Conformally symmetric vacuum solutions of the gravitational field equations in the brane-world models

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A class of exact solutions of the gravitational field equations in the vacuum on the brane are obtained by assuming the existence of a conformal Killing vector field, with non-static and non-central symmetry. In this case the general solution of the field equations can be obtained in a parametric form in terms of the Bessel functions. The behavior of the basic physical parameters describing the non-local effects generated by the gravitational field of the bulk (dark radiation and dark pressure) is also considered in detail, and the equation of state satisfied at infinity by these quantities is derived. As a physical application of the obtained solutions we consider the behavior of the angular velocity of a test particle moving in a stable circular orbit. The tangential velocity of the particle is a monotonically increasing function of the radial distance and, in the limit of large values of the radial coordinate, tends to a constant value, which is independent on the parameters describing the model. Therefore a brane geometry admitting a one-parameter group of conformal motions may provide an explanation for the dynamics of the neutral hydrogen clouds at large distances from the galactic center, which is usually explained by postulating the existence of the dark matter.

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I. INTRODUCTION

In two recent papers [1], [2] several classes of conformally symmetric solutions of the static gravitational field equations in the brane world scenario [3], in which our Universe is identified to a domain wall in a 5-dimensional anti-de Sitter space-time, have been obtained. The solutions have been derived by using some methods from Lie group theory. As a group of admissible transformations the one-parameter group of conformal motions has been considered. More exactly, as a starting point it has been assumed that the metric tensor $g_{\mu\nu}$ on the brane has the property $L_{\xi}g_{\mu\nu} = \psi(r) g_{\mu\nu}$, where the left-hand side is the Lie derivative of the metric tensor, describing the gravitational field in vacuum on the brane, with respect to the vector field $\xi^\mu$, and $\psi$, the conformal factor, is an arbitrary function of the radial coordinate $r$.

The assumption that in the brane world model the vacuum outside a matter distribution has a self-similar structure is not at all arbitrary; it has a strong theoretical and even observational basis. The assumption that the space-time admits, beside the spherical symmetry, a one-parameter group of conformal motions, is a particular case of the geometric self-similarity, a property of the metric which has extensively been investigated in the literature (for a review of this important subject see [4]). Geometric similarity should be distinguished from physical similarity, which is a property of the matter fields. These two properties are not necessarily equivalent. A "similarity hypothesis", which assumes that in a variety of physical situations, solutions of strongly non-linear differential equations may evolve naturally to a self-similar form, even if they start out in a more complicated form, has also been proposed [5]. The expansion of the Universe from the big bang and the collapse of a star to a singularity tend to self-similarity in some circumstances. Spherically symmetric cosmological fluctuations might naturally evolve from complex initial conditions via the Einstein equations to a self-similar form. Self-similar asymptotics can be obtained in a wide range of contexts in fluid dynamics and for orthogonally transitive $G_2$ cosmological models. The presence of the voids in universe and the large scale structure can be described by self-similar asymptotic solutions of the Friedmann equations [6]. All hypersurface homogeneous locally rotationally symmetric spacetimes, which admit conformal symmetries have been determined in [7], and the symmetry vectors were explicitly obtained. Hence conformal symmetry is generically possible, it is a mathematical property of the solutions of many non-linear differential equations and in some physical/astrophysical phenomena this property has been effectively observed.

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Due to the correction terms coming from the extra dimensions, at very high energies significant deviations from the Einstein theory occur in the brane world models 8 (for a recent review of the brane world theories see 9). Gravity is largely modified at the electro-weak scale 1 TeV. The cosmological implications of the brane world theories have been extensively investigated in the physical literature.

For standard general relativistic spherical compact objects the exterior space-time is described by the Schwarzschild metric. In the five dimensional brane world models, the high energy corrections to the energy density, together with the Weyl stresses from bulk gravitons, imply that on the brane the exterior metric of a static star is no longer the Schwarzschild metric 10, 11. The presence of the Weyl stresses also means that the matching conditions do not have a unique solution on the brane; the knowledge of the five-dimensional Weyl tensor is needed as a minimum condition for uniqueness. The static vacuum gravitational field equations on the brane depend on the generally unknown Weyl stresses, which can be expressed in terms of two functions, called the dark radiation $U$ and the dark pressure $P$ terms (the projections of the Weyl curvature of the bulk, generating non-local brane stresses) 12, 13.

Generally, the vacuum field equations on the brane can be reduced to a system of two ordinary differential equations, which describe all the geometric properties of the vacuum as functions of the dark pressure and dark radiation terms 1. In order to close the system of vacuum field equations on the brane a functional relation between these two quantities is necessary. Hence a first possible approach to the study of the vacuum brane consists in adopting an explicit equation of state for the dark pressure as a function of the dark radiation. This method has been systematically considered in 1, where some classes of exact solutions of the vacuum gravitational field equations on the brane have been derived and a homology theorem for spherically symmetric vacuum on the brane has also been proven.

By using various assumptions and ansatze several classes of exact vacuum or interior solutions of the gravitational field equations on the brane have been obtained in 11 (the solution has the mathematical form of the Reissner-Nordstrom solution, in which a tidal Weyl parameter plays the role of the electric charge of the general relativistic field equations on the brane, in 11 (the exterior solution also matches a constant density interior) and in 12.

A different approach, which avoids the consideration of some ad hoc equations of state for the dark pressure consists in assuming that the vacuum brane has some particular symmetries, and to investigate these symmetries by using Lie group techniques 1, 2. As for the vector field $\xi^\mu$ generating the symmetry one can assume that it is static and spherically symmetric 1 or non-static and spherically symmetric 2. In both cases the gravitational field equations, describing the static vacuum brane with geometric self-similarity can be integrated in Schwarzschild coordinates, and several classes of exact solutions, corresponding to a brane admitting a one-parameter group of motions, can be obtained. The general solution of the field equations depends on some arbitrary integration constants. The main advantage of imposing geometric self-similarity via a group of conformal motions is that this condition also uniquely fixes the mathematical form of the dark radiation and dark pressure terms, respectively, which describe the non-local effects induced by the gravitational field of the bulk. Thus there is no need to impose an arbitrary relation between the dark radiation and the dark pressure.

