Sign changing solutions of the Brezis-Nirenberg problem in the Hyperbolic space.

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Abstract

In this article we will study the existence and nonexistence of sign changing solutions for the Brezis-Nirenberg type problem in the Hyperbolic space. We will also establish sharp asymptotic estimates for the solutions and the compactness properties of solutions.

1 Introduction

In this article we will study the equation

\[- \Delta_{B^N} u - \lambda u = |u|^{2^*-2} u, u \in H^1(B^N)\]  \hspace{1cm} (1.1)

where \(\lambda < \left(\frac{N-1}{2}\right)^2\) and \(H^1(B^N)\) denotes the Sobolev space on the disc model of the Hyperbolic space \(B^N\), \(\Delta_{B^N}\) denotes the Laplace Beltrami operator on \(B^N\) (see the Appendix for definitions) and \(2^* = \frac{2N}{N-2}\) is the critical Sobolev exponent and \(N \geq 3\).

Though equation (1.1) is a natural generalization of the well known Brezis-Nirenberg equation ([3]) to the Hyperbolic space, it came to prominence with the discovery of its connection with various other equations like Hardy-Sobolev-Mazya equations([7],[8],[10]) and Grushin equations([1]). Existence and uniqueness of positive finite energy solutions to (1.1) has been thoroughly investigated in [10], in fact for the general nonlinearity \(|u|^{p-2} u\) with \(2 < p \leq \frac{2N}{N-2}\) for \(N \geq 3\) and \(p > 1\) for \(N=2\). It is shown in [10] that (1.1) has a positive solution iff \(\frac{N(N-2)}{4} < \lambda < \left(\frac{N-1}{2}\right)^2, N \geq 4\) and the solution is unique up to hyperbolic isometry. The problem also exhibits a low dimensional phenomenon (nonexistence of positive solution for \(N = 3\) for any \(\lambda\)), which also implies that the best constant in the Sobolev inequality in

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the 3-dimensional hyperbolic space is the same as the corresponding one in the Euclidean space ([2]). Existence and nonexistence of positive solutions to the above problem in geodesic balls of the hyperbolic space have been studied in [13].

In this article we mainly discuss the sign changing solutions of (1.1). In the subcritical case, i.e., when the nonlinear term is $|u|^{p-2}u$ with $2 < p < \frac{2N}{N-2}$, the problem admits infinitely many sign changing solutions ([6]) for any $\lambda < \left(\frac{N-1}{N}\right)^2$. It is also shown in [6] that (1.1) has a pair of radial sign changing solution when $N \geq 7$. Radial sign changing solutions of (1.1) without the finite energy assumption for the case $\lambda = 0$ has been studied in [4]. So many questions remains unanswered in the critical case. First of all is the restriction on $\lambda$ for the existence of a positive solution is required for the existence of sign changing solutions as well? One may expect so as the condition is coming from a Pohozaev obstruction which is applicable to sign changing solutions as well. However we can not apply directly the Pohozaev identity as we do not know the behaviour of solutions near infinity. We establish asymptotic estimates for the solutions (see Theorem 2.1) and prove:

**Theorem 1.1.** The Eq. (1.1) does not have a solution if $\lambda \leq \frac{N(N-2)}{4}$.

There has been an extensive study of the Brezis-Nirenberg problem in the past two decades in bounded domains of the Euclidean space and also on compact Riemannian manifolds(see [9], [15], [16] and the references therein). One of the important result obtained is the existence of infinitely many sign changing solutions when the dimension $N \geq 7$ ([9], [15]). In all these approaches one of the main tool used is the compactness of the Brezis-Nirenberg problem established by Solimini([9]) in higher dimensions.

In the hyperbolic case, we prove the following compactness theorem for radial solutions:

**Theorem 1.2.** Let $N \geq 7$ and $\mathcal{A}$ be a bounded subset of $H^1(\mathbb{B}^N)$ consisting of radial solutions of (3.1) for a fixed $\lambda$ and $p$ varying in $(2, 2^*)$, then there exists a constant $C$ depending only on $\mathcal{A}$ such that

$$|u(x)| \leq C(1 - |x|^2)^{\frac{N-1}{2}}$$

(1.2)

holds for all $u \in \mathcal{A}$.

With the help of above theorem we prove:

**Theorem 1.3.** The Eq. (1.1) has infinitely many non-trivial radial sign changing solutions if $N \geq 7$ and $\frac{N(N-2)}{4} < \lambda < \left(\frac{N-1}{2}\right)^2$. 

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We divide the paper into four sections. In Section 2, we will prove the asymptotic estimates on the solutions, Section 3 is devoted to the compactness properties, Sections 4 and 5 will respectively prove the nonexistence and existence results and in section 6, we recall the definitions and embeddings of Sobolev spaces on the hyperbolic space.

**Notations.** We will denote by $H^1(\mathbb{B}^N)$ the Sobolev space with respect to the hyperbolic metric and $H^1_0(\mathbb{B}^N)$ will denote the Euclidean Sobolev space on the unit disc. We will denote the hyperbolic volume by $dV_{\mathbb{B}^N}$.

### 2 Asymptotic estimates

From the standard elliptic theory we know that the solutions of (1.1) are in $C^2(\mathbb{B}^N)$. But we do not have any information on the nature of solutions as $x \to \infty$ (equivalently as $|x| \to 1$). If $u$ is a positive solution of (1.1), by moving plane method $u$ is radial with respect to a point and the exact behaviour of $u$ as $x \to \infty$ has been obtained in [10] by analysing the corresponding ode. But there is no reason to expect every solution to be radial (especially the sign changing ones) and hence the above mentioned approach does not help in finding apriori estimates in the general case. In this section we will prove the following asymptotic estimate which plays a major role in the proof of Theorem 1.1.

**Theorem 2.1.** Let $u$ be a solution of (1.1), then $|u(x)| + |\nabla_{\mathbb{B}^N} u(x)|^2 \to 0$ as $x \to \infty$ in $\mathbb{B}^N$. If $\lambda \leq \frac{N(N-2)}{4}$ then $|u(x)| \leq C \left(1-|x|^2\right)^\alpha$ where $c_\lambda = \min\left\{\frac{(N-1)+\sqrt{(N-1)^2-4\lambda}}{2}, \frac{N+2}{2}\right\}$.

We will prove this theorem in several steps. First a few propositions.

**Proposition 2.2.** Let $v \in D^{1,2}(\mathbb{R}^N_+)$ be a weak solution of the equation

$$-\Delta v + \eta \frac{v}{x_N} = (f)v + gv$$  \hspace{1cm} (2.1)

where $\eta \geq 0$, $f \in L^\infty_{loc}(\mathbb{R}^N_+)$ and $g \in L^q_{loc}(\mathbb{R}^N_+)$ for some $q > \frac{n}{2}$, then $v \in L^\infty_{loc}(\mathbb{R}^N_+)$.  

**Remark 2.3.** Note that in the above proposition $f$ is only assumed to be in $L^\infty_{loc}$, by a weak solution we mean $v$ satisfies

$$\int_{\mathbb{R}^N_+} \nabla v \nabla \psi + \int_{\mathbb{R}^N_+} \eta \frac{v \psi}{x_N} = -\int_{\mathbb{R}^N_+} f \psi v + \int_{\mathbb{R}^N_+} g v \psi, \forall \psi \in C^\infty_c(\mathbb{R}^N_+)$$  \hspace{1cm} (2.2)
Proof. We will prove the theorem using Moser Iteration. Fix a point $x_0 \in \partial \mathbb{R}^N_+$ and $R > 0$. Define $\tilde{v} = v^+ + 1$ and

$$v_m = \begin{cases} 
\tilde{v} & \text{if } v < m \\
1 + m & \text{if } v \geq m 
\end{cases}$$

For $\beta > 0$ define the test function $w = w_\beta$ as $w_\beta = \varphi^2(v_m^{2\beta} \tilde{v} - 1)$ where $\varphi \in C_0^\infty(\mathbb{R}^N)$, $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $B(x_0, r_{i+1})$, $\text{supp} \varphi \subseteq B(x_0, r_i)$, $R < r_{i+1} < r_i < 2R$ and $|\nabla \varphi| \leq \frac{C}{r_i - r_{i+1}}$ where $C$ is independent of $\varphi$.

Then $0 \leq w \in H_0^1(\mathbb{R}^N)$ and using $\varphi = w$ as the test function in (2.2), we get

$$\int_{\mathbb{R}^N_+} \nabla v \nabla w + \int_{\mathbb{R}^N_+} \eta \frac{vw}{x_N} = -\int_{\mathbb{R}^N_+} fw + \int_{\mathbb{R}^N_+} gw$$

(2.3)

Now substituting $w$ and observing that $\int_{\mathbb{R}^N_+} \eta \frac{vw}{x_N} \geq 0$ we get

L.H.S $\geq$

$$\int_{\mathbb{R}^N_+} [v_m^{2\beta} \varphi^2 \nabla v \nabla \tilde{v} + 2\beta v_m^{2\beta-1} \tilde{v} \varphi^2 \nabla \tilde{v} \nabla v_m + 2\varphi(v_m^{2\beta} \tilde{v} - 1) \nabla \tilde{v} \nabla \varphi] dx$$

In the support of 1st integral $\nabla v = \nabla \tilde{v}$, and in the support of 2nd integral $v_m = \tilde{v}, \nabla v_m = \nabla \tilde{v}$. Therefore using Cauchy- Schwartz along with the above fact we get

L.H.S $\geq$

$$\frac{1}{2} \int_{\mathbb{R}^N_+} v_m^{2\beta} \varphi^2 |\nabla \tilde{v}|^2 dx + 2\beta \int_{\mathbb{R}^N_+} v_m^{2\beta} \varphi^2 |\nabla v_m|^2 dx$$

$$-2 \int_{\mathbb{R}^N_+} |\nabla \varphi|^2 v_m^{2\beta} \tilde{v}^2 dx$$

(2.4)

