Halo structure, masses of dark objects and parallax microlensing

D. Marković
Theoretical Astrophysics Center, Juliane Maries Vej 30, DK-2100 Copenhagen Ø, Denmark

ABSTRACT
We study the use of parallax microlensing to separate the effects of the mass function of dark massive halo objects (MHOs or ‘machos’) on the one hand and their spatial distribution and kinematics on the other. This disentanglement is supposed to allow a much better determination of the two than could be achieved entirely on the basis of the durations of events. We restrict our treatment to the same class of power-law spherical models for the halo of MHOs studied in a previous paper (Markovic & Sommer-Larsen 1997). Whereas the duration-based error in the average MHO mass, \( \bar{\mu} \equiv \bar{M}/M_\odot \) exceeds (at \( N = 100 \) events) \( \bar{\mu} \) by a factor of 2 or more, parallax microlensing remarkably brings it down to 15-20% of \( \bar{\mu} \), regardless of the shape of the mass function. In addition, the slope \( \alpha \) of the mass function, \( dn/d\mu \propto \mu^\alpha \), can be inferred relatively accurately (\( \sigma_\alpha < 0.4 \)) for a broader range, \(-3 < \alpha < 0\). The improvement in the inference of the halo structure is also significant: the index \( \gamma \) of the density profile (\( \rho \sim R^{-\gamma} \)) can be obtained with the error \( \sigma_\gamma < 0.4 \) for a broader range, \(-3 < \gamma < 0\).

Key words: Microlensing – Galactic halo – Macho mass function.

1 INTRODUCTION AND OVERVIEW
A statistical analysis by Alcock et al. (1996) of the 2-year microlensing data (6 or 8 events) obtained by the MA-CHO project in the direction of the Large Magellanic Cloud (LMC) indicated that the massive dark halo objects (MHOs or ‘machos’) responsible for the microlensing events could account for 30-100% of the total mass in the halo of our Galaxy. According to their analysis, typical (average) mass of the MHOs should lie in the range \( 0.1 - 0.6 M_\odot \). The more recent 4-year data (14 events, Axelrod 1997) yield similar ranges of the inferred quantities. Apart from the statistical error due to the relatively small number of events, our ignorance regarding the structure of the halo of massive objects (i.e., their spatial distribution and kinematics) leads to rather large uncertainties in the inferred masses.

This last source of error is not likely to be extinguished if one relies only on the measurement of event durations \( T = R_E/v_n \) (\( R_E \) is the Einstein radius, \( v_n \) is the MHO’s velocity orthogonal to the line of sight). Indeed, as shown by Markovic & Sommer-Larsen (1997, paper I), for the number of events \( N < 1000 \) the halo structure cannot be constrained sufficiently to allow a determination of the average mass \( \bar{\mu} \equiv M/M_\odot \) of the MHOs to better than a factor of about 2. Furthermore, paper I discussed only a limited class of spherical haloes; the results of Evans (1997), based on a far wider variety of halo models, imply that the range of \( \bar{\mu} \) (at virtually arbitrary \( N \)) could in principle extend from 0.1 to 1.

The duration \( T \) is, however, not the only relevant quantity that can be obtained from a microlensing event. For instance, photometric (Gould 1994a; Nemiroff & Wickramasinghe 1994) or spectroscopic (Maoz & Gould 1994) methods have been proposed to measure the proper motion of the lens \( v_n/zD \), where \( D \) is the Earth-source distance, \( z \equiv D_L/D \), and \( D_L \) is the Earth-lense distance. Another approach [discussed by Grieger, Kayser & Refsdal (1986) in the context of quasar astronomy] is parallax microlensing, i.e., observing magnification through telescopes displaced from each other by about 1AU. More recently, Gould (1994b) studied and advocated the use of parallax microlensing to obtain more in-
formation regarding the position and velocities of the lenses (and consequently reduce the uncertainty of their masses).

The utility of the parallaxes stems primarily from the fact that the delay \( \tau \) between the maximal magnifications in the two detectors (one on the Earth and the other on a satellite in a heliocentric orbit) does not depend on the mass of the MHO crossing the two lines of sight to a source. Additional information is contained in the two maximal magnifications determined by the impact parameters \( u_1 \) and \( u_2 \) measured in units of the Einstein radius. Breaking a 4-fold degeneracy (Gould 1994b; see also Section 2 of this paper) by observing from a second satellite would allow us to obtain the so-called reduced transverse velocity \( \tilde{v} \equiv v_\alpha / (1 - z) \) of the MHO. A sufficiently large number of such measurements would then presumably put tight constraints on the structure of the halo. However, even in absence of a second satellite, the relative motion of the first satellite and the Earth could suffice to reduce the ambiguity to (at most) a 2-fold degeneracy in the direction of \( \tilde{v} \) for a majority of events (Gould 1995; Bouteux & Gould 1996).

[According to Han & Gould (1995) this should (in the case of galactic bulge microlensing) permit measurement of individual MHO masses to an accuracy of about 0.2 on the logarithmic scale.]

In this paper we explore quantitatively the extent to which one could expect parallax microlensing to help constrain the mass function of the MHOs as well as their spatial distribution and kinematics. The assumptions of the present paper are similar to those of paper I: for convenience we again limit ourselves to a class of spherical halo models (see section 3) described by a set of 5 parameters (the singular isothermal sphere is a particularly simple member of this class). On the other hand, the mass function is assumed to be a simple power law, \( dn / dp \propto \mu^\alpha \) characterized by three parameters (independent of the position in the halo): the average mass \( \mu \), slope \( \alpha \) and range \( \beta \) on the logarithmic scale.

Although lacking somewhat in generality, this framework will allow a straightforward application of the apparatus of statistical parameter estimation: the errors of maximum likelihood inference of the mass function and halo parameters can be estimated from the sensitivity of the distributions of directly measurable quantities to small shifts in the underlying parameters (see paper I and Section 5 of the present paper). We call such estimates the Cramer errors of inference. Finally, section 6 contains basic conclusions of this paper along with a bit of speculation regarding their more general validity.