As a possible physical application of the conformally symmetric solutions with non-static symmetry of the spherically symmetric gravitational field equations in the vacuum on the brane the behavior of the angular velocity $v_\psi$ of the test particles in stable circular orbits has been considered in 2. In this case the conformal factor $\psi$, together with two constants of integration, uniquely determines the rotational velocity of the particle. In the limit of large radial distances and for a particular set of values of the integration constants the angular velocity tends to a constant value. This behavior is typical for massive particles (hydrogen clouds) outside galaxies 13, and is usually explained by postulating the existence of the dark matter. Thus, the rotational galactic curves can be naturally explained in brane world models without introducing any additional hypothesis. The galaxy is embedded in a modified, spherically symmetric geometry, generated by the non-zero contribution of the Weyl tensor from the bulk. The extra-terms, which can be described in terms of the dark radiation term $U$ and the dark pressure term $P$, act as a "matter" distribution outside the galaxy. The existence of the dark radiation term generates an equivalent mass term $M_U$, which is linearly increasing with the distance, and proportional to the baryonic mass of the galaxy, $M_U(r) \approx M_B(r/r_0)$. The particles moving in this geometry feel the gravitational effects of $U$, which can be expressed in terms of an equivalent mass. Moreover, the limiting value $v_{\psi,\infty}$ of the angular velocity can be obtained as function of the total baryonic mass $M_B$ and radius $r_0$ of the galaxy as $v_{\psi,\infty} \approx (2/\sqrt{3})\sqrt{G M_B/r_0 + (1/12\sqrt{3})(G M_B/r_0)^{3/2}}$. For a galaxy with baryonic mass of the order $10^9 M_\odot$ and radius of the order of $r_0 \approx 70$ kpc, we have $v_{\psi,\infty} \approx 287 \text{ km/s}$, which is of the same order of magnitude as the observed value of the angular velocity of the galactic rotation curves. In the framework of this model all the relevant physical parameters (metric tensor components, dark radiation and dark pressure terms) can be obtained as function of the tangential velocity, and hence they can be determined observationally.

It is the purpose of the present paper to consider vacuum spacetimes on the brane admitting a one parameter group of conformal motions, with the vector field generating the motion being non-static and having a more general, non-central symmetry. More exactly, we assume that the vector field $\xi$ generating the conformal symmetry has the general non-static and non-spherically symmetric form $\xi = \xi^0(t, r) \partial/\partial t + \xi^1(t, r) \partial/\partial r + \xi^2(\theta, \phi) \partial/\partial \theta + \xi^3(\theta, \phi) \partial/\partial \phi$, that is, we also introduce an explicit dependence of $\xi$ on the angular coordinates $\theta$ and $\phi$. With this assumption
the metric tensor components can be obtained as functions of the conformal factor $\psi$ and of the radial coordinate $r$. The gravitational field equations describing the conformally symmetric vacuum brane with general symmetry can be reduced to a single differential-integral equation, whose solution can be obtained in terms of the modified Bessel functions. The expressions for the metric coefficients, the dark energy and the dark pressure can be expressed in an exact parametric form.

As a physical application of the derived solutions we consider the behavior of a test particle in a stable circular orbit on the conformally symmetric brane. Similar to the spherically symmetric non-static case, the angular velocity is always a monotonically increasing function of the radial distance and tends, in the limit of large distances, to a constant value. This behavior strongly suggest the possibility that a self similar geometry on the brane, generated by a general, non-static and non-centrally symmetric group of conformal motions, may provide an explanation for the dynamics of the neutral hydrogen clouds at large distances from the galactic center, which is usually explained by postulating the existence of the dark matter.

The present paper is organized as follows. The basic equations describing the spherically symmetric gravitational field equations in the vacuum on the static brane are derived in Section II. The general form of the metric tensor and the field equations for vacuum brane space-times admitting a one parameter group of conformal motions, with non-static and non-central conformal symmetry, are derived in Section III. In Section IV the general solution of the gravitational field equations is obtained. The behavior of the angular velocity of a test particle in stable circular motion is considered in Section V. We conclude and discuss our results in Section VI.

II. THE FIELD EQUATIONS FOR A STATIC, SPHERICALLY SYMMETRIC VACUUM BRANE

We start by considering a 5-dimensional space-time (the bulk), with a single 4-dimensional brane, on which gravity is confined. The 4–D brane world $((^4)M, g_{\mu\nu})$ is located at a hypersurface $\{B (X^4) = 0\}$ in the 5–D bulk space-time $((^5)M, g_{AB})$, of which coordinates are described by $X^4, A = 0, 1, ..., 4$. The action of the system is given by

$$ S = S_{bulk} + S_{brane}, $$

where

$$ S_{bulk} = \int_{(5)M} \sqrt{-g} \left( \frac{1}{2k_5^2} R + L_{m} + \Lambda_5 \right) d^5 X, $$

and

$$ S_{brane} = \int_{(4)M} \sqrt{-g} \left( \frac{1}{k_5^2} K^\pm + L_{brane} (g_{\alpha\beta}, \psi) + \lambda_0 \right) d^4 x, $$

where $k_5^2 = 8\pi G_5$ is the 5-D gravitational constant, $R$ and $L_m$ are the 5-D scalar curvature and matter Lagrangian in the bulk, respectively. $x^\mu, \mu = 0, 1, 2, 3$ are the induced 4–D coordinates of the brane, $K^\pm$ is the trace of the extrinsic curvature on either side of the brane and $L_{brane} (g_{\alpha\beta}, \psi)$ is the 4–D Lagrangian, which is given by a generic functional of the brane metric $g_{\alpha\beta}$ and of the matter fields $\psi$. In the following capital Latin indices run in the range 0, ..., 4, while Greek indices take the values 0, ..., 3. $\Lambda_5$ and $\Lambda$ are the negative vacuum energy densities in the bulk and in the brane, respectively.

The Einstein field equations in the bulk are given by

$$ (^5)G_{IJ} = k_5^2 (^5)T_{IJ}, \quad (^5)T_{IJ} = -\Lambda_5 (^5)g_{IJ} + \delta(B) \left[ -\lambda_0 (^5)g_{IJ} + T_{IJ} \right], $$

where

$$ (^5)T_{IJ} \equiv -2\frac{\delta L_m}{\delta g_{IJ}} + (^5)g_{IJ} L_m, $$

is the energy-momentum tensor of bulk matter fields, while $T_{\mu\nu}$ is the energy-momentum tensor localized on the brane and which is defined by

$$ T_{\mu\nu} \equiv -2\frac{\delta L_{brane}}{\delta g_{\mu\nu}} + g_{\mu\nu} L_{brane}. $$

The $\delta (B)$ denotes the localization of brane contribution. In the 5-D space-time a brane is a fixed point of the $Z_2$ symmetry. The basic equations on the brane are obtained by projection of the variables onto the brane world, because
we assume that the gravity on the brane is confined. The induced 4-D metric is \( g_{IJ} = (5) g_{IJ} - n_l n_J \), where \( n_l \) is the space-like unit vector field normal to the brane hypersurface \((4) M\). In the following we assume that all the matter fields, except gravitation, are confined to the brane. This implies that \((5) L_m = 0\).

