Effective theory of excitations in a Feshbach resonant superfluid

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In the last few years, the rapid development of experiments of atomic Fermi gas using Feshbach resonance has revitalized theoretical interest of strongly interacting many-body system. Recent experiments, for instance, the collective excitation $^{19}$F and heat capacity $^{8}$Be, point to the need of a dynamical theory beyond mean-field BEC-BCS crossover theory. A basic difficulty with the crossover theory is that perturbative expansion is not justifiable if going to next order for strong interaction. Various efforts have been made on the basis of hydrodynamic theory $^{1}$, $^{2}$, $^{3}$, $^{4}$, $^{5}$, which in general explain the frequency of collective modes particularly well on the resonance point. A disadvantage of hydrodynamic theory is that fermionic excitation is hard to incorporate beyond the density and velocity field approximation. In this paper, I develop an effective field theory for the resonant superfluid that retains both fermionic and bosonic excitations explicitly based on broken symmetry. An advantage of this theory over a pure bosonic theory is that perturbative expansion is not justifiable if going to next order for strong interaction.

For attractive interaction $g$, one can perform an exact Hubbard-Stratonovich transformation to introduce the auxiliary pair fields, $\Delta(x)$, and its complex conjugate. Then, the Lagrangian becomes

$$L = \int d^3x \left\{ \psi_\sigma^* \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi_\sigma + \frac{1}{2} \frac{\Delta^2}{g^2} + \text{c.c.} \right\}. \quad (2)$$

Up to this point, no approximation has been made. One may proceed then to derive the gap and chemical equations, following the BCS-BEC crossover theory procedure (for instance, Ref. $^{11}$ and references therein). But we are interested in the dynamics of low energy excitations about the superfluid state.

The atomic gas is in strong coupling regime, having a scattering length greater than or comparable to the Fermi wavelength $k_F^{-1}$ ($|k_F a| \gtrsim 1$). Conventional perturbation theory does not directly apply, at least not in a standard, controllable manner. To have a controlled theory for the description of quantum dynamics beyond mean field theory, one may seek for available different approach.

The broken symmetry of the superfluid state provides such an approach—effective field theory $^{12}$. The atomic gas conserves the total number of particles. This conservation law corresponds to a U(1) symmetry of the phase transformation. For the above model, the U(1) symmetry transformation is

$$\psi_\sigma(x) \rightarrow e^{i\alpha} \psi_\sigma(x), \quad \Delta(x) \rightarrow e^{i2\alpha} \Delta(x), \quad (3)$$

with $\alpha$ an arbitrary, constant phase.

Recall that the superfluid state is a Bose-Einstein condensate of fermionic atom pairs, which breaks the U(1) symmetry. The order parameter acquires a finite value below the superfluid phase transition temperature, $\langle \Delta(x) \rangle = \Delta_0 \neq 0$. Fluctuations about this superfluid ground state are excitations.

Here I derive an effective theory of the phase fluctuation, directly based on the broken symmetry. We start by treating $\Delta_0$—the groundstate or thermal mean value of the order parameter—as an input parameter. It may
be taken directly from experiments; of course it can also be taken from the result of Monte Carlo, or any mean field crossover theory but that will require further justification. We are concerned with phase fluctuations in space and time, so write the order parameter field

\[ \Delta(x) = \Delta_0 e^{i\theta(x)} \]  

(4)

(The factor 2 is inserted to make it explicit that the pair field carries two units of mass.) We have set to ignore the amplitude fluctuation for the purpose of simplicity; this is justifiable for it is known gapped and, as can be checked below, does not directly coupled to phase except this is justifiable for it is known gapped and, as can be checked below, does not directly coupled to phase except

\[ \psi_\sigma(x) = \tilde{\psi}_\sigma(x) e^{i\theta(x)}, \quad \psi_\sigma^*(x) = \tilde{\psi}_\sigma^*(x) e^{-i\theta(x)}. \]  

(5)

The transformation is designed to eliminate the phase fluctuation \( \theta(x) \) dependence from the off-diagonal pairing potential terms, \( (\psi_\sigma^* \psi_\tau^* \Delta + c.c.) \). As a result, the \( \theta(x) \)-dependence can only arise from the kinematic terms of the fermion theory sector. Collecting all terms of \( \tilde{\psi} \) and \( \theta \), we have

\[ \mathcal{L}_{\tilde{\psi}{\theta}} = \tilde{\psi}_\sigma^*(D_\nu - \mu - \frac{\nabla^2}{2m}) \tilde{\psi}_\sigma + (\Delta_0 \tilde{\psi}_\uparrow \tilde{\psi}_\downarrow + c.c.) \]  

(6)

where \( D_\nu = \partial_\nu + i\partial_\nu \theta(x) \), \( \nu = \tau, x, y, z \). The U(1) symmetry guarantees the phase field to appear only through derivatives, since energy cannot depend on a constant phase. The form of coupling between the phase and fermion fields is exact from the model. This theory is no perturbation, apart from excluding the amplitude fluctuation.

