FINITE PRESENTABILITY OF UNIVERSAL CENTRAL EXTENSIONS
OF $\mathfrak{sl}_n$, II

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Abstract. In this note we connect finite presentability of a Jordan algebra to finite presentability of its Tits-Kantor-Koecher algebra. Through this we complete our discussion of finite presentability of universal central extensions of $\mathfrak{sl}_n(A)$, $A$ a $k$-algebra, initiated in [12], and answer a question raised by Shestakov-Zelmanov [11] in the positive.

Throughout this note all algebras are considered over a field $k$ containing $\frac{1}{2}$.

1. Introduction

Let $\mathcal{V}$ be a variety (of universal algebras) in the sense of [7], [13]. An algebra $A \in \mathcal{V}$ is said to be finitely presented (f.p.) if it can be presented in $\mathcal{V}$ by finitely many generators and finitely many relations.

Definition 1.1. A $k$-algebra $J$ satisfying the identities

\begin{enumerate}
  \item $xy = yx$,
  \item $(x^2y)x = x^2(yx)$
\end{enumerate}

for all $x, y \in J$ is called a Jordan algebra.

Remark 1.2. $J$ as above is sometimes referred to as a linear Jordan algebra, in contrast to the concept of quadratic Jordan algebras. These two concepts are equivalent when $\frac{1}{2} \in k$.

Example 1.3. An associative $k$-algebra $A$ admits a canonical Jordan product given by $x \circ y = \frac{1}{2}(xy + yx)$. This new product on $A$ makes it a Jordan algebra, denoted by $A^{(+)}$.

J.M. Osborn (see [6]) showed that for a finitely generated associative algebra $A$ the Jordan algebra $A^{(+)}$ is finitely generated.

In [11], Shestakov and Zelmanov considered the question whether for a finitely presented associative algebra $A$ the Jordan algebra $A^{(+)}$ is finitely presented. They proved (among other things) that

\begin{enumerate}
  \item the Jordan algebra $k\langle x, y \rangle^{(+)}$, where $k\langle x, y \rangle$ is the free associative algebra of rank 2, is not finitely presented;
  \item let $A$ be a finitely presented associative algebra and let $M_n(A)$ be the algebra of $n \times n$ matrices over $A$, and $n \geq 3$, then the Jordan algebra $M_n(A)^{(+) is finitely presented.}
\end{enumerate}

For the borderline case of $M_2(A)$ they asked if the Jordan algebra $M_2(A)^{(+) is finitely presented.
In this paper we give a positive answer to this question.

**Theorem 1.4.** Let $A$ be a finitely presented associative $k$-algebra. Then the Jordan algebra $M_2(A)^{(+)}$ is finitely presented.

The proof of this theorem is based on the paper [12] on universal central extensions of Lie algebras $\mathfrak{sl}_n(A)$.

**Definition 1.5.** Let $A$ be a finitely presented associative algebra. $\mathfrak{sl}_n(A)$ is the Lie algebra generated by off-diagonal matrix units among $n \times n$ matrices; forming a subalgebra of $\mathfrak{gl}_n(A)$. Equivalently, $\mathfrak{sl}_n(R) = \{X \in \mathfrak{gl}_n(A) \mid \text{tr}(X) \in [A, A]\}$.

**Definition 1.6.** Let $L, g$ be $k$-Lie algebras where $g$ is perfect. A surjective Lie homomorphism $\pi : L \to g$ is called a central extension of $g$ if $\ker(\pi)$ is central in $L$. This $\pi : L \to g$ is called a universal central extension if there exists a unique homomorphism $\phi : L \to \hat{g}$ from any other central extension $p : M \to g$ of $g$. In other words, $f = p \circ \phi$. The universal central extension of $g$ is customarily denoted as $\hat{g}$. Perfectness of $g$ guarantees the existence of $\hat{g}$, which is necessarily perfect (see [10]).

In [12] it was shown that for a finitely presented associative algebra $A$ and $n \geq 3$, the Lie algebra $\hat{\mathfrak{sl}}_n(A)$ is finitely presented. As a consequence of the Lie-Jordan correspondence results of this paper, we may sharpen this result:

**Theorem 1.7.** The universal central extension of $\mathfrak{sl}_2(k\langle x, y \rangle)$ is not finitely presented as a Lie algebra.

**Remark 1.8.** The result in [12] holds without the restriction $\text{char}(F) \neq 2$.

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2. Finite Presentation of Jordan systems

**Jordan triple systems**

A Jordan algebra $J$ is equipped with a Jordan triple product $\{a, b, c\} = (ab)c + a(bc) - b(ac)$. If $J = A^{(+)}$, where $A$ is an associative algebra, the $\{a, b, c\} = \frac{1}{2}(abc + cba)$.

