One-loop Quantum Holography
for Higher Dimensional Black Holes

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Abstract: The one-loop quantum corrections to the free energy associated with scalar field in a higher dimensional static curved space-time is investigated making use of the conformal transformation method. For a space-time with bifurcate horizon, horizon divergences are accounted for choosing the Planck length as natural cutoff. The leading term in the high temperature quantum correction satisfies holographically the "area law", like the tree level Bekenstein-Hawking term. Furthermore it is stressed that only for the asymptotically AdS black holes one may have a microscopic interpretation of the entropy also at quantum level.

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I. Recently, the issue concerning the microscopic explanation of the Bekenstein-Hawking formula for the black hole entropy [1, 2] has been vastly discussed in literature. Among the several approaches, we would like to recall the investigations related to (2+1)-dimensional BTZ black hole [3], the stringy approach (see, for example, [4]), the Matrix theory approach [5], the loop gravity approach [6], the induced gravity approach [7] and the new approach appeared in [8].

There have also been some attempts to compute semiclassically the quantum corrections to the Bekenstein-Hawking classical entropy for the 4-dimensional Schwarzschild black hole. However, so far all the evaluations have been plagued by the appearance of divergences, first noticed in ref. [9] (see also [10, 11, 12, 13, 14, 15]).

In the 4-dimensional black hole case, the physical origin of these divergences can be traced back to the equivalence principle according to which, in a static space-time with canonical horizons, a system in thermal equilibrium has a local Tolman temperature given by \( T(x) = T/\sqrt{-g_{00}(x)} \), \( T \) being the generic asymptotic temperature. Since, roughly speaking, very near the horizon a static space-time may be regarded as a Rindler-like space-time, one gets for the Tolman temperature \( T(\rho) = T/\rho \), \( \rho \) being the distance from the horizon. As a consequence, omitting the multiplicative constant, the total entropy reads

\[
S \sim \int dx \int_{\epsilon}^{\infty} T^3(\rho) \, d\rho = \frac{AT^3}{2e^2},
\]

(1)

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where $A$ is the area of the horizon and $\varepsilon$ is the horizon cutoff. These considerations suggest the use of the optical metric $\bar{g}_{\mu\nu} = g_{\mu\nu}/|g_{00}|$, conformally related to the original one, and one of the purposes of this paper is to implement this idea, and try to shed some light on the holographic property [14] of black hole space-times, which has been discussed recently in [17, 18] within the AdS/CFT correspondence [19].

In particular, Barbon and Rabinovici [18] have explained how the internal dimensions remain invisible to the thermodynamics of the 4D conformal field theory living on the boundary of $AdS_5$. The dimensionality of a spacetime is revealed in the free energy of massless fields, which scales as $T^d$ in a $d$-dimensional space. By allowing the phase transition to an anti-de Sitter black hole, they have shown that the region where the local temperature exceeds the Kaluza-Klein threshold (i.e. $T > 1/R_c$, where $R_c$ is the radius of compact internal dimensions), is roughly inside the event horizon and thus is cut out from the Euclidean section, which is where the free energy is computed. This gives one-loop holography, but only if one choses to renormalize away the horizon divergences.

From the point of view of the low energy effective field theory, this is unsatisfactory, because it requires a tree level bare entropy with no statistical interpretation. Instead, we can rely on the correspondence between ultraviolet effects in the bulk with infrared effects in the boundary theory [17], to argue that there should be no horizon divergences, because there should be no infrared divergences in a field theory on a compact space. Hence, we shall keep the horizon contribution and show that it scales holographically.

II. To begin with, we consider a scalar field on a $(N+1)$-dimensional static space-time with the metric (signature $-++...+$), $D = N + 1$.

\[ ds^2 = g_{00}(x)(dx^0)^2 + g_{ij}(x)dx^idx^j, \quad x = \{x^j\}, \quad i, j = 1, ..., N. \]  

The restriction to scalar fields may be justified noting that the horizon divergences we are going to discuss here should be independent on the specific features of the bulk (supergravity) theory.

