THE INVISCID LIMIT OF THIRD-ORDER LINEAR AND NONLINEAR ACOUSTIC EQUATIONS

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ABSTRACT. We analyze the behavior of third-order in time linear and nonlinear sound waves in thermally relaxing fluids and gases as the sound diffusivity vanishes. The nonlinear acoustic propagation is modeled by the Jordan–Moore–Gibson–Thompson equation both in its Westervelt and in its Kuznetsov-type forms, that is, including quadratic nonlinearities of the type \((u^2)_{tt}\) and \((u^2 + |\nabla u|^2)_{tt}\). As it turns out, sufficiently smooth solutions of these equations converge in the energy norm to the solutions of the corresponding inviscid models at a linear rate. Numerical experiments illustrate our theoretical findings.

1. Introduction

The present work focuses on the limiting behavior of linear and nonlinear equations that describe the motion of sound waves through thermally relaxing media as the diffusivity of sound vanishes. In modeling, the need to combine thermal relaxation with the nonlinear and dissipative effects leads to third-order in time equations. They have a general form of

\[
\tau u_{ttt} + u_{tt} - (\delta + \tau c^2)\Delta u_t - c^2\Delta u = f(u, u_t, u_{tt}, \nabla u, \nabla u_t),
\]

where the function \(u = u(x, t)\) may denote the acoustic pressure or acoustic velocity potential. The parameter \(\tau > 0\) denotes the thermal relaxation time, \(c > 0\) the speed of sound, and the coefficient \(\delta\) is often referred to as the diffusivity of sound [13, §3]. The parameter \(\delta\) is relatively small in fluids and gases, which motivates our research into the behavior of solutions of (1.1) as \(\delta \to 0^+\).

Third-order models of nonlinear acoustics in the form of (1.1) originate from using a general temperature law within the governing system of equations, which includes conservation laws and constitutive equations of the medium [14]. By the standard Fourier temperature law, a thermal disturbance at one point has an instantaneous effect elsewhere in the medium [12, 24], which may lead to a paradox of diffusion in waves. The Maxwell–Cattaneo law, on the other hand, introduces a time lag between the temperature gradient and the heat flux induced by it [21, 36]:

\[
\tau q_t + q = -K \nabla \theta.
\]

The heat then propagates in time via thermal waves, which is often referred to as the second-sound phenomenon in the literature [14, 36]. Employing the Maxwell–Cattaneo temperature law within the governing equations of sound propagation leads to third-order in time models [13] that avoid the paradox of diffusion and

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instead have the expected hyperbolic character.

Third-order wave motion may also originate from the presence of molecular relaxation when the pressure-density relation of the medium is not satisfied exactly, but up to a memory term; see [28, §1.1] and [34, §4]. Such relaxation mechanisms typically occur in media with “impurities”; this can be, for example, water with micro-bubbles, seawater, chemically reacting fluids, or a mixture of gases. As such, they arise in various applications. For instance, micro-bubbles are often used as a contrast agent in ultrasonic imaging [10]. They are also known to increase the speed and efficacy of the focused ultrasound treatments [37].

In this work, we investigate convergence of solutions to equations (1.1) as $\delta \to 0^+$, leading to the propagation of sound through relaxing, inviscid media. We consider the limiting behavior in the linearized versions of (1.1) on smooth bounded domains as well as in the nonlinear PDE, which is often referred to as the Jordan–Moore–Gibson–Thompson equation in the literature. In particular, we take into account two types of nonlinearities, which have a physical motivation:

\[
(1.2) \quad f(u, u_t, u_{tt}) = \frac{1}{2}(ku^2)_{tt} = kuu_{tt} + ku_t^2,
\]

and

\[
(1.3) \quad f(u_t, u_{tt}, \nabla u, \nabla u_t) = \frac{1}{2}(\kappa u_t^2 + \sigma|\nabla u|^2)_{tt} = \kappa u_{tt} + \sigma \nabla u \cdot \nabla u_t.
\]

Our main results pertain to the convergence of solutions to (1.1) with homogeneous Dirichlet boundary conditions in the energy norm to their inviscid counterparts as the sound diffusivity vanishes. It turns out that sufficiently smooth solutions of (1.1) converge to the solutions of the inviscid equation at a linear rate:

\[
\|u^{(\delta)} - u\|_E \lesssim \delta \quad \text{as} \quad \delta \to 0^+.
\]

The smoothness requirements are naturally higher in the presence of the quadratic gradient nonlinearity (1.3). We refer to upcoming Theorems 3.1, 4.2, and 6.2 for details.

We organize the remaining of our exposition as follows. In Section 2, we provide more information on the mathematical modeling of ultrasonic waves in thermally relaxing fluidic media and give an overview of the related mathematical results. Section 3 deals with a linearized version of (1.1). In particular, we derive a uniform bound with respect to $\delta$ for its solutions and prove a limiting result for the vanishing sound diffusivity. Section 4 extends the investigation to the JMGT equation with the nonlinearity in form of (1.2) by employing Banach’s Fixed-point theorem. In Section 5, we return to a linearized problem to derive a higher-order energy bound that is uniform with respect to $\delta$. This is needed so that in Section 6 we can analyze the limiting behavior of the JMGT equation with the right-hand side in the form of (1.3). Finally, in Section 7 we provide the results of numerical experiments which illustrate our theory and the convergence rate $O(\delta)$.

2. Mathematical modeling and related work

The Jordan–Moore–Gibson–Thompson (JMGT) equation, given by

\[
(2.1) \quad \tau \psi_{ttt} + \psi_{tt} - (\delta + \tau c^2)\Delta \psi_t - c^2 \Delta \psi = \frac{1}{2}(\kappa \psi_t^2 + \sigma|\nabla \psi|^2)_t,
\]

is a model of ultrasonic propagation that accounts for thermal relaxation, dissipation due to viscosity, and nonlinear effects of quadratic type. It is expressed in terms of the acoustic velocity potential $\psi = \psi(x, t)$; we refer to [14] for its derivation. The
parameter \( \tau > 0 \) represents the relaxation time of the heat flux and the coefficient \( c > 0 \) stands for the speed of sound. The nonlinear parameters on the right-hand side are typically \( \sigma = 2 \) and \( \kappa = \frac{B}{A} c^2 \), where \( B/A \) is the parameter of nonlinearity that arises from the pressure-density relation in a given medium. For the purposes of our analysis, it is sufficient to assume that \( \sigma, \kappa \in \mathbb{R} \).

Local nonlinear effects in sound propagation can often be neglected if the propagation distance is sufficiently large in terms of the number of wavelengths. In such cases, employing the approximation \( |\nabla \psi|^2 \approx \frac{1}{c^2} \psi_t^2 \) in (2.1) leads to

\[
\tau \psi_{ttt} + \psi_{tt} - c^2 \Delta \psi - (\delta + \tau c^2) \Delta \psi_t = \frac{1}{c^2} \left( 1 + \frac{B}{2A} \right) (\psi_t^2)_t.
\]

This equation is frequently expressed in terms of the acoustic pressure. Formally differentiating (2.2) with respect to time and then using the relation \( p = \varrho \psi_t \), where \( \varrho \) is the medium density, yields

\[
\tau p_{utt} + p_{tt} - (\delta + \tau c^2) \Delta p_t - c^2 \Delta p = \frac{1}{2} (kp^2)_tt,
\]

where now \( k = \frac{1}{\varrho c^2} \left( 1 + \frac{B}{2A} \right) \). Again, for our theoretical purposes, we can take \( k \in \mathbb{R} \).

Letting \( \tau \to 0 \) in (2.1) and (2.3) leads to the classical second-order models of nonlinear acoustics – Kuznetsov’s [22] and Westervelt’s [49] equations. For this reason and to distinguish different third-order equations, we will henceforth refer to (2.1) and (2.3) as the JMGT–Kuznetsov and JMGT–Westervelt equation, respectively.

A linear version of these equations is often referred to as the Moore–Gibson–Thompson (MGT) equation [27, 40]:

\[
\tau p_{utt} + \alpha p_{tt} - (\delta + \tau c^2) \Delta p_t - c^2 \Delta p = 0.
\]

Early mathematical investigations of third-order in time linear PDEs include the studies on classical solutions and Cauchy problems by Varlamov in [41, 43, 46], and Renno in [32]. Matkowsky and Reiss studied the asymptotic expansion of solutions to the linear problem as \( \tau \to 0^+ \) in [26]. Further studies on the theory of singular perturbation can be found in [2, 42]. In [47], Varlamov and Nesterov analyzed the linear equation with spatially varying coefficients with results on the existence and uniqueness of classical solutions and their asymptotic expansion as \( \tau \to 0^+ \). Nonlinear third-order propagation was seemingly first investigated by Varlamov in [44, 45] in terms of existence and asymptotic behavior of classical solutions with a heuristically motivated right-hand side nonlinearity given by \( \Delta(u^2) \).

A more recent mathematical research on third-order ultrasonic waves was initiated in [15] with a semigroup approach employed in the well-posedness and stability analysis of the linear equation (2.4). As concluded in [15], exponential stability of solutions requires that

\[
\gamma := \alpha - \frac{\tau c^2}{\delta + \tau c^2} > 0.
\]

The problem is unstable when \( \gamma < 0 \) and marginally stable when \( \gamma = 0 \). We note that in this work we focus on the short-time behavior, and so no assumptions on the sign of \( \gamma \) are made.

