Maximum likelihood estimator for mixed fractional Vasicek process

Chunhao Cai
School of Mathematics, Shanghai University of Finance and Economics, Shanghai, China.
caichunhao@mail.shufe.edu.cn

Yinzhong Huang
School of Mathematics, Shanghai University of Finance and Economics, Shanghai, China.
1307111968@qq.com

Weilin Xiao
School of Management, Zhejiang University, Zhejiang, China.
wlxiao@zju.edu.cn

Abstract
In this paper, we will study asymptotical properties of the unknown parameter in the drift terms of the mixed fractional Brownian motion Vasicek process. The fundamental martingale and Laplace transform will be the main tools for our analysis. At the same time, we complete the strong consistency of MLE of the drift parameter in mixed fractional Brownian motion which has not analyzed in [3].

Keywords: MLE, mixed fractional Brownian motion, Vasicek model, Laplace Transform

1. Introduction
The standard Vasicek model was proposed and studied by O. Vasicek [10] in 1977 for the purpose of interest rate modeling. It is described by the following stochastic differential equation

$$dX_t = (\alpha - \beta X_t)dt + \gamma dW_t, \quad t \in [0,T],$$

where $\alpha, \beta, \gamma \in \mathbb{R}^+$ and $W = W_t, \quad t \in [0,T]$ is a standard Wiener process. As we know the ratio $\alpha/\beta$ is the long-term ratio and the constant $\gamma$ represents the stochastic volatility.

When we change the standard Brownian motion with a fractional Brownian motion, it becomes a fractional Vasicek model which has various financial applications presented for example in [8] and [9]. In the financial model, parameter estimation for the stochastic process is also a very interesting topic, we can refer to Kutoyants [7] for the statistical inference with continuous observation for the diffusion process. In the fractional Vasicek model, the estimation for the unknown parameters $\alpha$ and $\beta$ can be found in Lohvonenko et al. [12]. This is the extension of the parametric estimation of the fractional Ornstein-Uhlenbeck process completed by Kleptsyna et al. [4].

These days, mixed fractional Brownian motion attracts the attention of many researchers. This process is first presented by Cheridito [1], he has proved that when the Hurst parameter $H > 3/4$, the market driven by mixed fractional Brownian motion has no arbitrary. Cai et al. [2] developed the theory of Cheridito using the filtering approach. On the other hand, Kleptsyna et al. [5] explained this process from the view of spectral theory. The maximum likelihood estimator for
the drift parameter of the ergodic mixed fractional Ornstein-Uhlenbeck process was constructed by Kleptsyna and Chigansky [3]. With the complicated calculus of the Laplace Transform and the limit presence of the eigenvalues and eigenfunctions of the covariance operator of the fractional Brownian motion [4] they have presented that the MLE for the drift parameter is asymptotically normal.

As far as we know, until now no body has considered the parametric estimation for the mixed fractional Vasicek model and in this paper we try to complete it. In addition, in the article [3] they have not consider the strong consistency of this drift parameter and we will add this conclusion in our paper.

Now let us define the mixed fractional Vasicek process

$$X_t = (X_t, t \in [0,T])$$

satisfying the following SDE:

$$dX_t = (\alpha - \beta X_t)dt + \gamma \xi_t, t \in [0,T], X_0 = 0.$$  \hspace{1cm} (2)

where $\alpha, \beta, \gamma > 0$, $\xi_t = W_t + B^H_t$, $t \in [0,T]$, $H \in (0,1)$ is a mixed fractional Brownian motion defined in [2]. Suppose that we can observe the whole trajectory of the process $X = (X_t, t \in [0,T])$, our purpose is to construct the Maximum Likelihood Estimators for $\alpha$ and $\beta$ for $H > 1/2$ is known. In this paper we suppose $\gamma$ is also known, if not it can be estimated using the quadratic variation of the process $X = (X_t, t \in [0,T])$ and without difference we suppose $\gamma = 1$.

