Hylomorphic solitons

Vieri Benci
Dipartimento di Matematica Applicata “U. Dini”
Università di Pisa
Via Filippo Buonarroti 1/c, 56127 Pisa, Italy
e-mail: benci@dma.unipi.it

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Abstract
This paper is devoted to the study of solitary waves and solitons whose existence is related to the ratio energy/charge. These solitary waves are called hylomorphic. This class includes the Q-balls, which are spherically symmetric solutions of the nonlinear Klein-Gordon equation (NKG), as well as solitary waves and vortices which occur, by the same mechanism, in the nonlinear Schrödinger equation and in gauge theories. This paper is devoted to the study of hylomorphic soliton. Mainly we will be interested in the very general principles which are at the base of their existence such as the Variational Principle, the Invariance Principle, the Noether theorem, the Hamilton-Jacobi theory etc. We give a general definition of hylomorphic solitons and an interpretation of their nature (swarm interpretation) which is very helpful in understanding their behavior. We apply these ideas to the Nonlinear Schrödinger Equation (NS) and to the Nonlinear Klein-Gordon Equation (NKG) respectively.

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Introduction

Roughly speaking a solitary wave is a solution of a field equation whose energy travels as a localized packet and which preserves this localization in time. A soliton is a solitary wave which exhibits some strong form of stability so that it has a particle-like behavior.

Today, we know (at least) three mechanism which might produce solitary waves and solitons:

- Complete integrability: e.g. Kortewg-de Vries equation
  \[ u_t + u_{xxx} + 6uu_x = 0 \]

- Topological constrains: e.g. Sine-Gordon equation
  \[ u_{tt} - u_{xx} + \sin u = 0 \]

- Ratio energy/charge: e.g. the following nonlinear Klein-Gordon equation
  \[ \psi_{tt} - \Delta \psi + \frac{\psi}{1 + |\psi|} = 0; \quad \psi \in \mathbb{C} \]

This paper is devoted to the third type of solitons which will be called hylomorphic solitons. This class of solitons includes the q-balls (see [20]) which are spherically symmetric solutions of as well as solitary waves and vortices which occur, by the same mechanism, in the nonlinear Schroedinger equation and in gauge theories.

The most general equations for which it is possible have hylomorphic solitons need to have the following features:

- **A-1.** The equations are variational namely they are the Euler-Lagrange equation relative to a Lagrangian density \( \mathcal{L} \).
• A-2. The equations are invariant for time translations, namely $L$ does not depend explicitly on $t$.

• A-3. The equations are invariant for a gauge action, namely $L$ does not depend explicitly on the phase of the field $\Psi$ which is supposed to be complex valued (or at least to have some complex valued component).

By Noether theorem A-1 and A-2 guarantee the conservation of energy, while A-1 and A-3 guarantee the conservation of another integral of motion which we call hylenic charge.

The existence of hylomorphic solitons is guaranteed by the interplay between energy and hylenic charge.

This paper is devoted to the study of hylomorphic solitons. Mainly we will be interested in the very general principles which are at the base of their existence such as the Variational Principle, the Invariance Principle, the Noether theorem, the Hamilton-Jacobi theory etc. These principles will be discussed in section 2. We will also give a proof of Noether theorem in the form needed for this study.

In section 3 we give a general definition of hylomorphic solitons and we give an interpretation of their nature (swarm interpretation) which is very helpful in understanding their behavior.

In sections 4 and 5 we apply these ideas to the Nonlinear Schrödinger Equation (NS) and to the Nonlinear Klein-Gordon Equation (NKG) respectively. The amount of results relative to NSE and NKG is huge (we just to cite some of the main papers: [25], [34], [37], [19], [14], [18], [36], [20] and the books [35], [38])); however, here we restrict ourselves to the aspects of these equations related to the ideas of the previous sections; in particular, we will emphasize their common "hylomorphic" nature which has been recently discovered. The technical aspects related to the study of these equations will be treated superficially, but we will refer the interested readers to the appropriate papers.

Another set of equations to which this kind of ideas can been applied are the Klein-Gordon-Maxwell Equations (NKGM) and also more general gauge theories such as the Yang-Mills equations. In this paper, we will not discuss these equation and we refer to [6], [7], [16], [23], [24], [32], [9], [10] and their references.

Another very interesting aspect of NS, NKG and NKGM is the fact that they admit hylomorphic vortices namely solitary waves having a nonvanishing angular momentum. The existence of these vortices is guaranteed by a mechanism similar to that of hylomorphic solitary waves. Here we will not discuss the hylomorphic vortices and we refer to [13], [8], [9], [5] and [10].

2 The general theory

2.1 The variational principle

The fundamental equation of Physics are the Euler-Lagrange equations of a suitable functional. This fact is quite surprising. There is no logical reason for this. It is just an empirical fact: all the fundamental equations which have been discovered until now derive from a variational principle.

For example, the equations of motion of $k$ particles whose positions at time $t$ are given by $x_j(t), x_j \in \mathbb{R}^3, j = 1, ..., k$ are obtained as the Euler-Lagrange
equations relative to the following functional

\[ S = \int \left( \sum_j \frac{m_j}{2} |\dot{x}_j|^2 - V(t, x_1, ..., x_k) \right) dt \]  

(2.1)

where \( m_j \) is the mass of the \( j \)-th particle and \( V \) is the potential energy of the system.

More generally, the equations of motion of a finite dimensional system whose generalized coordinates are \( q_j(t), \ j = 1, ..., k \) are obtained as the Euler Lagrange equations relative to the following functional

\[ S = \int \mathcal{L}(t, q_1, ..., q_k, \dot{q}_1, ..., \dot{q}_k) \, dt \]

Also the Dynamics of fields can be determined by the Variational Principle. From a mathematical point of view a field is a function \( u : \mathbb{R}^{N+1} \rightarrow \mathbb{R}^k, \ u = (u_1, ..., u_k) \).

where \( \mathbb{R}^{N+1} \) is the space-time continuum and \( \mathbb{R}^k \) is called the internal parameters space. Of course, in physical problems, the space dimension \( N \) is 1, 2 or 3. The space and time coordinates will be respectively denoted by \( x = (x_1, ..., x_N) \) and \( t \) respectively. The function \( u(t, x) \) describes the internal state of the ether (or vacuum) at the point \( x \) and time \( t \).

From a mathematical point of view, assumption A-1 states that the field equations are obtained by the variation of the action functional defined as follows:

\[ S = \int \int \mathcal{L}(t, x, u, \nabla u, \partial_t u) \, dx \, dt. \]  

(2.2)

The function \( \mathcal{L} \) is called Lagrangian density function but in the following, as usual, we will call it just Lagrangian function.

If \( u \) is a scalar function, the variation of (2.2) gives the following equation:

\[ \sum_{i=0}^N \frac{\partial}{\partial x_i} \left( \frac{\partial \mathcal{L}}{\partial u_{x_i}} \right) - \frac{\partial \mathcal{L}}{\partial u} = 0 \]  

(2.3)

If \( u = (u_1, ..., u_k) \), the Euler-Lagrange equations take the same form provided that we have use the convention that

\[ \frac{\partial \mathcal{L}}{\partial u_{x_i}} = \left( \frac{\partial \mathcal{L}}{\partial u_{1,x_i}}, ..., \frac{\partial \mathcal{L}}{\partial u_{k,x_i}} \right), \quad \frac{\partial \mathcal{L}}{\partial u} = \left( \frac{\partial \mathcal{L}}{\partial u_1}, ..., \frac{\partial \mathcal{L}}{\partial u_k} \right). \]

So, if \( u \) has \( k \) components ( \( k > 1 \) ) then eq. (2.3) is equivalent to the \( k \) equations:

\[ \sum_{i=0}^N \frac{\partial}{\partial x_i} \left( \frac{\partial \mathcal{L}}{\partial u_{x_i}} \right) - \frac{\partial \mathcal{L}}{\partial u_{\ell}} = 0, \quad \ell = 1, ..., k \]  

(2.4)

\(^1\)We use the convention to use Greek letters \( \psi, \Psi \) etc. to denote complex valued functions and Latin letters \( u, v, \ldots \) to denote real valued function.
2.2 The invariance principle

A functional $J$ is called invariant for a representation $T_g$ of a Lie group if

$$J(T_g u) = J(u). \tag{2.5}$$

Now, let us consider the variational equation

$$\begin{cases}
    u \in X \\
    F(u) = 0
\end{cases}$$

where $F(u) = dJ(u)$. If $J$ is invariant, given any solution $u$, we have that also $T_g u$ is a solution.

We need to be careful in the interpretation of (2.5). In fact, if $u$ belongs to some function space $\mathcal{F}(\Omega, V)$, $(\Omega \subset \mathbb{R}^{N+1}$ and $V$ is a finite dimensional vector space), it might happen that $T_g u \notin \mathcal{F}(\Omega, V)$. For example, if

$$(T_h u)(x) = u(x - h), \ h \in \mathbb{R}^{N+1}$$

we have that $T_h u \in \mathcal{F}(\Omega - h, V)$.

Thus we are led to give the following definition: we say the Lagrangian $L(t, x, u, \nabla u, \partial_t u)$ is invariant with respect to the representation $T_g$ if

$$L(t, x, u, \nabla u, \partial_t u) = L(t', x', u', \nabla u', \partial_t u') \tag{2.6}$$

where $u'(t', x') = T_g u(t, x)$.

In this case, the equation (2.5) need to be interpreted as follows:

$$\int_{T_g \Omega} L(t', x', u', \nabla u', \partial_t u') \, dx dt = \int_{\Omega} L(t, x, u, \nabla u, \partial_t u) \, dx dt \tag{2.7}$$

where

$$T_g \Omega := \{(t', x') \in \mathbb{R}^{N+1} : (t, x) \in \Omega\}.$$ 

We say that a Lagrangian $L$ is invariant for the action $T_g$ if (2.7) holds for all bonded sets $\Omega \subset \mathbb{R}^N$.

2.3 The Poincaré invariance

The fundamental equations of Physics are invariant for the Poincaré group: it is the basic principle on which the special theory of relativity is based.

The Poincaré group $\mathcal{P}$ is a generalization of the isometry group $\mathcal{E}$. The isometry group $\mathcal{E}$ in $\mathbb{R}^N$ is the group of transformation which preserves the quadratic form

$$|x|^2 := \sum_{i=1}^{N} x_i^2$$

i.e. the Euclidean norm and hence the Euclidean distance

$$d_E(x, y) = \sqrt{\sum_{i=1}^{N} |x_i - y_i|^2};$$

namely, if $g \in \mathcal{E}$,

$$d_E(gx, gy) = d_E(x, y).$$
If we identify the physical space with $\mathbb{R}^3$, the isometry group is also called the congruence group. Roughly speaking, the content of Euclidean geometry is the study of the properties of geometric objects which are preserved by the congruence group.

The Poincaré group $\mathfrak{P}$, by definition, is the transformation group in $\mathbb{R}^{N+1}$ which preserves the quadratic form

$$|x|^2_M = -x_0^2 + \sum_{i=1}^{N} x_i^2$$

which is induced by the Minkowski bilinear form

$$\langle x, y \rangle_M = -x_0 y_0 + \sum_{i=1}^{N} x_i y_i$$

The Minkowski vectors $v = (v_0, \ldots, v_N) \equiv (v_0, v)$ are classified according to their causal nature as follows:

- a vector is called space-like if $\langle v, v \rangle_M > 0$,
- a vector is called space-like if light-like if $\langle v, v \rangle_M = 0$,
- a vector is called space-like if time-like if $\langle v, v \rangle_M < 0$.

The causal nature is not changed by a Poincaré transformation, and hence it is not a transitive group (as the isometry group): space and time are mixed, but not so much.

In the real world we have $N = 3$ and the Poincaré group is a 10 parameter Lie group generated by the following one-parameter transformations:

- Space translations in the directions $x, y, z$:

  $$
  \begin{align*}
  x' &= x + x_0 \\
  y' &= y \\
  z' &= z \\
  t' &= t
  \end{align*}
  \quad
  \begin{align*}
  x' &= x \\
  y' &= y + y_0 \\
  z' &= z \\
  t' &= t
  \end{align*}
  \quad
  \begin{align*}
  x' &= x \\
  y' &= y \\
  z' &= z + z_0 \\
  t' &= t
  \end{align*}
  $$

  This invariance guarantees that space is homogeneous, namely that the laws of physics are independent of space: if an experiment is performed here or there, it gives the same results.

- Space rotations:

  $$
  \begin{align*}
  x' &= x \\
  y' &= y \cos \theta_1 - z \sin \theta_1 \\
  z' &= y \sin \theta_1 + z \cos \theta_1 \\
  t' &= t
  \end{align*}
  \quad
  \begin{align*}
  x' &= x \cos \theta_2 - z \sin \theta_2 \\
  y' &= y \\
  z' &= z \sin \theta_2 + z \cos \theta_2 \\
  t' &= t
  \end{align*}
  \quad
  \begin{align*}
  x' &= x \cos \theta_3 - y \sin \theta_3 \\
  y' &= x \sin \theta_3 + y \cos \theta_3 \\
  z' &= z \\
  t' &= t
  \end{align*}
  $$

  This invariance guarantees that space is isotropic, namely that the laws of physics are independent of orientation.
- Time translations:
  \[ x' = x, \quad y' = y, \quad z' = z, \quad t' = t + t_0. \]

  This invariance guarantees that time is isotropic; namely that the laws of
  physics are independent of time: if an experiment is performed earlier or
  later, it gives the same results.

- Lorentz boosts:
  \[
  x' = \gamma (x - v_1 t), \quad y' = \gamma (y - v_2 t), \quad z' = \gamma (z - v_3 t), \quad t' = \gamma (t - v_3 z),
  \]

  where
  \[ \gamma = \frac{1}{\sqrt{1 - v^2}}. \]

  with \( v = v_i, \ i = 1, 2, 3. \) This invariance is an empirical fact and, as it will be
  shown in section 5.5, it implies the remarkable facts of the theory of relativity
  such as the space contraction, the time dilation and the equality between mass
  and energy.

  The Lorentz group is the 6 parameters Lie group generated by the space
  rotations and the lorentz boosts (plus the time inversion, \( t \rightarrow -t, \) and
  the parity inversion \( (x, y, z) \rightarrow (-x, -y, -z) \)). Clearly it is a linear subgroup of
  \( GL(6). \)

  The Poincaré group is the 10 parameters Lie group generated by the Lorentz
  group and the space-time translations. Then it is a subgroup of the affine group
  in \( \mathbb{R}^6. \)

  The Poicaré group acts on a scalar field \( \psi \) by the following representation:
  \[ (T_g \psi) (t, x) = \psi (t', x'), \quad (t', x') = g (t, x) \]

  The simplest equation invariant for this representation of the Poincaré group
  is the D’Alembert equation:
  \[ \Box \psi = 0 \] (2.10)

  where
  \[ \Box \psi = \frac{\partial^2 \psi}{\partial t^2} - \Delta \psi \quad \text{and} \quad \Delta \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}. \]

  The D’Alembert equation is the simplest variational field equation invariant
  for the Poincaré group.

  In fact it is obtained from the variation of the action
  \[ S_0 = -\frac{1}{2} \int \langle d\psi, d\psi \rangle_M \ dx \ dt = \frac{1}{2} \int \left[ |\partial_t \psi|^2 - |\nabla \psi|^2 \right] \ dx \ dt. \] (2.11)

  In this case, the Lagrangian function is given by
  \[ L_0 = -\frac{1}{2} \langle d\psi, d\psi \rangle_M = \frac{1}{2} |\partial_t \psi|^2 - \frac{1}{2} |\nabla \psi|^2 \] (2.12)

  It is easy to check that if \( \psi \) is a solution of this equation, then also \( T_g \psi \) is a
  solutions of the equation for every \( g \in \mathcal{P}. \)
2.4 The Galileo invariance

The Galileo group \( \mathcal{G} \) as the Poincaré group is a transformation group on the space-time \( \mathbb{R}^4 \). The Galileo group, by definition, if the set of transformations which preserves the time intervals and the Euclidean distance between simultaneous points. More precisely, an affine transformation \( g \in \mathcal{G} \) if, given two points \((t_1, x_1)\) and \((t_2, x_2)\), we have that

\[
t'_1 - t'_2 = t_1 - t_2
\]

and

\[t_1 = t_2 \Rightarrow (t'_1 = t'_2 \text{ and } d_E (x'_1 - x'_2) = d_E (x_1 - x_2))\]

where \((t'_1, x'_1) = g (t_1, x_1)\) and \(d_E(x, y)\) is the Euclidean distance.

Thus the Galileo group is a 10 parameters Lie group generated by the space-time translation the space rotations but the Lorentz boosts (2.8) are replaced be the Galilean transformations namely by the tranformations

\[
x' = x - v_1 t \quad x' = x \quad x' = x
\]

\[
y' = y \quad y' = y - v_2 t \quad y' = y
\]

\[
z' = z \quad z' = z \quad z' = z - v_3 t
\]

\[
t' = t \quad t' = t \quad t' = t
\]

(2.13)

The equations of classical mechanics are invariant for the Galileo group. We are interested in field equations which are invariant for a representation of the Galileo group.