Since the results of [15], the interest in the qualitative behavior of third-order acoustic equations has flourished significantly, and these linear and nonlinear models represent by now a very active area of research. We thus provide the reader with a selection of relevant works here on the analysis of linear [2, 7, 9, 25, 24, 30] and nonlinear third-order acoustic equations [6, 16, 31]. We also note that the limiting
behavior of solutions for vanishing thermal relaxation time \( \tau \) (and fixed \( \delta > 0 \)) has been studied in, e.g., [4, 17]. Taking into account the effects of both thermal and molecular relaxation leads to third-order equations with memory, which have also been a topic of extensive research recently; see, for example, [1, 8, 23] and the references given therein. To the best of our knowledge, the present work is the first dealing with the convergence of solutions to third-order equations with the vanishing sound diffusivity \( \delta \).

2.1. Notation and auxiliary results. Before proceeding, let us briefly set the notation and go over commonly used inequalities and embedding results. To simplify notation, we often omit the spatial domain and the time interval when writing norms; in other words, \( \| \cdot \|_{L^p(L^q)} \) denotes the norm on \( L^p(0,T;L^q(\Omega)) \).

Throughout the paper, we make the following regularity assumption concerning the spatial domain:

\(^{(A_1)}\) \( \Omega \subset \mathbb{R}^d \) is an open, bounded, and \( C^{1,1} \) regular or polygonal and convex set, where \( d \in \{1,2,3\} \).

When stating solution spaces for \( p \) and \( \psi \), we denote

\[
H^2_\Phi(\Omega) = H^1_0(\Omega) \cap H^2(\Omega),
\]

\[
H^3_\Phi(\Omega) = \{ \psi \in H^3(\Omega) : \text{tr}_{\partial \Omega} \psi = 0, \ \text{tr}_{\partial \Omega} \Delta \psi = 0 \}.
\]

In the analysis, we will need to rely on the boundedness of the operator \((-\Delta)^{-1} : L^2(\Omega) \to H^2_\Phi(\Omega) \). We point out that since we do not need a stronger elliptic regularity result than this, we do not introduce stronger regularity assumptions than given in \(^{(A_1)}\) on \( \Omega \).

Additionally, we will often use the Poincaré–Friedrichs inequality and the continuous embeddings \( H^1(\Omega) \hookrightarrow L^4(\Omega) \) and \( H^2(\Omega) \hookrightarrow L^\infty(\Omega) \):

\[
\|v\|_{L^4(\Omega)} \leq C_{H^1,L^4} \|\nabla v\|_{L^2(\Omega)}, \quad v \in H^1_0(\Omega)
\]

\[
\|v\|_{L^\infty(\Omega)} \leq C_{H^2,L^\infty} \|\Delta v\|_{L^2(\Omega)}, \quad v \in H^2_\Phi(\Omega).
\]

We occasionally use \( x \lesssim y \) to denote \( x \leq Cy \), where the generic constant \( C > 0 \) does not depend on the sound diffusivity \( \delta \), but may depend on other medium parameters and the final time \( T \).

3. The linearized JMGT–Westervelt equation

We begin by investigating an initial boundary-value problem for a linearization of the JMGT–Westervelt equation:

\[
\begin{aligned}
\tau p_{ttt} + \alpha p_{tt} - (\delta + \tau \xi^2) \Delta p_t - \xi^2 \Delta p - \mu p_t - \eta p &= f, \\
p_{\partial \Omega} &= 0,
\end{aligned}
\]

with initial conditions

\[
(p,p_t,p_{tt})|_{t=0} = (p_0,p_1,p_2).
\]

In particular, we wish to derive an energy bound for (3.1a) that is uniform with respect to \( \delta \) and will later allow us to derive the corresponding bound for the nonlinear equation. To this end we make the following regularity assumptions.

\(^{(A_2)}\) The coefficients \( \alpha, \mu, \) and \( \eta \) are sufficiently smooth

\[
\alpha \in W^{1,\infty}(0,T;H^2_\Phi(\Omega)), \ \mu \in L^\infty(0,T;H^2_\Phi(\Omega)), \ \eta \in L^\infty(0,T;H^1_0(\Omega)),
\]
and uniformly bounded
\[ \|\alpha\|_{W^{1,\infty}(H^2_0(\Omega))}, \|\mu\|_{L^\infty(H^2_0(\Omega))}, \|\nabla \eta\|_{L^\infty(L^2)} \leq R, \]
where \( R \) is a positive constant independent of \( \delta \). This assumption further implies that
\[ \underline{\alpha} \leq \alpha \leq \overline{\alpha} \]
for \( \underline{\alpha} = -C H^2 L^\infty R \) and \( \overline{\alpha} = C H^2 L^\infty R \).

(A3) The initial conditions (\ref{3.1b}) satisfy
\[ (p_0, p_1, p_2) \in X^W_0 = H^1_\delta(\Omega) \times H^2_\delta(\Omega) \times H^1_0(\Omega). \]

(A4) The source term \( f \) satisfies
\[ f \in L^2(0, T; H^1_0(\Omega)). \]

Note that since possibly \( \underline{\alpha} < 0 \), we do not make a non-degeneracy assumption on the coefficient \( \alpha \) here. As already mentioned, this stems from the fact that we are interested in the short-time behavior of solutions. For an estimate in weaker norms, we will replace (A3) and (A4) by the following weaker assumptions.

(\(\sim\)A3) The initial conditions (\ref{3.1b}) satisfy
\[ (p_0, p_1, p_2) \in H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega). \]

(\(\sim\)A4) The source term \( f \) satisfies
\[ f \in L^2(0, T; L^2(\Omega)). \]

We next prove a well-posedness result with a uniform bound in \( \delta \) for (\ref{3.1a}).

**Proposition 3.1.** Let assumptions (A1)–(A4) hold and let \( \tau, \epsilon > 0 \), and \( \delta \in [0, \delta] \). Then initial boundary-value problem (\ref{3.1}) has a unique solution
\[ p \in X^W = W^{3,\infty}(0, T; L^2(\Omega)) \cap W^{2,\infty}(0, T; H^1_0(\Omega)) \cap W^{1,\infty}(0, T; H^2_0(\Omega)). \]

Furthermore, this solution satisfies
\[ \|\nabla p(t)\|_{H^2}^2 + \|\Delta p(t)\|_{L^2}^2 + \|\Delta p(t)\|_{L^2}^2 \leq C(\tau) \exp(K(\tau)(R^2 + 1)T) \left( \|\nabla p(t)\|_{H^2}^2 + \|\Delta p(t)\|_{L^2}^2 + \|\Delta p(t)\|_{L^2}^2 + \|\nabla f\|_{L^2(L^2)}^2 \right), \]
where the constants \( C(\tau) \) and \( K(\tau) \) tend to infinity as \( \tau \to 0^+ \), but are independent of \( R \), the final time \( T \), and the sound diffusivity \( \delta \).

If instead of assumptions (A3) and (A4), only (\(\sim\)A3) and (\(\sim\)A4) hold, any solution \( p \) of (\ref{3.1}) (if it exists) satisfies the estimate
\[ \|p(t)\|_{L^2}^2 + \|\nabla p(t)\|_{L^2}^2 + \|\Delta p(t)\|_{L^2}^2 \leq C(\tau) \exp(K(\tau)(R^2 + 1)T) \left( \|p(t)\|_{L^2}^2 + \|\nabla p(t)\|_{L^2}^2 + \|\nabla f\|_{L^2(L^2)}^2 \right). \]

**Proof.** The proof can be conducted by employing smooth Faedo–Galerkin approximations in space. In particular, we can project the problem onto the span \( V_n \) of the first \( n \) eigenfunctions of the Dirichlet Laplacian pointwise in time; cf. \[11, 33\]. We will focus here on deriving the crucial energy bound for the Galerkin approximations \( p^n \) and refer to, for example, \[17, 18\] for details regarding the application of the Faedo–Galerkin procedure in nonlinear acoustics. To ease the notation, we
drop the superscript \( n \) below and use just \( p \).

In view of the fact that this linear equation is close to a wave equation for \( z = \tau p_t + p \), we test it with \( -\Delta(\tau p_t + p)_t \) and use integration by parts with respect to space. Noting that \( p = \Delta p = 0 \) on \( \partial \Omega \) for sufficiently smooth Galerkin approximations, we can rely on the following identities:

\[
(\nabla[\alpha p_t], \nabla p_t)_{L^2} = (\alpha \nabla p_t, \nabla p_t)_{L^2} + (\nabla \alpha p_t, \nabla p_t)_{L^2},
\]
\[
(\nabla[\alpha p_t], \nabla p_t)_{L^2} = \frac{1}{2} \frac{d}{dt} (\alpha \nabla p_t, \nabla p_t)_{L^2} + (\nabla \alpha p_t - \frac{1}{2} \alpha t \nabla p_t, \nabla p_t)_{L^2},
\]
\[
\nabla[\mu p_t] = \nabla \mu p_t + \mu \nabla p_t.
\]

We also have

\[
(\Delta p, \Delta p_t)_{L^2} = - \|\Delta p_t\|^2_{L^2} + \frac{d}{dt} (\Delta p, \Delta p_t)_{L^2},
\]
\[
(\nabla p_{ut}, \nabla p_t)_{L^2} = - \|\nabla p_t\|^2_{L^2} + \frac{d}{dt} (\nabla p_{ut}, \nabla p_t)_{L^2}.
\]

Taking into account these identities after testing with \( -\Delta(\tau p_t + p)_t \) leads to the energy identity

\[
\frac{1}{2} \tau^2 \frac{d}{dt} \|\nabla p_t\|^2_{L^2} + \tau (\alpha \nabla p_t, \nabla p_t)_{L^2} - \|\nabla p_t\|^2_{L^2} \\
+ \frac{1}{2} \tau (\delta + \tau \epsilon^2) \frac{d}{dt} \|\Delta p_t\|^2_{L^2} + \frac{1}{2} \frac{d}{dt} (\alpha \nabla p_t, \nabla p_t)_{L^2} + \delta \|\Delta p_t\|^2_{L^2}
\]
\[
= (f_1(p) + f_2(p) + \nabla f, \nabla(\tau p_t + p_t))_{L^2} - \frac{1}{2} (\alpha t \nabla p_t, \nabla p_t)_{L^2} \\
- \tau \epsilon^2 \frac{d}{dt} (\Delta p, \Delta p_t)_{L^2} - \tau \frac{d}{dt} (\nabla p_{ut}, \nabla p_t)_{L^2},
\]

where we have introduced the following short-hand notation:

\[
f_1(p) = p_t \nabla \mu + \mu \nabla p_t + p_t \nabla \alpha, \quad f_2(p) = p \nabla \eta + \eta \nabla p.
\]

