Nonequilibrium stationary states of 3D self-gravitating systems

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Three dimensional self-gravitating systems do not evolve to thermodynamic equilibrium, but become trapped in nonequilibrium quasistationary states. In this Letter we present a theory which allows us to a priori predict the particle distribution in a final quasistationary state to which a self-gravitating system will evolve from an initial condition which is isotropic in particle velocities and satisfies a virial constraint $2K = -U$, where $K$ is the total kinetic energy and $U$ is the potential energy of the system.

Unlike systems with short-range forces which relax to thermodynamic equilibrium starting from an arbitrary initial condition, systems with long-range interactions become trapped in nonequilibrium quasistationary states (QSS) the lifetime of which diverges with the number of particles $N$. For interaction potentials unbounded from above, the QSS have been observed to have a characteristic core-halo structure. The extent of the halo is determined by the parametric resonances which arise from the collective density oscillations during the relaxation process. The dynamics of 3D self-gravitating systems, however, is significantly more complex due to the existence of unbound states. Indeed, Newton’s gravitational potential is bounded from above, so that the parametric resonances may actually transfer enough energy to allow some particles to completely escape from the gravitational cluster. This makes the study of 3D self-gravitating systems particularly challenging. Recently, however, it was shown that if the initial particle distribution function is isotropic in velocity and satisfies the VC, relaxation to equilibrium will then proceed adiabatically. In the thermodynamic limit, each particle of the gravitational cluster will evolve under the action of a quasistatic mean-field potential and the phase-mixing of particle trajectories will lead to a nonequilibrium QSS. In this Letter we will show that it is possible to a priori predict the density and the velocity distribution functions within the QSS to which a 3D gravitational system will evolve if the initial distribution is isotropic in particle velocities and satisfies VC.

The virial theorem requires that a stationary gravitational system must have $2K = -U$, where $K$ is the total kinetic energy and $U$ is the potential energy. This, however does not mean that an arbitrary initial distribution which satisfies the VC will remain stationary. To be stationary, a distribution function must have a priori predict the density and the velocity distribution functions within the QSS to which a 3D gravitational system will evolve if the initial distribution is isotropic in particle velocities and satisfies VC.

Consider a spherically symmetric — in both positions and velocities — initial phase space particle distribution. We will work in the thermodynamic limit $N \to \infty$, $m \to 0$, while $mN = M$, where $N$ is the total number of particles, $m$ is the mass of each particle, and $M$ is the total mass of the gravitational system. At $t = 0$ the particles are distributed in accordance with the initial distribution $f_0(r, p)$ inside an infinite 3D configuration space. We would like to predict the distribution function for the system when it relaxes to a QSS. It is easy to see that in the thermodynamic limit the positional correlations between the particles vanish and all the dynamics is controlled by the mean-field potential. Furthermore, if the initial distribution is such that the VC is satisfied, the mean-field potential should vary adiabatically and the energy of each particle should change little. Since the mean-field potential is a nonlinear function of position, the particles on the energy shell $\{\mathcal{E}, \mathcal{E} + d\mathcal{E}\}$ with slightly distinct one-particle energies $\mathcal{E}$ will have incommensurate orbital frequencies. This means that after a transient period, the phase-mixing will result in a uniform particle distribution over the energy shell. The particle distribution in the final QSS can then be obtained by a coarse-graining of the initial distribution over the phase space available to the particle dynamics, taking into account the conservation of the angular momentum of each particle, given the spherical symmetry of the mean-field potential.

Consider an arbitrary initial particle distribution $f_0(r, p)$
that satisfies VC. For $t > 0$ the particles will evolve under the action of an external adiabatically varying potential $\varphi(r, t)$ which will eventually converge to some $\psi(r)$. Our approach will be to construct a coarse-grained distribution for particles evolving directly under the action of the static potential $\psi(r)$ which will then be calculated self-consistently \[31,33\]. Clearly such an approximation will only work if the variation of $\varphi(r, t)$ is adiabatic and no resonances are excited. This is precisely the case for the initial distributions which are isotropic in velocity and satisfy VC \[29\].

