Models $WD_n$ in the presence of disorder and the coupled models.

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Abstract.

We have studied the conformal models $WD_n^{(p)}$, $n = 3, 4, 5, ...$, in the presence of disorder which couples to the energy operator of the model. In the limit of $p \gg 1$, where $p$ is the corresponding minimal model index, the problem could be analyzed by means of the perturbative renormalization group, with $\epsilon$-expansion in $\epsilon = \frac{1}{p}$. We have found that the disorder makes to flow the model $WD_n^{(p)}$ to the model $WD_n^{(p-1)}$ without disorder. In the related problem of $N$ coupled regular $WD_n^{(p)}$ models (no disorder), coupled by their energy operators, we find a flow to the fixed point of $N$ decoupled $WD_n^{(p-1)}$. But in addition we find in this case two new fixed points which could be reached by a fine tuning of the initial values of the couplings. The corresponding critical theories realize the permutational symmetry in a non-trivial way, like this is known to be the case for coupled Potts models, and they could not be identified with the presently known conformal models.

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1 Introduction

With the present knowledge of the effect of disorder on the critical behavior of statistical systems it could be fair to say that any extra theoretical model will be interesting in which these effects are nontrivial and can be studied reliably.

In this respect we find the set of conformal theory models $WD_n^{(p)}$, $n = 3, 4, 5, \ldots$, to be interesting. Here $p$ is the index of the corresponding unitary series of minimal models. These models are based on the corresponding $W$-chiral algebras. They have been defined in the paper [1].

The simplest model in this series $WD_3^{(p)}$ will be sufficient to give a preliminary discussion of our approach and results.

The model itself, which has not much been used in statistical physics applications, will be described in the next Section. For the moment we shall observe that a particular primary operator in this theory, which is naturally identified with the energy operator, $\varepsilon(x)$, possesses conformal dimension

$$\Delta_{\varepsilon} = \frac{1}{2}$$

in the limiting theory $WD_3^{(p)}$, $p \rightarrow \infty$. Physical dimension of $\varepsilon$ will be twice bigger, $\Delta_{\varepsilon}^{ph} = 2\Delta_{\varepsilon} = 1$. For $p$ finite, $\Delta_{\varepsilon}$ is of the form:

$$\Delta_{\varepsilon} = \frac{1}{2} - \frac{5}{2}\epsilon$$

where

$$\epsilon = \frac{1}{p + 1}$$

If we couple disorder to $\varepsilon(x)$:

$$\int d^2x \ m(x)\varepsilon(x)$$

with $m(x)$ being random valued, then the disorder produced interaction of the replicated theory:

$$g \int d^2x \sum_{a \neq b}^N \varepsilon_a(x)\varepsilon_b(x)$$
will be marginal in the limit \( p \to \infty \), and will be slightly relevant in the model \( W D_3^{(p)} \) with \( p \) large but finite. In this case the effect of disorder can reliably studied by the perturbative renormalization group, by developing in \( \epsilon \).

We observe also that the problem formulated in this way will be defined better, theoretically, compared to the much used Potts model, or the minimal conformal theory models, with disorder [2,3]. In this latter case the disorder is marginal in the case of the Ising model [4], and the \( \epsilon \) parameter of the Potts model, in which, similarly, the disorder is slightly relevant, corresponds to the deviation of the central charge, or of the number of components, from the Ising case. The point of difference is that in this case there are no unitary models close to Ising which would correspond to infinitesimal values of \( \epsilon \). Ising model is not the accumulation point of the unitary series. Next to Ising, the 3-states Potts model, lies at numerically small but finite distance in \( \epsilon \). For this reason, the perturbative expansion in \( \epsilon \) is likely to give numerical results, in case of Potts model with disorder, not analytic ones.

This is different in case of the model \( W D_n^{(p)} \) where the disorder is marginal at the accumulation point of unitary series, \( p = \infty \). In this case for any small value of \( \epsilon \sim 1/p \) there would be a physical, unitary model behind. In this sense, \( \epsilon \) will be a well defined analytic parameter. One could expect that the perturbative expansion in \( \epsilon \) would provide in this case an analytic expansion in \( \epsilon = 1/(p+1) \) of unknown yet exact functions (central charge, dimensions of operators). Like this is actually the case for the minimal model perturbed with the operator \( \phi_{3,1} \) [5,6]. For these reasons we consider the model to be interesting.

Another important point of difference with the Potts model is that in the operator product decomposition of \( \{e(x)e(x')\} \) there appears an extra slightly relevant operator, we shall note it \( \phi(x) \). The operator algebra has the form:

\[
\{e(x)e(x')\} = \frac{1}{|x-x'|^{2\Delta_k}}(I + ...) + \frac{D_{e e}^\phi}{|x-x'|^{2\Delta_k-\Delta_\phi}}(\phi(x) + ...) + \frac{D_{e e}^\phi'}{|x-x'|^{2\Delta_k-\Delta_\phi'}}(\phi'(x) + ...)
\] (1.6)

The detailed information on this decomposition shall be given in the sections 3. For
the moment we shall observe only that one of the operators in (1.6), let's say \( \phi'(x) \), is irrelevant, so that it can be dropped in the RG calculation. While another, \( \phi(x) \), is slightly relevant \( \Delta \phi = 1 - A\epsilon \). This requires, in the RG calculation, to add this extra operator to the perturbative action, so that the replicated theory will have the perturbation of the form:

\[
A_{\text{pert}} = g \int d^2 x \sum_{a \neq b}^{N} \varepsilon_a(x) \varepsilon_b(x) + \lambda \int d^2 x \sum_{a=1}^{N} \phi_a(x)
\]

(1.7)

with the initial values, in the RG sense, \( g = -g_0, \ g_0 > 0 \), which is due to disorder, and \( \lambda = \lambda_0 = 0 \), initially zero, but produced further spontaneously by the interactions.

As will be shown in the next Sections, the theory with \( A_{\text{pert}} \) in (1.7) has a stable fixed point at

\[
\lambda_\ast \neq 0, \ g_\ast = 0
\]

(1.8)

and it corresponds to \( N \) decoupled models \( WD_3^{(p-1)} \), so that, in the limit \( N = 0 \) which corresponds to problem with disorder, one gets a flow:

\[
WD_3^{(p)} \text{ with disorder} \to WD_3^{(p-1)} \text{ regular, without disorder}
\]

(1.9)

One gets this flow for \( N = 0 \), in the coefficients of the \( \beta \) - function, and for the initial conditions \( g_0 < 0, \ \lambda_0 = 0 \). There are no other fixed points in this case (\( N = 0 \)) in addition to (1.8) and the trivial fixed point, unstable, \( g = \lambda = 0 \). Yet in the related problem of \( N \) coupled regular models (no disorder), for \( N = 3 \) or bigger, the \( \beta \) - function contains, in addition to (1.8), two other fixed points, both stable in one direction and instable in the other. They correspond to multicriticality. To reach them one has to have both initial couplings \( g_0, \ \lambda_0 \neq 0 \), and, in addition their ratio have to be fine tuned. These extra fixed points are nontrivial, in the sense of a nontrivial realization at these points of a permutational symmetry. They are also nontrivial in the sense that the corresponding couplings \( (g_\ast^{(1)}, \lambda_\ast^{(1)}), \ (g_\ast^{(2)}, \lambda_\ast^{(2)}) \) are all non-vanishing. We expect that these points would not be identified with any presently known conformal field theory, like this appears to be the case for the coupled Potts models [2,7]. As has been argued above, we expect that in this case, of coupled \( WD_3^{(p)} \) and more generally \( WD_n^{(p)} \) models, our calculations provide first analytic corrections in \( \epsilon \sim \frac{1}{p} \) to the development of yet unknown
exact functions of $\epsilon$, or $p$. In case of the Potts models, the corresponding corrections in $\epsilon$ have to be considered as only numerical.

The rest of the paper is organized as follows. In Section 2 we introduce the Coulomb-Gas representation of the $WD_3^{(p)}$ model. Then, in Section 3, we study the effects of weak bond randomness on this model and the related problem of $N$ coupled $WD_3^{(p)}$ models. The renormalization of the couplings and of the energy operators are computed at one loop order. In Section 4 we study the behavior of the RG-flow in the case of disordered system and of $N$ coupled systems; we show in particular the presence of non trivial fixed points in both cases. We give the analytic expansion in $\epsilon$ of the central charge and of the dimension of energy operators at these points. In Section 5 all the results we have found for the $WD_3^{(p)}$ model are generalized for the whole family of $WD_n^{(p)}$ models. We discuss what has been obtained in the Conclusions.

2 Coulomb gas of the model $WD_3^{(p)}$

To make our presentation self-contained, and also because the models $\{WD_n^{(p)}\}$ have not much been used so far in the statistical physics applications, we shall reproduce in this Section the results of the paper [1], by giving the details on a particular model of our interest, $WD_3^{(p)}$.

This model could be realized by a 3-component Coulomb gas, with the stress-energy operator $T(z)$ taking the form:

$$T(z) = -\frac{1}{4} : \partial \vec{\varphi}(z) \partial \vec{\varphi}(z) : + i \vec{\alpha}_0 \partial^2 \vec{\varphi}$$

(2.1)

Here $\vec{\varphi}(z) = \{\varphi_1(z), \varphi_2(z), \varphi_3(z)\}$ is a set of three free fields put into a vector, with the correlation functions normalized as:

$$\langle \varphi_a(z, \bar{z}) \varphi_b(z', \bar{z}') \rangle = 2 \log \frac{1}{|z - z'|^2} \delta_{ab}$$

$$= (2 \log \frac{1}{z - z'} + 2 \log \frac{1}{\bar{z} - \bar{z}'}) \delta_{ab}$$

(2.2)

In the following, and in the above formula (2.1), we often suppress the dependance of $\varphi_a(z, \bar{z})$, on $\bar{z}$, which will be implicit.
The vector $\vec{\alpha}_0$ in (2.1) corresponds to a presence of the background charge operator, $V_{-2\vec{\alpha}_0}(R)$, putted at infinity, $R \to \infty$ [8].

