Geometric Phase, Bundle Classification, and Group Representation

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Abstract

The line bundles which arise in the holonomy interpretations of the geometric phase display curious similarities to those encountered in the statement of the Borel-Weil-Bott theorem of the representation theory. The remarkable relation of the geometric phase to the classification of complex line bundles provides the necessary tools for establishing the relevance of the Borel-Weil-Bott theorem to Berry’s adiabatic phase. This enables one to define a set of topological charges for arbitrary compact connected semisimple dynamical Lie groups. In this paper, the problem of the determination of the parameter space of the Hamiltonian is also addressed. A simple topological argument is presented to indicate the relation between the Riemannian structure on the parameter space and Berry’s connection. The results about the fibre bundles and group theory are used to introduce a procedure to reduce the problem of the non-adiabatic (geometric) phase to Berry’s adiabatic phase for cranked Hamiltonians. Finally, the possible relevance of the topological charges of the geometric phase to those of the non-abelian monopoles is pointed out.
1 Introduction

In the past ten years, since the revival of the geometric phase, [1, 2], by Berry [3], the subject has attracted the attention of many physicists. The main reason for the unusual popularity of this remarkably simple subject, particularly among the theoretical physicists, has been its rich mathematical and physical foundations.

Recently, it was shown that the two holonomy interpretations of Berry’s phase were linked via the theory of universal bundles, [4, 5]. This remarkable coincidence of the physics of geometric phase and the mathematics of fibre bundles enables one to set up a convenient framework to analyze the non-adiabatic phase [5]. In the present paper, the results of [5] are briefly reviewed and their generalization to arbitrary finite dimensional unitary systems are presented.

In section 2, it is shown how the study of the standard example of a spin in a precessing magnetic field directs one to the Borel-Weil-Bott (BWB) theorem of the representation theory of compact semisimple Lie groups. In section 3, the relation of BWB theorem to the phenomenon of geometric phase is discussed in a general setting. Section 4 is devoted to a discussion of the relation of Berry’s connection and the Riemannian geometry of the parameter space. Section 5 includes the discussion of the reduction of the non-adiabatic phase problem to the adiabatic one for the cranked Hamiltonians. Section 6 consists of a short account on the classification of the parameter spaces and the topology of non-abelian monopoles. Section 7 includes the conclusions. A short proof of a result of Floquet theory is presented in the appendix.
2 Bundle Classification and the Holonomy Interpretations of the Geometric Phase

There are two mathematical interpretations of Berry’s (adiabatic) phase. These are due to Simon [6], and Aharonov and Anandan [7]. I shall refer to these two approaches by “BS” and “AA” which are the abbreviations of “Berry-Simon” and “Aharonov-Anandan”, respectively.

In the BS approach, one constructs a line bundle $L$ over the space $M$ of the parameters of the system. Then, $L$ is endowed with a particular connection which reproduces Berry’s phase as the holonomy of the closed loop in the parameter space.

Let us consider a quantum mechanical system whose evolution is governed by a parameter dependent Hamiltonian:

$$ H = H(x) , \quad x \in M . $$

Assume that for all $x \in M$ the spectrum of $H(x)$ is discrete and that there are no level crossings. Then, locally one can choose a set of orthonormal basic eigenstate vectors $\{ |n,x\rangle \}$. As functions of $x$, $|n,x\rangle$ are smooth and single valued. By definition, they satisfy:

$$ H(x)|n,x\rangle = E_n(x)|n,x\rangle , \quad (1) $$

where $E_n(x)$ are the corresponding energy eigenvalues. The Hamiltonian is made explicitly time dependent by interpreting time $t$ as the parameter of a curve

$$ C : [0,T] \ni t \rightarrow x(t) \in M , \quad (2) $$

and setting

$$ H(t) := H(x(t)) , \quad t \in [0,T] . \quad (3) $$

Then, each closed curve $C$ in $M$ defines a periodic Hamiltonian with period $T$. I shall discuss only the evolution of nondegenerate cyclic states with period $T$. 

Under the adiabatic approximation the initial eigenstates undergo cyclic evolutions, [3]. If $|\psi_n(t)\rangle$ denotes the evolving state vector, i.e., the solution of the Schrödinger equation:

$$H(t)|\psi_n(t)\rangle = i\frac{d}{dt}|\psi_n(t)\rangle$$ (4)

then

$$|\psi_n(0)\rangle := |n, x(0)\rangle ,$$

then

$$|\psi_n(T)\rangle\langle\psi_n(T)| \simeq |\psi_n(0)\rangle\langle\psi_n(0)| .$$ (5)

After a cycle is completed, the state vector gains a phase factor which consists of a dynamical ($e^{i\omega}$) and a geometric ($e^{i\gamma}$) part

$$|\psi_n(T)\rangle = e^{i(\omega+\gamma)}|\psi_n(0)\rangle ,$$ (6)

where

$$\omega := -\int_0^T E_n(x(t)) dt ,$$

and

$$e^{i\gamma} := \exp \oint_C A$$ (7)

$$A := -\langle n, x|d|n, x\rangle = -\langle n, x|\frac{\partial}{\partial x^{\mu}}|n, x\rangle dx^{\mu} .$$ (8)

The one-form $A$ is known as Berry’s connection one-form,[3].

In [6], Simon showed that $A$ could be interpreted as a connection one-form on a (spectral) line bundle $L$ over $M$,

$$\mathbb{C} \longrightarrow L \longrightarrow M ,$$ (9)

whose fibres are given by the energy eigenrays in the Hilbert space $\mathcal{H}$

$$L_x := \{z|n, x\rangle : z \in \mathbb{C}\} .$$ (10)

Thus, in the BS approach Berry’s phase is identified with the holonomy of the loop $C \subset M$ defined by the connection one-form $A$ of eq. (8).
In the AA approach one considers a complex line bundle $E$, or alternatively the associated $U(1)$-principal bundle, over the projective Hilbert space $P(\mathcal{H}) = \mathbb{C}P^N$, $N := \text{dim}(\mathcal{H}) - 1$:

$$\mathbb{C} \to E \to P(\mathcal{H}) .$$

The fibres are the rays, i.e., $\forall \eta = |\eta\rangle\langle \eta| \in P(\mathcal{H})$

$$E_\eta := \{ z|\eta\rangle : z \in \mathbb{C} \} .$$

The AA connection one-form $A$ is then viewed as a connection one-form on $E$ and the geometric phase is identified with the corresponding holonomy of loops

$$C : [0, T] \ni t \to \eta(t) \in P(\mathcal{H}) ,$$

in $P(\mathcal{H})$. In the adiabatic approximation one approximates $\eta(t)$ by $\psi_n(t)$ of eq. (4).

