COMMUTATORS OF AUTOMORPHIC COMPOSITION OPERATORS WITH ADJOUTS

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Abstract. In this paper, we investigate the compactness of the commutator \( [C_\psi^*, C_\varphi] \) on the Hardy space \( H^2(B_N) \) or the weighted Bergman space \( A^2_s(B_N) \) \((s > -1)\), when \( \varphi \) and \( \psi \) are automorphisms of the unit ball \( B_N \). We obtain that \( [C_\psi^*, C_\varphi] \) is compact if and only if \( \varphi \) and \( \psi \) commute and they are both unitary. This generalizes the corresponding result in one variable. Moreover, our technique is different and simpler. In addition, we also discuss the commutator \( [C_\psi^*, C_\varphi] \) on the Dirichlet space \( \mathcal{D}(B_N) \), where \( \varphi \) and \( \psi \) are linear fractional self-maps or both automorphisms of \( B_N \).

1. Introduction

Let \( B_N \) denote the unit ball of \( \mathbb{C}^N \) and let \( \varphi \) be a holomorphic self-map of \( B_N \). We define the composition operator \( C_\varphi \) by \( C_\varphi(f) = f \circ \varphi \), where \( f \) is analytic in \( B_N \).

In this paper, we are interested in characterizing the compactness of the commutator \( [C_\psi^*, C_\varphi] = C_\psi^* C_\varphi - C_\varphi C_\psi^* \) on some classical function spaces, when \( \varphi \) and \( \psi \) are automorphisms of \( B_N \). The motivation comes from a recent work of Clifford et al. \cite{4}. They considered when the commutator \( [C_\psi^*, C_\varphi] \) is non-trivially compact on the Hardy space \( H^2(D) \) for linear fractional self-maps \( \varphi \) and \( \psi \) of the unit disk \( D \). Here, non-trivially compact means that \( [C_\psi^*, C_\varphi] \) is compact but nonzero, moreover, both \( C_\psi^* C_\varphi \) and \( C_\varphi C_\psi^* \) are not compact. In particular, when \( \varphi \) and \( \psi \) are automorphisms of \( D \), they showed that \( [C_\psi^*, C_\varphi] \) is non-trivially compact if and only if both maps are rotations. All results were extended by MacCluer et al. \cite{15} to the weighted Bergman space \( A^2_s(D) \) \((s > -1)\).

In Section 3, we first investigate the commutator \( [C_\psi^*, C_\varphi] \) on the Hardy space \( H^2(B_N) \) and the weighted Bergman space \( A^2_s(B_N) \) \((s > -1)\), where \( \varphi \) and \( \psi \) are automorphisms of \( B_N \), neither \( \varphi \) nor \( \psi \) is the identity. In this case, we will prove that \( [C_\psi^*, C_\varphi] \) is compact if and only if \( \varphi \) and \( \psi \) commute and both maps are unitary. This generalizes the result of automorphisms case in the unit disk to the unit ball. In order to deduce that \( \varphi \) and \( \psi \) commute when \( [C_\psi^*, C_\varphi] \) is compact on \( A^2_s(D) \), MacCluer et al. \cite{15} had done lots of complicated calculations. It is difficult for us to use this similar idea. So in Section 3, we will find another simpler technique to solve similar problem in higher dimensions, which involved in the application of the semi-multiplication property for Toeplitz operators.

Furthermore, we will extend to discuss the compactness of \( [C_\psi^*, C_\varphi] \) on the Dirichlet space \( \mathcal{D}(B_N) \) in Section 4. Based on the adjoint formula for \( C_\psi^* \) on \( \mathcal{D}(B_N) \), we obtain the necessary and sufficient condition for \( [C_\psi^*, C_\varphi] \) to be compact in terms of linear fractional self-maps \( \varphi \) and \( \psi \) of \( B_N \). As an immediate result, for automorphisms \( \varphi \) and

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1
ψ of $B_N$, we find that $[C^s_w, C^s_\varphi]$ is compact on $\mathcal{D}(B_N)$ if and only if $\varphi$ and $\psi$ commute. More specially, when $\varphi$ and $\psi$ are linear fractional self-maps of $D$, the condition for $[C^s_w, C^s_\varphi]$ to be compact on the Dirichlet space $\mathcal{D}(D)$ is the same as that on the Hardy space $H^2(D)$ and the weighted Bergman space $A^2_s(D)$ ($s > -1$).

2. Preliminaries

Here we collect some necessary background information.

2.1. Analytic function spaces. Let $\partial B_N$ denote the boundary of the unit ball $B_N$. The Hardy space $H^2(B_N)$ is defined by

$$H^2(B_N) = \{ f \text{ analytic in } B_N : ||f||^2 \equiv \sup_{0 < r < 1} \int_{\partial B_N} |f(r\zeta)|^2 d\sigma(\zeta) < \infty \},$$

where $d\sigma$ denotes the normalized surface measure on $\partial B_N$. The weighted Bergman space $A^2_s(B_N)$, for $s > -1$, is defined by

$$A^2_s(B_N) = \{ f \text{ analytic in } B_N : ||f||^2_s \equiv \int_{B_N} |f(z)|^2 dv_s(z) < \infty \},$$

where $dv_s(z) = \frac{\Gamma(N+s+1)}{N!} (1-|z|^2)^s dv(z)$ and $dv$ denotes the normalized volume measure on $B_N$. In this paper, we will often use $\mathcal{H}$ to denote the Hardy space $H^2(B_N)$ or the weighted Bergman space $A^2_s(B_N)$. It is well known that both the Hardy space and the weighted Bergman space are the reproducing kernel Hilbert spaces, where the reproducing kernel is given by $K_w(z) = (1- < z, w >)^{-t}$ for $z, w \in B_N$, with $t = N$ for $H^2(B_N)$ and $t = N + s + 1$ for $A^2_s(B_N)$. So the the normalized reproducing kernel is given by

$$k_w(z) = \frac{K_w(z)}{||K_w||_\mathcal{H}} = \frac{(1- |w|^2)^{t/2}}{(1- < z, w >)^t}.$$

A multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$ is an $N$-tuple of non-negative integers $\alpha_i$. The total order of a multi-index is given by $|\alpha| = |\alpha_1| + \cdots + |\alpha_N|$. Let $\alpha! = \alpha_1! \cdots \alpha_N!$ and $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$ for $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$. An analytic function $f$ in $B_N$ has a power series representation

$$f(z) = \sum_\alpha c_\alpha z^\alpha,$$

where the sum is over all multi-indexes.

The Dirichlet space $\mathcal{D}(B_N)$ is defined as

$$\mathcal{D}(B_N) = \{ f(z) = \sum_\alpha c_\alpha z^\alpha \text{ analytic in } B_N : ||f||^2_{\mathcal{D}} \equiv \sum_\alpha |c_\alpha|^2 |\alpha|^{\alpha!} ||< \alpha || < \infty \},$$

where the quantity $||.||_{\mathcal{D}}$ defines a semi-norm on $\mathcal{D}(B_N)$. We equip it with the norm

$$||f||^2_{\mathcal{D}} = ||f(0)||^2 + ||f||^2_{\mathcal{D}_0}$$

and the inner product $< \cdot, \cdot >_{\mathcal{D}}$. So the reproducing kernel for $\mathcal{D}(B_N)$ is given by

$$K_w(z) = 1 + \log \frac{1}{1- < z, w >}, \quad z, w \in B_N.$$

When acting on the reproducing kernel, composition operators and Toeplitz operators have the following adjoint property:

$$C^*_\varphi T^*_h K_w = \overline{h(w)} K_{\varphi(w)}, \quad w \in B_N$$

for all analytic self-maps $\varphi$ of $B_N$ and $h \in L^\infty(B_N)$. 
2.2. **Adjoint formula.** Given a bounded measurable complex-valued function \( b \) on \( \partial B_N \) (or \( B_N \)), the Toeplitz operator \( T_b \) on \( H^2(B_N) \) (or \( A^2_N(B_N) \)) is defined by

\[
T_b f = P(bf),
\]

where \( P \) is the orthogonal projection from \( L^2(\partial B_N) \) (or \( L^2(B_N, dv_s) \)) onto \( H^2(B_N) \) (or \( A^2_N(B_N) \)). If \( b \) is analytic, then \( T_b \) is a multiplication by \( b \).