Note that

\[
\|f_1(p)\|_{L^2} \leq (C^2_{H^1, L^4} + C^2_{H^2, L^\infty}) \|\Delta p_t\|_{L^2} \|\nabla p_t\|_{L^2} + C^2_{H^1, L^4} \|\Delta \alpha\|_{L^2} \|\nabla p_t\|_{L^2},
\]
\[
\|f_2(p)\|_{L^2} \leq (C^2_{H^2, L^\infty} + C^2_{H^1, L^4}) \|\nabla \eta\|_{L^2} \|\Delta p\|_{L^2}.
\]

We next integrate (3.5) with respect to time and use Young’s inequality in the estimates

\[
\tau \epsilon^2 (\Delta p(t), \Delta p_t(t))_{L^2} \leq \frac{1}{2} \tau^2 \epsilon^2 |\Delta p_t(t)|^2_{L^2} + \frac{1}{2} \epsilon^2 \|\Delta p(t)\|^2_{L^2},
\]
\[
\tau (\nabla p_{ut}(t), \nabla p_t(t))_{L^2} \leq \frac{\epsilon}{2} \tau \|\nabla p_{ut}(t)\|^2_{L^2} + \frac{1}{2 \epsilon} \|\nabla p_t(t)\|^2_{L^2},
\]

where \( \epsilon \in (0, 1) \). We can also rely on the following inequality:

\[
\tau \int^t_0 (f_1(p) + f_2(p) + \nabla f, \nabla p_{ut})_{L^2} \, ds
\]
\[
\leq \tau \int^t_0 \|\nabla p_{ut}\|^2_{L^2} \, ds + \frac{\tau}{2} \int^t_0 \|f_1(p) + f_2(p)\|^2_{L^2} \, ds + \frac{\tau}{2} \int^t_0 \|\nabla f\|^2_{L^2} \, ds,
\]
as well as
\[
\int_0^t (f_1(p) + f_2(p) - \frac{1}{2} \alpha_t + \nabla f, \nabla p_t)_{L^2} \, ds
\]
\[
\leq \frac{1}{2} \int_0^t \|\nabla p_t\|_{L^2}^2 \, ds + \int_0^t \|f_1(p) + f_2(p) - \frac{1}{2} \alpha_t\|_{L^2}^2 \, ds + \int_0^t \|\nabla f\|_{L^2}^2 \, ds.
\]
In this manner, after integration with respect to time in (3.3), we arrive at the following energy estimate:
\[
\frac{\tau^2}{2} \|\nabla p(t)\|_{L^2}^2 + \frac{1}{2} \tau \|\Delta p(t)\|_{L^2}^2 + \delta \int_0^t \|\Delta p_t\|_{L^2}^2 \, ds + \frac{1}{2} \tau c^2 \|\Delta p(t)\|_{L^2}^2
\]
\[
\leq \frac{1}{2} \tau^2 \|\nabla p(t)\|_{L^2}^2 + \frac{1}{2} \tau (\delta + \tau c^2) \|\Delta p(t)\|_{L^2}^2 + \tau c^2 (\Delta p(0), \Delta p_t(0))_{L^2}
\]
\[
+ \frac{1}{2} \tau \|\nabla p(t)\|_{L^2}^2 + \tau (\|p\|_{L^\infty}) + 1 \int_0^t \|\nabla p\|_{L^2}^2 \, ds + \frac{1}{2} \left( \frac{1}{\epsilon} - \alpha \right) \|\nabla p(t)\|_{L^2}^2
\]
\[
+ \int_0^t \left( \frac{\tau}{2} \|f_1(p) + f_2(p)\|_{L^2}^2 + \|f_1(p) + f_2(p) - \frac{1}{2} \alpha_t \nabla p_t\|_{L^2}^2 \right) \, ds
\]
\[
+ \left( \frac{\tau}{2} + 1 \right) \int_0^t \|\nabla f\|_{L^2}^2 \, ds.
\]
We already have estimates of the $f_1$ and $f_2$ terms in (3.6). Further, 
\[
\|\alpha_t \nabla p_t\|_{L^2} \leq C_{H^2,L^\infty} \|\Delta \alpha_t\|_{L^2} \|\nabla p_t\|_{L^2},
\]
and we also know that 
\[
\|\nabla p_t\|_{L^2}^2 = \|\nabla p_t(0) + \int_0^t \nabla p\, ds\|_{L^2}^2 \leq 2 \|\nabla p_t(0)\|_{L^2}^2 + 2t \int_0^t \|\nabla p\|_{L^2}^2 \, ds.
\]
Gronwall’s inequality therefore yields 
\[
\tau^2 \|\nabla p(t)\|_{L^2}^2 + \tau^2 c^2 \|\Delta p_t(0)\|_{L^2}^2 + \tau^2 c^2 \|\Delta p(t)\|_{L^2}^2
\]
\[
\leq C \exp(K(\tau)(R^2 + 1)T) \left( \|\nabla p(0)\|_{L^2}^2 + \|\Delta p_t(0)\|_{L^2}^2 + \|\Delta p(0)\|_{L^2}^2 + \|\nabla f\|_{L^2(L^2)}^2 \right),
\]
with a constant $C > 0$ that only depends on the medium parameters $\tau$, $c^2$, and the embedding constants $C_{H^1,L^2}$ and $C_{H^2,L^\infty}$, but is independent of $\delta \in [0, \bar{\delta}]$. This further yields (3.3), at first in its semi-discrete version.

Additionally, from the (Galerkin-discretized) PDE, we obtain a bound on the third time derivative:
\[
\|p_{tt}(t)\|_{L^2}^2 = -\frac{1}{\tau} \|p_{tt}(t), \alpha(t)p_t(t) - (\delta + \tau c^2) \Delta p_t(t) - c^2 \Delta p(t)
\]
\[
- \mu(t)p_t(0) - \eta(t)p(t) - f(t) \|_{L^2}^2 \leq \frac{1}{\tau} \|p_{tt}(t)\|_{L^2}^2 + \|\alpha(t)p_t(t) - (\delta + \tau c^2) \Delta p_t(t) - c^2 \Delta p(t)
\]
\[
- \mu(t)p_t(0) - \eta(t)p(t) - f(t) \|_{L^2}^2.
\]
By the semi-discrete version of (3.3), the above inequality implies that also 
\[
\|p_{tt}\|_{L^\infty(L^2)} \leq \bar{C}(T, R, \tau, c) \left( \|\nabla p_{tt}(0)\|_{L^2}^2 + \|\Delta p_t(0)\|_{L^2}^2 + \|\Delta p(0)\|_{L^2}^2 + \|\nabla f\|_{L^2(L^2)}^2 \right)
\]
and the PDE is satisfied in an $L^\infty(0, T; L^2(\Omega))$ sense. The obtained semi-discrete bounds carry over to the solution of (3.1) via standard compactness arguments;
Theorem 3.1. Under the conditions of Proposition 3.1, the family of solutions \{p^{(\delta)}\}_{\delta > 0} to the linearized JMGT–Westervelt equation converges in the topology induced by the energy norm for the wave equation to a solution \(p\) of the inviscid linearized JMGT–Westervelt equation as \(\delta \to 0^+\) at a linear rate. In other words,

\[
\|p^{(\delta)} - p\|_{E} \lesssim \delta \quad \text{as} \quad \delta \to 0^+.
\]

Proof. Let \(\delta, \delta' \in [0, \overline{\delta}]\). Furthermore, let \(p^{(\delta)}\) and \(p^{(\delta')}\) be the solutions of (3.1) with the sound diffusivity \(\delta\) and \(\delta'\), respectively. We follow the general strategy of [18, 20, 33, 38] by proving that \(\{\overline{p}\}\) is a Cauchy sequence in suitable topology, where \(\overline{p} = p^{(\delta)} - p^{(\delta')}\).

We note that \(\overline{p}\) solves the equation

\[
\tau \overline{p}_{tt} + \alpha \overline{p}_{tt} - (\delta + \tau c^2)\Delta \overline{p} - c^2 \Delta \overline{p} - \mu \overline{p}_t - \eta \overline{p} = -(\delta - \delta')\Delta p^{(\delta')}.
\]
supplemented by zero initial conditions. Applying estimate (3.4) in Proposition 3.1 with 
\( f = -(\delta - \delta') \Delta p_t^{(\delta')} \) directly yields

\[
E[p](t) \leq \frac{1}{2} C(\tau) \exp(K(\tau)(R^2 + 1) T) |\delta - \delta'|^2 R^2
\]

with energy \( E[p] \) defined as in (3.7). Here we have used the bound

\[
R^2 = C(\tau) \exp(K(\tau)(R^2 + 1) T)
\]

on \( \|\Delta p_t^{(\delta')}\|_{L^2(L^2)} \), resulting from (3.3). The stated rate is obtained by setting \( \delta' \) to zero. \( \square \)

Note that from Theorem 3.1, we also obtain a convergence result for the linear Moore–Gibson–Thompson equation (2.4) by setting \( \mu \) and \( \eta \) to zero.