In addition to $T(z)$, the model contains two other operators in its chiral algebra: $W_3(z)$, $W_4(z)$, with conformal dimensions 3 and 4. They could also be expressed in terms of polynomials in derivatives of fields $\{\varphi_a(z)\}$ [1]. The expressions are longer, but they will not actually be needed in the present analysis. Important is that $W_3(z)$ and $W_4(z)$ exist and classify, together with $T(z)$, all the fields (operators) of the model in terms of primaries and descendants of the chiral algebra. Then the usual methods, combined with the available Coulomb gas representation, define dimensions of primary operators (Kac formula) and their correlation functions. In this sense the conformal theories of $WD_{n}^{(p)}$ are fully defined [1].

In the Coulomb gas of the model $WD_{3}^{(p)}$ there are 3 screening operators “+”, and 3 screening operators “−”:

$$V_a^+(z) \equiv V_{\vec{\alpha}_a^+}(z) = e^{i\vec{\alpha}_a^+ \vec{\varphi}(z)}$$

(2.3)

$$V_a^-(z) \equiv V_{\vec{\alpha}_a^-}(z) = e^{i\vec{\alpha}_a^- \vec{\varphi}(z)}$$

(2.4)

$$\vec{\alpha}_a^\pm = \alpha_\pm \vec{e}_a^\pm, \quad (\vec{e}_a^\pm)^2 = 1$$

(2.5)

The “length” of the screening operator vectors, $\alpha_\pm$, are fixed by the condition that the conformal dimension of $V_a^\pm(z)$ is equal to 1. In general, for a vertex operator

$$V_\vec{\alpha}(z) = e^{i\vec{\alpha} \vec{\varphi}}$$

(2.6)

its conformal dimension with respect to the stress-energy operator $T(z)$, eq.(2.1), is given by:

$$\Delta_\vec{\alpha} \equiv \Delta(V_\vec{\alpha}) = \vec{\alpha}^2 - 2\vec{\alpha}\vec{\alpha}_0 = \alpha^2 - 2\alpha\alpha_0 \cos \theta$$

(2.7)

Here $\Theta$ is the angle between $\vec{\beta}_0$ and $\vec{\alpha}$. For the geometry of screenings of $WD_{3}^{(p)}$ the angle between $\{\vec{\alpha}_a\}$ and $\vec{\alpha}_0$ is the same, Fig.1. One gets, as a condition $\Delta(V_a^\pm) = 1$,

$$\alpha_a^2 - 2\alpha_a\alpha_0 \cos \Theta = 1$$

(2.8)

$$(\alpha_a)_{1,2} \equiv \alpha_\pm = \alpha_0 \cos \Theta \pm \sqrt{\alpha_0^2 \cos \Theta + 1}$$

(2.9)
The unit vectors \( \{ \vec{e}_a \} \) shown in the Fig.1 correspond to simple roots of the classical Lie algebra \( D_3 \). As it was said above, the vector \( \vec{\alpha}_0 \) is supposed to be “equally distant” from \( \vec{e}_1, \vec{e}_2, \vec{e}_3 \). Assuming this condition, for which we will not go into details here, \( \cos \Theta \) will be defined by the geometry in Fig.1. One finds:

\[
\cos \Theta = \frac{1}{\sqrt{10}} \tag{2.10}
\]

By the formula (2.9), the lengths \( \alpha_{\pm} \) of the screenings in eq.(2.5) will be functions of \( \alpha_0 \) only. This implies that in the model \( WD_{D_3}^\text{(p)} \) the orientational geometry is fixed, but there is one free parameter in the lengths of the vectors: the screenings \( \{ \vec{\alpha}_a^{(\pm)} \} \) and the background charge \( \vec{\alpha}_0 \).

The primary operators of the model are represented by the vertex operators

\[
V_\beta(z) = e^{i\vec{\beta} \vec{\omega}(z)} \tag{2.11}
\]

The allowed values of the vectors \( \vec{\beta} \) are defined by the degeneracy condition of the modules of \( V_\beta(z) \) with respect to the chiral algebra. They are found to be given by [1]:

\[
\vec{\beta} = \vec{\beta}(n'_1, n_1)(n'_2, n_2)(n'_3, n_3) = \sum_{a=1}^{3} \left( \frac{1 - n'_a}{2} \alpha_- + \frac{1 - n_a}{2} \alpha_+ \right) \vec{\omega}_a \tag{2.12}
\]

Here \( \{ \vec{\omega}_a \} \) is a set of three vectors which are dual to the unit vectors of the screenings \( \{ \vec{e}_a \} \):

\[
(\vec{\omega}_a, \vec{e}_b) = \delta_{ab} \tag{2.13}
\]

Using the general formula for the dimensions of vertex operators, eq.(2.7), one then gets from (2.12) the Kac formula of the model which defines the set of dimensions of the primary operators \( \{ \phi(n'_1, n_1, n'_2, n_2)(n'_3, n_3)(z) \} \):

\[
\phi(n'_1, n_1)(n'_2, n_2)(n'_3, n_3)(z) \propto V_{\vec{\beta}(n'_1, n_1)(n'_2, n_2)(n'_3, n_3)} \tag{2.14}
\]

\[
\Delta(n'_1, n_1, n'_2, n_2, n'_3, n_3) = \beta^2(\ldots) - 2\vec{\alpha}_0 \vec{\beta}(\ldots)
\]

\[
= \sum_a (u(n'_a, n_a))^2 \left( \frac{\vec{\omega}_a^2}{4} + 2 \sum_{a<b} u(n'_a, n_a) u(n'_b, n_b) \frac{(\vec{\omega}_a \cdot \vec{\omega}_b)}{4} \right.
\]

\[
- (\alpha_+ + \alpha_-) \sum_{a,b} u(n'_a, n_a) \frac{(\vec{\omega}_a \cdot \vec{\omega}_b)}{2} \tag{2.15}
\]

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Here
\[ u(n'_a, n_a) = (1 - n'_a)\alpha_- + (1 - n_a)\alpha_+ \quad (2.16) \]

In (15) we have used in addition the decomposition:
\[ 2\vec{\alpha}_0 = (\alpha_+ + \alpha_-) \sum_a \vec{\omega}_a \quad (2.17) \]

which is easily verified by multiplying both sides with a vector \( \vec{e}_b \) and using:
\[ 2\vec{\alpha}_0 \vec{e}_b = 2\alpha_0 \cos \Theta = \alpha_+ + \alpha_- \quad (2.18) \]

by eq.(2.9), and \( (\vec{\omega}_a \vec{e}_b) = \delta_{ab} \), eq.(2.13). The unit vectors \( \{\vec{e}_a\} \), which are defined by the Fig.1, when expressed in components will be given by:
\[ \vec{e}_1 = \frac{1}{\sqrt{2}}(0, -1, 1), \quad \vec{e}_2 = \frac{1}{\sqrt{2}}(1, 1, 0), \quad \vec{e}_3 = \frac{1}{\sqrt{2}}(-1, 1, 2) \quad (2.19) \]

The dual vectors \( \{\vec{\omega}_a\} \), defined by the eq.(2.13), are found to be equal to:
\[ \vec{\omega}_1 = \sqrt{2}(0, 0, 1), \quad \vec{\omega}_2 = \frac{1}{\sqrt{2}}(1, 1, 1), \quad \vec{\omega}_3 = \frac{1}{\sqrt{2}}(-1, 1, 1) \quad (2.20) \]

Then one finds:
\[ (\vec{\omega}_a, \vec{\omega}_b) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & \frac{3}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{3}{2} \end{pmatrix} \quad (2.21) \]

Finally, the Kac formula of the model \( WD_3^{(p)} \) takes the following form:
\[ \Delta(n'_1, n_1; n'_2, n_2; n'_3, n_3) = \frac{1}{2}(u(n'_1, n_1))^2 + \frac{3}{8}(u(n'_2, n_2))^2 + \frac{3}{8}(u(n'_3, n'_3))^2 \]
\[ + \frac{1}{2}u(n'_1, n_1)u(n'_2, n_2) + \frac{1}{2}u(n'_1, n_1)u(n'_3, n_3) + \frac{1}{4}u(n'_2, n_2)u(n'_3, n_3) \]
\[ - (\alpha_+ + \alpha_-)(2u(n'_1, n_1) + \frac{3}{2}u(n'_2, n_2) + \frac{3}{2}u(n'_3, n_3)) \quad (2.22) \]

\( u(n'_a, n_a) \) are defined by eq.(2.16). One checks that
\[ \sum_b (\vec{\omega}_a, \vec{\omega}_b) = (4, 3, 3) \quad (2.23) \]

which we have used in the last term in (2.22).
As it was mentioned above, the model contains one free parameter. This will be either $\alpha_0$, or $\alpha_+$, or $\alpha_-$, or the central charge of the model, which, for $T(z)$ in eq.(2.1), will be given by:

$$c = 3 - 24\alpha_0^2$$  \hspace{1cm} (2.24)

One observes that the lengths of the screenings “+” and the screenings “−”, $\alpha_+$ and $\alpha_-$ in eq.(2.9), are related:

$$\alpha_+ + \alpha_- = 2\alpha_0 \cos \Theta$$  \hspace{1cm} (2.25)

$$\alpha_+ \alpha_- = -1$$  \hspace{1cm} (2.26)

Similar to the case of the basic conformal theory of [9], one gets a unitary set of models if the parameter $(\alpha_+)^2$ is restricted to the discrete values:

$$(\alpha_+)^2 = \frac{p + 1}{p}$$  \hspace{1cm} (2.27)

By the relation (2.26) one will have also

$$(\alpha_-)^2 = \frac{p}{p + 1}$$  \hspace{1cm} (2.28)

For $WD_3^{(p)}$, $p = 5, 6, ...$ [1]. In the following we shall restrict ourselves to this unitary series. This explains for the extra index $p$ of $WD_3^{(p)}$.

