On the Cauchy problem for

$$D_t^2 - D_x \left( b(t) a(x) \right) D_x$$

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Abstract

We consider the Cauchy problem for second order differential operators with two independent variables $P = D_t^2 - D_x \left( b(t) a(x) \right) D_x$. Assuming that $b(t)$ is a nonnegative $C^{n,\alpha}$ function and $a(x)$ is a nonnegative Gevrey function of order $1 < s < 1 + (n + \alpha)/2$ we prove that the Cauchy problem for $P$ is well-posed in the Gevrey class of any order $s'$ with $1 < s < s' < 1 + (n + \alpha)/2$.

Keywords: Cauchy problem, well-posedness, Weyl-Hörmander calculus, Gevrey classes.

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1 Introduction

We are concerned with the Cauchy problem for second order differential operators with two independent variables

$$(1.1) \quad P = D_t^2 - D_x \left( b(t) a(x) \right) D_x, \quad D_t = \frac{1}{i} \frac{\partial}{\partial t}, \quad D_x = \frac{1}{i} \frac{\partial}{\partial x}$$

with a nonnegative $b(t) \in C^{n,\alpha}([0, T])$, with $n \geq 0$ integer and $0 \leq \alpha \leq 1$ and a nonnegative Gevrey function $a(x)$ such that

$$(1.2) \quad \begin{cases} Pu = 0 \text{ in } (t, x) \in [0, T'] \times \mathbb{R}, \\ D_t^j u(0, x) = u_j(x) \text{ for } j = 0, 1 \end{cases}$$

where $0 < T' \leq T$. In the special case that $a(x)$ is positive constant the well-posedness in the Gevrey class of order $1 < s' < 1 + (n + \alpha)/2$ is proved in [2]. Moreover this result is optimal in the sense that there exists $0 \leq b(t) \in C^{n,\alpha}([0, T])$ such that the Cauchy problem for $D_t^2 - b(t) D_x^2$ is not well-posed in the Gevrey class of order $s' > 1 + (n + \alpha)/2$ (for more details see [2 Theorem 3]).

We denote by $G^s(\mathbb{R})$ the set of functions which are uniformly Gevrey $s$ on $\mathbb{R}$,

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Remark 1.1. Theorem 1.3.\]

Denote $\mathcal{G}_0^s(\mathbb{R}) = \mathcal{G}^s(\mathbb{R}) \cap C_0^\infty(\mathbb{R})$. In this paper we prove

**Theorem 1.1.** Assume $0 \leq b(t) \in C^{n,\alpha}([0, T])$ and $0 \leq a(x) \in \mathcal{G}^s(\mathbb{R})$. Assume $1 < s < s' < 1 + (n + \alpha)/2$ then the Cauchy problem \([1.2]\) for $P$ is well-posed in $\mathcal{G}^s(\mathbb{R})$. More precisely there exists $\theta_0 > 0$ such that for any $1 < s < s' < 1 + (n + \alpha)/2$ and $0 < \tau (\leq T)$ there is $\mu > 0$ such that for any $u_j \in \mathcal{G}_0^s(\mathbb{R})$ with finite $\sum_{j=0}^1 \| (D)_{\mu}^{1-j} \text{Op}(e^{\tau(\xi)^{s'}}) u_j \|$ and any $0 < \tau' < \tau$ there exists a unique solution $u \in C^1([0, \tau'/\theta_0]; \mathcal{G}_0^s(\mathbb{R}))$ to \([1.2]\) with finite $\sum_{j=0}^1 \| (D)_{\mu}^{1-j} \text{Op}(e^{\tau'(\nu_0(t)(\xi)^{s'})} D_x^2 u(t)) \|$.

For a more detailed statement, see Proposition 5.1 below. For case $b(t) \in C^{n,\alpha}$ with $n = 0, 1$ the result is a special case of a more general result [7, Théorème 1.3].

**Remark 1.1.** From the Plancherel’s theorem one has for $\tau > 0$

$$\| (D)_{\mu}^{1} \text{Op}(e^{\tau(\xi)^{s'}}) v \|^2 = \sum_{n=0}^{\infty} \frac{(2\tau)^n}{n!} \| (D)_{\mu}^{1} (D)_\mu^n v \|^2.$$ 

**Corollary 1.1.** Assume $b(t) \in C^{n,\alpha}([0, T])$ and $0 \leq a(x) \in \mathcal{G}^s(\mathbb{R})$ with $1 < s$ which is constant for large $|x|$. Then the Cauchy problem for $P$ is well-posed in $\mathcal{G}^s(\mathbb{R})$ for any $s' > s$.

In [3] they constructed a nonnegative $C^\infty$ function $b(t)$ such that the Cauchy problem for $D_x^2 - b(t) D_x^2$ is not $C^\infty$ well-posed. Our proof of Theorem 1.1 is based on the energy method. To explain the idea we consider the case $a(x) = 1$. After Fourier transform with respect to $x$ it suffices to consider $D_t^2 + b(t) \xi^2$. A very naive idea to obtain energy estimates is to employ a weighted energy

$$E(u) = e^{-\Lambda(t, \xi)} \left\{ |\partial_t \hat{u}(t, \xi)|^2 + (b(t) + |\xi|^{-2\delta}) \xi^2 |\hat{u}(t, \xi)|^2 \right\}$$

with weight $e^{-\Lambda}$ where $\delta > 0$ to be chosen suitably. In $dE(u)/dt$ we have a term $e^{-\Lambda(t, \xi)} b'(t) \xi^2 |\hat{u}(t, \xi)|^2$ which is the hardest term to manage because $b'(t)$ may change the sign although $b(t) \geq 0$ is assumed. A way to cancel the term is to choose $\Lambda(t, \xi)$ so that

$$\Lambda(t, \xi) = \int_0^t \frac{|b'(s)|}{b(s) + |\xi|^{-2\delta}} ds$$

since $dE(u)/dt$ supplies the term $-e^{-\Lambda} |b'(t)| \xi^2 |\hat{u}(t, \xi)|^2$ which obviously controls $e^{-\Lambda} b'(t) \xi^2 |\hat{u}(t, \xi)|^2$. For general $D_x^2 - D_x h(t, x) D_x$ this suggests to choose

$$\Lambda(t, x, \xi) = \int_0^t \frac{|\partial_x h(s, x)|}{h(s, x) + |\xi|^{-2\delta}} ds$$
which, however, does not work as a symbol of pseudodifferential operator for 
the lack of regularity in \( x \) (see however [8]). If \( h(t, x) = b(t)a(x) \), thanks to 
this special form, the weight \( \Lambda(t, x, \xi) \) works as a symbol of pseudodifferential 
operator with the metric \( g = (a(x) + |\xi|^{-2\delta})^{-1}dx^2 + |\xi|^{-2}d\xi^2 \), which is equivalent to 
the one defining the symbol class \( S_{1, \delta} \) on the set \( \{a(x) = 0\} \). However, since 
an expected weight is not \( \text{Op}(\pm \Lambda) \) but \( \text{Op}(e^{\pm \Lambda}) \) we must provide a calculus 
including \( \text{Op}(e^{\pm \Lambda}) \). Moreover in order to obtain energy estimates in the Gevrey 
class of order \( s' \) we need to choose \( \delta = 1 - 1/s' \) which could be very close to 1 and the larger \( \delta \) gives worse commutators against \( \Lambda \) because \( \partial_x \Lambda \) looses the factor \( (a(x) + |\xi|^{-2\delta})^{-1/2} \) on the symbol level, which is \( |\xi|^\delta \) if \( a(x) = 0 \). 
To manage such commutators we take advantage of pseudodifferential calculus 
with the metric \( g \) (see [8], [4]).

2 Metrics, symbols and weights

In this section we introduce a metric defining a class of symbols of pseudodifferential 
operators and weights which we use throughout the paper. For Weyl-Hörmander calculus of 
pseudodifferential operators we refer to [5], [6]. We 
denote

\[
\langle \xi \rangle_{\mu} = (\mu^{-2} + |\xi|^{2})^{1/2}
\]

where \( 0 < \mu \ll 1 \) is a small parameter. Assume that \( 0 \leq a(x) \in \mathcal{G}^s(\mathbb{R}) \) and with 
\( 0 \leq \delta < 1 \) we set

\[
\phi(x, \xi) = a(x) + \langle \xi \rangle_{\mu}^{-2\delta}.
\]

Throughout the paper all constants are assumed to be independent of \( \mu \) unless 
otherwise stated.

Lemma 2.1. There exist \( A > 0, C > 0 \) such that

\[
|\frac{\partial^k}{\partial x^k} \phi(x, \xi)|/\phi \leq CA^{k+l}(k + l)!(1 + (k + l)^{s-1}\langle \xi \rangle_{\mu}^{-\delta})^{k+l}\langle \xi \rangle_{\mu}^{-l-\delta k}
\]

for any \( k, l \in \mathbb{N} \).

Proof. Since \( a(x) \) is nonnegative note that \( |\frac{\partial^k}{\partial x^k} a(x)| \leq Ca^{1-j/2} \leq C\phi^{1-j/2} \leq C\phi \langle \xi \rangle_{\mu}^{\delta} \) for \( j = 0, 1, 2 \) and

\[
|\frac{\partial^j}{\partial \xi^j} \langle \xi \rangle_{\mu}^{-2\delta}| \leq CA^{j}(\langle \xi \rangle_{\mu}^{-2\delta})^{j-1} \leq CA^{j}j!\phi \langle \xi \rangle_{\mu}^{-j}
\]

for any \( j \in \mathbb{N} \). Then for \( k + l \leq 2 \) the estimate (2.1) holds clearly. For any \( k \geq 3 \) we have

\[
k!^s \langle \xi \rangle_{\mu}^{2\delta} \leq (2^{2(s-1)})^{k}(1 + k^{s-1}\langle \xi \rangle_{\mu}^{-\delta})^{k}k!\langle \xi \rangle_{\mu}^{2\delta}
\]

which proves with \( A_1 = 2^{2(s-1)} \) that for \( k \geq 3 \)

\[
|\frac{\partial^k}{\partial x^k} a(x)| \leq CA^{k}k!^s \phi \langle \xi \rangle_{\mu}^{2\delta} \leq C(AA_1)^{k}k!(1 + k^{s-1}\langle \xi \rangle_{\mu}^{-\delta})^{k}\phi \langle \xi \rangle_{\mu}^{2\delta}.
\]

This together with (2.2) proves the assertion for any \( k, l \in \mathbb{N} \). \( \square \)
Assume \( 0 \leq b(t) \in C^{n,\alpha}([0, T]) \) and consider

\[
(2.3) \quad \Lambda(t, x, \xi) = \int_0^t \frac{|b'(s)|\phi(x, \xi)}{b(s)\phi(x, \xi) + \langle \xi \rangle_{\mu}^{-2\delta}} ds.
\]

Here recall [2, Lemma 1].

**Lemma 2.2.** Assume that \( f(t, \eta) \geq 0 \) and \( f(t, \eta) \in C^{n,\alpha}([0, T]) \) for any \( \eta \in \Omega \). Then one has

\[
\int_0^t \frac{|\partial_t f(s, \eta)|}{f(s, \eta) + r} ds \leq C_N B^{1/(n+\alpha)} r^{-1/(n+\alpha)}
\]

for \( (t, \eta) \in [0, T] \times \Omega \) where \( B = \sup_{\eta \in \Omega} \|f(t, \eta)\|_{C^{n,\alpha}([0, T])} \).

Denote \( N = n + \alpha \) and assume \( 1 < s < s' < 1 + N/2 \). Define \( \kappa > 0 \) and \( \tilde{\kappa} \) by

\[
0 < \kappa = \frac{1}{s'}, \quad \tilde{\kappa} = \frac{2\delta}{N}
\]

so that \( s\kappa < s'\kappa = 1 \) and we put

\[
0 < \delta = 1 - \kappa < 1.
\]

**Lemma 2.3.** We have \( \tilde{\kappa} < \kappa \) and hence \( s\kappa < s\kappa = s(1 - \delta) < 1 \).

**Proof.** Since \( s' < 1 + N/2 \) then \( \delta = 1 - 1/s' < N/(2 + N) \). Thus we have

\[
\kappa = 1 - \delta > \frac{2}{2 + N} = \frac{N}{2 + N} > \frac{2\delta}{N} = \tilde{\kappa}
\]

which proves the assertion. \( \square \)

**Corollary 2.1.** There is \( B \) such that

\[
(2.4) \quad \Lambda(t, x, \xi) \leq B\langle \xi \rangle_{\mu}^{\tilde{\kappa}}.
\]

**Proof.** Thanks to Lemma 2.2 the proof is clear. \( \square \)

**Lemma 2.4.** There are \( A > 0, C > 0 \) such that

\[
|\partial_x^k \partial_{\xi}^l \Lambda(t, x, \xi)| \leq CA^{k+l}(k + l)!(1 + (k + l)s^{-1}\langle \xi \rangle_{\mu}^{-\delta})^{k+l}\Lambda(\langle \xi \rangle_{\mu}^{-l+2\delta k})
\]

for any \( k, l \in \mathbb{N} \).

**Proof.** Note that with \( \Phi = b(s)\phi(x, \xi) + \langle \xi \rangle_{\mu}^{-2\delta} \) one has

\[
(2.5) \quad |\partial_x^k \partial_{\xi}^l \Phi| \leq CA^{k+l}(k + l)!(1 + (k + l)s^{-1}\langle \xi \rangle_{\mu}^{-\delta})^{k+l}\Phi^{-l+2\delta k}.
\]

Indeed we have

\[
|\partial_x^k \partial_{\xi}^l \Phi| \leq CA^{k+l}(k + l)!(1 + (k + l)s^{-1}\langle \xi \rangle_{\mu}^{-\delta})^{k+l}(b(s)\phi(\langle \xi \rangle_{\mu}^{-l+2\delta k}) + \langle \xi \rangle_{\mu}^{-2\delta}(\langle \xi \rangle_{\mu}^{-1}) ) \leq CA^{k+l}(k + l)!(1 + (k + l)s^{-1}\langle \xi \rangle_{\mu}^{-\delta})^{k+l}\Phi^{-l+2\delta k}.
\]
Lemma 4.1. Therefore we see

\[ |\partial_x^i \partial_\xi^j \frac{\partial'}{\partial \Phi} \Phi| \leq |\partial' \Phi| C A^{k+l}(k + l)! (1 + (k + l)^{s-1} \lambda_\mu^{-\delta})^{k+l+1} \partial_\lambda \Lambda(x, \xi) \langle \xi \rangle^{(k+l+1)(s-1)-1} \]

which proves the assertion.

\[ |\partial_x^i \partial_\xi^j \partial \Lambda(t, x, \xi)| \leq C A^{k+l}(k + l)! (1 + (k + l)^{s-1} \lambda_\mu^{-\delta})^{k+l+1} \partial_\lambda \Lambda(t, x, \xi) \langle \xi \rangle^{(k+l+1)(s-1)-1} \]

for any \( k, l \in \mathbb{N} \).

Taking Lemma 2.5 into account we study \( e^\psi \) with \( \psi \) satisfying

\[ |\partial_x^i \partial_\xi^j \partial \Lambda(t, x, \xi)| \leq C A^{k+l}(k + l)! (1 + (k + l)^{s-1} \lambda_\mu^{-\delta})^{k+l+1} \lambda(x, \xi) \langle \xi \rangle^{(k+l+1)(s-1)-1} \]

for any \( k, l \in \mathbb{N} \) with some \( \lambda(x, \xi) > 0 \) and \( C > 0 \). Define \( \omega^i_j \ (i, j \in \mathbb{N}) \) by

\[ \partial_x^i \partial_\xi^j e^{\psi(x, \xi)} = \omega^i_j (x, \xi) e^{\psi(x, \xi)} \]

Lemma 2.6. There exist \( B > 0, C > 0, A > 0 \) such that one has

\[ |(\omega^i_j)^{(k)}_{(l)}| \leq C A^{i+j+k+l} \langle \xi \rangle^{-(i+k)+\delta(j+l)} \]

for any \( i, j, k, l \in \mathbb{N} \) where \( (\omega^i_j)^{(k)}_{(l)} = \partial_x^i \partial_\xi^j \omega^i_j \).

Proof. We first note that one can find \( C > 0, B > 0 \) such that for \( i + j = 1 \)

\[ |\partial_x^i \partial_\xi^i \partial \psi| \leq C^{k+l}(k + l)! (1 + (k + l)^{s-1} \lambda_\mu^{-\delta})^{k+l+1} (B \lambda) \langle \xi \rangle^{-(k+i)+\delta(l+j)} \]

for any \( k, l \in \mathbb{N} \). Indeed from (2.6)

\[ |\partial_x^i \partial_\xi^i \partial \psi| \leq C^{k+l+1}(k + l)! (1 + (k + l+1)^{s-1} \lambda_\mu^{-\delta})^{k+l+1} \times \lambda \langle \xi \rangle^{-(k+i)+\delta(l+j)} \]

\[ \leq C^{k+l+1} C_1^{k+l+1}(k + l)! (1 + (k + l)^{s-1} \lambda_\mu^{-\delta})^{k+l+1} \lambda \langle \xi \rangle^{-(k+i)+\delta(l+j)} \]

with some \( C_1 > 0 \). Thus replacing \( C \) by \( CC_1 \) and choosing \( B = CC_1 \) we get (2.7). Assume that there exist \( C > 0, A_1 > 0, A_2 > 0 \) such that for \( i + j \leq n \) we have for any \( k, l \in \mathbb{N} \)

\[ |(\omega^i_j)^{(k)}_{(l)}| \leq C A_2^{i+j} A_1^{k+l} \langle \xi \rangle^{-(i+k)+\delta(j+l)} \]

\[ \times \sum_{p=0}^{i+j-1} (B \lambda)^{i+j-p} (k + l + p)! (1 + (k + l + p)^{s-1} \lambda_\mu^{-\delta})^{k+l+p} \]

(2.8)
When \( i + j = 1 \) the estimate (2.8) holds thanks to (2.7) for any \( k, l \in \mathbb{N} \) because 
\( \omega_j = \partial_i^j \eta_j \). When \( e_1 + e_2 = 1 \) we have 
\[ \omega_j^{i+e_1} = \omega_j^{(e_1)} + \psi^{(e_1)} \omega_j^{i} \]
then it follows that

\[ |\omega_j^{i+e_1}(k)| = |\omega_j^{(k) \omega_j^{i+e_1}} + \sum \binom{k}{k'} \binom{l}{l'} \psi^{(e_1+k') \omega_j^{(k-k')}} | \]

\[ \leq CA_1^{k+l+1} A_2^{i+j} \langle \xi \rangle^{-(i+k+e_1) + \delta (j+l+e_2)} \sum p=0 \ (B \lambda)^{i+j-p} \]

+ \sum \binom{k}{k'} \binom{l}{l'} \langle B \lambda \rangle C^{k'+l'+1} (k'+l') \langle 1 + (k'+l')^{s-1} \langle \xi \rangle^{-(i+k-k')} + (j+l-l'+p) \rangle \]

\[ \times \langle 1 + (k-k' + l-l' + p)^{s-1} \langle \xi \rangle^{-(i+k-k')} + (j+l-l'+p) \rangle \]

We denote the right-hand side by \( \langle \xi \rangle^{-(i+k+e_1) + \delta (j+l+e_2)} (I_1 + I_2) \) where

\[ I_1 = CA_1^{k+l+1} A_2^{i+j} \sum p=0 \sum_{i+j=p} (B \lambda)^{i+j-p} \]

\[ \times (k+l+1+p) ! (1 + (k+l+1+p)^{s-1} \langle \xi \rangle^{-(i+k+1+p)} \]

and the remaining part called \( I_2 \). We consider the sum over \( k', l' \) in \( I_2 \):

\[ \sum \binom{k}{k'} \binom{l}{l'} \langle B \lambda \rangle C^{k'+l'+1} (k'+l') \langle 1 + (k'+l')^{s-1} \langle \xi \rangle^{-(i+k-k')} + (j+l-l'+p) \rangle \]

\[ \times \langle (k-k' + l-l' + p)^{s-1} \langle \xi \rangle^{-(i+k-k')} + (j+l-l'+p) \rangle \]

which is bounded by

\[ CA_1^{k+l} \sum_{k', l'} \langle k' \rangle^{k'+l'} \langle l' \rangle^{k'+l'} \langle 1 + (k'+l')^{s-1} \langle \xi \rangle^{-(i+k-k')} + (j+l-l'+p) \rangle \]

\[ \times \langle (k-k' + l-l' + p)^{s-1} \langle \xi \rangle^{-(i+k-k')} + (j+l-l'+p) \rangle \]

\[ \leq 2^{k'+l'} \langle k+l+p \rangle ! \]

because

\[ \binom{k}{k'} \binom{l}{l'} (k'+l')! (k-k' + l-l' + p)! \leq 2^{k'+l'} (k+l+p)! \]

\[ \leq 2^{k'+l'} \langle k+l+p \rangle ! \]
Thus we obtain

\[ (1 + (k' + l')^{s-1} \langle \xi \rangle^{-\delta}_{\mu})^{k' + l'} (1 + (k - k' + l - l' + p)^{s-1} \langle \xi \rangle^{-\delta}_{\mu})^{k - k' + l - l' + p} \]

\[ \leq (1 + (k + l + p)^{s-1} \langle \xi \rangle^{-\delta}_{\mu})^{k' + l'} (1 + (k + l + p)^{s-1} \langle \xi \rangle^{-\delta}_{\mu})^{k - k' + l - l' + p} \]

\[ \leq (1 + (k + l + p)^{s-1} \langle \xi \rangle^{-\delta}_{\mu})^{k + l + p} \]

then (2.9) is estimated by

\[ CA_{1}^{k+l}(1 + (k + l + p)^{s-1} \langle \xi \rangle^{-\delta}_{\mu})^{k + l + p} \sum_{k' = 0}^{k} (2C/A_{1})^{k'} \sum_{l' = 0}^{l} (2C/A_{1})^{l'} \]

\[ \leq CA_{1}^{k+l+2}(A_{1} - 2C)^{-2}(1 + (k + l + p)^{s-1} \langle \xi \rangle^{-\delta}_{\mu})^{k + l + p}. \]

Thus we obtain

\[ I_{2} \leq C_{2}^{i+j} A_{2}^{k+l+2} A_{1}^{k+l}(A_{1} - 2C)^{-2} \sum_{p=0}^{i+j-1} (B\lambda)^{i+j+1-p} (k + l + p)! (1 + (k + l + p)^{s-1} \langle \xi \rangle^{-\delta}_{\mu})^{k + l + p}, \]

\[ I_{1} \leq CA_{1}^{k+l+1} A_{2}^{i+j} \sum_{p=1}^{i+j} (B\lambda)^{i+j+1-p} (k + l + p)! \]

\[ \times (1 + (k + l + p)^{s-1} \langle \xi \rangle^{-\delta}_{\mu})^{k + l + p}. \]

Therefore we have proved that

\[ I_{1} + I_{2} \leq C_{2}^{i+j+1} A_{1}^{k+l} (A_{1}A_{2}^{-1} + CA_{1}^{2} A_{2}^{-1} (A_{1} - 2C)^{-2}) \]

\[ \times \sum_{p=0}^{i+j} (B\lambda)^{i+j+1-p} (k + l + p)! (1 + (k + l + p)^{s-1} \langle \xi \rangle^{-\delta}_{\mu})^{k + l + p}. \]

Choosing \( A_{1}, A_{2} \) such that \( A_{1}A_{2}^{-1} + CA_{1}^{2} A_{2}^{-1} (A_{1} - 2C)^{-2} \leq 1 \) the estimate (2.8) holds for \( i + j = n + 1 \).

