Induced star arboricity and injective edge-coloring of graphs

Baya Ferdjallah$^{1,4}$, Samia Kerdjoudj$^{1,5}$, and André Raspaud$^{2}$

$^1$LIFORCE, Faculty of Mathematics, USTHB, BP 32 El-Alia, Bab-Ezzouar 16111, Algiers, Algeria
$^2$LaBRI, Université de Bordeaux, 351 cours de la Libération, 33405 Talence Cedex, France
$^4$Université de Boumerdès, Avenue de l’indépendance, 35000, Boumerdès, Algeria
$^5$Université de Blida 1, Route de Soumâa BP 270, Blida, Algeria

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Abstract

The induced star arboricity (isa($G$)) of a graph $G$ is a new parameter introduced in 2019 by Axenovich et al.\cite{axenovich2019induced}, it is defined as the smallest number of induced star-forests covering the edges of $G$. An injective edge-coloring $c$ of a graph $G$ is an edge coloring such that if $e_1$, $e_2$ and $e_3$ are three consecutive edges in $G$ (they are consecutive if they form a path or a cycle of length three), then $e_1$ and $e_3$ receive different colors. The minimum integer $k$ such that $G$ has an injective $k$-edge-coloring is called injective chromatic index of $G$ ($\chi'_{inj}(G)$). This parameter was introduced in 2015 by Cardoso et al.\cite{cardoso2015injective} motivated by the Packet Radio Network problem. They proved that computing $\chi'_{inj}(G)$ of a graph $G$ is NP-hard and they gave the injective chromatic index of some classes of graphs. In our paper we first prove that for any graph $G$ we have isa($G$)$=\chi'_{inj}(G)$. We give some new upper bounds for this last parameter and we prove that if $G$ is a subcubic graph with maximum average degree less than $\frac{16}{7}$ (resp. $\frac{8}{5}$, 3), then $G$ admits an injective edge-coloring with at most 4 (resp. 6, 7) colors. Moreover, we establish a tight upper bound for subcubic outerplanar graphs. For this purpose, we do not use the discharging method but rather we use the fundamental structural properties of the subcubic graphs and outerplanar subcubic graphs. All our results can be immediately translated in terms of induced star arboricity.

1 Introduction

All the graphs we consider are finite and simple. For a graph $G$, we denote by $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ its vertex set, edge set, minimum degree and maximum degree,
respectively.

A proper vertex (respectively, edge) coloring of $G$ is an assignment of colors to the vertices (respectively, edges) of $G$ such that no two adjacent vertices (respectively, edges) receive the same color.

Three edges $e_1$, $e_2$ and $e_3$ in a graph $G$ are consecutive if they form a path (in this order) or a cycle of length three. An injective edge-coloring of a graph $G = (V, E)$ is a coloring $c$ of the edges of $G$ such that if $e_1$, $e_2$ and $e_3$ are consecutive edges in $G$, then $c(e_1) \neq c(e_3)$. In other words, every two edges at distance exactly 2 or belonging to a triangle do not use the same color. The injective chromatic index of $G$, denoted by $\chi'_{inj}(G)$, is the minimum number of colors needed for an injective edge-coloring of $G$. Note that an injective edge-coloring is not necessarily a proper edge coloring. This notion was introduced in 2015 by Cardoso et al. [12] motivated by the Packet Radio Network problem. They proved that computing $\chi'_{inj}(G)$ of a graph $G$ is NP-hard and they gave the injective chromatic index of some classes of graphs (path, cycle, wheel, Petersen graph and complete bipartite graph).

**Proposition 1.1** [12] Let $P_n$ (resp. $C_n$) be a path (resp. a cycle) with $n$ vertices. Then

1. $\chi'_{inj}(P_n) = 2$, for $n \geq 4$.

2. $\chi'_{inj}(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \text{(mod} 4) \\ 3 & \text{otherwise.} \end{cases}$

For the case of tree, they gave the following bound:

**Proposition 1.2** [12] For any tree of order $n \geq 2$, $1 \leq \chi'_{inj}(G) \leq 3$.

In 2019, Axenovich et al. [3] introduced the notion of star arboricity of a graph $G$, denoted by $\text{isa}(G)$, it is defined as the smallest number of induced star-forests covering the edges of $G$. They prove that if $F$ is the class of planar graphs, then $18 \leq \text{isa}(F) \leq 30$. Actually, this work has been done in the more general setting of induced arboricity of graphs. The induced arboricity of a graph $G$, denoted by $\text{ia}(G)$, is the smallest number $k$ such that the edges of $G$ can be covered with $k$ induced forests in $G$.

The authors provide bounds on $\text{ia}(F)$ when $F$ is the class of planar graphs, the class of $d$-degenerate graphs, or the class of graphs having tree-width at most $d$. They prove that if $F$ is the class of planar graphs, then $8 \leq \text{ia}(F) \leq 10$. They establish similar results for induced star arboricities of classes of graphs.

**Proposition 1.3** Let $G$ be a graph then $\chi'_{inj}(G) = \text{isa}(G)$.

**Proof.**

1. We fist prove that $\chi'_{inj}(G) \leq \text{isa}(G)$. Assume that $\text{isa}(G) = k$ and that we have the $k$ induced star-forests $\{F_1, \cdots, F_k\}$ covering the edges of $G$. We define a coloring $c$ of the edges of $G$ as follows: we color the edges of the induced star-forest $F_i$ with the color $i$ (for $i \in \{1, \cdots, k\}$). It easy to see that if three edges $e_1$, $e_2$ and $e_3$ of the graph $G$ are consecutive then $c(e_1) \neq c(e_3)$. Hence, $c$ is an injective edge-coloring of $G$. We deduce that: $\chi'_{inj}(G) \leq \text{isa}(G)$. 


2. Now we prove that $\text{isa}(G) \leq \chi'_{\text{inj}}(G)$. Assume that $G$ has an injective edge-coloring with $k$ colors. The graph induced by each color class $i$ ($i \in \{1, \cdots, k\}$) is an induced star forest $F_i$, because of the condition that if three edges $e_1$, $e_2$ and $e_3$ in a graph $G$ are consecutive then $c(e_1) \neq c(e_3)$. Then the $k$ obtained induced star-forests cover the edges of $G$. We deduce that: $\text{isa}(G) \leq \chi'_{\text{inj}}(G)$.

In what follows we will use the terminology of injective edge-coloring, it is easier to handle colors.

A vertex of degree $k$ is called a $k$-vertex, and a $k$-neighbor of a vertex $v$ is a $k$-vertex adjacent to $v$. The maximum average degree of a graph $G$, denoted $\text{mad}(G)$, is defined to be the maximum average over all subgraphs $H$ of $G$, i.e $\text{mad}(G) = \max \{ \frac{2|E(H)|}{|V(H)|} : H \subseteq G \}$. We say a graph $G$ is subcubic if $\Delta(G) \leq 3$.

In 2018, Bu and Qi [9] studied the injective chromatic index of graphs with maximum degree 3 and bounded maximum average degrees.

