RNS and Pure Spinors Equivalence for Type I Tree Level Amplitudes Involving up to Four Fermions

G. Alencar a, M. O. Tahim a, R. R. Landim b and R.N. Costa Filho b

a Universidade Estadual do Ceará, Faculdade de Educação, Ciências e Letras do Sertão Central- R. Epitcio Pessoa, 2554, 63.900-000 Quixadá, Ceará, Brazil.

b Departamento de Física, Universidade Federal do Ceará- Caixa Postal 6030, Campus do Pici, 60455-760, Fortaleza, Ceará, Brazil.

Abstract

In this paper we give a proof of the equivalence between the RNS and Pure Spinor formalism for Type I tree level amplitudes involving up to four fermions. This result have been obtained previously for amplitudes involving only closed or open amplitudes and here we extend it to include Type I amplitudes. For this we first prove the cyclic symmetry of Type I superstring in a way that it is also valid for the Pure Spinor formalism. The technique used is applied to simplify the calculation of the tree level three point amplitude previously computed. As a byproduct, we are able to calculate the gauge anomaly amplitude of type I Superstrings in this formalism.
1 Introduction

The covariant quantization of superstring theory have been an unsolved problem for a long time. The covariant quantization presents manifest supersymmetry and gives us an explicit covariant prescription to compute tree-level and higher loop scattering amplitudes for an arbitrary number of states [1]. This is important for understanding the low energy limit of superstrings through the construction of effective actions corresponding to such amplitudes. Besides, it is possible to make explicit calculations in a specific background, a powerful characteristic that is applied to understand the AdS/CFT correspondence from the string theory side [2–8]. Beyond this background, the formalism have been used to study sigma models in [9–11]. This new formalism keeps all the good properties of Ramond-Neveu-Schwarz and Green-Schwarz and does not have its undesired characteristics. In the Ramond-Neveu-Schwarz formalism, when the number of loops in computations of scattering amplitudes are increased, more and more spin structures have to be considered making the computations very long. On the other hand, in the Green-Schwarz formalism the quantization is only possible in the light-cone gauge and the amplitude computations involve non-covariant operators at the interaction points.

In order to establish the power of this method it is important to check its consistency by simply comparing its results to those coming from the standard Ramond-Neveu-Schwarz (RNS) an Green-Schwarz (GS) formulations. In this sense, several important results related to scattering amplitude computations were achieved. The tree-level amplitudes were shown to be equivalent with the RNS computations in [12], for amplitudes containing any number of bosons and up to four fermions. Years later, the multiloop prescription was given [13, 14] showing the equivalence up to two-loop level. The explicit calculations of equivalence of results for one- and two-loop amplitudes with the minimal and non-minimal formalisms, the computation of the gauge variation of the massless six-point open string amplitude, obtaining the kinematic factor related to the anomaly cancellation, between other important related results were established [15–20]. More recently, the tree level amplitude of six massless open strings was obtained [22]. A recursive formula for super Yang-Mills color-ordered n - point tree amplitudes based on the cohomology of pure spinor superspace in ten space-time dimensions was constructed [23]. Another example of the power of the formalism is the computation of the coefficient of the massless one-loop and two-loop four-
point amplitude from first principles [24,25], which was not possible with RNS and GS.

The interesting thing here is that type I supergravity can be described in quantum language with the use of superspace, therefore solving two important problems. We know, for example, that all the cubic terms are encoded in an unique and simple superspace expression [26], showing the great simplification brought by the formalism. Hence, many terms of the effective action are obtained when we take this expression and expand in components. From the standpoint of field theory, the effective action for the type I supergravity is obtained from the global super Yang-Mills action by imposing local supersymmetry. This procedure generates many compensation terms [27] that are interpreted as interaction terms. From the standpoint of superstrings, all these interaction terms must come out naturally from amplitude computations. The interaction terms of the effective action for type I supergravity have some interesting properties. A very peculiar one is the fact that there is a coupling between the Kalb-Ramond and two photons. This term is necessary inorder to guarantee local supersymmetry. In order to keep gauge invariance, the Kalb-Ramond field must have a unusual transformation under U(1) symmetry. This coupling will become very important for the mixed anomaly cancellation in the SO(32) theory.

In all of these calculations we strongly use the cyclic symmetry property of scattering amplitudes. This means basically that the tree amplitudes do not depend on which of the vertex operators are chosen to be unintegrated. In a more specific result [26] this symmetry was used to compute the effective action for type I supergravity using the pure spinor formalism for superstrings. That work regarded the tree level approximation for Type I superstring amplitudes. If we identify the fermionic and bosonic vertex operators and use the cyclic symmetry, then that computation can be simplified. As we will see, the bosonic vertex operators contributes to zero or one thetas and therefore will always increase the number of thetas. However the expression for the fermionic integrated vertex operator acts as a theta derivative, and therefore contributes to $-1$ in the computation of the five thetas, a fact which can make the computations very long. On the other hand the unintegrated contributes with two thetas. Therefore the strategy is to use the cyclic symmetry to always choose the fermion operators as unintegrated ones.

In this paper, after an explanation about the cyclic symmetry for Type I tree level amplitudes, we are interested in showing the equivalence between the RNS and Pure Spinor formalism for Type I amplitudes involving up to
four fermions. We also show how the techniques used here can simplify the
computation of the three point tree level amplitude. As a byproduct, the
computation of the gauge anomaly amplitude of type I Superstrings from
the viewpoint of pure spinors is obtained. The organization of the work is as
follows: in the second section we give a review of the scattering amplitude
prescription in the pure spinor formalism. In the third section we give the
proof of cyclic symmetry. In the fourth, fifth and sixth sections we do explicit
computations of amplitude equivalences. In the seventh section we simplify
the computation for the three point type I amplitude, discussing the appear-
ance of the well known gauge anomaly in this model. In the final section we
present conclusions and perspectives.

2 The Pure Spinor Formalism

2.1 Tree-level Amplitudes in Type I Pure Spinor For-
malism

The pure spinor formalism \[1\] of superstrings contains the usual bo-
sonic \(X\) field, the spinor field \(\theta\) and a pure spinor \(\lambda\) and its respective conjugate
momenta \(p, \omega\). The Lorentz generators for the ghosts \(\lambda\) are

\[N^{nm} = \frac{\alpha'}{4} (\lambda \gamma^{mn} \omega)\]

and the OPEs are given by

\[p_\alpha (z_1) \theta^\beta (z_2) = -\frac{\delta^\beta_\alpha}{z_2 - z_1}, \quad \bar{p}_\alpha (\bar{z}_1) \bar{\theta}^\beta (\bar{z}_2) = -\frac{\delta^\beta_\alpha}{\bar{z}_2 - \bar{z}_1}\]

Here we must be careful because we will consider Type I amplitudes with
Riemann surface given by the upper half complex plane. Therefore we also
have the following OPEs

\[p_\alpha (z_1) \bar{\theta}^\beta (\bar{z}_2) = -\frac{\delta^\beta_\alpha}{\bar{z}_2 - z_1}, \quad \bar{p}_\alpha (\bar{z}_1) \theta^\beta (z_2) = -\frac{\delta^\beta_\alpha}{z_2 - \bar{z}_1}\]

due to the mixing between the left and right movers. For the \(X_\mu\) field we have

\[: X^\mu (z_1) X_\nu (z_2) : \sim -\frac{\alpha'}{2} \eta^\mu_\nu \left[ \ln |z_1 - z_2|^2 + \ln |z_1 - \bar{z}_2|^2 \right]. \quad (1)\]
This formalism is symmetric under a BRST transformation for the left and right moving sectors, with generators given by

\[ Q_c = \int dz \lambda^\alpha d^c_\alpha, \quad \bar{Q}_c = \int dz \bar{\lambda}^\alpha \bar{d}^c_\alpha, \quad \text{(2)} \]

with the definition

\[ d^c_\alpha = \frac{\alpha'}{2} p_\alpha - \theta \gamma^m \partial x_m - \frac{1}{8} \gamma^m_{\alpha\beta} \gamma_m \delta \theta \theta^\beta \theta^\delta \partial \theta^\eta, \]

and the similarly for \( \bar{d}^c_\alpha \). With this we can compute the following important OPEs

\[ d^c_\alpha (z_i) V (z_j) \sim - \frac{\alpha'}{2} \frac{D^V_\alpha}{z_j - z_i} - \frac{\alpha'}{2} \frac{\bar{D}^V_\alpha}{\bar{z}_j - \bar{z}_i}, \quad \text{(3)} \]

and

\[ \bar{d}^c_\alpha (z_i) V (z_j) \sim - \frac{\alpha'}{2} \frac{\bar{D}^V_\alpha}{\bar{z}_j - \bar{z}_i} - \frac{\alpha'}{2} \frac{\bar{D}^V_\alpha}{\bar{z}_j - \bar{z}_i}, \quad \text{(4)} \]

\[ d_\alpha (z_i) d_\beta (z_j) \sim \frac{\alpha'}{2} \frac{\gamma^m_{\alpha\beta} \Pi_m}{z_j - z_i}. \]

where

\[ D_\alpha = \frac{\alpha'}{2} \partial_\alpha + \theta \gamma^m \partial_m, \quad \bar{D}_\alpha = \frac{\alpha'}{2} \bar{\partial}_\alpha + \bar{\theta} \gamma^m \partial_m. \]

