$P − V$ criticality of AdS black hole in $f(R)$ gravity

Songbai Chen*, Xiaofang Liu, Changqing Liu
Institute of Physics and Department of Physics, Hunan Normal University, Changsha, Hunan 410081, People’s Republic of China
Key Laboratory of Low Dimensional Quantum Structures and Quantum Control of Ministry of Education, Hunan Normal University, Changsha, Hunan 410081, People’s Republic of China

Abstract

We study thermodynamics of a charged AdS black hole in the special $f(R)$ correction with the constant Ricci scalar curvature. Our results show that the $f(R)$ correction influences the Gibbs free energy and the phase transition of system. The ratio $\rho_c$ occurred at the critical point increases monotonically with the derivative term $f'(R_0)$. We also disclose that the critical exponents are the same as those of the liquid-gas phase transition in the Van der Waals model, which does not depend on the $f(R)$ correction considered here.

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* csb3752@hunnu.edu.cn
I. INTRODUCTION

$f(R)$ gravity is a kind of important modified gravity theory, which generalizes Einstein’s gravity by adding higher powers of the scalar curvature $R$, the Riemann and Ricci tensors, or their derivatives in the usual lagrangian formulation [1–3]. Without unknown forms of dark energy or dark matter, $f(R)$ gravity can explain the accelerating expansion of the current Universe. It can also mimic the inflation and structure formation in the early Universe. Thus, a lot of efforts have been focused in the study of $f(R)$ gravity including not only the applications of $f(R)$ theories on gravitation and cosmology, but also many observational and experimental methods to distinguish them from general relativity [2, 3].

It is of interest to extend the study of $f(R)$ gravity to the black hole physics, which could provide us some features of black holes differed from that obtained in Einstein’s gravity. In general, the black hole solution is difficult to be obtained in the $f(R)$ theory since the field equations of $f(R)$ gravity become more complicated than that in Einstein’s gravity, even without the presence of a matter field. With the condition that the energy-momentum tensor is traceless for the matter field, Moon et al [4] obtain an exact analytical four-dimensional solution from $R + f(R)$ theory coupled to a matter field, which is later extended to the rotating cases [5]. The higher dimensional charged solution [6] are obtained only in the case of power-Maxwell field with $d = 4p$, where $p$ is the power of conformally invariant Maxwell lagrangian. Moreover, some other solutions of black hole in $f(R)$ gravity have been also constructed in [7–11].

Thermodynamical properties of the AdS black hole has been a subject of intense study for the past decades because that it is dual to a thermal state on the conformal boundary in terms of the AdS/CFT correspondence. The duality has been recently applied to study the strongly correlated condensed matter physics from the gravitational dual. Recently, Kubiznak et al [12] investigated the critical behavior of charged AdS black holes by treating the cosmological constant as a thermodynamic pressure and its conjugate quantity as a thermodynamic volume [13]. It is found that in the system of charged AdS black hole the small-large black hole phase transition possesses the same critical behavior of liquid-gas phase transitions in the Van der Waals model. The similar critical behaviors are disclosed in the spacetimes of a rotating AdS black hole [14] and a high dimensional Reissner-Nordström-AdS black hole [15]. The same qualitative properties are also found in the AdS black hole spacetime with the Born-Infeld electrodynamics [14, 16], with the power-Maxwell field [17] and with Gauss-Bonnet correction [18].

The aim of this Letter is to study the thermodynamics of a charged AdS black hole [4] in the special $f(R)$
correction in which the Ricci scalar curvature remains the constant \( R = R_0 \) and to probe the effects of the \( f(R) \) correction on thermodynamical quantities and the small-large black hole phase transition. Finally, we discuss the analogy of black hole with Van der Waals liquid-gas system.

We firstly review the four-dimensional charged AdS black hole obtained by Moon et al.\cite{4} in the \( R + f(R) \) gravity with the constant Ricci scalar curvature. The action describing a four-dimensional charged AdS black hole in the \( R + f(R) \) gravity can be expressed as \cite{4}

\[
S = \int_{\mathcal{M}} d^4x \sqrt{-g}[R + f(R) - F_{\mu\nu}F^{\mu\nu}],
\]

where \( R \) is the Ricci scalar curvature and \( f(R) \) is an arbitrary function of \( R \). \( F_{\mu\nu} \) is electromagnetic field tensor which is related to the electromagnetic potential \( A_\mu \) by \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \).