Assuming a metric of the form \( d s^2 = (n_l n_l + g_{IJ}) d x^I d x^J \), with \( n_l d x^l = d \chi \) the unit normal to the \( \chi \) =constant hypersurfaces and \( g_{IJ} \) the induced metric on \( \chi \) =constant hypersurfaces, the effective four-dimensional gravitational equations on the brane (the Gauss equation), take the form \([3]\): 

\[
G_{\mu \nu} = -\Lambda g_{\mu \nu} + k_5^2 T_{\mu \nu} + k_5^4 S_{\mu \nu} - E_{\mu \nu},
\]

where \( S_{\mu \nu} \) is the local quadratic energy-momentum correction 

\[
S_{\mu \nu} = \frac{1}{12} T T_{\mu \nu} - \frac{1}{4} T_{\mu}^{\alpha} T_{\nu \alpha} + \frac{1}{24} g_{\mu \nu} (3T^{\alpha \beta} T_{\alpha \beta} - T^2),
\]

and \( E_{\mu \nu} \) is the non-local effect from the free bulk gravitational field, the transmitted projection of the bulk Weyl tensor \( C_{IAB} \), \( E_I = C_{IAB} n^A n^B \), with the property \( E_{IJ} \rightarrow E_{\mu \nu} \delta_I^\mu \delta_J^\nu \) as \( \chi \rightarrow 0 \). We have also denoted \( k_4^2 = 8\pi G \), with \( G \) the usual four-dimensional gravitational constant.

The four-dimensional cosmological constant, \( \Lambda \), and the four-dimensional coupling constant, \( k_4 \), are given by \( k_4 = k_5 (\Lambda_5 + k_5^2 \lambda_5^2 / 6) \) and \( k_4 = k_5 \lambda_6 / 6 \), respectively. In the limit \( \lambda_6^{-1} \rightarrow 0 \) we recover standard general relativity.

The Einstein equation in the bulk and the Codazzi equation also imply the conservation of the energy-momentum tensor of the matter on the brane, \( D_{\nu} E_{\mu \nu} = 0 \), where \( D_{\nu} \) denotes the brane covariant derivative. Moreover, from the contracted Bianchi identities on the brane it follows that the projected Weyl tensor should obey the constraint

\[
\delta_{\mu \nu} = \frac{1}{3} D_{\mu} u_{\nu} + A_{\mu} u_{\nu} + \frac{4}{3} U A_{\mu} + D^\nu P_{\mu \nu} + A^\nu P_{\mu \nu} = 0,
\]

where \( k = k_5 / k_4 \), \( h_{\mu \nu} = g_{\mu \nu} + u_{\mu} u_{\nu} \) projects orthogonal to \( u^\mu \), the "dark radiation" term \( U = -k^4 E_{\mu \nu} u^\mu u^\nu \) is a scalar, \( Q_{\mu} = k^4 h_{\mu}^\alpha E_{\alpha \beta} \) a spatial vector and \( P_{\mu \nu} = -k^4 [h_{(\mu} h_{\nu)}^\alpha - \frac{1}{3} h_{\mu \nu} h^{\alpha \beta}] E_{\alpha \beta} \) a spatial, symmetric and trace-free tensor.

In the case of the vacuum state we have \( \rho = p = 0 \), \( T_{\mu \nu} = 0 \) and consequently \( S_{\mu \nu} = 0 \). Therefore the field equations describing a static brane take the form

\[
R_{\mu \nu} = -E_{\mu \nu} + 4\Lambda,
\]

with the trace \( R \) of the Ricci tensor \( R_{\mu \nu} \) satisfying the condition \( R = R^\mu_\mu = E^\mu_\mu = 0 \).

In the vacuum case \( E_{\mu \nu} \), satisfies the constraint \( D_{\nu} E_{\mu \nu} = 0 \). In an inertial frame at any point on the brane we have \( u^\mu = \delta_0^\mu \) and \( h_{\mu \nu} = \text{diag}(0,1,1,1) \). In a static vacuum \( Q_{\mu} = 0 \) and the constraint for \( E_{\mu \nu} \) takes the form \([11]\)

\[
\frac{1}{3} D_{\mu} U + \frac{4}{3} U A_{\mu} + D^\nu P_{\mu \nu} + A^\nu P_{\mu \nu} = 0,
\]

where \( D_{\mu} \) is the projection (orthogonal to \( u^\mu \)) of the covariant derivative and \( A_{\mu} = u^\nu D_{\nu} u_{\mu} \) is the 4-acceleration. In the static spherically symmetric case we may chose \( A_{\mu} = A(r) r_{\mu} \) and \( P_{\mu \nu} = P(r) \left( r_{\mu} r_{\nu} - \frac{1}{2} h_{\mu \nu} \right) \), where \( A(r) \) and \( P(r) \) (the "dark pressure") are some scalar functions of the radial distance \( r \), and \( r_{\mu} \) is a unit radial vector \([10]\).

We chose the static spherically symmetric metric on the brane in the standard form

\[
ds^2 = -e^{(r)} d t^2 + e^{(r)} d r^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\]

Then the gravitational field equations and the effective energy-momentum tensor conservation equation in the vacuum take the form \([1], [2]\):

\[
e^{-\lambda} \left( \frac{1}{r^2} \frac{\lambda'}{r} \right) + \frac{1}{r^2} = \frac{48\pi G}{k^4 \lambda_6} U + \Lambda,
\]

\[
e^{-\lambda} \left( \frac{u'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = \frac{16\pi G}{k^4 \lambda_6} (U + 2P) - \Lambda.
\]
\[ e^{-\lambda} \left( \nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2} \right) = \frac{32\pi G}{k^4\lambda_b} (U - P) - 2\Lambda, \]  

(15)

\[ \nu' = -\frac{U' + 2P'}{2U + P} - \frac{6P}{r(2U + P)}, \]  

(16)

where we denoted \( ' = d/dr \). In the following we denote \( \beta = 16\pi G/k^4\lambda_b \).

III. FIELD EQUATIONS ON A VACUUM BRANE WITH NON-STATIC AND NON-CENTRAL CONFORMAL SYMMETRY

The system of the field equations for the vacuum on the brane is under-determined. A functional relation between the dark energy \( U \) and the dark pressure \( P \) must be specified in order to solve the equations. An alternative method, which avoids \( ad \) hoc \ specifications, is to assume that the brane is mapped conformally onto itself along the direction \( \xi \), so that

\[ L_\xi g_{\mu\nu} = g_{\mu\nu,\lambda} \xi^\lambda + g_{\lambda\nu,\mu} \xi^\lambda + g_{\mu\lambda,\nu} \psi g_{\lambda\nu}, \]  

(17)

where \( \psi \) is the conformal factor. In the following we shall assume that \( \psi \) is a function of the radial coordinate only, \( \psi = \psi(r) \). As for the choice of \( \xi \), in the framework of the standard general relativity, Herrera and Ponce de Leon \cite{14} assumed that the vector field generating the conformal symmetry is static and spherically symmetric,

\[ \xi = \xi^0(r) \frac{\partial}{\partial t} + \xi^1(r) \frac{\partial}{\partial r}. \]  

(18)

Using this form of the conformal vector in Eqs. (17) one obtains

\[ \xi^0 = A, \xi^1 = \frac{B}{2} r \exp \left( -\frac{\lambda}{2} \right), \]  

(19)

and

\[ \psi (r) = B \exp \left( -\frac{\lambda}{2} \right) \exp (\nu) = C^2 r^2, \]  

(20)

respectively, where \( A, B, C \) are constants. \( A \) may be set to zero since \( A \partial/\partial t \) is a Killing vector and \( B \) may be set to 1 by a rescaling \( \xi \rightarrow B^{-1} \xi, \psi \rightarrow B^{-1} \psi \), which leaves Eqs. (17) invariant.