In this paper, we will often use the semi-multiplication property for Toeplitz operators mod \( \mathcal{K} \), where \( \mathcal{K} \) denotes the ideal of compact operators. That is, if \( b \in L^\infty(\partial B_N) \) (or \( L^2(B_N) \)) and \( h \in C(B_N) \), then \( T_bT_h - T_{bh} \) is compact on \( H^2(B_N) \) (or \( A^2_N(B_N) \)). This result on the Hardy space \( H^2(B_N) \) is Proposition 1.4 in [17]; the Bergman space version comes from the proof of Theorem 1 in [5]; similar to the proof of Theorem 1 in [5], this property also can be extended to the weighted Bergman space \( A^2_N(B_N) \) (\( s > -1 \)). In the unit disk \( D \), this fact has been well described in page 73 of [15].

**Theorem A.** Suppose that

\[
\varphi(z) = \frac{az + b}{cz + d}
\]

is a linear fractional self-map of \( D \) and \( ad - bc \neq 0 \). Then the adjoint of \( C_\varphi \) acting on the Hardy space \( H^2(D) \) or the weighted Bergman space \( A^2_s(D) \) (\( s > -1 \)) is given by

\[
C^*_\varphi = T_g C_\sigma T_h^*,
\]

where

\[
\sigma(z) = \frac{\overline{az - c}}{\overline{bz + d}}
\]

is the Krein adjoint of \( \varphi \), \( g(z) = \overline{(bz + d)}^{-t} \) and \( h(z) = (cz + d)^t \) are in \( H^\infty \) with \( t = 1 \) for \( H^2(D) \) and \( t = s + 2 \) for \( A^2_s(D) \).

This adjoint formula of \( C_\varphi \) was first established on \( H^2(D) \) by Cowen [6] and was generalized to \( A^2_s(D) \) (\( s > -1 \)) by Hurst [12]. In the unit ball \( B_N \), Cowen and MacCluer [8] obtained the following similar adjoint formula for \( C_\varphi \) when acting on \( H^2(B_N) \) or \( A^2_N(B_N) \) (\( s > -1 \)).

**Theorem B.** Let

\[
\varphi(z) = \frac{Az + B}{< z, C > + d}
\]

be a linear fractional self-map of \( B_N \), where \( A \) is an \( N \times N \) matrix, \( B \) and \( C \) are \( N \times 1 \) matrices, and \( d \) is a scalar. Then on the space \( \mathcal{H} \),

\[
C^*_\varphi = T_g C_\sigma T_h^*,
\]

where

\[
\sigma(z) = \frac{A^* z - C}{< z, -B > + d^t}
\]

\( g(z) = ( < z, -B > + d^t )^{-t} \) and \( h(z) = ( < z, C > + d)^t \) with \( t = N \) when \( \mathcal{H} = H^2(B_N) \) and \( t = N + s + 1 \) when \( \mathcal{H} = A^2_s(B_N) \) (\( s > -1 \)).

We will refer to the functions \( g, h \) and \( \sigma \) as the auxiliary functions of \( \varphi \) when they connected by the equation \( C^*_\varphi = T_g C_\sigma T_h^* \). We also need frequently use the property: if \( \varphi \in \text{Aut}(B_N) \) (or \( \text{Aut}(D) \)) then \( \sigma = \varphi^{-1} \), where \( \text{Aut}(B_N) \) (or \( \text{Aut}(D) \)) denotes the set of automorphisms of \( B_N \) (or \( D \)).
Moreover, we have a different adjoint formula for $C_\phi$ on the Dirichlet space $D(B_N)$ as the following.

**Theorem C.** (Theorem 7 in [18]) Let $\phi$ be a linear fractional self-map of $B_N$ and $K_\omega$ be the reproducing kernel of $D(B_N)$. Then for $f \in D(B_N)$, we have

$$C_\phi^* f = f(0)K_\phi(0) + C_\sigma f - f(\sigma(0)),$$

where $\sigma$ is the Krein adjoint of $\phi$.

This means that the adjoint of $C_\phi$ on $D(B_N)$ can be identified as another composition operator and a rank 2 operator. For the unit disk case, this adjoint formula on the Dirichlet space $D(D)$ was given by Gallardo-Gutiérrez and Montes-Rodríguez (see Theorem 3.3 in [10]).

It is well known that when $\phi$ is a linear fractional self-map of $B_N$, then $C_\phi$ is compact on the space mentioned in this paper if and only if $||\phi||_\infty < 1$. Let $\phi$ be a linear fractional self-map of $D$ with a fixed point $\omega \in \partial D$, if $\omega$ is the only fixed point for $\phi$, we say that $\phi$ is parabolic. If $\phi$ has an additional fixed point, we call $\phi$ hyperbolic.

### 2.3. Julia-Carathéodory theorem.

Given $\zeta \in \partial B_N$, a continuous function $\gamma : [0, 1] \to B_N$ with $\lim \gamma(t) = \zeta$ is said to a restricted $\zeta$-curve if

$$\lim_{t \to 1} \frac{|\gamma(t) - \gamma(t), \zeta|}{1 - |\gamma(t), \zeta|} = 0 \quad \text{and} \quad \sup_{0 \leq t < 1} \frac{|\zeta - \gamma(t), \zeta|}{1 - |\gamma(t), \zeta|} < \infty.$$

If $\lim_{z \to \zeta} f(\gamma(t)) = f(\zeta)$ for every restricted $\zeta$-curve $\gamma$, we say that $f : B_N \to \mathbb{C}$ has restricted limit and write $R \lim_{z \to \zeta} f(z) = f(\zeta)$.

Let $\phi$ be a holomorphic self-map of $B_N$. We say that $\phi$ has finite angular derivative at $\zeta \in \partial B_N$, if there exists a point $\eta \in \partial B_N$ so that

$$A_\phi(\zeta) = R \lim_{z \to \zeta} \frac{1 - \phi(z), \eta}{1 - \phi(z, \eta)}$$

exists. We write $\phi_\eta := \phi, \eta >$ and $D_\zeta = \frac{\partial}{\partial \zeta}$ for the directional derivative in the direction of $\zeta$, and we put

$$d_\phi(\zeta) = \lim_{z \to \zeta} \frac{1 - |\phi(z)|}{1 - |z|}.$$

The following is the Julia-Carathéodory theorem for the ball (see Theorem 8.5.6 in [19] or Theorem 2.2 in [3])

**Theorem D.** Let $\phi$ be a holomorphic self-map of $B_N$ and $\zeta \in \partial B_N$. The following statements are equivalent:

1. $\phi$ has finite angular derivative at $\zeta$.
2. $d_\phi(\zeta) < \infty$.
3. $\phi$ has restricted limit $\eta \in \partial B_N$ at $\zeta$ and $D_\zeta \phi_\eta(\zeta) = \phi'(\zeta)\zeta, \eta >$ has finite restricted limit at $\zeta$.

