Semiclassical simulation of the double-slit experiments with single photons

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Many quantum phenomena, traditionally described by quantum electrodynamics, can be calculated within the framework of so-called semiclassical theory, in which atoms are described by the wave equations of quantum mechanics (Schrödinger, Dirac, etc.), while light is described by classical electrodynamics without quantization of the radiation. These phenomena include the photoelectric effect, the Compton effect, the Lamb shift, radiative effects, spontaneous emission, the Hanbury Brown and Twiss effect, etc. In this paper, I show that the double-slit experiment can also be calculated in detail in terms of semiclassical theory if we take into account the discrete (atomic) structure of matter. I show that the results of a semiclassical simulation of the “linear” double-slit experiment coincide with predictions of wave theory only for low-intensity light and for short exposure time, while stricter dependences exist for long-term exposure. The semiclassical theory is used for calculation of the “nonlinear” double-slit experiment with an intense laser beam in which multiphoton and tunnel excitation of atoms on a photographic plate occurs. The Born rule for light is derived from the semiclassical theory.

Subject Index A22, A60

1. Introduction

We know that light has both wave- and particle-like properties: in some experiments (diffraction and interference experiments), light behaves as a classical electromagnetic wave, while, in other experiments (interaction with matter), light behaves as a flux of particles, i.e., photons. This property of light is called the wave–particle duality. The duality is manifested in the observation that light interacting with matter induces discrete events (clicks of a detector or the appearance of spots on a photographic plate), which are interpreted as the interactions of single photons. After prolonged exposure, these discrete events merge into a single continuous pattern, well described by classical electrodynamics or even classical optics [1,2].

Compatibility between the wave and corpuscular properties of light in quantum theory is achieved by using a probabilistic interpretation of optical phenomena: the probability \( p \) of finding a photon at some point in space is proportional to the intensity of the classical light wave \( I \sim E^2 \) at this point, calculated on the basis of the methods of wave optics [3]:

\[
p \sim E^2,
\]

where \( E \) is the strength of the electric field of the classical light wave.
Equation (1) is a mathematical formulation of the wave–particle duality because the probability \( p \) refers to the particle (photon), while the intensity \( I \sim E^2 \) refers to the wave, which, in many cases, can be calculated by the methods of classical wave optics.

For brevity, we will call Eq. (1) “the Born rule for light” by analogy with the same type of Born rule describing the probability of finding a nonrelativistic quantum particle.

The direct application of expression (1) to the interpretation of the results of experiments leads to paradoxes.

Most clearly and impressively, the paradoxes associated with the wave–particle duality of light are manifested in Young’s double-slit experiment [1,2]: the closing of one of the slits in the double-slit experiment changes the “behavior” of the photons that pass through the open slit. Experiments indicate that the wave properties of photons cannot be explained as a result of the interaction of photons in the beam because they are manifested, even for a very weak light source, when “photons fly alone” [1,2]. This behavior indicates that each individual photon should have wave properties. As a result, the representation that the photon interferes with itself has appeared.

Numerous attempts to explain these experiments by unusual (nonclassical) motion of point photons [4–12] cannot be considered successful.

At the same time, we know that many quantum phenomena, traditionally described by quantum electrodynamics, can in fact be explained within the framework of so-called semiclassical theory, in which atoms are described by the wave equations of quantum mechanics (Schrödinger, Dirac, etc.), while light is described by classical electrodynamics without quantization of the radiation. These phenomena include the photoelectric effect [13,14], the Compton effect [15–18], the Lamb shift [19–22], radiative effects [20–23], spontaneous emission [19,22,23], the Hanbury Brown and Twiss effect [24,25], etc.

In this paper, I show that the double-slit experiment can also be calculated in terms of semiclassical theory if we take into account the discrete structure of matter (a detecting screen) and the specific nature of the interaction of light with matter, which is described by the Schrödinger equation (or other wave equation of quantum mechanics).

This method is also used for direct numerical simulation of a “nonlinear” double-slit experiment when so-called multiphoton or/and tunnel excitation of atoms on a photographic plate occurs.

Note that the term “photographic plate” is used in this paper as a synonym for a detector that allows detection of discrete events that are interpreted as a result of the action of the photons.

2. Linear double-slit experiment

Let us consider a weak classical electromagnetic wave that passes through a diffractive device, such as a diffraction grating or a system of slits in an opaque screen, and impinges on the surface of the photoactive substance (conditionally, a photographic plate), the atoms of which can be excited by light.