From the last OPE we see that

\[ Q^2_c = - \frac{\alpha'}{2} \int \lambda^\alpha \lambda^\beta \gamma^m_{\alpha\beta} \Pi_m, \quad \bar{Q}^2_c = - \frac{\alpha'}{2} \int \bar{\lambda}^\alpha \bar{\lambda}^\beta \gamma^m_{\alpha\beta} \bar{\Pi}_m, \quad Q_c \bar{Q}_c = 0, \]

and therefore the BRST operators are nilpotent only if they satisfy the condition

\[ \lambda \gamma^m \lambda = \bar{\lambda} \gamma^m \bar{\lambda} = 0. \quad \text{(5)} \]

A spinor that satisfies the above condition was called a pure spinor by Cartan \[28\]. The only OPE involving ghost fields which will be needed in this work is \[\Pi\].
\[ N^{mn}(z_1) \lambda^\alpha(z_2) = \frac{\alpha'}{4(z_2 - z_1)} (\lambda \gamma^{mn})^\alpha. \] (6)

With the BRST operator we can construct the vertex operator. The fixed one is defined as the cohomology with ghost number +1. The most simple object of ghost number one is given by

\[ V = g'_o \lambda^\alpha A_\alpha(z, \theta), \]

where \( A_\alpha \) is a spinorial superfield. The physical state condition

\[ QV = 0, \]

give us the equations of motion

\[ D_\alpha A_\beta + D_\beta A_\alpha = \gamma^m_{\alpha\beta} A_m, \]

(7)

and \( A_m \) is a vector superfield. These are the right equations of supergravity. A fact that will be used in this work is the definition of the integrated operators. For the open string case this is defined as \([Q_o, U] = \dot{V}\) and for the closed \(\{Q_c, \bar{Q}_c, U\} = \partial \bar{\partial} V\). The important fact here is that, for Type I strings case, we have the definition

\[ Q_o U(z, \bar{z}) = Q_o(e^{ikX} U(\theta) \bar{U}(\bar{\theta})) = \{ \partial(e^{ikX} V(\theta)) \bar{U}(\bar{\theta}) + U(\theta) \bar{\partial}(e^{ikX} \bar{V}(\bar{\theta})) \} \]

(8)

what give us a ‘half integrated” operator. In the above expression \( U(\theta) \bar{U}(\bar{\theta}) \) are functions that depend only on \( \theta \) and \( \bar{\theta} \) respectively.

The integrated vertex operator for the open string, in its turn, has ghost number zero and is

\[ g'_o \int dy_3 \left( \partial \theta^\alpha A_\alpha + A_m \Pi^m + d_\alpha W^\alpha + \frac{1}{2} N^{mn} \mathcal{F}_{mn} \right), \]

where \( N^{mn} \) was defined before and \( A_\alpha, A_m \) and \( d_\alpha \) are defined above. \( W^\alpha \) and \( \mathcal{F}_{mn} \) are field strengths given by

\[ W^\alpha = \frac{1}{10} \gamma^\alpha_{\beta\gamma} D_\beta A^m, \quad \mathcal{F}_{mn} = 2 \partial_m A_n |. \]

When necessary, the superfields will be expanded in components. The vertex operator for the closed string is given by the product of two open string
operators $\lambda^\alpha A_\alpha \bar{\lambda}^\alpha \bar{A}_\alpha$. The prescription to compute tree-level closed string amplitudes with the pure spinor formalism is given by

$$A_n = \langle V^1(z_1)V^2(z_2)V^3(z_3) \int d^2 z_4 U_4(z_4) \ldots \int d^2 z_N U_N(z_N) \rangle$$  \hspace{1cm} (9)

and after the necessary $\theta$-expansions of superfields and use of OPEs, the integration of the zero-modes of $\lambda^\alpha$ and $\theta^\alpha$ is carried out by taking only the terms which contain three $\lambda$s and five $\theta$s in the correlator which are proportional to the pure spinor measure

$$\langle (\lambda \gamma^m \theta) (\lambda \gamma^n \theta) (\lambda \gamma^p \theta) (\theta \gamma_{mnp} \theta) \rangle = c.$$  

Here the normalization can be chosen in order to obtain the right results for comparison with other formalisms. We also have use two different values for $c$ in order to compare our result with the RNS formalism and with a previous computation of the three point function in the pure spinor formalism.

### 2.2 Bosonic and Fermionic Vertex Operators

As said in the introduction, we must identify the bosonic and fermionic vertex operators. These have been found in [12] and we need to be careful with the factors of $\alpha'$. These operators are given by

$$U^B = a^m[\partial x_m - i k^n (N_{mn} - \alpha' \frac{1}{4} p_{mn} \theta) + 2 \theta^s \ldots] e^{ik \cdot x}, \quad U^F = -\xi^\alpha [\alpha' \frac{1}{2} p_{\alpha} + \ldots] e^{ik \cdot x},$$

$$V^B = a^m[\frac{1}{2} \lambda m \theta + 3 \theta^s \ldots] e^{ik \cdot x}, \quad V^F = \xi^\alpha [\frac{1}{3} (\lambda \gamma m \theta)(\gamma m \theta)_{\alpha} + 4 \theta^s \ldots] e^{ik \cdot x},$$

where in the above expressions, $U, V$ stands for integrated and unintegrated operators respectively. From the above expression we can see that the fermionic integrated vertex operator acts as a theta derivative, and therefore contributes to $-1$ in the computation of the five thetas, what can make the computations very long. On the other hand the unintegrated contributes with two thetas. Therefore the strategy here is to use the cyclic symmetry to always choose the fermion operators as unintegrated ones. The bosonic vertex operators contributes to zero or one thetas and therefore will always increase the number of thetas. With this strategy we will see that only
the above order in thetas are needed. As a simplification we must use the definitions

\[ b_m \equiv \lambda \gamma^m \theta, \quad M_{mn} \equiv N_{mn} - \frac{\alpha'}{4} (p \gamma_{mn} \theta) \]

and

\[ f_\alpha \equiv (\lambda \gamma^m \theta)(\gamma_m \theta)_\alpha. \]

With this we have

\[ U^B = a^m [\partial x_m - i k^n M_{mn} + 2 \theta s...] e^{ik \cdot x}, \quad U^F = -\xi^a \left[ \frac{\alpha'}{2} p_\alpha + ... \right] e^{ik \cdot x}, \]

\[ V^B = a^m \left[ \frac{1}{2} b_m + 3 \theta s... \right] e^{ik \cdot x}, \quad V^F = \xi^a \left[ \frac{1}{3} f_\alpha + 4 \theta s... \right] e^{ik \cdot x}. \]

and using our previous OPEs we get

\[ M_{mn}(z_1) f_\alpha(z_2) \rightarrow \frac{\alpha'(\gamma_{mn}) \alpha^\beta f_\beta(z_2)}{4(z_1 - z_2)}, \quad M_{mn}(z_1) b_p(z_2) \rightarrow \alpha' \frac{\eta_{mp} b_m(z_2) - \eta_{mp} b_n(z_2)}{2(z_1 - z_2)}. \]

The other OPEs needed in this work are

\[ M_{kl}(z_1) M_{mn}(z_2) = \frac{\alpha' \eta_{m[k} M_{l]n}(z_2) - \eta_{m[k} M_{l]m}(z_2)}{z_1 - z_2} + \frac{\alpha^2 \eta_{kn} \eta_{lm} - \eta_{km} \eta_{ln}}{4(z_1 - z_2)^2} \]

and

\[ M_{mn}(z_1) p_\alpha(z_2) \rightarrow \frac{\alpha'(\gamma_{mn}) \alpha^\beta p_\beta(z_2)}{4(z_1 - z_2)}. \]

These OPEs will be needed for comparison with the respective ones in the RNS formalism.

3 Cyclic Symmetry for Type I Superstring Amplitudes

In this section we present a proof of the superstring cyclic symmetry for tree level Type I amplitudes that is valid for the pure spinor formalism. As said in the introduction, the Type I effective action for supergravity was recently...
computed in the framework of pure spinor superstrings and the tree-level Type I amplitudes (for one closed and two open strings) were calculated. In these calculations, the cyclic symmetry was used as a tool but a proof was not presented. This exists within the BRST formalism such that it also applies to pure spinors, but it only works for the open string or closed string amplitudes [12]. We extend this to the general case including Type I strings. This symmetry will also be needed to prove the RNS and Pure Spinor equivalence for amplitudes involving up to four fermions.

