Parallel transport of $Hom$-complexes and the Lovász conjecture

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Abstract

The groupoid of projectivities, introduced by M. Joswig [17], serves as a basis for a construction of parallel transport of graph and more general $Hom$-complexes. In this framework we develop a general conceptual approach to the Lovász conjecture, recently resolved by E. Babson and D. Kozlov in [4], and extend their result from graphs to the case of simplicial complexes.

1 Introduction

Hidden in the background of the Babson and Kozlov proof of Lovász conjecture [4] [20] are interesting topological and combinatorial concepts and structures associated to graphs and graph homomorphisms. The proof itself runs in two phases, each phase divided in several steps, often involving a detailed case analysis. For this reason the underlying secondary structures may not be visible or immediately recognized under the layers of intricate technical details. Recall that the crux of Babson and Kozlov approach is a skilful and technically quite involved application of spectral sequences. One of the classical applications of this method is to the (co)homology of fibered spaces which by the nature are spaces which allow some form of transport from one fibre to another.

In this paper we focus on one of these secondary structures which can be, somewhat informally, described as the “parallel transport” of graph complexes over graphs.

The introduction of this structure and recognition of its role leads to a great simplification of the proof of Lovász conjecture in some cases. Another of its features, aside from offering a conceptual “explanation” for the success of one of the technical approaches of Babson and Kozlov, is its potential for generating other statements of this type.

The “parallel transport” of graph complexes introduced here seems to be a novel concept. However the groupoids (groups of projectivities) used in its definition have already appeared in geometric combinatorics in the work of M. Joswig [17], see also [18] [19] [20], where they have been applied to toric manifolds, branched coverings over $S^3$, colorings of simple polytopes etc.

It is an exciting “coincidence” that there have been other recent developments in geometric and algebraic combinatorics where groupoids and associated objects and constructions were implicitly used, [21] [22] [23] [24]. These are not isolated examples...
of course. In particular one should be fully aware of a rich and deep combinatorics already present in numerous categorical constructions related to groupoids and their applications in geometry and mathematical physics.

2 The Lovász conjecture

One of central themes in topological combinatorics, after the landmark paper of Laszlo Lovász [21] where he proved the classical Kneser conjecture, has been the study and applications of graph complexes.

The underlying theme is to explore how the topological complexity of a graph complex $X(G)$ reflects in the combinatorial complexity of the graph $G$ itself. The results one is usually interested in come in the form of inequalities $\alpha(X(G)) \leq \xi(G)$, or equivalently in the form of implications

$$\alpha(X(G)) \geq p \Rightarrow \xi(G) \geq q,$$

where $\alpha(X(G))$ is a topological invariant of $X(G)$, while $\xi(G)$ is a combinatorial invariant of the graph $G$.

The most interesting candidate for the invariant $\xi$ has been the chromatic number $\chi(G)$ of $G$, while the role of the invariant $\alpha$ was played by the “connectedness” of $X(G)$, its equivariant index, the height of an associated characteristic cohomology class etc., see [20] [22] [23] [28] for recent accounts.

The famous result of Lovász quoted above is today usually formulated in the form of an implication

$$\text{Hom}(K_2, G) \text{ is } k\text{-connected } \Rightarrow \chi(G) \geq k + 3,$$  \hspace{1cm} (1)

where $\text{Hom}(K_2, G)$ is the so called “box complex” of $G$. The box complex is a special case of a general graph complex $\text{Hom}(H, G)$ (also introduced by L. Lovász), a cell complex which functorially depends on the input graphs $H$ and $G$.

An outstanding conjecture in this area, refereed to as “Lovász conjecture”, was that one obtains a better bound if the graph $K_2$ in (1) is replaced by an odd cycle $C_{2r+1}$. More precisely Lovász conjectured that

$$\text{Hom}(C_{2r+1}, G) \text{ is } k\text{-connected } \Rightarrow \chi(G) \geq k + 4.$$  \hspace{1cm} (2)

This conjecture was confirmed by Babson and Kozlov in [4], see also [21] for a more detailed exposition.

Our objective is to develop methods which both offer a simplified approach to the proof of implication (2), at least in the case when $k$ is odd, and providing new insight, open a possibility of proving similar results for other classes of (hyper)graphs and simplicial complexes.

An example of such a result is Theorem 22. One of its corollaries is the following implication,

$$\text{Hom}(\Gamma, K) \text{ is } k\text{-connected } \Rightarrow \chi(K) \geq k + d + 3$$  \hspace{1cm} (3)

which, under a suitable assumption on the complex $\Gamma$ and the assumption that integer $k$ is odd, extends (2) to the case of pure $d$-dimensional simplicial complexes.
3 Parallel transport of $Hom$-complexes

3.1 Generalities about “parallel transport”

In order to avoid any ambiguities, we briefly clarify what is in this paper meant by a “parallel transport” on a “bundle” of spaces.

A “bundle” is a map $\phi : X \to S$. We assume that $S$ is a set and that $X(i) := \phi^{-1}(i)$ is a topological space, so a bundle is just a collection of spaces (fibres) $X(i)$ parameterized by $S$. If all spaces $X(i)$ are homeomorphic to a fixed “model” space, this space is referred to as the fiber of the bundle $\phi$.

Suppose that $G$ is a groupoid on $S$ as the set of objects. In other words $G = (Ob(G), Mor(G))$ is a small category where $Ob(G) = S$, such that all morphisms $\alpha \in Mor(G)$ are invertible.

A “connection” or “parallel transport” on the bundle $\mathcal{X} = \{X(i)\}_{i \in S}$ is a functor $\mathcal{P} : G \to Top$ such $X(i) = \mathcal{P}(i)$ for each $i \in S$.

Informally speaking, the groupoid $G$ provides a “road map” on $S$, while the functor $\mathcal{P}$ defines the associated transport from one fibre to another.

Sometimes it is convenient to view the bundle $\mathcal{X} = \{X(i)\}_{i \in S}$ as a map $\mathcal{X} : S \to Top$. Then to define a “connection” on this bundle is equivalent to enriching the map $\mathcal{X}$ to a functor $\mathcal{P} : G \to Top$.

3.2 Natural bundles and groupoids over simplicial complexes

Suppose that $K$ and $L$ are finite simplicial complexes and let $k$ be an integer such that $0 \leq k \leq \dim(K)$. Let $S_k = S_k(K)$ be the set of all $k$-dimensional simplices in $K$. Define a bundle $\mathcal{F}^L_K : S_k \to Top$ by the formula

$$\mathcal{F}^L_K(\sigma) = Hom(\sigma, L) \cong L^{k+1}$$

where $Hom(\sigma, L)$ is one of the $Hom$-complexes introduced in Section 5.1 and $L^{k+1}$ is the complex well known in topological combinatorics as the deleted product of $L$, Chapter 6. A typical cell in $L^{k+1}$ is of the form $e = \sigma_0 \times \sigma_1 \times \ldots \times \sigma_k \in L^{k+1}$ where $\{\sigma_i\}_{i=0}^k$ is a collection of non-empty simplices in $L$ such that if $i \neq j$ then $\sigma_i \cap \sigma_j = \emptyset$. The corresponding cell in $Hom(\sigma, L)$ is a function $\eta : V(\sigma) \to L \setminus \{\emptyset\}$, where $V(\sigma) = \sigma^{(0)}$ is the set of all vertices of $\sigma$, and if $v_1 \neq v_2$ then $\eta(v_1) \cap \eta(v_2) = \emptyset$.