We shall next be interested in this model in the limit of $p$ large, $p \gg 1$. First, when we take $p \to \infty$, the Kac formula (2.22) simplifies and takes the form:

$$\alpha_+ = 1, \quad \alpha = -1$$  \hspace{1cm} (2.29)

$$u(n'_a, n_a) = n'_a - n_a$$  \hspace{1cm} (2.30)

$$\Delta(n'_1, n_1)(n'_2, n_2)(n'_3, n_3) = \frac{1}{2}(n'_1 - n_1)^2 + \frac{3}{8}(n'_2 - n_2)^2 + \frac{3}{8}(n'_3 - n_3)^2$$

$$+ \frac{1}{2}(n'_1 - n_1)(n'_2 - n_2) + \frac{1}{2}(n'_1 - n_1)(n'_3 - n_3) + \frac{1}{4}(n'_2 - n_2)(n'_3 - n_3)$$  \hspace{1cm} (2.31)

In general, not specifically for $p$ large or infinite, the model contains a symmetry, in the set of its primary operators and their dimensions, w.r.t. permutation of indexes 2 and 3. This corresponds to the reflectional $Z_2$ symmetry of the geometry of the screening vectors $\{\vec{e}_a\}$ in Fig.1.
In the set of dimensions $\Delta_{(n'_1, n_1)(n'_2, n_2)(n'_3, n_3)}$ there is a subset which is not $Z_2$ degenerate and which corresponds to the singlet operators. For instance, singlet operators are $\{\phi_{(n'_1, n_1)(1,1)}\}$. The lowest dimension operator in the singlet sector, next to the identity operator $I = \phi_{(1,1)(1,1)(1,1)}(\Delta_{(1,1)(1,1)(1,1)} = 0$, by eq. (2.22)), will be the operator

$$\phi_{(2,1)(1,1)(1,1)}(x)$$

(2.32)

It is natural to identify it with the energy operator of the model,

$$\varepsilon(x) = \phi_{(2,1)(1,1)(1,1)}(x)$$

(2.33)

If we look now at the limiting case of $p \to \infty$, we shall find, by the formula (2.31), that

$$\Delta_\varepsilon(p \to \infty) = \frac{1}{2}$$

(2.34)

For $p$ finite, by the formula (2.22), one finds:

$$\Delta_\varepsilon = \Delta_{(2,1)(1,1)(1,1)} = \frac{5}{2} \alpha_+^2 - 2$$

(2.35)

Substituting, by eq. (2.28),

$$\alpha_+^2 = \frac{p}{p + 1} = 1 - \frac{1}{p + 1} = 1 - \epsilon$$

(2.36)

where we have defined

$$\epsilon = \frac{1}{p + 1}$$

(2.37)

one obtains:

$$\Delta_\varepsilon = \frac{1}{2} - \frac{5}{2} \epsilon$$

(2.38)

In our present study we are going to couple disorder to the energy operator $\varepsilon(x)$, defined in eq. (2.33). In the related problem of coupled regular models, we shall couple the models between themselves by their energy operators:

$$A_{\text{pert}} = g \sum_{a \neq b}^N \int d^2x \, \varepsilon_a(x) \varepsilon_b(x)$$

(2.39)

As has been discussed in the introduction, in both cases one gets a problem with a slightly relevant perturbation, if $\Delta_\varepsilon$ is given by the eq. (2.38) with $\epsilon$ small, or $p$ large, eq. (2.37). These problems will next be studied by the methods of the perturbative RG.
3 1-loop RG equations for $WD_3^{(p)}$

3.1 Renormalization of the couplings

We initially consider $N$ regular $WD_3^{(p)}$ models coupled by the energy-energy interaction (2.39):

$$A = \sum_{a}^{N} A_{0}^{(a)} + g_{0} \sum_{a \neq b}^{N} \int d^{2}x \, \varepsilon_{a}(x)\varepsilon_{b}(x)$$

(3.1)

$A_{0}^{(a)}$ being the conformal action corresponding to a single $WD_3^{(p)}$ model. The action (3.1) describes a conformal field theory perturbed by a slightly relevant term quadratic in the energy operator; such a problem can be reliably studied by means of perturbative RG with $\epsilon$-expansion. The RG scheme requires that all the relevant terms produced by the energy-energy interaction have to be added to (3.1) and the algebra of the enlarged set of perturbing fields have not to present other relevant operators. As shown below, the O.P.E. of $\varepsilon(x)\varepsilon(x')$ contains, apart from identity, one and only one (slightly) relevant operator, namely $\phi_{(1,1)(2,1)(2,1)} \equiv \phi$, whose dimension is $\Delta_{(1,1)(2,1)(2,1)} = 1 - 4\epsilon$. This implies that in the perturbative computation there will be diagrams (generated by terms in (2.39) with the same replica index) which produce, apart from trivial or irrelevant contributions, the term:

$$\sum_{a}^{N} \int d^{2}x \, \phi_{a}(x)$$

(3.2)

We therefore consider the more general action:

$$A = \sum_{a}^{N} A_{0}^{(a)} + g_{0} \sum_{a \neq b}^{N} \int d^{2}x \, \varepsilon_{a}(x)\varepsilon_{b}(x) + \lambda_{0} \sum_{a}^{N} \int d^{2}x \, \phi_{a}(x)$$

(3.3)

The problem of a single $WD_3^{(p)}$ model perturbed by $\phi$ has been considered in [1], where the existence of a non trivial infrared fixed point to which the system flows has been shown. The conformal field theory associated to this point corresponds to the $WD_3^{(p-1)}$ model.

In order to investigate the energy algebra, we study the four point correlation function $G(x) \equiv \langle \varepsilon(0)\varepsilon(x)\varepsilon(1)\varepsilon(\infty) \rangle$, which can be decomposed in a sum over s-channel diagrams corresponding to insertions of different operators. Introducing the Coulomb
gas representation (cfr. section 2) for $G(x)$ and defining for simplicity $\tilde{\beta}_{(1,1)(1,1)} = \tilde{\beta}_\epsilon$, $2\tilde{\alpha}_0 - \tilde{\beta}_{(2,1)(1,1)} = \tilde{\beta}_\epsilon$, $\tilde{\beta}_{(1,1)(2,1)} = \tilde{\beta}_\phi$, and $2\tilde{\alpha}_0 - \tilde{\beta}_{(1,1)(2,1)} = \tilde{\beta}_\phi$. we can write:

$$G(x) \propto \int \cdots \int \langle V_{\tilde{\beta}_\epsilon} (0) V_{\tilde{\beta}_\epsilon} (1) V_{\tilde{\beta}_\epsilon} (\infty) V_{\tilde{\beta}_\epsilon} (\mu_1) V_{\tilde{\beta}_\epsilon} (\mu_2) V_{\tilde{\beta}_\epsilon} (\xi) V_{\tilde{\beta}_\epsilon} (\nu) \rangle d^2 \mu_1 d^2 \mu_2 d^2 \xi d^2 \nu$$

(3.4)

where the screenings $V_a^-$, defined in (2.3), are integrated over the 2D plane and their number is determined by the charge neutrality condition:

$$2\beta_\epsilon + \alpha_- (m_1 \bar{\epsilon}_1 + m_2 \bar{\epsilon}_2 + m_3 \bar{\epsilon}_3) = 0$$

(3.5)

which is satisfied for $m_1 = 2$, $m_2 = 1$, $m_3 = 1$.

Each leading term in $x \to 0$ limit originates in the integral representation (3.4) when a particular number $q_a$ of screenings $V_a^-$ (by eq.(3.5) $q_1 \leq 2$, $q_2 \leq 1$, $q_3 \leq 1$) approaches the origin. Since in this limit

$$V_{\tilde{\beta}_\epsilon} (x) V_{\tilde{\beta}_\epsilon} (0) \prod_a (V_a^-)^{q_a} \to V_{2\tilde{\beta}_\epsilon + \alpha_- \sum_a q_a \bar{\epsilon}_a} (0) + \cdots,$$

(3.6)

the intermediate state vertex $V_{2\tilde{\beta}_\epsilon + \alpha_- \sum_a q_a \bar{\epsilon}_a}$ corresponds to a primary field which belongs to the energy algebra. In particular we have noticed the presence of the vertex $V_{2\tilde{\beta}_\epsilon + \alpha_- \bar{\epsilon}_1}$, which corresponds to the primary operator $\phi$, according to

$$2\tilde{\beta}_\epsilon + \alpha_- \bar{\epsilon}_1 = -\alpha_- \left( \frac{\bar{\omega}_2}{2} + \frac{\bar{\omega}_3}{2} \right)$$

(3.7)

In appendix A we shall show in detail that $\phi$ is the only relevant operator in the energy product decomposition and that the algebra of $\phi$ and $\epsilon$, apart from irrelevant terms, is:

$$\epsilon (x) \epsilon (0) = \frac{I}{|x|^{4\Delta_\epsilon}} + \frac{D_{\epsilon \epsilon}^\phi}{|x|^{4\Delta_\epsilon - 2\Delta_\phi}} \phi + \cdots$$

$$\phi (x) \phi (0) = \frac{I}{|x|^{4\Delta_\phi}} + \frac{D_{\phi \phi}^\phi}{|x|^{4\Delta_\phi}} \phi + \cdots$$

$$\phi (x) \epsilon (0) = \frac{D_{\epsilon \phi}^\phi}{|x|^{2\Delta_\phi}} \phi + \cdots;$$

(3.8)

$D_{\epsilon \epsilon}^\phi$ and $D_{\phi \phi}^\phi$ are the associated structure constants of the $WD_3^{(p)}$ model.

At one loop order the main RG characteristics of action (3.3), e.g. $\beta$-functions or renormalization of operators, are easily obtained from eq.(3.8) (see appendix C). The
renormalized coupling constants \( g(r) \) and \( \lambda(r) \) (\( r \) is the short distance cut-off) have been computed, and the correspondent \( \beta \)-functions take the form:

\[
\beta_g \equiv \frac{\partial g(r)}{\partial \ln r} = (2 - 4\Delta_\varepsilon)g - (N - 2)g^2 - 4D^\phi_{\varepsilon\varepsilon}\lambda g
\]

\[
\beta_\lambda \equiv \frac{\partial \lambda(r)}{\partial \ln r} = (2 - 2\Delta_\phi)\lambda - \frac{D^\phi_{\phi\phi}}{2}\lambda^2 - \frac{N - 1}{2}D^\phi_{\varepsilon\varepsilon}g^2
\]

where the coupling constants have been redefined as \( g \rightarrow g/4\pi \) and \( \lambda \rightarrow \lambda/2\pi \). In order to study the coupling flow induced by eq.(3.9) (section 4) the structure constants have been determined (see appendix B):

\[
D^\phi_{\varepsilon\varepsilon} = \sqrt{\frac{5}{3}} + 0(\epsilon), \quad D^\phi_{\phi\phi} = \frac{8}{\sqrt{15}} + 0(\epsilon)
\]

In section 4 we will investigate the properties of equations (3.9) for \( N = 0 \) (disorder) and \( N \geq 2 \) (coupled systems).