2 DIFFERENTIAL PARALLAX MICROLENSING RATES

As observed from the Earth, a lens of mass \( M = \mu M_\odot \), crossing the Earth-source line of sight at distance \( zD \) (0 \( \leq z \leq 1 \) from the Earth and with the impact parameter \( u_1 R_E \),
\[
[\tilde{R}_E = r_E \sqrt{\alpha \gamma} (1 - z)] , r_E \equiv 2 \sqrt{GM_\odot D/c^2} \approx 3.2 \times 10^9 \text{ km}]
\]
will magnify the star by the maximum factor \( A_{\text{max}} = (u_1^2 + 2)/(u_1 \sqrt{u_1^2 + 4}) \). On the other hand, a satellite will detect maximum magnification, determined by the impact parameter \( u_2 R_E \) relative to the satellite-source line of sight, with the time shift \( \tau \) from the moment of the Earth-observed maximum magnification.

The geometric relations between the observable quantities, \( T \equiv R_E/v_o \) (event duration as observed by both the Earth-based observer and the satellite), \( \tau, u_1 \) and \( u_2 \), are easily derived from Fig. 1. If \( r \) is the component of the Earth-satellite vector orthogonal to the line of sight, its source-centered projection onto the plane (also orthogonal to the line of sight) is

\[
a = \frac{r(1 - z)}{R_E} = \frac{r}{R_E} \frac{1}{\sqrt{1 - z}} \sqrt{1 - \frac{1}{z}} \quad (1)
\]

(measured in units of the Einstein radius \( R_E \)).

The points '1' and '2' in Fig. 1 denote intersections of the Earth-source and satellite-source lines of sight respectively with the lens plane. The lens’ trajectory along the unit vector \( \mathbf{v} = (\cos \phi, \sin \phi) \) (again, projected on the lens plane), shown as the solid arrow, crosses the 1-2 line at distance \( u R_E \) from the parallel line (dotted) drawn through the midpoint between 1 and 2. Consequently,

\[
p = a \cos \phi = \frac{r(1 - z) \cos \phi}{v_o} \quad T = \frac{\tau}{T} \quad (2)
\]

is an observable quantity. On the other hand, \( w \equiv a \sin \phi \) generally cannot be obtained unambiguously from \( u_1 \) and \( u_2 \) only; the 4-fold degeneracy is illustrated in Fig. 2. A secure way of breaking this degeneracy would be to use a second satellite [the line tangent to three circles of radii \( u_1, u_2 \) and \( u_3 \) (the last measured from the second satellite) is unique]. However, as Gould (1995) has shown, the motion of the Earth and the single satellite relative to the line of sight should allow us to resolve the ambiguity — at least regarding the magnitude of \( w \) — in most cases. Nevertheless, in this paper we will discuss both the ‘degenerate’ (the 4-fold ambiguity unresolved) and the ‘resolved’ (\( w \) uniquely determined) parallax microlensing.

![Figure 1](image1.png)

**Figure 1.** A parallax microlensing event projected onto the lens plane (orthogonal to the line of sight). All distances are given in units of the Einstein radius \( R_E \).

![Figure 2](image2.png)

**Figure 2.** Four possibilities corresponding to a pair of values \( u_1 \geq 0, u_2 \geq 0 \) in the case of degenerate parallax microlensing.

The lens’ rate of crossing near the line of sight per single source and a single (number density near the Sun \( n_o = 1 \)) lens is

\[
\Gamma = \int d\mu \frac{d\nu_o}{du} \int DH(z) dz \int R_E(z) du \int f_o(\nu_o, \phi) \nu_o^2 dv_o d\phi,
\]

where \( H(z) [H(0) = 1] \) is the MHOs’ halo density profile along the line of sight, \( f_o \left[ \int f_o(\nu_o, \phi) \nu_o dv_o d\phi = 1 \right] \) is the \( z \)-dependent, 2-dimensional distribution of velocities projected orthogonal to the line of sight and \( \frac{d\nu_o}{du} \left[ \int (d\nu_o/du) d\mu = 1 \right] \) is the \( z \)-independent mass function of the MHOs.

Using

\[
z = \frac{(r/R_E)^2}{\mu a^2 + (r/R_E)^2},
\]

rewriting the lens plane area element

\[
d\mu dw = a a d\phi = \frac{1}{2} (\frac{dz}{da^2})^{-1} dz d\phi,
\]

and switching to integration over \( T \) (\( d\nu_o/duT = -v_o/T \)) we obtain

\[
\chi(T, p, w) = \frac{dT}{dt dw dw}
\]

\[
= \int d\mu \frac{d\nu_o}{du} DH(z)R_E(z) f_o(\nu_o, \phi) \nu_o^3 \frac{2(r/R_E)^2 \mu}{\mu a^2 + (r/R_E)^2},
\]

where \( \nu_o = r E \sqrt{\mu} \sqrt{z(1 - z)/T} \) and \( \tan \phi = w/p \).

By contrast with resolved parallaxes, where \( \chi(T, p, w) \) is of more immediate relevance, measuring \( u_1 = |u + w/2| \) and \( |u - w/2| \) (see Fig. 1) is not sufficient to uniquely determine \( w \) in the case of degenerate parallaxes. Inserting ‘dummy’ integration \( \int dw_1 \delta(u_1 - |u + w/2|) \int dw_2 \delta(u_2 - |u - w/2|) \) in the rate \( \chi \) and using the identity \( d(u + w/2) \wedge d(u - w/2) = \ldots \)
\begin{align*}
\Psi(T, p, u_1, u_2) & \equiv \frac{dT}{dT dp du_1 du_2} \\
& = \chi(T, p, u_1 + u_2) + \chi(T, p, -u_1 - u_2) \\
& + \chi(T, p, u_1 - u_2) + \chi(T, p, -u_1 + u_2),
\end{align*}

expressing the differential rate in terms of variables accessible to degenerate parallax microcosmic detection.

