Abstract. The Lie algebra of pseudodifferential symbols on the circle has a nontrivial central extension (by the “logarithmic” 2-cocycle) generalizing the Virasoro algebra. The corresponding extended subalgebra of integral operators generates the Lie group of classical symbols of all real (or complex) degrees. It turns out that this group has a natural Poisson-Lie structure whose restriction to differential operators of an arbitrary integer order coincides with the second Adler-Gelfand-Dickey structure. Moreover, for any real (or complex) $\alpha$ there exists a hierarchy of completely integrable equations on the degree $\alpha$ pseudodifferential symbols, and this hierarchy for $\alpha = 1$ coincides with the KP one, and for an integer $\alpha = n \geq 2$ and purely differential symbol gives the $n$-KdV-hierarchy.
1. We begin with a description of our main object, the ring $\mathfrak{G}$ of pseudodifferential symbols and of its central extension.

This ring consists of formal series $A(x, D) = \sum_{i=-\infty}^{n} a_i(x)D^i$ with respect to $D (= d/dx)$ where $a_i \in C^\infty(S^1, \mathbb{R} \text{ or } \mathbb{C})$. The multiplication law in $\mathfrak{G}$ is given by the Leibnitz rule for multiplication of symbols: $A(x, \xi) \circ B(x, \xi) = \sum \frac{1}{n!} A^{(n)}(x, \xi) B^{(n)}(x, \xi)$ where $A^{(n)} = d^n/d\xi^n A(x, \xi)$, $B^{(n)} = d^n/dx^n B(x, \xi)$, and the Lie algebra structure on $\mathfrak{G}$ is natural: $[A, B] = A \circ B - B \circ A$. Recall also that the operator $\text{res} : \mathfrak{G} \to C^\infty(S^1)$ is defined by $\text{res} \left( \sum a_i(x)D^i \right) = a_{-1}(x)$.

Consider the formal expression $\log D$. Certainly, $\log D \notin \mathfrak{G}$, but for any $A \in \mathfrak{G}$ the formal commutator $[\log D, A] = \log D \circ A - A \circ \log D$ is an element of $\mathfrak{G}$. Thus $\log D$ acts on $\mathfrak{G}$ by commutation $[\log D, \bullet]$, and explicitly this action is given by: $[\log D, A] = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} A^{(k)} D^{-k}$.

**Theorem 1.** [10] The following 2-cocycle

$$c(A, B) = \int \text{res} \left( [\log D, A] \circ B \right) = \int \text{res} \left( \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} A^{(k)} D^{-k} \circ B \right)$$

(1)

gives a nontrivial central extension of the Lie algebra $\mathfrak{G}$ (here $A$ and $B$ are arbitrary pseudodifferential symbols on $\mathbb{R}$ or $S^1$). The restrictions of this cocycle to the subalgebra $\mathfrak{G}_{DO}$ of differential operators and to the Lie algebra of vector fields $\text{Vect}$ are nontrivial.

For $\mathfrak{G}_{DO}$ the restriction gives the Kac-Petersen cocycle [9], for $\text{Vect}(S^1)$ we get the Gelfand-Fuchs cocycle which defines the Virasoro algebra.

**Remark.** The other 2-cocycle on $\psi DS(S^1)$ is given by $c'(A, B) = \int \text{res} \left( [x, A] \circ B \right)$. One can define the universal extension of $\psi DS(S^1)$ keeping the symmetry of $x$ and $D$.

2. Let $\tilde{\mathfrak{G}}$ be the “double extension” of $\mathfrak{G}$, i.e. we extend the algebra $\mathfrak{G}$ by the 2-cocycle $c(L, M)$ and by the symbol $\log D : \tilde{\mathfrak{G}} = \left\{ \left( \sum_{j=-\infty}^{n} a_j(x)D^j + \lambda \log D, c \right) \right\}$. Here $c$ lies in the center, the commutators of the rest are given by the Leibnitz rule above, $\lambda \in \mathbb{R}$ or $\mathbb{C}$ and, finally, $c(\log D, M) = 0$ for $M \in \mathfrak{G}$.

