Group-Theoretical Quantization of 2+1 Gravity in the Metric-Torus Sector

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Abstract

A symmetry based quantization method of reparametrization invariant systems is described; it will work for all systems that possess complete sets of perennials whose Lie algebras close and which generate a sufficiently large symmetry groups. The construction leads to a quantum theory including a Hilbert space, a complete system of operator observables and a unitary time evolution. The method is applied to the 2+1 gravity. The paper is restricted to the metric-torus sector, zero cosmological constant Λ and it makes strong use of the so-called homogeneous gauge; the chosen algebra of perennials is that due to Martin. Two frequent problems are tackled. First, the Lie algebra of perennials does not generate a group of symmetries. The notion of group completion of a reparametrization invariant system is introduced so that the group does act; the group completion of the physical phase space of our model is shown to add only some limit points to it so that the ranges of observables are not unduly changed. Second, a relatively large number of relations between observables exists; they are transferred to the quantum theory by the well-known methods due to Kostant and Kirillov. In this way, a uniqueness of the physical representation of some extension of Martin’s algebra is shown. The Hamiltonian is defined by a systematic procedure due to Dirac; for the torus sector, the result coincides with that by Moncrief. The construction may be extensible to higher genera and non-zero Λ of the 2+1 gravity, because some complete sets of perennials are well-known and there are no obstructions to the closure of the algebra.
1 Introduction

The (2+1)-dimensional gravity is a popular model. It has been utilized as a laboratory for studies of specific problems in quantum gravity (for reviews, see [1], [2]). Our aim is to illustrate a symmetry based method of quantization of the so-called reparametrization invariant systems (RIS); a RIS is a constrained system whose dynamics is completely determined by its constraints (the value of the Hamiltonian is zero). For this purpose, 2+1 gravity seems particularly interesting, because it is a generally covariant RIS and the structure of its constraints is very similar to that of the general relativity.

The (technical) starting point of the method is a choice of a set of functions on the phase space that satisfy three conditions. First, they should have vanishing Poisson brackets with all constraints. Such quantities have been called “first-class” by Dirac [3] and they are often called “observables” today. The name “observables” is, however, not justified in the case of RIS’s. This difficulty, which is connected to the problem of time, has been first noticed by Kuchař [4], who has introduced a special name, perennials, for the first-class quantities of RIS’s. An attempt at a mathematical formulation of Kuchař’s ideas can be found in [5], where a construction of observables from perennials is described that necessarily makes use of some additional (time) structures (cf. Sec. 2). The second condition on the set is that it separates the classical solutions. This means that for two different solutions there is at least one function from the set that takes on different values at these solutions. This property is called completeness, it was first introduced by Bergmann [6] and it was later used by Ashtekar [7] as one of the basic properties within the algebraic quantization method. The third condition on the set is that it forms a closed algebra with respect to linear combinations and Poisson brackets. Thus, it will be a Lie algebra $g$.

However, availability of perennials in the general relativity, or even their existence, has been questioned [4]. Indeed, one can invent RIS’s that possess no perennials. However, these are mathematical constructs whose physics is strange: they cannot be reduced, even locally, and their physical phase spaces are not manifolds. It turn out that physically reasonable systems do possess complete systems of perennials (for proof, see [3]). True, perennials of a special kind, for example those that are linear in momenta or are local in certain sense are already shown to be absent in the general relativity [8], [9]. We shall show, however, how these proofs are transcended for the 2+1 gravity model (to which the proof methods of Kuchař, Anderson and Torre may be applicable). The third condition, that the perennials form a Lie algebra, might also be a source of problems; there are finite dimensional symplectic manifolds that do not admit a complete finite set of perennials whose algebra closes. However, it seems that such systems are again rather artificial with strange physics
and that they also would be very difficult to quantize by any existing method.

The complete algebra of perennials plays a twofold rôle in the theory. First, as already mentioned, observables can be constructed from them. We assume that these observables comprise the most important and directly measurable properties of the system—this condition, although a little vague one from the mathematical point of view, should influence the choice of the algebra $g$. Second, a group $G$ of symmetries is generated by $g$, if some conditions are satisfied. Such a group, if it exists, can strongly simplify the task of finding suitable representation of the algebra $g$ (that is, quantizing the system). The corresponding methods are those of the so-called group quantization (see [10] for a review); a modification of the methods suitable for the RIS’s has been suggested by Rovelli [11]. The existence of the group $G$ can however also help to solve the problem of time that afflicts the quantization of RIS’s. The idea of this use for $G$ stems from an old paper by Dirac [12]; in this paper, a time evolution has been constructed for a system of relativistic particles on the Minkowski spacetime. Dirac’s ideas have been extended to general RIS’s and developed to a coherent theory in [13] and [5]. Although already published, this theory is far from being well-known. We give, therefore, a short pedagogic exposition in Sec. 2 to make the paper self-contained.

The group $G$ of symmetries is obtained from the algebra $g$ in two steps. First, each Lie algebra determines a unique (simply connected) abstract Lie group $G$. Second, the Hamiltonian vector fields of the elements of the algebra determine an action of $G$ on the phase space, if the vector fields are complete—we shall meet a prominent example of incomplete Hamiltonian vector fields in the 2+1 gravity model. The action is then unique; let us call it Hamiltonian action of $G$. In some cases, the center of $G$ will contain a non-trivial subgroup $G_c$, whose elements act trivially. Two cases must be distinguished: isolated elements of $G$ in $G_c$ and Lie subgroups of $G$ in $G_c$. One can simplify $G$ by taking the quotient with respect to the isolated elements, but not with respect to the Lie part of $G_c$ [10].

The 2+1 gravity model can be considered as a dynamics of an ISO(1,2) affine connection of a three-dimensional manifold $M$ [14]. If some conditions are satisfied, then the affine connection defines a Lorentzian metric on $M$ and this property is preserved by the dynamics (we shall cut out the singularities!). We assume that Cauchy surface $\Sigma$ is the torus ($S^1 \times S^1$) and that the metric is well-defined; in such a case, we speak of the metric-torus sector. This sector alone gives a solvable but unexpectedly interesting model so that all the general points above can be illustrated in a rather non-trivial way. We shall limit ourselves to this sector and we will work in a particular gauge, the so-called homogeneous gauge, in which the three-metric depends only on the time coordinate. Let us remark that there is no reason why our method should not work for higher genera. One can try to use the loop variables
described in [15] or [16]; the topology of the physical phase spaces for all higher genera is contractible and there are global Darboux coordinates; thus, there are no obstructions to the existence of a complete system of functions with a closed algebra.

In the present paper, we will choose the complete set of perennials that has been published by Martin [13]. The perennials are directly related to the topological degrees of freedom of the system and can be expressed as a kind of Wilson loop variables. For the metric-torus sector, Moncrief observed that Martin’s perennials form the six-dimensional algebra $g$ isomorph to iso(1,2). We shall introduce new canonical coordinates that are adapted to Martin’s perennials; the constraint becomes formally the mass-shell condition for a rest-mass-zero relativistic particle. Thus, further four perennials can be immediately written down; together with the old ones, they form a ten-dimensional algebra so(2,3). The corresponding group $G$ is the conformal group of three-dimensional flat spacetime; $G$ is isomorph to SO(2,3).

The problem of existence of the Hamiltonian action for $G$ is non-trivial. Our calculation will reveal that only a subgroup $G_0 \subset G$ with the structure of SO(1,2)$\times\mathbb{R}$ acts on the phase space of the system. It is easy to observe, however, that the phase space can be extended so that the whole group $G$ has the Hamiltonian action on the extended space. We call such extensions group completions. A minimal group completion is unique under some quite general conditions. It turns out that the minimal group completion of the physical phase space consists, in our case, of adding “relatively few points” in such a way that the ranges of observables are not changed except for adding some boundary points to them.

The metric-torus sector has two degrees of freedom. Thus, a ten-dimensional algebra like $g$ will exhibit six independent relations. Using the Kostant-Kirillov method, we find that the group $G$ does not possess any physical representation. This has to do with Van Hove theorem: the physical phase space is too small and the algebra $g$ of functions is too large to be represented without deformation. We find, however, that one of its maximal subgroups, $G_1$, which is seven-dimensional, and isomorph to $(\text{SO}(1,2)\times\mathbb{R}) \otimes_S \mathbb{R}^3$, possesses a unique physical representation and we calculate the form of the operators representing $g_1$. Here, “$\otimes_S$” denotes the semidirect product of groups. Three independent classical relations for $G_1$ can be written down in terms of (generalized) Casimir operators.

For the construction of time evolution à la Dirac, we can use only the four-dimensional subgroup $G_0$; then the Hamiltonian action of $G_0$ on the constraint surface (and so on the classical spacetimes) provides the interpretation of the corresponding unitary transformations in the Hilbert space. It turns out that this action “goes in time direction” and so a time evolution can be constructed. The candidates for the Hamiltonian that generates everywhere time evolution towards future form a three-dimensional family. Most of these operators are unbounded from below. Thus,
the condition that an operator generates time evolution towards the future does not necessarily guarantee that the operator has a non-negative spectrum. However, there is exactly one Hamiltonian that is bounded from below (and it is even non-negative). The positive Hamiltonian coincides with the Hamiltonian written down by Moncrief [23] and the time coincides with the constant mean external curvature.

Some surprise is that the quantum mechanics we have constructed is not equivalent to the “ordinary” quantum mechanics of the rest-mass-zero free particle in three dimensional Minkowski spacetime, in spite of the fact that the algebra of observables is so(2,3) like for the particle, and that we managed to introduce new variables in which the constraint coincides with the ordinary mass-shell condition for such a particle. The explanation is that the global structure of the torus configuration space is very different from that of the particle (which is the three-dimensional Minkowski spacetime): the former is only a subset of the latter, namely the inside of the light cone of the origin. The points outside of the light cone correspond to timelike two-surfaces evolving in a spacelike direction (the signature of the spacetime remains +1).

The plan of the paper is as follows. In Sec. 2, we briefly describe the mathematical apparatus that has turned out advantageous for the study of RIS’s, their observable properties and their time evolution. No detail and proofs are given, because these can be found in already published papers [13] and [3]. In Sec. 3, we list our starting assumptions and equations concerning the 2+1 gravity model. They are mostly taken over from [22] and [23], where more detail can be found. One non-trivial but plausible assumption is that the so-called homogeneous gauge can be chosen in which the model becomes finite-dimensional ([22], [24]).

In Sec. 4, we study the problem of action of the group, define the group completion of a RIS and the weak action of the group. We derive the group completion of the model and prove that the group completed physical phase space contains the original one as an open dense subset. We define the action of $G$ on the completed physical phase space, give the form of all observables obtained from the algebra $\mathfrak{g}$ on the physical phase space and list all independent relations.

Sec. 5 describes an application of Dirac’s time evolution idea to our model. We find that the dynamics is much more unique than in the case of free relativistic particle studied by Dirac (who found three inequivalent “forms of relativistic dynamics”: we find only one “form”. Finally, in Sec. 6, after a brief description of the Kostant-Kirillov method, we derive the physical representation.
2 Example: the relativistic particle

We consider a free relativistic particle of mass \( m > 0 \) on the four-dimensional Minkowski spacetime with coordinates \( x^\mu \) and the metric \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \). This is one of the systems studied originally by Dirac [12]. Let the conjugate momenta be \( p_\mu \) and the constraint be \( \mathcal{H} = (1/2)(p \cdot p + m^2) \). We shall use the abbreviation \( A \cdot B := \eta_{\mu\nu}A^\mu B^\nu \) throughout the paper.

The manifold \( \mathbb{R}^8 \) with the coordinates \( x^\mu \) and \( p_\mu \), and with the symplectic form \( \tilde{\Omega} = dp_\mu \wedge dx^\mu \) will be called extended phase space and denoted by \( \tilde{\Gamma} \). The submanifold \( \Gamma \) of \( \tilde{\Gamma} \) defined by \( \mathcal{H} = 0 \) and \( p_0 < 0 \) is the constraint surface. The function \( \mathcal{H} \) is the so-called super-Hamiltonian. It defines \( \Gamma \), generates reparametrizations along the particle trajectories and generates the dynamics:

\[
\dot{x}^\mu = \mathcal{N}\{x^\mu, \mathcal{H}\}, \quad \dot{p}_\mu = \mathcal{N}\{p_\mu, \mathcal{H}\},
\]

where \( \mathcal{N} \) is an arbitrary function, the so-called lapse. The arcs that are determined by maximal solutions of the equations \eqref{eq:example} will be called c-orbits. A c-orbit will be typically denoted by \( \gamma \). c-orbits represent classical solutions. The quotient space \( \bar{\Gamma} = \Gamma/\gamma \) will be called physical phase space; we will assume that \( \bar{\Gamma} \) is a manifold. In this case, there is a symplectic form \( \bar{\Omega} \) on \( \bar{\Gamma} \) that is uniquely determined by \( \tilde{\Omega} \). In our example, \( \Gamma \) is seven-dimensional, c-orbits are one-dimensional, so \( \bar{\Gamma} \) is six-dimensional. The dimension of the physical space is the double of the physical degrees of freedom.