Given the Galileo tranformation \( g_\nu : \mathbb{R}^4 \to \mathbb{R}^4 \) defined by

\[g_\nu (t, x) = (t, x - \nu t)\]

we consider the representation \( T_{g_\nu} : L^2 (\mathbb{R}^4, \mathbb{C}) \to L^2 (\mathbb{R}^4, \mathbb{C}) \) defined by

\[
(T_{g_\nu} \psi) (t, x) = \psi (t, x - \nu t) e^{i(\nu \cdot x - \frac{1}{2} \nu^2 t)}, \tag{2.14}
\]

The Shroedinger equation for a free particle (of mass 1)

\[
i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \Delta \psi
\]

is the simplest second order equation invariant for a representation of the Galileo group on the space of complex valued vector fields \( L^2 (\mathbb{R}^4, \mathbb{C}) \).

2.5 The Gauge invariance

Take a function

\[\psi : \mathbb{R}^4 \to V\]

and assume that on \( V \) acts the representation \( T_g \) of some group \((G, \circ)\). This action induces two possible action on \( \psi \):

- a global action: \( \psi (x) \mapsto T_g \psi (x) \) where \( g \in G \)
- a local action: \( \psi (x) \mapsto T_{g(x)} \psi (x) \) where \( g(x) \) is a smooth function with values in \( G \).
In the second case, we have a representation of the infinite dimensional group
\[ \mathfrak{g} = C^\infty (\mathbb{R}^4, G) \]
equipped with the group operation
\[(g \circ h)(x) = g(x) \circ h(x)\]
If a Lagrangian \( \mathcal{L} \) satisfies the following condition,
\[ \mathcal{L}(t, x, \psi, \nabla \psi, \partial_t \psi) = \mathcal{L}(t, x, T_g \psi, \nabla (T_g \psi), \partial_t (T_g \psi)), \quad g \in G \]
we say that it is invariant for a local action of the group \( G \), or for a trivial gauge action of the group \( G \); if \( \mathcal{L} \) satisfies the following condition,
\[ \mathcal{L}(t, x, \psi, \nabla \psi, \partial_t \psi) = \mathcal{L}(t, x, T_g(x) \psi, \nabla (T_g(x) \psi), \partial_t (T_g(x) \psi)), \quad g(x) \in \mathfrak{g} \]
we say that it is invariant for a local action of the group \( G \), or for a gauge action of the group \( \mathfrak{g} \).

Let us consider two simple examples: the functional
\[ \int \mathcal{L} (\nabla u) \, dx , \quad u \in \mathbb{R} \]
is invariant for a global action of the group \((\mathbb{R}, +)\). In fact, if we set \( T_r u = u + r \), \( r \in \mathbb{R} \), we have that
\[ \mathcal{L} (\nabla u) = \mathcal{L} (\nabla (T_r u)) . \]
Next, consider the functional
\[ \int \mathcal{L} (d\alpha) \, dx \]
where \( \alpha \) is a 1-form and \( d \) is the exterior derivative of \( \alpha \). In this case, \( \mathcal{L} (d\alpha) \) is not only invariant for a trivial action of \((\mathbb{R}, +)\), but also for the local action
\[ T_g(x) \alpha = \alpha + dg(x), \quad g(x) \in \mathfrak{g} := C^\infty (\mathbb{R}^4, \mathbb{R}) \]
in fact
\[ \mathcal{L} (d (\alpha + dg(x))) = \mathcal{L} (d\alpha) . \]

The simplest gauge invariance can be obtained taking a complex valued scalar field
\[ \psi : \mathbb{R}^4 \to \mathbb{C}, \]
and to consider the group \( S^1 = \{ e^{i\theta} : \theta \in \mathbb{R} \} \) and the following representation
\[ \psi \mapsto e^{i\theta} \psi \quad (2.15) \]
The Schrödinger equation and the Klein-Gordon equation are invariant for the global action \((2.15)\). The Klein-Gordon-Maxwell equations are invariant for a local action \((2.15)\). For a discussion of these aspects of KGM, we refer to \[9\].
2.6 Noether’s theorem

In this section we will give a proof of Noether theorem stated in a suitable form for the applications considered in this paper.

First of all, we need the following lemma

**Lemma 1.** Let \( \rho : \mathbb{R}^{N+1} \rightarrow \mathbb{R} \) and \( J : \mathbb{R}^{N+1} \rightarrow \mathbb{R}^N \) be two smooth functions defined on the "space-time". Assume that they satisfy the continuity equation

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0 \tag{2.16}
\]

and that for all \( t \)

\[
\rho(\cdot, t), \frac{\partial \rho}{\partial t}(\cdot, t) \text{ and } J(\cdot, t) \text{ are in } L^1(\mathbb{R}^3) \tag{2.17}
\]

Then for all \( t \)

\[
\frac{d}{dt} \int_{\mathbb{R}^N} \rho(x, t)dx = 0 \tag{2.18}
\]

**Proof.** Let

\[
B_R = \{ x \in \mathbb{R}^3 : |x| < R \} , \quad R > 0
\]

then, integrating on \( B_R \), we get

\[
\int_{B_R} \frac{\partial \rho}{\partial t}dx = - \int_{B_R} \nabla \cdot J dx = - \int_{\partial B_R} (J \cdot n)d\sigma \tag{2.19}
\]

where \( n \) denotes the outward normal to the boundary \( \partial B_R \) of \( B_R \). Then

\[
\left| \int_{B_R} \frac{\partial \rho}{\partial t}dx \right| \leq \int_{\partial B_R} |J \cdot n| d\sigma \tag{2.20}
\]

Since \( \frac{\partial \rho}{\partial t}(\cdot, t) \) is in \( L^1(\mathbb{R}^3) \), there exists \( \lim_{R \to \infty} \left| \int_{B_R} \frac{\partial \rho}{\partial t}dx \right| \), and we have to prove that this limit is 0. Arguing by contradiction we assume that

\[
\lim_{R \to \infty} \left| \int_{B_R} \frac{\partial \rho}{\partial t}dx \right| = \alpha > 0 \tag{2.21}
\]

then, by (2.20) and (2.21), the map \( \varphi \) defined by

\[
\varphi(R) = \int_{\partial B_R} |J \cdot n| d\sigma
\]

is not integrable in \((0, +\infty)\) and

\[
\int_{\mathbb{R}^N} |J \cdot n| dx = \int_0^{+\infty} \varphi(R)dR = +\infty
\]

which contradicts assumption (2.17).

\[ \square \]

Suppose that a Lagrangian is invariant for the action \( T_g \) of some Lie group \( G \). We denote by \( T_{g(\lambda)} (\lambda \in \mathbb{R}) \) the action of a one-parameter subgroup \( \{ g(\lambda) \}_{\lambda \in \mathbb{R}} \).
Notice that this subgroup is isomorphic either to $S^1$ or $\mathbb{R}$. We use the following notation:

$$u_\lambda = T_{g(\lambda)}u$$  \hspace{1cm} (2.22)

and, if the group acts also on the independent variables, we set

$$t_\lambda = T_{g(\lambda)}t$$  \hspace{1cm} (2.23)

$$x_\lambda = T_{g(\lambda)}x$$  \hspace{1cm} (2.24)

For example, consider the first of the Lorentz transformation (2.8); in this case the parameter $\lambda$ is the first component of the velocity $v$, and we have

$$t_v = \frac{t - vx}{\sqrt{1 - v^2}}$$

$$x_{1,v} = \frac{x_1 - vt}{\sqrt{1 - v^2}}$$

$$u_v = u(t_v, x_{1,v}, x_2, x_3)$$

Lemma 2. If $\mathcal{L}$ is invariant with respect to a one parameter group $g(\lambda)$, then, using the notation (2.22), (2.23), (2.24), we have that

$$\frac{d}{d\lambda} \bigg|_{\lambda=0} \int_{\Omega} \mathcal{L}(t_\lambda, x_\lambda, u_\lambda, \nabla u_\lambda, \partial_t u_\lambda) \varphi(t_\lambda, x_\lambda) \, dxdt = 0$$

where $\Omega = [t_0, t_1] \times \mathbb{R}^N$ and $\varphi \in D(\Omega)$.

Proof. We approximate $\varphi$ with a function $\varphi_\varepsilon$ defined by

$$\varphi_\varepsilon = \sum_j j\varepsilon \chi_{\Omega_j}$$

where $\chi_{\Omega_j}$ is the characteristic function of

$$\Omega_j := \{(t, x) \in \mathbb{R}^{N+1} : j\varepsilon \leq \varphi(t, x) < (j + 1)\varepsilon\}$$

If $\lambda$ is small, so that the support of $\varphi(t_\lambda, x_\lambda)$ is contained in $\Omega$, we have that

$$\int_{\Omega} \mathcal{L}(t_\lambda, x_\lambda, u_\lambda, \nabla u_\lambda, \partial_t u_\lambda) \varphi_\varepsilon(t_\lambda, x_\lambda) \, dxdt = \sum_j \int_{T_j \Omega_j} \mathcal{L}(t_\lambda, x_\lambda, u_\lambda, \nabla u_\lambda, \partial_t u_\lambda) \, dxdt$$

and hence, by (2.24),

$$\sum_j \int_{T_j \Omega_j} \mathcal{L}(t_\lambda, x_\lambda, u_\lambda, \nabla u_\lambda, \partial_t u_\lambda) \, dxdt = \sum_j \int_{\Omega_j} \mathcal{L}(t, x, u, \nabla u, \partial_t u) \, dxdt$$

$$= \int_{\Omega} \mathcal{L}(t, x, u, \nabla u, \partial_t u) \varphi_\varepsilon(t, x) \, dxdt$$

and so,

$$\int_{\Omega} \mathcal{L}(t_\lambda, x_\lambda, u_\lambda, \nabla u_\lambda, \partial_t u_\lambda) \varphi_\varepsilon(t_\lambda, x_\lambda) \, dxdt = \int_{\Omega} \mathcal{L}(t, x, u, \nabla u, \partial_t u) \varphi(t, x) \, dxdt$$

$^2D(\Omega)$ denotes the space of infinitely differentiable functions with compact support in $\Omega$.  

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Taking the limit for $\varepsilon \to 0$, we get that
\[
\int_{\Omega} \mathcal{L}(t, x, u, \nabla u, \partial_t u) \varphi(t, x) \, dx \, dt = \int_{\Omega} \mathcal{L}(t, x, u, \nabla u, \partial_t u) \varphi(t, x) \, dx \, dt
\]
and we get the conclusion.

The above lemma is useful since the introduction of the compact support function $\varphi$ allows us to work on a fixed domain $\Omega$ and hence we do not have to consider the variation of the domain $T_{\lambda} \Omega$.

**Theorem 3.** Let $\mathcal{L}$ be invariant with respect to a one parameter group $g(\lambda)$, and let $u = u_\lambda$ be a smooth solution of the Euler-Lagrange equation (2.3). Using the notation (2.22), (2.23), (2.24), we set
\[
\rho = \left( \frac{\partial \mathcal{L}}{\partial u_{x,i}} \frac{\partial u_\lambda}{\partial \lambda} - \mathcal{L} \frac{\partial t_\lambda}{\partial \lambda} \right)_{\lambda=0}
\]
and
\[
J = \sum_{i=1}^{N} \left( \frac{\partial \mathcal{L}}{\partial u_{x,i}} \frac{\partial u_\lambda}{\partial \lambda} - \mathcal{L} \frac{\partial x^i_\lambda}{\partial \lambda} \right)_{\lambda=0} e_i.
\]
Then we get the continuity equation
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0.
\]

**Proof.** By the invariance of the Lagrangian and lemma 2, we have:
\[
\frac{d}{d\lambda} \int \mathcal{L} \varphi \, dx \, dt = 0 \quad \forall \varphi \in \mathcal{D}(\Omega)
\]
and hence
\[
\int \left( \frac{d\mathcal{L}}{d\lambda} \varphi + \mathcal{L} \frac{d\varphi}{d\lambda} \right) \, dx \, dt = 0 \quad \forall \varphi \in \mathcal{D}(\Omega)
\]
Let us compute each derivative; in this computation, we write $u, x, t$ instead of $u_\lambda, x_\lambda, t_\lambda$ to make it readable and we use the notation $x^0 = t$:
\[
\frac{d\mathcal{L}}{d\lambda} = \sum_{i=0}^{N} \frac{\partial \mathcal{L}}{\partial u_{x,i}} \frac{\partial^2 u}{\partial \lambda \partial x^i} + \frac{\partial \mathcal{L}}{\partial u} \frac{\partial u}{\partial \lambda}
\]
\[
= \sum_{i=0}^{N} \left[ \frac{\partial}{\partial x^i} \left( \frac{\partial \mathcal{L}}{\partial u_{x,i}} \frac{\partial u}{\partial \lambda} \right) - \frac{\partial}{\partial x^i} \left( \frac{\partial \mathcal{L}}{\partial u_{x,i}} \frac{\partial u}{\partial \lambda} \right) \right] + \frac{\partial \mathcal{L}}{\partial u} \frac{\partial u}{\partial \lambda}
\]
\[
= \sum_{i=0}^{N} \frac{\partial}{\partial x^i} \left( \frac{\partial \mathcal{L}}{\partial u_{x,i}} \frac{\partial u}{\partial \lambda} \right) - \left[ \sum_{i=0}^{N} \frac{\partial}{\partial x^i} \left( \frac{\partial \mathcal{L}}{\partial u_{x,i}} \right) \right] \frac{\partial u}{\partial \lambda}
\]
Then, by equation (2.28)
\[
\frac{d\mathcal{L}}{d\lambda} = \sum_{i=0}^{N} \frac{\partial}{\partial x^i} \left( \frac{\partial \mathcal{L}}{\partial u_{x,i}} \frac{\partial u}{\partial \lambda} \right).
\]
Also
\[ \frac{d\varphi}{d\lambda} = \sum_{i=0}^{N} \frac{\partial \varphi}{\partial x^i} \frac{\partial x^i}{\partial \lambda} \]

Then, by (2.28), (2.29) and the above equality, we have that
\[ \int \left[ \sum_{i=0}^{N} \frac{\partial}{\partial x^i} \left( \frac{\partial \mathcal{L}}{\partial u_{x^i}} \frac{\partial u}{\partial \lambda} + \mathcal{L} \sum_{i=0}^{N} \frac{\partial \varphi}{\partial x^i} \frac{\partial x^i}{\partial \lambda} \right) \right] dx dt = 0 \quad (2.30) \]

Moreover, since \( \varphi \) has compact support, by the divergence theorem we have that
\[ \int \sum_{i=0}^{N} \frac{\partial}{\partial x^i} \left( \mathcal{L} \frac{\partial x^i}{\partial \lambda} \right) \varphi \, dx dt + \int \mathcal{L} \sum_{i=0}^{N} \frac{\partial \varphi}{\partial x^i} \frac{\partial x^i}{\partial \lambda} dx dt = 0 \]

By this equality and (2.30), we get that
\[ 0 = \int \sum_{i=0}^{N} \frac{\partial}{\partial x^i} \left( \mathcal{L} \frac{\partial x^i}{\partial \lambda} \right) \varphi \, dx dt - \int \sum_{i=0}^{N} \frac{\partial}{\partial x^i} \left( \mathcal{L} \frac{\partial x^i}{\partial \lambda} \right) \varphi \, dx dt \]
\[ = \int \left[ \sum_{i=0}^{N} \frac{\partial}{\partial x^i} \left( \frac{\partial \mathcal{L}}{\partial u_{x^i}} \frac{\partial u}{\partial \lambda} - \mathcal{L} \frac{\partial x^i}{\partial \lambda} \right) \varphi \right] dx dt \]

By the arbitrariness of \( \varphi \) we get
\[ \sum_{i=0}^{N} \frac{\partial}{\partial x^i} \left( \frac{\partial \mathcal{L}}{\partial u_{x^i}} \frac{\partial u}{\partial \lambda} - \mathcal{L} \frac{\partial x^i}{\partial \lambda} \right) = 0 \]
or,
\[ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial u_{x^i}} \frac{\partial u}{\partial \lambda} - \mathcal{L} \frac{\partial x^i}{\partial \lambda} \right) + \sum_{i=1}^{N} \frac{\partial}{\partial x^i} \left( \frac{\partial \mathcal{L}}{\partial u_{x^i}} \frac{\partial u}{\partial \lambda} - \mathcal{L} \frac{\partial x^i}{\partial \lambda} \right) \]

Then the functions (2.25) and (2.26) satisfy the continuity equation (2.27).

Then by lemma 1 and Th. 3, we have the following result:

**Theorem 4. (Noether’s theorem)** Let \( \mathcal{L} \) be invariant with respect to a one parameter group \( g(\lambda) \), and let \( u \) be a smooth solution of the Euler-Lagrange equation (2.3). Suppose that \( u \) decays sufficiently fast so that (2.17) holds. Then, using the notation (2.22), (2.23), (2.24), we have that
\[ I(u) = \int \left( \frac{\partial \mathcal{L}}{\partial u_{\lambda,t}} \frac{\partial u_{\lambda}}{\partial \lambda} - \mathcal{L} \frac{\partial u_{\lambda}}{\partial \lambda} \right) \lambda=0 dx \]
is an integral of motion.
2.7 Conservation laws

Now, using Noether theorem\(^4\), we can compute the main integral of motion. They are due to the homogeneity of time and the homogeneity and isotropy of space which provide the invariance with respect to the time translations, space translations and space rotations. We consider the case in which \(\mathcal{L}\) depends on a complex valued scalar function \(\psi\). The computation can be done setting \(\psi = u_1 + iu_2\), and considering \(\mathcal{L}\) as function of \(u = (u_1, u_2)\).

