PRESENTATIONS FOR GENERALIZED NILHECKE ALGEBRAS

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ABSTRACT. In this note presentations are given for the nilHecke algebras implicit in the work of Bressler and Evens on Schubert calculus for generalized cohomology theories. Such algebras do not usually satisfy the braid relation. Here the obstruction to the braid relation being satisfied is given explicitly and a connection with the deformation theory of algebras is made.

1. Introduction

The cohomology ring of the flag variety $H^*(\text{SL}(n, \mathbb{C})/B)$ can be described both in terms of Chern classes and in terms of classes dual to Schubert cells. The nilHecke algebra can be used to relate these two descriptions, see [4, 5]. In [2, 3] the authors Bressler and Evens constructed a version of Schubert calculus for generalized cohomology theories.

In this note we give a presentation for the nilHecke algebras implicit in the work of Bressler and Evens which we call the generalized nilHecke algebras. The principal result in [3] is that the only generalized nilHecke algebras for which the braid relation is satisfied are those associated to singular cohomology and complex K-theory; cohomology theories with polynomial formal group laws. The presentation in section 3 gives a formula for the obstruction to the braid relation being satisfied in terms of the formal group law associated to the cohomology theory.

Using this presentation one can see more precisely how the structure of the nilHecke algebra changes under perturbation. Orientable cohomology theories form a space $\mathcal{M}$ over which it is natural to view the generalized nilHecke algebras as a sheaf; the germ at a given point being the algebra associated to a formal group law. We observe that this geometric language is compatible with the language of deformation theory of associative algebras.

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2. Complex Oriented Cohomology Theories

In this section we recall some of the ideas necessary to understand the main result. The existence of a theory of Chern classes is sufficient for a generalized cohomology theory to support a definition in our case. Such theories are called multiplicative complex oriented cohomologies.

2.1. Generalized Cohomology. Here we discuss the class of generalized cohomology theories which satisfy the properties necessary for the construction. Our exposition will avoid the language of spectra.

**Definition 2.2.** A generalized cohomology theory is a contravariant functor $h^*: \text{HTop} \to \text{gr-Ring}$ from the homotopy category of topological spaces to the category of graded commutative rings which satisfies the Eilenberg-Steenrod axioms minus the dimension axiom:

1. If $X = \bigsqcup \alpha X_\alpha$ then $h^n(X) \cong \prod \alpha h^n(X_\alpha)$.
2. If $X = A \cup B$ then there is a long exact sequence,
   $$\cdots \to h^n(X) \to h^n(A) \oplus h^n(B) \to h^n(A \cap B) \to h^{n+1}(X) \to \cdots$$

Where $h^n$ is used to denote the $n$th graded component of $h^*$.

If $h^*(X)$ is a graded commutative ring for each space $X$ then $h^*$ is called multiplicative. A cohomology theory $h^*$ is oriented if there exists a chosen class $y \in h^2(\mathbb{C}P^1)$ such that

$$h^*(\mathbb{C}P^\infty) \cong h^*(pt)[[x_h]]$$

and $x_h = i^*(y)$ where $i: \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$ is the standard inclusion. In what follows, by a good cohomology theory, we mean an oriented multiplicative cohomology theory. Unless otherwise noted, if the reader sees a cohomology theory then it shall be good.

With the $h^*$ cohomology of $\mathbb{C}P^\infty$ fixed, it now follows that

$$h^*(BU(n)) \cong h^*(\mathbb{C}P^\infty)^{\otimes n} \cong h^*(pt)[[x_1, \ldots, x_n]].$$

This isomorphism depends on the choice of orientation; the class $x_h \in h^*(\mathbb{C}P^\infty)$ orienting the cohomology functor $h^*$.

The space $BU$ is the universal classifying space for complex vector bundles. By definition, $BU = \text{colim}_n BU(n)$ where $BU(n) = \text{Gr}(n, \mathbb{C}^\infty)$ is the Grassmannian of $n$-planes in $\mathbb{C}^\infty$. The cohomology ring $H^*(BU)$ is home for Chern classes in the usual sense. A splitting principle argument constructs a theory of Chern classes for each $h^*$ (see [11] 16.2). In particular,

$$h^*(BU(n)) \cong h^*(BT^n)^{\Sigma_n} \cong h^*(pt)[[x_1, \ldots, x_n]]^{\Sigma_n} \cong h^*(pt)[[c_1, \ldots, c_n]].$$
where $\Sigma_n$ is the symmetric group.

