A Quantitative Central Limit Theorem for the Excursion Area of Random Spherical Harmonics over Subdomains of $S^2$

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Abstract

In recent years, considerable interest has been drawn by the analysis of geometric functionals for the excursion sets of random eigenfunctions on the unit sphere (spherical harmonics). In this paper, we extend those results to proper subsets of the sphere $S^2$, i.e., spherical caps, focusing in particular on the excursion area. Precisely, we show that the asymptotic behaviour of the excursion area is dominated by the so-called second-order chaos component, and we exploit this result to establish a Quantitative Central Limit Theorem, in the high energy limit. These results generalize analogous findings for the full sphere; their proofs, however, require more sophisticated techniques, in particular a careful analysis (of some independent interest) for smooth approximations of the indicator function for spherical caps subsets.

• Keywords and Phrases: Gaussian Eigenfunctions, Spherical Harmonics, Excursion Area, Quantitative Central Limit Theorem, Wiener-chaos expansion, Clebsch-Gordan coefficients.

• AMS Classification: 42C10, 33C55, 60B10.

1 Introduction and background results

Let $S^2$ be the unit 2-dimensional sphere and consider the Helmholtz equation

$$\Delta_{S^2} T_\ell + \lambda_\ell T_\ell = 0, \quad T_\ell : S^2 \to \mathbb{R},$$

where $\Delta_{S^2}$ is the Laplace-Beltrami operator on $S^2$, defined as usual as

$$\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi < 2\pi,$$

and $\lambda_\ell = \ell (\ell + 1)$, $\ell = 0, 1, \ldots$. For a given eigenvalue $\lambda_\ell$, the corresponding eigenspace is the $(2\ell + 1)$-dimensional space of spherical harmonics of degree $\ell$. A standard, complex-valued $L^2$ basis $\{Y_{\ell m}(\cdot)\}_{m=-\ell, \ldots, \ell}$ can be defined as (see [16] pag. 64)

$$Y_{\ell m}(\theta, \varphi) := \sqrt{\frac{2\ell + 1}{4\pi}} \sqrt{\frac{(\ell - m)!}{(\ell + m)!}} P_{\ell m}(\cos \theta) \exp(i m \varphi), \quad \text{for } m \geq 0,$$

$$Y_{\ell m}(\theta, \varphi) := (-1)^m \overline{Y}_{\ell,-m}(\theta, \varphi), \quad \text{for } m < 0,$$

where $P_{\ell m}(\cdot)$ denotes the associated Legendre functions. We can hence consider random eigenfunctions of the form

$$T_\ell(x) = \sqrt{\frac{4\pi}{2\ell + 1}} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x),$$

where the coefficients $\{a_{\ell m}\}$ are independent, safe for the condition $a_{\ell m} = (-1)^m \overline{a}_{\ell,-m}$; for $m \neq 0$ they are standard complex-valued Gaussian variables, while $a_{\ell 0}$ is a standard real-valued Gaussian variable.
The random fields \(\{T_\ell(x), x \in S^2\}\) are Gaussian and isotropic, namely the probability laws of \(T_\ell(\cdot)\) and \(T_\ell(g)\) are the same for any rotation \(g \in SO(3)\). Also, we have that
\[
\mathbb{E}[T_\ell(x)] = 0, \text{ and } \mathbb{E}[T_\ell(x)^2] = 1,
\]
where \(P_\ell\) are the Legendre polynomials and \(d(x, y)\) is the spherical geodesic distance between \(x\) and \(y\), i.e.
\[
d(x, y) = \arccos((x, y)).
\]
The analysis of random eigenfunctions on the sphere or on other compact manifolds (such as the torus) has been recently considered in many papers, due to strong motivations arising from Cosmology and Quantum Mechanics, see i.e., [16], [14], [15], [32], [34] and [33]. Many papers have focussed on the geometry of the \(z\)-excursion sets, which are defined for \(z \in \mathbb{R}\) as
\[
A_z(T_\ell, S^2) := \{x \in S^2 : T_\ell(x) > z\},
\]
see for instance [21], [23], [22], [31]. More precisely, a natural tool to characterize the geometry of \(\{A_z(T_\ell, S^2)\}\) is provided by the so-called Lipschitz-Killing curvatures (see i.e., [2]), which in the 2-dimensional case correspond to the area of \(A_z(T_\ell, S^2)\) (we shall write as \(L_2(A_z(T_\ell, S^2))\), (half) the boundary length \(\partial A_z(T_\ell)\) (i.e., the length of level curves \(T_\ell^{-1}(z)\), written \(L_1(A_z(T_\ell, S^2))\), and their Euler-Poincaré characteristic, i.e., the difference between the number of connected regions and the number of “holes” (written \(L_0(A_z(T_\ell, S^2))\)).

In order to characterize the stochastic properties of these functionals, the first step clearly is the evaluation of their expected values. This goal can be achieved by means of the celebrated Gaussian Kinematic Formula (see [2]), which yields, respectively,
\[
\mathbb{E}[L_0(A_z(T_\ell, S^2))] = 2\{1 - \Phi(u)\} + \frac{\lambda_T z e^{-z^2/2}}{2 \sqrt{(2\pi)^3}} 4\pi,
\]
for the Euler-Poincaré characteristic,
\[
\mathbb{E}[L_1(A_z(T_\ell, S^2))] = \pi \times \sqrt{\frac{\lambda_T}{2}} e^{-z^2/2},
\]
for (half) the boundary length, and
\[
\mathbb{E}[L_2(A_z(T_\ell, S^2))] = 4\pi \times \{1 - \Phi(z)\},
\]
for the excursion area; note that \(\lambda_T = \ell(\ell + 1) = P_\ell'(1)\) represents the derivative of the covariance function at the origin.

The next step in the investigation of the random properties for these functionals is the derivation of their variances and hence their limiting distributions. A crucial step to achieve these results is to note that all these statistics can be written as nonlinear functionals of the random fields itself and their spatial derivatives. For instance, the excursion area can be expressed by
\[
S_\ell(z) = \int_{S^2} 1_{\{z, +\infty\}}(T_\ell(x)) \, dx,
\]
where \(1_A(\cdot)\) is, as usual, the indicator function of the set \(A\), which takes value one if the condition in the argument is satified, zero otherwise; likewise, using a Kac-Rice argument (see [3], [2]) the length of level curves can be written as
\[
L_\ell(z) = \int_{S^2} \delta_z(T_\ell(x)) ||\nabla T_\ell(x)|| \, dx,
\]
and a related expression can be given for the Euler-Poincaré characteristic (see [8]). Starting from these expressions, it is possible to compute explicitly the expansion of Lipschitz-Killing curvatures into the
orthonormal system generated by Hermite polynomials; for instance, in the case of the excursion area it can be readily shown that (see [21, 23, 18])

\[ S_\ell(u) = \sum_{q=0}^{\infty} \frac{J_q(u)}{q!} \int_{\mathbb{S}^2} H_q(T_\ell(x)) \, dx, \]

the equality holding in the \(L^2(\Omega)\)-sense; we recall here the standard definition of Hermite polynomials, i.e., \(H_0 \equiv 1\) and, for \(q \geq 1\), (see for instance [20])

\[ H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q} e^{-\frac{x^2}{2}}, \ x \in \mathbb{R}. \]

The coefficients \(\{J_q(\cdot)\}\) have the analytic expressions \(J_0(u) = \Phi(u), J_1(u) = -\phi(u), J_2(u) = -u\phi(u), J_3(u) = (1-u^2)\phi(u)\) and in general

\[ J_q(u) = -H_{q-1}(u)\phi(u), \ q = 1, 2, 3... \]

where \(\phi(\cdot)\) and \(\Phi(\cdot)\) are the density function and the distribution function of a standard Gaussian variable (21, 23). As in [23], we denote

\[ h_{\ell,q} = \int_{\mathbb{S}^2} H_q(T_\ell(x)) \, dx \quad q = 1, 2, \ldots, \]

and we can hence write

\[ S_\ell(z) = \sum_{q=0}^{\infty} \frac{J_q(z)}{q!} h_{\ell,q} \text{ in } L^2(\Omega). \]

It can be readily verified that the term corresponding to \(q = 1\) in (1.1), (1.2) are identically equal to zero for every \(\ell \geq 1\); indeed we have that

\[ h_{\ell,1} := \int_{\mathbb{S}^2} \sqrt{\frac{4\pi}{2\ell+1}} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x) \, dx = \sqrt{\frac{4\pi}{2\ell+1}} \sum_{m=-\ell}^{\ell} a_{\ell m} \int_{\mathbb{S}^2} Y_{\ell m}(x) \, dx = 0. \]

The crucial step in [21, 18] is then to show that a single term, corresponding to \(q = 2\), has asymptotically (in the high-energy regime \(\ell \to \infty\)) a dominating role in the expansion, i.e.,

\[ \text{Var}(S_\ell) = \left\{ \frac{J_2(z)}{2} \right\}^2 \text{Var}(h_{\ell,2}) + o(\text{Var}(S_\ell)), \text{ as } \ell \to \infty, \]

so that both the asymptotic variance and the Central Limit Theorem can be established by simply considering the behaviour of this single term. Similar expansions can be derived for the boundary length and the Euler-Poincaré characteristic (see [5, 19, 37, 3]), thus leading to the following asymptotic expressions for the variances (see [8, 23, 18, 31]):

\[ \lim_{\ell \to \infty} \text{Var}(L_0(A_2(T_\ell, S^2))) = \frac{1}{4}(H_3(z) + H'_3(z))^2 \phi(z)^2, \]

\[ \lim_{\ell \to \infty} \ell^{-1} \text{Var}(L_1(A_2(T_\ell, S^2))) = \frac{1}{2} \sqrt{\frac{\pi}{8}}(H_2(z) + H'_2(z))^2 \phi(z)^2, \]

\[ \lim_{\ell \to \infty} \ell \text{Var}[L_2(A_2(T_\ell, S^2))] = (H_1(z) + H'_1(z))^2 \phi(z)^2. \]

See also [11, 38, 18, 37, 7, 28, 27, 17, 10] for related results on the torus and on the plane, and [19, 4, 8, 18, 21, 32, 34] for other works concerning the geometry of random eigenfunctions on compact manifolds. A common features of all these statistics is the disappearance of the leading term at the zero level \(z = 0\) (the so-called Berry’s cancellation phenomenon, investigated in [17, 22, 35]). In the case of the excursion area, at \(z = 0\) all the odd-order chaoses become relevant, and the Central Limit Theorem can be established as in [19, 8, 17]. For other functionals (nodal length), at \(z = 0\) the fourth
order chaos plays the role of the dominant term. Along the same lines, it has been possible to establish Quantitative Central Limit Theorems for the asymptotic fluctuations in the high-energy regime. To report these results, we need to introduce some more notation. Recall that the Wasserstein $d_W$ distance between random variables $Z, N$ is defined by

$$d_W(Z, N) := \sup_{h \in Lip(1)} |E[h(Z)] - E[h(N)]|$$

where $Lip(1)$ denotes the set of Lipschitz functions with bounding constant equal to 1. It should be noted that the functionals $\{h_{\ell, q}\}$ belong to the so-called Wiener chaoses of order $q$, defined as the space spanned by linear combinations of Hermite polynomials of order $q$; as such, they belong to the domain of application for the so-called Stein-Malliavin method, leading to very neat characterizations for Quantitative Central Limit Theorems (see i.e., [29], [26]). More precisely, we have that (Theorem 5.2.6, pag.99 [26])

$$d_W\left(\frac{h_{\ell, q}}{\sqrt{\text{Var}(h_{\ell, q})}}, Z\right) \leq 2\sqrt{\frac{q - 1}{3q}} \left(\frac{\text{cum}_4(h_{\ell, q})}{\text{Var}(h_{\ell, q})}\right)^{1/2},$$

where $Z \sim N(0, 1)$ and $\text{cum}_4(Y) := EY^4 - 3EY^2$ denotes the fourth-order cumulant of a random variable $Y$. In words, this means that in these circumstances to prove a Quantitative Central Limit Theorem for standardized sequences it is enough to show that their fourth-order moment goes to 3. This approach was used to establish Quantitative Central Limit Theorems in [21], [18] (see also [23], [19], [8], [17]), i.e., for $z \neq 0$,

$$d_W\left(\frac{S\ell(z) - E[S\ell(z)]}{\sqrt{\text{Var}(S\ell(z))}}, Z\right) = O(\ell^{-1/2}),$$

as $\ell \to \infty$, entailing as a Corollary that

$$\frac{S\ell(z) - E[S\ell(z)]}{\sqrt{\text{Var}(S\ell(z))}} \to_d Z, \ z \neq 0$$

d denoting the convergence in distribution.

### 1.1 Main Result

In this paper we extend and generalize some of the previous results, considering the case of the excursion area evaluated on a spherical cap rather than the full sphere. More precisely, we shall focus on a symmetric spherical cap $B$ of radius $r < \pi$, which we take without loss of generality to be centred around the North Pole $N = (0, 0)$, i.e.,

$$B = \{x \in S^2 : 0 \leq \theta_x \leq r, \ 0 \leq \varphi_x \leq 2\pi\}. \tag{1.5}$$

We shall then consider the excursion set

$$A\ell(z, B) = \{x \in B : T\ell(x) > z\},$$

and in particular the excursion area

$$S\ell(B, z) := \int_B 1_{\{T\ell(x) > z\}}(T\ell(x)) \, dx.$$  

Our main result is a Quantitative Central Limit Theorem of the form

**Theorem 1.1.** For every $z \neq 0$, as $\ell \to \infty$, we have that

$$d_W\left(\frac{S\ell(B, z)}{\sqrt{\text{Var}(S\ell(B, z))}}, Z\right) = O\left(\frac{1}{\sqrt{\ell}}\right),$$

where $Z \sim N(0, 1)$.  


The main steps in the proof of this result are described in the next section. We anticipate that our main ideas are broadly similar to those previously exploited in the related literature: namely, we compute the $L^2$-expansion into Hermite polynomials and we show that the second order term is the dominating one. Along these similarities, we stress however that there exist as well very important differences, which we list below as follows:

- While the first-order chaos term is identically zero in the case of the full sphere (see (1.3)), this result does no longer hold on subdomains and a careful analysis is needed to show that the corresponding terms are of lower stochastic order. Here we shall also require the properties of a smooth approximation for the indicator function of the spherical cap, whose construction is of some independent interest.

- The second-order chaos term is still the leading one in the $L^2$ expansion, and it decays to zero with the same rate $\ell^{-1}$ as in the full spherical case. However, the normalizing constants are different, and they can be given a natural interpretation as the relative area of the region under consideration.

- It is still possible to show that a (Quantitative) Central Limit Theorem holds. However the proof is entirely different from the one exploited in the case of the full sphere, and indeed much more challenging. In fact, due to Parseval’s identity, in the case of the full sphere the second-order chaos boils down to a simple sum of independent and identically distributed random variables, so that the Central Limit Theorem, even in its Quantitative version, is almost immediate. Here, on the contrary, these identities no longer hold, and it thus becomes necessary to exploit the full power of Stein-Malliavin results (see [26] and [29]) by means of a careful computation of fourth-order cumulants. In particular, the latter result requires the investigation of complex cross-sums of so-called Clebsch-Gordan coefficients (see [36], [16]), which arise from integrals of multiple products of spherical harmonics. Finally, it is remarkable that the asymptotic rate of convergence in the Quantitative Central Limit Theorem turns out to be identical to the full spherical case.

