Volume-preserving mean curvature flow
for tubes in symmetric spaces

Naoyuki Koike

Abstract
In this paper, we investigate the volume-preserving mean curvature flow starting from a tube (of nonconstant radius) over a compact closed domain of a reflective submanifold in a symmetric space. We prove that the tubeness is preserved along the flow under certain conditions.

1 Introduction

Let \( f_t \)’s \((t \in [0, T])\) be a one-parameter \( C^\infty \)-family of immersions of an \( n \)-dimensional compact manifold \( M \) into an \((n+1)\)-dimensional Riemannian manifold \( \overline{M} \), where \( T \) is a positive constant or \( T = \infty \). Define a map \( \tilde{f} : M \times [0, T) \to \overline{M} \) by \( \tilde{f}(x, t) = f_t(x) \) \(((x, t) \in M \times [0, T))\). Denote by \( \pi_M \) the natural projection of \( M \times [0, T) \) onto \( M \). For a vector bundle \( E \) over \( M \), denote by \( \pi_M^* E \) the induced bundle of \( E \) by \( \pi_M \). Also, denote by \( H_t, g_t \) and \( N_t \) the mean curvature, the induced metric and the outward unit normal vector of \( f_t \), respectively. Define the function \( H \) over \( M \times [0, T) \) by \( H(x, t) := (H_t)_x \) \(((x, t) \in M \times [0, T))\), the section \( g \) of \( \pi_M^* (T(0,2)M) \) by \( g(x, t) := (g_t)_x \) \(((x, t) \in M \times [0, T))\) and the section \( N \) of \( \tilde{f}^*(T\overline{M}) \) by \( N(x, t) := (N_t)_x \) \(((x, t) \in M \times [0, T))\), where \( T(0,2)M \) is the tensor bundle of degree \((0,2)\) of \( M \) and \( T\overline{M} \) is the tangent bundle of \( \overline{M} \). The average mean curvature \( \overline{H} : [0, T) \to \mathbb{R} \) is defined by

\[
\overline{H}_t := \frac{\int_M H_t \, dv_g}{\int_M dv_g},
\]

where \( dv_g \) is the volume element of \( g_t \). The flow \( f_t \)’s \((0 \leq t < T)\) is called a volume-preserving mean curvature flow if it satisfies

\[
\tilde{f}_* \left( \frac{\partial}{\partial t} \right) = (\overline{H} - H)N.
\]
In particular, if \( f_t \)'s are embeddings, then we call \( M_t := f_t(M)'s (0 \in [0, T)) \) rather than \( f_t's (0 \in [0, T)) \) a volume-preserving mean curvature flow. Note that, if \( M \) has no boundary and if \( f \) is an embedding, then, along this flow, the volume of \((M, g_t)\) decreases but the volume of the domain \( D_t \) surrounded by \( f_t(M) \) is preserved invariantly.

First we shall recall the result by M. Athanassenas ([A1,2]). Let \( P_i (i = 1, 2) \) be affine hyperplanes in the \((n + 1)\)-dimensional Euclidean space \( \mathbb{R}^{n+1} \) meeting a affine line \( l \) orthogonally and \( E \) a closed domain of \( \mathbb{R}^{n+1} \) with \( \partial E = P_1 \cup P_2 \). Also, let \( M \) be a hypersurface of revolution in \( \mathbb{R}^{n+1} \) such that \( M \subset E, \partial M \subset P_1 \cup P_2 \) and that \( M \) meets \( P_1 \) and \( P_2 \) orthogonally. Let \( D \) be the closed domain surrounded by \( P_1, P_2 \) and \( M \), and \( d \) the distance between \( P_1 \) and \( P_2 \). She ([A1,2]) proved the following fact.

**Fact 1.** Let \( M_t (0 \leq t < T) \) be the volume-preserving mean curvature flow starting from \( M \). Assume that \( M_t \) meets \( P_1 \) and \( P_2 \) orthogonally for all \( t \in [0, T) \). Then the following statements (i) and (ii) hold:

(i) \( M_t (t \in [0, T)) \) remain to be hypersurfaces of revolution.

(ii) If \( \text{Vol}(M) \leq \frac{\text{Vol}(D)}{d} \) holds, then \( T = \infty \) and as \( t \to \infty \), the flow \( M_t \) converges to the cylinder \( C \) such that the volume of the closed domain surrounded by \( P_1, P_2 \) and \( C \) is equal to \( \text{Vol}(D) \).

E. Cabezas-Rivas and V. Miquel ([CM1,2,3]) proved the similar result in certain kinds of rotationally symmetric spaces. Let \( \overline{M} \) be an \((n + 1)\)-dimensional rotationally symmetric space (i.e., \( SO(n) \) acts on \( \overline{M} \) isometrically and its fixed point set is a one-dimensional submanifold). Note that real space forms are rotationally symmetric spaces. Denote by \( l \) the fixed point set of the action, which is an one-dimensional totally geodesic submanifold in \( \overline{M} \). Let \( P_i (i = 1, 2) \) totally geodesic hypersurfaces (or equidistant hypersurfaces) in \( \overline{M} \) meeting \( l \) orthogonally and \( E \) a closed domain of \( \overline{M} \) with \( \partial E = P_1 \cup P_2 \). An embedded hypersurface \( M \) in \( \overline{M} \) is called a hypersurface of revolution if it is invariant with respect to the \( SO(n) \)-action. Let \( M \) be a hypersurface of revolution in \( \overline{M} \) such that \( M \subset E, \partial M \subset P_1 \cup P_2 \) and that \( M \) meets \( P_1 \) and \( P_2 \) orthogonally. Let \( D \) be the closed domain surrounded by \( P_1, P_2 \) and \( M \), and \( d \) the distance between \( P_1 \) and \( P_2 \). They ([CM1,2,3]) proved the following fact.

**Fact 2.** Let \( M_t (0 \leq t < T) \) be the volume-preserving mean curvature flow starting from \( M \). Assume that \( M_t \) meets \( P_1 \) and \( P_2 \) orthogonally for all \( t \in [0, T) \). Then the following statements (i) and (ii) hold:

(i) \( M_t (t \in [0, T)) \) remain to be hypersurfaces of revolution.
(ii) If $\text{Vol}(M) \leq C$ holds, where $C$ is a constant depending on $\text{Vol}(D)$ and $d$, then $T = \infty$ and, as $t \to \infty$, the flow $M_t$ ($t \in [0,T]$) converges to a hypersurface of revolution $C$ of constant mean curvature such that the volume of the closed domain surrounded by $P_1, P_2$ and $C$ is equal to $\text{Vol}(D)$.

A symmetric space of compact type (resp. non-compact type) is a naturally reductive Riemannian homogeneous space $M$ such that, for each point $p$ of $M$, there exists an isometry of $M$ having $p$ as an isolated fixed point and that the isometry group of $M$ is a semi-simple Lie group each of whose irreducible factors is compact (resp. not compact) (see [He]). Note that symmetric spaces of compact type other than a sphere and symmetric spaces of non-compact type other than a hyperbolic space are not rotationally symmetric. An equifocal submanifold in a (general) symmetric space is a compact submanifold (without boundary) satisfying the following conditions:

(E-i) the normal holonomy group of $M$ is trivial,

(E-ii) $M$ has a flat section, that is, for each $x \in M$, $\Sigma_x := \exp^\perp(T_x^\perp M)$ is totally geodesic and the induced metric on $\Sigma_x$ is flat, where $T_x^\perp M$ is the normal space of $M$ at $x$ and $\exp^\perp$ is the normal exponential map of $M$.

