LENS SPACES ISOSPECTRAL ON P-FORMS FOR EVERY P

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Abstract. To every lens space \( L \) we associate a congruence lattice \( \mathcal{L} \) in \( \mathbb{Z}^m \), showing that two lens spaces \( L \) and \( L' \) are isospectral on functions if and only if the associated lattices \( \mathcal{L} \) and \( \mathcal{L}' \) are isospectral with respect to \( \|\cdot\|_1 \). We prove that \( L \) and \( L' \) are isospectral on \( p \)-forms for every \( p \) if and only if \( \mathcal{L} \) and \( \mathcal{L}' \) are \( \|\cdot\|_1 \)-isospectral and satisfy a stronger condition (we call it \( \|\cdot\|_1^* \)-isospectrality). By constructing such congruence lattices we give infinitely many pairs of 5-dimensional lens spaces that are \( p \)-isospectral for all \( p \) but are not strongly isospectral.

We also give such examples in arbitrarily high dimensions. We show that \( \|\cdot\|_1^* \)-isospectrality can be detected by calculation of finitely many numbers, which allows us to get all such examples for low values of the parameters, using computer programs.

1. Introduction

Two compact Riemannian manifolds \( M, M' \) are said to be \( p \)-isospectral if the spectra of their Hodge-Laplace operators acting on \( p \)-forms are the same. Clearly, if \( M \) and \( M' \) are isometric then they are \( p \)-isospectral for all \( p \). The first examples of manifolds that are \( p \)-isospectral for every \( p \) and are not isometric were two isospectral tori, constructed by using isospectral lattices of dimension \( n = 16 \) (see [Mi1]). After this example, many others have been given, showing different connections between the spectra and the geometry of a Riemannian manifold (e.g. [Vi], [Ik1], [Go2], [Sch]).

In [Su] Sunada gave a general method —later extended in [DG], [Be]— to produce isospectral manifolds. By using this method many new examples were constructed, but the resulting manifolds are always strongly isospectral, that is they are isospectral for every natural strongly elliptic operator acting on sections of a natural vector bundle \( E \) over \( M \), in particular they are \( p \)-isospectral for all \( p \). In the context of spherical space forms, A. Ikeda [Ik2], P. Gilkey [Gi], J. A. Wolf [Wo2] applied the method to produce new strongly isospectral examples.

The construction of pairs of manifolds that are \( p \)-isospectral for some values of \( p \) only, has been a subject of interest for some time. The first such pair was given by C. Gordon [Go1] who constructed Heisenberg nilmanifolds that are 0-isospectral and are not 1-isospectral. Among other examples we mention those given in [Gi] for nilmanifolds and in [MR1] [MR2] in the

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context of compact flat manifolds. In the context of spherical space forms, A. Ikeda [IK3] gave, for each given \( p_0 \), families of lens spaces that are \( p \)-isospectral for every \( 0 \leq p \leq p_0 \), but are not \( p_0 + 1 \)-isospectral. Furthermore, in [GM], examples are given of lens spaces that are \( p \)-isospectral for some values of \( p \) only by computer methods, based on Ikeda’s approach.

However, there were no known examples of compact Riemannian manifolds that are \( p \)-isospectral for every \( p \) but are not strongly isospectral. This led us to wonder whether one could distinguish between these two notions. This problem has been present for some time (see for instance J. A. Wolf [Wo2, p. 323]). The main goal of this paper is to answer the question by constructing pairs of lens spaces that are isospectral on \( p \)-forms for all \( p \), but are not strongly isospectral.

Our approach consists of using representations of the compact Lie group \( G = \text{SO}(2m) \) and invariants to derive a formula for the multiplicity of each eigenvalue of \( \Delta_p \), the Hodge-Laplace operator on \( p \)-forms on the lens space \( L \), in terms of the multiplicities of the weights of representations of \( G \) and the number of solutions of certain congruences, depending on the data of the lens space \( L(q; s_1, \ldots, s_m) \) (see Theorem 3.7).

It turns out that one can associate to each lens space \( L(q; s_1, \ldots, s_m) \) the congruence lattice in \( \mathbb{Z}^m \) defined by

\[
L(q; s_1, \ldots, s_m) = \{(a_1, \ldots, a_m) \in \mathbb{Z}^m : a_1 s_1 + \cdots + a_m s_m \equiv 0 \pmod{q}\}.
\]

We prove that two lens spaces \( L(q; s_1, \ldots, s_m) \) and \( L(q'; s_1', \ldots, s_m') \) are 0-isospectral if and only if the lattices \( L(q; s_1, \ldots, s_m) \) and \( L(q'; s_1', \ldots, s_m') \) are isospectral with respect to \( \|\cdot\|_1 \) (Theorem 3.8). Remarkably, it turns out that \( L(q; s_1, \ldots, s_m) \) and \( L(q'; s_1', \ldots, s_m') \) are \( p \)-isospectral for every \( p \), if and only if the lattices are \( \|\cdot\|_1^* \)-isospectral, that is, for each \( k \in \mathbb{N} \) and \( 0 \leq \ell \leq m \), there are the same number of elements \( \mu \) in each lattice having \( \|\mu\|_1 = k \) and exactly \( \ell \) coordinates equal to zero.

We are now left with the task of exhibiting pairs of \( \|\cdot\|_1^* \)-isospectral congruence lattices. This we do in Section 5 in the case when \( m = 3 \). The proof is rather elaborate and is based on a subdivision of the lattice into layers, then splitting each layer into finitely many regions and then giving \( \|\cdot\|_1 \)-preserving bijective correspondences between the regions. In this way we construct an infinite two-parameter family of pairs of \( \|\cdot\|_1^* \)-isospectral non-isometric lattices in \( \mathbb{Z}^3 \) (Theorem 5.6).

In Section 4 we define, for any congruence lattice \( \mathcal{L} \), finitely many numbers \( N_{c, \ell}^{\mathcal{L}}(k) \) that count the number of lattice points of a fixed norm \( k \) in a small cube and having exactly \( \ell \) zero coordinates. These numbers determine the \( p \)-spectrum of the associated lens space. In particular, we can decide with finitely many computations, whether two lens spaces are \( p \)-isospectral for all \( p \). We implemented an algorithm in Sage [Sa] obtaining all examples in dimensions \( n = 5 \) and 7 for values of \( q \) up to 300 and 150 respectively (see Tables 1 and 2 in Section 6). We also include some questions on the nature of the existing examples.
In Section 7 we summarize the results on lens spaces that are a consequence of the results on the previous sections. The pairs of 3-dimensional $\parallel \cdot \parallel^*$-isospectral congruence lattices in Section 5 produce an infinite family of 5-dimensional lens spaces that are $p$-isospectral for all $p$. This allows to obtain lens spaces that are $p$-isospectral for all $p$ in arbitrarily high dimensions, by using a result of Ikeda (Theorem 7.1). We point out that the resulting lens spaces are homotopically equivalent but cannot be homeomorphic to each other (see Lemma 7.5). Also, by taking $M = L \times S^k$ and $M' = L' \times S^k$, where $L, L'$ is a $p$-isospectral pair as above and $S^k$ is a $k$-sphere, one obtains non-strongly isospectral manifolds that are $p$-isospectral for every $p$ in any given dimension $n \geq 5$.

It is well known that two non-isometric lens spaces cannot be strongly isospectral (see Proposition 7.6). In Section 7 we consider the simplest pair of 5-dimensional lens spaces $L = L(49; 1, 6, 15)$ and $L' = L(49; 1, 6, 20)$ that are $p$-isospectral for all $p$ and give many representations $\tau$ of $K = SO(5)$ for which the associated elliptic operators do not have the same spectrum. This gives a direct proof of the fact that these lens spaces are not strongly isospectral. Actually, there should be only few representations $\tau$ of $SO(5)$ such that $L$ and $L'$ are $\tau$-isospectral, i.e. these lens spaces are very far from being strongly isospectral.

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2. Preliminaries

Let $(G, K)$ be a Riemannian symmetric pair of compact type and let $X = G/K$ be the associated Riemannian compact symmetric space. We shall assume that $G$ is semisimple. Let $\Gamma$ be a discrete (hence finite) subgroup of the compact group $G$. We are mostly interested in the case when $\Gamma$ acts freely on $X$, thus $\Gamma \setminus X$ inherits a locally $G$-invariant Riemannian structure. However, we could relax this last condition and work in the context of good orbifolds.

2.1. Homogeneous vector bundles. For each finite dimensional unitary representation $(\tau, W_\tau)$ of $K$, we denote by $E_\tau$ the associated homogeneous vector bundle (see [Wa, §5.2] or [LMR §2.1])

$$E_\tau = G \times_\tau W_\tau \longrightarrow X = G/K.$$  

We identify the space of smooth sections $\Gamma^\infty(E_\tau)$ of $E_\tau$ with the space $C^\infty(G; \tau)$ of smooth functions $f : G \to W_\tau$ such that $f(xk) = \tau(k)^{-1}f(x)$ for all $k \in K$ and $x \in G$. The space of $L^2$-sections of $E_\tau$ decomposes as

$$L^2(E_\tau) = \sum_{\pi \in \widehat{G}} V_\pi \otimes \text{Hom}_K(V_\pi, W_\tau).$$

The left action of $G$ on $L^2(E_\tau)$ given by $(g \cdot f)(x) = f(g^{-1}x)$ preserves every component of the orthogonal sum and acts on $V_\pi$ by the representation $\pi$,.
i.e. $g \cdot (v \otimes A) = (\pi(g) v) \otimes A$. Thus, the space of $L^2$-sections of the vector bundle $\Gamma \backslash E_\tau$, which coincides with the elements in $L^2(E_\tau)$ invariant by $\Gamma$, decomposes as

$$L^2(\Gamma \backslash E_\tau) = \sum_{\pi \in \hat{G}} V^\Gamma_\pi \otimes \text{Hom}_K(V_\pi, W_\tau).$$

The Laplace operator $\Delta_\tau$ acting on smooth sections $\Gamma^\infty(E_\tau)$ of $E_\tau$ coincides with the action of the Casimir element $C \in U(g)$. This operator commutes with the left action of $G$, thus $\Delta_\tau$ induces a differential operator $\Delta_{\tau, G}$ acting on smooth sections of the vector bundle $\Gamma \backslash E_\tau$ over $\Gamma \backslash X$. Moreover, the action of $C$ on each summand $V^\Gamma_\pi \otimes \text{Hom}_K(V_\pi, W_\tau)$ is given by the scalar

$$\lambda(C, \pi) = \langle \Lambda + \rho, \Lambda + \rho \rangle - \langle \rho, \rho \rangle,$$

where $\Lambda$ is the highest weight of $\pi$, $\rho$ is half the sum of the positive roots of $g_C$ and $\langle \cdot, \cdot \rangle$ denotes the inner product on $h^*$ induced by the multiple of the Killing which gives the Riemannian metric. In particular, the multiplicity $d_\lambda(\tau, \Gamma)$ of $\lambda \in \mathbb{R}$ in the spectrum of $\Delta_{\tau, G}$ is

$$d_\lambda(\tau, \Gamma) = \sum_{\pi \in \hat{G} : \lambda(C, \pi) = \lambda} \dim V^\Gamma_\pi [\tau : \pi],$$

where $[\tau : \pi] = \dim(\text{Hom}_K(V_\pi, W_\tau)).$

The Laplace type operator $\Delta_{\tau, \Gamma}$ acting on sections of the vector bundle $\Gamma \backslash E_\tau$ over $\Gamma \backslash X$ defines a natural differential operator (see [Pe2, p. 265]).

**Definition 2.1.** Let $(G, K)$ be a Riemannian symmetric pair, let $X = G/K$ and let $\tau$ be a finite dimensional representation of $K$. Let $\Gamma$ and $\Gamma'$ be two finite subgroups of $G$. The spaces $\Gamma \backslash X$ and $\Gamma' \backslash X$ are said to be $\tau$-isospectral if the Laplace type operators $\Delta_{\tau, \Gamma}$ and $\Delta_{\tau, \Gamma'}$ have the same spectrum.

The manifolds $\Gamma \backslash X$ and $\Gamma' \backslash X$ are said to be strongly isospectral if for any strongly elliptic natural operator $D$ acting on sections of a natural bundle $E$ over $X$, the associated operators $D_\Gamma$ and $D_{\Gamma'}$ acting on sections of the bundles $\Gamma \backslash E$ and $\Gamma' \backslash E$ have the same spectrum. The vector bundles $E_\tau$ and the operators $\Delta_{\tau, \Gamma}$ and $\Delta_{\tau, \Gamma'}$ satisfy these conditions for every finite dimensional representation $\tau$ of $K$, thus strong isospectrality implies in particular $\tau$-isospectrality for every $\tau \in \hat{K}$.

**2.2. Spherical space forms.** For simplicity, from now on we restrict our attention to the symmetric pair $(G, K) = (\text{SO}(2m), \text{SO}(2m - 1))$, thus $X = S^{2m-1}$ is an odd-dimensional sphere. This is the main case for spherical space forms, that is, when $\Gamma$ acts freely on a sphere.

We fix the standard maximal torus in $G$,

$$T = \left\{ t = \text{diag} \left( \begin{bmatrix} \cos(2\pi \theta_1) & \sin(2\pi \theta_1) \\ -\sin(2\pi \theta_1) & \cos(2\pi \theta_1) \end{bmatrix}, \ldots, \begin{bmatrix} \cos(2\pi \theta_m) & \sin(2\pi \theta_m) \\ -\sin(2\pi \theta_m) & \cos(2\pi \theta_m) \end{bmatrix} \right) : \theta \in \mathbb{R}^m \right\}.$$
The Lie algebra of $T$ is given by
\begin{equation}
\mathfrak{h}_0 = \left\{ H = \text{diag} \left( \begin{array}{ccc}
0 & 2\pi\theta_1 & 0 \\
-2\pi\theta_1 & 0 & 0 \\
0 & -2\pi\theta_m & 2\pi\theta_m
\end{array} \right), \ldots, \begin{array}{ccc}
0 & 2\pi\theta_1 & 0 \\
-2\pi\theta_1 & 0 & 0 \\
0 & -2\pi\theta_m & 2\pi\theta_m
\end{array} \right) : \theta \in \mathbb{R}^m \right\}.
\end{equation}

Note that $t = \exp(H)$ if $t \in T$ and $H \in \mathfrak{h}_0$ as above. The Cartan subalgebra $\mathfrak{h} := \mathfrak{h}_0 \otimes \mathbb{R}$ is given as in (2.2) with $\theta_1, \ldots, \theta_m \in \mathbb{C}$, and in this case we let $\varepsilon_j \in \mathfrak{h}^*$ be given by $\varepsilon_j(H) = -2\pi i \theta_j$ for any $1 \leq j \leq m$. The weight lattice for $G = \text{SO}(2m)$ is $P(G) = \bigoplus_{j=1}^{m} \mathbb{Z}\varepsilon_j$.

We fix the standard system of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{h}) = \{ \varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq m \}$, with system of simple roots $\{ \varepsilon_j - \varepsilon_{j+1} : 1 \leq j \leq m-1 \} \cup \{ \varepsilon_{m-1} + \varepsilon_m \}$ and dominant weights of the form $\sum_{j=1}^{m} a_j \varepsilon_j \in P(G)$ such that $a_1 \geq \cdots \geq a_{m-1} \geq |a_m|$.

If $K = \{ k \in \text{SO}(2m) : k \varepsilon_{2m} = \varepsilon_{2m} \} \simeq \text{SO}(2m-1)$, then we take the maximal torus $T \cap K$, thus the Cartan subalgebra associated $\mathfrak{h}_K$ can be seen as included in $\mathfrak{h}$. Under this convention, the positive roots are $\{ \varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq m-1 \} \cup \{ \varepsilon_i : 1 \leq i \leq m-1 \}$, the simple roots are $\{ \varepsilon_j - \varepsilon_{j+1} : 1 \leq j \leq m-2 \} \cup \{ \varepsilon_{m-1} \}$, the weight lattice of $K$ is $P(K) = \bigoplus_{j=1}^{m} \mathbb{Z}\varepsilon_j$ and $\mu = \sum_{j=1}^{m-1} a_j \varepsilon_j \in P(K)$ is dominant if and only if $a_1 \geq \cdots \geq a_{m-1} \geq 0$.

Let $\tau_p$ denote the irreducible representation of $K$ with highest weight $\varepsilon_1 + \cdots + \varepsilon_p$ for $0 \leq p \leq m - 1$. The associated Laplace operator $\Delta_p$ acting on smooth sections of the vector bundle $E_{\tau_p}$ over $S^{2m-1}$ coincides with the Hodge-Laplace operator $\Delta_p$ acting on $p$-forms. As usual, we call $p$-spectrum the spectrum of $\Delta_p$ and we write $p$-isospectral in place of $\tau_p$-isospectral. If $\Gamma \subset O(2m)$ acts freely on $S^{2m-1}$ then $\Gamma \subset \text{SO}(2m)$, that is, $\Gamma \backslash S^{2m-1}$ is orientable. Hence, the $p$-spectrum of $\Gamma \backslash S^{2m-1}$ coincides with the $(2m-1-p)$-spectrum for every $0 \leq p \leq 2m - 1$.

We next describe the $p$-spectrum of any odd-dimensional spherical space form $\Gamma \backslash S^{2m-1}$ in terms of $\Gamma$-invariants. We first introduce some more notation. Let $\mathcal{E}_0 = \{ 0 \}$ and
\begin{equation}
\mathcal{E}_p = \{ \lambda_{k,p} := k^2 + k(2m - 2) + (p - 1)(2m - 1 - p) : k \in \mathbb{N} \}
\end{equation}
for $1 \leq p \leq m$. A basic useful fact is that $\mathcal{E}_p$ and $\mathcal{E}_{p+1}$ are disjoint for every $0 \leq p \leq m - 1$ (see for instance [IT] Rmk. after Thm. 4.2) [Ik3, Rmk. 1.14] and [LMR, Thm. 1.1]). Let
\begin{equation}
\Lambda_p = \begin{cases}
0 & \text{if } p = 0, \\
\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_p & \text{if } 1 \leq p \leq m.
\end{cases}
\end{equation}

Let $\pi_{k,p}$ denote the irreducible representation of $\text{SO}(2m)$ with highest weight $k \varepsilon_1 + \Lambda_p$ for $0 \leq p < m$, and let $\pi_{k,m}$ denote the sum of the irreducible representations with highest weights $k \varepsilon_1 + \Lambda_m$ and $k \varepsilon_1 + \Xi_m$, where $\Xi_m = \varepsilon_1 + \cdots + \varepsilon_{m-1} - \varepsilon_m$. We will usually write $\pi_{k,\varepsilon_1}$ and $\pi_{\Lambda_p}$ in place of $\pi_{k,0}$ and $\pi_{0,p}$, respectively.

