GRADED LOCAL COHOMOLOGY: ATTACHED AND ASSOCIATED PRIMES, ASYMPTOTIC BEHAVIORS

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ABSTRACT. Assume that $R = \bigoplus_{i \in \mathbb{N}_0} R_i$ is a homogeneous graded Noetherian ring, and that $M$ is a $\mathbb{Z}$-graded $R$-module, where $\mathbb{N}_0$ (resp. $\mathbb{Z}$) denote the set all non-negative integers (resp. integers). The set of all homogeneous attached prime ideals of the top non-vanishing local cohomology module of a finitely generated module $M$, $H^c_{R_+}(M)$, with respect to the irrelevant ideal $R_+ := \bigoplus_{i \geq 1} R_i$ and the set of associated primes of $H^c_{R_+}(M)$ is studied. The asymptotic behavior of $\text{Hom}_R(R/R_+, H^c_{R_+}(M))$ for $s \geq f(M)$ is discussed, where $f(M)$ is the finiteness dimension of $M$. It is shown that $H^c_{R_+}(M)$ is tame if $H^c_{R_+}(M)$ is Artinian for all $i > b$.

1. INTRODUCTION

Throughout $R = \bigoplus_{i \in \mathbb{N}_0} R_i$ is a homogeneous positively graded Noetherian ring, so that $R = R_0[x_1, \ldots, x_t]$ for some $x_1, \ldots, x_t \in R_1$, $R_+ := \bigoplus_{i \geq 1} R_i$ is the irrelevant ideal of $R$, and $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a graded $R$-module which is finitely generated whenever it is explicitly stated. Denote by $H^i_{R_+}(M)$ the $ith$ local cohomology module of $M$ with respect to $R_+$. It is well-known that $H^i_{R_+}(M)$ inherits a natural grading and its $n$th components $H^i_{R_+}(M)_n$ is finitely generated $R_0$-module for all $n$ and is zero for all $n \gg 0$.

Set $c := \text{cd}(M) = \sup\{i \in \mathbb{Z} \mid (H^c_{R_+}(M)) \neq 0\}$ to be the cohomological dimension of a finitely generated module $M$ with respect to $R_+$. In section 2 we study $\text{Att}_R(H^c_{R_+}(M))$, the set all homogeneous attached prime ideals of $H^c_{R_+}(M)$ and we will show that if $(R_0, m_0)$ is local, then the set of maximal elements of it, $\text{Max}(\text{Att}_R(H^c_{R_+}(M)))$, is a finite set which is equal to $\text{Att}_R(H^c_{R_+}(M/m_0))$.

In section 3, we study $\text{Ass}_R(H^s_{R_+}(M))$, for $s \geq 0$. We first show that if $K$ is a module (not necessarily finite) over a Noetherian ring, then for each ideal $a$ of $S$ and each non-negative integer $s$, we have

$$\text{Ass}_S(H^s_a(K)) \subseteq \bigcup_{0 \leq j < s} \text{Ass}_S(\text{Ext}^s_S(S/a, H^j_a(K))/L_j) \cup \text{Ass}_S(\text{Ext}^s_S(S/a, K)/L)$$

for some submodules $L, L_0, \ldots, L_{s-1}$ of their appropriate modules. When $M$ is a graded $R$-module $(R = R_0[R_i]$ is, as usual, a homogeneous Noetherian graded ring) and the modules

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Ext$^s_R(R/R_+, M)$, Ext$^{s-j+1}_R(R/R_+, H^i_{R_+}(M))$, $0 \leq j < s$, are weakly Laskerian (see Definition 3.3), then Ass$_R(H^i_{R_+}(M))$ is a finite set.

Section 4 is devoted to asymptotic behavior of the modules $H^i_{R_+}(M)$ and Hom$_R(R/R_+, H^i_{R_+}(M))$. We examine the asymptotic stability of associated primes, the asymptotic stability of supports, and tameness of Hom$_R(R/R_+, H^i_{R_+}(M))$ for graded module $M$. This gives a generalization of the result that $H^i_{R_+}(M)$ is tame, where $f := f(M) = \inf \{i | H^i_{R_+}(M) \text{ is not finitely generated} \}$ (cf. [BRS, Theorem 3.6(a)]). It is shown in [Br, Theorem 4.8 (e)] that $H^\dim_R(M)$ is tame. We show that $H^s_{R_+}(M)$ is tame if $H^s_{R_+}(M)$ is Artinian $R$–module for all $i > h$. This, in particular, implies that $H^\dim_R(M)^{-1}$ is tame.