**Remark 1** (Perturbation of the wave speed). For potential comparisons with the results on regularizing perturbations of second-order wave equations in [35], we again view (3.1a) as a second-order wave equation for \( z = \tau p_t + p \). For simplicity, we set \( \alpha = 1, \mu = \eta = f = 0 \), which yields

\[
z_{tt} - c^2 \Delta z - \frac{\delta}{\tau} \Delta z + \frac{\delta}{\tau} \Delta p = 0.
\]

However, neither the parabolic perturbation \( u_{tt} - c^2 \Delta u - \epsilon \Delta u_t = 0 \) considered in [35, Proposition 7] nor the non-singular perturbation \( u_{tt} - c^2 \Delta u - \epsilon u = 0 \) tackled in [35, Proposition 9] apply here. In fact, equation (3.9) shows that we do not deal with damping, but rather with a perturbation of the wave speed \( c \).

**Remark 2** (Heterogeneous media). All statements in this section remain valid when besides \( \alpha, \mu, \eta, \tau, \delta, k, \) and \( c \) are space (and possibly time) dependent coefficients, as long as they are sufficiently smooth and \( c \) and \( \tau \) are bounded away from zero. In particular, an inspection of the proofs shows that it suffices to have an \( L^\infty \) sound speed satisfying

\[
0 < \underline{c} \leq c(x) \leq \bar{c}
\]

for reproducing the higher-order energy estimate (3.3). This allows for the practically relevant setting of piecewise-constant sound speed. Note that for obtaining the lower-order energy estimate (3.4), integration by parts with respect to space requires existence and a certain integrability of the gradient of \( c^2 \), though.

4. Uniform bounds for the JMGT–Westervelt equation and the inviscid limit

We next wish to extend the study of the limiting behavior to the solutions of the nonlinear JMGT–Westervelt equation given by (2.3). To derive \( \delta \)-independent bounds on solutions to (2.3), we employ a fixed-point argument on the mapping

\[
T^W : q \mapsto p,
\]

where \( p \) solves the linearized equation

\[
\tau p_{tt} + \alpha p_t - (\delta + \tau c^2) \Delta p_t - c^2 \Delta p - k q p_t = 0
\]

with homogeneous Dirichlet data and initial conditions

\[
(p,p_t,p_{tt})|_{t=0} = (p_0,p_1,p_2).
\]
Above, we have set $\alpha = 1 - kq$ and taken
\[ q \in B_R^W := \{ q \in X^W : \| q \|_{X^W} \leq R \text{ and } q(0) = p_0, q_t(0) = p_1, q_{tt}(0) = p_2 \}, \]
with some appropriately chosen radius $R > 0$; recall that $X^W$ is defined in (3.2). More precisely, crucial for our well-posedness proof will be existence of $R > 0$, such that the bounds
\[ (4.2) \quad \sqrt{C(\tau) \exp(K(\tau)(R^2 + 1)T)} \quad r \leq R \]
and
\[ (4.3) \quad \theta = \left( C_{H^2, L^\infty} + C_{H^2, L^4}^2 \right) C(\tau) \exp(K(\tau)(R^2 + 1)T) \| k|\sqrt{T} < 1 \]
hold. By showing that $T^W$ is a self-mapping and contraction on $B_R^W$, we obtain the following result.

**Theorem 4.1.** Let assumption $(A_1)$ hold and let $\tau, c > 0$, and $k \in \mathbb{R}$. Furthermore, let $r$ and $T$ be such that assumptions (4.2) and (4.3) hold for some $R > 0$. Then for any initial data $(p_0, p_1, p_2) \in X_0^W$ satisfying
\[ \| \nabla p_0 \|^2_{L^2} + \| \Delta p_0 \|^2_{L^2} + \| \Delta p_0 \|^2_{L^2} \leq r^2 \]
and any $\delta \in [0, \tilde{\delta}]$, there exists a unique solution $p \in X^W$ of problem
\[
\begin{cases}
\tau p_{tt} + p_{tt} - (\delta + \tau c^2)\Delta p_t - c^2 \Delta p = \frac{k}{2}(p^2)_{tt} & \text{in } \Omega \times (0, T), \\
 p = 0 & \text{on } \partial \Omega \times (0, T), \\
p(0, p_0, p_t) = (p_0, p_1, p_2) & \text{in } \Omega \times \{0\},
\end{cases}
\]
where $X^W$ is defined in (3.2). The solution $p$ satisfies the estimate
\[
\| \nabla p(t) \|^2_{L^2} + \| \Delta p(t) \|^2_{L^2} + \| \Delta p(t) \|^2_{L^2} \leq C(\tau) \exp(K(\tau)(R^2 + 1)T) \left( \| \nabla p(0) \|^2_{L^2} + \| \Delta p(0) \|^2_{L^2} + \| \Delta p(0) \|^2_{L^2} \right),
\]
where the constants $C(\tau)$ and $K(\tau)$ tend to infinity as $\tau \to 0^+$, but are independent of $\delta$.

**Proof.** The proof follows by relying on the Banach Fixed-point theorem in combination with the linear result of Proposition 3.1. Indeed, by employing estimate (4.3) with $\alpha = 1 - kq, \mu = kq_t, \eta = 0, f = 0$, and using
\[ W^{1, \infty}(0, T; H^2_0(\Omega)) \subseteq X^W, \]
one immediately sees that $T^W$ is a well-defined self-mapping on $B_R^W$, provided (4.2) holds.

We next prove that $T^W$ is strictly contractive with respect to the norm on the weaker space $E$, defined in (3.8). To this end, we take $q^{(1)}$ and $q^{(2)}$ in $B_R^W$ and use the short-hand notation $p^{(1)} = T^W q^{(1)}$ and $p^{(2)} = T^W q^{(2)}$. We also introduce the differences
\[ \overline{p} = p^{(1)} - p^{(2)}, \quad \overline{q} = q^{(1)} - q^{(2)}. \]
Then we know that $\overline{p}$ solves the linear equation
\[ \tau \overline{p}_{tt} + (1 - kq^{(1)})\overline{p}_{tt} - c^2 \Delta \overline{p} - b \Delta \overline{p}_t - kq_t^{(1)} \overline{p}_t = kq_t^{(1)} p_1^{(2)} + kq_t^{(2)} \overline{p}_t \]
and has zero initial conditions. Estimate (3.4) with $\alpha = 1 - kq^{(1)}, \mu = kq_t^{(1)}, \eta = 0$, and $f = kq_t^{(2)} + kq_t^{(2)}$ yields the bound
\[ \| \overline{p} \|_E \leq \sqrt{C(T, R, \tau)} \| f \|_{L^2(\Omega)} \]
\[
\leq \sqrt{C(\tau) \exp(K(\tau)(R^2 + 1)T)} |k| \left( C_{H^2,L^\infty} \|\Delta p_t^{(2)}\|_{L^\infty(L^2)} \|\overline{q}_t\|_{L^2(L^2)} + C_{H^1,L^4}^2 \|\nabla p_t^{(2)}\|_{L^\infty(L^2)} \|\nabla \overline{q}\|_{L^2(L^2)} \right)
\leq \theta \|\overline{q}\|_E.
\]

Above we have estimated
\[
\|\Delta p_t^{(2)}\|_{L^\infty(L^2)} + \|\nabla p_t^{(2)}\|_{L^\infty(L^2)} \leq \sqrt{C(\tau) \exp(K(\tau)(R^2 + 1)T)} r
\]
and, since \(\overline{q}(0) = 0\) and \(\overline{q}_t(0) = 0\), we have relied on the bound
\[
\|\overline{q}_t\|_{L^2(L^2)} + \|\nabla \overline{q}\|_{L^2(L^2)} \leq \sqrt{T} \|\overline{q}\|_E.
\]

Thus we obtain strict contractivity provided condition (4.3) holds.

Note that the space \(B^W_R\) with the metric induced by the norm \(\| \cdot \|_E\) is a closed subset of a complete normed space. This follows from \(B^W_R\) being a ball and thus weakly closed in \(X^W\). Therefore, any Cauchy sequence with respect to the norm \(\| \cdot \|_E\) converges to some \(y \in E\) and has a weakly-star in \(X^W\) convergent subsequence with limit \(x \in B^W_R\). Since the limits are unique, it must be \(y = x \in B^W_R\).

The claim then follows by Banach’s Fixed-point theorem, which at first yields a unique solution in \(B^W_R\). Uniqueness in \(X^W\) follows by arguing by contradiction and employing the stability estimate on the resulting equation for the difference of two solutions; see, for example, [53, Theorem 2.5.1] for similar arguments.

**Remark 3** (Small data or short time for the JMGT–Westervelt equation). To satisfy conditions (4.2) and (4.3), we can either make the radius \(r\) of the initial data small or the final time \(T\) short. More precisely, abbreviating
\[
C(\Omega, \tau, k) = \left( C_{H^2,L^\infty} + C_{H^1,L^4}^2 \right) C(\tau)|k|,
\]
on the one hand for given \(T\), we can read these two conditions as
\[
r \leq \frac{R}{\sqrt{C(\tau) \exp(K(\tau)(R^2 + 1)T)}}
\]
(4.4)
\[
r < \frac{1}{C(\Omega, \tau, k) \exp(K(\tau)(R^2 + 1)T) \sqrt{T}}
\]
for some \(R > 0\). They can always be satisfied by, e.g., setting \(R = 1\) and choosing \(r > 0\) small enough. Note that this smallness condition can be weakened by maximizing the right-hand sides in (4.4) with respect to \(R\).