**Example:** It is well known that if $L \cong \sigma^m = \Delta^{[m+1]}$ is a $m$-dimensional simplex, then the associated deleted square $(\sigma^m)^2_\Delta$ is homeomorphic to a $(m-1)$-dimensional sphere. In other words, $\mathcal{F}^\sigma_S : S_1(K) \to Top$ is a spherical bundle naturally associated to the simplicial complex $K$.

Our next goal, in the spirit of Section 6.1, is to identify a groupoid on the set $S_k$ which acts on the bundle $\mathcal{F}^L_K$, i.e. a groupoid which provides a parallel transport of fibres of the bundle $\mathcal{F}^L_K$. It is a pleasant coincidence that this groupoid has already appeared in geometric combinatorics [16] [17]. Indeed, the *groups of projectivities* M. Joswig introduced and studied in these papers are just the vertex or isotropy groups of a groupoid which we call the $k$-th groupoid of projectivities of $K$ and denote by $G^L_K(K)$. In these and in subsequent papers [13] [14] [15], the groups of projectivities found
interesting applications to toric manifolds, branched coverings over $S^3$, colorings of simple polytopes, etc. Here is a summary of this construction.

Two $k$-dimensional simplices $\sigma_0$ and $\sigma_1$ in $K$ are called adjacent if they share a common $(k - 1)$-dimensional face $\tau$. A perspectivity from $\sigma_0$ to $\sigma_1$ is the unique non-degenerated simplicial map $\overrightarrow{\sigma_0\sigma_1} = (\sigma_0, \sigma_1): \sigma_0 \rightarrow \sigma_1$ which leaves the simplex $\tau$ point-wise fixed. In the special case when $\sigma_0 = \sigma_1$, the perspectivity $(\sigma_0, \sigma_0): \sigma_0 \rightarrow \sigma_0$ is the identity map $I_{\sigma_0}$.

A projectivity between two, not necessarily adjacent, simplices $\sigma_0$ and $\sigma_n$ is a composition of perspectivities

$$(p) = \overrightarrow{\sigma_0\sigma_1} \ast \overrightarrow{\sigma_1\sigma_2} \ast \ldots \ast \overrightarrow{\sigma_{n-1}\sigma_n}$$

where $p = (\sigma_0, \sigma_1, \ldots, \sigma_n)$ is a path of $k$-dimensional simplices in $K$ such that $\sigma_{i-1}$ and $\sigma_i$ share a common $(k - 1)$-dimensional face $\tau_i$.

**Caveat:** Here we adopt a useful convention that $(x)(f \ast g) = (g \circ f)(x)$ for each two composable maps $f$ and $g$. The notation $f \ast g$ is often given priority over the usual $g \circ f$ if we want to emphasize that the functions act on the points from the right, that is if the arrows in the associated formulas point from left to the right.

**Definition 1** [16] [17] The $k$-th groupoid of projectivities $\mathcal{G}_k^P(K)$ of a simplicial complex $K$, or the $P_k$-groupoid associated to $K$, is the small category

$$\mathcal{G}_k^P(K) = (\text{Ob}(\mathcal{G}_k^P(K)), \text{Mor}(\mathcal{G}_k^P(K)))$$

which has the set $S_k = \text{Ob}(\mathcal{G}_k^P(K))$ of all $k$-dimensional simplices for the set of objects, and for each two simplices $\sigma_0, \sigma_1 \in S_k$, the associated morphism set $\text{Mor}_\mathcal{G}_k^P(K)(\sigma_0, \sigma_1)$ is the collection of all projectivities from $\sigma_0$ to $\sigma_1$. The associated point (isotropy) groups

$$\Pi_k(K, \sigma_0) := \text{Mor}_\mathcal{G}_k^P(K)(\sigma_0, \sigma_0)$$

are called the groups of projectivities or the combinatorial holonomy groups of $K$.

**Proposition 2** For each finite simplicial complex $K$ and an auxiliary “coefficient” complex $L$, there exists a canonical connection $\mathcal{P}^L = \mathcal{P}_K^L$ on the bundle $\mathcal{F}_k$. In other words the function $\mathcal{F}_k^L: S_k \rightarrow \text{Top}$ can be enriched (extended) to a functor

$$\mathcal{F}_k^L: S_k \rightarrow \text{Top}.$$  

**Proof:** If $\overrightarrow{\sigma_0\sigma_1}$ is a perspectivity from $\sigma_0$ to $\sigma_1$ and if $\eta: V(\sigma_1) \rightarrow 2^{V(L)} \setminus \{\emptyset\}$ is a cell in $\text{Hom}(\sigma, L)$, then $\mathcal{P}^L: \mathcal{F}_k^L(\sigma_1) \rightarrow \mathcal{F}_k^L(\sigma_0)$ is the map defined by $\mathcal{P}^L(\overrightarrow{\sigma_0\sigma_1})(\eta) := \overrightarrow{\sigma_0\sigma_1} \ast \eta$. More generally, if $p = (\sigma_0) = \overrightarrow{\sigma_0\sigma_1} \ast \overrightarrow{\sigma_1\sigma_2} \ast \ldots \ast \overrightarrow{\sigma_{n-1}\sigma_n}$ is a projectivity between $\sigma_0$ and $\sigma_n$, then

$$\mathcal{P}^L(p) = \mathcal{P}^L(\overrightarrow{\sigma_0\sigma_1}) \ast \mathcal{P}^L(\overrightarrow{\sigma_1\sigma_2}) \ast \ldots \ast \mathcal{P}^L(\overrightarrow{\sigma_{n-1}\sigma_n})$$

(5)

or in other words

$$\mathcal{P}^L((p))(\eta) = \overrightarrow{\sigma_0\sigma_1} \ast \overrightarrow{\sigma_1\sigma_2} \ast \ldots \ast \overrightarrow{\sigma_{n-1}\sigma_n} \ast \eta.$$  

(6)

It is clear from the construction that the map $\mathcal{P}^L((p))$ depends only on the projectivity $(p)$ and not on the associated path $p$. \qed
3.3 Parallel transport of graph complexes

The main motivation for introducing the parallel transport of $Hom$-complexes is the Lovász conjecture and its ramifications. This is the reason why the case of graphs and the graph complexes deserves a special attention. Additional justification for emphasizing graphs comes from the fact that graph complexes $Hom(G, H)$ have been studied in numerous papers and today form a well established part of graph theory and topological combinatorics. The situation with simplicial complexes is quite the opposite. In order to extend the theory of $Hom$-complexes from graphs to the category of simplicial complexes, many concepts should be generalized and the corresponding facts established in a more general setting. One is supposed to recognize the main driving forces and to isolate the most desirable features of the theory. A result should be a dictionary/glossary of associated concepts, cf. Table 1. Consequently, Section 3.3 should be viewed as an important preliminary step, leading to the more general theory developed in Sections 5 and 6.

