A UNIQUENESS THEOREM FOR ASYMPTOTICALLY CYLINDRICAL SHRINKING RICCI SOLITONS

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ABSTRACT. We prove that a shrinking gradient Ricci soliton which agrees to infinite order at spatial infinity with one of the standard cylindrical metrics on $S^k \times \mathbb{R}^{n-k}$ for $k \geq 2$ along some end must be isometric to the cylinder on that end. When the underlying manifold is complete, it must be globally isometric either to the cylinder or (when $k = n-1$) to its $\mathbb{Z}_2$-quotient.

1. Introduction

A shrinking Ricci soliton is a Riemannian manifold $(M, g)$ for which

\begin{equation}
2 \text{Rc}(g) + \mathcal{L}_X g = g
\end{equation}

for some smooth vector field $X$ on $M$. The soliton is gradient if $X = \nabla f$ for some $f \in C^\infty(M)$. When $(M, g)$ is complete and of bounded curvature, it is always possible to find $f$ such that $X - \nabla f$ is Killing \cite{40, 42}, and so, for most applications, there is no loss of generality in considering only gradient solitons. Below, we will assume that all shrinking solitons (or, simply, shrinkers) are gradient and are normalized to satisfy

\begin{equation}
\text{Rc}(g) + \nabla \nabla f = \frac{g}{2}, \quad R + |\nabla f|^2 = f.
\end{equation}

The contracted second Bianchi identity implies that $\nabla (R + |\nabla f|^2 - f) \equiv 0$ on any gradient shrinker which satisfies the first equation, so it is always possible to achieve the latter normalization by adding a constant to $f$ on each connected component of $M$. We will denote a soliton structure by $(M, g, \nabla f)$ or $(M, g, f)$ when we wish to emphasize the vector field or the potential, and otherwise simply identify it with the underlying manifold.

Shrinking solitons are generalizations of positive Einstein metrics and arise as model spaces in the theory of smooth metric measure spaces. We are interested in their connection to the Ricci flow

\begin{equation}
\frac{\partial}{\partial t} g = -2 \text{Rc}(g),
\end{equation}

where they correspond to shrinking self-similar solutions, the generalized fixed points of the equation which move only under the natural actions of $\mathbb{R}_+$ and Diff$(M)$.
on the space of metrics on $M$. When the manifold $(M, g)$ is complete, the vector field $\nabla f$ is complete [50], and the system

\[
\begin{cases}
\frac{\partial \phi}{\partial t} &= -\frac{1}{t} \nabla f \circ \phi \\
\phi_{-1} &= \text{Id}
\end{cases}
\]

may be solved to obtain a family of diffeomorphisms $\phi_t : M \to M$ defined for $t \in (-\infty, 0)$. The rescaled pull-backs $g(t) = -t \phi_t^* g$ of the original metric then solve (1.3) on $M \times (-\infty, 0)$.

The study of shrinkers is an important component of the analysis of the singular behavior of solutions to the Ricci flow. Solutions to the Ricci flow which develop a singularity at a finite time $T$ are expected to “generically” satisfy a so-called Type-I curvature bound $\sup_{M \times [0, T]} (T - t) |R_m| < \infty$. From the work of Hamilton [21], Perelman [42], Šešum [45], Naber [40], and Enders, Müller (Buzano), and Topping [16], it is now known that, about any point in the high-curvature region of a Type-I singular solution, one can extract a sequence of blow-ups converging to a complete nontrivial shrinking gradient Ricci soliton. In this sense, shrinkers represent potential models for the geometry of a solution in the neighborhood of a developing singularity. It is a fundamental problem to understand what possible forms they may take.

1.1. The classification problem for shrinking Ricci solitons. Shrinking solitons are completely classified in dimensions two and three. Hamilton [20] proved that the only complete two-dimensional shrinkers are the flat plane $\mathbb{R}^2$ with the Gaussian soliton structure and the standard round metrics on $S^2$ and $\mathbb{R}P^2$. The combined results of Hamilton [21], Ivey [24], Perelman [42], Ni-Wallach [41], and Cao-Chen-Zhu [5] show that the only complete three-dimensional shrinkers are the Gaussian soliton on $\mathbb{R}^3$ and finite quotients of the round sphere $S^3$ and standard round cylinder $S^2 \times \mathbb{R}$. These classifications are aided by the presence of some additional a priori structure peculiar to those dimensions: in dimension two, orientable gradient solitons are necessarily rotationally symmetric (the application of the complex structure to $\nabla f$ is a Killing vector field) and in dimension three, complete shrinkers are necessarily of nonnegative sectional curvature (on account of the Hamilton-Ivey estimate [21, 24]).

In higher dimensions, the class of shrinking solitons (which includes all Einstein manifolds with positive scalar curvature) is simply too large to hope for an exhaustive classification. The three-dimensional classification has nevertheless been extended to a variety of restricted classes. For example, the work of Cao-Wang-Zhang [7], Eminenti-LaNave-Mantegazza [15], Fernández-López and García-Río [19], Munteanu-Sesum [33], Ni-Wallach [41], Petersen-Wylie [44], and Zhang [51], has shown that the only complete shrinkers with vanishing (even harmonic) Weyl tensor are either the Gaussian soliton $\mathbb{R}^n$ or finite quotients of $S^n$ or $S^{n-1} \times \mathbb{R}$. In this direction, Cao-Chen [4] have also obtained a classification for solitons with vanishing Bach tensor.

Some partial classifications are also now known for shrinkers subject to an additional curvature positivity condition. By a theorem of Chen [9] (cf. [13]), any complete shrinker must have nonnegative scalar curvature, however, there are examples beginning in dimension four with Ricci curvatures of mixed sign [18]. As corollaries of the work of Böhm-Wilking [11], Brendle [2], and Brendle-Schoen [3], it is known that any compact shrinker whose curvature operator is 2-positive or which
satisfies the so-called PIC1 condition must be a quotient of the round sphere. In four dimensions, Li, Ni, and K. Wang [30] have shown recently that a complete gradient shrinker with positive isotropic curvature must be a quotient of the standard sphere $S^4$ or standard cylinder $S^3 \times \mathbb{R}$. In another direction, Munteanu and J. Wang [39] (generalizing results of Perelman [43] and Naber [40] in dimensions three and four) have shown that any complete shrinker with positive sectional curvature must be compact.

There are a variety of other results, too many to adequately summarize here, concerning the geometric properties of shrinkers in all dimensions; we refer the reader to [6], [8], [22], [23], [31], [34], and the references therein.

1.2. Complete noncompact shrinking solitons. The formal resemblance of the shrinking soliton equation (1.2) to the condition of nonnegative Ricci curvature suggests that a complete noncompact shrinking soliton would have to balance strong and competing tendencies toward incompleteness and reducibility. A growing body of evidence appears to support the expectation that the possibilities for the asymptotic geometry of a complete shrinker are extremely restricted.

All known examples of complete noncompact shrinking solitons fit one of two descriptions. Either they split (at least locally) as products or have a single end smoothly asymptotic to a regular cone. Examples of this latter type are scarce. The first are due to Feldman-Ilmanen-Knopf [18], who constructed a family of complete shrinkers on the tautological line bundle of $\mathbb{C}P^{n-1}$ for $n \geq 2$. Their examples are Kähler with a $U(n)$-symmetry and Ricci curvatures of mixed sign. Dancer-Wang [14] and Yang [49] have further generalized their construction to line bundles over products of Kähler-Einstein metrics with positive scalar curvature. These examples, too, have quadratic curvature decay and a single asymptotically conical end.

In four dimensions, it is conjectured that any complete shrinker must fit one of these two descriptions, at least asymptotically. The recent work of Munteanu-Wang [36, 37, 38] has framed this possible dichotomy in terms of the scalar curvature. In [36, 37], the authors show that, if the scalar curvature tends to zero at spatial infinity, then every end of $(M^4, g)$ must be smoothly asymptotic to a cone. In [38], they show that if, instead, the curvature remains bounded below by a positive constant, then either every end of $(M^4, g)$ is smoothly asymptotic to a quotient of $S^3 \times \mathbb{R}$, or, along any sequence of points $x_i$ going to infinity along an integral curve of $\nabla f$, the sequence of pointed manifolds $(M^4, g, x_i)$ will subconverge in the smooth Cheeger-Gromov sense to a quotient of $S^2 \times \mathbb{R}^2$. (See also [12] for a general splitting criterion for limits of pointed sequences of shrinkers.) The expectation that the scalar curvature must satisfy exactly one of these alternatives is confirmed in [38] when $(M^4, g)$ is Kähler and the scalar curvature is bounded.

The primary link between the dichotomy proposed above and a potential classification of complete noncompact four-dimensional solitons is a question of uniqueness of interest in all dimensions, namely, to what extent is a shrinker determined by its asymptotic geometry? The authors have previously considered this question in the asymptotically conical case in [28]. There it is shown that, if two shrinkers are $C^2$-asymptotic to the same cone on some ends of each, then the shrinkers must be isometric to each other on some neighborhood of infinity of those ends. This result, an analog of a theorem of the second author for asymptotically conical self-shrinkers to the mean curvature flow [46], reduces the classification of asymptotically conical shrinking solitons to that of the potential asymptotic cones. At present, there are
few restrictions known to hold on the cones which admit an asymptotic shrinker. Lott-Wilson [32] have shown that there are at least no formal obstructions to the existence of a shrinker or an expander asymptotic to any regular cone, and it is a consequence of the uniqueness result in [28] that any isometry of the cone must correspond to an isometry of the shrinker. The first author has also shown in [27] that if the cone is Kähler the shrinker must also be Kähler.

1.3. Asymptotically cylindrical shrinking Ricci solitons. In this paper, we address the above question of uniqueness in the complementary case of asymptotically cylindrical geometries. In order to state the main result, we need to establish some notation.

For each \( k \geq 2 \), we will write \( \mathcal{C}^k = S^k \times \mathbb{R}^{n-k} \) and let \((\mathcal{C}^k, g_k, f_k)\) denote the standard cylindrical soliton structure with the normalizations implied by (1.2). Thus,

\[
g_k = (2(k-1)\hat{g}) \oplus \hat{g}, \quad f_k(\theta, z) = \frac{|z|^2}{4} + \frac{k}{2},
\]

where \( \hat{g} \) is the round metric on \( S^k \) of constant sectional curvature 1 and \( \hat{g} \) is the Euclidean metric on \( \mathbb{R}^{n-k} \). For each \( r > 0 \), let \( \mathcal{C}_r^k \) denote the set

\[
\mathcal{C}_r^k = \left\{ \begin{array}{ll}
S^k \times (\mathbb{R}^{n-k} \setminus B_r(0)) & \quad 2 \leq k < n-1 \\
n^{n-1} \times (r, \infty) & \quad k = n-1.
\end{array} \right.
\]

By an end of a Riemannian manifold \((M, g)\), we will mean an unbounded connected component of the complement of a compact set in \( M \).

**Definition 1.1.** Let \( r > 0 \). We will say that \((\mathcal{C}_r^k, \hat{g})\) is strongly asymptotic to \((\mathcal{C}^k, g_k)\) if, for all \( l, m \geq 0 \),

\[
\sup_{\mathcal{C}_r^k} |z|^4 |\nabla^{(m)}_{g_k} (\hat{g} - g_k)(\theta, z)| < \infty.
\]

We will say that \((\tilde{M}, \hat{g}, \hat{f})\) is asymptotically cylindrical to \((\mathcal{C}^k, g_k)\) to the end \( V \subset (\tilde{M}, \hat{g}) \) if there exists \( r > 0 \) and a diffeomorphism \( \Psi : \mathcal{C}_r^k \to V \) such that \((\mathcal{C}_r^k, \Psi^* \hat{g})\) is strongly asymptotic to \((\mathcal{C}^k, g_k)\).

The purpose of this paper is to prove the following local uniqueness result.

**Theorem 1.2.** Suppose \((\tilde{M}, \hat{g}, \hat{f})\) is a shrinking gradient Ricci soliton for which \((\tilde{M}, \hat{g})\) is strongly asymptotic to \((\mathcal{C}^k, g_k)\) along the end \( V \subset (\tilde{M}, \hat{g}) \) for some \( k \geq 2 \). Then \((V, \hat{g}|_V)\) is isometric to \((\mathcal{C}^k, g_k|_{\mathcal{C}_r^k})\) for some \( r \).

The hypothesis of infinite order decay should be understood in terms of the locality of the entire statement to the end \( V \). In particular, the manifold \((\tilde{M}, \hat{g})\) is neither assumed to be complete nor to satisfy any a priori restriction on the number of its topological ends. There is some reason to believe that, in this generality, the infinite order decay of \( \hat{g} - g_k \) is actually necessary. The second author has previously established an analogous uniqueness theorem for embedded self-shrinkers to the mean-curvature flow which are asymptotic of infinite order to one of the standard cylinders [47]. This paper includes the construction of a family of non-rotationally-symmetric self-shrinkers over cylindrical ends \( S^{n-1} \times (a, \infty) \to \mathbb{R}^{n+1} \) which decay to the cylinder at arbitrarily high polynomial rates, showing that the assumption of infinite order decay is effectively optimal in this case.

When the underlying manifold \((\tilde{M}, \hat{g})\) is complete, however, Theorem 1.2 implies that it must be globally isometric to a quotient of \((\mathcal{C}^k, g_k)\).
Corollary 1.3. Suppose that, in addition to the assumptions in Theorem 1.2, the manifold \((M, \tilde{g})\) is complete. Then, either \((M, \tilde{g})\) is isometric to \((C^k, g_k)\), or \(k = n - 1\) and \((M, \tilde{g})\) is isometric to the quotient \((C^{n-1}, g_{n-1})/\Gamma\) where \(\Gamma = \{\text{Id}, \gamma\}\) and \(\gamma(\theta, z) = (-\theta, -z)\).

The techniques of this paper are rather specialized to address the local problem of uniqueness in Theorem 1.2. We expect that it should be possible to weaken or eliminate entirely the assumption on the rate of convergence to the cylinder when the manifold is complete. We have also not attempted here to optimize the decay assumption in terms of the number of derivatives on \(\tilde{g} - g_k\). An inspection of the proof shows that we require (1.4) to hold for finitely many \(m\); by an interpolation argument it is enough to assume that the derivatives \(\nabla^{[m]}(\tilde{g} - g_k)\) are merely bounded.

1.4. Overview of the proof. As in \[28\], \[47\], our basic strategy is to use the correspondence between shrinkers and self-similar solutions to (1.3) to transform Theorem 1.2 into an equivalent problem of unique continuation for solutions to the Ricci flow, which we ultimately treat with the method of Carleman inequalities. The resulting singular problem of backward uniqueness, for a nonlinear weakly parabolic system, is substantially more complicated than those addressed in either \[28\], where the solutions extend smoothly to the terminal time slice, or in \[47\] where the analysis reduces to that of a solution to a single scalar parabolic inequality. Our implementation of this strategy involves several new ingredients to overcome obstacles not present in these previous applications. We summarize the major steps of our argument now.

For simplicity, in the following discussion, we will assume that \(k \geq 2\) is fixed and suppress the subscript \(k\) in our notation, writing \(C = C^k\), \(C_r = C^k_r\), \(g = g_k\), and \(f = f_k\), and using \(| \cdot | = | \cdot |_{g_k}\) and \(\nabla = \nabla_{g_k}\) to denote the norms and connections induced by \(g\) and its Levi-Civita connection on tensor bundles over \(C\).

1.4.1. Normalizing the soliton structure. It is sufficient to prove Theorem 1.2 in the case that \(\tilde{g}\) and \(\tilde{f}\) are actually defined on \(C_{r_0}\) for some \(r_0 > 0\), that is, when \((C_{r_0}, \tilde{g})\) is strongly asymptotic to \((C, g)\). The first step then is to put the soliton structure \((C_{r_0}, \tilde{g}, \tilde{f})\) into a more canonical form. The hypotheses of Theorem 1.2 only explicitly constrain the asymptotic behavior of \(\tilde{g}\), and (even with the normalizations implicit in (1.2)) do not fully determine \(\tilde{X} = \tilde{\nabla} \tilde{f}\) nor imply, even, that the difference of \(\tilde{X}\) and \(X = \nabla f\) vanishes at spatial infinity.

At the same time, these hypotheses do not permit \(\tilde{X}\) much flexibility. In Proposition 2.2 we first show that we can arrange for \(\tilde{X} - X\) to vanish to infinite order at infinity by pulling back \(\tilde{g}\) and \(\tilde{X}\) by an appropriate translation on the Euclidean factor. Having made this adjustment, we show in Theorem 2.5 that it is possible to construct a further injective diffeomorphism \(\Phi : C_{r_1} \to C_{r_0}\) for some \(r_1 > r_0\) such that \(\Phi^* \tilde{X} = X\) and for which \((C_{r_1}, \Phi^* \tilde{g})\) is still strongly asymptotic to \((C, g)\). We postpone the details of the construction of the map \(\Phi\) to Appendix A.

1.4.2. Reducing to a problem of backward uniqueness. Having reduced Theorem 1.2 to the case that \(\tilde{X}\) and \(X\) coincide on \(C_{r_1}\) for some \(r_1 > 0\), we next recast it as a problem of parabolic unique continuation for solutions to the Ricci flow. The
family of diffeomorphisms $\Psi : \mathcal{C}_r \times (0, 1] \to \mathcal{C}_r$, given by $\Psi_\tau(\theta, z) = (\theta, z/\sqrt{\tau})$ solve

$$\frac{\partial \Psi}{\partial \tau} = -\frac{1}{\tau} X \circ \Psi, \quad \Psi_1 = \text{Id},$$

and (since $X = \nabla f = \tilde{\nabla} \tilde{f}$), we may use them to construct from $\tilde{g}$ and $g$ self-similar families of metrics

$$\tilde{g}(\tau) = \tau \Psi_\tau^* \tilde{g}, \quad g(\tau) = \tau \Psi_\tau^* g = (2(k-1)\tau \tilde{g}) \oplus \bar{g},$$

solving the backward Ricci flow

$$\frac{\partial g}{\partial \tau} = 2 \text{Rc}(g)$$

on $\mathcal{C}_r$, for $\tau \in (0, 1]$. On account of the normalization we have already performed, these solutions have the advantage that their difference $h(\tau) = (\tilde{g} - g)(\tau)$ is itself self-similar. Since $(\mathcal{C}_r, \tilde{g})$ is strongly asymptotic to $(\mathcal{C}, g)$, the tensor $h$ will vanish to infinite order as $|z| \to \infty$ and $\tau \searrow 0$ in the sense that

$$\sup_{\mathcal{C}_r \times (0, 1]} \frac{|z|^{2l}}{\tau^l} |\nabla^{(m)} h|((\theta, z, \tau)) < \infty$$

for all $l, m \geq 0$. Here and below, we write $|\cdot| = |\cdot|_{g(\tau)}$ and $\nabla = \nabla_{g(\tau)}$ (in fact, the connection $\nabla_{g(\tau)}$ of the evolving cylinder is independent of time).

To prove Theorem 1.2, it is enough to show then that $h(\tau_0) \equiv 0$ on $\mathcal{C}_r$ for some $\tau_0$ and $r > 0$. Indeed it then follows that $h(1) = \tau_0^{-1} (\Psi_{\tau_0}^{-1})^* h(1)$ vanishes on $\mathcal{C}_{r'}$ for $r' = r/\sqrt{\tau_0}$, and hence, by a continuation argument, that $\tilde{g}$ and $g$ are isometric on $\mathcal{C}_{r_0}$. We give this parabolic restatement in Theorem 3.2 and verify that it indeed implies Theorem 1.2 at the end of Section 3.

1.4.3. Prolonging the system. To prove Theorem 3.2, one must first address the lack of strict parabolicity of equation (1.3). The degeneracy of the equation, a consequence of its diffeomorphism invariance, is not rectifiable here by the use of DeTurck’s trick as it is in the problem of forward uniqueness of solutions to the Ricci flow: the diffeomorphisms needed to pass to a problem of backward uniqueness for the strictly parabolic Ricci-DeTurck flow are naturally solutions to an ill-posed terminal-value problem for a harmonic map-type heat flow. See, e.g., [26] for a discussion of these and related issues.

To circumvent the degeneracy of (1.3), we follow the method used by the first author in [26] and encode the vanishing of $h$ in terms of the vanishing of solutions to a prolonged “PDE-ODE” system of mixed differential inequalities. One important difference between our implementation of this device and the implementations in [26] and [28] is that the system used in these latter references is too coarse to keep track of the blow-up which is occurring anisotropically in our problem. A critical part of our approach is to parlay the infinite order decay that we assume on $h$ and its derivatives into an exponential-quadratic rate, and the Carleman inequalities we use for this purpose can absorb only a limited amount of blow up from the coefficients on the lower order terms on the right hand side.

Thus we use two systems: a “basic” system, which is simpler to work with and is suitable for ultimately establishing the vanishing of $h$, and a more elaborate “refined” system, with which we can track the blow-up rate of individual components of $\nabla \tilde{Rm}$ sufficiently well to verify the exponential decay of $h$ and its derivatives. The basic system is equivalent to those considered in [26] [28], and consists of the
sections \( X = \widetilde{\nabla}Rm = \widetilde{\nabla}g - \nabla Rm \) and \( Y = (h, \nabla h, \nabla \nabla h) \). These sections satisfy a system of inequalities of the form
\[
| (D_r + \Delta) X | \leq \frac{B}{\tau} |X| + B|Y|, \quad |D_r Y| \leq B \left( |X| + |\nabla X| \right) + \frac{B}{\tau} |Y|
\]
for some constant \( B \) on \( C_r \times (0,1] \). Here, \( Rm = Rm(g(\tau)), \widetilde{Rm} = Rm(\tilde{g}(\tau)), \nabla = \nabla \tilde{g}(\tau), \) and \( \Delta = \Delta \tilde{g}(\tau), \) and \( D_r \) indicates a derivative taken relative to evolving \( g \)-orthogonal frames. We describe this system in Section 4.

The defect of this basic system for our purposes is that the Carleman estimate (6.7), which we use to establish the exponential-quadratic space-time decay of \( X \) and \( Y \), cannot directly absorb the coefficient of \( \tau^{-1} \) which appears on the right side of the equation for \( X \). In Section 5 we will replace the parabolic component \( X \) of our system with a more elaborate choice \( W = (W^0, W^1, \ldots, W^5) \) which consists of selected components of \( \widetilde{\nabla}Rm \) relative to the \( g \)-orthogonal splitting \( TM = TS^k \oplus TR^{n-k} \) rescaled by powers of \( \tau \). The components \( W^j \) are chosen so that they satisfy inequalities of the form
\[
| (D_r + \Delta) W^i | \leq B \tau^{\beta} (|W| + |Y|) + B \sum_{j<i} \tau^{-\gamma_j} |W^j|
\]
for some constants \( B, \gamma_j \geq 0 \) and \( \beta > 0 \). We will exploit the strict triangular structure of the singular terms in (1.6) to control the unbounded coefficients on the right side of the equation for any \( W^i \) by suitably weighted applications of the inequalities for \( W^{i'} \) for \( i' < i \).

1.4.4. Promoting the rate of decay to exponential. The Carleman inequalities (7.4), (7.10), with which we will ultimately prove the vanishing of \( X \) and \( Y \), involve a weight which, for large \( |z| \) and small \( \tau \), grows on the order of \( \exp(C|z|^{2\delta} / \tau^\delta) \) for some \( \delta \in (0,1) \). In order to apply these inequalities, we first need to verify that \( X \) and \( Y \) decay rapidly enough to be integrable against this weight. To this end, in Theorem 5.1 (proven in Section 4) we show that there are constants \( N_0, N_1 > 0 \), such that
\[
\int_0^1 \int_{A_{r,2r}} \left( |X|^2 + |\nabla X|^2 + |Y|^2 \right) e^{-\frac{N_0 \tau^2}{\tau^\delta}} d\mu_{\tilde{g}(\tau)} d\tau \leq N_1,
\]
for all sufficiently large \( r \), where \( A_{r,2r} = C_r \setminus \overline{C_{2r}} \). This argument, including the derivation of the system (1.6) above, is the most involved in the paper.

We first establish the decay of \( W \) and \( Y \) by an inductive argument, using the Carleman inequality (6.7) in tandem with (6.8) and (6.9) to obtain successively upper bounds of the form \( CL^m \tau^{-2m} m! \) on the weighted \( L^2 \)-norms of \( W \) and \( Y \) on \( S^k \times B_{r}(z_0) \) for small \( r \) and \( z_0 \in C_{r_0} \). These estimates involve a weight approximately of the form \( \tau^{-m} \exp(-|z - z_0|^2 / 4\tau) \) localized about \( z_0 \). Since the components of \( W \) are merely rescaled components of \( \widetilde{\nabla}Rm \), the estimates on \( W \) directly yield corresponding estimates for \( X \), which can be summed and rescaled to obtain the asserted rate of exponential decay. The primary inequality (6.7) is analogous to one established in [47] and is ultimately modeled on the inequality proven in [47] for an application to solutions to linear parabolic inequalities on Euclidean half-spaces.

1.4.5. Establishing the vanishing of \( X \) and \( Y \). In Section 4 we work again with the basic system \( (X, Y) \), and, having proven that they decay sufficiently rapidly, show, using Carleman inequalities analogous to those in [26] and [47], with a family
of exponentially growing weights, that they must vanish identically. The argument here is modeled fairly closely on the corresponding argument in [47], with some modifications to handle the ODE component $Y$. It is in this part of the argument where we make essential use of the self-similarity of $h$ (and hence of $X$ and $Y$). The Carleman inequalities needed here and above in the proof of the exponential decay of $X$ and $Y$ are proven in Section 8.

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2. Normalizing the soliton

For the rest of the paper, we will fix $1 < k < n$, and, as we did in the latter part of the introduction, write simply $\mathcal{C} = \mathcal{C}_k = S^k \times \mathbb{R}^{n-k}$ and $\mathcal{C}_r = \mathcal{C}_r^k$ for $r > 0$, and continue to denote by $g = g_k = (2((k - 1)\hat{g}) \oplus \bar{g})$, $f(\theta, z) = f_k(\theta, z) = \frac{|z|^2}{4} + \frac{k}{2}$, the metric and potential of the normalized cylindrical soliton structure on $\mathcal{C}$. We will also define $A_{a,b} = C_a \setminus C_b$, $S_r = \left\{ S^k \times \partial B_r(0) \quad \begin{array}{ll} k < n - 1 \\ S^{n-1} \times \{r\} \quad k = n - 1 \end{array} \right\}$ for $a, b, r > 0$.

We will often use spherical coordinates on the Euclidean factor of $C_a$ to identify it with $S^k \times S^{n-k-1} \times (a, \infty)$ via $(\theta, z) \mapsto (\theta, \sigma, r)$ where $\sigma = z/|z|$ and $r = |z|$. For simplicity, we will also continue to use the unadorned notation $|\cdot| = |\cdot|_g$, $\nabla = \nabla_g$ for the norms and connections induced by $g$ and its Levi-Civita connection on the tensor bundles $T^{(a,b)}C$.

2.1. Some preliminary estimates. To prove Theorem 1.2 it suffices to consider the situation that $M = V = C_0$ for some $r_0 > 0$ and $(C_{r_0}, \tilde{g})$ is strongly asymptotic to $(\mathcal{C}, g)$. We first record some simple consequences of the asymptotic cylindricity of the soliton metric $\tilde{g}$.

Lemma 2.1. Suppose that $(C_{r_0}, \tilde{g}, \tilde{f})$ is a shrinking Ricci soliton where

$$\sup_{C_{r_0}} r^3 |\nabla^{(m)}(\tilde{g} - g)| < \infty$$

for $m = 0, 1, 2$. Then, there exists $r_1 \geq r_0$, and constants $k_0, K_0 > 0$, such that

$$\frac{1}{2}g \leq \tilde{g} \leq 2g, \quad |\tilde{\nabla} \tilde{f}| \leq K_0(r + 1), \quad |\nabla \tilde{f}| \leq K_0(r + 1),$$

and

$$\frac{1}{8}r^2 \leq \tilde{f} \leq \frac{1}{4}(r + k_0)^2$$

on $C_{r_1}$.

The quadratic bounds on $\tilde{f}$ in (2.3), sufficient for our purposes, have been established in sharper form for general complete shrinking solitons by Cao-Zhou [6].
Proof. It is immediate from (2.1) that the inequalities $(1/2)g \leq \tilde{g} \leq 2g$ and $\tilde{R} \geq k/4$ will hold on $\mathcal{C}_a$ provided $a \geq r_0$ is chosen large enough. Combined with the identity $\tilde{R} + |\nabla \tilde{f}|_{\tilde{g}}^2 = \tilde{f}$, we may then see that $\tilde{f} \geq k/4$ and $|\nabla \tilde{f}|^2 \leq 2|\nabla \tilde{f}|_{\tilde{g}}^2 \leq 2\tilde{f}$ on the same set. Integrating along along integral curves of $\frac{\partial}{\partial r}$ we then see that

$$\tilde{f}^{1/2}(\theta, \sigma, r) - \tilde{f}^{1/2}(\theta, \sigma, a) \leq \int_a^r |\nabla \tilde{f}^{1/2}| \leq r - a.$$  

for all $(\theta, \sigma) \in S^k \times S^{n-k-1}$. In particular, $|\nabla \tilde{f}| \leq 2|\nabla \tilde{f}|_{\tilde{g}} \leq 4(r + K)$ on $\mathcal{C}_a$ for some $K$ depending on $\sup_{\mathcal{C}_a} \tilde{f}$, proving the last two inequalities in (2.2) if $r_1 \geq a$.

Next, using the soliton equation, we have

$$\nabla_i \nabla_j \tilde{f} = \nabla_i \nabla_j \tilde{f} - \nabla_i \nabla_j \tilde{R} + \tilde{g}_{ij} = (\tilde{\Gamma}^k_{ij} - \Gamma^k_{ij}) \nabla_k \tilde{f} - (\tilde{R}_{ij} - R_{ij}) + \frac{1}{2}(\tilde{g}_{ij} - g_{ij}) - R_{ij} + \tilde{g}_{ij},$$

where $A^k_{ij}$ and $S_{ij}$ are polynomials in $g^{-1}$, $\tilde{g}^{-1}$, and $\nabla^{(m)}(\tilde{g} - g)$ for $m \leq 2$. So, using (2.4) and that $|\nabla \tilde{f}| \leq 4(r + K)$, we have

$$\frac{1}{2} - \frac{K}{r^2} \leq \frac{\partial^2 \tilde{f}}{\partial r^2} \leq \frac{1}{2} + \frac{K}{r^2},$$

for some possibly larger $K$. Integrating both inequalities in (2.4) along integral curves of $\frac{\partial}{\partial r}$ starting at $\mathcal{C}_a$, we obtain

$$\frac{r}{2} - K' \leq \left\langle \nabla \tilde{f}, \frac{\partial}{\partial r} \right\rangle \leq \frac{r}{2} + K'$$

for some $K' > 0$ depending on $a$. Hence

$$\frac{r^2}{4} - K'r - \frac{r^2}{4} \leq \tilde{f}(\theta, \sigma, r) \leq \frac{r^2}{4} + K'r + \tilde{f}(\theta, \sigma, r_1) \leq \frac{r^2}{4} + K'r + (r_1 + K'')^2$$

for any $r_1 \geq a$ and some $K''$ depending on $a$. Here we have used (2.3) to estimate $\tilde{f}(\theta, \sigma, r_1)$. Choosing then $r_1 \geq a$ large enough to ensure that the left side is larger than $r^2/8$ on $\mathcal{C}_{r_1}$, and then choosing $k_0$ large enough depending on $r_1$ to bound the right hand side by $(r + k_0)^2/4$, we obtain (2.3).

2.2. Correcting the vector field by a translation. The implicit normalizations in (1.2) together with the assumption that $(\mathcal{C}_{r_0}, \tilde{g})$ is strongly asymptotic to $(\mathcal{C}, g)$ do not quite determine the gradient vector field $\tilde{\nabla} \tilde{f}$ of a soliton structure $(\mathcal{C}_{r_0}, \tilde{g}, \tilde{f})$.

In general, $\tilde{\nabla} \tilde{f} - \nabla f$ need not even decay to infinite order. For example, the soliton structure $(\mathcal{C}, g, f_{z_0})$ with the potential

$$f_{z_0}(\theta, z) = \frac{|z - z_0|^2}{4} + \frac{k}{2}$$

satisfies (1.2) for any $z_0 \in \mathbb{R}^{n-k}$, but the difference

$$\nabla f - \nabla f_{z_0} = \sum_{i=1}^{n-k} z_i \frac{\partial}{\partial z_i}$$

is constant. Of course, the two soliton structures here can be made to agree by pulling back one by a translation of the Euclidean factor. A similar adjustment can
be made in our situation: by pulling back \( \tilde{g} \) and \( \tilde{f} \) by an appropriate translation of \( \mathbb{R}^{n-k} \), we can arrange that \( \nabla \tilde{f} - \nabla f \) decays to infinite order at infinity.