3.1 Cyclic Symmetry for Open Strings

For the sake of simplicity, we first give a proof for the cyclic symmetry for open strings tree amplitudes, a result that is similar to the one given in [12] for closed strings. The amplitude in this case is given by

\[
\langle V_1(y_1)V_2(y_2)V_3(y_3) \int_{y_3}^{y_1} dy_4 U_4(y_4) \int_{y_4}^{y_1} dy_5 U_5(y_5) \cdots \int_{y_{N-1}}^{y_1} d^2 y_N U_N(y_N) \rangle.
\]

(12)

Considering the delta function representation \( \dot{\Theta}(y - y') = \delta(y - y') \), where \( \Theta \) is the Heaviside step function, and using \([Q, U(y)] = V\) we obtain

\[
\int_{y_2}^{y_3} dy \{ \Theta(y - y_2) - \Theta(y - y_3) + \Theta(y_3 - y_4) \} [Q, U_3(y)] = V_3(y_3) - V_3(y_2).
\]

(13)

Now using the cancelled propagator argument (CPA) [29], the second term in the rhs can be disregarded and we get

\[
\langle V_1(y_1)V_2(y_2) \int_{y_2}^{y_3} dy \int_{y_3}^{y_1} dy_4 dy \{ \Theta(y - y_2) - \Theta(y - y_3) + \Theta(y_3 - y_4) \} \times
\]

\[
\times [Q, U_3(y)] U_4(y_4) \int_{y_4}^{y_1} dy_5 U_5(y_5) \cdots \int_{y_{N-1}}^{y_1} d^2 y_N U_N(y_N) \rangle \tag{14}
\]

for the amplitude.

Now we pull off the BRST operator to all the other vertex. Again the CPA cancels the contribution of all of them and we are left only with

\[
\langle V_1(y_1)V_2(y_2) \int_{y_2}^{y_3} dy U_3(y) \int_{y_3}^{y_1} dy_4 \{ \Theta(y - y_2) - \Theta(y - y_3) + \Theta(y_3 - y_4) \} \times
\]

\[
\times [Q, U_4(y_4)] \int_{y_4}^{y_1} dy_5 U_5(y_5) \cdots \int_{y_{N-1}}^{y_1} d^2 y_N U_N(y_N) \rangle \tag{15}
\]
leading to
\[
\langle V_1(y_1) V_2(y_2) \int_{y_2}^{y_3} dy U_3(y) V_4(y_3) \int_{y_4}^{y_1} dy_5 U_5(y_5) \cdots \int_{y_{N-1}}^{y_1} d^2 y_N U_N(y_N) \rangle. 
\] (16)

The same argument applied here will be used to the case of Type I amplitudes.

### 3.2 Cyclic Symmetry for Type I Strings

Here we give the proof for the cyclic symmetry for Type I amplitudes. We use the same sort of argument used previously for the open string with the \( \Theta \) function. We must point that this method is necessary here because differently from [12] we can have a half fixed operator. This does not happen in the non-mixing string case. Using the steps above we write the amplitude involving three open fixed operators as
\[
\langle V_1(y_1) V_2(y_2) V_3(y_3) \int dz \bar{d} \int d^2 z_4 U_4(z_4, \bar{z}_4) \times \int d^2 z_5 U_5(z_5, \bar{z}_5) \cdots \int d^2 z_N U_N(z_N, \bar{z}_N) \rangle. 
\] (17)

The dots can include any number of open or closed strings. In the above amplitude we consider only three open strings fixed and all the closed strings integrated, because here we are concerned only with the exchange between open and closed strings. Using the same argument as before we get
\[
\langle V_1(y_1) V_2(y_2) \int dy \int dz_4 d\bar{z}_4 U_4(z_4) \{ \Theta(y - y_2) - \Theta(y - y_3) + \ln(|z_3 - z_4|^2) \} \times [Q, U_3(y)] U_4(\bar{z}_4) \int d^2 z_5 U_5(z_5, \bar{z}_5) \cdots \int d^2 z_N U_N(z_N, \bar{z}_N) \rangle. 
\] (18)

The function \( \ln(|z_3 - z_4|^2) \) was chosen such that we have a well defined real \( y \) integration. Using BRST invariance of the amplitude to pull it off to the other vertex operators and using again the CPA, the only remaining term is the one coming from \( U_4(z_4, \bar{z}_4) \). Therefore because of (8) we get
\[
\langle V_1(y_1) V_2(y_2) \int dy \int dz_4 d\bar{z}_4 U_3(y) \{ \Theta(y - y_2) - \Theta(y - y_3) + \ln(|z_3 - z_4|^2) \} \times \{ \partial(e^{ikX} V(\theta)) \bar{U}(\bar{\theta}) + U(\theta) \partial(e^{ikX} \bar{V}(\bar{\theta})) \} \int d^2 z_5 U_5(z_5, \bar{z}_5) \cdots \int d^2 z_N U_N(z_N, \bar{z}_N) \rangle. 
\] (19)
Performing the integration we finally get
\[ \langle V_1(y_1)V_2(y_2) \rangle \int_{y_2}^{y_1} dy U(y) \left[ \int dz_4 V(\theta)\bar{U}(\bar{\theta})e^{ikX} + \int dz_4 U(\theta)\bar{V}(\bar{\theta})e^{ikX} \right] \times \int d^2z_5 U_5(z_5, \bar{z}_5) \cdots \int d^2z_N U_N(z_N, \bar{z}_N) \]. \quad (20)

From the above result we can see the fact that we can fix only “one half” of the closed string. Now we can perform the same steps for another open string. For each of the above terms we obtain
\[ \langle V_1(y_1) \int_{y_1}^{y_2} dy' \int d\tilde{z}_4 [Q, U_2(y')]\{\Theta(y' - y_2) - \Theta(y' - y_1) + \ln(|z_3 - z_4|^2)\} \times V(\theta)\bar{U}(\bar{\theta})e^{ikX} \int_{y_2}^{y_1} dy U_3(y) \int d^2z_5 U_5(z_5, \bar{z}_5) \cdots \int d^2z_N U_N(z_N, \bar{z}_N) \rangle. \]
and using the fact that \( Q^2 = 0 \) we get, after pulling of the BRST operator
\[ \langle V_1(y_1) \int_{y_1}^{y_2} dy' U_2(y') \int d\tilde{z}_4 \{\Theta(y' - y_2) - \Theta(y' - y_1) + \ln(|z_3 - z_4|^2)\} V(\theta)\bar{V}(\bar{\theta})e^{ikX} \int_{y_2}^{y_1} dy U_3(y) \int d^2z_5 U_5(z_5, \bar{z}_5) \cdots \int d^2z_N U_N(z_N, \bar{z}_N) \rangle. \]
Using the same arguments as before we finally get
\[ \langle V_1(y_1) \int_{y_1}^{y_2} dy' U_2(y') \int_{y_2}^{y_1} dy U_3(y) V(\theta)\bar{V}(\bar{\theta})e^{ikX}(z_3, \bar{z}_3) \times \int d^2z_5 U_5(z_5, \bar{z}_5) \cdots \int d^2z_N U_N(z_N, \bar{z}_N) \rangle. \quad (21) \]
and we end our proof. This symmetry will be used to establish the equivalence with RNS and to simplify the computation of the three point tree level Type I amplitude.

4 Equivalence for Amplitudes involving four fermions

In this section we show the equivalence for amplitudes involving four fermions. Here and in the next two sections we choose the measure constant \( c = 2880 \).
in order to compare the results with the RNS formalism. In another section we must choose $c = 1$ for comparison with previous computations of the three point function. The general amplitude is given by

$$A = \left\langle VVV \int U \int U \right\rangle$$

The case with four open or two closed string fermionic states would reduce to the case already studied at [12]. Therefore we must consider the case with one fermionic closed and two fermionic open strings. With the help of cyclic symmetry we can choose the closed and one open string fixed. Therefore the amplitude can be written as

$$A = \langle \xi_1^{\alpha} \tilde{\xi}^{\beta} V^{F}_\alpha (z_1) \tilde{V}^{F}_\beta (\tilde{z}_1) \xi_2^{\gamma} V^{F}_\gamma (y_2) \int d\psi_3 \xi_3 \psi_3 \times$$

$$\int d^2 z_4 h^m_4 \tilde{h}_4^n U^B_m (z_4) \tilde{U}^B_n (\tilde{z}_4) \int d\psi_5 a^p U_p \ldots \int d\psi_5 a^q U_q e^{i \sum_{r=1}^N k_r \cdot x (z_r)} \rangle,$$

where the dots mean an arbitrary number of closed or open bosonic strings. Now we must use the explicit shape of the vertex operators and look for the terms with five thetas. For this, note that $p_i$ acts as a theta derivative. Because the fixed fermionic operators contribute with at least six thetas, all the integrated bosonic operators must contribute with zero thetas. With this, the amplitude is given by

$$A = -\frac{\alpha'}{2 \times 27} \langle \xi_1^{\alpha} \tilde{\xi}^{\beta} f_\alpha (z_1) \tilde{f}_\beta (\tilde{z}_1) \xi_2^{\gamma} f_\gamma (y_2) \int d\psi_3 \xi_3 \psi_3 \times$$

$$\int d^2 z_4 h^m_4 \tilde{h}_4^n (\partial x_{m_4} - ik^{p_4} M_{m_4 p_4}) (\partial x_{n_4} - ik^{q_4} M_{n_4 q_4}) \times$$

$$\int d\psi_5 a^m_5 (\partial x_{m_5} - ik^{p_5} M_{m_5 p_5}) \ldots \int d\psi_N a^m_N (\partial x_{m_n} - ik^{p_n} M_{m_n p_n}) e^{i \sum_{r=1}^N k_r \cdot x (z_r)} \rangle.$$