For simplicity, let us assume that the photographic plate represents a single layer of atoms, in which the atoms are placed randomly with surface density \( n \) (number of atoms per unit surface). The atoms are quantum objects, which are described by the Schrödinger equation, whereas the light is considered as a classical electromagnetic wave. In this case, the light intensity can be calculated using classical optics and is considered to be known on the surface of the photographic plate.

We are interested in the interaction of a weak light wave with matter, when the photons impinge on the surface of the photographic plate one by one. It is these conditions, realized in the double-slit experiments with feeble light [1,2], that demonstrate the wave properties of individual photons.
The probability of excitation of any atom on the surface of a photographic plate by a classical electromagnetic wave for time $\Delta t$ is described by Fermi’s golden rule:

$$w\Delta t = bI\Delta t,$$

(2)

where $w$ is the probability of excitation of atoms per unit time; $I \sim E^2$ is the intensity of the classical electromagnetic wave at the location of the atom; and $b$ is a constant depending on the characteristics of the atom and the frequency of the incident electromagnetic wave.

Note that, unlike the Born rule (1), expression (2) is not a postulate; rather, it follows directly from the perturbation theory [26].

Relation (2) is sufficient for direct semiclassical simulation of the interaction of light with a photographic plate. In these calculations, we assume that the excitation (ionization) of an atom is perceived as blackening of the appropriate point of the photographic plate.

The calculation proceeds quite trivially using the Monte Carlo method: at each time, the probability of excitation of each unexcited atom will be calculated. The calculation for each atom is continued until its excitation occurs.

The rate of atomic excitation $w$ does not depend on the concentration of atoms and is determined only by the intensity of the radiation at a given point on the screen. If the radiation intensity does not change with time, then the law of excitation of atoms will be similar to that of radioactive decay. In particular, the probability of excitation at time $t$ of an atom located at a point on the screen with a given value $w$ is

$$P_+(t) = 1 - \exp(-wt).$$

(3)

Taking into account (2), one obtains

$$P_+(t) = 1 - \exp(-bIt).$$

(4)

Let us introduce the nondimensional time

$$\tau = bI_0 t,$$

(5)

where $I_0$ is the maximum intensity of light on the screen (photographic plate). The nondimensional time (5) plays the role of exposure time.

In this case, the probability of excitation of the atom for time $t$ at the points on the screen having the intensity of light $I$ is

$$P_+(t) = 1 - \exp(- (I/I_0) \tau).$$

(6)

For the double-slit experiment, the distribution of the light intensity on the screen is given by the well-known expression [27]

$$I(z) = I_0 \cos^2 \left( \frac{d}{b} x \right) \left( \frac{\sin x}{x} \right)^2,$$

(7)

where

$$x = \frac{\pi b}{\lambda H} z, \quad z = H \sin \theta;$$

$b$ is the width of the slits; $d$ is the distance between the slits; $\lambda$ is the wavelength of light; $H$ is the distance from the slits to the screen; and $\theta$ is the angular coordinate.

For the calculation of the double-slit experiment using the Monte Carlo method, we can create a system of randomly and uniformly distributed points $i = 1, \ldots, N$ in a given area $L_x \times L_y$, simulating the screen. These points are considered as the atoms of the material of the screen. We can use
the average distance between atoms as a length scale; in these units, the concentration of the atoms on the surface of the screen is equal to unity.

At each moment of time \( \tau \) for each as-yet unexcited atom \( i \), the probability \( P_{+i} \) is calculated by expressions (6) and (7); at the same time, the random number \( R_i \in [0, 1], i = 1, \ldots, N \), is generated by a random number generator. If the condition \( R_i \leq P_{+i} \) is satisfied, then the given atom is considered to be excited, and it is depicted by a black dot. Unexcited atoms are not depicted.

The results of the calculations of the process of the “accumulation of photons” on the screen for different moments of time \( \tau \) at \( \frac{\pi b}{\lambda H} = 0.03 \) and \( d/b = 5 \) are shown in Fig. 1 (left). The markers

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**Fig. 1.** Interference pattern buildup (left) and the corresponding distribution functions of dots on the screen (right) for different exposure times \( \tau \), obtained using the Monte Carlo simulation of the interaction of light with the detecting screen. a) \( \tau = 0.02 \) (100 “photons”); b) \( \tau = 0.1 \) (424 “photons”); c) \( \tau = 1 \) (3452 “photons”); d) \( \tau = 10 \) (11 600 “photons”); e) \( \tau = 30 \) (14 530 “photons”). The markers on the graphs (right) show the histograms obtained by treating the corresponding picture on the left; the lines show the theoretical dependence (7), predicted by classical optics.
on the graphs on the right are histograms obtained by treating the corresponding picture on the left; the lines on the graphs on the right show the theoretical dependence (7), predicted by classical optics.