An important notion is that of transversal surface. This is any submanifold \( \Gamma_i \) of \( \Gamma \) such that each c-orbit intersects \( \Gamma_i \) exactly once and in a transversal direction (any vector at an intersection point that is simultaneously tangential to the c-orbit and to the transversal surface is necessarily the zero vector). The importance of transversal surfaces for the description of time evolution in relativistic theories has been recognized by Dirac [12]. Each transversal surface \( \Gamma_i \) carries a unique symplectic form \( \Omega_i \), the pull-back of \( \Omega \) to \( \Gamma_i \). The symplectic space \((\Gamma_i, \Omega_i)\) can be identified with \((\bar{\Gamma}, \bar{\Omega})\) by the map that sends each c-orbit \( \gamma \) from \( \Gamma \) to the intersection point of \( \gamma \) with \( \Gamma_i \). An example of transversal surface, which we denote by \( \Gamma_0 \), is given by the equations \( x^0 = 0 \) and \( \mathcal{H} = 0 \). As coordinates on \( \Gamma_0 \), the functions \( x^1, x^2, x^3, p_1, p_2 \) and \( p_3 \) can be chosen; in these coordinates, \( \Omega_0 = dp_k \wedge dx^k \). As we can see, \( \Gamma_0 \) defines a particular time instant, namely \( x^0 = 0 \). This is a general property of transversal surfaces; for example, in the general relativity, a transversal surface defines a unique Cauchy surface in any generic spacetime solution (ie. the solution that admits no symmetry).

The Poincaré group (in fact, only the component of identity thereof) \( \text{ISO}(1,3) \) acts on \( \bar{\Gamma} \) in the usual way, leaving both \( \bar{\Omega} \) and \( \Gamma \) invariant. Such transformations in \( \bar{\Gamma} \) are called symmetries. The action is generated by ten functions \( p_\mu, J_k := \epsilon_{klm}x lp_m \).
and $K_k := x_k p_0 - x_0 p_k$, $k = 1, 2, 3$ (the indices are lowered and raised by the metric). The functions have vanishing Poisson brackets with $H$ (as $H$ is an invariant), so they are perennials. They form a Lie algebra that is isomorph to iso(1,3) with respect to linear combinations and Poisson brackets, and the set separates c-orbits. Thus, we have an example of a complete Lie algebra $\mathfrak{g}$ of perennials.

In this case, the group that is determined by $\mathfrak{g}$ is $\text{Sl}(2, \mathbb{C}) \otimes_\mathbb{R} \mathbb{R}^4$. The Hamiltonian action of the group is just the above action of ISO(1,3): The center of $\text{Sl}(2, \mathbb{C})$ contains only one element that is different from identity, which acts trivially, so we can restrict ourselves to the group ISO(1,3).

Important general properties of symmetries are (see [11] and [13]): a) a symmetry sends c-orbits onto c-orbits and b) it sends transversal surfaces onto transversal surfaces. The property b) is crucial to the construction of dynamics à la Dirac [12]: by the transformations of the group $G$, a time instant is sent into another time instant. Dirac also proposed to choose maximally symmetric (with respect to $G$) transversal surfaces. Such a choice not only minimizes the number of “different time instants”, but also simplifies the operator observables [12]. There are three inequivalent maximally symmetric transversal surfaces for the relativistic particle: a) the spacelike plane $x^0 = 0$, b) the pair of hyperboloids $x \cdot x = 1$ and c) the null plane $x^0 - x^1 = 0$. This leads to Dirac’s “three forms of dynamics”.

At the first sight, it seems that perennials are observables. Thus, $p_k$ are the momenta and $J_k$ are the angular momenta of the particle. Such an association is, however, more difficult for $-p_0$ and $K_k$. It is clearly not reasonable to put $-p_0$ equal to the energy, because only $\sqrt{p^2 + m^2}$ will lead to the observed spectrum of the particle. Of course, $\sqrt{p^2 + m^2}$ is the value of $-p_0$ at the constraint surface $\Gamma$. More problems are encountered, if we try to interpret $K_k$ as an observable. We can utilize $p_\mu$ together with $K_k$ to form the functions

$$X^k_t := \frac{K_k}{p_0} + \frac{p_k}{p_0}$$

for any constant $t$: as functions of perennials, $X^k_t$ are themselves perennials. They can be interpreted as coordinates of the particle at the time $x^0 = t$. It turns out, that this is quite general way of forming observables from perennials; for it to work, some “time structure” is necessary. Here, we have used the (standard) family of transversal surfaces $\Gamma_t$ defined by the equations $x^0 = t$, as well as the family of maps generated by $p_0$ and sending $\Gamma_t$ to $\Gamma_{t'}$ for each pair $(t, t')$. One easily verifies that

$$\frac{\partial X^k_t}{\partial t} + \{X^k_t, p_0\} = 0.$$

The class $\{X^k_t | t \in \mathbb{R}\}$ can, therefore, be considered as a Heisenberg observable, and so an observable turns out to be a class of perennials. The apparently trivial
relation between perennials and observables for the particular perennials $p_k$ and $J_k$ in this case (the functions in the class are $t$-independent and coincide in form with the original perennials) is due to the fact that these perennials are connected in a special way to the standard time structure (represented by $\Gamma_t$ and $p_0$) that is determined in the phase space by an inertial frame: $p_k$ and $J_k$ have vanishing Poisson brackets with $p_0$ and they leave (via Poisson bracket) each time instant $\Gamma_t$ invariant. A systematic theory is given in [5].

An important technical tool for the present paper will be projection to a transversal surface. One can project perennials and symmetries. The projection $o_1$ of a perennial $o$ to the transversal surface $\Gamma_1$ is a function on $\Gamma_1$ defined as the pullback of $o$ to $\Gamma_1$: $o_1 := o|_{\Gamma_1}$. This projection preserves three operations: linear combinations, multiplications of functions and Poisson brackets (of perennials). In particular, for the Poisson brackets, we have: $$\{o_1, o'_1\}_1 = \{o, o'\},$$ where $\{\cdot, \cdot\}_1$ is the Poisson bracket in $(\Gamma_1, \Omega_1)$, $\{\cdot, \cdot\}$ that of $(\tilde{\Gamma}, \tilde{\Omega})$, $o_1$ the projection of $o$ and $o'_1$ that of $o'$ to $\Gamma_1$ (for the proof, see [13]). Projections of perennials to a given transversal surface can (sometimes!) be considered as Schrödinger observables (for more detail, see [5]).

Similarly to perennials, symmetries can also be projected. Let $\phi$ be a symmetry and $p \in \Gamma_1$. Then, the projection $\phi_1 : \Gamma_1 \to \Gamma_1$ of $\phi$ to $\Gamma_1$ is defined by $\{\phi_1(p)\} := \gamma_{\phi(p)} \cap \Gamma_1$, where $\{a\}$ is the set with the element $a$ and $\gamma_q$ is the c-orbit through the point $q$. One can show that $\phi_1$ preserves $\Omega_1$ as well as the group multiplication of symmetries; thus, the projection of a group of symmetries is a group of symplectic maps [3]. Moreover, if $o$ generates a one-dimensional group of symmetries $\phi_t$, then $o_1$ generates a one-dimensional group of symplectic maps $\psi_t$ and it holds that $\psi_t = \phi_{t1}$ for all $t$, where $\phi_{t1}$ is the projection of $\phi_t$ to $\Gamma_1$ [3].

Finally, let $\Gamma_1$ and $\Gamma_2$ be two transversal surfaces, $p \in \Gamma_1$ and $\rho : \Gamma_1 \to \Gamma_2$ defined by $\{\rho(p)\} := \gamma_p \cap \Gamma_2$. Then, the pull-back of $\Omega_2$ by $\rho$ is $\Omega_1$ and the pull-back of $o_2$ is $o_1$ for any perennial $o$ with projection $o_1$ to $\Gamma_1$ and $o_2$ to $\Gamma_2$. Thus, $\rho$ realizes the equivalence of all $\Gamma_i$'s [3].

Another important notion that we can illustrate with the relativistic particle model is that of relation. The projections of the ten perennials $p_\mu$, $J_k$ and $K_k$ to the transversal surface $\Gamma_0$ are linearly independent. However, there must be four functional relations between them, because $\Gamma_0$ is only six-dimensional. There cannot be more independent relations, as the functions form a complete system. The relations can be written as

$$p^2 = -m^2, \quad \epsilon^{\mu\nu\rho\sigma} p_\mu J_{\rho\sigma} = 0,$$

where $J_{\rho\sigma} := \epsilon_{\rho\sigma\mu\nu} x^\mu p^\nu$. These relations have a close connection to the values of Casimir elements for this particular representation of the Poincaré group. Pohlmayer [17] has discussed the general case.
In the quantum version, we shall try to preserve these relations so that they become similar relations between operators (that may be deformed: some additional terms proportional to $\hbar$ may appear). There also can be spectral conditions like $p_0 < 0$, that should be satisfied in the quantum theory. A unitary representation of the group $G$ that preserves the relations between its generators and that satisfies the spectral condition is called physical representation.

3 The homogeneous gauge

In this section, we return to the 2+1 gravity and briefly summarize our starting assumptions and equations. More detail can be found in [22], [23]. We shall consider only the metric-torus sector of the 2+1 gravity system. In arbitrary coordinates $x^1$ and $x^2$ on an arbitrary spacelike surfaces $t = \text{const}$ with the manifold structure $\Sigma$, the metric of the spacetime $\mathcal{M}$ has the form

$$ds^2 = -N^2 dt^2 + g_{ab}(dx^a - N^a dt)(dx^b - N^b dt)$$

and the ADM action for the model reads

$$S = \int_{\mathbb{R}} dt \int_{\Sigma} d^2 x \left( \pi_{ab} \frac{\partial g_{ab}}{\partial t} - N \mathcal{H} - N^a \mathcal{H}_a \right),$$

where

$$\mathcal{H} = -\frac{1}{\sqrt{g}} [\pi_{ab} \pi_{ab} - (\pi^a)^2] + \sqrt{g} \mathcal{R},$$

$$\mathcal{H}_a = -2 \nabla_b \pi_{a}^b,$$

$\nabla_a$ is the covariant derivative associated with the metric $g_{ab}$ and $\mathcal{R}$ is the curvature scalar of the two-surfaces $t = \text{const}$.

The analysis of the model simplifies enormously, if one can choose the so-called “homogeneous gauge”. It is the choice of coordinates such that the fields $g_{ab}$ and $\pi_{ab}$ are independent of $x^1$ and $x^2$ [24], [23]. A rigorous proof that such coordinates exist for each classical solution of the model has not yet been published. If each solution, however, admits at least one Cauchy surface of constant mean curvature (CMC), which seems to be a very plausible conjecture, then the existence can be shown [22]. The existence of the CMC surface has been proved by L. Andersson for genera higher than 1 and there are some ideas even for genus 1 [20]. We will assume, that there is such a Cauchy surface; in any case, one can consider this as a part of the definition of the system, and if the conjecture is invalid, then this system will not be completely equivalent to the 2+1 gravity.

In the homogeneous gauge, the metric can be taken in the form [23]:

$$ds^2 = -N(t)^2 dt^2 + e^{2\rho(t)} (dx^1)^2 + e^{2\rho(t)} (dx^2 + \beta(t) dx^1)^2. \quad (2)$$
A straightforward calculation then leads to the action

\[ S = \int dt \left( p_\mu \dot{\mu} + p_\nu \dot{\nu} + p_\beta \dot{\beta} - N\mathcal{H} \right), \]

where

\[ \mathcal{H} = \frac{1}{2} (e^{-\mu - \nu} p_\mu p_\nu - e^{\mu - 3\nu} p_\beta^2). \]

Moncrief then performs two canonical transformations; the first is:

\[ q_1 = \nu - \mu, \quad q_2 = \beta, \quad q_3 = \nu + \mu, \]

\[ p_\mu = -p_1 + p_3, \quad p_\nu = p_1 + p_3, \quad p_\beta = p_2, \]

so that the super-Hamiltonian becomes

\[ \mathcal{H} = \frac{1}{2} e^{-q_3} (p_3^2 - p_1^2 - e^{-2q_3} p_2^2). \]

These coordinates are advantageous for visualisation of the geometry of the system. The extended phase space is \( \tilde{\Gamma} = T^* \mathbb{R}^3 \) with the canonical coordinates \( q^1, q^2, q^3, p_1, p_2 \) and \( p_3 \); the meaning of \( q^3 \) and \( p_3 \) is

\[ q^3 = \log \sqrt{g}, \quad p_3 = \frac{\sqrt{g}}{N} \frac{\partial q^3}{\partial t}. \]

The constraint surface \( \Gamma \) is the light cone in the momentum space:

\[ p_3^2 - p_1^2 - e^{-2q_3} p_2^2 = 0. \]

The \( p_3 > 0 \) half of the cone represents expanding, the \( p_3 < 0 \) half contracting, and the cusp \( p_3 = 0 \) represents the static tori—solutions with higher symmetry. We shall adhere to the convention that the coordinate \( t \) on the tori spacetimes is future oriented for \( N > 0 \).