Moreover, when these conditions hold, the following statements hold:

4. $D_\zeta \phi_\eta(\zeta) = \phi'(\zeta)\zeta, \eta >$ has restricted limit at $\zeta$ with $D_\zeta \phi_\eta(\zeta) = d_\phi(\zeta)$.
5. $A_\phi(\zeta) = d_\phi(\zeta)$.
6. $\phi_{\eta_0}(\zeta)$ has restricted limit 0 at $\zeta$ for any $\eta_0 \perp \zeta \zeta \in \partial B_N$ orthogonal to $\eta$. 

3. The commutator on the Hardy space and the weighted Bergman space

In this section, we discuss the compactness of the commutator \([C_\psi^*, C_\varphi]\) on the Hardy space \(H^2(B_N)\) and the weighted Bergman space \(A^2_s(B_N)\) \((s > -1)\), when \(\varphi\) and \(\psi\) are automorphisms of \(B_N\). We will show the following main theorem.

**Theorem 3.1.** Let \(\varphi\) and \(\psi\) be automorphisms of \(B_N\), neither \(\varphi\) nor \(\psi\) is the identity. The commutator \([C_\psi^*, C_\varphi]\) is compact on \(H^2(B_N)\) or \(A^2_s(B_N)\) \((s > -1)\) if and only if \(\varphi\) and \(\psi\) commute and both maps are unitary.

Before we give a proof for Theorem 3.1, we need some useful lemmas. The first lemma is analogous to Lemma 3.2 of [4] and Lemma 4.3 of [15]. But the result has been improved. Recall that for \(w \in B_N\), the function \(k_w = K_w/\|K_w\|_\mathcal{H}\).

**Lemma 3.2.** Assume that \(\varphi\) and \(\psi\) are holomorphic self-maps of \(B_N\). Suppose that there exist points \(\zeta_1\) and \(\zeta_2\) on \(\partial B_N\) such that \(\varphi(\zeta_1) = \psi(\zeta_2) = \omega \in \partial B_N\) and \(A_\varphi(\zeta_1)\) and \(A_\psi(\zeta_2)\) exist. Then on the space \(\mathcal{H}\),

\[
\lim_{r \to 1} \frac{1}{<C_\psi^* k_{r\zeta_2}, C_\psi^* k_{r\zeta_1}>} = \left( \frac{2}{d_\varphi(\zeta_1) + d_\psi(\zeta_2)} \right)^t > 0,
\]

where \(t = N\) when \(\mathcal{H} = H^2(B_N)\) and \(t = N + s + 1\) when \(\mathcal{H} = A^2_s(B_N)\) \((s > -1)\).

**Proof.** Let \(U\) be a unitary map of \(B_N\) so that \(U\omega = e_1\). Then \(U\varphi(\zeta_1) = U\psi(\zeta_2) = U\omega = e_1\). Write \(\phi = U\varphi\) and \(\rho = U\psi\), we see that

\[
<C_\phi^* k_{r\zeta_2}, C_\phi^* k_{r\zeta_1}> = C_\phi^* C_\psi^* k_{r\zeta_2}, C_\psi^* C_\varphi^* k_{r\zeta_1} > = C_\psi^* k_{r\zeta_2}, C_\psi^* k_{r\zeta_1} >.
\]

Thus we may assume \(\omega = e_1\). Note that

\[
\frac{1}{<C_\psi^* k_{r\zeta_2}, C_\psi^* k_{r\zeta_1}>} = \frac{\|K_{r\zeta_2}\|\|K_{r\zeta_1}\|_\mathcal{H}}{<K_{\psi(r\zeta_2)}, K_{\psi(r\zeta_1)}>} = \left( \frac{1 - <\varphi(r\zeta_1), \psi(r\zeta_2)>}{1 - r^2} \right)^t
\]

and

\[
\frac{1 - <\varphi(r\zeta_1), \psi(r\zeta_2)>}{1 - r^2} = \frac{1 - |\psi(r\zeta_2)|^2}{1 - r^2} + \frac{1 - <\varphi(r\zeta_1), \psi(r\zeta_2)>}{1 - r^2} = \frac{1 - |\psi(r\zeta_2)|^2}{1 - r^2} + \frac{<\psi(r\zeta_2) - \varphi(r\zeta_1), \psi(r\zeta_2)>}{1 - r^2}.
\]

Now, we calculate that

\[
\frac{<\psi(r\zeta_2) - \varphi(r\zeta_1), \psi(r\zeta_2)>}{1 - r^2} = \frac{\psi_1(r\zeta_2) - \varphi_1(r\zeta_1)}{1 - r^2} \cdot \frac{\psi_1(r\zeta_2)}{1 - r^2} + \sum_{j=2}^{\infty} \frac{\psi_j(r\zeta_2) - \varphi_j(r\zeta_1)}{1 - r^2} \cdot \frac{\psi_j(r\zeta_2)}{1 - r^2}
\]

\[
= \left( \frac{1 - <\varphi(r\zeta_1), e_1>}{1 - r\zeta_1, \zeta_1} - \frac{1 - <\psi(r\zeta_2), e_1>}{1 - r\zeta_2, \zeta_2} \right) \frac{\psi_1(r\zeta_2)}{1 + r} + \sum_{j=2}^{\infty} \left( \frac{\psi_j(r\zeta_2) - \varphi_j(r\zeta_1)}{1 - r\zeta_2, \zeta_2} - \frac{\varphi_j(r\zeta_1)}{1 - r\zeta_1, \zeta_1} \right) \frac{\psi_j(r\zeta_2)}{1 + r}.
\]

Since \(\varphi\) and \(\psi\) have finite angular derivatives respectively at \(\zeta_1\) and \(\zeta_2\), by Theorem D(5), we get that

\[
\lim_{r \to 1} \frac{1 - <\varphi(r\zeta_1), e_1>}{1 - r\zeta_1, \zeta_1} = A_\varphi(\zeta_1) = d_\varphi(\zeta_1)
\]
and
\[ \lim_{r \to 1} \frac{1 - <\psi(\zeta_2), e_1>}{1 - <\zeta_2, \zeta_2>} = A_\psi(\zeta_2) = d_\psi(\zeta_2). \]
Moreover, by Theorem D(6),
\[ \lim_{r \to 1} \frac{\varphi_j(r\zeta_2)}{1 - <\zeta_1, \zeta_1>} = 0 \quad \text{and} \quad \lim_{r \to 1} \frac{\psi_j(r\zeta_2)}{1 - <\zeta_2, \zeta_2>} = 0 \]
hold for \(2 \leq j \leq N\). Therefore,
\[ \lim_{r \to 1} \frac{1}{C^*_{\varphi}k_{r\zeta_2}, C^*_{\varphi}k_{r\zeta_1}} = \lim_{r \to 1} \left( \frac{1 - <\varphi(r\zeta_2), \psi(r\zeta_2)>}{1 - r^2} \right)^t \]
\[ = \lim_{r \to 1} \left( \frac{1 - |\psi(r\zeta_2)|^2}{1 - r^2} + \frac{<\psi(r\zeta_2) - \varphi(r\zeta_1), \psi(r\zeta_2)>}{1 - r^2} \right)^t \]
\[ = \left[ d_\varphi(\zeta_2) + \frac{1}{2}(d_\varphi(\zeta_1) - d_\psi(\zeta_2)) \right]^t \]
\[ = \left( \frac{d_\varphi(\zeta_1) + d_\psi(\zeta_2)}{2} \right)^t, \]
and we obtain the desire conclusion. \(\square\)