**Proposition 2.2.** Let \( p \geq 2 \) and suppose that \( (C_{r_0}, \tilde{g}, \tilde{f}) \) satisfies (1.2) and
\[
\sup_{C_{r_0}} r^l |\nabla^{(m)}(\tilde{g} - g)| < \infty
\]
for all \( l \geq 0 \) and \( m \leq p \). Then, there is a constant vector field \( V \) tangent to the \( \mathbb{R}^{n-k} \) factor such that
\[
\tilde{f} = r \frac{\partial}{\partial r} + V + E
\]
where \( E \) satisfies
\[
\sup_{C_{r_0}} r^l |\nabla^{(m)} E| < \infty
\]
for all \( l \geq 0 \) and \( 0 \leq m \leq p - 1 \).

**Proof.** Let \( X = \nabla f = \frac{\partial}{\partial r} \) and \( \tilde{X} = \nabla \tilde{f} \). From (1.2), we compute that
\[
\nabla \tilde{X}^j = \nabla X^j + (\Gamma^j_{ik} - \tilde{\Gamma}^j_{ik}) \tilde{X}^k
\]
\[
= \nabla X^j + (g^j_{ik} R_{ik} - \tilde{g}^j_{ik} \tilde{R}_{ik}) + (\Gamma^j_{ik} - \tilde{\Gamma}^j_{ik}) \tilde{X}^k.
\]

Using (2.6) and that \( |\tilde{X}| \leq K_0(r + 1) \) from Lemma 2.1, we see that \( W = \tilde{X} - X \) satisfies
\[
\sup_{C_{r_0}} r^l |\nabla^{(m)} W| < \infty
\]
for all \( l \geq 0 \) and \( 1 \leq m \leq p - 1 \).

Fix any \( q = (\theta, z) \in C_{r_0} \), and let \( \{F_{q,i}\}_{i=1}^n \) be any orthonormal basis for \( T_q C \).
Extend this basis by parallel transport to a frame \( \{F_{q,i}(r)\}_{i=1}^n \) along the radial line \( \gamma_q(r) = (\theta, rz/|z|) \). For any \( |z| \leq r_1 \leq r_2 \), and any \( l \geq 0 \), we have
\[
|\langle W, F_{q,i}\rangle(\gamma_q(r_2)) - \langle W, F_{q,i}\rangle(\gamma_q(r_1))| \leq \int_{r_1}^{r_2} |\nabla W|(\gamma_q(r)) dr \leq \frac{M_l}{r_1^l}
\]
for some \( M_l \), and it follows that
\[
\lim_{r \to \infty} \langle W, F_{q,i}\rangle(\gamma_q(r)) = V_i(q) < \infty
\]
for some numbers \( V^i(q) \) for each \( i = 1, 2, \ldots, n \). Define \( V(q) = V^i(q) F_{q,i} \in T_q C \) and suppose we repeat this process starting from another orthonormal basis \( \{\tilde{F}_{q,i}\}_{i=1}^n \). Then \( \tilde{F}_{q,i}(r) = A^i_j F_{q,j}(r) \) for some fixed orthogonal transformation \( A \), and
\[
\tilde{V}^i(q) = \lim_{r \to \infty} \langle W, \tilde{F}_{q,i}\rangle(\gamma_q(r)) = (A^T)^i_j V^j(q)
\]
so the limit \( V(q) = \tilde{V}(q) \) depends only on \( q \). Taking such a limit at each \( q \) thus defines a (rough) vector field on \( C_{r_0} \).

By construction, for all \( \theta \) and \( \sigma \) and all \( r_0 \leq r_1 \leq r_2 \), the value of \( V(\theta, \sigma, r_2) \) will coincide with that of the parallel transport of \( V(\theta, \sigma, r_1) \) along the radial line connecting \( (\theta, \sigma, r_1) \) and \( (\theta, \sigma, r_2) \). We claim that \( V \) is actually parallel. To see this, fix any \( (\theta, \sigma) \) and \( (\tilde{\theta}, \tilde{\sigma}) \) in \( S^k \times S^{n-k-1} \) and any \( r_1 \geq r_0 \). For \( r \geq r_1 \), consider the points \( q_r = (\theta, \sigma, r) \) and \( \tilde{q}_r = (\tilde{\theta}, \tilde{\sigma}, r) \).

Let \( \alpha : [0, L] \to S^k \times S^{n-k-1} \) be a unit-speed geodesic with \( \alpha(0) = (\sigma, \theta) \) and \( \alpha(L) = (\tilde{\theta}, \tilde{\sigma}) \) and, for \( r \geq r_1 \), define \( \lambda_r(s) = (\alpha(s), r) \in S^r \). On the cylinder, the
path \( \lambda_r \) will have length bounded by \( C(r+1) \) for some \( C \). Let \( P_{r,s} : T_p C \to T_{\lambda_r(s)} C \) denote parallel translation along \( \lambda_r \). The vector field \( W \) is bounded on account of the decay of \( |\nabla W| \), and, by the definition of \( V \) and equation (2.10), we have
\[
|V - W| \leq \frac{M_l}{r^l}
\]
for each \( l \) for some constant \( M_l \). Hence,
\[
|P_{r,L}(V(q_r)) - W(\tilde{q}_r)|^2
\]
\[
= |V(q_r) - W(q_r)|^2 + 2 \int_0^L \langle (D_{\frac{d}{ds}} W)(\lambda_r(s)), P_{r,s}(V(q_r)) - W(\lambda_r(s)) \rangle \, ds
\]
\[
\leq |V(q_r) - W(q_r)|^2 + 2 \int_0^L |\nabla W||V(q_r)| + |W(\lambda_r(s))| \, ds \leq \frac{M_l}{r^l},
\]
for some \( M_l \). So
\[
|P_{r,L}(V(q_r)) - W(\tilde{q}_r)| \leq \frac{M_l}{r^l},
\]
using (2.10) again. But, since \( g \) is cylindrical and \( V \) is parallel along radial lines,
\[
|P_{r,L}(V(q_r)) - V(\tilde{q}_r)| = |P_{r,L}(V(q_r)) - V(\tilde{q}_r)|.
\]
Consequently, we have \( P_{r,L}(V(q_r)) = V(\tilde{q}_r) \) upon sending \( r \to \infty \). It follows that \( V \) is parallel. In particular, \( V \) is smooth and tangent everywhere to the \( \mathbb{R}^{n-k} \) factor, where it is represented by a constant vector.

2.3. Aligning the vector fields. Motivated by Proposition 2.2, we update our notion of asymptotic cylindricity to involve the entire soliton structure.

**Definition 2.3.** We will say that \((C_{r_0}, \tilde{g}, \tilde{X})\) is strongly asymptotic to \((C, g, X)\) as a soliton if
\[
\sup_{C_{r_0}} |z|^l \left( |\nabla^{(m)}(\tilde{g} - g)| + |\nabla^{(m)}(\tilde{X} - X)| \right) < \infty
\]
for all \( l, m \geq 0 \).

We may then restate Proposition 2.2 as follows.

**Proposition 2.4.** Suppose that \((C_{r_0}, \tilde{g}, \tilde{f})\) is a gradient shrinking soliton for which \((C_{r_0}, \tilde{g})\) is strongly asymptotic to \((C, g)\). Then, there is \( r_1 \geq r_0 \) and a translation \( \tau_{z_0}(\theta, z) = (\theta, z - z_0) \) such that \((C_{r_1}, \tau_{z_0}^* \tilde{g}, \tau_{z_0}^* (\tilde{f}))\) is strongly asymptotic to \((C, g, \nabla f)\) as a soliton.

**Proof.** Let \( \tilde{X} = \tilde{V} \) and \( X = \nabla f \). By Proposition 2.2, we may write \( \tilde{X} = X + V + E \) for some constant vector field \( V \) tangent to the \( \mathbb{R}^{n-k} \) factor and \( E \) satisfying
\[
\sup_{C_{r_0}} |z|^l |\nabla^{(m)} E|(\theta, z) < \infty
\]
for all \( l, m \geq 0 \).

Let us write the components of \( V \) as \( V^i = z^i_0 / 2 \), and define the translation map \( \tau_{z_0} : C \to C \) by \( \tau_{z_0}(\theta, z) = (\theta, z - z_0) \). Provided \( r_1 > r_0 + |z_0| \), we will have \( \tau_{z_0}(S^k \times B_{r_2}(0)) \subset S^k \times B_{r_1}(0) \). Since \( \tau_{z_0} \) is an isometry of \( g \), the restriction of \( \tau_{z_0}^* \tilde{g} \) to \( C_{r_1} \) will continue to be strongly asymptotic to \( g \), but we will now have in addition that
\[
\tau_{z_0}^* \tilde{X}(\theta, z) = X(\theta, z - z_0) + V + E(\theta, z - z_0) = X(\theta, z) + \tilde{E}(\theta, z)
\]
where \( \hat{E}(\theta, z) = E(\theta, z - z_0) \) satisfies
\[
\sup_{C_1} |z|^l |\nabla^{(m)} \hat{E}|(\theta, z) < \infty
\]
for all \( l, m \geq 0 \). \( \square \)

In fact, after adjusting metric and potential by a further diffeomorphism, we can arrange that the gradient vector field of \((C_{r_0}, \hat{g}, \hat{f})\) actually coincides with the standard cylindrical vector field.

**Theorem 2.5.** Suppose \((C_{r_0}, \hat{g}, \hat{f})\) is strongly asymptotic to \((C_{r_0}, g, f)\). Then there is \( r_1 \geq r_0 \) and an injective local diffeomorphism \( \Phi : C_{r_1} \to C_{r_0} \) for which \( C_{2r_1} \subset \Phi(C_{r_1}) \), \((C_{r_1}, \Phi^* \hat{g})\) is strongly asymptotic to \((C, g)\), and
\[
\Phi^*(\tilde{\nabla} \hat{f}) = \nabla f = r \frac{\partial}{\partial r}
\]
on \( C_{r_1} \).

The construction of the map \( \Phi \) is straightforward but conceptually independent of the rest of the paper. We postpone its proof until Appendix A.

3. Reduction to a Problem of Parabolic Unique Continuation

In this section, we recast Theorem 1.2 as a problem of uniqueness for the backward Ricci flow, using the correspondence between soliton structures and self-similar solutions discussed in the introduction. The following proposition summarizes this correspondence for an asymptotically cylindrical soliton and a cylinder which share the same gradient vector field.

**Proposition 3.1.** Write \( X = \nabla f \) and suppose that \((C_{r_0}, \tilde{g}, X)\) is strongly asymptotic to \((C_{r_0}, g, X)\). Let \( \Psi : C_{r_0} \times (0, 1] \to C_{r_0} \) be the map \( \Psi(\theta, z, \tau) = (\theta, z/\sqrt{\tau}) \) and put \( \Psi(\tau) = \Psi(\cdot, \cdot, \tau) \). Then the families of metrics
\[
g(\tau) = \tau \Psi_r^* g = (2(k - 1)\tau \tilde{g} \circ \tilde{g}), \quad \tilde{g}(\tau) = \tau \Psi_r^* \tilde{g},
\]
solve (1.5) on \( C_{r_0} \times (0, 1] \), and \( h(\tau) = (\tilde{g} - g)(\tau) = \tau \Psi_r^* h(1) \) satisfies
\[
\sup_{C_{r_0} \times (0, 1]} \frac{|z|^{2l}}{\tau^{l}} |\nabla^{(m)} h(\tau)| g(\tau) < \infty
\]
for each \( l, m \geq 0 \).

**Proof.** Since the map \( \Psi \) satisfies
\[
\frac{\partial \Psi}{\partial \tau}(\theta, z, \tau) = -\frac{1}{\tau}(X \circ \Psi)(\theta, z, \tau), \quad \Psi(\theta, z, 1) = (\theta, z),
\]
and \( \tilde{\nabla} \hat{f} = X = \nabla f \), a standard calculation shows that \( g(\tau) = \tau \Psi_r^* \hat{g} \) and \( \tilde{g}(\tau) = \tau \Psi_r^* \tilde{g} \) solve (1.5) (see, e.g., [10]). Equation (3.1) follows then by scaling: fixing \( l, m \geq 0 \), we have
\[
\frac{|z|^{2l}}{\tau^{l}} |\nabla^{(m)} h(\tau)| g(\tau) |_{g(\tau)}(\theta, z, \tau) = \frac{|z|^{2l}}{\tau^{l+\frac{m}{2}}} |\nabla^{(m)} h| \left( \theta, \frac{z}{\sqrt{\tau}} \right) < \infty
\]
on \( C_{r_0} \) by our assumption on \( h \). \( \square \)

Going forward, we will write simply
\[
\hat{g} = \tilde{g}(\tau), \quad g = g(\tau), \quad h = h(\tau), \quad |\cdot| = |\cdot|_{g(\tau)}, \quad \nabla = \nabla_{g(\tau)}.
\]
3.1. A reformulation of Theorem 1.2

The purpose of the subsequent sections will be to prove the following theorem which states that any solution \( \tilde{g} = \hat{g}(\tau) \) to (1.3) satisfying the assumptions of Proposition 3.1 must be isometric to the standard shrinking cylinder.

**Theorem 3.2.** Suppose \( \tilde{g}(\tau) = \tau \Psi^*_z g(1) \) is a self-similar solution to (1.3) on \( C_{r_0} \times (0, 1) \) for some \( r_0 > 0 \), where \( \Psi : C_{r_0} \times (0, 1) \to C_{r_0} \) is the map \( \Psi_z(\theta, z) = (\theta, z/\sqrt{\tau}) \), and \( g = g(\tau) = (2(k-1)\tau \hat{g}) \oplus \bar{g} \). If, for all \( l, m \geq 0 \), there exist constants \( M_{l,m} > 0 \) such that \( h = g - \hat{g} \) satisfies

\[
\sup_{C_{r_0} \times (0, 1)} \frac{|z|^{2l}}{\tau^l} |\nabla^{(m)} h| \leq M_{l,m},
\]

then \( h \equiv 0 \) on \( C_{r_1} \times (0, \tau_0) \) for some \( r_1 \geq r_0 \) and \( 0 < \tau_0 \leq 1 \).

In fact, \( g(\tau) \) and \( \hat{g}(\tau) \) will be isometric on all of \( C_{r_0} \times (0, 1] \). We prove Theorem 3.2 in Section 6. For now, we note that it indeed implies Theorem 1.2.

**Proof of Theorem 1.2 assuming Theorem 3.2.** Let \((\tilde{M}, \tilde{g}, \tilde{f})\) be a shrinking Ricci soliton for which \((\tilde{M}, \tilde{g})\) is strongly asymptotic to \((C, g)\) along the end \( V \subset (\tilde{M}, \tilde{g}) \). Then, for some \( r_0 > 0 \), there is a diffeomorphism \( \varphi : C_{r_0} \to V \) such that \((C_{r_0}, \varphi^* g)\) is strongly asymptotic to \((C, \tilde{g})\). By Proposition 2.4 there is \( r_1 > r_0 \) and an injective local diffeomorphism \( \psi : C_{r_1} \to C_{r_0} \) such that \((C_{r_1}, (\varphi \circ \psi)^* \tilde{g}, (\varphi \circ \psi)^* \nabla f)\) is strongly asymptotic to \((C, g, \nabla f)\) as a soliton structure. Finally, by Theorem 2.5 there is \( r_2 > r_1 \) and an injective local diffeomorphism \( \Phi : C_{r_2} \to C_{r_1} \) such that \((C_{r_2}, (\varphi \circ \psi \circ \Phi)^* \tilde{g}, \nabla f)\) is strongly asymptotic to \((C, g, \nabla f)\).

Write \( \hat{g} = (\varphi \circ \psi \circ \Phi)^* \tilde{g} \). Using Proposition 3.1 we can construct a self-similar solution \( \hat{g}(\tau) = \tau \Psi_r \hat{g}(1) \) on \( C_{r_1} \times (0, 1] \) from \( \hat{g} = \hat{g}(1) \) and \( \nabla f \) for which \( h = \hat{g} - g \) satisfies

\[
\sup_{C_{r_2} \times (0, 1]} \frac{|z|^{2l}}{\tau^l} |\nabla^{(m)} h| < \infty
\]

for all \( l, m \geq 0 \).

Theorem 3.2 then says that \( h \equiv 0 \) on \( C_{r_3} \times (0, \tau_0) \) for some \( \tau_0 > 0 \) and \( r_3 \geq r_2 \). Fixing any \( a \in (0, \tau_0] \), we then have \( \hat{g}(a) = a \Psi^*_a \hat{g}(1) = a \Psi^*_a g(1) = g(a) \) on \( C_{r_3} \), so \( \hat{g} = (\varphi \circ \psi \circ \Phi)^* \hat{g} = g \) on \( C_{r_3} \), where \( r_4 = r_3/\sqrt{a} \). However, as Ricci solitons, both \( \hat{g} \) and \( g \) are real-analytic relative to atlases consisting of their own geodesic normal coordinate charts [24]. Any isometry on \( C_{r_3} \) can be extended to an isometry on \( C_{r_2} \) by continuation along paths so in fact \( \hat{g} \) and \( g \) are isometric on \( C_{r_2} \). Likewise, \( \varphi^* \hat{g} \) and \( g \) are isometric on \( C_{r_0} \), that is, \((V, \hat{g})\) and \((C_{r_0}, g)\) are isometric. \( \square \)

**Proof of Corollary 1.3.** Suppose now that \((\tilde{M}, \tilde{g})\) is complete. By Theorem 1.2 \((V, \hat{g})\) is isometric to \((C_{r_0}, g)\) for some \( r_0 > 0 \). Then the lift \((M', g')\) of \((\tilde{M}, \tilde{g})\) to the universal cover \( M' \) of \( M \) is complete, real-analytic (see, e.g., [23]), and isometric to \((C, g)\) on an open set. Since \( C \) and \( M' \) are simply connected, it follows that \((M', g')\) is globally isometric to \((C, g)\). So \((M, \tilde{g})\) must be a quotient of \((C, g)\) by a discrete subgroup \( \Gamma \) of isometries acting freely and properly on \( C \).

To identify this quotient, let \( \pi : C \to \tilde{M} \) be the covering map, and consider \( V' = \pi^{-1}(V) \). By [15], the fundamental group of \( \tilde{M} \) is finite, so \( \pi \) is proper, and we may write \( V' \) as the disjoint union of finitely many connected components \( V'_i \), \( i = 1, 2, \ldots, N \). Each \( V'_i \) is itself an end of \((C, g)\), and, since \( V \) is open and simply connected, the restriction of \( \pi \) to any \( V'_i \) is a diffeomorphism.
When $2 \leq k < n - 1$, we must have $N = 1$ since $(\mathcal{C}, g)$ is connected at infinity. Thus $\pi : \mathcal{C} \rightarrow \tilde{M}$ is a diffeomorphism and $\Gamma = \{ \text{Id} \}$ in this case. Similarly, when $k = n - 1$, $(\mathcal{C}, g)$ has two ends, and we must have $N \leq 2$ and $|\Gamma| \leq 2$. Any isometry $\gamma$ of $(\mathcal{C}, g)$ must take the form $\gamma(\theta, r) = (F(\theta), G(r))$, and, in our situation, both $F$ and $G$ must have order at most two. Then either $G(r) = r$ or $G(r) = -r + c$ for some $c$. If $G(r) = r$, then either $\gamma = \text{Id}$ or $F(\theta) = -\theta$. The latter is impossible, however, since $\mathbb{R}P^{n-1} \times \mathbb{R}$ has no end isometric to $S^{n-1} \times (a, \infty)$ for any $a$. If instead $G(r) = -r + c$ for some $c$, then $\gamma$ fixes $S^{n-1} \times \{c/2\}$, and we must have $F(\theta) = -\theta$, if $\gamma$ is not to fix any points. Thus, when $k = n - 1$, either $\Gamma = \{ \text{Id} \}$ or $\Gamma = \{ \text{Id}, \gamma \}$ where $\gamma(\theta, r) = (-\theta, -r + c)$ is a reflection on both factors. □

4. THE BASIC SYSTEM

Next we transform Theorem 3.2 into a problem that we can treat with Carleman inequalities. Following the method of [26], we will first define a simple “PDE-ODE” system whose components satisfy a coupled system of mixed parabolic and ordinary differential inequalities amenable to the application of the Carleman inequalities (7.9) and (7.10) in Section 7. These estimates involve a weight which grows like $\exp(|z|^2/\tau^\delta)$ for $\tau$ near 0. In order to even be able to apply these estimates, we will need to first verify that the components of our system decay fast enough to be integrable against these weights. For this, as we discussed in the introduction, we will introduce a second, finer system later in Section 5.

4.1. The setting. First we need to establish some notation. Here, as before, $\bar{g}(\tau) = (2(k - 1)\bar{g}) \oplus \bar{g}$ will represent the normalized shrinking cylindrical solution to (1.5) on $\mathcal{C} \times (0, \infty)$. We will use $\bar{g}$ and $\nabla$ as the reference metric and connection in our computations. The structural properties of the systems we describe will make no use of the self-similarity of $\bar{g}$, so, except within the context of the last assertion in Proposition 4.1, we will assume in this section only that $\bar{g} = \tilde{g}(\tau)$ is a solution to the backward Ricci flow (1.5) on $\mathcal{C}_{r_0} \times (0, 1]$ for which $\bar{h} = \tilde{h} - \bar{g}$ satisfies

$$\sup_{\mathcal{C}_{r_0}} \frac{|z|^{2l}}{\tau^l} |\nabla^{(m)} \bar{h}|(\theta, z, \tau) < \infty$$

for all $l, m \geq 0$.

It will be convenient to introduce the operator

$$D_\tau = \frac{\partial}{\partial \tau} - R^g_p \Lambda^q_p$$

acting on families of $(k, l)$ tensors $V = V(\tau)$, where

$$\Lambda^q_p(V)_{b_1b_2...b_k}^{a_1a_2...a_l} = V_{p_1p_2...p_k}^{a_1a_2...a_l} + V_{b_1b_2...b_k}^{b_1b_2...b_k} + \cdots + V_{b_1b_2...b_k}^{a_1a_2...a_l} - V_{b_1b_2...b_k}^{b_1b_2...b_k} - V_{b_1b_2...b_k}^{a_1a_2...a_l} - \cdots - V_{b_1b_2...b_k}^{a_1a_2...a_l}.$$

Here $R^g_q = g^{pr} R_{rq}$. (We have two metrics lurking in the background here, so to avoid confusion, we will only implicitly raise and lower indices with the metric $g$, and explicitly include any instances of $\tilde{g}$ and $\tilde{g}^{-1}$.) When $\{e_i(\tau)\}_{i=1}^n$ is a smooth family of local orthonormal frames evolving so as to remain orthonormal relative to $g(\tau)$, the components of $D_\tau V$ express the total derivatives

$$D_\tau V_{b_1b_2...b_k}^{a_1a_2...a_l} = \frac{\partial}{\partial \tau} \left( V(e_{b_1}, e_{b_2}, \cdots, e_{b_k}, e_{a_1}^*, e_{a_2}^*, \cdots, e_{a_l}^*) \right).$$

In particular, $D_\tau g \equiv 0$. 

4.2. Definition of the system. Now consider the bundles
\[ \mathcal{X} = T^{(5,0)}(\mathcal{C}), \quad \mathcal{Y} = T^{(2,0)}(\mathcal{C}) \oplus T^{(3,0)}(\mathcal{C}) \oplus T^{(4,0)}(\mathcal{C}) \]
over \( \mathcal{C} \) equipped with the smooth families of metrics and connections induced by \( g \). Let \( \mathbf{X} \) and \( \mathbf{Y} \) be the family of sections of \( \mathcal{X} \) and \( \mathcal{Y} \) over \( \mathcal{C}_{r_0} \times (0,1] \) defined by
\[ (4.2) \quad \mathbf{X} = \check{\nabla} \check{Rm} - \check{\nabla} \check{Rm} - \nabla \check{Rm}, \quad \mathbf{Y} = (Y_0, Y_1, Y_2) = (h, \nabla h, \nabla \nabla h). \]
The system \((\mathbf{X}, \mathbf{Y})\) is equivalent to that considered in [20], [28]. The components of \( \mathbf{Y} \) are chosen to ensure that, together, \( \mathbf{X} \) and \( \mathbf{Y} \) satisfy a closed system of differential inequalities.

**Proposition 4.1.** Let \( \mathbf{X} \) and \( \mathbf{Y} \) denote the sections of \( \mathcal{X} \) and \( \mathcal{Y} \) defined above. There is a constant \( B > 0 \) such that
\[ (4.3) \quad |(D_\tau + \Delta)\mathbf{X}| \leq \frac{B}{\tau}|\mathbf{X}| + B|\mathbf{Y}| \]
\[ (4.4) \quad |D_\tau \mathbf{Y}| \leq B(|\mathbf{X}| + |\nabla \mathbf{X}|) + \frac{B}{\tau}|\mathbf{Y}| \]
on \( \mathcal{C}_{r_0} \times (0,1] \), and, for each \( l, m \geq 0 \), constants \( M_{l,m} \) such that
\[ \sup_{\mathcal{C}_{r_0} \times (0,1]} \frac{\tau^m}{\tau^l} \left( |\nabla^{(m)} \mathbf{X}| + |\nabla^{(m)} \mathbf{Y}| \right) \leq M_{l,m}. \]
Moreover, when \( h(\tau) = \tau \Psi^*_\tau(h(1)) \) as in Theorem 3.2, \( \mathbf{X} \) and \( \mathbf{Y} \) are self-similar in the sense that
\[ (4.5) \quad \mathbf{X}(\tau) = \tau \Psi^*_\tau(\mathbf{X}(1)), \quad \mathbf{Y}(\tau) = \tau \Psi^*_\tau(\mathbf{Y}(1)). \]

The decay \[ (4.3) \] and self-similarity \[ (4.5) \] of \( \mathbf{X} \) and \( \mathbf{Y} \) follow from the corresponding properties of \( h \), and the observation that the components of \( \mathbf{X} \) and \( \mathbf{Y} \) scale the same as \( h \). The verification of \[ (4.3) \] is close to that of Lemma 3.1 in [28]; see Proposition 4.3 below. We include some of the computations on which it relies since we will need them in any case when we refine this system in the next section.

4.2.1. Evolution equations. Here and below we will use \( V \ast W \) to denote linear combinations of contractions of \( V \otimes W \) or \( \check{V} \otimes \check{W} \) for any tensors \( \check{V} \) and \( \check{W} \) identified to \( V \) and \( W \) via the isomorphisms \( TC \rightarrow T^*C \) and \( T^*C \rightarrow TC \) induced by \( g \). The coefficients in these linear combinations are understood to be bounded by dimensional constants.

We will first recall standard formulas for the difference of the Levi-Civita connections and curvature tensors of different metrics.

**Lemma 4.2.** Let \( g, \check{g} \) be any two metrics and \( h = g - \check{g} \). Then
\[ (4.6) \quad \check{g}^{ij} - g^{ij} = -\check{g}^{1a}g^{jb}p_{ab} = \check{g}^{-1} \ast h, \]
\[ (4.7) \quad \nabla_k \check{g}^{ij} = -\check{g}^{1a}g^{jb} \nabla_k h_{ab} = \check{g}^{-2} \ast \nabla h, \]
\[ (4.8) \quad \check{\nabla} \check{R} - \check{R} = \nabla \nabla h + \check{g}^{-1} \ast (\nabla h)^2 + \check{R} \ast h, \]
where \( \check{R} \) and \( \check{\nabla} \) denote the \((4,0)\) curvature tensors of \( g \) and \( \check{g} \). In addition,
\[ (4.9) \quad \check{\nabla} V - \nabla V = \check{g}^{-1} \ast \nabla h \ast V, \]
\[ (4.10) \quad \check{\Delta} V - \Delta V = \check{g}^{-2} \ast \nabla h \ast \nabla V + \check{g}^{-3} \ast (\nabla h)^2 \ast V + \check{g}^{-2} \ast \nabla \nabla h \ast V \]
\[ + \check{g}^{-1} \ast h \ast \nabla \nabla V, \]

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for any tensor $V$ of rank at least 1.

Now, from e.g., [10], recall the standard evolution equations
\[
\frac{\partial}{\partial \tau} \bar{\Gamma}^k_{ij} = \bar{g}^{mk} \left( \bar{\nabla}_i \bar{R}_{jm} + \bar{\nabla}_j \bar{R}_{im} - \bar{\nabla}_m \bar{R}_{ij} \right),
\]
and
\[
\left( \frac{\partial}{\partial \tau} + \Delta \right) \bar{R}_{ijkl} = -2(\bar{B}_{ijkl} - \bar{B}_{ijk} + \bar{B}_{ikjl} - \bar{B}_{ijlk}) + \bar{g}^{pq} \left( \bar{R}_{ap} \bar{R}_{qijkl} + \bar{R}_{jp} \bar{R}_{iqkl} + \bar{R}_{kp} \bar{R}_{ijql} + \bar{R}_{lp} \bar{R}_{ijqk} \right),
\]
where
\[
\bar{B}_{ijkl} = -\bar{g}^{pr} \bar{g}^{qs} \bar{R}_{prijq} \bar{R}_{rkl}.
\]
Combining these equations with a bit of further computation, one obtains the following equation for the evolution of $\nabla Rm$.

**Lemma 4.3.** If $\bar{g}$ satisfies (1.5), then
\[
\left( \frac{\partial}{\partial \tau} + \Delta \right) \nabla_a \bar{R}_{ijkl} = -2 \nabla_a \left( \bar{B}_{ijkl} - \bar{B}_{ijk} + \bar{B}_{ikjl} - \bar{B}_{ijlk} \right) + 2\bar{g}^{pr} \bar{g}^{qs} \left( \bar{R}_{iqap} \nabla_r \bar{R}_{sjkl} + \bar{R}_{jqap} \nabla_r \bar{R}_{iskl} + \bar{R}_{kpqap} \nabla_r \bar{R}_{ijkl} \right) + \bar{g}^{pq} \left( \bar{R}_{ap} \nabla_a \bar{R}_{ijkl} + \bar{R}_{jp} \nabla_a \bar{R}_{iqkl} + \bar{R}_{kp} \nabla_a \bar{R}_{ijql} + \bar{R}_{lp} \nabla_a \bar{R}_{ijqk} \right),
\]
Note that, according to our normalization, the curvature tensor of the cylindrical metric $g$ satisfies
\[
|Rm|^2 = \frac{k}{2(k-1)\tau^2}.
\]
The first assertion in Proposition 4.1 is now a consequence of the decay assumption (4.1), Lemma 4.2, and the following schematic evolution equations.

**Proposition 4.4.** The tensors $h$ and $\nabla Rm$ satisfy
\[
D_r h = \bar{g}^{-1} * \nabla \nabla h + \bar{g}^{-2} * (\nabla h)^2 + \bar{g}^{-1} * Rm * h,
\]
\[
D_r \nabla h = \bar{g}^{-1} * \nabla \nabla Rm + \bar{g}^{-2} * \nabla h * \nabla \nabla h + \bar{g}^{-3} * (\nabla h)^3
\]
\[
+ \bar{g}^{-2} * Rm * h * \nabla h + Rm * \nabla h,
\]
\[
D_r \nabla \nabla h = \bar{g}^{-2} * \nabla h * \nabla Rm + \bar{g}^{-1} * \nabla \nabla Rm + \bar{g}^{-3} * \nabla \nabla h * (\nabla h)^2 + \bar{g}^{-4} * (\nabla h)^4 + \bar{g}^{-2} * (\nabla \nabla h)^2 + \bar{g}^{-3} * Rm * h * (\nabla h)^2
\]
\[
+ \bar{g}^{-2} * Rm * h * \nabla \nabla h + \bar{g}^{-3} * Rm * (\nabla h)^2 + \bar{g}^{-2} * Rm * \nabla \nabla h + Rm * \nabla \nabla h,
\]
and
\[
(D_r + \Delta) \nabla Rm = \bar{g}^{-1} * \nabla (3) Rm * h + \bar{g}^{-1} * \nabla \nabla Rm * \nabla h
\]
\[
+ \bar{g}^{-2} * \nabla Rm * (\nabla h)^2 + \bar{g}^{-1} * \nabla Rm * \nabla \nabla h + \bar{g}^{-2} * (Rm - Rm) * \nabla Rm
\]
\[
+ \bar{g}^{-2} * h * Rm * \nabla Rm + \bar{g}^{-2} * Rm * \nabla Rm.
\]
Proof. For \( (4.11) \), we have

\[
D_r h_{ij} = 2 \tilde{R}_{ij} - R_i^p \tilde{g}_{pj} - R_j^p \tilde{g}_{ip} = 2 (\dot{R}_{ij} - R_{ij}) - R_i^p h_{pj} - R_j^p h_{ip},
\]

which yields the desired expression after applying \( (4.8) \) to the first term on the right. Equations \( (4.12) \) and \( (4.13) \) follow similarly, using that the Levi-Civita connection is time-independent.

