Discontinuity gravity modes in hybrid stars: assessing the role of rapid and slow phase conversions

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Discontinuity gravity modes may arise in perturbed quark-hadron hybrid stars when a sharp density jump exists in the stellar interior and are a potential fingerprint to infer the existence of quark matter cores in compact objects. When a hybrid star is perturbed, conversion reactions may occur at the quark-hadron interface and may have a key role in global stellar properties such as the dynamic stability and the quasi-normal mode spectrum. In this work we study the role of the conversion rate at the interface. To this end, we first derive the junction conditions that hold at the sharp interface of a non-radially perturbed hybrid star in the case of slow and rapid conversions. Then, we analyse the discontinuity $g$-mode in both cases. For rapid conversions, the discontinuity $g$-mode has zero frequency because a displaced fluid element near the phase splitting surface adjusts almost immediately its composition to its surroundings and gravity cannot provide a buoyancy force. For slow conversions, a $g$-mode exists and its properties are analysed here using modern hadronic and quark equations of state. Moreover, it has been shown recently that in the case of slow conversions an extended branch of stable hybrid configurations arises for which $\partial M/\partial \epsilon_c < 0$. We show that $g$-modes of the standard branch (that is, the one with $\partial M/\partial \epsilon_c > 0$) have frequencies and damping times in agreement with previous results in the literature. However, $g$-modes of the extended branch have significantly larger frequencies (in the range $1 – 2\, \text{kHz}$) and much shorter damping times (few seconds in some cases). We discuss the detectability of $g$-mode GWs with present and planned GW observatories.

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I. INTRODUCTION

Since their discovery more than 50 years ago, neutron stars (NS) have attracted much attention because of their extreme physical properties. Such interest has been highly boosted recently by the direct detection of gravitational waves (GWs) from the NS merger GW170817 [1] and its electromagnetic counterparts GRB170817A and AT2017gfo [2] which imposed a new set of observational constraints on some key properties of these objects [3–7].

It is known that NSs contain matter under extreme conditions, but the exact nature of their deep interiors is still one of the key unsolved issues in the area. It is therefore very important to identify astrophysical signatures that can be unequivocally associated with specific internal aspect of NSs. For instance, if a sharp density discontinuity is present inside a NSs, it has been proposed that the so-called gravity pulsation mode (hereafter $g$-mode) would arise when the star is perturbed (see Refs. [8–10]). This is very relevant to understand the interiors of the so-called hybrid stars, composed by a quark matter core and external hadronic layers. In fact, since the $g$-mode of perturbed quark-hadron hybrid stars is expected to emit GWs in the ballpark of 0.5 kHz [11–14] this could work as a smoking gun to infer the presence of quark matter inside some nearby compact objects.

In this work, we will investigate $g$-modes of quark-hadron hybrid stars assuming that the quark matter core is separated from the external hadronic layers of the star by a sharp interface with a density jump across which thermal, mechanical and chemical equilibrium is maintained. The assumption of a sharp interface is expected to be correct if charge screening and surface effects are high enough and inhibit the formation of a mixed phase of quarks and hadrons (see [15–19] and references therein).

A key aspect of a sharp interface that deserves a detailed investigation is its behaviour under small perturbations. When a hybrid star is perturbed, fluid elements all along the stellar interior are displaced from their equilibrium positions. In particular, fluid elements in the neighbourhood of the quark-hadron interface can be periodically compressed and rarified and their pressures may become higher or lower than the phase transition pressure, making possible a phase conversion. However, the quark-hadron conversion in a compact star involves a quite complex mechanism where strong interactions, Coulomb screening, etc. play a significant role (see [19] and references therein); thus, a fluid element oscillating around the interface will not necessarily undergo a phase conversion. In fact, the probability that such conversion occurs depends on the nucleation timescale, which at present is a model dependent quantity with an uncertain value (see [19–21] and references therein).

In recent works [22, 23] we analysed the stability of hybrid stars under small radial perturbations, focusing on two limiting cases: slow and rapid phase conversions at the sharp interface. Slow conversions involve the stretch and squashing of volume elements near the quark-hadron interface without their change of nature (nucleation timescales much larger than those of perturbations). Rapid conversions imply a practically imme-
ate conversion of volume elements from one phase to the other and vice-versa in the vicinity of the discontinuity upon any perturbation [22, 24]. One of the main conclusions emerging from the analysis is that the usual static stability condition $\partial M / \partial \epsilon_r \geq 0$, where $\epsilon_r$ is the central density of a star whose total mass is $M$, always remains true if phase conversions are rapid but breaks down in general if they are slow. As a consequence, an additional branch of stable HS configurations is possible in the case of slow phase conversions [22, 23].

The main goal of the present work is to study the role of slow and rapid phase conversions on discontinuity $g$-modes of zero-temperature hybrid stars. The paper is organised as follows. In Section II we present the relativistic structure equations, we review the role of slow and rapid phase conversions on the stability of static configurations under small radial perturbations, and present the non-radial oscillation equations that will be employed in this work. In Section III we derive the junction conditions that hold at the sharp interface of a hybrid star when non-radial oscillations occur. In Section IV we analyse the physical mechanism that suppresses the existence of discontinuity $g$-modes when phase conversions at the interface are rapid. In Section V we calculate the properties of $g$-modes in the case of slow phase conversions using specific equations of state (EOS) for hadronic and quark matter. Special attention in given to $g$-modes of objects located at the extended branch of hybrid stars. Finally, in Section VI we summarise our results and explore some of their astrophysical consequences.

