Doped Heisenberg chains: Spin-$S$ generalizations of the supersymmetric $t$–$J$ model

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A family of exactly solvable models describing a spin $S$ Heisenberg chain doped with mobile spin-$(S - 1/2)$ carriers is constructed from $gl(2|1)$-invariant solutions of the Yang-Baxter equation. The models are generalizations of the supersymmetric $t$–$J$ model which is obtained for $S = 1/2$. We solve the model by means of the algebraic Bethe Ansatz and present results for the zero temperature and thermodynamic properties. At low temperatures the models show spin charge separation, i.e. contain contributions of a free bosonic theory in the charge sector and an $SU(2)$-invariant theory describing the magnetic excitations. For small carrier concentration the latter can be decomposed further into an $SU(2)$ level-$2S$ Wess-Zumino-Novikov-Witten model and the minimal unitary model $\mathcal{M}_p$ with $p = 2S + 1$.

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1 Introduction

Doping of antiferromagnetic Mott insulators causes frustration which has a profound effect on the magnetic properties of such systems. High temperature superconductivity, charge ordering and anomalous transport properties have been observed in doped transition metal oxides with perovskite structure (see Ref. [1] for a review). Numerical studies of one-dimensional models proposed for doped Nickel-oxides and Manganites for such systems showed strong tendencies toward ferromagnetism and phase separation. For a better understanding of these phenomena requires to take into account strong electronic correlations. A commonly used starting point for a description of such systems is the ferromagnetic Kondo Hamiltonian [2, 3]

\[ H = -t \sum_{\langle ij \rangle} \left( c_{i\sigma}^\dagger c_{j\sigma} + h.c. \right) - J_H \sum_i \sigma_i \cdot S_i \]

(1.1)

where the first term is the kinetic energy given in terms of canonical fermionic creation and annihilation operators \( c_{i\sigma}^\dagger \) and \( c_{i\sigma} \) for the itinerant electrons on site \( i \) with spin \( \sigma = \uparrow, \downarrow \) and the second is the ferromagnetic Hund’s rule coupling between the spins \( \sigma_i^k = \sum_{\alpha\beta} c_{i\alpha}^\dagger (\sigma^k)_{\alpha\beta} c_{i\beta} \), \( k = x, y, z \), of the itinerant electrons to a localized spin \( S_n \). A Coulombic repulsion to suppress double occupancy in the itinerant band is implicit. Large Hund coupling \( J_H \) favours the alignment of the itinerant and the localized spin, i.e. spin eigenstates with the maximum allowed total spin. Hence, for local spins of length \( S - 1/2 \) the degrees of freedom that need to be kept for an effective theory of the low lying modes in this model are spin-\( S \) “spins” and spin-\( S - 1/2 \) “holes”. Within the double-exchange Hamiltonian [2-3] classical “background spins” \( S_i \) are used to approximate these holes. However, the non-trivial phases arising in a full quantum mechanical treatment of these spin degrees of freedom may induce low-energy modes which are essential for an understanding of the magnetic properties of these systems [4].

To obtain an effective lattice model on the 4\( S + 1 \) dimensional local Hilbertspace one has to eliminate the other allowed spin configurations in a perturbative analysis for \( J_H \gg t \) [4, 5]. To leading order in \( J_H \) the effective Hamiltonian of the resulting model is simply the projection of (1.1) onto the states listed above. \( SU(2) \)-invariance implies that this operator can be written as a polynomial of spin-operators:

\[ \mathcal{H}_{\text{eff}} \approx -t \sum_{\langle ij \rangle} P_{ij} Q_S(y_{ij}) \quad y_{ij} = S_i \cdot S_j / S(S - 1/2) \]

(1.2)

where the operator \( P_{ij} \) permutes the states on sites \( i \) and \( j \) thereby allowing the holes to move. For large but finite \( J_H \) one finds additional antiferromagnetic Heisenberg exchange terms.
Hamiltonian operators of this type have been used as a starting point for studies the phase diagram of doped transition metal oxides by numerical diagonalization of small clusters \[1,3,5\].

In this paper we introduce a class of exactly solvable models in one spatial dimension which generalize the supersymmetric \(t-J\) model \[7,8\] and a model for doped spin-1 chains \[10\]. In spite of the appearance of several additional couplings guaranteeing integrability these models may provide further insights into the peculiar properties of these compounds. In the following section we shall use the framework of the Quantum Inverse Scattering Method (QISM) to construct families of vertex models making use of so called ‘atypical’ representations of the super Lie algebra \(gl(2|1)\). The spectra of the corresponding commuting transfer matrices are obtained by means of the algebraic Bethe Ansatz (BA). In Sect. \[3\] we derive local Hamiltonians (i.e. operators involving nearest neighbour interactions on the lattice only) similar to the ones discussed above using a fusion procedure. In Sec. \[4\] integral equations determining the spectra and thermodynamic properties of these models are obtained from the BA equations. From these equations we obtain the phase diagram of the doped spin chains in a magnetic field for low temperatures \(T \ll H\) and the low temperature properties for vanishing field \(H = 0\) in Sects. \[5\] and \[6\]. We conclude with some remarks on a possible \(SU(2)\)-invariant effective field theory description for the low energy/low temperature sector of these systems.

2 Construction of the models

Below we will construct a class of integrable vertex models from solutions to the Yang Baxter equation (YBE) which are invariant under the action of the graded Lie algebra \(gl(2|1)\) \[11,12\]. The nine generators of \(gl(2|1)\) are classified into even \((1, S^z, S^\pm, B)\) and odd \((V_\pm, W_\pm)\) ones depending on their parity w.r.t. grading. The even generators are the spin operators \(S^\alpha\) form a \(SU(2)\) subalgebra with commutation relations 
\[
[S^z, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = 2S^z \quad \text{and the } U(1) 
\]
charge \(B\) which commutes with the \(S^\alpha\): \([B, S^\pm] = [B, S^z] = 0\). The commutators between even and odd generators of the algebra are
\[
[S^z, V_\pm] = \pm \frac{1}{2} V_\pm, \quad [S^\pm, V_\pm] = 0, \quad [S^\mp, V_\mp] = V_\mp, \quad [B, V_\pm] = \frac{1}{2} V_\pm, \\
[S^z, W_\pm] = \pm \frac{1}{2} W_\pm, \quad [S^\pm, W_\pm] = 0, \quad [S^\mp, W_\mp] = W_\mp, \quad [B, W_\pm] = -\frac{1}{2} W_\pm, \quad (2.1)
\]
and the odd generators satisfy anticommutation rules
\[
\{V_\pm, V_\mp\} = \{V_\pm, V_\mp\} = \{W_\pm, W_\mp\} = \{W_\pm, W_\mp\} = 0, \quad \{V_\pm, W_\pm\} = \pm \frac{1}{2} S^\pm, \quad \{V_\mp, W_\mp\} = \frac{1}{2} (-S^z \pm B). \quad (2.2)
\]
The ‘typical’ representations $[b, s]$ of this algebra can be characterized by the eigenvalues of operators $B$ and $S^2$ on the multiplet with largest total $SU(2)$-spin $[11, 12]$. Their dimension is $8S$ and they can be decomposed into two spin-$(S - 1/2)$ multiplets with charge $b \pm 1/2$ and a spin-$S$ and a spin-$(S - 1)$ multiplet with charge $b$ with respect to the $SU(2)$-subalgebra of $gl(2|1)$. In the following we shall be particularly interested in the $(4S + 1)$-dimensional so-called ‘atypical’ representations $[S]_+$ which contain two multiplets of spin $S$ and $(S - 1/2)$ and corresponding charges $b = S$ and $b = S + 1/2$. Choosing a basis $\{|b, s, m\}\rangle$ in which $B$, $S^2$ and $S^z$ are diagonal, the nonvanishing matrix elements of the fermionic operators are

\[
\langle S + \frac{1}{2}, S - \frac{1}{2}, m \pm \frac{1}{2} | V_{\pm} | S, S, m \rangle = \pm \sqrt{\frac{S + m}{2}}.
\]

\[
\langle S, S, m | W_{\pm} | S + \frac{1}{2}, S - \frac{1}{2}, m \mp \frac{1}{2} \rangle = \sqrt{\frac{S \pm m}{2}}.
\]

(2.3)

Tensor products of atypical representation can be decomposed as

\[
[S]_+ \otimes [S']_+ = [S + S']_+ \oplus \left[ S + S' + \frac{1}{2}, S + S' - \frac{1}{2} \right] \oplus \cdots \oplus \left[ S + S' + \frac{1}{2}, |S - S'| + \frac{1}{2} \right].
\]

(2.4)

The irreducible components in this tensor product can be identified by the action of the quadratic Casimir of the algebra

\[
K_2 = S^2 - B^2 - V_- V_+ + V_+ V_- - V_- W_+ + V_+ W_-
\]

(2.5)

which has eigenvalues 0 on $[s]_+$ and $(s^2 - b^2)$ on $[b, s]$.

