

TURÁN TYPE INEQUALITIES FOR DUNKL KERNEL AND 
\( q \)-DUNKL KERNEL

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Abstract. We prove Turán type inequalities for Dunkl kernel. We provide a 
\( q \)-integral representation for the \( q \)-Dunkl kernel. Using a \( q \)-version of Schwartz 
inequality, we get a Turán type inequalities for \( q \)-Dunkl kernel.

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1. Introduction

In 1941, Paul Turán established the famous Turán inequality for Legendre polynomials.

\[ P_{n-1}(x)P_{n+1}(x) < |P_n(x)|^2, \quad |x| < 1, \quad n = 1, 2, \ldots \]

In 1948, Gabor Szegő presented elegant proofs of Turán inequality for Legendre polynomials and extended the result to Gegenbauer, Laguerre and Hermite polynomials. After 1948 analogous results were obtained by several authors for a large class of orthogonal polynomials and special functions (for example Bessel, \( q \)-Bessel, modified Bessel, polygamma, Riemann Ζeta functions). In 1981 one of the PhD student of P. Turán, L. Alpár [1] in Turán’s bibliography mentioned that the above Turán inequality had a wide ranging effect. Actually, the Turán type inequalities have a more extensive literature and recently the results have been applied in problems arising from many fields such as information theory, economic theory and biophysics. Recently it has been shown by Á. Baricz [2, 4, 5] that the Gauss and Kummer hypergeometric functions, as well as the generalized hypergeometric functions satisfy some Turán type inequalities. For deep study about this subject we refer to [2, 4, 5, 10, 13, 14].

In this paper our aim is to provide some new Turán type inequalities for Dunkl kernel and \( q \)-Dunkl kernel.

Our paper is organized as follows: in section 2, we present some preliminary results and notations that will be useful in the sequel. In section 3, using the series expansion of the Dunkl kernel \( E_\nu(\lambda, x) \), we prove that the function \( \nu \longmapsto E_\nu(\lambda, x) \) is log-convex on \((0, \infty)\). In particular we deduce some Turán type inequalities for the Dunkl kernel. Using an integral representation, we show analogous results for the
normalized Dunkl kernel $\widetilde{E}_\nu(\lambda, x)$. In section 4, using the series expansion of the q-
Dunkl kernel $E_\nu(x, q^2)$, we prove that the function $\nu \mapsto E_\nu(x, q^2)$ is log-convex on $]0, \infty[$, in particular we deduce some Turán type inequalities for the q-Dunkl kernel. We establish a q-integral representation for q-Dunkl kernel. Using a q-version of Schwartz inequality, we deduce some Turán type inequalities for the the normalized q-Dunkl kernel. As application, in section 5, we give some hyperbolic Jordan’s type inequalities for hyperbolic functions.

2. Notations and preliminaries

The Euler gamma function $\Gamma(z)$ is defined for $\mathcal{R}(z) > 0$, by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$$ 

The psi(or digamma) function $\psi(z)$ is the logarithmic derivative of $\Gamma(z)$, that is,

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$ 

It's well known that the digamma function satisfies

$$\psi(x) = -\gamma + (x - 1) \sum_{k \geq 0} \frac{1}{(k + 1)(x + k)}, \quad x > 0,$$

where $\gamma$ is the Euler constant. Thus the digamma function is concave in $]0, \infty[$.

Throughout the section 4, we will fix $q \in ]0, 1[$. We recall some usual notions and notations used in the q-theory (see [10] and [13]). We refer to the book by G. Gasper and M. Rahman [10], for the notations, definitions and properties of the q-shifted factorials and q-hypergeometric functions. We note

$$\mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}.$$ 

The q-derivative $D_q f$ of a function $f$ is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad \text{if } x \neq 0,$$

$(D_q f)(0) = f'(0)$ provided $f'(0)$ exists. If $f$ is differentiable, then $(D_q f)(x)$ tends to $f'(x)$ as $q$ tends to 1.