In order to simplify the exposition we assume, without a serious loss of generality, that all graphs $G = (V(G), E(G))$ are without loops and multiple edges. In short, graphs are 1-dimensional simplicial complexes. Let $G_{xy} \cong K_2$ be the restriction of $G$ on the edge $xy \in E(G)$.

Following the definitions from Section 3.2 the map $F^H : E(G) \to Top$, where $F^H(xy) := F^H_{xy} = Hom(G_{xy}, H)$, can be thought of as a “bundle” over the graph $G$, with $F^H_{xy} = Hom(G_{xy}, H)$ in the role of the “fibre” over the edge $xy$. More generally, given a class $C$ of subgraphs of $G$, say the subtrees, the chains, the $k$-cliques etc., one can define an associated “bundle” $F^H_C : C \to Top$ by a similar formula $F^H_C(\Gamma) := Hom(\Gamma, H)$, where $\Gamma \in C$.

The parallel transport $P^H$, for a given graph (1-dimensional, simplicial complex) $H$, is a specialization of the parallel transport $P^L$ introduced in Section 3.2. For example if $e_1 \rightarrow e_2$ is the perspectivity between adjacent edges $e_1 = x_0x_1$ and $e_2 = x_1x_2$ in $G$, and if $\eta : \{x_1, x_2\} \to 2^{V(H)} \setminus \{\emptyset\}$ is a cell in $F^H_{x_1x_2} = Hom(G_{x_1x_2}, H)$, then $\eta' := P^H(e_1 \rightarrow e_2)(\eta) : \{x_0, x_1\} \to 2^{V(H)} \setminus \{\emptyset\}$ is defined by $\eta'(x_0) := \eta(x_2)$ and $\eta'(x_1) := \eta(x_1)$.

Fundamental observation: The construction of the connections $P^L$, respectively $P^H$, are quite natural and elementary but it is Proposition 4 respectively its more general relative Proposition 17 that serve as an actual justification for the introduction of these objects. Proposition 4 allows us to analyze the parallel transport of homotopy types of maps from the complex $Hom(G, H)$ to complexes $Hom(G_e, H)$, where $e \in E(G)$, providing a key for a resolution of the Lovász conjecture in the case when $k$ is an odd integer.

Implicit in the proof of Proposition 4 is the theory of folds of graphs and the analysis of natural morphisms between graph complexes $Hom(T, H)$, where $T$ is a tree, as developed in [3] [18] [19]. This theory is one of essential ingredients in the Babson
and Kozlov spectral sequence approach to the solution of Lovász conjecture. Some of these results are summarized in Proposition \[3\] in the form suitable for application to Proposition \[4\].

As usual \(L_m\) is the graph-chain of vertex-length \(m\), while \(L_{x_1 \ldots x_m}\) is the graph isomorphic to \(L_m\) defined on a linearly ordered set of vertices \(x_1, \ldots, x_m\). In this context the “flip” is a generic name for the automorphism \(\sigma : L_{x_1 \ldots x_m} \to L_{x_1 \ldots x_m}\) of the graph-chain such that \(\sigma(x_j) = x_{m-j+1}\) for each \(j\).

**Proposition 3** Suppose that \(e_1 = \overrightarrow{x_0x_1}\) and \(e_2 = \overrightarrow{x_1x_2}\) are two distinct, adjacent edges in the graph \(G\). Let \(\sigma : L_{x_0x_1x_2} \to L_{x_0x_1x_2}\) be the flip automorphism of \(L_{x_0x_1x_2}\) and \(\hat{\sigma}\) the associated auto-homeomorphism of \(\text{Hom}(L_{x_0x_1x_2}, H)\). Suppose that \(\gamma_{ij} : L_{x_i x_j} \to L_{x_0x_1x_2}\) is an obvious embedding and \(\hat{\gamma}_{ij}\) the associated maps of graph complexes. Then,

(a) the induced map \(\hat{\sigma} : \text{Hom}(L_{x_0x_1x_2}, H) \to \text{Hom}(L_{x_0x_1x_2}, H)\) is homotopic to the identity map \(I\), and

(b) the diagram

\[
\begin{array}{ccc}
\text{Hom}(L_{x_0x_1x_2}, H) & \xrightarrow{=} & \text{Hom}(L_{x_0x_1x_2}, H) \\
\gamma_{01} \downarrow & & \downarrow \hat{\gamma}_{12} \\
\text{Hom}(L_{x_0x_1}, H) & \overset{\mathcal{P}^H(e_1 e_2)}{\leftarrow} & \text{Hom}(L_{x_1x_2}, H)
\end{array}
\]

is commutative up to homotopy.

**Proof:** Both statements are corollaries of Babson and Kozlov analysis of complexes \(\text{Hom}(T, H)\), where \(T\) is a tree, and morphisms \(\hat{\sigma} : \text{Hom}(T, H) \to \text{Hom}(T', H)\), where \(T'\) is a subtree of \(T\) and \(e : T' \to T\) the associated embedding.

Our starting point is an observation that both \(L_{x_0x_1}\) and \(L_{x_1x_2}\) are retracts of the graph \(L_{x_0x_1x_2}\) in the category of graphs and graph homomorphisms. The retraction homomorphisms \(\phi_{ij} : L_{x_0x_1x_2} \to L_{x_i x_j}\), where \(\phi_{01}(x_0) = x_0, \phi_{01}(x_1) = x_1, \phi_{01}(x_2) = x_0\) and \(\phi_{12}(x_0) = x_2, \phi_{12}(x_1) = x_1, \phi_{12}(x_2) = x_2\) are examples of *foldings* of graphs. By the general theory \[3\] \[18\], the maps \(\gamma_{ij} : \text{Hom}(L_{x_0x_1x_2}, H) \to \text{Hom}(L_{x_i x_j}, H)\) and \(\hat{\phi}_{ij} : \text{Hom}(L_{x_i x_j}, H) \to \text{Hom}(L_{x_0x_1x_2}, H)\) are homotopy equivalences. Actually \(\gamma_{ij}\) is a deformation retraction and \(\hat{\phi}_{ij}\) is the associated embedding such that \(\gamma_{ij} \circ \hat{\phi}_{ij} = I\) is the identity map.

