\title{Z^0 Decay into Charmonium via Charm Quark Fragmentation}

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\section*{Abstract}

In decays of the Z^0, the dominant mechanism for the direct production of charmonium states is the decay of the Z^0 into a charm quark or antiquark followed by its fragmentation into the charmonium state. We calculate the fragmentation functions describing the splitting of charm quarks into S-wave charmonium states to leading order in the QCD coupling constant. Leading logarithms of M_Z/m_c are summed up using Altarelli-Parisi evolution equations. Our analytic result agrees with the complete leading order calculation of the rate for Z^0 \rightarrow \psi\bar{c}c. We also use our fragmentation functions to calculate the production rate of heavy quarkonium states in W^\pm, top quark, and Higgs decays.
Introduction

Among the rare decay modes of the $Z^0$ predicted by the Standard Gauge Theory are ones whose final states include charmonium. Of particular importance are the $^3S_1$ charmonium states $J/\psi$ and $\psi'$, since their decays into lepton pairs provide easily identifiable experimental signatures. The dominant production mechanism for $\psi$ and $\psi'$ is the decay of $B$ hadrons; in fact, this serves as a signature for $B$ hadron production in $Z^0$ decay. The direct production of $\psi$ and $\psi'$ is therefore important in $Z^0$ decays as a background to $B$ physics. It is also of interest in its own right, since it involves both short distance and long distance aspects of quantum chromodynamics (QCD). The production of a charm quark and antiquark with small relative momentum in $Z^0$ decay is a short distance process with a characteristic length scale that can range from $1/M_Z$ to as large as $1/m_c$. The subsequent formation of a bound state from the $c\bar{c}$ pair is a long distance process involving all the complications of nonperturbative QCD. The methods of perturbative QCD can be used to calculate the production rates provided that it is possible to systematically separate the short distance effects from the long distance effects.

Most previous work on charmonium production in $Z^0$ decay \cite{1, 2, 3} has focused on short distance processes in which the $c\bar{c}$ pair that form the $\psi$ is produced with a transverse separation of order $1/M_Z$. Long distance effects involved in the formation of the bound state are factored into the nonrelativistic radial wavefunction at the origin $R(0)$. The best example of a short distance process is $Z^0 \to \psi gg$, which has a branching fraction of about $10^{-7}$. This small branching fraction can be partly attributed to a factor of $|R(0)|^2/(m_cM_Z^2)$, which represents the probability for a $c\bar{c}$ pair that is produced in a region of size $1/(m_cM_Z^2)$ to form a bound state. This probability factor suppresses the branching fractions for short distance processes by $m_c^2/M_Z^2$, so that they can be neglected in the limit $M_Z/m_c \to \infty$.

As pointed out by Kühn and Schneider \cite{4}, the direct production of charmonium in $Z^0$ decay will be dominated not by short distance processes but by fragmentation processes. The fragmentation mechanism is the decay of the $Z^0$ into a final state that includes a high energy quark or gluon, followed by the splitting of that parton into the charmonium state plus other partons. In the fragmentation mechanism, the $c$ and $\bar{c}$ that form the charmonium
state are produced with a separation of order $1/m_c$. The probability that they form a bound state is proportional to $|R(0)|^2/m_c^3$. The branching ratio for such a process is therefore not suppressed by the factor $m_c^2/M_Z^2$ associated with short distance processes. The fragmentation of a parton is described by a fragmentation function $D(z, \mu)$, which gives the probability for a parton with invariant mass less than $\mu$ to split into the charmonium state with longitudinal momentum fraction $z$. It was recently shown that the fragmentation functions for the splitting of partons into heavy quarkonium states can be calculated using perturbative QCD \cite{5}. The fragmentation functions $D_g \rightarrow \psi(z, \mu)$ and $D_g \rightarrow \eta_c(z, \mu)$ that describe the splitting of gluons into S-wave quarkonium states were calculate to leading order in $\alpha_s$ at the scale $\mu = m_c$. They were evolved to larger scales $\mu$ by using Altarelli-Parisi evolution equations, which sum up leading logarithms of $\mu/m_c$. The production of $\psi$ in $Z^0$ decay from the splitting of virtual gluons has been considered by Hagiwara, Martin, and Stirling \cite{6}, but they did not organize the calculation in terms of fragmentation functions and were thus unable to sum up leading logarithms of $M_Z/m_c$.