Choosing an irreducible $d$-dimensional representation of $gl(2|1)$ acting on a quantum space $V \sim \mathbb{C}^d$, it is straightforward to verify that the $\mathcal{L}$-operator

\[
\mathcal{L}(\mu) = \begin{pmatrix}
\mu + 2iB & i\sqrt{2}W_- & i\sqrt{2}W_+ \\
i\sqrt{2}V_+ & \mu + i(B + S^z) & -iS^+ \\
-i\sqrt{2}V_- & -iS^- & \mu + i(B - S^z)
\end{pmatrix}.
\]

(2.6)

written as a matrix in the three-dimensional matrix space $\mathcal{M}$ solves the Yang-Baxter equation

\[
R(\lambda - \mu) (\mathcal{L}(\lambda) \otimes \mathcal{L}(\mu)) = (\mathcal{L}(\mu) \otimes \mathcal{L}(\lambda)) R(\lambda - \mu)
\]

(2.7)

with the $R$-matrix

\[
R(\lambda) = b(\lambda)I + a(\lambda)\Pi, \quad a(\lambda) = \frac{\lambda}{\lambda + i}, \quad b(\lambda) = \frac{i}{\lambda + i}.
\]

(2.8)
Here $I$ and $\Pi$ are the unit and graded permutation operator acting on the tensor product $M_1 \otimes M_2$ of matrix spaces in which $\mathcal{L}$-operators act in (2.7). Assigning Grassmann parities $\epsilon_i \in \{0, 1\}$ to the basis of these spaces the matrix elements of $\Pi$ are

$$\Pi_{i_2 j_2}^{i_1 j_1} = (-1)^{\epsilon_1 \epsilon_2} \delta_{i_1 j_2} \delta_{i_2 j_1}. \quad (2.9)$$

Similarly, the matrix elements of the operators acting on the tensor product of these spaces pick up signs $(A \otimes B)_{i_2 j_2}^{i_1 j_1} = (-1)^{\epsilon_2 (\epsilon_1 + \epsilon_j)} A_{i_1 j_1} B_{i_2 j_2}$ due to the grading of the basis. Considering the $\mathcal{L}$-operator as a linear operator acting on the tensor-product of spaces $\mathcal{M} \otimes \mathcal{V}$ with the fundamental three-dimensional representation $[1/2]_+$ acting on the matrix space its $gl(2|1)$-invariance can be established by rewriting (2.6) as $\mu - i K_2$ in terms of the Casimir operator (2.5) on the tensor product (up to a shift of the spectral parameter).

The intertwining relation (2.7) implies that the monodromy matrix, defined as the matrix product

$$\mathcal{T}(\lambda) = \mathcal{L}_L(\lambda) \mathcal{L}_{L-1}(\lambda) \cdots \mathcal{L}_1(\lambda) \quad (2.10)$$

of $\mathcal{L}_n$-operators (2.6) with entries acting on different quantum spaces $\mathcal{V}_n$ satisfies a Yang-Baxter equation with the same $R$-matrix (2.8):

$$R(\lambda - \mu) (\mathcal{T}(\lambda) \otimes \mathcal{T}(\mu)) = (\mathcal{T}(\mu) \otimes \mathcal{T}(\lambda)) R(\lambda - \mu). \quad (2.11)$$

As an immediate consequence of this identity the transfer matrix given by the matrix super trace of $\mathcal{T}$

$$t_3(\mu) = sTr (\mathcal{T}(\mu)) = \sum_{i=1}^3 (-1)^{\epsilon_i} [\mathcal{T}(\mu)]^{iii} \quad (2.12)$$

commutes for different values of the spectral parameter $\mu$ thus being the generating functional for a family of commuting operators on the graded tensor product of $L$ quantum spaces which we will identify below with the Hilbert space of an integrable spin chain. The subscript to the transfer matrix is used to label the dimension of the matrix space of the corresponding monodromy matrix.

The spectrum of this transfer matrix is obtained by means of the algebraic Bethe Ansatz (ABA) [14]. As a consequence of the grading different sets of Bethe Ansatz equations (BAE) follow from different orderings of the basis [13]. We now restrict ourselves to representations $[S]_+$ in the quantum spaces $\mathcal{V}_n$ where we choose the state $|0\rangle_n \equiv |S, S, S\rangle_n$ as our reference state. The two other possible sets of BAE for this model are given in Appendix A, their equivalence is shown in Appendix B.
Since the ABA for the transfer matrix (2.12) is completely analogous to the case considered in [13, 14, 15], we only sketch the main steps leading to the BAE and the spectrum. The action of (2.6) on the reference state is

$$\mathcal{L}_n(\mu)|0\rangle_n = \begin{pmatrix} \mu + 2iS & 0 & 0 \\ 0 & \mu + 2iS & 0 \\ -i\sqrt{2}V & -iS_n & \mu \end{pmatrix} |0\rangle_n .$$  

(2.13)

Similarly, acting with the monodromy matrix (2.10) on the state $|\Omega_S\rangle = |0\rangle_L \otimes \cdots \otimes |0\rangle_1$ we get

$$\mathcal{T}(\mu)|\Omega_S\rangle = \begin{pmatrix} (\mu + 2iS)^L & 0 & 0 \\ 0 & (\mu + 2iS)^L & 0 \\ C_1(\mu) & C_2(\mu) & \mu^L \end{pmatrix} |\Omega_S\rangle .$$  

(2.14)

Hence, $|\Omega_S\rangle$ is an eigenstate of the transfer matrix (2.12) with eigenvalue ($-\mu^L$) (we have chosen the grading $\epsilon_1 = 0$, $\epsilon_2 = \epsilon_3 = 1$ in the matrix space here). The operators $C_1(\lambda)$ and $C_2(\lambda)$ create a hole and lower the spin of the system respectively. For eigenstates of $t_{3S}(\lambda)$ with $N_h$ holes (generating sites with spin $S - 1/2$) and magnetization $M^z = LS - \frac{1}{2}N_h - N_\downarrow$ we make the Ansatz

$$|\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n|F\rangle = C_{a_1}(\tilde{\lambda}_1) \cdots C_{a_n}(\tilde{\lambda}_n)|\Omega_S\rangle F^{a_n \cdots a_1}$$  

(2.15)

where $n = N_h + N_\downarrow$. Using the algebra of the operators in (2.11) we are led to an eigenvalue problem for the amplitudes $F^{a_n \cdots a_1}$ which is solved by a second Bethe Ansatz parametrized by 'hole rapidities' $\{\tilde{\nu}_\alpha\}_{a=1}^{N_h}$. Finally, we find that (2.15) is an eigenstate of (2.12) with eigenvalue

$$\Lambda_3 \left( \mu | \{\tilde{\lambda}_j\}_{j=1}^{N_h+N_\downarrow}, \{\tilde{\nu}_\alpha\}_{\alpha=1}^{N_h} \right) = -\mu^L \prod_{j=1}^{N_h+N_\downarrow} \frac{\mu - \tilde{\lambda}_j + i}{\mu - \tilde{\lambda}_j} + (\mu + 2iS)^L \prod_{\alpha=1}^{N_h} \frac{\mu - \tilde{\nu}_\alpha + i}{\mu - \tilde{\nu}_\alpha} \left\{ 1 - \prod_{j=1}^{N_h+N_\downarrow} \frac{\tilde{\lambda}_j - \mu + i}{\tilde{\lambda}_j - \mu} \right\}$$  

(2.16)

provided the spectral parameters $\tilde{\lambda}_j \equiv \lambda_j - iS$ and $\tilde{\nu}_\alpha \equiv \nu_\alpha - iS + i/2$ satisfy the following set of BAE

$$\left( \frac{\lambda_j + iS}{\lambda_j - iS} \right)^L = \prod_{k \neq j}^{N_h+N_\downarrow} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i} \prod_{\alpha=1}^{N_h} \frac{\lambda_j - \nu_\alpha - \frac{i}{2}}{\lambda_j - \nu_\alpha + \frac{i}{2}},$$

$$j = 1, \ldots, N_h + N_\downarrow$$

$$1 = \prod_{k=1}^{N_h+N_\downarrow} \frac{\nu_\alpha - \lambda_k + \frac{i}{2}}{\nu_\alpha - \lambda_k - \frac{i}{2}}, \quad \alpha = 1, \ldots, N_h .$$  

(2.17)
3 Doped spin chains

Choosing the fundamental three dimensional representation $[1/2]_+$ of $gl(2|1)$ in (2.6), the $\mathcal{L}$-operator taken at $\mu = -i$ becomes a graded permutation operator on the tensor product $\mathcal{M} \otimes \mathcal{V}$ of matrix and quantum space. Hence the transfer matrix (2.12) generates a translation by one site on the lattice for this value of the spectral parameter. Expanding the logarithm of $t_3(\mu)$ about this shift point we can construct a Hamiltonian with nearest neighbour interactions only (2.12)

$$-i \frac{\partial}{\partial \mu} \ln t_3(\mu) \bigg|_{\mu = -i} = \sum_{n=1}^{L} \ldots$$

which is the supersymmetric $t-J$ model (see e.g. [13]). In this case, Eqs. (2.17) are known as Sutherland’s form of the BAE for this model [8].