Noting that \( \sum_{p=0}^{i+j-1} (B\lambda)^{i+j-p} (k + l + p)! (1 + (k + l + p)^{s-1} \langle \xi \rangle^{-\delta}_{\mu})^{k + l + p} \) is bounded by

\[ C_{1}^{k+l} B\lambda (k + l)! (1 + (k + l)^{s-1} \langle \xi \rangle^{-\delta}_{\mu})^{k + l} \sum_{p=0}^{i+j-1} C_{1}^{p} (B\lambda)^{i+j+1-p} (p + p^{s} \langle \xi \rangle^{-\delta}_{\mu})^{p} \]

\[ \leq C_{2}^{k+l+i+j} B\lambda (k + l)! (1 + (k + l)^{s-1} \langle \xi \rangle^{-\delta}_{\mu})^{k + l} \times (B\lambda + (i + j - 1) + (i + j - 1)^{s} \langle \xi \rangle^{-\delta}_{\mu})^{i+j-1} \]

if \( i + j \geq 1 \) we have the following lemma (see [9 Lemma 5.1]):

**Lemma 2.7.** Assume that \( \psi \) satisfies (2.6). Then we have

\[ |D_{x}^{i} D_{y}^{j} e^{\psi}| \leq C_{2}^{i+j} (B\lambda + i + j + (i + j)^{s} \langle \xi \rangle^{-\delta}_{\mu})^{i+j} \langle \xi \rangle^{-\delta}_{\mu}^{i+j} e^{\psi} \]

(2.10)
and \((i + j \geq 1)\)

\[
|\partial_\xi^k \partial_\mu^l (\omega_j^s)| \leq C^{i+j+k+l} B\lambda (B\lambda + (i + j - 1) + (i + j - 1)^s (\xi)^{-\delta})^{i+j-1} \times (k + l)! (1 + (k + l)^s (\xi)^{-\delta})^{k+l} (\xi)^{-(i+k)+\delta(j+l)}.
\]

We apply these results to \(\psi = \pm \Lambda\) and \(\omega_j^s = T_j^s\) to obtain

**Corollary 2.2.** There exist \(A > 0, C > 0\) such that

\[
|\partial_\xi^k \partial_\mu^l (\epsilon^\pm |) \leq CA^{k+l+1} (\xi)^{\delta} + (i + j) + (k + l)^s (\xi)^{-\delta})^{k+l} (\xi)^{-(i+k)+\delta(j+l)}
\]

and for any \(i, j (i + j \geq 1)\)

\[
|\partial_\xi^k \partial_\mu^l (T_j^s)| \leq CA^{k+l+1} (1 + (k + l)^s (\xi)^{-\delta})^{k+l} \times (\xi)^{\delta} + (i + j - 1) + (i + j - 1)^s (\xi)^{-\delta}^{i+j-1} (\xi)^{-(i+k)+\delta(j+l)}.
\]

We introduce some symbol classes [R], Definition 4.1 (see also [S]):

**Definition 2.1.** Let \(0 \leq \delta < 1\) and \(1 < s\). Let \(m(x, \xi; \mu)\) be a positive function with a small parameter \(\mu > 0\). We say that \(a(x, \xi; \mu) \in C^\infty (\mathbb{R} \times \mathbb{R})\) belongs to \(S^{(s)}_\delta (m)\) if for any \(\varepsilon > 0\) there exist \(C > 0, A > 0\) independent of \(0 < \mu \leq 1\) such that for all \(i, j \in \mathbb{N}\) one has

\[
|\partial_\xi^i \partial_\mu^j a(x, \xi, \mu)| \leq C A^{i+j} ((i + j)^{1+\varepsilon} + (i + j)^s (\xi)^{-\delta})^{i+j} (\xi)^{-(i+k)+\delta(j+l)} m(x, \xi, \mu).
\]

**Definition 2.2.** Let \(s > 1\) and \(m(x, \xi, \mu)\) be a positive function with a small parameter \(\mu > 0\). We say that \(a(x, \xi, \mu) \in C^\infty (\mathbb{R} \times \mathbb{R})\) belongs to \(S^{(s)}_\delta (m)\) if there exist \(C > 0, A > 0\) independent of \(\mu\) such that one has

\[
|\partial_\xi^i \partial_\mu^j a(x, \xi, \mu)| \leq C A^{i+j} ((i + j)^{s(i+j)} m(x, \xi, \mu)
\]

for any \(i, j \in \mathbb{N}\).

We often write \(a(x, \xi)\) for \(a(x, \xi; \mu)\) dropping \(\mu\). Now we introduce the metric (see [1])

\[
g(dx, d\xi) = g_{x, \xi}(dx, d\xi) = \phi(x, \xi)^{-1} dx^2 + (\xi)^{-2} d\xi^2
\]

and \(g_\delta(dx, d\xi) = (\xi)^{2\delta} dx^2 + (\xi)^{-2} d\xi^2\). It is clear that \(g \leq g_\delta\) and \(g_\delta\) is the metric defining the class \(S^{(s)}_{1, \delta}\) for any fixed \(\mu > 0\).

**Lemma 2.8.** Assume \(0 < \delta < 1\). Then the metric \(g\) is slowly varying and \(\sigma\) temperate (uniformly in \(\mu\)) and

\[
h^2 = \sup g / g^\sigma = (\xi)^{-2} \phi^{-1} \leq (\xi)^{2\delta - 2} = (\xi)^{-2(1-\delta)} \leq 1
\]

that is, \(g\) is an admissible metric.

**Lemma 2.9.** \(\phi^{\pm 1}\) are \(g\) continuous and \(\sigma, g\) temperate (uniformly in \(\mu\)), that is \(g\)-admissible weights.
Proof. Proofs of the above two lemmas are found in [4].

Definition 2.3. We denote $S_\phi(m) = S(m, g)$ and $S_\delta(m) = S(m, g_\delta)$ for a $g$-admissible and $g_\delta$-admissible (uniformly in $\mu$) weight $m(x, \xi, \mu) > 0$ respectively. By $a \in \mu^c S_\phi(m)$ (resp. $a \in \mu^c S_\delta(m)$) we mean that $\mu^c a \in S_\phi(m)$ (resp. $\mu^{-c} a \in S_\delta(m)$) uniformly in small $\mu > 0$.

Lemma 2.10. For any $k, l \in \mathbb{N}$ there is $C > 0$ such that

$$|\partial_x^k \partial_\xi^l \Lambda(t, x, \xi)| \leq CA\phi^{-k/2} \langle \xi \rangle^{-l}_\mu.$$

Proof. Note that with $\Phi = b(s)\phi(x, \xi) + \langle \xi \rangle^{-2\delta}_\mu$ one has

$$(2.15) \quad |\partial_x^k \partial_\xi^l \Phi(s, x, \xi)^{-1}| \leq C_{kl}\Phi(s, x, \xi)^{-1}\phi^{-k/2} \langle \xi \rangle^{-l}_\mu.$$

Indeed we have

$$|\partial_x^k \partial_\xi^l \Phi(s, x, \xi)| \leq C(b(s)\phi(x, \xi)\phi^{-k/2} \langle \xi \rangle^{-1}_\mu + \langle \xi \rangle^{-2\delta}_\mu \langle \xi \rangle^{-l}_\mu)
\leq C\Phi(s, x, \xi)\phi^{-k/2} \langle \xi \rangle^{-l}_\mu.$$

Then from $\Phi^{-1}\Phi = 1$ one obtains the assertion for $\Phi^{-1}$. Therefore we see

$$(2.16) \quad |\partial_x^k \partial_\xi^l \frac{b'(s)\phi(x, \xi)}{\Phi(s, x, \xi)}| \leq C\frac{|b'(s)|\phi(x, \xi)}{\Phi(s, x, \xi)}\phi^{-k/2} \langle \xi \rangle^{-l}_\mu$$

which proves the assertion.

Lemma 2.11. For any $k, l \in \mathbb{N}$ there exist $C > 0$ such that

$$|\partial_x^k \partial_\xi^l \phi(x, \xi)/\phi \leq C\phi^{-k/2} \langle \xi \rangle^{-l}_\mu.$$

that is $\phi \in S_\phi(\phi)$.

Proof. Note that $|\partial_x a(x)| \leq C\sqrt{a} \leq C\phi^{1/2}$ since $a(x) \geq 0$ and for $k \geq 2$

$$|\partial_x^k a(x)| \leq C_k \phi^{1-k/2}$$

since $\phi^{-1} \geq c$ with some $c > 0$. Taking (2.12) into account the proof is clear.

Lemma 2.12. For any $k, l \in \mathbb{N}$ there is $C > 0$ such that

$$|\partial_x^k \partial_\xi^l \partial_\eta \Lambda(t, x, \xi)| \leq C\partial_\eta \Lambda(t, x, \xi)\phi^{-k/2} \langle \xi \rangle^{-l}_\mu.$$

Lemma 2.13. For any $i, j, k \in \mathbb{N}$ there exists $C > 0$ such that

$$(2.17) \quad |\partial_\xi^i \partial_x^j T_j^i| \leq C \langle \xi \rangle^{\delta(i+j)/2} \phi^{-j/2} \langle \xi \rangle^{-(i+k)}$$

that is $T_j^i \in S_\phi(\langle \xi \rangle^{\delta(i+j)} \phi^{-j/2} \langle \xi \rangle^{-(i+k)})$.

Since $\phi^{-1} \leq \langle \xi \rangle^{\delta}$ we have $S_\phi(\langle \xi \rangle^m) \subset S_\delta(\langle \xi \rangle^m).$
Lemma 2.14. We have $\Lambda \in S_0((\langle \xi \rangle^c_\mu)^2) \cap S_0((\langle \xi \rangle^c_\mu)^{1/2^k})$ and $\partial_t \Lambda \in S_0((\sqrt{\langle \xi \rangle_\mu})^2) \cap S_0((\langle \xi \rangle^c_\mu)^{1/2^k})$.

Proof. The first assertion follows from Lemmas 2.4, 2.10 and Corollary 2.1. Noting

$$\partial_t \Lambda(t, x, \xi) = \phi^{1/2} \left[ \frac{b(t)\phi^{1/2}}{b(t) + \langle \xi \rangle^{1/2}_\mu} \right] \leq \phi^{1/2}$$

the second assertion follows from Lemmas 2.5 and 2.12.

Lemma 2.15. We have $e^{-\Lambda} \in S_0((\langle \xi \rangle)^c_\mu(1)) \subset S_0(1)$.

Proof. Thanks to Lemmas 2.4 and 2.7 one has

$$|\partial^{i+j}\lambda| \leq CA^{i+j}(\Lambda + (i + j)\alpha(\langle \xi \rangle^{c_\mu}_\mu))$$

Since $\Lambda > 0$ one has $A^{i+j}e^{-\Lambda} \leq A^{i+j}(i + j)^{i+j}(\langle \xi \rangle^{c_\mu}_\mu)^{i+j}$ and hence the right-hand side is bounded by

$$CA^{i+j}(i + j)^{i+j}(\langle \xi \rangle^{c_\mu}_\mu)^{i+j}$$

which proves the assertion.

Lemma 2.16. Let $c > 0$ and $0 < \kappa < 1$. Then for any $0 < c' < c$ and $s > 1$, $\delta > 0$ one has $e^{-c(\langle \xi \rangle^c_\mu)} \in S_0((e^{-c'}(\langle \xi \rangle^c_\mu)^{1/2^k})$.

Proof. Since $|\partial^{l}_{\xi}(\langle \xi \rangle^c_\mu)| \leq C^{l+1}!(\langle \xi \rangle^{c_\mu}_\mu)^{-l}$ then from Lemma 2.7 there exists $C_1 > 0$ such that

$$|\partial^{l}_{\xi}(\langle \xi \rangle^c_\mu)| \leq C^{l+1}!(\langle \xi \rangle^{c_\mu}_\mu)^{-l}$$

For any $c' < c$ there is $A > 0$ such that $(\langle \xi \rangle^{c_\mu}_\mu)^{1/2^k} \leq A^{l+1}l!$ and hence we have

$$|\partial^{l}_{\xi}(\langle \xi \rangle^c_\mu)| \leq C^{l+1}!(\langle \xi \rangle^{c_\mu}_\mu)^{-l}$$

which proves the assertion.

3 Composition of PDO’s acting in the Gevrey classes

We recall several facts on compositions of pseudodifferential operators including $\text{Op}(e^{\pm \Lambda})$ whose proofs are given in Appendix. We denote by $\text{Op}(a)$ the Weyl quantization of $a(x, \xi)$ (see [5]). Here we recall

$$\delta = 1 - \kappa, \quad \kappa > \kappa, \quad s \kappa < 1.$$
Proposition 3.2. Assume \( b(x, \xi) \in S^{(\nu)}_\delta((\xi)_\mu^m) \). Then we have

\[
(b \pm \Lambda)^\# e^{\pm \Lambda} = b + b^\pm + R^\pm
\]

where \( b^\pm \in S^{(\nu)}_\delta((\xi)_\mu^{m-(\kappa-\tilde{\kappa})}) \) and \( R^\pm \in S^{(\tilde{s})}_{0,0}(e^{-c(\xi)_\mu^s}) \) with some \( c > 0, \tilde{s} \) and \( \tilde{\kappa} \) such that \( \tilde{s} \tilde{\kappa} < 1 \) and \( \tilde{\kappa} > \tilde{\kappa} \).

Corollary 3.1. There is \( \tilde{s} > 1 \) such that

\[
e^{\pm \Lambda} # e^{\pm \Lambda} = 1 \in S^{(\tilde{s})}_\delta((\xi)_\mu^{-(\kappa-\tilde{\kappa})}) \subset \mu^{\kappa-\tilde{\kappa}} S_\delta(1).
\]

Corollary 3.2. Assume \( b(x, \xi) \in S^{(\nu)}_\delta((\xi)_\mu^m) \) then

\[
(b \pm \Lambda)^\# e^{\pm \Lambda} \in S_\delta((\xi)_\mu^m).
\]

Proof. Since \( S^{(\tilde{s})}_{0,0}(e^{-c(\xi)_\mu^s}) \subset S_\delta((\xi)_\mu^k) \) for any \( k \) the proof is immediate. \( \square \)

Proposition 3.3. Assume \( b(x, \xi) \in S^{(\nu)}_\delta((\xi)_\mu^m) \) and \( a(x, \xi) \in S^{(\nu)}_\delta((\xi)_\mu^d) \) then

\[
(b \pm \Lambda)^\# a = \sum_{k+l<N} \frac{(-1)^k}{(2l)^{k+l}k!l!} (\partial_x^k \partial^l_\xi (b \pm \Lambda)) \partial_x^l \partial^k_\xi a + q_N e^{-\Lambda} + R_N,
\]

\[
a^\# (b \pm \Lambda) = \sum_{k+l<N} \frac{(-1)^k}{(2l)^{k+l}k!l!} \partial_x^l \partial^k_\xi a (\partial_x^l \partial^k_\xi (b \pm \Lambda)) + \tilde{q}_N e^{-\Lambda} + \tilde{R}_N
\]

with \( q, \tilde{q} \in S^{(\nu)}_\delta((\xi)_\mu^{m+d-(\kappa-\tilde{\kappa})N}) \) and \( R_N, \tilde{R}_N \in S^{(\tilde{s})}_{0,0}(e^{-c(\xi)_\mu^s}) \) with some \( \tilde{s} > 1, \tilde{\kappa} > 0 \) and \( c > 0 \) satisfying \( \tilde{s} \tilde{\kappa} < 1 \) and \( \tilde{\kappa} > \tilde{\kappa} \).

Corollary 3.3. Assume \( b(x, \xi) \in S^{(\nu)}_\delta((\xi)_\mu^m) \) and \( a(x, \xi) \in S^{(\nu)}_\delta((\xi)_\mu^d) \) then

\[
(b \pm \Lambda)^\# a = c e^{-\Lambda} + r, \quad a^\# (b \pm \Lambda) = \hat{c} e^{-\Lambda} + \hat{r}
\]

with \( c, \hat{c} \in S^{(\nu)}_\delta((\xi)_\mu^{m+d}) \) and \( r, \hat{r} \in S^{(\tilde{s})}_{0,0}(e^{-c(\xi)_\mu^s}) \) with some \( \tilde{s} > 1, \tilde{\kappa} > 0 \) and \( c > 0 \) satisfying \( \tilde{s} \tilde{\kappa} < 1 \) and \( \tilde{\kappa} > \tilde{\kappa} \).

Proof. Write

\[
(\partial_x^k \partial^l_\xi (b \pm \Lambda)) \partial_x^l \partial^k_\xi a = c_{k,l} e^{-\Lambda}.
\]

Then to prove the assertion it suffices to note \( c_{k,l} \in S^{(\nu)}((\xi)_\mu^{m+d}) \) which can be seen from Corollary 2.2 because \( \kappa > \tilde{\kappa} \). \( \square \)

Proposition 3.4. Assume \( p \in S^{(\nu)}_{0,0}(e^{-c(\xi)_\mu^s}) \) with \( \tilde{s} \tilde{\kappa} < 1, \tilde{s} \tilde{\kappa} > \tilde{\kappa} \) and \( c > 0 \). Then

\[
p^\# e^{\Lambda}, \quad e^{\Lambda} \# p \in S^{(s^*)}_\delta((e^{-c(\xi)_\mu^s})^*),
\]

with some \( s^* > 1 \) and \( \kappa^* > \tilde{\kappa} \). In particular we have \( p^\# e^{\Lambda}, e^{\Lambda} \# p \in S_\delta((\xi)_\mu^l) \) for any \( l \in \mathbb{R} \).
Corollary 3.4. Assume \( b(x, \xi) \in S^{(s)}_{\delta}(\langle \xi \rangle^m) \) and \( a(x, \xi) \in S^{(s)}_{\delta}(\langle \xi \rangle^d) \) then
\[
\langle be^{-\Lambda}\#a\#e^\Lambda, e^\Lambda\#a\#(be^{-\Lambda}) \rangle \in S_{\delta}(\langle \xi \rangle^{m+d}).
\]

Proof. Thanks to Corollary 3.3 one can write \( \langle be^{-\Lambda}\#a\#e^\Lambda, e^\Lambda\#a\#(be^{-\Lambda}) \rangle = \langle ce^{-\Lambda}\#e^\Lambda + r\#e^\Lambda, e^\Lambda\#a\#(ce^{-\Lambda}) \rangle \) where \( s\bar{k} < 1, \bar{k} > \bar{k} \) and \( c > 0 \). Write
\[
\langle ce^{-\Lambda}\#e^\Lambda + r\#e^\Lambda, e^\Lambda\#a\#(ce^{-\Lambda}) \rangle \text{ in virtue of Corollary 3.2,}
\]

then in virtue of Corollary 3.2 one has \( \langle ce^{-\Lambda}\#e^\Lambda + r\#e^\Lambda, e^\Lambda\#a\#(ce^{-\Lambda}) \rangle \in S_{\delta}(\langle \xi \rangle^{m+d}) \). On the other hand from Proposition 3.4 it follows that \( r\#e^\Lambda \in S_{\delta}(\langle \xi \rangle^{m+d}) \) and hence the assertion.

\[\square\]

Lemma 3.1. Assume \( p \in S^{(s)}_{\delta}(\langle \xi \rangle^m) \) then there is \( C > 0 \) such that
\[
\|\text{Op}(pe^{-\Lambda})u\| \leq C\|\langle D \rangle_{\mu}^m\text{Op}(e^{-\Lambda})u\|.
\]

Proof. From Corollary 3.1 it follows that
\[
e^{-\Lambda}\#e^\Lambda = 1 - R, \quad e^\Lambda\#e^{-\Lambda} = 1 - \tilde{R}
\]

with \( R, \tilde{R} \in \mu^{\kappa-\bar{k}}S_{\delta}(1) \). Choosing \( \mu > 0 \) small the inverse \( (1 - \text{Op}(R))^{-1} \) and \( (1 - \text{Op}(\tilde{R}))^{-1} \) are well-defined as a bounded operator on \( L^2 \). Thanks to \[\] (see also \[\]) there exist \( K \) and \( \tilde{K} \in S^{1}_{\delta,\delta} \) such that \( \text{Op}(K) = (1 - \text{Op}(R))^{-1} \) and \( \text{Op}(\tilde{K}) = (1 - \text{Op}(\tilde{R}))^{-1} \). Therefore one has
\[
\langle D \rangle_{\mu}^m\text{Op}(e^{-\Lambda})\text{Op}(e^\Lambda) = 1, \quad \text{Op}(e^\Lambda)\text{Op}(e^{-\Lambda})\text{Op}(\tilde{K}) = 1.
\]

Remark that
\[
\langle D \rangle_{\mu}^m\text{Op}(e^{-\Lambda})\text{Op}(e^\Lambda) = 1, \quad \text{Op}(e^{-\Lambda})\text{Op}(\tilde{K})\text{Op}(e^\Lambda) = 1.
\]

Set \( \langle pe^{-\Lambda}\#e^\Lambda = q \) which belongs to \( S_{\delta}(\langle \xi \rangle^m) \) by Corollary 3.2. Note that
\[
\text{Op}(pe^{-\Lambda}) = \text{Op}(pe^{-\Lambda})\text{Op}(e^\Lambda)\text{Op}(e^{-\Lambda}) = \text{Op}(q)\text{Op}(K)\text{Op}(e^{-\Lambda})
\]

and hence
\[
\|\text{Op}(pe^{-\Lambda})u\| = \|\text{Op}(q)\text{Op}(K)(\langle D \rangle_{\mu}^m\text{Op}(e^{-\Lambda})\text{Op}(e^\Lambda))u\| \leq C\|\langle D \rangle_{\mu}^m\text{Op}(e^{-\Lambda})u\|
\]

since \( q\#K\#(\langle \xi \rangle^m) = S_{\delta}(1) \). This proves the assertion.

\[\square\]

Corollary 3.5. For any \( k, l \in \mathbb{R} \) there is \( C > 0 \) such that
\[
\|\langle \xi \rangle_{\mu}^k\text{Op}(e^{-\Lambda})\text{Op}(e^{-\Lambda})\text{Op}(e^{-\Lambda})\text{Op}(e^{-\Lambda})u\| \leq C\|\langle D \rangle_{\mu}^{k+l}\text{Op}(e^{-\Lambda})u\|.
\]

Proof. Using \[\] write
\[
\langle D \rangle_{\mu}^k\text{Op}(e^{-\Lambda})\text{Op}(e^{-\Lambda})\text{Op}(e^{-\Lambda})\text{Op}(e^{-\Lambda})u\rangle = \langle D \rangle_{\mu}^k\langle \text{Op}(e^{-\Lambda})\text{Op}(e^{-\Lambda})\text{Op}(e^{-\Lambda})\text{Op}(e^{-\Lambda})u\rangle \text{Op}(K)\text{Op}(e^{-\Lambda}).
\]

Thanks to Corollary 3.4 we see \( \langle \xi \rangle_{\mu}^l\#e^\Lambda \in S_{\delta}(\langle \xi \rangle_{\mu}^l) \) and hence
\[
\langle \xi \rangle_{\mu}^k\#\langle \xi \rangle_{\mu}^l\#e^\Lambda \#K \in S_{\delta}(\langle \xi \rangle_{\mu}^{k+l}).
\]

This shows the second inequality. The first inequality follows from the second one.

\[\square\]
Lemma 3.2. Assume \( R \in S_{0,0}^{(s)}(e^{-c(\xi^s)}) \) with \( s\bar{k} < 1 \), \( \bar{k} > \bar{k} \) and \( c > 0 \) and \( p \in S_{0,0}^{(s)}((\xi)^{m_s}) \). Then for any \( k, l \in \mathbb{R} \) there is \( C \) such that
\[
\|\langle D \rangle_{\mu}^k \text{Op}(R)\text{Op}(p)u\| \leq C\mu \|\langle D \rangle_{\mu}^l \text{Op}(e^{-\Lambda})u\|.
\]

Proof. We first show that for any \( k, l \in \mathbb{R} \) we have
\[
\|\langle D \rangle_{\mu}^k \text{Op}(R)u\| \leq C\mu \|\langle D \rangle_{\mu}^l \text{Op}(e^{-\Lambda})u\|.
\]
Indeed using (3.3) we write \( \text{Op}(R) = \text{Op}(R)\text{Op}(e^{\Lambda})\text{Op}(K)\text{Op}(e^{-\Lambda}) \). Thanks to Proposition 3.4 it follows that \( R\# e^{\Lambda} \in S_{0}((\xi)^{l_{k-1}}) \) for any \( l \in \mathbb{R} \). Therefore one has \( \langle \xi \rangle_{\mu}^k \# (R\# e^{\Lambda}) \in S_{0}((\xi)^{l_{k-1}}) \). Then \( \langle \xi \rangle_{\mu}^k \# (R\# e^{\Lambda}) \# K \in S_{0}((\xi)^{l_{k-1}}) \subseteq \mu S_{0}((\xi)^{l_{k}}) \) for any \( l \) and hence (3.4). Therefore one has
\[
\|\langle D \rangle_{\mu}^l \text{Op}(R)\text{Op}(p)u\| \leq C\mu \|\langle D \rangle_{\mu}^l \text{Op}(e^{-\Lambda})\text{Op}(p)u\|.
\]
From Corollary 3.3 one can write
\[
\langle \xi \rangle_{\mu}^{l_{m}} \# (e^{-\Lambda} \# p) = \langle \xi \rangle_{\mu}^{l_{m}} \langle \xi \rangle_{\mu} \langle \xi \rangle_{\mu}^{l_{m}} \# r
\]
where \( q \in S_{0}((\xi)^{m_s}) \) and \( r \in S_{0}((e^{-c(\xi^s)})^{s}) \) with \( s\bar{k} < 1 \) and \( \bar{k} > \bar{k} \). By Corollary 3.3 again it follows that \( \langle \xi \rangle_{\mu}^{l_{m}} \# (qe^{-\Lambda}) = c e^{-\Lambda} + R \) where \( c \in S_{0}((\xi)^{m_s}) \) and \( R \in S_{0}((e^{-c(\xi^s)})^{s}) \) with possibly different \( s, \bar{k} \) satisfying the same conditions. Therefore we have
\[
\|\langle D \rangle_{\mu}^{l_{m}} \text{Op}(qe^{-\Lambda})u\| \leq C\mu \|\langle D \rangle_{\mu}^{l} \text{Op}(e^{-\Lambda})u\|
\]
thanks to Lemma 3.1. Applying 3.4 again one obtains
\[
\|\langle D \rangle_{\mu}^{l_{m}} \text{Op}(r)u\| \leq C\mu \|\langle D \rangle_{\mu}^{l} \text{Op}(e^{-\Lambda})u\|
\]
and we conclude the assertion. 