The structure of the constraint surface \( \Gamma \) and of the physical phase space \( \tilde{\Gamma} = \Gamma / \gamma \) is spoilt by the points of higher symmetry [21]. The c-orbits with \( p_3 \neq 0 \) are curves, those with \( p_3 = 0 \) are just points. As the static solutions form a set of measure zero, we can cut them away. Thus, we consider only that part of the system that satisfies the condition

\[ p_3 \neq 0. \]

The new system will have a disconnected phase space \( \tilde{\Gamma}' = \tilde{\Gamma}'_+ \cup \tilde{\Gamma}'_-- \), a constraint surface \( \Gamma' = \Gamma'_+ \cup \Gamma'_- \) and a physical phase space \( \tilde{\Gamma}' = \tilde{\Gamma}'_+ \cup \tilde{\Gamma}'_- \), where \( \tilde{\Gamma}'_+ := \{ x \in \tilde{\Gamma} \mid p_3 > 0 \} \), \( \Gamma'_+ := \Gamma \cap \tilde{\Gamma}'_+ \), and \( \Gamma'_- := \Gamma'_-/\gamma \) (similarly for \( p_3 < 0 \)).

After this truncation, Moncrief performs the second transformation:

\[ T = \ln |p_3| - q^3, \quad p_T = -p_3, \]
the other variables remaining the same; the super-Hamiltonian then reads
\[ H = -\frac{e^T}{2p_T}(p_T^2 - p_1^2 - e^{-2q_1}p_2^2). \]  
(3)

The meaning of the variable \( T \) is given by
\[ g^{ab}K_{ab} = \frac{1}{2N}g^{ab}\frac{\partial g_{ab}}{\partial t} = e\epsilon^T, \]
where \( \epsilon = \pm 1 \) and the sign is determined by \( \epsilon = \text{sign} p_3 = -\text{sign} p_T \). Thus, \( \epsilon \) is just the sign of the CMC of the surface \( t = \text{const}. \)

Martin’s \([15]\) constants of motion (perennials) are in these coordinates given by \([23]\):
\[
C_1 = -\frac{\epsilon}{2}e^T\{(e^{-q_1} + (q^2)e^{q_1})(p_T + p_1) - 2e^{-q_1}p_1 + 2q^2e^{-q_1}p_2\}, \quad (4) \\
C_2 = -\frac{\epsilon}{2}e^T[ e^{q_1}(p_T + p_1)], \quad (5) \\
C_3 = -\frac{\epsilon}{2}e^T[ e^{q_1}(p_T + p_1) + e^{-q_1}p_2], \quad (6) \\
C_4 = \frac{1}{2}\{[e^{-2q_1} - (q^2)^2]p_2 + 2q^2p_1\}, \quad (7) \\
C_5 = \frac{1}{2}p_2, \quad (8) \\
C_6 = p_1 - q^2p_2. \quad (9)
\]

They can be expressed as a kind of loop integrals by means of the original fields \( g_{ab}(x) \) and \( \pi^{ab}(x) \) \([13]\).

Moncrief observed that the Poisson brackets of the variables \( P_\mu \) and \( J^\mu \) defined by
\[
P_0 := \frac{1}{2}(C_1 + C_2), \quad P_1 := \frac{1}{2}(C_1 - C_2), \quad P_2 := C_3, \quad (10)
\]
and
\[
J^0 := -C_4 + C_5, \quad J^1 := -C_4 - C_5, \quad J^2 = -C_6,
\]
form a Lie algebra isomorphic to \( \text{iso}(1,2) \): if we introduce the abbreviations
\[
A := A^\mu P_\mu, \quad C := C_\mu J^\mu,
\]
then
\[
\{A, A'\} = 0, \quad \{A, C\} = (\varepsilon^{\rho\mu\nu} A_\mu C_\nu)P_\rho, \quad \{C, C'\} = (\varepsilon_{\rho\mu\nu}C^\mu C^\nu)J^\rho. \quad (11)
\]

Here, we raise and lower the indices of \( X^\mu \) and \( P_\mu \) by the Minkowskian three-metric \( \text{diag}(-1,1,1) \), \( \varepsilon^{\rho\mu\nu} \) and \( \varepsilon_{\rho\mu\nu} \) are the usual antisymmetric symbols (\( \varepsilon_{\rho\mu\nu} \) is not \( \varepsilon^{\rho\mu\nu} \) with lowered indices).
The formulas (4)–(9) imply the following four equations:

\[ C_1 C_2 - C_3^2 = \frac{1}{4} e^{2T} (p_T^2 - p_1^2 - e^{-2q_1^1} p_2^2), \]  
(12)

\[ \{ C_4, H' \} = \{ C_5, H' \} = \{ C_6, H' \} = 0, \]

where

\[ H' = p_T^2 - p_1^2 - e^{-2q_1^1} p_2^2. \]

Thus, all \( C \)'s are perennials.

There also are some discrete symmetries that originate from non-uniqueness of the metric (2) for a given torus geometry (class group transformations, see e.g. [25]). We shall not discuss the question whether these transformations are to be considered as symmetries or as gauge transformations. If \( X_1 \) and \( X_2 \) form a basis of a lattice in \( \mathbf{E}^2 \) defining the torus, then the metric has the form

\[ g_{11} = X_1 \cdot X_1, \quad g_{12} = X_1 \cdot X_2, \quad g_{22} = X_2 \cdot X_2, \]

where \( a \cdot b \) denotes the scalar product of the vectors \( a \) and \( b \) in \( \mathbf{E}^3 \). The following two transformations of the basis generate the whole group of the discrete transformations ("large diffeomorphisms"):

\[ X'_1 = X_2, \quad X'_2 = X_1, \]

so that

\[ g'_{11} = g_{22}, \quad g'_{12} = g_{12}, \quad g'_{22} = g_{11}, \]  
(13)

and

\[ X'_1 = X_1, \quad X'_2 = X_1 + X_2, \]

so that

\[ g'_{11} = g_{11}, \quad g'_{12} = g_{11} + g_{12}, \quad g'_{22} = g_{11} + 2g_{12} + g_{22}, \]  
(14)

Using Eq. (2), we obtain from Eq. (13) for \( \mu, \nu \) and \( \beta \):

\[ e^{2\mu'} = \frac{e^{2(\mu+\nu)}}{e^{2\mu} + \beta^2 e^{2\nu}}, \quad e^{2\nu'} = e^{2\mu} + \beta^2 e^{2\nu}, \]

\[ \beta' = \beta \frac{e^{2\nu}}{e^{2\mu} + \beta^2 e^{2\nu}}. \]

The corresponding transformation of \( q^1, q^2 \) and \( q^3 \) is

\[ e^{q_1'} = e^{q_1 \left[ (q^2)^2 + e^{-2q_1^1} \right]}, \]  
(15)

\[ q^2' = \frac{q^2}{(q^2)^2 + e^{-2q_1^1}}, \]  
(16)

\[ q^3' = q^3. \]  
(17)
This gives for the momenta
\[ p'_1 = \frac{(q^2)^2 - e^{-2q^1}}{(q^2)^2 + e^{-2q^1}} p_1 + \frac{2q^2 e^{-2q^1}}{(q^2)^2 + e^{-2q^1}} p_2, \]
\[ p'_2 = 2q^2 p_1 - [(q^2)^2 - e^{-2q^1}] p_2. \]

One can then easily verify that
\[ p'_1^2 + e^{-2q^1} p'_2^2 = p_1^2 + e^{-2q^1} p_2^2, \]
so that the super-Hamiltonian (3) is invariant.

The transformation (14) gives
\[ \mu' = \mu, \quad \nu' = \nu, \quad \beta' = 1 + \beta. \]

Thus,
\[ q'^1 = q^1, \quad q'^2 = 1 + q^2, \quad q'^3 = q^3, \]
and
\[ p'_1 = p_1, \quad p'_2 = p_2, \quad p'_3 = p_3. \]

Again, \( \mathcal{H} \) is conserved.

The transformation (13)–(17) implies
\[ C'_1 = C_2, \quad C'_2 = C_1, \quad C'_3 = C_3, \]
and (18) implies
\[ C'_1 = C_1 + C_2 + 2C_3, \quad C'_2 = C_2, \quad C'_3 = C_2 + C_3. \]

These are both integral transformations with determinant 1 as one expects for loop variables, if the loops are just permuted or linearly combined.

4 Group comletion of the phase space

4.1 Completion by ISO(1,2)

In this section, we will investigate which transformations are generated by the perennials \( P_\mu \) and \( J^\mu \) in the phase space of the system. This task will be simplified, if we use coordinates that are adapted to the perennials in the following sense. \( C_1, C_2 \) and \( C_3 \) action via Poisson brackets in the phase space can be projected to the configuration space spanned by \( T, q^1 \) and \( q^2 \), and the projections are the vector fields \( \hat{C}_1, \hat{C}_2 \) and \( \hat{C}_3 \) given by replacing \( p_T, p_1 \) and \( p_2 \) by \( \partial/\partial T, \partial/\partial q^1 \) and \( \partial/\partial q^2 \) in the expressions (4)–(6). The perennials \( C_1, C_2 \) and \( C_3 \) have vanishing Poisson
brackets with each other, so \( \hat{C}_1, \hat{C}_2 \) and \( \hat{C}_3 \) will define a holonomous frame; the corresponding coordinates are the desired ones. Let us first simplify these vectors by the transformation

\[
\begin{align*}
    u &= T, \quad v = q^1 - T, \quad y = q^2,
  
\end{align*}
\]

so that

\[
\begin{align*}
    \hat{C}_1 &= -\frac{\epsilon}{2} e^{-v} [ (1 + y^2 e^{2u+2v}) \partial_u + 2y \partial_y - 2 \partial_v ], \\
    \hat{C}_2 &= -\frac{\epsilon}{2} e^{2u+v} \partial_u, \\
    \hat{C}_3 &= -\frac{\epsilon}{2} e^{2u+v} (y \partial_u + e^{-2u-2v} \partial_y).
\end{align*}
\]

Now, we look for pairs of independent functions that are annihilated by each of these differential operators. The method of characteristics suggests that we study integral curves of the vector fields. The integral curve of \( \hat{C}_1 \) is defined by:

\[
\begin{align*}
    \dot{u} &= A(1 + y^2 e^{2u+2v}), \quad \dot{y} = A(2y), \quad \dot{v} = A(-2),
  
\end{align*}
\]

where \( A = -\epsilon e^{-v}/2 \). Thus,

\[
\begin{align*}
    \frac{1}{y} \dot{y} + \dot{v} &= 0,
    
    -2e^{-2u-v} \dot{u} + 2ye^v \dot{y} + ( -e^{-2u-v} + y^2 e^v ) \dot{v} &= 0,
\end{align*}
\]

and we have:

\[
\begin{align*}
    \hat{C}_1 (y e^v) &= 0, \\
    \hat{C}_1 (e^{-2u-v} + y^2 e^v) &= 0.
\end{align*}
\]

The integral curve of \( \hat{C}_1 \) satisfies

\[
\begin{align*}
    \dot{u} &= -\frac{1}{2} e^{2u+v}, \quad \dot{y} = 0, \quad \dot{v} = 0,
  
\end{align*}
\]

hence,

\[
\begin{align*}
    \hat{C}_2 v &= 0, \\
    \hat{C}_2 y &= 0.
\end{align*}
\]

Similarly for \( \hat{C}_3 \):

\[
\begin{align*}
    \dot{u} &= By, \quad \dot{y} = Be^{-2u-2v}, \quad \dot{v} = 0,
  
\end{align*}
\]

where \( B = -(\epsilon/2)e^{2u+v} \). Thus,

\[
\begin{align*}
    -e^{-2u-2v} \dot{u} + y \dot{y} &= 0,
\end{align*}
\]

and

\[
\begin{align*}
    \hat{C}_3 v &= 0, \\
    \hat{C}_3 (e^{-2u-2v} + y^2) &= 0.
\end{align*}
\]

The pair of independent functions we have found for each vector field determines all functions that are annihilated by the field. We can easily find three independent functions such that each vector field annihilates exactly two of them. The results can be summarized in the following table.
\[ \begin{array}{c|cc}
\hat{C}_1 & ye^v, & e^{-2u-v} + y^2e^v, \\
\hat{C}_2 & v, & ye^v, \\
\hat{C}_3 & v, & e^{-2u-v} + y^2e^v. 
\end{array} \]

Hence, the following transformation will simplify the vector fields:

\[ \xi = e^{-2u-v} + y^2e^v, \quad \eta = v, \quad \zeta = ye^v. \]

Indeed, we obtain that

\[ \hat{C}_1 = \epsilon e^{-\eta} \partial_\eta, \quad \hat{C}_2 = \epsilon \partial_\xi, \quad \hat{C}_3 = -\frac{\epsilon}{2} \partial_\zeta. \]

Finally, the transformation

\[ X^0 = \epsilon(e^\eta + \xi), \quad X^1 = \epsilon(e^\eta - \xi), \quad X^2 = -2\epsilon\zeta, \]

gives

\[ \dot{P}_\rho = \frac{\partial}{\partial X^\rho}, \quad \rho = 0, 1, 2. \]

Composing all the transformations, we express \( X^\rho \) by means of the original variables \( T, q^1 \) and \( q^2 \):

\[ \begin{align*}
X^0 &= \epsilon e^{-T}[e^{q^1} + e^{-q^1} + (q^2)^2 e^{q^1}], \\
X^1 &= \epsilon e^{-T}[e^{q^1} - e^{-q^1} - (q^2)^2 e^{q^1}], \\
X^2 &= \epsilon e^{-T}[-2q^2 e^{q^1}].
\end{align*} \quad (19) \]