- **Energy.** Energy, by definition, is the quantity which is preserved by the time invariance of the Lagrangian; it has the following form

\[
E = \text{Re} \int \left( \frac{\partial \mathcal{L}}{\partial (\partial_t \psi)} \cdot \overline{\partial_t \psi} - \mathcal{L} \right) dx \tag{2.31}
\]

- **Momentum.** Momentum, by definition, is the quantity which is preserved by the space invariance of the Lagrangian; the invariance for translations in the \(x_i\) direction gives the following invariant:

\[
P_i = -\text{Re} \int \frac{\partial \mathcal{L}}{\partial (\partial_t \psi)} \cdot \overline{\partial_x \psi} dx
\]

The numbers \(P_i\) are the components of the vector

\[
P = -\text{Re} \int \frac{\partial \mathcal{L}}{\partial (\partial_t \psi)} \cdot \nabla \psi dx \tag{2.32}
\]

- **Angular momentum.** The angular momentum, by definition, is the quantity which is preserved by virtue of the invariance under space rotations of the Lagrangian with respect to the origin

\[
M = \text{Re} \int \frac{\partial \mathcal{L}}{\partial (\partial_t \psi)} \cdot (\mathbf{x} \times \nabla \psi) dx.
\]

**Proof.** First, we compute \(M_3\). Setting \(\mathbf{x} = (x, y, z)\), we have that

\[
\begin{align*}
x_\lambda &= x \cos \lambda - y \sin \lambda \\
y_\lambda &= x \sin \lambda + y \cos \lambda \\
z_\lambda &= z \\
t_\lambda &= t
\end{align*}
\]

then, setting \(\psi = u_1 + iu_2\),

\[
\left( \frac{\partial \mathcal{L}}{\partial u_{1,t}} \frac{\partial \psi}{\partial \lambda} + \frac{\partial \mathcal{L}}{\partial u_{2,t}} \frac{\partial \psi}{\partial \lambda} - \mathcal{L} \frac{\partial t}{\partial \lambda} \right)_{\lambda=0} = \text{Re} \left[ \frac{\partial \mathcal{L}}{\partial (\partial_t \psi)} \left( \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \right) \right]_{\lambda=0}
\]

Analogously, we have

\[
\begin{align*}
M_1 &= \text{Re} \int \frac{\partial \mathcal{L}}{\partial (\partial_t \psi)} \left( \frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial z} \right) dx \\
M_2 &= \text{Re} \int \frac{\partial \mathcal{L}}{\partial (\partial_t \psi)} \left( \frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial x} \right) dx
\end{align*}
\]
Then we get the conclusion.

\[\square\]

**Hylenic charge.** The hylenic charge, by definition, is the quantity which is preserved by the trivial gauge action (2.15). The charge has the following expression

\[
\mathcal{H} = \text{Im} \int \frac{\partial \mathcal{L}}{\partial (\partial_t \psi)} \cdot \overline{\psi} \, dx
\]  

(2.33)

**Proof.** We have that \(\psi_\lambda = \psi e^{i\lambda}\); then

\[
\text{Re} \left( \frac{\partial \mathcal{L}}{\partial (\psi_\lambda)} \cdot \frac{\partial}{\partial \lambda} \psi_\lambda - \mathcal{L} \frac{\partial}{\partial \lambda} \right)_{\lambda=0} = \text{Re} \left( \frac{\partial \mathcal{L}}{\partial (\partial_t \psi)} \cdot \overline{\psi} \right)
\]

\[
= \text{Im} \left( \frac{\partial \mathcal{L}}{\partial (\partial_t \psi)} \cdot \overline{\psi} \right)
\]

\[\square\]

### 2.8 The Hamilton-Jacobi theory

In order to understand the motion of hylomorphic solitary waves, it is necessary to know the basic notions of the Hamilton-Jacobi formulation of the laws of Mechanics. Then, in this section, we will briefly recall these notions.

The Lagrangian formulation of the laws of the Mechanics assumes a function

\[
\mathcal{L} = \mathcal{L}(t, q, \dot{q})
\]

of the generalized coordinates of the system \(q = (q_1, ..., q_k)\), of their derivatives \(\dot{q} = (\dot{q}_1, ..., \dot{q}_k)\) and of time. The trajectories \(q(t)\) such that \(q(t_0) = x_0\) and \(q(t_1) = x_1\) are the critical points of the action functional

\[
\mathcal{S}(q) = \int_{t_0}^{t_1} \mathcal{L}(t, q, \dot{q}) \, dt
\]  

(2.34)

defined on the space

\[
\mathcal{C}^1_{x_0, x_1}[t_0, t_1] = \{ q \in C^1[t_0, t_1] : q(t_0) = x_0 \text{ and } q(t_1) = x_1 \}
\]

Thus a trajectory \(q(t)\) satisfies the "Euler-Lagrange" equations:

\[
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0, \quad j = 1, ..., k
\]  

(2.35)

However, this is not the only formulation of the laws of Mechanics. An other very important formulation can be obtained as a first order system provided that

\[
\frac{\partial^2 \mathcal{L}}{\partial q_j^2} > 0.
\]  

(2.36)

In this case, we set

\[
p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j}(t, q, \dot{q})
\]  

(2.37)
By (2.36), we have that the function
\[ \dot{q}_j \mapsto \frac{\partial \mathcal{L}}{\partial \dot{q}_j}(t, q, \dot{q}) \]
is smoothly invertible and hence there exists a smooth function \( F \) such that (2.37) can be rewritten as follows:
\[ \dot{q} = F(t, q, p) \quad (2.38) \]
Now, we can define the Hamiltonian function as follows:
\[ H(t, q, p) = \langle p, \dot{q} \rangle - \mathcal{L}(q, \dot{q}, t)_{\dot{q}=F(p)} \]
where
\[ \langle p, \dot{q} \rangle = \sum_{j=1}^{k} p_j \dot{q}_j \]
denotes the paring between the tangent space (to the space of the \( q' \)s) and the relative cotangent space.

Then, the action (2.34) can be rewritten as follows
\[ S(p, q) = \int_{t_0}^{t_1} \left[ \langle p, \dot{q} \rangle - \mathcal{H}(t, q, p) \right] dt \]
and the relative "Euler-Lagrange" equations take the form
\[ \dot{q} = \frac{\partial \mathcal{H}}{\partial p}(t, q, p) \]
\[ \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}(t, q, p) \]
This is the Hamiltonian formulation of the laws of Mechanics and the above equations are called Hamilton equations. An other equivalent formulation of the laws of Dynamics is given by the Hamilton-Jacobi theory which uses notions both of the Lagrangian formulation and the Hamiltonian one. It reduces the laws of Mechanics to a partial differential equation and to a first order ordinary differential equation. The Hamilton-Jacobi theory has been very useful to relate the laws of Optics to Dynamics. For us, it essential if we want to understand the motion of solitons regarded as material particles.

The starting point is the definition of a function \( S = S(t, x) \) called action. We fix once for ever a point \((t_0, x_0)\) and a point \((t, x)\) which will be considered variable. Moreover, we set
\[ S(t, x) = \int_{t_0}^{t} \mathcal{L}(t, q_x, \dot{q}_x) \, dt \quad (2.39) \]
where \( q_x \) is a critical point of (2.34) on the space \( C^1_{x_0, x} [t_0, t] \); in general, this point is not unique; however, if (2.36) holds, it is possible to prove the uniqueness of the minimum provided that \((t, x)\) is sufficiently close to \((t_0, x_0)\) \((t \neq t_0)\). Hence, there exists an open set \( \Omega \) in which the function (2.39) is well defined. The function (2.39) is called action as the functional (2.34). However, even if they
have formal similar definitions, they are quite different objects: $S$ is a functions of $k + 1$ variables defined in an open set $\Omega \subset \mathbb{R}^{N+1}$ while $S$ is a functional defined in the function space $C^1_{\bar{t}_0, x}[t_0, t]$.

The Hamilton-Jacobi theory states that the function $S$, in $\Omega$, satisfies the following partial differential equation:

$$\partial_t S + \mathcal{H}(t, x, \nabla S) = 0 \quad (2.40)$$

Moreover, this result can be inverted in the sense stated by the following theorem:

**Theorem 5.** Let $S$ be a solution of eq. $(2.40)$ in $\Omega$ and let $q = (q_1, ..., q_k)$ be a solution of the following Cauchy problem for $(t, x) \in \Omega$:

$$\frac{\partial L}{\partial \dot{q}_j}(t, q_j, \dot{q}_j) = \nabla S(t, q_j), \ j = 1, ..., k \quad (2.41)$$

$$q(\bar{t}) = \bar{x}$$

with $(\bar{t}, \bar{x}) \in \Omega$. Then, $q$ satisfies eq. $(2.35)$ with initial conditions

$$q(\bar{t}) = \bar{x}$$

$$\dot{q}(\bar{t}) = F(\bar{t}, \bar{x}, \nabla S(\bar{t}, \bar{x}))$$

where $F$ is given by $(2.38)$.

The proof of this theorem can be found in any book of Classical Mechanics; for example in the beautiful book of Landau and Lifchitz [30].

Notice that the above Cauchy problem is well posed, at least for small times, since, by $(2.38)$, eq. $(2.41)$ gets the form

$$\dot{q} = F(t, q, \nabla S(t, q))$$

where $F$ is the smooth function given by $(2.38)$.

Thus, we can say that the equation of motions $(2.35)$ are equivalent to the set of equations,

$$\partial_t S + \mathcal{H}(t, x, \nabla S) = 0 \quad (2.42)$$

$$\dot{q} = F(t, q, \nabla S(t, q)) \quad (2.43)$$

If $L$ does not depend on $t$, then $\mathcal{H}$ is a constant of motion (namely it is the energy of the system). In this case, by eq. $(2.40)$, $\partial_t S = -h$, namely it does not depend on time and it represents the energy of the system with the sign changed. In this case, eq. $(2.40)$ takes the form

$$\mathcal{H}(x, \nabla S) = h.$$

Let see some examples:

**Newtonian dynamics:**

$$\mathcal{L}(t, q, \dot{q}) = \frac{1}{2}m\dot{q}^2 - V(q)$$

Then

$$p = m\dot{q}$$
\[ \mathcal{H} = \frac{1}{2m} p^2 + V(q) \]

and equations (2.42,2.43) take the form

\[ \partial_t S + \frac{1}{2m} |\nabla S|^2 + V(x) = 0 \] (2.44)

\[ \dot{q} = \frac{1}{m} \nabla S(t, q) \] (2.45)

**Relativistic dynamics:** The Lagrangian of a relativistic particle is given by:

\[ L(t, q, \dot{q}) = -m_0 \sqrt{1 - \dot{q}^2} \]

where \( m_0 \) is a parameter. We refer to Landau-Lifchitz [31] for a very elegant deduction of this Lagrangian from the Mikowsky geometry of space-time.

Then

\[ p = \frac{\partial L}{\partial \dot{q}} = \frac{m_0}{\sqrt{1 - \dot{q}^2}} \dot{q} = \gamma m_0 \dot{q} \] (2.46)

with

\[ \gamma = \frac{1}{\sqrt{1 - \dot{q}^2}} \]

and equation (2.38) becomes

\[ \dot{q} = \frac{p}{\sqrt{m_0^2 + p^2}} \] (2.47)

Since the mass of a particle is defined by the equation \( m = p/\dot{q} \), we will get that the mass changes with velocity

\[ m = \gamma m_0 \]

and the interpretation of \( m_0 \) as rest mass. The Hamiltonian is:

\[ \mathcal{H} = p\dot{q} + m_0 \sqrt{1 - \dot{q}^2} = \frac{m_0}{\sqrt{1 - \dot{q}^2}} \dot{q}^2 + m_0 \sqrt{1 - \dot{q}^2} = \frac{m_0}{\sqrt{1 - \dot{q}^2}} = \gamma m_0 \]

Since the Lagrangian in independent of time, the Hamiltonian represents the energy and this gives the Einstein equation:

\[ \mathcal{E} = \mathcal{H} = m = \gamma m_0 \] (2.48)

Now let express \( \mathcal{H} \) as function of \( p \). Using eq. (2.47) we get

\[ \mathcal{H}(p, q) = \frac{m_0}{\sqrt{1 - \frac{p^2}{m_0^2 + p^2}}} = \sqrt{m_0^2 + p^2} \] (2.49)

and the equations (2.42,2.43) take the form

\[ \partial_t S + \sqrt{m_0^2 + |\nabla S|^2} = 0 \] (2.50)

\[ \dot{q} = \frac{\nabla S}{\sqrt{m_0^2 + |\nabla S|^2}} \] (2.51)
3 Hylomorphic solitary waves and solitons

3.1 An abstract definition of solitary waves and solitons

Solitary waves and solitons are particular states of a dynamical system described by one or more partial differential equations. Thus, we assume that the states of this system are described by one or more fields which mathematically are represented by functions

$$\Psi : \mathbb{R}^N \rightarrow V$$ (3.1)

where $V$ is a vector space with norm $| \cdot |_V$ which is called the internal parameters space. We assume the system to be deterministic; this means that it can be described as a dynamical system $(X, U)$ where $X$ is the set of the states and $U : \mathbb{R} \times X \rightarrow X$ is the time evolution map. If $\Psi_0(x) \in X$, the evolution of the system will be described by the function

$$\Psi (t, x) = U_t \Psi_0(x)$$ (3.2)

Now we can give a formal definition of solitary wave:

**Definition 6.** A state $\Psi_0 \in X$, is called solitary wave if its evolution has the following form:

$$U_t \Psi_0(x) = h_t \Psi_0(g_t x)$$

where $h_t$ and $g_t$ are transformations of $V$ and $\mathbb{R}^N$ respectively.

For example, consider a solution of a field equation which has the following form

$$\Psi (t, x) = e^{-i(k \cdot x - \omega t)} \Psi_0(x - vt); \quad \Psi_0 \in L^2(\mathbb{R}^N, \mathbb{C})$$

then the conditions of the above definition are satisfied with $h_t \psi = e^{-i(k \cdot x - \omega t)} \psi$ ($\psi \in \mathbb{C}$) and $g_t x = x - vt$.

The solitons are solitary waves characterized by some form of stability. To define them at this level of abstractness, we need to recall some well known notion in the theory of dynamical systems.

**Definition 7.** Let $X$ be a metric space and let $(X, U)$ be a dynamical system. An invariant set $\Gamma \subset X$ is called stable, if

$$\forall \varepsilon > 0, \exists \delta > 0, \quad \forall \Psi \in X, \quad d(\Psi, \Gamma) \leq \delta, \quad \text{implies that} \quad \forall t \in \mathbb{R}, \quad d(U_t \Psi, \Gamma) \leq \varepsilon$$

**Definition 8.** A state $\Psi_0$ is called orbitally stable if there exists a finite dimensional invariant stable manifold $\Gamma$ such that $\Psi_0 \in \Gamma$.

The above definition needs some explanation. Since $\Gamma$ is invariant, $U_t \Psi_0 \in \Gamma$ for every time. Thus, since $\Gamma$ is finite dimensional, the evolution of $\Psi_0$ is described by a finite number of parameters. Thus the dynamical system $(\Gamma, U)$ behaves as a point in a finite dimensional phase space. By the stability of $\Gamma$, a small perturbation of $\Psi_0$ remains close to $\Gamma$. However, in this case, its evolution depends on an infinite number of parameters. Then, as time goes...
on, the evolution of the perturbed system might become very different from $U_t\Psi_0$. Thus, this system appears as a finite dimensional system with a small perturbation. We refer to section (4.7) where this fact will be seen in details in a concrete case.

**Definition 9.** A state $\Psi_0 \in X$, $\Psi_0 \neq 0$, is called soliton if it is a orbitally stable solitary wave.

In general, $\dim(\Gamma) > N$ and hence, the "state" of a soliton is described by $N$ parameters which define its position and other parameters which define its "internal state".

### 3.2 Definition of hylomorphic solitons

Now let us assume that our system satisfies assumptions A-1, A-2 and A-3 of pag 2. By Noether theorem, assumptions A-1 and A-2 of guarantee the conservation of the energy $E(\Psi)$ (see (2.31)) while A-1 and A-3 guarantee the conservation of the hylenic charge $H(\Psi)$ (see (2.33)).

**Definition 10.** A stationary wave $\Psi_0$ is called hylomorphic wave if

$$E(\Psi_0) = c_\sigma := \min_{\Psi \in M_\sigma} E(\Psi) \text{ for some } \sigma \in \mathbb{R}$$

(3.3)

where

$$M_\sigma = \{ \Psi \in X : H(\Psi) = \sigma \}$$

Moreover $\Psi_0$ is called hylomorphic soliton if it satisfies definition 9 namely if

$$\Gamma_\sigma = \{ \Psi_0 \in X : E(\Psi) = c_\sigma \}$$

(3.4)

is a finite dimensional stable manifold.

Observe that the energy and the charge are constant of the motion, then $\Gamma_\sigma$ is an invariant set. Then the above definition is consistent with definition 9.