Lemma 3.3. Assume \( p \in S_{0}^{(s)}((\xi)^{m_1}) \) and \( q \in S_{0}^{(s)}((\xi)^{m_2}) \). Then for any \( k \in \mathbb{R} \) there exists \( C > 0 \) such that
\[
\|\langle D \rangle_{\mu}^{k} \text{Op}(qe^{-\Lambda})\text{Op}(p)u\| \leq C\mu \|\langle D \rangle_{\mu}^{k+m_1+m_2} \text{Op}(e^{-\Lambda})u\|
\]
\[
\leq C\mu \|\langle D \rangle_{\mu}^{k+m_1+m_2+1} \text{Op}(e^{-\Lambda})u\|
\]

Proof. Thanks to Corollary 3.3 one can write \( \langle \xi \rangle_{\mu}^{k} \# (qe^{-\Lambda}) = c e^{-\Lambda} + R \) where \( c \in S_{0}^{(s)}((\xi)^{k+m_2}) \) and \( r \in S_{0}^{(s_2)}((e^{-c(\xi^s)})^{s_1}) \) with \( s_1 \bar{k} < 1 \) and \( \bar{k} > \bar{k} \). Thus we have
\[
\langle \xi \rangle_{\mu}^{k} \# (qe^{-\Lambda}) \# p = (c e^{-\Lambda}) \# p + r \# p
\]
and Corollary 3.3 proves that \( (c e^{-\Lambda}) \# p = \check{c} e^{-\Lambda} + R \) where \( \check{c} \in S_{0}^{(s)}((\xi)^{k+m_1+m_2}) \) and \( R \in S_{0}^{(s_2)}((e^{-c(\xi^s)})^{s_2}) \) with \( s_2 \bar{k} < 1 \) and \( \bar{k} > \bar{k} \). Since
\[
\|\text{Op}(r)\text{Op}(p)u\| \leq C\mu \|\langle D \rangle_{\mu}^{k+m_1+m_2} \text{Op}(e^{-\Lambda})u\|
\]
in virtue of Lemma 3.2 the proof follows from Lemmas 3.1 and Lemma 3.2.

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Lemma 3.4. Assume $p \in S_0(m) \cap S_0^{(s)}(\langle \xi \rangle^*_\mu)$ where $m$ is $g$-admissible, $l \in \mathbb{R}$. Then for any $M \in \mathbb{N}$ one can write

$$pe^{-\Lambda} = (p + p_1 + p_2)\#e^{-\Lambda} + qe^{-\Lambda} + r,$$

$$pe^{-\Lambda} = (p + \tilde{p}_1 + \tilde{p}_2) + qe^{-\Lambda} + \tilde{r}$$

with $p_j, \tilde{p}_j \in S_0(m\phi^{-j/2}(\langle \xi \rangle^{-j+\bar{k}})) \cap S_0^{(s)}(\langle \xi \rangle^{1-(\kappa-\bar{k})})$ where $p_1, \tilde{p}_1$ are pure imaginary and $q, \tilde{q} \in S_0^{(s)}(\langle \xi \rangle^{-M})$, $r, \tilde{r} \in S_0^{(s)}(e^{-c\langle \xi \rangle^*_\mu})$ with some $\bar{s} > 1$, $\bar{k} > 0$ and $c > 0$ such that $\bar{s}k < 1$ and $\bar{k} > \bar{k}$.

Proof. Thanks to Proposition 3.3 we can write $pe^{-\Lambda} = p\#e^{-\Lambda} + p_1e^{-\Lambda} + p_2e^{-\Lambda} + qe^{-\Lambda} + r$ where $q \in S_0^{(s)}(\langle \xi \rangle^{-M})$ choosing $N$ such that $-l + (\kappa - \bar{k})N \geq M$ and $r \in S_0^{(s)}(e^{-c\langle \xi \rangle^*_\mu})$ with $\bar{s}k < 1$ and $\bar{k} > \bar{k}$. It follows from Lemma 2.13 that $p_j \in S_0(m\phi^{-j/2}(\langle \xi \rangle^{-j+\bar{k}}) \cap S_0^{(s)}(\langle \xi \rangle^{1-(\kappa-\bar{k})})$. Note that $p_1$ is pure imaginary. Repeating the same arguments we have $p_1e^{-\Lambda} = p_1\#e^{-\Lambda} + \tilde{p}_1e^{-\Lambda} + \tilde{q}e^{-\Lambda} + \tilde{r}$ so that

$$pe^{-\Lambda} = (p + p_1)\#e^{-\Lambda} + p'te^{-\Lambda} + q'e^{-\Lambda} + r'$$

where $p'_2 \in S_0(m\phi^{-1}(\langle \xi \rangle^{2-\bar{k}}) \cap S_0^{(s)}(\langle \xi \rangle^{-2+(\kappa-\bar{k})})$ and $q' \in S_0^{(s)}(\langle \xi \rangle^{-M})$ and $r' \in S_0^{(s)}(e^{-c\langle \xi \rangle^*_\mu})$ with possibly different $\bar{s}, \bar{k}, c > 0$ such that $\bar{s}k < 1$ and $\bar{k} > \bar{k}$. Repeating the same argument to $p'_2e^{-\Lambda}$ we conclude the assertion. The second assertion can be proved similarly.

4 Energy estimates

Instead of $P = D^2_t - b(t)D_x a(x)D_x$ we study

$$Pu = D^2_t u - b(t)\langle D \rangle_{\mu} a(x)\langle D \rangle_{\mu} u$$

which differs from $\tilde{P}$ only by a zero-th order term which is irrelevant in the arguments proving Proposition 3.4. Assume $0 \leq a(x) \in G_0^s(\mathbb{R})$ and consider

$$P^s = Op(e^{(\tau - \theta t)\langle \xi \rangle^*_\mu})P \text{Op}(e^{-(\tau - \theta t)\langle \xi \rangle^*_\mu})$$

where $\tau > 0$ is a fixed positive constant and $\theta > 0$ is a positive parameter where $0 \leq \theta t \leq \tau$. Thanks to Proposition 3.4 we have

$$P^s = (D_t - i\theta \langle D \rangle_{\mu}^2 - b(t)\langle D \rangle_{\mu} a(x)\langle D \rangle_{\mu}$$

$$= A^2 - b(t)\langle D \rangle_{\mu} a(x)\langle D \rangle_{\mu} + R$$

with $A = D_t - i\theta \langle D \rangle_{\mu}^2$ where $R = b(t)\langle D \rangle_{\mu} \text{Op}(q + r)\langle D \rangle_{\mu}$ with $q(x, \xi) \in S_0^{(s)}(\langle \xi \rangle^{-2+2\kappa})$ and $r(x, \xi) \in S_0^{(s)}(e^{-c\langle \xi \rangle^*_\mu})$ with some $c > 0$ and $a_1(t, x, \xi) = (\tau - \theta t)D_x a(x)\partial_\xi \langle \xi \rangle_{\mu}^\kappa$.
In order to prove Theorem 1.1 we derive an apriori estimate for $P^x$. We now introduce the energy:

$$E(u) = \|\text{Op}(e^{-\Lambda})Au\|^2 + \text{Re}(b(t)a(\cdot)\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u, \text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u)$$

$$+ \|\langle D \rangle_{\mu}^\ast\text{Op}(e^{-\Lambda})u\|^2$$

with $\Lambda(t, x, \xi)$ defined by (2.3) where it is clear that

$$E(u) \geq \|\text{Op}(e^{-\Lambda})Au\|^2 + \|\langle D \rangle_{\mu}^\ast\text{Op}(e^{-\Lambda})u\|^2. \tag{4.2}$$

It is easy to check that

$$\frac{d}{dt}\|\text{Op}(e^{-\Lambda})Au\|^2 = -2\text{Re}(\text{Op}(\Lambda'e^{-\Lambda})Au, \text{Op}(e^{-\Lambda})Au)$$

$$-2\theta\text{Re}(\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}^\ast\text{Op}(e^{-\Lambda})u, \text{Op}(e^{-\Lambda})Au)$$

$$-2\text{Im}(\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}b(t)a\langle D \rangle_{\mu}u, \text{Op}(e^{-\Lambda})Au)$$

$$-2\text{Im}(\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}b(t)\text{Op}(a_1)\langle D \rangle_{\mu}u, \text{Op}(e^{-\Lambda})Au)$$

$$+ 2\text{Im}(\text{Op}(e^{-\Lambda})\Lambda u, \text{Op}(e^{-\Lambda})Au) - 2\text{Im}(\text{Op}(e^{-\Lambda})P^2u, \text{Op}(e^{-\Lambda})Au) \tag{4.3}$$

where $\Lambda' = \partial_\xi \Lambda$. We have also

$$\frac{d}{dt}\|\langle D \rangle_{\mu}^\ast\text{Op}(e^{-\Lambda})u\|^2 = -2\text{Re}(\langle D \rangle_{\mu}^\ast\text{Op}(\Lambda'e^{-\Lambda})u, \langle D \rangle_{\mu}^\ast\text{Op}(e^{-\Lambda})u)$$

$$-2\text{Im}(\langle D \rangle_{\mu}^\ast\text{Op}(e^{-\Lambda})u, \langle D \rangle_{\mu}^\ast\text{Op}(e^{-\Lambda})u)$$

$$-2\theta\text{Re}(\langle D \rangle_{\mu}^\ast\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}^\ast\text{Op}(e^{-\Lambda})u, \langle D \rangle_{\mu}^\ast\text{Op}(e^{-\Lambda})u). \tag{4.4}$$

On the other hand we have

$$\frac{d}{dt}\text{Re}(b(t)a\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u, \text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u)$$

$$= \text{Re}(b'(t)a(\cdot)\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u, \text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u)$$

$$-2\text{Re}(b(t)a(\cdot)\text{Op}(\Lambda'e^{-\Lambda})\langle D \rangle_{\mu}u, \text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u)$$

$$-2\text{Im}(b(t)a(\cdot)\text{Op}(e^{-\Lambda})A\langle D \rangle_{\mu}u, \text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u)$$

$$-2\theta\text{Re}(b(t)a(\cdot)\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}^{1+\varepsilon}u, \text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u). \tag{4.5}$$

We denote

$$\mathcal{E}_1 = \text{Re}(b(t)a(\cdot)\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}^{1+\varepsilon}u, \text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u)$$

$$+ \text{Re}(\langle D \rangle_{\mu}^\ast\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}^\ast\text{Op}(e^{-\Lambda})u), \langle D \rangle_{\mu}^\ast\text{Op}(e^{-\Lambda})u),$$

$$\mathcal{E}_2 = \text{Re}(\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}^\ast Au, \text{Op}(e^{-\Lambda})Au)$$

and

$$\mathcal{H} = \text{Im}\left\{(\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}b(t)a\langle D \rangle_{\mu}u, \text{Op}(e^{-\Lambda})Au)$$

$$+ (b(t)a\text{Op}(e^{-\Lambda})A\langle D \rangle_{\mu}u, \text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u)\right\},$$

$$\mathcal{K} = -2\text{Re}\left\{(b(t)a\text{Op}(\Lambda'e^{-\Lambda})\langle D \rangle_{\mu}u, \text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u)$$

$$+ (\langle D \rangle_{\mu}^\ast\text{Op}(\Lambda'e^{-\Lambda})u, \langle D \rangle_{\mu}^\ast\text{Op}(e^{-\Lambda})u)\right\}$$

$$+ \text{Re}(b'(t)a\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u, \text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u).$$
Then one can write
\begin{equation}
\frac{dE}{dt} = -2\theta \mathcal{E}_1 - 2\theta \mathcal{E}_2 - 2\mathcal{H} + \mathcal{K} - 2\Re (\text{Op}(\lambda e^{-\Lambda})Au, \text{Op}(e^{-\Lambda})Au)
\end{equation}
(4.6)
\begin{align*}
&-2\Im (\langle D \rangle_{\mu}^\kappa \text{Op}(e^{-\Lambda})Au, \langle D \rangle_{\mu}^\kappa \text{Op}(e^{-\Lambda})u) \\
&-2\Im (\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu} b(t) \text{Op}(a_1)\langle D \rangle_{\mu} u, \text{Op}(e^{-\Lambda})Au) \\
&+2\Im (\text{Op}(e^{-\Lambda})Ru, \text{Op}(e^{-\Lambda})Au) - 2\Im (\text{Op}(e^{-\Lambda})P^\mu u, \text{Op}(e^{-\Lambda})Au).
\end{align*}

In estimating $dE/dt$, a term which is bounded by
\begin{equation}
C \mu^\epsilon (\|\langle D \rangle_{\mu}^{\kappa/2} \text{Op}(e^{-\Lambda})u\|^2 + \|\langle D \rangle_{\mu}^{\kappa/2} \text{Op}(e^{-\Lambda})Au\|^2)
\end{equation}
(4.7)
with some $\epsilon > 0$ and $C > 0$ is irrelevant, choosing $\mu > 0$ small (see Lemmas 3.2 and 4.0 below) then we write
\[ A \leq B \]
if the inequality $A \leq B$ holds modulo terms bounded by (4.7). If $A = B$ holds modulo terms bounded by (4.7) we denote $A \simeq B$.

We start with estimating $\mathcal{E}_1$.

**Lemma 4.1.** We have
\[ \mathcal{E}_1 \simeq \Re (b(t) \text{Op}(\phi(\xi)_{\mu}^\kappa)\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu} u, \text{Op}(e^{-\Lambda})\langle D \rangle_{\mu} u) \]
\[ \simeq \|\text{Op}(\sqrt{b(t)}\sqrt{\phi(\xi)_{\mu}^{\kappa/2}})\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu} u\|^2. \]

**Proof.** Note that $\langle \xi \rangle_{\mu}^\kappa \in S_{\phi}(\langle \xi \rangle_{\mu}^\kappa) \cap S_\delta^{(s)}(\langle \xi \rangle_{\mu}^\kappa)$. From Proposition 3.3 taking $N$ large, one can write
\[ e^{-\Lambda} # \langle \xi \rangle_{\mu}^\kappa = (\langle \xi \rangle_{\mu}^\kappa + a_1 + a_2)e^{-\Lambda} + q e^{-\Lambda} + r, \quad r \in S_0 M(e^{-c(\xi)_{\mu}^\kappa}) \]
where $a_j \in S_{\phi}(\langle \xi \rangle_{\mu}^{\kappa-j+\kappa j} \phi^{-j/2}) \cap S_\delta^{(s)}(\langle \xi \rangle_{\mu}^{-M}(\kappa-j-\kappa))$ by Proposition 3.3 and Lemma 2.13 and $q \in S_\delta^{(s)}(\langle \xi \rangle_{\mu}^{-M})$ where $s \kappa < 1$, $\kappa > \kappa$ and $a_j$ is pure imaginary. Thanks to Lemma 3.4 one can write
\[ (\langle \xi \rangle_{\mu}^\kappa + a_1 + a_2)e^{-\Lambda} = (\langle \xi \rangle_{\mu}^\kappa + A_1 + A_2)\# e^{-\Lambda} + q e^{-\Lambda} + q' \]
where $q'$ and $r'$ enjoy the same properties as $q$ and $r$ and hence
\begin{equation}
(4.8) \quad e^{-\Lambda} \# \langle \xi \rangle_{\mu}^\kappa = (\langle \xi \rangle_{\mu}^\kappa + A_1 + A_2)\# e^{-\Lambda} + \tilde{q} e^{-\Lambda} + \tilde{r}
\end{equation}
where $A_j \in S_{\phi}(\langle \xi \rangle_{\mu}^{\kappa+j+j \phi^{-j/2}})$ and that $\tilde{q} \in S_\delta^{(s)}(\langle \xi \rangle_{\mu}^{-M})$, $\tilde{r} \in S_0 M(e^{-c(\xi)_{\mu}^\kappa})$ with possibly different $\tilde{s}$, $\tilde{\kappa}$ such that $\tilde{s} \tilde{\kappa} < 1$ and $\tilde{s} > \tilde{\kappa}$. Note that $A_1$ is pure imaginary and then
\begin{equation}
(4.9) \quad a \# (\langle \xi \rangle_{\mu}^\kappa + A_1 + A_2) = a \langle \xi \rangle_{\mu}^\kappa + \tilde{A}_1 + \tilde{A}_2
\end{equation}

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where $\tilde{A}_1 \in S_\phi((\xi)_{\mu}^{n+\kappa-1})$ is pure imaginary and $\tilde{A}_2 \in S_\delta((\xi)_{\mu}^{n+2\kappa-2})$ because $a \in S_\phi(\phi)$. Therefore from (4.8) and (4.9) one can write

$$a\#e^{-A} \#(\xi)_{\mu}^n = Q\#e^{-A} + a\#(\tilde{q}e^{-A} + \tilde{r}), \quad \tilde{r} \in S_{0,0}(e^{-C(\xi)_{\mu}})$$

with $\bar{\kappa} < 1$ and $\tilde{\kappa} > \bar{\kappa}$ where

$$\text{Re} \ Q - a(x)(\xi)_{\mu}^n \in S_\delta((\xi)_{\mu}^{n+2\kappa+2\bar{\kappa}}) \subset \mu^{2(\kappa-\bar{\kappa})}S_\delta((\xi)_{\mu}^{n+2\kappa})$$

since $\tilde{A}_1$ is pure imaginary. Therefore taking Lemma 3.1 and Corollary 3.5 into account one has

$$\text{Re}(\text{Op}(e^{-A}))(D)_{\mu}^{1+\kappa}u, \text{Op}(e^{-A})(D)_{\mu}u) \simeq \text{Re}(\text{Op}(\phi(\xi)_{\mu}^n)\text{Op}(e^{-A})(D)_{\mu}u, \text{Op}(e^{-A})(D)_{\mu}u).$$

On the other hand, thanks to Lemmas 3.2 and 3.3 one has

$$\|a\text{Op}(\tilde{q}e^{-A} + \tilde{r})(D)_{\mu}u, \text{Op}(e^{-A})(D)_{\mu}u) \leq C\mu\|\text{Op}(e^{-A})u\|^2$$

choosing $M \geq 2$. From (4.8) one obtains

$$\langle \xi \rangle_{\mu}^{2\kappa} \#e^{-A} \#(\xi)_{\mu}^n = (\langle \xi \rangle_{\mu}^{3\kappa} + \tilde{A}_1 + \tilde{A}_2)\#e^{-A} + \langle \xi \rangle_{\mu}^{2\kappa} \#(\tilde{q}e^{-A} + \tilde{r})$$

where $\tilde{A}_1 \in S_\delta((\xi)_{\mu}^{2\kappa+\bar{\kappa}})$ is pure imaginary and $\tilde{A}_2 \in S_\delta((\xi)_{\mu}^{\kappa+2\bar{\kappa}}) \subset \mu^{2(\kappa-\bar{\kappa})}S_\delta((\xi)_{\mu}^{3\kappa})$ because $\kappa + \bar{\kappa} = 1$ and $\phi^{-\gamma/2} \in S_\phi((\xi)_{\mu}^{\bar{\kappa}})$. Therefore

$$\text{Re}(\langle D \rangle_{\mu}^n\text{Op}(e^{-A})(D)_{\mu}u, \langle D \rangle_{\mu}^n\text{Op}(e^{-A})u) \simeq \|\langle D \rangle_{\mu}^{3\kappa/2}\text{Op}(e^{-A})u\|^2.$$

Repeating the same arguments proving (4.8) one can write

$$e^{-A} \#(\xi)_{\mu}^{-1} = (\langle \xi \rangle_{\mu}^{-1} + A_1)\#e^{-A} + \tilde{q}e^{-A} + \tilde{r}$$

where $A_1 \in S_\phi((\xi)_{\mu}^{\kappa-2}\phi^{-1/2}) \subset S_\delta((\xi)_{\mu}^{\kappa-2+\bar{\kappa}})$. Thus we have

$$\|\langle D \rangle_{\mu}^{3\kappa/2}\text{Op}(e^{-A})u\| = \|\langle D \rangle_{\mu}^{3\kappa/2}\text{Op}(e^{-A})\langle D \rangle_{\mu}^{-1}u\| \approx \|\langle D \rangle_{\mu}^{3\kappa/2-1}\text{Op}(e^{-A})(D)_{\mu}u, \text{Op}(e^{-A})(D)_{\mu}u)\|$$

because $\langle \xi \rangle_{\mu}^{3\kappa/2} \#A_1 \in S_\delta((\xi)_{\mu}^{\kappa/2+\bar{\kappa}+1}) \subset \mu^{\kappa+\bar{\kappa}}S_\delta((\xi)_{\mu}^{3\kappa/2-1})$. Since $3\kappa - 2 = -2\delta + \kappa$ and $a(x)\langle \xi \rangle_{\mu}^{\kappa} + \langle \xi \rangle_{\mu}^{2\kappa+\kappa} = (a(x) + \langle \xi \rangle_{\mu}^{-2\kappa})\langle \xi \rangle_{\mu}^{\kappa} = \phi(\xi)_{\mu}^\kappa$ we have the first assertion.

