Virasoro action on Schur function expansions, skew Young tableaux and random walks

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The aim of this paper is to show that the Virasoro action on two-dimensional Fourier series in Schur polynomials and the backward/forward equations for random walks are close allies!

Some matrix integrals over $U(n)$ are known to satisfy a sl(2,$\mathbb{R}$)-algebra of Virasoro constraints \([2]\) \((k = -1, 0, 1)\)

$$\mathbb{V}_k(t, s) \int_{U(n)} e^{\text{tr}V(M, \bar{M})} dM = 0, \text{ with } V(x, y) := \sum_{i=1}^{\infty} (t_j x^j - s_j y^j) \quad (1.0.1)$$

where

\begin{align*}
\mathbb{V}_{-1}(t, s) &= V_{-1}(t) - V_{1}(s) + n \left( t_1 + \frac{\partial}{\partial s_1} \right) \\
\mathbb{V}_0(t, s) &= V_0(t) - V_0(s) \\
\mathbb{V}_1(t, s) &= -V_{-1}(s) + V_1(t) + n \left( s_1 + \frac{\partial}{\partial t_1} \right),
\end{align*}

involving standard Virasoro operators\(^1\) in the variables $t_1, t_2, \ldots$ for all $k \in \mathbb{Z}$,

$$V_k(t) = \frac{1}{2} \sum_{i+j=k} \frac{\partial^2}{\partial t_i \partial t_j} + \sum_{-i+j=k} it_i \frac{\partial}{\partial t_j} + \frac{1}{2} \sum_{-i-j=k} (it_i)(jt_j). \quad (1.0.4)$$

Appropriate shifts of the $t_i$'s and $s_i$'s in the matrix integral appearing in (1.0.1) lead to the following matrix integrals, already considered by several

\(^1\)For $k = -1, 0, 1$, they take on the following simple form:

$$V_k(t) := \sum_{i \geq \max(k+1, 1)} (i-k)t_{i-k} \frac{\partial}{\partial t_i}, \quad \text{for } k = -1, 0, 1. \quad (1.0.3)$$
authors: [9, 4, 5, 14]

| shifts | matrix integrals |
|--------|------------------|
| 1: $i t_i \mapsto i t_i + z \delta_{i1}$ $i s_i \mapsto i s_i - z \delta_{i1}$ | $I_1 = \int_{U(n)} e^{z \text{Tr}(M + \bar{M})} e^{\text{Tr}V(M)} dM$ |
| 2: $i t_i \mapsto i t_i - q (-1)^i$ $i s_i \mapsto i s_i - z \delta_{i1}$ | $I_2 = \int_{U(n)} \det(I + M)^q e^{z \text{Tr} \bar{M}} e^{\text{Tr}V(M)} dM$ |
| 3: $i t_i \mapsto i t_i - p (-z)^i$ $i s_i \mapsto i s_i + q (-z)^i$ | $I_3 = \int_{U(n)} \det(I + z M)^p \det(I + z \bar{M})^q e^{\text{Tr}V(M)} dM$ |

Clearly, applying the shifts 1, 2 or 3 to $V_k$, as in table 1 to (1.0.2), lead to the Virasoro constraints

$$V_k|_{\text{shifted}} (I_i) = 0 \quad \text{for } k = -1, 0, 1$$

for the corresponding matrix integrals above. Consider the generators $\tilde{V}_z$ in the span of $\tilde{V}_{-1}, \tilde{V}_0, \tilde{V}_1|_{\text{shifted}}$, involving finite sums of $V_k(t), V_k(s), \partial/\partial t_k, \partial/\partial s_k, t_k, s_k$. They are given by the $\tilde{V}_z$ in Table 2:

| shifts | $\tilde{V}_z(t, s)$ | $\tilde{V}_\Lambda(t, s)$ | $\bar{L}_\Lambda$ |
|--------|------------------|------------------|------------------|
| 1: | $\tilde{V}_0, \tilde{V}_\pm |_{i t_i \mapsto i t_i \pm z \delta_{i1}}$ $i s_i \mapsto i s_i \pm z \delta_{i1}$ | $\tilde{V}_z |_{z \mapsto k \Lambda_k^{-1}}$ | $\bar{L}_0^{(1)}$, $\bar{L}_\pm^{(1)}$ |
| 2: | $\pm(\tilde{V}_0 + \tilde{V}_\pm) |_{i t_i \mapsto i t_i - q (-1)^i}$ $i s_i \mapsto i s_i - z \delta_{i1}$ | $\tilde{V}_z |_{z \mapsto k \Lambda_k^{-1}}$ | $\bar{L}_\pm^{(2)}$ |
| 3: | $(\tilde{V}_{-1} + (z + z^{-1}) \tilde{V}_0 + \tilde{V}_1) |_{i t_i \mapsto i t_i - p (-z)^i}$ $i s_i \mapsto i s_i + q (-z)^i$ | $\tilde{V}_z |_{z \mapsto k \Lambda_k^{-1}}$ | $\bar{L}_\Lambda^{(3)}$ |

It is also useful to replace $z$ in the column $\tilde{V}_z$ of Table 2 by -roughly speaking- the operator $\Lambda_k^{-1}$, where

$$\Lambda_k^{-1} f(k) = f(k - 1), \quad k \in \mathbb{Z}.$$ (1.0.5)
The operators $\tilde{V}_\Lambda(t, s)$ act on functions of $(t, s, k) \in \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{Z}$. Let them act on two-dimensional Fourier series in Schur polynomials $s_\lambda(t)$ and $s_\mu(s)$ with partitions $\lambda$ and $\mu$ having first column bounded by $n$ and with arbitrary coefficients $\tilde{b}_{\lambda\mu}^{(k)}$. It turns out that the functions obtained can be expressed again in terms of double Fourier series, with coefficients $\tilde{L}_{\lambda\mu}(\tilde{b}_{\lambda\mu}^{(k)})$, which are linear difference operators of finite order on the coefficients $\tilde{b}_{\lambda\mu}^{(k)}$:

$$\tilde{V}_\Lambda(t, s) \sum_{\lambda, \mu} \tilde{b}_{\lambda\mu}^{(k)} s_\lambda(t) s_\mu(-s) = \sum_{\lambda, \mu} \tilde{L}_{\lambda\mu}(\tilde{b}_{\lambda\mu}^{(k)}) s_\lambda(t) s_\mu(-s) \quad (1.0.6)$$

The surprise is that the expressions $\tilde{L}_{\lambda\mu}$, appearing in the right hand column of table 2 and given explicitly in section 6, are precisely the difference equations for the transition probabilities for certain random walks, naturally generated by the matrix integrals above. This circle of ideas will now be explained.

As a first ingredient, the three matrix integrals containing Schur polynomials $s_\lambda(t)$ admit the following expansions in $z$:

$$\int_{U(n)} s_\lambda(M)s_\mu(\bar{M}) \left\{ \begin{array}{l}
e^{z \text{Tr}(M+\bar{M})} \\
det(I+M)^q e^{z \text{Tr} M} \\
det(I+zM)^p \det(I+z\bar{M})^q \end{array} \right\} dM = \sum_{k=0}^{\infty} \frac{z^k}{k!} \tilde{b}_{\lambda\mu}^{(k)}$$

whereas the following matrix integrals below admit (double) Fourier expansions:

---

[2] Throughout the paper, the $s_\lambda(t)$’s denote Schur polynomials for a partition $\lambda$, expressed in terms of the symmetric functions $k t_k = \sum_i x_i^k$, and not in terms of the $x_k$ themselves. The elementary Schur polynomials $s_k(t)$ are defined by $e^{\sum_i t_i z^i} = \sum_0^\infty s_k(t) z^k$ and $s_k(t) = 0$ for $k < 0$. Given a unitary matrix $M$, we shall also use the notation $s_\lambda(M)$ to denote a symmetric function of the eigenvalues $x_1, \ldots, x_n$ of the unitary matrix $M$ and thus in the notation of the present paper

$$s_\lambda(M) := s_\lambda(\text{Tr} M, \frac{1}{2} \text{Tr} M^2, \frac{1}{3} \text{Tr} M^3, \ldots).$$
sions in Schur polynomials:

\[
\int_{U(n)} \left\{ \begin{array}{l}
(\text{Tr}(M + \bar{M}))^k \\
(\text{Tr} \bar{M})^k \det(I + M)^q \\
k!s_k(\ldots, -\frac{1}{i}\text{Tr}(p(-M)^i + q(-\bar{M})^i), \ldots)
\end{array} \right\} e^{\text{Tr} V(M, \bar{M})} dM
\]

\[
= \sum_{\lambda, \mu} \tilde{b}^{(k)}_{\lambda \mu} s_\lambda(t) s_\mu(-s),
\]

whose coefficients are given by the same \( \tilde{b}^{(k)}_{\lambda \mu} \). This is to say that the \( \tilde{b}^{(k)}_{\lambda \mu} \)'s appear in expansions of two different integrals.

It is useful here to make the change of variables, from partitions \( \lambda \) and \( \mu \), with first columns smaller than \( n \), to strictly increasing sets of integers \( x \) (initial position of the random walk) and \( y \) (final position of the random walk) in \( \mathbb{Z}_{\geq 0} \), defined by

\[
\begin{align*}
x & := (x_1 < x_2 < \ldots < x_n) = (0 + \lambda_n, 1 + \lambda_{n-1}, \ldots, n - 1 + \lambda_1) \\
y & := (y_1 < y_2 < \ldots < y_n) = (0 + \mu_n, 1 + \mu_{n-1}, \ldots, n - 1 + \mu_1)
\end{align*}
\]

(1.0.7)

So, the partitions \( \lambda \) and \( \mu \) measure the discrepancy from close packing 0, 1, \ldots, \( n - 1 \) for \( x \) and \( y \)!

So the new expressions

\[
\tilde{b}_{xy}^{(k)} = \tilde{b}^{(k)}_{\lambda \mu}
\]

and

\[
L_\Lambda := \tilde{L}_\Lambda
\]

satisfy the difference relations

\[
L_\Lambda (\tilde{b}_{xy}^{(k)}) = 0
\]

(1.0.8)
where the $b_{xy}^{(k)}$ have the following interpretation in terms of walks:\footnote{The number of “effective moves” counts the actual steps taken by all walkers; i.e., two walkers walking simultaneously counts for two moves, a walker not walking contributes nothing!}

\begin{align}
\text{Case 1} & \\
\quad b_{xy}^{(k)} &= \# \begin{cases}
\text{ways that } n \text{ non-intersecting walkers in } \mathbb{Z} \text{ move during } k \text{ instants from } x_1 < x_2 < ... < x_n \text{ to } y_1 < y_2 < ... < y_n, \\
\quad \text{where at each instant exactly one walker moves}
\quad \text{either one step to the left, or one step to the right}
\quad \text{leading to } k \text{ effective moves}.
\end{cases} \\
\quad (1.0.9)
\end{align}

\begin{align}
\text{Case 2} & \\
\quad b_{xy}^{(k)} &= \# \begin{cases}
\text{ways that } n \text{ non-intersecting walkers move during } q + k \text{ instants from } x_1 < ... < x_n \text{ to } y_1 < ... < y_n, \text{ where at the instants 1 to } q, \text{ walkers may move one step to the right, or stay put, and at the instants } q + 1, ..., q + k \text{ exactly one walker moves one step to the left, with}
\quad \text{total } \# \{\text{effective moves}\} = 2k + \sum_{i=1}^{n} (y_i - x_i).
\end{cases} \\
\quad (1.0.10)
\end{align}

\begin{align}
\text{Case 3} & \\
\quad b_{xy}^{(k)} &= k! \# \begin{cases}
\text{ways that } n \text{ non-intersecting walkers move during } p + q \text{ instants from } x_1 < ... < x_n \text{ to } y_1 < ... < y_n, \text{ where at the instants 1 to } p, \text{ walkers may move one step to the right, or stay put, and at the instants } p + 1, ..., p + q \text{ walkers may move one step to the left or stay put},
\quad \text{with total } \# \{\text{effective moves}\} = k.
\end{cases} \\
\quad (1.0.11)
\end{align}

Case 1 leads, in particular, to a forward and backward equation for the transition probability, as shown in section 7,
ways that \( n \) non-intersecting walkers in \( \mathbb{Z} \) move during \( k \) instants from \( x_1 < x_2 < ... < x_n \) to \( y_1 < y_2 < ... < y_n \), where at each instant exactly one walker moves either one step to the left, or one step to the right

\[
P(k, x, y) = \begin{pmatrix}
\text{ways that } n \text{ non-intersecting walkers in } \mathbb{Z} \text{ move during } k \text{ instants from } x_1 < x_2 < ... < x_n \text{ to } y_1 < y_2 < ... < y_n, \\
\text{where at each instant exactly one walker moves either one step to the left, or one step to the right}
\end{pmatrix}
\]

\[
P(k, x, y) = \frac{b^{(k)}_{xy}}{(2n)^k},
\]

namely,

\[
A_i P(k, x, y) = 0, \quad (1.0.12)
\]

where the \( A_i \) are the same difference operators as the operator \( L_\Lambda \) in (1.0.8), except for the division by \( 2n \), which accounts for considering the probability rather than the \( b_{xy} \)'s:

\[
A_1 := \sum_{i=1}^{n} \left( \frac{k}{2n} \Lambda_k^{-1} \partial^+_{2x} + x_i \partial^-_{x} + \partial^+_{y} y_i - (x_i - y_i) \right)
\]

\[
A_2 := \sum_{i=1}^{n} \left( \frac{k}{2n} \Lambda_k^{-1} \partial^+_{2x} + y_i \partial^-_{x} + \partial^+_{x} x_i - (y_i - x_i) \right) \quad (1.0.14)
\]

The operators \( A_1 \) and \( A_2 \) are the \textit{forward and backward random walk equation}, because \( A_1 \) essentially involves the end points \( y \), whereas \( A_2 \) involves the initial points \( x \).

\textbf{Remark:} The transition probabilities for a random walk in \( \mathbb{Z}^n \) absorbed at the boundary of the Weyl chamber \( z_1 < z_2 < ... < z_n \), with equally likely steps \( \pm e_1, \ldots, \pm e_n \) (studied, e.g. in ([10],[11])) also satisfy the same backward and forward difference equations (1.0.12).

In a subsequent paper, we show the following limit theorem: Let the spacings between the \( n \) walkers and the number of steps \( k \) grow larger, with

\[
\begin{align*}
\partial^+_{\alpha x}, f & := f(k, x + \alpha e, y) - f(k, x, y) \\
\partial^-_{\alpha x}, f & := f(k, x, y) - f(k, x - \alpha e, y) \\
\Lambda_k^{-1} f & := f(k - 1, x, y)
\end{align*}
\]
an appropriate rescaling; then one finds, for fixed $n$, setting $\varepsilon := \sqrt{\frac{t}{k}}$ and letting $k \to \infty$ or, what is the same, letting $\varepsilon \to 0$,

$$A_1\Big|_{x=\frac{\tilde{x}}{\varepsilon \sqrt{t}}, y=\frac{\tilde{y}}{\varepsilon \sqrt{t}}} = \frac{t^{(n+1)/2}}{\varepsilon} \sum_{i=1}^{n} \left( \frac{\partial}{\partial \tilde{y}_i} - \frac{\tilde{x}_i - \tilde{y}_i}{t} \right)$$

$$- 2t^{(n+2)/2} \left( \frac{\partial}{\partial t} - \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2}{\partial \tilde{y}_i^2} \right) + O(\varepsilon)$$

$$A_2\Big|_{x=\frac{\tilde{x}}{\varepsilon \sqrt{t}}, y=\frac{\tilde{y}}{\varepsilon \sqrt{t}}} = \text{same expression, but with } \tilde{x} \leftrightarrow \tilde{y}$$

This expansion in $\varepsilon$ is valid, when acting on an appropriate function space. The term $O(1)$ in the $\varepsilon$-expansion of $A_1$ contains precisely the forward diffusion equation for Brownian motion in the Weyl chamber $\{\tilde{y}_1 < \ldots < \tilde{y}_n\} \subset \mathbb{R}^n$. The term $O(1)$ in the $\varepsilon$-expansion of $A_2$ contains the corresponding backward equation. This Brownian motion in the Weyl chamber is tantamount to the motion of $n$ non-intersecting Brownian motions, i.e., who are killed as soon as they collide. It is related to Dyson’s Brownian motion ([6]).