Comparing Fig. 1 with the real picture of the “accumulation of photons” on the photographic plate in the double-slit experiment [2] we see that the calculated pattern is consistent with the experimental interference pattern and the semiclassical theory reproduces the results of these experiments: at short exposure times or at low light intensities, a random system of dots appears on the screen, which can be interpreted as the locations of the “fall of the photons” on the screen, although no photons are considered in our model. With increasing exposure time or light intensity, these dots form clear interference patterns corresponding to the theoretical distribution of intensity, as follows from wave optics. Closing one of the slits, we obtain, in accordance with the considered calculation scheme, exactly the picture predicted by classical optics. In other words, we do not find the paradox related to the “wave–particle duality” of light.

Note that, according to expression (6), a change in light intensity $I_0$ does not result in changes in the pattern in Fig. 1; only the time scale is changed: at high light intensity, the same pattern is reached at a shorter exposure time. At high light intensity or at a long exposure time, the pattern on the photographic plate is different from the predictions of wave optics. This means that the simple Born rule (1) for light in these cases ceases to work, and we need to use the more common rule.

Let the total number of atoms in a selected volume $V$ of matter (e.g., on a selected surface of the photographic plate) be equal to $N$ and be uniformly distributed thereon. Let us choose a small volume $dV$, in which $dN = \frac{N}{V} dV$ atoms are located. Then, for time $t$ within volume $dV$, $dN_+ = P_+ dN$
atoms would be excited:

\[ dN_+ = P_+ \frac{N}{V} dV. \]  

(8)

Because we interpret the excitation of the atoms as photons hitting the atoms, the number of photons entering the volume \( dV \) would be determined by expression (8). Accordingly, the probability of a photon entering into a given volume \( dV \) is

\[ pdV = \frac{P_+}{\int P_+ dV} dV, \]  

(9)

where the integral is taken over the entire volume (e.g., the entire surface of the photographic plate).

Thus, the probability density of detecting a photon at a given point in space is given by

\[ p = \frac{P_+}{\int P_+ dV} \sim P_+. \]  

(10)

Then, for an arbitrary exposure time, the probability “of detecting a photon” at a given point of the photographic plate is determined by expression (10), which, taking into account expression (6), can be written as

\[ \frac{p}{p(0)} = \frac{1 - \exp(- (I/I_0) \tau)}{1 - \exp(-\tau)}. \]  

(11)

where \( p(0) \) is the probability “of detecting a photon” in the middle of the photographic plate (at \( I = I_0 \)). The ratio \( p/p(0) \) in the double-slit experiments with “single photons” plays the same role as the ratio \( I/I_0 \) in the optical experiments. In particular, at short exposure times \( \tau \ll 1 \), we obtain from expression (11) the Born rule (1): \( p/p(0) \approx I/I_0 \).

The above analysis leads to the following conclusions: (i) the Born rule for light (1) is a trivial consequence of quantum mechanics if the electromagnetic wave is considered as a classical field, while the matter (detector) is considered as consisting of discrete atoms; (ii) the Born rule (1) is valid only for weak electromagnetic waves and a relatively short exposure time \( \tau \ll 1 \). For strong electromagnetic waves or for a longer exposure time for which this condition is not satisfied, the Born rule is violated and should be replaced by the stricter rule (11).