So far in this section we have ignored the question of the minimum amplification necessary for successful detection of a microcosmic event. In particular, microcosmic amplification produces sufficient magnification only in the Earth- (or satellite-) based detector, i.e., the magnification in the other detector (say ‘2’) could be too small for a reliable determination of \( \tau \) and \( u_2 \). To deal with this possibility, we will require \( u_1 < u_{th}, u_2 < u_{th} \) for a detectable parallax (double, i.e. in both detectors) microcosmic event, while single events will correspond to \( u_1 < u_{th}, u_2 > u_{th} \) or \( u_1 > u_{th}, u_2 < u_{th} \), where \( u_{th} \) is a certain threshold value. The detection rate of events (both single and double) in one (say ‘1’) detector is thus

\[ P(T, u_1) \equiv \frac{dT^{(1)}}{dT du_1} \]

\[ = \int_{0}^{u_{th}} du_2 \int_{-\infty}^{\infty} dp \Psi(T, p, u_1, u_2) \]

\[ = \Omega^{(1)}(T, u_1), \]

where

\[ \Omega^{(1)}(T, u_1) \equiv \frac{dT^{(1)}}{dT du_1} = \int_{0}^{u_{th}} du_2 \int_{-\infty}^{\infty} dp \Psi(T, p, u_1, u_2) \]

is the differential detection rate for single events, expressed in terms of the only available measurable, \( T \) and \( u_1 \). Of course, \( \Omega^{(1)}(T, u) = \Omega^{(2)}(T, u) = \Omega(T, u) \). The differential rate \( P(T, u) \), introduced in equation \( 8 \), is

\[ P(T, u) = 2Dr E \int dp \frac{dn_0}{d\mu} \left( \frac{\mu}{D} \right)^2 \int_{0}^{1} dz H(z) |z(1-z)|^2 \]

\[ \times \int_{0}^{2\pi} d\phi f_0(v_u, \phi). \]

If we assume (as we will in the present paper) that the MHO mass function can be well approximated by a simple power law

\[ \frac{dn_0}{d\mu} = \frac{1}{C_0(\alpha)} \mu_0^{\alpha}, \]

where \( \beta = \log_{10}(\mu_{\text{max}}/\mu_{\text{min}}) \), \( \mu_{\text{max}} \) and \( \mu_{\text{min}} \) are the upper and lower bounds of the mass range, \( \mu_0 = \sqrt{\mu_{\text{max}} \mu_{\text{min}}} \) and

\[ C_0(\alpha) = \begin{cases} 
\beta \ln 10 & \alpha = -1, \\
\frac{1}{10^\frac{\beta}{\alpha+1}} \left[ 10^{\beta(\alpha+1)/2} - 10^{-\beta(\alpha+1)/2} \right] & \alpha \neq -1,
\end{cases} \]

the one-detector rate \( P(T, u) \) simplifies to

\[ P(T, u) = 2Dr E \frac{T^{2(\alpha+1)}}{C_0(\alpha) \mu_0^{\alpha+1}} \int_{\mu_0 - \mu_{\text{th}}^2/2}^{\mu + \mu_{\text{th}}^2/2} y^\alpha F(y) dy, \]

where \( y \equiv \mu/T^2 \) and

\[ F(y) = y^2 \int_{0}^{1} dz [z(1-z)]^2 H(z) \]

\[ \times \int_{0}^{2\pi} d\phi f_0 \left[ r_E \sqrt{z(1-z)} y, \phi \right]. \]

In the rest of the paper we will use a ‘composite’ notion of microcosmic event including double (parallax) events and single events detected only in detector ‘1’ or ‘2’. The composite probability distribution function for degenerate parallax microcosmic can then be obtained by introducing the normalising constant \( A \)

\[ \Psi(T, p, u_1, u_2) = \frac{1}{A} \Psi(T, p, u_1, u_2), \]

\[ \Omega(T, u) = \frac{1}{A} \Omega(T, u), \]

so that

\[ \int_{0}^{\infty} dT \int_{-\infty}^{u_{th}} dp \int_{0}^{u_{th}} du_1 \int_{0}^{u_{th}} du_2 \Psi(T, p, u_1, u_2) \]

\[ + 2 \int_{0}^{\infty} dT \int_{0}^{u_{th}} du_1 \Omega(T, u) = 1. \]

In order to take account of the detection condition \( u_1 < u_{th}, u_2 < u_{th} \) in the case of resolved parallax microcosmic detection, we multiply \( \chi(T, p, w) \) by the range of \( u \) (see Fig. 1) for which the double event detection condition is satisfied

\[ \chi(T, p, w) \rightarrow (2u_{th} - |w|) \chi(T, p, w), \]

\( (|w| < 2u_{th}), \) and thus obtain the differential detection rate of events characterised by the observables \( T, p, \) and \( w \). The relevant probability distribution for the detectable double events is then

\[ \tilde{\chi}(T, p, w) = \frac{1}{A} (2u_{th} - |w|) \chi(T, p, w), \]

where \( A \) has the same value as in the degenerate parallax case [one can show \( \int_{0}^{u_{th}} du_1 \int_{0}^{u_{th}} du_2 \Psi(T, p, u_1, u_2) = \int_{-2u_{th}}^{2u_{th}} dw (2u_{th} - |w|) \chi(T, p, w) \)], thus yielding the normalisation

\[ \int_{0}^{\infty} dT \int_{-\infty}^{\infty} dp \int_{-2u_{th}}^{2u_{th}} dw \tilde{\chi}(T, p, w) \]

\[ + 2 \int_{0}^{\infty} dT \int_{0}^{u_{th}} dw \tilde{\Omega}(T, u) = 1. \]

3 MODELS OF MHO DISTRIBUTION AND KINEMATICS

In this paper we will consider a range of spherically symmetric models of the massive halo objects’ distribution and velocities. Probably the most commonly used model is the isothermal sphere with the velocity dispersion constant throughout the halo and the density profile which is well approximated by
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Figure 3. Anisotropy parameter $\beta$ and velocity dispersion for the CS halo model as functions of the distance $d$ from the Earth in the direction of LMC; $\sigma_t$ is given by the solid line, $\sigma_r = \sigma_j$ by the dotted line and $\sigma_i$ [see paragraph following equation (24)] by the dashed line. The straight solid lines correspond to the singular isothermal sphere, SIS $(\beta = 0, \sigma = 156$ km/s $)$. $\rho(R) = \rho_0 \frac{a^2 + R_\odot^2}{a^2 + R^2}$, where $a \approx 5$ kpc is the ‘core’ radius and $R_\odot = 8.5$ kpc is the distance of the Sun from the galactic centre. Assuming that the total (luminous + dark matter) halo density is distributed according to expression (20), one obtains the observed (approximately) flat rotation curve for the galaxy.

