digraph $D$, for a semicomplete digraph $D$, and for a symmetric digraph $D$. In particular, Sun et al. proved that for all fixed integers $k \geq 2$ and $\ell \geq 2$ it is NP-complete to decide whether $\kappa_S(D) \geq \ell$ for an arbitrary digraph $D$ and a vertex set $S$ of $D$ of size $k$. Since deciding the same problem for generalized $k$-connectivity of undirected graphs is polynomial time solvable [5], it is clear that computing strong subgraph $k$-connectivity is somewhat harder than computing generalized $k$-connectivity.

We will postpone discussion of further results from [8] until Subsection 1.4 and now overview new results obtained in this paper. First, we improve the following tight bound used in [8]

$$
\kappa_k(D) \leq \min\{\delta^-(D), \delta^+(D)\}
$$

(1)

for a digraph $D$, where $\delta^-(D)$ and $\delta^+(D)$ are the minimum in-degree and out-degree of $D$, respectively. We will show a new sharp bound $\kappa_k(D) \leq \kappa(D)$, where $\kappa(D)$ is the strong connectivity of $D$. Note that $\kappa(D) \leq \min\{\delta^-(D), \delta^+(D)\}$. Interestingly, for undirected graphs $G$, $\kappa_k(G) \leq \kappa(G)$ holds only for $k \leq 6$ [4,6].

In what follows, $n$ will denote the number of vertices of the digraph under consideration.

A digraph $D = (V(D), A(D))$ is called minimally strong subgraph $(k, \ell)$-connected if $\kappa_k(D) \geq \ell$ but for any arc $e \in A(D)$, $\kappa_k(D - e) \leq \ell - 1$. Let $\mathcal{F}(n, k, \ell)$ be the set of all minimally strong subgraph $(k, \ell)$-connected digraphs with order $n$. We define

$$
F(n, k, \ell) = \max\{|A(D)| \mid D \in \mathcal{F}(n, k, \ell)\}
$$

and

$$
f(n, k, \ell) = \min\{|A(D)| \mid D \in \mathcal{F}(n, k, \ell)\}.
$$
We further define
\[
Ex(n, k, \ell) = \{ D \mid D \in F(n, k, \ell), |A(D)| = F(n, k, \ell) \}
\]
and
\[
ex(n, k, \ell) = \{ D \mid D \in F(n, k, \ell), |A(D)| = f(n, k, \ell) \}.
\]

Using the Hamilton cycle decomposition theorem of Tillson [10], Theorem 3.1, it is not hard to see \( f(n, k, n-1) = F(n, k, n-1) = n(n-1) \) and that the only extremal digraph is the complete digraph on \( n \) vertices. However, computing \( f(n, k, n-2) \) and \( F(n, k, n-2) \) appears to be harder. In Theorem 3.5, we characterize minimally strong subgraph \((2, n-2)\)-connected digraphs. The characterization implies that \( f(n, 2, n-2) = n(n-1) - 2\lfloor n/2 \rfloor \), \( F(n, 2, n-2) = n(n-1) - 3 \). We will also prove the lower bound \( f(n, k, \ell) \geq n\ell \) and describe some cases when \( f(n, k, \ell) = n\ell \). Finally, we will show that \( F(n, n, \ell) \leq 2\ell(n-1) \) and \( F(n, k, 1) = 2(n-1) \). We leave it as an open problem to obtain a sharp upper bound on \( F(n, k, \ell) \) for every \( k \geq 2 \) and \( \ell \geq 2 \).

1.1 Algorithms and Complexity Results

Let \( k \geq 2 \) and \( \ell \geq 2 \) be fixed integers. By reduction from the Directed 2-Linkage problem, Sun et al. [8] proved that deciding whether \( \kappa_S(D) \geq \ell \) is NP-complete for a \( k \)-subset \( S \) of \( V(D) \). Thomassen [9] showed that for every positive integer \( p \) there are digraphs which are strongly \( p \)-connected, but which contain a pair of vertices not belonging to the same cycle. This implies that for every positive integer \( p \) there are digraphs \( D \) such that \( \kappa_2(D) = 1 \) [8].