II. BASIC EQUATIONS

A. Equilibrium configurations of hybrid stars

In order to calculate the oscillation modes of a compact star, its equilibrium structure must be determined first. In the following, we assume that compact objects are made up of layers of a perfect fluid, whose stress-energy tensor is $T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu + pg_{\mu\nu}$, where $u^\mu$ is the fluid’s four-velocity and $g_{\mu\nu}$ is the metric tensor. The general static spherically symmetric line element which describes the geometry of a star can be written as $ds^2 = -c^2\nu(r)dt^2 + e^{2\lambda(r)}dr^2 + r^2(\Theta^2 + \sin^2 \Theta d\phi^2)$ where $\nu(r)$ and $\lambda(r)$ are metric functions with respect to $r$. A mass function $M(r)$ is defined as $M(r) = r(1-e^{-2\lambda})/2$. Solving Einstein’s equations, we obtain the stellar structure equations (Tolman-Oppenheimer-Volkoff (TOV) equations [25])

$$\frac{dM}{dr} = 4\pi r^2 \epsilon, \quad (1)$$
$$\frac{dp}{dr} = -\frac{(\epsilon + p)(M + 4\pi r^3 p)}{r(r - 2M)}, \quad (2)$$
$$\frac{d\nu}{dr} = -\frac{1}{r} \frac{dp}{dr}. \quad (3)$$

where $p(r)$, $\epsilon(r)$, and $M(r)$, are the pressure, the energy density, and the gravitational mass, respectively, as measured in a proper frame of reference.

The boundary conditions for an equilibrium stellar model are

$$M(r = 0) = 0, \quad (4)$$
$$p(r = R) = 0, \quad (5)$$
$$\nu(r = R) = \frac{1}{2} \ln \left(1 - 2M(R)/R\right). \quad (6)$$

where $R$ and $M(R)$ are the star radius and total mass, respectively.

The above equations must be supplemented with an EOS which for cold catalyzed matter has the form $p = \rho(c)$. Moreover, since we focus on quark-hadron hybrid stars with a sharp density discontinuity in thermal, chemical and mechanical equilibrium, the EOS has the generic form:

$$\epsilon(p) = \begin{cases} 
\epsilon_H(p) & p < p_t \\
\epsilon_Q(p) & p > p_t 
\end{cases} \quad (7)$$

where the subscript $H$ refers to hadrons, the subscript $Q$ to quarks, and $p_t$ is the pressure at the density discontinuity (transition pressure). Chemical equilibrium at the interface is guaranteed by the condition

$$g_H(p_t) = g_Q(p_t), \quad (8)$$

where $g$ is the Gibbs free energy per baryon defined as $g = (\sum_i \mu_i n_i)/n_B$, being $\mu_i$ the chemical potential of particle species $i$, $n_i$ their number density, and $n_B = \frac{1}{3} \sum_i n_i$ the baryon number density of each phase.

B. Conversion speed at the interface and the dynamical stability of hybrid stars

The stability of an equilibrium stellar configuration can be analysed by its response to small radial perturbations [26]. When a stable configuration is perturbed, fluid elements all along the stellar interior oscillate around their equilibrium positions, compressing and expanding periodically. On the contrary, in the case of an unstable configuration, small perturbations grow without limit, leading to the collapse or disruption of the star. In the case of hybrid stars, special care must be taken with fluid elements close to the quark-hadron interface, because, as the fluid oscillates, their pressures become alternatively higher and lower than the phase transition pressure $p_t$. Such compression and decompression around the interface leads to two essentially different behaviours depending on the speed of the quark-hadron nucleation mechanism (which depends on many poorly known microphysical details). If the nucleation timescale is much shorter than those of perturbations (rapid conversions) fluid elements will convert almost immediately from one phase to the other as their pressure go alternatively beyond and below $p_t$. On the contrary, if the nucleation timescale
is much larger than those of perturbations (slow con-
versions) the motion around the interface involves only the
stretch and squash of volume elements without any phase
transition.

In a recent work [22], we have shown that in spite of
being physically complex, the nature of the conversion
can be mathematically summarised into simple
junction conditions on the radial fluid displacement χ and the
corresponding Lagrangian perturbation of the pressure Δp
at the phase-splitting surface.

For slow conversions, the jump of χ and Δp across the
interface should always be null:

\[ [\chi^+] = \chi^+ - \chi^- = 0, \quad [\Delta p^+] = \Delta p^+ - \Delta p^- = 0. \quad (9) \]

In the latter equation and in the rest of this work,
\([x]^+ = x^+ - x^-\) indicates the jump of quantity \(x\) across
the interface, being \(x^+\) the value of \(x\) just above the
interface (i.e. at the hadronic side) and \(x^-\) the value just
below it (i.e. at the quark side).

For rapid phase transitions it was found that

\[ [\chi']^+ = \Delta p \left( \frac{1}{p_0} \right)^+, \quad [\Delta p]^+ = 0. \quad (10) \]

where \(p_0' \equiv dp_0/dr\) is the pressure gradient of the
background pressure at the interface.

The formalism of small radial perturbations of spheri-
cally symmetric stars [26] shows that, if the fundamental
oscillation frequency is a real number (\(\omega_0^2 > 0\)), then
any radial perturbation of the star will produce oscillatory
fluid displacements and the stellar configuration is
dynamically stable. However, in many cases, an equiva-
lent and much simpler static condition can be derived from
the latter one. In fact, it is widely known that for a
hydrostatic configuration of a spherically symmetric
cold-catalysed one-phase star it holds

\[ \partial M/\partial \epsilon_c < 0 \quad \Rightarrow \quad \text{unstable configuration}, \quad (11) \]

where \(\epsilon_c\) is the central density of a star whose total mass
is \(M\) [27]. However, Eq. (11) is not necessarily true
if matter is non-catalysed [28] or if the system contains
multiple phases separated by rapid density discontinu-
ities [22, 23, 29].

