The wrapping hull and a unified framework
for volume estimation

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Abstract

This paper develops a unified framework for estimating the volume of a set in \( \mathbb{R}^d \) based on observations of points uniformly distributed over the set. The framework applies to all classes of sets satisfying one simple axiom: a class is assumed to be intersection stable. No further hypotheses on the boundary of the set are imposed; in particular, the convex sets and the so-called weakly-convex sets are covered by the framework. The approach rests upon a homogeneous Poisson point process model. We introduce the so-called wrapping hull, a generalization of the convex hull, and prove that it is a sufficient and complete statistic. The proposed estimator of the volume is simply the volume of the wrapping hull scaled with an appropriate factor. It is shown to be consistent for all classes of sets satisfying the axiom and mimics an unbiased estimator with uniformly minimal variance. The construction and proofs hinge upon an interplay between probabilistic and geometric arguments. The tractability of the framework is numerically confirmed in a variety of examples.

Keywords: volume estimation, wrapping hull, Poisson point process, \( r \)-convex sets, UMVU, stopping set

MSC code: 60G55, 62G05, 62M30

1 Introduction

The problem of estimating the support of a density has received a large amount of attention in the statistical literature since the 1980s partly because of vast areas of applications in image analysis, signal processing and econometrics. The fundamental results in this area were obtained in Devroye and Wise (1980); Korostelev and Tsybakov (1993a,b, 1994); Cuevas and Fraiman (1997); Cholaquidis, Fraiman, Lugosi, and Pateiro-López (2016). Furthermore, see Tsybakov (1997); Walther (1997); Rigollet and Vert (2009); Mason and Polonik (2009); Cholaquidis, Fraiman, Mordecki,
and Papalardo (2017) for a more general problem of estimating the level sets of a density. In particular, Korostelev and Tsybakov (1993a) established the minimax optimal rates for estimating the support of a density having a Hölder-continuous boundary in the Hausdorff and symmetric difference metrics and constructed an estimator which attains the optimal rates. The case of convex support estimation was first studied in Korostelev and Tsybakov (1994, 1993b), where it was shown that the convex hull $\hat{C}$ of the sample points, which is the maximum likelihood estimator, is rate-optimal for estimating the support set $C$ in the Hausdorff and symmetric difference metrics.

The volume of a set is one of its most basic characteristics. Surprisingly, as it was shown in Korostelev and Tsybakov (1993a,b), the volume of a rate-optimal estimator of the set is not necessarily a rate-optimal estimator of the volume. The first fully rate-optimal estimators of the volume of a convex support with smooth boundary and a support with Hölder-continuous boundary were constructed by Gayraud (1997) based on three-fold sample splitting. A more efficient and flexible estimator of the volume of a convex set with no assumption on the boundary was recently proposed in Baldin and Reiß (2016). In fact the proposed estimator is simple to compute and non-asymptotically efficient. The problem of calculating the volume of a convex set has also attracted mathematicians working in computer science and computational geometry, see Dyer, Frieze, and Kannan (1991); Kannan, Lovász, and Simonovits (1997); Lovász and Vempala (2006). The setting is slightly different: an experimenter uses a sampling algorithm that generates points over the space that either fall inside or outside a set (this information is obtained via queries to the oracle). We refer to Vempala (2010) for a recent survey of the existing fast randomised algorithms for calculating the volume of a convex set. To the best of our knowledge, no efficient estimators of the volume of a more general than convex class of sets have been rigorously studied.

1.1 Main contribution and the structure

In the present paper, we combine techniques from statistics and stochastic geometry to build a general framework for estimating the volume of a set. We focus on the Poisson point process (PPP) observation model with intensity $\lambda > 0$ on a set $A$. We thus observe

$X_1, \ldots, X_N \overset{i.i.d.}{\sim} U(A), \quad N \sim \text{Poiss}(\lambda |A|),$

where $(X_n), N$ are independent and $|A|$ denotes the volume or the Lebesgue measure of the set $A$. The set $A$ is meant to belong to a class $\mathcal{A}$ satisfying one simple assumption: the class is assumed to be intersection stable, see Section 2 for a concise definition. The classes of sets covered by the assumption hence include:

- convex sets;
- weakly-convex sets;
- star-shaped sets with a Hölder-continuous boundary;
- concentric sets;
- polytopes with fixed directions of outer unit normal vectors;
- compact sets.
In Section 2, we introduce the so-called wrapping hull \( \hat{A} \), which can informally be described as the minimal set from the class that contains the data points \( X_1, ..., X_N \). It is then used in Section 3 to construct the so-called oracle estimator for the volume of a set belonging to one of the aforementioned classes when the intensity \( \lambda \) of the process is known. The oracle estimator is shown to be uniformly of minimum variance among unbiased estimators (UMVU). Section 4 is devoted to the estimation of the intensity and derives a fully data-driven estimator of the volume,

\[
\hat{\vartheta} = \frac{N + 1}{N_\circ + 1} |\hat{A}|,
\]

where \( N_\circ \) is the number of data points lying in the interior of the wrapping hull. Figure 1 illustrates an example in which a naive estimator \( |\hat{A}| \) significantly underestimates the true volume \( |A| \) even in the case when the class of sets is known whereas the estimator \( \hat{\vartheta} \) produces a rather striking performance, see Section 6 for a more detailed numerical study\(^1\). The mean squared risk of the estimator is shown to mimic the mean squared risk of the oracle estimator. Although the main object of analysis is the PPP model, the key results transfer to the so-called uniform model, cf. Section 4.1, using “Poissonisation”. Section 5 further establishes the rates of convergence of the oracle estimator and the estimator \( \hat{\vartheta} \) in (1.1) for the considered classes of sets satisfying the assumption. Theorem 4.4 states a generalized Efron’s inequality for the wrapping hull, cf. Efron (1965), which reduces the analysis of the mean squared error of the estimator \( \hat{\vartheta} \) to the distributional characteristics of the missing volume \( |A \setminus \hat{A}| \), a uniform lower bound on its expectation and a uniform deviation inequality. Interestingly, a uniform lower bound on the expectation of the missing volume has not even been established for the class of convex sets. We therefore es-

\(^1\)The simulations were implemented using the R package “spatstat” by Baddeley and Turner (2005).
establish the rates of convergence only for a relatively simple class of polytopes with fixed directions of outer unit normal vectors in Section 5.4. A more general question is beyond the scope of the present paper and left to future research. In volume estimation of weakly-convex sets in Section 5.1 there is a further peculiar question of adaptation to a smoothing parameter. We suggest an adaptation procedure inspired by Lepski’s method, cf. Lepskii (1992), and study it numerically in Section 6. Our numerical results in Section 6, mainly devoted to volume estimation for the weakly-convex sets, in particular, demonstrate that overestimating the smoothing parameter may have a significant cost for volume estimation. Some of the technical lemmata are deferred to the Appendix. Finally, we encounter and state a variety of new open questions in stochastic geometry, which we barely begin to nibble at the edges. Interestingly enough, the framework was mentioned in a seminal paper Kendall (1974) by David Kendall in the Statslab at the University of Cambridge, but has never been fully explored.

1.1.1 A simple one-dimensional example

Let \( X_1, ..., X_n \) be a sample of i.i.d. points drawn from the uniform distribution \( U(a, b) \), and let \( X_{(1)}, ..., X_{(n)} \) denote the order statistics, so that \( X_{(1)} < ... < X_{(n)} \). It holds by symmetry that the expected length of the interval \( (X_{(1)}, X_{(n)}) \) satisfies

\[
\mathbb{E}[|X_{(n)} - X_{(1)}|] = \frac{(n - 1)}{(n + 1)}(b - a) .
\]

(1.2)

An objective of statistical inference is to estimate the length of the interval, when the location of the points \( a \) and \( b \) is assumed to be unknown. A naive estimator,

\[
\hat{l}_{\text{naive}} := X_{(n)} - X_{(1)} ,
\]

clearly underestimates the length. A more attractive idea is to somehow dilate the interval \( (X_{(1)}, X_{(n)}) \) and take the length of the dilated interval as an estimator. There are at least two viable dilations: 1) add and subtract some fixed vectors from the end points \( X_{(n)} \) and \( X_{(1)} \) (additive dilation) and 2) dilate the interval \( (X_{(1)}, X_{(n)}) \) from its centre \( (X_{(n)} + X_{(1)})/2 \) with some scaling factor (multiplicative dilation). In the one-dimensional case, both dilations are equivalent. It follows from (1.2) that a reasonable additive dilation factor is \( 2(X_{(n)} - X_{(1)})/(n - 1) \) which yields an estimator for the volume,

\[
\hat{l}_1 := \frac{(n + 1)}{(n - 1)}(X_{(n)} - X_{(1)}) .
\]

This estimator is not only unbiased, \( \mathbb{E}[\hat{l}_1] = b - a \), but also, as we shall see in Section 3 and Section 4, is minimax optimal. We also refer to Moore (1984) for a comprehensive literature review of set estimation in the one-dimensional case.

