Global well-posedness for the Euler alignment system with mildly singular interactions

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Abstract

We consider the Euler alignment system with mildly singular interaction kernels. When the local repulsion term is of the fractional type, global in time existence of smooth solutions was proved in [16, 22–24]. Here, we consider a class of less singular interaction kernels and establish the global regularity of solutions as long as the interaction kernels are not integrable. The proof relies on modulus of continuity estimates for a class of parabolic integro-differential equations with a drift and mildly singular kernels.

Keywords: Euler alignment system, integro-differential equations, fractional dissipation

1. Introduction

The Euler alignment system. The Cucker–Smale model [14]

\[
\dot{x}_i = v_i, \quad \dot{v}_i = \frac{1}{N} \sum_{j=1}^{N} \psi(|x_i - x_j|)(v_j - v_i)
\] (1.1)

describes the dynamics of a flock of \(N\) individuals (birds, fish, etc.) that tend to align their velocities locally. Here, \(x_i\) and \(v_i\) are the position and the velocity of the \(i\)th individual in the flock. The non-negative ‘influence function’ \(\psi(r) \geq 0\), measures the strength of the alignment and is a decreasing function of \(r\). By now, it is one of the standard models for the flocking phenomenon—emergence of self-organized groups (flocks) that move as a group—see [9, 15, 28] for a review.

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The Euler alignment system

\begin{align}
\partial_t \rho + \partial_x (\rho u) &= 0 \quad (1.2) \\
\partial_t u + u \partial_x u &= \int_{\mathbb{R}} \psi(|x-y|)(u(t,y) - u(t,x))\rho(t,y)dy, \quad (1.3)
\end{align}

is a hydrodynamic limit of the Cucker–Smale system (1.1), in the regime where the number \(N\) of the individuals is very large, and they are already locally aligned, so that their evolution may be described in terms of a local density \(\rho(t,x)\) and a local velocity \(u(t,x)\). In the absence of the local alignment, when \(\psi \equiv 0\), the velocity equation (1.3) is simply the in viscous Burgers’ equation that may develop a discontinuity in \(u(t,x)\) in a finite time. In terms of the flocking dynamics, this corresponds to a collision of two flocks that can easily happen when there is no local tendency to align. On the intuitive level, a positive interaction kernel \(\psi > 0\) promotes a local alignment and fights the shock creation. However, it was shown in [7, 26] that if the kernel \(\psi(r)\) is Lipschitz, then the solutions of the Euler alignment system may still develop a discontinuity in \(u(t,x)\) in a finite time, though the class of the initial conditions that lead to a discontinuity is smaller than for the Burgers’ equation, so even a Lipschitz interaction kernel \(\psi(r)\) has some regularizing effect. More precisely, if \(\psi(r)\) is Lipschitz, then solutions of the Euler alignment system remain regular if and only if the initial conditions \(u_0(x) = u(0, x)\) and \(\rho_0(x) = \rho(0, x) \geq 0\) satisfy

\begin{equation}
\partial_x u_0(x) \geq - (\psi \ast \rho_0)(x) \text{ for all } x \in \mathbb{R}. \quad (1.4)
\end{equation}

Otherwise, \(u(t,x)\) develops a discontinuity in a finite time. This is a natural generalization of the classical criterion for regularity of the solutions of the Burgers’ equation with \(\psi = 0\). Note that \(\rho_0 \geq 0\) from physical considerations.

Recently there has been an increased interest in alignment kernels \(\psi(r)\) that are singular at \(r \downarrow 0\), so that the local alignment effect is much stronger than for the Lipschitz kernels, both for the Cucker–Smale and the Euler alignment systems—see [8, 16, 19–24] and references therein. In light of the regularity condition (1.4) for the Lipschitz interaction kernels, it is natural to conjecture that solutions of the Euler alignment systems (1.2) and (1.3) remain smooth for all times \(t > 0\), provided that the interaction kernel \(\psi \not\equiv 0\) is not integrable, as then (1.4) holds automatically, as long as \(\rho_0 \not\equiv 0\). In this direction, the global existence of smooth solutions of (1.2) and (1.3) for singular interaction kernels of the form \(\psi(x) = C|x|^{-1-\alpha}\), with \(\alpha \in (0, 2)\) was proved in [16, 22–24]. We note that the particular scaling properties of such kernels are important for the regularity proofs, especially in [16]. We also mention that a qualitatively similar nonlinearly enhanced regularizing effect happens also in nonlinear porous medium problems and Keller–Segel equations [1–6].

The main results

In this paper, we consider the Euler alignment systems (1.2) and (1.3) with \(2\pi\)-periodic initial conditions \(\rho(0, x) = \rho_0(x), \ u(0, x) = u_0(x)\), such that \(\rho_0(x) \geq c_0 > 0\) for all \(x \in \mathbb{R}\), and establish the global regularity of the solutions for a general class of interaction kernels \(\psi(r)\) that are not integrable but blow-up much slower than \(r^{-1-\alpha}\) as \(r \downarrow 0\). We make the following assumptions on the interaction kernel \(\psi\):

\begin{enumerate}
\item \(\int_{\mathbb{R}} \psi(r)dr = \infty\)
\item \(\psi(r) \leq C_0 r^{-1-\alpha}\) as \(r \downarrow 0\),
\item \(\psi(r) \geq C_1 r^{-1-\alpha}\) as \(r \downarrow 0\),
\end{enumerate}
For any $\alpha > 0$, $\psi(r)$ is less singular than $1/r^{1+\alpha}$ but more singular than $1/r^{1-\alpha}$, so that there exists $c_\alpha > 0$ such that
\[
\frac{1}{c_\alpha r^{1-\alpha}} \leq \psi(r) \leq \frac{c_\alpha}{r^{1+\alpha}} \quad \text{for all } 0 < r \leq 1,
\]
and $\psi(r)$ is not integrable:
\[
M(r) := \int_r^\infty \psi(y)dy \to +\infty \quad \text{as } r \to 0.
\]
It follows from (1.5) that $M(r)$ is less singular than $1/r^\alpha$ for any $\alpha > 0$:
\[
\lim_{r \to 0} r^\alpha M(r) = 0.
\]

The function $\psi(r)$ is symmetric, decreasing and satisfies the Hörmander–Mikhlin type condition: there exists $C > 0$ so that
\[
|r\psi'(r)| \leq C\psi(r),
\]
and also that a doubling condition holds:
\[
\psi(r) \leq C\psi(2r) \quad \text{for all } r > 0.
\]

We also assume that there exists $r_0 \leq 1$ such that
\[
\frac{r\psi(r)}{M(r)} \text{ is non-decreasing for } 0 < r < r_0,
\]
and that there exists $\gamma \in (0, 1/2]$ and $r_0 > 0$ such that
\[
r^\gamma M(r) \text{ is non-decreasing for } 0 \leq r \leq r_0.
\]
The last assumption is almost automatic since both $r^\gamma M(r) > 0$ for all $r > 0$ and (1.7) holds. It follows that $r M(r)$ is also non-decreasing. Note that we do not need to assume that $m(r) = r\psi(r)$ is singular at $r = 0$ as in [16, 23]. The Hörmander–Mikhlin condition is used in the proof of lemma 2.4, a version of the Constantin–Vicol nonlocal maximum principle, that allows us to control the density $\rho(t, x)$ in the $L^\infty$-norm, ensuring that the dissipative term is, indeed, dissipating. One may reasonably say that our assumptions cover most ‘well-behaved’ non-integrable influence functions $\psi(r)$.

A typical example of $\psi(r)$ that satisfies our assumptions is
\[
\psi(r) = \frac{1}{r^{p} \log(1/r)^p} \quad \text{for } 0 < p \leq 1 \text{ and } 0 < r \leq 1,
\]
with enough regularity and decay, such as, for instance, a geometric decay $r^{-(1+\beta)}$ with some suitable power $\beta > 0$, for $r > 1$. One can easily verify that (1.12) satisfies conditions (a)-(c) by noticing that
\[
M(r) = \log(\log(1/r)) + O(1/r^\beta) \quad \text{for } r > 1.
\]
for $p = 1$ and
\[ M(r) = (\log(1/r))^{1-p} + O(1/r^3)I_{\{r>1\}} \]
for $p \in (0, 1)$.

The main result of this paper is the following theorem.

**Theorem 1.1.** Under the above assumptions, the Euler alignment systems (1.2) and (1.3) with periodic smooth initial conditions $(\rho_0, u_0)$ such that $\rho_0(x) \geq c_0 > 0$, has a unique global in time smooth solution $\rho(t, x), u(t, x)$.

The strict positivity of the density is needed for ‘unconditional’ regularity: if there are regions such that $\rho_0(x) = 0$ then a Burgers’-like mechanism may lead to blow up even for fractional-type influence kernels [27]. Let us also mention that when the influence kernel is integrable, the finite time blow-up scenario for Lipschitz influence kernels in [7] still applies, even without the assumption that the influence function $\psi$ is Lipschitz. Indeed, since $\psi$ only shows up in the convolutions, and the quantities $\rho$ and $\partial_x \rho$ that $\psi$ convolves with, by proof of contradiction, are assumed to stay bounded, the proof applies for $\psi \in L^1(\mathbb{R})$ as well. In that sense, theorem 1.1 is reasonably sharp, except for our assumptions above that $\psi$ is not just non-integrable but also ‘nicely-behaved’.

In order to explain the proof of theorem 1.1, we recall that the Euler alignment systems (1.2) and (1.3) can be reformulated as
\[ \partial_t \rho + u \partial_x \rho + \rho \mathcal{L} \rho = -G \rho, \tag{1.13} \]
with $G := \partial_t u - \mathcal{L} \rho$, and the operator $\mathcal{L}$ given by
\[ \mathcal{L} f(x) := \int_{\mathbb{R}} \psi(x-y)f(y)dy. \tag{1.14} \]
As in [16, 22–24], one may show that both the density $\rho$ and the function $G$ are uniformly bounded. Thus, (1.13) may be thought of as an integro-differential equation for $\rho(t, x)$ of the form
\[ \partial_t q + v(t, x) \cdot \nabla q + \mathcal{L} q = f(t, x), \tag{1.15} \]
with a bounded function $f(t, x)$ and an operator $\mathcal{L}$ of the form
\[ \mathcal{L} q(t, x) = \int_{\mathbb{R}^d} (q(t, x) - q(t, x+z))k(x, z, t)dz, \tag{1.16} \]
with a kernel $k(x, z, t)$ that obeys bounds similar to $\psi$, when considered as a function of $z$. When $k(x, z, t) = C |z|^{-d-\alpha}, \alpha \in (0, 2)$, the operator $\mathcal{L}$ is the standard fractional Laplacian, and Hölder estimates for such time dependent fractional diffusion equations with a drift have been obtained in [25] using purely analytic techniques. For more general kernels, closer to our assumptions, elliptic estimates in the absence of a drift are provided in [18] applying a combination of analytical and probabilistic methods. These estimates were extended in [12] to time dependent equations with a drift, using a purely probabilistic approach. Both [12, 18] assume that $m(x) = |x| \psi(x)$ varies regularly at zero with index $\alpha \in \mathbb{R}$, in the sense that for every $\lambda > 0$,
\[ \lim_{r \to 0} \frac{m(r)}{m(\lambda r)} = \lambda^\alpha, \tag{1.17} \]
and rely on several properties derived from this assumption. Here, we present an alternative analytical approach to the Hölder estimates for the parabolic equations with a drift, and a weaker than fractional dissipation, based on combining the methods in [18] with a version of the quantitative comparison principle in [25]. This allows us to relax (1.17) to assumptions (1.5), (1.9)–(1.11), and obtain the following Hölder regularity estimate for the solutions to the Euler alignment system, that leads to theorem 1.1.