For \( (4.14) \), observe that, by Lemma 4.3,

\[
\bigg( \nabla_{\tau} - \frac{\Delta}{\tau} \bigg) \nabla_a \tilde{R}_{ijkl} = -2 \nabla_a \left( \tilde{B}_{ijkl} - \tilde{B}_{ijlk} + \tilde{B}_{ilkj} - \tilde{B}_{ijkl} \right) + 2 \tilde{g}^{ip} \tilde{g}^{qs} \left( \tilde{R}_{iqap} \nabla_r \tilde{R}_{sjkl} + \tilde{R}_{jqap} \nabla_r \tilde{R}_{iskl} + \tilde{R}_{kpap} \nabla_r \tilde{R}_{ijkl} + \tilde{R}_{qapq} \nabla_r \tilde{R}_{ijkl} \right),
\]

\[
\left( \tilde{g}^{ap} \tilde{R}_{ap} - R^2 \right) \nabla_a \tilde{R}_{ijkl} + \left( \tilde{g}^{ap} \tilde{R}_{ip} - R^2 \right) \nabla_a \tilde{R}_{ijkl} + \left( \tilde{g}^{ap} \tilde{R}_{jp} - R^2 \right) \nabla_a \tilde{R}_{ijkl} + \left( \tilde{g}^{ap} \tilde{R}_{kp} - R^2 \right) \nabla_a \tilde{R}_{ijkl}.
\]

The desired expression then follows from \( (4.10) \) and the observation that the terms on the left on the first two lines are all of the schematic form \( \tilde{g}^{-2} \ast \text{Rm} \ast \nabla \text{Rm} \). \( \square \)

5. Exponential decay: A refined system

In order to apply the Carleman inequalities in Section 7, we need to show that \( X \) and \( Y \) vanish near spatial infinity and \( \tau = 0 \) at least an exponential rate. The goal of the next two sections will be to prove the following local estimate, which establishes their uniform exponential decay on regions of fixed size. We will write

\[
\mathcal{D}_r (z_0) = S^k \times B_r (z_0),
\]

for \( r > 0 \) and \( z_0 \in \mathbb{R}^{n-k} \), and use the shorthand \( dm = d\mu g(\tau) \, d\tau \).

**Theorem 5.1.** There exist positive constants \( N_0, N_1 \) depending only on \( n, k, r_0 \) and finitely many of the constants \( M_{l,m} \) from \( (4.1) \) such that

\[
\int_0^1 \int_{\mathcal{D}_1 (z_0)} (|X|^2 + |
abla X|^2 + |Y|^2) e^{\frac{N_0}{\tau}} \, dm \leq N_1,
\]

for any \( z_0 \in \mathbb{R}^{n-k} \setminus \overline{B_{r_0} (0)} \).

In Proposition 7.2 we will use the self-similarity of \( X \) and \( Y \) to rewrite this estimate as a measure of the space-time vanishing rate of the sections. However, the self-similarity of \( X \) and \( Y \) will not be used in the proof of Theorem 5.1 or elsewhere in the the next two sections.

In contrast with [28], we are not able to use the system of inequalities \( (4.3) \) to prove Theorem 5.1 directly, since the Carleman estimates (Theorems 6.2 and 6.3 below) cannot absorb the coefficients of \( X \) on the right side of \( (4.3) \) which blow up at a rate proportional to \( 1/\tau \). This difficulty is, however, at least in part an artifact of the coarse way in which we have estimated the reaction terms in the evolution equation for \( \nabla \text{Rm} \). We will now analyze the algebraic structure of these terms more carefully and introduce a replacement for \( X \) with which we may track the vanishing of the components of \( \nabla \text{Rm} \) relative to the splitting of \( TC \) individually. We will define
5.1. Notational conventions. We will not make use the self-similarity of \( \tilde{g} \) in the computations below, so for the rest of this section, \( \tilde{g} = \tilde{g}(\tau) \) will simply represent a smooth solution to (1.5) on \( C_{r_0} \times (0,1] \) satisfying (4.1). We will continue to use \( g = g(\tau) \) to represent the normalized shrinking cylindrical solution on \( C \times (0,1] \).

Let \( \mathcal{H} \) and \( \mathcal{K} \) denote the subbundles of \( TC \) with fibers \( \mathcal{H}(\theta,z) = T(\theta,z)(\mathbb{S}^k \times \{z\}) \) and \( \mathcal{K}(\theta,z) = T(\theta,z)(\{\theta\} \times \mathbb{R}^{n-k}) \), and let \( \bar{P} : TC \to \mathcal{H} \) and \( \tilde{P} : TC \to \mathcal{K} \) denote the corresponding \( g \)-orthogonal projections onto these subbundles. The projections \( \bar{P} \) and \( \tilde{P} \) are smooth, globally defined, families of \((1,1)\)-tensor fields on \( C \times (0,1] \) satisfying
\[
\bar{P}^2 = \bar{P}, \quad \tilde{P}^2 = \tilde{P}, \quad \bar{P} + \tilde{P} = \text{Id}_{TC}, \quad g(\bar{P}, \tilde{P}) = 0
\]
and
\[
\nabla \bar{P} = \nabla \tilde{P} = 0, \quad \frac{\partial}{\partial \tau} \bar{P} = \frac{\partial}{\partial \tau} \tilde{P} = D_\tau \bar{P} = D_\tau \tilde{P} = 0.
\]

Using \( \bar{P} \) and \( \tilde{P} \), we can track the components of any tensor relative to the splitting \( TC = \mathcal{H} \oplus \mathcal{K} \). We will use a notational system of underlined and barred indices to distinguish these components. Underlined indices will denote components acting on directions tangent to the spherical factor and barred indices will denote components acting on directions tangent to the Euclidean factor. Thus, for example, we will write
\[
\tilde{R}_{ab} = \bar{R}_{ij} \bar{P}^i_a \bar{P}^j_b, \quad \bar{R}_{ab} = \bar{R}_{ij} \tilde{P}^i_a \bar{P}^j_b, \quad \tilde{R}_{\bar{a}\bar{b}} = \bar{R}_{ij} \bar{P}^i_a \tilde{P}^j_b, \quad \bar{R}_{\bar{a}\bar{b}} = \bar{R}_{ij} \tilde{P}^i_a \tilde{P}^j_b.
\]

An unadorned index will represent an unmodified component, e.g.,
\[
\tilde{R}_{ab} = \tilde{R}_{ij} P^i_a P^j_b.
\]

We emphasize that each of the above expressions represent globally defined tensor fields and that the underlined and barred indices denote modifications to the tensor field, not the expression of the components of the tensor relative to a particular local frame.

Since we will not usually need to carefully examine the algebraic structure of terms that are quadratic or better in \( h \) or its derivatives, and it will be useful to introduce an economical notation for tensors with rapid spacetime decay.

**Notation 5.2.** We will use the expression \( o(\infty) \) to denote any family of tensors \( V \) that vanishes to infinite order in space and time in the sense that
\[
\sup_{C_{r_0} \times (0,1]} \left( \frac{|z|^2}{\tau^l} \right) |V|(\theta, z, \tau) < \infty
\]
for all \( l \geq 0 \). Here \( | \cdot | = | \cdot |_{g(\tau)} \) as before.

Finally, we will also use a repeated index to denote a contraction with the metric \( g \), and write out explicitly any contraction with \( \tilde{g} \).

5.2. The gradient of the scalar curvature. We begin our analysis by examining the evolution of the differential of the scalar curvature. In this and the calculations that follow, we will focus our attention on the structure of the linearization of the reaction terms based at the cylindrical solution \( g \).
Proposition 5.3. The differential $\nabla \tilde{R}$ of the scalar curvature of $\tilde{g}$ satisfies
\begin{equation}
(D_\tau + \Delta) \nabla_a \tilde{R} = -4\tilde{g}^{pq} \tilde{g}^{rs} \nabla_a \tilde{R}_{pr} \tilde{R}_{qs} + \tilde{g}^{pq} \tilde{R}_{ap} \nabla_q \tilde{R},
\end{equation}
on $C_{r_0} \times (0, 1]$.

Proof. From the standard formula
\begin{equation}
\left( \frac{\partial}{\partial \tau} + \Delta \right) \tilde{R} = -2|\nabla \tilde{R}|^2,
\end{equation}
we have
\begin{equation}
\left( \frac{\partial}{\partial \tau} + \Delta \right) \nabla_a \tilde{R} = -4\tilde{g}^{pq} \tilde{g}^{rs} \nabla_a \tilde{R}_{pr} \tilde{R}_{qs} + \tilde{g}^{pq} \tilde{R}_{ap} \nabla_q \tilde{R},
\end{equation}
and then
\begin{equation}
(D_\tau + \Delta) \nabla_a \tilde{R} = \Delta \nabla_a \tilde{R} - \Delta \nabla_a \tilde{R} - 4\tilde{g}^{pq} \tilde{g}^{rs} \nabla_a \tilde{R}_{pr} \tilde{R}_{qs} + (\tilde{g}^{pq} \tilde{R}_{ap} - g^{pq} R_{ap}) \nabla_q \tilde{R}
= o(\infty) * (h + \nabla h + \nabla \tilde{R}c) - 4\nabla_a \tilde{R}_{pq} \tilde{R}_{pq}
= o(\infty) * (h + \nabla h + \nabla \tilde{R}c) - 2\nabla_a \tilde{R}_{pq} \tilde{R}_{pq}.
\end{equation}
Using (4.10) and (5.3), and the fact that $R_{ij} = (1/2\tau) \tilde{P}_{ij}$, where $\tilde{P}_{ij} = g_{jk} \tilde{P}_i^k = g_{ij}$, we may rewrite this as
\begin{equation} \begin{aligned}
(D_\tau + \Delta) \nabla_a \tilde{R} &= \Delta \nabla_a \tilde{R} - \Delta \nabla_a \tilde{R} - 4\tilde{g}^{pq} \tilde{g}^{rs} \nabla_a \tilde{R}_{pr} \tilde{R}_{qs} + (\tilde{g}^{pq} \tilde{R}_{ap} - g^{pq} R_{ap}) \nabla_q \tilde{R}
&= o(\infty) * (h + \nabla h + \nabla \tilde{R}c) - 4\nabla_a \tilde{R}_{pq} \tilde{R}_{pq}
&= o(\infty) * (h + \nabla h + \nabla \tilde{R}c) - 2\nabla_a \tilde{R}_{pq} \tilde{R}_{pq}.
\end{aligned} \end{equation}
and, using our indexing convention, again as
\begin{equation} \begin{aligned}
(D_\tau + \Delta) \nabla_a \tilde{R} &= o(\infty) * (h + \nabla h + \nabla \tilde{R}c) - \frac{2}{\tau} \nabla_a \tilde{R}_{pp}
&= o(\infty) * (h + \nabla h + \nabla \tilde{R}c) - \frac{2}{\tau} \nabla_a \tilde{R}_{pp} + \frac{2}{\tau} \nabla_a \tilde{R}_{pp}
&= o(\infty) * (h + \nabla h + \nabla \tilde{R}c) - \frac{2}{\tau} \nabla_a \tilde{R} + \frac{2}{\tau} \nabla_a \tilde{R}_{pp}.
\end{aligned} \end{equation}
Here, to obtain the second line in the above computation, we used that
\begin{equation}
\nabla_a \tilde{R}_{pp} = g^{pq} \nabla_a \tilde{R}_{pq} = (g^{pq} - \tilde{g}^{pq}) \nabla_a \tilde{R}_{pq} + \nabla_a \tilde{R} = o(\infty) * \nabla \tilde{R}c + \nabla_a \tilde{R}.
\end{equation}
We then multiply $\nabla \tilde{R}$ by $\tau^2$ so that an application of $D_\tau$ will pick off the second term on the right in (5.3). This yields equation (5.2). \hfill \Box

5.2.1. A remark on the strategy. In the computation above to obtain (5.3), we have traded the singular term proportional to $\nabla_a \tilde{R}_{pp}$ for a singular term proportional to $\nabla_a \tilde{R}_{pp}$, exchanging a tensor with two underlined indices for one with two bar red indices. Although we have not eliminated the singular coefficient, we have reassigned it from a primarily spherical component of $\nabla \tilde{R}c$ to a primarily Euclidean one.

The computations for $\nabla \tilde{R}c$ and $\nabla \tilde{R}m$ that follow are essentially just more elaborate versions of this “under for over” exchange, with the goal of rearranging appropriately rescaled components of $\nabla \tilde{R}$, $\nabla \tilde{R}c$, and $\nabla \tilde{R}m$ into a system whose singular part has a strictly triangular structure. This structure will allow us to transfer the blow-up in the equations for the spherical and mixed components of the system to the equations of components with fewer spherical directions. At the end of the line
are the principally Euclidean components of $\nabla R_{m}$ which satisfy evolution equations with reaction terms that are quadratic or better in the other elements of the system, and which can absorb the blow-up that we have sent in their direction.

5.3. Decomposition of $\nabla \tilde{R}_{c}$. We next examine the evolution of the covariant derivative of the Ricci tensor. Define

$$\tilde{G}_{ijkl} = \tilde{\nabla}_{i} \tilde{R}_{jk} - \tilde{\nabla}_{j} \tilde{R}_{ik}.$$ 

Proposition 5.4. The components of $\nabla \tilde{R}_{c}$ satisfy the equations

\begin{align*}
(5.4) & \quad |(D_{\tau} + \Delta) \tilde{\nabla}_{a} \tilde{R}_{jk}| \lesssim |\tilde{\nabla}_{a} \tilde{R}_{ijkl}| \\
(5.5) & \quad |(D_{\tau} + \Delta) \tilde{\nabla}_{a} \tilde{R}_{jk}| \lesssim \tau^{\frac{1}{2}} |(\tilde{\nabla}_{a} \tilde{R}) + |\tilde{\nabla}_{a} \tilde{R}_{jk}| + |\tilde{\nabla}_{a} \tilde{R}_{ijkl}|| \\
(5.6) & \quad |(D_{\tau} + \Delta) \tilde{\nabla}_{a} \tilde{R}_{jk}| \lesssim \tau^{\frac{1}{2}} |(\tilde{\nabla}_{a} \tilde{R}) + |\tilde{\nabla}_{a} \tilde{R}_{jk}| + |\tilde{\nabla}_{a} \tilde{R}_{ijkl}|| \\
(5.7) & \quad |(D_{\tau} + \Delta) \tilde{\nabla}_{a} \tilde{R}_{jk}| \lesssim \tau^{\frac{1}{2}} |(\tilde{\nabla}_{a} \tilde{R}) + |\tilde{\nabla}_{a} \tilde{R}_{jk}| + |\tilde{\nabla}_{a} \tilde{R}_{ijkl}||. \\
(5.8) & \quad \text{where the notation } |U| \lesssim |V| \text{ indicates that} \\
& \quad |U| \leq o(\infty) \left(|h| + |\nabla h| + |\nabla R_{m}| + C|V|\right)

for some constant $C = C(n) > 0$. The components of the tensor $\tilde{G}_{ajk}$ satisfy

\begin{align*}
(5.9) & \quad |(D_{\tau} + \Delta) \tilde{G}_{ajk}| \lesssim \frac{1}{\tau^{\frac{1}{2}}} |\tilde{\nabla}_{a} \tilde{R}_{ijkl}| \\
(5.10) & \quad |(D_{\tau} + \Delta) \tilde{G}_{ajk}| \lesssim |\tilde{\nabla}_{a} \tilde{R}| + |\tilde{\nabla}_{a} \tilde{R}_{jk}| + |\tilde{\nabla}_{a} \tilde{R}_{ijk}| + |\tilde{\nabla}_{a} \tilde{R}_{ijkl}|. \\
\end{align*}

Proof. Starting from the equation

$$\left(\frac{\partial}{\partial \tau} + \Delta \right) \tilde{R}_{jk} = -2\tilde{g}^{pr} \tilde{g}^{qs} \tilde{R}_{jpqk} \tilde{R}_{rs} + 2\tilde{g}^{pr} \tilde{R}_{jpr} \tilde{R}_{qs},$$

we obtain

$$\left(\frac{\partial}{\partial \tau} + \Delta \right) \tilde{\nabla}_{a} \tilde{R}_{jk} = \tilde{g}^{pq} \left(\tilde{R}_{ap} \tilde{\nabla}_{a} \tilde{R}_{jk} + \tilde{R}_{jp} \tilde{\nabla}_{a} \tilde{R}_{pk} + \tilde{R}_{kp} \tilde{\nabla}_{a} \tilde{R}_{jq}\right)$$

$$- 2\tilde{g}^{pr} \tilde{g}^{qs} \left(\tilde{\nabla}_{a} \tilde{R}_{jpqk} \tilde{R}_{rs} + \tilde{R}_{jpqk} \tilde{\nabla}_{a} \tilde{R}_{rs} + \tilde{R}_{pajq} \tilde{\nabla}_{r} \tilde{R}_{sk} + \tilde{R}_{pajq} \tilde{\nabla}_{k} \tilde{R}_{rs}\right),$$

and hence

$$\left(\frac{\partial}{\partial \tau} + \Delta \right) \tilde{\nabla}_{a} \tilde{R}_{jk} = \Delta \tilde{\nabla}_{a} \tilde{R}_{jk} - \tilde{\nabla}_{a} \tilde{R}_{jk}$$

$$+ (\tilde{g}^{pq} \tilde{R}_{ap} - \tilde{R}_{ap}^{q}) \tilde{\nabla}_{a} \tilde{R}_{jk} + (\tilde{g}^{pq} \tilde{R}_{jp} - \tilde{R}_{jp}^{q}) \tilde{\nabla}_{a} \tilde{R}_{pk} + (\tilde{g}^{pq} \tilde{R}_{kp} - \tilde{R}_{kp}^{q}) \tilde{\nabla}_{a} \tilde{R}_{jq}\right)$$

$$- 2\tilde{g}^{pr} \tilde{g}^{qs} \left(\tilde{\nabla}_{a} \tilde{R}_{jpqk} \tilde{R}_{rs} + \tilde{R}_{jpqk} \tilde{\nabla}_{a} \tilde{R}_{rs} + \tilde{R}_{pajq} \tilde{\nabla}_{r} \tilde{R}_{sk} + \tilde{R}_{pajq} \tilde{\nabla}_{k} \tilde{R}_{rs}\right).$$

So, in view of (4.11) and (4.10), we have

$$\left(\frac{\partial}{\partial \tau} + \Delta \right) \tilde{\nabla}_{a} \tilde{R}_{jk} = o(\infty) \cdot (h + \nabla h + \nabla \tilde{R}_{m}) + E_{ajk},$$

where

$$E_{ajk} = -2 \left(\tilde{\nabla}_{a} \tilde{R}_{pq} \tilde{R}_{jpqk} + \tilde{\nabla}_{p} \tilde{R}_{ajq} \tilde{R}_{paqk} + \tilde{\nabla}_{a} \tilde{R}_{pq} \tilde{R}_{paqj} + \tilde{\nabla}_{a} \tilde{R}_{jpqk} \tilde{R}_{pq}\right).$$
Now, according to our normalization, on the evolving cylinder we have
\[ R_{ijkl} = \frac{1}{2(k-1)\tau} (\tilde{P}_{ij} \tilde{P}_{jk} - \tilde{P}_{ik} \tilde{P}_{jl}), \quad R_{ij} = \frac{1}{2\tau} \tilde{P}_{ij}, \]
so (5.11) becomes
\[
E_{a_{jk}} = -\frac{1}{\tau} \tilde{\nabla}_a \tilde{R}_{ppjk} \tilde{P}_{pq} + \frac{1}{(k-1)\tau} \tilde{\nabla}_a \tilde{R}_{pq} (\tilde{P}_{jq} \tilde{P}_{pk} - \tilde{P}_{jk} \tilde{P}_{pq}) \\
+ \frac{1}{(k-1)\tau} \tilde{\nabla}_p \tilde{R}_{pq} (\tilde{P}_{jq} \tilde{P}_{qp} - \tilde{P}_{jq} \tilde{P}_{pq}) \\
+ \frac{1}{(k-1)\tau} \tilde{\nabla}_p \tilde{R}_{qpk} (\tilde{P}_{jq} \tilde{P}_{aj} - \tilde{P}_{jk} \tilde{P}_{pq}) \\
= -\frac{1}{\tau} \tilde{\nabla}_a \tilde{R}_{ppjk} + \frac{1}{(k-1)\tau} (\tilde{\nabla}_a \tilde{R}_{jk} - \tilde{P}_{jk} \tilde{\nabla}_a \tilde{R}_{pp}) \\
+ \frac{1}{(k-1)\tau} (\tilde{\nabla}_a \tilde{R}_{aj} - \tilde{\nabla}_a \tilde{R}_{pj} \tilde{P}_{ak}) + \frac{1}{(k-1)\tau} (\tilde{\nabla}_a \tilde{R}_{ak} - \tilde{\nabla}_a \tilde{R}_{pk} \tilde{P}_{aj}).
\tag{5.12}
\]
Computing as in the proof of Proposition 5.3 we see that
\[ \tilde{\nabla}_a \tilde{R}_{ppjk} = o(\infty) * \tilde{\nabla} Rm + \tilde{\nabla}_a \tilde{R}_{jk} - \tilde{\nabla}_a \tilde{R}_{jppk}, \]
and
\[
\tilde{\nabla}_a \tilde{R}_{pp} = o(\infty) * \tilde{\nabla} Rc + \tilde{\nabla}_a \tilde{R} - \tilde{\nabla}_a \tilde{R}_{pp}, \\
\tilde{\nabla}_a \tilde{R}_{pj} = o(\infty) * \tilde{\nabla} Rc + \frac{1}{2} \tilde{\nabla}_a \tilde{R} - \tilde{\nabla}_a \tilde{R}_{pj}.
\]
Returning, then, to (5.12) and putting things together, we obtain
\[
E_{a_{jk}} = o(\infty) * \tilde{\nabla} Rm - \frac{1}{\tau} \tilde{\nabla}_a \tilde{R}_{jk} + \frac{1}{\tau} \tilde{\nabla}_a \tilde{R}_{jppk} \\
+ \frac{1}{(k-1)\tau} \left( \tilde{\nabla}_a \tilde{R}_{jk} + \tilde{\nabla}_k \tilde{R}_{aj} + \tilde{\nabla}_j \tilde{R}_{ak} \right) \\
+ \frac{\tilde{P}_{jk}}{(k-1)\tau} \left( \tilde{\nabla}_a \tilde{R}_{pp} - \tilde{\nabla}_a \tilde{R} \right) + \frac{\tilde{P}_{pk}}{(k-1)\tau} \left( \tilde{\nabla}_a \tilde{R}_{pp} - \frac{1}{2} \tilde{\nabla}_j \tilde{R} \right) \\
= o(\infty) * \tilde{\nabla} Rm + F_{a_{jk}},
\tag{5.13}
\]
where, by inspection, the components of the tensor \( F_{a_{jk}} \) satisfy
\[
F_{a_{jk}} = -\frac{1}{\tau} \tilde{\nabla}_a \tilde{R}_{jk} + \frac{1}{\tau} \tilde{\nabla}_a \tilde{R}_{jppk} \\
F_{a_{jk}} = -\frac{1}{\tau} \frac{(k-2)}{(k-1)} \tilde{\nabla}_a \tilde{R}_{jk} + \frac{1}{\tau} \tilde{\nabla}_a \tilde{R}_{jppk} + \frac{\tilde{P}_{jk}}{(k-1)\tau} \left( \tilde{\nabla}_a \tilde{R}_{pp} - \tilde{\nabla}_a \tilde{R} \right) \\
F_{a_{jk}} = -\frac{1}{\tau} \tilde{\nabla}_a \tilde{R}_{jk} + \frac{1}{\tau} \tilde{\nabla}_a \tilde{R}_{jppk} \\
F_{a_{jk}} = -\frac{1}{\tau} \frac{(k-2)}{(k-1)} \tilde{\nabla}_a \tilde{R}_{jk} - \frac{1}{(k-1)\tau} \tilde{\nabla}_a \tilde{R}_{jppk} + \frac{1}{\tau} \tilde{\nabla}_a \tilde{R}_{jppk} \\
+ \frac{\tilde{P}_{jk}}{(k-1)\tau} \left( \tilde{\nabla}_a \tilde{R}_{pp} - \frac{1}{2} \tilde{\nabla}_k \tilde{R} \right). 
\tag{5.17}
\]
\[ F_{a j k} = -\frac{1}{\tau} \left( k - \frac{4}{k - 1} \right) \tilde{\nabla}_a \tilde{R}_{j k} + \frac{1}{(k - 1)\tau} (\tilde{G}_{j a k} + \tilde{G}_{k a j}) + \frac{1}{\tau} \tilde{\nabla}_a \tilde{R}_{j \bar{p} \bar{k}} \]
\[ + \frac{\hat{P}_{j k}}{(k - 1)\tau} (\tilde{\nabla}_a \tilde{R}_{\bar{p} \bar{k}} - \tilde{\nabla}_a \tilde{R}) + \frac{\hat{P}_{a k}}{(k - 1)\tau} \left( \tilde{\nabla}_a \tilde{R}_{\bar{p} \bar{j}} - \frac{1}{2} \tilde{\nabla}_a \tilde{R} \right) \]
\[ + \frac{\hat{P}_{a j}}{(k - 1)\tau} \left( \tilde{\nabla}_a \tilde{R}_{\bar{p} \bar{k}} - \frac{1}{2} \tilde{\nabla}_a \tilde{R} \right) \]  
(5.18)

The relations (5.4) - (5.8) then follow directly from the identities (5.14) - (5.18) for \( F_{a j k} \). For example, using that \( D_\tau \hat{P} = D_\tau \tilde{P} = \Delta \hat{P} = \Delta \tilde{P} = 0 \), we have

\[ (D_\tau + \Delta) \tilde{\nabla}_a \tilde{R}_{j k} = \hat{P}_p \hat{P}^q (D_\tau + \Delta) \tilde{\nabla}_p \tilde{R}_{j k} = o(\infty) \ast (h + \nabla h + \tilde{\nabla} \tilde{R} m) + F_{a j k}. \]

Then, using (5.14), we see that

\[ (D_\tau + \Delta)(\tau \tilde{\nabla}_a \tilde{R}_{j k}) = o(\infty) \ast (h + \nabla h + \tilde{\nabla} \tilde{R} m) + \tilde{\nabla}_a \tilde{R}_{j k} + \tau F_{a j k} = o(\infty) \ast (h + \nabla h + \tilde{\nabla} \tilde{R} m) + \tilde{\nabla}_a \tilde{R}_{j \bar{p} \bar{k}}, \]

which implies (5.4). Relations (5.4) - (5.8) can be verified similarly. For (5.7), we use the second Bianchi identity in (5.17).

The identities (5.9) - (5.10) follow in the same way from the identities

\[ F_{a j \bar{k}} - F_{a \bar{k} j} = -\frac{1}{\tau} \left( \frac{k}{k - 1} \right) \bar{\tilde{G}}_{j \bar{a} k} - \frac{1}{\tau} \tilde{\nabla}_p \tilde{R}_{j \bar{a} \bar{k}}, \]

\[ F_{a j \bar{k}} - F_{a \bar{k} j} = -\frac{1}{\tau} \bar{\tilde{G}}_{j \bar{a} k} + \frac{1}{\tau} \bar{\tilde{\nabla}}_p \tilde{R}_{j \bar{a} \bar{k} \bar{p}} + \frac{\hat{P}_{a k}}{(k - 1)\tau} \left( \tilde{\nabla}_p \tilde{R}_{j \bar{p} \bar{k}} - \tilde{\nabla}_j \tilde{R}_{\bar{p} \bar{k}} + \frac{1}{2} \tilde{\nabla}_j \tilde{R} \right) \]

\[ - \frac{\hat{P}_{j k}}{(k - 1)\tau} \left( \tilde{\nabla}_a \tilde{R}_{\bar{p} \bar{k}} - \tilde{\nabla}_a \tilde{R}_{\bar{p} \bar{j}} + \frac{1}{2} \tilde{\nabla}_a \tilde{R} \right), \]

which are consequences of (5.17) and (5.18) and the second Bianchi identity.

5.4. Decomposition of \( \tilde{\nabla} \tilde{R} m \). Now we examine the components of the full covariant derivative of \( \tilde{R} m \). We will only need expressions for sufficiently many of the components to obtain a closed system of inequalities.

**Proposition 5.5.** The components of \( \tilde{\nabla} \tilde{R} m \) satisfy

\[ |(D_\tau + \Delta) \tilde{\nabla}_a \tilde{R}_{j \bar{k} \bar{i}}| \lesssim 0 \]  
(5.19)

\[ |(D_\tau + \Delta) \tilde{\nabla}_a \tilde{R}_{j \bar{i} \bar{k} \bar{l}}| \lesssim 0 \]  
(5.20)

\[ |(D_\tau + \Delta) \tau \tilde{\nabla}_a \tilde{R}_{j \bar{k} \bar{l}}| \lesssim 0 \]  
(5.21)

\[ |(D_\tau + \Delta) \tau \tilde{\nabla}_a \tilde{R}_{j \bar{k} \bar{i}| \lesssim \tau \frac{k}{k - 1} \left( |\tilde{\nabla}_a \tilde{R}_{j \bar{k} \bar{i}}| + |\tilde{\nabla}_a \tilde{R}_{j \bar{i} \bar{k}}| + |\tilde{\nabla}_a \tilde{R}_{i \bar{k} \bar{j}}| \right) \]  
(5.22)

\[ |(D_\tau + \Delta) \tilde{\nabla}_a \tilde{R}_{j \bar{k} \bar{i}| \lesssim \tau^{-1} \left( |\tilde{\nabla}_a \tilde{R}_{j \bar{k} \bar{i}}| + |\tilde{\nabla}_a \tilde{R}_{j \bar{i} \bar{k}}| + |\tilde{\nabla}_a \tilde{R}_{i \bar{k} \bar{j}}| \right) \]  
(5.23)

\[ |(D_\tau + \Delta) \tilde{\nabla}_a \tilde{R}_{j \bar{k} \bar{i}| \lesssim \tau^{-1} \left( |\tilde{\nabla}_a \tilde{R}_{j \bar{k} \bar{i}}| + |\tilde{\nabla}_a \tilde{R}_{j \bar{i} \bar{k}}| + |\tilde{\nabla}_a \tilde{R}_{i \bar{k} \bar{j}}| \right) \]  
(5.24)

\[ |(D_\tau + \Delta) \tau \tilde{\nabla}_a \tilde{R}_{j \bar{k} \bar{i}| \lesssim \tau^{-1} \left( |\tilde{\nabla}_a \tilde{R}_{j \bar{k} \bar{i}}| + |\tilde{\nabla}_a \tilde{R}_{j \bar{i} \bar{k}}| + |\tilde{\nabla}_a \tilde{R}_{i \bar{k} \bar{j}}| \right) \]  
(5.25)
\[(D_\tau + \Delta)\tau \nabla_a \tilde{R}_{ijkl} \lesssim \tau^{-\frac{1}{p+1}} \left( |\nabla_a \hat{R}_{ijkl}| + |\nabla_a \tilde{R}_{ijkl}| + |\nabla_a \tilde{R}_{ij}| \right) \tag{5.26}\]

\[(D_\tau + \Delta)\tau \nabla_a \tilde{R}_{ijkl} \lesssim \tau^{-\frac{1}{p+1}} \left( |\nabla_a \hat{R}_{ijkl}| + |\nabla_a \tilde{R}_{ij}| \right) \tag{5.27}\]

where, here, by \(|U| \lesssim |V|\), we mean
\[|U| \lesssim \rho(\infty)(|h| + |\nabla h| + |\nabla Rm| + C|V|)\]
for some constant \(C = C(n) > 0\).