The $p$-spectrum of $\Gamma \backslash S^{2m-1}$ can be described as follows.
**Proposition 2.2.** Let $\Gamma \backslash S^{2m-1}$ be a spherical space form. If $\lambda$ is an eigenvalue of $\Delta_{p,\Gamma}$ then $\lambda \in \mathcal{E}_p \cup \mathcal{E}_{p+1}$. Its multiplicity is given by

$$d_\lambda(0, \Gamma) = \dim V^\Gamma_{\pi_{k,1}},$$

if $\lambda = k^2 + k(2m - 2) \in \mathcal{E}_0 \cup \mathcal{E}_1$, for $p = 0$,

$$d_\lambda(p, \Gamma) = \begin{cases} 
\dim V^\Gamma_{\pi_{k,p}} & \text{if } \lambda = \lambda_{k,p} \in \mathcal{E}_p, \\
\dim V^\Gamma_{\pi_{k,p+1}} & \text{if } \lambda = \lambda_{k,p+1} \in \mathcal{E}_{p+1},
\end{cases}$$

for $1 \leq p \leq m - 1$.

When $\Gamma = 1$ this description appears in [IT], the case for general $\Gamma$ involves only minor modifications (see [LMR] Thm. 1.1). In the notation in [LMR] we have that $n_\Gamma(\pi_{k,p}) = \dim V^\Gamma_{\pi_{k,p}}$, by Frobenius reciprocity.

The following proposition follows from [IT] Thm. 4.2 (see also [Ik3, Prop. 2.1]). It will be a useful tool to prove one of the main results in the next section.

**Proposition 2.3.** Let $\Gamma$ and $\Gamma'$ be finite subgroups of $\mathrm{SO}(2m)$. Then

(i) $\Gamma \backslash S^{2m-1}$ and $\Gamma' \backslash S^{2m-1}$ are 0-isospectral if and only if, for every $k \in \mathbb{N}$,

$$\dim V^\Gamma_{\pi_{k,1}} = \dim V'^\Gamma_{\pi_{k,1}}.$$  

More generally, for any $0 \leq p \leq m - 1$, $\Gamma \backslash S^{2m-1}$ and $\Gamma' \backslash S^{2m-1}$ are $p$-isospectral if and only if, for every $k \in \mathbb{N}$,

$$\dim V^\Gamma_{\pi_{k,p}} = \dim V'^\Gamma_{\pi_{k,p}} \text{ and } \dim V^\Gamma_{\pi_{k,p+1}} = \dim V'^\Gamma_{\pi_{k,p+1}}.$$

(ii) $\Gamma \backslash S^{2m-1}$ and $\Gamma' \backslash S^{2m-1}$ are $p$-isospectral for every $p$, if and only if

$$\dim V^\Gamma_{\pi_{k,p}} = \dim V'^\Gamma_{\pi_{k,p}}$$

for every $1 \leq p \leq m$ and every $k \in \mathbb{N}$.

**Proof.** By Proposition 2.2 if $\lambda \in \mathbb{R}$ is an eigenvalue of $\Delta_{p,\Gamma}$ then $\lambda \in \mathcal{E}_p \cup \mathcal{E}_{p+1}$ for some $k \in \mathbb{N}$ and its multiplicity is $\dim V^\Gamma_{\pi_{k,p}}$ or $\dim V^\Gamma_{\pi_{k,p+1}}$ depending on whether $\lambda$ is in $\mathcal{E}_p$ or in $\mathcal{E}_{p+1}$. Since $\mathcal{E}_p \cap \mathcal{E}_{p+1}$ is empty, then (i) follows. When $p = 0$, $\pi_{k,0}$ and $\pi_{k,1}$ are the irreducible representations of $\mathrm{SO}(2m)$ with highest weight $k\varepsilon_1$ and $(k+1)\varepsilon_1$ respectively.

Item (ii) follows from (i) since $p$-isospectrality for $0 \leq p \leq m - 1$ implies $p$-isospectrality for every $p$.

3. ISOSPECTRALITY CONDITIONS FOR LENS SPACES

This section contains the first main result in this paper that gives characterizations of lens spaces that are 0-isospectral, or $p$-isospectral for every $p$ (see Theorem 3.8).

Odd dimensional lens spaces can be described as follows: for each $q \in \mathbb{N}$ and $s_1, \ldots, s_m \in \mathbb{Z}$ coprime to $q$, denote

$$L(q; s_1, \ldots, s_m) = \langle \gamma \rangle \backslash S^{2m-1}$$

where

$$\gamma = \text{diag} \left( \begin{bmatrix} \cos(2\pi s_1/q) & \sin(2\pi s_1/q) \\ -\sin(2\pi s_1/q) & \cos(2\pi s_1/q) \end{bmatrix}, \ldots, \begin{bmatrix} \cos(2\pi s_m/q) & \sin(2\pi s_m/q) \\ -\sin(2\pi s_m/q) & \cos(2\pi s_m/q) \end{bmatrix} \right)$$

(3.1)
The element $\gamma$ generates a cyclic group of order $q$ in $\text{SO}(2m)$ that acts freely on $S^{2m-1}$. Sometimes we shall abbreviate $L(q; s)$ in place of $L(q; s_1, \ldots, s_m)$, where $s$ stands for the vector $s = (s_1, \ldots, s_m)$. The following fact is well known (see [Co, Ch. V] or [Mi2, §12]).

**Proposition 3.1.** Let $L = L(q; s)$ and $L' = L(q; s')$ be lens spaces. Then the following assertions are equivalent.

1. $L$ is isometric to $L'$.
2. $L$ is diffeomorphic to $L'$.
3. $L$ is homeomorphic to $L'$.
4. There exist $t \in \mathbb{Z}$ coprime to $q$ and $\epsilon \in \{\pm 1\}^m$ such that $(s_1, \ldots, s_m)$ is a permutation of $(\epsilon_1 ts'_1, \ldots, \epsilon_m ts'_m)$ (mod $q$).

For $q \in \mathbb{N}$ and $s_1, \ldots, s_m \in \mathbb{Z}$ coprime to $q$, we consider the following congruence lattice

$$L(q; s_1, \ldots, s_m) = \{(a_1, \ldots, a_m) \in \mathbb{Z}^m : a_1s_1 + \cdots + a_m s_m \equiv 0 \pmod{q}\}.$$

We will often write $\mathcal{L}(q; s) = \mathcal{L}(q; s_1, \ldots, s_m)$, where $s = (s_1, \ldots, s_m)$ and we call $\mathcal{L}(q; s)$ the congruence lattice associated to the lens space $L(q; s)$. Also, we identify the weight lattice $P(\text{SO}(2m)) = \bigoplus_{j=1}^n \mathbb{Z} e_j$ with the correspondence $\sum_{j=1}^n a_j e_j \mapsto (a_1, \ldots, a_m)$. For $\mu = \sum_{j=1}^m a_j e_j \in \mathbb{Z}^m$, we set $||\mu||_1 = \sum_{j=1}^m |a_j|$.

**Proposition 3.2.** Let $L(q; s)$, $L(q; s')$ be lens spaces with $\mathcal{L}(q; s)$ and $\mathcal{L}(q; s')$ the associated lattices. Then, $L(q; s)$ and $L(q; s')$ are isometric if and only if $\mathcal{L}(q; s)$ and $\mathcal{L}(q; s')$ are $||\cdot||_1$-isometric.

**Proof.** By Proposition 3.1, $L$ and $L'$ are isometric if and only if there exist $t$ coprime to $q$ and $\varphi$, a composition of permutations and changes of signs, such that $\varphi(ts) = \varphi(ts_1, \ldots, ts_m) = (s'_1, \ldots, s'_m) = s'$. It is clear that the congruence lattices associated to the parameters $s_1, \ldots, s_m$ and $ts_1, \ldots, ts_m$ are the same. On the other hand, $\mathcal{L}(q, \varphi(s)) = \varphi(\mathcal{L}(q, s))$ and $\varphi$ is an isometry of $\mathbb{R}^n$ with respect to $||\cdot||_1$.

The converse assertion follows from the fact that every $||\cdot||_1$-linear isometry of $\mathbb{R}^n$ is a composition of permutations and changes of signs that is proved as follows. If $T$ is a $||\cdot||_1$-linear isometry of $\mathbb{R}^n$, then for each $1 \leq k \leq n$, $T(\varepsilon_k) = \sum_{j=1}^n c_{k,j} \varepsilon_j$ with $\sum |c_{k,j}| = 1$. We claim that $c_{k,j} \neq 0$ for at most one value of $j$. Otherwise, there are $h, k, \ell$ such that $c_{k,h} \varepsilon_h, c_{k,\ell} \varepsilon_\ell \neq 0$. Hence $|c_{k,h} + \delta c_{h,\ell}| < |c_{k,h}| + |c_{k,\ell}|$, for $\delta = 1$ or $\delta = -1$. Thus, for this choice of $\delta$ we have $2 = ||T(\varepsilon_k) + \delta T(\varepsilon_h)||_1 = \sum_{j=1}^n |c_{k,j} + \delta c_{h,j}| < \sum_{j=1}^n |c_{k,j}| + |c_{k,j}| = 2$, a contradiction. \hfill \square

**Lemma 3.3.** Let $\Gamma = (\gamma)$ where $\gamma$ is as in (3.2). Let $L = L(q; s_1, \ldots, s_m)$ be the corresponding lens space and let $\mathcal{L} = \mathcal{L}(q; s_1, \ldots, s_m)$ be the associated lattice. If $(\pi, V_\pi)$ is a finite dimensional representation of $\text{SO}(2m)$, then

$$(3.4) \quad \dim V_\pi^T = \sum_{\mu \in \mathcal{L}} m_\pi(\mu),$$
where \( m_\pi(\mu) \) denotes the multiplicity of the weight \( \mu \) in \( \pi \).

Proof. One has that \( V_\pi = \oplus_{\mu \in P(G)} V_\pi(\mu) \), where \( V_\pi(\mu) \) is the \( \mu \)-weight space, i.e. the space of vectors \( v \) such that \( \pi(h)v = h^\mu v \) for every \( h \in T \). Thus, \( V_\pi^\Gamma = \oplus_{\mu \in P(G)} V_\pi(\mu)^\Gamma \). Now, \( v \in V_\pi(\mu) \), \( v \neq 0 \), is \( \Gamma \)-invariant if and only if \( \gamma^\mu = 1 \), hence

\[
\dim V_\pi^\Gamma = \sum_{\mu : \gamma^\mu = 1} m_\pi(\mu).
\]

We let

\[
H_\gamma = \text{diag}\left(\left(\begin{array}{cc} 0 & 2\pi s_1/q \\ -2\pi s_1/q & 0 \end{array}\right), \ldots, \left(\begin{array}{cc} 0 & 2\pi s_m/q \\ -2\pi s_m/q & 0 \end{array}\right)\right),
\]

thus \( \exp(H_\gamma) = \gamma \). If \( \mu = \sum_{j=1}^m a_j \varepsilon_j \in P(\text{SO}(2m)) \) then

\[
\gamma^\mu = e^{\mu(H_\gamma)} = e^{-2\pi i \left( \frac{2\pi s_1 + \cdots + 2\pi s_m}{q} \right)} = 1,
\]

if and only if \( a_1 s_1 + \cdots + a_m s_m \equiv 0 \pmod q \), that is, \( \mu \in \mathcal{L} \).

Let \( \mathcal{L} \) be an arbitrary sublattice of \( \mathbb{Z}^m \). For \( \mu \in \mathbb{Z}^m \) we set \( Z(\mu) = \#\{ j : 1 \leq j \leq m, a_j = 0 \} \). We denote, for any \( 0 \leq \ell \leq m \) and any \( k \in \mathbb{N}_0 \),

\[
N_\mathcal{L}(k) = \#\{ \mu \in \mathcal{L} : \|\mu\|_1 = k \},
\]

\[
N_\mathcal{L}(k, \ell) = \#\{ \mu \in \mathcal{L} : \|\mu\|_1 = k, Z(\mu) = \ell \}.
\]

That is, \( N_\mathcal{L}(k, \ell) \) counts the number of lattice points \( \mu \) in \( \mathcal{L} \) of \( 1 \)-norm \( k \) that lie in exactly one of the \( \binom{m}{\ell} \) coordinate subspaces of dimension \( m - \ell \).

**Definition 3.4.** Let \( \mathcal{L} \) and \( \mathcal{L}' \) be sublattices of \( \mathbb{Z}^m \).

(i) \( \mathcal{L} \) and \( \mathcal{L}' \) are said to be \( \|\cdot\|_1 \)-isospectral if \( N_\mathcal{L}(k) = N_{\mathcal{L}'}(k) \) for every \( k \in \mathbb{N} \).

(ii) \( \mathcal{L} \) and \( \mathcal{L}' \) are said to be \( \|\cdot\|_* \)-isospectral if \( N_\mathcal{L}(k, \ell) = N_{\mathcal{L}'}(k, \ell) \) for every \( k \in \mathbb{N} \) and every \( 0 \leq \ell \leq m \).

In order to express the \( p \)-spectrum of lens spaces in terms of properties of the \( 1 \)-norms in the associated lattices, we will need two useful lemmas on weight multiplicities. The first lemma follows from known facts but we could not find it stated in the form we need it, so we include a short proof here. Recall that \( \Lambda_p \) is given by (2.3) and \( \pi_{0,p} = \pi_{\Lambda_p} \) is the exterior representation of \( \text{SO}(2m) \) on \( \Lambda^p \mathbb{C}^{2m} \) for \( 0 \leq p \leq m \).

**Lemma 3.5.** Let \( k \in \mathbb{N} \) and \( 0 \leq p \leq m \). If \( \mu = \sum_{j=1}^m a_j \varepsilon_m \in \mathbb{Z}^m \) we have

\[
m_{\pi_{k+1}}(\mu) = \begin{cases} (r + m - 2) \varepsilon(r - m - 2) & \text{if } \|\mu\|_1 = k - 2r \text{ with } r \in \mathbb{N}_0, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
m_{\pi_{\Lambda_p}}(\mu) = \begin{cases} (m - p + 2r) \varepsilon(r - m - 2) & \text{if } \|\mu\|_1 = p - 2r \text{ with } r \in \mathbb{N}_0, \text{ and } |a_j| \leq 1 \forall j, \\ 0 & \text{otherwise}. \end{cases}
\]
Proof. It is well known that the representation $\pi_{k,1}$ can be realized in the space of harmonic homogeneous polynomials $\mathcal{H}_k$ of degree $k$. Moreover, $\mathcal{P}_k = \mathcal{H}_k \oplus \mathcal{P}_{k-2}$ where $\mathcal{P}_k$ denotes the space of homogeneous polynomials of degree $k$, thus

\begin{equation}
\tag{3.9}
m_{\pi_{k,1}}(\mu) = m_{\mathcal{P}_k}(\mu) - m_{\mathcal{P}_{k-2}}(\mu).
\end{equation}

In order to find the weights of $\mathcal{P}_k$, we set $f_j(x) = x_{2j-1} + ix_{2j}$, $f_{j+m} = x_{2j-1} - ix_{2j} \in \mathcal{P}_1$ for each $1 \leq j \leq m$. It can be easily seen that the polynomials $f_1^i \ldots f_{2m}^j$ with $\sum_{j=1}^{2m} l_j = k$ form a basis of $\mathcal{P}_k$ given by weight vectors. Indeed, $h \in T$ acts on $f_1^i \ldots f_{2m}^j$ by multiplication by $h^\mu$ where $\mu = \sum_{j=1}^{2m} l_j = k$. It follows that $\mu = \sum_{j=1}^{2m} a_j \varepsilon_j \in \mathbb{Z}^m$ is a weight of $\mathcal{P}_k$ if and only if there are $l_1, \ldots, l_{2m} \in \mathbb{N}_0$ such that $a_j = l_j - l_{j+m}$ and $\sum_{j=1}^{2m} l_j = k$. Furthermore, one checks that the last condition is equivalent to $k - \|\mu\|_1 = 2r$ with $r \in \mathbb{N}_0$. Hence, $m_{\mathcal{P}_k}(\mu)$ equals the number of different ways one can write $r$ as a sum of $m$ different nonnegative integers, which equals $\binom{r+m-1}{m-1}$. This implies that

$$m_{\mathcal{P}_k}(\mu) = \begin{cases} \binom{r+m-1}{m-1} & \text{if } r = \frac{1}{2}(k - \|\mu\|_1) \in \mathbb{N}_0, \\ 0 & \text{otherwise.} \end{cases}$$

This formula and (3.9) imply (3.7).

We now prove the second assertion. The representation $\pi_{\Lambda_p}$ can be realized as the complexified $p$-exterior representation $\bigwedge^p (\mathbb{C}^{2m})$ with the canonical action of $\mathrm{SO}(2m)$. Let $\{e_1, \ldots, e_{2m}\}$ denote the canonical basis of $\mathbb{C}^{2m}$. For $1 \leq j \leq m$, we set $v_j = e_{2j-1} - i e_{2j}$ and $v_{j+m} = e_{2j-1} + i e_{2j}$. Hence $\{v_1, \ldots, v_{2m}\}$ is also a basis of $\mathbb{C}^{2m}$ and

\begin{equation}
\tag{3.10}
\{v_{i_1} \wedge \cdots \wedge v_{i_p} : 1 \leq i_1 < i_2 < \cdots < i_p \leq 2m\}
\end{equation}

is a basis of $\bigwedge^p (\mathbb{C}^{2m})$. For $I = \{1 \leq i_1 < i_2 < \cdots < i_p \leq 2m\}$ we write $\omega_I = v_{i_1} \wedge \cdots \wedge v_{i_p}$.

One can check that $h \in T$ acts on $\omega_I$ by multiplication by $h^\mu$ where $\mu = \sum_{j=1}^{2m} a_j \varepsilon_j$ is given by

$$a_j = \begin{cases} 1 & \text{if } j \in I \text{ and } j + m \notin I, \\ -1 & \text{if } j \notin I \text{ and } j + m \in I, \\ 0 & \text{if both } j, j + m \in I, \text{ or } j, j + m \notin I. \end{cases}$$

Thus, an arbitrary element $\mu = \sum_{j=1}^{2m} a_j \varepsilon_j \in \mathbb{Z}^m$ is a weight of $\bigwedge^p (\mathbb{C}^{2m})$ if and only if $|a_j| \leq 1$ for all $j$ and $p - \|\mu\|_1 \in 2\mathbb{N}_0$.