On the other hand, for given \(r > 0\), conditions (4.2) and (4.3) can be satisfied by choosing \(T\) short enough so that
\[
T < (C(\Omega, \tau, k) r)^{-2},
\]
(4.5)
\[
T \leq \min \left\{ 1, \min \{ \ln(R^2/(r^2 C(\tau)), \ln(1/(r C(\Omega, \tau, k))) \} / (K(\tau(R^2 + 1)) \right\}
\]
for some \(R > r \sqrt{C(\tau)}\). Again, \(R\) can simply be fixed to, for example,
\[
R = r \sqrt{C(\tau)} + 1.
\]

A more sophisticated approach would be to optimize it to make the upper bounds in (4.5) as large as possible. The short-time setting might be more preferable having in mind applications of nonlinear ultrasonic waves, where the data are often smooth, but not necessarily small; see, for example, [14, §14.6] for the numerical modeling of high-intensity ultrasonic waves used in lithotripsy.
We are now ready to prove a convergence result for the nonlinear equation (2.3) as the sound diffusivity vanishes.

**Theorem 4.2.** Under the conditions of Theorem 4.1, the family of solutions \( \{p^{(\delta)}\}_{\delta > 0} \) to the JMGT–Westervelt equation converges in the topology induced by the energy norm (3.7) to a solution \( p \) of the inviscid JMGT–Westervelt equation as \( \delta \to 0^+ \) at a linear rate. In other words,

\[
\|p^{(\delta)} - p\|_E \lesssim \delta \quad \text{as} \quad \delta \to 0^+.
\]

**Proof.** Let \( \delta, \delta' \in [0, \overline{\delta}] \). Again we use the short-hand notations \( p^{(\delta)} \) and \( p^{(\delta')} \) for the solutions of (3.1) with \( \delta \) and \( \delta' \), as well as \( \overline{\mathbf{p}} = p^{(\delta)} - p^{(\delta')} \), and prove that \( \{\overline{\mathbf{p}}^{(\delta)}\} \) is a Cauchy sequence in the topology induced by (3.7).

Note that \( \overline{\mathbf{p}} \) solves the equation

\[
\tau \overline{\mathbf{p}}_{tt} + (1 - kp^{(\delta)})(\overline{\mathbf{p}}_t - (\delta + \tau c^2)\Delta \overline{\mathbf{p}}) - c^2 \Delta \overline{\mathbf{p}} = k\overline{\mathbf{p}}_t(p^{(\delta)}_t + p^{(\delta')}_t) + k\tau \overline{\mathbf{p}}_{tt}^{(\delta')} - (\delta - \delta')\Delta p^{(\delta')}_t,
\]

supplemented by zero initial conditions. Applying estimate (3.4) in Proposition 3.1 with \( \alpha = 1 - kp^{(\delta)} \), \( \mu = k(p^{(\delta)}_t + p^{(\delta')}_t) \), \( \eta = k\overline{\mathbf{p}}_t^{(\delta')} \), and the right-hand side \( f = -(\delta - \delta')\Delta p^{(\delta')}_t \) directly yields

\[
E[p](t) \leq \frac{1}{2} \mathcal{C}(\tau) \exp(K(\tau)(R^2 + 1)T) |\delta - \delta'|^2 R^2
\]

with energy \( E[p] \) defined as in (3.7). The stated rate is obtained by setting \( \delta' \) to zero. \( \square \)

5. **The Linearized JMGT–Kuznetsov Equation**

We continue by investigating a linearization of the JMGT–Kuznetsov equation given by

\[
(5.1a) \quad \tau \psi_{tt} + \alpha \psi_t - (\delta + \tau c^2) \Delta \psi_t - c^2 \Delta \psi = \sigma \nabla \phi \cdot \nabla \psi_t
\]

with homogeneous Dirichlet boundary conditions and initial conditions

\[
(5.1b) \quad (\psi, \psi_t, \psi_{tt})|_{t=0} = (\psi_0, \psi_1, \psi_2),
\]

where now the coefficient in front of the second time derivative is given by

\[
\alpha = 1 - \kappa \phi_t.
\]

Since the JMGT–Kuznetsov equation has a quadratic gradient nonlinearity, we will need to obtain uniform bounds for \( \|\nabla \psi\|_{L^\infty(L^\infty)} \) and \( \|\nabla \psi_t\|_{L^\infty(L^\infty)} \) in the course of the analysis. Our goal in this section is thus to derive a higher-order energy bound for the linearization (5.1) that is uniform with respect to \( \delta \) and will later allow us to derive the corresponding bound for the nonlinear equation. To this end, we strengthen our previous assumptions on the regularity of the coefficients and initial data.

\( (A_2) \) The coefficient \( \phi \) is sufficiently smooth so that

\[
\phi \in L^\infty(0, T; H^2_0(\Omega)) \cap W^{1,\infty}(0, T; H^2_0(\Omega)),
\]

and uniformly bounded

\[
\|\phi\|_{W^{1,\infty}(H^2_0(\Omega))}, |\phi|_{L^\infty(H^2_0(\Omega))} \leq R,
\]
for some positive constant $R$, independent of $\delta$. This further implies

$$\alpha \in W^{1,\infty}(0, T; H^2_0(\Omega)) \subseteq L^\infty(0, T; L^\infty(\Omega)) \text{ with } \underline{\alpha} \leq \alpha \leq \overline{\alpha}$$

for $\underline{\alpha} = 1 - |\kappa|C_{H^2,L^\infty}R$ and $\overline{\alpha} = 1 + |\kappa|C_{H^2,L^\infty}R$.

(A3) The initial conditions (5.1b) satisfy

$$(\psi_0, \psi_1, \psi_2) \in X_0^K = H^3_0(\Omega) \times H^3_0(\Omega) \times H^3_0(\Omega).$$

Note that also here we do not impose a non-degeneracy assumption on $\alpha$.

**Proposition 5.1.** Let $\delta \in [0, 3]$ and $\tau, c > 0$. Furthermore, let assumptions (A1) and (A2)–(A3) hold. Then problem (5.1) has a unique solution

$$\psi \in X^K = W^{3,\infty}(0, T; H^3_0(\Omega)) \cap W^{2,\infty}(0, T; H^3_0(\Omega)) \cap W^{1,\infty}(0, T; H^3_0(\Omega)),$$

which satisfies

$$\sup_{t \in (0, T)} \| \nabla \psi_{tt}(t) \|_{L^2}^2 + \sup_{t \in (0, T)} \| \Delta \psi_{tt}(t) \|_{L^2}^2 + \sup_{t \in (0, T)} \| \nabla \Delta \psi_{tt}(t) \|_{L^2}^2$$

$$\leq C(T, \tau, R)(\| \Delta \psi_0 \|_{L^2}^2 + \| \nabla \Delta \psi_1 \|_{L^2}^2 + \| \nabla \Delta \psi_2 \|_{L^2}^2),$$

where constant $C(T, \tau, R)$ tends to infinity as $T \to \infty$ or $\tau \to 0^+$, but is independent of $\delta$.

**Proof.** The proof can be carried out as before by employing smooth Faedo–Galerkin approximations in space, where we project the problem onto the span $V_n$ of the first $n$ eigenfunctions of the Dirichlet Laplacian pointwise in time. We will again focus our attention on deriving the crucial energy bound for the Galerkin approximations $\psi^n$. For ease of notation, we drop the superscript $n$ below.

We test (5.1a) with $\Delta^2 \psi_{tt}$ and integrate in space. We can rely on the fact that $\psi = \Delta \psi = 0$ on $\partial \Omega$ for our Galerkin approximations. Therefore, the following identities hold:

$$(\alpha \psi_{tt}, \Delta^2 \psi_{tt})_{L^2} = (\Delta[\alpha \psi_{tt}], \Delta \psi_{tt})_{L^2} = (\alpha \Delta \psi_{tt} + \psi_{tt} \Delta \alpha + \nabla \alpha \cdot \nabla \psi_{tt}, \Delta \psi_{tt})_{L^2},$$

and

$$-c^2(\Delta \psi, \Delta^2 \psi_{tt})_{L^2} = c^2 \frac{d}{dt}(\nabla \Delta \psi, \nabla \Delta \psi_{tt})_{L^2} - c^2\| \nabla \Delta \psi_1 \|_{L^2}^2.$$

We thus arrive at the following energy identity:

$$\frac{1}{2} \frac{d}{dt} \| \Delta \psi_{tt} \|_{L^2}^2 + \frac{1}{2}(\delta + \tau c^2) \frac{d}{dt} \| \nabla \Delta \psi \|_{L^2}^2$$

$$= - (\alpha \Delta \psi_{tt}, \psi_{tt} \Delta \alpha + \nabla \alpha \cdot \nabla \psi_{tt}, \Delta \psi_{tt})_{L^2} - c^2 \frac{d}{dt}(\nabla \Delta \psi, \nabla \Delta \psi_{tt})_{L^2}$$

$$+ c^2\| \nabla \Delta \psi_1 \|_{L^2}^2 + \sigma(\nabla \phi \cdot \nabla \psi_{tt}, \Delta^2 \psi_{tt})_{L^2}.$$