The above negative results motivate studying strong subgraph \( k \)-connectivity for special classes of digraphs. In [8], Sun et al. showed that the problem of deciding whether \( \kappa_k(D) \geq \ell \) for every semicomplete digraphs is polynomial-time solvable for fixed \( k \) and \( \ell \). The main tool used in their proof is a recent Directed \( k \)-Linkage theorem of Chudnovsky, Scott and Seymour [2].

A digraph \( D \) is symmetric if for every arc \( xy \) of \( D \), \( D \) also contains the arc \( yx \). In other words, a symmetric digraph \( D \) can be obtained from its underlying undirected graph \( G \) by replacing each edge of \( G \) with the corresponding arcs of both directions, that is, \( D = \vec{G} \). Sun et al. [8] showed that for any connected graph \( G \), the parameter \( \kappa_2(\vec{G}) \) can be computed in polynomial time. This result is best possible in the following sense, unless \( P=NP \). Let \( D \) be a symmetric digraph and \( k \geq 3 \) a fixed integer. Then it is NP-complete to decide whether \( \kappa_S(D) \geq \ell \) for \( S \subseteq V(D) \) with \( |S| = k \) [8].

2 New sharp upper bound of \( \kappa_k(D) \)

To prove a new bound on \( \kappa_k(D) \) in Theorem 2.2, we will use the following proposition of Sun et al. [8].

**Proposition 2.1** Let \( 2 \leq k \leq n \). For a strong digraph \( D \) of order \( n \), we have
\[
1 \leq \kappa_k(D) \leq n - 1.
\]
Moreover, both bounds are sharp, and the upper bound holds if and only if 
\(D \equiv \overrightarrow{K}_n, 2 \leq k \leq n\) and \(k \not\in \{4, 6\}\).

**Theorem 2.2** For \(k \in \{2, \ldots, n\}\) and \(n \geq \kappa(D) + k\), we have

\[
\kappa_k(D) \leq \kappa(D).
\]

Moreover, the bound is sharp.

**Proof:** For \(k = 2\), assume that \(\kappa(D) = \kappa(x, y)\) for some \(\{x, y\} \subseteq V(D)\). It follows from the strong subgraph connectivity definition that \(\kappa(x, y) \leq \kappa(x, y)\), so \(\kappa(D) \leq \kappa(x, y) \leq \kappa(D)\).

We now consider the case of \(k \geq 3\). If \(\kappa(D) = n - 1\), then we have \(\kappa_k(D) \leq n - 1 = \kappa(D)\) by Proposition 2.1. If \(\kappa(D) = n - 2\), then there two vertices, say \(u\) and \(v\), such that \(uv \not\in A(D)\). So we have \(\kappa_k(D) \leq n - 2 = \kappa(D)\) by Proposition 2.1. If \(1 \leq \kappa(D) \leq n - 3\), then there exists a \(\kappa(D)\)-vertex cut, say \(Q\), for two vertices \(u, v\) in \(D\) such that there is no \(u - v\) path in \(D - Q\). Let \(S = \{u, v\} \cup S'\) where \(S' \subseteq V(D) \setminus (Q \cup \{u, v\})\) and \(|S'| = k - 2\). Since \(u\) and \(v\) are in different strong components of \(D - Q\), any strong subgraph containing \(S\) in \(D\) must contain a vertex in \(Q\). By the definition of \(\kappa_S(D)\) and \(\kappa_k(D)\), we have \(\kappa_k(D) \leq \kappa_S(D) \leq |Q| = \kappa(D)\).

For the sharpness of the bound, consider the following digraph \(D\). Let \(D\) be a symmetric digraph whose underlying undirected graph is \(K_k \cup \overrightarrow{K}_{n-k}\) \((n \geq 3k)\), i.e. the graph obtained from disjoint graphs \(K_k\) and \(\overrightarrow{K}_{n-k}\) by adding all edges between the vertices in \(K_k\) and \(\overrightarrow{K}_{n-k}\).