The q-Jackson integrals from 0 to $a$, from 0 to $\infty$ and in a generic interval $[a, b]$ are defined by (see [11])

$$\int_0^a f(x)d_q x = (1 - q)a \sum_{n=0}^\infty f(a q^n)q^n,$$

$$\int_0^\infty f(x)d_q x = (1 - q) \sum_{n=-\infty}^\infty f(q^n)q^n,$$
provided the sums converge absolutely, and
\[ \int_a^b f(x)d_q x = \int_0^b f(x)d_q x - \int_0^a f(x)d_q x. \]
The improper integral is defined in the following way (see [13])
\[ \int_0^\infty f(x)d_q x = (1-q) \sum_{n=-\infty}^\infty f \left( \frac{q^n}{A} \right) \frac{q^n}{A}. \]
We remark that for \( n \in \mathbb{Z} \), we have
\[ \int_0^\infty q^n f(x)d_q x = \int_0^\infty f(x)d_q x. \]

\( q \)-Analogues of the exponential functions (see [10, 13]) is given by:
\[ e(q, z) = \psi_0(0; -; q, z) = \sum_{n=0}^\infty \frac{(1-q)^n}{(q; q)_n} z^n = \frac{1}{(z; q)^\infty}. \]
For the convergence of the series, we need \( |z| < 1 \); however, due to its product representation, \( e_q \) is continuous to a meromorphic function on \( \mathbb{C} \) and has simple poles at \( z = q^{-n}, \ n \in \mathbb{N} \).

Jackson [11] defined a \( q \)-analogous of the Gamma function by
\[ \Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x}, \quad x \neq 0, -1, -2, \ldots. \]
It is well known that it satisfies:
\[ \Gamma_q(x+1) = \frac{1-q^x}{1-q} \Gamma_q(x), \quad \Gamma_q(1) = 1 \text{ and } \lim_{q \to 1^-} \Gamma_q(x) = \Gamma(x), \ \Re(x) > 0. \]
The \( q \)-psi (or \( q \)-digamma) function is defined as the Logarithmic \( q \)-derivative of the \( q \)-gamma function:
\[ \psi_q(x) = \frac{\Gamma'_q(x)}{\Gamma_q(x)}, \]
and satisfies:
\[ \psi_q(x) = -\text{Log}(1-q) + \text{Log}(q) \sum_{n=1}^\infty \frac{q^{nx}}{1-q^n}; \quad q \in ]0, 1[, \]
where \( \text{Log}(x) \) means \( \text{Log}_e(x) \).

The \( q \)-modified Bessel function of first kind is defined by (see [6]):
\[ I_{\nu}^{(1)} ((1-q^2)z; q^2) = \frac{1}{\Gamma_q(\nu + 1)} \sum_{k=0}^\infty \frac{(1-q^2)^{2k} z^{2k+\nu}}{2^{2k+\nu}(q^2; q^2)_k (q^{2\nu+1}; q^2)_k}, \quad |z| < \frac{1}{1-q^2}. \]
The normalized $q$-modified Bessel function of the first kind is defined by:

$$I_\nu(z; q^2) = (1 + q^2)^\nu \frac{\Gamma(\nu + 1)}{z^\nu} I^{(1)}_\nu(2(1-q)z; q^2).$$

### 3. Turán Type Inequalities for Dunkl Kernel

We recall that the Dunkl operator is defined for $f \in C^1(\mathbb{R})$ by:

$$T_\nu f(x) = f'(x) + \nu \frac{f(x) - f(-x)}{2}, \quad \nu > 0.$$  

For $\lambda \in \mathbb{C}$, the Dunkl kernel $E_\nu(\lambda, \cdot)$ on $\mathbb{R}$ was introduced by C. Dunkl in [9] and is given by:

$$E_\nu(\lambda, x) = j_{\nu-\frac{1}{2}}(i\lambda x) + \frac{\lambda x}{2\nu + 1} j_{\nu+\frac{1}{2}}(i\lambda x),$$

where $j_\alpha$ is the normalized Bessel function of the first kind of order $\alpha$. The Dunkl kernel $E_\nu(\lambda, x)$ is the unique solution on $\mathbb{R}$ of the initial problem associated to Dunkl operator:

$$T_\nu f(x) = \lambda f(x), \quad f(0) = 1, \quad x \in \mathbb{R}.$$  

**Lemma 1.** (see [19]). For $\lambda, x \in \mathbb{R}$ and $\nu > 0$, the Dunkl kernel $E_\nu(\lambda, x)$ admits the series expansions

$$E_\nu(\lambda, x) = \sum_{n=0}^{\infty} \frac{(\lambda x)^n}{b_n(\nu)},$$

where

$$b_{2n}(\nu) = 2^{2n} n! \frac{\Gamma(n + \nu + 1)}{\Gamma(\nu + 1)}, \quad b_{2n+1}(\nu) = 2(\nu + 1)b_{2n}(\nu + 1).$$

**Lemma 2.** For $\lambda, x \in \mathbb{R}$ and $\nu > 0$, the Dunkl kernel $E_\nu(\lambda, x)$ admits the following integral representation

$$E_\nu(\lambda, x) = c(\nu) \int_{-1}^{1} e^{\lambda x t} (1 - t^2)^{\nu-1}(1 + t) dt,$$

where $c(\nu) = \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(\nu)}$.