2. Attached Primes

Let $S$ be a commutative ring and $K$ be an $S$–module. A prime ideal $p$ of $S$ is called an attached prime of $K$ whenever there exists a submodule $L$ of $K$ such that $p = L : S K$ (c.f. [MS]). Note that when $K$ has a secondary representation $K = K_1 + ... + K_r$, say, then $p = \sqrt{0 : K_i}$ for some $i \in \{1, ..., r\}$. In this case the set of attached primes of $K$ is uniquely defined (c.f. [BS, 7.2]).

For the graded module $M$, if $p$ is an attached prime of $M$ then it is straightforward to see that $p^*$, the homogeneous prime ideal generated by all homogeneous elements of $p$, is also an attached prime of $M$. As usual we denote by $^*\text{Spec} R$ the set of all homogeneous prime ideal of $R$. For a subset $T$ of Spec $R$, we denote $^*T = T \cap ^*\text{Spec} R$. Note that if $M$ is an Artinian graded $R$–module, then $^*\text{Att}_R(M) = \text{Att}_R(M)$ (cf. [Ri, corollary 1.6]). For the graded $R$–module $M$, denote by $c := \text{cd}(M)$, the cohomological dimension of $M$ with respect to $R_+$, i.e. $c = \sup \{i | H^i_{R_+}(M) \neq 0 \}$.

When $M$ is a finitely generated $R$–module it is shown, in [BH, 6.2.7] and [Br, Proposition 3.4(a)], that $c = \sup \{\text{dim}_R(M/m_0 M) | m_0 \text{ is a maximal element of Spec } R_0 \}$.

In this section, we assume that $M$ is a finitely generated graded $R$–module and that $R_0$ is not necessarily local and study the set $^*\text{Att}_R(H^i_{R_+}(M))$. We will show that the set $\text{Max}(^*\text{Att}_R(H^i_{R_+}(M)))$, of maximal elements of $^*\text{Att}_R(H^i_{R_+}(M))$ is a finite set if $R_0$ is local (see Corollary 2.2).

2.1. Theorem. Assume that $M$ is a finitely generated $R$–module. Then

\[ \bigcup_{p \in ^*\text{Att}_R(H^i_{R_+}(M))} p = \bigcup_{p \in ^*\text{Att}_R(H^i_{R_+}(M)/m_0 M), m_0 \in \text{Max}(R_0)} p. \]

Proof. We prove it in three steps.

Step 1. We show that $\bigcup_{p \in ^*\text{Att}_R(H^i_{R_+}(M)/m_0 M), m_0 \in \text{Max}(R_0)} p \subseteq \bigcup_{p \in ^*\text{Att}_R(H^i_{R_+}(M))} p$. Note that, by definition of $c$, for any $m_0 \in \text{Max}(R_0)$, $H^i_{R_+}(M/m_0 M)$ is either zero or an Artinian graded $R$–module and thus, by [Ri, Corollary 1.6], each element of $\text{Att}_R(H^i_{R_+}(M/m_0 M))$, if there exists any, is a homogeneous ideal. Choose $m_0 \in \text{Max}(R_0)$ with $c = \text{dim}_R(M/m_0 M)$. By the exact sequence $0 \rightarrow m_0 M \rightarrow M \rightarrow M/m_0 M \rightarrow 0$ we get an epimorphism $H^i_{R_+}(M) \rightarrow H^i_{R_+}(M/m_0 M)$ and so $\text{Att}_R(H^i_{R_+}(M/m_0 M)) \subseteq ^*\text{Att}_R(H^i_{R_+}(M))$. Thus our claim is clear.