To prove the second assertion we note

$$\langle \phi^{1/2}(\xi)_{\mu}^{\kappa/2} \#(\phi^{1/2}(\xi)_{\mu}^{\kappa/2} - \phi(\xi)_{\mu}^\kappa) \in S_\phi((\xi)_{\mu}^{2\kappa}) \subset \mu^{2\kappa}S_\delta((\xi)_{\mu}^{2\kappa+3\kappa})$$

since $\phi^{1/2}(\xi)_{\mu}^{2\kappa} \in S_\phi(\phi^{1/2}(\xi)_{\mu}^{2\kappa})$ then using Corollary 3.5 we have

$$\|\sqrt{b(t)}\text{Op}(\sqrt{\phi}(\xi)_{\mu}^{\kappa/2})\text{Op}(e^{-A})(D)_{\mu}u\| \approx (b(t)\text{Op}(\phi(\xi)_{\mu}^\kappa)\text{Op}(e^{-A})(D)_{\mu}u, \text{Op}(e^{-A})(D)_{\mu}u)$$

which proves the second assertion. □
Lemma 4.2. We have
\[ \mathcal{E}_1 \geq (1 - C\mu^{\kappa-\tilde{k}})\|\langle D \rangle_{\mu}^{3\kappa/2}\text{Op}(e^{-\Lambda})u\|^2. \]

Proof. Since \( 0 \leq a(x)\langle \xi \rangle_\mu^\kappa \in S(\langle \xi \rangle_\mu^\kappa, dx^2 + \langle \xi \rangle_\mu^2 dt^2) \) then from the Fefferman-Phong inequality it follows that
\[ \text{Re}(\text{Op}(a(\langle \xi \rangle_\mu^\kappa))\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u, \text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u) \]
\[ \geq -C\|\langle \xi \rangle_{\mu}^{\kappa/2-1}\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u\|^2 \geq -C\|\langle \xi \rangle_{\mu}^{\kappa/2}\text{Op}(e^{-\Lambda})u\|^2. \]

The rest of the proof is clear from Lemma 4.0 \[ \square \]

Lemma 4.3. Assume that \( T \in S_\phi(\sqrt{\phi}(\langle \xi \rangle_\mu^{\kappa/2}) \) is real or pure imaginary. Then there is \( C > 0 \) such that
\[ \| b(t)\text{Op}(T)\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u\|^2 \leq C\mathcal{E}_1. \]

Proof. We may assume that \( T \) is real. Note \( \bar{T}#T = T^2 + R \) with \( R \in S_\phi(\langle \xi \rangle_{\mu}^{-2+\kappa}) \subset \mu^{2\kappa}S_\delta(\langle \xi \rangle_{\mu}^{-2+3\kappa}) \) and by Corollary 3.3 one has
\[ (\text{Op}(R)\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u, \text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u) \simeq 0. \]

Write \( C\phi(\langle \xi \rangle_{\mu}^\kappa - T^2 = Cq^2 \) with \( q = \langle \xi \rangle_{\mu}^{\kappa/2}\sqrt{\phi}\sqrt{1 - C^{-1}T^2\langle \xi \rangle_{\mu}^{-\kappa}\phi^{-1}} \) where \( q \in S_\phi(\langle \xi \rangle_{\mu}^{\kappa/2}\sqrt{\phi}) \) if we take a large \( C > 0 \). Since \( \phi \in S_\phi(\tilde{\phi}) \) we have
\[ q\#q - q^2 = r \in S_\phi(\langle \xi \rangle_{\mu}^{\kappa-2}) \subset \mu^{2\kappa}S_\delta(\langle \xi \rangle_{\mu}^{-2+3\kappa}) \]
and \( |(\text{Op}(r)\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}, \text{Op}(e^{-\Lambda})\langle D \rangle_{\mu})| \simeq 0 \) by Corollary 3.3. The rest of the proof is clear. \[ \square \]

Proposition 4.1. There exists \( C > 0 \) such that
\[ |\mathcal{H}| \lesssim C\mu^{\kappa-\tilde{k}}\mathcal{E}_1. \]

Proof. We first write
\[ \mathcal{H} = \text{Im} (b(t)\text{Op}(e^{-\Lambda})A\langle D \rangle_{\mu}u, [a, \text{Op}(e^{-\Lambda})]\langle D \rangle_{\mu}u) \]
\[ + \text{Im} (b(t)|\text{Op}(e^{-\Lambda}), \langle D \rangle_{\mu}|a\langle D \rangle_{\mu}u, \text{Op}(e^{-\Lambda})Au) \]
\[ + \text{Im} (b(t)\text{Op}(e^{-\Lambda})a\langle D \rangle_{\mu}u, [\langle D \rangle_{\mu}, \text{Op}(e^{-\Lambda})]Au) \]
\[ = \text{Im} \mathcal{H}_1 + \text{Im} \mathcal{H}_2 + \text{Im} \mathcal{H}_3 \]
because \( \langle D \rangle_{\mu}A = A\langle D \rangle_{\mu} \). Then to prove the proposition it suffices to prove there is \( C > 0 \) such that
\[ (4.11) \quad |\text{Im} \mathcal{H}_j| \lesssim C\mu^{\kappa-\tilde{k}}\mathcal{E}_1 \]

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with some $C > 0$ for $j = 1, 2, 3$. We first study

$$\mathcal{H}_1 = (b(t)\text{Op}(e^{-\Lambda})A\langle D\rangle_{\mu} u, [a, \text{Op}(e^{-\Lambda})]\langle D\rangle_{\mu} u).$$

From Proposition 3.3 and Lemma 3.3 one can write

$$(4.12) \quad e^{-\Lambda} # a - a # e^{-\Lambda} = (B_1 + B_2) # e^{-\Lambda} + q e^{-\Lambda} + r$$

where $q \in S_\delta((\xi)_\mu^M)$ and $r \in S_{\delta,0}(e^{-c(\xi)_\mu^R})$ with $\hat{s} \kappa < 1$ and $\hat{s} \kappa > \hat{s} \kappa$. Note that $B_1 \in S_{\phi}(\phi^1 - j/2(\xi)_\mu^{\kappa - j}) \cap S_{\delta}^s(1)$ since $a \in S_{\phi}(\phi) \cap S_{\delta}^s(1)$. Note that

$$\|\langle \text{Op}(e^{-\Lambda})A\langle D\rangle_{\mu} u, \text{Op}(B_2)\text{Op}(e^{-\Lambda})\langle D\rangle_{\mu} u\rangle\| \leq \|(\langle D\rangle_{\mu}^{1+\kappa/2}\text{Op}(e^{-\Lambda})\langle D\rangle_{\mu} A u\| + \|\langle D\rangle_{\mu}^{1-\kappa/2}\text{Op}(B_2)\text{Op}(e^{-\Lambda})\langle D\rangle_{\mu} u\| \approx 0$$

by Corollary 3.5 because $B_2 \in S_{\phi}(\langle \xi \rangle^2 \mu^2) \subset \mu^{2(\kappa - \kappa)}S_{\delta}(\langle \xi \rangle^2 \mu^2)$. Noting that $\langle \xi \rangle^{1-\kappa/2} # B_1 \in S\phi(\sqrt{\delta}(\xi)^{\kappa - \kappa}/2) \subset \mu^{\kappa - \kappa}S_{\phi}(\sqrt{\delta}(\xi)^{\kappa/2})$ from Lemma 4.3 one has

$$\|b(t)\langle D\rangle_{\mu} B_1\text{Op}(e^{-\Lambda})\langle D\rangle_{\mu} u\|^2 \leq C\mu^{2(\kappa - \kappa)}\mathcal{E}_1$$

with some $C > 0$. Therefore (4.11) for $j = 1$ is proved. We turn to study

$$\mathcal{H}_2 = (b(t)|\text{Op}(e^{-\Lambda}), \langle D\rangle_{\mu} a\langle D\rangle_{\mu} u, \text{Op}(e^{-\Lambda})Au)$$

$$= (b(t)\langle D\rangle_{\mu}^{1-\kappa/2}\text{Op}(e^{-\Lambda}), \langle D\rangle_{\mu} a\langle D\rangle_{\mu} u, \langle D\rangle_{\mu}^{\kappa/2}\text{Op}(e^{-\Lambda})Au)$$

where the right-hand side is estimated by

$$C\|\langle D\rangle_{\mu}^{\kappa/2}\text{Op}(e^{-\Lambda})Au\|^2 + C\|b(t)\langle D\rangle_{\mu}^{\kappa/2}[\text{Op}(e^{-\Lambda}), \langle D\rangle_{\mu} a\langle D\rangle_{\mu} u]\|^2$$

where $e = (\kappa - \kappa)/2$. We now estimate $\|b(t)\langle D\rangle_{\mu}^{\kappa/2}[\text{Op}(e^{-\Lambda}), \langle D\rangle_{\mu} a\langle D\rangle_{\mu} u]\|$. Taking $N$ large so that $(\kappa - \kappa)N > M + 1$ in Proposition 3.3 one can write

$$(4.13) \quad e^{-\Lambda} # \langle \xi \rangle - \langle \xi \rangle # e^{-\Lambda} = (A_1 + A_2) # e^{-\Lambda} + q e^{-\Lambda} + R$$

where $A_j \in S\phi(\langle \xi \rangle)_{\mu}^{j+1-\phi^j/2} \cap S_{\delta}^s(\langle \xi \rangle^{1-\kappa} - \kappa)$, $q \in S_{\delta}^s(\langle \xi \rangle^{M})$ and $R \in S_{\delta,0}(\langle \xi \rangle_{\mu}^R)$ with $\hat{s} \kappa < 1$, $\hat{s} \kappa > \hat{s} \kappa$. It follows from Lemmas 3.2 and 3.3 that $\|\langle D\rangle_{\mu}^{\kappa/2+\epsilon}\text{Op}(q e^{-\Lambda} + R)\rangle\langle D\rangle_{\mu} u\| \approx 0$. From (4.12) one can write $e^{-\Lambda} # a = (a + B_1 + B_2) # e^{-\Lambda} + q e^{-\Lambda} + r$ then

$$(A_1 + A_2) # e^{-\Lambda} # a = (\tilde{A}_1 + \tilde{A}_2) # e^{-\Lambda} + (A_1 + A_2) # (q e^{-\Lambda} + r)$$

where $\tilde{A}_j \in S\phi(\phi^1 - j/2(\xi)^{\kappa - j + 1})$. It follows from Lemma 3.3 and Lemma 3.2 that

$$\|\langle D\rangle_{\mu}^{\kappa/2+\epsilon}\text{Op}(A_1 + A_2)\text{Op}(q e^{-\Lambda} + r)\rangle\langle D\rangle_{\mu} u\| \approx 0.$$
It remains to estimate $\|b(t)(\mathcal{D})_{\mu}^{-\kappa/2+\epsilon}\text{Op}(\tilde{\mathcal{A}}_1)\text{Op}(e^{-\Lambda})\langle \mathcal{D} \rangle_{\mu} u\|$. Since $\langle \xi \rangle_{\mu}^{-\kappa/2+\epsilon} \# \tilde{\mathcal{A}}_1 \in S_\phi(\phi^{1/2} \langle \xi \rangle_{\mu}^{-\kappa/2}) \subset \mu^* S_\phi(\phi^{1/2} \langle \xi \rangle_{\mu}^{-\kappa/2})$ we conclude (4.11) for $j = 2$ from Lemma 4.3. We finally consider $\text{Im} \mathcal{H}_3$. From (4.13) one can write

$$[\text{Op}(e^{-\Lambda}), \langle \mathcal{D} \rangle_{\mu}] = \text{Op}(A_1 + A_2)\text{Op}(e^{-\Lambda}) + \text{Op}(qe^{-\Lambda} + R).$$

Thanks to Lemmas 3.2 and 3.3 one has

$$\|(\text{Op}(e^{-\Lambda})a\langle \mathcal{D} \rangle_{\mu} u, \text{Op}(qe^{-\Lambda} + R)Au)\| \leq \|(\langle \mathcal{D} \rangle_{\mu}^{-1}\text{Op}(e^{-\Lambda})a\langle \mathcal{D} \rangle_{\mu} u\| \|(\langle \mathcal{D} \rangle_{\mu}^{-1}\text{Op}(e^{-\Lambda})Au\| \simeq 0.$$

Write

$$\begin{align*}
(b(t)a\text{Op}(e^{-\Lambda})\langle \mathcal{D} \rangle_{\mu} u, \text{Op}(A_1 + A_2)\text{Op}(e^{-\Lambda})Au) \\
= (b(t)[\text{Op}(e^{-\Lambda}), a]\langle \mathcal{D} \rangle_{\mu} u, \text{Op}(A_1 + A_2)\text{Op}(e^{-\Lambda})Au) \\
+ (b(t)a\text{Op}(e^{-\Lambda})\langle \mathcal{D} \rangle_{\mu} u, \text{Op}(A_1 + A_2)\text{Op}(e^{-\Lambda})Au).
\end{align*}$$

It is clear from Corollary 3.5 that

$$\|(b(t)a\text{Op}(e^{-\Lambda})\langle \mathcal{D} \rangle_{\mu} u, \text{Op}(A_2)\text{Op}(e^{-\Lambda})Au)\| \leq \|(b(t)\langle \mathcal{D} \rangle_{\mu}^{\kappa/2}\text{Op}(\tilde{\mathcal{A}}_2)a\text{Op}(e^{-\Lambda})\langle \mathcal{D} \rangle_{\mu} u\| \|(\langle \mathcal{D} \rangle_{\mu}^{\kappa/2}\text{Op}(e^{-\Lambda})Au\| \simeq 0$$

because $\langle \xi \rangle_{\mu}^{-\kappa/2} \# \tilde{\mathcal{A}} \# a \in S_\phi(\langle \xi \rangle_{\mu}^{-\kappa/2 - 1}) \subset \mu^* \tilde{\mathcal{A}} S_\phi(\langle \xi \rangle_{\mu}^{3\kappa/2 - 1})$. Note that

$$2(b(t)a\text{Op}(e^{-\Lambda})\langle \mathcal{D} \rangle_{\mu} u, \text{Op}(A_1)\text{Op}(e^{-\Lambda})Au) \leq \|(b(t)\langle \mathcal{D} \rangle_{\mu}^{\kappa/2+\epsilon}\text{Op}(\tilde{\mathcal{A}}_1)a\text{Op}(e^{-\Lambda})\langle \mathcal{D} \rangle_{\mu} u\|^2 + \|(\langle \mathcal{D} \rangle_{\mu}^{\kappa/2+\epsilon}\text{Op}(e^{-\Lambda})Au\|^2.$$

Since $\langle \xi \rangle_{\mu}^{-\kappa/2+\epsilon} \# \tilde{\mathcal{A}}_1 \# a \in S_\phi(\sqrt{\phi}(\langle \xi \rangle_{\mu}^{-\kappa/2+\epsilon})) \subset \mu^* S_\phi(\sqrt{\phi}(\langle \xi \rangle_{\mu}^{\kappa/2}))$ one can conclude from Lemma 4.3.

$$(b(t)a\text{Op}(e^{-\Lambda})\langle \mathcal{D} \rangle_{\mu} u, \text{Op}(A_1 + A_2)\text{Op}(e^{-\Lambda})Au) \leq C\mu^{\kappa-\tilde{\kappa}}\mathcal{E}$$

with some $C > 0$. Using (4.12) we see that

$$\|(\text{Op}(e^{-\Lambda}), a]\langle \mathcal{D} \rangle_{\mu} u, \text{Op}(A_1 + A_2)\text{Op}(e^{-\Lambda})Au\| \simeq 0$$

in virtue of Lemmas 3.2 and 5.6 and Corollary 5.5. Thus we get (4.11) for $j = 3$. □

**Proposition 4.2.** We have

$$\mathcal{K} \leq C\mu^{3\kappa/2-\tilde{\kappa}}\mathcal{E}_1$$

with some $C > 0$. 

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Since \( \Lambda' = \partial_t \Lambda \in S_\phi(\sqrt{\phi(\xi_{\mu}^{1-\kappa})}) \cap S_\delta(\xi_{\mu}^{1-\kappa}) \) by Lemma 2.14 applying
Lemma 3.4 we get

\[
\Lambda' e^{-\Lambda} = (\Lambda' + \lambda_1 + \lambda_2)\# e^{-\Lambda} + q e^{-\Lambda} + r
\]

with \( \lambda_j \in S_\phi(\phi^{(j-1)/2}(\xi_{\mu}^{(j-1)-\kappa + \bar{\kappa}^j})) \cap S_\delta(\xi_{\mu}^{(1-\kappa - (\kappa - \bar{\kappa})^j}), j = 1, 2 \) and \( q \in S_\delta(\xi_{\mu}^{1-\kappa}) \), \( r \in S_\delta(\xi_{\mu}^{1-\kappa}) \) with \( \bar{\kappa} < 1, \bar{\kappa} > \kappa \). Note that \( \lambda_1 \) is pure
imaginary. Since \( a \in S_\phi(\phi) \) one can write

\[
a\#(\Lambda' + \lambda_1 + \lambda_2) = a\Lambda' + Q_1 + Q_2
\]

where \( Q_2 \in S_\phi(\phi^{(3-j)/2}(\xi_{\mu}^{(j-1)-\kappa + \bar{\kappa}^j})) \) and \( Q_1 \) is pure
imaginary. Note that

\[
(a \text{ Op}(qe^{-\Lambda} + r)(D)_\mu u, \text{ Op}(e^{-\Lambda})(D)_\mu u) \simeq 0
\]

by Lemmas 3.3 and 3.2 therefore one has

\[
\Re(b(t)a\text{ Op}(\Lambda' e^{-\Lambda})(D)_\mu u, \text{ Op}(e^{-\Lambda})(D)_\mu u)
\]

\[
\simeq (\text{ Op}(b(t)a\Lambda')(D)_\mu u, \text{ Op}(e^{-\Lambda})(D)_\mu u)
\]

\[
+ \Re(\text{ Op}(b(t)Q_2)(D)_\mu u, \text{ Op}(e^{-\Lambda})(D)_\mu u).
\]

Using 4.14 one has \( (\langle \langle D \rangle_{\mu}^\kappa \text{ Op}(\Lambda' e^{-\Lambda})u, (D)_\mu^\kappa \text{ Op}(e^{-\Lambda})u \rangle) \simeq (\langle \langle D \rangle_{\mu}^{2\kappa - 1} \text{ Op}(\Lambda' + \lambda_1 + \lambda_2) \text{ Op}(e^{-\Lambda})u, (D)_\mu^\kappa \text{ Op}(e^{-\Lambda})u \rangle) \) by

\[
(\langle \langle D \rangle_{\mu}^\kappa \text{ Op}(\Lambda' + \lambda_1 + \lambda_2) \text{ Op}(e^{-\Lambda})u, (D)_\mu^\kappa \text{ Op}(e^{-\Lambda})u \rangle)
\]

\[
\simeq (\langle \langle D \rangle_{\mu}^{2\kappa - 1} \text{ Op}(\Lambda' + \lambda_1 + \lambda_2) e^{-\Lambda} u, \text{ Op}(e^{-\Lambda})(D)_\mu u \rangle).
\]

For \( \Lambda' \in S_\phi(\sqrt{\phi(\xi_{\mu}^{1-\kappa})}) \subset \mu^{2\kappa - \bar{\kappa}} \delta(\xi_{\mu}^{3\kappa}) \), and \( \lambda_j \in S_\phi(\phi^{(j-1)/2}(\xi_{\mu}^{(1-\kappa - (\kappa - \bar{\kappa})^j})) \cap A_j \subset S_\delta(\xi_{\mu}^{(1-\kappa - (\kappa - \bar{\kappa})^j)}). \) Thus one has

\[
(\langle \langle D \rangle_{\mu}^\kappa \text{ Op}(\Lambda' e^{-\Lambda})u, (D)_\mu^\kappa \text{ Op}(e^{-\Lambda})u \rangle)
\]

\[
\simeq (\langle \langle D \rangle_{\mu}^{2\kappa - 1} \text{ Op}(\Lambda' + \lambda_1 + \lambda_2) e^{-\Lambda} u, \text{ Op}(e^{-\Lambda})(D)_\mu u \rangle).
\]

Repeating the same argument one has

\[
(\langle \langle D \rangle_{\mu}^{2\kappa - 1} \text{ Op}(\Lambda' + \lambda_1 + \lambda_2)(D)_\mu^{-1}(\langle \langle D \rangle_{\mu}^{-1} \text{ Op}(e^{-\Lambda})u, \text{ Op}(e^{-\Lambda})(D)_\mu u \rangle) \simeq 0
\]

and hence

\[
(\langle \langle D \rangle_{\mu}^\kappa \text{ Op}(\Lambda' e^{-\Lambda})u, (D)_\mu^\kappa \text{ Op}(e^{-\Lambda})u \rangle)
\]

\[
\simeq (\langle \langle D \rangle_{\mu}^{2\kappa - 1} \text{ Op}(\Lambda' + \lambda_1 + \lambda_2)(D)_\mu^{-1} \text{ Op}(e^{-\Lambda})(D)_\mu u, \text{ Op}(e^{-\Lambda})(D)_\mu u \rangle).
\]

Since \( \xi_{\mu}^{2\kappa - 1}(\Lambda' + \lambda_1 + \lambda_2) \# \xi_{\mu}^{-1} - \xi_{\mu}^{2\kappa - 2} \Lambda' \in S_\delta(\xi_{\mu}^{2\kappa - 2 + \kappa + \bar{\kappa}}) \) which is contained in
\( \mu^{2\kappa - \bar{\kappa}} \delta(\xi_{\mu}^{2\kappa + 3\kappa}) \) we obtain

\[
(\langle \langle D \rangle_{\mu}^\kappa \text{ Op}(\Lambda' e^{-\Lambda})u, (D)_\mu^\kappa \text{ Op}(e^{-\Lambda})u \rangle)
\]

\[
\simeq (\text{ Op}(\Lambda' \xi_{\mu}^{2\kappa - 2} \text{ Op}(e^{-\Lambda})(D)_\mu u, \text{ Op}(e^{-\Lambda})(D)_\mu u).\
\]

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We now study the sum:

$$(\text{Op}(b(t)a\Lambda')\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u, \text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u) + (\text{Op}(\Lambda'(\xi)_{\mu}^{2\kappa-2})\text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u, \text{Op}(e^{-\Lambda})\langle D \rangle_{\mu}u).$$

Noting

$$b(t)a(x)\Lambda' + (\xi)_{\mu}^{2\kappa-2}\Lambda' = \frac{|b'(t)|\phi (b(t)a(x) + (\xi)_{\mu}^{-2\delta})}{b(t)\phi + (\xi)_{\mu}^{-2\delta}}$$

we write

$$\Delta = 2(b(t)a(x)\Lambda' + (\xi)_{\mu}^{2\kappa-2}\Lambda') - b'(t)a(x) = \frac{F}{b(t)\phi(x, \xi) + (\xi)_{\mu}^{-2\delta}}$$

where

$$F = 2|b'(t)|\phi (b(t)a(x) + (\xi)_{\mu}^{-2\delta}) - b'(t)a(x)b(t)\phi - (\xi)_{\mu}^{-2\delta}b'(t)a(x)$$

$$= |b'(t)|\phi (b(t)a(x) + (\xi)_{\mu}^{-2\delta}) + \left\{|b'(t)|b(t)a(x)\phi + |b'(t)|\phi (\xi)_{\mu}^{-2\delta} - b'(t)b(t)a(x)\phi - b'(t)a(x)(\xi)_{\mu}^{-2\delta}\right\}$$

$$\geq |b'(t)|\phi (b(t)a(x) + (\xi)_{\mu}^{-2\delta})$$

since $\phi \geq a(x)$. Therefore there exists $c > 0$ such that

$$\frac{F}{b(t)\phi + (\xi)_{\mu}^{-2\delta}} \geq \frac{|b'(t)|\phi (b(t)a(x) + (\xi)_{\mu}^{-2\delta})}{b(t)\phi + (\xi)_{\mu}^{-2\delta}}$$

$$= \frac{|b'(t)|\phi (b(t)a(x) + (\xi)_{\mu}^{-2\delta})}{b(t)a(x) + (b(t) + 1)(\xi)_{\mu}^{-2\delta}} \geq c|b'(t)|\phi$$

because $0 \leq b(t) \leq C$. On the other hand it is clear that $F \leq 3|b'(t)|\phi (b(t)\phi + (\xi)_{\mu}^{-2\delta})$ for $a(x) \leq \phi$ and hence

$$c|b'(t)|\phi \leq \frac{F}{b(t)\phi + (\xi)_{\mu}^{-2\delta}} \leq 3|b'(t)|\phi.$$

**Lemma 4.4.** We have

$$\left(\frac{|b'(t)|^{-1}\phi^{-1}F}{b(t)\phi + (\xi)_{\mu}^{-2\delta}}\right)^{1/2} \in S_{\phi}(1)$$

uniformly in $t$ with $|b'(t)| \neq 0$. That is, for any $k, l$ there exists $C_{kl}$ such that

$$|\partial_x^k \partial_{t}^{l}\left(\frac{|b'(t)|^{-1}\phi^{-1}F}{b(t)\phi + (\xi)_{\mu}^{-2\delta}}\right)^{1/2}| \leq C_{kl}\phi^{-k/2}(\xi)_{\mu}^{-l}$$

for any $t \in [0, T]$ with $b'(t) \neq 0$. 