The RHS of (2.3) can be estimated by

$$|\int_{\mathbb{R}^N_+} f[2(v_m^{2\beta} \tilde{v} - 1)\varphi_{x_i} + \varphi^2 v_m^{2\beta} (\tilde{v})_{x_i} + 2\beta \varphi^2 v_m^{2\beta-1} \tilde{v} (v_m)_{x_i}]| + |\int_{\mathbb{R}^N_+} g \varphi^2 v_m^{2\beta} \tilde{v}^2|$$

$$\leq \frac{1}{4} \int_{\mathbb{R}^N_+} v_m^{2\beta} \varphi^2 |\nabla \tilde{v}|^2 dx + \beta \int_{\mathbb{R}^N_+} v_m^{2\beta} \varphi^2 |\nabla v_m|^2 dx + C \int_{\mathbb{R}^N_+} |\nabla \varphi|^2 v_m^{2\beta} \tilde{v} dx$$

$$+ C(1 + \beta) \int_{\mathbb{R}^N_+} v_m^{2\beta} \varphi^2 + \int_{\mathbb{R}^N_+} |g| \varphi^2 v_m^{2\beta} \tilde{v}^2$$
where $C$ is a constant depending on the $L^\infty$ norm of $f$ on $B(x_0, 2R)$. Since $\tilde{v} \geq 1$ we can estimate the RHS as

$$ \text{R.H.S} \leq \frac{1}{4} \int_{\mathbb{R}^N_+} v_m^{2\theta} \varphi^2 |\nabla \tilde{v}|^2 dx + \beta \int_{\mathbb{R}^N_+} v_m^{2\theta} \varphi^2 |\nabla v_m|^2 dx$$

$$+ C \frac{\beta}{(r_i - r_{i+1})^2} \int_{\mathbb{R}^N_+} v_m^{2\theta} \tilde{v}^2 dx + \int_{\mathbb{R}^N_+} |g|^2 v_m^{2\theta} \tilde{v}^2 \quad (2.5)$$

Using the estimates (2.4) and (2.5) in (2.3) we get

$$\frac{1}{2} \int_{\mathbb{R}^N_+} v_m^{2\theta} \varphi^2 |\nabla \tilde{v}|^2 dx + 2\beta \int_{\mathbb{R}^N_+} v_m^{2\theta} \varphi^2 |\nabla v_m|^2 dx$$

$$\leq C \frac{1 + \beta}{(r_i - r_{i+1})} \int_{\mathbb{R}^N_+} v_m^{2\theta} \tilde{v}^2 dx + \int_{\mathbb{R}^N_+} |g|^2 v_m^{2\theta} \tilde{v}^2 \quad (2.6)$$

Defining $\bar{w} = v_m^{\frac{\theta}{2}} \tilde{v}$, (2.6) becomes

$$\frac{1}{4(1 + 2\beta)} \int_{\mathbb{R}^N_+} |\nabla (\varphi \bar{w})|^2 dx \leq C \frac{(1 + \beta)}{(r_i - r_{i+1})} \int_{\mathbb{R}^N_+} (\varphi \bar{w})^2 dx + \int_{\mathbb{R}^N_+} |g|(\varphi \bar{w})^2$$

$$\leq C \frac{1 + \beta}{(r_i - r_{i+1})} \int_{\mathbb{R}^N_+} (\varphi \bar{w})^2 dx + C \left( \int_{\mathbb{R}^N_+} |\varphi \bar{w}|^q \right)^{\frac{2}{q}} \quad (2.7)$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Since $q > \frac{N}{2} \Rightarrow q' < \frac{N}{N-2} = \frac{N}{2}$ (say). Now let $\frac{1}{q} = \theta + \frac{1-q}{r}$, then using interpolation inequality we get

$$|(\varphi \bar{w})^2|_{L^{q'}} \leq \varepsilon (1 - \theta) |(\varphi \bar{w})^2|_{L^r} + C \varepsilon^{-\frac{1-q}{r}} |(\varphi \bar{w})^2|_{L^1}, \forall \varepsilon$$

where $\theta$ depends on $N, q'$. Note that $2r = 2^*$. Therefore

$$|(\varphi \bar{w})^2|_{L^{q'}} = |\varphi \bar{w}|^2_{L^{q'}}(\mathbb{R}^N) \leq C |\nabla (\varphi \bar{w})|^2_{L^2(\mathbb{R}^N)}$$

Hence

$$|(\varphi \bar{w})^2|_{L^{q'}} \leq C \varepsilon |\nabla (\varphi \bar{w})|^2_{L^2(\mathbb{R}^N)} + C \varepsilon^{-\frac{q}{r}} |(\varphi \bar{w})^2|_{L^1(\mathbb{R}^N)}$$

Now choosing $\varepsilon$ suitably and substituting in (2.7), we get

$$\int_{\mathbb{R}^N_+} |\nabla (\varphi \bar{w})|^2 dx \leq C (1 + \beta)^{\alpha} \frac{1}{(r_i - r_{i+1})^2} \int_{B(x_0, r_i)} \bar{w}^2 dx$$

Now using the Sobolev inequality in the above expression we get

$$\left( \int_{B(x_0, r_{i+1})} \bar{w}^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} \leq C (1 + \beta)^{\alpha} \frac{1}{(r_i - r_{i+1})^2} \int_{B(x_0, r_i)} \bar{w}^2 dx$$
Now using $\chi = \frac{N}{N-2} > 1$, $\tilde{v} = v_m^\beta \tilde{v}$, $v_m \leq \tilde{v}$ and $\gamma = 2(\beta + 1)$ we get

$$
\left( \int_{B(x_0,r_{i+1})} v_m^{\gamma} dx \right)^{\frac{1}{\gamma}} \leq \left[ \frac{C(1 + \beta)^2}{(r_i - r_{i+1})^2} \right]^{\frac{1}{\gamma}} \left( \int_{B(x_0,r_i)} \tilde{v}^\gamma dx \right)^{\frac{1}{\gamma}}
$$

Now letting $m \to \infty$ we get

$$
\left( \int_{B(x_0,r_{i+1})} \tilde{v}^{\gamma} dx \right)^{\frac{1}{\gamma}} \leq \left[ \frac{C(1 + \beta)^2}{(r_i - r_{i+1})^2} \right]^{\frac{1}{\gamma}} \left( \int_{B(x_0,r_i)} \tilde{v}^\gamma dx \right)^{\frac{1}{\gamma}}
$$

provided $|\tilde{v}|_{L^\gamma(B(x_0,r_{i+1})}$ is finite. $C$ is a positive constant independent of $\gamma$. Now we will complete the proof by iterating the above relation. Let us take $\gamma = 2, 2\chi, 2\chi^2 \ldots$ i.e., $\gamma_i = 2\chi^i$ for $i = 0, 1, 2, \ldots r_{i+1} = R + \frac{R}{2^i}$. Hence for $\gamma = \gamma_i$ we get

$$
\left( \int_{B(x_0,r_{i+1})} \tilde{v}^{\gamma_i} dx \right)^{\frac{1}{\gamma_i}} \leq C^{i+1} \left( \int_{B(x_0,r_i)} \tilde{v}^{\gamma_{i+1}} dx \right)^{\frac{1}{\gamma_{i+1}}}
$$

where $C > 1$ is a constant depends on $R, N, ||g||_{L^\gamma(B(x_0,2R))}$ and $||f||_{L^\infty(B(x_0,2R))}$. Now by iteration we obtain

$$
\left( \int_{B(x_0,r_{i+1})} \tilde{v}^{\gamma_i} dx \right)^{\frac{1}{\gamma_i}} \leq C^{\sum \frac{i}{\gamma_{i+1}}} \left( \int_{B(x_0,r_i)} \tilde{v}^{\gamma_{i+1}} dx \right)^{\frac{1}{\gamma_{i+1}}}
$$

letting $i \to \infty$ we obtain

$$
\sup_{B(x_0,R)} \tilde{v} \leq C ||\tilde{v}||_{L^\gamma(B(x_0,2R))}.
$$

This proves the local boundedness of $u^+$. Similarly we get the boundedness of $u^-$ as $-u$ also satisfies the same equation with $-f$ in place of $f$. This proves the theorem.

**Proposition 2.4.** Let $u \in D^{1,2}(\mathbb{R}^N_+)$ be a weak solution of the problem

$$
- \Delta u + \frac{\eta u}{x_N} = |u|^{2^*-2} u
$$

with $\eta \geq 0$, then $u_{x_i}, u_{x_ix_i} \in D^{1,2}(\mathbb{R}^N_+)$ for all $1 \leq i < N$.

**Proof.** Using Moser iteration as in Brezis-Kato (See [13], Appendix B, Lemma B3) we can show that $u \in L^q_{\text{loc}}(\mathbb{R}^N_+)$ for some $q > \frac{2N}{N-2}$. Thus from Proposition 2.2 with $f = 0$ and $g = |u|^{2^*-2}$ we get $|u(x)| \leq M$ for all $|x| \leq 1$ for some $M > 0$. Since the Kelvin transform of $u$ given by $u(x) = u(|x|^{-1} x)$ also satisfies (2.8) we get $u \in L^\infty(\mathbb{R}^N_+)$. 

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We will show that \( u_{x_i} \in D^{1,2}(\mathbb{R}^N_+) \) by the method of difference quotients. The case of \( u_{x_i} \) follows exactly as in Theorem 2.1 of [5]. To prove the estimate on \( u_{x_i} \), first note that by standard elliptic theory \( u \in C^3(\mathbb{R}^N_+) \). Differentiating (2.8) with respect to \( x_i \) we see that

\[
- \Delta u_{x_i} + \frac{u_{x_i}}{x_N} = (2^* - 1)|u|^{2^* - 2}u_{x_i}
\]  

(2.9)

Thus, we have for any \( w \in D^{1,2}(\mathbb{R}^N_+) \)

\[
\int \nabla u_{x_i} \nabla w + \eta \int \frac{u_{x_i} w}{x_N^2} = (2^* - 1) \int |u|^{2^* - 2}u_{x_i} w
\]  

(2.10)

For \( |h| > 0 \) and \( i < N \), define \( w = -D_i^{-h}(D_i^{h} u_{x_i}) \) where \( D_i^{h} \) denotes the difference quotient

\[
D_i^{h} u_{x_i}(x) = \frac{u_{x_i}(x + he_i) - u_{x_i}(x)}{h}
\]

For this choice of \( w \) the L.H.S of (2.10) simplifies to \( \int |\nabla (D_i^{h} u_{x_i})|^2 + \eta \int \frac{(D_i^{h} u_{x_i})^2}{x_N^2} \) while R.H.S can be estimated as

\[
| \int |u|^{2^* - 2}u_{x_i} w | = | \int \frac{\partial}{\partial x_i} D_i^{h}(|u|^{2^* - 2}u)D_i^{h} u_{x_i} |
\]

\[
= | - \int D_i^{h}(|u|^{2^* - 2}u) \frac{\partial}{\partial x_i} (D_i^{h} u_{x_i}) |
\]

\[
\leq \int |D_i^{h}(|u|^{2^* - 2}u)||\nabla (D_i^{h} u_{x_i}) |
\]

\[
\leq C \int | |u|^{2^* - 2}(x + he_i) + |u|^{2^* - 2}(x)||D_i^{h} u||\nabla (D_i^{h} u_{x_i}) |
\]

\[
\leq C \int |D_i^{h} u||\nabla (D_i^{h} u_{x_i}) |
\]

\[
\leq C \varepsilon \int |\nabla (D_i^{h} u_{x_i})|^2 + \frac{C}{4 \varepsilon} \int |D_i^{h} u|^2
\]