The class group transformations in terms of the coordinates \( X \) read:

\[ \begin{align*}
X^{\prime 0} &= (3/2)X^0 + (1/2)X^1 - X^2, \\
X^{\prime 1} &= -(1/2)X^0 + (1/2)X^1 + X^2, \\
X^{\prime 2} &= -X^0 - X^1 + X^2,
\end{align*} \]

which is an element of \( \text{SO}(1,2) \), and

\[ X^{\prime 0} = X^0, \quad X^{\prime 1} = -X^1, \quad X^{\prime 2} = X^2. \]

The relations (19)–(21) together with (14)–(16) and (10) define a symplectic embedding \( \iota : \tilde{\Gamma}' \rightarrow T^*M^3 \) of the phase space \( \tilde{\Gamma}' \) of our system, each component of which is spanned by the coordinates \( T, q^1, q^2, p_t, p_1 \) and \( p_2 \), into the cotangent bundle \( T^*M^3 \) of the three-dimensional Minkowski space \( M^3 \) with the coordinates \( (X^\mu, P_\mu) \). A very important point is that the image \( \iota(\tilde{\Gamma}') \) is only a proper subset of \( T^*M^3 \), namely the cotangent bundle of the inside of the light cone of the origin. Indeed, calculating \( X \cdot X \) from (19)–(21), we obtain the identity

\[ X \cdot X = -4e^{-2T} = -\frac{4}{\tau^2}. \]
At the points of the light cone, the CMC is infinite; this surface represents the singularity of the torus dynamics. For the CMC $\tau$, we obtain

$$\tau = \frac{2\epsilon}{\sqrt{-X \cdot X}}. \quad (22)$$

From Eq. (19), it follows that $\epsilon X^0 > 0$. Thus $\tilde{\Gamma}'_+ (\tilde{\Gamma}'_-)$ is mapped on the inside of the future (past) light cone. Moreover, Eqs. (4)–(6) and (10) yield

$$P_0 = \frac{\epsilon e^T}{4} (V_T p_T - V_1 p_1 - e^{-2q^1} V_2 p_2),$$

where

$$V_T = e^{q^1} + e^{-q^1} + (q^2)^2 e^{q^1}, \quad V_1 = e^{q^1} - e^{-q^1} + (q^2)^2 e^{q^1}, \quad V_2 = 2q^2 e^{q^1}.$$

we easily verify the identity:

$$-V_T^2 + V_1^2 + e^{-2q^1} V_2^2 = -4.$$

Thus, $(V_T, V_1, V_2)$ is a “timelike vector” oriented towards future ($V_T > 0$) and $(p_T, p_1, p_2)$ is a “null vector” (at the constraint surface, see (3)). Their “scalar product” $-V_T p_T + V_1 p_1 + e^{-2q^1} V_2 p_2$ is, therefore, negative (positive) if $p_T > 0$ ($p_T < 0$). As the sign of $p_T$ and of $\epsilon$ are correlated, it follows that

$$P_0 < 0$$

everywhere at the constraint surface. This, together with the Eqs. (10) and (12) imply that the points of the constraint surface satisfy the conditions

$$P \cdot P = 0, \quad P_0 < 0 \quad (23)$$

with respect to the new variables $(X^\mu, P_\mu)$. Let us denote by $\mathcal{P}$ the set of points in the momentum space with the coordinates $P_\mu$ that satisfy Eq. (23).

The transformation inverse to (19)–(21) is well-defined only inside the light cone, and can be written with respect to the coordinates as follows

$$e^T = \frac{2}{\sqrt{-X \cdot X}}, \quad e^{q^1} = \epsilon \frac{X^0 + X^1}{\sqrt{-X \cdot X}}, \quad q^2 = -\frac{X^2}{X^0 + X^1}.$$ 

As $X^0 + X^1$ is positive (negative) inside the future (past) half cone, $e^{q^1}$ will be always positive. If we try to extend the transformation to the whole of $\mathbb{M}^3$, then we discover the meaning of the points lying outside the light cone. A simple calculation leads to the following complex transformation of the original variables:

$$t = it', \quad \nu = \nu' + \frac{i\pi}{2},$$
\( \mu \) and \( \beta \) remaining unchanged, where \( t' \) and \( \nu' \) are real. Then, all new momenta are real, but
\[
e^{2T} = -p_3^2 e^{-2\mu-2\nu'}, \quad e^{2q_1} = -e^{-2\mu+2\nu'},
\]
so that \( e^{2T} \) and \( e^{2q_1} \) become negative as necessary. Thus, we obtain spacetimes with the metric
\[
d s^2 = N^2(t') dt'^2 + e^{2\mu t'} (dx^1)^2 - e^{2\nu'(t')} [dx^2 + \beta(t') dx^1]^2,
\]
which have the Lorentz signature, but are acausal. A complete null geodesic crossing from the inside to the outside of the light cone represents an analytic three-dimensional spacetime analogous to the Taub-NUT solution (see e.g. [26]) with a Cauchy horizon and the Taub-NUT-like incompleteness at the cross point, if the singularity is viewed as a lightlike torus.

Martin’s perennials are push-forwarded by \( \iota \) just into the usual generators of the Poincaré group in the three-dimensional Minkowski spacetime \( M^3 \): \( P_\rho \) and \( J^{\rho} = \varepsilon^{\rho \mu \nu} X_\mu P_\nu \). The corresponding group action is not transitive on \( T^* M^3 \): the orbits are classified by the well-known invariants \( P \cdot P \) and sign \( P_0 \), if \( P \cdot P \leq 0 \). However, each orbit of the group \( \text{ISO}(1,2) \) intersects \( \iota(\tilde{\Gamma}') \), so no superfluous orbit has been added. Inside of the light cone, only the subgroup \( \text{SO}(1,2) \) acts; \( P \)'s do not define any group action on \( \iota(\tilde{\Gamma}') \), because the corresponding vector fields are badly incomplete there.

We can interpret our construction as follows.

The system of six Martin’s functions form a complete algebra perennials; only three of them, the generators of \( \text{SO}(1,2) \), can be integrated to give a group action on the phase space \( \tilde{\Gamma}' \); the three Hamiltonian vector fields corresponding to the perennials \( P_\rho \) are incomplete in \( \tilde{\Gamma}' \). However, the Lie algebra generated by the six perennials defines a group, \( \text{ISO}(1,2) \). With the standard symplectic form, \( \Omega_{\text{ISO}} = dP_\mu \wedge dX^\mu \) of cotangent bundles, \( T^* M^3 \) is a phase space, on which this group does act. There is a map, \( \iota \), that sends \( \tilde{\Gamma}' \) in \( T^* M^3 \); \( \iota \) is a symplectic imbedding and it pushes forwards Martin’s perennials into the generators of the action of \( \text{ISO}(1,2) \) on \( T^* M^3 \). Thus, the map \( \iota \) is equivariant for the two actions of the subgroup \( \text{SO}(1,2) \) of \( \text{ISO}(1,2) \). Such a space \( T^* M^3 \) together with such a map \( \iota \) can be called \textit{minimal group completion of the phase space} \( \tilde{\Gamma}' \) \textit{corresponding to the complete Lie algebra of Martin’s perennials}. The completion is minimal in the sense, that there is no smaller completion (subspace of \( T^* M^3 \)), because each orbit of \( \text{ISO}(1,2) \) in \( T^* M^3 \) intersects \( \iota(\tilde{\Gamma}') \).

The constraint surface \( \Gamma' \) will be mapped to the subset of \( T^* M^3 \) given by \( X \cdot X < 0 \) and Eq. (23). The group \( \text{ISO}(1,2) \) does not act on \( \iota(\Gamma') \) even if the generators of the group are tangential to \( \iota(\Gamma') \): again, the translations are incomplete within this surface. However, there is a unique completion of \( \iota(\Gamma') \) in \( T^* M^3 \) on which the group acts, which we call \( \Gamma_{\text{ISO}} \). This surface is determined just by the equations (23).
Clearly, \((T^*M^3, \Omega_{ISO}, \Gamma_{ISO})\) is a reparametrization invariant system that defines the corresponding c-orbits: they coincide with the maximal null geodesics in \(M^3\). Moreover, the image \(\iota(\gamma)\) of each c-orbit \(\gamma\) in \(\Gamma'\) lies completely within some of the c-orbits of \((T^*M^3, \Omega_{ISO}, \Gamma_{ISO})\) namely in that null geodesic that extends \(\iota(\gamma)\). Thus, we also have a well-defined map, \(\bar{\iota} : \tilde{\Gamma}' \mapsto \tilde{\Gamma}_{ISO}\), where \(\tilde{\Gamma}_{ISO}\) is the physical phase space of \((T^*M^3, \Omega_{ISO}, \Gamma_{ISO})\).

In this sense, we can speak about a group completion of the reparametrization invariant system.

The completion constructed above has an important property which makes them interesting for physics. Let \((M, \Omega)\) be a symplectic manifold and let a set \(g\) of functions form a Lie algebra with respect to linear combinations and Poisson brackets. Let \(G\) be the (abstract) simply connected group that is determined by \(g\). Let \((M_G, \Omega_G, \iota)\) be a minimal group completion of \((M, \Omega)\) by \(G\). Thus, \(G\) acts on \((M_G, \Omega_G)\) as a group of symplectic diffeomorphisms. Let the image \(\iota(M)\) be an open dense subset of \(M_G\). Then, we say that the group \(G\) has a weak action on \((M, \Omega)\).

For the above construction, we shall show the theorem:

**Theorem 1** The group \(ISO(1,2)\) has a weak action on the physical phase space \(\tilde{\Gamma}'\).

**Proof** Consider the two surfaces \(\Sigma_{\pm}\) defined by \(X \cdot X = 1\) and \(\pm X^0 > 0\), respectively, inside the light cone of the origin in \(M^3\); the manifolds \(\Sigma_{\pm} \times P\) are global transversal surfaces in \(\iota(\Gamma_{\pm})\) and \(\iota(\Gamma_{\mp})\), as they are intersected by all null geodesics inside of the light cone. Thus, \(\tilde{\Gamma}'\) can be identified with \((\Sigma_+ \times P) \cup (\Sigma_- \times P)\). Next consider the surface \(\Sigma\) in \(M^3\) defined by \(X^0 = 0\). Clearly, the manifold \(\Sigma \times P\) is a global transversal surface for the group completed system, because it is intersected by any null geodesic at exactly one point; we can identify \(\tilde{\Gamma}_{ISO}\) with \(\Sigma \times P\).

Every point of \(\Sigma_{\pm} \times P\) defines a unique null geodesic; this geodesic intersects \(\Sigma \times P\) at precisely one point. Thus we have a well-defined map \(\rho : (\Sigma_+ \times P) \cup (\Sigma_- \times P) \mapsto \Sigma \times P\) (the map \(\rho\) and its properties have been described in Sec. 2). It is easy to see that \(\bar{\iota}\) can be identified with \(\rho\) and so we have to show that \(\rho((\Sigma_+ \times P) \cup (\Sigma_- \times P)\) is open and dense in \(\Sigma \times P\).

