Lie symmetries methods in boundary crossing problems for diffusion processes

Dmitry Muravey*

Abstract
This paper deals with boundary crossing problems for diffusion processes. Using PDE and Lie symmetry group methods we extend known identities for diffusion processes. Our main result is the explicit connection between boundary crossing identities and Lie symmetries. For time-homogeneous diffusion with unit variance we find necessary and sufficient conditions of symmetries existence. We show that if drift function solves one of 4 Ricatti equations then corresponded PDE has 4 or 6 parametric group of symmetries. For each case we show that these groups generate two-parametric transformation from one boundary to the parametric family and prove uniqueness of these transforms. Also we extend identities for one-sided boundary to the two-sided case.

Contents

1 Introduction 3

2 Lie group symmetries method 4

3 Connection between Lie symmetry and boundary crossing identity 5

4 Symmetries for time homogeneous diffusions 7
   4.1 Necessary and sufficient conditions of symmetries’ existence 7
   4.2 Closed form solutions of drift equations 8
   4.3 Explicit formulas for symmetries 10
   4.4 Two parametric boundary transforms: closed form formulas and uniqueness 13

5 Boundary transforms via change of measure 15
   5.1 Boundary transforms via change of process to Brownian motion and Bessel process 15
   5.2 Unified approach to compute closed form densities for Brownian motion and Bessel process 17

6 Closed form formulas for some diffusion processes 17
   6.1 $2M − X$ process 17
   6.2 Pearson diffusion 18
   6.3 Ornstein-Uhlenbeck process 18
   6.4 Bessel process with drift 19
   6.5 Radial Ornstein-Uhlenbeck process 19

7 Identities for two sided boundaries 20
   7.1 Connections to the Lie symmetries 20
   7.2 Wedge probabilities for Brownian motion 21
   7.3 About one generalization of double square-root boundary 22

8 Remarks about time inhomogeneous diffusions 23

9 Concluding remarks 24

*e-mail:d.muravey87@gmail.com.
## 10 Proof routines

| Section | Description | Page |
|---------|-------------|------|
| 10.1    | Routines from proof of Theorem 3 | 25   |
| 10.2    | Routines from proof of Theorem 4 | 26   |
| 10.3    | Routines from proof of Theorem 5 | 27   |
| 10.4    | Routines from proof of Theorem 6 | 29   |
| 10.5    | Routines from proof of Theorem 7 | 30   |
| 10.6    | Routines from proof of Theorem 8 | 30   |
| 10.7    | Routines from proof of Theorem 9 | 31   |
| 10.8    | Technical Lemma from proof of Theorem 10 | 32   |
| 10.9    | Technical Lemma from proof of Theorem 11 | 32   |
| 10.10   | Routines from proof of Theorem 12 | 33   |
| 10.11   | Routines from proof of Theorem 13 | 33   |
| 10.12   | Routines from proof of Theorem 14 | 36   |
| 10.13   | Routines from proof of Theorem 15 | 37   |
| 10.14   | Routines from proof of Theorem 16 | 37   |
| 10.15   | Routines from proof of Theorem 17 | 38   |
| 10.16   | Routines from example 6.2 | 38   |
1 Introduction

Let $b(t)$ be a smooth function of time, $W_t$ be a standard Brownian motion and $X_t$ be a solution to the following stochastic differential equation

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \quad X_0 = x_0 \geq b(0).$$

Assume that functions $\mu(x, t)$ and $\sigma(x, t)$ are smooth functions such that SDE (1) has a unique non-explosive solution for any initial value $x_0$. We define the first passage time $\tau_b$ of the diffusion process $X_t$ to the curved boundary $b(t)$:

$$\tau_b = \inf\{t > 0; X_t \leq b(t)\}. \quad (2)$$

Denote by $\mathbb{P}_{x_0}$ probability conditional on the process $X_t$ started at $X_0 = x_0$. We omit subscript $x_0$ if process $X_t$ starts at zero, i.e. $x_0 = 0$. Define function $u^b_{x_0}(x, t)$ as

$$u^b_{x_0}(x, t) = \frac{\partial}{\partial x}\mathbb{P}_{x_0}(X_t \leq x, \tau_b > t). \quad (3)$$

Due to standard arguments from probability theory function $u^b_{x_0}(x, t)$ solves Cauchy problem for parabolic partial differential equation

$$\mathcal{L}_{x,t}u^b_{x_0} = 0, \quad u^b_{x_0}(b(t), t) = 0, \quad u^b_{x_0}(x, 0) = \delta(x - x_0), \quad (4)$$

where $\delta(x - x_0)$ is a Dirac measure at $x_0$ and differential operator $\mathcal{L}_{x,t}$ (subscripts $x$ and $t$ denotes differentiation variables) defined as

$$\mathcal{L}_{x,t} = \frac{1}{2} \frac{\partial^2}{\partial x^2}\sigma^2(x, t) - \frac{\partial}{\partial x}\mu(x, t) - \frac{\partial}{\partial t} = \frac{\sigma^2(x, t)}{2} \frac{\partial^2}{\partial x^2} + \eta(x, t) \frac{\partial}{\partial x} + \lambda(x, t) - \frac{\partial}{\partial t}. \quad (5)$$

where

$$\eta(x, t) = \frac{\partial}{\partial x}\sigma^2(x, t) - \mu(x, t), \quad \lambda(x, t) = \frac{1}{2} \frac{\partial^2}{\partial x^2}\sigma^2(x, t) - \frac{\partial}{\partial x}\mu(x, t). \quad (6)$$

Superscript $b$ in (3) accentuates association with boundary $b(t)$. Where scripts are no present, function $u$ is considered only as a formal solution of equation $\mathcal{L}_{x,t}$. If boundary $b(t)$ is sufficiently regular, the problem (4) has unique solution and probability $\mathbb{P}_{x_0}(\tau_b > t)$ and density $\rho^b_{x_0}(t)$ of the moment $\tau_b$ have the following representations (we also omit subscript $x_0$ in density notation if $x_0 = 0$):

$$\mathbb{P}_{x_0}(\tau_b > t) = \int_{b(t)}^{\infty} u^b_{x_0}(x, t)dx, \quad \rho^b_{x_0}(t) = \frac{\partial}{\partial t}\mathbb{P}_{x_0}(\tau_b > t) = \frac{\partial}{\partial x}\left(\frac{\sigma^2(x, t)u^b_{x_0}(x, t)}{2}\right)_{x=b(t)}. \quad (7)$$

The most popular analytic techniques to analyze stopping times $\tau_b$ are martingale theory, measure/time changes, integral equations, and PDE theory. Martingale methods were used in [22, 23, 8] etc. Applications of PDE theory to the boundary crossing problems can be found in [20]. Papers [19, 26] were devoted to connections between first passage times and integral equations.

In this paper we use Lie symmetries to analyze stopping times [2]. The main contribution of this paper is the explicit connection formula between group symmetries and boundary crossing identities - this result was formulated and proved in Theorem 1. Obtained formulas generalize known boundary crossing identities from [2] and [4]. Derived explicit transformations of first hitting time density from one boundary to the parametric families of boundaries depending only on group symmetry’ parameters. To the best of our knowledge these results are not observed or discussed in the literature. Alili and Patie in their paper [5] mentioned about connection between boundary identities for the Brownian Motion and symmetries of heat equation but they are not presented explicit relations between symmetries and identities. However they used Lie method to prove uniqueness of their two-parametric transformation.

For time-homogeneous diffusion with unit variance ($\sigma(X_t, t) \equiv 1$) we obtain necessary and sufficient conditions of existence of non-trivial symmetries. This is core result of Theorem 2: if drift function $\mu(X_t)$ solves one of 4 families of Ricatti’ equations [12] then Lie symmetries’ exist. Moreover, corresponded PDE can have
only 4 or 6 symmetries. Theorem 3 provides explicit solutions for each Ricatti equation in terms of special functions, including Airy, parabolic cylinder, Bessel and Whittaker functions. Based on these results we derive all symmetries for each drift family in explicit form, for details see Theorems 14, 15, 16, 17, 18, 19 and 20. Each symmetry can give transformation from one boundary to the one-parametric family. In Theorems 12 and 13 we derive two-parametric transformations from one boundary to the parametric family and prove its uniqueness. Families with 6 groups of symmetries have 2 two-parametric transforms: first transform is the generalization of Alili and Patie two-parametric transformation for Brownian Motion and second adds linear or hyperbolic drift (its depends on drift type) to the boundary. Families with 4 group of symmetry do not have transformation with linear or hyperbolic drift. We illustrate applications of obtained results on the many diffusion processes including Pearson diffusion, Ornstein-Uhlenbeck process, radial Ornstein-Uhlenbeck process, Bessel processes with drift and etc. Theorem 16 is the boundary transformations with change of measure. For each drift function and any boundary we show that first hitting time density can be explicitly represented in terms of first hitting time density for Brownian Motion or Bessel process. Explicit formulas are presented in Theorems 17, 18, 19 and 20. Based on these results we propose unified approach to derive almost all known closed form formulas for first hitting times densities of Brownian motion and Bessel process to the curved boundary. Theorem 24 provides identities for two-sided boundary crossing problems. In contrast to one sided case there are no similar simple relations between first hitting time densities. The identities are represented in terms of solutions of PDE boundary problems. Using these results we present alternative PDE based derivation of the Anderson’ formula for two slopping lines. Obtained results generalizes formulas for wedge probabilities from [15] to the finite horizon case. In addition we present an examples of time-homogeneous process with known group of symmetries. This example shows that Lie method can also be applicable to the time in-homogeneous processes.

2 Lie group symmetries method

A Lie group symmetry of a differential equation is a transformation which maps solutions to solutions. Application of this technique allows to construct complex solutions from trivial solutions. Here we briefly discuss the main idea of Lie method, for more details see Olver book [24]. Let \( u(x, t) \) be a solution of equation \( \mathcal{L}_{x,t}u = 0 \) and \( \mathbf{v} \) be a vector field of the form

\[
\mathbf{v} = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \varphi(x, t, u)\partial_u
\]

where \( \partial_x \) denotes partial derivative \( \partial/\partial_x \) etc. Functions \( \xi, \tau, \) and \( \varphi \) are sought to guarantee that field \( \mathbf{v} \) generates a symmetry of PDE \( \mathcal{L}_{x,t}u = 0 \). According to the Lie’s theorem, vector field \( \mathbf{v} \) generates local group of symmetries if and only if its second prolongation \( \mathbf{pr}^2\mathbf{v}[\mathcal{L}_{x,t}u] = 0 \) (See Chapter 2 of [24] for the explicit formulas of prolongations).

Action of the vector field \( \mathbf{v} \) gives triplet \((\mathcal{X}, \mathcal{T}, \mathcal{U})\) which can be obtained by the exponentiation procedure (see Chapter 1, [24])

\[
e^{\mathbf{rv}}(x, t, u) = (\mathcal{X}, \mathcal{T}, \mathcal{U}).
\]

It means that triplet \((\mathcal{X}, \mathcal{T}, \mathcal{U})\) solves the following ODE system

\[
\begin{align*}
d\mathcal{X}/de &= \xi(\mathcal{X}, \mathcal{T}, \mathcal{U}), & \mathcal{X}|_{e=0} &= x, \\
d\mathcal{T}/de &= \tau(\mathcal{X}, \mathcal{T}, \mathcal{U}), & \mathcal{T}|_{e=0} &= t, \\
d\mathcal{U}/de &= \varphi(\mathcal{X}, \mathcal{T}, \mathcal{U}), & \mathcal{U}|_{e=0} &= u.
\end{align*}
\]

**Example 1.** Consider the heat equation \( u_{xx}/2 = u_t \). The vector field \( x\partial_x + t^2\partial_t - \frac{(x^2+1)}{2}u\partial_u \) generates group of symmetry for the heat equation (see Chapter 2 [24]). System (2) has the form

\[
\begin{align*}
d\mathcal{X}/de &= \mathcal{X}\mathcal{T}, & \mathcal{X}|_{e=0} &= x, \\
d\mathcal{T}/de &= \mathcal{T}^2, & \mathcal{T}|_{e=0} &= t, \\
d\mathcal{U}/de &= -\mathcal{U}(\mathcal{X}^2 + \mathcal{T})/2, & \mathcal{U}|_{e=0} &= u.
\end{align*}
\]

We firstly solve equation for \( \mathcal{T} \), then solve equation for \( \mathcal{X} \), and in the end solve equation for \( \mathcal{U} \). In the result we have

\[
\mathcal{X} = \frac{x}{1 - et}, \quad \mathcal{T} = \frac{t}{1 - et}, \quad \mathcal{U} = \sqrt{1 - et}e^{-\frac{x^2 + 1}{2(1 - et)}} u(x, t).
\]
It means that \( U \) as a function from variables \( X \) and \( T \), i.e., \( U = U(X, T) \) is a solution of heat equation. The inverse transform of variables
\[
x = \frac{\mathcal{X}}{1 + \epsilon T}, \quad t = \frac{\mathcal{T}}{1 + \epsilon T}, \quad 1 - \epsilon t = \frac{1}{1 + \epsilon T}
\]
turns out to
\[
U(X, T) = \frac{1}{\sqrt{1 + \epsilon T}} e^{-\frac{\epsilon x^2}{2T}} u \left( \frac{\mathcal{X}}{1 + \epsilon T}, \frac{\mathcal{T}}{1 + \epsilon T} \right).
\]
Let us note that \( U(X, T) \) still be a solution of heat equation if we set \( u \equiv 1 \), because \( u \equiv 1 \) is a solution of heat equation. If we apply obvious symmetries \((X, T, U) \rightarrow (X, T + 1/\epsilon, U)\) and \((X, T, U) \rightarrow (X, T, U\sqrt{T} / \sqrt{2\pi})\) we obtain fundamental solution for the heat equation
\[
U(X, T) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}}.
\]

The relations between Lie group symmetries and fundamental solutions were discussed in [10].

For the second order parabolic operator it is easy to see (see Chapter 2 of [24]) that component \( \xi \) is independent from \( u \) \((\xi(x, t, u) = \xi(x, t))\), \( \tau \) depends only on \( t \) \((\tau(x, t, u) = \tau(t))\) and \( \varphi(x, t, u) \) is linear function \((\varphi(x, t, u) = \alpha(x, t)u + \beta(x, t))\). For parabolic operator [5] functions \( \xi, \tau, \alpha \) and \( \beta \) solve the following system of PDEs:
\[
\begin{align*}
\frac{1}{2} \frac{\partial}{\partial t} (\sigma^2 \tau) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} \xi & = 0, \\
\mathcal{L}_{x,t} \xi - \lambda \xi - \xi \frac{\partial}{\partial x} (\eta \tau) - \sigma^2 \frac{\partial}{\partial x} \alpha & = 0, \\
\mathcal{L}_{x,t} \alpha - \lambda \alpha + \xi \frac{\partial}{\partial x} \lambda + \frac{\partial}{\partial t} (\lambda \tau) & = 0, \\
\mathcal{L}_{x,t} \beta & = 0.
\end{align*}
\]
The last equation means that \( \beta(x, t) \) is an arbitrary solution of the equation [11]. The corresponded symmetry is \((x, t, u) \rightarrow (x, t, u + \beta(x, t))\). Note that system \([11]\) always has trivial solution \( \tau \equiv \xi \equiv 0 \) and \( \alpha \equiv C \), where \( C \) is an arbitrary constant. This field generates trivial symmetry \((x, t, u) \rightarrow (x, t, \epsilon u)\). Both symmetries are consequence of linearity of differential operator \( \mathcal{L} \). In general case there are no known techniques to find a non-trivial solutions of system \([11]\).

3 Connection between Lie symmetry and boundary crossing identity

**Theorem 1.** Suppose that equation \( \mathcal{L}_{x,t} u = 0 \) associated with diffusion process [11] has a non-trivial Lie symmetry
\[
s : (x, t, u(x, t)) \rightarrow (\mathcal{X}(x, t), \mathcal{T}(t), f(x, t)U(X, T)), \quad \mathcal{L}_{X,T} U = 0, \quad T(0) = 0, \quad (11)
\]
Let \( g(t) \) be a solution of equation \( \mathcal{X}(g(t), t) = b(T) \) and \( \Psi(\mathcal{Y}) \) be a solution of equation \( \mathcal{X}(\Psi(\mathcal{Y}), 0) = \mathcal{Y} \):
\[
g(t) : \mathcal{X}(g(t), t) = b(T), \quad \Psi(\mathcal{Y}) : \mathcal{X}(\Psi(\mathcal{Y}), 0) = \mathcal{Y}. \quad (12)
\]
The density function of the first passage time to the boundary \( g(t) \) is represented by
\[
\rho_{x_0}^g(t) = \frac{\sigma^2(g(t), t)}{\sigma^2(b(T(t)), T(t))} \frac{f(g(t), t) \partial \mathcal{X}/\partial x|_{x=g(t)}}{f(x_0, 0) \Psi'(\mathcal{X}(x_0, 0))} \delta(\mathcal{X}(x_0, 0)(T(t))). \quad (13)
\]
**Proof.** Let function \( U(X, T) \) be a solution of Cauchy problem
\[
\begin{align*}
\mathcal{L}_{X,T} U & = 0, \\
U(b(T), T) & = 0, \\
U(X, 0) & = \delta(\Psi(X) - x_0)/f(\Psi(X), 0).
\end{align*}
\]
Hence function \( u^g_{x_0}(x, t) = f(x, t)\mathcal{U}(X, T) \) solves the problem
\[
\begin{align*}
\mathcal{L}_{x, t} u^g_{x_0} &= 0, \\
u^g_{x_0}(g(t), t) &= 0, \\
u^g_{x_0}(x, 0) &= \delta(x-x_0).
\end{align*}
\] (14)

We can represent function \( \mathcal{U} \) in the following form:
\[
\mathcal{U}(X, T) = \int_{b(0)}^{\infty} u_{X_0}^b(X, T) W(X_0, 0) dX_0
= \int_{b(0)}^{\infty} u_{X_0}^b(X, T) \delta(\Psi(X_0) - x_0) f(\Psi(X_0), 0) dX_0
= \int_{b(0)}^{\infty} u_{X_0}^b(X, T) \delta(\Psi(X_0) - x_0) \frac{\psi(X_0) f(\Psi(X_0), 0)}{\psi'(X_0)} |_{\psi(X_0)=x_0}
= \frac{u_{X_0}^b(X, T)}{\psi'(X)|_{X(x_0), f(x_0), 0}}.
\]

The density \( \rho^{T_g}_{x_0}(t) \) is represented by
\[
\rho^{T_g}_{x_0}(t) = \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\sigma^2(x, t) u^g_{x_0}(x, t; x_0)}{\Psi'(X)|_{X=x_0, f(x_0), 0}} \right) |_{x=g(t)}
= \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\sigma^2(x, t) u_{X_0}^b(X, T) f(x, t)}{\sigma^2(X, T) \Psi'(X)|_{X(x_0, f(x_0), 0)}} \right) |_{x=g(t)}
= \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\sigma^2(x, t) f(x, t)}{\sigma^2(X, T) \Psi'(X)|_{X(x_0, f(x_0), 0)}} \right) |_{x=g(t)}.
\]

\textbf{Example 2.} We illustrate application of the main formula (13) on the well-known Bachelier-Lévy formula. This formula gives the first passage time of Brownian motion to the linear boundary \( g(t) = \alpha + \beta t \). We present two transformations of well-known density correspondent to the straight line, i.e. \( b(t) = \alpha, \rho^{T_g}_{x_0} = \frac{|\alpha|}{\sqrt{2\pi t^3}} e^{-\frac{\alpha^2}{2t}} \) which gives us density of the first passage time to the slopping line \( g(t) \).

Recall one parametric symmetries for the heat equation (see Chapter 2 of [12]):
\[
\begin{align*}
s_1 &: u(x, t) = \frac{1}{\sqrt{1+\epsilon t}} e^{-\frac{\epsilon^2 x^2}{1+\epsilon t}} \mathcal{U} \left( \frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t} \right), \\
s_2 &: u(x, t) = \mathcal{U} e^{\epsilon x, e^{2\epsilon t}}.
\end{align*}
\] (15)
\[ s_1 : \ u(x,t) = e^{-\epsilon x + \epsilon^2 t/2}U(x-\epsilon t,t), \]
\[ s_4 : \ u(x,t) = U(x+\epsilon,t) \]
\[ s_5 : \ u(x,t) = U(x,t+\epsilon), \]
\[ s_6 : \ u(x,t) = e^{\epsilon}U(x,t), \]
\[ s_3 : \ u(x,t) = \beta(x,t) + U(x,t). \]

Symmetry \( s_4 \) with \( \epsilon = \beta \) yields

\[ u(x,t) = e^{-\beta x + \beta^2 t/2}U(x-\beta t,t). \]

New boundary \( g(t) \) is a solution of equation \( X(g(t),t) = g(t) - \beta t = \alpha \). Application formula \( (13) \) turns out to identity \( (f(0,0) = 1, dX/dx \equiv 1, \Psi' \equiv 1) \)

\[ \rho^{s_1}(t) = \frac{|\alpha|}{\sqrt{2\pi t^3}}e^{-\frac{\alpha^2}{2t}}e^{-\beta(\alpha+\beta)t + \beta^2 t/2} = \frac{|\alpha|}{\sqrt{2\pi t^3}}e^{-\frac{\alpha^2}{2t^2} - \frac{\beta^2}{2t}}. \]

Another derivation is based on application of composition of symmetries \( s_1 \) and \( s_5 \) to the boundary \( b(t) = \beta \)

with density \( \rho^{s_1} = \frac{|\beta|}{\sqrt{2\pi t^3}}e^{-\frac{\beta^2}{2t}} \)

\[ u(x,t) = \frac{e^{-\frac{(x^2+\alpha^2)}{2t}}}{\sqrt{t+\alpha/\beta}}U \left( \frac{x}{t+\alpha/\beta}, \frac{t}{(t+\alpha/\beta)\alpha/\beta} \right). \]

Transformed boundary \( g(t) = (t+\alpha/\beta)b(T(t)) = \alpha + \beta t \) and \( \Psi' = \alpha/\beta, f(x_0,0) = \sqrt{\beta/\alpha} \). Hence \( \rho^{s_1}(t) \) is equal

\[ \rho^{s_1}(t) = \frac{\sqrt{\beta/\alpha}}{\sqrt{(t+\alpha/\beta)^3}}e^{-\frac{t^2+\alpha^2}{2t}}\rho^{s_1} \left( \frac{t}{(t+\alpha/\beta)\alpha/\beta} \right) = \frac{|\alpha|}{\sqrt{2\pi t^3}}e^{-\frac{\alpha^2}{2t^2} - \frac{\beta^2}{2t}}. \]

4 Symmetries for time homogeneous diffusions

In this section we consider only time-homogeneous diffusion process, i.e. \( \mu(x,t) \equiv \mu(x) \) and \( \sigma(x,t) \equiv \sigma(x) \).