1.1.2 Estimation of the volume of a convex set in high dimensions

The one-dimensional model is useful to grasp the main ideas of volume estimation, yet it is not widely used in real applications. The two-dimensional model already covers several important applications in image analysis and signal processing. Here, we observe the points \( X_1, ..., X_n \) drawn uniformly over a set \( C \subseteq \mathbb{R}^2 \) and an objective is to recover the volume \( V_C \) of the set and the set itself. Let us assume that
Figure 2: The points $X_1, \ldots, X_n$ drawn uniformly over a set $C$, the convex hull of the points $\hat{C} = \text{conv}(X_1, \ldots, X_n)$ and the dilated hull estimator $\tilde{C}$.

$C$ belongs to the class of convex sets. The wrapping hull is then simply the convex hull $\hat{C}$ of the points $\hat{C} = \text{conv}(X_1, \ldots, X_n)$. Analogously to the one-dimensional case, it is natural to consider the volume $|\hat{C}|$ of the convex hull as a baseline estimator for the volume $V_C$ of the set $C$. It is quite intuitive that this estimator performs quite poorly because it always underestimates the true volume and it should therefore be dilated as in the one-dimensional case. Section 3 and Section 4 show that an optimal estimator has the following form

$$\hat{V}_{\text{opt}} = \frac{n + 1}{n_o + 1} |\hat{C}|,$$

(1.3)

where $n_o$ is the number of purple points in Figure 2 that lie in the interior of the convex hull $\hat{C}$. Note that $\hat{V}_{\text{opt}}$ is the volume of the “dilated” hull $\tilde{C}$, the set obtained by dilating the convex hull with the same factor from the centre of gravity $\hat{x}_0$ of the convex hull:

$$\tilde{C} = \left\{ \hat{x}_0 + \left( \frac{n + 1}{n_o + 1} \right)^{1/d} (x - \hat{x}_0) \mid x \in \hat{C} \right\},$$

which can in fact be used to estimate the set $C$ itself. Similarly, the same estimators for the volume and the set itself can be used in higher dimensions.

The uniform model of a fixed number of points drawn uniformly over a convex set $C$ has been extensively studied in stochastic geometry. The focus of study is rather on understanding the distributional characteristics of key functionals like the volume of $\hat{C}$, the number of vertices of $\hat{C}$ and the distance between $\hat{C}$ and $C$. The main references here are Bárány and Larman (1988); Reitzner (2003, 2005); Vu (2005); Pardon (2011). The Poisson point process (PPP) model studied in the present paper is closely related to the uniform model. Using Poissonisation and de-Poissonisation techniques, this model exhibits asymptotic properties like the uniform model, see e.g. the references above and Section 4.1. However the geometric properties of the PPP model are much more fecund for conducting statistical inference, see Reiß and Selk (2017); Baldin and Reiß (2016), where the techniques from the Poisson point processes theory were successfully employed for estimation of linear functionals in a one-sided regression model and estimation of the volume of a convex set.
1.1.3 How fast can we estimate $\pi$?

There are quite a few ways how one can calculate the number $\pi$, see Arndt and Haenel (2001). We here discuss one interesting way based on the Monte Carlo simulations of independent uniformly distributed random variables. It is a toy illustrative application of volume estimation in sampling theory. Let us draw the points $X_1, \ldots, X_N$ from the uniform distribution over the square $[0,1] \times [0,1]$ and count the number of points $n$ which fall inside the circle centred at the origin of radius 1. Let $\hat{\pi} := n/N$ denote the ratio of the points inside the circle to the total number of points. It approximately equals $\pi/4$, because it is an unbiased estimator:

$$E[\hat{\pi}] = \frac{1}{N}E[n] = \frac{1}{N}E\left[\sum_{i=1}^{N} 1(X_i \in C)\right] = \frac{\pi}{4},$$

and therefore its mean squared risk is governed by the variance:

$$E\left[(\hat{\pi} - \pi)^2\right] = \text{Var}(\hat{\pi}) = \frac{1}{N^2} \text{Var}\left(\sum_{i=1}^{N} 1(X_i \in C)\right) = \frac{1}{N} \frac{\pi}{4} \left(1 - \frac{\pi}{4}\right).$$

It turns out $\hat{\pi}$ is even a maximum likelihood estimator. Surprisingly, we are able to estimate $\pi$ with a much faster rate based on the data points in this experiment. Following (1.3), we define our properly scaled estimator as

$$\hat{\pi}_{\text{opt}} = 4 \frac{n + 1}{n_o + 1} |\hat{C}|,$$

where $n_o$ is the number of points lying inside the convex hull $\hat{C}$ of the points lying inside the circle. Theorem 3.1 and Theorem 4.2 to follow state that the rate of convergence of the mean squared risk of the estimator $\hat{\pi}_{\text{opt}}$ satisfies $E\left[(\hat{\pi}_{\text{opt}} - \pi)^2\right] = \mathcal{O}(N^{-5/3})$, see Figure 3 for a numerical comparison of the two estimators. Note that both estimators can well be computed in polynomial time.

1.2 Relationship to the work on volume estimation of a convex set

Some of the theoretical results in the present paper are underpinned by the results for the convex case, cf. Section 3 and Theorem 4.2 in Section 4, and the corresponding results in Baldin and Reiß (2016). In fact, a key observation igniting the development of the present framework is that estimation of the volume of convex sets can in fact solely rely upon the property of convex sets being stable under taking intersections rather than convexity itself. This striking observation appears to have a substantial value for volume estimation in a variety of scenarios far beyond convexity. Volume estimation for some of the classes covered by the framework, in particular, the weakly-convex sets, has been long seen as notoriously difficult with standard geometric arguments, see the references in Section 5.

Not violating the flow of the paper, we shall therefore omit some of the proofs of the statements that are deduced from the proofs of the corresponding statements for the convex case. The proof of the result that the wrapping hull is a complete statistic in Theorem 3.3 is slightly simplified compared to the proof of the theorem that the convex hull is a complete statistic in Baldin and Reiß (2016) and hinges upon a measure-theoretic result in stochastic geometry. In contrast to the special
case of convex sets, the present paper further argues that the designed estimator \( \hat{\vartheta} \) is in fact adaptive as its rate explicitly depends on the rate of convergence of the missing volume \( |A \setminus \hat{A}| \). This result rests upon Efron’s inequality proved in Section 4.2. Section 4.1 explicitly states that the same estimator is minimax optimal in the uniform model. Section 5 appears to convey the most noticeable value for applications as it provides efficient data-driven estimators and clearly outlines the steps of deriving explicit rates of convergence for specific classes of intersection stable sets.

2 Poisson point process theory and the wrapping hull

In this section, we slightly digress on Poisson point processes, introduce the main notions and collect recently developed mathematical tools. Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space, fix a convex compact set \( E \) in \( \mathbb{R}^d \) and equip it with the Borel \( \sigma \)-algebra \( \mathcal{E} \) with respect to the Euclidean metric \( \rho \). Without loss of generality one may assume that \( E = [0, 1]^d \). By a point process \( \mathcal{N} \) on \( E \) we mean an integer-valued random measure, or a kernel, from the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) into \( (E, \mathcal{E}) \). Thus, \( \mathcal{N} \) is a mapping from \( \Omega \times \mathcal{E} \) into \( \{0, 1, \ldots\} \) such that \( \mathcal{N}(\omega, \bullet) \) is an integer-valued measure for fixed \( \omega \in \Omega \) and \( \mathcal{N}(\bullet, B) \) is an integer-valued random variable for fixed \( B \in \mathcal{E} \). For convenience, we write \( \mathcal{N}(B) = \mathcal{N}(\bullet, B) \).

Let \( K \) be the set of all compact subsets of \( E \) equipped with its Borel \( \sigma \)-algebra \( \mathcal{B}_K \) with respect to the Hausdorff-metric \( \rho_H \) defined for two non-empty compact sets \( A \) and \( B \) by

\[
\rho_H(A, B) = \max \left( \sup_{x \in B} \rho(x, A), \sup_{x \in A} \rho(x, B) \right).
\]

It is known, see Theorem C.5 in Molchanov (2006), that the Borel \( \sigma \)-algebra \( \mathcal{B}_K \) coincides with the \( \sigma \)-algebra \( \sigma([B], B \in K) \) with \( [B]_K = \{A \in K : A \subseteq B\} \).
Moreover, the space \((K, \rho_H)\) is Polish.