(E-iii) for each parallel normal vector field $v$ of $M$, the focal radii of $M$ along the normal geodesic $\gamma_{v_x}(t)$ with $\gamma_{v_x}'(0) = v_x$ are independent of the choice of $x \in M$, where $\gamma_{v_x}'(0)$ is the velocity vector of $\gamma_{v_x}$ at $0$.

In [Ko3], we showed that the mean curvature flow starting from an equifocal submanifold in a symmetric space of compact type collapses to one of its focal submanifolds in finite time. In [Ko5], we showed that the mean curvature flow starting from a certain kind of (not necessarily compact) submanifold satisfying the above conditions (E-i), (E-ii) and (E-iii) in a symmetric space of non-compact type collapses to one of its focal submanifolds in finite time. The following question arise naturally:

**Question.** In what case, does the volume-preserving mean curvature flow starting from a submanifold in a symmetric space of compact type (or non-compact type) converges to a submanifold satisfying the above conditions (E-i), (E-ii) and (E-iii)?

Let $M$ be an equifocal hypersurface in a rank $l(\geq 2)$ symmetric space $M$ of compact type or non-compact type. Then it admits a reflective focal submanifold $F$ and it is a tube (of constant radius) over $F$, where the “reflectivity of submanifold” means that the submanifold is a connected component of the fixed point set of an involutive isometry of $M$ and a “tube of constant radius ($r(> 0)$) over $F$” means the image of $t_r(F) := \{ \xi \in T^\perp F \mid ||\xi|| = r \}$ by the normal exponential map $\exp^\perp$ of $F$ under
the assumption that the restriction $\exp^\perp|_{t_r(F)}$ of $\exp^\perp$ to $t_r(F)$ is an embedding, where $T^\perp F$ is the normal bundle of $F$ and $|| \cdot ||$ is the norm of $(\cdot)$. Any reflective submanifold in a symmetric space $\overline{M}$ of compact type or non-compact type is a singular orbit of a Hermann action (i.e., the action of the symmetric subgroup of the isometry group of $\overline{M}$) (see [KT]). Note that even if T. Kimura and M. Tanaka ([KT]) proved this fact in compact type case, the proof is valid in non-compact type case. From this fact, it is shown that $M$ is curvature-adapted, where “the curvature-adaptedness” means that, for any point $x \in M$ and any normal vector $v$ of $M$ at $x$, $R(\cdot, v)v$ preserves the tangent space $T_xM$ of $M$ at $x$ invariantly, and that the restriction $R(\cdot, v)v|_{T_xM}$ of $R(\cdot, v)v$ to $T_xM$ and the shape operator $A_v$ commute to each other ($R$ : the curvature tensor of $M$). For a non-constant positive-valued function $r$ over $F$, the image of $t_r(F) := \{ \xi \in T^\perp F | ||\xi|| = r(\pi(\xi)) \}$ by $\exp^\perp$ is called the tube of non-constant radius $r$ over $F$ in the case where the restriction $\exp^\perp|_{t_r(F)}$ of $\exp^\perp$ to $t_r(F)$ is an embedding, where $\pi$ is the bundle projection of $T^\perp F$. Note that $\exp^\perp|_{t_r(F)}$ is an embedding for a non-constant positive-valued function $r$ over $F$ such that max $r$ is sufficiently small because $F$ is homogeneous. Since $F$ is reflective, so is also the normal umbrella $F^\perp_x := \exp^\perp(T^\perp_x F)$ of $F$ at $x$ and hence $F^\perp_x$ is a symmetric space. If $F^\perp_x$ is a rank one symmetric space, then tube over $F$ of constant radius satisfies the above conditions (E-i), (E-ii) and (E-iii). Hence, when $F^\perp_x$ is of rank one, it is very interesting to investigate in what case the volume-preserving mean curvature flow starting from a tube of non-constant radius over a reflective submanifold in $\overline{M}$ converges to a hypersurface satisfying the above conditions (E-i), (E-ii) and (E-iii). Under this motivation, we prove a result similar to those of M. Athanassenas ([A1,2]) and E. Cabezas-Rivas and V. Miquel ([CM1,2,3]) in this paper.

The setting in this paper is as follows.

**Setting.** Let $F$ be a reflective submanifold in a rank $l(\geq 2)$ symmetric space $\overline{M}$ of compact type or non-compact type and $B$ be a compact closed domain in $F$ with smooth boundary which is star-shaped with respect to some $x_0 \in B$ and does not intersect with the cut locus of $x_0$ in $F$. Assume that the normal umbrellas of $F$ are rank one symmetric spaces. Set $P := \bigcup_{x \in \partial B} F^\perp_x$ and denote by $E$ the closed domain in $\overline{M}$ surrounded by $P$. Let $M := t_{r_0}(B)$ and $f := \exp^\perp|_{t_{r_0}(B)}$, where $r_0$ is a non-constant positive function over $B$ with $r_0 < r_{\text{cut}}$ ($r_{\text{cut}}$ : the cut radius of $F^\perp_x$). Assume that $f$ is an embedding. Denote by $D$ the closed domain surrounded by $P$ and $f(M)$.
Theorem A. Let $f$ be as in the above setting and $f_t (0 \leq t < T)$ be the volume-preserving mean curvature flow starting from $f$. Assume that the following conditions hold:

(i) $(\text{grad } r_t)_x$ belongs to a common eigenspace of the family $\{R(\cdot, \xi)\xi \in T^\perp_x B\}$ for all $(x, t) \in B \times [0, T)$, where $r_t$ is the radius function of $M_t$ (i.e., $M_t = \exp^\perp (t r_t(B))$).

(ii) $\text{grad } r_t = 0$ and $\nabla \text{grad } r_t = 0$ hold along $\partial B$, where $\nabla$ is the Riemannian connection of the induced metric on $B$ for all $t \in [0, T)$.

Then $M_t (t \in [0, T))$ remain to be tubes over $B$ such that the volume of the closed domain surrounded by $M_t$ and $P$ is equal to $\text{Vol}(D)$.

Remark 1.1. (i) At least one of singular orbits of any Hermann action of cohomogeneity one on any rank $l (\geq 2)$ symmetric space $\overline{M}$ of compact type or non-compact type is a reflective submanifold whose normal umbrellas are rank one symmetric spaces and tubes of constant radius over the reflective singular orbit satisfy the above conditions (E-i), (E-ii) and (E-iii) (i.e., of constant mean curvature). Note that, when $\overline{M}$ is of compact type, the Hermann action has exactly two singular orbits and, when $\overline{M}$ is of non-compact type, the Hermann action has the only one singular orbit. Hermann actions of cohomogeneity one on irreducible symmetric spaces of compact type or non-compact type are classified in [BT].

(ii) See Section 7 about examples of a reflective $F$ of a symmetric space $\overline{M}$ admitting a non-trivial common eigenspace of the family $\{R(\cdot, \xi)\xi \in T^\perp_x F\}$.

(iii) At least one of singular orbits of any Hermann action of cohomogeneity greater than one on any symmetric space $\overline{M}$ of compact type or non-compact type is a reflective submanifold but tubes of constant radius over the reflective singular
orbit do not satisfy the above conditions (E-i), (E-ii) and (E-iii) (i.e., not of constant mean curvature).

From Theorem A, we can derive the following results.

**Corollary B.** Let \( F \) be either a Helgason sphere (which is a meridian) or the corresponding polar of a rank one symmetric space \( \overline{M} \) of compact type, and \( B \) and \( f \) be as in the above setting. Assume that \( r_t \ (0 \leq t < T) \) satisfy the boundary condition (ii) in Theorem A. Then \( M_t \ (t \in [0, T)) \) remain to be tubes over \( B \) such that the volume of the closed domain surrounded by \( M_t \) and \( P \) is equal to \( \text{Vol}(D) \).