Without loss of generality we assume that \( \sigma(x) \equiv 1 \). If \( \sigma(x) \neq 1 \) we can apply Lamberti transform (see e.g. [8]):

\[ dU_t = \nu(U_t)dt + \sigma(U_t)dW_t, \quad F(y) = \int_y^\infty \frac{du}{\sigma(u)} \quad X_t = F(U_t); \]
\[ dX_t = \mu(X_t)dt + dW_t, \quad \mu(y) = \left( \frac{\nu}{\sigma} - \frac{\sigma'}{2} \right) \circ F^{-1}(y). \]

Corresponded differential operator \( \mathcal{L}_{x,t} \) has the form

\[ \mathcal{L}_{x,t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} - \mu(x) \frac{\partial}{\partial x} - \mu'(x) - \frac{\partial}{\partial t}. \]

4.1 Necessary and sufficient conditions of symmetries’ existence

Necessary and sufficient conditions of existence of non-trivial symmetry of differential operator \( \mathcal{L} \) are in the next theorem.

Theorem 2. Equation

\[ \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - \mu(x) \frac{\partial u}{\partial x} - \mu'(x)u - \frac{\partial u}{\partial t} = 0. \]

has non-trivial Lie symmetry \( \mathcal{F} \) if and only if drift function \( \mu(x) \) is a solution of one of the following families of Ricatti equations:

\[ \mathcal{F}_1 : \mu' + \mu^2 = 4Bx + 2C, \]
\[ \mathcal{F}_2 : \mu' + \mu^2 = Ax^2 + 4Bx + 2C, \]
\[ \mathcal{F}_3 : \mu' + \mu^2 = 2C + \frac{D}{x^2}, \]
\[ \mathcal{F}_4 : \mu' + \mu^2 = Ax^2 + 2C + \frac{D}{x^2}. \]

where \( A, B, C \) and \( D \) are arbitrary constants.
Proof. Craddock and Lennox considered in [11] Lie symmetries for operator
\[ \sigma x^2 \frac{\partial^2 u}{\partial x^2} + f(x) \frac{\partial u}{\partial x} - g(x)u - \frac{\partial u}{\partial t} = 0. \] (20)
They proved that if \( f(x) \) solves one of the families of Riccati equations with coefficients depending on function \( g \), then operator (20) has non-trivial Lie symmetries. Differential operator \( \mathcal{L} \) from [11] is the special case of (20) with specifications \( \gamma = 0, \sigma = 1/2, f(x) = -\mu(x) \) and \( g(x) = \mu'(x) \). For readers’ convenience we present proof from [11] with our specifications. According to [11, p. 129]) components of vector field \( v \) allow the following representation
\[ \xi(x,t) = \frac{x}{2} \tau_t + \rho, \quad \varphi(x,t,u) = \alpha(x,t)u + \beta(x,t), \] (21)
where \( \tau, \rho \) and \( \eta \) are functions of \( t \) only and solve the equation
\[ -\frac{x^2}{4} \tau_{tt} - x \rho_t + \eta_t = \frac{\tau_{tt}}{4} - L(x) \tau_t + K(x) \rho. \] (22)
Functions \( K(x) \) and \( L(x) \) are defined analogical to [11, p. 129, formula 2.3]
\[ L(x) = \frac{x}{4} \left( \mu' + \mu^2 \right)' + \frac{\mu' + \mu^2}{2}, \quad K(x) = -\frac{(\mu' + \mu^2)^2}{2}. \]
Equation (22) has non-trivial solution if and only if function \( L(x) \) is quadratic (in this case function \( K(x) \) is linear) i.e.
\[ L(x) = \frac{x}{4} \left( \mu' + \mu^2 \right) + \frac{\mu' + \mu^2}{2} = Ax^2 + 3Bx + C. \]
For function \( \zeta(x) = \mu' + \mu^2 \) we have first order ODE
\[ \zeta' + \frac{2}{x} \zeta = 4Ax + 12B + \frac{4C}{x}, \]
which can be simply solved by standard techniques
\[ \mu' + \mu^2 = \frac{D}{x^2} + \frac{1}{x^2} \int \left( 4Ax^3 + 12Bx^2 + 4Cx^2 \right) dx = Ax^2 + 4Bx + 2C + \frac{D}{x^2}, \]
for \( K(x) \) and \( L(x) \) we have the following formulas
\[ L(x) = Ax^2 + 3Bx + C, \quad K(x) = -\left( Ax + 2B - \frac{D}{x^2} \right). \] (23)
Substitution of (23) in (22) yields
\[ -\frac{x^2}{4} \tau_{tt} - x \rho_t + \eta_t = \frac{\tau_{tt}}{4} - \left( Ax^2 + 3Bx + C \right) \tau_t - \left( Ax + 2B - \frac{D}{x^2} \right) \rho. \] (24)
Comparison of the corresponding powers turns out to the following ODE system
\begin{align*}
\tau_{tt} - 4A\tau_t &= 0, \quad \rho_t - A\rho = 3B\tau_t, \\
\eta_t &= -\left( \frac{1}{4} \tau_{tt} + C\tau_t + 2B\rho \right), \\
D\rho &= 0.
\end{align*} (25)
It is easy to see that condition \( D = 0 \) leads to \( B = 0 \). Hence we have two families of Riccati equations for \( \mu \):
\begin{align*}
\mu' + \mu^2 &= Ax^2 + 4Bx + 2C, \\
\mu' + \mu^2 &= Ax^2 + 2C + \frac{D}{x^2}.
\end{align*}
For convenience we arrange it into 4 families (19).

Remark 1. Condition (19) is equivalent to existence of Lie symmetries of differential operator \( \mathcal{G} = \frac{\partial}{\partial t} \), where \( \mathcal{G} \) is the infinitesimal generator of diffusion process (19). Application of Proposition 2.1 from [17] with specifications \( \gamma = 0, \sigma = 1/2, f(x) = \mu(x), g(x) = 0 \) to the operator \( \mathcal{G} = \frac{\partial}{\partial t} \) gives the same 4 families of Ricatti equations.
4.2 Closed form solutions of drift equations

Each Ricatti equation from (19) can be explicitly solved. For all equations we find solution $\mu(x)$ in the following form

$$\mu(x) = \ln' (\theta_{\mathcal{F}_n}(x)), \quad \theta_{\mathcal{F}_n}(x) = e^{\int \mu(x) dx}, \quad \mu \in \mathcal{F}_n, \quad n = 1, 4,$$

function $\theta_{\mathcal{F}_n}(x)$ makes linearization of Ricatti equation

$$\theta_{\mathcal{F}_n}'(x) - (\mu'(x) + \mu^2(x)) \theta_{\mathcal{F}_n}(x) = 0. \quad (27)$$

In the next theorem we present explicit formulas for each $\theta_{\mathcal{F}_n}(x)$ in terms of elementary and special functions, for definitions and properties see [1].

**Theorem 3.** Each function $\theta_{\mathcal{F}_n}(x)$ $n = 1, 4$ is represented by the following formulas:

$$\begin{align*}
\theta_{\mathcal{F}_1}(x) &= \left\{ \begin{array}{ll}
c_1 Ai \left( \sqrt{4B} \left[ x + C/2B \right] \right) + c_2 Bi \left( \sqrt{4B} \left[ x + C/2B \right] \right), & B \neq 0, \\
c_1 \sinh \left( \sqrt{2C} x \right) + c_2 \cosh \left( \sqrt{2C} x \right), & B = 0,
\end{array} \right.
\quad (30)

\theta_{\mathcal{F}_2}(x) &= c_1 D_{2v} \left( \sqrt{2A^{1/4}} \left[ x + 2B/4 \right] \right) + c_2 D_{2v} \left( -\sqrt{2A^{1/4}} \left[ x + 2B/4 \right] \right), \quad \nu = \frac{1}{2\sqrt{A}} \left( \frac{2B^2}{A} - \frac{\sqrt{A}}{2} - C \right),

\theta_{\mathcal{F}_3}(x) &= c_1 D_{1/2} \left( \sqrt{2A^{1/4}} \left[ x + 1/2 \right] \right) + c_2 D_{1/2} \left( -\sqrt{2A^{1/4}} \left[ x + 1/2 \right] \right), \quad C \neq 0,

\theta_{\mathcal{F}_4}(x) &= c_1 M_{\lambda,\mu} \left( \sqrt{Ax^2} \right) / \sqrt{x} + c_2 W_{\lambda,\mu} \left( \frac{\sqrt{Ax^2}}{2} \right) / \sqrt{x},
\end{align*}$$

where $Ai(z)$ and $Bi(z)$ are Airy functions, $D_{\nu}(z)$ and $D_{-\nu}(z)$ are parabolic cylinder functions, $I_{\nu}(z)$ and $K_{\nu}(z)$ are modified Bessel functions, $M_{\lambda,\mu}(z)$ and $W_{\lambda,\mu}(z)$ are Whittaker functions and $c_1$ and $c_2$ are arbitrary constants.

**Proof.** Each equation (27) can be reduced to the one of ODEs:

$$\begin{align*}
\frac{d^2 Z}{dx^2} - \lambda Z &= 0, \quad Z(z) = c_1 \sinh(\sqrt{x}) + c_2 \cosh(\sqrt{x}), \quad (28)

\frac{d^2 Z}{dx^2} - \frac{\lambda}{z} Z &= 0, \quad Z(z) = c_1 z^{1/2+\sqrt{\lambda+1/4}} + c_2 z^{1/2-\sqrt{\lambda+1/4}}, \quad (29)

\frac{d^2 Z}{dz^2} - z Z &= 0, \quad Z(z) = c_1 Ai(z) + c_2 Bi(z), \quad (30)

\frac{d^2 Z}{dz^2} + \frac{1}{z} \frac{dZ}{dz} + \left( 1 + \frac{\mu^2}{z^2} \right) Z &= 0, \quad Z(z) = c_1 I_{\nu}(z) + c_2 K_{\nu}(z), \quad (31)

\frac{d^2 Z}{dz^2} + \left( p + 1 + \frac{1/4 - \mu^2}{z^2} \right) Z &= 0, \quad Z(z) = c_1 D_{p}(z) + c_2 D_{p}(-z), \quad (32)

\frac{d^2 Z}{dz^2} + \left( \frac{1}{4} + \frac{\lambda}{z} + \frac{1/4 - \mu^2}{z^2} \right) Z &= 0, \quad Z(z) = c_1 M_{\lambda,\mu}(z) + c_2 W_{\lambda,\mu}(z), \quad (33)
\end{align*}$$

All of equations exclude (28) and (29) can not be solved in elementary functions. Linearly independent solutions of (30), (31) and (32) are Airy, modified Bessel, parabolic cylinder and Whittaker functions respectively, for definitions and properties see [1]. Let us mention that Airy, modified Bessel and parabolic cylinder functions can be represented in terms of Whittaker functions.

Explicit transformations which leads to the one of equations (28–33) are presented in subsection 10.1. □

**Remark 2.** Constants $c_1$ and $c_2$ are set to guarantee the existence of unique and non-explosive solution of SDE (16).
4.3 Explicit formulas for symmetries

**Theorem 4.** All symmetries of differential equation (4) with drift functions $\mu(x) \in \mathcal{F}_1$ are generated by the following vector fields or its composition:

\[
\begin{align*}
\mathbf{v}_1 &\ = \ (xt + Bt^3) \partial_x + t^2 \partial_t - u \left( \frac{x^2}{2} + 3Bxt^2 + \frac{t}{2} + Ct^2 + \frac{B^2}{2} t^4 - \mu(x)(xt + Bt^3) \right) \partial_u, \\
\mathbf{v}_2 &\ = \ (\frac{x}{2} + \frac{Bt^2}{2}) \partial_x + t \partial_t - u \left( 3Bxt + Ct + B^2 t^3 - \mu(x) \left( \frac{t}{2} + \frac{3Bt^2}{2} \right) \right) \partial_u, \\
\mathbf{v}_3 &\ = \ t \partial_x - u \left( Bt^2 + x - t \mu(x) \right) \partial_u, \\
\mathbf{v}_4 &\ = \ \partial_x - u \left( 2Bt - \mu(x) \right) \partial_u, \\
\mathbf{v}_5 &\ = \ \partial_t, \\
\mathbf{v}_6 &\ = \ u \partial_u, \\
\mathbf{v}_7 &\ = \ \beta(x,t) \partial_u.
\end{align*}
\]

**Proof.** Proof is a routine solving of system (25) with $A = 0$. Each linearly independent solution gives vector field from (54). All routines can be found in subsection 10.3. We group of all terms with respect to the constants $C_i$, $i = 1, 6$ (each constant correspond to the each vector field from (54)) and use the following representation for $\alpha(x,t)$

\[
\alpha(x,t) = - \int \frac{\partial \xi}{\partial t} (x,t) dx + \mu \xi + \eta.
\]

**Theorem 5.** Each vector field from (54) generates symmetry

\[
\begin{align*}
\mathbf{s}_1 : u(x,t) &\ = \ f_1^*(x,t) \mathcal{U} \left( \frac{x - Bt^2}{1 + e^t} + \frac{Bt^2}{(1 + e^t)^2} \right), \\
\mathbf{s}_2 : u(x,t) &\ = \ f_2^*(x,t) \mathcal{U} \left( (x - Bt^2) e^{-e^t/2} + Bt^2 e^{-2e^t}, te^{-t} \right), \\
\mathbf{s}_3 : u(x,t) &\ = \ f_3^*(x,t) \mathcal{U}(x - et, t), \\
\mathbf{s}_4 : u(x,t) &\ = \ (x - \epsilon, t, f_4^*(x,t) \mathcal{U}(X, T)), \\
\mathbf{s}_5 : u(x,t) &\ = \ \mathcal{U}(x,t + \epsilon), \\
\mathbf{s}_6 : u(x,t) &\ = \ \epsilon^t \mathcal{U}(x,t), \\
\mathbf{s}_7 : u(x,t) &\ = \ \beta(x,t) + \mathcal{U}(x,t).
\end{align*}
\]

where functions $f_n^*(x,t)$, $n = 1, 4$ defined as

\[
\begin{align*}
f_1^*(x,t) &\ = \ \frac{\theta_{x_1}(x)}{\theta_{x_1}((x - Bt^2) e^{-e^t/2} + Bt^2 e^{-2e^t})} \exp \left\{ 2Bt (x - Bt^2) \left[ e^{-3e} - 1 \right] + \frac{4B^2 t^3}{3} [e^{-3e} - 1] + Ct [\epsilon^{-e} - 1] \right\}, \\
f_2^*(x,t) &\ = \ \frac{\theta_{x_1}(x)}{\theta_{x_1}(x - et)} \exp \left\{ \frac{2t^2}{3} - \epsilon \left( x + Bt^2 \right) \right\}, \\
f_3^*(x,t) &\ = \ \frac{\theta_{x_1}(x)}{\theta_{x_1}(x - et)} \exp \left\{ -2B t e^t \right\}, \\
f_4^*(x,t) &\ = \ \frac{\theta_{x_1}(x)}{\theta_{x_1}(x - et)} \exp \left\{ \frac{2B t^2}{3} \left[ - \epsilon x - 1 \right] + Ct [\epsilon^{-e} - 1] \right\}.
\end{align*}
\]

**Proof.** Proof of this theorem consists of only explicit exponentiation (49) of each field (54). Routine solutions of corresponded systems (10) are in subsection 10.3.

**Theorem 6.** All symmetries of differential equation (7) with drift functions $\mu(x) \in \mathcal{F}_2$ are generated by the following vector fields or its composition:

\[
\begin{align*}
\mathbf{v}_1 &\ = \ (x \sqrt{A} + \frac{2B}{\sqrt{A}}) \cosh \left( 2\sqrt{A} t \right) \partial_x + \sinh \left( 2\sqrt{A} t \right) \partial_t \\
\end{align*}
\]
- \( u \left( \frac{x^2 A + 4 B x + 2 B^2}{A} + C \right) \sinh \left( 2 \sqrt{A t} \right) + \left( \frac{\sqrt{A}}{2} - \frac{2 B}{\sqrt{A}} \right) u(x) \cosh \left( 2 \sqrt{A t} \right) \partial_u, \)

\[ v_2 = \left( x \sqrt{A} + \frac{2 B}{\sqrt{A}} \right) \sinh \left( 2 \sqrt{A t} \right) \partial_x + \cosh \left( 2 \sqrt{A t} \right) \partial_t \]

- \( u \left( \frac{x^2 A + 4 B x + 2 B^2}{A} + C \right) \cosh \left( 2 \sqrt{A t} \right) + \left( \frac{\sqrt{A}}{2} - \frac{2 B}{\sqrt{A}} \right) u(x) \sinh \left( 2 \sqrt{A t} \right) \partial_u, \)

\[ v_3 = \sinh \left( \sqrt{A t} \right) \partial_x - u \left( \left( \sqrt{A} x + \frac{2 B}{\sqrt{A}} \right) \cosh \left( \sqrt{A t} \right) - \sinh \left( \sqrt{A t} \right) u(x) \right) \partial_u, \]

\[ v_4 = \cosh \left( \sqrt{A t} \right) \partial_x - u \left( \left( \sqrt{A} x + \frac{2 B}{\sqrt{A}} \right) \sinh \left( \sqrt{A t} \right) - \cosh \left( \sqrt{A t} \right) u(x) \right) \partial_u, \]

\[ v_5 = \partial_t, \quad v_6 = u \partial_u, \quad v_7 = \beta(x, t) \partial_u. \]

**Proof.** Proof is fully analogous to the proof of theorem 3. Formulas of all terms used in field components \( \tau, \xi, \) and \( \alpha \) can be found in 10.3. \( \square \)

**Theorem 7.** Vector fields \( \nu^a \) generate the following symmetries:

\[ s_1 : u(x, t) = f_1^a(x, t) \partial_t \left( \frac{x + 2 B/A}{\sqrt{1 + e^{2 \sqrt{A t}}}} - 2 B/A, \frac{1}{2 \sqrt{A}} \ln \left( \frac{1}{1 + \sqrt{e^{2 \sqrt{A t}}} - 1} \right) \right), \quad (37) \]

\[ s_2 : u(x, t) = f_2^a(x, t) \partial_t \left( \frac{x + 2 B/A}{\sqrt{1 + e^{-2 \sqrt{A t}}}} - 2 B/A, \frac{1}{2 \sqrt{A}} \ln \left( e^{2 \sqrt{A t}} \right) \right), \]

\[ s_3 : u(x, t) = f_3^a(x, t) \partial_t \left( x - e^{\sqrt{A t}}, t \right), \]

\[ s_4 : u(x, t) = f_4^a(x, t) \partial_t \left( x - e^{-\sqrt{A t}}, t \right), \]

\[ s_5 : u(x, t) = U(x, t + \epsilon), \]

\[ s_6 : u(x, t) = e^\epsilon U(x, t), \]

\[ s_7 : u(x, t) = \beta(x, t) + U(x, t) \]

Functions \( f_n^a(x, t), n = 1, 4 \) are defined by the following formulas

\[ f_2^1(x, t) = \frac{\theta_{\varphi_2}(x)}{\theta_{\varphi_2}(x + 2 B/A - 2 B/A)} \left( 1 + \epsilon e^{2 \sqrt{A t}} \right)^{1/2} \exp \left\{ - \frac{\sqrt{A}}{2} (x + 2 B/A)^2 \left( 1 - \frac{1}{1 + \epsilon e^{2 \sqrt{A t}}} \right) \right\}, \quad (38) \]

\[ f_3^1(x, t) = \frac{\theta_{\varphi_2}(x)}{\theta_{\varphi_2}(x - e^{\sqrt{A t}})} \exp \left\{ \frac{\sqrt{A}}{2} e^{2 \sqrt{A t}} - \sqrt{A} e^{\sqrt{A t}} (x + 2 B/A) \right\}, \]

\[ f_4^1(x, t) = \frac{\theta_{\varphi_2}(x)}{\theta_{\varphi_2}(x - e^{-\sqrt{A t}})} \exp \left\{ \frac{\sqrt{A}}{2} e^{-2 \sqrt{A t}} + \sqrt{A} e^{-\sqrt{A t}} (x + 2 B/A) \right\}, \]

constant \( \nu \) is equal

\[ \nu = \frac{1}{2 \sqrt{A}} \left( \frac{2 B^2}{A} - \frac{\sqrt{A}}{2} - C \right). \]

**Proof.** Proof of this theorem consists of only explicit exponentiation \( \nu \) of each field. \( \square \) Routine solutions of corresponded systems \( 10.3 \) are in 10.5
Theorem 8. All symmetries of differential equation (3) with drift functions \( \mu(x) \in \mathcal{F}_3 \) are generated by the following vector fields or its composition:

\[
\begin{align*}
\mathbf{v}_1 &= x t \partial_x + t^2 \partial_t - u \left( \frac{x^2}{2} + \frac{t}{2} + C t^2 - x t \mu(x) \right) \partial_u, \\
\mathbf{v}_2 &= \frac{x}{2} \partial_x + t \partial_t - u \left( C t - \frac{\nu}{2} \mu(x) \right) \partial_u, \\
\mathbf{v}_3 &= \partial_t, \\
\mathbf{v}_4 &= u \partial_u, \\
\mathbf{v}_5 &= \beta(x,t) \partial_u.
\end{align*}
\] (39)

Proof. These fields can be derived from routines for (33) with \( B = 0, C_3 = 0, C_4 = 0 \) and further re-numeration \( C_5 \rightarrow C_3 \) and \( C_6 \rightarrow C_4 \).

