Entanglement Entropy of BTZ Black Hole and Conformal Anomaly

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ABSTRACT

We study the logarithmic divergence term of the entropy of scalar fields propagating in BTZ black hole space-time. The logarithmic divergence term is related to the conformal anomalies and its coefficient is proportional to the “a” and “c” type anomalies. The (2+1) dimensional massive theory is obtained from (3+1) dimensional free massless theory via dimensional reduction. We obtained the divergence term of (2+1) dimensional massive theory after integrating over the masses. The mass term does not affect the area law. The logarithmic coefficient of divergence term is directly related to the \( c_{(-1)} \), which we calculate numerically using the entanglement entropy approach. The numerical results are in agreement with the analytic results.

keywords: Scalar field in BTZ black hole Space-time, Entanglement Entropy, Logarithmic Corrections, Conformal Anomaly.
I. INTRODUCTION

The black hole entropy, also known as Bekenstein-Hawking entropy was first proposed by Bekenstein [1-4] and in later years, calculation of black hole temperature and Hawking radiation [5] by Hawking, strengthen its roots. The Bekenstein-Hawking entropy formula shows that the entropy of a black hole is directly proportional to the area of the horizon. The black hole entropy is a key feature to study the thermodynamical behavior of black holes [6-8].

The entanglement entropy is the source of the quantum information and it measures the correlation between subsystems and separated by the boundary called the entangling surface [9, 10]. The entanglement entropy depends upon the geometry of the boundary, but not on the properties of subsystems. It is defined by the von Neumann entropy \( S = - Tr [\rho \ln \rho] \), where \( \rho \) is reduced matrix. Let us consider a system divided in two subsystems “A” and “B”, separated by the entangling surface. The states of the subsystem “A” is formed by the degrees of freedom defined in region “A” and states of the sub-system “B” are defined in region “B”. If we trace the degrees of freedom of the subsystem “A”, we will obtained the density matrix of subsystem “B”. In this consequence, the vacuum state becomes mixed state and this mixing reflects the presence of quantum entanglement between the two subsystems.

The logarithmic divergence term in entropy of black holes appear due to the infinite number of states near the horizon and these divergences scales by the size of the black holes. These logarithmic divergence term is related to the conformal anomalies. In even dimensions, conformal field theory (CFT) contains a divergence term, but in odd dimensions there is no divergence term across the entangling surface [11, 12]. The coefficient of logarithmic term is proportional to the conformal anomaly [13] (a and c type anomaly). For the spherical system, the results of “a” type anomaly can be extended in any dimension [14, 15], but “c” type anomaly can not be extended in higher dimensional theory [16].

In this paper, we study the logarithmic contribution to the entropy for scalar fields by using the dimensional reduction technique. In this technique, the coefficient of logarithmic divergence term in (2+1) dimensional massive theory can be obtained via dimensional reduction of (3+1) dimensional massless theory. The entangling surface in (3+1) dimension is of form \( \Sigma_2 = \Sigma_1 \times S^1 \). The entanglement entropy is calculated numerically using the
entanglement entropy method, where the reduced density matrix written in terms of correlators \cite{17}. The reduced density matrix in terms of correlators is well known for scalar fields and obeys the Wick’s theorem. This theorem fixes the \( (\rho = c \exp(-\mathcal{H})) \), where \( \mathcal{H} \) is a Hamiltonian, which is quadratic in the fields. We calculate the density matrix for scalar fields in BTZ space-time and explicitly diagonalize the reduced density matrix in order to estimate the entanglement entropy \cite{18}.

The massive scalar fields of \((2+1)\) dimensional space-time are related to the coefficient of logarithmic term \((\log \epsilon)\) corresponding to the \((3+1)\) dimensional massless theory \cite{19–21}. The coefficient of the logarithmic contribution is a linear combination of central charge, which is given by the relation \cite{22},

\[
S(\Sigma, M) = c_0 + c_1 MR + \sum_{n=0}^{\infty} \frac{c_{-1-2n}}{(MR)^{2n+1}}.
\]

We calculate these terms for scalar fields in the background of BTZ black hole numerically using the entanglement entropy approach.