The production rate of $\psi$ via the process $Z^0 \rightarrow \psi c\bar{c}$ has been calculated by Barger, Cheung, and Keung \cite{7} with a rather surprising result: it has a branching fraction of about $10^{-5}$. This is almost two orders of magnitude larger than $Z^0 \rightarrow \psi gg$, in spite of the fact that both rates are the same order in $\alpha_s$. An explanation for the relatively large branching fraction of $Z^0 \rightarrow \psi c\bar{c}$ was provided in Ref. \cite{5}, where it was pointed out that this process includes a fragmentation contribution that is not suppressed by a factor of $m_c^2/M_Z^2$. This contribution can be factored into the rate for the $Z^0$ to decay into a $c\bar{c}$ pair multiplied by the probability for the $c$ or $\bar{c}$ to fragment into $\psi$.

In this paper, we calculate the fragmentation functions $D_{c \rightarrow \psi}(z, \mu)$ and $D_{c \rightarrow \eta_c}(z, \mu)$ for a charm quark to split into an S-wave charmonium state. The fragmentation functions at the scale $\mu = m_c$ are calculated to leading order in $\alpha_s(m_c)$. Altarelli-Parisi equations are used to evolve them up to the scale $\mu = M_Z/2$ appropriate for $Z^0$ decay. Our simple analytic result for $Z^0 \rightarrow \psi c\bar{c}$ agrees with the complete leading order calculation of Barger, Cheung, and Keung. We also use our fragmentation functions to calculate the direct production rates for $\psi$ in $W^\pm$ decays and for $\Upsilon$ in top quark and Higgs decays.
Z⁰ Decay via Fragmentation

The fragmentation contribution to the inclusive decay rate of the Z⁰ into charmonium is the term that survives in the limit \( M_Z/m_c \to \infty \). The general form of the fragmentation contribution to the differential decay rate for the production of a \( \psi \) of energy \( E \) is

\[
d\Gamma(Z^0 \to \psi(E) + X) = \sum_i \int_0^1 dz \, d\hat{\Gamma}(Z^0 \to i(E/z) + X, \mu) \, D_{i \to \psi}(z, \mu) \, , \tag{1}
\]

where the sum is over partons of type \( i \) and \( z \) is the longitudinal momentum fraction of the \( \psi \) relative to the parton. The physical interpretation of (1) is that a \( \psi \) of energy \( E \) can be produced by first producing a parton \( i \) of larger energy \( E/z \) which subsequently splits into a \( \psi \) carrying a fraction \( z \) of the parton energy. The expression (1) for the differential decay rate has a factored form: all the dependence on the energy \( E \) is in the parton subprocess decay rate \( \hat{\Gamma} \), while all the dependence on the charm quark mass \( m_c \) is in the fragmentation function \( D_{i \to \psi} \). To maintain this factored form in spite of the logarithms of \( M_Z/m_c \) that arise in perturbation theory, a factorization scale \( \mu \) must be introduced. The dependence on the arbitrary scale \( \mu \) cancels between the two factors. Large logarithms of \( E/\mu \) in the subprocess decay rate \( \hat{\Gamma} \) can be avoided by choosing \( \mu \) on the order of \( E \). Large logarithms of \( \mu/m_c \) then necessarily appear in the fragmentation functions \( D_{i \to \psi}(z, \mu) \), but they can be summed up by solving the evolution equations [8]

\[
\mu \frac{\partial}{\partial \mu} D_{i \to \psi}(z, \mu) = \sum_j \int_z^1 dy \, \frac{1}{y} \, P_{i \to j}(z/y, \mu) \, D_{j \to \psi}(y, \mu) \, , \tag{2}
\]

where \( P_{i \to j}(x, \mu) \) is the Altarelli-Parisi function for the splitting of the parton of type \( i \) into a parton of type \( j \) with longitudinal momentum fraction \( x \). For example, the \( c \to c \) splitting function for a charm quark with energy much greater than its mass is the usual splitting function for quarks:

\[
P_{c \to c}(x, \mu) = \frac{\alpha_s(\mu)}{2\pi} \left[ \frac{4}{3} \frac{1 + x^2}{(1 - x)_+} + 2 \, \delta(1 - x) \right] . \tag{3}
\]