Let \(A \subseteq K\) be a family of compact subsets of \(E\) fulfilling the following assumption

**Assumption 2.1.** \(A\) is closed under taking arbitrary intersections and \(\emptyset, E \in A\).

Then the metric subspace \((A, \rho_H)\) has the induced Borel \(\sigma\)-algebra \(\mathcal{B}_A = A \cap \mathcal{B}_K = \{A \cap K : K \in \mathcal{B}_K\}\), which thus coincides with the \(\sigma\)-algebra \(A = \sigma([B], B \in A)\) where \([B] = \{A \in A : A \subseteq B\}\). It turns out there is a fascinating connection between the families of sets satisfying Assumption 2.1 and the trapping systems introduced in the groundbreaking work of Kendall (1974) on the theory of random sets, see also Section 7.2 in Molchanov (2006).

We call a point process \(N : \Omega \rightarrow M(E)\) a Poisson point process (PPP) of intensity \(\lambda > 0\) on \(A \in A\), if

- for any \(B \in \mathcal{E}\), we have \(N(B) \sim \text{Poiss}(\lambda|A \cap B|)\), where \(|A \cap B|\) denotes the Lebesgue measure of \(A \cap B\);
- for all mutually disjoint sets \(B_1, \ldots, B_n \in \mathcal{E}\), the random variables \(N(B_1), \ldots, N(B_n)\) are independent.

For statistical inference, we assume the Poisson point process to be defined on a set of non zero Lebesgue measure, i.e. \(|A| > \delta\) for \(\delta > 0\).

Interestingly, one can view \((N(K), K \in K)\) as a set-indexed stochastic process. It has no direct order and its natural filtration is defined by

\[
\mathcal{F}_K \overset{\text{def}}{=} \sigma(\{N(U); U \subseteq K, U \in K\})
\]

for any \(K \in K\). The properties of the filtration \((\mathcal{F}_K, K \in K)\) are well studied, cf Zuyev (1999). By construction, the restriction \(N_K = N(\cdot \cap K)\) of the point process \(N\) onto \(K \in K\) is \(\mathcal{F}_K\)-measurable (in fact, \(\mathcal{F}_K = \sigma(\{N_K(U); U \in K\})\)). Moreover, it can be easily seen that \(N_K\) is a Poisson point process in \(M\), cf. the Restriction Theorem in Kingman (1992). Furthermore, we say that a set-indexed, \((\mathcal{F}_K)\)-adapted integrable process \((X_K, K \in K)\) is a martingale if \(E[X_B|\mathcal{F}_A] = X_A\) holds for any \(A, B \in K\) with \(A \subseteq B\). By the independence of increments, the process

\[
M_K \overset{\text{def}}{=} N(K) - \lambda|K|, \quad K \in K,
\]

is clearly a martingale with respect to its natural filtration \((\mathcal{F}_K, K \in K)\).

A random compact set \(K\) is a measurable mapping \(K : (M, \mathcal{M}) \rightarrow (K, \mathcal{B}_K)\). A random compact set \(K\) is called an \(\mathcal{F}_K\)-stopping set if \(\{K \subseteq K\} \in \mathcal{F}_K\) for all \(K \in K\). The sigma-algebra of \(K\)-history is defined as \(\mathcal{F}_K = \{A \in \mathcal{F}_K : A \cap \{K \subseteq K\} \in \mathcal{F}_K \text{ for all } K \in K\}\), where \(\mathcal{F}_K = \sigma(\mathcal{F}_K; \mathcal{F}_K)\). We introduce the wrapping hull of the PPP points on a set \(A \in A\), which is served as a set-estimator of \(A\).

**Definition 2.2.** The \(A\)-wrapping hull (or simply the wrapping hull) of the PPP points is a mapping \(\hat{A} : M \rightarrow A\) defined as

\[
\hat{A} \overset{\text{def}}{=} \text{wrap}_A\{X_1, \ldots, X_N\} \overset{\text{def}}{=} \bigcap\{A \in A : X_i \in A, \forall i = 1, \ldots, N\}.
\]

For a set \(A \subseteq E\) let \(A^c\) denote its complement.
Lemma 2.3. The set $\hat{K} \text{ def } \overline{A^c}$, the closure of the complement of the wrapping hull, is an $(\mathcal{F}_K)$-stopping set.

The proof of Lemma 2.3 essentially repeats the steps of the proof of an analogous statement for the convex hull of the PPP points given in Lemma 2.2 in Baldin and Reiß (2016). A striking observation is the following corollary, which rests upon the optional sampling theorem for set-indexed martingales, cf. Zuyev (1999).

Corollary 2.4. The number of points $N_\partial$ lying on the wrapping hull $\hat{A}$ and the missing volume $|A \setminus \hat{A}|$ satisfy the relation

$$E[N_\partial] = \lambda E[|A \setminus \hat{A}|].$$

Corollary 2.4. The number of points $N_\partial$ lying on the wrapping hull $\hat{A}$ and the missing volume $|A \setminus \hat{A}|$ satisfy the relation

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We shall define the likelihood function for the PPP model. Note that we can evaluate the probability

$$P_A(\hat{A} \in B) = \sum_{k=0}^{\infty} \frac{e^{-\lambda|A|} \lambda^k}{k!} \int_{A^k} 1(\text{wrap}_A\{x_1, \ldots, x_k\} \in B) d(x_1, \ldots, x_k)$$

for $B \in \mathfrak{B}_A$. Usually, we only write the subscript $A$ or sometimes $(A, \lambda)$ when different probability distributions are considered simultaneously. The likelihood function

$$L(A, N) = \frac{dP_{A, \lambda}}{dP_{E, \lambda_0}}(N) = e^{\lambda_0|E| - \lambda|A|}(\lambda/\lambda_0)^N 1(\hat{A} \subseteq A),$$

(2.1)

cf. Thm. 1.3 in Kutoyants (1998). For the last line, we have used that a point set is in $A$ if and only if its wrapping hull $\hat{A} = \text{wrap}_A\{X_1, \ldots, X_N\}$ is contained in $A$.

The following theorem is an essential ingredient for deriving our estimators. The statement of the theorem is quite intuitive and already used in Privault (2012). Its proof is similar to the proof of Theorem 3.1 in Baldin and Reiß (2016).

Theorem 2.5. Set $\hat{K} \text{ def } \overline{A^c}$. The number $N_\partial$ of points on the boundary of the wrapping hull $\hat{A}$ is measurable with respect to the sigma-algebra of $\hat{K}$-history $\mathcal{F}_{\hat{K}}$. The number of points in the interior of the wrapping hull $N_\circ$ is, conditionally on $\mathcal{F}_{\hat{K}}$, Poisson-distributed:

$$N_\circ \mid \mathcal{F}_{\hat{K}} \sim \text{Poiss}(\lambda_0) \text{ with } \lambda_0 \text{ def } \lambda|\hat{A}|.$$

(2.2)

In addition, we have $\mathcal{F}_{\hat{K}} = \sigma(\hat{A})$, where the latter is the sigma-algebra $\sigma(\{\hat{A} \subseteq B, B \in \mathcal{A}\})$ completed with the null sets in $\mathfrak{F}$.

We shall further use the following short notation: $N = N(E)$ denotes the total number of points, $N_\circ = N(\hat{A}^c)$ the number of points in the interior of the wrapping hull $\hat{A}$ and $N_\partial = N(\partial \hat{A}) = N(\partial \hat{K})$ the number of points on the boundary of the wrapping hull. For asymptotic bounds we write $f(x) = O(g(x))$ or $f(x) \lesssim g(x)$ if $f(x)$ is bounded by a constant multiple of $g(x)$ and $f(x) \sim g(x)$ if $f(x) \lesssim g(x)$ as well as $g(x) \lesssim f(x)$.
3 Oracle case: known intensity $\lambda$

In the case when $\lambda$ is known one can just estimate the volume $|A|$ by $N/\lambda$, which is an unbiased estimator, whose mean squared risk is given by

$$E[(N/\lambda - |A|)^2] = Var(N/\lambda) = |A|/\lambda,$$

(3.1)

thus implying $O(\lambda^{-1/2})$-rate of convergence. This rate can be improved. As we shall see in Theorem 3.3, the wrapping hull is a complete and sufficient statistic thus allowing to construct the unique best unbiased estimator of the volume in virtue of the Lehmann-Scheffé theorem. In view of Theorem 2.5 we thus derive our oracle estimator

$$\hat{\vartheta}_{\text{oracle}} = E\left[\frac{N}{\lambda} \big| \mathcal{F}_K\right] = E\left[\frac{N_o + N_0}{\lambda} \big| \mathcal{F}_K\right] = \frac{N_o}{\lambda} + |\hat{A}|$$

(3.2)

The following result is fundamental in characterising the rate of convergence of the risk of the oracle estimator.

