Hopf-type Cyclic Cohomology
via Karoubi Operator

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Abstract

In this paper we propose still another approach to the Hopf-type cohomologies of a Hopf algebra $\mathcal{H}$, based on the notion of the universal differential calculus on $\mathcal{H}$. Few remarks, concerning the possible generalizations and applications of this approach are made.

1 Introduction

In the original papers of A. Connes and H. Moscovici (see [1, 2, 3]) the explicit structure of cyclic module defining the so-called Hopf-type cyclic cohomologies of a Hopf algebra was given. Later, in his paper M. Crainic showed that this cyclic module could be obtained as the space of coinvariants of the Hopf algebra’s action on some other cyclic module. Some further generalizations and developments, we know of, were made in the papers [16], [17], [18].

The purpose of this paper is to describe the Hopf-type cohomologies of a Hopf algebra in the terms of a subcomodule of the so-called algebra of non-commutative differential forms, associated with the Hopf algebra. It turns

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out, that for any "modular pair in involution", \((\delta, \sigma)\) one can associate a subcomplex of this differential algebra, stable under Karoubi operator \(\kappa\) (see papers [5, 6] and [20]) or its twisted version \(\kappa_\xi\), see (61) (also [15]).

In addition to giving a new point of view on this homology theory, this approach seems to have some virtues of its own. For instance, one can try to define some similar sort of cyclic cohomologies, when modular pair is substituted for a more general object. Besides this, it can be used to establish bridges between this cyclic cohomology theory and the Hopf-Galois theory, developed in the papers of T. Brzezinski, M. Đurđević, P. Hajac, S. Majid \([8, 9, 10, 11, 12, 13]\) and others. Only few remarks, concerning this subject are made here, since we postpone deeper discussions to a paper to follow.

Let’s, first of all recall the construction of the Hopf-type cohomologies, due to A. Connes and H. Moscovici. Here and below \(\mathcal{H}\) will denote a Hopf algebra over a field of characteristic 0 (\(\mathbb{C}\) is our main example). Let \(m, \Delta, 1, \epsilon\) and \(S\) be the multiplication, comultiplication (or diagonal), unit, counit and antipode of \(\mathcal{H}\) respectively. Below we shall usually miss \(m\) in our formulae, and use the standard (Sweedler, [19]) notation to write down the diagonal:

\[
\Delta(h) = \sum h_{(1)} \otimes h_{(2)}. \tag{1}
\]

One says, that \((\sigma, \delta)\), where \(\sigma\) is a group-like element in \(\mathcal{H}\) and \(\delta : \mathcal{H} \to \mathbb{C}\) an algebraic character, and \(\delta(\sigma) = 1\), is a modular pair in involution, if

\[
S^2_\delta(h) = \sigma h \sigma^{-1}, \quad h \in \mathcal{H}, \tag{1}
\]

where

\[
S_\delta(h) = \sum \delta(h_{(1)})S(h_{(2)}). \tag{2}
\]

This is equivalent to \((\sigma S_\delta)^2 = 1\).

Given a modular pair in involution one can define the (co)cyclic module \(\mathcal{H}_{(\delta, \sigma)}\). Recall, that cyclic category is self-dual, hence it is not necessary to distinguish very carefully between cyclic and cocyclic objects. So, one puts: \((\mathcal{H}_{(\delta, \sigma)})_n = \mathcal{H}^\otimes n\), and the cyclic structure maps are defined as follows:

\[
\delta_i : \mathcal{H}_{n}^\sharp \to \mathcal{H}_{n+1}^\sharp, \quad i = 0, \ldots, n + 1 \tag{3}
\]

\[
\sigma_i : \mathcal{H}_{n}^\sharp \to \mathcal{H}_{n-1}^\sharp, \quad i = 1, \ldots, n, \tag{4}
\]

\[
\tau_n : \mathcal{H}_{n}^\sharp \to \mathcal{H}_{n}^\sharp. \tag{5}
\]
are given by
\[ \delta_i(h_1, \ldots, h_n) = \begin{cases} (1, h_1, \ldots, h_n), & i = 0, \\ (h_1, \ldots, \Delta(h_i), \ldots, h_n), & 1 \leq i \leq n, \\ (h_1, \ldots, h_n, \sigma), & i = n + 1; \end{cases} \]  
(6)

\[ \sigma_i(h_1, \ldots, h_n) = \epsilon(h_i)(h_1, \ldots, h_{i-1}, \hat{h}_i, h_{i+1}, \ldots, h_n), \]  
\[ 1 \leq i \leq n, \]  
(7)

\[ \tau_n(h_1, h_2, \ldots, h_n) = S_\delta(h_1) \cdot (h_2, \ldots, h_n, \sigma). \]  
(8)

Here in the last formula we assume, that \( \mathcal{H} \) acts on its own tensor power as follows:
\[ h \cdot (h_1, \ldots, h_n) = (h_{(1)} h_1, \ldots, h_{(n)} h_n). \]

For any cocyclic module one can define its cyclic, negative cyclic and periodic cyclic cohomology. To this end one has to consider the cyclic, negative and periodic complexes respectively (see, for example the book of Loday \[14\] and paper [3]). For instance, periodic cohomology are defined by the following super-complex:
\[ CP_i = \prod_{n \equiv i \pmod{2}} \mathcal{H}_n^2, \quad i = 0, 1; \]  
(9)

equipped with differentials \( b : \mathcal{H}_n^2 \to \mathcal{H}_{n+1}^2 \) and \( B : \mathcal{H}_n^2 \to \mathcal{H}_{n-1}^2 \), defined as follows
\[ b = \sum_i (-1)^i \delta_i, \]  
(10)

\[ B = N \circ (1 - \tau_{n+1}) \circ \tilde{\sigma}_0, \]  
(11)

where
\[ \tilde{\sigma}_0 = \sigma_n \circ \tau_n. \]  
(12)

Recall, that we deal with cocyclic module here, so the usual formulae for differentials in mixed complexes, associated with such an object, are inverted.
2 Special case: \((\delta, \sigma) = (\epsilon, 1)\)

Let \(\mathcal{H}\) be the given (unital) Hopf algebra. We shall denote by \(\Omega(\mathcal{H})\) the universal unital differential graded algebra, generated by \(\mathcal{H}\). Recall, that

\[
\Omega(\mathcal{H}) = \bigoplus_{n \geq 0} \Omega_n(\mathcal{H}); \quad \Omega_0(\mathcal{H}) = \mathcal{H}; \quad \Omega_1(\mathcal{H}) = \ker(m: \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}); \quad \Omega_n(\mathcal{H}) = \underbrace{\Omega_1(\mathcal{H}) \otimes_\mathcal{H} \ldots \otimes_\mathcal{H} \Omega_1(\mathcal{H})}_n.
\]

The differential \(d: \mathcal{H} \to \Omega_1(\mathcal{H})\) is given by

\[
d(x) = 1 \otimes x - x \otimes 1,
\]

and one can prove that any element \(\theta\) in \(\Omega_n(\mathcal{H})\) can be written down in the form

\[
\theta = \sum_i a^i_0 da^i_1 da^i_2 \ldots da^i_n, \quad a^i_j \in \mathcal{H}
\]

