Multicriticality in a one-dimensional topological band insulator

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A central tenet in the theory of quantum phase transitions (QPTs) is that a nonanalyticity in the ground-state energy in the thermodynamic limit implies a QPT. Here we report on a finding that challenges this assertion. As a case study we take a phase diagram of a one-dimensional band insulator with spin-orbit coupled electrons, supporting trivial and topological gapped phases separated by intersecting critical surfaces. The intersections define multicritical lines across which the ground-state energy becomes nonanalytical, concurrent with a closing of the band gap, but with no phase transition taking place.

Our current understanding of continuous QPTs (excluding infinite-order QPTs of Berezinskii-Kosterlitz-Thouless type [3, 4]) is that they entail a change of symmetry or topology of a ground state. Symmetry-breaking QPTs are described by the Landau-Ginzburg-Wilson paradigm [5] or that of “deconfined quantum criticality” [6], being key to a number of theories of quantum matter, implying universal scaling behavior of observables at low temperatures [1, 7]. Topological QPTs in turn can be broadly classified into transitions between symmetry-protected [9] or topologically ordered [8] phases, characterized by a topological invariant or ground state degeneracy.

In both symmetry-breaking and topological QPTs, the competition between two incompatible ground states is thought to be the cause of the nonanalyticity in the ground-state energy [7]. In this Letter we report on a finding which calls for a reexamination of this assumption. Studying a model of non-interacting spin-orbit coupled electrons on a one-dimensional (1D) lattice, we uncover a phase diagram consisting of topologically nontrivial and trivial gapped phases separated by multiple intersecting gapless (critical) surfaces. The crossing between two such surfaces defines a multicritical line with an unexpected property: The ground-state energy develops a nonanalyticity in the thermodynamic limit, accompanied by a closing of the energy gap to the first excited level, suggestive of a continuous QPT. Yet there is no change of symmetry or topology across the multicritical line.

Our results suggest that the present understanding of quantum criticality is incomplete. If the nonanalyticity of the ground-state energy does imply a phase transition, as presumed from conventional theory, then it is necessary to widen the characterization of a zero-temperature phase beyond the current schemes of symmetry and topology. The other option is that multicriticality occasionally allows for a nonanalyticity in the ground-state energy with no phase transition taking place. We shall argue that this second scenario is the more likely one, and may be understood in terms of a proximity-induced “spurious” transition occurring when one and the same phase gets pinched off by a multicritical line.

Model — We consider a 1D lattice with $N$ sites populated by electrons with nearest-neighbor hopping and subject to Dresselhaus and Rashba spin-orbit interactions, the latter being spatially modulated. The corresponding tight-binding Hamiltonian writes:

$$H = \sum_{n=1}^{N} \sum_{\alpha,\alpha'} h_{\alpha\alpha'}(n) c_{n,\alpha}^\dagger c_{n+1,\alpha'} + \text{H.c.}, \quad (1)$$

where $c_{n,\alpha}^\dagger (c_{n,\alpha})$ is the creation (annihilation) operator for an electron at site $n$ with spin projection $\alpha = \uparrow, \downarrow$ along a $z$-quantization axis. The matrix elements are given by $h_{\alpha\alpha'}(n) = -i\delta_{\alpha\alpha'} - i\gamma_D \sigma^x_{\alpha\alpha'} - i\gamma_R(n) \sigma^y_{\alpha\alpha'}$, with $\sigma^x(y)$ the $x(y)$ Pauli matrix and the real parameters $\gamma_D$, and $\gamma_R(n)$ being the amplitudes of hopping, Dresselhaus, and Rashba spin-orbit coupling respectively. With this choice of basis, the chain is along the $x$-axis. The Rashba parameter is spatially modulated as $\gamma_R(n) = \gamma_R + \gamma'_R \cos(2\pi q n + \phi)$, with $2\pi q/a$ being the wave number ($a$ is the lattice spacing) and $\phi$ the phase of the modulation with respect to the underlying lattice. This Hamiltonian belongs to the class of generalized Aubry-André-Harper models [13, 14]. In Ref. 15 a similar Hamiltonian was derived from an effective description of a curved quantum wire. A different realization, with an added periodic chemical potential and electron interactions, may be obtained by gating a quantum wire with an array of nano-sized electrodes [16].

With periodic boundary conditions, $H$ is translation invariant on a lattice with $r = 1/q$ sites per unit
FIG. 1. (Color online) Winding number $W$ and local winding numbers $\bar{W}_\pm$ in the three-dimensional $(\gamma_{\text{eff}}, \theta, \phi)$ parameter space of the model. The phase diagram consists of topologically nontrivial insulating phases (dotted) where $W = 2$ and trivial insulating phases (empty) where $W = 0$, separated by critical surfaces $A$ and $B$ where the spectrum displays a pair of gap-closing nodes symmetrically located in the Brillouin zone with $\bar{W}_\pm = -1$ (blue), $\bar{W}_\pm = 1$ (orange), and $\bar{W}_\pm = 0$ (yellow), with the latter defining the multicritical lines of the model. In the inset, paths that connect two regions with $W = 0$ are shown on a large-$\gamma_{\text{eff}}$ cross section of the phase diagram. Along the curved paths the system undergoes two consecutive second-order topological QPTs, the first at the critical line $B$ and the second at $A$. As the paths approach the limiting straight line that crosses the multicritical point, the second-order topological QPTs become closer in parameter space and eventually merge into a fourth-order nonanalyticity at the multicritical point.