For those who are familiar with gauge theory, the time and spatial derivatives of \( \theta(x) \) together behave like a four component gauge field. After the transformation \[ \tilde{\psi}_\sigma \], the fermions \( \psi_\sigma \) see a constant off-diagonal pairing potential \( \Delta(x) = \Delta_0 \) but interact with the (Goldstone) phase field \( \theta(x) \) by a means similar to ‘gauge’ coupling.

*Landau superfluid hydrodynamics.* The effective theory [6] has a very strong implication for superfluid hydrodynamics. To reveal that, let us collect all terms of first order in \( \theta \) from \( \mathcal{L}_{\tilde{\psi}{\theta}} \):

\[ + \hat{n}(x) i\partial_\nu \theta(x) + \nabla \theta(x) \cdot \hat{J}(x) \]  

(7)

where \( \hat{n}(x) = \tilde{\psi}_\sigma^*(x) \tilde{\psi}_\sigma(x) = \psi_\sigma^*(x) \psi_\sigma(x) \) and \( \hat{J}(x) = -\frac{i}{m} (\tilde{\psi}_\sigma^* \nabla \tilde{\psi}_\sigma - \nabla \tilde{\psi}_\sigma^* \tilde{\psi}_\sigma) \) are the fermion density and current operators. The first term implies that the fermion *total* density and the order parameter phase are canonically conjugate. The amplitude fluctuation of the order parameter is separate. [This can be checked by shifting \( \Delta_0 \) by a quantum fluctuation \( \Delta'(x), \Delta_0 \to \Delta_0 + \Delta'(x). \)]

To derive the hydrodynamic theory, we add to \( \mathcal{L}_{\tilde{\psi}{\theta}} \) the term

\[ \Delta \mathcal{L} = -i\lambda (\rho - \psi_\sigma^0 \psi_\sigma) \]  

(8)

and do path integrals over the new fields \( \rho(x), \lambda(x) \). This is allowed because the path-integral over \( \lambda(x) \) gives a \( \delta \)-functional, which sets the equation

\[ \rho(x) = \psi_\sigma^*(x) \psi_\sigma(x), \]  

(9)

and the path integral over \( \rho(x) \) gives a constant. The equation justifies \( \rho(x) \) as fermion density field. The added term \( \Delta \mathcal{L} \) is designed to extract the effective density field. Up to now the transformation of \( \mathcal{L}_{\tilde{\psi}{\theta}} \) is exact. Performing path integrals over fermion fields produces the effective action:

\[ S = S_{\rho\theta} + S_\lambda \]  

(10)

\[ S_{\rho\theta} = \int_x \left[ i\partial_\rho \theta + \left( \frac{\nabla \theta}{2m} \right)^2 \right] + \int_x \partial_j (x - x') \partial_{x'} \theta \partial_x \theta \right) \]  

(11)

\[ S_\lambda = \int_x i\lambda \Delta - \frac{i}{2} \int_x \partial_j (x - x') \lambda(x) \lambda(x') \]

where ‘...’ stands for higher powers of \( \theta \) and/or \( \lambda \) fields. Here, \( \mathcal{D} \) and \( \mathcal{P} \) are the density and current correlation functions defined in the homogeneous ground state (so the subscript ‘0’):

\[ \mathcal{D}(x - x') = -\langle \hat{n}(x) \hat{n}(x') \rangle_0 \]  

(12)

\[ \mathcal{P}_{ij}(x - x') = -\langle \hat{J}_i(x) \hat{J}_j(x') \rangle_0, \quad i, j = x, y, z. \]  

(13)

This is a low energy (derivative) expansion of the collective fields, rather than a perturbation in some coupling constant. The derivative nature of \( \lambda(x) \) is less obvious, but immediately becomes manifest when examining the quantum equation of motion,

\[ \frac{\delta S}{\delta \rho} = 0 \Rightarrow \lambda = -\partial_\tau \theta + \frac{i}{2} (\nabla \theta)^2. \]

Therefore the power expansion on \( \lambda \) is equivalent to a low energy expansion. The field \( \lambda \) is auxiliary in nature. We can calculate the effective couplings of \( \theta \) and \( \rho \) mediated by \( \lambda(x) \) perturbatively in low energy limit. The resulting effective action is the equivalent of Landau superfluid hydrodynamics:

\[ S'_{\rho,\theta} = S_{\rho\theta} - \frac{1}{2} \int_{x'x} \mathcal{D}(x - x')^{-1} \rho(x) \rho(x') \]  

(14)

where \( S_{\rho\theta} \) is the same as given above.