For representation different from $[1/2]_+$ it is not possible to construct a local Hamiltonian directly from the $\mathcal{L}$-operators (2.6) (they can be used to construct $t$–$J$ models perturbed by integrable impurities though [17–19]). To obtain a homogenous lattice model such as (3.1) new $\mathcal{L}$-operators have to be found which act on tensor products of matrix and quantum spaces of the same dimension. Their $gl(2|1)$-invariance implies that they can be expressed as sum over multiples of the projectors on irreducible components of the tensor product of representations in the two spaces. Noting that the spin multiplets at charge $(S + S')$ in the tensor product (2.4) are just the ones obtained by adding two spins of length $S$, $S'$ (and similarly spins $(S - 1/2)$, $(S' - 1/2)$ at charge $(S + S' + 1)$) we find linear operators $\mathcal{L}^{(SS')}(\mu)$ acting on spaces carrying atypical representations $[S]_+$ and $[S']_+$ from the $gl(2)$-invariant ones constructed in Ref. [20], namely:

$$\mathcal{L}^{(SS')}(\mu) = -\prod_{k=|S-S'|+1}^{S+S'} \frac{\mu - ik}{\mu + ik} \mathcal{P}[S+S']_+ - \sum_{m=|S-S'|}^{S+S'-1} \prod_{k=|S-S'|+1}^{m} \frac{\mu - ik}{\mu + ik} \mathcal{P}[S+S'+1, m+1].$$

(3.2)

Here $\mathcal{P}_\Lambda$ is a projector on the $gl(2|1)$-multiplet $\Lambda$ in the tensor product $[S]_+ \otimes [S']_+$. Choosing one of the representations to be $[1/2]_+$ and comparing this expression with (2.6) we find

$$\mathcal{L}^{(1/2,S)}(\mu) = -\frac{1}{\mu + i(S + 1/2)} \mathcal{L}(\mu - i(S + 1/2)).$$

(3.3)

The new $\mathcal{L}$-operators satisfy the YBEs

$$R_{S_1S_2}(\lambda - \mu) \left( \mathcal{L}^{(S_1S_3)}(\lambda) \otimes \mathcal{L}^{(S_2S_3)}(\mu) \right) = \left( \mathcal{L}^{(S_2S_3)}(\mu) \otimes \mathcal{L}^{(S_1S_3)}(\lambda) \right) R_{S_1S_2}(\lambda - \mu)$$

(3.4)

where $R_{SS'} = \Pi \mathcal{L}^{(SS')}$. As a consequence of this relation the transfer matrices of the corresponding vertex models

$$t^{(S_0S)}(\mu) = sTr_0 \left( \mathcal{L}_L^{(S_0S)}(\mu) \cdots \mathcal{L}_1^{(S_0S)}(\mu) \right)$$

(3.5)
(the product of \(L\)-operators and the super trace are taken in the \((4S_0 + 1)\)-dimensional matrix space) commute, i.e.  
\[ [t^{(S_0 S)}(\mu), t^{(S_1 S)}(\lambda)] = 0 \]  
for all \(S_0, S_1\). From (3.2) we observe that  
\(L^{SS}(\mu = 0) \propto \Pi\). As in (3.1) an integrable Hamiltonian with nearest neighbour interactions on the lattice can be constructed by taking the logarithmic derivative of \(\ln t^{(SS)}(\mu)\) at \(\mu = 0\).

The eigenstates of the transfer matrices \(t^{(S')}(\mu)\) are parametrized by the roots of the BAE (2.17). To compute their eigenvalues we need the so-called fusion relations between these operators for different \(S'\) which are obtained from considering tensor products of different matrix spaces.

Starting with the YBE (3.4) for \(S_1 = S_2 = 1/2\) we observe that choosing \(\lambda - \mu = -i\) the matrix \(R_{\frac{1}{2}-\frac{1}{2}}\) is proportional to a projector onto the five-dimensional subrepresentation \([1]_+\) of the tensor product \([1/2]_+ \otimes [1/2]_+\). This implies that the \(L\)-operator \(\tilde{L}(\mu) = L^{(1/2,S)}(\mu - i/2) \otimes L^{(1/2,S)}(\mu + i/2)\) satisfies the condition

\[ P_{[1]+} \tilde{L}(\mu) P_{[3/2,1/2]} = 0 \]  
(3.6)

Consequently, it can be rewritten as

\[ \tilde{L}(\mu) \sim \begin{pmatrix} L^{(3/2,1/2,2)}(\mu) & \ast \\ 0 & L^{(1,1)}(\mu) \end{pmatrix} \]  
(3.7)

after a proper reordering of the basis in \(\mathcal{M}_1 \otimes \mathcal{M}_2\). Here \(L^{(3/2,1/2,2)}(\mu)\) is a \(4 \times 4\) matrix acting on the \([3/2,1/2]\) component of this tensor product. Building a monodromy matrix from \(L\) copies of (3.7) we obtain the fusion relation for the corresponding transfer matrices

\[ \tilde{t}(\mu) \equiv \text{sTr} \left( \tilde{L}_L(\mu) \cdots \tilde{L}_1(\mu) \right) = t^{(1/2,S)}(\mu - i/2) t^{(1/2,S)}(\mu + i/2) = t^{(1,1)}(\mu) + t^{(3/2,1/2,2)}(\mu) \]  
(3.8)

and an equivalent equation for their eigenvalues \(\Lambda^{(S)}(\mu)\). Since there are still two unknown functions of \(\mu\) on the RHS of this equation it is not possible to determine the spectrum of the new transfer matrices from (3.8) alone. As additional information we make use of the fact that the eigenvalues of the transfer matrix are analytical functions of \(\mu\), i.e. the residues at their simple poles vanish as a consequence of the BAE (2.17). Complemented by the trivial action of \(t^{(1,1)}(\mu)\) and \(t^{(3/2,1/2,2)}(\mu)\) on the pseudo vacuum \(|\Omega_S\rangle\) this allows to compute \(\Lambda^{(1,S)}\) with the result (see e.g. [21])

\[ \Lambda^{(1,S)}(\mu \{ \lambda_j \}_{j=1}^{N_h+N_\downarrow}, \{ \nu_\alpha \}_{\alpha=1}^{N_h}) = \left( \frac{\mu - i S}{\mu + i S} \frac{\mu - i (S + 1)}{\mu + i (S + 1)} \right)^L \prod_{j=1}^{N_h+N_\downarrow} \frac{\mu - \lambda_j + i}{\mu - \lambda_j - i} \]
The exchange and kinetic part of the Hamiltonian expressed in terms of spin operators $H$ can permute the spins on sites $ij$.

As observed above, the local vertex operator $L^{(1,1)}(\mu = 0)$ becomes a graded permutation operator on the tensor product of the five-dimensional matrix space and the quantum space in which the representation $[S = 1]_+$ is acting. Hence we can proceed as for the case of the fundamental representation above and obtain the local lattice Hamiltonian for a spin-1 chain doped with $S = 1/2$ holes introduced in Ref. [14]:

$$H^{(1)} = -i \frac{\partial}{\partial \mu} \ln \left( t^{(1,1)}(\mu) \right) \bigg|_{\mu=0} - 3L = \sum_{n=1}^{L} \left\{ H_{n,n+1}^{\text{exch}} + H_{n,n+1}^{\text{hopp}} \right\} - N_h.$$ (3.10)

The exchange and kinetic part of the Hamiltonian expressed in terms of spin operators $S_i$ with $S_i^2 = S_i(S_i + 1)$ with $S_i = 1$ or $1/2$ read

$$H_{ij}^{\text{exch}} = \frac{1}{2} \left( \frac{1}{S_i S_j} S_i \cdot S_j - 1 + \delta_{S_i S_j, 1} \left( 1 - (S_i \cdot S_j)^2 \right) \right) ,$$

$$H_{ij}^{\text{hopp}} = - (1 - \delta_{S_i, S_j}) \mathcal{P}_{ij} (S_i \cdot S_j) .$$

$\mathcal{P}_{ij}$ permutes the spins on sites $i$ and $j$. The corresponding eigenvalues of (3.10) are obtained from (3.9): adding an external magnetic field $H$ and a chemical potential

$$E^{(1)}(\{\lambda_j\}, \{\nu_\alpha\}) = HM^z - \mu N_h$$

$$= \sum_{k=1}^{N_h+N_i} \left( H - \frac{2}{\lambda_k^2 + 1} \right) - \sum_{\alpha=1}^{N_h} \left( \mu + \frac{1}{2}H \right) - LH$$ (3.11)

(we have added an external magnetic $H$ field and a (hole) chemical potential $\mu$ to the Hamiltonian).