The part (a) of the proposition is an immediate consequence of the fact that \(\phi_{01} \circ \sigma \circ \gamma_{01} : L_{x_0x_1} \to L_{x_0x_1}\) is an identity map. It follows that \(\gamma_{01} \circ \hat{\sigma} \circ \hat{\phi}_{01} = I\), and in light of the fact that \(\gamma_{01}\) and \(\hat{\phi}_{01}\) are homotopy inverses to each other, we conclude that \(\hat{\sigma} \simeq I\).

For the part (b) we begin by an observation that \(\phi_{12} \circ \sigma \circ \gamma_{01} = \overrightarrow{e_1 e_2}\). Then, \(\mathcal{P}^H(e_1 e_2) = \gamma_{01} \circ \hat{\sigma} \circ \hat{\phi}_{12}\), and as a consequence of \(\hat{\sigma} \simeq I\) and the fact that \(\phi_{12} \circ \gamma_{12} \simeq I\), we conclude that

\[
\mathcal{P}^H(e_1 e_2) \circ \gamma_{12} = \gamma_{01} \circ \hat{\sigma} \circ \hat{\phi}_{12} \circ \gamma_{12} \simeq \gamma_{01}.
\]

\[\square\]
Proposition 4 Suppose that \( x_0, x_1, x_2 \) are distinct vertices in \( G \) such that \( x_0 x_1, x_1 x_2 \in E(G) \). Let \( \alpha_{ij} : G_{x_i x_j} \to G \) be the inclusion map of graphs and \( \hat{\alpha}_{ij} \) the associated map of \( \text{Hom}(\cdot, H) \) complexes. Then the following diagram commutes up to a homotopy,

\[
\begin{array}{c}
\text{Hom}(G, H) \xrightarrow{=} \text{Hom}(G, H) \\
\downarrow \hat{\alpha}_{01} & \downarrow \hat{\alpha}_{12} \\
\text{Hom}(G_{x_0 x_1}, H) & \text{Hom}(G_{x_1 x_2}, H) \\
\end{array}
\]

(7)

Proof: The diagram (7) can be factored as

\[
\begin{array}{c}
\text{Hom}(G, H) \xrightarrow{=} \text{Hom}(G, H) \\
\downarrow \hat{\beta} & \downarrow \hat{\beta} \\
\text{Hom}(G_{x_0 x_1 x_2}, H) & \text{Hom}(G_{x_0 x_1 x_2}, H) \\
\downarrow \hat{\gamma}_{01} & \downarrow \hat{\gamma}_{12} \\
\text{Hom}(G_{x_0 x_1}, H) & \text{Hom}(G_{x_1 x_2}, H) \\
\end{array}
\]

(8)

where \( \beta \) and \( \gamma_{ij} \) are obvious inclusions of indicated graphs such that \( \alpha_{ij} = \beta \circ \gamma_{ij} \). Then the result is a direct consequence of Proposition 4 part (b). \( \square \)

4 Lovász-Babson-Kozlov result for odd \( k \)

The proof 4 of Lovász conjecture splits into two main branches, corresponding to the parity of a parameter \( n \), where \( n \) is an integer which enters the stage as the size of the vertex set of the complete graph \( K_n \).

The first branch relies on Theorem 2.3. (loc. cit.), more precisely on part (b) of this result, while the second branch is founded on Theorem 2.6. Both theorems are about the topology of the graph complex \( \text{Hom}(C_{2r+1}, K_n) \). Theorem 2.3. (b) is a statement about the height of the first Stiefel-Whitney class, or equivalently the Conner-Floyd \( \mathbb{Z}_2 \)-index 10 of the \( \mathbb{Z}_2 \)-space \( \text{Hom}(C_{2r+1}, K_n) \). Theorem 2.6. claims that for \( n \) even, \( 2_{K_n} \) is a zero homomorphism where

\[
\iota_{K_n} : \tilde{H}^*(\text{Hom}(K_2, K_n); \mathbb{Z}) \to \tilde{H}^*(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z})
\]

(9)

is the homomorphism associated to the continuous map \( \iota_{K_n} : \text{Hom}(C_{2r+1}, K_n) \to \text{Hom}(K_2, K_n) \), which in turn comes from the inclusion \( K_2 \hookrightarrow C_{2r+1} \).

The central idea of our paper is an observation that Theorem 2.6. can be incorporated into a more general scheme, involving the “parallel transport” of graph complexes over graphs.

Theorem 5 Suppose that \( \alpha : K_2 \to C_{2r+1} \) is an inclusion map, \( \beta : K_2 \to K_2 \) a nontrivial automorphism of \( K_2 \), and

\[
\hat{\alpha} : \text{Hom}(C_{2r+1}, H) \to \text{Hom}(K_2, H), \quad \hat{\beta} : \text{Hom}(K_2, H) \to \text{Hom}(K_2, H)
\]

The proof 5 of the theorem is a direct consequence of the previous results. \( \square \)
the associated maps of graph complexes. Then the following diagram is commutative up to a homotopy

\[
\begin{array}{c}
\text{Hom}(C_{2r+1}, H) \\
\hat{\alpha} \downarrow \\
\text{Hom}(K_2, H) \\
\end{array} \xrightarrow{=} \begin{array}{c}
\text{Hom}(C_{2r+1}, H) \\
\hat{\alpha} \\
\text{Hom}(K_2, H) \\
\end{array}
\] (10)

**Proof:** Assume that the consecutive vertices of \( G = C_{2r+1} \) are \( x_0, x_1, \ldots, x_{2r} \) and let \( e_i = x_i - x_{i-1} \) be the associated sequence of edges where by convention \( e_{2r+1} = x_{2r} - x_0 \).

Identify the graph \( K_2 \) to the subgraph \( G_{x_0 x_1} \) of \( G = C_{2r+1} \).

By iterating Proposition 4 we observe that the diagram

\[
\begin{array}{c}
\text{Hom}(C_{2r+1}, H) \\
\hat{\alpha} \downarrow \\
\text{Hom}(G_{x_0 x_1}, H) \\
\end{array} \xrightarrow{p_H(\tilde{p})} \begin{array}{c}
\text{Hom}(C_{2r+1}, H) \\
\hat{\alpha} \\
\text{Hom}(G_{x_0 x_1}, H) \\
\end{array}
\] (11)

is commutative up to a homotopy, where \( \tilde{p} = e_1 e_2 \ast \ldots \ast e_{2r+1} e_1 \). The proof is completed by the observation that \( p = \beta \) in the groupoid \( G^P(G) \).

Theorem 2.6. from [4], the key for the proof of Lovász conjecture for odd \( k \), is an immediate consequence of Theorem 5.