We next integrate in time and estimate the terms arising on the right-hand side. First we note that

$$\int_0^t (\alpha \Delta \psi_{tt} + \psi_{tt} \Delta \alpha + \nabla \alpha \cdot \nabla \psi_{tt}, \Delta \psi_{tt})_{L^2} \, dt$$

$$\leq \{ \overline{\alpha} \| \nabla \phi \|_{L^\infty(L^2)} + R \| \psi_{tt} \|_{L^\infty(L^2)} \} \| \Delta \psi_{tt} \|_{L^2(L^2)}$$

$$\leq \{ \overline{\alpha} \| \nabla \phi \|_{L^\infty(L^2)} + R \| \psi_{tt} \|_{L^\infty(L^2)} + R \| \nabla \psi_{tt} \|_{L^2(L^1)} \} \| \Delta \psi_{tt} \|_{L^2(L^2)}.$$
where we have utilized uniform boundedness of $\alpha$ on account of assumption (A2). Further,
\[
c^2(\nabla \Delta \psi(t), \nabla \Delta \psi(t))_{L^2} \leq \sqrt{T} c^2 \|\nabla \Delta \psi(t)\|_{L^2(L^2)} \|\nabla \Delta \psi(t)\|_{L^2}
\leq \frac{1}{2e} T c^2 \|\nabla \Delta \psi(t)\|_{L^2(L^2)}^2 + \frac{e}{2} \|\nabla \Delta \psi(t)\|_{L^2}^2.
\]
Since $\nabla \phi \cdot \nabla \psi = 0$ on $\partial \Omega$ by the semi-discrete PDE, we have
\[
(\nabla \phi \cdot \nabla \psi(t), \Delta^2 \psi(t))_{L^2} = (\Delta [\nabla \phi \cdot \nabla \psi(t)], \Delta \psi(t))_{L^2}
= (\nabla \Delta \phi \cdot \nabla \psi(t) + 2D^2 \phi : D^2 \psi(t) + \nabla \phi \cdot \nabla \Delta \psi(t), \Delta \psi(t))_{L^2}
\]
where $D^2 v = (\partial_{x_i} \partial_{x_j} v)_{i,j}$ denotes the Hessian. By elliptic regularity, the Hessian satisfies
\[
\|D^2 v\|_{L^2} \leq C_H \|\Delta v\|_{L^2}
\]
and so we can rely on the following bound:
\[
\|D^2 v\|_{L^4} \leq C_{H^1,L^4}(\|D^2 v\|_{L^2} + \|D^2 v\|_{L^2}) \leq C_{H^1,L^4} C_H (\|\nabla \Delta v\|_{L^2} + \|\Delta v\|_{L^2}).
\]
Thus, we have
\[
\left| \int_0^t (\nabla \phi \cdot \nabla \psi(t), \Delta^2 \psi(t))_{L^2} \, ds \right|
\leq \Delta \psi(t) \|D^2 v\|_{L^2(L^2)} \left\{ \|\nabla \Delta \phi\|_{L^2(L^2)} \|\nabla \psi(t)\|_{L^2(L^2)} + \|\nabla \phi\|_{L^2(L^2)} \|\nabla \Delta \psi(t)\|_{L^2(L^2)} + \right.
\left. 2C^2_{H^1,L^4} C^2_H (\|\nabla \Delta \phi\|_{L^2(L^2)} + \|\nabla \phi\|_{L^2(L^2)}) (\|\nabla \Delta \psi(t)\|_{L^2(L^2)} + \|\Delta \psi(t)\|_{L^2(L^2)}) \right\}.
\]
Relying on the assumption (A2) on the uniform boundedness of $\phi$ yields
\[
\left| \int_0^t (\nabla \phi \cdot \nabla \psi(t), \Delta^2 \psi(t))_{L^2} \, ds \right|
\leq \Delta \psi(t) \|D^2 v\|_{L^2(L^2)} R \left\{ \|\nabla \psi(t)\|_{L^2(L^2)} + \|\nabla \Delta \psi(t)\|_{L^2(L^2)} + \right.
\left. 4C^2_{H^1,L^4} C^2_H (\|\nabla \Delta \psi(t)\|_{L^2(L^2)} + \|\Delta \psi(t)\|_{L^2(L^2)}) \right\}.
\]
Let $\bar{T} \leq T$. Employing the derived bounds within (5.1) after integration in time leads to
\[
\frac{1}{2} \tau \sup_{t \in (0, \bar{T})} \|\Delta \psi(t)\|_{L^2(L^2)}^2 + \frac{1}{2} (\tau c^2 - \varepsilon) \sup_{t \in (0, \bar{T})} \|\nabla \Delta \psi(t)\|_{L^2(L^2)}^2
\leq \|\Delta \psi(t)\|_{L^2(L^2)}^2 + \|\nabla \Delta \psi(t)\|_{L^2(L^2)}^2 + \|\Delta \psi(t)\|_{L^2(L^2)}^2
\leq \left( \frac{\tau^2}{2} + R^2 \right) \|\nabla \psi(t)\|_{L^2(L^2)}^2 + \left( \frac{\tau^2}{2} + R^2 \right) \|\nabla \Delta \psi(t)\|_{L^2(L^2)}^2 + \left( \frac{\tau^2}{2} + R^2 \right) \|\Delta \psi(t)\|_{L^2(L^2)}^2.
\]
Note that we can obtain a bound on $\|\nabla \psi(t)\|_{L^2}$ by relying on elliptic regularity
\[
\|\nabla \psi(t)\|_{L^2} \leq \psi(t)_{H^2} \leq C \|\Delta \psi(t)\|_{L^2}.
\]
An application of Gronwall’s inequality yields
\[
\sup_{t \in (0, \bar{T})} \|\Delta \psi(t)\|_{L^2(L^2)}^2 + \sup_{t \in (0, \bar{T})} \|\nabla \Delta \psi(t)\|_{L^2(L^2)}^2
\leq C(T, \tau, R) (\|\Delta \psi(t)\|_{L^2(L^2)}^2 + \|\nabla \Delta \psi(t)\|_{L^2(L^2)}^2 + \|\Delta \Delta \psi(t)\|_{L^2(L^2)}).
\]
We can obtain an additional bound on the third-time derivative by testing with $-\Delta \psi_{tt}$. At first this leads to
\[
\tau \| \nabla \psi_{tt}(t) \|_{L^2} \leq \| \nabla \left\{ \alpha(t) \psi_{tt}(t) - (\delta + \tau c^2) \Delta \psi(t) - c^2 \Delta \psi(t) - \sigma \nabla \phi(t) \cdot \nabla \psi(t) \right\} \|_{L^2},
\]
from which a uniform bound on $\| \nabla \psi_{tt} \|_{L^\infty(L^2)}$ follows by (5.5), the assumptions on $\phi$, and $\delta \in [0, \overline{\delta}]$. Standard compactness arguments allow to carry over the result to the solution of (5.1). Note that from $\psi \in X^K$, it follows that
\[
\psi \in C([0, T]; H^3_0(\Omega)), \quad \psi_t \in C_w([0, T]; H^3_0(\Omega)), \quad \psi_{tt} \in C_w([0, T]; H^3_0(\Omega)),
\]
and so $\psi_1$ and $\psi_2$ are attained weakly; cf. [39, Lemma 3.3].

For $f \in L^2(0, T; L^2(\Omega))$, we also briefly consider the linearization with a source term
\[
(5.6) \quad \tau \psi_{tt} + \alpha \psi_{tt} - (\delta + \tau c^2) \Delta \psi_t - c^2 \Delta \psi = \sigma \nabla \phi \cdot \nabla \psi_t + f,
\]
under the same assumptions on $\phi$. Formally testing with $\psi_{tt}$ and integrating over space and time leads to
\[
\frac{\tau}{2} \| \psi_{tt}(s) \|_{L^2}^2 \bigg|_0^t + \frac{\delta + \tau c^2}{2} \| \nabla \psi_{tt}(s) \|_{L^2}^2 \bigg|_0^t = -\int_0^t \int_\Omega \alpha \psi_{tt}^2 \, dx \, ds - c^2 \int_0^t \int_\Omega \nabla \psi \cdot \nabla \psi_{tt} \, dx \, ds
+ \sigma \int_0^t \int_\Omega (\nabla \phi \cdot \nabla \psi_t) \psi_{tt} \, dx \, ds + \int_0^t \int_\Omega f \psi_{tt} \, dx \, ds,
\]
where $\alpha = 1 - \kappa \phi_t$. The first term on the right-hand side of (6.1) can be treated as before by first integrating by parts with respect to time
\[
-c^2 \int_0^t \int_\Omega \nabla \psi \cdot \nabla \psi_{tt} \, dx \, ds = -c^2 (\nabla \psi \cdot \nabla \psi_t)_{L^2} \bigg|_0^t + c^2 \| \nabla \psi_t \|_{L^2(L^2)}^2,
\]
and then employing Hölder’s and Young’s inequalities together with
\[
\| \nabla \psi \|_{L^\infty(L^2)} \leq \sqrt{T} \| \nabla \psi_t \|_{L^2(L^2)} + \| \nabla \psi_0 \|_{L^2}.
\]
Additionally, we have
\[
\left| \int_0^t \int_\Omega (\nabla \phi \cdot \nabla \psi_t) \psi_{tt} \, dx \, ds \right| \leq \| \nabla \phi \|_{L^\infty(L^\infty)} \| \nabla \psi_t \|_{L^2(L^2)} \| \psi_{tt} \|_{L^2(L^2)},
\]
and
\[
\left| \int_0^t \int_\Omega f \psi_{tt} \, dx \, ds \right| \leq \| f \|_{L^2(L^2)} \| \psi_{tt} \|_{L^2(L^2)}.
\]
Employing these bounds and using Gronwall’s inequality gives
\[
(5.7) \quad \| \psi_{tt}(t) \|_{L^2}^2 + \| \nabla \psi_{tt}(t) \|_{L^2}^2 \leq C(T, \tau, R)(\| \psi_2 \|_{L^2}^2 + \| \nabla \psi_1 \|_{L^2}^2 + \| \nabla \psi_0 \|_{L^2}^2) \| f \|_{L^2(L^2)}^2,
\]
which we will rely on in the upcoming proof.
6. Uniform bounds for the JMGT–Kuznetsova equation and the inviscid limit

The goal of this section is to investigate the behavior of solutions to equation (2.1) as $\delta \to 0^+$. As before, our work plan is to derive uniform bounds for a linearization and then relate them to the nonlinear model via a fixed-point argument. We thus introduce the mapping

$$\mathcal{T}^K : \phi \mapsto \psi,$$

where we take $\phi$ from a suitably chosen ball in the space $X^K$ and $\psi$ as the solution of the linearized equation (5.1) with initial data

$$(\psi(0), \psi_t(0), \psi_{tt}) = (\phi(0), \phi_t(0), \phi_{tt}(0)) = (\psi_0, \psi_1, \psi_2)$$

and $\alpha = 1 - \kappa \phi_t$. We next prove a small-data well-posedness result for (2.1).