This paper is organized as follows; we study the scalar field propagating in BTZ black hole space-time in section (II). We have given the brief review of free massive theory in section (III) and numerical calculations for logarithmic contribution to the entanglement entropy have been done in section (IV). We present our results and its physical implication of entropy for scalar fields in BTZ black hole space-time in section (V).

II. SCALAR FIELDS IN BTZ BLACK HOLE SPACE-TIME

Let us consider the action of the \((2+1)\) dimensional gravity with cosmological constant \cite{23, 24},

\[
S = \frac{1}{2\pi} \int d^3x \sqrt{-g} [R + 2\Lambda],
\]

where \( \Lambda = -\frac{1}{l^2} \) is the cosmological constant. The BTZ black hole is a solution of \((2+1)\) dimensional gravity with negative cosmological constant. The metric of BTZ black hole is given by the equation;

\[
ds^2 = -(-M + r^2/l^2)dt^2 + r^2d\phi^2 + \frac{1}{(-M + r^2/l^2 + J^2/4r^2)}dr^2 - Jdtd\phi,
\]

where \(-\infty < t < \infty\) and \(0 \leq \phi \leq 2\pi\).
The metric of the BTZ black hole in term of proper distance \( r^2 = r_+^2 \cosh^2 \rho + r_-^2 \sinh^2 \rho \) is written as,

\[
d s^2 = u^2 \, d t^2 + d \rho^2 + l^2 (u^2 + M) d \phi^2 - J \, d t \, d \phi,
\]

(4)

where \( r(\rho)^2 = l^2 (u^2 + M) \) and \( r_+ \) and \( r_- \) are inner and outer horizon of the black hole respectively.

The action of massive scalar field in the background of BTZ black hole is written as,

\[
S = -\frac{1}{2} \int dt \sqrt{-g} (g^{\mu \nu} \partial_\mu \Phi \partial_\nu \Phi + \mu^2 \Phi^2)
\]

(5)

where \( \sqrt{-g} \) and \( g^{\mu \nu} \) are the determinant and the metric element of the BTZ black hole (4).

The field \( \Phi \) can be decomposed using the separation of variables as,

\[
\Phi(t, \rho, \phi) = \sum_m \phi_m(t, \rho) e^{i m \phi}.
\]

(6)

This decomposition of \( \Phi \) manifest the cylindrical symmetry of the system. Substituting the value of \( \Phi \) in action (5), we get the expression for action as,

\[
S = -\frac{1}{2} \int dt \left\{ \sqrt{\frac{\sqrt{M + u^2}}{[u^2 + \frac{J^2}{4(u^2 + M)}]^{1/2}}} \dot{\Phi}_m^2 + u \sqrt{\left[ (u^2 + M) + \frac{J^2}{4u^2} \right]} \left( \partial_\rho \Phi_m^2 \right) + \frac{u^2 m^2}{u \sqrt{\left[ (u^2 + M) + \frac{J^2}{4u^2} \right]}} \Phi_m^2 - \frac{i J m}{2 u \sqrt{\left[ (u^2 + M) + \frac{J^2}{4u^2} \right]}} \Phi_m \dot{\Phi}_m - \frac{\mu^2 \Phi_m^2}{2 \left[ (u^2 + M) + \frac{J^2}{4u^2} \right]} \right\}.
\]

(7)

The conjugate momentum \( \pi_m \) corresponding to field \( \Phi_m \), is given by,

\[
\pi_m = \sqrt{\frac{\sqrt{M + u^2}}{[u^2 + \frac{J^2}{4(u^2 + M)}]^{1/2}}} \dot{\Phi}_m + \frac{i J m}{2 u \sqrt{\left[ (u^2 + M) + \frac{J^2}{4u^2} \right]}} \Phi_m.
\]