This form of \( \xi \) gives the most general \( \xi \) invariant under the Killing symmetries, that is \( [\partial/\partial t, \xi] = 0 = [X_\alpha, \xi] \), where \( X_\alpha \) generates \( SO(3) \). The corresponding form of the metric, obtained by imposing static conformal symmetry, has been used to study the properties of compact anisotropic general relativistic objects \cite{14} and to investigate the properties of charged strange stars \cite{15}. The general solution of the vacuum brane gravitational field equations for this choice of \( \xi \) has been obtained in \cite{1}.

A more general conformal symmetry has been proposed by Maartens and Maharajah \cite{16}, which generalizes the isotropic conformal vector \( t\partial/\partial t + r\partial/\partial r \) of the Minkowski space-time, but weakens the static symmetry of \( \xi \) in Eq. (18):

\[ \xi = \xi^0(t, r) \frac{\partial}{\partial t} + \xi^1(t, r) \frac{\partial}{\partial r}. \]  

(21)

With this choice the conformal vector is non-static and spherically symmetric. With the assumption that the conformal factor \( \psi \) is static, \( \psi = \psi (r) \) and with this form of \( \xi \) Eqs. (17) give \cite{16}:

\[ \xi^0 = A + \frac{1}{2} \frac{k}{B} t, \]  

(22)

\[ e^\nu = C^2 r^2 \exp \left( -2kB^{-1} \int \frac{dr}{r\psi} \right), \]  

(23)
\[ \psi = Be^{-\lambda/2}, \quad (24) \]

where \( k \) is a separation constant and \( A, B \) and \( C \) are integration constants. Without any loss of generality we can chose \( A = 0 \). Thus in this model we obtain for the vector field \( \xi \) the expression

\[ \xi = \frac{1}{2} \frac{k}{B} \frac{\partial}{\partial t} + \frac{r\psi(r)}{2} \frac{\partial}{\partial r}. \quad (25) \]

In the framework of the standard general relativity the static solutions of the gravitational field equations with metric tensor components given by Eqs. (23) and (24) have been obtained, for anisotropic and charged compact objects, by Maartens and Maharaj [16]. All the solutions of the vacuum gravitational field equations on the brane having the conformal symmetry generated by the vector \( \xi \) given by Eq. (25) have been obtained in [2].

In the following we consider a generalization of the conformal vector \( \xi \) by relaxing the condition of the spherical symmetry. Hence we assume a more general form of the vector field \( \xi \), generating the conformal symmetry, namely

\[ \xi = \xi^0(t, r) \frac{\partial}{\partial t} + \xi^1(t, r) \frac{\partial}{\partial r} + \xi^2(\theta, \phi) \frac{\partial}{\partial \theta} + \xi^3(\theta, \phi) \frac{\partial}{\partial \phi}. \quad (26) \]

We still assume that the conformal factor is spherically symmetric, that is, we take \( \psi = \psi(r) \). With the use of this form of \( \xi \), Eqs. (17) give

\[ \nu' \xi^1 + 2 \frac{\partial \xi^0}{\partial t} = \psi(r), \quad (27) \]

\[ \lambda' \xi^1 + 2 \frac{\partial \xi^1}{\partial r} = \psi(r), \quad (28) \]

\[ \frac{\xi^1}{r} + \frac{\partial \xi^2}{\partial \theta} = \frac{\psi(r)}{2}, \quad (29) \]

\[ \frac{\xi^1}{r} \frac{\cos \theta}{\sin \theta} \xi^2 + \frac{\partial \xi^3}{\partial \phi} = \frac{\psi(r)}{2}. \quad (30) \]

From Eq. (29) it follows that

\[ \xi^1(r) = \frac{r(\psi - \alpha)}{2}, \quad (31) \]

\[ \xi^2(\theta, \phi) = \frac{\alpha}{2} \theta + \frac{dF(\phi)}{d\phi}, \quad (32) \]

where \( \alpha \) is an arbitrary separation constant and \( F \) an arbitrary function of the azimuthal angle \( \phi \). Then Eq. (30) gives

\[ \xi^3(\theta, \phi) = \frac{\alpha}{2} (1 - \theta \cot \theta) \phi - \cot \theta F(\phi) + G(\theta), \quad (33) \]

where \( G(\theta) \) is an arbitrary integration function. From Eq. (27) we obtain

\[ \xi^0(t) = \frac{k}{2} t + A, \quad (34) \]

where \( k \) is a separation constant and \( A \) is an integration constant, which can be taken to be zero without any loss of generality, \( A = 0 \). Hence the metric tensor component \( \exp(\nu) \) is given by

\[ \exp(\nu) = C^2 r^2 \exp \left[ 2(\alpha - k) \int \frac{dr}{r(\psi - \alpha)} \right]. \quad (35) \]
where $C$ is an arbitrary integration constant. Finally, the metric tensor component $\exp(\lambda)$ can be obtained in the form

$$\exp(\lambda) = \frac{B^2}{(\psi - \alpha)^2} \exp\left[2\alpha \int \frac{dr}{r(\psi - \alpha)}\right],$$

(36)

where $B$ is an arbitrary constant of integration.

Therefore the requirement of the conformal symmetry on the brane completely fixes the form of the metric tensor, as a function of the conformal factor $\psi$. As for the vector field $\xi$, it is given by

$$\xi = \frac{k}{2} \frac{\partial}{\partial t} + \frac{r(\psi - \alpha)}{2} \frac{\partial}{\partial r} + \left[\frac{\alpha}{2} \theta + \frac{dF(\phi)}{d\phi}\right] \frac{\partial}{\partial \theta} + \left[\frac{\alpha}{2} (1 - \theta \cot \theta) \phi - \cot \theta F(\phi) + G(\theta)\right] \frac{\partial}{\partial \phi}.$$  

(37)

With the use of the representations given by Eqs. (35) and (36) for the metric tensor components, the gravitational field equations (13)-(15) on the vacuum brane with general, non-central conformal symmetry, take the form

$$-\frac{(\psi - \alpha)^2}{B^2} e^{-2\alpha \int \frac{dr}{r(\psi - \alpha)}} \left[\frac{2\psi'}{r(\psi - \alpha)} - \frac{2\alpha}{r^2(\psi - \alpha)} \right] + \frac{1}{r^2} = 3\beta U + \Lambda,$$