Now it’s clear, that

\[
d\theta = \sum_i da^i_0 da^i_1 da^i_2 \ldots da^i_n.
\]

So far, the coalgebra structure hasn’t yet come to the scene. In effect, one can define the universal differential algebra, associated to any unital algebra \(\mathcal{A}\) in precisely same way. But now, since \(\mathcal{H}\) is in fact a Hopf algebra, one can define left- and right-coactions of \(\mathcal{H}\) on \(\Omega(\mathcal{H})\). Namely, put

\[
\Delta_R(\theta) = \sum_i a^i_0 da^i_1 da^i_2 \ldots da^i_n, \quad a^i_j \in \mathcal{H}
\]

and

\[
\Delta_L(\theta) = \sum_i a^i_0 da^i_1 da^i_2 \ldots da^i_n, \quad a^i_j \in \mathcal{H}
\]

The fact, that these formulae really determine a well-defined maps follows from the universal properties of \(\Omega(\mathcal{H})\).
Moreover, (17) and (18) define the right- and left-Hopf-comodule algebra structures on $\Omega(\mathcal{H})$. That is, the map $\Delta_R : \Omega(\mathcal{H}) \to \Omega(\mathcal{H}) \otimes \mathcal{H}$ is an algebra morphism, and similarly for $\Delta_L$. In particular, $\Omega(\mathcal{H})$ is a left and right Hopf module over the Hopf algebra $\mathcal{H} = \Omega_0(\mathcal{H})$. In a general case conditions that formulas (17) and (18) define such structures impose an additional restrictions on the structure of a given differential calculus $\Omega'(\mathcal{H})$. Differential calculi verifying these restrictions are called bicovariant. This matter is accurately explained in [7], where the general definition of a bicovariant differential calculus on a Hopf algebra is given.

Observe, see (16), that both maps (17) and (18) preserve differential. Hence, in particular, the subspaces of left and right coinvariants are differential graded algebras in $\Omega(\mathcal{H})$. Let’s describe explicitly the structure of these subalgebras. For instance, take $\Omega^R(\mathcal{H}) = \Omega(\mathcal{H})^{co\mathcal{H}}$. It is shown in [7] and [9, 10] that the map $\pi^R : \mathcal{H} \to \Omega^R_1(\mathcal{H}), \ h \mapsto da(1) \cdot S(a(2))$ (19) identifies the space $\Omega^R_1(\mathcal{H})$ with ker $\epsilon$. Moreover, one can show, that $\Omega(\mathcal{H}) \cong \Omega^R(\mathcal{H}) \otimes \mathcal{H},$ and $\Omega^R_n(\mathcal{H}) \cong \Omega^R_1(\mathcal{H})^\otimes n \cong (\ker \epsilon)^\otimes n$. (20)

In terms of these isomorphisms, one can write down the differential and left and right actions of $\mathcal{H}$ on the bimodule $\Omega(\mathcal{H})$ as follows:

$$dh = dh_{(1)} \cdot S(h_{(2)})h_{(3)} = \pi^R(h_{(1)}) \otimes h_{(2)},$$

$$d\pi^R(h) = dh_{(1)}dS(h_{(2)}) = -dh_{(1)}S(h_{(2)})dh_{(3)}S(h_{(4)}) = \pi^R(h_{(1)}) \otimes \pi^R(h_{(2)}),$$

$$a \cdot \pi^R(h) = a_{(1)} \pi^R(h)S(a_{(2)})a_{(3)} = \pi^R(a_{(1)}h - \epsilon(h)a_{(1)}) \otimes a_{(2)},$$

and the right action of $\mathcal{H}$ on $\Omega_1(\mathcal{H}) = \Omega^R(\mathcal{H}) \otimes \mathcal{H}$ is trivial. In particular, if $h$ is in ker $\epsilon$, then formula (23) gives the following description of the left action of $\mathcal{H}$ on bimodule $\Omega_1(\mathcal{H}) \cong \ker \epsilon \otimes \mathcal{H}$:

$$a \cdot (h_1 \otimes h_0) = a_{(1)}h_1 \otimes a_{(2)}h_0, \quad a, h_0 \in \mathcal{H}, \ h_1 \in \ker \epsilon.$$ (24)

This formula extends in a natural way to the $n$–th graded component of $\Omega(\mathcal{H})$, $\Omega_n(\mathcal{H}) \cong (\ker \epsilon)^{\otimes n} \otimes \mathcal{H}$, namely

$$a \cdot (h_1 \otimes h_2 \otimes \cdots \otimes h_n \otimes h) = a_{(1)}h_1 \otimes a_{(2)}h_2 \otimes \cdots \otimes a_{(n)}h_n \otimes a_{(n+1)}h,$$ (25)

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where \( h_1, \ldots, h_n \in \ker \epsilon, \ h, a \in \mathcal{H}. \)

Let now \( b, \) and \( \kappa \) be the usual Hochschild differential and Karoubi operator on \( \Omega(\mathcal{H}) \), defined as follows (see [3]):

\[
b(\omega da) = (-1)^{|\omega|}(\omega a - a \omega),
\]

(26)

where \( a \in \mathcal{H}, \ \omega \in \Omega(\mathcal{H}), \) and

\[
\kappa = 1 - bd - db.
\]

(27)

Explicitly one can show, that

\[
\kappa(\omega da) = (-1)^{|\omega|} da \omega.
\]

This is the usual way these operators are introduced. For our purposes it would be useful to consider a little bit different operators, \( b', \ \kappa' : \)

\[
b'(da\omega) = a \omega - \omega a,
\]

(28)

\[
\kappa'(da\omega) = b' d + db' - 1.
\]

(29)

Explicitly

\[
\kappa'(da\omega) = (-1)^{|\omega|} da \omega.
\]

(30)

Let \( B' = \sum \kappa^i \circ d. \) Then \( B' \) corresponds to the operator \( B \) from [3]. and these operators verify all the usual properties of \( b \) and \( \kappa \) and \( B, \) see §3 of [3]. This can be proven by a slight modification of the reasoning used in the quoted paper in the usual setting. Hence, we conclude, that \( b' \) and \( B' \) induce the structure of cyclic module on \( \Omega(\mathcal{H}). \) Moreover, formulae (28) and (30) show that

\[
\kappa' = \kappa^{-1},
\]

(31)

and

\[
b' = \pm b \circ \kappa'
\]

(32)

The following theorem is the main result of this section.
Theorem 1. Let $\mathcal{H}$ be a Hopf algebra and the modular pair $(1, \epsilon)$ is such, that the conditions of [4] are satisfied (i.e. $S^2 = 1$) then $\Omega^R(\mathcal{H})$ is a differential graded subalgebra in $\Omega(\mathcal{H})$, stable under $b$ and Karoubi operator $\kappa$ (and, $b'$ and $\kappa'$) and hence is a mixed subcomplex in $(\Omega(\mathcal{H}), b, B)$. The same is true about $\Omega^L(\mathcal{H})$. Moreover, periodic cohomologies of $\Omega^R(\mathcal{H})$ with the mixed complex structure, induced from $(\Omega(\mathcal{H}), b', B')$, are naturally isomorphic to the periodic Hopf-type cohomologies $HP^*_{e,1}(\mathcal{H})$ of the Hopf algebra $\mathcal{H}$.