Remark 1. In this paper we suppose $H > 1/2$ is known. In fact, the estimation of $H$ in the mixed fractional Brownian motion is not an easy work. Until now, we only found some results in [14]. Different from [13], the mixed fractional Brownian motion is not self-similar and the property of LAN for mfBm will be our further study.

2. Preliminary: Mixed fractional Brownian motion

From [2], let us define the mixed fractional Brownian motion $\xi = (\xi_t, t \in [0,T])$:

$$\xi_t = W_t + B^H_t$$

where $W = (W_t, t \in [0,T])$ is the standard Brownian motion and $B^H = (B^H_t, t \in [0,T])$ is the independent fractional Brownian motion (fBm) with the Hurst exponent $H \in (0,1)$ that is a centered Gaussian process with covariance function

$$K(s,t) = \mathbf{E}B^H_t B^H_s = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H-1} \right).$$

Let $\mathcal{F}_t = (\mathcal{F}^\xi_t, t \in [0,T])$ and consider the filtering process

$$M_t = \mathbf{E}(B_t | \mathcal{F}^\xi_t), \ t \in [0,T].$$

The process $M$ is an $\mathcal{F}^\xi$-martingale and admits the representation

$$M_t = \int_0^t g(s,t)dX_s, \ \langle M \rangle_t = \int_0^t g(s,t)ds, \ t \geq 0,$$  \hspace{1cm} (3)

where the kernel $g(s,t)$ solves integro-differential equation

$$g(s,t) + H \frac{d}{ds} \int_0^t g(r,t)|r - s|^{2H-1}\text{sign}(s-r)dr = 1, \ 0 < s < t \leq T.$$
for $H > 1/2$, the equation $g(s, t)$ is a Wiener-Hopfner equation:

$$g(s, t) + H(2H - 1) \int_0^t g(r, t)|r - s|^{2H - 2} dr = 1, \ 0 \leq s \leq t \leq T.$$ 

Moreover, from [2] we have the following theorem:

**Theorem 2.1.** For $H > 1/2$, the quadratic variation of the $\mathcal{F}^\xi$-martingale $M$ is

$$\langle M \rangle_t = \int_0^t g^2(s, s)ds \quad (4)$$

and moreover,

$$\xi_t = \int_0^t G(s, t)dM_s, \ t \in [0, T] \quad (5)$$

where $G(s, t)$ is defined in the equation (2.20) of [2].

The equality in (4) suggests that the martingale $M$ admits innovation type representation, which can be used to analyse the structure of the mixed fBm with stochastic drift and to derive an analogue of Girsanov’s theorem which will be the key tool for constructing the maximum likelihood estimator:

**Corollary 2.2.** Consider a process $Y = (Y_t, t \in [0, T])$ defined by

$$Y_t = \int_0^t f(s)ds + \xi_t, \ t \in [0, T]$$

where $f = (f(t), t \in [0, T])$ is a process with continuous path and $\mathbb{E} \int_0^T |f(t)|dt < \infty$, adapted to a filtration $\mathcal{G} = (\mathcal{G}_t)$ with respect to which $M$ is a martingale. Then $Y$ admits the representation

$$Y_t = \int_0^t G(s, t)dZ_s$$

with $G$ defined in [5] and the process $Z = (Z_t, t \in [0, T])$

$$Z_t = \int_0^t g(s, t)dY_s, \ t \in [0, T]$$

is a $\mathcal{G}$-martingale with Doob-Meyer decomposition

$$Z_t = M_t + \int_0^t \Phi(s)d\langle M \rangle_s$$

where

$$\Phi(t) = \frac{d}{d\langle M \rangle_t} \int_0^t g(s, t)f(s)ds.$$ 

In particular, $\mathcal{F}Y = \mathcal{F}^Z_t$, $\mathbb{P}$ - a.s. for all $t \in [0, T]$ and, if

$$\mathbb{E} \exp \left\{ - \int_0^T \Phi(t)dM_t - \frac{1}{2} \int_0^T \Phi^2(t)d\langle M \rangle_t \right\} = 1,$$
then the measures $\mu^\xi$ and $\mu^Y$ are equivalent and the corresponding Radon-Nikodym derivative is given by
\[
\frac{d\mu^Y}{d\mu^\xi}(Y) = \exp \left\{ \int_0^T \hat{\Phi}(t)dZ_t - \frac{1}{2} \int_0^T \hat{\Phi}^2(t)d\langle M \rangle_t \right\},
\]
where $\hat{\Phi}(t) = E(\Phi(t)|\mathcal{F}^Y_t)$.