**Lemma 3.3.** Assume that \(\varphi\) and \(\psi\) are holomorphic self-maps of \(D\). Suppose that there exist points \(\zeta_1\) and \(\zeta_2\) on \(\partial D\) such that \(\varphi(\zeta_1) = \psi(\zeta_2) = \omega \in \partial D\) and the angular derivatives \(\varphi'(\zeta_1)\) and \(\psi'(\zeta_2)\) exist. Then on \(H^2(D)\) or \(A^2_s(D)\) \((s > -1)\),
\[ \lim_{r \to 1} < C^*_{\psi}k_{r\zeta_2}, C^*_{\varphi}k_{r\zeta_1}> = \left( \frac{2}{|\varphi'(\zeta_1)| + |\psi'(\zeta_2)|} \right)^t > 0, \]
where
\[ k_w(z) = \left( \frac{\sqrt{1 - |w|^2}}{1 - \overline{z}w} \right)^t, \quad z, w \in D \]
is the normalized reproducing kernel with \(t = 1\) for \(H^2(D)\) and \(t = s + 2\) for \(A^2_s(D)\).

This is an easy corollary of Lemma 3.2. In fact, Lemma 3.3 can also be immediately obtained if we notice that \(\zeta_1 \varphi'(\zeta_1)\omega = |\varphi'(\zeta_1)|\) and \(\zeta_2 \psi'(\zeta_2)\omega = |\psi'(\zeta_2)|\) from the Julia-Caratheodory Theorem of the unit disk (see Theorem 2.44 of [7]). Because Lemma 3.2 of [4] and Lemma 4.3 of [15] have given that
\[ \lim_{r \to 1} < C^*_{\psi}k_{r\zeta_2}, C^*_{\varphi}k_{r\zeta_1}> = \left( \frac{2\omega}{2\omega|\psi'(\zeta_1)| + |\zeta_1\varphi'(\zeta_1) - \zeta_2\psi'(\zeta_2)|} \right)^t, \]
we only need multiple \(\varpi\) on numerator and denominator of the above fraction to deduce the limit in Lemma 3.3.

We now show that when \(\varphi\) and \(\psi\) are automorphisms of \(D\), in order for the commutator \([C^*_{\psi}, C^*_{\varphi}]\) to be compact, the inducing maps \(\varphi\) and \(\psi\) must commute. This result on the Hardy space \(H^2(D)\) is Lemma 5.1 of [4], and it on the weighted Bergman space \(A^2_s(D)\) is Theorem 4.6 of [15]. Their technique is similar, but on \(A^2_s(D)\), some calculations involved are very complex. It is almost impossible to do similar calculations in higher dimensions. We have to find other method which can be used on \(H^2(D)\) and \(A^2_s(D)\) simultaneously and which can also be extended to the unit ball.

**Lemma 3.4.** Assume that \(\varphi\) and \(\psi\) are automorphisms of \(D\). If the commutator \([C^*_{\psi}, C^*_{\varphi}]\) is compact on \(H^2(D)\) or \(A^2_s(D)\) \((s > -1)\), then \(\varphi\) and \(\psi\) commute.
Proof. This is Lemma 5.1 in [4] and Theorem 4.6 in [15]. We will give another simpler proof and we only focus on the Hardy space $H^2(D)$. Set
\[
\varphi(z) = \frac{a_1z + b_1}{c_1z + d_1} \quad \text{and} \quad \psi(z) = \frac{a_2z + b_2}{c_2z + d_2}
\]
with normalization $a_1d_1 - c_1b_1 = 1$ and $a_2d_2 - c_2b_2 = 1$. By Theorem A, we have
\[
C_\varphi = T_{h_1}^* C_{\sigma_1}^* T_{g_1}^* \quad \text{and} \quad C_\psi = T_{h_2}^* C_{\sigma_2}^* T_{g_2}^*.
\]
where $g_1, h_1, \sigma_1$ and $g_2, h_2, \sigma_2$ are respectively the auxiliary functions for $\varphi$ and $\psi$ on Theorem A. Since $\varphi$ and $\psi$ are automorphisms of $D$, we have $\sigma_1 = \varphi^{-1}$ and $\sigma_2 = \psi^{-1}$.

For $w \in D$, let $K_w$ denote the reproducing kernel and $k_w$ be the normalized reproducing kernel respectively given by
\[
K_w(z) = \frac{1}{1 - zw} \quad \text{and} \quad k_w(z) = \frac{K_w(z)}{||K_w||} = \frac{\sqrt{1 - |w|^2}}{1 - zw},
\]
where $|| \cdot ||$ denotes the norm of $H^2(D)$. Now, using the formula $C_\phi^* T_b^* K_w = b(w)K_\phi(w)$, we get that
\[
<C_\varphi K_z, C_\psi K_w> = <T_{h_1} C_{\sigma_1}^* T_{g_1}^* K_z, T_{h_2} C_{\sigma_2}^* T_{g_2}^* K_w> = g_1(z) g_2(w) <T_{h_1} K_{\sigma_1(z)}, T_{h_2} K_{\sigma_2(w)}> = g_1(z) g_2(w) <K_{\sigma_1(z)}, T_{h_1}^* T_{h_2} K_{\sigma_2(w)}>.
\]

Since $h_1(z) = c_1z + d_1$ and $h_2(z) = c_2z + d_2$ are in $H^\infty$, using the semi-multiplicative property for Toeplitz operators mod $K$ as mentioned in Section 2, we see
\[
T_{h_1}^* T_{h_2} = T_{h_1}^* T_{h_2} = T_{h_1} T_{h_2} = T_{h_1} T_{h_2} = T_{h_1} + T_{h_2} T_{h_1} + L = T_{h_2} T_{h_1} + L,
\]
where $L$ is a compact operator on $H^2(D)$. It follows that
\[
<C_\varphi K_z, C_\psi K_w> = g_1(z) g_2(w) <K_{\sigma_1(z)}, T_{h_1}^* T_{h_2} K_{\sigma_2(w)}> + g_1(z) g_2(w) <K_{\sigma_1(z)}, L K_{\sigma_2(w)}> + g_1(z) g_2(w) <K_{\sigma_1(z)}, L K_{\sigma_2(w)}>
\]
\[
= g_1(z) g_2(w) <K_{\sigma_1(z)}, T_{h_1}^* K_{\sigma_2(w)}> + g_1(z) g_2(w) <K_{\sigma_1(z)}, L K_{\sigma_2(w)}> + g_1(z) g_2(w) <K_{\sigma_1(z)}, L K_{\sigma_2(w)}>
\]
\[
= g_1(z) g_2(w) h_2 \circ \sigma_1(z) \bar{h}_2 \circ \sigma_2(w) <K_{\sigma_1(z)}, K_{\sigma_2(w)}>.
\]