Step 2. We prove that $\bigcup_{p \in ^*\text{Att}_R(H^i_{R_+}(M))} p \subseteq \bigcup_{p \in ^*\text{Supp}_R(M), \text{cd}(R/p) = c} p$. Set $X = \{p \in \text{Ass}_R(M) | \text{cd}(R/p) = c \}$. Thus there exists a graded submodule $N$ of $M$ such that $\text{Ass}_R(M/N) = X$ and $\text{Ass}_R(N) = \text{Ass}_R(M) \setminus X$ (one may choose $N$ to be a maximal element of the set $\{T | T$ is a graded
submodule of $M$ such that $\operatorname{Ass}_R T \subseteq \operatorname{Ass}_R (M) \setminus X$ (see [Bo, page 263, Proposition 4] for non–graded case). Consider the exact sequence $H^c_{R,+} (N) \to H^c_{R,+} (M) \to H^c_{R,+} (M/N) \to 0$ of graded modules. Note that $\text{cd}(N) \leq c$ (c.f. [Br, Corollary 3.5]) and $\operatorname{Ass}_R (N) \subseteq \operatorname{Ass}_R (M)$. Therefore it follows that $H^c_{R,+} (N) = 0$ and $H^c_{R,+} (M) \cong H^c_{R,+} (M/N)$. Now, choose $r$ to be a homogeneous element of $R$ such that $r \not\in \bigcup_{p \in \operatorname{Supp}_R (M), \text{cd}(R/p) = c} p$. It follows from definition of $N$ that $r$ is not a zero divisor on $M/N$. Consider the exact sequence $0 \to M/N[-l] \to M/N \to \frac{M/N}{r(M/N)} \to 0$, $(l := \deg r)$, from which we have the exact sequence $H^c_{R,+} (M/N)[-l] \to H^c_{R,+} (M/N) \to H^c_{R,+} (\frac{M/N}{r(M/N)}) \to 0$. If $H^c_{R,+} (\frac{M/N}{r(M/N)}) \neq 0$, then $\text{cd}(\frac{M/N}{r(M/N)}) = c$ and so $\text{cd}(R/q) = c$ for some $q \in \operatorname{Ass}_R (\frac{M/N}{r(M/N)})$. As $q \in \operatorname{Supp}_R (M)$ and $r \in q$, this gives a contradiction. Therefore we have $H^c_{R,+} (M/N) = rH^c_{R,+} (M/N)$ which implies that $r \not\in \bigcup_{p \in \operatorname{Att}_R (H^c_{R,+} (M/N))} p$. This completes step 2 because $H^c_{R,+} (M) \cong H^c_{R,+} (M/N)$.

**Step 3.** Finally we show $\text{cd}(R/p) = c, p \in \bigcup_{m_0 \in \operatorname{Max}(R_0), p \in \operatorname{Att}_R (H^c_{R,+} (M/m_0 M))} p$. Choose $p \in \operatorname{Supp}_R (M)$ with $\text{cd}(R/p) = c$. Hence $\dim R(\frac{R/p}{m_0(R/p)}) = c$ for some $m_0 \in \operatorname{Max}(R_0)$ and so there exists $q \in \operatorname{Supp}_R (M/m_0 M)$ such that $m_0 R + p \subseteq q$ and $\dim R(q) = c$. This implies that $q \in \operatorname{Att}_R (H^c_{R,+} (M/m_0 M))$ (see [MS], or [DY1, Theorem A] for non–local case). This is step 3 and so the proof is complete. \hfill \square

The following corollary is in contrast to the result [Br, Corollary 3.9] which states that $\operatorname{Min}(\operatorname{Ass}_R (H^c_{R,+} (M)))$ is a finite set.

**2.2. Corollary.** Assume that $(R_0, m_0)$ is a local ring and that $M$ is a finitely generated graded $R$–module. Then

$$\operatorname{Max}(\operatorname{Att}_R (H^c_{R,+} (M))) = \operatorname{Att}_R (H^c_{R,+} (M/m_0 M)).$$

In particular, $\operatorname{Max}(\operatorname{Att}_R (H^c_{R,+} (M)))$ is a finite set and it depends only on $\operatorname{Supp}_R (M)$.