These two interpretations of Berry’s phase turn out to be linked via the theory of universal bundles. It is shown in [4, 5] that $E$ (with $N \to \infty$) is indeed the universal classifying line bundle [8, 9, 10], and as a result of the classification theorem for complex line bundles [9, 8, 11], every complex line bundle can be obtained as a pullback bundle from $E$. In particular, there is a smooth map

$$f : M \to P(\mathcal{H})$$

such that

$$L = f^*(E) .$$

The map $f$ is simply given by

$$f(x) := |n, x\rangle\langle n, x| .$$

Furthermore, the fact that the phase is obtained from either of $A$ or $\mathcal{A}$ is a consequence of the theory of universal connections [12, 13]. In fact, the
AA connection $\mathcal{A}$ is precisely the universal connection which yields all connections on all complex line bundles as pullback connections. In particular, Berry’s connection on $L$ is given by

$$A = f^*(\mathcal{A}) .$$  

(17)

These results are exploited in [5] to explore the quantum dynamics of Berry’s original example:

$$H(x) = b \bar{x}.\vec{J}, \quad \bar{x} \in S^2 \subset \mathbb{R}^3 ,$$  

(18)

where $b$ is the Larmor frequency, $\bar{x}$ is the direction of the magnetic field, and $\vec{J} = (J_i), i = 1, 2, 3,$ are the generators of rotations, $J_i \in so(3) = su(2)$. In [5], it is shown that if one considers the case of precessing magnetic field, i.e., precessing $\bar{x}$, about a fixed axis then one can promote Simon’s construction to the non-adiabatic case, namely, define a non-adiabatic analog of Berry’s connection and identify the non-adiabatic phase with its holonomy. This can be done in general unless the frequency of precession, $\omega$, becomes equal to $b$.

In the northern hemisphere the non-adiabatic connection $\tilde{A}$ is given by

$$\tilde{A} = ik(1 - \cos \tilde{\theta}) \, d\phi ,$$  

(19)

where $k$ labels an eigenvalue of $H(x)$ (alternatively an eigenvalue of $J_3$), and

$$\cos \tilde{\theta} := \frac{\cos \theta - \nu}{\sqrt{\nu^2 - 2\nu \cos \theta + 1}} ,$$  

(20)

$$\nu := \frac{\omega}{b} .$$  

(21)

Here $(\theta, \phi)$ are the spherical coordinates ($\theta \in [0, \pi]$), and $\nu$ is the “slowness parameter,” [14]. The adiabatic limit is characterized by $\nu \rightarrow 0$. In this limit $\tilde{A}$ approaches to Berry’s connection

$$A = ik(1 - \cos \theta) \, d\phi .$$  

(22)
The topology of a line bundle on $S^2$ is determined by its first Chern number
\[ c_1 := \frac{i}{2\pi} \int_{S^2} \Omega, \tag{23} \]
where $\Omega$ is the curvature two-form. For line bundles, the curvature two-form is obtained from the connection one-form by taking its ordinary exterior derivative \[15\]. A simple calculation shows that taking $\Omega = d\tilde{A}$ results in
\[ c_1 = -2k \quad \text{for} \quad \nu < 1. \tag{24} \]
This is quite remarkable since the fact that $c_1$ is an integer agrees with the fact that $k$ is a half-integer. The first statement is an algebraic topological result, whereas the second is related to group theory. One of the best known mathematical results that links these two disciplines is the celebrated Borel-Weil-Bott (BWB) theorem \[16, 17, 18, 19\].

Eq. (24) may also be viewed as an example of a topological quantization of angular momentum. In the language of magnetic monopoles, which are relevant to the adiabatic case, $k = -c_1/2$ corresponds to the product of the electric and magnetic charges \[20, 21\].

### 3 Borel-Weil-Bott Theorem and the Berry-Simon Line bundles

The BWB theorem constructs all the finite dimensional irreducible representations (irreps.) of semisimple compact Lie groups from the irreps. of their maximal tori. The construction is as follows.

Let $G$ be a semisimple compact Lie group and $T$ be a maximal torus. Let $\mathcal{G}$ and $\mathcal{T}$ be the Lie algebras of $G$ and $T$, respectively. $G$ can be viewed as a principal bundle over the quotient space $G/T$, \[22\]:
\[ T \rightarrow G \rightarrow G/T. \tag{25} \]
The homogeneous space $G/T$ can be shown to have a canonical complex structure \[17\]. Since $T$ is abelian, its irreps. are one dimensional \[22\]. Thus, each irrep. $\Lambda$ of $T$ defines an associated complex line bundle $L_\Lambda$ to (25):

$$C \rightarrow L_\Lambda \rightarrow G/T .$$

(26)

Now, consider a $\Lambda$ whose corresponding line bundle $L_\Lambda$ is an ample (positive) line bundle. Then, $L_\Lambda$ has the structure of a holomorphic line bundle. BWB theorem asserts that all the irreps. of $G$ are realized on the spaces of holomorphic sections of ample (positive) line bundles, $L_\Lambda$. In particular, the space $\mathcal{H}_\Lambda$ of the holomorphic sections of $L_\Lambda$ provides the irrep. of $G$ with maximal weight $\Lambda$, \[18, 17, 19\].

The simplest nontrivial example of the application of BWB theorem is for $G = SU(2)$. In this case, $T = U(1) = S^1$ and $G/T = S^2 = \mathbb{CP}^1$. The bundle (25) is the Hopf bundle, \[22\]:

$$U(1) = S^1 \rightarrow SU(2) = S^3 \rightarrow S^2 .$$

(27)

$\Lambda$ takes nonnegative half-integers. It is usually denoted by $j$ in QM. It is a common knowledge that $j = 0, \frac{1}{2}, 1, \ldots$ yield all the irreps. of $SU(2)$ and that the $j$-representation has dimension $2j + 1$. The dimension of the space $\mathcal{H}_\Lambda$ can be given by an index theorem \[18, 19\]. For $SU(2)$ it is obtained by the Riemann-Roch theorem in the context of the theory of Riemann surfaces. The result is

$$dim (\mathcal{H}_\Lambda) = c(L_\Lambda) = 1 + c_1(L_\Lambda) ,$$

(28)

where $c$ and $c_1$ denote the total and the first Chern numbers of $L_\Lambda$. This means that one must have:

$$c_1(L_\Lambda) = 2j .$$

(29)

Combining (24) and (29), one recovers the line bundle $L_\Lambda$ as Simon’s line bundle $L$ of (9) for $k = -j$. 
In the rest of this section, I shall try to show that there is a general relationship between the constructions used in the BWB theorem and those encountered in BS interpretation of Berry’s phase. To proceed in this direction, let us consider the generalization of (18) to an arbitrary compact semisimple Lie group, namely consider:

\[ H(x) = \epsilon \sum_{i=1}^{d} x^i J_i , \quad (x^i) \in \mathbb{R}^d - \{0\} . \]

