A $q$-MULTISUM IDENTITY ARISING FROM FINITE CHAIN RING PROBABILITIES

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Abstract. In this note, we prove a general identity between a $q$-multisum $B_N(q)$ and a sum of $N^2$ products of quotients of theta functions. The $q$-multisum $B_N(q)$ recently arose in the computation of a probability involving modules over finite chain rings.

1. Introduction

Probabilistic proofs of classical $q$-series identities constitute an intriguing part of the literature in combinatorics. A prominent example of this perspective concerns the Andrews-Gordon identities [1, 10] which state for $1 \leq i \leq k$ and $k \geq 2$

$$
\sum_{n_1, \ldots, n_k \geq 0} q^{N_1^2 + \cdots + N_k^2 + N_1 + \cdots + N_{k-1}} \frac{1}{(q)_{n_1} \cdots (q)_{n_k}} = \prod_{s=1}^{\infty} \frac{1}{1 - q^s},
$$

where $N_j = n_j + \cdots + n_{k-1}$. Here and throughout, we use the standard $q$-hypergeometric (or "$q$-Pochhammer symbol") notation

$$(a)_n = (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k),$$

valid for $n \in \mathbb{N} \cup \{\infty\}$. In [9], Fulman uses a Markov chain on the nonnegative integers to prove the extreme cases $i = 1$ and $i = k$ of (1.1). Chapman [3] cleverly extends Fulman’s methods to prove (1.1) in full generality. In [4], Cohen explicitly computes probability laws of $p^\ell$-ranks of finite abelian groups to give a group-theoretic proof of (1.1). For a generalization of this computation, see [5]. In this note, we are interested in a recent probability computation with a ring-theoretic flavor as it leads to an expression similar to the left-hand side of (1.1).

Our focus is on finite chain rings, a notion we now briefly recall (for further details, see Section 2 in both [2] and [12]). A ring is called a left (resp. right) chain ring if its lattice of left (resp. right) ideals forms a chain. Any finite chain ring is a local ring, i.e., it has a unique maximal ideal which coincides with its radical. Let $\mathcal{R}$ be a finite chain ring with radical $\mathcal{N}$, $q$ be the order of the residue field $\mathcal{R}/\mathcal{N}$ and $N$ be the index of nilpotency of $\mathcal{N}$. Recently, the authors of [2] expressed the density $\psi(n, k, q, N)$ of free submodules $\mathcal{M}$ of $\mathcal{R}^n$ (over $\mathcal{R}$) of length $k := \log_q(|\mathcal{M}|)$ as $n \to \infty$ as the reciprocal of the $q$-multisum (replacing $1/q$ in their notation with $q$)
\[
B_N(q) := \sum_{K_2 \ldots K_N \geq 0 \atop N|K_2 + \cdots + K_N} \frac{q^{K_2 + \cdots + K_N - (K_2 + \cdots + K_N)^2/N}}{(q)_{K_2} \cdots (q)_{K_N}},
\]

where \(N \geq 2\) is an integer and \(K_i = \sum_{j=2}^i k_j\). Upper and lower bounds for \(B_N(q)\) are obtained and then used to show (under suitable conditions) that \(\psi(n, k, q, N)\) is at least \(1 - \epsilon\) where \(0 < \epsilon < 1\) (see Theorems 6 and 8, respectively, in [2]). Moreover, we have

\[
B_2(q) = \prod_{s = \pm 2, \pm 3, \pm 4, \pm 5 \pmod{16}} \frac{1}{1 - q^s},
\]

which is (S.83) in [15]. In view of (1.1) and (1.3), the authors in [2] posed the following (slightly rewritten) problem.

**Problem 1.1.** Determine whether \(B_N(q)\) can be expressed as a product of \(q\)-Pochhammer symbols.

The purpose of this note is to solve Problem 1.1. It turns out that the solution is slightly more involved than either (1.1) or (1.3), namely \(B_N(q)\) is a sum of \(N^2\) products of quotients of theta functions (but not a single product of \(q\)-Pochhammer symbols, for general \(N\)). Before stating our main result, we recall some further standard notation:

\[
\begin{align*}
    j(x; q) &:= (x)_\infty (q/x)_\infty (q)_\infty, \\
    j(x_1, x_2, \ldots, x_n; q) &:= j(x_1; q) j(x_2; q) \cdots j(x_n; q), \\
    J_{a,m} &:= j(q^a; q^m), \\
    \overline{J}_{a,m} &:= j(-q^a; q^m), \\
    J_m &:= J_{m,3m} = (q^m; q^m)_\infty.
\end{align*}
\]

Note that these quantities are products of \(q\)-Pochhammer symbols. Our main result is now the following.