Proof. The fact, that $\Omega^R(\mathcal{H})$ (and $\Omega^L(\mathcal{H})$ as well) is a differential graded subalgebra in $\Omega(\mathcal{H})$ follows directly from the discussion above. Now we shall prove the second statement of this theorem. We shall confine our attention to $\Omega^R(\mathcal{H})$. (In the case of $\Omega^L(\mathcal{H})$ reasoning is absolutely similar.)

First of all let’s note, that it’s enough to prove the stability of $\Omega^R(\mathcal{H})$ under the action of $b'$ and $\kappa'$ (just look at formulae (21) and (22)). So, let’s start with proving, that $\Omega^R(\mathcal{H})$ is stable under $b'$. To this end we shall directly compute the image of an element $\omega \in \Omega^R(\mathcal{H})$ under $b'$. First, let $\omega$ belong to $\Omega^R_1(\mathcal{H})$. We compute:

$$b'(\omega) = b'(da(1)S(a(2))) = a(1)S(a(2)) - S(a(2))a(1) = \epsilon(a) \cdot 1 - S(a(2))S^2(a(1)) = \epsilon(a) \cdot 1 - S(S(a(1))a(2))$$

$$= \epsilon(a) \cdot 1 - \epsilon(a) \cdot 1 = 0,$$

which is, of course, a right-coinvariant element. Here we’ve used formula (28) and the possibility to represent any element in $\Omega^R_1(\mathcal{H})$ as the image of some $a \in \mathcal{H}$ under the map $\pi^R$ from (13).

Now, if the element $\omega$ belongs to $\Omega^R_n(\mathcal{H})$ we use the identification (24) and formula (25) to compute

$$b'(\omega) = b'(\pi^R(a_1) \otimes \pi^R(a_2) \otimes \cdots \otimes \pi^R(a_n))$$

$$= b'(d(a_1,1)S(a_1,2)\omega') = a_1(1)S(a_1,2)\omega' - S(a_1,2)\omega'a_1(1)$$

$$= -\left(\pi^R(S(a_1,n+1)a_2) \otimes \cdots \otimes \pi^R(S(a_1,3)a_n)\right)S(a_1,2)\omega'a_1(1)$$

$$= -\left(\pi^R(S(a_1,n+1)a_2) \otimes \cdots \otimes \pi^R(S(a_1,3)a_n)\right)(S(S(a_1,2)a_n))$$

$$= -\left(\pi^R(S(a_1,n+1)a_2) \otimes \cdots \otimes \pi^R(S(a_1,3)a_n)\right)S(S(a_1,2)a_n))$$

$$= -\pi^R(S(a_1,n-1)a_2) \otimes \pi^R(S(a_1,n)a_3) \otimes \cdots \otimes \pi^R(S(a_1,1)a_n).$$

(34)
Here \( a_i \in \ker \epsilon, \ i = 1, \ldots, n \), and we denote for brevity \( \omega' = \pi^R(a_2) \otimes \pi^R(a_3) \otimes \cdots \otimes \pi^R(a_n) \). Clearly, \( b'(\omega) \) lies in \( \Omega^R_{n-1}(\mathcal{H}) \).

Of course, since the inverse of Karoubi operator \( \kappa' \) is written down in terms of \( d \) and \( b' \), (see (29)) one can conclude, that \( \Omega^R(\mathcal{H}) \) is stable under its action. So, the second statement of our theorem is proved.

However below we shall need the explicit formula for this operator written down in terms of identification (20). We use (30) and find (we stick to the notation explained after (29)):

\[
\kappa'(da_{1,1})S(a_{1,2})\omega' da_{1,1} = (-1)^{\lceil \omega' \rceil}S(a_{1,2})\omega' da_{1,1} \\
= (-1)^{\lceil \omega' \rceil}\left(\pi^R(S(a_{1,n+1})a_2) \otimes \cdots \otimes \pi^R(S(a_{1,3})a_n)\right) (S(a_{1,2})da_{1,1}) \\
= (-1)^{\lceil \omega' \rceil}\left(\pi^R(S(a_{1,n+1})a_2) \otimes \cdots \otimes \pi^R(S(a_{1,3})a_n)\right) (S(a_{1,2})dS(a_{1,1})) \\
= (-1)^{\lceil \omega' \rceil + 1}\left(\pi^R(S(a_{1,n})a_2) \otimes \cdots \otimes \pi^R(S(a_{1,3})a_n)\right) \pi^R(S(a_{1,1})) \\
= (-1)^n \pi^R(S(a_{1,n})a_2) \otimes \cdots \otimes \pi^R(S(a_{1,2})a_n) \otimes \pi^R(S(a_{1,1}) - \epsilon(a_{1,1})).
\]

In other words, this can be written down as

\[
\kappa' : \ker \epsilon^{\otimes n} \to \ker \epsilon^{\otimes n}, \\
(h_1, h_2, \ldots, h_n) \mapsto (-1)^n \text{proj}'(S(a_1) \cdot (h_2, \ldots, h_n, 1)).
\]

(36)

Here \( \text{proj}' \) denotes the standard projection \( \text{proj}' : \mathcal{H}^{\otimes n} \to \ker \epsilon^{\otimes n} \), which sends each component \( h_i \) to \( h_i - \epsilon(h_i) \).

Now, we believe, the similarity between these formulae and the structure of cyclic module, introduced by A. Connes and H. Moscovici is conspicuous. For instance, the cyclic operator \( \tau \) of this module is given by

\[ \tau(h_1, h_2, \ldots, h_n) = S(h_1)(h_2, \ldots, h_n, 1), \]

that is it coincides with \( \kappa' \), up to the sign nd projection on the kernel of counit.

In the view of this observation, let’s finally show, that the cohomology of the induced sub mixed complex \((\Omega^R(\mathcal{H}), b', B')\) coincide with the Hopf-type cohomologies of Connes and Moscovici.

To this end we first consider the cyclic object \( \mathcal{H}_{(\epsilon, 1)}^\sharp \), defined in [2] and [4], see section 1 above. Let \( \tilde{b}, \tilde{B} \), be the differentials (10), (12) and (12) (we
use tilde here to distinguish these maps from the Cuntz-Quillen’s operators on $\Omega(\mathcal{H})$, introduced above).

Recall, that (in this special case)

$$\delta = \sum (-1)^i \delta_i,$$  \hfill (37)

where

$$\delta_i(h_1, h_2, \ldots, h_n) = \begin{cases} (1, h_1, h_2, \ldots, h_n), & i = 0, \\ (h_1, \ldots, \Delta(h_i), \ldots, h_n), & 1 \leq i \leq n, \\ (h_1, h_2, \ldots, h_n, 1), & i = n + 1, \end{cases} \hfill (38)$$

$$\tilde{B} = N \circ \tilde{\sigma}_0 \circ (1 - \tau_{n+1}) \hfill (39)$$

where

$$\tilde{\sigma}_0(h_1, h_2, \ldots, h_n) = S(h_1) \cdot (h_2, \ldots, h_n),$$

$$N = \sum_{i=0}^{n} (-1)^i \tau_i.$$  

Now, consider a slightly different mixed complex $(\tilde{\mathcal{H}}^\sharp_{(\epsilon,1)}, \tilde{\mathcal{B}}', \tilde{\mathcal{B}}')$, where $(\tilde{\mathcal{H}}^\sharp_{(\epsilon,1)})_n = \ker \epsilon^\otimes n$ and $\tilde{b}' = \text{proj}' \circ \tilde{b}, \tilde{B}' = \text{proj}' \circ \tilde{B}$.