Let $\mu = \sum_{j=1}^{2m} a_j \varepsilon_j \in \mathbb{Z}^m$ be such that $|a_j| \leq 1$ for all $j$ and $r = \frac{1}{2}(p - \|\mu\|_1) \in \mathbb{N}_0$. Let $I_\mu = \{i : 1 \leq i \leq m, a_i = 1\} \cup \{i : m + 1 \leq i \leq 2m, a_{i-m} = -1\}$. Thus $I_\mu$ has $p - 2r$ elements. It is a simple matter to check that $\omega_I$ is a weight vector with weight $\mu$ if and only if $I$ has $p$ elements, $I_\mu \subseteq I$ and $I$ satisfies that $j \in I \setminus I_\mu \iff j + m \in I \setminus I_\mu$ for $1 \leq j \leq m$. One can check that there are $\binom{m-p+2r}{r}$ choices for $I$, hence the claim follows. \qed
The second lemma is crucial in the proof of Theorem 3.8(ii). We recall that \( \pi_{k,p} \) is the irreducible representation of SO(2\(m \)) with highest weight \( k\varepsilon_1 + \Lambda_p \) if \( p < m \) and, when \( p = m \), the sum of the irreducible representation with highest weights \( k\varepsilon_1 + \Lambda_p \) and \( k\varepsilon_1 + \Lambda_p \).

**Lemma 3.6.** Let \( \mu, \mu' \in P(\text{SO}(2m)) \simeq \mathbb{Z}^m \). If \( \|\mu\|_1 = \|\mu'\|_1 \) and \( Z(\mu) = Z(\mu') \) then \( m_{\pi_{k,p}}(\mu) = m_{\pi_{k,p}}(\mu') \) for every \( k \in \mathbb{N} \) and every \( 1 \leq p \leq m \).

**Proof.** We say that a finite dimensional representation \( \sigma \) of SO(2\(m \)) satisfies condition (\( \ast \)) if \( m_{\sigma}(\mu) = m_{\sigma}(\mu') \) for every \( \mu \) and \( \mu' \) such that \( \|\mu\|_1 = \|\mu'\|_1 \) and \( Z(\mu) = Z(\mu') \). By Lemma 3.5, \( \pi_{k\varepsilon_1} \) and \( \pi_{\Lambda_p} \) satisfy (\( \ast \)) for every \( k \in \mathbb{N} \).

We next show that \( \sigma := \pi_{k\varepsilon_1} \otimes \pi_{\Lambda_p} \) also satisfies (\( \ast \)). Let \( \mu = \sum_{i=1}^m a_i \varepsilon_i \) and \( \mu' = \sum_{i=1}^m a'_i \varepsilon_i \) be such that \( \|\mu\|_1 = \|\mu'\|_1 \) and \( Z(\mu) = Z(\mu') \). We fix a bijection \( \rho : [1, m] \to [1, m] \) so that \( a'_i \neq 0 \) if and only if \( a_{\rho(i)} \neq 0 \). We have that

\[
(3.11) \quad m_{\sigma}(\mu) = \sum_{\eta \in \mathbb{Z}^m} m_{\pi_{\Lambda_p}}(\eta) m_{\pi_{k\varepsilon_1}}(\mu - \eta)
\]

and a similar expression for \( m_{\sigma}(\mu') \) (see for instance [Kn, Ex. V.14]). Both sums are already over the weights of \( \pi_{\Lambda_p} \), that is, over the weights \( \eta = \sum_{i=1}^m b_i \varepsilon_i \) such that \( |b_i| \leq 1 \) for all \( i \) and \( \|\eta\|_1 = p - 2r \) for some \( r \in \mathbb{N} \), by Lemma 3.5. To each such \( \eta \) we make correspond \( \eta' = \sum_{i=1}^m b'_i \varepsilon_i \) defined by \( b'_i = b_{\rho(i)} \) for every \( i \) such that \( a'_i = 0 \) and \( b'_i = \text{sg}(a_{\rho(i)}) \text{sg}(a'_i) b'_{\rho(i)} \) for every \( i \) such that \( a'_i \neq 0 \). One can check that \( \|\eta\|_1 = \|\eta'\|_1 \), \( Z(\eta) = Z(\eta') \) and furthermore \( \|\mu - \eta\|_1 = \|\mu' - \eta'\|_1 \), thus \( m_{\pi_{\Lambda_p}}(\eta) m_{\pi_{k\varepsilon_1}}(\mu - \eta) = m_{\pi_{\Lambda_p}}(\eta') m_{\pi_{k\varepsilon_1}}(\mu' - \eta') \). By (3.11) we have that \( m_{\sigma}(\mu) = m_{\sigma}(\mu') \) as asserted.

By Steinberg’s formula (see for instance [Kn, Ex. 17-Ch. IX]), the representation \( \sigma \) decomposes as

\[
(3.12) \quad \chi_{\sigma} = \sum_{\mu} m_{\pi_{\Lambda_p}}(\mu) \text{sgn}(\mu + k\varepsilon_1 + \rho) \chi_{(\mu + k\varepsilon_1 + \rho)^V - \rho},
\]

where \( \chi_{\sigma} \) denotes the character of the representation \( \sigma \), \( \eta^V \) denotes the only dominant weight in the same Weyl orbit as \( \eta \), and

\[
\text{sgn}(\mu) = \begin{cases} 
0 & \text{if } \omega \mu = \mu \text{ for some nontrivial } \omega \in W, \\
\text{sg}(\omega) & \text{otherwise, where } \omega \mu \text{ is dominant.}
\end{cases}
\]

Note that the sum in (3.12) is over the weights of \( \pi_{k\varepsilon_1} \), which are described in (3.8). Moreover, the character of the representation \( \pi_{k,p} \) appears in the sum in the right-hand side in (3.12) and this is the only time it does, hence \( \pi_{k,p} \) appears exactly once in the decomposition of \( \sigma \). Now the proof of the lemma follows by an inductive argument in \( k \) and \( p \) by checking that any other irreducible representation \( \pi_{k',p'} \) that appears in (3.12) satisfies \( k' < k \), or else \( k' = k \) and \( p' < p \), thus \( \pi_{k',p'} \) satisfies (\( \ast \)) by the strong inductive hypothesis. Finally, since \( \sigma \) also satisfies (\( \ast \)) then \( \pi_{k,p} \) also does. \( \square \)
The next theorem gives an explicit formula for \( \dim V_{\pi_{k,p}}^\Gamma \) in terms of weight multiplicities \( m_{\pi_{k,p}}(\mu) \) and number of lattice points \( N_\ell(k, \ell) \), when \( L = \Gamma \backslash S^{2m-1} \) is a lens space with associated congruence lattice \( \mathcal{L} \). In particular, by Proposition 2.2, it determines the \( p \)-spectrum of the lens space \( L \).

**Theorem 3.7.** Let \( L = \Gamma \backslash S^{2m-1} \) be a lens space with associated lattice \( \mathcal{L} \) and let \( k \in \mathbb{N} \) and \( 0 \leq p \leq m \). Then

\[
\dim V_{\pi_{k,p}}^\Gamma = \sum_{r=0}^{\lfloor (k+p)/2 \rfloor} \sum_{\ell=0}^{m} m_{\pi_{k,p}}(\mu_{r,\ell}) N_\ell(k + p - 2r, \ell),
\]

where \( \mu_{r,\ell} \) is any weight such that \( Z(\mu_{r,\ell}) = \ell \) and \( \|\mu_{r,\ell}\|_1 = k + p - 2r \).

In the particular case when \( p = 0 \) we have that

\[
\dim V_{\pi_{k,1}}^\Gamma = \sum_{r=0}^{\lfloor k/2 \rfloor} \binom{r + m - 2}{m - 2} N_\ell(k - 2r).
\]

**Proof.** By Lemma 3.3 we have that

\[
\dim V_{\pi_{k,p}}^\Gamma = \sum_{\mu \in \mathcal{L}} m_{\pi_{k,p}}(\mu).
\]

The sum is finite since it is a sum over the weights \( \mu \) of \( \pi_{k,p} \). These weights are of the form \( k\varepsilon_1 + \Lambda_p - \nu \) with \( \nu \) a sum of positive roots, if \( p < m \), and of the form \( k\varepsilon_1 + \Lambda_m - \nu \) or \( k\varepsilon_1 + \Lambda_m - \nu \), if \( p = m \). Since \( \|\alpha\|_1 = \|\varepsilon_i \pm \varepsilon_j\|_1 = 2 \) for every positive root \( \alpha \) of \( \mathfrak{so}(2m, \mathbb{C}) \) (see (2.2), then \( \|k\varepsilon_1 + \Lambda_p\|_1 - \|\mu\|_1 = k + p - \|\mu\|_1 \in 2\mathbb{N}_0 \) if \( m_{\pi_{k,p}}(\mu) > 0 \). Hence

\[
\dim V_{\pi_{k,p}}^\Gamma = \sum_{r=0}^{\lfloor (k+p)/2 \rfloor} \sum_{\ell=0}^{m} \sum_{\mu \in \mathcal{L} : Z(\mu) = \ell, \|\mu\|_1 = k + p - 2r} m_{\pi_{k,p}}(\mu).
\]

Since, by Lemma 3.6, the value of \( m_{\pi_{k,p}}(\mu) \) depends only on \( \|\mu\|_1 \) and \( Z(\mu) \), the last sum equals the number of weights \( \mu \) such that \( \|\mu\|_1 = k + p - 2r \) and \( Z(\mu) = \ell \), times the multiplicity of any such weight. This proves (3.13).

In the case when \( p = 0 \), the multiplicity \( m_{\pi_{k,1}}(\mu) \) is as given in (3.7). Thus

\[
\dim V_{\pi_{k,1}}^\Gamma = \sum_{r=0}^{\lfloor k/2 \rfloor} \sum_{\mu \in \mathcal{L} : \|\mu\|_1 = k - 2r} \binom{r + m - 2}{m - 2} = \sum_{r=0}^{\lfloor k/2 \rfloor} \binom{r + m - 2}{m - 2} N_\ell(k - 2r).
\]

This completes the proof. \( \square \)

We now state the first main result in this paper.

**Theorem 3.8.** Let \( L = \Gamma \backslash S^{2m-1} \) and \( L' = \Gamma' \backslash S^{2m-1} \) be lens spaces with associated congruence lattices \( \mathcal{L} \) and \( \mathcal{L}' \) respectively. Then

(i) \( L \) and \( L' \) are 0-isospectral if and only if \( \mathcal{L} \) and \( \mathcal{L}' \) are \( \|\cdot\|_1 \)-isospectral.
(ii) $L$ and $L'$ are $p$-isospectral for all $p$ if and only if $L$ and $L'$ are $\|\cdot\|_1$-isospectral.

Proof. Proposition 2.3(i) (resp. (ii)) says that $L$ and $L'$ are 0-isospectral (resp. $p$-isospectral for all $p$) if and only if, for every $k \in \mathbb{N}$, $\dim V_{\pi_{k+1}}^{\Gamma} = \dim V_{\pi_{k+1}}^{\Gamma'}$ (resp. $\dim V_{\pi_{k+1}}^{\Gamma} = \dim V_{\pi_{k+1}}^{\Gamma'}$ for every $k \in \mathbb{N}$ and every $1 \leq p \leq m$). Hence, in the converse direction, (i) and (ii) follow immediately from (3.14) and (3.13) respectively.

We now assume that $L$ and $L'$ are 0-isospectral. We shall prove by induction that
\begin{equation}
N_L(k) = N_{L'}(k)
\end{equation}
for every $k \in \mathbb{N}$. The case $k = 0$ is clear, since both sides are equal to one. Suppose that (3.15) holds for every $k < k_0$. By (3.14) we have that
\begin{equation*}
\sum_{r \geq 0} \left( \begin{array}{c} r + m - 2 \\ m - 2 \end{array} \right) N_L(k_0 - 2r) = \sum_{r \geq 0} \left( \begin{array}{c} r + m - 2 \\ m - 2 \end{array} \right) N_{L'}(k_0 - 2r).
\end{equation*}
All the terms with $r > 0$ on both sides are equal by assumption, hence this equality implies that also $N_L(k_0) = N_{L'}(k_0)$. This proves (i).

We next prove (ii). Assume that $L$ and $L'$ are $p$-isospectral for all $p$. We shall prove that
\begin{equation}
N_L(k, \ell) = N_{L'}(k, \ell) \quad \forall \ell : 0 \leq \ell \leq m,
\end{equation}
for every $k \in \mathbb{N}$. We use an inductive argument on $k$. The case $k = 0$ is again clear. We suppose that (3.16) holds for every $k < k_0$. For each $1 \leq p \leq m$, if we let $k = k_0 - p$, then, by (3.13), since $L$ and $L'$ are $p$-isospectral, we have that
\begin{equation*}
\sum_{r \geq 0} \sum_{\ell=0}^m m_{\pi_{k+1}}(\mu_r, \ell) N_L(k_0 - 2r, \ell) = \sum_{r \geq 0} \sum_{\ell=0}^m m_{\pi_{k+1}}(\mu_r, \ell) N_{L'}(k_0 - 2r, \ell),
\end{equation*}
where $\mu_{r, \ell}$ is any weight satisfying $\|\mu_{r, \ell}\|_1 = k_0 - 2r$ and $Z(\mu_{r, \ell}) = \ell$. By assumption, all terms in both sides with $r > 0$ coincide. Thus
\begin{equation*}
\sum_{\ell=0}^{m-1} m_{\pi_{k+1}}(\mu_0, \ell) N_L(k_0, \ell) = \sum_{\ell=0}^{m-1} m_{\pi_{k+1}}(\mu_0, \ell) N_{L'}(k_0, \ell).
\end{equation*}
Note that the terms $\ell = m$ in both sides have been deleted since they are both equal to zero.

To prove our claim it suffices to show that the $m \times m$-matrix $(m_{\pi_{k+1}}(\mu_0, \ell))_{p, \ell}$ with $p = 1, \ldots, m$ and $\ell = 0, \ldots, m - 1$ is invertible. We claim that this matrix has 1’s on the anti-diagonal and it is ‘upper-triangular’ with respect to the anti-diagonal, hence it has determinant $\pm 1$.

Now, the element $\mu_{0, \ell}$ is any weight in $\mathbb{Z}^m$ such that $\|\mu_{0, \ell}\|_1 = k_0$ and $Z(\mu_{r, \ell}) = \ell$, thus we may pick
\[\mu_{0, \ell} = (k_0 - m + \ell + 1) \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{m-\ell}.\]
If \( m - \ell = p \) (i.e. \((p, \ell)\) is on the antidiagonal), then \( \mu_{0, \ell} = k\varepsilon_1 + \Lambda_p \). If \( p < m \), then \( \pi_{k,p} \) has highest weight \( k\varepsilon_1 + \Lambda_p \), hence \( m_{\pi_{k,p}}(\mu_{0,\ell}) = 1 \). On the other hand, if \( m - \ell < p \) then \( \mu_{0,\ell} \) cannot be a weight since \( k\varepsilon_1 + \Lambda_p - \mu_{0,\ell} \) is not a sum of positive roots since the coefficient of \( \varepsilon_1 \) equals \( m - \ell - p < 0 \).

The case \( p = m \) is very similar and its verification is left to the reader. □

**Remark 3.9.** A pair of non-isometric lens spaces given by Theorem 3.8 cannot be strongly isospectral. This comes from the general fact that two strongly isospectral lens spaces are necessarily isometric (see Proposition [7, Thm. 3.4]).

**Remark 3.10.** Ikeda in [Ik1] gave many pairs of non-isometric lens spaces that are 0-isospectral. The simplest such pair is \( L(11; 1, 2, 3) \) and \( L(11; 1, 2, 4) \) in dimension 5. In light of Theorem 3.8(i), the associated congruence 3-dimensional lattices \( L = \mathcal{L}(11; 1, 2, 3) \) and \( L' = \mathcal{L}(11; 1, 2, 4) \) must be \( \|\cdot\|_1 \)-isospectral. However, it is a simple matter to check that \( L \) and \( L' \) are not \( \|\cdot\|_1^* \)-isospectral. In fact, it is easy to see that \( \pm(2, -1, 0) \) and \( \pm(1, 1 - 1) \) are the only vectors in \( L \) with 1-norm equal to 3, while \( \pm(2, -1, 0) \) and \( \pm(0, 2, -1) \) are those with 1-norm equal to 3 lying in \( L' \). This implies that \( N_L(3, 0) = 2 \neq N_{L'}(3, 0) = 0 \) and \( N_L(3, 1) = 2 \neq N_{L'}(3, 1) = 4 \), proving the assertion.

As we shall see in Section 5, there exist infinitely many pairs of congruence lattices that are \( \|\cdot\|_1^* \)-isospectral in dimension \( m = 3 \). Such examples do not exist for \( m = 2 \), by Theorem 3.8(i), since Ikeda and Yamamoto showed that two 0-isospectral 3-dimensional lens spaces are necessarily isometric ([IY], [Ya]).

In the relevant paper [Ik3], Ikeda produced, for each given \( p_0 \), pairs of lens spaces that are \( p \)-isospectral for every \( 0 \leq p \leq p_0 \) but are not \( p_0 + 1 \) isospectral. Also, he proved that any pair in his set of examples cannot be \( p \)-isospectral for all \( p \) (see [Ik3, Thm. 3.4]).

4. **Finiteness conditions**

In this section we give a necessary and sufficient condition for two \( m \)-dimensional \( q \)-congruence lattices to be \( \|\cdot\|_1^* \)-isospectral, in terms of the equality, for the two lattices, of a finite set of numbers of cardinality at most \( \binom{m+1}{2} q \). In light of the connection with lens spaces in Theorem 3.8(ii), one can thus check with finitely many computations whether two lens spaces are \( p \)-isospectral for all \( p \). In Section 6 we will show many examples of \( \|\cdot\|_1^* \)-isospectral lattices found with a computer. Furthermore, this finite set determines every individual \( p \)-spectrum (see Section 7).

We first need to introduce some notions and notations. For \( q \in \mathbb{N} \) we set

\[
C(q) = \left\{ \sum_j a_j\varepsilon_j \in \mathbb{Z}^m : |a_j| < q, \forall j \right\}.
\]

An element in \( C(q) \) will be called \( q \)-reduced. We define an equivalence relation in \( \mathbb{Z}^m \) as follows: if \( \mu = \sum_j a_j\varepsilon_j, \mu' = \sum_j a_j'\varepsilon_j \in \mathbb{Z}^m \) then \( \mu \sim \mu' \)
if and only if $\mu - \mu' \in (q\mathbb{Z})^m$ and $a_ja_j' \geq 0$ for every $j$ such that $a_j \neq 0 \pmod{q}$. This relation induces an equivalence relation in $\mathbb{Z}^m$ and also in any $q$-congruence lattice $\mathcal{L}$ since $q\mathbb{Z}^m \subset \mathcal{L}$. Furthermore, $C(q)$ and $C(q) \cap \mathcal{L}$ give a complete set of representatives of $\sim$ on $\mathbb{Z}^m$ and $\mathcal{L}$ respectively. We now consider the number of $q$-reduced elements $\mathcal{L}$ with a fixed norm and a fixed number of zeros.