The difference equations (1.0.8) above for the three cases (1.0.9), (1.0.10) and (1.0.11) are based on replacing the operators $\partial/\partial t_n$ and multiplication by $nt_n$ in the Murnaghan-Nakayama rule,

$$nt_n s_\lambda(t) = \sum_{\mu: \lambda \in B(n)} (-1)^{ht(\lambda \setminus \mu)} s_\mu(t)$$

$$\frac{\partial}{\partial t_n} s_\lambda(t) = \sum_{\mu: \lambda \in B(n)} (-1)^{ht(\lambda \setminus \mu)} s_\mu(t), \quad (1.0.15)$$

by the action of the Virasoro algebra:

$$V_{-n}s_\lambda = \sum_{\mu: \lambda \in B(n)} d_{\lambda \mu}^{(-n)} s_\mu$$

$$V_n s_\lambda = \sum_{\lambda \setminus \mu \in B(n)} d_{\mu \lambda}^{(-n)} s_\mu \quad (1.0.16)$$
with \((n \geq 1)\)

\[
d_{\lambda \mu}^{(-n)} = \sum_{i \geq 1} \sum_{\nu \text{ such that } \\
\lambda \setminus \nu \in B(i) \\
\mu \setminus \nu \in B(n + i) \\
\lambda \setminus \nu \subset \mu \setminus \nu} (-1)^{ht(\lambda \setminus \nu) + ht(\mu \setminus \nu)}
\]

\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{\nu \text{ such that } \\
\nu \setminus \lambda \in B(i) \\
\mu \setminus \nu \in B(n - i)} (-1)^{ht(\nu \setminus \lambda) + ht(\mu \setminus \nu)}. \tag{1.0.17}
\]

This will be shown in section 5. To explain the notation, \(h \in B(i)\) denotes a border-strip (i.e., a connected skew-shape \(\lambda \setminus \mu\) containing \(i\) boxes, with no \(2 \times 2\) square) and the height \(ht\) of a border strip \(h\) is defined as

\[
ht h := \#\{\text{rows in } h\} - 1. \tag{1.0.18}
\]

In view of the infinite sum in the Virasoro algebra (1.0.4), one would expect \(V_n s_{\lambda}\) to be expressible as an infinite sum of Schur polynomials. This is not so: acting with Virasoro \(V_{-n}\) (resp. \(V_n\)) leads to the same precise sum as acting with \(nt_n\) (resp. \(\partial/\partial t_n\)), except the coefficients (1.0.17) are different from the ones in (1.0.15). This is to say the two operators have the same band structure or locality!

# 2 Non-intersecting walks and partitions

Consider \(m\) walkers on \(\mathbb{Z}\) departing from position \(x_1, \ldots, x_m\), and ending up at \(y_1, \ldots, y_m\), such that at each instant, only one walker moves either one step to the left, or one step to the right, with all possible moves equally likely. The main statement of this section is to connect the transition probabilities of these walks with pairs of skew Young tableaux and other combinatorial formulas. They will play a crucial role in section 4. Such connections have been known in various situations in the combinatorial literature; see R. Stanley [13] (p. 313), P. Forrester [8], D. Grabiner & P. Magyar [11, 12], J. Baik [3]:
Theorem 2.1

\[
P \left( \text{that } m \text{ walkers in } \mathbb{Z}, \right. \\
\left. \text{go from } x_1, \ldots, x_m \text{ to } y_1, \ldots, y_m \text{ in } T \text{ steps,} \right. \\
\left. \text{and do not intersect} \right) \\
= \frac{1}{(2m)^T} \left( \begin{array}{c} T \\ \mathbf{T}_L \mathbf{T}_R \end{array} \right) \sum_{\lambda \text{ with } \lambda \supseteq \mu, \nu} f^{\lambda \setminus \mu} f^{\lambda \setminus \nu} \\
= \frac{1}{(2m)^T} \sum_{w \in W} (-1)^{\sigma(w)} \left( \sum_{i=1}^{m} (u_i + u_i^{-1}) \right)^T \left| \begin{array}{c} u_1^{y_1 - w(x_1)} \ldots u_m^{y_m - w(x_m)} \end{array} \right| \\
= \frac{1}{(2m)^T} \sum_{w \in W} (-1)^{\sigma(w)} \sum_{\sum_{i=1}^{m} k_i = T/2} \left( k_1 + \frac{y_1 - x_w(1)}{2}, k_1 - \frac{y_1 - x_w(1)}{2}, \ldots, k_m + \frac{y_m - x_w(m)}{2}, k_m + \frac{y_m - x_w(m)}{2} \right) \\
\text{where } \mu, \nu \text{ are fixed partitions defined by the points } x_i \text{ and } y_i, \\
\mu_k = k - 1 - x_k, \quad \nu_k = k - 1 - y_k \\
T_L = \frac{1}{2}(T + \sum_{i=1}^{m} (x_i - y_i)) = \frac{1}{2}(T - |\mu| + |\nu|) \\
T_R = \frac{1}{2}(T - \sum_{i=1}^{m} (x_i - y_i)) = \frac{1}{2}(T + |\mu| - |\nu|) \\
T = T_L + T_R, \quad \sum_{i=1}^{m} (x_i - y_i) = T_L - T_R.
\]

Proof: will follow from Propositions 2.3 and 2.5, as given in the subsequent subsections.
2.1 Non-intersecting walks and skew-tableaux

Proposition 2.2 There is a $1 - 1$ correspondence between

\[
\begin{cases}
\text{ways that } m \text{ non-intersecting walkers move from } x_1 < \ldots < x_m & \\
\text{to } y_1 < \ldots < y_m, \text{ where at each instant } 1, \ldots, T_L \text{ one walker moves one step to the left, and at each instant } T_L + 1, \ldots, T_L + T_R \text{ one walker moves one step to the right} \end{cases}
\quad \iff \quad \begin{cases}
\text{all couples } (P, Q) \text{ of standard skew-tableaux of arbitrary shape } \lambda \setminus \mu \text{ and } \lambda \setminus \nu, \text{ given fixed partitions } \mu, \nu, \\
\text{with } |\lambda \setminus \mu| = T_L, |\lambda \setminus \nu| = T_R, \\
\text{filled with numbers } 1, \ldots, T_L \text{ and } 1, \ldots, T_R \\
\text{with } \lambda_1^T \leq m
\end{cases},
\]

and so, for this walk,

\[
\# \left\{ \begin{array}{l}
\text{ways that } m \text{ non-intersecting walkers go } T_L \text{ steps to the left and then } T_R \text{ steps to right, from } x_1 < \ldots < x_m \\
\text{and then } T_R \text{ steps to right, from } x_1 < \ldots < x_m \\
to y_1 < \ldots < y_m
\end{array} \right\} = \sum_{\lambda \supset \mu, \nu, \lambda_1^T \leq m} f^{\lambda \setminus \mu} f^{\lambda \setminus \nu},
\]

where $\mu, \nu$ are fixed partitions defined by the points $x_i$ and $y_i$, and $\lambda \supset \mu, \nu$, such that

\[
\begin{align*}
\mu_k &:= k - 1 - x_k \\
\nu_k &:= k - 1 - y_k \\
|\lambda| &= \frac{1}{2} (T + |\mu| + |\nu|), \ T = T_L + T_R.
\end{align*}
\]

Proof: Consider two Young diagrams $\mu, \nu$ and $\lambda$ such that

\[
\mu \subset \lambda, \quad \nu \subset \lambda
\]

and two standard skew-tableaux

\[
(P, Q) = \begin{cases}
P \text{ standard skew-Young tableaux of shape } = \lambda \setminus \mu \\
filled \text{ with numbers } 1, \ldots, |\lambda \setminus \mu| \\
Q \text{ standard skew-Young tableaux of shape } = \lambda \setminus \nu \\
filled \text{ with numbers } 1, \ldots, |\lambda \setminus \nu|.
\end{cases}
\]
To the standard skew-Young tableaux $P$ of shape $\lambda \setminus \mu$, we associate $m$ walkers starting at

$$x_1 = -\mu_1 + 0 < \ldots < x_k = -\mu_k + k - 1 < \ldots < x_m = -\mu_m + m - 1,$$

so that $x_k - x_{k-1} = \mu_{k-1} - \mu_k + 1$ and requiring the $k^{th}$ walker, starting at $x_k = -\mu_k + k - 1$, to move to the left only, at instants

$$c_{ki} = \text{content of box } (k, i) \in P,$$

and thus he has made, in the end, $\lambda_k - \mu_k$ steps to the left. So, at each instant exactly one walker is moving and this during a time-span $T_L = |\lambda \setminus \mu|$, until the $m$ walkers reach the position

$$-\lambda_1 + 0 < \ldots < -\lambda_k + k - 1 < \ldots < -\lambda_m + m - 1.$$

The fact that the skew-tableau $P$ is standard implies that the walkers have never intersected, as one sees by imagining $\mu$ filled in a standard fashion with the numbers $0, -1, \ldots, -|\mu| + 1$, thus yielding for each walker $k$, a path from $x_k = -\mu_k + k - 1$ to $-\lambda_k + k - 1$, not intersecting the paths of the neighboring walkers.

In the same way, to $Q$, we associate $m$ walkers starting at

$$y_1 = -\nu_1 + 0 < \ldots < y_k = -\nu_k + k - 1 < \ldots < y_m = -\nu_m + m - 1.$$
moving left at instants \( c_k \), each making, in the end \( \lambda_k - \nu_k \) steps to the left, until the \( m \) walkers reach, after time \( T_R = |\lambda\nu| \) the position

\[-\lambda_1 + 0 < ... < -\lambda_k + k - 1 < ... < -\lambda_m + m - 1,\]
the same position as before, without having ever intersected.

Now we assemble the two walks. It yields a walk with \( m \) non-intersecting walkers, going from \((x_1 < \ldots < x_m)\) to \((y_1 < \ldots < y_m)\), moving first \( T_L \) steps to the left and then \( T_R \) steps to the right, obtained by reversing the second walk (associated with \( Q \)) such that the final position of the first walk (moving left) is the starting position of the second walk (moving right). Therefore, we have the total number of steps

\[T = T_L + T_R = 2|\lambda| - |\mu| - |\nu|,\]
from which Proposition 2.2 follows.

Consider now \( m \) walkers on \( \mathbb{Z} \) departing from position \( x_1, \ldots, x_m \), and ending up at \( y_1, \ldots, y_m \), as in Theorem 2.1.

**Proposition 2.3**

\[
P \left( \begin{array}{c}
\text{that } m \text{ walkers in } \mathbb{Z}, \\
\text{go from } x_1, \ldots, x_m \text{ to} \\
y_1, \ldots, y_m \text{ in } T \text{ steps,} \\
\text{and do not intersect}
\end{array} \right) = \frac{1}{(2m)^2} \binom{T}{T_L, T_R} \sum_{\lambda \vdash n \mu, \nu \vdash n} f^{\lambda\mu} f^{\lambda\nu}
\]

where \( \mu, \nu \) are fixed partitions defined by the \( x_i \) and \( y_i \)'s, as in Theorem 2.1.

**Corollary 2.4**

\[
P \left( \begin{array}{c}
\text{that } m \text{ walkers in } \mathbb{Z}, \\
\text{go from and return} \\
to } 1, \ldots, m \text{ in } 2n \text{ steps,} \\
\text{and do not intersect}
\end{array} \right) = \frac{1}{(2m)^{2n}} \binom{2n}{n}^{2n} \sum_{\lambda \vdash n} (f^{\lambda})^2
\]

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Proof of Proposition 2.3: At first, notice that \( m \) walkers in \( \mathbb{Z} \), obeying this rule, is tantamount to a random walk in \( \mathbb{Z}^m \), where at each point the only moves are

\[ \pm e_1, \ldots, \pm e_m. \]

That is to say the walk has at each point \( 2m \) possibilities and thus at time \( T \) the walk has

\[ (2m)^T \]  

(2.1.4)

places to go.

Returning to the \( \mathbb{Z} \)-picture, we now associate to a given walk a sequence of \( T_L \) L’s and \( T_R \) R’s:

\[ L \ R \ R \ R \ L \ R \ L \ L \ R \ldots \ R , \]

(2.1.5)

thus recording the nature of the move, left or right, at the first instant, at the second instant, etc...

If the \( k^{th} \) walker is to go from \( x_k \) to \( y_k \), then

\[ y_k - x_k = \# \left\{ \text{right moves for } k^{th} \text{ walker} \right\} - \# \left\{ \text{left moves for } k^{th} \text{ walker} \right\} \]

and so, if

\[ T_L := \# \left\{ \text{left moves for all } m \text{ walkers} \right\} \quad \text{and} \quad T_R := \# \left\{ \text{right moves for all } m \text{ walkers} \right\} , \]

we have, since at each instant exactly one walker moves,

\[ T_R + T_L = T \]
\[ T_R - T_L = \sum_{i=1}^{m} (y_k - x_k) , \]

from which

\[ T = \frac{1}{2} \left( T \pm \sum_{i=1}^{m} (x_k - y_k) \right) . \]

Next, we show there is a canonical way to map a walk, corresponding to (2.1.5) into one with left moves only during times 1, \ldots, \( T_L \) and then right moves during times \( T_L +1, \ldots, T_L + T_R = T \), thus corresponding to a sequence

\[ L \ L \ L \ldots \ L \quad R \ R \ R \ldots \ R . \]

(2.1.6)
Indeed, in a typical sequence, as (2.1.5),
\[ \overbrace{L \, R \, R \, R \, R \, L} \, R \, L \, L \, R \ldots \, R \, , \]  
consider the first sequence \( R \, L \) (underlined) you encounter, in reading from left to right. It corresponds to one of the following three configurations (in the left column),

\[
\begin{array}{ccc}
L & \backslash & \quad | \quad | \\
R & | & | \vert \quad \Rightarrow \quad R & | & | \vert \\
L & \vert & | \quad \vert \quad \vert \quad \Rightarrow \quad L & \vert & | \quad \vert \quad \vert \\
R & \vert & | \quad \vert \quad \vert & \Rightarrow & R & \vert & | \quad \vert \quad \vert \\
\end{array}
\]

which then can be transformed into a new configuration \( L \, R \), with same beginning and end, thus yielding a new sequence; in the third case the reflection occurs the first place it can. These moves have been considered by Forrester in [8]. So, by the moves above, the original configuration (2.1.5) can be transformed in a new one. In the new sequence, pick again the first sequence \( RL \), reading from left to right, and use again one of the moves. So, this leads again to a new sequence, etc...

\[
\begin{array}{ccc}
L & \quad R \quad R \quad R \quad R \quad L \quad R \quad L \quad L \quad R \ldots \, R \\
L & \quad R \quad R \quad L \quad R \quad L \quad L \quad R \ldots \, R \\
L & \quad L \quad R \quad R \quad L \quad R \quad L \quad L \quad R \ldots \, R \\
L & \quad L \quad R \quad R \quad R \quad L \quad L \quad L \quad R \ldots \, R \\
\end{array}
\]

Since this procedure is invertible, it gives a \textit{one-to-one} map between all the left-right walks corresponding to a given sequence, with \( T_L \) \( L \)'s and \( T_R \) \( R \)'s

\[
\begin{array}{ccc}
L & \quad L \quad L \quad \ldots \, L \quad \underbrace{R \quad R \quad R \quad R \quad \ldots \, R} \\
T_L & \quad R \quad R \quad R \quad R \quad \ldots \, R \quad T_R \\
\end{array}
\]  

(2.1.8)
and all the walks corresponding to

\[
\begin{array}{cccc}
T_L & L & L & \ldots & L \\
\hline
L & L & L & \ldots & L \\
T_R & R & R & \ldots & R
\end{array}
\]  \quad (2.1.10)

On the one hand, corresponding to this sequence, \( m \) walkers can walk in

\[
\sum_{\lambda^T \leq m} f^{\lambda \mu} f^{\lambda \nu} \]

\quad (2.1.11)

ways, as follows from Proposition 2.2. On the other hand, there are \( \binom{T}{T_L, T_R} \) sequences of \( T_L \) L’s and \( T_R \) R’s, which combined with (2.1.11) yields the result.

\[\square\]

Proof of Corollary 2.4: In this situation, we have close packing, and thus \( \mu_k = \nu_k = 0 \) for all \( k \), and so \( \mu = \nu = \emptyset \) and \( T_L = T_R = T/2 \). With these data, (2.1.3) is an immediate consequence of (2.1.2).

\[\square\]

2.2 Non-intersecting walks and D. André’s principle

We now prove the second and third formulae of Theorem 2.1, namely

Proposition 2.5

\[
P\left( \text{that } m \text{ walkers in } \mathbb{Z}, \ \text{go from } x_1, \ldots, x_m \text{ to } y_1, \ldots, y_m \text{ in } T \text{ steps, and do not intersect} \right)
\]

\[
= \frac{1}{(2m)^T} \sum_{w \in W} (-1)^{\sigma(w)} \left( \sum_{i=1}^{m} (u_i + u_i^{-1}) \right)^T_{u_1^{y_1-x_1}, \ldots, u_m^{y_m-x_m}}
\]

\[
= \frac{1}{(2m)^T} \sum_{w \in W} (-1)^{\sigma(w)}
\]

\[
\sum_{k_i \geq 0, \sum_{i=1}^{m} k_i = T/2} \binom{y_1-x_{(1)}}{2}, \binom{y_1-x_{(1)}}{2}, \ldots, \binom{y_m-x_{(m)}}{2}, \binom{y_m-x_{(m)}}{2}.
\]

