TRUNCATIONS OF A RANDOM UNITARY MATRIX AND YOUNG TABLEAUX

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Abstract. Let $U$ be a matrix chosen randomly, with respect to Haar measure, from the unitary group $U(d)$. We express the moments of the trace of any submatrix of $U$ as a sum over partitions whose terms count certain standard and semistandard Young tableaux. Using this combinatorial interpretation, we obtain a simple closed form for the moments of an individual entry of a random unitary matrix and use this to deduce that the rescaled entries converge in moments to standard complex Gaussian random variables. In addition, we recover a well-known theorem of E. Rains which shows that the moments of the trace of a random unitary matrix enumerate permutations with restricted increasing subsequence length.

1. Introduction

Consider the unitary group $U(d)$ as a probability space under normalized Haar measure. Given a random variable $X : U(d) \to \mathbb{C}$, its expected value is defined to be

$$\mathbb{E}_{U(d)}(X) = \int_{U(d)} X dU.$$

When studying a random variable $X$, one often wishes to know its moments

$$\mathbb{E}_{U(d)}(X^m \overline{X}^n),$$

since in many situations the moments of $X$ uniquely determine its distribution. $X$ is called a polynomial random variable if there is a polynomial $f \in \mathbb{C}[x_{11}, \ldots, x_{dd}]$ such that

$$X(U) = f(u_{11}, \ldots, u_{dd})$$

for all $U \in U(d)$. If $X$ is a polynomial random variable, then

$$\mathbb{E}_{U(d)}(X^m \overline{X}^n) = 0 \text{ if } m \neq n,$$
(see [1]), so knowledge of the moments of \( X \) reduces to knowledge of the quantities

\[
E_{U(d)}(|X|^{2n}).
\]

Recently, certain polynomial random variables on the unitary group have been shown to possess interesting combinatorial properties. For a random matrix \( U \in U(d) \), write its characteristic polynomial as

\[
det(U - zI) = (-1)^d \sum_{j=0}^{d} (-1)^j \text{Sc}_j(U) z^{d-j}.
\]

\( \text{Sc}_j(U) \) is called the \( j \)-th secular coefficient of \( U \). In particular,

\[
\text{Sc}_1(U) = \text{Tr}(U) \quad \text{and} \quad \text{Sc}_d(U) = \text{det}(U).
\]

It was shown by Rains in [5] that

\[
E_{U(d)}(|\text{Tr}(U)|^{2n}) = \sum_{\lambda \vdash n, \ell(\lambda) \leq d} (f^\lambda)^2,
\]

the number of pairs of semistandard Young tableaux on the same Young diagram, over all possible Young diagrams with \( n \) boxes and at most \( d \) rows. By the Schensted correspondence, this is equal to the number of permutations in the symmetric group \( S_n \) with no increasing subsequence of length greater than \( d \).

The other secular coefficients of a random unitary matrix also encode interesting combinatorial information, as shown by Diaconis and Gamburd in [2]. For \( jn \leq d \), [2] proves that

\[
E_{U(d)}(|\text{Sc}_j(U)|^{2n}) = H_n(j),
\]

where \( H_n(j) \) is the number of \( n \times n \) matrices whose entries are non-negative integers and whose rows and columns all sum to \( j \) (“magic squares”). The method of proof given in [2] is simple and elegant, and consists of two main ingredients:

- The secular coefficients of \( U \) are the elementary symmetric functions applied to the eigenvalues of \( U \).
- The elementary symmetric functions can be written as linear combinations of Schur functions, which are the irreducible characters of the unitary group and thus satisfy orthogonality relations.