Theorem 9. Each vector field from (39) generates symmetry

\[
\begin{align*}
s_1 : u(x,t) &= \frac{\theta_{\mathcal{F}_1}(x) \exp \left\{ \left[ \frac{x^2}{4} + \mathcal{C} \right] \left[ \frac{t}{1 + \varepsilon} - t \right] \right\}}{\theta_{\mathcal{F}_1}(x/(1 + \varepsilon t))} \mathcal{U} \left( \frac{x}{1 + \varepsilon t} - \frac{t}{1 + \varepsilon t} \right), \\
s_2 : u(x,t) &= \frac{\theta_{\mathcal{F}_1}(x) \exp \left\{ \left[ \frac{x^2}{4} + \mathcal{C} \right] \left[ \frac{t}{1 + \varepsilon} - t \right] \right\}}{\theta_{\mathcal{F}_1}(x/(1 + \varepsilon t))} \mathcal{U} \left( x e^{-\varepsilon/2} - \varepsilon \right), \\
s_3 : u(x,t) &= \mathcal{U}(x,t + \varepsilon), \\
s_4 : u(x,t) &= \mathcal{U}(x,t), \\
s_5 : u(x,t) &= \beta(x,t) + \mathcal{U}(x,t).
\end{align*}
\] (40)

Proof. Vector fields (39) are special case of (33) with \( B = 0 \) and without \( \mathbf{v}_3 \) and \( \mathbf{v}_4 \). Hence we can set \( B = 0 \) in all formulas (39) and exclude symmetries \( s_3 \) and \( s_4 \).

Theorem 10. All symmetries of differential equation (3) with drift functions \( \mu(x) \in \mathcal{F}_4 \) are generated by the following vector fields or its composition:

\[
\begin{align*}
\mathbf{v}_1 &= x \sqrt{A} \cosh \left( 2 \sqrt{A} t \right) \partial_x + \sinh \left( 2 \sqrt{A} t \right) \partial_t \\
&\quad - u \left( x^2 A + C \right) \sinh \left( 2 \sqrt{A} t \right) - \left( \sqrt{A} \right) x \sqrt{A} \mu(x) \cosh \left( 2 \sqrt{A} t \right) \partial_u, \\
\mathbf{v}_2 &= x \sqrt{A} \sinh \left( 2 \sqrt{A} t \right) \partial_x + \cosh \left( 2 \sqrt{A} t \right) \partial_t \\
&\quad - u \left( x^2 A + C \right) \cosh \left( 2 \sqrt{A} t \right) - \left( \sqrt{A} \right) x \sqrt{A} \mu(x) \sinh \left( 2 \sqrt{A} t \right) \partial_u, \\
\mathbf{v}_3 &= \partial_t, \\
\mathbf{v}_4 &= u \partial_u, \\
\mathbf{v}_5 &= \beta(x,t) \partial_u.
\end{align*}
\] (41)

Theorem 11. Vector fields (42) generate the following symmetries:

\[
\begin{align*}
s_1 : u(x,t) &= \mathcal{F}_1(x,t) \mathcal{U} \left( \frac{x}{\sqrt{1 + \varepsilon e^{2 \sqrt{A} t}}} \right) \frac{1}{2 \sqrt{A}} \ln \left( \frac{1}{\varepsilon e^{2 \sqrt{A} t}} \right), \\
s_2 : u(x,t) &= \mathcal{F}_2(x,t) \mathcal{U} \left( \frac{x}{\sqrt{1 + \varepsilon e^{-2 \sqrt{A} t}}} \right) \frac{1}{2 \sqrt{A}} \ln \left( \varepsilon e^{2 \sqrt{A} t} \right), \\
s_3 : u(x,t) &= \mathcal{U}(x,t + \varepsilon), \\
s_4 : u(x,t) &= \varepsilon \mathcal{U}(x,t), \\
s_5 : u(x,t) &= \beta(x,t) + \mathcal{U}(x,t).
\end{align*}
\] (42)

Functions

\[
\begin{align*}
f_1(x,t) &= \frac{\theta_{\mathcal{F}_2}(x) \left( 1 + \varepsilon e^{2 \sqrt{A} t} \right)^\nu \exp \left\{ -\frac{\sqrt{A} x^2}{2} \left( 1 - \frac{1}{1 + \varepsilon e^{2 \sqrt{A} t}} \right) \right\}}{\theta_{\mathcal{F}_2} \left( \frac{x}{\sqrt{1 + \varepsilon e^{2 \sqrt{A} t}}} \right)}.
\end{align*}
\] (43)
\[ f_2^2(x,t) = \frac{\theta_{g_1}(x)}{\sqrt{1 + e^{-2\sqrt{A}}}} \left(1 + e^{-2\sqrt{A}}\right)^{1/2} \exp \left\{ \frac{\sqrt{A}x^2}{2} \left(1 - \frac{1}{1 + e^{-2\sqrt{A}}}\right) \right\}. \]

constant \( \nu \) is equal
\[ \nu = -\frac{1}{4} - \frac{C}{2\sqrt{A}}. \]

**Proof.** Proof is fully analogical to the proof of Theorem 9. \( \square \)

### 4.4 Two parametric boundary transforms: closed form formulas and uniqueness

In theorems 5, 7, 9 and 11 we proved that for corresponded parametric families there are only 4 or 6 group of symmetries and presented it in explicit form. Each symmetry transforms curved boundary \( b(t) \) to the one-parametric family \( g(t) \) depending on parameter \( \epsilon \). Allili and Patie found in [2] transformation from curved boundary to the two-parametric family of boundaries for Brownian motion and proved in [3] its uniqueness. In this section we generalize their results any process with drift from \( \mathcal{F}_1 \) to \( \mathcal{F}_4 \). We present these transformations in explicit form and prove uniqueness. For families with 6 group of symmetries \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) we have another one transformation which adds linear or hyperbolic drift to the boundary. Note that fields \( v_6 \) in [23, 30], \( v_4 \) in [23, 31], \( v_5 \) and \( v_3 \) in all families do not generate any transformation of boundaries because they do not have action on spatial and time variables. Symmetry generated by the field \( v_5 \) (or \( v_3 \)) can be applicable only to satisfy necessary condition \( T(0) = 0 \). Field \( v_3 \) or \( v_4 \) in [23] and [30] does not transform time variable \( t \) and adds linear (for \( \mathcal{F}_1 \)) or hyperbolic (for \( \mathcal{F}_2 \)) drift to the boundary \( b(t) \).

**Theorem 12.** All symmetries of the form \([17]\) for drift functions \( \mu \in \mathcal{F}_1 \) are represented as one of the following two-parametric symmetries or its composition

\[ s_f^{\alpha,\beta}: u(x,t) = f_{\mathcal{F}_1}^{\alpha,\beta}(x,t)U \left( \frac{(x - Bt^2)}{1 + \alpha \beta t} + \frac{Ba^4t^2}{(1 + \alpha \beta t)^2}, \frac{\alpha^2t}{1 + \alpha \beta t} \right) \]  \hspace{1cm} (44)

\[ s_h^{\alpha,\beta}: u(x,t) = h_{\mathcal{F}_1}^{\alpha,\beta}(x,t)U(x - \alpha - \beta t). \]

where functions \( f_{\mathcal{F}_1}^{\alpha,\beta}(x,t) \) and \( h_{\mathcal{F}_1}^{\alpha,\beta}(x,t) \)

\[ \begin{align*}
  f_{\mathcal{F}_1}^{\alpha,\beta}(x,t) &= \frac{1}{\theta_{\mathcal{F}_1}(x)} e^{\frac{\alpha x^2 - 2\alpha \beta x + \beta^2}{1 + \alpha \beta t} - Ct} = \frac{1}{\theta_{\mathcal{F}_1}(x)} \exp \left\{-\frac{\beta t}{2} \right\}, \quad \mathcal{X} = \frac{(x - Bt^2)}{1 + \alpha \beta t} + \frac{Ba^4t^2}{(1 + \alpha \beta t)^2}, \quad T = \frac{\alpha^2t}{1 + \alpha \beta t},
  \\
  h_{\mathcal{F}_1}^{\alpha,\beta}(x,t) &= \frac{\theta_{\mathcal{F}_1}(x)}{\theta_{\mathcal{F}_1}(x - \alpha - \beta t)} \exp \left\{-\frac{2\beta t}{2} \right\}.
\end{align*} \]

Transformation \( s_f^{\alpha,\beta} \) maps boundary \( b(t) \) to the following families of boundaries

\[ g^{\alpha,\beta}(t) = \frac{1}{\alpha} \left[ (1 + \alpha \beta) b \left( \frac{\alpha^2 t}{1 + \alpha \beta t} \right) + Bt^2 \left( \frac{\alpha^4}{1 + \alpha \beta t} + 1 \right) \right] \]  \hspace{1cm} (45)

The density function of the first hitting time of boundary \( g^{\alpha,\beta}(t) \) are represented as

\[ \rho_{x_0}^{g^{\alpha,\beta}}(t) = \left( \frac{2B^2t^2}{3} \right) \left[ 1 - \frac{\alpha^6}{(1 + \alpha \beta t)^3} + C t \frac{\alpha^2}{1 + \alpha \beta t} - 1 \right] \]  \hspace{1cm} (46)

Transformation \( s_h^{\alpha,\beta} \) adds linear drift to the boundary \( b(t) \)

\[ g^{\alpha,\beta}(t) = \frac{1}{\alpha} \left[ (1 + \alpha \beta) b \left( \frac{\alpha^2 t}{1 + \alpha \beta t} \right) + Bt^2 \left( \frac{\alpha^4}{1 + \alpha \beta t} + 1 \right) \right] \]  \hspace{1cm} (47)

The density function of the first hitting time of boundary \( g^{\alpha,\beta}(t) \) are represented as

\[ \rho_{x_0}^{g^{\alpha,\beta}}(t) = \frac{\theta_{\mathcal{F}_1}(x_0 - \alpha) \theta_{\mathcal{F}_1}(b(t) + \alpha + \beta t)}{\theta_{\mathcal{F}_1}(x_0) \theta_{\mathcal{F}_1}(b(t))} e^{\beta x_0 + \alpha \beta - b(t) - (\beta^2/2 + 2B \alpha) t - \beta B t^2} \rho_{x_0 - \alpha}(t) \]  \hspace{1cm} (48)
Proof. Two-parametric symmetry $s^\alpha_\beta$ is generated by the following composition of vector fields from $e^{-2\ln\alpha\beta} \circ e^{\alpha\beta\psi_1}$, for details see subsection 10.6. Uniqueness of transform is equivalent to that any repeated action of $s_1$ or $s_2$ holds structure of transform, i.e. $\forall \alpha, \beta, \epsilon, \exists \alpha, \beta$: $s_1(s_2) : \left( \frac{x - B t^2}{1 + \alpha \beta t} + \frac{\alpha^2 t}{1 + \alpha \beta t} f^\alpha_\beta(x, t) \right) \rightarrow \left( \frac{x - B t^2}{1 + \alpha \beta t} + \frac{\alpha^4 t}{(1 + \alpha \beta t)^2} + \frac{\alpha^2 t}{1 + \alpha \beta t} f^\alpha_\beta(x, t) \right)$.

For details see 10.7. The uniqueness of transform $s^\alpha_\beta$ is trivial.

Theorem 13. All symmetries of the form (17) for drift functions $\mu \in \mathcal{R}_2$ are represented as one of the following two-parametric symmetries or its composition

$$s^\alpha_\beta : u(x, t) = f^\alpha_\beta(x, t) \mathcal{U} \left( \frac{\alpha x + \beta t}{\sqrt{T_+ T_-}} \right).$$

$$s^\alpha_\beta : u(x, t) = h^\alpha_\beta(x, t) \mathcal{U}(x - \alpha e^{\sqrt{T_+ - T_-}}, t).$$

where $T_-, T_+, \chi_\alpha, \beta$ and $\chi_0^{a, b}$ are defined as

$$T_- = \beta + 1 + \alpha e^{-2\sqrt{T_+}}, \quad T_+ = \alpha + 1 + \beta e^{2\sqrt{T_+}}.$$

$$\chi_\alpha = \frac{x + 2B/A}{\sqrt{T_+ T_-}} + \frac{2B}{A}, \quad \chi_0^{a, b} = \frac{x_0 + 2B/A}{\sqrt{\alpha + \beta + 1} - \frac{2B}{A}}.$$

functions $f^\alpha_\beta(x, t)$ and $h^\alpha_\beta(x, t)$ are equal

$$f^\alpha_\beta(x, t) = \frac{\theta_\mathcal{F}_2(x) T_+^{1/2 - \nu} \exp \left\{-\sqrt{\frac{T_+}{T_-}} (2\chi_\alpha^{a, b} + 2B/A)^2 (\beta + 1) T_+ - (\alpha + 1) T_- \right\} \theta_\mathcal{F}_2(\chi_\alpha^{a, b})}{\theta_\mathcal{F}_2(\chi_\alpha^{a, b})}.$$

$$h^\alpha_\beta(x, t) = \frac{\exp \left\{-\sqrt{\frac{T_+}{T_-}} (\alpha e^{\sqrt{T_+ - T_-}} + \beta e^{\sqrt{T_+ - T_-}}) + (x + \frac{2B}{A}) \left( \alpha e^{\sqrt{T_+ - T_-}} + \beta e^{\sqrt{T_+ - T_-}} \right) \right\}}{\theta_\mathcal{F}_2(\chi_\alpha^{a, b})}.$$

Symmetry $s^\alpha_\beta$ maps boundary $b(t)$ to the boundaries $g^{\alpha, \beta}(t)$ defined as

$$g^{\alpha, \beta}(t) = \sqrt{T_+ T_-} \left[ b \left( t + \frac{T_- - \ln T_+}{\sqrt{T_+ T_-}} \right) + \frac{2B}{A} \right] - \frac{2B}{A}.$$

The density corresponded to the boundary $g^{\alpha, \beta}(t)$ is equal

$$\rho^{\alpha, \beta}(t) = \frac{f^\alpha_\beta(x, t) \theta_\mathcal{F}_2(\chi_0^{a, b}) (t + \ln T_- - \ln T_+)}\sqrt{T_+ T_-},$$

Symmetry $s^\alpha_\beta$ adds hyperbolic drift $\alpha e^{\sqrt{T_+}} + \beta e^{\sqrt{T_-}}$ to the boundary $b(t)$

$$\rho^{\alpha, \beta}(t) = \frac{h^\alpha_\beta(b(t) - \alpha e^{\sqrt{T_+}} - \beta e^{\sqrt{T_-}} t, t)}{h^\alpha_\beta(x(0), 0)} \rho^{\alpha, \beta}(t).$$

Proof. Proof is analogous to the proof of Theorem 12. Two-parametric symmetry $s^\alpha_\beta$ is generated by the following composition of vector fields from $e^{-2\ln\alpha\beta} \circ e^{\alpha\beta\psi_1}$, for details see subsection 10.6. Uniqueness of transform is equivalent to that any repeated action of $s_1$ or $s_2$ holds structure of transform, i.e. $\forall \alpha, \beta, \epsilon, \exists \alpha, \beta$: $s_1(s_2) : \left( \frac{x - B t^2}{1 + \alpha \beta t} + \frac{\alpha^2 t}{1 + \alpha \beta t} f^\alpha_\beta(x, t) \right) \rightarrow \left( \frac{x - B t^2}{1 + \alpha \beta t} + \frac{\alpha^4 t}{(1 + \alpha \beta t)^2} + \frac{\alpha^2 t}{1 + \alpha \beta t} f^\alpha_\beta(x, t) \right)$.

Let us note that direct application of $s_1$ or $s_2$ does not hold condition $T(0) = 0$. In order to satisfy it we apply $s_0$ after $s_1$ or $s_2$. Details of exponentiation $53$ can be found in 10.10. Uniqueness check is presented in 10.11.
Theorem 14. All symmetries of the form (11) for drift functions \( \mu \in \mathcal{P}_3 \) are two-parametric symmetries of the form
\[
s^\alpha_\beta: u(x, t) = \frac{1}{\sqrt{1 + \alpha \beta t}} \theta_{x, \alpha \beta t} \left( \frac{ax}{1 + \alpha \beta t} \right) \exp \left\{ -\frac{\alpha \beta x^2}{2(1 + \alpha \beta t)} + Ct \left[ \frac{\alpha^2}{1 + \alpha \beta t} - 1 \right] \right\} \mathcal{U} \left( \frac{\alpha x}{1 + \alpha \beta t}, \frac{\alpha^2 t}{1 + \alpha \beta t} \right)
\]
(56)
Transformation \( s^\alpha_\beta \) maps boundary \( b(t) \) to the following families of boundaries
\[
g^{\alpha_\beta}(t) = \frac{1 + \alpha \beta t}{\alpha} \left( \frac{\alpha^2 t}{1 + \alpha \beta t} \right).
\]
(57)
The density function of the first hitting time of boundary \( g(t) \) are represented as
\[
\rho^\alpha_\beta(t) = \frac{\alpha^2}{(1 + \alpha \beta t)^{3/2}} \frac{\theta_{x, \alpha t} \theta_{x, \alpha t} (g(t))}{\theta_{x, \alpha t} (b(1 + \alpha \beta t))} \exp \left\{ \frac{\alpha \beta x_0^2}{2} - \frac{\alpha \beta (1 + \alpha \beta t)}{2 \alpha^2} b^2 \left( \frac{\alpha^2 t}{1 + \alpha \beta t} \right) \right\}
\]
(58)
\[
\cdot \exp \left\{ Ct \left[ \frac{\alpha^2}{1 + \alpha \beta t} - 1 \right] \right\} \rho_{x_0} (\frac{\alpha^2 t}{1 + \alpha \beta t})
\]

Theorem 15. All symmetries of the form (11) for drift functions \( \mu \in \mathcal{P}_4 \) are two-parametric symmetries of the form
\[
s^\alpha_\beta: u(x, t) = f^\alpha_\beta(x, t) \mathcal{U} \left( \frac{x \sqrt{1 + \alpha + \beta}}{\sqrt{T_+ T_-}}, \frac{t + \ln T_- - \ln T_+}{2 \sqrt{A}} \right)
\]
(59)
where function \( f^\alpha_\beta \) is represented by
\[
f^\alpha_\beta_{x, \alpha \beta t} (x, t) = \frac{\theta_{x, \alpha \beta t} (x)}{\theta_{x, \alpha \beta t} (x \sqrt{1 + \alpha + \beta / \sqrt{T_+ T_-}})} \frac{T_+}{(T_+ - 1)^{3/2}} \exp \left\{ -x^2 \sqrt{A} \left[ (\beta + 1) T_+ - (\alpha + 1) T_- \right] \right\}
\]
variables \( T_- \) and \( T_+ \) and constant \( \nu \) are defined as
\[
T_- = \beta + 1 + e^{-2 \sqrt{A}}; \quad T_+ = \alpha + 1 + e^{2 \sqrt{A}}, \quad \nu = -1 + \frac{C}{2 \sqrt{A}}
\]
The density corresponded to the boundary \( g^{\alpha_\beta}(t) \) is equal
\[
\rho^\alpha_\beta(t) = \frac{f^\alpha_\beta_{x, \alpha \beta t} (g(t), t) \rho_{x_0}^\alpha_T \frac{T_+ - T_-}{2 \sqrt{A}} \left( t + \ln T_- - \ln T_+ \right)}{\sqrt{T_+ T_-}}.
\]
(60)

5 Boundary transforms via change of measure

5.1 Boundary transforms via change of process to Brownian motion and Bessel process

Theorem 1 provides explicit relation between first hitting times densities for different boundaries. In this section we allow to change both boundaries and process. Let \( Y \) be another diffusion process
\[
dY = \nu (Y_t, t) dt + \Sigma (Y_t, t) dW_t, \quad Y_0 = y_0.
\]
(62)
differential operator \( \mathcal{D}_{x,t} \) are defined analogical to \( \mathcal{L}_{x,t} \). Define as \( \rho^\alpha_\beta_{y_0} (t; Y) \) the density of first passage time of diffusion process \( Y \) started from \( y_0 \) to the curved boundary \( b(t) \). We use the following notations in case of \( Y \) be a Brownian motion or Bessel process: \( \rho^\alpha_\beta_{y_0} (t; BM), \rho^\alpha_\beta_{y_0} (t; BES_n) \). Next theorem presents explicit relation between first hitting time densities of processes \( X \) and \( Y \).
Theorem 16. Suppose that equation $\mathcal{L}_{x,t}u = 0$ associated with diffusion process $[\mathcal{L}]$ has the following identity
\[ u(x, t) = f(x, t)\mathcal{U}(X(x, t), T(t)), \quad \mathcal{T} \mathcal{U} = 0, \quad T(0) = 0, \quad (63) \]
Let $g(t)$ be a solution of equation $X(g(t), t) = b(T)$ and $\Psi(\Psi)$ be a solution of equation $X(\Psi(\Psi), 0) = \mathcal{Y}$:
\[ g(t) : X(g(t), t) = b(T), \quad \Psi(\Psi) : X(\Psi(\Psi), 0) = \mathcal{Y}. \quad (64) \]
The density function of the first passage time of the process $Y_t$ to the boundary $g(t)$ is represented by
\[ \rho_{g_t}^n(t) = \frac{\sigma^2(g(t), t)}{\sigma^2(b(T(t)), T(t))} \frac{f(g(t), t) \partial X/\partial x|_{x=g(t)}}{f(x_0, 0) \Psi'(X(x_0, 0))} \rho_{X(x_0, 0)}^T(T(t); Y). \quad (65) \]