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Proof: Consider again the random walk in $\mathbb{Z}^m$, where at each point the only moves are 

$$\pm e_1, \ldots, \pm e_m.$$ 

Then 

$$\# \left\{ \text{ways to walk in } T \text{ steps} \right\}$$

from $x = (x_1, \ldots, x_m)$ to $y = (y_1, \ldots, y_m)$ in $\mathbb{Z}^m$

$$= \left( \sum_{i=1}^m (u_i + u_i^{-1}) \right)^T \Bigg|_{u_1^{y_1-x_1} \cdots u_m^{y_m-x_m}}$$

$$= \sum_{a_i, b_i \geq 0}^{m} \sum_{a_i+b_i=T} \left( \begin{array}{cc} a_1 & b_1 \\ b_1 & a_m \\ a_m & b_m \end{array} \right) u_1^{a_1} u_1^{-b_1} \cdots u_m^{a_m} u_m^{-b_m} \Bigg|_{u_1^{y_1-x_1} \cdots u_m^{y_m-x_m}}$$

$$= \sum_{a_i+b_i=T} \left( \begin{array}{cc} a_1 & b_1 \\ b_1 & a_m \\ a_m & b_m \end{array} \right)$$

$$= \sum_{k_i \geq 0 \sum k_i = T/2} \left( k_1 + \frac{y_1-x_1}{2}, k_1 - \frac{y_1-x_1}{2}, \ldots, k_m + \frac{y_m-x_m}{2}, k_m - \frac{y_m-x_m}{2} \right) \right.$$ 

ending the proof of Proposition 2.5. 

\[\Box\]

3 A matrix integral

The main statement of this section is to prove
Proposition 3.1 The following matrix integral admits a “Fourier” expansion

\[ \int_{U(n)} e^{\sum_{i=1}^{\infty} \text{Tr}(t(0)^{ij} M^{ij} - s(0)^{ij} \bar{M}^{ij})} \sum_{i=1}^{\infty} \text{Tr}(t_s^{ij} M^{ij} - s_{\bar{s}}^{ij} \bar{M}^{ij})} dM \]

\[ = \sum_{\lambda, \mu \text{ such that } \lambda \leq n} a_{\lambda \mu}(t(0), s(0)) s_{\lambda}(t) s_{\mu}(-s) \tag{3.0.1} \]

with Fourier coefficients, taking on many different forms:

\[ a_{\lambda \mu}(t(0), s(0)) = \det \left( \int_{S^1} u^{\lambda - \ell - \mu + k} \sum_{i=1}^{\infty} (t(0)^{ij} u^{i-j} s(0)^{ij}) \frac{du}{2\pi i u} \right)_{1 \leq \ell, k \leq n} \]

\[ = \sum_{\nu \text{ such that } \nu \geq \lambda, \mu} \left( s_{\nu}(t(0)) s_{\nu}(s(0)) \right) \]

\[ = \int_{U(n)} s_{\lambda}(M) s_{\mu}(\bar{M}) e^{\sum_{i=1}^{\infty} \text{Tr}(t(0)^{ij} M^{ij} - s(0)^{ij} \bar{M}^{ij})} dM. \tag{3.0.2} \]

The proof of this Proposition will be given later in this section. Specializing the \( t(0) \)'s and \( s(0) \)'s will lead to several examples, discussed in the next section (section 4).

Consider the \((t, s)\)-dependent semi-infinite matrix

\[ m_{\infty}(t, s) = (\mu_{ij}(t, s))_{0 \leq i, j < \infty}, \tag{3.0.3} \]

evolving according to the equations (here \( \Lambda \) is the shift matrix, with zeroes everywhere, except for 1's just above the diagonal, i.e., \( (\Lambda v)_n = v_{n+1} \))

\[ \frac{\partial m_{\infty}}{\partial t_k} = \Lambda^k m_{\infty} \quad \text{and} \quad \frac{\partial m_{\infty}}{\partial s_k} = -m_{\infty}(\Lambda^T)^k, \tag{3.0.4} \]

with given initial condition \( m_{\infty}(0, 0) \). According to [1], the unique (formal) solution to this problem is given by

\[ m_{\infty}(t, s) = e^{\sum_{i=1}^{\infty} t_i \Lambda^i} m_{\infty}(0, 0) e^{-\sum_{i=1}^{\infty} s_i \Lambda^T_i}, \tag{3.0.5} \]

where

\[ e^{\sum_{i=1}^{\infty} t_i \Lambda^i} = \sum_{0}^{\infty} \Lambda^i s_i(t) = \left( s_{j-i}(t) \right)_{1 \leq i < \infty, 1 \leq j < \infty}, \]

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is a matrix of Schur polynomials\(^5\) \(s_i(t)\), of which a truncated version is given by the following \(n \times \infty\) submatrix:

\[
E_n(t) = \begin{pmatrix}
1 & s_1(t) & s_2(t) & \ldots & s_{n-1}(t) & s_n(t) & \ldots \\
0 & 1 & s_1(t) & s_2(t) & \ldots & s_{n-2}(t) & s_{n-1}(t) & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & s_1(t) & s_2(t) & \ldots \\
0 & 0 & 0 & \ldots & 1 & s_1(t) & \ldots \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
(s_j-i(t))_{1 \leq i \leq n, 1 \leq j < \infty}
\end{pmatrix}
\]  \(3.0.6\)

Then the \(n \times n\) upper-left corner \(m_n(t, s) := (\mu_{ij}(t, s))_{0 \leq i, j \leq n-1}\) of \(m_\infty(t, s)\) is given by

\[
m_n(t, s) = E_n(t) \ m_\infty(0, 0) \ E_n^\top(-s).
\]  \(3.0.7\)

**Lemma 3.2** Given the semi-infinite initial condition \(m_\infty(0, 0)\), and given integers

\[1 \leq a_1 < \ldots < a_n\] and \[1 \leq b_1 < \ldots < b_n,\]

the following determinants have a “Fourier” expansion in Schur and skew-Schur polynomials\(^6\)

\[
det(\mu_{k,\ell}(t, s))_{1 \leq k, \ell \leq n} = \sum_{\lambda, \nu, 1 \leq i \leq n} \det(m^{\lambda,\nu}(0, 0)) s_\lambda(t) s_\nu(-s), \text{ for } n > 0,
\]  \(3.0.8\)

\[
det(\mu_{a_k, b_\ell}(t, s))_{1 \leq k, \ell \leq n} = \sum_{\lambda \supset a, \nu \supset b, 1 \leq i \leq n} \det(m^{\lambda,\nu}(0, 0)) s_{\lambda \setminus a}(t) s_{\nu \setminus b}(-s) \text{ for } n > 0,
\]  \(3.0.9\)

with Fourier coefficients, involving the matrices

\[
m^{\lambda,\nu}(0, 0) := (\mu_{i-n+j, i-n+j}(0, 0))_{1 \leq i, j \leq n} \text{ for } \lambda^\top_1, \nu^\top_1 \leq n,
\]  \(3.0.10\)

\(^5\)See footnote 2 .

\(^6\)The sum below is taken over all Young diagrams \(\lambda\) and \(\nu\), with the first columns \(\lambda^\top_1\) and \(\nu^\top_1 \leq n.\)
and where $\alpha$ and $\beta$ are partitions defined by
\[ a_j = \alpha_{n-j+1} + j \quad \text{and} \quad b_j = \beta_{n-j+1} + j. \] (3.0.11)

**Proof:** Note that every strictly increasing sequence $1 \leq k_1 < \ldots < k_n < \infty$ of integers can be mapped into a Young diagram $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$, by setting $k_j = j + \lambda_{n+1-j}$; and similarly with the increasing sequences $\ell_j, a_j, b_j$.

So, this leads to partitions $\lambda, \nu, \alpha, \beta$:
\[ k_j = \lambda_{n-j+1} + j \quad \ell_j = \nu_{n-j+1} + j \quad a_j = \alpha_{n-j+1} + j \quad b_j = \beta_{n-j+1} + j. \]

It is also useful to relabel the index $i$ with $1 \leq i \leq n$, by setting $i' := n - i + 1$, also with $1 \leq i' \leq n$ and so on with the other indices.

Applying the Cauchy-Binet formula twice, and setting
\[ A_{a_1a_2\ldots a_n} := \text{matrix formed with the rows } a_1 \ldots a_n \text{ of } A, \]

the expression (3.0.7) leads to:
\[
\begin{align*}
\det \left( m_n(t, s)_{a_i b_j} \right)_{1 \leq i, j \leq n} &= \det \left( E(t)_{a_1 a_2 \ldots a_n} m_\infty(0, 0) \left( E_n(-s)_{b_1 b_2 \ldots b_n} \right)^\top \right) \\
&= \sum_{1 \leq k_1 < \ldots < k_n < \infty} \det \left( p_k \right)_{1 \leq i, j \leq n} \det \left( \left( m_\infty(0, 0) E(-s)_{b_1 b_2 \ldots b_n} \right)^\top \right)_{1 \leq i, r \leq n} \\
&= \sum_{1 \leq k_1 < \ldots < k_n < \infty} \det \left( p_k \right)_{1 \leq i, j \leq n} \det \left( \left( \mu_{k_i-1,j-1} \right)_{1 \leq i \leq n, 1 \leq j < \infty} \left( p_i - b_r(-s) \right)_{1 \leq s < \infty} \right)_{1 \leq i, r \leq n} \\
&= \sum_{1 \leq k_1 < \ldots < k_n < \infty} \det \left( p_k \right)_{1 \leq i, j \leq n} \sum_{1 \leq \ell_1 < \ldots < \ell_n < \infty} \det \left( \mu_{k_i-1,\ell_j-1} \right)_{1 \leq i, j \leq n} \det \left( p_{\ell_j} - b_r(-s) \right)_{1 \leq j, r \leq n}
\end{align*}
\]
\[
\sum_{\lambda, \nu} \det \left( \mu_{\lambda_j \nu_j - j' + n} \right)_{1 \leq j, j' \leq n} = \sum_{\lambda, \nu} \det \left( \mu_{\lambda_j \nu_j} \right)_{1 \leq j, j' \leq n} \leq n \det \left( \mu_{\lambda_i \nu_i} \right)_{1 \leq i, j \leq n} \leq 1 \leq n \nu \leq n \leq n
\]

establishing Lemma 3.2.

\[\begin{align*}
\text{Proof of Proposition 3.1:} & \quad \text{In the integral} \\
\int_{U(n)} e^{\sum_{j=1}^{\infty} \text{Tr}(t_j M - s_j \tilde{M})} dM, \\
& \text{the shifts } t_i \mapsto t_i + t_i^{(0)}, s_i \mapsto s_i + s_i^{(0)} \text{ lead to a Toeplitz matrix of the form} \\
\int_{U(n)} e^{\sum_{j=1}^{\infty} \text{Tr}(t_j^{(0)} M - s_j^{(0)} \tilde{M})} e^{\sum_{j=1}^{\infty} \text{Tr}(t_j M - s_j \tilde{M})} dM \\
& \quad \int_{U(n)} e^{\text{Tr}(V(M))} e^{\sum_{j=1}^{\infty} \text{Tr}(t_j M - s_j \tilde{M})} dM \\
& \quad = \det \left( \int_{S^1} u^{\ell-k} e^{V(u)} e^{\sum_{j=1}^{\infty} (t_j u^j - s_j u^{-j})} \frac{du}{2\pi i u} \right)_{1 \leq \ell, k \leq n}
\end{align*}\]

with
\[V(u) := \sum_{j=1}^{\infty} (t_j^{(0)} u^j - s_j^{(0)} u^{-j}) \quad (3.0.12)\]

The following matrix of integrals has the form (3.0.5), using footnote 2 and (3.0.6),
\[\begin{align*}
& \left( \int_{S^1} u^{\ell-k} e^{V(u)} e^{\sum_{j=1}^{\infty} (t_j u^j - s_j u^{-j})} \frac{du}{2\pi i u} \right)_{1 \leq \ell, k \leq n} \\
& \quad = \left( \sum_{\alpha, \beta=0}^{\infty} s_\alpha(t)s_\beta(-s) \int_{S^1} u^{\ell-k+\alpha-\beta} e^{V(u)} \frac{du}{2\pi i u} \right)_{1 \leq \ell, k \leq n} \\
& \quad = E_n(t) \left( \int_{S^1} u^{\ell-k} e^{V(u)} \frac{du}{2\pi i u} \right)_{1 \leq \ell, k \leq \infty} E_n(-s)^T
\end{align*}\]
or, alternatively, the matrix above satisfies the differential equations (3.0.4), with initial condition

\[
\left( \int_{S^1} u^{\ell-k} e^{\lambda(u)} \frac{du}{2\pi i u} \right)_{1 \leq \ell, k \leq n}.
\]

Now, using Lemma 3.2, its determinant admits a Fourier expansion in Schur polynomials

\[
\int_{U(n)} e^{\sum_{t=1}^{\infty} \text{Tr}(t_j(0) M_j - s_j(0) M_j)} e^{\sum_{t=1}^{\infty} \text{Tr}(t_j M_j - s_j M_j)} dM = \sum_{\lambda, \mu \text{ such that } \lambda^T \mu \leq n} a_{\lambda\mu}(t(0), s(0)) s_\lambda(t) s_\mu(-s),
\]

where the Fourier coefficients can be expressed in two different ways, first as a determinant, using (3.0.8), and secondly as a Fourier series, to be explained below,

\[
a_{\lambda\mu}(t(0), s(0)) = \det \left( \int_{S^1} u^{\lambda-\mu+k} e^{\sum_{t=1}^{\infty} (t_j(0) u - s_j(0) u)} \frac{du}{2\pi i u} \right)_{1 \leq \ell, k \leq n} = \sum_{\nu \text{ with } \nu^T \leq n} s_\nu(t(0)) s_{\nu\lambda}(s(0)),
\]

To prove the second expression above, we apply (3.0.9) of Lemma 3.2. Indeed, switching points of view (i.e., \( t \to t(0), s \to s(0) \)), the initial condition for the differential equation (3.0.4) is given here by

\[
\left( \int_{S^1} u^{\ell-k} e^{\sum_{t=1}^{\infty} (t_j(0) u - s_j(0) u)} \frac{du}{2\pi i u} \right)_{1 \leq \ell, k \leq n} \bigg|_{t(0)=s(0)=0} = I_n,
\]

which implies that in the sum (3.0.9), the Fourier coefficients all vanish, except when \( \lambda = \nu \), for which they equal 1. Moreover, the sequences \( 1 \leq a_1 < a_2 < \ldots < a_n \) and \( 1 \leq b_1 < b_2 < \ldots < b_n \) are given by

\[
\begin{align*}
1 & \leq \lambda_n - n + c < \lambda_{n-1} - (n - 1) + c < \ldots < \lambda_1 - 1 + c \\
1 & \leq \mu_n - n + c < \mu_{n-1} - (n - 1) + c < \ldots < \mu_1 - 1 + c
\end{align*}
\]
with $c$ the smallest integer for which the inequality $\leq$ is satisfied; e.g.,

$$a_j = \lambda_{n-j+1} - (n - j + 1) + c = \lambda_{n-j+1} + j - (n + 1 - c).$$

The orthonormality of the Schur polynomials $\langle s_{\lambda}, s_{\mu} \rangle = \delta_{\lambda\mu}$ allows for an alternative way of expressing the “Fourier” coefficients $a_{\lambda\mu}(t^{(0)}, s^{(0)})$ in (3.0.2):

$$a_{\lambda\mu}(t^{(0)}, s^{(0)}) = \left. \sum_{\alpha, \beta \text{ such that } \alpha^\top 1, \beta^\top 1 \leq n} a_{\alpha\beta}(t^{(0)}, s^{(0)}) s_{\alpha}(t) s_{\beta}(-s) \right|_{t=0, s=0}$$

$$= \sum_{\alpha, \beta \text{ such that } \alpha^\top 1, \beta^\top 1 \leq n} a_{\alpha\beta}(t^{(0)}, s^{(0)}) s_{\alpha}(t) s_{\beta}(-s)$$

$$= s_{\lambda}(\tilde{\partial}_t) s_{\mu}(-\tilde{\partial}_s) \int_{U(n)} e^{\sum_{j=1}^{\infty} \text{Tr}(t^{(0)} M^j - s^{(0)} \bar{M}^j)} e^{\sum_{j=1}^{\infty} \text{Tr}(t^{(0)} M^j - s^{(0)} \bar{M}^j)} dM$$

$$= \int_{U(n)} s_{\lambda}(M) s_{\mu}(\bar{M}) e^{\sum_{j=1}^{\infty} \text{Tr}(t^{(0)} M^j - s^{(0)} \bar{M}^j)} dM,$$

upon using the identity $\langle f(t), g(t) \rangle := f(\tilde{\partial}_t) g(t) \bigg|_{t=0}, \quad \text{with } \tilde{\partial}_t := \left( \frac{\partial}{\partial t_1}, \frac{1}{2} \frac{\partial}{\partial t_2}, \frac{1}{3} \frac{\partial}{\partial t_3}, \ldots \right)$ (3.0.14)

$$s_{\lambda}(\tilde{\partial}_t) e^{\sum_{j=1}^{\infty} t_j M^j} \bigg|_{t=0} = s_{\lambda} \left( \text{Tr} M, \frac{1}{2} \text{Tr} M^2, \frac{1}{3} \text{Tr} M^3, \ldots \right) =: s_{\lambda}(M),$$

thus ending the proof of Proposition 3.1. ■

---

7. where the inner-product is defined by

$$\langle f(t), g(t) \rangle := f(\tilde{\partial}_t) g(t) \bigg|_{t=0}, \quad \text{with } \tilde{\partial}_t := \left( \frac{\partial}{\partial t_1}, \frac{1}{2} \frac{\partial}{\partial t_2}, \frac{1}{3} \frac{\partial}{\partial t_3}, \ldots \right)$$