**Proof.** Note that the set $\operatorname{Att}_R (H^c_{R,+} (M/m_0 M))$ is a subset of the associated height of $M/m_0 M$ so that it is a finite set and there are no containment relations among its elements (c.f. [DY1, Theorem A]). Hence, by Theorem 2.1, and the right exactness of $H^c_{R,+} (-)$, we get the result. \hfill \square

Now we present an example of a finitely generated graded module $M$ such that $\operatorname{Att}_R (H^c_{R,+} (M))$ is not a finite set. Assume that $R = R_0 [X]$ is the polynomial ring over a Noetherian local ring $R_0$ with $\dim R_0 > 1$. As $X$ is a non–zero divisor on $R$, it is trivial that $H^1_{(X)} (R) \cong R_X / R$. It is easy to see that $p_0 [X] \in \operatorname{Att}_R (H^1_{(X)} (R))$ for any $p_0 \in \operatorname{Spec} R_0$ so that $\operatorname{Att}_R (H^1_{(X)} (R))$ is an infinite set.

### 3. Associated Primes, non–graded and graded cases

For a graded $R$–module $M$, we study the set of associated primes of $H^c_{R,+} (M)$. In this section we first bring some results for non–graded case and then we state them for our main purpose. We assume that $S$ is a Noetherian ring, $K$ is an $S$–module and $a$ is an ideal of $S$. The aim of the following result is to show that, for any non–negative integer $t$, the set of associated primes of $H^c_{t}(K)$ depends on the set of the associated primes of some quotients of the modules $\text{Ext}_S^{t}(S/a, K)$ and $\text{Ext}_S^{t+j+1}(S/a, H^c_{t}(K))$, $0 \leq j < t$ (compare with [DM, Theorem 2.5]).
3.1. **Theorem.** Assume that $S$ is a Noetherian ring, $a$ is an ideal of $S$ and that $K$ is an $S$–module. Then, for each non–negative integer $t$, there exist a submodule $L$ of $\text{Ext}_S^t(S/a, K)$ and submodules $L_j$ of $\text{Ext}_S^{t-j+1}(S/a, \text{H}_a^j(K))/L$, $0 \leq j < t$, such that

$$\text{Ass}_S(\text{H}_a^t(K)) \subseteq \bigcup_{0 \leq j < t} \text{Ass}_S(\text{Ext}_S^{t-j+1}(S/a, \text{H}_a^j(K))/L_j) \cup \text{Ass}_S(\text{Ext}_S^t(S/a, K)/L).$$

**Proof.** The result is clear for $t = 0$. Assume that $t > 0$ and that $t - 1$ is settled. Note that $\text{H}_a^j(K) \cong \text{H}_a^j(K/\Gamma_a(K))$ for all $j > 0$. Assume that $E$ is an injective hull of $K/\Gamma_a(K)$ and set $N := E/(K/\Gamma_a(K))$. Hence we have $\text{H}_a^j(N) \cong \text{H}_a^{j+1}(K)$ and $\text{Ext}_S^j(S/a, N) \cong \text{Ext}_S^{j+1}(S/a, K/\Gamma_a(K))$ for all $i \geq 0$. By our induction hypothesis, there exist submodules $T$ and $L_{j+1}$ of the appropriate modules such that

\begin{equation}
\text{Ass}_S(\text{H}_a^{t-1}(N)) \subseteq \bigcup_{0 \leq j < t-1} \text{Ass}_S(\frac{\text{Ext}_S^{t-1-j+1}(S/a, \text{H}_a^j(N))}{L_{j+1}}) \cup \text{Ass}_S(\frac{\text{Ext}_S^{t-1}(S/a, N)}{T}).
\end{equation}

Note that from the exact sequence $\text{Ext}_S^j(S/a, K) \xrightarrow{f} \text{Ext}_S^j(S/a, K/\Gamma_a(K)) \xrightarrow{g} \text{Ext}_S^{j+1}(S/a, \Gamma_a(K))$ we have the induced exact sequence

$$\text{Ext}_S^j(S/a, K)/f^{-1}(T) \xrightarrow{f} \text{Ext}_S^j(S/a, K/\Gamma_a(K))/T \xrightarrow{g} \text{Ext}_S^{j+1}(S/a, \Gamma_a(K))/g(T).$$

Therefore, there are submodules $L$ and $L_0$ of the appropriate modules such that

\begin{equation}
\text{Ass}_S(\frac{\text{Ext}_S^j(S/a, K/\Gamma_a(K))}{T}) \subseteq \text{Ass}_S(\frac{\text{Ext}_S^j(S/a, K)}{L}) \cup \text{Ass}_S(\frac{\text{Ext}_S^{j+1}(S/a, \Gamma_a(K))}{L_0}).
\end{equation}