Let \(V(D) = W \cup U\), where \(W = V(K_k) = \{w_i \mid 1 \leq i \leq k\}\) and 
\(U = V(\overrightarrow{K}_{n-k}) = \{u_j \mid 1 \leq j \leq n - k\}\). Let \(S\) be any \(k\)-subset of vertices of \(V(D)\) such that \(|S \cap U| = s\) \((s \leq k)\) and \(|S \cap W| = k - s\). Without loss of generality, let \(w_i \in S\) for \(1 \leq i \leq k - s\) and \(u_j \in S\) for \(1 \leq j \leq s\). For \(1 \leq i \leq k - s\), let \(D_i\) be the symmetric subgraph of \(D\) whose underlying undirected graph is the tree \(T_i\) with edge set

\[
\{w_iu_1, w_iu_2, \ldots, w_iz_s, u_{k+i}w_1, u_{k+i}w_2, \ldots, u_{k+i}w_{k-s}\}.
\]

For \(k - s + 1 \leq j \leq k\), let \(D_j\) be the symmetric subgraph of \(D\) whose underlying undirected graph is the tree \(T_j\) with edge set

\[
\{w_ju_1, w_ju_2, \ldots, w_ju_s, w_jw_1, w_jw_2, \ldots, w_jw_{k-s}\}.
\]

Observe that \(\{D_i \mid 1 \leq i \leq k - s\} \cup \{D_j \mid k - s + 1 \leq j \leq k\}\) is a set of \(k\) internally disjoint strong subgraph containing \(S\), so \(\kappa_S(D) \geq k\), and then \(\kappa_k(D) \geq k\). Combining this with the bound that \(\kappa_k(D) \leq \kappa(D)\) and the fact that \(\kappa(D) \leq \min\{\delta^+(D), \delta^-(D)\} = k\), we can get \(\kappa_k(D) = \kappa(D) = k\).  □

3 Minimally strong subgraph \((k, \ell)\)-connected digraphs

Below we will use the following Hamilton cycle decomposition theorem of Tillson.
Theorem 3.1 ([10]) The arcs of $\overrightarrow{K}_n$ can be decomposed into Hamiltonian cycles if and only if $n \neq 4, 6$.

The following observation will be used in the sequel.

Proposition 3.2 ([8]) If $D'$ is a strong spanning digraph of a strong digraph $D$, then $\kappa_k(D') \leq \kappa_k(D)$.

By the definition of a minimally strong subgraph $(k, \ell)$-connected digraph, we can get the following observation.

Proposition 3.3 A digraph $D$ is minimally strong subgraph $(k, \ell)$-connected if and only if $\kappa_k(D) = \ell$ and $\kappa_k(D - e) = \ell - 1$ for any arc $e \in A(D)$.

Proof: The direction “if” is clear by definition, and we only need to prove the direction “only if”. Let $D$ be a minimally strong subgraph $(k, \ell)$-connected digraph. By definition, we have $\kappa_k(D) \geq \ell$ and $\kappa_k(D - e) \leq \ell - 1$ for any arc $e \in A(D)$. Then for any set $S \subseteq V(D)$ with $|S| = k$, there is a set $D$ of $\ell$ internally disjoint strong subgraphs containing $S$. As $e$ must belong to one and only one element of $D$, we are done. $\square$

A digraph $D$ is minimally strong if $D$ is strong but $D - e$ is not for every arc $e$ of $D$.

Proposition 3.4 The following assertions hold:
(i) A digraph $D$ is minimally strong subgraph $(k, 1)$-connected if and only if $D$ is minimally strong digraph;
(ii) For $k \neq 4, 6$, a digraph $D$ is minimally strong subgraph $(k, n - 1)$-connected if and only if $D \cong \overrightarrow{K}_n$.

Proof: To prove (i), it suffices to show that a digraph $D$ is strong if and only if $\kappa_k(D) \geq 1$. If $D$ is strong, then for every vertex set $S$ of size $k$, $D$ has a strong subgraph containing $S$. If $\kappa_k(D) \geq 1$, for each vertex set $S$ of size $k$ construct $D_S$, a strong subgraph of $D$ containing $S$. The union of all $D_k$ is a strong subgraph of $D$ as there are sets $S_1, S_2, \ldots, S_p$ such that the union of $S_1, S_2, \ldots, S_p$ is $V(D)$ and for each $i \in [p - 1]$, $D_{S_i}$ and $D_{S_{i+1}}$ share a common vertex.