(38)

$$\frac{(\psi - \alpha)^2}{B^2} e^{-2\alpha \int \frac{dr}{r(\psi - \alpha)}} \left[\frac{2(\alpha - k)}{r^2(\psi - \alpha)} + \frac{3}{r^2}\right] - \frac{1}{r^2} = \beta (U + 2P) - \Lambda,$$

(39)

$$\frac{(\psi - \alpha)^2}{B^2} e^{-2\alpha \int \frac{dr}{r(\psi - \alpha)}} \left[\frac{2\psi'}{r(\psi - \alpha)} - \frac{2k}{r^2(\psi - \alpha)} + \frac{k(\alpha - k)}{r^2(\psi - \alpha)^2} + \frac{1}{r^2}\right] = \beta (U - P) - \Lambda.$$  

(40)

**IV. GENERAL SOLUTION OF THE GRAVITATIONAL FIELD EQUATIONS FOR THE VACUUM BRANE WITH NON-CENTRAL CONFORMAL SYMMETRY**

By multiplying Eq. (40) with 2, adding the obtained equation to Eq. (39) and equating the resulting equation with Eq. (38) gives the following differential-integral equation, satisfied by the conformal factor $\psi(r)$:

$$3r(\psi - \alpha)\psi' + 3\psi^2 - 3(k + 2\alpha)\psi + 3\alpha^2 + 2\alpha k + k^2 - B^2 e^{2\alpha \int \frac{dr}{r(\psi - \alpha)}} - 4\Lambda B^2 e^{2\alpha \int \frac{dr}{r(\psi - \alpha)} r^2} = 0.$$  

(41)

In order to solve Eq. (41) we introduce a new variable $u$ defined as

$$\ln u = \alpha \int \frac{dr}{r(\psi - \alpha)}.$$  

(42)

In terms of $u$ Eq. (41) can be expressed in the form

$$-3\frac{\alpha^2}{r} u^2 u'' + 3\alpha (\alpha - k) \frac{1}{r} u' + k^2 - k\alpha - B^2 u^2 - 4\Lambda B^2 u^2 r^2 = 0.$$  

(43)

With the use of the identities $1/u' = dr/du$ and $u''/u'^3 = -d^2 r/du^2$, Eq. (43) can be transformed into the following differential equation:

$$u^2 \frac{d^2 r}{du^2} - m u \frac{dr}{du} - (l^2 u^2 - n^2) r - 12\Lambda u^2 r^3 = 0,$$

(44)

where we denoted

$$m = \frac{k - \alpha}{\alpha}, l^2 = \frac{B^2}{3\alpha^2}, n^2 = \frac{k(k - \alpha)}{3\alpha^2}.$$  

(45)

In the following we assume that we can neglect the nonlinear term in Eq. (44). This approximation is valid in the case of very small numerical values of the parameters $l$ and $\Lambda$ and for small values of $r$. Therefore the basic equation
describing the geometry of the vacuum brane with geometric self-similarity with the effect of the cosmological constant ignored is given by the following linear, Bessel type differential equation:

\[ u^2 \frac{d^2 r}{du^2} - mu \frac{dr}{du} - (l^2 u^2 - n^2) r = 0, \]  

(46)

The general solution of Eq. 46 is given by

\[ r^\star (u) = u^s [C_1 I_p (lu) + C_2 K_p (lu)], \]  

(47)

where \( C_1 \) and \( C_2 \) are arbitrary constants of integration, \( s = (1 + m)/2 = k/2\alpha \), \( p = \sqrt{(1+m)^2 - 4n^2}/2 = \sqrt{s(2-s)/3} \), and \( I_p (x) \) and \( K_p (x) \) are the modified Bessel functions of the first and second kind, respectively, satisfying the differential equation \( x^2 y'' + xy' - (x^2 + p^2) y = 0 \) \[17\]. For all values of \( p \), \( I_p (x) \) and \( K_p (x) \) are linearly independent. \( I_p (x) \) and \( K_p (x) \) are real and positive when \( p > -1 \) and \( x > 0 \). Since by definition \( l > 0 \), it follows that the parameter \( u \) must be non-negative, \( u \geq 0 \). On the other hand the condition that \( p \) must be a real number, \( p \in \mathbb{R} \), imposes the constraint \( 0 < s \leq 2 \) on the numerical parameter \( s \).

To obtain a correct physical description, the function \( r(u) \) must be an increasing function of the variable \( u \) and obey the conditions \( \lim_{u \to 0} r(u) = 0 \) and \( \lim_{u \to \infty} r(u) = \infty \), respectively. For small values of the arguments and for fixed \( p \) the modified Bessel functions behave like \( I_p (x) \sim (x/2)^p \Gamma (p + 1) \) and \( K_p (x) \sim \Gamma (p) (\pi x^2)^{-p} / 2 \), respectively, where \( \Gamma \) is the Euler gamma function, defined as \( \Gamma (z) = \int_0^\infty t^{z-1} e^{-t} dt \). Hence in the limit of small \( u \) we obtain \( \lim_{u \to 0} r(u) \sim u^s \left[ C_1 (l/2)^p u^p \Gamma (p + 1) + C_2 \Gamma (p) (l/2)^{-p} u^{-p}/2 \right] \). In order to obtain \( \lim_{u \to 0} r(u) = 0 \), the condition \( s > p \) must necessarily hold. This condition is satisfied for values of \( s \) so that \( s > 1/2 \).

In the opposite limit of large values of the argument, the modified Bessel functions \( I_p (x) \) and \( K_p (x) \) have the asymptotic behaviors \( I_p (x) \sim \exp (x) / \sqrt{2\pi x} \) and \( K_p (x) \sim \exp (-x) \), respectively \[17\]. Hence \( \lim_{u \to \infty} r(u) = \lim_{u \to \infty} \left[ (C_1/\sqrt{2\pi l}) u^{s-1/2} \exp (lu) \right] = \infty \). To obtain the correct form of the limit it is also necessary that \( C_1 > 0 \).

The numerical values of the parameter \( s \) are restricted to the range \( s \in (1/2, 2] \).

Therefore the general solution of the gravitational field equations on the vacuum brane with non-static and non-central conformal symmetry can be expressed in an exact parametric form, with \( u \) taken as parameter, and given by

\[ e^\nu = C^2 u^{2(1-s)} \left[ C_1 I_p (lu) + C_2 K_p (lu) \right]^2, \]  

(48)

\[ e^\lambda = \frac{3l^2 u^{2s} \left[ C_1 I_p (lu) + C_2 K_p (lu) \right]^2}{(du)^2}, \]  

(49)

\[ 3\beta U = \frac{1}{3l^2} \left\{ 1 + \frac{1}{l^2} \left[ \left( \frac{d}{du} \ln r \right)^2 + \frac{2(2s - 1)}{u} \left( \frac{d}{du} \ln r \right) + \frac{2n^2}{u^2} \right] \right\}, \]  

(50)

\[ 3\beta P = \frac{1}{3l^2} \left[ 4 \left( \frac{d}{du} \ln r \right)^2 - \frac{2(2s - 1)}{u} \left( \frac{d}{du} \ln r \right) + l^2 - \frac{n^2}{u^2} \right] - \frac{2}{r^2}. \]  

(51)

The variation of the metric coefficient \( \exp (\nu) / C^2 C^2 \) is represented, as a function of \( r/C_2 \), in Fig. 1

For the chosen values of the parameters \( \exp (\nu) \) is a decreasing function of the radial distance \( r \). The variation of the metric function \( \exp (\lambda) / 3l^2 \) is represented in Fig. 2.