Since $\psi(r)$ is static and spherically symmetric, the energy and the angular momentum of each particle will be preserved. The nonlinearity of $\psi(r)$ will lead to phase-mixing of particle trajectories with the same energy and angular momentum. The number of particles with energy between $E$ and $E + dE$ and the square of the angular momentum between $\ell^2$ and $\ell^2 + d\ell^2$ is $n(E, \ell^2) dE d\ell^2$ and is conserved throughout dynamics. In the QSS these particles will spread over the phase space volume $g(E, \ell^2) dE d\ell^2$, so that the coarse-grained distribution function for the QSS will be

$$f(E, \ell^2) = \frac{n(E, \ell^2)}{g(E, \ell^2)} \quad (1)$$

The self-consistent potential $\psi(r)$ must satisfy the Poisson equation,

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\psi}{dr} \right) = 4\pi G m \rho(r) \quad (2)$$

where

$$\rho(r) = \int d^3p \ f \left[ \mathbf{r}(\mathbf{r}, \mathbf{p}), \ell^2(\mathbf{r}, \mathbf{p}) \right] \quad (3)$$

is the asymptotic particle density. This gives us a closed set of equations which can be used to calculate the distribution function in the QSS. To simplify the notation we will scale all the distances to an arbitrary length scale $L_0$, time to $\sqrt{L_0/GM}$, the potential to $GM/L_0$, and the energy to $GM^2/L_0$.

Because of the conservation of the angular momentum of each particle, it is convenient to work with the canonical positions $(r, \theta, \phi)$ and conjugate momenta $(p_r, p_\theta, p_\phi)$. Note that in terms of these variables the invariant phase space measure is $d^3rd^3p = dr d\theta d\phi dp_r dp_\theta dp_\phi$. The particle energy and square modulus of the angular momentum are

$$\epsilon(r, \theta, p_r, p_\theta, p_\phi) = \frac{1}{2} \left( p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) + \psi(r), \quad (4)$$

$$\ell^2(\theta, p_\theta, p_\phi) = p_\phi^2 + \frac{p_\theta^2}{\sin^2 \theta} \quad (5)$$

respectively. The density of states $g(E, \ell^2)$ is

$$g(E, \ell^2) = \int dp_r dp_\theta dp_\phi \int dr d\theta d\phi \delta \left[ \ell^2 - \ell^2(\theta, p_\theta, p_\phi) \right] \times \delta \left[ E - \epsilon(r, \theta, p_r, p_\theta, p_\phi) \right] \quad (6)$$

and the particle phase space density $n(E, \ell^2)$ is

$$n(E, \ell^2) = \int dp_r dp_\theta dp_\phi \int dr d\theta d\phi \delta \left[ \ell^2 - \ell^2(\theta, p_\theta, p_\phi) \right] \times \delta \left[ E - \epsilon(r, \theta, p_r, p_\theta, p_\phi) \right] \times f_0 \left( r, \sqrt{\frac{p_r^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta}} \right). \quad (7)$$

Integration over all the variables in Eqs. (6) and (7), other than $dr$, can be performed with the help of a Dirac delta function identity

$$\delta[f(x)] = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}, \quad (8)$$

where $x_i$ is the $i$’th root of $f(x)$. Carrying out the integration we obtain the coarse-grained distribution function for the QSS,

$$f(E, \ell^2) = \frac{1}{N(r)} \int dr \int \frac{\Theta - \frac{\ell^2}{2} - \psi(r)}{\sqrt{\frac{\Theta - \ell^2 - \psi(r)}{\ell^2(\Theta - \psi(r))}}}, \quad (9)$$

where $\Theta$ is the Heaviside step function. The coarse-grained distribution function depends on position and momentum only through the conserved quantities $E$ and $\ell^2$; therefore, it is automatically a stationary solution of the Vlasov equation.