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Proof. Assume \( b'(t) \neq 0 \). Set \( A = |b'(t)|^{-1} \) and \( B = b(t) \). We first note that

\[
|\partial_x^k \partial_t^j (B \phi + \langle \xi \rangle_{\mu}^{-2\delta})| \leq C_{kl} (B \phi + \langle \xi \rangle_{\mu}^{-2\delta}) \phi^{-k/2} \langle \xi \rangle_{\mu}^{-l}
\]

because \( \phi, \langle \xi \rangle_{\mu}^{-2\delta} \in S(\phi, g) \) from which it follows that

\[
|\partial_x^k \partial_t^j (B \phi + \langle \xi \rangle_{\mu}^{-2\delta})^{-1}| \leq C_{kl} (B \phi + \langle \xi \rangle_{\mu}^{-2\delta})^{-1} \phi^{-k/2} \langle \xi \rangle_{\mu}^{-l}.
\]

Since

\[
A\phi^{-1} F = 2(Ba(x) + \langle \xi \rangle_{\mu}^{-2\delta}) - \frac{b'(t)}{|b'(t)|} \left( Ba(x) + \phi^{-1} a \langle \xi \rangle_{\mu}^{-2\delta} \right)
\]

and \( a(x) \in S(\phi, g) \) it is easy to see

\[
|\partial_x^k \partial_t^j A\phi^{-1} F| \leq C_{kl} (B \phi + \langle \xi \rangle_{\mu}^{-2\delta}) \phi^{-k/2} \langle \xi \rangle_{\mu}^{-l}.
\]

Therefore we conclude that

\[
(4.18) \quad |\partial_x^k \partial_t^j (A\phi^{-1} F/(B \phi + \langle \xi \rangle_{\mu}^{-2\delta}))| \leq C_{kl} \phi^{-k/2} \langle \xi \rangle_{\mu}^{-l}
\]

where \( C_{kl} \) is independent of \( A \) and \( B \). Since \( c \leq A\phi^{-1} F/(B \phi + \langle \xi \rangle_{\mu}^{-2\delta}) \leq 3 \) the proof follows from (4.18).

Define \( \Gamma \) by

\[
\Gamma = \begin{cases} 
|b'(t)|^{1/2} \phi^{1/2} \left( \frac{|b'(t)|^{-1} \phi^{-1} F}{b(t) \phi + \langle \xi \rangle_{\mu}^{-2\delta}} \right)^{1/2} & \text{if } b'(t) \neq 0 \\
0 & \text{if } b'(t) = 0
\end{cases}
\]

then thanks to Lemma 4.4 we have \( \Gamma \in S_{\sqrt{\phi}}(\sqrt{\phi}) \) uniformly in \( t \). Since \( \Delta = \Gamma^2 \) we can write

\[
\Delta = \Gamma \# \Gamma + R
\]

where \( R \in S_{\phi}(\langle \xi \rangle_{\mu}^{-2}) \) uniformly in \( t \) from which one has

\[
(Op(\Delta) Op(e^{-\Lambda}) (D)_{\mu} u, Op(e^{-\Lambda}) (D)_{\mu} u) \geq -C \| (D)_{\mu}^{-1} Op(e^{-\Lambda}) \langle \xi \rangle_{\mu}^{2\delta} \| u^2
\]

which proves

\[
(4.19) \quad -2(Op(b(t)a\Lambda' + \langle \xi \rangle_{\mu}^{2\delta-2} \Lambda') Op(e^{-\Lambda}) (D)_{\mu} u, Op(e^{-\Lambda}) (D)_{\mu} u) + Re(b'(t)a Op(e^{-\Lambda}) (D)_{\mu} u, Op(e^{-\Lambda}) (D)_{\mu} u) \leq 0.
\]

We get back to estimate the second term on the right-hand side of (4.15) which is estimated as

\[
|\langle Op(b(t)Q_2) Op(e^{-\Lambda}) (D)_{\mu} u, Op(e^{-\Lambda}) (D)_{\mu} u \rangle| \leq C (|b(t) Q_2 | D)_{\mu}^{-\tilde{\gamma}} Op(e^{-\Lambda}) (D)_{\mu} u^2 + |\langle D)_{\mu}^{-1+\tilde{\gamma}} Op(e^{-\Lambda}) \rangle (D)_{\mu} u|^2).
\]

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Since $\langle \xi \rangle_\mu^{\tilde{\kappa}} \# Q_2 \in S_\phi(\sqrt{\phi}(\xi_\mu^{\tilde{\kappa}+\tilde{\kappa}}) \subset \mu^{3\kappa/2-\tilde{\kappa}} S_\phi(\sqrt{\phi}(\xi_\mu^{\tilde{\kappa}+\tilde{\kappa}}))$ then from Lemma 4.3 it follows that

\begin{equation}
(4.20) \quad \text{Re}(\text{Op}(b(t)Q_2)e^{-\Lambda} \langle D \rangle_\mu u, e^{-\Lambda} \langle D \rangle_\mu u) \leq C \mu^{3\kappa/2-\tilde{\kappa}} E_1
\end{equation}

with some $C > 0$. We end the proof of Proposition 4.2 combining (4.15), (4.17), (4.19) and (4.20).

**Lemma 4.5.** There exists $C > 0$ such that

\[ (\text{Op}(\Lambda') \text{Op}(e^{-\Lambda})u, \text{Op}(e^{-\Lambda})u) \geq -C \mu^\kappa ||\text{Op}(e^{-\Lambda})u||^2. \]

**Proof.** Recall $\Lambda' \in S_\phi(\rho^{1/2}(\xi_\mu^{\beta}))$ by Lemma 2.14. Since $h = \left( \sup g/g^* \right)^{1/2} = \phi^{-1/2}(\xi_\mu^{\beta})^{-1} \leq \mu^\kappa \phi^{-1/2}(\xi_\mu^{\beta})$ by (2.14) then we see that $\mu^{-\kappa} \Lambda' \in S_\phi(1/h)$ hence the sharp Gårding inequality (see [5] Theorem 18.6.2) proves the assertion. \(\square\)

**Lemma 4.6.** There exists $C > 0$ such that

\begin{align*}
(4.21) & \quad \text{Re}(\text{Op}(\Lambda' e^{-\Lambda})Au, \text{Op}(e^{-\Lambda})Au) \geq -C \mu^\kappa ||\text{Op}(e^{-\Lambda})Au||^2, \\
(4.22) & \quad \mathcal{E}_2 \geq (1 - C \mu^\kappa) \|\langle D \rangle_\mu^{3/2} \text{Op}(e^{-\Lambda})Au\|^2, \\
(4.23) & \quad \text{Re}(\langle D \rangle_\mu^{3/2} \text{Op}(e^{-\Lambda})\langle D \rangle_\mu^{\kappa} u, \langle D \rangle_\mu^{\kappa} \text{Op}(e^{-\Lambda})u) \\
& \quad \geq (1 - C \mu^\kappa) \|\langle D \rangle_\mu^{3/2} \text{Op}(e^{-\Lambda})u\|^2.
\end{align*}

**Proof.** From (4.14) one has

\[ \Lambda' e^{-\Lambda} = (\Lambda' + \lambda) \# e^{-\Lambda} + qe^{-\Lambda} + r, \quad \lambda \in S_\phi(\langle \xi_\mu^\kappa \rangle) \]

where $q \in S_\phi(\langle \xi_\mu^\kappa \rangle)$ and $r \in S_\phi(\langle \xi_\mu^\kappa \rangle)$ with $\bar{\kappa} < 1$, $\bar{\kappa} > \bar{\kappa}$. Noting $\lambda \in \mu^\kappa S_\phi(1)$ then (4.21) follows from Lemma 4.3 and Lemmas 3.2 and 3.3.

We next consider $\mathcal{E}_2 = \text{Re}(\text{Op}(e^{-\Lambda})\langle D \rangle_\mu^{3/2} Au, \text{Op}(e^{-\Lambda})Au)$. From the same arguments proving (4.13) one can write

\[ e^{-\Lambda} \# \langle \xi_\mu^\kappa \rangle = (\langle \xi_\mu^\kappa + A_1 \rangle) \# e^{-\Lambda} + qe^{-\Lambda} + R \]

where $A_1 \in S_\phi(\langle \xi_\mu^\kappa \rangle)$ and $q \in S_\phi(\langle \xi_\mu^\kappa \rangle)$ and $R \in S_\phi(\langle \xi_\mu^\kappa \rangle)$ with $\tilde{\kappa} < 1$, $\bar{\kappa} > \bar{\kappa}$. Noting

\[ \text{Re}(\text{Op}(A_1)\text{Op}(e^{-\Lambda})Au, \text{Op}(e^{-\Lambda})Au) \geq \|\langle D \rangle_\mu^{3/2} \text{Op}(e^{-\Lambda})Au, \text{Op}(e^{-\Lambda})Au\|^2 \]

and applying Lemmas 3.3 and 3.2 to $\text{Op}(qe^{-\Lambda} + r)$ we conclude (4.22). The last assertion is proved similarly. \(\square\)

**Proposition 4.3.** We have

\[ |2\text{Im}(b(t)\text{Op}(e^{-\Lambda})\langle D \rangle_\mu \text{Op}(a_1)\langle D \rangle_\mu u, \text{Op}(e^{-\Lambda})Au)| \leq \tau \sqrt{c^* (\mathcal{E}_1 + \mathcal{E}_2)} \]

where

\begin{equation}
(4.24) \quad c^* = 2 \sup_{t \in [0,T]} b(t) \sup_{x \in \mathbb{R}} \partial^2_x u(x).
\end{equation}
Proof. Recalling (4.13) and taking Lemmas 3.2 and 3.3 into account we see

\[
\left\langle (D)_{\mu}^{-\kappa/2} \text{Op}(e^{-A})(D)_{\mu} \text{Op}(a_1)(D)_{\mu} u \right\rangle \approx \left\langle (D)_{\mu}^{-\kappa/2} \text{Op}(\langle \xi \rangle_{\mu} + A_1 + A_2) \text{Op}(e^{-A}) \text{Op}(a_1)(D)_{\mu} u \right\rangle^2.
\]

because \( a_1 = (\tau - \theta t)D_xa_0\partial_\xi \langle \xi \rangle_{\mu}^\kappa \in S_e(\phi^{1/2}(\xi)^{1-\kappa}) \cap S^e(\langle \xi \rangle^{1-\kappa}) \). From Proposition 3.3 and Lemma 3.4 we have

\((4.25)\quad a_1\#e^{-A} - e^{-A} \#a_1 = (b_1 + b_2)\#e^{-A} + qe^{-A} + R\)

where \( b_j \in S_e(\phi^{-j/2}(\xi)^{\kappa j + \kappa - 1 - j}) \cap S^e(\langle \xi \rangle^{(\kappa - \kappa_j)j}) \) and \( q \in S^e(\langle \xi \rangle^{-M}) \), \( R \in S^e(\langle e^{-c(\xi)^{\kappa_j}} \rangle) \) with \( \bar{s} < 1, \bar{r} > \bar{s} \). Repeating the same arguments as before we conclude that

\[
\left\langle (D)_{\mu}^{-\kappa/2} \text{Op}(e^{-A})(D)_{\mu} \text{Op}(a_1)(D)_{\mu} u \right\rangle \approx \left\langle (D)_{\mu}^{-\kappa/2} \text{Op}(\langle \xi \rangle_{\mu} + A_1 + A_2) \text{Op}(a_1 + b_1 + b_2) \text{Op}(e^{-A}) \text{Op}(a_1)(D)_{\mu} u \right\rangle^2.
\]

Since \( b_j \in S_e(\phi^{-j/2}(\xi)^{\kappa j + \kappa - 1 - j}) \) and \( A_j \in S_e(\phi^{-j/2}(\xi)^{\kappa j + j - 1}) \) one has

\[
\langle \xi \rangle_{\mu}^{-\kappa/2} \#(\langle \xi \rangle_{\mu} + A_1 + A_2)\#(a_1 + b_1 + b_2) = \langle \xi \rangle_{\mu}^{-\kappa/2} a_1 + B
\]

with \( B \in S_e(\langle \xi \rangle_{\mu}^{\kappa j + \kappa - 1}) \subset \mu^{\kappa - \kappa} S_e(\langle \xi \rangle_{\mu}^{1+3\kappa/2}) \) and \( \| \text{Op}(B) \text{Op}(e^{-A}) \text{Op}(a_1)(D)_{\mu} u \| ^2 \approx 0 \) in virtue of Corollary 3.3. Noticing \( \langle \xi \rangle_{\mu}^{1 - \kappa/2} a_1 \in S_e(\sqrt{\phi}(\xi)^{\kappa/2}) \) and hence

\[
\langle \xi \rangle_{\mu}^{1 - \kappa/2} a_1\#(\langle \xi \rangle_{\mu}^{1 - \kappa/2} a_1) - \langle \xi \rangle_{\mu}^{2 - \kappa} a_1^2 \in S_e(\langle \xi \rangle_{\mu}^{\kappa - 2}) \]

we have

\[
\| b(t) \text{Op}(\langle \xi \rangle_{\mu}^{1 - \kappa/2} a_1) \text{Op}(e^{-A}) \text{Op}(a_1)(D)_{\mu} u \| ^2 \approx \langle (b(t)^2 \langle \xi \rangle_{\mu}^{2 - \kappa} a_1^2) \text{Op}(e^{-A}) \text{Op}(a_1)(D)_{\mu} u, \text{Op}(e^{-A}) \text{Op}(a_1)(D)_{\mu} u \rangle.
\]

Note that \( \tau^2 c^* \ell a(\phi(\xi)^{\kappa} a_1 x) \geq b(t) \langle \xi \rangle_{\mu}^{2 - \kappa} a_1^2 = b(t) \langle \xi \rangle_{\mu}^{2 - \kappa} (\tau - \theta t) \partial_\xi a(x) \partial_\xi \langle \xi \rangle_{\mu} \rangle^2 \) because \( |\partial_\xi \langle \xi \rangle_{\mu} | \leq \langle \xi \rangle_{\mu}^{-1} \) and \( c^* a(x) \geq b(t) \langle \partial_\xi a(x) \rangle^2 \) for \( (t, x) \in [0, T] \times \mathbb{R} \) by Glaeser’s inequality. Since \( 0 \leq \tau^2 c^* \ell a(\phi(\xi)^{\kappa} a_1 x) + b(t) \langle \xi \rangle_{\mu}^{2 - \kappa} a_1^2 \in S_e(\langle \xi \rangle_{\mu}^{\kappa} d\nu + \langle \xi \rangle_{\mu}^{\kappa} d\nu^2) \) then from the Fefferman-Phong inequality it follows that

\[
\tau^2 c^* \text{Re}(b(t) \text{Op}(\phi(\xi)^{\kappa} a_1 x) w, w) - \langle (b(t)^2 \langle \xi \rangle_{\mu}^{2 - \kappa} a_1^2) w, w \rangle \geq -C \langle \langle \xi \rangle_{\mu}^{2 - 1} w \rangle^2 \approx 0
\]

where \( w = \text{Op}(e^{-A}) \text{Op}(a_1)(D)_{\mu} u \). Thanks to Lemmas 4.1 and 4.0 one has

\[
|2 \text{Re}(b(t) \text{Op}(e^{-A}) \text{Op}(a_1)(D)_{\mu} u, \text{Op}(e^{-A}) Au)| \\
\leq \langle (\sqrt{\tau^2 c^*})^{-1} b(t) \langle \xi \rangle_{\mu}^{\kappa/2} \text{Op}(e^{-A}) \text{Op}(a_1)(D)_{\mu} u \rangle^2 \\
+ \tau \sqrt{c^*} \langle \langle \xi \rangle_{\mu}^{\kappa/2} \text{Op}(e^{-A}) Au \rangle^2 \leq \tau \sqrt{c^*} (\langle \xi_1 + \xi_2 \rangle)
\]

which proves the assertion.
We study $R$ in (4.1). Recall $R = b(t)\langle D \rangle_{\mu} \text{Op}(q + r)\langle D \rangle_{\mu}$. One has
\[
\|\langle \text{Op}(e^{-\Lambda})\rangle_{\mu} \text{Op}(q)\langle D \rangle_{\mu} u, \text{Op}(e^{-\Lambda})Au\| 
\leq C\|\langle D \rangle_{\mu}^{1-\kappa/2} \text{Op}(e^{-\Lambda})\langle D \rangle_{\mu} u\|\|\langle D \rangle_{\mu}^{\kappa/2} \text{Op}(e^{-\Lambda})Au\|.
\]
From Corollary 5.3 one can write $e^{-\Lambda} \# q = c e^{-\Lambda} + \tilde{r}$ with $c \in S_{0}^{(s)}(\langle \xi \rangle_{\mu}^{-2+2\kappa})$ and $\tilde{r} \in S^{(s)}(e^{-c\langle \xi \rangle_{\mu}^{2}})$ with $\tilde{c} < 1, \tilde{\kappa} > \kappa$. Recalling $c \in \mu^{-\tilde{c}}S_{0}(\langle \xi \rangle_{\mu}^{-3/2-2})$ thanks to Lemmas 3.2 and 3.3 one concludes that the right-hand-side is $\simeq 0$. On the other hand note that
\[
\|\langle \text{Op}(e^{-\Lambda})\rangle_{\mu} \text{Op}(r)\langle D \rangle_{\mu} u, \text{Op}(e^{-\Lambda})Au\| 
\leq C\|\langle D \rangle_{\mu} \text{Op}(r)\langle D \rangle_{\mu} u\|\|\text{Op}(e^{-\Lambda})Au\|
\]
since $e^{-\Lambda} \in S_{0}(1)$ by Lemma 2.15. We apply Lemma 3.2 to conclude that the right-hand side is $\simeq 0$. In conclusion we have
\[
(4.26) \quad \|\langle \text{Op}(e^{-\Lambda})\rangle_{\mu} \text{Op}(R) u, \text{Op}(e^{-\Lambda})Au\| \simeq 0.
\]
Finally we note that
\[
-2\text{Im} \langle \langle D \rangle_{\mu}^{3/2} \text{Op}(e^{-\Lambda})Au, \langle D \rangle_{\mu}^{3/2} \text{Op}(e^{-\Lambda})u\rangle 
\leq \|\langle D \rangle_{\mu}^{3/2} \text{Op}(e^{-\Lambda})u\|^{2} + \|\langle D \rangle_{\mu}^{3/2} \text{Op}(e^{-\Lambda})Au\|^{2} \leq \mathcal{E}_{1} + \mathcal{E}_{2}
\]
and
\[
-2\text{Im} \langle \text{Op}(e^{-\Lambda})P^{2}u, \text{Op}(e^{-\Lambda})Au\rangle \leq \|\text{Op}(e^{-\Lambda})P^{2}u\|^{2} + E
\]
by (4.2). From (4.6) and Propositions 4.1, 4.2 and 4.3 and Lemma 4.6 we conclude that
\[
(4.27) \quad \frac{dE}{dt} \leq -(2\theta - \tau \sqrt{c} - 1 - C\mu^{\epsilon})(\mathcal{E}_{1} + \mathcal{E}_{2}) + E + \|\text{Op}(e^{-\Lambda})P^{2}u\|^{2}
\]
with some $\epsilon > 0$. We now fix $\theta_{0}$ and $\mu > 0$ such that
\[
(4.28) \quad 2\theta_{0} > T\sqrt{c} + 1, \quad 2\theta_{0} - T\sqrt{c} - 1 - C\mu^{\epsilon} \geq 0.
\]
Then from (4.27) it follows that for $0 \leq t \leq \tau/\theta_{0}$
\[
E(u; t) \leq e^{T}E(u; 0) + e^{T} \int_{0}^{t} \|\text{Op}(e^{-\Lambda})P^{2}u\|^{2}ds.
\]
Since $\Lambda(0) = 0$ and $E(u; 0) \leq \|\langle D \rangle_{\mu}^{\kappa}u(0)\|^{2} + C\|\langle D \rangle_{\mu} u(0)\|^{2}$ this shows that
\[
\|\langle D \rangle_{\mu}^{\kappa} \text{Op}(e^{-\Lambda})u(t)\|^{2} + \|\text{Op}(e^{-\Lambda})Au(t)\|^{2} \leq C \left\{ \|\langle D \rangle_{\mu} u(0)\|^{2} + \|D_{t}u(0)\|^{2} \right\} + C \int_{0}^{t} \|\text{Op}(e^{-(\tau - \theta_{0})\langle \xi \rangle_{\mu}^{2}})P \text{Op}(e^{-\langle \xi \rangle_{\mu}^{2}})u(s)\|^{2}ds.
\]
Replacing $u$ by $\text{Op}(e^{-(\tau - \theta_{0})\langle \xi \rangle_{\mu}^{2}})u(s)$ we obtain
Lemma 4.7. For any $\tau < \tau'$ and $s \in \mathbb{R}$ there exists $C > 0$ such that

$$
\| (D)_{\mu}^s e^{(\tau-\theta_0 t)(D)_{\mu}^s} u \| \leq C \| (D)_{\mu}^s e^{(\tau-\theta_0 t)(D)_{\mu}^s} u \|
$$

for $0 \leq t \leq \tau'/\theta_0$.

Proof. Thanks to Lemma 2.16 we have $e^{-(\tau-\tau')(\xi)_\mu} \in S_\delta^{(s)} (e^{-c\xi}_\mu)$ where $0 < c < \tau - \tau'$. Denote

$$
p = e^{-(\tau-\tau')(\xi)_\mu} \# e^A
$$

then Proposition 3.1 shows that $p \in S_\delta((\xi)_\mu^l)$ for any $l \in \mathbb{R}$. Repeating the same arguments as in the proof of Lemma 3.1 we have $\text{Op}(e^{-(\tau-\tau')(\xi)_\mu}) = \text{Op}(p)\text{Op}(K)\text{Op}(e^{-A})$ since $1 = \text{Op}(e^A)\text{Op}(K)\text{Op}(e^{-A})$ and hence

$$
\| \text{Op}(e^{-(\tau-\tau')(\xi)_\mu}) u \| = \| \text{Op}(p)\text{Op}(K)\text{Op}(e^{-A}) u \| \leq C \| \text{Op}(e^{-A}) u \|
$$

which proves the first inequality replacing $u$ by $\text{Op}(e^{(\tau-\theta_0 t)(\xi)_\mu}) (D)_{\mu}^s u$ and taking Corollary 3.5 into account. On the other hand since $e^{-A} \in S_\delta(1)$ by Lemma 2.16 the second inequality is clear from Corollary 3.5.

Thanks to Lemma 4.7 it follows from Proposition 3.4 that

Theorem 4.1. For any $0 < \tau' < \tau$ there exists $C > 0$ such that one has

$$
\sum_{j=0}^{1} \| (D)_{\mu}^{(1-j)s} \text{Op}(e^{(\tau-\theta_0 t)(\xi)_\mu}) D^j_t u(t) \|^2 \leq C \sum_{j=0}^{1} \| (D)_{\mu}^{1-j} \text{Op}(e^{(\tau-\theta_0 t)(\xi)_\mu}) D^j_t u(0) \|^2
$$

$$
+ C \int_0^t \| \text{Op}(e^{(\tau-\theta_0 s)(\xi)_\mu}) P u(s) \|^2 ds
$$

for $0 \leq t \leq \tau'/\theta_0$.