**Definition 4.1.** Let $\mathcal{L}$ be a $q$-congruence lattice as in (3.3). For any $k \in \mathbb{N}_0$ and $0 \leq \ell \leq m$, we set

$$N^{\text{red}}_{\mathcal{L}}(k, \ell) = \# \{ \mu \in C(q) \cap \mathcal{L} : \|\mu\|_1 = k, \ Z(\mu) = \ell \}.$$ 

We note that any element $\mu \in \mathbb{Z}^m$ lying in the regular tetrahedron $\|\mu\|_1 < q$ is $q$-reduced, thus $N_{\mathcal{L}}(k, \ell) = N^{\text{red}}_{\mathcal{L}}(k, \ell)$ for every $k < q$. Also, for each of the $m - \ell$ nonzero coordinates $a_i$ of a $q$-reduced element one has $|a_i| \leq q - 1$, thus $N^{\text{red}}_{\mathcal{L}}(k, \ell) = 0$ for every $k > (m - \ell)(q - 1)$. Hence, the total number of possibly nonzero numbers $N^{\text{red}}_{\mathcal{L}}(k, \ell)$ is at most $(m^{\ell+1})q$.

We have mentioned above that every element in a $q$-congruence lattice $\mathcal{L}$ is equivalent to one and only one $q$-reduced element in $\mathcal{L}$. As one should expect, the finite set of $N^{\text{red}}_{\mathcal{L}}(k, \ell)$'s determines the numbers $N_{\mathcal{L}}(k, \ell)$, for every $k, \ell$. That is, if $N^{\text{red}}_{\mathcal{L}}(k, \ell) = N^{\text{red}}_{\mathcal{L}'}(k, \ell)$ for every $k$ and $\ell$, then $\mathcal{L}$ and $\mathcal{L}'$ are $\|\cdot\|_1$-isospectral. The next theorem shows this fact by giving an explicit formula for $N_{\mathcal{L}}(k, \ell)$ in terms of the $N^{\text{red}}_{\mathcal{L}}(k, \ell)$. This formula will allow us also prove that the numbers $N_{\mathcal{L}}(k, \ell)$ determine the numbers $N^{\text{red}}_{\mathcal{L}}(k, \ell)$.

**Theorem 4.2.** Let $\mathcal{L}$ and $\mathcal{L}'$ be two $q$-congruence lattices as in (3.3).

(i) If $k = \alpha q + r \in \mathbb{N}$ with $0 \leq r < q$, then

$$N_{\mathcal{L}}(k, \ell) = \sum_{s=0}^{m-\ell} 2^s \binom{\ell + s}{s} \sum_{t=s}^{\alpha} \binom{t - s + m - \ell - 1}{m - \ell - 1} N^{\text{red}}_{\mathcal{L}}(k - tq, \ell + s).$$

(ii) $N_{\mathcal{L}}(k, \ell) = N_{\mathcal{L}'}(k, \ell)$ for every $k$ and $\ell$ if and only if $N^{\text{red}}_{\mathcal{L}}(k, \ell) = N^{\text{red}}_{\mathcal{L}'}(k, \ell)$ for every $k$ and $\ell$.

**Proof.** We begin by proving (i). We fix $0 \leq r < q$ and we write $k = \alpha q + r$ for some $\alpha \in \mathbb{N}_0$. When $\alpha = 0$ (4.4) reduces to the identity $N_{\mathcal{L}}(r, \ell) = N^{\text{red}}_{\mathcal{L}}(r, \ell)$, which is valid. For convenience, in the rest of this proof, we say that $\mu$ is of type $(k, \ell)$ if $\|\mu\|_1 = k$ and $Z(\mu) = \ell$.

Assume now that $\alpha = 1$. In this case (4.4) reduces to

$$N_{\mathcal{L}}(q + r, \ell) = N^{\text{red}}_{\mathcal{L}}(q + r, \ell) + (m - \ell)N^{\text{red}}_{\mathcal{L}}(r, \ell) + 2(\ell + 1)N^{\text{red}}_{\mathcal{L}}(r, \ell + 1).$$

There are three terms in the right hand side. Also, if $\mu$ is an element of type $(q + r, \ell)$ and $\mu_0$ is the only element in $C(q)$ such that $\mu \sim \mu_0$, then there are three possible different types for $\mu_0$, namely $(q + r, \ell)$, $(r, \ell)$ and $(r, \ell + 1)$. We next check the correspondence between the three terms and the three types, in the same order that are given.
The first term corresponds to the elements in $L$ of type $(q+r, \ell)$ which are already reduced. The second term corresponds to the elements in $L$ that are equivalent to a reduced element of type $(r, \ell)$. Indeed, if $\mu = \sum_i a_i e_i \in L \cap C(q)$ is of type $(r, \ell)$, then for each nonzero coordinate $i$ of $\mu$ (there are $m-\ell$ of them), the element $\mu + a_i q e_i$ has type $(q+r, \ell)$ and lies in the lattice, since $\pm q e_i \in q\mathbb{Z}^m \subseteq L$. Regarding the third term, for each $\mu \in L \cap C(q)$ of type $(r, \ell + 1)$ and each zero coordinate $i$ of $\mu$ (there are $\ell + 1$ of them), the element $\mu + q e_i$ has type $(q+r, \ell)$.

The detailed description done in the particular case $\alpha = 1$ will help to understand the general case. Let $\mu \in L$ of type $(k, \ell)$ and denote by $\mu_0$ the only element in $C(q) \cap L$ such that $\mu \sim \mu_0$. One can check that $\mu_0$ is of type $(k-tq, \ell+s)$ for some $0 \leq s \leq m-\ell$ and some $s \leq t \leq \alpha$.

Assume that $\mu_0$ is of type $(k-tq, \ell+s)$. For each choice of $s$ zero coordinates, $i_1, \ldots, i_s$, of $\mu_0$, the element $\mu_1 := \mu_0 \pm q e_{i_1} \pm \cdots \pm q e_{i_s}$ has type $(k-tq + sq, \ell)$. There are $2^s \binom{\ell+s}{s}$ different ways to choose $\mu_1$ from $\mu_0$. Now, it remains to add $\pm q$ (depending on the sign of the coordinate) $(t-s)$-times in the $m-\ell$ nonzero coordinates. This can be done in as many ways as the number of ordered partitions of $t-s$ into $m-\ell$ parts, that is, the number of ways of writing $t-s \in \mathbb{N}_0$ as a sum of $m-\ell$ non-negative integers. This equals $\binom{t-s+m-\ell-1}{m-\ell-1}$ and establishes formula $4.1$.

We next prove (ii). We first assume that $N_L^\ast(k, \ell) = N_L^\ast(k, \ell)$ for every $k$ and $\ell$. We write $k = \alpha q + r$ with $0 \leq r < q$. We argue by induction on $\alpha$. When $\alpha = 0$, $N_L(k, \ell) = N_L^\ast(k, \ell)$ and similarly for $L'$, thus $N_L^\ast(k, \ell) = N_L^\ast(k, \ell)$ for every $k < q$.

We assume that $N_L^\ast(k, \ell) = N_L^\ast(k, \ell)$ holds for every $k = \alpha q + r$ with $\alpha < \alpha_0 \in \mathbb{N}$. Clearly, $N_L^\ast(\alpha q + r, m) = N_L^\ast(\alpha(q + r, m) = 0$. We now proceed by induction on $\ell$ decreasing from $m$ to 0. Suppose that $N_L^\ast(\alpha q + r, \ell) = N_L^\ast(\alpha(q + r, \ell)$ for every $\ell > \ell_0$. By $4.1$, $N_L(\alpha q + r, \ell_0)$ can be written as a linear combination of the $N_L^\ast(\alpha q + r, \ell)$ for $\alpha \leq \alpha_0$ and $\ell \geq \ell_0$, and similarly for $N_L^\ast(\alpha(q + r, \ell_0)$. By the inductive hypothesis, we obtain that $N_L(\alpha q + r, \ell_0) = N_L^\ast(\alpha q + r, \ell_0)$ as asserted.

The converse assertion in (ii) follows immediately from $4.1$.

5. Construction of $\|\cdot\|_1^\ast$-isospectral lattices

The goal of this section is to construct an infinite two-parameter family of pairs of $\|\cdot\|_1^\ast$-isospectral lattices in $\mathbb{Z}^m$ for $m = 3$. Together with Theorem 3.8(2), this will produce infinitely many pairs of non-isometric 5-dimensional lens spaces, isospectral on $p$-forms for every $p$. Although the computations in Section 4 show that our construction does not give all of the examples for $m = 3$, the list given there shows that many of them can be obtained by a slight variation of the method used in this section.

We recall that it was shown by Ikeda-Yamamoto that such pairs cannot exist in dimension $m = 2$ ([IY, Ya]).
Throughout this section, we fix \( r, t \in \mathbb{N} \) and consider (see (3.3)) the lattices
\[
\mathcal{L} = \mathcal{L}(q; 1, rt - 1, 2rt + 1), \quad \mathcal{L}' = \mathcal{L}(q; 1, rt - 1, 3rt - 1).
\]
In other words, \( \mathcal{L} \) and \( \mathcal{L}' \) are defined by the following equations
\[
\mathcal{L} : \quad a + (rt - 1)b + (2rt + 1)c \equiv 0 \pmod{r^2t},
\]
\[
\mathcal{L}' : \quad (rt - 1)a' + b' + (3rt - 1)c' \equiv 0 \pmod{r^2t}.
\]
One can check, by using Propositions 3.1 and 3.2, that \( \mathcal{L} \) and \( \mathcal{L}' \) are not \( \|\cdot\|_1 \)-isospectral for \( r \geq 7 \). Proposition 6.4 gives a proof for a more general family.

We recall that \( \mathcal{L} \) and \( \mathcal{L}' \) are said to be \( \|\cdot\|_1^* \)-isospectral if \( N_{\mathcal{L}}(k, \ell) = N_{\mathcal{L}'}(k, \ell) \) for every \( k \in \mathbb{N} \) and every \( 0 \leq \ell \leq m = 3 \) (where \( N_{\mathcal{L}}(k, \ell) \) is as in (3.6)). We next prove that this equality holds easily for \( \ell = 1, 2, 3 \).

**Lemma 5.1.** For any \( \ell = 1, 2, 3 \) and any \( k \in \mathbb{N} \), one has that \( N_{\mathcal{L}}(k, \ell) = N_{\mathcal{L}'}(k, \ell) \).

**Proof.** The assertion is clear for \( \ell = 3 \). Also, it is easy to check for \( \ell = 2 \) since the elements \( (qs, 0, 0), (0, qs, 0), (0, 0, qs) \) for \( s \in \mathbb{Z}, s \neq 0 \) are the only ones in both lattices with two coordinates equal to zero.

For \( \ell = 1 \), using the definitions, it is not hard to give a \( \|\cdot\|_1 \)-preserving bijection between the sets \( \{ \eta \in \mathcal{L} : Z(\eta) = 1 \} \) and \( \{ \eta' \in \mathcal{L}' : Z(\eta') = 1 \} \). Namely one has
\[
(a, b, 0) \in \mathcal{L} \iff (b, a, 0) \in \mathcal{L}',
\]
\[
(a, 0, c) \in \mathcal{L} \iff (c, 0, a) \in \mathcal{L}',
\]
\[
(0, b, c) \in \mathcal{L} \iff (0, c, b) \in \mathcal{L}'.
\]
The first equivalence is clear by (5.2). To check the second equivalence we note that, since \( \gcd(3rt - 1, r^2t) = 1 \), then \( a + (2rt + 1)c \equiv 0 \pmod{r^2t} \) is equivalent to \( (3rt - 1)a + (3rt - 1)(2rt + 1)c \equiv (3rt - 1)a + (rt - 1)c \equiv 0 \pmod{r^2t} \). Similarly, the third equivalence holds since \( (rt - 1)b + (2rt + 1)c \equiv 0 \pmod{r^2t} \) is valid if and only if \( -(2rt - 1)(rt - 1)b - (rt - 1)(2rt + 1)c \equiv (3rt - 1)b + c \equiv 0 \pmod{r^2t} \).

It thus remains to prove that \( N_{\mathcal{L}}(k, 0) = N_{\mathcal{L}'}(k, 0) \) for every \( k \). By Lemma 5.1, this is equivalent to showing that \( \mathcal{L} \) and \( \mathcal{L}' \) are \( \|\cdot\|_1 \)-isospectral, since \( N_{\mathcal{L}}(k) = \sum_{\ell=0}^{3} N_{\mathcal{L}}(k, \ell) \). This will need quite some work and will be our task for the rest of this section.

**Remark 5.2.** We note that, remarkably, in light of Theorem 3.8(i), the previous lemma allows to reduce the verification that the associated lens spaces \( L \) and \( L' \) are \( p \)-isospectral for all \( p \), to prove that they are just 0-isospectral.

The following description will be useful.
Proposition 5.3. We have that \( \mathcal{L} = \bigcup_{h \in \mathbb{Z}} \mathcal{L}_h \) and \( \mathcal{L}' = \bigcup_{h \in \mathbb{Z}} \mathcal{L}'_h \), disjoint unions, where

\[
\mathcal{L}_h = \{(a, b, c) \in \mathbb{Z}^3 : a - b + c = rth, \ a + 3c + h \equiv 0 \pmod{r}\},
\]

\[
\mathcal{L}'_h = \{(a', b', c') \in \mathbb{Z}^3 : a' - b' + c' = rth, \ a' + 3c' - h \equiv 0 \pmod{r}\}.
\]

Proof. By (5.2), if \((a, b, c) \in \mathcal{L}\) then

\[
(a - b + c + rt(b + 2c) \equiv 0 \pmod{r^2}.
\]

Hence \(rt|a - b + c\) and \(r|b + 2c + h\) where \(h = (a - b + c)/rt\). Or, equivalently, \(rt|a - b + c\) and \(r|h + a + 3c\). A similar argument works for \(\mathcal{L}'\). This proves the lemma.

The planar sections \(\mathcal{L}_h\) and \(\mathcal{L}'_h\) will be called layers. From Proposition 5.3, it follows easily that \(\mu \mapsto -\mu\) maps bijectively \(\mathcal{L}_h\) into \(\mathcal{L}_h\) and \(\mathcal{L}'_h\) into \(\mathcal{L}'_h\) for every \(h \in \mathbb{N}\) and preserves \(\|\cdot\|_1\). We will denote by \(w + \mathcal{L}_h\) the set \(\{w + v : v \in \mathcal{L}_h\}\) for any \(w \in \mathcal{L}\).

Lemma 5.4. Let \(h, s \in \mathbb{Z}\), \(w \in \mathcal{L}_1\), \(w' \in \mathcal{L}'_1\). Then \(sw \in \mathcal{L}_s\), \(sw' \in \mathcal{L}'_s\). Furthermore,

\[
sw + \mathcal{L}_h = \mathcal{L}_{s+h} \quad \text{and} \quad sw' + \mathcal{L}'_h = \mathcal{L}'_{s+h}
\]

and for any \(k \in \mathbb{Z}\),

\[
\mathcal{L}_{rk} = \mathcal{L}'_{rk}.
\]

Proof. The first two assertions and (5.4) follow immediately from Proposition 5.3. It is also clear from Proposition 5.3 that, if \(v_1 \in \mathcal{L}_h\), for \(i = 1, 2\), then \(v_1 + v_2 \in \mathcal{L}_{h_1 + h_2}\). This implies that \(sw + \mathcal{L}_h \subset \mathcal{L}_{s+h}\) and \(-sw + \mathcal{L}_{s+h} \subset \mathcal{L}_h\), as asserted. The verification for \(\mathcal{L}'\) is the same.

The next definition will be used in the proof of \(\|\cdot\|_1\)-isospectrality of \(\mathcal{L}\) and \(\mathcal{L}'\).

Definition 5.5. Given two lattices \(\mathcal{L}\) and \(\mathcal{L}'\), we say that two subsets \(A \subset \mathcal{L}\), \(A' \subset \mathcal{L}'\) are \(\|\cdot\|_1\)-isospectral (denoted \(A \approx A'\)) if there is a \(\|\cdot\|_1\)-preserving bijection between \(A\) and \(A'\).

We now state the main result of this section.

Theorem 5.6. For any \(r, t\) odd positive integers, \(r \not\equiv 0 \pmod{3}\), \(q = r^2t\), the lattices \(\mathcal{L}(q; 1, rt - 1, 2rt + 1)\) and \(\mathcal{L}(q; 1, rt - 1, 3rt - 1)\) in (5.2) are \(\|\cdot\|_1\)-isospectral.

Proof. For simplicity, in what follows we shall just write \(\mathcal{L} = \mathcal{L}(q; 1, rt - 1, 2rt + 1)\) and \(\mathcal{L}' = \mathcal{L}(q; 1, rt - 1, 3rt - 1)\). By Lemma 5.1 it only remains to prove that \(\mathcal{L}\) and \(\mathcal{L}'\) are \(\|\cdot\|_1\)-isospectral.

To facilitate the reading of the (rather long) proof we will carry it out in the case \(r = 7\), showing that \(\mathcal{L}(49; 1, 7t - 1, 14t + 1)\) and \(\mathcal{L}(49; 1, 7t - 1, 21t - 1)\) are \(\|\cdot\|_1\)-isospectral for any odd \(t\). The proof for general \(r\) is exactly the same by replacing \(7\) by \(r\) and \(2\) by the inverse of \(-3 \pmod{r}\).
By Proposition 5.3, we have that

\[(5.5) \quad \mathcal{L} = \bigcup_{k \in \mathbb{Z}} \left( \mathcal{L}_{7k} \cup \bigcup_{s=1}^{s=3} (\mathcal{L}_{7k+s} \cup \mathcal{L}_{7k-s}) \right) \]

and similarly for \( \mathcal{L}' \). Our goal will be to show that

\[(5.6) \quad \mathcal{L}_{7k+s} \cup \mathcal{L}_{7k-s} \approx \mathcal{L}'_{7k+s} \cup \mathcal{L}'_{7k-s} \]

for every \( k \in \mathbb{N}_0 \) and any \( s = 1, 2, 3 \). This will imply that \( \mathcal{L} \approx \mathcal{L}' \) by Proposition 5.3.

