The Young matroid: a multiset extension of the Catalan matroid to arbitrary Young diagrams

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Abstract

Introduced by Ardila (J. Combin. Theory Ser. A, 2003), the Catalan matroid is obtained by defining the bases of the matroid using Dyck paths from \((0,0)\) to \((n,n)\). Further research has gone into the topic, with variants like lattice path matroids (introduced by Bonin, de Mier, and Noy (J. Combin. Theory Ser. A, 2003)) and shifted matroids (introduced independently by Klivans (2003), and Ardila (J. Combin. Theory Ser. A, 2003)) being studied intensively. In this short note, we introduce the Young matroid, an extension of the Catalan matroid, where the bases are defined using the standard Young tableaux of a fixed shape. This extension necessarily involves the consideration of independent multisets and multiset bases.

Notations. \(\mathbb{Z}\) denotes the set of all integers, \(\mathbb{N}\) denotes the set of all nonnegative integers, and \(\mathbb{Z}^+\) denotes the set of all positive integers. For any \(a, b \in \mathbb{Z}\), we will abuse notation and denote by \([a, b]\), the set of all integers between \(a\) and \(b\). For any \(n \in \mathbb{Z}^+\), \([n]\) denotes the set \([1, n]\).

1 Introduction

The Catalan numbers are a sequence of positive integers of immense combinatorial interest, with a plethora of research studying its features, and its connections with other combinatorial objects. (See Stanley [Sta15] for a detailed account of Catalan numbers, and several pointers to related literature.) Our investigation begins by noting a matroid construction by Ardila [Ard03], called the Catalan matroid.

Let \(n \in \mathbb{Z}^+\), and consider Dyck paths from \((0,0)\) to \((n,n)\) (lattice paths in \(\mathbb{Z}^2\) comprising of only north and east steps, such that the vertices of the paths are on or below the line \(Y = X\).) Note that each Dyck path has exactly \(2n\) steps, and we label the \(i\)-th step of a Dyck path by \(i\). Further
note that with this labeling, each Dyck path has exactly \( n \) east steps, and \( n \) north steps; the sets of east steps and north steps partition \([2n]\). For each Dyck path \( P \), let \( E(P) \) be the set of east steps in \( P \). Ardila [Ard03] defined the Catalan matroid \( C_n \) to be a matroid over the ground set \([2n]\), with the collection of bases \( \{E(P) : P \text{ is a Dyck path}\} \).

Two interesting extensions of the Catalan matroid have been explored in the literature.

- Bonin, de Mier, and Noy [BdMN03] introduced lattice path matroids: Fix an upper lattice path and a lower lattice path between \((0,0)\) and \((m,n)\), and lying inside the rectangle \([0,m] \times [0,n]\) (for some fixed \( m, n \in \mathbb{Z}^+ \)), and consider all lattice paths that lie between the upper and lower lattice paths. These correspond to the collection of bases of a matroid, analogous to the Catalan matroid.

Lattice path matroids have been studied intensively in the past couple of decades (for instance, [BdM06, Bon10, Sch10, Sch11, MT15, CTY15]).

- Klivans [Kli03a] and Ardila [Ard03] independently arrived at shifted matroids, which are matroids corresponding to shifted independence complexes. Studying the algebraic and combinatorial properties of shifted matroids, also in conjunction with shifted independence complexes, has been an interesting line of enquiry (for instance, [Kli03b, Duv05, Dun19]).

In this work, we will consider another extension of the Catalan matroid in the following sense. Note that there is a one-to-one correspondence between Dyck paths from \((0,0)\) to \((n,n)\) and standard Young tableaux of shape \((n,n)\). We are interested in the following question.

**Question 1.1.** Is there an extension of the Catalan matroid to Young diagrams of arbitrary shape? More specifically, for a fixed Young diagram with shape \( \lambda \), do the standard Young tableaux (SYTs) of shape \( \lambda \) induce a matroid structure (as bases or otherwise)?

Of course, there need not be a unique answer to Question 1.1. Indeed, one answer is a special case of shifted matroids. Ardila [Ard03] showed that for any fixed Young diagram of shape \( \lambda \), the sets defined by elements on the first row of all SYTs of shape \( \lambda \) define bases of a shifted matroid. However, we note two features (more precisely, a lack thereof) in this construction.