cell. A Fourier transform yields the single-particle Bloch Hamiltonian represented by a $2r \times 2r$ matrix $\mathcal{H}(k) = \text{diag}(Q(k),Q^T(k))$, with $Q(k)$ an $r \times r$ matrix. The formalism is detailed in the Supplemental Material [17]. Choosing, for example, $r = 4$,

$$Q(k) = \begin{bmatrix} A_1 & e^{-ikA_1^T} \\ A_2 & A_3 \end{bmatrix},$$

where $A_n$, $n = 1, ..., 4$, are the $2 \times 2$ matrices

$$A_n = \begin{bmatrix} \alpha_n^+ & \beta_n \\ \beta_n^+ & \alpha_n^- \end{bmatrix}$$

with diagonal [off-diagonal] entries given by spin-conserving [spin-flipping] hopping amplitudes. With $t = \gamma_R = 1$, one finds that $\alpha_n^+ = -1 \mp i\gamma_{\text{eff}} \mp i\cos(\theta)\cos(\pi n/2 + \phi)$ and $\beta_n = i\sin(\theta)\cos(\pi n/2 + \phi)$, with $\gamma_{\text{eff}} = \sqrt{\gamma_R^2 + \gamma_D^2}$ and $\theta = \arctan(\gamma_D/\gamma_R)$. By this, the model is fully parametrized by $\gamma_{\text{eff}}, \theta$, and $\phi$.

When $r$ is an even integer, the $2r \times 2r$ Bloch Hamiltonian $\mathcal{H}(k)$ belongs to symmetry class CII of the Altland-Zirnbauer classification [9], being invariant under chiral and time-reversal symmetry [17]. Differently, the two-band version of the model, with $r = 1$ site per unit cell, supports only time-reversal symmetry and, thus, belongs to class AII which is trivial in 1D [9]. Generating a non-trivial topology from a trivial two-band model by increasing the number of bands is an interesting possibility surfaced by the present model.

**Topological phase diagram** — The gapped phases of the half-filled $2r$-band model in (S1) (with even $r$) are distinguished by the $2Z$ winding number of class CII [9], call it $W$. For the points in parameter space where the band gap closes, the half-filled gapless spectrum can be characterized by ‘local’ winding numbers $\bar{W}_\pm$ [18] (with $\pm$ here labelling two gap-closing points symmetrically located in the Brillouin zone (BZ)), analogous to how a Weyl node in a semimetal is characterized by a topological charge [19]. For details, see Ref. 17.

Fig. 1 shows the result of a numerical computation of $W$ and $\bar{W}_\pm$ in the three-dimensional $(\gamma_{\text{eff}}, \theta, \phi)$ parameter space when $r = 4$. We constrain $\theta \in [0, \pi]$ and $\phi \in [0, \pi/2]$ since the phase diagram is periodic with period $\pi$ along $\theta$ and $\pi/2$ along $\phi$. The phase diagram consists of topologically nontrivial (trivial) gapped phases — the dotted (empty) regions where $W = 2$ ($W = 0$) — separated by critical surfaces colored in orange (blue) if the corresponding gap-closing nodes carry local winding numbers $\bar{W}_\pm = 1$ ($\bar{W}_\pm = -1$). The intersections of the critical surfaces, depicted in yellow, define multicritical lines along which $\bar{W}_\pm = 0$.

The critical surfaces come in two types: the plane $A$ at $\phi = \pi/4$, and the surfaces $B$ which are curved towards $\theta = 0, \pi$ for small $\gamma_{\text{eff}}$ and become flat for large $\gamma_{\text{eff}}$. The reason for choosing $r = 4$ can now be explained: This choice provides the minimal realization of the considered class of models supporting a multicritical phase diagram. Indeed, with $r = 4$ sites per unit cell, the model acquires an off-centered mirror symmetry when $\phi = \pi/4$ and this additional symmetry forces the band gap to close at zero energy [17]. Multicritical lines are thus generated from
the crossings of the plane $A$ (defined by $\phi = \pi/4$) and the curved critical surfaces $B$ of the model. We note that while $A$ implies symmetry-enforced band touchings in the BZ, the band touchings associated with the surfaces $B$ are accidental, with no extra symmetry enforcing them [17]. The closing of the gap at $A$ and $B$ happens through the formation of a pair of time-reversal symmetric zero-energy nodes defining the apexes of two 1D Dirac cones in the BZ and carrying equal local winding numbers $W_\pm = W$. 

**Multicriticality** — Having resolved the phase diagram, we come to our main result: the anomalous critical behavior across the multicritical lines in Fig. 1.