Now, (1) and (2) imply the claim. \qed

3.2. **Corollary.** Assume that $R = \bigoplus_{i \in \mathbb{N}_0} R_i$ is a homogeneous Noetherian graded ring (i.e. $R = R_0[R_1]$) and $M$ is a graded $R$–module. Then, for each non–negative integer $s$, there exist graded submodules $L$ of $\text{Ext}_R^s(R/R_+, M)$ and $L_j$ of $\text{Ext}_R^{s-j+1}(R/R_+, \text{H}_{R_+}^j(M))$, $0 \leq j < s$, such that

$$\text{Ass}_R(\text{H}_{R_+}^s(M)) \subseteq \bigcup_{0 \leq j < s} \text{Ass}_R(\frac{\text{Ext}_R^{s-j+1}(R/R_+, \text{H}_{R_+}^j(M))}{L_j}) \cup \text{Ass}_R(\frac{\text{Ext}_R^s(R/R_+, M)}{L_0}).$$

**Proof.** The proof is similar to that of Theorem 3.1. One might take into consideration that in the proof we replace injective hull of $M/\Gamma_{R_+}(M)$ by $\ast$injective hull of $M/\Gamma_{R_+}(M)$ and note that all modules are graded and all homomorphisms are homogeneous. \qed

3.3. **Definition.** (see [DM, Definition 2.1]) A graded $R$–module $M$ is called $\ast$weakly Laskerian if for each graded submodule $N$ of $M$, $\text{Ass}_R(M/N)$ is a finite set.

3.4. **Corollary.** Assume that $R$ is a homogeneous Noetherian graded ring, $M$ is a graded $R$–module and that $s$ is a non–negative integer. If all the modules $\text{Ext}_R^s(R/R_+, M)$ and $\text{Ext}_R^{s-j+1}(R/R_+, \text{H}_{R_+}^j(M))$, $0 \leq j < s$, are $\ast$weakly Laskerian, then $\text{Ass}_R(\text{H}_{R_+}^s(M))$ is a finite set.

**Proof.** It follows from Corollary 3.2. \qed
4. Asymptotic behaviors

Assume that \( M \) is a graded \( R \)–module. The module \( M \) is said to have the property of asymptotic stability of associated primes (resp. asymptotic stability of supports) if there exists an integer \( n_0 \) such that \( \text{Ass}_{R_0}(M_n) = \text{Ass}_{R_0}(M_{n_0}) \) for all \( n \leq n_0 \) (resp. \( \text{Supp}_{R_0}(M_n) = \text{Supp}_{R_0}(M_{n_0}) \) for all \( n \leq n_0 \)). The module \( M \) is called tame if \( M_i = 0 \) for all \( i \ll 0 \) or else \( M_i \neq 0 \) for all \( i \ll 0 \). Here, we study the above asymptotic behaviors of \( \text{Hom}_R(R/R_+, H^i_{R_+}(M)) \). In this connection, there are three open problems.

4.1. Problems. (cf. [Br, Problems 6.1, Problem 7.1, and Problem 4.3]) Let \( i \in \mathbb{N}_0 \) and let \( M \) be a finitely generated graded \( R \)–module.

(i) \( H^i_{R_+}(M) \) has the property of asymptotic stability of associated primes.

(ii) \( H^i_{R_+}(M) \) has the property of asymptotic stability of supports.

(iii) \( H^i_{R_+}(M) \) is tame.

Note that (i) implies (ii) and (ii) implies (iii). In this section we investigate the above questions for the module \( \text{Hom}_R(R/R_+, H^i_{R_+}(M)) \). We note that for a graded \( R \)–module \( M \), \( \text{Ext}^i_R(R/R_+, M) \) has a natural grading and \( \text{Ext}^i_R(R/R_+, M) \cong \text{Ext}^i_R(R/R_+, M) \) for all \( i \geq 0 \) (see [BS, Proposition 12.2.7]). We first note the following easy lemma.

4.2. Lemma. Assume that \( R \) is a homogeneous Noetherian graded ring and that \( M \) is a graded \( R \)–module. Then the following statements hold.

