Khovanov homotopy types and the Dold-Thom functor

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Abstract. We show that the spectrum constructed by Everitt and Turner as a possible Khovanov homotopy type is a product of Eilenberg-MacLane spaces and is thus determined by Khovanov homology. By using the Dold-Thom functor it can therefore be obtained from the Khovanov homotopy type constructed by Lipshitz and Sarkar.

A Khovanov homotopy type is a way of associating a (stable) space to each link $L$ so that the classical invariants of the space yield the Khovanov homology of $L$. There are two recent constructions of Khovanov homotopy types, using different techniques and giving different results [3, 6]. In [3] homotopy limits were employed to build an $\Omega$-spectrum $X_L = \{X_k(L)\}$ with the following properties:

(i). the homotopy type is a link invariant, and  
(ii). the homotopy groups are Khovanov homology: 

\[ \pi_i(X_k(L)) = Kh^{-i}(L). \]

The main goal of this note is to prove the following result.

Theorem 1. Each of the spaces $X_k(L)$ is homotopy equivalent to a product of Eilenberg-MacLane spaces.

In [6] the programme of Cohen, Jones and Segal [2] was generalized to produce a suspension spectrum $X_{Kh}(L)$ with the following properties:

(i). the homotopy type is a link invariant, and  
(ii). the reduced cohomology is Khovanov homology: 

\[ \tilde{H}^i(X_{Kh}(L)) = Kh^i(L). \]

As a corollary we obtain that $X_\bullet(L)$ is homotopy equivalent to the infinite symmetric product of $X_{Kh}(L)$.

To prove Theorem 1 we use the explicit model, due to McCord [8], of the Eilenberg-MacLane spaces. Given a monoid $G$ and a based topological space $X$, let $B(G,X)$ denote the set of maps

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Proposition 1. [8 Proposition 6.7] McCord’s construction is a bifunctor

\[ B(-, -) : \text{Ab} \times \text{Top}_* \rightarrow \text{AbTop} \]

Furthermore, as special case of [8 Theorem 11.4], for an abelian group \( G \) the space \( B(G, S^n) \) is the Eilenberg-MacLane space \( K(G, n) \). Thus we may take as the Eilenberg-MacLane space functor:

\[ B(-, -) : \text{Ab} \rightarrow \text{AbTop} \]

Conversely, the following is [4 Corollary 4K.7, p. 483] (apparently originally due to Moore; cf. [8 p. 295]):

Proposition 2. A path-connected, commutative topological monoid is a product of Eilenberg-MacLane spaces.

The spaces \( X_k(L) \) are built as homotopy limits of diagrams of spaces. Recall that given a small category \( C \) and a (covariant) functor \( D : C \rightarrow \text{Top}_* \) (a diagram), that \( \text{holim}_C D \) is constructed as follows (see, e.g., [1 Section 11.5] or the concise notes [9 Section 3.7]). Consider the product

\[
\prod_{\sigma \in N(C)} \text{Hom}(\Delta^n, D(c_{\sigma})) = \prod_{n \geq 0} \prod_{\alpha_1, \ldots, \alpha_n : \alpha_i \neq \text{Id}} \text{Hom}(\Delta^n, D(c_{\sigma}))
\]

(1)

where \( N(C) \) is the subset of the nerve of \( C \) consisting of all sequences of composable morphisms \( \sigma = (c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} c_n) \) in which none of the morphisms are identity maps, and \( \text{Hom} \) denotes the space of continuous maps from the standard \( n \)-simplex. The homotopy limit \( \text{holim}_C D \) is the subspace of this product consisting of those tuples \( (f_\sigma)_{\sigma \in N(C)} \) such that the following diagrams commute:

\[
\begin{array}{ccc}
\Delta^{n-1} & \xrightarrow{f_{0, \sigma}} & D(c_{\sigma}) \\
\alpha_i & \downarrow & \downarrow \text{Id} \\
\Delta^n & \xrightarrow{f_\sigma} & D(c_{\sigma})
\end{array}
\]

(2)

for each \( 0 < i < n \), and

\[
\begin{array}{ccc}
\Delta^{n-1} & \xrightarrow{f_{0, \sigma}} & D(c_{\sigma}) \\
\alpha_i & \downarrow & \downarrow \text{Id} \\
\Delta^n & \xrightarrow{f_\sigma} & D(c_{\sigma})
\end{array}
\]

\[
\begin{array}{ccc}
\Delta^{n-1} & \xrightarrow{f_{0, \sigma}} & D(c_{n-1}) \\
\alpha_i & \downarrow & \downarrow \text{Id} \\
\Delta^n & \xrightarrow{f_\sigma} & D(c_n)
\end{array}
\]

(3)

and

\[
\begin{array}{ccc}
\Delta^{n-1} & \xrightarrow{f_{0, \sigma}} & D(c_{n-1}) \\
\alpha_i & \downarrow & \downarrow \text{Id} \\
\Delta^n & \xrightarrow{f_\sigma} & D(c_n)
\end{array}
\]

corresponding to the cases \( i = 0 \) and \( i = n \), respectively. Here the map \( d^i \) denotes the \( i \)-th face inclusion. \( d_0, d_n \) similarly.