**Lemma 2.** The natural projection from $\mathcal{H}^\sharp_{(\epsilon,1)}$ to $\tilde{\mathcal{H}}^\sharp_{(\epsilon,1)}$ induces isomorphism on cyclic cohomology.

**Proof.** This is a direct consequence of the fact that this projection yields an isomorphism of the Hochschild homologies of these two complexes (i.e. their homologies with respect to the differentials $\tilde{b}$ and $\tilde{b}'$), which is a standard fact of homology algebra (in fact, the latter complex is just the normalization of the former one with respect to the degeneracy operators $\sigma_i$).

Now, as we’ve observed above, the cyclic structures on $\Omega^R(\mathcal{H})$, induced by Karoubi operator and that on $\mathcal{H}^\sharp_{(\epsilon,1)}$, induced from $\mathcal{H}^\sharp_{(\epsilon,1)}$ coincide. In effect, we already know, that

$$\Omega^R_n(\mathcal{H}) \cong \ker \epsilon^\otimes n = (\mathcal{H}^\sharp_{(\epsilon,1)})_n.$$

The only problem is, that under this isomorphism, differential $\tilde{b}'$ on $\mathcal{H}^\sharp_{(\epsilon,1)}$ corresponds to differential $d$ on $\Omega^R(\mathcal{H})$ (not to $b'$, or $B'$). Other differentials
also play different roles in these cyclic modules. Indeed, it is easy to see, that $\tilde{B}'$ of $\tilde{\mathcal{H}}_{\kappa,1}^\sharp$ corresponds to the operator $\sum (\kappa')^j b'$ in $\Omega^R(\mathcal{H})$, while $B' = (\sum \kappa')^j d$ in $\Omega^R(\mathcal{H})$.

To cure this problem, recall, (§3) that the super-complex $(\Omega(\mathcal{H}), b + B)$ is quasi-isomorphic to the subcomplex $P\Omega(\mathcal{H})$, on which

$$\kappa = 1 - \frac{1}{n(n + 1)} bB,$$

and hence

$$B = (n + 1)d$$

on $P\Omega_n(\mathcal{H})$. Here $P$ is the corresponding projection. Clearly, the same is true about $\kappa'$, $b'$, see formulae (31), (32).

Since the quasi-isomorphism $P$ and chain homotopy $Gd$ (this pair is called special deformation retraction in [6], §3) is expressed in terms of $\kappa'$ and $d$, we conclude, that $\Omega^R(\mathcal{H})$ is quasi-isomorphic to

$$P\Omega^R(\mathcal{H}) \overset{\text{def}}{=} P\Omega(\mathcal{H}) \bigcap \Omega^R(\mathcal{H}).$$

So, we see, that homology of $(\Omega^R(\mathcal{H}), b', B')$ is equal to the homology of $(P\Omega^R(\mathcal{H}), b', \deg \circ d)$. Here $\deg$ is the operator, which multiplies the degree $n$ homogeneous elements by $n$.

On the other hand, consider the map $Gb'$. It is easy to see, that pair $(P, Gb')$ verifies all the properties of special deformation retraction for the $(\Omega^R(\mathcal{H}), d, \sum (\kappa')^j b')$. Recall, that the operator $\sum (\kappa')^j b'$ is the image of $\tilde{B}'$ under the above isomorphism. In fact one just repeats the reasoning from [3], p.391. So, we conclude this time, that periodic cohomology of $(\Omega^R(\mathcal{H}), d, \sum (\kappa')^j b')$ equals the periodic cohomology of $P\Omega^R(\mathcal{H})$ with induced differentials. And from (31) it follows, that $\sum (\kappa')^j b' = \deg \circ b'$ on $P\Omega^R(\mathcal{H})$.

Now it is enough to observe, that in both cases we obtain the periodic cohomology of mixed complex $(P\Omega^R(\mathcal{H}), d, b')$ (which is the cohomology of the super-complex $(P\Omega(\mathcal{H}), d + b')$).

By a slight modification of these reasoning we obtain the following
**Corollary 3.** Periodic cohomology of the mixed complex $\Omega^R(\mathcal{H})$ with differentials, induced from $b$ and $B$, is isomorphic to the periodic Hopf-type cohomology of $\mathcal{H}$.

**Proof.** Just observe, that super complex $(\Omega^R(\mathcal{H}), b + B)$ is as before quasi-isomorphic to $(P\Omega^R(\mathcal{H}), d + b)$, and from (32) and (41) we conclude, that at this subcomplex $b' = b$.

Now note, that, since the space of left-coinvariants in $\Omega(\mathcal{H})$ is also closed under the Hochschild boundary $b$ and Karoubi operator $\kappa$, one can consider the corresponding mixed subcomplex and its periodic cohomologies.

**Proposition 4.** The antipode $S$ of the Hopf algebra $\mathcal{H}$ induces an isomorphism of periodic complex of the mixed complex $(\Omega^L(\mathcal{H}), b, B)$ to the periodic complex of $(\Omega^R(\mathcal{H}), b', B')$.

**Proof.** Observe, that, in virtue of the universal properties of $\Omega(\mathcal{H})$, antipode $S$ can be extended to an anti-automorphism of $\Omega(\mathcal{H})$. Since $S^2 = 1$ in $\mathcal{H}$, the same equation holds for this extension. Hence we get an involutive anti-automorphism of the universal differential calculus of $\mathcal{H}$. Now a straightforward computation shows, that this map intertwines the right and left $\mathcal{H}$-comodule structures, and differentials $b, \kappa$ and $b', \kappa'$ in the mixed complex.

**Corollary 5.** Periodic cohomology of $(\Omega^L(\mathcal{H}), b, B)$ is canonically isomorphic to the periodic Hopf-type cohomology of $\mathcal{H}$.

**Remark.** Note, that the universality property of $\Omega(\mathcal{H})$ implies that, in fact, all the maps, defined at the level of $\mathcal{H}$ can be extended to this differential calculus. Thus, one can introduce the structure of differential graded Hopf algebra on $\Omega(\mathcal{H})$.

### 3 General case: arbitrary $\delta$ and $\sigma$

In this section we shall investigate the case of a general modular pair in involution $(\delta, \sigma)$. We shall reduce this case to a variant of the construction, we’ve just considered.
First recall, that for any character \( \xi \) of a Hopf algebra \( \mathcal{H} \) one can introduce the following endomorphism (and even automorphism) of \( \mathcal{H} \):

\[
\tilde{\xi} : \mathcal{H} \rightarrow \mathcal{H}, \quad \tilde{\xi}(a) = a \star \xi \overset{\text{def}}{=} \sum a_{(1)} \xi(a_{(2)}).
\] (43)

The inverse of \( \tilde{\xi} \) is given by the right convolution with \( \xi^{-1} \overset{\text{def}}{=} \xi \circ S \).