See [CN] about the definitions of a polar and a meridian of a symmetric space.

**Corollary C.** Let \( F \) be the reflective submanifold in a rank one symmetric space \( \overline{M} \) of non-compact type given as a orbit of the dual action of a Hermann action having a Helgason sphere or the corresponding polar of the compact dual symmetric space \( \overline{M}^* \) as an orbit, and \( B \) and \( f \) be as in the above setting. Assume that \( r_t \ (0 \leq t < T) \) satisfy the boundary condition (ii) in Theorem A. Then \( M_t \ (t \in [0, T)) \) remain to be tubes over \( B \) such that the volume of the closed domain surrounded by \( M_t \) and \( P \) is equal to \( \text{Vol}(D) \).

Remark 1.2. If \( F \) is as in Corollaries B and C, then the condition (i) in Theorem A automatically holds. In fact, \( R(\cdot, \xi)\xi|_{T_x F} \) is then a constant-multiple of the identity transformation of \( T_x F \) for all \( x \in F \) and all \( \xi \in T^\perp_x F \).

In the future, under the above setting, we want to derive the result similar to the statement (ii) of Facts 1 and 2. For its purpose, we must show the uniformly boundedness of the norms of the \( i \)-th covariant derivatives \( \nabla^i A_t \ (i \in \mathbb{N} \cup \{0\}) \) of the shape operator \( A_t \) of \( f_t \) (see the proofs of Theorems 16 and 17 of [CM3] in detail). However, we will need further long and delicate discussions as in Sections 6 and 7 (Page 198-202) of [CM3] to show their uniformly boundedness.

2 The mean curvature of a tube over a reflective submanifold

In this section, we shall calculate the mean curvature of a tube over a reflective submanifold in a symmetric space of compact type or non-compact type. Let \( \overline{M} = \overline{G/K} \) be a symmetric space of compact type or non-compact type, where \( G \) is the identity component of the isometry group of \( \overline{M} \) and \( K \) is the isotropy group of \( G \) at
some point \( p_0 \) of \( \overline{M} \). Let \( F \) be a reflective submanifold in \( \overline{M} \) such that the normal umbrellas \( \Sigma_x \)'s (\( x \in F \)) are symmetric spaces of rank one. Denote by \( g \) and \( k \) the Lie algebras of \( G \) and \( K \), respectively. Also, let \( \theta \) be the Cartan involution of \( g \) with \( (\text{Fix} \theta)_0 \subset K \subset \text{Fix} \theta \) and set \( \mathfrak{p} := \text{Ker}(\theta + \text{id}) \), which is identified with the tangent space of \( T_{p_0} \overline{M} \) of \( \overline{M} \) at \( p_0 \). Without loss of generality, we may assume that \( p_0 \) belongs to \( F \). Set \( \mathfrak{p}' := T_{p_0}F \) and \( \mathfrak{p}' \perp := T_{p_0} \perp F \). Take a maximal abelian subspace \( b \) of \( \mathfrak{p}' \perp \) and a maximal abelian subspace \( a \) of \( \mathfrak{p} \) including \( b \). Note that the dimension of \( b \) is equal to 1 because the normal umbrellas of \( F \) is symmetric spaces of rank one by the assumption. For each \( \alpha \in a^* \) and \( \beta \in b^* \), we define a subspace \( \mathfrak{p}_\alpha \) and \( \mathfrak{p}_\beta \) of \( \mathfrak{p} \) by

\[
\mathfrak{p}_\alpha := \{ Y \in \mathfrak{p} \mid \text{ad}(X)^2(Y) = -\varepsilon_\alpha(X)^2 Y \text{ for all } X \in a \}
\]

and

\[
\mathfrak{p}_\beta := \{ Y \in \mathfrak{p} \mid \text{ad}(X)^2(Y) = -\varepsilon_\beta(X)^2 Y \text{ for all } X \in b \},
\]

respectively, where \( \text{ad} \) is the adjoint representation of \( g \), \( a^* \) (resp. \( b^* \)) is the dual space of \( a \) (resp. \( b \)) and \( \varepsilon \) is given by

\[
\varepsilon := \begin{cases} 
1 & \text{when } \overline{M} \text{ is of compact type} \\
-1 & \text{when } \overline{M} \text{ is of non-compact type} 
\end{cases}
\]

Define a subset \( \Delta \) of \( a^* \) by

\[
\Delta := \{ \alpha \in a^* \mid \mathfrak{p}_\alpha \neq \{0\} \},
\]

and subsets \( \Delta' \) and \( \Delta'_V \) of \( b^* \) by

\[
\Delta' := \{ \beta \in b^* \mid \mathfrak{p}_\beta \neq \{0\} \}
\]

and

\[
\Delta'_V := \{ \beta \in b^* \mid \mathfrak{p}_\beta \cap \mathfrak{p}' \neq \{0\} \}.
\]

The systems \( \Delta \) and \( \Delta'_V \) are root systems and \( \Delta' = \{ \alpha_b \mid \alpha \in \Delta \} \) holds. Let \( \Delta_+ \) (resp. \( (\Delta'_V)_+ \)) be the positive root system of \( \Delta \) (resp. \( \Delta'_V \)) with respect to some lexicographic ordering of \( a^* \) (resp. \( b^* \)) and \( \Delta'_+ \) be the positive subsystem of \( \Delta' \) with respect to the lexicographic ordering of \( b^* \), where we take one compatible with the lexicographic ordering of \( b^* \) as the lexicographic ordering of \( a^* \). Also we have the following root space decomposition:

\[
\mathfrak{p} = a + \sum_{\alpha \in \Delta_+} \mathfrak{p}_\alpha = \mathfrak{z}(b) + \sum_{\beta \in \Delta'_+} \mathfrak{p}_\beta,
\]
where \( \mathfrak{z}_p(b) \) is the centralizer of \( b \) in \( p \). For convenience, we set \( p_0 := \mathfrak{z}_p(b) \). Since the normal umbrellas of \( F \) are symmetric spaces of rank one, \( \dim b = 1 \) and this root system \( \Delta'_V \) is of \( (a_1) \)-type or \( (b_0) \)-type. Hence \( (\Delta'_{V})_+ \) is described as

\[
(\Delta'_{V})_+ = \left\{ \begin{array}{cl} \{ \beta \} & (\Delta'_V : (a_1)\text{-type}) \\ \{ \beta, 2\beta \} & (\Delta'_V : (b_0)\text{-type}) \end{array} \right. 
\]

for some \( \beta(\neq 0) \in b^* \). However, in general, we may describe as \( (\Delta'_V)_+ = \{ \beta, 2\beta \} \) by interpreting as \( p_{2\beta} = \{0\} \) when \( \Delta'_V \) is of \( (a_1) \). The system \( \Delta'_+ \) is described as

\[
\Delta'_+ = \{ k\beta \mid k \in \mathbb{K} \}
\]

for some finite subset \( \mathbb{K} \) of \( \mathbb{R}_+ \). Set \( b := |\beta(X_0)| \) for a unit vector \( X_0 \) of \( b \). Since \( F \) is curvature-adapted, \( p' \) and \( p'^\perp \) are \( \text{ad}(X)^2 \)-invariant for each \( X \in b \). Hence we have the following direct sum decompositions:

\[
p' = p_0 \cap p' + \sum_{k \in \mathbb{K}} (p_{k\beta} \cap p')
\]

and

\[
(p')^\perp = b + \sum_{k=1}^{2} (p_{k\beta} \cap (p')^\perp).
\]