**Theorem 1.** For $\lambda, x \in \mathbb{R}^+$, the function $\nu \mapsto E_\nu(\lambda, x)$ is log-convex on $]0, \infty[$ i.e.

$$E_{\alpha \nu_1 + (1-\alpha)\nu_2}(\lambda, x) \leq [E_{\nu_1}(\lambda, x)]^\alpha [E_{\nu_2}(\lambda, x)]^{1-\alpha}, \quad \forall \nu_1 > 0, \nu_2 > 0, \quad \forall \alpha \in [0, 1].$$

In particular, we get Turán type inequalities for Dunkl kernel:

$$[E_{\nu+1}(\lambda, x)]^2 \leq E_{\nu}(\lambda, x) E_{\nu+2}(\lambda, x), \quad \forall \nu > 0.$$
Proof. To show the Log-convexity of the function $\nu \mapsto E_\nu(\lambda, x)$, we just need to show the Log-convexity of each term of its series expansion and then, we use the fact that sums of Log-convex functions are Log-convex too. Let $n \geq 0$, since the function $\psi$ is concave on $]0, \infty[$, we get:

$$\frac{d^2}{d\nu^2}\left[\log\left(\frac{1}{b_{2n}(\nu)}\right)\right] = \psi'(\nu + 1) - \psi'(\nu + n + 1) \geq 0$$

and

$$\frac{d^2}{d\nu^2}\left[\log\left(\frac{1}{b_{2n+1}(\nu)}\right)\right] = \psi'(\nu + 2) - \psi'(\nu + n + 2) + \frac{1}{(\nu + 1)^2} \geq 0.$$ 

Thus for $\nu_1, \nu_2 > 0$, $\alpha \in [0, 1]$:

$$E_{\alpha\nu_1 + (1-\alpha)\nu_2}(\lambda, x) \leq [E_{\nu_1}(\lambda, x)]^\alpha [E_{\nu_2}(\lambda, x)]^{1-\alpha}.$$ 

In particular, for $\nu > 0$, $\nu_1 = \nu$, $\nu_2 = \nu + 2$ and $\alpha = \frac{1}{2}$, Turán type inequality for Dunkl Kernel is deduced. ■

Definition 1. For $\lambda, x \in \mathbb{R}$ and $\nu > 0$, the normalized Dunkl kernel is defined by

$$\tilde{E}_\nu(\lambda, x) = \frac{1}{c_\nu}E_\nu(\lambda, x),$$

where $c_\nu$ is given by (7).

Theorem 2. For $\lambda, x \in \mathbb{R}$, the function $\nu \mapsto \tilde{E}_\nu(\lambda, x)$ is log-convex on $]0, \infty[$ i.e.

$$\tilde{E}_{\alpha\nu_1 + (1-\alpha)\nu_2}(\lambda, x) \leq [\tilde{E}_{\nu_1}(\lambda, x)]^\alpha [\tilde{E}_{\nu_2}(\lambda, x)]^{1-\alpha}, \forall \nu_1 > 0, \nu_2 > 0, \forall \alpha \in [0, 1].$$

In particular, we get Turán type inequalities for normalized Dunkl kernel:

$$[\tilde{E}_{\nu+1}(\lambda, x)]^2 \leq \tilde{E}_\nu(\lambda, x)\tilde{E}_{\nu+2}(\lambda, x), \forall \nu > 0.$$ 