The map \( \tilde{\xi} \) is, in an evident way, a morphism of algebras, but it does not respect the coalgebra structure on \( \mathcal{H} \). That is \( \Delta(\tilde{\xi}(a)) \neq \sum \tilde{\xi}(a_{(1)}) \otimes \tilde{\xi}(a_{(2)}) \).

In fact, this equation is substituted for the following two:

\[
\Delta(\tilde{\xi}(a)) = \sum a_{(1)} \otimes \tilde{\xi}(a_{(2)}),
\] (44)

and

\[
\Delta(\tilde{\xi}(a)) = \sum \tilde{\xi}(a_{(1)}) \otimes Ad_\xi(a_{(2)}),
\] (45)

where

\[
Ad_\xi(a) = \xi^{-1} \star a \star \xi = \sum \xi(S(a_{(1)})) a_{(2)} \xi(a_{(3)}).
\] (46)

It is easy to see, that \( Ad \) defines an action of the group of characters of \( \mathcal{H} \) on \( \mathcal{H} \) by Hopf algebra homomorphisms. One calls it the adjoint action.

Since \( \Omega(\mathcal{H}) \) is universal differential calculus, we conclude, that homomorphism \( \tilde{\xi} \) can be extended to higher degree forms. By abuse of notation we shall denote this map by the same symbol \( \tilde{\xi} \). Remark, that equations (44) and (45) are fulfilled, in a slightly different form, for this new map, too. Namely:

\[
\Delta_R(\tilde{\xi}(\omega)) = (Id \otimes \tilde{\xi}) \Delta_R(\omega) = (\tilde{\xi} \otimes Ad_\xi) \Delta_R(\omega), \quad \omega \in \Omega(\mathcal{H}).
\] (47)

Meanwhile the left coaction remains unchanged:

\[
\Delta_L(\tilde{\xi}(\omega)) = (Id \otimes \tilde{\xi}) \Delta_L(\omega).
\] (48)

In fact, the formulae (47) and (48) are particular cases of the following observation. As it is remarked above, \( \Omega(\mathcal{H}) \) is a differential graded Hopf algebra. Its diagonal map we shall denote by \( \tilde{\Delta} \). If once again by abuse of notation, \( Ad_\xi \) denotes the automorphism of \( \Omega(\mathcal{H}) \), induced by the appropriate automorphism of \( \mathcal{H} \), then the formulas (44) and (45) hold with \( \tilde{\Delta} \) substituted for \( \Delta \).
Note, that, since $\tilde{\xi}$ is a map of differential graded algebras, one can use it to define a new differential structure on $\Omega(\mathcal{H})$. Namely, put
\[
d_{\xi}(\omega) \overset{\text{def}}{=} d(\tilde{\xi}(\omega)) = \tilde{\xi}(d\omega)
\] (49)

One easily checks the following statement, compare [6]:

**Proposition 6.** (i) Differential $d_{\xi}$ verifies the following equation
\[
d_{\xi}(\omega_1\omega_2) = d_{\xi}(\omega_1)\tilde{\xi}(\omega_2) + (-1)^{|\omega_1|}\tilde{\xi}(\omega_1)d_{\xi}(\omega_2).
\] (50)

(ii) Algebra $\Omega(\mathcal{H})$, equipped with the differential $d_{\xi}$ is the universal example of $\tilde{\xi}-$differential calculi on $\mathcal{H}$, that is, of such graded algebras $\Omega$, that

1. $\Omega_0 = \mathcal{H}$;
2. $\Omega$ is equipped with a degree 1 map $d_\Omega$, $d_\Omega^2 = 0$, called differential;
3. automorphism $\tilde{\xi}$ of $\mathcal{H}$ extends to a degree 0 automorphism of $\Omega$, commuting with $d_\Omega$;
4. its differential $d_\Omega$ verifies (50).

(iii) Any element $\theta$ in $\Omega_n(\mathcal{H})$ can in a unique way be represented in the form
\[
\theta = \sum_i a^i_0d_{\xi}(a^i_1)d_{\xi}(a^i_2)\ldots d_{\xi}(a^i_n),
\] (51)

for some $a^i_\alpha \in \mathcal{H}$.

All this is checked by a straightforward inspection of definitions. Below we will denote the universal differential calculus $\Omega(\mathcal{H})$ with differential $d_{\xi}$ by $\Omega_{\xi}(\mathcal{H})$. We shall also use the presentation of part (iii) to write down the elements of $\Omega_{\xi}(\mathcal{H})$.

Now it is natural to write down the left and right coactions of $\mathcal{H}$ on $\Omega(\mathcal{H})$ in terms of the formula (51). By virtue of the formulae (47) and (48), one gets
\[
\Delta_R(d_{\xi}(\omega)) = (d_{\xi} \otimes Ad_{\xi})\Delta_R(\omega),
\] (52)
\[
\Delta_L(d_{\xi}(\omega)) = (Id \otimes d_{\xi})\Delta_L(\omega).
\] (53)
Hence, formulae (17) and (18) become
\[
\Delta_R(\theta) = \sum_i a^i_{0,(1)} d_\xi a^i_{1,(1)} a^i_{2,(1)} \ldots d_\xi a^i_{n,(1)} \otimes Ad_\xi (a^i_{0,(2)} a^i_{1,(2)} a^i_{2,(2)} \ldots a^i_{n,(2)}),
\]
\[
\Delta_L(\theta) = \sum_i a^i_{0,(1)} a^i_{1,(1)} a^i_{2,(1)} \ldots a^i_{n,(1)} \otimes a^i_{0,(2)} d_\xi a^i_{1,(2)} d_\xi a^i_{2,(2)} \ldots d_\xi a^i_{n,(2)}.
\]
(54) (55)

Here we’ve used the fact, that \( Ad_\xi \) is a Hopf algebra homomorphism.

To put short the above considerations, one can say, that one can consider the universal \( \xi \)-differential algebra \( \Omega_\xi(\mathcal{H}) \), which consists of linear combinations of elements of the form \( a_0 d_\xi a_1 d_\xi a_2 \ldots d_\xi a_n \), and on which the Hopf algebra \( \mathcal{H} \) coacts on both sides by formulae (54) and (55). We shall use this notation below, though it is not absolutely necessary, since it is just another way to speak about the universal calculus \( \Omega(\mathcal{H}) \).

As before, one can consider the spaces of right- and left-coinvariants in \( \Omega_\xi(\mathcal{H}) \). For instance, the space of right ones, \( \Omega^R_\xi(\mathcal{H}) \) consists of the tensor powers of the space spanned by elements
\[
\pi^R(a) = d(a_{(1)}) S(a_{(2)}) = d_\xi (a_{(1)}) \xi^{-1}(a_{(2)}) S(a_{(3)}) = d_\xi (a_{(1)}) S_{\xi^{-1}}(a_{(2)}), \quad a \in \ker \epsilon.
\]
(56)

As before, \( \Omega^R_\xi(\mathcal{H}) \) is a d.g. subalgebra in \( \Omega_\xi(\mathcal{H}) \). This follows directly from (54).