If $h^*$ is a good cohomology theory oriented by $x_h \in h^*(\mathbb{C}P^\infty)$ then to any $U(n)$-bundle $E \to X$ one can assign classes $c_i(E) \in h^i(X)$ which satisfy the following properties

1. $c_0(E) = 1$.
2. Naturality: $c_i(f^*E) = f^*c_i(E)$ for $f : Y \to X$.
3. Whitney sum:
   $$c_n(E \oplus E') = \sum_{i+j=n} c_i(E) \cdot c_j(E')$$
4. Let $\gamma \to \mathbb{C}P^\infty$ be the Hopf $U(1)$-bundle over $\mathbb{C}P^\infty$ then $c_1(\gamma) = x_h$.

If the functor $h^*$ is complex cobordism then it is common to refer to the classes, $c_i$, as Connor-Floyd classes.

The properties (1)-(4) above are those that hold uniformly with respect to the cohomology theories under consideration. There is one property which changes as we change the cohomology theory, if $L$ and $L'$ are line bundles on a space $X$ then the first Chern class of their tensor product $c_1(L \otimes L')$ depends on $h^*$:

$$c_1(L \otimes L') = F(c_1(L), c_1(L'))$$

The function $F(x, y) \in h^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong h^*(pt)[[x, y]]$ is given by $F(x, y) = \otimes^*(x_h)$. Here the map

$$\otimes : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$$

is determined by tensor product of complex line bundles and the universal property of $\mathbb{C}P^\infty \simeq BU(1)$.

**Definition 2.3.** (formal group law) A formal group law is a commutative ring $R$ together with $F(x, y) \in R[[x, y]]$ such that

1. $F(0, x) = x$ and $F(x, 0) = x$
2. $F(x, y) = F(y, x)$
3. $F(x, F(y, z)) = F(F(x, y), z)$

for all $x, y, z \in R$.

A grading will be useful in our context. A formal group law is graded if $R$ is a graded commutative ring, $|x| = 2$, $|y| = 2$ and

4. $F(x, y)$ is homogeneous of degree 2 in $R[[x, y]]$.

All formal group laws will be graded unless otherwise stated.

The series $F(x, y)$ defined above satisfies these properties because $\otimes$ is unital, commutative and associative. We call the pair $h^*(pt)$ and $F(x, y) \in h^*(pt)[[x, y]]$ the formal group law associated to $h^*$. 
A homomorphism $\phi : F \to G$ between two formal group laws $F, G \in R[[x, y]]$ is a power series $\phi \in R[[x]]$ which satisfies,

$$\phi(F(x, y)) = G(\phi(x), \phi(y)).$$

For every formal group law there exists a unique series $\text{inv}(x) \in R[[x]]$ such that

$$F(x, \text{inv}(x)) = 0$$

(see [10]). We adopt the notation,

$$x +_F y = F(x, y) \quad \text{and} \quad x -_F y = F(x, \text{inv}(y)).$$

A formal group law $F$ is symmetric if it satisfies

$$x_i -_F x_{i+1} = -(x_{i+1} -_F x_i).$$

Symmetric formal group laws have simpler presentations. See remark section 3.3.

There are many examples of formal group laws. If $C$ is any 1-dimensional algebraic group then expanding the multiplication at identity as a power series yields a formal group law. When $R = \mathbb{Z}$ the formal group law associated to singular cohomology $H^*$ is

$$x +_H y = x + y.$$ 

More examples are explored in section 2.3.

Suppose that $F \in R[[x, y]]$ and $G \in S[[x, y]]$ are formal group laws.

$$F(x, y) = \sum_{i,j} c_{ij} x^i y^j \quad \text{and} \quad G(x, y) = \sum_{i,j} d_{ij} x^i y^j$$

If $\varphi : R \to S$ is a ring homomorphism such that $d_{ij} = \varphi(c_{ij})$ then we say that $\varphi(F) = G$. Given $F_U \in L[[x, y]]$, if for all formal group laws $G \in S[[x, y]]$ there exists a ring homomorphism $\varphi : L \to S$ such that $\varphi(F_U) = G$ then we say that $F_U$ is universal.