- It remains true that the leading term in the variance expansion vanishes in the “nodal” case $z = 0$, i.e., some form of the Berry’s cancellation phenomenon (see [5], [31], [37]) applies to subdomains of the sphere as well.

1.2 Plan of the paper

In Section 2 we briefly explain the main ideas in our argument to prove the main result, while Section 3 discuss the construction of a smooth approximation to the indicator function and its asymptotic properties. The proof of the Central Limit Theorem is given in Section 4 where the asymptotic behavior of the Chaos components of the excursion area are given. Further technical computations are collected in the Appendix.

In the sequel, given any two positive sequence $a_n, b_n$, we shall write $a_n \sim b_n$ if $\lim_{n \to \infty} a_n / b_n = 1$. Also we shall write $a_n \ll b_n$ or $a_n = O(b_n)$ when the sequence $a_n / b_n$ is asymptotically bounded.

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2 On the proof of the main result

From now on $B$ will denote the spherical cap defined in (1.5). As mentioned earlier, in order to study the excursion area, we start by writing it as the functional

$$S_\ell(B, z) = \int_B 1_{\{T_\ell(x) > z\}}(T_\ell(x)) dx$$
and then, exploiting the $L^2$-expansion into Wiener Chaoses, we have

$$1_{\{T_\epsilon(x) \leq z\}}(T_\epsilon(x)) = \sum_{q=0}^{\infty} \frac{J_q(z)}{q!} H_q(T_\epsilon(x)), \quad (2.1)$$

meaning that

$$\lim_{Q \to \infty} E[\left| \sum_{q=0}^{Q} \frac{J_q(z)}{q!} H_q(T_\epsilon(x)) - 1_{\{T_\epsilon(x) \leq z\}}(T_\epsilon(x)) \right|^2] = 0.$$ 

Because of the linearity of the integral and Jensen inequality, one has

$$\int_B 1_{\{T_\epsilon(x) > z\}}(T_\epsilon(x)) \, dx = \lim_{Q \to \infty} \sum_{q=0}^{Q} \frac{J_q(z)}{q!} \int_B H_q(T_\epsilon(x)) \, dx = \lim_{Q \to \infty} \sum_{q=0}^{Q} \frac{J_q(z)}{q!} h_{\epsilon,q}(B)$$

with

$$h_{\epsilon,q}(B) = \int_B H_q(T_\epsilon(x)) \, dx;$$

Indeed,

$$E \left[ \left| \sum_{q=0}^{Q} \frac{J_q(u)}{q!} \int_B H_q(T_\epsilon(x)) \, dx - \int_B 1_{\{T_\epsilon(x) \leq u\}} \right|^2 \right] \leq E \left[ \int_B \sum_{q=0}^{Q} \frac{J_q(u)}{q!} H_q(T_\epsilon(x)) \, dx - 1_{\{T_\epsilon(x) \leq u\}} \right]^2 \, dx$$

which goes to zero for (2.1). We can hence write

$$\int_B 1_{\{T_\epsilon(x) > z\}} \, dx = \int_B \{1 - \Phi(z)\} \, dx + \int_B \phi(z)H_1(T_\epsilon(x)) \, dx + \int_B z\phi(z)\frac{1}{2}H_2(T_\epsilon(x)) \, dx$$

$$+ \sum_{q=3}^{\infty} \frac{J_q(z)}{q!} H_q(T_\epsilon(x)) \, dx, \quad (2.2)$$

in the $L^2(\Omega)$–convergence sense. The same holds for the variance thanks to the continuity of the norm. Indeed

$$\text{Var} \left( \int_B 1_{\{T_\epsilon(x) > z\}}(T_\epsilon(x)) \, dx \right) = E \left[ \left( \int_B 1_{\{T_\epsilon(x) > z\}}(T_\epsilon(x)) \, dx \right)^2 \right] =$$

$$= \left( \int_B 1_{\{T_\epsilon(x) > z\}}(T_\epsilon(x)) \, dx \right) \left( \int_B 1_{\{T_\epsilon(x) > z\}}(T_\epsilon(x)) \, dx \right)_{L^2(\Omega)} =$$

$$= \lim_{Q \to \infty} \sum_{q=0}^{Q} \frac{J_q(z)}{q!} \int_B H_q(T_\epsilon(x)) \, dx \sum_{q=0}^{Q} \frac{J_q(z)}{q!} \int_B H_q(T_\epsilon(x)) \, dx. \quad (2.3)$$

Hence the following expansion holds in $L^2(\Omega)$ sense:

$$\text{Var} \left( \int_B 1_{\{T_\epsilon(x) > z\}} \, dx \right) = 0 + \phi(z)^2 \text{Var} \left( \int_B H_1(T_\epsilon(x)) \, dx \right) + \frac{\phi(z)^2}{4} \text{Var} \left( \int_B H_2(T_\epsilon(x)) \, dx \right)$$

$$+ \text{Var} \left( \sum_{q=3}^{\infty} \frac{J_q(z)}{q!} H_q(T_\epsilon(x)) \, dx \right). \quad (2.4)$$

The Quantitative Central Limit Theorem is established by the analysis of the asymptotic behaviour for each of these terms; here below we give a summary of the results we obtained for the singular components. In the sequel, we shall need an approximation of the indicator function, which we shall label $1_{B,\epsilon}(x)$, see Section 3 for more details.
Proposition 2.1. Let $B$ the spherical cap defined in (1.5), under the assumption (4.1), the variance of the first chaotic component of (2.4) is

$$\text{Var} \left( \int_{S^2} 1_B(x)T_\ell(x) \, dx \right) = o \left( \frac{1}{\ell} \right)$$

as $\ell \to \infty$.

To establish this result, we write the variance as

$$\text{Var} \left( \int_{S^2} 1_B(x)T_\ell(x) \, dx \right) = \text{Var} \left( \int_{S^2} (1_B(x) - 1_B,\varepsilon(x))T_\ell(x) \, dx \right) + \text{Var} \left( \int_{S^2} 1_B,\varepsilon(x)T_\ell(x) \, dx \right) +$$

$$+ E \left[ \int_{S^2} (1_B,\varepsilon(x) - 1_B(x))1_B,\varepsilon(y)T_\ell(x)T_\ell(y) \, dx \right].$$

The second integral will be computed to be

$$\text{Var} \left( \int_{S^2} 1_B,\varepsilon(x)T_\ell(x) \, dx \right) = \frac{4\pi}{2\ell + 1} b^2_{\ell,\varepsilon},$$

where $b_{\ell,\varepsilon}$ are the Fourier coefficients of $1_B,\varepsilon(x)$ given below in Theorem 3.2, the former and the latter terms in (2.5) will be proved to be of order $\frac{1}{\ell^{3/2}}$, for $\varepsilon \to 0$. Hence, choosing an appropriate sequence $\varepsilon = \varepsilon_\ell$, the thesis of the Proposition 2.1 will follow.

As far as the second chaos is concerned, the following proposition will be proved.

Proposition 2.2. Under the hypothesis of Proposition 2.1, the variance of the second chaotic projection of the area of the excursion region $B$ is

$$\text{Var} \left( \int_B H_2(T_\ell(x)) \, dx \right) = 8\pi \sum_{\ell_1} b^2_{\ell_1,\varepsilon_\ell} \frac{1}{2\ell_1 + 1} \left( C^{\ell_1,0}_0 \right)^2 + o \left( \frac{1}{\ell} \right),$$

as $\ell \to \infty$, where $\{C^{\ell_1,0}_0\}$ are the Clebsch-Gordan coefficients (139, chapter 8 or the Appendix below).

The idea of the proof is similar as the one given in Proposition 2.1. More precisely, we write

$$\text{Var} \left( \int_{S^2} H_2(T_\ell(x)) \, dx \right) = \text{Var} \left( \int_{S^2} (1_B(x) - 1_B,\varepsilon(x))H_2(T_\ell(x)) \, dx \right) + \text{Var} \left( \int_{S^2} 1_B,\varepsilon(x)H_2(T_\ell(x)) \, dx \right) +$$

$$+ 2E \left[ \int_{S^2 \times S^2} 1_B,\varepsilon(x)(1_B(y) - 1_B,\varepsilon(y))H_2(T_\ell(x))H_2(T_\ell(y)) \, dxdy \right].$$

The first integral can be shown to be smaller than $\frac{\text{Const} \cdot \varepsilon}{2\ell + 1}$ (see below (4.16)), which is a $o \left( \frac{1}{\ell} \right)$ since $\varepsilon \to 0$; the same bound holds for the third integral in view of the Cauchy-Schwarz inequality. Likewise, the second integral can be shown to be

$$\text{Var} \left[ \int_{S^2} 1_B,\varepsilon(x)H_2(T_\ell(x)) \, dx \right] = 8\pi \sum_{\ell_1} b^2_{\ell_1,\varepsilon_\ell} \frac{1}{2\ell_1 + 1} \left( C^{\ell_1,0}_0 \right)^2$$

and it is possible to show that (2.8) is asymptotic to $\frac{1}{\ell}$. The proof of the second equality in (2.8) is based on manipulations of spherical harmonics and their integrals. More precisely, we make use of the addition formula (see for example [10], eq.(3.42) pag.66)

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(x)Y_{\ell m}(y) = \frac{2\ell + 1}{4\pi} P_\ell (\cos \theta);$$

(2.9)
moreover, recalling that
\[ Y_{\ell_0}(\theta, \varphi) = \sqrt{\frac{2\ell + 1}{4\pi}} P_\ell(\cos \theta), \]
using the expansion
\[ 1_{B,\varepsilon}(x) = \sum_{\ell=0}^{\infty} b_{\ell,\varepsilon} Y_{\ell_0}(x) \]
and replacing these formulae in the left hand side in (2.8), we obtain the so-called Gaunt integral of spherical harmonics ([16] eq.(3.64) pag.81) which can be computed by the following relation:
\[ \int_{S^2} Y_{\ell_1,m_1}(x) Y_{\ell_2,m_2}(x) Y_{\ell_3,m_3}(x) d\sigma(x) = \sqrt{\frac{(2\ell_1 + 1)(2\ell_2 + 1)}{4\pi(2\ell_3 + 1)}} \left( C_{\ell_3}^{\ell_1} \right)^0 \left( C_{\ell_3}^{\ell_2} \right)^0 \left( C_{\ell_3}^{\ell_1 \ell_2} \right)^0, \]
for all \( \ell_1, \ell_2, \ell_3 \). Finally, the proof is completed by a careful analysis of properties for the Clebsch-Gordan coefficients, most of which are reported in the Appendix.

The next important step in our argument is to establish the Quantitative Central Limit Theorem. This argument requires two steps; first we need to show that the variance of all higher-order chaoses for \( q \geq 3 \) is of smaller order; this can be done quite simply by some rather easy majorizations, which allow to show that all these terms are of order \( o\left(\frac{1}{\ell^3}\right) \). On the other hand, since the second term is the leading component, we need to compute its fourth-order cumulant to be able to apply Theorem 5.2.6 of Nourdin and Peccati [26] and hence to establish asymptotic Gaussianity. More precisely, we shall show that

Proposition 2.3. Under the assumption of Proposition 2.1, the fourth cumulant of the second chaotic component of the expansion in (2.2) satisfy
\[ \text{cum}_4(h_{\ell,2}(B)) = O\left(\frac{1}{\ell^3}\right), \]
as \( \ell \to \infty \).

To establish Proposition 2.3 it is sufficient to compute the fourth-order cumulant of
\[ \int_{S^2} 1_{B,\varepsilon}(x) H_2(T_\ell(x)) dx \]
because
\[ E \left[ \int_{S^2} 1_{B,\varepsilon}(x) H_2(T_\ell(x)) dx - \int_{S^2} 1_B(x) H_2(T_\ell(x)) dx \right]^2 = o\left(\frac{1}{\ell}\right) \]
as \( \varepsilon \to 0 \).

Our approach here is different from the one used in related circumstances by for instance [22], [9], [30]; indeed these papers use an approximation of Legendre polynomials known as Hilb’s asymptotics: however this approximation turned out not to be efficient enough in the present framework. Hence we need to exploit a different argument, i.e., we compute the exact values of the multiple integrals for spherical harmonics by means of Gaunt integrals (see [16]) and Clebsch-Gordan coefficients. Lengthy computations allow then to show that
\[ \frac{\text{cum}_4(h_{\ell,2}(B))}{\text{Var}(h_{\ell,2}(B))^2} = O\left(\frac{1}{\ell}\right). \]
From Theorem 5.2.6 [26], the following bound holds
\[ d_W\left( \frac{h_{\ell,2}(B)}{\sqrt{\text{Var}(h_{\ell,2}(B))}}, Z \right) \leq \sqrt{\frac{1}{6} \left( \frac{\text{cum}_4(h_{\ell,2}(B))}{\text{Var}(h_{\ell,2}(B))^2} \right)} \]
and hence, for the second chaotic component the standard CLT follows:
\[ \frac{h_{\ell,2}(B)}{\sqrt{\text{Var}(h_{\ell,2}(B))}} \to_d Z, \]
where \( Z \sim N(0,1) \). Indeed, we prove a stronger result than the standard Central Limit Theorem, in that we are able to provide rates of convergence in Wasserstein distance, similarly to those obtained for the full sphere given in [18], see Theorem 1.1.
3 Construction of a mollifier for the characteristic function

This section can be considered of some independent interest; it describes a method to construct an approximation of the indicator function, i.e., it gives an explicit expression for the function \( 1_{B, \varepsilon} \), already mentioned, converging to the indicator function \( 1_B(\cdot) \) in \( L^1(S^2) \).

For any fixed \( M > 0, M \in \mathbb{N} \), a general method to construct a function \( \phi(\cdot) \in C^M \), can be given by the B-splines approach (see [16], pag.250), as follows. First of all, recall that the Bernstein polynomials are defined as

\[
B_i^{(n)}(t) := \binom{n}{i} t^i (1 - t)^{n-i},
\]

where \( t \in [0, 1], i = 0, \ldots, n \) and \( n = 1, 2, \ldots \). Then, we can define polynomials

\[
q_{2k+1}(t) := \sum_{i=0}^{k} B_i^{(2k+1)}(t);
\]

one has that \( q_{2k+1}(0) = 1 \) and \( q_{2k+1}(1) = 0 \). Moreover,

\[
q_{2k+1}^{(m)}(1) = q_{2k+1}^{(m)}(0) = 0 \quad \text{for} \quad m = 1, \ldots, k.
\]

Hence, let \( r \in (0, \pi) \) and \( \theta \in [0, \pi) \), for any \( \varepsilon > 0 \) we set

\[
t := \frac{\theta - (r - \varepsilon)}{r - (r - \varepsilon)} \in [0, 1]
\]

and define the function

\[
\phi_{r, \varepsilon}(\theta) := \begin{cases} 
1 & \text{if } \theta \in [0, r - \varepsilon) \\
q_{2k+1}(t) = q_{2k+1}\left(\frac{t-r+\varepsilon}{\varepsilon}\right) & \text{if } \theta \in [r - \varepsilon, r] \\
0 & \text{if } \theta \in [r, \pi) 
\end{cases}
\]

with \( \theta \in (0, \pi) \).