**Proof.** From the proof of Lemma 4.3, we have
\[
(D_\tau + \Delta)\nabla_a \tilde{R}_{ijkl} = -2\nabla_a \left( \tilde{B}_{ijkl} - \hat{B}_{ijkl} + \hat{B}_{ikjl} - \hat{B}_{i} \right) + 2\tilde{g}^{pr}\tilde{g}^{qs} \left( \tilde{R}_{r q p} \nabla_r \tilde{R}_{ijkl} + \tilde{R}_{q p r} \nabla_r \tilde{R}_{ijkl} + \tilde{R}_{k q p} \nabla_r \tilde{R}_{ijkl} + \tilde{R}_{k q p} \nabla_r \tilde{R}_{ijkl} \right) + \left( \tilde{g}^{pq} \tilde{R}_{k p q} \nabla_k \tilde{R}_{ijkl} + (\tilde{g}^{pq} \tilde{R}_{k p q} - \tilde{R}_{k q}^p) \nabla_k \tilde{R}_{ijkl} + (\tilde{g}^{pq} \tilde{R}_{k p q} - \tilde{R}_{k q}^p) \nabla_k \tilde{R}_{ijkl} \right) \tag{5.28}\]
and so
\[(D_\tau + \Delta)\nabla_a \tilde{R}_{ijkl} = o(\infty) \ast (h + \nabla h + \nabla Rm) + J_{aijkl} + L_{aijkl}, \tag{5.28}\]
where
\[J_{aijkl} = -2\nabla_a \left( \tilde{B}_{ijkl} - \hat{B}_{ijkl} + \hat{B}_{ikjl} - \hat{B}_{i} \right), \tag{5.28}\]
and
\[L_{aijkl} = 2 \left( \tilde{R}_{r q p} \nabla_r \tilde{R}_{ijkl} + \tilde{R}_{q p r} \nabla_r \tilde{R}_{ijkl} + \tilde{R}_{k q p} \nabla_r \tilde{R}_{ijkl} + \tilde{R}_{k q p} \nabla_r \tilde{R}_{ijkl} \right). \tag{5.28}\]

As in the proof of Proposition 5.4, the inequalities (5.19)-(5.27) follow from (5.28) and the expressions (5.29)-(5.33) and (5.35)-(5.41) for the corresponding components of the tensors \(J\) and \(L\) below. We also use the Bianchi identities to estimate \(|\nabla_a \hat{R}_{ijkl}| \leq C|\nabla_a \hat{R}_{ijkl}|\) and \(|\nabla_a \tilde{R}_{ijkl}| \leq C|\nabla_a \tilde{R}_{ijkl}|\) in (5.26), \(|\nabla_a \hat{R}_{ijkl}| \leq C|\nabla_a \tilde{R}_{ijkl}|\) and \(|\nabla_a \tilde{R}_{ijkl}| \leq C|\nabla_a \tilde{R}_{ijkl}|\) in (5.27), and \(|\nabla_a \tilde{R}_{ijkl}| \leq C|\nabla_a \tilde{R}_{ijkl}|\) in (5.27).

We consider the tensor \(J_{aijkl}\) first.

**Proposition 5.6.** The components of the tensor
\[J_{aijkl} = -2\nabla_a \left( \tilde{B}_{ijkl} - \hat{B}_{ijkl} + \hat{B}_{ikjl} - \hat{B}_{i} \right) \tag{5.29}\]
satisfy the relations
\[(5.29) \quad J_{aijkl} \simeq 0 \tag{5.29}\]
\[(5.30) \quad J_{aijkl} \simeq -\frac{1}{(k - 1)\tau} \nabla_a \tilde{R}_{ijkl} \tag{5.30}\]
\[(5.31) \quad J_{aijkl} \simeq \frac{1}{(k - 1)\tau} \left( \nabla_a \tilde{R}_{ijkl} + \nabla_a \tilde{R}_{dijkl} + (\nabla_a \tilde{R}_{jlpk} - \nabla_a \tilde{R}_{jk}) \tilde{P}_{dl} \right) \tag{5.31}\]
\[ J_{aijkl} \simeq \frac{1}{(k-1)\tau} \left( (\bar{\nabla}_a \bar{R}_{l\bar{i}} - \bar{\nabla}_a \bar{R}_{j\bar{p}\bar{k}}) \bar{P}_{\bar{p}k} - (\bar{\nabla}_a \bar{R}_{l\bar{j}} - \bar{\nabla}_a \bar{R}_{i\bar{p}k}) \bar{P}_{\bar{p}l} \right) \]
\[ J_{aijkl} \simeq \frac{1}{(k-1)\tau} \left( (\bar{\nabla}_a \bar{R}_{l\bar{i}} - \bar{\nabla}_a \bar{R}_{j\bar{p}\bar{k}}) \bar{P}_{\bar{p}k} + (\bar{\nabla}_a \bar{R}_{l\bar{j}} - \bar{\nabla}_a \bar{R}_{i\bar{p}k}) \bar{P}_{\bar{p}l} \right) \]
\[ - \frac{1}{(k-1)\tau} \left( (\bar{\nabla}_a \bar{R}_{i\bar{l}} - \bar{\nabla}_a \bar{R}_{j\bar{p}p}) \bar{P}_{\bar{p}l} + (\bar{\nabla}_a \bar{R}_{i\bar{j}} - \bar{\nabla}_a \bar{R}_{j\bar{p}p}) \bar{P}_{\bar{p}j} \right), \]

where, here, \( U \simeq V \) signifies that

\[ U = o(\infty) * \bar{\nabla} \bar{R}m + V. \]

**Proof.** We first compute that

\[
\bar{\nabla}_a \bar{B}_{ijkl} = -\bar{g}^{pr} \bar{g}^{qs} \left( \bar{\nabla}_a \bar{R}_{pjrq} \bar{R}_{rkl} + \bar{R}_{pjrq} \bar{\nabla}_a \bar{R}_{rkl} \right) \\
= o(\infty) * \bar{\nabla} \bar{R}m - \bar{\nabla}_a \bar{R}_{pjrq} \bar{R}_{pkl} - \bar{\nabla}_a \bar{R}_{pkl} \bar{R}_{pjrq} \\
= o(\infty) * \bar{\nabla} \bar{R}m - \frac{1}{2(k-1)\tau} \bar{\nabla}_a \bar{R}_{pjrq} (\bar{P}_{pq} \bar{P}_{kl} - \bar{P}_{pl} \bar{P}_{qk}) \\
- \frac{1}{2(k-1)\tau} \bar{\nabla}_a \bar{R}_{pkl} (\bar{P}_{pq} \bar{P}_{ij} - \bar{P}_{pj} \bar{P}_{iq}) \\
= o(\infty) * \bar{\nabla} \bar{R}m - \frac{1}{2(k-1)\tau} \left( \bar{\nabla}_a \bar{R}_{ipjj} \bar{P}_{kl} + \bar{\nabla}_a \bar{R}_{kkpp} \bar{P}_{ij} \right) \\
+ \frac{1}{2(k-1)\tau} \left( \bar{\nabla}_a \bar{R}_{ijjk} + \bar{\nabla}_a \bar{R}_{jijkl} \right)
\]

for any \( a, i, j, k, l \). Permuting the indices in this identity and summing, we obtain

\[ J_{aijkl} = o(\infty) * \bar{\nabla} \bar{R}m \\
+ \frac{1}{(k-1)\tau} \left( \bar{\nabla}_a \bar{R}_{ipjj} \bar{P}_{kl} + \bar{\nabla}_a \bar{R}_{kkpp} \bar{P}_{ij} - \bar{\nabla}_a \bar{R}_{ijjk} - \bar{\nabla}_a \bar{R}_{jijkl} \right) \\
- \frac{1}{(k-1)\tau} \left( \bar{\nabla}_a \bar{R}_{ipjj} \bar{P}_{kl} + \bar{\nabla}_a \bar{R}_{kkpp} \bar{P}_{ij} - \bar{\nabla}_a \bar{R}_{ijjk} - \bar{\nabla}_a \bar{R}_{jijkl} \right) \\
+ \frac{1}{(k-1)\tau} \left( \bar{\nabla}_a \bar{R}_{ipjj} \bar{P}_{kl} + \bar{\nabla}_a \bar{R}_{kkpp} \bar{P}_{ij} - \bar{\nabla}_a \bar{R}_{ijjk} - \bar{\nabla}_a \bar{R}_{jijkl} \right), \]

that is,

\[ J_{aijkl} = o(\infty) * \bar{\nabla} \bar{R}m - \frac{1}{(k-1)\tau} (\text{tr}_p(\bar{\nabla}_a \bar{R}m) \odot \bar{P})_{ijkl} \\
+ \frac{1}{(k-1)\tau} \left( \bar{\nabla}_a \bar{R}_{ikjl} + \bar{\nabla}_a \bar{R}_{jikl} - \bar{\nabla}_a \bar{R}_{ijjk} - \bar{\nabla}_a \bar{R}_{jijkl} \right) \\
+ \frac{1}{(k-1)\tau} \left( \bar{\nabla}_a \bar{R}_{ikjl} + \bar{\nabla}_a \bar{R}_{jikl} - \bar{\nabla}_a \bar{R}_{ijjk} - \bar{\nabla}_a \bar{R}_{jijkl} \right) \\
= o(\infty) * \bar{\nabla} \bar{R}m - \frac{1}{(k-1)\tau} \left( (\text{tr}_p(\bar{\nabla}_a \bar{R}m) \odot \bar{P})_{ijkl} + \bar{\nabla}_a \bar{R}_{ijjk} + \bar{\nabla}_a \bar{R}_{jijkl} \right) \\
+ \frac{1}{(k-1)\tau} \left( \bar{\nabla}_a \bar{R}_{ikjl} + \bar{\nabla}_a \bar{R}_{jikl} - \bar{\nabla}_a \bar{R}_{ijjk} - \bar{\nabla}_a \bar{R}_{jijkl} \right), \]

where

\[ \text{tr}_p(\bar{\nabla}_a \bar{R}m)_{ij} = \bar{\nabla}_a \bar{R}_{ipjj}, \]
and $U \otimes V$ denotes the Kulkarni-Nomizu product

$$(U \otimes V)_{ijkl} = U_{il}V_{jk} + U_{jk}V_{il} - U_{ik}V_{jl} - U_{jl}V_{ik}.$$ 

A case by case examination of (5.34), using the first Bianchi identity and the observation that

$$\text{tr}_P(\bar{\nabla}_a \bar{R}_m)_{ij} = \bar{\nabla}_a \bar{R}_{ippp} = o(\infty) * \bar{\nabla} \bar{R}_m + \bar{\nabla}_a \bar{R}_{ij} - \bar{\nabla}_a \bar{R}_{ippp},$$ yields (5.29) - (5.33).

Now we perform a similar analysis for the tensor $L_{ijkl}$ in (5.28).

**Proposition 5.7.** The components of the tensor

$$L_{ijkl} = 2 \left( R_{iqap} \bar{\nabla}_p \bar{R}_{aqkl} + R_{jqap} \bar{\nabla}_p \bar{R}_{aqkl} + R_{kpq} \bar{\nabla}_p \bar{R}_{iliq} + R_{lqp} \bar{\nabla}_p \bar{R}_{iliq} \right)$$

satisfy the relations

(5.35) $L_{aijkl} \simeq 0$

(5.36) $L_{aijki} \simeq 0$

(5.37) $L_{aijkl} \simeq \frac{1}{(k-1)^\tau} \left( \bar{\nabla}_a \bar{R}_{ijkl} + \bar{\nabla}_j \bar{R}_{aijk} \right)$

$$+ \frac{\bar{P}_{ia}}{(k-1)^\tau} \left( \bar{\nabla}_b \bar{R}_{pjki} - \bar{\nabla}_i \bar{R}_{kj} + \bar{\nabla}_k \bar{R}_{jl} \right)$$

(5.38) $L_{aijkl} \simeq \frac{1}{(k-1)^\tau} \bar{\nabla}_a \bar{R}_{ijkl} + \frac{\bar{P}_{ia}}{(k-1)^\tau} \left( \bar{\nabla}_b \bar{R}_{pjkki} - \bar{\nabla}_i \bar{R}_{kj} + \bar{\nabla}_k \bar{R}_{jl} \right)$

$$- \frac{\bar{P}_{ja}}{(k-1)^\tau} \left( \bar{\nabla}_b \bar{R}_{pikij} - \bar{\nabla}_j \bar{R}_{ablk} + \bar{\nabla}_k \bar{R}_{abl} \right)$$

(5.39) $L_{aijkl} \simeq \frac{1}{(k-1)^\tau} \left( 2\bar{\nabla}_a \bar{R}_{ijkl} - \bar{\nabla}_j \bar{R}_{ailk} - \bar{\nabla}_k \bar{R}_{abl} \right)$

$$+ \frac{\bar{P}_{ia}}{(k-1)^\tau} \left( \bar{\nabla}_b \bar{R}_{pjkki} - \bar{\nabla}_i \bar{R}_{kj} + \bar{\nabla}_k \bar{R}_{jl} \right)$$

$$- \frac{\bar{P}_{ja}}{(k-1)^\tau} \left( \bar{\nabla}_b \bar{R}_{pikij} - \bar{\nabla}_j \bar{R}_{ablk} + \bar{\nabla}_k \bar{R}_{abl} \right)$$

(5.40) $L_{aijkl} \simeq \frac{1}{(k-1)^\tau} \left( 2\bar{\nabla}_a \bar{R}_{ijkl} + \bar{\nabla}_j \bar{R}_{aikl} \right)$

$$+ \frac{\bar{P}_{ia}}{(k-1)^\tau} \left( \bar{\nabla}_b \bar{R}_{pjkki} - \bar{\nabla}_i \bar{R}_{kj} + \bar{\nabla}_k \bar{R}_{jl} \right)$$

$$- \frac{\bar{P}_{ja}}{(k-1)^\tau} \left( \bar{\nabla}_b \bar{R}_{pikij} - \bar{\nabla}_j \bar{R}_{ablk} + \bar{\nabla}_l \bar{R}_{abl} \right)$$
\[ L_{aijkl} \simeq \frac{2}{(k-1)^{\tau}} \nabla_a \tilde{R}_{ijkl} + \frac{\dot{P}_{ia}}{(k-1)^{\tau}} \left( \nabla_b \tilde{R}_{pjkl} - \nabla_l \tilde{R}_{ijk} + \nabla_k \tilde{R}_{lj} \right) + \frac{\dot{P}_{ja}}{(k-1)^{\tau}} \left( \nabla_b \tilde{R}_{pikl} - \nabla_l \tilde{R}_{ik} + \nabla_k \tilde{R}_{jl} \right) + \frac{\dot{P}_{ka}}{(k-1)^{\tau}} \left( \nabla_b \tilde{R}_{plij} - \nabla_l \tilde{R}_{ik} + \nabla_k \tilde{R}_{jl} \right) - \frac{\dot{P}_{la}}{(k-1)^{\tau}} \left( \nabla_b \tilde{R}_{pklj} - \nabla_j \tilde{R}_{ik} + \nabla_l \tilde{R}_{kj} \right), \]

(5.41)

where here \( U \simeq V \) signifies \( U = o(\% \nabla Rm + V. \)

**Proof.** Note that

\[ R_{aqap} \tilde{\nabla}_p \tilde{R}_{qjkl} = \frac{1}{2(k-1)^{\tau}} \left( \dot{P}_{qi} - \dot{P}_{qa} \right) \tilde{\nabla}_p \tilde{R}_{ajkl} = \frac{1}{2(k-1)^{\tau}} \left( \tilde{\nabla}_i \tilde{R}_{ajkl} - \dot{P}_{ia} \tilde{\nabla}_p \tilde{R}_{pjkl} \right), \]

and so

\[ L_{aijkl} = 2 \left( R_{aqap} \tilde{\nabla}_p \tilde{R}_{qjkl} - R_{jqap} \tilde{\nabla}_p \tilde{R}_{qijkl} + R_{kpap} \tilde{\nabla}_p \tilde{R}_{qijkl} - R_{kqap} \tilde{\nabla}_p \tilde{R}_{qijkl} \right) \]

\[ = \frac{1}{(k-1)^{\tau}} \left( \tilde{\nabla}_i \tilde{R}_{ajkl} - \tilde{\nabla}_j \tilde{R}_{aki} - \dot{P}_{ia} \tilde{\nabla}_p \tilde{R}_{pjkl} + \dot{P}_{ja} \tilde{\nabla}_p \tilde{R}_{pijkl} \right) + \frac{1}{(k-1)^{\tau}} \left( \tilde{\nabla}_k \tilde{R}_{aij} - \tilde{\nabla}_l \tilde{R}_{akij} - \dot{P}_{ka} \tilde{\nabla}_p \tilde{R}_{pkij} + \dot{P}_{la} \tilde{\nabla}_p \tilde{R}_{pijkl} \right). \]

Using the identity

\[ \tilde{\nabla}_p \tilde{R}_{ijkl} = \nabla_i \tilde{R}_{kjl} - \nabla_k \tilde{R}_{ijl}, \]

we may rewrite the terms in the above equation of the form \( \tilde{\nabla}_p \tilde{R}_{ijkl} \) as

\[ \tilde{\nabla}_p \tilde{R}_{ijkl} = o(\% \nabla Rm + \nabla_i \tilde{R}_{kjl} - \nabla_k \tilde{R}_{ijl} - \tilde{\nabla}_p \tilde{R}_{ijkl} \).

The relations (5.33) - (5.41) then follow from a case-by-case inspection of the above identity for \( L_{aijkl} \) with the Bianchi identities. \( \square \)

**5.5. Assembling the components of the system.** Using Propositions 5.3, 5.4 and 5.5 we now organize the rescaled components of \( \tilde{\nabla} R, \tilde{\nabla} Rc, \) and \( \tilde{\nabla} Rm \) into groupings which satisfy a closed system of inequalities whose singular part has a triangular structure.

Define \( W = (W^0, W^1, \ldots, W^5) \) by

\[ W^0 = (\nabla_a \tilde{R}_{ijkl}, \nabla_a \tilde{R}_{ijkl}^{\tau_c}, \tau^c \nabla_a \tilde{R}_{ijkl}), \quad W^1 = (\tau \nabla_a \tilde{R}_{ijl}, \tau \nabla_a \tilde{R}_{jkl}, \tau^{1+c} G_{ij}), \]

\[ W^2 = (\tau^2 \nabla_a \tilde{R}_{ijkl}, \tau^c \nabla_a \tilde{R}_{ijkl}, \tau \nabla_a \tilde{R}_{ijkl}), \]

\[ W^3 = (\tau^{1-c} \nabla_a \tilde{R}_{ijkl}, \tau^{1-c} \nabla_a \tilde{R}_{ijkl}, \tau \nabla_a \tilde{R}_{ijkl}), \]

\[ W^4 = (\tau^{1-c} \nabla_a \tilde{R}_{ijkl}, \tau \nabla_a \tilde{R}_{ijkl}, \tau^{1-c} \nabla_a \tilde{R}_{ijkl}), \quad W^5 = (\tau^{2-c} \nabla_a \tilde{R}_{ijkl}, \tau \nabla_a \tilde{R}_{ijkl}, \tau^{2-c} \nabla_a \tilde{R}_{ijkl}), \]

where \( c = 1/(k-1) \).
Proposition 5.8. The components $W^i$ of $W$ satisfy the system

\[
|(D_\tau + \Delta)W^0| \lesssim 0
\]
\[
|(D_\tau + \Delta)W^1| \lesssim |W^0|
\]
\[
|(D_\tau + \Delta)W^2| \lesssim \tau^{-(1+2\varepsilon)}|W^0| + \tau^{-(2+\varepsilon)}|W^1|
\]
\[
|(D_\tau + \Delta)W^3| \lesssim \tau^{-(1+3\varepsilon)}|W^0| + \tau^{-(2+3\varepsilon)}|W^1| + (\tau^{-(1+3\varepsilon)} + \tau^{-(2+3\varepsilon)})|W^2|
\]
\[
|(D_\tau + \Delta)W^4| \lesssim \tau^{-(1+3\varepsilon)}|W^1| + \tau^{-(2+3\varepsilon)}|W^2| + (\tau^{-(1+3\varepsilon)} + \tau^{-(2+3\varepsilon)})|W^3|
\]
\[
|(D_\tau + \Delta)W^5| \lesssim \tau^{-(1+\varepsilon)}|W^3| + |\tau^{-(2+\varepsilon)}|W^4| + (\tau^{-(1+2\varepsilon)} + \tau^{-(2+\varepsilon)})|W^5|
\]

on $C_{r_0} \times (0, 1]$. Here, $c = 1/(k - 1)$ as before, and $|U| \lesssim |V|$ means that

\[
|U| \leq |o(\infty)|(|h| + |\nabla h| + |\nabla \tilde{Rm}|) + C|V|
\]

for some constant $C = C(n) > 0$. Moreover, we have

\[
(5.43)\quad |\nabla \tilde{Rm}| + |\nabla \nabla \tilde{Rm}| \leq C(|W| + |\nabla W|)
\]

on $C_{r_0} \times (0, 1]$ for some $C = C(n)$.

Proof. Let us observe that (5.43) is satisfied first. Using the symmetries of $\nabla \tilde{Rm}$ and the Bianchi identities, we have

\[
|\nabla_a \tilde{R}_{ijkl}| \leq C(|\nabla_a \tilde{R}_{ijkl}| + |\nabla_a \tilde{R}_{ijkl}| + |\nabla_a \tilde{R}_{ijkl}| + |\nabla_a \tilde{R}_{ijkl}|) \\
\leq C(|W^0| + \tau^4|W^2| + \tau^{3\varepsilon}|W^3| + \tau^{2\varepsilon}|W^5|)
\]

for some $C = C(n) > 0$. Similarly, $|\nabla \nabla \tilde{Rm}|$ can be controlled by the sum of $|\nabla W^0|$, $|\nabla W^2|$, $|\nabla W^3|$, and $|\nabla W^5|$, so (5.43) follows.

Now we verify the system of inequalities satisfied by the components of $W$. Denoting the components of $W^i$ by $W^{i,j}$, we first see from (5.19)-(5.21) that

\[
|(D_\tau + \Delta)W^{0,j}| \lesssim 0
\]

for $j = 0, 1, 2$. The inequality for $W^0$ follows. Next, from (5.4) and (5.6), we have

\[
|(D_\tau + \Delta)W^{1,0}| \lesssim |\nabla_a \tilde{R}_{ijkl}| = |W^{0,0}|, \quad |(D_\tau + \Delta)W^{1,1}| \lesssim |\nabla_a \tilde{R}_{ijkl}| = |W^{0,1}|
\]

and, from (5.5), that

\[
|(D_\tau + \Delta)W^{1,2}| \lesssim \tau^4|\nabla_a \tilde{R}_{ijkl}| = |W^{0,2}|
\]

Taken together, these inequalities yield the relation for $W^1$.

For $W^2$, we start with (5.22), which implies

\[
|(D_\tau + \Delta)W^{2,0}| \lesssim \tau |\nabla_a \tilde{R}_{ij}| = |W^{1,0}|
\]

Then (5.22) and (5.23) yield, respectively, that

\[
|(D_\tau + \Delta)W^{2,1}| \lesssim \tau^{-(1+\varepsilon)}(|\nabla_a \tilde{R}_{ijkl}| + |\nabla_a \tilde{R}_{ijkl}| + |\nabla_a \tilde{R}_{ijkl}|) \\
\lesssim \tau^{-(1+\varepsilon)}(|W^{0,0}| + \tau^{-(1+2\varepsilon)}|W^{0,2}| + \tau^{-(2+\varepsilon)}|W^{1,0}|)
\]

and

\[
|(D_\tau + \Delta)W^{2,2}| \lesssim \tau^{-1}(|\nabla_a \tilde{R}_{ijkl}| + |\nabla_a \tilde{R}_{ij}|) \lesssim \tau^{-1}|W^{0,1}| + \tau^{-2}|W^{1,1}|
\]

and the desired inequality for $W^2$ follows.
Similarly, using (5.5) and (5.7), we see that
\[ |(D_\tau + \Delta)W^{3,0}| \lesssim \tau^{-c}(|\nabla a \tilde{R}| + |\nabla a \tilde{R}_{jk}| + |\nabla a \tilde{R}_{ijkl}|) \]
and
\[ |(D_\tau + \Delta)W^{3,1}| \lesssim \tau^{-c}(|\nabla a \tilde{R}| + |\tilde{G}_{ajk}| + |\nabla a \tilde{R}_{jk}| + |\nabla a \tilde{R}_{ijkl}|) \]
\[ \lesssim \tau^{-(1+c)|W^{1,0}| + \tau^{-(2+c)|W^{2,0}|} + |W^{2,1}|, \]
while, using (5.10) and (5.25), we see that
\[ |(D_\tau + \Delta)W^{3,2}| \lesssim |\nabla a \tilde{R}| + |\nabla a \tilde{R}_{jk}| + |\nabla a \tilde{R}_{ijkl}| \]
\[ \lesssim \tau^{-1}|W^{1,0}| + \tau^{-1}|W^{1,1}| + \tau^{-2}|W^{2,0}| + |W^{2,2}|, \]
and
\[ |(D_\tau + \Delta)W^{3,3}| \lesssim \tau^{-(1+3c)} \left( |\nabla a \tilde{R}_{ijkl}| + |\nabla a \tilde{R}_{ijkl}| + |\nabla a \tilde{R}_{ijkl}| + |\nabla a \tilde{R}_{ijkl}| \right) \]
\[ \lesssim \tau^{-(1+3c)}(|W^{0,1}| + |W^{2,2}|) + \tau^{-2+3c}(|W^{1,0}| + |W^{1,1}|). \]
Combining these relations yields the inequality for $W^3$.

Next, from (5.8) and (5.24), we have
\[ |(D_\tau + \Delta)W^{4,0}| \lesssim \tau^{-3c} \left( |\nabla a \tilde{R}| + |\tilde{G}_{ajk}| + |\nabla a \tilde{R}_{jk}| + |\nabla a \tilde{R}_{ijkl}| \right) \]
\[ \lesssim \tau^{-(2+3c)|W^{2,0}| + \tau^{-(1+3c)}(|W^{1,0}| + |W^{1,1}| + |W^{3,2}|) + |W^{3,3}|, \]
and
\[ |(D_\tau + \Delta)W^{4,1}| \lesssim \tau^{-1} \left( |\nabla a \tilde{R}_{ijkl}| + |\nabla a \tilde{R}_{ijkl}| \right) \lesssim \tau^{-(1-c)|W^{2,1}| + \tau^{-(2-c)}|W^{3,0}|, \]
which yield the inequality for $W^4$.

Finally, to obtain the inequality for $W^5$, we use
\[ |(D_\tau + \Delta)W^{5,0}| \lesssim \tau^{-(1+2c)} \left( |\nabla a \tilde{R}_{ijkl}| + |\nabla a \tilde{R}_{ijkl}| + |\nabla a \tilde{R}_{ijkl}| \right) \]
\[ \lesssim \tau^{-(1+c)|W^{2,1}| + \tau^{-(2+c)}|W^{3,0}| + \tau^{-(2+c)}|W^{3,1}| + \tau^{-(1+2c)|W^4,1|}, \]
from (5.20), and
\[ |(D_\tau + \Delta)W^{5,1}| \lesssim \tau^{-(1+2c)} \left( |\nabla a \tilde{R}_{ijkl}| + |\nabla a \tilde{R}_{ijkl}| \right) \]
\[ \lesssim \tau^{-(1-c)|W^{3,3}| + \tau^{-(2-c)}|W^{4,0}|, \]
from (5.27).

The largest exponent of $\tau$ which appears in the denominator of the coefficients of $|W^0|$ on the right side of the above relations is $\gamma = 2 + 3/(k - 1)$. Unwinding the notation $\lesssim$, we thus see that, for all $\beta > 0$, there is $B_0 = B_0(\beta)$ depending on finitely many of the constants $M_{l,m}$ in (1.1) such that
\[ (5.44) \quad |(D_\tau + \Delta)W^i| \leq B_0 \tau^\beta (|W| + |Y|) + B_0 \sum_{j=0}^{i-1} \tau^{-\gamma} |W^j|. \]
and, in view of (4.3) and (5.43),
\[ |D_\tau Y| \leq B_0(|W| + |\nabla W|) + B_0 \tau^{-1} |Y| \]
on $C_\tau \times (0, 1]$.

6. Exponential Decay: The induction argument

The system (5.44)-(5.45) for the sections $W$ and $Y$ has the advantage over the system (4.3) for $X$ and $Y$ that the singular terms in the equations (5.44) appear in a strictly triangular form. In this section, we will prove decay estimates for general systems with this triangular structure, and use these estimates to deduce Theorem 5.1. These estimates will use the weights
\[ \sigma(\tau) = \tau e^{\frac{\tau}{2}}, \quad G_{z_0}(z, \tau) = e^{-\frac{|z - z_0|^2}{\tau}}, \]
for fixed $z_0 \in \mathbb{R}^{n-k}$. Note that $\sigma$ is comparable to $\tau$ in the sense that
\[ \tau \leq \sigma(\tau) \leq e^\frac{\tau}{2} \tau \]
for $0 \leq \tau \leq T$, and that $\sigma'(\tau) > 0$ and $\sigma(\tau) \leq 1$ on $[0, T]$ as long as $T \leq 1$.

**Proposition 6.1.** Let the bundles $W = \oplus_{i=0}^q T^{(k,i)}(\mathcal{C})$ and $Y = \oplus_{i=0}^q T^{(k,i)}(\mathcal{C})$ be equipped with the family of metrics and connections induced by $g = g(\tau)$. Suppose that $W = (W^0, \ldots, W^q)$ and $Y = (Y^0, \ldots, Y^q)$ are families of sections of $W$ and $Y$ over $C_\tau \times (0, 1]$ satisfying the following two conditions:

(a) There are nonnegative constants $\beta, \gamma, \mu, \text{ and } B$ such that
\[ |(D_\tau + \Delta)W^i| \leq B \tau^\beta (|W| + |Y|) + B \sum_{j=0}^{i-1} \tau^{-\gamma} |W^j| \]
\[ |D_\tau Y| \leq B \tau^{-\mu} (|W| + |\nabla W|) + B \tau^{-1} |Y| \]
for each $i = 0, \ldots, q$ on $C_\tau \times (0, 1]$.

(b) For each $l \geq 0$,
\[ \sup_{C_\tau \times (0, 1]} \frac{|z|^{2l}}{\tau^l} (|W| + |\nabla W| + |Y|) \leq M_l \]
for some constant $M_l \geq 0$.

Then, there are positive constants $\beta_0 = \beta_0(k,n,q,\gamma,\mu)$ and $\lambda_0 = \lambda_0(k,n,\mu)$, and $L_0$, $K_0$, and $T_0 \leq 1$ depending on $k$, $n$, $\gamma$, $\mu$, $B$, and finitely many of the constants $M_l$, such that, if $\beta \geq \beta_0$, the inequality
\[ \int_0^T \int_{D_\tau(z_0)} \left( \tau |W|^2 + \tau^2 |\nabla W|^2 + \tau^{\lambda_0} |Y|^2 \right) \sigma^{-m} G_{z_0} \, dm \leq K_0 L_0^m r^{-2m} m! \]
holds for all $m \geq 0$ and all $0 < \tau^2 \leq T \leq T_0$ and $z_0$ with $B_{4r}(z_0) \subset \mathbb{R}^{n-k} \setminus B_{r_0}(0)$.

The point is that the constants $\beta_0$, $\lambda_0$, $L_0$, $K_0$, and $T_0$ do not depend on $m$. 

6.1. Proof of Theorem 5.1. We will prove Proposition 6.1 by an induction argument in the next subsection. First we show that it indeed implies Theorem 5.1.

Proof of Theorem 5.1 assuming Proposition 6.1. By choosing \( B_0 = B_0(\beta) \) appropriately large in (5.44) and (5.45) we may assume that (6.3) is satisfied with \( \gamma = 2 + 3/(k-1) \) and \( \mu = 0 \). Let \( z_0 \in \mathbb{R}^{n-k} \setminus \overline{B_{r_0}(0)} \) and \( 0 < T \leq T_0 \). Since \( r_0 > 1 \), we are assured that \( B_r(w) \subset \mathbb{R}^{n-k} \setminus \overline{B_{r_0}(0)} \) whenever \( w \in B_2(z_0) \) and \( 0 \leq r \leq \sqrt{T} \leq 1 \). At any such \( w \), we may then combine (5.43) with (6.5) to obtain that, for all \( r \leq \sqrt{T} \) and \( m \geq 0 \), the inequality

\[
\int_0^T \int_{D_r(w)} r^p (|X|^2 + |\nabla X|^2 + |Y|^2) \sigma^{-m} G_w \, dm \leq K_0 L_m r^{-2m} m!
\]

holds for some \( N = N(B_0, K_0) \) and fixed integer \( p = \max \{ \lambda_0, 2 \} \).