**Theorem 1.4** [9] Let $G$ be a subcubic graph.

1. If $\text{mad}(G) < \frac{5}{2}$ then $\chi'_{\text{inj}}(G) \leq 5$.
2. If $\text{mad}(G) < \frac{18}{7}$ then $\chi'_{\text{inj}}(G) \leq 6$

We improve the result of Theorem 1.4. In particular, we show that

**Theorem 1.5** Let $G$ be a subcubic graph.

1. If $\text{mad}(G) < \frac{16}{7}$ then $\chi'_{\text{inj}}(G) \leq 4$.
2. If $\text{mad}(G) < \frac{8}{3}$ then $\chi'_{\text{inj}}(G) \leq 6$
3. If $\text{mad}(G) < 3$ then $\chi'_{\text{inj}}(G) \leq 7$

Since for any planar graph $G$ with girth at least $g$, we have $\text{mad}(G) < \frac{2g}{g-2}$, the following corollary, can be easily derived from Theorem 1.5.

**Corollary 1.6** Let $G$ be a subcubic planar graph with a girth $g$.

1. If $g \geq 16$ then $\chi'_{\text{inj}}(G) \leq 4$.
2. If $g \geq 8$ then $\chi'_{\text{inj}}(G) \leq 6$.
3. If $g \geq 6$ then $\chi'_{\text{inj}}(G) \leq 7$

We have also study the injective chromatic index of subcubic outerplanar graph. An outerplanar graph is a graph with a planar drawing for which all vertices belong to the outer face of the drawing.

We prove the following result:

**Theorem 1.7** If $G$ is an outerplanar graph with maximum degree $\Delta(G) = 3$, then $\chi'_{\text{inj}}(G) \leq 5$. Moreover, the bound is tight.
2 General upper bound of $\chi'_{inj}(G)$

A strong $k$-edge-coloring of a graph $G$ is an assignment of $k$ colors to the edges of $G$ in such a way that any two edges at distance at most two are assigned distinct colors. The minimum number of colors for which a strong edge coloring of $G$ exists is the strong chromatic index of $G$, denoted $\chi'_s(G)$. This notion was introduced by Fouquet and Jolivet \[15, 16\]. In 1985, Erdős and Nesetril conjectured that the strong chromatic index of a graph is at most $\frac{5}{2}\Delta^2(G)$ if $\Delta(G)$ is even and $\frac{1}{4}(5\Delta^2(G) - 2\Delta(G) + 1)$ if $\Delta(G)$ is odd. Andersen \[1\] proved the conjecture for $\Delta(G) = 3$. When $\Delta$ is large enough, Molloy and Reed \[21\] showed that $\chi'_s(G) \leq 1.998\Delta^2(G)$. This result was improved by Bruhn and Joos \[8\] who proved $\chi'_s(G) \leq 1.93\Delta^2(G)$ for $\Delta$ large enough. Let us give an immediate remark : for any graph $G$, we have :

$$\chi'_{inj}(G) \leq \chi'_s(G)$$

An alternate way of looking at the injective chromatic index of a graph $G$ is to consider the graph $G^{(s)}$ obtained from $G$ as follow : $V(G^{(s)}) = E(G)$ and two vertices of $G^{(s)}$ are adjacent if the edges of $G$ corresponding to these two vertices of $G^{(s)}$ are at distance 2 in $G$ or in a triangle. Then,

$$\chi'_{inj}(G) = \chi(G^{(s)}) \quad (1)$$

We recall that, by Proposition \[11\] if $C_n$ is a cycle with $n$ vertices then $\chi_{inj}(C_n) = 2$ if $n \equiv 0 \mod 4$ and 3 otherwise.

**Theorem 2.1** Let $G$ be a graph which is not a cycle with any maximum degree $\Delta(G)$. Then $\chi'_{inj}(G) \leq 2(\Delta(G) - 1)^2$.

**Proof.** Let $G = (V(G), E(G))$ be a graph with maximum degree $\Delta(G)$. Consider the graph $G^{(s)}$ defined above.

$G^{(s)}$ has a maximum degree at most $2(\Delta(G) - 1)^2$. According to the Brook’s theorem $2(\Delta(G) - 1)^2$ colors are enough to properly color the vertices of $G^{(s)}$, except the case where $G^{(s)}$ is a complete graph or an odd cycle. If $G^{(s)}$ is not a complete graph or an odd cycle the proof is done. $G^{(s)}$ is an odd cycle (or an union of disjoint odd or even cycles) then $G$ is a cycle witch is not possible by hypothesis. Now if $G^{(s)}$ is a complete graph with degree $2(\Delta(G) - 1)^2$, then $G$ has $2(\Delta(G) - 1)^2 + 1$ edges. Consider a vertex $v$ of $G$ with degree $\Delta(G)$, each two incident edges to $x$ must be in a same triangle then $G$ is a complete graph with degree $\Delta(G)$. A contradiction.

**Corollary 2.2** If $G$ is a subcubic graph then $\chi'_{inj}(G) \leq 8$.

2.1 Injective chromatic index and acyclic chromatic number

A proper vertex coloring of a graph $G$ is acyclic if there is no bicolored cycle in $G$. In other words, the union of any two color classes induces a forest. The acyclic chromatic number, denoted by $\chi_a(G)$, of a graph $G$ is the smallest integer $k$ such that $G$ has an acyclic $k$-coloring. This notion was introduced by Grünbaum \[17\] in 1973 and studied by Mitchem \[20\], Albertson and Berman \[2\] and Kostochka \[18\]. In 1979, Borodin \[6\] confirmed the conjecture of Grünbaum by proving
Theorem 2.3  Every planar graph is acyclically 5-colorable.

In Theorem 2.4 we give an upper bound of $\chi'_{\text{inj}}(G)$ in term of $\chi_a(G)$ (see also [3]).

Theorem 2.4 Let $G$ be a graph with $\chi_a(G) = k$. Then $\chi'_{\text{inj}}(G) \leq \frac{3k(k-1)}{2}$.

Proof. Let $G$ be a graph and $\phi$ an acyclic coloring of $G$ with $k$ colors. Consider $H_{i,j}$ the graph induced by the union of two color classes $i$ and $j$ in $\phi$, for $i \neq j$ and $i, j \in \{1, 2, \cdots, k\}$. Note that $H_{i,j}$ is a forest. By Proposition 1.2, there exist an injective edge-coloring of $H_{i,j}$ with 3 colors. After coloring each of the $\frac{k(k-1)}{2}$ graphs $H_{i,j}$, for $i \neq j$ and $i, j \in \{1, 2, \cdots, k\}$, we obtain an injective edge-coloring of $G$ with $\frac{3k(k-1)}{2}$ colors.

□

Theorem 2.3, Theorem 2.4 yields the following (see also [3])

Corollary 2.5 For every planar graph $G$, $\chi'_{\text{inj}}(G) \leq 30$.