**Theorem 1.2.** Suppose the above assumptions (1.5), (1.6) and (1.9)–(1.11) hold, and let \( \rho(t, x) \) be a solution to the Euler alignment systems (1.2) and (1.3). There exists \( \beta \in (0, 1) \), \( r_0 > 0 \) and a sufficiently small constant \( c' > 0 \) such that for any \( 0 < t \leq 1 \) and \( |x - y| \leq \min(r_0, M^{-1}(c'/t)) \), we have

\[
|\rho(t, x) - \rho(t, y)| \leq K_0 t^{-\beta} [M(|x - y|)]^{-\beta},
\]

with a constant \( K_0 \) that depends only on the initial conditions \( \rho_0 \) and \( u_0 \).

Let us note that, compared to [25], we need to work with the advection \( u(t, x) \) that is not Lipschitz but only \( M \)-Lipschitz in space: there exists \( C > 0 \) such that

\[
|u(t, x) - u(t, y)| \leq C|x - y|M(|x - y|), \text{ for } |x - y| \leq 1.
\]

This is similar to log-Lipschitz velocities in the Yudovich theory for the Euler equation.

A word on notation: we note by \( C \) a word on notation: we note by \( C > 0 \) universal constants that may change from line to line. For important constants, we denote as \( c', C', C_1 \), etc. to distinguish them. For higher order derivatives in \( x \), we use \( (n) \) to denote, for example, \( \rho^{(n)}(t, x) = \partial^n_x \rho(t, x) \). The torus we use here is \( \mathbb{T} = [-\pi, \pi] \).

## 2. Preliminaries

In this section, we establish some preliminary results for the proof of theorem 1.1.

### 2.1. A reformulation of the Euler alignment system

Let us first recall a convenient reformulation of the Euler alignment system. Applying the operator \( \mathcal{L} \) to (1.2) gives:

\[
0 = \partial_t (\mathcal{L} \rho) + \partial_x (\mathcal{L} (\rho u)) = \partial_t (\mathcal{L} \rho) + \partial_x \left( \int_{\mathbb{R}} \psi(x - y) [\rho(x) u(x) - \rho(y) u(y)] \, dy \right) \tag{2.1}
\]

\[
= \partial_t (\mathcal{L} \rho) + \partial_x \left( \int_{\mathbb{R}} \psi(x - y) [u(x) - u(y)] \rho(y) \, dy \right) + \partial_x \left( u(x) \int_{\mathbb{R}} \psi(x - y) [\rho(x) - \rho(y)] \, dy \right) \\
= \partial_t (\mathcal{L} \rho) + \partial_x \left( \int_{\mathbb{R}} \psi(x - y) [u(x) - u(y)] \rho(y) \, dy \right) + \partial_x (u(x) \mathcal{L} \rho(x)).
\]

Next, we apply \( \partial_x \) to (1.3) to get

\[
\partial_t (\partial_x u) + \partial_x (u \partial_x u) = \partial_x \left( \int_{\mathbb{R}} \psi(x - y) [u(y) - u(x)] \rho(y) \, dy \right). \tag{2.2}
\]

Thus, if we set

\[
G(t, x) := \partial_t u(t, x) - \mathcal{L} \rho(t, x), \tag{2.3}
\]
then, subtracting (2.1) from (2.2), the Euler alignment systems (1.2) and (1.3) can be recast into a system of equations for $\rho$ and $G$

$$\partial_t \rho + \partial_x (\rho u) = 0 \quad (2.4)$$
$$\partial_t G + \partial_x (G u) = 0. \quad (2.5)$$

The velocity field $u$ can be recovered from (2.3) up to a constant. In order to find the constant, note that the averages of $\rho$ and $G$ over $\mathbb{T}$ are preserved in time by (2.4) and (2.5):

$$\kappa := \frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(t, x) \, dx \quad \text{and} \quad \nu := \frac{1}{2\pi} \int_{-\pi}^{\pi} G(t, x) \, dx. \quad (2.6)$$

Therefore, the functions

$$\theta(t, x) := \rho(t, x) - \kappa \quad \text{and} \quad \tilde{\theta}(t, x) = G(t, x) - \nu \quad (2.7)$$

have periodic mean-zero primitive functions $\Phi$ and $\Psi$, respectively:

$$\theta(t, x) = \partial_x \Phi(t, x) \quad \text{and} \quad \tilde{\theta}(t, x) = \partial_x \Psi(t, x). \quad (2.8)$$

Then, $u$ can be written as

$$u(t, x) = L\Phi(t, x) + \Psi(t, x) + I_0(t). \quad (2.9)$$

As in [16], we find that

$$I_0(t) = \frac{1}{2\pi \kappa} \left[ \int_{-\pi}^{\pi} \rho_0(x) u_0(x) \, dx - \int_{-\pi}^{\pi} \rho(t, x) \Psi(t, x) \, dx \right]. \quad (2.10)$$

Note that the function $F = G/\rho$ satisfies

$$\partial_t F + u \partial_x F = 0, \quad (2.11)$$

whence

$$\|F(t, \cdot)\|_{L^\infty} \leq \|F_0\|_{L^\infty}. \quad (2.12)$$

Combining (2.3) and (2.4), we have the following equation for the density

$$\partial_t \rho + u \partial_x \rho + \rho L \rho = -G \rho. \quad (2.13)$$

This equation will be the starting point for our analysis below.

### 2.2. Some properties of the influence kernel

Here, we prove some basic properties of the influence kernel that follow from our assumptions on $\psi(r)$.

**Lemma 2.1.** The function $M(r)$ also satisfies the doubling condition: there exists $C > 0$ so that

$$M(r) \leq CM(2r) \quad \text{for all} \quad r > 0. \quad (2.14)$$
Proof. This is easily seen from a change of variables, using the doubling condition (1.9) on the function \( \psi(r) \):
\[
M(r) = \int_{r}^{\infty} \psi(x) \, dx = \frac{1}{2} \int_{2r}^{\infty} \psi(y/2) \, dy \leq \frac{C}{2} \int_{2r}^{\infty} \psi(y) \, dy = \frac{C}{2} M(2r).
\]
\[\square\]

Lemma 2.2. There exist \( C_1 \) and \( C_2 \) so that for all \( k > 1 \), we have
\[
M(\frac{r}{k}) \leq C_1 C_2^k [M(r)]^k \quad \text{for all} \quad 0 < r < r_0,
\]
with \( r_0 \) as in (1.10).

Proof. Let us define
\[
p(y) := \log[M(e^{-y})].
\]
There exists \( y_0 \geq 0 \) so that the function \( p(y) \) is increasing and concave for \( y \geq y_0 \) because
\[
p'(y) = -\frac{M'(e^{-y})e^{-y}}{M(e^{-y})} = \frac{\psi(e^{-y})e^{-y}}{M(e^{-y})} \geq 0,
\]
and
\[
p''(y) = -\left(\frac{\psi(e^{-y})e^{-y}}{M(e^{-y})}\right)'e^{-y} \leq 0, \quad \text{for} \quad y \geq y_0 = \log r_0^{-1},
\]
due to assumption (1.10). It follows that the function
\[
\frac{p(y) - p(y_0)}{y - y_0}
\]
is strictly decreasing for \( y \geq y_0 \). Hence, for all \( k > 1 \) we have
\[
\frac{p(y_0 + k(y - y_0)) - p(y_0)}{k(y - y_0)} \leq \frac{p(y_0 + y - y_0) - p(y_0)}{y - y_0}, \quad \text{(2.16)}
\]
so that
\[
p(y_0 + k(y - y_0)) \leq kp(y) + p(y_0) - kp(y_0). \quad \text{(2.17)}
\]

Going back to the function \( M \), this says
\[
M(e^{-y_0 - k(y - y_0)}) \leq C^{1-k}[M(e^{-y})]^k, \quad C = e^{p(y_0)} = M(e^{-y_0}) = M(r_0), \quad \text{(2.18)}
\]
which is
\[
M\left(\frac{r^k}{r_0^{k-1}}\right) \leq \frac{1}{[M(r_0)]^{k-1}}[M(r)]^k, \quad 0 < r < r_0, \quad \text{(2.19)}
\]
or
\[
M(x^k) \leq \frac{1}{[M(r_0)]^{k-1}}[M(r_0^{1-k} x)]^k \leq \frac{1}{[M(r_0)]^{k-1}}[M(r_0 x)]^k \leq \frac{C_0^k}{[M(r_0)]^{k-1}}[M(x)]^k, \quad \text{(2.20)}
\]
for all \( 0 < x < r_0 \). We used the doubling property in the last inequality above.
2.3. A pointwise bound on the density

We first obtain uniform bounds on the density \( \rho(t, x) \).

**Proposition 2.3.** There exist \( c_0 > 0 \) and \( C_0 < +\infty \) that depend only on the initial conditions \( u_0(x) \) and \( \rho_0(x) \) so that

\[
0 < c_0 \leq \rho(t,x) \leq C_0, \quad x \in \mathbb{T}, \; t \geq 0.
\]

(2.21)

The proof is a combination of the Constantin–Vicol maximum nonlocal principle used in [23] in the case when \( m(x) \) is singular at \( x = 0 \) with the strategy of [16]. The function \( \Phi(t,x) \) defined in (2.7) and (2.8) satisfies a uniform bound

\[
\|\Phi(t, \cdot)\|_{L^\infty} \leq \|\theta(t, \cdot)\|_{L^1} \leq C\|\rho_0\|_{L^1}.
\]

(2.22)

We have the following version of the Constantin–Vicol nonlocal maximum principle.