3. Transformation of Model and Main Results

Let us define
\[
Z_t = \int_0^t g(s,t)dX_s, \quad Q_t = \frac{d}{d\langle M \rangle_t} \int_0^t g(s,t)X_sds, \quad t \in [0,T]
\]
then for $\gamma = 1$ our observation will be $Z = (Z_t, t \in [0,T])$ where $Z_t$ satisfies the following equation:
\[
dZ_t = (\alpha - \beta Q_t)d\langle M \rangle_t + dM_t, \quad t \in [0,T].
\]
and the explicit expression of the likelihood function:
\[
L_T(\alpha, \beta, Z_T) = \exp \left( \int_0^T (\alpha - \beta Q_t)dZ_t - \frac{1}{2} \int_0^T (\alpha - \beta Q_t)^2d\langle M \rangle_t \right)
\]
Denote the log-likelihood equation $\Lambda(Z_T) = \log L_T(\alpha, \beta, Z_T)$ From the partial derivative of $\Lambda(Z_T)$ with respect to the $\alpha$ and $\beta$:
\[
\begin{aligned}
\frac{\partial \Lambda(Z_T)}{\partial \alpha} &= Z_T - \alpha \langle M \rangle_T + \beta \int_0^T Q_t d\langle M \rangle_t = 0 \\
\frac{\partial \Lambda(Z_T)}{\partial \beta} &= -\int_0^T Q_t dZ_t + \alpha \int_0^T Q_t d\langle M \rangle_t - \beta \int_0^T Q_t^2 d\langle M \rangle_t
\end{aligned}
\]
The maximum likelihood estimator $\hat{\alpha}_T$ and $\hat{\beta}_T$ is solution of equation of (8) and when we check the second partial derivative of $\Lambda(Z_T)$ and by the Cauchy-Schwarz inequality, the maximization can be confirmed. Now the solution of (8) gives us:
\[
\hat{\alpha}_T = \frac{\int_0^T Q_t dS_t \int_0^T Q_t d\langle M \rangle_t - Z_T \int_0^T Q_t^2 d\langle M \rangle_t}{\left( \int_0^T Q_t d\langle M \rangle_t \right)^2 - \langle M \rangle_T \int_0^T Q_t^2 d\langle M \rangle_t}
\]
and
\[
\hat{\beta}_T = \frac{\langle M \rangle_T \int_0^T Q_t dZ_t - Z_T \int_0^T Q_t d\langle M \rangle_t}{\left( \int_0^T Q_t d\langle M \rangle_t \right)^2 - \langle M \rangle_T \int_0^T Q_t^2 d\langle M \rangle_t}
\]
From the expression of $Z = (Z_t, t \in [0,T])$ we have the error of the maximum likelihood estimator:
\[
\hat{\alpha}_T - \alpha = \frac{\int_0^T Q_t dM_t \int_0^T Q_t d\langle M \rangle_t - M_T \int_0^T Q_t^2 d\langle M \rangle_t}{\left( \int_0^T Q_t d\langle M \rangle_t \right)^2 - \langle M \rangle_T \int_0^T Q_t^2 d\langle M \rangle_t}
\]
\[ \hat{\beta}_T - \beta = \frac{(M)_T \int_0^T Q_t dM_t - M_T \int_0^T Q_t \langle M \rangle_t}{\left( \int_0^T Q_t \langle M \rangle_t \right)^2} \] (11)

Such as in [12] we present the asymptotic normality of \( \hat{\alpha}_T \) and \( \hat{\beta}_T \):

**Theorem 3.1.** For \( H > 1/2 \)

\[ \sqrt{T} (\hat{\beta}_T - \beta) \overset{d}{\to} \mathcal{N}(0, \frac{1}{2\beta}) \] (12)

and

\[ T^{1-H}(\hat{\alpha}_T - \alpha) \overset{d}{\to} \mathcal{N}(0, v_H), \quad v_H = \frac{2H \Gamma(H + 1/2) \Gamma(3-2H)}{\Gamma(3/2 - H)} \] (13)

**Remark 2.** We can see that this result is the same as the model driven by the pure fractional Brownian motion, this corresponds for the linear regression and Ornstein-Uhlenbeck model.