Proof. Proof is fully analogous to the proof of Theorem 1

In previous section we proved that operator $\mathcal{L}$ has Lie symmetries if and only if drift $\mu(x)$ solves one of 4 Ricatti equations. For families $\mathcal{F}_1$ and $\mathcal{F}_2$ operator $\mathcal{L}$ has 6 groups of symmetries and for $\mathcal{F}_3$ and $\mathcal{F}_4$ has 4 groups. It is known that PDE of the form $[5]$ with 6 groups of symmetries can be reduced to the heat equation (e.g. see $[17]$)
\[ 1 \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0, \]
and with 4 groups to the equation
\[ 1 \frac{\partial^2 u}{\partial x^2} - \frac{Q}{2} u - \frac{\partial u}{\partial t} = 0 \]
It means that we can represent first hitting time density for any process from $\mathcal{F}_1$ and $\mathcal{F}_2$ in terms of hitting time of Brownian motion and for processes from $\mathcal{F}_3$ and $\mathcal{F}_4$ in terms of hitting time of Bessel process. Next theorems provides explicit transformations for each family $\mathcal{F}_n$, $n = 1, 2, 3, 4$. All proofs of Theorems above are consist only of derivation of identity $[63]$ where $\mathcal{U}(\mathcal{X}, T)$ solves one of the following equations (left equation is corresponded to the Brownian motion, right equation is corresponded to the $n$-dimensional Bessel process)
\[ 1 \frac{\partial^2 U}{\partial X^2} - \frac{\partial U}{\partial T} = 0, \quad 1 \frac{\partial^2 U}{\partial X^2} - \frac{Q}{2} u - \frac{\partial U}{\partial T} = 0 \]
For all families $\mathcal{F}_n$ we apply first the following transform $u(x, t) = \theta_{\mathcal{F}_n}(x) G(x, t)$ which turns out to the following equation for $G(x, t)$
\[ 1 \frac{\partial^2 G}{\partial x^2} - \frac{\mu' + \mu^2}{2} G - \frac{\partial G}{\partial t} = 0. \]
All routines for the next 4 Theorems are presented in subsections $10.12, 10.13, 10.14$ and $10.15$

Theorem 17. Let $X_t$ be a stochastic process with drift $\mu(x) \in \mathcal{F}_1$. The density $\rho^\tau$ of the first hitting time $\tau_t$ to the curved boundary $b(t)$ can be represented in terms of first hitting time of Brownian Motion to the curve $b(t) + Bt^2$:
\[ \rho_{b_t}^\tau(t) = \frac{\theta_{\mathcal{F}_1}(b(t))}{\theta_{\mathcal{F}_1}(x_0)} e^{2Bt^2 - Ct - 2Bt(b(t))} \rho_{b_0}^{\tau+b^2}(t; BM) \quad (66) \]

Theorem 18. Let $X_t$ be a stochastic process with drift $\mu(x) \in \mathcal{F}_2$. The density $\rho^\tau$ of the first hitting time $\tau_t$ to the curved boundary $b(t)$ can be represented in terms of first hitting time of Brownian Motion to the curve $g(t)$:
\[ g(t) = \sqrt{4A} \left( b \left( \frac{\ln \left( t + 1 \right)}{2\sqrt{A}} \right) + 2B/A \right) \sqrt{t + 1} \]
\[ \rho_{b_t}^\tau(t) = 2\sqrt{A} \frac{\theta_{\mathcal{F}_1}(b(t))}{\theta_{\mathcal{F}_1}(x_0)} e^{-\frac{\pi^2(n+2B/A)^2}{4} - \frac{\pi^2(n+2B/A)^2}{4} + [\frac{4B}{A} + \frac{4B}{A} - C]} e^{\frac{\pi^2(x_0(n+2B/A)^2}{4(4A(A(x_0+2B/A))} \left( e^{2\sqrt{A}t - 1}; BM \right) \quad (67) \]

Theorem 19. Let $X_t$ be a stochastic process with drift $\mu(x) \in \mathcal{F}_3$. The density $\rho^\tau$ of the first hitting time $\tau_t$ to the curved boundary $b(t)$ can be represented in terms of first hitting time of Bessel process to the same curve $b(t)$:
\[ \rho_{b_t}^\tau(t) = \frac{\theta_{\mathcal{F}_3}(b(t))}{\theta_{\mathcal{F}_3}(x_0)} e^{-Ct} \rho_{b_0}^{\tau+b^2}(t; BM) \quad (68) \]
Theorem 20. Let $X_t$ be a stochastic process with drift $\mu(x) \in \mathcal{F}_t$. The density $\rho^{b_1}$ of the first hitting time $\tau_b$ to the curved boundary $b(t)$ can be represented in terms of first hitting time of Bessel process to the curve $g(t)$:

$$g(t) = \sqrt{4Ab} \left( \frac{\ln(t+1)}{2A} \right) \sqrt{t+1},$$

$$\rho^{b_1}_{x_0}(t) = 2\sqrt{A_x} \frac{n-1}{b(t) \tan \frac{\theta}{x_0}} \frac{\partial}{\partial x_0} \left[ \pi \right] \theta \left[ \frac{\pi}{2} \right] + \frac{[\tau_{b_1} - c]}{\theta} \rho^{b_1}_{\sqrt{4A_x^2} g(t)} \left( e^{2\sqrt{A_t}} - 1; BES(n) \right), \quad n = 3 + 2\sqrt{4D+1} \ (69)$$

5.2 Unified approach to compute closed form densities for Brownian motion and Bessel process

One of the way to obtain closed form formulas for stopping time densities is the series expansion method presented in [21]. This technique is applicable only for time-homogeneous diffusion and constant-level boundaries. In these cases Laplace transform of [1] turns out to the Sturm - Liouville problem for ODE. For Brownian Motion we have only two curved boundaries which are equivalent to the straight lines for some time-homogeneous diffusion processes. There are well-known quadratic (see [18] and [28]) and square-root boundaries (see [22]). Theorems 17 and 18 provides explicit transformations to the processes with drift from $\mathcal{F}_1$ or $\mathcal{F}_2$. Let us mention that transformation (66) can be used for two-sided square root $\pm \sqrt{t + c}$ boundaries, transformation (68) can be used to the two sided boundaries $a t^2 + b$.

Hence the unified approach to obtain almost all closed-form formulas (except Daniels’ boundary, see [12]) for Brownian motion is the combination of spectral expansion method and transformations (66) and further application of two-parametric transformation from [14].

For Bessel process we also have only two transforms to the processes with drift from $\mathcal{F}_3$ and $\mathcal{F}_4$. These transformations corresponded to linear and square root boundaries (see [10], [13], [16]) Let us also mention that Theorem 2 proves no existence of any attainable transformations (68) to time-homogeneous processes for Brownian Motion and for Bessel process.

6 Closed form formulas for some diffusion processes

6.1 $2M - X$ process

Let $W_t^{(\omega)}$ be a Brownian motion with drift $\omega$ and $M_t^{(\omega)} = \sup \{ W_s : s \leq t \}$. The process $X_t = 2M_t^{(\omega)} - W_t^{(\omega)}$ ([27]) can be represented as a solution of SDE

$$dX_t = |\omega| \cosh (|\omega| X_t) \, dt + dW_t, \quad X_0 = x_0.$$ Drift function $\mu = |\omega| \cosh (|\omega| x)$ be in the family $\mathcal{F}_1$ with parameters $B = 0, C = |\omega|^2/2$. Function $\theta(x)$ is equal

$$\theta(x) = e^{\int_0^x \mu(x') \, dx'} = \sinh (|\omega| x).$$

Application of transform $s_f^{\alpha, \beta}$ yields

$$g(t) = \frac{1 + \alpha \beta t}{\alpha} b \left( \frac{\alpha^2 t}{1 + \alpha \beta t} \right).$$

$$\rho^{b_1}_{x_0}(t) = \frac{\alpha^2 \sinh (\alpha x_0) \sinh \left( \frac{|\omega| (1 + \alpha \beta t)}{\alpha} \right) \sinh \left( \frac{\alpha^2 t}{1 + \alpha \beta t} \right)}{(1 + \alpha \beta t)^{3/2} \sinh (|\omega| x_0) \sinh \left( \frac{|\omega| b}{1 + \alpha \beta t} \right)} e^{\frac{\alpha^2 b}{2} \left( \frac{\alpha^2 t}{1 + \alpha \beta t} \right) - \frac{\alpha \beta t (1 + \alpha \beta t) b^2}{2} \left( \frac{\alpha^2 t}{1 + \alpha \beta t} \right) + \frac{\alpha^2 \beta^2 t}{2} \left( \frac{\alpha^2 t}{1 + \alpha \beta t} \right) - \frac{\alpha^2 \beta^2 t}{2} \left( \frac{\alpha^2 t}{1 + \alpha \beta t} \right) \rho^{b_1}_{0}(t) \left( \frac{\alpha^2 t}{1 + \alpha \beta t} \right)}.$$ Application of transform $s_f^{\alpha, \beta}$ yields

$$\rho^{b_1}_{x_0}(t) = \frac{\sinh (|\omega| (x_0 - \alpha)) \sinh (|\omega| (\alpha + \beta t + b(t)))}{\sinh (|\omega| x_0) \sinh (|\omega| b(t))} e^{\beta x_0 + \alpha \beta - \beta t - \beta^2 t^2/2} \rho^{b_1}_{x_0 - \alpha}(t), \quad g(t) = \alpha + \beta t + b(t).$$

Using relation (66) we can derive explicit formulas for the constant level, stopping line, quadratic, and square-root boundary from analogical formulas for Brownian Motion.
6.2 Pearson diffusion

Pearson diffusion is defined as a solution of the following SDE:
\[ dU_t = (\alpha U_t + \beta) dt + \sqrt{\alpha U_t^2 + b U_t + c} dW_t, \quad a > 0. \]

Suppose that \( d = b^2 - 4ac \neq 0 \). Application of Lambert’s transform turns out to stationary process \( X_t \):
\[ X_t = \int \frac{du}{\sqrt{\alpha u^2 + bu + c}|_{u=U_t}} = \frac{1}{\sqrt{\alpha}} \ln \left( \sqrt{\alpha U_t^2 + b U_t + c + \sqrt{\alpha U_t + b \sqrt{\alpha}}} \right) \]
\[ U_t = -\frac{b}{2\sqrt{\alpha}} + \frac{e^{\sqrt{\pi}X_t} - de^{-\sqrt{\pi}X_t}}{2\sqrt{\alpha}}, \quad \sqrt{\alpha U_t^2 + b U_t + c} = \frac{e^{\sqrt{\pi}X_t} + de^{-\sqrt{\pi}X_t}}{2}. \]

Let \( d = -(b^2 - 4ac) \) and \( d > 0 \). Pearson diffusion has a symmetries only in a following special case (for more details see [10, 16]):
\[ dU_t = \frac{3}{4} (\alpha U_t + b) dt + \sqrt{\alpha U_t^2 + b U_t + c} dW_t, \]
\[ dX_t = \mu(X_t) dt + dW_t, \quad \mu(X_t) = \sqrt{\alpha} e^{\sqrt{\pi}X_t} + de^{-\sqrt{\pi}X_t}, \quad \mu' + \mu^2 = a. \]

Function \( \mu \) also be in the family \( \mathcal{F}_1 \) with parameters \( B = 0, C = a/2, \) function \( \theta(x) \) is equal
\[ \theta(x) = e^{\int \mu(x) dx} = e^{\sqrt{\alpha}x - de^{-\sqrt{\alpha}x}} = (1 - d) \cosh(\sqrt{\alpha}x) + (1 + d) \sinh(\sqrt{\alpha}x) \]

We represent \( \theta(x) \) in the following form:
\[ \theta(x) = \sqrt{4d} \sinh(\sqrt{\alpha}x + \omega), \quad \omega = \arccosh\left( \sqrt{\frac{1 - d}{4d}} \right). \]

\[ \rho_{x_0}^T(t) = \frac{\alpha^2 \sinh(\sqrt{\alpha}x_0 + \omega) \sinh(\sqrt{\alpha}x_0 + \omega + \left( \frac{\sqrt{\alpha} + \frac{1}{2\sqrt{\alpha}} b}{1 + \frac{1}{\sqrt{\alpha}}} \right)) + e^{-\frac{\alpha b^2}{2\sqrt{\alpha}}} \frac{\alpha b^2}{2\sqrt{\alpha}} + \frac{\alpha^2}{2\sqrt{\alpha} + b} \rho_{x_0}(\sqrt{\alpha} b + \omega)}{1 + \alpha \beta(t)^{1/2} \sinh(\sqrt{\alpha}x_0 + \omega) \sinh(\sqrt{\alpha} x_0 + \omega + \left( \frac{\sqrt{\alpha} + \frac{1}{2\sqrt{\alpha}} b}{1 + \frac{1}{\sqrt{\alpha}}} \right)) + e^{-\frac{\alpha b^2}{2\sqrt{\alpha}}} \frac{\alpha b^2}{2\sqrt{\alpha}} + \frac{\alpha^2}{2\sqrt{\alpha} + b} \rho_{x_0}(\sqrt{\alpha} b + \omega)} \rho_{x_0}(\sqrt{\alpha} b + \omega + \left( \frac{\sqrt{\alpha} + \frac{1}{2\sqrt{\alpha}} b}{1 + \frac{1}{\sqrt{\alpha}}} \right)) + e^{-\frac{\alpha b^2}{2\sqrt{\alpha}}} \frac{\alpha b^2}{2\sqrt{\alpha}} + \frac{\alpha^2}{2\sqrt{\alpha} + b} \rho_{x_0}(\sqrt{\alpha} b + \omega)}{1 + \alpha \beta(t)^{1/2} \sinh(\sqrt{\alpha}x_0 + \omega) \sinh(\sqrt{\alpha} x_0 + \omega + \left( \frac{\sqrt{\alpha} + \frac{1}{2\sqrt{\alpha}} b}{1 + \frac{1}{\sqrt{\alpha}}} \right)) + e^{-\frac{\alpha b^2}{2\sqrt{\alpha}}} \frac{\alpha b^2}{2\sqrt{\alpha}} + \frac{\alpha^2}{2\sqrt{\alpha} + b} \rho_{x_0}(\sqrt{\alpha} b + \omega)} \rho_{x_0}(\sqrt{\alpha} b + \omega + \left( \frac{\sqrt{\alpha} + \frac{1}{2\sqrt{\alpha}} b}{1 + \frac{1}{\sqrt{\alpha}}} \right)) + e^{-\frac{\alpha b^2}{2\sqrt{\alpha}}} \frac{\alpha b^2}{2\sqrt{\alpha}} + \frac{\alpha^2}{2\sqrt{\alpha} + b} \rho_{x_0}(\sqrt{\alpha} b + \omega)}. \]

Application of transform \( s_h^x \) yields
\[ g(t) = \alpha + \beta t + b(t). \]

Using relation \([66]\) we can derive explicit formulas for the constant level, slopping line, quadratic, and square-boundary formulas from analogous formulas for Brownian Motion.

6.3 Ornstein - Uhlenbeck process

Let \( X_t \) be Ornstein-Uhlenbeck process with zero mean and unit variance:
\[ dX_t = -\kappa X_t dt + dW_t, \quad \mu(x) = -\kappa x, \]

drift function \( \mu(x) = -\kappa x \) solves Riccati equation \( \mu' + \mu^2 = -\kappa + \kappa^2 x^2 \) \( (\mu \in \mathcal{F}_2) \) constants \( A = \kappa^2, B = 0, C = -\kappa/2, \nu = 0 \). Function \( \theta \) is equal
\[ \theta(x) = e^{\int \mu(x) dx} = e^{-\kappa x^2/2}. \]

Apply symmetry \( s_f^{\alpha, \beta} \).
\[ \mathcal{T}_+ = \beta + 1 + \alpha e^{-2\alpha t}, \quad \mathcal{T}_- = \alpha + 1 + \beta e^{2\beta t}, \quad \mathcal{X}^{\alpha, \beta} = \frac{2 \sqrt{\alpha + \beta + 1}}{\sqrt{\mathcal{T}_+ \mathcal{T}_-}} \]

18
function \( f_{x_2}^{\alpha,\beta} \) is equal
\[
f_{x_2}^{\alpha,\beta}(x, t) = \mathcal{T}_-^{1/2} \exp \left\{ -\frac{\kappa x^2 (T_+(\beta + 1)T_- - (\alpha + 1)T_-(1 + \alpha + \beta))}{2T_+T_-} \right\}
\]
\[
j_{x_2}^{\alpha,\beta}(x, 0) = (1 + \alpha + \beta)^{1/2} \exp \left\{ -\frac{\kappa x^2(\beta - \alpha)}{2(\alpha + \beta + 1)} \right\}
\]
\[
\rho_{x_0}^\gamma = \exp \left\{ \frac{\kappa x^2(\beta - \alpha)}{2(\alpha + \beta + 1)} \right\} \frac{\kappa x^2(T_+(\beta + 1)T_- - (\alpha + 1)T_-(1 + \alpha + \beta))}{2T_+T_-} \rho_{x_0}^\gamma \left( t + \frac{\ln T_- - \ln T_+}{2\kappa} \right)
\]

Second symmetry adds hyperbolic drift: \( g(t) = b(t) + \alpha e^{\kappa t} + \beta e^{-\kappa t} \)
\[
h_{x_2}^{\alpha,\beta}(x, t) = \exp \left\{ \kappa\alpha^2 e^{2\kappa t} - 2\kappa x_0 \alpha e^{\kappa t} \right\}, \quad h_{x_2}^{\alpha,\beta}(x_0, 0) = \exp \left\{ \kappa\alpha^2 - 2\kappa x_0 \alpha \right\}
\]
\[
\rho_{x_0}^\gamma(t) = e^{-\kappa\alpha^2(e^{2\kappa t} + 1) + 2\alpha \omega(e^{\kappa t} - x_0)} \rho_{x_0}^{-\alpha - \beta}(t)
\]

Let us mention that we can obtain closed form density of first hitting time to the hyperbolic boundary \( a e^{-\kappa t} + b e^{\kappa t} + \gamma \) by using symmetry \( s_h^{\alpha,\beta} \) to the constant level boundary \( b(t) = \gamma \). This stopping time were studied in [9]. The explicit formulas for the first hitting time density of Ornstein-Uhlenbeck process to the constant level boundary can be found in [9].

### 6.4 Bessel process with drift

Let \( X_t \) be a Bessel process with drift:
\[
dX_t = \left( \frac{\omega + 1/2}{X_t} + \kappa \frac{I_{\omega+1}(\kappa X_t)}{I_{\omega}(\kappa X_t)} \right) dt + dW_t, \quad a > 0, \quad X_0 = x_0.
\]

where \( I_{\omega}(z) \) is the modified Bessel function. These processes arise from the radial part of multidimensional Brownian motion and were studied by Pitman and Yor in [25]. Drift coefficient \( \mu \) is equal
\[
\mu(x) = \frac{\omega + 1/2}{x} + \kappa \frac{I_{\omega+1}(\kappa x)}{I_{\omega}(\kappa x)}.
\]

Corresponded function \( \theta(x) \) is equal
\[
\theta(x) = \sqrt{x} I_{\omega}(\kappa x).
\]

Hence \( C = \kappa^2/2, \ D = \omega^2 - 1/4 \). It is easy to see if we apply the following recurrence relation between modified Bessel functions
\[
\frac{d}{dz} I_{\omega}(z) = I_{\omega+1}(z) + \frac{\omega}{z} I_{\omega}(z).
\]
\[
\rho_{x_0}^\gamma(t) = \alpha^{3/2} \sqrt{b} e^{\omega t} \frac{I_{\omega}(\kappa x_0)}{I_{\omega}(\kappa x)} \rho_{x_0}^\gamma(t) e^{\frac{\alpha^2 t}{\omega^2}} \frac{\alpha^2 + \alpha^2 t}{\omega^2} \left( \frac{\alpha^2}{\omega^2} \right)^{1/2} \rho_{x_0}^\gamma \left( \frac{\alpha^2 t}{1 + \alpha^2} \right)
\]

### 6.5 Radial Ornstein-Uhlenbeck process

Let \( X_t \) be a radial Ornstein-Uhlenbeck process
\[
dX_t = \left( \frac{\omega + 1/2}{X_t} - \kappa X_t \right) dt + dW_t.
\]

Drift \( \mu(x) \) solves Ricatti equation
\[
\mu' + \mu^2 = \kappa^2 x + \frac{(\omega + 1/2)^2 - (\omega + 1/2)}{x^2} + 2\omega \kappa
\]
hence $\mu \in \mathcal{F}_1$, $A = \kappa^2$, $C = \omega \kappa$, $D = \omega^2 - 1/4$. Corresponded function $\theta(x)$ is equal

$$\theta(x) = x^{\omega + 1/2} e^{-\kappa x^2 / 2}.$$

Apply symmetry $s_f^{\alpha, \beta}$.

$$T_+ = \beta + 1 + \alpha e^{-2\kappa t} \quad T_- = \alpha + 1 + \beta e^{2\kappa t}, \quad \chi^{\alpha, \beta} = x^{\sqrt{\alpha + \beta + 1}} \sqrt{T_+ T_-}.$$

$$g^{\alpha, \beta}(t) = \frac{\sqrt{T_+ T_-}}{\sqrt{1 + \alpha + \beta}} \left( t + \frac{\ln T_+ - \ln T_-}{2\sqrt{A}} \right).$$

function $f^{\alpha, \beta}$ is equal ($\nu = -1/4 - \omega/2$)

$$f^{\alpha, \beta}(x, t) = \frac{T_-^{\omega + 1}}{(1 + \alpha + \beta)^{\omega/2 + 1/4}} \exp \left\{ -\frac{\kappa x^2 (T_+ T_- + (\beta + 1)T_+ - (\alpha + 1)T_- - (1 + \alpha + \beta))}{2T_+ T_-} \right\}$$

$$f^{\alpha, \beta}(x_0, 0) = (1 + \alpha + \beta)^{\omega/2 + 3/4} \exp \left\{ -\frac{\kappa x^2_0 (\beta - \alpha)}{2(\alpha + \beta + 1)} \right\}$$