The function \( \phi_{r, \varepsilon}(\theta) \) is a \( 2k + 1 \)-degree polynomial, so \( \phi_{r, \varepsilon} \in C^M \) for \( M < k + 1/2 \) and \( \phi_{r, \varepsilon}(r - \varepsilon) = q_{2k+1}(0) = 1 \).
Remark 1. The indicator function $1_B(x), x \in S^2$ can be written in spherical coordinates as $1_B(\theta, \varphi)$ with $\theta \in [0, \pi)$ and $\varphi \in [0, 2\pi]$ but it only depends on the angle $\theta$, namely,
\[
1_B(\theta, \varphi) = \begin{cases} 
1 & \theta \leq r \ 
0 & \text{otherwise}
\end{cases}
= 1_B(\theta).
\]

Defining $1_{B, \varepsilon}(\theta) := \phi_{r, \varepsilon}(\theta)$, it is easily to see that, as $\varepsilon \to 0$, $1_{B, \varepsilon}(\cdot) \to 1_B(\cdot)$ in $L^1(S^2)$. In fact,
\[
\int_{S^2} |1_B(x) - 1_{B, \varepsilon}(x)| dx = 2\pi \int_0^\pi |1_B(\theta) - 1_{B, \varepsilon}(\theta)| \sin \theta d\theta
\leq 2\pi \int_{r-\varepsilon}^r q_{2k+1} \left(\frac{\theta - \cos r + \varepsilon}{\varepsilon}\right) \sin \theta d\theta \leq 2\pi \varepsilon \to 0,
\]
as $\varepsilon \to 0$.

Now we focus on the function $\phi_{r, \varepsilon}(\cdot)$. As denoted in [13], we define $k_{r, \varepsilon}(\mu) := \phi_{r, \varepsilon}(\arccos \mu)$ with $\mu \in [-1, 1]$. Now recall that any function $u \in L^2(-1, 1)$ can be expanded in the $L^2(-1, 1)$ convergent Fourier-Legendre series as
\[
u = \sum_{\ell=0}^\infty u_{\ell} \frac{2\ell + 1}{2} P_\ell = \sum_{\ell=0}^\infty u_{\ell} \frac{4\pi}{2} \sqrt{\frac{2\ell + 1}{4\pi}} \sqrt{\frac{2\ell + 1}{4\pi}} P_\ell = \sum_{\ell=0}^\infty 2\pi u_{\ell} \sqrt{\frac{2\ell + 1}{4\pi}} Y_\ell = \sum_{\ell=0}^\infty b_{\ell} Y_\ell,
\]
with
\[
u_{\ell} = \int_{-1}^1 u(x) P_\ell(x) dx
\]
and hence
\[
b_{\ell} = 2\pi u_{\ell} \sqrt{\frac{2\ell + 1}{4\pi}} = \int_{-1}^1 u(x) Y_\ell(x) dx;
\]
thus we can expand $k_{r, \varepsilon}$ in such a series and its Fourier coefficients are
\[
b_{\ell, \varepsilon} = \sqrt{\frac{2\ell + 1}{4\pi}} \int_{-1}^1 k_{r, \varepsilon}(\mu) P_\ell(\mu) d\mu.
\]

Remark 2. For $\ell = 0$, it is easy to see that $b_{0, \varepsilon}$ is bounded above and below by two positive constants. Actually, by definition,
\[
b_{0, \varepsilon} = \sqrt{\frac{1}{4\pi}} \int_{-1}^1 k_{r, \varepsilon}(\theta) d\theta = \frac{1}{\sqrt{4\pi}} \int_{-1}^1 \phi_{r, \varepsilon}(\arccos \theta) d\theta;
\]
changing coordinates $\arccos \theta = x$, one has
\[
b_{0, \varepsilon} = \frac{1}{\sqrt{4\pi}} \int_0^\pi \phi_{r, \varepsilon}(x) \sin x dx
\]
\[
= \frac{1}{\sqrt{4\pi}} \int_0^{r-\varepsilon} \sin x dx + \frac{1}{\sqrt{4\pi}} \int_r^{r+\varepsilon} q_{2k+1}(x) \sin x dx
\]
\[
\geq \frac{1}{\sqrt{4\pi}} \int_0^{r-\varepsilon} \sin x dx = \frac{1}{\sqrt{4\pi}} (1 - \arccos(r - \varepsilon)) \geq \frac{1 - r + \varepsilon}{\sqrt{4\pi}} \geq \frac{1 - r}{\sqrt{4\pi}}
\]
and since
\[
|\phi_{r, \varepsilon}(\theta)| \leq 1,
\]
it is immediate to conclude that
\[
\frac{1 - r}{\sqrt{4\pi}} \leq b_{0, \varepsilon} \leq \frac{1}{\sqrt{\pi}}.
\]

The main result of this section is given in the proposition below, which yields a bound for the Fourier coefficients $b_{\ell, \varepsilon}$.
Proposition 3.1. For any fixed $M \in \mathbb{N}$ and $r \in (0, \pi)$, there exists a constant $K_{M,r}$ such that

$$|b_{r,\epsilon}^r| \leq \min \left\{ b_{0,\epsilon}^r, \frac{K_{M,r}}{\epsilon^{M-\frac{1}{2}+2M+1}} \right\}.$$  

In order to prove Proposition 3.1, we get a bound for the $M$-derivative of $k_{r,\epsilon}$. Since $k_{r,\epsilon}(\mu)$ is a composite function, Faà di Bruno’s formula holds:

$$D^M (\phi_{r,\epsilon}(\text{arccos} \mu)) = M! \sum_{\nu_1+\cdots+\nu_M=M} \frac{(D^\nu \phi_{r,\epsilon})(\text{arccos} \mu)}{\nu_1! \cdots \nu_M!} D^{\nu_1} \text{arccos} \mu \cdots D^{\nu_M} \text{arccos} \mu,$$  

(3.4)

where the second sum is computed on all the possible integer values of $h_1, \ldots, h_M \geq 1$ with sum equal to $M$. We note that this sum is bounded by a constant which depends on $r$; indeed, the arccos is a $C^\infty$ function in each compact subset of $(-1, 1)$ and since outside $[r-\epsilon, r]$ all the derivatives of $\phi$ are zero and $r \neq \pi$, $\mu$ is always different from $+1$ and $-1$; hence the second sum of (3.4) is bounded. As far as the first sum is concerned in (3.4), it is possible to compute it explicitly

$$= \sum_{\nu_1+\cdots+\nu_M=M} \frac{1}{\nu_1! \cdots \nu_M!} \left[ D^\nu \phi_{r,\epsilon}(\text{arccos} \frac{\mu - r + \epsilon}{\epsilon}) \right] (\text{arccos} \frac{\mu - r + \epsilon}{\epsilon})^{\nu_1} \cdots (\text{arccos} \frac{\mu - r + \epsilon}{\epsilon})^{\nu_M} \left[ D^{\nu_1} \text{arccos} \frac{\mu - r + \epsilon}{\epsilon} \right] \left[ D^{\nu_2} \text{arccos} \frac{\mu - r + \epsilon}{\epsilon} \right] \cdots \left[ D^{\nu_M} \text{arccos} \frac{\mu - r + \epsilon}{\epsilon} \right]$$  

(3.5)

Since $\sum_{i=0}^k \binom{2k+1}{i} (\text{arccos} \frac{\mu - r + \epsilon}{\epsilon})^i (r - \text{arccos} \mu)^{2k+1-i}$ is a polynomial in the compact domain $[r-\epsilon, r]$, we can bound (3.5) by $C_{M,r}$, where $C_{M,r}$ is a constant depending on $r$ and $M$. The absolute value of (3.4) satisfy then

$$D^M \phi_{r,\epsilon}(\text{arccos} \mu) \leq \frac{M!C_{M,r}}{\epsilon^{2M+1}}.$$  

(3.6)

We are hence in the position to prove Proposition 3.1.

Proof of Proposition 3.1. We recall the following property of the Legendre polynomials (see for instance 11)

$$(2\ell+1)P_\ell(x) = \frac{d}{dx} \left[ P_{\ell+1}(x) - P_{\ell-1}(x) \right],$$  

(3.7)

and we substitute it in the definition of $b_{r,\epsilon}^r$ to obtain, integrating by parts,

$$\int_{-1}^1 k_{r,\epsilon}(x)P_\ell(x) \, dx = \left[ k_{r,\epsilon}(x) P_{\ell+1}(x) - P_{\ell-1}(x) \right]\frac{d}{dx} \left[ \int_{-1}^1 \frac{d}{dx} k_{r,\epsilon}(x) P_{\ell+1}(x) - P_{\ell-1}(x) \right] \, dx$$  

$$= \frac{1}{2\ell+1} \int_{-1}^1 \frac{d}{dx} k_{r,\epsilon}(x) P_{\ell+1}(x) \, dx - \frac{1}{2\ell+1} \int_{-1}^1 \frac{d}{dx} k_{r,\epsilon}(x) P_{\ell-1}(x) \, dx.$$  

(3.8)

Applying again (3.7) to $P_{\ell+1}$ and to $P_{\ell-1}$ in the place of $P_\ell$ and integrating by parts, one has that (3.8)
Let \(\ell\) and consider Example 3.3.

For any \(M > 0\) and \(\varepsilon > 0\) there exists a function \(1_{B,\varepsilon} \in C^M\) which converges to the indicator function \(1_B(x)\) in \(L^1(S^2)\), as \(\varepsilon \to 0\), such that the coefficients \(b_{\ell,\varepsilon}\) of the Fourier expansion

\[
1_{B,\varepsilon}(\theta) = \sum_{\ell=0}^{\infty} b_{\ell,\varepsilon} \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos \theta), \quad b_{\ell,\varepsilon} = \sqrt{\frac{2\ell+1}{4\pi}} \int_{-1}^{1} 1_{B,\varepsilon}(\arccos x)Y_\ell(x) \, dx
\]

satisfy the condition

\[
|b_{\ell,\varepsilon}| \leq \min \left\{ b_{0,\varepsilon}, \frac{K_{M,\varepsilon}}{\ell^{M-\frac{3}{2}} \varepsilon^{2M+1}} \right\}
\]

as \(\ell \to \infty\), where

\[
K_{M,\varepsilon} = \sqrt{\frac{3}{4\pi}} M! 2^M C M_{\varepsilon, r}.
\]

In the end, this section can be summarised in the theorem below.

**Theorem 3.2.** Let \(B \subset S^2\) be a spherical cap of radius \(r \in (0, \pi)\), parametrized by \(\theta \in [0, r], \varphi \in [0, 2\pi]\). For any \(M > 0\) and \(\varepsilon > 0\) there exists a function \(1_{B,\varepsilon} \in C^M\) which converges to the indicator function \(1_B(x)\) in \(L^1(S^2)\), as \(\varepsilon \to 0\), such that the coefficients \(b_{\ell,\varepsilon}\) of the Fourier expansion

\[
1_{B,\varepsilon}(\theta) = \sum_{\ell=0}^{\infty} b_{\ell,\varepsilon} \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos \theta), \quad b_{\ell,\varepsilon} = \sqrt{\frac{2\ell+1}{4\pi}} \int_{-1}^{1} 1_{B,\varepsilon}(\arccos x)Y_\ell(x) \, dx
\]

satisfy the condition

\[
|b_{\ell,\varepsilon}| \leq \min \left\{ b_{0,\varepsilon}, \frac{K_{M,\varepsilon}}{\ell^{M-\frac{3}{2}} \varepsilon^{2M+1}} \right\}
\]

as \(\ell \to \infty\), where

\[
K_{M,\varepsilon} = \sqrt{\frac{3}{4\pi}} M! 2^M C M_{\varepsilon, r}.
\]

**Example 3.3.** Let us consider \(k = 1\), then \(n = 2k+1 = 3\), \(M = 1\) and \(B_0(t) = (1-t)^3, B_1(t) = 3t(1-t)^2\). It follows that

\[
q(t) = B_0(t) + B_1(t) = 2t^3 - 3t^2 + 1
\]

and

\[
q'(t) = 6t^2 - 6t.
\]

Hence, the first derivative of \(k_{\varepsilon,\varphi}(\mu), \mu \in [-1, 1]\) is

\[
\frac{d}{d\mu} k(\mu) = \frac{d}{d\mu} \phi(\arccos \mu) = \phi'(\arccos \mu) \frac{-1}{\sqrt{1-\mu^2}} = \left[ \frac{6}{\varepsilon^3} (\arccos \mu - r + \varepsilon)^2 - \frac{6}{\varepsilon^3} (\arccos \mu - r + \varepsilon) \right] \frac{-1}{\sqrt{1-\mu^2}} \quad (3.10)
\]

\[
= \frac{6}{\varepsilon^3} (\arccos \mu - r + \varepsilon)(\arccos \mu - r) \frac{-1}{\sqrt{1-\mu^2}}
\]
Accordingly for \( b_{\ell,\varepsilon} \) one obtains

\[
|b_{\ell,\varepsilon}| \leq \frac{1}{2\ell + 1} \left( \frac{2\ell + 1}{4\pi} \right) \int_{-1}^{1} \frac{d}{d\mu} k(\mu)(P_{\ell+1}(x) - P_{\ell-1}(x)) \, dx \leq \frac{C_r}{\ell^{3/2} \varepsilon^3}.
\]

We give some values of \( b_{\ell,\varepsilon} \) in figure 1; the graphic was realized choosing the parameters as \( \varepsilon = \frac{1}{2} \) and \( r = \frac{\pi}{4} \).

![Figure 1: First values of \( b_{\ell,\varepsilon} \) varying \( \ell \).](image)

**Example 3.4.** Choosing \( k = 2 \), one has \( n = 5, M = 2 \) and \( B_0(t) = (1 - t)^5, B_1(t) = 5t(1 - t)^4 \) and \( B_2(t) = 10t^2(1 - t)^3 \). One finds that

\[
q(t) = -6t^5 + 15t^4 - 10t^3 + 1,
\]

\[
q'(t) = -30t^4 + 60t^3 - 30t^2
\]

and

\[
q''(t) = -120t^3 + 180t^2 - 60t.
\]

Then, the first and the second derivatives of \( k_{\varepsilon,r}(\mu) \) are respectively

\[
\frac{d}{d\mu} k(\mu) = \left[ -30\left( \frac{\arccos \mu - r + \varepsilon}{\varepsilon} \right)^4 + 60\left( \frac{\arccos \mu - r + \varepsilon}{\varepsilon} \right)^3 - 30\left( \frac{\arccos \mu - r + \varepsilon}{\varepsilon} \right)^2 \right] \frac{1}{\varepsilon} \frac{-1}{\varepsilon \sqrt{1 - \mu^2}};
\]

\[
\frac{d^2}{d\mu^2} k(\mu) = \left[ -120\left( \frac{\arccos \mu - r + \varepsilon}{\varepsilon} \right)^3 + 180\left( \frac{\arccos \mu - r + \varepsilon}{\varepsilon} \right)^2 - 60\frac{\arccos \mu - r + \varepsilon}{\varepsilon} \right] \frac{1}{\varepsilon^2} \frac{1}{1 - \mu^2} + \left[ -30\left( \frac{\arccos \mu - r + \varepsilon}{\varepsilon} \right)^4 + 60\left( \frac{\arccos \mu - r + \varepsilon}{\varepsilon} \right)^3 - 30\left( \frac{\arccos \mu - r + \varepsilon}{\varepsilon} \right)^2 \right] \frac{1}{\varepsilon} \frac{1}{\varepsilon \sqrt{1 - \mu^2}} \frac{-1}{1 - \mu^2} + \frac{\arccos \mu - r + \varepsilon}{\varepsilon^5} \left[ -120(\arccos \mu - r + \varepsilon)^2 + 180(\arccos \mu - r + \varepsilon)\varepsilon - 60\varepsilon^2 \right] \frac{1}{1 - \mu^2} + \frac{1}{1 - \mu^2} \left( \frac{-\mu}{\sqrt{1 - \mu^2}} \right).
\]

(3.11)
Hence

\[ \left| \frac{d^2}{d\mu^2} k(\mu) \right| \leq \frac{C_{\epsilon}}{\epsilon^5} \]

and

\[ |b_{\ell,\epsilon}| \leq \frac{C_{\epsilon}}{\ell^{3/2} \epsilon^5}. \]

**Remark 3.** In the table and the graphs below, we compare \( b_{\ell,\epsilon} \), for \( \ell = 1, 2, 3, 4, 5 \), for different values of \( \epsilon \) and the assumptions of the example 3.4

| \( \ell \) | \( b_{\ell,1/2} \) | \( b_{\ell,1/4} \) | \( b_{\ell,1/8} \) | \( b_{\ell,1/10} \) |
|---|---|---|---|---|
| 1 | 0.132269 | 0.188425 | 0.218866 | 0.225059 |
| 2 | 0.111278 | 0.147981 | 0.163897 | 0.166747 |
| 3 | 0.084363 | 0.0983641 | 0.0987674 | 0.0982093 |
| 4 | 0.055163 | 0.0493925 | 0.0381274 | 0.0352638 |
| 5 | 0.0294925 | 0.00959262 | -0.00638063 | -0.0097985 |

**Figure 2:** \( b_{\ell,\epsilon} \) varying \( \ell \), for \( \epsilon = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{10} \).