By choosing \( C \varepsilon < 1 \) we have

\[
\int_{\mathbb{R}^N_+} |\nabla (D_i^{h} u_{x_i})|^2 \leq C \int_{\mathbb{R}^N_+} |\nabla u|^2 \leq C
\]

where \( C \) is independent of \( h \) and this implies \( \int |\nabla u_{x_i}| \leq M \) and this completes the proof. \( \square \)
Proposition 2.5. Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function bounded in $(0, 1)$ and $\eta > 0$ be a positive constant and $v$ solves the ODE

$$- \frac{d^2 v}{dr^2} + \eta \frac{v}{r^2} = f(r), \quad v(0) = 0$$  \hspace{1cm} (2.11)

then there exist constants $C_1, C_2$ depending only on the $L^\infty$ norm of $f|_{(0,1)}$ such that

$$|v(r)| \leq C_1 r^{\frac{1+\sqrt{4\eta+1}}{2}} + C_2 r^2$$  \hspace{1cm} (2.12)

holds for all $r \in (0, 1)$.

Proof. Let $v(r) = \theta(\log r)$ then the equation (2.11) transforms to

$$\frac{d^2 \theta}{dt^2}(t) + \frac{d}{dt}(t) - \eta \theta(t) = -e^{2t} f(e^t)$$  \hspace{1cm} (2.13)

$$\theta(t) \to 0 \quad \text{as} \quad t \to -\infty$$

Using the method of variation of parameters we can write

$$\theta(t) = \theta_c(t) + \theta_p(t)$$

where $\theta_c(t)$ is the Complementary Function given by

$$\theta_c(t) = C_1 e^{m_1 t} + C_2 e^{m_2 t}$$  \hspace{1cm} (2.14)

with $m_1 = \frac{1+\sqrt{4\eta+1}}{2}$ and $m_2 = \frac{1-\sqrt{4\eta+1}}{2}$.

$\theta_p$ is a particular integral given by

$$\theta_p(t) = v_1(t)e^{m_1 t} + v_2(t)e^{m_2 t}$$  \hspace{1cm} (2.15)

where

$$v_1(t) = v_1(0) + \frac{1}{\sqrt{4\eta+1}} \int_0^t e^{(m_2+1)s} f(e^s) ds$$

$$v_2(t) = \frac{1}{\sqrt{4\eta+1}} \int_{-\infty}^t e^{(m_1+1)s} f(e^s) ds$$

From the expressions for $v_1$ we get

$$v_1(t) \leq v_1(0) + \frac{1}{\sqrt{4\eta+1}} \frac{M}{m_2 + 1} = \frac{M}{\sqrt{4\eta+1}(m_2 + 1)} e^{(m_2+1)t}$$  \hspace{1cm} (2.16)

where $|f| \leq M$ on $(0, 1)$. Thus,

$$|v_1(t)e^{m_1 t}| \leq C_1 e^{m_1 t} + C_2 e^{2t}$$  \hspace{1cm} (2.17)
Similarly from the expression of \( v_2(t) \) we have
\[
|v_2(t)| \leq \frac{M}{\sqrt{4\eta + 1}} \int_{-\infty}^{t} e^{(m_1+1)s}ds = \frac{M}{\sqrt{4\eta + 1}(m_1 + 1)}e^{(m_1+1)t} \tag{2.18}
\]
Thus,
\[
|v_2(t)e^{m_2t}| \leq \frac{M}{\sqrt{4\eta + 1}(m_1 + 1)}e^{2t} \tag{2.19}
\]
Since \( \theta(t) \to 0 \) as \( t \to -\infty \), using (2.17) and (2.19) we get \( C_2 = 0 \) in (2.14).

Using these informations we get
\[
|\theta(t)| \leq C_1 e^{m_1t} + \tilde{C}_2 e^{2t} \tag{2.20}
\]
for all \( t < 0 \). Changing the variable as \( r = e^t \) proves the proposition. \( \square \)

**Proof of Theorem 2.1** Let \( M \) be the isometry between \( \mathbb{B}^N \) and the upper half space model \( \mathbb{H}^N \) given by
\[
M(x) := \left( \frac{2x'}{(1 + x_N)^2 + |x'|^2}, \frac{1 - |x|^2}{(1 + x_N)^2 + |x'|^2} \right) \tag{2.21}
\]
where a point in \( \mathbb{H}^N \) is denoted by \( x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \). Then \( \tilde{u} = u \circ M : \mathbb{H}^N \to \mathbb{R} \) satisfies the equation (note \( M^{-1} = M \))
\[
-\Delta_{\mathbb{H}^N} \tilde{u} - \lambda \tilde{u} = |\tilde{u}|^{2^* - 2} \tilde{u}, \tilde{u} \in H^1(\mathbb{H}^N) \tag{2.22}
\]
where \( \Delta_{\mathbb{H}^N} \) is the Laplace-Beltrami operator in \( \mathbb{H}^N \) given by
\[
\Delta_{\mathbb{H}^N} u = x_N^2 \Delta u - (N - 2)x_N u_{x_N} \tag{2.23}
\]
Making a conformal change of the metric, defining \( v(x) = x_N^{-\frac{N-2}{4}} u(x) \), \( v \) satisfies the equation (2.8) with \( \eta = (\frac{N(N-2)}{4} - \lambda) \geq 0 \).

From standard elliptic theory we know that \( v \in C^{3,\alpha}_{\text{loc}}(\mathbb{R}^N_+) \). Moreover Proposition 2.4 tells us that \( v_{x_i}, v_{x_i x_i} \in D^{1,2}(\mathbb{R}^N_+) \) for all \( 1 \leq i < N \) and \( v \) is bounded. Next we claim that \( v_{x_i} \) and \( v_{x_i x_i} \) are locally bounded for \( 1 < i < N \).

We know that \( v_{x_i} \) satisfies (2.21). Applying Proposition 2.2 with with \( f = 0, g = (2^* - 1)|v|^{2^*-2} \), we get \( v_{x_i} \) is locally bounded.

Since \( v_{x_i} \) is in \( D^{1,2}(\mathbb{R}^N_+) \) we get
\[
\int_{\mathbb{R}^N_+} \nabla v_{x_i} \nabla \varphi + \eta \int_{\mathbb{R}^N_+} \frac{v_{x_i} \varphi}{x_N^2} = \int_{\mathbb{R}^N_+} (2^* - 1)|v|^{2^*-2} v_{x_i} \varphi
\]

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for all $\varphi \in C_c^\infty(\mathbb{R}^N_+)$. Taking $\varphi_{x_i}$ instead of $\varphi$ and an integration by parts gives
\[
\int_{\mathbb{R}^N_+} \nabla v_{x_i} \cdot \nabla \varphi + \eta \int_{\mathbb{R}^N_+} \frac{v_{x_i} \varphi}{x^2} = -(2^*-1) \int_{\mathbb{R}^N_+} |v|^{2^*-2} v_x \varphi_x,
\]
This shows that $v_{x_i}$ satisfies (2.21) with $f = (2^*-1)|v|^{2^*-2} v_x$, and $g = 0$. This proves the local boundedness of $v_{x_i}$ for $i < N$.

Now we will estimate the solution $v$. Fix a point $x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$, and define $v(r) = v(x', r)$ for $r > 0$. Then $v$ satisfies the ODE (2.11) with $f(r) = \sum_{i<N} v_{x_i}(x', r) + |v|^{2^*-2} v(x', r)$. Thus from Proposition 2.4 we get
\[
|v(x', r)| \leq C_1 r^{\frac{1+\sqrt{2m+1}}{2}} + C_2 r^2
\] (2.24)
Since $C_1$ and $C_2$ depends only on the local bound on $f$, from the uniform bound of $\sum_{i<N} v_{x_i}(x', r) + |v|^{2^*-2} v(x', r)$ on compact subsets of $\mathbb{R}^N_+$ we get the above estimate locally in $\mathbb{R}^N_+$, in particular
\[
|v(x)| \leq C_1 r^{\frac{1+\sqrt{2m+1}}{2}} + C_2 x^{2}_N \quad \forall x \in B(0, 1) \cap \mathbb{R}^N_+ \quad (2.25)
\]
To get a global bound on $v$, first observe that if $v$ is a solution of (2.8) then its Kelvin transform $\tilde{v}(x) := \frac{1}{|x|^{N-2}} v\left(\frac{x}{|x|^2}\right)$ also solves (2.8). So $\tilde{v}$ also satisfies the estimate (2.25). Hence we have,
\[
|v(x)| \leq C_1 \frac{x^{m_1}_N}{|x|^{N-2+2m_1}} + C_2 \frac{x^{2}_N}{|x|^{N-2+4}} \quad \forall x \in (B(0, 1))^c \cap \mathbb{R}^N_+ \quad (2.26)
\]
So, combining (2.25) and (2.26) we have,
\[
|v(x)| \leq C_1 \frac{x^{m_1}_N}{(1+|x|^2)^{\frac{N-2+2m_1}{2}}} + C_2 \frac{x^{2}_N}{(1+|x|^2)^{\frac{N-2+4}{2}}} \quad \forall x \in \mathbb{R}^N_+ \quad (2.27)
\]
Now recall that $\tilde{u} = x^{-2}_N v$ and hence $\tilde{u}$ satisfies the estimate
\[
|u(x)| \leq C_1 \frac{x^{m_1+(N-2)/2}_N}{((1+x_N)^2 + |x'|^2)^{\frac{N-2+2m_1}{2}}} + C_2 \frac{x^{2+(N-2)/2}_N}{((1+x_N)^2 + |x'|^2)^{\frac{N-2+4}{2}}} \quad (2.28)
\]
For a point $\xi \in \mathbb{R}^N$, let $M(\xi) = x$ then \(\frac{1-|\xi|^2}{2} = \frac{x_N}{(1+x_N)^2 + |x'|^2}\). Since $u = \tilde{u} \circ M$ we get
\[
|u(\xi)| = |u(x', x_N)| \leq C_1 \frac{x^{m_1+(N-2)/2}_N}{((1+x_N)^2 + |x'|^2)^{\frac{N-2+2m_1}{2}}} + C_2 \frac{x^{2+(N-2)/2}_N}{((1+x_N)^2 + |x'|^2)^{\frac{N-2+4}{2}}} \quad (2.29)
\]
where $|\xi| < 1$. Now putting the value of $m_1$ and $m_2$ we get

$$|u(\xi)| \leq C \left[ \left( \frac{1 - |\xi|^2}{2} \right)^{(\frac{N-1}{2})+\sqrt{(\frac{N-1}{2})^2-4\lambda}} + \left( \frac{1 - |\xi|^2}{2} \right)^{\frac{N+2}{2}} \right]$$

This proves the theorem.