8. Thus $s_{\lambda}(M)$ is viewed as a symmetric function of the eigenvalues $x_1, \ldots, x_n$ of the unitary matrix $M$
4 Matrix integrals and Random Walks

In this section, we consider some interesting special cases, based on special values of $t$ and $s$, for which the skew Schur polynomials take on the following form:

\[
\left. s_{\lambda \backslash \alpha}(t) \right|_{t_i = u \delta_{i1}} = u^{|\lambda \backslash \alpha|} s_{\lambda}(1, 0, \ldots)
\]

\[
= \frac{u^{|\lambda \backslash \alpha|}}{|\lambda \backslash \alpha|!} \# \begin{cases} 
\text{standard skew-tableaux} \\
\text{of shape } \lambda \backslash \alpha, \text{ filled}
\end{cases}
\text{with numbers } 1, \ldots, |\lambda \backslash \alpha| \}
\]

\[
= \frac{u^{|\lambda \backslash \alpha|}}{|\lambda \backslash \alpha|!} f_{\lambda \backslash \alpha}
\]

and

\[
\left. s_{\lambda \backslash \alpha}(t) \right|_{t_i = qu} = u^{|\lambda \backslash \alpha|} s_{\lambda \backslash \alpha}(q, q^2, q^3, \ldots)
\]

\[
= u^{|\lambda \backslash \alpha|} \# \begin{cases} 
\text{semi-standard skew-tableaux} \\
\text{of shape } \lambda \backslash \alpha, \text{ filled}
\end{cases}
\text{with numbers } 1, \ldots, q \}
\]

We now study the three integrals, appearing in the introduction:

4.1 Integral 1: \[
\int_{U(n)} s_{\lambda}(M) s_{\mu}(\bar{M}) e^{z \text{Tr}(M + \bar{M})} dM
\]

A generating function for

\[
b_{xy}^{(k)} = \# \begin{cases} 
\text{ways that } n \text{ non-intersecting walkers in } \mathbb{Z} \text{ move during}
\end{cases}
\text{k instants from } x_1 < x_2 < \ldots < x_n \text{ to } y_1 < y_2 < \ldots < y_n,
\text{where at each instant exactly one walker moves}
\text{either one step to the left, or one step to the right}
\end{cases}
\]

is given by the matrix integral

\[
\sum_{k \geq 0} \frac{z^k}{k!} b_{xy}^{(k)} = \int_{U(n)} s_{\lambda}(M) s_{\mu}(\bar{M}) e^{z \text{Tr}(M + \bar{M})} dM =: a_{\lambda \mu}(z).
\]
Furthermore, a generating function for the \(a_{\lambda\mu}\)'s is given by

\[
\sum_{\lambda,\mu \text{ such that } \lambda_1^\top,\mu_1^\top \leq n} a_{\lambda\mu}(z)s_\lambda(t)s_\mu(-s) = \int_{U(n)} e^z \Tr(M+\bar{M}) e^{\sum_{1}^{\infty} \Tr(t_iM'-s_i\bar{M}')} \, dM,
\]

where

\[
\mu_{n-k+1} := x_k - k + 1, \quad \lambda_{n-k+1} := y_k - k + 1. \quad \text{for } k = 1, \ldots, n. \quad (4.1.2)
\]

**Proof:** Consider the locus

\[
\mathcal{L}_1 = \{ \text{all } t^{(0)}_k = s^{(0)}_k = 0, \text{ except } t^{(0)}_1 = z, s^{(0)}_1 = -z \}.
\]

Then, since

\[
e^{\sum_{1}^{\infty} (t^{(0)}_i - s^{(0)}_i)} \bigg|_{\mathcal{L}_1} = e^{z(u+u^{-1})},
\]

we have, using (3.0.1),

\[
\int_{U(n)} e^z \Tr(M+\bar{M}) e^{\sum_{1}^{\infty} \Tr(t_iM'-s_i\bar{M}')} \, dM = \sum_{\lambda,\mu \text{ such that } \lambda_1^\top,\mu_1^\top \leq n} a_{\lambda\mu}(z)s_\lambda(t)s_\mu(-s), \quad (4.1.3)
\]

with

\[
a_{\lambda\mu}(z) = \int_{U(n)} s_\lambda(M)s_\mu(\bar{M}) e^z \Tr(M+\bar{M}) \, dM
\]

\[
= \det \left( \oint_{S^1} u^\lambda -\ell - \mu_k + k e^{z(u+u^{-1})} \frac{du}{2\pi iu} \right)_{1\leq\ell,k\leq n}
\]

\[
= \sum_{\nu \text{ with } \nu_1^\top \leq n} s_{\nu\setminus\lambda}(t^{(0)})s_{\nu\setminus\mu}(-s^{(0)}) \bigg|_{\mathcal{L}_1}
\]

\[
= \sum_{\nu \text{ with } \nu_1^\top \leq n} \frac{z^{[\nu\setminus\lambda]}}{[\nu\setminus\lambda]!} f_{\nu\setminus\lambda} \frac{z^{[\nu\setminus\mu]}}{[\nu\setminus\mu]!} f_{\nu\setminus\mu}
\]

\[
= \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{k!}{k_1! k_2!} \sum_{\nu \text{ with } \nu_1^\top \leq n} f_{\nu\setminus\lambda} f_{\nu\setminus\mu}
\]
\[
\sum_{k \geq 0} \frac{z^k}{k!} \# \left\{ \text{ways that } n \text{ non-intersecting walkers in } \mathbb{Z} \text{ move in } k \text{ steps from } x_1 < x_2 < \ldots < x_n \text{ to } y_1 < y_2 < \ldots < y_n \right\}
\]
as a consequence of (2.1.2), where
\[
k_1 = \frac{1}{2}(k - |\lambda| + |\mu|), \quad k_2 = \frac{1}{2}(k + |\lambda| - |\mu|).
\]

An alternative way of proving the final formula is to invoke the D. André reflection principle. Indeed
\[
a_{\lambda \mu}(z) = \det(m^{\lambda, \mu}) = \det \left( e^{z(u+w-1)} \right)_{1 \leq i,j \leq n} \text{ using (3.0.2) and (4.1.2)}
\]
\[
= \sum_{w \in W} (-1)^{\sigma(w)} \prod_{i=1}^{n} \left. e^{z(u_i+u_i-1)} \right|_{u_i=x_{w(i)}}
\]
\[
= \sum_{k=0}^{\infty} \# \left\{ \text{walks of } k \text{ steps from } x \rightarrow y \text{ in } \mathbb{Z}^n \text{ within } \{u_1 < \ldots < u_n\} \right\} \frac{z^k}{k!}, \text{ using Theorem 2.1}
\]
\[
= \sum_{k \geq 0} \# \left\{ \text{ways that } n \text{ non-intersecting walkers move in } k \text{ steps from } x_1 < x_2 < \ldots < x_n \text{ to } y_1 < y_2 < \ldots < y_n \right\} \frac{z^k}{k!}
\]
\[
= \sum_{k \geq 0} b_{x,y}^{(k)} \frac{z^k}{k!},
\]

### 4.2 Integral 2:
\[
\int_{U(n)} s_\lambda(M)s_\mu(\bar{M}) \det(I + M)^q e^{z \text{Tr} \bar{M}} dM
\]

A generating function for
\[
b_{x,y}^{(k)} = \# \left\{ \text{ways that } n \text{ non-intersecting walkers move during } q + k \text{ instants from } x_1 < \ldots < x_n \text{ to } y_1 < \ldots < y_n, \text{ where at the instants 1 to } q, \text{ walkers may move one step to the right, or stay put, and at the instants } q + 1, \ldots, q + k \text{ exactly one walker moves one step to the left, with total } \#\{\text{effective moves}\} = 2k + \sum_{i=1}^{n} (y_i - x_i). \right\}
\]

(4.2.1)
is given by the matrix integral
\[ \sum_{k \geq 0} \frac{z^k}{k!} b^{(k)}_{xy} = \int_{U(n)} s_\lambda(M) s_\mu(\bar{M}) \det(I + M)^q e^{z \text{Tr} \, M} dM =: a_{\lambda \mu}(z). \]

Furthermore, a generating function for the \( a_{\lambda \mu} \)’s is given by
\[ \sum_{\lambda, \mu \text{ such that } \lambda_1, \mu_1 \leq n} a_{\lambda \mu}(z) s_\lambda(t) s_\mu(-s) = \int_{U(n)} \det(I + M)^q e^{z \text{Tr} \, M} e^{\sum_{i=1}^\infty \text{Tr}(t_i M^i-s_i \bar{M}^i)} dM, \]

with
\[ \mu_{n-j+1} := x_j - j + 1, \quad \lambda_{n-j+1} := y_j - j + 1, \quad \text{for } j = 1, \ldots, n. \]

**Proof:** Consider the locus
\[ \mathcal{L}_2 = \{ \text{all } it_{\lambda}^{(0)} = -q(-1)^i, is_{\lambda}^{(0)} = -z \delta_{i1} \}. \]

Then, using
\[ e^{\sum_{i=1}^\infty t_i u^i} \bigg|_{it_i = qx^i} = e^{q \sum_{i=1}^\infty \frac{(x^i)^i}{i}} = (1 - xu)^{-q}, \]

we have by (3.0.1) that
\[ \int_{U(n)} \det(I + M)^q e^{z \text{Tr} \, M} e^{\sum_{i=1}^\infty \text{Tr}(t_i M^i-s_i \bar{M}^i)} dM = \sum_{\lambda, \mu \text{ such that } \lambda_1, \mu_1 \leq n} a_{\lambda \mu}(z) s_\lambda(t) s_\mu(-s). \]

Using
\[ s_{(\lambda \setminus \alpha)^\tau}(t) = (-1)^{|\lambda \setminus \alpha|} s_{\lambda \setminus \alpha}(-t), \]
\[ s_{\lambda \setminus \alpha}(-t_1, t_2, -t_3, \ldots) = (-1)^{|\lambda \setminus \alpha|} s_{\lambda \setminus \alpha}(t), \]

the Fourier coefficients \( a_{\lambda \mu}(z) \) of (3.0.2) have the following interpretation:
\[ a_{\alpha \beta}(z) = \det \left( \int_{S^1} u^{\alpha_i - \ell - \beta_k + k} e^{\sum_{i=1}^\infty (t_j^{(0)} u^j - s_j^{(0)} u^{-j})} \frac{du}{2 \pi i u} \right)_{1 \leq \ell, k \leq n} \bigg|_{it_i = -q(-1)^i, is_i = -z \delta_{i1}} \]
\[ = \det \left( \int_{S^1} u^{\alpha_i - \ell - \beta_k + k} (1 + u)^q e^{zu^{-1}} \frac{du}{2 \pi i u} \right)_{1 \leq \ell, k \leq n} \]

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\[
\sum_{\lambda \supseteq \alpha, \beta} s_{\lambda}(t) \big|_{t_i = -q(-1)^i} s_{\lambda}(s) \big|_{s_i = -z \delta_{i1}}
\]

\[
= \sum_{\lambda \supseteq \alpha, \beta} s_{(\lambda \setminus \alpha)^\top} (-t) \big|_{t_i = -q(-1)^i} s_{(\lambda \setminus \beta)^\top} (s) \big|_{s_i = -z \delta_{i1}} (-1)^{|\lambda \setminus \alpha| + |\lambda \setminus \beta|} f_{\lambda \setminus \beta}
\]

\[
= \sum_{\lambda \supseteq \alpha, \beta} s_{(\lambda \setminus \alpha)^\top} (t) \big|_{t_i = q(-1)^i} (-1)^{|\lambda \setminus \alpha|} z^{(\lambda \setminus \beta)} |\lambda \setminus \beta|! f_{\lambda \setminus \beta}
\]

\[
= \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{\lambda \supseteq \alpha, \beta \atop \lambda_1 \leq n} s_{(\lambda \setminus \alpha)^\top} \left( q, \frac{q}{2}, \frac{q}{3}, \ldots \right) f_{\lambda \setminus \beta}
\]

\[
= \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{\lambda \supseteq \alpha, \beta \atop \lambda_1 \leq n} \left\{ \begin{array}{l}
\text{all (P, Q), with } P \text{ semi-standard and } \\
Q \text{ standard skew-tableaux of arbitrary shape } (\lambda \setminus \alpha)^\top \text{ and } (\lambda \setminus \beta)^\top, \text{ filled with numbers 1, \ldots, } q \text{ and } 1, \ldots, k, \\
\text{with } \lambda \vdash k + |\beta|, \lambda_1 \leq n \text{ and fixed } \alpha, \beta
\end{array} \right\}
\]

\[
= \sum_{k=0}^{\infty} \frac{z^k}{k!} b_{xy}^{(k)}, \quad (4.2.2)
\]

upon setting

\[x_j := \alpha_{n-j+1} + j - 1 \quad \text{and} \quad y_j := \beta_{n-j+1} + j - 1.\]

Then\footnote{The number of “effective steps” counts the actual steps taken by all walkers; i.e., two walkers walk simultaneously count for two steps, a walker not walking contributes nothing!}

\[
\text{Total } \# \{ \text{effective steps} \} = T_R + T_L = 2|\lambda| - |\alpha| - |\beta|
\]

\[
= 2(|\lambda| - |\beta|) - |\alpha| + |\beta|
\]

\[
= 2k + \sum_{1}^{n} (y_i - x_i),
\]

proving statement (4.2.1).
The last equality in (4.2.2) requires a little explanation. We refer here to figure 1 in section 2.1, but somewhat modified, with \( \mu \to \alpha \) and \( \nu \to \beta \). Namely, the left diagram corresponds to right steps and the right diagram to left steps. The last row corresponds to the first walker \( x_1 = \alpha_n \) and the first to the last walker \( x_n = \alpha_1 + n - 1 \). With this slight change, a semi-standard skew-tableau of shape \( (\lambda \setminus \alpha)^\top \), with \( \lambda^\top_1 \leq n \), filled with numbers \( 1, \ldots, q \), is tantamount to a skew-partition \( \lambda \setminus \alpha \), with \( \lambda^\top_1 \leq n \), filled with numbers among \( 1, \ldots, q \), which are strictly increasing from left to right and non-decreasing from top to bottom; not all numbers need to be used. If some number \( 1 \leq r \leq q \) is not used in the skew Young diagram, then no walker moves at that instant. So, the pair \( (P, Q) \), as in figure 1, corresponds to a walk departing from \( x_k = \alpha_{n-k+1} + k - 1 \) and ending up at \( y_k = \beta_{n-k+1} + k - 1 \). The most-left walker moves to the right only at the moments indicated by the integer appearing in the first row, the second walker moves right at the moments indicated in the second row, etc... Thus, corresponding to \( P \), two walkers can simultaneously move right!

4.3 Integral 3:

\[
\int_{U(n)} s_\lambda(M)s_\mu(\bar{M}) \det(I + zM)^p \det(I + z\bar{M})^q dM
\]

A generating function for

\[
\begin{align*}
\binom{\lambda}{\mu}^{(k)}_{xy} &= k! \# \left\{ \begin{array}{l}
\text{ways that } n \text{ non-intersecting walkers move during } p + q \\
\text{instants from } x_1 < \ldots < x_n \text{ to } y_1 < \ldots < y_n, \text{ where at the instants } 1 \text{ to } p, \text{ walkers may move one step to the right, or stay put, and at the instants } p + 1, \ldots, p + q \text{ walkers may move one step to the left or stay put, with total } \# \{ \text{effective moves} \} = k.
\end{array} \right. 
\end{align*}
\]

is given by the matrix integral

\[
\sum_{k \geq 0} \frac{z^k}{k!} \binom{\lambda}{\mu}^{(k)}_{xy} = \int_{U(n)} s_\lambda(M)s_\mu(\bar{M}) \det(I + zM)^p \det(I + z\bar{M})^q dM =: a_{\lambda\mu}(z).
\]
Furthermore, a generating function for the $a_{\lambda\mu}$'s is given by

$$
\sum_{\lambda, \mu \text{ such that } \lambda_1^\top \mu_1^\top \leq n} a_{\lambda\mu}(z)s_\lambda(t)s_\mu(-s)
= \int_{U(n)} \det(I + zM)^p \det(I + z\bar{M})^q e^{\sum_{i=1}^\infty \text{Tr}(t_iM^i - s_i\bar{M}^i)} dM,
$$

with

$$
\mu_{n-j+1} := x_j - j + 1, \quad \lambda_{n-j+1} := y_j - j + 1,
$$

for $j = 1, \ldots, n$.