The boundary condition on the evolution equation (2) is the initial fragmentation function \( D_{i \to \psi}(z, m_c) \) at the scale \( m_c \). As shown in Ref. [3], it can be calculated perturbatively as a series in \( \alpha_s(m_c) \).
We can easily count the order in $\alpha_s$ for the fragmentation contributions to $\psi$ production in $Z^0$ decay. The subprocess rate $\hat{\Gamma}$ for producing gluons is of order $\alpha_s$, while that for producing quarks is of order 1. The fragmentation function for a gluon to split into $\psi$, which was calculated in Ref. \cite{5}, is proportional to $\alpha_s^3$. A light quark can split into a $\psi$ only by radiating a gluon which splits into a $\psi$, so its fragmentation function is of order $\alpha_s^4$. In contrast, the fragmentation function for a charm quark to split into a $\psi$, which will be calculated explicitly below, is only of order $\alpha_s^2$. Thus the fragmentation of charm quarks into $\psi$ dominates by two powers of $\alpha_s$ over the fragmentation of light quarks or gluons.

Keeping only the charm quark and antiquark contributions to (3), the energy distribution of the $\psi$ reduces at leading order in $\alpha_s$ to

$$\frac{d\Gamma}{dz}(Z^0 \to \psi(E) + X) = 2 \hat{\Gamma}(Z^0 \to c\bar{c}) D_{c \to \psi}(z, M_Z/2), \ z = \frac{2E}{M_Z}. \quad (4)$$

This fragmentation formula is of course applicable only for a $\psi$ of energy $E$ that is a significant fraction $z$ of the energy $M_Z/2$ of the charm quark and much greater than the mass $M_\psi$ of the $\psi$. In (4), the factor of 2 accounts for the contribution from the fragmentation of the $\bar{c}$. We have set the factorization scale $\mu$ to $M_Z/2$ to avoid large logarithms from higher orders in perturbation theory. At leading order in $\alpha_s$, only the diagonal term in the evolution equation (2) survives:

$$\mu \frac{\partial}{\partial \mu} D_{c \to \psi}(z, \mu) = \int_1^1 \frac{dy}{y} P_{c \to c}(z/y, \mu) D_{c \to \psi}(y, \mu). \quad (5)$$

Integrating (4) over the energy, the total rate for inclusive $\psi$-production is

$$\Gamma(Z^0 \to \psi + X) = 2 \hat{\Gamma}(Z^0 \to c\bar{c}) \int_0^1 dz D_{c \to \psi}(z, m_c). \quad (6)$$

We have set the fragmentation scale equal to $m_c$ by exploiting the fact that at leading order in $\alpha_s$ the Altarelli-Parisi splitting function \cite{3} satisfies $\int_0^1 dx P_{c \to c}(x, \mu) = 0$. The evolution equation (5) then implies that the fragmentation probability $\int_0^1 dz D_{c \to \psi}(z, \mu)$ does not evolve with the scale $\mu$.

**Fragmentation function for $c \to \psi$**
We proceed to calculate the initial fragmentation function $D_{c \rightarrow \psi}(z, m_c)$ for a charm quark to split into a $\psi$ to leading order in $\alpha_s(m_c)$. Our strategy is to isolate the contribution $\Gamma_1$ to the decay rate for $Z^0 \rightarrow \psi c \bar{c}$ that arises from the fragmentation of the charm quark. We can then obtain the fragmentation probability $\int_0^1 dz D(z)$ by dividing $\Gamma_1$ by the rate $\Gamma_0$ for $Z^0 \rightarrow c \bar{c}$:

$$\Gamma_0 = \frac{1}{2M_Z} \int [d\bar{q}] [dq] (2\pi)^4 \delta^4(Z - \bar{q} - q) \frac{1}{3} \sum |A_0|^2,$$