**Theorem 2.4.** (Lazard) There exists a universal formal group law $F_U \in L[[x, y]]$ and the associated ring $L$ is a polynomial ring on infinitely many generators,

$$L \cong \mathbb{Z}[u_1, u_2, u_3, \ldots]$$

where $|u_i| = -2i$.

To prove existence one defines $L = \mathbb{Z}[[c_{ij}]]/I$ and $F_U \in L[[x, y]]$ to be

$$F_U(x, y) = \sum_{i,j} c_{ij} x^i y^j.$$ 

The ideal $I$ is generated by relations (1)–(3) in definition 2.3. $L$ is graded by $|c_{ij}| = -2(i+j-1)$. Lazard proved that $L$ is isomorphic to a polynomial ring on generators, $u_n$, in even degree (see [10]).
2.5. **Complex Cobordism and Genera.** In this section we briefly recall the work of Quillen and others on complex cobordism (see [9, 8, 1]).

There is a cohomology theory $MU^*$ defined so that for a manifold $X$ of dimension $r$, $MU^{r-l}(X)$ is spanned by cobordism classes of maps $f : M \to X$ where $f$ is smooth, proper and $M$ is an $l$-dimensional manifold equipped with a stable almost complex structure. If $s \in MU^*(X)$, $s : M \to X$ and $g : Y \to X$ then $g^*(s) = M \times_X Y \to Y \in MU^*(Y)$. If $L \to X$ is a $U(1)$-bundle over $X$ then the zero section of the Thom space $X^L$, $s : X \to X^L$ determines a class $s \in MU^2(X^L)$ and $MU^*$ is oriented by the choice $c_1(L) = s^*(s) \in MU^2(X)$.

Note that $MU^*(pt)$ is spanned by compact stably almost complex manifolds modulo the cobordism relation.

**Theorem 2.6.** (Quillen) Complex cobordism of a point is isomorphic to the Lazard ring,

$$MU^*(pt) \cong L$$

The induced isomorphism,

$$\text{Hom}(L, R) \cong \text{Hom}(MU^*(pt), R)$$

implies that the set of formal group laws in $R$ is in bijection with the set of $R$-valued cobordism invariants of complex manifolds. Such invariants are called $R$-genera.

Thom showed (see [12, 1]) that

$$MU^*(pt) \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^1, \mathbb{C}P^2, \ldots]$$

with $|\mathbb{C}P^n| = -2i$, so that a genus $\rho : MU^*(pt) \to R$ is determined by its values on complex projective spaces. Note, $u_i \in L$ of theorem 2.4 is not equal to $\mathbb{C}P^i$ above.

Below are examples of genera and their formal group laws.

**Examples.** (Formal Group Laws and Genera)

1. The genus $\rho$ associated to homology is determined by $\rho(\mathbb{C}P^n) = 0$.

2. A graded Todd genus determined by $\text{Td}(\mathbb{C}P^n) = \beta^n \in \mathbb{Z}[\beta]$ is associated to the formal group law $F_K(x, y) = x + y - \beta xy$ for connective complex K-theory.

3. Hirzebruch’s $\chi_a$-genus is defined using Dolbeault cohomology,

$$\chi_a(M) = \sum_{p,q}(-1)^p a^q \dim H^{p,q}(M) \in \mathbb{Z}[a]$$

If $M$ is $\mathbb{C}P^n$ then $\chi_a(\mathbb{C}P^n)$ is the cyclotomic polynomial $1-a+a^2-\cdots \pm a^n$ and the formal group law is

$$F_{\chi_a}(x, y) = \frac{x + y + (a - 1)xy}{1 + axy}.$$
This formal group law doesn’t quite satisfy our grading conventions. Later we will instead we will consider the more general,

$$F_{\alpha,\beta}(x, y) = \frac{x + y - \beta xy}{1 + \alpha xy} \in \mathbb{Z}[\alpha, \beta][[x, y]].$$

The grading is obtained by placing $\alpha$ in degree $-4$ and $\beta$ in degree $-2$.