We first check the character of the topological QPTs across $A$ - Fig. 2(a) - and $B$ - Fig. 2(b) - when going between a trivial $W = 0$ phase and a topological $W = 2$ phase: In both cases the second derivatives of the ground-state energy of a half-filled chain develops a cusp at the critical value of the control parameter $\theta$ or $\phi$. The response becomes progressively sharper with system size $M$, signaling a precursor of a continuous second-order QPT in the thermodynamic limit. This result is consistent with a recent claim that topological QPTs of one-dimensional noninteracting band insulators, as well as superconductors, are always of second-order [10].

By instead crossing a multicritical line along a path which connects two regions in the phase diagram with the *same* $W$, one obtains a different result. Now the derivatives of the ground-state energy are smooth up to third order [17], but with a sharp cusp developing in the fourth derivative for large $M$ when hitting the multicritical line; cf. Fig. 2(c) for a $W = 2 \rightarrow 2$ transition and Fig. 2(d) for a $W = 0 \rightarrow 0$ transition.

Fig. 3 shows the band gap $\Delta$ on the $\theta \times \phi$ plane for $\gamma_{\text{eff}} = 10$, with the values of the winding number $W$ of the gapped phases indicated on the gap surface. The gap displays a conical profile close to the topological QPTs connecting phases with different values of $W$. In contrast, the fourth-order nonanalyticity across a multicritical line linking phases with equal $W$ correlates with a parabolic gap-closing, as obtained from analyzing a cross section of Fig. 3 along the path defined in Fig. 2(c).

The transitions are signaled also by a divergence in the localization length $\xi$ of the topological edge states in the $W = 2$ phases. For transitions away from (through) a multicritical point, $\xi$ scales linearly (parabolically) with the inverse distance to the critical point [20], implying linear scaling between $\Delta$ and $\xi$ and a dynamical critical exponent $z = 1$. These results confirm predictions from a recent formulation of scaling laws in topological QPTs [33], suggesting that such formulation may also be useful to describe topological phase diagrams where the topological invariant is not a thorough marker of criticality.

**Discussion** — In the conventional theory of continuous QPTs, a nonanalyticity in the ground-state energy does not happen except at a critical point separating phases which are distinguished either by symmetry or by topology [1, 7]. The present simple model of a topological band insulator defies this notion: The ground-state energy becomes nonanalytical at multicritical lines in the phase diagram, with associated closing of the band gap and divergence of a localization length, and yet there is no change of symmetry or in the topological invariant. A spontaneous symmetry breaking is excluded by the fact that an insulating ground state — whether in a topologically trivial or nontrivial phase — is unique when periodic boundary conditions are imposed, with the many-particle ground state formed by a Slater determinant of the single-particle states of the filled bands. With the two regions carrying the same winding numbers $W$, there is clearly no change of topology as defined by the Altland-Zirnbauer (AZ) classification [9]. Therefore, if conventional theory is strictly true, and having excluded — on general grounds — a change of symmetry, it follows that the equal-$W$ phases connected by the multicritical line must be distinguishable by a new topological invariant beyond the AZ scheme [9].

As a case in point, it is well known that by adding space group symmetries to the AZ scheme produces a
and fitting the data we obtain a parabolic gap-closing: \( \Delta = W^2(c) \) parametrizing a fourth-order section of the above gap surface along the path defined in Fig. QPTs connecting phases with different closing of the gap is conical for the second-order topological indicated on the gap surface. The figure indicates that the \( \gamma \) conspire to produce one at higher order, falsely signaling But how do the two second-order nonanalyticities then nonanalyticities at the two intersecting critical surfaces. At the multicritical line as a "remnant" of the normal mation \[36\]. For the type of phase diagram as in Fig. can be folded on top of the other via a gauge transfor- dundancy is clearly of no interest; here one of the phases a critical point connecting equal phases is a result of a re- true "in-phase" critical points. Phase diagrams in which occasionally develop a critical behavior under- taking a QPT? The conventional theory of multicritical behavior in symmetry-breaking QPTs is not of much help here. Such transitions are driven by quantum fluctuations in a local order parameter, governed by the competing renormalization-group (RG) fixed points for the critical lines that meet at multicriticality. This produces a multicritical behavior with new critical and crossover exponents, or the appearance of a first-order transition, but not one of higher order \[24\]. As for QPTs of topologi- cal band insulators, as in the present case, the transitions are not caused by disordering fluctuations in a local order parameter, but instead by a rearrangement of the phase structure of the ground state. This opens up for phenomena not encountered in symmetry-breaking QPTs, the unexpected interplay between topology and multicriticality uncovered in our work being an example. While our finding has been established for a particular class of Hamiltonians, we expect that similar results can be derived for other models exhibiting topological multicriticality; the Haldane model for a Chern insulator \[21\] being a case in point \[20\]. It is an open problem to explore the full implications of our finding, maybe taking off from recent attempts to formulate an RG approach to QPTs between symmetry-protected topological phases \[25–35\].