For each \( p \in \overline{M} \), we choose a shortest geodesic \( \gamma_{p_{\text{pop}} : [0,1] \to \overline{M}} \) with \( \gamma_{p_{\text{pop}}}(0) = p_0 \) and \( \gamma_{p_{\text{pop}}}(1) = p \), where the choice of \( \gamma_{p_{\text{pop}}} \) is not unique in the case where \( p \) belongs to the cut locus of \( p_0 \). Denote by \( \tau_p \) the parallel translation along \( \gamma_{p_{\text{pop}}} \). For \( w \in T_p\overline{M} \), we define linear transformations \( D_{w}^{co} \) and \( D_{w}^{si} \) of \( T_p\overline{M} \) by

\[
D_{w}^{co} := \tau_p \circ \cos(\text{iad}(\tau_p^{-1}w)) \circ \tau_p^{-1}
\]

and

\[
D_{w}^{si} := \tau_p \ast \frac{\sin(\text{iad}(\tau_p^{-1}w))}{\text{iad}(\tau_p^{-1}w)} \circ \tau_p^{-1},
\]

respectively, where \( i \) is the imaginary unit. Let \( r \) be a positive-valued function over \( F \) and \( B \) a compact closed domain in \( F \). Set \( M := t_r(B) \) and \( f := \exp^+ \mid_{t_r(B)} \). Assume that \( f \) is an embedding. Denote by \( N \) the outward unit normal vector field of \( M \) and \( A \) the shape operator of \( M \) with respect to \( -N \). Fix \( x \in B \) and \( \xi \in M \cap T_x B \). Without loss of generality, we may assume that \( \tau_x^{-1}\xi \in b \). Denote by \( \gamma_{\xi} \) the normal geodesic of \( B \) whose initial vector is equal to \( \xi \) and \( P_{\gamma_{\xi}} \) the parallel translation along \( \gamma_{\xi}[0,1] \). The vertical subspace \( V_{\xi} \) and the horizontal subspace \( H_{\xi} \) at \( \xi \) are defined by \( V_{\xi} := T_{\xi}(M \cap T_x B) \) and \( H_{\xi} := \{ X_{\xi} \mid X \in T_x B \} \), respectively,
where \( \tilde{X}_\xi \) is the natural lift of \( X \) to \( \xi \) (see [Ko1] about the definition of the natural lift). Take \( v \in \mathcal{V}_\xi \) with \( (P_{\gamma_\xi} \circ \tau_x^{-1})(v) \in p_{k\beta} \) and \( X \in T_B \) with \( \tau_x^{-1}X \in p_{k\beta} \), where \( k \in \mathcal{K} \cup \{0\} \). According to (i) of Theorem A in [Ko1], we have

\[
Av = \frac{1}{\sqrt{1 + ||(D_{\xi}^\circ)\mathcal{T}^{-1}(\text{grad} r)_x||^2}} \times \left( \frac{\sqrt{\varepsilon}kb}{\tan(\sqrt{\varepsilon}kbr(x))} v + ((P_{\gamma_\xi} \circ \tau_x)(Z_v(1)))_T \right),
\]

where grad \( r \) is the gradient vector field of \( r \) (with respect to the induced metric on \( B \)), \((\cdot)_T\) is the \( TM \)-component of \((\cdot)\), \( \sqrt{\varepsilon} \) is given by

\[
\sqrt{\varepsilon} := \begin{cases} 1 & \text{when } \overline{M} \text{ is of compact type} \\ i & \text{when } \overline{M} \text{ is of non-compact type} \end{cases}
\]

and \( Z_v(\rho : \mathbb{R} \to p) \) is the solution of the following differential equation:

\[
Z''(s) = \text{ad}(a^\ast_{\varepsilon_1} \xi^2)(Z(s)) - \frac{2\cos^2(s\sqrt{\varepsilon}kbr(x))}{\cos^2(\sqrt{\varepsilon}kbr(x))} \left[ [\tau_x - 1_\xi, \tau_x^{-1}\text{grad} \, r], \tau_x^{-1}v_0 \right] + 2\tan^2(\sqrt{\varepsilon}kbr(x)) \left[ [\tau_x^{-1}_\xi, \tau_x^{-1}v_0], \tau_x^{-1}\text{grad} \, r \right]
\]

satisfying the following initial condition:

\[
Z(0) = \tau_x^{-1}(((D_{\xi}^\circ)^{-2})_{\ast \xi}v)(\text{grad} \, r)_x,
\]

and

\[
Z'(0) = 0,
\]

where \( v_0 \) is the element of \( T^\perp_B \) corresponding to \( v \) under the identification of \( T\xi(T^\perp_B) \) and \( T^\perp_B \), \((D_{\xi}^\circ)^{-2})_{\ast \xi}v \) is the differential of \((D_{\xi}^\circ)^{-2} \) at \( \xi \) (which is regarded as a map from \( T^\perp_B \) to \( T^\perp_B \otimes T_B \)) and \((D_{\xi}^\circ)^{-2})_{\ast \xi}v \) is regarded as an element of \( T^\perp_B \otimes T_B \) under the natural identification of \( T_{(D_{\xi}^\circ)^{-2}}(T^\perp_B \otimes T_B) \) and \( T^\perp_B \otimes T_B \). According to (ii) of Theorem A in [Ko1], we have

\[
A\tilde{X}_\xi = \frac{\sqrt{\varepsilon}kb \tan(\sqrt{\varepsilon}kbr(x))}{\sqrt{1 + ||(D_{\xi}^\circ)^{-1}(\text{grad} r)_x||^2}} \tilde{X}_\xi
\]

\[
+ \frac{(Xr)\sqrt{\varepsilon}kb \tan(\sqrt{\varepsilon}kbr(x))}{(1 + ||(D_{\xi}^\circ)^{-1}(\text{grad} r)_x||^2)^{3/2}} ((D_{\xi}^\circ)^{-2}(\text{grad} r)_x)_\xi
\]

\[
- \frac{1}{1 + ||(D_{\xi}^\circ)^{-1}(\text{grad} r)_x||^2} ((P_{\gamma_\xi} \circ \tau_x)(Z_{X\xi}(1)))_T,
\]

9
where $Z_{X, \xi}(\mathbb{R} \to p)$ is the solution of the following differential equation:

\[(2.6)\]

\[
Z''(s) = \text{ad}(\tau_x^{-1} \xi)^2(Z(s)) + \frac{2Xr}{r(x)} \text{ad}(\tau_x^{-1} \xi)^2(\tau_x^{-1}(D_{\xi}^co \circ (D_{\xi}^co)^{-2}((\text{grad } r)_x)))
\]

\[
-2 \left[\tau_x^{-1} \xi, \tau_x^{-1}(D_{\xi}^co \circ (D_{\xi}^co)^{-2}((\text{grad } r)_x)), \tau_x^{-1} \frac{dD_{\xi}^co}{ds} X\right]
\]

\[
-2 \left[\tau_x^{-1} \xi, \tau_x^{-1}(D_{\xi}^co X), \tau_x^{-1} \left(\frac{dD_{\xi}^co}{ds} \circ (D_{\xi}^co)^{-2}((\text{grad } r)_x)\right)\right]
\]

satisfying the following initial condition:

\[(2.7)\]

\[
Z(0) = \tau_x^{-1}\nabla^\pi_M \left(\left(D_{\xi}^co\right)^{-2}((\text{grad } r)_x)\right)
\]

and

\[(2.8)\]

\[
Z'(0) = -[[\tau_x^{-1} \xi, \tau_x^{-1} X], \tau_x^{-1}(D_{\xi}^co)^{-2}((\text{grad } r)_x)],
\]

where $\nabla^\pi_M$ is the covariant derivative along $\pi_M$ induced from the Riemannian connection $\nabla$ of the induced metric on $F$. Here we give the table of the correspondence between the above notations and the notations in [Ko1].

| The notation in [Ko1] | The notations in this paper | Remark |
|----------------------|----------------------------|--------|
| $M$                  | $B$                        |        |
| $\varepsilon$        | $r$                        |        |
| $t_{\varepsilon}(M)$ | $M(=t_{r}(B))$             |        |
| $A_{\xi}$            | $A$                        |        |
| $B_{\xi}$            | $D_{\xi}^co$               | by the reflectivity of $F$ |
| $g_{\varepsilon}$    | $\tau_x$                  |        |
| $\mu(g_{\varepsilon}^{-1} \xi)$ | $\sqrt{\varepsilon} kbr(x)$ |        |

Table 1.