(30)

Here, \( J_i \) are the generators of \( G \), and \( \epsilon \) is a constant with the dimension of energy. Since \( H(x) \) is assumed to be hermitian, \( J_i \) must be represented by hermitian matrices. In other words, the group \( G \) is in a unitary representation. In this sense, the example of \( G = U(N) \) plays a universal role.\(^1\)

The system described by eq. (30) is studied in [23] and [24]. In [23], it is argued that in general there are unitary operators \( U(t) \) which diagonalize the instantaneous Hamiltonian:

\[ H(t) = U(t) H_D(t) U(t)\dagger . \]

(31)

In view of eq. (3), one has

\[ U(t) = U(x(t)) , \]

(32)

where

\[ x(t) = (x^i(t)) \in \mathcal{G} - \{0\} = \mathbb{R}^d - \{0\} , \]

(33)

are the points of the loop in the parameter space. In fact, one can show that the parameter space “is not” \( \mathbb{R}^d - \{0\} \) but a submanifold of this space, namely the flag manifold \( G/T \).

To see this, let me first introduce the root system of \( \mathcal{G} \) associated with \( \Upsilon \) and the corresponding Cartan decomposition:

\[ \mathcal{G}_C = \Upsilon_C \oplus_{\alpha} \mathcal{G}_\alpha , \]

(34)

\(^1\)This reminds one of the Peter-Weyl theorem, [19, 22].
where the subscript $\mathbb{C}$ means *complexification* and $\alpha$ stand for the roots. Let $l$ denote the rank of $G$, $\{H_i\}_{i=1,2,\ldots,l}$ and $E_\alpha$ be bases of $\Upsilon$ and $G_\alpha$, respectively \cite{23, 22, 18, 17}. Then, one has

\begin{align*}
[H_i, H_j] &= 0 \\
[H_i, E_\alpha] &\propto E_\alpha \\
[E_\alpha, E_{-\alpha}] &\propto H_\alpha \in \Upsilon \\
[E_\alpha, E_\beta] &\propto E_{\alpha+\beta} \quad \text{for } \beta \neq -\alpha .
\end{align*}

(35)

Any group element can be obtained as a product of the exponentials of the generators of the algebra. In particular

$$U(t) = \exp \left[ i \sum_\alpha \chi_\alpha(t) E_\alpha \right] \exp \left[ i \sum_i \chi_i(t) H_i \right] .$$

(36)

Since any diagonal element commutes with $H_i$’s, it belongs to $\Upsilon$. Hence, one has

$$H_D(t) = \sum_i b_i(t) H_i .$$

(37)

Substituting eq. (37) in eq. (36) and using the resulting equation to simplify eq. (31), one obtains

\begin{align*}
H(t) &= e^{i \sum_\alpha \chi_\alpha(t) E_\alpha} H_D(t) e^{-i \sum_\alpha \chi_\alpha(t) E_\alpha} \\
&= e^{i \sum_{\alpha>0} [z_\alpha(t) E_\alpha + z_\alpha^*(t) E_{-\alpha}] H_D(t) e^{-i \sum_{\alpha>0} [z_\alpha^*(t) E_\alpha + z_\alpha(t) E_{-\alpha}]} .
\end{align*}

(38)

(39)

In eqs. (38) and (39) $\chi_\alpha \in \mathbb{R}$ and $z_\alpha \in \mathbb{C}$ are time dependent parameters. It is shown in \cite{23} that in general the geometric phase is given in terms of $\chi_\alpha$’s or alternatively in terms of $z_\alpha$’s, and it does not depend on $H_D(t)$. It is not difficult to see that indeed $\chi_\alpha$ correspond to the coordinates of the points of the flag manifold $G/T$. Alternatively, one can use the complex coordinates $z_\alpha$. This is reminiscent of the fact that $G/T$ has a canonical complex structure, \cite{17}. This completes the proof of the claim that the true
The parameter space of the system described by (30) is $G/T$, or a submanifold of $G/T$. I will come back to this point in section 6. The fact that $G/T$ can be viewed as embedded in $G$ is useful because it allows one to work with the global cartesian coordinate systems on $G = \mathbb{R}^d$, [24]. A natural embedding of $G/T$ is provided by taking a regular (non-degenerate) element $H_0$ of $\Upsilon$ and considering the Adjoint action of $G$ on $G$. The orbit corresponding to $H_0$ is a copy of $G/T$. Thus, one might note that in eq. (30)
\[ x = (x^i) \in G/T \subset \mathbb{R}^d. \] (40)

The fact that the phase information is encoded in $U(t)$ of eq. (31) can be used to simplify the problem, namely one can restrict to the case where the $H_D(t) = H_D(0) = H_0$ is kept constant, i.e.,
\[ H_D = \sum_i b_i H_i =: H_0 \in \Upsilon , \quad b_i = \text{const.}. \] (41)

The Hilbert space $\mathcal{H}$ of the quantum state vectors provides the representation space. It can be decomposed into irrep. spaces. I shall assume that $\mathcal{H}$ (or the subspace of $\mathcal{H}$ relevant to the geometric phase) corresponds to an irrep. with maximal weight $\Lambda$, [18]. The weights are the simultaneous eigenvectors of $H_i$’s, [25]. They are conveniently denoted by $|\lambda_1, \ldots, \lambda_l\rangle$, or collectively by $|\lambda\rangle$, where
\[ H_i |\lambda\rangle = \lambda_i |\lambda\rangle , \quad \forall i = 1, \ldots, l. \] (42)

Clearly, the weight vectors $|\lambda\rangle$ are the eigenstate vectors of the initial Hamiltonian. Here, I have set $U(0) = 1$ in eq. (31), [23]. In general, this can be achieved by appropriately choosing the maximal torus $T$. Thus, one has
\[ H(x(0)) = H_D = H_0 , \] (43)
and
\[ H_D |\lambda\rangle = \sum_{i=1}^l b_i \lambda_i |\lambda\rangle . \] (44)
Making the dependence of $H_D$ ($H_0$) on the initial point $x_0 := x(0)$ explicit, one can write eq. (44) in the form

$$H_0(x_0)|λ, x_0⟩ = E_λ(x_0)|λ, x_0⟩$$

(45)

$$E_λ(x_0) := \sum_{i=1}^{l} b_i \lambda_i(x_0).$$

The weight vectors $|λ, x_0⟩$ are precisely the eigenvectors $|n, x_0⟩$ of the instantaneous Hamiltonian $H_0(x_0)$. Since $x_0$ can be chosen arbitrarily, one can simply drop the subscript “0”, i.e., replace $x_0$ by $x$ and $H_0(x_0)$ by $H(x)$.