To proceed to higher $S$ we iterate the procedure used above: for $S_1 = 1/2$ and $S_2 = S' - 1/2$ arbitrary we use the fact that $R_{\frac{1}{2}, S - \frac{1}{2}}(\mu = -iS') \propto \mathcal{P}_{[S']_+}$ in the YBE (3.4). This leads to the fusion relation

$$t^{(1/2,S)} \left( \mu - i \left( S' - \frac{1}{2} \right) \right) t^{(S'-1/2,S)} \left( \mu + \frac{i}{2} \right) = t^{(S',S)}(\mu) + t^{(S'+1/2,S'-1/2,S)}(\mu)$$ (3.12)

which allows to determine the eigenvalues of $t^{(S',S)}(\mu)$ from the known ones of $t^{(1/2,S)}(\mu)$ and $t^{(S'-1/2,S)}(\mu)$ as

$$\Lambda^{(S',S)} \left( x^{N_h+N_i}, \{\lambda_j\}^{N_h}, \{\nu_\alpha\}^{N_h} \right) = \left( \frac{S+S'}{k=S-S'+1} \mu - ik \right)^{L_{N_h+N_i}} \prod_{j=1}^{N_h} \frac{\mu - \lambda_j + iS'}{\mu - \lambda_j - iS'} .$$
The terms in braces are determined by the fusion equations together with the vanishing of the residues at the simple poles of $\Lambda^{(S',S)}(\mu)$ due to the BAE (2.17). They do not contribute to the spectrum of the nearest neighbour spin chain Hamiltonian

$$\mathcal{H}^{(S)} = -i \frac{\partial}{\partial \mu} \ln \left( t^{(S,S)}(\mu) \right) \bigg|_{\mu=0} + \text{const.}$$

(3.14)

whose eigenvalues are

$$E^{(S)} \left( \{\lambda_j\}, \{\nu_\alpha\} \right) - HM^z - \mu N_h$$

$$= \sum_{k=1}^{N_h+N_i} \left( H - \frac{2S}{\lambda_k^2 + S^2} \right) - \sum_{\alpha=1}^{N_h} \left( \mu + \frac{1}{2} H \right) - LH.$$ (3.15)

### 4 Thermodynamic Bethe Ansatz

To study the thermodynamics of the doped spin chains (3.14) we have to analyze the BAE (2.17). In the thermodynamic limit $L \to \infty$ general solutions are known to consist of real hole rapidities $\nu_\alpha$ and complex $n$-strings of spin-rapidities $\lambda_j^n$. Rewriting the BAE in terms of the real variables $\nu_\alpha$ and $\lambda_j^n$ and taking the logarithm we obtain

$$z_c(\nu_\alpha) = \frac{I_\alpha}{L}, \quad z_s^{(n)} \left( \lambda_j^{(n)} \right) = \frac{J_j^{(n)}}{L},$$

(4.2)

where $J_j^{(n)}$ and $I_\alpha$ are integers (or half-odd integers) and the functions $z_i$ are given as

$$2\pi z_s^{(n)}(\lambda) = \theta_{n,2S}(\lambda) - \frac{1}{L} \sum_{m=1}^{\infty} \sum_{j=1}^{M_m} \Xi_{nm} \left( \lambda - \lambda_j^{(m)} \right) + \frac{1}{L} \sum_{\alpha=1}^{N_h} \theta_n \left( \lambda - \nu_\alpha \right)$$

$$2\pi z_c(\nu) = \frac{1}{L} \sum_{n=1}^{\infty} \sum_{j=1}^{M_n} \theta_n \left( \nu - \lambda_j^{(n)} \right)$$

(4.3)

where $\theta_n(x) = 2 \arctan(2x/n)$ and

$$\theta_{nm}(x) = \theta_{m+n-1}(x) + \theta_{m+n-3}(x) + \ldots + \theta_{|m-n|+1}(x),$$

$$\Xi_{nm}(x) = \theta_{n+m}(x) + 2\theta_{n+m-2}(x) + \ldots + 2\theta_{|n-m|+2}(x) + (1 - \delta_{nm}) \theta_{|n-m|}(x).$$ (4.4)
The quantum numbers $J_j^{(n)}$ and $I_\alpha$ in (4.2) uniquely determine a particular eigenstate of the system. The asymptotic behaviour of the functions (4.3) determine their possible values. This allows to introduce densities $\rho(\nu)$ of the hole rapidities, $\sigma_n(\lambda)$ of the $\lambda$-strings and the corresponding hole densities $\tilde{\rho}(\nu)$ and $\tilde{\sigma}_n(\lambda)$ with

$$\sigma_n(x) + \tilde{\sigma}_n(x) = \frac{\partial}{\partial x} z_s^{(n)}(x), \quad \rho(x) + \tilde{\rho}(x) = \frac{\partial}{\partial x} z_c(x). \quad (4.5)$$

In the thermodynamic limit $L \to \infty$ with $N_h/L$ and $M_n/L$ held fixed these equations become linear integral equations

$$\tilde{\sigma}_n(x) = (A_{n,2S} * s)(x) - \sum_m (A_{nm} * \sigma_m)(x) + (a_n * \rho)(x),$$

$$\rho(x) + \tilde{\rho}(x) = \sum_n (a_n * \sigma_n)(x). \quad (4.6)$$

Here, $(f * g)(x)$ denotes a convolution, $2\pi a_n(x) = \theta_n(x) = 4n/(4x^2 + n^2)$, $s(x) = 1/(2 \cosh \pi x)$ and

$$A_{nm}(x) = \frac{1}{2\pi} \Xi_{nm}'(x) + \delta_{nm} \delta(x). \quad (4.7)$$

Similarly, the energy $E/L$ in the thermodynamic limit can be rewritten in terms of the densities

$$E/L = \sum_{n=1}^{\infty} \int dx \left( \epsilon_n^{(0)}(x) + n H \right) \sigma_n(x) - \int dx \left( \mu + \frac{1}{2} H \right) \rho(x) \quad (4.8)$$

where $\epsilon_n^{(0)}(x) = -2\pi (A_{n,2S} * s)(x)$ are the bare energies of the $\lambda$-strings.

At finite temperature the equilibrium state is obtained by minimization of the free energy $F = E - TS$ by variation of $\sigma_n$ and $\rho$. Here $S$ is the combinatorical entropy $22$

$$S/L = \sum_{n=1}^{\infty} \int dx \left\{ (\sigma_n + \tilde{\sigma}_n) \ln (\sigma_n + \tilde{\sigma}_n) - \sigma_n \ln \sigma_n - \tilde{\sigma}_n \ln \tilde{\sigma}_n \right\}$$

$$+ \int dx \left\{ (\rho + \tilde{\rho}) \ln (\rho + \tilde{\rho}) - \rho \ln \rho - \tilde{\rho} \ln \tilde{\rho} \right\}. \quad (4.9)$$

As a result we obtain the thermodynamic Bethe ansatz (TBA) equations for the energies $\epsilon_n = T \ln(\tilde{\sigma}_n/\sigma_n)$ of $\lambda$-strings and $\kappa = T \ln(\tilde{\rho}/\rho)$ for the hole rapidities

$$\epsilon_n(x) - \frac{T}{2\pi} \sum_m \Xi_{nm}' \ln \left[ 1 + e^{-\epsilon_m/T} \right](x) + Ta_n \ln \left[ 1 + e^{-\kappa/T} \right](x) = \epsilon_n^{(0)}(x) + nH$$

$$\kappa(x) + T \sum_m a_m \ln \left[ 1 + e^{-\epsilon_m/T} \right](x) = -\left( \mu + \frac{1}{2} H \right) \quad (4.10)$$

An alternative form of these equations can be obtained by using the identity $\sum_k (C_{nk} * A_{km})(x) = \delta_{nm} \delta(x)$ with

$$C_{nm}(x) = \delta_{nm} \delta(x) - (\delta_{n+1,m} + \delta_{n-1,m}) s(x). \quad (4.11)$$
This allows to rewrite the integral eqs. (4.6) as
\[ \delta_{n,2S} s(x) = \sigma_n(x) + (C_{nm} * \tilde{\sigma}_m)(x) - \delta_{n,1} (s * \rho)(x) \]
\[ (a_{2S} * s)(x) = \tilde{\rho}(x) + [(1 + a_2)^{-1} * \rho](x) + (s * \tilde{\sigma}_1)(x). \quad (4.12) \]