**Definition 2.1** (See [9]). A vector space $V$ over a field $k$ containing $\frac{1}{2}$ is called a Jordan triple system if it admits a trilinear product $\{\ldots\} : V^3 \to V$ that is symmetric in the outer variables, while satisfying the identities

1. $\{a, b, \{a, c, a\}\} = \{a, \{b, a, c\}, a\}$,
2. $\{\{a, b, a\}, b, c\} = \{a, \{b, a, b\}, c\}$,
3. $\{a, \{b, \{a, c, a\}, b\}, a\} = \{\{a, b, a\}, c, \{a, b, a\}\}$

for any $a, b, c \in V$.

**Remark 2.2.** When $\frac{1}{2} \in k$, the three defining identities of a Jordan triple system may be merged in to $\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{a, b, \{c, d, e\}\}$. 

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2. Finite Presentation of Jordan systems
It is easy to see that every Jordan algebra is a Jordan triple system with respect to the Jordan triple product \{ , , \}.

**TKK construction for a Jordan triple system.**

For a Jordan algebra \( J \) we let \( K(J) \) be the Tits-Kantor-Koecher (abreviated TKK) Lie algebra of \( J \) viewed as a Jordan triple system.

**Definition 2.3** (See [11]. See also [2, Section 5], [1]). Let \( T \) be a Jordan triple system. Let \( \{e_u, u \in I\} \) be a basis of the vector space \( T \). Let

\[
\{e_u, e_v, e_w\} = \sum \gamma_{uvw}^t e_u,
\]

where \( t, u, v, w \in I; \gamma_{uvw}^t \in k \). The Lie algebra \( K(T) \) is presented by generators \( x_u^\pm, u \in I \) and relations

\[
(K1) \quad [x_u^\sigma, x_v^\sigma] = 0,
\]

\[
(K2) \quad [[x_u^\sigma, x_v^{-\sigma}], x_w^\sigma] - \sum \gamma_{uvw}^t e_t = 0,
\]

where \( t, u, v, w \in I; \sigma = \pm \). This Lie algebra is called the (Universal) Tits-Kantor-Koecher Lie algebra associated to \( T \). It is obvious that this construction is basis-independent.

**Remark 2.4.** The above (universal) TKK construction \( T \rightarrow K(T) \) is a functor: see for example [3]. In other words, any homomorphism of Jordan Triple systems \( T_1 \rightarrow T_2 \) gives rise to a homomorphism \( K(T_1) \rightarrow K(T_2) \) of Lie algebras. A quick proof follows from the definition of \( K(T) \) above: choose a basis \( B_1 \cup B_2 \) of \( T_1 \) such that \( \phi(B_1) \) is linearly independent while \( \phi(B_2) = 0 \). Extend \( \phi(B_1) \) to a basis of \( T_2 \) and it is clear that we may extend \( \phi \).

**Lemma 2.5.** Let \( T \) be a Jordan triple system. If the Lie algebra \( K(T) \) is finitely presented then the Jordan triple system is finitely presented as well.

**Proof.** Let \( \{e_u, u \in I\} \) be a basis of the vector space \( T \) and let \( \{e_u, e_v, e_w\} = \sum \gamma_{uvw}^t e_u \), where \( t, u, v, w \in I; \gamma_{uvw}^t \in k \). Define \( K(T) \) as in Definition 2.3. For a subset \( S \subseteq I \), let \( R(S) \) be the set of those relations from (K1),(K2) that have all indices \( t, u, v, w \) lying in \( S \).

By our assumption there exists a finite subset \( S \subseteq I \) such that the algebra \( K(T) \) is generated by \( x_u^\pm, u \in S \), and presented by the set of relations \( R(S) \). Let \( \tilde{T} \) be the Jordan triple system presented by generators \( y_u, u \in S \) and the set of relations

\[
R_J(S) : \{y_u, y_v, y_w\} - \sum \gamma_{uvw}^t y_t = 0,
\]

where \( u, v, w, t \in S \). We claim that the mapping \( y_u \mapsto e_u, u \in S \), extends to an isomorphism \( \tilde{T} \cong T \). Since the elements \( e_u, u \in S \) satisfies the relations from \( R_J(S) \) it follows that the mapping \( y_u \mapsto e_u, u \in S \), extends to a homomorphism \( \tilde{T} \rightarrow T \). This homomorphism gives rise to a homomorphism \( K(\tilde{T}) \rightarrow K(T) \).

Consider the Lie algebra \( K(\tilde{T}) \). Since the elements \( y_u^\pm \in K(\tilde{T}), u \in S \) satisfy the relations \( R(S) \), the mapping \( x_u^\pm \rightarrow y_u^\pm, u \in S \) extends to a homomorphism \( K(T) \rightarrow K(\tilde{T}) \). Hence the homomorphism \( K(\tilde{T}) \rightarrow K(T) \) is an isomorphism. This implies that the homomorphism \( \tilde{T} \rightarrow T \) is bijective, hence an isomorphism. □
Lemma 2.6. Let $J$ be a unital Jordan algebra. If $J$ is finitely presented as a Jordan triple system, then $J$ is finitely presented as a Jordan algebra.