We note that the decay of the coefficients \( b_{\ell,\epsilon} \) is actually faster than given by our upper bound.

**Remark 4.** For the coefficients \( b_{\ell,\epsilon} \) to go to zero, the condition

\[ \ell^{M-1/2} \epsilon^{2M+1} \to \infty, \]

as \( \ell \to \infty \) and \( \epsilon \to 0 \), has to be satisfied.

**Remark 5.** It is quite natural to compare our result with the work of Lang and Schwab in [13]. They define the space \( V^\eta(-1, 1) \) as the closures of \( H^\eta(-1, 1) \), where \( H^\eta(-1, 1) \) is the standard Sobolev spaces, with respect to the weighted norms \( ||u||_{V^\eta(-1, 1)} := \sum_{j=0}^{\infty} ||u||_{H^j(-1, 1)}^2 \), where for \( j \in \mathbb{N}_0 \),

\[ ||u||_{H^\eta(-1, 1)}^2 := \int_{-1}^{1} \left| \frac{\partial^j}{\partial \mu^j} u(\mu) \right|^2 (1 - \mu^2)^\eta \ d\mu, \]

is a seminorm. Denoted as \((\frac{2\ell+1}{2}(1+\ell^2\eta)), \ell \in \mathbb{N}_0 \) the sequence of weights, the authors in [13] show an isomorphism between the spaces \( V^\eta(-1, 1) \) and the spaces of the weights \( \ell_\eta := \ell^2((\frac{2\ell+1}{2}(1+\ell^2\eta)), \ell \in \mathbb{N}. \)

Precisely, if \( u \in L^2(-1, 1) \) and \( \eta \in R_+ \) be given; then \( u \in V^\eta(-1, 1) \) if and only if

\[ \sum_{\ell=0}^{\infty} u^2 \frac{2\ell+1}{2}(1+\ell^2\eta) < \infty; \]
i.e.,

\[ \|u\|_{V^2((-1,1),1)}^2 \sim \sum_{\ell=0}^{\infty} u^2 \frac{2\ell+1}{2} (1 + \ell^2) \]

is an equivalent norm in \( V^2((-1,1)) \). In other words, it states that for \( k(\mu) \in V^2((-1,1),n \in \mathbb{N}_0) \), the sequence \( (\ell^{n+1/2}A_\ell, \ell \geq n) \), with \( A_\ell = 2\pi u_\ell \), is in \( \ell^2(\mathbb{N}_0) \) if and only if \( (1 - \mu^2)^{n/2} \frac{d}{d\mu} k(\mu) \) is in \( L^2((-1,1)) \); namely,

\[ \frac{1}{(4\pi)^2} \sum_{\ell \geq n} A^2 \frac{2\ell+1}{2} \ell^{2n} < +\infty \]

if and only if

\[ \int_{-1}^{1} \left| \frac{d^n}{d\mu^n} k(\mu) \right|^2 (1 - \mu^2)^n d\mu < \infty. \]

More explicitly, in their proof (pag.13 [13]) they get that

\[ \int_{-1}^{1} \left| \frac{d^n}{d\mu^n} k(\mu) \right|^2 (1 - \mu^2)^n d\mu = \sum_{\ell \geq n} A^2 \frac{2\ell+1}{2(4\pi)^2} \frac{(\ell + n)!}{(\ell - n)!} \]  

and

\[ c_1(n) \ell^{2n} \leq \frac{(\ell + n)!}{(\ell - n)!} \leq c_2(n) \ell^{2n}. \]

Although it is possible to compute explicitly the integral on the left hand side of (3.12), this would be sufficient only for a bound on the tail behavior of the series, while we require a full control on any term \( A^2 \).

**Remark 6.** We refer to [12] for the broadly similar construction of a “spherical bump function”. Also, our proposal is in some sense symmetric to so-called needlets, a form of spherical wavelets which is now quite popular in the literature (see i.e., [24], [25] and Chapter 10 of [16]). Indeed, in the standard needlet construction one considers spherical functions with compact support in the harmonic domain and nearly -exponential decay in the real domain, whereas here the converse is studied: functions with compact support in the real domain and polynomial decays in the harmonic space.
4 Proof of the main result

Here we finally prove Theorem 1.1 as described in the introduction we do that by means of the study of the single terms of the chaotic projection (2.4). We divide in small different subsections the results obtained for each components. From now on, \( 1_{B,\varepsilon} \) is the function given in Remark 4.1 and \( \varepsilon > 0 \) is such that

\[
\ell^{M-1}\varepsilon^{M+1} > \ell^2 \quad \text{as} \quad \ell \to \infty.
\]  

(4.1)

The reason of this assumption will be clear later, in the proof of Proposition 2.1.

4.0.1 First chaotic component

The variance of the first chaotic component, i.e., Proposition 2.1 follows as a corollary of the lemma below.

Lemma 4.1. For any \( \varepsilon > 0 \), satisfying (4.1),

\[
\text{Var} \left( \int_{S^2} 1_{B}(x)T_{\ell}(x) \, dx \right) = \frac{4\pi}{2\ell + 1} b_{\ell,\varepsilon}^2 + O(\ell^{-1/2}\varepsilon^{3/2}),
\]  

(4.2)

as \( \ell \to 0 \), where \( b_{\ell,\varepsilon} \) are the Fourier coefficients of \( 1_{B,\varepsilon}(x) \).

Proof of Lemma 4.1 The first chaotic projection can be written as

\[
\int_{B} T_{\ell}(x) \, dx = \int_{S^2} [1_B(x) - 1_{B,\varepsilon}(x)]T_{\ell}(x) \, dx + \int_{S^2} 1_{B,\varepsilon}(x)T_{\ell}(x) \, dx
\]

and consequently, its variance as

\[
\text{Var} \left( \int_{B} T_{\ell}(x) \, dx \right) = \text{Var} \left( \int_{S^2} [1_B(x) - 1_{B,\varepsilon}(x)]T_{\ell}(x) \, dx \right) + \text{Var} \left( \int_{S^2} 1_{B,\varepsilon}(x)T_{\ell}(x) \, dx \right)
\]

(4.3)

\[
+ 2E \left[ \int_{S^2 \times S^2} (1_B(x) - 1_{B,\varepsilon}(x))1_{B,\varepsilon}(y)T_{\ell}(x)T_{\ell}(y) \, dx \, dy \right].
\]

For the first variance of (4.3) it holds that

\[
\text{Var} \left( \int_{S^2} (1_B(x) - 1_{B,\varepsilon}(x))T_{\ell}(x) \, dx \right) = \int_{S^2 \times S^2} (1_B(x) - 1_{B,\varepsilon}(x))(1_B(y) - 1_{B,\varepsilon}(y))E[T_{\ell}(x)T_{\ell}(y)] \, dx \, dy
\]

\[
\leq \int_{S^2} |1_B(x) - 1_{B,\varepsilon}(x)| \left( \int_{S^2} |1_B(y) - 1_{B,\varepsilon}(y)| P_{\ell}(|\langle x, y \rangle|) \, dy \right) \, dx
\]

(4.4)

and applying the Cauchy-Schwarz inequality to the second integral, (4.4) is bounded by

\[
\leq \int_{S^2} |1_B(x) - 1_{B,\varepsilon}(x)| \left( \int_{S^2} |1_B(y) - 1_{B,\varepsilon}(y)|^2 \, dy \right)^{1/2} \left( \int_{S^2} |P_{\ell}(|\langle x, y \rangle|)|^2 \, dy \right)^{1/2} \, dx
\]

(4.5)

\[
\leq \sqrt{\frac{2}{2\ell + 1}} 2\pi \sqrt{2\pi \varepsilon \sqrt{\varepsilon}};
\]

the third term in (4.3) is as small as this one by Cauchy-Schwarz inequality. Concerning the second variance in (4.3), one has

\[
\text{Var} \left( \int_{S^2} 1_{B,\varepsilon}(x)T_{\ell}(x) \, dx \right) = E \left[ \left( \int_{S^2} 1_{B,\varepsilon}(x)T_{\ell}(x) \, dx \right)^2 \right] = \int_{S^2 \times S^2} 1_{B,\varepsilon}(x)1_{B,\varepsilon}(y)E[T_{\ell}(x)T_{\ell}(y)] \, dx \, dy
\]

\[
= \int_{S^2 \times S^2} 1_{B,\varepsilon}(x)1_{B,\varepsilon}(y)P_{\ell}(|\langle x, y \rangle|) \, dx \, dy.
\]  

(4.6)
Through the addition formula \[ \sum_{m=-\ell}^{\ell} Y_{\ell m}(x)Y_{\ell m}(y) = \frac{2\ell + 1}{4\pi} P_\ell((x, y)) \]
and the expansion
\[ 1_{B,\varepsilon}(x) = \sum_{\ell=0}^{\infty} b_{\ell,\varepsilon} Y_{\ell 0}(x), \quad (4.7) \]
it is possible to write (4.6) as
\[ \int_{S^2_x \times S^2_y} \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} b_{\ell,\varepsilon} \sum_{\ell_2=0}^{\infty} b_{\ell_2,\varepsilon} \int_{S^2_x \times S^2_y} Y_{\ell m}(x)Y_{\ell m}(y) \delta_{\ell_2,0}(y) d\mu_d = \int_{S^2_y} Y_{\ell m}(y)Y_{\ell 0}(y) d\mu_d. \quad (4.8) \]
Condition (4.1) implies that the series \( \sum_{\ell=0}^{\ell} b_{\ell,\varepsilon} Y_{\ell 0}(x) \) is absolutely convergent; indeed
\[ \sum \left| b'_{\ell,\varepsilon} \right| |Y_\ell(x)| \sim \sum \left| b'_{\ell,\varepsilon} \right| \sqrt{\ell} < \sum \frac{1}{\ell^2} < \infty, \]
and so we can exchange the series with the integral to derive that (4.8) equals to
\[ \int_{S^2} Y_{\ell m}(x)Y_{\ell m'}(x) d\mu_d = \delta_{\ell,\ell'} \delta_{m, m'}, \quad (4.10) \]
reduces (4.9) to
\[ \int_{S^2} Y_{\ell m}(x)Y_{\ell 0}(x) d\mu_d = \frac{4\pi}{2\ell + 1} b_{\ell,\varepsilon} = \delta_{\ell,0} \delta_{m,0} \quad (4.11) \]
and then the variance (4.6) is
\[ \text{Var} \left( \int_{S^2} 1_{B,\varepsilon}(x)T_\ell(x) d\mu_d \right) = \frac{4\pi}{2\ell + 1} b_{\ell,\varepsilon}; \]
thus (4.11), (4.4) lead to the thesis of the lemma.

Now, Proposition 2.1 is proven by choosing a sequence \( \varepsilon = \varepsilon_\ell \) such that \( \frac{1}{\sqrt{\ell}} \varepsilon_\ell = o\left( \frac{1}{\ell} \right) \).

Remark 7. It is easy to see that
\[ \left| 1_{B,\varepsilon} \right| L_2^2 (S^2) \leq m(S^2) = 4\pi, \]
indeed
\[ \left| 1_{B,\varepsilon} \right| L_2^2 (S^2) = \int_{S^2} \left| 1_{B,\varepsilon}(x) \right|^2 dx = \int_{S^2} \sum_{\ell=0}^{\infty} b_{\ell,\varepsilon} Y_{\ell 0}(x) \sum_{\ell'=0}^{\infty} b_{\ell',\varepsilon} Y_{\ell' 0}(x) dx = \sum_{\ell=0}^{\infty} b_{\ell,\varepsilon} \sum_{\ell'=0}^{\infty} b_{\ell',\varepsilon} \delta_{\ell,\ell'} = \sum_{\ell} b_{\ell,\varepsilon}^2. \quad (4.12) \]
Remark 8. If we set for instance $\varepsilon_\ell = \frac{1}{\ell^\alpha}$, with $\alpha > 0, \alpha \in \mathbb{R}$, the assumption \ref{4.1} becomes
\[
\ell^{M-1} \varepsilon^{-(2M+1)\alpha} > \ell^2 \iff M - 1 - \alpha(2M + 1) > 2 \iff \alpha < \frac{M-3}{2M+1}
\]
and
\[
\frac{1}{\sqrt{\ell^2}} \sqrt{\varepsilon} = \frac{\varepsilon^{-1/2}}{\ell^{\alpha/2}} = o \left( \frac{1}{\ell} \right) \iff \frac{3}{2} \alpha - \frac{1}{2} > 0 \iff \alpha > \frac{1}{3}.
\]
the condition \ref{4.1} is then satisfied for $\alpha > 0$.