In [3], when \( \alpha = 0 \) we only prove the asymptotical normality of the estimator error \( \hat{\beta}_T \), but there exists one problem, is \( \hat{\beta}_T \) converges into \( \beta \) almost surely? In this work, we prove that the this convergence almost surely is true for all \( \alpha \in \mathbb{R} \) and the key point of the demonstration is the same as well as \( \alpha = 0 \).

**Theorem 3.2.** For \( H > 1/2 \) the estimator \( \hat{\beta}_T \) converges to \( \beta \) almost surely for all \( \alpha \in \mathbb{R} \), that is

\[ \lim_{T \to \infty} \hat{\beta}_T = \beta, \text{ a.s.} \]

4. Proof of the Main results

4.1. Proof of Theorem 3.1

First of all, we look at the asymptotical normality of \( \hat{\beta}_T \), we rewrite the equation (11)

\[ \sqrt{T} (\hat{\beta}_T - \beta) = \frac{1}{\sqrt{T}} \int_0^T Q_t dM_t - \frac{M_T}{\langle M \rangle_T} \frac{1}{\sqrt{T}} \int_0^T Q_t \langle M \rangle_t}{\left( \frac{1}{\sqrt{T}} \int_0^T Q_t \langle M \rangle_t \right)} - \frac{1}{T} \int_0^T Q_t^2 d\langle M \rangle_t} \]

when \( M_T \) is a centered Gaussian random variable with variation \( \langle M \rangle_T \) then with Lemma 5.3 and Lemma 5.4 we have

\[ \frac{M_T}{\langle M \rangle_T} \frac{1}{\sqrt{T}} \int_0^T Q_t \langle M \rangle_t} \overset{P}{\to} 0 \]

and

\[ \left( \frac{1}{\sqrt{T}} \int_0^T Q_t \langle M \rangle_t} \right)^2 \overset{P}{\to} 0 \]

then the convergence in distribution of (12) can be achieved directly from Lemma 5.5.

Now we look at the convergence of \( \hat{\alpha}_T \), from (10) we have

\[ T^{1-H}(\hat{\alpha}_T - \alpha) = \frac{1}{\sqrt{T}} \int_0^T Q_t dM_t \frac{T^{1-H}}{(M)_T \sqrt{T}} \int_0^T Q_t \langle M \rangle_t} - \frac{T^{1-H}}{(M)_T} \frac{1}{\sqrt{T}} \int_0^T Q_t^2 d\langle M \rangle_t} \]

\[ \left( \frac{1}{\sqrt{T}} \int_0^T Q_t \langle M \rangle_t} \right)^2 - \frac{1}{T} \int_0^T Q_t^2 d\langle M \rangle_t} \]
it is obvious that
\[
\frac{1}{\sqrt{T}} \int_0^T Q_t dM_t \xrightarrow{\mathbb{P}} 0
\]
and
\[
\left( \frac{1}{\sqrt{T} \langle M \rangle_T} \int_0^T Q_t d\langle M \rangle_t \right)^2 \xrightarrow{\mathbb{P}} 0.
\]
now there exists only the term of
\[
T^{1-H} \frac{M_T}{\langle M \rangle_T} \xrightarrow{d} \mathcal{N}(0, v_H), \quad v_H = \frac{2H\Gamma(H+1/2)\Gamma(3-2H)}{\Gamma(3/2-H)}
\]
which has been proved in [3].