The Poisson equation can be rewritten as

$$r^2 \frac{d^2 \psi}{dr^2} + 2r \frac{d\psi}{dr} = N(r) \quad (10)$$

where $N(r) = 4\pi r^2 \rho(r)$, or

$$N(r) = \int dp_r dp_\theta dp_\phi \int d\theta d\phi f(E, \ell^2). \quad (11)$$

Multiplying Eq (11) by the identity

$$\int d(\ell^2) \delta \left( \ell^2 - p_\theta^2 - \frac{p_\phi^2}{\sin^2 \theta} \right) = 1, \quad (12)$$

and changing the order of integration, we can write

$$N(r) = \int d(\ell^2) \int dp_r dp_\theta dp_\phi \int d\theta d\phi \delta \left( \ell^2 - p_\theta^2 - \frac{p_\phi^2}{\sin^2 \theta} \right) \times f \left( \frac{p_\phi^2}{2} - \frac{p_\theta^2}{2r^2} + \frac{p_\phi^2}{2r^2 \sin^2 \theta} + \psi(r), p_\phi^2 + \frac{p_\theta^2}{2r^2} \right). \quad (13)$$

The integration over the variables $p_\theta, p_\phi, \theta, \phi$ can now be performed explicitly with the help of Eq. (8). Finally, changing the integration variable from $p_r$ to $E$, Eq. (13) simplifies to

$$N(r) = 8\pi^2 \int_0^\infty d(\ell^2) \int_0^{\infty} dE \int f(E, \ell^2) \frac{\Theta \left[ E - \frac{\ell^2}{2} - \psi(r) \right]}{\sqrt{2 \left( E - \frac{\ell^2}{2} - \psi(r) \right)}} \quad (14)$$
where the lower limit of integration is $\mathcal{E}_0 = \frac{p^2}{2M} + \psi(r)$ and $f(\mathcal{E}, \ell^2)$ is given by Eq. (9). Substituting Eq. (14) into Eq. (10), we find an integro-differential equation for the gravitational potential $\psi(r)$ in the QSS. Eq. (10) can be solved numerically using Picard iteration. Once the gravitational potential is known, the coarse-grained distribution function can be easily calculated by performing the integration in Eq. (9).

We next validated the proposed theory by comparing the marginal position and velocity distribution functions $N(r)$ and $N(p)$ to explicit molecular dynamics (MD) simulations of a 3D self-gravitating system of $N$ particles. The simulations were performed using a version of particle-in-cell (PIC) algorithm, in which each particle interacts with a mean-field potential produced by all other particles. In the absence of ROI this simulations produce identical particle distributions in QSS as calculated using traditional binary interaction methods, but are three orders of magnitude faster. This allows us to easily reach the QSS (34). The density distribution $N(r)$ is given by Eq. (14). To obtain the momentum distribution we first calculate the distribution

$$N(p_r) = \int dr \, dp_\theta \, dp_\phi \int d\theta \, d\phi \, f(\mathcal{E}, \ell^2),$$

where $\mathcal{E} = \frac{p_r^2}{2M} + \frac{p_\theta^2}{2\ell^2} + \psi(r)$ and $\ell^2 = \frac{p_\phi^2}{\sin^2 \theta}$. The change of variable from $p_r$ to the modulus of momentum $p$ can be performed with the help of Eq. (12) and the identity

$$\int dp^2 \, \delta(p^2 - p_r^2 - \ell^2) = 1$$

yielding,

$$N(p) = 8\pi^2 p \int_0^{\infty} dp_\theta \int_0^{\infty} dp_\phi \int d\ell^2 \, f(\mathcal{E}, \ell^2) \, \Theta \left( p^2 - \ell^2 \right) \sqrt{p^2 - \ell^2}$$

where $\mathcal{E} = \frac{p^2}{2M} + \psi(r)$.