**Theorem 6.1.** Let assumption (A1) hold and let $\tau, c > 0$ and $k \in \mathbb{R}$. Furthermore, let $T > 0$ be given. Then there exists $r > 0$ such that for any initial data $(\psi_0, \psi_1, \psi_2) \in X^K_0$ satisfying

$$\|\psi_2\|^2_{H^2} + \|\psi_1\|^2_{H^3} + \|\psi_0\|^2_{H^3} \leq r^2,$$

and any $\delta \in [0, \overline{\delta}]$, there exists a unique solution $\psi \in X^K$ of problem

$$\begin{cases}
\tau \psi_{tt} + \psi_t - (\delta + \tau c^2)\Delta \psi_t - c^2 \Delta \psi = \frac{1}{2}(\kappa \psi_t^2 + \sigma |\nabla \psi|^2)_t & \text{in } \Omega \times (0, T), \\
\psi = 0 & \text{on } \partial \Omega \times (0, T), \\
(\psi, \psi_t, \psi_{tt}) = (\psi_0, \psi_1, \psi_2) & \text{in } \Omega \times \{0\},
\end{cases}$$

where $X^K$ is defined in (5.2). Furthermore, the solution $\psi$ fulfills the estimate

$$\sup_{t \in (0, T)} \|\nabla \psi_{tt}(t)\|^2_{L^2} + \sup_{t \in (0, T)} \|\Delta \psi_{tt}(t)\|^2_{L^2} + \sup_{t \in (0, T)} \|\nabla \Delta \psi_{tt}(t)\|^2_{L^2} \leq C(T, \tau)(\|\Delta \psi_2\|^2_{L^2} + \|\nabla \Delta \psi_1\|^2_{L^2} + \|\nabla \Delta \psi_0\|^2_{L^2}),$$

where the constant $C(T, \tau)$ tends to infinity as $T \to \infty$ or $\tau \to 0^+$, but is independent of $\delta$.

**Proof.** Let $R > 0$. Take $\phi \in B^K_R$, where

$$B^K_R = \{ \phi \in X^K : \|\phi\|_{X^K} \leq R, (\phi(0), \phi_t(0), \phi_{tt}(0)) = (\psi_0, \psi_1, \psi_2) \}.$$

Then

$$\|\phi\|_{W^{1, \infty}(H^2_0(\Omega))}, \|\phi\|_{L^{\infty}(H^3_0(\Omega))} \leq R.$$

Thus, on account of Proposition 5.1, the mapping $\mathcal{T}^K$ is well-defined. Furthermore, by (5.3), it is a self-mapping provided $r$ is chosen so that

$$0 < r \leq R/\sqrt{C(T, \tau, R)}.$$

To show strict contractivity in the energy norm, we take $\phi^{(1)}, \phi^{(2)} \in B^K_R$ and set $\psi^{(1)} = \mathcal{T}^K\phi^{(1)}$ and $\psi^{(2)} = \mathcal{T}^K\phi^{(2)}$. Then $\overline{\psi} = \psi^{(1)} - \psi^{(2)}$ solves

$$\begin{cases}
\tau \overline{\psi}_{tt} + (1 - \kappa \phi^{(1)}_t)\overline{\psi}_{tt} - (\delta + \tau c^2)\Delta \overline{\psi}_t - c^2 \Delta \overline{\psi} = \overline{\phi}_t \overline{\psi}_{tt} + \sigma |\nabla \overline{\phi}| \cdot |\nabla \overline{\psi}_{tt}| + \sigma |\nabla \phi^{(1)}| \cdot |\nabla \overline{\psi}| + \sigma |\nabla \phi^{(2)}| \cdot |\nabla \overline{\psi}|
\end{cases}$$

and satisfies zero initial conditions. This equation corresponds to (5.6) if we set

$$f = \overline{\phi}_t \overline{\psi}_{tt} + \sigma |\nabla \overline{\phi}| \cdot |\nabla \overline{\psi}_{tt}| + \sigma |\nabla \phi^{(1)}| \cdot |\nabla \overline{\psi}| + \sigma |\nabla \phi^{(2)}| \cdot |\nabla \overline{\psi}|.$$
together with \( \alpha = 1 - \kappa \phi_1(1) \). Using estimate (6.7) thus yields
\[
\| \overline{\nu}_t(t) \|_{L^2}^2 + \| \nabla \overline{\nu}_t(t) \|_{L^2}^2 + \| \nabla \nabla \psi(t) \|_{L^2}^2 \leq C(T, \tau, R) \| f \|_{L^2(L^2)}^2.
\]
It remains to estimate the \( f \) term. By Hölder’s inequality, we have
\[
\| f \|_{L^2(L^2)} \leq |\kappa| \| \psi_t(2) \|_{L^\infty(L^4)} \| \phi_t \|_{L^2(L^4)} + |\sigma| \| \nabla \psi_t(1) \|_{L^\infty(L^\infty)} \| \nabla \phi \|_{L^2(L^2)}
\] 
\[
+ |\sigma| \| \nabla \psi_t(2) \|_{L^2(L^2)} \| \nabla \phi(2) \|_{L^\infty(L^\infty)}.
\]
Furthermore,
\[
|\kappa| \| \psi_t(2) \|_{L^\infty(L^4)} \| \phi_t \|_{L^2(L^4)} \leq |\kappa| C_{H^1, L^4}^2 \| \nabla \psi_t(2) \|_{L^\infty(L^2)} T \| \nabla \phi_t \|_{L^\infty(L^\infty)}.
\]
By noting that \( \| \nabla \phi \|_{L^2(L^2)} \leq T \| \nabla \phi_t \|_{L^2(L^2)} \) and that we have the uniform bound
\[
\| \nabla \phi(2) \|_{L^\infty(L^\infty)} \lesssim R,
\]
it further follows that
\[
\| f \|_{L^2(L^2)}^2 \lesssim 2 \kappa^2 C_{H^1, L^4}^2 \| \nabla \psi_t(2) \|_{L^\infty(L^2)}^2 T^2 \| \nabla \phi_t \|_{L^\infty(L^2)}^2 
\] 
\[
+ 2 \sigma^2 \| \nabla \psi_t(1) \|_{L^\infty(L^\infty)} T^2 \| \nabla \phi_t \|_{L^2(L^2)}^2 + 2 \sigma^2 R^2 \| \nabla \phi_t \|_{L^2(L^2)}^2.
\]
Employing this bound in (6.1) and relying on Gronwall’s inequality leads to
\[
\sup_{t \in (0, T)} \| \psi_t(t) \|_{L^2} + \| \nabla \psi_t(t) \|_{L^2} \leq C(T, R)(\| \nabla \psi_t(2) \|_{L^\infty(L^2)} + \| \nabla \psi_t(1) \|_{L^\infty(L^\infty)}) T \sup_{t \in (0, T)} \| \nabla \phi_t(t) \|_{L^2}.
\]
Note that \( \sup_{t \in (0, T)} \| \nabla \phi(t) \|_{L^2} \leq \| \phi \|_E \). By (6.3), we know that
\[
\| \nabla \psi_t(2) \|_{L^\infty(L^2)} + \| \nabla \psi_t(1) \|_{L^\infty(L^\infty)} \leq \sqrt{\tilde{C}(T, \tau, R)} r.
\]
for some \( \tilde{C}(T, \tau, R) > 0 \) independent of \( \delta \). Thus we can achieve strict contractivity of \( T^K \) in the energy norm by reducing \( r \). We can reason as in the proof of Theorem 4.1 concerning \( B_R^K \) being closed with respect to the metric induced by \( \| \cdot \|_E \) and arrive at our claim by employing the Banach Fixed-point theorem.

**Remark 4 (Small data or short time for the JMGT–Kuznetsov equation).** Like in the JMGT–Westervelt case (cf. Remark 3), instead of choosing the maximal magnitude of the initial data \( r \) small enough for fixed final time \( T \), we could have also achieved the self-mapping and contractivity properties needed for proving Theorem 6.1 by choosing \( T \) small enough, given initial data that is smooth but of arbitrary size. As mentioned in Remark 3, the latter scenario might be preferable in high-intensity ultrasound applications.

We conclude our theoretical investigations by proving a convergence result for the JMGT equation in potential form.