Using the relation \( H = \pi \dot{\Phi} - L \), the Hamiltonian of the system can be written as,

\[
H = \frac{1}{2} \int d\rho \left\{ \frac{u^2 + \frac{J^2}{4(u^2 + M)}}{M + u^2} \right\}^{1/2} \pi_m^2 + \frac{1}{2} \int d\rho d\rho' \left[ u \sqrt{\left[ (u^2 + M) + \frac{J^2}{4u^2} \right]} \partial_\rho (\Phi_m)^2 + \frac{u^2 m^2}{u \sqrt{\left[ (u^2 + M) + \frac{J^2}{4u^2} \right]}} \Phi_m^2 - \frac{i J m}{2 \left[ (u^2 + M) + \frac{J^2}{4u^2} \right]} \pi_m + \mu^2 \Phi_m^2 \right].
\]

(8)

To diagonalize the Hamiltonian, we define a new momentum as,

\[
\tilde{\pi}_m = \pi_m - \frac{i J m}{u \sqrt{\left[ (u^2 + M) + \frac{J^2}{4u^2} \right]}} \Phi_m.
\]

(9)
This new momentum $\tilde{\pi}$ forms a canonical pair with the field $\Phi$ and the commutation relation is given by the relation,

$$\{\phi_m(\rho), \tilde{\pi}_m(\rho)\} = \frac{iJm}{u \sqrt{[(u^2 + M) + \frac{J^2}{4a^2}]}} \delta_{m,m'} \delta(\rho - \rho').$$  \hspace{1cm} (10)

Now, we define a new variable $\psi$, which is related to field $\Phi$ through the relation,

$$\psi_m(t, \rho) = \left(\frac{u^2 + \frac{J^2}{4(u^2 + M)}}{(M + u^2)}\right)^{1/4} \Phi_m(t, \rho).$$  \hspace{1cm} (11)

This redefinition greatly simplifies our expressions and calculations.

Using the definition of new momentum $\tilde{\pi}$ (9) and redefined field $\psi$, the diagonalized Hamiltonian of the scalar field in the BTZ background space-time is [25–27] is given by,

$$H = \frac{1}{2} \int d\rho \tilde{\pi}^2_m(\rho) + \frac{1}{2} \int d\rho d\rho' u \sqrt{[(u^2 + M) + \frac{J^2}{4u^2}]} \left(\partial_\rho \sqrt{\frac{\sqrt{\left[u^2 + \frac{J^2}{4(u^2 + M)}\right]} \psi_m}}\right)^2$$

$$+ m^2 \frac{u^2 + \frac{J^2}{4(u^2 + M)}}{(M + u^2)} \psi^2_m + \mu^2 \Phi_m.$$  \hspace{1cm} (12)

To study the entanglement entropy of the system numerically, we discretized the system using the following relations,

$$\rho \rightarrow (A - 1/2)a$$

$$\delta(\rho - \rho') \rightarrow \delta_{AB}/a,$$  \hspace{1cm} (13)

where $A, B = 1, 2,...,N$ and “$a$” is UV cut-off length. We regain the continuum by taking the limit $a \rightarrow 0$ and $N \rightarrow \infty$ while the size of the system remains fixed. The corresponding Hamiltonian of the scalar fields in BTZ black hole space-time of the discretized system is given by,

$$H = \frac{1}{2a} \delta^{AB} p_A p_B + \frac{1}{2} \sum_{A,B=1}^{N} V_{AB} q^A q^B.$$  \hspace{1cm} (14)

with replacements,

$$\psi_m(\rho) \rightarrow q^A,$$

$$\tilde{\pi}_m(\rho) \rightarrow p_A/a,$$

$$V(\rho, \rho') \rightarrow V_{AB}/a^2.$$  \hspace{1cm} (15)
and $V_{AB}^m \psi_m^A \psi_m^B$ is expressed as

$$V_{AB}^m \psi_m^A \psi_m^B = a \sum_{A=1}^N \left[ u_{A+\frac{1}{2}} \sqrt{(u_{A+\frac{1}{2}}^2 + M)} + \frac{J^2}{4u_{A+\frac{1}{2}}^2} \sqrt{u_{A+1}^2 + \frac{J^2}{4(u_{A+1}^2 + M)}} \psi_{m+1}^A \psi_{m}^B - \sqrt{\frac{u_A^2 + \frac{J^2}{4(u_A^2 + M)}}{u_A^2 + m}} \psi_m^A \right]^2 + m^2 \frac{u_A^2 + \frac{J^2}{4(u_A^2 + M)}}{u_A^2 + M} \psi_m^A \mu^2 \Phi_m^2]. \quad (16)$$

The reduced density matrix of the system can be written in the exponential form [17],

$$\rho_V = K e^{-H} \quad (17)$$

where $H$ is the hermitian matrix to be identified with the Hamiltonian of the system (12).