Similarly, we find for the energy of the corresponding state
\[ E/L = E_0^{(S)}/L - \int dx [2\pi(a_{2S} * s) + \mu] \rho(x) + \int dx 2\pi s(x) \tilde{\sigma}_{2S}(x) \]
\[ - \lim_{n \to \infty} H n \int dx \tilde{\sigma}_n(x) \]
where \( E_0^{(S)} \) is the ground state energy of the spin-\( S \) Takhtajan–Babujian chain in a vanishing magnetic field \( [23] \)
\[ E_0^{(S)} = \begin{cases} -\sum_{k=1}^{S} \frac{2}{2k-1} & \text{for integer } S \\ -2 \ln 2 - \sum_{k=1}^{S-1/2} \frac{1}{k} & \text{for half-integer } S \end{cases} \quad (4.14) \]

Finally, an equivalent form of the TBA equations (4.10) is
\[ \epsilon_n(x) = T (s * \ln [1 + e^{\epsilon_n/T}][1 + e^{\epsilon_{n+1}/T}]) (x) \]
\[ -2\pi \delta_{n,2S} s(x) - \delta_{n,1} T (s * \ln [1 + e^{-\kappa/T}]) (x), \quad (4.15) \]
subject to the condition \( \lim_{n \to \infty} (\epsilon_n/n) = H \) and
\[ - [2\pi a_{2S} * s(x) + \mu] - T (s * \ln [1 + e^{\epsilon_1/T}]) (x) = \kappa(x) + T (R * \ln [1 + e^{-\kappa/T}]) (x) \]
where \( R = a_2 * (1 + a_2)^{-1} \).

In terms of the solutions to these equations the free energy reads
\[ F/L = E_0^{(S)}/L - T \int dx s(x) \ln [1 + e^{\epsilon_2S(x)/T}] - T \int dx (a_{2S} * s)(x) \ln [1 + e^{-\kappa(x)/T}] . \quad (4.17) \]

5 Zero temperature phases in a magnetic field

In the limit \( T \to 0 \) the TBA eqs. (4.10) become linear integral equations. As a consequence of (4.13) only \( \epsilon_1(x), \epsilon_{2S}(x) \) and \( \kappa(x) \) can take negative values for certain \( x \). Hence we have to solve three coupled integral equations for these quantities which in turn determine all other dressed energies
\[ \epsilon_n(x) + \frac{1}{2\pi} \left\{ \Xi'_{n1} * \epsilon_1^{(-)} + \Xi'_{n,2S} * \epsilon_{2S}^{(-)} \right\} (x) - a_n * \kappa^{(-)}(x) = \epsilon_n^{(0)}(x) + nH \]
\[ \kappa(x) - \left\{ a_1 * \epsilon_1^{(-)} + a_{2S} * \epsilon_{2S}^{(-)} \right\} (x) = - \left( \mu + \frac{1}{2} H \right), \quad (5.1) \]
where \( f^{(\pm)}(x) = \theta(\pm f(x)) f(x) \).

To discuss the solutions of these equations further we have to distinguish various cases:
5.1 \( \mu > H/2 \)

In this regime we have \( \kappa(x) < 0 \) and \( \epsilon_1(x) < 0 \) for all \( x \). This allows to express these functions in terms of the remaining unknown function \( \epsilon_{2S}(x) \). From (5.1) we find

\[
\kappa(x) = -2\mu + \left( a_1 * \epsilon_1^{(0)} \right)(x)
\]

\[
\epsilon_1(x) = \epsilon_1^{(0)}(x) - \left( \mu - \frac{H}{2} \right) - \left( a_{2S-1} * \epsilon_{2S}^{(-)} \right)(x)
\]

\[
\epsilon_{n>1}(x) + \frac{1}{2\pi} \left( \Xi'_{n-1,2S-1} * \epsilon_{2S}^{(-)} \right)(x) = -2\pi \left( A_{n-1,2S-1} * s \right)(x) + (n - 1)H
\]

The last set of equations can be identified with the integral equations for the dressed energies of the spin-\( S - 1/2 \) Takhtajan–Babujian model, hence this regime corresponds to the completely doped case (i.e. \( N_h = L \) holes). For magnetic field \( H > H^{(S-1/2)} \) with

\[
H^{(\sigma)} > \frac{2}{\sigma} \sum_{k=1}^{2\sigma} \frac{1}{2k-1}
\]

the system is in a ferromagnetically saturated state with maximal magnetization \( M^z = L(S - 1/2) \).

5.2 \(-H/2 < \mu < H/2\)

Here we find from (5.1) that \( \kappa(x) \equiv \kappa^{(-)}(x) < 0 \) for all \( x \) while \( \epsilon_1(x) \) can take non-negative values. Eliminating \( \kappa(x) \) from the integral equations for \( \epsilon_1 \) and \( \epsilon_{2S} \) we obtain

\[
\epsilon_1(x) + \left\{ a_{2S-1} * \epsilon_{2S}^{(-)} \right\}(x) = \epsilon_1^{(0)}(x) - \mu + \frac{1}{2}H,
\]

\[
\epsilon_{2S}(x) + \left\{ a_{2S-1} * \epsilon_1^{(-)} + 2 \sum_{k=1}^{2S-1} a_{2k} * \epsilon_{2S}^{(-)} \right\}(x) = \epsilon_{2S}^{(0)}(x) - \mu + \frac{4S - 1}{2}H.
\]

In this regime we find \( \epsilon_1 > 0 \) and \( \epsilon_{2S} > 0 \) for

\[
\mu < \min \left\{ \frac{4S - 1}{2}H - 4 \sum_{k=1}^{2S} \frac{1}{2k-1}, \frac{1}{2}H - \frac{2}{S} \right\}.
\]

Positive dressed energies for the \( \lambda \)-strings imply \( M_n = 0 \) and from (4.6) we find that \( N_h = 0 \) in this region. Hence for \( T \to 0 \) (5.3) belongs to the ferromagnetically saturated phase of the undoped system, namely the spin-\( S \) Takhtajan-Babujian model. This phase exists for magnetic fields \( H > H^{(S)} \).

Increasing the hole chemical potential to values \( \mu > H/2 - 2/S \) holes are added but the ground state continues to be fully polarized: For the dressed energies this corresponds to \( \epsilon_{2S} > 0 \).
and while the real spin rapidities fill all states with negative $\epsilon_1(x)$. As a consequence of the free fermionic nature of this state these dressed energies can be expressed in terms of their free values

$$\epsilon_1(x) = \epsilon_1^{(0)}(x) - \left(\mu - \frac{H}{2}\right).$$  \hfill (5.4)

The lower boundary of this phase in the $\mu$–$H$ plane is given by the condition $\min_x \{\epsilon_{2S}(x)\} = 0$.

For magnetic fields $(4S - 1/2)H < \mu + 4 \sum_{k=1}^{2S} 1/(2k - 1)$ the ground state is a filled sea of $\lambda$-strings of length $2S$ with negative energy

$$\epsilon_{2S}(x) + \left\{\sum_{k=1}^{2S-1} a_{2k} \star \epsilon_{2S}^{(-)}\right\}(x) = \epsilon_{2S}^{(0)}(x) + \left(2S - \frac{1}{2}\right)H - \mu.$$  \hfill (5.5)

The other dressed energies $\kappa(x) < 0$ and $\epsilon_{n\neq2S}(x) > 0$ can be expressed in terms of the solution of this equation. In this region of parameters the system has a finite concentration of holes and overturned spins. Although one might naively expect two types of massless excitations in such situation only one such branch with dispersion (5.5) is realized in this system which turns out to describe the charge excitations. Hence spin excitations are gapped in this regime [24].

**5.3 $\mu < -H/2$**

Again we find several phases that can be characterized by the configurations of strings present in the ground state: For $H > H^{(S)}$ all dressed energies are positive corresponding to completely polarized undoped state.

For smaller magnetic fields $\epsilon_{2S}(x)$ takes negative values in some interval to be determined from

$$\epsilon_n(x) + \frac{1}{2\pi} \left\{\Xi_n \star \epsilon_{2S}^{(-)}\right\}(x) = \epsilon_n^{(0)}(x) + nH.$$  \hfill (5.6)

These are the Bethe ansatz equations of the spin-$S$ Takhtajan–Babujian chain. This phase becomes unstable against hole creation for chemical potentials

$$\mu > \int \text{d}a_{2S}(x)\epsilon_{2S}^{(-)}(x) - \frac{1}{2}H \rightarrow \psi \left(\frac{2S + 1}{4}\right) - \psi \left(\frac{2S + 3}{4}\right) \quad \text{for } H = 0$$  \hfill (5.7)

($\psi(x)$ is the digamma function). Beyond this line the ground state is built from a filled sea of $\lambda$-strings with energies $\epsilon_{2S} < 0$ and another sea of charge rapidities with energies $\kappa(x) < 0$. Increasing the chemical potential further negative energy solutions for the real spin rapidities $\epsilon_1$ appear giving rise to a third condensate of Bethe rapidities.