*Effective theory of phase fluctuations.* Single fermion excitations are gapped at all energies below \( \Delta_0 \). The Goldstone boson \( \theta(x) \) interacts with fermion through particle-hole pairs. Therefore, for collective excitations of energy below \( 2\Delta_0 \), we can integrate out all fermionic degrees of freedom and have an effective theory for \( \theta \) alone. Starting from the theory \[ \tilde{\psi}_\sigma \], we write it into a quadrature of fermion fields in the Nambu spinor space, i.e.,

\[ \int d^3x d\tau \langle \psi_\uparrow(x), \psi_\downarrow(x) \rangle K_{x,x'} (\psi_\uparrow(x'), \psi_\downarrow(x')) \]  

(15)
with
\[ K_{x,x'} = [K^0(x) + K^\delta(x)] \delta(x - x') \delta(\tau - \tau'), \tag{16} \]
\[ K^0(x) = i \partial_\tau - \tau_3 \left( \frac{\nabla^2}{2m} + \mu \right) + \tau_1 \Delta_0, \tag{17} \]
\[ K^\delta(x) = \frac{1}{2} \left( \nu + \partial_\theta \cdot \nabla \right) \frac{\nu - \nabla \theta \cdot \nabla}{2m} + \tau_3 i \partial_\tau \theta, \tag{18} \]
where \( \{1, \tau_1, \tau_2, \tau_3\} \) are the identity and Pauli matrices for the Nambu spinor space. Integrating out fermion fields gives the effective action for the phase mode \( S_\theta = \int d^3 x \, \text{tr} \ln K_{x,x} \).

After some straightforward calculation, the effective action of the theory is found to be (in momentum-frequency space)
\[ S_\theta = \sum_q \left[ - \frac{\mathcal{D}(q)}{2} \omega^2 + \frac{\mathcal{P}(q)}{2} q^2 + \frac{n_0}{2m} q^2 \right] \theta(q)^* \theta(q) + O\left( \frac{q^4}{(2\Delta_0)^2} \right) \text{ (higher powers of } \theta) \quad \tag{19} \]
where \( n_0 \) is the total fermion density and
\[ \mathcal{D}(q) = \frac{1}{\beta V} \sum_p \text{tr} \left( \tau_3 G(p) \tau_3 G(p + q) \right), \quad q \equiv (q, i\omega) \]
\[ \mathcal{P}(q) = \frac{1}{\beta V} \sum_p \text{tr} \left\{ G(p - \frac{q}{2}) G(p + \frac{q}{2}) \right\}, \quad p \parallel \parallel q, \]
\[ G(p) = \frac{i\omega \tau_3 + \tau_1 \Delta_0}{(i\omega - E_{p})(i\omega - E_{p'})}, \quad p \equiv (p, i\omega'), \]
with \( \epsilon_p = \frac{\nu^2}{2m} - \mu \) and \( E_p = \sqrt{\epsilon_p^2 + \Delta_0^2} \). \( E_p \) is the fermionic quasiparticle spectrum. We are working in the imaginary time formalism: \( \omega \) and \( \omega' \) are the Matsubara frequency for bosons and fermions, respectively; \( \beta V \) = space-time volume; and \( G(p) \) is the fermion propagator. For readers who are familiar with diagrammatic calculation, \( \mathcal{D}(q) \) and \( \mathcal{P}(q) \) are directly related to the density and current correlation functions defined in Eqs. \( 12 \) and \( 13 \) by Fourier transformation.

One can directly evaluate \( \mathcal{D}(q) \) and \( \mathcal{P}(q) \). The calculation is tedious but otherwise elementary. At zero temperature \( T = 0 \), the result is
\[ \left( \begin{array}{c} \mathcal{D}(q) \\ \mathcal{P}(q) \end{array} \right) = \int_p \left( \begin{array}{c} \frac{1}{\beta V} p_x^2 \\ 0 \end{array} \right) \left[ 1 - \epsilon_+ \epsilon_- \pm \Delta_0^2 \right] \frac{E_+ - E_-}{(i\omega - E_+ - E_-)(i\omega + E_+ + E_-)}, \tag{20} \]
where \( \int_p = \int d^3 p \frac{d^3 p}{(2\pi)^3} \), and the subscript \( \pm \) stands for \( \epsilon_\pm \equiv \epsilon_p \pm q/2 \) and likewise for \( E_\pm \). (Note the important sign difference in the ‘coherence factor’ between \( \mathcal{D} \) and \( \mathcal{P} \).)