### 3.2 Renormalization of the energy operator

In order to study the effect of the perturbation on the energy operators, we need to compute the renormalized operators \( \varepsilon'_{a} \), which are expressed via the \( N \times N \) matrix \( [Z_\varepsilon]_{ab} \) by:

\[
\varepsilon'_a = \sum_b [Z_\varepsilon]_{ab} \varepsilon_b
\]

As usual, we proceed perturbatively: computing contributions from each coupling term and rewriting bare quantities in term of renormalized ones, we have determined up to the first order the matrix \( \gamma_{ab} \equiv d\ln[Z]_{ab}/d\ln(r) \) (see appendix C). The new critical exponents at the fixed points \( g_*, \lambda_* \) of the RG flow will be given by the eigenvalues of the dimension matrix \( \Delta_{ab} \equiv \Delta_\varepsilon\delta_{ab} - \gamma_{ab}(g_*, \lambda_*) \) which takes the form:

\[
\Delta_{ab} = \begin{bmatrix}
\Delta_{\varepsilon_1} + D^\phi_{\varepsilon\phi}\lambda_* & g_* & g_* & \cdots & g_* \\
g_* & \Delta_{\varepsilon_2} + D^\phi_{\varepsilon\phi}\lambda_* & g_* & \cdots & g_* \\
g_* & g_* & \Delta_{\varepsilon_3} + D^\phi_{\varepsilon\phi}\lambda_* & \cdots & g_* \\
\cdots & \cdots & g_* & \cdots & \cdots \\
g_* & g_* & \cdots & \cdots & \Delta_{\varepsilon_N} + D^\phi_{\varepsilon\phi}\lambda_*
\end{bmatrix}
\]
It is straightforward to see that the symmetric combination \( \epsilon_s = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_N \) and the antisymmetric ones \( \epsilon_{a_1} = \epsilon_1 - \epsilon_2 , \epsilon_{a_2} = \epsilon_2 - \epsilon_3 \ldots \) are eigenvectors of (3.12) and their dimensions turn out to be:

\[
\Delta_{\epsilon_s}(g_*, \lambda_*) = \Delta_\epsilon + D_\epsilon \lambda_* + (N - 1)g_* \\
\Delta_{\epsilon_{a_1}}(g_*, \lambda_*) = \Delta_{\epsilon_{a_2}} = \ldots = \Delta_\epsilon + D_\epsilon \lambda_* - g_* 
\]

(3.13)

In the next section we shall show that equations (3.9) admit fixed points with \( g_* \neq 0 \) at which \( N \) identical systems remain coupled; we expect therefore that the correspondent conformal field theory realizes in a non trivial way the permutation symmetry of \( N \) identical objects. The splitting of dimensions in the energy sector \( \Delta_{\epsilon_s} \neq \Delta_{\epsilon_a} \) indicates that this is indeed the case. Although a similar result was obtained for \( N \) coupled Potts models, in this case we have access to the analytic expansion in \( \epsilon \) of the dimensions of the energy operators and of the central charge (see next section) of new conformal field theories.

4 RG-flow and central charge

4.1 Fixed point structure and couplings flow

The first step in the study of RG-flow is to find all points \( g_*, \lambda_* \) such that \( \beta_g(g_*, \lambda_*) = \beta_\lambda(g_*, \lambda_*) = 0 \).

The equations (3.9) have been obtained for a generic number \( N \) of coupled models. We consider firstly the quenched case which is obtained in the \( N \to 0 \) limit. The \( \beta \)-functions, defined in (3.9), take in this limit the form:

\[
\beta_g = 10\epsilon g + 2g^2 - 2\sqrt{\frac{5}{3}} \lambda g \\
\beta_\lambda = 8\epsilon \lambda - \frac{4}{\sqrt{15}} \lambda^2 + \frac{1}{2} \sqrt{\frac{5}{3}} g^2 
\]

(4.1)

The RG flow has, apart from the trivial solution \( g = \lambda = 0 \), one stationary point:

\[
\lambda_* = 2\sqrt{15} \epsilon , \quad g_* = 0 
\]

(4.2)
which is stable in all directions. This point has already been found in [1]: the associated critical theory is described by a pure $WD_3^{(p-1)}$ model. Taking as initial conditions $\lambda_0 = 0$ and $g_0 < 0$, and assuming that higher loop corrections will not change the qualitative behavior of the flow near the two fixed points, the system flows toward point (4.2) as supported by the numerical calculations shown in Fig. 2.

Initial conditions $\lambda_0 = 0$ and $g_0 < 0$ correspond exactly to the disorder produced couplings; so, the addition of a small bond randomness will drive the model $WD_3^{(p)}$ to the model $WD_3^{(p-1)}$ without disorder.

We consider now $N \geq 2$, i.e. the case of coupled models: by (3.9) and (3.10) the $\beta$-functions are:

$$\beta_g = 10\epsilon g - (N - 2)g^2 - 2\sqrt{\frac{5}{3}}\lambda g$$
$$\beta_\lambda = 8\epsilon \lambda - \frac{4}{\sqrt{15}}\lambda^2 - \frac{N - 1}{2}\sqrt{\frac{5}{3}}g^2$$

(4.3)

In addition to point (4.2), which remains a point of attraction of the RG-flow since the respective RG-eigenvalues don’t depend on $N$, we have two new fixed points:

$$\lambda^{(1)}_* = \sqrt{15} \left( 1 - (N - 2)\sqrt{\frac{6}{N - 1 + 6N^2}} \right) \epsilon, \quad g^{(1)}_* = 10\sqrt{\frac{6}{N - 1 + 6N^2}} \epsilon$$

(4.4)

$$\lambda^{(2)}_* = \sqrt{15} \left( 1 + (N - 2)\sqrt{\frac{6}{N - 1 + 6N^2}} \right) \epsilon, \quad g^{(2)}_* = -10\sqrt{\frac{6}{N - 1 + 6N^2}} \epsilon$$

(4.5)

The stationary points (4.4) and (4.5) constitute node points of the flow and they can be reached only by a fine tuning of the initial conditions. Studying in detail the flow diagram of (4.3) (shown for $N = 3$ in Fig.3) we see that with the initial conditions $\lambda_0 = 0$ and $g_0 \neq 0$ the system will flow far from our fixed points toward either a massive theory or another fixed point which cannot be seen at this order in perturbation theory.

---

1 It could be mentioned that a particular case of $N = 2$ coupled $WD_n^{(p)}$ theories, with $g_0 \neq 0, \lambda_0 = 0$, have been considered previously in the paper [11], with a conclusion that the theory is integrable. We only want to mention again that according to our RG analysis the second coupling, $\lambda \neq 0$, will have to be admitted eventually into the action (see the corresponding flows in Fig.3).

This is in contrast with two coupled Virasoro algebra minimal models $M_p$ which were shown to be
other hand, in the region $\lambda > 0$ and of order $\epsilon$, the situation is rather different. There are two solutions of eq.(4.3), say $g_+(\lambda) > 0$ and $g_-(-\lambda) < 0$, which are attracted respectively by the node point (4.4) and (4.5); if we take $\lambda_0 > 0$ and $g_-(\lambda_0) < g_0 < g_+(\lambda_0)$, the system will flow toward the stable point (4.2). In this case the infrared limit of the system will be described by $N$ decoupled $WD_{2}^{(p-1)}$ models.

It’s important to note that, although quite similar, the flow is not symmetric along the $\lambda$ axis, and this will be at the origin of the difference between the values of the central charge at the two fixed points. This asymmetry can be explained by a simple physical argument. Indeed, when $g > 0$, the coupling of different models is always frustrated, while for $g < 0$ the $N$-system can arrange itself in a kind of ferromagnetic configuration in order to minimize the energy. For $N = 2$ (i.e. no frustration) we recover the symmetry $g \rightarrow -g$, and the central charge has the same value at the two new fixed points (see next section).

Inserting the values of the structure constants (3.10) in eq. (3.13), we have at the fixed points $(g^*_1, \lambda^*_1)$ and $(g^*_2, \lambda^*_2)$:

$$\Delta_{\epsilon}^{(1),(2)} = \Delta_{\epsilon} + 5 \left( 1 \pm N \sqrt{\frac{6}{N - 1 + 6N^2}} \right) \epsilon$$

where $\Delta_{\epsilon}^{(1)} \equiv \Delta_{\epsilon,a}^{(1)}(g^*_1, \lambda^*_1)$ and $\Delta_{\epsilon}^{(2)} \equiv \Delta_{\epsilon,a}^{(2)}(g^*_2, \lambda^*_2)$. By eq. (4.6) we have $\Delta_{\epsilon}^{(1)} = \Delta_{\epsilon,a}^{(2)}$ and $\Delta_{\epsilon,a}^{(1)} = \Delta_{\epsilon,a}^{(2)}$ for all $N$. This result can be easily explained for $N = 2$: in this case the two models are equivalent under the replacements $g \rightarrow -g$ and $\epsilon_1 \rightarrow -\epsilon_1$. On the other hand, for $N > 2$ no reason can be given, as it can be seen from the multiplicity of the antisymmetric energy combinations. We believe therefore that these equalities are accidental and they originate from the simplicity of the one-loop order computation.

integrable [12] and studied in detail in [11]. In this last case the original action, with a coupling term $g_0$ only, is stable with respect to RG evolution.

Then, if the coupling $\lambda \neq 0$ has to be taken into the action, for the two coupled $WD_{n}^{(p)}$ theories, eq.(3.3), the natural question would be if the presence of the second perturbative term will modify essentially the analysis of integrability.
4.2 Central charge

In the previous section we have shown that the RG-flow of the quenched system ($N = 0$) and of the coupled systems ($N \geq 2$) exhibits non-trivial fixed points. A simple way for computing the central charge of the associated conformal theories is given by the Zamolodchikov’s c-theorem [5]; the theorem provides us with a function of the couplings $c(g, \lambda)$, to be defined below, whose value $c(g_*, \lambda_*)$ at the fixed point $(g_*, \lambda_*)$ is equal to the central charge of the corresponding critical theory.