(i). If \( \text{Hom}_R(R/R_+, M) \) is tame, then \( M \) is tame.

(ii). Assume that \((R_0, m_0)\) is local and that \( m^* := m_0 + R_+ \) is the maximal ideal of \( R \). Assume that, for the graded \( R \)–module \( M \), each component \( M_i \) is a finite \( R_0 \)–module. If \( M/m^*M \) is Artinian, then \( M \) is tame.

Proof. (i). Assume that there is \( n_0 \in \mathbb{Z} \) such that \( \text{Hom}_R(R/R_+, M)_n \) is either zero for all \( n \) with \( n \leq n_0 \) or is non–zero for all \( n \) with \( n \leq n_0 \). Now assume that \( M_k = 0 \) for some \( k \leq n_0 \). We show that \( M_i = 0 \) for all \( i \leq k \). It is enough to show that \( M_{k-1} = 0 \). As \( R_1 M_{k-1} \subseteq M_k = 0 \), we have \( M_{k-1} \subseteq 0 \). By Nakayama Lemma we get \( M_k = R_1 M_{k-1} \) for all \( k \ll 0 \). This implies that \( M \) is tame.

4.3. Definition. A graded module \( M \) over a homogeneous graded ring \( R \) is called asymptotically zero if \( M_n = 0 \) for all \( n \ll 0 \).

All finitely generated graded \( R \)–modules are asymptotically zero. Now, we are ready to present our main results of this section. We put these results in the following theorem.

4.4. Theorem. Assume that \( M \) is a graded \( R \)–module and that \( s \) is a fixed non–negative integer such that the modules

\[ \text{Ext}^{s-j}_{R}(R/R_+, H^i_{R_+}(M)), \text{Ext}^{s-j+1}_{R}(R/R_+, H^i_{R_+}(M)), j = 0, 1, \ldots, s-1 \]

are asymptotically zero (e.g. they might be finitely generated). Then the following statements hold.

(i) The module \( \text{Ext}^i_{R}(R/R_+, M) \) has the property of asymptotic stability of associated primes if and only if \( \text{Hom}_R(R/R_+, H^i_{R_+}(M)) \) has the property of asymptotic stability of associated primes.
(ii) The module $\text{Ext}_R^s(R/R_+, M)$ has the property of asymptotic stability of supports if and only if $\text{Hom}_R(R/R_+, H^i_{R_+}(M))$ has the property of asymptotic stability of supports.

(iii) The module $\text{Ext}_R^s(R/R_+, M)$ is tame if and only if $\text{Hom}_R(R/R_+, H^i_{R_+}(M))$ is tame.

Proof. The proofs of (i), (ii), and (iii) are essentially similar therefore we give a proof for (iii) only. The proof is inspired by that of [DY2, Theorem 6.3.9]. The case $s = 0$ is trivial because we have $\text{Hom}_R(R/R_+, \Gamma_{R_+}(M)) = \text{Hom}_R(R/R_+, M)$. Assume that $s > 0$ and that the case $s - 1$ is settled. Denote the $^*$ injective hull of $M/\Gamma_{R_+}(M)$ by $E$ and denote $N := E/(E/\Gamma_{R_+}(M))$. Therefore we have the exact sequence

$$0 \rightarrow M/\Gamma_{R_+}(M) \rightarrow E \rightarrow N \rightarrow 0,$$

which implies the isomorphisms $H^i_{R_+}(N) \cong H^i_{R_+}(M)$ and

$$\text{Ext}_R^j(R/R_+, N) \cong \text{Ext}_R^{j+1}(R/R_+, M/\Gamma_{R_+}(M))$$

for all $j \geq 0$ (note that $\text{Hom}_R(R/R_+, E) = 0 = \Gamma_{R_+}(E)$). Thus, for all $j \geq 0$, we have the isomorphisms

$$\text{Ext}_R^{s-1-j}(R/R_+, H^i_{R_+}(N)) \cong \text{Ext}_R^{s-(j+1)}(R/R_+, H^i_{R_+}(M))$$

and

$$\text{Ext}_R^{s-1-j+1}(R/R_+, H^i_{R_+}(N)) \cong \text{Ext}_R^{s-(j+1)+1}(R/R_+, H^i_{R_+}(M)).$$