Borodin, Kostochka and Woodall [5] proved that if $G$ is a planar graph with a girth $g \geq 5$ (respectively, $g \geq 7$) then $\chi_a(G) \leq 4$ (respectivement, $\chi_a(G) \leq 3$).

Corollary 2.6 For every planar graph $G$ with girth $g$,

1. If $g \geq 5$ then $\chi'_{\text{inj}}(G) \leq 18$.
2. If $g \geq 7$ then $\chi'_{\text{inj}}(G) \leq 9$.

2.2 Injective chromatic index and star chromatic number.

A star coloring of $G$ is a proper vertex coloring of $G$ such that the union of any two color classes induces a star forest in $G$, i.e. every connected component of this union is a star. The star chromatic number of $G$, denoted by $\chi_{st}(G)$, is the smallest integer $k$ for which $G$ admits a star coloring. This notion was first mentioned by Grünbaum [17] in 1973 (see also [14]).

Theorem 2.7 Let $G$ be a graph with $\chi_{st}(G) = k$. Then $\chi'_{\text{inj}}(G) \leq \frac{k(k-1)}{2}$.

Proof. Let $G$ be a graph and $\phi$ a star coloring of $G$ with $k$ colors. Consider $H_{i,j}$ the graph induced by the union of two color classes $i$ and $j$ in $\phi$, for $i \neq j$ and $i, j \in \{1, 2, \cdots, k\}$. Note that $H_{i,j}$ is a star forest. We have an injective edge-coloring of $H_{i,j}$ by coloring every edge of $H_{i,j}$ with the same color. After coloring each of the $\frac{k(k-1)}{2}$ graphs $H_{i,j}$, for $i \neq j$ and $i, j \in \{1, 2, \cdots, k\}$, we obtain an injective edge-coloring of $G$ with $\frac{k(k-1)}{2}$ colors.

□

This last bound is also interesting in terms of induced star arboricity of graphs.

In [10] it is prove that if $G$ has a girth $g \geq 13$ then $\chi_{st}(G) \leq 4$.

Theorem 2.7 yields the following corollary:

Corollary 2.8 If $G$ is a planar graph with girth $g \geq 13$ then $\chi'_{\text{inj}}(G) \leq 6$.
In the reverse direction we have:

**Theorem 2.9** Let $G$ be a graph with $\chi'_{inj}(G) = k$. Then $\chi_{st}(G) \leq 2^k$.

**Proof.** Let $G$ be a graph and $C = \{1, 2, \cdots, k\}$ be a set of $k$ colors. We assume that $G$ is a connected graph and not an isolated edge.
Assume that $G$ has an injective edge-coloring $\phi$ using $C$.
We define a vertex coloring $\psi$ of $G$ as follows.

- For every $v \in V(G)$ and $d_G(v) \geq 2$, let $\psi(v) = (x_1, x_2, \cdots, x_k)$ such that, for every $i \in \{1, \cdots, k\}$,
  
  $$x_i = \begin{cases} 
  1 & \text{if an edge of color } i \text{ is incident with the vertex } v \\
  0 & \text{otherwise}
  \end{cases}$$

- For every $v \in V(G)$ and $d_G(v) = 1$, let $\psi(v) = (0, 0, \cdots, 0)$.

$\psi$ is a proper vertex coloring. Indeed, if $\psi(v) = \psi(u)$, where $uv \in E(G)$ then, there exist two edges $uu_1$ and $vv_1$ in $G$, such that $\phi(uu_1) = \phi(vv_1)$ which contradicts the coloring $\phi$. So it remains to show that there is no 2-colored paths with four vertices in $G$. By contrary, assume that there exist a 2-colored paths with four vertices $uwvx$ in $G$. Then $\psi(u) = \psi(w)$ and by definition of $\phi$, we can deduce that $\phi(uv) = \phi(vw) = \alpha$. In this case, it is easy to see that $\alpha \notin \phi(x)$, where $\phi(x)$ denote the set of colors used on the edges incident with $x$. Hence, $\psi(v) \neq \psi(x)$. We conclude that there is no 2-colored paths with four vertices in $\psi$. Therefore, $\psi$ is a star coloring of $G$. The total number of used colors is at most $2^k$.

**Remark 1** Let $G$ be a graph with minimum degree at least 2. If $\chi'_{inj}(G) = k$ then $\chi_{st}(G) \leq 2^k - 1$.

Since any vertex is incident with a least two colored edges, $\psi$ is not using the color $(0, 0, \cdots, 0)$ ($G$ does not contain vertices of degree 1). The total number of used colors is at most $2^k - 1$.

### 3 Injective chromatic index of bipartite graph

Cardoso et al. [12] considered the injective edge-coloring of bipartite graphs. They showed that

**Proposition 3.1** If $G$ is a bipartite graph with bipartiton $V(G) = V_1 \cup V_2$ and $G$ has no isolated vertices, then $\chi'_{inj}(G) \leq \min\{|V_1|, |V_2|\}$.

The above bound is attained for every complete bipartite graph $K_{p,q}$. We give now an other useful bound in terms of maximums degrees.
Proposition 3.2 If \( G = (V_1 \cup V_2, E) \) is a bipartite graph and \( G \) has no isolated vertices. Then
\[
\chi'_{inj}(G) \leq \begin{cases} 
\min\{\Delta V_1(\Delta V_2 - 1), \Delta V_2(\Delta V_1 - 1)\} + 1 & \text{if } \Delta(G) \geq 3 \\
3 & \text{if } \Delta(G) = 2
\end{cases}
\]
Where \( \Delta V_1 = \max\{d(u); u \in V_1\} \) and \( \Delta V_2 = \max\{d(u); u \in V_2\} \).

Proof. Let \( G = (V_1 \cup V_2, E) \) be a bipartite graph. If \( \Delta(G) = 2 \) then by Proposition 1.1, \( \chi'_{inj}(G) \leq 3 \).

Assume that \( \Delta(G) \geq 3 \). Let \( G' = (V', E') \) be the graph obtained from \( G \) as follows: \( |V'| = |V_1| \) and two vertices of \( G' \) are adjacent if they are at distance two. Hence, the maximum degree of \( G' \) is at most \( \Delta V_1(\Delta V_2 - 1) \). We claim that \( G' \) has a proper vertex coloring \( \phi \) with \( \Delta V_1(\Delta V_2 - 1) \). Indeed, if \( G' \) is not a cycle or a complete graph, then by Brooks theorem \( \chi(G') \leq \Delta V_1(\Delta V_2 - 1) \), otherwise, if \( G' \) is the complete graph \( K_{\Delta V_1(\Delta V_2 - 1)} \) then \( \chi(G') = \Delta V_1(\Delta V_2 - 1) + 1 \). Otherwise, if \( G' \) is an odd cycle, then \( \chi(G') = 3 \leq \Delta V_1(\Delta V_2 - 1) + 1 \).