Denote by $(\cdot)_k$ the $\tau_x(p_{k,3})$-component of $(\cdot) \in T_x B$, where $k \in \mathcal{K} \cup \{0\}$. Then we have

\[(2.9)\]

\[
(D_{\xi}^co)^{-j}((\text{grad } r)_x) = \sum_{k \in \mathcal{K} \cup \{0\}} \frac{1}{\cos^j(\sqrt{\varepsilon} kbr(x))}((\text{grad } r)_x)_k \quad (j = 1, 2)
\]

and

\[(2.10)\]

\[
((D_{\xi}^co)^{-2}(v))((\text{grad } r)_x) = \sum_{k \in \mathcal{K} \cup \{0\}} \frac{2\sqrt{\varepsilon} kbr(x) \sin(\sqrt{\varepsilon} kbr(x))}{\cos^3(\sqrt{\varepsilon} kbr(x))}((\text{grad } r)_x)_k.
\]
Since $F$ is reflective, it is an orbit of a Hermann action (i.e., the action of a symmetric subgroup of $G$) and hence it is homogeneous. Let $c : I \to B$ be the (homogeneous) geodesic in $B$ with $c'(0) = X$ and $\hat{\xi}$ the normal vector field of $B$ along $c$ such that $\hat{\xi}(0) = \xi$, $\frac{\hat{\xi}}{||\hat{\xi}||}$ is parallel (with respect to the normal connection) and that $||\hat{\xi}(t)|| = r(\pi(c(t)))$ for all $t$ in the domain of $\hat{\xi}$. This curve $\hat{\xi}$ is regarded as a curve in $M$ with $\xi'(0) = \hat{X}_\xi$. Since $c$ is a homogeneous curve, it is described as $c(t) = \tilde{a}(t)x$ ($t \in I$) for some curve $\tilde{a} : I \to G$. Then, since $\frac{\hat{\xi}}{||\hat{\xi}||}$ is parallel and $F$ is a submanifold with section (i.e., with Lie triple systematic normal bundle in the sense of [K1]) by the reflectivity of $F$, we can show

$$
(2.11) \quad \text{Span}\{\hat{\xi}(t)\} = (\tilde{a}(t)_* (\text{Span}\{\xi\})) = (\tilde{a}(t)_* \circ \tau_\pi)(b) \quad (t \in I)
$$

(see Theorem 5.5.12 of [PT]) and

$$
(2.12) \quad T_{\tilde{c}(t)}B = \sum_{k \in K \cup \{0\}} (T_{\tilde{c}(t)}B \cap (\tilde{a}(t)_* \circ \tau_\pi)(p_{k\beta})).
$$

Denote by $(\cdot)_k$ the $(\tilde{a}(t)_* \circ \tau_\pi)(p_{k\beta})$-component of $(\cdot) \in T_{\tilde{c}(t)}B$. Then we can show

$$
\nabla_{\hat{X}_\xi}^\pi ((D_{\xi}^\rho)^{-1} \frac{\text{grad}}{r}) = \left. \frac{\nabla}{dt} \right|_{t=0} (D_{\xi}^\rho)^{-1} ((\text{grad} r)_{\tilde{c}(t)})
= \sum_{k \in K \cup \{0\}} \left( \frac{\nabla}{dt} \right|_{t=0} \left( \frac{1}{\cos^2(\sqrt{2} k \beta ((\tilde{a}(t)_* \circ \tau_\pi)^{-1}(\hat{\xi}(t))))} ((\text{grad} r)_{\tilde{c}(t)})_k \right)
= \sum_{k \in K \cup \{0\}} \left( \frac{\nabla}{dt} \right|_{t=0} \left( \frac{1}{\cos^2(\sqrt{2} k \beta (r(c(t))))} ((\text{grad} r)_{\tilde{c}(t)})_k \right)
= \sum_{k \in K \cup \{0\}} \left( \frac{2(Xr)\sqrt{2}k \sin(\sqrt{2} k \beta (r(x)))}{\cos^3(\sqrt{2} k \beta (r(x)))} ((\text{grad} r)_k x \right)
+ \frac{1}{\cos^2(\sqrt{2} k \beta (r(x)))} \left. \frac{\nabla}{dt} \right|_{t=0} ((\text{grad} r)_{\tilde{c}(t)})_k \right),
$$

(2.13)

where $\frac{\nabla}{dt}$ is the covariant derivative along $c$ with respect to $\nabla$. Since $F$ is a submanifold with section, we have

$$
(2.14) \quad [[p', p'^\perp], p'^\perp] \subset p', \quad [[p'^\perp, p'^\perp], p'] \subset p' \quad \text{and} \quad [[p'^\perp, p'], p'] \subset p'^\perp.
$$

Hence, since $Z_c$ satisfies (2.2), we have

$$
Z''_c(s) \equiv \text{ad}(\tau_x^{-1}\xi)^2 Z_v(s) \quad (\text{mod} \ p')
$$

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Also, since $Z_v$ satisfies (2.3), it follows from (2.10) that $Z_v(0) \equiv 0 \pmod{p'}$. Furthermore, since $Z_v$ satisfies (2.4), we have $Z'_v(0) = 0$. Hence we can show $Z_v(s) \equiv 0 \pmod{p'}$. Therefore, form (2.1), we obtain

\[
Av \equiv \frac{\sqrt{\varepsilon k b}}{\tan(\sqrt{\varepsilon k b}r(x))} \sqrt{1 + ||(D_{\xi}^z)^{-1}(\text{grad }r)_{x}||^2} v \pmod{p'}.
\]

On the other hand, since $Z_{X,\xi}$ satisfies (2.6), it follows from the third relation in (2.14) that

\[
Z''_{X,\xi}(s) \equiv \text{ad}(\tau^{-1}_x \xi)^2 Z_{X,\xi}(s)
+ \frac{2Xr}{r(x)} \text{ad}(\tau^{-1}_x \xi)^2 (\tau^{-1}_x ((D_{\xi}^z \circ (D_{\xi}^x)^{-2})(\text{grad }r)_{x}))
\pmod{p'\perp}.
\]

For simplicity, set

\[
Y_{X,\xi}(s) := \frac{2Xr}{r(x)} \text{ad}(\tau^{-1}_x \xi)^2 (\tau^{-1}_x ((D_{\xi}^z \circ (D_{\xi}^x)^{-2})(\text{grad }r)_{x})).
\]

Also, since $Z_{X,\xi}$ satisfies (2.7), it follows from (2.13) that

\[
Z_{X,\xi}(0) = \tau^{-1}_x \nabla^{|\varepsilon|/\ell}_X ((D_{\xi}^z)^{-2} \text{grad }r) \in p'.
\]

Furthermore, since $Z'_{X,\xi}(0)$ satisfies (2.8), it follows from the third relation in (2.14) that

\[
Z'_{X,\xi}(0) \equiv 0 \pmod{p'\perp}.
\]