We first consider a waterbag initial distribution,

$$f_0(r, p) = \eta \, \Theta \left( r_m^2 - r^2 \right) \, \Theta \left( p_m^2 - p^2 \right).$$

where $\eta = 9/(16\pi^2 r_m^3 p_m^3)$ is the normalization constant. We will measure all the lengths in units of $r_m$, which is equivalent to setting $r_m = 1$. The VC requires that $2K = -U$, where

$$K = \frac{1}{2} \int d^3r \, d^3p \, f_0(r, p) \, p^2$$

is the kinetic energy and

$$U = \frac{1}{2} \int d^3r \, d^3p \, f_0(r, p) \, \psi_0(r)$$

is the potential energy of the system. The potential $\psi_0(r)$ for the initial waterbag distribution is

$$\psi_0(r) = \begin{cases} \frac{r^2 - \frac{3}{2}}{2} & \text{if } r < 1 \\ \frac{1}{r} & \text{if } r \geq 1. \end{cases}$$

Using Eqs (18) and (21) to calculate $K$ and $U$, the VC reduces to $p_m = 1$. In Fig 1 we plot the joint distribution function $f(\mathcal{E}, \ell^2)$ for the QSS.

The marginal distribution functions can be calculated using Eqs (14) and (17) together with Eq. (9). Fig 2 shows the position and velocity distributions $N(r)$ and $N(p)$ predicted by the integrable model. The symbols are the results of MD simulations. The initial $t = 0$ density and momentum distributions are plotted with dashed lines – an initial waterbag distribution is given by Eq (18).

One particularly nice feature of the present theory is that it can be easily used to predict the final QSS for any initial distribution as long as it satisfies VC. We next study a parabolic initial distribution, given by

$$f_0(r, p) = \eta \left( 1 - r^2 \right) \Theta \left( 1 - r^2 \right) \Theta \left( p_m^2 - p^2 \right)$$

with $\eta = 45/(32\pi^2 p_m^3)$. The VC for this distribution is $p_m = 5/\sqrt{21}$. The marginal distributions predicted by the theory are compared with simulations in Fig 3. Once again the agreement is very good. For strongly inhomogeneous initial distributions, VC is not enough to completely prevent the temporal dynamics of the mean-field potential. That is, even if we restrict one moment of the distribution function, other moments might still have sufficiently strong dynamics to excite...
parametric resonances. Indeed, we find that for very strongly inhomogeneous initial distributions, there is some discrepancy between the theory and the simulations. Nevertheless even in these extreme cases the theory remains quite accurate [34].

We have presented a theory that is able to predict the particle distribution in the final QSS to which a 3D self-gravitating system will relax from an initial condition. The theory can be used for initial distributions which are isotropic in particle velocity and satisfy the VC. It is interesting to compare and contrast our approach with the theory of violent relaxation developed by Lynden-Bell (LB). The statistical mechanics of LB theory is very different from the predictions of LB theory [17–19, 36, 37]. It was recently observed, however, that for more complex inhomogeneous or multilevel distributions LB theory failed even when the initial distribution function satisfied VC [29, 38]. The failure of LB theory can now be attributed to the almost complete absence of ergodicity and mixing when the initial distribution satisfies VC. The evolution of the mean-field potential of such systems is almost adiabatic and the dynamics is closer to integrable than to ergodic [29]. The relaxation to QSS is the result of phase-mixing of particles on the same energy shells and not a consequence of ergodicity over the full energy surface. Indeed for 3D gravitational systems LB theory fails to accurately account for either ergodicity or density distributions, as can be seen in Fig. 4 even for the initial virial waterbag distribution, Eq. (18). Furthermore, LB theory is very difficult to extent to more complicated initial conditions than a one-level waterbag distribution, while the present approach can, in principle, be used for any arbitrary distribution as long as it satisfies VC. The goal of the future work will be to extend the theory presented in this Letter to initial distributions which do not satisfy VC. Parametric resonances and particle evaporation, however, make this a very difficult task.

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