**Theorem 3.1.** For known intensity $\lambda > 0$, the oracle estimator $\hat{\vartheta}_{\text{oracle}}$ is unbiased and of minimal variance among all unbiased estimators (UMVU) in the PPP model with parameter class $A$. It satisfies

$$E[(\hat{\vartheta}_{\text{oracle}} - |A|)^2] = Var(\hat{\vartheta}_{\text{oracle}}) = \frac{E[|A \setminus \hat{A}|]}{\lambda}.$$

**Remark 3.2.** The theorem asserts that the rate of convergence of $\hat{\vartheta}_{\text{oracle}}$ is in fact faster than $\lambda^{-1/2}$ for all classes of sets $A$ satisfying Assumption (2.1).

**Proof.** By the tower property of conditional expectation, the estimator $\hat{\vartheta}_{\text{oracle}}$ is unbiased, $E[\hat{\vartheta}_{\text{oracle}}] = |A|$. Using law of total variance, we derive

$$Var(\hat{\vartheta}_{\text{oracle}}) = Var\left(E\left[\frac{N}{\lambda} \big| \mathcal{F}_K\right]\right) = Var\left(E\left[\frac{N_o + N_0}{\lambda} \big| \mathcal{F}_K\right]\right) = \frac{|A|}{\lambda} - E\left[Var\left(\frac{N_o + N_0}{\lambda} \big| \mathcal{F}_K\right)\right] = \frac{E[|A \setminus \hat{A}|]}{\lambda}.$$

Theorem 3.3 below affirms that the wrapping hull $\hat{A}$ is a complete and sufficient statistic such that by the Lehmann-Scheffé theorem, the estimator $\hat{\vartheta}_{\text{oracle}}$ has the UMVU property.

**Theorem 3.3.** For known intensity $\lambda > 0$, the wrapping hull is a complete and sufficient statistic.

The proof of Theorem 3.3 is deferred to the Appendix. As a result of Theorem 3.1, the performance of the estimator $\hat{\vartheta}_{\text{oracle}}$ of the volume is reduced to the analysis of the performance of the wrapping hull estimator of the set itself, which clearly depends on the geometric properties of classes of sets satisfying Assumption 2.1.

The minimax lower bounds on the rate of convergence of the risk of estimating the volume of a set $A \in \mathcal{A}$ are often easier to establish for concrete classes of sets using the so-called hypercube argument, cf. Gayraud (1997). Interestingly, the following general bound on the minimax optimal rate holds.
Theorem 3.4. The minimax optimal rate of estimating the volume of a set $A \in A$ satisfies
\[
\lambda^{-2} \lesssim \inf_{\hat{\vartheta}} \sup_{A \in A} \mathbb{E}[(\hat{\vartheta} - |A|)^2] \lesssim \lambda^{-1}, \tag{3.3}
\]
where the infimum extends over all estimators $\hat{\vartheta}$ in the Poisson point process model with intensity $\lambda$.

Remark 3.5. The rate $O(\lambda^{-1})$ is minimax for estimating the volume in some parametric classes of sets, in particular, the class of concentric sets, whereas the rate $O(\lambda^{-1/2})$ is established for estimating the volume in the class of compact sets, see Section 5.

Proof. The upper bound in (3.3) follows directly from Theorem 3.1. The lower bound is obtained by reducing the minimax risk to the Bayes risk and then lower-bounding the Bayes risk at its minimum. These steps are fairly standard, cf. Korostelev and Tsybakov (1993b), and we hence omit them here. \qed

4 Data-driven estimator of the volume

The main ingredient to deriving the estimator of $\lambda$ is the fact that the closure of the complement of the $A$-wrapping hull $\hat{K} \defeq \overline{A^c}$ is in fact an $(F_K)$-stopping set according to Lemma 2.3. Moreover in analogy with a time-indexed Poisson process, our problem boils down to the estimation of the intensity of a time-indexed Poisson process starting from an unknown origin. To see this, recall that according to Theorem 2.5, the number of points $N_0$ lying inside the wrapping hull $\hat{A}$ is Poisson-distributed with intensity $\lambda_0 \defeq \lambda|\hat{A}|$ provided that $|\hat{A}| > 0$:

\[ N_0 \mid F_K \sim \text{Poiss}({\lambda}_0). \]

We aim to find an estimator for $\lambda^{-1}_0$. On the event $|\hat{A}| > 0$, we follow the idea developed in Baldin and Reiß (2016). That is to say, we use that the first jump time $\tau$ of a time-indexed Poisson process $(Y_t, t > 0)$ with intensity $\nu > 0$ is $\text{Exp}(\nu)$-distributed and hence $\mathbb{E}[\tau] = \nu^{-1}$. Using the memoryless property of the exponential distribution, we then have

\[ \mathbb{E}[\tau \mid Y_1 = m] = \frac{1}{m+1} \mathbf{1}(m \geq 1) + (1 + \nu^{-1}) \mathbf{1}(m = 0). \]

Therefore, we conclude that

\[ \hat{\mu}(N_0, \lambda_0) \defeq \begin{cases} (N_0 + 1)^{-1}, & \text{for } N_0 \geq 1, \\
1 + \lambda_0^{-1}, & \text{for } N_0 = 0 \end{cases} \]

satisfies $\mathbb{E}[\hat{\mu}(N_0, \lambda_0) \mid F_K] = \lambda_0^{-1}$ provided that $|\hat{A}| > 0$. Omitting the term depending on $\lambda_0$ in the unlikely event $N_0 = 0$, we derive our final estimator:

\[ \hat{\vartheta} = |\hat{A}| + \frac{N_0}{N_0 + 1} |\hat{A}| = \frac{N + 1}{N_0 + 1} |\hat{A}|. \tag{4.1} \]
**Remark 4.1.** As it follows from Definition 2.2, a wrapping hull $\hat{A}$ may consist of disjoint sets, in which case the number of points $N_{o,k}$ lying inside a piece $\hat{A}_k$ satisfies $N_{o,k} |F_k \sim \text{Poiss}(\lambda|\hat{A}_k|)$ due to the homogeneity of the Poisson point process. This fact can further be used to estimate $\lambda^{-1}$ locally. However, in the homogeneous case, we prefer to use the total number of points to estimate the intensity.

Note that a more explicit bound can be derived using the Cauchy-Schwarz inequality given a bound on the expected number of points $N_{\partial}$ lying on the wrapping hull $\hat{A}$ and a bound on the moments of the missing volume $|A \setminus \hat{A}|$. This clearly depends on a considered class of sets satisfying Assumption 2.1. The following very general oracle inequality holds for the mean squared error of the estimator $\hat{\vartheta}$.

**Theorem 4.2.** The following oracle inequality for the risk of the estimator $\hat{\vartheta}$ holds for all $A \in \mathcal{A}$ whenever $\lambda|A| \geq 1$:

$$
\mathbb{E}[(\hat{\vartheta} - |A|)^2]^{1/2} \leq (1 + c_\alpha(\lambda, A)) \text{Var}(\hat{\vartheta}_{\text{oracle}})^{1/2} + r(\lambda, A),
$$

where

$$
\alpha(\lambda, A) := \frac{1}{|A|} \left( \frac{\text{Var}(|A \setminus \hat{A}|)}{\mathbb{E}[|A \setminus \hat{A}|]} + \mathbb{E}[|A \setminus \hat{A}|] \right),
$$

$$
r(\lambda, A) := c_1 \lambda^{-1} \mathbb{E}[N_{\partial}^2]^{1/4} \mathbb{P}(|A \setminus \hat{A}| \geq |A|/2)^{1/4},
$$

with some numeric constants $c, c_1 > 0$. In particular, $\alpha(\lambda, A)$ is bounded by some universal constant.

### 4.1 Volume estimation in the uniform model

In the PPP model, the data we observe are uniformly distributed points over a set in some given class and the number of points is a realisation of a Poisson random variable. The uniform model,

$$
X_1, \ldots, X_n \overset{i.i.d.}{\sim} U(A), \quad A \in \mathcal{A},
$$

is closely related to the PPP model and assumes that the number of points $n$ is fixed. In stochastic geometry, the objects studied in the PPP model typically exhibit a similar asymptotic behaviour in the uniform model and vice versa, see e.g. Pardon (2011) and references therein for a study of the functionals of the convex hull. This section examines which results of the present paper derived in the PPP model remain true in the uniform model.