In particular, we have shown in [22, 23], that the stan-
dard stability criterion of Eq. (11) remains always true
for rapid phase transitions but breaks down in general for
slow phase transitions. In fact, for slow transitions the
frequency of the fundamental mode can be a real num-
ber (indicating stability) even for some branches of stellar
models that verify \(\partial M/\partial \epsilon_c < 0\). Thus, in the case of slow
conversions, new branches of stable stellar configurations
could arise.

C. Non-radial oscillations

In this work we are interested in the influence of some
microphysical properties of matter on the GWs emitted
by a compact object; therefore, we consider only even-
parity perturbations, which are coupled to the fluid.
Following the formalism of Detweiler and Lindblom [30],
the perturbed metric can be written as

\[ ds^2 = -e^{2\nu}(1 + r^l H_0 Y_m^l e^{i\omega t}) dt^2 \\
- 2i\omega r^{l+1} H_1 Y_m^l e^{i\omega t} drd\theta \\
+ e^{2\lambda}(1 - r^l H_0 Y_m^l e^{i\omega t}) dr^2 \\
+ r^2 (1 - r^l KY_m^l e^{i\omega t}) (d\theta^2 + \sin^2\theta d\phi^2). \quad (12) \]

The small amplitude motion of the perturbed configu-
ration is described by the Lagrangian 3-vector fluid dis-
placement \(\xi\), which can be represented in terms of per-
urbation functions \(W(r)\) and \(V(r)\) as

\[ \xi^r = r^{-1}e^{-\lambda}W(r)Y_m^l e^{i\omega t}, \quad (13) \]
\[ \xi^\theta = -r^{-1}V(r)\partial Y_m^l e^{i\omega t}, \quad (14) \]
\[ \xi^\phi = -r^l(r\sin\theta)^{-1}V(r)\partial Y_m^l e^{i\omega t}. \quad (15) \]

Thorne and Campolattaro [31, 32] showed that the five
perturbation functions \(H_0, H_1, K, V,\) and \(W\) are not all
independent. They are related by

\[ [\text{3M} + \frac{1}{2}(l + 2)(l - 1)r + 4\pi e^p] H_0 = 8\pi r^2 e^{-\nu}X \\
- [\frac{1}{2}(l + 1)(M + 4\pi e^p) - \omega^2 r^2 e^{-2(\lambda + \nu)}] H_1 \\
+ [\frac{3}{2}(l + 2)(l - 1)r - \omega^2 r^2 e^{-2\nu} \\
- r^{-1}e^{2\lambda}(M + 4\pi e^p)(3M - r + 4\pi e^p)] K. \quad (16) \]

Later, Lindblom and Detweiler [30, 33] introduced a
new function \(X(r)\) to replace \(V(r)\). The perturbation
equations can be written in terms of \(X(r)\) as

\[ K_{rr} = r^{-1}H_0 + \frac{1}{2}(l + 1)r^{-1}H_1 - \frac{1}{2}(l + 1)r^{-1} - \nu, \quad (17) \]
\[ H_{1,rr} = -r^{-1}[l + 1 + 2M e^{2\lambda}r^{-1} + 4\pi e^2 e^{2\lambda}(p - \epsilon)]H_1 + r^{-1}e^{2\lambda}[H_0 + K - 16\pi (\epsilon + p)V], \quad (18) \]
\[ W_{rr} = -(l + 1)r^{-1}W + re^{\lambda}[(\gamma p)^{-1} - e^{-\nu}X - l(l + 1)r^{-2}V + \frac{1}{2}H_0 + K], \quad (19) \]
\[ X_{rr} = -l^{-1}r^4X + (\epsilon + p)\nu^r\{\frac{1}{2}(r^{-1} - \nu, \nu)H_0 + \frac{1}{2}[\nu^p e^{2\nu} + \frac{1}{2}(r^{-1} - \nu)H_1 + \frac{1}{2}(3\nu^p - r^{-1})K \\
-l(l + 1)\nu^p - r^{-2}V - r^{-1}[4\pi (\epsilon + p)e^{\nu} + \omega^2 e^{\lambda - 2\nu} - r^2(r^{-1} - \nu^p)]X\}. \quad (20) \]
The adiabatic index, $\gamma$, is defined by
\[ \gamma = \frac{(\epsilon + p)}{p} \Delta p \Delta e, \tag{21} \]
and we can use Eq. (16) to eliminate $H_0$. Furthermore, $X(r)$ is related to $W(r)$ and $V(r)$ by
\[ X = \omega^2 (\epsilon + p) e^{-\nu} V - r^{-1} p_r e^{(\nu-\lambda)} W + \frac{1}{2} (\epsilon + p) e^\nu H_0. \tag{22} \]
Equations (17)–(20) form a fourth-order system of linear differential equations, and for each appropriate $l$ and $\omega$ there exist four linearly independent solutions.

The boundary conditions to be satisfied are: the perturbation functions must be finite everywhere, especially at $r=0$ where the system becomes singular; and the perturbed pressure must vanish at the surface of the star $r = R$ at any time, implying $\Delta p(r = R) = 0$. We can write [34]
\[ \Delta p = -r^l e^{-\nu} X, \tag{23} \]
therefore $\Delta p(r = R) = 0$ implies $X(r = R) = 0$. For a given set of $l$ and $\omega$, there is only one solution which satisfies all of the boundary conditions.

To numerically solve the equations, we expand the solutions at $r = 0$ and $r = R$ as suggested by Lindblom and Detweiler [30, 33] and corrected by L and Suen [35]. Concerning the surface of the star, we do not employ a polytropic atmosphere as in some previous works, i.e. we simply adopt $X = 0$ at the point where the pressure is effectively zero (we compared our calculations with previous works that considered polytropic atmospheres and we obtained the same results). Notice that until now we have discussed only the boundary conditions for single-phase stars; for hybrid stars we will dedicate a special section.