Let $\Psi_0$ be a hylomorphic stationary wave as in Def. 10 and set

$$V(\Psi) = (E(\Psi) - c_\sigma)^2 + (H(\Psi) - \sigma)^2;$$

(3.5)

then we have that

**Proposition 11.** Let $\Psi_0$ and $\Gamma_\sigma$ be as in Def. 10 and let $\Psi_n$ be any sequence in the phase space $X$ which is supposed a metric space with metric $d$. Then if

$$V(\Psi_n) \to 0 \Rightarrow d(\Psi_n, \Gamma_\sigma) \to 0$$

(3.6)

$\Psi_0$ is a hylomorphic soliton.

**Proof.** By (3.6), it is immediate to check that $V$ is a Liapunov function relative to the flow $U_t$ defined by 3.2. Then, by the well known Liapunov theorem, it follows that $\Gamma_\sigma$ is stable and by the definitions 8 and 9 the conclusion follows.

$\square$
If $\Psi_0$ is a stationary wave of a dynamical system relative to a Lagrangian which is invariant for the Lorentz or the Galileo group, then it is possible to obtain a travelling wave just making a Lorentz boost (see (2.8)) or a Galilean transformation (see (2.13)). More precisely, let $T_v$ be the representation of a Lorentz boost (or a Galilean transformation) relative to our system and let

$$\Psi(t, x) = U_t \Psi_0(x)$$

be the evolution of our stationary wave $\Psi_0(x)$; then

$$\Psi'(t', x') = T_v \Psi(t, x)$$

is a solution of our equation which moves in time; then $\Psi_v(x) := \Psi'(t', x')|_{t=0}$ is a travelling wave at the time $t = 0$. At pag. 36 and pag. 49 we will see as this principle works in some particular cases.

Obviously, if $\Psi_0(x)$ is a hylomorphic standing wave $\Psi_v(x)$ will be called hylomorphic travelling wave; moreover, if $\Psi_0(x)$ is orbitally stable, also $\Psi_v(x)$ is orbitally stable and hence is a hylomorphic soliton.

### 3.3 Structure of hylomorphic solitons

Now, as it happens in Th. 4, we assume $E$ and $H$ to be local quantities, namely, given $\Psi \in X$, there exist the density functions $\rho_{E, \Psi}(x)$ and $\rho_{H, \Psi}(x) \in L^1(\mathbb{R}^N)$ such that

$$E(\Psi) = \int \rho_{E, \Psi}(x) \, dx \quad \text{and} \quad H(\Psi) = \int \rho_{H, \Psi}(x) \, dx$$

Energy and hylenic density allow to define the density of binding energy as follows:

$$\beta(t, x) = \beta_\Psi(t, x) = [E_0 \cdot |\rho_{H, \Psi}(t, x)| - \rho_{E, \Psi}(t, x)]^+ \quad (3.7)$$

The support of the binding energy density is called bound matter region; more precisely we have the following definition

**Definition 12.** Given any configuration $\Psi$, we define the bound matter region as follows

$$\Sigma(\Psi) = \{x : \beta_\Psi(t, x) \neq 0\}.$$ 

If $\Psi_0$ is a soliton, the set $\Sigma(\Psi_0)$ is called support of the soliton at time $t$.

Thus a hylomorphic soliton $\Psi_0$ consists of bound matter localized in a precise region of the space, namely $\Sigma(\Psi_0)$. This fact gives the name to this type of soliton from the Greek words "hyle" = "matter" and "morphe" = "form".

We now set

$$E_0 = \lim_{\varepsilon \to 0} \inf_{\Psi \in X_\varepsilon} \frac{E(\Psi)}{H(\Psi)} \quad (3.8)$$

where

$$X_\varepsilon = \{\Psi \in X : \forall x, \, |\Psi(x)|_V < \varepsilon\}. \quad (3.9)$$
Now suppose that there is a state $\Psi$ which satisfies the inequality

$$\frac{\mathcal{E} (\Psi)}{|\mathcal{H} (\Psi)|} < E_0$$

(3.10)

which will be called hylomorphy condition. The quantity

$$\Lambda (\Psi) = \frac{\mathcal{E} (\Psi)}{|\mathcal{H} (\Psi)|},$$

(3.11)

is also an invariant of motion and will be called hylomorphy ratio.

Notice that, by def. (3.8) and (3.11), for any hylomorphic solitary wave $\Psi_0$, we have that

$$\Lambda (\Psi_0) = \frac{c_0}{\sigma} \leq \frac{E_0}{\sigma}$$

However, if (3.10) holds, we have that

$$\Lambda (\Psi_0) < \frac{E_0}{\sigma}$$

Actually the hylomorphy condition (3.10) seems to be a necessary condition in order to have hylomorphic solitons.

The hylomorphy condition (3.10) guarantees the presence of bound matter even if no soliton is present:

**Proposition 13.** If $\Lambda (\Psi(0, \cdot)) < E_0$, then for all $t \in \mathbb{R}$

$$\Sigma(\Psi(t, \cdot)) \neq \emptyset$$

**Proof.** To fix the ideas, assume $\mathcal{H} (\Psi) > 0$.

$$\int \beta(t, x) = \int |E_0 \rho_{\mathcal{H}, U, \Psi} (x)| - \rho_{\mathcal{E}, U, \Psi} (x)|^+\geq \int E_0 \rho_{\mathcal{H}, U, \Psi} (x) - \rho_{\mathcal{E}, U, \Psi} (x) = E_0 \mathcal{H} (U, \Psi) - \mathcal{E} (U, \Psi) = E_0 \mathcal{H} (\Psi) - \mathcal{E} (\Psi) = \mathcal{H} (\Psi) \left[ E_0 - \frac{\mathcal{E} (\Psi)}{\mathcal{H} (\Psi)} \right] = \mathcal{H} (\Psi) [E_0 - \Lambda (\Psi)] > 0$$

$\square$

If $\Psi(x)$ is a finite energy field usually it disperses as time goes on, namely

$$\lim_{t \to \infty} \|U_t \Psi(x)\|_{L^\infty(\mathbb{R}^N, V)} = 0$$

However, if $\Lambda (\Psi) < E_0$, this is not the case:
Proposition 14. If $\Lambda (\Psi) < E_0$, then

$$\min \lim_{t \to \infty} \| U_t \Psi \|_{L^\infty(\mathbb{R}^N, V)} = \delta > 0$$

Proof: To fix the idea, set $\Lambda (\Psi) = E_0 - a$, $a > 0$. We argue indirectly and assume that, for every $\varepsilon > 0$, there exists $\bar{t}$ such that

$$\| U_{\bar{t}} \Psi \|_{L^\infty(\mathbb{R}^N, V)} < \varepsilon$$

namely, $U_{\bar{t}} \Psi \in X_\varepsilon$ where $X_\varepsilon$ is defined by (3.3). Then, by (3.8), if $\varepsilon$ is sufficiently small

$$\Lambda (U_{\bar{t}} \Psi) = \frac{\mathcal{E} (U_{\bar{t}} \Psi)}{|\mathcal{H} (U_{\bar{t}} \Psi)|} \geq E_0 - \frac{a}{2}$$

Since $\Lambda (U_{\bar{t}} \Psi) = \Lambda (\Psi)$ we get a contradiction. □

Thus if $\Lambda (\Psi) < E_0$, by the above propositions the field $\Psi$ and the bond matter field $\beta \Psi$ will not disperse but will form bumps of matter which eventually might lead to the formation of one or more hylomorphic soliton. This remark shows the importance of the hylomorphy condition (3.10).

3.4 The swarm interpretation of hylomorphic solitons

Clearly the physical interpretation of hylomorphic solitons depends on the model which we are considering. However we can always assume a conventional interpretations which we will call swarm interpretation since the soliton is regarded as a swarm of particles bound together. In each particular physical situation this interpretation, might have or might not have any physical meaning; in any case it represents a pictorial way of thinking of the mathematical phenomena which occur. This interpretation is consistent with the names and the definitions given in the previous section.

We assume that $\Psi$ is a field which describes a fluid consisting of particles; the particles density is given by the function $\rho_H(t, x) = \rho_{H, \Psi}(t, x)$ which, of course satisfies a continuity equation

$$\partial_t \rho_H + \nabla \cdot J_H = 0$$

where $J_H$ is the flow of particles. Hence $H$ is the total number of particles. Notice that $H$ does not need to be an integer number and you may assume that fractional particle can exist or that $H$ is a sort of limit valid for a very large number of particle as it happens in fluid-dynamics. Also, in some equations as for example in NKG, $H$ can be negative; in this case we assume the existence of antiparticle.

Thus, the hylomorphy ratio

$$\Lambda (\Psi) = \frac{\mathcal{E} (\Psi)}{|\mathcal{H} (\Psi)|}$$
represents the average energy of each particle (or antiparticle). The number \( E_0 \) defined by (3.8) is interpreted as the rest energy of each particle when they do not interact with each other. If \( \Lambda (\Psi) > E_0 \), then the average energy of each particle is bigger than the rest energy; if \( \Lambda (\Psi) < E_0 \), the opposite occurs and this fact means that particles act with each other with an attractive force.

If the particles would be at rest and they would not act on each other, their energy density would be
\[
E_0 \cdot |\rho_H(t, x)|;
\]
but their energy density is \( \rho_E(t, x) \); if
\[
\rho_E(t, x) < E_0 \cdot |\rho_H(t, x)|;
\]
then, in the point \( x \) at time \( t \), the particles attract each other with a force which is stronger that the repulsive forces; this explains the name density of bond energy given to \( \beta(t, x) \) in (3.7).

Thus a soliton relative to the state \( \Psi \) can be considered as a "rigid" object occupying the region of space \( \Sigma (\Psi) \) (cf. Def. 12); it consists of particles which stick with each other; the energy to destroy the soliton is given by
\[
\int \beta(t, x) dx = \int_{\Sigma(\Psi)} (E_0 |\rho_H(t, x)| - \rho_E(t, x)) dx
\]
However out of \( \Sigma (\Psi) \) the energy density is bigger than \( E_0 \cdot |\rho_H(t, x)| \); thus the total energy necessary to reduce the soliton to isolated particles is give by
\[
E_0 \cdot |\mathcal{H}(\Psi)| - \mathcal{E}(\Psi).
\]

As it is shown in Prop. 13 there are states \( \Psi \) such that \( \Sigma (\Psi) \neq 0 \) but they are not necessarily solitons. In these states, \( \Sigma (\Psi) \) is a region where the particles stick with each other but they do not have reached a stable configurations; the shape of \( \Sigma (\Psi) \) might changes with time. In may concrete situations, such states may evolve toward one or more solitons. In these cases we say that the solitons are asymptotically stable. The study of asymptotical stability is a problem quite involved. We refer to [29], [15], [21] and [22] and their references.

4 The nonlinear Schroedinger equation

4.1 General features of NS

The Schroedinger equation for a particle which moves in a potential \( V(x) \) is given by
\[
i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \Delta \psi + V(x) \psi
\]
We are interested to the nonlinear Schroedinger equation:
\[
i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \Delta \psi + \frac{1}{2} W'(\psi) + V(x) \psi \quad (4.1)
\]
where
\[
W'(\psi) = \frac{\partial W}{\partial \psi_1} + i \frac{\partial W}{\partial \psi_2} \quad (4.2)
\]
namely
\[ W'(\psi) = F'(|\psi|) \frac{\psi}{|\psi|}. \]
for some smooth function \( F : [0, \infty) \rightarrow \mathbb{R} \).

We always assume that
\[ W(0) = W'(0) = 0 \]
If \( V(x) = 0 \), then we get the equation
\[ i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \Delta \psi + \frac{1}{2} W'(\psi); \quad (NS) \]
this equation can be considered as the simplest equation which is variational
and invariant for a representation of the Galileo group.

First of all let us check that it is variational:

**Proposition 15.** Equation (4.1) is the Euler-Lagrange equation relative to the
Lagrangian density
\[ L = \text{Re} \left( i \partial_t \overline{\psi} \psi \right) - \frac{1}{2} |\nabla \psi|^2 - W(\psi) - V(x) |\psi|^2 \]  

**(Proof)** Set
\[ S(\psi) = S_1(\psi) + S_2(\psi) \]
\[ S_1(\psi) = \int \text{Re} \left( i \partial_t \overline{\psi} \psi \right) dx dt; \quad S_2(\psi) = -\int \left[ \frac{1}{2} |\nabla \psi|^2 + W(\psi) + V(x) |\psi|^2 \right] dx dt \]
and set \( \psi = u_1 + iu_2 \). We have
\[ S_1(\psi) = \int \text{Re} \left( i \partial_t \overline{\psi} \psi \right) dx dt \]
\[ = \int \text{Re} \left[ (i \partial_t u_1 - \partial_t u_2) (u_1 - iu_2) \right] dx dt \]
\[ = \int (\partial_t u_1 u_2 - \partial_t u_2 u_1) dx dt \]
Then, if \( \varphi = v_1 + iv_2 \)
\[ dS_1(\psi)[\varphi] = \int (\partial_t u_1 v_2 + \partial_t v_1 u_2 - \partial_t u_2 v_1 - \partial_t v_2 u_1) dx dt \]
\[ = \int (2 \partial_t u_1 v_2 - 2 \partial_t u_2 v_1) dx dt \]
\[ = \int \text{Re} \left[ 2i (\partial_t u_1 + \partial_t u_2) (v_1 - iv_2) \right] \]
\[ = \int \text{Re} (2i \partial_t \psi \varphi) \]
Then
\[ dS_2(\psi)[\varphi] = -\int \left[ \text{Re} \left( \nabla \psi, \nabla \varphi \right) + \text{Re} \left( W'(\psi) \overline{\varphi} \right) + 2 \text{Re} \left( V(x) \psi \overline{\varphi} \right) + \right] dx dt \]
\[ = -\int \left[ \text{Re} \left( -\Delta \psi \overline{\varphi} \right) + \text{Re} \left( W'(\psi) \overline{\varphi} \right) + 2 \text{Re} \left( V(x) \psi \overline{\varphi} \right) + \right] dx dt \]
\[ = -\int \text{Re} \left[ (\Delta \psi + W'(\psi) + 2V(x) \psi) \overline{\varphi} \right] dx dt. \]
\[ dS(\psi)[\varphi] = \int \text{Re} [(2i\partial_t \psi + \Delta \psi + W'(\psi) + 2V(x)\psi) \varphi] \, dx \, dt \]

So, the critical points of \( S \), satisfy the equation
\[ 2i\partial_t \psi + \Delta \psi - W'(\psi) - 2V(x)\psi = 0 \]
which is equivalent to (4.1).

\[ \square \]

Sometimes it is useful to write \( \psi \) in polar form
\[ \psi(t,x) = u(t,x)e^{iS(t,x)}. \quad (4.4) \]

where \( u(t,x) \in \mathbb{R}^+ \) and \( S(t,x) \in \mathbb{R}/(2\pi\mathbb{Z}). \) Thus the state of the system \( \psi \) is uniquely defined by the couple of variables \((u,S)\). Using these variables, the action \( S = \int L \, dx \, dt \) takes the form
\[ S(u,S) = -\int \left[ \frac{1}{2} |\nabla u|^2 + W(u) + \left( \partial_t S + \frac{1}{2} |\nabla S|^2 + V(x) \right) u^2 \right] \, dx \]
and equation (4.1) becomes:
\[ -\frac{1}{2} \Delta u + \frac{1}{2} W'(u) + \left( \partial_t S + \frac{1}{2} |\nabla S|^2 + V(x) \right) u = 0 \quad (4.5) \]
\[ \partial_t (u^2) + \nabla : (u^2 \nabla S) = 0 \quad (4.6) \]

### 4.2 First integrals of NS and the hylenic ratio

The results of section 2.7 and easy computations show that the integral of motion of (4.1) are given by the following expression:

- **Energy.** We have
\[ \mathcal{E}(\psi) = \int \left[ \frac{1}{2} |\nabla \psi|^2 + W(\psi) + V(x) |\psi|^2 \right] \, dx \quad (4.7) \]

Using (4.4) we get:
\[ \mathcal{E}(\psi) = \int \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) \, dx + \int \left( \frac{1}{2} |\nabla S|^2 + V(x) \right) u^2 \, dx \quad (4.8) \]

- **Momentum.** The momentum is constant in time if the Lagrangian is space-translation invariant; this happens when \( V \) is a constant. In particular, in eq. [NS] we have that
\[ \mathbf{P} = \text{Im} \int \nabla \psi \overline{\psi} \, dx \quad (4.9) \]

Using (4.4) we get:
\[ \mathbf{P} = \int u^2 \nabla S \, dx \quad (4.10) \]
Angular momentum. If we assume that $V$ is a constant, the angular momentum is an integral of motion and, in NS, we have

$$M = \text{Im} \int \mathbf{x} \times \nabla \psi \bar{\psi} \, dx$$  \hspace{1cm} (4.11)

Using (4.4) we get:

$$M = \int \mathbf{x} \times \nabla S u^2 \, dx$$  \hspace{1cm} (4.12)

Hylenic charge. Here the hylenic charge has the following expression

$$H(\psi) = \int |\psi|^2 \, dx = \int u^2 \, dx$$  \hspace{1cm} (4.13)

Barycenter (or Hylecenter) velocity. The quantity preserved by the Galileo transformation (2.14) is the following

$$K = \int \mathbf{x} u^2 \, dx - t P$$  \hspace{1cm} (4.14)

Here we make the explicit computation since it is more difficult than the computations for the other first integrals.