We define as $\Theta(x)$ the trace of the stress-energy tensor. It’s well known that $\Theta(x)$, which is zero at the fixed point, is proportional to the perturbing terms of the theory. In our case we have from the action (3.3):

$$\Theta(x) = \frac{\beta_g}{8} \sum_{a \neq b} \varepsilon_a(x) \varepsilon_b(x) + \frac{\beta_\lambda}{4} \sum_c \phi_c(x)$$

(4.7)

where the renormalization of the couplings $g \to g/4\pi$ and $\lambda \to \lambda/2\pi$ is taken into account. The function $c(g, \lambda)$ is then completely determined by the following equations [5]:

$$\beta_g \frac{\partial c(g, \lambda)}{\partial g} + \beta_\lambda \frac{\partial c(g, \lambda)}{\partial \lambda} = -24 < \Theta(0)\Theta(1) >$$

$$c(0,0) = c_{\text{pure}}$$

(4.8)

with $c_{\text{pure}}$ and $< \Theta(0)\Theta(1) >$ respectively the central charge and the $\Theta$ field two-point correlation function of the unperturbed theory. Inserting eq.(4.7) and eq.(3.9) into eq.(4.8) and using the following relations:

$$\sum_{a \neq b, c \neq d} \sum_{a=1}^{N} \sum_{b=1}^{N} < \varepsilon_a(1) \varepsilon_b(1) \varepsilon_c(0) \varepsilon_d(0) > = 2N(N-1)$$

$$\sum_{c \neq d, a=1}^{N} \sum_{a=1}^{N} < \varepsilon_c(0) \varepsilon_d(0) \phi_a(1) > = 0$$

$$\sum_{a=1}^{N} \sum_{b=1}^{N} < \phi_a(1) \phi_b(0) > = N,$$

(4.9)

it’s straightforward to verify that the function

$$c(g, \lambda) = c_{\text{pure}} - N \left( \frac{3}{2} \Delta_\varepsilon g^2 + 3 \Delta_\phi \varepsilon \lambda^2 - \frac{(N-1)(N-2)}{4} g^3 - \frac{D_\phi}{4} \lambda^3 - \frac{3}{4} (N-1) D_\varepsilon g^2 \lambda \right)$$

(4.10)
satisfies eq.(4.8). The function (4.10) has been obtained when a number \( N \) of \( WD_3^{(p)} \) models is considered and so \( c_{\text{pure}} = Nc(WD_3^{(p)}) \), where \( c(WD_3^{(p)}) \) is the central charge of a single \( WD_3^{(p)} \) model, calculated in [1]:

\[
c(WD_3^{(p)}) = 3 \left( 1 - \frac{20}{p(p+1)} \right)
\] (4.11)

In order to compute the central charge in the case of the disordered problem, we normalize the function \( c(g, \lambda) \) by \( N \) and then we take the limit \( N \to 0 \); in fact, in terms of replicas this is exactly the limit in which the related quenched free energy is obtained. Using the values of the structure constants (3.10), the central charge \( c_{\text{dis.}} \) at the infrared stable point (4.2) turns out to be:

\[
c_{\text{dis.}} = \lim_{N \to 0} \frac{c(0, 2\sqrt{15} \epsilon)}{N} = c(WD_3^{(p)}) - 120\epsilon^3 \approx c(WD_3^{(p-1)})
\] (4.12)

This result is consistent with what has already been said in the previous section: the infra-red behavior of a \( WD_3^{(p)} \) model with disorder coupled to the energy operator is described by a \( WD_3^{(p-1)} \) model.

In the related problem of \( N \) coupled models, the central charge \( c_{\text{coupl.}} \) at the fixed point (4.2) is \( c_{\text{coupl.}} = c(0, 2\sqrt{15} \epsilon) = Nc(WD_3^{(p-1)}) \), i.e. the correspondent critical theory is described by \( N \) decoupled \( WD_3^{(p-1)} \) models.

Finally we have access to the analytic expansion up to the third order in \( \epsilon \) of the central charges \( c_1 \) and \( c_2 \) at the new fixed points \( (g^{(1)}_*, \lambda^{(1)}_*) \) and \( (g^{(2)}_*, \lambda^{(2)}_*) \):

\[
c_{1,2} \simeq Nc(WD_3^{(p)}) - N \left[ 60 \left( 1 \mp (N - 2)\sqrt{\frac{6}{N - 1 + 6N^2}} \right) \epsilon^3 \right]
\] (4.13)

The eq.(4.13) represents the first analytic result for a new series of conformal theories which in addition to the \( W \)-symmetry present a non-trivial representation of the permutation symmetry.

## 5 Analysis of the \( WD_n^{(p)} \) models

We shall show that the results we have obtained in the previous sections still hold in the case of \( WD_n^{(p)} \) models. The construction of the Coulomb-Gas representation of these
theories is a direct generalization of what has been presented in Section 2. The $WD_n^{(p)}$ model presents, in addition to the conformal symmetry, a series of additional symmetries which are generated by a series of local currents $\{W_{2k}(x)\}_{k=1,...,n-1}$ and $W_n(x)$ with dimension $\Delta_{2k} = 2k$ and $\Delta_n = n$ respectively. It can be represented by an $n$-component Coulomb Gas. The (2.3), (2.4) and (2.5) still define the screenings “+” and “−”, with unit vectors $\vec{e}_a$ which lie on an $n$-dimensional space and correspond to simple roots of Lie algebra $D_n$. The primary operators are represented by vertex operators $V_{\vec{\beta}}$ with

$$\vec{\beta} \equiv \vec{\beta}(n'_1,n_1)(n'_2,n_2)...(n'_n,n_n) = \sum_{a=1}^{n}(\frac{1-n'_a}{2}\alpha_- + \frac{1-n_a}{2}\alpha_+)\vec{\omega}_a$$

(5.1)

where the set of dual vectors $\{\vec{\omega}_a\}$, defined by (2.13), have the quadratic form matrix $F_{ab} \equiv (\vec{\omega}_a,\vec{\omega}_b)$:

$$F_{ab} = 2a, \quad a \leq b < n - 1; \quad F_{an-1} = F_{an} = a, \quad a < n - 1;$$

$$F_{nn} = F_{n-1n-1} = \frac{n}{2}; \quad F_{n-1m} = \frac{n-2}{2}$$

(5.2)

The central charge $c(WD_n^{(p)})$ of the $WD_n^{(p)}$ model has been calculated in [1]:

$$c(WD_n^{(p)}) = n\left(1 - \frac{(2n-2)(2n-1)}{p(p+1)}\right)$$

(5.3)

Using (2.15) and (5.2) the dimension of primary operators can be easily calculated. In particular the dimension of the operator $\Phi_{(2,1),(1,1),...,(1,1)}$, naturally identified with the energy operator $\varepsilon$ (Section 2), is:

$$\Delta_{\varepsilon} \equiv \Delta_{(2,1),(1,1),...,(1,1)} = \frac{1}{2} - \frac{2n-1}{2}\varepsilon$$

(5.4)

Therefore the disorder induced interaction ($\sim \varepsilon \varepsilon$) is slightly relevant: we can study the $WD_n^{(p)}$ model with disorder coupled to the energy operator and the related problem of $N$-coupled systems using the same technique as the one we have exploited in the case of a $WD_3^{(p)}$ model. The analogy with this case goes further: we show in appendix 1 that the operator $\phi_{(1,1)(2,1),(1,1),...,(1,1)} \equiv \phi$ with dimension

$$\Delta_{\phi} \equiv \Delta_{(1,1),(2,1),(1,1),...,(1,1)} = 1 - 2(n-1)\varepsilon$$

(5.5)

is, apart from the identity, the only relevant operator in the O.P.E. of $\varepsilon(x')\varepsilon(x)$. The enlarged algebra of the fields $\phi$ and $\varepsilon$ is given by eq.(3.8), where $D_{\varepsilon\varepsilon}^{\phi}$ and $D_{\phi\phi}^{\varepsilon}$ are now
the related structure constant of the $WD_n^{(p)}$ model. We consider thus the action (3.3),
where in this case $A_y^{(a)}$ describes a single $WD_n^{(p)}$ model: the correspondent $\beta$-functions
take the form (3.9) with the values of the structure constants (up to the first order in $\epsilon$)
calculated in appendix 1:

$$D^\phi_{\epsilon\epsilon} = \sqrt{\frac{2n-1}{n}} + O(\epsilon), \quad D^\phi_{\phi\phi} = \frac{4(n-1)}{\sqrt{n(2n-1)}} + O(\epsilon) \quad (5.6)$$

In the case $N = 0$, in addition to the unstable trivial fixed point $(0, 0)$, there is one stable
fixed point

$$g_* = 0, \quad \lambda_* = 2\sqrt{n(2n-1)} \epsilon \quad (5.7)$$

For $N \geq 2$, there are two other supplementary fixed points, both node points of the RG
flow:

$$\lambda_*^{(1)} = \sqrt{n(2n-1)} \epsilon \left( 1 - (N-2) \sqrt{\frac{n(n-1)}{N-1 + n(n-1)N^2}} \right)$$

$$g_*^{(1)} = +2(2n-1) \epsilon \sqrt{\frac{n(n-1)}{N-1 + n(n-1)N^2}} \quad (5.8)$$

$$\lambda_*^{(2)} = \sqrt{n(2n-1)} \epsilon \left( 1 + (N-2) \sqrt{\frac{n(n-1)}{N-1 + n(n-1)N^2}} \right)$$

$$g_*^{(2)} = -2(2n-1) \epsilon \sqrt{\frac{n(n-1)}{N-1 + n(n-1)N^2}} \quad (5.9)$$

In order to compute the central charges at the fixed points for the disorder problem and
for the coupled models, we can directly apply the result (4.9), provided the new values of
the structure constants (5.6) and of the dimension of operators (5.4) and (5.5) are taken
into account. For the quenched problem, the central charge $c^{(n)}_{\text{dis.}}$ at the stable fixed point
(5.7) is

$$c^{(n)}_{\text{dis.}} = \lim_{N \to 0} \frac{c(0, 2\sqrt{n(2n-1)} \epsilon)}{N} = c(WD_n^{(p)}) - 4n(n-1)(2n-1)\epsilon^3 \approx c(WD_n^{(p-1)}) \quad (5.10)$$

The correspondent field theory is described by the $WD_n^{(p-1)}$ model, as found already in
[1]. In the coupled model, the central charge $c^{(n)}_{\text{coupl.}}$ at the same fixed point is $c^{(n)}_{\text{coupl.}} =$
\( N c(W D_{n}^{(p-1)}) \) and the associated critical theory is described by \( N \) decoupled \( W D_{n}^{(p-1)} \) models.