We define an edge coloring \( \phi' \) of \( G \) with \( \Delta V_1(\Delta V_2 - 1) \) colors as follows: for any edge \( uv \) of \( G \) incident with a vertex \( v \in V_1 \), we set \( \phi'(uv) = \phi(v) \). It easy to see that for any three consecutive edges \( e_1, e_2 \) and \( e_3 \), we have \( \phi'(e_1) \neq \phi'(e_3) \). Hence, \( \phi' \) is an injective edge-coloring of \( G \) and \( \chi_{inj}(G) \leq \Delta V_1(\Delta V_2 - 1) + 1 \).

Similarly, let \( G'' = (V'', E'') \) be the graph obtained from \( G \) as follows: \( |V''| = |V_2| \) and two vertices of \( G'' \) are adjacent if they are at distance two. Hence, the maximum degree of \( G'' \) is at most \( \Delta V_2(\Delta V_1 - 1) \). By a similar above argument, we can deduce that \( \chi'_{inj}(G) \leq \Delta V_2(\Delta V_1 - 1) + 1 \).

We conclude that \( \chi'_{inj}(G) \leq \min\{\Delta V_1(\Delta V_2 - 1), \Delta V_2(\Delta V_1 - 1)\} + 1 \). \( \square \)

We obtain the following easy corollary.

Corollary 3.3 For any bipartite subcubic graph \( G \), we have \( \chi'_{inj}(G) \leq 7 \).

4 Proof of Theorem 1.5

For an edge coloring \( \phi \) of a graph \( G \) and a vertex \( v \in V(G) \), \( \phi(v) \) denotes the set of colors used on the edges incident with \( v \). An edge \( uv \) is weak if at least one of \( u \) and \( v \) is a vertex of degree 1. A vertex \( u \) is weak if at least one of edges incident with \( u \) is weak.

4.1 Proof of Theorem 1.5

Let \( C = \{1, 2, 3, 4\} \) be a set of four colors. Suppose that the Theorem 1.5 is not true. Let \( H \) be a counterexample minimizing \( |E(H)| + |V(H)| \): \( H \) is not injective edge-colorable with four colors, \( \text{mad}(H) < \frac{7}{2} \) and for any edge \( e \), \( \chi'_{inj}(H \setminus e) \leq 4 \).

Lemma 4.1 The minimal counterexample \( H \) satisfies the following properties:

1. \( H \) does not contain a weak 2-vertex.
2. $H$ does not contain a 3-vertex adjacent to two 1-vertices.

Proof.

1. Suppose that $H$ contains a 2-vertex $u$ adjacent to a 1-vertex $u_1$. Let $u_2$ be the second neighbor of $u$. By minimality of $H$, the graph $H' = H \setminus \{u_1u\}$ has an injective edge-coloring $\phi$ using $C$. We can view $\phi$ as a partial injective edge-coloring of $H$. We color $u_1u$ with a color $\alpha$ distinct from colors of edges adjacent to $uu_2$ (we have at most 2 forbidden colors). Then we get an injective edge-coloring of $H$. We get a contradiction.

2. Suppose that $H$ contains a 3-vertex $u$ with $N(u) = \{u_1, u_2, u_3\}$, where $d(u_1) = d(u_2) = 1$. By minimality of $H$, the graph $H' = H \setminus \{u_1u\}$ has an injective edge-coloring $\phi$ using $C$. We can view $\phi$ as a partial injective edge-coloring of $H$. We color $u_1u$ with a color $\alpha$ distinct from colors of edges adjacent to $uu_3$ (we have at most 2 forbidden colors). Then we get an injective edge-coloring of $H$. We get a contradiction.

Let $H^*$ denote the graph obtained from $H$ by deleting all vertices of degree 1. If $H$ does not contain a 1-vertex, then $H^* = H$. Since $H^* \subseteq H$, $\text{mad}(H^*) < \frac{16}{7}$ and by the Lemma 4.1 $H^*$ does not contain a 1-vertex.

Lemma 4.2 $H^*$ satisfies the following properties:

1. $H^*$ has no 3-cycle $uvw$ such that $d_{H^*}(v) = d_{H^*}(w) = 2$.

2. $H^*$ has no cycle $xuvwx$ such that $d_{H^*}(v) = d_{H^*}(v) = d_{H^*}(w) = 2$.

3. $H^*$ has no path $xuvwv$ such that $d_{H^*}(u) = d_{H^*}(v) = d_{H^*}(w) = 2$.

4. $H^*$ does not contain a 3-vertex adjacent to three 2-vertices such that two of them have 2-neighbors in $H^*$.

Proof.

1. Suppose that $H$ contains a 3-cycle $uvw$ such that $d_{H^*}(v) = d_{H^*}(w) = 2$. If $x \in \{v, w\}$ has a 1-neighbor in $H$, denote this neighbor by $x'$. Let $N_H(u) = \{w, v, y\}$. Consider $H' = H \setminus \{v, w\}$. By minimality of $H$, $H'$ admits an injective edge-coloring $\phi$ using $C$. $\phi$ is a partial injective edge-coloring of $H$. We will extend $\phi$ to $H$. We color the edges $uv, uw, vw, vv'$ and $ww'$, in this order, as follows:

- $uv$ with any color $\alpha$ distinct from colors of edges adjacent to $uy$ (we have at most 2 forbidden colors).
- $uw$ with any color $\beta$ distinct from $\alpha$ and distinct from colors of edges adjacent to $uy$ (we have at most 3 forbidden colors).
- $vw$ with any color $\gamma$ distinct from $\alpha, \beta$ and $\phi(uy)$ (we have at most 3 forbidden colors).
2. Suppose that $H$ contains a cycle $xuwvx$ such that $d_{H^*}(u) = d_{H^*}(v) = d_{H^*}(w) = 2$. If $s \in \{u, v, w\}$ has a 1-neighbor in $H$, denote this neighbor by $s'$, otherwise $s'$ does not exist. Let $z$ be the third neighbor of $x$ in $H$, if exists. By the minimality of $H$, graph $H' = H \setminus \{v, u\}$ has an injective edge-coloring $\phi$ using $C$. We view $\phi$ as a partial injective edge-coloring of $H$. We will extend $\phi$ to $H$. We color the edges $xu, uv, vv', uu'$ and $wv$, in this order, as follows:

- $xu$ with any color $\alpha$ distinct from $\phi(xw)$ and distinct from colors of edges adjacent to $xy$ (we have at most 3 forbidden colors).
- $uv$ with any color $\beta$ distinct from $\phi(xy), \phi(xw)$ and $\phi(wu')$ (we have at most 3 forbidden colors).
- $vw'$ with any color $\gamma$ distinct from $\alpha, \phi(xw)$ and $\phi(wu')$ (we have at most 3 forbidden colors).
- $uw'$ with any color $\lambda$ distinct from $\gamma, \phi(xy)$ and $\phi(xw)$ (we have at most 3 forbidden colors).
- $uv'$ with any color $\xi$ distinct from $\alpha, \lambda$ and $\phi(xy)$ (we have at most 3 forbidden colors).

So, we obtain an injective edge-coloring of $H$. We get a contradiction.