In addition to the usual coinvariants, one can consider the space of elements \( \theta \), such that
\[
\Delta_R(\theta) = \theta \otimes \sigma
\]
(57)

for some group-like element \( \sigma \). We shall call such elements (right) \( \sigma \)-coinvariants. Let \( \Omega^R_\sigma(\mathcal{H}) \) (respectively \( \Omega^R_{\xi,\sigma}(\mathcal{H}) \)) denote the space of right \( \sigma \)-coinvariants in \( \Omega(\mathcal{H}) \) (resp. in \( \Omega_\xi(\mathcal{H}) \)).

Clearly, since \( \Delta_R \) commutes with differential \( d \), \( \Omega^R_\sigma(\mathcal{H}) \) is d.g. subalgebra in \( \Omega(\mathcal{H}) \). Similar statement holds for \( d_\xi \), \( \Omega^R_{\xi,\sigma}(\mathcal{H}) \) and \( \Omega_\xi(\mathcal{H}) \).

**Proposition 7.** The differential \( d_\xi \) maps the space of \( \sigma \)-coinvariants into itself. Moreover, the space \( \Omega^R_{\xi,\sigma}(\mathcal{H}) \) is a differential graded \( \Omega^R_\xi(\mathcal{H}) \)- sub-bimodule in \( \Omega_\xi(\mathcal{H}) \).
Proof. Note, that the right multiplication by $\sigma$ establishes an isomorphism between the space of (right) coinvariants and the space of (right) $\sigma$-coinvariants. The inverse is given by the multiplication by $\sigma^{-1} \overset{\text{def}}{=} S(\sigma)$. Hence, any element in $\Omega_{\xi,\sigma}^R(\mathcal{H})$ is representible in the form

$$\theta = \theta' \cdot \sigma, \quad (58)$$

for a suitable $\theta' \in \Omega_{\sigma}^R(\mathcal{H})$. Hence, it is enough to show, that $d_\xi(\sigma) \in \Omega_{\xi,\sigma}^R(\mathcal{H})$. We compute:

$$d_\xi(\sigma) = d(\tilde{\xi}(\sigma)) = d(\sigma \xi(\sigma)), \quad (59)$$

Which is, clearly, (right) $\sigma$-coinvariant, since $d$ commutes with coaction. Here we’ve used the fact, that $\sigma$ is group-like, i.e. $\Delta(\sigma) = \sigma \otimes \sigma$.

Finally, the fact that $\Omega_{\xi,\sigma}^R(\mathcal{H}) = \Omega_{\sigma}^R(\mathcal{H})$ is a left $\Omega_{\xi}^R(\mathcal{H})$-module is a consequence of the presentation (58). Since the left multiplication by $\sigma^{\pm 1}$ also establishes an isomorphism between $\Omega_{\xi,\sigma}^R(\mathcal{H})$ and $\Omega_{\xi}^R(\mathcal{H})$, the conclusion follows.

Let’s now define the $\xi$-twisted cyclic structure on $\Omega_{\xi}(\mathcal{H}) = \Omega(\mathcal{H})$, that is the analogs of Hochschild operator $b$ (or $b'$) and Karoubi operator $\kappa$ (or $\kappa'$). In other words, let’s use the presentation (51) to define the following operators on $\Omega(\mathcal{H})$. Put (compare (24)-(30))

$$b_\xi(\omega d_\xi a) = (-1)^{|\omega|}(\omega \tilde{\xi}(a) - a \omega), \quad (60)$$

$$\kappa_\xi = 1 - b_\xi d - db_\xi, \quad (61)$$

or explicitly

$$\kappa_\xi(\omega d_\xi a) = (-1)^{|\omega|}da \omega. \quad (62)$$

And, similarly

$$b'_\xi(d_\xi a \omega) = \tilde{\xi}(a)\omega - \omega a, \quad (63)$$

$$\kappa'_\xi(d_\xi a \omega) = (-1)^{|\omega|}\omega da. \quad (64)$$

It is clear, that operators $b_\xi$ and $b'_\xi$ are well defined, since $\tilde{\xi}(1) = 1$. Also observe, that

$$\kappa'_\xi = \kappa^{-1}_\xi, \quad (65)$$

$$b'_\xi = b_\xi \kappa'_\xi \quad (66)$$
Once again, one easily checks that these operators verify all the properties of the standard ones, listed in [6], §3, only few modifications should be made. In fact, the following proposition holds (compare [6], §3).

**Proposition 8.**

(i) \[ b_\xi^2 = (b'_\xi)^2 = 0. \]

(ii) Following operators commute

\[
[b_\xi, \kappa_\xi] = [d, \kappa_\xi] = [d_\xi, \kappa_\xi] = 0, \\
[\xi, \kappa_\xi] = [\xi, b_\xi] = 0
\]

Moreover, on elements of \((\Omega_\xi)_n(\mathcal{H})\) one has the following identities:

(iii) \[ \kappa_\xi^{n+1}d_\xi = \bar{\xi}^{-1}d_\xi = d \]

(iv) \[ \kappa_\xi^n = \bar{\xi}^{-1} + b_\xi \kappa_\xi^n d. \]

(v) \[ \kappa_\xi^n b_\xi = \bar{\xi}^{-1}b_\xi. \]

(vi) \[ \kappa_\xi^{n+1} = \bar{\xi}^{-1}(1 - db_\xi). \]

(vii) \[ (\kappa_\xi^n - \bar{\xi}^{-1})(\kappa_\xi^{n+1} - \bar{\xi}^{-1}) = 0. \]

(viii) Let

\[ B_\xi = \sum_{j=0}^{n} \kappa_\xi^j d_\xi, \quad (67) \]

then \[ B_\xi d_\xi = d_\xi B_\xi = B_\xi^2 = 0. \]

(ix) \[ \kappa_\xi^{n(n+1)} - 1 = b_\xi B_\xi = -B_\xi b_\xi. \]

**Proof.** Part \(i\) is checked by a direct inspection of formulas. Part \(ii\) follows from part \(i\), (61) and the fact, that \(\bar{\xi}\) is a d.g. algebra automorphism, and hence it commutes with \(b_\xi\) and \(d_\xi\) (and consequently with \(\kappa_\xi\), too). All the rest is obtained by mimicking the reasoning of the cited paper, taking in
consideration the fact, that $\tilde{\xi}$ commutes with all the operators, introduced above. For instance: let’s prove part (iv). We compute, using formula (62) and the definitions of $b_\xi$ and $d_\xi$:

$$\kappa^n_\xi(a_0d_\xi a_1 \ldots d_\xi a_n) = da_1 \ldots d_\xi a_n a_0$$

$$= \tilde{\xi}^{-1}(d_\xi a_1 \ldots d_\xi a_n \tilde{\xi}(a_0))$$

$$= \tilde{\xi}^{-1}(a_0 d_\xi a_1 \ldots d_\xi a_n + (-1)^n b_\xi(d_\xi a_1 \ldots d_\xi a_n d_\xi a_0))$$

$$= \tilde{\xi}^{-1}(a_0 d_\xi a_1 \ldots d_\xi a_n + (-1)^n b_\xi \kappa^n_\xi(d_\xi a_0 d_\xi \tilde{\xi}(a_1) \ldots d_\xi \tilde{\xi}(a_n)))$$

$$= \tilde{\xi}^{-1}(a_0 d_\xi a_1 \ldots d_\xi a_n + b_\xi \kappa^n_\xi \tilde{\xi}(a_0 d_\xi a_1 \ldots d_\xi a_n))$$

$$= (\tilde{\xi}^{-1} + b_\xi \kappa^n_\xi \tilde{\xi})(a_0 d_\xi a_1 \ldots d_\xi a_n),$$

since $d_\xi = d \tilde{\xi}$.