The two point correlators of $X_{AB}$ and $P_{AB}$ are,

$$\langle \psi_A \psi_B \rangle = X_{AB} = \frac{1}{2} (V^{-1/2})_{AB}$$

$$\langle \pi_A \pi_B \rangle = P_{AB} = \frac{1}{2} (V^{1/2})_{AB}$$

$$\langle \psi_A \pi_B \rangle = \langle \pi_A \psi_B \rangle = \frac{i}{2} \delta_{AB}. \quad (18)$$

where $XP \geq 1/4$ [17].

The position and momentums can express as the linear combinations of creation and annihilation operators,

$$\psi_A = \alpha^{*}_{AB} a_B^\dagger + \alpha_{AB} a_B, \quad (19)$$

$$\pi_A = -\beta^{*}_{AB} a_B^\dagger + \beta_{AB} a_B. \quad (20)$$

The commutation relation of these operators is written as,

$$\alpha^{*} \beta^T + \alpha \beta^i = -1 \quad (21)$$

Using the relation (19) and (20) and substituting in (17), we can write the density matrix as,

$$\rho = K \exp(- \sum M_{AB} \psi_A \psi_B + N_{AB} \pi_A \pi_B) \quad (22)$$

where,

$$M = \frac{1}{4} (\alpha^{-1})^T \epsilon \alpha^{-1} = \frac{P}{2C} \log \left( \frac{C + \frac{1}{2}}{C - \frac{1}{2}} \right),$$

$$N = \alpha \epsilon \alpha^T = \frac{1}{2C} \log \left( \frac{C + \frac{1}{2}}{C - \frac{1}{2}} \right) X. \quad (23)$$

[1] we are discretized the equation (16) using the identity $\partial \rho \rho g(\rho) = \frac{1}{a} f_{A+1/2}(g_{A+1} - g_A)$. 
The entropy of the system is given by,

\[ S_{\text{ent}} = \text{tr}(C + \frac{1}{2}) \log(C + \frac{1}{2}) - \text{tr}(C - \frac{1}{2}) \log(C - \frac{1}{2}). \]

where \( C = \sqrt{XP} \) and \( XP \geq \frac{1}{4} \). We have two cases for the value of \( C \), one is \( \geq \frac{1}{2} \) and other is \( < \frac{1}{2} \). We study the two case of the first conditions, which are given as follows,

- for \( C = \frac{1}{2} \), we have pure state and entropy of the system is zero.
- for \( C > \frac{1}{2} \), we get mixed state and the entropy will be finite.

The total entropy of the system is given by the sum of all the modes “\( m \)” \[21\],

\[ S = \lim_{N \to \infty} S(n_B, N) = S_0 + 2 \sum_{m=1}^{\infty} S^m_{\text{ent}}, \tag{24} \]

where \( S(n_B, N) \) is the entanglement entropy of the total system \( N \) with partition size \( n_B \). The \( S_0 \) is the entropy of the system at \( (m = 0) \), and \( S^m_{\text{ent}} \) is the entropy of the subsystem for given value azimuthal quantum number “\( m \)”. The equation (24) is infinite series, and it converges for large value of “\( m \)”. We truncate the series depending on the accuracy, which we require.