The MHO mass distribution, however, need not follow that of the total halo mass. We may, for instance, follow the hints provided by recent observations (Sommer-Larsen, Flynn & Christensen 1994, Sommer-Larsen et al. 1997) of the blue horizontal branch field stars (BHBFS) in the outer halo. These observations imply that the velocity dispersion changes from $\beta \equiv 1 - \sigma_t^2/\sigma_r^2 > 0$ ($\sigma_t$ and $\sigma_r$ are velocity dispersions respectively in the radial and tangential direction relative to the Galactic centre) at smaller distances $R$ from the centre of the Galaxy to $\beta < 0$ at larger distances. The radial velocity dispersion is well described by the analytic fit $\sigma_r^2 = \sigma_o^2 + \sigma_\perp^2 \left[ \frac{1}{2} - \frac{1}{\pi} \tan^{-1} \left( \frac{R - r_o}{l} \right) \right]$, where the best agreement with the observations is achieved with $\sigma_o = 80$ km/s $^{-1}$, $\sigma_\perp = 145$ km/s $^{-1}$, $r_o = 10.5$ kpc and $l = 5.5$ kpc (these are the values used in paper I and the present one; more recent values, based on a larger sample of stars, are given in Sommer-Larsen et al. 1997). The BHBFS halo is close to spherical with the density that is well modeled by the power law $\rho = \rho_o (R_\odot/R)^\gamma$, where $\gamma \approx 3.4$.

The Jeans’ equation for spherical systems (Binney & Tremaine 1987) yields the tangential velocity dispersion $\sigma_t^2 = \frac{1}{2} V_o^2 \left( \frac{\gamma}{2} - 1 \right) \sigma_r^2 + \frac{r}{2} \frac{d\sigma_t^2}{dR}$, where $V_o = \left( -R d\Phi/dR \right)^{1/2}$ is the (roughly constant) rotation velocity. This tangential dispersion is smaller than in the case of an isothermal sphere ($\gamma = 2, \sigma_t = \text{const.}$) with the same $V_o$ (see Fig. 3).

We will, following paper I, model the velocity distribution by the Gaussian $f(v_r, v_\theta, v_\phi) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sigma_r \sigma_\phi \sigma_\theta} \exp \left[ -\frac{1}{2} \left( \frac{v_r^2}{\sigma_r^2} + \frac{v_\phi^2 + v_\theta^2}{\sigma_\phi^2} \right) \right]$, where $\sigma_r$ and $\sigma_\phi$ are given by equations (21) and (22) for power-law density profiles. The relevant distribution used, e.g., in the rate (3) of velocities orthogonal to the line of sight is then $f_o(v_n, \phi) = \frac{1}{2\pi \sigma_r \sigma_\phi} \exp \left[ -\frac{1}{2} \left( \frac{v_n \cos(\phi + \phi_o) + s_i}{\sigma_i} \right)^2 \right. \left. + \frac{[v_n \sin(\phi + \phi_o) + s_j]^2}{\sigma_j} \right]$. For the above equation we have introduced orthonormal vectors in the plane orthogonal to the line of sight: $i$ is in the plane determined by the Sun, LMC and the Galactic centre (GC) and points in the general direction of GC; $j = k \times i$, where $k$ points along the Sun-LMC line of sight. Thus, $s_i$ and $s_j$ are the corresponding components of the local ($z$-dependent) velocity of the line of sight relative to the Sun; $V_o = 156$ km/s $^{-1}$ (approximately).

Figure 4. Differential rate $d\Gamma/dT$ (normalised) for the SIS (solid line) and CS (short-dashed) halo models with Sun’s and LMC motion neglected. For the dotted line one assumes the CS model and takes into account the motion of the Sun and LMC. In all cases the mass function is a delta function centered on $\mu = 0.4$. The best agreement with the observations is achieved with $\sigma_r = 156$ km/s $^{-1}$, $\sigma_i = 145$ km/s $^{-1}$, $r_o = 10.5$ kpc and $l = 5.5$ kpc (these are the values used in paper I and the present one; more recent values, based on a larger sample of stars, are given in Sommer-Larsen et al. 1997). The BHBFS halo is close to spherical with the density that is well modeled by the power law $\rho = \rho_o (R_\odot/R)^\gamma$, where $\gamma \approx 3.4$.
As in paper I, the halo model corresponding to BHBFS with the power-law density profile $\gamma = 3.4$ and the dispersion given by (21) and Jeans' equation will be called the 'concentrated sphere' ('CS'). More generally, we will assume for our study that the MHO halo can be described by a member of a class of models specified by five parameters: $\gamma$, $\sigma_o$, $\sigma_t$, $r_o$, and $l$. For instance, the model with $\gamma = 2$ and constant velocity dispersion $\sigma_t = \sigma_t = V_c/\sqrt{2} = 156$ km/s (i.e., $\sigma_o = V_c/\sqrt{2}$, $\sigma_+ = 0$) is just the familiar singular isothermal sphere (SIS).

4 DISTRIBUTION OF MEASURABLE QUANTITIES

In order to understand basic features of the distribution functions $\chi(T, p, w)$ [and consequently $\Psi(T, p, w_1, w_2)$] and $\Omega(T, u)$ we will at first limit ourselves to the relatively simple case of the SIS halo, neglect the motion of the Sun (i.e., the detectors) and the LMC and assume that all MHOs have the same mass, $\mu = \bar{\mu}$ (see Fig. 4).
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Figure 6. Plots of $\chi$ with the CS halo (Sun’s and LMC motion taken into account) and a delta mass function ($\mu = 0.4$) at $T = 15.1$ (top) and $T = 81.4$ (bottom) days.

With the above assumptions

$$\chi(T, p, w) \propto \frac{(1 - z)^2 z^4 H(z)}{T^4} \exp \left[ -\frac{1}{2} \frac{r_E^2 \mu}{\sigma^2} \frac{z(1 - z)}{T^2} \right],$$

(25)

where $z(p, w)$ is given by equation (4). At sufficiently small $a^2 = p^2 + w^2$, a MHO is near LMC, $z \approx 1$, $1 - z \approx (r_E/r)^2 \mu a^2$ and we thus have

$$\chi \sim a^4 \exp \left[ -\frac{1}{2} \left( \frac{r_E^2 \mu}{r \sigma T} \right)^2 a^2 \right].$$

(26)

Near the origin of the $p, w$ plane $\chi$ grows at first as $a^4$ (from $\chi = 0$ at $a = 0$) to a maximum value at $a_1 = 2r \sigma T / r_E^2 \mu$ (see the ‘funnel’ emerging from $w = 0$ on the right of Fig. 5, which is a projection along the $p$-axis of the 3D plot on the left), which for the values of the SIS model parameters and $\mu = 0.4$, $r = 2$ AU takes on value $a_1 = T/508$ d.