At long-term exposure, the diffraction pattern will qualitatively appear as that predicted by wave optics, i.e., a system of periodic fringes, but quantitatively, it will be significantly different from both the predictions of wave optics (the lines in Fig. 1, right) and the predictions based on the simple Born rule (1). Quantitative agreement with the wave theory will be observed only at relatively short exposure times. Note that quantitative disagreement with the wave theory is also observed at very short exposure times (Figs. 1(a) and (b)); however, for a long exposure time, this difference is due to the approximate nature of the Born rule (1); for short exposure times, this difference is connected to the random scatter due to the small number of recorded events. If one performs a large number of similar tests with a short exposure time and averages the results of these tests, then, for \( \tau \ll 1 \), the obtained pattern will exactly correspond to the predictions of classical optics (7) and the Born rule for light (1). This conclusion is confirmed by Fig. 2 (left), which presents the results of the calculations for \( \tau = 0.1 \) averaged over 10 statistical realizations. A similar result corresponding to the long exposure (\( \tau = 5 \)) is shown in Fig. 2 (right). The lines in Fig. 2 (right) show expression (11) corresponding to \( \tau = 0 \) (it coincides with expression (7)) and \( \tau = 5 \). In the latter case, the distribution of dots on the screen is found to be significantly different from that predicted by classical optics (7) and the Born rule (1); however, the distribution is well described by the general expression (11).
Fig. 2. Comparison of the results of Monte Carlo simulations using expression (6) for $\tau = 0.1$ (left) and $\tau = 5$ (right) and averaged over 10 statistical realizations (markers), with expression (7) (predicted by classical optics (line $\tau = 0$)) and with dependence (11) (predicted by classical electrodynamics, taking into account expression (2) (line $\tau = 5$)). The data shown in Fig. 1 correspond to one particular test.

Fig. 3. Normalized distribution functions of the number of "photons" calculated using expression (11) for different exposure times $\tau$.

For comparison, Fig. 3 shows the normalized distribution function calculated by expression (11) for different exposure times $\tau$.

Note that the theoretical dependence (7) (line $\tau = 0$ in Figs. 1–3) describes the distribution of dots on the screen, which must be obtained if we assume that the probability of finding a photon at a given point is determined by the Born rule (1). In this case, the normalized distribution of dots on the screen should not depend on the exposure time (at least, as long as the fraction of excited atoms on the surface of the screen is small). In contrast, according to the semiclassical theory, when light is considered as a classical electromagnetic wave, the distribution of dots on the screen (11) substantially depends on the exposure time, and, at $\tau > 1.5$, the distinction from the pure photon theory (7) will already be clearly visible (Figs. 2 and 3). This effect can be observed experimentally.
3. Nonlinear double-slit experiment

3.1. Interaction of an intense laser field with an atom

If photon energy is less than atom’s ionization (excitation) potential \((\hbar \omega < U_i)\), it is assumed that ionization (excitation) of the atom can occur only after simultaneous absorption of \(K\) photons so that \(K \hbar \omega > U_i\), where \(\omega\) is the laser carrier frequency, \(U_i\) is the ionization (excitation) potential of the atom. The minimum number of photons required to be absorbed for atom ionization (excitation) is

\[
K = \langle \frac{U_i}{\hbar \omega} + 1 \rangle,
\]

where \(\langle \ldots \rangle\) denotes the integer part.

Such a process can be observed only in an intense laser field.

Interaction of an intense laser field with an atom is described by the Keldysh theory [28], which predicts the multiphoton and tunnel ionization of atoms. The Keldysh theory is a semiclassical one in which atoms are described by the Schrödinger equation, while light is considered as a continuous classical electromagnetic wave. Nevertheless, the continuous solution obtained for the Schrödinger equation is interpreted from the standpoint of “photonic” representations, which allow for the conclusion that, under certain conditions, “multiphoton” ionization of an atom occurs when the atom “simultaneously absorbs several photons”. Such an interpretation is based on the fact that the solution of the Schrödinger equation contains components with a phase factor \(n \hbar \omega\), where \(n = 1, 2, \ldots\), which is interpreted as a result of the simultaneous absorption of \(n\) “photons”, although the radiation is not quantized in the Keldysh theory.

The behavior of an atom in high laser intensity \((>10^{13} \text{ W/cm}^2)\) is usually analyzed on the basis of the “adiabaticity” Keldysh parameter, given as

\[
\gamma = \frac{\omega \sqrt{2m_e U_i}}{e E_p},
\]

where \(m_e\) is the electron rest mass, \(E_p\) the time-dependent amplitude of the linearly polarized laser field, and \(e\) the electron charge [28].

The Keldysh theory was the first one able to describe atom ionization by an alternating field in the low-intensity regime \(\gamma \gg 1\) as well as in the high-intensity regime \(\gamma \ll 1\). The former regime refers to multiphoton ionization (MPI), through which the electron is freed as the atom absorbs \(K = \langle \frac{U_i}{\hbar \omega} + 1 \rangle\) photons. The latter corresponds to the tunnel regime, for which the electron leaves the ion by passing through the Coulomb barrier. It is usually considered that, for \(K \gg 1\), the probability of simultaneous absorption of such a large number of photons is negligible and ionization (excitation) of an atom is only possible as a result of the tunnel effect.