(a) Since only the entries in the first row of an SYT contribute to the corresponding basis, if the Young diagram of shape \( \lambda \) has three or more rows, then there is no one-to-one correspondence between the bases of the matroid and the SYTs of shape \( \lambda \).

(b) Ardila [Ard03] showed that the Catalan matroid is self-dual. However, if the Young diagram of shape \( \lambda \) has three or more rows, then the dual of the corresponding shifted matroid is not a matroid of the same kind; it does not correspond to a shifted matroid coming from a Young diagram (of any shape).

We introduce a matroid construction that remedies both the above deficiencies, by considering independent multisets and multiset bases.

### 1.1 Multiset matroids

Note that in a matroid over a ground set \([n]\), the independent sets and the bases are subsets of \([n]\), in other words, these are elements of the Boolean lattice \(2^{[n]}\). It might be interesting to note that
good matroid-like constructions are possible over other lattices too. Indeed, there is the notion of a \( q \)-matroid (introduced by Jurrius and Pellikaan [JP16]) which is essentially a \textit{matroid over the subspace lattice}\(^1\), and there is the notion of a sum-matroid (introduced by Panja, Pratihar, and Randrianarisoa [PPR19]) which is essentially a \textit{matroid over a finite product of subspace lattices}. There are also other extensions like polymatroids, demimatroids, and more, which are not necessarily matroid-like constructions over some lattice. We consider a different extension of the notion of matroid, which is very close to the classical matroid, as well as the \( q \)-matroid. In a nutshell, we consider \textit{matroids over a grid lattice} (a finite product of finite chains).

We interpret a \textit{multiset} (with \( n \) coordinates) as a function \( A : [n] \to \mathbb{N} \). Let us introduce some notations, which are an abuse of notations for sets. For a multiset \( A : [n] \to \mathbb{N} \), we will denote \( |A| = \sum_{i \in [n]} A(i) \). Further, for any \( i \in [n] \),

- we denote ‘\( i \in A \)’ if \( A(i) \geq 1 \).
- we define multisets \( A \cup i, A \setminus i : [n] \to \mathbb{N} \) by

\[
(A \cup i)(j) = \begin{cases} 
A(i) & \text{if } j \neq i, \\
A(i) + 1 & \text{if } j = i,
\end{cases}
\]

and \( (A \setminus i)(j) = \begin{cases} 
A(i) - 1 & \text{if } A(i) \geq 1, \\
A(i) & \text{if } A(i) = 0.
\end{cases} \)

In addition, for multisets \( A, B : [n] \to \mathbb{N} \), we define multisets \( A \cup B, A \cap B : [n] \to \mathbb{N} \) by

\[
(A \cup B)(i) = \max\{A(i), B(i)\}, \quad \text{for all } i \in [n],
\]

and \( (A \cap B)(i) = \min\{A(i), B(i)\}, \quad \text{for all } i \in [n]. \)

We will only consider \textbf{uniform grids}, that is, finite grids of the form \([0, k_1 - 1] \times \cdots \times [0, k_n - 1]\). Again, to keep the terminology as close as possible to the set case, we interpret a uniform grid as follows: For \( k_1, \ldots, k_n \in \mathbb{N} \), we define the \textbf{uniform grid}

\[
G(k_1, \ldots, k_n) = \{ A : [n] \to \mathbb{N} \mid A(i) \leq k_i - 1, \text{ for all } i \in [n] \}.
\]

With the above terminologies and notations, let us now define matroids over uniform grids. We mention three (cryptomorphically) equivalent definitions – in terms of rank function, bases, and independent sets. The proofs of the cryptomorphisms are very similar to that in the case of the Boolean lattice, and so we omit these. Throughout, in all general statements henceforth, let us denote \( G := G(k_1, \ldots, k_n) \).

**Matroids over grids: Rank function.** A matroid over \( G \) is defined by a \textbf{rank function} \( r : G \to \mathbb{N} \) satisfying

(a) \( 0 \leq r(A) \leq |A|, \text{ for all } A \in G. \)

(b) \( \text{if } A, B \in G, A \subseteq B, \text{ then } r(A) \leq r(B). \)

(c) \( \text{for any } A, B \in G, \text{ we have } r(A \cup B) + r(A \cap B) \leq r(A) + r(B). \)

\(^1\)The subspace lattice is the lattice of all subspaces of \( \mathbb{F}_q^n \) under inclusion.
Matroids over grids: Bases. A matroid over \( G \) is defined by a collection of bases \( B \subseteq G \) satisfying

(a) \( B \neq \emptyset \).