In the RNS case the amplitude is given by [30]
\[ A_{RNS} = -\langle \xi_1^\alpha c e^{-\frac{\phi}{2} \Sigma_\alpha(z_1)} \xi_2^\beta c e^{-\frac{\phi}{2} \Sigma_\beta(z_1)} \xi_3^\gamma c e^{-\frac{\phi}{2} \Sigma_\gamma(y_2)} \int dy_3 \xi_3^\delta e^{-\frac{\phi}{2} \Sigma_\delta(y_3)} \int d^2 z_4 h_4^{m4} h_4^{n4} (\partial x_m(z_4) - i\alpha' k_4^{\mu_4} \bar{\psi_4} \psi_4 (z_4)) (\partial x_n(z_4) - i\alpha' k_4^{\nu_4} \bar{\psi_4} \psi_4 (z_4)) \times \]

\[ \int dy_5 a_5^{n5} (\partial x_n(z_5) - i\alpha' k_5^{\nu_5} \bar{\psi_5} \psi_5 (z_5)) \times \int dz_N a_N^{mN} (\partial x_n(z_N) - i\alpha' k_N^{\nu_N} \bar{\psi_N} \psi_N (z_N)) e^{i \sum_{r=1}^N k_r x(z_r)} \]  

(24)

and all objects above have been defined before. The OPEs between the \( x \) fields are the same in both formalisms, therefore we just need to prove the equivalence for the part independent of \( x \). It has been argued before in [12] that all the OPEs between the bosonic integrated operators and the fermionic operators are the same in both formalisms. This is also valid here when we consider the OPEs given in section 2 and the fact that here we have a mixing between left and right movers. This guarantee that the dependence on \( z_4...z_N \) are the same. In order to prove the equivalence we just have to show that

\[ \frac{1}{27} \langle f_\alpha(z_1) \bar{f}_\beta(\bar{z}_1) f_\gamma(y_2) p_\delta(y_3) \rangle = \langle c e^{-\frac{\phi}{2} \Sigma_\alpha(z_1)} c e^{-\frac{\phi}{2} \Sigma_\beta(\bar{z}_1)} c e^{-\frac{\phi}{2} \Sigma_\gamma(y_2)} e^{-\frac{\phi}{2} \Sigma_\delta(y_3)} \rangle. \]  

(25)

The right hand side can be computed easily and results in [30]

\[ \langle c e^{-\frac{\phi}{2} \Sigma_\alpha(z_1)} c e^{-\frac{\phi}{2} \Sigma_\beta(\bar{z}_1)} c e^{-\frac{\phi}{2} \Sigma_\gamma(y_2)} c e^{-\frac{\phi}{2} \Sigma_\delta(y_3)} \rangle = \left\{ \begin{array}{c} \gamma_{\alpha\beta}(\gamma_{m\gamma \alpha})(z_1) \gamma_{\delta\gamma}(\gamma_{n\gamma \delta})(\bar{z}_1) + \gamma_{\beta\delta}(\gamma_{m\gamma \beta})(z_1) \gamma_{\alpha\gamma}(\gamma_{n\gamma \alpha})(\bar{z}_1) + \gamma_{\alpha\beta}(\gamma_{m\gamma \alpha})(z_1) \gamma_{\delta\gamma}(\gamma_{n\gamma \delta})(\bar{z}_1) \end{array} \right\}. \]  

(26)

For the left hand side, following the same lines of reasoning of [12], we must analyze the poles of \( p_\delta(y_3) \). Here we must be careful with the mixing of left an right movers. That is why, beyond the pole coming from \( y_3 \to z_1 \) we also have a pole in \( y_3 \to \bar{z}_1 \). Therefore, in the Type I case we have the following residue

\[ \frac{1}{27} \left\{ \left[ (\gamma_{\alpha\delta}(\gamma_{m\gamma \theta})(z_1) - (\gamma_{m\lambda})_\delta(\gamma_{m\theta})_\alpha(z_1))(\lambda \gamma_{\theta})(\gamma_{n\theta})_\beta(\bar{z}_1) + (\lambda \gamma_{\theta})(\gamma_{m\theta})_\alpha(z_1)[\gamma_{\delta\beta}(\lambda \gamma_{n\theta})(\bar{z}_1) - (\lambda \gamma_{\theta})(\gamma_{n\theta})_\beta(\bar{z}_1)] \right] \right\} \]  

(27)

12
The above expression can be simplified if we use

\[-(\gamma_m\lambda)_\delta(\gamma^m\theta)_\alpha(z_1)(\bar{\lambda}\gamma^n\bar{\theta})(\gamma_n\bar{\theta})_\beta(\bar{z}_1) - (\lambda\gamma_m\theta)(\gamma^m\theta)_\alpha(z_1)(\bar{\lambda}\gamma^n\bar{\theta})_\delta(\gamma_n\bar{\theta})_\beta(\bar{z}_1)\]

\[= \lambda D\{\frac{1}{2}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\delta(\bar{\lambda}\gamma^n\bar{\theta})(\gamma_n\bar{\theta})_\beta\} + \lambda\bar{D}\{\frac{1}{2}(\lambda\gamma_m\theta)(\gamma^m\theta)_\alpha(\gamma_m\bar{\theta})_\beta\} +\]

\[+ \frac{1}{2}\gamma^m_{\alpha\delta}(\lambda\gamma_m\theta)(\bar{\lambda}\gamma^n\bar{\theta})(\gamma_n\bar{\theta})_\beta + \frac{1}{2}(\lambda\gamma_m\theta)(\gamma^m\theta)_\alpha\gamma^m_{\alpha\delta}(\bar{\lambda}\gamma_m\bar{\theta})\]

and we obtain that the lhs is given by

\[\frac{2}{\alpha'}Q_\alpha\{\frac{1}{2}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\delta(\bar{\lambda}\gamma^n\bar{\theta})(\gamma_n\bar{\theta})_\beta + \frac{1}{2}(\lambda\gamma_m\theta)(\gamma^m\theta)_\delta(\gamma_m\bar{\theta})_\beta\} +\]

\[+ \frac{1}{2}\gamma^m_{\alpha\delta}(\lambda\gamma_m\theta)(\bar{\lambda}\gamma^n\bar{\theta})(\gamma_n\bar{\theta})_\beta + \frac{1}{2}(\lambda\gamma_m\theta)(\gamma^m\theta)_\alpha\gamma^m_{\alpha\delta}(\bar{\lambda}\gamma_m\bar{\theta})\]

Due to the BRST invariance of the amplitude we can always exchange

\[-(\gamma_m\lambda)_\delta(\gamma^m\theta)_\alpha(z_1)(\bar{\lambda}\gamma^n\bar{\theta})(\gamma_n\bar{\theta})_\beta(\bar{z}_1) - (\lambda\gamma_m\theta)(\gamma^m\theta)_\alpha(z_1)(\bar{\lambda}\gamma^n\bar{\theta})_\delta(\gamma_n\bar{\theta})_\beta(\bar{z}_1)\]

\[\rightarrow \frac{1}{2}\gamma^m_{\alpha\delta}(\lambda\gamma_m\theta)(\bar{\lambda}\gamma^n\bar{\theta})(\gamma_n\bar{\theta})_\beta + \frac{1}{2}(\lambda\gamma_m\theta)(\gamma^m\theta)_\alpha\gamma^m_{\alpha\delta}(\bar{\lambda}\gamma_m\bar{\theta}).\]

Using the above result our expression above is simplified to give

\[\frac{\gamma^m_{\alpha\delta}}{18}(\lambda\gamma_n\theta)(z_1)(\bar{\lambda}\gamma^n\bar{\theta})(\gamma_n\bar{\theta})_\beta(\bar{z}_1)(\lambda\gamma^p\theta)(\gamma^p\theta)_\gamma(y_2))\]

\[+ \frac{\gamma^m_{\alpha\delta}}{18}(\lambda\gamma_m\theta)(\gamma^m\theta)_\alpha(z_1)(\bar{\lambda}\gamma_n\bar{\theta})(\bar{z}_1)(\lambda\gamma^p\theta)(\gamma^p\theta)_\gamma(y_2))\]

\[\equiv \frac{\gamma^m_{\alpha\delta}}{18}(G_n)_{\beta\gamma} + \frac{\gamma^m_{\beta\gamma}}{18}(H_n)_{\alpha\gamma}\]

and each of the above terms is easily computed. First of all the quantity in brackets is Lorentz invariant and therefore must be proportional to \((\gamma_n)_{\beta\gamma}\). If we use our measure we finally obtain \((G_n)_{\beta\gamma} = 18(\gamma_n)_{\beta\gamma}\) and \((H_n)_{\alpha\gamma} = 18(\gamma_n)_{\alpha\gamma}\). Therefore the residue when \(y_3 \rightarrow z_1\) is the same as in the RNS case. The argument for the \(y_2\) pole is identical to that with only open string and is unnecessary here. The above result can be summarized in the identity

\[\frac{1}{27}\langle f_\alpha(z_1)f_\beta(\bar{z}_1)f_\gamma(y_2)p_3(y_3) \rangle = \{\frac{\gamma^m_{\alpha\delta}\gamma_m\beta\gamma}{z_1 - y_3} + \frac{\gamma^m_{\beta\delta}\gamma_m\gamma\alpha}{\bar{z}_1 - y_3} + \frac{\gamma^m_{\gamma\delta}\gamma_m\alpha\beta}{y_2 - y_3}\}.\]