The algebra $\tilde{\mathfrak{G}}$ (as well as $\mathfrak{G}$, cf.[7]) has an $ad$-invariant nondegenerate inner product (“Killing form”): $((A + \lambda \log D, c), (B + \mu \log D, d)) = 2 \int \text{res}(A \circ B) + \lambda d + \mu c$ for $A, B \in \mathfrak{G}$ (the $ad$-invariance of this form is an immediate corollary of the definition of $\tilde{\mathfrak{G}}$). Moreover, this algebra has two remarkable subalgebras: 1) $\tilde{\mathfrak{G}}_{DO}$ which is the algebra of centrally extended differential operators ($\sum_{j \geq 0} a_j(x)D^j, c$), where $c$ is the two cocycle
\[ c(A, B) = \int \text{res} ([\log D, A] \circ B) \] and \(2) \tilde{\mathfrak{G}}_{\text{Int}} \) which is the algebra of integral symbols together with \( \log D : \{ \sum_{j=-\infty}^{-1} a_j(x)D^j + \lambda \log D \} \).

**Theorem 2.** \( (\tilde{\mathfrak{G}}, \tilde{\mathfrak{G}}_{\text{DO}}, \tilde{\mathfrak{G}}_{\text{Int}}) \) is a Manin triple (or, equivalently, \( \tilde{\mathfrak{G}}_{\text{Int}} \) is a Lie bialgebra).

The proof consists of the observation that \( \tilde{\mathfrak{G}} = \tilde{\mathfrak{G}}_{\text{DO}} \oplus \tilde{\mathfrak{G}}_{\text{Int}} \) and the subalgebras \( \tilde{\mathfrak{G}}_- = \tilde{\mathfrak{G}}_{\text{Int}} \) and \( \tilde{\mathfrak{G}}_+ = \tilde{\mathfrak{G}}_{\text{DO}} \) are isotropic with respect to the inner product discussed above.

**Corollary.** The Lie group \( \tilde{\mathfrak{G}}_{\text{Int}} \) corresponding to the Lie bialgebra \( \tilde{\mathfrak{G}}_{\text{Int}} \) has a natural Poisson-Lie structure (the next section contains a detailed description of this group).

This structure can be defined by the following bivector \( r \in \tilde{\mathfrak{G}} \land \tilde{\mathfrak{G}} : \langle r, \tilde{\alpha}_+ \land \tilde{\alpha}_- \rangle = -(r, \tilde{\alpha}_- \land \tilde{\alpha}_+) = (a_+, a_-) \). Here \( a_{\pm} \in \tilde{\mathfrak{G}}_{\pm} \), by \( \tilde{\alpha} \) we denote the element of \( \tilde{\mathfrak{g}}^* \) dual to \( a \in \tilde{\mathfrak{g}} : \langle \tilde{\alpha}, \ast \rangle = (a, \ast) \), where \( \langle \ast, \ast \rangle \) is the natural pairing and \( \langle \ast, \ast \rangle \) is the Killing form. If we identify the space \( \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}} \) with \( \text{Hom}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}) \) using the pairing on \( \mathfrak{g} \) then the bivector \( r \) corresponds to the skew symmetric operator \( \tilde{\tau} \) such that \( \tilde{\tau}|_{\tilde{\mathfrak{g}}_-} = -1, \tilde{\tau}|_{\tilde{\mathfrak{g}}_+} = 1. \)

To recall following [3],[12],[8] explicit formulae for the group Lie-Poisson structure let \( \tilde{G}, \tilde{G}_+ \) and \( \tilde{G}_- \) be the Lie groups corresponding to the Lie algebras \( \mathfrak{g}, \mathfrak{g}_+ \) and \( \mathfrak{g}_- \), and let \( \theta \) and \( \zeta \) be cotangent vectors to \( \tilde{G}_- \) at a point \( g \in \tilde{G}_- \) \( (\theta, \zeta) \in T^*_g \tilde{G}_- \). Extend them arbitrarily to cotangent vectors (at \( g \)) to the larger group \( \tilde{G} \supset \tilde{G}_- \) (we denote these extensions by the letters \( \theta' \) and \( \zeta' \)).