**Corollary 6** ([4], T.2.6.) If \( n \) is even then \( 2 \cdot \iota^*_K \) is a 0-map where \( \iota^*_K \) is the map described in line (9).

**Proof:** It is sufficient to observe that for \( H = K_n \), the complex \( \text{Hom}(K_2, K_n) \cong S^{n-2} \) is an even dimensional sphere such that the automorphism \( \hat{\beta} \) from the diagram (10) is essentially an antipodal map. It follows that \( \hat{\beta} \) changes the orientation of \( \text{Hom}(K_2, K_n) \) and as a consequence \( \iota^*_K = -\iota^*_K \).

\section{5 Generalizations and ramifications}

In this section we extend the results from Section 3.3 to the case of simplicial complexes. This generalization is based on the following basic principles.

Graphs are viewed as 1-dimensional simplicial complexes. Graph homomorphisms are special cases of non-degenerated simplicial maps of simplicial complexes, [15] [17]. The definition of \( \text{Hom}(G, H) \) is extended to the case of \( \text{Hom} \)-complexes \( \text{Hom}(K, L) \) of simplicial complexes \( K \) and \( L \). The groupoids needed for the definition of the parallel transport of \( \text{Hom} \)-complexes are already introduced by Joswig in [17], see Section 3.2 for a summary. Theory of folds for graph complexes [3] [18] is extended in Section 5.4 to the case of \( \text{Hom} \)-complexes in sufficient generality to allow “parallel transport” of homotopy types of maps between graph complexes. This development eventually leads to Theorem 22 which extends Theorem 5 to the case of \( \text{Hom} \)-complexes \( \text{Hom}(K, L) \) and represents the currently final stage in the evolution of Theorem 2.6. from [4].
| Dictionary |
|-------------|
| graphs      | simplicial complexes |
| trees       | tree-like complexes |
| foldings of graphs | vertex collapsing of complexes |
| graph homomorphisms | non-degenerated simplicial maps |
| \(\text{Hom}(G,H)\) | \(\text{Hom}(K,L)\) |
| chromatic number \(\chi(G)\) | chromatic number \(\chi(K)\) |

Table 1: Graphs vs. simplicial complexes.

5.1 From \(\text{Hom}(G,H)\) to \(\text{Hom}(K,L)\)

Suppose that \(K \subset 2^{V(K)}\) and \(L \subset 2^{V(L)}\) are two (finite) simplicial complexes, on the sets of vertices \(V(K)\) and \(V(L)\) respectively.

**Definition 7** A simplicial map \(f : K \to L\) is non-degenerated if it is injective on simplices. The set of all non-degenerated simplicial maps from \(K\) to \(L\) is denoted by \(\text{Hom}_0(K,L)\).

**Definition 8** \(\text{Hom}(K,L)\) is a cell complex with the cells indexed by the functions \(\eta : V(K) \to 2^{V(L)} \setminus \{\emptyset\}\) such that

1. for each two vertices \(u \neq v\), if \(\{u, v\} \in K\) then \(\eta(u) \cap \eta(v) = \emptyset\),
2. for each simplex \(\sigma \in K\), the join \(\ast_{v \in V(\sigma)} \eta(v) \subset \Delta^{V(L)}\) of all sets (0-dimensional complexes) \(\eta(v)\), where \(v\) is a vertex of \(\sigma\), is a subcomplex of \(L\).

More precisely, each function \(\eta\) satisfying conditions (1) and (2) defines a cell \(c_\eta := \prod_{v \in V(K)} \Delta^{\eta(v)}\) in \(\text{Hom}(K,L) \subset \prod_{v \in V(K)} \Delta^{V(L)}\) where by definition \(\Delta^S\) is an (abstract) simplex spanned by vertices in \(S\).

We have already used in Section 3.2 the fact that if \(K = \Delta^{[m]}\) is a \((m - 1)\)-dimensional simplex spanned by \([m]\) as the set of vertices, then \(\text{Hom}(\Delta^{[m]}, L) \cong L^{m}_\Delta\) is the deleted product of \(L\) \((22)\). The following example shows that \(\text{Hom}(G,H)\) is a special case of \(\text{Hom}(K,L)\).

**Example 9** The definition of \(\text{Hom}(K,L)\) is a natural extension of \(\text{Hom}(G,H)\) and reduces to it if \(K\) and \(L\) are 1-dimensional complexes. Moreover,

\[
\text{Hom}(G,H) \cong \text{Hom}(\text{Clique}(G), \text{Clique}(H))
\]

where \(\text{Clique}(\Gamma)\) is the simplicial complex of all cliques in a graph \(\Gamma\).

**Remark 10** The set \(\text{Hom}_0(K,L)\) is easily identified as the 0-dimensional skeleton of the cell-complex \(\text{Hom}(K,L)\). Moreover, the reader familiar with \((20)\) can easily check that \(\text{Hom}(K,L)\) is determined by the family \(M = \text{Hom}_0(K,L)\) in the sense of Definition 2.2.1. from that paper.
5.2 Functoriality of $\text{Hom}(K, L)$

The construction of $\text{Hom}(K, L)$ is functorial in the sense that if $f : K \to K'$ is a non-degenerated simplicial map of complexes $K$ and $K'$, then there is an associated continuous map $\tilde{f} : \text{Hom}(K', L) \to \text{Hom}(K, L)$ of $\text{Hom}$-complexes. Indeed, if $\eta : V(K') \to 2^{V(L)} \setminus \{\emptyset\}$ is a multi-valued function indexing a cell in $\text{Hom}(K', L)$, then it is not difficult to check that $\eta \circ f : V(K) \to 2^{V(L)} \setminus \{\emptyset\}$ is a cell in $\text{Hom}(K, L)$.

Perhaps even more important is the functoriality of $\text{Hom}(K, L)$ with respect to the second variable since this implies the functoriality of the bundle $\mathcal{F}_k^L$.

Proposition 11 Suppose that $g : L \to L'$ is a non-degenerated, simplicial map of simplicial complexes $L$ and $L'$. Then there exists an associated map

$$\tilde{g} : \text{Hom}(K, L) \to \text{Hom}(K, L').$$

Proof: Assume that $\eta : V(K) \to 2^{V(L)} \setminus \{\emptyset\}$ is a cell in $\text{Hom}(K, L)$. Then $g \circ \eta : V(K) \to 2^{V(L')} \setminus \{\emptyset\}$ is a cell in $\text{Hom}(K, L')$. Suppose $u$ and $v$ are distinct vertices in $V(K)$. By assumption $\eta(u) \cap \eta(v) = \emptyset$. We deduce from here that $g(\eta(u)) \cap g(\eta(v)) \neq \emptyset$, otherwise $g$ would be a degenerated simplicial map.