We set \( u = (7t - 1)/2 \in \mathbb{N} \) since \( t \) is odd. We have that

\[ w := (u + 1, 0, u) \in \mathcal{L}_{1} \quad \text{and} \quad w' := (u, 0, u + 1) \in \mathcal{L}'_{1}. \]

Since, by (5.3),

\[(5.7) \quad \mathcal{L}_{7k+s} = \pm sw \mathcal{L}_{7k}, \quad \mathcal{L}'_{7k+s} = \pm sw' \mathcal{L}'_{7k} \]

we will always consider the elements in \( \mathcal{L}_{7k+s} \) and \( \mathcal{L}'_{7k+s} \) written as \( \lambda \pm sw \) and \( \lambda' \pm sw' \) with \( \lambda = (a, b, c) \in \mathcal{L}_{7k} \) and \( \lambda' = (a', b', c') \in \mathcal{L}'_{7k} \) respectively. By Proposition 5.3 this implies that

\[
\begin{align*}
  b &= a + c + 49kt, & a + 3c &\equiv 0 \pmod{7}, \\
  b' &= a' + c' + 49kt, & a' + 3c' &\equiv 0 \pmod{7}.
\end{align*}
\]

By projecting \( \mathcal{L}_{7k} \) into the first and third coordinates \( a, c \), we obtain a planar lattice that is independent of \( k \). This lattice will be denoted by \( \Pi \); it is the set of \((a, c) \in \mathbb{Z} \times \mathbb{Z}\) such that \( a + 3c \equiv 0 \pmod{7} \). For the proof, we shall give two different partitions of \( \Pi \), denoted \(+\) and \(-\), which induce a partition on \( \mathcal{L}_{7k+s} \) and \( \mathcal{L}_{7k-s} \) respectively, and similarly for \( \mathcal{L}'_{7k+s} \). Our task will be to give \( \| \cdot \|_1 \)-preserving bijections between the sets occurring in the partitions of \( \mathcal{L}_{7k+s} \cup \mathcal{L}_{7k-s} \) and \( \mathcal{L}'_{7k+s} \cup \mathcal{L}'_{7k-s} \).

We partition \( \Pi \) into several regions: quadrants, semistrips and rectangles. The notation will use the symbols \( N, S, E, W \) alluding to the four cardinal directions. The quadrants:

\[
\begin{align*}
  Q^+_SE &= \{c \leq -s(u+1), s(u+1) \leq a\}, & Q^-NE &= \{s(u+1) \leq a, s(u+1) \leq c\}, \\
  Q^+_SW &= \{a, c \leq -s(u+1)\}, & Q^-SW &= \{a \leq su, c \leq su\}, \\
  Q^+_NE &= \{-su \leq a, -su \leq c\}, & Q^-SW &= \{a \leq -s(u+1), s(u+1) \leq c\}. \\
\end{align*}
\]
The semistrips:

\[ S_+^+ = \{ c \leq s(u+1), -su \leq a < s(u+1) \}, \]
\[ S_+^0 = \{ a \leq s(u+1), -su \leq c < s(u+1) \}, \]
\[ S_0^- = \{ s(u+1) \leq a, -s(u+1) < c \leq su \}, \]
\[ S_1^- = \{ s(u+1) \geq c, -s(u+1) < a \leq su \}, \]
\[ T_1^+ = \{ c \leq -s(u+1), -su < a < -s \}, \]
\[ T_1^0 = \{ a \leq -s(u+1), -su < c < -s \}, \]
\[ T_0^- = \{ s(u+1) \leq a, -s(u+1) < c \leq -su \}, \]
\[ T_0^0 = \{ a \leq -s(u+1), -su < c < -s(u+1) \}, \]
\[ T_1^- = \{ c \leq -s(u+1), su < a < s(u+1) \}, \]
\[ T_0^- = \{ s(u+1) \leq a, su < c < s(u+1) \}, \]
\[ T_1^- = \{ s(u+1) \leq c, su < a < s(u+1) \}. \]

The rectangles:

\[ R_1^+ = \{ -s(u+1) < a < su, -su \leq c < s(u+1) \}, \]
\[ R_0^+ = \{ -su \leq a < s(u+1), -s(u+1) < c < -su \}, \]
\[ R_0^- = \{ -s(u+1) < a \leq su, su < c < s(u+1) \}, \]
\[ R_1^- = \{ su < a < s(u+1), -s(u+1) < c \leq su \}, \]
\[ C^- = \{ su < a, c < s(u+1) \}, \quad C^+ = \{ -s(u+1) < a, c < -su \}. \]

To have a visual understanding of the regions see Figures 1 and 2. We note that the partitions + and − are symmetric with respect to the origin, e.g. 
\[ Q_{NE}^- = -S_{SW}^+ \text{ and } S_{NE}^- = -S_{SW}^+. \]

Exactly the same regions are defined in the case of \( \Pi' = \Pi \), adding ‘ in each case. We split \( \Pi \) in two different ways depending on whether we are looking at \( L_{7k+s} \) or at \( L_{7k-s}, s = 1, 2, 3 \). We shall use the following disjoint splittings.

\[ \Pi = Q_{NW}^+ \cup Q_{SW}^- \cup Q_{SE}^+ \cup Q_{NE}^- \cup S_{SW}^+ \cup S_{NE}^- \cup T_{NW}^+ \cup T_{SW}^- \cup T_{SE}^+ \cup T_{NE}^- \cup T_{NW}^- \cup T_{SW}^+ \cup T_{SE}^- \cup T_{NE}^+ \cup T_{NW}^+ \cup T_{SW}^- \cup T_{SE}^+ \cup T_{NE}^- \cup C^+ \cup C^- \]

\[ \Pi = Q_{NW}^- \cup Q_{SW}^+ \cup Q_{SE}^- \cup Q_{NE}^+ \cup S_{SW}^- \cup S_{NE}^+ \cup T_{NW}^- \cup T_{SW}^+ \cup T_{SE}^- \cup T_{NE}^+ \cup T_{NW}^+ \cup T_{SW}^- \cup T_{SE}^+ \cup T_{NE}^- \cup R_{NW}^+ \cup R_{SW}^- \cup R_{SE}^+ \cup R_{NE}^- \cup R_{SW}^+ \cup R_{SE}^- \cup R_{NE}^+ \cup R_{NW}^- \cup R_{SW}^+ \cup R_{SE}^- \cup R_{NE}^+ \cup R_{SW}^- \cup R_{SE}^+ \cup R_{NE}^- \cup C^+ \cup C^- . \]

For \( \Pi' \) we use the same regions, denoted with ‘.

We shall show that all eight quadrants (\( Q \)'s in \( \Pi \)) are in a bijective correspondence with the eight quadrants in \( \Pi' \), all four semistrips (\( S \)'s in \( \Pi \)) are in correspondence with all four semistrips in \( \Pi' \) and four thin semistrips (\( T \)'s) in \( \Pi \) are in correspondence with four thin strips in \( \Pi' \). Finally, the remaining thin semistrips, rectangles and small squares in \( \Pi \), will be put in correspondence with a similar union in \( \Pi' \) (see (5.23) and (5.24)). We begin
by showing that

\begin{align}
Q_{NE}^\pm &\approx Q_{NE}', \quad Q_{SW}^\pm \approx Q_{SW}', \quad Q_{NW}^\pm \approx Q_{NW}', \quad Q_{SE}^\pm \approx Q_{SE}', \\
S_W^\pm &\approx S_S^\pm, \quad S_S^\pm \approx S_S^\pm', \quad S_E^- \approx S_E^-, \quad S_N^- \approx S_N^-', \\
T_S^\pm &\approx T_S^\pm', \quad T_W^\pm \approx T_W^\pm'.
\end{align}

(5.8)

We fix $k \in \mathbb{N}$ and $s \in \{1, 2, 3\}$. The elements of $\mathcal{L}_{7k} = \mathcal{L}_{7k}'$ can be written as

\begin{align}
\lambda = (a, a + c - 49kt, c) \quad \text{for} \quad a + 3c &\equiv 0 \pmod{7}.
\end{align}

Then, by (5.7), the elements in $\mathcal{L}_{7k \pm s}$ and $\mathcal{L}_{7k \pm s}'$ have the form

\begin{align}
\lambda \pm sw = (a \pm s(u + 1), a + c - 49kt, c \pm su) \\
\lambda \pm sw' = (a \pm su, a + c - 49kt, c \pm s(u + 1)),
\end{align}

(5.10) (5.11)
with $a, c \in \mathbb{Z}$ such that $a + 3c \equiv 0 \pmod{7}$, and their 1-norms are given by

\begin{align}
\|\lambda \pm sw\|_1 &= |a \pm s(u + 1)| + |a + c - 49kt| + |c \pm su| \\
\|\lambda \pm sw'\|_1 &= |a \pm su| + |a + c - 49kt| + |c \pm s(u + 1)|.
\end{align}

We note that these numbers are the distances with respect to $\|\cdot\|_1$ between $(a, c)$ and $(s(u + 1), su)$ plus $|b|$, with $b = a + c - 49kt$.

We first deal with the quadrants. Let $\lambda = (a, b, c)$ and $\lambda' = (a', b', c')$ in $\mathcal{L}_{7k} = \mathcal{L}_{7k}'$ with $k \in \mathbb{N}$.

- $Q_{NE}^{\pm} \approx Q_{NE}^{\pm}$. Assume that $(a, c) \in Q_{NE}^{\pm}$ and $(a', c') \in Q_{NE}^{\pm}$, i.e. $a, a' \geq -su$ and $c, c' \geq -su$. 
These inequalities imply that the first and third term in (5.12) and (5.13) with + are non-negative, hence
\[
\|
\lambda + sw\|_1 = a + s(u + 1) + |b| + c + su, \\
\|
\lambda' + sw'\|_1 = a' + su + |b'| + c' + s(u + 1).
\]
Hence, if we let \((a, c) = (a', c')\), then \(b = b'\) and \(\|
\lambda + sw\|_1 = \|
\lambda' + sw'\|_1\), thus the identity map is a bijection between \(Q_{NE}^+\) and \(Q_{NE}^-\) that preserves \(\|
\cdot\|_1\).

In the case of \(Q_{NE}^- \approx Q_{NE}^+\), since the first and third terms in (5.12) and (5.13), with −, are again non-negative, the numbers \(\|
\lambda - sw\|_1\) and \(\|
\lambda - sw'\|_1\) are again equal to each other, by letting \((a, c) = (a', c')\). The \(\|
\cdot\|_1\)-preserving bijection is again the identity map.

- \(Q_{SW}^+ \approx Q_{SW}^-\). This case is similar to the previous one, but now the first and third terms in (5.12) and (5.13) are non-positive. This implies again that if \((a, c) = (a', c')\), then \(\|
\lambda + sw\|_1 = \|
\lambda' + sw'\|_1\) and the identity map again does the job.

- \(Q_{NW}^+ \approx Q_{NW}^-\). Assume that \((a, c) \in Q_{NW}^+\) and \((a', c') \in Q_{NW}^-,\), i.e. \(a, a' \leq -s(u + 1)\) and \(c, c' \geq s(u + 1)\).

By (5.10) and (5.11) we have that
\[
\|
\lambda + sw\|_1 = -a + |b| + c - s, \\
\|
\lambda' + sw'\|_1 = -a' + |b'| + c' - s,
\]
thus the identity map from \(Q_{NW}^+\) to \(Q_{NW}^-\) is an isometry. In the remaining case, the identity map from \(Q_{NW}^-\) to \(Q_{NW}^+\) is an isometry.

- \(Q_{SE}^+ \approx Q_{SE}^-\). The proof is entirely similar to the previous one. It is omitted.

Since in (5.10) and (5.11) the parameters \(a\) and \(c\) satisfy \(a + 3c \equiv 0\) (mod 7), it will be useful to introduce the following notation
\[
(a, c) = ([−3c], c) = (a, [2a]), \text{ where } [x]
\]
denotes the set of numbers in a given interval such that \([x] \equiv x\) (mod 7).

Under this convention, (5.10) and (5.11) look as follows
\[
\begin{align}
\|
\lambda + sw\|_1 &= |a + s(u + 1)| + |a + 2a - 49kt| + |2a| + su \quad (5.14) \\
\|
\lambda + sw\|_1 &= \|−3c + s(u + 1)\| + \|−3c\| + c - 49kt + |c + su|, \\ \\
\|
\lambda' + sw'\|_1 &= |a' + su| + |a' + 2a' - 49kt| + |2a'| + s(u + 1) \\
\|
\lambda' + sw'\|_1 &= \|−3c' + su\| + \|−3c'\| + c' - 49kt + |c'| + s(u + 1)|. \\
\end{align}
\]

Let \(\lambda = (a, b, c)\) and \(\lambda' = (a', b', c')\) in \(L_{7k} = L_{7k}^+\) with \(k \in \mathbb{N}\).

- \(S_{SW}^+ \approx S_{NW}^+\). Assume that \((a, c) \in S_{SW}^+\) and \((a', c') \in S_{NW}^+\), thus
\[
−su \leq a < s(u + 1), \quad c \leq −s(u + 1), \\
−su \leq c' < s(u + 1), \quad a' \leq −s(u + 1).
\]

We write \((a, c) = ([−3c], c)\) and \((a', c') = (a', [2a'])\). Note that, for fixed \(c\) and \(a'\), the elements in \([−3c]\) and \([2a]\) live in the interval \([−su, s(u + 1)]\) of length \(s(u + 1) − (−su) = 7st\), thus, in each case, there are \(st\) of them.
By (5.16) and (5.17) we have that
\[ \|\lambda + sw\|_1 = ([−3c] + s(u + 1)) − ([−3c] + c − 49kt) − (c + su) \]
\[ = 49kt − 2c + s, \]
\[ \|\lambda' + sw'\|_1 = −(a' + su) − (a' + [2a'] − 49kt) + ([2a'] + s(u + 1)) \]
\[ = 49kt − 2a' + s. \]

Hence, the bijection from \( S^+_W \) onto \( S^+_V \) sending \((a, c)\) to \((a', c')\) with \( c = a' \) and the \( s t \) numbers \( a \) in \([−3c]\) to the \( s t \) numbers \( c' \) in \([2a']\), preserves \( \|\cdot\|_1 \).

- \( S^+_W \approx S^+_V \). An entirely similar argument as before shows that \( S^+_W \approx S^+_V \).
- \( T^+_S \approx T^+_V \). Assume that \((a, c) \in T^+_S \) and \((a', c') \in T^+_V \), thus \( −s(u + 1) < a < −su \), \( su < a' < s(u + 1) \) and \( c, c' \leq −s(u + 1) \).
  
  We write \((a, c) = ([−3c], c) \) and \((a', c') = ([−3c'], c') \). By (5.16) and (5.18),
  
  \[ \|\lambda + sw\|_1 = ([−3c] + s(u + 1)) − ([−3c] + c − 49kt) − (c + su) \]
  \[ = 49kt − 2c + s, \]
  \[ \|\lambda' + sw'\|_1 = ([−3c'] − su) − ([−3c'] + c' − 49kt) − (c' − s(u + 1)) \]
  \[ = 49kt − 2c' + s. \]

The correspondence between these sets is given by \( c = c' \) since for each \( c \leq −s(u + 1) \), \( a \) is in \([−3c]\) and satisfies \( −s(u + 1) < a < −su \) if and only if \( a' = a + s(2u + 1) = a + 7ts \) is in \([−3c']\) and satisfies \( su < a' < s(u + 1) \).

- \( T^-_S \approx T^-_V \). Assume that \((a, c) \in T^-_S \) and \((a', c') \in T^+_V \), thus \( su < a < s(u + 1) \), \( −s(u + 1) < a' < −su \) and \( c, c' \leq −s(u + 1) \).
  
  We write \((a, c) = ([−3c], c) \) and \((a', c') = ([−3c'], c') \). By (5.16) and (5.18),
  
  \[ \|\lambda − sw\|_1 = −([−3c] − s(u + 1)) − ([−3c] + c − 49kt) − (c − su) \]
  \[ = 49kt − 2c − 2[−3c] + s(2u + 1) \]
  \[ = 49kt − 2c − 2[−3c] + 7ts, \]
  \[ \|\lambda' + sw'\|_1 = −([−3c'] + su) − ([−3c'] + c' − 49kt) − (c' + s(u + 1)) \]
  \[ = 49kt − 2c' − 2[−3c'] − s(2u + 1) \]
  \[ = 49kt − 2c' − 2[−3c'] − 7ts. \]

Note that if \( c = c' \) then \([−3c] \in (su, s(u + 1))\) if and only if \([−3c'] \in (−s(u + 1), −su)\); moreover \([−3c] = [−3c'] + 2.7ts \), which shows that the bijection \( c = c' \) is \( \|\cdot\|_1 \)-preserving between these regions.

- \( T^+_W \approx T^-_W \). The proof in this case follows by the same argument as in \( T^-_S \approx T^+_V \) since one obtains \( \|\lambda + sw\|_1 = 49kt − 2a − 2[2a] − 7ts \) and \( \|\lambda' + sw'\|_1 = 49kt − 2a' − 2[2a'] + 7ts \).
- \( T^-_W \approx T^+_W \). The proof in this case follows by the same argument as in \( T^-_S \approx T^+_V \) since one obtains \( \|\lambda − sw\|_1 = 49kt − 2a + s \) and \( \|\lambda' + sw'\|_1 = 49kt − 2a' + s \).
\( S_{E} \approx S_{N} \). This case will need a more elaborate argument. Assume that 
\((a, c) \in S_{E}^{+}\) and \((a', c') \in S_{N}^{+}\), thus \(a, c' \geq s(u + 1)\), 
\(-s(u + 1) < c < su\) and 
\(-s(u + 1) < a' < su\). By (5.12) and (5.13),
\[
\| \lambda - sw \|_1 = (a - s(u + 1)) + |a + c - 49kt| - (c - su),
\]
\[
= a - c - s + |a + c - 49kt|,
\]
(5.19) 
\[
\| \lambda' - sw' \|_1 = -(a' - su) + |a' + c' - 49kt| + (c' - s(u + 1)),
\]
\[
= -a' + c' - s + |a' + c' - 49kt|.
\]

At this point, it is convenient to split the half-line \([s(u + 1), \infty)\] where \(a\) and \(c'\) lie in three disjoint intervals, namely

\[
(5.21) \quad [s(u + 1), 49kt - su) \cup [49kt - su, 49kt + s(u + 1)) \cup [49kt + s(u + 1), \infty)
\]

with corresponding splittings of \( S_{E}^{+} \) and \( S_{N}^{+} \):

\[
S_{E}^{-} = S_{E,1}^{-} \cup S_{E,2}^{-} \cup S_{E,3}^{-}, \quad S_{N}^{-} = S_{N,1}^{-} \cup S_{N,2}^{-} \cup S_{N,3}^{-}.
\]

We will show that \( S_{E,i}^{-} \approx S_{N,i}^{-} \) for \( i = 1, 2, 3 \) by using a different argument in each case.

- \( S_{E,3}^{-} \approx S_{N,3}^{-} \). In this case we have that \( a, c' \geq 49kt + s(u + 1)\). By (5.19) and (5.20) we obtain

\[
\| \lambda - sw \|_1 = -49kt + 2a - s, \quad \| \lambda' - sw' \|_1 = -49kt + 2c' - s.
\]

Hence, the bijection that sends \((a, c)\) to \((a', c')\) with \(a = c'\) and the \(st\) numbers \(c\) in \([2a]\) to the \(st\) numbers \(a'\) in \([-3c']\), preserves \(\|\cdot\|_1\).

- \( S_{E,1}^{-} \approx S_{N,1}^{-} \). In this case we have that \(s(u + 1) \leq a, c' < 49kt - su\).

