Effective hydrodynamic field theory and condensation picture of topological insulators

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While many features of topological band insulators are commonly discussed at the level of single-particle electron wave functions, such as the gapless Dirac spectrum at their boundary, it remains elusive to develop a hydrodynamic or collective description of fermionic topological band insulators in 3+1 dimensions. As the Chern-Simons theory for the 2+1-dimensional quantum Hall effect, such a hydrodynamic effective field theory provides a universal description of topological band insulators, even in the presence of interactions, and that of putative fractional topological insulators. In this paper, we undertake this task by using the functional bosonization. The effective field theory in the functional bosonization is written in terms of a two-form gauge field, which couples to a $U(1)$ gauge field that arises by gauging the continuous symmetry of the target system (the $U(1)$ particle number conservation). Integrating over the $U(1)$ gauge field by using the electromagnetic duality, the resulting theory describes topological band insulators as a condensation phase of the $U(1)$ gauge theory (or as a monopole condensation phase of the dual gauge field). The hydrodynamic description, and the implication of its duality, of the surface of topological insulators are also discussed. We also touch upon the hydrodynamic theory of fractional topological insulators by using the parton construction.

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I. INTRODUCTION

The recent discoveries of time-reversal symmetric topological band insulators in two and three dimensions have greatly extended our understanding on topological phenomena in condensed matter physics.4–4 They clearly demonstrate how topological phases beyond the physics of the quantum Hall effect (quantum Hall effect) emerges at the level of single particle physics. It remains, however, to be understood the effects of interactions; we need to understand topological states of matter where electron-electron interactions are not necessarily weak, or even those topological phases that arise precisely because of strong correlations.

One possible approach to address these questions is to develop collective or hydrodynamic descriptions of topological (band) insulators.5,6 These coarse-grained descriptions are to be contrasted with more microscopic descriptions which heavily rely on free electrons or nearly free quasiparticles. In fact, a hydrodynamic picture was developed for the quantum Hall effect, and involves the
Chern-Simons gauge theory. Once such effective description of the low-energy physics is established, it is likely to be robust against interactions, and has a wider range of applicability than the non-interacting microscopic system. The Chern-Simons field theory in the context of the fractional quantum Hall effect has been used as a vital tool to describe and predict quasiparticle statistics, ground state degeneracy, the properties of the gapless edge states, etc. (For a review see, e.g. Ref. [10].)

The purpose of the paper is to develop a hydrodynamic effective field theory description of topological insulators. The method of our choice is the functional bosonization procedure. The functional bosonization is a recipe to derive an effective action, which reproduces the correlation functions of the conserved currents (“hydrodynamic modes”) of the system. The functional bosonization approach relies on the gauge invariance of the original, microscopic system, e.g., the $U(1)$ gauge invariance of the conserve electromagnetic charge. The resulting effective field theory contains a dynamical gauge field whose gauge group is determined by the symmetry of the microscopic system. In this sense, this procedure may also be thought of as a procedure which is akin to gauging, a useful technique to study symmetry-protected topological phases in general.

By making use of the functional bosonization procedure, in Ref. [6] a new quantum field theory description of (both non-interacting and interacting) topological insulators in two and three dimensions was proposed. It consists of the BF topological field theory supplemented with an axion term. BF topological field theory has played a key role in the description of topological phases of matter ranging from superconductors to topological insulators. This is a first step toward understanding and describing the fractional topological insulator in three dimensions. As expected, the effective field theory reproduces all universal properties of topological band insulators, such as the topological electromagnetic effect. In addition, once written in terms of hydrodynamic degrees of freedom, there is a natural way (at least at the level of field theories) to incorporate the effects of interactions, in particular, the fractionalization of electrons. With the working hypothesis of electron fractionalization (i.e., the parton construction), the effective field theory predicts, for example, the fractionalized version of the topological magnetoelectric effect, and non-trivial ground state degeneracy when the system is put on a manifold with non-trivial topology. Such predictions can be compared with future numerical studies and experiments. Furthermore, the field theory description is a natural generalization of the Chern-Simons hydrodynamic field theory for the fractional quantum Hall effect, and hints a clue to generalize important theoretical ideas, such as the particle-vortex duality, statistical transmutation by flux attachment, parton construction, and anomalies, among others.

Guided by these previous works, we further continue to develop a hydrodynamic description of both interacting and non-interacting topological insulators in 3+1 space-time dimensions. While for the case of non-interacting topological insulators, this may be a mere rewriting of the non-interacting theory, it would give us a theoretical framework to discuss weakly or moderately interacting topological insulators, and putative fractional topologically insulators. We follow the spirit of our previous work, and try to develop understandings in terms of the hydrodynamic degrees of freedom – we will make use of the hydrodynamic effective field theory. In particular, we discuss significant issues that were left out in our previous papers.

The outline of the paper and the main results are summarized as follows:

Firstly, as noted in Ref. [6] (see also Ref. [27]), the BF-theory with the axion term is not yet written solely in terms of hydrodynamic degrees of freedom. The theory includes a $U(1)$ gauge field $a_\mu$ which is not directly tied to hydrodynamic variables (densities) and can be thought of as a higher dimensional analogues of “statistical gauge fields”, which appear in the composite particle theories of quantum Hall liquid. (See Sec. II A). In this paper, we complete our mission of deriving effective field theories written solely in terms of hydrodynamic degrees of freedom by integrating over the statistical gauge field (Sec. II B).

Along the course of implementing these technical steps, we will also note that the integration of the statistical gauge field can be viewed as a procedure which effective implements the electromagnetic duality of the Maxwell gauge field (Sec. II C). This allows us to develop some physical picture of topological insulators; By making a comparison with Julia-Toulouse approach to defect condensation, we will show a topological band as well as trivial insulator phase can be viewed as a Higgs phase of the statistical gauge field. This is in analogy with the interpretation of the quantum Hall effect in the composite boson theory where the quantum Hall liquid is viewed as arising by the condensation of composite bosons. (For a similar condensation picture for bosonic topological insulators, see Refs. [29 and 30].)

Furthermore, we will implement another aspect which was not fully discussed in the previous work, in particular the compact nature of the gauge field, in Section III. In the spirit of the functional bosonization approach, the method of our choice to derive the hydrodynamic field theory, we rely on the gauge invariance of the original, microscopic theories, e.g., the $U(1)$ gauge invariance associated to the charge conservation. The functional bosonization of Refs. [11 and 12] is a recipe that allow to derive an effective action, which reproduces the correlation functions of the current associated to the gauge invariance. In the presence of monopoles, i.e., if one were interested in the response of the system to the introduction of monopoles, the $U(1)$ gauge field must be treated as a compact variable. The compact nature of the $U(1)$ gauge field can be made explicit by considering the monopole gauge transformations. They are
discrete two-form gauge transformations, which originate from the arbitrariness of the location of the Dirac strings emanating from monopoles. (See, e.g., Ref. [33].) The system must be invariant under the monopoles gauge transformations in order for the precise locations of the Dirac strings not to affect physics. Following the spirit of the functional bosonization, one can derive a hydrodynamic theory for the collective variables associated to the monopole gauge invariance. We will show how this procedure can be implemented. Once the compact nature of the gauge field is fully incorporated, the resulting bosonized theory has much resemblance with the Cardy-Rabinovici theory and the description of the condensed phase of the Abelian-Higgs model in Ref. [35].

Subsequently, we will also discuss the boundary (surface) of topological insulators in terms of the hydrodynamic effective field theory in Section IV. As in the bulk, the statistical gauge field can be integrated over to obtain a hydrodynamic effective field theory. This process, as in the bulk, can be viewed as an implementation of the electromagnetic duality, and relates two different 2+1 dimensional theories with and without the statistical gauge field theory. This surface duality is essentially the bosonized version of the recently proposed duality between the free Dirac fermion and QED$_3$ in Refs. [37–39]. In addition, the resulting hydrodynamic theory is compared with the Fradkin-Kivelson theory in Ref. [40], which enjoys PSL(2, $\mathbb{Z}$) duality symmetry.

Finally, in Section V, we discuss putative fractional topological insulators by using the parton construction. Assuming the electron fractionalization into partons, we used the functional bosonization to derive the bulk and surface hydrodynamic theories of fractional topological insulators. We show the resulting theory in the bulk is the $\mathbb{Z}_K$ Cardy-Rabinovici theory with $K > 1$. We conclude in Section VI by discussing open problems.

II. FUNCTIONAL BOSONIZATION, ELECTROMAGNETIC DUALITY, AND THE JULIA-TOULOUSE APPROACH

In this section, we review the functional bosonization of $D = 3 + 1$-dimensional topological insulators presented in Ref. [6]. For technical simplicity, we will focus on topological insulators in symmetry class AIII in $D = 3 + 1$, characterized by an integer-valued topological invariant, the three-dimensional winding number $\nu$, and protected by chiral symmetry. This topological insulator is somewhat analogous to the time-reversal symmetric topological insulator in symmetry class AII, in that it supports a Dirac fermion surface state, and has a nontrivial axion-electrodynamics response to the external electromagnetic field. The difference is, however, that the latter is characterized by a $\mathbb{Z}_2$ topological invariant, rather than an integer topological invariant. To capture the $\mathbb{Z}_2$ nature of topological insulators in AII class, one needs to consider a dimensional reduction from a one higher dimension $D = 4 + 1$, whereas topological insulators in class AIII in $D = 3 + 1$ can be studied on its own. An example of topological insulators in AIII class can be found in Ref. 42 which discusses a lattice tight-binding model description. Topological insulators in symmetry class DIII in $D = 3 + 1$ can also be studied in a similar way.