The BS line bundle, in this case, is obtained as the pullback bundle from the universal classifying bundle $E$,

$$L^\text{BS}_λ := f^*(E),$$

(46)

induced by the map

$$f : M \ni x \rightarrow |λ, x⟩⟨λ, x| \in \mathcal{P}(\mathcal{H}) \subset \mathbb{C}P^\infty.$$ 

Recalling some basic facts about the flag manifolds and their relation to projective spaces, [18], one finds that in fact $L^\text{BS}_λ$ corresponds to the line bundle $L_λ$ of the BWB theorem, if the weight vector $|λ, x_0⟩$ is chosen to be the maximal weight $Λ$ of the representation. First, let us recall, [18, 17], that flag manifolds are projective varieties, i.e., there exist embeddings of $M$ into $\mathbb{C}P^\infty$

$$i : M \hookrightarrow \mathbb{C}P^\infty.$$ 

(47)

Indeed, one can obtain $M = G/T$ as a unique closed orbit of the action of $G$ on $\mathcal{P}(\mathbb{C}^{N+1}) = \mathbb{C}P^N$, for some $(N+1)$-dimensional irrep., [18, §23.3]. The line bundle $L_λ$ is then the restriction (pullback under the identity map) of $E$:

$$L_λ = i^*(E).$$

(48)
Let \(|v_0\rangle\) be a nonzero vector in the representation (Hilbert) space of the \(\Lambda\)-representation of \(G\), \(G_{\mathbb{C}}\) be the complexification of \(G\) and consider the map 
\[ \Phi : G_{\mathbb{C}} \longrightarrow \mathcal{P}(\mathcal{H}) , \]
defined by 
\[ \Phi(\tilde{g}) := [U(\tilde{g})|v_0\rangle] = U(\tilde{g})|v_0\rangle\langle v_0|U(\tilde{g})^\dagger. \] (49)
Here \(U(\tilde{g})\) is the representation of \(\tilde{g} \in G_{\mathbb{C}}\) and \([U(\tilde{g})|v_0\rangle]\) denotes the ray passing through \(U(\tilde{g})|v_0\rangle\). \(\Phi\) is clearly not one-to-one. Let \(P\) be the closed subgroup of \(G_{\mathbb{C}}\) defined by
\[ P := \left\{ \tilde{h} \in G_{\mathbb{C}} : U(\tilde{h})|v_0\rangle = c|v_0\rangle, \text{ for some } c \in \mathbb{C} - \{0\} \right\} . \] (50)
By construction the map \(\Phi\) induces a one-to-one map on \(G_{\mathbb{C}}/P\):
\[ \hat{\Phi} : G_{\mathbb{C}}/P \longrightarrow \mathcal{P}(\mathcal{H}) . \] (51)
Now, let us choose 
\[ |v_0\rangle := |\Lambda, x_0\rangle , \] (52)
and denote by \(B\) the Borel subgroup of \(G_{\mathbb{C}}\) generated by \(H_i\) and \(E_{\alpha > 0}\). Then, \(B \subset P\) and consequently \(G_{\mathbb{C}}/P\) is a compact submanifold (subvariety) of \(G_{\mathbb{C}}/B\). However, one has the identity
\[ G_{\mathbb{C}}/B = G/T , \]
where by equality I mean the diffeomorphism of homogeneous spaces, \([7]\). Thus, in general \(G_{\mathbb{C}}/P \subset G/T\).

The extreme case is when \(P = B\), i.e., \(M = G_{\mathbb{C}}/P = G/T\). However, in general \(B\) may be a proper subgroup of \(P\), in which case the parameter manifold can be restricted to the submanifold \(G_{\mathbb{C}}/P\) of \(G/T\). This depends on the representation, i.e., on \(\Lambda\).
Let us consider the general case, i.e., $M = G_{\mathbb{C}}/P$. The basic vectors $|\lambda, x\rangle$ are parametrized by the points of $G_{\mathbb{C}}/P \subset G/T$ and the map $f$ of (10) becomes

$$f : G_{\mathbb{C}}/P \ni x \longrightarrow |\lambda, x\rangle \langle \lambda, x| \in \mathcal{P}(\mathcal{H}) . \quad \text{(53)}$$

In view of the fact that $G_{\mathbb{C}}/P \subset G/T$, one may work with the representative of $x = [g] \in G/T$ rather than $x = [\tilde{g}] \in G_{\mathbb{C}}/P$ for the parameters $x$. The next logical step is to compare the map $\hat{\Phi}$ of (51) with $f$. Let $x \in M \subset G/T$, then every eigenstate vector $|\lambda, x\rangle$ can be obtained by the action of $G$ on a nonzero vector. In particular, there is a $g_x \in G$ such that

$$|\lambda, x\rangle = U(g_x) |\lambda, x_0\rangle . \quad \text{(54)}$$

Combining eqs. (52), (53), (54), and specializing to $\lambda = \Lambda$, one finds

$$f(x) = U(g_x) |v_0\rangle \langle v_0| U(g_x) = [U(g_x)|v_0]\rangle . \quad \text{(55)}$$

Recalling the procedure according to which $x$ is assigned to represent the parameter, (40), of the system (30), one can identify $[g_x] \in G_{\mathbb{C}}/P \subset G/T$ with $x$, i.e.,

$$U(g_x) \equiv U(x) ,$$

and consequently

$$f(x) = [U(x)|v_0]\rangle = \hat{\Phi}(x) . \quad \text{(56)}$$

For the special case of $P = B$, the map $\hat{\Phi}$ becomes the map $i$ of (17). Thus, according to eqs. (18) and (56) the following identity is established:

$$L_\Lambda = f^*(E) . \quad \text{(57)}$$

Comparing eq. (51) with eq. (10), one arrives at the desired result, namely that the bundle $L_\Lambda$ of the BWB theorem is identical to the $BS$ bundle $L_\Lambda^{BS}$. In particular, the dimension of the irrep., i.e., the Hilbert space $\mathcal{H}$ is given by the number of the linearly independent holomorphic sections of $L_\Lambda^{BS}$. The latter is a topological invariant of $L_\Lambda^{BS}$.
It is well-known that the topology of a complex line bundle is uniquely determined by its first Chern class \( \hat{c}_1 \). \[28\]. \( \hat{c}_1 \) is represented by a closed differential 2-form on \( M \). It can be characterized by a set of \((p := \dim H_2(M, \mathbb{Z}))\) integers by integrating it over \( p \) compact 2-dimensional submanifolds of \( M \) which are called the 2-cells of \( M \). For example, if \( G = SU(2) \), \( M = S^2 \) and the space \( S^2 \) is the only 2-cell. Therefore, \( \hat{c}_1 \) is determined by a single integer \( c_1 \) via eq. \[23\].