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[1] S. Sachdev, *Quantum Phase Transitions*, 2nd edition (Cambridge University Press, 2011).
[2] M. Kastner, Rev. Mod. Phys. 80, 167 (2008).
[3] V. L. Berezinskii, Sov. Phys. JETP 34, 610 (1972).
[4] J. M. Kosterlitz and D. J. Thouless, J. Phys. C: Sol. State Phys. 6, 1181 (1973).
[5] P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, 1995).
[6] T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, and M. P. A. Fisher, Science 303, 1490 (2004).
[7] M. Continentino, *Quantum Scaling in Many-Body Systems: An Approach to Quantum Phase Transitions*, 2nd edition (Cambridge University Press, 2017).
[8] X.-G. Wen, Rev. Mod. Phys. 89, 041004 (2017).
[9] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, Rev. Mod. Phys. 88, 035005 (2016).
[10] S. N. Kempkes, A. Quelle, and C. Morais Smith, Sci. Rep. 6, 38530 (2016).
[11] W. P. Su, J. R. Schrieffer, and A. J. Heeger, Phys. Rev. Lett. 42, 1698 (1979).
[12] A. Y. Kitaev, Phys.-Usp. 44, 131 (2001).
[13] P. G. Harper, Proc. Phys. Soc. London, Ser. A 68, 874 (1955).
[14] S. Aubry and G. André, Ann. Isr. Phys. Soc. 3, 133 (1980).
[15] P. Gentile, M. Cuoco and C. Ortix, Phys. Rev. Lett. 115, 256801 (2015).
[16] M. Malard, I. Grusha, G. I. Japaridze and H. Johannesson, Phys. Rev. B 84, 075466 (2011).
[17] M. Malard, D. Brandao, P. E. de Brito, and H. Johannesson, Supplemental Material.
[18] L. Li and S. Chen, Phys. Rev. B 92, 085118 (2015).
[19] N. P. Armitage, E. J. Mele, and A. Vishwanath, Rev. Mod. Phys. 90, 015001 (2018).
[20] M. Malard, P. E. de Brito, W. Chen, and H. Johannesson, in progress.
[21] F. D. M. Haldane, Phys. Rev. Lett. 61, 2015 (1988).
[22] Y. Ando and L. Fu, Annu. Rev. Cond. Matt. Phys. 6, 361 (2015).
[23] L. Carr (ed.), Understanding Quantum Phase Transitions (CRC Press, 2010).
[24] M. E. Fisher, in Multicritical Phenomena, R. Pynn and A. Skjeltorp (eds.) (Plenum Press, 1984).
[25] M. A. Continentino, F. Deus, and H. Caldas, Phys. Lett. A 378, 1561 (2014).
[26] B. Roy, P. Goswami, and J. D. Sau, Phys. Rev. B 94, 041101(R) (2016).
[27] W. Chen, J. Phys.: Condens. Matter, 28, 055601 (2016).
[28] W. Chen, M. Sigrist, and A. P. Schnyder, J. Phys.: Condens. Matter 28, 365501 (2016).
[29] W. Chen, M. Legner, A. Rüegg, and M. Sigrist, Phys. Rev. B 95, 075116 (2017).
[30] M. A. Continentino, Physica B: Cond. Matter 505, A1 (2017).
[31] M. A. Griffith and M. A. Continentino, Phys. Rev. E 97, 012107 (2018).
[32] E. P. L. van Nieuwenburg, A. P. Schnyder, and W. Chen, Phys. Rev. B 97, 155151 (2018).
[33] W. Chen and A. P. Schnyder, New. J. Phys. 21, 073003 (2019).
[34] S. Rufo, N. Lopes, M. A. Continentino, and M. A. R. Griffith, Phys. Rev. B 100, 195432 (2019).
[35] M. A. Continentino, S. Rufo, and G. M. Rufo, arXiv:1903.00758
[36] A simple realization of such a phase diagram is given by the SSH model [11]. The model is typically defined for real and positive hopping amplitudes only, for if these were allowed to be negative (or complex) the corresponding phases could be absorbed through a trivial redefinition of the basis states in the Hamiltonian. Allowing for negative hopping amplitudes produces a phase diagram with a critical point between equal phases at the origin, removable simply by folding the extended phase diagram back onto the region of positive amplitudes.
Supplemental material to
Multicriticality in a one-dimensional topological band insulator

Model
In this section the Bloch Hamiltonian presented in the accompanying Letter [S1] is derived. For that, let us reintroduce the original tight-binding model in position space, Eq. (1) in Ref. [S1]:

\[
H = \sum_{n=1}^{N} \sum_{\alpha,\alpha'} h_{\alpha\alpha'}(n) c_{n,\alpha}^\dagger c_{n+1,\alpha'} + \text{H.c.},
\]  

(S1)

where

\[
h_{\alpha\alpha'}(n) = -t\delta_{\alpha\alpha'} - i\gamma_D \sigma_\alpha^n \sigma_{\alpha'}^n - i\gamma_R(n) \sigma_y^n,
\]  

(S2)

with \(c_{n,\alpha}^\dagger (c_{n,\alpha})\) the creation (annihilation) operator for an electron at site \(n\) with spin projection \(\alpha = \uparrow, \downarrow\) along a \(z\)-quantization axis, \(\sigma^{(y)}\) the \(x (y)\) Pauli matrix, and the real parameters \(t, \gamma_D, \text{ and } \gamma_R(n) = \gamma_R + \gamma_R' \cos(2\pi qn + \phi)\) the amplitudes of hopping, Dresselhaus spin-orbit coupling, and the spatially modulated Rashba spin-orbit coupling respectively; \(2\pi q/a\) is the wave number \((a \text{ being the lattice spacing}) \text{ and } \phi\) is the phase of the modulation.