(4) There is a formal group law associated to any 1-dimensional algebraic group. If $Q(x) = 1 - 2\delta x^2 + \epsilon x^4$ then Euler determined a two parameter family of formal group laws

$$F(x, y) = \frac{x \sqrt{Q(y)} + y \sqrt{Q(x)}}{1 - \epsilon x^2 y^2}$$

associated to Jacobi quartics. $F$ is defined over the ring $\mathbb{Z}[\frac{1}{2}, \epsilon, \delta]$ in which $\delta$ has degree $-4$ and $\epsilon$ has degree $-8$. The values of the associated genus $\rho$ are determined by power series $\log_F(x)$ where

$$\log_F(x) = \int_0^x \frac{dy}{\sqrt{Q(y)}} \quad \text{and} \quad \rho(\mathbb{C}P^n) = \frac{\log_{F(n+1)}(0)}{n!}.$$

Given a genus $\rho : MU^*(pt) \to R$, one can hope to define a cohomology theory $h^*$ by extending coefficients:

$$h^*(X) = MU^*(X) \otimes_{MU^*(pt)} R.$$ 

Doing this can sometimes recover the functor $h^*$ with formal group law determined by $\rho$. In general, it is necessary for $R$ to be flat for Mayer-Vietoris long exact sequence to hold \(\text{(2.2 (2))}\), the Landweber exactness theorem gives more general criteria (see [17]).

3. nilHecke Algebra Presentations

In this section we derive a presentation for the nilHecke algebra $G_{NH}$ associated to each formal group law $F \in R[[x, y]]$. As we vary the formal group law $F$, the algebras $G_{NH}$ are related: $G_{NH}$ is the germ of a sheaf of algebras $G_{\mathcal{N}H_n}$ on the moduli space $\mathcal{M}$ of formal group laws.

3.1. nilHecke Algebra. In this section we briefly recall the definition of the nilHecke algebra in order to provide some framework for comparison.

**Definition 3.2.** (nilHecke algebra $NH_n$) The nilHecke algebra $NH_n$ is the graded ring generated by operators $x_i$ in degree 2, $1 \leq i \leq n$, and $\partial_j$ in degree $-2$, $1 \leq j < n$, subject to the relations:

$$\partial_i^2 = 0, \quad \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$$

$$x_i \partial_i - \partial_i x_{i+1} = 1, \quad \partial_i x_i - x_{i+1} \partial_i = 1.$$
The operators also satisfy far commutativity relations,
\[
\partial_i x_j = x_j \partial_i \quad \text{if } |i - j| > 1, \quad \partial_i \partial_j = \partial_j \partial_i \quad \text{if } |i - j| > 1,
\]
\[
x_i x_j = x_j x_i \quad \text{for } 1 \leq i, j \leq n.
\]

The ring of coinvariants \( P_n = H^*(\text{Fl}_n; \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_n]_{\Sigma_n} \) is a module over the nilHecke algebra \( \text{NH}_n \) and serves as a kind of defining representation. The action of \( \text{NH}_n \) on \( P_n \) is defined by letting \( x_i \) act by multiplication by \( x_i \) and \( \partial_j \) act on \( f \in P_n \) by
\[
\partial_j(f) = \frac{f - s_j(f)}{x_j - x_{j+1}}.
\]

One can check that \( \partial_j(\sigma) = 0 \) if \( \sigma \) is a symmetric polynomial and that the other relations above are satisfied.

### 3.3. Generalized nilHecke Algebra.

In this section we determine a presentation for the algebra \( \text{GNH}_n \). The algebra \( \text{GNH}_n \) is defined implicitly in [2, 3] as the algebra generated by convolutions, \( \pi^* \circ \pi_* \), associated to Bott-Samuelson resolutions of the full flag variety. Such operations allow one to interpolate between the Schubert basis of the cohomology of the full flag variety \( h^*(\text{Fl}_n) \) and its Borel description:
\[
h^*(\text{Fl}_n) \cong h^*(pt)[[x_1, \ldots, x_n]]_{\Sigma_n}
\]
an idea going back to [4, 5].

While for some purposes it is useful to have a geometric description of an algebraic construction. For other purposes an abstract presentation is often welcome.

Throughout this section fix a graded formal group law \( F \in R[[x, y]] \).

Recall from section 2.1 that \( x_i - F x_{i+1} \) denotes the power series \( F(x_i, \text{inv}(x_{i+1})) \in R[[x_i, x_{i+1}]] \). The multiplicative inverse \( 1/(x_i - F x_{i+1}) \) of \( x_i - F x_{i+1} \) always exists if \( R \) has no 2-torision (see [2]).