The second condition from Definition 8 is checked by a similar argument. □

5.3 Chromatic number $\chi(K)$ and its relatives

The chromatic number $\chi(K)$ of a simplicial complex $K$ is

$$\inf\{m \in \mathbb{N} \mid \text{Hom}_0(K, \Delta^m) \neq \emptyset\}.$$ 

In other words $\chi(K)$ is the minimum number $m$ such that there exists a non-degenerated simplicial map $f : K \to \Delta^m$. It is not difficult to check that $\chi(K) = \chi(G_K)$ where $G_K = (K^{(0)}, K^{(1)})$ is the vertex-edge graph of the complex $K$. In particular $\chi(K)$ reduces to the usual chromatic number if $K$ is a graph, that is if $K$ is a 1-dimensional simplicial complex.

Aside from the usual chromatic number $\chi(G)$, there are many related colorful graph invariants [12, 20]. Among the best known are the fractional chromatic number $\chi_f(G)$ and the circular chromatic number $\chi_c(G)$ of $G$. These and other related invariants are conveniently defined in terms of graph homomorphisms into graphs chosen from a suitable family $\mathcal{F} = \{G_i\}_{i \in I}$ of test graphs. Motivated by this, we offer an extension of the chromatic number $\chi(K)$ in hope that some genuine invariants of simplicial complexes objects arise this way.

Definition 12 Suppose that $\mathcal{F} = \{T_i \mid i \in I\}$ is a family of “test” simplicial complexes and let $\phi : I \to \mathbb{R}$ is a real-valued function. A $T_i$-coloring of $K$ is just a non-degenerated simplicial map $f : K \to T_i$ and $\chi_{(\mathcal{F}, \phi)}(K)$, the $(\mathcal{F}, \phi)$-chromatic number of $K$, is defined as the infimum of all weights $\phi(i)$ over all $T_i$-colorings,

$$\chi_{(\mathcal{F}, \phi)}(K) := \inf\{\phi(i) \mid \text{Hom}_0(K, T_i) \neq \emptyset\}.$$
5.4 Tree-like simplicial complexes

The tree-like or vertex collapsible complexes are intended to play in the theory of $Hom(K, L)$-complexes the role similar to the role of trees in the theory of graph complexes $Hom(G, H)$.

A pure, $d$-dimensional simplicial complex $K$ is shellable \[8, 27\], if there is a linear order $F_1, F_2, \ldots, F_m$ on the set of its facets, such that for each $j \geq 2$, the complex $F_j \cap (\bigcup_{i<j} F_i)$ is a pure $(d - 1)$-dimensional subcomplex of the simplex $F_j$. The restriction $R_j$ of the facet $F_j$ is the minimal new face added to the complex $\bigcup_{k<j} F_k$ by the addition of the facet $F_j$. Let $r_j := \dim(R_j) \in \{0, 1, \ldots, d\}$ be the type of the facet $F_j$. If $r_j \neq d$ for each $j$ then the complex $K$ is collapsible. The collapsing process is just the shelling order read in the opposite direction. From this point of view, $R_j$ can be described as a free face in the complex $\bigcup_{i<j} F_i$, and the process of removing all faces $F$ such that $R_j \subset F \subset F_j$ is called an elementary $r_j$-collapse.

**Definition 13** A pure $d$-dimensional simplicial complex $K$ is called tree-like or vertex collapsible if it is collapsible to a $d$-simplex with the use of elementary $0$-collapses alone. In other words $K$ is shellable and for each $j \geq 2$, the intersection $F_j \cap (\bigcup_{i<j} F_i)$ is a proper face of $F_j$.

In order to establish analogs of Propositions 3 and 4 for complexes $Hom(K, L)$, we prove a result which shows that elementary vertex collapsing provides a good substitute and a partial generalization for the concept of “foldings” of graphs used in \[8, 18\] in the theory of graph complexes $Hom(G, H)$.

**Proposition 14** Suppose that the simplicial complex $K'$ is obtained from $K$ by an elementary vertex collapse. In other words we assume that $K = \sigma \cup K'$, $\sigma \cap K' = \sigma'$, where $\sigma$ is a simplex in $K$ and $\sigma'$ a facet of $\sigma$. Assume that $\sigma'$ is not maximal in $K'$, i.e. that for some simplex $\sigma'' \in K'$ and a vertex $u \in \sigma''$, $\sigma' = \sigma'' \setminus \{u\}$. Then for any simplicial complex $L$, the inclusion map $\gamma : K' \to K$ induces a homotopy equivalence

$$\tilde{\gamma} : Hom(K, L) \to Hom(K', L).$$

**Proof:** Let $\{v\} = \sigma \setminus \sigma'$. Aside from the inclusion map $\gamma : K' \to K$, there is a retraction (folding) map $\rho : K \to K'$, where $\rho(v) = u$ and $\rho|_{K'} = I_{K'}$. Since $\rho \circ \gamma = I_{K'}$, we observe that $\tilde{\gamma} \circ \hat{\rho} = Id_{K'}$ is the identity map on $Hom(K', L)$, i.e. the complex $Hom(K', L)$ is a retract of the complex $Hom(K, L)$. It remains to be shown that $\hat{\rho} \circ \tilde{\gamma} \simeq Id_K$ is homotopic to the identity map on $Hom(K, L)$.

Note that if $\eta \in Hom(K, L)$ then $\eta' := \hat{\rho} \circ \tilde{\gamma}(\eta)$ is the function defined by

$$\eta'(w) = \begin{cases} 
\eta(w), & \text{if } w \neq v \\
\eta(u), & \text{if } w = v.
\end{cases}$$

Let $\omega : Hom(K, L) \to Hom(K, L)$ be the map defined by

$$\omega(\eta)(w) = \begin{cases} 
\eta(w), & \text{if } w \neq v \\
\eta(u) \cup \eta(v), & \text{if } w = v.
\end{cases}$$
Note that $\omega$ is well defined since if a vertex $x$ is adjacent to $v$ it is also adjacent to $u$, hence the condition $\omega(\eta)(v) \cap \eta(x) = \emptyset$ is a consequence of $\eta(u) \cap \eta(x) = \emptyset = \eta(v) \cap \eta(x)$.

Since for each $\eta \in \text{Hom}(K, L)$ and each vertex $x \in K$,

$$\eta(x) \subset \omega(\eta)(x) \supset \hat{\rho} \circ \hat{\gamma}(\eta)(x),$$

by the Order Homotopy Theorem \cite{7, 24, 25} all three maps $\text{Id}_K, \omega$ and $\hat{\rho} \circ \hat{\gamma}$ are homotopic. This completes the proof of the proposition. □

**Corollary 15** If $T$ is a $d$-dimensional, tree-like simplicial complex than $\text{Hom}(T, L)$ has the same homotopy type as the deleted join $\text{Hom}(\Delta^d, L) = L_{\Delta}^{d+1}$.