(b) \(|A| = |B|\), for all \( A, B \in B \).

(c) for any \( A, B \in B, A \neq B \), and for any \( i \in A \), there exists \( j \in B \) such that \( A \setminus i \cup j \in B \).

Matroids over grids: Independent sets. A matroid over \( G \) is defined by a collection of independent sets \( I \subseteq G \) satisfying

(a) \( I \neq \emptyset \).

(b) if \( A \in I, B \subseteq A \), then \( B \in I \).

(c) if \( A, B \in I, |B| < |A| \), then there exists \( i \in A \setminus B \) such that \( B \cup i \in I \).

Dual matroid. Let us also define the dual of a matroid over \( G \), via the collection of bases. If \( B \) defines a matroid over \( G \), as a collection of bases, then the dual matroid is a matroid over \( G \), defined by the collection of bases \( B^* = \{ A^* \in G : A \in B \} \), where we define \( A^*(i) = k_i - 1 - A(i) \), for all \( i \in [n] \).

1.2 The Young matroid

We will stick to the notation \( \lambda = (\lambda_0, \ldots, \lambda_{h-1}) \) for the shape \( \lambda \) of a Young diagram. Note that we then have \( \lambda_0 \geq \cdots \geq \lambda_{h-1} > 0 \), that is, the corresponding Young diagram has \( h \) rows. Also denote \( |\lambda| := \sum_{j \geq 0} \lambda_j \). Further, let \( \text{SYT}(\lambda) \) denote the set of all standard Young tableaux of shape \( \lambda \).

Let \( \mathcal{S}_h \) denote the permutation group on the integer interval \([0, h-1]\). Consider a Young diagram with shape \( \lambda = (\lambda_0, \ldots, \lambda_{h-1}) \), and \(|\lambda| = n \). For any \( T \in \text{SYT}(\lambda) \), and \( \sigma \in \mathcal{S}_h \), we define a multiset \( T_\sigma \in \mathcal{G}(\underbrace{(h, \ldots, h)}_{\text{n times}}) \) as follows: For \( i \in [n] \), define

\[ T_\sigma(i) := \sigma(j), \quad \text{where } j \in [0, h-1] \text{ is unique such that } i \text{ occurs in the } j\text{-th row of } T. \]

Given a Young diagram of shape \( \lambda \) with \( h \) rows, and a permutation \( \sigma \in \mathcal{S}_h \), we define the Young matroid with parameters \( (\lambda, \sigma) \) by the collection of bases

\[ \mathcal{Y}(\lambda, \sigma) = \{ T_\sigma : T \in \text{SYT}(\lambda) \}. \]

If we take \( \lambda = (n, n) \), \( h = 2 \), and \( \sigma = (1 \ 0) \) (transposition on two symbols, 0 and 1), then the Young matroid \( \mathcal{Y}(\lambda, \sigma) \) is exactly the Catalan matroid \( \mathcal{C}_n \).

Our main result is, indeed, that the collection of bases given above defines a multiset matroid.

**Theorem 1.2.** For any Young diagram of shape \( \lambda \) with \( h \) rows, and \( \sigma \in \mathcal{S}_h \), the set \( \mathcal{Y}(\lambda, \sigma) \) is a collection of bases of a matroid over the grid \( \mathcal{G}(\underbrace{(h, \ldots, h)}_{\text{n times}}) \).
Interpretation in terms of lattice paths

Even though the Young matroid generalizes the Catalan matroid to arbitrary Young diagrams, we see that there is a simple interpretation of the bases in terms of higher-dimensional lattice paths. Fix a Young diagram of shape \( \lambda \) with \( h \) rows, and let \( |\lambda| = n \). For \( t \in [0, h - 1] \), let \( e_t \) denote the \( t \)-th standard unit vector (having 1 at the \( t \)-th component, and 0 elsewhere) in \( \mathbb{Z}^{[0,h-1]} \). Consider lattice paths in \( \mathbb{Z}^{[0,h-1]} \) from \((0, \ldots, 0)\) to \((\lambda_0, \ldots, \lambda_{h-1})\) with each step in the direction of one of the \( e_t \)-s. Then each such lattice path has exactly \( n \) steps. We refer to the steps as the first step, second step, or more generally, the \( i \)-th step, in the obvious way.