(7)

where $Z$, $\bar{q}$, and $q$ are the 4-momenta of the $Z^0$, $\bar{c}$, and $c$, and $[dq] = d^3q/(16\pi^3q_0)$ is the Lorentz-invariant phase space element. The square of the amplitude $A_0$ for $Z^0 \rightarrow c \bar{c}$, averaged over initial spins and summed over final spins and colors, is

$$\frac{1}{3} \sum |A_0|^2 = \left(-g^{\alpha\beta} + \frac{Z^\alpha Z^\beta}{M_Z^2}\right) \text{tr} \left( \Gamma_\alpha (\bar{q} - m_c) \Gamma_\beta (\bar{q} + m_c) \right),$$

(8)

where $\Gamma_\alpha$ is the $Z^0c\bar{c}$ vertex whose explicit form is not required. In the limit $M_Z >> m_c$, the factors of $m_c$ in the trace can be neglected.

The rate for the decay $Z^0 \rightarrow \psi c \bar{c}$ is

$$\Gamma_1 = \frac{1}{2M_Z} \int [d\bar{q}] [dp] [dp'] (2\pi)^4 \delta^4(Z - \bar{q} - p - p') \frac{1}{3} \sum |A_1|^2,$$

(9)

where $\bar{q}$, $p$, and $p'$ are the 4-momenta of the $\bar{c}$, $\psi$, and $c$. The four Feynman diagrams that contribute to the amplitude $A_1$ at leading order in $\alpha_s$ are shown in Figure 1. The contributions to the process $Z^0 \rightarrow \psi c \bar{c}$ that correspond to the fragmentation of the charm quark come from the region of phase space in which the $\psi - c$ system has large momentum $q = p + p'$ of order $M_Z$ and small invariant mass $s = q^2$ of order $m_c^2$. To facilitate the extraction of the fragmentation probability, we write the 3-body phase space for the outgoing particles in an iterated form by introducing integrals over $q$ and over $s$:

$$\int [d\bar{q}] [dp] [dp'] (2\pi)^4 \delta^4(Z - \bar{q} - p - p')$$

$$= \int \frac{ds}{2\pi} \int [d\bar{q}] [dq] (2\pi)^4 \delta^4(Z - \bar{q} - q) \int [dp] [dp'] (2\pi)^4 \delta^4(q - p - p').$$

(10)
We also express the two-body phase space integral over $p$ and $p'$ in terms of the longitudinal momentum fraction $z$ of the $\psi$. In a frame in which the virtual charm quark has the 4-momentum $q = (q_0,0,0,q_3)$, the longitudinal momentum fraction of the $\psi$ is $z = (p_0 + p_3)/(q_0 + q_3)$ and its transverse momentum is $\vec{p}_\perp = (p_1,p_2)$. Expressed in terms of these variables, the Lorentz invariant phase space element is $[dp] = dzd^2\vec{p}_\perp/(16\pi^3 z)$. Integrating over the 4-momentum $p'$ and over $\vec{p}_\perp$, the 2-body phase space integral reduces to

$$\int [dp][dp'] (2\pi)^4 \delta^4(q - p - p') = \frac{1}{8\pi} \int_0^1 dz \theta \left( s - \frac{4m_c^2}{z} - \frac{m_c^2}{1-z} \right).$$

We have set $M_\psi = 2m_c$, which is accurate up to relativistic corrections. If $s = q^2$ is of order $m_c^2$, the delta function $\delta^4(q - p - p')$ constrains $\vec{p}_\perp$ to be of order $m_c$. From the mass-shell condition, the component $p_0 - p_3 = (p_0^2 + 4m_c^2)/(p_0 + p_3)$ is of order $m_c^2/M_Z$. Thus, to leading order in $m_c/M_Z$, we can set $p = zq$.