**Proof:** Consider the locus:

$$
\mathcal{L}_3 = \{ \text{all } it_i = -p(-z)^i, is_i = q(-z)^i \}.
$$

Then, using

$$
e^{\sum_{i=1}^\infty (t_iu^i - s_iu^{-i})} \bigg|_{\mathcal{L}_3} = (1 + zu)^p(1 + zu^{-1})^q,
$$

we have by (3.0.2),

$$
a_{\alpha\beta}(z) = \det \left( \int_{S^1} u^{\alpha_\ell - \beta_k + k}(1 + zu)^p(1 + zu^{-1})^q \frac{du}{2\pi i u} \right)_{1 \leq \ell, k \leq n}
= \sum_{\lambda \text{ with } \lambda_1^\top \leq n} s_{(\lambda\alpha)}(t)s_{(\lambda\beta)}(-s) \bigg|_{\mathcal{L}_3}
= \sum_{\lambda \text{ with } \lambda_1^\top \leq n} s_{(\lambda\alpha)^\top}(-t)(-1)^{|\lambda\alpha|^t}s_{(\lambda\beta)^\top}(s)(-1)^{|\lambda\beta|^s} \bigg|_{\mathcal{L}_3}
= \sum_{\lambda \text{ with } \lambda_1^\top \leq n} s_{(\lambda\alpha)^\top} \left( p, \frac{p}{2}, \frac{p}{3}, \ldots \right) z^{(|\lambda\alpha|^t)}s_{(\lambda\beta)^\top} \left( q, \frac{q}{2}, \frac{q}{3}, \ldots \right) z^{(|\lambda\beta|^s)}
= \sum_{k=0}^\infty z^k \sum_{\lambda \text{ with } \lambda_1^\top \leq n \atop |\lambda| = \frac{k}{2}(|\lambda\alpha|^t + |\lambda\beta|^s)} \lambda_{\alpha}^\top \left( p, \frac{p}{2}, \ldots \right) \lambda_{\beta}^\top \left( q, \frac{q}{2}, \ldots \right)
$$

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as given in (4.3.1).

5 The action of Virasoro on Schur polynomials

A border-strip $h \in B(i)$ is a connected skew-shape $\lambda \setminus \mu$ containing $i$ boxes, with no $2 \times 2$ square. The height of a border strip $h$ is defined as

$$\text{ht } h := \#\{\text{rows in } h\} - 1. \quad (5.0.1)$$

Consider the Virasoro operator in the variables $t_1, t_2, \ldots$ for $k \in \mathbb{Z}$,

$$V_k(t) = \frac{1}{2} \sum_{i+j=k} \frac{\partial^2}{\partial t_i \partial t_j} + \sum_{-i+j=k} it_i \frac{\partial}{\partial t_j} + \frac{1}{2} \sum_{-i-j=k} (it_i)(jt_j) \quad (5.0.2)$$

In this section we study the action of the Virasoro operators on Schur polynomials $s_\lambda$. From the Murnaghan-Nakayama rule [13], stated in Proposition 5.2, it follows that (Corollary 5.3)

$$nt_n s_\lambda(t) = \sum_{\mu : \lambda \setminus \mu \in B(n)} (-1)^{\text{ht}(\mu \setminus \lambda)} s_\mu(t)$$

$$\frac{\partial}{\partial t_n} s_\lambda(t) = \sum_{\lambda \mu : \mu \in B(n)} (-1)^{\text{ht}(\lambda \setminus \mu)} s_\mu(t). \quad (5.0.3)$$

In view of this, one would expect $V_n s_\lambda$ to be expressible as an infinite sum of Schur polynomials. This is not so: acting with Virasoro leads to the same precise sum, except for different coefficients:
Theorem 5.1 The Virasoro operator acts as follows on Schur polynomials

\[ V_{-n}s_{\lambda} = \sum_{\mu: \lambda \in B(n)} d_{\lambda\mu}^{(-n)} s_{\mu} \]  \hspace{1cm} (5.0.4)

\[ V_{n}s_{\lambda} = \sum_{\mu: \lambda \in B(n)} d_{\mu\lambda}^{(-n)} s_{\mu} \]  \hspace{1cm} (5.0.5)

with \((n \geq 1)\)

\[ d_{\lambda\mu}^{(-n)} = \sum_{i \geq 1} \sum_{\nu: \lambda \in B(i) \land \mu \in B(n+i) \land \lambda \subseteq \mu} (-1)^{ht(\lambda|\nu) + ht(\mu|\nu)} \]

\[ + \frac{1}{2} \sum_{i=1}^{n} \sum_{\nu: \nu \in B(i) \land \mu \in B(n-i)} (-1)^{ht(\nu|\lambda) + ht(\mu|\nu)}. \]  \hspace{1cm} (5.0.6)

We need a few combinatorial preliminaries. Given

\[ \alpha := (\alpha_1, \alpha_2, \ldots), \quad \alpha_i \in \mathbb{Z}_{\geq 0}, \]  \hspace{1cm} (5.0.7)

define

\[ T := \text{border-strip tableau of shape } \lambda \setminus \mu \text{ and type } \alpha, \quad \sum \alpha_i = |\lambda \setminus \mu|, \]

is an assignment of integers \((\geq 0)\) to the boxes of \(\lambda \setminus \mu\), such that

(1) the positive integers are weakly increasing from left to right and top to bottom

(2) the integer \(i\) appears \(\alpha_i\) times

(3) \{squares containing \(i\}\} is a border strip \(B_i\).
Define
\[ \text{ht } B_i = \#\{\text{rows in } B_i\} - 1 \]
\[ \text{ht } T = \sum_i \text{ht } B_i \]
\[ \chi^{\lambda\mu}(\alpha) = \sum \left\{ \begin{array}{c}
\text{all border-strip tableaux } T \\
\text{of shape } \lambda\mu \text{ and type } \alpha
\end{array} \right\} (-1)^{\text{ht } T} \quad (5.0.8) \]

Given \(\alpha\) as in (5.0.7), define
\[ p_{\alpha} = p_{\alpha_1} p_{\alpha_2} \ldots = \left( \sum_i x_1^{\alpha_1} \right) \left( \sum_i x_1^{\alpha_2} \right) \ldots = \alpha_1 t_{\alpha_1} \alpha_2 t_{\alpha_2} \ldots \]

The Schur polynomials \(s(t)\) form an orthonormal basis in the space of symmetric functions, for the customary inner-product (3.0.14) in footnote 6, between symmetric functions, where \(k t_k = \sum_{i \geq 1} x_i^k:\)
\[ \langle s_{\lambda}, s_{\mu} \rangle = \delta_{\lambda\mu}. \quad (5.0.9) \]

The identity
\[ \langle it_i s_{\lambda}, s_{\mu} \rangle = \langle s_{\lambda}, \frac{\partial}{\partial t_i} s_{\mu} \rangle \quad (5.0.10) \]
shows that, with regard to the inner-product (3.0.14),
\[ \left( \frac{\partial}{\partial t_i} \right)^T = it_i, \quad (5.0.11) \]
and so the matrices representing \(it_i\) and \(\partial/\partial t_i\) in the orthonormal basis are transpose of each other. Also, for \(k \geq 0,\)
\[ V_k^T = \left( \frac{1}{2} \sum_{i+j=k} \frac{\partial^2}{\partial t_i \partial t_j} + \sum_{-i-j=k} it_i \frac{\partial}{\partial t_j} + \frac{1}{2} \sum_{-i-j=k} it_i j t_j \right)^T \]
\[ = \frac{1}{2} \sum_{i+j=k} it_i j t_j + \sum_{-i-j=k} j t_j \frac{\partial}{\partial t_i} + \frac{1}{2} \sum_{-i-j=k} \frac{\partial^2}{\partial t_i \partial t_j} \]
\[ = V_{-k}^T; \quad (5.0.12) \]
hence also the matrices associated with \(V_k\) and \(V_{-k}\) are transpose of each other.

We now have: 33
Proposition 5.2 (Murnaghan-Nakayama rule) [13] We have
\[ p_\alpha s_\lambda = \sum_{\mu} \chi^{\mu \backslash \lambda}(\alpha)s_\mu \] (5.0.13)

Corollary 5.3 The following holds:
\[ it_i s_\lambda(t) = \sum_{\mu \backslash \lambda \in B(i)} (-1)^{ht(\mu \backslash \lambda)}s_\mu(t) \]
\[ \frac{\partial}{\partial t_i}s_\lambda(t) = \sum_{\lambda \backslash \mu \in B(i)} (-1)^{ht(\lambda \backslash \mu)}s_\mu(t). \] (5.0.14)

Example: For a partition \( \lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0) \), Corollary 5.3 implies:
\[ \frac{\partial}{\partial t_1}s_\lambda(t) = \sum_{\nu \text{ such that } |\lambda \backslash \nu|=1} s_\nu(t) = \sum_{1 \leq i \text{ such that } \lambda_{i+1} < \lambda_i} s_{\lambda-e_i}(t) = \sum_{1}^{n} s_{\lambda-e_i}(t), \]
\[ t_1s_\lambda(t) = \sum_{\nu \text{ such that } |\nu \backslash \lambda|=1} s_\nu(t) = \sum_{1 \leq i \text{ such that } \lambda_i < \lambda_{i-1}} s_{\lambda+e_i}(t) = \sum_{1}^{n+1} s_{\lambda+e_i}(t) \]
\[ \frac{\partial}{\partial t_2}s_\lambda(t) = \sum_{\lambda \backslash \nu \in B(2)} (-1)^{ht(\lambda \backslash \nu)} s_\nu(t) = \sum_{1}^{n} s_{\lambda-2e_i}(t) - \sum_{1}^{n-1} s_{\lambda-e_i-e_i+1}(t), \]
with \( B(2) = \{ \begin{array}{c} \square \\square \end{array}, \begin{array}{c} \square \\square \\square \end{array} \} \).

where \( e_i = (0, \ldots, 0, \underbrace{1}_{i}, 0, \ldots)^T \). Note that the right most sum in each of the expressions means “wherever it makes sense!”; i.e., you sum over \( i \), wherever \( \lambda - e_i, \lambda + e_i, \lambda - 2e_i \) and \( \lambda - e_i - e_i+1 \) are again partitions.
Proof of Corollary 5.3: The first identity follows by applying the Murnaghan-Nakayama rule to \( p_i = \sum_{k \geq 1} x_k^i = it_i \). In that case,

\[
\chi^{\mu\setminus\lambda}(i) = \sum_{\substack{\text{border-strip tableaux } T \\ \text{of shape } \mu\setminus\lambda \text{ and type } i}} (-1)^{htT} \nonumber
\]

= \((-1)^{ht(\mu\setminus\lambda)}\), with \( \mu\setminus\lambda \in B(i) \),

since the only border-strip tableau \( T \) of shape \( \mu\setminus\lambda \) and type \( i \) is a border-strip \( \mu\setminus\lambda \in B(i) \).

To prove the second relation (5.0.14), let

\[
\frac{\partial}{\partial t_i} s_\mu(t) = \sum_\nu c_{\mu\nu}s_\nu(t). \nonumber
\]

Then using the duality (5.0.11), the result follows immediately, by taking the transpose of the first relation (5.0.14).

Proof of Theorem 5.1: At first, it suffices to prove (5.0.4) for \( n \geq 1 \); the second identity (5.0.5) follows immediately from the duality (5.0.12).

**Step 1:** From (5.0.2), (5.0.9), (5.0.10) and from Corollary 5.3, it follows that

\[
d^{(-n)}_{\lambda \mu} = \langle V_{-n}s_\lambda, s_\mu \rangle \nonumber
\]

\[
= \sum_{i \geq 1} \left\langle \frac{\partial}{\partial t_i} s_\lambda, \frac{\partial}{\partial t_{n+i}} s_\mu \right\rangle + \frac{1}{2} \sum_{i=1}^{n-1} \left\langle it_is_\lambda, \frac{\partial}{\partial t_{n-i}} s_\mu \right\rangle \nonumber
\]

\[
= \sum_{i \geq 1} \left( \sum_{\lambda \setminus \nu \in B(i)} (-1)^{ht(\lambda\setminus\nu)} s_\nu, \sum_{\mu\setminus\nu' \in B(i+n)} (-1)^{ht(\mu\setminus\nu')} s_{\nu'} \right) \nonumber
\]

\[
+ \frac{1}{2} \sum_{i=1}^{n-1} \left( \sum_{\lambda \setminus \nu \in B(i)} (-1)^{ht(\lambda\setminus\nu)} s_\nu, \sum_{\mu\setminus\nu' \in B(n-i)} (-1)^{ht(\mu\setminus\nu')} s_{\nu'} \right) \nonumber
\]
\[ V_{-n}s_{\lambda} = \sum_{\mu} \langle V_{-n}s_{\lambda}, s_{\mu} \rangle s_{\mu} \]

\[ = \sum_{\mu} s_{\mu} \sum_{i \geq 1} \frac{1}{2} \sum_{\nu \text{ such that } \lambda \setminus \nu \in B(i) \text{ and } \mu \setminus \nu \in B(i + n)} (-1)^{ht(\lambda \setminus \nu) + ht(\mu \setminus \nu)} \]

\[ + \frac{1}{2} \sum_{i=1}^{n-1} \sum_{\nu \text{ such that } \nu \setminus \lambda \in B(i) \text{ and } \mu \setminus \nu \in B(n - i)} (-1)^{ht(\nu \setminus \lambda) + ht(\mu \setminus \nu)} \]

\[ = \sum_{\mu} \sum_{i \geq 1} \frac{1}{2} \sum_{h_i \in B(i), h_{n+i} \in B(n + i)} (-1)^{ht(h_i) + ht(h_{n+i})} \]

\[ \lambda \setminus h_i = \mu \setminus h_{n+i} \text{ for some partition } \nu \]

\[ = \sum_{\mu} \sum_{i=1}^{n-1} \frac{1}{2} \sum_{h_i \in B(i), h_{n-i} \in B(n - i)} (-1)^{ht(h_i) + ht(h_{n-i})} \]

\[ \lambda \setminus h_i = \mu \setminus h_{n-i} = \nu \text{ for some partition } \nu \]

\[ (5.0.15) \]

**Step 2:** Having shown that

\[ V_{-n}s_{\lambda} = \sum_{\mu} d_{\lambda \mu}^{(-n)} s_{\mu}, \]

we now prove that

\[ V_{-n}s_{\lambda} = \sum_{\mu \geq \lambda} d_{\lambda \mu}^{(-n)} s_{\mu}. \]

The first sum in (5.0.15) contains a summation over \( h_i \in B(i), h_{n+i} \in B(n + i) \), such that \( \lambda \setminus h_i = \mu \setminus h_{n+i} = \) some partition \( \nu \). Therefore \( \mu = \lambda \setminus h_i + h_{n+i} \). We distinguish two cases:
(i) \( \lambda \subset \mu \), in which case \( h_i \subset h_{i+n} \)

(ii) \( \lambda \not\subset \mu \Leftrightarrow h_i \not\subset h_{i+n} \)

Assume we are in case (ii). Then

either (a) \( h_i \cap h_{i+n} = \emptyset \)

or (b) \( h_i \cap h_{i+n} = b \neq \emptyset \)

so in case (ii,b), we have

\[
\begin{align*}
h_i \setminus b &= \alpha \neq \phi, \quad h_{i+n} \setminus b &= \beta \neq \phi
\end{align*}
\]

and thus \( \alpha \) and \( \beta \) are border-strips such that

\[
h_i = \alpha + b \quad \text{and} \quad h_{i+n} = \beta + b.
\]

Therefore, since \( \lambda \setminus h_i = \mu \setminus h_{n+i} \), we have found border strips \( \alpha \) and \( \beta \), such that

\[
\lambda \setminus \alpha = \mu \setminus \beta \quad \alpha \in B(i - |b|), \quad \beta \in B(n + i - |b|)
\]

and

\[
\lambda \setminus (\alpha + b) = \mu \setminus (\beta + b) = \text{some partition } \nu, \quad \alpha + b \in B(i), \quad \beta + b \in B(n + i).
\]  

Hence, in the first sum of (5.0.15), the coefficient of \( s_\mu \) contains

\[
(-1)^{ht(\alpha)+ht(\beta)} + (-1)^{ht(\alpha+b)+ht(\beta+b)} = 0,
\]

and therefore does not contribute. The point is that, if, say, \( \beta \) is above \( \alpha \), then the first statement in (5.0.16) forces \( \beta \) and \( b \) to have boxes lying on the same row, and \( \alpha \) and \( b \) on the same column; hence

\[
\begin{align*}
ht(\beta + b) &= ht(\beta) + ht(b) \\
ht(\alpha + b) &= ht(\alpha) + ht(b) + 1
\end{align*}
\]

and thus

\[
(ht(\alpha + b) + ht(\beta + b)) - (ht(\alpha) + ht(\beta)) = 2ht(b) + 1,
\]

which is odd.
As an illustration, we give the following example. Let

\[ \lambda = \begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{array} \quad \mu = \begin{array}{c}
\ast \ast \\
\ast \ast \\
\ast \ast \\
\ast \ast \\
\end{array} \]

\[ h_7 = \begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{array} \quad h_{10} = \begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{array} \]

and \( h_7 \cap h_{10} = b = \begin{array}{c}
\ast \ast \\
\ast \ast \\
\ast \ast \\
\ast \ast \\
\end{array} \) \hspace{1cm} (5.0.17)

\[ \alpha = h_7 - b = h_2 = \begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{array} \quad \text{and} \quad \beta = h_{10} - b = h_5 = \begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{array} \]

Clearly

\[ \lambda - h_7 = \mu - h_{10}. \]

where \( h_7 \) and \( h_{10} \) are border strips, having a border strip \( b \) in common (\( b \) is given by the boxes containing stars \( \ast \)). In the first sum of (5.0.15) above, the couples of border-strips \( (h_2, h_5) \) and \( (h_7, h_{10}) \) contribute to the sum, as follows:

\[ (-1)^{ht(\alpha)+ht(\beta)} + (-1)^{ht(\alpha+b)+ht(\beta+b)} = (-1)^{ht(h_2)+ht(h_5)} + (-1)^{ht(h_7)+ht(h_{10})} = (-1)^3 + (-1)^5 + 5 = 0, \]

and thus

\[ (ht(h_7) + ht(h_{10})) - (ht(h_2) + ht(h_5)) = 2ht(b) + 1 = 7. \quad (5.0.18) \]
In case (ii,a), we consider $h_i$ and $h_{n+i}$ ($h_i \cap h_{n+i} = \emptyset$) with the unique connector $b$, which make $h_i + h_{n+i} + b$ a border strip and again, by the same reason given before,

$$(-1)^{ht(h_i)+ht(h_{n+i})} + (-1)^{ht(h_i+b)+ht(h_{n+i}+b)} = 0,$$

(5.0.19)

In Figure 2, consider $h_2$ and $h_5$ with $h_2 \cap h_5 = \emptyset$. The border strip $b$, as in Figure 2, is the unique connector of $h_2$ and $h_5$. So (5.0.19) is illustrated again by formula (5.0.18).