From the cases considered above we obtain the qualitative zero temperature phase diagram of the doped spin-$S$ system in the $\mu$–$H$ plane presented in Fig. [1](a). Using Eqs. (5.6) the
corresponding phase boundaries can be given as a function of the hole concentration \( x = N_h/L \). For \( S = 1 \) this is shown in Fig. 1(b).

### 6 Low-temperature thermodynamics at \( H = 0 \)

Further simplification is possible in the case of finite doping in a vanishing magnetic field which corresponds to chemical potentials \( \mu \in \psi((2S + 1)/4) - \psi((2S + 3)/4), 0 \). Furthermore, \( \epsilon_1(x) < 0 \), \( \epsilon_{2S}(x) < 0 \) for all \( x \) while \( \kappa(x) < 0 \) for \( |x| < Q \) in this regime. All other dressed energies vanish in this limit. Eliminating the \( \epsilon_n \) from the equation for the energy of the holes we obtain

\[
- [2\pi a_{2S} * s(x) + \mu] = \kappa(x) - \int_{-Q}^{Q} dy R(x-y) \kappa(y) \tag{6.1}
\]

where the Fermi point \( Q \) is determined by the condition \( \kappa(\pm Q) = 0 \). Similarly, the density of hole rapidities \( \rho_0(x) \) in this regime is given by

\[
\rho_0(x) - \int_{-Q}^{Q} dy R(x-y) \rho_0(y) = a_{2S} * s(x) . \tag{6.2}
\]

Excitations with charge rapidities near \( \pm Q \) are massless. The velocity of this charge mode can be obtained from the dispersion (6.1)

\[
v = \frac{1}{2\pi \rho_0(Q)} \frac{\partial \kappa_0}{\partial x} \bigg|_{x=Q} . \tag{6.3}
\]

Similarly, one has massless excitations near \( x = \pm \infty \) in the magnetic sector with energies \( \epsilon_1(x) \) and \( \epsilon_{2S}(x) \) with velocities

\[
v_{2S} = \lim_{x \to \infty} \frac{\epsilon'_{2S}(x)}{2\pi \sigma_{2S}(x)} = \pi , \quad v_1 = \lim_{x \to \infty} \frac{\epsilon'_1(x)}{2\pi \sigma_1(x)} = -\frac{1}{2} \frac{\int_{-Q}^{Q} dy e^{\pi y} \kappa_0(y)}{\int_{-Q}^{Q} dy e^{\pi y} \rho_0(y)} . \tag{6.4}
\]

As a consequence of the behaviour of the dressed energies as \( H \to 0 \) we can replace \( \kappa \) in Eq. (1.13) by its zero temperature value \( \kappa_0(x) \) and the driving terms by their asymptotics to obtain the leading low temperature behaviour. As a result we get

\[
\epsilon_n(x) = T S * \ln[1 + e^{\epsilon_{n-1}(x)/T}[1 + e^{\epsilon_{n+1}(x)/T}] - 2\pi \delta_{n,2S} e^{-\pi|x|} - 2\pi A \delta_{n,1} e^{-\pi|x|} \tag{6.5}
\]

where \( 2\pi A = -\int_{-Q}^{Q} dy e^{\pi y} \kappa_0(y) \).

To move further we have to separate the contributions to the free energy stemming from the charge-sector from those due to the \( \epsilon_n \). Considering low temperatures again the leading
contributions to $\kappa$ come from the vicinity of the Fermi wave vectors $\pm Q$. In this region one can safely neglect contributions to Eq. (4.16) from $\epsilon_1$ and rewrite it as

$$- [2\pi a_{2S} * s(x) + \mu] - TR \ln[1 + e^{-|\kappa(x)|/T}]$$

$$= \kappa(x) - \int^{Q}_{-Q} dy R(x - y) \kappa(y)$$

(6.6)

where $Q$ is determined by the condition $\kappa(\pm Q) = 0$. Using the procedure introduced by Takahashi [25], we can rewrite the free energy (4.17) as $F/L = E(S)/L + f_c + f_s$ where

$$f_c = - T \int dx \rho_0(x) \ln [1 + e^{-|\kappa(x)|/T}] \approx - \pi T^2/6v,$$

(6.7)

$$f_s = - T \int dx s(x) \ln [1 + e^{\epsilon_2S(x)/T}] - T \int dx (s * \rho_0)(x) \ln [1 + e^{\epsilon_1(x)/T}].$$

(6.8)

Now the thermodynamics is described by Eq. (6.7) representing a scalar bosonic mode (the charge sector) and by Eqs. (6.5) and (6.8) for the spin sector.

At low temperatures the spin contribution $f_s$ is dominated by contributions from the regions $\nu_{2S} \exp(-\pi|x|) \sim T$ where $|\epsilon_{2S}| \sim T$ and the second one by the regions $\nu_1 \exp(-\pi|x|) \sim T$. The leading temperature dependence of $f_s$ at low $T$ can be obtained by rewriting (5.5) for large positive $x$ as

$$\varphi_n(x) = s \ln[1 + e^{\varphi_{n-1}}][1 + e^{\varphi_{n+1}}] - \delta_{n,2S} e^{-\pi x} - A \delta_{n,1} e^{-\pi x}$$

(6.9)

in terms of the $T$-independent functions

$$\varphi_n(x) = \frac{1}{T} \epsilon_n \left(x - \frac{1}{\pi} \ln \frac{T}{2\pi}\right).$$

In the low-$T$ limit we can replace $s(x)$ and $s * \rho_0(x)$ in (6.8) by their asymptotics to obtain the free energy

$$f_s \approx - \frac{\pi T^2}{6} \left(\frac{c_{2S}}{v_{2S}} + \frac{c_1}{v_1}\right).$$

(6.10)

Such an expression is characteristic of a system two with massless excitations with velocities (6.4). In cases where these excitations can be characterized by different observable quantum numbers the coefficients $c_i$ are the central charges of the underlying Virasoro-algebra thus determining the universality class of the system. In this case they are given in terms of the solutions of (5.9) by

$$c_{2S} = \frac{6}{\pi} \int dx e^{-\pi x} \ln [1 + e^{\varphi_{2S}(x)}], \quad c_1 = \frac{6}{\pi} \int dx (A e^{-\pi x} \ln [1 + e^{\varphi_1(x)}]).$$

(6.11)

Using standard methods [23, 26, 27] for the analysis of the TBA equations we find that their sum can be written as

$$c_{2S} + c_1 = \frac{6}{\pi^2} \sum_n \left[\mathcal{L} \left(\frac{e^{\varphi_n(x)}}{1 + e^{\varphi_n(x)}}\right)\right]_{x=-\infty}^\infty$$

(6.12)
where \( \mathcal{L}(x) \) is Rogers dilogarithm function

\[
\mathcal{L}(x) = -\frac{1}{2} \int_0^x dt \left[ \ln \frac{1}{1-t} + \ln(1-t) \right].
\]

Hence, the \( c_{2S} + c_1 \) is completely determined by the asymptotic behaviour of the solutions of (3.9) as \( x \to \pm \infty \):

\[
\lim_{x \to \infty} \varphi_n(x) = \ln \left( (n+1)^2 - 1 \right),
\]

\[
\lim_{x \to -\infty} \varphi_n(x) = \begin{cases} 
\ln \left( \left( (n-2S+1)^2 - 1 \right) \right) & \text{for } n > 2S \\
-\infty & \text{for } n = 2S \\
\ln \left( \frac{\sin^2(\pi n/2S+1)}{\sin^2(\pi/2S+1)} - 1 \right) & \text{for } 1 < n < 2S 
\end{cases}
\]

giving

\[
c_{2S} + c_1 = 2 \frac{4S - 1}{2S + 1}
\]

independent of the doping (i.e. \( A \)).