Keldysh theory [28] is limited to hydrogen-like atoms in their fundamental electronic state and does not consider the Coulomb interaction between the leaving electron and the residual ion.

Later, Perelomov et al. [29,30] developed a more accurate model (the PPT model). They included the Coulomb interaction between the ion and the electron, when the latter leaves the atomic core, and considered any atomic bound states as initial. The resulting rate expressed in atomic units (a.u.), \(m_e = e = \hbar = a_B = 1\) [31], is

\[
\begin{align*}
\omega = 4\sqrt{2} & \left| C_{n^*,l^*} \right|^2 \left( \frac{2E_0}{E_p \sqrt{1 + \gamma^2}} \right)^{2n^* - 3/2 - |m|} \frac{f(l, m)}{|m|!} \exp \left( -2\nu \left[ \sinh^{-1} \gamma - \frac{\gamma \sqrt{1 + \gamma^2}}{1 + 2\gamma^2} \right] \right), \\
& \times U_i \frac{\gamma^2}{1 + \gamma^2} \sum_{k \geq 0} \exp \left( -\alpha(k + \nu_0 - \nu) \right) \Phi_m \left( \sqrt{\beta(k + \nu_0 - \nu)} \right)
\end{align*}
\]

(14)
where

\[ E_0 = (2U_i)^{3/2}, \quad \gamma = \omega \sqrt{2U_i/E_p}, \quad v = \tilde{U}_i/(\hbar \omega)_{\text{a.u.}}, \quad \beta = \frac{2\gamma}{\sqrt{1 + \gamma^2}}, \]

\[ \alpha = 2 \left[ \sinh^{-1} \gamma - \frac{\gamma}{\sqrt{1 + \gamma^2}} \right], \quad \nu_0 = (v + 1), \quad \tilde{U}_i \equiv U_i + U_p, \]

\[ \sinh^{-1} \gamma = \arcsinh \gamma = \ln \left( \gamma + \sqrt{1 + \gamma^2} \right), \quad \Phi_m(z) = \int_0^z (z^2 - y^2)^{|m|} \exp (y^2 - z^2) dy \]

\[ U_p \equiv \frac{e^2 E_p^2}{4m_e \omega^2} \]

is the electron ponderomotive energy; \( n^* = \frac{Z}{\sqrt{2U_i}} \) is the effective quantum number; \( Z \) is the residual ion charge; and \( l^* = n^* - 1 \) and \( n, l, m \) are the principal quantum number, the orbital momentum, and the magnetic quantum number, respectively. The factors \( |C_{n^*,l^*}| \) and \( f(l, m) \) are

\[ |C_{n^*,l^*}|^2 = \frac{2^{2n^*}}{n^* \Gamma(n^* + l^* + 1) \Gamma(n^* - l^*)} \]

\[ f(l, m) = \frac{(2l + 1)(l + |m|)!}{2^{2|m|}|m|!(l - |m|)!}. \] (15)

The PPT rate, Eq. (14), holds to describe photoionization of atoms. It can lead to some discrepancy when it is applied to molecular systems, because the coefficients \( |C_{n^*,l^*}| \), originally evaluated from atomic wavefunctions, cannot reproduce molecular peculiarities. To overcome such limitations, the molecular tunneling theory was extended by Tong et al. [32] by plugging molecular coefficients into the tunnel limit of the PPT formula and prolonging the latter to low-intensity MPI regimes analytically.

Applying the Keldysh theory to the double-slit experiment, we assume that, passing through the double slit and interfering, an intense laser beam (regardless of its intensity) behaves as a classical (linear) electromagnetic wave. It is assumed that the double-slit experiment is carried out in a vacuum, so the nonlinear effects of electromagnetic wave propagation between the double slit and photographic plate are absent. Obviously, in this case, the difference between linear and nonlinear double-slit experiments will only occur as a result of the interaction of light waves with atoms of photographic plates, which is described by the relations (2) and (14), respectively. Thus, it is assumed that, as for the linear double-slit experiment, in the nonlinear double-slit experiment, the intensity of light on the surface of the photographic plate is described by the classical expression (7).