Let us introduce the coordinates \(u\) and \(v\) at the surfaces \(\Sigma\) by

\[
\begin{align*}
X^0 &= \epsilon \sqrt{1 + u_1^2 + u_2^2}, \\
X^1 &= u_1, \\
X^2 &= u_2,
\end{align*}
\]

and the coordinates \(x_1\) and \(x_2\) at \(\Sigma\) by \(X^1 = x_1\) and \(X^2 = x_2\). Let us consider null geodesics with a definite three-momenta of the form

\[
P_\mu = p(1, \cos \alpha, \sin \alpha),
\]

(24)
where \( p \) is a (negative) number. The null geodesic with the momenta (24) starting at the point \((x_1, x_2)\) of \( \Sigma \) will intersect \( \Sigma_\epsilon \) at the point

\[
\begin{align*}
t &= \epsilon \sqrt{1 + u_1^2 + u_2^2}, \\
u_1 &= x_1 + \epsilon \cos \alpha \sqrt{1 + u_1^2 + u_2^2}, \\
u_2 &= x_2 + \epsilon \sin \alpha \sqrt{1 + u_1^2 + u_2^2},
\end{align*}
\]

Solving for \((x_1, x_2)\), we obtain a description of \( \rho \) in terms of the coordinates \( u_1, u_2, \epsilon, x_1 \) and \( x_2' \):

\[
\begin{align*}
x_1 &= u_1 - \epsilon \cos \alpha \sqrt{1 + u_1^2 + u_2^2}, \\
x_2 &= u_2 - \epsilon \sin \alpha \sqrt{1 + u_1^2 + u_2^2}.
\end{align*}
\]

(25) (26)

To see which part of \( \Sigma \) is hit, we introduce new variables:

\[
\begin{align*}
x_1' &= x_1 \cos \alpha + x_2 \sin \alpha, \\
x_2' &= -x_1 \sin \alpha + x_2 \cos \alpha, \\
u_1' &= u_1 \cos \alpha + u_2 \sin \alpha, \\
u_2' &= -u_1 \sin \alpha + u_2 \cos \alpha,
\end{align*}
\]

and observe that

\[
u_1^2 + u_2^2 = u_1'^2 + u_2'^2.
\]

Then, Eqs. (25) and (26) are equivalent to

\[
\begin{align*}
x_1' &= u_1' - \epsilon \sqrt{1 + u_1'^2 + u_2'^2}, \\
x_2' &= u_2'.
\end{align*}
\]

(27)

Thus, as \( u_2' \) runs through \( \mathbb{R} \), so does \( x_2' \). For a fixed \( x_2' \),

\[
x_1' = u_1' - \epsilon \sqrt{1 + u_1'^2 + x_2'^2},
\]

hence

\[
\frac{dx_1'}{du_1'} = 1 - \epsilon \frac{u_1'}{\sqrt{1 + u_1'^2 + x_2'^2}} > 0, \quad \epsilon = \pm 1.
\]

For \( \epsilon = +1 \), \( x_1' \rightarrow -\infty \) as \( u_1' \rightarrow -\infty \). On the other side, we obtain

\[
\lim_{u_1' \rightarrow -\infty} x_1' = \lim_{u_1' \rightarrow -\infty} \frac{-1 - x_2'^2}{u_1' + \sqrt{1 + u_1'^2 + x_2'^2}} = 0.
\]

For \( \epsilon = -1 \), \( x_1' \rightarrow \infty \) as \( u_1' \rightarrow \infty \). On the other side, we obtain

\[
\lim_{u_1' = -\infty} x_1' = \lim_{u_1' = -\infty} \frac{1 + x_2'^2}{|u_1'| + \sqrt{1 + u_1'^2 + x_2'^2}} = 0.
\]
Thus, $\Sigma_+$ is mapped to $x_1 \cos \alpha + x_2 \sin \alpha < 0$ and $\Sigma_-$ is mapped to $x_1 \cos \alpha + x_2 \sin \alpha > 0$. Only the straight line $x_1 \cos \alpha + x_2 \sin \alpha = 0$ is missing from each surface $P_1 = \text{const}$, $P_2 = \text{const}$. It follows that $\rho(\hat{\Gamma}')$ is open and dense, Q.E.D.

There is a standard way of construction of group completions, if the Lie algebra of observables is complete. Let $(M, \Omega)$ be a symplectic manifold and let $\mathfrak{g}$ be a Lie algebra generated by a complete system of functions on $M$. Let $G$ be the unique simply connected group whose Lie algebra coincides with $\mathfrak{g}$. Let $\mathfrak{g}^*$ be the dual linear space to $\mathfrak{g}$ and let $\text{ad}^*$ be the co-adjoint representation of $G$ on $\mathfrak{g}^*$. The orbits of the action $\text{ad}^*$ of $G$ in $\mathfrak{g}^*$ are homogeneous symplectic spaces of the group $G$ according to a beautiful result of Kirillov [19]. Moreover, from a basis of $\mathfrak{g}$, a (basis-independent) map $\Pi : M \mapsto \omega$ of $M$ into an orbit $\omega$ can be constructed; $\Pi$ is the so-called momentum map. Then, the manifold $\omega$ with Kirillov’s symplectic structure and with $\Pi$ as $\iota$ is the desired (minimal) completion.

As an example consider the manifold $M$ with the coordinates $q$ and $p$ given by $q^2 + p^2 < 1$, equipped with the symplectic form $\Omega = dp \wedge dq$. Let the algebra $\mathfrak{g}$ be generated by the functions $q$, $p$ and 1 (constant function). The corresponding group is the three-dimensional Heisenberg group defined on $\mathbb{R}^3$ with group law

$$(a_1, b_1, r_1) \cdot (a_2, b_2, r_2) = (a_1 + a_2, b_1 + b_2, r_1 + r_2 + \frac{b_1a_2 - b_2a_1}{2}).$$

The space dual to the algebra is $\mathbb{R}^3$ with the coordinates $A$, $B$ and $R$, the orbits of the group are the planes $R = \text{const}$ and the momentum map is

$$A = q, \quad B = p, \quad R = 1.$$ 

Thus, the orbit $\omega$ on which $M$ is mapped is given by the equation $R = 1$. The image of $M$ is the disk $A^2 + B^2 < 1$, and the group does not act even weakly.

4.2 SO(2,3) completion

There is a motivation to look for further symmetries: the subgroup $\text{SO}(1,2)$ that acts on $\hat{\Gamma}'$ is too small to define a time evolution à la Dirac. The simple form of the reparametrization invariant system $(T^*\mathbb{M}^3, \Omega_{\text{ISO}}, \Gamma_{\text{ISO}})$ allows us to see immediately that there are more perennials than just the generators of the three-dimensional Poincaré group: we have also the conformal isometries. The so-called dilatation is generated by

$$D := X^\mu P_\mu, \quad (28)$$

and the so-called conformal accelerations are generated by

$$Q_0 := (X \cdot X)P_0 + 2X^0(X \cdot P), \quad (29)$$
$$Q_1 := (X \cdot X)P_1 - 2X^1(X \cdot P), \quad (30)$$
$$Q_2 := (X \cdot X)P_2 - 2X^2(X \cdot P). \quad (31)$$

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It is easy to verify that the Poisson brackets of these variables with $P \cdot P$ weakly vanish. Let us denote $B^\mu Q_\mu$ by $B$. Then, the Lie algebra of $A$, $B$, $C$ and $D$ is given by Eqs. (11) and

$$\{A, B\} = 2(A \cdot B)D - 2(\varepsilon_{\rho\mu\nu}A^\mu B^\nu)J^\rho, \quad \{B, C\} = (\varepsilon_{\rho\mu\nu}B^\mu C^\nu)Q_\rho,$$

$$\{A, D\} = -A, \quad \{B, B'\} = 0, \quad \{B, D\} = B, \quad \{C, D\} = 0. \quad (32)$$

This is the Lie algebra of the group $SO(2,3)$. Indeed, let $Z^0$, $Z^1$, $Z^3$, $W^0$ and $W^1$ be coordinates in $\mathbb{R}^5$ with the metric

$$dS^2 = -(Z^0)^2 - (dW^0)^2 + (dZ^1)^2 + (dZ^2)^2 + (dW^1)^2; \quad (34)$$

to obtain the algebra (11), (32) and (33), we have just to identify:

$$J^\rho \mapsto \varepsilon^ {\rho\mu\nu}Z_\mu \frac{\partial}{\partial Z^\nu}, \quad D \mapsto W_0 \frac{\partial}{\partial W^1} - W_1 \frac{\partial}{\partial W^0},$$

$$P_\rho \mapsto \left( Z_\rho \frac{\partial}{\partial W^0} - W_0 \frac{\partial}{\partial Z_\rho} \right) - \left( Z_\rho \frac{\partial}{\partial W^1} - W_1 \frac{\partial}{\partial Z_\rho} \right),$$

$$Q_\rho \mapsto \left( Z_\rho \frac{\partial}{\partial W^0} - W_0 \frac{\partial}{\partial Z_\rho} \right) + \left( Z_\rho \frac{\partial}{\partial W^1} - W_1 \frac{\partial}{\partial Z_\rho} \right),$$

where the indices are lowered by the metric (34).

The Hamiltonian vector field corresponding to the dilatation is complete not only within $T^\ast \mathbb{M}^3$, but even within $\iota(\tilde{\Gamma}')$. The Hamiltonian vector fields corresponding to $Q$’s are incomplete within $T^\ast \mathbb{M}^3$; however, there is still a chance to construct additional observables from $Q$’s, so we have to construct the next completion. This completion is well-known: it is the cotangent bundle $T^\ast \tilde{\mathbb{M}}^3$ of the compactified Minkowski spacetime $\tilde{\mathbb{M}}^3 \quad [27]$. Let us briefly describe the construction, because we shall need some details of it.

Consider the three-dimensional Einstein cosmology spacetime $\mathbb{M}^3_E$ with the coordinates $\tau$, $\vartheta$ and $\varphi$ and the metric

$$ds^2 = -d\tau^2 + d\vartheta^2 + \sin^2 \vartheta d\varphi^2.$$

The null geodesics that start at the point $\tau = \tau_0$, $\vartheta = 0$, are given by

$$\tau = \tau_0 + \lambda, \quad \vartheta = \lambda, \quad \varphi = \text{const}.$$

These geodesic form a null cone that refocuses at the point $\tau = \tau_0 + \pi$, $\vartheta = \pi$. Similarly, all null geodesics through the point $\tau = \tau_0$, $\vartheta = \pi$, have the form

$$\tau = \tau_0 + \lambda, \quad \vartheta = \pi - \lambda, \quad \varphi = \text{const},$$

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and they refocus at $\tau = \tau_0 + \pi, \vartheta = 0$. As it is well-known, (e.g. [27]), the compactified Minkowski spacetime $\bar{M}^3$ is obtained from $M^3_E$ by the identification of each point $(\tau, \vartheta, \varphi)$ with the point $(\tau + \pi, \pi - \vartheta, \varphi + \pi)$. There is a conformal isometry $\phi$ that sends $M^3_E$ into $\bar{M}^3$ such that the set $\phi(M^3_E)$ lies in the future of the null cone between the points $(\tau = -\pi, \vartheta = 0)$ and $(\tau = 0, \vartheta = \pi)$, and in the past of the null cone between $(\tau = 0, \vartheta = \pi)$ and $(\tau = \pi, \vartheta = 0)$.

Another copy of Minkowski spacetime lies between the points $(\tau = -\pi, \vartheta = \pi), (\tau = 0, \vartheta = 0)$ and $(\tau = \pi, \vartheta = \pi)$. By the above identification and the two conformal isometries, the point $X^\mu$ of the first Minkowski spacetime will be mapped to the point $Y^\mu$ of the second one given by

$$Y^\mu = \frac{X^\mu}{X \cdot X},$$

if the inertial coordinates $X^\mu$ and $Y^\mu$ are chosen properly. We will make some use of these two patches of $\bar{M}^3$; the fact that they do not cover $\bar{M}^3$ will not be important. Let us call them $U$ and $V$.

The map $\phi$ is a diffeomorphism of three-dimensional manifolds and it can be extended to an isomorphism $\phi_{\text{cot}}$ of the cotangent bundles of these manifolds. As the conformal group SO(2,3) acts transitively in $T^*\mathbb{M}^3$, the triad $(T^*\mathbb{M}^3, \Omega_{\text{SO}}, \phi_{\text{cot}})$ is the desired minimal SO(2,3) completion; the form $\Omega_{\text{SO}}$ is the standard symplectic form of cotangent bundles; observe that it is exact. The set $\phi_{\text{cot}}(T^*\mathbb{M}^3)$ is equal to $T^*U$, and so it is open and dense in $T^*\mathbb{M}^3$. Thus, SO(2,3) acts weakly on $T^*\mathbb{M}^3$.

Let the canonical coordinates in the cotangent bundles $T^*U$ and $T^*V$ be $X^\mu, P_\mu$ and $Y^\mu, Q_\mu$, respectively. Then the transformation (35) between $X^\mu$ and $Y^\mu$ leads to the following transformation between $P_\mu$ and $Q_\mu$:

$$Q_\mu = (X \cdot X)P_\mu - 2(X \cdot P)X_\mu, \quad (36)$$

The inverse map is

$$X^\mu = \frac{Y^\mu}{Y \cdot Y}, \quad P_\mu = (X \cdot X)Q_\mu - 2(X \cdot Q)Y_\mu, \quad (37)$$

Comparison with Eqs. (29)–(31) shows that the use of the letter $Q$ for this coordinate will not lead to any confusion with the notation for the generators of conformal accelerations. From the transformation formulas (36) and (37), we easily verify that the symplectic form $\Omega_{\text{SO}}$ at the points where the patches overlap satisfies

$$dP_\mu \wedge dX^\mu = dQ_\mu \wedge dY^\mu.$$
within each chart $T^*U_i$, where the metric $g_{\mu\nu}$ is the metric of the conformal chart $U_i$, $P_\mu$ the canonical coordinate in the fibers of $T^*U_i$ and we allow only charts that have the same time orientation.