Performing a rotation of basis that diagonalizes the uniform part of \(H\) in spin space:

\[
d_{n,+} = \frac{1}{\sqrt{2}} (e^{-i\theta/2} c_{n,\uparrow} - ie^{i\theta/2} c_{n,\downarrow}),
\]

\[
d_{n,-} = \frac{1}{\sqrt{2}} (-ie^{-i\theta/2} c_{n,\uparrow} + e^{i\theta/2} c_{n,\downarrow}),
\]

with \(\theta = \arctan(\gamma_D/\gamma_R)\), Eqs. (S1)-(S2) take on the form

\[
H = \sum_{n=1}^{N} \sum_{\tau} \left[ \alpha_n^{\tau} d_{n,\tau}^\dagger d_{n+1,\tau} + \beta_n d_{n,\tau}^\dagger d_{n+1,-\tau} \right] + \text{H.c.},
\]

(S3)

where \(\tau = \pm\) labels the spin projections along the new quantization axis determined by the combination of Dresselhaus and Rashba couplings and where the strength \(\alpha_n^{\tau}\) of the spin-conserving and \(\beta_n\) of the spin-flipping hopping are given by

\[
\alpha_n^{\tau} = -(t + i\tau \gamma_{\text{eff}}) - i\tau \gamma_R' \cos(\theta) \cos(2\pi qn + \phi),
\]

(S4)

\[
\beta_n = i\gamma_R' \sin(\theta) \cos(2\pi qn + \phi).
\]

(S5)

Imposing periodic boundary conditions, \(H\) is translation invariant on a lattice with \(M N/r\) unit cells with \(r = 1/q\) sites per unit cell. Eq. (S3) can thus be rewritten in terms of intra-cell and inter-cell contributions as

\[
H = \sum_{m=1}^{M} \left[ \sum_{n=1}^{N} \sum_{\tau = \pm} H_{\text{intra}} + \sum_{\tau = \pm} H_{\text{inter}} \right] + \text{H.c.},
\]

(S6)

with

\[
H_{\text{intra}} = \alpha_n^{\tau} d_{m,n}^\dagger d_{m,n+1}^{\tau} + \beta_n d_{m,n}^\dagger d_{m,n+1}^{-\tau},
\]

(S7)

\[
H_{\text{inter}} = \alpha_{\tau} d_{m+1,n}^\dagger d_{m+1,n+1}^{\tau} + \beta_{\tau} d_{m+1,n}^\dagger d_{m+1,n+1}^{-\tau},
\]

(S8)

where \(d_{m,n}^\dagger (d_{m,n})\) creates (annihilates) a particle at site \(n\) in unit cell \(m\) with spin projection \(\tau\). Fig. S1 illustrates the spin-conserving and the spin-flipping hoppings within and across a unit cell.

By Fourier transforming the electron operators with respect to the unit cell position coordinate \(m\),

\[
d_{m,n} = \frac{1}{\sqrt{M}} \sum_{k = -\pi}^{\pi} d_{k,n}^e e^{ikm},
\]

(S9)
where \( k = k_j = \pm 2\pi j/M, \ j = 0, 1, ..., M/2, \) Eqs. (S6)-(S8) turn into

\[
H = \sum_{k=-\pi}^{\pi} \sum_{n,n'=1}^{r} \sum_{\tau,\tau'=\pm} d_{k,n}^{\dagger} \mathcal{H}_{n\tau,n'\tau'}(k) d_{k,n'}
\]  
(S10)

with

\[
\mathcal{H}_{n\tau,n'\tau'}(k) = \alpha_{n}^{\tau} \delta_{n',n+1}\delta_{\tau',\tau} + \beta_{n} \delta_{n',n+1}\delta_{\tau',-\tau} + \alpha_{n-1}^{\tau} \delta_{n',n-1}\delta_{\tau',\tau} + \beta_{n-1}^{\tau} \delta_{n',n-1}\delta_{\tau',-\tau} + \alpha_{n}^{\tau} e^{-ik} \delta_{n,1}\delta_{\tau',-\tau} + \beta_{n}^{\tau} e^{-ik} \delta_{n,1}\delta_{\tau',\tau} + \alpha_{n-1}^{\tau} e^{+ik} \delta_{n,1}\delta_{\tau',\tau} + \beta_{n-1}^{\tau} e^{+ik} \delta_{n,1}\delta_{\tau',-\tau}.
\]  
(S11)

We proceed by partitioning the lattice into two sublattices - one formed out from the odd-labelled intra-cell sites and the other from the even-labelled ones - and defining a \( 2r \)-dimensional row spinor \( d_{k}^{l} \) which groups the creation operators in the following way:

\[
d_{k}^{l} = (d_{k,1}^{+}, d_{k,1}^{-}, ..., d_{k,r-1}^{+}, d_{k,r-1}^{-}, d_{k,2}^{+}, d_{k,2}^{-}, ..., d_{k,r}^{+}, d_{k,r}^{-}),
\]  
(S12)

where the first (last) \( r \) entries receive the creation operators defined on the sublattice of odd (even) intra-cell sites, with operators for up and down spins at the same site placed next to each other. A \( 2r \)-dimensional column spinor \( d_{k} \) is defined by grouping the annihilation operators in the same way.