Now suppose that we are given a random matrix \( U \in U(d) \), and we want to calculate the moments of a single entry of \( U \)

\[
E_{U(d)}(|u_{ij}|^{2n})
\]
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(note that since the permutation matrices are in $U(d)$ and Haar measure is translation invariant, all entries of $U$ are equidistributed). More generally, for a positive integer $k$ with $1 \leq k \leq d$, let $U_k$ denote the $k \times k$ upper left corner of $U$. We could ask about the moments of the matrix $U_k$

$$E_{U(d)}(|\text{Tr}(U_k)|^{2n})$$

(again, since the permutation matrices are in $U(d)$, the traces of any two $k \times k$ submatrices of $U$ are equidistributed). The methods used in [3], [2] will not work in this situation, since we are no longer dealing with the eigenvalues of a unitary matrix, but rather the eigenvalues of submatrices of a random unitary matrix. However, quite surprisingly, there is a simple combinatorial formula for these moments.

**Theorem 1.** Let $U$ be a matrix chosen randomly with respect to Haar measure from the unitary group $U(d)$, and let $U_k$ be its $k \times k$ upper left corner. We have

$$E_{U(d)}(|\text{Tr}(U_k)|^{2n}) = \sum_{\lambda \vdash n, \ell(\lambda) \leq k} (f^{\lambda})^2 \frac{s_{\lambda,k}(1)}{s_{\lambda,d}(1)}.$$

Here, for a positive integer $r$, $s_{\lambda,r}(1)$ denotes the number of semistandard Young tableaux on the shape $\lambda$ with entries from the set $[r] = \{1, \ldots, r\}$.

The notation $s_{\lambda,r}(1)$ is shorthand for the value of the Schur function $s_\lambda$ obtained by setting the first $r$ variables equal to 1 and making the remaining variables 0

$$s_\lambda(1, 1, \ldots, 1, 0, 0, \ldots).$$

The matrix $U_k$ is called a truncation of $U$. Truncations of random unitary matrices where first studied in [4] from an analytical point of view, where it was shown that $U_k$ is a contraction (i.e. all of its eigenvalues lie in the closed unit disc in $\mathbb{C}$), and the joint probability density function of the eigenvalue was found to be

$$C_{d,k} \prod_{i<j} |z_i - z_j|^2 \prod_{j=1}^k (1 - |z_j|^2)^{d-k-1}.$$  

$C_{d,k}$ is a normalization constant that was found in [4] to be

$$C_{d,k} = \frac{1}{\pi^{k/2}} \frac{1}{k!} \prod_{j=0}^{k-1} \binom{d-k+j-1}{j} (d-k+j).$$

Note that Rains’s result is an immediate corollary of Theorem 1; it is simply the case $k = d$. 


Corollary 1.1. In the special case $k = d$, we have

$$
\mathbb{E}_{U(d)}(\|\text{Tr}(U)\|^{2n}) = \sum_{\lambda \vdash n, d(\lambda) \leq d} (f^\lambda)^2.
$$

Proof. When $k = d$,

$$
\frac{s_{\lambda,k}(1)}{s_{\lambda,d}(1)} = \frac{s_{\lambda,d}(1)}{s_{\lambda,d}(1)} = 1.
$$

\[\square\]

Rains’s theorem is the extreme case $k = d$ of Theorem 1. The other extreme $k = 1$ is also of interest, since this corresponds to the computation of the moments of a single entry of a random unitary matrix.

Corollary 1.2. In the special case $k = 1$, we have

$$
\mathbb{E}_{U(d)}(|u_{ij}|^{2n}) = \frac{n!}{d(d+1)\ldots(d+n-1)} = \left(\frac{d+n-1}{n}\right)^{-1}.
$$

Proof. For $k = 1$ the only contribution to the sum is made by the single partition whose diagram is a row of $n$ boxes. Thus we have

$$
\int_{U(d)} |u_{ij}|^{2n} dU = \frac{1}{s_{n,d}(1)}.
$$

The generalized hook length formula asserts that

$$
\begin{align*}
\ &s_{\lambda,d}(1) = \prod_{\square \in \lambda} \frac{d + c(\square)}{h(\square)}, \\
\end{align*}
$$

where $c(\square)$ is the content of the box, and $h(\square)$ is its hook length (see [8]). For the single row partition of $n$, this gives

$$
\begin{align*}
\ &s_{n,d}(1) = \frac{d(d+1)\ldots(d+n-1)}{n!},
\end{align*}
$$

and the result follows.

\[\square\]

Explicitly knowing the moments of $u_{ij}$ makes it easy to determine its limiting distribution. Recall that if $x, y$ are Gaussian random variables with mean 0 and variance $1/2$, then the random variable $z = x + iy$ is called a standard complex Gaussian.