5 Proof of Theorem 1.1

We prove a detailed version of Theorem 1.1.
Proposition 5.1. Assume $0 \leq b(t) \in C^{n,\alpha}([0, T])$ and $0 \leq a(x) \in G^\gamma(x)$. There exists $\theta_0 > 0$ such that for any $1 < s < s' < 1 + (n + \alpha)/2$ and $0 < \tau \leq T$ there is $\mu > 0$ such that for any $u_j \in G^\alpha_0(\mathbb{R})$ with finite $\sum_{j=0}^1 \|D_j^{(1-j)\mu} \text{Op}(e^{(\tau-\theta_0\epsilon)}\xi)^\alpha_\mu D_j^2 u(t)\|^2$ and any $0 < \tau' < \tau$ there exist $C > 0$ and a unique $u \in C^1([0, \tau'/\theta_0]; G^\alpha_0(\mathbb{R}))$, solution to \eqref{1.2}, such that

\begin{equation}
\sum_{j=0}^1 \|D_j^{(1-j)\mu} \text{Op}(e^{(\tau-\theta_0\epsilon)}\xi)^\alpha_\mu D_j^2 u(t)\|^2 \\
\leq C \sum_{j=0}^1 \|D_j^{(1-j)\mu} \text{Op}(e^{(\tau-\theta_0\epsilon)}\xi)^\alpha_\mu D_j^2 u(0)\|^2.
\end{equation}

Moreover if $\cup_{j=0}^1 \text{supp} u_j \subset \{|x| \leq R\}$ then $\text{supp} u(t, \cdot) \subset \{|x| \leq R + \epsilon t\}$.

Proof. Take $\chi(x) \in G_0^\alpha(\mathbb{R})$ such that $0 \leq \chi(x) \leq 1$ and $1$ in $|x| \leq 1$ and $0$ for $|x| \geq 2$. We define $0 \leq a_\nu(x) \in G_0^\alpha(\mathbb{R})$ by $a_\nu(x) = a(x)\chi(\nu x)$. Then it is easy to see that there are $C > 0$, $A > 0$ independent of $0 < \nu \leq 1$ such that

$$|\partial^k_x a_\nu(x)| \leq CA^k k! \forall x \in \mathbb{R}, \quad k = 0, 1, \ldots.$$ 

Since $C$ and $A$ are independent of $\nu$ it can be seen that one can take $\theta_0$ and $\mu$ verifying \eqref{1.28} which are independent of $\nu$. Introducing a small positive parameter $\epsilon > 0$ we set

$$b_\epsilon(t) = b(t) + \epsilon, \quad a_{\nu, \epsilon}(x) = a_\nu(x) + \epsilon$$

and consider $P_{\nu, \epsilon} = D_\epsilon^2 - D_x b_\epsilon(t) a_{\nu, \epsilon}(x) D_x$. Noting $a_{\nu, \epsilon}(x) = a_\nu(x)$, $0 \leq a_{\nu, \epsilon}(x) \leq a_{\nu, \epsilon}(x)$ and $b_\epsilon(t) = b_\epsilon(t)$, $0 \leq b_\epsilon(t) \leq b_\epsilon(t)$ it is not difficult to examine that Theorem 4.4 holds uniformly in $0 < \nu \leq 1$ and $0 < \epsilon \leq \epsilon_0$. Since $P_{\nu, \epsilon}$ is strictly hyperbolic and $b_\epsilon(t) a_{\nu, \epsilon}(x) \leq \epsilon^2$ for $(t, x) \in [0, T] \times \mathbb{R}$ with $\epsilon > 0$ independent of $0 < \epsilon \leq \epsilon_0$ and $0 < \nu \leq 1$ the Cauchy problem for $P_{\nu, \epsilon}$

$$\left\{ \begin{array}{l}
P_{\nu, \epsilon} u_{\nu, \epsilon} = 0 \quad \text{in} \quad (t, x) \in [0, T] \times \mathbb{R}, \\
P^j_{\nu, \epsilon} u_{\nu, \epsilon}(0, x) = u_{j}(x) \quad \text{for} \quad j = 0, 1
\end{array} \right.$$ 

with $u_j(x) \in G^\alpha_0(\mathbb{R})$ supported in $|x| \leq R$ has a unique solution $u_{\nu, \epsilon}$ such that

$$\text{supp} u_{\nu, \epsilon}(t, \cdot) \subset \{x \mid |x| \leq R + \epsilon t\}.$$ 

From Theorem 4.4 we conclude that

$$\sup_{0 \leq t \leq \tau'/\theta_0} \sum_{j=0}^1 \|D_j^{(1-j)\mu} \text{Op}(e^{(\tau-\theta_0\epsilon)}\xi)^\alpha_\mu D_j^2 u_{\nu, \epsilon}(t)\|$$

is uniformly bounded in $0 < \epsilon \leq \epsilon_0$ provided $\sum_{j=0}^1 \|D_j^{(1-j)\mu} \text{Op}(e^{(\tau-\theta_0\epsilon)}\xi)^\alpha_\mu u_j\|$ is finite. Therefore by the standard argument one can prove that there exists $u_\nu \in C^1([0, \tau'); G^\alpha_0(\mathbb{R}))$ satisfying

$$\left\{ \begin{array}{l}
D^j_\epsilon u_\nu - D_x (b(t) a_\nu(x)) D_x u_\nu = 0 \quad \text{in} \quad (t, x) \in [0, T] \times \mathbb{R}, \\
D^j_\epsilon u_\nu(0, x) = u_{j}(x) \quad \text{for} \quad j = 0, 1
\end{array} \right.$$ 


and the energy estimate \[5.1\]. Since \( D_x(b(t)a_{\nu}(x))D_xu_{\nu} = D_x(b(t)a(x))D_xu_{\nu} \),
taking \( \nu^{-1} > R + \epsilon T \) then \( u_{\nu} \) is a desired solution to the Cauchy problem \[1.2\].

It remains to prove the uniqueness. Let \( 0 < \bar{T} < \tau/\theta_0 \). Since \( P^* = P \) we conclude from Proposition \[5.1\] that the Cauchy problem for \( P^* \) with reversed time direction has a solution with finite propagation speed, that is there is \( \mu > 0 \)
such that for any \( \phi \in \mathcal{G}_0^\prime(\mathbb{R}) \) with finite \( \sum_{j=0}^{\mu} \| \langle D \rangle_{\mu}^{\tau+(\xi)} \|_2 \phi \| \) there exists a solution \( v \in C^1([0, \bar{T}); \mathcal{G}_0^2(\mathbb{R})) \) to the adjoint Cauchy problem

\[
\begin{cases}
P^*v = 0 \text{ in } (t, x) \in [0, \bar{T}] \times \mathbb{R}, \\
D_t^j v(\bar{T}, x) = \phi_j(x) \text{ for } j = 0, 1
\end{cases}
\]

with finite propagation speed \( \hat{c} \). Now assume that \( u \in C^1([0, \tau'/\theta_0]; \mathcal{G}_0^2(\mathbb{R})) \) is a solution to \[1.2\] with \( u_t(x) = 0 \) and \( T' = \tau'/\theta_0 \). Then following the Holmgren’s arguments we have

\[
0 = \int_0^{\bar{T}} (Pu, v) dt = -i(D_tu(\bar{T}), \phi_0) - i(u(\bar{T}), \phi_1) + \int_0^{\bar{T}} (u, P^*v) dt
= -i(D_tu(\bar{T}), \phi_0) - i(u(\bar{T}), \phi_1).
\]

Since \( \phi_j \) are arbitrarily we conclude that \( u(\bar{T}) = 0 \).

\[\square\]

6 Appendix

In this section we give the proofs of Propositions \[3.1\], \[3.2\], \[3.3\] and \[3.4\] where for the convenience of applications to operators of the form \[1.1\], the assertions are stated in one dimensional space, while the dimension is irrelevant we prove the statements in \( n \) dimensional space.

6.1 Composition \((be^\psi)\#e^{-\psi}\)

**Proposition 6.1.** Assume \( \psi \in S_1^{(s)}(\langle \xi \rangle^\delta_{\mu}) \) and \( b \in S_0^{(s)}(\langle \xi \rangle^\mu) \) where \( 1 - \delta > \hat{\kappa} \) and \( s(1 - \delta) < 1 \). Then we have

\[
(b e^\psi)\#e^{-\psi} = b + \hat{b} + R
\]

where \( \hat{b} \in S_1^{(s)}(\langle \xi \rangle^\mu^{-\delta-\kappa}) \) and \( R \in S_0^{(s)}(e^{-c \langle \xi \rangle^\mu}) \) with some \( \hat{s} > 1, \hat{\kappa} > 0, c > 0 \) such that \( \hat{s}\hat{\kappa} < 1 \) and \( \kappa > \hat{\kappa} \).

**Corollary 6.1.** Assume \( \psi \in S_1^{(s)}(\langle \xi \rangle^\delta_{\mu}) \) where \( 1 - \delta > \hat{\kappa} \) and \( s(1 - \delta) < 1 \). Then one has with some \( \hat{s} > 1 \)

\[
e^\psi \#e^{-\psi} - 1 \in S_1^{(s)}(\langle \xi \rangle^{-\delta-\kappa}_{\mu}) \subset \mu^{1-\delta-\kappa} S_0(1).
\]

**Proof.** We apply Proposition \[6.1\] with \( b = 1 \) to get \( e^\psi \#e^{-\psi} - 1 = r + R \) with \( r \in S_0^{(s)}(\langle \xi \rangle^{-\delta-\kappa}_{\mu}) \) and \( R \in S_0^{(s)}(e^{-c \langle \xi \rangle^\mu}) \). We note that if \( R \in S_0^{(s)}(e^{-c \langle \xi \rangle^\mu}) \)
with $c > 0$ then for any $0 < c' < c$ and $\delta \geq 0$ one has $R \in S^{(\delta)}_\delta(e^{-c'}(\xi)^n_\mu)$ with some $\tilde{s} > 1$. Indeed since

$$e^{-c(\xi)^n_\mu} \leq |e^{-|\alpha|\langle \xi \rangle} - |\alpha + \beta| \langle \xi \rangle^{(1+\delta)|\alpha + \beta|} e^{-c - c' |\xi|^n_\mu}$$

$$\leq CA^{(\alpha + \beta)} |\langle \xi \rangle|^{-(1+\delta)/\kappa} \langle \xi \rangle^{-(\alpha + \beta)|\alpha + \beta| |\alpha| e^{-c'(\xi)^n_\mu}}$$

it follows that

$$|\alpha + \beta|^{\delta |\alpha + \beta|} e^{-c(\xi)^n_\mu} \leq CA^{(\alpha + \beta)} (|\alpha + \beta|^{\delta (1+\delta)/\kappa} |\xi|^{-\delta/\kappa})^{(\alpha + \beta)} |\langle \xi \rangle|^{-|\alpha| e^{-c'(\xi)^n_\mu}}$$

that is $R \in S^{(\delta)}_\delta(e^{-c'(\xi)^n_\mu})$ with $\tilde{s} = \tilde{s} + (1 + \delta)/\kappa$. Since $S^{(\delta)}_\delta(\langle \xi \rangle^{(-\delta - \delta)/\kappa}) \subset S^{(\delta)}_\delta(\langle \xi \rangle^{(-1-\delta - \delta)/\kappa})$ and $\langle \xi \rangle^{(-1-\delta - \delta)/\kappa} \leq \mu^{1-\delta - \delta}$ we get the assertion. \[\square\]

**Proof of Proposition 6.1.** The idea of the proof is same as the proof of [9] Theorem 5.1. Consider

$$(be^{\psi})#e^{-\psi} = \int e^{-2i\sigma(Y,Z)} b(X + Y)e^{\psi(X+Y) - \psi(X+Z)}dYdZ$$

$$= b + \int e^{-2i\sigma(Y,Z)} b(X + Y)(e^{\psi(X+Y) - \psi(X+Z)} - 1)dYdZ$$

where $Y = (y, \eta)$, $Z = (z, \zeta)$ and $\sigma(Y, Z) = \langle \eta, z \rangle - \langle y, \zeta \rangle$. After the change of variables $Z \rightarrow Z + Y$ the integral on the right-hand side turns to

$$(6.1) \int e^{-2i\sigma(Y,Z)} b(X + Y)(e^{\psi(X+Y) - \psi(X+Y+Z)} - 1)dYdZ.$$
Lemma 6.1. Notations being as above. We have

\[ |\frac{\partial}{\partial \xi}(\frac{\partial}{\partial \eta})\tilde{b}(X, Y, Z)| \leq CA^{[\alpha + \beta]}(|\alpha + \beta|^{1+\varepsilon} + |\alpha + \beta|^{\delta})^{[\alpha + \beta]} \times \langle \xi \rangle^{\varepsilon} (1 + \langle \xi \rangle^{\delta} g_{X}^{1/2}(Z)) \langle \xi \rangle^{m-1+\delta} \langle \xi \rangle^{-[\alpha + \beta]} \nabla X. \]

Proof. Note that \( \tilde{b} \) consists of terms such as

\[ \tilde{b} = \frac{\partial^{2}}{\partial \xi^{2}} \left( \frac{\partial}{\partial \eta}(\frac{\partial}{\partial \xi})\psi(X + Y + \theta Z) \right) \]

where \( |e + f| = 1 \). The assertion for \( \tilde{b} \) is easy. As for \( \tilde{b}_{2} \) write

\[ \frac{\partial^{2}}{\partial \xi^{2}} \left( \frac{\partial}{\partial \xi}(\frac{\partial}{\partial \eta})\psi(X + Y + \theta Z) \right) = \sum_{j} \int_{0}^{1} \left( \frac{\partial^{2}}{\partial \xi^{2}} \frac{\partial}{\partial \xi} \psi(X + Y + s \theta Z) \right) \]

then on the support of \( \tilde{\chi}_{0} \) it is easy to see

\[ |\frac{\partial^{2}}{\partial \xi^{2}} \left( \frac{\partial}{\partial \xi}(\frac{\partial}{\partial \eta})\psi(X + Y + \theta Z) \right)| \leq CA^{[\alpha + \beta] + 2}(|\alpha + \beta| + 2 + (|\alpha + \beta| + 2)^{\delta} \langle \xi \rangle^{\delta} + 2^{\delta} g_{X}^{1/2}(Z)). \]

Thus we get the assertion for \( \tilde{b}_{2} \).

Lemma 6.2. Let \( \Psi(X, Y, Z) = \psi(X + Y) - \psi(X + Y + \theta Z) \) then on the support of \( \tilde{\chi}_{0} \) one has

\[ |\Psi(X, Y, Z)| \leq C \langle \xi \rangle^{\delta} g_{X}^{1/2}(Z) \]

and

\[ |\frac{\partial^{2}}{\partial (x, y)} \left( \frac{\partial}{\partial \eta} \psi \right)| \leq CA^{[\alpha + \beta] \langle \xi \rangle^{\delta} - [\alpha + \beta]} \times \langle \xi \rangle^{\delta} g_{X}^{1/2}(Z) + |\alpha + \beta| + |\alpha + \beta|^{\delta} g_{X}^{1/2}(Z). \]

Proof. It suffices to repeat the proof of Lemma 2.7. \( \square \)

Introducing the following differential operators and symbols

\[
\begin{align*}
L &= 1 + 4^{-1} \langle \xi \rangle^{2} |\theta X|^{2} + 4^{-1} \langle \xi \rangle^{2} |\theta Y|^{2}, \\
M &= 1 + 4^{-1} \langle \xi \rangle^{2} |\partial X|^{2} + 4^{-1} \langle \xi \rangle^{2} |\partial Y|^{2}, \\
\Phi &= 1 + \langle \xi \rangle^{2} |z|^{2} + \langle \xi \rangle^{2} |\eta|^{2} = 1 + \langle \xi \rangle^{2+\delta} g_{X}(Z), \\
\Theta &= 1 + \langle \xi \rangle^{2} |y|^{2} + \langle \xi \rangle^{2} |\eta|^{2} = 1 + g_{X}(Y)
\end{align*}
\]
so that $\Phi^{-N} L^N e^{-2i\sigma(Y,Z)} = e^{-2i\sigma(Y,Z)}$ and $\Theta^{-\ell} M^\ell e^{-2i\sigma(Y,Z)} = e^{-2i\sigma(Y,Z)}$ we make integration by parts in (6.1). Let $F = \hat{b}(X,Y,Z)e^\Phi$ where $\hat{b}$ is given in Lemma 6.3 and consider

$$
(6.4) \quad \partial_\xi \Phi^\alpha \int e^{-2i\sigma(Y,Z)} F dY dZ = \int e^{-2i\sigma(Y,Z)} L^N \Phi^{-N} M^\ell \Theta^{-\ell} (\partial_\xi^2 \Phi^\alpha F) dY dZ.
$$

Here we note that $|\partial_{\xi_{\alpha\beta}} \Phi^\ell| \leq CA_{|\alpha+\beta|}|\alpha+\beta|!(\xi_\mu)^{-|\alpha+\delta|\beta}\Theta^{-\ell}$. Applying Lemmas 6.1 and 6.2 we can estimate $|L^N \Phi^{-N} M^\ell \Theta^{-\ell} (\partial_\xi^2 \Phi^\alpha F)|$ by

$$
(6.5) \quad C A_1^{2N+|\alpha+\beta|} \Phi^{-N} \Theta^{-\ell} (\xi_\mu)^{-\delta-\kappa} (1 + \langle \xi_\mu \rangle g_X^{1/2} (Z)) (\xi_\mu)^{1/2} (Z)
$$

$$
+ (2N + |\alpha + \beta|)^{1+\varepsilon} + (2N + |\alpha + \beta|)^{s} (\xi_\mu)^{-\delta} 2N + |\alpha+\beta| e^\Phi (\xi_\mu)^{m-|\alpha+\delta|\beta}.
$$

Here we recall the following easy lemma [9 Lemma 5.3].

**Lemma 6.3.** Let $A \geq 0$, $B \geq 0$. Then there exists $C > 0$ independent of $n,m \in \mathbb{N}$, $A, B$ such that

$$
(A + (n + m)^{1+\varepsilon} + (n + m)^s B)^{n+m} \leq C^{n+m} (A + n^{1+\varepsilon} + n^s B)^n (A + m^{1+\varepsilon} + m^s B)^m.
$$

From Lemma 6.3 the right-hand side of (6.5) can be estimated by

$$
C A_1^{2N+|\alpha+\beta|} \Phi^{-N} \Theta^{-\ell} (\xi_\mu)^{-\delta-\kappa} (1 + \langle \xi_\mu \rangle g_X^{1/2} (Z))
$$

$$
\times \langle (\xi_\mu)^{1/2} (Z) + (2N)^{1+\varepsilon} + \langle \xi_\mu \rangle + (2N)^{s} (\xi_\mu)^{-\delta} \rangle 2N
$$

$$
\times \langle (\xi_\mu)^{1/2} (Z) + |\alpha + \beta|^1 + |\alpha + \beta|^s (\xi_\mu)^{-\delta} \rangle^{|\alpha+\beta|} (\xi_\mu)^{m-|\alpha+\delta|\beta}.
$$

Writing

$$
(6.6) \quad A_1^{2N} \Phi^{-N} \langle (\xi_\mu)^{1/2} (Z) + (2N)^{1+\varepsilon} + (2N)^{s} (\xi_\mu)^{-\delta} \rangle 2N
$$

$$
= \left( \frac{A_1 (\xi_\mu)^{1/2} (Z)}{\Phi^{1/2}} + \frac{A_1 (2N)^{1+\varepsilon}}{\Phi^{1/2}} + \frac{A_1 (2N)^{s} (\xi_\mu)^{-\delta}}{\Phi^{1/2}} \right) 2N
$$

we choose $N = N(Z,\xi)$ so that $2N = \bar{c} \Phi^{1/2(1+\varepsilon)}$ with a small $\bar{c} > 0$. Assuming $1 + \varepsilon < s$ without restrictions one has

$$
\Phi^{s/2(1+\varepsilon)-1/2} (\xi_\mu)^{-\delta} = (1 + \langle \xi_\mu \rangle^{2(1-\delta)} g_X (Z))^{s/2(1+\varepsilon)-1/2} (\xi_\mu)^{-\delta}
$$

$$
\leq C \langle \xi_\mu \rangle^{s(1+\varepsilon)/(1-\delta) - 1}(1-\delta) = C \langle \xi_\mu \rangle^{s(1-\delta)/(1+\varepsilon) - 1} \leq C \mu^c
$$

with some $c > 0$ on the support of $\tilde{\chi}_0$ because $g_X (Z)$ is bounded there and $s(1 - \delta) < 1$. Then the right-hand side of (6.5) is bounded by

$$
(\bar{c} (\mu^{1-\delta+\kappa} + \mu^{1+\varepsilon} + \mu^c)) 2N \leq C e^{-c_1 \Phi^{1/2(1+\varepsilon)}}
$$

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choosing \( \bar{c} \) and \( \mu > 0 \) small such that \( A_1(\mu^{1-\delta+\bar{\kappa}}+c_1^{1+\varepsilon}+\mu^\varepsilon) < 1 \). Assuming \( \varepsilon > 0 \) small so that \( 1-\delta > (1+\varepsilon)\bar{\kappa} \) then since \( g_X^{1/2}(Z) \leq Cg_X^{1/2(1+\varepsilon)}(Z) \) on the support of \( \tilde{\chi}_0 \) we have

\[
\Phi^{1/2(1+\varepsilon)} \geq c|\xi|^{(1-\delta)/(1+\varepsilon)}g_X^{1/2}(Z) \geq c\mu^{(1-\delta)/(1+\varepsilon)}|\xi|^{\bar{\kappa}}g_X^{1/2}(Z).
\]

Then one has

\[
(\|\xi\|^{\bar{\kappa}}g_X^{1/2}(Z) + |\alpha + \beta|^{1+\varepsilon} + |\alpha + \beta|^s(\xi)_\mu^{-\delta})^{\alpha + \beta}|e^{-c\Phi^{1/2(1+\varepsilon)}}
\leq CA^{\alpha+\beta}(|\alpha + \beta|^{1+\varepsilon} + |\alpha + \beta|^s(\xi)_\mu^{-\delta})|\alpha + \beta|^{\bar{\kappa}}e^{-c\Phi^{1/2(1+\varepsilon)}}.
\]

Noting \( e^{-c\Phi^{1/2(1+\varepsilon)}} \leq C\Phi^{-\ell} \) and Lemma 6.2 and (6.7) we have

\[
\left|L^N\Phi^{-N}M^\ell\Theta^{-\ell}(\partial_\xi^\beta \partial_\mu^\alpha F)\right| \leq \mu^\kappa CA^{\alpha+\beta}(|\alpha + \beta|^{1+\varepsilon} + |\alpha + \beta|^s(\xi)_\mu^{-\delta})(\xi)_\mu^{-\delta-\bar{\kappa}}|\alpha + \beta|\Theta^{-\ell} \Phi^{-\ell}.
\]

Finally choosing \( \ell > (n+1)/2 \) and recalling \( \int \Theta^{-\ell} \Phi^{-\ell} dYdZ = C \) we conclude that

\[
\left|\partial_\xi^\beta \partial_\mu^\alpha \int e^{-2i\sigma(Y,Z)} F \tilde{\chi}_0 dYdZ \right|
\leq CA^{\alpha+\beta}(|\alpha + \beta|^{1+\varepsilon} + |\alpha + \beta|^s(\xi)_\mu^{-\delta})(\xi)_\mu^{-\delta-\bar{\kappa}}|\alpha + \beta|\Theta^{-\ell} \Phi^{-\ell}.
\]

Denoting \( F = b(X+Y)(e^{\psi(X+Y)}-\psi(X+Y+Z)-1)\tilde{\chi}_1 \) we next consider

\[
\partial_\xi^\beta \partial_\mu^\alpha \int e^{-2i\sigma(Y,Z)} F dYdZ
= \int e^{-2i\sigma(Y,Z)}(\langle \xi \rangle^2 |D_\eta|^2 + |D_\zeta|^2)L^N(\langle \xi \rangle^2 |z|^2 + |y|^2)^{-N} \partial_\zeta^\beta \partial_\eta^\alpha F dYdZ.
\]