**Lemma 2.4.** Let \( \rho(x) \) be a smooth periodic function attaining its maximum at a point \( \bar{x} \in \mathbb{T} \). There exists a positive constant \( c, \tilde{c} \) such that either

\[
\mathcal{L}_p \rho(\bar{x}) \geq c \theta(\bar{x}) M \left( \frac{\|\Phi\|_{L^\infty}}{\theta(\bar{x})} \right), \quad \text{or} \quad \theta(\bar{x}) \leq \tilde{c} \|\Phi\|_{L^\infty}.
\]

(2.23)

**Proof.** The proof is very similar to [13]. Let \( \chi(x) \) be a radially non-decreasing smooth cut-off function such that \( \chi(x) = 0 \) for \( |x| \leq 1/2 \) and \( \chi(x) = 1 \) for \( |x| \geq 1 \). We have, by the periodicity of \( \rho \) and \( \psi(y) \) being decreasing and even, for any \( R \in (0, \pi) \):

\[
\mathcal{L}_p \rho(\bar{x}) = \sum_{j \in \mathbb{Z}} \int_{\mathbb{T}} (\rho(\bar{x}) - \rho(\bar{x} + y)) \psi(y + 2\pi j) \, dy \geq \int_{\mathbb{T}} (\rho(\bar{x}) - \rho(\bar{x} + y)) \psi(y) \chi \left( \frac{y}{R} \right) \, dy
\]

\[
= \int_{\mathbb{T}} (\theta(\bar{x}) - \theta(\bar{x} + y)) \psi(y) \chi \left( \frac{y}{R} \right) \, dy
\]

\[
\geq 2\theta(\bar{x}) \int_{\mathbb{T}} \psi(y) \, dy - \int_{\mathbb{T}} \Phi(\bar{x} + y) \left| \partial_y \left[ \psi(y) \chi \left( \frac{y}{R} \right) \right] \right| \, dy.
\]

We used integration by parts in the last integral above. The Hörmander–Mikhlin condition (1.8) implies

\[
\left| \partial_y \left[ \psi(y) \chi \left( \frac{y}{R} \right) \right] \right| = \left| \psi'(y) \chi \left( \frac{y}{R} \right) + \frac{1}{R} \psi(y) \chi' \left( \frac{y}{R} \right) \right| \leq C \frac{\psi(y)}{|y|} \chi \left( \frac{y}{R} \right) + \frac{\psi(y)}{R} \chi' \left( \frac{y}{R} \right).
\]

(2.24)

Therefore, we get

\[
\mathcal{L}_p \rho(\bar{x}) \geq 2\theta(\bar{x}) \int_{\mathbb{T}} \psi(y) \, dy - C \|\Phi\|_{L^\infty} \int_{\mathbb{T}} \left( \frac{\psi(y)}{|y|} \chi \left( \frac{y}{R} \right) + \frac{\psi(y)}{R} \chi' \left( \frac{y}{R} \right) \right) \, dy
\]

\[
\geq 2\theta(\bar{x})(M(R) - M(\pi)) - C \|\Phi\|_{L^\infty} \int_{R/2}^{\mathbb{T}} \psi(y) \, dy
\]

\[
\geq 2\theta(\bar{x})(M(R) - M(\pi)) - C \frac{\|\Phi\|_{L^\infty}}{R} M(R/2) \geq \theta(\bar{x}) M(R) - C \frac{\|\Phi\|_{L^\infty}}{R} M(R).
\]
provided that $R$ is sufficiently small: $R < R_0$ with $R_0$ independent of the function $\rho$. We used (2.14) in the last inequality above. If we have

$$R_\Phi = \frac{2C\|\Phi\|_{L^\infty}}{\theta(\bar{x})} < R_0,$$

(2.25)

then we can take $R = R_\Phi$, leading to

$$\mathcal{L}\rho(\bar{x}) \geq \frac{\theta(\bar{x})}{2} \left( \frac{C\|\Phi\|_{L^\infty}}{\theta(\bar{x})} \right).$$

On the other hand, if (2.25) does not hold, then we have

$$\theta(\bar{x}) \leq \tilde{c}\|\Phi\|_{L^\infty}.$$  

Proof of proposition 2.3. As $\rho$ satisfies

$$\partial_t \rho + u \partial_x \rho = -\rho \partial_x u,$$  

(2.26)

in order to prove the upper bound $\|\rho(t, \cdot)\|_{L^\infty} \leq C_0$, with some $C_0 > \|\rho_0\|_{L^\infty}$, it is sufficient to show that $\partial_x u(t, \bar{x}) > 0$ if $\rho(t, \bar{x}) \geq C_0$. Moreover, we only need to consider the situation when

$$\theta(t, \bar{x}) \geq C\|\rho_0\|_{L^1},$$  

(2.27)

with a sufficiently large constant $C > 0$ for otherwise we are done. In other words, because of (2.22), we may assume that the first alternative in (2.23) holds. In particular, if $C > 10$, it follows from (2.27) that

$$\frac{1}{2}\rho(t, \bar{x}) \leq \theta(t, \bar{x}) \leq \rho(t, \bar{x}).$$  

(2.28)

Using the function $F = G/\rho$, as in (2.11), the $L^\infty$-bound (2.12) on $F$ and the first alternative in (2.23), together with (2.28), imply that

$$\partial_x u(t, \bar{x}) = F(t, \bar{x})\rho(t, \bar{x}) + \mathcal{L}\rho(t, \bar{x}) \geq c\theta(\bar{x})M(\frac{\|\Phi\|_{L^\infty}}{c\theta(\bar{x})}) - \|F_0\|_{L^\infty}\rho(t, \bar{x})$$

$$\geq \frac{c}{2}\rho(t, \bar{x})M(\frac{\|\Phi\|_{L^\infty}}{c\rho(t, \bar{x})}) - \|F_0\|_{L^\infty}\rho(t, \bar{x})$$

since $M(x)$ is a decreasing function. It follows that $\partial_x u(t, \bar{x}) > 0$ when $\rho(t, \bar{x})$ is large enough because of the uniform bound (2.22) on $\Phi$ and the singularity of $M(r)$ as $r \to 0$. This proves the upper bound in proposition 2.3.

For the uniform positive lower bound, we let $\bar{x}$ be a minimal point so that

$$\rho(t, \bar{x}) = \min_{x \in \mathbb{T}} \rho(t, x),$$

and from (2.13) and $F = G/\rho$ we have

$$\partial_t \rho + u \partial_x \rho = -F\rho^2 - \rho\mathcal{L}\rho.$$
Therefore, at the minimal point we have

$$
\partial_t \rho(t, \mathbf{x}) \geq -\|F_0\|_{L^\infty} \rho(\mathbf{x})^2 + \rho(\mathbf{x}) \sum_{j \in \mathbb{Z}} \int_T (\rho(\mathbf{x}) - \rho(\mathbf{x} + y)) \psi(y + 2\pi j) dy
$$

$$
\geq -\|F_0\|_{L^\infty} \rho(\mathbf{x})^2 + \rho(\mathbf{x}) \int_T (\rho(\mathbf{x}) + y - \rho(\mathbf{x})) \psi(y) dy
$$

$$
\geq -\|F_0\|_{L^\infty} \rho(\mathbf{x})^2 + \rho(\mathbf{x}) \psi(\pi) (2\pi c - 2\pi \rho(\mathbf{x}))
$$

$$
= -\|F_0\|_{L^\infty} + 2\pi \psi(\pi) \rho(\mathbf{x})^2 + 2\pi \psi(\pi) < \rho(\mathbf{x}).
$$

It follows that $\partial_t \rho(t, \mathbf{x}) > 0$ if $\rho(\mathbf{x})$ is sufficiently small, thus there exists $c_0 > 0$ such that $\rho(t, \mathbf{x}) \geq c_0$. □

The uniform bound (2.12) on $F = G/\rho$ and lemma 2.1 give an upper bound on $G$:

$$
|G(t, x)| \leq C|\rho(t, x)| \leq \tilde{C}.
$$

(2.29)

The function $Q = \partial_t F/\rho$ also satisfies the transport equation

$$
\partial_t Q + u \partial_x Q = 0,
$$

(2.30)

which gives the bounds

$$
|\partial_x F(t, x)| \leq C|\rho(t, x)| \leq C,
$$

$$
|\partial_x G(t, x)| \leq |\partial_x F|/\rho + |F||\partial_x \rho| \leq C|\rho(t, x)|^2 + |\rho_x(t, x)| \leq C(1 + |\rho_x(t, x)|).
$$

(2.31)

The arguments leading to (2.29)–(2.31) can be iterated to obtain a higher order control of $G$: the function $\partial_t Q/\rho$ satisfies the transport equation, and so on, leading to the hierarchical point-wise bounds as in [22, 24]:

$$
|G^{(n)}(t, x)| \leq C_n(|\rho^{(n)}(t, x)| + |\rho^{(n-1)}(t, x)| + \cdots + |\rho(t, x)|).
$$

(2.32)

3. The proof of theorem 1.2

3.1. A Hölder regularity result for a class of linear integro-differential equations

We first investigate the Hölder estimates for solutions to a class of integro-differential equations of the form

$$
\partial_t q + v(t, x) \cdot \nabla q + \mathcal{L}q = 0,
$$

(3.1)

in $\mathbb{R}^d$ with an operator $\mathcal{L}$ given by

$$
\mathcal{L}q(t, x) = \int_{\mathbb{R}^d} (q(t, x) - q(t, x + z)) k(x, z, t) dz,
$$

(3.2)

and a kernel $k(x, z, t)$ such that there exist $C > 0$ and a function $\eta(r)$ such that

$$
\frac{1}{C} \eta(|z|) \leq k(x, z, t) \leq C \eta(|z|).
$$

(3.3)
We assume here \( \eta(r) \) satisfies the following properties, as in our assumptions on \( \psi(r) \). First, we suppose that for any \( \alpha \in (0, 1) \), there exists \( c_\alpha \) such that
\[
\eta(r) \geq \frac{1}{c_\alpha r^{d-\alpha}} \quad \text{for all } 0 < r \leq 1,
\]
and
\[
M_\eta(r) := \int_r^\infty \eta(s)s^{d-1} \, ds \to +\infty \text{ as } r \to 0.
\]
(3.4)

We also assume that the function \( M_\eta(r) \) satisfies (1.10) in the \( d \)-dimensional form
\[
\frac{r^d \eta(r)}{M_\eta(r)} \text{ is non-decreasing for } 0 \leq r \leq r_0,
\]
and (1.11), which, as we recall, implies
\[
sM_\eta(s) \leq rM_\eta(r) \quad \text{for } 0 < s \leq r \leq r_0.
\]
(3.5)

We also assume that the drift \( \nu(t, x) \) grows at most linearly at infinity, and is \( M_\eta \)-Lipschitz continuous in \( x \) for each \( t > 0 \), in the sense that there exist \( C, C' > 0 \) such that
\[
\frac{\nu(t, x)}{1 + |x|} \leq C, \quad \text{for all } 0 \leq t, \text{ and } x \in \mathbb{R},
\]
(3.6)
\[
|\nu(t, x) - \nu(t, y)| \leq C'|x - y|M_\eta(|x - y|) \quad \text{for } t \geq 0 \text{ and } |x - y| \leq 1.
\]
(3.8)

Our goal in this section is to show the following by an analytical approach.