A. Functional bosonization

We start from the partition function in the presence of an external gauge field, $Z[A^{ex}]$, where $A^{ex}$ is an external $U(1)$ gauge field associated to the electromagnetic $U(1)$ gauge invariance. The partition function is invariant under the electromagnetic gauge transformations,

$$Z[A^{ex} + a] = Z[A^{ex}], \quad \text{where} \quad a = d\phi. \quad (2.1)$$

By making use of the gauge invariance, $Z[A^{ex} + a] = Z[A^{ex}]$ with $a = d\phi$, one can average the partition function over $a$:

$$Z[A^{ex}] = \mathcal{N} \int \mathcal{D}[a, b] Z[A^{ex} + a] \times \exp \left( \frac{i}{2\pi} \int_{M_4} b \wedge da \right), \quad (2.2)$$

where $\mathcal{N}$ is a normalization constant, and $M_4$ is the spacetime manifold of our interest. Here, the totally antisymmetric rank two tensor $b = (1/2)\epsilon_{\mu\nu}dx^\mu \wedge dx^\nu$ is introduced such that the integration over $b$ enforces the pure gauge condition $da = 0$. With a shift $a \rightarrow a - A^{ex}$, the partition function is given by

$$Z[A^{ex}] = \mathcal{N} \int \mathcal{D}[a, b] Z[a] \times \exp \frac{i}{2\pi} \int_{M_4} b \wedge (da - dA^{ex}). \quad (2.3)$$

So far, we have not assumed any microscopic details except for the electromagnetic $U(1)$ gauge invariance. i.e., the underlying system can be topologically trivial or nontrivial, and may or may not include interactions. We now specialize to the case of non-interacting topological band insulators, which can be described, at low energies, by a theory of free massive Dirac fermions. In this case, the partition function $Z[a]$ can be evaluated by integrating over fermions in the presence of background gauge fields $a$. The effective action can be expanded in terms of the inverse band gap, and written as

$$Z[a] \propto \exp(-W[a]), \quad (2.4)$$

where $W[a]$ has the form

$$W[a] = \frac{1}{g^2} \int da \wedge *da + \frac{i\theta}{8\pi^2} \int da \wedge da + \cdots \quad (2.5)$$

Here, $*$ represents the Hodge dual, and $g$ is the effective coupling constant for the Maxwell term and $\theta$ is the electromagnetic polarizability (the theta angle).
angle is quantized, $\theta = \pi \times \text{(integer)}$ in the presence of time-reversal symmetry (AII) or CT symmetry (AIII).

To summarize, the bosonized partition function (in the Euclidean signature) is given by

$$Z[A^{ex}] = \mathcal{N} \int \mathcal{D}[a, b] \ e^{-S[a, b]}, \quad (2.6)$$

where the effective Euclidean action $S[a, b]$ is

$$S[a, b] = -\frac{i}{2\pi} \int b \wedge (da - dA^{ex}) + \frac{\tau_2}{4\pi} \int da \wedge \ast da + \frac{i\tau_1}{4\pi} \int da \wedge da, \quad (2.7)$$

where we have introduced a complex coupling by

$$\tau = i\tau_2 + \tau_1 = \frac{4\pi}{g^2} + \frac{\theta}{2\pi}. \quad (2.8)$$

These steps of the functional bosonization, starting from the $U(1)$ gauge invariance in Eq. (2.1) to the final bosonized action of Eq. (2.7), were carried out in the previous paper, Ref. 6. We now raise two issues, which we will discuss in the rest of the paper.

1. **Role of the fields $b_{\mu\nu}$ and $a_\mu$**

The bosonized effective theory consists of the path integral over two kinds of gauge fields, a vector (gauge) field $a_\mu$ and an anti-symmetric tensor field (Kalb-Ramond) $b_{\mu\nu}$. The first issue pertains to the role played by these two gauge fields in the functional bosonization.

From the functional derivative of $\ln Z[A^{ex}]$ with respect to the external gauge field $A^{ex}$, one establishes the identification $\epsilon^{\mu\nu\rho\lambda} \partial_\rho b_{\mu\lambda}$ as the electromagnetic $U(1)$ current $j^\mu$ (“bosonization dictionary”),

$$j^\mu := \frac{\delta}{\delta A^{ex}_\mu} \ln Z[A^{ex}] = \frac{1}{2\pi} \epsilon^{\mu\nu\rho\lambda} \partial_\rho b_{\mu\lambda} \quad (2.9)$$

(in the Minkowski signature). On the other hand, the gauge field $a_\mu$ does not appear to be related to any physical quantity. In fact, for the case of the $D = 2 + 1$ dimensional quantum Hall effect or Chern insulators, a comparison between the functional bosonization and the composite boson theory shows that $a_\mu$ plays a role similar to the “statistical gauge field” of the theories of the fractional quantum Hall effect.\(^{43,44}\) In the composite boson theory, this statistical gauge field is introduced to change the fermionic statistics of electrons into bosonic statistics to form “composite bosons” out of electrons.

In the composite boson theory of the quantum Hall effect\(^{43}\) (both integer and fractional quantum Hall effect), the composite bosons undergo condensation, and, as a consequence, the statistical gauge field acquires a mass by a Higgs mechanism. To be more precise, the Meissner effect occurs the combination of the statistical and the electromagnetic $U(1)$ gauge fields. In the condensed phase, one can make use of the boson-vortex duality in $D = 2 + 1$ dimensions to rewrite the theory written in terms of the composite boson field and the statistical gauge field – into the theory written in terms of the “vortex gauge field” interacting with the statistical gauge field. One can then integrate over the statistical gauge field to end up with the single-component hydrodynamic Chern-Simons theory of the vortex gauge field. This vortex gauge field corresponds to the gauge field $b_\mu$ in the functional bosonization. In passing, we note that the composite particle approach is possible only when the magnetic field is non-zero, and cannot be applied to the trivial band insulator, while the functional bosonization does apply to both trivial and non-trivial band insulators as well.

Following our discussion in $D = 2 + 1$, we can interpret $a_\mu$ in $D = 3 + 1$ as a counterpart of the statistical gauge field $a_\mu$ in $D = 2 + 1$ dimensions. On the other hand, $b_{\mu\nu}$ is directly related to the electric charge current, and hence it is a natural hydrodynamic variable, analogous to the hydrodynamic gauge field of the quantum Hall effect in $2+1$ dimensions.\(^{9}\) Following $D = 2 + 1$, it is desirable to integrate over statistical gauge fields $a_\mu$ to obtain the theory written solely in terms of the hydrodynamic variable.

2. **Condensation picture**

The above comparison with the composite particle theory (the composite boson theory) of the quantum Hall effect leads to our second issue. In the quantum Hall effect or in Chern insulators, the insulator phases are interpreted as a phase where composite bosons condense. On the other hand, it is not clear (yet) what is the physical picture suggested by the effective bosonized action of Eq. (2.7). It is highly desirable to establish a physical interpretation of (topological) insulator phases within the functional bosonization. In the following, we will address these issues.

3. **Compact v.s. non-compact $U(1)$ gauge invariance**

Before proceeding, it is important to emphasize that we have treated both $A^{ex}$ and $a$ as non-compact $U(1)$ gauge fields. This may be justified, in the spirit of the functional bosonization, if we are interested only in the response of the system to smooth configurations of $A^{ex}$; The bosonized action of Eq. (2.7) is capable of describing the system’s response to smooth configurations of $A^{ex}$, which does not include monopoles. It is however well-known that one of the defining properties of topological insulators is their response to monopoles.\(^{45}\) For this reason, it is desirable to develop the functional bosonization scheme which fully takes into account the presence of monopoles in $A^{ex}$ (and $a$ as well). Note that, as a consequence of the presence of monopoles, electric charges in
the system are quantized in the unit of the elementary magnetic charge. This is the Dirac quantization condition.

Instead of considering monopoles in $A^{\text{ex}}$, one can impose the quantization of electric charges from the outset. This in turn makes the gauge field $A^{\text{ex}}$ (and $a$ as well) an angular variable, and the corresponding gauge group is compact $U(1)$. (Note also that if the system of interest is defined on a lattice, a compact $U(1)$ gauge field can be introduced naturally, by defining the gauge field on links. However, the existence of an underlying lattice is neither necessary nor sufficient to discuss the compact $U(1)$ gauge theory.) Because of the compact nature of the gauge field, discontinuous configurations of $A^{\text{ex}}$ (and $a$) are allowed, and hence monopoles exist in the compact $U(1)$ gauge theory.

These two mechanisms of charge quantization, one because of the Dirac quantization condition in the presence of monopoles, and the other in which it is enforced from the outset, may seem logically independent. These two points of view, however, are essentially the same. Nevertheless, details of the bosonization procedure differ slightly depending on which points of view we take; if we take the gauge group to be a compact $U(1)$, or if we merely allow the presence of monopoles in the system. The latter point of view can be implemented as the monopole gauge theory, as we will discuss.

The reason why we have emphasized compact v.s. non-compact nature of the gauge fields is that this is closely related to the condensation picture of topological (as well as ordinary) insulators; condensed phases of the $U(1)$ gauge theory, which can possibly coupled to matter fields of various kinds, may be described as condensation of electric charges (Higgs phases), condensation of magnetic charges (confined phases), or condensation of both magnetic and electric charges (oblique confinement). To be able to access and discuss these phases, one of which may describe (topological) insulator phases, it is well-adviced to keep the compact nature of the gauge fields and monopoles in $A^{\text{ex}}$ and $a$. In fact, as our analysis below will reveal, topological as well as ordinary band insulator phases can be identified as the Higgs phase of the gauge field $a$ or the monopole condensation phase of the dual gauge field of $a$.

4. The Julia-Toulouse mechanism

In the next section, we will integrate over the “statistical” gauge field $a_{\mu}$. We first attempt this in a direct way (see below), and then make a connection to the electromagnetic duality (S-duality). While we have emphasized the importance of including compact nature of the $U(1)$ gauge field, we will first proceed with the “non-compact version” of the bosonized action of Eq. (2.7): We postpone to discuss the compact $U(1)$ gauge field in Sec. III.

As we will demonstrate, the $b$ field, once the compact nature of the gauge field is implemented, is treated as a discrete variables. Even so, however, if defects (monopoles) of the dual gauge field condense, the $b$ field can be treated as a continuum variable: From the point of view of the dual gauge field, $db$ represents monopole currents (recall that $db$ represents electric currents in terms of the original gauge field $A^{\text{ex}}$ and $a$). Once monopoles in the dual gauge field proliferate, $db$ can be treated as a continuum variable. This is nothing but the Julia-Toulouse approach to defect condensation.

In Appendix C, we give a short summary of the Julia-Toulouse approach following the work of Quevedo and Trugenberger.