**Proposition.** [3],[12],[8] The Poisson structure on \( \tilde{G}_- \) is defined by the following formula

\[ \pi_{\tilde{G}_-}(\theta, \zeta) = ((\theta')_+, (\zeta')_+) - ((Ad^*_g \theta')_+, Ad^*_g \zeta'). \]

**3.** In this section we describe the structure and geometry of the Lie group \( \tilde{G}_{\text{Int}} \) corresponding to the Lie algebra \( \tilde{\mathfrak{G}}_{\text{Int}} \) of integral symbols extended by the log \( D \).

While the Lie algebra \( \tilde{\mathfrak{G}}_{\text{DO}} \) of differential operators does not seem to have a reasonable Lie group, the dual part of the Manin triple, i.e. the algebra \( \tilde{\mathfrak{G}}_{\text{Int}} \) admits the Lie group with the following simple description.

**Definition.** Classical Volterra’s pseudodifferential symbol \( (\psi DS) \) is an expression of the form \( P = \left( 1 + \sum_{k=-\infty}^{-1} u_k(x)D^k \right) \circ D^\alpha \), where \( \alpha \in \mathbb{R} \) or \( \mathbb{C} \), \( D = d/dx \), \( u_k \in C^\infty (\mathbb{R} \text{ or } S^1) \). Call the (real or complex) number \( \alpha \) the degree of the symbol \( P \). The multiplication of the
symbols is uniquely defined by the commutation relation 
\[ [D^{\alpha}, u(x)] = \sum_{\ell \geq 1} \left( \frac{\alpha}{\ell} \right) u^{(\ell)}(x) D^{\alpha-\ell} \]
where \( \left( \frac{\alpha}{\ell} \right) = \frac{\alpha(\alpha-1)\cdots(\alpha-\ell+1)}{\ell!} \). All such symbols form a group with respect to this product.

Define on the set of \( \psi DS \) the topology as the standard topology on the line in the direction of \( \alpha \) and the topology of the projective limit along the variable \( k \). For an individual \( k \) we consider a usual \( C^\infty(\mathbb{R}) \) (or \( C^\infty(S^1) \)) topology on the coefficients \( u_k \). Then the basic neighborhoods of a point \( P(0) \) are the sets of \( P \)'s such that 
\[ |\alpha - \alpha(0)| < \varepsilon, \\
|u_k(x) - u_k^{(0)}(x)| < \varphi(x), \ k = 0, \ldots, \ell \]
for fixed \( \varepsilon, \ell \) and a fixed positive function \( \varphi(x) \).

**Theorem 3.** The set of classical Volterra’s symbols equipped with this topology forms a Lie group (denote it as \( \tilde{G}_{\text{Int}} \)). The corresponding Lie algebra coincides with \( \tilde{G}_{\text{Int}} \).

**Remark.** It is evident that the Lie algebra for the subgroup \( \{(1+ \sum_{k=-\infty}^{-1} u_k D^k)\} \) is \( \mathfrak{g}_{\text{int}} \) consisting of integral symbols \( \sum_{k=-\infty}^{-1} u_k D^k \). Heuristically, the tangent vector to \( D^{\alpha} \) can be obtained by differentiation of this 1-parameter subgroup with respect to \( \alpha \) at \( \alpha = 0 \):
\[ \frac{d}{d\alpha} \bigg|_{\alpha=0} D^{\alpha} = \log D \circ D^{\alpha} \bigg|_{\alpha=0} = \log D. \]

To define the exponential map \( \tilde{\mathfrak{g}}_{\text{Int}} \to \tilde{G}_{\text{Int}} \) let us fix an integral symbol \( A \). The would be exponent \( P(t) = \exp(t(\lambda \log D + A)) \) of the symbol \( \lambda \log D + A \) should satisfy the equation \( \left( \frac{d}{dt} P(t) \right) \circ (P(t))^{-1} = \lambda \log D + A \).