3. Assume now $H^*$ contains a path $xuwvy$ such that $d_{H^*}(u) = d_{H^*}(v) = d_{H^*}(w) = 2$. If $s \in \{u, v, w\}$ has a 1-neighbor in $H$, denote this neighbor by $s'$, otherwise $s'$ does not exist. Let $N_H(y) = \{w, y_1, y_2\}$ and $N_H(x) = \{u, x_1, x_2\}$. By the minimality of $H$, graph $H' = H \setminus \{uv, vw, wy, uw', vu', wv\}$ has an injective edge-coloring $\phi$ using $C$. We view $\phi$ as a partial injective edge-coloring of $H$. We will extend $\phi$ to $H$. First, we color the edges $uv, vw$ and $vu'$, in this order, as follows:

- $uv$ with any color $\alpha \notin \{\phi(x_1), \phi(x_2), \phi(wy)\}$.
- $vw$ with any color $\beta \notin \{\phi(y_1), \phi(y_2), \phi(xu)\}$.
- $vu'$ with any color $\gamma \notin \{\phi(wy), \alpha, \phi(xu)\}$.

Hence, we get a partial injective edge-coloring $\phi_1$ of $H$. We will now color $wu'$:

(a) If there exist a color $\lambda$ distinct from $\phi_1(y_1), \phi_1(y_2), \alpha$ and $\gamma$ then, we set $\phi(wu') = \lambda$ and we color $uu'$ as follow

i. If we can color $uu'$ with some color $\xi \notin \{\phi'(x_1), \phi'(x_2), \beta, \gamma\}$, then we get a contradiction.
4. Suppose that $H$ such that $x, y, z$. Let $s$ be a partial injective edge-coloring of $H$. We have obtained an injective coloring of $H$. Thus, we get a contradiction.

(b) Otherwise, w.l.o.g., we may assume that $\alpha = 1, \gamma = 3, \phi(y_1y_1) = 4$ and $\phi(y_1y_2) = 2$.

i. If $\beta \neq \gamma$ then, by construction, $\beta = 1$ and $1 \notin \{\phi'(wy), \phi'(xu)\}$. Hence, we recolor $vw'$ with the color 1 and assign color 3 to $ww'$. Next, we color $uu'$ with any color $\xi$ distinct from 1, $\phi'(xx_1)$ and $\phi'(xx_2)$. Thus, we get a contradiction.

ii. Otherwise, if $\beta = \gamma = 3$ then $\phi'(wy) \notin \{1, 3\}$ and $1 \notin \{\phi'(xx_1), \phi'(xx_2)\}$.

- If $\phi'(xu) = 1$, we uncolor $vw'$ and we set $\phi'(uu') = 1, \phi'(ww') = 3$ and we color $vw$ with a color $\gamma' \notin \{\phi'(wy), 3, 1\}$. Thus, we get a contradiction.

- If $\phi'(xu) \neq 1$ then, we uncolor the edges $vv'$ and $vw$. We set $\phi'(ww') = 3, \phi'(vw) = \phi'(vv) = 1$ and $\phi'(uu') = \xi' \notin \{\phi'(xx_1), \phi'(xx_2), 1\}$. Thus, we get a contradiction.

4. Suppose that $H^*$ contains a 3-vertex $u$ adjacent to three 2-vertices $x, y$ and $z$ such that $x$ (respectively, $y$ and $z$) has a 2-neighbor $x_1$ (respectively, $y_1$ and $z_1$). By Lemma 4.2.2, $x_1 \neq y_1, x_1 \neq z_1$ and $y_1 \neq z_1$. Let $v$ (respectively, $w$ and $t$) denote the second neighbor in $H^*$ of $x_1$ (respectively, $y_1$ and $z_1$). For each $s \in \{x, y, z, x_1, y_1, z_1\}$, if $s$ has a (unique) 1-neighbor in $H$, then we denote this neighbor by $s'$. Let $H' = H \setminus \{ux, uy, uz, xx_1, yy_1, zz_1, x_1x_1, yy_1y_1, zz', z_1z_1'\}$. By the minimality of $H$, graph $H'$ has an injective edge-coloring $\phi$ using $C$. We view $\phi$ as a partial injective edge-coloring of $H$. We extend $\phi$ to $H$. We color, in the order

- $ux, uy$ and $uz$ with the same color $\alpha$ distinct from $\phi(z_1t), \phi(y_1w)$ and $\phi(x_1v)$.
- $z_1z_1'$ and $zz_1$ with some color $\beta_1$ distinct from $\alpha$ and distinct from colors of the edges adjacent to $z_1t$ (we have at most 3 forbidden colors).
- $y_1y_1'$ and $yy_1$ with some color $\beta_2$ distinct from $\alpha$ and distinct from colors of the edges adjacent to $y_1w$ (we have at most 3 forbidden colors).
- $x_1x_1'$ and $xx_1$ with some color $\beta_3$ distinct from $\alpha$ and distinct from colors of the edges adjacent to $x_1v$ (we have at most 3 forbidden colors).
- $zz'$ with some color $\gamma_1$ distinct from $\alpha, \beta_1$ and $\phi(z_1t)$.
- $yy'$ with some color $\gamma_2$ distinct from $\alpha, \beta_2$ and $\phi(y_1w)$.
- $xx'$ with some color $\gamma_3$ distinct from $\alpha, \beta_3$ and $\phi(x_1v)$.

We have obtained an injective coloring of $H$. Thus, we get a contradiction.

$\Box$
For \( j \in \{1, 2, 3\} \), let \( V_j \) be denote the set of vertices of degree \( j \) in \( H^* \) and \( |V_j| = n_j \).

By Lemma 4.1, \( n_1 = 0 \) and by Lemma 4.2, \( n_2 \leq \frac{n_1}{2} + 2n_3 \). It follows that,

\[
n_3 \geq \frac{2n_2}{5}.
\]

(2)

Since \( \text{mad}(H^*) \geq \frac{2|E(H^*)|}{|V(H^*)|} \), where \( 2|E(H^*)| = 3n_3 + 2n_2 \), we get

\[
\text{mad}(H^*) \geq 3 - \frac{n_2}{n_3 + n_2}.
\]

(3)

By equation (2), we may deduce that \( \frac{n_2}{n_3 + n_2} \leq \frac{5}{7} \), which yields \( \text{mad}(H^*) \geq \frac{16}{7} \). We get a contradiction.

This completes the proof of Theorem 1.5.1.

4.2 Proof of Theorem 1.5.2

Let \( C = \{1, 2, 3, 4, 5, 6\} \) be a set of six colors. Suppose that the Theorem 1.5.2 is not true. Let \( H \) be a counterexample minimizing \( |E(H) \cup V(H)| \): \( H \) is not injective edge-colorable with four colors, \( \text{mad}(H) < \frac{8}{3} \) and for any edge \( e \), \( \chi'_{\text{inj}}(H \setminus e) \leq 6 \).

Lemma 4.3 The minimal counterexample \( H \) satisfies the following properties:

1. \( H \) does not contain a 1-vertex.
2. \( H \) does not contain two adjacent 2-vertices.
3. \( H \) does not contain a 3-vertex adjacent to two 2-vertices.