\subsection{Second chaotic component}
To the aim of proving the Proposition \ref{2.2} we introduce the two lemmas below, whose proofs can be found in the Appendix.

\begin{lemma}
Under the assumptions of Proposition \ref{2.2} one has that
\[
\text{Var} \left( \int_{S^2} 1_{B,\varepsilon}(x) H_2(T_\ell(x)) \, dx \right) = 8\pi \sum_{\ell_1} b^{2}_{\ell_1;\varepsilon} \frac{1}{2\ell + 1} \left( C^{\ell,0}_{\ell_0;0} \right)^2,
\]
where $\{C^{\ell,0}_{\ell_0;0}\}$ are the Clebsch-Gordan coefficients (see \cite{36} or the Appendix).
\end{lemma}

\begin{lemma}
There exist two strictly positive constants $c_1$ and $c_2$ such that
\[
\frac{c_1}{\ell} \leq \text{Var} \left( \int_{S^2} 1_{B,\varepsilon}(x) H_2(T_\ell(x)) \, dx \right) \leq \frac{c_2}{\ell}
\]
as $\ell \to \infty$.
\end{lemma}

\begin{proof}[Proof of Proposition \ref{2.2}]
The variance of the second chaotic component can be written as
\[
\text{Var} \left[ \int_{S^2} 1_B(x) H_2(T_\ell(x)) \, dx \right] = E \left[ \int_{S^2} (1_B(x) - 1_{B,\varepsilon}(x)) H_2(T_\ell(x)) \, dx \right]^2 + \text{Var} \left[ \int_{S^2} 1_{B,\varepsilon}(x) H_2(T_\ell(x)) \, dx \right] + 2E \left[ \int_{S^2} 1_{B,\varepsilon}(x) \left( 1_B(y) - 1_{B,\varepsilon}(y) \right) H_2(T_\ell(x)) H_2(T_\ell(y)) \, dx \, dy \right].
\]
The first integral in \ref{4.15} is
\[
E \left[ \int_{S^2} (1_B(x) - 1_{B,\varepsilon}(x)) H_2(T_\ell(x)) \, dx \right]^2 = \int_{S^2 \times S^2} (1_B(x) - 1_{B,\varepsilon}(x))(1_B(y) - 1_{B,\varepsilon}(y)) E[H_2(T_\ell(x)) H_2(T_\ell(y))] \, dy \, dx \leq 2 \int_{S^2 \times S^2} |1_B(x) - 1_{B,\varepsilon}(x)|^2 P_\ell^2((x,y)) \, dy \, dx \leq 2C \varepsilon \frac{2}{2\ell + 1},
\]
where $C = 2\pi$ has already been computed in \ref{4.4}, and for the Cauchy-Schwarz inequality, the same bound holds for the third integral in \ref{4.15}. Then, Lemma \ref{4.2} with Lemma \ref{4.3} conclude the proof.
\end{proof}

\begin{remark}
Let us consider the case of the full sphere $B = S^2$, i.e. $1_B(\cdot) = 1_{S^2}(\cdot)$; in this case the approximating function $1_{B,\varepsilon}(\cdot)$ is not necessary. Indeed, the only term of the Fourier expansion of the indicator function $1_{S^2}(\cdot)$ is $\ell_1 = 0$, moreover,
\[
C^{00}_{00;0} = \frac{1}{\sqrt{2\ell + 1}} \quad \text{(see \ref{B.13}),}
\]
\end{remark}
and then
\[ b_0 = 2\pi \int_0^\pi \frac{1}{\sqrt{4\pi}} \sin \theta d\theta = \frac{4\pi}{\sqrt{4\pi}} = \sqrt{4\pi}, \]
so that
\[ \text{Var} \left( \int_B H_2(T_\ell(x)) \, dx \right) = 2! \int_{S^2 \times S^2} P_t(|x,y|)^2 \, dx dy = 2 \cdot 4\pi \frac{1}{2\ell + 1} \sim 16\pi^2 \frac{1}{\ell}, \]
that is exactly the value obtained in Proposition 2.1 of [21].

4.0.3 Terms of the chaotic components for \( q \geq 3 \)

The variance of the third term of (2.2) can be bounded by its absolute value, in fact, considering the full sphere instead of the spherical cap \( B \), one has
\[
\text{Var} \left( \int_B J_3(u) H_3(T_\ell(x)) \, dx \right) = \frac{J_3(u)^2}{3!} \int_B \int_{S^2} P_t(|x,y|)^3 \, dx dy \leq \frac{J_3(u)^2}{3!} \int_B \int_{S^2} |P_t(|x,y|)|^3 \, dx dy
\]
\[
= \frac{J_3(u)^2}{3!} 2\pi m(B) \int_0^{\pi/2} |P_t(\cos \theta)|^3 \sin \theta d\theta = \frac{J_3(u)^2}{3!} 2\pi m(B) \int_0^1 |P_t(x)|^3 \, dx;
\]
the Cauchy-Schwarz inequality implies that (4.17) is
\[
\leq \left( \frac{J_3(u)^2}{3!} 2\pi m(B) \right)^{1/2} \left( \int_0^1 P_t(x)^2 \, dx \right)^{1/2} \left( \int_0^1 P_t(x)^4 \, dx \right)^{1/2} \quad (4.18)
\]
and since it has been proved in [21] and [18] that \( \int_0^1 P_t(x)^2 \, dx = O(\ell^{-1}) \) and \( \int_0^1 P_t(x)^4 \, dx = O\left( \frac{\log \ell}{\ell^2} \right) \), (4.18) has order \( O\left( \frac{\log \ell}{\ell^2} \right) \), as \( \ell \to \infty \).

Likewise, for the variance of the fourth chaotic projection of (2.4), one obtains that
\[
\text{Var} \left( \int_B J_4(u) H_4(T_\ell(x)) \, dx \right) = \frac{J_4(u)^2}{4!} \int_B \int_{S^2} P_t(|x,y|)^4 \, dx dy
\]
\[
\leq \frac{J_4(u)^2}{4!} \int_B \int_{S^2} P_t(|x,y|)^4 \, dx dy
\]
\[
= \frac{J_4(u)^2}{4!} m(B) 2\pi \int_0^1 |P_t(x)|^4 \, dx
\]
which behaves as \( \frac{\log \ell}{\ell^2} \), as \( \ell \to \infty \) [21].

Eventually, for the remaining terms of (2.2), in the same way we get
\[
\text{Var} \left( \int_B \sum_{q=5}^{\infty} \frac{J_q(u)}{q!} H_q(T_\ell(x)) \, dx \right) = E \left[ \int_B \sum_{q=5}^{\infty} \frac{J_q(u)}{q!} H_q(T_\ell(x)) \, dx \right]^2
\]
\[
= \sum_{q=5}^{\infty} \frac{J_q(u)^2}{(q!)^2} \int_B \int_B E[H_q(T_\ell(x))H_q(T_\ell(y))] \, dx dy = \sum_{q=5}^{\infty} \frac{J_q(u)^2}{(q!)^2} \int_B \int_{S^2} |P_t(|x,y|)|^q \, dx dy
\]
\[
\leq \sum_{q=5}^{\infty} \frac{J_q(u)^2}{q!} \int_B \int_{S^2} |P_t(|x,y|)|^q \, dx dy \leq \sum_{q=5}^{\infty} \frac{J_q(u)^2}{q!} \int_B \int_{S^2} |P_t(\cos \theta)|^q \sin \theta d\theta = \sum_{q=5}^{\infty} \frac{J_q(u)^2}{q!} 2\pi m(B) \int_0^1 |P_t(x)|^q \, dx
\]
and \( \int_0^1 |P_t(x)|^q \, dx = O\left( \frac{1}{\ell^q} \right) \) (Lemma 5.7 [21], Proposition 1.1 [18]).
4.0.4 Fourth cumulant of the second chaotic component

At this point, we have established that the second chaos component is the leading term and thus, as anticipated in the introduction, we investigate its fourth cumulant with the purpose of establishing the Central Limit Theorem. In order to prove Proposition 2.3, we observe that (2.6) can be written as

$$\mathbb{E} \left[ \int_{S^2} 1_{B,\varepsilon}(x) H_2(T_\ell(x)) \, dx - \int_{S^2} 1_{B}(x) H_2(T_\ell(x)) \, dx \right]^2 = o\left( \frac{1}{\ell^4} \right) \text{ as } \varepsilon \to 0,$$

so that we can simply study the fourth cumulant of

$$h_{2,\ell}^*(B) = \int_{S^2} 1_{B,\varepsilon}(x) H_2(T_\ell(x)) \, dx,$$

given by

$$\text{cum}(h_{2,\ell}^*(B)) = \int_{S^2} 1_{B,\varepsilon}(x) \int_{S^2} 1_{B,\varepsilon}(z) \int_{S^2} P_\ell(\langle x, y \rangle) P_\ell(\langle y, z \rangle) 1_{B,\varepsilon}(y) \, dy \cdot \int_{S^2} P_\ell(\langle z, w \rangle) P_\ell(\langle w, x \rangle) 1_{B,\varepsilon}(w) \, dw \, dx \, dz. \quad (4.19)$$

Proposition 2.3 follows by showing that (4.19) is an $O\left( \frac{1}{\ell^3} \right)$; such a proof is collected in the Appendix and the tools are, as for the other results already given in this paper, the Gaunt integral and the properties of the Clebsch-Gordan coefficients.

4.0.5 Quantitative Central Limit Theorem

Finally, we prove Theorem 1.1; the argument is quite similar to the one for the full sphere given in [18].

**Proof of Theorem 1.1.** As in [18],

$$S_\ell(M) = \int_B M(T_\ell(x)) \, dx,$$

with

$$M(T_\ell(x)) = 1_{(T_\ell(x)) > 1};$$

now, consider the chaos expansion

$$S_\ell(M) = \int_B \sum_{q=1}^{\infty} \frac{J_q(M) H_q(T_\ell(x))}{q!} \, dx,$$

which we write as

$$S_\ell(M) = J_1(M) h_{\ell,1}(B) + \frac{J_2(M)}{2} h_{\ell,2}(B) + \frac{J_3(M)}{3!} h_{\ell,3}(B) + \frac{J_4(M)}{4!} h_{\ell,4}(B) + \int_B \sum_{q=5}^{\infty} \frac{J_q(M) H_q(T_\ell(x))}{q!} \, dx$$

$$= S_\ell(M; 1) + S_\ell(M; 2)$$

where

$$S_\ell(M; 1) = J_1(M) h_{\ell,1}(B) + \frac{J_2(M)}{2} h_{\ell,2}(B) + \frac{J_3(M)}{3!} h_{\ell,3}(B) + \frac{J_4(M)}{4!} h_{\ell,4}(B),$$

$$S_\ell(M; 2) = \int_B \sum_{q=5}^{\infty} \frac{J_q(M) H_q(T_\ell(x))}{q!} \, dx.$$
Hence, one has that
\[
d_W\left( \frac{S_t(M)}{\sqrt{\text{Var}[S_t(M)]}}, N(0,1) \right) \leq d_W\left( \frac{S_t(M)}{\sqrt{\text{Var}[S_t(M)]}}, \frac{S_t(M;1)}{\sqrt{\text{Var}[S_t(M)]}} \right) + \\
+ d_W\left( \frac{S_t(M;1)}{\sqrt{\text{Var}[S_t(M)]}}, N\left( 0, \frac{\text{Var}[S_t(M;1)]}{\text{Var}[S_t(M)]} \right) \right) + d_W\left( N\left( 0, \frac{\text{Var}[S_t(M;1)]}{\text{Var}[S_t(M)]} \right), N(0,1) \right) \leq \\
\leq \frac{1}{\sqrt{\text{Var}[S_t(M)]}} E\left[ \int_B \left( \sum_{q=5}^{\infty} \frac{J_q(M)H_q(T_q(x))}{q!} \right)^2 dx \right]^{1/2} + \\
+ d_W\left( \frac{S_t(M;1)}{\sqrt{\text{Var}[S_t(M)]}}, N\left( 0, \frac{\text{Var}[S_t(M;1)]}{\text{Var}[S_t(M)]} \right) \right) + d_W\left( N\left( 0, \frac{\text{Var}[S_t(M;1)]}{\text{Var}[S_t(M)]} \right), N(0,1) \right).
\]

(4.20)

We have seen that
\[
\text{Var}(S_t(M;2)) \ll \frac{1}{\ell^2}
\]
and since \( \text{Var}(S_t(M)) \) has the same asymptotic order as the Second Chaos, we have that
\[
\frac{\text{Var}(S_t(M;2))}{\text{Var}(S_t(M))} \ll \frac{1}{\ell};
\]

moreover, the triangular inequality gives
\[
d_W\left( \frac{S_t(M;1)}{\sqrt{\text{Var}[S_t(M)]}}, N\left( 0, \frac{\text{Var}[S_t(M;1)]}{\text{Var}[S_t(M)]} \right) \right) \leq d_W\left( \frac{J_3(M)}{2\sqrt{\text{Var}[S_t(M)]}} h_{\ell;2}(B), N\left( 0, \frac{\text{Var}[S_t(M;1)]}{\text{Var}[S_t(M)]} \right) \right) + \\
+ d_W\left( J_3(M) \frac{h_{\ell;1}(B) + \frac{J_3(M)}{3!\sqrt{\text{Var}[S_t(M)]}} h_{\ell;2}(B) + \frac{J_4(M)}{4!\sqrt{\text{Var}[S_t(M)]}} h_{\ell;4}(B)}{3!\sqrt{\text{Var}[S_t(M)]}}, N\left( 0, \frac{\text{Var}[S_t(M;1)]}{\text{Var}[S_t(M)]} \right) \right).
\]

(4.21)

Thus for the fourth moment Theorem [20], the first term in (4.21) is \( O\left( \frac{1}{\sqrt{\ell}} \right) \), while the second one is
\[
d_W\left( \frac{J_1(M)}{\sqrt{\text{Var}[S_t(M)]}} h_{\ell;1}(B), 0 \right) + d_W\left( \frac{J_3(M)}{3!\sqrt{\text{Var}[S_t(M)]}} h_{\ell;3}(B), 0 \right) + d_W\left( \frac{J_4(M)}{4!\sqrt{\text{Var}[S_t(M)]}} h_{\ell;4}(B), 0 \right); \]

(4.22)

since
\[
d_W\left( \frac{J_1(M)}{\sqrt{\text{Var}[S_t(M)]}} h_{\ell;1}(B), 0 \right) \leq \sqrt{\text{E}}\left[ \left( \frac{J_1(M)}{\sqrt{\text{Var}[S_t(M)]}} h_{\ell;1}(B) \right)^2 \right] = o\left( \frac{1}{\sqrt{\ell}} \right),
\]
\[
d_W\left( \frac{J_3(M)}{3!\sqrt{\text{Var}[S_t(M)]}} h_{\ell;3}(B), 0 \right) \leq \sqrt{\text{E}}\left[ \left( \frac{J_3(M)}{3!\sqrt{\text{Var}[S_t(M)]}} h_{\ell;3}(B) \right)^2 \right] = O\left( \frac{\sqrt{\log \ell}}{\ell^{1/2}} \right)
\]
and
\[
d_W\left( \frac{J_4(M)}{4!\sqrt{\text{Var}[S_t(M)]}} h_{\ell;4}(B), 0 \right) \leq \sqrt{\text{E}}\left[ \left( \frac{J_4(M)}{4!\sqrt{\text{Var}[S_t(M)]}} h_{\ell;4}(B) \right)^2 \right] = O\left( \frac{\log \ell}{\ell^2} \right),
\]

one has that
\[
d_W\left( \frac{S_t(M;1)}{\sqrt{\text{Var}[S_t(M)]}}, N\left( 0, \frac{\text{Var}[S_t(M;1)]}{\text{Var}[S_t(M)]} \right) \right) = O\left( \frac{1}{\sqrt{\ell}} \right).
\]
Finally, Proposition 3.6.1 in [26] leads to
\[ dW \left( N \left( 0, \frac{\text{Var}(S_\ell(M; 1))}{\text{Var}(S_\ell(M))} \right), N(0, 1) \right) \leq \sqrt{\frac{2}{\pi}} \frac{\text{Var}(S_\ell(M; 1))}{\text{Var}(S_\ell(M))} - 1 = O \left( \frac{1}{\ell} \right) \]
and the thesis of the theorem follows.

\[ \square \]

A Technical details

In this section we give all the technical details of the proofs of the propositions and the lemmas whereby the main result has been proved.