4.2. Proof of Theorem 4.2

First of all, let us prove that for \( \alpha = 0 \) the result is true, in fact when \( \alpha = 0 \),
\[
\hat{\beta}_T - \beta = \frac{\int_0^T Q_t^U t^T dM_t}{\int_0^T (Q_t^U)^2 d\langle M \rangle_t}
\]
Due to the strong law of large numbers, to get the convergence almost surely, we only need to prove
that
\[
\int_0^T (Q_t^U)^2 d\langle M \rangle_t \xrightarrow{a.s.} \infty
\]
from the Appendix of [15] we have
\[
\lim_{T \to \infty} K_T(\mu) = \lim_{T \to \infty} \frac{1}{T} \log \mathbb{E} \exp \left( -\mu \int_0^T (Q_t^U)^2 d\langle M \rangle_t \right) = \frac{\beta}{2} - \sqrt{\frac{\beta^2}{4} + \frac{\mu}{2}}
\]
for all \( \mu > -\frac{\beta^2}{T} \). When we take \( \mu > 0 \) then the limit of the Laplace transform:
\[
\lim_{T \to \infty} \mathbb{E} \left( -\mu \int_0^T (Q_t^U)^2 d\langle M \rangle_t \right) = 0
\]
which achieves the equation (14). Now we return to the case of \( \alpha \neq 0 \), in this situation, we have
\[
\hat{\beta}_T - \beta = \frac{\langle M \rangle_T \int_0^T Q_t dM_t}{\left( \int_0^T Q_t d\langle M \rangle_t \right)^2 - \langle M \rangle_T \int_0^T Q_t^2 d\langle M \rangle_t} - \frac{M_T}{\langle M \rangle_T} \frac{\int_0^T Q_t d\langle M \rangle_t}{\left( \frac{1}{\sqrt{T} \langle M \rangle_T} \int_0^T Q_t d\langle M \rangle_t \right)^2} + \int_0^T Q_t^2 d\langle M \rangle_t
\]
For the first term,
From the proof of Lemma 5.5 and equation (14) we know
\[
\frac{\int_0^T Q_t^2 d\langle M \rangle_t}{\int_0^T Q_t dM_t} \to \infty.
\]
and with the previous proof we know
\[
\left( \frac{\int_0^T Q_t d\langle M \rangle_t}{\langle M \rangle_T \int_0^T Q_t dM_t} \right)^2
\]
is bounded, so the first term tends to 0 almost surely as well as the second term which achieves the proof.

5. Appendix: Auxiliary Results

From \cite{3} we have

Lemma 5.1. For \( H > 1/2 \), we have
\[
\frac{d}{dT} (M)_T \sim T^{1-2H}, \quad \left( \frac{d}{dT} \log \frac{d}{dT} (M)_T \right)^2 \sim T^{-2}, \quad T \to \infty.
\]

The next Lemma is the relation between \( \alpha = 0 \) and \( \alpha \neq 0 \):

Lemma 5.2. Let \( U = (U_t, 0 \leq t \leq T) \) be a mixed fractional Ornstein-Uhlenbeck process with drift parameter \( \beta \):
\[
dU_t = -\beta U_t dt + d\xi_t, \quad t \in [0, T], \quad U_0 = 0,
\]
then
\[
X_t = \frac{\alpha}{\beta} - \frac{\alpha}{\beta} e^{-\beta t} + U_t, \quad t \in [0, T]
\]
and we have the development of \( Q_t \) with
\[
Q_t = \frac{\alpha}{\beta} - \frac{\alpha}{\beta} V(t) + Q_t^U
\]
where
\[
V(t) = \frac{d}{d(M)_t} \int_0^t g(s, t)e^{-\beta s} ds, \quad Q_t^U = \frac{d}{d(M)_t} \int_0^t g(s, t)U_s ds
\]
The followings will be some limit results:

Lemma 5.3. For \( H \in (0, 1) \) and \( H \neq 1/2 \) we have
\[
\int_0^T V(t) d(M)_t \sim Const
\]
for \( T \to \infty \).
Proof. From the definition of the function of $g(s,t)$ we know that $0 < g(s,t) < 1$, then
\[
\int_0^T V(t)d\langle M \rangle_t = \int_0^T \frac{d}{d(M)_t} \int_0^t g(s,t)e^{-\beta s} dsd(M)_t = \int_0^T g(s,T)e^{-\beta s} ds \leq \int_0^T e^{-\beta s} ds
\]
which completes the proof. \qed