Here to an important point regarding the perturbative expansion. The expansion is sorted by the powers of momentum and frequency of the field \( \theta(q, i\omega) \). Unlike the Gaussian fluctuation theories, the present approach does not require the phase fluctuation field itself small but slowly varying in space and time. This approach is also quantitative: The low energy and momentum scales are set by \( \Delta_0 \) and \( \Delta_0/v_F \), respectively; One can improve the theory order by order and estimate the next order correction. A more complete development of this approach will be given in a future study. The essential point of why this approach works is that the phase fluctuation (Goldstone) field must appear through derivatives dictated by the broken U(1) symmetry.

For those familiar with field theory, I should be careful about approximation in the expansion of \( 19 \). The general form is no approximation, but the coefficients \( \mathcal{D} \) and \( \mathcal{P} \), however, could not be derived in any ‘honest’ calculation for strongly interacting gas if we had included the amplitude fluctuation too. So the coefficients should be treated as good as mean field approximation. Nowadays, a more typical application of Weinberg effective field theory in particle physics is to directly construct ingredients of proper symmetry transformation, write a general form of effective Lagrangian, and match the unknown coefficients with experiments rather than derive. We will not discuss that general approach here.

The \( \omega_q \ll 2\Delta_0 \) limit. This corresponds to the limit of long wavelength, \( q \to 0 \). Expanding the \( \mathcal{D} \) and \( \mathcal{P} \) of Eq. \( 20 \) in powers of \( q^2 \), we get the leading order
\[ \mathcal{D}(q = 0) = - \int d^3 p \sum_{\mathcal{P}} \frac{\Delta^2}{(2\pi)^3} \frac{\Delta}{v_F} = \frac{\Delta^2 m k_F}{2}, \tag{21} \]
\[ \mathcal{P}(q = 0) = 0, \tag{22} \]
where \( \Delta = \Delta_0/\epsilon_F \), \( \mu = \mu \epsilon_F \), and \( \mathcal{P} = \int_0^\infty dx \frac{\Delta}{(x^2 - \mu^2 + \Delta^2)^{1/2}} \), \( \epsilon_F \) is defined as the Fermi energy at the non-interacting limit. The difference between \( \mathcal{D} \) and \( \mathcal{P} \) can be seen from the coherence factor in their expressions. In the case of \( \mathcal{P}(q) \), the factor is exactly vanishing at \( q = 0 \).

Then the effective action of the phase mode becomes
\[ S_\theta = \frac{1}{2} \sum_q \left[ \frac{\Delta^2 m k_F}{2 v_F^2} \frac{\omega^2}{2} + \frac{\Delta^2}{(2\Delta_0)^2} \right] \theta(q) \theta(q) + O\left( \frac{q^4}{(2\Delta_0)^2} \right). \tag{23} \]

The energy spectrum of the collective mode is obtained by analytical continuation \( i\omega \to \omega_q + i0^+ \) and by finding the zero of the coefficient of the quadratic \( \theta \)-term. The spectrum is
\[ \omega_q = v_s |q| + O\left( \frac{|q|^3}{\Delta^3} \right), \quad v_s = \frac{\epsilon_F}{\sqrt{3\Delta^2}}, \tag{24} \]
where \( v_s \) is the sound velocity.

For small \( \Delta_0 \) compared with the Fermi energy \( \epsilon_F \) (BCS limit), \( \mathcal{F} \approx \frac{\Delta_0}{\epsilon_F} \left[ 1 + O(\Delta_0^3) \right] \). Using the relation \( k_F = (3\pi^2 n_0)^{1/3} \) and \( v_F = k_F/m_F \), this reproduces the sound velocity \( v_s = v_F / \sqrt{3} \) the well-known result in the BCS limit obtained in the theories of Bogoliubov, Anderson, and others later on \( 11, 13 \).
Specific Heat. Recently, Kinast et al \[8\] reported a measurement of the heat capacity of a strongly interacting Fermi gas of $^6$Li atoms, slightly detuned to the BCS side from the resonance. The heat capacity reveals a kink at a temperature about $T = 0.27T_F$, arguably an evidence of superfluid phase transition. While the crossover theory employed in the paper agrees with the data in general, it deviates from the data noticeably at low temperature below the transition, as shown in their figure of energy input versus temperature.