In general, the following modification of eq. \[23\] provides the necessary integers

\[
c_a^1 = \hat{c}_1(\sigma_a) := \frac{i}{2\pi} \int_{\sigma_a} \Omega ,
\]

where \( \sigma_a \) is the \( a \)-th 2-cell \((a = 1, \cdots, p)\), \( c_a^1 \) is the first Chern number associated with \( \sigma_a \), and \( \Omega \) is the curvature 2-form of the line bundle.

For the case of the BWB-BS line bundle, \( c_a^1 \) determine the irreps. On the other hand, the irreps. are given by the maximal weight \( \Lambda \) of the representation. The latter can be written as a linear combination of the so called fundamental weights, \[18, \S 14.1\], with non-negative integer coefficients. Let us denote these by \( \Lambda_b, b = 1, \cdots, l \). Then,

\[
\Lambda = \sum_{b=1}^l k_b \Lambda_b , \quad k_b \in \mathbb{Z}^+ \cup \{0\} .
\]

This means that to determine the \( k_b \)'s and hence the irrep., one needs precisely \( l \) “independent” first Chern numbers. These are obtained by integrating \[58\] over the 2-cells of \( G/T \). The 2-cells are \( l \) copies of \( S^2 \) which correspond to the canonical \( SU(2) \) subgroups of \( G \). These are generated by the triplets of the generators \((E_\alpha, E_{-\alpha}, H_\alpha)\), where \( \alpha \)'s are the \( l \) simple roots of \( G \), and \( E_\alpha \) and \( H_\alpha \) are as in eq. \[35\]. Denoting these \( SU(2) \) subgroups and their maximal tori by \( G_a \) and \( T_a \), respectively, the 2-cells are given by

\[
\sigma_a := G_a/T_a = SU(2)/U(1) = S^2 .
\]
The restriction of the curvature 2-form $\Omega$ on $\sigma_a$ yields Berry’s curvature 2-form, [3]. Integrating these 2-forms on $\sigma_a$ gives rise to $l$ identities of the form (24). Incidentally, in view of the relevance of the system of eq. (18) to magnetic monopoles, [21], (30) corresponds to a generalized magnetic monopole whose charge has a vectorial character with integer components. I shall return to the discussion of monopoles in section 6.

4 Berry’s Connection and the Riemannian geometry of the Parameter Manifold

One of the rather interesting facts about the geometric phase is that the AA connection $\mathcal{A}$ is related to the Fubini-Study metric on the projective space $\mathbb{C}P^N$, [27]. In the language of fibre bundles, the Riemannian geometry of a manifold $X$ means the geometry of its tangent bundle $TX$. In particular, the Riemannian metric (the Levi Civita connection) is a metric (resp. a connection) on $TX$. The statement that the AA connection is related to the Riemannian geometry of $\mathbb{C}P^N$ is equivalent to say that the universal (AA) bundle

$$E : \mathbb{C} \longrightarrow E \longrightarrow \mathbb{C}P^N$$

is related to the tangent bundle

$$T\mathbb{C}P^N : \mathfrak{g}^N \longrightarrow T\mathbb{C}P^N \longrightarrow \mathbb{C}P^N.$$

This is easy to show topologically. The precise relation is demonstrated in the form of the following identity:

$$Det \left[ T\mathbb{C}P^N \right] = E^* \otimes E^* , \quad (61)$$

where $Det$ means the determinant bundle:

$$Det \left[ T\mathbb{C}P^N \right] := \underbrace{T\mathbb{C}P^N \wedge \cdots \wedge T\mathbb{C}P^N}_{N\text{-times}},$$

16
\( \wedge \) stands for the wedge product of the vector bundles, \( E^* \) is the dual line bundle to \( E \), and \( \otimes \) is the tensor product, [5]. To see the validity of eq. (61), it is sufficient to examine the first Chern classes of both sides. In fact, since \( \mathbb{C}P^N \) has a single 2-cell, namely \( \mathbb{C}P^1 = S^2 \), one can simply compare the first Chern numbers. It is well-known, [10], that

\[
c_1(E) = -1. \tag{62}
\]

Furthermore, for any vector bundle \( V \)

\[
\hat{c}_1[\text{Det} V] = \hat{c}_1[V]. \tag{63}
\]

Also it is not difficult to show that

\[
c_1(T\mathbb{C}P^N) = c_1(T\mathbb{C}P^1) = \chi(S^2) = 2, \tag{64}
\]

where \( \chi \) stands for the Euler-Poincaré characteristic. Eqs. (63) and (64) imply that

\[
c_1[\text{Det} T\mathbb{C}P^N] = 2. \]

The last equality together with the fact that

\[
c_1(E^*) = -c_1(E)
\]

and eq. (62) are sufficient to establish the validity of eq. (61).

The existence of this relationship between the AA connection and the Riemannian metric on \( \mathbb{C}P^N \) has triggered the investigation of a similar pattern in the BS approach, [28]. In [28], the authors discuss the case of a general Hamiltonian with a dynamical group \( G \) and a parameter space \( G/H \) where \( H \) is a closed subgroup of symmetries of the Hamiltonain. The analysis presented above seems to include all these cases. In the following section, I will show that the system of eq. (30) has a universal character. In other words, all the cases discussed in [28] can be reduced to the one given by (30). In all these cases the parameter space, \( G/H \), is a submanifold of \( FU(m) := U(m)/T^m \),
\( T^m := [U(1)]^m \), which is itself embedded into \( CP^\infty \). Hence, the results of [28] are expected because

- the BS bundle (connection) is the pullback (restriction) of the universal bundle \( E \);

- \( E \) is related to \( TCP^N \), via eq. (61).

5 Reduction of the Non-Adiabatic Phase to the Adiabatic Phase for the Cranked Hamiltonians

Let us consider an arbitrary \( m \times m \) Hamiltonian \( H \) acting on \( \mathcal{H} = \mathbb{C}^m \). \( H \) can be viewed as an element of the (real) vector space of all complex \( m \times m \) dimensional hermitian matrices. It is very easy to compute the real dimension of this space and find out that it is equal to \( m^2 \). Thus, \( H \) can be written as a linear combination of \( m^2 \) linearly independent hermitian matrices. Incidentally, the generators \( J_i \) of \( U(m) \) form a set of \( m^2 \) such matrices. This simply indicates that one can always express \( H \) in the form of eq. (30). This may be seen as a realization of the Peter-Weyl theorem, [13]. The particular representation of \( H \) given by eq. (30) with \( G = U(m) \) for some \( m \in \mathbb{Z}^+ \) might not be a practical choice. For example, the quadratic Hamiltonian

\[
H = \sum_{i,j=1}^{3} Q_{ij} \sigma_i \otimes \sigma_j ,
\]

with \( \sigma_i \) being Pauli matrices, [28, 29], is more manageable in this form than in the form of eq. (30) with \( J_i \) chosen to be the generators of \( U(4) \). However, in principle one can always use the linear representation, eq. (30).