The individual values of the \( c_i \) are easily calculated for small and large doping corresponding to \( A \to 0 \) and \( A \to \infty \), respectively. In these cases the regions contributing to the integrals (6.11) are well separated and the functions \( \varphi_n(x) \) take constant values in between. For small doping \( (A \ll 1) \) we find \( \varphi_n(x) = \varphi_n^{(0)} \) for \( \ln A \ll \pi x \ll 0 \) with

\[
\varphi_n^{(0)} = \begin{cases} 
\ln \left( (n-2S+1)^2 - 1 \right) & \text{for } n > 2S \\
-\infty & \text{for } n = 2S \\
\ln \left( \frac{\sin^2(\pi n/2S+1)}{\sin^2(\pi/2S+1)} - 1 \right) & \text{for } 1 \leq n < 2S 
\end{cases}
\]

Hence, near \( x \approx 0 \) they behave as in the undoped system giving the central charge \( 3S/(S+1) \) of the \( SU(2)_2 \) WZNW model. In the region around \( x \approx \ln A \) the \( \varphi_{n<2S} \) are solutions of the finite set of TBA of the minimal unitary model \( \mathcal{M}_p \) [27] with central charge \( c_1 = 1 - 6/(p(p+1)) \) where \( p = 2S+1 \) (this is the Ising model for \( S = 1 \) (see [1]), tricritical Ising model for \( S = 3/2 \), three-state Potts model for \( S = 2 \), tricritical three state Potts model for \( S = 5/2 \) and so forth). Putting everything together we find the leading contribution to the spin part (3.8) to the free energy at small doping

\[
f_s = -\frac{\pi T^2}{6\nu_{2S}} \frac{3S}{S + 1} - \frac{\pi T^2}{6\nu_1} \left\{ 1 - \frac{3}{(S+1)(2S+1)} \right\}.
\]

Proceeding in an analogous way in the limit of large doping \( (A \gg 1) \) corresponding to a spin-\((S - 1/2)\) chain doped with spin-\(S\) carriers we find

\[
\varphi_n^{(0)} = \begin{cases} 
\ln (n^2 - 1) & \text{for } n > 1 \\
-\infty & \text{for } n = 1
\end{cases}
\]
for $0 \ll \pi x \ll \ln A$. In this limit the low temperature contributions to $f_s$ can be written as the sum of a $SU(2)_1$ and a $SU(2)_{2S-1}$ WZNW model, the latter being the well known continuum limit of the pure spin-$(S - 1/2)$ Takhtajan-Babujian model:

$$f_s = \frac{\pi T^2}{6v_{2S}} \frac{6S - 3}{2S + 1} - \frac{\pi T^2}{6v_1}. \quad (6.17)$$

For finite values of $A$ the coefficients $c_{2S}$ and $c_1$ in (6.10) have to be determined numerically. They are found to interpolate smoothly between their limiting values in (6.15) and (6.17). For $S = 1$ and $S = 3$ their doping dependence is shown in Figure 4.

7 Summary and Conclusion

To summarize, we have introduced a class of integrable models describing a magnetic system which upon doping interpolates between the integrable spin-$S$ and $S - 1/2$ Takhtajan-Babujian chains. These models arise when considering vertex models invariant under the action of the graded Lie algebra $gl(2|\mathbb{1})$ with the local quantum spaces carrying the ‘atypical’ higher-spin representations $[S]_+$. Their solution by means of the algebraic Bethe Ansatz allows for a detailed study of their low temperature phase diagram. The spectrum of low lying excitations is described in terms of the dressed energies satisfying the TBA equations (4.15) and (4.16). Without an external magnetic field the critical degrees of freedom separate into charge and magnetic modes as is well known in the Tomonaga-Luttinger liquid models for one-dimensional correlated electrons (see e.g. Refs. [28, 29]). Different from these models, however, one finds two branches of low lying modes in the magnetic sector which at small (large) doping can be identified with higher level $SU(2)_k$ WZW models and a minimal model (free boson). The WZW models have to be present in order to reproduce the well understood critical behaviour of the undoped and completely doped limiting cases. The second gapless magnetic mode, however, is quite peculiar: its appearance in the low energy of the undoped system is crucial to allow for the smooth crossover between the limiting cases (6.15) and (6.17) subject to the constraint $c_{2S} + c_1 = \text{const.}$

The low-$T$ behaviour of the $S = 1$ integrable model has motivated the proposition of an effective field theory of four (real) Majorana fermions as a possible starting point for studies of perturbations around the integrable model [11]. While free field representations could be used for the constituents of the undoped model, interaction terms between the two sectors had to be introduced to reproduce the change of the coefficients $c_{2S}$ and $c_1$ with the hole concentration observed in the exact solution. The possible form of this interaction term is constrained by the
$SU(2)$-symmetry of the model without a magnetic field. A similar construction of an $SU(2)$-invariant effective low energy field theory for the $S > 1$ models introduced here is possible by using the fact that the minimal models can be obtained within a GKO coset construction applied to \cite{30,31}

\[
\frac{SU(2)_{2S-1} \otimes SU(2)_1}{SU(2)_{2S}}.
\]

(7.1)

In fact, the observed change in the conformal weights attributed to the magnetic modes between the limiting cases of the undoped and the completely doped system appear to be just a ‘adiabatic’ realization of this construction

\[
SU(2)_{2S} \otimes \mathcal{M}_{2S+1} \longrightarrow SU(2)_{2S-1} \otimes SU(2)_1.
\]

(7.2)

On the other hand, taking the limit $H \to 0$ starting from the phase discussed in Section 5.3 one may obtain a different field theoretical description of the $SU(2)$-symmetric phase: There the critical degrees of freedom can be described in terms of two free bosons each contributing $c = 1$ to the sum $c_{2S} + c_1$. For $S = 1$ this should give a complete description of the massless magnetic modes. It is likely that the apparent difference between the $H \to 0$ limit and the $H = 0$ model can be understood as a rotation in the space of the effective fields (note that no physical field couples to one of the magnetic modes alone) \cite{32}. For $S > 1$ one has $c_{2S} + c_1 > 2$ from (6.13). Here, the difference to the finite field critical properties is similar to the one observed in the Takhtajan-Babujian models \cite{33}: it is due to the appearence of gap for parafermionic degrees of freedom in the critical theory for any non-zero magnetic field.

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A Two more Bethe Ansätze

As mentioned above, the grading of the underlying algebra leads to different (though equivalent) sets of Bethe Ansatz equations (BAE) when using different orderings of the basis \( B \).

A second Bethe Ansatz can be obtained by starting from the pseudo vacuum \( |\Omega_S\rangle \) by choosing a different reference state in the auxiliary eigenvalue problem for the amplitudes \( F^{a_1 \cdots a_l} \). Following Refs. [13, 15, 16] the roots of

\[
\left( \frac{\lambda_j + iS}{\lambda_j - iS} \right)^L = \prod_{\alpha=1}^{N_\mu} \frac{\lambda_j - \nu_\alpha + \frac{i}{2}}{\lambda_j - \nu_\alpha - \frac{i}{2}}, \quad j = 1, \ldots, N_\mu + N_\downarrow,
\]

\[
1 = \prod_{k=1}^{N_\mu + N_\uparrow} \frac{\nu_\alpha - \lambda_k + \frac{i}{2}}{\nu_\alpha - \lambda_k - \frac{i}{2}}, \quad \alpha = 1, \ldots, N_\downarrow.
\]

are found to parametrize eigenstates of the transfer matrix (2.12) with eigenvalues (\( \tilde{\lambda}_j \equiv \lambda_j - iS \), \( \tilde{\nu}_\alpha \equiv \nu_\alpha - iS - i/2 \))

\[
\Lambda_2 \left( \mu | \{ \tilde{\lambda}_j \}_{j=1}^{N_\mu + N_\uparrow}, \{ \tilde{\nu}_\alpha \}_{\alpha=1}^{N_\mu} \right) = \mu^L \prod_{j=1}^{N_\mu + N_\uparrow} \frac{\mu - \tilde{\lambda}_j + i}{\mu - \lambda_j}
\]

\[
+ (\mu + 2iS)^L \prod_{\alpha=1}^{N_\mu} \frac{\tilde{\nu}_\alpha - \mu + i}{\tilde{\nu}_\alpha - \mu} \left\{ 1 - \prod_{j=1}^{N_\mu + N_\uparrow} \frac{\mu - \tilde{\lambda}_j + i}{\mu - \lambda_j} \right\}.
\]

(The spectrum of the other transfer matrices defined in (3.3) can be obtained from this expression by means of the fusion equations derived in Section 3).