### 3 Compactness of solutions

In this section we will study the compactness properties of solutions of the equation

$$- \Delta_{\mathbb{B}^N} u - \lambda u = |u|^{p-2} u, \quad u \in H^1_r(\mathbb{B}^N) \quad (3.1)$$

where $H^1_r(\mathbb{B}^N)$ denotes the subspace of $H^1(\mathbb{B}^N)$ consisting of radial functions, $p \in (2, 2^*]$ and $\frac{N(N-2)}{4} < \lambda < \left( \frac{N-2}{2} \right)^2$.

First recall the following radial estimate (see [6], Theorem 3.1 for a proof):

**Lemma 3.1.** Let $\mathcal{A}$ be a bounded subset of $H^1_r(\mathbb{B}^N)$, then there exists a constant $C$ depending only on $\mathcal{A}$ such that

$$|u(x)| \leq C|x|^{-N \frac{1}{2}} (1 - |x|^2)^{\frac{N-1}{2}} \quad (3.2)$$

holds for all $u \in \mathcal{A}$.

This estimate gives us control over the radial functions away from the origin. The main result we prove in this section rules out blow-up at the origin in higher dimensions if members of $\mathcal{A}$ are in addition solutions of (3.1).

**Theorem 3.2.** Let $N \geq 7$ and $\mathcal{A}$ be a bounded subset of $H^1_r(\mathbb{B}^N)$ consisting of solutions of (3.1) for a fixed $\lambda$ and $p$ varying in $(2, 2^*]$, then there exists a constant $C$ depending only on $\mathcal{A}$ such that

$$|u(x)| \leq C (1 - |x|^2)^{\frac{N-1}{4}} \quad (3.3)$$

holds for all $u \in \mathcal{A}$.

As a corollary we have the following compactness theorem:

**Corollary 3.3.** Let $N \geq 7$ and $u_n$ be a sequence of solutions of (3.1) with $p = p_n \in (2, 2^*]$. Suppose $p_n \to p_0 \in (2, 2^*]$ and $u_n$ is bounded in $H^1_r(\mathbb{B}^N)$, then up to a subsequence $u_n \to u$ in $H^1_r(\mathbb{B}^N)$ and $u$ solves (3.1) with $p = p_0$. Moreover $u_n \to u$ in $C(\mathbb{B}^N)$.  

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Proof. Since \( u_n \) is bounded in \( H^1_r(\mathbb{B}^N) \) up to a subsequence we may assume that \( u_n \) converges weakly and pointwise a.e. to \( u \in H^1_r(\mathbb{B}^N) \). We can immediately see that \( u \) solves (3.1) with \( p = p_0 \) and hence

\[
\int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u|^2 dV_{\mathbb{B}^N} - \lambda \int_{\mathbb{B}^N} u^2 dV_{\mathbb{B}^N} = \int_{\mathbb{B}^N} |u|^{p_0} dV_{\mathbb{B}^N}
\]

Since \( u_n \) solves (3.1) with \( p = p_n \) we get

\[
\int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u_n|^2 dV_{\mathbb{B}^N} - \lambda \int_{\mathbb{B}^N} u_n^2 dV_{\mathbb{B}^N} = \int_{\mathbb{B}^N} |u_n|^{p_n} dV_{\mathbb{B}^N}
\]

Using the estimate in Theorem 3.2 and dominated convergence theorem we get the RHS converges to \( \int_{\mathbb{B}^N} |u|^{p_0} dV_{\mathbb{B}^N} \). Combining we get

\[
\int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u_n|^2 dV_{\mathbb{B}^N} - \lambda \int_{\mathbb{B}^N} u_n^2 dV_{\mathbb{B}^N} \to \int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u|^2 dV_{\mathbb{B}^N} - \lambda \int_{\mathbb{B}^N} u^2 dV_{\mathbb{B}^N}
\]

and hence in \( H^1_r(\mathbb{B}^N) \) thanks to Lemma 6.1. Now the convergence in \( C(\mathbb{B}^N) \) follows by standard elliptic estimates and the decay estimate.

**Proof of Theorem 3.2** Suppose the theorem is not true, then there exists \( u_n \in A \) such that \( \max_{x \in \mathbb{B}^N} |u_n(x)| = |u_n(x_n)| \to \infty \) where \( u_n \) satisfies (3.1) with \( p = p_n \) and we assume \( p_n \to p_0 \in (2, 2^*) \). From Lemma 3.1 it is clear that \( x_n \to 0 \). We will show that this leads to a contradiction.

Define, \( v_n(x) = \left( \frac{2}{1 - |x|^2} \right)^{\frac{N-2}{2}} u_n \), then \( v_n \) is a bounded sequence in the Euclidean Sobolev space \( H^1_0(\mathbb{B}^N) \) and solves the Euclidean equation

\[
- \Delta v_n - \tilde{\lambda} \left( \frac{2}{1 - |x|^2} \right)^2 v_n = |v_n|^{p_n-2}v_n \left( \frac{2}{1 - |x|^2} \right)^{q_n} , \quad v_n \in H^1_0(\mathbb{B}^N) \quad (3.4)
\]

where \( \tilde{\lambda} = \lambda - \frac{N(N-2)}{4} > 0 \) and \( q_n = \frac{2N-p_n(N-2)}{2} \). Using Lemma 3.1

\[
|v_n(x)| \leq C|x|^{-\frac{N-2}{2}} \left( 1 - |x|^2 \right)^{\frac{1}{2}}, \quad \forall n \quad (3.5)
\]

Also if \( p_0 < 2^* \) by standard elliptic estimates we get \( \max_{|x| \leq \frac{1}{2}} |v_n(x)| \leq C < \infty, \forall n \), which is impossible as \( |v_n(x_n)| \to \infty \). Therefore \( p_0 = 2^* \).

Since \( v_n \) is bounded in \( H^1_0(\mathbb{B}^N) \) we may assume up to a subsequence \( v_n \) converges weakly and pointwise a.e to \( v \) in \( H^1_0(\mathbb{B}^N) \). If this convergence is
strong then by standard Brezis-Kato type arguments we get \( \max |v_n(x)| \leq C < \infty, \forall n \). Therefore \( v_n \not\to v \) in \( H^1_0(B^N) \). However \( v \) solves

\[
-\Delta v - \lambda \left( \frac{2}{1 - |x|^2} \right)^2 v = |v|^{2^* - 2} v
\]

Choose a cut off function \( \varphi \in C_0^\infty(\mathbb{R}^N) \) such that \( \varphi = 1 \) in \( B_1 = \{ x \in \mathbb{B}^N : |x| < \frac{1}{2} \} \), \( \varphi = 0 \) in \( B_2 = \{ x \in \mathbb{B}^N : |x| \geq \frac{5}{6} \} \) and \( 0 \leq \varphi \leq 1 \). Let \( w_n = \varphi v_n, \tilde{w}_n = (1 - \varphi)v_n \), then \( v_n = w_n + \tilde{w}_n \). Multiplying (3.4) by \( (1 - \varphi)^2 v_n \) and integrating by parts we get

\[
\int_{\mathbb{B}^N} |\nabla \tilde{w}_n|^2 - \lambda \left( \frac{2}{1 - |x|^2} \right)^2 |\tilde{w}_n|^2 = \int_{\mathbb{B}^N} |v_n|^{p_n - 2} (\tilde{w}_n)^2 \left( \frac{2}{1 - |x|^2} \right)^{q_n} + \int_{\mathbb{B}^N} v_n^2 |\nabla \varphi|^2
\]

Using the estimate (3.5) and applying dominated convergence theorem we easily see that the RHS converges to \( \int_{\mathbb{B}^N} |v|^{2^* - 2} (\tilde{w})^2 + \int_{\mathbb{B}^N} v^2 |\nabla \varphi|^2 \), where \( \tilde{w} = (1 - \varphi)v \).

Multiplying (3.6) by \( (1 - \varphi)^2 v \) and integrating by parts we get

\[
\int_{\mathbb{B}^N} |\nabla \tilde{w}|^2 - \lambda \left( \frac{2}{1 - |x|^2} \right)^2 |\tilde{w}|^2 = \int_{\mathbb{B}^N} |v|^{2^* - 2} (\tilde{w})^2 + \int_{\mathbb{B}^N} v^2 |\nabla \varphi|^2
\]

Thus

\[
\int_{\mathbb{B}^N} |\nabla \tilde{w}_n|^2 - \lambda \left( \frac{2}{1 - |x|^2} \right)^2 |\tilde{w}_n|^2 \to \int_{\mathbb{B}^N} |\nabla \tilde{w}|^2 - \lambda \left( \frac{2}{1 - |x|^2} \right)^2 |\tilde{w}|^2
\]

and hence \( \tilde{w}_n \to \tilde{w} \) in \( H^1_0(B^N) \). Therefore \( w_n \) converges weakly to \( w = \varphi v \) but not strongly.