At the new fixed points (5.8) and (5.9) the central charges \( c_{1,2}^{(n)} \) turn out to be:

\[
c_{1,2}^{(n)} = N c(W D_{n}^{(p)}) - 2 N n(n-1)(2n-1) \left[ 1 \mp (N-2) \sqrt{\frac{n(n-1)}{N-1+n(n-1)N^2}} \right] \epsilon^3 \tag{5.11}
\]

As the RG-flow behavior is the same for all \( n \geq 3 \), the results we have shown in section 4.1 for the disordered \( W D_{3}^{(p)} \) model and for the system of \( N \) coupled \( W D_{3}^{(p)} \) models are valid for the whole series of \( W D_{n}^{(p)} \) conformal theories.

Finally we give the direct generalization of eq.(4.6):

\[
\Delta_{\varepsilon}^{(1),(2)} = \Delta_{\varepsilon} + (2n-1) \left( 1 \mp N \sqrt{\frac{n(n-1)}{N-1+n(n-1)N^2}} \right) \epsilon \\
\Delta_{\varepsilon_a}^{(1),(2)} = \Delta_{\varepsilon} + (2n-1) \left( 1 \mp N \sqrt{\frac{n(n-1)}{N-1+n(n-1)N^2}} \right) \epsilon \tag{5.12}
\]

### 6 Conclusions

As has been stressed throughout the paper, for every small value of \( \epsilon \):

\[
\epsilon = \frac{1}{p+1}, \quad p \gg 1 \tag{6.1}
\]

we have a unitary model, \( W D_{n}^{(p)} \), associated to it. When a set of \( N \geq 2 \) of such models are coupled and brought, by fine tuning of the couplings \( g \) and \( \lambda \), to one of the two newly found fixed points, (5.8) or (5.9), the corresponding critical theory should also be described by a unitary conformal field theory. As we have a fixed point for every \( p \), we should have a unitary series of new conformal theories, with \( p \) being a parameter, accumulating towards \( p = \infty \). Moreover, we should have a unitary series for every \( n \), of \( W D_{n}^{(p)} \), \( n = 3, 4, 5, \ldots \).

These new theories, presently unknown, have to incorporate into them the permutational symmetry \( S_{N}, N = 2, 3, \ldots \). This symmetry has to be incorporated into the chiral
algebra of these theories. As the symmetry is discrete, the natural suggestion would be that it should be represented by parafermionic currents.

For $N \geq 3$, the group $S_N$ is non-Abelian. Looking at the expression of the central charge, eq.(5.11), we observe that the correction term which we have calculated is non-rational for $N \geq 3$.

If we accept the idea of non-Abelian parafermionic conformal theories, we have to accept also that these theories possess an infinite series of unitary models, labeled by $p$, according to the arguments given above. And then the formula (5.11) indicates that the central charge for these unitary models takes non-rational values. This feature is unusual.

Being more precise, the argument for rational or non-rational values of the central charge $c$, and the dimensions of the operators, could be given as follows.

If one assumes that $c$, which is a function of $p$, takes rational values for all integer values of $p$, this then requires that the function $c(p)$ should have a simple rational form, like $c(p) = Q(p)/P(p)$, where $Q(p), P(p)$ are two polynomials of $p$ with rational valued coefficients. If the function $c(p)$ of such a form is developed in a series of $1/p$, the coefficients will all be rational.

Still, it should be admitted that we don’t know in fact if the perturbative expansion in $\epsilon \sim 1/p$, which we define in this paper, represents a convergent series. The series might also be asymptotic, as it is often the case in perturbative expansions in field theory. This then allows for two possibilities.

The first possibility will be that the series is convergent, representing an analytic function $c(p)$, like this is two case for the perturbed minimal models in [5].

In this case, according to the argument given above, one could actually judge on rationality- non rationality with the coefficients of the expansion.

Saying it again, for having rational values of $c(p)$, for all integer $p$, a complicated function $c(p)$ will not be allowed.

Then the coefficients of the expansion will also have to be rational.
The second possibility will be that the series in $\epsilon$ is not convergent, that it is only asymptotic. In this case our arguments do not apply and the exact values of the central charge might well be rational, in spite of the irrational coefficients of the expansion.

In view of this second possibility, our conclusions on non-rationality should not be viewed as definite.

Construction of the corresponding exact conformal field theories represents a theoretical challenge.

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A Appendix 1

A.1 Relevant Operators in the O.P.E. of $\varepsilon(x')\varepsilon(x)$

In section 3 we have explained how to deduce the O.P.E. of $\varepsilon(x')\varepsilon(x)$ from the integral representation of the energy four-point correlation function: within the set of vectors 

$$\{2\vec{\beta}_\varepsilon + \alpha_\varepsilon \sum_{a=1}^3 q_a \vec{e}_a\}$$

where $q_1 = 0,1,2$, $q_2 = 0,1$, and $q_3 = 0,1$, we search for the ones which decompose into positive integer numbers of $\{\alpha_\varepsilon \vec{\omega}_a\}$, basic vectors of the representation lattice, i.e.:

$$2\vec{\beta}_\varepsilon + \alpha_\varepsilon \sum_{a=1}^3 q_a \vec{e}_a = \alpha_\varepsilon \sum_{a=1}^3 \frac{1-n'_a}{2} \vec{\omega}_a$$ (A.1)

with $n'_a \geq 1$. In this case the vertex operator $V_{2\vec{\beta}_\varepsilon + \alpha_\varepsilon \sum_{a=1}^3 q_a \vec{e}_a}$ belongs to the physical sector of the Kac table and the correspondent primary operator $\phi_{(\alpha'_1,1)(\alpha'_2,1)(\alpha'_3,1)}$ appears in the energy algebra. Using eq.(2.19) and eq.(2.20), the condition (A.1) is equivalent to:

$$n'_1 = 3 - 2q_1 + q_3 + q_2$$

$$n'_2 = 1 - 2q_2 + q_1$$

$$n'_3 = 1 - 2q_3 + q_1$$ (A.2)

Testing all the possible set of values $(q_1, q_2, q_3)$, it’s easy to see that the system (A.2) admits only three solutions; they correspond to the identity operator (for $q_1 = 2, q_3 = 1, q_2 = 1$), to the operator $\phi$ (for $q_1 = 1, q_3 = 0, q_2 = 0$) and to the operator $\phi_{(3,1)(1,1)(1,1)}$ (for $q_1 = q_2 = q_3 = 0$). According to eq.(2.31) $\Delta_{(3,1)(1,1)(1,1)} = 2$: the only relevant operators in the energy algebra are the identity and $\phi$.

Now, in order to verify that the algebra formed by these two fields doesn’t contain any other relevant operator, we have to check the O.P.E. of $\phi(x')\phi(x)$. Following the same procedure used before, it’s easy to verify that, apart from identity, the $\phi$ field is the only relevant operator in the $\phi$-algebra, as already discussed in [1]. We consider in this case the integral representation of the four-point correlation function $G'(x) \equiv$
where the variables $\mu_1, \mu_2, \xi_1, \xi_2, \nu_1, \nu_2$ are integrated over the 2D plane. The charge neutrality is satisfied according to $-2\bar{\beta}_\phi + 2\alpha_-(\bar{\epsilon}_1 + \bar{\epsilon}_2 + \bar{\epsilon}_3) = 0$. As discussed above all the operators $\phi(n'_{1,1})$ present in the O.P.E. of $\phi(x') \phi(x)$ are given by the condition

$$2\bar{\beta}_\phi + \alpha_+ \sum_{a=1}^{3} q_a \bar{\epsilon}_a = \alpha_+ \sum_{a=1}^{3} \frac{1 - n'_a}{2} \bar{\omega}_a$$

where $q_1, q_2, q_3$ can assume the values 0, 1, 2; eq. (A.4) is equivalent to:

$$n'_1 = 1 - 2q_1 + q_3 + q_2$$

$$n'_2 = 3 - 2q_2 + q_1$$

$$n'_3 = 3 + q_1 - 2q_3$$

By eq. (A.5) we have determined, apart from the identity, four primary operators: $\phi(1,1)(1,1)(1,1)$, $\phi(2,1)(1,1)(1,1), \phi(1,2)(1,1)(1,1)$ and $\phi$; the only field whose dimension is smaller than the unity is $\phi$.

### A.2 Relevant Operators in the O.P.E. of $\varepsilon(x') \varepsilon(x)$ for the $WD_n^{(p)}$ model with $n \geq 4$

As already discussed in section 5, the energy operator $\varepsilon$ of the $WD_n^{(p)}$ model is associated to the primary operator $\phi(2,1)(1,1)_{(1,1)}$. We show below that in the correspondent algebra there is, apart from the identity, only one relevant primary operator, namely $\phi(1,1)(2,1)(1,1)_{(1,1)} \equiv \phi$. In the following we define for simplicity $\bar{\beta}_{(2,1)(1,1)_{(1,1)}} \equiv \bar{\beta}_\varepsilon$, $2\bar{\omega}_0 - \bar{\beta}_{(2,1)(1,1)_{(1,1)}} \equiv \bar{\beta}_\varepsilon$, $\bar{\beta}_{(1,1)(2,1)(1,1)_{(1,1)}} \equiv \bar{\beta}_\phi$, and $2\bar{\omega}_0 - \bar{\beta}_{(1,1)(2,1)(1,1)_{(1,1)}} \equiv \bar{\beta}_\phi$.