Alternatively, we may start from the fully polarized state \( |\Omega_{S - \frac{1}{2}}\rangle = \prod_{n=1}^{L} |S + \frac{1}{2}, S - \frac{1}{2}, S - \frac{1}{2}\rangle \rangle \) with maximal number of spin-(\( S - \frac{1}{2} \))-holes in the system. Now, eigenstates of (2.12) obtained by adding \( N_e \) particles to \( |\Omega_{S - \frac{1}{2}}\rangle \) and lowering the spin by \( N_\uparrow \) are parametrized by solutions of the third set of BAE [34]

\[
\left( \frac{\lambda_j + iS}{\lambda_j - iS} \right)^L = \prod_{\alpha=1}^{N_e} \frac{\lambda_j - \nu_\alpha + \frac{i}{2}}{\lambda_j - \nu_\alpha - \frac{i}{2}}, \quad j = 1, \ldots, N_e,
\]

\[
\left( \frac{\nu_\alpha + i(S - \frac{1}{2})}{\nu_\alpha - i(S - \frac{1}{2})} \right)^L = \prod_{\beta \neq \alpha}^{N_e} \frac{\nu_\alpha - \nu_\beta + i}{\nu_\alpha - \nu_\beta - i} \prod_{j=1}^{N_e} \frac{\nu_\alpha - \lambda_j - \frac{i}{2}}{\nu_\alpha - \lambda_j + \frac{i}{2}},
\]

\[
\alpha = 1, \ldots, N_\uparrow.
\]

The corresponding eigenvalues are (\( \tilde{\lambda}_j \equiv \lambda_j - iS - i, \tilde{\nu}_\alpha \equiv \nu_\alpha - iS - i/2 \))

\[
\Lambda_3 \left( \mu | \{ \tilde{\lambda}_j \}_{j=1}^{N_e}, \{ \tilde{\nu}_\alpha \}_{\alpha=1}^{N_e} \right) = -(\mu + 2iS + i)^L \prod_{j=1}^{N_e} \frac{\tilde{\lambda}_j - \mu + i}{\tilde{\lambda}_j - \mu}
\]

\[
+ (\mu + 2iS)^L \prod_{\alpha=1}^{N_e} \frac{\tilde{\nu}_\alpha - \mu + i}{\tilde{\nu}_\alpha - \mu} + (\mu + i)^L \prod_{j=1}^{N_e} \frac{\tilde{\lambda}_j - \mu + i}{\tilde{\lambda}_j - \mu} \prod_{\alpha=1}^{N_e} \frac{\mu - \tilde{\nu}_\alpha + i}{\mu - \tilde{\nu}_\alpha}.
\]
For $S = \frac{1}{2}$ Eqs. (A.3) become Lai’s BAE for the supersymmetric $t$-$J$ model [7, 9].

### B  Equivalence of the Bethe Ansätze

In this appendix the equivalence of the sets (2.17) and (A.3) of Bethe Ansatz equations starting from the fully polarized state of spin-$S$ and $S-1/2$ multiplets respectively is shown by means of a particle-hole transformation in the space of the rapidities.

Following Refs. [35, 36] we rewrite the second set of Eqs. (2.17) as $P(\nu_\alpha) = 0$ with the polynomial

$$P(\omega) = \prod_{k=1}^{N_h+N_i} \left( \omega - \lambda_k - \frac{i}{2} \right) - \prod_{k=1}^{N_h+N_i} \left( \omega - \lambda_k + \frac{i}{2} \right). \tag{B.1}$$

According to (2.17) the first $N_h$ of the $N_h+N_i$ roots of $P(\omega)$ can be identified as $\omega_\alpha = \nu_\alpha$, $\alpha = 1, \ldots, N_h$. Labelling the remaining $N_i$ ones as $\nu'_\alpha$ we have

$$\sum_{\alpha=1}^{N_h} \ln \left( \frac{\lambda_\ell - \nu_\alpha + \frac{i}{2}}{\lambda_\ell - \nu_\alpha - \frac{i}{2}} \right) = \sum_{\alpha=1}^{N_h} \frac{1}{2\pi i} \oint_{C_\alpha} d\omega \ln \left( \frac{\lambda_\ell - \omega + \frac{i}{2}}{\lambda_\ell - \omega - \frac{i}{2}} \right) \frac{d\omega}{\omega} \ln P(z)$$

$$= - \sum_{\alpha=1}^{N_i} \ln \left( \frac{\lambda_\ell - \nu'_\alpha + \frac{i}{2}}{\lambda_\ell - \nu'_\alpha - \frac{i}{2}} \right) + \ln \left( \frac{P(z_n)}{P(z_p)} \right), \tag{B.2}$$

where $C_\alpha$ is a contour enclosing $\nu_\alpha$ and $z_{n,p} = \lambda_\ell \pm i/2$ are the end points of the branch cut of the logarithm in (B.2). From the definition (B.1) we have

$$P(\lambda_\ell \pm \frac{i}{2}) = \pm \prod_{k=1}^{N_h+N_i} (\lambda_\ell - \lambda_k \pm i). \tag{B.3}$$

which – when used in (B.2) – implies that

$$\prod_{\alpha=1}^{N_h} \frac{\lambda_\ell - \nu_\alpha + \frac{i}{2}}{\lambda_\ell - \nu_\alpha - \frac{i}{2}} = - \prod_{\alpha=1}^{N_i} \frac{\lambda_\ell - \nu'_\alpha - \frac{i}{2}}{\lambda_\ell - \nu'_\alpha + \frac{i}{2}} \prod_{k=1}^{N_h+N_i} \frac{\lambda_\ell - \lambda_k + i}{\lambda_\ell - \lambda_k - i}. \tag{B.4}$$

Using this identity in the first of Eqs. (2.17) we obtain

$$\left( \frac{\lambda_\ell + iS}{\lambda_\ell - iS} \right)^L = \prod_{\alpha=1}^{N_i} \frac{\lambda_\ell - \nu'_\alpha + \frac{i}{2}}{\lambda_\ell - \nu'_\alpha - \frac{i}{2}}. \tag{B.5}$$

We continue by rewriting these equations as $Q(\lambda_\ell) = 0$ with

$$Q(\omega) = (\omega + iS)^L \prod_{\alpha=1}^{N_i} \left( \omega - \nu'_\alpha - \frac{i}{2} \right) - (\omega - iS)^L \prod_{\alpha=1}^{N_i} \left( \omega - \nu'_\alpha + \frac{i}{2} \right). \tag{B.6}$$
Similar as above we can identify the first $N_h + N_i$ roots of this polynomial of degree $L + N_i$ with $\lambda_j$ and denote the remaining $L - N_h \equiv N_e$ ones by $\lambda'_j$. Proceeding as in (B.2) we obtain

$$\sum_{k=1}^{N_h+N_i} \ln \left( \frac{\nu'_{\alpha} - \lambda_k + \frac{i}{2}}{\nu'_{\alpha} - \lambda_k - \frac{i}{2}} \right) = \sum_{k=1}^{N_h+N_i} \frac{1}{2\pi i} \oint_{C_k} \frac{d\ln (\nu'_{\alpha} - z + \frac{i}{2})}{d\ln \nu'_{\alpha} - z - \frac{i}{2}} \ln Q(z)$$

$$= -\sum_{k=1}^{N_e} \ln \left( \frac{\nu'_{\alpha} - \lambda'_k + \frac{i}{2}}{\nu'_{\alpha} - \lambda'_k - \frac{i}{2}} \right) + \ln \left( \frac{Q(\nu'_{\alpha} + \frac{i}{2})}{Q(\nu'_{\alpha} - \frac{i}{2})} \right).$$

(B.7)

Exponentiating this equation we obtain

$$\prod_{k=1}^{N_h+N_i} \frac{\nu'_{\alpha} - \lambda_k + \frac{i}{2}}{\nu'_{\alpha} - \lambda_k - \frac{i}{2}} = \prod_{k=1}^{N_e} \frac{\nu'_{\alpha} - \lambda'_k - \frac{i}{2}}{\nu'_{\alpha} - \lambda'_k + \frac{i}{2}} \frac{Q(\nu'_{\alpha} + \frac{i}{2})}{Q(\nu'_{\alpha} - \frac{i}{2})}. \quad (B.8)$$

Using this and the fact that $\nu'_\alpha$ solve the second set of Eqs. (2.17) together with the definition (B.6) the unprimed variables can be eliminated and we find

$$\left( \frac{\nu'_\alpha + i(S - \frac{1}{2})}{\nu'_\alpha - i(S - \frac{1}{2})} \right)^L = -\prod_{\beta=1}^{N_i} \frac{\nu'_\alpha - \nu'_\beta + i}{\nu'_\alpha - \nu'_\beta - i} \prod_{k=1}^{N_e} \frac{\nu'_\alpha - \lambda'_k - \frac{i}{2}}{\nu'_\alpha - \lambda'_k + \frac{i}{2}}. \quad (B.9)$$

Comparing Eqs. (B.5) and (B.9) with the Bethe Ansatz equations (A.3) the equivalence of the latter with (2.17) becomes evident. The proof of equivalence with (A.1) is completely analogous.

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Figure 1: (a) Schematic phase diagram of the doped spin chain in the $\mu$–$H$ plane: The bold line denotes the transition to a fully polarized state, interpolating between the saturation fields (5.2) for the spin $S$ and $S' = S - 1/2$ Takhtajan–Babujian chains. The left (right) shaded region corresponds to the undoped (completely doped) regime. (b) Phase diagram of the doped $S = 1$ chain as a function of hole concentration $x$. 
Figure 2: Dependency of $c_1$ (dashed line) and $c_{2S}$ (full line) on the concentration $x$ of holes for (a) $S = 1$, (b) $S = 3$. 