Proof. Let elements $a_1, \ldots, a_n \in J$ generate $J$ as a Jordan triple system. Then they clearly generate $J$ as a Jordan algebra, according to the formula $\{a, b, c\} = (ab)c + a(bc) - b(ac)$.

Let $\mathfrak{T}$ and $\mathfrak{J}$ be the free Jordan triple system and the free Jordan algebra on the set of free generators $x_u$, $u \geq 1$, respectively. Since $\mathfrak{J}$ is a Jordan triple system with respect to the Jordan triple product there exists a natural homomorphism $\mathfrak{T} \to \mathfrak{J}$, $a \mapsto \tilde{a}$, that extends the identical mapping $x_u \mapsto x_u$, $i \geq 1$.

Let $R \subset \mathfrak{T}$ be a finite subset, all elements from $R$ become zero when evaluated at $a_u$, $1 \leq u \leq n$, and $R$ defines $J$ as a Jordan triple system. Since elements $a_1, \ldots, a_n$ generate $J$ as a Jordan triple system, there exists an element $\omega(x_1, \ldots, x_n) \in \mathfrak{T}$ such that $\omega(a_1, \ldots, a_n) = 1$.

Let $P = R \cup \{\omega(x_1, \ldots, x_n)^2 - \omega(x_1, \ldots, x_n)\} \subset \mathfrak{J}$. Consider the Jordan algebra $\tilde{J} = \langle x_1, \ldots, x_n | P = (0) \rangle$. Our aim is to show that $\tilde{J} \cong J$.

Since the generators $a_1, \ldots, a_n$ satisfy the relations $P$ it follows that there exists a surjective homomorphism $\tilde{J} \twoheadrightarrow J$, $\varphi(x_i) = x_i$, $1 \leq i \leq n$. We claim that the elements $x_1, \ldots, x_n$ generate the Jordan algebra $\tilde{J}$ as a Jordan triple system: Indeed, let $\tilde{J}'$ be the Jordan triple system generated by $x_1, \ldots, x_n$ in $\tilde{J}$. The relations $P$ imply that the element $\omega(x_1, \ldots, x_n)$ is an identity element of the algebra $\tilde{J}$ and $\omega(x_1, \ldots, x_n) \in \tilde{J}'$. If $a, b \in \tilde{J}'$ then $ab = \{a, \omega(x_1, \ldots, x_n), b\} \in \tilde{J}'$, which implies $\tilde{J}' \cong \tilde{J}$.

Since the generator $x_1, \ldots, x_n$ of the Jordan triple system $\tilde{J}$ satisfy the relations $R$ it follows that there exists a homomorphism of Jordan triple systems $J \xrightarrow{\psi} \tilde{J}$ that extends $\psi(x_u) = x_u$, $1 \leq u \leq n$. This implies that $\varphi, \psi$ are isomorphisms of Jordan triple systems. In particular, $\varphi$ is a bijection. Hence $\varphi$ is an isomorphism of Jordan algebras.

Lemmas 2.5, 2.6 imply the following proposition:

Proposition 2.7. Let $J$ be a Jordan algebra with 1. Then, if its TKK Lie algebra $K(J)$ is finitely presented, then $J$ is finitely presented.

3. Non-finite presentation of $\hat{sl}_2(A)$

Theorem 3.1. Let $k$ be a field where $\text{char}(k) \neq 2$. Then $\hat{sl}_2(k\langle x, y \rangle)$ is not finitely presented.

The universal central extension of $sl_2(A)$ was discussed in Kassel-Loday [8]; see also [4]. To elaborate:
Theorem 3.2 (See [4]). \( \hat{\text{sl}}_2(A) \) admits a presentation where the Lie algebra is generated by \{\( X_{12}(a), X_{21}(a), T(a, b) \mid a, b \in A \}\}, subjecting to the relations

\[
X_{ij}(aa + \beta b) = \alpha X_{ij}(a) + \beta X_{ij}(b),
T(a, b) = [X_{12}(a), X_{21}(b)],
[T(a, b), X_{12}(c)] = X_{12}(abc + cba),
[T(a, b), X_{21}(c)] = -X_{12}(bac + cab),
[X_{ij}(A), X_{ij}(A)] = 0.
\]

for all \( 1 \leq i \neq j \leq 2 \), \( a, b, c \in A \), \( \alpha, \beta \in k \).