**Proposition 3.1.** Assume that (3.3)–(3.8) hold, and let \( q(t, x) \) be a solution to (3.1). There exists \( \beta \in (0, 1), r_0 > 0 \) and a sufficiently small constant \( c' > 0 \) such that for any \( 0 < t \leq 1 \) and \( |x - y| \leq \min(r_0, M_\eta^{-1}(c'/t)) \), we have
\[
|q(t, x) - q(t, y)| \leq Kt^{-\beta}[M_\eta(|x - y|)]^{-\beta} \sup_{0 \leq \zeta \leq 1, \zeta \in \mathbb{R}^d} |q(t, x)|.
\]
(3.9)

The first step in the proof of proposition 3.1 is to re-center: fix \( (t_0, x_0) \) with \( 0 < t_0 < 1 \) and \( x_0 \in \mathbb{R}^d \), and write
\[
\tilde{q}(t, x) = q(t_0 + t, x_0 + x - A(t)) - q(t_0, x_0)
\]
with the function \( A(t) \) to be determined. Note that \( \tilde{q}(0, 0) = 0 \), as long as \( A(0) = 0 \). The function \( \tilde{q}(t, x) \) satisfies
\[
\partial_t \tilde{q} + \tilde{v}(t, x) \cdot \nabla \tilde{q} + \mathcal{L} \tilde{q} = 0,
\]
(3.10)
with
\[
\tilde{v}(t, x) = v(t + t_0, x + x_0 - A(t)) + A'(t),
\]
(3.11)
and
\[
\mathcal{L} \tilde{q}(t, x) = \int_{\mathbb{R}^d} (\tilde{q}(t, x) - \tilde{q}(t, x + z)) \tilde{k}(t, x, z) \, dz
\]
(3.12)
where
\[
\tilde{k}(t, x, z) = k(t + t_0, x + x_0 - A(t), z).
\]

Note that the function \(\tilde{k}(x, z, t)\) still satisfies the bounds (3.3) by \(\eta(z)\) from above and below. We choose \(A(t)\) as a solution to
\[
A'(t) = -v(t_0 + t, x_0 - A(t)), \quad A(0) = 0,
\]
so that \(\tilde{v}(t, 0) = 0\). A solution to (3.13) exists due to the continuity of \(v\) in \(x\). Note that by the \(M_\eta\)-Lipschitz continuity (3.8) of \(v(t, x)\) in \(x\) and the choice of \(A(t)\) we have
\[
|\tilde{v}(t, x)| = |v(t + t_0, x + x_0 - A(t)) - v(t_0 + t, x_0 - A(t))| \leq C|x|M_\eta(|x|), \quad |x| \leq 1,
\]
and because of (3.7), \(\tilde{v}(t, x)\) is sub linear at infinity:
\[
|\tilde{v}(t, x)| \leq C_0(1 + |x|),
\]
with a constant \(C_0\) that depends on \(x_0\) and \(t_0\).

To use a De Giorgi-type argument, given \(r > 0\), we define an \(M_\eta\)-parabolic cylinder
\[
Q(r) := (-c'/M_\eta(r), 0) \times B(r).
\]

Here, \(B(r) := \{ x \in \mathbb{R}^d : |x| < r \}\) is a ball in \(\mathbb{R}^d\), and \(c' > 0\) is sufficiently small to be chosen later. Proposition 3.1 is a consequence of the following lemma.

**Lemma 3.2.** There exist a sufficiently large constant \(K > 0\) and a sufficiently small constant \(c' > 0\) that do not depend on \((t_0, x_0)\), and \(r_0 > 0\) so that for all \(0 < r < r_0\) we have
\[
|\tilde{q}(t, x)| \leq K[M_\eta(r)]^\beta \left( \frac{|t|}{c'} + \frac{1}{[M_\eta(|x|)]} \right)^\beta \sup_{-c'/M_\eta(r) \leq t \leq 0, x \in \mathbb{R}^d} |\tilde{q}(t, x)|, \quad \text{for all } (t, x) \in Q(r).
\]

(3.17)

In terms of the function \(q(t, x)\), (3.17) says that
\[
|q(t + t_0, x + x_0 - A(t)) - q(t_0, x_0)| \leq 2K[M_\eta(r)]^\beta \left( \frac{|t|}{c'} + \frac{1}{[M_\eta(|x|)]} \right)^\beta \sup_{t_0 - c'/M_\eta(t) \leq t \leq t_0, x \in \mathbb{R}^d} |q(t, x)|,
\]

(3.18)

for \((t, x) \in Q(r)\). Taking \(r = M_\eta^{-1}(c'/|t_0|)\) and setting \(t' = t_0 + t\) and \(x' = x_0 + x - A(t)\) in (3.18) gives
\[
|q(t', x') - q(t_0, x_0)| \leq 2K \left( \frac{|t' - t_0|}{c'} + \frac{1}{M_\eta(|x' - x_0 - A(t)|)} \right)^\beta \sup_{0 \leq t \leq t_0, x \in \mathbb{R}^d} |q(t, x)|,
\]

(3.19)
for $0 \leq t' \leq t_0$, and $|x' - x_0 + A(t)| \leq M_q^{-1}(c'/|t_0|)$. It follows from (3.15), that there exists $\lambda_0 > 0$ which depends on $(t_0, x_0)$ such that

$$|A(t)| \leq \lambda_0 e^{c|t_0|} - 1,$$

with $C_0$ as in (3.15), thus

$$|q(t', x') - q(t_0, x_0)| \leq \frac{2K}{|t_0|^q} \left( \frac{|t' - t_0|}{c'} + \frac{1}{M_q(|x' - x_0| + \lambda_0 (e^{c|t_0|} - 1))} \right)^\beta \sup_{0 \leq t \leq t_0, x \in \mathbb{R}^d} |q(t, x)|,$$

for $t' \in (0, t_0]$ and

$$|x' - x_0| + \lambda_0 (e^{c|t_0|} - 1) \leq M_q^{-1}(c'/|t_0|).$$

(3.20)

Setting $t' = t_0$ in (3.20) and (3.21) finishes the proof of proposition 3.1. Note that both constants $C_0$ and $\lambda_0$ that depend on $(t_0, x_0)$ disappear when $t' = t_0$. Thus, we only need to prove lemma 3.2.

**Proof of lemma 3.2.** The proof uses a De Giorgi type argument. We fix $r > 0$, and normalize $\tilde{q}(t, x)$, setting

$$w(t, x) = \frac{\tilde{q}(t, x)}{2\|\tilde{q}(t, x)\|_{L^\infty([-c/M_q(t), 0] \times \mathbb{R}^d)}}.$$

We also define a decreasing sequence of radii $r_n$ as

$$r_n = M_q^{-1}(a^n M_q(r_n)),$$

with some $a > 2$ to be specified later. Note that

$$M_q(r_n) = a M_q(r_{n-1}),$$

thus $M_q(r_n) \to +\infty$, and $r_n \to 0$, as $n \to \infty$.

**Lemma 3.3.** There exists $b \in (1, 2)$ and $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$ we have

$$\text{osc}_{Q(r_n)} w(t, x) \leq b^{1-n}.$$  

(3.23)

Lemma 3.2 is an immediate consequence of lemma 3.3. As $r_n \to 0$, for any $(t, x) \in \bar{Q}(r)$, there exists $n \in \mathbb{N}$ such that $(t, x) \in Q(r_n) \setminus Q(r_{n+1})$, so that

$$\text{either } \frac{c'}{M_q(r_{n+1})} < |t| \leq \frac{c'}{M_q(r_n)} \text{ or } r_{n+1} \leq |x| < r_n.$$  

Thus we have

$$|w(t, x)| = |w(t, x) - w(0, 0)| \leq \text{osc}_{Q(r_n)} w(t, x) \leq b^{1-n}$$

$$= b \left(\frac{1}{a^n}\right) \leq b \left(\frac{M_q(r)}{M_q(r_{n+1})}\right) \leq b [M_q(r)]^{b/a} \left(\frac{|t|}{c'} + \frac{1}{M_q(|x|)}\right)^{b/a}.$$  

thus (3.17) holds with $\beta = \log b / \log a$, finishing the proof of lemma 3.2.
Proof of lemma 3.3. We prove (3.23) by induction. For \( n = 1 \), it holds automatically since \(|w(t,x)| \leq 1/2\) for all \((t,x) \in Q(r)\). Suppose that (3.23) holds for \(1 \leq k \leq n\) and set

\[
m := \frac{1}{2} \left( \sup_{Q(r_n)} w(t,x) + \inf_{Q(r_n)} w(t,x) \right),
\]

so that the function

\[
\tilde{w}(t,x) := 2b^{n-1}(w(t,x) - m),
\]

satisfies \(|\tilde{w}(t,x)| \leq 1\) for \((t,x) \in Q(r_n)\). It is convenient to set

\[
\varphi(r) = M_0^{-1}(a^{-1}M_0(r))
\]

so that \(r_n = \varphi(r_{n+1})\), and the measure

\[
\mu(dy) = \frac{\eta(|y|)}{M_0(|y|)} \, dy.
\]

To proceed with the induction argument, we need to find \(a > 2\), so that

\[
(-c'/M_0(r_n), -c'/M_0(r_{n+1})] = (-ac'/M_0(r_{n+1}), -c'M_0(r_{n+1})]
\]

\[
\supset (-2c'/M_0(r_{n+1}), -c'M_0(r_{n+1})],
\]

and \(\theta \in (0,1)\), so that if \(|\tilde{w}(t,x)| \leq 1\) on \(Q(r_n)\), and

\[
\mu \left( (t,x) \in (-2c'/M_0(r_{n+1}), -c'/M_0(r_{n+1})] \times [B(\varphi(r_{n+1})) \setminus B(r_{n+1})] : \tilde{w}(t,x) \leq 0 \right) \\
\geq \frac{1}{2} \mu \left( (-2c'/M_0(r_{n+1}), -c'/M_0(r_{n+1})] \times [B(\varphi(r_{n+1})) \setminus B(r_{n+1})] \right),
\]

then we have \(\tilde{w} \leq 1 - \theta\) on \(Q(r_{n+1})\). This requires the following lemma on lowering the maximum.

Lemma 3.4. Fix \(0 < r \leq r_0\), with \(M_0(r_0) \geq 1\), and fix \(\delta > 0\). Assume that \(w(t,x) \leq 1\) for all \((t,x) \in Q(\varphi(r))\) and the function \(w(t,x)\) satisfies

\[
\dot{\partial}_t w(t,x) + v(t,x) \cdot \nabla w(t,x) + Lw(t,x) = 0, \quad (t,x) \in Q(\varphi(r)),
\]

with

\[
|v(t,x)| \leq C\lambda|\nabla w(t,x)| \leq 1,
\]

and the operator \(L\) as in (3.2)–(3.6). There exists a constant \(c' > 0\) sufficiently small, and another constant \(\theta \in (0,1)\), so that if

\[
\mu \left( \{w(t,x) \leq 0\} \cap (-2c'/M_0(r), -c'/M_0(r)] \times B(\varphi(r)) \setminus B(r) \right) \geq \delta > 0,
\]

then

\[
w \leq 1 - \theta \text{ in } Q(r).
\]
The conclusion of lemma 3.2 follows from lemma 3.3. Indeed, this lemma implies that if (3.25) holds, then we have \( w(t, x) \leq 1 - \theta \) on \( Q(r_{n+1}) \), so that
\[
u(t, x) \leq \frac{1 - \theta}{2} b^{1-n} + m \quad \text{for} \quad (t, x) \in Q(r_{n+1}).
\]
As \( \inf_{Q(r_{n+1})} w(t, x) \geq \inf_{Q(r)} w(t, x) \), then the oscillation of \( w(t, x) \) on \( Q(r_{n+1}) \) is bounded by
\[
\text{osc}_{Q(r_{n+1})} w(t, x) \leq \frac{1 - \theta}{2} b^{1-n} + m - \inf_{Q(r_{n+1})} w(t, x) \\
\leq \frac{1 - \theta}{2} b^{1-n} + m - \inf_{Q(r_{n+1})} w(t, x) = \frac{1 - \theta}{2} b^{1-n} + \frac{1}{2} \left( \sup_{Q(r)} w(t, x) - \inf_{Q(r_{n+1})} w(t, x) \right) \\
= \frac{1 - \theta}{2} b^{1-n} + \frac{1}{2} \text{osc}_{Q(r)} w(t, x) \leq \frac{1 - \theta}{2} b^{1-n} + \frac{1}{2} b^{1-n} = \frac{1 - \theta}{2} b^{1-n} = b^{-n},
\]
if we choose
\[
b = \frac{2}{2 - \theta} \in (1, 2).
\]
We used the induction assumption (3.23) in the last inequality above. If (3.25) does not hold, then we have
\[
\mu \left( \{ \tilde{w}(t, x) > 0 \} \cap \left( -2c'/M_y(r_{n+1}), -c'/M_y(r_{n+1}) \right] \times \left( B(\varphi(r_{n+1})) \setminus B(r_{n+1}) \right) \right) \\
\geq \frac{1}{2} \mu \left( -2c'/M_y(r_{n+1}), -c'/M_y(r_{n+1}) \right] \times \left( B(\varphi(r_{n+1})) \setminus B(r_{n+1}) \right),
\]
and we can repeat the argument above for \( -\tilde{w}(t, x) \). This finishes the proof of lemma 3.3.