Therefore, for all $j = 0, \ldots, s - 2$, the modules

$$\text{Ext}_R^{s-1-j}(R/R_+, H^i_{R_+}(N)), \text{Ext}_R^{s-1-j+1}(R/R_+, H^i_{R_+}(N))$$

are asymptotically zero. We now prepare the requirement for the induction step $s - 1$ for $N$. By the exact sequence $0 \rightarrow \Gamma_{R_+}(M) \rightarrow M \rightarrow M/\Gamma_{R_+}(M) \rightarrow 0$ we have the exact sequence of graded modules with homogeneous homomorphisms

$$\text{Ext}_R^s(R/R_+, \Gamma_{R_+}(M)) \rightarrow \text{Ext}_R^s(R/R_+, M) \rightarrow \text{Ext}_R^s(R/R_+, M/\Gamma_{R_+}(M)) \rightarrow \text{Ext}_R^{s+1}(R/R_+, \Gamma_{R_+}(M)).$$

By our hypothesis for $s$ and $j = 0$, there exists $t \in \mathbb{Z}$ such that the modules $\text{Ext}_R^n(R/R_+, \Gamma_{R_+}(M))$ and $\text{Ext}_R^{n+1}(R/R_+, \Gamma_{R_+}(M))$ are zero for all $n \leq t$. Therefore, for each $n \leq t$, we have an $R_0$-isomorphism

$$\text{Ext}_R^n(R/R_+, M) \cong \text{Ext}_R^n(R/R_+, M/\Gamma_{R_+}(M)).$$

Now, it follows, from (†) and (‡), that $\text{Ext}_R(R/R_+, M)$ is tame if and only if $\text{Ext}_R^{s-1}(R/R_+, N)$ is tame, which is also equivalent to say that $\text{Hom}_R(R/R_+, H^s_{R_+}(N))$ is tame, by our induction hypothesis. This statement is also equivalent to say $\text{Hom}_R(R/R_+, H^i_{R_+}(M))$ is tame.

4.5. Corollary. (see [Br, Theorem 4.8(b)] and [DY3, Theorem 2.1]) Assume that $M$ is a finitely generated graded $R$-module and that $s$ is a fixed non–negative integer such that for each $i < s$, $H^i_{R_+}(M)$ is graded $R_+$-cfinite, i.e. $\text{Ext}_R^i(R/R_+, H^i_{R_+}(M))$ is finitely generated for all $j$. Then $\text{Hom}_R(R/R_+, H^s_{R_+}(M))$ is asymptotically zero. In particular, $\text{Hom}_R(R/R_+, H^s_{R_+}(M))$ is finitely generated and so $H^s_{R_+}(M)$ is tame.

Let $M$ be a finitely generated $R$–module and $n := \dim_R(M)$. In [Br, Theorem 4.8 (c)], Brodmann showed that $H^s_{R_+}(M)$ is tame. This clearly implies that $H^s_{R_+}(M)$ is tame (this is also a consequence of the fact that $H^s_{R_+}(M)$ is an Artinian module). Brodmann also showed that $H^s_{R_+}(M)$ is tame for all $i \in \Psi(M)$ , where $\Psi(M) = \{\text{height}(p)/(0 :_R M)) | p \in \text{Min}(0 : R M) + R_+ \}$ (see [Br,
Theorem 4.8 (d)). Note that when \( R_0 \) is a field and \( M = R_0[x] \), we have \( \Psi(M) = \{ \dim_R(M) \} \). Therefore it would be significant to see explicitly that \( H^{q(M)}_{R_+}(M) \) is Artinian. This result is a consequence of a more general one (see Corollary 4.8).

In the remainder of this section we assume that \( M \) is a graded module over the homogeneous Noetherian graded ring \( R = \bigoplus_{i \in \mathbb{N}_0} R_i \).

4.6. Definition. (See [DY3, Definition 3.1]) Define \( q(M) := \sup \{ i | H^i_{R_+}(M) \text{ is not Artinian} \} \). If \( H^i_{R_+}(M) \) is Artinian for all \( i \), we write \( q(M) = -\infty \).

4.7. Theorem. Assume that the base ring \( R_0 \) is local with maximal ideal \( m_0 \) and that \( M \) is a finitely generated graded \( R \)-module. Then

\[
H^{q(M)}_{R_+}(M)/m^t H^{q(M)}_{R_+}(M)
\]

is Artinian.