\section{Identities for two sided boundaries}

\subsection{Connections to the Lie symmetries}

Let $b_l(t)$ and $b_u(t)$ are sufficiently smooth functions and $b_l(t) < b_u(t), \forall t \geq 0$. We define the first passage time

$$\tau_{b_l, b_u} = \inf \{ t > 0; X_t \leq b_l(t) \text{ or } X_t \geq b_u(t) \}. \quad \text{(70)}$$

According to the one-sided case we introduce function $u_{b_l, b_u}(x, t)$

$$u_{b_l, b_u}(x, t) = \frac{\partial}{\partial x} \Psi_{x_0} (X_t \leq x, \tau_{b_l, b_u} > t) \quad \text{(71)}$$

which solves the following PDE

$$\mathcal{L}_{x, t} u_{b_l, b_u} = 0,$$

$$u_{b_l, b_u}(b_l(t), t) = 0,$$

$$u_{b_l, b_u}(b_u(t), t) = 0,$$

$$u_{b_l, b_u}(x, 0) = \delta(x - x_0),$$

where $\delta(x - x_0)$ is a Dirac delta function and $\mathcal{L}_{x, t}$ is defined in \[
\text{(43)}\]. Define probability and density of this stopping time

$$\mathbb{P}_{x_0}(\tau_{b_l, b_u} > t) = \int_{b_l(t)}^{b_u(t)} u_{b_l, b_u}^x(x, t)dx, \quad \rho_{x_0, \tau_{b_l, b_u}}(t) = \frac{\partial}{\partial x} \mathbb{P}_{x_0}(\tau_{b_l, b_u} > t) = \frac{\partial}{\partial x} \left( \frac{\sigma^2 u_{b_l, b_u}^x}{2} \right) \bigg|_{x = b_l(t)}. \quad \text{(73)}$$

\textbf{Theorem 21.} Suppose that equation $\mathcal{L}_{x, t} u = 0$ generated by diffusion process \[
\text{(4)}\] has a non-trivial Lie symmetry \[
\text{(77)}\]. Let $u_{b_l, b_u}$ be a solution of Cauchy problem \[
\text{(77)}\], and functions $g_l(t)$, $g_u(t)$, $\Psi(\mathcal{Y})$ solve equations $X(g_l(t), t) = b_l(T(t))$, $X(g_u(t), t) = b_u(T(t))$ and $X(\Psi(\mathcal{Y}), 0) = \mathcal{Y}$ respectively. Define functions $\Theta_{x_0}^{\mathcal{Y}}(t)$ and $\mathcal{Y}(\mathcal{Y})$ with parameters $x_0$ and function $\mathcal{Y}(t)$

$$\Theta_{x_0}^{\mathcal{Y}}(t) = \frac{\partial}{\partial x} \left( \frac{\sigma^2(x(t))u_{b_l, b_u}^x(x(t), t)}{2} \right) \bigg|_{x = \mathcal{Y}(t)}, \quad \mathcal{Y}(t) = t = f(\mathcal{Y}(t), t) \sigma^2(\mathcal{Y}(t), t) \frac{\partial X}{\partial x} \bigg|_{x = \mathcal{Y}(t)} \quad \text{(74)}$$

The densities corresponding to the original and transformed boundaries have the following representations

$$\rho_{x_0, \mathcal{Y}(t)}^{b_l, b_u}(t) = \Theta_{x_0}^{\mathcal{Y}}(t) - \Theta_{x_0}^{b_l(t)}(t),$$

$$\rho_{x_0, \mathcal{Y}(t)}^{g_l, g_u}(t) = \frac{1}{f(x_0, 0) \Psi(\mathcal{Y}(x_0, 0))} \left( \mathcal{Y}(t) \Theta_{x_0}^{\mathcal{Y}}(t) - \mathcal{Y}(t) \Theta_{x_0}^{b_l(t)}(t) \right). \quad \text{(75)}$$

\textbf{Proof.} Proof is fully analogical to the proof of theorem \[
\text{(4)}\]. \hfill \square
7.2 Wedge probabilities for Brownian motion

We illustrate theorem (21) by derivation of Anderson’ formula [7] for first passage times of Brownian motion to the two slopping lines. Also we generalize alternative representation of Doob’ formula [13] from [15] to the case of Anderson’ formula.

Consider the following stopping times \( \tau_{a_1,a_2} = \inf \{ t > 0; W_t \leq -a_1 \text{ or } W_t \geq a_2 \} \) and \( \tau_{a_1,b_1,a_2+b_2} = \inf \{ t > 0; W_t \leq -a_1 t - b_1 \text{ or } W_t \geq a_2 t + b_2 \} \) corresponded to the two straight and slopping lines respectively. Let function \( u_{x_0,a_1,a_2}(t,x,0) \) solves PDE

\[
\begin{align*}
& \mathcal{L}_x t u_{x_0}^{a_1,a_2} = 0, \\
& u_{x_0}^{a_1,a_2}(-a_1, t) = 0, \\
& u_{x_0}^{a_1,a_2}(a_2, t) = 0, \\
& u_{x_0}^{a_1,a_2}(x, 0) = \delta(x - x_0).
\end{align*}
\] (76)

There are two approaches to obtain solution of the problem (76): series expansions approach and Laplace transform. Let us mention that comparison of obtained solutions is a one of approaches to prove Poisson’s summation formula (this formula is sufficiently used in [15])

\[
\sum_{n=-\infty}^{+\infty} e^{-\frac{x^2}{2} \pi n^2} \cos (\pi n v / u) = \sqrt{\frac{2u}{\pi}} e^{-\frac{x^2}{2u}} \sum_{n=-\infty}^{+\infty} e^{-2nu^2} \cosh (2nv).
\]

Consider in the detail series expansion approach. We seek solution \( u_{x_0,a_1,a_2}^{a_1,a_2}(x,t) \) in the form

\[
u_{x_0,a_1,a_2}^{a_1,a_2}(x,t) = \sum_{n=0}^{\infty} A_n \sin (\omega_n(x + \alpha)) e^{-\frac{x^2}{2} \pi t}.
\]

The phase shift \( \alpha \) and frequencies \( \omega_n \) are set to satisfy boundary conditions \( u_{x_0,a_1,a_2}^{a_1,a_2}(-a_1, t) = u_{x_0,a_1,a_2}^{a_1,a_2}(a_2, t) = 0 \), i.e. \( \alpha = a_1 \), and \( \omega_n = \pi n / l, l = a_1 + a_2 \). Hence at the moment \( t = 0 \) function \( u_{x_0,a_1,a_2}^{a_1,a_2}(x, 0) \) is Fourier series on the segment \([-a_1,a_2]\). Calculation of coefficients \( A_n \) turns out to the following series expansion

\[
u_{x_0,a_1,a_2}^{a_1,a_2}(x,t) = \frac{2}{l} \sum_{n=0}^{\infty} \sin \left( \frac{n\pi}{l}(x + a_1) \right) \sin \left( \frac{n\pi}{l}(x_0 + a_1) \right) e^{-\frac{x^2}{2} \pi t} \] (77)

or

\[
u_{x_0,a_1,a_2}^{a_1,a_2}(x,t) = \frac{1}{2l} \sum_{n=-\infty}^{+\infty} \left( \cos \left( \frac{n\pi}{l}(x - x_0) \right) - \cos \left( \frac{n\pi}{l}(x + x_0 + 2a_1) \right) \right) e^{-\frac{x^2}{2} \pi t}. \] (78)

Application of Laplace transform reduces original Cauchy problem (76) to the boundary problem for linear ODE

\[
\begin{align*}
\frac{1}{2} \frac{d^2 w}{dx^2} + \zeta w &= -\delta(x - x_0), \\
w(-a_1; \zeta) &= 0, \\
w(a_2; \zeta) &= 0.
\end{align*}
\] (79)

where \( w(x; \zeta) \) is image of transformation, i.e.

\[
w(x; \zeta) = \int_{0}^{\infty} e^{-\zeta t} u_{x_0,a_1,a_2}^{a_1,a_2}(x,t) dt, \quad u_{x_0,a_1,a_2}^{a_1,a_2}(x,t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\xi t} w(x; \zeta) d\zeta.
\]

Problem (29) can be easily solved explicitly by the method of constants’ variation. Laplace transform inversion yields another representation of \( u_{x_0,a_1,a_2}^{a_1,a_2}(x,t) \)

\[
u_{x_0,a_1,a_2}^{a_1,a_2}(x,t) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \left[ e^{-\frac{(x-x_0+2\pi n)^2}{2t}} - e^{-\frac{(x+x_0+2\pi n)^2}{2t}} \right]
\].
The densities $\rho_{x,t}^{a_1}(t)$ and $\Theta_{x,t}^{a_2}(t)$ are equal

$$
\Theta_{x,t}^{a_1}(t) = \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{+\infty} (x_0 + a_1 + 2nt)e^{-\frac{(x_0+a_1+2nt)^2}{2t}} = \frac{\pi}{t^2} \sum_{n=0}^{\infty} n \sin\left(\frac{n\pi}{t}(x_0 + a_1)\right) e^{-\frac{x_0^2}{2t}},
$$

$$
\Theta_{x,t}^{a_2}(t) = \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{+\infty} (x_0 + a_1 + (2n+1)t)e^{-\frac{(x_0+a_1+(2n+1)t)^2}{2t}} = \frac{\pi}{t^2} \sum_{n=0}^{\infty} (-1)^n n \sin\left(\frac{n\pi}{t}(x_0 + a_1)\right) e^{-\frac{(x_0+a_1)^2}{2t}}.
$$

Consider the composition of symmetries $s_1$, $s_2$ and $s_4$ for the heat equation \[15\] and set $x_0 = 0$

$$(x,t,u) \rightarrow (X, T, f(x,t)\mathcal{U}(X,T)), \quad f(x,t) = \frac{1}{\sqrt{1 + \delta}} e^{-\frac{(x+\kappa)^2}{2(t+\delta)}}, \quad X(x,t) = \frac{x + \kappa}{t + \delta}, \quad T(t) = \frac{1}{\delta} - \frac{1}{t + \delta},$$

where $\kappa = (a_2b_1 - a_1b_2)/(a_1 + a_2)$ and $\delta = (b_1 + b_2)/(a_1 + a_2)$. Solutions of equations $X(x,t) = -a_1$ and $X(x,t) = a_2$ are $x = -a_1 t - b_1$ and $x = a_2 t + b_2$ respectively

$$
f(-a_1 t - b_1, t) = \frac{e^{-\frac{(a_1+b_1+a_2)^2}{2(t+\delta)}}}{\sqrt{1 + \delta}}, \quad f(a_2 t + b_2, t) = \frac{e^{-\frac{(a_2+b_2+a_1)^2}{2(t+\delta)}}}{\sqrt{1 + \delta}},
$$

$$
f(0,0) = \frac{\exp\left(\frac{-2}{\delta}\right)}{\sqrt{1 + \delta}}, \quad X(0,0) = \kappa/\delta.
$$

The densities $\rho_{-a_1,a_2}(t)$ and $\rho_{-a_1-t_1,a_2+t_2}(t)$ are equal

$$
\rho_{-a_1,a_2}(t) = \Theta_{x,t}^{a_2}(t) - \Theta_{x,t}^{a_1}(t),
$$

$$
\rho_{-a_1-t_1,a_2+t_2}(t) = \frac{e^{\frac{-2}{\delta}}}{\sqrt{1 + \delta}} \left[ \Theta_{x,t}^{a_2}(t) \left( \frac{1}{\delta} - \frac{1}{t + \delta} \right) - \Theta_{x,t}^{a_1}(t) \left( \frac{1}{\delta} - \frac{1}{t + \delta} \right) \right].
$$

The Anderson formula for $P(\tau_{-a_1-t_1,a_2+t_2} > t)$ can be derived by using first representation in \[80\] and integrating of density \[31\]. If we tends $t$ to infinity we obtain Doob’s formula \[13\]. If we choose second representation in \[80\] we obtain new formula for wedge probability $P(\tau_{-a_1-t_1,a_2+t_2} > t)$. Taking the limit $t \rightarrow \infty$ turns out to Drouilhet and Ycart formula \[13\]. To the best of our knowledge, second branches of formulas \[80\] \[31\] has not been observed in the literature. Let us mention that we can also obtain wedge probabilities for any process with drift function from $\mathcal{F}_1$ and $B = 0$. Also we can simply generalize formula \[44\] to the case of double sided boundaries.

### 7.3 About one generalization of double square-root boundary

Consider first passage time of Brownian motion to the double square root boundaries $b_1 = -b_1 \sqrt{t+c}$ and $b_2 = b_2 \sqrt{t+c}$. For $b_1 = b_2$ the explicit formula for the density (Mellin transform of density function) is known and firstly was derived by Shepp in \[29\].

$$\mathcal{L}_{x,t}^{b_1,b_2} = 0,$$

$$u_{x_0}^{b_1,b_2}(-b_1 \sqrt{t+c}, t) = 0,$$

$$u_{x_0}^{b_1,b_2}(b_2 \sqrt{t+c}, t) = 0,$$

$$u_{x_0}^{b_1,b_2}(x, 0) = \delta(x - x_0),$$

According to the paper \[22\] we introduce the following change of variables:

$$z = \frac{x}{\sqrt{t+c}}, \quad \zeta = \frac{1}{2} \ln\left(\frac{t+c}{c}\right), \quad u_{x_0}^{b_1,b_2} = U(z, \zeta),$$

$$\frac{\partial^2 U}{\partial z^2} = \frac{1}{t+c} \frac{\partial^2 U}{\partial z^2}, \quad \frac{\partial U}{\partial t} = -\frac{\partial U}{\partial z} \frac{\partial z}{\partial t} + \frac{\partial U}{\partial \zeta} \frac{\partial \zeta}{\partial t} = \frac{1}{2(t+c)} \left( -z \frac{\partial U}{\partial z} + \frac{\partial U}{\partial \zeta} \right).$$

In the result we have the following PDE:

$$\frac{\partial^2 U}{\partial z^2} + z \frac{\partial U}{\partial z} \frac{\partial U}{\partial \zeta} = 0,$$
This PDE problem can be solved explicitly in terms of parabolic cylinder functions by using Laplace transform with respect to the variable $\zeta$. Let us mention that it equivalent to the Mellin transform with respect to the original time variable $t$.

Application of symmetries turns out to the closed form formula for the density of stopping time

$$
\tau = \inf\{t > 0; W_t \leq a t - b_1 \sqrt{t^2 + pt + q} \text{ or } W_t \geq a t + b_2 \sqrt{t^2 + pt + q}\}.
$$

8 Remarks about time inhomogeneous diffusions

As we mention above the main difficulties of Lie method arises in solving of PDE system (10). In previous sections we considered in the details time-homogeneous diffusions and the logical question is: can we perform Lie method for time-inhomogeneous diffusion? Here we present an example of time inhomogeneous drift and find symmetries in explicit form.

Consider radial Ornstein Uhlenbeck process (70) with time varying $\gamma = \gamma(t)$ assumed to guarantee existence of unique and non-explosive solution:

$$
\frac{dX_t}{X_t} = \left(\frac{2\nu + 1}{2X_t} - \gamma(t)X_t\right)dt + dW_t, \quad \mu(x,t) = \frac{2\nu + 1}{2x} - \gamma(t)x.
$$

Note that equation for $\xi(x,t)$ does not depend on drift $\mu(x,t)$, hence $\xi(x,t)$ can be represented by formula from (21). The main system (10) has the form (we also set $\rho(t) \equiv 0$)

$$
\frac{x}{2} = \xi,
$$

$$
-\mu\xi_x + \mu_x\xi - \xi_t + \mu_t \tau + \mu \tau_t = \alpha_x,
$$

$$
\frac{1}{2}\alpha_{xx} - \mu \alpha_x - \alpha_t - \xi \mu_{xx} - \mu x \tau_t - \mu \tau_t = \frac{1}{2}.
$$

All quantities used in (22) are equal

$$
\xi = \frac{x}{2} \tau_t, \quad \xi_x = \frac{\tau_t}{2}, \quad \xi_{xx} = 0, \quad \xi_t = \frac{x}{2} \tau_{tt},
$$

$$
\mu = \frac{\nu + 1/2}{x} - \gamma x, \quad \mu_x = \frac{\nu + 1/2}{x^2} - \gamma, \quad \mu_{xx} = \frac{2(\nu + 1/2)}{x^3}, \quad \mu_t = -\gamma \xi x, \quad \mu \tau_t = -\gamma t.
$$

Hence

$$
\alpha_x = \frac{\tau_t}{2} \left(\frac{\nu + 1/2}{x} + \gamma x\right) - \frac{x}{2} \tau_{tt} - \gamma \xi \tau_t + \left(\frac{\nu + 1/2}{x} - \gamma x\right) \tau_t
$$

$$
= -x \left(\frac{\tau_{tt}}{2} + (\gamma \tau) t\right) = -x \omega, \quad \omega(t) = \frac{\tau_{tt}}{2} + (\gamma \tau) t.
$$

Function $\alpha$ and its partial derivatives are equal

$$
\alpha = -\frac{x^2}{2} \omega + \eta, \quad \alpha_x = -x \omega, \quad \alpha_{xx} = -\omega, \quad \alpha_t = -\frac{x^2}{2} \omega t + \eta t.
$$

Substitutions in third line of (22) turns out to

$$
\frac{\omega}{2} + x \omega \left(\frac{\nu + 1/2}{x} - \gamma x\right) + \frac{x^2}{2} \omega_1 - \eta_t - \frac{\tau_t}{2} \frac{2(\nu + 1/2)}{x^3} \tau_t + \frac{x^2}{2} \omega \tau_t + \gamma \tau_t = 0,
$$

$$
-\frac{\omega}{2} + (\nu + 1/2) \omega - x^2 \gamma \omega + \frac{x^2}{2} \omega t - \eta_t + (\gamma \tau)_t = 0.
$$

Comparison of the corresponded powers of $x$ yields the following system

$$
\omega_t - 2\gamma \omega = 0,
$$
\[\tau t + 2(\gamma \tau)_t = 2\omega, \quad \eta_t = \nu \omega + (\gamma \tau)_t.\]

First equation has solution
\[\omega = C_1 e^{2 \int_0^t \gamma(s)ds}.\]

Second equation can be reduced to the first order ODE
\[\tau_t + 2\gamma \tau = 2 \int \omega dt = C_1 \int_0^t 2e^{2 \int_0^t \gamma(p)dp} ds + C_2,\]
therefore
\[
\begin{align*}
\tau &= C_1 \int_0^t 2 \left[ e^{-2 \int_0^t \gamma(r)dr} \int_0^s e^{2 \int_0^r \gamma(r)dr} dp \right] ds \\
&+ C_2 \int_0^t e^{-2 \int_0^t \gamma(p)dp} ds + C_3 e^{-2 \int_0^t \gamma(s)ds} \\
\tau_t &= 2 \int \omega dt + C_2 - \gamma \tau = \\
&= C_1 \left( 2 \int_0^t e^{2 \int_0^t \gamma(p)dp} ds - \gamma \int_0^t 2 \left[ e^{-2 \int_0^t \gamma(r)dr} \int_0^s e^{2 \int_0^r \gamma(r)dr} dr \right] ds \right) \\
&+ C_2 \left( 1 - \gamma \int_0^t e^{-2 \int_0^t \gamma(p)dp} ds \right) - C_3 \gamma e^{-2 \int_0^t \gamma(s)ds} \\
\eta &= \nu \int \omega dt + \gamma \tau = \\
&= C_1 \left( \nu \int_0^t e^{2 \int_0^t \gamma(p)dp} ds + \gamma \int_0^t 2 \left[ e^{-2 \int_0^t \gamma(r)dr} \int_0^s e^{2 \int_0^r \gamma(r)dr} dr \right] ds \right) \\
&+ C_2 \left( \gamma \int_0^t e^{-2 \int_0^t \gamma(p)dp} ds \right) + C_3 \gamma e^{-2 \int_0^t \gamma(s)ds} + C_4.
\end{align*}
\]

Application of formulas for \(\alpha\) and \(\xi\) turns out to the following group of symmetries
\[
\begin{align*}
\mathbf{v}_1 &= x \left( \int_0^t e^{2 \int_0^t \gamma(p)dp} ds - \gamma \int_0^t \left[ e^{-2 \int_0^t \gamma(r)dr} \int_0^r e^{2 \int_0^s \gamma(r)dr} dp \right] ds \right) \partial_x \\
&+ 2 \int_0^t \left[ e^{-2 \int_0^t \gamma(r)dr} \int_0^r e^{2 \int_0^s \gamma(r)dr} dp \right] ds \partial_t \\
&+ u \left( x \gamma e^{-2 \int_0^t \gamma(s)ds} + \nu \int_0^t e^{2 \int_0^t \gamma(p)dp} ds + 2 \gamma \int_0^t \left[ e^{-2 \int_0^t \gamma(r)dr} \int_0^r e^{2 \int_0^s \gamma(r)dr} dp \right] ds \right) \partial_u, \\
\mathbf{v}_2 &= x \left( 1 - \gamma \int_0^t e^{-2 \int_0^t \gamma(p)dp} ds \right) \partial_x + \int_0^t e^{-2 \int_0^t \gamma(p)dp} ds \partial_t + \gamma \int_0^t e^{2 \int_0^t \gamma(p)dp} ds \partial_u \\
\mathbf{v}_3 &= -\gamma x e^{-2 \int_0^t \gamma(s)ds} \partial_x + e^{-2 \int_0^t \gamma(s)ds} \partial_t + \gamma u e^{-2 \int_0^t \gamma(s)ds} \partial_u, \\
\mathbf{v}_4 &= u \partial_u, \\
\mathbf{v}_5 &= u(x, t) + \beta(x, t).
\end{align*}
\]

### 9 Concluding remarks

In this paper we present several applications of Lie symmetries method to the boundary crossing problems. We emphasize that proposed symmetries method can be applicable to the time-inhomogeneous diffusions but the calculation of symmetries is very complicated problem. Also we do not know any known symmetry’ existence conditions like Theorem 2. From our point of view the main advantage of presented technique is the simple generalization to boundary crossing problem for multidimensional diffusion processes.