Varying the action (1), one can obtain the equations of motion for gravitational field \( g_{\mu\nu} \) and the gauge field \( A_\mu \)

\[
R_{\mu\nu}[1 + f'(R)] - \frac{1}{2}g_{\mu\nu}[R + f(R)] + (g_{\mu\nu} \nabla^2 - \nabla_\mu \nabla_\nu)f'(R) = T_{\mu\nu}.
\]

and

\[
\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0,
\]

respectively. In order to obtain an analytical solution of the equation (2), Moon et al.\cite{4} consider only the case of the constant Ricci scalar curvature \( R = R_0 = \text{const} \) for the sake of convenience. In this simple case, Eq. (2) can be rewritten as

\[
R_{\mu\nu}[1 + f'(R_0)] - \frac{g_{\mu\nu}}{4} R_0[1 + f'(R_0)] = T_{\mu\nu},
\]

since the Maxwell energy-momentum tensor is traceless in the four-dimensional spacetime. The general metric for a four-dimensional charged static spherically symmetric black hole can be expressed as

\[
ds^2 = -N(r)dt^2 + \frac{dr^2}{N(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2).
\]

Solving Eq. (3) in the four-dimensional static spacetime \cite{5}, one can get the Maxwell field

\[
F_{tr} = \frac{q}{r^2}.
\]

Inserting Eqs. (5) and (6) into Eq. (4), one can find that the metric \cite{3} has the form

\[
N(r) = 1 - \frac{2m}{r} + \frac{q^2}{br^2} - \frac{R_0}{12} r^2.
\]
Here $b = [1 + f'(R_0)]$. The parameters $m$ and $q$ are related to the ADM mass $M$ and the electric charge $Q$ of the black hole by

$$M = mb, \quad Q = \frac{q}{\sqrt{b}}$$  \hspace{1cm} (8)$$

and the electric potential $\Phi$ at horizon $r_+$ are

$$\Phi = \frac{\sqrt{b}q}{r_+}$$  \hspace{1cm} (9)$$

Thus, the $f(R)$ correction modifies the mass $M$ and the electric charge $Q$ of black hole and the corresponding thermodynamic potentials. Treating the Ricci scalar curvature as $R_0 = -\frac{12}{r^2} = -4\Lambda$, one can find that the spacetime described by solution $(7)$ is asymptotically AdS. In other words, the Ricci scalar curvature $R_0$ is negative in the AdS background. The Hawking temperature $T$ with the outer event horizon $r = r_+$ is

$$T = \frac{N'(r_+)}{4\pi} \bigg|_{r=r_+} = \frac{1}{4\pi r_+} \left[ 1 - \frac{q^2}{r_+^2 b} - \frac{R_0 r_+^2}{4} \right],$$  \hspace{1cm} (10)$$

and the entropy of the black hole is

$$S = \pi r_+^2 b.$$  \hspace{1cm} (11)$$

For an AdS black hole, one can get both the differential and integral formulas of the first law of thermodynamics by treating the cosmological constant as a variable related to the thermodynamic pressure [13]. Motivated by this spirit, we here interpret $R_0$ as a thermodynamic pressure $P$, which is given by

$$P = -\frac{bR_0}{32\pi},$$  \hspace{1cm} (12)$$

and the corresponding volume $V$ is

$$V = \frac{4\pi r_+^3}{3}.$$  \hspace{1cm} (13)$$

Thus, for the charged AdS black hole [5], the Smarr formula can be expressed as

$$M = 2TS + \Phi Q - 2PV,$$  \hspace{1cm} (14)$$

which is consistent with the usual charged AdS black hole without $f(R)$ correction. The differential form the first law of thermodynamics becomes

$$d\left( \frac{M}{b} \right) = T d\left( \frac{S}{b} \right) + \left( \frac{\Phi}{b} \right) dQ + V d\left( \frac{P}{b} \right).$$  \hspace{1cm} (15)$$

Obviously, it recovers the standard first law of thermodynamics for the black hole as $b = 1$. From Eqs. (12) and (13), we find that $R_0$ and $f'(R_0)$ modifies the thermodynamic pressure $P$ of the system and the
differential form the first law of thermodynamics. Thus, it is expected that the \( f(R) \) correction will influence the thermodynamical phase transition of the AdS black hole and could lead to some new features for the thermodynamics of black hole.