The integral representation of the correspondent four-point correlation function $G(x) \equiv \langle \varepsilon(0) \varepsilon(x) \varepsilon(1) \varepsilon(\infty) \rangle$ is:

$$G(x) \propto \int \cdots \int \langle V_{\bar{\beta}_\varepsilon}(0)V_{\bar{\beta}_\varepsilon}(1)V_{\bar{\beta}_\varepsilon}(\infty) V_{1}^{-} \cdots V_{m_1}^{-} V_{2}^{-} \cdots V_{m_2}^{-} \cdots V_{n}^{-} \cdots \rangle$$

(A.6)
where the screenings $V_a^-$ with $a = 1, \cdots, n$ are integrated over the 2D plane. Using the quadratic form (5.2) it’s easy to see that the charge neutrality condition

$$-2\vec{\beta}_z + \alpha_- \sum_a^n m_a \vec{e}_a = 0$$  \hspace{1cm} (A.7)

imposes $m_a = 2$ for $a = 1, \cdots, n - 2$ and $m_{n-1} = m_n = 1$. For the same arguments discussed in the previous section, we search for the vectors $\vec{\beta}(n'_1,1)(n'_2,1)\cdots(n'_n,1)$ with $n'_a \geq 1$ (see 5.1) such that:

$$2\vec{\beta}_z + \alpha_- \sum_{a=1}^n q_a \vec{e}_a = \vec{\beta}(n'_1,1)(n'_2,1)\cdots(n'_n,1)$$  \hspace{1cm} (A.8)

where $q_a = 0, 1, 2$ for $a = 1, \cdots, n - 2$, $q_{n-1} = 0, 1$ and $q_n = 0, 1$. Using eq.(5.2), the condition (A.8) is equivalent to the following system of equations:

$$n'_1 = 3 - 2q_1 + q_2$$
$$n'_a = 1 - 2q_a + q_{a-1} + q_{a+1} \quad 2 \leq a < n - 2$$
$$n'_{n-2} = 1 - 2q_{n-2} + q_{n-3} + q_{n-1} + q_n$$
$$n'_{n-1} = 1 - 2q_{n-1} + q_{n-2}$$
$$n'_n = 1 - 2q_n + q_{n-2}$$  \hspace{1cm} (A.9)

Eq.(A.9) gives us all the possible primary operators $\phi(n'_1,1)(n'_2,1)\cdots(n'_n,1)$ present in the energy algebra. Once again, testing all the possible set of values $(q_1, \cdots, q_n)$ it’s easy to verify that, apart from the identity, there are only two solutions: they correspond to the operator $\phi(3,1)(1,1)\cdots(1,1)$ (for $q_a = 0$), with dimension greater than unity, and to the operator $\phi(1,1)(2,1)(1,1)\cdots(1,1) \equiv \phi$ (for $q_1 = 1$ and $q_a = 0, a \geq 2$), whose dimension is given by eq(5.5). In the operator product decomposition $\phi(x')\phi(x)$, we have checked that the only relevant field is $\phi$ itself: by studying the correspondent four-point correlation function we obtain the following system:

$$n'_1 = 1 - 2q_1 + q_2$$
$$n'_2 = 3 - 2q_2 + q_1 + q_3$$
$$n'_i = 1 - 2q_i + q_{i-1} + q_{i+1} \quad 3 \leq i \leq n - 3$$
\[ n'_{n-2} = 1 - 2q_{n-2} + q_{n-3} + q_{n-1} + q_n \]
\[ n'_{n-1} = 1 - 2q_{n-1} + q_{n-2} \]
\[ n'_n = 1 - 2q_n + q_{n-2} \]  \hspace{1cm} (A.10)

with the integers \( q_1, q_{n-1}, q_n = 0, 1, 2 \) and \( q_a = 0, 1, 2, 3, 4 \) for \( a = 2, \ldots, n-2 \). According to eq.(A.10), the \( \phi \) field is the only operator with dimension smaller than unity which appears in the \( \phi \)-algebra.

### B Computation of the structure constants \( D^{\phi}_{\varepsilon\varepsilon} \) and \( D^{\phi}_{\phi\phi} \) for the \( WD_n^{(p)} \) models

The structure constants \( D^{\phi}_{\varepsilon\varepsilon} \) and \( D^{\phi}_{\phi\phi} \) are determined by two three-point correlation functions of the unperturbed \( WD_n^{(p)} \) model:

\[
D^{\phi}_{\varepsilon\varepsilon} = \langle \varepsilon(0)\varepsilon(1)\phi(\infty) \rangle \\
D^{\phi}_{\phi\phi} = \langle \phi(0)\phi(1)\phi(\infty) \rangle 
\]  \hspace{1cm} (B.1)

where \( \phi \) corresponds to the primary operator \( \phi_{(1,1)(2,1)(2,1)} \) for the \( WD_3^{(p)} \) model and to the primary operator \( \phi_{(1,1)(2,1)(1,1)\cdots(1,1)} \) for the \( WD_n^{(p)} \) model with \( n \geq 4 \). In order to compute their values using the Coulomb-Gas representation, we have to take into account that the vertex operators \( V_{\beta\varepsilon}, V_{\bar{\beta}\varepsilon}, V_{\beta\phi} \) and \( V_{\bar{\beta}\phi} \) can acquire non-trivial normalization factors \( N_{\varepsilon}, N_{\bar{\varepsilon}}, N_{\phi} \) and \( N_{\bar{\phi}} \) such that:

\[
\varepsilon(x) = N_{\varepsilon}^{-1}V_{\beta\varepsilon}(x) = N_{\varepsilon}^{-1}V_{\bar{\beta}\varepsilon}(x) \\
\phi(x) = N_{\phi}^{-1}V_{\beta\phi}(x) = N_{\phi}^{-1}V_{\bar{\beta}\phi}(x) 
\]  \hspace{1cm} (B.2)

After imposing the fields normalizations \( \langle \varepsilon(1)\varepsilon(0) \rangle = N_{\varepsilon}N_{\bar{\varepsilon}} = 1 \) and \( \langle \phi(1)\phi(0) \rangle = N_{\phi}N_{\bar{\phi}} = 1 \), we need to compute four integrals:

\[
I_1 \equiv \prod_i^n \left( \frac{(-)^{m_i^{(1)}}}{m_i^{(1)}!} \right) \int \cdots \int \langle V_{\beta\varepsilon}(0)V_{\beta\phi}(1)V_{\bar{\beta}\varepsilon}(\infty) V_{\beta\varepsilon} \cdots V_i \cdots V_n \rangle = N_{\phi}D^{\phi}_{\varepsilon\varepsilon} 
\]  \hspace{1cm} (B.3)
\[ I_2 \equiv \prod_i \left( \frac{(-)^i}{m_i^{(2)}} \right) \int \cdots \int \langle V_{\bar{\beta}_e}(0)V_{\bar{\beta}_o}(1)V_{\bar{\beta}_o}(\infty) V_1 \cdots V_n \rangle = N_\varepsilon^2 N_\phi^{-1} D^\phi_\varepsilon \] (B.4)

\[ I_3 \equiv \prod_i \left( \frac{(-)^i}{m_i^{(3)}} \right) \int \cdots \int \langle V_{\bar{\beta}_e}(0)V_{\bar{\beta}_o}(1)V_{\bar{\beta}_o}(\infty) V_1 \cdots V_n \rangle = N_\phi D^\phi_\varepsilon \] (B.5)

\[ I_4 \equiv \prod_i \left( \frac{(-)^i}{m_i^{(4)}} \right) \int \cdots \int \langle V_{\bar{\beta}_e}(0)V_{\bar{\beta}_o}(1)V_{\bar{\beta}_o}(\infty) V_1 \cdots V_n \rangle = N_\varepsilon^2 \] (B.6)

where the integration over the 2D plane of the screenings, whose number is fixed by the charge neutrality condition, is intended. In eq. (B.6) the vertex \( V_{\bar{\beta}_o} \) corresponds to the identity operator \( I \) and the relative normalization constant \( N_{\bar{\beta}_o} = N_I = 1 \). According to the general form of the correlation function of \( N \) vertex operator \( V_{\bar{\beta}_i} \):

\[ \langle V_{\bar{\beta}_i}(\xi_1) \cdots V_{\bar{\beta}_N}(\xi_N) \rangle = \prod_{i<j} |\xi_i - \xi_j|^4 \] (B.7)

and using the formula

\[
\begin{align*}
C(\alpha, \beta) &= \int d^2 \xi |\xi|^{2\alpha} |\xi - 1|^{2\beta} \\
&= \frac{\pi}{\Gamma(1 + \alpha) \Gamma(1 + \beta) \Gamma(-1 - \alpha - \beta)} \Gamma(-\alpha) \Gamma(-\beta) \Gamma(2 + \alpha + \beta)
\end{align*}
\]

(B.8)

and

\[
\begin{align*}
K(\alpha, \rho) &= \int d^2 \xi d^2 \zeta |\xi|^{2\alpha} |\xi - 1|^{2\alpha} |\zeta|^{2\alpha} |\zeta - 1|^{2\alpha} |\xi - \zeta|^{4\rho} \\
&= \frac{2\pi^2 \Gamma(2\rho) \Gamma(1 - \rho)}{\Gamma(\rho) \Gamma(1 - 2\rho)} \prod_{i=0}^{1} \frac{\Gamma^2(1 + \alpha + i\rho) \Gamma(-1 - 2\alpha - (1 + i)\rho)}{\Gamma^2(-\alpha - i\rho) \Gamma(2 + 2\alpha + (1 + i)\rho)}
\end{align*}
\]

(B.9)

where \( \Gamma(x) \) is the Gamma-function, the integrals (B.3)-(B.6) have been determined. Eq.(B.8) and eq.(B.9) represent special cases of a family of integrals calculated in [8].