In order to connect NS pulsations with GWs detected on terrestrial laboratories, we need to know how the oscillation propagates until it reaches a distant observer. In general, outside the star the perturbed metric describes a combination of outgoing and incoming GWs; however, we are particularly interested in purely outgoing radiation, representing the quasi-normal modes (QNMs) of the stellar model.

Outside the star the fluid quantities vanish and the perturbation equations reduce to the Zerilli equation [36–38]
\[ \frac{d^2 Z}{dr^*} + [\omega^2 - V(r^*)] Z = 0, \tag{24} \]
where the effective potential $V(r^*)$ is given by
\[ V(r^*) = \frac{2(1 - 2M/r)}{r^3(nr + 3M)^2} \left[ n^2(n + 1)r^3 + 3n^2Mr^2 + 9nMr^2 + 9M^2 \right], \tag{25} \]
and $r^*$ is the “tortoise” coordinate, which can be written in terms of $r$ as
\[ r^* = r + 2M \log \left( \frac{r}{2M} - 1 \right), \tag{26} \]
with $n = (l - 1)(l + 2)/2$.

In terms of $H_0(r)$ and $K(r)$, the Zerilli function $Z(r^*)$ and its derivative are
\[ Z(r^*) = \frac{k(r)K(r) - a(r)H_0(r) - b(r)K(r)}{k(r)g(r) - h(r)}, \tag{27} \]
\[ \frac{dZ(r^*)}{dr^*} = \frac{h(r)K(r) - a(r)g(r)H_0(r) - b(r)g(r)K(r)}{h(r) - k(r)g(r)}, \tag{28} \]
where [35]
\[ a(r) = -(nr + 3M)/[\omega^2r^2 - (n + 1)M/r], \tag{29} \]
\[ b(r) = \frac{nrM(n - 2M)}{(r - 2M)[\omega^2r^2 - (n + 1)M/r]}, \tag{30} \]
\[ g(r) = \frac{n(n + 1)^2 + 3nM^2}{r^2(nr + 3M)}, \tag{31} \]
\[ h(r) = \frac{(-nr^2 + 3nMr + 3M^2)}{(r - 2M)(nr + 3M)}, \tag{32} \]
\[ k(r) = -r^2/(r - 2M). \tag{33} \]

The Zerilli equation has two linearly independent solutions $Z_+(r^*)$ and $Z_-(r^*)$. They correspond to incoming and outgoing GWs respectively and the general solution for $Z(r^*)$ is given by their linear combination
\[ Z(r^*) = A(\omega)Z_-(r^*) + B(\omega)Z_+(r^*). \tag{34} \]

At large radius, one can expand $Z_+$ and $Z_-$ as
\[ Z_-(r^*) = e^{-i\omega r^*} \sum_{j=0}^{\infty} \beta_j r^{-j}, \tag{35} \]
\[ Z_+(r^*) = e^{i\omega r^*} \sum_{j=0}^{\infty} \beta_j r^{-j}, \tag{36} \]
where $\overline{\beta}_j$ is the complex conjugate of $\beta_j$. Replacing Eq. (35) (keeping terms to $j = 2$) into Eq. (24), one obtains [35]
\[ \beta_1 = -i(n + 1)\omega^{-1}\beta_0, \tag{37} \]
\[ \beta_2 = -\omega^2 \left[ \frac{1}{2}n(n + 1) - \frac{3}{2}M\omega \left( 1 + \frac{2}{n} \right) \right] \beta_0. \tag{38} \]

III. OSCILLATING HYBRID STARS: JUNCTION CONDITIONS AT THE INTERFACE

In recent papers [22, 29], we derived the junction conditions at the interface of a radially perturbed hybrid star in the presence of slow and rapid phase conversions. In this section, we derive the junction conditions that hold at the sharp splitting surface of a hybrid star when the object is perturbed non-radially. We treat separately slow (Sec. IIIA) and rapid (Sec. IIIB) phase conversions at the interface.
A. Slow transitions

When a hybrid star is perturbed, fluid elements in the neighbourhood of the sharp quark-hadron interface can be radially displaced and their pressures may become higher or lower than the phase transition pressure. However, a fluid element oscillating around the interface will not necessarily undergo a phase conversion. In fact, if the timescale of the process transforming one phase into another is much larger than the oscillation period (slow transitions), volume elements near the interface will simply comove with the splitting surface without changing their nature. In such a case there is no mass transfer across the interface. Since it is always possible to track down the elements near the splitting surface, then $\xi^r$ must be continuous across the interface, i.e. $[\xi^r]^+ = \xi^{r+} - \xi^{r-} = 0$. Since $\lambda$ and $r$ are continuous across the surface, we obtain from Eq. (13) the first junction condition for slow transitions:

$$[W]^+ = 0.$$  

Additionally, when a hybrid star oscillates, the pressure on one side of the phase discontinuity keeps always the same value as on the other side, even if such value is different from the equilibrium one. As a consequence, the Lagrangian change of the pressure must be continuous across the interface. Therefore, the second junction condition for slow transitions reads:

$$[\Delta p]^+ = 0.$$  

Notice that these junction conditions have already been used in several previous works (see e.g. [11, 13, 34] and references therein) but without examining the role of the conversion speed at the interface.

B. Rapid transitions

Rapid phase transitions happen when the characteristic timescale of the process transforming one phase into the other is much smaller than the timescale of the perturbations. As a limiting case we consider that a volume element near the phase-splitting boundary $\Sigma$ is converted instantaneously from one phase to another when, due to perturbations, its pressure changes alternatively below and above the transition pressure $p_t$. Since conversion rates are very fast, the pressure at the surface $\Sigma$ is always the same as for the unperturbed configuration, i.e. $[p]^+ = 0$ and, therefore,

$$[\Delta p]^+ = 0$$  

across the interface $\Sigma$.