**Theorem 4.** The exponential map \( \exp : \tilde{\mathfrak{g}}_{\text{Int}} \to \tilde{G}_{\text{Int}} \) given by the relation \( t(\lambda \log D + A) \mapsto P(t) \) is well-defined on the entire Lie algebra \( \mathfrak{g}_{\text{Int}} \) and for fixed \( t \neq 0 \) it is a bijection between \( \tilde{\mathfrak{g}}_{\text{Int}} \) and \( \tilde{G}_{\text{Int}} \).

**Remark.** The group \( \tilde{G}_{\text{Int}} \) is an infinite dimensional analog of a unipotent group. Analogously to the finite dimensional case where the exponential map is one-to-one and the exponential series consists of a finite number of terms, in our situation for \( \psi DS \) every coefficient in \( P = \exp(\log D + A) \) is defined by a finite number of terms of the symbol \( A \).

4. In this section we show how the general Poisson-Lie group techniques (see [3, 12, 8]) can be applied to the group of pseudodifferential symbols. As a corollary of these constructions we obtain the Gelfand-Dickey- (also called Adler-Gelfand-Dickey- or generalized KdV-)structures ([1],[5]) on the symbols.

**Definition.** The (second generalized) Gelfand-Dickey Poisson structure on \( \tilde{G}_{\text{Int}} = \{(1+ \sum_{k=-\infty}^{-1} u_k(x) D^k) \circ D^{\alpha}\} \) is defined as follows:

a) The value of the Poisson bracket of two functions at the given point is determined by the restriction of these functions on the subset \( \alpha = \text{const.} \).
b) The subset $\alpha = \text{const}$ is an affine space, so we can identify the tangent space to this subset with the set of operators of the form $\delta L = (\sum_{k=-\infty}^{-1} \delta u_k D^k) D^\alpha$. We can also identify the cotangent space with the space of operators of the form $X = D^{-\alpha} \overline{X}$, where $\overline{X}$ is a differential operator, using the pairing $F_X(\delta L) \overset{\text{def}}{=} \langle X, \delta L \rangle = \int \text{res} \delta L \circ X$.

c) Now it is sufficient to define the bracket on linear functionals, and

$$\{F_X, F_Y\}_L = F_Y(V_{F_X}(L)), \quad V_{F_X} \text{ is the following Hamiltonian mapping}$$

$$V_{F_X}(L) \mapsto V_{F_X}(L) \text{ (from cotangent space } D^{-\alpha} \circ \mathfrak{S}_{DO} \text{ to the tangent space } \mathfrak{S}_{\text{Int}} \circ D^\alpha):$$

$$V_{F_X}(L) = (LX)_+ L - L(XL)_+ .$$

**Remark.** Usually this definition is given only in the case when $\alpha$ is a fixed positive integer and $L$ is a differential operator, cf. [1], [5], [7].

**Theorem 5.** The Poisson structure $\pi_{\tilde{G}_{\text{Int}}}$ (described in section 2) on the Poisson-Lie group $\tilde{G}_{\text{Int}}$ coincides with the second generalized Gelfand-Dickey structure.

**Corollary.** The function $\alpha = \deg L$ of degree of $\psi DS$ is the Casimir function, i.e. $\{\deg L, \varphi(L)\} = 0$ for any function $\varphi(L)$ on $\tilde{G}_{\text{Int}}$. Hence we can restrict the bracket to any level $\deg L = \text{const}$.

**Remark.** This is an analogue of the $GL_n$-Gelfand-Dickey structure. The analog of the $SL_n$-structure on the submanifold $u_{-1}(x) \equiv 0$ is a result of the Poisson reduction along the action on $\tilde{G}_{\text{Int}}$ of the multiplicative group of nonvanishing functions.