**Proof of lemma 3.4.** Let \( g(r) \), \( r > 0 \), be a radially smooth non-increasing function such that \( g(r) > 0 \) for \( 0 \leq r < 2 \), with \( g(r) = 1 \) for \( 0 \leq r \leq 1 \), and \( g(r) = 0 \) for \( r \geq 2 \). We set
\[
g_r(t, x) = g \left( \frac{|x| + C_r r M_y(r)}{r} \right) = g \left( \frac{|x| - C_r r M_y(r)}{r} \right),
\]
with a large constant \( C_r \) to be chosen later. Condition (3.6) implies that
\[
\frac{M_y(2r + 2C_r c r)}{M_y(r)} \geq \frac{1}{2(1 + C_r c)} \geq \frac{1}{a}
\]
if we choose \( a \) sufficiently large. As
\[
M_y(\varphi(r)) = a^{-1} M_y(r),
\]
it follows that
\[
\varphi(r) \geq 2r + C_r r M_y(r) \quad \text{for} \quad -2c'/M_y(r) \leq t \leq 0.
\]
We also set
\[
\mu(t) = \mu \left( \{ w(t, x) \leq 0 \} \cap \left( B(\varphi(r)) \setminus B(r) \right) \right), \tag{3.30}
\]
and
\[
\bar{\mu}_t = \mu \left( \{ w \leq 0 \} \cap \left( -2c'/M_y(r), -c'/M_y(r) \right] \times \left( B(\varphi(r)) \setminus B(r) \right) \right) \geq \delta, \tag{3.31}
\]
by (3.27).

Our goal will be to show that
\[ w(t, x) \leq 1 - \zeta(t)\varphi(t, x) \text{ in } Q(r). \]  
(3.32)
The function \( \zeta(t) \) in (3.32) obeys the ODE
\[ \zeta'(t) = \sigma \mu(t) - C_1 M_\eta(r) \zeta(t), \quad -2c'/M_\eta(r) \leq t \leq 0, \]  
(3.33)
with the initial condition
\[ \zeta \left( -\frac{2c'}{M_\eta(r)} \right) = 0. \]
To ensure that \( \zeta(t) \) is \( C^1 \) on \( (-ac'/M_\eta(r), 0] \), we extend it to \( t \leq -2c'/M_\eta(r) \) so that
\[ \lim_{t \to (-2c'/M_\eta(r))^-} \zeta(t) = \sigma \mu(-2c'/M_\eta(r)), \]
and \( \zeta(t) \leq 0 \) for \( t \leq -2c'/M_\eta(r) \). The solution to (3.33) is
\[ \zeta(t) = \int_{-2c'/M_\eta(r)}^t \sigma \mu(s)e^{-C_1 M_\eta(r)(t-s)} \, ds. \]
Hence, we have a lower bound
\[ \zeta(t) \geq \sigma e^{-2c'C_1} \int_{-2c'/M_\eta(r)}^t \mu(s)ds \geq \sigma e^{-2c'C_1} \bar{\mu} \geq \sigma e^{-2c'C_1} \delta, \]  
(3.34)
for all \( t \in [-c'/M_\eta(r), 0] \). Going back to (3.32), it follows that
\[ w(t, x) \leq 1 - \sigma e^{-2c'C_1} \delta \text{ in } Q(r), \]  
(3.35)
thus (3.28) holds with \( \theta = \sigma e^{-2c'C_1} \delta \) if we choose a small \( \sigma \) and a sufficiently large \( C_1 \).

The proof of (3.32) is by contradiction. Suppose that
\[ w(t, x) > 1 - \zeta(t)\varphi(t, x) \]  
(3.36)
at some point \( (t, x) \in Q(r) \). Let \( (t_0, x_0) \) be the maximal point of the function
\[ w(t, x) + \zeta(t)\varphi(t, x), \]
so that, in particular,
\[ w(t_0, x_0) + \zeta(t_0)\varphi(t_0, x_0) \geq 1. \]  
(3.37)
As a consequence of (3.37) and the assumption that \( w(t, x) \leq 1, t_0 \) must be in the time interval where \( \zeta(t) > 0 \) and \( (t_0, x_0) \) must be in the support of \( \varphi \), hence \( t_0 \in (-2c'/M_\eta(r), 0] \) and
\[ |x_0| < 2r(1 + C_c c'). \]  
(3.38)
Thus, at \( (t_0, x_0) \) we have
\[ \partial_t w(t_0, x_0) + \zeta'(t_0)\partial_t \varphi(t_0, x_0) + \zeta(t_0)\partial_t \varphi(t_0, x_0) \geq 0, \]
\[ \nabla w(t_0, x_0) + \zeta(t_0)\nabla \varphi(t_0, x_0) = 0. \]
This gives a lower bound

\[
0 = \partial_t w(t_0, x_0) + v(t_0, x_0) \cdot \nabla w(t_0, x_0) + \mathcal{L}w(t_0, x_0)
\geq -\zeta'(t_0)\varphi(t_0, x_0) + C_v r M_\delta(r)\zeta(t_0)\|\nabla \varphi(t_0, x_0)\| - \zeta(t_0) v(t_0, x_0) \cdot \nabla \varphi(t_0, x_0) + \mathcal{L}w(t_0, x_0)
\geq -\zeta'(t_0)\varphi(t_0, x_0) + (C_v r M_\delta(r) - C'|x_0|M_\delta(|x_0|)) \zeta(t_0)\|\nabla \varphi(t_0, x_0)\| + \mathcal{L}w(t_0, x_0).
\]

(3.39)

When \(|x_0| < r\), then \((t_0, x_0)\) satisfies

\[
\|x_0\| - C_v r M_\delta(r)|t_0| < r,
\]

(3.40)
given \(C_v r M_\delta(r)|t_0| < r\), which requires

\[
C_v c' < 1/2.
\]

(3.41)

Condition (3.41) holds if we pick \(c'\) small. With that, whenever \(|x_0| < r\), \(\varphi(t_0, x_0) = 1\) and thus the term in (3.39) with \(\nabla \varphi(t_0, x_0) = 0\) disappears. So we may consider the more difficult case \(|x_0| \geq r\). Combining (3.38) and \(M_\delta(|x_0|) \leq M_\delta(r)\), we have

\[
|x_0|M_\delta(|x_0|) \leq 2r(1 + C_v c'|M_\delta(r).
\]

Therefore, it gives

\[
C_v r M_\delta(r) - C'|x_0|M_\delta(|x_0|) \geq C_v r M_\delta(r) - 2rC'(1 + C_v c')M_\delta(r)
\]

\[
= C_v r M_\delta(r) \left[ 1 - \frac{2C'(1 + C_v c')}{C_v} \right] \geq 0,
\]

if we take \(C_v \geq 4C\) and \(c' < 1/(8C)\), so that

\[
\frac{2C'(1 + C_v c')}{C_v} = \frac{2C}{C_v} + 2C c' < 1.
\]

(3.42)

Note that we can ensure that (3.42) holds while keeping (3.41) intact, if we first choose \(C_v\) large and then \(c'\) small.

Now, no matter where \(x_0\) locates, (3.39) becomes

\[
0 = \partial_t w(t_0, x_0) + v(t_0, x_0) \cdot \nabla w(t_0, x_0) + \mathcal{L}w(t_0, x_0) \geq -\zeta'(t_0)\varphi(t_0, x_0) + \mathcal{L}w(t_0, x_0).
\]

(3.43)

To get a contradiction, we will need a lower bound on \(\mathcal{L}w(t_0, x_0)\) in the right side of (3.43).

First, we write

\[
\mathcal{L} \varphi(t, x) = \int_{\mathbb{R}^d} [\varphi(t, x) - \varphi(t, x + \varepsilon)]k(x, z, t)dz
\]

\[
= \int_{\mathbb{R}^d} \left[ \varphi \left( \frac{|x| - C_v r M_\delta(r)|t|}{r} \right) - \varphi \left( \frac{|x + \varepsilon| - C_v r M_\delta(r)|t|}{r} \right) \right] k(x, z, t)dz,
\]

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hence

\[ \mathcal{L}\partial_t(t, r) = \int_{\mathbb{R}^d} [\partial_t(t, r) - \partial_r(t, r \pm z)] k(r, z, t) dz \]

\[ = r^{-d} \int_{\mathbb{R}^d} \left[ \partial \left( \frac{|r \pm z| - C_r M_\eta(r)|t|}{r} \right) - \partial \left( \frac{|r| - C_r M_\eta(r)|t|}{r} \right) \right] k(r, r \pm z, t) dz, \]

\[ = r^{-d} \int_{\mathbb{R}^d} \left[ \partial(|r| - C_r M_\eta(r)|t|) - \partial(|r \pm z| - C_r M_\eta(r)|t|) \right] k(r, r \pm z, t) dz. \]

Let us set \( \tilde{\partial}(t, x) = \partial(|x| - C_r M_\eta(r)|t|) \), so that

\[ \mathcal{L}\partial_t(t, r) = r^{-d} \int_{\mathbb{R}^d} \left[ \tilde{\partial}(t, |y|) - \tilde{\partial}(t, |y + z|) \right] k(r, r \pm z, t) dz. \]  \hspace{1cm} (3.44)

Note that if \( \partial_t(t, r) = 0 \), then \( \tilde{\partial}(t, y) = 0 \), and (3.44) becomes

\[ \mathcal{L}\partial_t(t, r) = -r^{-d} \int_{\mathbb{R}^d} \tilde{\partial}(t, |y + z|) k(r, r \pm z, t) dz \leq -C r^{-d} \int_{\mathbb{R}^d} \tilde{\partial}(t, |y + z|) \eta(rz) dz \]

\[ \leq -C r^{-d} \int_{\mathbb{R}^d} \tilde{\partial}(t, |y + z|) \frac{1}{c_\alpha |rz|^{\alpha - d}} dz \]

\[ = -\frac{C}{c_\alpha} r^{-2\alpha + d} \int_{\mathbb{R}^d} \tilde{\partial}(t, |y + z| - C_r M_\eta(r)|t|) |z|^{\alpha - d} dz. \]  \hspace{1cm} (3.45)