It follows that the projection of the c-orbits to the configuration space $T^*\bar{M}^3$ are complete null geodesics; they are closed curves (topologically $S^1$). The $\phi$-images of the c-orbits of the system $(T^*\bar{M}^3, \Omega_{ISO}, \Gamma_{ISO})$ are the null geodesics in the chart $U_i$; each of them is completed by one point in $\bar{M}^3$. The null geodesics in $\bar{M}^3$ that do not contain any $\phi$-images form the light cone of the origin of the chart $V$ ($Y^\mu = 0$). Hence, the surface $\Gamma_0$ given by the equations $X^0 = 0, P_0 = -\sqrt{p_1^2 + p_2^2}$ within the chart $T^*U$, and by $Y^0 = 0, Q_0 = -\sqrt{Q_1^2 + Q_2^2}$ within the chart $T^*V$ is a global transversal surface. We can introduce coordinates $(x_1, x_2) \in \mathbb{R}^2, (p_1, p_2) \in \mathbb{R}^2 \setminus \{0\}$ in $\Gamma_0 \cap T^*U$ and $(y_1, y_2) \in \mathbb{R}^2, (q_1, q_2) \in \mathbb{R}^2 \setminus \{0\}$ in $\Gamma_0 \cap T^*V$ such that the imbedding equations are

$$
\begin{align*}
X^0 &= 0, & X^1 &= x_1, & X^2 &= x_2, \\
P_0 &= -\sqrt{p_1^2 + p_2^2}, & P_1 &= p_1, & P_2 &= p_2,
\end{align*}
$$

and

$$
\begin{align*}
Y^0 &= 0, & Y^1 &= y_1, & Y^2 &= y_2, \\
Q_0 &= -\sqrt{q_1^2 + q_2^2}, & Q_1 &= q_1, & Q_2 &= q_2.
\end{align*}
$$

One easily verifies that $\Gamma_0$ defined in this way is a smooth surface and that the transformation formulas (35), (36) and (37) imply the relations

$$
x_k = \frac{y_k}{y_1^2 + y_2^2}, \quad p_k = (y_1^2 + y_2^2)q_k - 2(y_1q_1 + y_2q_2)y_k.
$$

In particular, Eq. (39) implies that $Q_0 < 0$ if $P_0 < 0$. The pull-back $\Omega_0$ of the symplectic form $\Omega_{SO}$ to $\Gamma_0$ is given in these coordinates by

$$
\Omega_0 = dp_k \wedge dx_k = dq_k \wedge dy_k,
$$

the last inequality following from Eqs. (39). $\Omega_0$ is exact.

The manifold $\Gamma_0$ is a bundle with the fiber $P \cong S^1 \times \mathbb{R}$ given by $x_k = \text{const}$ or $y_k = \text{const}$. The base space is $S^2$, and the coordinates $(x_1, x_2)$ and $(y_1, y_2)$ are nothing but the two stereographic projection charts of $S^2$. The symplectic space $(\Gamma_0, \Omega_0)$ is a homogeneous symplectic space of the group $SO(2,3)$, which acts on $\Gamma_0$ by projection of symmetries (see Sec. 2): each element of the conformal group maps null geodesics in null geodesics. $(\Gamma_0, \Omega_0)$ can be identified with the group completed physical phase space $\bar{\Gamma}_{SO}$. The composition $\bar{\phi} \circ \bar{\iota}$ of the ISO(1,2)-completion and the SO(2,3)-completion gives the image $\bar{\phi}(\bar{\iota}(\Gamma'))$ as an open dense subspace of $\Gamma_0$; thus, the conformal group $SO(2,3)$ acts weakly (and transitively) on $\Gamma'$.

The generators of the action of $SO(2,3)$ on $\Gamma_0$ are Hamiltonian vector fields of the projections of the perennials $P_\mu, Q_\mu, J^\mu$ and $D$ from $T^*\bar{M}^3$ to $\Gamma_0$ (see Sec. 2).
In the patch \((x_k, p_k)\), the projections coincide with the functions
\[
\begin{aligned}
\bar{P}_0 &:= -\sqrt{\mathbf{p} \cdot \mathbf{p}}, & \bar{Q}_0 &:= -(\mathbf{x} \cdot \mathbf{x})\sqrt{\mathbf{p} \cdot \mathbf{p}}, & \bar{J}^0 &:= x_1 p_2 - x_2 p_1, \\
\bar{P}_1 &:= p_1, & \bar{Q}_1 &:= (\mathbf{x} \cdot \mathbf{x}) p_1 - 2(\mathbf{x} \cdot \mathbf{p}) x_1, & \bar{J}^1 &:= -x_2 \sqrt{\mathbf{p} \cdot \mathbf{p}}, \\
\bar{P}_2 &:= p_2, & \bar{Q}_2 &:= (\mathbf{x} \cdot \mathbf{x}) p_2 - 2(\mathbf{x} \cdot \mathbf{p}) x_2, & \bar{J}^2 &:= x_1 \sqrt{\mathbf{p} \cdot \mathbf{p}}, \\
\bar{D} &:= \mathbf{x} \cdot \mathbf{p}.
\end{aligned}
\] (40)

These functions will be considered as observables. Here, we have used the abbreviation \(u \cdot v := u_1 v_1 + u_2 v_2\). The expressions within the other patch, \((y_k, q_k)\), are analogous, one just have to exchange \(P\)'s and \(Q\), write \(y\) for \(x\) and \(q\) for \(p\). Via Poisson brackets, the functions generate the Lie algebra of SO(2,3). They are ten functions of four variables; thus, there will be six relations. These relations can be systematically written down, if we solve the definitions of \(\bar{P}_1, \bar{P}_2, \bar{J}^1, \) and \(\bar{J}^2\) for \(p_1, p_2, x_1\) and \(x_2\) and substitute the results into the other definitions:

\[
\begin{aligned}
\bar{P}_0^2 &= \bar{P}_1^2 + \bar{P}_2^2, & \bar{J}^0 \bar{P}_0 + \bar{J}^1 \bar{P}_1 + \bar{J}^2 \bar{P}_2 &= 0, \\
\bar{Q}_0^2 &= \bar{Q}_1^2 + \bar{Q}_2^2, & \bar{D} &= \frac{\bar{P}_1 \bar{J}^2 - \bar{P}_2 \bar{J}^1}{\sqrt{\mathbf{p} \cdot \mathbf{p}}}, \\
\bar{Q}_1 &= \frac{\bar{P}_1}{\mathbf{p} \cdot \mathbf{p}} [\bar{J}^1]^2 - (\bar{J}^2)^2] + \frac{\bar{P}_2}{\mathbf{p} \cdot \mathbf{p}} [2\bar{J}^1 \bar{J}^2], \\
\bar{Q}_2 &= \frac{\bar{P}_1}{\mathbf{p} \cdot \mathbf{p}} [2\bar{J}^1 \bar{J}^2] - \frac{\bar{P}_2}{\mathbf{p} \cdot \mathbf{p}} [(\bar{J}^1)^2 - (\bar{J}^2)^2].
\end{aligned}
\] (41–43)

The two Eqs. (41) are relations concerning also ISO(1,2) alone and the four relations (42)–(44) can be used to calculate the remaining generators of SO(2,3). A quadratic relation follows

\[- (\bar{J}^0)^2 + (\bar{J}^1)^2 + (\bar{J}^2)^2 = \bar{D}^2.\] (45)

The Poisson brackets of the four quadratic expressions \(P \cdot P, Q \cdot Q, J \cdot P\) and \(-D^2 + J \cdot J\) with the generators \(P_\mu, Q_\mu, J_\mu\) and \(D\) are mostly vanishing or proportional to \(P \cdot P\); they are (generalized) Casimirs of some subgroups.

As it was explained in Sec. 4, the action of SO(2,3) on \(\Gamma_0\) is generated by the Hamiltonian vector fields of the observables (39). This defines a linear map from the Lie algebra \(\text{so}(2,3)\) into vector fields on \(\Gamma_0\). We can describe the map, if we choose a basis of \(\text{so}(2,3)\) and list the images of the elements of the basis. Let the basis be

\[ (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{J}^0, \mathcal{J}^1, \mathcal{J}^2, \mathcal{D}). \] (46)

Here, we denote the abstract elements of the Lie algebra by upper case calligraphic letters to distinguish them from the corresponding perennials or vector fields. The association with the vector fields is:

\[ \mathcal{P}_0 \mapsto -\frac{p_k}{\sqrt{\mathbf{p} \cdot \mathbf{p}}} \frac{\partial}{\partial x_k}, \quad \mathcal{P}_k \mapsto \frac{\partial}{\partial x_k}, \quad \mathcal{Q}_k \mapsto \bar{Q}_k, \quad \mathcal{J}^0 \mapsto \bar{J}^0, \quad \mathcal{J}^1 \mapsto \bar{J}^1, \quad \mathcal{J}^2 \mapsto \bar{J}^2, \quad \mathcal{D} \mapsto \bar{D}. \] (47)
\begin{align}
Q_0 & \mapsto -\frac{x \cdot x}{\sqrt{p \cdot p}} p_k \frac{\partial}{\partial x_k} + 2\sqrt{p \cdot p} x_k \frac{\partial}{\partial p_k}, \\
Q_k & \mapsto [(x \cdot x)\delta_{kl} - 2x_kx_l] \frac{\partial}{\partial x_l} + [(x \cdot p)\delta_{kl} + x_kp_l - x_lp_k] \frac{\partial}{\partial p_l}, \\
J^0 & \mapsto -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - p_2 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial p_2}, \\
J^1 & \mapsto -\frac{x_2p_k}{\sqrt{p \cdot p}} \frac{\partial}{\partial x_k} + \sqrt{p \cdot p} \frac{\partial}{\partial p_2}, \\
J^2 & \mapsto \frac{x_1p_k}{\sqrt{p \cdot p}} \frac{\partial}{\partial x_k} - \sqrt{p \cdot p} \frac{\partial}{\partial p_1}, \\
D & \mapsto x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - p_1 \frac{\partial}{\partial p_1} - p_2 \frac{\partial}{\partial p_2}.
\end{align}

We shall need the form of these vector fields for the construction of the physical representation of the group in Sec. 5.

5 The Hamiltonian

In this section, we study the time evolution of the 2+1 gravity model. We are going to apply Dirac’s idea: a choice of transversal surfaces of maximal symmetry, and a comparison of different time levels using symmetry operations.

There are two problems that prevent a straightforward application. First, the group SO(2,3) is too large to have a representation that satisfies all conditions on physical representation (see the next section). Second, only a four-dimensional subgroup of SO(2,3) has the action on the phase space of the system that is associated with the corresponding Lie algebra of perennials.

The largest subgroups that have physical representations are G_1 and G_2 with the structure \((SO(1,2) \times \mathbb{R}) \otimes_\mathbb{R} \mathbb{R}^3\). G_1 is generated by \(J^0, J^1, J^2, D, P_0, P_1\) and \(P_2\) and G_2 by \(J^0, J^1, J^2, D, Q_0, Q_1\) and \(Q_2\). Consider G_1. The corresponding group completion of the 2+1 gravity coincides with the ISO(1,2)-completion that was constructed in the previous section, because the only additional element, the dilatation \(D\), acts on \(\tilde{\Gamma}'\). Let us, therefore, restrict ourselves to G_1.

The group G_1 acts weakly on the physical phase space \(\tilde{\Gamma}'\). This is important for the quantum generators of the group to have suitable spectra. On the other hand, the weak action is not sufficient for the construction of a time evolution according to Dirac. Let us explain these two points.

Consider the two-dimensional disk example described at the end of the previous section. The Heisenberg group did not act even weakly. On the disk, the values of the classical observables \(q\) and \(p\) are bounded: \(q^2 + p^2 < 1\). On the group completion,
which is a whole plain, the values of \( q \) fill up the interval \((-\infty, \infty)\) and similarly for \( p \). This follows from the structure of the Lie algebra (see, e.g. [10]). If we represent \( q \) and \( p \) by self-adjoint operators satisfying the canonical commutation relations, then this structure forces the spectra again to fill up the whole real axis. The corresponding quantum mechanics contains semiclassical wave packets with average values of \( \hat{q} \) and \( \hat{p} \) that lie far away from possible classical values. Let us call this the problem of ranges. On the other hand, if a group acts weakly, then the only change of the range of classical values that is motivated is an addition of some limit points. This would happen in any case in the quantum mechanics, because the spectrum of any self-adjoint operator is a closed subset of \( \mathbb{R} \). In this respect, the weak action does not differ from an ordinary action.

However, for Dirac’s idea to work, we have to find a family of maximally symmetric transversal surfaces and a sufficiently large subset of the symmetry group so that all such surfaces can be obtained from one by the action of the subset. We emphasize that this has to work within the constraint surface of the classical theory so that we can interpret the transformations. Clearly, for these purposes, the weak action is not adequate. First, the maximally symmetric surfaces will lie in the group completed constraint surface, but not, in general, inside that of the system; the intersection of such surfaces with the constraint surface of the system will not be, in general, globally transversal. Second, an image of a globally transversal surface by an element of the group that has only a weak action will not, in general, lie inside the constraint surface. It is easy to construct examples of this kind for the action of \( G_1 \). This means that only the subgroup \( G_0 \) is at our disposal for Dirac’s construction.

The group \( G_0 = \text{SO}(1,2) \times \mathbb{R} \) generated by \( \mathcal{J}^0, \mathcal{J}^1, \mathcal{J}^2 \) and \( \mathcal{D} \) is a cartesian product of two simple groups. Its three-dimensional subgroups are of two types:

1. \( \text{SO}(1,2) \), which is the unique subgroup of this type, because it is a normal subgroup,

2. subgroups that leave a null plane invariant; an example is the subgroup of the plane \( X^0 + X^1 = 0 \), which is generated by \( \mathcal{J}^0 + \mathcal{J}^1, \mathcal{J}^2 \) and \( \mathcal{D} \). All other subgroups of this type are (group-) similar to this one.