Proof. Let us compute $K_1$ using Th. 3 in this case the parameter $\lambda$ is $v$ the first component of the velocity $\mathbf{v} = (v, 0, 0)$ which appears in (2.13). In this case the computations are easier if we use polar coordinates: we have

$$L = - \left[ \frac{1}{2} |\nabla u|^2 + W(u) + \left( \partial_t S + \frac{1}{2} |\nabla S|^2 \right) u^2 \right]$$

Moreover we have that

$$x_v = x - vt$$
$$t_v = t$$

and recalling (2.14), we have that the representation $T_{g_v}$ acts on a state $\psi$ of the system as follows

$$(T_{g_v}) \psi(t, x) = \psi(t, x - vt) e^{i(\mathbf{v} \cdot \mathbf{x} - \frac{1}{2} v^2 t)} = u(t, x - vt) e^{iS(t, x - vt)} e^{i(\mathbf{v} \cdot \mathbf{x} - \frac{1}{2} v^2 t)}$$

namely

$$u_v(t, x) = u(t, x - vt)$$
$$S_v(t, x) = S(t, x - vt) + vx_1 - \frac{1}{2} v^2 t$$

Then, by (2.25)

$$\rho_{K_1} = \left[ \frac{\partial L}{\partial u_v} \frac{\partial u_v}{\partial v} + \frac{\partial L}{\partial S_v} \frac{\partial S_v}{\partial v} - L \frac{\partial t_v}{\partial v} \right]_{v=0} = \left[ u_v^2 \frac{\partial S_v}{\partial v} \right]_{v=0} = \left( x_1 - t \frac{\partial S_v}{\partial x_1} \right) u^2$$

27
Thus
\[ K_1 = \int x_1 u^2 dx - t \int \frac{\partial S_v}{\partial x_1} u^2 dx = \int x_1 u^2 dx - tP_1 \]

\[ \square \]

The three components of \( K \) are the integrals of motion relative to the Galileo invariance. Let us interpret this fact in a more meaningful way. If we derive both sides of (4.14) with respect to \( t \), we get
\[ 0 = \frac{d}{dt} \left( \int x u^2 dx \right) - P \quad (4.15) \]

If we define the barycenter (or hylecenter) as follows
\[ q := \int x u^2 dx \int u^2 dx = \frac{\int x u^2 dx}{\mathcal{H}} \quad (4.16) \]

By (4.15) we get
\[ \dot{q} = \frac{P}{\mathcal{H}} = \text{const.} \quad (4.17) \]

Then, the three integrals of motion relative to the Galileo invariance imply that the three components of \( \dot{q} \) are constant.

From now on, we will study the existence of solitary waves assuming \( V(x) = 0 \). If \( W \in C^2 \), we can write
\[ W(s) = \frac{1}{2} a s^2 + N(s) \quad (4.18) \]
where \( N''(s) = 0 \).

Now let us compute the limit (3.8).

**Theorem 16.** Assume that \( W \) is given by (4.18) and that \( E_0 \) is given by (3.8). Then
\[ E_0 = \frac{1}{2} a. \quad (4.19) \]

**Proof.** We have that
\[ \Lambda(\Psi) = \frac{\mathcal{E}(\Psi)}{\mathcal{H}(\Psi)} = \frac{\int \frac{1}{2} \left( |\nabla u|^2 + |\nabla S|^2 u^2 \right) + W(\psi) dx}{\int u^2 dx} \geq \frac{\int \frac{1}{2} a u^2 + N(u) dx}{\int u^2 dx} = \frac{1}{2} a + \frac{\int N(u) dx}{\int u^2 dx} \]

Then since \( N(u) = O(u^3) \) for \( u \to 0 \), we have that
\[ \Lambda(\Psi) = \lim_{\varepsilon \to 0} \inf_{\Psi \in X_\varepsilon} \frac{\mathcal{E}(\Psi)}{\mathcal{H}(\Psi)} \geq \frac{1}{2} a. \]

In order to prove the opposite inequality, take \( \Psi_{\varepsilon,R} = \varepsilon u_R \) where \( u_R(x) \) is defined as follows
\[ u_R(x) = \begin{cases} 
1 & \text{if } |x| < R \\
0 & \text{if } |x| > R + 1 \\
1 + R - |x| & \text{if } R < |x| < R + 1 
\end{cases} \]
Then
\[
\inf_{\Psi \in X} \frac{\mathcal{E}(\Psi)}{\mathcal{H}(\Psi)} \leq \frac{\varepsilon^2 \int \left[ \frac{1}{2} |\nabla u_R|^2 + \frac{1}{2\varepsilon^2} W(\varepsilon u) \right] dx}{\varepsilon^2 \int u_R^2 dx}
\]
\[
= \frac{\int \left[ \frac{1}{2} |\nabla u_R|^2 + \frac{1}{2} a u_R^2 + \frac{1}{2\varepsilon^2} N(\varepsilon u) \right] dx}{\int u_R^2 dx}
\]
\[
\leq \frac{1}{2} a + \frac{1}{2} \int |\nabla u_R|^2 dx + \frac{1}{2} \int \frac{1}{\varepsilon^2} N(\varepsilon u) dx
\]
\[
= 1 + O \left( \frac{1}{R} \right) + O(\varepsilon)
\]
and so, we have that
\[
\inf_{\Psi \in X} \frac{\mathcal{E}(\Psi)}{\mathcal{H}(\Psi)} \leq \frac{1}{2} a
\]

\[
\square
\]

By (4.18) and (4.19), we can write
\[
W(s) = E_0 s^2 + N(s), \quad N(0) = N'(0) = N''(0) = 0 \tag{4.20}
\]

It is not restrictive to assume that
\[
E_0 = 0
\]

In fact, if \( \psi_0(t, x) \) is a solution of the equation
\[
i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \Delta \psi + E_0 \psi + \frac{1}{2} N'(\psi) \tag{4.21}
\]
then
\[
\psi_1(t, x) = \psi_0(t, x) e^{-iE_0 t}
\]
satisfies the equation
\[
i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \Delta \psi + \frac{1}{2} N'(\psi) \tag{4.22}
\]

So equation (4.21) and (4.22) have the same dynamics except for a phase factor. This fact, in the swarm model, can be interpreted saying that the internal energy of a particle (namely \( E_0 \)) does not affect the dynamics of the particle.

### 4.3 Swarm interpretation of NS

Before giving the swarm interpretation to equation [NS], we will write it with the usual physical constants \( m \) and \( \hbar \):
\[
i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + \frac{1}{2} W'(\psi) + V(x) \psi \tag{4.23}
\]

Here \( m \) has the dimension of mass and \( \hbar \), the Plank constant, has the dimension of action.
The polar form of a state $\psi$ is written as follows

$$
\psi(t, x) = u(t, x)e^{iS(t, x)/\hbar}
$$

(4.24)

and equations (4.15) and (4.16) become

$$
-\frac{\hbar^2}{2m}\Delta u + \frac{1}{2}W'(u) + \left( \partial_t S + \frac{1}{2m}|\nabla S|^2 + V(x) \right) u = 0
$$

(4.25)

$$
\partial_t \left( u^2 \right) + \nabla \cdot \left( u^2 \frac{\nabla S}{m} \right) = 0
$$

(4.26)

The continuity equation (3.12) for NS is given by (4.26). This equation allows us to interpret the matter field to be a fluid composed by particles whose density is given by

$$
\rho_H = u^2
$$

and which move in the velocity field

$$
v = \frac{\nabla S}{m}.
$$

(4.27)

So equation (4.26) reads

$$
\partial_t \rho_H + \nabla \cdot (\rho_H v) = 0
$$

If

$$
-\frac{\hbar^2}{2m}\Delta u + \frac{1}{2}W'(u) \ll u,
$$

(4.28)

equation (4.25) can be approximated by

$$
\partial_t S + \frac{1}{2m}|\nabla S|^2 + V = 0.
$$

(4.29)

This is the Hamilton-Jacobi equation of a particle of mass $m$ in a potential field $V$ (cf. eq. 2.44). The trajectory $q(t)$ of each particle, by (4.27) satisfies the equation

$$
\dot{q} = \frac{\nabla S}{m}
$$

(4.30)

(cf. eq. 2.45).

If we do not assume (4.28), equation (4.29) needs to be replaced by

$$
\partial_t S + \frac{1}{2m}|\nabla S|^2 + V + Q(u) = 0
$$

(4.31)

with

$$
Q(u) = -\frac{\left(\hbar^2/m\right)\Delta u + W'(u)}{2u} = E_0 + -\frac{\left(\hbar^2/m\right)\Delta u + N'(u)}{2u}
$$

(4.32)

The term $Q(u)$ can be regarded as a field describing a sort of interaction between particles. The term $E_0$ does not affect the dynamics, while the other terms might contribute to the formations of hylomorphic solitons.
Given a wave of the form \( (4.24) \), the local frequency and the local wave number are defined as follows:

\[
\omega(t, x) = -\frac{\partial_t S(t, x)}{\hbar}, \\
k(t, x) = \frac{\nabla S(t, x)}{\hbar};
\]

the energy of each particle moving according to \( (4.29) \), is given by

\[
E = -\partial_t S
\]
and its momentum is given by

\[
p = \nabla S;
\]

thus we have that

\[
E = \hbar \omega \\
p = \hbar k;
\]

these two equations are the De Broglie relation. It is interesting to observe that they have been deduced by the swarm interpretation of the Schrödinger equation.

We recall again that the swarm interpretation is just a useful pictorial way to look at to NS. In physical models, in general, there are different interpretations. Here we will mention very shortly some of them.

In the traditional model of quantum mechanics for one particle, we have \( W = 0 \) and \( \rho_H \) is interpreted as a probability density of the position of this particle.

One of the most important applications of NS is in the Bose-Einstein condensate. In this case \( W(\psi) = \frac{U_0}{4} |\psi|^4 \) (where \( U_0 \) is a constant) and NS takes the name of Gross-Pitaevski equation. Here \( |\psi|^2 \) is interpreted as the particle density which in this case are bosons (as, for example, atoms).

### 4.4 An existence result for an elliptic equation

Many existence theorems of solitary waves reduce to the following elliptic equation in \( \mathbb{R}^N \)

\[
\begin{cases}
-\Delta u + G'(u) = 0 \\
u > 0
\end{cases}
(4.33)
\]

where \( G \) is a \( C^1 \)-function with \( G(0) = 0 \).

This equation has been studied by many authors (see e.g. [37], [19], [14] and their bibliography). In particular, in [14], there are necessary and "almost sufficient" conditions for the existence of "finite energy" solutions.

Since this equation is the basic equation for the existence of solitary waves, we give an existence proof. This proof is a variant of the proof in [14]. It is simpler but it uses slightly more restrictive assumptions.

Equation \( (4.33) \) is the Euler-Lagrange equation relative to the functional

\[
J(u) = \frac{1}{2} \int |\nabla u|^2 \, dx + \int G(u) \, dx,
\]

We assume that \( G : \mathbb{R}^+ \to \mathbb{R} \) satisfies the following assumptions:
• (G-i) \(G(0) = G'(0) = 0\).
• (G-ii) \(G'(s) \geq c_1 s - c_2 s^{p-1}\), \(2 < p < 2^*\), \(s > 0\), \(c_1, c_2 > 0\);
• (G-iii) \(\exists s_0 \in \mathbb{R}^+ : G(s_0) < 0\).

**Theorem 17.** Assume that \(G\) satisfies (G-i), (G-ii), (G-iii). Then eq. (4.33) has a nontrivial finite energy solution.

In order to prove the above theorem, first of all we define an auxiliary function \(\bar{G}\) as follows:

If

\[|G'(s)| \leq c_3 + c_4 |s|^{p-1}\]  \hspace{1cm} (4.34)

(where \(p\) is defined by (G-ii)), we set

\[
\bar{G}(s) = \begin{cases} 
G(s) & \text{for } s \geq 0 \\
0 & \text{for } s \leq 0
\end{cases}
\]

If \(G\) does not satisfy (4.34), then, by (G-ii), there exist \(s_1 > s_0\), such that \(G'(s_1) > 0\). In this case we set

\[
\bar{G}(s) = \begin{cases} 
G(s) & \text{for } 0 \leq s \leq s_1; \\
G(s_1) + G'(s_1) (s - s_1) & \text{for } s \geq s_1; \\
0 & \text{for } s \leq 0.
\end{cases}
\]

In any case, \(\bar{G}\) satisfies the assumptions (G-i), (G-ii) (G-iii) and the following ones:

• (G-iii) \(|\bar{G}'(s)| \leq c_3 + c_4 |s|^{p-1}\), \(2 \leq p < 2^*\)
• (G-v) \(\forall s < 0, \bar{G}(s) = 0\)

**Lemma 18.** Let \(u \in H^1\) be a solution of the following equation:

\[-\Delta u + \bar{G}'(u) = 0\]  \hspace{1cm} (4.35)

Then, \(u\) is positive and it is a is a solution of (4.33).

**Proof.** The fact that \(u\) is positive is a straightforward consequence of the maximum principle and the fact that \(\bar{G}'(u) = 0\) for \(u \leq 0\). Then if (4.34) holds, \(\bar{G}'(u) = G'(u)\) and hence \(u\) is a solution of (4.33). If (4.34) does not hold, by the maximum principle it follows that \(u(x) \leq s_1\); then, also in this case \(\bar{G}'(u) = G'(u)\) and \(u\) is a solution of (4.33).

\[\square\]

Thus, by the above lemma it is not restrictive to assume that \(G\) satisfies (G-i), (G-iii), (G-iii) and (G-v), since otherwise we can work with \(\bar{G}\).

We set

\[H^1_r = \{ u \in H^1(\mathbb{R}^N) : u = u(|x|) \}\]

**Lemma 19.** There exists \(\tilde{u} \in H^1_r\) such that

\[\int G(\tilde{u}) \, dx < 0\]
Proof. We set

\[ u_R(x) = \begin{cases} 
  s_0 & \text{for } |x| < R; \\
  s_0 - s_0(|x| - R) & \text{for } R \leq |x| \leq R + 1; \\
  0 & \text{for } |x| > R + 1.
\] (4.36)

Thus we have

\[
\int_{\mathbb{R}^N} G(u) \, dx 
\leq \left[ \max_{|s| \leq [R, R+1]} G(s) \right] \int_R^{R+1} r^{N-1} \, dr + \int_0^R G(s_0) r^{N-1} \, dr 
\leq C_1 [(R+1)^N - R^N] + \frac{1}{N} G(s_0) R^N 
\leq C_2 R^{N-1} + \frac{1}{N} G(s_0) R^N
\]

where \( C_1, C_2 \) are positive constant. By (G-iii), \( G(s_0) < 0 \); hence, for \( R \) sufficiently large, \( \int G(u_R) \, dx < 0 \)
□

Lemma 20. If \( u \in D^3 \), is a radially symmetric function, then

\[ |u(x)| \leq C_N \frac{\|u\|_{H^1}}{|x|^\frac{N}{2}} \]

Proof. We have that

\[ \frac{d}{dr} (r^{N-1} u^2) = 2r^{N-1} u \frac{du}{dr} + (N-1) r^{N-2} u^2 \geq 2r^{N-1} u \frac{du}{dr} \]

Then, integrating over \((R, +\infty)\) we get

\[ -R^{N-1} u(R)^2 \geq 2 \int_R^{+\infty} r^{N-1} \frac{du}{dr} \, dr \]

and so

\[ R^{N-1} u(R)^2 \leq 2 \int_R^{+\infty} \left| \frac{du}{dr} \right| r^{N-1} \, dr \leq \int_R^{+\infty} \left( \left| \frac{du}{dr} \right|^2 + u^2 \right) r^{N-1} \, dr \leq C_N \|u\|_{H^1}^2 \]

□

Theorem 21. (Strauss [37]) For \( p \in (2, 2^* \) the embedding

\[ H^1_r \rightarrow L^p \]

is compact.

\[ ^3 \text{Here } D \text{ denotes the space of } C^\infty \text{-functions with compact support.} \]
Proof. Let \( u_n \to 0 \) weakly in \( H^1_r \); we need to prove that \( u_n \to 0 \) strong in \( L^p \). Since \( u_n \to 0 \), then there is a constant \( M \) such that \( \| u_n \|_{H^1} \leq M \); so by lemma 20 we have that

\[
\int_{\mathbb{R}^N - B_R} |u_n|^p \, dx \leq \| u_n \|_{L^p(\mathbb{R}^N - B_R)}^{p-2} \int_{\mathbb{R}^N - B_R} |u_n|^2 \, dx \quad (4.37)
\]

\[
\leq \left( \frac{\| u \|_{L^2}^2}{R^2} \right)^{p-2} \int_{\mathbb{R}^N - B_R} |u_n|^2 \, dx \quad (4.38)
\]

\[
\leq \left( \frac{M}{R^2} \right)^{p-2} \| u \|_{L^2}^2 \leq \frac{M^p}{R^\alpha} \quad (4.39)
\]

where

\[
\alpha = \frac{(N-1)(p-2)}{2} > 0.
\]

Since \( u_n \to 0 \) strongly in \( L^p(B_R) \), by (4.37), we have that

\[
\liminf_{n \to \infty} \int |u_n|^p \, dx = \liminf_{n \to \infty} \| u_n \|_{L^p(B_R)}^p + \liminf_{n \to \infty} \| u_n \|_{L^p(\mathbb{R}^N - B_R)}^p \leq \frac{M^p}{R^\alpha}
\]

By the arbitrariness of \( R \), it follows that this limit is 0.

\[\square\]

Proof of Th. 17. Take a function \( \beta \in C^1(\mathbb{R}) \) such that

- (\( \beta \)-i) \( \forall s \in \mathbb{R}, \ 0 \leq \beta(s) \leq 1 \)
- (\( \beta \)-ii) \( \forall s < 0, \ \beta(s) > 0 \)
- (\( \beta \)-iii) \( \forall s \geq 0, \ \beta(s) = 0 \)
- (\( \beta \)-iii) \( \forall s \in \mathbb{R}, \ \beta'(s) < 0 \).