The one-loop partition function is given by (we perform the Wick rotation $x_0 = -i\tau$, thus all differential operators one is dealing with will be elliptic)

\[ Z = \int d[\phi] \exp \left( -\frac{1}{2} \int \phi L_D \phi d^Dx \right), \]  

where $\phi$ is a scalar density of weight $-1/2$ and $L_D$ is a Laplace-like operator that has the form

\[ L_D = -\Delta_D + m^2 + \xi R. \]  

Here $\Delta_D$ is the Laplace-Beltrami operator acting in $D$- dimensional space-time, $m$ (the mass) and $\xi$ are arbitrary parameters and $R$ is the scalar curvature.

We recall method of the conformal transformation [20]. This method is useful because it permits to compute all physical quantities in an ultrastatic manifold (called the optical manifold [21]). The ultrastatic Euclidean metric $\bar{g}_{\mu\nu}$ is related to the static one by the conformal transformation

\[ \bar{g}_{\mu\nu}(x) = e^{2\sigma(x)}g_{\mu\nu}(x), \]  

with $\sigma(x) = -\frac{1}{2} \ln g_{00}$. In this manner, $\bar{g}_{00} = 1$ and $\bar{g}_{ij} = g_{ij}/g_{00}$ (Euclidean optical metric).

With regard to the one-loop partition function, it is possible to show that
\[ \bar{Z} = J[g, \bar{g}] Z , \] (6)

where \( J[g, \bar{g}] \) is the Jacobian of the conformal transformation. Such a Jacobian can be explicitly computed, but here we shall need only its structural form. Using \( \zeta \)-function regularization for the determinant of the second order differential operator we get

\[
\ln Z = \ln \bar{Z} - \ln J[g, \bar{g}] = \frac{1}{2} \frac{d}{ds} \zeta(s|\bar{L}_D\ell^2)|_{s=0} - \ln J[g, \bar{g}] ,
\] (7)

where \( \ell \) is an arbitrary parameter necessary to adjust the dimensions, the function \( \zeta(s|\bar{L}_D\ell^2) \) associated with the operator \( \bar{L}_D \), which explicitly reads

\[
\bar{L}_D = e^{-\sigma} D_D e^{-\sigma} = -\partial^2_\tau - \Delta_N + \xi_D \bar{R} + e^{-2\sigma}\left[ m^2 + (\xi - \xi_D) R \right] = -\partial^2_\tau + \bar{L}_N .
\] (8)

In Eq. (8), \( \xi_D = (N - 1)/4N \) and

\[
\bar{R} = e^{-2\sigma}\left[ R - 2N\Delta_N \sigma - N(N - 1)g^{\mu\nu}\partial_\mu \sigma \partial_\nu \sigma \right] .
\] (9)

Now we formally have to deal with an ultrastatic space-time. For a scalar field in thermal equilibrium at finite temperature \( T = 1/\beta \) the partition function \( \bar{Z}_\beta \) can be obtained, within the path integral approach, after the Wick rotation \( \tau = ix^0 \) imposing on the field a periodicity in \( \tau \). Thus one obtains [20, 14, 22]

\[
\ln \bar{Z}_\beta = -\frac{\beta}{2} \left[ \text{PP} \zeta\left(-\frac{3}{2}|\bar{L}_N \right) + (2 - 2\ln 2\ell) \text{Res} \zeta\left(-\frac{1}{2}|\bar{L}_N \right) \right] + \lim_{s \to 0} \frac{d}{ds} \frac{\beta}{\sqrt{4\pi} \Gamma(s)} \sum_{n=1}^\infty \int_0^\infty t^{s-3/2} e^{-n^2\beta^2/4t} \text{Tr} e^{-t\bar{L}_N} dt ,
\] (10)

where PP and Res stand for the principal part and for the residue of the zeta function. The free energy is related to the canonical partition function by means of the equation

\[
F(\beta) = -\frac{1}{\beta} \ln Z_\beta = -\frac{1}{\beta} (\ln \bar{Z}_\beta - \ln J[g, \bar{g}]) .
\] (11)

Since we are considering a static space-time the quantity \( \ln J[g, \bar{g}] \) depends linearly on \( \beta \) and according to Eq. (11) it gives no contribution to the entropy, which has the form

\[
S_\beta = \beta^2 \partial_\beta F_\beta ,
\] (12)

where \( F_\beta \) is the temperature dependent part (statistical sum) of \( F(\beta) \).