At large $a$, $z \approx (r/r_E)^2 / \mu a^2$ (close to the detector at $z = 0$) and thus

$$\chi \sim \frac{1}{a^8} \exp \left[ -\frac{1}{2} \left( \frac{r}{\sigma T} \right)^2 \frac{1}{a^2} \right].$$

(27)

This expression reaches a maximum at $a_2 = (1/2^{3/2}) r / \sigma T = 7.9$ d/T, and falls off as $\chi \propto a^{-8}$ for $a > a_2$.

For $T < T_o$, where

$$T_o \equiv \frac{1}{2^{7/4}} \frac{r_E \sqrt{\mu}}{\sigma} = 63$$

(28)
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Figure 7. The fraction $\Omega(T, u)/P(T, u)$ of events detected in one detector (i.e., with $u < u_{th} = 1$) that are not detected in the other detector. The upper panel gives the plots for the SIS halo model (without Sun’s or LMC motion) and the bottom for the CS (with Sun’s and LMC motion) model, both with a delta mass function, $\mu = 0$.

4. The lines are labeled with event durations $T$ in days. The total (i.e., integrated over $T$) fraction of single events out of all events detected in a detector is 12% in the upper case and 20% in the bottom case: the CS model gives more events at low $z$, where $R_E/r(1 - z)$ is smaller.

Figure 8. The lines tangential to both the circles of radius $R_E$ and $uR_E$ are drawn to find the sector of size $2(\phi_1 + \phi_2)$ from which the events at $u (< u_{th} = 1)$ in the right-hand side detector are not detectable by the one on the left-hand side. The distance between the centres of the circles is $r(1 - z)$.

...
with increasing $u$ to the coordinate $(d/dz)$ typical of a constant area under the curve. Figure 9. Normalised (i.e. of a constant area under the curve) typical times $d$ from ($\Gamma$ dotted) and $\Omega$ (with Sun’s and LMC motion; top) and CS (with Sun’s and LMC motion; bottom) halo models. In both cases the mass function is delta at $\Omega = 1$. The 166d and 361d curves are virtually indistinguishable.

Fig. 7. Naturally, the $\Omega/P$ curves are depressed as $T$ [and thus $R_E/(1-z)$] is increased.

For MHOs passing at larger $z > 0.02$ (corresponding to typical times $T \approx R_E/\sqrt{2}a > r/\sqrt{2}a \approx 15$d) the Einstein radius will be longer than the projected Earth–satellite distance $r(1-z)$. This means that an event of a small impact parameter $u < 1 - r(1-z)/R_E$, with respect to one detector, will inevitably be detected $[\Omega(T,u)/P(T,u) = 0]$ by the other one. For $u > 1 - r(1-z)/R_E$, the fraction detectable in only one detector rises approximately as

$$\frac{\Omega(T,u)}{P(T,u)} = \frac{1}{\pi} \cos^{-1} \frac{R_E}{r(1-z)}(1-u)$$

with increasing $u$ and reaches $1/2$ at $u = u_{th} = 1$.

The correspondence between the event durations and typical $z$’s is illustrated in Fig. 9. The differential rate $dT/dTdz$ is relatively sharply peaked if plotted with respect to the coordinate $(r_E/r)(\mu z/(1-z))^{1/2} = R_E/r(1-z) = 1/a$ instead of $z$ [the area element below a curve in Fig. 9 differs from $(dT/dTdz)dz$ only by a constant factor]; we used this fact in the above discussion to relate a narrow range of $z$ (or, rather, simply a single value of $z$) to each $T$.

5 INFERENCE OF THE MHO MASS FUNCTION AND THE HALO STRUCTURE

Even at first sight it seems obvious that the increased amount of information obtained from parallax microlensing measurements should allow a more reliable determination of the structure of the halo and the MHO mass function than just the measurements of event durations. Indeed, the measurable quantity $pT = r(1-z)\cos \phi/v_0$ (equation [2]) involves only the kinematic properties of a MHO (i.e., excluding its mass). In the case of resolved parallaxes, an additional quantity, $w$, is measured and this gives us the 2-dimensional velocity projected on the observer’s plane (the so-called reduced velocity; see, e.g., Gould 1994b)

$$\tilde{v} \equiv \frac{1}{1-z}v_u = \frac{r}{T}a = \frac{r}{T}p^2 + w^2(p,w).$$

Although the 4-fold ambiguity of the degenerate parallaxes reduces the quality of available information, both types of parallax microlensing would constrain the halo independently of the MHO masses; this should make it possible to separate the effects of the masses and determine the MHO mass function.

In order to assess the information gain due to parallax microlensing we will investigate simulated maximum likelihood inference of the five parameters ($c_1 \equiv \gamma$, $c_2 \equiv \sigma_\alpha$, $c_3 \equiv \sigma_\phi$, $c_4 \equiv r_{\alpha}$, $c_5 \equiv \Gamma$) of the halo model and the three parameters ($c_6 \equiv \mu$, $c_7 \equiv \alpha$, $c_8 \equiv \beta$) of the mass function based on the normalised probability distributions $\hat{\chi}(T,p,w)$, $\hat{\Omega}(T,u)$ (equation [3]) for resolved and $\hat{\Psi}(T,p,u_1,u_2)$, $\hat{\Omega}(T,u)$ (equation [4]) for degenerate parallax measurables.

Ideally, one would prefer to obtain an answer through a series of Monte-Carlo simulations, but the enormous numerical task of computing the above 3-dimensional distributions $\hat{\chi}$ for many points in the 8-dimensional parameter space compels us to seek more economical alternatives. For a sufficiently large number $N$ of detected events, the average errors $\sigma_{\chi} \equiv \left\langle (c_\mu - c_{\chi})^2 \right\rangle^{1/2}$ of the parameters ($c_{\chi}$) is the ‘real’ value of a parameter) can be shown (see, e.g., paper I) to approach asymptotically the Cramer limit

$$\langle (c_\mu - c_{\chi})^2 \rangle \rightarrow C_{\mu\nu},$$

where $C_{\mu\nu}$ is the inverse of the information matrix

$$I_{\mu\nu}^{(N)} = N \int \int \int \int dT dp dv dw \Psi_\mu \Psi_\nu \Omega + 2N \int \int dv dw \frac{\hat{\Omega}_\mu \hat{\Omega}_\nu}{\hat{\Omega}}$$

($\mu$ denotes derivative with respect to the $\mu$’th parameter) for degenerate parallaxes and by analogy for the resolved parallaxes.