Part (ii) follows from Proposition [2.1] $\square$

The following result characterizes minimally strong subgraph $(2, n - 2)$-connected digraphs.

Theorem 3.5 A digraph $D$ is minimally strong subgraph $(2, n - 2)$-connected if and only if $D$ is a digraph obtained from the complete digraph $\overrightarrow{K}_n$ by deleting an arc set $M$ such that $\overrightarrow{K}_n[M]$ is a 3-cycle or a union of $\lceil n/2 \rceil$ vertex-disjoint 2-cycles. In particular, we have $f(n, 2, n - 2) = n(n - 1) - 2 \lceil n/2 \rceil$, $F(n, 2, n - 2) = n(n - 1) - 3$. 


**Proof:** Let $D \cong \overrightarrow{K}_n - M$ be a digraph obtained from the complete digraph $\overrightarrow{K}_n$ by deleting an arc set $M$. Let $V(D) = \{u_i \mid 1 \leq i \leq n\}$.

Firstly, we will consider the case that $\kappa_2(D) = n - 2$. By (1), we have $\kappa_2(D) = \min\{\delta^+(D), \delta^-(D)\} = n - 2$. Let $S = \{u, v\} \subseteq V(D)$; we just consider the case that $u = u_1$, $v = u_2$ since the other cases are similar. Let $D_1$ be a subdigraph of $D$ with $V(D_1) = \{u_1, u_2, u_3\}$ and $A(D_1) = \{u_1 u_3, u_3 u_2, u_2 u_1\}$; for $2 \leq i \leq n - 2$, let $D_i$ be a subdigraph of $D$ with $V(D_i) = \{u_1, u_2, u_{i+2}\}$ and $A(D_i) = \{u_1 u_{i+2}, u_2 u_{i+2}, u_{i+2} u_1, u_{i+2} u_2\}$. Clearly, $\{D_i \mid 1 \leq i \leq n - 2\}$ is a set of $n - 2$ internally disjoint strong subgraphs containing $S$, so $\kappa_S(D) \geq n - 2$ and $\kappa_2(D) \geq n - 2$. Hence, $\kappa_2(D) = n - 2$.

For any $e \in A(D)$, without loss of generality, one of the two digraphs in Figure 1 is a subgraph of $\overrightarrow{K}_n[M \cup \{e\}]$, so if the following claim holds, then we must have $\kappa_2(D - e) \leq \kappa_2(D') \leq n - 3$ by Proposition 3.2, and so $D$ is minimally strong subgraph $(2, n - 2)$-connected. Now it suffices to prove the following claim.

![Figure 1: Two graphs for Claim 1.](image)

**Claim 1.** If $\overrightarrow{K}_n[M']$ is isomorphic to one of two graphs in Figure 1, then $\kappa_2(D') \leq n - 3$, where $D' = \overrightarrow{K}_n - M'$.

**Proof of Claim 1.** We first show that $\kappa_2(D') \leq n - 3$ if $M'$ is the digraph of Figure 1(a). Let $S = \{u_2, u_4\}$; we will prove that $\kappa_S(D') \leq n - 3$, and then we are done. Suppose that $\kappa_S(D') \geq n - 2$, then there exists a set of $n - 2$ internally disjoint strong subgraphs containing $S$, say $\{D_i \mid 1 \leq i \leq n - 2\}$. If both of the two arcs $u_2 u_4$ and $u_4 u_2$ belong to the same $D_i$, say $D_1$, then for $2 \leq i \leq n - 2$, each $D_i$ contains at least one vertex and at most two vertices of $\{u_i \mid 1 \leq i \leq n, i \neq 2, 4\}$. Furthermore, there is at most one $D_i$, say $D_2$, contains (exactly) two vertices of $\{u_i \mid 1 \leq i \leq n, i \neq 2, 4\}$. We just consider the case that $u_1 u_3 \in V(D_2)$ since the other cases are similar. In this case, we must have that each vertex of $\{u_i \mid 5 \leq i \leq n\}$ belongs to exactly one digraph from $\{D_i \mid 3 \leq i \leq n - 2\}$ and vice versa. However, this is impossible since the vertex set $\{u_2, u_4, u_5\}$ cannot induce a strong subgraph of $D'$ containing $S$, a contradiction.