**Theorem 6.2.** Let the assumptions of Theorem 6.1 hold. Then the family of solutions \( \{ \psi(\delta) \}_{\delta > 0} \) to the JMGT–Kuznetsov equation (2.1) converges in the topology induced by the energy norm for the wave equation at a linear rate to the solution \( \psi \) of the inviscid JMGT–Kuznetsov equation as \( \delta \to 0^+ \). In other words,
\[
\| \psi(\delta) - \psi \|_E \lesssim \delta \quad \text{as} \quad \delta \to 0^+.
\]
Theorem 6.1. In particular, we have

\[ \psi = \psi(\delta) - \psi(\delta') \]

where now \( \alpha \) refer to [17, §] for details. The three parameters within the scheme are chosen as

\[ \alpha = 1 - \kappa \psi_t^{(\delta)} \]

We can proceed similarly to the proof of contractivity in Theorem 6.1. In particular, we have

\[ \left| \int_0^t \int_\Omega (\psi_t^{(\delta')})^r \psi_t \psi_t \, dx \, ds \right| \leq |\alpha| \|\psi_t^{(\delta')}\|_{L^\infty(\Omega)} \|\psi_t\|_{L^2(\Omega)} \|\psi_t\|_{L^2(\Omega)}. \]

as well as

\[ \left| \sigma \int_0^T \int_\Omega (\nabla \psi_t \cdot \nabla \psi_t^{(\delta')} + \nabla \psi_t \cdot \nabla \psi_t^{(\delta')} \psi_t \psi_t \, dx \, ds \right| \]

\[ \leq |\sigma| \|\nabla \psi_t^{(\delta')}\|_{L^\infty(\Omega)} \|\nabla \psi_t\|_{L^2(\Omega)} \|\psi_t\|_{L^2(\Omega)}^2 \]

\[ + |\sigma| \|\nabla \psi_t\|_{L^2(\Omega)} \|\psi_t^{(\delta')}\|_{L^\infty(\Omega)} \|\psi_t\|_{L^2(\Omega)}^2. \]

The \( \delta - \delta' \) term can be estimated as follows:

\[ \left| \int_0^T \int_\Omega (\Delta \psi_t^{(\delta')} \psi_t) \, dx \, ds \right| \leq |\delta - \delta'| \|\Delta \psi_t^{(\delta')}\|_{L^2(\Omega)} \|\psi_t\|_{L^2(\Omega)}. \]

Employing these bounds in (6.2) leads to

\[ \sup_{t \in (0, T)} E(\psi)(t) \lesssim |\delta - \delta'|^2 \|\Delta \psi_t^{(\delta')}\|_{L^2(\Omega)}^2. \]

Setting \( \delta' \) to zero yields the claimed convergence rate. \( \square \)

Note that from Theorems 6.1 and 6.2, we also obtain well-posedness and convergence results for the JMGW–Westervelt equation in potential form (2.24) by setting \( \kappa = \frac{1}{\tau} (1 + \frac{\beta}{a}) \) and \( \sigma = 0. \)

7. Numerical results

In this section, we illustrate some of our previous theoretical results numerically by employing a Matlab implementation. For discretization in the spatial variable, we use continuous piecewise linear finite elements on a uniform discretization of the computational domain with mesh size \( h \). In time, we rely on a Newmark discretization for third-order in time equations realized as a predictor-corrector scheme; we refer to [17, §8] for details. The three parameters within the scheme are chosen as
Having in mind our discussion in Remark 1 regarding the perturbation of the speed of sound, we heuristically choose the time step $\Delta t$ so that
\[
\left( c + \sqrt{\frac{\delta}{\tau}} \right) \Delta t \leq CFL \cdot h
\]
with $CFL = 0.1$. The nonlinearity is resolved via a fixed-point iteration, where we treat the whole nonlinear term as the previous iterate and set the tolerance to $TOL = 10^{-8}$.

We consider sound propagation through water, where the speed of sound is taken to be $c = 1500 \text{ m/s}$ and density $\rho = 1000 \text{ kg/m}^3$; cf. [19, §5]. The parameter of nonlinearity is set to $B/A = 5$. Values of the nonlinearity parameter in different media can be found, for example, in [3]. For sea water, at least two molecular relaxation processes are known to be pronounced, with molecular relaxation times $\tau_1 = 1 \times 10^{-3} \text{ s}$ and $\tau_2 = 1.5 \times 10^{-5} \text{ s}$; see [28, §1]. We choose to adopt the same values for the thermal relaxation time in our numerical experiments and additionally test with $\tau = \tau_0 = 1.5 \times 10^{-7} \text{ s}$ to observe the effects of the dissipation and nonlinearity when $\tau$ is relatively small.

### 7.1. Nonlinear propagation in a channel.

We first consider nonlinear propagation in a narrow channel, as modeled by the JMGT–Westervelt equation [23] with $p = p(x, t)$ and $x \in \Omega = (0, 0.4)$. We set initial data to
\[
(p, p_t, p_{tt})|_{t=0} = \left( A \exp\left( -\frac{(x - 0.2)^2}{2\sigma^2} \right), 0, 0 \right),
\]
with $A = 100 \text{ MPa}$ and $\sigma = 0.01$, which corresponds to a Gaussian-like initial pressure distribution centered around $x = 0.2$. The sound diffusivity $\delta$ is expected to be relatively small in water with values in the interval $[10^{-9}, 10^{-4}] \text{ m}^2/\text{s}$; cf. [19, §5] and [48, §5]. To demonstrate the influence of this parameter on the solutions of the equation, we test the problem in an exaggerated setting by choosing $\delta \in \{0, 1\} \text{ m}^2/\text{s}$.

To resolve the nonlinear behavior, we employ 600 elements in space and choose the time step according to (7.1).

### Figures 1–3

Figures 1–3 depict the acoustic pressure at final time $T = 70 \mu s$ for two different values of $\delta$. The difference between the linear and nonlinear pressure distribution is

\[
\begin{array}{c}
\text{(left) Pressure at final time for different values of $\delta$ and fixed, small $\tau$} \\
\text{(right) Linear and nonlinear pressure distribution at final time with $\delta = 0$}
\end{array}
\]
also displayed in inviscid media (where $\delta = 0$) on the right.

In Figure 1 the thermal relaxation parameter is taken to be relatively small, $\tau = 1.5 \times 10^{-7}$ s. Thus, we expect the behavior as observed in the corresponding second-order model. Indeed, we see that increasing $\delta$ leads to the damping of the amplitude and subduing the nonlinear steepening of the wavefront. For a rigorous study into the behavior of third-order acoustic models as $\tau \to 0^+$ with $\delta > 0$ fixed, we refer to, for example, [4, 17].

In Figure 2, the thermal relaxation time is set to $1.5 \times 10^{-5}$ s, which appears to be in an intermediate range in terms of the displayed effects. Here increasing $\delta$ influences the amplitude of the wave and its propagation speed. We also note that in this parameter regime, the nonlinear effects are subdued compared to the case of having a shorter relaxation time.

In Figure 3, the thermal relaxation time is set to $1.5 \times 10^{-3}$ s, which appears to be in an intermediate range in terms of the displayed effects. Here increasing $\delta$ influences the amplitude of the wave and its propagation speed. We also note that in this parameter regime, the nonlinear effects are subdued compared to the case of having a shorter relaxation time.

Figure 3 displays the pressure distribution when the thermal relaxation time is relatively large, $\tau = 1 \times 10^{-3}$ s. Here the effects of thermal relaxation appear to
overtake both the effects of dissipation and nonlinearity. In fact, in this setting it might be more sensible to observe \( z = \tau u_t + u \), whose values we also plot at final time in Figure 4. We see that the effects of increasing \( \delta \) are practically negligible.

Our parameter study in Figures 1–4 suggests that a deeper theoretical investigation into the interplay among \( \delta \), \( \tau \), and the nonlinear parameters is of interest.

For the convergence study, we take \( \delta \in [0, 10^{-2}] \) m²/s. In the experiments we conducted the same rate of convergence was obtained for the three values of the thermal relaxation parameter considered before; we thus present here only the case \( \tau = 1.5 \times 10^{-5} \) s. The plot of the relative error in the energy norm

\[
err = \frac{\|p_h(\delta) - p_h\|_E}{\|p_h\|_E}
\]

is given in Figure 5 on the left, where

\[
\|p_h\|_E = \sup_{t \in (0, T)} \|p_{h,tt}(t)\|_{L^2} + \sup_{t \in (0, T)} \|\nabla p_{h,t}(t)\|_{L^2}.
\]
We observe a linear rate of convergence, as expected on account of Theorem 4.2.

7.2. Linear propagation with an external source. We also illustrate our convergence results for the linear MGT equation (2.4) in a two-dimensional setting with $\alpha = 1$ and a source term. For $\Omega = (0, 0.5) \times (0, 0.5)$, we take the source term to be

$$f(x, y, t) = A \exp \left( -\frac{(x-x_0)^2}{2\sigma_x^2} - \frac{(y-y_0)^2}{2\sigma_y^2} \right) \sin(wt),$$

where $A$, $x_0 = y_0 = 0.25$, $\sigma_x = 0.02$, and $\sigma_y = 0.01$. Furthermore, the frequency is set to $w = 2\pi f$ with $f = 2 \cdot 10^4$. The initial conditions $(p_0, p_1, p_2)$ are assumed to be zero. We employ a uniform triangular mesh with mesh size $h = 0.01$ to discretize the computational domain. The time step is then chosen according to (7.1). We take again the medium parameters of water and set the thermal relaxation time to $\tau = 1.5 \times 10^{-5}$ s. Figure 6 provides snapshots of the approximate acoustic pressure in inviscid media, where $\delta = 0$, until the final time $T = 1.5 \times 10^{-4}$ s.

![Snapshots of the approximate acoustic pressure](image)

**Figure 6.** Linear evolution of the acoustic pressure over time in thermally relaxing, inviscid media in the presence of an external source of sound.

To perform the convergence study, we take $\delta \in [0, 10^{-2}]$ m$^2$/s and compute the relative error in the energy norm according to (7.2). The plot is given in Figure 5 on the right. We observe again a linear convergence rate with respect to $\delta$, as we expected based on the result of Theorem 3.1.
Discussion and outlook

In this paper, we have investigated the limit as the diffusivity of sound $\delta$ tends to zero in the third-order JMGT equation, its version containing a quadratic gradient nonlinearity, and its linearization given by the MGT equation. The analysis also included well-posedness of the respective limiting equations as well as uniform $\delta$-independent energy estimates.

From our estimates and even more clearly from our numerical experiments, it is apparent that there is a rather involved interplay among the diffusivity parameter $\delta$, the relaxation time $\tau$, and the nonlinearity (determined by the parameters $\kappa$ and $\sigma$). Analytic and further numerical studies of this interaction will, therefore, be the subject of further research.

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