2PN/RM gauge invariance in Brans–Dicke-like scalar–tensor theories

Olivier Minazzoli

Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Drive, Pasadena, CA 91109-0899, USA

E-mail: ominazzo@caltech.edu

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Abstract
In this note, we study the 2PN/RM gauge invariance structure of Brans–Dicke-like scalar–tensor theories (STT) without potential. Since the spherical isotropic metric plays an important role in the literature, its 2PN/RM STT version is deduced from the general equations given by Minazzoli and Chauvineau (2011 Class. Quantum Grav. 28 085010), by using the invariance structure properties. It is found that the second-order Eddington parameter $\epsilon$ can be written in terms of the usual post-Newtonian parameter $\gamma$ and $\beta$ as $\epsilon = 4/3\gamma^2 + 4/3\beta - 1/6\gamma - 3/2$.

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1. Introduction
Because of their expected accuracies, most projects relying on precise measurements of the characteristics of the propagation of light in the solar system require the knowledge of the space-time metric at the $c^{-4}$ level\(^1\) in order to have sufficiently accurate equations describing the various observables (see, for instance, Klioner and Zschocke (2010), Minazzoli and Chauvineau (2011), Deng and Xie (2012) and references therein).

In a recent work, Minazzoli and Chauvineau (2011) give the 2PN/RM metric of scalar–tensor theories (STT) without potential, in a set of coordinates that respect the weak spatial isotropy condition (WSIC: $g_{ij} \propto \delta_{ij} + O(c^{-4})$, (Damour et al 1991)); otherwise arbitrary. Although eventually such kind of dynamical metric would have to be taken into account for accurate solar system calculations, many works exploring the 2PN/RM phenomenology simplify the problem by considering the spherical case in isotropic coordinates (Epstein and

\(^1\) The $c^{-4}$ metric is also called the 2PN/RM metric, where PN/RM stands for post-Newtonian/relativistic motion. It means that the development of the post-Newtonian metric is developed to the order that has to be taken into account when dealing with test particles with relativistic velocities only (Minazzoli and Chauvineau 2009, 2011).
Shapiro 1980, Fischbach and Freeman 1980, Richter and Matzner 1983, Turyshnev et al 2004, Plowman and Hellings 2006, Ashby and Bertotti 2010, Teyssandier 2012). This allows one to explore the main features of the 2PN/RM phenomenology, without the complication of a more realistic dynamical metric as given for instance by Minazzoli and Chauvineau (2011). Hence, a parameterized 2PN/RM metric is often considered, with a new parameter \( \epsilon \) entering in the \( c^{-4} \)-space–space part of the metric.

In this note, we first study the gauge invariance structure left in the 2PN/RM field equations, which are such that the coordinate system respects the WSIC only. Then, we use this analysis in order to derive the value of the second-order post-Newtonian parameter \( \epsilon \) in the case of STT without potential. This last calculation illustrates the method that uses the gauge-invariant structure of the field equations in order to find specific gauges.

### 2. Field equations

We start with the usual STT action without potential in the Jordan representation (Minazzoli and Chauvineau 2011):

\[
S = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g} \left[ \Phi R - \frac{\omega(\Phi)}{\Phi} g^{\alpha \beta} \partial_\alpha \Phi \partial_\beta \Phi \right] + \int d^4x \sqrt{-g} \mathcal{L}_m(\Psi, g_{\mu \nu}).
\]

Here, \( g \) is the metric determinant, \( R \) is the Ricci scalar constructed from the physical metric \( g_{\mu \nu} \), \( \mathcal{L}_m \) is the material Lagrangian and \( \Psi \) represents the non-gravitational fields. From there, one deduces the following 2PN/RM metric as well as the corresponding field equations (Minazzoli and Chauvineau 2011):

\[
g_{00} = -1 + \frac{2W}{c^2} - \frac{\beta}{c^4} \frac{2W^2}{c^4} + O(c^{-5}),
\]

\[
g_{0i} = -(\gamma + 1) \frac{2W_i}{c^4} + O(c^{-5}),
\]

\[
g_{ij} = \delta_{ij} \left\{ 1 + \gamma \frac{2W}{c^2} + (\gamma^2 + \beta - 1) \frac{2W^2}{c^4} \right\} + (\gamma + 1) \frac{2W_{ij}}{c^4} + O(c^{-5}),
\]

with

\[
\gamma \equiv \frac{\omega_0 + 1}{\omega_0 + 2}, \quad \beta \equiv 1 + \frac{\omega_0}{(2\omega_0 + 3)(2\omega_0 + 4)^2}, \quad G_{\text{eff}} \equiv \frac{2\omega_0 + 4}{2\omega_0 + 3} G,
\]

and

\[
\Box W + \frac{1 + 2\beta - 3\gamma}{c^2} W \Delta W + \frac{2}{c^2} (1 + \gamma) \partial_\alpha J = -4\pi G_{\text{eff}} \Sigma + O(c^{-3}),
\]