In terms of these integrals, the structure constants are :

\[ D^\phi_{\varepsilon\phi} = I_3 \sqrt{\frac{I_2}{I_1 I_4}} \]

\[ D^\varepsilon_{\phi\phi} = \sqrt{\frac{I_1 I_2}{I_4}} \] (B.10)
B.1 Computation of $I_1$

The charge neutrality condition for $I_1$ is:

$$-\bar{\beta}_\phi + \alpha_\epsilon \sum_{a=1}^{n} m_a^{(1)} \bar{e}_a = 0$$ \hspace{1cm} (B.11)

It is satisfied for $m_{1(1)} = m_{n-1(1)} = m_{n(1)} = 1$, $m_{a(1)} = 2$, $a = 2, \ldots, n - 2$. Using eq. (B.7) and taking into account all the scalar products, easily computed from the quadratic form (5.2), the integral $I_1$ takes the form:

$$I_1 = -2^{3-n} \cdot \prod_{a=2}^{n-2} \prod_{k=1}^{2} d^2 \xi_{a(k)} d^2 \xi_{n-1} d^2 \xi_n \times$$

$$\times |\xi_1|^{-2\alpha_2} \prod_{k=1}^{2} |\xi_{2(k)} - \xi_1|^{-2\alpha_2} |\xi_{2(k)} - 1|^{-2\alpha_2} \prod_{a=2}^{n-3} \prod_{k,l=1}^{2} \xi_{a(k)} - \xi_{a+1}^{(l)}|^{-2\alpha_2} \times$$

$$\times \prod_{a=2}^{n-2} |\xi_{a(1)} - \xi_{a(2)}|^{4\alpha_2} \prod_{k=1}^{2} |\xi_{n-2(k)} - \xi_{n-2}|^{-2\alpha_2}$$ \hspace{1cm} (B.12)

where $\alpha_2 = p/(p+1) \approx 1 - \epsilon$. Integrating over the variables $\xi_n$, and $\xi_{n-1}$ we obtain by eq. (B.8):

$$I_1 = -2^{3-n} C^2(-\alpha_\epsilon, -\alpha_\epsilon) \cdot \prod_{a=1}^{n-2} \prod_{k=1}^{2} d^2 \xi_{a(k)} \times$$

$$\times |\xi_1|^{-2\alpha_2} \prod_{k=1}^{2} |\xi_{2(k)} - \xi_1|^{-2\alpha_2} |\xi_{2(k)} - 1|^{-2\alpha_2} \prod_{a=2}^{n-3} \prod_{k,l=1}^{2} \xi_{a(k)} - \xi_{a+1}^{(l)}|^{-2\alpha_2} \times$$

$$\times \prod_{a=2}^{n-3} |\xi_{a(1)} - \xi_{a(2)}|^{4\alpha_2} \prod_{k=1}^{2} |\xi_{n-2(k)} - \xi_{n-2}|^{4-4\alpha_2}$$ \hspace{1cm} (B.13)

Then using eq. (B.9) we integrate in order over the couple of variables $\xi_{n-2(k)}, \ldots, \xi_{2(k)}$ and over the variable $\xi_1$; the result is:

$$I_1 = -2^{3-n} C^2(-\alpha_\epsilon, -\alpha_\epsilon) C(-\alpha_\epsilon, 2\alpha_\epsilon + 2n) \prod_{i=1}^{n-3} K(-\alpha_\epsilon, i(1 - \alpha_\epsilon))$$ \hspace{1cm} (B.14)

B.2 Computation of $I_2$

According to :

$$-2\bar{\beta}_\epsilon + \bar{\beta}_\phi + \alpha_\epsilon \bar{e}_1 = 0$$ \hspace{1cm} (B.15)
we have \( m_1^{(2)} = 1 \) and \( m_a^{(2)} = 0 \) for \( a = 2, \ldots, n \); the integral (B.4) reads:

\[
I_2 = - \int d^2 \xi_1 \langle V_{\vec{\beta}_z}(0)V_{\vec{\beta}_z}(1)V_{\vec{\beta}_z}(\infty) V_1^- (\xi_1) \rangle = - \int d^2 \xi_1 |\xi|^{-2\alpha^2} |\xi_1 - 1|^{-2\alpha^2} = -C(-\alpha^2, -\alpha^2^2)
\]

\[(B.16)\]

**B.3 Computation of \( I_3 \)**

The integral (B.5) must satisfy the charge neutrality condition (B.11); it takes the form:

\[
I_3 = -2^{3-n} \int \cdots \int d^2 \xi_1 \prod_{a=2}^{n-2} \prod_{k=1}^{2} d^2 \xi_a^k d^2 \xi_{n-1} d^2 \xi_n \times
\]

\[
\times \prod_{k=1}^{2} |\xi_a^{(k)} - \xi_1|^{-2\alpha^2} |\xi_2^{(k)} - 1|^{-2\alpha^2} |\xi_2^{(k)} - 2|^{-2\alpha^2} \prod_{a=2}^{n-3} \prod_{k,l=1}^{2} |\xi_a^{(k)} - \xi_{a+1}^{(l)}|^{-2\alpha^2} \times
\]

\[
\times \prod_{a=2}^{n-2} |\xi_a^{(1)} - \xi_a^{(2)}|^{-4\alpha^2} \prod_{k=1}^{2} |\xi_{n-2}^{(k)} - \xi_{n-1}^{(k)}|^{-2\alpha^2} |\xi_{n-2}^{(k)} - \xi_n^{(k)}|^{-2\alpha^2}
\]

\[(B.17)\]

Following the same order of integrations used in the computation of the integral (B.3), with the difference that we integrate first over the variable \( \xi_1 \) and then over the couple of variables \( \xi_2^{(1)}, \xi_2^{(2)} \), we obtain:

\[
I_3 = -2^{3-n} C^3 (-\alpha^2, -\alpha^2^2) K (-\alpha^2, \frac{2n - 5}{2} - (n - 2)\alpha^2) \prod_{i=1}^{n-4} K (-\alpha^2, i(1 - \alpha^2)) \]

\[(B.18)\]

**B.4 Computation of \( I_4 \)**

According to the charge neutrality condition:

\[
- 2\vec{\beta}_z^* \sum_{a=1}^{n} m_a^{(4)} \varepsilon_a = 0
\]

we have \( m_a^{(4)} = 2 \) for \( a = 1, \ldots , n - 2 \), \( m_{n-1}^{(4)} = m_{n}^{(4)} = 1 \). The integral (B.6) has the form:

\[
I_4 = 2^{2-n} \int \cdots \int \prod_{a=1}^{n-2} \prod_{k=1}^{2} d^2 \xi_a^{(k)} d^2 \xi_{n-1} d^2 \xi_n \times
\]

\[
\times \prod_{k=1}^{2} |\xi_1^{(k)} - 1|^{-2\alpha^2} |\xi_1^{(k)} - 2\alpha^2 \prod_{a=1}^{n-3} \prod_{k,l=1}^{2} |\xi_a^{(k)} - \xi_{a+1}^{(l)}|^{-2\alpha^2} \times
\]

\[
30
\]
\[
\times \prod_{a=1}^{n-2} |\xi_a^{(1)} - \xi_a^{(2)}|^{4\alpha^2} \prod_{k=1}^2 |\xi_n^{(k)} - \xi_{n-1}|^{-2\alpha^2} |\xi_n^{(k)} - \xi_n|^{-2\alpha^2} \tag{B.20}
\]

Integrating over the variables \(\xi_n, \xi_{n-1}, \xi_{n-2}, \ldots, \xi_1\), the integral (B.6) assumes the value:

\[
I_4 = C^2(-\alpha_-^2, -\alpha_-^2) \prod_{i=1}^{n-2} K(-\alpha_-^2, i(1 - \alpha_-^2)) \tag{B.21}
\]

Developing in \(\epsilon\) the integrals we have computed and using eq. (B.10), the eq. (5.6) are obtained.

## C  RG Equations and Renormalization of energy operators

Using the O.P.E.(3.8) and the dimensions of the energy and \(\phi\) fields (eq.(5.4) and (5.5)), the operator algebra of the perturbing terms is:

\[
\sum_{a \neq b}^{N} (\epsilon_a \epsilon_b)(x) \sum_{c \neq d}^{N} (\epsilon_c \epsilon_d)(y) \rightarrow 4(N - 2)|x - y|^{-2 + 2(n-1)\epsilon} \sum_{a \neq b}^{N} (\epsilon_a \epsilon_b)(y) + \\
+ 4(N - 1)D_{\epsilon\epsilon}^\phi |x - y|^{-2 + 4\epsilon} \sum_{a=1}^{N} \phi_a(y) + \ldots
\]

\[
\sum_{a \neq b}^{N} (\epsilon_a \epsilon_b)(x) \sum_{a=1}^{N} \phi_a(y) \rightarrow 2D_{\epsilon\epsilon}^\phi |x - y|^{-2 + 2(n-1)} \sum_{a \neq b}^{N} (\epsilon_a \epsilon_b)(y) + \ldots
\]

\[
\sum_{a=1}^{N} \phi_a(x) \sum_{b=1}^{N} \phi_a(y) \rightarrow D_{\phi\phi}^\phi |x - y|^{-2 + 2(n-1)} \sum_{a=1}^{N} \phi_a(y) + \ldots \tag{C.1}
\]

where we have omitted the irrelevant terms. By eq.(C.1) the 1-loop RG-equations can be easily obtained (see for example [8] or [10]):

\[
\beta_g = 2(2n - 1)\epsilon g - 4\pi(N - 2)g^2 - 4\pi D_{\epsilon\epsilon}^\phi \lambda g
\]

\[
\beta_\lambda = 2(n - 1)\epsilon \lambda - \pi D_{\phi\phi}^\phi \lambda^2 - 2\pi(N - 1)D_{\epsilon\epsilon}^\phi g^2 \tag{C.2}
\]

With the redefinitions \(g \rightarrow g/(4\pi)\) and \(\lambda \rightarrow \lambda/(2\pi)\), we find the eq.(3.9).
Similarly the renormalized energy operators $\varepsilon'_c(x)$ (3.11) can be computed at the first order from the following operator product decomposition

$$\sum_{a \neq b}^N (\varepsilon_a \varepsilon_b)(x) \varepsilon_c(y) \rightarrow 2|x - y|^{-2 + 2(2n-1)\epsilon} \sum_{a \neq c}^N \varepsilon_a(y) + \cdots$$

$$\sum_a^N \phi_a(x) \varepsilon_c(y) \rightarrow D_{\varepsilon \varepsilon}^\phi |x - y|^{-2 + 2(n-1)\epsilon} \varepsilon_c(y) + \cdots \quad (C.3)$$
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Figure 1: Screening geometry of the $WD_3^{(p)}$ model.
Figure 2: RG-flow of the $W_D^3$ model with disorder.
Figure 3: RG-trajectories of $N = 3$ coupled $W D_3^{(p)}$ models.