In our numerical experiment we assume that the parameters of the concentrated sphere (CS) describe accurately the MHO halo. The Cramer errors of the inference of the mass-function parameters for $N = 100$ are shown in Fig. 10 as functions of the mass-function slope $\alpha$. The resolved parallax errors obtained with the assumption that both the halo structure and the mass function parameters were unknown (and thus variable in the maximum likelihood fitting) are given as solid lines. For comparison, we also show the Cramer-limit errors (dotted lines) for measurements of
event durations only. In addition, we plot the resolved parallax errors (dashed lines) and the errors based on event durations only (dot-dash) assuming that the halo parameters are known precisely (and thus not varied for the maximum likelihood fit). In all computations we take into account the motion of the Sun and LMC, assume \( r = 2 \) and that the Earth-satellite segment is in the ecliptic and orthogonal to the Earth-LMC line of sight (see the Appendix).

The most striking feature of these plots is a significant reduction of the error in \( \bar{\mu} \) brought about by parallax measurements. As shown in paper I, by changing the parameters of the halo model one can match closely event duration distribution curves corresponding to widely different average masses. This is reflected in the large errors in \( \bar{\mu} \) if the halo parameters are allowed to vary in addition to the parameters of the mass function. The extra information provided by parallaxes effectively allows us to constrain the halo as to bring \( \sigma_\beta \) down to the values comparable to those of inference with a fixed (i.e., ‘known’) halo model. (Notice that the gain in accuracy due to parallaxes for a known halo structure is modest by comparison.)

In paper I we have concluded that for a certain range of \( \alpha \) (very roughly, \( -2 \lesssim \alpha \lesssim 0 \)) the inference of \( \alpha \) and \( \beta \) is relatively weakly affected by the uncertainty of the halo structure. This again is manifested in errors of the same (small) order near \( \alpha = -1.5 \) in all four cases shown in Fig. 10. However, away from \( \alpha = -1.5 \), \( \sigma_\alpha \) and \( \sigma_\beta \) grow large for event duration-based inference. Here again, the parallaxes restrain this growth significantly (especially for \( \sigma_\alpha \)) and keep the errors down in the range characteristic of the fixed-halo inference. Interestingly, degenerate parallax errors \( \sigma_\beta, \sigma_\alpha \) and \( \sigma_\beta \) are only a few percent smaller than the resolved parallax values (and we omit the corresponding plots).

As suggested above, the significant improvement in the accuracy of the mass function determination should be ascribed to the disentanglement of the halo structure from the MHO masses. This disentanglement is obvious from the plots in Fig. 11 of the correlation matrix

\[
\frac{C_{\mu\nu}}{\sqrt{C_{\mu\mu}C_{\nu\nu}}} \quad (34)
\]

The off-diagonal spikes ‘coupling’ the halo with the mass function are noticeably less pronounced in the parallax case (right). [Notice, e.g., the strong correlation between \( \bar{\mu} \) and \( \gamma \) in the duration based inference (left) and its significant drop in the parallax case (right).]

Further comparison of the two plots of Fig. 11 also shows a significant reduction in the correlation among some of the halo parameters due to the parallax measurements. Does this imply that the halo structure itself could be inferable? Figure 12 indeed displays a remarkable suppression of the Cramer errors of the halo parameters’ inference: the (resolved) parallax errors at least are of the same order of magnitude as the parameters themselves or smaller.\[\dagger\] In particular, the density profile index \( \gamma \) can be determined rather accurately. [The difference between the degenerate and resolved parallax errors is more pronounced for the halo model parameters than for the mass function; still, the degenerate parallax errors are larger than the resolved parallax ones by at most 20%. The measurement of \( p \) and the information contained in \( u_1, u_2 \) and \( \Omega(T, u) \) are thus sufficient to constrain the halo even if the determination of \( w \) is subject to the 4-fold ambiguity.]

\[\dagger\] As discussed in paper I, the Cramer errors can serve as reliable estimates of actual errors only if they are small in comparison with the corresponding parameters that one is trying to infer. Typically, we would expect the nonlinear dependence of the distribution function of the observables on the underlying parameters to make the actual errors smaller that the Cramer estimates when these are relatively large. In this case the actual errors would fall slower than \( N^{-1/2} \) and approach the Cramer limit from below.
The residual but still considerable correlations among the errors of the halo parameters indicate that some combinations (functions) of them (together with the mass function parameters) may be possible to determine with a significantly higher accuracy. In order to investigate this possibility we examine the eigen-values and eigen-vectors of the relative error matrix for the CS halo and $\bar{\mu} = 0.4$, $\alpha = -1.5$ and $\beta = 2$.

We illustrate the above point by a specific example. Table 1 contains the square roots (‘eigen-errors’) of the eigen-values along with the eigen-vectors of the relative error matrix for the CS halo and $\bar{\mu} = 0.4$, $\alpha = -1.5$ and $\beta = 2$ (same as for Fig. 11). We immediately notice that the eigen-errors for the eigen-‘modes’ are small ($\sigma_{(5)} < 1$) except for a single mode ($\sigma_{(5)} = 3.316$). This, the 5’th mode ‘mixes’ rather strongly $c_2 = \sigma_\alpha$, $c_3 = l$ and to a lesser degree $c_4 = \sigma_\sigma$ (see the corresponding components of $V_{(5)}$). Its mass function components, $V_{(5)}$, are very small which explains why its large associated eigen-error does not lead to large errors of the mass function parameters. On the other hand, although there is an eigen-vector, $V_{(3)}$, that strongly mixes the halo and the mass function parameters, its relatively small eigen-error, $\sigma_{(3)} = 0.126$ contributes little to the errors of the mass function parameters and thus gives at most moderate correlations between these and the halo-model errors.