For time-homogeneous diffusion processes we derive necessary and sufficient conditions of symmetry’ existence, obtain unique two parametric transformations of boundaries \([14], [19], [20], [21]\) and present connections with first hitting time of for Brownian motion \([19], [21]\) and with first hitting time for Bessel process \((?), (??)\). To the best of our knowledge these results are not observed or discussed in the literature. Also we present simple and elegant method to derive closed-form formulas for first passage time densities for Brownian Motion and Bessel process.
10 Proof routines

10.1 Routines from proof of Theorem 3

Family \( \mathcal{F}_1 \):

\[
\frac{d^2 \theta}{dx^2} - (4Bx + 2C) \theta = 0, \quad z = (4B)^{1/3} x + 2C (4B)^{-2/3}, \quad B \neq 0,
\]

\[
\frac{d^2 \theta}{dz^2} = (4B)^{2/3} \frac{d \theta}{dz}, \quad 4Bx + 2C = (4B)^{2/3} z,
\]

\[
\frac{d \theta}{dx} - (4Bx + 2C) \theta = \frac{d \theta}{dz} - z \theta = 0,
\]

\[
\theta(x) = c_1 A t \left( (4B)^{1/3} x + 2C (4B)^{-2/3} \right) + c_2 B t \left( (4B)^{1/3} x + 2C (4B)^{-2/3} \right).
\]

Family \( \mathcal{F}_2 \):

\[
\frac{d^2 \theta}{dx^2} - (Ax^2 + 4Bx + 2C) \theta = 0, \quad \frac{d^2 \theta}{dz^2} - \left( A \left( x + \frac{2B}{A} \right)^2 + 2C - \frac{4B^2}{A} \right) \theta = 0,
\]

\[
z = \sqrt[4]{2A^{1/4}} \left( x + \frac{2B}{A} \right), \quad \frac{d^2 \theta}{dx^2} = 2\sqrt{A} \frac{d^2 \theta}{dz^2}, \quad A \left( x + \frac{2B}{A} \right)^2 + 2C - \frac{4B^2}{A} = 2\sqrt{A} \left( \frac{2B^2}{A^{3/2}} \right),
\]

\[
\frac{d^2 \theta}{dz^2} + \left( \frac{2B^2}{A^{3/2}} - \frac{C}{\sqrt{A}} \left( 1 + \frac{1}{2} \right) - \frac{z^2}{4} \right) \theta = 0,
\]

\[
\theta(x) = c_1 D \frac{2B^2}{A^{3/2}} - \frac{C}{\sqrt{A}} - \frac{1}{2} \left( 2\sqrt{2A^{1/4}} x + \frac{2\sqrt{2B}}{A^{3/4}} \right) + c_2 D \frac{2B^2}{A^{3/2}} - \frac{C}{\sqrt{A}} - \frac{1}{2} \left( -2\sqrt{2A^{1/4}} x \right).
\]

Family \( \mathcal{F}_3 \):

\[
\frac{d^2 \theta}{dx^2} - \left( 2C + \frac{D}{x^2} \right) \theta = 0,
\]

\[
\theta(x) = \sqrt{2V}(x), \quad \frac{d^2 \theta}{dx^2} = \sqrt{2} \left( \frac{dV}{dx^2} + \frac{1}{x} \frac{dV}{dx} - \frac{V}{4x^2} \right), \quad z = \sqrt{2C}, \quad C \neq 0.
\]

\[
\frac{d^2 \theta}{dx^2} = \left( 2C + \frac{D}{x^2} \right) \theta = \frac{dV}{dx} \frac{d\theta}{dx} + \frac{1}{x} \frac{dV}{dx} - \left( 2C + \frac{1}{4} + D \right) \frac{V}{z^2} \frac{d\theta}{dz} + \frac{dV}{dz} \frac{d\theta}{dz} - \left( 1 + \frac{1}{4} + D \right) V = 0,
\]

\[
\theta(x) = c_1 \sqrt{2} \frac{V}{\sqrt{4C}} \left( \sqrt{2C} x \right) + c_2 \sqrt{2} \frac{K}{\sqrt{4C}} \left( \sqrt{2C} x \right).
\]

Family \( \mathcal{F}_4 \):

\[
\frac{d^2 \theta}{dx^2} - \left( Ax^2 + 2C + \frac{D}{x^2} \right) \theta = 0, \quad z = x^2\sqrt{A}, \quad Ax^2 + 2C + \frac{D}{x^2} = 4\sqrt{A} \left( \frac{z}{4} + \frac{C}{2\sqrt{A}} + \frac{D}{4z} \right),
\]

\[
\frac{d^2 \theta}{dz^2} = \frac{d^2 \theta}{dx^2} \left( \frac{dz}{dx} \right)^2 + \frac{d\theta}{dx} \frac{d^2 z}{dx^2} = 4\sqrt{A} \frac{d^2 \theta}{dz^2} + 2\sqrt{A} \frac{d\theta}{dz},
\]

\[
\frac{d^2 \theta}{dz^2} - \left( Ax^2 + 2C + \frac{D}{x^2} \right) \theta = \frac{d^2 \theta}{dx^2} + \frac{1}{2z} \frac{d^2 \theta}{dz^2} + \left( \frac{1}{4} - \frac{C}{2\sqrt{A}} - \frac{D}{4z^2} \right) \theta = 0,
\]

\[
\theta = z^{-1/4} V(z), \quad \frac{d^2 \theta}{dz^2} = z^{-1/4} \left( \frac{d^2 V}{dz^2} - \frac{1}{2z} \frac{dV}{dz} + \frac{5}{16z^2} \right), \quad \frac{d\theta}{dz} = z^{-1/4} \left( \frac{1}{2z} \frac{dV}{dz} - \frac{1}{8z^2} \frac{dV}{dz^2} \right),
\]

\[
\frac{d^2 V}{dz^2} + \left( -\frac{1}{4} - \frac{C}{2\sqrt{A}} + \frac{3 - 4D}{16z^2} \right) V = 0, \quad V = c_1 M \frac{\sqrt{A} x^2}{\sqrt{x}} + c_2 W \frac{\sqrt{A} x^2}{\sqrt{x}}.
\]

\[
\theta(x) = c_1 M \frac{\sqrt{A} x^2}{\sqrt{x}} + c_2 W \frac{\sqrt{A} x^2}{\sqrt{x}}.
\]
10.2 Routines from proof of Theorem 4

\[ \tau(t) = C_1 t^2 + C_2 t + C_5, \]
\[ \tau_t(t) = 2C_1 t + C_2, \]
\[ \frac{1}{4} \tau_{tt}(t) = \frac{1}{2} C_1, \]
\[ \rho(t) = 3B \int \left( \int \tau_{tt} \right) dt = C_1 B t^3 + C_2 \frac{3B^2}{2} t^2 + C_3 t + C_4, \]
\[ \int \rho(t) dt = C_1 B t^4 + C_2 \frac{B^2}{2} t^3 + C_3 \frac{1}{2} t^2 + C_4 t + C_6, \]
\[ \eta(t) = - \int \left( \frac{1}{4} \tau_{tt} + C \tau_t + 2B \rho \right) dt, \]
\[ = -C_1 \left( \frac{t}{2} + C t^2 + \frac{B^2}{2} t^4 \right) - C_2 \left( C t + B^2 t^3 \right) - C_3 B t^2 - C_4 B t + C_6, \]
\[ \xi(x, t) = C_1 \left( x t + B t^3 \right) + C_2 \left( x + \frac{3B^2}{2} t^2 \right) + C_3 t + C_4, \]
\[ \frac{\partial \xi(x, t)}{\partial t} = C_1 \left( x + 3B t^2 \right) + C_2 3B t + C_3, \]
\[ \int \frac{\partial \xi(x, t)}{\partial t} dx = C_1 \left( \frac{x^2}{2} + 3B t^2 x \right) + C_3 B t x + C_3 x. \]

10.3 Routines from proof of Theorem 6

\[ \tau(t) = C_1 \sinh \left( 2\sqrt{A} t \right) + C_2 \cosh \left( 2\sqrt{A} t \right) + C_5, \]
\[ \tau_t(t) = C_1 2\sqrt{A} \cosh \left( 2\sqrt{A} t \right) + C_2 2\sqrt{A} \sinh \left( 2\sqrt{A} t \right), \]
\[ \frac{1}{4} \tau_{tt}(t) = C_1 A \sinh \left( \sqrt{A} t \right) + C_2 A \cosh \left( \sqrt{A} t \right), \]
\[ \rho_{tt} - A \rho = C_1 6B \sqrt{A} \cosh \left( 2\sqrt{A} t \right) + C_2 6B \sqrt{A} \sinh \left( 2\sqrt{A} t \right), \]
\[ \rho(t) = C_1 \frac{2B}{\sqrt{A}} \cosh \left( 2\sqrt{A} t \right) + C_2 \frac{2B}{\sqrt{A}} \sinh \left( 2\sqrt{A} t \right) + C_3 \sinh \left( \sqrt{A} t \right) + C_4 \cosh \left( \sqrt{A} t \right), \]
\[ \int \rho(t) dt = C_1 \frac{B}{A} \sinh \left( \sqrt{A} t \right) + C_2 \frac{B}{A} \cosh \left( \sqrt{A} t \right) + \]
\[ + C_3 \frac{1}{\sqrt{A}} \cosh \left( \sqrt{A} t \right) + C_4 \frac{1}{\sqrt{A}} \sinh \left( \sqrt{A} t \right) + C_6, \]
\[ \eta(t) = - \int \left( \frac{1}{4} \tau_{tt} + C \tau_t + 2B \rho \right) dt, \]
\[ = -C_1 \left( \frac{\sqrt{A}}{2} \cosh \left( 2\sqrt{A} t \right) + C \sinh \left( 2\sqrt{A} t \right) + \frac{2B^2}{A} \sinh \left( (2\sqrt{A} t) \right) \right) \]
\[ - C_2 \left( \frac{\sqrt{A}}{2} \sinh \left( 2\sqrt{A} t \right) + C \cosh \left( 2\sqrt{A} t \right) + \frac{2B^2}{A} \cosh \left( (2\sqrt{A} t) \right) \right) \]
\[ - C_3 \frac{2B}{\sqrt{A}} \cosh \left( (\sqrt{A} t) \right) - C_4 \frac{2B}{\sqrt{A}} \sinh \left( (\sqrt{A} t) \right) + C_6, \]
\[ \xi(x, t) = C_1 \cosh \left( 2\sqrt{A} t \right) \left( x\sqrt{A} + \frac{2B}{\sqrt{A}} \right) + C_2 \sinh \left( 2\sqrt{A} t \right) \left( x\sqrt{A} + \frac{2B}{\sqrt{A}} \right) \]
\[ + C_3 \sinh \left( \sqrt{A} t \right) + C_4 \cosh \left( \sqrt{A} t \right), \]
\[ \frac{\partial \xi(x, t)}{\partial t} = C_1 \sinh \left( 2\sqrt{A} t \right) \left( 2Ax + 4B \right) + C_2 \cosh \left( 2\sqrt{A} t \right) \left( 2Ax + 4B \right), \]

26
\[
\int \frac{\partial \xi(x,t)}{\partial t} \, dx = C_1 \sinh \left(2\sqrt{At}\right) (Ax^2 + 4Bx) + C_2 \cosh \left(2\sqrt{At}\right) (Ax^2 + 4Bx).
\]

+ \ C_3 \cosh \left(\sqrt{At}\right) \sqrt{Ax} + C_4 \sinh \left(\sqrt{At}\right) \sqrt{Ax}.

10.4 Routines from proof of Theorem 5

Symmetry \ s_1 :

\[
\frac{dT}{de} = T^2, \quad T|_{e=0} = t, \\
T = \frac{t}{1 - et}, \\
\{ \begin{array}{l}
t = \frac{T}{1 + eT}, \\
1 - et = \frac{1}{1 + eT}, \\
\frac{t}{1 - et} = T
\end{array} \}
\]

\[
\frac{dX}{de} = X + BT^3,
\]

\[
X = \frac{1}{1 - et} + \frac{1}{(1 - et)^2} \int \frac{Bt^3}{(1 - et)^2} \, de,
\]

\[
X = \frac{1}{1 - et} + \frac{1}{(1 - et)^2} \int \frac{Bt^3}{(1 - et)^2} \, de.
\]

\[
\{ \begin{array}{l}
x - Bt^2 = \frac{X - BT^2}{1 + eT}, \\
x = \frac{X - BT^2}{1 + eT} + \frac{BT^2}{1 + eT}
\end{array} \}
\]

\[
- \frac{d\ln \mathcal{U}}{de} = \frac{X^2}{2} + 3BXT^2 + \frac{T}{2} + CT^2 + \frac{B^2}{2}T^4 + \mu(X) (X + BT^3)
\]

\[
= \frac{1}{2} \left( \frac{x - Bt^2}{1 - et} + \frac{Bt^3}{(1 - et)^2} \right)^2 + \frac{3Bt^2}{(1 - et)} \left( \frac{x - Bt^2}{1 - et} + \frac{Bt^2}{(1 - et)^2} \right)
\]

+ \frac{1}{2} \frac{t}{1 - et} + \frac{Ct^2}{2(1 - et)^2} + \frac{Bt^4}{2(1 - et)^3} - \mu(X) \frac{dX}{de}
\]

\[
= \frac{1}{2} \frac{t}{1 - et} + \frac{Bt^4}{(1 - et)^2} - \mu(X) \frac{dX}{de}.
\]

\[
\ln \mathcal{U} = \dot{C} + \frac{1}{2} \frac{t}{t(1 - et)} + \frac{2(x - Bt^2)Bt}{(1 - et)^2} + \frac{4B^2t^1}{(1 - et)^2} - \frac{1}{2} \ln(1 - et) + \frac{Ct}{1 - et} - \int \mu(X)\,dx - \int \mu(X)\,dX.
\]

\[
\frac{(x - Bt^2)^2}{2t} \left[ \frac{1}{1 - et} - 1 \right] + 2(x - Bt^2)Bt \left[ \frac{1}{1 - et} \right] - \frac{1}{2} \ln(1 - et) + \frac{Ct}{1 - et} - \int \mu(x)\,dx - \int \mu(X)\,dX.
\]
\[
\begin{aligned}
4B^2 t^2/3 \left[ \frac{1}{(1 - e t)^3} - 1 \right] - \frac{1}{2} \ln(1 - e t) &= 4B^2/3 \left[ \frac{T^3}{(1 + e T)^3} \right] + \frac{1}{2} \ln(1 + e T) \\
\frac{Ct}{1 - e t} - Ct + \int \mu(x) dx - \int \mu(X) dX &= C \left[ T - \frac{T^3}{1 + e T} \right] + \ln \theta_{\mathcal{F}_1}(X - B T^2) + \frac{B T^2}{(1 + e T)^2} - \ln \theta_{\mathcal{F}_1}(X)
\end{aligned}
\]

\text{Symmetry } s_2

\[
\frac{dT}{de} = T, \quad T|_{e=0} = t,
\]
\[
\begin{aligned}
\{ \frac{dX}{de} & = \frac{X}{2} + \frac{3B}{2} e^t t^2, \quad X|_{e=0} = x, \\
X & = \mathcal{C}e^{e/2} + \frac{3B^2 e^{e/2}}{2} \int e^{e/2} de \\
& = (x - B t^2) e^{e/2} + B t^2 e x.
\end{aligned}
\]
\[
\begin{aligned}
\{ x - B t^2 & = (X - B T^2) e^{-e/2}, \quad x = (X - B T^2) e^{-e/2} + B T^2 e x \\
- \frac{d \ln U}{de} & = 3 B X T + C T + B^2 T^3 - \mu(X) \left( \frac{X}{2} + \frac{3B}{2} T^2 \right) \\
& = 3 B t e^t \left( x - B t^2 \right) e^{-e/2} + C t e^t + B t^2 e x - \mu(X) \frac{dX}{de} \\
- \ln U & = \mathcal{C} + 2 B t e^{e/2} \left( x - B t^2 \right) + \frac{4}{3} B^2 t^3 e x + C t e^t - \int \mu(X) dX \\
& = - \ln u + 2 B t \left( x - B t^2 \right) \left( e^{e/2} - 1 \right) + \frac{4}{3} B^2 t^3 (e x - 1) + C t (e^t - 1) + \int \mu(x) dx - \int \mu(X) dX.
\end{aligned}
\]

\text{Symmetry } s_3

\[
\begin{aligned}
\{ \frac{X}{e} & = x + e t, x = X - e t, \quad T \equiv t \}, \\
- \frac{d \ln U}{de} & = B T^2 + X - T \mu(X) = B t^2 + x + e t - \mu(X) \frac{dX}{de}.
\end{aligned}
\]
\[
\begin{aligned}
U(X, T) & = u(x, t) \exp \left\{ -e \left( x + B t^2 \right) - e^t t^2 \right\} \mathcal{G}_1(x + e t) \\
& = u(X - e T, T) \frac{\mathcal{G}_1(X)}{\mathcal{G}_1(X - e t)} \exp \left\{ -e \left( X + B T^2 \right) + e^t t^2 / 2 \right\}.
\end{aligned}
\]

\text{Symmetry } s_4

\[
\begin{aligned}
\{ \frac{X}{e} & = x + e, x = X - e, \quad T \equiv t \}, \\
- \frac{d \ln U}{de} & = 2 B T - \mu(X) = 2 B t - \mu(X) \frac{dX}{de}.
\end{aligned}
\]
\[
\begin{aligned}
U(X, T) & = u(x, t) \exp \left\{ -2 e B t \right\} \mathcal{G}_1(x + e) \\
& = u(X - e T, T) \frac{\mathcal{G}_1(X)}{\mathcal{G}_1(X - e t)} \exp \left\{ -2 e B t \right\}.
\end{aligned}
\]
10.5 Routines from proof of Theorem 7

Symmetries $s_1$ and $s_2$ are generated by the following combinations of vector fields (+ is corresponded to the $s_1$):

\[
\begin{align*}
\mathbf{v}_1 \pm \mathbf{v}_2 & = \frac{e^{\pm \sqrt{A} A} \sqrt{2}}{2 \sqrt{A}} \left[ (x \sqrt{A} + \frac{B}{A}) \partial_x \pm \partial_t - u \left( \frac{e^{\pm A^{\pm \sqrt{A} A}}}{2 \sqrt{A}} \left( x + \frac{B}{A} \right)^2 \right) \right] \\
\mathbf{v}_1 \pm \mathbf{v}_2 & = \frac{e^{\pm \sqrt{A} A} \sqrt{2}}{2 \sqrt{A}} \left[ \frac{1}{2} (x + \frac{B}{A}) \partial_x \pm \frac{1}{2 \sqrt{A}} \partial_t - u \left( \frac{e^{\pm A^{\pm \sqrt{A} A}}}{2 \sqrt{A}} \left( x + \frac{B}{A} \right)^2 \right) - \frac{1}{4} \left( -\nu + \frac{1}{4} \right) - \frac{1}{2} \frac{e^{\pm A^{\pm \sqrt{A} A}}}{2 \sqrt{A}} \mu(x) \right] \partial_u \\
\frac{dX}{de} & = \frac{e^{\pm \sqrt{A} A} \sqrt{2}}{2 \sqrt{A}} \left( X + \frac{B}{A} \right), \quad X|_{e=0} = x \\
- \frac{d\ln U}{de} & = \frac{e^{\pm \sqrt{A} A} \sqrt{2}}{2 \sqrt{A}} \left( X + \frac{B}{A} \right)^2 \left( -\frac{1}{4} \pm \left( -\nu + \frac{1}{4} \right) - \frac{1}{2} \frac{e^{\pm A^{\pm \sqrt{A} A}}}{2 \sqrt{A}} \mu(X) \right), \quad U|_{e=0} = u \\
- \ln U & = \frac{\sqrt{A}}{2} \left( x + \frac{B}{A} \right)^2 \left[ \frac{e^{\pm \sqrt{A} A}}{\sqrt{A}} \left( -\nu + \frac{1}{4} \right) \right] + \frac{1}{4} \ln \left( e^{\pm \sqrt{A} A} \right) + \ln \left( \theta_{\mathcal{G}_1}(X) \right) \\
& = - \ln u \left( x + \frac{B}{A} \right)^2 \left[ \frac{e^{\pm \sqrt{A} A}}{\sqrt{A}} \left( -\nu + \frac{1}{4} \right) \right] + \frac{1}{4} \ln \left( e^{\pm \sqrt{A} A} \right) + \ln \left( \theta_{\mathcal{G}_1}(X) \right) \\
& = \ln \left( \theta_{\mathcal{G}_1} \left( \frac{X + \frac{B}{A}}{1 + e^{\pm \sqrt{A} A}} - \frac{2B}{A} \right) / \theta_{\mathcal{G}_1}(X) \right) \\
& = \ln \left( \theta_{\mathcal{G}_1} \left( \frac{X + \frac{B}{A}}{1 + e^{\pm \sqrt{A} A}} - \frac{2B}{A} \right) / \theta_{\mathcal{G}_1}(X) \right).
\end{align*}
\]