Using that \( \sigma(\tau) \leq \sqrt{T} \), we then have

\[
\frac{1}{(m-p)!} \int_0^T \int_{D_r(w)} (|X|^2 + |\nabla X|^2 + |Y|^2) \left( \frac{r^2}{4L \tau} \right)^{m-p} G_w \, dm \leq \left( \frac{N'}{r^{2p}} \right)^m\
\]

for some \( L = L(L_0) \geq 1 \) and \( N' = N'(N) \). Summing both sides of this inequality over all \( m \geq p \) yields

\[
\int_0^T \int_{D_r(w)} (|X|^2 + |\nabla X|^2 + |Y|^2) e^{\frac{r^2 - (1 - \frac{s}{m}) \sigma}{4L \tau}} \, dm \leq N'' r^{-2p},
\]

for some \( N'' = N''(p, N') \), and, consequently, that

\[
(6.6) \quad \int_0^T \int_{D_r(w)} (|X|^2 + |\nabla X|^2 + |Y|^2) e^{\frac{r^2}{4L \tau}} \, dm \leq N'' r^{-2p}.
\]

Returning to the statement of Theorem 5.1, consider first the interval \([0, T]\) where \( T = \min \{ 1, T_0 \} \). We may cover \( D_1(z_0) \) with finitely many sets of the form \( D_r(w_i), i = 1, \ldots, \nu \), where \( r = \sqrt{T}/(2\sqrt{T}) \) and \( w_i \in B_2(z_0) \). This can be done so that the number of sets satisfies \( \nu \leq C(L/T)^{(n-k)/2} \) for some dimensional constant \( C \). Since \( B_{4r}(w_i) \subset \mathbb{R}^{n-k} \setminus \overline{B_{r_0}(0)} \) for each \( i \), we may apply the estimate in (6.6) to obtain that

\[
\int_0^T \int_{D_1(z_0)} (|X|^2 + |\nabla X|^2 + |Y|^2) e^{\frac{r^2}{4L \tau}} \, dm \leq N'' L^{\frac{1}{m}} T^{-p - \frac{1}{2}}.
\]

When \( T_0 = 1 \), we are done. Otherwise, if \( T_0 < 1 \), we may obtain an estimate of the same form on \([T_0, 1]\) since

\[
\int_{T_0}^1 \int_{D_1(z_0)} (|X|^2 + |\nabla X|^2 + |Y|^2) e^{\frac{r^2}{4L \tau}} \, dm \leq N'''(1 - T_0) e^{\frac{r^2}{4L \tau}}
\]

for some \( N''' \) depending on \( M_{0,m} \) for \( m \leq 4 \). Combining this estimate with the one above on \([0, T_0]\) proves (5.41). \( \square \)

6.2. Three Carleman-type estimates. We will prove Proposition 6.1 by induction on the degree \( m \) of polynomial decay. The induction step is based primarily on the application of the following Carleman-type estimates to \( W \) and \( Y \). The estimates apply to arbitrary compactly supported families of sections of bundles \( Z \) of the form \( Z = \bigoplus T^{(k, l)} \mathcal{C} \) on \( \mathcal{C} \times (0, 1] \) with metrics and connections induced by \( g = g(\tau) \).
The first Carleman estimate will be applied to a suitably cut-off version of the “PDE” component $W$ of our system. A similar estimate was proven by the second author in [47], following [17].

**Theorem 6.2.** Assume $0 < T \leq 2$. Then, for any $\alpha \geq 1$ and $z_0 \in \mathbb{R}^{n-k}$, the estimate

\begin{equation}
\int_0^T \int_D \tau^{2\alpha} \sigma |z|^2 G_{z_0} \, dm \leq 10 \int_0^T \int_D \tau^2 |D_{\tau} G_{z_0}|^2 \, dm
\end{equation}

holds for any smooth family of sections $Z$ of $Z$ with compact support in $C \times (0, T)$.

We will use the next two estimates to control the “ODE” component $Y$.

**Theorem 6.3.** Assume $0 < T \leq 2$ and let $D, U \subset C$ be open sets such that $D$ is precompact and $\overline{D} \subset U$. For any $\lambda > 0$, there is $\alpha_0 = \alpha_0(\lambda, k) \geq 1$ such that, for all $\alpha \geq \alpha_0$ and $z_0 \in \mathbb{R}^{n-k}$ the estimates

\begin{equation}
2\alpha \int_0^T \int_D \tau^\lambda \sigma^{-2\alpha} |z|^2 G_{z_0} \, dm \leq \int_0^T \int_D \tau^{\lambda - 1} \sigma^{-2\alpha} |z - z_0|^2 |z|^2 G_{z_0} \, dm
\end{equation}

and

\begin{equation}
\alpha^2 \int_0^T \int_D \tau^\lambda \sigma^{-2\alpha} |z|^2 \, dm \leq 16 \int_0^T \int_D \tau^{\lambda + 2} \sigma^{-2\alpha} |D_{\tau}z|^2 \, dm
\end{equation}

hold for all smooth families of sections $Z$ of $Z$ over $U \times (0, T)$ with $\text{supp} Z \subset U \times [a, b]$ for some $0 < a < b < T$.

We will prove Theorems 6.2 and 6.3 in Section 8.2 below.

### 6.3. A delocalization procedure

The normal strategy would be to simultaneously apply the Gaussian-localized inequalities (6.7) and (6.8) to suitably cut-off versions of $W$ and $Y$ and sum the result to deduce the decay estimate needed for the induction step. However (6.8) is too lossy to allow us to do this in a single application. We will need to supplement it with estimates of $W$ and $Y$ relative to the purely time-dependent weight $\sigma$ on regions of spacetime where $|z - z_0|^2 / \tau > cm$ for some $c$. In fact, the weakness of (6.8) is the reason we need to employ an induction argument at all. By contrast, in [28], where the background metric converges smoothly to a conical metric as $\tau \to 0$, and in [47], in which the analysis reduces to that of a scalar function satisfying a strictly parabolic inequality, the exponential decay can be deduced in a single step.

In the proof of Proposition 6.1 in the next subsection we will use the following two technical lemmas to relate the localized estimates to the unlocalized estimates. We will use the first of these lemmas to convert Gaussian-weighted $L^2$-bounds on $W$, $\nabla W$ and $Y$ on sets $D_r(z)$ of a fixed radius $r$ into slightly weaker bounds minus the Gaussian weights on sets $D_s(z)$ with $s \ll r$. The proof is by an elementary covering argument.

**Lemma 6.4.** Suppose $F$ is a smooth positive function on $C_{r_0} \times (0, T')$ for some $0 < T' \leq 1$ with $|F| \leq M$. For all $\epsilon \in (0, 1/4)$ and $\alpha > (n-k)/2$, there exists a constant $C_\alpha = C_\alpha(n, k)$ such that whenever, for some integer $m \geq 0$, the inequality

\begin{equation}
\int_0^T \int_{D_r(z_0)} F \sigma^{-m} G_{z_0} \, dm \leq N L^m r^{-2m} m!
\end{equation}
holds for some \( N \geq M \) and \( L \geq (4e)^{-2} \) and all \( r, T, z_0 \) satisfying \( 0 < r^2 \leq T \leq T' \) and \( B_{4r}(z_0) \subset \mathbb{R}^{n-k} \setminus B_{r_0}(0) \), the inequality

\[
\int_0^T \int_{D_{4r}(z_0)} \tau^a F \sigma^{-m} \, dm \leq C a N L^m ((1 - \epsilon)t)^{-2m} m!
\]

holds for the same such \( r, T, \) and \( z_0 \).

**Proof.** Fix \( \epsilon \in (0, 1/4) \) and \( \alpha > (n - k)/2 \) and suppose the inequality (6.10) holds for some \( m \geq 0 \) and \( L \geq 1/(4e)^2 \) and \( N \geq M \), for all \( 0 < r^2 \leq T \leq T' \) and all \( z_0 \in \mathbb{R}^{n-k} \) with \( B_{4r}(z_0) \subset \mathbb{R}^{n-k} \setminus B_{r_0}(0) \).

Now let us fix such a specific \( r, T, \) and \( z_0 \) and verify that (6.11) continues to hold. We begin by splitting up the integral to obtain

\[
\int_0^T \int_{D_{4r}(z_0)} \tau^a F \sigma^{-m} \, dm = \left( \int_0^{16e^2r^2} + \int_{16e^2r^2}^T \right) \int_{D_{4r}(z_0)} \tau^a F \sigma^{-m} \, dm
\]

for some \( C = C(n, k) \). To estimate the first term in (6.12), observe that, for any \( 0 < s \leq 4e r \), we can cover \( B_{4r}(z_0) \) by a collection of balls \( \{B_{s}(w_i)\}_{i=1}^\nu \) with \( w_i \in B_{4r}(z_0) \). The \( w_i \) can be chosen so that their total number will satisfy the bound

\[
\nu(s) \leq c \left( \frac{4e r}{s} \right)^{n-k}
\]

for some \( c = c(n, k) \). We now define \( s_j = 4er/2^j \) and \( \nu_j = \nu(s_j) \) for \( j = 0, 1, 2, \ldots \) and apply this observation to choose collections \( \{w_{i,j}\}_{i=1}^{\nu_j} \subset B_{4r}(z_0) \) of such points.

Since \( w_{i,j} \in B_{4r}(z_0) \),

\[
B_{4(1-\epsilon)r}(w_{i,j}) \subset B_{4r}(z_0) \subset \mathbb{R}^{n-k} \setminus B_{r_0}(0)
\]

and so the estimate (6.10) for \( F \) is valid over \( B_{(1-\epsilon)r}(w_{i,j}) \). In particular, for each \( w_{i,j} \), \( j \geq 1 \), we have

\[
\int_{s_j}^{s_{j-1}} \int_{D_{s_j}(w_{i,j})} \tau^a F \sigma^{-m} \, dm \leq e^\chi s_j^{2a} \int_{s_j}^{s_{j-1}} \int_{D_{s_j}(w_{i,j})} \tau^a F \sigma^{-m} G_{w_{i,j}} \, dm
\]

\[
\leq e^\chi \left( \frac{8e r}{2^j} \right)^{2a} \int_0^T \int_{D_{(1-\epsilon)r}(w_{i,j})} \tau^a F \sigma^{-m} G_{w_{i,j}} \, dm
\]

\[
\leq e^\chi \left( \frac{1}{4e} \right)^{j} \frac{(8e r)^{2a} N L^m m!}{((1 - \epsilon)t)^{2m}}.
\]

(In the second inequality, we have used that \( s_j \leq 2er < (1 - \epsilon)r \) since \( \epsilon < 1/4 \).) We then may apply (6.13) to obtain that

\[
\int_{s_j}^{s_{j-1}} \int_{D_{4r}(z_0)} \tau^a F \sigma^{-m} \, dm \leq \sum_{i=1}^{\nu_j} \int_{s_j}^{s_{j-1}} \int_{D_{s_j}(w_{i,j})} \tau^a F \sigma^{-m} \, dm
\]

\[
\leq \left( \frac{1}{22a-n+k} \right)^{j} e^\chi \left( \frac{8e r)^{2a} N L^m m!}{((1 - \epsilon)t)^{2m}}.
\]
for each $j \geq 1$. Summing over $j$, we see that
\[
\int_0^{16\varepsilon^2 r^2} \int_{\mathcal{D}_{4r}(z_0)} \tau^a F\sigma^{-m} \, dm = \sum_{j=1}^{\infty} \int_{s_j^2}^{s_{j+1}} \int_{\mathcal{D}_{4r}(z_0)} \tau^a F\sigma^{-m} \, dm \leq C_a' \frac{NL^m m!}{((1 - \varepsilon)r)^{2m}},
\]
for some $C_a' = C_a'(n, k)$, and then, combining with (6.12), that
\[
\int_0^{T} \int_{\mathcal{D}_{4r}(z_0)} \tau^a F\sigma^{-m} \, dm \leq \frac{C_a' NL^m m!}{(1 - \varepsilon)^{2m}} + \frac{CM}{(4\varepsilon r)^{2m}} \leq (C + C_a') \frac{NL^m m!}{((1 - \varepsilon)r)^{2m}}
\]

since we have assumed that $L \geq 1/(4\varepsilon)^2$ and $N \geq M$. So (6.11) holds with the choice $C_a = C_a' + C$.

6.4. Advancing the unlocalized bounds. For the next lemma, we return to the setting of the statement of Proposition 6.1 and let $W$ and $Y$ be families of sections of $W$ and $\nabla W$ over $C_0 \times (0, 1]$ satisfying (6.3) and (6.4) for some constants $\beta$, $\mu$, $B$, and $M_1$. We will use this lemma to convert $L^2$-bounds with time-dependent weights of degree $m$ on $W$ and $\nabla W$ into corresponding bounds of degree $m + 1$ on $Y$. This is a simple application of the estimate (6.9).

**Lemma 6.5.** Fix $\alpha \geq 0$ and $\lambda \geq 2\mu + a$. There is an integer $m_0 \geq 0$ depending on $\lambda$, $k$, $B$, and $M_0$, such that whenever, for some $m \geq m_0$, $L \geq 2$, and $N \geq 1$, the inequality
\[
\int_0^{T} \int_{D_r(z_0)} (\tau^a |W|^2 + \tau^{a+1} |\nabla W|^2) \sigma^{-m} \, dm \leq NL^m r^{-2m} m!
\]
holds for some $r$, $T$, $z_0$ satisfying $0 < r^2 \leq T \leq 1$ and $B_{2r}(z_0) \subset \mathbb{R}^{n-k} \setminus B_{r_0}(0)$, the inequality
\[
\int_0^{T} \int_{D_r(z_0)} \tau^a |Y|^2 \sigma^{-(m+1)} \, dm \leq NL^m r^{-2m} (m - 1)!
\]
also holds for the same $r$, $T$, and $z_0$.

**Proof.** For now, we will regard $m_0$ as having some fixed large integral value, and will set lower bounds for it over the course of the proof. Suppose that (6.15) holds for some $m \geq m_0$ and $L \geq 2$, and $N \geq 1$ at some $r$, $T$, $z_0$ satisfying $0 < r^2 \leq T \leq 1$ and $B_{2r}(z_0) \subset \mathbb{R}^{n-k} \setminus B_{r_0}(0)$.

For any $0 < \varepsilon < T/4$, let $\xi_\varepsilon \in C_0^\infty(\mathbb{R})$ be a bump function with support in $(\varepsilon, 3T/4)$ which is identically one on $[2\varepsilon, T/2]$ and satisfies $|\xi'_{\varepsilon}| \leq C \varepsilon^{-1}$ on $[\varepsilon, 2\varepsilon]$ and $|\xi''_{\varepsilon}| \leq C T^{-1}$ on $[T/2, 3T/4]$. Here and below, $C$ will denote a series of positive constants depending at most on $n$ and $k$.

Define $W_\varepsilon = \xi_\varepsilon W$ and $Y_\varepsilon = \xi_\varepsilon Y$. Then, by (6.3),
\[
|D_{r} Y_\varepsilon|^2 \leq CB^2 \tau^{-2} |Y_\varepsilon|^2 + CB^2 \tau^{-2\mu} (|W_\varepsilon|^2 + \tau |\nabla W_\varepsilon|^2) + C |\xi'_{\varepsilon}|^2 |Y|^2.
\]

The first constraint that we will impose on $m_0$ will be that $m_0 \geq 2\alpha_0(k, \lambda)$, where $\alpha_0$ is as in Theorem 6.3. We may then apply (6.9) with $D = D_{r}(z_0)$, $Z = Y_\varepsilon$, and

\[
\int_0^{T} \int_{D_r(z_0)} (\tau^a |W_\varepsilon|^2 + \tau^{a+1} |\nabla W_\varepsilon|^2) \sigma^{-m} \, dm \leq NL^m r^{-2m} m!
\]

and
\[
\int_0^{T} \int_{D_r(z_0)} \tau^a |Y_\varepsilon|^2 \sigma^{-(m+1)} \, dm \leq NL^m r^{-2m} (m - 1)!
\]
\[ \alpha = (m+1)/2 \] to obtain
\[
\int_0^T \int_{D_r(z_0)} \tau^\lambda \sigma^{-(m+1)} |\mathbf{Y}_\epsilon|^2 \, dm \leq \frac{CB^2}{(m+1)^2} \int_0^T \int_{D_r(z_0)} \tau^\lambda \sigma^{-(m+1)} |\mathbf{Y}_\epsilon|^2 \, dm \\
+ \frac{CB^2}{(m+1)^2} \int_0^T \int_{D_r(z_0)} \tau^{\lambda-2\mu} \sigma^{-(m+1)} (r |\mathbf{W}_\epsilon|^2 + \tau^2 |\nabla \mathbf{W}_\epsilon|^2) \, dm \\
+ \frac{CB^2}{\epsilon^2(m+1)^2} \int_0^T \int_{D_r(z_0)} \sigma^{-(m+1)} \tau^{\lambda+2} |\mathbf{Y}_\epsilon|^2 \, dm \\
+ \frac{CB^2}{T^2(m+1)^2} \int_{\frac{T}{2}}^T \int_{D_r(z_0)} \sigma^{-(m+1)} \tau^{\lambda+2} |\mathbf{Y}_\epsilon|^2 \, dm.
\]

Provided \( m_0 \) is taken greater still, say \( m_0 > \sqrt{2CB} \), we may hide the first term on the right in the term on the left. Having done this, we see that, by our decay assumption (6.4), all of the integrands on the right are integrable on \((0,T]\), and when we send \( \epsilon \downarrow 0 \), the third term will tend to 0. Doing so thus yields
\[
\int_0^T \int_{D_r(z_0)} \tau^\lambda \sigma^{-(m+1)} |\mathbf{Y}_\epsilon|^2 \, dm \\
\leq \frac{CB^2}{(m+1)^2} \int_0^{\frac{T}{2}} \int_{D_r(z_0)} \sigma^{-(m+1)} \tau^{\lambda-2\mu} (r |\mathbf{W}_\epsilon|^2 + \tau^2 |\nabla \mathbf{W}_\epsilon|^2) \, dm \\
+ \frac{CB^2}{T^2(m+1)^2} \int_{\frac{T}{2}}^T \int_{D_r(z_0)} \sigma^{-(m+1)} \tau^{\lambda+2} |\mathbf{Y}_\epsilon|^2 \, dm.
\]

Since we assume \( \lambda \geq 2\mu + a \), we may use (6.13) (and that \( \tau \leq \sigma \)) to estimate
\[
\int_0^{\frac{T}{2}} \int_{D_r(z_0)} \sigma^{-(m+1)} \tau^{\lambda-2\mu} (r |\mathbf{W}_\epsilon|^2 + \tau^2 |\nabla \mathbf{W}_\epsilon|^2) \, dm \\
\leq \int_0^{\frac{T}{2}} \int_{D_r(z_0)} \sigma^{-m} (r^a |\mathbf{W}_\epsilon|^2 + \tau^{a+1} |\nabla \mathbf{W}_\epsilon|^2) \, dm \leq NL^{m} r^{-2m} m!.
\]

We may also estimate directly that
\[
\int_{\frac{T}{2}}^T \int_{D_r(z_0)} \sigma^{-(m+1)} \tau^{\lambda+2} |\mathbf{Y}_\epsilon|^2 \, dm \leq CM_0^2 r^{-m} T^{\lambda+\frac{k}{2}+2-m} \leq CM_0^2 T^{2m} r^{-2m}.
\]

Putting these two pieces together, we thus obtain
\[
\int_0^{\frac{T}{2}} \int_{D_r(z_0)} \tau^\lambda \sigma^{-(m+1)} |\mathbf{Y}_\epsilon|^2 \, dm \leq CB^2 \left( \frac{1 + M_0^2}{m+1} \right) NL^{m} r^{-2m} (m-1)!
\]

On the other hand,
\[
\int_{\frac{T}{2}}^T \int_{D_r(z_0)} \tau^\lambda \sigma^{-(m+1)} |\mathbf{Y}_\epsilon|^2 \, dm \leq CM_0^2 T^{2m} r^{-2m} \leq CM_0^2 NL^{m} r^{-2m},
\]

which when added to the previous inequality yields (6.10), provided \( m_0 \) is chosen larger still to ensure
\[ m_0 \geq 1 + C(B^2 + (1 + B^2)M_0^2). \]

This completes the proof. \( \square \)
6.5. The induction argument. In this section we assemble a proof of Proposition 6.1 using Lemmas 6.4 and 6.5. We will use the notation
\[ \mathcal{A}_{r,s}(z_0) = D_s(z_0) \setminus B_r(z_0) = S^k \times (B_s(z_0) \setminus B_r(z_0)) \]
for \( 0 < r < s \) and \( z_0 \in \mathbb{R}^{n-k} \). Note that \( \mathcal{A}_{r,s}(0) = \mathcal{A}_{r,s} \).

Proof of Proposition 6.1. Define \( \lambda_0 = 2\mu + (n-k)/2 + 2 \) and fix any \( b > \lambda_0/2 \). Then choose \( \beta_0 = (q+1)b + q\gamma \), and let \( m_0 = m_0(\lambda_0,k,B,M_0) \) be the constant guaranteed by Lemma 6.5. For now, we will regard \( K_0, L_0 > 0 \) and \( T_0 \leq 1 \) as fixed constants and \( m_1 \geq m_0 \) as a fixed large integer, and specify their values over the course of the proof.

Using the assumption (6.4), we may assume there is a constant \( M \geq 1 \) depending on \( m_1, n, k \) and finitely many of the constants \( M_l \) such that
\[
\sup_{c_{r_0} \times (0,1]} \tau^{-m_1} (|W|^2 + |
abla W|^2 + |Y|^2)
\leq \frac{1}{2}
\int_0^1 \int_{c_{r_0}} \sigma^{-m_1} (|W|^2 + |
abla W|^2 + |Y|^2) \, dm \leq M.
\]
In particular, for any \( w, r, \) and \( T \) satisfying \( 0 < r^2 \leq T \leq T_0 \) and \( B_{4r}(w) \subset \mathbb{R}^{n-k} \setminus B_{r_0}(0) \), we will have
\[
\int_0^T \int_{D_{r}(w)} \sigma^{-m_1} (\tau |W|^2 + \tau^2 |\nabla W|^2 + \tau^{\lambda_0} |Y|^2) \, G_w \, dm \leq M.
\]
So, assuming initially that \( K_0 \geq M \) and \( L_0 \geq 1 \), at least, the inequality (6.5) with \( m \leq m_1 \) will hold for all such \( r, w, \) and \( T \).

Proceeding by induction, let \( m > m_1 \) and assume that (6.5) holds for all \( p \) with \( p \leq m-1 \). Fix \( r, z_0, \) and \( T \) satisfying \( 0 < r^2 \leq T \leq T_0 \) and \( B_{4r}(z_0) \subset \mathbb{R}^{n-k} \setminus B_{r_0}(0) \). We will show that the inequality also holds with the exponent \( m \) for \( r, z_0, \) and \( T \). Below we will use \( C \) to denote a sequence of nonnegative constants depending at most on \( n \) and \( k \), and use \( K \) to denote a sequence which may depend as well on \( \beta, \mu, B, \) and \( M \).

We start by applying the Carleman inequality (6.7) to a fixed component \( W^i \) of \( W \) suitably cut-off in space and time. Let \( \phi \in C^{\infty}(\mathbb{R}^{n-k}) \) be a smooth bump function with support in \( B_{2r}(z_0) \) which is identically one on \( B_r(z_0) \). Regarding \( \phi \) as a function on \( C \) that is independent of \( \theta \in S^k \), we have \( \phi \equiv 1 \) on \( D_r(z_0) \) and \( \text{supp}(\phi) \subset D_{2r}(z_0) \). For each \( \epsilon < T/4 \), let \( \xi_\epsilon \in C^{\infty}(\mathbb{R}) \) be a bump function with support in \((\epsilon, 3T/4)\) which is identically one on \([2\epsilon, T/2)\). These functions may be chosen to satisfy the inequalities
\[
r|\nabla \phi| + r^2 |\Delta \phi| \leq C, \quad \epsilon |\xi_\epsilon|^2 [\chi_{(\epsilon, 2\epsilon)} + T] \leq C
\]
for some \( C \). (Note that \( |\nabla \phi|_{(\theta, z, \tau)} = |\nabla \phi|_{(\theta, z)} \) and \( (\Delta \phi)(\theta, z, \tau) = (\Delta \phi)(z, \cdot, \cdot) \).

Then define \( W_\epsilon = \phi \xi_\epsilon W \) and \( Y_\epsilon = \phi \xi_\epsilon Y \). Using (6.3), we compute that
\[
|(D_\tau + \Delta) W_\epsilon|^2 \leq CB^2 \tau^{-3} (|W|^2 + |Y|^2) + CB \sum_{j=0}^{i-1} \tau^{-2j} |W_\epsilon|^2 \
+ C \xi_\epsilon^2 (|\nabla \phi|^2 |\nabla W|^2 + |\Delta \phi|^2 |W|^2) + C \phi^2 (\xi_\epsilon^2 |W|^2)
\]
for each $i = 0, \ldots, q$. For each $i$, define $\nu_i = (q - i)(\gamma + b)$ and apply the Carleman estimate (6.7) to $W_\epsilon^i$ with $\alpha_i = m/2 + \nu_i$ to obtain

$$
\int \int \sigma^{2\alpha_i} \tau(\alpha_i |W_\epsilon^i|^2 + \tau |\nabla W_\epsilon^i|^2) G_{z_0} \, dm
\leq K \sum_{j < i} \int \int \tau^{2-2\gamma} \sigma^{-2\alpha_i} |W_\epsilon^j|^2 G_{z_0} \, dm
+ K \sum_{j > i} \int \int \tau^{2\beta+2} \sigma^{-2\alpha_i} |W_\epsilon^j|^2 G_{z_0} \, dm
+ K \int \int \tau^{2\beta+2} \sigma^{-2\alpha_i} |Y_\epsilon|^2 G_{z_0} \, dm
+ C \int \int \tau^{2-2\gamma} \sigma^{-2\alpha_i} (|W_\epsilon|^2 + |\nabla W_\epsilon|^2) G_{z_0} \, dm
+ C \int \int \tau^{2-2\gamma} \sigma^{-2\alpha_i} |W_\epsilon|^2 G_{z_0} \, dm
+ C \int \int \tau^{2-2\gamma} \sigma^{-2\alpha_i} |Y_\epsilon|^2 G_{z_0} \, dm.
$$

(6.17)

For the integrals in the first term on the right, we have immediately that

$$
\int \int \tau^{2-2\gamma} \sigma^{-2\alpha_i} |W_\epsilon^j|^2 G_{z_0} \, dm \leq K \int \int \tau^{2} \sigma^{-2(\alpha_j - b)} |W_\epsilon^j|^2 G_{z_0} \, dm,
$$

using (6.2) and that $\alpha_j \geq \alpha_i + \gamma + b$ for $j < i$. For the integrals in the second term, our choice of $\beta_0$ ensures that

$$
(\beta - \alpha_i) - (b - \alpha_j) \geq (q + i - j)(\gamma + b) \geq 0,
$$

and hence $\sigma^{2(\beta - \alpha_i)} \leq \sigma^{2(b - \alpha_j)}$, for all $0 \leq i \leq j \leq q$ and $\beta \geq \beta_0$. Thus

$$
\int \int \tau^{2\beta+2} \sigma^{-2\alpha_i} |W_\epsilon^j|^2 G_{z_0} \, dm \leq \int \int \tau^{2} \sigma^{-2(\alpha_j - b)} |W_\epsilon^j|^2 G_{z_0} \, dm
$$

for $i \leq j$, again using (6.2). Therefore, we may combine the first two terms to obtain

$$
\sum_{j < i} \int \int \tau^{2-2\gamma} \sigma^{-2\alpha_i} |W_\epsilon^j|^2 G_{z_0} \, dm + \sum_{j > i} \int \int \tau^{2\beta+2} \sigma^{-2\alpha_i} |W_\epsilon^j|^2 G_{z_0} \, dm
\leq KT_0^{2b+1} \sum_{j=0}^q \int \tau \sigma^{-2\alpha_j} |W_\epsilon|^2 G_{z_0} \, dm.
$$

Equation (6.18) also shows that $\beta - \alpha_i \geq b - \alpha_q = b - m/2$ for all $i$, so that we can estimate the third term in (6.17) by

$$
\int \int \tau^{2\beta+2} \sigma^{-2\alpha_i} |Y_\epsilon|^2 G_{z_0} \, dm \leq \int \int \tau^{2} \sigma^{2b-m} |Y_\epsilon|^2 G_{z_0} \, dm.
$$
Returning to (6.17), using that \(\sigma^{-2\alpha_i} \leq \tau^{-2\nu_0}\sigma^{-m}\) in the last three terms, and summing over \(i\), we obtain that

\[
\sum_{i=0}^{q} \int \sigma^{-2\alpha_i} \tau(\alpha_i|W_\epsilon^i|^2 + \tau|\nabla W_\epsilon^i|^2)G_{z_0} \, dm
\]

\[
\leq KT_0^{2b+1} \sum_{j=0}^{q} \int \tau \sigma^{-2\alpha_j}|W_j|^2 G_{z_0} \, dm
\]

\[
+ K \int \tau^2 \sigma^{2b-m}|Y_\epsilon|^2 G_{z_0} \, dm
\]

\[
+ \frac{C}{r^2} \int_{\epsilon}^{2\epsilon} \int_{A_{r,2r}(z_0)} \tau^{2-2\nu_0}\sigma^{-m}(|W|^2 + |\nabla W|^2)G_{z_0} \, dm
\]

\[
+ \frac{C}{\epsilon^2} \int_{\epsilon}^{2\epsilon} \int_{D_{2r}(z_0)} \tau^{2-2\nu_0}\sigma^{-m}|W|^2 G_{z_0} \, dm
\]

\[
+ \frac{C}{T^2} \int_{\epsilon}^{2\epsilon} \int_{D_{2r}(z_0)} \tau^{2-2\nu_0}\sigma^{-m}|W|^2 G_{z_0} \, dm.
\]

(6.19)

If \(T_0\) is sufficiently small (depending on \(K\) and \(b\)), we may bring the first term on the right side over to the left and split the domain of integration of the second term to obtain that

\[
\sum_{i=0}^{q} \int \sigma^{-2\alpha_i} (\tau|W_\epsilon^i|^2 + \tau^2|\nabla W_\epsilon^i|^2)G_{z_0} \, dm
\]

\[
\leq K \int_{2\epsilon}^{\infty} \int_{D_{r}(z_0)} \tau^2 \sigma^{2b-m}|Y_\epsilon|^2 G_{z_0} \, dm
\]

\[
+ \frac{K}{r^2} \int_{\epsilon}^{2\epsilon} \int_{A_{r,2r}(z_0)} \tau^{2-2\nu_0}\sigma^{-m}(|W|^2 + |\nabla W|^2 + |Y|^2)G_{z_0} \, dm
\]

\[
+ \frac{K}{\epsilon^2} \int_{\epsilon}^{2\epsilon} \int_{D_{2r}(z_0)} \tau^{2-2\nu_0}\sigma^{-m}|W|^2 + |Y|^2) G_{z_0} \, dm
\]

\[
+ \frac{K}{T^2} \int_{\epsilon}^{2\epsilon} \int_{D_{2r}(z_0)} \tau^{2-2\nu_0}\sigma^{-m}(|W|^2 + |Y|^2) G_{z_0} \, dm.
\]

(6.20)

Now, using our decay assumption (6.4), we may send \(\epsilon \to 0\) in (6.20) and the third term on the right will vanish. Using that

\[
\int \int \sigma^{-m}(\tau|W_\epsilon|^2 + \tau^2|\nabla W_\epsilon|^2)G_{z_0} \, dm = \int \int \sigma^{-2\alpha_i}(\tau|W_\epsilon^i|^2 + \tau^2|\nabla W_\epsilon^i|^2)G_{z_0} \, dm
\]

\[
\leq \sum_{i=0}^{q} \int \sigma^{-2\alpha_i}(\tau|W_\epsilon^i|^2 + \tau^2|\nabla W_\epsilon^i|^2)G_{z_0} \, dm,
\]
we obtain

\[
\int_0^T \int_{D_r(z_0)} \sigma^{-m}(\tau|W|^2 + \tau^2|\nabla W|^2)G_{z_0} \, dm
\]

\[
\leq KT_0^2 \int_0^T \int_{D_r(z_0)} \tau^{\lambda_0}\sigma^{-m}|Y|^2G_{z_0} \, dm
\]

\[
+ \frac{K}{T^2} \int_0^{2T} \int_{D_{2r}(z_0)} \tau^{2-\mu}\sigma^{-m}(|W|^2 + |\nabla W|^2 + |Y|^2)G_{z_0} \, dm
\]

(6.21)

In the first term on the right, we have used that \(2b < \lambda_0\).