Consider a Young matroid \( Y(\lambda, \sigma) \). For every \( T \in SYT(\lambda) \), we have the basis \( T_\sigma \in Y(\lambda, \sigma) \), and in the corresponding lattice path from \((0, \ldots, 0)\) to \((\lambda_0, \ldots, \lambda_{h-1})\), the \( i \)-th step is along the direction \( e_{T_\sigma(i)} \). Further, every vertex \((v_0, \ldots, v_{h-1})\) in the lattice path satisfies \( v_0 \geq \cdots \geq v_{h-1} \). In addition, for each \( j \in [0, h - 1] \), the lattice path has exactly \( \lambda_j \) steps along the direction of \( e_j \). It is easy to check that the converse is also true. Thus there is a one-to-one correspondence between elements of \( Y(\lambda, \sigma) \) and lattice paths in \( \mathbb{Z}^{[0,h-1]} \) between \((0, \ldots, 0)\) and \((\lambda_0, \ldots, \lambda_{h-1})\) satisfying

1. for each \( j \in [0, h - 1] \), there are exactly \( j \) steps along the direction of \( e_j \).
2. each vertex \((v_0, \ldots, v_{h-1})\) in the lattice path satisfies \( v_0 \geq \cdots \geq v_{h-1} \).

1.3 Isomorphism and duality

Let us quickly mention what we mean by isomorphism.

We describe here some isomorphism properties as well as duality properties of Young matroids.

Fix a Young diagram of shape \( \lambda \) with \( h \) rows. For any \( \sigma \in S^l_h \) and \( i \in [0, h - 1] \), let \( orb(\sigma, i) = \{ \sigma(i), \sigma^2(i), \ldots \} \) (the orbit of \( i \) under the action of \( \sigma \)). We say \( \sigma \in S^l_h \) is \( \lambda \)-compatible if

\[ i, j \in [0, h - 1], j \in orb(\sigma, i) \implies \lambda_j = \lambda_i. \]

Thus a \( \lambda \)-compatible permutation corresponds to exactly those permutations of the rows of the Young diagram with shape \( \lambda \), such that the resultant is again the same Young diagram. We then define \( \sigma, \tau \in S^l_h \) to be \( \lambda \)-synonyms, denoted by \( \sigma \sim_\lambda \tau \), if \( \sigma\alpha = \tau \) for some \( \lambda \)-compatible \( \alpha \in S^l_h \). It is intuitive that if \( \sigma \sim_\lambda \tau \), then the Young matroids \( Y(\lambda, \sigma) \) and \( Y(\lambda, \tau) \) have essentially the same structure. We formalize this as follows.

**Proposition 1.3.** For any Young diagram of shape \( \lambda \) with \( h \) rows, and \( \sigma, \tau \in S^l_h \), we have

\[ Y(\lambda, \sigma) \simeq Y(\lambda, \tau) \iff \sigma \sim_\lambda \tau. \]

The duals of Young matroids can also be characterized in a simple manner. In particular, the dual of every Young matroid is a Young matroid. For any \( \sigma = a_0 \cdots a_{h-1} \in S^l_h \) as a word, define \( \sigma^* = (h - 1 - a_0) \cdots (h - 1 - a_{h-1}) \) as a word. We have the following.

**Proposition 1.4.** In the description of matroids via collections of bases, for any Young diagram of shape \( \lambda \) with \( h \) rows, and \( \sigma \in S^l_h \), we have

\[ Y(\lambda, \sigma)^* = Y(\lambda, \sigma^*). \]
2 Proofs

We give here an outline of the proofs of our results. (We defer detailed proofs to a later version.)

- Note that given a basis of the Young matroid \(Y(\lambda, \sigma)\), when the corresponding lattice path is projected to any two coordinates, we get a Dyck path. The basis exchange axiom for \(Y(\lambda, \sigma)\) then follows immediately by using a projection to suitable pair of coordinates, followed by the arguments of either Ardila [Ard03] or Bonin et al. [BdMN03]. This would prove Theorem 1.2.