Also \( w_n \in H^1_0(B^N_k) \) satisfies the equation

\[
-\Delta w_n - \lambda \left( \frac{2}{1 - |x|^2} \right)^2 w_n = |v_n|^{p_n - 2} w_n \left( \frac{2}{1 - |x|^2} \right)^{q_n} - 2\langle \nabla v_n, \nabla \varphi \rangle - v_n \Delta \varphi,
\]

Thus proceeding exactly as in Lemma 6.2 of [9], we see that up to a subsequence \( w_n \) is a concentrating sequence. i.e., there exists a positive integer \( k \) and \( \varphi_i \in D^{1,2}(\mathbb{R}^N), i = 1, \ldots, k \) satisfying \( -\Delta \varphi_i = |\varphi_i|^{2^* - 2} \varphi_i \), \( k \) sequence of positive real numbers \( \epsilon^i_n \) and \( y^i_n \in \mathbb{B}^N, y^i_n \to 0, i = 1, \ldots, k \) such that

\[
w_n - \sum_{i=1}^{k} \varphi_{i,n} \to w \quad \text{in} \quad L^2(\mathbb{R}^N)
\]
where \( \varphi_{i,n}(x) = [\epsilon_{n}^{i}]^{\frac{2-N}{2}} \varphi_{i}(x-y_{n}^{i}) \). Moreover using (3.5) we see that that \( |w_{n}| \) (extended by zero out of \( B_{2}^{c} \)) solves

\[
-\Delta |w_{n}| \leq b|w_{n}|^{2^*-1} + A
\]

in the sense of distributions where \( b \) and \( A \) are constants independent of \( n \). i.e, \( |w_{n}| \) is a controlled sequence in the sense of Solimini (9). Thus if we let \( \epsilon_{n} = \epsilon_{n}^{i} \), \( y_{n} = y_{n}^{i} \) where \( i \) is chosen such that \( \limsup_{n \to \infty} \epsilon_{n}^{j} \leq C \) for all \( j = 1, ..., k \), we have from Proposition 3.1 and Corollary 4.1 of [9]

**Lemma 3.4.** Then there exists a constant \( C > 0 \) such that

\[
\sup_{n \in \mathbb{N}} \max_{(C+1)\sqrt{\epsilon_{n}} \leq |x| \leq (C+4)\sqrt{\epsilon_{n}}} |w_{n}(x)| < \infty
\]

Moreover there exist \( t_{n} \in [C+2, C+3] \) such that,

\[
\int_{\partial B_{n}} |\nabla w_{n}|^{2} \leq C\epsilon_{n}^{\frac{N-3}{2}}
\]

where \( B_{n} = B(y_{n}, t_{n}^{1/2}) \).

With this estimate and a local Phozaev identity we will arrive at a contradiction. First we will derive the local Pohozaev identity. Let us denote the outward normal to \( \partial B_{n} \) by \( \vec{n} \).

Multiplying (3.4) by \( x.\nabla v_{n} \), and integrating by parts over \( B_{n} \) we get,

\[
\int_{B_{n}} \nabla v_{n}.\nabla (x.\nabla v_{n}) - \tilde{\lambda} \int_{B_{n}} \left( \frac{2}{1-|x|^{2}} \right)^{q_{n}} v_{n}(x.\nabla v_{n})
\]

\[
= \int_{B_{n}} |v_{n}|^{p_{n}-2} v_{n} \left( \frac{2}{1-|x|^{2}} \right)^{q_{n}}
\]

The RHS of (3.11) can be simplified as

\[
\frac{1}{p_{n}} \int_{B_{n}} (|v_{n}|^{p_{n}})_{x} \left( \frac{2}{1-|x|^{2}} \right)^{q_{n}} = \frac{1}{p_{n}} \int_{B_{n}} |v_{n}|^{p_{n}} \left( \frac{2}{1-|x|^{2}} \right)^{q_{n}} (x.\vec{n})
\]

\[
-\frac{N}{p_{n}} \int_{B_{n}} |v_{n}|^{p_{n}} \left( \frac{2}{1-|x|^{2}} \right)^{q_{n}} - \frac{q_{n}}{p_{n}} \int_{B_{n}} |v_{n}|^{p_{n}} |x|^{2} \left( \frac{2}{1-|x|^{2}} \right)^{q_{n+1}}
\]

By direct calculation and integration by parts, LHS of (3.11) simplifies as

\[
\text{LHS} = -\int_{\partial B_{n}} (\nabla v_{n}) (\nabla w_{n} \vec{n}) + \frac{1}{2} \int_{\partial B_{n}} \left| \nabla v_{n} \right|^{2} (x.\vec{n}) + \frac{2-N}{2} \int_{\partial B_{n}} \left| \nabla v_{n} \right|^{2}
\]

\[
\frac{\tilde{\lambda}N}{2} \int_{B_{n}} \left( \frac{2}{1-|x|^{2}} \right)^{2} v_{n}^{2} + \frac{\tilde{\lambda}}{2} \int_{B_{n}} \left( \frac{2}{1-|x|^{2}} \right)^{3} |x|^{2} v_{n}^{2}
\]

\[
-\frac{\tilde{\lambda}}{2} \int_{\partial B_{n}} \left( \frac{2}{1-|x|^{2}} \right)^{2} v_{n}^{2} (x.\vec{n})
\]

(3.13)
Now from the equation (3.4) we have

\[
\int_{B_n} |\nabla v_n|^2 \, dx - \tilde{\lambda} \int_{B_n} \left( \frac{2}{1 - |x|^2} \right)^2 v_n^2 \, dx = \int_{B_n} |v_n|^{p_n} \left( \frac{2}{1 - |x|^2} \right)^{q_n} + \int_{\partial B_n} (\nabla v \cdot \vec{n}) v \quad (3.14)
\]

Substituting (3.12) and (3.13) in (3.11) and using (3.14), we get

\[
\left( \frac{N}{p_n} - \frac{N}{2n} \right) \int_{B_n} |v|^{p_n} \left( \frac{2}{1 - |x|^2} \right)^{q_n} (x, \vec{n}) + \frac{N}{2n} \int_{\partial B_n} (\nabla v_n, \vec{n}) v_n + \int_{\partial B_n} (\nabla v_n, x)(\nabla v_n, \vec{n}) - \frac{1}{2} \int_{B_n} |\nabla v_n|^2 (x, \vec{n}) - \lambda \int_{B_n} \left( \frac{2}{1 - |x|^2} \right)^{q_n} |v_n|^{p_n} \cdot \vec{n}^2 \quad (3.15)
\]

Ignoring the positive term on the left and the negative term on the right we get

\[
\tilde{\lambda} \int_{B_n} \left( \frac{2}{1 - |x|^2} \right)^2 |v_n|^2 \leq \frac{1}{p_n} \int_{\partial B_n} |v_n|^{p_n} \left( \frac{2}{1 - |x|^2} \right)^{q_n} (x, \vec{n}) + \frac{N}{2n} \int_{\partial B_n} (\nabla v_n, \vec{n}) v_n + \int_{\partial B_n} (\nabla v_n, x)(\nabla v_n, \vec{n}) - \frac{1}{2} \int_{\partial B_n} |\nabla v_n|^2 (x, \vec{n}) + \frac{\tilde{\lambda}}{2} \int_{\partial B_n} \left( \frac{2}{1 - |x|^2} \right)^2 v_n^2 (x, \vec{n}) \quad (3.16)
\]

Using Lemma 3.4 we can easily show that the RHS of (3.16) is less than or equal to \( C_1 \epsilon_n^2 \) for some \( C_1 \) independent of \( n \). Also using the decomposition (3.7) we can see that LHS \( \geq C_2 \epsilon_n^2 \). We omit the details as the proof is exactly the same as the one given in the proof of Lemma 6.1 in [9]. Thus \( \epsilon_n^2 \leq C \epsilon_n^2 \) where \( C \) is independent of \( n \). This is impossible if \( N \geq 7 \). This completes the proof of Theorem 3.2.

4 Nonexistence

In this section we will prove Theorem 1.1. The proof is based on the Pohozaev identity. The difficulty of applying this identity is because of blowing
up nature of the Riemannian metric on the boundary of the Hyperbolic ball model. So we need to have some decay estimate on the solution of the equation (1.1) to counter the blow up nature of the Hyperbolic metric on the boundary. We will use the asymptotic estimate derived in Section 2 to show the nonexistence of the solution.

First we will convert the equation of the Hyperbolic ball model to Euclidean Ball model by multiplying with the Conformal factor. If \( u \) solves (1.1), then \( v = \left( \frac{2}{1 - |x|^2} \right)^\frac{N-2}{2} u \) solves the Euclidean equation

\[
- \Delta v - \tilde{\lambda} \left( \frac{2}{1 - |x|^2} \right)^2 v = |v|^{2^*-2}v, \quad v \in H^1_0(B^N) \tag{4.1}
\]

where \( \tilde{\lambda} = (\lambda - \frac{N(N-2)}{4}) \). We will show that for \( \tilde{\lambda} \leq 0 \) i.e., \( \lambda \leq \frac{N(N-2)}{4} \) has no solution for the Eq.(4.1). When \( \tilde{\lambda} = 0 \) from the standard Pohozaev identity we know that the equation has no solution. So it is enough to consider the case when \( \tilde{\lambda} < 0 \). Before proving the theorem we will establish a gradient estimate.

For \( \varepsilon > 0 \) define \( A_\varepsilon := \{ x \in \mathbb{B}^N : 1 - 2\varepsilon < |x| < 1 - \varepsilon \} \)

**Proposition 4.1.** If \( v \) satisfies Eq.(4.1) with \( \tilde{\lambda} \leq 0 \), then

\[
\int_{A_\varepsilon} |\nabla v|^2 = O(\varepsilon^\alpha) \tag{4.2}
\]

where \( \alpha \) is a constant strictly greater than 1

**Proof.** For \( \varepsilon > 0 \) we define a smooth function

\[
\psi_\varepsilon(x) = \begin{cases} 
1 & \text{if } 1 - 2\varepsilon < |x| \leq 1 - \varepsilon \\
0 & \text{if } |x| \in [1 - 3\varepsilon, 1 - \varepsilon]^c 
\end{cases}
\]

such that \(|\Delta \psi_\varepsilon(x)| \leq \frac{1}{\varepsilon^2} \).