For each $k \in K \cup \{0\}$, define $(Z_{X,\xi})_k : \mathbb{R} \to p_{k\beta}$ by $(Z_{X,\xi})_k(s) := Z_{X,\xi}(s)_k (s \in \mathbb{R})$. From (2.16), (2.17) and (2.18), we can derive

\[
(Z_{X,\xi})_k(s) \equiv \cos(s\sqrt{\varepsilon k b}r(x)) \tau^{-1}_x \nabla^{|\varepsilon|/\ell}_X ((D_{\xi}^z)^{-2}(\text{grad }r)_k)
+ \left( \frac{\cos(s\sqrt{\varepsilon k b}r(x)) - 1}{(s\sqrt{\varepsilon k b}r(x))^2} + \frac{\sin(s\sqrt{\varepsilon k b}r(x))}{(s\sqrt{\varepsilon k b}r(x))^3} \right) (Y_{X,\xi}(s))_k \pmod{p'\perp}
\]

On the other hand, according to Lemma 3.3 in [Ko1], we have

\[
((P_{\xi} \circ a_{*p_0})(Z_{X,\xi}(1)))T = ((D_{\xi}^z)^{-1}(a_{*p_0}Z_{X,\xi}(1)))
- \frac{((D_{\xi}^z)^{-1}(a_{*p_0}Z_{X,\xi}(1)))r}{1 + ||(D_{\xi}^z)^{-1}(\text{grad }r)_{x}||^2((D_{\xi}^z)^{-2}(\text{grad }r)_{x})_\xi}.
\]

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Also, according to (1.1) and (1.2) in [Ko1], we have

\[ \exp_\xi^+(\vec{X}_\xi) = J_{\xi,X}(1) + \frac{Xr}{r(x)}\gamma'_\xi(1) = P_{\gamma}(D^\xi(X)) + \frac{Xr}{r(x)}\gamma'_\xi(1) \]

\[ = \cos(\sqrt{\varepsilon kbr}(x))P_{\gamma}(X) + \frac{Xr}{r(x)}\gamma'_\xi(1), \]

where \( J_{\xi,X} \) is the Jacobi field along \( \gamma_\xi \) with \( J_{\xi,X}(0) = X \) and \( J'_{\xi,X}(0) = 0 \). From (2.5), (2.19), (2.20) and (2.21), we can derive

\[ A \vec{X}_\xi \equiv \frac{1}{(||X||^2 \cos^2(\sqrt{\varepsilon kbr}(x)) + (Xr)^2)\sqrt{1 + ||(D^\xi)^{-1}(\text{grad} r)_x||^2}} \]

\[ \times (||X||^2 \sqrt{\varepsilon kb\sin(\sqrt{\varepsilon kbr}(x)) \cos(\sqrt{\varepsilon kbr}(x))}) \]

\[ + \left( \cos(\sqrt{\varepsilon kbr}(x)) - 1 + \frac{\sin(\sqrt{\varepsilon kbr}(x))}{\sqrt{\varepsilon kbr}(x)} \right) \frac{2(Xr)^2}{r(x)} \]

\[ + 2(Xr)^2 \sqrt{\varepsilon kb \tan(\sqrt{\varepsilon kbr}(x)) + \langle \text{grad} r, X \rangle} \]

\[ \vec{X}_\xi \mod T^\perp_x \text{Span}\{\vec{X}_\xi\} \]

Set \( m^V := \dim V \), \( m^H := \dim H \), \( m_k^V := \dim(p_{k\beta} \cap p^\perp) \) and \( m_k^H := \dim(p_{k\beta} \cap p') \) \((k = 0, 1, 2)\).

**Assumption.** Assume that \( \tau^{-1}_x((\text{grad} r)_x) \in p_{k0\beta} \cap p' \) for some \( k_0 \in \{0, 1, 2\} \).

Set \( W_k := p_{k\beta} \cap p' \) \((k \neq k_0)\) and \( W_{k_0} := (p_{k0\beta} \cap p') \oplus \text{Span}\{(\text{grad} r)_x\} \). For a subspace \( W \) of \( H_\xi \), we define \( \tilde{W}_\xi \) by \( \tilde{W}_\xi := \{ \tilde{X}_\xi | X \in W \} \).

According to (2.15) and (2.22), we can show the following fact.

**Proposition 2.1.** Under the above assumption, the mean curvature \( H_\xi \) of \( M \) at \( \xi \)
is given by
\[ H_\xi = \frac{\cos(\sqrt{x} k_0 b r(x))}{\sqrt{\cos^2(\sqrt{x} k_0 b r(x)) + ||(\text{grad } r)_x||^2}} \]
\[ \times \left\{ \sum_{k \in K \cup \{0\}} \frac{1}{\cos^2(\sqrt{x} k b r(x))} \text{Tr}(\text{pr}_{W_k} \circ (\nabla \text{grad } r)_x | W_k) \right. \]
\[ + 2 \left( \sum_{k=0}^{m^V_k} \tan(\sqrt{x} k b r(x)) \sum_{k \in K \setminus \{k_0\}} m^H_k \sqrt{x} k b \tan(\sqrt{x} k b r(x)) \right) \]
\[ + (m^H_{k_0} - 1) \sqrt{x} k_0 b \tan(\sqrt{x} k_0 b r(x)) + \frac{\cos^2(\sqrt{x} k_0 b r(x)) + ||(\text{grad } r)_x||^2}{r(x)} \frac{1}{\cos^2(\sqrt{x} k_0 b r(x))} \]
\[ \times \left\{ \left( \cos(\sqrt{x} k_0 b r(x)) - 1 + \frac{\sin(\sqrt{x} k_0 b r(x))}{\sqrt{x} k_0 b r(x)} \right) \frac{||r(x)||}{r(x)} \cos^2(\sqrt{x} k_0 b r(x)) \right. \]
\[ + 1 \left( (m^H_{k_0} - 1) \sqrt{x} k_0 b \tan(\sqrt{x} k_0 b r(x)) + \frac{\cos^2(\sqrt{x} k_0 b r(x)) + ||(\text{grad } r)_x||^2}{r(x)} \frac{1}{\cos^2(\sqrt{x} k_0 b r(x))} \right) \}

Proof. From (2.15), we have
\[ \text{Tr}(A|_{\frac{\sqrt{x} k b}{\tan(\sqrt{x} k b r(x))}})_\xi = \frac{\cos(\sqrt{x} k_0 b r(x))}{\sqrt{\cos^2(\sqrt{x} k_0 b r(x)) + ||(\text{grad } r)_x||^2}} \]
\[ \sum_{k=0}^{m^V_k} \frac{m^V_k \sqrt{x} k b}{\tan(\sqrt{x} k b r(x))}, \]
where \( \frac{\sqrt{x} k b}{\tan(\sqrt{x} k b r(x))} \) means \( \frac{1}{r(x)} \) in case of \( k = 0 \). According to (2.21), the horizontal subspace \( H_\xi \) is orthogonally decomposed as
\[ H_\xi = \left( \bigoplus_{k \in K \cup \{0\}} \overline{(W_k)_\xi} \right) \oplus \text{Span}\{(\text{grad } r)_x\}. \]