Corollary 1.3. As $d \to \infty$, the random variable $\sqrt{d}u_{ij}$ converges in distribution to a standard complex Gaussian random variable.
Proof. It is well-known (see for instance [4]) that the moments of a standard complex Gaussian $z$ are given by
\[ E(z^m z^n) = \delta_{mn} n! \]

Corollary 1.2 shows that
\[ E_U((\sqrt{d}u_{ij})^m (\sqrt{d}\overline{u}_{ij})^n) = \delta_{mn} \frac{d^n n!}{d(d+1)\ldots(d+n-1)} \]
\[ = \delta_{mn} \frac{d^n n!}{(1)(1+\frac{1}{d})\ldots(1+\frac{n-1}{d})} \]
\[ \to \delta_{mn} n!. \]

In order to prove Theorem 1, one needs to connect unitary expectations to symmetric function theory by some method other than applying symmetric functions to eigenvalues. This can be done using the Weingarten function introduced in [1], which is a powerful tool for computing the moments of polynomial random variables on the unitary group. The Weingarten function has already been used in free probability theory to prove asymptotic freeness results for random unitary matrices (see the recent book [3] for a clear account of this).

2. The Weingarten Function

For any positive integers $d$ and $n$, define a function $Wg(d, n, \cdot) : S_n \to \mathbb{Q}$ by
\[ Wg(d, n, \sigma) := \frac{1}{n!^2} \sum_{\lambda \vdash n, \ell(\lambda) \leq d} \frac{(f^\lambda)^2}{s_{\lambda, d}(1)} \chi^\lambda(\sigma), \]
where $\chi^\lambda$ is the irreducible character of $S_n$ labeled by $\lambda$.

The following integration formula was proved in [1].

**Theorem 2.** Let $i, j, i', j' : [n] \to [d]$ be any functions. Then
\[ \int_{U(d)} u_{i(1)j(1)} \ldots u_{i(n)j(n)} \overline{u}_{i(1)j(1)} \ldots \overline{u}_{i(n)j(n)} dU \]
\[ = \sum_{\sigma, \tau \in S_n} \delta_{i(1)\tau(1)} \ldots \delta_{i(n)\tau(n)} \delta_{j(1)\tau(1)} \ldots \delta_{j(n)\tau(n)} Wg(d, n, \tau \sigma^{-1}), \]
where $\delta$ is the Kronecker delta.

We can succinctly express the moments of $\text{Tr}(U_k)$ in terms of the Weingarten function as follows:
Lemma 1. For any positive integers $n$ and $k$, where $1 \leq k \leq d$, we have
\[
\mathbb{E}_{U(d)}(|\text{Tr}(U_k)|^{2n}) = n! \sum_{\alpha} \binom{n}{\alpha} \sum_{\sigma \in S_\alpha} \text{Wg}(d, n, \sigma),
\]
where the outer sum runs over all weak $k$-part compositions $\alpha$ of $n$, and the inner sum runs over all permutations in the Young subgroup $S_\alpha$ of $S_n$.