Proof.

1. Assume that \( H \) has a 1-vertex \( u \) adjacent to some \( v \). Let \( N_H(v) = \{u, u_1, u_2\} \). By the minimality of \( H \), the graph \( H \setminus \{u\} \) has an injective edge-coloring \( \phi \) using \( C \). \( \phi \) is a partial injective edge-coloring of \( H \). We extend \( \phi \) to \( H \) by coloring \( uv \) with any color \( \alpha \) distinct from the colors of the edges adjacent to \( vu \) and \( vu_2 \) (we have at most 4 forbidden colors).

2. Suppose that \( H \) contains a 2-vertex \( u \) adjacent to a 2-vertex \( v \). Let \( t \) and \( w \) be the other neighbours of \( u \) and \( v \) respectively. Let \( N_H(w) = \{v, w_1, w_2\} \). By minimality of \( H \), the graph \( H' = H \setminus \{v\} \) has an injective edge-coloring \( \phi \) using \( C \). We extend \( \phi \) to \( H \). First, we color \( uv \) with any color \( \alpha \) distinct from the edges adjacent to \( tu \) and \( vw \) (we have at most 4 forbidden colors). Next, we color \( vw \) with any color \( \beta \) distinct from \( \phi(ut) \) and \( \phi(wv) \) (we have at most 5 forbidden colors).

3. Suppose \( H \) contains a 3-vertex \( u \) adjacent to two 2-vertices \( v \) and \( w \) whose second neighbors are \( y \) and \( z \), respectively. By minimality of \( H \), there exists an injective edge-coloring \( \phi \) of \( H' = H \setminus \{uw, uv\} \) using \( C \). Let \( x \) be the third neighbor of \( u \) and \( N_H(y) = \{v, y_1, y_2\} \), \( N_H(z) = \{w, z_1, z_2\} \). We will extend \( \phi \) to \( H \) by coloring, in the order,
• vy with some color α different from φ(xu) and different from the color of the edges adjacent to yy1 and yy2.
• wz with some color β different from φ(xu) and different from the color of the edges adjacent to zz1 and zz2.
• uw with some color γ different from β, φ(yy1), φ(yy2) and different from the color of the edges adjacent to ux.
• uw with some color λ different from α, φ(zz1), φ(zz2) and different from the color of the edges adjacent to ux.

Thus, we can extend φ to H, which is a contradiction.

□

For j ∈ {1, 2, 3}, let Vj be denote the set of vertices of degree j in H and |Vj| = nj. By Lemma 4.3.1, n1 = 0, by Lemma 4.3.2 every v ∈ V2 has two neighbors in V3, and by Lemma 4.3.3 every v ∈ V3 has at most one neighbor in V2. It follows that,

\[ n_3 \geq 2n_2. \] 

Since \( \text{mad}(H) \geq \frac{2|E(H)|}{|V(H)|} \), where \( 2|E(H)| = 3n_3 + 2n_2 \), we get

\[ \text{mad}(H) \geq 3 - \frac{n_2}{n_3 + n_2}. \] 

By equation (3), we may deduce that \( \frac{n_2}{n_3 + n_2} \leq \frac{1}{3} \). Thus, equation (3) yields \( \text{mad}(H) \geq \frac{8}{3} \). We get a contradiction.

This completes the proof of Theorem 1.5.2.

4.3 Proof of Theorem 1.5.3

Let \( C = \{1, 2, \cdots, 7\} \) be a set of seven colors. Suppose to the contrary that the Theorem 1.5.3 is not true. Let H be a counterexample minimizing |E(H)| + |V(H)|: H is not injective edge-colorable with seven colors, mad(H) < 3 and for any vertex u, \( \chi'_{inj}(H \setminus u) \leq 7 \).

Assume that H has a 1-vertex u adjacent to some v. Let \( N_H(v) = \{u, u_1, u_2\} \).

By the minimality of H, the graph H \ \{u\} has an injective edge-coloring \( \phi \) using C. \( \phi \) is a partial injective 7-edge-coloring of H. We extend \( \phi \) to H by coloring uv with any color α distinct from the edges adjacent to vu1 and vu2 (we have at most 4 forbidden colors). So, \( \delta(H) \geq 2 \).

Suppose now that H has a 2-vertex u adjacent to v and w. Let \( N_H(v) = \{v, v_1, v_2\} \) and \( N_H(w) = \{v, w_1, w_2\} \). By the minimality of H, graph H \ \{u\} has an injective edge-coloring \( \phi \) using C. \( \phi \) is a partial injective 7-edge-coloring of H. We will extend \( \phi \) to H. First, we color the edge uv with any color α distinct from \( \phi(wv_1) \), \( \phi(wv_2) \) and distinct from colors of edges adjacent to vv1 and vv2 (we have at most 6 forbidden colors). Next, we color the edge uw with any color β distinct from \( \phi(vw_1) \), \( \phi(vw_2) \) and distinct from colors of edges adjacent to vw1 and vw2 (we have at most 6 forbidden colors). Thus, we can extend a coloring \( \phi \) to H, which is a contradiction.

Therefore, H is a 3-regular graph which contradicts the hypothesis mad(G) < 3.
5 Outerplanar graph

A path $P = v_1v_2\cdots v_k$ is called a simple path in $G$ if $v_2, \ldots, v_{k-1}$ are all 2-vertices in $G$. The length of a path is the number of its edges.

It is easy to see that the greedy coloring with $k + 1$ colors of a $k$-tree gives an acyclic coloring (see [7]). Moreover, it is well-known that any outerplanar graph is a partial 2-tree [14]. Hence, if $G$ is an outerplanar graph then $\chi_a(G) \leq 3$. By Theorem 2.4 we have

Corollary 5.1 If $G$ is an outerplanar graph then $\chi_{inj}'(G) \leq 9$.

For the proof of Theorem 1.7, we will use a well known property of outerplanar graphs:

Lemma 5.2 Every outerplanar graph $G$ with $\delta(G) \geq 2$ contains a cycle $xv_1\cdots v_kyx$, where $v_1, \ldots, v_k$ are 2-vertices, $k \geq 1$, $d_G(x) \geq 2$ and $d_G(y) \geq 2$.

We first proof a strong property for the easy case.

Lemma 5.3 Let $G$ be the union of two cycles $C$ and $C'$ such that this two cycles have exactly one edge in common. Then $G$ has an injective 4-edge-coloring such that in every simple path of length three, exactly two colors appear.

Proof. Let $G$ be the union of two cycles $C = xv_1\cdots v_kyx$ and $C' = xv_1\cdots w_jyx$ such that this two cycles have exactly one edge $xy$ in common. We define an injective 4-edge-coloring $\phi$ of $G$ with colors $\{\alpha, \beta, \gamma, \lambda\}$ such that in every simple path of length three has two colors as follows.