Fix $\omega \in \partial D$, since $\varphi$ and $\psi$ are automorphisms, there exist $\zeta_1$ and $\zeta_2$ on $\partial D$ such that $\varphi(\zeta_1) = \psi(\zeta_2) = \omega$. Setting $z = r \zeta_2$ and $w = r \zeta_1$, then
\[
\lim_{r \to 1^-} <C_\varphi k_{r \zeta_2}, C_\psi k_{r \zeta_1}> = \lim_{r \to 1^-} <C_\varphi \frac{K_{r \zeta_2}}{||K_{r \zeta_2}||}, C_\psi \frac{K_{r \zeta_1}}{||K_{r \zeta_1}||}>
\]
\[
= \lim_{r \to 1^-} (1 - r^2) <C_\varphi K_{r \zeta_2}, C_\psi K_{r \zeta_1}>
\]
\[
= \lim_{r \to 1^-} (1 - r^2) g_1(r \zeta_2) g_2(r \zeta_1) h_2 \circ \sigma_1(r \zeta_2) \bar{h}_1 \circ \sigma_2(r \zeta_1) <K_{\sigma_1(r \zeta_2)}, K_{\sigma_2(r \zeta_1)}> + \lim_{r \to 1^-} (1 - r^2) g_1(r \zeta_2) g_2(r \zeta_1) <K_{\sigma_1(r \zeta_2)}, L K_{\sigma_2(r \zeta_1)}> = I + II.
Note that \( \{k_w\} \) is a weakly convergent sequence as \(|w| \to 1\) with \( ||k_w|| = 1\) and \(L\) is compact, we see that

\[
\lim_{|w| \to 1} \sqrt{1 - |w|^2} ||LK_w|| = \lim_{|w| \to 1} ||Lk_w|| = 0.
\]

Which gives that

\[
\lim_{r \to 1^-} \sqrt{1 - r^2} ||LK_{\sigma_2(r\xi_1)}|| = \lim_{r \to 1^-} \sqrt{1 - |\sigma_2(r\xi_1)|^2} ||LK_{\sigma_2(r\xi_1)}|| = 0 \cdot \frac{1}{|\sigma_2'(r\xi_1)|^{1/2}} = 0.
\]

Hence,

\[
\lim_{r \to 1^-} (1 - r^2)||g_1(r\xi_2)\overline{g_2(r\xi_1)} < K_{\sigma_1(r\xi_2)}, LK_{\sigma_2(r\xi_1)} > | \leq \lim_{r \to 1^-} (1 - r^2)||g_1(r\xi_2)\overline{g_2(r\xi_1)}| \cdot ||K_{\sigma_1(r\xi_2)}|| \cdot ||LK_{\sigma_2(r\xi_1)}|| = \lim_{r \to 1^-} |g_1(r\xi_2)\overline{g_2(r\xi_1)}| \sqrt{1 - r^2} ||LK_{\sigma_2(r\xi_1)}|| = |g_1(\xi_2)\overline{g_2(\xi_1)}| \frac{1}{|\sigma_2'(\xi_1)|^{1/2}} \cdot 0 = 0
\]

and so \( II = 0 \).

Now, using \( \sigma_1 = \varphi^{-1} \) and \( \sigma_2 = \psi^{-1} \), we calculate that

\[
I = \lim_{r \to 1^-} (1 - r^2)g_1(r\xi_2)\overline{g_2(r\xi_1)} h_2 \circ \sigma_1(r\xi_2) h_1 \circ \sigma_2(r\xi_1) \leq K_{\sigma_1(r\xi_2)}, K_{\sigma_2(r\xi_1)} > \]

\[
= \lim_{r \to 1^-} h_1 \circ \psi^{-1}(r\xi_1) g_1(r\xi_2) h_2 \circ \varphi^{-1}(r\xi_2) g_2(r\xi_1) \frac{1 - r^2}{1 - |\psi^{-1}(r\xi_1), \varphi^{-1}(r\xi_2)|}.
\]

It is easy to see that

\[
\lim_{r \to 1^-} \frac{1 - r^2}{1 - |\psi^{-1}(r\xi_1), \varphi^{-1}(r\xi_2)|} = 0
\]

unless \( \varphi^{-1}(\xi_2) = \psi^{-1}(\xi_1) \). Next, we assume \( \varphi^{-1}(\xi_2) = \psi^{-1}(\xi_1) \). By Lemma 3.3, we get

\[
\lim_{r \to 1^-} \frac{1 - r^2}{1 - |\psi^{-1}(r\xi_1), \varphi^{-1}(r\xi_2)|} = \lim_{r \to 1^-} \frac{1 - r^2}{1 - |C_{\varphi^{-1}kr\xi_2}, C_{\psi^{-1}kr\xi_1}|} = \frac{2}{|((\varphi^{-1})'(\xi_1))| + |((\psi^{-1})'(\xi_2))|}.
\]

On the other hand, an easy computation shows that

\[
h_1(z)g_1 \circ \varphi(z) = \frac{c_1z + d_1}{a_1 - c_1\varphi(z)} = \frac{c_1z + d_1}{a_1} = \frac{c_1z + d_1}{a_1} \frac{a_1z + b_1}{a_1z + d_1} = \frac{|c_1z + d_1|^2}{a_1d_1 - c_1b_1} = |c_1z + d_1|^2
\]

and similarly

\[
h_2(z)g_2 \circ \psi(z) = |c_2z + d_2|^2.
\]
Using Lemma 3.3 again, we know that
\[ h_1 \circ \varphi^{-1}(\zeta_1) \; g_1(\zeta_2) = h_1 \circ \varphi^{-1}(\zeta_2) \; g_1 \circ \varphi \circ \varphi^{-1}(\zeta_2) \]
\[ = |c_1 \varphi^{-1}(\zeta_2) + d_1|^2 = \left| c_1 \frac{d_1 \zeta_2 - b_1}{-c_1 \zeta_2 + a_1} + d_1 \right|^2 \]
\[ = \frac{1}{|a_1 - c_1 \zeta_2|^2} = |(\varphi^{-1})'(\zeta_2)| \]
and
\[ h_2 \circ \varphi^{-1}(\zeta_2) \; g_2(\zeta_1) = h_2 \circ \varphi^{-1}(\zeta_1) \; g_2 \circ \psi \circ \varphi^{-1}(\zeta_1) \]
\[ = |c_2 \psi^{-1}(\zeta_1) + d_2|^2 = |(\psi^{-1})'(\zeta_1)|. \]
As a result, we show that
\[ I = |(\varphi^{-1})'(\zeta_2)|||\psi^{-1})'(\zeta_1)| = \frac{2}{|(|\varphi^{-1})'(\zeta_2)| + |\psi^{-1})'(\zeta_1)|} \]
if \( \varphi^{-1}(\zeta_2) = \psi^{-1}(\zeta_1) \); Otherwise \( I = 0 \). Moreover, since \( \varphi(\zeta_1) = \psi(\zeta_2) = \omega \in \partial D \), if \( \varphi^{-1}(\zeta_2) = \psi^{-1}(\zeta_1) \) then we have
\[ \psi^{-1} \circ \varphi^{-1}(\omega) = \varphi^{-1} \circ \psi^{-1}(\omega). \]
This means \( \varphi^{-1} \) and \( \psi^{-1} \) commute and hence \( \varphi \) and \( \psi \) commute. Immediately, we get
\[ (\psi^{-1})'(\varphi^{-1}(\omega))(\varphi^{-1})'(\omega) = (\varphi^{-1})'(\psi^{-1}(\omega))(\psi^{-1})'(\omega), \]
i.e.
\[ (\psi^{-1})'(\zeta_1) \frac{1}{\varphi'(\zeta_1)} = (\varphi^{-1})'(\zeta_2) \frac{1}{\psi'(\zeta_2)}. \]
Thus, the above discussions deduce that
\[ \lim_{r \to 1^-} < C_\varphi k_{r \zeta_2}, C_\psi k_{r \zeta_1} > = I + II \]
\[ = |(\varphi^{-1})'(\zeta_2)|||\psi^{-1})'(\zeta_1)| = \frac{2}{|(|\varphi^{-1})'(\zeta_2)| + |\psi^{-1})'(\zeta_1)|} + 0 \]
\[ = \left| (\psi^{-1})'(\zeta_1) \frac{\psi'(\zeta_2)}{\varphi'(\zeta_1)} \right|||\psi^{-1})'(\zeta_1)| = \frac{2}{|(\psi^{-1})'(\zeta_1)| + |\psi'(\zeta_2)|} \]
under the condition of \( \varphi^{-1}(\zeta_2) = \psi^{-1}(\zeta_1) \). Otherwise, it is zero.
At last, by our hypothesis that \( [C^*_\psi, C_\varphi] \) is compact on \( H^2(D) \), we see that
\[ 0 = \lim_{r \to 1^-} \left\| [C^*_\psi, C_\varphi] k_{r \zeta_2} \right\| \geq \lim_{r \to 1^-} \left| < [C^*_\psi, C_\varphi] k_{r \zeta_2}, k_{r \zeta_1} > \right| \]
\[ = \lim_{r \to 1^-} \left| < C_\varphi k_{r \zeta_2}, C_\psi k_{r \zeta_1} > - < C^*_\psi k_{r \zeta_2}, C^*_\varphi k_{r \zeta_1} > \right|. \]
Using Lemma 3.3 again, we know that
\[ \lim_{r \to 1^-} < C^*_\psi k_{r \zeta_2}, C^*_\varphi k_{r \zeta_1} > = \frac{2}{|\varphi'(\zeta_1)| + |\psi'(\zeta_2)|}. \]
It follows
\[ \lim_{r \to 1^-} < C_\varphi k_{r \zeta_2}, C_\psi k_{r \zeta_1} > = \lim_{r \to 1^-} < C^*_\psi k_{r \zeta_2}, C^*_\varphi k_{r \zeta_1} > \neq 0. \]
Combining this with the previous conclusion, we must have \( \varphi^{-1}(\zeta_2) = \psi^{-1}(\zeta_1) \). Moreover, the above equality implies
\[
| (\psi^{-1})'(\zeta_1) \psi'(\zeta_2) | = 1.
\]
(We don’t know how to use this conclusion, but it maybe have some independent interest.) Note that we have obtained that \( \varphi \) and \( \psi \) commute from \( \varphi^{-1}(\zeta_2) = \psi^{-1}(\zeta_1) \). Therefore, \([C_\psi^*, C_\varphi]\) is compact must deduce that \( \varphi \) and \( \psi \) commute. □