From the Hawking temperature (10) and the pressure (12) of the system, one can get the equation of state \( P = P(T, V) \),

\[
P = \frac{bT}{2r_+} - \frac{b}{8\pi r_+^2} + \frac{q^2}{8\pi r_+^4},
\]

for a fixed charge \( q \). Here \( r_+ \) is a function of the thermodynamic volume \( V \). As doing in Refs. [12], one can find that the geometric quantities \( P \) and \( T \) can be translated into physical pressure and temperature of system by using dimensional analysis and \( l_p^2 = G\hbar/c^3 \)

\[
[\text{Press}] = \frac{\hbar c}{l_p^2} P, \quad [\text{Temp}] = \frac{\hbar c}{k} T.
\]

And then the physical pressure and physical temperature are given by

\[
\text{Press} = \frac{\hbar c}{l_p^2} P = \frac{\hbar c}{l_p^2} \frac{T}{2r_+} + \cdots = \frac{k\text{Temp}}{2l_p^2} + \cdots.
\]

In order to compare it with the Van der Waals equation [12, 14], we must identify the specific volume \( v \) with the horizon radius \( r_+ \) of the black hole by

\[
v = 2r_+ l_p^2.
\]

After finishing these operations, one can find that the equation of state (16) can be rewritten as

\[
P = \frac{bT}{v} - \frac{b}{2\pi v^2} + \frac{2q^2}{\pi v^4}.
\]

In Fig.(1) , we plot the \( P - V \) diagram for a four-dimensional charged AdS black hole in the \( f(R) \) with the different \( f'(R_0) \) for fixed \( q = 1 \). Obviously, there exist a small-large black hole phase transition in this system. Fig.(1) tells us that with increase of \( f'(R_0) \) the critical temperature \( T_c \) increases.

The critical points in \( P - V \) diagram can be obtained from

\[
\frac{\partial P}{\partial v} = 0, \quad \frac{\partial^2 P}{\partial v^2} = 0,
\]

which yields

\[
v_c = \frac{2q\sqrt{6}}{\sqrt{b}}, \quad T_c = \frac{\sqrt{6b}}{18\pi q}, \quad P_c = \frac{b^2}{96\pi q^2}.
\]
These formulas tell us that with increase of $f'(R_0)$ both of the critical temperature $T_c$ and the pressure $P_c$ increase, but the critical volume $v_c$ decreases, which is consistent with that shown in Fig. (1). In order to ensure the occurrence of the phase transition, all of the critical values $P_c$, $T_c$ and $v_c$ must be positive, which implies that the condition $b = 1 + f'(R_0) > 0$ should be satisfied for the black hole.

The ratio occurred at the critical point is given by

$$\rho_c = \frac{P_c v_c}{T_c} = \frac{3b}{8}$$

(23)

for the charged AdS black hole in the $f(R)$ gravity. Clearly, the $f(R)$ correction changes the ratio $\rho_c$ occurred at the critical point. It depends on the concrete form of $f(R)$ gravity. With increase of $f'(R_0)$, the ratio $\rho_c$ increases monotonically. When the derivative term $f'(R_0)$ disappears, one can find that the ratio $\rho_c$ is recovered to that in the usual four-dimensional charged AdS black hole spacetime. From Eq.(23), one can find that the charged AdS black hole in the $f(R)$ gravity possesses the similar feature of the Van der-Waals fluid.