Let us apply this formalism to scalar fields in a \( D \)-dimensional static space-time with metric

\[
ds^2 = A(r) d\tau^2 + A(r)^{-1} dr^2 + r^2 d\Sigma_{d-1} + dE_{D-d-1} ,
\] (13)

where we are using polar coordinates, \( r \) being the radial one and \( d\Sigma_{d-1} \) and \( dE_{D-d-1} \)s are the metrics of two \((d-1)\)-dimensional and \((D - d - 1)\)-dimensional Einstein spaces. Particularly interesting are the two \( D \)-dimensional space-times relevant in the CFT/AdS correspondence [7]: \( X_1 = \text{AdS}_{d+1} \times M_{D-d-1} \) and \( X_2 = \text{AdS}^{BH}_{d+1} \times M_{D-d-1} \). The first contains the periodically identified AdS space and the second one the Schwarzschild AdS black hole, and \( M_{D-d-1} \) is a suitable compact \((D - d - 1)\)-dimensional manifold. However, we would like to consider a more general class of metrics, which may be defined by the function \( A(r) \) and by the related \((d-1)\)-dimensional manifold, namely
(I). The AdS space,
\[ A(r) = \left(1 + \frac{r^2}{l^2}\right), \quad d\Sigma_{d-1} = d\Omega_{d-1}. \] (14)

(II). The Schwarzschild black hole,
\[ A(r) = \left(1 - \frac{C_d M}{r^{d-2}}\right), \quad d\Sigma_{d-1} = d\Omega_{d-1}. \] (15)

(III). The Schwarzschild-AdS black hole [23, 24],
\[ A(r) = \left(1 + \frac{r^2}{l^2} - \frac{C_d M}{r^{d-2}}\right), \quad d\Sigma_{d-1} = d\Omega_{d-1}. \] (16)

(IV). The toroidal AdS black hole [25, 26, 27, 28],
\[ A(r) = \left(\frac{r^2}{l^2} - \frac{C_d M}{r^{d-2}}\right), \quad d\Sigma_{d-1} = dT_{d-1}. \] (17)

(V). The hyperbolic AdS black hole [25, 26, 27, 28],
\[ A(r) = \left(-1 + \frac{r^2}{l^2} - \frac{C_d M - \frac{2}{d} \left((d-2)l^2\right)^{\frac{d-2}{2}}}{r^{d-2}}\right), \quad d\Sigma_{d-1} = dH_{d-1}. \] (18)

In Eqs. (15) - (18) \( M \) is the mass of the black hole, the cosmological constant \( \Lambda \) is given by \(|\Lambda| = 1/l^2\), \( C_d \) is a normalization constant depending on the \((d+1)-\)dimensional Newton Constant and \( d\Omega_{d-1}, dT_{d-1} \) and \( dH_{d-1} \) are the metric of the \((d-1)\)-dimensional sphere, torus and compact hyperbolic manifold respectively.

Generally speaking, the optical metric reads
\[ ds^2 = d\tau^2 + \frac{dr^2}{A(r)^2} + \frac{r^2}{A(r)} d\Sigma_{d-1} + \frac{1}{A(r)} dE_{D-d-1}. \] (19)

When a metric has an event horizon and the black hole is not extremal, the localization of the horizon is given by the simple positive root of \( g^{11}, \) namely \( A(r_+) = 0. \) Thus one can make use of the near-horizon approximation, which is valid for large black hole mass. As a result the optical metric may be approximated by (see, for example, Refs. [3, 22])
\[ ds^2 \simeq d\theta^2 + \frac{d\rho^2}{\rho^2} + \frac{r^2}{\rho^2} d\Sigma_{d-1} + \frac{1}{\rho^2} dE_{D-d-1}, \] (20)

where
\[ \rho = 2(r - r_+)^{1/2} \left[ \frac{d}{dr} A(r)|_{r=r_+} \right]^{-1/2}, \quad \theta = \frac{\tau}{2} \frac{d}{dr} A(r)|_{r=r_+}. \] (21)
In this not extremal case, one gets in a standard way the inverse of the Hawking temperature requiring the absence of the conical singularity, namely
\[
\beta_H = 4\pi \left(\frac{dA(r)/dr}{r} \right)_{r=r_+}^{-1}.
\]

In order to study the quantum properties of matter fields, it is sufficient to investigate the kernel of the operator \(e^{-tL_N}\) and use the Eq. (10). We shall to assume the high temperature expansion or, in the case of black hole, the Rindler-like approximation.