The metric function \( \exp (\lambda) \) is a monotonically increasing function of the coordinate \( r \). In the limit of large values of \( r \), \( \exp (\lambda) \) tends to a constant value. The variation of the dark radiation term \( 3\beta C^2 U \) is represented, as a function of \( r/C_2 \), in Fig. 3.
The dark radiation term is positive for all values of the radial coordinate $r$, $U(r) \geq 0$, $\forall r \in (0, \infty)$. In the limit of large $r$, $U$ tends to zero, $\lim_{r \to \infty} U(r) = 0$. The variation of the absolute value of the dark pressure $3\beta C_2^2 |P|$ as a function of $r/C_2$ is represented in Fig. 4.

In the present model the dark pressure is negative, satisfying the condition $P(r) \leq 0$, $\forall r \in (0, \infty)$. In the large time limit, similar to the dark radiation term, the dark pressure also tends to zero, $\lim_{r \to \infty} P(r) = 0$.

In the limit of small $r$, corresponding to small values of the parameter $u$, we have $r \approx \left(\frac{C_2}{C_2/2} \Gamma(p) \left(\frac{l}{2}\right)^{p}\right)^{\frac{1}{s-p}}, r \to 0$.

Therefore in the limit of small radial distances the metric coefficients $e^{\nu}$ and $e^{\lambda}$ behave as

$$e^{\nu} \approx C^2 \left[\frac{2}{C_2 \Gamma(p)} \left(\frac{l}{2}\right)^{p} \right]^{\frac{1}{s-p}} r^{\frac{1}{s-p}}, r \to 0,$$

and

$$e^{\lambda} \approx 3l^2 \left[\frac{2}{C_2 \Gamma(p)} \left(\frac{l}{2}\right)^{p} \right]^{\frac{1}{s-p}} r^{\frac{2}{s-p}} \left\{1 + \frac{\Gamma(p+1)}{\Gamma(p)} + \left(\frac{1}{2}\right)^2 \frac{\Gamma(p-1)}{\Gamma(p)} \left[\frac{2}{C_2 \Gamma(p)} \left(\frac{l}{2}\right)^{p}\right]^{\frac{1}{s-p}} r^{\frac{2}{s-p}}\right\}, r \to 0.$$
FIG. 3: Variation, as a function of the radial distance \( r/C_2 \), of the dark radiation term \( 3\beta C_2^2 U \) (in a logarithmic scale) for a static, conformally symmetric vacuum space-time on the brane, for \( l = 10^{-9} \), \( R_0 = C_1/C_2 = 10^{-3} \) and different values of \( s \): \( s = 1.1 \) (solid curve), \( s = 1.2 \) (dotted curve), \( s = 1.3 \) (dashed curve) and \( s = 1.4 \) (long dashed curve).

FIG. 4: Variation, as a function of the radial distance \( r/C_2 \), of the dark pressure term \( 3\beta C_2^2 |P| \) for a static, conformally symmetric vacuum space-time on the brane, for \( l = 10^{-9} \), \( R_0 = C_1/C_2 = 10^{-3} \) and different values of \( s \): \( s = 1.1 \) (solid curve), \( s = 1.2 \) (dotted curve), \( s = 1.3 \) (dashed curve) and \( s = 1.4 \) (long dashed curve).

respectively.

In the limit of large \( r \) the dependence of the metric functions on the radial distance can be given only in a parametric form. Since for \( u \to \infty \) we have \( r = \left( C_1/\sqrt{2\pi l} \right) u^{s-1/2} \exp(lu) \), the asymptotic behavior of \( \exp(\nu) \) and \( \exp(\lambda) \) is given, for large \( r \), in the parametric form, with \( u \) taken as parameter

\[
r \approx \frac{C_1}{\sqrt{2\pi l}} u^{s-1/2} e^{lu}, e^\nu \approx \frac{C_2^2}{2\pi l} u^{1-2s} e^{2lu}, r \to \infty, \tag{55}
\]

and

\[
r \approx \frac{C_1}{\sqrt{2\pi l}} u^{s-1/2} e^{lu}, e^\lambda \approx \frac{3l^2 u^2}{(s+lu)^2}, r \to \infty. \tag{56}
\]

For \( r \to 0 \) the radial distance dependence of the dark radiation \( U \) is given by

\[
3\beta U(r) \approx \frac{1}{3\nu^2} \left( 1 + \frac{K_U}{r^{s-p}} \right), r \to 0, \tag{57}
\]

where

\[
K_U = \frac{2n^2 + [s + \Gamma(p+1)/\Gamma(p)]^2 + 2(2s-1) [s + \Gamma(p+1)/\Gamma(p)]}{l^2 \left[ \frac{\Gamma}{\zeta(2)(\frac{1}{2})^p} \right]^{s-p}}. \tag{58}
\]
In the limit of large \( r \), \( r \to \infty \), the dark radiation behaves as

\[
3\beta U(r) \approx \frac{2}{3r^2}, r \to \infty.
\]

(59)

In the limit of small \( r \) we find for the dark pressure

\[
3\beta P(r) \approx -\frac{2}{3r^2} \left( 1 + \frac{K_P}{r^2} \right), r \to 0,
\]

(60)

where

\[
K_P = n^2 - 4 \left[ s + \Gamma(p+1)/\Gamma(p) \right]^2 + 2(2s-1) \left[ s + \Gamma(p+1)/\Gamma(p) \right].
\]

(61)

For \( r \to \infty \) the dark pressure has the asymptotic limit

\[
3\beta P(r) \approx -\frac{1}{3r^2}, r \to \infty.
\]

(62)

Therefore at infinity the dark pressure and the dark radiation obey the equation of state

\[
U + 2P = 0.
\]

(63)

V. **STABLE CIRCULAR ORBITS IN VACUUM BRANE SPACE-TIMES WITH GENERAL CONFORMAL SYMMETRY**

As a physical application of the conformally symmetric brane metric \([12]\) generated by a vector field \( \xi \) with arbitrary symmetry we consider now the problem of constructing stable circular timelike geodesic orbits. The motion of a test particle in the gravitational field can be described by the Lagrangian \([18]\)

\[
2L = \left( \frac{ds}{dt} \right)^2 = -e^{\nu(r)} \left( \frac{dt}{d\tau} \right)^2 + e^{\lambda(r)} \left( \frac{dr}{d\tau} \right)^2 + r^2 \left( \frac{d\Omega}{d\tau} \right)^2,
\]

(64)

where we denoted by \( \tau \) the affine parameter along the geodesics. In the timelike case \( \tau \) corresponds to the proper time. The equation giving the tangential velocity of the body has been derived in \([18]\) and \([19]\), and is given by

\[
v_{tg}^2 = \frac{r\nu'}{2}.
\]

(65)

Thus, the rotational velocity of the test body is determined by the metric coefficient \( \exp (\nu) \) only.