Now we estimate the component \(Y\). With \(\xi_e\) defined as before, it follows from (6.3) that

\[
|D_r(\xi_e Y)|^2 \leq CB^2\xi_e^2 \left(\tau^{-2\mu}(|W|^2 + |\nabla W|^2) + \tau^{-2}|Y|^2\right) + C|\xi_e|^2|Y|
\]

Since \(m \geq m_1 \geq m_0\), we may apply the Carleman estimate (6.8) to \(Z = \xi_e Y\) on \(D_r(z_0)\) with \(\alpha = m/2\) to obtain that

\[
m \int_0^T \int_{D_r(z_0)} \tau^{\lambda_0}\sigma^{-m}\xi_e^2|Y|^2G_{z_0} \, dm
\]

\[
\leq \int_0^T \int_{D_r(z_0)} \tau^{\lambda_0-1}\sigma^{-m}|z - z_0|^2\xi_e^2|Y|^2G_{z_0} \, dm
\]

\[
+ \frac{K}{m} \int_0^T \int_{D_r(z_0)} \tau^{\lambda_0-2}\sigma^{-m}\xi_e^2(|W|^2 + |\nabla W|^2)G_{z_0} \, dm
\]

(6.22)

\[
+ \frac{K}{m} \int_0^T \int_{D_r(z_0)} \tau^{\lambda_0-2}\sigma^{-m}\xi_e^2|Y|^2G_{z_0} \, dm
\]

\[
+ \frac{C}{\epsilon^2m} \int_0^{2\epsilon} \int_{D_r(z_0)} \tau^{\lambda_0+2}\sigma^{-m}|Y|^2G_{z_0} \, dm
\]

\[
+ \frac{C}{T^2m^2} \int_0^{2T} \int_{D_r(z_0)} \tau^{\lambda_0+2}\sigma^{-m}|Y|^2G_{z_0} \, dm.
\]

Provided \(m_1\) is chosen large enough to satisfy that \(K/m_1^2 < 1/2\) we may hide the third term on the right in the left-hand side. Then sending \(\epsilon \searrow 0\), and using that \(\lambda_0 > 2\mu\), we arrive at the inequality

\[
\int_0^T \int_{D_r(z_0)} \tau^{\lambda_0}\sigma^{-m}|Y|^2G_{z_0} \, dm
\]

\[
\leq \frac{2}{m} \int_0^{2T} \int_{D_r(z_0)} \tau^{\lambda_0-1}\sigma^{-m}|z - z_0|^2|Y|^2G_{z_0} \, dm
\]

(6.23)

\[
+ \frac{K}{m^2} \int_0^T \int_{D_r(z_0)} \tau^2\sigma^{-m}(|W|^2 + |\nabla W|^2)G_{z_0} \, dm
\]

\[
+ \frac{K}{T^2m^2} \int_0^{2T} \int_{D_r(z_0)} \tau^2\sigma^{-m}(|W|^2 + |\nabla W|^2 + |Y|^2)G_{z_0} \, dm.
\]
that we have $G$ and split the domain of integration into three spacetime regions:

Also, by (6.4), $(6.24)$ may be estimated from above by

Adding $(6.21)$ to $(6.23)$, we see that if $m_1$ is taken large enough and $T_0$ small enough (depending on $K$) we may bring some terms from the right to the left and arrive at the inequality

We now estimate each term on the right side of $(6.24)$ in turn. For the first, note that we have $G_{z_0}(z, \tau) \leq e^{-\frac{2m}{\tau^2}}$ on $A_{r,2r}(z_0) \times (0,3T/4)$ and, hence, by Stirling’s formula,

Also, by $(6.24)$, $(|W|^2 + |\nabla W|^2 + |Y|^2)^{\tau^{-2m_0}} \leq M$ on $C_{T_0} \times (0, T)$, provided $m_1 \geq 2m_0$. So the first term on the right side of $(6.24)$ may be estimated from above by

For the second term, we simply note that

The third term in $(6.24)$ will require more work. First, we fix some $0 < \delta < 1/4$ and split the domain of integration into three spacetime regions:

Here we have also absorbed part of the second term on the right of $(6.22)$ into the last term of $(6.23)$.

We now estimate each term on the right side of $(6.24)$ in turn. For the first, note that we have $G_{z_0}(z, \tau) \leq e^{-\frac{2m}{\tau^2}}$ on $A_{r,2r}(z_0) \times (0,3T/4)$ and, hence, by Stirling’s formula,

Also, by $(6.24)$, $(|W|^2 + |\nabla W|^2 + |Y|^2)^{\tau^{-2m_0}} \leq M$ on $C_{T_0} \times (0, T)$, provided $m_1 \geq 2m_0$. So the first term on the right side of $(6.24)$ may be estimated from above by

For the second term, we simply note that

The third term in $(6.24)$ will require more work. First, we fix some $0 < \delta < 1/4$ and split the domain of integration into three spacetime regions:

Here we have also absorbed part of the second term on the right of $(6.22)$ into the last term of $(6.23)$.

We now estimate each term on the right side of $(6.24)$ in turn. For the first, note that we have $G_{z_0}(z, \tau) \leq e^{-\frac{2m}{\tau^2}}$ on $A_{r,2r}(z_0) \times (0,3T/4)$ and, hence, by Stirling’s formula,
The first and second terms in (6.27) can be estimated exactly as their counterparts in (6.24) above, to yield

\[
\frac{4}{m} \int_0^T \int_{D_{4\delta r}(z_0)} \tau^{\lambda_0 - 1} \sigma^{-m} |z - z_0|^2 |\nabla| G_{z_0} \leq K 2^m r^{-2m}
\]

and

\[
\frac{4}{m} \int_0^T \int_{D_{4\delta r}(z_0)} \tau^{\lambda_0 - 1} \sigma^{-m} |z - z_0|^2 |Y|^2 G_{z_0} \ dm \leq K (2\delta r)^{-2m} (m - 1)!. \tag{6.29}
\]

To estimate the third term on the right of (6.27), we will split the domain of integration into two further spacetime regions

\[
\Omega = (D_{4\delta r}(z_0) \times (0, T/2)) \cap \{ |z - z_0|^2 \leq \frac{m \tau}{8} \}, \quad \Omega' = (D_{4\delta r}(z_0) \times (0, T/2)) \cap \Omega^c.
\]

Then we have $|z - z_0|^2 G_{z_0} / r \leq (m/8)e^{-m/32}$ on $\Omega'$, provided at least that $m_1 \geq 32$, and so

\[
\frac{4}{m} \int_0^T \int_{D_{4\delta r}(z_0)} \tau^{\lambda_0 - 1} \sigma^{-m} |z - z_0|^2 |Y|^2 G_{z_0} \ dm
\]

\[
\leq \frac{1}{2} \int_0^T \int_{\Omega} \tau^{\lambda_0} \sigma^{-m} |Y|^2 G_{z_0} \ dm + \frac{e^{-m/32}}{2} \int_0^T \int_{\Omega'} \tau^{\lambda_0} \sigma^{-m} |Y|^2 \ dm
\]

\[
\leq \frac{1}{2} \int_0^T \int_{D_{r}(z_0)} \tau^{\lambda_0} \sigma^{-m} |Y|^2 G_{z_0} \ dm + \frac{e^{-m/32}}{2} \int_0^T \int_{D_{4\delta r}(z_0)} \tau^{\lambda_0} \sigma^{-m} |Y|^2 \ dm.
\]

Putting things together, we see that the third term on the right side of (6.24) admits the bound

\[
\frac{4}{m} \int_0^T \int_{D_{r}(z_0)} \tau^{\lambda_0 - 1} \sigma^{-m} |z - z_0|^2 |Y|^2 G_{z_0} \ dm
\]

\[
\leq \frac{1}{2} \int_0^T \int_{D_{r}(z_0)} \tau^{\lambda_0} \sigma^{-m} |Y|^2 G_{z_0} \ dm
\]

\[
+ \frac{e^{-m/32}}{2} \int_0^T \int_{D_{4\delta r}(z_0)} \tau^{\lambda_0} \sigma^{-m} |Y|^2 \ dm + K 2^m \delta^{-2m} r^{-2m} (m - 1)!
\]

for any $\delta \in (0, 1/4)$. Incorporating (6.25), (6.28), and (6.30) into (6.24) then yields

\[
\int_0^T \int_{D_{r}(z_0)} \sigma^{-m} (\tau |W|^2 + \tau^2 |\nabla W|^2 + \tau^{\lambda_0} |Y|^2) G_{z_0} \ dm
\]

\[
\leq e^{-3/2} \int_0^T \int_{D_{4\delta r}(z_0)} \tau^{\lambda_0} \sigma^{-m} |Y|^2 \ dm + K_0 2^m \delta^{-2m} r^{-2m} (m - 1)!
\]

provided $K_0$ is sufficiently large.

To estimate the first term on the right, apply Lemma 6.3 with $F = \tau |W|^2 + \tau^2 |\nabla W|^2$ and $\alpha = \lambda_0 - 2\mu - 1 = 1 + (n - k)/2$. Pick $\delta$ so small that

\[0 < \delta < 1 - e^{-\frac{3}{4}}.\]
Then, since we already have assumed that $K_0 \geq M$, if, in addition, $L_0 \geq 1/(4\delta)^2$, the lemma together with the induction hypothesis implies that
\[
\int_0^T \int_{D_{4T}(z_0)} \sigma^{-(m-1)} \left( \tau^{a+1} |W|^2 + 2\tau^{a+2} |\nabla W|^2 \right) \leq CK_0 \left( \frac{L_0}{(1-\delta)^2 r^2} \right)^{m-1} (m-1)!
\]
for some $C = C_0$ (which, with our choice of $a$, only depends on $n$ and $k$). Then, since $m_1 \geq m_0$, provided $L_0 \geq 2$, we may apply Lemma 6.3 with $\lambda = \lambda_0$ and $a + 1$ in place of $a$ to obtain that
\[
e^{-\frac{m}{4T}} \int_0^T \int_{D_{4T}(z_0)} \tau^{\lambda_0} \sigma^{-m} |Y|^2 \, dm \leq CK_0 e^{-\frac{m}{4T}} \left( \frac{L_0}{(1-\delta)^2 r^2} \right)^{m-1} (m-2)!
\]
\[
\leq CK_0 L_0^{m-1} r^{-2m} (m-2)!
\]
Returning to (6.31), we see that
\[
\int_0^T \int_{D_r(z_0)} \sigma^{-m} (\tau |W|^2 + \tau^2 |\nabla W|^2 + \tau^{\lambda_0} |Y|^2) G_{z_0} \, dm
\]
\[
\leq \frac{K_0 L_0^m (m-1)!}{r^{2m}} \left( \frac{C}{L_0} + \left( \frac{4}{\delta L_0} \right)^m \right)
\]
\[
\leq \frac{K_0 L_0^m (m-1)!}{2r^{2m}}
\]
provided $L_0$ is taken large enough depending on $C$ and the universal constant $\delta$. On the other hand,
\[
\int_0^T \int_{D_r(z_0)} \sigma^{-m} (\tau |W|^2 + \tau^2 |\nabla W|^2 + \tau^{\lambda_0} |Y|^2) G_{z_0} \, dm \leq CM 2^m r^{-2m}.
\]
Summing these two inequalities completes the proof provided $K_0$ and $L_0$ are taken larger still.

\section{7. Backward uniqueness}

In this section, we will prove Theorem 5.2 via an analysis of the system composed of $X = \nabla Rm$ and $Y = (Y^0, Y^1, Y^2)$ from Section 4. Our analysis will only make use of the following properties of $X$ and $Y$.

(1) There exists a constant $B$ such that
\[
|(D_\tau + \Delta)X| \leq B \tau^{-1} |X| + B |Y|
\]
\[
|D_\tau Y| \leq B(|X| + |\nabla X|) + B \tau^{-1} |Y|
\]
on $C_{r_0} \times (0, 1]$.

(2) The sections $X$ and $Y$ are self-similar in the sense that, if $\overline{X} = X|_{C_{r_0} \times \{1\}}$ and $\overline{Y} = Y|_{C_{r_0} \times \{1\}}$, and $\Psi_\tau(\theta, z) = (\theta, z/\sqrt{\tau})$, then
\[
X = \tau \Psi_\tau \overline{X}, \quad Y = \tau \Psi_\tau \overline{Y},
\]
and
\[
|X|^2 = \tau^3 |\overline{X}|_{\phi(1)}^2 \circ \Psi_\tau, \quad |Y|^2 = \sum_{i=0}^2 \tau^i |\overline{Y}|_{\phi(1)}^2 \circ \Psi_\tau.
\]
(3) There is a constant $M_0$ such that
\begin{equation}
\sup_{C_{r_0} \times (0,1]} (|X|^2 + |\nabla X|^2 + |Y|^2) \leq M_0.
\end{equation}

(4) There are constants $N_2$, $N_3 > 0$ and $r_1 \geq r_0$ such that
\begin{equation}
\int_0^1 \int_{A_{r, 2r}} (|X|^2 + |\nabla X|^2 + |Y|^2) e^{\frac{N_2 r^2}{2}} \, dm \leq N_3
\end{equation}
for all $r \geq r_1$.

The precise exponents of the scale factors in (2) are not important for the analysis; we require only that $X$ and $Y$ are self-similar and satisfy some relationship akin to (7.2). We will show that these four conditions imply that $X$ and $Y$ must vanish identically on $C_{r_2} \times (0, T_1)$ for some $r_2 \geq r_1$ and $0 < T_1 \leq 1$.

**Theorem 7.1.** Suppose that $X$ and $Y$ are smooth sections of $\mathcal{X}$ and $\mathcal{Y}$ defined on $C_{r_0} \times (0, 1]$ satisfying conditions (1) - (4) above. Then there exists $r_2 > 0$ and $0 < T_1 \leq 1$ such that $X \equiv 0$ and $Y \equiv 0$ on $C_{r_2} \times (0, T_1]$.

We have already seen in Proposition 4.1 that $X$ and $Y$ defined by (4.2) satisfy (1) - (3). The following proposition, which is essentially a corollary of Theorem 5.1, shows that they also satisfy the exponential decay estimate in the precise form given in (4). Theorem 5.2 is thus a consequence of Theorem 7.1.

### 7.1. Space-time exponential decay revisited.

Using the self-similarity of $X$ and $Y$ and the reference metric $g$, Theorem 5.1 implies that $X$ and $Y$ also decay in space at an exponential-quadratic rate.

**Proposition 7.2.** There exist $N_2$ and $N_3$ (depending on $N_0$, $N_1$, and $r_0$) such that
\begin{equation}
\int_0^1 \int_{A_{r, 2r}} (|X|^2 + |\nabla X|^2 + |Y|^2) e^{\frac{N_2 r^2}{2}} \, dm \leq N_3
\end{equation}
or any $r \geq 16r_0$.

**Proof.** For simplicity, let $r_1 = 16r_0$. The set $A_{r_1, 2r_1}$ can be covered by a finite collection of sets of the form $D_1(z_i)$ where $z_i \in \mathbb{R}^{n-k} \setminus \overline{B}_{r_1/2}(0)$ and so we obtain from Theorem 5.1 the inequality
\begin{equation}
\int_0^1 \int_{A_{r, 2r}} (|X|^2 + |\nabla X|^2 + |Y|^2) e^{\frac{N_2 r^2}{2}} \, dm \leq C N_1 r_0^{n-k}
\end{equation}
for some $C = C(n,k)$.

Now fix $r \geq r_1$. Then
\[ |X|^2(\theta, z, \tau) = \tau^{-3} |\nabla \text{Rm}|^2(\Psi_\tau(\theta, z), 1), \quad d\mu_g(\tau) = \tau^k \, d\mu_{g(1)}, \]
and so, for any $0 < a < 1$, by a change of variables,
\[ \int_a^1 \int_{A_{r, 2r}} |X|^2 e^{\frac{N_2 r^2}{2}} \, dm = \left( \frac{r}{r_1} \right)^{n-4} \int_a^{r_1^2} \int_{A_{r_1, 2r_1}} |X|^2 e^{\frac{N_2 r^2}{2}} \, dm. \]
Taking $N_2 = N_0/(2r_1^2) = N_0/(512r_0^2)$, then, and sending $a \to 0$, we obtain
\[ \int_0^1 \int_{A_{r, 2r}} |X|^2 e^{\frac{N_2 r^2}{2}} \, dm \leq e^{-\frac{N_0 r^2}{2r_1^2}} \left( \frac{r}{r_1} \right)^{n-4} \int_0^{r_1^2} \int_{A_{r_1, 2r_1}} |X|^2 e^{\frac{N_0 r^2}{2r_1^2}} \, dm \]

\[ \leq N \int_{0}^{1} \int_{A_{r_{1}, 2r_{1}}} |X|^{2} e^{\frac{N}{r}} \, dm \]

for some \( N = N(N_{0}) \). The estimate \((7.5)\) for \( X \) then follows from \((7.6)\). Analogous scaling arguments prove \((7.5)\) for the other terms in the integrand. \( \square \)

### 7.2. Carleman estimates

To prove Theorem 7.1 we will use two Carleman-type inequalities with weights that grow at an approximately exponential-quadratic rate at infinity. Following \([47]\), for \( \alpha > 0 \), \( 0 < T \leq 1 \), and \( \delta \in (7/8, 1) \), we define \( \phi_{\alpha} : \mathcal{C} \times (0, \infty) \to \mathbb{R} \) by

\[ \phi_{\alpha}(\theta, z, \tau) = \alpha \eta(\tau) \left( \frac{|z|^{2}}{\tau} \right)^{\delta}, \]

and \( \eta : [0, T] \to [0, 1] \) by

\[ \eta(\tau) = \begin{cases} 
1 & \text{if } \tau \in [0, \tau_{0}] \\
1 - \frac{1}{32} \delta(4\delta - 3) \left( \frac{\tau}{\tau_{0}} - 1 \right)^{2} & \text{if } \tau \in [\tau_{0}, 2\tau_{0}] \\
1 + \frac{1}{32} \delta(4\delta - 3) \left( 3 - \frac{\tau}{\tau_{0}} \right) & \text{if } \tau \in [2\tau_{0}, T],
\end{cases} \]

where

\[ \tau_{0} = \frac{2\delta(4\delta - 3)T}{3\delta(4\delta - 3) + 32} \]

The function \( \eta \) has been engineered to be monotone decreasing, identically one near \( \tau = 0 \), and proportional to \( T - \tau \) near \( \tau = T \) with \( \eta(T) = 0 \).

Below, \( Z \) will denote an bundle of the form \( \bigoplus T^{(k, I_{1}, I_{2})} \mathcal{C} \) equipped with the family of metrics and connections induced by \( g(\tau) \).

**Theorem 7.3.** For any \( \delta \in (7/8, 1) \) and \( T \leq 1 \), there exists \( r_{3} \geq 1 \) depending on \( n, k, \) and \( \delta \) such that, for all smooth families of sections \( Z \) of the bundle \( Z \) with support compactly contained in \( \mathcal{C}_{r_{3}} \times (0, T) \), we have the inequality

\[ \int_{0}^{T} \int_{\mathcal{C}_{r}} \left( \frac{\alpha}{\tau} |Z|^{2} + \tau |\nabla Z|^{2} \right) e^{2\phi_{\alpha}} \, dm \leq 10 \int_{0}^{T} \int_{\mathcal{C}_{r}} \tau^{2} |D_{\tau} + \Delta Z|^{2} e^{2\phi_{\alpha}} \, dm \]

for all \( \alpha > 0 \) and \( r \geq r_{3} \).

We will apply this estimate to the PDE component \( X \) of our system. To control the ODE component \( Y \), we will use the following matching estimate.

**Theorem 7.4.** For any \( \delta \in (7/8, 1) \), and \( T \leq 1 \) there exists \( r_{4} > 0 \), depending on \( n, k, \) and \( \delta \), such that, for all smooth families of sections \( Z \) of \( Z \) with support compactly contained in \( \mathcal{C}_{r} \times (0, T) \), we have the inequality

\[ \int_{0}^{T} \int_{\mathcal{C}_{r}} \frac{\alpha}{\tau} |Z|^{2} e^{2\phi_{\alpha}} \, dm \leq \int_{0}^{T} \int_{\mathcal{C}_{r}} \tau^{2} |D_{\tau} Z|^{2} e^{2\phi_{\alpha}} \, dm \]

for all \( \alpha \geq 1 \) and \( r \geq r_{4} \).

We will prove Theorems 7.3 and 7.4 in Section 8. For now, we will take them for granted and use them to prove Theorem 7.1.

**Proof of Theorem 7.1.** Our argument is a modification of that of Theorem 3.3 in \([47]\). Let \( r_{2} \geq \max \{ r_{1}, r_{3}, r_{4} \} \) and fix some \( R \geq r_{2} \) and \( 0 < T \leq 1 \).

We will need two cutoff functions. For all \( \alpha > 8 \) and \( 0 < \epsilon < T/8 \), let \( \chi_{\alpha, \epsilon} \) be a smooth bump function on \( [0, 1] \) with support in \( (\epsilon, T - T/\alpha) \) satisfying \( \chi_{\alpha, \epsilon} \equiv 1 \) on
\[2\varepsilon, T - 2T/\alpha], \quad |\chi'_{\alpha,c}| \leq 2/\varepsilon \text{ on } (\varepsilon, 2\varepsilon), \quad \text{and } |\chi'_{\alpha,c}| \leq 2\alpha/T \text{ on } (T - 2T/\alpha, T - T/\alpha)\].

For the spatial cutoff, choose, for each \(r > R + 1\), a bump function \(\psi_r\) on \(\mathbb{R}^{n-k}\) with support in \(B_r(0) \setminus B_{R+1}(0)\) which satisfies \(\psi_r \equiv 1\) on \(B_r(0) \setminus B_{R+1}(0)\) and the bounds \(|\nabla \psi_r| + |\Delta \psi_r| \leq C\). We regard \(\psi_r = \psi_r(\theta, z)\) as a function on \(C\) which is independent of \(\theta\), in which case \(|\nabla \psi_r| = |\nabla \psi_r|_T\) and \(\Delta \psi_r = \Delta \psi_r\).

Now define \(X_{\alpha,c,r} = \chi_{\alpha,c,\psi_r}X\) and \(Y_{\alpha,c,r} = \chi_{\alpha,c,\psi_r}Y\). From (7.11), we have

\[
|\langle D_r + \Delta \rangle X_{\alpha,c,r} | \leq B \tau^{-1} |X_{\alpha,c,r}| + B|Y_{\alpha,c,r}| + \psi_r|\chi_{\alpha,c}||X|
+ 2\chi_{\alpha,c,\psi_r}(|\nabla \psi_r| + |\Delta \psi_r|)(|X| + |\nabla X|)
\]

\[
|\langle D_r \rangle Y_{\alpha,c,r} | \leq B(|X_{\alpha,c,r}| + |\nabla X_{\alpha,c,r}|) + B\tau^{-1}|Y_{\alpha,c,r}| + \psi_r|\chi_{\alpha,c}||Y|
+ B\chi_{\alpha,c,\psi_r}|\nabla \psi_r||X|
\]

on \(C_R \times (0, T]\).

Applying the inequalities (7.10) and (7.11) to \(X_{\alpha,c,r}\) and \(Y_{\alpha,c,r}\) and summing the result, we arrive at the inequality

\[
\int_0^T \int_{C_R} (\alpha \tau^\delta (|X_{\alpha,c,r}|^2 + |Y_{\alpha,c,r}|^2) + \tau|\nabla X_{\alpha,c,r}|^2) e^{2\phi_\alpha} \, dm
\]

\[
\leq K \int_0^T \int_{C_R} (|X_{\alpha,c,r}|^2 + |Y_{\alpha,c,r}|^2 + \tau^2|\nabla X_{\alpha,c,r}|^2) e^{2\phi_\alpha} \, dm
+ \frac{C}{\varepsilon} \int_\varepsilon^{2\varepsilon} \int_{A_{R,2r}} \tau^2 (|X|^2 + |Y|^2 + |\nabla X|^2) e^{2\phi_\alpha} \, dm
+ \frac{C\alpha^2}{T^2} \int_{T - \frac{2T}{\alpha}}^{T - \frac{2T}{\alpha} + 1} \int_{A_{R,R+1}} \tau^2 (|X|^2 + |\nabla X|^2) e^{2\phi_\alpha} \, dm
+ K \int_\varepsilon^{2\varepsilon} \int_{A_{R,2r}} \tau^2 (|X|^2 + |\nabla X|^2) e^{2\phi_\alpha} \, dm
+ K \int_{T - \frac{2T}{\alpha}}^{T - \frac{2T}{\alpha} + 1} \int_{A_{R,R+1}} \tau^2 (|X|^2 + |\nabla X|^2) e^{2\phi_\alpha} \, dm.
\]

Here and below, we use \(C\) to denote a constant depending at most on \(n\) and \(k\), and \(K\) a constant depending possibly in addition on \(\delta, B, M_0, N_2,\) and \(N_3\).

Now, provided \(T\) is chosen small enough (depending on \(n, k, B\) and \(\delta\)) we can hide the first term on the right in the term on the left at the expense of enlarging the constants on the right, say, by a factor of two. And, using the decay estimate (7.3), we can estimate the second term on the right via

\[
\frac{1}{\varepsilon^2} \int_\varepsilon^{2\varepsilon} \int_{A_{R,2r}} \tau^2 (|X|^2 + |Y|^2 + |\nabla X|^2) e^{2\phi_\alpha} \, dm
\]

\[
\leq 4\varepsilon \left(2\varepsilon (\frac{A_{R,2r}}{\varepsilon})^k - N_2 R^2\right) \int_\varepsilon^{2\varepsilon} \int_{A_{R,2r}} (|X|^2 + |Y|^2 + |\nabla X|^2) e^{\frac{N_2 R^2}{\tau}} \, dm
\]

\[
\leq K_{\alpha,r} e^{-\frac{N_2 R^2}{\tau}}
\]

for some \(K_{\alpha,r}\) depending on \(\alpha, \delta, r, R, N_2,\) and \(N_3\). Hence, this term tends to 0 as \(\varepsilon \searrow 0\) for any fixed \(\alpha\) and \(r\).
Similarly, on $A_{R+1} \times (0, T)$ and $A_{r, 2r} \times (0, T)$, we have $e^{2\phi_\alpha} \leq K_\alpha e^{\frac{N\alpha r^2}{2}}$ and $e^{2\phi_\alpha} \leq K_\alpha e^{\frac{N\alpha r^2}{2}}$, respectively, for some $K_\alpha$ depending on $\alpha$ and $\delta$, so, using (7.4), we see that the fourth and fifth terms on the right converge to finite values as $\epsilon \searrow 0$.

After taking this limit, we obtain from (7.11) that

$$
\int_0^T \int_{A_{r, 2r}} (\alpha \tau^{-\delta}(|X|^2 + |Y|^2) + \tau |\nabla X|^2) e^{2\phi_\alpha} \, dm
\leq C\alpha^2 \int_{T-2T}^T \int_{A_{r, 2r}} \tau^2 \left(|X|^2 + |Y|^2 + |\nabla X|^2\right) e^{2\phi_\alpha} \, dm
$$

(7.12)

$$
+ K \int_0^T \int_{A_{R+1}} \tau^2 \left(|X|^2 + |\nabla X|^2\right) e^{2\phi_\alpha} \, dm
+ K \int_0^T \int_{A_{r, 2r}} \tau^2 \left(|X|^2 + |\nabla X|^2\right) e^{2\phi_\alpha} \, dm.
$$

Estimating as above, we see also that

$$
\int_0^T \int_{A_{r, 2r}} \tau^2 \left(|X|^2 + |\nabla X|^2\right) e^{2\phi_\alpha} \, dm \leq K_\alpha e^{\frac{N\alpha r^2}{2}},
$$

so the last term on the right of (7.12) tends to zero as $r \to \infty$. The first term on the right of (7.12) can also be seen to be bounded above independently of $r$; we will verify this now and further show that it is bounded independently of $\alpha$.

Let us assume from now on that $\alpha \geq \alpha_1$ where $\alpha_1 = \alpha_1(\delta)$ is large enough that $T - 2T/\alpha_1 \geq 2\tau_0$. (The constant $\tau_0$ here is from the definition of $\eta$ in (7.8).) Then $\eta(T) = c_0(T - \tau)/T$ on the interval $[T - 2T/\alpha, T - T/\alpha]$ for some constant $c_0 = c_0(\delta)$ and, consequently, $\phi_\alpha \leq 2c_0|z|^{2\delta}/\tau^2$ for $\tau$ in the same range. Choosing $m$ so large that $2^m R \geq r$, we may estimate that

$$
\int_{T-2T}^T \int_{A_{r, 2r}} \tau^2 \left(|X|^2 + |Y|^2 + |\nabla X|^2\right) e^{2\phi_\alpha} \, dm
\leq \int_{T-2T}^T \int_{A_{r, 2r}} \left(|X|^2 + |Y|^2 + |\nabla X|^2\right) e^{\frac{4\alpha_1|z|^{2\delta}}{\tau^2}} \, dm
\leq K \int_{2\tau_0}^T \int_{A_{r, 2m+1}} \left(|X|^2 + |Y|^2 + |\nabla X|^2\right) e^{\frac{N\alpha_1|z|^{2\delta}}{\tau^2}} \, dm
\leq K \sum_{l=0}^\infty \left\{e^{-\frac{N\alpha_1|z|^{2\delta}}{8\tau^2}} \int_0^T \int_{A_{2^l r, 2^{l+1} r}} \left(|X|^2 + |Y|^2 + |\nabla X|^2\right) e^{\frac{N\alpha_1|z|^{2\delta}}{\tau^2}} \, dm\right\}
\leq K
$$

for any $\alpha \geq \alpha_1$ and $r \geq r_2$. Thus we may take the limit as $r \to \infty$ on both sides of (7.12) to obtain that

$$
\int_0^T \int_{A_{r+1}} (\alpha \tau^{-\delta}(|X|^2 + |Y|^2) + \tau |\nabla X|^2) e^{2\phi_\alpha} \, dm
\leq \frac{K\alpha^2}{T^2} + K \int_0^T \int_{A_{R+1}} \tau^2 \left(|X|^2 + |\nabla X|^2\right) e^{2\phi_\alpha} \, dm.
$$

(7.13)
8.1. Integral identities. In this subsection, we will use \( g = g(\tau) \) to denote an arbitrary solution to (1.5) on a smooth manifold \( M = M^n \) for \( \tau \in (0, T) \), and \( Z \) to denote a tensor bundle over \( M \). We will use \( \nabla = \nabla_{g(\tau)} \) and \( d\mu = d\mu_{g(\tau)} \) to represent the Levi-Civita connection and Riemannian density associated to \( g \), and define the operator \( D_\tau \) in terms of \( g \) as in Section 4. We will also continue to use the shorthand \( dm = d\mu_{g(\tau)} \, d\tau \).

Let \( \phi : M \times (0, T) \to \mathbb{R} \) be a smooth positive function and consider the operator

\[
\mathcal{L} = \tau e^\phi (D_\tau + \Delta) e^{-\phi}
\]

acting on smooth families of sections of \( Z \). Explicitly, then, we have

\[
\mathcal{L}V = \tau \left( (\nabla \phi)^2 - \frac{\partial \phi}{\partial \tau} - \Delta \phi \right) V + \tau (D_\tau + \Delta) V - 2\tau \nabla_{\nabla \phi} V,
\]
Proof. For the time-being, write 
\( y\) yields the identity 
effectively, the above identity will yield an estimate of the 
\( L^2 \)
writing \( \Phi \)
and the formal \( L^2(dm) \)-adjoint of \( L \) is given by 
\[
L^* V = \tau \left( |\nabla \phi|^2 + \Delta \phi - \frac{\partial \phi}{\partial \tau} - \frac{1}{\tau} R \right) V - \tau(D \tau - \Delta) V + 2 \tau \nabla \phi \nabla V.
\]
Writing \( L \) in terms of its symmetric and antisymmetric parts 
\[
SV = \frac{L V + L^* V}{2} = \tau \left( |\nabla \phi|^2 - \frac{\partial \phi}{\partial \tau} - \frac{R}{2} - \frac{1}{2 \tau} \right) V + \tau \Delta V
\]
\[
AV = \frac{L V - L^* V}{2} = \tau \left( \frac{R}{2} - \Delta \phi + \frac{1}{2 \tau} \right) V + \tau D \tau V - 2 \tau \nabla \phi \nabla V,
\]
yields the identity 
\[
\int \int \tau^2 |D \tau Z + \Delta Z|^2 e^{2 \phi} \, dm = \int \int |L V|^2 \, dm
\]
\[
(8.1)
\]
for any smooth family \( Z = e^{- \phi} V \) of sections of \( Z \) with compact support in \( \mathcal{C} \times (0, T) \).

Provided (with a judicious choice of \( \phi \)) we can estimate the commutator \([S, A] \)
effectively, the above identity will yield an estimate of the \( L^2 \)-norm of \((D \tau + \Delta) Z\)
from below by that of \( Z \). The following identity will be the basis of our estimate.