\[
\Delta W_i - \partial_i J = -4\pi G_{\text{eff}} \Sigma^i + O(c^{-2}),
\]

\[
\Delta W_{ij} + \partial_i W \partial_j W + (1 - \beta) \delta_{ij} \Delta W - \partial_i J_j - \partial_j J_i - 2\gamma \delta_{ij} \partial_t J = -4\pi G_{\text{eff}} \Sigma^{ij} + O(c^{-1}),
\]

where one has set

\[
J = \partial_t W + \partial_k W_k + O(c^{-2}),
\]

\[
J_i = \partial_k W_{ik} - \frac{1}{2} \partial_i W_{kk} + \partial_t W_i - \frac{1 - \gamma}{2} \partial_i P,
\]

with

\[
\Delta P + \frac{2\beta - 1 - \gamma}{c^2} W \Delta W - 2\partial_t J = -4\pi G_{\text{eff}} \frac{\Sigma_{kk}}{3\gamma - 1} + O(c^{-1}),
\]

\[2\] In our model, we assume that \( g_{\mu \nu} \) is the physical metric, in the sense that it is the one that describes actual time and length as measured by clocks and rods in our experiments (Esposito-Farèse 2004).
while for the matter part of the equations, one has
\[ \Sigma = \frac{1}{c^2}(T^{00} + \gamma T^{kk}), \Sigma^i = \frac{1}{c}T^{0i}, \Sigma^{ij} = T^{ij} - \gamma T^{kk}\delta_{ij}. \]

3. The gauge invariance structure

As in the general relativity (GR) case, the diffeomorphism invariance leaves four degrees of freedom (dof) left unconstrained in the field equations. Therefore, there is a gauge-invariance-like behavior of such field equations. Because of the WSIC \((g_{ij} \propto \delta_{ij} + O(c^{-4}))\) imposed to the metric at the 1PN level, the 1PN field equation gauge freedom is characterized by an arbitrary scalar function \(\lambda\) only. Indeed, the WSIC in the Jordan representation follows from the strong spatial isotropy condition (SSIC: \(g_{00}g_{ij} = -\delta_{ij} + O(c^{-4}), (\text{Damour et al} 1991)\)) in the Einstein representation (Minazzoli and Chauvineau 2011). Therefore, it fixes the gauge freedom corresponding to the spatial dof that would appear in the field equations otherwise and leaves only the gauge freedom corresponding the choice of time coordinates. One has
\[ W' = W - \frac{1}{c^2}\partial_t \lambda, \quad W'_i = W_i + \frac{1}{2(1 + \gamma)}\partial_i \lambda. \]
At the 2PN/RM level, the metric in general can no longer be put in an isotropic form, and the spatial dof reappear anew in the field equations, as in GR (Minazzoli and Chauvineau 2009). Therefore, at this level, the gauge invariance is characterized by an additional arbitrary 3-vector \(A_i\), such that
\[ W''_{ij} = W_{ij} + C_iA_j + C_jA_i + \frac{\gamma}{1 + \gamma}\delta_{ij}\partial_t \lambda. \]
But to be complete, one also has to take into account the invariance of the equation on the scalar field \(P\):
\[ P' = P + \frac{1}{1 + \gamma}\partial_t \lambda. \]
This scalar field is a leftover of the scalar field \(\Phi\) in the field equations at the \(c^{-4}\) level. It is due to the fact that the scalar field equation’s source is proportional to the trace of the stress–energy tensor instead of being proportional to \(\Sigma\).

Equations (6)–(8) represent the 2PN/RM gauge invariance structure of the scalar–tensor field equations in a set of coordinates that respect the WSIC.

4. The spherical isotropic case

The spherical isotropic case has an important place in the literature related to the 2PN/RM metric. Indeed, it looks like the simplest metric that one can use in order to derive the 2PN/RM phenomenology characterized for instance by the time transfer or deviation angle equations. In this case—corresponding to a spherical source at the center of the coordinates—the metric in various alternative theories would write, according to Epstein and Shapiro (1980):
\[ g_{00} = -1 + \frac{2W}{c^2} - \beta\frac{2W^2}{c^4} + O(c^{-5}), \]
\[ g_{0i} = O(c^{-5}), \]
\[ g_{ij} = \delta_{ij}\left(1 + \gamma\frac{2W}{c^2} + \frac{3\epsilon}{2}\frac{W^2}{c^4}\right) + O(c^{-5}). \]
with \( W' = GM/r' \), where \( G \) is the effective gravitational constant, \( M \) is the mass of the source and \( r' \) is the radial coordinate in the isotropic system of coordinates. We dub \( \epsilon \) the second-order Eddington parameter—equal to 1 in GR. However, one should note that there is no reason to expect that a vector–tensor theory, for instance, would not break the spherical symmetry of the problem because of the local direction of the space part of the vector field. Therefore, such a metric seems to be useful for a very restricted set of alternative theories—namely, probably only STT. However, while in our opinion one should use the metric in the general form, such as the one given by equations (2), it still seems interesting to give the value of the \( \epsilon \) parameter in STT—mainly because most of the papers studying some aspects of the 2PN/RM phenomenology use the metric in the form given by equations (9)–(11) (Epstein and Shapiro 1980, Fischbach and Freeman 1980, Richter and Matzner 1983, Turyshev et al 2004, Plowman and Hellings 2006, Teyssandier 2010, Ashby and Bertotti 2010, Teyssandier 2012).