In the case of the motion of a test particle in a conformally symmetric, static spherically symmetric space-time, with a general non-static vector field generating the symmetry, the metric coefficient \( \exp (\nu) \) is given by Eq. \([18]\). Therefore as a function of the conformal factor \( \psi \) the angular velocity \( v_{tg} \) is given by the simple expression

\[
v_{tg}^2 = 1 - \frac{\alpha(2s - 1)}{\psi - \alpha}.
\]

(66)

Eq. \([66]\) gives a simple physical interpretation of the conformal factor \( \psi \) in terms of the tangential velocity, \( \psi = \alpha \left[ 1 + (2s - 1) \left( 1 - v_{tg}^2 \right)^{-1} \right] \). Hence the conformal factor is proportional to the tangential velocity of a test particle in the gravitational field of a brane with general conformal symmetry. As a function of the parameter \( u \), \( v_{tg} \) is given by

\[
v_{tg}^2 = 1 - \frac{2s - 1}{u \frac{d}{du} \ln r}.
\]

(67)

Together with Eq. \([47]\), Eq. \([67]\) gives the parametric representation of the tangential velocity of a test particle in a stable circular orbit on the radial distance \( r \). By using the parametric dependence of \( r \) we obtain the following explicit representation for \( v_{tg} \) as a function of \( u \):

\[
v_{tg}^2 (u) = 1 - \frac{2s - 1}{s + \frac{1}{2} C_1 I_{p-1}(lu) + I_{p+1}(lu) + C_2 [K_{p-1}(lu) + K_{p+1}(lu)] - C_1 I_{p}(lu) + C_2 K_{p}(lu)}.
\]

(68)
The variation of \( v_{tg} \) as a function of the radial distance \( r \) is represented, for different values of the constant \( s \), in Fig. 5.

The tangential velocity of a particle in a stable circular orbit in a conformally symmetric geometry on the brane, with arbitrary conformal factor, is a monotonically increasing function of the distance. In the limit of small radial distances, by using the approximate expressions of the Bessel functions for small values of the argument we obtain the following explicit dependence of the tangential velocity on the radial distance:

\[
v_{tg}^2(r) \approx 1 - \frac{2s - 1}{s + \Gamma(p+1)/\Gamma(p)} \left( \frac{2}{C_2} \right)^{\frac{1}{2} - \frac{1}{p}} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{p}\right)}{\Gamma\left(\frac{3}{2} + \frac{1}{p}\right)} \Gamma(p-1) \Gamma\left(\frac{p-2}{p}\right) \Gamma\left(\frac{p-2}{p}\right) r^{\frac{p-2}{p}}, \quad r \to 0.
\] (69)

The angular velocity is a monotonically increasing function of \( r \). For small values of \( r \), one can neglect the term containing the radial coordinate with respect to the other terms, giving

\[
v_{tg}^2 \approx 1 - \frac{2s - 1}{s + \Gamma(p+1)/\Gamma(p)} = \text{constant}.
\]

Therefore at small distances from the origin the angular velocity of a test particle moving on the conformally symmetric brane has a constant value.

In the limit of large distances we obtain the following parametric representation for \( v_{tg} \):

\[
r \approx \frac{C_1}{\sqrt{2\pi l}} u^{s-1/2} e^{lu}, \quad v_{tg}^2(u) \approx 1 - \frac{2s - 1}{s + lu}.
\] (70)

In the limit of large \( r \) the tangential velocity of a test particle in a stable circular orbit in a vacuum brane with general conformal symmetry tends to the speed of light \( c \). This type of behavior, corresponding to an increase in the velocity of test particles in a stable circular orbit, is very similar to the behavior of hydrogen clouds outside spiral galaxies, and which is usually attributed to the presence of the dark matter. Therefore, based on this analogy, we shall assume that the conformally symmetric solution of the gravitational field equations on the brane describes the gravitational field outside a galaxy.

The metric coefficient \( e^\lambda \) can be expressed as a function of the tangential velocity and the parameter \( u \) as

\[
e^\lambda = 3 \left( \frac{l}{2s - 1} \right)^2 u^2 (1 - v_{tg}^2)^2.
\] (71)

The field equation can be immediately integrated to give the following representation for the metric tensor component \( e^{-\lambda} \),

\[
e^{-\lambda} = 1 - \frac{2GM_U(r)}{r},
\] (72)

where

\[
M_U(r) = \frac{3\beta}{2G} \int_0^r U(\rho)\rho^2 d\rho,
\] (73)
is the mass corresponding to the dark radiation term \( U(r) \) (the "dark mass"). By using Eq. (49) we obtain the following parametric representation for \( M_U(r) \):

\[
M_U(r) = \frac{r}{2G} \left[ 1 - \frac{1}{3l^2} \left( \frac{d}{du} \ln r \right)^2 \right].
\] (74)

The dark mass can be also expressed in terms of the tangential velocity of a particle in a stable circular orbit as

\[
M_U(r) = \frac{r}{2G} \left[ 1 - \frac{(2s - 1)^2}{3l^2} \frac{1}{u^2 (1 - v_{tg})^2} \right].
\] (75)

In the limit of small \( r \) the dark mass behaves like

\[
M_U(r) \approx \frac{r}{2G} \left[ 1 - \frac{1}{3l^2} \left( s + \frac{\Gamma(p + 1)}{\Gamma(p)} \right)^2 \left( \frac{2}{C_2 \Gamma(p)} \left( \frac{l}{2} \right)^p \right)^{-\frac{3}{2}} \frac{1}{r^{-\frac{3}{2}}} \right].
\] (76)

In the limit of large \( r \) \((u \to \infty)\) we obtain

\[
M_U(r) = \frac{r}{3G}.
\] (77)

Therefore for large \( r \) the dark mass is linearly increasing with the distance to the galactic center.