In this way, we have shown the equivalence with the RNS formalism for amplitudes involving four fermions. Now we must consider the case involving two fermions.
5 Equivalence for Amplitudes Involving Two Fermions

As before, the general amplitude is given by

$$\mathcal{A} = \langle VVV \int U \cdots \int U \rangle.$$ 

The case with two open or one closed string fermionic states would be reduced to the case already studied at [12]. Therefore we must consider the case with one fermionic closed string, which involves one fermion for each moving sector. With the cyclic symmetry for Type I strings we can choose the closed and one open string fixed. This will give the contribution of four thetas. The other unintegrated vertex operator is bosonic and therefore we get one more theta, which fulfills five thetas. The last vertex operator is bosonic and integrated and therefore contributes with zero thetas as expected. Therefore, the amplitude can be written as

$$\mathcal{A} = \langle \xi^{\alpha} \tilde{\xi}^{\beta} V^F_{\alpha}(z_1) V^F_{\beta}(\bar{z}_1) b^m_2 V^B_{m_2}(y_2) \int dy_3 a^{m_3} U_{m_3}(y_3) \int d^2 z_4 h^m \tilde{h}^n U^B_{m}(z_4) U^B_{n}(\bar{z}_4) \times \int dy_5 a^{p_5} U_{p_5} \cdots \int dy_N a^{p_N} U_{p_N} e^{\sum_{r=1}^{N} k_{r} \cdot x(z_r)} \rangle,$$

where the dots mean an arbitrary number of closed or open bosonic strings. Now we must use the explicit shape of the vertex operators and look for the terms with five thetas. To achieve this, note that the fixed operators already contribute with at least five thetas. Therefore, all the integrated bosonic operators must contribute with zero thetas. With this, the amplitude is given by

$$\mathcal{A} = -\frac{\alpha'}{18} \langle \xi^{\alpha} \tilde{\xi}^{\beta} f_{\alpha}(z_1) \bar{f}_{\beta}(\bar{z}_1) a^m b^{m_2} \int dy_3 a^{m_3}(\partial x_{m_3} - ik^{p_3} M_{m_3}) (y_3) \times \int d^2 z_4 h^m h^m_4 \tilde{h}^n \tilde{h}^{n_4} (\partial x_{m_4} - ik^{p_4} M_{m_4}) (\partial x_{n_4} - ik^{q_4} M_{n_4}) \int dy_5 a^{m_5} (\partial x_{m_5} - ik^{p_5} M_{m_5}) \cdots \int dy_N a^{m_N} (\partial x_{m_N} - ik^{p_N} M_{m_N}) \rangle.$$

In the RNS case the amplitude is given by
The OPEs between the \( x \) fields are again the same in both formalisms, therefore we just need to prove equivalence for the \( x \) independent part. Like in the last section, we use the fact that it has been argued before in \([12]\) that all the OPEs between the bosonic integrated operators and the fermionic operators are the same in both formalisms. This is also valid here when we consider that all the OPEs, in both formalisms, have a mixing between left and right movers. This guarantee that the dependence on \( z_4 \), \( \ldots \), \( z_N \) are the same, and to prove the equivalence we just need to show that

\[
\frac{1}{18} \langle f_\alpha(z_1) \bar{f}_\beta(\bar{z}_1) b_m(z_3) \rangle = \langle ce^{-\frac{k}{2} \Sigma_\alpha(z_1)} \bar{c}e^{-\frac{k}{2} \Sigma_\beta(\bar{z}_1)} ce^{-\phi} \psi_m(y_2) \rangle. \tag{34}
\]

The RNS result for the right hand side is given by \((\gamma_m)_{\alpha\beta}\). The left hand side is given by

\[
\frac{1}{18} \langle (\lambda \gamma^m \theta)(\gamma_p \theta)(z_1)(\bar{\lambda} \gamma^m \bar{\theta})(\gamma_m \bar{\theta})(\bar{z}_1)(\lambda \gamma_n \theta)(y_2) \rangle = \frac{1}{18} (G_n)_{\beta\gamma} = (\gamma_n)_{\beta\gamma} \tag{35}\]

and therefore we have shown the equivalence between the amplitudes. Next section we consider the last case, that is, amplitudes with zero fermions.

### 6 Equivalence for Amplitudes Involving Zero Fermions

Now we arrive to the last and by far the more involved point. The case with only open or only closed string bosonic states would also be reduced to the
case already studied at [12]. Therefore we must consider the case with at least one closed and one open string. With the cyclic symmetry for Type I strings we can choose the closed and one open string fixed. Therefore, the amplitude can be written as

$$A = \langle h^{m_1} \tilde{h}^{m_1} V^B_{m_1n_1}(z_1, \bar{z}_1) b_2^{m_2} V^B_{m_2}(y_2) \int dy_3 a^{m_3} U^B_{m_3}(y_3) \int d^2 z_4 h^{n_4} \tilde{h}^{n_4} U^B_{m_4}(z_4) \tilde{U}^B_{n_4}(\bar{z}_4) \times \int dy_5 a^{m_5} U_{m_5} \cdots \int dy_n a^{m_n} U_{m_n} \rangle.$$  

The strategy in this case is to use supersymmetry to simplify the computations. The supersymmetry transformations are given by

$$\{q_\alpha, V^B_m\} = \frac{i}{2} k^\alpha (\gamma_{mn})_{\alpha \beta} V^F_\beta + Q(\Omega_{mn}),$$

$$[q_\alpha, U^B_m] = \frac{i}{2} k^\alpha (\gamma_{mn})_{\alpha \beta} U^F_\beta - \partial(\Omega_{mn}), \quad \{q_\alpha, U^F_\beta\} = \gamma_{\alpha \beta} U^B_m + \partial(\Sigma_{\alpha \beta}),$$

$$[q_\alpha, V^F_{m_1n_1}] = \frac{i}{2} k^\alpha (\gamma_{m_1n_1})_{\alpha \beta} V^F_{\beta n_1} + \frac{i}{2} k^\alpha (\gamma_{n_1})_{\alpha \beta} V^F_{m_1 \beta} + Q(\Omega_{m_1n_1}),$$

$$[q_\alpha, V^F_{\beta n_1}] = \gamma_{\alpha \beta} V^B_{m_1n_1} + \frac{i}{2} k^\alpha (\gamma_{n_1})_{\alpha \beta} \delta V^F_{n_1} + Q(\Sigma_{\alpha \beta n_1}). \quad (36)$$

we can get

$$V^B_{m_1n_1} = \frac{1}{16} \gamma_{m_1n_1}[q_\alpha, V^F_{\beta n_1}] - \frac{i}{32} \gamma_{m_1n_1} \frac{k^\alpha (\gamma_{n_1})_{\alpha \beta} \delta V^F_{n_1} + Q(\Sigma_{\alpha \beta n_1})}. \quad (37)$$

The above expression can be used in the amplitude to replace the fixed closed string. Let us analyze the second term of the right side, which is more simple. After using the argument that the OPEs between the $x$ field is the same for the RNS and PS formalism, we must prove that the part independent of $x$ is equivalent. However, note that in this case we can factorize the left and right movers of $V^B_{\alpha \beta}(z, \bar{z}) = V^F_\alpha(z) V^F_\beta(\bar{z})$ and we get the same case as that with two fermions, but with a different polarization, namely

$$A_2 = -\frac{i}{32} \langle h^{m_1} \tilde{h}^{n_1} \kappa^{p_1}(\gamma_{m_1p_1})_\alpha \gamma V^F_\alpha(z_1) \gamma_{\alpha \beta} V^F_\beta(\bar{z}_1) b_2^{m_2} V^F_{m_2}(y_2) \int dy_3 a^{m_3} U^B_{m_3}(y_3) \times \int d^2 z_4 h^{n_4} \tilde{h}^{n_4} U^B_{m_4}(z_4) \tilde{U}^B_{n_4}(\bar{z}_4) \int dy_5 a^{m_5} U_{m_5} \cdots \int dy_n a^{m_n} U_{m_n} \rangle.$$
For the first term we use the fact that the amplitude is supersymmetric, what will give rise to many terms. We also use again the fact that we just need to consider the part independent of $x$ to use $V_{\beta n_1} = \tilde{V}_\beta^F(z)V_{n_1}^B(z)$. For example, when the operator acts in $y_2$ we get