- To prove Proposition 1.3, it is enough to consider \(\sigma\) being a transposition \((j, j+1)\) and \(\tau\) being the identity. Then we have \(\lambda_j = \lambda_{j+1}\). The proof then follows by projecting to the \((j, j+1)\) coordinates, and following the arguments in Ardila [Ard03].

- Proposition 1.4 follows quite easily by tracing the definition of dual matroid.

References

[Ard03] Federico Ardila. The catalan matroid. *Journal of Combinatorial Theory, Series A*, 104(1):49–62, 2003. https://doi.org/10.1016/S0097-3165(03)00121-3.

[BdM06] Joseph Bonin and Anna de Mier. Lattice path matroids: structural properties. *European Journal of Combinatorics*, 27(5):701–738, 2006. https://doi.org/10.1016/j.ejc.2005.01.008.

[BdMN03] Joseph Bonin, Anna de Mier, and Marc Noy. Lattice path matroids: enumerative aspects and tutte polynomials. *Journal of Combinatorial Theory, Series A*, 104(1):63–94, 2003. https://doi.org/10.1016/S0097-3165(03)00122-5.

[Bon10] Joseph Bonin. Lattice path matroids: the excluded minors. *Journal of Combinatorial Theory, Series B*, 100(6):585–599, 2010. https://doi.org/10.1016/j.jctb.2010.05.001.

[CTY15] Emma Cohen, Prasad Tetali, and Damir Yeliussizov. Lattice path matroids: negative correlation and fast mixing. *arXiv Preprint*, 2015. https://arxiv.org/abs/1505.06710.

[Dun19] Chad Kenneth Duna. *Matroid Independence Polytopes and Their Ehrhart Theory*. PhD thesis, University of Kansas, 2019. https://kuscholarworks.ku.edu/bitstream/handle/1808/29877/Duna_ku_0099D_16618_DATA_1.pdf.

[Duv05] Art M. Duval. A common recursion for laplacians of matroids and shifted simplicial complexes. *Documenta Mathematica*, 10:583–618, 2005. https://www.emis.de/journals/DMJDMV/vol-10/18.html.

[JP16] Relinde Jurrius and Ruud Pellikaan. Defining the q-analogue of a matroid. *arXiv Preprint*, 2016. https://arxiv.org/abs/1610.09250.

[Kli03a] Caroline Klivans. Shifted matroid complexes. *preprint*, 2003. https://www.dam.brown.edu/people/cklivans/matroids_cjk.pdf.
[Kli03b] Caroline Jane Klivans. *Combinatorial properties of shifted complexes*. PhD thesis, Massachusetts Institute of Technology, 2003. [https://dspace.mit.edu/bitstream/handle/1721.1/29358/52769866-MIT.pdf](https://dspace.mit.edu/bitstream/handle/1721.1/29358/52769866-MIT.pdf).

[MT15] Jason Morton and Jacob Turner. Computing the tutte polynomial of lattice path matroids using determinantal circuits. *Theoretical Computer Science*, 598:150–156, 2015. [https://doi.org/10.1016/j.tcs.2015.07.042](https://doi.org/10.1016/j.tcs.2015.07.042).

[PPR19] Avijit Panja, Rakhi Pratihar, and Tovohery Hajatiana Randrianarisoa. Some matroids related to sum-rank metric codes. *arXiv Preprint*, 2019. [https://arxiv.org/abs/1912.09984](https://arxiv.org/abs/1912.09984).

[Sch10] Jay Schweig. On the $h$-vector of a lattice path matroid. *The Electronic Journal of Combinatorics*, 17(N3):6, 2010. [https://doi.org/10.37236/452](https://doi.org/10.37236/452).

[Sch11] Jay Schweig. Toric ideals of lattice path matroids and polymatroids. *Journal of Pure and Applied Algebra*, 215(11):2660–2665, 2011. [https://doi.org/10.1016/j.jpaa.2011.03.010](https://doi.org/10.1016/j.jpaa.2011.03.010).

[Sta15] Richard P Stanley. *Catalan numbers*. Cambridge University Press, 2015. [https://doi.org/10.1017/CBO9781139871495](https://doi.org/10.1017/CBO9781139871495).