Lemma 5.4. For $H > 1/2$ we have
\[
\frac{1}{\sqrt{T}} \int_0^T Q_{t}^U d\langle M \rangle_t \overset{P}{\rightarrow} 0
\]
Proof. From the definition of $Q_t$ we have:
\[
E \left( \int_0^T Q_{t}^U d\langle M \rangle_t \right)^2 = E \left( \int_0^T g(t,T)U_t dt \right)^2 = \int_0^T \int_0^T g(s,T)g(t,T)E(U_sU_t)dtds \\
\leq \int_0^T e^{-2\beta(T-t)} dt + C_{H,\beta}H(2H-1) \int_0^T \int_0^T g(t,T)g(s,T)|t-s|^{2H-2} dtds \\
= \int_0^T e^{-2\beta(T-t)} dt + C_{H,\beta} \int_0^T (1-g(s,T)) g(s,T) ds \\
= \frac{1}{2\beta}(1-e^{-2\beta T}) + 2C_{H,\beta}(M)_T.
\]

Lemma 5.5. Let $H > 1/2$, we have the following convergences:
\[
\frac{1}{T} \int_0^T Q_{t}^2 d\langle M \rangle_t \overset{P}{\rightarrow} \frac{1}{2\beta}, \ T \rightarrow \infty. \tag{18}
\]
and from the theorem of the convergence of martingale we have:
\[
\frac{1}{\sqrt{T}} \int_0^T Q_t dM_t \overset{d}{\rightarrow} \mathcal{N} \left( 0, \frac{1}{2\beta} \right), \ T \rightarrow \infty. \tag{19}
\]
Proof. From the definition of $Q_t$ we have:
\[
Q_t^2 = \left( \frac{\alpha}{\beta} - \frac{\alpha}{\beta} V(t) + Q_t^U \right)^2 = \left( \frac{\alpha}{\beta} \right)^2 V^2(t) + (Q_t^U)^2 - 2 \left( \frac{\alpha}{\beta} \right)^2 V(t) + \frac{2\alpha}{\beta} Q_t^U - \frac{2\alpha}{\beta} V(t)Q_t^U.
\]
We will analyze these terms one by one, first of all from (8), we have
\[
\frac{1}{T} \int_0^T (Q_t^U)^2 d\langle M \rangle_t \overset{P}{\rightarrow} \frac{1}{2\beta}, \ T \rightarrow \infty. \tag{20}
\]
and it is to see that
\[ \frac{1}{T} \int_0^T \left( \frac{\alpha}{\beta} \right)^2 d\langle M \rangle_t = \frac{(M)_T}{T} \to 0. \] (21)

On the other hand from Lemma 5.3
\[ \frac{1}{T} \int_0^T V(t) d\langle M \rangle_t \to 0, \] (22)

Now, we will look at the term for \( V^2(t) \), in fact
\[ V^2(t) = \frac{1}{d\langle M \rangle_t} \int_0^t g(s,t) e^{-\beta s} ds = \frac{1}{g^2(t,t)} \left( g(t,t) e^{-\beta t} + \int_0^t \dot{g}(s,t) e^{-\beta s} ds \right) \] (23)

where \( \dot{g}(s,t) = \frac{\partial}{\partial t} g(s,t) \). With Cauchy-Schwarz inequality we have
\[ \int_0^t \dot{g}(s,t) e^{-\beta s} ds \leq \sqrt{\int_0^t \dot{g}^2(s,t) ds} \sqrt{\int_0^t e^{-2\beta s} ds} \leq C_1 \sqrt{\int_0^t s^{4H-4} ds} \sqrt{\int_0^t e^{-2\beta s} ds} \] (24)

where \( C_1 \) is a constant and the last inequality comes from the fact that
\[ \dot{g}(s,t) + H(2H-1) \int_0^t \dot{g}(r,t) |r-s|^{2H-2} = -H(2H-1) g(t,t) |s-t|^{2H-2}, \quad s \in (0,t), \ t > 0. \]