As done in [22], we will use only physical considerations to deduce the appropriate boundary condition for $\xi^r$ at $\Sigma$. We will demand that $\Sigma$ is well-localized, i.e., $[r_2]^+ = 0$, where the $r_2^+$ is the radial position of the phase-splitting surface with respect to the radial coordinates above and below it, respectively. In equilibrium $\Sigma$ is at the position $r_2^+ = R_0$. In the perturbed configuration, we should generically have $r_2^+ = R_0 + A(r_2^+, \theta, \phi, t)$, where $A^\pm \equiv A(r_2^\pm, \theta, \phi, t)$ are unknowns and of the order of $\xi^r$.

Furthermore, the transition pressure is the equilibrium one, so at $r = r_2^+$ we have:

$$p(r_2^+; \theta, \phi, t) = p_0(R_0).$$  

On the left-hand side of Eq. (42) we can use the definition of the Lagrangian displacement of the pressure $p(r, \theta, \phi, t) = p_0(r) + \Delta p(r, \theta, \phi, t) - \xi^r p_0$, where $p_0(r)$ stands for the pressure at $r$ in the unperturbed configuration. Thus, we can write:

$$p(r_2^+; \theta, \phi, t) = p_0(r_2^+) + \Delta p(r_2^+, \theta, \phi, t) - \xi^r(r_2^+, \theta, \phi, t)p_0(r_2^+).$$  

Additionally, we can expand in series the quantity $p_0(r_2^+)$ on the righthand side of the latter equation:

$$p_0(r_2^+) \simeq p_0(R_0) + p_0(r_2^+) A(r_2^+, \theta, \phi, t).$$  

Replacing Eqs. (43) and (44) into Eq. (42), it follows that:

$$p_0(R_0) + p_0(r_2^+) A(r_2^+, \theta, \phi, t) + \Delta p(r_2^+, \theta, \phi, t) - \xi^r(r_2^+, \theta, \phi, t)p_0(r_2^+) = p_0(R_0).$$

In this equation, we eliminate $p_0(R_0)$, we use Eq. (13), we write $A$ and $\Delta p$ in terms of spherical harmonics, and we find:

$$p_0(r_2^+) A(r_2^+, \theta, \phi, \omega) e^{i\omega t} + \Delta p(r_2^+) Y_{lm} Y_{lm}^* e^{i\omega t} - r_2^{l-1} e^{-\lambda} W(r_2^+) Y_{lm} Y_{lm}^* e^{i\omega t} p_0(r_2^+) = 0.$$  

Simplifying, we obtain

$$A(r_2^+) = -\frac{\Delta p(r_2^+)}{p_0(r_2^+)} + r_2^{-l-1} e^{-\lambda} W(r_2^+).$$  

Now, from the condition $[r_2]^+ \equiv r_2^+ - R_2 = 0$, we have $[A(r)]^+ \equiv A(r_2^+) - A(r_2^-) = 0$, and then Eq. (47) reads:

$$[W(r)]^+ = r_2^{-l+1} e^{\lambda} \left[\frac{\Delta p(r)^+}{p_0}\right]^-.$$  

Equations (48) and (41) are the junction conditions at the quark-hadron interface for rapid phase transitions. Notice the similarity of the radial case [22] with the non-radial one. This occurs because of the freedom in writing the angular dependence through spherical harmonics.

Another form of the latter junction condition can be found replacing Eqs. (2), (23) and (22) into Eq. (48) and taking into account that $H_0$ is continuous through the interface in view of the metric continuity:

$$[V(r)]^+ = 0.$$  

IV. NONEXISTENCE OF DISCONTINUITY GRAVITY MODES IN HYBRID STARS WITH RAPID PHASE CONVERSIONS

Gravity modes are a consequence of buoyancy in a gravitational field and are intrinsically related with convective instabilities in stars. When a fluid element undergoes a small radial displacement outward, the star’s gravity provides a force to restore the displaced element to its original location if the displaced element’s density is greater than that of the unperturbed fluid in the surroundings. When the displaced fluid element is of equal or lower density than the unperturbed fluid, gravity provides either no force (marginal stability) or a force to increase the displacement (instability to convection) \[9\]. A similar analysis is valid for a fluid element undergoing a small radial displacement downward.

To first order in small quantities, the relativistic buoyancy force per unit volume acting on a fluid element displaced a small radial distance \(\delta r\) is

\[
f \equiv g(\epsilon + p)Ae^\lambda \delta r
\]

where \(-g\) is the gravitational acceleration in the radial direction measured by a stationary observer at \(r\), \(\gamma_0 = \frac{(\epsilon + p)}{\rho} \frac{dp}{d\rho}\) is the adiabatic index in the unperturbed configuration and \(\gamma\) is given in Eq. (21). The quantity \(A\) is the relativistic convective stability discriminant defined by

\[
A \equiv e^{-\lambda} \frac{dp}{d\rho} \left( \frac{1}{\gamma_0} - \frac{1}{\gamma} \right).
\]

At the stellar layers where \(A < 0\), the local buoyancy force is restoring and the star is stable against convection. Conversely, the star is neutrally stable where \(A = 0\) and unstable where \(A > 0\). The buoyancy force density \(f\) causes local fluid oscillations that are characterised by the relativistic Brunt-Visl frequency \(N^2 = -Ag\), which specifies the locally measured frequency with which a fluid element oscillates around its equilibrium position. Different stability regimes can be identified by looking at the sign of \(N^2\) \[9\].