Proof. This is really just a calculation. We expand
\[
|\text{Tr}(U_k)|^{2n} = (u_{11} + \cdots + u_{kk})^{n} = \sum_{\alpha} \sum_{\beta} \binom{n}{\alpha} \binom{n}{\beta} u^\alpha \overline{u}^\beta,
\]
where we are summing over all pairs of weak $k$-part compositions of $n$, $\alpha = (a_1, \ldots, a_k)$ and $\beta = (b_1, \ldots, b_k)$. We are using multi-index notation,
\[
\binom{n}{\alpha} = \binom{n}{a_1, \ldots, a_k} \quad u^\alpha = u_{a_1}^{a_1} \cdots u_{a_k}^{a_k} \quad \overline{u}^\beta = \overline{u}_{b_1}^{b_1} \cdots \overline{u}_{b_k}^{b_k}
\]
Hence
\[
\mathbb{E}_{U(d)}(|\text{Tr}(U_k)|^{2n}) = \sum_{\alpha} \sum_{\beta} \binom{n}{\alpha} \binom{n}{\beta} \mathbb{E}_{U(d)}(u^\alpha \overline{u}^\beta).
\]
We will use the Weingarten integration formula to evaluate the expectation $\mathbb{E}_{U(d)}(u^\alpha \overline{u}^\beta)$ for a fixed pair of compositions $\alpha, \beta$. Implicitly define coordinate functions $i_\alpha, j_\alpha, i_\beta, j_\beta : [n] \to [d]$ by setting
\[
\int_{U(d)} u_{i_\alpha(1)} j_{\alpha}(1) \cdots u_{i_\alpha(n)} j_{\alpha}(n) \overline{u}_{i_\beta(1)} j_{\beta}(1) \cdots \overline{u}_{i_\beta(n)} j_{\beta}(n) \, dU
:= \int_{U(d)} u_{a_1}^{a_1} \cdots u_{a_k}^{a_k} \overline{u}_{b_1}^{b_1} \cdots \overline{u}_{b_k}^{b_k} \, dU
= \mathbb{E}_{U(d)}(u^\alpha \overline{u}^\beta).
\]
Applying the Weingarten integration formula, we have
\[
\mathbb{E}_{U(d)}(u^\alpha u^\beta) = \sum_{\sigma, \tau \in S_n} \delta_{i_\alpha(1)i_\beta(\sigma(1))} \cdots \delta_{i_\alpha(n)i_\beta(\sigma(n))} \delta_{j_\alpha(1)j_\beta(\tau(1))} \cdots \delta_{j_\alpha(n)j_\beta(\tau(n))} \ Wg(d, n, \sigma^{-1}).
\]

Since we are only taking entries from the diagonal, we have \(i_\alpha = j_\alpha\) and \(i_\beta = j_\beta\). Moreover, the level sets of these functions are easy to read off:

\[
i_\alpha^{-1}(1) = [1, a_1]
\]
\[
i_\alpha^{-1}(2) = [a_1 + 1, a_1 + a_2]
\]
\[\vdots\]
\[
i_\alpha^{-1}(k) = [n - a_k + 1, n]
\]

and

\[
i_\beta^{-1}(1) = [1, b_1]
\]
\[
i_\beta^{-1}(2) = [b_1 + 1, b_1 + b_2]
\]
\[\vdots\]
\[
i_\beta^{-1}(k) = [n - b_k + 1, n]
\]

Hence
\[
\prod_{r=1}^{a_1} \delta_{i_\alpha(r)i_\beta(\sigma(r))} = \prod_{r=1}^{a_1} \delta_{1i_\beta(\sigma(r))}
\]
\[
\prod_{r=a_1+1}^{a_1+a_2} \delta_{i_\alpha(r)i_\beta(\sigma(r))} = \prod_{r=a_1+1}^{a_1+a_2} \delta_{2i_\beta(\sigma(r))}
\]
\[\vdots\]
\[
\prod_{r=n-a_k+1}^{n} \delta_{i_\alpha(r)i_\beta(\sigma(r))} = \prod_{r=n-a_k+1}^{n} \delta_{ki_\beta(\sigma(r))},
\]

Thus in order for the product
\[
\delta_{i_\alpha(1)i_\beta(\sigma(1))} \cdots \delta_{i_\alpha(n)i_\beta(\sigma(n))}
\]
to be nonzero, we see that \(\sigma\) must bijectively map the interval \([1, a_1]\) onto the interval \([1, b_1]\), and \(\sigma\) must also bijectively map the interval \([a_1 + 1, a_1 + a_2]\) onto the interval \([b_1 + 1, b_1 + b_2]\), etc. Similarly, in order for the product
\[
\delta_{j_\alpha(1)j_\beta(\tau(1))} \cdots \delta_{j_\alpha(n)j_\beta(\tau(n))}
\]
to be nonzero, we see that \( \tau \) must bijectively map the interval \([1, a_1]\) onto the interval \([1, b_1]\), and \( \sigma \) must also bijectively map the interval \([a_1 + 1, a_1 + a_2]\) onto the interval \([b_1 + 1, b_1 + b_2]\), etc. Thus we see that the expectation \( E_U(d)(u^n \bar{w}^\tau) \) is zero unless:

- \( \alpha = \beta \), i.e. these two are the same weak \( k \)-part composition of \( n \),
- \( \sigma, \tau \) are both in the Young subgroup \( S_\alpha \), i.e. the subgroup of permutations in \( S_n \) that permute the first \( a_1 \) symbols amongst themselves, the next \( a_2 \) symbols amongst themselves, etc.

Thus

\[
E_U(d)(|\text{Tr}(U_k)|^{2n}) = \sum_\alpha \binom{n}{\alpha}^2 E_U(d)(u^{\alpha} \bar{w}^\tau)
\]

\[
= \sum_\alpha \binom{n}{\alpha}^2 \sum_{\sigma, \tau \in S_\alpha} Wg(d, n, \tau\sigma^{-1})
\]

\[
= \sum_\alpha \binom{n}{\alpha}^2 \alpha! \sum_{\sigma \in S_\alpha} Wg(d, n, \sigma)
\]

\[
= n! \sum_\alpha \binom{n}{\alpha} \sum_{\sigma \in S_\alpha} Wg(d, n, \sigma)
\]

\[
\square
\]

3. Proof of the Main Theorem

We are now in a position to prove Theorem 1.

Proof. Having proved that

\[
E_U(d)(|\text{Tr}(U_k)|^{2n}) = n! \sum_\alpha \binom{n}{\alpha} \sum_{\sigma \in S_\alpha} Wg(d, n, \sigma),
\]

we will work with the sum on the right. Plugging in the definition of the Weingarten function, we have

\[
n! \sum_\alpha \binom{n}{\alpha} \sum_{\sigma \in S_\alpha} Wg(d, n, \sigma) = n! \sum_\alpha \binom{n}{\alpha} \sum_{\sigma \in S_\alpha} \frac{1}{n!^2} \sum_{\lambda \vdash n, \ell(\lambda) \leq d} \frac{(f_\lambda)^2}{s_{\lambda, d}(1)} \chi^\lambda(\sigma).
\]

Changing order of summation, this becomes

\[
\sum_{\lambda \vdash n, \ell(\lambda) \leq d} (f_\lambda)^2 \frac{1}{s_{\lambda, d}(1)} \sum_\alpha \frac{1}{\alpha!} \sum_{\sigma \in S_\alpha} \chi^\lambda(\sigma) = \sum_{\lambda \vdash n, \ell(\lambda) \leq d} (f_\lambda)^2 \frac{1}{s_{\lambda, d}(1)} \sum_\alpha \langle 1, \chi^\lambda \rangle_{S_\alpha},
\]
where the inner product $\langle \cdot, \cdot \rangle_{S_\alpha}$ is the averaged dot product on the space $CF(S_\alpha)$ of complex-valued class functions on the group $S_\alpha$. Note that this sum may be written as

$$\sum_{\lambda \vdash n, \ell(\lambda) \leq d} (f^\lambda)^2 \frac{1}{s_{\lambda,d}(1)} \sum_\alpha \langle 1, \chi^\lambda \downarrow_{S_{\alpha}}^{S_n} \rangle_{S_\alpha},$$

where $\chi^\lambda \downarrow_{S_{\alpha}}^{S_n}$ is the restriction of the irreducible character $\chi^\lambda$ of $S_n$ to the subgroup $S_\alpha$. Now, the function which is identically 1 is the character of the trivial representation of $S_\alpha$. Thus we may apply Frobenius reciprocity (see for instance [6]):

$$\langle 1, \chi^\lambda \downarrow_{S_{\alpha}}^{S_n} \rangle_{S_\alpha} = \langle 1 \uparrow_{S_\alpha}^{S_n}, \chi^\lambda \rangle_{S_n},$$

where $1 \uparrow_{S_\alpha}^{S_n}$ is the induction of the trivial character of $S_\alpha$ to $S_n$.