For third and forth symmetries we apply combinations $\mathbf{v}_3 \pm \mathbf{v}_4$ (+ is corresponded to the third symmetry)

\[
\begin{align*}
\mathbf{s}_3 \text{ and } \mathbf{s}_4 : \\
\end{align*}
\]
\[ \begin{align*}
X &= x + e^{\pm\sqrt{T}}u, \\
10.6 \text{ Routines from proof of Theorem 12:} \\
- \ln \mathcal{U} &= - \ln u \pm \frac{\sqrt{A}}{2} e^{2e^{\pm\sqrt{T}t}} \pm \sqrt{A}e(x + 2B/A) e^{\pm\sqrt{T}t} - \ln \theta_{\mathcal{F}_x}(x + e^{\pm\sqrt{T}t}) + \ln \theta_{\mathcal{F}_x}(x), \\
- \ln \mathcal{U} &= - \ln u \pm \frac{\sqrt{A}}{2} e^{2e^{\pm\sqrt{T}t}} \pm \sqrt{A}e(x + 2B/A) e^{\pm\sqrt{T}t} - \ln \theta_{\mathcal{F}_x}(x + e^{\pm\sqrt{T}t}) + \ln \theta_{\mathcal{F}_x}(x - e^{\pm\sqrt{T}t}).
\end{align*} \]

10.6 Routines from proof of Theorem 12

\[ u(x,t) = e^{-2\ln \alpha \text{v}_2} \circ e^{\alpha \text{v}_1} \mathcal{U}(X, T) \]
\[ = e^{-2\ln \alpha \text{v}_2} f_{\alpha \beta}^{\alpha \beta}(x,t) \mathcal{U} \left( \frac{x-Bt^2}{1+\alpha \beta t} + \frac{Bt^2}{(1+\alpha \beta t)^2} \right) \]
\[ = f_{\alpha \beta}^{\alpha \beta} \left( \frac{x-Bt^2}{1+\alpha \beta t} + \frac{Bt^2}{(1+\alpha \beta t)^2} \right) \mathcal{U} \left( \frac{x-Bt^2}{1+\alpha \beta t} + \frac{B\alpha^4 t^2}{(1+\alpha \beta t)^2} \right) \]
\[ = \frac{1}{\sqrt{1+\alpha \beta t}} \theta_{\mathcal{F}_x}(x) \mathcal{U} \left( \frac{x-Bt^2}{1+\alpha \beta t} + \frac{B\alpha^4 t^2}{(1+\alpha \beta t)^2} \right) \]
\[ \cdot \exp \left\{ \left[ \frac{(x-Bt^2)^2}{2t^2} + C \right] \left[ \frac{t}{1+\alpha \beta t} - t \right] + \frac{2B(x-Bt^2)}{t} \left[ \frac{t^2}{1+\alpha \beta t} - t^2 \right] + \frac{4B^2}{3} \left[ \frac{t^3}{1+\alpha \beta t} - t^3 \right] \right\} \]
\[ \cdot \exp \left\{ \frac{2B(x-Bt^2)}{t} \left[ \frac{t^3}{(1+\alpha \beta t)^3} - t^3 \right] + \frac{4B^2}{3} \left[ \frac{t^6}{(1+\alpha \beta t)^6} - t^6 \right] \right\} \mathcal{U} \left( \frac{x-Bt^2}{1+\alpha \beta t} + \frac{B\alpha^4 t^2}{(1+\alpha \beta t)^2} \right) \]
\[ \cdot \mathcal{U} \left( \frac{x-Bt^2}{1+\alpha \beta t} + \frac{B\alpha^4 t^2}{(1+\alpha \beta t)^2} \right). \]

Hence function \( f_{\mathcal{F}_x}^{\alpha \beta}(x,t) \) is equal

\[ f_{\mathcal{F}_x}^{\alpha \beta}(x,t) = \frac{1}{\sqrt{1+\alpha \beta t}} \theta_{\mathcal{F}_x}(x) \mathcal{U} \left( \frac{x-Bt^2}{1+\alpha \beta t} + \frac{B\alpha^4 t^2}{(1+\alpha \beta t)^2} \right) \]
\[ \cdot \exp \left\{ \frac{(x-Bt^2)^2}{2t^2} \left[ \frac{t}{1+\alpha \beta t} - t \right] + \frac{2B(x-Bt^2)}{t} \left[ \frac{t^2}{1+\alpha \beta t} - t^2 \right] + \frac{4B^2}{3} \left[ \frac{t^3}{1+\alpha \beta t} - t^3 \right] \right\} \]
\[ \cdot \exp \left\{ \frac{2B(x-Bt^2)}{t} \left[ \frac{t^3}{(1+\alpha \beta t)^3} - t^3 \right] + \frac{4B^2}{3} \left[ \frac{t^6}{(1+\alpha \beta t)^6} - t^6 \right] \right\} \mathcal{U} \left( \frac{x-Bt^2}{1+\alpha \beta t} + \frac{B\alpha^4 t^2}{(1+\alpha \beta t)^2} \right) \]

10.7 Routines from proof of Theorem 12

\[ \textbf{v}_1: \alpha \frac{x-Bt^2}{1+\alpha \beta t} + \frac{\alpha^4 Bt^2}{(1+\alpha \beta t)^2} \rightarrow \left( \alpha \frac{x-Bt^2}{1+\alpha \beta t} + \frac{\alpha^4 Bt^2}{(1+\alpha \beta t)^2} \right) \left[ 1 + \frac{\epsilon \alpha^2 t}{1+\alpha \beta t} \right] \]
\[ - \frac{\alpha^4 Bt^2}{(1+\alpha \beta t)^2} \left[ 1 + \frac{\epsilon \alpha^2 t}{1+\alpha \beta t} \right] + \frac{\alpha^4 Bt^2}{(1+\alpha \beta t)^2} \left[ 1 + \frac{\epsilon \alpha^2 t}{1+\alpha \beta t} \right]^2 \]
\[ 
\]
\[
\frac{\alpha^2 t}{1 + \alpha \beta t} = \frac{\alpha x - Bt^2}{1 + \alpha(\beta + \alpha) t} + \frac{\alpha^4 Bt^2}{(1 + \alpha(\beta + \alpha) t)^2},
\]
\[
\frac{\alpha^2 t}{1 + \alpha \beta t} = \frac{\alpha x - Bt^2}{1 + \alpha \beta t} + \frac{\alpha^4 Bt^2}{(1 + \alpha \beta t)^2}, \quad \hat{\alpha} = \alpha, \quad \hat{\beta} = \beta + \alpha,
\]
\[
f_{\mathcal{F}_1}(x,t) \rightarrow f_{\mathcal{F}_1}(x,t)f_1^2 \left( \frac{\alpha x - Bt^2}{1 + \alpha \beta t} + \frac{\alpha^4 Bt^2}{(1 + \alpha \beta t)^2} ; \frac{\alpha^2 t}{1 + \alpha \beta t} \right)
\]
\[
v_2 \rightarrow \frac{\alpha x - Bt^2}{1 + \alpha \beta t} + \frac{\alpha^4 Bt^2}{(1 + \alpha \beta t)^2} \rightarrow \alpha e^{-\epsilon/2} \frac{x - Bt^2}{1 + \alpha \beta t} + \frac{e^{-\epsilon/2} \alpha^4 Bt^2}{(1 + \alpha \beta t)^2} - B \frac{e^{-\epsilon/2} \alpha^2 t}{(1 + \alpha \beta t)^2} + e^{-2\epsilon} B \frac{\alpha^4 t^2}{(1 + \alpha \beta t)^2}.
\]
\[
\frac{\alpha^2 t}{1 + \alpha \beta t} \rightarrow e^{-\epsilon/2} \frac{x - Bt^2}{1 + \alpha \beta t} + \frac{e^{-\epsilon/2} \alpha^4 Bt^2}{(1 + \alpha \beta t)^2} - B \frac{e^{-\epsilon/2} \alpha^2 t}{(1 + \alpha \beta t)^2} + e^{-2\epsilon} B \frac{\alpha^4 t^2}{(1 + \alpha \beta t)^2}, \quad \hat{\alpha} = e^{-\epsilon/2} \alpha, \quad \hat{\beta} = e^{\epsilon/2} \beta
\]
Then we re-arrange terms in arguments of exponents and obtain second formula.

Proof. At first we apply new variables $\mathcal{X}$ and $\mathcal{T}$ in the first formula from Lemma

$$f_{\mathcal{X}_1}(x, t) = \frac{1}{\sqrt{1 + \alpha \beta t}} \frac{\theta_{\mathcal{X}_1} (x)}{\theta_{\mathcal{X}_1} (\mathcal{X})} \exp \left\{ -\frac{(x-Bt)^2}{2t} - 2Bt(x-Bt^2) - \frac{4B^2t^2}{3} - Cx \right\} \exp \left\{ -\frac{(x-Bt^2)^2}{2t} - 2Bt(x-Bt^2) - \frac{4B^2t^2}{3} - Cx \right\}$$

where $\mathcal{X}$ and $\mathcal{T}$ are defined as

$$\mathcal{X} = \frac{x-Bt^2}{1+\alpha \beta t}, \quad \mathcal{T} = \frac{\alpha^2 t}{1+\alpha \beta t}.$$

Then we re-arrange terms in arguments of exponents and obtain second formula.

10.8 Technical Lemma from proof of Theorem 12

**Lemma 1.** Function $f_{\mathcal{X}_1}^{\alpha, \beta}(x, t)$ from (82) has two equivalent representations

$$f_{\mathcal{X}_1}^{\alpha, \beta}(x, t) = \frac{\theta_{\mathcal{X}_1}(x)}{\theta_{\mathcal{X}_1}(\mathcal{X})} \exp \left\{ -\frac{\alpha^2(x-Bt^2)^2}{2(1+\alpha \beta t)} \right\} \exp \left\{ -\frac{\alpha^2(x-Bt^2)^2}{2(1+\alpha \beta t)^2} - \frac{4B^2t^2}{3(1+\alpha \beta t)^2} \right\}$$

where $\mathcal{X}$ and $\mathcal{T}$ are defined as

$$\mathcal{X} = \frac{x-Bt^2}{1+\alpha \beta t}, \quad \mathcal{T} = \frac{\alpha^2 t}{1+\alpha \beta t}.$$

Proof. At first we apply new variables $\mathcal{X}$, and $\mathcal{T}$ in the first formula from Lemma

$$f_{\mathcal{X}_1}^{\alpha, \beta}(x, t) = \frac{1}{\sqrt{1 + \alpha \beta t}} \frac{\theta_{\mathcal{X}_1} (x)}{\theta_{\mathcal{X}_1} (\mathcal{X})} \exp \left\{ -\frac{(x-Bt^2)^2}{2t} - 2Bt(x-Bt^2) - \frac{4B^2t^2}{3} - Cx \right\} \exp \left\{ -\frac{(x-Bt^2)^2}{2t} - 2Bt(x-Bt^2) - \frac{4B^2t^2}{3} - Cx \right\}$$

Then we re-arrange terms in arguments of exponents and obtain second formula.

10.9 Technical Lemma from proof of Theorem 13

**Lemma 2.** Function $f_{\mathcal{X}_1}^{\alpha, \beta}(x, t)$ from (87) has two equivalent representations

$$f_{\mathcal{X}_1}^{\alpha, \beta}(x, t) = \frac{\theta_{\mathcal{X}_1}(x)}{\theta_{\mathcal{X}_1}(\mathcal{X})} \left( \frac{\alpha + \beta e^{2\sqrt{\mathcal{T}}}}{\beta + 1 + \alpha e^{-2\sqrt{\mathcal{T}}}} \right)^{\nu}$$

$$\cdot \exp \left\{ \frac{\sqrt{\mathcal{X}}}{2} (x + 2B/A)^2 \frac{\alpha + \beta e^{-2\sqrt{\mathcal{T}}}}{\beta + 1 + \alpha e^{-2\sqrt{\mathcal{T}}}} - \frac{\beta (\alpha + 1)e^{2\sqrt{\mathcal{T}}}}{1 + \alpha + \beta e^{2\sqrt{\mathcal{T}}}} \right\}$$

$$f_{\mathcal{X}_1}^{\alpha, \beta}(x, t) = \frac{\theta_{\mathcal{X}_1}(x)}{\theta_{\mathcal{X}_1}(\mathcal{X})} \left( \frac{\alpha + \beta e^{2\sqrt{\mathcal{T}}}}{\beta + 1 + \alpha e^{-2\sqrt{\mathcal{T}}}} \right)^{\nu}$$

$$\cdot \exp \left\{ \frac{\sqrt{\mathcal{X}}}{2} (x + 2B/A)^2 \frac{\alpha + \beta e^{-2\sqrt{\mathcal{T}}}}{\beta + 1 + \alpha e^{-2\sqrt{\mathcal{T}}}} - \frac{\beta (\alpha + 1)e^{2\sqrt{\mathcal{T}}}}{1 + \alpha + \beta e^{2\sqrt{\mathcal{T}}}} \right\}$$

$$f_{\mathcal{X}_1}^{\alpha, \beta}(x, t) = \frac{\theta_{\mathcal{X}_1}(x)}{\theta_{\mathcal{X}_1}(\mathcal{X})} \left( \frac{\alpha + \beta e^{2\sqrt{\mathcal{T}}}}{\beta + 1 + \alpha e^{-2\sqrt{\mathcal{T}}}} \right)^{\nu}$$

$$\cdot \exp \left\{ \frac{\sqrt{\mathcal{X}}}{2} (x + 2B/A)^2 \frac{\alpha + \beta e^{-2\sqrt{\mathcal{T}}}}{\beta + 1 + \alpha e^{-2\sqrt{\mathcal{T}}}} - \frac{\beta (\alpha + 1)e^{2\sqrt{\mathcal{T}}}}{1 + \alpha + \beta e^{2\sqrt{\mathcal{T}}}} \right\}$$
10.10 Routines from proof of Theorem 13

\[ u(x,t) = e^{\ln\left(\frac{1}{1+\alpha+\beta}\right)} v_0 \circ e^{2\sqrt{At}} \ln\left(\frac{1+\beta+\alpha}{1+\alpha}\right) v_0 \circ e^{\frac{\alpha}{\beta+1}(v_1-v_2)} \circ e^{\frac{\alpha}{\beta+1}(v_1+v_2)} \mathcal{U}(\alpha, t) \]

\[ = f_1^{(\alpha+\beta+1)}(x,t) e^{\frac{\alpha}{\beta+1}(v_1-v_2)} \mathcal{U}(\alpha, t) \]

\[ = f_1^{(\alpha+\beta+1)}(x,t) f_2^{(\alpha+\beta+1)}(x,t) \mathcal{U}(\alpha, t) \]

Next we apply technical Lemma 2 and obtain main formula for symmetry \( s^\alpha,\beta \) from (52).

10.11 Routines from proof of Theorem 13

\[
\frac{(x+2B/A)^{1+\alpha+\beta}}{1+\alpha+\beta e^{2\sqrt{At}}} \sqrt{1+\beta+ae^{-2\sqrt{At}}} \rightarrow \frac{(x+2B/A)^{1+\alpha+\beta}}{1+\alpha+\beta e^{2\sqrt{At}}} \sqrt{1+\beta+ae^{-2\sqrt{At}}} \]

\[ = e^{\sqrt{At}} \frac{(x+2B/A)^{1+\alpha+\beta}}{1+\alpha+\beta e^{2\sqrt{At}}} \sqrt{1+\beta+ae^{-2\sqrt{At}}} \]

\[ = \frac{(x+2B/A)^{1+\alpha+\beta}}{1+\alpha+\beta e^{2\sqrt{At}}} \sqrt{(1+\beta)(1+\alpha)\beta e^{-2\sqrt{At}}} \]

\[ = \frac{(x+2B/A)^{1+\alpha+\beta}}{1+\alpha+\beta e^{2\sqrt{At}}} \sqrt{1+\beta+ae^{-2\sqrt{At}}} \]

\[ = \frac{(x+2B/A)^{1+\alpha+\beta}}{1+\beta+ae^{-2\sqrt{At}}} \]

\[ \alpha = \alpha(e+1), \quad \beta = \beta + e \]

\[ = \frac{1}{2\sqrt{A}} \ln \left( \frac{\beta+1+ae^{-2\sqrt{At}}}{\beta+(1+\alpha)e^{-2\sqrt{At}}} \right) \]
\[
\begin{align*}
\frac{1}{2\sqrt{A}} & \ln \left( \frac{\beta + 1 + \alpha e^{-2\sqrt{At}}}{\beta e + \beta + \epsilon + (\alpha e + \alpha + 1)e^{-2\sqrt{At}}} \right) \quad \text{[apply 5th symmetry]} \\
\rightarrow \frac{1}{2\sqrt{A}} & \ln \left( \frac{\beta + 1 + \alpha e^{-2\sqrt{At}}}{\beta e + \beta + \epsilon + (\alpha e + \alpha + 1)e^{-2\sqrt{At}}} \right) + \frac{1}{2\sqrt{A}} \ln(\epsilon + 1) \\
= \frac{1}{2\sqrt{A}} & \ln \left( \frac{(\beta + 1)(\epsilon + 1) + (\epsilon + 1)\alpha e^{-2\sqrt{At}}}{\beta e + \beta + \epsilon + (\alpha e + \alpha + 1)e^{-2\sqrt{At}}} \right) \\
= \frac{1}{2\sqrt{A}} & \ln \left( \frac{\hat{\beta} + 1 + \hat{\alpha}e^{-2\sqrt{At}}}{\hat{\beta} + (1 + \hat{\alpha})e^{-2\sqrt{At}}} \right)
\end{align*}
\]

Check action of function \( f_{\mathcal{F}_2}^{\alpha,\beta}(x,t) \):

\[
f_{\mathcal{F}_2}^{\alpha,\beta}(x,t) \Lambda_1 \left( \frac{(x + 2B/A)\sqrt{1 + \alpha + \beta}}{\sqrt{1 + \alpha + \beta e^{2\sqrt{At}}} \sqrt{1 + \beta + \alpha e^{-2\sqrt{At}}} - \frac{2B}{A}} \right) \Lambda_2 \left( \frac{(x + 2B/A)\sqrt{1 + \alpha + \beta}}{\sqrt{1 + \alpha + \beta e^{2\sqrt{At}}} \sqrt{1 + \beta + \alpha e^{-2\sqrt{At}}} - \frac{2B}{A}} \right) = \frac{2B}{A} \Lambda_1 \left( \frac{(x + 2B/A)\sqrt{1 + \alpha + \beta}}{\sqrt{1 + \alpha + \beta e^{2\sqrt{At}}} \sqrt{1 + \beta + \alpha e^{-2\sqrt{At}}} - \frac{2B}{A}} \right) = R_1(x,t) R_2(x,t) e^{\frac{2B}{A}[(x + 2B/A)^2 - (R_3(x,t) - 1)],}
\]

where

\[
R_1(x,t) = \frac{\theta_{\mathcal{F}_2}(x)}{\theta_{\mathcal{F}_2}(x)} \frac{\theta_{\mathcal{F}_2} \left( \frac{(x + 2B/A)\sqrt{1 + \alpha + \beta}}{\sqrt{1 + \alpha + \beta e^{2\sqrt{At}}} \sqrt{1 + \beta + \alpha e^{-2\sqrt{At}}} - \frac{2B}{A}} \right)}{\theta_{\mathcal{F}_2} \left( \frac{(x + 2B/A)\sqrt{1 + \alpha + \beta}}{\sqrt{1 + \alpha + \beta e^{2\sqrt{At}}} \sqrt{1 + \beta + \alpha e^{-2\sqrt{At}}} - \frac{2B}{A}} \right)}
\]

\[
R_2(x,t) = \alpha + \beta e^{2\sqrt{At}} \nu \beta + (1 + \beta) e^{-2\sqrt{At}} \nu
\]

\[
= \frac{\alpha + \beta e^{2\sqrt{At}} \nu}{\beta + 1 + \alpha e^{-2\sqrt{At}}} \nu^{-1/2} \left( 1 + \epsilon \beta + 1 + \alpha e^{-2\sqrt{At}} \right)^{-\nu/2}
\]

\[
R_3(x,t) = \frac{2}{\alpha + 1 + \beta e^{2\sqrt{At}}} - 2 \left( \frac{\alpha + 1 + \beta e^{2\sqrt{At}}}{(\alpha + 1 + \beta e^{2\sqrt{At}}) \left( 1 + \beta + \alpha e^{-2\sqrt{At}} \right)} \right) + \frac{\alpha + 1 + \beta e^{2\sqrt{At}}}{(\alpha + 1 + \beta e^{2\sqrt{At}}) \left( 1 + \beta + \alpha e^{-2\sqrt{At}} \right)}
\]

\[
= \frac{2}{\alpha + 1 + \beta e^{2\sqrt{At}}} \left( \frac{\alpha + 1 + \beta e^{2\sqrt{At}}}{\left( \alpha + 1 + \beta e^{2\sqrt{At}} \right) \left( 1 + \beta + \alpha e^{-2\sqrt{At}} \right)} \right) + \frac{\alpha + 1 + \beta e^{2\sqrt{At}}}{\left( \alpha + 1 + \beta e^{2\sqrt{At}} \right) \left( 1 + \beta + \alpha e^{-2\sqrt{At}} \right)}
\]

\[
= \frac{2}{\alpha + 1 + \beta e^{2\sqrt{At}}} + \frac{\alpha + 1 + \beta e^{2\sqrt{At}}}{\left( \alpha + 1 + \beta e^{2\sqrt{At}} \right) \left( 1 + \beta + \alpha e^{-2\sqrt{At}} \right)}
\]

34
\begin{align*}
&= 2 \frac{\hat{\alpha} e^{-2\sqrt{\beta t}}}{(1 + \hat{\beta} + \hat{\alpha} e^{-2\sqrt{\beta t}})} + \frac{\hat{\alpha} + \hat{\beta} + 1}{(1 + \hat{\beta} + \hat{\alpha} e^{-2\sqrt{\beta t}})} \\
&= 2 \frac{\hat{\alpha} e^{-2\sqrt{\beta t}}}{(1 + \hat{\beta} + \hat{\alpha} e^{-2\sqrt{\beta t}})} \left( \frac{\hat{\alpha} + \hat{\beta} + 1}{(1 + \hat{\beta} + \hat{\alpha} e^{-2\sqrt{\beta t}})} \right) - \frac{\hat{\alpha} + \hat{\beta} + 1}{(1 + \hat{\beta} + \hat{\alpha} e^{-2\sqrt{\beta t}})} \\
&= 2(\hat{\alpha} + 1) \left( \frac{\hat{\alpha} + \hat{\beta} + 1}{(1 + \hat{\beta} + \hat{\alpha} e^{-2\sqrt{\beta t}})} \right) - \frac{\hat{\alpha} + \hat{\beta} + 1}{(1 + \hat{\beta} + \hat{\alpha} e^{-2\sqrt{\beta t}})} \\
&= \frac{2(\hat{\alpha} + 1)}{(\hat{\alpha} + 1 + \hat{\beta} e^{-2\sqrt{\beta t}})} \left( \frac{\hat{\alpha} + \hat{\beta} + 1}{(1 + \hat{\beta} + \hat{\alpha} e^{-2\sqrt{\beta t}})} \right) - \frac{\hat{\alpha} + \hat{\beta} + 1}{(1 + \hat{\beta} + \hat{\alpha} e^{-2\sqrt{\beta t}})}
\end{align*}