Due to \( |t| < 2c_\gamma M_\eta(r) \) and (3.41), the time dependent shift in \( \eta \) is of \( O(1) \). Because \( \tilde{\partial} \) is compactly supported, we may pick sufficiently large \( a \) to make \( |rz| < 1 \) guaranteed, so that we can apply the lower bound of \( \eta \) to factor \( r \) out from the integral. Observe that when \( t = 0 \) and \( |y| = 2 \), the right side of (3.45) is strictly negative. It follows that there exists a universal constant \( c_1 > 0 \) so that we still have

\[ \mathcal{L}\partial_t(t, x) \leq 0 \quad \text{provided that} \quad \partial_t(t, x) \leq c_1. \]  \hspace{1cm} (3.46)

We will consider two cases for a lower bound on \( \mathcal{L}w(t_0, x_0) \): if \( \partial_t(t_0, x_0) > c_1 \), then we will show that

\[ \mathcal{L}w(t_0, x_0) \geq -C \zeta(t_0) M_\eta(r_0) + \sigma M_\eta(r_0) \mu(t_0). \]  \hspace{1cm} (3.47)

Using this estimate, together with the ODE (3.33) for \( \zeta(t) \) in (3.43) gives

\[ 0 \geq \sigma \mu(t_0) \left( M_\eta(r) - \partial_t(t_0, x_0) \right) + M_\eta(r) \zeta(t_0) (C_1 \partial_t(t_0, x_0) - C). \]  \hspace{1cm} (3.48)

The condition

\[ M_\eta(r) \geq M_\eta(r_0) \geq 1 \]  \hspace{1cm} (3.49)

in lemma 3.4 ensures that the first term in the right side of (3.48) is non-negative. Moreover, since \( \partial_t(t_0, x_0) > c_1 \) and \( \zeta(t_0) > 0 \), we get a contradiction if we choose \( C_1 \) in (3.33) large enough.

When \( \partial_t(t_0, x_0) \leq c_1 \), we will obtain the following lower bound for \( \mathcal{L}w(t_0, x_0) \):

\[ \mathcal{L}w(t_0, x_0) \geq \sigma M_\eta(r) \mu(t_0). \]  \hspace{1cm} (3.50)
Again, using this estimate, together with (3.33) in (3.43) gives

\[ 0 \geq \sigma \mu(t_0) \left( M_\eta(r) - \varphi_r(t_0, x_0) \right) + M_\eta(r) \zeta(t_0) C_1 \varphi_r(t_0, x_0). \]  

This is a contradiction to (3.49) and \( \varphi_r(t_0, x_0) < 1 \) for any \( C_1 \geq 0 \).

It remains to show that estimates (3.47) and (3.50) hold in their domains of validity. We will drop \( t_0 \) in \( \varphi_r(t_0, x_0) \) as it does not affect the following computation. Since the function \( w + \zeta \varphi_r \) obtains its maximum at \( x_0 \), we have

\[ w(t_0, x_0) - w(t_0, x_0 + y) \geq -\zeta(t_0) (\varphi_r(x_0) - \varphi_r(x_0 + y)), \]

for all \( y \in \mathbb{R}^d \). Note that if \( w(t_0, x_0 + z) \leq 0 \), then, because of (3.37), we have

\[ w(t_0, x_0) + \zeta(t_0) \varphi_r(x_0) - w(t_0, x_0 + z) - \zeta(t_0) \varphi_r(x_0 + z) \geq 1 - \zeta(t_0) \geq \frac{1}{2}, \]

if we choose \( c_0 \) in (3.33) to be sufficiently small. Let us introduce the ‘good’ set

\[ G := \{ w \leq 0 \} \cap (B(\varphi(r))) \setminus B(r), \]

and write

\[
\mathcal{L} w(t_0, x_0) = \int_{\mathbb{R}^d} (w(t_0, x_0) - w(t_0, x_0 + y)) k(x_0, y, t_0) \, dy \\
= \int_{\mathbb{R}^d} (w(t_0, x_0) + \zeta(t_0) [\varphi_r(x_0) - \varphi_r(x_0 + y)] - w(t_0, x_0 + y)) k(x_0, y, t_0) \, dy \\
- \zeta(t_0) \int_{\mathbb{R}^d} [\varphi_r(x_0) - \varphi_r(x_0 + y)] k(x_0, y, t_0) \, dy \\
= \int_{x_0 + y \in G} (w(t_0, x_0) + \zeta(t_0) \varphi_r(x_0) - \zeta(t_0) \varphi_r(x_0 + y) - w(t_0, x_0 + y)) k(x_0, y, t_0) \, dy \\
+ \int_{x_0 + y \notin G} (w(t_0, x_0) + \zeta(t_0) \varphi_r(x_0) - \zeta(t_0) \varphi_r(x_0 + y) - w(t_0, x_0 + y)) k(x_0, y, t_0) \, dy \\
- \zeta(t_0) \int_{\mathbb{R}^d} [\varphi_r(x_0) - \varphi_r(x_0 + y)] k(x_0, y, t_0) \, dy \\
\geq -\zeta(t_0) \int_{\mathbb{R}^d} (\varphi_r(x_0) - \varphi_r(x_0 + y)) k(x_0, y, t_0) \, dy \\
+ \int_{x_0 + y \in G} (w(t_0, x_0) + \zeta(t_0) \varphi_r(x_0) - w(t_0, x_0 + y) - \zeta(t_0) \varphi_r(x_0 + y)) k(x_0, y, t_0) \, dy \\
\geq -\zeta(t_0) \int_{\mathbb{R}^d} (\varphi_r(x_0) - \varphi_r(x_0 + y)) k(x_0, y, t_0) \, dy + \frac{1}{2C} \int_{x_0 + y \notin G} \eta(y) \, dy =: J_1 + J_2.
\]  

(3.53)
We used (3.52) in the last two steps above. To bound \( J_1 \), when \( \varphi_1(t_0, x_0) > c_1 \) we write

\[
J_1 \geq -C \zeta(t_0) \left( \int_{|y| \leq r} \| \nabla \varphi \|_{L^\infty} |y| \eta(y) \, dy + 2 \| \varphi \|_{L^\infty} \int_{|y| \geq r} \eta(y) \, dy \right) \\
\geq -C \zeta(t_0) \left( \frac{1}{r} \int_{|y| \leq r} |y| \eta(y) \, dy + M_\eta(r) \right) \geq -C \zeta(t_0) M_\eta(r),
\]

(3.54)

because assumption (3.5) implies that there exists \( C \) such that

\[
|y|^d \eta(y) \leq CM_\eta(|y|) \quad \text{for all} \quad 0 \leq |y| \leq r_0,
\]

(3.55)

and we can take \( \gamma \in (0, 1/2) \) as in the assumption (1.11) to have \( |y|^\gamma M_\eta(|y|) \) be an increasing function for \( 0 \leq |y| < r_0 \), so that the first term in the last line satisfies

\[
\int_{|y| \leq r} |y| \eta(y) \, dy \leq C \int_{|y| \leq r} M_\eta(|y|) \, dy = C \int_{|y| \leq r} |y|^\gamma M_\eta(|y|) \, dy \leq Cr^\gamma M_\eta(r) \int_{|y| \leq r} \frac{1}{|y|^\gamma + \nu-1} \, dy = Cr^\gamma M_\eta(r). \]

(3.56)

When \( \varphi_1(t_0, x_0) \leq c_1 \), then we simply have \( J_1 \geq 0 \) because of (3.46). For a bound on \( J_2 \), note that \( |x_0| < 2r(1 + C_\nu c') < 3r \), and we use the inequality

\[
|y| \leq |x_0 + y| + |x_0| \leq |x_0 + y| + 3r \leq 4|x_0 + y|
\]

for \( |x_0 + y| \geq r \). Assumption (3.6) gives

\[
\eta(y) \geq \eta(4|x_0 + y|) \geq C \eta(|x_0 + y|),
\]

thus

\[
J_2 \geq C \int_{x_0 + y \in G} \eta(x_0 + y) \, dy = C \int_{y \in G} \eta(y) \, dy = C \int_{y \in G} M_\eta(|y|) \mu(dy)
\]

\[
\geq CM_\eta(\rho(r)) \mu(G) = \frac{C}{a} M_\eta(r) \mu(t) \geq \sigma M_\eta(r) \mu(t),
\]

(3.57)

by choosing \( \sigma \) small so that \( \sigma \leq C/a \). The above estimates for \( J_1 \) and \( J_2 \) lead to (3.47) and (3.50) in their respective cases.

It is straightforward to extend proposition 3.9 to equations with a forcing in the following way.

**Lemma 3.5.** Under the above assumptions, solutions of

\[
\partial_t q + v(t, x) \cdot \nabla q + \mathcal{L} q = f(t, x),
\]

(3.58)

with a uniformly bounded function \( f(t, x) \) satisfy

\[
|q(t, x) - q(t, y)| \leq C t^{-\beta} \left[ M_{\eta}(|x - y|) \right]^{-\beta} \left[ \sup_{0 \leq s \leq 1, x \in \mathbb{R}^d} |q(s, x)| + t \sup_{0 \leq s \leq 1, x \in \mathbb{R}^d} |f(s, x)| \right] .
\]

(3.59)
Proof. This is a simple consequence of the Duhamel formula. Let \( K(t, x) \) be the Green’s function for the operator \( v \cdot \nabla + L \), so that the solution \( q(t, x) \) to (3.1) with the initial condition \( q(0, x) = q_0(x) \) is
\[
q(t, x) = \int_{\mathbb{R}^d} K(t, x - y)q_0(y)dy.
\]

Then (3.9) implies that for \( 0 < t \leq 1 \) and \( |x_1 - x_2| \leq r_0 \), we have
\[
\frac{|q(t, x_1) - q(t, x_2)|}{[M(|x_1 - x_2|)]^{-\beta}} \leq \left| \int_{\mathbb{R}^d} (K(t, x_1 - y) - K(t, x_2 - y))\beta q_0(y)dy \right| \leq C\|q_0\|_{\infty}.
\]

It follows that the kernel
\[
\tilde{K}(x_1, x_2, y, t) := \frac{K(t, x_1 - y) - K(t, x_2 - y)}{[M(|x_1 - x_2|)]^{-\beta}}
\]
satisfies
\[
\sup_{|x_1 - x_2| \leq r_0, 0 < t \leq 1} \int_{\mathbb{R}^d} \tilde{K}(x_1, x_2, y, t)dy \leq C. \tag{3.60}
\]

Let now \( q(t, x) \) be solution to (3.58) with \( q(0, x) = 0 \). It is given by the Duhamel formula
\[
q(t, x) = \int_0^t \int_{\mathbb{R}^d} K(t - s, x - y)f(s, y)dyds,
\]
and we can write, using (3.60):
\[
\frac{|q(t, x_1) - q(t, x_2)|}{[M(|x_1 - x_2|)]^{-\beta}} \leq \int_0^t \int_{\mathbb{R}^d} \frac{|K(t - s, x_1 - y) - K(t - s, x_2 - y)|}{[M(|x_1 - x_2|)]^{-\beta}}f(s, y)dyds \leq \|f\|_{L^\infty} \int_0^t \frac{1}{(t - s)^\beta} \int_{\mathbb{R}^d} |\tilde{K}(x_1, x_2, y, t - s)|dyds \leq Ct^{-\beta}\|f\|_{L^\infty},
\]
and (3.59) follows.
\[\square\]

3.2. The end of the proof of theorem 1.2

In our case, \( \rho(t, x) \) satisfies (2.13), which is of the form (3.58) with \( k(x, z, t) = \psi(z)\rho(t, x) \), with a uniformly bounded forcing term \( G\rho \) in the right side, due to proposition 2.3 and (2.29). As \( \rho(t, x) \) obeys the uniform upper and lower bounds in proposition 2.3, the bounds (3.3) on the kernel \( k(x, z, t) \) hold with \( \eta(z) = \psi(z) \). Assumptions (3.4) and (3.5) are then simply (1.6) and (1.10) respectively, while (3.6) holds due to the monotonicity of \( rM(r) \) for \( 0 \leq x \leq r_0 \), see (1.11) and the comment following it.