We conclude that only case (i) remains, i.e., where $\lambda \subset \mu$ in which case $h_i \subset h_{i+n}$. Therefore

$$\mu = (\lambda \setminus h_i) + h_{n+i} = \lambda + (h_{n+i} \setminus h_i), \quad \text{with} \quad |\mu \setminus \lambda| = |h_{n+i} \setminus h_i| = n.$$  

(5.0.20)

In the second summation of (5.0.15), it is obvious that

$$\lambda \subset \mu, \quad h_i \cap h_{n-i} = \emptyset \quad \text{and} \quad |\mu \setminus \lambda| = n.$$  

(5.0.21)

So, altogether, we have shown that

$$V_{-n} s_\lambda = \sum_{\mu \setminus \lambda \in B(n), |\mu \setminus \lambda| = n} d_{\lambda \mu}^{(-n)} s_\mu.$$  

Step 3: Finally, we show that

$$V_{-n} s_\lambda = \sum_{\mu \setminus \lambda \in B(n)} d_{\lambda \mu}^{(-n)} s_\mu.$$  

Indeed, if $\mu \setminus \lambda \notin B(n)$, then, by (5.0.15), we have two possible contributions to the coefficient of $s_\mu$; we first deal with the case as in (5.0.20), and

$$\lambda \subset (\lambda \setminus h_i) + h_{i+n} = \mu, \quad \text{with} \quad \mu - \lambda \notin B(n)$$

Then $\mu \setminus \lambda = h_{i+n} \setminus h_i = \sigma$ is not a border strip, while $h_i + \sigma$ is a border strip and hence connected. Thus $\sigma$ must come in exactly two border strips; say

$$h_{i+n} \setminus h_i = \alpha + \beta,$$  

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and thus we are precisely in the situation of (5.0.21), with either $h_i' = \alpha$, $h_{n-i'} = \beta$, or $h_i' = \beta$, $h_{n-i'} = \alpha$.

$$\lambda \subset \mu, \ h_{i'} \cap h_{n-i'} = \emptyset \text{ and } \mu \setminus \lambda = h_{i'} + h_{n-i'}.$$ 

That puts us in the situation of the second contribution to $s_{\mu}$ in (5.0.15).

In Figure 3 below, $\lambda$ is the Young diagram to the left of the bold face line and $\mu$ denotes everything; $\mu \setminus \lambda$ consists of the two border strips $\alpha$ and $\beta$ below, while $h_{n+i}$ is the large border strip bordering $\lambda$ on the right.
In the first contribution to \( s_\mu \) in (5.0.15),
\[
\lambda \setminus h_i = \text{partition} = \mu \setminus h_{n+i}
\]
contributes
\[
(-1)^{ht(h_i)+ht(h_{i+n})}.
\] (5.0.22)

In the second contribution to \( s_\mu \) in (5.0.15),
\[
\mu \setminus h_{n-\nu'} = \nu, \quad \nu \setminus h_{\nu'} = \lambda
\]
contributes, upon first setting \( \alpha = h_{\nu'}, \quad \beta = h_{n-\nu'} \), and then setting \( \alpha = h_{n-\nu}, \quad \beta = h_{\nu} \), yielding
\[
2 \left( \frac{1}{2} \right) (-1)^{ht(h_{\nu'})+ht(h_{n-\nu'})}.
\] (5.0.23)

The sum of (5.0.22) and (5.0.23) vanishes, the point being that
\[
ht(h_i) + ht(h_{i+n}) = ht(h_i) + ht(h_{\alpha+h_i+\beta})
\]
\[
= ht(h_i) + ht(h_{\alpha}) + ht(h_{\beta}) + ht(h_i) + 1
\]
\[
= ht(h_{\nu'}) + ht(h_{n-\nu'}) + 2 ht(h_i) + 1;
\] (5.0.24)

this is a consequence of the fact that the upper strip \( \alpha \) and \( h_i \) must have a cell on the same row, where they meet, since \( \mu \setminus (\alpha + \beta) = \lambda \) is a partition. For the same reason, \( \beta \) and \( h_i \) cannot have a cell on the same row. Thus the two contributions to \( s_\mu \) in (5.0.15) must cancel out in unique pairs, if \( \mu \setminus \lambda \notin B(n) \). This ends the proof of Theorem 5.1.

**Corollary 5.4** For a partition \( \lambda \) with at most \( n \) rows, the following holds:
\[
V_0 s_\lambda = |\lambda| \ s_\lambda
\] (5.0.25)
\[
V_{-1} s_\lambda = \sum_{1 \leq i \text{ such that } \lambda_i < \lambda_{i-1}} (\lambda_i - i + 1) \ s_{\lambda+e_i} = \sum_{i=1}^{n+1} (\lambda_i - i + 1) \ s_{\lambda+e_i}
\] (5.0.26)
\[
V_1 s_\lambda = \sum_{1 \leq i \text{ such that } \lambda_i+1 < \lambda_i} (\lambda_i - i) \ s_{\lambda-e_i} = \sum_{i=1}^{n} (\lambda_i - i) \ s_{\lambda-e_i}
\] (5.0.27)
where $e_j$ stands for a box added to the right of the $j$th row of $\lambda$. The second sum in (5.0.26) and (5.0.27) refers to a sum over all $i$, with the understanding that $s_{\lambda \pm e_i} = 0$, if $\lambda \pm e_i$ is not a Young diagram.

Remark: Compare with

\[
\frac{\partial}{\partial t^1} s_\lambda = \sum_{i=1}^{n+1} s_{\lambda + e_i} \tag{5.0.28}
\]

\[
\frac{\partial}{\partial t^2} s_\lambda(t) = \sum_{i=1}^{n} s_{\lambda - 2e_i(t)} - \sum_{i=1}^{n-1} s_{\lambda - e_i - e_{i+1}(t)} \tag{5.0.29}
\]

Proof: To see identity (5.0.25), observe that for monomials

\[
V_0(\prod_j t_j^{r_j}) = \sum_i i t_i \frac{\partial}{\partial t_i}(\prod_j t_j^{r_j}) = (\sum r_j) \prod_j t_j^{r_j} \tag{5.0.31}
\]

and so, since Schur polynomials have the form

\[
s_{\lambda} = \sum_{\sum j r_j = |\lambda|} \alpha_r \prod_j t_j^{r_j},
\]

the result follows.

We now turn to the identity (5.0.26). From the identity in Theorem 5.1, it follows that the second sum in (5.0.15) vanishes and so

\[
V_{-1} s_\lambda = \sum_{\mu : \lambda \in B(1)} d_{\lambda \mu} s_\mu = \sum_{j \geq 1 \text{ such that } \lambda_{j-1} > \lambda_j} d_{\lambda, \lambda + e_j} s_{\lambda + e_j}
\]
\[= \sum_{j \geq 1 \text{ such that } \lambda_{j-1} > \lambda_j} s_{\lambda + e_j} \sum_{i \geq 1} \sum_{h_i \in B(i)} \sum_{h_{i+1} \in B(i+1)} (-1)^{\text{ht}(h_i) + \text{ht}(h_{i+1})}
\]

In the sum above
\[h_{i+1} = h_i + \begin{cases} \text{one box, appearing} \\ \text{at the right most place} \\ \text{in the } j\text{th row of } h_{i+1} \end{cases}\]

Consider now a \(\lambda\) such that \(\lambda \setminus h_i = (\lambda + e_j) \setminus h_{i+1}\) = some partition. Given the rightmost box \(b\) in the \(j\)th row such that \(\lambda_{j-1} > \lambda_j\); the extra box \(e_j\) is precisely added to the right of this box. Two kinds of \(h_i\), satisfying these requirements can be formed.

- All possible border strips \(h\) of \(\lambda\), containing that box \(b\) and contained in the rows \(j, j + 1, \ldots\) of \(\lambda\), such that \(\lambda \setminus h = \nu\) remains a partition; see figure 3 for an example. Then \(h \in B(i)\) for some \(i \geq 1\). Writing \(h_i := h\) and letting \(h_{i+1} := h_i + e_j \in B(i)\), we have that \(\lambda \setminus h_i = (\lambda + e_j) \setminus h_{i+1}\) and \(\text{ht}(h_{i+1}) = \text{ht}(h_i)\).

- All border strips \(h\) of \(\lambda\), containing that box \(b\) and contained in the rows \(1, \ldots, j\) of \(\lambda\), such that \(\lambda \setminus h = \nu\) remains a partition; see figure 4 for an example. Then \(h \in B(i)\) for some \(i \geq 1\). Writing \(h_i := h\) and letting \(h_{i+1} := h_i + e_j \in B(i)\), we have that \(\lambda \setminus h_i = (\lambda + e_j) \setminus h_{i+1}\) and \(\text{ht}(h_{i+1}) = \text{ht}(h_i) + 1\).

**Example:** Consider
\[
\begin{array}{|c|c|c|c|c|c|}
\hline
 & & & & & \\
 & & & & & \\
 & & & & & \\
 & & & & & \\
 & & & & & \\
 & & & & & \\
\hline
\end{array}
\]
\[
\lambda = (\ ) = (7, 6, 6, 4, 1, 1)
\]
\[ \lambda + e_j = 7, 6, 6, 5, 1, 1 \]

- On the one hand, the different \( h_i \)'s, corresponding to a + sign in (5.0.32), are generated as follows:

- On the other hand, the different \( h_i \)'s, corresponding to a − sign in (5.0.32), are generated by:

| \( \lambda \backslash h_i \) | \( (\lambda + e_j) \backslash h_{i+1} \) | \((-1)^{\text{ht}(h_i) + \text{ht}(h_{i+1})}\) |
|-----------------------------|---------------------------------|----------------------------------|
| \( \lambda \)              | \( (\lambda + e_j) \)           | \( + \)                           |
| \( \lambda \)              | \( (\lambda + e_j) \)           | \( + \)                           |
| \( \lambda \)              | \( (\lambda + e_j) \)           | \( + \)                           |
| \( \lambda \)              | \( (\lambda + e_j) \)           | \( + \)                           |
| \( \lambda \)              | \( (\lambda + e_j) \)           | \( + \)                           |

| \( \lambda \)              | \( (\lambda + e_j) \)           | \( + \)                           |
| \( \lambda \)              | \( (\lambda + e_j) \)           | \( + \)                           |
| \( \lambda \)              | \( (\lambda + e_j) \)           | \( + \)                           |
| \( \lambda \)              | \( (\lambda + e_j) \)           | \( + \)                           |
| \( \lambda \)              | \( (\lambda + e_j) \)           | \( + \)                           |

figure 4

They correspond respectively to

\[ \lambda \backslash h_i = (\lambda + e_j) \backslash h_{i+1} \]

\[ (-1)^{\text{ht}(h_i) + \text{ht}(h_{i+1})} = 1 \]
They correspond respectively to
\[
\lambda \backslash h_i = (\lambda + e_j) \backslash h_{i+1} \quad \begin{array}{c|c}
(\lambda \backslash h_i) = (\lambda + e_j) \backslash h_{i+1} & (-1)^{ht(h_i) + ht(h_{i+1})} \\
\lambda \backslash \Box & = -1 \\
\lambda \backslash \Box & = -1 \\
\lambda \backslash \Box & = -1 \\
\end{array}
\]

The point is that in the first case each additional \( h \) corresponds to adding to the previously added \( h \) a new box to the immediate left of the previous \( h \) (in the \( j \)th row) and as many boxes as there are in \( \lambda \) below the new box added, thus yielding \( \lambda_j \) cases in all, each contributing +1, since \( ht(h_i) = ht(h_{i+1}) \). In the second case, we add to the previously added \( h \) one box above the left most box of \( h \) and then all the boxes in \( \lambda \) to the right of that new box, yielding \( j - 1 \) cases, each contributing -1, since \( ht(h_i) = ht(h_{i+1}) - 1 \). So, the total sum equals \( \lambda_j - (j - 1) = \#\{1\} - \#\{(-1)\} = 4 - 3 = 1 \). For this example, one computes

\[
V_{-1}s_{(7,6,6,4,1,1),1} = 7 \, s_{(8,6,6,4,1,1)} + 5 \, s_{(7,7,6,4,1,1)} + s_{(7,6,6,5,1,1)} - 3 \, s_{(7,6,6,4,2,1)} - 6 \, s_{(7,6,6,4,1,1,1)},
\]

corresponding to the different ways to add a box \( e_j \) to the partition \( \lambda \). This ends the proof of (5.0.26), whereas identity (5.0.27) follows from (5.0.26) by taking the transpose. ■
6 Virasoro actions, Fourier series and difference equations

Remember the notation (1.0.2) for

\[ V_{-1} = V_{-1}(t) - V_1(s) + n \left( t_1 + \frac{\partial}{\partial s_1} \right) \]
\[ V_0 = V_0(t) - V_0(s) \]
\[ V_1 = -V_{-1}(s) + V_1(t) + n \left( s_1 + \frac{\partial}{\partial t_1} \right) . \]  

with the \( V_k \)-operators as in (1.0.3); e.g.,

\[ V_0(t) := \sum_{i \geq 1} it_i \frac{\partial}{\partial t_i} \]
\[ V_{-1}(t) := \sum_{i \geq 1} (i + 1)t_{i+1} \frac{\partial}{\partial t_i}, \quad V_1(t) := \sum_{i \geq 2} (i - 1)t_{i-1} \frac{\partial}{\partial t_i}. \]  

In [2], we have shown the following:

**Proposition 6.1** For all integers \( n \geq 0 \), the integrals

\[ \tau_n(t, s) := \int_{U(n)} e^{\sum_{i=1}^{\infty} Tr(t_iM_i - s_iM_i)} dM \]
\[ = \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} \left( e^{\sum_{i=1}^{\infty} (t_i z_i - s_i z_i) \frac{dz_k}{2\pi i z_k}} \right) \]  

satisfy the following three Virasoro constraints:

\[ \forall \ i \tau_n = 0 \quad \text{for} \ i = -1, 0, 1. \]

Two strictly increasing sets of integers \( x = (x_1 < x_2 < \ldots < x_n) \) and \( y = (y_1 < y_2 < \ldots < y_n) \) in \( \mathbb{Z}_{\geq 0} \) are equivalent to two partitions \( \lambda = (\lambda_1 \geq \ldots \lambda_n) \) and \( \mu = (\mu_1 \geq \ldots \mu_n) \), with \( \lambda_1^T \leq n \) and \( \mu_1^T \leq n \), by setting \( x_i = \lambda_{n-i+1} + (i - 1) \) and \( y_i = \mu_{n-i+1} + (i - 1) \), as in (1.0.7).
6.1 Integral 1