In both the circumstances, the leading term of the heat-kernel turns to be \(20\), \(15\)

\[
\text{Tr} e^{-tL_N} \simeq \frac{\bar{V}_N}{(4\pi)^{N/2}}, \tag{22}
\]

where \(\bar{V}_N\) is the optical volume of the whole spatial section. From Eq. (10) the corresponding off-shell free energy reads

\[
\bar{F}_\beta \simeq -\frac{\bar{V}_N}{\beta^D}. \tag{23}
\]

For a space-time without event horizon (the AdS space) there are no horizon divergences and the free energy has a leading term \(\beta^{-D}\), i.e. no holographic reduction of the dimensionality is present.

The situation is different for black hole space-times. The optical volume, formally given by

\[
\bar{V}_N = V_{d-1}V_{D-d-1} \int_{r_+}^{\infty} dr \frac{r^{d-1}}{A(r)^{D/2}}, \tag{24}
\]

is divergent, because of the non integrable singularity at \(r = r_+\). Thus one has to introduce a cutoff parameter, i.e. \(r_+ = \varepsilon\). In fact, making use of the near-horizon approximation, it is possible to show that the leading term in the off-shell free energy is \(22\)

\[
\bar{F}_\beta \simeq -\frac{r_+^{d-1}V_{d-1}V_{D-d-1}}{\varepsilon^{D-2}} \left(\frac{\beta_H}{\beta}\right)^D. \tag{25}
\]

The off-shell entropy can be computed by means of Eq. (12) and the result is

\[
S_\beta \simeq r_+^{d-1}V_{d-1}V_{D-d-1} \left(\frac{\beta_H}{\beta}\right)^D. \tag{26}
\]

From a phenomenological point of view, well known stringy arguments suggest to choose \(\varepsilon \simeq l_P\) (of the order of the \(D\)-dimensional Planck length) \(3\). Furthermore, \(G_D \simeq l_D^{-2}\) and, and recalling the relation between the Newton constants in different dimensions

\[
\frac{V_{D-d-1}}{G_D} = \frac{1}{G_{d+1}} \tag{27}
\]

and finally going on-shell at \(\beta = \beta_H\), one obtains the "area law-like" expression also for the one-loop quantum correction:

\[
S \simeq r_+^{d-1}V_{d-1} \frac{1}{G_{d+1}}, \tag{28}
\]

where the proportionality factor depends on species of fields. As a consequence, going on-shell, one obtains the holographic reduction of the dimensionality, in agreement with the results of ref.
even though the dimensionality reduction mechanism is completely different. In fact, in our approach, the dimensionality reduction is due to the presence of quantum horizon divergences.

We conclude with some remarks. First, as far as the computation of the quantum black hole entropy is concerned, the "internal" $D - d - 1$-dimensional manifold does not contribute. Second, the class of $D$-dimensional black hole we have considered and for which we have estimated the one-loop quantum correction to the entropy, can be divided into two subclasses: the first one, example (II) (Schwarzschild black hole), corresponds to non-negative cosmological constant and the second one, examples (I), (III) − (V) (toroidal, Schwarzschild-AdS and hyperbolic AdS black holes), corresponds to a negative cosmological constant. For all solutions, one can easily computed a relationship between the $r_+$ and the Hawking temperature, eliminating the mass of the black hole. This is not a trivial task, particularly for the hyperbolic AdS black holes (see [22] and Eq. 18). Omitting the details, it turns out that for the first type of black holes $r_+ \simeq \beta_H$, and for the second type $r_+ \simeq \beta_H^{-1}$. As a consequence, the entropy for the Schwarzschild black hole goes like

$$S \simeq T^{-d+1},$$

related to negative specific heat and the instability of this black hole. With regard to the asymptotically AdS black holes,

$$S \simeq T^{d-1},$$

which, in turn, is related to the positivity of the their specific heat and their stability.

As a consequence, only for the stable AdS black hole it seems to exist, also at quantum level, a microscopic explanation of the black hole entropy via the AdS/CFT correspondence. In fact (conformal) quantum fields living on the horizon (boundary) of the black hole have a statistical entropy with $S \simeq T^{d-1}$ as leading term.

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