It is relatively straightforward to show that the wrapping hull remains a sufficient and complete statistic in the uniform model with slightly adjusted arguments of the proof of Theorem 3.3. It is unknown however whether there exists an UMVU estimator in the uniform model. Nevertheless, an estimator

$$
\hat{\vartheta}_{\text{unif},n} := \frac{n + 1}{n_o + 1}|\hat{A}|,
$$

where $n_o$ is the number of points lying inside the wrapping hull, inherits the same rate of convergence as the final estimator $\hat{\vartheta}$ in (4.1) in the PPP model due to the following result.
Proposition 4.3 (Poissonisation). Let \( n = |\lambda A| > 0 \) with \( A \) being any set in a class \( \mathbf{A} \). Then letting \( \lambda \to \infty \) the following asymptotic equivalence result holds true

\[
\mathbb{E}[\hat{\vartheta}_{unif,n} - |A|^2] \sim \mathbb{E}[\hat{\vartheta} - |A|^2], \quad \forall A \in \mathbf{A}.
\] (4.2)

Furthermore, the minimax lower bounds satisfy

\[
\inf_{\hat{\vartheta}} \sup_{A \in \mathbf{A}} \mathbb{E}[\hat{\vartheta} - |A|^2] \sim \inf_{\hat{\vartheta}_\lambda} \sup_{A \in \mathbf{A}} \mathbb{E}[\hat{\vartheta}_\lambda - |A|^2],
\] (4.3)

where the infimum on the left-hand side extends over all estimators in the uniform model, whereas the infimum on the right-hand side extends over all estimators in the Poisson point process model.

Proof. We only prove (4.3) here. (4.2) can then be proved exploiting similar arguments. Let us first show the inequality "\( \gtrsim \)". Assume it does not hold and that there exists an estimator \( \hat{\vartheta}'_n \) in the uniform model with the rate of convergence faster than the minimax optimal rate in the PPP model. Then for the estimator \( \hat{\vartheta}'_N \) we have for any \( A \in \mathbf{A} \)

\[
\mathbb{E}[(\hat{\vartheta}'_N - |A|^2)] = \sum_{k=1}^{[2\lambda |A|]} \mathbb{E}[(\hat{\vartheta}'_N - |A|^2) \mid N = k] \mathbb{P}(N = k)
\]

\[
\leq \sum_{k=[\lambda |A|/2]}^{[2\lambda |A|]} \mathbb{E}[(\hat{\vartheta}'_N - |A|^2) \mid N = k] \mathbb{P}(N = k) + c_2 \exp(-c_3 n)
\]

\[
\leq c_1 \mathbb{E}[(\hat{\vartheta}'_n - |A|^2)] + c_2 \exp(-c_3 n),
\]

for some constants \( c_1, c_2, c_3 > 0 \) using Bennett’s inequality, a contradiction in view of Theorem 3.4. The other direction follows using the same technique. \( \square \)

4.2 Efron’s inequality for the wrapping hull

In this section, we show that the rate of convergence of the risk for the estimator \( \hat{\vartheta}' \) in Theorem 4.2 hinges in fact upon only a deviation of the missing volume \( |A \setminus \hat{A}| \). More than 50 years ago Efron showed in Efron (1965) that the moments of the number of the points \( N_{\partial,k} \) lying on the boundary of a convex hull \( \hat{C}_k \) in the uniform model \( X_1, \ldots, X_k \overset{i.i.d.}{\sim} U(C) \), with \( C \subseteq \mathbb{R}^d \) being a convex set, satisfies the identity

\[
\mathbb{E}[N_{\partial,k}^q] = \sum_{r=1}^{q} n(k, q, r) \mathbb{E}[|\hat{C}_{k-r}|^r],
\]

where \( n(k, q, r) \) is the number of \( q \)-tuples from \( 1, \ldots, k \) having exactly \( r \) different values, \( n(k, q, r) = \binom{k}{r} \sum_{m=1}^{r} (-1)^{r-m} \binom{r}{m} m^q \). This yields a striking dimension-free asymptotic equivalence result,

\[
\mathbb{E}[N_{\partial,k}^q] \sim k^q \mathbb{E}[|C \setminus \hat{C}_k|^q].
\] (4.4)

We here extend a one-sided version of this results to the wrapping hull.
Proposition 4.4 (Efron’s inequality for the wrapping hull). Let \( A \) be any class satisfying Assumption 2.1 and \( \hat{A} \) be the corresponding wrapping hull of the PPP points of intensity \( \lambda > 0 \) over a set \( A \in A \). Then the following asymptotic inequality holds

\[
\mathbb{E}[N^q_A] \lesssim \lambda^q \mathbb{E}[[A \setminus \hat{A}]^q],
\]

provided that the probability of observing \( q \) points lying on the boundary of the wrapping hull \( \hat{A} \) is non-zero.

Remark 4.5. It follows by Jensen’s inequality and Corollary 2.4, that \( \mathbb{E}[N^q_0] \geq \mathbb{E}[N^q_A]^q = \lambda^q \mathbb{E}[[A \setminus \hat{A}]^q]^q \). For some examples, like the class of convex sets, this in fact implies \( \mathbb{E}[N^q_A] \sim \lambda^q \mathbb{E}[[A \setminus \hat{A}]^q] \).

Remark 4.6. Identities that relate the functionals of the convex hull of the points distributed uniformly over a convex set are thoroughly studied in stochastic geometry, see Pardon (2011); Buchta (2013); Beermann and Reitzner (2015).

Proof. Let us first consider the uniform model and then transfer the result to the PPP model using Poissonisation. We follow Efron’s idea, see also Beermann and Reitzner (2015); Brunel (2014), that

\[
\mathbb{E}[A \setminus \hat{A}_k^q] = |A|^q \mathbb{P}(X_{k+1} \notin \hat{A}_k^{(i)}, ..., X_{k+q} \notin \hat{A}_k^{(i)})
\geq |A|^q \mathbb{P}(X_{k+1} \in \partial \hat{A}_k^{(i)}, ..., X_{k+q} \in \partial \hat{A}_k^{(i)})
= \frac{|A|^q}{(k+q)^q} \sum \mathbb{1}(X_{i_1} \in \partial \hat{A}_k^{(i)}, ..., X_{i_q} \in \partial \hat{A}_k^{(i)})
= \frac{|A|^q}{(k+q)^q} \mathbb{E}[N^q_{\partial,k}^q],
\]

the sum being taken over all tuples \( (i_1, ..., i_q) \) from the integers \( 1, ..., k+q \). Rearranging the terms, this entails \( \mathbb{E}[N^q_{\partial,k}] \lesssim k^q \mathbb{E}[[A \setminus \hat{A}_k]^q] \).

Using Poissonisation, we further derive for the PPP model,

\[
\mathbb{E}[A \setminus \hat{A}_k^q] = \sum_{k=1}^{\infty} \mathbb{E}(\mathbb{E}[A \setminus \hat{A}_k^q | N = k]) \mathbb{P}(N = k)
\geq \sum_{k=1}^{\infty} (2\lambda |A|)^{-q} \mathbb{E}[N^q_{\partial,k}] \mathbb{P}(N = k) + \sum_{k=|\mathbb{E}[N^q_{\partial,k}]|}^{\infty} (k^{-q} - (2\lambda |A|)^{-q}) \mathbb{E}[N^q_{\partial,k}] \mathbb{P}(N = k)
= (2\lambda |A|)^{-q} \mathbb{E}[N^q_0] + \sum_{k=|\mathbb{E}[N^q_{\partial,k}]|}^{\infty} (k^{-q} - (2\lambda |A|)^{-q}) \mathbb{E}[N^q_{\partial,k}] \mathbb{P}(N = k)
\]

with the absolute value of second sum being bounded using the Cauchy-Schwarz inequality and large deviations by

\[
c_1 \mathbb{E}[N^q_{\partial,k}^{1/2}] \mathbb{P}(N \geq (2\lambda |A|)^{1/2}) \leq c_1 \mathbb{E}[N^q_0^{1/2}] \exp(-c_2 n),
\]

for some constants \( c_1, c_2 > 0 \). Thus, (4.4) follows. \( \square \)
Proposition 4.4 and Theorem 4.2 immediately suggest the following bound for the remainder term in the oracle inequality,
\[ r(\lambda, A) \leq c \mathbb{E}[|A \setminus \hat{A}|^{1/4}] \mathbb{P}(|A \setminus \hat{A}| \geq |A|/2)^{1/4}, \]
for some numeric constant \( c > 0 \). Therefore, the oracle inequality in Theorem 4.2 hinges upon only two probabilistic results:

- the ratio \( \text{Var}(|A \setminus \hat{A}|)/\mathbb{E}[|A \setminus \hat{A}|] \) of the moments of the missing volume,
- a uniform deviation inequality for the missing volume.