5.5 Parallel transport of homotopy types of maps

As in the case of graph complexes, the real justification for the introduction of the parallel transport of $\text{Hom}$-complexes comes from the fact that it preserves the homotopy type of the maps $\text{Hom}(K, L) \to \text{Hom}(\sigma, L)$. As in Section \ref{section_3.3} as a preliminary step we prove an analogue of Proposition \ref{prop_3.3}.

**Proposition 16** Suppose that $\sigma_1$ and $\sigma_2$ are two distinct, adjacent $k$-dimensional simplices in a finite simplicial complex $K$ which share a common $(k-1)$-dimensional simplex $\tau$. Let $\Sigma = \sigma_1 \cup \sigma_2$. Let $\alpha: \Sigma \to \Sigma$ be the automorphism of $\Sigma$ which interchanges simplices $\sigma_1$ and $\sigma_2$ keeping the common face $\tau$ point-wise fixed.

Suppose that $\gamma_i: \sigma_i \to \Sigma$ is an obvious embedding and $\hat{\gamma}_i$ the associated maps of $\text{Hom}$-complexes. Then,

(a) the induced map $\hat{\alpha}: \text{Hom}(\Sigma, L) \to \text{Hom}(\Sigma, L)$ is homotopic to the identity map $I_{\Sigma}$, and

(b) the diagram

$$\begin{array}{ccc}
\text{Hom}(\Sigma, L) & \xrightarrow{=} & \text{Hom}(\Sigma, L) \\
\gamma_1 \downarrow & & \downarrow \gamma_2 \\
\text{Hom}(\sigma_1, L) & \xleftarrow{\mathcal{P}^H(\sigma_1 \sigma_2)} & \text{Hom}(\sigma_2, L)
\end{array}$$

is commutative up to homotopy.

**Proof:** By Proposition \ref{prop_4.4} both maps $\hat{\gamma}_i: \text{Hom}(\Sigma, L) \to \text{Hom}(\sigma_i, L)$ for $i = 1, 2$ are homotopy equivalences. Let $\rho_1: \Sigma \to \sigma_1$ and $\rho_2: \Sigma \to \sigma_2$ be the folding maps. Then $\rho_i \circ \gamma_i = I_{\sigma_i}$, $\hat{\gamma}_i \circ \hat{\rho}_i = I$ and we conclude that $\hat{\rho}_i: \text{Hom}(\sigma_i, L) \to \text{Hom}(\Sigma, L)$ is also a homotopy equivalence.

Part (a) of the proposition follows from the fact that $\rho_1 \circ \alpha \circ \gamma_1 = I_{\sigma_1}$ is an identity map. Indeed, an immediate consequence is that $\gamma_1 \circ \alpha \circ \hat{\rho}_1 = I: \text{Hom}(\sigma_1, L) \to \text{Hom}(\sigma_1, L)$ is also an identity map and, in light of the fact that $\hat{\gamma}_1$ and $\hat{\rho}_1$ are homotopy inverses to each other, we deduce that $\hat{\alpha} \simeq I$.

For the part (b) we begin by an observation that $\rho_2 \circ \alpha \circ \gamma_1 = \sigma_1 \sigma_2$. Then, $\mathcal{P}^H(\sigma_1 \sigma_2) = \gamma_1 \circ \alpha \circ \rho_2$, and as a consequence of $\hat{\alpha} \simeq I$ and the fact that $\hat{\rho}_2 \circ \hat{\gamma}_2 \simeq I$, we conclude that $\mathcal{P}^H(\sigma_1 \sigma_2) \circ \hat{\gamma}_2 = \gamma_1 \circ \alpha \circ \rho_2 \circ \hat{\gamma}_2 \simeq \hat{\gamma}_1$. □
Proposition 17 Suppose that \( K \) and \( L \) are finite simplicial complexes and \( \sigma_1, \sigma_2 \) a pair of adjacent (distinct), \( k \)-dimensional simplices in \( K \). Let \( \alpha_i : \sigma_i \to K \) be the embedding of \( \sigma_i \) in \( K \) and \( \tilde{\alpha}_i : \text{Hom}(K, L) \to \text{Hom}(\sigma_i, L) \) the associated map of \( \text{Hom} \)-complexes. Then the following diagram commutes up to a homotopy.

\[
\begin{array}{ccc}
\text{Hom}(K, L) & \overset{=}{\longrightarrow} & \text{Hom}(K, L) \\
\tilde{\alpha}_1 \downarrow & & \downarrow \tilde{\alpha}_2 \\
\text{Hom}(\sigma_1, L) & \overset{p_L(\tilde{e}_1\tilde{e}_2)}{\leftarrow} & \text{Hom}(\sigma_2, L)
\end{array}
\]

(12)

Proof: Let \( \Sigma := \sigma_1 \cup \sigma_2 \), \( \tau := \sigma_1 \cap \sigma_2 \). Then \( \alpha_i = \beta \circ \gamma_i \) where \( \beta : \Sigma \to K \) and \( \gamma_i : \sigma_i \to \Sigma \) are natural embeddings of complexes. The diagram (12) can be factored as

\[
\begin{array}{ccc}
\text{Hom}(K, L) & \overset{=}{\longrightarrow} & \text{Hom}(K, L) \\
\tilde{\beta} \downarrow & & \downarrow \tilde{\beta} \\
\text{Hom}(\Sigma, L) & \overset{=}{\longrightarrow} & \text{Hom}(\Sigma, L) \\
\tilde{\gamma}_1 \downarrow & & \downarrow \tilde{\gamma}_2 \\
\text{Hom}(\sigma_1, L) & \overset{p_L(\tilde{\sigma}_1\tilde{\sigma}_2)}{\leftarrow} & \text{Hom}(\sigma_2, L)
\end{array}
\]

(13)

Then the result is a direct consequence of Proposition 16 part (b). \( \square \)

Corollary 18 Suppose that \( K \) and \( L \) are finite simplicial complexes, \( \sigma \) a \( k \)-dimensional simplex in \( K \) and \( \alpha : \sigma \to K \) the associated embedding. Let \( \tau \in \Pi(K, \sigma) \). Then the following diagram commutes up to a homotopy.

\[
\begin{array}{ccc}
\text{Hom}(K, L) & \overset{=}{\longrightarrow} & \text{Hom}(K, L) \\
\tilde{\alpha} \downarrow & & \downarrow \tilde{\alpha} \\
\text{Hom}(\sigma, L) & \overset{\tilde{\tau}}{\leftarrow} & \text{Hom}(\sigma, L)
\end{array}
\]

(14)

6 Main results

In this section we prove the promised extension of the Lovász-Babson-Kozlov theorem. The graphs are replaced by pure \( d \)-dimensional simplicial complexes, while the role of the odd cycle \( C_{2r+1} \) is played by a complex \( \Gamma \) which has some special symmetry properties in the sense of the following definition.