So we now assume that each $D_i$ contains at most one of $u_2 u_4$ and $u_4 u_2$. Without loss of generality, we may assume that $u_2 u_4 \in A(D_1)$ and $u_4 u_2 \in A(D_2)$. In this case, we must have that each vertex of $\{u_i \mid 1 \leq i \leq n, i \neq 2, 4\}$ belongs to exactly one digraph from $\{D_i \mid 1 \leq i \leq n - 2\}$ and vice versa. However, this is also impossible since the vertex set $\{u_2, u_4, u_5\}$ cannot induce a strong subgraph of $D'$ containing $S$, a contradiction.

Hence, we have $\kappa_2(D') \leq n - 3$ in this case. For the case that $M'$ is
the digraph of Figure 1 (b), we can choose \( S = \{ u_2, u_3 \} \) and prove that \( \kappa_S(D') \leq n - 3 \) with a similar argument, and so \( \kappa_2(D') \leq n - 3 \) in this case. This completes the proof of the claim.

Secondly, we consider the case that \( \overrightarrow{K}_n[M] \) is a union of \( \lfloor n/2 \rfloor \) vertex-disjoint 2-cycles. Without loss of generality, we may assume that \( M = \{u_{2i-1}u_{2i}, u_{2i}u_{2i+1} \mid 1 \leq i \leq \lfloor n/2 \rfloor \} \). We just consider the case that \( S = \{ u_1, u_3 \} \) since the other cases are similar. In this case, let \( D_1 \) be the subgraph of \( D \) with \( V(D_1) = \{ u_1, u_3 \} \) and \( A(D_1) = \{ u_1u_3, u_3u_1 \} \); let \( D_2 \) be the subgraph of \( D \) with \( V(D_2) = \{ u_1, u_2, u_3, u_4 \} \) and \( A(D_2) = \{ u_1u_4, u_4u_1, u_2u_4, u_4u_2, u_2u_3, u_3u_2 \} \); for \( 3 \leq i \leq n - 2 \), let \( D_i \) be the subgraph of \( D \) with \( V(D_i) = \{ u_1, u_2, u_{i+2} \} \) and \( A(D_i) = \{ u_1u_{i+2}, u_{i+2}u_1, u_{i+2}u_3 \} \). Clearly, \( \{ D_i \mid 1 \leq i \leq n - 2 \} \) is a set of \( n - 2 \) internally disjoint strong subgraphs containing \( S \), so \( \kappa_S(D) \geq n - 2 \) and then \( \kappa_2(D) \geq n - 2 \). By (1), we have \( \kappa_2(D) \leq \min \{ \delta^+(D), \delta^-(D) \} = n - 2 \). Hence, \( \kappa_2(D) = n - 2 \). Let \( e \in A(D) \); clearly \( e \) must be incident with at least one vertex of \( \{ u_i \mid 1 \leq i \leq \lfloor n/2 \rfloor \} \). Then we have that \( \kappa_2(D - e) \leq \min \{ \delta^+(D - e), \delta^-(D - e) \} = n - 3 \) by (1). Hence, \( D \) is minimally strong subgraph \( (2, n - 2) \)-connected.

Now let \( D \) be minimally strong subgraph \( (2, n - 2) \)-connected. By Proposition 2.3, we have that \( D \not\cong \overrightarrow{K}_n[M] \), that is, \( D \) can be obtained from a complete digraph \( \overrightarrow{K}_n \) by deleting a nonempty arc set \( M \). To end our argument, we need the following three claims. Let us start from a simple yet useful observation.

**Proposition 3.6** No pair of arcs in \( M \) has a common head or tail.

**Proof of Proposition 3.6** By (1) no pair of arcs in \( M \) has a common head or tail, as otherwise we would have \( \kappa_2(D) \leq n - 3 \).