(i) For the fixed partitions $\lambda$ and $\mu$, with at most $n$ rows, the integral

$$a_{\lambda\mu}(z) := \int_{U(n)} s_{\lambda}(M)s_{\mu}(\bar{M})e^{z \text{Tr}(M+\bar{M})}dM = \sum_0^\infty \frac{z^k}{k!} b_{xy}^{(k)}$$  \hspace{1cm} (6.1.1)

satisfies the difference equations, with $L^{(1)}_{0 \pm} := L^{(1)}_{0 \pm} (\lambda, \mu, z)$,

$$L^{(1)}_-(a_{\lambda\mu}) := \sum_{1 \leq i \leq n} (a_{\lambda-e_i,\mu}(\lambda_i - i + n) - a_{\lambda,\mu+e_i}(\mu_i - i + n + 1))$$

$$+ z \left( n a_{\lambda\mu} - \sum_1^n a_{\lambda,\mu+2e_i} + \sum_1^n a_{\lambda,\mu+e_i+e_{i+1}} \right) = 0$$  \hspace{1cm} (6.1.2)

$$L^{(1)}_0(a_{\lambda\mu}) := (|\lambda| - |\mu|) a_{\lambda,\mu} + z \sum_{1 \leq i \leq n} (a_{\lambda+e_i,\mu} - a_{\lambda,\mu+e_i}) = 0$$  \hspace{1cm} (6.1.3)

$$L^{(1)}_+(a_{\lambda\mu}) = 0,$$  \hspace{1cm} (6.1.4)

where

$$L^{(1)}_-(\lambda, \mu) := L^{(1)}_+(\mu, \lambda).$$

(ii) Moreover,

$$b_{xy}^{(k)} := \# \left\{ \text{ways that } n \text{ non-intersecting walkers} \right. \right.$$  \hspace{1cm} (6.1.5)

in $\mathbb{Z}$ move from $x_1 < x_2 < ... < x_n$ to $y_1 < y_2 < ... < y_n$ in $k$ steps

satisfies the difference equations, alluded to in table 2 of the introduction, with $L^{(1)}_{0 \pm} := L^{(1)}_{0 \pm} (x, y, \Lambda_k)$,

$$L^{(1)}_-(b_{xy}^{(k)}) := - \sum_{1 \leq i \leq n} \left( (y_i + 1)b_{x,y+e_i}^{(k)} - x_i b_{x-e_i,y}^{(k)} \right)$$

$$+ k \left( nb_{x,y}^{(k-1)} - \sum_{1 \leq i \leq n} b_{x,y+2e_i}^{(k-1)} + \sum_{1 \leq i \leq n} b_{x,y+e_i+e_{i+1}}^{(k-1)} \right) = 0$$  \hspace{1cm} (6.1.6)
\[ L_0^{(1)}(b_{xy}^{(k)}) := -b_{xy}^{(k)} \sum_{1 \leq i \leq n} (y_i - x_i) + k \left( \sum_{1 \leq i \leq n} b_{x+\epsilon_i,y}^{(k-1)} - \sum_{1 \leq i \leq n} b_{x,y+\epsilon_i}^{(k-1)} \right) = 0 \]  

(6.1.7)

\[ L_+^{(1)}(b_{xy}^{(k)}) = 0, \text{ where } L_+^{(1)}(x, y) := L_-^{(1)}(y, x). \]  

(6.1.8)

**Remark:** In the formulae above, the integer \( b_{xy}^{(k)} = 0 \), when the strict inequalities \( x_1 < x_2 < \ldots < x_n \) and \( y_1 < y_2 < \ldots < y_n \) are not satisfied.

**Proof:** Applying the shifts \( t_1 \mapsto t_1 + z, \ s_1 \mapsto s_1 - z \) to the Virasoro constraints of Proposition 6.1, and to the matrix integral (6.0.3) lead to the following equations, for \( k = -1, 0, 1 \),

\[
0 = \mathbb{V}_k \int_{U(n)} e^{z \text{Tr}(M+\bar{M})} e^{\sum_{i=1}^{\infty} \text{Tr}(t_i M^{i-1} - s_i \bar{M}^{i-1})} dM
\]

\[
= \mathbb{V}_k \sum_{\lambda, \mu \text{ such that } \lambda^T_1 \cdot \mu^T_1 \leq n} a_{\lambda \mu}(z) s_{\lambda}(t) s_{\mu}(-s),
\]

where

\[
\mathbb{V}_{-1} = V_{-1}(t) - V_1(-s) + n \left( t_1 + \frac{\partial}{\partial s_1} \right) + z \left( n + \frac{\partial}{\partial s_2} \right)
\]

\[
\mathbb{V}_0 = V_0(t) - V_0(-s) + z \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial s_1} \right)
\]

\[
\mathbb{V}_1 = -V_{-1}(s) + V_1(-t) + n \left( s_1 + \frac{\partial}{\partial t_1} \right) - z \left( n - \frac{\partial}{\partial t_2} \right).
\]

(6.1.9)

Using the action (5.0.28), (5.0.30), (5.0.26), (5.0.27) of \( t_1, \partial/\partial t_2, V_{-1} \) and \( V_1 \)
on Schur polynomials, one computes

\[ 0 = \mathbb{V}_{-1}(t, s) \sum_{\lambda_1^\top \mu_1^\top \leq n} a_{\lambda \mu}(z)s_{\lambda}(t)s_{\mu}(-s) \]

\[ = \sum_{\lambda_1^\top \mu_1^\top \leq n} a_{\lambda \mu'} \begin{cases} \sum_{i=1}^{n+1} (\lambda_i' - i + 1)s_{\lambda' + e_i}(t)s_{\mu'}(-s) \\ -s_{\lambda}(t) \sum_{i=1}^{n}(\mu_i' - i)s_{\mu' - e_i}(-s) \\ +n \left( \sum_{i=1}^{n+1} s_{\lambda' + e_i}(t)s_{\mu'}(-s) - s_{\lambda}(t) \sum_{i=1}^{n} s_{\mu' - e_i}(-s) \right) \\ +zn \ s_{\lambda}(t)s_{\mu'}(-s) \\ -zs_{\lambda'}(t) \left( \sum_{i=1}^{n} s_{\mu' - 2e_i}(-s) - \sum_{i=1}^{n-1} s_{\mu' - e_i - e_{i+1}}(-s) \right) \end{cases} \]

\[ = \sum_{\lambda_1^\top \leq n} s_{\lambda}(t)s_{\mu}(-s) \mathcal{L}_{-1}^{(1)}(a_{\lambda \mu}(z)) \hspace{1cm} (**). \]

The coefficients \( \mathcal{L}_{-1}^{(1)}(a_{\lambda \mu}(z)) \) of \( s_{\lambda}(t)s_{\mu}(-s) \) in the formula above are linear expressions in the \( a_{\lambda \mu}(z) \), given by the difference equation (6.1.2). Also notice that, from the expression (**), it would seem one would have to include in the sum (**) partitions \( \lambda \) of the form \( \lambda = \lambda' + e_{n+1} \), with \( \lambda_1^\top = n \), so that \( \lambda_i^\top = n+1 \). However, the coefficient of such terms vanish by visual inspection!

The Virasoro duality

\[ \mathbb{V}_1 = -\mathbb{V}_{-1} \big|_{t \rightarrow -s} \]

implies at once expression (6.1.4).
Finally,

\[
0 = \mathbb{V}_0 \sum_{\lambda^1, \mu^1 \leq n} a_{\lambda \mu}(z) s_\lambda(t) s_\mu(-s)
\]

\[
= \sum_{\lambda^1, \mu^1 \leq n} a_{\lambda \mu} \left\{ \left( \mathbb{V}_0(t) s_\lambda(t) \right) s_\mu(-s) - s_\lambda(t) \left( \mathbb{V}_0(-s) s_\mu(-s) \right) \right\} + z \left( \left( \frac{\partial}{\partial t_1} s_\lambda(t) \right) s_\mu(-s) - s_\lambda(t) \left( \frac{\partial}{\partial (-s_1)} s_\mu(-s) \right) \right)
\]

\[
= \sum_{\lambda^1, \mu^1 \leq n} a_{\lambda \mu} \left\{ \left( |\lambda| - |\mu| \right) s_\lambda(t) s_\mu(-s) \right\} + z \left( \sum_{1}^{n} s_{\lambda - e_i(t)}(t) s_\mu(-s) - z s_\lambda(t) \left( \sum_{1}^{n} s_{\mu - e_i}(-s) \right) \right)
\]

\[
= \sum_{\lambda^1 \leq n \atop \mu^1 \leq n} \sum_{\lambda^0, \mu^0 \leq n} a_{\lambda \mu} \left( \sum_{k \geq 0} b_{xy}^{(k)} \frac{z^k}{k!} \right) \left( \sum_{k \geq 0} L_{xy}^{(1)} \left( b_{xy}^{(k)} \frac{z^k}{k!} \right) \right).
\]

In this expression, the coefficient \(L_{x}^{(1)}(a_{\lambda \mu}(z))\) of \(s_\lambda(t) s_\mu(-s)\) is precisely equation (6.1.3).

Then setting

\[
a_{\lambda \mu}(z) = \sum_{k \geq 0} b_{xy}^{(k)} \frac{z^k}{k!}
\]

in (6.1.2), (6.1.3) and (6.1.4), one finds, using the map \(x_{n+1-i} = \lambda_i + n - i,\)
\(y_{n+1-i} = \mu_i + n - i,\)

\[
0 = L_{x}^{(1)}(a_{\lambda \mu}(z)) = L_{xy}^{(1)} \left( \sum_{k \geq 0} b_{xy}^{(k)} \frac{z^k}{k!} \right) = \sum_{k \geq 0} L_{x}^{(1)} (b_{xy}^{(k)} \frac{z^k}{k!}),
\]

leading to (6.1.6), (6.1.7) and (6.1.8).

\[\square\]

6.2 Integral 2

(i) For the fixed partitions \(\lambda\) and \(\mu\), with at most \(n\) rows, the integral

\[
a_{\lambda \mu}(z) := \int_{U(n)} s_\lambda(M) s_\mu(\bar{M}) \det(I + M)^{q e^z \text{Tr} \bar{M}} dM = \sum_{0}^{\infty} b_{xy}^{(k)} \frac{z^k}{k!}
\]
satisfies the difference equations, with \( \mathcal{L}^{(2)}_{\pm} := \mathcal{L}^{(2)}_{\pm}(\lambda, \mu, z) \),

\[
\mathcal{L}^{(2)}_{-}(a_{\lambda \mu}) := \sum_{i=1}^{n} a_{\lambda_{-i+1}, \mu_{i}}(\lambda_{i} - i + n) - \sum_{i=1}^{n} a_{\lambda_{i}+\mu_{i}}(\mu_{i} - i + n + z + 1) + a_{\lambda \mu}(|\lambda| - |\mu| + nz) - z \left( \sum_{i=1}^{n} a_{\lambda_{i}+2e_{i}} - \sum_{i=1}^{n-1} a_{\lambda_{i}+\mu_{i}+e_{i+1}} \right) = 0
\] (6.2.2)

\[
\mathcal{L}^{(2)}_{+}(a_{\lambda \mu}) := \sum_{i=1}^{n} a_{\lambda_{i-1}, \mu_{i}}(\mu_{i} - i + n) - \sum_{i=1}^{n} a_{\lambda_{i}+\mu_{i}}(\lambda_{i} - i + n + q + 1) + a_{\lambda \mu}(|\mu| - |\lambda| + nz) + z \sum_{i=1}^{n} a_{\lambda_{i}+\mu_{i}} = 0.
\] (6.2.3)

(ii) The coefficients \( b_{xy}^{(k)} \) satisfy, with \( L^{(2)}_{\pm} := L^{(2)}_{\pm}(x, y, \Lambda_{k}) \),

\[
L^{(2)}_{-}(b_{xy}^{(k)}) := \sum_{i=1}^{n} \left( x_{i} b_{x_{i}-1, y_{i}}^{(k)} - (y_{i} + 1) b_{x_{i}+1, y_{i}}^{(k)} + (x_{i} - y_{i} + q) b_{x_{i}+e_{i}, y_{i}}^{(k)} \right) + k \left( -\sum_{i=1}^{n} (b_{x_{i}+e_{i}+1, y_{i}}^{(k-1)} - b_{x_{i}+2e_{i}+1, y_{i}}^{(k-1)}) + \sum_{i=1}^{n-1} b_{x_{i}+e_{i}+1, y_{i}+1}^{(k-1)} \right) = 0
\] (6.2.4)

\[
L^{(2)}_{+}(b_{xy}^{(k)}) := \sum_{i=1}^{n} \left( y_{i} b_{x_{i}+1, y_{i}}^{(k)} - (x_{i} + 1 + q) b_{x_{i}+e_{i}+1, y_{i}}^{(k)} + (y_{i} - x_{i}) b_{x_{i}+e_{i}, y_{i}}^{(k)} \right) + k \sum_{i=1}^{n} \left( b_{x_{i}+1, y_{i}}^{(k-1)} + b_{x_{i}+e_{i}, y_{i}+1}^{(k-1)} \right) = 0.
\] (6.2.5)

Proof: Applying the shifts \( it_{i} \mapsto it_{i} - q(-1)^{i}, is_{i} \mapsto is_{i} - z\delta_{i1} \) to the Virasoro constraints and the matrix integral of Proposition 6.1, leads to the
equations

\[ 0 = \mathbb{V}_k \int_{U(n)} \det(I + M)^q e^{z \text{Tr} \tilde{M} e^{\sum_{i=1}^{\infty} \text{Tr}(t_i M_i - s_i M_i)} dM \]

\[ = \mathbb{V}_k \sum_{\lambda, \mu \text{ such that } \lambda_1^{\prime}, \mu_1^{\prime} \leq n} a_{\lambda \mu}(z) s_{\lambda}(t) s_{\mu}(-s), \]

where

\[ \mathbb{V}_{-1} = V_{-1}(t) - V_1(-s) + n\left(t_1 + q + \frac{\partial}{\partial s_1}\right) + q \sum_{i \geq 1} (-1)^i \frac{\partial}{\partial t_i} + z \frac{\partial}{\partial s_2} \]

\[ \mathbb{V}_0 = V_0(t) - V_0(-s) - q \sum_{i \geq 1} (-1)^i \frac{\partial}{\partial t_i} + z \frac{\partial}{\partial s_1} \] (6.2.6)

\[ \mathbb{V}_1 = -V_{-1}(s) + V_1(-t) + n\left(s_1 - z + \frac{\partial}{\partial t_1}\right) + q \sum_{i \geq 2} (-1)^i \frac{\partial}{\partial t_i}. \]

The only linear combinations in the span of \( \mathbb{V}_{-1}, \mathbb{V}_0, \mathbb{V}_1, \) involving finite sums of \( V_k(t), V_k(s), \partial/\partial t_k, \partial/\partial s_k, t_k, s_k \) are as follows:

\[ \mathbb{V}_{-1} + \mathbb{V}_0 = \left( V_{-1}(t) + nt_1 \right) - \left( V_1(-s) + (n + z) \frac{\partial}{\partial (-s_1)} \right) \]

\[ + V_0(t) - V_0(-s) + nq - z \frac{\partial}{\partial (-s_2)} \]

\[ -\mathbb{V}_0 - \mathbb{V}_1 = \left( V_{-1}(-s) + n(-s_1) \right) - \left( V_1(t) + (n + q) \frac{\partial}{\partial t_1} \right) \]

\[ + V_0(-s) - V_0(t) + nz + z \frac{\partial}{\partial (-s_1)} \]
and thus, using Corollary 5.3 and 5.4, compute

$$0 = (V_{-1} + V_0) \sum_{\lambda, \mu \text{ such that } \lambda^T \preceq n} a_{\lambda \mu}(z) s_{\lambda}(t) s_{\mu}(-s)$$

$$= \sum_{\lambda', \mu' \preceq n} a_{\lambda' \mu'} \begin{cases} 
(V_{-1}(t) + nt_1) s_{\lambda'}(t) s_{\mu'}(-s) \\
- s_{\lambda'}(t) \left( V_1(-s) + (n + z) \frac{\partial}{\partial(-s_1)} \right) s_{\mu'}(-s) \\
+ (V_0(t)s_{\lambda'}(t)) s_{\mu'}(-s) - s_{\lambda'}(t)(V_0(-s)s_{\mu'}(-s)) \\
+ nq s_{\lambda'}(t)s_{\mu'}(-s) - z s_{\lambda'}(t) \frac{\partial}{\partial(-s_2)} s_{\mu'}(-s) \\
\sum_{1}^{n+1}(\lambda_i' - i + n + 1) s_{\lambda'-e_i}(t) s_{\mu'}(-s) \\
- s_{\lambda'}(t) \sum_{1}^{n}(\mu_i' - i + n + z) s_{\mu'-e_i}(-s) \\
+ (|\lambda'| - |\mu'| + nq) s_{\lambda'}(t)s_{\mu'}(-s) \\
- z s_{\lambda'}(t) \left( \sum_{1}^{n} s_{\mu'-e_i}(-s) - \sum_{1}^{n-1} s_{\mu'-e_i-e_{i+1}}(-s) \right) 
\end{cases}$$

$$= \sum_{\lambda^T \preceq n} s_{\lambda}(t)s_{\mu}(-s) L^{(2)}_{-}(a_{\lambda \mu}(z)),$$

implying the vanishing of all the coefficients $L^{(2)}_{-}(a_{\lambda \mu}(z))$ of $s_{\lambda}(t)s_{\mu}(-s)$, leading to the difference equation (6.2.2). The same remark as for integral 1 holds for this case.
The remaining identities are obtained in a similar fashion:

\[
0 = -(\mathcal{V}_1 + \mathcal{V}_0) r_n
\]

\[
= \sum_{\lambda', \mu' \leq n} a_{\lambda', \mu'} \left\{ 
\begin{aligned}
& \begin{cases}
(\lambda', \mu', \lambda_1^T, \mu_1^T) \
\lambda_1^T, \mu_1^T \leq n
\end{cases} \\
& s_{\lambda'}(t)(V_1(-s) - ns) \cdot s_{\mu'}(-s) \\
& \quad \quad - (V_1(t) + (n + q) \frac{\partial}{\partial t}) \cdot s_{\lambda'}(t) \cdot s_{\mu'}(-s) \\
& \quad \quad + (s_{\lambda'}(t))V_0(-s) \cdot s_{\mu'}(-s) - V_0(t) \cdot s_{\lambda'}(t) \cdot s_{\mu'}(-s) \\
& \quad \quad + nz s_{\lambda'}(t) \cdot s_{\mu'}(-s) + t s_{\lambda'}(t) \frac{\partial}{\partial t} \cdot s_{\mu'}(-s) \\
\end{aligned}
\right\}
\]

\[
= \sum_{\lambda', \mu' \leq n} a_{\lambda', \mu'} \left\{ 
\begin{aligned}
& \quad \quad \sum_{i}^{n+1} s_{\lambda'}(t)(\mu_i' - i + 1 + n) \cdot s_{\mu' + \epsilon_i}(-s) \\
& \quad \quad - \sum_{i}^{n} (\lambda_i' - i + n + q) s_{\lambda' - \epsilon_i}(-s) \\
& \quad \quad + (|\mu'| - |\lambda'| + nz s_{\lambda'}(t) \cdot s_{\mu'}(-s) \\
& \quad \quad + nz s_{\lambda'}(t) \sum_{i}^{n} s_{\mu' - \epsilon_i}(-s) \\
\end{aligned}
\right\}
\]

\[
= \sum_{\lambda_1^T, \mu_1^T \leq n+1} \sum_{\mu_1^T \leq n} s_{\lambda}(t) \cdot s_{\mu}(-s) \mathcal{L}^{(2)}(a_{\lambda\mu}(z))
\]

implying \( \mathcal{L}^{(2)}(a_{\lambda\mu}(z)) = 0 \), and thus (6.2.3). Identities (6.2.4) and (6.2.5) follow by the precise same method as for integral 1. 

