Predicting the CMB temperature

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Abstract

Big-bang nucleosynthesis (BBN), a pillar of modern cosmology, begins with the trailblazing 1948 paper of Alpher, Bethe and Gamow [1] which proposed nucleosynthesis during an early ($t \sim 1 - 1000$ sec), radiation-dominated phase of the Universe to explain the abundances of the chemical elements. While their model was flawed, they called attention to the importance of a hot beginning and the radiation now known as the cosmic microwave background (CMB). In subsequent papers they made estimates for the temperature of the CMB, from 5 K to 50 K, all based upon wrong physics. To illustrate the prediction that could have been made and to elucidate the key physics underpinning BBN, I show that in the absence of a detailed model of BBN and the desired abundances, at best one could have estimated an upper limit to the CMB temperature of between 10 K and 60 K, predicated upon the assumption of some nucleosynthesis.

The landmark 1948 paper [1, 2] was motivated by the failure of equilibrium processes in stars to account for the pattern of the abundances of the elements. It broke new ground with its early radiation-dominated hot phase of cosmic evolution and led to the prediction of a relic radiation of this hot phase: Because the energy density in photons decreases as $R^{-4}$ while that in matter decreases only as $R^{-3}$, even a hot, radiation-dominated beginning eventually yields to a cool matter-dominated present, with only the ember of the hot beginning with us today (here $R$ is the FRW cosmic scale factor).

Essentially all of the nuclear physics in their model was wrong, and it took almost 20 years to get it all right, in the equally important paper by Wagoner, Fowler and Hoyle [3, 4].

\footnote{Before this paper, Hoyle and Tayler [4] and Peebles [5] wrote papers estimating the $^4$He produced in the big bang, but without the full nuclear physics.}
Gamow and his colleagues made various physically-wrong predictions for the temperature of the CMB, which were largely ignored. In part, because they didn’t think it was detectable and buried such predictions deep in their papers. As is well known, the all-important relic radiation of the hot big bang was discovered accidentally by Penzias and Wilson in 1965, while the Princeton group led by Dicke, having re-discovered the idea of a hot big bang, were mounting an experiment to search for the CMB.

My goal in this short note is to briefly explain what could have been predicted for the CMB temperature, with the correct physics and “back of the envelope” style. Of course, a detailed computer calculation, like that of [3], and the primordial abundances would have allowed one to precisely predict the CMB temperature. Today’s variant on that theme is one of the important consistency tests of cosmology: namely given the CMB temperature and the abundances, do the predictions match?

Here is the top-level story: Early on ($T \gg 1$ MeV and $t \ll 1$ sec) thermal equilibrium is established. Free nucleons are thermodynamically favored and weak interactions keep the ratio of neutrons to protons at its equilibrium value, $n/p = \exp[-\Delta m/T]$, close to unity ($\Delta m \simeq 1.3$ MeV). Very late ($T \ll 0.01$ MeV and $t \gg 10^4$ sec) nuclei are thermodynamically favored, any free neutrons have decayed (neutron mean life $\tau_n \sim 900$ sec) and nuclear reactions have ceased owing to Coulomb barriers.

What happens in between the “very early” and the “very late,” which involves non-equilibrium nuclear physics, determines the outcome of BBN. And that outcome only depends upon the baryon-to-photon ratio $\eta$, whose value today is known to be $6.03 \times 10^{-10}$.

Three things are crucial in the all-important “in-between phase,” where there is neither thermal equilibrium nor frozen out nuclear reactions. First, the weak interactions freeze out around $T \sim 1$ MeV and the ratio of neutrons-to-protons freezes in at a value of about $1/7$, rather than exponentially decreasing to zero. This means the first step of any BBN is an EM interaction, $n + p \to d + \gamma$, and not a weak interaction for which there would not be enough time.