The surfaces in \( \Gamma' \) symmetric with respect to \( \text{SO}(1,2) \) satisfy \(- (X^0)^2 + (X^1)^2 + (X^2)^2 = \text{const} \) (\( \text{const} < 0 \)) have two components (with \( X^0 > 0 \) and \( X^0 < 0 \)) and form one-dimensional family. The components coincide with the CMC surfaces, and the union of both components is globally transversal. On the other hand, the projection of the surfaces to \( M^3 \) that are invariant with respect to the null plane groups are of course these null planes; the surfaces do not lie inside the null cone \( \iota(\tilde{\Gamma'}) \) and their intersections with \( \iota(\tilde{\Gamma'}) \) are not transversal. We summarize the results:
Theorem 2 The 2+1 gravity model possesses a unique one-dimensional family of maximally symmetric globally transversal surfaces. Each such surface has two components that are CMC surfaces with opposite values of CMC.

The second step of the construction is to find a subgroup that would carry us along the family of the CMC surfaces. Thus, it must be a subgroup whose elements are representatives of all classes of $G_0/SO(1,2)$. However, $G_0/SO(1,2) = \mathbb{R}$, so the desired subgroup is generated by just one element, which must have the form

$$\mathcal{D} + a\mathcal{J}^0 + b\mathcal{J}^1 + c\mathcal{J}^2,$$

where $a$, $b$ and $c$ are three arbitrary reals. We can normalize the generator in this way, as the overall factor does not change the subgroup, and the factor in front of $\mathcal{D}$ must be non-zero: a nontrivial motion of the CMC surface is generated just by the $\mathcal{D}$ term. Let us study how it acts on the CMC $\tau$. Using equations (22) and (28), we find that

$$\{\tau, \mathcal{D}\} = -\frac{2\epsilon}{\sqrt{-X \cdot X}}.$$

Thus, for $\epsilon > 0$, the action of $\mathcal{D}$ diminishes $\tau$, and for $\epsilon < 0$, it enlarges $\tau$. $\epsilon > 0$ ($\epsilon < 0$) means that we are in the future (past) light cone of the origin. In the future light cone, we have expanding tori ($\tau > 0$) and they expand from the “big bang” $\tau = \infty$ to the maximal expansion state $\tau = 0$. Thus, $\mathcal{D}$ generates evolution towards future here. In the past light cone ($\tau < 0$), we have contracting tori that start at the maximal expansion state $\tau = 0$ and finish at the “big crunch” $\tau = -\infty$. Thus $\mathcal{D}$ generates an evolution towards past. Luckily enough, there is a smooth Hamiltonian $H$ that evolves everything towards the future: $H = \mathcal{D}$ inside the future light cone and $H = -\mathcal{D}$ inside the past light cone. Let us calculate the corresponding perennial $\bar{H}$ on the physical phase space $\bar{\Gamma}_{\text{ISO}}$. From the proof of the theorem 1, it follows that the portion $\bar{\Gamma}_+^\text{I}$ of the physical phase space that corresponds to the future half of the light cone is given by $x_1 \cos \alpha + x_2 \sin \alpha < 0$ and that $\bar{\Gamma}_-^\text{I}$ corresponding the past one by $x_1 \cos \alpha + x_2 \sin \alpha > 0$. However, Eq. (24) implies that

$$p_1 = p \cos \alpha, \quad p_2 = p \sin \alpha,$$

where $p < 0$. Here $x_1$, $x_2$, $p_1$ and $p_2$ are the coordinates that we have chosen in the physical phase space $\bar{\Gamma}$ (cf. Eq. (38)). Then, from the last Eq. of (40) it follows that $\bar{\mathcal{D}}$ is positive for the expanding tori and negative for the contracting ones, or

$$\bar{H} = |\bar{\mathcal{D}}|.$$  \hspace{1cm} (54)

In the general case, we start from the function

$$\bar{\mathcal{D}} + a\bar{\mathcal{J}}^0 + b\bar{\mathcal{J}}^1 + c\bar{\mathcal{J}}^2.$$
As it is only the $D$-part which leads to changes in $\tau$, the Hamiltonian that evolves towards the future corresponding to the above function is
\[ \bar{H} = \text{sign}(\bar{D})(\bar{D} + a\bar{J}^0 + b\bar{J}^1 + c\bar{J}^2). \]

Eqs. (40) lead to
\[ \bar{H} = \text{sign}(\mathbf{x} \cdot \mathbf{p})[\mathbf{x} \cdot \mathbf{p} + a(x_1p_2 - x_2p_1) + bx_2\sqrt{\mathbf{p} \cdot \mathbf{p}} - cx_1\sqrt{\mathbf{p} \cdot \mathbf{p}}]. \]

Let us change the variables $x_1$, $x_2$, $p_1$ and $p_2$ to $\bar{D}$, $\bar{J}^0$, $\mathbf{p}$, $\alpha$; we obtain
\[ \bar{H} = |\bar{D}|(1 + b\sin \alpha - c\cos \alpha) + \text{sign}(\bar{D})\bar{J}^0(a + b\cos \alpha + c\sin \alpha). \] (55)

We can see that $\bar{H}$ is unbounded from below except for the case that $a = b = c = 0$, because $\bar{J}^0$ can take on an arbitrary values independently of $\bar{D}$ and $\alpha$. Hence, (54) is the only one from the three-dimensional family of possible candidates for a Hamiltonian that is bounded from below (and even positive). This is, of course, nothing but an intriguing observation: there is no a priori reason for the generator of the time evolution, even if it evolves towards the future, to be bounded from below or positive, unless it plays simultaneously another role, for example that of the total energy of the system. We also observe that the dynamics simplifies strongly if we choose (54) in comparison with all other candidates: $\bar{J}^\mu$ become time independent, and $\bar{P}^\mu$ just scale with time. The next comment is that the choice (54) leads to the dynamics that has been obtained by Moncrief [23]. Finally, it is easy to see that there will be no problem to define the quantum mechanical operator $\hat{H}$ from $\bar{H}$, if the operators $\hat{D}$ and $\hat{J}^\mu$ are given, because $\hat{D}$ will commute with all $\hat{J}$'s. The corresponding problem of ranges will be automatically solved, if we define $|\hat{H}|$ by the spectral theorem.

6 The physical representation

The physical representation of the algebra $\mathfrak{so}(2,3)$ would map each element of the algebra to a linear operator on a common invariant dense domain $K_0$ in a Hilbert space $\mathcal{K}$; the map $R$ must satisfy the following conditions:

1. $R$ is linear, $R(1) = \text{id}$, and
\[ \frac{i}{\hbar}R(\{X, Y\}) = R(X)R(Y) - R(Y)R(X) \]
for all $X, Y \in \mathfrak{so}(2,3)$ on $\mathcal{K}_0$,

2. the operators $R(X)$ for all $X \in \mathfrak{so}(2,3)$ are essentially self-adjoint on $\mathcal{K}_0$,
3. the problem of ranges is satisfactorily solved,

4. the operators $R(X)$ for all $X \in \text{so}(2,3)$ satisfy algebraic relations that go over to (41)–(44) in the classical limit.

In general, the group method of quantization of an algebra $\mathfrak{g}$ of observables on a symplectic manifold $(M, \Omega)$ is to find a unitary representation on a Hilbert space $\mathcal{K}$ of the group $G$ corresponding to the algebra. Then, the generators of the group action on $\mathcal{K}$ satisfy automatically the conditions 1 and 2, but a part of the condition 3 ($P_0 < 0$) and the condition 4 can pose problems.

In this section, we are going to use an old idea of finding the physical representation by the group way: the Kostant-Kirillov method of orbits. This method works quite generally for finite systems. Let us briefly describe the steps of the method (for more detail see [19], [18]).

The method of orbits is based on the momentum map $\Pi$ determined by the algebra of observables $\mathfrak{g}$ (in this way, the relations and ranges are encoded). $\Pi(M)$ is a particular orbit $\omega$ of $G$ in the linear space $\mathfrak{g}^*$ dual to the Lie algebra $\mathfrak{g}$, where the group acts via the co-adjoint representation.

The method starts with a choice of a point $F \in \omega$ and with calculating the stabilizer $G_F \subset G$ of $F$. Then, the subalgebra $\mathfrak{n}_F$ called subordinate to $F$ must be found satisfying the conditions:

1. $\langle F, [X, Y] \rangle = 0 \quad \forall X, Y \in \mathfrak{n}_F$,
2. $\text{codim}_\mathfrak{g} \mathfrak{n}_F = (1/2)\dim \omega$,
3. Pukanszky’s condition: let $\mathfrak{n}_F^\perp$ be the subspace of $\mathfrak{g}^*$ that annihilates $\mathfrak{n}_F$; then, $F + \mathfrak{n}_F^\perp \subset \omega$.

One can show that $\mathfrak{g}_F \subset \mathfrak{n}_F$. The subalgebra $\mathfrak{n}_F$ generates a subgroup $N_F$ of $G$ and one must find a one-dimensional unitary representation $R_n$ of $N_F$ such that

$$R_n(\exp X) = \exp(\langle F, X \rangle)$$

in a neighbourhood of the identity of $N_F$. Such a representation will exist, if Kirillov’s symplectic form of $\omega$ is integral (its integral over any 2-cycle is an integer). The physical representation is then just the unitary representation of $G$ induced by $N_F$ (see [19], [28]).

In our case, $M = \Gamma_0$ and as the group we take first $\text{SO}(2,3)$. The momentum map $\Pi$ will be described in terms of a coordinate system in $\mathfrak{g}^*$; the coordinate system will be associated with the basis that is dual to (16); let the corresponding coordinates...
be \((\xi_\mu, \zeta_\mu, \theta^\mu, \delta)\). Then, the momentum map is given by

\[
\begin{align*}
\xi_0 &= -\sqrt{\mathbf{P} \cdot \mathbf{P}}, & \xi_0 &= -(\mathbf{x} \cdot \mathbf{x})\sqrt{\mathbf{P} \cdot \mathbf{P}}, & \theta^0 &= x_1p_2 - x_2p_1, \\
\xi_1 &= p_1, & \xi_1 &= (\mathbf{x} \cdot \mathbf{x})p_1 - 2(\mathbf{x} \cdot \mathbf{p})x_1, & \theta^1 &= -x_2\sqrt{\mathbf{P} \cdot \mathbf{P}}, \\
\xi_2 &= p_2, & \xi_2 &= (\mathbf{x} \cdot \mathbf{x})p_2 - 2(\mathbf{x} \cdot \mathbf{p})x_2, & \theta^2 &= x_1\sqrt{\mathbf{P} \cdot \mathbf{P}}, \\
\delta &= \mathbf{x} \cdot \mathbf{p}.
\end{align*}
\]

(56)

As the group \(\text{SO}(2,3)\) has a trivial center, the momentum map is a symplectic isomorphism and the homogeneous symplectic space \((\Gamma_0, \Omega_0)\) of \(G\) can be identified with the orbit \(\omega\). Then, the Kirillov symplectic form \(\Omega_0\) is exact and so it is trivially integral.

Let us choose the point \(F\) corresponding to the point \(u \in \Gamma_0\) that is given by the values of coordinates \(x_1 = x_2 = 0, p_1 = 1\) and \(p_2 = 0\). From Eqs. (56), we calculate the coordinates of \(F\) to be \((-1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)\). The map of \(\mathfrak{g}\) into \(T_F\omega\) is given by the values of the vector fields \((17) - (53)\) at the point \(u\):

\[
\begin{align*}
\mathcal{P}_0 &\mapsto -\frac{\partial}{\partial x_1}, & \mathcal{P}_k &\mapsto \frac{\partial}{\partial x_k}, \\
\mathcal{Q}_0 &\mapsto 0, & \mathcal{Q}_k &\mapsto 0, \\
\mathcal{J}^0 &\mapsto \frac{\partial}{\partial p_2}, & \mathcal{J}^1 &\mapsto \frac{\partial}{\partial p_2}, & \mathcal{J}^2 &\mapsto -\frac{\partial}{\partial p_1}, \\
\mathcal{D} &\mapsto -\frac{\partial}{\partial p_1}.
\end{align*}
\]

(57)  (58)  (59)  (60)

The kernel of the map is, therefore, the subalgebra \(\mathfrak{g}_F\) generated by \(\mathcal{P}_0 + \mathcal{P}_1, \mathcal{J}^0 - \mathcal{J}^1, \mathcal{D} - \mathcal{J}^2\) and \(\mathcal{Q}_\mu, \, \mu = 0, 1, 2\). The algebra \(\mathfrak{g}_F\) has to be extended to \(\mathfrak{n}_F\). Thus, we have to find a two-dimensional subspace of \(\mathfrak{g}/\mathfrak{g}_F\) which is invariant with respect to \(\mathfrak{g}_F\). \(\mathfrak{g}/\mathfrak{g}_F\) is four-dimensional; we choose \([\mathcal{P}_1], [\mathcal{P}_2], [\mathcal{J}^1]\) and \([\mathcal{J}^2]\) as its basis. The action of \(Y \in \mathfrak{g}_F\) on \(\mathfrak{g}/\mathfrak{g}_F\) is given by \([X] \mapsto \pi([X, Y])\), where \(X\) is a representant of an element of the basis of \(\mathfrak{g}/\mathfrak{g}_F\), \([X]\) is the corresponding class and \(\pi\) is the projector from \(\mathfrak{g}\) to \(\mathfrak{g}/\mathfrak{g}_F\). A straightforward calculation gives

\[
\begin{array}{ccccccc}
| & \mathcal{P}_0 + \mathcal{P}_1 & \mathcal{J}^0 - \mathcal{J}^1 & \mathcal{D} - \mathcal{J}^2 & \mathcal{Q}_0 & \mathcal{Q}_1 & \mathcal{Q}_2 \\
\hline
[\mathcal{P}_1] & 0 & -[\mathcal{P}_2] & 0 & [2\mathcal{J}^2] & [2\mathcal{J}^2] & [-2\mathcal{J}^1] \\
[\mathcal{P}_2] & 0 & 0 & -[\mathcal{P}_2] & [-2\mathcal{J}^1] & [2\mathcal{J}^1] & [2\mathcal{J}^2] \\
[\mathcal{J}^1] & [\mathcal{J}^2] & 0 & 0 & 0 & 0 & 0 \\
[\mathcal{J}^2] & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Our task is to find a two-dimensional (in general complex) common invariant subspace of all six transformations. The abelian subalgebra generated by \(\mathcal{Q}_0, \mathcal{Q}_1\) and \(\mathcal{Q}_2\) is represented by triangular matrices. They have a common invariant subspace.
$T$ spanned by $[\mathcal{J}^1]$ and $[\mathcal{J}^2]$; there is no complex linear combination $[a\mathcal{P}_1 + b\mathcal{P}_2]$ that would be mapped by all $\mathcal{Q}_\mu$, $\mu = 0, 1, 2$, to a one-dimensional subspace of $T$. Thus, $T$ is the only two-dimensional invariant subspace of all $\mathcal{Q}_\mu$, $\mu = 0, 1, 2$. However, this subspace is not invariant with respect to $\mathcal{P}_0 + \mathcal{P}_1$. Hence, no subordinate algebra $\mathfrak{n}_F$ exists for the whole group $\text{SO}(2,3)$.