Since \( |\psi(X+Y)| + |\psi(X+Y+Z)| \) is bounded by \( C(\xi)_\mu \) and \( C^{-1} \leq (\xi + \eta)_\mu / (\xi)_\mu \), \( (\xi + \zeta)_\mu / (\xi)_\mu \leq C \) with some \( C > 0 \) on the support of \( \tilde{\chi}_1 \) thanks to Lemma 6.2 it is not difficult to show

\[
\left|\langle \xi \rangle^2 |D_\eta|^2 + |D_\zeta|^2 \right| L^N \partial_\zeta^\beta \partial_\eta^\alpha F \leq CA^{2N+|\alpha + \beta|}(\xi)_\mu^{-\delta}|\alpha + \beta|\Theta^{-\ell} \Phi^{-\ell}.
\]

Choose \( 2N = c_1(\xi)_\mu^{-\delta}/(1+\varepsilon) \) with small \( c_1 > 0 \) so that

\[
A^{2N} \langle \xi \rangle_\mu^{2(1-\delta)N} \left( C(\xi)_\mu^{\bar{\kappa}} + (2N)^{1+\varepsilon} + (2N)^s(\xi)_\mu^{-\delta}\right)^2
= \left( AC(\xi)_\mu^{\bar{\kappa}} + Ac_1^{1+\varepsilon} + Ac_1^{s(1-\delta)/(1+\varepsilon)}(\xi)_\mu^{-\delta}\right)^{2N}
\]

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is bounded by \( C e^{-c(\xi(\mu)^{(1-\delta)/(1+\varepsilon)})} \) choosing \( \mu \) small which is possible because \( \bar{k} < 1 - \delta \) and \( s(1 - \delta) < 1 \). Note that \( \langle \xi \rangle^{\frac{\alpha + \beta}{\mu}} e^{-c(\xi(\mu)^{(1-\delta)/(1+\varepsilon)})} \) is bounded by

\[
CA^{\alpha + \beta} \left( |\alpha + \beta(1+\varepsilon)\bar{k}/(1-\delta)\right)^{\alpha + \beta} e^{-c'(\xi(\mu)^{(1-\delta)/(1+\varepsilon)})}
\]

\[
\leq CA^{\alpha + \beta} |\alpha + \beta(1+\varepsilon)\bar{k}/(1-\delta)\alpha + \beta e^{-c'(\xi(\mu)^{(1-\delta)/(1+\varepsilon)})}
\]

for \( \bar{k}/(1-\delta) < 1 \) where \( 0 < c' < c \). Since \(|y|^2 + \langle \xi \rangle^{2\delta} \geq 1 \) if \( \tilde{\chi}_1 \neq 0 \) then

\[
\langle \xi \rangle^{\alpha + \beta} \int (|y|^2 + \langle \xi \rangle^{2\delta} |z|^2)^{-N} \tilde{\chi}_1 dYdZ \leq C.
\]

Noting \( \langle \xi \rangle^{2n-n\delta} e^{-c'(\xi(\mu)^{(1-\delta)/(1+\varepsilon)})} \leq C \) we conclude that \( (6.30) \) is also bounded by the right-hand side of \( (6.29) \): Therefore recalling \( \tilde{\chi} = \tilde{\chi}_0 + \tilde{\chi}_1 \) we have

**Lemma 6.4.** Let \( \tilde{\chi} = \chi(\langle \eta \rangle^{-1})(\langle \xi \rangle^{-1}) \). Then for any \( \varepsilon > 0 \) there exist \( C > 0 \), \( A > 0 \) such that we have

\[
\left| \frac{\partial^\beta}{\partial t^\beta} \int e^{-2i\sigma(Y,Z)} b(X + Y) (e^{\psi(X+Y)-\psi(X+Y+Z)} - 1) \tilde{\chi} dYdZ \right|
\]

\[
\leq CA^{\alpha + \beta} (|\alpha + \beta|^{1+\varepsilon} + |\alpha + \beta|^\delta \langle \xi \rangle^{-\delta})(\xi)^{-\alpha + \beta} (\xi)^{-1-\delta} - \eta \langle \xi \rangle^{-m-|\alpha + \beta|\delta} \beta).
\]

We turn to \( \int e^{-2i\sigma(Y,Z)} b(X + Y) (e^{\psi(X+Y)-\psi(X+Y+Z)} - 1)(1 - \chi)dYdZ \). Making the change of variables \( Z \rightarrow Z - Y \) we come back to the original coordinates. Denote \( \chi(\langle \xi \rangle^{-1})(\langle \eta \rangle + \langle \xi \rangle^{-1})(\xi - \eta) \) \( \tilde{\chi}_1 \) and write

\[
1 - \tilde{\chi}_1 = (1 - \tilde{\chi}_1)(\langle \eta \rangle^{-1}\langle \xi \rangle) + (1 - \tilde{\chi}_1)(1 - \chi(\langle \eta \rangle^{-1}\langle \xi \rangle))) = \tilde{\chi}_2 + \tilde{\chi}_3.
\]

Note that on the support of \( \tilde{\chi}_2 \) we have \( C(\eta) \geq \langle \xi \rangle \) and \( C(\eta) \geq \langle \xi \rangle \). Similarly on the support of \( \tilde{\chi}_3 \) one has \( C(\xi) \geq \langle \eta \rangle \) and \( C(\xi) \geq \langle \xi \rangle \). Denoting \( F = b(X + Y)(e^{\psi(X+Y)-\psi(X+Z)} - 1) \) we consider

\[
\frac{\partial^\beta}{\partial t^\beta} \int e^{-2i\sigma(Y,Z)} F \tilde{\chi}_1 dYdZ = \int e^{-2i\sigma(Y,Z)} \langle \eta \rangle - 2N \langle \xi \rangle^{-2N} \langle \xi \rangle^{-2N}
\]

\[
\times (D_2)_{2Z} (D_y)_2^N (y)^{-2\ell} (z)^{-2\ell} (D_\xi)^{2\ell} (D_\eta)^{2\ell} \frac{\partial^\beta}{\partial t^\beta} F \tilde{\chi}_2 dYdZ.
\]

For case \( \tilde{\chi}_2 \) we choose \( N_1 = \ell, N_2 = N \). Since \( |\psi(X+Y)| + |\psi(X+Z)| \leq C(\eta) \) on the support of \( \tilde{\chi}_2 \) it is not difficult to see that

\[
|\langle \xi \rangle^{-2N} \langle \xi \rangle^{-2\ell} (D_2)^{2\ell} (D_y)^{2\ell} (y)^{-2\ell} (z)^{-2\ell} (D_\xi)^{2\ell} (D_\eta)^{2\ell} \frac{\partial^\beta}{\partial t^\beta} F \tilde{\chi}_2 |
\]

is bounded by

\[
Ce^{A^{2N+|\alpha + \beta|}(\eta)^{-2N} \langle \xi \rangle^{-2\ell} (y)^{-2\ell} (z)^{-2\ell} \langle \xi \rangle^{-|\alpha + \beta|\delta + 2\ell}}
\]

\[
\times (C(\eta)^{\delta + \delta} + N^{1+\varepsilon}(\eta)^{\delta} + N^2 (C(\eta)^{\delta + \delta}) + |\alpha + \beta|^{1+\varepsilon}(\eta)^{\delta} + |\alpha + \beta|^\delta)(\alpha + \beta)^{2N} e^{C(\eta)^{\delta}}
\]

(6.11)
because

\begin{align}
(C\langle\omega\rangle)^{\tilde{k}} + (2k)^{1+\varepsilon} + (2k)^{\delta} \langle\omega\rangle^{-\delta})^{2k} \langle\omega\rangle^{2\delta l} \\
\leq (C\langle\omega\rangle)^{\tilde{k}+\delta} + (2k)^{1+\varepsilon} \langle\omega\rangle^\delta + (2k)^{\delta}^{2k}
\end{align}

for any \( l, k \in \mathbb{N}, l \leq k \) and any \( \omega \), in particular \( \omega = \xi + \eta, \omega = \xi + \zeta \). Here writing

\begin{align}
A^{2N} \langle\eta\rangle^{-2N} \left( C\langle\eta\rangle^{\tilde{k}+\delta} + (2N)^{1+\varepsilon} \langle\eta\rangle^\delta + (2N)^\delta \right)^{2N}
= \left( \frac{AC\langle\eta\rangle^{\tilde{k}+\delta}}{\langle\eta\rangle} + \frac{A(2N)^{1+\varepsilon}}{\langle\eta\rangle^{1-\delta}} + \frac{A(2N)^\delta}{\langle\eta\rangle} \right)^{2N}
\end{align}

we take \( 2N = c_1 \langle\eta\rangle^{(1-\delta)/(1+\varepsilon)} \) with small \( c_1 > 0 \) so that the right-hand side is bounded by \( Ce^{-c\langle\eta\rangle^{(1-\delta)/(1+\varepsilon)}} \) with some \( c > 0 \) since \( s(1-\delta) < 1 \). Since

\begin{align}
\langle\eta\rangle^{\delta} |\alpha+\beta| e^{-c\langle\eta\rangle^{(1-\delta)/(1+\varepsilon)}}
\leq CA_1^{[\alpha+\beta]} \left( |\alpha+\beta|^{\delta(1+\varepsilon)/(1-\delta)} \right) e^{-c_1 \langle\eta\rangle^{(1-\delta)/(1+\varepsilon)}}
\langle\eta\rangle^{(\tilde{k}+\delta)} |\alpha+\beta| e^{-c\langle\eta\rangle^{(1-\delta)/(1+\varepsilon)}}
\leq CA_1^{[\alpha+\beta]} \left( |\alpha+\beta|^{(1+\varepsilon)(\tilde{k}+\delta)/(1-\delta)} \right) e^{-c_1 \langle\eta\rangle^{(1-\delta)/(1+\varepsilon)}}
(0 < c_1 < c)
\end{align}

and

\begin{align}
(1+\varepsilon)\delta + 1 + \varepsilon = \frac{1+\varepsilon}{1-\delta}, \quad (1+\varepsilon)(\tilde{k}+\delta) = \frac{1+\varepsilon}{1-\delta}
\end{align}

one sees that \( \text{(6.11)} \) is bounded by

\[ C_1 A_1^{[\alpha+\beta]} \langle\zeta\rangle^{-2\ell} \langle y\rangle^{-2\ell} (z)^{-2\ell} (|\alpha+\beta|^{(1+\varepsilon)/(1-\delta)}) |\alpha+\beta| e^{-c_1 \langle\eta\rangle^{(1-\delta)/(1+\varepsilon)}}. \]

For case \( \hat{\chi}_3 \) choosing \( N_1 = N \) and \( N_2 = \ell \) we can prove that

\[ \| \langle\eta\rangle^{-2\ell} \langle\zeta\rangle^{-2N} (D_2)^{-2\ell} (D_g)^{-2N} (y)^{-2\ell} (z)^{-2\ell} (D_\zeta)^{2\ell} (D_\eta)^{2\ell} \partial_x^\beta \eta^\rho F \hat{\chi}_3 \]

is bounded by

\[ C_1 A_1^{2N} + |\alpha+\beta| \langle\eta\rangle^{-2\ell} \langle\zeta\rangle^{-2N} (y)^{-2\ell} (z)^{-2\ell} (\zeta)^{m+2\delta \ell + 6\delta \ell}
\times (C\langle\zeta\rangle^{\tilde{k}+\delta} + N^{1+\varepsilon} \zeta^\delta + N^\delta)^{2N} (C\zeta)^{\tilde{k}+\delta}
+ |\alpha+\beta|^{1+\varepsilon} \zeta^\delta + |\alpha+\beta|^\delta e^{-C\langle\zeta\rangle^\delta}. \]

Therefore repeating the same arguments, choosing \( 2N = c_1 \langle\zeta\rangle^{(1-\delta)/(1+\varepsilon)} \) now, this is bounded by

\[ C_1 A_1^{[\alpha+\beta]} \langle\zeta\rangle^{-2\ell} (z)^{-2\ell} (|\alpha+\beta|^{(1+\varepsilon)/(1-\delta)}) |\alpha+\beta| e^{-c_1 \langle\zeta\rangle^{(1-\delta)/(1+\varepsilon)}}. \]

Thus recalling that \( \langle\xi\rangle \leq C\langle\eta\rangle, \langle\xi\rangle \leq C\langle\zeta\rangle \) on the support of \( \hat{\chi}_2, \hat{\chi}_3 \) we get
Lemma 6.5. We have
\[ \left| \partial_\xi^\beta \partial_\zeta^\alpha \int e^{-2i\sigma(Y,Z)} b(X+Y) e^{\psi(X+Y) - \psi(X+Z)} - 1 \right| (1 - \chi_1) dY dZ \leq CA^{\alpha + \beta} |(\alpha + \beta)(1+\epsilon)/(1-\delta)|^{\alpha + \beta} e^{-c_1(\xi)(1-\delta)/(1+\epsilon)}. \]

We choose \( \bar{\epsilon} > 0 \) such that \( 1 - \delta > (1 + \bar{\epsilon})\bar{k} \) and set \( \bar{k} = (1 - \delta)/(1 + \bar{\epsilon}) \). With \( \bar{s} = (1 + \bar{\epsilon})/(1 - \delta) (\bar{s} > s) \) we finish the proof of Proposition 6.1.

6.2 Composition \((be)\# a\)

**Proposition 6.2.** Assume \( \psi \in S_\delta'(\langle \xi \rangle_\mu^*) \) and \( a \in S_\delta'(\langle \xi \rangle_\mu^{m_2}) \), \( b \in S_\delta'(\langle \xi \rangle_\mu^{m_1}) \) where \( 1 - \delta > \bar{k} \) and \( (1 - \delta)s < 1 \). Then for any \( p \in \mathbb{N} \) we have
\[
(be)^\# a = \sum_{|\alpha| < p} \frac{(-1)^{|\beta|}}{(2i)^{|\alpha + \beta|} \alpha! \beta!} a^{(\beta)}(be)^{(\alpha)}(\beta) + r_p e^\psi + R_p
\]
where \( r_p \in \mathcal{S}_\delta^{(s)} (\langle \xi \rangle_\mu^{m_1 + m_2 - (1 - \delta - \bar{k})p}) \) and \( R_p \in \mathcal{S}_\delta (e^{-c(\xi)^s}) \) with some \( \bar{s} > 1 \), \( \bar{k} \) and \( c > 0 \) satisfying \( \bar{s} \bar{k} < 1 \) and \( \bar{k} > \bar{\bar{k}} \). For \( a^\# (be)^\# \) similar assertion holds, where \( (-1)^{|\beta|} \) is replaced by \( (-1)^{|\alpha|} \).

**Proof of Proposition 6.2.** Write
\[
(be)^\# a = \int e^{-2i\sigma(Y,Z)} b(X+Y) e^{\psi(X+Y)} a(X+Z) dY dZ
\]
with \( Y = (y, \eta) \), \( Z = (z, \zeta) \) and replace \( a(X+Z) \) by its Taylor expansion
\[
\sum_{|\alpha| < p} \frac{1}{\alpha!} \partial_\alpha a(X) Z^\alpha + \sum_{|\alpha| = p} \frac{p}{\alpha!} \int_0^1 (1 - \theta)^{p-1} \partial_\alpha a(X + \theta Z) d\theta \cdot Z^\alpha
\]
\[
= \sum_{|\alpha| < p} \frac{1}{\alpha!} \partial_\alpha a(X) Z^\alpha + R_p
\]
where
\[
R_p = \sum_{|\alpha| = p} \frac{p}{\alpha!} \int_0^1 (1 - \theta)^{p-1} \partial_\alpha a(X + \theta Z) d\theta \cdot Z^\alpha.
\]
Here note that the first term on the left-hand side yields
\[
\sum_{|\alpha| < p} \frac{1}{\alpha!} \int e^{-2i\sigma(Y,Z)} b(X+Y) e^{\psi(X+Y)} \partial_\alpha a(X) Z^\alpha dY dZ
\]
\[
= \sum_{|\mu + \nu| < p} \frac{(-1)^{|\nu|}}{(2i)^{|\mu + \nu|} \mu! \nu!} (\partial_\xi^\mu \partial_\zeta^\nu (b(X) e^{\psi(X)})) \partial_\alpha \partial_\xi^\mu \partial_\zeta^\nu a(X).
\]
Consider the remainder term

\[
\int e^{-2i\sigma(Y,Z)} b(X + Y) e^{\psi(X+Y)} R_\mu dY dZ = p \sum_{|\mu + \nu| = p} C_{\mu,\nu} \int_0^1 (1 - \theta)^{p-1} \times \int e^{-2i\sigma(Y,Z)} \partial_\mu^p \partial_\nu^p (b(X + Y) e^{\psi(X+Y)}) \partial_\xi^p \partial_\sigma^p a(X + \theta Z) d\theta dY dZ.
\]

with \(C_{\mu,\nu} = (-1)^{||\nu||}/((2i)^{||\mu + \nu||} \nu! \mu!).\) Denoting

\[
F = \partial_\nu^p \partial_\mu^p (b(X + Y) e^{\psi(X+Y)}) \partial_\xi^p \partial_\sigma^p a(X + \theta Z)
\]

and \(\hat{\chi} = \chi(\langle \eta \rangle_{\mu}^{-1}) \chi(\langle \xi \rangle_{\mu}^{-1}), \tilde{\chi} = \chi(\langle \xi \rangle_{\mu}^{\delta} |y|/6) \chi(|z|/6)\) we write \(\int e^{-2i\sigma(Y,Z)} F dY dZ\) as

\[
e^{\psi(X)} \left( \int e^{-2i\sigma(Y,Z)} F e^{-\psi(X)} \{ \tilde{\chi} \hat{\chi} + (1 - \tilde{\chi}) \hat{\chi} \} dY dZ + \int e^{-2i\sigma(Y,Z)} F (1 - \tilde{\chi}) dY dZ. \right)
\]

Denote \(\hat{\chi}_0 = \chi \hat{\chi}\) as before.

**Lemma 6.6.** Let \(\Psi(X,Y) = \psi(X + Y) - \psi(X)\) then on the support of \(\hat{\chi}_0\) one has

\[
|\Psi(X,Y)| \leq C\langle \xi \rangle_{\mu}^\delta g_X^{1/2}(Y).
\]

Assume \(a_i \in S_\delta^{(s)}(m_i), i = 1, 2\) then

\[
|\partial_\nu^p \partial_\mu^p (a_1(X + Y) e^{\Psi(X,Y)} a_2(X + Z) \hat{\chi}_0)| \leq CA^{\alpha + \beta} \langle \xi \rangle_{\mu}^{-|\alpha + \delta|} \langle \xi \rangle_{\mu}^{-|\beta|} \times (\langle \xi \rangle_{\mu}^{\delta} g_X^{1/2}(Y) + |\alpha + \beta|^{1 + \varepsilon} + |\alpha + \beta|^{\varepsilon} \langle \xi \rangle_{\mu}^{-\delta})^{\alpha + \beta} m_1 m_2 e^\Psi.
\]

Denoting

\[
\tilde{F} = F e^{-\psi(X)} = \partial_\nu^p \partial_\mu^p (b(X + Y) e^{\Psi(X,Y)}) \partial_\xi^p \partial_\sigma^p a(X + \theta Z)
\]

and applying Lemma 6.6 we obtain

**Corollary 6.2.** One has

\[
|\partial_\nu^p \partial_\mu^p (a_1(X + Y) e^{\Psi(X,Y)} a_2(X + Z) \tilde{\chi}_0)| \leq CA^{\alpha + \beta + \hat{\alpha} + \hat{\beta}} \langle \xi \rangle_{\mu}^{\delta} \langle \xi \rangle_{\mu}^{-|\alpha + \hat{\alpha}|} \times (\langle \xi \rangle_{\mu}^{\delta} + |\hat{\alpha} + \hat{\beta}|^{1 + \varepsilon} + |\alpha + \beta|^{\varepsilon} \langle \xi \rangle_{\mu}^{-\delta})^{\hat{\alpha} + \hat{\beta}} \times (\langle \xi \rangle_{\mu}^{\delta} g_X^{1/2}(Y) + |\alpha + \beta|^{1 + \varepsilon} + |\alpha + \beta|^{\varepsilon} \langle \xi \rangle_{\mu}^{-\delta})^{\alpha + \beta} \times (\langle \xi \rangle_{\mu}^{\delta} + p^{1 + \varepsilon} + p^\delta \langle \xi \rangle_{\mu}^{-\delta})^{p(1 + \varepsilon) + p^\delta \langle \xi \rangle_{\mu}^{-\delta}} \langle \xi \rangle_{\mu}^{-(1 - \delta) p} \langle \xi \rangle_{\mu}^{m_1 + m_2 e^\Psi}.
\]

Repeating similar arguments proving Lemma 6.4 one has
Lemma 6.7. We have
\[ \left| \frac{\partial^2 f}{\partial t^2} \right| \leq \mu^2 c \int e^{-2\epsilon t} R_p e^{-\psi(X)} \hat{\chi} dY dZ \]

by Lemma 6.7. We have
\[ \left| \frac{\partial^2 f}{\partial t^2} \right| \leq C A |(\alpha + \beta)/(1 - \delta)\rangle |(\alpha + \beta)/(1 - \epsilon)\rangle e^{-c(1 - \delta)/(1 + \epsilon)} \]

From (6.15) and Lemmas 6.7 and 6.8 we end the proof of Proposition 6.2.

6.3 Composition \((p e^{-\psi}) \# e^\psi\)

Proposition 6.3. Assume \(\psi \in S_3(\xi)\) where \(\hat{\kappa} < 1\) and \(\hat{\kappa} = \kappa \hat{\kappa} < 1\) with \(\hat{\kappa} = \kappa > \kappa\), \(c > 0\). Then we have \(p \# e^\psi, e^\psi \# p \in S_3(\xi)\) with \(s^* > 1\) and \(\kappa^* > 0\) such that \(\kappa^* \kappa^* < 1\).

Corollary 6.3. Assume \(\psi \in S_3(\xi)\) and \(p \in S_0(\xi)\) with \(\hat{\kappa} < 1\) and \(\kappa > \kappa, c > 0\) then \(p \# e^\psi, e^\psi \# p \in S_3(\xi)\) for any \(l \in \mathbb{R}\).

Remark 6.1. As we observed in the proof of Corollary 6.3 if \(p \in S_0(\xi)\) with \(s^* > 1\) then for any \(0 < c' < c\) and \(\delta \geq 0\) one has \(p \in S_3(\xi)\).