A.1 Proof of Lemma 4.2

Proof. The left hand side of (4.13) is given by
\[ \text{Var} \left( \int_{S^2} 1_{B; \varepsilon}(x) H_2(T_\ell(x)) \, dx \right) = E \left[ \left( \int_{S^2} 1_{B; \varepsilon}(x) H_2(T_\ell(x)) \, dx \right)^2 \right] \]
\[ = \int_{S^2 \times S^2} 1_{B; \varepsilon}(x) 1_{B; \varepsilon}(y) E[H_2(T_\ell(x))H_2(T_\ell(y))] \, dxdy \]
\[ = 2! \int_{S^2 \times S^2} 1_{B; \varepsilon}(x) 1_{B; \varepsilon}(y) E[T_\ell(x)T_\ell(y)]^2 \, dxdy \]
\[ = 2! \int_{S^2 \times S^2} 1_{B; \varepsilon}(x) 1_{B; \varepsilon}(y) P_\ell(\langle x, y \rangle)^2 \, dxdy. \]

Along the same lines as the proof of (4.2), we replace \( P_\ell^2(\langle x, y \rangle) \) with \( P_\ell(\langle x, y \rangle) \) and (4.7) to obtain
\[ P_\ell(\langle x, y \rangle)^2 = \left( \frac{4\pi}{2\ell + 1} \right)^2 \sum_{m_1 = -\ell}^{\ell} \sum_{m_2 = -\ell}^{\ell} Y_{\ell m_1}(x) Y_{\ell m_1}(y) Y_{\ell m_2}(x) Y_{\ell m_2}(y) \]
\[ \text{(A.2)} \]
and
\[ \int_{S^2 \times S^2} P_\ell(\langle x, y \rangle)^2 1_{B; \varepsilon}(x) 1_{B; \varepsilon}(y) \, dxdy = \]
\[ \int_{S^2 \times S^2} \left( \frac{4\pi}{2\ell + 1} \right)^2 \sum_{m_1} \sum_{m_2} Y_{\ell m_1}(x) Y_{\ell m_2}(x) Y_{\ell m_2}(y) Y_{\ell m_1}(y) \cdot \sum_{\ell_1} \sum_{\ell_2} b_{\ell_1, \varepsilon} b_{\ell_2, \varepsilon} Y_{\ell_1 0}(x) Y_{\ell_2 0}(y) \, dxdy \]
\[ \text{(A.3)} \]
\[ = \left( \frac{4\pi}{2\ell + 1} \right)^2 \sum_{\ell_1} \sum_{\ell_2} \sum_{m_1} \sum_{m_2} b_{\ell_1, \varepsilon} b_{\ell_2, \varepsilon} \int_{S^2} Y_{\ell_1 m_1}(x) Y_{\ell_2 0}(x) Y_{\ell_2 m_2}(x) \, dx \cdot \int_{S^2} Y_{\ell_1 m_1}(y) Y_{\ell_2 0}(y) Y_{\ell_2 m_2}(y) \, dy. \]
\[ \text{(A.4)} \]
we already justified the exchange between the series and the integral in Lemma 4.1. (A.4) is known as a Gaunt integral and it is given in [16] by the following relation:
\[ \int_{S^2} Y_{\ell_1 m_1}(x) Y_{\ell_2 m_2}(x) \, d\sigma(x) = \sqrt{\frac{(2\ell_1 + 1)(2\ell_2 + 1)}{4\pi(2\ell_3 + 1)}} \frac{c_{\ell_1 m_1 \ell_2 m_2}}{c_{\ell_1 0 \ell_2 0}}, \]
\[ \text{(A.5)} \]
for all \( \ell_1, \ell_2, \ell_3 \), with the convention that \( C_{\ell_1 m_1 \ell_2 m_2} = 0 \) for those integers \( \ell_1, \ell_2, \ell_3 \) not verifying the triangle conditions. Replacing it in (A.4), one has
Recalling that the Clebsch-Gordan coefficients are related to the Wigner 3j coefficients by the identities (see [16] and Appendix):

\[
\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{\ell_3+m_3} \frac{1}{\sqrt{2\ell_3+1}} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{A.7}
\]

and using their permutation property of columns

\[
\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{\ell_1+\ell_2+\ell_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \tag{A.9}
\]

it follows that

\[
\begin{align*}
C_{\ell m_1 \ell_1 0}^{\ell_m 2} &= (-1)^{\ell-\ell_1+m_2} \sqrt{2\ell+1} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ m_1 & m_2 & -m_2 \end{pmatrix} \\
&= (-1)^{\ell-\ell_1+m_2} \sqrt{2\ell+1} (-1)^{\ell+\ell_1+\ell} \begin{pmatrix} \ell & \ell & \ell_1 \\ m_1 & -m_2 & 0 \end{pmatrix} \\
&= (-1)^{\ell+m_2+2\ell} \sqrt{2\ell+1} (-1)^{\ell+2\ell} \frac{1}{\sqrt{2\ell_1+1}} C_{\ell-m_1 \ell m_2}^{\ell_1 0} \\
&= (-1)^{\ell+m_2+\ell_1} \sqrt{2\ell+1} \sqrt{2\ell_1+1} C_{\ell-m_1 \ell m_2}^{\ell_1 0} \tag{A.10}
\end{align*}
\]

and

\[
C_{\ell m_2 \ell_2 0}^{\ell_m 1} = (-1)^{\ell+\ell_1+m_2} \sqrt{2\ell+1} \frac{1}{\sqrt{2\ell_2+1}} C_{\ell-m_2 \ell m_1}^{\ell_2 0},
\]

so equation (A.6) is equal to

\[
\left( \frac{4\pi}{2\ell+1} \right)^2 \sum_{\ell_1} \sum_{\ell_2} \sum_{m_1} \sum_{m_2} b_{\ell_1,2} \sqrt{2\ell_1+1} C_{\ell_0 \ell_1 0}^{\ell 0} \sum_{\ell_2} b_{\ell_2,2} \sqrt{2\ell_2+1} C_{\ell_0 \ell_2 0}^{\ell 0} \cdot
\]

\[
\sum_{m_1 m_2} (-1)^{m_1+m_2} (-1)^{\ell_1+\ell_2} \frac{1}{\sqrt{2\ell_1+1}} \frac{1}{\sqrt{2\ell_2+1}} C_{\ell-m_1 \ell m_2}^{\ell_1 0} C_{\ell-m_2 \ell m_1}^{\ell_2 0} = \tag{A.11}
\]

\[
\begin{align*}
\sum_{m_1 m_2} (-1)^{m_1+m_2} (-1)^{\ell_1+\ell_2} \frac{1}{\sqrt{2\ell_1+1}} \frac{1}{\sqrt{2\ell_2+1}} C_{\ell-m_1 \ell m_2}^{\ell_1 0} C_{\ell-m_2 \ell m_1}^{\ell_2 0} = \\
\sum_{m_1 m_2} (-1)^{m_1+m_2} C_{\ell-m_1 \ell m_2}^{\ell_1 0} C_{\ell-m_2 \ell m_1}^{\ell_2 0}
\end{align*}
\]

and for the triangular condition

\[m_1 - m_2 = 0 \Rightarrow m_1 = m_2\]

and the unitary relation [36]:

\[
\sum_{m_1 m_2} C_{j_1 m_1 j_2 m_2}^{j m} C_{j_1 m_1 j_2 m_2}^{j m'} = \delta^j_j \delta^m_m, \tag{A.12}
\]

23
Proof. The variance in (A.13) is bounded from below by a single term of the series on the right hand side of (A.13), i.e.,

$$\frac{4\pi}{2\ell + 1} \left\{ \sum_{\ell_1} (-1)^{\ell_1} b_{\ell_1;\epsilon} C_{0\ell_0 \ell_1}^{\ell_0} \sum_{\ell_2} (-1)^{\ell_2} b_{\ell_2;\epsilon} C_{\ell_1 \ell_2}^{\ell_0} \right\} \delta^{\ell_2}_{\ell_1},$$

(A.13)

As in (A.10) one has

$$C_{0\ell_0 \ell_1}^{\ell_0} = (-1)^{\ell - \ell_1} \sqrt{2\ell + 1} \left( \begin{array}{ccc} \ell & \ell_1 & \ell \\ 0 & 0 & 0 \end{array} \right)$$

$$= (-1)^{\ell - \ell_1} \sqrt{2\ell + 1} (-1)^{2\ell_1 + \ell} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & \ell_1 \end{array} \right)$$

$$= (-1)^{\ell + \ell_1} \sqrt{2\ell + 1} (-1)^{\ell_1 + 2\ell} \frac{1}{\sqrt{2\ell_1 + 1}} C_{\ell_1 \ell_0}^{\ell_0}$$

(A.14)

and then (A.13) is

$$\frac{4\pi}{2\ell + 1} \left\{ \sum_{\ell_1} (-1)^{\ell_1} b_{\ell_1;\epsilon} (-1)^{\ell + \ell_1} \sqrt{2\ell + 1} \frac{1}{\sqrt{2\ell_1 + 1}} C_{0\ell_0 \ell_1}^{\ell_0} \sum_{\ell_2} (-1)^{\ell_2} (-1)^{\ell_2} \frac{1}{\sqrt{2\ell_2 + 1}} C_{\ell_2 \ell_0}^{\ell_0} \right\} \delta^{\ell_2}_{\ell_1}$$

$$= 4\pi \sum_{\ell_1} b_{\ell_1;\epsilon} \frac{1}{2\ell_1 + 1} C_{0\ell_0}^{\ell_0} \sum_{\ell_2} \frac{1}{2\ell_2 + 1} b_{\ell_2;\epsilon} C_{\ell_1 \ell_2}^{\ell_0}$$

$$= 4\pi \left\{ \sum_{\ell_1} b_{\ell_1;\epsilon} \frac{1}{2\ell_1 + 1} C_{0\ell_0}^{\ell_0} \right\}^2$$

$$= 4\pi \sum_{\ell_1} b_{\ell_1;\epsilon}^2 \frac{1}{2\ell_1 + 1} \left( C_{\ell_1 \ell_0}^{\ell_0} \right)^2,$$

(A.15)

where the last step is due to the previous property (A.12) with $m_1 = m_2 = m_3 = 0$; finally Lemma 4.2 is proven.

A.2 Proof of Lemma 4.3

Proof. The variance in (A.13) is bounded from below by a single term of the series on the right hand side of (A.13), i.e.,

$$8\pi b_{\ell_1;\epsilon}^2 \frac{1}{2\ell_1 + 1} \left( C_{\ell_1 \ell_0}^{\ell_0} \right)^2,$$

for a fixed $\ell_1$ of the sum; for instance $\ell_1 = 0$, i.e.,

$$\text{Var} \left[ \int_{S^2} A_{\ell_1}(x) H_2(T_{\ell_1}(x)) \, dx \right] = 8\pi \sum_{\ell_1} b_{\ell_1;\epsilon}^2 \frac{1}{2\ell_1 + 1} \left( C_{\ell_1 \ell_0}^{\ell_0} \right)^2 \geq 8\pi b_{\ell_1;\epsilon}^2 \left( C_{\ell_1 \ell_0}^{\ell_0} \right)^2 = 8\pi b_{\ell_1;\epsilon}^2 \frac{1}{2\ell_1 + 1},$$

by the property

$$C_{\ell_1 m_1 \ell_2 m_2}^{00} = (-1)^{\ell_1 - m_1} \frac{\delta^{\ell_2}_{\ell_1} \delta^{-m_2}}{\sqrt{2\ell_1 + 1}}$$

(A.16)

(see [36]). To find an upper bound, it is sufficient to recall that for any $\ell_1, \ell_2, \ell_3$,

$$\left| \begin{array}{ccc} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{array} \right| \leq |\max\{2\ell_1 + 1, 2\ell_2 + 1, 2\ell_3 + 1\}|^{-1/2}$$

(A.17)

(see [16] pag.110) so that

$$|C_{\ell_1 m_1 \ell_2 m_2}^{\ell_3 m_3}| \leq \sqrt{2\ell_1 + 1 |\max\{2\ell_1 + 1, 2\ell_2 + 1, 2\ell_3 + 1\}|^{-1/2}}$$

(A.18)
and then, it is easy to see that
\[ 8\pi \sum_{\ell_1} b_{\ell_1;\varepsilon}^2 \frac{1}{2\ell_1 + 1} \left( C_{\ell_1 0 0 0}^{\ell_1;\varepsilon} \right)^2 \leq 8\pi \sum_{\ell_1} b_{\ell_1;\varepsilon}^2 \frac{1}{2\ell_1 + 1} \frac{2\ell_1 + 1}{2\ell_1 + 1} = \frac{8\pi}{2\ell_1 + 1} \sum_{\ell_1} b_{\ell_1;\varepsilon}^2. \]

The series is finite by Remark 7. In conclusion, (4.13) is bounded above and below by
\[ 8\pi b_{0;\varepsilon}^2 \frac{1}{2\ell + 1} \leq 8\pi \sum_{\ell_1} b_{\ell_1;\varepsilon}^2 \frac{1}{2\ell_1 + 1} \left( C_{\ell_1 0 0 0}^{\ell_1;\varepsilon} \right)^2 \leq \frac{8\pi}{2\ell_1 + 1} \sum_{\ell_1} b_{\ell_1;\varepsilon}^2 \leq \frac{8\pi}{2\ell + 1} m_1 S^2 = \frac{8\pi}{2\ell + 1} 4\pi \tag{A.19} \]
and because of Remark 2 the lemma is proven. \( \square \)

### A.3 Proof of Proposition 2.3

**Proof.** The purpose here is to compute the fourth cumulant in (4.19); putting together (2.9) and (2.11) in the last equation, we obtain four Gaunt integrals and formula (3.64) in [16] implies that (2.3) is

\[ \text{Proof.} \]

\[ \text{A.3 Proof of Proposition 2.3} \]

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\[ (4.19) \]

\[ (2.9) \]

\[ (2.11) \]

\[ \text{the triangular condition implies that } m_1 = m_2 = m_3 = m_4, \text{ so that (A.20) is} \]

\[ (A.20) \]

\[ (A.21) \]

\[ \text{Besides, for the symmetry properties (B.5 or [36]), one has that} \]

\[ C_{\ell m_1 \ell_1 0}^{\ell m_1} = (-1)^{\ell - m_1} \sqrt{\frac{2\ell + 1}{2\ell_1 + 1}} C_{\ell m_1 \ell_1 0}^{\ell m_1} \]

\[ \text{and then} \]

\[ \sum_{m_1 = -\ell}^\ell C_{\ell m_1 \ell_1 0}^{\ell m_1} C_{\ell m_1 \ell_2 0}^{\ell m_1} C_{\ell m_1 \ell_3 0}^{\ell m_1} C_{\ell m_1 \ell_4 0}^{\ell m_1} = \]

\[ \sum_{m_1 = -\ell}^\ell (-1)^{4(\ell - m_1)} \sqrt{\frac{(2\ell + 1)^4}{(2\ell_1 + 1)(2\ell_2 + 1)(2\ell_3 + 1)(2\ell_4 + 1)}} C_{\ell m_1 \ell_1 0}^{\ell m_1} C_{\ell m_1 \ell_2 0}^{\ell m_1} C_{\ell m_1 \ell_3 0}^{\ell m_1} C_{\ell m_1 \ell_4 0}^{\ell m_1}. \tag{A.22} \]
By (B.14) and (B.15) \((36)\), \((A.22)\) becomes

\[
= \left(\frac{4\pi}{2\ell + 1}\right)^2 \sum_{\ell_1 = -\ell}^{\ell} b_{\ell_1 z} C_{00\ell_1}^{\ell_0} \sum_{\ell_2 = 0}^{\ell} b_{\ell_2 z} C_{00\ell_2}^{\ell_0} \sum_{\ell_3 = -\ell}^{\ell} b_{\ell_3 z} C_{00\ell_3}^{\ell_0} \sum_{\ell_4 = -\ell}^{\ell} b_{\ell_4 z} C_{00\ell_4}^{\ell_0} \prod_{\ell_1, \ell_2, \ell_3, \ell_4} C_{\ell_1\ell_2\ell_3\ell_4}^{HL00} C_{\ell_2\ell_3\ell_4}^{HL00} \left\{ \frac{\ell}{\ell_1} \frac{\ell}{\ell_2} \frac{\ell}{\ell_3} \right\},
\]