Both results are fairly involved and we shall only discuss here how to derive them for some simple classes of sets satisfying Assumption 2.1.

5 Classes of sets satisfying Assumption 2.1

This section collects some examples of classes of sets that satisfy Assumption 2.1. Note that the class of all convex sets \( \mathbb{C}_{\text{conv}} \) satisfies the assumption and was extensively studied in Baldin and Reiß (2016). The most involved statements in the inference on convex sets were underpinned by the abundance of results from stochastic geometry on moment bounds and deviation inequalities for the missing volume, see Lemma 4.6 in Baldin and Reiß (2016). In particular, the ratio \( \text{Var}(|C \setminus \hat{C}|)/\mathbb{E}[|C \setminus \hat{C}|] \sim 1/\lambda \) is established in Pardon (2011) for all convex sets \( C \) in dimensions \( d = 1, 2 \). In dimensions \( d > 2 \), one can bound the ratio only for some subsets of the class of convex sets. Thus, for a convex set \( C \) with \( C^2 \)-boundary of positive curvature, it is known thanks to Reitzner (2005) that \( \text{Var}(|C \setminus \hat{C}|) \lesssim \lambda^{-(d+3)/(d+1)} \). The lower bound for the first moment, \( \mathbb{E}[|C \setminus \hat{C}|] \gtrsim \lambda^{-2/(d+1)} \), was shown in Schütt (1994). For a polytope \( C \), the upper bound \( \text{Var}(|C \setminus \hat{C}|) \lesssim \lambda^{-2}(\log \lambda)^{d-1} \) was obtained in Bárány and Reitzner (2010), while the lower bound for the first moment, \( \mathbb{E}[|C \setminus \hat{C}|] \gtrsim \lambda^{-1}(\log \lambda)^{d-1} \), was proved in Bárány and Larman (1988). A uniform deviation inequality for convex sets obtained in Thm. 1 in Brunel (2013) allows to derive sharp upper bounds on the moments of the missing volume. The proof of the deviation inequality exploited a bound on the entropy of convex sets. It remains an intriguing open question in stochastic geometry whether \( \lambda \text{Var}(|C \setminus \hat{C}|) \sim \mathbb{E}[|C \setminus \hat{C}|] \) holds universally for all convex sets in arbitrary dimensions. Some of the classes of sets we consider here are much larger, yet very little has been known about them in the mathematical literature.

5.1 \( r \)-convex sets

We denote by \( B(x, r) \subseteq \mathbb{R}^d \) (resp. \( B_o(x, r) \)) the closed (resp. open) ball with centre \( x \) and radius \( r \).

**Definition 5.1.** A compact set \( C_r \) in \( \mathbb{E} \subseteq \mathbb{R}^d \) is called \( r \)-convex for \( r > 0 \), if its complement is the union of all open Euclidean balls of diameter \( r \) that are disjoint to \( C_r \), i.e. if
\[
C_r = \bigcap_{B^c_o(x, r) \cap C_r = \emptyset} B^c_o(x, r).
\]

We denote the class of \( r \)-convex sets by \( \mathbb{C}_r \).
Note that an \( r \)-convex set fulfills the outside rolling ball condition, i.e. for all \( y \in \partial C_r \) there is a closed ball \( B(x, r) \) such that \( y \in \partial B(x, r) \) and \( B_o(x, r) \cap C_r = \emptyset \). Heuristically this means that one can “roll” a ball of radius \( r \) freely over the boundary of a set. Note that according to the definition, \( r \)-convex sets can have “holes” and do not need to be connected, see Figure 4 for some examples. In the terminology of Kendall (1974), \( C_r \in C_r \) means that the set \( C_r \) is trapped by the balls of radius \( r \). The \( r \)-convex sets were introduced in Perkal (1956) and presumably independently in Efimov and Stechkin (1959); see Walther (1997); Cuevas, Fraiman, and Pateiro-López (2012) and references therein for a recent work on estimation of \( r \)-convex sets. In the literature, much more attention has been devoted to the sets satisfying the so-called inside and outside rolling ball condition, when both \( C_r \) and \( C_{c_r} \) are \( r \)-convex, see Mammen and Tsybakov (1995); Walther (1995). The reason probably is that sets with smooth boundaries (with no angles) are sometimes easier to handle with geometric arguments, see Pateiro-López (2008).

The \( C_r \)-wrapping hull is defined by

\[
\hat{C}_r := \bigcap_{B_o(x,r) \subset \{X_1, \ldots, X_N\} = \emptyset} B_c(x,r)
\]

and often called the \( r \)-convex hull in the literature. Thus the oracle estimator in (3.2) has the following form

\[
\hat{\vartheta}_{r,\text{oracle}} := \frac{N_o}{\lambda} + |\hat{C}_r|,
\]

where \( N_o \) is the number of sample points lying on the \( r \)-convex hull \( \hat{C}_r \). In order to investigate the performance of this estimator according to Theorem 3.1 it suffices to study \( \sup_{C_r \in C_r} \mathbb{E}[|C_r \setminus \hat{C}_r|] \). In fact the following result holds and it is a consequence of Theorem 3.1.

**Theorem 5.2.** For known intensity \( \lambda > 0 \), the worst case mean squared error of the oracle estimator \( \hat{\vartheta}_{r,\text{oracle}} \) over the parameter class \( C_r \) decays as \( \lambda \uparrow \infty \) like

\[
\limsup_{\lambda \to \infty} \sup_{C_r \in C_r} \mathbb{E}[|C_r \setminus \hat{C}_r|]/\lambda \leq \frac{(d+3)}{d+1}.
\]

**Remark 5.3.** Note that the class of convex sets \( C_{\text{conv}} \) belongs to \( C_r \) for all \( r > 0 \) and thus using Theorem 3.4 in Baldin and Reiß (2016) we have a lower bound on the rate of convergence,

\[
\inf_{\lambda} \lambda^{(d+3)/(d+1)} \sup_{C_r \in C_r} \mathbb{E}[|C_r \setminus \hat{\vartheta}_\lambda|^2] \geq \inf_{\lambda} \lambda^{(d+3)/(d+1)} \sup_{C \in C_{\text{conv}}} \mathbb{E}[|C \setminus \hat{\vartheta}_\lambda|^2] > 0,
\]

where the infimum extends over all estimators \( \hat{\vartheta}_\lambda \) in the PPP model with intensity \( \lambda \). Furthermore, the rate \( \lambda^{-(d+3)/(d+1)} \) is achieved up to a logarithmic factor for sets \( C_r \in C_r \) with a smooth boundary following Pateiro-López (2008).

Following Section 4, the mean squared error of the estimator

\[
\hat{\vartheta}_r := \frac{N + 1}{N_o + 1} |\hat{C}_r|,
\]
satisfies the oracle inequality in Theorem 4.2. Upper bounding the functions \( \alpha(\lambda, C_r) \) and \( r(\lambda, C_r) \) and establishing an exact rate of convergence of the risk of the estimator \( \hat{\vartheta} \) requires sharp probabilistic bounds similarly to the convex case and is beyond the scope of the present paper. In order to compute \( \hat{\vartheta} \) in practice, it is crucial to know the radius \( r \). It is clear that \( C_{r_2} \subseteq C_{r_1} \) for all \( r_2 > r_1 > 0 \).

Define the true radius \( r^* \) corresponding to a set \( C \) as

\[
r^* = \sup \{ r > 0 : C \in C_r \}.
\]

It is intuitively evident that it is better to underestimate the radius \( r^* \), because \( C_{r'} \subseteq C_r \) for all \( r' > r > 0 \). On the other hand, there is a typical bias-variance tradeoff in choosing the optimal parameter \( r \). We prefer to use large values of \( r \) when the number of sample points is scarce, and can afford to use small values \( r < r^* \) when there is an abundance of the sample points. We here propose a procedure for estimating \( r^* \) based on Lepski’s method Lepski (1992), see also a more accessible reference, Section 8.2.1 in Giné and Nickl (2016). Fix some \( R \in \mathbb{R}_+ \) and \( K \in \mathbb{N} \) and break the interval \( (0, R) \) down into \( K + 1 \) pieces \( 0 < r_1 < \ldots < r_K < R \) of equal length. Define an estimator for \( \hat{r} \) as

\[
\hat{r} := \inf \{ r_{k-1} \mid \exists k' \leq k : |\hat{\vartheta}_{r_{k-1}} - \hat{\vartheta}_{r_{k'}}| > \kappa_n \} \wedge r_K ,
\]

with \( \kappa_n = N \delta / n^2 \). This calibration is suggested by Theorem 3.1 in view of Corollary 2.4. The asymptotic behaviour of the estimator \( \hat{\vartheta} \) depends on an exact deviation inequality on the missing volume \( |C_r \setminus \hat{C}_r| \). This question however is beyond the scope of the present paper. We provide a numerical study of this adaptation procedure in Section 6.