A zero temperature hybrid star is chemically homogeneous everywhere except at the quark-hadron interface where the density changes discontinuously. Thus, a buoyancy force (and a \(g\)-mode) can only be expected at the interface, where we could have \(\gamma \neq \gamma_0\). In fact, this is the case when conversions at the interface are slow: the adiabatic index \(\gamma_0\) governing the pressure-density relation is zero at the discontinuity, the adiabatic index \(\gamma\) governing the perturbations remains finite there, \(A\) is different from zero, and a \(g\)-mode arises. However, in the case of rapid conversions, a displaced fluid element adjusts almost immediately its composition to its surroundings when it is pushed to the other side of the discontinuity. In this case, \(\gamma_0\) is still zero at the interface, and so is \(\gamma\) since complete thermodynamic equilibrium is maintained at all times with the unperturbed fluid. Therefore, we have \(\gamma = \gamma_0 = 0\), which implies \(A = 0\), and there is no restoring force. In other words, since the displaced fluid element adjusts immediately its thermodynamic state to its surroundings, its density will be always the same as in the unperturbed fluid and gravity cannot provide a restoring force. As a consequence, the discontinuity \(g\)-mode has zero frequency for rapid transitions.

V. DISCONTINUITY GRAVITY MODES IN THE CASE OF SLOW PHASE CONVERSIONS

In this section, we study the \(g\)-mode arising from a quark-hadron phase discontinuity when phase conversions are slow. We first describe the EOSs adopted for both phases and then present the results of our calculations.

A. Equations of State

1. Hadronic Matter

For hadronic matter we use an EOS based on nuclear interactions derived from chiral effective field theory (EFT), combined with constrains arising from the recent observation of high mass pulsars. In recent years, the development of chiral EFT has provided the framework for a systematic expansion for nuclear forces at low momenta allowing to constrain the properties of neutron-rich matter up to nuclear saturation density to a high degree. However, our knowledge of the EOS at densities greater than one to two times the saturation density is still insufficient due to limitations on both laboratory experiments and theoretical methods. Fortunately, the recent detection of very massive pulsars \[39-41\] with \(\approx 2M_{\odot}\) puts stringent constraints on the nuclear EOS at supranuclear densities. Moreover, with the advent of GW observations of binary neutron star mergers \[1, 2, 42\] even tighter constraints are expected for the near future.

The EOS at subnuclear densities can be extended in a general way to higher densities using piecewise polytropic EOSs and requiring non-violation of causality and consistency with the observation of \(2M_{\odot}\) pulsars. In Ref. \[43\], hadronic matter at densities above \(\rho_1 = 1.1\rho_0\) \((\rho_0 = 2.7 \times 10^{14}\text{g/cm}^3)\) is described by a set of three polytropes which are valid, respectively, in three consecutive density regions. This general polytropic extension leads to a very large number of EOSs, which verify the physical and observational constraints mentioned above. For use in astrophysical simulations, Ref. \[43\] provides detailed numerical tables for three representative EOS labeled as soft, intermediate and stiff. In order to ensure that our hybrid configurations verify the \(2M_{\odot}\) constrain we adopt only the intermediate and stiff parametrizations of Ref. \[43\]. Below \(\rho_{\text{crust}} = \rho_0/2\) we use the Baym, Pethick and Sutherland EOS \[44\]. For more details, see Ref. \[43\].
2. Quark Matter

For quark matter we consider a generic MIT bag model which is defined by the following grand thermodynamic potential \[\Omega = -\frac{3}{4\pi^2}a_4\mu^4 + \frac{3}{4\pi^2}a_2\mu^2 + B,\] (52)

where \(\mu = (\mu_u + \mu_d + \mu_s)/3\) is the quark chemical potential and \(a_4, a_2, B\) are free parameters independent of \(\mu\). Since the quark matter EOS is used here essentially in the high density regime, we have neglected the electron contribution (see discussion in Ref. [22]).

The above phenomenological model is interesting because it allows exploring several aspects of dense quark matter. The influence of strong interactions on the pressure of the free-quark Fermi sea is roughly taken into account by the parameter \(a_4\), where \(0 \leq a_4 \leq 1\), and \(a_4 = 1\) indicates no correction to the ideal gas [45]. The standard MIT bag model is obtained for \(a_4 = 1\) and \(a_2 = m_s^2\), being \(m_s\) the mass of the strange quark. The effect of the color superconductivity phenomenon in the Color Flavor Locked (CFL) phase can be explored setting \(a_2 = m_s^2 - 4\Delta^2\), being \(\Delta\) the energy gap associated with quark pairing [22, 45]. The bag constant \(B\) is related to the confinement of quarks, representing in a phenomenological way the vacuum energy [46].

From Eq. (52), we can obtain all thermodynamic quantities, such as the pressure \(p = -\Omega\), the baryon number density:

\[n_B = -\frac{1}{3} \frac{\partial \Omega}{\partial \mu} = \frac{1}{\pi^2} a_4 \mu^3 - \frac{1}{2\pi^2} a_2 \mu,\] (53)

and the energy density

\[\epsilon = \Omega + 3\mu n_B = \frac{9}{4\pi^2} a_4 \mu^4 - \frac{3}{4\pi^2} a_2 \mu^2 + B.\] (54)

The chemical potential can be written as a function of pressure,

\[\mu^2 = \frac{1}{2} \left[\frac{a_2 + a_2 \sqrt{1 + \frac{16\pi^2 a_4}{a_2^2} (\epsilon - B)}}{3a_4}\right],\] (55)

which allows finding the EOS \(p = p(\epsilon)\):

\[p = \frac{\epsilon - 4B}{3} - \frac{a_2^2}{12\pi^2 a_4} \left[1 + \sqrt{1 + \frac{16\pi^2 a_4}{a_2^2} (\epsilon - B)}\right].\] (56)

Depending on the values of \(a_2, a_4\) and \(B\), either hybrid stars or strange stars may be described by this model. For more details see Ref. [22] and references therein.