In the next section, we integrate over the non-compact statistical gauge field $a_{\mu}$ in Eq. (2.7). This will reveal the electromagnetic duality, which, together with the Julia-Toulouse approach, allows us to discuss the Higgs phase as well as the confined phase of the theory with a compact $U(1)$ gauge field $a$. In fact, the BF coupling in the effective action of Eq. (2.7), by construction, enforces $da = 0$ everywhere, which is indicative of the Higgs phase. The compact nature of the $U(1)$ gauge fields and monopoles will be discussed in Sec. III, where the $b$ field is treated as a discrete variable. By making a comparison with the Cardy-Ravinovici theory, we will confirm the condensation picture even when the compact nature of the $U(1)$ gauge field is taken into account.

B. Integration over the statistical gauge field

Since the Euclidean action of Eq. (2.7) is quadratic, the integration over the gauge field $a_{\mu}$ can be done by using its equation of motion:

$$i \frac{\partial}{\partial \lambda} b_{\mu \nu} \varepsilon^{\mu \nu \lambda \kappa} + \tau_2 \partial_{\lambda} f_{\kappa \lambda}[a] + i \frac{\tau_1}{2} \varepsilon^{\mu \nu \lambda \rho} \partial_{\mu} f_{\rho \lambda}[a] = 0,$$

(2.10)

where $f[a]$ is the field strength of $a$. While the last term is identically zero (the Bianchi identity), we will keep this term to be consistent with the derivation using the electromagnetic duality. This equation of motion can be solved by

$$b_{\mu \nu} = \tau_1 f_{\mu \nu} + \tau_2 \frac{i}{2} \varepsilon^{\mu \nu \lambda \rho} f_{\lambda \rho}. $$

(2.11)

This solution, however, is not unique as one could add any term which vanishes when acted with the operator $\varepsilon^{\mu \nu \lambda \kappa} \partial_{\lambda}$. Hence, starting from the solution given above, one can generate the family of solutions

$$b_{\mu \nu} \rightarrow b_{\mu \nu} + \partial_{\mu} v_{\nu} - \partial_{\nu} v_{\mu},$$

(2.12)

since

$$\varepsilon^{\mu \nu \lambda \kappa} \partial_{\lambda} (\partial_{\mu} v_{\nu} - \partial_{\nu} v_{\mu}) = 0.$$ 

(2.13)
To eliminate $a_\mu$, we plug the solution of the equation of motion into the action to obtain

$$S = \frac{\tau_2}{4\pi} \int_{\mathcal{M}_4} (b + dv) \wedge \ast (b + dv)$$
$$+ \frac{-i\tau_1}{4\pi} \int_{\mathcal{M}_4} (b + dv) \wedge (b + dv)$$

where we introduced the dual coupling as

$$\tilde{\tau} = i\tilde{\tau}_2 + \tilde{\tau}_1 = -\frac{1}{\tau} \tag{2.15}$$

where

$$\tilde{\tau}_1 = -\frac{\tau_1}{\tau_1^2 + \tau_2^2}, \quad \tilde{\tau}_2 = \frac{\tau_2}{\tau_1^2 + \tau_2^2} \tag{2.16}$$

are, respectively, the real and the imaginary parts of the dual coupling.

\section*{C. Electromagnetic duality}

We will now give more transparent derivation of the hydrodynamic action (2.14) by invoking the electromagnetic duality. In particular, we will show the one-form $v$ can be interpreted as a dual gauge field to $a_\mu$.

\subsection*{1. Review of the electromagnetic duality in the vacuum}

As a warm up, let us follow Ref. 36 and review the electromagnetic duality of the Maxwell theory in the vacuum which is described by the Euclidean action

$$S = \frac{\tau_2}{4\pi} \int da \wedge \ast da + \frac{i\tau_1}{4\pi} \int da \wedge da. \tag{2.17}$$

To this end, the Maxwell action is expanded by introducing a two-form gauge field $u_{\mu\nu}$ and a one-form gauge field $v_\mu$ as

$$S = \frac{i}{2\pi} \int dv \wedge u$$
$$+ \frac{\tau_2}{4\pi} \int (da - u) \wedge \ast (da - u)$$
$$+ \frac{i\tau_1}{4\pi} \int (da - u) \wedge (da - u). \tag{2.18}$$

In addition to its invariance under the electromagnetic $U(1)$ gauge transformations, this theory is also invariant under the following two-form gauge invariance

$$a_\mu \to a_\mu + \xi_\mu,$$
$$u_{\mu\nu} \to u_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu. \tag{2.19}$$

The extended theory is equivalent to the Maxwell theory, as can be seen upon integrating over $v$ to set $du = 0$. Alternatively, we can first gauge away $a_\mu$ by using a two-form gauge transformation, $a_\mu \to a_\mu + (-a_\mu)$ and $u_{\mu\nu} \to u_{\mu\nu} - \partial_\mu a_\nu - \partial_\nu a_\mu$, and consider

$$S = \int_{\mathcal{M}_4} \left[ \frac{i}{2\pi} dv \wedge u + \frac{\tau_2}{4\pi} u \wedge \ast u + \frac{i\tau_1}{4\pi} u \wedge u \right]. \tag{2.20}$$

One can then integrate over the two-form gauge field $u_{\lambda\rho}$ to get the dual action,

$$S = \int_{\mathcal{M}_4} \left[ \frac{\tau_2}{4\pi} dv \wedge \ast dv + \frac{i\tau_1}{4\pi} (dv \wedge dv) \right]. \tag{2.21}$$

We have thus established the duality (electromagnetic duality or S-duality)

$$a_\mu \leftrightarrow v_\mu, \quad \tau \leftrightarrow \tau = -\frac{1}{\tau}. \tag{2.22}$$

By combining the S-duality of Eq. (2.22) with the periodicity of the theta angle,

$$\tau \to \tau = \tau + n, \quad n \in \mathbb{Z}, \tag{2.23}$$

one can generate the full $SL(2, \mathbb{Z})$ (actually, $PSL(2, \mathbb{Z})$) group of duality transformations, which consists of the following set of modular transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1. \tag{2.24}$$

\subsection*{2. Integrating over the statistical gauge field}

We now go back to our bosonized Lagrangian, and integrate over the statistical gauge field. Our bosonized action differs from the Maxwell theory, Eq. (2.17), by the presence of $b_{\mu\nu}$. Following the discussion on the electromagnetic duality, we introduce two-form and one-form gauge fields, $u_{\mu\nu}, v_\mu$,

$$S = -\frac{i}{2\pi} \int b \wedge (da - u) + \frac{i}{2\pi} \int dv \wedge u$$
$$+ \frac{\tau_2}{4\pi} \int (da - u) \wedge \ast (da - u)$$
$$+ \frac{i\tau_1}{4\pi} \int (da - u) \wedge (da - u). \tag{2.25}$$

Integration over $v$ and $b$ sets $du = 0$ and also $da - u = 0$, and hence the theory is in some sense trivial since after substituting these, the action vanishes identically.

Even in the presence of $b$, the duality transformation presented above can still be carried out and one obtains

$$S = \frac{i}{2\pi} \int b \wedge dA^{\alpha\beta} - \frac{i}{2\pi} \int (b + dv) \wedge U^{\alpha\beta}$$
$$+ \frac{\tau_2}{4\pi} \int (b + dv) \wedge \ast (b + dv)$$
$$+ \frac{i\tau_1}{4\pi} \int (b + dv) \wedge (b + dv). \tag{2.26}$$
where we have reinstated the external sources. Here $U^{\text{ex}}$ is the external monopole gauge field, which will be discussed in detail in the next section. We have thus eliminated statistical gauge fields $a_\mu$ and $u_{\mu\nu}$ and express the theory in terms hydrodynamic degrees of freedom. A similar theory is presented, for example, in Refs. [47–49].

Our final hydrodynamic theory, Eq. (2.26), written in terms of $v$ and $b$, is fully gapped. This should be expected since our original theory is trivial in the sense that upon integration over $b$ and $v$, it sets $du = 0$ and also $da - u = 0$. After substituting these, the action simply vanishes. To see why the theory is fully gapped, we first note that the hydrodynamic theory of Eq. (2.26) is invariant under 2-form gauge transformations $b_{\mu\nu} \rightarrow b_{\mu\nu} + \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$ and $v_\mu \rightarrow v_\mu - \xi_\mu$. By making use of this gauge invariance, we can gauge away the one-form gauge field $v_\mu$ to obtain an action in terms of $b_{\mu\nu}$. The resulting theory is clearly gapped with no propagating degrees of freedom. If one wishes, it would also be possible to add a gauge invariant kinetic term for $b_{\mu\nu}$. We are assuming that we are in the phase where the massive Klein-Gordon equation with the mass $m_1 = \Lambda$ is locally never obeyed since our original theory is trivial in the sense of charges in the statistical gauge field. Observe also that in our example the topological current is given by $J_c = \int da \wedge \delta(N_2)$. By the bosonization rule, this may simply be identified as a charge current. Thus, the conservation of the current, in this interpretation, is because we enforce the theory to be in the monopole condensation phase of $v$.

III. FUNCTIONAL BOSONIZATION WITH MONOPOLES

As advocated, we now make a further step by developing the functional bosonization that can deal with the compactness of $A^{\text{ex}}$. As discussed in Sec. II A, the compact nature of the $U(1)$ gauge field can be incorporated by demanding the quantization of charges or by introducing monopoles into the theory. We will implement functional bosonization in terms both of these point of view. The resulting bosonized theory will be compared with the Cardy-Rabinovici theory.\textsuperscript{34}

A. Bosonization with compact $U(1)$

Let us recall the starting point of the bosonization, Eq. (2.2), in which the flat connection condition $da = 0$ is imposed by the auxiliary field $b$. Instead of imposing $da = 0$ strictly, we can impose $da \equiv 0 \mod 2\pi$ locally (i.e., for all plaquettes if we work on a lattice). If so, the auxiliary two-form gauge field $b_{\mu\nu}$ must be a discrete variable. (This is standard in abelian compact gauge theories on a lattice, see, e.g., Refs. [33, 50, and 51].) To see this, we consider the generalized Poisson summation formula:

$$
\sum_{N_{D-p}} \exp\left(2\pi i \int_{M_D} \delta_p(N_{D-p}) \wedge A_{D-p}\right) = \sum_{Q_p} \delta(A_{D-p} - \delta_{D-p}(Q_p)),
$$

valid for an arbitrary $D - p$ form $A_{D-p}$. Here, the delta function form $\delta_{D-p}(N_p)$ is a $D - p$ form associated to a $d$-dimensional submanifold of spacetime $M_D$, and defined by the relation

$$
\int_{M_D} A_p \wedge \delta_{D-p}(N_p) = \int_{N_p} A_p
$$

for an arbitrary $p$-form $A_p$. Useful properties of the delta forms are summarized in Appendix. In the generalized Poisson identity, Eq. (3.1), the summation $\sum_{N_{D-p}}$ runs over all possible $D - p$ dimensional submanifolds of $M_D$.