5.2 Compact sets

Interestingly the class of all compact sets \( K \) of non-zero Lebesgue measure satisfies Assumption 2.1 as well. The richness of this class makes it foremost for conducting statistical inference, yet very little has been proposed and studied so far. Estimation of this class of sets was studied in Devroye and Wise (1980), where it was shown that the union of small Euclidean balls centred at the points of the sample is a consistent estimator of a compact set. The \( K \)-wrapping hull is just the union of sample points and so \( N_{\partial K} = N \) and \( |\hat{K}| = 0 \) a.s. Hence for the oracle estimator in (3.2) we have

\[
\hat{\vartheta}_{K,\text{oracle}} := \frac{N}{\lambda}.
\]

This estimator is unbiased and from (3.1) the following result immediately follows.

**Lemma 5.4.** For known intensity \( \lambda > 0 \), the worst case mean squared error of the oracle estimator \( \hat{\vartheta}_{K,\text{oracle}} \) over the parameter class \( K \) decays as \( \lambda \uparrow \infty \) like \( \lambda^{-1} \):

\[
\sup_{K \in \mathcal{K}} \frac{1}{|K|} \mathbb{E} \left[ (\hat{\vartheta}_{K,\text{oracle}} - |K|)^2 \right] = \frac{1}{\lambda}.
\]

It seems impossible without imposing further structure on the class \( K \) to estimate \( \lambda \) in this scenario.
5.3 Polytopes

It was noted in Baldin and Reiβ (2016), that the estimator of the volume based on the $C_{\text{conv}}$-wrapping hull estimator (the convex hull $\hat{C}$) is adaptive to the class of polytopes $P$ (see Remark 3.3). In fact, the estimator

$$\hat{\vartheta}_{P,\text{oracle}} := \frac{N_\partial}{\lambda} + |\hat{C}|$$

satisfies

**Lemma 5.5.** For known intensity $\lambda > 0$, the worst case mean squared error of the oracle estimator $\hat{\vartheta}_{P,\text{oracle}}$ over the parameter class $P$ decays as $\lambda \uparrow \infty$ like $
lambda^{-2}(\log(\lambda))^{d-1}:

$$\limsup_{\lambda \to \infty} \lambda^2(\log(\lambda))^{1-d} \sup_{P \in P, |P| > 0} \mathbb{E}[(\hat{\vartheta}_{P,\text{oracle}} - |P|)^2] < \infty.$$$$

We stress here, however, that the class polytopes $P$ does not satisfy Assumption 2.1. The framework applies to the class of convex sets $C$ and Lemma 5.5 only allows to improve the rate for the subclass of $C$. The class $P$ is stable only under finite intersections; taking arbitrary (possibly uncountable) intersections, one can obtain an element not lying in the class.

5.4 Polytopes with fixed directions of outer unit normal vectors

The class of polytopes $P_{S_k}$ with fixed directions $S_k = \{u_1, \ldots, u_k\}$ of outer unit normal vectors $u_k$ belonging to the unit sphere $S^{d-1}$ provides another interesting example of intersection stable sets. We assume the class is well-defined in the sense that there exists a polytope $P_{S_k}$ whose outer unit normal vectors are exactly $\{u_1, \ldots, u_k\}$. Without loss of generality we may assume $E = P_{S_k}$. The $P_{S_k}$-wrapping hull $\hat{P}$ is a polytope with at most $k$ facets and is given by

$$\hat{P} := \bigcap_{P \in P_{S_k}, (X_1, \ldots, X_N) \in P} P.$$

The oracle estimator and the data-driven estimator are thus defined as

$$\hat{\vartheta}_{P_{S_k},\text{oracle}} := \frac{N_\partial}{\lambda} + |\hat{P}|, \quad \hat{\vartheta}_{P_{S_k}} = \frac{N + 1}{N_\partial + 1} |\hat{P}|,$$

where the number of points lying on the boundary of the wrapping hull $N_\partial$ is equal to the number of facets of the wrapping hull and hence upper-bounded by $k$. According to the general scheme, the rate of convergence of the risk for the oracle estimator $\hat{\vartheta}_{P_{S_k},\text{oracle}}$ and the final estimator $\hat{\vartheta}_{P_{S_k}}$ rests upon a deviation inequality for the missing volume and is established in the following theorem which is proved in the Appendix.

**Theorem 5.6.** The worst case mean squared error of the estimator $\hat{\vartheta}_{P_{S_k}}$ over the parameter class $P_{S_k}$ satisfies:

$$\sup_{P \in P_{S_k}} \mathbb{E}[(\hat{\vartheta}_{P_{S_k}} - |P|)^2] \lesssim \frac{kW(\lambda/k)}{\lambda^2},$$

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In this section, we illustrate the performance of the proposed estimators for a class of \(r\)-convex sets. Our primal example of an \(r\)-convex set for simulations is the annulus \(C_r^* = B(0.5, 0.5) \setminus B(0.5, 0.25)\). Thus clearly \(C_r^* \subseteq C_r\) for all \(0 < r \leq 0.25\). Figure 4 depicts the \(r\)-convex hull estimator (5.1) for \(r = 0.01, 0.04, 0.2, 1\) based on the observations of the PPP with \(\lambda = 300\). An important observation is that once the value of \(r\) is larger than the true radius \(r^*\) of an \(r\)-convex set, the \(r\)-convex hull essentially misses the “holes” of radius \(r^*\). One should bear this in mind when using large values of \(r\) for constructing the oracle estimator when the number of observation points is small. This subtle issue is depicted in Figure 5, where the root mean squared error of the oracle estimator \(\hat{\theta}_{r,\text{oracle}}\) coincides with the RMSE of \(\hat{\theta}_{K,\text{oracle}}\) and equals \(|C_r^*|/\lambda\) (red line in Figure 5)!

Finally we depict RMSE estimates for the oracle estimator \(\hat{\theta}_{r,\text{oracle}}\) for different \(r\) in Figure 6. One can clearly see the regions of decreasing value of the RMSE, the fairly flat value of RMSE and the jump when \(r\) becomes larger than the true parameter \(r^*\). Table 1 further collects the Monte Carlo estimates of the number of points \(N_o\) lying inside the wrapping hull, the number of points \(N_b\) on the boundary of the wrapping hull and the number of isolated points \(N_{iso}\) of the boundary of the wrapping hull. For analyzing the performance of the adaptive estimator proposed in Section 5.1, we break the interval \([0.06, 0.5]\) into pieces of length 0.02, compute the estimates of the radius \(\hat{\mu}\) from (5.2) and the estimates of the RMSE of \(\hat{\theta}_r\) based on 200 Monte Carlo iterations in Table 2.

### 6.1 Appendix

#### 6.1.1 Proof of Theorem 3.3

Sufficiency follows from the Neyman factorisation criterion applied to the likelihood function (2.1), while completeness follows by definition provided that we show

\[
\forall A \in \mathcal{A} : \mathbb{E}_A[T(\hat{A})] = 0 \implies T(\hat{A}) = 0 \quad \mathbb{P}_E - a.s.
\]

for any \(\mathcal{A}\)-measurable function \(T : A \to \mathbb{R}\). From the likelihood in (2.1) for \(\lambda = \lambda_0\), we derive

\[
\mathbb{E}_A[T(\hat{A})] = \mathbb{E}_E[T(\hat{A}) \exp(\lambda |E \setminus A|) 1(\hat{A} \subseteq A)].
\]

Since \(\exp(\lambda |E \setminus A|)\) is deterministic, we have \(\forall A \in \mathcal{A}\)

\[
\mathbb{E}_A[T(\hat{A})] = 0 \implies \mathbb{E}_E[T(\hat{A}) 1(\hat{A} \subseteq A)] = 0.
\]

Splitting \(T = T^+ - T^-\) with non-negative \(\mathcal{A}\)-measurable functions \(T^+\) and \(T^-\), we infer that the measures \(\mu^\pm(B) = \mathbb{E}_E[T^\pm(\hat{A}) 1(\hat{A} \subseteq B)], B \in \mathcal{A}\), agree on \(\{[B] \mid B \in \mathcal{A}\}\).
Figure 4: The four weakly-convex set with $r^* = 0.25$ (blue), observations of the PPP with $\lambda = 300$ (points) and their $r$-convex hulls for different values of $r$ (black).