In matrix form the Hamiltonian (S10) thus becomes

\[
H = \sum_{k} d_{k}^{\dagger} \mathcal{H}(k) d_{k},
\]  
(S13)

with the entries of the \( 2r \times 2r \) Bloch matrix \( \mathcal{H}(k) \) given by Eq. (S11). In the chosen spinor representation (S12),

\[
\mathcal{H}(k) = \frac{\sigma_z}{2} \otimes [Q(k) + Q^{\dagger}(k)] + i\frac{\sigma_y}{2} \otimes [Q(k) - Q^{\dagger}(k)]
\]  
(S14)

or, equivalently,

\[
\mathcal{H}(k) = \begin{bmatrix} 0 & Q(k) \\ Q^{\dagger}(k) & 0 \end{bmatrix},
\]  
(S15)

with the \( r \times r \) matrix \( Q(k) \) given by

\[
Q(k) = \begin{bmatrix} A_{1} & 0 & 0 & ... & 0 & zA_{r-1}^{*} \\ A_{2}^{*} & A_{3} & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & A_{r-2}^{*} & A_{r-1} \end{bmatrix},
\]  
(S16)

where \( z = e^{-ik} \) and

\[
A_{n} = \begin{bmatrix} \alpha_{n}^{+} & \beta_{n} \\ \beta_{n} & \alpha_{n} \end{bmatrix}
\]  
(S17)

are \( 2 \times 2 \) matrices whose diagonal (off-diagonal) entries are given by the spin-conserving (-flipping) hopping amplitudes of Eqs. (S4)-(S5). For a chain with \( r = 4 \) sites per unit cell, \( Q(k) \) is thus given by Eq. (2) in the accompanying Letter [S1].
Symmetry Class

A single-particle Bloch Hamiltonian \( \mathcal{H}(k) \) is invariant under chiral \((S)\), time reversal \((T)\), and particle-hole \((C)\) transformations if it satisfies the following invariance relations:

\[
S \mathcal{H}(k) S^{-1} = -\mathcal{H}(k),
\]

\[
T \mathcal{H}(k) T^{-1} = \mathcal{H}^*(-k),
\]

\[
C \mathcal{H}(k) C^{-1} = -\mathcal{H}^*(-k),
\]

where \( S \), \( T \) and \( C \) are matrices representing \( S \) and the unitary parts of \( T \) and \( C \) respectively [S2].

Here we define the chiral transformation as \( S = \mathcal{P}_o - \mathcal{P}_e \), where \( \mathcal{P}_o \) (\( \mathcal{P}_e \)) is the projector onto the sublattice of odd (even) intra-cell sites. Given this definition, \( S \) satisfies the following invariance relations:

\[
\mathcal{S} \mathcal{H}(k) \mathcal{S}^{-1} = \mathcal{H}(k),
\]

\[
\mathcal{T} \mathcal{H}(k) \mathcal{T}^{-1} = \mathcal{H}^*(k),
\]

\[
\mathcal{C} \mathcal{H}(k) \mathcal{C}^{-1} = -\mathcal{H}^*(k),
\]

where \( \mathcal{S} \), \( \mathcal{T} \) and \( \mathcal{C} \) are matrices representing \( S \) and the unitary parts of \( T \) and \( C \) respectively [S2].

Going to \( T \)-symmetry, \( \mathcal{T} \) in Eq. (S19) is the matrix performing a spin flip which, in the representation (S12), reads

\[
\mathcal{T} = \mathbb{I}_{r \times r} \otimes (-i\sigma_y), \quad \mathcal{T}^2 = -\mathbb{I},
\]

as required for a spinful electron. The symmetry relation (S19) applied to Eqs. (S15)-(S16) thus implies:

\[
(-i\sigma_y) A_n(i\sigma_y) = A_n^*,
\]

which is indeed satisfied with \( A_n \) given by Eq. (S17) and Eqs. (S4)-(S5).

As for \( C \)-symmetry, \( S = \mathcal{T} \mathcal{C} \) [S2] implies that \( \mathcal{C} = -\mathcal{T} \mathcal{S} \). Having fulfilled equalities (S18)-(S19), one checks that Eq. (S20) is then also satisfied. With the above expressions for \( S \) and \( T \), we have that

\[
\mathcal{C} = \sigma_z \otimes \mathbb{I}_{r/2 \times r/2} \otimes (i\sigma_y),
\]

and hence \( \mathcal{C}^2 = -\mathbb{I} \).