We proceed to isolate the contribution to the amplitude $A_1$ from the fragmentation of the charm quark. In covariant gauges, this contribution comes from both of the diagrams in Figure 1a and 1b, while the diagrams in Figures 1c and 1d contain contributions from $\bar{c}$ fragmentation. In the axial gauge associated with the 4-vector $\vec{q}$, the contribution from fragmentation of the charm quark comes only from the diagram shown in Figure 1a. The amplitude for Figure 1a in this gauge can be reduced to

$$A_1 = \frac{4g^2 R(0)}{3\sqrt{6\pi}m_c} \varepsilon_\alpha(Z) \varepsilon^{\dagger}_\mu(p) \frac{1}{(s - m_c^2)^2} \bar{u}(p') \left( 2m_c \gamma^\mu (\not{q} + m_c) \right. \left. + \frac{s - m_c^2}{\not{q} \cdot (2q - p)} \not{q} \gamma^\mu (\not{p} + 2m_c) \right) \Gamma^\alpha v(\not{q}).$$

We have used standard covariant Feynman rules [1] for projecting the amplitude for production of a $c\bar{c}$ pair with equal 4-momenta $p/2$ onto the amplitude for production of a $\psi$ with 4-momentum $p$. The parameter $R(0)$ is the value of the nonrelativistic radial wavefunction at the origin. Averaging over initial spins and summing over final spins and colors, the square of the amplitude reduces to

$$\frac{1}{3} \sum |A_1|^2 = \frac{128\pi \alpha_s^2 |R(0)|^2}{27m_c} \frac{1}{(s - m_c^2)^4} \left(-g^{\alpha\beta} + \frac{Z^\alpha Z^\beta}{M_Z^2}\right) \text{tr} \left( \Gamma_\alpha (\not{q} - m_c) \Gamma_\beta D \right),$$

6
where $D$ is a Dirac matrix that depends on $\bar{q}, q,$ and $p$. We need only keep the terms in $D$ for which the Dirac trace in (13) is of order $m_c^4M_Z^2$. While $\bar{q}, q,$ and $p$ all have components of order $M_Z$, $s = q^2$ is of order $m_c^2$ in the fragmentation region. Simplifying the Dirac matrix by dropping terms which are suppressed by powers of $m_c/M_Z$, it reduces to

$$D = (s^2 - 2m_c^2 s - 47m_c^4) \bar{q} - (s - m_c^2)(s - 9m_c^2) p$$

$$+ 4 \frac{s - m_c^2}{\bar{q} \cdot (2q - p)} (s + 7m_c^2) \bar{q} \cdot q p - \bar{q} \cdot p \bar{q} - 8m_c^2 \bar{q} \cdot q \bar{q}$$

$$+ 12 \left( \frac{s - m_c^2}{\bar{q} \cdot (2q - p)} \right)^2 \bar{q} \cdot p \bar{q} \cdot (q - p) p .$$

We have exploited the fact that while $p$ and $q$ are both of order $M_Z$, their product $pq$ is only of order $m_cM_Z$. The coefficients of $p$ and $q$ in (14) are all manifestly of order $m_c^4$, so we can substitute $p = zq$ for all the remaining factors of $p$. The Dirac trace in (12) is then proportional to $\text{tr}(\Gamma_\alpha \bar{q} \Gamma_\beta q)$. It is now easy to divide $\Gamma_1$ by the decay rate $\Gamma_0$ given in (7) to obtain the fragmentation probability:

$$\int^1_0 dz D_{c \rightarrow \psi}(z) = \frac{8\alpha_s^2|R(0)|^2}{27\pi m_c} \int^\infty_0 ds \frac{1}{(s - m_c^2)^4} \int^1_0 dz \theta \left( s - \frac{4m_c^2}{z} - \frac{m_c^2}{1 - z} \right)$$

$$\left( (s^2 - 2m_c^2 s - 47m_c^4) - z(s - m_c^2)(s - 9m_c^2) + 4 \frac{z(1 - z)}{2 - z} s(s - m_c^2) \right.$$

$$- 4 \frac{8 - 7z - 5z^2}{2 - z} m_c^2(s - m_c^2) + 12 \frac{z^2(1 - z)}{(2 - z)^2} (s - m_c^2)^2 \left). \right) \qquad \text{(15)}$$

Note that the upper limit on the integral over $s$ has been increased to $\infty$. Since the integrand behaves like $1/s^2$ at large $s$, this only changes the integral by an amount of order $m_c^2/M_Z^2$, which we have been systematically neglecting. Evaluating the integral over $s$ in (15), we obtain our final expression for the initial fragmentation function:

$$D_{c \rightarrow \psi}(z, 3m_c) = \frac{64}{27\pi} \alpha_s(3m_c)^2 \frac{|R(0)|^2}{M^{3}_\psi} \frac{z(1 - z)^2(16 - 32z + 72z^2 - 32z^3 + 5z^4)}{(2 - z)^6} . \quad \text{(16)}$$