$$A_1 = -\frac{1}{16}\langle \hat{h}^{m_1}n^1 V_{n_1}^B(z_1)\gamma^{\alpha\beta}_{m_1}V_F^F(\tilde{z}_1)b_2^{m_2}\{q_\alpha, V_{m_2}^B(y_2)\} \int dy_3 a^{m_3} U_{m_3}(y_3) \times$$

$$\int d^2z_4 h^{m_4}n^4 U_{m_4}^B(z_4)\tilde{U}_{\beta n_1}^B(\tilde{z}_4) \int dy_5 a^{m_5} U_{m_5} \cdots \int dy_n a^{m_n} U_{m_n} \rangle,$$

and using again the previous supersymmetry transformation the expression above turns to be the same as the one used to compute the amplitude with two fermions. They only differ in the polarizations. When $q_\alpha$ acts in an integrated operator, $y_3$ for example, we get

$$A = -\frac{1}{16}\langle \hat{h}^{m_1}n^1 V_{m_1}^B(z_1)\gamma^{\alpha\beta}_{m_1}V_F^F(\tilde{z}_1)b_2^{m_2} V_{m_2}^B(y_2) \int dy_3 a^{m_3}\{q_\alpha, U_{m_3}^B(y_3)\} \times$$

$$\int d^2z_4 h^{m_4}n^4 U_{m_4}^B(z_4)\tilde{U}_{m_4}^B(\tilde{z}_4) \int dy_5 a^{m_5} U_{m_5} \cdots \int dy_n a^{m_n} U_{m_n} \rangle,$$

that give us

$$A = -\frac{1}{16}\langle h^{m_1}n^1 V_{m_1}^B(z_1)\gamma^{\alpha\beta}_{m_1}V_F^F(\tilde{z}_1)b_2^{m_2} V_{m_2}^B(y_2) \int dy_3 a^{m_3}\frac{i}{2} k^F_{m_3} (\gamma_{m_3\beta_3})_\alpha^\sigma U^F_\sigma(y_3) \times$$

$$\int d^2z_4 h^{m_4}n^4 U_{m_4}^B(z_4)\tilde{U}_{m_4}^B(\tilde{z}_4) \int dy_5 a^{m_5} U_{m_5} \cdots \int dy_n a^{m_n} U_{m_n} \rangle.$$

Using now the cyclic symmetry we can exchange $V_{m_2}^B(y_2) \int dy_3 U^F_\sigma(y_3)$ for $\int dy_3 V_{m_2}^B(y_3) V^F_\sigma(y_2)$. Then again this amplitude is related to the case with two fermions of the last section. Therefore, any pure spinor amplitude is related to a combination of RNS amplitudes involving $N-2$ bosons. We should show here that this amplitude is related to the $N$ bosons amplitude in this formalism. At this point is must be clear that the argument used in [12] is also valid here. We must focus on an application of these ideas to simplify the tree level three point amplitude in the next section.
7  Simplifying the Three Point Type I Amplitude

In this section we must use the ideas of the last ones to simplify the computation of the three point amplitude computed in [26]. With this technique we will see that no computer program will be needed and so we gain a considerable simplification. For the comparison with the previous result we must choose the measure constant $c = 1$. With this, the identities of the last sections are modified to

$$\frac{1}{27} \langle f_\alpha(z_1)f_\beta(\bar{z}_1)f_\gamma(y_2)p_\delta(y_3) \rangle = \frac{1}{2880} \left\{ \frac{\gamma_{m\delta}\gamma_{m\beta\gamma}}{z_1 - y_3} + \frac{\gamma_{m\delta}\gamma_{m\gamma\alpha}}{\bar{z}_1 - y_3} + \frac{\gamma_{m\delta}\gamma_{m\alpha\beta}}{y_2 - y_3} \right\} \quad (38)$$

and

$$\frac{1}{18} \langle f_\alpha(z_1)f_\beta(\bar{z}_1)b_m(z_3) \rangle = \frac{(\gamma_n)_{\beta\gamma}}{2880} \quad (39)$$

which we must use throughout this section.

7.1  Kalb-Ramond and Two Photinos

In this case we will always have a fermion vertex operator integrated, but using the above identity the result will be obtained directly. In Type I superstring fermion-fermion contribution will give us the Kalb-Ramond field. Therefore this will give us the amplitude for one Kalb-Ramond and two Photinos. This is the simplest amplitude, and using our previous definitions this is given by

$$\mathcal{A} = -\frac{\alpha'}{2 \times 27} \langle \xi^\alpha f_\alpha \bar{\xi}^\beta f_\beta \xi_2^\gamma f_\gamma \int dy \xi_3^\delta p_\delta \rangle.$$

Fixing now the position of the operators, the above amplitude can be written as

$$\mathcal{A} = -\frac{\alpha'}{2 \times 27} \xi^\alpha \bar{\xi}^\beta \xi_2^\gamma \xi_3^\delta \int dy \langle f_\alpha(ia)f_\beta(-ia)f_\gamma(\infty)p_\delta(y) \rangle$$

and from the above identity we get

18
\[ A = -\frac{\alpha'}{2 \times 2880} \xi^\alpha \tilde{\xi}^\beta \xi_2 \xi_3 \int dy \left[ \frac{\gamma^m_{\alpha \delta} \gamma_{m \beta \gamma}}{i a - y} + \frac{\gamma^m_{\beta \delta} \gamma_{m \gamma \alpha}}{-i a - y} \right]. \]

Using residues we arrive at

\[ A = \frac{2 \alpha'}{2880} \pi i \xi^\alpha \tilde{\xi}^\beta \xi_2 \xi_3 \gamma^m_{\alpha \delta} \gamma_{m \beta \gamma}, \]

and using the fact that, in type I superstring

\[ \xi^\alpha \tilde{\xi}^\beta = \frac{1}{96} \gamma^\alpha_{abc} H^{abc} \]

we obtain

\[ A = \frac{i \pi \alpha'}{48 \times 2880} \gamma^\alpha_{abc} H^{abc} \xi_2 \xi_3 \gamma^m_{\alpha \delta} \gamma_{m \beta \gamma}. \]

Using now the identity

\[ \gamma^a \gamma^{bcd} \gamma_a = -4 \gamma^{bcd} \]

we obtain

\[ A = -\frac{i \pi \alpha'}{720 \times 48} H^{abc} \xi_2 \xi_3 \gamma^\alpha_{abc} \xi_4, \]

which agrees with the previous computation for the Kalb-Ramond and two photinos with pure spinor formalism [26].

### 7.2 Two Fermions

In this subsection we will consider all the possibilities that contain two fermions. These are: Kalb-Ramond and two photons; one gravitino/dilatino, one photon and one photino; one graviton/dilaton and two photinos.
7.2.1 Kalb Ramond and two Photons

In this case we have

\[ A = \frac{1}{18} \xi^\alpha \tilde{\xi}^\beta a_2^\alpha a_3^\beta \int dy \langle f_\alpha(ia) f_\beta(-ia) b_n^2(\infty) \rangle \]

and here there is one thing that must be observed. Note that when we contract \( \dot{x} \) of the integrated open string with the unintegrated closed string we get a null result because

\[ \int dy \langle f_\alpha(ia) f_\beta(-ia) b_n^2(\infty) \rangle = 0. \]

and this will be used throughout this whole section. Therefore in the above expression we are left only with the second term and using the OPEs we obtain

\[ A = \frac{1}{18} \xi^\alpha \tilde{\xi}^\beta a_2^\alpha a_3^\beta \times \int dy \langle \left[ -\frac{i\alpha' k_1^m}{4(y - ia)} - \frac{i\alpha' k_1^m}{4(y + ia)} \right] f_\alpha(ia) f_\beta(-ia)] b_n(\infty) \rangle. \]

Using residues we arrive at

\[ A = -\frac{\pi\alpha'}{18} \xi^\alpha \tilde{\xi}^\beta a_2^\alpha a_3^\beta \left\{ \frac{1}{2} k^p_3(\gamma_{mp})^\sigma f_\sigma(ia) f_\beta(-ia) + \frac{1}{2} k^p_3(\gamma_{mp})^\sigma f_\alpha(ia) f_\sigma(-ia) \right\} b_n(\infty) \]

and finally our measure give us

\[ A = -\frac{\pi\alpha'}{18 \times 160} \xi^\alpha \tilde{\xi}^\beta a_2^\alpha a_3^\beta \left\{ \frac{1}{2} k^p_3(\gamma_{mp})^\sigma (\gamma_n)^\sigma_{\sigma\beta} + \frac{1}{2} k^p_3(\gamma_{mp})^\sigma (\gamma_n)^\sigma_{\sigma\alpha} \right\}. \]

Now using the identity

\[ \gamma_{mp} \gamma_n + \gamma_n \gamma_{mp} = 2 \gamma_{mp}, \]

we arrive at
\[ A = -\frac{\pi\alpha'}{18 \times 160} \xi^\alpha \tilde{\xi}^\beta a_2^a a_3^m k_3^p (\gamma_{mnp})_{\alpha\beta}. \]

From the above we can get the amplitudes for a Kalb-Ramond and two Photons. Using

\[ \xi^\alpha \tilde{\xi}^\beta = \frac{1}{96} \gamma_{abc} H_{abc} \]

we finally get

\[ A = -\frac{\pi}{720 \times 4 \times 96} H^{abc} a_n^a a_m^b k_p^c \gamma_{abc} \gamma_{\alpha\beta} = \frac{i\pi}{720} \left( \frac{1}{8} a_2^a F_{bc}^3 H^{abc} \right) \]

a result which agrees with the previous computation for the Kalb-Ramond and two photons with pure spinor formalism [26].