**Remark.** This viewpoint can be useful to describe the relation of the $W_n$- to $W_\infty$-algebras appeared recently in theoretical physics (see e.g. [11]). The classical $W_n$-algebras (or $SL_n$-Gelfand-Dickey algebras) are the Poisson algebras of functions on the sections of the subgroup of $\tilde{G}_{\text{Int}}$ (for which $\{u_{-1}(x) \equiv 0\}$) by hyperplanes $\alpha = n$. Let us linearize this quadratic Poisson structure at the identity element of $\tilde{G}_{\text{Int}}$. This linear Poisson structure on $\mathfrak{S}_{\text{Int}}$ is exactly the Lie algebra structure on $\mathfrak{S}_{DO}$, i.e. on the centrally extended Lie algebra of all differential operators, and it has been recognized as $W_{1+\infty}$ ([2]). The condition $u_{-1}(x) \equiv 0$ implies discarding of functions, i.e. of zero order differential operators in $\mathfrak{S}_{DO}$ and gives exactly the $W_\infty$-algebra.

5. Finally, we describe integrable hierarchies related to the construction discussed.

Let us fix $\alpha \neq 0$. The corresponding “hyperplane” $L_\alpha = \{L = (1 + \sum_{k=-\infty}^{-1} u_k D^k) \circ D^\alpha\}$ in $\tilde{G}_{\text{Int}}$ is a Poisson submanifold ($\alpha$ is a Casimir functional). Consider the following infinite sequence of evolution equations on the coefficients of $L \in L_\alpha$:
\[
\frac{\partial L}{\partial t_m} = \left[ L, (L^{m/\alpha})_+ \right], \quad m = 1, 2, \ldots
\]  

(2)

Here the derivative \( \frac{\partial}{\partial t_m} \) denotes \( m \)th flow of the hierarchy. An arbitrary power of any operator in \( \tilde{G} \text{Int} \) is well-defined due to the “good properties” of the exponential map \( \tilde{G} \text{Int} \to \tilde{G} \text{Int} \) (theorem 4). Indeed, the operator \( L \) defines uniquely the 1-parameter subgroup passing through it, and the power of \( L \) is the parameter along this subgroup.

In particular, the degree of the operator \( L^{m/\alpha} \) is equal to \( m \), i.e. to an integer, and thus the operation + (taking the differential part) does make sense.

**Theorem 6.**
1. For any positive integer \( m \) the equation (2) defines an evolution on \( L_\alpha \) (i.e. the vector \( [L, (L^{m/\alpha})_+] \) is tangent to \( L_\alpha \)).

2. For any \( m \) the equation is Hamiltonian on \( L_\alpha \) with respect to the (second generalized) Gelfand-Dickey bracket and the Hamiltonian function \( H_m(L) = \frac{\alpha}{m} \int \text{res}(L^{m/\alpha}) \).

3. The sequence \( \{H_m\}, m \in \mathbb{N} \) is an infinite set of integrals in involution with respect to the second Gelfand-Dickey structure.

3’. Any two flows of the hierarchy (2) commute.

**Remark.** For \( \alpha = 1 \) we get nothing else but the standard KP-hierarchy on \( L_1 = \{(D + u_{-1}(x)D^0 + u_{-2}(x)D^{-1} + \cdots)\} \) (usually to get the SL-case instead of the GL-one we put \( u_{-1} \equiv 0 \)). For integer \( \alpha = n \geq 2 \) we get the \( n \)th KdV-hierarchy. Usually the classical KdV-flows are considered only on the differential part of the \( L_n \) (such a Poisson submanifold in \( L_\alpha \) exists for integer \( \alpha \) only) rather than on the entire space \( L_n \). Thus we may restrict ourselves by \( u_{n-1} \equiv u_{n-2} \equiv \cdots \equiv 0 \).

Thus within the framework of the Poisson-Lie group of \( \psi DS \) it is natural to define KP-KdV-hierarchies for any real (or complex) \( \alpha \) (let us call them fractional- or \( \alpha \)-KP-KdV-hierarchies). This construction can be viewed as a natural generalization of the approach of Gelfand and Dickey [4] to the \( n \)th root of an integer degree \( n \) differential operator. It would be interesting to find a relation of our construction of integrable hierarchies to that in [6]. Proofs and applications of the Poisson-Lie structure to the Poisson properties of the Miura transform and of the dressing action on \( \psi DS \) will appear elsewhere.

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