Thus, the following sum over the discrete auxiliary field $b$

$$
\sum_{b=\delta(M_2)} \exp\left(iq_\tau \int da \wedge b\right)
$$

enforces $da$ to be given in terms of the delta function for some manifold $N_2$ as:

$$
da = 2\pi q_\tau^{-1} \delta(N_2).
$$

To summarize, analogously to Eq. (2.7), the bosonized partition function/action is given by

$$
Z[A^{\text{ex}}] = \mathcal{N} \int \mathcal{D}[a] \sum_{b=\delta(M_2)} \exp(-S),
$$

where the Euclidean action $S$ now is

$$
S = -iq_\tau \int b \wedge (da - dA^{\text{ex}}) + \frac{\tau_2}{4\pi} \int da \wedge \ast da + \frac{i\tau_1}{4\pi} \int da \wedge da.
$$
Physical observables, e.g. $F_2$, are invariant under a monopole gauge transformation generated by

$$A_1 \to A_1 + \eta_1, \quad \Sigma_2 \to \Sigma_2 + \sigma_2, \quad (3.15)$$

where the one-form $\eta_1$ and the two-form $\sigma_2$ are given in terms of a 3d manifold $\mathcal{M}_3$ as

$$\eta_1 = q_m \delta_1(\mathcal{M}_3),$$

$$\sigma_2 = \delta_2(\partial \mathcal{M}_3). \quad (3.16)$$

The monopole gauge invariance, for example, directly leads to the Dirac quantization condition of the electric and magnetic charge. The minimal coupling between the $U(1)$ gauge field and the electric current

$$S_{min} = q_e \int A_1 \wedge J_{e3} (3.17)$$

is transformed, by monopole gauge transformations, into

$$q_e \int A_1 \wedge J_{e3} \to q_e \int A_1 \wedge J_{e3} + q_e \int \eta_1 \wedge J_{e3}. \quad (3.18)$$

Demanding the invariance of $\exp(iS_{min})$ under the monopole gauge transformations leads to the Dirac quantization condition of the electric and magnetic charges,

$$q_e q_m = 2\pi \times \text{(integer)}, \quad (3.19)$$

where we used that for arbitrary surfaces $\mathcal{M}_p$ and $\mathcal{N}_{D-p}$,

$$\int \delta_{D-p}(\mathcal{M}_p) \wedge \delta_p(\mathcal{N}_{D-p}) = I(\mathcal{M}_p, \mathcal{N}_{D-p}) \quad (3.20)$$

where $I(\mathcal{M}_p, \mathcal{N}_{D-p})$ is the intersection number, which is an integer.

2. Bosonization with monopole gauge invariance

The presence of monopoles demands the introduction of monopole gauge invariance. Following the spirit of the functional bosonization, one can bosonize the conserved current associated with the monopole gauge invariance. Let us start from the partition function in the presence of external gauge fields,

$$Z[A^{ex}, U^{ex}], \quad (3.21)$$

where $A^{ex}$ is an external $U(1)$ gauge field associated to the electromagnetic $U(1)$ gauge invariance, and $U^{ex}$ is an external gauge field associated to the monopole gauge invariance. The partition function is invariant under the following two types of gauge transformations: (i) Electromagnetic gauge transformations,

$$Z[A^{ex} + a, U^{ex}] = Z[A^{ex}, U^{ex}],$$

where $a = d\phi$. \hfill (3.22)
(ii) Monopole gauge transformations,
\[ Z[A^{ex} + \xi, U^{ex} + u] = Z[A^{ex}, U^{ex}] \]
where \[ d\xi + q_m u = 0. \] (3.23)

Here, \( u \) is a “compact” variable given by
\[ u = \delta_2(M_2), \] (3.24)
where \( M_2 \) has no boundaries, \( \partial M_2 = 0 \). If the topology of the spacetime is trivial, by Poincare’s lemma, \( M_2 \) can be written as \( M_2 = \partial M_3 \).

The details of the functional bosonization of \( Z[A^{ex}, U^{ex}] \) with these gauge invariances are presented in Appendix B. The final hydrodynamic theory is given by
\[ Z[A^{ex}, U^{ex}] = \mathcal{N} \int \mathcal{D}[b, v] \sum_{\delta(N_2)} \exp(-S), \] (3.25)
where
\[ S = i \int (q_e b + dv) \wedge q_m U^{ex} + i q_e \int b \wedge dA^{ex} + \frac{\tau_2}{4\pi} \int \alpha \wedge \star\alpha + \frac{i \tau_1}{4\pi} \int \alpha \wedge \alpha \] (3.26)
where
\[ \frac{\alpha}{2\pi} = q_e b + dv + 2\pi q_m^{-1} w. \] (3.27)

The form of the final action is almost identical to the calculations without monopoles that we did in Sec. II. The only modification is the appearance of the discrete variable \( w = \delta(N_2) \).

C. Comparison with the Cardy-Rabinovici theory

To develop a physical interpretation of the final bosonized action, let us make a comparison with the Cardy-Rabinovici theory.\(^{34}\) The Cardy-Rabinovici theory is defined on a four-dimensional hypercubic lattice. Its partition function is given by
\[ Z = \text{Tr}_{\phi,n,s} \prod_r \delta[\Delta \mu n_\mu(r)] \exp(-S), \] (3.28)
where \( \phi_\mu (\mu = 1, \ldots, 4) \) is a compact \( U(1) \) gauge field (an angular variable) defined on the links of the lattice, and \( n_\mu \) and \( s_{\mu\nu} \) are integer-valued fields defined respectively on links and plaquettes, respectively. The integer-valued two-form gauge field \( s_{\mu\nu} \) amounts to allowing multivalued configurations of the gauge field. The sum on \( s_{\mu\nu} \) corresponds to a sum over topologically non-trivial configurations with magnetic monopoles.\(^{30}\) In fact, the monopole current is given explicitly by
\[ m_\mu = \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} \Delta_\nu \delta_{\lambda\sigma}. \] (3.29)
where \( \Delta_\mu \) is the lattice difference operator in the \( \mu \)-direction. On the other hand, we interpret \( n_\mu \) as the electric current of a charge field. The discrete delta function \( \delta[\Delta \mu n_\mu(r)] \) enforces current conservation. The Boltzmann weight is given by
\[ S = -iK \sum_L n_\mu \phi_\mu + \frac{1}{2g^2} \sum_p \Gamma_{\mu\nu} \Gamma_{\mu\nu} \\
- \frac{iK\theta}{32\pi^2} \sum_{r,r'} f(r-r') \epsilon_{\mu\nu\lambda\sigma} \Gamma_{\mu\nu}(r) \Gamma_{\lambda\sigma}(r'), \] (3.30)
where \( \Gamma_{\mu\nu} = \Delta_\mu \phi_\nu - \Delta_\nu \phi_\mu - 2\pi s_{\mu\nu} \) is the field strength. The second and third terms are the Maxwell and axion terms, respectively. (The precise nature of the smearing function \( f(r-r') \) is not important here.) The sum over \( n_\mu \) has the effect of constraining \( \phi_\mu \) to take its values restricted to the abelian cyclic group \( \mathbb{Z}_K \), \( \phi_\mu = (2\pi/K) k_\mu \). Because the sum over \( n_\mu \) is constrained, we can always add any total divergence to \( \phi_\mu \). Thus, the restriction to \( \phi_\mu = (2\pi/K) k_\mu \) represents a partial fixing of the gauge.

The action is quadratic in \( \phi_\mu \), so we may integrate it out to obtain the Coulomb gas representation of interacting electric and magnetic currents:
\[ Z = \text{Tr}_{n,s} \prod_r \delta[\Delta \mu n_\mu(r)] \exp(-S), \] (3.31)
where
\[ S = \frac{2\pi^2}{g^2} \sum_{r,r'} m_\mu(r) G(r-r') m_\mu(r') + \frac{1}{2} K^2 g^2 \sum_{r,r'} n_\mu(r) G(r-r') n_\mu(r') - iK \sum_{r,r'} m_\mu(r) \Theta_{\mu\nu}(r-r') n_\nu(r'). \] (3.32)

Here, \( G \) is the lattice Green function and
\[ n_\mu(r) = n_\mu(r) + \frac{\theta}{2\pi} m_\mu(r) \] (3.33)
is the electric current, modified to include the induced electric charges of the magnetic monopoles due to the Witten effect. The first two terms in the new action describe the Coulomb interactions of a gas of electric and magnetic charges. The last term represents the statistical interaction (the Aharonov-Bohm effect) between an electric current \( K g m_\mu \) and the Dirac string of a magnetic monopole current \( 2\pi g^{-1} m_\mu \). \( \Theta_{\mu\nu} \), the antisymmetric matrix appearing in the last term, is essentially the “angle” between the two currents, in the plane perpendicular to \( m_\mu \) and \( n_\mu \).

The duality of the model can conveniently be described by the complex coupling \( \zeta = \theta/(2\pi) + i(2\pi)/(K g^2) \); Under the duality, \( \zeta \rightarrow -1/\zeta \) and \( \zeta \rightarrow \zeta + 1 \).

By comparing entropy and free energy, Cardy and Rabinovici showed that for certain parameter ranges, there are phases in this model where a condensate of current
loops carrying electric and magnetic charges in the ratio $-p/q$ is formed. For example, setting $m = 0$, we obtain the electric charge condensation. Observe that in this case, the theta term drops out from the action. On the other hand, setting $n = 0$, we obtain the monopole condensation. For generic values of $p$ and $q$, the resulting phase is an example of oblique confinement.