Define power series \( s, t \in R[[x_i, x_{i+1}]] \) to be
\[
s(x_i, x_{i+1}) = \frac{1}{x_i - F x_{i+1}} + \frac{1}{x_{i+1} - F x_i} \quad \text{and} \quad t(x_i, x_{i+1}) = \frac{x_i - x_{i+1}}{x_i - F x_{i+1}}
\]
and let \( l, r \in R[[x_i, x_{i+1}, x_{i+2}]] \) to be the power series
\[
\frac{1}{(x_{i+2} - F x_{i+1})(x_{i+2} - F x_i)} - \frac{1}{(x_i - F x_{i+1})(x_{i+2} - F x_i)} - \frac{1}{(x_{i+1} - F x_i)(x_{i+2} - F x_{i+1})}
\]
\[
\frac{1}{(x_{i+1} - F x_{i+2})(x_{i+2} - F x_i)} + \frac{1}{(x_{i+2} - F x_{i+1})(x_{i+1} - F x_i)} - \frac{1}{(x_{i+1} - F x_i)(x_{i+2} - F x_i)}.
\]

The functions \( s, t, l \) and \( r \) are homogeneous of degree \(-2, 0, -4 \text{ and } -4\) respectively.
Definition 3.4. (GHN\textsubscript{n}) Let \( R \) be a graded commutative ring and \( F \in R[[x,y]] \) a formal group law (see section 2.1 definition 2.3). The generalized nilHecke algebra \( \text{GHN}_n \) associated to this formal group law is the graded \( R \)-algebra generated by operators \( x_i \) in degree 2, \( 1 \leq i \leq n \) and \( \partial_j \) in degree \(-2\), \( 1 \leq j < n \), subject to the relations enumerated below.

The two quadratic relations are deformed by the \( s \) and \( t \) functions,
\[
\partial_i^2 = s(x_i, x_{i+1})\partial_i, \quad x_i\partial_i - \partial_i x_{i+1} = t(x_i, x_{i+1}) \cdot 1, \\
\partial_i x_i - x_{i+1}\partial_i = t(x_i, x_{i+1}) \cdot 1.
\]

The Reidemeister III type relation is deformed by the \( l \) and \( r \) functions,
\[
\partial_{i+1}\partial_i\partial_{i+1} - \partial_i\partial_{i+1}\partial_i = l(x_i, x_{i+1}, x_{i+2})\partial_i + r(x_i, x_{i+1}, x_{i+2})\partial_{i+1}.
\]

The operators also satisfy far commutativity relations,
\[
\partial_i x_j = x_j\partial_i \quad \text{if} \quad |i - j| > 1, \quad \partial_i\partial_j = \partial_j\partial_i \quad \text{if} \quad |i - j| > 1, \\
x_i x_j = x_j x_i \quad \text{for} \quad 1 \leq i, j \leq n.
\]

All of the relations above are homogenous with respect to the grading.

Remark. Recall that a symmetric formal group law \( F \) satisfies \( x_i - F x_{i+1} = -(x_{i+1} - F x_i) \). When this is so the first two relations above become
\[
\partial_i^2 = 0 \quad \text{and} \quad \partial_{i+1}\partial_i\partial_{i+1} - \partial_i\partial_{i+1}\partial_i = l(x_i, x_{i+1}, x_{i+2})(\partial_i - \partial_{i+1})
\]
because \( r(x_i, x_{i+1}, x_{i+2}) = -l(x_i, x_{i+1}, x_{i+2}) \).

3.4.1. Graphical representation of \( \text{GHN}_n \). In our graphical calculus the generators \( x_i \) are represented by a dot on the \( i \)th horizontal strand appearing in a picture reading from left to right and the generators \( \partial_j \) are illustrated by a singular crossing between the \( j \)th and \( j+1 \)st vertical strands.