Hence for function \( f^{\alpha,\beta}_{\theta^2}(x, t) \) we have the following relation

\[ f^{\alpha,\beta}_{\theta^2}(x, t) \rightarrow (1 + e)^{\nu - 1/2} f^{\hat{\alpha},\hat{\beta}}_{\theta^2}(x, t). \]

We use symmetry \( s_6 \) with parameter \( \frac{2\ln(1+\epsilon)}{\nu + 1} \) and have

\[ f^{\alpha,\beta}_{\theta^2}(x, t) \rightarrow f^{\hat{\alpha},\hat{\beta}}_{\theta^2}(x, t). \]

Application of second symmetry yields

\[ \frac{(x + 2B/A)\sqrt{1 + \alpha + \beta}}{\sqrt{1 + \alpha + \beta e^{2\sqrt{\beta t}}} \sqrt{1 + \beta + \alpha e^{-2\sqrt{\beta t}}}} \rightarrow \frac{(x + 2B/A)\sqrt{1 + \alpha + \beta}}{\sqrt{1 + \alpha + \beta e^{2\sqrt{\beta t}}} \sqrt{1 + \beta + \alpha e^{-2\sqrt{\beta t}}}} \frac{(1 + \epsilon) e^{-2\sqrt{\beta t}}}{\sqrt{1 + \beta + \alpha e^{-2\sqrt{\beta t}}} \sqrt{1 + \beta + \alpha e^{-2\sqrt{\beta t}}}} \]

\[ = \frac{(x + 2B/A)\sqrt{1 + \alpha + \beta}}{\sqrt{1 + \alpha + \beta e^{2\sqrt{\beta t}}} \sqrt{1 + \beta + \beta e + (\alpha + \alpha + \epsilon) e^{-2\sqrt{\beta t}}}} \]

\[ = \frac{(x + 2B/A)\sqrt{1 + \alpha + \beta}}{\sqrt{1 + \alpha + \beta e^{2\sqrt{\beta t}}} \sqrt{1 + \beta + (1 + \alpha + \epsilon) e^{-2\sqrt{\beta t}}}} \frac{\epsilon}{\sqrt{1 + \beta + (1 + \alpha + \epsilon) e^{-2\sqrt{\beta t}}}} \frac{1}{\sqrt{1 + \beta + (1 + \alpha + \epsilon) e^{-2\sqrt{\beta t}}}} \frac{1}{\sqrt{1 + \beta + (1 + \alpha + \epsilon) e^{-2\sqrt{\beta t}}}} \]

\[ = \frac{1}{2\sqrt{\beta}} \ln \left( \frac{\beta + 1 + \alpha e^{-2\sqrt{\beta t}}}{\beta + (1 + \alpha) e^{-2\sqrt{\beta t}}} \right) \]

\[ \rightarrow \frac{1}{2\sqrt{\beta}} \ln \left( \frac{\beta + 1 + \alpha e^{-2\sqrt{\beta t}}}{\beta + (1 + \alpha) e^{-2\sqrt{\beta t}}} \right) \frac{1}{\sqrt{1 + \beta + (1 + \alpha + \epsilon) e^{-2\sqrt{\beta t}}}} \frac{1}{\sqrt{1 + \beta + (1 + \alpha + \epsilon) e^{-2\sqrt{\beta t}}}} \frac{1}{\sqrt{1 + \beta + (1 + \alpha + \epsilon) e^{-2\sqrt{\beta t}}}} \]

\[ = \frac{1}{2\sqrt{\beta}} \ln \left( \frac{\beta + 1 + \alpha e^{-2\sqrt{\beta t}}}{\beta + (1 + \alpha) e^{-2\sqrt{\beta t}}} \right) \frac{1}{\sqrt{1 + \beta + (1 + \alpha + \epsilon) e^{-2\sqrt{\beta t}}}} \frac{1}{\sqrt{1 + \beta + (1 + \alpha + \epsilon) e^{-2\sqrt{\beta t}}}} \frac{1}{\sqrt{1 + \beta + (1 + \alpha + \epsilon) e^{-2\sqrt{\beta t}}}} \]

Check action on function \( f^{\alpha,\beta}_{\beta^2}(x, t) \):

\[ f^{\alpha,\beta}_{\beta^2}(x, t) f^2 \left( \frac{(x + 2B/A)\sqrt{1 + \alpha + \beta}}{\sqrt{1 + \alpha + \beta e^{2\sqrt{\beta t}}} \sqrt{1 + \beta + \alpha e^{-2\sqrt{\beta t}}}} \frac{2B}{A} \frac{1}{2\sqrt{\beta}} \ln \left( \frac{\beta + 1 + \alpha e^{-2\sqrt{\beta t}}}{\beta + (1 + \alpha) e^{-2\sqrt{\beta t}}} \right) \right) = \]

35
where

\[ R_1(x, t) = \frac{\theta_{\mathcal{F}_2}(x)}{\theta_{\mathcal{F}_2}\left(\frac{(x+2B/A)\sqrt{1+\alpha+\beta}}{\sqrt{1+\alpha+\beta e^{-2\sqrt{At}}} - 2B/A}\right)} \]

\[ R_2(x, t) = \frac{(\alpha + 1 + \beta e^{2\sqrt{At}})^{\nu}}{(\beta + 1 + \alpha e^{-2\sqrt{At}})^{\nu-1/2}} \left(1 + \frac{\beta + (1 + \alpha)e^{2\sqrt{At}}}{\beta + 1 + \alpha e^{-2\sqrt{At}}}\right)^{-\nu+1/2} \]

\[ R_3(x, t) = 2 \frac{\alpha + 1}{\alpha + 1 + \beta e^{2\sqrt{At}}} \frac{\hat{\alpha} + \hat{\beta} + 1}{\hat{\alpha} + 1 + \hat{\beta} e^{2\sqrt{At}}} = 2 \frac{\hat{\alpha} + 1}{\hat{\alpha} + 1 + \beta e^{2\sqrt{At}}} \frac{\hat{\alpha} + \hat{\beta} + 1}{\hat{\alpha} + 1 + \hat{\beta} e^{2\sqrt{At}}} \left(1 + \hat{\beta} + \hat{\alpha} e^{-2\sqrt{At}}\right). \]

Hence for function \( f_{\alpha,\beta}^{\mathcal{F}_2}(x, t) \) we have the following relation

\[ f_{\alpha,\beta}^{\mathcal{F}_2}(x, t) \rightarrow (1 + \epsilon)^{-\nu} f_{\hat{\alpha},\hat{\beta}}^{\mathcal{F}_2}(x, t). \]

We use symmetry \( s_0 \) with parameter \( \frac{\nu \ln(1+\epsilon)}{2} \) and have

\[ f_{\alpha,\beta}^{\mathcal{F}_2}(x, t) \rightarrow f_{\hat{\alpha},\hat{\beta}}^{\mathcal{F}_2}(x, t). \]

### 10.12 Routines from proof of Theorem \textbf{17}

\[ 1 \frac{\partial^2 G}{\partial x^2} - (2Bx + C) G - \frac{\partial G}{\partial t} = 0, \]

\[ G = e^{-\frac{2\beta}{B}t^2 - Ct - 2Bxt}H(x, t), \]

\[ \frac{\partial^2 G}{\partial x^2} = e^{-\frac{2\beta}{B}t^2 - Ct - 2Bxt} \left(\frac{\partial^2 H}{\partial x^2} - 4Bt \frac{\partial H}{\partial x} + 4B^2 t^2 H\right), \]

\[ \frac{\partial G}{\partial t} = e^{-\frac{2\beta}{B}t^2 - Ct - 2Bxt} \left(\frac{\partial H}{\partial t} - [C + 2Bx + 2B^2 t^2] H\right), \]

\[ \frac{1}{2} \frac{\partial^2 H}{\partial x^2} - 2Bl \frac{\partial H}{\partial x} - \frac{\partial H}{\partial t} = 0, \]
10.13 Routines from proof of Theorem 18

\[ \frac{\partial^2 H}{\partial x^2} = \frac{\partial^2 U}{\partial x^2}, \quad \frac{\partial H}{\partial x} = \frac{\partial U}{\partial x}, \quad \frac{\partial H}{\partial t} = \frac{\partial U}{\partial t} - 2Bt \frac{\partial U}{\partial x}, \]

\[ u(x,t) = \theta_{\mathcal{F}_x}(x) e^{-2Bx^2 t - Ct - 2Bxt} U(x - Bt^2, t), \quad 1 + \frac{\partial^2 U}{2 \partial x^2} - \frac{\partial U}{\partial t} = 0. \]

10.14 Routines from proof of Theorem 19

\[ 1 + \frac{\partial^2 G}{2 \partial x^2} - \frac{1}{2} \left( A(x + 2B/A)^2 + 2C - \frac{4B^2}{A} \right) G \frac{\partial G}{\partial t} = 0. \]

\[ z = (4A)^{1/4} \left( x + \frac{2B}{A} \right), \quad z^2 = 2\sqrt{A} \left( x + \frac{2B}{A} \right)^2, \quad G(x,t) = H(z,t). \]

\[ \sqrt{A} \frac{\partial^2 H}{\partial z^2} + \sqrt{A} \left( \frac{z^2}{4} + \frac{1}{\sqrt{A}} \left[ \frac{2B^2}{A} - C \right] \right) G \frac{\partial G}{\partial t} = 0. \]

\[ \frac{\partial^2 H}{\partial z^2} + \left( \frac{z^2}{4} + \frac{1}{\sqrt{A}} \left[ \frac{2B^2}{A} - C \right] \right) H - \frac{1}{2} \frac{\partial G}{\partial t} = 0. \]

\[ H(x,t) = e^{\left[ \frac{4 + 2B/A - C}{2} \right] t} e^{2z/4} V(z,t). \]

\[ \frac{\partial^2 V}{\partial z^2} + z \frac{\partial V}{\partial z} - \frac{1}{\sqrt{A}} \frac{\partial V}{\partial t} = 0. \]

\[ V(z,t) = U(X,T), \quad X = ze^{\mathcal{R} T}, \quad T = e^{2\mathcal{R} T} - 1. \]

\[ \frac{\partial^2 V}{\partial z^2} = e^{2\mathcal{R} T} \frac{\partial^2 U}{\partial X^2}, \quad \frac{\partial V}{\partial z} = X \frac{\partial U}{\partial x}, \quad \frac{\partial V}{\partial t} = \sqrt{AX} \frac{\partial U}{\partial t} + 2\sqrt{A} \frac{\partial U}{\partial T}. \]

\[ u(x,t) = \theta_{\mathcal{F}_x}(x) e^{\left[ \frac{4 + 2B/A - C}{2} \right] t} \left( \sqrt{4A} (x + 2B/A) e^{\mathcal{R} T}, e^{2\mathcal{R} T} - 1 \right), \quad 1 + \frac{\partial^2 U}{2 \partial x^2} - \frac{\partial U}{\partial T} = 0. \]
10.15 Routines from proof of Theorem 20

\[ \frac{1}{2} \frac{\partial^2 G}{\partial x^2} = \left( \frac{Ax^2}{2} + C + \frac{D}{2x^2} \right) G - \frac{\partial G}{\partial t} = 0, \]

\[ G(x, t) = e^{-Ct} H(z, t), \quad \sqrt{4Ax}, \]

\[ \frac{\partial^2 H}{\partial z^2} - \left[ \frac{z^2}{4} + \frac{D}{z^2} \right] H - \frac{1}{\sqrt{A}} \frac{\partial H}{\partial t} = 0 \]

\[ H(z, t) = z^\omega e^{-z^2/4} W(z, t) \]

\[ \frac{\partial^2 W}{\partial z^2} = z^\omega e^{z^2/4} \left( \frac{\partial^2 W}{\partial z^2} + \left[ z + \frac{2\omega}{z} \right] \frac{\partial W}{\partial z} + \left[ \frac{z^2}{4} + \omega + \frac{1}{2} + \frac{\omega^2 - \omega}{z^2} \right] W \right) \]

\[ \frac{\partial^2 W}{\partial z^2} \left( z + \frac{2\omega}{z} \right) \frac{\partial W}{\partial z} + \left[ \frac{1}{2} + \frac{\omega^2 - \omega - D}{z^2} \right] W - \frac{1}{\sqrt{A}} \frac{\partial W}{\partial t} = 0. \]

\[ W = e^{(\omega + \frac{1}{2}) \sqrt{A} t} \mathcal{U}(x, T), \quad \mathcal{X} = ze^{\sqrt{A} t}, \quad T = e^{2\sqrt{A} t} - 1 \]

\[ \frac{\partial^2 \mathcal{U}}{\partial z^2} = e^{2\sqrt{A} t} \frac{\partial^2 \mathcal{U}}{\partial x^2} + \left( z + \frac{2\omega}{z} \right) \frac{\partial \mathcal{U}}{\partial x} \left( \frac{1}{\sqrt{A}} \frac{\partial \mathcal{U}}{\partial t} - \frac{1}{\sqrt{A}} \frac{\partial \mathcal{U}}{\partial t} \right) + \frac{2\omega - n - 1}{2}, \quad \omega^2 - \omega - D = \frac{n - 1}{2}. \]

\[ u(x, t) = \theta_{\mathcal{F}_1}(x) \left( \sqrt{4Ax} e^{\frac{1}{4} e^{2\sqrt{A} t} - C t + (\frac{n}{2}) \sqrt{A} t} \right) \mathcal{U} \left( \sqrt{4Ax} e^{\sqrt{A} t}, e^{2\sqrt{A} t} - 1 \right) \]

10.16 Routines from example 6.2

\[ \mu(x) = \begin{cases} \frac{\alpha u + \beta}{\sqrt{au^2 + bu + c}} & = \frac{2au + b}{4\sqrt{au^2 + bu + c}} \\ \frac{\beta - b/4}{\sqrt{au^2 + bu + c}} & + \frac{(\alpha - a/2) u}{\sqrt{au^2 + bu + c}} \end{cases} \]

\[ = \frac{2\beta - ab/a}{\sqrt{e^{2\sqrt{a} x} + de^{-\sqrt{a} x}}} + \frac{\alpha - a/2}{\sqrt{e^{2\sqrt{a} x} + de^{-\sqrt{a} x}}} \]

\[ = \phi \left( e^{\sqrt{a} x} + de^{-\sqrt{a} x} \right) + \psi \left( e^{\sqrt{a} x} - de^{-\sqrt{a} x} \right), \quad \phi = 2\beta - ab/a, \quad \psi = \frac{\alpha - a/2}{\sqrt{a}}. \]

\[ \mu' = \psi \sqrt{a} - \phi \sqrt{a}, \quad \mu^2 = \begin{cases} \frac{\phi^2}{(e^{\sqrt{a} x} + de^{-\sqrt{a} x})^2} + 2\phi \psi \left( e^{\sqrt{a} x} - de^{-\sqrt{a} x} \right) \left( e^{\sqrt{a} x} + de^{-\sqrt{a} x} \right)^{-2} \quad & \text{if } \phi \neq 0, \\ \psi^2 \left( e^{\sqrt{a} x} - de^{-\sqrt{a} x} \right) \left( e^{\sqrt{a} x} + de^{-\sqrt{a} x} \right)^{-2} \quad & \text{if } \phi = 0. \end{cases} \]

Condition \( \mu(x) \in \mathcal{F}_1 \) is equivalent to \( \phi \) and \( \psi \) are solve the following system

\[ \phi \left( 2\psi - \sqrt{a} \right) = 0, \quad \phi^2 + 4\alpha \psi \left( \psi - \sqrt{a} \right) = 0. \]

First equation has two solutions: \( \phi = 0 \) and \( 2\psi = \sqrt{a} \). In the case of \( \psi = 0 \) the second equation reduces to \( \alpha \psi \left( \psi - \sqrt{a} \right) = 0 \). We suppose \( \psi \neq 0 \) (in this case \( \mu \equiv 0 \)) to avoid trivial problem. Case \( \alpha = 0 \) is not interesting because in this case Pearson diffusion is Geometric Brownian motion, hence we set \( \psi = \sqrt{a} \). If \( \phi \neq 0 \) and
\[ \psi = \sqrt{a}/2 \] the term \( \alpha \psi (\psi - \sqrt{a}) < 0 \) and second equation does not have real solutions. It turns out to the following specifications for the Pearson diffusion:

\[ \psi = \frac{\alpha - a/2}{\sqrt{a}} = \sqrt{a}, \Rightarrow \alpha = \frac{3}{2} a, \]
\[ \phi = 2 \beta - \frac{ab}{a} = 0, \Rightarrow \beta = \frac{3}{4} b. \]

References

[1] Abramowitz M., Stegun I. (1972) Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables.

[2] Alili L., Patie P. (2005) On the first crossing times of a Brownian motion and a family of continuous curves. C. R. Acad. Sci. Paris, Ser I 340 (2005) 225-228.

[3] Alili L., Pedersen J.L., Patie P. Representations of the first hitting time density of an Ornstein - Uhlenbeck process, Stochastic Models 21(4): 967-980, 2005.

[4] Alili L., Patie P. (2010) Boundary-Crossing Identities for Diffusions Having the Time-Inversion Property. Journal of Theoretical Probability. March 2010, Volume 23, Issue 1, pp. 65-85.

[5] Alili L., Patie P. (2014) Boundary crossing identities for Brownian motion and some nonlinear ode’s. Proc. Amer.Math.Soc. 142 no. 11, 3811-3824, 2014.

[6] Alili L., Matsumoto H. Further studies on square-root boundaries fro Bessel process. Electron. Commun. Probab. Vol. 23 (2018) paper no. 39.

[7] Anderson T.W. A modification of sequential probability ration test to reduce sample size. Ann. Math. Statist., 31(1):165-197, 1960.

[8] Borovkov K., Downes A.N. (2010) On boundary crossing probabilities for diffusion processes, Stochastic Processes and thier Applications 120 (2010), 105-129.

[9] Bounocore A., Nobile A.G., Ricciardi L.M. A new integral equation for the evaluation of first-passage-time for diffusion-processes. J.Appl.Probab. 27, 1 (1990), 102-114.

[10] Craddock M. Fundamental solutions, transition densities and the integration of Lie symmetries. J. Differential equations 246 (2009) 2538-2560.

[11] Craddock M., Lennox K.A. The calculation of expectations for classes of diffusion processes by Lie symmetry methods (2009) The annals of Applied Probability 2009, Vol. 19, No. 1, 127-157.

[12] Daniels H.E. Minimum of a stationary Markov process superimposed on a U -shaped trend. J. Appl. Probab. 6, 2 (1969), 399-408.

[13] Doob J.L. Heuristic approach to the Kolmogorov-Smirnov theorems. Ann. Math. Statist., 20(3):393-403, 1949.

[14] DeLong D. Crossing probabilities for a square root boundary by a Bessel process. Comm. Statist. A-Theory methods 10 (1981) no. 21, 2197-2213.

[15] Drouilhet R., Ycart B., (2016) Computing wedge probabilities. Preprint https://arxiv.org/pdf/1612.05764.

[16] Enriquez N., Sabot C., Yor M. Renewal series and square-root boundary for Bessel process, Elect. Comm. in Proba., 13 (2008), 649-652.

[17] Goard J., fundamental solutions to Kolmogorov equations via reduction to canonical form. J. Appl. Math. Decis. Sci. 2006 24.

[18] Groeneboom P., Brownian motion with a parabolic drift and Airy functions. Probab Th. Rel. Fields 81, 79-109 (1989).
[19] Jaimungal S., Kreinin A., A. Valov (2009) Integral equations and the first passage time of brownian motions. Preprint. https://arxiv.org/pdf/0902.2569.pdf.
[20] Lerche H.R. (1986) Boundary crossing of Brownian motion (Lecture notes in statist. 40). Springer, Berlin.
[21] Linetsky V. Computing hitting time densities for CIR and OU diffusions: Applications to mean-reverting models. J. Comput. Finance 7 (1-22), 2004.
[22] Novikov A.A. (1971) On stopping times for a Wiener process. Theory Prob. Appl. 16, 458-465.
[23] Novikov A., Frishling V. and Korzakhia N. Approximations of boundary crossing probabilities for a Brownian motion. J. Appl. Probab., 36(4):1019-1030, 1999.
[24] Olver P.J. (1993) Applications of Lie Groups to Differential Equations, 2nd ed. Graduate Texts in Mathematics 107. Springer, New York.
[25] Pitman J., Yor M. (1981) Bessel processes and infinitely divisible laws. Stochastic Integrals (Proc. Sympos., Univ. Durham, Durham, 1980). Lecture Notes in Math. 851 285-370. Springer, Berlin.
[26] Peskir G. (2002) On the integral equations arising in the first passage problem for Brownian motion. Journal Integral Equations. Appl., 397-423.
[27] Rogers L.C.G. and Pitman J. 1981 Markov functions, Ann. Probab. 9 no 4 573-582.
[28] Salminen P. On the first hitting time and the last exit time for a Brownian motion to/from a moving boundary. Adv. in Appl. Probab. 20(2):411-426, 1988.
[29] Shepp L.A. (1967) A first passage problem for the Wiener process. Ann. Math. Statist. 38, 1912-1914.