In fact, if we just want to have a unitary representation of $\text{SO}(2,3)$ by complex functions on a two-dimensional manifold $M$ (this reflects the fact that we have two physical degrees of freedom), then such a representation will determine a definite action of $\text{SO}(2,3)$ on $M$ that will be transitive, or else the representation will not be irreducible. Then, $M = \text{SO}(2,3)/G_M$, where $G_M$ is a stabilizer of a point of $M$. Thus, $\text{SO}(2,3)$ had to admit an 8-dimensional subgroup. However, there is no such subgroup $[28]$.

There is, however, $\mathfrak{n}_F$, if we restrict ourselves to some subgroup of $\text{SO}(2,3)$: the largest are $G_1$, generated by $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \mathcal{J}^0, \mathcal{J}^1, \mathcal{J}^2, \mathcal{D})$ and $G_2$ generated by $(\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{J}^0, \mathcal{J}^1, \mathcal{J}^2, \mathcal{D})$. $\Gamma_0$ is still a space where $G_1$ or $G_2$ act; they do not act transitively, however: the points with $x_1 = x_2 = 0$ are invariant with respect to $G_2$ and those with $y_1 = y_2 = 0$ with respect to $G_1$. These points have to be cut out in respective cases. Thus, the coordinate patch $(x, y^2)$ is a homogeneous space of $G_1$ and $(y, x^2)$ that of $G_2$. The action of the group $G_1$ on $T^*U$ is the same as that of $G_2$ on $T^*V$ in the respective coordinates. Let us consider $G_1$ and $\mathcal{M}^3$ only. In fact, only this case is a group extension of our original system, namely a $G_1$-extension.

From the corresponding part of the table, we can see immediately that there are two different invariant subspaces: $T_1$ spanned by $[\mathcal{P}_1]$ and $[\mathcal{P}_2]$ and $T_2$ spanned by $[\mathcal{P}_2]$ and $[\mathcal{J}^2]$. It is easy to see that there are no others. From $T_2$, we obtain the subalgebra $\mathfrak{n}_{2F}$ generated by $\mathcal{P}_0 + \mathcal{P}_1, \mathcal{P}_2, \mathcal{J}^0 - \mathcal{J}^1, \mathcal{J}^2$ and $\mathcal{D}$; $\mathfrak{n}_{2F}$ satisfies the conditions 1 and 2, but it does not satisfy Pukanszky’s condition. Indeed, the subspace $\mathfrak{n}_{2F}$ that annihilates it has the form $(a, -a, 0, b, b, 0, 0)$ where $(a, b) \in \mathbb{R}^2$.

The subset $F + \mathfrak{n}_{2F}$ is given by $(-1 + a, 1 - a, 0, b, b, 0, 0)$. This will lie in $\omega$ if the equations

$$
-x_1p_2 - x_2p_1 = b, \quad -x_2\sqrt{p \cdot p} = b, \quad x_1\sqrt{p \cdot p} = 0,
$$

have solutions for any $a$ and $b$. However, the first equation implies that $a < 1$. Thus, this algebra is not admissible. Similar calculation for $T_1$ shows that the corresponding algebra $\mathfrak{n}_{1F}$ satisfies all three conditions and so it is the only possibility. Let us concentrate on $\mathfrak{n}_{1F}$, which is generated by $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \mathcal{J}^0 - \mathcal{J}^1$ and $\mathcal{D} - \mathcal{J}^2$. 
The map of the Lie algebra $\mathfrak{g}_1$ of the group $G_1$ into $T_F\omega$ given by Eqs. (47) and (50)–(53) sends $\mathfrak{n}_{1F}$ on the subspace $E_F \in T_F\omega$ spanned by the vectors

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}.$$  

We can use the action $\text{ad}^*$ of the group $G_1$ to bring the subspace from the point $F$ to any other point of $\omega$; this is a well-defined procedure, because $\mathfrak{n}_{1F}$ is invariant with respect to the stabilizer $G_{1F}$ of $F$. The result is the subspace spanned by (61) at any point $(x_1, x_2, p_1, p_2)$ of $\omega$. Indeed, we can use the four one-dimensional subgroups of $G_1$ generated by $P_1$, $P_2$, $J^0$ and $J^2$. The corresponding vector fields given by Eqs. (47), (50) and (52) describe the action of these generators on $\omega$; their projections to the submanifold $x_1 = x_1^0$, $x_2 = x_2^0$ for any constant $x_1^0$ and $x_2^0$ is independent of $x_1^0$ and $x_2^0$. Thus, the curve defined by

$$x_1 = \gamma_1(t), \quad x_2 = \gamma_2(t), \quad p_1 = p_1^0, \quad p_2 = p_2^0,$$

with the real constants $p_1^0$ and $p_2^0$ will be mapped onto a curve of the form

$$x_1 = \gamma'_1(t), \quad x_2 = \gamma'_2(t), \quad p_1 = p_1^0, \quad p_2 = p_2^0$$

by any element of the group. Hence, the subspace $E_F$ will be mapped to the subspace $E_{F'}$ spanned by the vectors (61) at any point $F'$ of $\omega$ with $x_1 = x_2 = 0$ and $p_1$ and $p_2$ arbitrary. The vector fields (61) (which are now used in their role of the action of $P_1$ and $P_2$) can easily be integrated; they generate the maps

$$(x_1, x_2, p_1, p_2) \mapsto (x_1 + a, x_2 + b, p_1, p_2)$$

with arbitrary $a$ and $b$. Thus, curves of the form (62) are mapped to

$$x_1 = \gamma_1(t) + a, \quad x_2 = \gamma_2(t) + b, \quad p_1 = p_1^0, \quad p_2 = p_2^0.$$  

Hence, $E_{F'}$ goes over to $E_{F''}$ spanned again by (61) in an arbitrary point $F''$ of $\omega$. The resulting subbundle of the tangent bundle is called polarization. It is an integrable subbundle, its integral manifolds $E$ being given by $p_1 = \text{const}$, $p_2 = \text{const}$.

At this stage, it is much quicker to guess the form of the operators representing the Lie algebra that to calculate the representation according to the general procedure by Kostant-Kirillov. The unitary representation of $G_1$ that we are going to construct is induced by the representation $R_n$ of the subgroup $N_{1F}$ that is generated by the subalgebra $\mathfrak{n}_{1F}$. Thus, the Hilbert space will be built from complex functions on the homogeneous space $G_{1F}/N_{1F}$. This may be identified with the manifold $\bar{\Gamma}_{\text{ISO}}/E$ that is just $\mathbb{R}^2 \setminus \{0\}$ with the coordinates $p_1$ and $p_2$ and which we have denoted by $P$ in Sec. 4.1. If we look at the formula for the induced representation (see e.g. [28], P. 31...
479, formula (15)), we can see that there will be three kinds of terms in the operators representing the Lie algebra of \( G_1 \). From the representation of \( N_{1F} \), multiplicative terms will come; they must clearly be multiplications by \(-\sqrt{\mathbf{P} \cdot \mathbf{P}}\), \( p_1 \) and \( p_2 \) for the operators \( \hat{P}_0, \hat{P}_1 \) and \( \hat{P}_2 \). From the action of \( G_1 \) on the classes \( G_1/N_{1F} \), differential operators come; they must be projections of the vector fields (17), (50)–(53) to the space \( \mathcal{P} \) multiplied by \(-i\):

\[
J_{\text{diff}}^0 = i p_2 \frac{\partial}{\partial p_1} - i p_1 \frac{\partial}{\partial p_2},
\]

\[
J_{\text{diff}}^1 = -i \sqrt{\mathbf{P} \cdot \mathbf{P}} \frac{\partial}{\partial p_2}, \quad J_{\text{diff}}^2 = i \sqrt{\mathbf{P} \cdot \mathbf{P}} \frac{\partial}{\partial p_1},
\]

\[
D_{\text{diff}} = i p_1 \frac{\partial}{\partial p_1} + i p_2 \frac{\partial}{\partial p_2}.
\]

Finally, there will be terms coming from the Radon-Nikodym derivative that will correct the differential operators. Such terms have the general form

\[
\frac{i}{2\sigma} (\xi_{\text{diff}} \sigma),
\]

where \( \sigma \) is a quasi-invariant measure on \( G_1/N_{1F} \). Different but equivalent measures will lead to unitarily equivalent representations. A choice that strongly simplifies the correction terms is

\[
\sigma = \frac{1}{\sqrt{\mathbf{P} \cdot \mathbf{P}}}
\]

Then, finally, the operators must have the form

\[
\hat{P}_0 \psi(p) = -\sqrt{\mathbf{P} \cdot \mathbf{P}} \psi(p), \quad \hat{P}_k \psi(p) = p_k \psi(p), \quad (63)
\]

\[
\hat{J}^0 \psi(p) = i p_2 \frac{\partial \psi(p)}{\partial p_1} - i p_1 \frac{\partial \psi(p)}{\partial p_2}, \quad (64)
\]

\[
\hat{J}^1 \psi(p) = -i \sqrt{\mathbf{P} \cdot \mathbf{P}} \frac{\partial \psi(p)}{\partial p_2}, \quad \hat{J}^2 = i \sqrt{\mathbf{P} \cdot \mathbf{P}} \frac{\partial \psi(p)}{\partial p_1}, \quad (65)
\]

\[
\hat{D} = i p_1 \frac{\partial \psi}{\partial p_1} + i p_2 \frac{\partial \psi}{\partial p_2} + \frac{i}{2} \psi. \quad (66)
\]

It is straightforward but tedious to verify that this guessed operators coincide with those that would follow from the full general construction of the representation.

An interesting question is, what happen with the relations. For the group \( G_1 \), we have only three relations, and we can take Eqs. (11) and (13). Composing the corresponding operators on a common invariant domain (say, \( C_0^\infty(\mathcal{P}) \)), we obtain easily:

\[
- \hat{P}_0^2 + \hat{P}_1^2 + \hat{P}_2^2 = 0, \quad (67)
\]

\[
\hat{J}^0 \hat{P}_0 + \hat{J}^1 \hat{P}_1 + \hat{J}^2 \hat{P}_2 = 0, \quad (68)
\]

\[
- \hat{D}^2 - (\hat{J}^0)^2 + (\hat{J}^1)^2 + (\hat{J}^2)^2 = \frac{1}{4}. \quad (69)
\]
Thus only the last relation has been deformed (of course, there will be $\hbar^2/4$ at the right hand side, if $\hbar$, which has been set equal 1, is restored). The left hand sides of Eqs. (67)–(69) are generalized Casimirs of the group $G_1$ in the following sense. Let $E(g_1)$ be the enveloping algebra of $g_1$, let $\mathcal{H} = -P^2_0 + P^2_1 + P^2_2$ represent the constraint and let $\mathcal{I}(\mathcal{H})$ be the ideal in $E(g_1)$ generated by $\mathcal{H}$. Then, the classes of the left hand sides of Eqs. (67)–(69) in $E(g_1)/\mathcal{I}(\mathcal{H})$ commute with all elements of $E(g_1)/\mathcal{I}(\mathcal{H})$.

It is interesting to observe that each state $\psi$ of the Hilbert space $\mathcal{K}$ of the physical representation must satisfy the equation

$$(-\hat{P}_0^2 + \hat{P}_1^2 + \hat{P}_2^2)\psi = 0,$$

which follows from Eq. (67). This is a point where the group method touches the operator constraint method of quantization, because Eq. (70) has the form of the operator constraint equation of our system (for more detail, see [17]).

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