### 7.2.2 One Gravitino/Dilatino, one Photon and one Photino

In order to compute this amplitude we must be careful with the construction of the fixed operator for the gravitino. We need to remember that this operator comes from the product \( \lambda \Lambda \bar{\lambda} \Lambda \). Therefore we are going to have two contributions, namely

\[ \frac{1}{18} \left( \xi^\alpha f_\alpha \tilde{h}^p \tilde{b}_p + \tilde{\xi}^\alpha \tilde{f}_\alpha h^p b_p \right) \]

but because only the zero modes contribute, this is equivalent to consider

\[ \frac{1}{18} (\xi^\alpha \tilde{h}^p + \tilde{\xi}^\alpha h^p) f_\alpha \tilde{b}_p = \frac{1}{9} \tilde{\psi}^{\alpha \rho} f_\alpha \tilde{b}_p. \]

We have considered above the same identification as previously done in [26]. With these considerations the amplitude is given by

\[ A = \frac{1}{9} \xi^\alpha h^p \xi_2^\beta a_3^m \int dy (f_\alpha (ia) b_p (-ia) f_\beta (\infty) [\dot{x}_m - i k_3^p M_{mn}]). \]
As said before, the first term of the integrated open string gives a null result. Using now the previous OPEs we get

$$\mathcal{A} = \frac{1}{9} \tilde{\psi}^{\alpha p} \xi_2^\beta a_3^m \int dy \left\{ \frac{-i\alpha'}{4(y - ia)} k_3^\alpha (\gamma_{mn})_{\sigma} f_{\sigma}(ia) b_p(-ia) f_{\beta}(\infty) - \right\}
$$

$$i\alpha' f_{\alpha}(ia) \frac{k_3^m \eta_{mp} b_m(-ia) - \eta_{mp} b_n(-ia)}{2y + ia} f_{\beta}(\infty)\right\}$$

which integrated with residues gives us

$$\mathcal{A} = -\frac{\pi\alpha'}{9} \tilde{\psi}^{\alpha p} \xi_2^\beta a_3^m$$

$$\langle \frac{1}{2} k_3^m (\gamma_{mn})_{\sigma} f_{\sigma}(ia) b_p(-ia) f_{\beta}(\infty) - f_{\alpha}(ia) k_3^m (\eta_{mp} b_m(-ia) - \eta_{mp} b_n(-ia)) f_{\beta}(\infty) \rangle,$$

and using our measure we arrive at

$$\mathcal{A} = -\frac{\pi\alpha'}{9 \times 160} \tilde{\psi}^{\alpha p} \xi_2^\beta a_3^m \left[ \frac{1}{2} k_3^m (\gamma_{mn})_{\sigma} \gamma_{\rho\sigma\beta} - k_3^n (\eta_{mp} \gamma_{\rho\alpha\beta} - \eta_{mp} \gamma_{\alpha\beta}) \right].$$

At this point we must remember that we can decompose

$$\tilde{\psi}_p^\alpha = \psi_p^{\alpha} + \frac{1}{10} \gamma_p^{\alpha \delta} \lambda_\delta$$

where

$$\psi_p^{\alpha} = (\tilde{\psi}_p^{\alpha} - \frac{1}{10} \gamma_p^{\alpha \delta} \gamma_{\delta\lambda} \tilde{\psi}_p^{\alpha}), \quad \lambda_\delta = \gamma_{\delta\lambda} \tilde{\psi}_p^{\alpha}$$

so that $\psi$ is the traceless gravitino and $\lambda$ is the dilatino. Using this fact, the above amplitude gives us two contributions.

For the gravitino we have

$$\mathcal{A} = -\frac{\pi\alpha'}{9 \times 160} \left[ \frac{1}{2} \psi^{\alpha p} \xi_2^\beta a_3^m k_3^m (\gamma_{mn})_{\sigma} \gamma_{\rho\sigma\beta} - \psi^{\alpha p} \xi_2^\beta a_3^m k_3^n (\eta_{mp} \gamma_{\rho\alpha\beta} - \eta_{mp} \gamma_{\alpha\beta}) \right]$$

and we make use of the identity
The second term of the lhs gives a null contribution because of the gamma traceless property of the gravitino. Therefore we are left with

\[ A = -\frac{\pi i \alpha'}{720} \eta_{mp} \gamma_m \gamma_p. \]

For the dilatino contribution, we just use the identity

\[ \gamma_p \gamma_p = 6 \gamma_{mn}. \]

to get

\[ A = -\frac{\pi i \alpha'}{720} \times 4 F_{mn} \gamma_{n \alpha \beta}. \]

which agrees with the previous computation for the one gravitino/dilatino, one photon and one photino with pure spinor formalism [20].

### 7.2.3 One Graviton/Dilaton and two Photinos

In this case we have

\[ A = -\frac{1}{18} h^\mu h_\nu \xi_2 \xi_3 \int dz \langle [\partial x - ik_1 \gamma_m M_{mn} b_p (-ia) f_\alpha (a) f_\beta (+\infty)] \rangle \]

performing the OPEs we obtain

\[ A = -\frac{1}{18} h^\mu h_\nu \xi_2 \xi_3 \int dz \langle \left[ \frac{i \alpha'}{z - a} k_1^2 f_\alpha (a) - ik_1 \alpha' (\gamma_m \gamma_m) \right] b_p (-ia) f_\beta (+\infty) \rangle. \]

Integrating with residues and using our measurement we get

\[ A = \frac{2 \pi \alpha'}{18 \times 160} h^\mu h_\nu \xi_2 \xi_3 [k_1^2 \gamma_\mu \gamma_\nu + \frac{1}{4} k_1^2 (\gamma_m \gamma_m) \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma]. \]
Now by using the equation of motion, the identity 

\[ \gamma_{mn} \gamma_p = \gamma_{mnp} - \eta_{np} \gamma_m + \eta_{mp} \gamma_n \]

and the fact that just the symmetric part of the polarization contributes, we get

\[ A = \frac{\pi \alpha'}{720 \times 2} \xi_2 \xi_3 k_1^2 (\gamma_p)_{\alpha \beta} = -\frac{\pi i \alpha'}{720 \times 2} \xi_2 \xi_3 \gamma_p \partial_m \xi_2 \]

which agrees with the previous computation for the One graviton/dilaton and two photinos with pure spinor formalism [26].

\section{7.3 One Graviton/Dilaton and two Photons}

This is the last case to be considered. The amplitude is given by

\[ A = h^p \tilde{h}^q a^r a^m \int dy \langle V^B_p (ia) \nabla^B_q (-ia) V^B_r (\infty) U^B_m (y) \rangle. \]

As argued before, this will give rise to three terms. The first one is given by

\[ A_1 = -\frac{1}{16} h^p \tilde{h}^q a^r a^m \int dy \langle i k_1 (\gamma_{ps})_{\alpha} \sigma^F \gamma^F (ia) \gamma^\alpha q \gamma^\beta q V^F (\infty) \rangle, \]

and using our previous result for the Kalb-Ramond and two photons we get

\[ A_1 = -\frac{\pi \alpha'}{16 \times 18 \times 160} h^p \tilde{h}^q a^r a^m \frac{i}{2} k_1^2 (\gamma_{ps})_{\alpha} \gamma^\alpha q \gamma^\beta q k_3 (\gamma_{mrt})_{\alpha \beta}. \]

Using now the identity

\[ Tr(\gamma_{ps} \gamma_{mrt}) = -96 \delta_{mrt} \]

and the fact that the metric is symmetric we see that the above gives a null result.