Proof. Using the integral representation (6) of the Dunkl Kernel and Hölder inequality we have, for $\nu_1, \nu_2 > 0$ and $\alpha \in [0, 1]$:

$$\tilde{E}_{\alpha\nu_1 + (1-\alpha)\nu_2}(\lambda, x) = \int_{-1}^{1} e^{\lambda xt}(1 - t^2)^{\nu_1 - 1}(1 + t)^{\alpha} (1 + t)^{\nu_2 - 1}(1 + t)dt$$

$$= \int_{-1}^{1} [e^{\lambda xt}(1 - t^2)]^{\nu_1 - 1}(1 + t)^{\alpha} [e^{\lambda xt}(1 - t^2)]^{\nu_2 - 1}(1 + t)dt$$

$$\leq \left[\int_{-1}^{1} e^{\lambda xt}(1 - t^2)^{\nu_1 - 1}(1 + t)dt\right]^{\alpha} \left[\int_{-1}^{1} e^{\lambda xt}(1 - t^2)^{\nu_2 - 1}(1 + t)dt\right]^{1-\alpha}$$

$$\leq \left[\tilde{E}_{\nu_1}(\lambda, x)\right]^\alpha \left[\tilde{E}_{\nu_2}(\lambda, x)\right]^{1-\alpha}.$$
In particular, for \(\nu > 0\), \(\nu_1 = \nu\), \(\nu_2 = \nu + 2\) and \(\alpha = \frac{1}{2}\), we get Turán type inequalities for the normalized Dunkl Kernel.

4. Turán Type Inequalities for \(q\)-Dunkl Kernel

We consider the \(q\)-Dunkl operator \(T_{q,\nu}\) defined by:

\[
T_{q,\nu} f(x) = D_q f(x) + \frac{[2\nu + 1]_q}{x} \left[ f(qx) - f(-qx) \right].
\]

We note that the \(q\)-Dunkl operator \(T_{q,\nu}\) tends to the Dunkl operator \(T_\nu\) as \(q \to 1^\pm\).

Definition 2. (\(q\)-Dunkl kernel)

For \(q \in [0, 1]\) and \(|x| < \frac{1}{(1 - q)^2}\), we define the \(q\)-Dunkl kernel by

\[
E_\nu(x; q^2) = I_\nu(x; q^2) + \frac{x}{(1 + q) [\nu + 1]_q} I_{\nu + 1}(x; q^2),
\]

where \(I_\nu(x; q^2)\) is the normalized \(q\)-modified Bessel function of the first kind.

Lemma 3. For \(q \in [0, 1]\), \(|\lambda| < \frac{1}{1 - q}\), the \(q\)-Dunkl kernel \(E_\nu(\lambda x; q^2)\) is the unique analytic solution of the \(q\)-problem

\[
T_{q,\nu} f(x) = \lambda f(x), \quad f(0) = 1.
\]

Lemma 4. For \(|x| < \frac{1}{(1 - q)^2}\), The \(q\)-Dunkl kernel \(E_\nu(x; q^2)\) admits the series expansions

\[
E_\nu(x; q^2) = \sum_{k=0}^{\infty} \frac{x^k}{b_k(\nu; q^2)},
\]

where

\[
b_{2k}(\nu; q^2) = \frac{(1 + q)^k \Gamma_{q^2}(k + 1) \Gamma_{q^2}(\nu + k + 1)}{\Gamma_{q^2}(\nu + 1)}
\]

\[
b_{2k+1}(\nu; q^2) = \frac{(1 + q)^{2k+1} \Gamma_{q^2}(k + 1) \Gamma_{q^2}(\nu + k + 2)}{\Gamma_{q^2}(\nu + 1)}
\]

Theorem 3. For \(q \in [0, 1]\) and \(x, \lambda \in [0, \frac{1}{1 - q}]\), the function \(\nu \mapsto E_\nu(\lambda x; q^2)\) is log-convex on \([0, \infty]\), i.e.

\[
E_{\alpha \nu_1 + (1-\alpha)\nu_2}(\lambda x; q^2) \leq [E_{\nu_1}(\lambda x; q^2)]^\alpha [E_{\nu_2}(\lambda x; q^2)]^{1-\alpha}, \quad \forall \nu_1, \nu_2 > 0, \quad \forall \alpha \in [0, 1].
\]
In particular, the Turán type inequalities for the $q$-Dunkl kernel holds:

\begin{equation}
[E_{\nu+1}(\lambda x; q^2)]^2 \leq E_\nu(\lambda x; q^2)E_{\nu+2}(\lambda x; q^2), \ \forall \nu > 0.
\end{equation}

**Proof.** As in the classical case, we establish the Log-convexity of the function
\(\nu \mapsto E_\nu(\lambda x; q^2)\) by proving the Log-convexity of each term of its series expansion, and then we use the fact that the sums of Log-convex functions are Log-convex too.