Actually, one can use the generators of \( SU(m) \) rather than \( U(m) \). This is emphasized in [23]. It can be directly justified by recalling that the \((m^2 - 1)\) generators of \( SU(m) \) are also linearly independent and these together with
the \((m \times m)\) identity matrix \(I\) provide a basis for the space of \((m \times m)\) hermitian matrices. The Hamiltonian \(H\) can then be written as a linear combination in this basis. Clearly, the term proportional to \(I\) does not contribute to the geometric phase. This is often used as an indication of the geometric nature of Berry’s phase, [30].

An advantage of the linear representation is that it allows one to use the knowledge about the universal bundles and BWB theorem directly. In particular, in some cases, it is possible to obtain the non-adiabatic analog of the BS line bundle and the connection \(A\). The first example of this is presented in [5]. In this section, I will show that since the above argument does not refer to the adiabaticity of the system, one can always reduce the Hamiltonian to the linear form. Moreover, if the time dependence of the corresponding linear Hamiltonian is realized by cranking of the initial Hamiltonian along a fixed direction [24], then one can obtain a non-adiabatic analog \(\tilde{A}\) of Berry’s connection \(A\) as a pullback connection one-form. The geometric phase is then identified with the associated holonomy of the loops in the space of parameters. This is quite remarkable because it means that one does not need to solve the Schrödinger equation, provided that the function \(F\) that induces \(\tilde{A}\) as a pullback one-form is given. Wang [24] has presented a procedure that essentially computes \(F\). Nevertheless, he does not even label this function nor does he implement the idea of universal bundles. Let us see how the conditions introduced in [5] are realized in for cranked Hamiltonians. These conditions are:

1. The cyclic states are the eigenstates of a hermitian operator \(\tilde{H}\) which depends parametrically on the points of the parameter manifold \(M\), i.e., the cyclic states are eigenstates of \(\tilde{H}(x_0)\) with \(x_0 = x(t = 0)\).

2. \(\tilde{H}\) is related to the Hamiltonian according to

\[
\tilde{H}(x) = H(F(x)) = (HoF)(x),
\]

(65)
where $F : M \to M$ is some smooth function, such that in the adiabatic limit, $F$ approaches to the identity map.

Let us first see how the first condition is fulfilled for any periodic Hamiltonian. According to a result of Floquet theory, [31], the time evolution operator for any periodic Hamiltonian is of the form

$$U(t) = Z(t) e^{it\tilde{H}} ,$$

(66)

where $\tilde{H}$ is a time independent hermitian operator and $Z$ is a periodic unitary operator with the same period as the Hamiltonian, i.e.,

$$Z(t + T) = Z(t) ,$$

$$Z(0) = 1 .$$

(67)

A simple proof of this statement, i.e., eq. (66), is presented in the appendix. Clearly, one has

$$U(T) = e^{iT\tilde{H}} ,$$

(68)

which justifies the first condition. The second condition can be seen to hold for the cranked Hamiltonians either by refering to the work of Wang [24] or following the argument used in the discussion of the transformation of the Hamiltonian into the linear form. The latter is quite straightforward. One simply starts by realizing that since $\tilde{H}$ is hermitian, it can also be written in the linear form:

$$\tilde{H}(x_0) = \sum_{i=1}^{d} \tilde{x}_0^i J_i ,$$

(69)

where $\tilde{x}_0 := (\tilde{x}_0^i) \in M$ must depend on the Hamiltonian (30), and consequently on $C \subset M$. However, for the cranked Hamiltonians the time dependence of the Hamiltonian is governed by the action of a one parameter subgroup of $G$, i.e., the operator $U(t)$ of eq. (32) is given by

$$U(t) := \exp [i \omega t n_\alpha E_\alpha] \quad \text{with} \quad n_\alpha = \text{const.} ,$$

20
where \( \omega \) and \((n, \alpha)\) are called the cranking rate and direction, respectively. It is clear that for such systems \( \tilde{x}_0 \) can only depend on the initial Hamiltonian and thus on \( x_0 \). The function \( F \) is defined by

\[
\tilde{x}_0 =: F(x_0) .
\]  

(70)

The only problem is that in some cases, depending on the value of the slowness parameter \( \nu (\omega) \), \( F \) may be discontinuous or even multi-valued. This happens in the case of eq. (18) for \( \nu = \omega/b = 1 \). But in the generic case \( F \) is smooth and the second condition holds as well. The non-adiabatic analog of the BS line bundle is then given by

\[
\tilde{L} := F^*(L) .
\]  

(71)

It is endowed with the non-adiabatic connection one-form

\[
\tilde{A} := F^*(A) .
\]  

(72)

For completeness, let me briefly review the arguments of [5] which lead to eqs. (71) and (72). The basic idea is that the existence of \( \tilde{H} \) which satisfies eq. (69) allows one to imitate Berry’s treatment of the adiabatic systems. The energy eigenstate vectors \( |n, x\rangle \) are replaced by the eigenstate vectors \( |\tilde{n}, x\rangle \) of \( \tilde{H}(x) \). In view of eq. (65), these are given by

\[
|\tilde{n}, x\rangle = |n, \tilde{x}\rangle = |n, F(x)\rangle .
\]  

(73)

The non-adiabatic line bundle \( \tilde{L} \) is obtained from the universal line bundle \( E \) via the non-adiabatic analog of the map \( f \) of eq. (14). Denoting the latter by \( \tilde{f} : M \to \mathcal{P}(\mathcal{H}) \), one has

\[
\tilde{f}(x) := |\tilde{n}, x\rangle \langle \tilde{n}, x| \\
= |n, F(x)\rangle \langle n, F(x)| \\
= (f o F)(x) .
\]
Then, using the functorial property of the pullback operation one shows that
\[ \tilde{L} = \tilde{f}^*(E) = (f \circ F)^*(E) = (F^* \circ f^*)(E) = F^*(L), \]  
where in the last equality eq. (15) is used. This proves eq. (71). The proof of eq. (72) is identical. An important observation is that unlike \( |n, x_0\rangle \) the initial state vectors \( |\tilde{n}, x_0\rangle \) undergo exact cyclic evolutions.