3. Hybrid Matter

In order to describe hybrid stars, we combine the hadronic and the quark EOSs described above. As mentioned in Sec. II A, we assume that matter has a first order quark-hadron phase transition with a sharp density discontinuity at the pressure \(p_t\). Once the model parameters are chosen, the transition pressure \(p_t\) is found by requiring that the Gibbs free energy per baryon of both phases is the same at \(p_t\): \(g_H(p_t) = g_Q(p_t)\); see Eq. (8). The quark phase is energetically preferred for \(p > p_t\) and the hadronic phase for \(p < p_t\). We have chosen the EOS parameters in order to allow the existence of hybrid stars with \(M > 2M_\odot\). The choice of parameters employed in the present paper is presented in Table I.

### TABLE I. Combinations of EOS’ parameters adopted to construct hybrid stars models.

| Hybrid model | Hadronic EOS | Quark EOS |
|--------------|--------------|-----------|
| Hyb-S1       | Stiff        | 92.55     |
| Hyb-S2       | Stiff        | 74.28     |
| Hyb-S3       | Stiff        | 74.28     |
| Hyb-I1       | Intermediate | 92.55     |
| Hyb-I2       | Intermediate | 41.16     |

![FIG. 1](image_url)

Mass–radius relationship (a) and stellar mass as a function of the central density (b) for the hybrid models of Table I. In both panels, round dots indicate the maximum mass and triangular dots mark the last stable hybrid configuration for which the frequency of the fundamental radial oscillation mode vanishes in the case of slow phase conversions. Extended stable branches begin at round dots and end at triangular dots (only for slow conversions). The upper horizontal band on panel (a) corresponds to the observed mass of the pulsar PSR J0348+0432 and the lower horizontal band to PSR J1614-2230 [40, 41].

B. Results

We have shown in previous works [22] that, in the case of slow phase conversions, a new branch of stable hybrid configurations arises for which \(\partial M/\partial \epsilon < 0\). Such extended branch begins at the maximum mass configu-
allow the existence of twin objects, i.e. couples are not smooth.

g
frequency and the damping time of

is around 8%. For the Hyb-I1 model at 2

and the one with the same mass in the extended brach radius between the hybrid star in the standard branch and their g-mode frequencies are in the range $1.2 - 2$ kHz (see Fig. 2a) while damping times can be as short as some seconds (see Fig. 2b).

Since frequencies around 2 kHz are typical of the fundamental mode of NSs, it is important to compare systematically the frequencies of both f- and g-modes of our hybrid configurations. As seen in Fig. 3, within each model $f_1$ is always larger than $f_g$ of a NS with the same gravitational mass (as it must be). But for some models, e.g. Hyb-S1, the difference between $f_1$ and $f_g$ is small, which may make difficult their observational discrimination. However, since $\tau_g$ is several orders of magnitude larger than $\tau_1$ (see Fig. 4), both modes would be clearly differentiated if damping times were observed.

A brief comment on some numerical issues is in order. Our calculations have been done using the standard algorithm of Lindblom and Detweiler [30, 33]. As already emphasised by Finn [8–10], both the effort and the error involved in the integration of g-modes may become large with such method. In fact, since in many cases the imaginary part of the eigenfrequency is fractionally too small compared to the real part, a small fractional error in the real frequency can seriously affect the estimate of the damping time. To circumvent this difficulty, we have first calculated the frequency of g-modes using the Cowling approximation [47, 48] and have used these results as initial values for the full calculation. With such approach, we were able to determine $f_g$ with high precision but in some cases it was difficult to resolve numerically the value of $\tau_g$ with arbitrary precision. As a consequence, some of the curves shown in Figs. 2b and 4 are not smooth. As a byproduct of our calculations, and just for sake of comparison, we show in the Appendix the values of $f_g$ obtained within the Cowling approximation.

Now, let us focus on the detectability of the modes calculated in this work. It is possible to estimate the minimum energy that must be released through a mode in order to be detected by a given GW observatory according to the formula [49, 50]

$$
\frac{E_{GW}}{M_{\odot}c^2} = 3.47 \times 10^{36} \left( \frac{S}{N} \right)^2 \frac{1 + 4Q^2}{4Q^2} \left( \frac{D}{10\text{ kpc}} \right)^2 \left( \frac{f}{1\text{ kHz}} \right)^2 \left( \frac{S_n}{\text{Hz}^{-1}} \right),
$$

where $E_{GW}$ is the energy emitted in the form of GWs, $S/N$ is the signal-to-noise ratio, $Q = \pi f \tau$ is the quality factor, $D$ the distance to the source, $f$ the frequency, $\tau$
the damping time and $S_n$ the noise power spectral density of the detector.

We consider a detector with $S_n^{1/2} \sim 2 \times 10^{-23} \text{ Hz}^{-1/2}$ which is representative of the Advanced LIGO-Virgo at ~kHz [1], and another one with $S_n^{1/2} \sim 10^{-24} \text{ Hz}^{-1/2}$ which is illustrative of the planned third-generation ground-based Einstein Observatory at the same frequencies [51]. Taking $S/N = 8$ we calculated the minimum energy $E_{GW}$ that a NS must release through a mode in order to be detected at a distance $D \sim 10$ kpc (NS in our Galaxy) and $D \sim 15$ Mpc (NS at the Virgo cluster).

Our results are shown in Fig. 5 and show that $E_{GW}$ for $g$-modes is lower than for $f$-modes. However, in order to assess the relevance of each mode in GW emission, one must analyse several factors, being the amount of energy that can be stored in a given mode the most important. Furthermore, the amount of energy that can be channeled through GWs, depends on other dissipative processes that take energy away from the star, e.g. neutrino diffusion and viscosity (for the case of a newly born, hot star). Numerical simulations of extremely energetic processes, like core collapse to a NS or binary coalescence leading to NS formation, indicate that the $f$-mode is the most excited [52]. However, further work should be done regarding these astrophysical simulations in view of the possible existence of the new extended stable branch discussed in this work. Since $g$-modes of this new branch have a significantly larger frequency, one may wonder whether they could carry more energy than $g$-modes of the standard branch.