Let \(k \geq 0,\) since \(\psi_q' q \) is decreasing on \([0, \infty[\), we get:
\[
\frac{d^2}{d\nu^2} \left[ \log \left( \frac{1}{b_{2k}(\nu; q^2)} \right) \right] = \psi_q'(\nu + 1) - \psi_q'(\nu + k + 1) \geq 0
\]
and
\[
\frac{d^2}{d\nu^2} \left[ \log \left( \frac{1}{b_{2k+1}(\nu; q^2)} \right) \right] = \psi_q'(\nu + 1) - \psi_q'(\nu + k + 2) \geq 0
\]
Consequently, the function \(\nu \mapsto E_\nu(\lambda x; q^2)\) is Log-convex on \([0, \infty[\) :
\[
E_{\nu_1+(1-\alpha)\nu_2}(\lambda x; q^2) \leq \left[ E_{\nu_1}(\lambda x; q^2) \right]^\alpha \left[ E_{\nu_2}(\lambda x; q^2) \right]^{1-\alpha}, \ \forall \nu_1, \nu_2 > 0, \ \forall \alpha \in [0, 1].
\]

In particular, for \(\alpha = \frac{1}{2}, \ \nu_1 = \nu, \ \nu_2 = \nu + 2,\) Turán type inequality for $q$-Dunkl Kernel holds.

\[\square\]

**Lemma 5.** For all \(q \in ]0, 1[, \ |x| < \frac{1}{(1-q)^2},\) the $q$-Dunkl kernel admits the $q$-integral representation :

\begin{equation}
E_\nu(x; q^2) = C(\nu, q^2) \int_{-1}^{1} W_\nu(t; q^2)(1 + t)E(q, (1 - q)tx)d_q t,
\end{equation}

where

\begin{equation}
C(\nu, q^2) = \frac{(1 + q)\Gamma_q(\nu + 1)}{2\Gamma_q(\frac{3}{2})\Gamma_q(\nu + \frac{1}{2})}
\end{equation}

and

\begin{equation}
W_\nu(x; q^2) = \frac{(x^2q^2; q^2)_\infty}{(x^2q^{2\nu+1}; q^2)_\infty}.
\end{equation}

**Proof.** Let \(q \in ]0, 1[, \ |x| < \frac{1}{(1-q)^2},\) as in [16], the normalized $q$-modified Bessel function of first kind admits the following integral representation :

\begin{equation}
I_\nu(x; q^2) = C(\nu, q^2) \int_{-1}^{1} W_\nu(t; q^2)E(q, (1 - q)tx)d_q t,
\end{equation}

where \(C(\nu, q^2); W_\nu(t; q^2)\) are given respectively by (21) and (22).

Knowing that,

\begin{equation}
D_q(E((1 - q)x; q)) = E((1 - q)x; q)
\end{equation}
and since

\[(25)\]

\[E_\nu(x; q^2) = I_\nu(x; q^2) + D_q(I_\nu(x; q^2)).\]

Using (23), (24) and (25), we get:

\[E_\nu(x; q^2) = I_\nu(x; q^2) + D_q(I_\nu(x; q^2))\]

\[= C(\nu, q^2) \int_{-1}^{1} W_\nu(t; q^2)E(q, (1-q)t)x)dt + C(\nu, q^2) \int_{-1}^{1} W_\nu(t; q^2)D_q[E(q, (1-q)t)]d_qt\]

\[= C(\nu, q^2) \int_{-1}^{1} W_\nu(t; q^2)E(q, (1-q)t)x)dt + C(\nu, q^2) \int_{-1}^{1} W_\nu(t; q^2)E(q, (1-q)t)x)dt.\]

\[\blacksquare\]

**Definition 3.** For all \(q \in ]0, 1[\) and \(|x| < \frac{1}{(1-q)^2}\), the normalized \(q\)-Dunkl kernel is defined by

\[(26)\]

\[\tilde{E}_\nu(x; q^2) = \frac{E_\nu(x; q^2)}{C(\nu, q^2)},\]

where \(C(\nu, q^2)\) is given by (21).