Now we define a \( C^1 \)-functional on \( H^1_r \) as follows:

\[
F(u) = \frac{1}{2} \int |\nabla u|^2 \, dx - b \beta \left( \int G(u) \, dx \right);
\]

here \( b \) is a positive constant defined by

\[
b = \frac{1}{b} \left[ \int |\nabla \bar{u}|^2 \, dx + 1 \right] / \beta \left( \int G(\bar{u}) \, dx \right)
\]

and \( \bar{u} \) is defined by lemma 19. This choice of \( b \), implies that

\[
F(\bar{u}) = -1
\]

Clearly \( F \) is a \( C^1 \) functional defined on \( H^1_r \).

Since \( \beta \) is bounded, then \( F(u) \) is bounded below and its infimum is a number less or equal to \(-1\). Then there exists a minimizing sequence \( u_n \). Also, \( \beta \) bounded implies \( \frac{1}{2} \int |\nabla u_n|^2 \, dx \) bounded and hence \( u_n \to w \) weakly in \( D^{1,2}_r \).
Since the infimum is negative we have that \( \beta \left( \int G(u_n) \, dx \right) < 0 \), and hence, by (\( \beta \)-iii),
\[
\int G(u_n) \, dx < 0.
\] (4.40)

By (G-ii), we have that there is a constant \( c_5 > 0 \) such that, for \( s \geq 0 \)
\[
G(s) \geq \int_{0}^{s} G'(t) \, dt \geq \int_{0}^{s} \left( c_1 t - c_2 t^{p-1} \right) \, dt
\geq \frac{1}{2} c_1 s^2 - \frac{c_2}{p} s^p \geq \frac{1}{2} c_4 s^2 - c_5 s^2^*
\]

By this inequality and (4.40), we get
\[
0 > \int G(u_n) \, dx \geq \frac{1}{2} c_4 \int |u_n|^2 \, dx - c_5 \int |u_n|^2^* \, dx
\geq \frac{1}{2} c_4 \| u_n \|_{L^2}^2 - c_5 \| \nabla u_n \|_{L^2}^2
\]

Thus since \( \| \nabla u_n \|_{L^2} \) is bounded, by the above inequality, also \( \| u_n \|_{L^2} \) is bounded and hence also \( \| u_n \|_{H^1} \) is bounded. So, by theorem 21, \( u_n \rightarrow w \) in \( L^p \) strong, where \( p \) is defined by (G-ii). By our assumptions, the functional
\[
u \rightarrow \int G(u) \, dx
\]
is continuous in \( L^p \). Then we have that \( \int G(u_n) \, dx \rightarrow \int G(w) \, dx \) and hence \( \beta \left( \int G(u_n) \, dx \right) \rightarrow \beta \left( \int G(w) \, dx \right) \). Since \( \frac{1}{2} \int |\nabla u|^2 \, dx \) is l.s.c. it follows that \( w \) is the minimum of \( F \). Thus \( w \) is different from 0 and it satisfies the following equation:
\[
-\Delta w + \lambda G'(w) = 0
\]
where
\[
\lambda = -b \beta' \left( \int G(w) \, dx \right)
\]
By (\( \beta \)-iii), we have that \( \lambda \geq 0 \), and since \( w \neq 0 \), we have that \( \lambda > 0 \). Now set
\[
u(x) = w \left( \frac{x}{\sqrt{\lambda}} \right)
\]
Clearly \( u \) satisfies equation (4.33)
\[
\square
\]

4.5 Standing waves and travelling waves

In any complex valued field theory, the simplest possible solitary waves are the stationary waves, namely finite energy solution having the following form
\[
\psi(t, x) = u(x) e^{-i\omega t}, \quad u \geq 0,
\] (4.41)

In particular, for the nonlinear Schroedinger equation (NS), substituting (4.41) in eq. (NS), we get
\[
-\Delta u + W'(u) = 2\omega u
\] (4.42)

Now we can apply the results of section 4.4 and get the following theorem:
Theorem 22. Assume that $W$ has the form (4.20) where $N$ satisfies the following assumptions:

$$N'(s) \geq -cs^{p-1} \text{ for any } s \geq 1,$$  \hfill (4.43)

$$\exists s_0 \in \mathbb{R}^+ : N(s_0) < 0. \hfill (4.44)$$

then eq. (NS) has finite energy solitary waves of the form (4.41) for every frequency $\omega \in (E_1, E_0)$ where $E_0$ is defined by (4.20) and

$$E_1 = \inf \{ a \in \mathbb{R} : \exists s \in \mathbb{R}^+, \ as^2 > E_0 s^2 + N(s) \}$$

Notice that, by virtue of (4.44), $E_1 < E_0$. Also, it is possible that $E_1 = -\infty$; for example this happens if

$$W(s) = -\frac{1}{p} |s|^p, \ 2 < p < 2^*$$

Proof. We want to apply theorem 17 to eq. (4.42); to this end we set

$$G(s) = W(s) - \omega s^2 = (E_0 - \omega) s^2 + N(s)$$

It is trivial to verify that, for every $\omega \in (E_1, E_0)$, $G$ satisfies the assumptions of Th. 17.

Now we will exploit the other symmetries of equation (NS) to produce other solutions. First of all, since (NS) is invariant for translations, for any $x_0 \in \mathbb{R}^N$, the function

$$\psi_{x_0}(t, x) = u(x - x_0) e^{-i\omega t} \hfill (4.46)$$

is a standing wave concentrated around the point $x_0$. The space rotations do not produce other solutions since $u(x)$ is rotationally invariant.

Since the Lagrangian (4.3) is invariant for the Galileo group, we can obtain other solutions: we can produce travelling waves just applying the transformation (2.14) to (4.46)

$$\psi_{x_0,v}(t, x) = u(x - x_0 - vt) e^{i(v \cdot x - Et)}, \quad E = \frac{1}{2} v^2 + \omega \hfill (4.47)$$

$\psi_{x_0,v}(t, x)$ is a solitary wave concentrated in the point $x_0 + vt$, an hence it travels with velocity $v$.

Finally, other solutions can be produced by the invariance (2.15); for $\theta \in [0, 2\pi)$, we have the solutions

$$\psi_{x_0,v,\theta}(t, x) = u(x - x_0 - vt) e^{i(v \cdot x - Et + \theta)} \hfill (4.48)$$

The invariance by time translations do not produce new solutions, since a time translation on $\psi_{x_0,v,\theta}$ produces a space and phase translation. Concluding, if we fix a charge $\sigma$, we obtain a radially symmetric solution of the form (4.41); by the invariance of the equation, this solution produces a 7-parameters family of solutions given by (4.47).

If we consider (NS) with the usual physical constants $m$ and $h$, namely equation (4.23) with $V = 0$, (4.47) takes the form

$$\psi_{x_0,v,\theta}(t, x) = u(x - x_0 - vt) e^{i(p \cdot x - Et)/h + \theta} \hfill (4.49)$$

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where

\[
\mathbf{p} = m \mathbf{v} \\
E = \frac{1}{2}mv^2 + \omega \hbar
\]

The meaning of these relations within the swarm interpretation is the following: \(\psi_{x_0,\mathcal{N},\theta}(t, x)\) is interpreted as a swarm of particles of mass \(m\); \(\mathbf{p}\) is the momentum of each particle and \(E\) is its energy: \(\frac{1}{2}mv^2\) is its kinetic energy and \(\omega \hbar\) is related to the nonlinear interaction of the particles.

### 4.6 Existence of solitons

Theorem 22 provides sufficient conditions for the existence of solitary waves. In order to prove the existence of hylomorphic solitons (cf. Def. 10), first of all it is necessary to assume that the Cauchy problem for \([\mathbf{NS}]\) is well posed namely, that it has a unique global solution which depends continuously on the initial data. For example, this is the case if \(W'\) is a globally Lipschitz function; we refer to the books [28] and [17] for more general conditions.

Moreover, it is necessary to investigate under which assumptions the energy

\[
\mathcal{E} = \int \left( \frac{1}{2} |\nabla \psi|^2 + W(\psi) \right) dx \\
= \int \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |\nabla S|^2 u^2 + W(u) \right) dx
\]

achieves the minimum on the manifold

\[\mathcal{M}_\sigma = \left\{ \psi \in H^1(\mathbb{R}^N, \mathbb{C}) : \int |\psi|^2 dx = \sigma \right\}\]

Clearly, if \(u\) minimizes the functional

\[J(u) = \int \frac{1}{2} |\nabla u|^2 + N(u) dx\]

on the manifold

\[\mathcal{M}_\sigma = \left\{ u \in H^1(\mathbb{R}^N) : \int |u|^2 dx = \sigma \right\}\]

then the set of all minimizers of \(\mathcal{E}\) is given by \(\psi(x) = u(x)e^{i\theta}, \theta \in [0, 2\pi)\)

We make the following assumptions on \(N\):

\[|N'(s)| \leq c_1 |s|^{q-1} + c_2 |s|^{p-1} \text{ for some } 2 < q < p < 2^* \tag{4.49}\]

\[N(s) \geq -c_1 s^2 - c_2 |s|^{\gamma} \text{ for some } c_1, c_2 \geq 0, \ \gamma < 2 + \frac{4}{N} \tag{4.50}\]

**Theorem 23.** Let \(N\) satisfy (4.49), (4.50) and (4.44). Then, \(\exists \bar{\sigma}\) such that, \(\forall \sigma > \bar{\sigma}\), \(J\) has a minimizer \(\bar{u}\) on \(\mathcal{M}_\sigma\) which is positive and radially symmetric around some point. Moreover, if the Cauchy problem is well posed \(\psi := u e^{i\theta}\) is a hylomorphic soliton.
In order to have stronger results, we can replace (4.44) with the following hypothesis

\[ N(s) < -s^\beta, \quad 2 < \beta < 2 + \frac{4}{N} \text{ for small } s. \]  

(4.51)

In this case we find the following results concerning the existence of the minimizer of \( J(u) \) for any \( \sigma \).

**Theorem 24.** If (4.49), (4.50) and (4.51) hold, then the same conclusions of Th.23 hold for every \( \sigma > 0 \).

In particular, for \( N = 3 \) we have

**Corollary 25.** Let \( N = 3 \). If (4.49) and (4.50) hold and \( N \in C^3 \), with \( N'''(0) < 0 \), then the same conclusions of Th.23 hold for every \( \sigma > 0 \).

For the proofs of Th.23, Th.24 and Th.25, we refer to [4]. Analogous results have been obtained by Cazenave and Lions [18] when \( W(u) = \frac{1}{p}|u|^p \).

### 4.7 Dynamics of solitons

In this section we will describe a recent result relative to the dynamics of solitons when a potential \( V(x) \) is present. As we will see, we will introduce three parameter, \( h, \alpha \) and \( \gamma \) in our equation and we obtain a meaningful result for particular values of these parameters. First of all, let us consider the following "unperturbed" Cauchy problem:

\[ i\hbar \frac{\partial \psi}{\partial t} = \frac{h^2}{2} \Delta \psi + \frac{1}{2h^\alpha} W'(h^\gamma \psi) \]  

(4.52)

\[ \psi(0, x) = \frac{1}{h^\gamma} U \left( \frac{x - q_0}{h^\beta} \right) e^{\frac{i}{h^{\alpha}} \cdot x} \]  

(4.53)

where

\[ \beta = 1 + \frac{\alpha - \gamma}{2} \]  

(4.54)

and \( U : \mathbb{R}^N \rightarrow \mathbb{R}, N \geq 2, \) is a positive, radially symmetric solution of the equation

\[ -\Delta U + W'(U) = 2\omega U \]  

(4.55)

with

\[ \|U\|_{L^2} = \sigma \]  

(4.56)

We set

\[ u_h(x) = h^{-\gamma} U \left( \frac{x}{h^\beta} \right) \]  

and establish a relation between \( \alpha, \beta \) and \( \gamma \) in order to have stationary solution of (4.52) of the form \( \psi(t, x) = u_h(x)e^{-i\omega t} \); replacing this expression in (4.52) we get

\[ -h^2 \Delta u_h + \frac{1}{h^{\alpha}} W'(h^\gamma u_h) = 2\omega u_h. \]  

(4.57)

If we take the explicit expression of \( u_h \), we get

\[ -h^{2-\gamma} \Delta \left[ U \left( \frac{x}{h^\beta} \right) \right] + \frac{1}{h^{\alpha}} W'(U \left( \frac{x}{h^\beta} \right)) = 2\omega h^{-\gamma} U \left( \frac{x}{h^\beta} \right) \]  

(4.58)
and hence, by rescaling the variable $x$,

$$-h^{2-\gamma-2\beta+\alpha} \Delta U(x) + W'(U(x)) = 2\omega h^{\alpha-\gamma} U(x).$$

Thus, comparing the above expression with (4.55), we get (4.54) and

$$\omega_h = \frac{\omega}{h^{\alpha-\gamma}}. \quad (4.58)$$

Using the arguments of section 4.3, we have that the solution of (4.52), (4.53) is given by

$$\psi(t, x) = \frac{1}{h^{\gamma}} U \left( \frac{x - q_0 - vt}{h^\beta} \right) e^{i \hat{v} \cdot (x - Et)} \quad (4.59)$$

with

$$E = \frac{1}{2} v^2 + \frac{\omega}{h^{\alpha-\gamma}}$$

Moreover if the problem (4.52), (4.53) is well posed this is the unique solution.

We can interpret this result saying that the barycenter $q(t)$ of the solution of (4.52), (4.53) (defined by (4.16)) satisfies the Cauchy problem

$$\begin{cases}
\ddot{q} = 0 \\
q(0) = q_0 \\
\dot{q}(0) = v
\end{cases}$$

Let us see what happens if the problem (4.52), (4.53) is perturbed, namely let us investigate the problem

$$\begin{cases}
i h \frac{\partial \psi}{\partial t} = -h^2 \Delta \psi + \frac{1}{2h^\alpha} W'(h^\gamma \psi) + V(x) \psi \\
\psi(0, x) = \varphi_h(x)
\end{cases} \quad (4.60)$$

where

$$\varphi_h(x) = \left[ \frac{1}{h^{\gamma}} (U + w_0) \left( \frac{x - q_0}{h^\beta} \right) \right] e^{i \hat{v} \cdot x} \quad (4.61)$$

and $w_0$ is small, namely there is a constant $C$ such that

$$\|w_0\|_{H^1} \leq Ch^{\alpha-\gamma};$$

$$\int_{\mathbb{R}^N} V(x)|w_0(x)|^2 dx \leq Ch^{\alpha-\gamma}.$$
Finally, we assume that
\[ \alpha > \gamma \]  
(4.62)

Then the barycenter \( q_h(t) \) of the solution of the problem (4.60) satisfies the following Cauchy problem:

\[
\begin{align*}
\ddot{q}_h(t) + \nabla V(q_h(t)) &= H_h(t) \\
q_h(0) &= q_0 \\
\dot{q}_h(0) &= v
\end{align*}
\]

where
\[ \sup_{t \in \mathbb{R}} |H_h(t)| \to 0 \quad \text{as} \quad h \to 0 \]

**Proof:** The proof of this theorem can be found in [12].

\[ \Box \]

**Remark 27.** The assumption \([V_2]\) is necessary if we want to identify the position of the soliton with the barycenter (4.16). Let us see why. Consider a soliton \( \psi(x) \) and a perturbation

\[ \psi_d(x) = \psi(x) + \varphi(x - d), \quad d \in \mathbb{R}^N \]

Even if \( \varphi(x) \ll \psi(x) \), when \( d \) is very large, the “position” of \( \psi(x) \) and the barycenter of \( \psi_d(x) \) are far from each other. In [12] (lemma 25), it has been proved that this situation cannot occur provided that \([V_2]\) hold.