As an example of application, I will use the effective field theory \[24\] to calculate the specific heat for temperature well below the superfluid critical temperature $T_c$ ($T \ll T_c \sim 2\Delta_0/k_B$). All fermionic quasiparticle excitations are gapped, so the only contribution to the entropy must be due to the collective superfluid phonons with a linear dispersion. The thermal energy per unit volume due to the phonons is $E = \frac{\pi^2(k_B T)^3}{30\pi^2}$. The coefficient factor (including $v_s$) is model dependent; but the $T^4$ power law is universal in the limit of $T \ll T_c$. Differentiating the thermal energy gives the specific heat, $c = \frac{2\pi^2}{3\sqrt{3}}k_B T^3$. By measuring the coefficients of the $T^3$ law, one can determine the sound velocity and check the theoretical result. Fig. 1 shows how $v_s$ and $c$ vary with $\Delta_0$ between weak and strong coupling limits—a result seeming not reported anywhere else yet.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1}
\caption{Sound velocity and specific heat in a resonant fermionic superfluid. The chemical potential is self-consistently determined when varying $\Delta_0$ with density fixed. Parameters: $k_F = (3\pi^2n_0)^{1/3}$; $\epsilon_F = k_F^2/(2m)$; $v_F = k_F/m$. The BEC limit (molecular side) is plotted to just show the trend; otherwise it is beyond the scope of validity of the present theory.}
\end{figure}

The $T^4$ power law of the energy differs from the empirical $E \propto T^{3.73}$ \[3\]. Bulgac \[14\] examined the role of collective excitations by focusing on the unitary regime and predicted a complicated $T$-dependence in different temperature regime, from exponential to $E \propto T^5$. Here I provide an explanation other than those discussed in Ref. 3, 14. Away from the limit of $T \ll T_c$, we now move to the temperature regime slightly below $T_c$ ($0 \ll T < T_c$). There can be preformed Cooper pairs but the fraction of condensed Cooper pairs is small. In our effective field theory description, that corresponds to a small amplitude of order parameter, $\Delta_0$. The superfluid phonon still exists but loses the linear dispersion characteristic. This fact is actually implicit in Eq. 24 for which the expansion ought to include nonlinear powers of $|q|/\Delta_0$ when $\Delta_0$ is relatively small. In fact, for temperature near $T_c$, the expansion should be done another way around, in terms of small $\Delta_0/|q|$ instead. At the end, the superfluid phonon and non-condensed Cooper pair, coupled together, all are expected to have a dispersion quadratic in momentum, $\omega_q \sim q^2$. Such a Bose gas gives a temperature dependence as $E \propto T^{5/2}$ by simple power counting. For strongly interacting gas, fermionic excitations are believed to have a pseudogap far greater than $\Delta_0$—sometimes called coherent superfluid gap. Therefore fermionic excitations do not alter the above power law. (There might be contributions due to gapless fermions on the trap edge if the effect of trap were included, but that seem not as important in the thermodynamic limit.) In summary, the power law is $T^4$ for $T/T_c \to 0$ and is argued to be $T^{5/2}$ for $0 \ll T \lesssim T_c$. Perhaps, in my opinion, an interpolation of these two limits can fit the data better, entirely consistent with the empirical $T^{3.73}$ law from a single-exponent fit.

All results apply on the fermionic side of a broad Feshbach resonance, when $\Delta_0 > 0$ (the larger the better, contrary to weak perturbation theory). One may easily generalize our approach to develop a more complete effective theory valid for the molecular side as well, by starting with a Bose-Fermi resonance model of both atoms and molecules.

Finally, I learned that the low energy expansion was used long ago to derive time-dependent Ginzburg-Landau theory for a BCS superconductor by Abrahams and Tsumeto \[15\]. But their expansion was done in a very different manner and assumed the (weak coupling) BCS limit which was quite alright at that time. Another useful difference is that the present effective theory, having retained fermions explicitly, is convenient to study, for example, the damping of collective modes due to decay into (fermionic) quasiparticles.

I am grateful for discussions with D. Son, J. Thomas, and F. Zhou. I thank the Institute for Nuclear Theory at the University of Washington for its hospitality through the Program on Quantum Liquids and Gases and the Department of Energy for partial support during the completion of this work.

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