Next, a simple calculation of nuclear statistical equilibrium shows that nuclei become favored over free nucleons at a temperature [10]:

$$T_{\text{nuclei}} \simeq \frac{B_A/(A - 1)}{\ln \eta^{-1} + 1.5 \ln (m/T)} \simeq 0.2 \text{ MeV}[1 + \ln \eta_{10}/35],$$

where $B_A$ is the binding energy of nucleus $A$, $m \simeq 1$ GeV is the nucleon mass, and $\eta_{10} \equiv \eta/10^{-10}$. Put simply, because of the very large number of photons per baryon (“high entropy”), nuclei are not thermodynamically favored until a temperature much, much less than a typical nuclear binding energy (here taken

\[\text{I count at least 13 papers co-authored by Alpher following the original 1948 paper, discussing aspects of BBN. The one that comes closest to getting the correct nuclear physics is [6], which sets up the modern equations for the neutron-to-proton ratio but stops there.}^{2}\]

\[\text{There are many accounts of the discovery of the CMB; Peebles has written a recent, very thorough one in his book, \textit{Cosmology’s Century}.}^{3}\]

\[\text{It does decrease slowly due to neutron decays.}^{4}\]
to be that of $^4$He, or about 7 MeV per nucleon). Further, the more photons per baryon – that is, smaller $\eta$ – the lower is the temperature at which nucleons become favored.

Third, any nucleosynthesis beyond deuterium involves charged-particle on charged-particle reactions, e.g., $d + d \rightarrow ^4$He + $\gamma$, and so Coulomb barriers are important. They suppress nuclear rates by an exponential factor; for $d(d,\gamma)^4$He, that factor is $\exp\left[{-2(T/\text{MeV})^{-1/3}}\right]$. Because of this, the nuclear reactions that are necessary to produce anything beyond deuterium will cease occurring (“freeze out”) at a temperature that can be estimated by comparing the expansion rate $H \sim T^2/m_{\text{pl}}$ to the nuclear reaction rate $\Gamma$ (see appendix for more details):

$$T_{\text{Coulomb}} \sim \frac{0.03 \text{ MeV}}{\left[1 + \ln \eta_{10}/7\right]^{3/2}}.$$  

(2)

Simply put, “higher entropy” – that is, smaller $\eta$ – means nuclear reactions freeze out earlier, and at a higher temperature.

Here is the crux of our back-of-the-envelope prediction: For there to be any nucleosynthesis beyond deuterium, nuclei must become thermodynamically favored before Coulomb barriers freeze out nuclear reactions, or $T_{\text{nuclei}} \geq T_{\text{Coulomb}}$. This corresponds to $\eta > 10^{-11}$ (see Fig. 1); and further, larger values of $\eta$ will result in more nucleosynthesis.

Finally, to turn lower limit for $\eta$ into an upper limit for $T_{\text{CMB}}$, we need the relationship between $\eta$ and the CMB temperature. In convenient units, it is:

$$\frac{T_{\text{CMB}}}{2.7255 \text{ K}} = \left(\frac{274 \Omega_B h^2}{\eta_{10}}\right)^{1/3},$$

where $h = H_0/100 \text{ km/s/Mpc}$ and $\Omega_B$ is the fraction of the critical density contributed by baryons.

Now comes the fun! What to use for $\Omega_B h^2$ with our lower bound to $\eta$, $\eta_{10} \geq 0.1$, to get an upper limit for the “predicted” CMB temperature. Here are three plausible choices

1. The well-determined present value, $\Omega_B h^2 = 0.022$ ($\rho_B \simeq 4 \times 10^{-31} \text{ g/cc}$).
   This leads to $T_{\text{CMB}} \leq 11 \text{ K}$.

2. A 1960-ish $h = 1$ and low-density universe $\Omega_B = 0.1$ ($\rho_B \simeq 2 \times 10^{-29} \text{ g/cc}$).
   This leads to $T_{\text{CMB}} \leq 18 \text{ K}$.

3. A 1950-ish $h = 2$ and high-density universe $\Omega_B = 1$ ($\rho_B \simeq 10^{-28} \text{ g/cc}$).
   This leads to $T_{\text{CMB}} \leq 61 \text{ K}$.