The final step in the proof relies on the characteristic map $ch^n : CF(S_n) \to \Lambda^n$, where $\Lambda^n$ is the inner product space of degree $n$ symmetric functions equipped with the Hall inner product $\langle \cdot, \cdot \rangle_{\Lambda^n}$ (see for [6] or [8]). The class function is an isometry, and has the following important properties:

$$ch_n(1 \uparrow_{S_\alpha}^{S_n}) = h_\alpha$$

$$ch_n(\chi^\lambda) = s_\lambda,$$

where $h_\alpha$ is the complete homogeneous symmetric function indexed by $\alpha$, and $s_\lambda$ is the Schur function indexed by $\lambda$. Thus we have

$$\langle 1 \uparrow_{S_\alpha}^{S_n}, \chi^\lambda \rangle_{S_n} = \langle ch_n(1 \uparrow_{S_\alpha}^{S_n}), ch_n(\chi^\lambda) \rangle_{S_n} = \langle h_\alpha, s_\lambda \rangle_{\Lambda^n}.$$

It is well-known that the Schur functions constitute an orthonormal basis for $\Lambda^n$, and that the coordinates of the complete homogeneous symmetric functions with respect to the basis of Schur functions are the Kostka numbers (see [8]). That is,

$$h_\alpha = \sum_{\mu \vdash n} K_{\mu \alpha} s_\mu,$$

where the Kostka number $K_{\mu \alpha}$ is by definition the number of semistandard Young tableaux on the diagram of $\mu$ with content vector $\alpha$. Thus,

$$\langle h_\alpha, s_\lambda \rangle_{\Lambda^n} = \sum_{\mu \vdash n} K_{\mu \alpha} \langle s_\mu, s_\lambda \rangle_{\Lambda^n} = K_{\lambda,\alpha}.$$
Thus we have

$$\sum_{\lambda \vdash n, \ell(\lambda) \leq d} \frac{(f^{\lambda})^2}{s_{\lambda,d}(1)} \sum_{\alpha} \langle 1, \chi^{\lambda} \rangle s_{\alpha} = \sum_{\lambda \vdash n, \ell(\lambda) \leq d} \frac{(f^{\lambda})^2}{s_{\lambda,d}(1)} \sum_{\alpha} K_{\lambda \alpha}$$

$$= \sum_{\lambda \vdash n, \ell(\lambda) \leq d} \frac{(f^{\lambda})^2 s_{\lambda,k}(1)}{s_{\lambda,d}(1)},$$

where the last equality follows from the fact that, by definition,

$$\sum_{\alpha} K_{\lambda \alpha} = s_{\lambda,k}(1),$$

since the sum runs over all weak $k$-part compositions of $n$.

Finally, we remark that if $\lambda$ is a partition of $n$ with $\ell(\lambda) > k$, then $s_{\lambda,k}(1) = 0$. Hence,

$$\sum_{\lambda \vdash n, \ell(\lambda) \leq d} \frac{(f^{\lambda})^2 s_{\lambda,k}(1)}{s_{\lambda,d}(1)} = \sum_{\lambda \vdash n, \ell(\lambda) \leq k} \frac{(f^{\lambda})^2 s_{\lambda,k}(1)}{s_{\lambda,d}(1)},$$

which proves our theorem.

\[\square\]

4. Conclusion

In this paper, we have only investigated the moments of the trace of a truncation of a random unitary matrix. It seems possible to analyze the moments of the other secular coefficients of a truncation by the same method, and it would be interesting to see what combinatorial interpretations can be given for the moments of these coefficients.

In [2], the moments of secular coefficients of random orthogonal and symplectic matrices are investigated, and results analogous to the unitary case are proved. In [1], a notion of Weingarten function is defined for the orthogonal and symplectic groups. It is likely possible to analyze the moments of secular coefficients of truncations of random orthogonal and symplectic matrices using the Weingarten function for these groups.

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