By (5.19) and (5.20) we obtain

\[
\| \lambda - sw \|_1 = 49kt - 2c - s, \quad \| \lambda' - sw' \|_1 = 49kt - 2a' - s.
\]

The numbers \(c\) and \(a'\) are in \([2a]\) and \([-3c']\) respectively. Furthermore, they are in the same interval \([-s(u + 1), su - 1]\) which has length \(su - (-s(u + 1))\) = 7ts.

If \(a\) (resp. \(c'\)) runs through 7 consecutive integers, then \(2a\) (resp. \(-3c'\)) runs through all congruence classes mod 7. Also, for \(a\) (resp. \(c'\)) running through any interval of length 7, the elements in the set \([2a]\) (resp. \([-3c']\)) take all 7st values in the interval \([-s(u + 1) + 1, su]\) once. Since the length of the interval \([s(u + 1), 49kt - su - 1]\) for \(a, c'\) is 49kt - 7st, a multiple of 7, this implies that \(S_{E,1}^{-} \approx S_{N,1}^{-}\).

- \( S_{E,2}^{-} \approx S_{N,2}^{-} \). We now consider the case of the ‘short’ interval, that is, when \(49kt - su \leq a, c' < 49kt + s(u + 1)\).

We write \(c\) as \([2a]\) and \(a'\) as \([-3c']\). Since the expressions \(a + [2a] - 49kt\) and \(c' + [-3c'] - 49kt\) change sign, it is now convenient to write \(a = 49kt + \varepsilon\) and \(c' = 49kt + \varepsilon'\) with \(\varepsilon, \varepsilon' \in I_2 = [-su, s(u + 1))\), thus \([2a] = [2\varepsilon]\) and \([-3c'] = [-3\varepsilon']\).
By (5.19) and (5.20) we obtain
\[ ||\lambda - sw||_1 = 49kt - s + f(\varepsilon), \]
\[ ||\lambda' - sw'||_1 = 49kt - s + g(\varepsilon'), \]
where \( f(\varepsilon) := \varepsilon - [2\varepsilon] + [\varepsilon + 2|\varepsilon|] \) and \( g(\varepsilon') := \varepsilon' - [-3\varepsilon'] + [3\varepsilon'] + \varepsilon' \). We note that \( f \) and \( g \) are multifactions, that is \( f(\varepsilon) \) and \( g(\varepsilon') \) are multisets for each \( \varepsilon, \varepsilon' \).

Thus, to prove that \( S_{E,2}^{-} \approx S_{N,2}^{-} \) it will be sufficient to prove the equality of the multisets:
\[ \{ \{ f(\varepsilon) : \varepsilon \in I_2 \} \} = \{ \{ g(\varepsilon') : \varepsilon' \in I_2 \} \}. \]

This will be a consequence of the following lemma, by choosing \( A = I_2 \).

**Lemma 5.7.** Let \( A \) be a finite interval in \( \mathbb{Z} \) with \( |A| \equiv 0 \pmod{7} \). For \( x \in A \) consider the multifactions given by \( f(x) = x - [2x] + [x + 2|x|] \) and \( g(x) = x - [-3x] + [x + -3x] \), where \([2x], [-3x] \) lie in \(-A\). Then
\[ \{ \{ f(x) : x \in A \} \} = \{ \{ g(x) : x \in A \} \}. \]

**Proof.** Denote by \( \bar{A} = \{ t \in A : t \equiv x \pmod{7} \} \) and let \( f(\bar{x}) = \{ \{ f(x) : x \in \bar{x} \} \} \). One can check the following identities of subsets of \(-A:\)

\[ -\bar{x} = [-x], \quad |t\bar{x}| = |\bar{x}| = |tx|. \]

This implies that
\[ f(\bar{x}) = 3x - [2|\bar{x}|] + |\bar{x}| + [2|\bar{x}|] \]
\[ = 3x - [\bar{x}] + [-\bar{x}] + [-\bar{x}] \]
\[ = -[4x] + \bar{x} + |4x - \bar{x}| \]
\[ = \bar{x} - [4x] + |\bar{x} + 4x| = g(\bar{x}). \]

Now, multiplication by 3 gives a bijection between congruence classes mod 7 in \( A \) and also from \( \bar{x} \) to \( 3\bar{x} \) for each \( x \), since \( |A| \) is a multiple of 7. The assertion in the lemma is now clear. \( \square \)

**Remark 5.8.** The proof of the lemma for general \( r \) is the same, replacing 7 by \( r \), 2 by \( \alpha \) and -3 by \( \beta \), where \( \alpha \beta \equiv 1 \pmod{r} \).

- \( S_N^{-} \approx S_N^{-} \). The argument is the same as that in the case of \( S_E^{-} \approx S_N^{-} \). It is omitted.

We have given the \( ||\cdot||_1 \)-preserving bijection in the regions in (5.8). It remains to consider the regions \( T_{E}^{\pm}, T_{N}^{\pm}, R_{N}^{-}, R_{E}^{-}, R_{S}^{\pm}, R_{W}^{\pm}, \) and the corresponding regions in \( \Pi' \). As before, we split the interval \( [s(u+1), \infty] \) into three intervals as in (5.21) and then split the thin strips \( T_{N}^{\pm}, T_{E}^{\pm}, T_{N}^{-}, T_{E}^{-}, T_{E}^{\pm} \) accordingly. Thus, \( T_{N}^{\pm} = \cup_{i=1}^{3} T_{N,i}^{\pm}, \quad T_{E}^{\pm} = \cup_{i=1}^{3} T_{E,i}^{\pm}, \) and similarly for \( T_{N}^{-} \) and \( T_{E}^{-} \). We will show that
\[ T_{E,3}^{\pm} \approx T_{E,3}^{\pm}, \quad T_{N,3}^{\pm} \approx T_{N,3}^{\pm}, \quad T_{E,2}^{-} \approx R_{N}^{-}, \quad T_{E,2}^{-} \approx R_{E}^{-}, \]
\[ R_{N}^{-} \approx T_{E,2}^{-}, \quad T_{E}^{-} \approx T_{N}^{-}, \quad R_{W}^{-} \approx R_{S}^{-}. \]
and also that
\[(5.24)\]
\[T_{E,1}^{-} \approx T_{N,1}^{-}, \quad T_{N,1}^{+} \cup T_{N,2}^{+} \approx T_{E,1}^{+} \cup T_{E,2}^{+}, \quad T_{N,1}^{-} \cup C^{-} \approx T_{E,1}^{-} \cup C^{-},\]
\[T_{E,1}^{+} \cup T_{E,2}^{+} \cup R_{S}^{+} \cup C^{+} \approx T_{N,1}^{+} \cup T_{N,2}^{+} \cup R_{W}^{+} \cup C^{+}.\]

\[\bullet \quad T_{E,3}^{+} \approx T_{E,3}^{-}. \quad \text{Assume that } (a, c) \in T_{E,3}^{+} \text{ and } (a', c') \in T_{E,3}^{-}, \text{ thus } a, a' \geq 49kt + s(u + 1) \text{ and } su < c < -su \text{ and } su < c' < s(u + 1),\]

We write \((a, c) = (a, [2a])\) and \((a', c') = (a', [2a'])\). By (5.15) and (5.17), one obtains that \(\|\lambda + sw\|_1 = -49kt + 2a + s\) and \(\|\lambda' - sw'\|_1 = -49kt + 2a' + s\).

The correspondence between these sets is again given by \(a = a'\) since for each \(a \geq 49kt + s(u + 1), \ c\) is in \([2a]\) and satisfies \(-s(u + 1) < c < -su\) if and only if \(c' = c + s(2u + 1) = c + 7ts\) is in \([2a']\) and satisfies \(su < c' < s(u + 1)\).

\[\bullet \quad T_{E,3}^{-} \approx T_{E,3}^{+}. \quad \text{Assume that } (a, c) \in T_{E,3}^{-} \text{ and } (a', c') \in T_{E,3}^{+}, \text{ thus } a, a' \geq 49kt + s(u + 1), \ su < c < s(u + 1) \text{ and } -s(u + 1) < c' < -su.\]

We write \((a, c) = (a, [2a])\) and \((a', c') = (a', [2a'])\). By (5.15) and (5.17), one obtains that \(\|\lambda - sw\|_1 = -49kt + 2a + 2[2a] + 7ts\) and \(\|\lambda' + sw'\|_1 = -49kt + 2a' + 2[2a'] + 7ts\).

The correspondence between these sets is again given by \(a = a'\) since, for each \(a = a' \geq 49kt + s(u + 1), \) the following holds: there exists \(c \in [2a]\) with \(su < c < s(u + 1)\) if and only if there is \(c' \in [2a']\) with \(-s(u + 1) < c' < -su\) and, when this happens, \(2c - 7ts = 2c' + 7ts, \) thus the corresponding \(1\)-norms are equal to each other.

\[\bullet \quad T_{N,3}^{+} \approx T_{N,3}^{-}. \quad \text{The proof in this case follows by the same argument as in } T_{E,3}^{-} \approx T_{E,3}^{+} \text{ since one obtains } \|\lambda + sw\|_1 = -49kt + 2c + 2[-3c] + 7ts \text{ and } \|\lambda' + sw'\|_1 = -49kt + 2c' + 2[-3c'] - 7ts.\]

\[\bullet \quad T_{N,3}^{-} \approx T_{N,3}^{+}. \quad \text{The proof in this case follows by the same argument as in } T_{E,3}^{+} \approx T_{E,3}^{-} \text{ since one obtains } \|\lambda + sw\|_1 = -49kt + 2c + s \text{ and } \|\lambda' + sw'\|_1 = -49kt + 2c' + s.\]

\[\bullet \quad R_{N}^{-} \approx T_{E,2}^{-}. \quad \text{Suppose that } (a, c) \in R_{N}^{-} \text{ and } (a', c') \in T_{E,2}^{-}, \text{ thus } \]

\[-s(u + 1) < a \leq su, \quad su < c < s(u + 1), \quad 49kt - su \leq a' < 49kt + s(u + 1), \quad su < c' < s(u + 1).\]

We write \((a, c) = (a, [2a])\) and \((a', c') = (a', [2a'])\). By (5.15) and (5.17), one obtains that

\[\|\lambda - sw\|_1 = 49kt - 2a + s, \quad \|\lambda' - sw'\|_1 = -49kt + 2a' + s.\]

If we write \(a = -\varepsilon\) and \(a' = 49kt + \varepsilon', \) then \(-su \leq \varepsilon < s(u + 1), \ -su \leq \varepsilon' < s(u + 1)\) and

\[\|\lambda - sw\|_1 = 49kt + 2\varepsilon + s, \quad \|\lambda' - sw'\|_1 = 49kt + 2\varepsilon' + s.\]

Thus, the correspondence \(\varepsilon = \varepsilon'\) gives a \(\|\|_1\)-preserving bijection.
We claim that the correspondence \( \varepsilon \) valid, by observing that 
\[
-\varepsilon < \frac{s(u + 1)}{2} < s(u + 1),
\]
\[
-s < s(u + 1),
\]
\[
-s < \varepsilon < s(u + 1).
\]
We write \((a, c) = (a, [2a])\) and \((a', c') = (a', [2a'])\). By (5.15) and (5.17), one obtains that
\[
\| \lambda - sw \|_1 = -49kt + 2a + 2[2a] - 7ts,
\]
\[
\| \lambda' - sw' \|_1 = 49kt - 2a' - 2[2a'] + 7ts.
\]
If we write \(a = 49kt + \varepsilon\) and \(a' = -\varepsilon\), then 
\[
-s < \varepsilon < s(u + 1),
\]
\[
-s < \varepsilon < s(u + 1),
\]
\[
\| \lambda' - sw' \|_1 = 49kt + 2\varepsilon + 2[2\varepsilon] - 7ts,
\]
\[
\| \lambda' - sw' \|_1 = 49kt + 2\varepsilon - 2[-2\varepsilon'] + 7ts.
\]
We claim that the correspondence \( \varepsilon = \varepsilon' \) gives a \( \| \cdot \|_1 \)-preserving bijection.
To prove this, it suffices to show that \(2[2\varepsilon] - 7ts = -2[-2\varepsilon'] + 7ts\) or, equivalently, that \(2[2\varepsilon] + [-2\varepsilon'] = 7ts\). One checks that the last equality is valid, by observing that \([-x] = [x] + 7ts\) when \([x] \in (su, s(u + 1))\).
Note that, for \(-su \leq \varepsilon < s(u + 1)\) (resp. \(-su < \varepsilon < s(u + 1)\)), if \(c\) is in \([2\varepsilon]\) (resp. if \(c'\) is in \([2\varepsilon']\)) and satisfies \(su < c < s(u + 1)\) (resp. \(su < c' < s(u + 1)\)) then 
\[-s < -2c - 7ts < s\] (resp. 
\[-s < -2c' - 7ts < s\]).
Hence, the correspondence \( \varepsilon = \varepsilon' \) gives a \( \| \cdot \|_1 \)-preserving bijection since the sets \(-s < -2[2\varepsilon] - 7ts < s\) and \(-s < -2[2\varepsilon'] + 7ts < s\) have the same cardinality.

- \( R_{E, 2}^- \approx T_{E, 2}^- \). The proof in this case follows by the same argument as in \( T_{E, 2}^- \approx R_N^- \) since one obtains 
\[
\| \lambda - sw \|_1 = 49kt - 2c + 2[2c] + 7ts
\]
and 
\[
\| \lambda' - sw' \|_1 = -49kt + 2c + 2[2c] + 7ts.
\]
- \( T_{N, 2}^- \approx R_{W}^- \). The proof in this case follows by the same argument as in \( R_{N}^- \approx T_{E, 2}^- \) since one obtains 
\[
\| \lambda - sw \|_1 = 49kt + 2c + s
\]
and 
\[
\| \lambda' - sw' \|_1 = 49kt - 2c + s.
\]
- \( R_{W}^+ \approx R_{S}^+ \). In this case the arguments in (5.16) and (5.17) are positive-negative-positive, so the corresponding norms are constant: 
\[
\| \lambda - sw \|_1 = \| \lambda' - sw' \|_1 = 49kt - 7ts.
\]
Since both rectangles have a side of length a multiple of 7, they have the same number of points in \( \Pi = \Pi' \), thus the bijection is clear.

The next lemma will be very useful in the last four cases, the most delicate
in the proof. We write the proof for general \( r \).

Lemma 5.9. Let \( r \) be odd, \((r, 3) = 1\). Let \( \alpha, \beta \in \mathbb{Z} \) with \( \alpha \beta \equiv 1 \pmod{r}\).
Let \( A = (su, s(u + 1)) \). Then
(i) \( \# \Pi \cap (-A \times A) = \# \Pi \cap (A \times -A) \). More generally, for any finite
intervals \( B, C \subset \mathbb{Z} \), one has \( \# \Pi \cap (B \times C) = \# \Pi \cap (B' \times C') \), where
\( B' = jr + B \) and \( C' = j'r + C \) with \( j, j' \in \mathbb{Z} \).
(ii) As multisets, \( \{x + [\alpha x] + |x + [\alpha x]|\} = \{x + [\beta x] + |x + [\beta x]|\} \), for \( x \in -A \) and \( [\alpha x], [\beta x] \in A \).

(iii) As multisets, \( \{x - [\alpha x] + |x + [\alpha x]|\} = \{x - [\beta x] + |x + [\beta x]|\} \) for \( x \in -A \) and \( [\alpha x], [\beta x] \in A \).

Proof. (i) The lattice \( \Pi \) is invariant by translations by vectors in \( r\mathbb{Z} \times r\mathbb{Z} \) and \( A \times -A \) is the translate of \( -A \times A \) by the vector \( (s(2u + 1), -s(2u + 1)) = (srt, -srt) \).

(ii) The translation in (i) takes the point \( (x, [\beta x]) \in -A \times A \) into the point \( ([\alpha x], x) \in A \times -A \) so that the expression \( x + [\beta x] + |x + [\beta x]| \), that we associate to the first point, equals the expression \( x + [\alpha x] + |x + [\alpha x]| \), associated to the second point. This establishes a bijection between the two multisets.

(iii) \( \Pi \) is invariant by the central symmetry (with respect to the origin). This symmetry will establish a bijection between both multisets by looking at the first multiset as the numbers \( x - y + |x + y| \) associated to the points \( (x, y) \in (\Pi \cap (-A \times A)) \) with \( y = [\alpha x] \) and at the second multiset as the numbers \( -y - (-x) + |-y + (-x)| \) associated to the points \( (-x, -y) \in (\Pi \cap (A \times -A)) \) with \( (-x, -y) = ([\beta x], x) \). Since both numbers are equal, this establishes a bijection between the two multisets. \( \square \)

- \( \mathcal{T}_{E, 1} \approx \mathcal{T}_{N, 1}^- \). Suppose that \( (a, c) \in \mathcal{T}_{E, 1} \) and \( (a', c') \in \mathcal{T}_{N, 1}^- \), thus \( s(u + 1) \leq a, c' < 49kt - su \) and \( su < c, a' < s(u + 1) \).

We write \( (a, c) = (a, [2a]) \) and \( (a', c') = ([3c'], c') \). We get norms

\[
\|\lambda - sw\|_1 = a + [2a] - 7ts + |a + [2a] - 49kt|, \\
\|\lambda' - sw'\|_1 = c' + [3c'] - 7ts + |c' + [3c'] - 49kt|.
\]

We now split the interval \( [s(u + 1), 49kt - su] = [s(u + 1), 49kt - s(u + 1)] \cup (49kt - s(u + 1), 49kt - su) \), where \( a \) and \( c' \) live. In the first subinterval we have \( a + [2a] - 49kt \leq 0 \) and \( c' + [3c'] - 49kt \leq 0 \), thus we get \( \|\lambda - sw\|_1 = \|\lambda' - sw'\|_1 = 49kt - 7ts \), a constant value.

We claim that the number of points of \( \Pi = \Pi' \) in the region \( [s(u + 1), 49kt - su] \times (su, s(u + 1)) \) is the same as in the region \( (su, s(u + 1)) \times [s(u + 1), 49kt - s(u + 1)] \). In the first place, we have that the number of points of \( \Pi \) in the region \( \mathcal{T}_{E, 1}^- \) is the same as the number of points of \( \Pi' \) in the region \( \mathcal{T}_{N, 1}^- \), since the length of the interval \( [s(u + 1), 49kt - su] \) is \( 49kt - s(2u + 1) = 49kt - 7ts \), a multiple of 7. Secondly, by Lemma 5.9(i), there are the same number of points in the small squares \( (49kt - s(u + 1), 49kt - su) \times (su, s(u + 1)) \) and \( (su, s(u + 1)) \times (49kt - s(u + 1), 49kt - su) \), both having sides of length \( s - 1 \). These two facts imply our claim.