**Theorem 1.2.** For all \(N \geq 2\), we have

\[
B_N(q) = \frac{1}{(q)_\infty^2} \sum_{r=0}^{N-1} \sum_{s=0}^{N-1} (-1)^{r+s+1} q^{(r+1)(r+3)/2} \cdot \frac{j((-1)^N q^{N(N+2)r+N(N+3)/2}; q^{N^2(N+2)})}{j((-1)^N q^{N(N+2)r+N(N+3)/2}; q^{N^2(N+2)})} \cdot \frac{j((-1)^N q^{N(N+2)(r+s)+N(N+3)}; q^{N^2(N+2)})}{j((-1)^N q^{N(N+2)(r+s)+N(N+3)}; q^{N^2(N+2)})}.
\]

Formula (1.4) is of interest for at least two reasons. First, Andrews-Gordon type \(q\)-multisums akin to (1.1) are typically evaluated as single infinite products using \(q\)-series methods such as Bailey pairs, the triple product identity or the quintuple product identity. Instances of \(q\)-multisums which evaluate to sums of infinite products seem to be less well-studied and thus certainly require further attention. For pertinent work involving character formulas of irreducible highest weight modules of Kac-Moody algebras of affine type, see [6] [7]. Second, in order to compute asymptotics or find congruences for the coefficients of \(q\)-multisums, one would ideally prefer a single infinite product expression. In lieu of this situation, sums of infinite products
are often still helpful. Indeed, contrarily to (1.2) which requires computing a \((N - 1)\)-fold sum, (1.4) only involves a double sum. As a comparison with Table 1 in [2], we explicitly compute \(B_N(q)\) for \(2 \leq N \leq 10\) and \(N = 100\) and \(1/q = 2, 3, 5, 7, 11\) to five decimals with Maple using (1.4). Table II above suggests that when \(q \to 0\), the limiting value of \(B_N(q)\) is 1. This statement is confirmed in [2, Corollary 10, (1)].

The paper is organized as follows. In Section 2, we recall one of the main results from [17], then prove Theorem 1.2. In Section 3, we make some concluding remarks.

### 2. Proof of Theorem 1.2

Before the proof of Theorem 1.2, we need to recall some background from the important work of Hickerson and Mortenson [17]. First, we employ the Hecke-type series

\[
f_{a,b,c}(x, y, q) := \left( \sum_{r,s \geq 0} - \sum_{r,s < 0} \right) (-1)^{r+s} x^r y^s a(r)_q + b(r) + c(s)_q. \tag{2.1}
\]

Next, consider the Appell-Lerch series

\[
m(x, q, z) := \frac{1}{j(z; q)} \sum_{r \in \mathbb{Z}} \left( -1 \right)^r q^{(z)_r} z^r, \tag{2.2}
\]

where \(x, z \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}\) with neither \(z\) nor \(xz\) an integral power of \(q\) in order to avoid poles. One of the main results in [17] expresses (2.1) in terms of (2.2). Let

\[
g_{a,b,c}(x, y, q, z_1, z_0) := \sum_{t=0}^{a-1} (-y)^t q^{(z_1)_t} j(q^{bt}; q^a) m \left( -q^{(b+1)_2} - c^{(a+1)_2} - t(b^2 - ac) \frac{(-y)^a}{(-x)^b} , q^{(b^2 - ac), z_0} \right) + \sum_{t=0}^{c-1} (-x)^t q^{(z_0)_t} j(q^{bt}; q^c) m \left( -q^{(b+1)_2} - a^{(c+1)_2} - t(b^2 - ac) \frac{(-x)^c}{(-y)^b} , q^{(b^2 - ac), z_1} \right), \tag{2.3}
\]
Following [17], we use the term “generic” to mean that the parameters do not cause poles in the Appell-Lerch sums or in the quotients of theta functions.

**Theorem 2.1** ([17], Theorem 1.3). Let \( n \) and \( p \) be positive integers with \( (n, p) = 1 \). For generic \( x, y \in \mathbb{C} \),

\[
    f_{n,n+p,n}(x, y, q) = gn_{n,n+p,n}(x, y, q, -1, -1) + \frac{1}{J_{0,np}(2n+p)}\theta_{n,p}(x, y, q),
\]

where

\[
    \theta_{n,p}(x, y, q) := \sum_{r^* = 0}^{p-1} \sum_{s^* = 0}^{n-1} q^{r-(n-1)/2} + (n+p)(r-(n-1)/2)(s+(n+1)/2) + n(2+(n+1)/2) (-x)^{r-(n-1)/2}
\]

\[
    \times \frac{(n+p)^{s+(n+1)/2} J_{p^2}(2n+p)^2}{p^{n+p} \gamma^2} \left(-q^{n+p} x^{n+p} \right) \times \left(q^{p(n+p)}(r+s) + (n+p) (xy)^p \right) \times \left(q^{2n+p} \right)^{2(n+p)}.
\]

Here, \( r := r^* + \{(n-1)/2 \} \) and \( s := s^* + \{(n-1)/2 \} \) with \( 0 \leq \{\alpha\} < 1 \) denoting the fractional part of \( \alpha \).

We can now prove Theorem 1.2.