(A.23)

and since the triangular condition implies \(j = 0\), (A.23) gives

\[
= \left(\frac{4\pi}{2\ell + 1}\right)^2 \sum_{\ell_1 = -\ell}^{\ell} b_{\ell_1 z} C_{00\ell_1}^{\ell_0} \sum_{\ell_2 = 0}^{\ell} b_{\ell_2 z} C_{00\ell_2}^{\ell_0} \sum_{\ell_3 = -\ell}^{\ell} b_{\ell_3 z} C_{00\ell_3}^{\ell_0} \sum_{\ell_4 = -\ell}^{\ell} b_{\ell_4 z} C_{00\ell_4}^{\ell_0} \prod_{\ell_1, \ell_2, \ell_3, \ell_4} C_{\ell_1\ell_2\ell_3\ell_4}^{HL00} \left\{ \frac{\ell}{\ell_1} \frac{\ell}{\ell_2} \frac{\ell}{\ell_3} \frac{\ell}{\ell_4} \right\}. \tag{A.24}
\]

In view of equation (B.5), (A.24) reduces to

\[
= (4\pi)^2 \sum_{\ell_1 = 0}^{\infty} b_{\ell_1 z} C_{00\ell_1}^{\ell_0} \sum_{\ell_2 = 0}^{\infty} b_{\ell_2 z} C_{00\ell_2}^{\ell_0} \sum_{\ell_3 = 0}^{\infty} b_{\ell_3 z} C_{00\ell_3}^{\ell_0} \sum_{\ell_4 = 0}^{\infty} b_{\ell_4 z} C_{00\ell_4}^{\ell_0} \prod_{k} C_{\ell_1\ell_2\ell_3\ell_4}^{HL00} \left\{ \frac{\ell}{\ell_1} \frac{\ell}{\ell_2} \frac{\ell}{\ell_3} \frac{\ell}{\ell_4} \right\}. \tag{A.25}
\]

In order to simplify the notation, we define this last expression as \(A_{\ell, k}(\ell_1, \ell_2, \ell_3, \ell_4)\). We split it in different cases and we study them separately; hence, we rewrite (A.25) as

\[
=A_{\ell, k}(\ell_1, \ell_2, \ell_3, \ell_4) + A_{\ell, k}(0, 0, 0, 0) + A_{\ell, k}(\ell_1, 0, \ell_3, \ell_4) + A_{\ell, k}(0, \ell_2, \ell_3, \ell_4) + A_{\ell, k}(\ell_1, \ell_2, 0, \ell_4) + A_{\ell, k}(0, \ell_2, \ell_3, 0) + 2A_{\ell, k}(0, 0, \ell_3, \ell_4) + 2A_{\ell, k}(0, 0, 0, \ell_4) + 2A_{\ell, k}(0, 0, 0, 0) + 2A_{\ell, k}(\ell_1, 0, \ell_3, \ell_4) + 2A_{\ell, k}(\ell_1, \ell_2, 0, 0).
\]

(A.26)

Note that

\[
A_{\ell, k}(\ell_1, \ell_2, \ell_3, \ell_4) :=
\]

\[
(4\pi)^2 \sum_{\ell_1 = 1}^{\infty} b_{\ell_1 z} C_{00\ell_1}^{\ell_0} \sum_{\ell_2 = 1}^{\infty} b_{\ell_2 z} C_{00\ell_2}^{\ell_0} \sum_{\ell_3 = 1}^{\infty} b_{\ell_3 z} C_{00\ell_3}^{\ell_0} \sum_{\ell_4 = 1}^{\infty} b_{\ell_4 z} C_{00\ell_4}^{\ell_0} \prod_{k} C_{\ell_1\ell_2\ell_3\ell_4}^{HL00} \left\{ \frac{\ell}{\ell_1} \frac{\ell}{\ell_2} \frac{\ell}{\ell_3} \frac{\ell}{\ell_4} \right\}, \tag{A.27}
\]

\[
A_{\ell, k}(0, 0, 0, 0) = +(4\pi)^2 b_{0 z} C_{0000}^{00} C_{0000}^{00} C_{0000}^{00} \left\{ \frac{\ell}{0} \frac{\ell}{0} \frac{\ell}{0} \right\},
\]

\[
A_{\ell, k}(0, \ell_2, \ell_3, \ell_4) = (4\pi)^2 b_{0 z} C_{0000}^{00} \sum_{\ell_2, \ell_3, \ell_4} b_{\ell_2 z} C_{00\ell_2}^{\ell_0} b_{\ell_3 z} C_{00\ell_3}^{\ell_0} b_{\ell_4 z} C_{00\ell_4}^{\ell_0} \prod_{k} C_{\ell_2\ell_3\ell_4}^{HL00} \left\{ \frac{\ell}{\ell_2} \frac{\ell}{\ell_3} \frac{\ell}{\ell_4} \right\},
\]

(10) and similar expressions hold for \(A_{\ell, k}(\ell_1, 0, \ell_3, \ell_4), A_{\ell, k}(\ell_1, \ell_2, 0, \ell_4), A_{\ell, k}(\ell_1, \ell_2, \ell_3, 0)\); for example

\[
A_{\ell, k}(0, 0, \ell_3, \ell_4) = (4\pi)^2 b_{0 z} C_{0000}^{00} \sum_{\ell_3, \ell_4} b_{\ell_3 z} C_{00\ell_3}^{\ell_0} b_{\ell_4 z} C_{00\ell_4}^{\ell_0} \prod_{k} C_{\ell_3\ell_4}^{HL00} \left\{ \frac{\ell}{0} \frac{\ell}{\ell_3} \frac{\ell}{\ell_4} \right\}.
\]
All the terms with three indexes among \( \ell_1, \ell_2, \ell_3, \ell_4 \) equal to zero, are zero for the triangular condition, in fact, if we look at the term

\[
3(4\pi)^2 (b_{0,2} C_{000}^{00})^3 \sum_{\ell_3 = 1}^{\infty} b_{\ell_3} C_{\ell_3000}^{\ell_300} \sum_k C_{k000}^{k000} C_{000}^{k00} \begin{pmatrix} \ell & \ell & 0 \\ \ell & \ell & 0 \\ 0 & \ell_3 & k \end{pmatrix},
\]

in the last sum the Clebsch-Gordan coefficient \( C_{k000}^{k000} \) is different from zero only if \( \ell_3 = 0 \), a possibility not allowed in the current series. As far as concerns \( A_{\ell,k}(0,0,0,0) \), for the symmetry properties of the 9j symbols \(^{[36]}\), one has

\[
\begin{pmatrix} \ell & \ell & 0 \\ \ell & \ell & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2\ell + 1}.
\]

Now we look at the term \( A_{\ell,k}(0,0,0,0) \); for the triangular condition the only term in the sum in \( k \) which does not vanish is \( k = \ell_2 \) and for the symmetry properties of the Wigner 9j coefficients and for \(^{[B.19]}\), it follows that

\[
\begin{pmatrix} \ell & \ell & 0 \\ \ell & \ell & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{(-1)^{\ell_3 + \ell_2}}{[(2\ell_2 + 1)(2\ell + 1)]^{1/2}} \begin{pmatrix} \ell_3 & \ell_4 & \ell_2 \\ \ell & \ell & 0 \\ \ell & \ell & 0 \end{pmatrix}.
\]

Likewise,

\[
\begin{pmatrix} \ell & \ell & 1 \\ \ell & \ell & 0 \\ \ell & \ell & 0 \end{pmatrix} = \frac{(-1)^{\ell_3 + \ell_1}}{[(2\ell_1 + 1)(2\ell + 1)]^{1/2}} \begin{pmatrix} \ell_3 & \ell_4 & \ell_1 \\ \ell & \ell & 0 \\ \ell & \ell & 0 \end{pmatrix} = \frac{(-1)^{\ell_3 + \ell_1}}{[(2\ell_1 + 1)(2\ell + 1)]^{1/2}} \begin{pmatrix} \ell & \ell & \ell \\ \ell & \ell & \ell \\ \ell & \ell & \ell \end{pmatrix},
\]

where the last equality is due to the invariance under permutation of the Wigner 6j coefficients. Similarly,

\[
\begin{pmatrix} \ell & \ell & 1 \\ \ell & \ell & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{(-1)^{\ell_2 + \ell_4}}{[(2\ell_4 + 1)(2\ell + 1)]^{1/2}} \begin{pmatrix} \ell_2 & \ell_4 & \ell_4 \\ \ell & \ell & \ell \\ \ell & \ell & \ell \end{pmatrix}
\]

and

\[
\begin{pmatrix} \ell & \ell & 1 \\ \ell & \ell & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{(-1)^{\ell_1 + \ell_3}}{[(2\ell_3 + 1)(2\ell + 1)]^{1/2}} \begin{pmatrix} \ell & \ell & \ell \\ \ell & \ell & \ell \\ \ell & \ell & \ell \end{pmatrix}.
\]

Regarding \( A_{\ell,k}(0,0,0,0) \), for the triangular condition, the only term of the sum in \( k \) which is non-zero is \( k = 0 \) and the symmetry properties of 9j symbol and the relation \(^{[B.21]}\) imply

\[
\begin{pmatrix} \ell & \ell & 0 \\ \ell & \ell & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{\delta_{\ell_3}^{\ell_4}}{[(2\ell_4 + 1)(2\ell + 1)]^{1/2}}.
\]

The same properties give

\[
\begin{pmatrix} \ell & \ell & 0 \\ \ell & \ell & 0 \\ \ell & \ell & 0 \end{pmatrix} = \frac{1}{[(2\ell_2 + 1)(2\ell + 1)]^{1/2}} \begin{pmatrix} \ell & \ell & \ell \\ 0 & 0 & 0 \end{pmatrix}.
\]

27
Therefore, remembering that \( \pi \) is odd, the series only run on even indices, this implies \( (-1)^{\ell_2+\ell_4} = 1 \) and \( (-1)^{\ell_3} = 1 \).

Finally, equation (A.26) reduces to

\[
\sum_{\ell} h^{\ell}_{\ell_2} = A_{\ell, k}(\ell_1, \ell_2, \ell_3, \ell_4) + A_{\ell, k}(0, 0, 0, 0) + 4A_{\ell, k}(0, \ell_2, \ell_3, \ell_4) + 12A_{\ell, k}(0, 0, \ell_3, \ell_4).
\]  

(A.30)
The aim now is to understand the asymptotic behavior of expression (A.30); by the results of the second chaotic component, it is easily seen that the last summand of (A.30) is $O\left(\frac{1}{\ell^2}\right)$, as $\ell \to \infty$. Concerning the third term of the same equation, because of (B.17), the following upper bound holds

$$\left\{ \ell_3 \ell_4 \ell_2 \ell \right\} \leq \frac{1}{\sqrt{2\ell + 1}} \min \left( \frac{1}{\sqrt{2\ell + 1}}, \frac{1}{\sqrt{2\ell + 3}}, \frac{1}{\sqrt{2\ell + 4}} \right)$$

(A.31)

and from (B.11),

$$|C_{\ell_00}| \leq \frac{\sqrt{2\ell + 1}}{\sqrt{2\ell + 1}}.$$  

(A.32)

taking the absolute value, one has that $A_{\ell,k}(0, \ell_2, \ell_3, \ell_4)$ is bounded by

$$\leq (4\pi)^2 b_{00} \left(\frac{-1}{2\ell + 1}\right) \sum_{\ell_2, \ell_3, \ell_4 = 1}^{\ell} b_{\ell_2} C_{0000}^{\ell_2} b_{\ell_3} C_{0000}^{\ell_3} b_{\ell_4} C_{0000}^{\ell_4} \frac{1}{\sqrt{2\ell + 1}} \left\{ \ell_3 \ell_4 \ell_2 \ell \right\}$$

$$\leq (4\pi)^2 b_{00} \left(\frac{1}{2\ell + 1}\right) \sum_{\ell_2, \ell_3, \ell_4 = 1}^{\ell} |b_{\ell_2} b_{\ell_3} b_{\ell_4}| \frac{\sqrt{2\ell + 1}}{\sqrt{2\ell + 1}} \frac{\sqrt{2\ell + 1}}{\sqrt{2\ell + 1}} \frac{\sqrt{2\ell + 1}}{\sqrt{2\ell + 1}}$$

$$\times \min \left( \frac{1}{\sqrt{2\ell + 1}}, \frac{1}{\sqrt{2\ell + 1}}, \frac{1}{\sqrt{2\ell + 1}} \right).$$

(A.33)

We already have discussed the absolute convergence of the series, which allows us to say that (A.33) is $O\left(\frac{1}{\ell^2}\right)$ as $\ell \to \infty$.