Figure 5: Monte Carlo RMSE estimates for the oracle estimator for the volume of the annulus $B(0.5, 0.5) \setminus B(0.5, 0.25)$ with respect to the sample size.
\[ r = 0.04 \]

| \( n = \lambda/|A| \) | \( N_o \) | \( N_\theta \) | \( N_{iso} \) | \( \text{RMSE}(\hat{\vartheta}_{r,oracle}) \) | \( \text{RMSE}(\hat{\vartheta}_r) \) | \( \text{RMSE}(\hat{\vartheta}_r)/\text{RMSE}(\hat{\vartheta}_{r,oracle}) \) |
|-----------------|-----|-----|-----|-----------------|-----------------|-----------------|
| 50              | 0.13 | 49.8 | 44  | 0.087           | 0.55            | 6.31            |
| 100             | 1.5  | 98.37| 66  | 0.059           | 0.33            | 4.65            |
| 200             | 23.6 | 175.8| 58  | 0.042           | 0.15            | 3.7             |
| 300             | 90.4 | 211  | 33  | 0.039           | 0.109           | 2.82            |
| 400             | 191.6| 210  | 17  | 0.043           | 0.085           | 1.97            |

\[ r = 0.1 \]

| \( n = \lambda/|A| \) | \( N_o \) | \( N_\theta \) | \( N_{iso} \) | \( \text{RMSE}(\hat{\vartheta}_{r,oracle}) \) | \( \text{RMSE}(\hat{\vartheta}_r) \) | \( \text{RMSE}(\hat{\vartheta}_r)/\text{RMSE}(\hat{\vartheta}_{r,oracle}) \) |
|-----------------|-----|-----|-----|-----------------|-----------------|-----------------|
| 50              | 8.5 | 42.16| 8.52| 0.071           | 0.138           | 1.94            |
| 100             | 45.9| 53.34| 1.37| 0.043           | 0.064           | 1.48            |
| 200             | 138.2| 60.95| 0.03| 0.021           | 0.027           | 1.26            |
| 300             | 233.4| 68.08| 0   | 0.015           | 0.018           | 1.20            |
| 400             | 326.1| 74.40| 0   | 0.013           | 0.015           | 1.15            |

\[ r = 0.25 \]

| \( n = \lambda/|A| \) | \( N_o \) | \( N_\theta \) | \( N_{iso} \) | \( \text{RMSE}(\hat{\vartheta}_{r,oracle}) \) | \( \text{RMSE}(\hat{\vartheta}_r) \) | \( \text{RMSE}(\hat{\vartheta}_r)/\text{RMSE}(\hat{\vartheta}_{r,oracle}) \) |
|-----------------|-----|-----|-----|-----------------|-----------------|-----------------|
| 50              | 24.75 | 24.21| 0.06| 0.061           | 0.085           | 1.39            |
| 100             | 68.58 | 29.60| 0   | 0.033           | 0.0405          | 1.20            |
| 200             | 163.75| 36.03| 0   | 0.018           | 0.019           | 1.04            |
| 300             | 261.44| 40.68| 0   | 0.0108          | 0.0124          | 1.13            |
| 400             | 357.41| 44.17| 0   | 0.0096          | 0.0104          | 1.076           |

\[ r = 0.3 \]

| \( n = \lambda/|A| \) | \( N_o \) | \( N_\theta \) | \( N_{iso} \) | \( \text{RMSE}(\hat{\vartheta}_{r,oracle}) \) | \( \text{RMSE}(\hat{\vartheta}_r) \) | \( \text{RMSE}(\hat{\vartheta}_r)/\text{RMSE}(\hat{\vartheta}_{r,oracle}) \) |
|-----------------|-----|-----|-----|-----------------|-----------------|-----------------|
| 50              | 30.71 | 18.70| 0   | 0.208           | 0.340           | 1.628           |
| 100             | 77.59 | 23.26| 0   | 0.2002          | 0.258           | 1.29            |
| 200             | 170.30| 29.39| 0   | 0.1982          | 0.232           | 1.17            |
| 300             | 265.17| 33.89| 0   | 0.1978          | 0.223           | 1.13            |
| 400             | 362.43| 37.89| 0   | 0.1987          | 0.219           | 1.10            |

Table 1: Monte Carlo RMSE estimates for the oracle estimator \( \hat{\vartheta}_{r,oracle} \) and for the fully data-driven estimator \( \hat{\vartheta}_r \) for the volume of the annulus \( A = B(0.5, 0.5) \setminus B(0.5, 0.25) \) with respect to \( r \) and \( n = \lambda |A| \), the number of points lying inside the wrapping hull \( N_o \), the number of points on the boundary of the wrapping hull \( N_\theta \) and the number of isolated points of the boundary of the wrapping hull \( N_{iso} \).

\[ r \]

| \( n = \lambda/|A| \) | \( \hat{\vartheta}_r \) | \( \text{RMSE}(\hat{\vartheta}_r) \) |
|-----------------|-----|-----------------|
| 50              | 0.088| 0.36            |
| 100             | 0.085| 0.160           |
| 200             | 0.084| 0.069           |
| 300             | 0.105| 0.033           |
| 400             | 0.125| 0.0182          |
| 500             | 0.149| 0.0123          |
| 1000            | 0.165| 0.0056          |

Table 2: Monte Carlo RMSE estimates for the adaptive estimator \( \hat{\vartheta}_r \) for the volume of the annulus \( A = B(0.5, 0.5) \setminus B(0.5, 0.25) \) with respect to \( n = \lambda |A| \).
Figure 6: Monte Carlo RMSE estimates for the oracle estimator for the volume of the annulus $B(0.5,0.5) \setminus B(0.5,0.25)$ with respect to $r$.

$A \setminus B$, where $[B] = \{ A \in A | A \subseteq B \}$. Since the brackets $[[B] | B \in A]$ generate the $\sigma$-algebra $\mathcal{A}$, the measures $\mu^\pm(B)$ agree on all sets in $\mathcal{A}$, in particular on $\{ T > 0 \}$ and $\{ T < 0 \}$, which entails $\mathbb{E}[T^+(\hat{A})] = \mathbb{E}[T^-(\hat{A})] = 0$. Thus, $T(\hat{A}) = 0$ holds $\mathbb{P}_\mathcal{E}$-a.s.

6.1.2 Proof of Theorem 5.6

Let us denote by $\rho_1(A,B) = |A \Delta B|$ the symmetric distance between two compact subsets $A$ and $B$ of the compact convex set $E$ in $\mathbb{R}^d$. Recall that an $\varepsilon$-net of the class $\mathbb{P}_{S_k}$ with respect to the metric $\rho_1$ is a collection $\{ P^1,...,P^{N_\varepsilon} \} \in \mathbb{P}_{S_k}$ such that for each $P \in \mathbb{P}_{S_k}$, there exists $i \in \{1,...,N_\varepsilon\}$ such that $\rho_1(P,P^i) \leq \varepsilon$. The $\varepsilon$-covering number $N(\mathbb{P}_{S_k},\rho_1,\varepsilon)$ is the cardinality of the smallest $\varepsilon$-net. The $\varepsilon$-entropy of the class $\mathbb{P}_{S_k}$ is defined by $H(\mathbb{P}_{S_k},\rho_1,\varepsilon) = \log_2 N(\mathbb{P}_{S_k},\rho_1,\varepsilon)$. Furthermore, it follows by dilation of a set that for $\hat{P} \in \mathbb{P}_{S_k}$ there exists $\hat{m} \in \{ 1,...,N_\varepsilon \}$ such that $\hat{P} \subseteq \hat{P}^{\hat{m}} \subseteq P$ and $\rho_1(\hat{P},\hat{P}^{\hat{m}}) \leq \varepsilon$ for some universal constant $c > 1$ and $\varepsilon$ small enough. We thus obtain for all $P \in \mathbb{P}_{S_k}$ and $x > 0$,

$$
\mathbb{P}( |P \setminus \hat{P}| > x/\lambda + 2\varepsilon ) \leq \mathbb{P}( |P \setminus P^{\hat{m}}| > x/\lambda + c\varepsilon )
\leq \sum_{m:|P \setminus P^{m}| > x/\lambda + c\varepsilon} \mathbb{P}( \mathcal{N}(P \setminus P^{m}) = 0 )
\leq \exp \left( -x - c\lambda\varepsilon + H(\mathbb{P}_{S_k},\rho_1,\varepsilon) \right) = e^{-x},
$$

plugging in $\varepsilon$ that solves $H(\mathbb{P}_{S_k},\rho_1,\varepsilon) = c\lambda\varepsilon$. The $\varepsilon$-covering number $N(\mathbb{P}_{S_k},\rho_1,\varepsilon)$ of the class $\mathbb{P}_{S_k}$ can be bounded by $(C/\varepsilon)^k$ for some universal constant $C > 1$. As a result, the asymptotic rate follows using Fubini’s theorem combined with Theorem 3.1 and Theorem 4.2.
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