VI. DISCUSSIONS AND FINAL REMARKS

In the present paper we have obtained a class of conformally symmetric solutions of the vacuum field equations in the brane world model, under the assumption of a non-static and non-central conformal symmetry, and we have discussed some of their physical properties. To obtain the solution we have made the crucial assumption of ignoring the possible effect of the cosmological constant on the geometrical structure of the vacuum on the brane. Mathematically, this means that the model can correctly describe the dynamics of particles at distances \( r \) from the galactic center satisfying the condition

\[
r << \sqrt{\frac{l^2 u^2 - n^2}{12l \Lambda u^2}}.
\] (78)

Taking into account the smallness of the value of the cosmological constant, the present model can be safely applied to describe the motion of particles outside galaxies. On the other hand, in the limit of large \( r \), both the dark radiation \( U \) and the dark pressure \( P \) are decreasing functions of \( r \), and at infinity the geometry and the dynamical behavior of the particles is determined by the cosmological constant only.

The behavior of the metric coefficients and of the angular velocity in the solution we have obtained depends on four arbitrary constants of integration \( l, s, C_1 \) and \( C_2 \). Their numerical values can be obtained by assuming the continuity of the metric coefficients \( \exp(\lambda) \) and \( \exp(\nu) \) across the vacuum boundary of the galaxy. For simplicity we assume that inside the "normal" (baryonic) luminous matter, with density \( \rho_B \), which form a galaxy, the non-local effects of the Weyl tensor can be neglected. We define the vacuum boundary \( r_0 \) of the galaxy (which, for simplicity, is assumed to have spherical symmetry) by the condition \( \rho_B (r_0) \approx 0 \).

Therefore at the vacuum boundary the metric coefficients are given by \( \exp(\nu) = \exp(-\lambda) = 1 - 2GM_B/r_0 \), where \( M_B = 4\pi \int_0^{r_0} \rho_B (r) r^2 dr \) is the total baryonic mass inside the radius \( r_0 \). For simplicity we shall also assume that the tangential velocity at the galactic boundary can be approximated by its Newtonian expression, \( v_{tg}^2 (r_0) \approx GM_B/r_0 \).

The continuity of the radius \( r \) through the surface \( r = r_0 \) gives the first condition which must be satisfied by the integration constants:

\[
r_0 = u_0 \left[ C_1 I_p (lu_0) + C_2 K_p (lu_0) \right],
\] (79)

where \( u_0 \) is the value of the parameter \( u \) corresponding to the vacuum boundary of the galaxy \( r = r_0 \). The continuity of \( \exp(\nu) \) at \( r = r_0 \) gives

\[
C^2 r_0^2 u_0^{2(1-2s)} = 1 - 2GM_B/r_0.
\] (80)
The continuity of the tangential velocity of a test particle across the galactic boundary fixes the value of the derivative $dr/du$ for $u = u_0$:

$$\left( \frac{dr}{du} \right)_{u=u_0} = \frac{r_0}{u_0} \frac{2s-1}{1 - \frac{GM_B}{r_0}}. \tag{81}$$

Then the continuity of $\exp(\lambda)$ gives

$$u_0^2 \left( 1 - \frac{GM_B}{r_0} \right)^2 \left( 1 - 2\frac{GM_B}{r_0} \right) = \frac{1}{3} \left( \frac{2s-1}{l} \right)^2. \tag{82}$$

Thus the continuity of the metric potentials gives the following compatibility condition, which must be satisfied by the constants $C$, $l$ and $s$:

$$C^2 \left[ \frac{12s-1}{3} \right] \frac{12^{1-2s}}{l} = \frac{1}{r_0^2} \left( 1 - \frac{GM_B}{r_0} \right)^{2(1-s)} \left( 1 - 2\frac{GM_B}{r_0} \right)^{1-2s}. \tag{83}$$

As a possible physical application of the obtained solutions we have considered the behavior of the angular velocity of a test particle in a stable circular orbit on the brane with non-static and non-central symmetry. The conformal factor $\psi$, together with two constants of integration, uniquely determines the rotational velocity of the particle. The angular velocity is always an increasing function of the radial distance $r$, which from a physical point of view can be considered as the distance from the galactic center. This behavior is independent on the numerical values of the parameters (separation and integration constants) of the model. In the limit of large radial distances the angular velocity tends to a constant value, which in the present case is the speed of light $c$. This general behavior is typical for massive particles (hydrogen clouds) outside galaxies. Thus the rotational galactic curves can be naturally explained in brane world models, without using the concept of dark matter.

It has long been known that Newtonian or general relativistic mechanics applied to the visible matter in galaxies and clusters does not correctly describe the dynamics of those systems. The rotation curves of spiral galaxies are one of the best evidences showing the problems Newtonian mechanics and/or standard general relativity has to face on the galactic/intergalactic scale. In these galaxies, neutral hydrogen clouds are observed at large distances from the center, much beyond the extent of the luminous matter. Assuming a non-relativistic Doppler effect and emission from stable circular orbits in a Newtonian gravitational field, the frequency shifts in the $21$ cm line hydrogen emission lines allows the measurement of the velocity of the clouds.

Observations show that the rotational velocities increase near the center of the galaxy and then remain nearly constant at a value of $v_{tgpc} \sim 200$ km/s. This leads to a mass profile $M(r) = rv_{tgpc}^2/G$. Consequently, the mass within a distance $r$ from the center of the galaxy increases linearly with $r$, even at large distances where very little luminous matter can be detected. This behavior of the galactic rotation curves is explained by postulating the existence of some dark (invisible) matter, distributed in a spherical halo around the galaxies. The dark matter is assumed to be a cold, pressureless medium. There are many possible candidates for dark matter, the most popular ones being the weakly interacting massive particles (WIMP).

However, despite more than 20 years of intense experimental and observational effort, up to now no non-gravitational evidence for dark matter has ever been found: no direct evidence of it and no annihilation radiation from it. Moreover, accelerator and reactor experiments do not support the physics (beyond the standard model) on which the dark matter hypothesis is based.

Therefore, it seems that the possibility that Einstein's (and the Newtonian) gravity breaks down at the scale of galaxies cannot be excluded a priori. Several theoretical models, based on a modification of Newton's law or of general relativity, have been proposed to explain the behavior of the galactic rotation curves. In the framework of the brane models, the role of the dark matter is played by the dark mass $M_U(r)$, which is the mass associated to the dark radiation term, having its physical origin in the five-dimensional bulk. As can be seen from Eq. (77), the dark mass is indeed linearly increasing with the distance to the galactic center. Thus the dark mass has a similar behavior to that observed in the case of the "dark matter" around the galaxies.

The main advantage of the brane world models as compared to the alternative explanations of the galactic rotation curves and of the dark matter is that it can give a systematic and coherent description of the Universe from galactic to cosmological scales. On the other hand, in the present model all the relevant physical quantities, including the dark energy and the dark pressure, which describe the non-local effects due to the gravitational field of the bulk, are expressed in terms of observable parameters (the baryonic mass and the radius of the galaxy). Therefore this opens the possibility of the testing of the brane world models by using astronomical and astrophysical observations at the galactic scale. A systematic comparison between the predictions of the model discussed in the present paper and the observational results will be considered in detail in a future publication.
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