In the proof of Lemma 3.4, replacing Lemma 3.3 by Lemma 3.2, we can deduce the similar result for the unit ball. We state it as the following and omit its proof.

**Lemma 3.5.** If \( \varphi, \psi \in \text{Aut}(B_N) \) and \([C_\psi^*, C_\varphi]\) is compact on \( H^2(B_N) \) or \( A^2_s(B_N) \) \((s > -1)\), then \( \varphi \) and \( \psi \) commute.

Next, we will give a complete proof for our main theorem. That is, we will show that when \( \varphi \) and \( \psi \) are automorphisms of \( B_N \), the compactness of \([C_\psi^*, C_\varphi]\) implies that both \( \varphi \) and \( \psi \) are unitary. In the proof of this similar conclusion, the technique treated on \( A^2_s(D) \) (see [15]) is different from that on \( H^2(D) \) (see [4]). On \( A^2_s(D) \), the polar decomposition of \( C_\varphi \) was used. I think which also holds on the unit ball, so we can use similar idea as on \( A^2_s(D) \) to complete the proof of this result. However, in order to exhibit the special property of composition operator \( C_\varphi \) when \( \varphi \) is an automorphism of \( B_N \), we try to use the following interesting lemmas to prove this result.

**Lemma 3.6.** Suppose that \( \varphi \) is an automorphism of \( B_N \). Then on the space \( \mathcal{H} \),
\[
C_\varphi^* = T_f C_\varphi^{-1} = T_f C_{\varphi^{-1}},
\]
where \( T_f \) is the Toeplitz operator with symbol
\[
f(z) = \left( \frac{1 - |\varphi(0)|^2}{|1 - \langle z, \varphi(0) \rangle|^2} \right)^t
\]
with \( t = N \) when \( \mathcal{H} = H^2(B_N) \) and \( t = N + s + 1 \) when \( \mathcal{H} = A^2_s(B_N) \) \((s > -1)\).

This result on the Hardy space \( H^2(B_N) \) and the Bergman space \( A^2_s(B_N) \) was established by Bourdon and MacCluer [2]. The extension to the weighted space \( A^2_s(B_N) \) can be obtained similarly when using the change of variables formula in Proposition 1.13 of [20]. For the case of the unit disk, please see Theorem 4.2 of [15].

**Lemma 3.7.** Let \( \varphi \) be an automorphism of \( B_N \). If \( f \) is continuous on \( \partial B_N \) or \( \overline{B_N} \), then
\[
C_\varphi T_f - T_{f \circ \varphi} C_\varphi
\]
is compact when acting on \( H^2(B_N) \) or \( A^2_s(B_N) \) \((s > -1)\).

**Proof.** Let \( \varphi \) be an automorphism of \( B_N \) and \( a = \varphi^{-1}(0) \). By Theorem 2.2.5 of [19], the identity
\[
1 - \langle \varphi(z), \zeta \rangle = 1 - \langle \varphi(z), \varphi \circ \varphi^{-1}(\zeta) \rangle = \frac{(1 - |a|^2)(1 - \langle z, \varphi^{-1}(\zeta) \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, \varphi^{-1}(\zeta) \rangle)}
\]
holds for all \( z \in \overline{B_N} \) and \( \zeta \in \partial B_N \). Now, using this identity, for any \( g \in H^2(B_N) \), we get that
\[
(C \phi T_f g)(z) = (T_f g)(\phi(z)) = \int_{\partial \overline{B_N}} \frac{1}{f(\zeta)g(\zeta)(1 - \langle \phi(z), \zeta \rangle)^{N}} d\sigma(\zeta)
\]
\[
= \int_{\partial \overline{B_N}} f(\zeta)g(\zeta) (1 - \langle z, a \rangle)^{N}(1 - \langle a, \phi^{-1}(\zeta) \rangle)^{N} d\sigma(\zeta)
\]
\[
= \int_{\partial \overline{B_N}} f \circ \phi(\eta)g \circ \phi(\eta) (1 - \langle z, \eta \rangle)^{N}(1 - \langle a, \eta \rangle)^{N} \cdot \frac{(1 - |a|^2)^{N}}{(1 - |\eta|^2)^{N}} d\sigma(\eta)
\]
\[
= (1 - \langle z, a \rangle)^{N} \int_{\partial \overline{B_N}} f \circ \phi(\eta)g \circ \phi(\eta) \frac{1}{1 - \langle \eta, a \rangle} \cdot \frac{1}{1 - \langle z, \eta \rangle} d\sigma(\eta)
\]
\[
= (1 - \langle z, a \rangle)^{N}(T_{f \circ \phi,K_\alpha} g \circ \phi)(z)
\]
\[
= (T_{1/K_\alpha}T_{f \circ \phi}T_{K_\alpha}C \phi g)(z),
\]
where we have used the change of variables formula (see Corollary 4.4 of [20]) and the kernel function
\[
K_\alpha(z) = \frac{1}{(1 - \langle z, a \rangle)^{N}}
\]
and the function \( 1/K_\alpha \) are analytic on \( \overline{B_N} \). Hence,
\[
C \phi T_f = T_{1/K_\alpha}T_{f \circ \phi}T_{K_\alpha}C \phi.
\]