3.2. Multiphoton double-slit experiment

Let us consider the case when

\[ \gamma_0 \gg 1, \] \hspace{1cm} (16)

where \( \gamma_0 \) is calculated by using the light intensity \( I_0 \) in the middle of the photographic plate. Obviously, in this case, the condition \( \gamma \gg 1 \) corresponding to the multiphoton regime is satisfied on the whole surface of the photographic plate.
In the multiphoton regime, the excitation rate is obtained by taking the limit \( \gamma \to +\infty \) in Eq. (14), which reduces to

\[
  w = \sigma^{(K)} I^K,
\]

where \( I \) is the light intensity and \( \sigma^{(K)} \) is the photoionization cross section

\[
  \sigma^{(K)} = 4\sqrt{2} \omega \left( \frac{U_i}{[\hbar \omega]_{\text{a.u.}}} \right)^{2K+3/2} \exp(2K - \frac{U_i}{[\hbar \omega]_{\text{a.u.}}}) \frac{E_0^{2K}}{E_0^{2K} - \frac{2U_i}{[\hbar \omega]_{\text{a.u.}}}}. \tag{18}
\]

For the multiphoton regime, taking into account expressions (2), (3), and (17), one obtains

\[
  P_+(t) = 1 - \exp\left( -\sigma^{(K)} I^K t \right). \tag{19}
\]

Let us introduce the nondimensional time

\[
  \tau = \sigma^{(K)} I_0^K t. \tag{20}
\]

In this case, the probability of excitation of the atom for time \( t \) at the points on the screen having the intensity of light \( I \) is

\[
  P_+(t) = 1 - \exp\left( -\frac{I}{I_0} K \tau \right). \tag{21}
\]

The method of Monte Carlo simulation of the nonlinear double-slit experiment is the same as for the “linear” one, described above.

Note that, in this case, the appearance of the spot on the photographic plate cannot be interpreted as a photon hitting the surface, since, according to the photon interpretation of the MPI regime, an atom needs to absorb \( K \) photons simultaneously in order to be excited. For this reason, we should talk not about photons hitting the photographic plate but about the occurrence of events—the appearance of spots on the photographic plates, which are a result of excitation of atoms (or groups of atoms) by a light wave.

The results of the calculations of the process of the “accumulation of events” on the screen in the nonlinear double-slit experiment for different moments of time \( \tau \) at \( \frac{\pi b}{2H} = 0.03 \) and \( d/b = 5 \) are shown in Fig. 4. It can be seen that the pattern of the accumulation of events in the multiphoton double-slit experiment is qualitatively and quantitatively different from the picture that was observed in the linear double-slit experiment (Fig. 1): the fringes are sharper and more uniform in the multiphoton double-slit experiment, and separated from each other by wide gaps in which events do not occur.

For an arbitrary exposure time, the probability of an event at a given point of the photographic plate is determined by expression (10), which, taking into account expression (21), can be written as

\[
  \frac{p}{p(0)} = \frac{1 - \exp\left( -\frac{I}{I_0} K \tau \right)}{1 - \exp(-\tau)}.
\]

Figure 5 shows the dependences calculated by using expression (22) for different exposure times \( \tau \) and two values of parameter \( K \).

### 3.3. “Tunnel” double-slit experiment

Another limiting case of the nonlinear double-slit experiment corresponds to

\[
  \gamma_0 \ll 1. \tag{23}
\]

The intensity of light transmitted through a double slit on the photographic plate varies from \( I_0 \) (in the middle of the photographic plate) to zero, so the Keldysh parameter in this case will change along
Fig. 4. Interference pattern buildup in the “multiphoton” double-slit experiment for different exposure times $\tau$, obtained using the semiclassical Monte Carlo simulation of the interaction of light with the detecting screen. $K = 5$; a) $\tau = 0.1$ (101 events); b) $\tau = 0.5$ (480 events); c) $\tau = 4$ (2059 events); d) $\tau = 100$ (5500 events); e) $\tau = 1000$ (7490 events); f) $\tau = 10^5$ (10 700 events).

Fig. 5. Normalized distribution functions of the number of events in the “multiphoton” double-slit experiment calculated using expression (22) for different exposure times $\tau$. a) $K = 5$; b) $K = 10$. 
the photographic plate in the range \([\gamma_0, \infty)\). Thus, in the regime (23), there will be regions on the photographic plate corresponding to both the tunnel ionization \((\gamma \ll 1)\) and multiphoton ionization \((\gamma \gg 1)\), including the intermediate modes \((\gamma \sim 1)\). We conclude that a pure “tunnel regime” on the entire surface of the photographic plates in the nonlinear double-slit experiment is impossible, and, if the condition (23) is satisfied, to calculate the nonlinear double-slit experiment it is necessary to use the general expression (14), which includes all possible modes of ionization (excitation) of atoms and molecules. The double-slit experiment corresponding to the condition (23) was conditionally named "tunnel", in order to emphasize that the tunnel excitation mode is realized in the middle of the photographic plates.