Table 1. Eigen-errors (for $N = 100$ events) and the corresponding eigen-vectors of the relative error matrix for the CS halo and $\bar{\mu} = 0.4$, $\alpha = -1.5$ and $\beta = 2$.

| $\sigma_{(1)}$ | 0.033 | $V_{(1)}$ | (0.7667, 0.2005, 0.5791, 0.1562, -0.1037, 0.0233, 0.0149, -0.0258) |
| $\sigma_{(2)}$ | 0.795 | $V_{(2)}$ | (0.0362, 0.7289, -0.0838, -0.5037, 0.4541, 0.0229, -0.0048, 0.0035) |
| $\sigma_{(3)}$ | 0.126 | $V_{(3)}$ | (-0.4744, 0.2243, -0.4946, 0.1677, -0.0685, 0.3439, -0.5733, 0.0175) |
| $\sigma_{(4)}$ | 0.383 | $V_{(4)}$ | (0.0511, 0.2676, -0.3076, 0.8129, 0.4111, 0.0083, -0.0103, -0.0338) |
| $\sigma_{(5)}$ | 3.316 | $V_{(5)}$ | (-0.0158, -0.5050, 0.3601, -0.0903, 0.7789, -0.0143, -0.0019, -0.0001) |
| $\sigma_{(6)}$ | 0.182 | $V_{(6)}$ | (0.3227, -0.1966, -0.3563, -0.1221, 0.0442, -0.0117, 0.5212, 0.0540) |
| $\sigma_{(7)}$ | 0.102 | $V_{(7)}$ | (-0.2675, 0.1085, 0.2431, 0.0859, -0.0271, 0.4702, 0.7796, -0.1434) |
| $\sigma_{(8)}$ | 0.081 | $V_{(8)}$ | (-0.0270, 0.0343, 0.0510, 0.0499, 0.0048, 0.0186, 0.1351, 0.9871) |

### Figure 11
Correlation matrices for Cramer errors of inference based on event durations only (left) and resolved parallaxes (right). The underlying halo model is CS and the parameters of the mass function function are $\bar{\mu} = 0.4$, $\alpha = -1.5$ and $\beta = 2$. 

The residuals $\sigma_{(5)}$ are very small and hence the above correlation matrices can be considered as diagonal. The residual but still considerable correlations among the errors of the halo parameters indicate that some combinations (functions) of them (together with the mass function parameters) may be possible to determine with a significantly higher accuracy. In order to investigate this possibility we examine the eigen-values and eigen-vectors of the relative error matrix for the CS halo and $\bar{\mu} = 0.4$, $\alpha = -1.5$ and $\beta = 2$. 

We illustrate the above point by a specific example. Table 1 contains the square roots (‘eigen-errors’) of the eigen-values along with the eigen-vectors of the relative error matrix for the CS halo and $\bar{\mu} = 0.4$, $\alpha = -1.5$ and $\beta = 2$ (same as for Fig. 11). We immediately notice that the eigen-errors for the eigen-‘modes’ are small ($\sigma_{(5)} < 1$) except for a single mode ($\sigma_{(5)} = 3.316$). This, the 5’th mode ‘mixes’ rather strongly $c_2 = \sigma_\alpha$, $c_3 = l$ and to a lesser degree $c_4 = \sigma_\sigma$ (see the corresponding components of $V_{(5)}$). Its mass function components, $V_{(5)}$, are very small which explains why its large associated eigen-error does not lead to large errors of the mass function parameters. On the other hand, although there is an eigen-vector, $V_{(3)}$, that strongly mixes the halo and the mass function parameters, its relatively small eigen-error, $\sigma_{(3)} = 0.126$ contributes little to the errors of the mass function parameters and thus gives at most moderate correlations between these and the halo-model errors.

In Fig. 13 we compare changes in the halo model caused by a displacement in the direction of the vector $V_{(1)}$ of a small eigen-error with a displacement along $V_{(5)}$. Notice that a much larger displacement ($\epsilon = 0.5$) is needed along $V_{(5)}$ to cause a discernible effect. In addition, these changes...
Figure 12. The Cramer-limit errors (at $\bar{\mu} = 0.4$, $\beta = 2$ and $N = 100$) of the halo-model parameters from resolved parallaxes (solid lines) and event durations only (dotted lines).

tend to be concentrated at larger distances, where the event rate (recall $\gamma = 3.4$) is smaller; microlensing is, not surprisingly, relatively insensitive to such displacements along $V_5$ in the parameter space. As a consequence, the large associated eigen-error, $\sigma_5$, is the leading culprit for the relatively large errors of $c_2 = \sigma_\sigma$, $c_5 = l$ and $c_3 = \sigma_+ \sigma$ evident in Fig. 12. By contrast, parallax microlensing constrains fairly well the displacements along the other directions. In particular, the eigen-modes closest to the parameters of the mass function ($V_6$, $V_7$ and $V_8$) all have rather small eigen-errors.

The coefficient $\epsilon = 0.5$ used in Fig. 13 for the displacement along $V_5$ is considerably smaller than the associated eigen-error $\sigma_5 = 3.3$ (see Table 1); the relatively small $\epsilon$ allows us to stay near the ‘linear’ range of shifts in the structure of the halo (the range of the validity of the Cramer errors). At $\epsilon = 3.3$ the structure would be changed radically, far beyond what one might expect on the basis of small linear displacements. This likely implies that actual errors would be smaller than the Cramer-limit estimate. The results of such strongly non-linear shifts can be studied only by means of Monte-carlo simulations which are beyond the scope of this paper.

The CS halo discussed so far can be regarded as a typical member of our class of spherical models. On the other hand, SIS is peculiar in that the parameters $r_o$ and $l$ can be arbitrary as long as $\sigma_+ = 0$. Might this ‘degeneracy’ lead to a large uncertainty in the inference of the halo structure?

To explore this issue we assume $\gamma = 2$, $\sigma_0 = 156$ km $\sigma_+ = 10$ km, $r_o = 10.5$ kpc and $l = 5.5$ kpc, only slightly deviating from SIS. The root-quadratic dependence of $\sigma_\beta$...
Table 2. Halo structure parameters and radial and tangential velocity dispersions after displacements $\sigma_{(i)V_{(i)}}$ in the parameter space.

|   | $\sigma_{(3)V_{(3)}}$ | $\sigma_{(4)V_{(4)}}$ | $\sigma_{(5)V_{(5)}}$ |
|---|----------------------|----------------------|----------------------|
| $\gamma$ | 2.02                 | 1.81                 | 1.77                 |
| $\sigma_o$ [km/s] | 170                  | 116                  | 15                   |
| $\sigma_t$ [km/s] | 81                   | 111                  | 272                  |
| $r_o$ [kpc] | $-1.7 \times 10^3$   | $-1.1 \times 10^3$  | $-2.5 \times 10^3$  |
| $l$ [kpc]   | $-0.6 \times 10^3$   | $-0.5 \times 10^3$  | $4.3 \times 10^3$   |
| $\sigma_t$ [km/s] | 186                  | 122                  | 158                  |
| $\sigma_t$ [km/s] | 155                  | 160                  | 164                  |

Moreover, some combinations of parameters describing the halo structure can be inferred rather accurately. Virtually all the uncertainty regarding the halo model is then localised in a few (precisely one in the CS-based example discussed in Section 5) ‘eigen-modes’ of the halo, i.e., those displacements in the halo parameter space that are left ‘loose’ by parallax microlensing while allowing other displacements to be independently (and tightly) constrained. These poorly constrained modes correspond to particularly small changes in the actual structure of the halo as given by density and velocity dispersion profiles. Unfortunately, their large associated eigen-errors may push us into a non-linear regime (inadequately charted by the Cramer limit) of deviations in the halo structure that can be properly explored only by time-consuming Monte-Carlo simulations.