Combining the equation (23), (24) and Lemma 5.1 we can easily obtain
\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T V^2(t) dt = 0 \] (25)

With Cauchy-Schwarz inequality and equation (19) and equation (25) we obtain:
\[ \left| \frac{1}{T} \int_0^T V(t) Q_t^U d\langle M \rangle_t \right| \leq \sqrt{\frac{1}{T} \int_0^T V^2(t) d\langle M \rangle_t} \frac{1}{T} \int_0^T (Q_t^U)^2 d\langle M \rangle_t} P \to 0. \] (26)

From the proof of Lemma 5.3 and Borel-Cantelli theorem
\[ \frac{1}{T} \int_0^T Q_t^U d\langle M \rangle_t \to 0, \ \text{a.s.} \] (27)

The convergence in probability of the equation (18) can be obtained by equation (20), (21), (22), (25), (26) and (27).

For the convergence of (19), as we know the process \( \int_0^T Q_s d\langle M \rangle, t \in [0,T] \) is a martingale and its quadratic variance is \( \int_0^T Q_s^2 d\langle M \rangle_s, t \in [0,T] \) and theorem of the convergence of martingale gives the proof of (19).
References

[1] P. Cheridito (2001) Mixed fractional Brownian motion, Bernoulli, 7, 913-934.
[2] C. Cai, P. Chigansky and M. Kleptsyna (2016) Mixed Gaussian process: A filtering approach, Annals of probability, 44(4),
[3] P. Chigansky and M. Kleptsyna (2018) Statistical analysis of mixed fractional Ornstein-Uhlenbeck process, Theory of Probability and Its Application, 3, 500-519.
[4] P. Chigansky and M. Kleptsyna (2018) Exact asymptotics in eigenproblems for fractional Brownian covariance operators, Stochastic Process and Their Applications, 128(6), 2007-2059.
[5] P. Chigansky, M. Kleptsyna and D. Marushkevych (2020) Mixed fractional Brownian motion: a spectral take, Journal of Mathematical Analysis and Application, 48(2), 123558.
[6] M. Kleptsyna and A. Le Breton (2002) Statistical analysis of the fractional Ornstein-Uhlenbeck type process, Statistical Inference for Stochastic Process, 5(3), 229-248.
[7] Y. A. Kutoyants (2004) Statistical inference for ergodic diffusion processes, Springer, London.
[8] F. Comte and E. Renault (1998) Long memory in continuous-time stochastic volatility models, Mathematical Finance, 8(4), 291-323
[9] A. Chronopoulou and F. G. Viens (2012) Stochastic volatility and option pricing with long-memory in discrete and continuous time, Quantitative Finance, 12(4), 635-649.
[10] Y. Hu and D. Nualart (2010) Parameter estimation for fractional Ornstein-Uhlenbeck processes, Statistics and Probability Letters, 80, 1030-1038.
[11] R. Liptser and A. Shiryaev (1974) Statistics of Random Process, Nauka, Moscow.
[12] S. Lohvinenko and K. Ralchenko (2017) Maximum likelihood estimation in the fractional vasicek model, Lithuanian Journal of Statistics, 56(1), 77-87.
[13] A. Brouste and M. Fukasawa (2018) Local asymptotic normality property for fractional Gaussian noise under high-frequency observations, Annals of Statistics, 46(5), 2045-2061.
[14] Y. Mishura, G. Shevchenko and M. Dozzi (2015) Statistical estimation by power variation in the mixed models, Statistical Inference for Stochastic Processes, 18(2), 151-175.
[15] D. Marushkevych (2016) Large deviation for drift parameter estimator of mixed fractional Ornstein-Uhlenbeck process, Modern Stochastic: Theory and Application, 3, 107-117.
[16] O. Vasicek (1977) An equilibrium characterization of the term structure, Journal of Finance Economics, 5(2), 177-188.