Let us now analyze the Gibbs free energy of the system. For a canonical ensemble, the Gibbs free energy is $G = M - TS$. This yields that for a charged AdS black hole in the $f(R)$ gravity the Gibbs free energy is

$$G = \frac{1}{4} \left[ br_+ - \frac{8\pi P r_+^3}{3} + \frac{3q^2}{r_+} \right].$$

(24)

We plot the change of the free energy $G$ with $T$ for fixed $q$ in Fig.(2). The presence of the characteristic "swallow tail" behavior of the free energy $G$ means that the small-large black hole transition occurred in the system is a first order phase transition. The two-phase existence is also shown in Fig. (3), in which the coexistence line is obtained by a fact that two phases share the same Gibbs free energy and temperature during the phase transition. The coexistence line can be also produced by Maxwell’s equal area law and the
FIG. 2: Gibbs free energy of a four-dimensional charged AdS black hole in the $f(R)$ gravity. The dashed, solid, and dot-dashed lines correspond to the cases $P/P_c = 1.5$, 1, and 0.2. The left, middle and right panels are for $f'(R_0) = -0.1$, 0, and 0.1, respectively. Here we set $q = 1$.

Figure 2: Gibbs free energy diagram for different values of $P/P_c$.

Clausius-Clapeyron equation. From Fig. (3), one can find that the coexistence line depends on the form of $f(R)$ and its slope increases with the derivative term $f'(R_0)$.

FIG. 3: Coexistence line of a four-dimensional charged AdS black hole in $f(R)$ gravity. The critical point is shown by a small circle at the end of the coexistence line. The left, middle and right panels are for $f'(R_0) = -0.5$, 0, and 0.5, respectively. Here we set $q = 1$.

The critical exponents are very important for the phase transition occurred in the thermodynamic system. Here, we will discuss the critical exponents $\alpha, \beta, \gamma, \delta$ for the charged AdS black hole in the $f(R)$ gravity. Similarly, the entropy (11) $S$ can be expressed as a function of $T$ and $V$,

$$S = S(T, V) = b \left(\frac{3V}{4\pi}\right)^{2/3}. \quad (25)$$

Obviously, it is independent of the temperature $T$, which yields directly that the specific heat vanishes $C_V = 0$ and the critical exponent $\alpha$ is zero. Adopting to the reduced thermodynamics

$$p = \frac{P}{P_c}, \quad \nu = \frac{v}{v_c}, \quad \tau = \frac{T}{T_c}, \quad (26)$$
one can find that the equation of state (16) can be rewritten as

\[ p = \frac{8\tau}{3\nu} - \frac{2}{\nu^2} + \frac{1}{3\nu^4}. \] (27)

It is the same as that of a charged AdS black hole without \( f(R) \) correction, which means that the \( f(R) \) correction with the constant Ricci scalar curvature does not change the critical exponents \( \alpha, \beta, \gamma, \delta \) for the system of black hole. In other words, the critical exponents for a charged AdS black hole in \( f(R) \) gravity with the constant Ricci scalar curvature coincide with those of the Van der Waals fluid and those of in the mean field theory.

In this Letter, we consider thermodynamics of a charged AdS black hole in the special \( f(R) \) correction in which the Ricci scalar curvature is the constant \( R = R_0 \). And then we study the effects of the \( f(R) \) correction on thermodynamical quantities and the phase transition. Our results show that the \( f(R) \) correction changes the equation of state of the system and influence the critical temperature \( T_c \), the volume \( v_c \), and the pressure \( P_c \). With increase of \( f'(R_0) \), both of the critical temperature \( T_c \) and the pressure \( P_c \) increase, but the critical volume \( v_c \) decreases. Moreover, we also find the ratio \( \rho_c \) occurred at the critical point depends on the form of \( f(R) \) and it increases monotonically with the derivative term \( f'(R_0) \). As the derivative term \( f'(R_0) \) disappears, it is recovered to that in the usual four-dimensional charged AdS black hole spacetime without \( f(R) \) correction. Moreover, we also find that the \( f(R_0) \) correction affects the Gibbs free energy and the slope of the two-phase coexistence line in the \( P-T \) plane. However, the critical exponents \( \alpha, \beta, \gamma, \delta \) are independent of the special \( f(R) \) correction and coincide with those of the Van der Waals fluid and of in the mean field theory, which could be the universal property for such kinds of phase transition.

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