The above results for \( S^2 \), \( T^2 \) and \( C^2 \) puts our 2\(r\)-band realization of the Hamiltonian (S13)-(S17), with \( r \in 2\mathbb{Z} \) sites per unit cell, in symmetry class CII of the Altland-Zirnbauer classification, with its gapped phases being labeled by a \( 2\mathbb{Z} \)-winding number [S2]. By the bulk-boundary correspondence, this implies that each edge of the chain hosts two zero-energy states, implying that the zero-energy level is four-fold degenerate. This is easy to verify numerically by computing eigenenergies and wave functions for the Hamiltonian in Eqs. (S1)-(S2) using open boundary conditions [S3].

Winding numbers

**Winding numbers.** The \( 2\mathbb{Z} \)-winding number of the gapped phases, denoted by \( W \) in Ref. [S1], is defined as the number of times that the complex number \( \det[Q(k)] \), with \( Q(k) \) given in Eqs. (S16)-(S17), winds around the origin of the complex plane as \( k \) sweeps through the Brillouin zone (BZ) from \( -\pi \) to \( \pi \) [S4].

Writing \( \det[Q(k)] = f(k) = R(k)e^{i\varphi(k)} \), it follows from the definition that \( W = -(2\pi)^{-1} \int dk \varphi \), or

\[
W = -\frac{1}{2\pi} \int_{-\pi}^{\pi} d_k \varphi \, dk,
\]

where \( d_k \equiv dk/dk \). The minus sign in Eq. (S21) is introduced to make \( W > 0 \) since \( \det[Q(k)] \) winds clockwise, i.e. \( d\varphi < 0 \).

Using that \( i\varphi = \ln[f(k)] - \ln[R(k)] \), we get

\[
W = -\frac{1}{2\pi i} \int_{-\pi}^{\pi} d_k \ln[f(k)] \, dk = -\frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{d_k f(k)}{f(k)} \, dk,
\]

since, differently from \( f(k) \), \( R(k) \) is a single-valued real function with \( R(-\pi) = R(\pi) \).

The function \( f \) depends on \( k \) through \( z = e^{-ik} \), i.e. \( f = f(z(k)) \) and thus

\[
W = -\frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{dz}{f(z)} \int f(z(k)) \, dz = \frac{1}{2\pi i} \int \frac{d_z f(z)}{f(z)} \, d_z,
\]

where the line integral is performed counter-clockwise around the unit circle in the complex plane.

Applying the Argument Principle of complex analysis, one finally arrives at

\[
W = N_z - N_p,
\]

(S24)
where $N_z$ ($N_p$) is the number of zeros (poles) of $f(z)$ inside the unit circle, counting also their degrees (orders).

From Eqs. (S16)-(S17) one finds that, for any $r \in 2\mathbb{Z}$, $f(z) = \det[Q(k)] = az^2 + bz + c$ (and thus $\det[Q(k)]$ indeed winds in the same clockwise direction as $z$), with $a, b, c$ constants depending on the parameters $\alpha_n$ and $\beta_n$. Therefore $N_p = 0$ and $W = N_z = 2, 1, 0$ if the zeros $z_{\pm} = (-b \pm \sqrt{b^2 - 4ac})/2a$ fall both inside, one inside and the other outside, or both outside the unit circle respectively.

Another useful formula to compute the winding number can be obtained by writing $\det[Q(k)] = h_x(k) + ih_y(k)$ and thus $W = -(2\pi)^{-1} \int d\varphi$ becomes

$$W = \frac{1}{2\pi} \int_{BZ} \frac{h_y dh_x - h_x dh_y}{h_x^2 + h_y^2},$$

(S25)

with $h_x$ and $h_y$ varying as $k$ sweeps through the BZ.

**Gapless surfaces.** There exist values of the microscopic parameters which enter $\alpha_n^\gamma$ and $\beta_n$ (cf. Eqs. (S4)-(S5)) - let us collectively call these values $\bar{g}$ - for which $\det[Q(k)] = 0$ at two points $k$ symmetrically located in the BZ - call them $k_{\pm}$. Due to Eq. (S15), also $\det[\mathcal{H}(k_{\pm})] = 0$, and hence $\mathcal{H}(k_{\pm})$ has at least one eigenvalue equal to zero. This fact, combined with chiral symmetry that forces the spectrum of $\mathcal{H}(k)$ to be symmetric around zero energy, implies that the gap between the bands immediately above and below zero energy closes at $k = k_{\pm}$. Moreover, since $\det[Q(k_{\pm})] = 0$, $W$ becomes undefined because the path traced out by $\det[Q(k)]$ on the complex plane cuts exactly through the origin when $k = k_{\pm}$.

Following the proposal in Ref. [S5], when the spectrum is gapless we define “local” winding numbers $W_{\pm}$ computed around each gap closing point ($\bar{g}, k_{\pm}$). Specifically, $W_{\pm}$ is defined as the number of times that $\det[Q(k)]$ winds around the origin of the complex plane as the point ($g, k$) moves, in the parameter-momentum space, around a small counter-clockwise contour $C_{\pm}$ centered at ($\bar{g}, k_{\pm}$), with the latter being the single gap-closing point inside $C_{\pm}$. Analogous to the calculation of $W$ in Eq. (S25), the invariants $W_{\pm}$ can thus be computed with the formula

$$W_{\pm} = \frac{1}{2\pi} \int_{C_{\pm}} \frac{h_y dh_x - h_x dh_y}{h_x^2 + h_y^2},$$

(S26)

with $h_x$ and $h_y$ varying as ($g, k$) winds around $C_{\pm}$.