On the other hand if we look at the points \( (a, [2a]) \) and \( ([3c'], c') \) in the small squares, the expressions \( a + [2a] - 49kt \) and \( c' + [3c'] - 49kt \) change sign. We reparametrize \( a = 49kt + \varepsilon \) and \( c' = 49kt + \varepsilon' \). Thus, we need to
compare
\[ \| \lambda - sw \|_1 = 49kt - 7ts + (\varepsilon + [2\varepsilon]) + |\varepsilon + [2\varepsilon]| \quad \text{and} \quad \| \lambda' - sw' \|_1 = 49kt - 7ts + (\varepsilon' + [-3\varepsilon']) + |\varepsilon' + [-3\varepsilon']|. \]

That these multisets are the same, follows from Lemma \ref{lem:isospectral} ii).

- \( T_{N,1}^+ \cup T_{N,2}^+ \approx T_{E,1}^+ \cup T_{E,2}^+ \): Suppose that \((a, c) \in T_{N,1}^+ \cup T_{N,2}^+\) and \((a', c') \in T_{E,1}^+ \cup T_{E,2}^+,\) thus \(-s(u+1) < a, c' < -su\) and \(s(u+1) \leq c, a' < 49kt + s(u+1)\).

Again we remove a small square at the end by splitting \([s(u+1), 49kt + su] \cup (49kt + su, 49kt + s(u+1))\). The argument is identical as that in the previous case, thus it will be omitted.

- \( T_{N,1}^+ \cup C^- \approx T_{E,1}^- \cup C'^- \): Suppose that \((a, c) \in T_{N,1}^+ \cup C^-\) and \((a', c') \in T_{E,1}^- \cup C'^-\), thus \(su < c, a' < 49kt - su\) and \(su < a, c' < s(u+1)\).

We write \((a, c) = ([3\varepsilon], c)\) and \((a', c') = (a', [2a'])\). We remove from the second interval a subinterval of length \(s-1\). Then \(su < c, a' \leq 49kt - s(u+1)\) of length \(49kt - s(2u+1)\), that is a multiple of 7, and \(49kt - s(u+1) < c, a' < 49kt - su\) of length \(s-1\).

The \(1\)-norms in the first subinterval equal respectively
\[ \| \lambda - sw \|_1 = 49kt + s - 2[-3c], \quad \| \lambda' - sw' \|_1 = 49kt + s - 2[2a']. \]

Since the length of the subinterval is a multiple of 7, these give the same set of numbers counted with multiplicities.

Relative to the second subinterval, by doing the substitution \(c = 49kt + x, a' = 49kt + x,\) with \(x\) in the interval \(-s(u+1) < x < -su\), we get two squares of side \(s - 1\), thus Lemma \ref{lem:isospectral} ii) applies directly to give \(1\)-isospectrality.

- \( T_{E,1}^+ \cup T_{E,2}^+ \cup R_{S}^+ \cup C^+ \approx T_{N,1}^+ \cup T_{N,2}^+ \cup R_{W}^+ \cup C'^+ \): This case is solved exactly as the case of \( T_{N,1}^- \cup C^- \approx T_{E,1}^- \cup C'^- \), by removing a square of side \(s - 1\) at the north and east ends, respectively, of each region. In these squares Lemma \ref{lem:isospectral} iii) applies. In the remaining part of the regions we argue as before, since it involves intervals with length a multiple of 7.

To sum up, we have proved that \eqref{eq:isospectral} holds for every \(k \geq 0\) and every \(s = 1, 2, 3\). For \(k < 0\) \eqref{eq:isospectral} also holds since \(L_h \approx L_{-h}\) for all \(h \in \mathbb{Z}\).

Finally, the case \(k = 0\) is simpler and will be omitted since the same type of arguments work. This proves that \(L \approx L'\) by \eqref{eq:isospectral}.

\begin{remark}
In Theorem \ref{thm:isospectral} for the purpose of the proof we have required \(rt\) to be odd and \(r\) coprime to 3. However, if \(rt\) is even and \(r \neq 0 \pmod{3}\), the theorem should equally hold with a very similar proof. Some variations are necessary since we do not have at our disposal the vectors \(w = (rt+1, 0, r, t+1)\) and \(w' = (rt+1, 0, r, t+1)\) in \(L\) and \(L'\). As we will see in Section \ref{sec:isospectral} the lattices \(L\) and \(L'\) defined by \eqref{eq:lattices} are \(\|\cdot\|_1\)-isospectral for many choices of \(r, t\) with \(rt\) even.

\begin{remark}
In Theorem \ref{thm:isospectral} we have proved that \(L_{7k+s} \cup L_{7k-s} \approx L'_{7k+s} \cup L'_{7k-s}\). However, with some more work one can prove that, actually, \(L_k \approx L'_k\) for every \(k \in \mathbb{Z}\).
\end{remark}
6. Computations and questions

In this section we shall use the finiteness theorem of Section 4 to produce, with the help of a computer, many examples of pairs of non-isometric lattices that are \( \| \cdot \|_1 \)-isospectral. In light of Theorem 3.8(ii), each such pair gives rise to a pair of non-isometric lens spaces that are \( p \)-isospectral for all \( p \).

We now explain the computational procedure to find \( \| \cdot \|_1 \)-isospectral lattices. For each \( m \) and \( q \), one finds first, by using Propositions 3.1 and 3.2, a complete list of non-isometric \( q \)-congruence lattices in \( \mathbb{Z}^m \). Then, for each lattice \( L \) in the list, one computes the (finitely many) numbers \( N^\text{red}_L(k, \ell) \) for \( 0 \leq \ell \leq m \) and \( 0 \leq k \leq (m - \ell)(q - 1) \). Next, for each pair of lattices, one compares their associated sets of numbers. Finally, the program puts together the lattices for which these numbers coincide. By Theorem 4.2, such lattices are mutually \( \| \cdot \|_1 \)-isospectral.

By the procedure outlined above, using the computer program Sage [Sa], we found all \( \| \cdot \|_1 \)-isospectral \( m \)-dimensional \( q \)-congruence lattices for \( m = 3 \), \( q \leq 300 \) and \( m = 4 \), \( q \leq 150 \) (see Tables 1 and 2). We point out that all such lattices come in pairs for these values of \( q \) and \( m \) (see Question 6.3). In the tables, the parameters \([s_1, \ldots, s_m]\) and \([s'_1, \ldots, s'_m]\) in a row indicate the corresponding \( \| \cdot \|_1 \)-isospectral lattices \( L(q; s_1, \ldots, s_m) \) and \( L(q; s'_1, \ldots, s'_m) \) as in (3.3).

Next we will attempt to explain in a unified manner the examples appearing in the tables. Let \( r \) and \( t \) be positive integers and set \( q = r^2t \), \( r > 1 \). We let \( \theta = rt - 1 \), considered as an element of \( (\mathbb{Z}/q\mathbb{Z})^\times \), the group of units of \( \mathbb{Z}/q\mathbb{Z} \). We will denote by \( \theta^{-1} \) the inverse of \( \theta \) (mod \( q \)). Clearly, for every \( k \in \mathbb{Z} \),

\[
\theta^k \equiv (-1)^{k+1}(krt - 1) \pmod{q}.
\]

In particular \( \theta^r \equiv (-1)^r \pmod{q} \). Hence, in \( (\mathbb{Z}/q\mathbb{Z})^\times \), \( \theta \) has order \( r \) if \( r \) is even and \( 2r \) if \( r \) is odd.

Thus, if \( \cong_1 \) denotes isometric in \( \| \cdot \|_1 \), then the pairs considered in Section 5 can be written in the form

\[
\begin{align*}
L &= \mathcal{L}(r^2t; 1, rt - 1, 2rt + 1) = \mathcal{L}(q; \theta^0, \theta^1, \theta^{-2}) \cong_1 \mathcal{L}(q; \theta^0, \theta^{-1}, \theta^{-3}), \\
L' &= \mathcal{L}(r^2t; 1, rt - 1, 3rt - 1) = \mathcal{L}(q; \theta^0, \theta^1, \theta^3).
\end{align*}
\]

Here \( \mathcal{L} \) is equal to \( \mathcal{L}(q; \theta^{-1}, \theta^0, \theta^{-3}) \), that is \( \| \cdot \|_1 \)-isometric to \( \mathcal{L}(q; \theta^0, \theta^{-1}, \theta^{-3}) \) via the map that flips the first two coordinates (see Propositions 3.1 and 3.2).

We note that all pairs in the tables have a description in terms of suitable powers of \( \theta \) for some choices of \( r \) and \( t \) such that \( q = r^2t \). For instance, the fifth example in Table 1 if we take \( r = 10 \) and \( t = 1 \) can be written as

\[
\begin{align*}
\mathcal{L}(100; 1, 9, 31) &= \mathcal{L}(q; \theta^0, \theta^1, \theta^{-3}) \cong_1 \mathcal{L}(q; \theta^0, \theta^{-1}, \theta^{-4}), \\
\mathcal{L}(100; 1, 9, 39) &= \mathcal{L}(q; \theta^0, \theta^1, \theta^4)
\end{align*}
\]
Theorem 5.6 (see (6.1)) responds to the following description:

| $q$ | $s_1$, $s_2$, $s_3$ | $s_1'$, $s_2'$, $s_3'$ |
|-----|------------------|------------------|
| 49  | 1, 6, 15         | 1, 6, 20         |
| 64  | 1, 7, 17         | 1, 7, 23         |
| 98  | 1, 13, 29        | 1, 13, 41        |
| 100 | 1, 9, 21         | 1, 9, 29         |
| 100 | 1, 9, 31         | 1, 9, 39         |
| 121 | 1, 10, 23        | 1, 10, 32        |
| 121 | 1, 10, 34        | 1, 10, 43        |
| 121 | 1, 10, 45        | 1, 10, 54        |
| 121 | 1, 21, 34        | 1, 21, 54        |
| 121 | 1, 21, 45        | 1, 21, 56        |
| 128 | 1, 15, 33        | 1, 15, 47        |
| 147 | 1, 20, 43        | 1, 20, 62        |
| 169 | 1, 12, 27        | 1, 12, 38        |
| 169 | 1, 12, 53        | 1, 12, 64        |
| 169 | 1, 12, 66        | 1, 12, 77        |
| 169 | 1, 25, 40        | 1, 25, 64        |
| 169 | 1, 25, 53        | 1, 25, 77        |
| 169 | 1, 38, 53        | 1, 38, 79        |
| 169 | 1, 12, 40        | 1, 12, 51        |
| 169 | 1, 25, 66        | 1, 25, 79        |
| 192 | 1, 25, 73        | 1, 49, 73        |
| 196 | 1, 13, 29        | 1, 13, 41        |
| 196 | 1, 13, 57        | 1, 13, 69        |
| 196 | 1, 41, 71        | 1, 41, 85        |
| 196 | 1, 13, 43        | 1, 13, 55        |
| 196 | 1, 13, 71        | 1, 13, 83        |
| 196 | 1, 27, 43        | 1, 27, 69        |
| 196 | 1, 27, 57        | 1, 27, 83        |
| 200 | 1, 19, 41        | 1, 19, 59        |
| 200 | 1, 19, 61        | 1, 19, 79        |
| 242 | 1, 21, 45        | 1, 21, 65        |

Pairs marked with * belong to the family given in Theorem 5.6.

and, the first pair in Table 2 if $r = 7$ and $t = 1$ becomes

\[ \mathcal{L}(49; 1, 6, 8, 20) = \mathcal{L}(q; \theta^0, \theta^1, \theta^{-1}, \theta^3) \cong_1 \mathcal{L}(q; \theta^0, \theta^1, \theta^2, \theta^4), \]
\[ \mathcal{L}(49; 1, 6, 8, 22) = \mathcal{L}(q; \theta^0, \theta^1, \theta^{-1}, \theta^{-3}) \cong_1 \mathcal{L}(q; \theta^0, \theta^{-1}, \theta^{-2}, \theta^{-4}). \]

We point out that all examples shown in Tables 1 and 2 and those in Theorem 5.6 (see (6.1)) respond to the following description:

\[ \mathcal{L}(q; \theta^{d_0}, \theta^{d_1}, \ldots, \theta^{d_{m-1}}) \text{ and } \mathcal{L}(q; \theta^{-d_0}, \theta^{-d_1}, \ldots, \theta^{-d_{m-1}}), \]
Table 2. Pairs of \(\|\cdot\|_{\theta}^\dagger\)-isospectral \(q\)-congruence lattices of dimension \(m = 4\) for \(q \leq 150\).

| \(q\) | \(s_1, s_2, s_3, s_4\) | \(s'_1, s'_2, s'_3, s'_4\) |
|---|---|---|
| 49 | [1, 6, 8, 20] | [1, 6, 8, 22] |
| 81 | [1, 8, 10, 26] | [1, 8, 10, 28] |
| 81 | [1, 8, 10, 35] | [1, 8, 10, 37] |
| 81 | [1, 13, 15, 41] | [1, 13, 15, 43] |
| 100 | [1, 9, 11, 29] | [1, 9, 11, 31] |
| 100 | [1, 9, 21, 39] | [1, 9, 29, 31] |
| 121 | [1, 10, 12, 32] | [1, 10, 12, 34] |
| 121 | [1, 10, 12, 54] | [1, 10, 12, 56] |
| 121 | [1, 10, 23, 56] | [1, 10, 32, 56] |
| 121 | [1, 10, 34, 54] | [1, 10, 43, 45] |
| 121 | [1, 121, 23, 54] | [1, 21, 23, 56] |
| 121 | [1, 10, 12, 43] | [1, 10, 12, 45] |
| 121 | [1, 10, 23, 43] | [1, 10, 32, 34] |
| 121 | [1, 10, 23, 45] | [1, 10, 32, 54] |
| 121 | [1, 10, 23, 54] | [1, 10, 32, 45] |
| 121 | [1, 10, 34, 54] | [1, 10, 43, 56] |
| 144 | [1, 11, 13, 47] | [1, 11, 13, 47] |
| 144 | [1, 11, 25, 59] | [1, 11, 35, 49] |
| 147 | [1, 20, 22, 62] | [1, 20, 22, 64] |

where \(q = r^2t\), \(r > 1\), \(\theta = rt - 1\) and \(0 = d_0 < d_1 < \cdots < d_{m-1} < r\). However, for some choices of \(m, r\) and \(t\), there are sequences \(0 = d_0 < d_1 < \cdots < d_{m-1} < r\) such that the lattices defined as in (6.4) are not \(\|\cdot\|_{\theta}^\dagger\)-isospectral. For example, this is the case when \(m = 3, r = 8, t = 1\) and \([d_0, d_1, d_2] = [0, 1, 4]\).

The following questions come up naturally.

**Question 6.1.** Give conditions on the sequence \(0 = d_0 < d_1 < \cdots < d_{m-1} < r\) for lattices as in (6.4) to be \(\|\cdot\|_{\theta}^\dagger\)-isospectral.

**Question 6.2.** Are there examples of \(\|\cdot\|_{\theta}^\dagger\)-isospectral lattices that are not of the type in (6.4) for some choice of \(\theta\)?

**Question 6.3.** Are there families of \(\|\cdot\|_{\theta}^\dagger\)-isospectral lattices having more than two elements?

We have carried out computations for small values of \(m\) and \(q\) and in this search we have not found yet any such family.

We next give a particular sequence as in (6.4) that is very likely to give \(\|\cdot\|_{\theta}^\dagger\)-isospectral pairs in all dimensions under rather general conditions on \(r\), for instance, if \(r\) is prime. This motivation makes it worthwhile showing
that this sequence always gives non-isometric lattices. We note that the pairs studied in Section 5 are part of this family for \( m = 3, r \geq 6 \) and any \( t \in \mathbb{N} \).

**Proposition 6.4.** Let \( m \geq 3, r \geq m + 3, t \in \mathbb{N} \) and set \( q = r^2t \) and \( \theta = rt - 1 \), then the \( q \)-congruence lattices

\[
\mathcal{L}(q; \theta^0, \theta^2, \theta^3, \ldots, \theta^m) \quad \text{and} \quad \mathcal{L}(q; \theta^{-2}, \theta^{-3}, \ldots, \theta^{-m})
\]

are not \( \|\|_1 \)-isometric.

*Proof.* In general, we associate to \( \mathcal{L}(q; \theta^0, \theta^1, \ldots, \theta^{d_m - 1}) \) with \( 0 = d_0 < d_1 < \cdots < d_m - 1 < r \) the following ordered partition of \( r \):

\[
r = (d_1 - d_0) + \cdots + (d_m - 1 - d_{m-2}) + (r - d_{m-1}).
\]

By using Propositions 3.1 and 3.2, one can check that two lens spaces with partitions \( r = a_1 + \cdots + a_m \) and \( r = b_1 + \cdots + b_m \) are \( \|\|_1 \)-isometric if and only if there is \( l \in \mathbb{Z} \) such that \( a_j = b_{j+l} \) for every \( j \), where the index \( j + l \) is taken in the interval \([0, m - 1]\) (mod \( m \)).

In our case, the ordered partitions for the lattices \( \mathcal{L}(q; \theta^0, \theta^2, \theta^3, \ldots, \theta^m) \) and \( \mathcal{L}(q; \theta^{-2}, \theta^{-3}, \ldots, \theta^{-m}) \) are

\[
r = 2 + 1 + \cdots + 1 + (r - m),
\]

\[
r = 1 + \cdots + 1 + 2 + (r - m).
\]

The assertion now follows since \( r - m \geq 3 \). \( \square \)

7. Lens spaces \( p \)-isospectral for every \( p \)

In this section we summarize the spectral properties of lens spaces that can be obtained from the results on congruence lattices in the previous sections, in light of the characterization in Theorem 3.8. We also study \( \tau \)-isospectrality, for some choices of \( \tau \), for the simplest example of lens spaces that are \( p \)-isospectral for all \( p \).

From Theorem 5.6 we obtain that the 5-dimensional lens spaces

\[
L(r^2t; 1, rt - 1, 2rt + 1) \quad \text{and} \quad L(r^2t; 1, rt - 1, 3rt - 1)
\]

are \( p \)-isospectral for all \( p \) for every \( r \) and \( t \) such that \( rt \) is odd and \( r \not\equiv 0 \) (mod \( 3 \)). These lens spaces are non-isometric for \( r \geq 7 \), by Proposition 6.4. Also, Tables 1 and 2 give more such pairs in dimensions 5 and 7 respectively. In higher dimensions we can state the following result.

**Theorem 7.1.** For any \( n_0 \geq 5 \), there are pairs of non-isometric lens spaces of dimension \( n \), with \( n > n_0 \), that are \( p \)-isospectral for all \( p \).