To see that the drift \( u(t, x) \) in (2.13) satisfies (3.7) and (3.8), we first recall the decomposition (2.9) that allows us to write
\[
u = u_1 + u_2 + I_0(t), \quad \text{with} \quad \partial_t u_1 = G - \nu, \quad \partial_t u_2 = L\rho. \tag{3.61}
\]
The uniform bound (2.29) on \( G \) implies that \( u_1 \) obeys both (3.7) and (3.8). As for \( u_2 \), it can be written as
\[
u_2(t, x) = \mathcal{L}\Phi = \int \psi(z)[\Phi(t, x) - \Phi(t, x + z)]dz, \tag{3.62}
\]
where $\Phi(t, x)$ is the mean-zero primitive of $\rho(t, x)$, as in (2.8). Since $\Phi$ is Lipschitz because of the uniform bound on $\rho$, the $L^\infty$-bound on $u_2$ follows from (1.5), and (3.7) holds. To verify (3.8), we note that for any $r > 0$ we can write

$$u_2(t, x) - u_2(t, y) = \int \psi(z)[\Phi(t, x) - \Phi(t, y) - \Phi(t, x + z) + \Phi(t, y + z)]dz$$

$$= \int_{|z| \leq r} + \int_{|z| > r} = I_1(r) + I_2(r).$$

(3.63)

These terms can be bounded as

$$|I_2(r)| \leq C|x - y| \int \psi(z)dz = C|x - y|M(r).$$

(3.64)

As for $I_1$, we use assumption (1.10), that implies

$$r\psi(r) \leq \gamma_0 M(r), \quad \gamma_0 := \frac{r\psi'(r_0)}{M(r_0)}, \quad \text{for all } 0 < r < r_0,$$

so that, as long as $r \in (0, r_0)$, we have, taking $\gamma \in (0, 1/2]$ as in (1.11), so that $z'M(z)$ is an increasing function for $z \in (0, r_0)$:

$$|I_1(r)| \leq C \int_{|z| \leq r} z'\psi(z)dz \leq C\gamma_0 r \int_{|z| \leq r} M(z)dz \leq C\gamma_0 r \int_{|z| \leq r} \frac{z'M(z)}{|z|} dz$$

$$\leq C\gamma_0 r^2 M(r) \int_{|z| \leq r} \frac{1}{|z|^2} dz = CrM(r).$$

(3.65)

Setting $r = |x - y|$ implies that (3.8) holds.

The forcing term $G\rho$ in (2.13) is uniformly bounded since

$$\|G\rho\|_{L^\infty} \leq C\|\rho\|_{L^\infty}^2,$$

by (2.29), thus lemma 3.4 finishes the proof.

\section*{4. Global existence of smooth solutions}

In this section, we prove theorem 1.1. We follow the strategy of [24] to show a uniform bound on $\rho_\tau$.

**Proposition 4.1.** For each $T > 0$ there exists $C_T > 0$ so that for all $0 \leq t \leq T$ we have $\|\rho_\tau(t, \cdot)\|_{L^\infty} \leq C_T$.

The global existence of smooth solutions in theorem 1.1 will then follow by a bootstrap argument. We begin the proof of proposition 4.1 with a nonlinear maximum principle for the operator $L$.

**Lemma 4.2.** Let $f \in C^1_0(\mathbb{R})$, and $x_0$ be the maximal point where $|f'(x_0)| = \max_x |f'(x)|$, then we have a lower bound

$$f'(x_0)Lf'(x_0) \geq \frac{1}{2} D^2 f'(x_0),$$

(4.1)
where
\[ Df'(x) := \int_{\mathbb{R}} |f'(x) - f'(x + z)|^2 \psi(z) dz. \] (4.2)

Proof. We have
\[
\begin{align*}
    f'(x_0) f'(x_0) &= \int_{\mathbb{R}} (f'(x_0)^2 - f'(x_0) f'(x_0 + z))^2 \psi(z) dz \\
    &= \frac{1}{2} \int_{\mathbb{R}} (f'(x_0)^2 - f'(x_0 + z)^2)^2 \psi(z) dz + \frac{1}{2} \int_{\mathbb{R}} (f'(x_0) - f'(x_0 + z))^2 \psi(z) dz \\
    &\geq \frac{1}{2} Df'(x_0).
\end{align*}
\]

The next step is a lower bound for \( Df'. \)

Lemma 4.3. There exists \( C > 0 \) that depends only on the kernel \( \psi \) so that for all \( f \in C^1_c(\mathbb{R}) \), we have a pointwise lower bound
\[ Df'(x) \geq |f'(x)|^2 M \left( \frac{C \|f\|_{L^\infty}}{|f'(x)|} \right), \quad \text{for all } x \in \mathbb{R}. \] (4.3)

Proof. As in the proof of lemma 2.4, let \( \chi(x) \) be a radially non-decreasing smooth cut-off function such that \( \chi(x) = 0 \) for \( |x| \leq 1/2 \) and \( \chi(x) = 1 \) for \( |x| \geq 1 \). We write, using (2.24):
\[
\begin{align*}
    Df'(x) &\geq \int_{\mathbb{R}} |f'(x) - f'(x + z)|^2 \psi(z) \chi(z/R) dz \\
    &\geq |f'(x)|^2 \int_{\mathbb{R}} \psi(z) \chi(z/R) dz - 2f'(x) \int_{\mathbb{R}} f'(x + z) \psi(z) \chi(z/R) dz \\
    &\geq |f'(x)|^2 \int_{|z| \geq R/2} \psi(z) dz + 2f'(x) \int_{|z| \geq R/2} f'(x + z) \partial_z \left( \psi(z) \chi(z/R) \right) dz \\
    &\geq 2|f'(x)|^2 M(R) - C |f'(x)| \frac{\|f\|_{L^\infty}}{R} \int_{|z| \geq R/2} \psi(z) dz \\
    &\geq 2|f'(x)|^2 M(R) - C |f'(x)| \|f\|_{L^\infty} \frac{M(R/2)}{R}.
\end{align*}
\]

The doubling condition (2.14) for \( M \) gives
\[ Df'(x) \geq 2|f'(x)|^2 M(R) - C |f'(x)| \|f\|_{L^\infty} \frac{M(R)}{R}. \]

Setting
\[ R = \frac{C \|f\|_{L^\infty}}{|f'(x)|}, \]
we conclude that
\[ Df'(x) \geq |f'(x)|^2 M \left( \frac{C \|f\|_{L^\infty}}{|f'(x)|} \right), \]
which is (4.3).
**Proof of proposition 4.1.** We first note that there exists a time $\tau_0$ that depends on the $\|\rho_0\|_{Lip}$ and $\|u_0\|_{Lip}$ such that for all $0 \leq t \leq \tau_0$ and all $x, y \in \mathbb{T}$ we have

$$|\rho(t, x) - \rho(t, y)| \leq \frac{|x - y|}{2} \|\rho_0\|_{Lip}. \quad (4.4)$$

Putting this estimate together with theorem 1.2, we conclude that there exists a constant $C > 0$ that depends on $\|\rho_0\|_{\infty}$ and $\tau_0$ so that we have

$$|\rho(t, x) - \rho(t, y)| \leq C[M(|x - y|)]^{-\beta}, \quad \text{for all } 0 \leq t \leq T \text{ and } x, y \in \mathbb{T}. \quad (4.5)$$

We take the derivative of equation (2.13) and use (2.3):

$$\partial_t \rho_\ast + (G + \mathcal{L}\rho)\rho_\ast + u\rho_{xx} + \rho_x G + \rho G_x = -\rho_x \mathcal{L}\rho - \rho \mathcal{L}\rho_x. \quad (4.6)$$

Multiplying (4.6) by $\rho_\ast$ and evaluating at the maximal point $x_\ast$ of $\rho_\ast$, so that $\rho_{xx}(x_\ast) = 0$, we obtain

$$\frac{1}{2} \partial_t |\rho_\ast(x_\ast)|^2 = - (G_x(x_\ast)\rho(x_\ast)\rho_\ast(x_\ast) + 2G(x_\ast)|\rho_\ast(x_\ast)|^2) \quad (4.7)$$

$$- 2|\rho_\ast(x_\ast)|^2 \mathcal{L}\rho(x_\ast) - \rho(x_\ast)\rho_\ast(x_\ast) \mathcal{L}\rho_x(x_\ast) = I + II + III.$$ 

By the uniform estimates (2.29) on $G$ and (2.31) on $G_x$, we can bound the first term in the right side as

$$I \leq |G_x(x_\ast)\rho(x_\ast)\rho_\ast(x_\ast) + 2G(x_\ast)|\rho_\ast(x_\ast)|^2| \leq C(1 + |\rho_\ast(x_\ast)|^2).$$

Lemma 4.2 together with a uniform lower bound on $\rho(t, x)$ in proposition 2.3 gives a bound for the dissipative term III in the right side of (4.7):

$$III \leq -\frac{1}{2} c_0 D\rho_\ast(x_\ast). \quad (4.8)$$

To estimate there term II in (4.7), we need to bound $\mathcal{L}\rho(x_\ast)$. We introduce a smooth symmetric cut-off function $\phi(x)$ such that $\phi(x) = 1$ for $|x| \leq 1$ and $\phi(x) = 0$ for $|x| \geq 2$, and write, for any $x$ and $0 < r < 1/2$:

$$\mathcal{L}\rho(x) = \int_{\mathbb{R}} \phi\left(\frac{x}{r}\right)(\rho(x) - \rho(x + z))\psi(z)dz + \int_{\mathbb{R}} (1 - \phi\left(\frac{x}{r}\right))(\rho(x) - \rho(x + z))\psi(z)dz$$

$$= II_1 + II_2. \quad (4.9)$$

The second term above can be estimated using the $M$-Hölder estimate (4.5) for $\rho$:

$$II_2 \leq C \int_{|z| > r} M(|z|)^{-\beta}\psi(z)dz \leq CM(r)^{1-\beta}. \quad (4.10)$$