Each dot is graded in degree 2 and each crossing in degree \(-2\). If we fix the convention that the first strand appearing in an illustration corresponds to the \( i \)th strand then the first collection of relations are depicted graphically below.

\[
\begin{align*}
\begin{array}{c}
\uparrow \\
\uparrow
\end{array}
& \quad \text{and} \quad \\
\begin{array}{c}
\uparrow \quad \downarrow \\
\uparrow
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\Updownarrow \\
\Updownarrow
\end{array}
& = s(x_i, x_{i+1}) \\
\begin{array}{c}
\Updownarrow \\
\Uparrow
\end{array}
& \quad \text{and} \quad \\
\begin{array}{c}
\Downarrow \\
\Updownarrow
\end{array}
& - \\
\begin{array}{c}
\Updownarrow \\
\Uparrow
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\Downarrow \\
\Updownarrow
\end{array}
& = t(x_i, x_{i+1}) \\
\begin{array}{c}
\Downarrow \\
\Uparrow
\end{array}
& \quad \text{and} \quad \\
\begin{array}{c}
\Downarrow \\
\Updownarrow
\end{array}
\end{align*}
\]
We picture the far commutativity relations in the usual way.

Theorem 3.5. The algebra defined by the presentation above is isomorphic to the algebra $\text{GNH}_n$.

Proof. The action of the algebra $\text{GNH}_n$ on $h^*(\text{Fl}_n)$ lifts to an action on the ring of power series, $h^*(B\text{Fl}_n) \cong h^*(pt)[[x_1, \ldots, x_n]]$.

$$h^*(B\text{Fl}_n) \longrightarrow h^*(\text{Fl}_n)$$

$$h^{*-2}(B\text{Fl}_n) \longrightarrow h^{*-2}(\text{Fl}_n)$$

Where operator $x_i \in \text{GNH}_n$ continues to act by multiplication and the operator $\partial_i$ acts on $f \in h^*(pt)[[x_1, \ldots, x_n]]$ by

$$\partial_i(f) = \frac{f}{x_i - F x_{i+1}} + \frac{s_i(f)}{x_{i+1} - F x_i}$$

where $s_i f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$, see [2]. For each $i$, $1 \leq i < n$, $\partial_i$ is a linear combination of operators from $h^*(pt)[\Sigma_n]$. Since

$$\Sigma_n = \langle s_1, \ldots, s_{n-1} | s_i^2 = 1, s_i s_j = s_j s_i \text{ for } |i - j| > 2, \text{ and } s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \rangle,$$
it suffices to check that the relations among \( \partial_i \)'s given above hold. In particular, the span of the set
\[
\{1, \partial_i, \partial_i \partial_{i+1}, \partial_i \partial_{i+1} \partial_i, \partial_i \partial_{i+1} \partial_i \partial_{i+1}, \partial_i \partial_{i+1} \partial_i \partial_{i+1} \partial_i \partial_{i+1} \}
\]
is linearly dependent and the Reidemeister III relation is obtained by solving for coefficients in the dependency relation. The Weyl-type commutation relation between \( x_i \pm 1 \) and \( \partial_i \) can be seen to hold in the same way. The far commutativity relations follow by observing that they hold for polynomials in \( x_i \) and the operators \( s_i \in \Sigma_n \) respectively.

\[\square\]

**Corollary 3.6.** There is a sheaf \( \mathcal{GNH}_n \) of nilHecke algebras defined on \( M \). If \( F \) is an \( R \)-point of \( M \) then the stalk of \( \mathcal{GNH}_n \) at \( F \) is the generalized nilHecke algebra associated to \( F \).

3.6.1. **Examples.** In this section we list some examples of presentations of the generalized nilHecke algebra \( \mathcal{GNH}_n \) associated to formal group laws and genera found in section 2.5.

(1) The genus \( \rho \) associated to homology is determined by \( \rho(\mathbb{C}P^n) = 0 \) corresponds to the formal group law
\[
F(x, y) = x + y.
\]
This yields the presentation for the nilHecke algebra \( \mathcal{NH}_n \) of section 3.1.