The next term is given by

\[ \text{...} \]
\[ A_2 = -\frac{1}{16} \tilde{h} q a_2 a_3^m \int dy \langle V^B_p(ia) \gamma_q^\alpha \gamma^F_\alpha \{ q_\alpha, V^B_r(\infty) \} U^B_m(y) \rangle \]

\[ = -\frac{1}{16} \hbar q a_2 a_3^m \int dy \langle V^B_p(ia) \gamma_q^\alpha \gamma^F_\alpha \{ q_\alpha, V^B_r(\infty) \} U^B_m(y) \rangle \]

and using our previous result we get

\[ A_2 = -\frac{\pi \alpha'}{16 \times 18 \times 160} \hbar q a_2 a_3^m \langle \gamma_q^\alpha \gamma^F_\alpha \{ q_\alpha, U^B_m(y) \} \rangle \]

Using the identities

\[ \gamma^q_{\gamma r s} \gamma^p_{\gamma r s} = -32 \delta^q_{p r} \delta^{mn} + 32 \delta^q_{p s} \delta^{mn} - 32 \delta^p_{q r} \delta^{mn} + 32 \delta^p_{q s} \delta^{mn} - 32 \delta^p_q \delta^m r s \]

and

\[ \gamma^q_{\gamma r s} \gamma^m = 32 \delta^m_{r s} \]

and using the fact that \( \hbar^p = 4\Phi \) we finally get

\[ A_2 = -\frac{\pi i \alpha'}{720} \left[ -\frac{1}{4} \hbar q F^p_{r q} F^3_{r p} - \frac{1}{8} \Phi F^p_{r q} F^3_{r s} \right] \]

The last term is given by

\[ A_3 = -\frac{1}{16} \hbar q a_2 a_3^m \int dy \langle V^B_p(ia) \gamma_q^\alpha \gamma^F_\alpha \{ q_\alpha, V^B_r(\infty) \} U^B_m(y) \rangle \]

\[ = -\frac{1}{16} \hbar q a_2 a_3^m \int dy \langle V^B_p(ia) \gamma_q^\alpha \gamma^F_\alpha \{ q_\alpha, V^B_r(\infty) \} U^F_m(y) \rangle \]

and using the cyclic symmetry we get
This is identical to our last term if we exchange $k_1 \leftrightarrow k_2$. Therefore we get

$$A_3 = \frac{\pi \alpha'}{720} \left[ -\frac{1}{4} h^p q_{3} F^2_{rp} - \frac{1}{8} \Phi F^2_{3 s} F^2_{rs} \right],$$

and adding the results we obtain

$$A_3 = \frac{\pi \alpha'}{720} \left[ -\frac{1}{2} h^p q_{3} F^2_{rp} - \frac{1}{4} \Phi F^2_{3 s} F^2_{rs} \right],$$

which finally agrees with the previous computation for the one graviton/dilaton and two photons with pure spinor formalism [26].

### 8 Conclusions

In this manuscript we have made a proof for the equivalence between the RNS and Pure Spinor formalism for a class of Type I tree level amplitudes. For this we have constructed a proof for the cyclic symmetry of superstring Type I amplitudes using a BRST approach. This symmetry was used previously in Ref. [26]. That was established for amplitudes involving up to four fermions. With this at hand, we have all the terms of the Type I effective action obtained using the RNS formalism which involves these kind of amplitudes. The amplitude involving one closed and two open strings was computed previously [26] with the Pure Spinor formalism. This result was shown to be correct by comparing with the effective action for Type I Supergravity. With the present result explicit computations are not needed anymore. This result also gives us a result of particular importance. Differently from amplitudes involving only closed or open strings, the Type I superstring has a gauge anomaly that is manifest already at tree level. Diagrams in which a two form is exchanged between two gauge fields on one side and four on the other side have to be considered for this cancellation. This amplitude involves two fermions and six bosons and so is obtained from our present result. Therefore we get as a byproduct of this manuscript the computation of this amplitude.
Recently, some important results regarding amplitude computations have been obtained. First we can mention a relation between disk amplitudes involving \( N_o \) open and \( N_c \) closed strings and disk amplitudes with only \( N_o + 2N_c \) open strings [31]. With this, and our results we can obtain many explicit amplitudes from the already known for Pure Spinor. Between these we can cite the six point amplitude computed recently [22], that could give us all the terms of the effective action involving one closed and four open or two closed and two open. Results of a more far reaching importance is the computation of any number of open strings at tree level [23]. With this we also gain as a byproduct the computation of any amplitude involving any number of open and closed string that contain up to four fermions. Posteriorly the result above was used to compute Supergravity and Super-Yang-Mills amplitudes. As a perspective of the present work the authors expect to reach similar results involving Type I Supergravity amplitudes.

Acknowledgment

The authors would like to thank Brenno Carlini Vallilo, Nathan Berkovits and C. R. Mafra for useful discussions. We would also like to thank: The Goethe-Institut Berlin for the hospitality. We acknowledge Linda Fuchs for her help with the manuscript, and the financial support provided by Fundação Cearense de Apoio ao Desenvolvimento Científico e Tecnológico (FUNCAP), the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and FUNCAP/CNPq/PRONEX.

References

[1] N. Berkovits, “Super-Poincare covariant quantization of the superstring,” JHEP 0004, 018 (2000) [arXiv:hep-th/0001035].

[2] N. Berkovits, “Quantum consistency of the superstring in AdS(5) x S**5 background,” JHEP 0503, 041 (2005) [arXiv:hep-th/0411170].

[3] B. C. Vallilo, “Flat currents in the classical AdS(5) x S**5 pure spinor superstring,” JHEP 0403, 037 (2004) [arXiv:hep-th/0307018].
[4] B. C. Vallilo, “One loop conformal invariance of the superstring in an AdS(5) x S5 background,” JHEP **0212**, 042 (2002) [arXiv:hep-th/0210064].

[5] M. Bianchi and J. Kluson, “Current Algebra of the Pure Spinor Superstring in AdS(5) x S(5),” JHEP **0608**, 030 (2006) [arXiv:hep-th/0606188].

[6] N. Berkovits, “A New Limit of the AdS(5) x S**5** Sigma Model,” JHEP **0708**, 011 (2007) [arXiv:hep-th/0703282].

[7] N. Berkovits and O. Chandia, “Superstring vertex operators in an AdS(5) x S**5** background,” Nucl. Phys. B **596**, 185 (2001) [arXiv:hep-th/0009168].

[8] O. A. Bedoya, L. I. Bevilaqua, A. Mikhailov and V. O. Rivelles, Nucl. Phys. B **848**, 155 (2011) [arXiv:1005.0049 [hep-th]].

[9] O. A. Bedoya and O. Chandia, JHEP **0701**, 042 (2007) [arXiv:hep-th/0609161].

[10] O. A. Bedoya, JHEP **0809**, 078 (2008) [arXiv:0807.3981 [hep-th]].

[11] O. A. Bedoya, [arXiv:0808.1755 [hep-th]].

[12] N. Berkovits and B. C. Vallilo, “Consistency of super-Poincare covariant superstring tree amplitudes,” JHEP **0007**, 015 (2000) [arXiv:hep-th/0004171].

[13] N. Berkovits, “Multi-loop amplitudes and vanishing theorems using the pure spinor formalism for the superstring,” JHEP **0409**, 047 (2004) [arXiv:hep-th/0406055].

[14] N. Berkovits, “Super-Poincare covariant two-loop superstring amplitudes,” JHEP **0601**, 005 (2006) [arXiv:hep-th/0503197].

[15] C. R. Mafra, “Four-point one-loop amplitude computation in the pure spinor formalism,” JHEP **0601**, 075 (2006) [arXiv:hep-th/0512052].

[16] N. Berkovits and C. R. Mafra, “Equivalence of two-loop superstring amplitudes in the pure spinor and RNS formalisms,” Phys. Rev. Lett. **96**, 011602 (2006) [arXiv:hep-th/0509234].
[17] C. R. Mafra, “Pure Spinor Superspace Identities for Massless Four-point Kinematic Factors,” JHEP 0804, 093 (2008) [arXiv:0801.0580 [hep-th]].

[18] N. Berkovits and C. R. Mafra, “Some superstring amplitude computations with the non-minimal pure spinor formalism,” JHEP 0611, 079 (2006) [arXiv:hep-th/0607187].

[19] C. R. Mafra, “Simplifying the Tree-level Superstring Massless Five-point Amplitude,” JHEP 1001, 007 (2010) [arXiv:0909.5206 [hep-th]].

[20] C. R. Mafra, “Towards Field Theory Amplitudes From the Cohomology of Pure Spinor JHEP 1011, 096 (2010) [arXiv:1007.3639 [hep-th]].

[21] I. Y. Park, JHEP 1009, 008 (2010) [arXiv:1003.5711 [hep-th]].

[22] C. R. Mafra, O. Schlotterer, S. Stieberger and D. Tsimpis, “Six Open String Disk Amplitude in Pure Spinor Superspace,” Nucl. Phys. B 846, 359 (2011) [arXiv:1011.0994 [hep-th]].

[23] C. R. Mafra, O. Schlotterer, S. Stieberger and D. Tsimpis, “A recursive formula for N-point SYM tree amplitudes,” [arXiv:1012.3981 [hep-th]].

[24] H. Gomez, JHEP 0912, 034 (2009) [arXiv:0910.3405 [hep-th]].

[25] H. Gomez and C. R. Mafra, JHEP 1005, 017 (2010) [arXiv:1003.0678 [hep-th]].

[26] G. Alencar, “Type I Supergravity Effective Action from Pure Spinor Formalism,” JHEP 0902, 025 (2009) [arXiv:0812.4201 [hep-th]].

[27] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, Nucl. Phys. B 195, 97 (1982).

[28] E. Cartan, “The theory of spinors,” (1966).

[29] J. Polchinski, “String theory. Vol. 1: An introduction to the bosonic string.”

[30] D. Friedan, E. J. Martinec, S. H. Shenker, Nucl. Phys. B271, 93 (1986).

[31] S. Stieberger, [arXiv:0907.2211 [hep-th]].