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The estimate for $H_1$ is more subtle. Let $\tilde{M}$, an odd extension of $M$, be the primitive of the even function $(-\psi(z))$, then integration by parts gives

$$H_1 = \int_{|z| \leq 2r} \phi(\frac{z}{r})(\rho(x) - \rho(x + z))(-\partial_z \tilde{M}(z))dz$$

$$\leq \frac{C}{r} \int_{r \leq |z| \leq 2r} |\rho(x) - \rho(x + z)|\tilde{M}(z)dz - \int_{|z| \leq 2r} \phi(\frac{z}{r})\rho_0(x + z)\tilde{M}(z)dz = H_{11} + H_{12}.$$  

(4.11)

Again by the $M$-Hölder estimate for $\rho$, we have, since $M(r)$ is decreasing

$$H_{11} \leq \frac{C}{r} \int_{r \leq |z| \leq 2r} |\rho(x) - \rho(x + z)|M(|z|)dz \leq \frac{C}{r} \int_{r \leq |z| \leq 2r} M(|z|)^{1-\beta}dz \leq CM(r)^{1-\beta}. $$ 

(4.12)

Moreover, since $\psi(z)$ is even and $\tilde{M}$ is odd, we can write

$$H_{12} = \int_{|z| \leq 2r} \phi(\frac{z}{r})(\rho_0(x) - \rho_0(x + z))\tilde{M}(z)dz$$

$$\leq C \int_{|z| \leq 2r} |\rho_0(x) - \rho_0(x + z)|\sqrt{\psi(z)M(|z|)}\sqrt{\psi(z)}dz$$

$$\leq C \left( \int_{|z| \leq 2r} |\rho_0(x) - \rho_0(x + z)|^2\psi(z)dz \right)^{1/2} \left( \int_{|z| \leq 2r} \frac{M(|z|)^2}{\psi(z)}dz \right)^{1/2}$$

(4.13)

$$= C \sqrt{D\rho_0(x)} \left( \int_{|z| \leq 2r} \frac{M(|z|)^2}{\psi(z)}dz \right)^{1/2}.$$ 

As a consequence of the lower bound on $\psi(r)$ in (1.5), and assumption (1.11) which implies that $x^{1/2}M(x)$ is non-decreasing, the integral in the right side of (4.13) can be bounded as

$$\int_{|z| \leq 2r} \frac{M(|z|)^2}{\psi(z)}dz \leq C \int_{|z| \leq 2r} M(|z|)^2|z|^{1-\alpha/2}dz$$

$$\leq C rM(2r)^2 \int_{|z| \leq 2r} |z|^{-\alpha/2}dz = C r^{2-\alpha/2}M(2r)^2 \leq Cr^{2-\alpha},$$

as it follows from (1.7) that

$$M(2r) \leq \frac{C}{(2r)^{\alpha/4}}.$$ 

We conclude that

$$H_{12} \leq C \sqrt{D\rho_0(x)r^{1-\alpha/2}}.$$ 

(4.14)

Going back to $II$, we have shown that

$$II \leq C|\rho_0(x_+)|^2M(r)^{1-\beta} + C|\rho_0(x_+)|^2 \sqrt{D\rho_0(x_+)r^{1-\alpha/2}}$$

$$\leq C|\rho_0(x_+)|^2M(r)^{1-\beta} + \frac{1}{4}c_0D\rho_0(x_+) + Cr^{2-\alpha}|\rho_0(x_+)|^4.$$ 

(4.15)
Collecting our bounds for the three terms in the right side of (4.7), and using lemma 4.2, we obtain
\[
\frac{1}{2}\partial_t|\rho_+(x)|^2 \leq C(1 + |\rho_+(x)|^2) + C|\rho_+(x)|^2 M(r)^{1-\beta} + Cr^{2-\alpha}|\rho_+(x)|^4 - \frac{1}{4}c_0|\rho_+(x)|^2 M(r)^{1-\beta} + Cr^{2-\alpha}|\rho_+(x)|^4 - \frac{1}{4}c_0|\rho_+(x)|^2 M(r)^{1-\beta}.
\]

(4.16)

Let us choose
\[
 r = \min\left(\frac{1}{2} r^2_C/(2-\alpha), \frac{C \|\rho\|_{L^\infty}}{|\rho_+(x)|}\right).
\]

(4.17)

It follows from lemma 2.2 that
\[
 M(r) \leq C(1 + M(r_c))^2/(2-\alpha).
\]

(4.18)

Using this estimate in (4.16), as well as the definition (4.17) of \( r \), we obtain
\[
\frac{1}{2}\partial_t|\rho_+(x)|^2 \leq C(1 + |\rho_+(x)|^2) + C|\rho_+(x)|^2 [1 + M(r_c)]^{2(1-\beta)/(2-\alpha)} - C|\rho_+(x)|^2 M(r_c).
\]

(4.19)

Let us set \( z(t) = |\rho_+(x_+(t))|^2 \), then (4.19) is
\[
\frac{dz}{dt} \leq C(1 + z) + Cz^2 \left[ M \left( \frac{C}{\sqrt{z}} \right) \right]^{2(1-\beta)/(2-\alpha)} - Cz M \left( \frac{C}{\sqrt{z}} \right).
\]

(4.20)

We choose \( \alpha < 2/\beta \) so that \( 2(1-\beta)/(2-\alpha) < 1 \), and then apply the Young’s inequality, to obtain
\[
\frac{dz}{dt} \leq C(1 + z) - \frac{Cz}{2} M \left( \frac{C}{\sqrt{z}} \right).
\]

(4.21)

As \( M(z) \to +\infty \) as \( z \to 0 \), the maximum principle implies that \( z(t) \) remains finite at all times, and the proof of proposition 4.1 is complete.

The proof of theorem 1.1 by bootstrapping. Proposition 4.1 tells that \( |\partial_t \rho(t, x)| \) stays bounded, hence so does \( |\partial_t G(t, x)| \) by (2.32). Therefore, \( |\partial_t u(t, x)| \) stays bounded because
\[
|\partial_t u(t, x)| \leq |G(t, x)| + |C\rho(t, x)| \\
\leq C + \int_{|z| \leq 1} |\rho(t, x) - \rho(t, x + z)| \psi(z) dz + \int_{|z| > 1} |\rho(t, x) - \rho(t, x + z)| \psi(z) dz \\
\leq C + \|\partial_t \rho(t, x)\|_{L^\infty} \int_{|z| \leq 1} |z| \psi(z) dz + 2\|\rho(t, x)\|_{L^\infty} M(1) \leq C,
\]

(4.22)
with $|z|\psi(z)$ being locally integrable by (1.5). Now we differentiate (2.13) in $x$ and rearrange it to be

$$
\partial_t (\rho_t) + u \partial_t (\rho_t) + \rho \mathcal{L}(\rho_t) = -G_{x1} \rho - G_{x2} \rho - \rho_t \mathcal{L} \rho - u_t \rho_t.
$$

(4.23)

Note that the right side of (4.23) is a bounded forcing term, thus (4.23), viewed as an equation for $\rho_t$, lies in the class of linear integro-differential equation (1.15) and $\rho_t$ satisfies the $M$-Hölder estimate (4.5). Now let us repeat the proof of proposition 4.1. We take the derivative of (4.6) and use (2.3) to get

$$
\partial_t \rho_{xx} + u \rho_{xx} = - (G_{xx} \rho + 3G_{x2} \rho + 3 \rho_{xx} \mathcal{L} \rho + 3G_{x} \rho_t)
$$

\(- 3 \rho_t \mathcal{L} \rho_t - \rho \mathcal{L} \rho_{xx}.
\)

(4.24)

Multiplying (4.24) by $\rho_{xx}$, at the maximal point $x_+$ of $\rho_{xx}$, so that $\rho_{xx}(x_+) = 0$, we obtain

$$
\frac{1}{2} \partial_t |\rho_{xx}(x_+)|^2 = - (G_{xx}(x_+) \rho(x_+) \rho_{xx}(x_+) + (3G(x_+) + 3 \mathcal{L}(x_+)) |\rho_{xx}(x_+)|^2 + 3G_{x}(x_+) \rho_t(x_+) \rho_{xx}(x_+)
$$

$$
- 3 \rho_{xx}(x_+) \rho_t(x_+) \mathcal{L} \rho_t(x_+) - \rho(x_+) \rho_{xx}(x_+) \mathcal{L} \rho_{xx}(x_+) = I' + II' + III'.
\)

(4.25)

We see that (4.25) has the same structure as (4.7). The estimate (2.32), and the boundedness of $u_t$ and $\mathcal{L} \rho$ give that

$$
I' \leq C(1 + |\rho_{xx}(x_+)|^2).
$$

(4.26)

Again lemma 4.2 and a uniform lower bound for $\rho(t, x)$ gives a bound

$$
II' \leq - \frac{1}{2} c_0 D \rho_{xx}(x_+).
$$

(4.27)

And for $II'$, follow the proof of proposition 4.1 and note that $\rho_t(x, t)$ is bounded, to get

$$
II' \leq C|\rho_{xx}(x_+)|M(r)^{1-\beta} + C|\rho_{xx}(x_+)| \sqrt{D \rho_{xx}(x_+)^{r^{1-\alpha}/2}}
$$

$$
\leq C|\rho_{xx}(x_+)|M(r)^{1-\beta} + \frac{1}{4} c_0 D \rho_{xx}(x_+)^2 + C r^{2-\alpha} |\rho_{xx}(x_+)|^2.
\)

(4.28)

We can now obtain an inequality similar to (4.21), which implies the that $|\partial_t^2 \rho(t, x)|$ stays bounded, and thus $|\partial_x^2 \rho(t, x)|$ stays bounded. The arguments above can be iterated due to two reasons: first, the analog of (4.23) for the higher order derivatives, is always in the form of an integro-differential equation

$$
\partial_t (\rho^{(n)}) + u \partial_t (\rho^{(n)}) + \rho \mathcal{L}(\rho^{(n)}) = f_n(G^{(k)}, \rho^{(k)}, u^{(k)}, \mathcal{L} \rho^{(k-1)}), \quad n \in \mathbb{N}, 1 \leq k \leq n,
$$

(4.29)

where $f_n$ is a polynomial function depending on $G^{(k)}, \rho^{(k)}, u^{(k)}, \mathcal{L} \rho^{(k-1)}$, $1 \leq k \leq n$, and thus $f_n$ is a bounded forcing term. Second, when we take $(n+1)$th derivative of (2.13) and use (2.3), it can always be put into the form

$$
\partial_t (\rho^{(n+1)}) + u \rho^{(n+2)} = -g(G^{(k)}, \rho^{(k)}, \mathcal{L} \rho^{(k-2)}) - C \mathcal{L} \rho^{(n)} - \rho \mathcal{L} \rho^{(n+1)}, \quad n \in \mathbb{N}, 2 \leq k \leq n + 1,
$$

(4.30)
and the first term \(|\rho| \leq C(1 + |\rho^{(n+1)}(t,x)|\)). In (4.30), we look at the maximal point of \(|\rho^{(n+1)}|\) and so on, to get that \(|\rho^{(n+1)}(t,x)|\) stays bounded, so is \(|u^{(n+1)}(t,x)|\). Estimates like in (4.21) also imply the local existence of \(\rho^{(n)}, u^{(n)}\). Bootstrapping, we conclude that \(\rho(t,x)\) and \(u(t,x)\) stay smooth for all times.  

□

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