Let us introduce the relative intensity of light on the photographic plate

\[ J = I/I_0, \]  

which for the double-slit experiment is described by the classical expression (7) independently of the intensity of the laser field.

Obviously,

\[ \gamma = \gamma_0 J^{-1/2}. \]  

Then the expression (14) can be rewritten in the form

\[ w = w_0\gamma_0^2 \Omega(J, \gamma_0), \]  

where

\[ \Omega(J, \gamma_0) = \left( J + \gamma_0^2 \right)^{-n^*+1/4-|m|} \exp\left( -2v \left[ \sinh^{-1} \gamma - \frac{\gamma_0\sqrt{J + \gamma_0^2}}{J + 2\gamma_0^2} \right] \right), \]  

\[ w_0 = 4\sqrt{2} \left| C_{n^*,l^*} \right|^2 U_i \left( \frac{2E_0}{E_{p0}} \right)^{2n^*+3/2-|m|} \frac{f(l, m)}{|m|!}, \]  

\[ \beta = \frac{2\gamma_0}{\sqrt{J + \gamma_0^2}}, \quad \sinh^{-1} \gamma = \ln \left( \gamma_0 + \sqrt{J + \gamma_0^2} \right) - \frac{1}{2} \ln J, \]  

\[ \alpha = 2 \left[ \sinh^{-1} \gamma - \frac{\gamma_0}{\sqrt{J + \gamma_0^2}} \right], \quad v = \left( 1 + J\gamma_0^{-2}/2 \right) K_0, \quad K_0 = U_i/[\hbar\omega]_{\text{a.u.}}; \]  

\( E_{p0} \) is the amplitude of the linearly polarized laser field in the middle of the photographic plate after diffraction on the double slit: \( I_0 = \frac{c}{8\pi} E_{p0}^2 \).

Taking into account expressions (3) and (26), for the probability of excitation of the atom for time \( t \) at the points on the photographic plate having the intensity of light \( I \), one can write

\[ P_+(t) = 1 - \exp\left( -w_0\gamma_0^2 \Omega(J, \gamma_0) t \right). \]  

Let us introduce the nondimensional time

\[ \tau = w_0\gamma_0^2 \Omega(1, \gamma_0) t. \]
Fig. 6. Interference pattern buildup in the “tunnel” double-slit experiment for different exposure times $\tau$, obtained using the semiclassical Monte Carlo simulation of the interaction of light with the detecting screen. $K_0 = 100, \gamma_0 = 0.1$: a) $\tau = 0.015$ (35 events); b) $\tau = 0.023$ (103 events); c) $\tau = 0.05$ (415 events); d) $\tau = 0.1$ (870 events); e) $\tau = 0.2$ (1341 events); f) $\tau = 0.5$ (1960 events); g) $\tau = 1.0$ (2309 events); h) $\tau = 2.0$ (2581 events).

In this case, one obtains

$$P_+(t) = 1 - \exp(-\Omega (J, \gamma_0)/\Omega (1, \gamma_0)\tau).$$

(32)

The method of Monte Carlo simulation of the nonlinear double-slit experiment in this case is the same as for the “linear” one, described above.

Note that, in this case, the appearance of a spot on the surface of the photographic plate cannot be interpreted as a photon hitting the surface, but we can talk about an event—the excitation of an atom (or groups of atoms) by the intense light wave.
Fig. 7. Comparison of the relative probability of excitation of an atom per unit time in the double-slit experiment for different regimes. Dependences 1–3 correspond to the tunnel regimes for $K_0 = 100$; 1) $\gamma_0 = 0.05$; 2) $\gamma_0 = 0.1$; 3) $\gamma_0 = 0.4$. Dependence 4 corresponds to the multiphoton regime ($\gamma_0 = \infty$) for $K_0 = 100$ and has been calculated by using expression $\Omega(J,\gamma_0)/\Omega(1,\gamma_0) = J^{K_0+1}$. Dependence 5 corresponds to the one-photon mode and has been calculated by using Fermi’s golden rule (2): $\Omega(J,\gamma_0)/\Omega(1,\gamma_0) = J$. The right picture shows in detail the dependences in the vicinity of the middle of the photographic plate.