Since \( f \) is continuous on \( \partial B_N \), by the semi-multiplicative property for Toeplitz operators mod \( \mathcal{K} \) in Section 2, we know that
\[
T_{1/K_\alpha}T_{f \circ \phi} = T_{1/K_\alpha}T_{f \circ \phi} + L = T_{f \circ \phi 1/K_\alpha} + L = T_{f \circ \phi}T_{1/K_\alpha} + L,
\]
where \( L \) is a compact operator on \( H^2(B_N) \). Therefore,
\[
C \phi T_f = T_{1/K_\alpha}T_{f \circ \phi}T_{K_\alpha}C \phi = (T_{f \circ \phi}T_{1/K_\alpha} + L)T_{K_\alpha}C \phi = T_{f \circ \phi}C \phi + L',
\]
where \( L' \) is compact. Applying similar technique, we can obtain the similar result for the weighted Bergman space and so we complete the proof. \( \square \)

Based on these lemmas, we will use another technique to prove Theorem 3.1, which generalizes Theorem 5.2 of [4] and Theorem 5.1 of [15] to the unit ball.

**Proof of Theorem 3.1.** We only need to prove the "only if" part and we only give a proof for the Hardy space case.

Assume that \( [C^*_\psi, C \phi] \) is compact. Since \( \psi \) is an automorphism of \( B_N \), by Lemma 3.6, we have
\[
C^*_\psi = T_f C_{\psi^{-1}}
\]
with
\[
f(z) = \left( \frac{1 - |\psi(0)|^2}{|1 - \langle z, \psi(0) \rangle|^2} \right)^{N}.
\]
Thus,
\[
[C^*_\psi, C \phi] = C^*_\psi C \phi - C \phi C^*_\psi = T_f C_{\psi^{-1}}C \phi - C \phi T_f C_{\psi^{-1}}.
\]
Now, using Lemma 3.5, \([C^*_\psi, C \phi] \) is compact means that \( \phi \) and \( \psi \) commute, i.e.
\[
\phi \circ \psi = \psi \circ \phi.
\]
This gives $\varphi = \psi \circ \varphi \circ \psi^{-1}$ and
\[
[C^*_\psi, C_\varphi]C_\psi = (T_f C_{\psi^{-1}}C_\varphi - C_\varphi T_f C_{\psi^{-1}})C_\psi
\]
\[
= T_f C_{\psi^{-1}}C_\varphi C_\psi - C_\varphi T_f C_{\psi^{-1}}C_\psi
\]
\[
= T_f C_{\psi^{-1}}C_\varphi C_\psi - C_\varphi T_f
\]
\[
= T_f C_\varphi - C_\varphi T_f.
\]

Note that $f$ is continuous on $\overline{B_N}$ and $\varphi$ is an automorphism of $B_N$. It follows from Lemma 3.7 that
\[
C_\varphi T_f \equiv T_{f \circ \varphi} C_\varphi \quad (\text{mod } \mathcal{K}).
\]
Therefore,
\[
[C^*_\psi, C_\varphi]C_\psi = T_f C_\varphi - C_\varphi T_f
\]
\[
\equiv T_f C_\varphi - T_{f \circ \varphi} C_\varphi \quad (\text{mod } \mathcal{K})
\]
\[
= T_{f - f \circ \varphi} C_\varphi \quad (\text{mod } \mathcal{K}).
\]
Finally, since $C_\psi$ and $C_\varphi$ are invertible, we see that $[C^*_\psi, C_\varphi]$ is compact if and only if $[C^*_\psi, C_\varphi]C_\psi$ is compact, which is equivalent to that $T_{f - f \circ \varphi}$ is compact. Applying Lemma 2 of [5], we see that $f - f \circ \varphi \equiv 0$ on $\overline{B_N}$. Since $\varphi$ is not the identity, the representation of $f$ gives that $f$ must be constant. Thus, we get $\psi(0) = 0$.

As we know, an automorphism $\phi$ of $B_N$ is a unitary transformation of $\mathbb{C}^N$ if and only if $\phi(0) = 0$ (see Lemma 1.1 of [20]). Together with the above discussion, we see that $\psi$ must be unitary. On the other hand, $[C^*_\psi, C_\varphi] = C^*_\psi C_\varphi - C_\varphi C^*_\psi$ is compact implies that
\[
(C^*_\psi C_\varphi - C_\varphi C^*_\psi)^* = C^*_\varphi C_\psi - C_\psi C^*_\varphi = [C^*_\psi, C_\varphi]
\]
is compact. So similar arguments deduce that $\varphi$ is also unitary. Consequently, all these give that both $\varphi$ and $\psi$ are unitary and they commute under the condition that $[C^*_\psi, C_\varphi]$ is compact. \[\square\]

4. The commutator on the Dirichlet space

In this section, we try to characterize the compactness of $[C^*_\psi, C_\varphi]$ on the Dirichlet space $\mathcal{D}(B_N)$, where $\varphi$ and $\psi$ are linear fractional self-maps of $B_N$. The following lemma is about compact difference of linear fractional composition operators on $\mathcal{D}(B_N)$, which will prove useful for our result.

**Lemma 4.1.** Suppose that $\varphi$ and $\psi$ are linear fractional self-maps of $B_N$. Then $C_\varphi - C_\psi$ is compact on $\mathcal{D}(B_N)$ if and only if $\varphi = \psi$ or both $C_\varphi$ and $C_\psi$ are compact.

**Proof.** This result for a special case has been pointed out by Pons [18], without proof. For completeness, we give a simple proof.

In [18], for real $s$, the weighted Dirichlet space $\mathcal{D}_s(B_N)$ is defined by
\[
\mathcal{D}_s(B_N) = \{ f(z) = \sum_\alpha c_\alpha z^\alpha \text{ analytic in } B_N : \sum_\alpha (|\alpha| + 1)^{1-s}|c_\alpha|^2 \omega_\alpha < \infty \},
\]
where
\[
\omega_\alpha = ||z^\alpha||^2 = \frac{(N-1)!|\alpha|}{(N-1 + |\alpha|)!}.
\]
On the other hand, let \( \beta(k) = (k + 1)^t \) for real \( t \), if \( f(z) = \sum \alpha c_\alpha z^\alpha = \sum_0^\infty f_k(z) \) is analytic in \( B_N \), then \( f \) belongs to the weighted Hardy space \( H^2(\beta, B_N) \) if and only if
\[
\sum_0^\infty ||f_k||^2 \beta(k)^2 = \sum_\alpha (|\alpha| + 1)^t |c_\alpha|^2 \omega_\alpha < \infty.
\]
Thus, the weighted Dirichlet space \( \mathcal{D}_s(B_N) \), in fact, is the weighted Hardy space \( H^2(\beta, B_N) \) with the weight \( \beta(k) = (k + 1)^{(1-s)/2} \). Using Proposition 2.4 of [13], we see that all linear fractional self-maps induce bounded composition operators on \( \mathcal{D}_s(B_N) \).