**Proposition 8.1.** If \( V \) is any smooth family of sections of \( Z \) with compact support in \( M \times (0, T) \), we have 
\[
\int \int \langle [S, A] V, V \rangle \, dm = \int \int \left( Q^{(1)}_\phi (\nabla V, \nabla V) + Q^{(2)}_\phi |V|^2 + Q^{(3)}_\phi (\nabla V, V) \right) \, dm
\]
where 
\[
Q^{(1)}_\phi (\nabla V, \nabla V) = 2 \tau^2 \left( 2 \nabla_i \nabla_j \phi - R_{ij} + \frac{g_{ij}}{2 \tau} \right) \langle \nabla_i V, \nabla_j V \rangle,
\]
\[
Q^{(2)}_\phi = \tau^2 \left( \frac{\partial^2 \phi}{\partial \tau^2} - \Delta^2 \phi - 2 \frac{\partial \phi}{\partial \tau} |\nabla \phi|^2 + \frac{1}{2} \left( \frac{\partial R}{\partial \tau} + \Delta R \right) - \langle \nabla R, \nabla \phi \rangle \right)
+ 2 \tau^2 \left( 2 \nabla \nabla \phi (\nabla \phi, \nabla \phi) - Rc(\nabla \phi, \nabla \phi) + |\nabla \phi|^2 \right) + \tau \left( \frac{\partial \phi}{\partial \tau} - 2 |\nabla \phi|^2 + \frac{R}{2} \right),
\]
and 
\[
Q^{(3)}_\phi (\nabla V, V) = -2 \tau^2 \left( \nabla_i R_{ja} - \nabla_j R_{ia} + 2 R_{ja} \nabla_i \phi \right) \langle \Lambda^j V, \nabla a V \rangle.
\]

**Proof.** For the time-being, write \( S \) and \( A \) as 
\[
S = \tau (\Delta + F \text{Id}), \quad A = \tau (D \tau - 2 \nabla \nabla + G \text{Id}).
\]
Then,
\[
S(A V) = \tau^2 \left( \Delta D \tau V - 2 \Delta (\nabla \nabla V) + \Delta (GV) + FD \tau V - 2 F (\nabla \nabla V) + FGV \right),
\]
and 
\[
A(S V) = \tau^2 \left( D \tau \Delta V + D \tau (F V) - 2 \nabla \nabla (\Delta V) - 2 \nabla \nabla (F V) + G \Delta V + FGV \right)
+ \tau (\Delta V + F V),
\]
we have the above identity to obtain that

\[\int \int \left( [\Delta, D_T] V + 2 \Delta V + 2 \langle \nabla \nabla \phi, \Delta \rangle V \right) dm \]

so

\[
[S, A] V = \tau^2 \left( [\Delta, D_T] V + 2 [\nabla \nabla \phi, \Delta] V + \left( 2 \langle \nabla F, \nabla \phi \rangle - \frac{\partial F}{\partial \tau} \right) V + \Delta GV \\
+ 2 \nabla \nabla GV - \frac{1}{\tau} (\Delta V + FV) \right).\]

Since \( V \) has compact support, we may integrate \( \langle [S, A] V, V \rangle \) over \( C \times (0, T) \) and integrate by parts in the integrals corresponding to the fourth and sixth terms of the above identity to obtain that

\[
\int \int \langle [S, A] V, V \rangle \, dm = \int \int \tau^2 \langle [\Delta, D_T] V + 2 [\nabla \nabla \phi, \Delta] V, V \rangle \, dm \\
+ \int \int \tau \langle |V|^2 \rangle \, dm + \int \int \left( \tau^2 \left( 2 \langle \nabla F, \nabla \phi \rangle - \frac{\partial F}{\partial \tau} \right) - \tau F \right) |V|^2 \, dm.
\]

We now simplify the commutator terms on the right side of (8.3). First,

\[
\int \int \tau^2 \langle [\Delta, D_T] V, V \rangle \, dm = \int \int \tau^2 \left( \langle [\nabla_a, D_T] \nabla_a V, V \rangle + \langle [D_T, \nabla_a] V, \nabla_a V \rangle \right) \, dm \\
= \int \int \tau^2 \left( \frac{1}{2} [\nabla_a, D_T] \nabla_a |V|^2 + 2 \langle [D_T, \nabla_a] V, \nabla_a V \rangle \right) \, dm,
\]

and since

\[
[\nabla_a, D_T] \nabla_a |V|^2 = R_{ab} \nabla_b \nabla_a |V|^2 + \langle \nabla_a R_{ac} - \nabla_c R_{aa} \rangle \nabla_a |V|^2 \\
= \nabla_b (R_{ab} \nabla_a |V|^2) - \langle \nabla R, \nabla |V|^2 \rangle,
\]

and

\[
[D_T, \nabla_a] V = -R_{ab} \nabla_b V - \langle \nabla_b R_{ac} - \nabla_c R_{ab} \rangle \Lambda^b_c(V),
\]

we have

\[
\int \int \tau^2 \langle [\Delta, D_T] V, V \rangle \, dm = \int \int \tau^2 \left( \frac{1}{2} \Delta R |V|^2 - 2 R_{ab} \langle \nabla_a V, \nabla_b V \rangle \\
- 2 \langle (\nabla_b R_{ac} - \nabla_c R_{ab}) \Lambda^b_c(V), \nabla_a V \rangle \right) \, dm.
\]

Likewise, for the second commutator term in (8.3), we compute that

\[
\int \int \tau^2 \langle \nabla \nabla \phi (\Delta V), V \rangle \, dm = - \int \int \tau^2 \left\{ \Delta \phi (\Delta V, V) + \langle \nabla \nabla \phi V, \Delta V \rangle \right\} \, dm
\]

and

\[
\int \int \tau^2 \langle (\nabla \nabla \phi) V, V \rangle \, dm = \int \int \tau^2 \left\{ \Delta \phi |\nabla V|^2 - 2 \langle [\nabla_a, \nabla_b] V, \nabla_a V \rangle \nabla_b \phi \\
- 2 \nabla_a \nabla \phi (\nabla_a V, \nabla_b V) - \langle \nabla \nabla \phi V, \Delta V \rangle \right\} \, dm.
\]

Using that

\[
\nabla_d \phi [\nabla_a, \nabla_d] V = -R_{bcad} \nabla_d \phi \Lambda^b_c(V),
\]

we then have

\[
2 \int \int \tau^2 \langle [\nabla \nabla \phi, \Delta] V, V \rangle \, dX = \int \int \tau^2 \left\{ 4 \nabla_a \nabla_b \phi (\nabla_a V, \nabla_b V) \\
- 4 R_{bcad} \nabla_d \phi (\Lambda^b_c(V), \nabla_a V) - \Delta^2 \phi |V|^2 \right\} \, dm.
\]
Now we expand the third term on the right of (8.3). Using that
\[ F = |\nabla \phi|^2 - \frac{\partial \phi}{\partial \tau} - \frac{R}{2} - \frac{1}{2\tau}, \]
we compute that
\[
2\langle \nabla F, \nabla \phi \rangle = 4\nabla \phi(\nabla \phi, \nabla \phi) - 2\left( \nabla \frac{\partial \phi}{\partial \tau}, \nabla \phi \right) - \langle \nabla R, \nabla \phi \rangle
\]
\[ = 4\nabla \phi(\nabla \phi, \nabla \phi) - 2\operatorname{Rc}(\nabla \phi, \nabla \phi) - \frac{\partial}{\partial \tau}|\nabla \phi|^2 - \langle \nabla R, \nabla \phi \rangle \]
and
\[
\frac{\partial F}{\partial \tau} = \frac{\partial}{\partial \tau}|\nabla \phi|^2 - \frac{\partial^2 \phi}{\partial \tau^2} - \frac{1}{2\tau} + \frac{1}{2\tau^2}
\]
so
\[
\iint \left( 2\tau^2\langle \nabla F, \nabla \phi \rangle - \tau^2\frac{\partial F}{\partial \tau} - \tau F \right) |V|^2 \, dm
\]
\[ = \iint 2\tau^2 \left( 2\nabla \phi(\nabla \phi, \nabla \phi) - \operatorname{Rc}(\nabla \phi, \nabla \phi) + \frac{|\nabla \phi|^2}{2\tau} \right) |V|^2 \, dm
\]
\[ + \iint \left\{ \tau^2 \left( \frac{\partial^2 \phi}{\partial \tau^2} - 2\frac{\partial}{\partial \tau}|\nabla \phi|^2 + \frac{1}{2\tau} \frac{\partial R}{\partial \tau} - \langle \nabla R, \nabla \phi \rangle \right) + \left( \frac{\partial^2 \phi}{\partial \tau^2} + \nabla \phi - \frac{R}{2} - 2\frac{\partial}{\partial \tau}|\nabla \phi|^2 \right) \right\} |V|^2 \, dm. \]
Combining the above identity with (8.3), (8.4), and (8.5), yields (8.2). □

**Remark 8.2.** When \( g(\tau) \) is a shrinking self-similar solution to (1.5) in the sense that \((M, g(1), f(1))\) satisfies (1.2) and \( g(\tau) = \tau \Psi^\tau g(1), f(\tau) = f(1) \circ \Psi^\tau \) where \( \frac{\partial \Psi}{\partial \tau} = -\tau^{-1}(\nabla g(1)f(1)) \circ \Psi \) and \( \Psi_1 = \text{Id} \), the quantities \( Q^{(i)}_\phi \), \( i = 1, 2, 3 \), on the right side of (8.2) vanish identically with the choice \( \phi = -\frac{f}{2} \). This can be seen immediately for \( Q^{(1)}_\phi \) and \( Q^{(3)}_\phi \) given the identities
\[
R_{ij} + \nabla_i \nabla_j f = \frac{g_{ij}}{2\tau}, \quad \nabla_i R_{jk} - \nabla_j R_{ik} = R^l_{ijk} \nabla_l f
\]
satisfied by \( g \) and \( f \) on \( M \times (0, T) \). The vanishing of \( Q^{(2)}_\phi \) follows from the additional identities
\[
\Delta f + R = \frac{n}{2\tau}, \quad \frac{\partial f}{\partial \tau} = -|\nabla f|^2, \quad \frac{\partial R}{\partial \tau} = -\langle \nabla R, \nabla f \rangle - \frac{R}{\tau},
\]
since
\[
Q^{(2)}_\phi = \frac{\tau^2}{2} \left( \left( \frac{\partial R}{\partial \tau} + \langle \nabla R, \nabla f \rangle + \frac{R}{\tau} \right) + \Delta (f + R) \right)
\]
\[ - \frac{\tau^2}{2} \left( \left( \frac{\partial}{\partial \tau} + \frac{1}{\tau} \right) \left( \frac{\partial f}{\partial \tau} + |\nabla f|^2 \right) + \left( \operatorname{Rc}(g) + \nabla \nabla f - \frac{g}{2\tau} \right) \langle \nabla f, \nabla f \rangle \right)
\]
\[ = 0. \]

We will use the simple energy estimate in the next proposition to control \( |\nabla Z| \) by \( |(D_\tau + \Delta)Z| \) in combination with our estimate for \( |Z| \).
Proposition 8.3. If $Z$ is any smooth family of sections of $Z$ with compact support in $M \times (0, T)$, then, for any $j$, $l \geq 0$, and $c > 0$,

\[
\int \int \tau^j |\nabla Z|^2 e^{2\phi} \, dm
\]

(8.6)

\[
\leq \int \int \tau^j \left( \Delta \phi + 2|\nabla \phi|^2 - \frac{\partial \phi}{\partial \tau} - \frac{R}{2} + \frac{c \tau^{1-l}}{2l} \right) |Z|^2 e^{2\phi} \, dm
\]

\[
+ \int \int \frac{\tau^{j+l}}{2c} |(D_\tau + \Delta)Z|^2 e^{2\phi} \, dm.
\]

Proof. Write $V = e^\phi Z$ as before and consider the identities

\[
\tau^j |\nabla V|^2 = \frac{1}{2} \left( \frac{\partial}{\partial \tau} + \Delta \right) (\tau^j |V|^2) - \frac{j \tau^{j-1}}{2} |V|^2 - \tau^j \langle (D_\tau + \Delta)V, V \rangle
\]

and

\[
\tau^j \langle (D_\tau + \Delta)V, V \rangle = \tau^{j-1} \langle L \nabla V, V \rangle + \tau^j \left( \Delta \phi + \frac{\partial \phi}{\partial \tau} - |\nabla \phi|^2 \right) |V|^2 + \tau^j \langle \nabla \phi, \nabla |V|^2 \rangle.
\]

Combining these identities, integrating over $M \times (0, T)$, and integrating by parts, we obtain

\[
\int \int \tau^j |\nabla V|^2 \, dm = \int \int \tau^j \left( |\nabla \phi|^2 - \frac{\partial \phi}{\partial \tau} - \frac{R}{2} - \frac{j}{2\tau} \right) |V|^2 \, dm - \int \int \tau^{j-1} \langle L \nabla V, V \rangle \, dm
\]

\[
\leq \int \int \tau^j \left( |\nabla \phi|^2 - \frac{\partial \phi}{\partial \tau} - \frac{R}{2} + \frac{c \tau^{1-l}}{2l} \right) |V|^2 \, dm
\]

(8.7)

\[
+ \int \int \frac{\tau^{j+l}}{2c} |(D_\tau + \Delta)Z|^2 e^{2\phi} \, dm
\]

for any $c > 0$ and $l \geq 0$. On the other hand,

\[
|\nabla V|^2 = e^{2\phi} (|\nabla Z|^2 + \langle \nabla \phi, \nabla |Z|^2 \rangle + |\nabla \phi|^2 |Z|^2),
\]

so

\[
\int \int \tau^j |\nabla Z|^2 \, dX = \int \int \tau^j |\nabla V|^2 \, dm + \int \int \tau^j (\Delta \phi + |\nabla \phi|^2) |Z|^2 e^{2\phi} \, dm.
\]

Combining (8.7) and (8.8), we obtain the desired inequality. \hfill \Box

8.2. Carleman estimates to imply exponential decay. For the rest of the section, we will specialize to the cylinder $M = C$ with $\Psi_\tau(\theta, z) = (\theta, z/\sqrt{\tau})$, and

\[
g(\tau) = \tau \Psi_\tau^* g(1) = (2(k-1)\tau \hat{g}) \oplus \hat{g}, \quad f_{z_0}(\theta, z, \tau) = f_{z_0}(\Psi_\tau(\theta, z), 1) = \frac{|z - z_0|^2}{4\tau} + \frac{k}{2},
\]

for $\tau > 0$ and some $z_0 \in \mathbb{R}^{n-k}$ as before.

8.2.1. An estimate for the PDE component. We start with the proof of Theorem 6.2. Following [17], [47], we define for $\alpha > 0$ and $z_0 \in \mathbb{R}^{n-k}$ the weight function $\varphi = \varphi_{\alpha, z_0} : C \times (0, \infty) \to \mathbb{R}$ by

\[
\varphi(z, \theta, \tau) = \frac{|z - z_0|^2}{8\tau} - \alpha \log \sigma(\tau) - \frac{1}{2} f_{z_0}(z, \theta, \tau) - \alpha \log \sigma(\tau) + \frac{k}{4}
\]

(8.9)

where $\sigma(\tau) = \tau e^{(T-\tau)/3}$. 

According to (8.1) and Proposition 8.1, we then have

$$Q^{(1)}_{\varphi} = 0, \quad Q^{(2)}_{\varphi} = -\alpha \tau (\tau (\log \sigma)^\prime + (\log \sigma)^\prime) = \frac{\alpha \tau}{3}, \quad Q^{(3)}_{\varphi} = 0.$$ 

According to (8.1) and Proposition 8.1 we then have

$$\alpha \int \tau |Z|^2 e^{2\varphi} \, dm \leq \int \tau^2 (D_\tau + \Delta)|Z|^2 e^{2\varphi} \, dm. \quad \text{(8.10)}$$

To incorporate the derivative of $Z$, we use Proposition 8.3 with $\phi = \varphi$, $j = 2$, $l = 1$, and $c = 2\alpha$. Using the soliton identities (see Remark 8.2), we can simplify the integrand of the first integral on the right of (8.10) to find

$$\tau^2 \left( \Delta \varphi + |\nabla \varphi|^2 - \frac{\partial \varphi}{\partial \tau} - \frac{R}{2} + \frac{\alpha}{\tau} \right)$$

$$= \tau^2 \left( -\frac{\Delta f_{\varphi_0}}{2} + \frac{|\nabla f_{\varphi_0}|^2}{4} + \frac{1}{2} \frac{\partial f_{\varphi_0}}{\partial \tau} + \alpha (\log \sigma)^\prime - \frac{R}{2} + \frac{\alpha}{\tau} \right)$$

$$= 2\alpha \tau - \tau \left( \frac{\alpha \tau}{3} + \frac{n}{4} \right),$$

and hence that

$$\int \tau^2 |\nabla Z|^2 e^{2\varphi} \, dm \leq 2\alpha \int \tau |Z|^2 e^{2\varphi} \, dm + \frac{T}{4\alpha} \int \tau^2 (D_\tau + \Delta)|Z|^2 e^{2\varphi} \, dm.$$

Combining this inequality with (8.10) and using that $T \leq 2$, we arrive at

$$\int \tau \left( \alpha |Z|^2 + \tau^2 |\nabla Z|^2 e^{2\varphi} \, dm \leq 10 \int \tau^2 (D_\tau + \Delta)|Z|^2 e^{2\varphi} \, dm.$$ 

This implies (6.7). \hfill \square

8.2.2. Estimates for the ODE component. Both of the Carleman-type estimates (6.8) and (6.9) are consequences of the simple identity

$$\frac{\partial}{\partial \tau} \left( \tau |Z|^2 e^{2\varphi} \, d\mu \right) = \tau^2 \left( \left( \frac{j}{\tau} + 2 \frac{\partial \phi}{\partial \tau} + R \right) |Z|^2 + 2 \langle D_\tau Z, Z \rangle \right) e^{2\varphi} \, d\mu \quad \text{(8.11)}$$

where $Z$ is a smooth family of tensor fields over $\mathcal{C}$, $j \geq 0$ is a fixed number, and $\phi : C \times (0, T) \to \mathbb{R}$ is an arbitrary smooth function.

Proof of Theorem 6.3. Again it suffices to consider the case that $Z$ is a tensor bundle over $\mathcal{C}$. Let $Z$ be a smooth family of sections of $\mathcal{Z}$ with compact support in $U \times (0, T)$ for some open $U \subset \mathcal{C}$. Let $D \subset C$ be any open set with $\overline{D} \subset U$ and fix $\alpha \geq 1$, $\lambda > 0$, and $z_0 \in \mathbb{R}^{n-k}$. (The support of $Z(\cdot, \tau)$ need not be contained in $\overline{D}$.)

For the first inequality (6.8), we apply (8.11) with $\phi = \varphi$ and $j = \lambda + 1$ at some fixed $p = (\theta, z)$, obtaining

$$\frac{\partial}{\partial \tau} \left( \tau^{\lambda+1} |Z|^2 e^{2\varphi} \, d\mu \right) = \tau^2 \left( \lambda + 1 + \frac{|z - z_0|^2}{4\tau} + \frac{k}{2} - \frac{2\alpha}{3}(3 - \tau) \right) |Z|^2$$

$$+ 2\tau^{\lambda+1} \langle D_\tau Z, Z \rangle \right) e^{2\varphi} \, d\mu.$$
Since $Z$ vanishes identically near $\tau = 0$ and $\tau = T$, we may integrate the above identity over $D \times [0, T]$ to obtain
\[
\int_0^T \int_D \tau^\lambda \left( \frac{8\alpha}{3} - 4\lambda - 2k - 4 \right) |Z|^2 e^{2\tau} \, dm 
\leq \int_0^T \int_D \tau^{\lambda - 1} (|z - z_0|^2 |Z|^2 + 8\tau^2 \langle D\tau Z, Z \rangle) e^{2\tau} \, dm.
\]
Estimating
\[
8\tau^2 \langle D\tau Z, Z \rangle \leq \frac{\alpha\tau}{3} |Z|^2 + \frac{4\tau^3}{\alpha} |D\tau Z|^2
\]
we see that
\[
2\alpha \int_0^T \int_D \tau^\lambda |Z|^2 e^{2\tau} \, dm \leq \int_0^T \int_D \tau^{\lambda - 1} \left( |z - z_0|^2 |Z|^2 + \frac{4\tau}{\alpha} |D\tau Z|^2 \right) e^{2\tau} \, dm.
\]
for $\alpha \geq \alpha'(k, \lambda)$ sufficiently large. This implies (6.8) for such $\alpha$ and $D$.

For (6.9), we apply (8.11) again with $p = (\theta, z)$, obtaining
\[
\frac{\partial}{\partial \tau} (\tau^{\lambda + 1} |Z|^2 \sigma^{-2\alpha} d\mu) = \left( \tau^\lambda \left( \lambda + 1 + \frac{k}{2} - \frac{2\alpha}{3} (3 - \tau) \right) |Z|^2 + 2\tau^{\lambda + 1} \langle D\tau Z, Z \rangle \right) \sigma^{-2\alpha} d\mu.
\]
Integrating over $D \times [0, T]$, we obtain
\[
\int_0^T \int_D \tau^\lambda \left( \frac{2\alpha}{3} - \lambda - \frac{k}{2} - 1 \right) |Z|^2 \sigma^{-2\alpha} \, dm \leq 2 \int_0^T \int_D \tau^{\lambda + 1} \langle D\tau Z, Z \rangle \sigma^{-2\alpha} \, dm.
\]
Since
\[
2\tau^{\lambda + 1} \langle D\tau Z, Z \rangle \leq \frac{\alpha\tau}{8} |Z|^2 + \frac{8\tau^{\lambda + 2}}{\alpha} |D\tau Z|^2,
\]
we have
\[
\frac{\alpha}{2} \int_0^T \int_D \tau^\lambda |Z|^2 \sigma^{-2\alpha} \, dm \leq \frac{8}{\alpha} \int_0^T \int_D \tau^{\lambda + 2} |D\tau Z|^2 \sigma^{-2\alpha} \, dm,
\]
provided $\alpha \geq \alpha''(k, \lambda)$ is sufficiently large. This implies (6.9) for such $\alpha$. Putting $\alpha_0 = \max\{\alpha', \alpha''\}$ finishes the proof. \qed

8.3. Carleman estimates to imply backward uniqueness. Now we prove the second set of Carleman-type estimates from Section 7. Here, as in [17], we fix some $0 < T \leq 1$ and construct our weight from the function $\phi_\alpha = \phi_{\alpha, \delta} : \mathcal{C} \times (0, T) \to \mathbb{R}$ given by
\[
\phi_\alpha(z, \theta, \tau) = \alpha \eta(\tau) \left( 4 \left( f_0(z, \theta, \tau) - \frac{k}{2} \right) \right)^\delta = \alpha \eta(\tau) \left( \frac{|z|^2}{\tau} \right)^\delta,
\]
as in [7,7] with $\eta : [0, T] \to [0, 1]$ defined as in (7.8). The function $\eta$ is piecewise-differentiable, twice weakly-differentiable, and satisfies the following inequalities.

Lemma 8.4 ([17]). The function $\eta$ is nonincreasing and satisfies
\[
0 \leq \eta \leq 1, \quad \delta \eta - \tau \eta' \geq 0, \quad \tau^2 \eta'' \geq -\frac{1}{4} \delta (4\delta - 3)
\]
for $\tau \in [0, T]$. 

\[
\delta \eta - \tau \eta' \geq 0, \quad \tau^2 \eta'' \geq -\frac{1}{4} \delta (4\delta - 3)
\]
These inequalities are verified in Lemma 2.5 of [47] for the function \( \tilde{\eta}(\tau) = \eta(\tau/T) \). They are invariant under rescaling of \( \tau \) and are hence also valid in our situation.

8.3.1. An estimate for the PDE component. To apply the integral identities in the preceding section, we first need to collect formulas for the various derivative expressions that appear in the quantities \( Q^{(i)}_{\phi_\alpha}, i = 1, 2, 3 \), in (8.2). The necessary expressions have already been computed in [47]. (The computations there, made relative to the Euclidean metric are valid for the evolving cylindrical metric here since \( \phi_\alpha \) is independent of the spherical variables.)

**Lemma 8.5** (Lemma 2.4, [47]). For any \( \alpha > 0 \), the derivatives of the function \( \phi_\alpha \) satisfy the expressions

\[
\nabla \phi_\alpha = \frac{2\alpha \delta \eta}{\tau^\delta} |z|^{2\delta - 2} z \\
|\nabla \phi_\alpha|^2 = \frac{4\alpha^2 \delta^2 \eta^2}{\tau^{2\delta}} |z|^{4\delta - 2} \\
\nabla \nabla \phi_\alpha = \frac{2\alpha \delta \eta}{\tau^\delta} |z|^{2\delta - 4} (|z|^2 \bar{P} + 2(\delta - 1)z \otimes z) \\
\Delta \phi_\alpha = \frac{2\alpha \delta (2(\delta - 1) + n - k) \eta}{\tau^\delta} |z|^{2\delta - 2} \\
\frac{\partial \phi_\alpha}{\partial \tau} = \frac{\alpha(\tau \eta' - \delta \eta)}{\tau^{\delta + 1}} |z|^{2\delta} \\
\frac{\partial^2 \phi_\alpha}{\partial \tau^2} = \frac{\alpha(\tau^2 \eta'' - 2 \delta \tau \eta' + \delta(\delta + 1) \eta)}{\tau^{\delta + 2}} |z|^{2\delta} \\
\frac{\partial}{\partial \tau} |\nabla \phi_\alpha|^2 = \frac{8\alpha^2 \delta^2 \eta(\tau \eta' - \delta \eta)}{\tau^{2\delta + 1}} |z|^{4\delta - 2} \\
\Delta^2 \phi_\alpha = \frac{4\alpha \delta (\delta - 1)(2(\delta - 1) + n - k)(2(\delta - 2) + n - k) \eta}{\tau^\delta} |z|^{2\delta - 4}
\]

on \( C_r \times (0, T) \) for any \( r > 0 \).

Above, in the first and third equations, we identify \( z \) with the differential of the function \( (\theta, z) \mapsto |z|^2/2 \) and, in the expression for \( \nabla \nabla \phi_\alpha \), we identify the endomorphism \( \bar{P} \) with the two-tensor \( P_{ij} = \bar{P}^i_k g_{kj} \). Now we prove Theorem 7.3.

**Proof of Theorem 7.3**. Fix \( \delta \in (7/8, 1) \) and \( T \in (0, 1] \), and let \( r \geq r_3 \) where \( r_3 \geq 1 \) is to be specified over the course of the proof. We will assume, as before, that \( Z \) is a fixed tensor bundle over \( C \). Let \( Z \) be a smooth family of sections of \( Z \) on \( C_r \) defined for \( \tau \in (0, T) \) and let \( V = e^{\phi_\alpha} Z \).

With an eye toward (8.2), let us define

\[
S_{\phi_\alpha} = \frac{g}{2\tau} - \text{Re}(g) + 2\nabla \nabla \phi_\alpha.
\]

Then, using Lemma 8.5, we have

\[
S_{\phi_\alpha} = \frac{\bar{P}}{2\tau} + 2\nabla \nabla \phi_\alpha = \frac{\bar{P}}{2\tau} + \frac{4\alpha \delta \eta}{\tau^\delta} |z|^{2\delta - 4} (|z|^2 \bar{P} + 2(\delta - 1)z \otimes z).
\]

Since \( \delta > 1/2 \), the second term, and hence the sum, is nonnegative definite when considered as a two-tensor on \( TC \) over \( C_r \). In particular, it follows that the quantity \( Q^{(1)}_{\phi_\alpha}(\nabla V; \nabla V) \) from (8.2) is nonnegative.
For the quantity $Q_{\phi_\alpha}^{(2)}$, we have similarly that
\[
Q_{\phi_\alpha}^{(2)} \geq \tau^2 \left( \frac{\partial^2 \phi_\alpha}{\partial \tau^2} - \Delta^2 \phi_\alpha - 2 \frac{\partial}{\partial \tau} |\nabla \phi_\alpha|^2 \right) + \tau \left( \frac{\partial \phi_\alpha}{\partial \tau} - 2|\nabla \phi_\alpha|^2 \right),
\]
where we have used that $\nabla R = 0$, $\Delta R = 0$, and $\frac{\partial R}{\partial \tau} + R/\tau = 0$. Now, two of the terms on the right are proportional to $\alpha^2$. Using Lemmas 8.4 and 8.5 we see that we may estimate them below by
\[
-2\tau \left( \frac{\partial}{\partial \tau} |\nabla \phi_\alpha|^2 + |\nabla \phi_\alpha|^2 \right) = -\frac{8\alpha^2 \delta^2 \eta |z|^{4\delta-2}}{\tau^{2\delta-1}} (2(\tau \eta' - \delta \eta) + \eta)
\]
\[
\geq \frac{6\alpha^2 \delta^2 \eta |z|^{4\delta-2}}{\tau^{2\delta-1}}.
\]
The remaining terms are proportional to $\alpha$, and we may estimate them similarly:
\[
\tau^2 \left( \frac{\partial^2 \phi_\alpha}{\partial \tau^2} - \Delta^2 \phi_\alpha \right) + \tau \frac{\partial \phi_\alpha}{\partial \tau}
\]
\[
= \frac{\alpha |z|^{2\delta}}{\tau^\delta} \left( \tau^2 \eta'' - 2\delta \tau \eta' + \delta(\delta + 1)\eta + (\tau \eta' - \delta \eta) - \frac{C(\delta, k, n) \tau^2 \eta}{|z|^4} \right)
\]
\[
\geq \frac{\alpha |z|^{2\delta}}{\tau^\delta} \left( \frac{3\delta \eta + (1 - 4\delta)\tau \eta'}{4} - \frac{C(\delta, k, n) \tau^2 \eta}{|z|^4} \right).
\]
So, if $r_3 = r_3(\delta, n, k)$ is taken sufficiently large, we will have
\[
Q_{\phi_\alpha}^{(2)} \geq \frac{\alpha |z|^{2\delta}}{2\tau^\delta} (\delta \eta - \tau \eta') + \frac{6\alpha^2 \delta^2 \eta |z|^{4\delta-2}}{\tau^{2\delta-1}},
\]
on $C \times (0, T)$.

Finally, $Q_{\phi_\alpha}^{(3)} = 0$ on the cylinder since $\nabla Rc = 0$ and $\text{Rm}(\cdot, \cdot, \cdot, \nabla \phi_\alpha) = 0$. Putting things together and using (8.1) and (8.2), we thus see that
\[
\int_0^T \int_{C_r} \left( \frac{\alpha |z|^{2\delta}}{2\tau^\delta} (\delta \eta - \tau \eta') + \frac{6\alpha^2 \delta^2 \eta |z|^{4\delta-2}}{\tau^{2\delta-1}} \right) |Z|^2 e^{2\phi_\alpha} \, dm
\]
\[
\leq \int_0^T \int_{C_r} \tau^2 |D_\tau Z + \Delta Z|^2 e^{2\phi_\alpha} \, dm,
\]
for all $\alpha > 0$ and $r \geq r_3$.

Now we use Proposition 8.2 to add in the gradient term. Taking $\phi = \phi_\alpha$ and $c = j = l = 1$ in (8.10) yields
\[
\int_0^T \int_{C_r} \tau |\nabla Z|^2 e^{2\phi_\alpha} \, dm
\]
\[
\leq \int_0^T \int_{C_r} \tau \left( \Delta \phi_\alpha + 2|\nabla \phi_\alpha|^2 - \frac{\partial \phi_\alpha}{\partial \tau} - \frac{R}{2} \right) |Z|^2 e^{2\phi_\alpha} \, dm
\]
\[
+ \int_0^T \int_{C_r} \frac{\tau^2}{2} (D_\tau + \Delta) Z|^2 e^{2\phi_\alpha} \, dm.
\]

By Lemma 8.5
\[
\tau \left( \Delta \phi_\alpha + 2|\nabla \phi_\alpha|^2 - \frac{\partial \phi_\alpha}{\partial \tau} - \frac{R}{2} \right)
\]
\[
= \frac{2\alpha \delta (2(\delta - 1) + n - k) \eta |z|^{2\delta-2} + 8\alpha^2 \delta^2 \eta^2 |z|^{4\delta-2} - \alpha(\tau \eta' - \delta \eta)}{\tau^{2\delta+1}} |z|^{2\delta} - \frac{k}{4\tau}
\]
We will assume below that \( (\delta \eta - \tau \eta') + \frac{\tau(n - k)}{|z|^2} \leq \frac{8\alpha^2 \delta^2 \eta^2 |z|^{d+2}}{\tau^{2d-1}} \),
\[
\leq \frac{\alpha |z|^{2 \delta}}{\tau^\delta} \left( (\delta \eta - \tau \eta') + \frac{\tau(n - k)}{|z|^2} \right) + \frac{8\alpha^2 \delta^2 \eta^2 |z|^{d+2}}{\tau^{2d-1}},
\]
for \( r_3 \) sufficiently large. Returning to (8.14) with this, multiplying both sides by \( 1 \), and combining the result with (8.13), we obtain
\[
\int_0^T \int_{C_r} \left( \left( \frac{\alpha \delta}{8 \tau^3} + \frac{\tau}{4} \right) |Z|^2 e^{2\phi_\alpha} + \frac{k}{2} \right) \delta^\tau |Z|^{2d} e^{2\phi_\alpha} dm \leq \int_0^T \int_{C_r} \tau^2 |D_r Z| + \Delta Z^2 e^{2\phi_\alpha} dm,
\]
for \( r \geq r_3 \) and all \( \alpha > 0 \). The estimate (7.9) follows. \( \square \)

8.3.2. An estimate for the ODE component. For the proof of the matching estimate for the ODE component, we again use the identity (8.11).