It remains to study the term $A_{\ell,k}(\ell_1, \ell_2, \ell_3, \ell_4)$: its absolute value can be bounded by

$$\sum_{\ell_3=1}^{\ell} |C_{\ell_0000}^{\ell_3} C_{\ell_0000}^{\ell_3} \sum_{\ell_4=1}^{\ell} \sum_{\ell_4=1}^{\ell} \sum_{\ell_4=1}^{\ell} |b_{\ell_4}| \sqrt{(2\ell + 1)(2\ell + 1)(2\ell + 1)(2\ell + 1)} \frac{1}{(2\ell + 2)^2}$$

$$\leq \left\{ \sum_{\ell_4=1}^{\ell} C_{\ell_0000}^{\ell_4} C_{\ell_0000}^{\ell_4} \left\{ \ell \ell \ell \ell_1 \right\} \right\} \leq \left\{ \ell \ell \ell \ell_2 \right\}.$$ 

(A.34)

For equation (B.15) one has

$$\sum_{s=0}^{(s+1)} \sqrt{(2\ell + 1)(2\ell + 1)(2\ell + 1)(2\ell + 1)}$$

$$= \frac{1}{\sqrt{(2\ell + 1)(2\ell + 1)(2\ell + 1)(2\ell + 1)}} \left\{ \ell \ell \ell \ell_1 \right\} \leq \left\{ \ell \ell \ell \ell_2 \right\}.$$ 

(A.35)

the triangular condition implies $\sigma = 0$, so that (A.35) is

$$= \frac{1}{\sqrt{(2\ell + 1)(2\ell + 1)(2\ell + 1)(2\ell + 1)}} \times \sum_s \sqrt{(2\ell + 1)(2\ell + 1)(2\ell + 1)(2\ell + 1)} \left\{ \ell \ell \ell \ell_1 \right\} \leq \left\{ \ell \ell \ell \ell_2 \right\}.$$ 

(A.36)

and thanks to the fact that
and since the series are absolutely convergent the proof is completed.

\[
\sum_{s} (2s + 1)C_{\ell_{1}000}^{000}C_{\ell_{0}000}^{000} \leq \left( \sum_{s} (2s + 1)C_{\ell_{1}000}^{000} \right)^{2} \left( \sum_{s} (2s + 1)C_{\ell_{0}000}^{000} \right)^{2}^{1/2}
\]

and since \(\sum_{s} (2s + 1)C_{\ell_{1}000}^{000}C_{\ell_{0}000}^{000} = 1\), and the permutations properties

\[
1 = \sum_{\ell} (C_{\ell,000}^{000})^{2} = \sum_{\ell} \left( \frac{2\ell + 1}{2\ell_{2} + 1} \right) (C_{\ell,000}^{000})^{2},
\]

it entails that

\[
\sum_{\ell} (2\ell + 1)(C_{\ell,000}^{000})^{2} = 2\ell_{2} + 1
\]

and then, the right hand side of (A.39) is equal to

\[
= \sqrt{2\ell_{4} + 1}\sqrt{2\ell_{2} + 1}.
\]

Eventually, \(A_{\ell,k}(\ell_{1}, \ell_{2}, \ell_{3}, \ell_{4})\) is smaller than

\[
(4\pi)^{2} \sum_{\ell_{1}=1}^{\infty} b_{\ell_{1},0}C_{\ell_{1}000}^{000} \sum_{\ell_{2}=1}^{\infty} b_{\ell_{2},0}C_{\ell_{2}000}^{000} \sum_{\ell_{3}=1}^{\infty} b_{\ell_{3},0}C_{\ell_{3}000}^{000} \sum_{\ell_{4}=1}^{\infty} b_{\ell_{4},0}C_{\ell_{4}000}^{000} \sum_{\ell_{4}000}^{\infty} b_{\ell_{4},0}C_{\ell_{4}000}^{000} C_{\ell_{4}000}^{000} \left\{ \ell \ell \ell \ell \right\} \leq (4\pi)^{2} \left( \frac{1}{2\ell + 1} \right) \cdot \min \left( \frac{1}{\sqrt{2\ell_{4} + 1}} \right)
\]

and since (A.32), one gets

\[
\sum_{s} (2s + 1)C_{\ell_{1}000}^{000}C_{\ell_{0}000}^{000} \leq \left( \sum_{s} (2s + 1)C_{\ell_{1}000}^{000} \right)^{2} \left( \sum_{s} (2s + 1)C_{\ell_{0}000}^{000} \right)^{2}^{1/2}
\]

In view of the Cauchy-Schwarz inequality, it follows that

\[
\min \left( \frac{1}{\sqrt{2\ell_{4} + 1}} \right)^{2} \cdot \min \left( \frac{1}{\sqrt{2\ell_{4} + 1}} \right) \cdot \min \left( \frac{1}{\sqrt{2\ell_{4} + 1}} \right) \leq 1.
\]
B  The Clebsch-Gordan coefficients

For completeness, in this Appendix we recall some basic facts and properties about the Clebsch-Gordan coefficients, which we used in our proofs above (we refer to [36] for further properties and details). The Clebsch-Gordan coefficients are important tools for the evaluation of multiple integrals of spherical harmonics. For \( SO(3) \) they are defined as the set \( \{ C_{\ell_1 \ell_2 \ell_3}^{m_1 m_2 m_3} \} \) of the elements of the unitary matrices \( C_{\ell_1 \ell_2} \) (chap 2.4.2 [16]); the Clebsch-Gordan coefficients vanish unless the Triangular condition

\[
|\ell_1 - \ell_2| \leq \ell_3 \leq \ell_1 + \ell_2,
\]

and the equation

\[
m_1 + m_2 = m_3
\]

are satisfied.

The following orthogonal conditions hold ([36]):

\[
\sum_{m_1 m_2} C_{\ell_1 \ell_2 \ell_3}^{m_1 m_2} C_{\ell_1 \ell_2 \ell_3}^{m_1 m_2} = \delta_{\ell_1} \delta_{\ell_2} \delta_{\ell_3}, \tag{B.1}
\]

\[
\sum_{\ell m} C_{\ell_1 \ell_2 \ell_3}^{m_1 m_2} C_{\ell_1 \ell_2 \ell_3}^{m_1 m_2} = \delta_{m_1} \delta_{m_2}. \tag{B.2}
\]

For \( m_1 + m_2 + m_3 = 0 \), an analytic expression is known:

\[
C_{\ell_1 \ell_2 \ell_3}^{m_1 m_2 m_3} := (-1)^{l_1 + m_1} \sqrt{2l_3 + 1} \left[ \frac{(l_1 + l_2 - l_3)!(l_1 - l_2 + l_3)!(l_1 - l_2 + l_3)!}{(l_1 + l_2 + l_3 + 1)!} \right]^{1/2} \times \left[ \frac{(l_3 + m_3)!(l_3 - m_3)!}{(l_1 + m_1)!(l_1 - m_1)!(l_2 + m_2)!(l_2 - m_2)!} \right]^{1/2} \times \sum_{z} z!((l_2 + l_3 - l_1 - z)!(l_3 + m_3 - z)!(l_1 - l_2 - m_3 + z)!)
\]

where the summation runs over all \( z \)'s such that the factorials are non-negative. When \( m_1 = m_2 = m_3 = 0 \), this expression becomes plainer

\[
C_{\ell_1 \ell_2 \ell_3}^{000} = \begin{cases} 
0, & \text{for } l_1 + l_2 + l_3 \text{ odd} \\
\frac{(-1)^{l_1 + l_2 + l_3}}{(l_1 + l_2 - l_3)/2!(l_1 - l_2 + l_3)/2!(l_1 + l_2 + l_3)/2!} \frac{(l_1 - l_2 - l_3)!(l_1 - l_2 + l_3)!}{(l_1 + l_2 + l_3)!} \end{cases} \left[ \frac{(l_1 + l_2 + l_3)!}{2l_3 + 1} \right]^{1/2}, \tag{B.4}
\]

We summarise below some basic properties.

- **Symmetry Properties:**

\[
C_{\ell_1 \ell_2 \ell_3}^{m_1 m_2 m_3} = (-1)^{\ell_1 + \ell_2 - \ell_3} C_{\ell_2 \ell_3 m_1}^{m_1 m_2} = (-1)^{\ell_1 - m_1} \sqrt{2l_3 + 1} C_{\ell_1 \ell_2 \ell_3 - m_3}^{m_1 m_2} = (-1)^{\ell_1 - m_1} \sqrt{2l_2 + 1} C_{\ell_1 \ell_2 \ell_3 - m_3}^{m_1 m_2} \tag{B.5}
\]

- **Mirror Properties:**

\[
C_{\ell_1 \ell_2 \ell_3}^{m_1 m_2 m_3} = (-1)^{\ell_1 + \ell_2 - \ell_3} C_{\ell_1 \ell_2 \ell_3 - m_3}^{m_1 m_2} \tag{B.6}
\]

\[
C_{\ell_1 \ell_2 \ell_3}^{m_1 m_2 m_3} = C_{\ell_1 \ell_2 \ell_3}^{m_2 m_1 m_3} \tag{B.7}
\]
B.1 Wigner 3j coefficients

Wigner 3j coefficients are related to the Clebsch-Gordan coefficients by the identities

\[
\begin{align*}
\frac{l_1}{m_1} & \quad \frac{l_2}{m_2} & \quad \frac{l_3}{m_3} &= (-1)^{\ell+3m_3} \frac{1}{\sqrt{2\ell_3 + 1}} C_{\ell_3}^{\ell_1 \ell_2 m_3} \\
C_{\ell_1 \ell_2 m_3}^{\ell_3 m_1 m_2} &= (-1)^{\ell_1 + \ell_2 + m_3} \sqrt{2\ell_3 + 1} \left( \begin{array}{ccc}
l_1 & l_2 & l_3 \\
m_1 & m_2 & m_3
\end{array} \right).
\end{align*}
\]

From [16], we have that for any \(\ell_1, \ell_2, \ell_3\), the following upper bound holds

\[
\left| \begin{array}{ccc}
l_1 & l_2 & l_3 \\
m_1 & m_2 & m_3
\end{array} \right| \leq \left[ \max\{2\ell_1 + 1, 2\ell_2 + 1, 2\ell_3 + 1\} \right]^{-1/2},
\]

then

\[
|C_{\ell_1 \ell_2 m_3}^{\ell_3 m_1 m_2}| \leq \sqrt{2\ell_3 + 1}\left[ \max\{2\ell_1 + 1, 2\ell_2 + 1, 2\ell_3 + 1\} \right]^{-1/2}
\]

As the Clebsch-Gordan coefficients they satisfy some symmetry properties; such as

\[
\left( \begin{array}{ccc}
l_1 & l_2 & l_3 \\
m_1 & m_2 & m_3
\end{array} \right) = (-1)^{\ell_1 + \ell_2 + \ell_3} \left( \begin{array}{ccc}
l_1 & l_2 & l_3 \\
3m_1 & 3m_2 & 3m_3
\end{array} \right).
\]

For special values of the arguments, namely if \(\ell_3 = 0\) or \(\ell_2 = 0\), one has explicit forms of these coefficients:

\[
C_{\ell_1 \ell_2 m_3}^{00} = (-1)^{\ell_1 - m_1} \frac{\delta_{\ell_1 \ell_2 m_3}}{\sqrt{2\ell_1 + 1}}
\]

and

\[
C_{\ell_1 \ell_2 m_3}^{\ell_3 m_1 m_2} = \delta_{\ell_1 \ell_2 m_3},
\]

for details see again [36]. Another property, involved in our computations is the sum of the products of four Clebsch-Gordan Coefficients:

\[
\sum_{\beta \gamma < \epsilon \phi} C_{\epsilon \beta \gamma}^{\alpha \beta \gamma} C_{\epsilon \phi \epsilon}^{\alpha \epsilon \phi} C_{\epsilon \phi \gamma}^{\alpha \epsilon \phi} C_{\epsilon \phi \epsilon}^{\alpha \epsilon \phi} = (-1)^{\alpha - \beta + \epsilon + \phi - \epsilon - \phi} \sum_{ssag} C_{\alpha \epsilon \phi}^{\epsilon \beta \gamma} C_{\epsilon \phi \gamma}^{\alpha \epsilon \phi} \left\{ \begin{array}{ccc}
b & c & a \\
s & f & d
\end{array} \right\} \left\{ \begin{array}{ccc}
b & e & g \\
s & j & k
\end{array} \right\}
\]

(see [33] for the last symbol).

B.2 Wigner 6j coefficients

The 6j symbol is invariant under any permutation of its columns or under interchange of the upper and lower arguments in each of any two columns:

\[
\begin{align*}
\{a & b & c\} = \{d & e & f\} & = \{a & c & b\} = \{e & d & f\} & = \{b & a & c\} = \{d & f & e\} & = \{c & b & a\} = \{f & e & d\} \\
\{a & e & f\} & = \{d & f & e\} & = \{c & a & b\} & = \{d & e & f\} & = \{b & c & d\} & = \{f & a & e\} & = \{b & d & c\} & = \{f & e & a\} \\
\{d & b & c\} & = \{d & c & b\} & = \{e & d & c\} & = \{e & c & d\} & = \{d & c & e\} & = \{e & c & d\} & = \{d & e & c\} & = \{e & d & c\} \\
\{a & b & f\} & = \{a & f & b\} & = \{b & a & f\} & = \{b & f & a\} & = \{f & a & b\} & = \{f & b & a\} & = \{a & f & b\} & = \{b & a & f\} \\
\{d & b & f\} & = \{d & f & b\} & = \{b & d & f\} & = \{b & f & d\} & = \{f & d & b\} & = \{f & b & d\} & = \{b & d & f\} & = \{b & f & d\} \\
\{a & e & c\} & = \{a & c & e\} & = \{e & a & c\} & = \{e & c & a\} & = \{e & c & a\} & = \{e & a & c\} & = \{e & c & a\} & = \{e & a & c\}
\end{align*}
\]

and the following upper bound holds [16]:

\[
\left| \begin{array}{ccc}
a & b & c \\
d & e & f
\end{array} \right| \leq \min \left( \frac{1}{\sqrt{(2c+1)(2f+1)}}, \frac{1}{\sqrt{(2a+1)(2d+1)}}, \frac{1}{\sqrt{(2b+1)(2e+1)}} \right)
\]

(32)
When one of the arguments is equal to zero, their expression reduces to

\[\begin{align*}
\{a & b & c \\
0 & e & f \} = (-1)^{a+b+c} \frac{\delta^e_c \delta^f_c}{\sqrt{(2b+1)(2c+1)}}, \\
\{a & 0 & c \\
de & e & f \} = (-1)^{a+d+c} \frac{\delta^e_d \delta^f_c}{\sqrt{(2a+1)(2d+1)}}, \\
\{a & b & c \\
d & 0 & f \} = (-1)^{a+b+d} \frac{\delta^f_d \delta^c_e}{\sqrt{(2a+1)(2c+1)}}.
\end{align*}\] (B.18)

**B.3 Wigner 9j coefficients**

Similarly to the Wigner 6j coefficients, when one of the arguments is equal to zero, they have an easier expression

\[\begin{align*}
\{a & b & c \\
g & e & f \} = ( -1)^{b+c+d+g} \frac{\delta^g_c \delta^h_e \delta^f_c}{[(2c+1)(2g+1)]^{1/2}} \\
& \{a & b & c \\
d & e & f \} \quad \{a & b & c \\
0 & 0 & 0 \} \quad \{a & b & c \\
d & e & f \} \quad \{a & b & c \\
g & h & j \}
\end{align*}\] (B.19)

Using symmetry properties, we get

\[\begin{align*}
\{0 & c & c \\
g & e & b \} = \{c & 0 & c \\
g & d & a \} = \{g & g & 0 \\
e & d & c \} = \{g & g & 0 \\
e & c & d \} = \{c & c & d \\
c & b & a \} = \{c & c & d \\
0 & g & g \} = \{c & c & d \\
c & b & a \} = \{c & c & d \\
0 & g & g \} = \frac{( -1)^{b+d+c+g}}{[(2c+1)(2g+1)][(2c+1)(2g+1)]^{1/2}} \quad \{a & b & c \\
0 & 0 & 0 \} \quad \{a & b & c \\
0 & 0 & 0 \} \quad \{a & b & c \\
0 & 0 & 0 \} \quad \{a & b & c \\
0 & 0 & 0 \}
\end{align*}\] (B.20)

\[\{0 & c & c \\
g & e & b \} = \{c & 0 & c \\
g & d & a \} = \{g & g & 0 \\
e & d & c \} = \{g & g & 0 \\
e & c & d \} = \{c & c & d \\
c & b & a \} = \{c & c & d \\
0 & g & g \} = \{c & c & d \\
c & b & a \} = \{c & c & d \\
0 & g & g \} = \frac{( -1)^{b+d+c+g}}{[(2c+1)(2g+1)][(2c+1)(2g+1)]^{1/2}} \quad \{a & b & c \\
0 & 0 & 0 \} \quad \{a & b & c \\
0 & 0 & 0 \} \quad \{a & b & c \\
0 & 0 & 0 \} \quad \{a & b & c \\
0 & 0 & 0 \}
\end{align*}\] (B.21)

to get all the symmetry properties see [36]; the one used in this paper is

\[\begin{align*}
\{a & b & c \\
d & e & f \} = \{a & b & c \\
g & h & j \} = \{a & d & g \\
b & e & h \} = \{c & f & j \}.
\end{align*}\] (B.22)

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