Suppose \( s_1 < s < s_2 < \infty \), complex interpolation theorem for the weighted Dirichlet space \( \mathcal{D}_s(B_N) \) tells us
\[
[\mathcal{D}_{s_1}, \mathcal{D}_{s_2}]_\theta = \mathcal{D}_s
\]
with \( s = (1-\theta)s_1 + \theta s_2 \) for \( \theta \in (0, 1) \) (see Proposition 1 of [18]). Choosing \( s_1 = -n \) and \( s_2 = 2 \), then for linear fractional self-maps \( \varphi \) and \( \psi \) of \( B_N \), the operator \( C_\varphi - C_\psi \) is bounded on \( \mathcal{D}_n(B_N) \) and \( \mathcal{D}_2(B_N) = A^2(B_N) \). Now, if \( C_\varphi - C_\psi \) is compact on \( \mathcal{D}(B_N) = \mathcal{D}_{1-n}(B_N) \). Since \( 1-n = (1-\theta)(-n) + \theta \cdot 2 \) with \( \theta = \frac{1}{\beta + 2} \), using the compactness theorem for interpolating operators (see Theorem 2.1 in [9]), then \( C_\varphi - C_\psi \) is also compact on \( H^2(B_N) = \mathcal{D}_1(B_N) \) with \( 1 = (1-\theta)(-n) + \theta \cdot 2 \) for \( \theta = \frac{n+1}{n+2} \). Here, all above spaces are identified to equal with an equivalent norm. Applying Theorem 2 of [11] or Theorem 3.1 of [14], we get that \( \varphi = \psi \) or both \( C_\varphi \) and \( C_\psi \) are compact. So we have shown one direction. Another direction is obvious and the proof is completed.

When \( \varphi \) is a linear fractional self-map of \( B_N \), the adjoint of composition operator \( C_\varphi \) on \( \mathcal{D}(B_N) \) is mainly determined by another composition operator. Together this adjoint property with Lemma 4.1, we will give the following condition for \( [C^*_\psi, C_\varphi] \) to be compact on \( \mathcal{D}(B_N) \).

**Theorem 4.2.** Let \( \varphi \) and \( \psi \) be linear fractional self-maps of \( B_N \). If \( [C^*_\psi, C_\varphi] \neq 0 \), then \( [C^*_\psi, C_\varphi] \) is non-trivially compact on \( \mathcal{D}(B_N) \) if and only if \( ||\psi||_\infty = ||\varphi||_\infty = 1 \) and \( \varphi \circ \sigma = \psi \circ \varphi \), where \( \sigma \) is the Krein adjoint of \( \psi \).

**Proof.** For \( w \in B_N \), let \( K_w \) denote the reproducing kernel of \( \mathcal{D}(B_N) \). Theorem C gives
\[
C^*_\psi f = f(0)K_{\psi(0)} + C_\sigma f - f(\sigma(0))
\]
for any \( f \in \mathcal{D}(B_N) \), where \( \sigma \) is the Krein adjoint of \( \psi \). Thus,
\[
[C^*_\psi, C_\varphi] f = (C^*_\psi C_\varphi - C_\varphi C^*_\psi) f = C^*_\psi C_\varphi f - C_\varphi C^*_\psi f = f \circ \varphi(0) K_{\psi(0)} - C_\sigma C_\varphi f - f \circ \varphi \circ \sigma(0) - f(0) K_{\psi(0)} \circ \varphi - C_\varphi C_\sigma f - f(\sigma(0)) = (C_\varphi C_\sigma - C_\varphi C_\psi) f + f \circ \varphi(0) K_{\psi(0)} + f(\sigma(0)) - f \circ \varphi \circ \sigma(0) - f(0) K_{\psi(0)} \circ \varphi.
\]
This implies that \( [C^*_\psi, C_\varphi] \) is compact on \( \mathcal{D}(B_N) \) if and only if
\[
[C_\varphi, C_\sigma] = C_\varphi C_\sigma - C_\sigma C_\varphi = C_{\varphi \circ \sigma} - C_{\varphi \circ \sigma} \circ \sigma\circ \sigma
\]
is compact.

It is easy to see that \( C^*_\psi C_\varphi \) is compact if and only if \( C_\sigma C_\varphi = C_{\varphi \circ \sigma} \) is compact, and the compactness of \( C_\varphi C^*_\psi \) is equivalent to the compactness of \( C_\varphi C_\sigma = C_{\sigma \circ \varphi} \). Since \( [C^*_\psi, C_\varphi] \neq 0 \), we get that \( [C^*_\psi, C_\varphi] \) is non-trivially compact on \( \mathcal{D}(B_N) \) if and only
if \( C_{\sigma \circ \varphi} - C_{\varphi \circ \sigma} \) is also non-trivially compact. By Lemma 4.1, this is equivalent to \( \varphi \circ \sigma = \sigma \circ \varphi \) and \( ||\varphi \circ \sigma||_{\infty} = ||\sigma \circ \varphi||_{\infty} = 1 \), i.e. \( ||\varphi||_{\infty} = ||\varphi||_{\infty} = 1 \). So we complete the proof. □

Note that if \( \psi \) is an automorphism of \( B_N \), then \( \sigma = \psi^{-1} \). Thus, \( \varphi \circ \sigma = \sigma \circ \varphi \) gives that \( \varphi \circ \psi^{-1} = \psi^{-1} \circ \varphi \). This is the same to \( \varphi \circ \psi = \psi \circ \varphi \). Immediately, as a corollary of Theorem 4.2, we obtain the following result.

**Theorem 4.3.** If \( \varphi \) and \( \psi \) are automorphisms of \( B_N \). Then \( [C_{\psi}, C_{\varphi}] \) is compact on \( \mathcal{D}(B_N) \) if and only if \( \varphi \) and \( \psi \) commute.

Using Theorem 4.2 and similar discussions in the proof of Theorem 3.1 in [15], we also obtain the following similar result on the Dirichlet space \( \mathcal{D}(D) \) as on the Hardy space \( H^2(D) \) and the weighted Bergman space \( A^2_s(D) \) \((s > -1)\).

**Theorem 4.4.** Suppose that \( \varphi \) and \( \psi \) are linear fractional self-maps of \( D \), and one of which is a non-automorphism. The commutator \( [C_{\psi}, C_{\varphi}] \) is non-trivially compact on \( \mathcal{D}(D) \) if and only if one of the following is true.

(i) \( \varphi \) and \( \psi \) are both parabolic with the same boundary fixed point,

(ii) \( \varphi \) and \( \psi \) are hyperbolic such that the fixed points of \( \varphi \) and \( \psi \) are \((\zeta, a)\) and \((\zeta, 1/\overline{a})\) respectively with \( \zeta \in \partial D \).

**Remark.** We first see the following example. Let

\[
\varphi(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (z_1, z_2/2), \quad z = (z_1, z_2) \in B_2
\]

and

\[
\psi(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (z_1, z_2/3), \quad z = (z_1, z_2) \in B_2.
\]

Thus,

\[
\sigma(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix}^* \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}
\]

and so

\[
\varphi \circ \sigma(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1/6 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \sigma \circ \varphi(z).
\]

Moreover, it is easy to check that

\[
[C_{\psi}, C_{\varphi}]f = (C_{\varphi} C_{\sigma} - C_{\sigma} C_{\varphi}) f + f \circ \varphi(0) K_{\psi(0)} + f(\sigma(0)) - f \circ \varphi \circ \sigma(0) - f(0) K_{\psi(0)} \circ \varphi = 0
\]

for any \( f \in \mathcal{D}(B_2) \). As a consequence, when \( \varphi \circ \sigma = \sigma \circ \varphi \) and \( ||\psi||_{\infty} = ||\varphi||_{\infty} = 1 \), it may happen that \( [C_{\psi}, C_{\varphi}] = 0 \). Hence, in Theorem 4.2, we need assume that \( [C_{\psi}, C_{\varphi}] \neq 0 \). However, from the proof of Theorem 3.1 in [15], this condition is not necessary for the unit disk.

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