Now, coming back to our bosonized action in Eq. (3.7), the dual gauge field $v$ plays the role of $\phi$, and the discrete variable $b$ the role of $s$. There is no counter part of the electric current $n$ in the bosonized action. After integrating over $v$, the bosonized action is written in terms of the discrete monopole charge $b$. By comparison with the Cardy-Rabinovici theory, the phase represented by the bosonized action is the monopole condensation of the dual gauge field $v$. This, in turn, implies that in terms of the original gauge field $a$, this phase is the Higgs phase of $a$.

On the other hand, the bosonized action (3.25) can be compared with the similar field theory presented in Ref. 35 In Ref. 35, the Higgs phase of the Abelian-Higgs model, where electric charges condense, is described, schematically, by the lagrangian:

$$S = \int \mathcal{D}[a] \left\{ \sum_{\Sigma_g = \delta(N_2)} \exp(i S) \right\},$$

where $\Sigma_g$ corresponds to the monopole gauge field for the electromagnetic $U(1)$ gauge field. The first term in the action is the kinetic term for the two-form gauge field $B_2$, which, for our purpose, can be simply dropped. We can make a correspondence $b \leftrightarrow B$, $w \leftrightarrow \Sigma_g$. In our case, $w$ is the monopole gauge field of the dual gauge field. This in turns means we are in the presence of the dual of the electric charge condensation — i.e. monopole condensation.

### IV. SURFACE THEORY

In this section, we will develop a hydrodynamic theory that describes the $D = 2 + 1$ dimensional surface of 3+1-dimensional topological insulators.

#### A. Functional bosonization on the surface

Here, we derive the hydrodynamic description for the surface of a topological insulator by the functional bosonization. The surface of a 3 + 1 D non-interacting topological insulator hosts gapless Dirac fermions which are described, schematically, by the lagrangian:

$$\mathcal{L} = \sum_{a=1}^{N_f} \bar{\psi}_a i (\partial_\mu - i A^{\text{ex}}_{\mu}) \gamma^\mu \psi_a,$$  \hspace{1cm} (4.1)

where $N_f$, the number of surface Dirac fermion flavors, is determined from the bulk topological invariant. In the following, we will focus on the case where the chemical potential is exactly at the Dirac point. One can try to apply the functional bosonization recipe developed in the preceding sections (see also Ref. [27]) to the surface Dirac fermion theory:

$$Z[A^{\text{ex}}] = N \int \mathcal{D}[a] \sum_{b = \delta(N_2)} Z[a] \exp \left[ i \int b \wedge (da - dA^{\text{ex}}) \right],$$  \hspace{1cm} (4.2)

where two one-form $U(1)$ gauge fields $b_\mu$ and $a_\mu$ are introduced. Here $b$ is a compact (discrete) variable (see the comment below). While the formal step leading to Eq. (4.2) is completely identical to the corresponding step in the bulk bosonization, compared with the functional bosonization in the gapped bulk, one cannot organize the integration over the gapless surface fermions in terms of the inverse gap expansion. Nevertheless, one can develop a systematic expansion if the number flavors $N_f$ of surface massless fermions is large. (See also Refs. 54 and 55.)

To the leading order in the $1/N_f$ expansion, the effective action $W[a]$, related to the fermion partition function as $Z[a] \propto \exp(-W[a])$, is given by

$$W[a] = \frac{g}{4\pi} \int d^3x f_{\nu\lambda}[a](\partial^2)^{-1/2} f^{\nu\lambda}[a]$$

$$+ \frac{f}{4\pi} \int d^3x \varepsilon^{\nu\lambda\mu} a_\mu \partial_\nu a_\lambda + \cdots$$  \hspace{1cm} (4.3)

where $\partial^2$ is the Laplacian in 2+1 dimensional Euclidean space-time, and $f$ and $g \sim N_f$ are parameters.

Summarizing, within the large $N_f$ expansion, the resulting bosonized theory can be written as

$$S = i \int d^3x b_\mu \epsilon^{\nu\lambda\mu} (f_{\nu\lambda}[a] - f_{\nu\lambda}[A^{\text{ex}}])$$

$$+ \frac{1}{2} \int d^3x a_\mu D^{\mu\nu} a_\nu,$$  \hspace{1cm} (4.4)

where

$$D^{\mu\nu} = \frac{1}{2\pi} \left[ g(\partial^2)^{-1/2} P^{\mu\nu} + f^{\nu\lambda\rho} \partial_\rho \right],$$

$$P^{\mu\nu} = -\partial^2 g^{\mu\nu} + \partial^\mu \partial^\nu.$$  \hspace{1cm} (4.5)

Here $g^{\mu\nu}$ is the metric of $\partial \mathcal{M}_4$. Observe that the kinetic term of the statistical gauge field $a_\mu$, generated by integrating over the gapless fermions, is non-local in spacetime. It is this non-locality that prevents condensation in the language of
the bosonized theory. This is necessary for the internal consistency of the functional bosonization since once the statistical gauge field is “Higgsed”, the bosonized theory is gapped, whereas the original surface theory is gapless. In turn, since we do not expect the condensation, it is preferred to work with the discrete hydrodynamic variable $b$. This should of course be contrasted with the Julia-Toulouse where the discrete nature of $b$ is immaterial once the condensation happens.

Upon integrating over $a_\mu$, we obtain the following effective action for the gauge field $b_\mu$,

$$S = i \int d^3 x \epsilon_{\mu\nu\lambda} f_{\nu\lambda} [A^\alpha + \frac{1}{2} \int d^3 x \, b_\mu \tilde{D}^{\mu\nu} b_\nu, \quad (4.6)$$

where $\tilde{D}^{\mu\nu}$ is the operator

$$\tilde{D}^{\mu\nu} = \frac{1}{2} \frac{1}{2\pi} \left[ \hat{g}(\partial^2)^{-1/2} p^{\mu\nu} + \hat{f} \epsilon^{\mu\nu\alpha} \partial_\alpha \right], \quad (4.7)$$

and $\hat{g}$ and $\hat{f}$ are the dual couplings

$$\hat{g} = \frac{g}{f^2 + g^2}, \quad \hat{f} = -\frac{f}{f^2 + g^2}. \quad (4.8)$$

The transformation $D^{\mu\nu} \rightarrow \tilde{D}^{\mu\nu}$ is precisely the duality transformation discussed in Ref. [40]. Below we will give a brief review of the results of Ref. [40]. A comparison of the bosonized surface theory with the Fradkin-Kivelson theory will also be given shortly.

Following the discussion of the bulk electromagnetic duality, we define a complex coupling

$$z = f + ig. \quad (4.9)$$

In terms of $z$, duality is the mapping

$$z \rightarrow \tilde{z} = -\frac{1}{z}. \quad (4.10)$$

Periodicity is the mapping

$$z \rightarrow \tilde{z} = z + n, \quad n \in \mathbb{Z}. \quad (4.11)$$

In addition, the charge conjugation is the operation

$$z \rightarrow \tilde{z} = -z^*. \quad (4.12)$$

These transformations can be combined to form the modular group.

$$z \rightarrow \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1. \quad (4.13)$$

Thus, similarly to the bulk, integration over the statistical gauge field is correlated with the duality transformation. The derivation of the hydrodynamic theory on the surface thus closely parallels the derivation in the bulk. Integration over the statistical gauge field thus entails to dualization. This connection between the bulk and surface duality transformations is essentially what was observed by Witten in Ref. [56]. It should be noted that in the above discussion the duality relates two different theories. Starting from the theory with two gauge fields $a$ and $b$, integration over the statistical gauge field leads to the theory solely written in terms of $b$ with the dualized coupling. This is similar to what was observed for the 2+1-dimensional Chern-Simons theory by Witten. That a duality relates two different theories is a common phenomenon in 2+1-dimensional field theories. This should be contrasted with the electromagnetic duality of the four-dimensional $U(1)$ gauge theory where the duality acts on the coupling constant of the theory, but do not modify the theory itself. More recently, a duality between the free Dirac fermion and the QED has been discussed in Refs. [37 and 38]. The 3d mirror symmetry, relating the $\mathcal{N} = 2$ supersymmetric QED in 2+1 dimensions to the so-called $XYZ$ model (the Wess-Zumino model), is another example.

To make a contact with the recently proposed duality between the free Dirac fermion in 2+1 dimensions and QED$_3$ in Refs. [37 and 38], note that the hydrodynamic surface theory in Eq. (4.6) is “designed”, by the functional bosonization receipt, to describe $N_f$ massless Dirac fermions at the 2+1 dimensional surface; The action of Eq. (4.6), within the large $N_f$ expansion, reproduces the correct effective action for $A^{\alpha\beta}$, which can be obtained by integrating out $b$. Now, going back to the theory before integrating over the statistical gauge field, one can interpret the action of Eq. (4.4) as describing a “matter field” in terms of the gauge field $a_\mu$, which couples to a dynamical gauge field $b$. This matter field, following our discussion just above on the action of Eq. (4.6), can be interpreted as a massless Dirac fermion, which is different from the original surface electric Dirac fermion. (Although one should note that $a_\mu$ is a compact and continuum variable, where as $b$ is a discrete variable.) Thus, the equivalence of the bosonized theory in Eq. (4.4) and the hydrodynamic theory Eq. (4.6) is exactly the particle-vortex duality discussed in Refs. [37 and 38]. In other words, $db$ represents the current associated to the massless Dirac fermion, while $da$ is associated to the dual massless Dirac fermion.