What have we learned? First, the cube root in the formula for the temperature really helps to reduce the range of predicted upper limits owing to imprecise knowledge of the cosmological parameters! Second, if we had more information, e.g., the need to explain a large primordial $^4$He abundance, but no heavier elements (a point made by Hoyle and Tayler [4] shortly before the discovery of the
Figure 1: Key temperatures during BBN vs. the baryon-to-photon ratio $\eta$: $T_F$, the freeze in of the n/p ratio; $T_{\text{nuclei}}$, the temperature below which nuclei are thermodynamically favored over free nucleons; $T_{\text{Coulomb}}$, the temperature below which charged-particle nuclear reactions cease occurring; and $T_n$, the temperature at which the age of the Universe is the lifetime of a free neutron. Any significant nucleosynthesis beyond deuterium requires $T_{\text{nuclei}} \geq T_{\text{Coulomb}}$, or $\eta \geq 10^{-11}$. The vertical line marked “time” shows the timeline of successful BBN: freeze in of the n/p ratio at $T = T_F$ $\rightarrow$ period of waiting until nuclei are favored $T_F > T > T_{\text{nuclei}}$ $\rightarrow$ nucleosynthesis $T_{\text{nuclei}} > T > T_{\text{Coulomb}}$ $\rightarrow$ frozen out nuclear reactions $T_{\text{Coulomb}} > T$ $\rightarrow$ any free neutrons remaining decay $T_n > T$.

CMB), we could have done better and made an actual prediction: In this case, more sophisticated estimates for $T_{\text{nuclei}}$ and $T_{\text{Coulomb}}$ would constrain $\eta_{10}$ to the interval $1 - 10$; for the 1960-ish values, that would predict $T_{\text{CMB}} = 3.8 - 8.2$ K.

I note that a better upper limit to $T_{\text{CMB}}$ was already known before the discovery of the CMB, at least to Hoyle: McKellar’s 1940 measurements of the population levels of interstellar CN and CH radicals constrained any “universal radiation field,” $T_{\text{CMB}} < 5$ K. In fact, Hoyle used McKellar’s result as an argument against Gamow’s hot big bang model in debates the two engaged in. Of course, there are also theoretical lower limits that one can derive to $\eta$, based upon the formation of structure.

I end by putting all of this in the larger context: The fog is thick at the frontiers of discovery in all science and thus confusion is often great, especially
true in cosmology. Gamow et al’s paper was ground-breaking and not surprisingly contained many mistakes (many ground-breaking papers do!). It deserves all the acclaim that it receives; however, the wrongheaded predictions for the CMB temperature that followed do not. Likewise, the missteps along the path to the accidental discovery of the CMB were many. In the end, it all worked out and the CMB has transformed cosmology into a precision science.

I thank Robert Wagoner and Robert Scherrer for their comments and insightful conversations.

Endnote. I have included a few figures beyond my hand drawn one: the history of measurements of $H_0$ (Fig. 2) and some useful figures from the classic paper by Wagoner, Fowler and Hoyle [3], Figs. 3, 4 and 5. And an Appendix follows with a few more mathematical details.

This note follows up the Physics Today article I wrote in 2008 [2], as well as conversations with Jim Peebles about the work of Gamow and his collaborators. Peebles has devoted 20 or so pages of his most recent book [9] to the predictions of Gamow and his collaborators, seeming to validate their prediction, a point of view I strongly disagree with. I must admit, I have trouble following his discussion, and I may be misinterpreting it. And further, some of their papers he discusses, bring in arguments based upon a crude model of the formation of galaxies and not BBN.

There are more than a dozen papers written by Alpher et al about BBN that one can wade through [1, 6, 12, 13]; for the reader simply interested in hearing retrospectively what they have to say about predicting the CMB temperature, I suggest [13].

The motivation for writing this note is to elucidate the underlying physics of BBN and how it alone could have been used to predict the CMB temperature – all with the benefit of 20/20 hindsight.