Differently from $W$ which is a global invariant involving a computation over the entire BZ, $W_{\pm}$ are local in the sense that they are defined in a small region in the parameter-momentum space. The accumulated winding number characterizing a gapless spectrum is given by adding up the individual contributions from the two gap-closing points in the BZ: $\bar{W} = W_{+} + W_{-}$.

**Band Structure and Critical Surfaces.**

As discussed in Ref. [S1], the critical (gapless) surfaces of the model come in two types: the plane $A$ at $\phi = \pi/4$, and the surfaces $B$ which are curved towards $\theta = 0, \pi$ for small $\gamma_{eff}$ and become flat for large $\gamma_{eff}$ (see Fig. 1 in Ref. [S1]). The intersections between the two types of surfaces define the multicritical lines which host the anomalous singularities identified in Ref. [S1].

When the model parameters ($\gamma_{eff}, \theta, \phi$) take values on the plane $A$, the band gap in $k$-space closes at zero energy, resulting in two band degeneracies at points $k_{\pm}$ symmetrically located in the BZ (cf. the discussion above). Formally, by adopting an analysis from Ref. [S6], one finds that for any value of $\gamma_{eff}$ and $\theta$, choosing $\phi = \pi/4$,

$$\mathcal{M}(k)\mathcal{H}(k)\mathcal{M}^{-1}(k) = \mathcal{H}(-k)$$

(S27)

where $\mathcal{H}(k)$ is the Bloch matrix in Eqs. (S15)-(S17), and $\mathcal{M}(k)$ is a matrix implementing a mirror transformation (composed by a site inversion and a spin flip). As follows from the analysis in Ref. [S6], the combination of this mirror ($M$) symmetry with chiral ($S$) and time-reversal ($T$) symmetry enforces that any point on the plane $A$ supports two zero-energy and time-reversal symmetric band degeneracies in the BZ. In other words, these band degeneracies can be moved around in the BZ, but not removed, by changing a single parameter, $\gamma_{eff}$ or $\theta$.

As for the critical surfaces of type $B$, when the model parameters take values on these surfaces, the band gap also closes at zero energy, again resulting in zero-energy and time-reversal symmetric band degeneracies in the BZ. However, different from the degeneracies enforced by the $M, S, T$ symmetries on the plane $A$, these degeneracies are *accidental* because they are removable by changing the value of a single parameter, $\gamma_{eff}$ or $\theta$. 
Three types of band structures with band degeneracies are shown in Fig. S2. Panel (a) ((b)) shows the spectrum for a point on a blue (orange) patch of the critical surface \( A \) (\( B \)) in Fig. 1 in Ref. [S1], i.e. where \( \bar{W}_\pm = -1 \) (\( \bar{W}_\pm = 1 \)). Fig. S2(c) shows the spectrum for a point on a multicritical line where \( A \) and \( B \) intersect each other and where \( \bar{W}_\pm = 0 \). In all cases, the gap closes at zero energy through the formation of a pair of time-reversal symmetric nodes, defining the apexes of two 1D Dirac cones and carrying equal winding numbers \( \bar{W}_+ = \bar{W}_- \). Figs. S2 (a)-(b) show a pictorial representation of the nodes as monopoles with associated fields.

**Ground state energy and its derivatives**

Figs. S3(a) and S3(b) depict the ground state energy and its smooth first derivative when going between a trivial \( W = 0 \) phase and a topological \( W = 2 \) phase across plane \( A \) and surface \( B \), respectively, for different values for the number \( M \) of unit cells. For paths connecting two regions in the phase diagram with the same \( W \), the derivatives of the ground-state energy are smooth up to third order, as seen in Fig. S4(a) for a \( W = 2 \rightarrow 2 \) transition and in Fig. S4(b) for a \( W = 0 \rightarrow 0 \) transition.
FIG. S4. (Color online) (a), (b): Ground state energy and its derivatives up to third-order for a (a) $W = 2 \to 2$ transition and (b) $W = 0 \to 0$ transition. The curves are shown for an increasing number of unit cells $M$.

[S1] M. Malard, D. Brandao, P. E. de Brito, and H. Johannesson, Accompanying Letter.
[S2] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, Rev. Mod. Phys. 88, 035005 (2016).
[S3] M. Malard, D. Brandao, P. E. de Brito, and H. Johannesson, unpublished.
[S4] J. K. Asbóth, L. Oroszlány, and A. Pállyi, A Short Course on Topological Insulators (Springer Verlag, 2016).
[S5] L. Li, C. Yang, and S. Chen, EPL 112, 10004 (2015).
[S6] M. Malard, P. E. de Brito, S. Östlund, and H. Johannesson, Phys. Rev. B 98, 165127 (2018).