Since \( v \) is a solution to the Eq.(4.1), then \( v \) is smooth away from the boundary of the Euclidean Ball and hence \( \psi_\varepsilon v \in C^2(B^N) \).Multiplying Eq. (4.1) by this test function and integrating by parts, we get

\[
\int_{B^N} \nabla v \nabla (\psi_\varepsilon v) - \tilde{\lambda} \int_{B^N} \left( \frac{2}{1 - |x|^2} \right)^2 \psi_\varepsilon v^2 = \int_{B^N} |v|^{2^*} \psi_\varepsilon \tag{4.3}
\]
By expanding we have
\[
\int_{1-3\varepsilon<|x|<1-\frac{3}{2}} (\nabla v \cdot \nabla \psi \varepsilon) v + \int_{1-3\varepsilon<|x|<1-\frac{3}{2}} |\nabla v|^2 \psi \varepsilon \\
\leq \int_{1-3\varepsilon<|x|<1-\frac{3}{2}} |v|^2 \psi \varepsilon + |\tilde{\lambda}| \int_{1-3\varepsilon<|x|<1-\frac{3}{2}} \left( \frac{2}{1-|x|^2} \right)^2 \psi \varepsilon v^2
\]
Rearranging the terms we have
\[
\int_{1-3\varepsilon<|x|<1-\frac{3}{2}} |\nabla v|^2 \psi \varepsilon \leq \int_{1-3\varepsilon<|x|<1-\frac{3}{2}} |v|^2 \psi \varepsilon \\
+ |\tilde{\lambda}| \int_{1-3\varepsilon<|x|<1-\frac{3}{2}} \left( \frac{2}{1-|x|^2} \right)^2 \psi \varepsilon v^2 - \int_{1-3\varepsilon<|x|<1-\frac{3}{2}} \nabla \left( \frac{1}{2} v^2 \right) \nabla \psi \varepsilon \\
= \int_{1-3\varepsilon<|x|<1-\frac{3}{2}} |v|^2 \psi \varepsilon + |\tilde{\lambda}| \int_{1-3\varepsilon<|x|<1-\frac{3}{2}} \left( \frac{2}{1-|x|^2} \right)^2 \psi \varepsilon v^2 \\
+ \int_{1-3\varepsilon<|x|<1-\frac{3}{2}} \left( \frac{1}{2} v^2 \right) \Delta \psi \varepsilon
\]
Then clearly we have
\[
\int_{A_\varepsilon} |\nabla v|^2 \leq \int_{1-3\varepsilon<|x|<1-\frac{3}{2}} |v|^2 + |\tilde{\lambda}| \int_{1-3\varepsilon<|x|<1-\frac{3}{2}} \left( \frac{2}{1-|x|^2} \right)^2 v^2 \\
+ \frac{c}{\varepsilon^2} \int_{1-3\varepsilon<|x|<1-\frac{3}{2}} v^2
\]
Now use the estimates on \( v \) from Section 2 to conclude
\[
\int_{A_\varepsilon} |\nabla v|^2 \leq O(\varepsilon^\alpha)
\]
where \( \alpha > 1 \).

\[\square\]

**Proof of Theorem 1.1.** We will prove the theorem using the Pohozaev identity. To make the test function smooth we introduce cut-off functions so that we are away from the boundary and then pass to the limit with the help of the asymptotic estimate proved.

For \( \varepsilon > 0 \), we define
\[
\varphi \varepsilon (x) = \begin{cases} 
1 & \text{if } |x| \leq 1 - 2\varepsilon \\
0 & \text{if } |x| \geq 1 - \varepsilon
\end{cases}
\]
Assume that (4.1) has a nontrivial solution \( v \), then \( v \) is smooth away from the boundary of the Euclidean Ball and hence \( (x \cdot \nabla) \varphi \varepsilon \in C^2_c (\mathbb{B}^N) \). Multiplying Eq. (4.1) by this test function and integrate by parts, we get
\[
\int_{\mathbb{B}^N} \nabla v \cdot \nabla ((x \cdot \nabla) \varphi \varepsilon ) + |\tilde{\lambda}| \int_{\mathbb{B}^N} \left( \frac{2}{1-|x|^2} \right)^2 v (x \cdot \nabla) \varphi \varepsilon
\]

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Now consider
\[
\text{LHS} = \int_{B_N} |v|^{2^*-2} (x, \nabla v) \varphi_\epsilon
\]
which is given by
\[
\begin{align*}
&= \int_{B_N} |v|^{2^*-2}(x, \nabla v) \varphi_\epsilon \\
&= \frac{1}{2^*} \int_{B_N} (\nabla(|v|^{2^*}) x) \varphi_\epsilon \\
&\quad - \frac{N}{2^*} \int_{B_N} |v|^{2^*} \varphi_\epsilon - \frac{1}{2^*} \int_{B_N} |v|^{2^*} [x, \nabla \varphi_\epsilon]
\end{align*}
\]

Using the monotone convergence theorem we get
\[
\lim_{\epsilon \to 0} -\frac{N}{2^*} \int_{B_N} |v|^{2^*} \varphi_\epsilon = -\frac{N}{2^*} \int_{B_N} |v|^{2^*} \tag{4.5}
\]

To estimate the 2nd term of RHS we need to use the estimate on \( v \) for \( \tilde{\lambda} \leq 0 \) which is given by
\[
|v(x)| \leq C_1[(1 - |x|^2) \frac{1+\sqrt{1-4\lambda}}{2} + (1 - |x|^2)^2] \tag{4.6}
\]

Now consider
\[
\left| \frac{1}{2^*} \int_{B_N} |v|^{2^*} [x, \nabla \varphi_\epsilon] \right| \leq \frac{c}{\epsilon} \int_{1-2\epsilon < |x| < 1-\epsilon} |v|^{2^*} \leq \frac{c}{\epsilon} |v|^{2^*} \epsilon^\alpha \tag{4.7}
\]

where \( \alpha > 1 \). Then letting \( \epsilon \to 0 \) in the above we have
\[
\lim_{\epsilon \to 0} R\text{H}S = -\frac{N}{2^*} \int_{B_N} |v|^{2^*} \tag{4.8}
\]

By direct calculation and integration by parts, LHS of (4.4) simplifies as
\[
\begin{align*}
\text{LHS} &= \int_{B_N} |\nabla v|^2 \varphi_\epsilon + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{2} \int_{B_N} (v_{x_i})^2 x_j \varphi_\epsilon x_j + \int_{B_N} \langle x, \nabla v \rangle \langle \nabla v, \nabla \varphi_\epsilon \rangle \\
&\quad + 2 \tilde{\lambda} \int_{B_N} \left( \frac{2}{1 - |x|^2} \right)^2 v^2 \varphi_\epsilon - |\tilde{\lambda}| \int_{B_N} \left( \frac{2}{1 - |x|^2} \right)^3 v^2 \varphi_\epsilon \\
&\quad - \frac{|\tilde{\lambda}| N}{2} \int_{B_N} \left( \frac{2}{1 - |x|^2} \right)^2 v^2 \varphi_\epsilon - |\tilde{\lambda}| \int_{B_N} \langle x, \nabla \varphi_\epsilon \rangle \left( \frac{2}{1 - |x|^2} \right)^2 v^2 \\
&\quad = -\frac{N-2}{2} \int_{B_N} |\nabla v|^2 \varphi_\epsilon - \frac{1}{2} \int_{B_N} |\nabla v|^2 \langle x, \nabla \varphi_\epsilon \rangle + \int_{B_N} \langle x, \nabla v \rangle \langle \nabla v, \nabla \varphi_\epsilon \rangle \\
&\quad + 2 \tilde{\lambda} \int_{B_N} \left( \frac{2}{1 - |x|^2} \right)^2 v^2 \varphi_\epsilon - |\tilde{\lambda}| \int_{B_N} \left( \frac{2}{1 - |x|^2} \right)^3 v^2 \varphi_\epsilon \\
&\quad - \frac{|\tilde{\lambda}| N}{2} \int_{B_N} \left( \frac{2}{1 - |x|^2} \right)^2 v^2 \varphi_\epsilon - |\tilde{\lambda}| \int_{B_N} \langle x, \nabla \varphi_\epsilon \rangle \left( \frac{2}{1 - |x|^2} \right)^2 v^2
\end{align*}
\]
Using the monotone convergence theorem we get
\[
\lim_{\varepsilon \to 0} - \frac{N - 2}{2} \int_{B_N} |\nabla v| \varphi_\varepsilon = - \frac{N - 2}{2} \int_{B_N} |\nabla v| \tag{4.9}
\]
Using the monotone convergence theorem we get
\[
\lim_{\varepsilon \to 0} 2 |\tilde{\lambda}| \int_{B_N} \left( \frac{2}{1 - |x|^2} \right)^2 v^2 \varphi_\varepsilon = 2 |\tilde{\lambda}| \int_{B_N} \left( \frac{2}{1 - |x|^2} \right)^2 v^2 \tag{4.10}
\]
Again using the monotone convergence theorem
\[
\lim_{\varepsilon \to 0} - |\tilde{\lambda}| \int_{B_N} \left( \frac{2}{1 - |x|^2} \right)^3 v^2 \varphi_\varepsilon = - |\tilde{\lambda}| \int_{B_N} \left( \frac{2}{1 - |x|^2} \right)^3 v^2 \tag{4.11}
\]
Similarly again using the monotone convergence theorem
\[
\lim_{\varepsilon \to 0} - \frac{|\tilde{\lambda}| N}{2} \int_{B_N} \left( \frac{2}{1 - |x|^2} \right)^2 v^2 \varphi_\varepsilon = - \frac{|\tilde{\lambda}| N}{2} \int_{B_N} \left( \frac{2}{1 - |x|^2} \right)^2 v^2 \tag{4.12}
\]
Now consider the term
\[
\left| - \frac{|\tilde{\lambda}|}{2} \int_{B_N} \langle x, \nabla \varphi_\varepsilon \rangle \left( \frac{2}{1 - |x|^2} \right)^2 v^2 \right| \leq \frac{C}{\varepsilon} \int_{1 - 2\varepsilon < |x| < 1 - \varepsilon} \left( \frac{2}{1 - |x|^2} \right)^2 v^2
\]
\[
\leq \frac{C}{\varepsilon} \int_{1 - 2\varepsilon < |x| < 1 - \varepsilon} \left( \frac{2}{1 - |x|^2} \right)^2 \left( (1 - |x|)^{1 + \sqrt{1 - 4\tilde{\lambda}}} + (1 - |x|)^4 \right) \leq \frac{C}{\varepsilon} \varepsilon^\alpha
\]
where \(\alpha > 1\). Then letting \(\varepsilon \to 0\) in the above we get
\[
\left| - \frac{|\tilde{\lambda}|}{2} \int_{B_N} \langle x, \nabla \varphi_\varepsilon \rangle \left( \frac{2}{1 - |x|^2} \right)^2 v^2 \right| \to 0 \tag{4.13}
\]
Now consider the remaining term
\[
\left| \int_{B_N} \langle x, \nabla v \rangle \langle \nabla v, \nabla \varphi_\varepsilon \rangle \right| \leq \int_{B_N} |\nabla v|^2 |\nabla \varphi_\varepsilon|
\leq \frac{C}{\varepsilon} \int_{1 - 2\varepsilon < |x| < 1 - \varepsilon} |\nabla v|^2
\]
Hence by the gradient estimate the above term goes to zero as \(\varepsilon \to 0\).