In order to obtain the metric in this class of coordinate system, one has to find a gauge transformation that kills the anisotropic terms in equation (3). This can be achieved by realizing that

\[
\frac{\partial}{\partial t}W' = \frac{1}{8} \partial_{ii}W^2 + \delta_{ij}U,
\]

where \( W' = GM/r' \), with \( r' \) being the original radial coordinate (i.e. in no specific coordinate system), and \( U = \frac{1}{4}(GM/r'^2) \). Therefore, since we are considering a static vacuum field solution, where \( \nabla W = O(c^{-2}) \), \( \partial J = 0 \) and \( \Sigma^{ij} = 0 \), equation (3) can be written as

\[
\nabla_{ij}W' + \frac{1}{8} \partial_{ii}W^2 + \delta_{ij}U - \partial_i J_j = O(c^{-1}),
\]

which can be re-written as

\[
\nabla_{ij}W + \partial_i \left( \nabla A_j + \frac{1}{16} (\partial_j W^2 - J_j) \right) + \partial_j \left( \nabla A_i + \frac{1}{16} (\partial_i W^2 - J_i) \right) = -\delta_{ij}U + O(c^{-1}),
\]

by using the gauge invariance of the field equation as well as the commutativity of partial derivatives. Then, by choosing the 3-vector gauge field \( A_i \) that satisfies the following equation: \( \nabla A_i = J_i - \frac{1}{16} (\partial_i W^2 - \partial_i W_k - \partial_k W_i) - \frac{1}{2} \partial_i W_k + \partial_i W - \frac{1}{2} \frac{\gamma}{\gamma - 1} \partial_i P - \frac{1}{16} \partial_i W^2 \)

which is obviously invertible, equation (12) can be re-written as

\[
W''_{ij} = -\delta_{ij} \frac{1}{4} \left( \frac{GM}{r'^2} \right)^2 + O(c^{-1}) \rightarrow W''_{ij} = -\delta_{ij} \frac{1}{8} W'^2 + O(c^{-1}),
\]

with \( r' = r + O(c^{-2}) \). From there, after injecting into the space–space component of the metric (2), one obtains

\[
\epsilon = \frac{4}{3} \gamma^2 + \frac{4}{3} \beta - \frac{\gamma}{6} - \frac{3}{2},
\]

It is the result found by Damour and Esposito-Farèse (1996) after they considered directly the 2PN spherical body solution in isotropic coordinates of the STT field equations (see their equation (5.8)). As a corollary, setting \( \gamma = 1 \) and \( \beta = 1 \) in (14) gives the corresponding transformation in the GR case.

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3 In contrast, if one wants to write the metric in the harmonic gauge (Minazzoli and Chauvineau 2011), one needs to impose: \( \nabla A_i = J_i \) as in GR (Minazzoli and Chauvineau 2009).
References

Ashby N and Bertotti B 2010 *Class. Quantum Grav.* **27** 145013
Damour T and Esposito-Farèse G 1996 *Phys. Rev.* D **53** 5541–78
Damour T, Soffel M and Xu C 1991 *Phys. Rev.* D **43** 3273–307
Deng X M and Xie Y 2012 *Phys. Rev.* D **86** 044007
Epstein R and Shapiro I I 1980 *Phys. Rev.* D **22** 2947–9
Esposito-Farèse G 2004 *Phi in the Sky: The Quest for Cosmological Scalar Fields* (American Institute of Physics Conference Series vol 736) ed C J A P Martins, P P Avelino, M S Costa, K Mack, M F Mota and M Parry (New York: AIP) pp 35–52
Fischbach E and Freeman B S 1980 *Phys. Rev.* D **22** 2950–2
Klioner S A and Zschocke S 2010 *Class. Quantum Grav.* **27** 075015
Minazzoli O and Chauvineau B 2009 *Phys. Rev.* D **79** 084027
Minazzoli O and Chauvineau B 2011 *Class. Quantum Grav.* **28** 085010
Plowman J E and Hellings R W 2006 *Class. Quantum Grav.* **23** 309–18
Richter G W and Matzner R A 1983 *Phys. Rev.* D **28** 3007–12
Teyssandier P 2010 *IAU Symposium* vol 261 ed S A Klioner P K Seidelmann and M H Soffel pp 103–11
Teyssandier P 2012 arXiv:1206.6309
Turyshev S G, Shao M and Nordtvedt K 2004 *Class. Quantum Grav.* **21** 2773–99