### III. FREE MASSIVE THEORY IN BTZ SPACE-TIME

In this section, we study the free massive theory in BTZ black hole space-time and the general structure of entanglement entropy. The entropy of massive fields in (2+1) dimensional massive theory can be obtained via dimensional reduction from (3+1) dimensional massless theory \[22, 28\]. The entanglement entropy of the system with logarithmic divergence is give by the equation,

\[ S = \frac{A(\Sigma_2)}{4\pi\epsilon^2} + s \ln \epsilon \tag{25} \]

The first term in above equation is Bekenstein-Hawking area law and second term is divergence term. The relation, which gives the general structure of logarithmically divergences term is given by the relation,

\[ s = (a_a^{\Sigma_2} + c_a^{\Sigma_2}) \tag{26} \]

[2] where \( m \) is azimuthal angular momentum
where,

\[ a^\Sigma_A = \frac{\pi}{8} \int_{\Sigma_2} (R_{ijij} - 2R_{ii} + R) \]  \hspace{1cm} (27)

\[ a^\Sigma_B = -\frac{\pi}{8} \int_{\Sigma_2} (R_{ijij} - R_{ii} + \frac{1}{3}R). \]  \hspace{1cm} (28)

Substituting the value of (27) in (26), we get the coefficient of divergence term, which is given by the relation,

\[ s = \frac{a}{180} \int_{\Sigma_2} R_{\Sigma_2} + \frac{c}{240\pi} \int_{\Sigma_2} (k_{i\mu} k_{j\nu} - \frac{1}{2}k_{i\mu} k_{j\nu}) \]  \hspace{1cm} (29)

where \( R_{\Sigma_2} \) is the intrinsic curvature of \( \Sigma_2 \) and it is proportional to the Euler number and it vanishes at the surface \( \Sigma_2 = \Sigma_1 \times S^1 \). The \( k_{i\mu} = -\gamma_{\alpha\beta} \partial_{\alpha} n_i^{\beta} \) and \( n_i^{\mu} \) with \( i = 1, 2 \) are orthogonal to \( \Sigma_2 \), and \( \gamma_{\mu\nu} = \delta_{\mu\nu} - n_i^{\mu} n_i^{\nu} \) is the induced metric on the surface. Simplifying (29), we get a relation between logarithmic divergence term and anomaly, given by,

\[ s = \frac{c}{240} \frac{L}{R}, \]  \hspace{1cm} (30)

where \( R \) is the radius of circle \( S^1 \) and \( L \) is the length in the compactified direction \( \phi \).

The EOM of scalar field is,

\[ \left( \partial_t^2 - \nabla^2 + \mu^2 \right) \Phi = 0. \]  \hspace{1cm} (31)

The Fourier decomposition of the field modes in the compactified direction can be written as,

\[ \Phi (t, r, \theta, \phi) = \phi (t, r, \theta) \exp \left( i \frac{2\pi m}{L} \phi \right). \]  \hspace{1cm} (32)

This decomposition of fields enables us to write the EOMs in form,

\[ \left( \partial_t^2 - \nabla^2 \right) \phi + \left( \frac{2\pi m}{L} \right)^2 \phi + \mu^2 \phi = 0 \]  \hspace{1cm} (33)

where

\[ M_m^2 = \mu^2 + \left( \frac{2\pi}{L} m \right)^2, \]  \hspace{1cm} (34)

here \( \mu \) is the mass of the free fields, and acts as infrared correlator and \( m \) is an azimuthal quantum number. In our study, we consider the free massless field \( \mu = 0 \). In this case, the equation (34) becomes [22],

\[ M_m^2 = \left( \frac{2\pi}{L} m \right)^2, \]  \hspace{1cm} (35)
The contribution of entanglement entropy of two dimension fields are given by the relation [21],

\[ S(\Sigma_2) = \sum_{m=-\infty}^{\infty} S(\Sigma_1, M_m), \]

\[ = \frac{2L}{\pi} \int_{0}^{\infty} dM S(\Sigma_1, M). \]  

(36)

where [28]

\[ S(\Sigma_2, M) = c_0 + c_1 MR + \sum_{n=0}^{\infty} \frac{c_{1-2n}}{(MR)^{2n+1}} \]  

(37)

Comparing the equation (29) and(37), we get

\[ s_s = c_{-1} \frac{L}{\pi R} \]  

(38)

The coefficient \( c_{-1} \) is related to the entanglement entropy of the free massless theory in (3+1)-dimensions and is directly related to the coefficient of \( (M \rho)^{-1} \). The coefficient \( c_{-1} \) is found \(-\frac{\pi}{240}\) for scalars.