**Theorem 4.** For \(q \in ]0, 1[\), \(|x| < \frac{1}{(1-q)^2}\), the normalized \(q\)-Dunkl kernel \(\tilde{E}_\nu(x; q^2)\) satisfy a Turán type inequality, i.e

\[(27)\]

\[\left[\tilde{E}_{\nu+1}(x; q^2)\right]^2 \leq \tilde{E}_\nu(x; q^2)\tilde{E}_{\nu+2}(x; q^2), \quad \forall \nu > 0.\]

**Proof.** Let \(q \in ]0, 1[\), \(\nu > 0\) and \(|x| < \frac{1}{(1-q)^2}\).

Using the relation:

\[(q^{2\nu+1}x^2; q^2)_{\infty} = (1-x^2q^{2\nu+1})(q^{2\nu+3}x^2; q^2)_{\infty},\]

we get:

\[(28)\]

\[W_{\nu+1}(x, q^2) < \left[W_{\nu}(x, q^2)\right]^\frac{1}{2} \left[W_{\nu+2}(x, q^2)\right]^\frac{1}{2},\]

where \(W_{\nu}(x, q^2)\) is given by (22).

Using (28), the \(q\)-version of Schwatz inequality and the \(q\)-integral representation of the \(q\)-Dunkl Kernel, we obtain:

for \(\nu > 0, q \in ]0, 1[\) and \(|x| < \frac{1}{(1-q)^2}\)

\[\tilde{E}_{\nu+1}(x; q^2) = \int_{-1}^{1} W_{\nu+1}(t; q^2)(1+t)E_q((1-q)t)x)dt.\]
\[ \leq \int_{-1}^{1} [W_\nu(t; q^2)(1 + t)E_q((1 - q)tx)]^{\frac{1}{2}} \times [W_{\nu+2}(t; q^2)(1 + t)E_q((1 - q)tx)]^{\frac{1}{2}} \, dq \, dt \]
\[ \leq \left[ \int_{-1}^{1} W_\nu(t; q^2)(1 + t)E_q((1 - q)tx) \right]^{\frac{3}{2}} \times \left[ \int_{-1}^{1} W_{\nu+2}(t; q^2)(1 + t)E_q((1 - q)tx) \right]^{\frac{3}{2}} \]
\[ \leq \left[ \tilde{E}_\nu(x; q^2) \right]^{\frac{1}{2}} \left[ \tilde{E}_{\nu+2}(x; q^2) \right]^{\frac{1}{2}} \]

5. Applications

**Theorem 5.** The following assertions are true:

(1) For \( \lambda, x \geq 0 \), the function \( \nu \mapsto \frac{E_{\nu+1}(\lambda, x)}{E_\nu(\lambda, x)} \) is increasing on \( ]0, \infty[ \).

(2) For \( q \in ]0, 1[ \) and \( x, \lambda \in ]0, \frac{1}{1-q}[ \), the function \( \nu \mapsto \frac{E_{\nu+1}(x, q^2)}{E_\nu(x, q^2)} \) is increasing on \( ]0, \infty[ \).

**Proof.** From Theorems 1 and 3, we deduce the Log-convexity of the functions \( \nu \mapsto E_\nu(\lambda, x) \) and \( \nu \mapsto E_\nu(x, q^2) \) on \( ]0, \infty[ \). Thus, the functions \( \nu \mapsto \log \left[ \frac{E_{\nu+1}(\lambda, x)}{E_\nu(\lambda, x)} \right] \) and \( \nu \mapsto \log \left[ \frac{E_{\nu+1}(x, q^2)}{E_\nu(x, q^2)} \right] \) are increasing on \( ]0, \infty[ \). Which completes the proof.

In the next corollary, we give some hyperbolic Jordan’s type inequalities for hyperbolic functions.

**Corollary 1.** The following inequalities are valid:

\[
(1 - x)e^x \leq \frac{\sinh x}{x} ; \quad x > 0
\]
\[
(1 + x)e^{-x} \leq \frac{\sinh x}{x} ; \quad x < 0.
\]

**Proof.** Since the function \( \nu \mapsto \frac{E_{\nu+1}(\lambda, x)}{E_\nu(\lambda, x)} \) is increasing on \( ]0, \infty[ \), we get:

\[
\frac{E_{\nu+1}(\lambda, x)}{E_\nu(\lambda, x)} \leq 1.
\]

By the definition of the Dunkl Kernel and since:

\[
j_{\frac{1}{2}}(ix) = \cosh x,
\]
\[ j_{-\frac{1}{2}}(ix) = \frac{\sinh x}{x}, \]
\[ j_{\frac{3}{2}}(ix) = -3\left(\frac{\sinh x}{x^3} - \frac{\cosh x}{x^2}\right) \]
we conclude. The second inequality is deduced by parity.

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