As usual, an involution \( \omega : X \to X \) is the same as a \( \mathbb{Z}_2 \)-action on \( X \). An involution on a simplicial complex \( \Gamma \) induces an involution on the complex \( \text{Hom}(\Gamma, L) \) for each simplicial complex \( L \). For all other standard facts and definitions related to \( \mathbb{Z}_2 \)-complexes, the reader is referred to [22].

Definition 19 A pure \( d \)-dimensional simplicial complex \( \Gamma \) is a \( \Phi_d \)-complex if it is a \( \mathbb{Z}_2 \)-complex with an invariant \( d \)-simplex \( \sigma = \{v_0, v_1, \ldots, v_d\} \) such that the restriction \( \tau := \omega|_{\sigma} \) of the involution \( \omega : \Gamma \to \Gamma \) on \( \sigma \) is a non-trivial element of the group \( \Pi(\Gamma, \sigma) \).
Remark 20 By definition, if $\Gamma$ is a $\Phi_d$-complex then the inclusion map $\alpha : \sigma \to \Gamma$ is $\mathbb{Z}_2$-equivariant, so the associated map $\hat{\alpha} : \text{Hom}(\Gamma, K) \to \text{Hom}(\sigma, K)$ is also $\mathbb{Z}_2$-equivariant for each complex $K$.

Example 21 The graph $C_{2r+1}$ is obviously an example of a $\Phi_1$-complex. Figure 1 displays four examples of $\Phi_2$-complexes, initial elements of two infinite series $\nabla_\mu$ and $\Sigma_\nu$, $\mu, \nu \in \mathbb{N}$. The complexes $\nabla_1$ and $\nabla_2$ etc. are obtained from two triangulated annuli, glued together along a common triangle $\sigma$. Similarly, the complexes $\Sigma_1, \Sigma_2, \ldots$, are obtained by gluing together two triangulated Möbius strips. The associated group of projectivities are $\Pi(\nabla_\mu, \sigma) = S_3$ and $\Pi(\Sigma_\nu, \sigma) = \mathbb{Z}_2$.

![Figure 1: Examples of $\Phi_2$-complexes.](image)

Theorem 22 Suppose that $\Gamma$ is a $\Phi_d$-complex in the sense of Definition 19 with an associated invariant simplex $\sigma = \{v_0, v_1, \ldots, v_d\}$. Suppose that $K$ is a pure $d$-dimensional simplicial complex. Then for $m$ even,

$$\text{Coind}_{\mathbb{Z}_2}(\text{Hom}(\Gamma, K)) \geq m \implies \chi(K) \geq m + d + 2. \quad (15)$$

Proof: By definition $\text{Coind}_{\mathbb{Z}_2}(\text{Hom}(\Gamma, K)) \geq m$ means that there exists a $\mathbb{Z}_2$-equivariant map $\mu : S^m \to \text{Hom}(\Gamma, K)$. Assume that $\chi(K) \leq m + d + 1$ which means that there exists a non-degenerated simplicial map $\phi : K \to \Delta^{[m+d+1]}$. By functoriality of the construction of $\text{Hom}$-complexes, Section 5.2, there is an induced $\mathbb{Z}_2$-equivariant map $\hat{\phi} : \text{Hom}(\Gamma, K) \to \text{Hom}(\Gamma, \Delta^{[m+d+1]})$ and similarly a map $\hat{\alpha} : \text{Hom}(\Gamma, \Delta^{[m+d+1]}) \to \text{Hom}(\sigma, \Delta^{[m+d+1]})$. By [13] Theorem 3.3.3., the complex

$$\text{Hom}(\sigma, \Delta^{[m+d+1]}) \cong \text{Hom}(K_{d+1}, K_{m+d+1})$$

is a wedge of $m$-dimensional spheres. Since $\text{Hom}(\sigma, \Delta^{[m+d+1]})$ is a free $\mathbb{Z}_2$-complex, we deduce that there exists a $\mathbb{Z}_2$-equivariant map $\tilde{\text{Hom}}(\sigma, \Delta^{[m+d+1]}) \to S^m$. All these maps can be arranged in the following sequence of $\mathbb{Z}_2$-equivariant maps

$$S^m \xrightarrow{\mu} \text{Hom}(\Gamma, K) \xrightarrow{\hat{\phi}} \text{Hom}(\Gamma, \Delta^{[m+d+1]}) \xrightarrow{\hat{\alpha}} \text{Hom}(\sigma, \Delta^{[m+d+1]}) \xrightarrow{\nu} S^m.$$
By Corollary 18, there is a homotopy equivalence $\hat{\alpha} \simeq \tau \circ \hat{\alpha}$. This is in contradiction with Proposition 23 which completes the proof of the theorem. □

**Proposition 23** Suppose that $f : X \rightarrow Y$ is a $\mathbb{Z}_2$-equivariant map of free $\mathbb{Z}_2$-complexes $X$ and $Y$ where $\mathbb{Z}_2 = \{1, \omega\}$. Assume that $\text{Coind}_{\mathbb{Z}_2}(X) \geq m \geq \text{Ind}_{\mathbb{Z}_2}(Y)$, where $m$ is an even integer. In other words our assumption is that there exist $\mathbb{Z}_2$-equivariant maps $\mu$ and $\nu$ such that

$$S^m \xrightarrow{\mu} X \xrightarrow{f} Y \xrightarrow{\nu} S^m.$$ 

Then the maps $f$ and $\omega \circ f$ are not homotopic.

**Proof:** If $f \simeq \omega \circ f : X \rightarrow Y$ then $\nu \circ f \circ \mu \simeq \nu \circ \omega \circ f \circ \mu : S^m \rightarrow S^m$ and by the equivariance of $\nu$, $\omega \circ g \simeq g : S^m \rightarrow S^m$ where $g := \nu \circ f \circ \mu$. It follows that

$$-\deg(g) = \deg(\omega)\deg(g) = \deg(\omega \circ g) = \deg(g),$$

i.e. $\deg(g) = 0$, which is in contradiction with a well known fact that a $\mathbb{Z}_2$-equivariant map $g : S^m \rightarrow S^m$ of even dimensional spheres must have an odd degree. □

**Corollary 24** Suppose that $\Gamma$ is a $\Phi_d$-complex with an associated invariant simplex $\sigma = \{v_0, v_1, \ldots, v_d\}$. Suppose that $K$ is a pure $d$-dimensional simplicial complex. Then for $k$ odd,

$$\text{Hom}(\Gamma, K) \text{ is } k\text{-connected } \Rightarrow \chi(K) \geq k + d + 3. \quad (16)$$

**Proof:** If $\text{Hom}(\Gamma, K)$ is $k$-connected then $\text{Coind}_{\mathbb{Z}_2}(\text{Hom}(\Gamma, K)) \geq k + 1$, hence the implication (16) is an immediate consequence of Theorem 22. □
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