We have set the scale $\mu$ in the fragmentation function and in the running coupling constant to $\mu = 3m_c$, which is the minimum value of the invariant mass $\sqrt{s}$ of the fragmenting charm
quark. We have also set $2m_c \to M_\psi$ in the denominator, which is accurate up to relativistic corrections. Integrating over $z$, we obtain the total fragmentation probability:

$$
\int_0^1 dz \, D_{c\to\psi}(z, 3m_c) = \frac{64}{27\pi} \alpha_s(3m_c)^2 \frac{|R(0)|^2}{M^3_\psi} \left( \frac{1189}{30} - 57 \log 2 \right). \quad (17)
$$

**Fragmentation function for $c \to \eta_c$**

The fragmentation function for a charm quark to split into the $^1S_0$ state of charmonium $\eta_c$ can be calculated in the same way as for $\psi$. The starting point is the expression (9) for the decay rate for $Z^0 \to \eta_c c\bar{c}$, except that the amplitude $A_1$ in (12) must be replaced by

$$
A_1 = \frac{4g^2 R(0)}{3\sqrt{6\pi m_c}} \epsilon_\mu(Z) \epsilon_\mu(p) \frac{1}{(s - m^2_c)^2} \bar{u}(p') \left( (\not{q} + 4m_c) \gamma_5 (\not{q} + m_c) + \frac{s - m^2_c}{\not{q} \cdot (2q - p)} \not{q} \gamma_5 (\not{q} + 2m_c) \right) \Gamma^\alpha v(q). \quad (18)
$$

The square of the amplitude has the form (13), except that the Dirac matrix $D$ reduces to

$$
D = (s + 3m^2_c)(s - 5m^2_c) \not{q} - (s - m^2_c)(s - 9m^2_c) \not{p} + 4 \frac{s - m^2_c}{\not{q} \cdot (2q - p)} \left( (s - m^2_c) \not{q} \cdot q - (s - 3m^2_c) \not{q} \cdot p \right) \not{p} + 4 \left( \frac{s - m^2_c}{\not{q} \cdot (2q - p)} \right)^2 \not{q} \cdot p \not{q} \cdot (q - p) \not{p}. \quad (19)
$$

Following the same path as in the $\psi$ calculation, we find that the initial fragmentation function for $\eta_c$ is

$$
D_{c\to\eta_c}(z, 3m_c) = \frac{64}{81\pi} \alpha_s(3m_c)^2 \frac{|R(0)|^2}{M^3_{\eta_c}} \frac{z(1 - z)^2(48 + 8z^2 - 8z^3 + 3z^4)}{(2 - z)^6}. \quad (20)
$$

Integrating over $z$, the fragmentation probability is

$$
\int_0^1 dz \, D_{c\to\eta_c}(z, 3m_c) = \frac{64}{27\pi} \alpha_s(3m_c)^2 \frac{|R(0)|^2}{M^3_{\eta_c}} \left( \frac{773}{30} - 37 \log 2 \right). \quad (21)
$$
Decay of $Z^0$ into Charmonium

From (6), the branching ratio for the decay of the $Z^0$ into $\psi$ relative to the decay into $c\bar{c}$ is

$$\frac{\Gamma(Z^0 \to \psi c\bar{c})}{\Gamma(Z^0 \to c\bar{c})} = 0.1870 \, \alpha_s(3m_c)^2 \frac{|R(0)|^2}{M_\psi^3}$$  (22)