**Proof of Theorem 1.2** The first step is to recognize \( B_N(q) \) in a different context. For \( N \geq 1 \), consider the vector function of level \( N \) of the affine Lie algebra \( A_1^{(1)} \) (e.g., see [14, 19])

\[
    C_{m,\ell}(q) = \sum_{(C^{-1}n)_1 \in \mathbb{Z}} \frac{q^{nC^{-1}(n-e)\ell}}{(q)_{n_1} \cdots (q)_{n_{N-1}}}, \quad (2.4)
\]

where \( n = (n_1, \ldots, n_{N-1}) \), \( e_i \) is the \( i \)-th standard unit vector in \( \mathbb{Z}^{N-1} \) (with \( e_0 = e_N = 0 \)), \( C \) is the \( A_{N-1} \) Cartan matrix whose inverse \( C^{-1} \) is given by

\[
    (C^{-1})_{i,j} = \min(i, j) - \frac{ij}{N},
\]

and \( (C^{-1}n)_1 \) is the first entry in the vector \( C^{-1}n \). A straightforward computation (see the proof of Theorem 5 in [2]) yields

\[
    B_N(q) = \sum_{\frac{m+p}{2N} + (C^{-1}n)_1 \in \mathbb{Z}} \frac{q^{nC^{-1}n\ell}}{(q)_{n_1} \cdots (q)_{n_{N-1}}}, \quad (2.5)
\]

Comparing (2.4) when \( \ell = 0 \) and \( m \) is divisible by \( 2N \) with (2.5), we have for all \( N \geq 2 \),

\[
    B_N(q) = q^{-m^2/4N} (q)_{\infty} C_{m,0}^N(q). \quad (2.6)
\]

Next, by Example 1.3 on page 386 of [17], we have

\[
    C_{m,0}^N(q) = \frac{1}{(q)_{\infty}} f_{1,N+1,1}(q^{1+m/2}, q^{1-m/2}, q).
\]
Thus from (2.6), we deduce that for all $N \geq 2$ and $m$ divisible by $2N$,

$$B_N(q) = \frac{q^{-m^2}}{(q)^2} f_{1,N+1,1}(q^{1+m/2}, q^{1-m/2}, q).$$  

By Theorem 2.1, we have

$$f_{1,N+1,1}(q^{1+m/2}, q^{1-m/2}, q) = g_{1,N+1,1}(q^{1+m/2}, q^{1-m/2}, q, -1, -1)$$

$$+ \frac{1}{J_{0,N(N+2)}} \theta_{1,N}(q^{1+m/2}, q^{1-m/2}, q).$$

Now, observe that

$$g_{1,N+1,1}(q^{1+m/2}, q^{1-m/2}, q, -1, -1) = 0$$

as there are no poles in the Appell-Lerch series

$$m(q^{N(N+1)/2+m(N+2)/2}, q^{N(N+2)}, -1)$$

and

$$m(q^{N(N+1)/2-m(N+2)/2}, q^{N(N+2)}, -1)$$

(inded, this is true whenever $m(N+2)/2 \neq \pm N(N+1)/2 \pmod{N(N+2)}$, which is always the case when $m \equiv 0 \pmod{2N}$) and $j(q^{1+m/2}; q) = j(q^{1-m/2}; q) = 0$. Thus,

$$B_N(q) = \frac{q^{-m^2}}{(q)^2} J_{0,N(N+2)} \theta_{1,N}(q^{1+m/2}, q^{1-m/2}, q).$$

We now take $m = 0$. The factor $q^{-m^2}$ disappears and $\theta_{1,N}(q, q, q)$ is given as in (1.4). This proves the result.

\[\Box\]

3. Concluding remarks

There are several avenues for further study. First, Table 1 suggests that as $N \to \infty$, the limiting value of $B_N(q)$ is strictly larger than $(q)_{\infty}$. This is a stronger statement than [2, Corollary 10, (2)]. Thus, it would be desirable to compute both asymptotics for $B_N(q)$ and the correct limiting value of $\psi(n, k, q, N)$ as $N \to \infty$. Second, for $N = 2, 3$ and 4, one can reduce the number of products of quotients of theta functions occurring in Theorem 1.2 by first invoking Theorems 1.9–1.11 in [17], then performing routine (yet possibly involved) simplifications [8]. In these cases, we require that $m \equiv 0 \pmod{2N}$, $m \not\equiv 0 \pmod{N(N+2)}$ and, if $m$ is odd, $m \not\equiv \pm(N+1) \pmod{2(N+2)}$. For example, one can recover (1.3) in this manner. The details are left to the interested reader. Third, given that (2.6) is a key step in the proof of Theorem 1.2, it is natural to wonder if string functions which generalize (2.4) (see [11, 13]) can also be realized in terms of computing an appropriate probability. For recent related works on string functions, see [16, 18]. Finally, can Theorem 1.2 be understood via Markov chains, group theory or, possibly, Hall-Littlewood functions [20]?
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