The following is well-known, but for completeness we give its (short) proof.
**Proposition 3.** Let $D: C \to \text{Top}$ be a diagram of topological abelian groups and continuous group homomorphisms. Then the homotopy limit of $D$ is a topological abelian group.

**Proof.** Pointwise addition makes the set $\text{Hom}(\Delta^n, D(c_n))$ into an abelian group, and the product in formula (1) is the product (topological abelian) group. It remains to see that the diagrams (2) and (3) describe a subgroup of this product. Suppose that tuples $(f_\sigma)$ and $(g_\sigma)$ make these diagrams commute. Then the first two diagrams automatically commute for the pointwise sum $(f_\sigma + g_\sigma)$. The third diagram for the pointwise sum becomes,

\[
\begin{array}{ccc}
\Delta^{n-1} & \longrightarrow & \Delta^{n-1} \times \Delta^{n-1} \\
\downarrow d^n & & \downarrow f_\sigma \times g_\sigma \\
\Delta^n & \longrightarrow & \Delta^n \times \Delta^n \\
\end{array}
\]

for which the first square obviously commutes, the second commutes since $f$ and $g$ are in the prescribed subspace and the third commutes from the fact that $D(\alpha_n)$ is a group homomorphism. The inverse operation is similarly seen to be closed, hence the subspace defined above is a subgroup. \(\square\)

**Proof (Proof of Theorem 1).** Let $L$ be an oriented link diagram with $c$ negative crossings. The space $X_k(L)$ is constructed as follows. Let $I$ denote the category with objects $\{0, 1\}$ and a single morphism from $0$ to $1$, and $I^n$ the product of $I$ with itself $n$ times. Let $0$ be the initial object in $I^n$, and let $P$ be the result of adjoining one more object to $I^n$ and a single morphism from the new object to every object except $0$.

In [3] it is shown that there is a functor $F: P \to \text{Ab}$ such that the $i^{th}$ derived functor of the inverse limit, $\lim_p^{-i} F$, is isomorphic to the $i^{th}$ unreduced Khovanov homology of $L$. The space $X_k(L)$ is constructed by composing this functor with the Eilenberg-MacLane space functor $K(-, k+c)$ and taking the homotopy limit of the resulting diagram of spaces.

We may now use the explicit model for Eilenberg-MacLane spaces given by McCord. By applying Proposition 1 we define a diagram $D: P \to \text{AbTop}$ as the composition

\[
P \xrightarrow{F} \text{Ab} \xrightarrow{B(-, S^{k+c})} \text{AbTop}.
\]

By the homotopy invariance property of the homotopy limit construction we have

$X_k(L) \simeq \text{holim}_P D$.

By Proposition 3 the homotopy limit on the right is itself a topological abelian group, and hence, by Proposition 2 a product of Eilenberg-MacLane spaces. \(\square\)

**Corollary 1.** The homotopy type of $X_\bullet(L)$ is determined by $\text{Kh}(L)$.

The spectrum $X_{\text{Kh}}(L) = \{\chi_{\text{Kh}}^{(k)}(L)\}$ constructed in [4] has the additional property that the cellular cochain complex of the space $\chi_{\text{Kh}}^{(k)}(L)$ is isomorphic to the Khovanov complex of $L$ (up to shift). It follows from the description of the Khovanov homology of the mirror image (see [5]) that

$\tilde{H}_i(X_{\text{Kh}}(L)) = \text{Kh}^{-i}(-L)$.
where $-L$ denotes the mirror of $L$. The infinite symmetric product $\text{Sym}^\infty \mathcal{X}_L^{(k)}(L)$ is seen from the Dold-Thom theorem to be

$$\text{Sym}^\infty \mathcal{X}_L^{(k)}(L) = \prod_n K(\tilde{H}_n(\mathcal{X}_L^{(k)}(L)), n)$$

from which we have the following.

**Corollary 2.** For large enough $k$, the space $X_k(-L)$ is homotopy equivalent to the infinite symmetric product $\text{Sym}^\infty \mathcal{X}_L^{(k)}(L)$.

We end by noting that the analogue or Theorem 1 for the spectra $\mathcal{X}_L(L)$ is not true. For all alternating knots $\mathcal{X}_L(L)$ is a wedge of Moore spaces [6], however there are examples of non-alternating knots for which $\mathcal{X}_L(L)$ is not a wedge of Moore spaces (see [7]).

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