Now we come to the main result of this paper. The following theorem is a straightforward generalization of the Theorem 1 of the section 2.

**Theorem 9.** Let $(\delta, \sigma)$ be a modular pair in involution. Let $\xi = \delta^{-1}$. Then the space $\Omega^R_{\xi, \sigma}(\mathcal{H})$ of $\sigma -$coinvariants in $\Omega_\xi(\mathcal{H})$ is stable under the Hochschild and Karoubi operators and periodic cohomology of the induced mixed complex is naturally isomorphic to the Hopf-type periodic cohomology $HP_{\delta, \sigma}(\mathcal{H})$ of A. Connes and H. Moscovici.

**Proof.** is obtained in a way, absolutely similar to the proof of Theorem 1. First of all, we establish the first part of this statement (once again we prefer to work with primed versions of cyclic operators).

Namely, let’s check, that $b_\xi(\omega) \in \Omega_\sigma(\mathcal{H})$ for all $\omega \in \Omega_\delta(\mathcal{H})$ (compare (34)). Recall, that $\xi = \delta^{-1}$:

$$b_\xi(\omega) = b_\xi(\pi^R(a_1) \otimes \cdots \otimes \pi^R(a_n) \sigma)$$

$$= b_\xi(d_\xi(a_{1,1}) S_\delta(a_{1,2}) \omega \sigma) = \tilde{\xi}(a_{1,1}) S_\delta(a_{1,2}) \omega \sigma - S_\delta(a_{1,2}) \omega \sigma a_{1,1}$$

$$= -\left(\pi^R(S(a_{1,n+1}) a_2) \otimes \cdots \otimes \pi^R(S(a_{1,3}) a_n)\right) (S_\delta(a_{1,2}) \sigma a_{1,1})$$

$$= -\left(\pi^R(S(a_{1,n+1}) a_2) \otimes \cdots \otimes \pi^R(S(a_{1,3}) a_n)\right) (S_\delta(a_{1,2}) S^2_\delta(a_{1,1}) \sigma)$$

$$= -\pi^R(S(a_{1,n+1}) a_2) \otimes \cdots \otimes \pi^R(S(a_{1,3}) a_n) \left(S_\delta(S_\delta(a_{1,1}) a_{1,2}) \right) \sigma$$

$$= -\pi^R(S(a_{1,n-1}) a_2) \otimes \pi^R(S(a_{1,n}) a_3) \otimes \cdots \otimes \pi^R(S_\delta(a_{1,1}) a_n) \otimes \sigma. \quad (68)$$
We’ve used the fact, that $S^2_\delta(a) = \sigma a \sigma^{-1}$, and the following properties of $S_\delta$:

\begin{align*}
S_\delta(ab) &= S_\delta(b)S_\delta(a); \\
\Delta(S_\delta(a)) &= S(h_1) \otimes S_\delta(h_2); \\
S_\delta(h_1)h_2 &= \delta(h).
\end{align*}

(69) (70) (71)

All this is proven by direct computations (see, e.g. [4]).

Similarly to the observation, following the equation (34), one concludes, that $\kappa'_\xi$ maps $\Omega_\sigma(H)$ to itself by a mere inspection of definitions. But we prefer to give an explicit proof here, too. We compute (c.f. (35)):

\begin{align*}
\kappa'_\xi(d_\sigma a_{1,1})S_\delta(a_{1,2})\omega'\sigma &= (-1)^{\mid\omega'\mid}S_\delta(a_{1,2})\omega'\sigma da_{1,1} \\
&= (-1)^{\mid\omega'\mid} \left( \pi^R(S(a_{1,n+1})a_2) \otimes \cdots \otimes \pi^R(S(a_{1,3})a_n) \right) (S_\delta(a_{1,2})\sigma da_{1,1}).
\end{align*}

(72)

Now, let’s consider the last term of this expression separately (we omit subscript 1 for the sake of brevity):

\begin{align*}
S_\delta(a_{2})\sigma da_{1,1} &= S_\delta(a_{3})\sigma \pi^R(a_{11})a_{2} \\
&= \pi^R(S(a_{4})\sigma a_{11} - S(a_{4})\sigma \epsilon(a_{11}))S_\delta(a_{3})\sigma a_{2} \\
&= \pi^R(S(a_{4})\sigma a_{11})S_\delta(a_{3})\sigma - \pi^R(S(a_{3})\sigma S_\delta(a_{2}))S_\delta(a_{11}) \sigma \\
&= \pi^R(S(a_{4})\sigma a_{11})S_\delta(S_\delta(a_{2}))S_\delta(a_{11}) \sigma \\
&= \pi^R(S(a_{4})\sigma a_{11})S_\delta(S_\delta(a_{2}))S_\delta(a_{11}) \sigma \\
&= \pi^R(S_\delta(a_{2}))S_\delta(a_{11}) \sigma - \pi^R(S_\delta(a_{2}))S_\delta(a_{11}) \sigma \\
&= \pi^R((S_\delta(a) - \delta(a)) \sigma) \\
&= -\pi^R((S_\delta(a) - \delta(a)) \sigma).
\end{align*}

(73)

Now, equations (72) and (73) show, that, identifying $(\Omega^R_\sigma)_n(H)$ with $(\ker \epsilon)^{\otimes n}$ (see proposition 7), one can write down the twisted Karoubi operator $\kappa'_\xi$ as follows:

\begin{align*}
\kappa_\xi(h_1, h_2, \ldots, h_n) &= \text{proj}'' S_\delta(h_1)(h_2, \ldots, h_n, \sigma),
\end{align*}

where proj$''$ is the following projection

\begin{align*}
\text{proj}''(h_1, h_2, \ldots, h_n) &= (h_1 - \epsilon(h_1) \cdot 1, h_2 - \epsilon(h_2) \cdot 1, \ldots, h_n - \epsilon(h_n) \sigma).
\end{align*}

(74)

The rest of the proof reproduces the reasoning of section 1. To make the analogy more evident, it is worth noting, that $\xi$ acts trivially on $\Omega^R_\sigma(H)$, since $\delta(\sigma) = 1$, and $\xi = \delta^{-1}$. \qed
4 Conclusions

Finally, we shall make few remarks, concerning the possible ways to generalize the Hopf-type cohomology.