Even so, the results presented in Fig. 5 look promising. Following a catastrophic astrophysical event such as a supernova collapse, a binary coalescence or a conversion of a hadronic star into a hybrid star, one expects that a strongly pulsating compact star will be created (if the event doesn’t end with the formation of a black hole). Although it is yet uncertain how much energy will be radiated through the oscillation modes, one can reasonably expect that the energy stored in stellar pulsations is some fraction of the kinetic energy of the formation.

FIG. 3. Frequency of $f$- and $g$-modes for the models considered in this work. The asterisk-lines indicate the frequency of $g$-modes, while solid lines represent $f$-modes. The curves for the $f$-mode include results for both, hadronic and hybrid stars.

FIG. 4. Damping times $\tau$ of $f$- and $g$-modes. Notice that $\tau_f$ and $\tau_g$ differ by several orders of magnitude.
event. In the case of a typical core collapse supernova, the total released energy is $\sim 10^{53}$ ergs while the kinetic energy of mass ejecta is $\sim 10^{44}$ ergs. Thus, the observation of $g$-mode GWs from a Milky Way event looks feasible, since Fig. 5a shows that the minimum detectable energy is in the range $\sim 10^{47}$–$10^{48}$ ergs for Advanced LIGO-Virgo. The Einstein Telescope, with a threshold in the range $\sim 10^{44}$–$10^{45}$ ergs for Galactic $g$-mode GWs is much more encouraging. Giant flares of Soft Gamma Repeaters (SGR) may be another detectable source of GWs. In the magnetar model, SGRs are highly magnetised NSs with surface magnetic fields around $10^{15}$ G. During giant flares, up to $\sim 10^{47}$ ergs may be released in $\gamma$-rays as a consequence of a strong rearrangement of the magnetic field probably leading to crustal deformations and cracking with the potential excitation of non-radial pulsation modes. According to Fig. 5a a detection of a galactic SGR with Advanced LIGO-Virgo requires the energy released in $g$-mode GWs to be of the same order of the one released in $\gamma$-rays. In the case of the Einstein Observatory, the minimum required energy is $100$–$1000$ times smaller. For completeness, the curves for sources in the Virgo cluster of galaxies are shown in Fig. 5b.

VI. SUMMARY AND CONCLUSIONS

In this paper we investigated the role of slow and rapid phase conversions on non-radial quasi-normal modes of hybrid stars. To this end, we derived the junction conditions that hold at the sharp interface of a perturbed hybrid star in the case of slow conversions (Eqs. (39) and (40)) and rapid conversions (Eqs. (41) and (48)).

After that, we focused on the discontinuity $g$-mode because of its relevance as a fingerprint of a sharp quark-hadron interface at the compact star interior.

In Section IV we analysed the physical mechanism that suppresses the existence of discontinuity $g$-modes when phase conversions at the interface are rapid. In this case, a displaced fluid element near the phase splitting surface adjusts almost immediately its composition to its surroundings when it is pushed to the other side of the discontinuity. Since it is always in equilibrium with its environment, its density will be always the same as in the unperturbed fluid and gravity cannot provide a restoring force. In fact, the relativistic buoyancy force per unit volume acting on a displaced fluid element (see Eq. (50)) vanishes for rapid conversions because the adiabatic index $\gamma_0$ governing the pressure-density relation and the adiabatic index $\gamma$ governing the perturbations are both zero at the discontinuity. Therefore, the discontinuity $g$-mode has zero frequency if phase conversions are rapid.

In the case of slow conversions, a buoyancy force and a $g$-mode arise at the interface because the adiabatic index $\gamma$ governing the perturbations remains finite there. In Section VB, $g$-modes were analysed using the EOSs for hadronic and quark matter presented in Sec. VA. Concerning slow conversions, notice that we have shown in previous works [22, 23] that a new branch of stable hybrid configurations arises for which $\partial M/\partial \epsilon_c < 0$. Such extended branch begins at the maximum mass configuration and extends up to the terminal configuration at which the frequency of the fundamental radial oscillation mode vanishes. Our results show that $g$-modes of the standard branch (that is, with $\partial M/\partial \epsilon_c > 0$) have frequencies and damping times in agreement with previous results in the literature [12, 34], i.e. frequencies in the range $f \sim 0.5$–$1$ kHz and very long damping times. However, for $g$-modes of the extended branch we obtain significantly larger frequencies (in the range $1$–$2$ kHz) and much shorter damping times (few seconds in some cases).

Finally, we discussed the detectability of $g$-mode GWs with present and planned GW observatories. The minimum released energy in $g$-mode GWs for a source at a galactic distance (10 kpc) is in the range $\sim 10^{47}$–$10^{48}$ ergs for Advanced LIGO-Virgo and in the range $\sim 10^{44}$–$10^{45}$ ergs for the Einstein Telescope. These results suggest that the detection of $g$-mode GWs from nearby core collapse supernova, compact star mergers and even SGRs is feasible, and that $g$-modes are a promising tool for the search of sharp quark-hadron discontinuities at the deep interior of compact stars.

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for the investigation of relativistic stars [48]. It arises when one neglects all metric perturbations in the full equations of Sec. II C and it strongly simplifies the calculation of the frequency of quasi-normal modes [13]. In our calculations, we first obtained the frequency of $g$-modes using the Cowling approximation and then we used these results as initial trial values for the full calculation. For completeness, we show in Fig. 6 the ratio between the frequency $f_g$ calculated within the full formalism and the frequency $f_g$, Cowling obtained within the Cowling approximation. For lower masses the approximation tends to be reasonably good but for larger ones the difference can be as large as $\sim 10\%$.

Appendix: Accuracy of the relativistic Cowling approximation

The Cowling approximation was first developed for the study of Newtonian stars [47] and subsequently adapted

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