Proof. Set \( n(M) := cd(M) - q(M) \) and we prove our claim by using induction on \( n(M) \). When \( n(M) = 0 \) the result is known (c.f. [Br, Theorem 2.3 (b)]) because, in this case, \( H^{q(M)}_{R_+}(M)/m^t H^{q(M)}_{R_+}(M) \) is a homomorphic image of \( H^{q(M)}_{R_+}(M)/m_0 H^{q(M)}_{R_+}(M) \). Assume that \( n(M) = n > 0 \) and we have proved the statement for any finitely generated graded \( R \)-module \( N \) with \( n(N) = n - 1 \). Thus we have \( cd(M) > 0 \). We may assume that \( q(M) > 0 \). Therefore, \( q(M) = q(M/\Gamma_{R_+}(M)) \), \( cd(M) = cd(M/\Gamma_{R_+}(M)) \), and \( n(M) = n(M/\Gamma_{R_+}(M)) \). Hence we may assume that \( M \) is \( R_+ \)-torsion free and there exists \( x \in R_1 \) which is non-zero-divisor on \( M \) and \( cd(M/xM) = cd(M) - 1 \) (c.f. [RS, 1.3.7]). The short exact sequence

\[
0 \rightarrow M[-1] \rightarrow M \rightarrow M/xM \rightarrow 0
\]

yields a long exact sequence

\[
\cdots \rightarrow H^{q(M)}_{R_+}(M)[-1] \rightarrow H^{q(M)}_{R_+}(M) \rightarrow H^{q(M)}_{R_+}(M/xM) \rightarrow H^{q(M)+1}_{R_+}(M)[-1] \rightarrow \cdots
\]

from which we have the exact sequence

\[
0 \rightarrow H^{q(M)}_{R_+}(M)/x H^{q(M)}_{R_+}(M) \rightarrow H^{q(M)}_{R_+}(M/xM) \rightarrow (0_{H^{q(M)+1}_{R_+}(M)} x)[-1] \rightarrow 0.
\]

Therefore we have the exact sequence

\[
\text{Tor}^R(R/m^t, (0_{H^{q(M)+1}_{R_+}(M)} x))[-1] \rightarrow H^{q(M)}_{R_+}(M)/m^t H^{q(M)}_{R_+}(M) \rightarrow H^{q(M)}_{R_+}(M)/m^t H^{q(M)}_{R_+}(M/xM) \rightarrow (0_{H^{q(M)+1}_{R_+}(M)} x)/m^t (0_{H^{q(M)+1}_{R_+}(M)} x)[-1].
\]

Note that the first and the last term in the above exact sequence are Artinian modules. If \( H^{q(M)}_{R_+}(M/xM) \) is Artinian then \( H^{q(M)}_{R_+}(M/xM)/m^t H^{q(M)}_{R_+}(M/xM) \) is Artinian and so

\[
H^{q(M)}_{R_+}(M)/m^t H^{q(M)}_{R_+}(M)
\]

is also Artinian. Now, we assume that \( H^{q(M)}_{R_+}(M/xM) \) is not Artinian. It follows that \( q(M/xM) = q(M) \) and hence \( n(M/xM) = cd(M/xM) = q(M/xM) = n(M) - 1 \). By our induction hypothesis, \( H^{q(M)}_{R_+}(M/xM)/m^t H^{q(M)}_{R_+}(M/xM) \) is Artinian. By the above exact sequence, the module

\[
H^{q(M)}_{R_+}(M)/m^t H^{q(M)}_{R_+}(M)
\]

is Artinian. \( \square \)
4.8. **Corollary.** With the assumptions as in Theorem 4.7, $H^i_{R_+}(M)$ is tame for all $i \geq q(M)$. In particular, the modules $H^{2d(M)}_{R_+}(M)$ and $H^{\dim R(M)-1}_{R_+}(M)$ are tame.

**Proof.** Note that $H^{\dim R(M)}_{R_+}(M)$ is Artinian (c.f. [BS, Theorem 7.1.6]). The claims follow by Theorem 4.7 and Lemma 4.2(ii). \hfill \Box

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