1. If $i \geq 2$ and $j \geq 2$ then first, we set $\phi(xv_1) = \phi(xw_1) = \phi(xy) = \alpha$ and $\phi(yw_j) = \beta$. Next, we assign to the ordered edges $w_jw_{j-1}, \ldots, w_2w_1$ of path $P$ the ordered colors $\beta\gamma\gamma\beta\gamma\beta\beta\beta\beta\ldots$ if $j$ is even and $\gamma\gamma\beta\gamma\beta\beta\beta\beta\beta\beta\ldots$ if $j$ is odd. Finally, we assign to the ordered edges $yv_i, v_iv_{i-1}, \ldots, v_2v_1$ of path $P'$ the ordered colors $\lambda\gamma\gamma\lambda\gamma\gamma\ldots$ if $j$ is odd and $\lambda\gamma\gamma\lambda\gamma\gamma\ldots$ if $j$ is even.

2. If $i = 1$ and $j \geq 2$ then we set $\phi(xv_1) = \phi(xw_1) = \alpha$, $\phi(yv_1) = \beta$ and $\phi(xy) = \gamma$. Next, we assign to the ordered edges $yw_j, w_jw_{j-1}, \ldots, w_2w_1$ of path $P$ the ordered colors $\beta\lambda\beta\lambda\beta\lambda\beta\beta\beta\ldots$ if $j$ is odd and $\lambda\beta\beta\lambda\beta\lambda\beta\beta\beta\beta\beta\ldots$ if $j$ is even.

3. If $i = 1$ and $j = 1$ then we set $\phi(xv_1) = \phi(xw_1) = \alpha$, $\phi(yv_1) = \phi(yw_1) = \beta$ and $\phi(xy) = \gamma$.

Now we prove the following stronger version of Theorem 1.7:

Theorem 5.4 If $G$ is an outerplanar graph with maximum degree $\Delta(G) = 3$, then $G$ has an injective 5-edge-coloring such that in every simple path of length three, exactly two colors appear. Moreover, the bound is tight.
Proof. First we prove the tightness of the bound. Let $G$ be the graph depicted in Figure 1. $G$ is an outerplanar graph with maximum degree 3 and $\chi'_{\text{inj}}(G) = 5$. Indeed, since each pair of the set $\{uu_1, uu_2, vv_1, vv_2\}$ have a common neighbor or are at distance one, in every injective edge-coloring $\phi$ of $G$, they must have four different colors. Let us then use colors 1, 2, 3, and 4 to color respectively $uu_1$, $uu_2$, $vv_1$, and $vv_2$. However, four colors are not enough to color $G$, because if we are only allowed 4 colors then by symmetry, w.l.o.g. we may assume that $\phi(uv) = 3$. Hence, $\phi(u_1u_2) = 4$. In this case, we have to set, in the order, $\phi(u_3u_5) = 3$, $\phi(u_4u_5) = 2$, $\phi(u_3u_4) = 1$ and $\phi(u_1u_5) = 4$. Thus, no color can be given to color $u_3u_2$. Consequently, $\chi'_{\text{inj}}(G) \geq 4$. Therefore, the bound is tight.

![Figure 1: An outerplanar graph $G$ with $\Delta(G) = 3$ and $\chi'_{\text{inj}}(G) = 5$](image)

Now the proof of Theorem 5.4.

Suppose to the contrary that Theorem 5.4 is not true and let $G$ be a counterexample minimizing $|V(G)| + |E(G)|$. $G$ is a connected graph.

Claim 5.5 $\delta(G) \geq 2$

Proof. Assume that $G$ has an edge, say $v_1v_2$, where $d_G(v_1) = 1$.

1. If $d_G(v_2) = 3$ then consider the graph $H = G \setminus \{v_1v_2\}$.

   (a) If $\Delta(H) = 2$, then $H$ is a cycle $C = v_2v_3v_4 \cdots v_i v_2$ or $H$ is a path. If $H$ is a path or triangle, then it is easy to see that $G$ has an injective edge-coloring using three colors, such that in every simple path of length three, which is a contradiction. So, we may assume that $H$ is a cycle of order at least 4. In this case, we define an injective 5-edge-coloring $\phi$ of $G$ as follow. First, we set $\phi(v_2v_3) = \phi(v_3v_4) = \alpha$, $\phi(v_1v_2) = \beta$, where $\beta \neq \alpha$. Next, we assign to the ordered edges $v_4v_5, v_5v_6, \cdots, v_{i-1}v_i, v_iv_2$ the ordered string $\gamma\gamma\gamma\gamma\lambda\lambda\cdots$ and we get a contradiction.

   (b) If $\Delta(H) = 3$, then by minimality of $G$, $H$ has an injective 5-edge-coloring $\phi$ such that in every simple path of length three, exactly two colors appear. Let $N_G(v_2) = \{v_1, w_1, w_2\}$. In this case, we extend $\phi$ to $G$ by setting $\phi(v_1v_2) = \alpha$, such that $\alpha$ is distinct from colors of edges adjacent to the edges $v_2w_1$ and $v_2w_2$, we get a contradiction.

2. If $d_G(v_2) \leq 2$ then consider the maximal simple path $P = v_1v_2 \cdots v_k$ in $G$, $k \geq 3$. Since $P$ is a maximal simple path and $\Delta(G) = 3$, we have $d_G(v_k) = 3$. Let $N(v_k) = \{v_{k-1}, w_1, w_2\}$. Consider the graph $H = G \setminus \{v_1, \cdots, v_{k-2}\}$. By minimality of $G$, $H$ has an injective 5-edge-coloring $\phi$ such that in every simple path of length three, exactly two colors appear. In this case, we assign
to the ordered edges \(v_{k-1}v_{k-2}, v_{k-2}v_{k-3}, \ldots, v_2v_1\) of path \(P\) the ordered colors \(\alpha\alpha\beta\alpha\beta\alpha\alpha\beta\cdots\), where \(\alpha \neq \beta\) and \(\alpha \notin \{\phi(v_kw_1), \phi(v_kw_2), \phi(v_kv_{k-1})\}\), we get a contradiction.

\[\square\]

In the rest of the proof, we prove that such a configuration given by Lemma 5.2 can be reduced.

**Claim 5.6** \(G\) does not contain a cycle \(xv_1 \cdots v_kyx\), where \(v_1, \ldots, v_k, y\) are 2-vertices, \(k \geq 1, d_G(x) = 3\).

**Proof.** Assume that \(G\) contains a cycle \(xv_1 \cdots v_kyx\), where \(v_1, \ldots, v_k, y\) are 2-vertices, \(k \geq 1, d_G(x) = 3\). Let \(N_G(x) = \{y, v_1, x_1\}\).

1. If \(k = 1\) then by minimality of \(G\), the graph \(H = G \setminus \{v_1y\}\) has an injective 5-edge-coloring \(\phi\) such that in every simple path of length three, exactly two colors appear. We can extend the coloring \(\phi\) as follow. First, we recolor the edges \(xy\) with a color \(\alpha\), such that \(\alpha\) is distinct from colors of edges adjacent to \(xx_1\) and distinct from \(\phi(xv_1)\). Next, we set \(\phi(yv_1) = \beta\), where \(\beta \notin \{\alpha, \phi(xv_1), \phi(xx_1)\}\). Hence, we can extend \(\phi\) to \(G\) and we get a contradiction.