Proof of Theorem 2.5. Fix \( \alpha \geq 1, 0 < T \leq 1 \), and let \( r \geq r_4 \) for some \( r_4 \) to be specified later. Let \( Z \) be a smooth family of sections the tensor bundle \( \mathcal{Z} \) with compact support in \( C_r \times (0, T) \). Starting from (8.11) with \( j = 1 \) and \( \phi = \phi_\alpha \), we have
\[
\frac{\partial}{\partial \tau} (\tau |Z|^2 e^{2\phi_\alpha} d\mu) = \tau \left( \left( \frac{1}{\tau} + 2 \frac{\partial \phi_\alpha}{\partial \tau} + \frac{k}{2} \right) |Z|^2 + 2\langle D_r Z, Z \rangle \right) e^{2\phi_\alpha} d\mu.
\]
By Lemma 8.5,
\[
\frac{\partial \phi_\alpha}{\partial \tau} = \alpha (\tau \eta' - \delta \eta) \tau^{-\delta-1} |z|^{2\delta} \leq -\alpha \delta \tau^{-\delta-1} |z|^{2\delta},
\]
so, integrating over \( C_r \times (0, T) \) and using Cauchy-Schwarz, we see that
\[
\int_0^T \int_{C_r} \tau^2 |D_r Z|^2 e^{2\phi_\alpha} dm \geq -\int_0^T \int_{C_r} \left( 2 \frac{\partial \phi_\alpha}{\partial \tau} + \frac{k + 4}{2} \right) |Z|^2 e^{2\phi_\alpha} dm \geq \int_0^T \int_{C_r} \left( \frac{2\alpha \delta |z|^2}{\tau^\delta} - \frac{k + 4}{2} \right) |Z|^2 e^{2\phi_\alpha} dm.
\]
Thus, provided \( r_4 = r_4(n, k, \delta) \) is sufficiently large, we will have
\[
\int_0^T \int_{C_r} \tau^2 |D_r Z|^2 e^{2\phi_\alpha} dm \geq \int_0^T \int_{C_r} \frac{\alpha \delta |z|^2}{\tau^\delta} |Z|^2 e^{2\phi_\alpha} dm
\]
as claimed. \( \square \)

Appendix A. Normalizing the soliton vector field

Now we prove Theorem 2.5 which provides the diffeomorphism \( \Phi \) we use to identify the soliton vector field with that of the standard cylindrical soliton structure. We will assume below that \( (\mathcal{C}_0, \tilde{g}, \nabla \tilde{f}) \) is strongly asymptotic to \( (\mathcal{C}, g, \nabla f) \) as a soliton structure and write, as before,
\[
h = \tilde{g} - g, \quad \tilde{X} = \nabla \tilde{f}, \quad X = \nabla f = \frac{r}{2} \frac{\partial}{\partial r}, \quad E = \tilde{X} - X.
\]
By assumption, there are constants \( M_{l,m} \) such that
\[
\sup_{\mathcal{C}_0} |z| \left\{ |\nabla^{(m)} h| + |\nabla^{(m)} E| \right\} \leq M_{l,m}
\]
for all \( l, m \geq 0 \).
Using the notation and terminology of [29], let $\Theta : D \subset C_{a_0} \times \mathbb{R} \to C$ be the maximal smooth flow of $\tilde{X}$. There are a variety of natural ways to construct an injective local diffeomorphism $S^k \times S^{n-k} \times (0, \infty) \to C$ from $\Theta$ by identifying $S^k \times S^{n-k-1}$ with an appropriate hypersurface in $C_{r_0}$ to which $\tilde{X}$ is nowhere tangent. Each of these local diffeomorphisms can be adjusted to pull $\tilde{X}$ back to $X$. The trick is to choose an identification for which it is convenient to see that the pull-back of $\tilde{g}$ by the map this identification produces is still strongly asymptotic to the cylindrical metric. Our strategy will be to construct a sequence of maps $\Phi(b)$ from the identifications of $S^k \times S^{n-k-1}$ with $S_b$ for values of $b$ tending to infinity. From this sequence, we will obtain a limit map which, in a certain sense, agrees with the identity to infinite order at spatial infinity.

A.1. A sequence of maps identifying the vector fields. To begin, let us use the infinite order agreement of $\tilde{X}$ and $X$ to choose $a_0$ so large that $a_0 > 2r_0$ and

$$
(A.2) \quad \left\langle \tilde{X}, \frac{\partial}{\partial r} \right\rangle_{(\theta, \sigma, r)} \geq \frac{r}{4}, \quad \left\langle \tilde{X}, \frac{\partial}{\partial r} \right\rangle_{(\theta, \sigma, r)} \geq \frac{r}{4},
$$
on $C_{a_0}$.

**Proposition A.1.** There exists a constant $a_1 \geq a_0$ with the property that, for each $b \geq a_1$, there is an injective local diffeomorphism $\Phi(b) : C_{a_1} \to C_{a_1/2}$ satisfying

$$
(A.3) \quad d\Phi(b)(\theta, \sigma, s) = \tilde{X}(\Phi(b)(\theta, \sigma, s)), \quad \phi(b) \big|_{S_b} = \text{Id}_{S_b},
$$

$$
(A.4) \quad C_{2a_1} \subset \Phi(b)(C_{a_1}),
$$

and

$$
(A.5) \quad \frac{s}{2} \leq r \circ \Phi(b)(\theta, \sigma, s) \leq 2s.
$$

Additionally, for each $l \geq 0$, there is a constant $C_l$ such that

$$
(A.6) \quad |r \circ \Phi(b)(\theta, \sigma, s) - s| \leq C_l \left| \frac{1}{s^l} - \frac{s}{b^{l+1}} \right|,
$$

$$
(A.7) \quad d_{S_b}(\pi(\theta, \sigma, s), \pi \circ \Phi(b)(\theta, \sigma, s)) \leq C_l \left| \frac{1}{s^l} - \frac{1}{b^l} \right|,
$$

for all $b, s \geq a_1$, where $d_{S_b}$ is the induced distance on $S_b$ and $\pi = \pi_s : C_{a_0} \rightarrow S_b$ is the projection $\pi_{s_{\theta}}(\theta, \sigma, r) = (\theta, \sigma, s)$.

**Proof.** By (A.2), $\tilde{X}$ is nowhere tangent to $S_a$ for $a \geq a_0$. We use this to construct a preliminary map $\tilde{\Phi}(b)$ following Theorem 9.20 of [29]. Let $\Theta : D \subset C_{a_0} \times \mathbb{R} \to C$ be the maximal smooth flow of $\tilde{X}$, and let $\Phi(b) = \Theta|_{O_b}$ where $O_b = D \cap (S_b \times \mathbb{R})$. By (A.2), $r$ is increasing along the integral curves of $\tilde{X}$, so the flow of $\tilde{X}$ preserves $C_{a_0}$. By (A.1), $|\tilde{X}| \leq M(r + 1)$ for some $M$, so the integral curves of $X$ starting at any point in $C_{a_0}$ exist for all positive $t$.

Fix some $a > a_0$. By the compactness of $S_a$, we will have $S_a \times (-\delta, \infty) \subset O_a$ for some $\delta > 0$, and this implies that, for all $b \geq a$, we will have $S_b \times (-\delta + \alpha(b)), \infty) \subset O_b$ where

$$
\alpha(b) = \inf \{ t \mid \Theta(S_a \times \{t\}) \cap S_b \neq \emptyset \}
$$

is the minimum time needed to reach $S_b$ via an integral curve of $\tilde{X}$ starting in $S_a$. 


Now, just as in [29], each $\tilde{\Phi}^{(b)}$ is a local diffeomorphism, and
\[
d\tilde{\Phi}^{(b)}_{(\theta,\sigma,t)} \left( \frac{\partial}{\partial t} \right) = \tilde{X}(\tilde{\Phi}^{(b)}(\theta,\sigma,t)), \quad \tilde{\Phi}^{(b)}(\theta,\sigma,0) = (\theta,\sigma,b).
\]
Provided $\delta$ is small enough, the restriction of $\tilde{\Phi}^{(b)}$ to $S_b \times (-\delta,\delta)$ will be injective and hence a diffeomorphism onto its image. But it is not hard to see that $\tilde{\Phi}^{(b)}$ is actually injective on all of $C_{b-(\alpha(b)+\delta)}$. Indeed, $\frac{d}{ds} r(\gamma(s)) \geq a_0/4 > 0$ along any integral curve $\gamma$ of $\tilde{X}$, so each point in the image of $\tilde{\Phi}^{(b)}$ lies on an integral curve which intersects $S_b$ in exactly one point. Following each point in the image along an integral curve to $S_b$ thus associates the point with a unique radial translation $t$ and a unique $(\theta,\sigma)$ such that $(\theta,\sigma,b) \in S_b$.

Now define
\[
\Phi^{(b)}(\theta,\sigma,2\ln(s/b)) \quad \text{for all } (\theta,\sigma,s) \text{ such that } (\theta,\sigma,2\ln(s/b)) \in \mathcal{O}_b.
\]
Then
\[
d\Phi^{(b)}_{(\theta,\sigma,s)} \left( \frac{s}{2} \frac{\partial}{\partial s} \right) = \tilde{X}(\Phi^{(b)}(\theta,\sigma,s)), \quad \Phi^{(b)}|_{S_b} = \text{Id}|_{S_b}
\]
and $\Phi^{(b)}$ is a diffeomorphism onto its image.

Now we consider the distortion of distance under $\Phi^{(b)}$. Fix $(\theta,\sigma) \in S^k \times S^{n-k-1}$. For all $s$ such that $\gamma^{(b)}(s) = \Phi^{(b)}(\theta,\sigma,s)$ is well-defined, we have from Proposition 2.2 that $r^{(b)}(s) = r(\gamma^{(b)}(s))$ satisfies
\[
\frac{d}{ds} \left( \frac{r^{(b)}(s)}{s} \right) = -\frac{r^{(b)}(s)}{s^2} + \frac{2}{s^2} \left\langle \tilde{X}, \frac{\partial}{\partial r} \right\rangle_{\gamma^{(b)}(s)} = \frac{2}{s^2} \left\langle E, \frac{\partial}{\partial r} \right\rangle_{\gamma^{(b)}(s)}.
\]
Integrating from $s$ to $b$, we find that
\[
\left| \frac{r^{(b)}(s)}{s} - 1 \right| \leq c \left| \int_s^b \frac{1}{t^2} dt \right|
\]
for some $c$ independent of $\theta,\sigma$, and, in particular, that
\[
-A \leq r^{(b)}(s) - s \leq A
\]
for all $s \leq b$ such that $\gamma^{(b)}(s)$ is defined. But $\gamma^{(b)}(s)$ will be defined at least as long as $r^{(b)}(s) > a_0$, and, so, at least for all $s > a_0 + c$. Choose $a_1 = 2(a_0 + c)$. Then $\Phi^{(b)}$ will be defined on $C_{a_1}$ and (A.9) says that, for $b \geq a_1$,
\[
r^{(b)}(a_1) \geq a_1 - c = 2a_0 + c > \frac{a_1}{2}.
\]
Consequently, $\Phi^{(b)}(C_{a_1}) \subset C_{a_1/2}$. Similarly,
\[
r^{(b)}(a_1) \leq a_1 + c \leq 2a_0 + 3c < 2a_1,
\]
so $C_{2a_1} \subset \Phi^{(b)}(C_{a_1})$. For $b, s \geq a_1$, we will also have
\[
\frac{s}{2} \leq a_0 + \frac{s}{2} \leq s - c \leq r^{(b)}(s) \leq s + c \leq 2s,
\]
which is (A.5). We may then estimate $|E \circ \Phi^{(b)}| \leq C_I r^{-1} \leq C_I 2^I s^{-I}$. Returning to (A.8) with this bound and integrating again along arbitrary paths with fixed $\theta,\sigma$ we obtain (A.6).
The estimate \([A.7]\) is proven in the same way. Fix \((\theta, \sigma) \in S^k \times S^{n-k-1}\) and \(s_0 \geq a_1\) and let \(p(s) = \pi_{s_0} \circ \Phi^{(b)}(\theta, \sigma, s)\). For any \(s\), we have
\[
p'(s) = d\pi_{s_0} \circ d\Phi^{(b)} \left( \frac{\partial}{\partial s} \right) = \frac{2}{s} d\pi_{s_0}(X(p(s))) = \frac{2}{s} d\pi_{s_0}(E(p(s))),
\]
while, by estimate \([A.6]\) above, we have \(|E(p(s))| \leq C_s s^{-l}\) for all \(l \geq 0\) for some \(C_l\) independent of \(\theta\) and \(\sigma\). But this is enough, since
\[
|d\pi_{s_0}(E(p(s)))|_{g_{s_0}} \leq \frac{s_0}{s} |E(p(s))|,
\]
and so
\[
d_{S_0}\left( (\theta, \sigma, s_0), \pi_{s_0} \circ \Phi^{(b)}(\theta, \sigma, s_0) \right) = d_{S_0}(p(b), p(s_0)) \leq \left| \int_{s_0}^b |p'(t)|_{g_{s_0}} \, dt \right|
\]
\[
\leq C \left| \int_{s_0}^b \frac{1}{t^{l+1}} \, dt \right|,
\]
and \([A.7]\) follows. \(\Box\)

A.2. Analysis of an associated system of ODE. Next we seek uniform derivative estimates to extract a limit from the family of maps \(\Phi^{(b)}\) as \(b \to \infty\). By \([A.6]-[A.7]\), these estimates can be obtained by an analysis of the local coordinate representations of \(\Phi^{(b)}\) relative to a fixed finite atlas on \(C_n\). Each of these coordinate representations will satisfy a system of equations with a common structure reflecting the infinite order agreement of \(\tilde{X}\) and \(X\) at spatial infinity. We analyze a general version of this system now.

Consider solutions
\[
\psi : U \times (s_0, \infty) \to W \subset \mathbb{R}^{n-1}, \quad r : U \times (s_0, \infty) \to (s_1, \infty),
\]
to the system
\[
\begin{align*}
\frac{\partial \psi}{\partial s} &= \frac{2}{s} E_{\psi}(\psi, r), \quad \psi(x, b) = x, \\
\frac{\partial r}{\partial s} &= \frac{r}{s} + \frac{2}{s} E_r(\psi, r), \quad r(x, b) = b,
\end{align*}
\]
where \(U \subset \mathbb{R}^{n-1}, V \subset \mathbb{R}^{n-1}\) are open sets and \(E = (E_{\psi}, E_r) : W \times (r_0, \infty) \to \mathbb{R}^n\) satisfies
\[
[A.11] \quad \left| \frac{\partial^{\mu+\nu} E}{\partial y^{\mu} \partial r^{\nu}} \right| (y, r) \leq \frac{C(\mu, \nu)}{r^l}
\]
for all \(l, p \geq 0\) and all multiindices \(\mu = (\mu_1, \ldots, \mu_{n-1})\).

Here in this subsection (and only here) we will write \(\Phi(x, s) = (\psi(x, s), r(x, s))\) and use \(\langle \cdot, \cdot \rangle\) and \(|\cdot|\) to denote the standard Euclidean inner product and norm on \(\mathbb{R}^n\). The collision of notation is intentional: in our eventual application, the neighborhoods \(U\) and \(V\) will correspond to the images of charts on coordinate neighborhoods of \(S^k \times S^{n-k-1}\). The maps \(\Phi\) and \(E\) will correspond to the coordinate representations of \(\Phi^{(b)}\) (for fixed \(b\)) and \(E\) relative to the associated charts on \(C\).

Our goal is to derive estimates on \(\Phi\) from this system on compact subsets of \(U \times (s_0, \infty)\) which are independent of \(b\).
Proposition A.2. For all \( k, l \geq 0 \), there is a constant \( C = C(k, l) \) independent of \( b \) such that

\[
\sup_{V \times (s_0, b]} s^t \left| \frac{\partial^{\mu+p}}{\partial x^p \partial s^q} (\Phi - \text{Id}) \right| (x, s) \leq C(k, l)
\]

for all \( \mu \) and \( p \geq 0 \) such that \( |\mu| + p = k \).

Proof. Let \( V \) be a precompact open set with with \( \overline{V} \subset U \). Fix \( x \in V \). Then

\[
\frac{\partial}{\partial s} \left( \frac{r(x, s)}{s} \right) = \frac{2}{s^2} E_r(\psi(x, s), r(x, s))
\]

so, using the bound \( |E(\psi, r)| \leq C \) we have

\[
1 - \frac{r(x, s)}{s} \leq C \int_s^b \frac{1}{t^2} dt \leq \frac{C}{s}
\]

and hence that \( |r(x, s) - s| \leq C \) for any \( x \) and any \( s_0 < s \leq b \).

For all \( s \) sufficiently large, we will also have that \( s/2 \leq r(x, s) \leq 2s \). Hence, for each \( l \), there is \( C_l \) such that \( |E(\psi(x, s), r(x, s))| \leq C_l s^{-l} \). Returning to (A.13), then we can estimate

\[
1 - \frac{r(x, s)}{s} \leq \int_s^b \frac{2}{t^2} |E(\psi(x, t), r(x, t))| dt \leq C_l \int_s^b \frac{1}{t^{l+2}} dt \leq \frac{C_l}{s^{l+1}}
\]

and hence that \( |r(x, s) - 1| \leq C_l s^{-l} \). Using now that \( r \) and \( s \) are comparable, we obtain similarly that

\[
|\psi(x, s) - x| \leq \int_s^b \frac{2}{t} |E(\psi(x, t), r(x, t))| dt \leq \frac{C_l}{s}.
\]

Now we estimate the first derivatives of \( \Phi \). Fix some \( l \geq 0 \). Above we have already estimated that

\[
\left| \frac{\partial \psi}{\partial s} \right| = \frac{2}{s} |E(\psi, r)| \leq \frac{C_l}{s^{l+1}}, \quad \left| \frac{\partial r}{\partial s} - 1 \right| \leq \left| \frac{r}{s} - 1 \right| + \frac{2}{s} |E_r(\psi, r)| \leq \frac{C_l}{s^{l+1}}.
\]

For the \( x \)-derivatives, it will be convenient to introduce the map

\[
F = \rho_x \circ \Phi : \mathbb{R}^{n-1} \times (s_0, \infty) \to \tilde{V} \times (0, \infty)
\]

where \( \rho_x(x, r) = (x, \lambda r) \), i.e., \( F(x, s) = (\psi(x, s), r(x, s)/s) \). Fix \( 1 \leq i \leq n-1 \). Then

\[
\frac{\partial}{\partial s} \frac{\partial F}{\partial x^i} = \frac{\partial}{\partial x^i} \left( \frac{2}{s} E_x \circ \Phi, \frac{2}{s^2} E_r \circ \Phi \right) = \frac{2}{s} (d\rho_x \circ dE) \frac{\partial \Phi}{\partial x^i}
\]

\[
= \frac{2}{s} (d\rho_x \circ dE \circ d\rho_s) \frac{\partial F}{\partial x^i}.
\]

Now, the matrix-valued function

\[
A = \frac{2}{s} (d\rho_x \circ dE \circ d\rho_s) = \begin{pmatrix} \frac{\partial E^a}{\partial x^r} & \frac{\partial E^a}{\partial y^r} \\ \frac{\partial E^a}{\partial x^r} & \frac{\partial E^a}{\partial y^r} \end{pmatrix}
\]

satisfies \( |A| \leq C_l s^{-(l+1)} \) for all \( l \), so the function \( \phi = \left| \frac{\partial F}{\partial x^r} - e_i \right|^2 \) satisfies

\[
\frac{\partial \phi}{\partial s} = 2 \left( A \frac{\partial F}{\partial x^r} + \frac{\partial E}{\partial y^r} \right) - 2 \left( A e_i \frac{\partial F}{\partial x^r} - e_i \right) \geq -3 |A| \phi - |A|.
\]
Fix $s_0 < s_1 \leq b$. Then, there is $C$ depending only on $l$ such that
\[
\frac{\partial \phi}{\partial s} \geq -C I s^{-2} (\phi + s_1^{-2l})
\]
for any $x$ and all $s \geq s_1$. Integrating from $s_1$ to $b$ yields
\[
\ln \left( \frac{\phi(x, b) + s_1^{-2l}}{\phi(x, s_1) + s_1^{-2l}} \right) \geq \frac{Cl}{b} - \frac{Cl}{s_1}
\]
which, since $\phi(x, b) = 0$, means that
\[
\phi(x, s_1) \leq e^{\frac{Cl}{s_1}} - e^{\frac{Cl}{b}} s_1^{-2l} \leq C l s_1^{-2l},
\]
where $C$ is independent of $s_1$. Since $s_1$ was arbitrary, we have
\[
\left| \frac{\partial \phi}{\partial x} - \delta \right| + \frac{1}{s} \left| \frac{\partial r}{\partial x} \right| \leq \frac{C l}{s}
\]
for all $s$, and the desired estimate follows.

The higher derivatives may be estimated similarly. We will give here the details only for the case $k = 2$. Fix again $l \geq 0$. From above, we have already seen that
\[
\frac{\partial^2 r}{\partial s^2} = \frac{\partial}{\partial s} \left( \frac{x + 2 s E_r}{s} \right) = \frac{2}{s} dE_r \frac{\partial \Phi}{\partial s},
\]
and
\[
\frac{\partial^2 \psi}{\partial s^2} = \frac{\partial}{\partial s} \left( \frac{2}{s} E_{\psi} \right) = -\frac{2}{s} E_{\psi} + \frac{2}{s} dE_{\psi} \frac{\partial \Phi}{\partial s},
\]
so
\[
\left| \frac{\partial^2 \psi}{\partial s^2} \right| + \left| \frac{\partial^2 r}{\partial s^2} \right| \leq \frac{C l}{s}
\]
for some $C_l$. Similarly,
\[
\left| \frac{\partial^2 r}{\partial x^i \partial s} \right| \leq \frac{1}{s} \left| \frac{\partial r}{\partial x^i} \right| + \frac{2}{s} \left| dE_{r} \frac{\partial \Phi}{\partial x^i} \right| \leq C l s^{-l},
\]
and
\[
\left| \frac{\partial^2 \psi}{\partial x^i \partial s} \right| \leq \frac{2}{s} \left| dE_{\psi} \frac{\partial \Phi}{\partial x^i} \right| \leq C l s^{-l}
\]
for any $i$.

For the pure $x$-derivatives, we again use the map $F$ and compute that
\[
\frac{\partial}{\partial s} \frac{\partial^2 F}{\partial x^i \partial x^j} = \frac{2}{s} (d\rho_{s} \circ dE \circ d\rho_{s}) \frac{\partial^2 F}{\partial x^i \partial x^j} + \frac{2}{s} \left( d\rho_{s} \circ d^2 E \right) \left( d\rho_{s} \frac{\partial F}{\partial x^i}, d\rho_{s} \frac{\partial F}{\partial x^j} \right)
\]
for any $i$ and $j$. Fixing any $x$ and integrating from $s$ to $b$, we may estimate as in the previous lemma that
\[
\left| \frac{\partial^2 F}{\partial x^i \partial x^j}(x, s) \right| \leq \frac{C l}{s^l}
\]
using that
\[
\frac{\partial^2 F}{\partial x^i \partial x^j}(x, b) = 0.
\]
The desired estimate on $\frac{\partial^2 \Phi}{\partial x^i \partial x^j}$ follows immediately. \qed
A.3. Convergence to a limit diffeomorphism. Now we are ready to extract a limit as \( b \to \infty \) from the family \( \Phi^{(b)} \) of local diffeomorphisms constructed in Proposition A.1. We first fix a finite coordinate atlas in order to import the estimates we have proven in the previous section to the cylinder.

It follows from the distance estimates (A.7) that we can cover \( S^k \times S^{n-k-1} \) by a finite collection \( \{ U^i_3 \}_{i=1}^N \) of products \( U^i_3 = B_3^k(p_i) \times B_3^{n-k-1}(q_i) \) of coordinate balls of radius \( \delta \) less than one fourth the injectivity radii of \( S^k \) and \( S^{n-k-1} \) with the property that

\[
\Phi^{(b)}(U^i_3 \times (a_2, \infty)) \subset U^i_{4\delta} \times (a_2/2, \infty)
\]

for all \( a_2 \geq a_1 \) sufficiently large (depending on \( \delta \)) and all \( b \geq a_2 \). Write \( \tilde{U}^i = U^i_3 \times (a_2, \infty) \) and \( \tilde{W}^i = U^i_{4\delta} \times (a_2/2, \infty) \) and consider the corresponding atlases \( \{(\tilde{U}^i, \tilde{\varphi}^i)\}_{i=1}^N \) and \( \{(\tilde{W}^i, \tilde{\varphi}^i)\}_{i=1}^N \) of \( C_{a_2} \) and \( C_{a_2'/2} \). Here we use \( \tilde{\varphi}^i \) to represent both the map \( \exp_{p_i}^{-1} \times \exp^{-1}_{q_i} \) on \( \tilde{W}^i \) and its restriction to \( \tilde{U}^i \).

Passing to the coordinate representations \( \tilde{\varphi}^i \circ \Phi^{(b)} \circ (\tilde{\varphi}^i)^{-1} \) and \( d\tilde{\varphi}^i(E) \circ (\tilde{\varphi}^i)^{-1} \) of \( \Phi^{(b)} \) and \( E \) (which we will continue to denote by the same symbols) we obtain a system of the form (A.10) on \( \tilde{\varphi}^i(\tilde{U}^i) \) with the bounds (A.11) for some \( C \) depending on \( U^i \); these bounds follow from (A.1) since the coordinate representation of \( g \) on \( \mathbb{R}^n \) satisfies

\[
C^{-1} \delta_{jk} \leq g_{jk}(y, s) \leq Cs^2 \delta_{jk}
\]

on \( \tilde{U}^i \) for some \( C > 0 \) depending only on \( i \), and we have bounds of the form

\[
|g^{(m)}(y, s)|_{jk} \leq C(i, m) \text{ on } \tilde{U}^i \text{ for all } m \geq 0.
\]

Here \( y = (\theta, \sigma) \).

From Proposition A.2 we obtain that, for fixed \( i \) and \( a_2 < s_1 < s_2 \), the \( C^k \)-norms of the coordinate representation of \( \Phi^{(b)} - \text{Id} \) are uniformly bounded on the compact set \( K = \overline{U}^i_3 \times [s_1, s_2] \subset \tilde{U}^i \) for each \( k \geq 0 \). From the Ascoli-Arzela theorem, then, there is a sequence \( b_j \to \infty \) such that \( \Phi^{(b_j)} \) converges in every \( C^k \)-norm to a smooth map \( \Phi^{(\infty)}_K \) on \( K \). Covering the annular regions \( A_j = A_{a_2'+1/j, a_2} \) by finitely many of the charts from this atlas, we can obtain a smooth limit \( \Phi^{(\infty)}_j \) on \( A_j \) for each \( j \); taking a further subsequence, we obtain a smooth limit \( \Phi = \Phi^{(\infty)} \) defined on all of \( C_{a_2} \). We record this statement and some additional observations in the following proposition.

**Proposition A.3.** Let \( a_2 \) be as in the discussion above. There exists \( a_3 \geq a_2 \) and a sequence \( b_j \to \infty \) such that \( \Phi^{(b_j)} \) converges locally smoothly as \( j \to \infty \) to a smooth map \( \Phi : \mathcal{C}_{a_3} \to \mathcal{C}_{a_3'/2} \) satisfying

(a) \( d\Phi_{(\theta, \sigma, s)}(X(\theta, \sigma, s)) = \dot{X} \circ \Phi(\theta, \sigma, s) \),

(b) \( \Phi \) is a diffeomorphism onto its image and \( \mathcal{C}_{2a_3} \subset \Phi(\mathcal{C}_{a_3}) \),

(c) On each coordinate neighborhood \( U = U^i_3 \) defined above, and for each \( k \), \( l \geq 0 \), there is \( C = C(i, k, l) \) such that, for all \( s > a_3 \),

\[
A.14 \quad |s^l \left\{ \| \Phi - \text{Id} \|_{C^k(\mathbb{R}^n \times [s, 2s])} + \| \Phi^s g - g \|_{C^k(\mathbb{R}^n \times [s, 2s])} \right\} \leq C
\]

relative to the Euclidean norm and connection.

**Proof.** We will assume for now just that \( a_3 \geq a_2 \) and prescribe a lower bound for its value as part of the argument. The identity in (a) follows from the \( C^1 \)-convergence of \( \Phi^{(b_j)} \) and (A.3). The second claim in (b) follows from (A.4), and the estimate on the first term in (A.14) in part (c) follows from Proposition A.2 and the discussion preceding the statement of this proposition. In particular, we
can choose $a_3$ sufficiently large so that $(1/2) \text{Id} \leq d\Phi \leq 2 \text{Id}$ on $U^i \times [a_3, \infty)$ for each $i$. In particular, $\Phi$ will be a local diffeomorphism on $C_{a_3}$. The argument that $\Phi$ is injective, goes then just as the corresponding argument for $\Phi^{(b)}$ in Proposition A.1. Here, as there, $r(s)$ is strictly increasing along the radial lines $s \mapsto (\theta(s), \sigma(s))$, and $\Phi$ is a diffeomorphism when restricted to $S_l \times (t - \epsilon, t + \epsilon)$ for some sufficiently large $l$ and sufficiently small $\epsilon$. Following the radial lines forward and backward as in the proof of Proposition A.1 we see that $\Phi$ must be injective on $C_{a_3}$, and hence a diffeomorphism onto its image. Using the $C^0$-comparison of $r \circ \Phi$ with $s$, we can also enlarge $a_3$ if necessary to ensure that $\Phi(C_{a_3}) \subset C_{a_3/2}$ and $C_{2a_3} \subset \Phi(C_{a_3})$.

Finally, the $C^k$ estimates on $\Phi^*g - g$ in (A.15) follow from the uniform estimates we have on the derivatives of the coordinate representations of $\Phi - \text{Id}$ and the metric $g$ on the neighborhoods $\tilde{U}^i$.

\section*{A.4. Proof of Theorem 2.5}

Now we assemble the proof of Theorem 2.5. Taking $r_1 = a_3$ (and recalling that $a_3 \geq a_0 \geq 2r_0$), Proposition A.3 gives us the existence of a map $\Phi : C_{r_1} \to C_{r_1/2} \subset C_0$ satisfying that $d\Phi(X) = X \circ \Phi$ and $C_{2r_1} \subset \Phi(C_{r_1})$. Moreover (patching together estimates using the local bounds on the Christoffel symbols), part (c) of that proposition ensures that

$$\sup_{C_{r_1}} s^l |\nabla^{(m)}(\Phi^*g - g)| < \infty$$

for all $l \geq 0$. Writing $\tilde{g} = \Phi^*g$ and $\tilde{\nabla}$ for the connection of $\tilde{g}$, we thus have

(A.15) $$\sup_{C_{r_1}} s^l |\nabla^{(m)}(\tilde{\nabla} - \Gamma)| < \infty,$$

and, consequently,

(A.16) $$\sup_{C_{r_1}} s^l |\nabla^{(m)}(\tilde{g} - g)| < \infty,$$

for all $l$ and $m$.

But then, for all $l$,

$$|\Phi^*\tilde{g} - g| \leq |\Phi^*\tilde{g} - \tilde{g}| + |\tilde{g} - g| \leq C|\Phi^*\tilde{g} - \tilde{g}| + |\tilde{g} - g|$$

$$= C|\tilde{g} - g| \circ \Phi + |\tilde{g} - g| \leq C_1s^{-l}$$

for some $C_1$, using that both $\tilde{g}$ and $\tilde{g}$ are strongly asymptotic to $g$ and that $r$ and $s$ are comparable. We can then proceed inductively, using (A.15) and (A.16) to estimate the covariant derivatives of $\tilde{g} - g$. For example, since

$$|\nabla(\Phi^*\tilde{g} - g)| \leq C|\tilde{\nabla} - \Gamma||\Phi^*\tilde{g} - g| + |\nabla(\Phi^*\tilde{g} - g)|$$

$$\leq C|\tilde{\nabla} - \Gamma||\Phi^*\tilde{g} - g| + C|\nabla(\Phi^*\tilde{g} - g)| + |\nabla(\tilde{g} - g)|$$

$$= C|\tilde{\nabla} - \Gamma||\Phi^*\tilde{g} - g| + C|\nabla(\tilde{g} - g)| \circ \Phi + |\nabla(\tilde{g} - g)|,$$

we see that we have a bound of the form $|\nabla(\Phi^*\tilde{g} - g)| \leq C_1s^{-l}$ for all $l$. We can argue similarly for the higher derivatives. This completes the proof.

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