**Remark 28.** We will give a rough explanation of the meaning of the assumption \( \alpha > \gamma \) which, in this approach to the problem, is crucial. The energy, using eq. (4.8) can be divided in two components: the internal energy

\[ J_h(u) = \int \left( \frac{h^2}{2} |\nabla u|^2 + W_h(u) \right) dx \]  
(4.63)

and the dynamical energy

\[ G(u, S) = \int \left( \frac{1}{2} |\nabla S|^2 + V(x) \right) u^2 dx \]  
(4.64)

which is composed by the kinetic energy \( \frac{1}{2} \int |\nabla S|^2 u^2 dx \) and the potential energy \( \int V(x) u^2 dx \). By our assumptions, the internal energy is bounded from below and the dynamical energy is positive. As \( h \to 0 \), we have that

\[ J_h(\psi_h) \approx h^{\beta - \alpha - \gamma} \]

and

\[ G(\psi_h) \approx ||\psi_h||_{L^2}^2 \approx h^{\beta - 2\gamma} \]

Then, we have that

\[ \frac{G(\psi_h)}{J_h(\psi_h)} \approx h^{\alpha - \gamma} \]

So the assumption \( \alpha - \gamma > 0 \) implies that, for \( h \ll 1, G(\psi_h) \ll J_h(\psi_h) \), namely the internal energy is bigger than the dynamical energy. This is the fact that guarantees the existence and the stability of the travelling soliton for any time.
We end this section with an heuristic proof of Th.26. This proof is not at all rigorous, but it helps to understand the underlying Dynamics. As in section 4.3 we interpret $\rho_H = u^2$ as the density of particles; then

$$H = \int \rho_H dx$$

is the total number of particles. By (4.31), each of these particle moves as a classical particle of mass $m = 1$ and hence, we can apply to the laws of classical dynamics. In particular the center of mass defined in (4.16) takes the following form:

$$q(t) = \frac{\int x m \rho_H dx}{\int m \rho_H dx} = \frac{\int x \rho_H dx}{\int \rho_H dx}. \quad (4.65)$$

The motion of the barycenter is not affected by the interaction between particles (namely by the term (4.32)), but only by the external forces, namely by $\nabla V$. The global external force acting on the swarm of particles is given by

$$\vec{F} = - \int \nabla V(x) \rho_H dx. \quad (4.66)$$

Thus the motion of the center of mass $q$ follows the Newton law

$$\vec{F} = M \ddot{q}, \quad (4.67)$$

where $M = \int m \rho_H dx$ is the total mass of the swarm; thus by (4.65), (4.66) and (4.67), we get

$$\ddot{q}(t) = - \frac{\int \nabla V \rho_H dx}{m \int \rho_H dx} = - \frac{\int \nabla V u^2 dx}{m \int u^2 dx}. \quad (4.68)$$

If we assume that the $u(t, x)$ and hence $\rho_H(t, x)$ is concentrated in the point $q(t)$, we have that

$$\int \nabla V u^2 dx \cong \nabla V (q(t)) \int u^2 dx$$

and so, we get

$$m \ddot{q}(t) \cong - \nabla V (q(t)). \quad (4.69)$$

Notice that the equation $m \ddot{q}(t) = - \nabla V (q(t))$ is the Newtonian form of the Hamilton-Jacobi equation (4.29).

### 5 The nonlinear Klein-Gordon equation

#### 5.1 General features of NKG

The D'Alambert equation,

$$\Box \psi = 0$$

is the simplest equation invariant for the Poincaré group, moreover it is invariant for the "gauge" transformation

$$\psi \mapsto \psi + c$$

Also, if $\psi$ is complex valued, it is invariant for the action (2.15). Thus, it satisfy assumptions A-1, A-2 and A-3, but it is linear and it does not produce solitary
waves. There exist only non-dispersive waves. Let us add to (2.12) a nonlinear term:
\[ \mathcal{L} = \frac{1}{2} |\partial_t \psi|^2 - \frac{1}{2} |\nabla \psi|^2 - W(\psi) \]  
where
\[ W : \mathbb{C} \to \mathbb{R} \]
satisfies the following assumption,
\[ W(e^{i\theta} \psi) = W(\psi) \]
namely \( W(\psi) = F(|\psi|) \) for some function \( F = \mathbb{R} \to \mathbb{R} \). This is simplest non-linear Lagrangian invariant for the Poincaré group and the trivial gauge action (2.15).

The equation of motion relative to the Lagrangian (5.1) is the following:
\[ \Box \psi + W'(\psi) = 0 \]  
(NKG)

where \( W'(\psi) \) is intended as in (4.2).

In the following sections we will see that equation (NKG) with suitable (but very general) assumptions on \( W \) produces a very rich model in which there are solitary waves and solitons. Moreover we will see that these solitons behave as relativistic particles.

If \( W'(\psi) \) is linear, namely \( W'(\psi) = m^2 \psi \), then eq. (NKG) reduces to the Klein-Gordon equation
\[ \Box \psi + m^2 \psi = 0 \]  
(5.2)

Among the solutions of the Klein-Gordon equations there are the \textit{wave packets} which behave as solitary waves but disperse in space as time goes on. On the contrary, if \( W \) has a nonlinear suitable component, the wave packets do not disperse and give hylomorphic solitons.

Sometimes, it will be useful to write \( \psi \) in polar form, namely
\[ \psi(t, x) = u(t, x)e^{iS(t, x)} \]  
(5.3)

In this case the action \( \int \mathcal{L} dx dt \) takes the form
\[ S(u, S) = \frac{1}{2} \int \left( \partial_t u^2 \right) - |\nabla u|^2 + \left[ \left( \partial_t S^2 \right) - |\nabla S|^2 \right] u^2 dx dt - \int W(u) dx dt = 0 \]  
(5.4)

and equation (NKG) becomes:
\[ \Box u - \left[ \left( \partial_t S^2 \right) - |\nabla S|^2 \right] u^2 + W'(u) = 0 \]  
(5.5)
\[ \partial_t \left( u^2 \partial_t S \right) - \nabla \cdot (u^2 \nabla S) = 0 \]  
(5.6)
5.2 First integrals of NKG and the hylenic ratio

Easy computations and the results of section 2.7, show that the integral of motion of NKG are given by the following expression:

- **Energy.** We get
  \[ \mathcal{E} = \int \left[ \frac{1}{2} |\partial_t \psi|^2 + \frac{1}{2} |\nabla \psi|^2 + W(\psi) \right] dx \]  \hspace{1cm} (5.7)

  Using (5.3) we get:
  \[ \mathcal{E} = \int \left[ \frac{1}{2} (\partial_t u)^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2} \left( (\partial_t S)^2 + |\nabla S|^2 \right) u^2 + W(u) \right] dx \]  \hspace{1cm} (5.8)

- **Momentum.** We have
  \[ \mathbf{P} = -\text{Re} \int \partial_t \psi \bar{\nabla} \psi \, dx \]  \hspace{1cm} (5.9)

  Using (5.3) we get:
  \[ \mathbf{P} = -\int \left( \partial_t u \nabla u + \partial_t S \nabla S u^2 \right) \, dx \]  \hspace{1cm} (5.10)

- **Angular momentum.** We have
  \[ \mathbf{M} = \text{Re} \int \mathbf{x} \times \nabla \psi \bar{\nabla} \psi \, dx \]  \hspace{1cm} (5.11)

  Using (5.3) we get:
  \[ \mathbf{M} = \int \left( \mathbf{x} \times \nabla S \partial_t S u^2 + \mathbf{x} \times \nabla u \partial_t u \right) \, dx \]  \hspace{1cm} (5.12)

- **Hylenic Charge.** We have
  \[ \mathcal{H} = \text{Im} \int \partial_t \psi \bar{\psi} \, dx \]  \hspace{1cm} (5.13)

  Using (5.3) we get:
  \[ \mathcal{H} = \int \partial_t S u^2 \, dx \]  \hspace{1cm} (5.14)

- **Ergocenter velocity.** If we take the Lagrangian (5.1), the quantity preserved by the Lorentz transformation, is the following
  \[ \mathbf{K} = i \mathbf{P} - \int \mathbf{x} \left[ \frac{1}{2} |\partial_t \psi|^2 + \frac{1}{2} |\nabla \psi|^2 + W(\psi) \right] dx \]  \hspace{1cm} (5.14)
The computation of $K$ is more involved than the previous ones and we will make it in details:

**Proof.** Let us compute $K_i$ using Th. 3; in this case the parameter $\lambda$ is the velocity $v_i$ which appears in (2.8); we have

$$
\rho K_i = \text{Re} \left( \frac{\partial L}{\partial \psi} \frac{\partial}{\partial v_i} \right) - L \frac{\partial t}{\partial v_i} = \text{Re} \left( \frac{\partial}{\partial \psi} \frac{\partial}{\partial v_i} \sum_{k=1}^{3} \frac{\partial \psi}{\partial x_k} \frac{\partial x_k}{\partial v_i} \right) - \left( \frac{1}{2} |\partial_t \psi|^2 - \frac{1}{2} |\nabla \psi|^2 - W(\psi) \right) \frac{\partial t}{\partial v_i}
$$

where the derivative with respect to $v_i$ need to be computed for $v_i = 0$.

Since for $k \neq i$, $\frac{\partial x_k}{\partial v_i} = 0$, we have that

$$
\rho K_i = \left| \frac{\partial_t \psi}{\partial v_i} \right|^2 + \text{Re} \left( \frac{\partial}{\partial \psi} \frac{\partial}{\partial v_i} \right) \frac{\partial x_i}{\partial v_i} - \left( \frac{1}{2} |\partial_t \psi|^2 - \frac{1}{2} |\nabla \psi|^2 - W(\psi) \right) \frac{\partial t}{\partial v_i}
$$

$$
= \rho E \frac{\partial t}{\partial v_i} - \rho P_i \cdot \frac{\partial x_i}{\partial v_i}
$$

Also we have

$$
\left( \frac{\partial t}{\partial v_i} \right)_{v_i = 0} = \left( \frac{\partial}{\partial v_i} \frac{t - v_i x}{\sqrt{1 - v_i^2}} \right)_{v_i = 0} = -x
$$

$$
\left( \frac{\partial x}{\partial v_i} \right)_{v_i = 0} = \left( \frac{\partial}{\partial v_i} \frac{x - v_i t}{\sqrt{1 - v_i^2}} \right)_{v_i = 0} = -t
$$

so that

$$
\rho K_i = -\rho E x_i + \rho P_i t
$$

Integrating in the space we get the $i$-th component of (5.14).

\[ \square \]

Let us interpret (5.14) in a more meaningful way. If we derive the terms of (5.14) with respect to $t$, we get

$$
P = \frac{d}{dt} \int x \left[ \frac{1}{2} |\partial_t \psi|^2 + \frac{1}{2} |\nabla \psi|^2 + W(\psi) \right] dx = \int x \rho E dx \quad (5.15)
$$

Now, we define the ergocenter (or barycenter) as follows

$$
Q := \int x \left[ \frac{1}{2} |\partial_t \psi|^2 + \frac{1}{2} |\nabla \psi|^2 + W(\psi) \right] dx = \int \frac{x \rho E dx}{\mathcal{E}} \quad (5.16)
$$

then, by the conservation of $\mathcal{E}$ and eq. (5.15), we get

$$
\dot{Q} = \frac{P}{\mathcal{E}} \quad (5.17)
$$
Concluding, the Poincaré group provides 10 independent integral of motions which are \( E, P, M, K \); they can be replaced by integral of motions \( E, P, M, \dot{Q} \) since also these quantities are independent.

Notice the difference between (4.14) and (5.14) and consequently the difference between the hylecenter (4.16) and the ergocenter (5.17). In NS the barycenter or "center of mass" or coincide with the hylecenter or "center of hylenic charge"; in NKG the barycenter coincide with the ergocenter or "center of energy".

The precise definition of mass and its meaning will be discussed at pag. 91.

We now assume that \( W \) is of class \( C^2 \) and we set

\[
W(s) = \frac{1}{2} m^2 s^2 + N(s)
\]

where \( m^2 = W''(0) \).

**Theorem 29.** If \( W \) is of class \( C^2 \), then we have that

\[
E_0 := \lim_{\varepsilon \to 0} \inf_{\Psi \in X_{\varepsilon}} \frac{\mathcal{E}(\Psi)}{|\mathcal{H}(\Psi)|} = m
\]

**Proof.** We have \( \Psi = (\psi, \psi_t) \equiv (u, u, \omega, k) \); then by (5.8), (5.13) and (??)

\[
\mathcal{E}(\Psi) = \frac{\int \left[ \frac{1}{2} (\partial_t u)^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2} [\partial_t S]^2 + (\nabla S)^2 \right] u^2 + W(u) \, dx}{\int |\omega u^2| \, dx} \geq \frac{\int \left[ \frac{1}{2} \omega^2 u^2 + \frac{1}{2} m^2 u^2 + N(u) \right] \, dx}{\int |\omega| u^2 \, dx}
\]

Since

\[
\int |\omega| u^2 \, dx \leq \left( \int \omega^2 u^2 \, dx \right)^{1/2} \cdot \left( \int u^2 \, dx \right)^{1/2}
\]

\[
= \frac{1}{m} \left( \int \omega^2 u^2 \, dx \right)^{1/2} \cdot \left( \int m^2 u^2 \, dx \right)^{1/2}
\]

\[
\leq \frac{1}{2m} \left[ \int \omega^2 u^2 \, dx + \int m^2 u^2 \, dx \right]
\]

\[
= \frac{1}{2m} \int (\omega^2 + m^2) u^2 \, dx
\]

we have that

\[
\frac{\mathcal{E}(\Psi)}{|\mathcal{H}(\Psi)|} \geq \frac{\int \left[ \frac{1}{2} \omega^2 u^2 + \frac{1}{2} m^2 u^2 + N(u) \right] \, dx}{2m \int (\omega^2 + m^2) u^2 \, dx} = m + \frac{\int N(u) \, dx}{2m \int (\omega^2 + m^2) u^2 \, dx}
\]

Then since \( N(u) = O(u^3) \) for \( u \to 0 \), we have that

\[
\lim_{\varepsilon \to 0} \inf_{\Psi \in X_{\varepsilon}} \frac{\mathcal{E}(\Psi)}{|\mathcal{H}(\Psi)|} \geq m
\]

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In order to prove the opposite inequality, take $\Psi_{\delta,R} = (\delta u_R, -i\delta u_R) \equiv (u, 0, 1, 0)$ where

$$\begin{align*}
u_R(x) = \begin{cases} 1 & \text{if } |x| < R \\ 0 & \text{if } |x| > R + 1 \\ 1 + R - |x| & \text{if } R < |x| < R + 1 \end{cases}
\end{align*} \quad (5.19)$$

Then

$$\inf_{\Psi \in \mathcal{X}} \frac{\mathcal{E}(\Psi)}{|\mathcal{H}(\Psi)|} \leq \frac{\mathcal{E}(\Psi_{\varepsilon,R})}{|\mathcal{H}(\Psi_{\varepsilon,R})|} = \frac{\varepsilon^2 \int \left[ \frac{1}{2} |\nabla u_R|^2 + \frac{1}{2} u_R^2 + \frac{1}{2 \varepsilon^2} W(\varepsilon u) \right] dx}{\varepsilon^2 \int u_R^2 dx} \leq 1 + \frac{1}{2} \frac{\int |\nabla u_R|^2 dx}{\int u_R^2 dx} + \frac{\int \frac{1}{2 \varepsilon^2} N(\varepsilon u) dx}{\int u_R^2 dx} = 1 + O\left(\frac{1}{R}\right) + O(\varepsilon)$$

\[ \square \]

5.3 Swarm interpretation of NKG

Before giving the swarm interpretation to equation NKG we will write it with the usual physical constants $c$, $m$ and $\hbar$:

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \Delta \psi + W'(\psi) = 0 \quad (5.20)$$

with

$$W(u) = \frac{m^2 c^4}{2 \hbar^2} u^2 + N(u)$$

Here $c$ has the dimension of a velocity (and represents the speed of light), $m$ has the dimension of mass and $\hbar$ is the Plank constant.

The polar form of $\psi$ is written as follows

$$\psi(t, x) = u(t, x)e^{iS(t, x)/\hbar} \quad (5.21)$$

and equations (5.5) and (5.6) become

$$\hbar^2 \left( \partial_t^2 u - c^2 \Delta u + N'(u) \right) + \left( -\partial_t^2 S + c^2 |\nabla S|^2 + m^2 c^4 \right) u = 0 \quad (5.22)$$

$$\partial_t (u^2 \partial_t S) - c^2 \nabla \cdot (u^2 \nabla S) = 0 \quad (5.23)$$

The continuity equation (3.12) for NKG is given by (5.23). This equation allows us to interpret the matter field to be a fluid composed by particles whose density is given by

$$\rho_H = -u^2 \partial_t S$$

and which move in the velocity field

$$\mathbf{v} = -\frac{\nabla S}{c^2 \partial_t S}. \quad (5.24)$$
If
\[ \hbar^2 (\partial_t^2 u - c^2 \Delta u + N'(u)) \ll \left( -\partial_t^2 S + c^2 |\nabla S|^2 + m^2 c^4 \right) u, \] (5.25)
namely, if \( \hbar \) is very small with respect to the other quantities involved, equation (5.22) can be approximated by
\[ \partial_t^2 S = c^2 |\nabla S|^2 + m^2 c^4. \] (5.26)
or
\[ \partial_t S + \sqrt{m^2 c^4 + c^2 |\nabla S|^2} = 0 \] (5.27)
This is the Hamilton-Jacobi equation of a free relativistic particle of rest mass \( m \) (cf. eq. 2.50) whose trajectory \( q(t) \), by (5.24) satisfies the equation
\[ \dot{q} = -\frac{\nabla S}{c^2 \partial_t S} \] (5.28)
(cf. eq. 2.51)
If we do not assume (4.28), equation (4.29) needs to be replaced by
\[ \partial_t S = \pm \sqrt{m^2 c^4 + c^2 |\nabla S|^2 + Q(u)} \]
with
\[ Q(u) = \hbar^2 \left( \partial_t^2 u - c^2 \Delta u + N'(u) \right) \]
The term \( Q(u) \) can be regarded as a field describing a sort of interaction between particles.