Now, The logarithmic divergence term of entropy is proportional to the mass term and given by the term \( c_1 \). The entropy of scalar field is given by [11],

\[ \Delta S_M = \gamma_d M^{d-1} A_{d-1}, \]  

(39)

where

\[ \gamma_d \equiv (-1)^{(d/2)}[12 (2\pi)^{(d-2)/2} (d-1)!!]^{-1}, \]  

(40)

and “A” is the area of event horizon \((A = 2\pi r_+)\), and \( \gamma_2 = -\frac{1}{12} \). The coefficient \( c_1 \) is linear with entropy and it is found \(-\frac{2\pi}{12}\).

IV. NUMERICAL ESTIMATION

The entropy of the scalar field for different masses in the range (.05 < M < .5) are computed numerically. The form of entropy for each mass is [21],

\[ S = c_0(M) + c_1(M) \rho + c_{-1}(M) \frac{1}{\rho} + \ldots, \]  

(41)
\[ c_1(M) = c_1^1(M) + c_1^0 + c_1^{-1} \frac{1}{M} \]
\[ c_{-1}(M) = c_{-1}^1(M) + c_{-1}^0 + c_{-1}^{-1} \frac{1}{M} \]

FIG. 1: The numerical calculation for \( S_{\text{ent}}(\text{massive}) \) of the scalar field in rotating BTZ space-time. \( S_{\text{ent}} \) is shown as a functions of \( r_+/a \) for different masses \( m=0.1, 0.3 \) and 0.5 . We have taken the lattice point \( N=200 \).

The value of coefficients \( c_1(M) \) and \( c_{-1}(M) \) are tabulated in the table (I) and shown in figure (11). If we calculated the value of coefficients, then we expand the \( c_1(M) \) and \( c_{-1}(M) \) in power of \( M \),

\[
\begin{align*}
    c_1(M) &= c_1^1(M) + c_1^0 + c_1^{-1} \frac{1}{M}, \\
    c_{-1}(M) &= c_{-1}^1(M) + c_{-1}^0 + c_{-1}^{-1} \frac{1}{M}
\end{align*}
\] (42)

The plot \( c_1 \) is the coefficient of term proportional to \( M \) and points correspond to the coefficient of term \( \rho \) in the fit with mass \( M \) as shown in figure (2) and The \( c_{-1} \) is shown as a functions of \( M \) and the coefficient of term proportional to \( (1/M) \) and points correspond to the coefficient of term \( (1/\rho) \) in the fit with mass range .05 < \( M < 0.5 \) as shown in figure (3). The co-efficients of \( c_1 \) and \( c_{-1} \) are found from the graph -0.503 and -0.0132 respectively. These results are consistent with the analytic results [11].
FIG. 2: The numerical calculation for $c_1$ of the scalar field in rotating BTZ space-time. $c_1$ is shown as a functions of $M$ and coefficient of term proportional to $M$ and points correspond to the coefficient of term $\rho$ in the fit with mass range $0.05 < M < 0.5$.

FIG. 3: The numerical calculation for $c_{-1}$ of the scalar field in rotating BTZ space-time. $c_{-1}$ is shown as a functions of $M$ and coefficient of term proportional to $(1/M)$ and points correspond to the coefficient of term $(1/\rho)$ in the fit with mass range $0.05 < M < 0.5$.

V. CONCLUSION

In this paper, we have studied the logarithmic divergence term of entanglement entropy for the scalar field propagating in the background of BTZ black hole numerically. The co-
efficient of divergence term $c_1$ and $c_{-1}$ calculated numerically. The logarithmic divergence term(s) of entanglement entropy is the linear combination of $c$ type anomaly. The term $c_1$ is obtained from the dimensional reduction of the theory and the term $c_{-1}$ is directly related to the coefficient of divergence term. The general structure of the coefficients is same as that found in (37). This is the agreement of our numerical results with analytical results [11]. We can also extend our results for the higher dimension theory. We have also studied the logarithmic divergence term of entanglement entropy for the scalar field propagating in the background of BTZ black hole numerically [29].

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