The value of the parameter $R(0)$ can be determined from the $\psi$ electronic width to be $|R(0)|^2 = (0.82 \text{ GeV})^3$. Taking $\alpha_s(3m_c) = 0.23$, we find that the branching ratio (22) is $1.8 \times 10^{-4}$. The simple result (22) agrees with the complete leading order calculation of $Z^0 \to \psi c\bar{c}$ in Ref. [7] after taking into account the differences in the values of $R(0)$, $\alpha_s$, and the charm quark mass. A slightly larger value for the wavefunction at the origin was used in Ref. [7]: $|R(0)|^2 = (0.92 \text{ GeV})^3$. It was also assumed implicitly in Ref. [7] that $Z^0 \to \psi c\bar{c}$ is a short distance process, so the running coupling constant was taken to be $\alpha_s(M_Z) \approx 0.15$. As we have shown, the dominant contribution comes from a fragmentation process, and the appropriate scale of the coupling constant is definitely on the order of $m_c$. Finally, instead of $M_\psi^3$ in the denominator of (22), the authors of Ref. [4] used $(2m_c)^3$ with $m_c = 1.35 \text{ GeV}$. The difference between $M_\psi$ and $2m_c$ is a relativistic correction, which we have consistently ignored in this analysis. Corrections to the fragmentation approximation are of order $(2M_\psi/M_Z)^2$ or about 0.4%, which is much smaller than the size of relativistic corrections and higher order perturbative corrections.

The rate for production of $\eta_c$ by fragmentation differs by less than 3% from that for $\psi$. From (21), we obtain

$$\frac{\Gamma(Z^0 \to \eta_c c\bar{c})}{\Gamma(Z^0 \to c\bar{c})} = 0.1814 \, \alpha_s(3m_c)^2 \frac{|R(0)|^2}{M_{\eta_c}^3}$$  (23)

This agrees with the calculation of Ref. [7] after taking into account the differences in the values of $\alpha_s$, $R(0)$, and $m_c$ and an apparent algebraic error of a factor of 3.

The energy distribution of the $\psi$'s produced by the fragmentation of charm quarks in $Z^0$ decay is given in (4). It is proportional to the fragmentation function evaluated at the scale $M_Z/2$. The initial fragmentation function (16) at the scale $3m_c$ is shown as a
solid line in Figure 1. It must be evolved up to the scale $M_Z/2$ using the Altarelli-Parisi equation \cite{5} in order to sum up the leading logarithms of $M_Z/m_c$ from higher order radiative corrections. The result is shown as the dotted line in Figure 2. The evolution softens the energy distribution, shifting the peak in the fragmentation function from $z = 0.75$ to $z = 0.68$. The energy distribution shown in Figure 2 should be accurate provided that the energy $E$ of the $\psi$ is large compared to its mass, or equivalently $z \gg 0.07$. The fragmentation function for $\eta_c$ production is also shown in Figure 2. It has a slightly softer distribution, but its behavior is otherwise similar to that for the $\psi$.

The expression \cite{22} also applies with minor modifications to the corresponding branching ratio for $\Upsilon$ production:

$$
\frac{\Gamma(Z^0 \rightarrow \Upsilon b\bar{b})}{\Gamma(Z^0 \rightarrow bb)} = 0.1870 \alpha_s(3m_b)^2 \frac{|R(0)|^2}{M_\Upsilon^2} \quad (24)
$$

where $R(0)$ is the radial wavefunction at the origin for the $\Upsilon$, which is determined from its electronic decay rate to be $|R(0)|^2 = (1.72 \text{ GeV})^3$. Taking $\alpha_s(3m_b) = 0.17$, we find that the branching ratio \cite{24} is $3.3 \times 10^{-5}$. The fragmentation approximation for $\Upsilon$ production in $Z^0$ decay is not as accurate as it is for $\psi$ production. Corrections are on the order of $(2M_\Upsilon/M_Z)^2$, which is about 4%.

**Decay of $W^\pm$ into $\psi$**

About 1/3 of the decays of the $W^+$ will proceed through the channel $W^+ \rightarrow c\bar{s}$. The mass of the $W$ is sufficiently large that the dominant production mechanism for charmonium will be $W^+ \rightarrow c\bar{s}$, followed by the fragmentation of the charm quark into charmonium. Fragmentation of the strange antiquark into $\psi$ is suppressed by a factor of $\alpha_s^2$. The branching ratio for decay into $\psi$ relative to decay into $c\bar{s}$ is therefore smaller than \cite{22} by a factor of 2:

$$
\frac{\Gamma(W^+ \rightarrow \psi c\bar{s})}{\Gamma(W^+ \rightarrow c\bar{s})} = 0.0935 \alpha_s(3m_c)^2 \frac{|R(0)|^2}{M_\psi^3}. \quad (25)
$$

Numerically this branching ratio is $9.2 \times 10^{-5}$. Our analytic calculation of the fragmentation contribution is consistent with the full leading order calculation of Ref. \cite{7}.