References

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Appendix: some technical details

Here are the key formulae and approximations used in my back-of-the-envelope calculations

\[ H^2 = \left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G \rho}{3} \approx \frac{8\pi G \rho_R}{3} \]

\[ H \approx \frac{10T^2}{m_{\text{pl}}} \]

\[ t = \frac{1}{2H} \approx \frac{1 \text{ sec}}{(T/\text{MeV})^2} \]

\[ \rho_R = g_\ast \frac{\pi^2}{90} T^4 \]

\[ \Gamma = n < \sigma v > \]

\[ n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3 \]

\[ n_B \approx \eta m_\gamma \]

During BBN the energy density is dominated by that in thermal, relativistic particles (photons, neutrinos and electron-positron pairs) with \( g_\ast = 10.75 \), and so the expansion rate \( H \approx 10T^2/m_{\text{pl}} \) (\( m_{\text{pl}} = 1.22 \times 10^{19} \text{ GeV} \) is the Planck mass). The reaction rate (per particle) is proportional to the number density of targets times the thermally-averaged cross section times relative velocity. Neglecting the factor of \( 11/4 \) increase in the number of photons when electron-positron pairs annihilate, the number density of baryons (nucleons) during BBN is just the baryon-to-photon ratio \( \eta \) times the number density of photons. Throughout, \( k_B = \hbar = c = 1 \) so that \( G = 1/m_{\text{pl}}^2 \).
The key concept underlying my calculations is “freeze out” of particle interactions. Namely, when a reaction rate key to adjusting a particle abundance cannot keep up with the rate at which the temperature is falling, \( \dot{T}/T = -\dot{R}/R = H \), it ceases to be relevant and the abundance freezes in. To a good approximation, a reaction rate is frozen out when \( \Gamma < H \), and freeze out occurs when \( \Gamma \equiv H \).

For example, the rate for the reactions that maintain thermal balance of neutrons and protons, \( p + e^- \leftrightarrow n + \nu \) etc., is \( \Gamma \simeq T^3 \times G_F^2 T^2 \simeq G_F^2 T^5 \). Comparing it to the expansion rate, it follows that the freeze of the neutron-to-proton ratio occurs

\[
T_F \simeq G_F^{-2/3} m_{pl}^{-1/3} \simeq 1 \text{ MeV}.
\]

The rate of any strong/EM charged-particle nuclear interactions will have a Coulomb barrier suppression factor,

\[
\exp[-2\tilde{A}^{1/3}(Z_1 Z_2)^{2/3}/(T/\text{MeV})^{1/3}],
\]

an overall nuclear cross section size, \( m^{-2} \simeq 2 \times 10^{-26} \text{ cm}^2 \), and a factor of \( \alpha_{EM}/\pi \) for an EM interaction. The controlling factor for freeze out is the exponential Coulomb barrier factor. And the Coulomb barrier is lowest for charge +1 on charge +1 and small \( \tilde{A} \equiv A_1 A_2/(A_1 + A_2) \). \( \tilde{A} \) is smallest for \( A_1 = 1 \) and \( A_2 = 2 \) \( (= 2/3) \) and \( A_1 = A_2 = 2 \) \( (= 1) \). To estimate a charged-particle nuclear rate, I have used \( d + d \rightarrow ^4\text{He} + \gamma \),

\[
\Gamma \sim \eta T^3 \frac{\alpha_{EM}/\pi}{m_{\pi}^2} \exp[-2(T/\text{MeV})^{-1/3}] .
\]

Comparing this rate to the expansion rate leads to Eq. (2).
Figure 2: Measurements of the Hubble constant from 1920 to 2000. All the major shifts involved adjustments to the distance scale. By 1970, most of the measurements were between 50 and 100 km/s/Mpc, but with unrealistically small error bars. The HST Key Project changed that with its 2000 determination, $H_0 = 72 \pm 2 \pm 6$ km/s/Mpc, with well quantified errors \[14\].
Figure 3: The BBN abundances calculated by Wagoner, Fowler and Hoyle [3]; while many of the cross sections and physics inputs have since been updated, the results are largely correct, especially when shown on a log-log plot. Note: their “little $h$” is not the same as mine; the rough correspondence to of their baryon mass density scale to $\eta$ is: $\eta \sim 10^{-12}/(\rho/10^{-33} \text{ g/cc})$. 

\[ \rho_b = h T_0^3 \text{gm cm}^{-3} \]

\[ T_0 = 3 \theta^* \text{K} \]
Figure 4: The network of nuclear reactions that leads to the production of $^4$He.

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**Figure 2A.—**Details of the reactions among the very light nuclei which are included in the calculation. All inverse reactions have also been included.
Figure 5: The complete network of nuclear reactions that leads to the BBN production of elements.