The results of calculations of the process of the “accumulation of events” on the screen in the considered case of the nonlinear double-slit experiment for different moments of time $\tau$ at $\frac{\gamma_0 b}{\lambda H} = 0.03$ and $d/b = 5$ are shown in Fig. 6. In this calculation, the values $n^* = 1$, $m = 0$, $\gamma_0 = 0.1$, $K_0 = 100$ were used.

It is interesting to compare the probabilities of excitation of the atoms per unit time in the double-slit experiment for different regimes: multiphoton, tunnel, and linear (one-photon) mode. Such a comparison is shown in Fig. 7. It can be seen that, for large $K_0 \gg 1$ and reasonable exposure times $\tau$ in the multiphoton mode ($\gamma_0 = \infty$), only one fringe will be observed in the middle of the photographic plate. However, the essential increase in the intensity of the laser beam and the transition to the tunnel mode ($\gamma_0 \ll 1$) will result in an increase of the number of fringes for the same period of observation, and, at a very high intensity of the laser beam ($\gamma_0 \to 0$), the interference pattern may be similar to that observed in the linear (one-photon) double-slit experiment.

When analyzing Fig. 7, it is necessary to keep in mind that the scales of nondimensional time are different for different modes, so that, even in the middle of the photographic plate, the real time of accumulation of the same number of events will be essentially different for different modes of the double-slit experiment.

4. Concluding remarks

We see that the “linear” double-slit experiments [1, 2] can be easily calculated within the framework of semiclassical theory, based on the Schrödinger equation and classical electrodynamics without the quantization of radiation. The application of Fermi’s golden rule (2) for explanation of the double-slit experiment does not lead to paradoxes in interpreting the experimental data, while the direct application of the Born rule (1) as the primary principle leads to a paradox related to the wave–particle duality of light.

Regarding the nonlinear double-slit experiment, it should be noted that there are very few studies devoted to this problem. For example, in Ref. [33], Young’s double-slit experiment was realized using intense and short laser pulses. The authors employed a 790 nm Ti:sapphire chirped pulse amplified system with 120 fs pulses of 50 mJ at 10 Hz repetition rate. The peak power reached by these pulses
was of terawatt order. After passing the double slit, the pulse had a high field energy (14 mJ), permitting the observation of nonlinear processes of the interaction of light with a detecting screen. This experiment was carried out in air in which the nonlinear Kerr effect occurred. This resulted in reduction of the beam diffraction due to the Kerr self-focusing and the nonlinear effects described in the present paper were not observed. To observe these effects, it is necessary to use a special detecting screen, the atoms of which have an excitation (ionization) potential $U_i \gg \hbar \omega$.

In this paper, the Keldysh semiclassical theory [28–32] is used for the description of the nonlinear double-slit experiment. Of course, it could also be explained by the nonlinear interaction of photons in a very intense laser beam, as in, e.g., Ref. [34]; however, we see that this can be done much more easily and more naturally in the framework of the semiclassical theory.

Of course, the interaction of an intense laser beam with the material of the double-slit device may result in additional nonlinear effects, which will affect the interference pattern. But the aim of this work is to show how the interference pattern buildup changes under the action of an intense laser beam due to the nonlinear effects of the interaction of light with the detecting screen. In calculating the real double-slit experiment with an intense laser beam, it is necessary to take into account the difference between the intensity of the diffracted beam due to the nonlinear interaction with the double-slit device from that described by the expression (7). However, even in this case, the calculation method will be the same as described in this paper. It should be noted that at present there are no other methods of calculating the nonlinear double-slit experiment.

The approach considered in this paper can be attributed to the class of event-by-event simulation. Another approach to event-based simulations is considered in Ref. [35] and references therein. It is based on a specific model of the detector, which possesses a memory and has a threshold behavior while light is considered as a flux of particles—photons that possess some “internal clocks”. An event-based model for such a detector cannot be derived from quantum theory. In contrast, the semiclassical theory considered in this paper is a direct consequence of the Schrödinger equation and, as we see, it is sufficient for the event-by-event description of the double-slit experiment. In addition, the semiclassical theory can be easily generalized to the case of an intense laser field, in which the nonlinear effects of the interaction of light with a photographic plate occur. In the case of the event-based model [35], it is not clear how it can be generalized to a very intense laser field.

Note also that the analysis carried out shows that, even in the linear double-slit experiment, the Born rule for light (1) is violated for the long exposure time and this effect can be observed experimentally.

Movies demonstrating the results described in this paper are provided as supplemental material [36].

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[36] See Supplementary Data for movies demonstrating the described results.