**Decay of Top Quark into $\Upsilon$**

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The top quark will probably decay almost exclusively into $W^+b$. If the top quark is heavy enough, the dominant production mechanism for bottomonium in top quark decay will be $t \to W^+b$, followed by the fragmentation of the $b$ quark into bottomonium. The branching fraction for the direct decay into the $^3S_1$ state $\Upsilon$ is one half of (24):

$$\frac{\Gamma(t \to W^+\Upsilon b)}{\Gamma(t \to W^+b)} = 0.0935 \alpha_s(3m_b)^2 \frac{|R(0)|^2}{M_\Upsilon^3}, \quad (26)$$

which has the numerical value $1.6 \times 10^{-5}$. The complete leading order calculation of the rate for $t \to W^+b\Upsilon$ gives a branching fraction of $4 \times 10^{-7}$ for a top quark with a mass of 100 GeV [7]. The fragmentation formula (26) does not apply to such a small value of the top quark mass, since the maximum momentum of the $\Upsilon - b$ system is only 13 GeV, too small for the decay rate to be dominated by fragmentation. The simple result (26) is a good approximation if the mass of the top quark is closer to 150 GeV.

**Decay of Higgs into $\Upsilon$**

If the Higgs mass is below the threshold for decay into $W$ pairs, than its dominant decay mode will be $H \to b\bar{b}$. The dominant production method for bottomonium in Higgs decay will be $H \to b\bar{b}$, followed by the fragmentation of the $b$ quark or antiquark into bottomonium. The branching fraction for the direct decay into the $\Upsilon$ is twice (26), because both the $b$ and $\bar{b}$ can fragment into $\Upsilon$:

$$\frac{\Gamma(H \to \Upsilon b\bar{b})}{\Gamma(H \to b\bar{b})} = 0.1870 \alpha_s(3m_b)^2 \frac{|R(0)|^2}{M_\Upsilon^3}, \quad (27)$$

This branching ratio is $3.3 \times 10^{-5}$, which is probably too small for this decay mode to be useful as a signal for an intermediate mass Higgs boson.

**Conclusions**

We have shown in this paper that the dominant mechanism for the direct production of charmonium in $Z^0$ decay is fragmentation, the production of a high energy charm quark or antiquark followed by its splitting into the charmonium state. Most previous calculations of charmonium production have considered only short-distance production mechanisms which
are suppressed by a factor of $m_c^2/M_Z^2$. We calculated the fragmentation functions $D(z, \mu)$ for charm quarks or antiquarks to split into S-wave charmonium states to leading order in $\alpha_s$. The fragmentation functions satisfy Altarelli-Parisi evolution equations which can be used to sum up large logarithms of $M_Z/m_c$. These fragmentation functions are universal, applying to the production of heavy quarkonium in any high energy process that can produce heavy quarks with energy large compared to their mass. We applied them to the production of charmonium and bottomonium in decays of the $Z^0$, $W^\pm$, top quark, and Higgs boson.

A complete calculation of the rate for $\psi$ production in $Z^0$ decay must include the production of the P-wave charmonium states $\chi_{cJ}$, followed by their radiative decays into $\psi$. In calculating the production of P-wave charmonium states, there are two distinct contributions that must be included at leading order in $\alpha_s$. The P-wave state can arise either from the production of a collinear $c\bar{c}$ pair in a color-singlet P-wave state, or from the production of a collinear $c\bar{c}$ pair in a color-octet S-wave state [9]. The calculation of the fragmentation functions for the splitting of charm quarks into P-wave charmonium states will be presented elsewhere [10].

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Figure Captions

1. The four Feynman diagrams for $Z^0 \to \psi \bar{c}c$ at leading order in $\alpha_s$.

2. The fragmentation functions $D_{c\to\psi}(z, \mu)$ and $D_{c\to\eta_c}(z, \mu)$ as a function of $z$ for $\mu = 3m_c$ (solid lines) and $\mu = M_Z/2$ (dotted lines).