**Claim 2.** \( |M| \geq 3 \).

**Proof of Claim 2.** Let \( |M| \leq 2 \). We may assume that \( |M| = 2 \) as the case of \( |M| = 1 \) can be considered in a similar and simpler way.

Let the arcs of \( M \) have no common vertices; without loss of generality, \( M = \{ u_1u_2, u_3u_4 \} \). Then \( \kappa_2(D - u_2u_1) = n - 2 \) as \( D - u_2u_1 \) is a supergraph of \( \overrightarrow{K}_n \) without a union of \( \lfloor n/2 \rfloor \) vertex-disjoint 2-cycles including the cycles \( u_1u_2u_1 \) and \( u_3u_4u_3 \). Thus, \( D \) is not minimally strong subgraph \( (2, n - 2) \)-connected. Let the arcs of \( M \) have no common vertex. By Proposition 3.6 without loss of generality, \( M = \{ u_1u_2, u_2u_3 \} \). Then \( \kappa_2(D - u_3u_1) = n - 2 \) as we showed in the beginning of the proof of this theorem. Thus, \( D \) is not minimally strong subgraph \( (2, n - 2) \)-connected. Now let the arcs of \( M \) have the same vertices, i.e., without loss of generality, \( M = \{ u_1u_2, u_2u_1 \} \). As above, \( \kappa_2(D - u_2u_1) = n - 2 \) and \( D \) is not minimally strong subgraph \( (2, n - 2) \)-connected.

**Claim 3.** If \( |M| = 3 \), then \( \overrightarrow{K}_n[M] \) is a 3-cycle.

**Proof of Claim 3.** Suppose that \( D \) is minimally strong subgraph \( (2, n - 2) \)-connected, but \( \overrightarrow{K}_n[M] \) is not a 3-cycle. By Proposition 3.6 no pair of arcs in \( M \) has a common head or tail. Thus, \( \overrightarrow{K}_n[M] \) must be isomorphic to one of graphs in Figures 1 and 2. If \( \overrightarrow{K}_n[M] \) is isomorphic to one of graphs in
Figure 1, then \( \kappa_2(D) \leq n - 3 \) by Claim 1 and so \( D \) is not minimally strong subgraph \((2, n - 2)\)-connected, a contradiction. For an arc set \( M_0 \) such that \( \overrightarrow{K}_n[M_0] \) is a union of \( \lfloor n/2 \rfloor \) vertex-disjoint 2-cycles, by the argument before, we know that \( \overrightarrow{K}_n - M_0 \) is minimally strong subgraph \((2, n - 2)\)-connected. For the case that \( \overrightarrow{K}_n - M_0 \) is isomorphic to (a) or (b) in Figure 2, we have that \( \overrightarrow{K}_n - M_0 \) is a proper subdigraph of \( \overrightarrow{K}_n[M] \) is a union of \( \lfloor n/2 \rfloor \) vertex-disjoint 2-cycles, by the argument before, we know that \( \overrightarrow{K}_n - M_0 \) is minimally strong subgraph \((2, n - 2)\)-connected. Hence, the claim holds.
Theorem 3.7 For $2 \leq k \leq n$, we have
\[ f(n, k, \ell) \geq n\ell. \]

Moreover, the following assertions hold:

(i) If $\ell = 1$, then $f(n, k, \ell) = n$; (ii) If $2 \leq \ell \leq n - 1$, then $f(n, n, \ell) = n\ell$ for $k = n \notin \{4, 6\}$; (iii) If $n$ is even and $\ell = n - 2$, then $f(n, 2, \ell) = n\ell$.

Proof: By [1], for all digraphs $D$ and $k \geq 2$ we have $\kappa_k(D) \leq \delta^+(D)$ and $\kappa_k(D) \leq \delta^-(D)$. Hence for each $D$ with $\kappa_k(D) = \ell$, we have that $\delta^+(D), \delta^-(D) \geq \ell$, so $|A(D)| \geq n\ell$ and then $f(n, k, \ell) \geq n\ell$.