Using (4.9), (4.10), (4.11), (4.12) and (4.13) we have
\[
\lim_{\varepsilon \to 0} \text{[LHS]} = - \frac{N - 2}{2} \int_{B_N} |\nabla v|^2 + 2 \tilde{\lambda} \int_{B_N} \left( \frac{2}{1 - |x|^2} \right)^2 v^2
\]
\[
- |\tilde{\lambda}| \int_{B_N} \left( \frac{2}{1 - |x|^2} \right)^3 v^2 - \frac{|\tilde{\lambda}| N}{2} \int_{B_N} \left( \frac{2}{1 - |x|^2} \right)^2 v^2 \tag{4.14}
\]
Substituting (4.8) and (4.14) in (4.4), and using Eq. (4.1), we get
\[
- 4 \tilde{\lambda} \int_{B_N} \left( \frac{1 + |x|^2}{1 - |x|^2} \right) v^2 = 0 \tag{4.15}
\]
which implies \(v = 0\).
5 Existence

In this section we will prove Theorem 1.3. In view of Corollary 3.3, it is enough to construct infinitely many solutions for subcritical problems which are bounded in $H^1_r(\mathbb{B}^N)$. Infinitely many solutions for the subcritical problem have been established in [6], however we do not have any idea about their boundedness. In this section we will prove the existence of sign changing solutions for the subcritical problem with an estimate on the Morse index from below by applying the abstract theorem of Schechter and Zou [15].

We fix $p_0 \in (2, 2^*)$ and choose a sequence $p_n \in (p_0, 2)$ such that $p_n \to 2^*$. Consider the problem

$$-\Delta_{B^N} u = \lambda u + |u|^{p_n-2} u, \quad u \in H^1_r(\mathbb{B}^N)$$

then we have :

**Theorem 5.1.** Fix $\lambda \in \left(\frac{N(N-2)}{4}, \frac{(N-1)^2}{2}\right)$, then for every $n$ the Equation (5.1) has infinitely many radial sign changing solutions $\{u^n_k\}_{n=1}^{\infty}$ such that for each $k$, the sequence $\{u^n_k\}_{n=1}^{\infty}$ is bounded in $H^1_r(\mathbb{B}^N)$ and the augmented Morse index of $u^n_k$ on the space $H^1_r(\mathbb{B}^N)$ is greater than or equal to $k$.

To prove the theorem we have to show that the functional

$$J_{n, \lambda}(u) = \frac{1}{2} \int_{\mathbb{B}^N} |\nabla u|^2 dV_{\mathbb{B}^N} - \frac{\lambda}{2} \int_{\mathbb{B}^N} u^2 dV_{\mathbb{B}^N} - \frac{1}{p_n} \int_{\mathbb{B}^N} |u|^{p_n} dV_{\mathbb{B}^N}$$

defined on $H^1(\mathbb{B}^N)$ has infinitely many critical points $\{u^n_k\}_{n=1}^{\infty}$. Because of the principle of symmetric criticality ([11]), enough to find the critical points of $J_{n, \lambda}$ on $H^1_r(\mathbb{B}^N)$. The augmented Morse index of $u^n_k$ on the space $H^1_r(\mathbb{B}^N)$ is the dimension of the largest subspace of $H^1_r(\mathbb{B}^N)$ where $J''_{n, \lambda}(u^n_k)$ is non-positive definite. We will prove the theorem by working with its conformal version (3.4). We will show that the functional,

$$G_{n, \lambda}(v) = \frac{1}{2} \int_{\mathbb{B}^N} |v|^2 - \frac{\hat{\lambda}}{2} \int_{\mathbb{B}^N} \left( \frac{2}{1 - |x|^2} \right)^2 v^2 - \frac{1}{p_n} \int_{\mathbb{B}^N} |v|^{p_n} \left( \frac{2}{1 - |x|^2} \right)^{\eta_n}$$

defined on $H^1_{0,r}(\mathbb{B}^N)$ satisfies all the assumptions of Theorem 2 of [15] where $\hat{\lambda}$ is as in (3.4).

Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots < \lambda_k \leq \ldots$ be the eigen values of $-\Delta$ on $H^1_{0,r}(\mathbb{B}^N)$ and $\varphi_k(x)$ be the eigen functions corresponding to $\lambda_k$. Denote $E_k := \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_k\}$. Then $H^1_{0,r}(\mathbb{B}^N) = \bigcup_{k=1}^{\infty} E_k$, dim$E_k = k$ and $E_k \subset E_{k+1}$.

For each $p_n \in (2, 2^*)$, we define

$$||u||_* = \left[ \int_{\mathbb{B}^N} |u|^{p_n} \left( \frac{2}{1 - |x|^2} \right)^{\eta_n} \right]^{\frac{1}{p_n}}, \quad u \in H^1_{0,r}(\mathbb{B}^N)$$
then from (6.2) we get \(||v||_* \leq C||v||\) for all \(v \in H^1_{0,r}(\mathbb{B}^N)\) for some constant \(C > 0\). Moreover using the radial estimate the embedding of \(H^1_{0,r}(\mathbb{B}^N)\) in to \((H^1_{0,r}(\mathbb{B}^N), ||.||_*)\) is compact.

Now define,
\[P := \{ v \in H^1_{0,r}(\mathbb{B}^N) : v \geq 0 \}\]
Also for \(\mu > 0\), define
\[D(\mu) := \{ v \in H^1_{0,r}(\mathbb{B}^N) : \text{dist}(v, P) < \mu \}, \quad D^* := D(\mu) \cup (-D(\mu)).\]

Also denote the set of all critical points by
\[K^\lambda_n := \{ v \in H^1_{0,r}(\mathbb{B}^N) : G^\prime_n(v) = 0 \}\]

Clearly \(G_n,\lambda \in C^2((H^1_{0,r}(\mathbb{B}^N), \mathbb{R})\) is an even functional which maps bounded sets to bounded sets in terms of the norm \(||.||\). The gradient \(G^\prime_n,\lambda\) is of the form \(G^\prime_n,\lambda(v) = v - K_n,\lambda(v)\), where \(K_n,\lambda : E \to E\) is a continuous operator.

Moreover

**Proposition 5.2.** For any \(\mu_0 > 0\) small enough, we have that \(K_n,\lambda(D(\mu_0)) \subset D(\mu) \subset D(\mu_0)\) for some \(\mu \in (0, \mu_0)\) for each \(n, \lambda\) with \(\frac{N(N-2)}{4} < \lambda < \frac{(N-1)^2}{4}\). Moreover, \(D(\mu_0) \cap K^\lambda_n \subset P\).

**Proof.** First note that \(K_n,\lambda(v)\) can be decomposed as \(K_n,\lambda(v) = L(v) + W(v)\) where \(L(v), W(v) \in E\) are the unique solutions of the equations
\[-\Delta(L(v)) = \tilde{\lambda}v \left( \frac{2}{1 - |x|^2} \right)^2 \quad \text{and} \quad -\Delta(W(v)) = |v|^{p_n - 2}v \left( \frac{2}{1 - |x|^2} \right)^{q_n}.\]

In other words, \(L(v)\) and \(W(v)\) are uniquely determine by the relations
\[
\langle Lv, u \rangle = \tilde{\lambda} \int_{\mathbb{B}^N} uv \left( \frac{2}{1 - |x|^2} \right)^2, \quad \langle W(v), u \rangle = \int_{\mathbb{B}^N} |v|^{p_n - 2}vu \left( \frac{2}{1 - |x|^2} \right)^{q_n} \quad (5.2)
\]

By Maximum Principle, \(L(v) \in P\) and \(W(v) \in P\) if \(v \in P\).

Now we will estimate \(||L(v)||\). We have
\[
\langle Lv, Lv \rangle = \tilde{\lambda} \int_{\mathbb{B}^N} vLv \left( \frac{2}{1 - |x|^2} \right)^2 dx \\
\leq \tilde{\lambda} \left( \int_{\mathbb{B}^N} \left( \frac{2}{1 - |x|^2} \right)^2 v^2 \right)^\frac{1}{2} \left( \int_{\mathbb{B}^N} \left( \frac{2}{1 - |x|^2} \right)^2 |Lv|^2 \right)^\frac{1}{2} \\
\leq 4\tilde{\lambda}||v||||Lv||
\]
	hanks{See Lemma (5.2). Thus \(||Lv|| \leq 4\tilde{\lambda}||v||\) where \(4\tilde{\lambda} < 1\). Let \(v \in H^1_{0,r}(\mathbb{B}^N)\) and \(u \in P\) be such that \(\text{dist}(v, P) = ||u - v||\), then
\[
\text{dist}(Lv, P) \leq ||Lv - Lu|| \leq 4\tilde{\lambda}||v - u|| \leq 4\tilde{\lambda}\text{dist}(v, P) \quad (5.3)
\]
Proposition 5.4. For any $\alpha$ and $G$, where $V$.

The proposition follows since

Proof. Since $E = \int |v|^{p_n-2}uW(v)^{-} \leq \|W(v)^{-}\| \leq \|W(v)\|^2 \leq \langle W(v), W(v)^{-}\rangle$

$$= \int_{B_N} |v|^{p_n-2}uW(v)^{-} \left( \frac{2}{1 - |x|^2} \right)^{q_n} \leq \int_{B_N} |v^{-}|^{p_n-1}W(v)^{-} \left( \frac{2}{1 - |x|^2} \right)^{q_n}$$

$$\leq \left( \int_{B_N} |v^{-}|^{p_n} \left( \frac{2}{1 - |x|^2} \right)^{q_n} \right) \left( \frac{p_n-1}{p_n} \right) \left( \int_{B_N} W(v)^{p_n} \left( \frac{2}{1 - |x|^2} \right)^{q_n} \right) \frac{1}{p_n}$$

$$\leq C \left( \int_{B_N} |v^{-}|^{p_n} \left( \frac{2}{1 - |x|^2} \right)^{q_n} \right) \left( \frac{p_n-1}{p_n} \right) \|W(v)^{-}\|$$

Now using

$$\int_{B_N} |v^{-}|^{p_n} \left( \frac{2}{1 - |x|^2} \right)^{q_n} = \min_{u \in P} \int_{B_N} |v - u|^{p_n} \left( \frac{2}{1 - |x|^2} \right)^{q_n} \leq C \min_{u \in P} ||v - u||$$

we get

$$\text{dist}(W(v), P) \leq C[\text{dist}(v, P)]^{p_n-1} \forall v \in H^1_0(B_N)$$

Choose $4\lambda < \nu < 1$. Then there exists $\mu_0$ such that, if $\mu \leq \mu_0$,

$$\text{dist}(W(v), P) \leq (\nu - 4\lambda) \text{dist}(v, P) \text{ for all } v \in D(\mu). \tag{5.4}$$

Fix $\mu \leq \mu_0$. Inequalities (5.3) and (5.4) yield

$$\text{dist}(K_{n, \lambda}(v), P) \leq \text{dist}(L(v), P) + \text{dist}(W(v), P) \leq \nu \text{dist}(v, P)$$

for all $v \in D(\mu)$. This proves the Proposition.