6 More on Parameter Spaces and Monopoles

In the discussion of the relation between the BS connection and the Riemannian structure on the parameter space, the parameter space is taken to be \( M = G/H \), for some arbitrary closed subgroup \( H \) of \( G \), [28]. It can be shown that all these cases are included in the analysis of the linear system eq. (30).

In section 3, I argued that depending on the (maximal weight \( \Lambda \) of the) irrep. of \( G \), \( M \) is of the form \( G_C/P \subset G/T \), where \( P \) is defined by eq. (50). Let us consider the Weyl chamber \( \mathcal{W} \) of \( \Upsilon^* \) with respect to which the positive and the negative roots are distinguished, [18]. If \( \Lambda \) happens to lie on at least one of the walls of \( \mathcal{W} \) then \( B \) is a proper subgroup of \( P \), otherwise \( P = B \). The universal character of the linear Hamiltonian is also realized in that all the homogeneous spaces of \( G \) can be obtained as \( G_C/P \) by choosing \( \Lambda \) appropriately. In fact, this is the basic idea of the classification of the compact homogeneous spaces of semisimple Lie groups. Therefore, in principle one should be able to reproduce the results of [28] using the relation of Berry’s phase to the theory of universal bundles.

Let us consider the group \( G = SU(3) \) in its defining (standard) representation. \( SU(3) \) is of rank \( l = 2 \). So any irrep. is given by two integers.
The standard representation is itself a fundamental representation, namely \((k_1 = 1, k_2 = 0)\), \([18]\). The maximal weight is on a wall of \(\mathcal{W}\) and the Borel subgroup of upper triangular matrices in \(SL(3, \mathbb{C}) = SU(3)\mathbb{C}\) is a proper subgroup of \(P\). The subgroup \(P\) of \(SL(3, \mathbb{C})\) consists of the elements of the form:

\[
\begin{bmatrix}
* & * & *
\end{bmatrix},
\]

where * are complex numbers, \([18]\). The parameter space is \(M = SL(3, \mathbb{C})/P = SU(3)/U(2) = \mathbb{CP}^2 = \mathcal{P}(\mathcal{H})\). It is interesting to see that in this case the parameter space \(M\) and projective Hilbert space \(\mathcal{P}(\mathcal{H})\) are identical. In fact, this is true for all \(SU(N + 1)\) groups. The defining representation corresponds to \((k_1 = 1, k_2 = \cdots = k_N = 0)\) and the parameter space is \(M = SU(N + 1)/U(N) = \mathbb{CP}^N = \mathcal{P}(\mathcal{H})\). Therefore the inducing map \(f\) maps \(\mathbb{CP}^N\) to itself for all \(N > 1\).

The situation is different for the octet representation of \(SU(3)\). In this case one has \(k_1 = k_2 = 1\). \(\Lambda\) lies in the interior of \(\mathcal{W}\), \(P = B\), and the parameter space is the full flag manifold \(M = SU(3)/U(1) \times U(1)\). The map \(f\) maps \(M\) into \(\mathcal{P}(\mathcal{H}) = \mathbb{CP}^7\). \([1]\)

For \(G = SU(2)\), it is well-known that the system of eq. (18) is related to the magnetic monopoles, \([21]\). The relation of monopoles to the gauge theories and their generalization to arbitrary compact semisimple gauge groups have been studied in the late seventies, \([20]\). These generalized monopoles are called \textit{non-abelian} or \textit{multi-monopoles} for general groups and \textit{color monopoles} for \(SU(3)\), \([32]\). They are topologically classified by an associated set of \(l\) integers where \(l\) is the rank. These are called the \textit{topological charges} of the monopole and they are defined as elements of the second homotopy group \(\pi_2(G/H)\), where \(H\) is the group of the symmetries of a ground state of the

\[\text{Note that this representation is 8-dimensional, i.e., the representation space for } SL(3, \mathbb{C}) \text{ is } \mathbb{C}^8. \text{ Hence } \mathcal{H} = \mathbb{C}^8.\]
Higgs fields (a minimum of Higgs potential), [20]. For $G = SU(3)$, there are two possibilities. Either,

$$\text{I}) \ H = U(2) \quad \text{or} \quad \text{II}) \ H = T = U(1) \times U(1).$$

These cases have been studied in almost every article written in this subject, e.g. see [33], [20] and references therein.

If $G$ is simply connected then a result of algebraic topology indicates that

$$\pi_2(G/H) = \pi_1(H).$$

Applying this result to $G = SU(3)$ one finds

$$\begin{align*}
\text{I}) & \quad \pi_2(SU(3)/U(2)) = \pi_1(U(2)) = \mathbb{Z} \\
\text{II}) & \quad \pi_2(SU(3)/U(1) \times U(1)) = \pi_1(U(1) \times U(1)) = \mathbb{Z} \oplus \mathbb{Z}.
\end{align*}$$

Thus, for I) and II) one has, respectively, one and two topological charges. This is precisely the case with the topological charges of the geometric phase defined earlier. The same correspondence holds for arbitrary compact, connected semisimple Lie groups.

The possible relevance of the topological charges of monopoles to the representations of the group have been conjectured by Goddard, et al., [34]. Although, the analysis of the present paper does not prove their conjecture, it provides a formula for the topological charges as integrals of the first Chern class, defined by Berry’s connection, over the 2-cells $\sigma_a$ of section 3. There is a simple topological explanation for the correspondence of the topological charges of the monopoles and those of the geometric phase. This can be summarized in the identity

$$\pi_2(G/H) = H_2(G/H, \mathbb{Z}),$$

where $H_2(\cdot, \mathbb{Z})$ denotes the second homology group. This identity is a consequence of Hurewicz theorem, [35], where one uses the fact that $\pi_1(G/H) = \mathbb{Z}$. \[24\]
$H_1(G/H) = 0$. The 2-cells $\sigma_a$ are indeed the generators of $H_2(G/T, \mathbb{Z})$. For $H \neq T$ some of them may be smashed to a point as is the case for $G = SU(3)$ and $H = U(2)$.

7 Conclusion

The relevance of the phenomenon of Berry’s phase to Borel-Weil-Bott theorem and specially to the theory of universal bundles is appealing not only for the aesthetical reasons but also for its allowing for a better understanding of the non-adiabatic phase. Moreover, it sheds light on a number of issues such as the determination of the appropriate parameter space and the relation between the geometry of the parameter space and the geometric structure of the phase. The BWB theorem leads to the definition of of a set of topological charges which determine the topology of the BS line bundles. These seem to be related, if not identical, to the topological charges of non-abelian monopoles. The integral nature of these charges is a consequence of the topological properties of the first Chern class. The latter is essentially the reason for the quantization of the charges of the monopoles.

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Appendix: A note on Floquet theory

The following is a proof which I learned its main idea from Prof. Pierre Cartier.