Given a wave of the form (5.21), the local frequency and the local wave number are defined as follows:
\[ \omega(t, x) = -\frac{\partial_t S(t, x)}{\hbar} \]
\[ k(t, x) = \frac{\nabla S(t, x)}{\hbar}; \]
the energy of each particle moving according to (5.27), is given by
\[ E = \partial_t S \]
and its momentum is given by
\[ p = \nabla S; \]
thus we have that
\[ E = \hbar \omega \]
\[ p = \hbar k; \]
these two equations are the De Broglie relation.

Thus, eq. (5.24) becomes
\[ \mathbf{v} = \frac{k}{\omega} = \frac{p}{E}. \] (5.29)
5.4 Existence of solitary waves and solitons in NKG

The easiest way to produce solitary waves of NKG consists in solving the static equation

$$-\Delta u + W'(u) = 0 \quad (5.30)$$

and setting

$$\psi_v(t, x) = \psi_v(t, x_1, x_2, x_3) = u \left( \frac{x_1 - vt}{\sqrt{1 - v^2}}, x_2, x_3 \right) \quad (5.31)$$

$\psi_v(t, x)$ is a solution of eq. NKG which represents a bump which travels in the $x_1$-direction with speed $v$.

Thus by Theorem 17, we obtain the following result:

**Theorem 30.** Assume that $W$ satisfies (G-i), (G-ii), (G-iii). Then eq. NKG has real valued solitary waves of the form (5.31).

However, it would be interesting to assume

$$W \geq 0; \quad (5.32)$$

in fact the energy of a solution of equation NKG is given by

$$E(\psi) = \int \left[ \frac{1}{2} |\partial_t \psi|^2 + \frac{1}{2} |\nabla \psi|^2 + W(\psi) \right] \, dx$$

Thus, (5.32) implies that every state $\psi$ has positive energy. In this case, the positivity of the energy, not only is an important request for the physical models related to this equation, but it provides good a priori estimates for the solutions of the relative Cauchy problem. These estimate allows to prove the existence and the well-posedness results under very general assumptions on $W$. Unfortunately Derrick [25], in a very well known paper, has proved that request (5.32) implies that equation (5.30) has only the trivial solution. His proof is based on the following equality (which in a different form was also found by Pohozaev).

The Derrick-Pohozaev identity (see e.g. ([7])) states that for any finite energy solution $u$ of eq. NKG it holds

$$\left( \frac{1}{N} - \frac{1}{2} \right) \int |\nabla u|^2 \, dx + \int W(u) \, dx = 0 \quad (5.33)$$

Clearly the above inequality and (5.32) imply that $u \equiv 0$ for $N \geq 2$.

However, we can try to prove the existence of solitons of eq. NKG (with assumption (5.32)) exploiting the possible existence of standing waves (as defined by (4.41)), since this fact is not prevented by eq. (5.33).

Substituting (4.41) in eq. NKG we get

$$-\Delta u + W'(u) = \omega^2 u \quad (5.34)$$

Since the Lagrangian (5.1) is invariant for the Lorentz group, we can obtain other solutions $\psi_1(t, x)$ just making a Lorentz transformation on it. Namely, if we take the velocity $v = (v, 0, 0)$, $|v| < 1$, and set

$$t' = \gamma (t - vx_1), \quad x_1' = \gamma (x_1 - vt), \quad x_2' = x_2, \quad x_3' = x_3 \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1 - v^2}}$$

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it turns out that $\psi_1(t, x) = \psi(t', x')$ is a solution of NKG.

In particular given a standing wave $\psi(t, x) = u(x)e^{-i\omega t}$, the function $\psi_\nu(t, x) := \psi(t', x')$ is a solitary wave which travels with velocity $v$. Thus, if $u(x) = u(x_1, x_2, x_3)$ is any solution of Eq. (5.34), then

$$\psi_\nu(t, x_1, x_2, x_3) = u(\gamma(x_1 - vt), x_2, x_3)e^{i(k \cdot x - \omega t)},$$

is a solution of Eq. NKG provided that

$$\omega = \gamma \omega_0 \quad \text{and} \quad k = \gamma \omega_0 v \quad (5.36)$$

Notice that (5.31) is a particular case of (5.35) when $\omega_0 = 0$.

We write $W$ in the form (5.18) and we make the following assumptions:

- (W-i) (Positivity) $W(s) \geq 0$
- (W-ii) (Nondegeneracy) $W = W(s) (s \geq 0)$ is $C^2$ near the origin with $W(0) = W''(0) = 0; W'''(0) = m^2 > 0$
- (W-iii) (Hylomorphy) $\exists s_0 : N(s_0) < 0$
- (W–iii) (Growth) there is a constant $c > 0$ such that

$$N'(s) \geq -c_1 s - c_2 s^{p-1}, \quad 2 < p < 2^*$$

Here there are some comments on assumptions (W-i), (W-ii), (W-iii).

(W-i) implies that the energy is positive; if this condition does not hold, it is possible to have solitary waves, but not hylomorphic waves (cf. Proposition 16 of [10]).

(W-ii) In order to have solitary waves it is necessary to have $W''(0) \geq 0$. There are some results also when $W''(0) = 0$ (null-mass case, see e.g. [14] and [2]), however the most interesting situation occurs when $W''(0) > 0$.

(W-iii) This is the crucial assumption which characterizes the potentials which might produce hylomorphic solitons. This assumption permits to have states $\Psi$ with hylomorphy ratio $\Lambda(\Psi) < m$. Actually, $N$ is the nonlinear term which, when it is negative, produces an attractive "force".

We have the following result:

**Theorem 31.** Assume that (W-i),...,(W-iii) hold, then Eq. NKG has finite energy solitary waves of the form $\psi(t, x) = u(x)e^{-i\omega t}$ for every frequency $\omega \in (m_0, m)$ where

$$m_0 = \inf \left\{ a \in \mathbb{R} : \exists u \in \mathbb{R}^+, \frac{1}{2}a^2 u^2 > W(u) \right\}$$

Notice that by (W-iii), $m_0 < m$, then the interval $(m_0, m)$ is not empty.

**Proof.** By the previous discussion, it is sufficient to show that equation (5.34) has a solution $u$ with finite energy. The solutions of finite energy of (5.34) are the critical points in the Sobolev space $H^1(\mathbb{R}^3)$ of the reduced action functional:

$$J(u) = \frac{1}{2} \int |\nabla u|^2 \, dx + \int G(u) \, dx, \quad G(u) = W(u) - \frac{1}{2}a^2 u^2$$

(5.37)
Now we apply theorem 17. It is easy to check that for every frequency \( \omega \in (m_0, m) \), the required assumptions are satisfied.

\[ \square \]

In [1] the existence of soliton is proved. The proof is quite involved and will not be discussed here; we only refer that it is necessary to strengthen assumption (W-iii) with the following one:

- (W-iii')(Growth condition) At least one of the following assumptions holds:
  
  - (a) there are constants \( a, b > 0 \), \( 2 < p < 2N/(N - 2) \) such that for any \( s > 0 \):
    
    \[
    |N'(s)| \leq as^{p-1} + bs^{2 - \frac{2}{p}}.
    \]
  
  - (b) \( \exists s_1 > s_0 : N'(s_1) \geq 0. \)

5.5 Dynamical properties

In this section we will show that the solitons and the solitary waves relative to eq. NKG behave as relativistic bodies. In fact, the relativistic effects like the space contraction, the time dilation and the equality between mass and energy are consequences of the variational principle A-1 and the invariance for the Poincaré group.

First of all observe that, by Eq. (5.35), the following theorem follows:

**Theorem 32.** Any moving solitary wave experiences a contraction in the direction of its movement of a factor \( 1/\gamma \) with \( \gamma = \frac{1}{\sqrt{1 - v^2}} \).

Thus the space contraction is a trivial fact. On the contrary, the time dilation needs a more subtle computation.

The standing waves of eq. can be considered as a clock. Let us denote by \( q(t) \) the position of our clock at the time \( t \).

If we assume that at \( t = 0 \), \( q(0) = (0, 0, 0) \), the motion of the clock is given by

\[
q(t) = (vt, 0, 0);
\]

then the behavior of the moving clock at the point \( q(t) \) is obtained replacing \( x \) by \( q(t) \) in Eq. (5.35):

\[
\psi_v(t,q(t)) = \psi_v(t,vt,0,0) = u(0,0,0)e^{i(k \cdot q(t) - \omega t)},
\]

taking into account eq. (5.36), we get

\[
k \cdot q(t) - \omega t = \gamma \omega_0 v_1 \cdot v_1 t - \omega t = (\gamma \omega_0 v_1^2 - \gamma \omega_0) t = \gamma \omega_0 (v_1^2 - 1) t = \frac{\omega_0 t}{\gamma}.
\]

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Then
\[ \psi_v(t, q(t)) = u(0, 0, 0) e^{-i \frac{\omega_0}{\gamma} t} \]
this equation shows that our moving clock is vibrating with a frequency
\[ \frac{\omega_0}{\gamma} = \omega_0 \sqrt{1 - v^2} \]
Since the intervals of time measured by a clock are inversely proportional to the frequency of the vibrations, we get that
\[ \Delta T = \frac{\Delta T_0}{\sqrt{1 - v^2}}. \]

Then we get the following

**Theorem 33.** A moving clock moves slower than a resting clock by a factor \( \gamma^{-1} \).

In classical mechanics the mass \( m \) can be defined as a quantity which relates the momentum \( P = (P_1, P_2, P_3) \) to the velocity \( v = (v_1, v_2, v_3) \) by the following formula
\[ P = m v \]
Since the momentum of a solitary wave is defined by (2.32), it is possible to define the mass of a solitary wave by the above formula and to compute it.

In the case of the Schrödinger equation, the velocity of a soliton is given by \( \dot{q} \) where \( q \) is defined by (4.10); then by (4.14) we have that
\[ \text{mass} = \frac{P}{\dot{q}} = \mathcal{H} \quad (5.38) \]
namely the mass of the soliton equals the hylenic charge. Actually, if we add the usual constant as in eq. (4.23), we get that
\[ \text{mass} = m \mathcal{H} \]
This equation is consistent with the swarm interpretation of section 4.3: it says that the soliton consists of a number \( \mathcal{H} \) of particles of mass \( m \).

In the case of the equation NKG, the velocity of a soliton is given by \( \dot{Q} \) where \( Q \) is defined by (5.16); then by (5.17) we have that
\[ \text{mass} = \frac{P}{\dot{Q}} = \mathcal{E} \]
namely the mass of the soliton equals its energy. If we add the usual constant as in eq. (5.20), we get that
\[ \text{mass} = \frac{\mathcal{E}}{c^2} \]
namely the celebrated Einstein equation. Namely, we have the following:

**Theorem 34.** The mass of a solitary wave of eq. NKG is proportional to its energy with the factor of proportionality \( c^{-2} \).
References

[1] J. Bellazzini, V. Benci, C. Bonanno, A.M. Micheletti, Solitons for the Nonlinear Klein-Gordon-Equation, to appear. (arXiv:0712.1103)

[2] M. Badiale, V. Benci, S. Rolando, A nonlinear elliptic equation with singular potential and applications to nonlinear field equations, J. Eur. Math. Soc., 9 (2007), 355–381

[3] J. Bellazzini, V. Benci, C. Bonanno, E. Sinibaldi, Hylomorphic solitons in the nonlinear Klein-Gordon equation., to appear. (arXiv:0810.5079)

[4] J. Bellazzini, V. Benci, M. Ghimenti, A.M. Micheletti, On the existence of the fundamental eigenvalue of an elliptic problem in $\mathbb{R}^N$, Adv. Nonlinear Stud. 7 (2007), 439–458

[5] Bellazzini J., Bonanno C., Nonlinear Schrödinger equations with strongly singular potentials, preprint.

[6] Benci V., Fortunato D., Solitary waves of the nonlinear Klein-Gordon field equation coupled with the Maxwell equations, Rev. Math. Phys. 14 (2002), 409–420.

[7] Benci V. Fortunato D., Solitary waves in classical field theory, in Nonlinear Analysis and Applications to Physical Sciences, V. Benci A. Masiello Eds Springer, Milano (2004), 1-50.

[8] Benci V. Fortunato D., Three dimensional vortices in Abelian Gauge Theories, Nonlinear Analysis T.M.A. (2008).

[9] V. Benci, D. Fortunato, Solitary waves in the nonlinear wave equation and in gauge theories, J. Fixed Point Theory Appl. 1 (2007), 61–86.

[10] Benci V. Fortunato D., Existence of hylomorphic solitary waves in Klein-Gordon and in Klein-Gordon-Maxwell equations, Rendiconti dell'Accademia Nazionale dei Lincei, to appear (arXiv:0903.3508).

[11] V. Benci, D. Fortunato, L. Pisani, Soliton like solution of a Lorentz invariant equation in dimension 3, Reviews in Mathematical Physics, 3 (1998), 315-344.

[12] V. Benci, M. Ghimenti, A.M. Micheletti, The Nonlinear Schroedinger equation: solitons dynamics, to appear (arXiv:0812.4152).

[13] Benci V., Visciglia N., Solitary waves with non vanishing angular momentum, Adv. Nonlinear Stud. 3 (2003), 151-160.

[14] H. Berestycki, P.L. Lions, Nonlinear Scalar Field Equations, I - Existence of a Ground State, Arch. Rat. Mech. Anal., 82 (4) (1983), 313-345.

[15] Buslaev, Vladimir S.; Sulem, Catherine, On asymptotic stability of solitary waves for nonlinear Schrödinger equations. Annales de l’institut Henri Poincaré (C) Analyse non linéaire, 20 no. 3 (2003), p. 419-475

52
[16] Cassani D., \textit{Existence and non-existence of solitary waves for the critical Klein-Gordon equation coupled with Maxwell’s equations}, Nonlinear Anal. 58 (2004), 733-747.

[17] Cazenave, T. \textit{Semilinear Schrödinger equations}, Courant Lecture Notes in Mathematics, vol. 10, New York University Courant Institute of Mathematical Sciences, New York, 2003.

[18] T. Cazenave and P.L. Lions, \textit{Orbital stability of standing waves for some nonlinear Schrödinger equations}, Comm. Math. Phys. 85 (1982), no. 4, 549–561.

[19] S. Coleman, V. Glaser, A. Martin, \textit{Action minima among solutions to a class of euclidean scalar field equation}, Commun. Math. Phys. 58 (1978), 211–221.

[20] S. Coleman, “Q-Balls”, Nucl. Phys. B262 (1985), 263–283; erratum: B269 (1986), 744–745

[21] Cuccagna, Scipio \textit{On asymptotic stability in 3D of kinks for the $\varphi^4$ model}. Trans. Amer. Math. Soc. 360 (2008), no. 5, 2581–2614.

[22] Cuccagna, Scipio; Mizumachi, Tetsu \textit{On asymptotically stability in energy space of ground states for nonlinear Schrödinger equations}. Comm. Math. Phys. 284 (2008), no. 1, 51–77.

[23] D’Aprile T., Mugnai D., \textit{Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations}, Proc. of Royal Soc. of Edinburgh, section A Mathematics, 134 (2004), 893-906.

[24] D’Aprile T., Mugnai D., \textit{Non-existence results for the coupled Klein-Gordon- Maxwell equations}, Advanced Nonlinear studies, 4 (2004), 307-322.3.

[25] C.H. Derrick, \textit{Comments on Nonlinear Wave Equations as Model for Elementary Particles}, Jour. Math. Phys. 5 (1964), 1252-1254.

[26] B. Gidas, W.M. Ni, L. Nirenberg, \textit{Symmetry and related properties via the maximum principle}, Comm. Math. Phys., 68 (1979), 209–243

[27] M. Grillakis, J. Shatah, W. Strauss, \textit{Stability theory of solitary waves in the presence of symmetry, I}, J. Funct. Anal. 74 (1987), 160–197

[28] Tosio Kato, \textit{Nonlinear Schrödinger equations}, Schrödinger operators (Sønderborg, 1988), Lecture Notes in Phys., vol. 345, Springer, Berlin, 1989, pp. 218–263.

[29] Komech, A.; Vainberg, B. \textit{On asymptotic stability of stationary solutions to nonlinear wave and Klein-Gordon equations}. Arch. Rational Mech. Anal. 134 (1996), no. 3, 227–248.

[30] Landau L., Lifchitz E., \textit{Mécanique}, Editions Mir, Moscow, 1966.

[31] Landau L., Lifchitz E., \textit{Théorie du Champ}, Editions Mir, Moscow, 1966.
[32] E. Long, Existence and stability of solitary waves in non-linear Klein-Gordon-Maxwell equations, Rev. Math. Phys. 18 (2006), 747-779.

[33] Pohozaev S. I., Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl., 165, (1965) 1408-1412.

[34] G. Rosen, Particle-like solutions to nonlinear complex scalar field theories with positive-definite energy densities, J. Math. Phys. 9 (1968), 996–998.

[35] Rubakov V., Classical theory of Gauge fields, Princeton University press, Princeton 2002.

[36] J. Shatah, Stable Standing waves of Nonlinear Klein-Gordon Equations, Comm. Math. Phys., 91, (1983), 313-327.

[37] W.A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. 55 (1977), 149-162.

[38] Y. Yang, Solitons in Field Theory and Nonlinear Analysis, Springer, New York, Berlin, 2000.