\[\text{6.3 Integral 3}\]

(i) For the fixed partitions \( \lambda \) and \( \mu \), with at most \( n \) rows, the integral

\[
a_{\lambda\mu}(z) := \int_{U(n)} s_{\lambda}(M)s_{\mu}(\bar{M}) \det(I + zM)^{p} \det(I + z\bar{M})^{q} dM = \sum_{0}^{\infty} b_{xy}(z)^{k}
\]

(6.3.1)
satisfies the difference equation, with $\mathcal{L}^{(3)} := \mathcal{L}^{(3)}(\lambda, \mu, z)$,

$$
\mathcal{L}^{(3)}(a_{\lambda\mu}) = \sum_{i=1}^{n} (a_{\lambda-e_i,\mu}(\lambda_i - i + n) + a_{\lambda+e_i,\mu}(\lambda_i - i + n + p + 1))

- \sum_{i=1}^{n} (a_{\lambda,\mu-e_i}(\mu_i - i + n) + a_{\lambda,\mu+e_i}(\mu_i - i + n + q + 1))

+ (z + z^{-1})(|\lambda| - |\mu|)a_{\lambda\mu} = 0.

(6.3.2)

(ii) The coefficients $b^{(k)}_{xy}$ satisfy, with $\mathcal{L}^{(3)} := \mathcal{L}^{(3)}(x, y, \Lambda_k)$,

$$
L^{(3)}(b^{(k)}_{xy}) = \sum_{i=1}^{n} \left( \begin{array}{c}
(x_i b^{(k)}_{x-e_i,y} + (x_i + p + 1)b^{(k)}_{x+e_i,y}) \\
- (y_i b^{(k)}_{x,y-e_i} + (y_i + q + 1)b^{(k)}_{x,y+e_i})
\end{array} \right)

+ \sum_{i=1}^{n} (x_i - y_i) (b^{(k-1)}_{xy} + b^{(k+1)}_{xy}) = 0.

(6.3.3)

Proof: Applying the shifts $it_i \mapsto it_i - p(-z)^i$, $is_i \mapsto is_i + q(-z)^i$ to the Virasoro constraints and the matrix integral of Proposition 6.1, leads to the equations

$$
V_{-1} = V_{-1}(t) - V_{1}(-s) + n \left( t_1 - \frac{\partial}{\partial(-s_1)} \right)

- p \sum_{i \geq 1} (-z)^{i+1} \frac{\partial}{\partial t_i} - q \sum_{i \geq 2} (-z)^{i-1} \frac{\partial}{\partial s_i}

V_0 = V_0(t) - V_0(-s) - p \sum_{i \geq 1} (-z)^i \frac{\partial}{\partial t_i} - q \sum_{i \geq 1} (-z)^i \frac{\partial}{\partial s_i}

V_1 = -V_{-1}(-s) + V_1(t) + n \left( s_1 + \frac{\partial}{\partial t_1} \right)

- p \sum_{i \geq 1} (-z)^{i+1} \frac{\partial}{\partial s_i} - q \sum_{i \geq 2} (-z)^{i-1} \frac{\partial}{\partial t_i}

(6.3.4)
Here the only linear combination in the span of $\mathbb{V}_{-1}, \mathbb{V}_0, \mathbb{V}_1$, involving finite sums of $V_k(t), V_k(s), \partial/\partial t_k, \partial/\partial s_k, t_k, s_k$ is the following expression:

$$
\mathbb{V}_{-1} + (z + z^{-1})\mathbb{V}_0 + \mathbb{V}_1 \\
= \left( V_{-1}(t) + V_1(t) + n \left( t_1 + \frac{\partial}{\partial t_1} \right) \right) \\
- \left( V_{-1}(-s) + V_1(-s) + n \left( -s_1 + \frac{\partial}{\partial(-s_1)} \right) \right) \\
+(z + z^{-1})(V_0(t) - V_0(-s)) + \left( p \frac{\partial}{\partial t_1} - q \frac{\partial}{\partial(-s_1)} \right),
$$

and so

$$
0 = (\mathbb{V}_{-1} + (z + z^{-1})\mathbb{V}_0 + \mathbb{V}_1) \tau_n
$$

$$
= \sum_{\lambda' \mu' \leq \lambda_1'} a_{\lambda' \mu'} \left\{ \begin{array}{l}
\left( V_{-1}(t) + V_1(t) + (z + z^{-1})V_0(t) \right) s_{\lambda'}(t)s_{\mu'}(-s) \\
- s_{\lambda'}(t) \left( \frac{V_{-1}(-s) + V_1(-s) + (z + z^{-1})V_0(-s)}{n(-s_1) + (n + q)\frac{\partial}{\partial(-s_1)}} \right) s_{\mu'}(-s)
\end{array} \right\}
$$

$$
= \sum_{\lambda' \mu' \leq \lambda_1'} a_{\lambda' \mu'} \left\{ \begin{array}{l}
\sum_{i=1}^{n+1} (\lambda'_i - i + n + 1) s_{\lambda'+e_i}(t) \\
+ \sum_{i=1}^{n} (\lambda'_i - i + n + p) s_{\lambda'-e_i}(t) \\
+ (z + z^{-1})|\lambda'| s_{\lambda'}(t')
\end{array} \right\}
$$

$$
= \sum_{\lambda' \mu' \leq \lambda_1'} a_{\lambda' \mu'} \left\{ \begin{array}{l}
\sum_{i=1}^{n+1} (\mu'_i - i + n + 1) s_{\mu'+e_i}(-s) \\
- s_{\lambda'}(-s) + \sum_{i=1}^{n} (\mu'_i - i + n + q) s_{\mu'-e_i}(-s) \\
+ (z + z^{-1})|\mu'| s_{\mu'}(-s)
\end{array} \right\}
$$

$$
= \sum_{\lambda' \mu' \leq \lambda_1'} s_{\lambda}(t)s_{\mu}(-s) L^{(3)}(a_{\lambda\mu})
$$

implying (6.3.2) and similarly (6.3.3).
6.4 The action of Virasoro on two-dimensional Fourier series

In this section we prove identity (1.0.6) of the introduction:

**Corollary 6.2** The action of the Virasoro operator \( \tilde{V}_{\Lambda} \) on “Fourier series”, with arbitrary coefficients, depending on an integer parameter \( k \), translates itself into a linear action of \( \tilde{L}_{\Lambda} \) on the coefficients:

\[
\tilde{V}_{\Lambda k} \sum_{\lambda, \mu, \text{such that } \lambda_1^1, \mu_1^1 \leq n} b^{(k)}_{\lambda\mu} s_\lambda(t) s_\mu(-s) = \sum_{\lambda, \mu, \text{such that } \lambda_1^1, \mu_1^1 \leq n} \tilde{L}_{\Lambda k} (\tilde{b}_{\lambda\mu}^{(k)}) s_\lambda(t) s_\mu(-s) \tag{6.4.1}
\]

where

\[
\tilde{V}_{\Lambda k}(t, s) \quad \tilde{L}_{\Lambda k}(\lambda, \mu)
\]

| \( \tilde{V}_{0}, \ \tilde{V}_{\pm 1} \) | \( t t_j \mapsto t t_j + k \Lambda_k^{-1} \delta_{i1} \) | \( L^{(1)}_0, L^{(1)}_\pm \) |
| --- | --- | --- |
| \( \pm (\tilde{V}_0 + \tilde{V}_{\pm 1}) \) | \( t t_j \mapsto t t_j + (-1)^j \delta_{i1} \) | \( L^{(2)}_\pm \) |
| \( (\tilde{V}_{-1} + (\Lambda_k + \Lambda_k^{-1}) \tilde{V}_0 + \tilde{V}_1) \) | \( t t_j \mapsto t t_j + (\Lambda_k + \Lambda_k^{-1})^{-1} \delta_{i1} \) | \( L^{(3)} \) |

The right column of \( \tilde{L}_{\Lambda k} \)'s are the precise equations satisfied by the three types of random walks considered in this paper.

**Proof:** Using the commutation relation,

\[
z \Lambda_k^{-1} z^k = \frac{z^k}{k!} \Lambda_k^{-1},
\]

one computes, on the one hand,

\[
\tilde{V}_z \sum_{\lambda, \mu, \text{such that } \lambda_1^1, \mu_1^1 \leq n} a_{\lambda\mu}(z) s_\lambda(t) s_\mu(-s) = \tilde{V}_z \sum_{\lambda, \mu, \text{such that } \lambda_1^1, \mu_1^1 \leq n} \left( \sum_{\lambda, \mu, \text{such that } \lambda_1^1, \mu_1^1 \leq n} \frac{z^k}{k!} \tilde{b}_{\lambda\mu}^{(k)} \right) s_\lambda(t) s_\mu(-s)
\]

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\[
\tilde{V}_z \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{\lambda,\mu \text{ such that } \lambda^1,\mu^1 \leq n} \tilde{b}_{\lambda\mu}^{(k)} s_{\lambda}(t) s_{\mu}(-s)
\]

\[
= \sum_{k=0}^{\infty} \frac{z^k}{k!} \tilde{V}_\Lambda \sum_{\lambda,\mu \text{ such that } \lambda^1,\mu^1 \leq n} \tilde{b}_{\lambda\mu}^{(k)} s_{\lambda}(t) s_{\mu}(-s)
\]

and, on the other hand,

\[
\tilde{V}_z \sum_{\lambda,\mu \text{ such that } \lambda^1,\mu^1 \leq n} a_{\lambda\mu}(z) s_{\lambda}(t) s_{\mu}(-s)
\]

\[
= \sum_{\lambda,\mu \text{ such that } \lambda^1,\mu^1 \leq n} \mathcal{L}(a_{\lambda\mu}(z)) s_{\lambda}(t) s_{\mu}(-s)
\]

\[
= \sum_{\lambda,\mu \text{ such that } \lambda^1,\mu^1 \leq n} \mathcal{L} \left( \frac{z^k}{k!} b_{xy}^{(k)} \right) s_{\lambda}(t) s_{\mu}(-s)
\]

\[
= \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{\lambda,\mu \text{ such that } \lambda^1,\mu^1 \leq n} L_{\lambda} \left( b_{xy}^{(k)} \right) s_{\lambda}(t) s_{\mu}(-s),
\]

using the argument in (6.1.10).

7 The discrete backward and forward equations for a random walk in a Weyl chamber

Remembering the definition of the difference operators (1.0.13) (see footnote 4), consider now the following difference operators:

\[\mathcal{A}_1 := \sum_{i=1}^{n} \left( \frac{k}{2n} \Lambda_k^{-1} \partial_{2y_i}^+ + x_i \partial_{x_i}^- + \partial_{y_i}^+ y_i - (x_i - y_i) \right)\]

\[\mathcal{A}_2 := \sum_{i=1}^{n} \left( \frac{k}{2n} \Lambda_k^{-1} \partial_{2x_i}^+ + y_i \partial_{y_i}^- + \partial_{x_i}^+ x_i - (y_i - x_i) \right)\]
Theorem 7.1 The probability

\[ P(k, x, y) := P \left( \text{that } n \text{ walkers in } \mathbb{Z}, \text{ go from } x_1, \ldots, x_n \text{ to } y_1, \ldots, y_n \text{ in } k \text{ steps, and do not intersect} \right) = \frac{b_{xy}^{(k)}}{(2n)^k} \]

satisfies

\[ \mathcal{A}_1 P(k, x, y) = 0 \quad (7.0.2) \]

Proof: Indeed

\[
(2n)^k \mathcal{A}_1 P(k, x, y) = k \sum_{i=1}^{n} \left( b_{x,y+e_i}^{(k-1)} - b_{x,y}^{(k-1)} \right) \\
+ \sum_{i=1}^{n} \left( x_i \left( b_{xy}^{(k)} - b_{x-e_i,y}^{(k)} \right) - x_i b_{xy}^{(k)} \right) \\
+ \sum_{i=1}^{n} \left( y_i + 1 \right) \left( b_{x,y+e_i}^{(k)} - b_{x,y}^{(k)} \right) + \left( y_i + 1 \right) b_{xy}^{(k)} \\
= k \left( \sum_{1 \leq i \leq n} b_{x,y+2e_i}^{(k-1)} - nb_{x,y}^{(k-1)} \right) \\
- \sum_{1 \leq i \leq n} \left( x_i b_{x-e_i,y}^{(k)} - (y_i + 1) b_{x,y+e_i}^{(k)} \right) \\
= -L_{-1}(x, y) b_{xy}^{(k)} = 0,
\]

using (6.1.6), insofar none of the final positions are adjacent. The second equation \( \mathcal{A}_2 P(k, x, y) = 0 \) follows immediately by the duality \( x \leftrightarrow y \). \hfill \blacksquare

References

[1] M. Adler and P. van Moerbeke: *String orthogonal Polynomials, String Equations and two-Toda Symmetries*, Comm. Pure and Appl. Math., **50**, 241–290 (1997).

[2] M. Adler and P. van Moerbeke: *Integrals over classical groups, random permutations, Toda and Toeplitz lattices*, Comm. Pure Appl. Math., **54**, 153–205, (2000) (arXiv: math.CO/9912143).
[3] Jinho Baik: *Random vicious walks and random matrices*. Comm. Pure Appl. Math. 53 (2000), no. 11, 1385–1410.

[4] A. Borodin and A. Okounkov: *A Fredholm determinant formula for Toeplitz determinants*, Integral Equations Operator Theory 37 (2000), no. 4, 386–396. math.CA/9907165

[5] A. Borodin and G. Olshanski: *Distributions on partitions, point processes, and the hypergeometric kernel*, Comm. Math. Phys. 211, 2, 335–358 (2000) math.RT/9904010

[6] F.J. Dyson: *A Brownian-Motion Model for the Eigenvalues of a Random Matrix*, Journal of Math. Phys. 3, 1191–1198 (1962)

[7] M. E. Fisher: *Walks, walls and wetting* J. Stat. Phys., 34:669 (1984)

[8] P. J. Forrester: *Random walks and random permutations*. J. Phys. A 34, 31, L417–L423 (2001) (arXiv:math.CO/9907037)

[9] I. M. Gessel: *Symmetric functions and P-recursiveness*, J. of Comb. Theory, Ser A, 53, 257–285 (1990)

[10] I. M. Gessel and Doron, Zeilberger: *Random walk in a Weyl chamber*. Proc. Amer. Math. Soc. 115, no. 1, 27–31 (1992)

[11] D. J. Grabiner and Peter Magyar: *Random walks in Weyl chambers and the decomposition of tensor powers*. J. Algebraic Combin. 2, 3, 239–260 (1993)

[12] D. J. Grabiner: *Random walk in an alcove of an affine Weyl group, and non-colliding random walks on an interval*. J. Combin. Theory Ser. A 97, no. 2, 285–306 (2002)

[13] R.S. Stanley, Enumerative combinatorics, vol. 2, Cambridge studies in advanced mathematics 62, Cambridge University Press 1999.

[14] C. A. Tracy and H. Widom: *On the distributions of the lengths of the longest monotone subsequences in random words*. Probab. Theory Related Fields 119, 3, 350–380 (2001) (arXiv:math.CO/ 9904042)