For the case that $\ell = 1$, let $D$ be a dicycle $\overrightarrow{C_n}$. Clearly, $D$ is minimally strong subgraph $(k, 1)$-connected, and we know $|A(D)| = n$, so $f(n, k, 1) = n$.

For the case that $k = n \notin \{4, 6\}$ and $2 \leq \ell \leq n - 1$, let $D \cong \overrightarrow{K_n}$. By Theorem 3.1, $D$ can be decomposed into $n - 1$ Hamiltonian cycles $H_i$ $(1 \leq i \leq n - 1)$. Let $D_\ell$ be the spanning subdigraph of $D$ with arc sets $A(D_\ell) = \bigcup_{1 \leq i \leq \ell} A(H_i)$. Clearly, we have $\kappa_n(D_\ell) \geq \ell$ for $2 \leq \ell \leq n - 1$. Furthermore, by [1], we have $\kappa_n(D_\ell) \leq \ell$ since the in-degree and out-degree of each vertex in $D_\ell$ are both $\ell$. Hence, $\kappa_n(D_\ell) = \ell$ for $2 \leq \ell \leq n - 1$. For any $e \in A(D_\ell)$, we have $\delta^+(D_\ell - e) = \delta^-(D_\ell - e) = \ell - 1$, so $\kappa_n(D_\ell - e) \leq \ell - 1$ by [1].

Thus, $D_\ell$ is minimally strong subgraph $(n, \ell)$-connected. As $|A(D_\ell)| = n\ell$, we have $f(n, n, \ell) \leq n\ell$. From the lower bound that $f(n, k, \ell) \geq n\ell$, we have $f(n, n, \ell) = n\ell$ for the case that $2 \leq \ell \leq n - 1, n \notin \{4, 6\}$.

Part (iii) follows directly from Theorem 3.5.

To prove two upper bounds on the number of arcs in a minimally strong subgraph $(k, \ell)$-connected digraph, we will use the following result, see e.g. [1].

Theorem 3.8 Every strong digraph $D$ on $n$ vertices has a strong spanning subgraph $H$ with at most $2n - 2$ arcs and equality holds only if $H$ is a symmetric digraph whose underlying undirected graph is a tree.

Proposition 3.9 We have (i) $F(n, n, \ell) \leq 2\ell(n - 1)$; (ii) For every $k \geq 2$, $F(n, k, 1) = 2(n - 1)$ and $Ex(n, k, 1)$ consists of symmetric digraphs whose underlying undirected graphs are trees.

Proof: (i) Let $D = (V, A)$ be a minimally strong subgraph $(n, \ell)$-connected digraph, and let $D_1, \ldots, D_\ell$ be arc-disjoint strong spanning subgraphs of $D$. Since $D$ is minimally strong subgraph $(n, \ell)$-connected and $D_1, \ldots, D_\ell$ are pairwise arc-disjoint, $|A| = \sum_{i=1}^{\ell} |A(D_i)|$. Thus, by Theorem 3.8 $|A| \leq 2\ell(n - 1)$. (ii) In the proof of Proposition 3.4 we showed that a digraph $D$ is strong if and only if $\kappa_k(D) \geq 1$. Now let $\kappa_k(D) \geq 1$ and a digraph $D$ has a minimal number of arcs. By Theorem 3.8 we have that $|A(D)| \leq 2(n - 1)$ and if $D \in Ex(n, k, 1)$ then $|A(D)| = 2(n - 1)$ and $D$ is a symmetric digraph whose underlying undirected graph is a tree.
4 Discussion

Perhaps, the most interesting result of this paper is the characterization of minimally strong subgraph \((2, n-2)\)-connected digraphs. As a simple consequence of the characterization, we can determine the values of \(f(n, 2, n-2)\) and \(F(n, 2, n-2)\). It would be interesting to determine \(f(n, k, n-2)\) and \(F(n, k, n-2)\) for every value of \(k \geq 3\). (Obtaining characterizations of all \((k, n-2)\)-connected digraphs for \(k \geq 3\) seems a very difficult problem.) It would also be interesting to find a sharp upper bound for \(F(n, k, \ell)\) for all \(k \geq 2\) and \(\ell \geq 2\).

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