B. Comparison with the Fradkin-Kivelson theory

Furthermore, the hydrodynamic theory Eq. (4.6) is nothing but the theory proposed and studied in Ref. [40]. We now give a brief overview of the Fradkin-Kivelson theory, and compare it with the bosonized surface theory. The Fradkin-Kivelson model is defined by

$$Z = \sum_{\{\ell_\mu\}} \prod_{\ell_\mu} \delta[\Delta_\mu^r \ell_\mu(r)] \exp(-\mathcal{S}[\ell]), \quad (4.14)$$

where $\ell_\mu$ is an integer-valued variable defined on a link of a three-dimensional lattice, and represents a conserved current, i.e. the worldlines of particles in 2+1-dimensional Euclidean space-time. Since the currents
form closed loops this theory is a theory of charged particles at charge neutrality. In other words, the theory preserves particle-hole symmetry – its importance has recently been discussed in Refs. [37, 38, and 58] in the context of the half-filled Landau level and its connection to the surface of topological insulators. The Boltzmann weight of a configuration of loops is given by

\[
S[\ell] = \frac{1}{2} \sum_{r,r'} \ell_\mu(r) G_{\mu\nu}(r - r') \ell_\nu(r') \\
+ \frac{i}{2} \sum_{r,R} \ell_\mu(r) K_{\mu\nu}(r, R) \ell_\nu(R) \\
+ \frac{i}{2} \sum_{r,r'} e(r - r') \ell_\mu(r) A_\mu(r) \\
+ \sum_{R,R'} h(R - R') \ell_\mu(R) B_\mu(R') \\
+ \frac{1}{2} \sum_{r,r'} A_\mu(r) \Pi^0_{\mu\nu}(r, r') A_\nu(r').
\] (4.15)

Here \( r \) and \( R \) represent sites on the 2+1 dimensional cubic lattice and on its dual lattice, respectively, and \( G_{\mu\nu} \) and \( K_{\mu\nu} \) are given in momentum space as

\[
G_{\mu\nu}(k) = \frac{g}{\sqrt{k^2}} \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right), \\
K_{\mu\nu}(k) = i f \varepsilon_{\mu\nu\lambda} \frac{k_\lambda}{k^2}.
\] (4.16)

where \( f \) and \( g \) are real coupling constants of the theory. Here \( A_\mu \) and \( B_\mu \) are, respectively, background gauge fields and their associated field strengths. The electric and magnetic charges of the particles are represented by \( e \) and \( h \) (in a point-split representation). An extension of this loop model was considered more recently by Geradts and Motrunich, in which two species of particles, each carrying either electric or magnetic charge (but not both), are introduced (as opposed to the Fradkin-Kivelson theory, in which electric and magnetic degrees of freedom are distributed between the original and dual lattices, respectively, by the point splitting prescription).

Recalling the bosonization dictionary \( \partial \propto \int \delta \), we see that the kernels \( G_{\mu\nu} \) and \( K_{\mu\nu} \) have the same structure as \( D^{\mu\nu} \) and \( \tilde{D}^{\mu\nu} \). Thus, within the large \( N_f \) expansion, the surface of topological insulators realizes the Fradkin-Kivelson theory in Ref. [40]. Alternatively, one can consider turning on the Coulomb interaction by promoting \( A^{\text{elect}} \) into the dynamical electromagnetic \( U(1) \) gauge field. The resulting theory, once “projected” to the surface, is given by the theory discussed in Ref. [40]. (See Ref. [60] for a related discussion.)

From this point of view, the non-locality of the action of the loop model of Ref. [40] can be understood as a consequence of being a theory of charged particles (with charges of both signs) that reside on a surface, interacting through a quantized Maxwell gauge field. The linking number represented by the odd-parity term in the action is a simply a Chern-Simons term on that surface or, equivalently, a \( \theta \) term in the 3+1-dimensional space-time. In other words, the loop model is equivalent to Witten’s modular-invariant theory\(^{36} \) on a four-manifold with a boundary that represents the 2+1-dimensional space-time with a charged massless fermion field on the boundary. This is precisely the theory of the surface states of the 3+1-dimensional topological insulator!

This remarkable duality between the free Dirac fermion theory (our starting theory) in Eq. (4.1), and the hydrodynamic theory whose action is given in Eq. (4.6) (i.e., the Fradkin-Kivelson theory), however, must be taken with care. The original theory is non-interacting, whereas the Fradkin-Kivelson theory is strongly interacting. The situation is somewhat similar to the proposed duality between the free Dirac fermion in 2+1 dimensions and QED\(^3 \)\(^{37,38} \) The duality may have to be regarded as a “weak” form of duality in the sense that there is a one-to-one correspondence at the level of operators (states) between in the two theories. The functional bosonization is in fact a prescription to map the set of correlation functions in one theory into another. On the other hand, whether or not the “stronger” form of duality holds, in which the two theories in the IR limit are actually identical, is a highly non-trivial dynamical question. It would be possible that the free-fermion fixed point exists among the fixed points of the Fradkin-Kivelson theory. (See the discussion below for a few interesting fixed points using the self-dual property of the Fradkin-Kivelson model.)

1. Implications of the self duality

We have so far discussed the duality which relates two different (2+1) dimensional theories. However, it is possible to have a duality that acts on the same theory in 2+1 dimensions. See, for example, a recent work by Xu and You. In Ref. [40], the non-local 2+1-dimensional theory (the Fradkin-Kivelson theory) and its duality was shown. (We emphasize that this duality within the Fradkin-Kivelson theory has nothing to do with the duality above discussed for the free 2+1-dimensional Dirac fermion and the Fradkin-Kivelson theory.) This self-duality can be used to constrain the possible values of the transport coefficients such as the diagonal and off-diagonal (Hall) conductance on the surface.\(^{40} \)

A duality in statistical mechanics and field theories, if exists, is a powerful tool that allows us to make a non-perturbative prediction on the structure of phase diagrams and the properties of the critical points even when strong interactions are present. A famous example is the Kramers-Wannier duality in the 2D classical Ising model that relates its high- and low-temperature phases.\(^{33,34,62-65} \)

We can follow the discussion developed in Ref. [40], where the phase diagram of the non-local Maxwell theory interacting with dynamical electric currents (or their dual magnetic currents) was discussed by using the du-
ality. By making use of the modular symmetry, the correlation functions and in particular the conductivities at the modular fixed points were exactly calculated.

Fixed points under the $PSL(2, \mathbb{Z})$ transformations can be found in the following way. We first note that the (non-abelian) modular group $PSL(2, \mathbb{Z})$ is generated by $S$ and $T$ with the relation $S^2 = e$ and $(ST)^3 = e$. This tells us that $PSL(2, \mathbb{Z})$ is essentially a free product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$. A point fixed by $S$ can be easily found:

$$z = i.$$ \hspace{1cm} (4.17)

In Ref. [40], this point is called the bosonic fixed point. Similarly, one can easily find a point fixed by $ST$ as

$$z = \frac{1}{2} + i\frac{\sqrt{3}}{2} = \rho.$$ \hspace{1cm} (4.18)

This is the fermionic fixed point of Ref. [40]. Furthermore, it is known that all the other fixed points can be found as an image of the bosonic $(i)$ or fermionic $(\rho)$ fixed point. Let $[i]$ and $[\rho]$ denote the sets of points of the upper half plane which are the images of $i$ and $\rho$; Also, let $[\infty]$ denote the sets of points of the upper half plane which are the images of $\infty$. $[\infty]$ is the set of points of the upper half place with $g = 0$ and $f$ rational. It can be shown that the set $C$ of all points of the upper half plane which are fixed points under a modular transformation) is $C = [i] \cup [\rho] \cup [\infty]$.

By making use of the modular symmetry, the correlation functions and in particular the conductivities at the bosonic and fermionic fixed points were exactly calculated in Ref. [40].

\section{V. Fractional Topological Insulators}

In this section, we discuss putative fractional topological insulators using the hydrodynamic effective field theory. We adopt the parton construction, in which we postulate that electrons are fractionalized, consist of K partons, and each parton is in its topological insulator phase. For each parton, we can apply functional bosonization to derive its hydrodynamic theory. Solving the constraints among parton densities, we will arrive at the hydrodynamic theory of fractional topological insulators. See Refs. [20–22] for previous studies of time-reversal symmetric fractional topological insulators in $D = 3 + 1$ in terms of the parton construction. While the parton construction may not be able to address questions regard to energetics, it can reveal expected topological properties of fractional topological insulators.

We write down the following action $S = \sum_{i=1}^{K} S^{(i)}$ for partons, where

$$S^i = - \int b^i \wedge (da^i - u^i - K^{-1} dA^{ex} + K^{-1} U^{ex})$$

$$+ \int dv^i \wedge (u^i - K^{-1} U^{ex})$$

$$- \frac{\tau_2}{4} \int (da^i - u^i) \wedge \ast (da^i - u^i)$$

$$+ \frac{\tau_1}{4} \int (da^i - u^i) \wedge (da^i - u^i) + \cdots$$ \hspace{1cm} (5.1)

Here, the parton densities are written in terms of the two-form gauge fields $b^i$ as $j^i = dB^i$, etc., and are subject to the constraint

$$db^1 = db^2 = \cdots = db^K = db,$$

$$dv^1 = dv^2 = \cdots = dv^K = dv,$$ \hspace{1cm} (5.2)

for all $i = 1, \ldots, K$. Solving the constraint, the resulting effective field theory is

$$S = - \int b \wedge \left[ \sum_i (da^i - u^i) - dA^{ex} + U^{ex} \right]$$

$$+ \int dv \wedge \left[ \sum_i u^i - U^{ex} \right]$$

$$+ \sum_i \int \left[ - \frac{\tau_2}{4} (da^i - u^i) \wedge \ast (da^i - u^i)$$

$$+ \frac{\tau_1}{4} (da^i - u^i) \wedge (da^i - u^i) \right].$$ \hspace{1cm} (5.3)

We can eliminate statistical gauge fields $a^i$ and $u^i$ one by one. Gauging away $a$, we obtain

$$S = + \int b \wedge dA^{ex} - \int (b + dv) \wedge U^{ex}$$

$$+ \int (b + dv) \wedge \sum_i u^i$$

$$+ \sum_i \int \left[ - \frac{\tau_2}{4} u^i \wedge \ast u^i + \frac{\tau_1}{4} u^i \wedge u^i \right].$$ \hspace{1cm} (5.4)

Integrating over $u^i$, the resulting theory is

$$S = i \int b \wedge dA^{ex} - i \int (b + dv) \wedge U^{ex}$$

$$+ \frac{\hat{\tau}_2 K}{2} \int (b + dv) \wedge \ast (b + dv)$$

$$+ \frac{i \hat{\tau}_1 K}{4} \int (b + dv) \wedge (b + dv).$$ \hspace{1cm} (5.5)

This final hydrodynamic action with fractionalization can then be related to the Cardy-Rabinovici theory with $K > 1$.