Let \( H = H(t) \) be a T-periodic selfadjoint operator serving as the nonconserved Hamiltonian of a quantum system, i.e.,

\[
H(t + T) = H(t) \quad \forall t \in [0, T].
\]

Let \( U(t) \) be the time evolution operator which satisfies the Schrödinger equation,

\[
\begin{align*}
\frac{d}{dt} U(t) & = -i H(t) U(t) \\
U(0) & = 1.
\end{align*}
\] (75)

**Theorem:** There exist a time independent selfadjoint operator \( \tilde{H} \) and a T-periodic unitary operator \( Z = Z(t) \) such that \( U(t) \) is of the form

\[
U(t) = Z(t) e^{i \tilde{H} t},
\] (76)

and

\[
Z(0) = Z(T) = 1.
\]

**Proof:** Let \( V(t) := U(t + T) \), then \( V \) satisfies the following schrödinger equation:

\[
\begin{align*}
\frac{d}{dt} V(t) & = -i H(t) V(t) \\
V(0) & = U(T) =: C.
\end{align*}
\] (77)
The operator $V'(t) := U(t) C$ satisfies eq. (77) as well. Then the uniqueness of the solution of this differential equation implies that $V(t) = V'(t)$, i.e.,

$$U(t + T) = U(t) U(T).$$  \hspace{1cm} (78)

One can easily show that (76) satisfies eq. (78). Thus it is the unique solution. In fact, it is not difficult to construct a pair $(Z(t), \tilde{H})$ which satisfies eq. (76).

Let $t = nT + t_0$ for some $n \in \mathbb{Z}$ and $t_0 < T$. $n$ and $t_0$ are uniquely determined for given $t$. Applying eq. (78) repeatedly, one has

$$U(t) = U(t_0) \left[ U(T) \right]^n = U(t_0) C^n.$$

Relabelling $C$ by $e^{i\tilde{H}'}$, and noting $n = \frac{t-t_0}{T}$, one obtains

$$U(t) = U(t_0) e^{i(t-t_0)\tilde{H}'} = U(t_0) e^{-\frac{i}{T}H'} e^{\frac{i}{T}H'} = Z(t) e^{it\tilde{H}},$$

where

$$Z(t) := U(t_0) e^{-\frac{i}{T}H'},$$

and

$$\tilde{H} := \frac{H'}{T}.$$  

Clearly, $\tilde{H}$ is selfadjoint and $Z(t)$ is unitary. Furthermore, $Z(t)$ satisfies

$$Z(t + T) = Z(t)$$

$$Z(0) = U(0) = 1,$$

by construction. One must however note that $\tilde{H}'$ is not unique, nor is the decomposition (76).
References

[1] S. Pancharatnam, Proc. Indian Acad. Sci. A44, 247 (1956).

[2] C. A. Mead and D. Truhlar, J. Chem. Phys. 70, 2284 (1979).

[3] M. V. Berry, Proc. R. Soc. Lond. A392, 45 (1984).

[4] A. Bohm, L. J. Boya, A. Mostafazadeh, and G. Rudolph, J. Geometry & Phys. 12, 13 (1993).

[5] A. Mostafazadeh and A. Bohm, J. Phys. A26, 5473 (1993).

[6] B. Simon, Phys. Rev. Lett. 51, 2167 (1983).

[7] Y. Aharonov and J. Anandan, Phys. Rev. Lett. 58, 1593 (1987).

[8] T. Eguchi, P. B. Gilkey, and A. J. Hanson, Phys. Rep. 66, 213 (1980).

[9] Y. Choquet-Bruhat and C. DeWitt-Morette, Analysis, Manifolds and Physics, Part II, North-Holland, Amsterdam (1989).

[10] M. Nakahara, Geometry, Topology and Physics, Adam Hilger, New York (1990).

[11] C. J. Isham, Modern Differential Geometry for Physicists, World Scientific, New Jersey (1989).

[12] M. Narasimhan and S. Ramanan, Amer. J. Math. 83, 563 (1961).

[13] R. Schlafly, Inv. Math. 59, 59 (1980).

[14] M. V. Berry, Proc. R. Soc. Lond. A414, 31 (1987).

[15] Y. Choquet-Bruhat, C. DeWitt-Morette with M. Dillard-Bleik, Analysis, Manifolds and Physics, Part I, North-Holland, Amsterdam (1989).
[16] R. Bott, “On Induced Representations,” in *The Mathematical Heritage of Herman Weyl*, Editor: R. O. Wells, Jr., Amer. Math. Soc., Rhode Island (1988); see also O. Alvarez, I. M. Singer, and P. Windey, Nucl. Phys. B337, 467 (1990).

[17] N. R. Wallach, *Harmonic Analysis on Homogeneous Spaces*, Marcel Dekker, New York (1973).

[18] W. Fulton and J. Harris, *Representation Theory*, Springer-Verlag, New York (1991).

[19] A. Pressley and G. Segal, *Loop Groups*, Oxford University Press, Oxford (1990).

[20] P. Goddard and D. Olive, Rep. Prog. Phys. 41, 91 (1978).

[21] I. J. Aitchison, Acta Phys. Polonica B18, no: 3, 207 (1987); M. G. Benedick, L. Gy. Feher, and Z. Horrath, J. Math. Phys. 30, 1727 (1989).

[22] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups*, Springer-Verlag, New York (1985).

[23] J. Anandan and L. Stodolsky, Phys. Rev. D35, 2597 (1987).

[24] S-J. Wang, Phys. Rev. A42, 5103 (1990).

[25] H. Georgi, *Lie Algebras in Particle Physics*, Addison-Wesley, New York (1982).

[26] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Princeton University Press, Princeton (1974).

[27] D. N. Page, Phys. Rev. A36, 3479 (1987); A. Bohm, L. J. Boya, and B. Kendrick, Phys. Rev. A43, 1206 (1991).
[28] S. Giller, C. Gonera, P. Kosinski, and P. Maslanka, Phys. Rev. A48, 907 (1993).

[29] H. Aradz and A. Babiuch, Acta. Phys. Polonica B20 no: 7, 579 (1989).

[30] A. Bohm, Quantum Mechanics: Foundations and Applications, Third Edition, Springer-Verlag, New York (1993).

[31] D. J. Moore, J. Phys. A23, L665 (1990).

[32] Y. M. Cho, Phys. Rev. Lett. 44, 1115 (1980).

[33] M. I. Monastyrskii and A. M. Perelomov, JETP Lett. 21, 43 (1975).

[34] P. Goddard, J. Nuyts, and D. Olive, Nucl. Phys. B125, 1 (1977).

[35] C. R. F. Maunder, Algebraic Topology, Van Nostrand Reinhold, London (1970).