From (2.22), we have
\[ \text{Tr}(A|_{\frac{\sqrt{x} k b}{(W_k)_\xi}})_\xi = \frac{\cos(\sqrt{x} k_0 b r(x))}{\sqrt{\cos^2(\sqrt{x} k_0 b r(x)) + ||(\text{grad } r)_x||^2}} \]
\[ \times \left( \frac{m^H_k \sqrt{x} k b \tan(\sqrt{x} k b r(x))}{(k \neq k_0)} \frac{\text{Tr}(\text{pr}_{W_k} \circ (\nabla \text{grad } r)_x | W_k)}{\cos^2(\sqrt{x} k_0 b r(x))} \right) \]
\[ \text{Tr}(A|_{\frac{\sqrt{x} k b}{(W_{k_0})_\xi}})_\xi = \frac{\cos(\sqrt{x} k_0 b r(x))}{\sqrt{\cos^2(\sqrt{x} k_0 b r(x)) + ||(\text{grad } r)_x||^2}} \]
\[ \times \left( (m^H_{k_0} - 1) \sqrt{x} k_0 b \tan(\sqrt{x} k_0 b r(x)) + \frac{\cos^2(\sqrt{x} k_0 b r(x)) + ||(\text{grad } r)_x||^2}{r(x)} \frac{1}{\cos^2(\sqrt{x} k_0 b r(x))} \right) \]
and

\[
A((\text{grad } r)_x)_\xi \equiv \frac{\cos(\sqrt{\varepsilon k_b} br(x))}{(\cos^2(\sqrt{\varepsilon k_b} br(x)) + ||(\text{grad } r)_x||^2)\sqrt{\cos^2(\sqrt{\varepsilon k_b} br(x)) + ||(\text{grad } r)_x||^2}} \\
\times \left( 2 \left( \cos(\sqrt{\varepsilon k_b} br(x)) - 1 + \frac{\sqrt{\varepsilon k_b} br(x)}{r(x)} \right) \right) \frac{\sqrt{\cos^2(\sqrt{\varepsilon k_b} br(x)) + ||(\text{grad } r)_x||^2}}{||((\text{grad } r)_x)_\xi||^2} \\
+ \frac{1}{||((\text{grad } r)_x||^2} \tilde{g}(\nabla (\text{grad } r)_x, r, (\text{grad } r)_x) \right) (\text{grad } r)_x}_\xi \\
(\mod T_x^\perp \text{Span}\{((\text{grad } r)_x)_\xi\}).
\]

From (2.24) and these relations, we obtain the desired relation. q.e.d.

3 The volume element of a tube over a reflective submanifold

We shall use the notations in Introduction and the previous section. In this section, we shall calculate the volume element of \(M\). First we recall the description of the Jacobi field in a symmetric space \(M\) of non-compact type. The Jacobi field \(J\) along the geodesic \(\gamma\) in \(M\) is described as

\[
(3.1) \quad J(t) = P_{\gamma|_0,\xi} \left( D_{t\gamma'(0)}^{co}(J(0)) + t D_{t\gamma'(0)}^{si}(J'(0)) \right).
\]

**Assumption.** \((\text{grad } r)_x\) belongs to a common eigenspace of the family \(\{R(\cdot, \xi)\}_\xi \in T_x^\perp B\) for all \(x \in B\).

Fix \(\xi \in M \cap T_x^\perp B\) and \(X \in T_x B\). Without loss of generality, we may assume that \(\tau^{-1}_x \xi \in b\). By the above assumption, we have \(\tau^{-1}_x ((\text{grad } r)_x) \in p_{k_0, \beta} \cap p'\) for some \(k_0 \in \mathcal{K} \cup \{0\}\). Let \(\tilde{S}(x, r(x))\) be the hypersphere of radius \(r(x)\) in \(T_x^\perp B\) centered the origin and \(S(x, r(x))\) the geodesic hypersphere of radius \(r(x)\) in \(F_x^\perp := \exp^\perp(T_x^\perp B)\) centered \(x\). Denote by \(dv(\cdot)\) the volume element of the induced metric on \((\cdot)\). Take \(v \in T_x \tilde{S}(x, r(x))\). Then, according to (3.1), we have

\[
(3.2) \quad \exp^\perp_x(v) = J_{\xi, v}(1) = P_{\xi} (D^{si}_\xi(v)),
\]

where \(J_{\xi, v}\) is the Jacobi field along \(\gamma_\xi\) with \(J_{\xi, v}(0) = 0\) and \(J_{\xi, v}'(0) = v\). Define a
function $\psi_r$ over $B$ by

$$
(3.3) \quad \psi_r(x) := \left( \prod_{k=0}^{2} \left( \frac{\sin(\sqrt{\varepsilon_k}kbr(x))}{\sqrt{\varepsilon_k}kbr(x)} \right)^{m_k} \right) \left( \prod_{k \in K \setminus \{k_0\}} \cos^{m_k^H}(\sqrt{\varepsilon_k}kbr(x)) \right) \\
\times \cos^{m_{k_0}^H-1}(\sqrt{\varepsilon_k}kbr(x)) \sqrt{\cos^2(\sqrt{\varepsilon_k}kbr(x)) + ||(\text{grad } r)_x||^2}.
$$

From (2.21) and (3.2), we can derive the following relation for the volume element of $M$.

**Proposition 3.1.** The volume element $dv_M$ is given by

$$
(3.4) \quad (dv_M)_\xi = \psi_r(x) \left( (\exp^\perp |\tilde{S}(x,r(x))|^{-1})^* dv_{\tilde{S}(x,r(x))} \wedge (\pi|_M)^* dv_B \right),
$$

where $\xi \in M \cap T^\perp_x B$.

From (3.4), we can derive the following relation for the volume of $M$.

**Proposition 3.2.** The volume $\text{Vol}(M)$ of $M$ and the average mean curvature $\overline{H}$ of $M$ are given by

$$
(3.5) \quad \text{Vol}(M) = v_{m^V} \int_B r^{m^V} \psi_r dv_B
$$

and

$$
(3.6) \quad \overline{H} = \frac{\int_B r^{m^V} \rho_r \psi_r dv_B}{\int_B r^{m^V} \psi_r dv_B},
$$

respectively, where $v_{m^V}$ is the volume of the $m^V$-dimensional Euclidean unit sphere and $\rho_r$ is the function over $B$ defined by assigning the value of the right-hand side of (2.23) to each $x \in B$.

### 4 The evolution of the radius function

Let $F, B, M = t_{r_0}(B)$, $f$ and $f_t$ be as in Theorem A. We use the notations in Introduction and Sections 1-2. Denote by $S^\perp B$ the unit normal bundle of $B$ and $S^\perp_x B$ the fibre of this bundle over $x \in B$. Define a positive-valued function $\tilde{r}_t : M \to \mathbb{R}$ $(t \in [0, T))$ and a map $w_t^1 : M \to S^\perp B$ $(t \in [0, T))$ by $f_t(\xi) = \exp^\perp(\tilde{r}_t(\xi)w_t^1(\xi))$ $(\xi \in M)$. Also, define a map $c_t : M \to B$ by $c_t(\xi) := \pi(w_t^1(\xi))$ $(\xi \in M)$ and a map $w_t : M \to T^\perp B$ $(t \in [0, T))$ by $w_t(\xi) := \tilde{r}_t(\xi)w_t^1(\xi)$ $(\xi \in M)$. Here we note that $c_t$
is surjective by the boundary condition in Theorem A, \( \bar{r}_0(\xi) = r_0(\pi(\xi)) \) and that \( c_0(\xi) = \pi(\xi) \) \((\xi \in M)\). Define a function \( \bar{r}_t \) over \( B \) by \( \bar{r}_t(x) := \bar{r}_t(\xi) \) \((x \in B)\) and a map \( \bar{c}_t : B \to B \) by \( \bar{c}_t(x) := c_t(\xi) \) \((x \in B)\), where \( \xi \) is an arbitrary element of \( M \cap S^+_B \). It is clear that they are well-defined. This map \( \bar{c}_t \) is not necessarily a diffeomorphism. In particular, if \( \bar{c}_t \) is a diffeomorphism, then \( M_t := \bar{f}_t(M) \) is equal to the tube \( \exp^\perp(t_\gamma(B)) \), where \( r_t := \bar{r}_t \circ \bar{c}_t^{-1} \). It is easy to show that, if \( c_t(\xi_1) = c_t(\xi_2) \), then \( \bar{r}_t(\xi_1) = \bar{r}_t(\xi_2) \) and \( \pi(\xi_1) = \pi(\xi_2) \) hold. Also, let \( a : M \times [0, T) \to G \) be a smooth map with \( a(\xi, t)p_0 = a_t(\xi) (\xi, t) \in M \times [0, T) \). In this section, we shall investigate the evolutions of the function \( r_t \) and \( \bar{c}_t \). Define \( \bar{f} : M \times [0, T) \to \overline{M}, r : B \times [0, T) \to \mathbb{R} \), \( w^1 : M \times [0, T) \to M \) and \( c : M \times [0, T) \to B \) by \( \bar{f}(\xi, t) := f_t(\xi), r(x, t) := \bar{r}_t(x), w^1(\xi, t) := w^1_t(\xi), w(\xi, t) := w_t(\xi) \) and \( c(\xi, t) := c_t(\xi) \), respectively, where \( \xi \in M, x \in B \) and \( t \in [0, T) \). Fix \((\xi_0, t_0) \in M \times [0, T) \) and set \( x_0 := \pi(\xi_0) \). Clearly we have