First of all, consider the special case, discussed in section 2. Since both $\Omega^R(H)$ and $\Omega^L(H)$ are closed under the mixed complex differentials of $\Omega(H)$, we conclude, that the subspace of bi-invariants is also a sub-mixed complex in $\Omega(H)$. Moreover, this subcomplex is stable under the involution $S$. The corresponding periodic and dihedral periodic (co)homologies we shall denote by $HP_{bi,1}(H)$ and $HD_{\sigma,1}(H)$ respectively. The same constructions allows one to define bi-invariant homology in the case of arbitrary modular pair $\delta, \sigma$. If $\sigma = 1$, one can reproduce the dihedral construction, too. What is the analog of dihedral (co)homology in the case of arbitrary $\sigma$ is not so evident.

Note, that if the Hopf algebra $H$ is cocommutative, the spaces of left- and right-(co)invariants coincide, so we see, that in this case bi-invariant cohomology is isomorphic to the Hopf-type one. In a generic case the answer is not clear. Besides this, it isn’t clear, whether it is possible to define this type of bi-invariant and dihedral homology in a $\Omega(H)$ independent way.

Another important observation is, that in order to define the twisted cyclic structure on $\Omega(H)$ (which is equivalent, up to a change of basis, to $\Omega_\xi(H)$), we didn’t really use the fact that isomorphism $\xi$ was the convolution with a character of $H$, nor even did we use the fact, that $H$ is a Hopf algebra. One can come along the same very line for any autmorphism $f$ of any algebra $A$, to define $f$—twisted cyclic operators on its universal differential calculus $\Omega(A)$. One can denote the corresponding cyclic (respectively negative cyclic, periodic cyclic, etc.) homology by $HC_f(A)$ (resp. $HC^-_f(A)$, $HP_f(A)$, etc.). For example, one can take automorphism

$$f : H \rightarrow H, \quad f(a) = \alpha \ast a \ast \beta$$

($\alpha$ and $\beta$ are characters of $H$). Then, if $(\sigma S_{\alpha,\beta})^2 = 1$ then, passing to $\sigma$—coinvariants one obtains the construction of $[\xi]$ ($S_{\alpha,\beta}^\delta$ is the evident generalization of the map $S^\delta$).

In fact, the homology $HC_f(A)$ can be defined in a quite $\Omega(A)$ independent way: see for example $[15]$. Namely, define the $f$—twisted cyclic module as
follows $CC_n(\mathcal{A}, f) = \mathcal{A}^{\otimes n+1}$ and

\[
\delta_i^f(h_0, h_1, \ldots, h_n) = \begin{cases} 
(h_0, \ldots, h_i h_{i+1}, \ldots, h_n), & i = 0, \ldots, n - 1, \\
(f(h_n)h_0, \ldots, h_{n-1}), & i = n,
\end{cases} \quad (75)
\]

\[
\sigma_i^f(h_0, h_1, \ldots, h_n) = (h_0, \ldots, h_{i-1}, 1, h_i, \ldots, h_n), \quad 1 \leq i \leq n, \quad (76)
\]

\[
\tau_n^f(h_0, h_1, \ldots, h_n) = (-1)^n (f(h_n), h_0, \ldots, h_n). \quad (77)
\]

Then all the usual equations of the cyclic operations are fulfilled for this ones, save that one should substitute the identity operator for an appropriate tensor power of $f$ in certaine formulae. Further, one defines the twisted homology theories in completely usual way, by means of the cyclic double complex.

One more way to generalize the constructions above is to use the remark in the end of section 2. Namely, traking into consideration the fact, that $\Omega(\mathcal{H})$ is a d.g. Hopf algebra, one can consider it as the input of Connes-Moscovici construction in the form presented in this paper. Then the universality property of $\Omega(\mathcal{H})$ guarantees, that $S_\delta$ extends to a homomorphism of this algebra, in such a way, that all the properties of this map are valid for the extension, too. What one obtains in this way, is a construction very similar to the non-commutative Weil complex of Crainic ([4]). On the other hand a very similar construction was introduced by Đurđević in the guise of universal characteristic classes construction of Galois-Hopf extensions. This matters will be a subject of thorough discussion in a following paper.

Finally, there are two more possible approaches to generalizing constructions, presented in this paper.

The first one consists of substituting the subcomodue $\Omega_R^R(\mathcal{H})$, determined by the modular pair for an arbitrary cyclically-stable one. Here one can plug in both the standard and $\xi$-twisted cyclic structures. For instance, if $\delta = \epsilon$, such stable subcomodules are in one-one correspondence with all subcoalgebras $\mathcal{H}'$ of $\mathcal{H}$, for which

\[ S(a_{(2)})\mathcal{H}'a_{(1)} \subseteq \mathcal{H}' \]

for all $a \in \mathcal{H}$. This is always the case, if $\mathcal{H}$ is commutative. And if $\mathcal{H}$ is co-commutative, this is equivalent to saying, that $\mathcal{H}'$ is stable under the adjoint action of $\mathcal{H}$ on itself.

The second construction seems to be even more general. It consists of the following idea: it is a well-known fact (see [21]), that one can obtain a good variant of cyclic-type homology, called the non-commutative De Rham complex.
homology from the universal differential calculus $\Omega(A)$ of an algebra $A$ by passing to the quotient space

$$\bar{\Omega}(A) \overset{\text{def}}{=} \Omega(A)/[\Omega(A), \Omega(A)],$$

where $[\Omega(A), \Omega(A)]$ is the subspace of graded commutators of elements of $\Omega(A)$. One easily checks, that the differential $d$ of $\Omega(A)$ descends to a differential in $\bar{\Omega}(A)$. A less trivial fact is, that the induced homology of $\bar{\Omega}(A)$ coincide with well-defined a subspace in the cyclic homology of $A$ (see the original paper of Karoubi, [20]).

Now, if we pass to the barred complex in the case of a Hopf algebra $H$, we can no more say, that $H$ acts on it. In fact, this is not the case, unless $H$ is commutative. But it is easy to see, that the space of commutators $[H, H]$ is a coideal in $H$, hence one can substitute $H$ for the coalgebra $\bar{H} = H/[H, H]$. Then $\bar{H}$ coacts on $\bar{\Omega}(H)$ on the right (and on the left, too) and it is possible to consider the space of coinvariants of this coaction, namely, the space of those elements $\bar{\omega} \in \bar{\Omega}(H)$, which are sent to $\bar{\omega} \otimes \bar{1}$, where $\bar{1}$ is the group-like element in $\bar{H}$ determined by $1 \in H$.

This construction seem to play an important role in the theory of characteristic classes of Galois-Hopf extensions, which will be an object of discussion in the next paper. Here we confine ourselves to the following remark.

An important application of the Hopf-type cohomology is the theory of characteristic classes of a Hopf-module algebra. On the other hand, to any Hopf-module algebra one can associate its smashed product with $H$, which is an example of Galois-Hopf extension of an algebra. In the next paper we shall investigate the relation between the Connes and Moscovici construction of characteristic classes of a Hopf-module algebra and various constructions of characteristic classes of Galois-Hopf extensions which exist.

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