More generally, we can consider different ways to split an electron into partons. This leads to an analogue of the
Such experiments can be studied via surface plasmon-polariton inelastic electron scattering. For example, Kogar and coworkers supported in part by an INSPIRE Grant at UIUC and DMR-1408713, and by the Alfred P. Sloan foundation. Yet another issue is the precise connection between the bosonized surface theory and the corresponding microscopic, fermionic surface theory, i.e., the surface Dirac fermions. In the case of the quantum Hall effect, an important prediction from the hydrodynamic Chern-Simons theory is the existence of the gapless chiral edge state. The vertex operators (bosonic exponentials) in the chiral edge theory then describe solitonic quasiparticle excitations, including electrons. Such “vertex operators” may be constructed within the bosonized theory for the surface of 3+1-dimensional topological insulators in Sec. IV. It would be then interesting to construct physical electron operators (with the Dirac dispersion) within the bosonized surface theory. It was argued in Ref. [5], based on a (different) topological field theory, that an effective field theory can give rise to a gapless fermionic surface state with a Fermi surface. In fact, within the non-local Maxwell theory in 2+1 dimensions, i.e., our surface theory, we can follow the approach developed by Marino to construct an fermion (electron) operator. It should however be noted that here we focused on the situation where the chemical potential is exactly at the Dirac point (since we focused on topological insulators in class AIII), whereas this is not typically the case (as in Dirac surface states realized in topological insulators in symmetry class AII).

Finally, more exotic physics that may occur in the presence of strong interactions can also be explored within the hydrodynamic effective field theory, including symmetry-respecting surface topological order and fractional topological insulators. For the latter, the hydrodynamic field theory in the presence of fractionalization discussed in Sec. V can provide a convenient platform to discuss, e.g., the fractionalized surface theory and its duality. We leave these issues for the future study.

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**Appendix A: $\delta$-function forms**

In this Appendix, we collect useful formulas involving the $\delta$-function forms. (The following discussion does not depend on the Euclidean/Minkowski signature of the
metric.) For an \( n \)-dimensional submanifold \( \mathcal{N} \) of \( \mathcal{M} \), we define a \((D-n)\)-form \( \delta_{D-n}(\mathcal{N}) \) by
\[
\int_{\mathcal{N}} A_n = \int_{\mathcal{M}} \delta_{D-n}(\mathcal{N}) \wedge A_n, \quad \forall A_n,
\]
where \( A_n \) is an arbitrary \( n \)-form on \( \mathcal{M} \).

If we flip the orientation of \( \mathcal{N} \),
\[
\delta_{D-n}(-\mathcal{N}) = -\delta_{D-n}(\mathcal{N}).
\]
More generally, for oriented submanifolds \( \mathcal{N}_i \),
\[
\delta\left(\sum_i c_i \mathcal{N}_i\right) = \sum_i c_i \delta(\mathcal{N}_i)
\]
where \( c_i \) is a coefficient.

Let \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) be a submanifold of \( \mathcal{M} \) with dimensions \( n_1 \) and \( n_2 \), respectively. Define \( d \) as
\[
d = n_1 + n_2 - D.
\]
When \( d \geq 0 \), \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) can have a \( d \)-dimensional intersection within \( \mathcal{M} \). By properly defining an orientation, we define \( I = \mathcal{N}_1 \neq \# \mathcal{N}_2 \). The orientation of \( I \) is defined to be consistent with
\[
\delta_{D-d}(I) = \delta_{D-n_1}(\mathcal{N}_1) \wedge \delta_{D-n_1}(\mathcal{N}_2).
\]
The exterior derivative of the delta form is given by
\[
\delta_{D-n+1}(\partial \mathcal{N}) = (-1)^{D-n+1} d \delta_{D-n}(\mathcal{N}).
\]

Appendix B: Details of Functional bosonization with monopole gauge invariance

In this Appendix, we derive the hydrodynamic theory of Eq. (3.25) by the functional bosonization. Our starting point is the two kinds of gauge invariance presented in Eqs. (3.22) and (3.23). By making use of the monopole gauge invariance, one can write
\[
Z[A^{ex}, U^{ex}] = \mathcal{N} \int \mathcal{D}[a] \sum_{u=\delta(\partial \mathcal{M}_3)} \delta(da + q_m u)
\times Z[A^{ex} + a, U^{ex} + u]
\]
where \( \sum_{u=\delta(\partial \mathcal{M}_3)} \) represents the sum over arbitrary submanifolds \( \mathcal{M}_3 \) of spacetime with the two form \( u \) given by \( u = \delta(\partial \mathcal{M}_3) \). The functional delta function can be converted into an integral over an auxiliary field \( b \),
\[
Z[A^{ex}, U^{ex}] = \mathcal{N} \int \mathcal{D}[a, b] \sum_{u=\delta(\partial \mathcal{M}_3)}
\times Z[A^{ex} + a, U^{ex} + u]
\times \exp iq_e \int b \wedge (da + q_m u).
\]
Shifting \( a \) and \( u \) as \( a \rightarrow a - A^{ex} \) and \( u \rightarrow u - U^{ex} \),
\[
Z[A^{ex}, U^{ex}] = \mathcal{N} \int \mathcal{D}[a, b] \sum_{u=\delta(\partial \mathcal{M}_3)} \partial \mathcal{M}_2 = \mathcal{L}_1
\times \exp iq_e \int b \wedge (da + q_m u - dA^{ex} - q_m U^{ex}).
\]
The summation over \( \mathcal{M}_2 \) is subjected to the constraint \( \partial \mathcal{M}_2 = \mathcal{L}_1 \), where \( \mathcal{L}_1 \) is related to the external monopole gauge field \( U^{ex} \) as \( dU^{ex} = \delta(\mathcal{L}_1) \).

Equation (2.3) is the analog of Eq. (2.3) in the presence of monopoles, and is the starting point of the functional bosonization. Before proceeding, we note, instead of imposing \( d\xi + q_m u = 0 \) strictly, we can impose \( d\xi + q_m u \equiv 0 \mod 2\pi/q_e \) for all plaquette if we work on a lattice. If so, the auxiliary field \( b \) must a discrete variable. By making use of the generalized Poisson identity, the sum
\[
\sum_{b=\delta(\mathcal{M}_2)} \exp iq_e \int (da + q_m u) \wedge b
\]
enforces \( da + q_m u \) is given in terms of the delta function for some manifold \( \mathcal{N}_2 \):
\[
da + q_m u = 2\pi q_e^{-1} \delta(\mathcal{N}_2).
\]
Here, we recall the Dirac quantization condition
\[
q_m q_e = 2\pi n \quad (2.6)
\]
where \( n \in \mathbb{Z} \). The continuum v.s. discrete summation over \( b \) depends on whether we assume the presence of an underlying lattice or not, but, ultimately, this is immaterial. In the following, we consider the continuum summation over \( b \), but it is always possible replace it with its discrete counter part.

Specializing now to topological insulators, the fermion partition function \( Z[a, u] \) is given by
\[
Z[a, u] \propto \exp(-W[a, u]),
\]
where \( W[a, u] \) now is
\[
W[a, u] = \frac{\tau_2}{4\pi} \int (da + q_m u) \wedge \star (da + q_m u)
+ i\frac{\tau_1}{4\pi} \int (da + q_m u) \wedge (da + q_m u) + \cdots.
\]
Then, the partition function is written as
\[
Z[A^{ex}, U^{ex}] = \mathcal{N} \int \mathcal{D}[a, b] \sum_{u=\delta(\mathcal{M}_2)} \exp(-S[a, b, u]),
\]
where
where the action $S[a, b, u]$ is given by
\[
S[a, b, u] = -i \int q_b \wedge (da + q_m u - dA^\text{ex} - q_m U^\text{ex}) \\
+ \frac{\tau_2}{4\pi} \int (da + q_m u) \wedge *(da + q_m u) \\
+ i\frac{\tau_1}{4\pi} \int (da + q_m u) \wedge (da + q_m u) + \cdots .
\]
(2.10)

The sum over $\mathcal{M}_2$ with the constraint $\partial \mathcal{M}_2 = \mathcal{L}_1$ can be converted into an unrestricted sum over $\mathcal{M}_2$, by introducing an auxiliary field $v$,
\[
Z[A^\text{ex}, U^\text{ex}] = N \int D[a, b, v] \sum_{u=\delta(\mathcal{M}_2)} \exp(-S[a, b, u, v]),
\]
where
\[
S[a, b, u, v] = -i \int dv \wedge (q_m u - q_m U^\text{ex}) \\
- iq_e \int b \wedge (da + q_m u - dA^\text{ex} - q_m U^\text{ex}) \\
+ \frac{i}{2\pi} \int (da + q_m u) \wedge \alpha \\
+ \frac{\tau_2}{4\pi} \int \alpha \wedge *\alpha + i\frac{\tau_1}{4\pi} \int \alpha \wedge \alpha.
\]
(2.12)

We now proceed to integrate over the statistical gauge fields $a$ and $u$, which do not couple to the external fields $A^\text{ex}$ and $U^\text{ex}$. We first integrate over $u$ and then $a$. To this end, we introduce an auxiliary field $\alpha$, with which the action is given by
\[
S = -i \int dv \wedge (q_m u - q_m U^\text{ex}) \\
- iq_e \int b \wedge (da + q_m u - dA^\text{ex} - q_m U^\text{ex}) \\
+ \frac{i}{2\pi} \int (da + q_m u) \wedge \alpha \\
+ \frac{\tau_2}{4\pi} \int \alpha \wedge *\alpha + i\frac{\tau_1}{4\pi} \int \alpha \wedge \alpha.
\]
(2.13)

By integrating over the auxiliary field $\alpha$, one recovers the action Eq.(2.12). By making use of the generalized Poisson identity, the sum over results in
\[
\sum_{u=\delta(\mathcal{M}_2)} \exp \int i \left(q_m dv + q_m q_e b - \frac{q_m}{2\pi} \alpha \right) \wedge u \\
\end{equation}
(2.14)

which enforces the constraint
\[
\frac{\alpha}{2\pi} = q_e b + dv + q_m^{-1}(2\pi)\delta(\mathcal{N}_2). \\
\]
(2.15)

Then, after integrating over $u$, we find
\[
S = i \int dv \wedge q_m U^\text{ex} \\
+ iq_e \int b \wedge (dA^\text{ex} + q_m U^\text{ex}) \\
+ \frac{i}{2\pi} \int da \wedge (dv + q_m^{-1}(2\pi)\delta(\mathcal{N}_2)) \\
+ \frac{\tau_2}{4\pi} \int \alpha \wedge *\alpha + i\frac{\tau_1}{4\pi} \int \alpha \wedge \alpha.
\]
(2.16)

Now, integrating over $a$ sets
\[
q_e d^2 v + q_m^{-1}(2\pi)d\delta(\mathcal{N}_2) = 0,
\]
(2.17)

which implies $\mathcal{N}_2 = \partial \mathcal{M}_3$. This completes the derivation of Eq. (3.25).