Proposition 5.3. For each $k$, $\lim_{||v|| \to \infty, v \in E_k} G_{n, \lambda}(v) = -\infty$

Proof: Since $E_k$ is finite dimensional, there exists a constant $C > 0$ such that $||v|| \leq C||v||_*$ for all $v \in E_k$. Thus

$$G_{n, \lambda}(v) \leq \frac{1}{2}||v||^2 - C||v||^{p_n}, \forall v \in E_k$$

Since $p_n > 2$, we have $\lim_{||v|| \to \infty, v \in E_k} G_{n, \lambda}(v) = -\infty$.

Proposition 5.4. For any $\alpha_1, \alpha_2 > 0$, there exist an $\alpha_3$ depending on $\alpha_1$ and $\alpha_2$ such that $||v|| \leq \alpha_3$ for all $v \in G_{n, \lambda}^\alpha \cap \{v \in H^1_0(B_N) : ||v||_* \leq \alpha_2\}$ where $G_{n, \lambda}^\alpha = \{v \in H^1_0(B_N) : G_{n, \lambda} \leq \alpha_1\}$

Proof. The proposition follows since

$$\frac{1}{2} - 4\lambda ||v||^2 \leq G_{n, \lambda}(v) + \frac{1}{p_n} ||v||^{p_n}$$
Proof of Theorem 5.1 The above discussions and Propositions 5.2, 5.3, 5.4 tells us that $G_{n,\lambda}$ satisfies all the conditions of Theorem 2 in [15]. Thus $G_{n,\lambda}$ has a sign changing critical point $v_k^n \in H^1_{0,r}(\mathbb{B}^N)$ at a level $C(n, \lambda, k)$ and $C(n, \lambda, k) \leq \sup_{E_{k+1}} G_{n,\lambda}$ and the augmented Morse index $m^*(v_k^n)$ of $v_k^n$ is $\geq k$. We claim that:

**Claim**: There exists a constant $T_1 > 0$ independent of $k$ and $n$ such that

$$\sup_{E_{k+1}} G_{n,\lambda} \leq T_1 \lambda_k^{*(\rho_0-\frac{2}{p_0})}$$

**Proof of claim**: The definition of $E_{k+1}$ implies that $\|v\|^2 \leq \lambda_{k+1} \|v\|_2^2$. Note that with $p_n > p_0$, we have $\|v\|_{p_0} \leq D_1 \|v\|_{p_n}$, where $D_1 > 0$ is a constant independent of $n$ and $k$. Therefore,

$$G_{n,\lambda}(v) \leq \frac{1}{2} \int_{\mathbb{B}^N} |\nabla v|^2 - \frac{\lambda}{2} \int_{\mathbb{B}^N} \left( \frac{2}{1-|x|^2} \right)^2 v^2 - \frac{1}{p_n} \int_{\mathbb{B}^N} |v|^{p_n}$$

$$\leq \frac{1}{2} \int_{\mathbb{B}^N} |\nabla v|^2 - \frac{1}{p_n} \int_{\mathbb{B}^N} |v|^{p_n}$$

$$\leq \frac{1}{2} \int_{\mathbb{B}^N} |\nabla v|^2 - D_2 \int_{\mathbb{B}^N} |v|^{p_0} + D_3$$

where $D_2 > 0, D_3 > 0$ are constant, independent of $n$ and $k$. Since there exist a constant $D_4 > 0$ such that $\|v\|_2 \leq D_4 \|v\|_{p_0}$, therefore we may have $D_5 > 0$ such that $\|v\|_{p_0} \leq D_5 \lambda_{k+1}^{p_0/2} \|v\|_{p_0}$ for all $v \in E_{k+1}$. Then

$$G_{n,\lambda}(v) \leq \frac{1}{2} \|v\|^2 - D_6 \lambda_{k+1}^{-p_0/2} \|v\|_{p_0} + D_3$$

$$\leq D_7 \lambda_{k+1}^{*(p_0-2)} \|v\|_{p_0} + D_3$$

$$\leq T_1 \lambda_{k+1}^{*(\rho_0-2)}$$

where $D_i (i = 1, \ldots, 7)$ and $T_1$ are positive constants independent of $k$ and $n$.

Also note that energy of any critical point of $G_{n,\lambda}$ is positive. Thus $G_{n,\lambda}(v_k^n) \in [0, T_1 \lambda_{k+1}^{2(p_0-2)}]$. This immediately implies that the sequence $\{v_k^n\}_{n=1}^\infty$ is bounded in $H^1_0(\mathbb{B}^N)$ for each $k$. Now $u_k^n = \left( \frac{4-|x|^2}{2} \right)^{\frac{N-2}{2}} v_k^n$ satisfies all the conclusions of Theorem 5.1. Moreover $J_{n,\lambda}(u_k^n) = G_{n,\lambda}(v_k^n)$.

Proof of Theorem 1.3 Using Corollary 3.3 and Theorem 5.1 we get a sequence $\{u_k\}_{k=1}^\infty$ of solutions of our original problem with energy $C(\lambda, k) \in [0, T_1 \lambda_{k+1}^{2(p_0-2)}]$. It remains to show that infinitely many $u_k$’s are different. This follows if we show that the energy of $u_k$ goes to infinity as $k \to \infty$. Suppose not, then $\lim_{k \to \infty} C(\lambda, k) = c' < \infty$. For any $k \in \mathbb{N}$ we may find an $n_k (\text{assume } n_k > k)$ such that $|C(n_k,\lambda,k) - C(\lambda, k)| < \frac{1}{k}$. It follows
that \( \lim_{k \to \infty} C(n_k, \lambda, k) = \lim_{k \to \infty} C(\lambda, k) = c' < \infty \). Hence, \( \{u_k^{n_k}\}_{k \in \mathbb{N}} \) is bounded in \( H^1_B(\mathbb{B}^N) \) and hence satisfies the uniform bound given by Theorem 3.2. Therefore the augmented Morse index of \( u_k^{n_k} \) remains bounded which contradicts the fact that the augmented Morse index of \( u_k^{n_k} \) is greater than or equal to \( k \). Thus \( \lim_{k \to \infty} C(\lambda, k) = \infty \) and hence infinitely many \( u_k \)'s are different. Moreover they are sign changing as the radial positive solutions are unique (see [10], Theorem 1.3). This completes the proof.

6 Appendix

Let \( \mathbb{B}^N := \{ x \in \mathbb{R}^N : |x| < 1 \} \) denotes the unit disc in \( \mathbb{R}^N \). The space \( \mathbb{B}^N \) endowed with the Riemannian metric \( g \) given by

\[
g_{ij} = \left( \frac{2}{1 - |x|^2} \right)^2 \delta_{ij} \]

is called the ball model of the Hyperbolic space. For more details on hyperbolic geometry we refer to [12].

We will denote the associated hyperbolic volume by \( dV_{\mathbb{B}^N} \) and is given by

\[
dV_{\mathbb{B}^N} = \left( \frac{2}{1 - |x|^2} \right)^N dx.
\]

The hyperbolic gradient \( \nabla_{\mathbb{B}^N} \) and the hyperbolic Laplacian \( \Delta_{\mathbb{B}^N} \) are given by

\[
\nabla_{\mathbb{B}^N} = \left( \frac{1 - |x|^2}{2} \right)^2 \nabla, \quad \Delta_{\mathbb{B}^N} = \left( \frac{1 - |x|^2}{2} \right)^2 \Delta + \left( N - 2 \right) \frac{1 - |x|^2}{2} < x, \nabla >
\]

Let \( H^1(\mathbb{B}^N) \) denotes the Sobolev space on \( \mathbb{B}^N \) with the above metric \( g \), then we have \( H^1(\mathbb{B}^N) \hookrightarrow L^p(\mathbb{B}^N) \) for \( 2 \leq p \leq \frac{2N}{N-2} \) when \( N \geq 3 \) and \( p \geq 2 \) when \( N = 2 \). In fact we have the following Poincaré-Sobolev inequality (see [10]):

For every \( N \geq 3 \) and every \( p \in (2, \frac{2N}{N-2}] \) there is an optimal constant \( S_{N,p,\lambda} > 0 \) such that

\[
S_{N,p,\lambda} \left( \int_{\mathbb{B}^N} |u|^p dV_{\mathbb{B}^N} \right)^{\frac{2}{p}} \leq \int_{\mathbb{B}^N} \left[ |\nabla_{\mathbb{B}^N} u|^2 - \frac{(n-1)^2}{4} u^2 \right] dV_{\mathbb{B}^N} \quad (6.1)
\]

for every \( u \in H^1(\mathbb{B}^N) \).

As an immediate consequence we get :

**Lemma 6.1.** For any \( \lambda < \frac{(n-1)^2}{4} \), \(|u|_\lambda \) defined by

\[
||u||^2_\lambda := \int_{\mathbb{B}^N} \left[ |\nabla_{\mathbb{B}^N} u|^2 - \lambda u^2 \right] dV_{\mathbb{B}^N}
\]

is an equivalent norm in \( H^1(\mathbb{B}^N) \).
Making a conformal change of the metric, we can get a Euclidean version of (6.1) on the Euclidean Sobolev space $H^1_0(\mathbb{B}^N)$.

Lemma 6.2. There is an optimal constant $S_{N,p,\lambda} > 0$ such that

$$S_{N,p,\lambda} \left( \int_{\mathbb{B}^N} |v|^p \left( \frac{2}{1-|x|^2} \right)^q dx \right)^{\frac{2}{p}} \leq \int_{\mathbb{B}^N} \left[ |\nabla v|^2 - \frac{1}{4} \left( \frac{2}{1-|x|^2} \right)^2 v^2 \right] dx \quad (6.2)$$

for every $v \in H^1_0(\mathbb{B}^N)$ where $p \in (2, \frac{2N}{N-2}]$ and $q = \frac{2N-p(N-2)}{2}$.

Proof. Put $u = \left( \frac{2}{1-|x|^2} \right)^{-\frac{N-2}{2}} v$ in (6.1) will establish the lemma.

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