*Proof.* We will apply Theorem 5.6 together with an extension of a duality result of Ikeda. For each \( q \in \mathbb{N} \) and \( n = 2m - 1 \) odd, denote by \( \mathcal{L}_0(q, m) \) the classes of non-isometric \( n \)-dimensional lens spaces \( L(q; s_1, \ldots, s_m) \) such that \( s_i \not\equiv \pm s_j \) (mod \( q \)) for all \( i \neq j \). Set \( h = (\phi(q) - 2m)/2 \), where \( \phi \) is the Euler function.
To each lens space $L = L(q; s_1, \ldots, s_m)$ in $\mathcal{L}_0(q, m)$, one associates the lens space $L = L(q; \bar{s}_1, \ldots, \bar{s}_h)$, where the parameters $\bar{s}_1, \ldots, \bar{s}_h$ are chosen so that the set $\{\pm s_1, \ldots, \pm s_m, \pm \bar{s}_1, \ldots, \pm \bar{s}_h\}$ exhausts the coprime classes module $q$. We thus obtain a new lens space $L$ of dimension $2h - 1 = \phi(q) - 2m - 1$.

By [Ik3] Thm. 3.6, if $q$ is prime, $L$ and $L'$ in $\mathcal{L}_0(q, m)$ are $p$-isospectral for all $p$ if and only if $L$ and $L'$ are $p$-isospectral for all $p$.

Now, for each $q = r^2$, $r$ an odd prime, $r \not\equiv 0 \pmod{3}$, in Theorem 5.6 we have obtained lens spaces $L$ and $L'$ in $\mathcal{L}_0(r^2, 3)$ that are $p$-isospectral for all $p$. Now, by an extension of Ikeda’s argument in [Ik3] Thm 3.6 for $q$ prime —to be sketched below— one can show that the associated lens spaces $L$ and $L'$ are $p$-isospectral for all $p$. These lens spaces have dimension $2h - 1 = \phi(r^2) - 7 = r^2 - r - 7$, a quantity that tends to infinity when $r$ does, thus the assertion in the theorem immediately follows.

We now explain why Ikeda’s argument works also in the case $q = r^2$, $r$ prime. One has that $L, L'$ are isospectral for every $p$ if and only if they have the same generating functions (see [Ik3] Thm 2.5)). Thus, one needs to show that the analogous sums for $L$ and $L'$ are equal to each other.

The generating function for $L$ is given as a sum over the elements in the cyclic group generated by $g$ (see [Ik3] Thm 2.5) which can be split into a subsum over $g^k$ with $(k, q) = 1$ plus a subsum over $g^k$ with $(k, q) = r$ plus a term corresponding to the identity element (i.e. $k = 0$) and similarly for the generating function for $L'$, with $g'$ in place of $g$. As asserted, the total sums are equal to each other for $L$ and $L'$.

It turns out that to prove the assertion for $L$ and $L'$ it suffices to show that the subsums just mentioned are equal to each other for $L$ and $L'$ (the contribution for $k = 0$ is the same in both cases). But it is not hard to show that this is true for the second subsums (hence also for the first ones) for the lens spaces corresponding to the lattices in Thm 5.6 by taking into account that both lattices are of the form $\mathcal{L}(q; p_1, \ldots, p_n)$ with $p_i \equiv \pm 1 \pmod{q}$. This concludes the proof.

**Remark 7.2.** We observe that the examples in (7.4) allow to obtain pairs of Riemannian manifolds in every dimension $n \geq 5$ that are $p$-isospectral for all $p$ and are not strongly isospectral. Indeed, for this purpose, we may just take $M = L \times S^k$ and $M' = L' \times S^k$, for any $k \in \mathbb{N}_0$, where $L, L'$ is any pair of non-isometric lens spaces in dimension $5$ satisfying $p$-isospectrality for every $p$.

**Remark 7.3.** In Section 4 we have seen that the finite set of $N^\text{red}_L(k, \ell)$ determines whether two $q$-congruence lattices are $\|\cdot\|_q$-isospectral, hence, it determines whether two lens spaces are $p$-isospectral for all $p$. Moreover, we point out that they also determine each individual $p$-spectrum of a lens space $L = \Gamma \backslash S^{2m-1}$ for $0 \leq p \leq n = 2m - 1$. Indeed, by Proposition 2.22, the multiplicities in the $p$-spectrum of $L$ depend only on the numbers $\dim V^\Gamma_{k, p}$ and, by expression (3.13), the numbers $\dim V^\Gamma_{k, p}$ are determined by the
If the discrete subgroup \( \Gamma \) of \( G \) acts possibly with fixed points on the compact symmetric space \( X \), then \( \Gamma \backslash X \) is a good orbifold. For instance, in our case, if we take \( L(q; s_1, \ldots, s_m) \) as in (3.1) with \( s_1, \ldots, s_m \) not necessarily coprime to \( q \), we obtain an orbifold lens space (see [Sh]).

Most of the results in this paper work also for orbifold lens spaces. For instance, the determination of the \( p \)-spectrum in Theorem 3.7 via Proposition 2.2 and the characterizations in Theorem 3.8 between lens spaces and congruence lattices. Furthermore, Section 4 works also for congruence lattices \( L(q; s_1, \ldots, s_m) \) without the assumption that the \( s_j \) are coprime to \( q \).

We now show that the lens spaces constructed in Section 5 are homotopically equivalent to each other. We note that they cannot be simply homotopically equivalent (see [Co, §31]) since in this case they would be homeomorphic.

**Lemma 7.5.** The lens spaces \( L(r^2t; 1, rt-1, 2rt+1) \), \( L(r^2t; 1, rt-1, 3rt-1) \), \( (r, 3) = 1 \), associated to the congruence lattices in Theorem 5.6 are homotopically equivalent to each other.

**Proof.** We have seen that these lens spaces can be written \( L = L(q; 1, \theta, \theta^{-2}) \) and \( L' = L(q; 1, \theta, \theta^3) \), where \( \theta = rt - 1 \). The condition for homotopy equivalence of \( L \) and \( L' \) (see [Co (29.6)]) is that \( \pm \theta^5 \equiv d^3 \mod r^2t \), for some \( d \in \mathbb{Z} \).

We note that if \( r \) is odd (resp. even), then \( (rt-1)r \equiv (-1)r \mod r^2t \). Thus, mod \( r^2t \), \( \theta \) has order 2 if \( r \) is odd and order 4 if \( r \) is even.

We claim that \( r \) divides \( \phi(r^2t) \). Indeed, we can write \( q = \prod_j p_j^{2v_{p_j}(r)+v_{p_j}(t)} \) a product over primes \( p_j \). We have

\[
\phi(q) = r \prod_j p_j^{v_{p_j}(r)+v_{p_j}(t)-1} (p_j - 1).
\]

This furthermore implies that if \( r \) is odd then \( 2r \) divides \( \phi(r^2t) \).

Assume first that \( rt \) is odd. Then \( H := \mathbb{Z}_{r^2t}^\times \) is a cyclic group of order \( \phi(r^2t) \). Thus, if \( \omega \) is a generator of this group, then \( \omega^{\phi(r^2t)/2r} \) has order 2r. Hence, since \( H \) is cyclic, \( \theta = \omega^{\pm h} \) for some \( h = j\phi(r^2t)/2r \) with \( (j, 2r) = 1 \).

If furthermore \( 3 \) divides \( \phi(r^2t) \), since \( (3, 2r) = 1 \), then \( \theta = \left( \omega^{\phi(r^2t)/2r} \right)^3 \), as asserted. If \( 3 \) does not divide \( \phi(r^2t) \), then the map \( x \mapsto x^3 \) is onto, hence \( \theta \) is again in the image. This proves the assertion for \( rt \) odd.

In case \( rt \) is even, then \( \mathbb{Z}_{r^2t}^\times \) is a cyclic group \( H \) times an abelian 2-group \( K \). Again the map \( x \mapsto x^3 \) in \( K \) is onto. By a similar argument as before we show that \( \theta \) is in the image of \( x \mapsto x^3 \) in \( H \). \( \square \)
Concerning the question of strong isospectrality, we have the following general fact which follows from well known results. We include a proof for completeness.

**Proposition 7.6.** If $L$ and $L'$ are strongly isospectral lens spaces, then they are isometric.

**Proof.** Assume first that $\Gamma \backslash S^{2m-1}$ and $\Gamma' \backslash S^{2m-1}$ are strongly isospectral spherical space forms, where $\Gamma$ and $\Gamma'$ are arbitrary finite subgroups of $O(2m)$ acting freely on $S^{2m-1}$. By Proposition 1 in [Pe1], the subgroups $\Gamma$ and $\Gamma'$ are representation equivalent, i.e. $L^2(\Gamma \backslash O(2m))$ and $L^2(\Gamma' \backslash O(2m))$ are equivalent representations of $O(2m)$. Hence, $\Gamma$ and $\Gamma'$ are almost conjugate in $O(2m)$ (see Lemma 2.12 in [Wo2]).

In our case, $L = \Gamma \backslash S^{2m-1}$ and $L' = \Gamma' \backslash S^{2m-1}$ are lens space with $\Gamma$ and $\Gamma'$ cyclic subgroups of $SO(2m)$. Since almost conjugate cyclic subgroups are necessarily conjugate, then $L$ and $L'$ are isometric. \qed

We end this section with complementary information on the non-strong isospectrality of our main examples in Section 5 in the case of the simplest example $L = L(49; 1, 6, 15)$ and $L' = L(49; 1, 6, 20)$. We recall that strongly isospectral spherical space forms are necessarily $\tau$-isospectral for every representation $\tau$ of $K = SO(2m-1)$ (see Subsection 2.1). We will illustrate the situation by exhibiting many choices of representations $\tau$ of $K$ such that the lens spaces are not $\tau$-isospectral. We denote by $\Gamma$ and $\Gamma'$ the finite cyclic subgroups of order $q = 49$ that satisfy $L = \Gamma \backslash S^5$ and $L' = \Gamma' \backslash S^5$.

**Lemma 7.7.** Let $\pi_0$ be the unitary irreducible representation of $SO(6)$ with highest weight $\Lambda_0 = 4 \varepsilon_1 + 3 \varepsilon_2$. Then $\dim V_{\pi_0}^T \neq \dim V_{\pi_0}^{T'}$.

**Proof.** By Lemma 3.3 we have that $\dim V_{\pi_0}^T = \sum_{\mu \in \mathcal{L}} m_{\pi_0}(\mu)$, where $\mathcal{L}$ is the associated congruence lattice given by (3.3) and similarly for $\mathcal{L}'$. We compute by using Sage [54] the weights of $\pi_0$ (i.e. $\mu \in \mathbb{Z}^m$ such that $m_{\pi_0}(\mu) > 0$) and their respective multiplicities:

| Weight | Multiplicity |
|--------|--------------|
| $4 \varepsilon_1 + 3 \varepsilon_2$ | 1 |
| $3 \varepsilon_1 + 2 \varepsilon_2 + 2 \varepsilon_3$ | 2 |
| $3 \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ | 4 |
| $2 \varepsilon_1 + \varepsilon_2$ | 9 |

(only the dominant weights are shown, since weights in the same Weyl orbit have the same multiplicity).

A weight $\mu = \sum a_i \varepsilon_i$ is in $\mathcal{L}$ (resp. $\mathcal{L}'$) if and only if $a_1 + 6 a_2 + 15 a_3 \equiv 0 \pmod{49}$ (resp. $a_1 + 6 a_2 + 20 a_3 \equiv 0 \pmod{49}$).

Hence, the weights of $\pi_0$ lying in $\mathcal{L}$ (resp. $\mathcal{L}'$) are $\pm(4 \varepsilon_1 + 3 \varepsilon_3)$, $\pm(-\varepsilon_1 + 2 \varepsilon_2 - 4 \varepsilon_3)$ and $\pm(3 \varepsilon_1 - 3 \varepsilon_2 + \varepsilon_3)$ (resp. $\pm(-3 \varepsilon_2 - 4 \varepsilon_3)$ and $\pm(-3 \varepsilon_1 + 2 \varepsilon_2 + 2 \varepsilon_3)$).

We thus obtain $\dim V_{\pi_0}^T = 2 + 2 + 2 + 4 = 8$ and $\dim V_{\pi_0}^{T'} = 2 + 4 = 6$. \qed
Proposition 7.8. The lens spaces $L = L(49; 1, 6, 15)$ and $L' = L(49; 1, 6, 20)$ are not $\tau$-isospectral for every irreducible representation $\tau$ of $\text{SO}(5)$ with highest weight of the form $b_1\varepsilon_1 + b_2\varepsilon_2$ where

$$4 \geq b_1 \geq 3 \geq b_2 \geq 0. \tag{7.2}$$

Proof. Let $\lambda_0 = \lambda(C, \pi_0) = \langle \Lambda_0 + \rho, \Lambda_0 + \rho \rangle - \langle \rho, \rho \rangle = (6^2 + 4^2) - (2^2 + 1^2) = 47$, the scalar for which the Casimir element $C$ acts on $\pi_0$ (see (2.1)). By (2.1), the multiplicity $d_{\lambda_0}(\tau, \Gamma)$ of the eigenvalue $\lambda_0$ of $\Delta_{\tau, \Gamma}$ is

$$d_{\lambda_0}(\tau, \Gamma) = \sum_{\pi} \dim V_\pi^\Gamma [\tau : \pi],$$

where the sum is over the irreducible representations $\pi$ of $\text{SO}(6)$ such that $\lambda(C, \pi) = \lambda_0 = 47$. A similar expression is valid for $d_{\lambda_0}(\tau, \Gamma')$.

Now let $\pi$ be an irreducible representation of $\text{SO}(6)$ with highest weight $\Lambda = a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3$ ($a_i \in \mathbb{Z}$ and $a_1 \geq a_2 \geq |a_3|$) and such that $\lambda(C, \pi) = \lambda_0$. Then we have that $(a_1 + 2)^2 + (a_2 + 1)^2 + a_3^2 = \langle \Lambda + \rho, \Lambda + \rho \rangle = \langle \Lambda_0 + \rho, \Lambda_0 + \rho \rangle = 52$. By taking congruences modulo 4, we see that the numbers $a_1 + 2 > a_2 + 1 > a_3$ are even. Hence

$$\left(\frac{a_1 + 2}{2}\right)^2 + \left(\frac{a_2 + 1}{2}\right)^2 + \left(\frac{a_3}{2}\right)^2 = 13.$$ 

It is easy to check that $13 = 3^2 + 2^2 + 0^2$ is the only way to write 52 as a sum of three squares, thus $a_1 = 4, a_2 = 3$ and $a_3 = 0$, therefore $\Lambda = \Lambda_0$.

The previous paragraph shows that

$$d_{\lambda_0}(\tau, \Gamma) = \dim V_{\pi_0}^\Gamma [\tau : \pi_0]$$

and similarly for $d_{\lambda_0}(\tau, \Gamma')$. Thus, by Lemma 7.7, the lens spaces $L$ and $L'$ cannot be $\tau$-isospectral for any $\tau$ such that $[\tau : \pi_0] > 0$. Finally, we see that the highest weights $b_1\varepsilon_1 + b_2\varepsilon_2$ of a representation $\tau$ such that $[\tau : \pi_0] > 0$ satisfy (7.2), by applying the branching law from $\text{SO}(6)$ to $\text{SO}(5)$ (see for instance [GW, Thm. 8.1.4]). \hfill \Box

Remark 7.9. We note that the assertion in Proposition 7.8 followed by comparing the multiplicities of $\lambda = \lambda(C, \pi)$ for only one choice of $\pi \in \hat{G}$ satisfying $\dim V_\pi^\Gamma \neq \dim V_\pi^{\Gamma'}$. By computer methods using Sage [Sa], we have checked that there are different choices of $\pi$ as in Lemma 7.7 that provide many other $\hat{K}$-types $\tau$ such that the lens spaces $L$ and $L'$ are not $\tau$-isospectral.

In Table 3 we show the representations $\pi$ of $\text{SO}(6)$ with highest weights $\Lambda = c_1\varepsilon_1 + c_2\varepsilon_2 + c_3\varepsilon_3$ satisfying $7 \geq c_1 \geq c_2 \geq c_3 \geq 0$ and $\dim V_\pi^\Gamma \neq \dim V_\pi^{\Gamma'}$. When $c_3 \neq 0, \pi$ denotes the sum of the irreducible representations with highest weight $\Lambda = c_1\varepsilon_1 + c_2\varepsilon_2 + c_3\varepsilon_3$ and $\overline{\Lambda} = c_1\varepsilon_1 + c_2\varepsilon_2 - c_3\varepsilon_3$, since $[\tau : \pi_{\overline{\Lambda}}] > 0$ if and only if $[\tau : \pi_{\Lambda}] > 0$ by the branching law, for any $\tau \in \hat{K}$. The column $\lambda$ shows the value of $\lambda(C, \pi_{\Lambda})$. 
Table 3. Irreducible representations of SO(6) with different invariant subspaces.

\[
\begin{array}{cccccc}
\left[ c_1, c_2, c_3 \right] & \dim V_\Gamma & \dim V_\Gamma' & \lambda & \left[ c_1, c_2, c_3 \right] & \dim V_\Gamma & \dim V_\Gamma' & \lambda \\
4, 3, 0 & 8 & 390 & [4, 4, 3] & 4 & 83 & 4 \\
6, 3, 0 & 54 & 418 & [6, 4, 3] & 82 & 111 & 6 \\
4, 4, 1 & 16 & 57 & [7, 4, 3] & 252 & 128 & 7 \\
6, 4, 1 & 140 & 85 & [7, 5, 3] & 254 & 139 & 5 \\
7, 5, 1 & 352 & 113 & [7, 7, 3] & 350 & 141 & 7 \\
7, 7, 1 & 248 & 144 & [5, 5, 4] & 204 & 176 & 7 \\
3, 2, 2 & 0 & 37 & [7, 5, 4] & 204 & 176 & 3 \\
7, 2, 2 & 84 & 93 & [6, 6, 4] & 84 & 172 & 7 \\
4, 3, 2 & 6 & 55 & [7, 7, 4] & 176 & 204 & 4 \\
6, 3, 2 & 76 & 83 & [6, 5, 5] & 40 & 220 & 6 \\
3, 3, 3 & 0 & 63 & [7, 7, 6] & 62 & 356 & 3 \\
7, 3, 3 & 104 & 119 & & & & \\
\end{array}
\]

For each \( \Lambda = c_1 \varepsilon_1 + c_2 \varepsilon_2 + c_3 \varepsilon_3 \) in Table 3 if there are no other dominant weights with the same Casimir scalar \( \lambda(C, \pi_\Lambda) \), one gets many \( \tau \in \hat{K} \) such that \( L \) and \( L' \) are not \( \tau \)-isospectral; namely all those having highest weights \( b_1 \varepsilon_1 + b_2 \varepsilon_2 \) such that \( c_1 \geq b_1 \geq c_2 \geq b_2 \geq c_3 \).

The previous remark leads us to make the following conjecture that roughly says that the lens spaces \( L \) and \( L' \) are ‘very far’ from being strongly isospectral.

**Conjecture 7.10.** There are only finitely many irreducible representations \( \tau \) of SO(5) such that \( L = L(49; 1, 6, 15) \) and \( L' = L(49; 1, 6, 20) \) are \( \tau \)-isospectral.

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