Although obtained on the basis of a limited class of halo models, the above conclusions should be relevant in a broader context. The halo structure/mass function disentanglement (and consequently accurate mass determination) as well as the inference of some properties of the halo structure itself should result from the significant enhancement of information (as elaborated in Section 5) due to parallax microlensing even if one allows for a much wider range of halo structures. For instance, in the more general case of non-spherical haloes, one may hope to constrain the density and velocity dispersions along the line of sight well enough to determine the mass function. The ‘uncertain’ modes would then describe possibly large changes in the halo structure away from the line of sight. Only parallax microlensing observations in several directions would presumably suffice to infer the overall structure of the halo. These conjectures need to be tested on other, more realistic or better dynamically founded models (e.g., those of Evans 1994) of the halo than the ones discussed in the present paper.

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Appendix: Position and motion of the Earth relative to the Galaxy and the Large Magellanic Cloud

This appendix contains formulae (with derivations) giving positions and proper motions in the rest frame of the Galaxy of points along the Earth-Large Magellanic Cloud line of sight.

We denote by \( \hat{x} \), \( \hat{y} \) and \( \hat{z} \) unit vectors directed, respectively, toward the Galactic centre (GC), tangentially along the rotation of the 'local standard of rest' (LSR) around the galactic centre, and toward the north Galactic pole. If the velocity of a source relative to the Sun is given in terms of Galactic-coordinate components \( v_{\text{rad}} \), \( v_l \) and \( v_b \), the conversion to the \( x \), \( y \), \( z \) components is

\[
\begin{align*}
\vec{v}'_x &= (\cos b \cos l v_{\text{rad}} - \sin b \cos l v_b - \sin l v_l) \hat{x} \\
&\quad + (\cos b \sin l v_{\text{rad}} - \sin b \sin l v_b + \cos l v_l) \hat{y} \\
&\quad + (\sin b v_{\text{rad}} + \cos b v_b) \hat{z}
\end{align*}
\]  

(A1)

in the rest frame of the Sun. From the 1990 data of B. Jones (as quoted in Greist 1991) the proper motion of the LMC is

\[
\begin{align*}
v_{\text{rad}} &= 250 \pm 5 \text{ km/s} \\
v_l &= -25 \pm 62 \text{ km/s} \\
v_b &= -31 \pm 62 \text{ km/s.}
\end{align*}
\]  

(A2)

Disregarding the error bars and using \( D = 55 \text{ kpc}, b = -32^\circ, l = 281^\circ \), [thus \( r_{\text{LMC}} = D \hat{k} = 55 \text{ kpc} (0.160, -0.825, -0.542); \hat{k} \cdot \hat{k} = 1 \)] we arrive at

\[
v_{\text{LMC}} = (44, -390, 146) \text{ km/s},
\]

Bearing in mind that the Sun’s velocity relative to the Galaxy is \( \vec{v}_{\odot} = (0, 220, 0) + (9, 11, 16) = (9, 231, 16) \text{ km/s} \), where the first and second terms are the velocity of LSR relative to the Galaxy and that of the Sun relative to LSR, the velocity of the LMC relative to the Galaxy is

\[
v_{\text{LMC}} = v_{\odot} + v_{\text{LMC}} = (53, -159, 162) \text{ km/s}. \]  

(A3)

[After completing the calculations described in this paper, the author learned about more recent values of Jones, Klemola & Lin (1994) who give \( v_{\text{LMC}} = (60 \pm 59, -155 \pm 25, 144 \pm 51) \text{ km/s}. \) On the other hand, Kroupa & Bastian (1997) give an estimate based, among other sources, on the Hipparcos data:

\[
v_{\text{LMC}} = (41 \pm 44, -200 \pm 31, 169 \pm 37) \text{ km/s}.
\]

These values do not differ sufficiently from velocity \( [A3] \) to lead to a significant change in the results reported in this paper.]

Along the Sun-LMC line of sight we can introduce auxiliary coordinates defined by the following orthonormal system: as above, \( \hat{k} \) is directed along the line of sight, \( \hat{i} \) lies in the Sun-GC-LMC plane and points in the direction of the GC and \( \hat{j} = \hat{k} \times \hat{i} \). Given the data listed above, \( \hat{i} = (0.987, 0.134, 0.088) \) and \( \hat{j} = (0, -0.549, 0.836) \). Each point on the line of sight moves relative to the galaxy with velocity \( s(z) = v_{\odot}(1 - z) + v_{\text{LMC}}z \) whose components along \( \hat{i} \) and \( \hat{j} \) are

\[
\begin{align*}
s_i &= v_{\odot i}(1 - z) + v_{\text{LMC}i}z \\
s_j &= v_{\odot j}(1 - z) + v_{\text{LMC}j}z
\end{align*}
\]  

(A4)

where \( v_{\odot i} = 41 \text{ km/s}, v_{\text{LMC}i} = 45 \text{ km/s}, v_{\odot j} = -113 \text{ km/s} \) and \( v_{\text{LMC}j} = 223 \text{ km/s} \).

In the above we have neglected the motion (revolution) of the Earth (detector '1') and the satellite (detector '2') around the Sun: one should in principle add the term \( \omega \times \vec{r}_0 \) (or the corresponding term for the satellite) to \( \vec{v}_{\odot} \). To avoid (at this stage) unnecessary complication we will keep disregarding the revolution around the Sun. For specific computations in this paper we assume that the '1-'2' line points along the unit vector \( \hat{r} \) directed along \( \hat{\omega} \times \hat{k} \). Since in galactic coordinates \( \hat{\omega} = (-0.095, 0.862, 0.498) \), (derived from data in Mihalas & Binney 1981) we find \( \hat{r} = (-0.652, 0.324, -0.685) \) and thus \( \hat{r} \cdot \hat{i} = -0.660 = \cos \phi_{\odot} \) and \( \hat{r} \cdot \hat{j} = -0.750 = \sin \phi_{\odot} \) corresponding to \( \phi_{\odot} = 229^\circ \).