1. An alternative derivation of Eq. (2.3)

Equation (2.3) can be derived in an alternative way as follows: Bosonizing the electromagnetic current, with a shift $a \rightarrow a - A^\text{ex}$, the partition function is given by
\[
Z[A^\text{ex}, U^\text{ex}] = N \int D[a, b] Z[a, U^\text{ex}] \\
\times \exp \frac{i}{2\pi} \int_{\mathcal{M}_4} b \wedge (da - dA^\text{ex}).
\]
(2.18)

We now make use of the monopole gauge invariance of $Z[a, U^\text{ex}]$, $Z[a, U^\text{ex}] = Z[a+\xi, U^\text{ex} + u]$ with $d\xi + q_m u = 0$. Following the bosonization of the electromagnetic $U(1)$ current above, we can average over $u$ and $\xi$ as
\[
Z[a, U^\text{ex}] = N \sum_{u, \xi} Z[a + \xi, U^\text{ex} + u]
\]
(2.19)

where the summation is over arbitrary boundaryless surfaces $\mathcal{M}_2$ and $\xi$, such that $u = \delta(\mathcal{M}_2)$ and $d\xi + q_m u = 0$. Implementing the latter condition by introducing an auxiliary field $q$,
\[
Z[a, U^\text{ex}] = N \int D[q] \sum_{u, \xi} Z[a + \xi, U^\text{ex} + u] \\
\times \exp \frac{i}{2\pi} \int_{\mathcal{M}_4} q \wedge (d\xi + q_m u),
\]
(2.20)

where $\sum_{u, \xi}$ is over arbitrary manifolds $\mathcal{M}_2$ and $\mathcal{N}_3$, respectively:
\[
\sum_u = \sum_{u \in \delta(\mathcal{M}_2)} \quad \sum_\xi = \sum_{\xi \in q_m \delta(\mathcal{N}_3)}.
\]
(2.21)

The integration over $q$ enforces the constraint $d\xi + q_m u = 0$, and reduces $\sum_{u, \xi}$ to
\[
\sum_{u \in \delta(\partial\mathcal{N}_3)}.
\]
(2.22)
I.e., the summation over $u_2$ is now given in terms of boundaryless manifolds. (In fact, $u$ doesn’t have to be discrete, and can be replaced by $\int D[u]$.) With a shift $a \rightarrow a - \xi$, the total partition function is

$$Z[A^{ex}, U^{ex}] = \mathcal{N} \int D[a, b, q] \sum_{u, \xi} Z[a, U^{ex} + u]$$

$$\times \exp \frac{i}{2\pi} \int b \wedge (da - d\xi - dA^{ex})$$

$$\times \exp \frac{i}{2\pi} \int q \wedge (d\xi + q_m u).$$

(2.23)

We first consider the summation over $\xi$:

$$\sum_{\xi \in q_m \delta(N_{u_3})} \exp \frac{i}{2\pi} \int (-b + q) \wedge d\xi$$

$$= \sum_{\xi \in q_m \delta(N_{u_3})} \exp \frac{i}{2\pi} \int d(b - q) \wedge \xi.$$  

(2.24)

By using the generalized Poisson identity,

$$\sum_{N_{D-p}} \exp 2\pi i \int_{\mathcal{M}_{D-p}} \delta_p(N_{D-p}) \wedge A_{D-p}$$

$$= \sum_{Q_p} \delta(A_{D-p} - \delta(Q_p))$$

(2.25)

the summation over $\xi$ sets

$$q_m d(b - q) = (2\pi)^2 \delta(Q_1).$$

(2.26)

While at this stage $Q_1$ appears to be arbitrary, since $d^2 = 0$, $\partial Q_1$ should be zero. (We have used the same discussion in defining the monopole gauge field $\Sigma$.) By the Poincaré lemma, the 2-form $b - q$ can be written in terms of a two-dimensional surface $P_2$ and a one-form $v$ as

$$b - q = q_m^{-1}(2\pi)^2 \delta(P_2) - q_m^{-1}(2\pi)^2 dv.$$  

(2.27)

(The minus here is purely a convention.) We can introduce $w$ as

$$w_2 = \delta_2(P_2), \quad dw_2 = \delta(Q_1).$$

(2.28)

By eliminating $q$,

$$Z[A^{ex}, U^{ex}] = \mathcal{N} \int D[a, b, v] \sum_{u, w} Z[a, U^{ex} + u]$$

$$\times \exp \frac{i}{2\pi} \int b \wedge (da - dA^{ex} + q_m u)$$

$$\times \exp -2\pi i \int (dv - w) \wedge u.$$  

(2.29)

Comments: (i) $w$ couples only to $u$ through:

$$\exp -2\pi i \int w \wedge u.$$  

(2.30)

This is always 1, so one can drop the sum over $w$. (If $u$ to be taken as a continuous variable rather than discrete, this summation sets $u$ to be a discrete variable.) (ii) $v$ couples only to $u$ through:

$$\exp -2\pi i \int dv \wedge u.$$  

(2.31)

Integration over $v$ sets $du = 0$, and hence, summation over $u$ becomes over boundaryless surfaces, as expected. Thus, the functional integral reduces to

$$Z[A^{ex}, U^{ex}] = \mathcal{N} \int D[a, b] \sum_{u \in \delta(M_3)} Z[a, U^{ex} + u]$$

$$\times \exp \frac{i}{2\pi} \int b \wedge (da - dA^{ex} + q_m u).$$

(2.32)

This is nothing but Eq. (2.3).

### Appendix C: Review: The Julia-Toulouse approach

In this appendix, we give a short summary of the Julia-Toulouse approach following the work of Quevedo and Trugenberger, which conveniently describes the $h$-form generalization of the Higgs mechanism ($h = 0, 1, \ldots$). We consider the class of field theories that contain compact ($h - 1$)-form $\phi_{h-1}$ with (generalized) gauge invariance under transformation

$$\phi_{h-1} \rightarrow \phi_{h-1} + d\lambda_{h-2}.$$  

(3.1)

The dynamics of the field $\phi_{h-1}$ may be described by the following generic low-energy effective action:

$$S = \int \frac{(-1)^{h-1}}{e^2} d\phi_{h-1} \wedge (\ast d\phi_{h-1}) + \kappa \int \phi_{h-1} \wedge \ast j_{h-1}$$

(3.2)

where $e$ and $\kappa$ are coupling constants, $j_{h-1}$ describes a conserved (tensor) current of fields whose dynamics is governed by the action $S_M$ (not included above). The spacetime dimension is $D = d + 1$. For example: when $h = 2, \phi_1 =: A$ is a one-form, the theory is nothing but the compact QED:

$$S = \int \frac{1}{e^2} dA \wedge (\ast dA) + \kappa \int A \wedge \ast j_1$$

(3.3)

When $h = 1$, on the other hand, $\phi_0 =: \phi$ is a scalar field, and the action is given by

$$S = \int \frac{1}{e^2} d\phi \wedge (\ast d\phi) + \kappa \int \phi \wedge \ast j_0.$$  

(3.4)

The Julia-Toulouse approach provides a prescription to write down an effective action describing a phase where defects in $\phi_{h-1}$ are condensed. The effective action is
given by
\[
S = \int \frac{(-1)^h}{A^2} d\omega_h \wedge \ast d\omega_h \\
+ \frac{(-1)^{h-1}}{e^2} \int (\omega_h - d\phi_{h-1}) \wedge \ast (\omega_h - d\phi_{h-1}) \\
+ \kappa \int (\omega_h - d\phi_{h-1}) \wedge \ast T_h, 
\]
(3.5)
where $\omega_h$ is a $h$-form gauge field, and $A$ is an energy scale associated to the condensation. The guiding principle in constructing the effective Lagrangian is the gauge invariance under the two gauge symmetries. The first gauge invariance is the invariance under the original gauge transformation, Eq. (3.1). In addition, we require invariance under the following gauge invariance:
\[
\omega_h \rightarrow \omega_h + d\psi_{h-1}, \quad \phi_{h-1} \rightarrow \phi_{h-1} + \psi_{h-1}. 
\]
(3.6)

Concurrently to this gauge invariance, in the last line of the effective action of Eq. (3.5), the original conserved $(h-1)$-form current $j$ is promoted to an $h$-form current $T_h$, which is given by $dT_h = j_{h-1}$.

The physical meanings of $\omega_h$ and the second gauge invariance are the following: We are interested in phases where topological defects in $\phi_{h-1}$ (which is a compact variable) condense. These topological defects can be characterized by an integer valued topological invariant
\[
\int_{S_h} \omega_h, \quad \omega_h = d\phi_{h-1}, 
\]
(3.7)
where $S_h$ is an $h$-dimensional sphere surrounding the singularity and $\omega_h$ is the topological density. If there is a single topological defect, $d\omega_h$ is zero almost everywhere, but at the defect, $d\omega_h \neq 0$ (delta function peak). For example, in the compact QED, $\omega$ is given by $\omega = dA$. If there is a magnetic monopole, $d\omega$ is zero otherwise. We further define the “topological current” by
\[
J_{d-h} = \ast (d\omega_h). 
\]
(3.8)
If there is no defect, $J_{d-h}$ is identically zero. If there is a defect, there will be a delta function singularity. If there are many defects, we can “smear” delta function singularity, and can treat them as a constant background. In this situation, the topological current $J_{d-h}$ is conserved. Writing the topological current in terms of $\omega_h$ as above, there is a redundancy: $\omega_h \rightarrow \omega_h + d\psi_{h-1}$ gives the same topological current. This is the origin of the emergent gauge symmetry in the condensed phase, which we have made use of to construct the effective action in the condensed phase.
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