(2) For connective K-theory, a graded Todd genus determined by \( \text{Td}(\mathbb{C}P^n) = \beta^n \in \mathbb{Z}[\beta] \) is associated to the formal group law
\[
F_K(x, y) = x + y - \beta xy
\]
yields an algebra \( \mathcal{GNH}_n \) defined over the ring \( R = \mathbb{Z}[\beta] \) in which \( \beta \) is placed in degree \(-2\). Direct computation yields the formula,
\[
x_i - F x_{i+1} = \frac{x_i - x_{i+1}}{1 + \beta x_{i+1}}.
\]
Using this we can compute the defining functions for \( \mathcal{GNH}_n \),
\[
s(x_i, x_{i+1}) = \beta \quad t(x_i, x_{i+1}) = 1 - \beta x_{i+1}
\]
l(\( x_i, x_{i+1}, x_{i+2} \)) = 0 \quad r(\( x_i, x_{i+1}, x_{i+2} \)) = 0
\]
Notice that \( l, r = 0 \) implies that the braid relation, \( \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \), holds in \( \mathcal{GNH}_n \). The formal group laws associated to singular homology and K-theory determine the only generalized nilHecke algebras \( \mathcal{GNH}_n \) for which this is true [2].

(3) Related to Hirzebruch’s \( \chi_a \)-genus defined in section 2.5 is the formal group law
\[
F_{\chi_a, \beta}(x, y) = \frac{x + y - \beta xy}{1 + \alpha xy}
\]
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defined over the graded ring \( \mathbb{Z}[\alpha, \beta] \) in which \( \alpha \) is placed in degree \(-4\) and \( \beta \) in degree \(-2\). Direct computation yields the formula,

\[
x_i - F x_{i+1} = \frac{x_i - x_{i+1}}{1 - \beta x_{i+1} - \alpha x_i x_{i+1}}
\]

Using this we can compute the defining functions for \( \text{GNH}_n \),

\[
\begin{align*}
  s(x_i, x_{i+1}) &= \beta \\
  t(x_i, x_{i+1}) &= 1 - \beta x_{i+1} - \alpha x_i x_{i+1} \\
  l(x_i, x_{i+1}, x_{i+2}) &= \alpha \\
  r(x_i, x_{i+1}, x_{i+2}) &= -\alpha
\end{align*}
\]

Setting \( \beta = 0 \) in this example yields a symmetric formal group law. Notice that the functions \( l, r \) do not depend on \( \beta \).

(4) There is an algebra \( \text{GNH}_n \) defined over \( \mathbb{Z}\left[\frac{1}{2}, \epsilon, \delta\right] \) using Euler’s formal group law in section 2.5.

There are many other formal group laws and associated generalized nilHecke algebras. The universal example is determined by complex cobordism and the universal formal group law.

We may think of a space as a functor of points \( \text{Ring} \to \text{Sets} \). For instance, the moduli space of formal group laws, \( \mathcal{M} \), is the functor \( \text{gr-Ring} \to \text{Sets} \) represented by the Lazard ring \( L \) (see theorem 2.4). The tangent bundle \( TX \) of a space \( X \) is the functor determined by the composition

\[
TX(R) = X(R \otimes \mathbb{Z}[\epsilon]/(\epsilon^2)).
\]

The fibered product,

\[
\begin{array}{ccc}
\pi^* \mathcal{GNH}_n & \longrightarrow & \mathcal{GNH}_n \\
\downarrow & & \downarrow \\
TM & \stackrel{\pi}{\longrightarrow} & \mathcal{M}
\end{array}
\]

yields a map \( \pi^* \mathcal{GNH}_n \to \mathcal{M} \). The fiber of \( \pi^* \mathcal{GNH}_n \) over a fixed formal group law \( F \in R[[x, y]] \) consists generalized nilHecke algebras defined over formal group laws \( F_i \in R[\epsilon]/(\epsilon^2)[[x, y]] \) of the form

\[
F_\epsilon(x, y) = F(x, y) + \epsilon F_1(x, y).
\]

There is a canonical map,

\[
\psi : (\pi^* \mathcal{GNH}_n)_F \to Z(C^2(\text{GNH}_n, \text{GNH}_n, \text{GNH}_n))
\]

from the fiber of the pullback of \( \mathcal{GNH}_n \) over a formal group law \( F \) to the second Hochschild cocycles of the GNH algebra fibering over \( F \). Moreover, if \( \phi : F_\epsilon \to F'_\epsilon \) is an isomorphism of tangential formal group laws then the associated cocycles are cohomologous. This implies that there is subspace of \( HH^2(\text{GNH}_n, \text{GNH}_n) \), the space of infinitesimal deformations of \( \text{GNH}_n \) \[6\], naturally associated to the infinitesimal horizontal perturbations of \( \text{GNH}_n \) in \( \mathcal{GNH}_n \).
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