Proof of Theorem 3.2. Define \( x(a) := X_{12}(a) \), and \( x_-(a) := X_{21}(\frac{1}{2}a) \). These new expressions give a new presentation of \( \hat{\text{sl}}_2(A) \), in generators \( \{x_+(a), x_-(a) \mid a \in A \} \):

\[
a \mapsto x_+(a) \text{ and } a \mapsto x_-(a) \text{ are } k \text{ linear maps},
[X_+(A), x_+(A)] = [x_-(A), x_-(A)] = 0,
[[x_+(a), x_-(b)], x_+(c)] = x_+(\frac{1}{2}(abc + cba)),
[[x_-(b), x_+(a)], x_-(c)] = x_-(\frac{1}{2}(bac + cab)),
\]

for all \( a, b, c \in A \). This gives \( \hat{\text{sl}}_2(A) \cong K(A^{(+)}), \) as this presentation defines \( K(A^{(+)}), \)

Now set \( A = k[x, y] \). According to [11], \( k[x, y]^{(+)} \) is not finitely presented as a Jordan algebra. Theorem 3.1 then implies that \( \hat{\text{sl}}_2(k[x, y]) \), being isomorphic to \( K(k[x, y]^{(+)}), \) is not finitely presented.

4. Finite presentation of \( M_2(A)^{(+)} \)

Lemma 4.1. Let \( k \) be a field where \( \text{char}(k) \neq 2 \), \( A \) a unital associative \( k \)-algebra. Then \( \hat{\text{sl}}_4(A) \cong K(M_2(A)^{(+)}). \)

Proof. According to [8, 9], \( \hat{\text{sl}}_4(A) \) admits the presentation with generating set \( \{X_{ij}(s) \mid s \in A, 1 \leq i \neq j \leq n \} \) and set of relations

\[
\alpha \mapsto X_{ij}(\alpha) \text{ is a } k \text{-linear map,}
[X_{ij}(\alpha), X_{jk}(\beta)] = X_{ik}(\alpha \beta), \text{ for distinct } i, j, k,
[X_{ij}(\alpha), X_{kl}(\beta)] = 0, \text{ for } j \neq k, i \neq l,
\]

for all \( \alpha, \beta \in A \). According to [12], \( \hat{\text{sl}}_4(A) \) is finitely presented.

Like in Theorem 3.1, we reorganize the presentation: the \( 2 \times 2 \) block partition of \( \text{sl}_4 \) allows us to identify \( X_{13}(A) \oplus X_{14}(A) \oplus X_{23}(A) \oplus X_{24}(A) \) to a copy of \( M_2(A)^{(+)} \) (" \( x_+ \)"), and \( X_{31}(A) \oplus X_{32}(A) \oplus X_{41}(A) \oplus X_{42}(A) \) to another copy of \( M_2(A)^{(+)} \) (" \( x_- \)"). Now identify through the \( (k \text{-linear}) \) assignment

\[
X_{13}(a) \to x_+(e_{11}(a)), \quad X_{14}(a) \to x_+(e_{12}(a)), \quad X_{23}(a) \to x_+(e_{12}(a)), \quad X_{24}(a) \to x_+(e_{22}(a))
\]
and
\[ \frac{1}{2}X_{31}(a) \to x_-(e_{11}(a)), \quad \frac{1}{2}X_{32}(a) \to x_-(e_{12}(a)), \quad \frac{1}{2}X_{41}(a) \to x_-(e_{12}(a)), \quad \frac{1}{2}X_{42}(a) \to x_-(e_{22}(a)), \]

where \( \oplus e_{ij}(A) \) is the Peirce decomposition of \( M_2(A) \), namely decomposing \( 2 \times 2 \) matrices into subspaces corresponding to the four entries.

Under these new expressions our defining presentation becomes

\[ U \mapsto x_+(U) \text{ and } U \mapsto x_-(U) \text{ are } k \text{ linear maps,} \]
\[ [x_+(M_2(A)), x_+(M_2(A))] = [x_-(M_2(A)), x_-(M_2(A))] = 0, \]
\[ [[x_+(U), x_-(V)], x_+(W)] = x_+(\frac{1}{2}(UVW + WVU)), \]
\[ [[x_-(V), x_+(U)], x_-(W)] = x_-(\frac{1}{2}(VUV + WUV)), \]

for all \( U, V, W \in M_2(A) \). Similar to Theorem 3.1, this gives \( \widehat{sl}_4(A) \cong K(M_2(A)^{(+)}). \) \( \square \)

**Theorem 4.2.** Notations be as before. If \( A \) is finitely presented as a \( k \)-algebra, then the special Jordan algebra \( M_2(A)^{(+)} \) is finitely presented.

**Proof.** Apply Proposition 2.7 to \( K(M_2(A)^{(+)})(\text{which is isomorphic to } \widehat{sl}_4(A)). \) \( \square \)

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