2. If \(k \geq 2\), then by minimality of \(G\), the graph \(H = G \setminus \{v_1v_2, \cdots, v_{k-1}v_k, v_ky\}\) has an injective 5-edge-coloring \(\phi\) such that in every simple path of length three, exactly two colors appear. We can extend the coloring \(\phi\) to \(G\) as follow. First, we recolor the edges \(xv_1\) and \(xy\) with the colors \(\alpha\) and \(\beta\) respectively such that \(\alpha \notin \phi(x_1)\) and \(\beta \notin \phi(x_1) \cup \{\alpha\}\). Next, we set \(\phi(yv_k) = \beta\) and we assign to the ordered edges \(v_1v_2, v_2v_3, v_3v_4, \cdots, v_{k-2}v_{k-1}, v_{k-1}v_k\) of path \(P\) the ordered colors \(\alpha\gamma\alpha\gamma\alpha\alpha\gamma\cdots\) if \(k\) is even and \(\gamma\gamma\alpha\gamma\alpha\gamma\gamma\cdots\) if \(k\) is odd, such that \(\gamma \notin \phi(x)\). Hence, we can extend \(\phi\) to \(G\) and we get a contradiction.

Hence, in each case, we can extend the coloring to \(G\), we get a contradiction.

\[\square\]

**Claim 5.7** \(G\) does not contain a cycle \(xv_1 \cdots v_kyx\), where \(v_1, \ldots, v_k\) are 2-vertices, \(k \geq 1, d_G(x) = d_G(y) = 3\).

**Proof.** Assume that \(G\) contains a cycle \(C = xv_1 \cdots v_kyx\), where \(v_1, \ldots, v_k\) are 2-vertices, \(k \geq 1, d_G(x) = d_G(y) = 3\). Let \(N_G(x) = \{y, v_1, x_1\}\) and \(N_G(y) = \{x, v_k, y_1\}\). Consider the graph \(H = G \setminus \{v_1, \cdots, v_k\}\), \(k \geq 1\). We claim that \(\Delta(H) = 3\). Indeed, if \(\Delta(H) = 2\) then \(G\) is a union of two cycles \(C_i\) and \(C_j\) such that they have exactly one edge in common. Therefore, by Lemma 5.3, \(G\) has an injective 4-edge-coloring such that in every simple path of length three, exactly two colors appear, which contradicts the choice of \(G\). Hence, \(\Delta(H) = 3\) and by minimality of \(G\), \(H\) has an injective edge-coloring \(\phi\) satisfying the hypothesis. We will extend \(\phi\) to an injective edge-coloring of \(G\) with the desired property as follows.

**Case 1:** Assume that \(x_1 = y_1\) and denote \(x_2\) the third neighbor of \(x_1\). Let \(\alpha, \beta, \gamma\) be the colors of the edges \(x_1x, xy,\) and \(yx_1\) respectively.

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Case 2: Assume that $x_1 \neq y_1$. W.l.o.g suppose that $\phi(xx_1) = \phi(xy) = \alpha$ and $\phi(yy_1) = \beta, \alpha \neq \beta$.

(a) If $k = 1$ then we can extend the coloring by setting $\phi(xv_1) = \gamma$ and $\phi(yv_1) = \lambda$, such that $\gamma$ is distinct from colors of edges incident with $xv_1$ and distinct from $\alpha$ and $\beta$, and $\lambda$ distinct from colors of edges incident with $yy_1$ and distinct from $\alpha$ and $\gamma$. Hence, we get a contradiction.

(b) If $k \geq 2$ then we color $xv_1$ with the color $\alpha$ and we consider the following cases:

- If $k$ is even then we assign to the ordered edges $xv_1, v_1v_2, v_2v_3, \ldots, v_{k-1}v_k$ of path $P$ the ordered colors $\lambda \alpha \alpha \lambda \alpha \alpha \cdots$, where $\lambda \notin \{\phi(x_1)\} \cup \{\beta\}$. The new partial edge coloring $\phi'$ is an injective 5-edge-coloring such that in every simple path of length three, exactly two colors appear.

Case 2: Assume that $x_1 \neq y_1$. W.l.o.g suppose that $\phi(xx_1) = \phi(xy) = \alpha$ and $\phi(yy_1) = \beta, \alpha \neq \beta$.

(a) If $k = 1$ then we can extend the coloring by setting $\phi(xv_1) = \gamma$ and $\phi(yv_1) = \lambda$, such that $\gamma$ is distinct from colors of edges incident with $xv_1$ and distinct from $\alpha$ and $\beta$, and $\lambda$ distinct from colors of edges incident with $yy_1$ and distinct from $\alpha$ and $\gamma$. Hence, we get a contradiction.

(b) If $k \geq 2$ then we color $xv_1$ with the color $\alpha$ and we consider the following cases:

- If $k$ is even then we assign to the ordered edges $yv_k, v_kv_{k-1}, \ldots, v_3v_2, v_2v_1$ of path $P$ the ordered colors $\gamma \beta \beta \gamma \beta \gamma \cdots$, where $\gamma \notin \{\phi(y_1)\} \cup \{\alpha\}$. Hence we can extend the coloring $\phi$ to $G$ and we get a contradiction.

- If $k$ is odd then we assign to the ordered edges $v_kv_2, v_2v_3, \ldots, v_{k-1}v_k$ of path $P$ the ordered colors $\lambda \alpha \alpha \lambda \alpha \alpha \cdots$, where $\lambda \notin \{\alpha, \beta, \gamma, \xi\}$. Hence we can extend the coloring $\phi$ to $G$ and we get a contradiction.

\[ \square \]

6 Open problem

While there exist cubic graphs with the injective chromatic index equal to 6 (See Figure 2), no example of a subcubic graph that would require 7 colors is known. Thus, we propose the following conjecture.
Conjecture 6.1 For every subcubic graph, $\chi_{\text{inj}}'(G) \leq 6$.

Figure 2: Two graphs having an injective chromatic index 6

It would be interesting to understand the list version of injective edge-coloring. An edge list $L$ for a graph $G$ is a mapping that assigns a finite set of colors to each edge of $G$. Given an edge list $L$ for a graph $G$, we shall say that $G$ is injective edge-choosable if for every edge if it has an injective edge-coloring $c$ such that $c(e) \in L(e)$ for every edge of $G$. The list-injective chromatic index, $\text{ch}_{\text{inj}}'(G)$, of a graph $G$ is the minimum $k$ such that for every edge list $L$ for $G$ with $|L(e)| = k$ for every $e \in E(G)$, $G$ is L-injective-edge colorable. Let us ask the following question.

Question 1 Is it true that $\text{ch}_{\text{inj}}'(G) = \chi_{\text{inj}}'(G)$ for every graph $G$?

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