Proof of Proposition 6.3 Consider
\[ p \# e^{-\psi} = \int e^{-2\epsilon t} R_p e^{-\psi(X)} e^{\psi(X + Z)} dY dZ \]

where \(Y = (y, \eta), Z = (z, \zeta)\). Let \(\hat{\chi}\) be as before. Write
\[ \int e^{-2\epsilon t} R_p e^{-\psi(X)} e^{\psi(X + Z)} \{\hat{\chi} + (1 - \hat{\chi})\} dY dZ \]
and consider
\[
\partial_x^2 \partial_\xi^2 \int e^{-2i\sigma(Y,Z)} p(X+Y) e^{\psi(X+Z)} \hat{\chi} dY dZ
\]
\[
= \int e^{-2i\sigma(Y,Z)} \langle \eta \rangle^{-2\ell} \langle \zeta \rangle^{-2\ell} \langle D_\xi \rangle^{2\ell} \langle D_\eta \rangle^{2\ell} 
\times \langle y \rangle^{-2\ell} \langle z \rangle^{-2\ell} \langle D_\xi \rangle^{2\ell} \langle D_\eta \rangle^{2\ell} \partial_x^2 \partial_\xi^2 F \hat{\chi} dY dZ
\]
where \( F = p(X+Y) e^{\psi(X+Z)} \hat{\chi} \). Since \( \psi \in S^{(4)}_\delta((\xi)_\mu^\kappa) \) and \( p \in S_{0,\delta}(e^{-c(\xi)_\mu^\kappa}) \) it is clear that
\[
|\partial_x^2 \partial_\xi^3 \psi| \leq CA^{[\alpha+\beta]}(\xi)_\mu^\kappa (|\alpha + \beta|^{1+\varepsilon} + |\alpha + \beta|^{s}(\xi)_\mu^\kappa)^{\delta|\alpha+\beta|},
\]
\[
|\partial_x^2 \partial_\xi^3 \beta| \leq CA^{[\alpha+\beta]}(|\alpha + \beta|^{1+\varepsilon} + |\alpha + \beta|^{s}(\xi)_\mu^\kappa)^{\delta|\alpha+\beta|} e^{-c(\xi)_\mu^\kappa}.
\]
Without restrictions we may assume \( \bar{s} \geq s \) and therefore
\[
|\langle \eta \rangle^{-2\ell} \langle \zeta \rangle^{-2\ell} \langle D_\xi \rangle^{2\ell} \langle D_\eta \rangle^{2\ell} \langle y \rangle^{-2\ell} \langle z \rangle^{-2\ell} \langle D_\xi \rangle^{2\ell} \langle D_\eta \rangle^{2\ell} \partial_x^2 \partial_\xi^2 F \hat{\chi}|
\]
is bounded by
\[
C_{\ell} A^{[\alpha+\beta]} \langle \eta \rangle^{-2\ell} \langle \zeta \rangle^{-2\ell} \langle y \rangle^{-2\ell} \langle z \rangle^{-2\ell} \langle (\xi)_\mu^\kappa \rangle (8\ell + |\alpha + \beta|)^{1+\varepsilon}
\]
\[
+ (8\ell + |\alpha + \beta|)^{\bar{s}}(\xi)_\mu^\kappa (8\ell + |\alpha + \beta|) e^{-c(\xi)_\mu^\kappa}.
\]
From Lemma 6.3 this can be estimated by
\[
C_{\ell} A^{[\alpha+\beta]} \langle \eta \rangle^{-2\ell} \langle \zeta \rangle^{-2\ell} \langle y \rangle^{-2\ell} \langle z \rangle^{-2\ell} \langle (\xi)_\mu^\kappa \rangle (8\ell + |\alpha + \beta|)^{1+\varepsilon}
\]
\[
\times (\xi)_\mu^\kappa (|\alpha + \beta|^{1+\varepsilon} + |\alpha + \beta|^{s}(\xi)_\mu^\kappa)^{\delta|\alpha+\beta|} e^{-c(\xi)_\mu^\kappa}.
\]
\[
\leq CA^{[\alpha+\beta]}|\alpha + \beta|^{\delta|\alpha+\beta|} e^{-c(\xi)_\mu^\kappa}.
\]
where \( \delta = \max\{(\bar{k} + \delta)/\bar{k}, 1 + \varepsilon + \delta/\bar{k}, \bar{s}\} \). Since \( \bar{k} < 1 - \delta, \bar{k} > \bar{k} \) and \( \bar{s} \bar{k} < 1 \) assuming \( \varepsilon \) such that \( (1 + \varepsilon)\bar{k} < 1 - \delta \) it is clear that \( \bar{s} \bar{k} < 1 \). Choosing \( \ell > (n + 1)/2 \) and recalling \( \int \Theta_{-\ell} \Phi_{-\ell} dY dZ = C \) we conclude
\[
(6.16) \quad \left| \partial_x^2 \partial_\xi^2 \int e^{-2i\sigma(Y,Z)} F \hat{\chi} dY dZ \right| \leq CA^{[\alpha+\beta]}|\alpha + \beta|^{\delta|\alpha+\beta|} e^{-c(\xi)_\mu^\kappa}.
\]
Denoting \( F = p(X+Y) e^{\psi(X+Z)} \) consider
\[
(6.17) \quad \partial_x^2 \partial_\xi^2 \int e^{-2i\sigma(Y,Z)} F \hat{\chi} dY dZ = \int e^{-2i\sigma(Y,Z)} \langle \eta \rangle^{-2N_2} \langle \zeta \rangle^{-2N_1}
\]
\[
\times \langle D_\xi \rangle^{2N_2} \langle D_\eta \rangle^{2N_1} \langle y \rangle^{-2\ell} \langle z \rangle^{-2\ell} \langle D_\xi \rangle^{2\ell} \langle D_\eta \rangle^{2\ell} \partial_x^2 \partial_\xi^2 F \hat{\chi} dY dZ
\]
\[39\]
where \( \hat{\chi}_1 = (1 - \hat{\chi})\chi(\langle \zeta \rangle \langle \eta \rangle^{-1}/4) \) and \( \hat{\chi}_2 = (1 - \hat{\chi})(1 - \chi(\langle \zeta \rangle \langle \eta \rangle^{-1}/4)) \). Note that there is \( C > 0 \) such that \( \langle \zeta \rangle \mu \leq C \langle \eta \rangle \) and \( \langle \zeta \rangle \leq C \langle \eta \rangle \) on the support of \( \hat{\chi}_1 \) and hence \( \langle \xi + \zeta \rangle \leq C \langle \eta \rangle \) there. Similarly on the support of \( \hat{\chi}_2 \) one has \( \langle \zeta \rangle \mu \leq C \langle \zeta \rangle \), \( \langle \eta \rangle \leq C \langle \zeta \rangle \) and \( \langle \xi + \zeta \rangle \leq C \langle \zeta \rangle \).

For case \( \hat{\chi}_1 \) we choose \( N_1 = \ell \), \( N_2 = N \). Since \( |\psi(X + Z)| \leq C \langle \eta \rangle^{\kappa} \) on the support of \( \hat{\chi}_1 \) it is not difficult to see that

\[
\left| \langle \eta \rangle^{-2N} \langle \zeta \rangle^{-2\ell} \langle D_z \rangle^{2N} \langle D_y \rangle^{2\ell} \langle \eta \rangle^{-2\ell} \langle D_z \rangle^{2\ell} \langle D_\eta \rangle^{2\ell} \partial_\eta^2 \partial_\zeta^2 F \hat{\chi}_1 \right|
\]

is bounded by

\[
Ce^{A_{2N}^2 + |\alpha + \beta|} \langle \eta \rangle^{-2N} \langle \zeta \rangle^{-2\ell} \langle \eta \rangle^{-2\ell} \langle \zeta \rangle^{-2\ell} \langle \eta \rangle^{m + 2\ell + 4\kappa \ell} \times (C\mu^{\kappa + \delta} + (2N)^{1+\varepsilon} \langle \eta \rangle^\delta) + (2N)^s \langle \eta \rangle^{2N} \times (C\mu^{\kappa + \delta} + |\alpha + \beta|^{1+\varepsilon} \langle \eta \rangle^\delta + |\alpha + \beta|^s)e^{C \langle \eta \rangle^{\kappa}}
\]

since \( \bar{s} \geq s \). Here writing

\[
A_{2N} \langle \eta \rangle^{-2N} \langle \zeta \rangle^{-2\ell} \langle \eta \rangle^{-2\ell} \langle \zeta \rangle^{-2\ell} \langle \eta \rangle^{m + 2\ell + 4\kappa \ell} \times (C\mu^{\kappa + \delta} + (2N)^{1+\varepsilon} \langle \eta \rangle^\delta) + (2N)^s \langle \eta \rangle^{2N}
\]

we choose \( 2N = c_1 \langle \eta \rangle^{(1-\delta)/(1+\varepsilon)} \) with small \( c_1 > 0 \) then the right-hand side is bounded by \( Ce^{c_1 \langle \eta \rangle^{(1-\delta)/(1+\varepsilon)}} \) because \( \delta(1 - \delta) < 1 \) and \( \kappa < 1 - \delta \). Thanks to \((6.13)\) and \((6.14)\) it follows that \((6.18)\) is bounded by

\[
Ce^{A_{1}^{[\alpha + \beta]} \langle \zeta \rangle^{-2\ell} \langle y \rangle^{-2\ell} \langle D_z \rangle^{2\ell} \langle D_y \rangle^{2\ell} \langle \eta \rangle^{-2\ell} \langle D_z \rangle^{2\ell} \langle D_\eta \rangle^{2\ell} \partial_\eta^2 \partial_\zeta^2 F \hat{\chi}_2 \rangle
\]

where \( \bar{s}_1 = \max\{1 + \varepsilon\}/(1 - \delta), \bar{s} \rangle \) and hence \( \bar{s}_1 \kappa < 1 \) assuming \( (1 + \varepsilon) \kappa < 1 - \delta \) which holds for small \( \varepsilon \).

For case \( \hat{\chi}_2 \) we choose \( N_1 = N \), \( N_2 = \ell \) in \((6.17)\). Since \( \eta \in S_{0,0}^{(3)}(e^{-c \langle \zeta \rangle^{\kappa}}) \) it follows that

\[
\left| \langle \eta \rangle^{-2\ell} \langle \zeta \rangle^{-2N} \langle D_z \rangle^{2\ell} \langle D_y \rangle^{2N} \langle \eta \rangle^{-2\ell} \langle D_z \rangle^{2\ell} \langle D_\eta \rangle^{2\ell} \partial_\eta^2 \partial_\zeta^2 F \hat{\chi}_2 \rangle
\]

is bounded by

\[
C_{\ell} A^{2N + |\alpha + \beta|} \langle \eta \rangle^{-2\ell} \langle \zeta \rangle^{-2\ell} \langle \eta \rangle^{-2\ell} \langle \zeta \rangle^{-2\ell} \langle \eta \rangle^{m + 2\ell + 4\kappa \ell} \times (2N)^{2\kappa N} (C\langle \zeta \rangle^{\kappa + \delta} + |\alpha + \beta|^{1+\varepsilon} \langle \zeta \rangle^\delta + |\alpha + \beta|^s)e^{C \langle \zeta \rangle^\kappa}.
\]

Choose \( 2N = c_1 \langle \zeta \rangle^{1/\bar{s}} \) then we have

\[
\langle \zeta \rangle^{-2N} (2N)^{2\kappa N} = (c_1^2)^{2N} \leq e^{-c \langle \zeta \rangle^{1/\bar{s}}}
\]

taking \( c_1 > 0 \) small. Here recall that \( 1/\bar{s} > \kappa \) by assumption. Noting

\[
\langle \zeta \rangle^{(\kappa + \delta)|\alpha + \beta|} e^{-c \langle \zeta \rangle^{1/\bar{s}}} \leq (1 + \alpha + \beta)^{\bar{s}(\kappa + \delta)} \langle \alpha + \beta \rangle^\kappa e^{-c \langle \zeta \rangle^{1/\bar{s}}}
\]

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it is easy to see that \( (6.19) \) is bounded by
\[
C e^{\alpha + \beta |\langle \eta \rangle| - 2t \langle y \rangle - 2t |\alpha + \beta^2| e^{-c_1(\zeta)^{1/5}}}
\]
where \( \delta_2 = \max\{\delta(\kappa + \delta), \delta \delta + 1 + \varepsilon, \delta \} \). Since \( \delta \kappa < 1 \) then \( (\delta \delta + 1 + \varepsilon) \kappa < \delta + (1 + \varepsilon) \kappa < 1 \) for small \( \varepsilon > 0 \) and hence \( \delta_2 \kappa < 1 \). Setting
\[
\left\{
\begin{array}{l}
s^* = \max\{\delta_1, \delta_2, \delta\}, \\
\kappa^* = \min\{(1 - \delta)/(1 + \varepsilon), 1/\delta, \kappa\}
\end{array}
\right.
\]
where \( s^* \kappa < 1 \) and \( \kappa^* > \kappa \) and recalling that \( \langle \xi \rangle_\mu \leq C \langle \eta \rangle, \langle \xi \rangle_\mu \leq C \langle \zeta \rangle \) on the support of \( \hat{\chi}_1 \) and \( \hat{\chi}_2 \) one can show
\[
(6.20) \quad \left| \frac{\partial_x^2 \partial_{\xi}^2}{\int e^{-2i\sigma(Y,2)} (1 - \hat{\chi}) dY dZ} \right| \leq C A^{\alpha + \beta |\langle \xi \rangle_\mu| - 2 + 2 \kappa} \langle \kappa \rangle_\mu^{-1}.
\]
Therefore combining \( (6.16) \) and \( (6.20) \) we obtain Proposition \( 6.3 \).

### 6.4 Composition \( e^{(D)^\kappa} a(x) e^{-(D)^\kappa} \)

**Proposition 6.4.** Assume \( 0 < \kappa < 1 \) and \( a(x) \in G^0_s(\mathbb{R}^n) \) with \( s > 1 \) and \( \kappa s < 1 \). Then the operator \( b(x, D) = e^{(D)^\kappa s} a(x) e^{-(D)^\kappa s} \) is a pseudodifferential operator with symbol given by
\[
(6.21) \quad b(x, \xi) = a(x) + \sum_{|\alpha| = 1} D_\alpha^s a(x) \langle \delta \xi \rangle_\mu^{\kappa} + q(x, \xi) + r(x, \xi)
\]
with \( q \in S^{(s)}_3 (\langle \xi \rangle_\mu^{-2 + 2s}) \) and \( r(x, \xi) \in S^{(s)}_{0,0} (e^{-c_1(\zeta)^{1/5}}) \) where \( \delta = 1 - \kappa \).

**Proof.** Write \( \phi(\xi) = \langle \xi \rangle_\mu^{\kappa} \) then
\[
b(x, \xi) = \int e^{-2i\sigma(Y,2)} e^{\phi(\xi + \eta) - \phi(\xi + \zeta)} a(x + y + z) dY dZ.
\]
The change of variables \( \tilde{y} = y + z, \tilde{z} = z - y, \tilde{\eta} = \eta - \zeta, \tilde{\zeta} = \zeta + \eta \) gives
\[
(6.22) \quad b(x, \xi) = (2\pi)^n \int e^{-i\tilde{y} \tilde{\eta}} e^{\phi(\xi + \tilde{\eta}) - \phi(\xi + \tilde{\zeta})} a(x + \tilde{y}) d\tilde{y} d\tilde{\eta}.
\]
Insert the Taylor expansion of \( a(X + Y) \)
\[
a(x + y) = \sum_{|\alpha| = 1} \frac{1}{\alpha!} D_\alpha^s a(x) (iy)^{\alpha} + 2 \sum_{|\alpha| = 2} \frac{(iy)^{\alpha}}{\alpha!} \int_0^1 (1 - \theta) D_\alpha^s a(x + \theta y) d\theta
\]
into \( (6.22) \) to get
\[
b(x, \xi) = \sum_{|\alpha| = 1} \frac{1}{\alpha!} \int e^{-i\eta \hat{\eta} e^{\phi(\xi + \tilde{\eta}) - \phi(\xi + \tilde{\zeta})}} D_\alpha^s a(x) (iy)^{\alpha} dy \eta
d\eta
\]
\[
+ 2 \sum_{|\alpha| = 2} \int e^{-i\eta \hat{\eta} e^{\phi(\xi + \tilde{\eta}) - \phi(\xi + \tilde{\zeta})}} (iy)^{\alpha} dy \eta \int_0^1 (1 - \theta) D_\alpha^s a(x + \theta y) d\theta.
\]
Choose $p$ so that $A > 0$. There exist constants $A > 0$ such that

$$
(6.24) \quad a(x) + \sum_{|\alpha| = 1} \partial_{\eta}^{\alpha} e^{\phi(\xi + \frac{\eta}{2}) - \phi(\xi - \frac{\eta}{2})} \bigg|_{\eta = 0} D_{\eta}^{\alpha} a(x)
$$

which is the second term on the right-hand side of (6.21). Note that

$$
\partial_{\xi}^{\alpha} \phi(\xi) \in S^{(s)}(\langle \xi \rangle^{-2+\kappa}), \quad |\alpha| = 1
$$

for any $\delta \geq 0$. After integration by parts, denoting $H_{\alpha}(\xi, \eta) = \partial_{\eta}^{\alpha} e^{\phi(\xi + \frac{\eta}{2}) - \phi(\xi - \frac{\eta}{2})}$ the second term on the right-hand side of (6.23) yields, up to a multiplicative constant,

$$
R = \int \sum_{|\alpha| = 2} e^{-i\eta \eta} H_{\alpha}(\xi, \eta) d\eta d\eta \int_{0}^{1} (1 - \theta) D_{\xi}^{\alpha} a(x + \theta y) d\theta
$$

$$
= \sum_{|\alpha| = 2} \int_{0}^{1} (1 - \theta) H_{\alpha}(\xi, \theta \eta) d\eta d\theta e^{-i\eta \eta} D_{\xi}^{\alpha} a(y) dy.
$$

Denote $E_{\alpha}(\eta) = \int e^{-i\eta \eta} D_{\xi}^{\alpha} a(y) dy$ then

**Lemma 6.9.** There exist $c > 0, C > 0$ such that $|E_{\alpha}(\eta)| \leq C e^{-c|\eta|^{1/\gamma}}$.

**Proof.** Integration by parts gives

$$
\eta^{\beta} E_{\alpha}(\eta) = \int e^{-i\eta \eta} D_{\xi}^{\alpha + \beta} a(y) dy.
$$

Then there exist constants $A > 0, C > 0$ such that $|E_{\alpha}(\eta)| \leq CA^{\beta} |\beta|! |\langle \eta \rangle|^{-|\beta|}$. Choose $p = |\beta|$ such that $p$ minimizes $Ap^{\lambda}(\eta)^{-p}$, that is $p \sim e^{-1} A^{-1/\gamma} |\eta|^{1/\gamma}$ so that $Ap^{\lambda}(\eta)^{-p} \lesssim e^{-A^{1/\gamma} |\eta|^{1/\gamma}}$.

Note that $H_{\alpha}(\xi, \eta)$ is a linear combination of terms

$$
\partial_{\xi}^{\hat{\beta}} \phi(\xi + \frac{\eta}{2}) \cdots \partial_{\xi}^{\hat{\beta}} \phi(\xi + \frac{\eta}{2}) \partial_{\xi}^{\hat{\beta}} \phi(\xi - \frac{\eta}{2}) \cdots \partial_{\xi}^{\hat{\beta}} \phi(\xi - \frac{\eta}{2}) e^{\phi(\xi + \frac{\eta}{2}) - \phi(\xi - \frac{\eta}{2})}
$$

$$
= K(\xi, \eta) e^{\phi(\xi + \frac{\eta}{2}) - \phi(\xi - \frac{\eta}{2})}
$$

where $\sum \beta_j = \beta, \sum \hat{\beta}_j = \hat{\beta}$ and $|\beta_j| \geq 1, |\hat{\beta}_j| \geq 1, \beta + \hat{\beta} = \alpha$. Let $\chi(r) \in \gamma^{(1+\epsilon)}(\mathbb{R})$ such that $\chi(r) = 1$ in $|r| < 1$ and $0$ for $r \geq 2$. Then we see

$$
(6.25) \quad |\partial_{\xi}^{\gamma} K(\xi, \eta)| \leq CA^{\gamma} |\gamma|! |\langle \xi \rangle|^{2\alpha - 2} |\langle \eta \rangle|^{-|\gamma|}
$$

on the support of $\hat{\chi} = \chi(\langle \xi \rangle^{-1} |\eta\rangle)$. Writing $H_{\alpha}(\xi, \eta) = C_{\alpha}(\xi, \eta) e^{\phi(\xi + \frac{\eta}{2}) - \phi(\xi - \frac{\eta}{2})}$ we have

$$
|\partial_{\xi}^{\gamma} (C_{\alpha}(\xi, \eta) \hat{\chi})| \leq CA^{\gamma} |\gamma|! |\langle \xi \rangle|^{2\alpha - 2} |\langle \eta \rangle|^{-|\gamma|}.
$$
On the other hand it is easy to check that

$$|\partial_x^2 (\phi(\xi + \eta/2) - \phi(\xi - \eta/2))| \leq CA^{\gamma_1}(\gamma_1)_{\mu}^{\kappa - 1}\eta\langle \xi \rangle_{\mu}^{-\gamma_1}$$

on the support of \( \tilde{\chi} \) and in particular

$$|\phi(\xi + \eta/2) - \phi(\xi - \eta/2)| \leq C\langle \xi \rangle_{\mu}^{\kappa - 1}\eta \leq C\langle \xi \rangle_{\mu}^{\kappa - 1}\eta \leq C'(\eta)^{\kappa}.$$

Therefore from Lemma 2.7 we conclude

$$|\partial_x^2 (H(\eta, \xi)\tilde{\chi})| \leq CA^{\gamma_1}(\gamma_1)_{\mu}^{\kappa - 2}\gamma_1(\gamma_1)_{\mu}^{\kappa - 1}\eta + |\gamma|^{1+\epsilon}\gamma_1(\gamma_1)_{\mu}^{\kappa - 1}\eta e^{C'(\eta)^{1/2}}.$$

Indeed since \( \kappa + \delta = 1 \) one has

$$\langle \xi \rangle_{\mu}^{\kappa - 1}\eta |\gamma|^1 e^{-c(\eta)^{1/2}} \leq \langle \xi \rangle_{\mu}^{\kappa - 1}\eta |\gamma|^1 e^{-c(\eta)^{1/2}} \leq CA^{\gamma_1}(\gamma_1)_{\mu}^{\kappa - 1}\eta |\gamma|^1 e^{-c(\eta)^{1/2}}.$$

With \( F_\alpha = (1 - \theta)H(\xi, \theta \eta)E_\alpha(\eta) \) denote

$$R = \sum_{|\alpha| = 3} \int_0^1 \int e^{ix\eta} F_\alpha \tilde{\chi} d\eta d\theta + \sum_{|\alpha| = 3} \int_0^1 \int e^{ix\eta} F_\alpha (1 - \tilde{\chi}) d\eta d\theta = R' + R''.$$

Thanks to Lemma 6.9 and 6.26, taking \( |\eta|^\beta e^{-c(\eta)^{1/2}} \leq CA^{\beta}|\beta|^\epsilon e^{-c(\eta)^{1/2}} \) into account \( (0 < \epsilon < C) \) one has

$$|\partial_x^2 R'| \leq CA^{\beta} \sum_{|\alpha| = 3} \int \int |\partial_x^2 H(\xi, \theta \eta)| \beta |\beta|^\epsilon e^{-c(\eta)^{1/2}} d\eta d\theta$$

$$\leq C\langle \xi \rangle_{\mu}^{2\kappa - 2}A^{\kappa + \beta}(\langle \gamma \rangle^{1+\epsilon} + |\gamma|^{\kappa}\langle \xi \rangle_{\mu}^{\kappa - \delta}| \beta |\beta|^{\kappa}\langle \xi \rangle_{\mu}^{\kappa - \delta}| \gamma |^{\kappa + \delta}|\beta|$$

$$\leq C\langle \xi \rangle_{\mu}^{2\kappa - 2}A^{\kappa + \beta}(|\beta|^\epsilon + |\gamma|^{} + |\beta|^{} + |\gamma|^{} \langle \xi \rangle_{\mu}^{\kappa - \delta}| \beta |^{\kappa + \delta}|\xi |^{\kappa + \delta}|\beta|$$

for any \( \delta \geq 0 \) that is \( R' \in S_3^{(s)} \langle \xi \rangle_{\mu}^{2\kappa - 2} \). On the other hand it is easy to check that

$$|\partial_x^2 (H(\xi, \theta \eta)(1 - \tilde{\chi})) e^{-c(\eta)^{1/2}} \leq CA^{\gamma_1}(\gamma_1)_{\mu}^{\kappa - 1}\eta e^{-c(\eta)^{1/2}}$$

and hence \( |\partial_x^3 R''| \) is bounded by \( CA^{\kappa + \beta}|\beta + \gamma|^{\kappa + \delta}|\xi |^{\kappa + \delta}|\beta| e^{-c(\eta)^{1/2}} \) so that \( R'' \in S_3^{(s)}(e^{-c(\eta)^{1/2}}) \). Therefore choosing \( q(x, \xi) = r + R' \) and \( R = R'' \) we end the proof of Proposition 6.4.
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