\[
(4.1) \quad \bar{f}_t \left( \frac{\partial}{\partial t} \right)_{(\xi_0, t_0)} = \frac{d}{dt} \bigg|_{t=t_0} \bar{f}(\xi_0, t) = \frac{d}{dt} \bigg|_{t=t_0} \exp^\perp(w(\xi_0, t)).
\]

Let \( J \) be the Jacobi field along the geodesic \( \gamma_w(\xi_0, t_0) \) (of direction \( w(\xi_0, t_0) \)) with \( J(0) = c_0 \left( \frac{\partial}{\partial t} \right)_{(\xi_0, t_0)} \) and \( J'(0) = \nabla^\perp \bigg|_{t=t_0} w(\xi_0, \cdot) \), where \( \nabla^\perp \) is the covariant derivative along the curve \( s \mapsto c(\xi_0, s) \) with respect to the Riemannian connection \( \nabla \) of

![Figure 2.](image-url)
This Jacobi field \( J \) is described as
\[
J(s) = P_{\gamma w(\xi_0,t_0)} |_{[0,s]} \left( D_{\gamma w(\xi_0,t_0)}^c \left( c_s \left( \frac{\partial}{\partial t} \right)_{(\xi_0,t_0)} \right) \right) + P_{\gamma w(\xi_0,t_0)} |_{[0,s]} \left( sD_{\gamma w(\xi_0,t_0)}^s \left( \frac{\nabla}{\partial t} \bigg|_{t=t_0} w(\xi_0,\cdot) \right) \right).
\]

According to (4.1), we have
\[
\tilde{f}_* \left( \frac{\partial}{\partial t} \right)_{(\xi_0,t_0)} = J(1)
\]
\[
= P_{\gamma w(\xi_0,t_0)} |_{[0,1]} \left( D_{w(\xi_0,t_0)}^c \left( c_s \left( \frac{\partial}{\partial t} \right)_{(\xi_0,t_0)} \right) \right) + P_{\gamma w(\xi_0,t_0)} |_{[0,1]} \left( D_{w(\xi_0,t_0)}^s \left( \frac{\nabla}{\partial t} \bigg|_{t=t_0} w(\xi_0,\cdot) \right) \right).
\]

On the other hand, we have
\[
\frac{\nabla}{\partial t} \bigg|_{t=t_0} w(\xi_0,\cdot) = \frac{d\tilde{r}(\xi_0,t)}{dt} \bigg|_{t=t_0} w^1(\xi_0,t_0) + \tilde{r}(\xi_0,t_0) \frac{\nabla}{\partial t} \bigg|_{t=t_0} w^1(\xi_0,t).
\]

From (4.2) and (4.3), we have
\[
\tilde{f}_* \left( \frac{\partial}{\partial t} \right)_{(\xi_0,t_0)} \equiv \frac{d\tilde{r}(\xi_0,t)}{dt} \bigg|_{t=t_0} P_{\gamma w(\xi_0,t_0)} |_{[0,1]} \left( w^1(\xi_0,t_0) \right) \mod \text{Span}\{P_{\gamma w(\xi_0,t_0)} |_{[0,1]}(w^1(\xi_0,t_0))\}^\perp.
\]

**Notation.** Set
\[
T_1 := \sup\{t' \in [0,T) \mid M_t := f_t(M) (0 \leq t \leq t') : \text{tubes over } B\}.
\]
(Note that \( \tilde{c}_t \) (0 \( \leq t < T_1 \)) are diffeomorphisms.)

Assume that \( t_0 < T_1 \). According to Lemma 3.2 in [Ko1], we have
\[
N(\xi_0,t_0) \equiv \frac{1}{\sqrt{1 + \| (D_{w(\xi_0,t_0)}^c)^{-1} (\text{grad } r_{t_0})_{c(\xi_0,t_0)} \|^2}} P_{\gamma w(\xi_0,t_0)} |_{[0,1]}(w^1(\xi_0,t_0)) \mod \text{Span}\{P_{\gamma w(\xi_0,t_0)} |_{[0,1]}(w^1(\xi_0,t_0))\}^\perp.
\]

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Hence, from (0.1), (0.2), (2.23) and (3.5), we obtain the following relation

\[ \tilde{J}_* \left( \frac{\partial}{\partial t} \right)_{(\xi, t_0)} \equiv \frac{\int_B r_{m'} \psi_{r_{i_0}} \Delta_{DB}}{\int_B r_{m'} \psi_{r_{i_0}} \Delta_{DB}} - \rho_{r_{i_0}}(c(\xi_0, t_0)) \]

\[ \sqrt{1 + \|\left( \frac{D_{w(\xi_0, t_0)}}{D_{w(\xi_0, t_0)}} \right)^{-1}(\text{grad } r_{i_0})_{c(\xi_0, t_0)}\|^2} \times P_{\gamma_{w(\xi_0, t_0)}} (w^1(\xi_0, t_0)) \]

\[ (\text{mod Span}\{ P_{\gamma_{w(\xi_0, t_0)}} (w^1(\xi_0, t_0)) \}^\perp), \]

where \( m' = \text{codim } F \). According to the condition (i) in Theorem A, we have \( \tau_{c(\xi_0, t_0)}^{-1}(\text{grad } r_{i_0})_{c(\xi_0, t_0)} \in P_{k_0, 0} \) for some \( k_0 \in K \cup \{0\} \). From (4.4), (4.6) and the arbitrariness of \((\xi_0, t_0)\), we can derive the following relation:

\[ \frac{\partial \tilde{\gamma}}{\partial t}(\xi, t) = \frac{\int_B r_{m'} \psi_{r_{i_0}} \Delta_{DB}}{\int_B r_{m'} \psi_{r_{i_0}} \Delta_{DB}} - \rho_{r_{i_0}}(c(\xi, t)) \]

\[ \sqrt{1 + \|\left( \frac{D_{w(\xi_0, t_0)}}{D_{w(\xi_0, t_0)}} \right)^{-1}(\text{grad } r_{i_0})_{c(\xi, t)}\|^2} \]

\[ \cos(\sqrt{k_0} b \tilde{\gamma}(\xi, t)) + \|\text{grad } r_{i_0}(\xi, t)\|^2 \]

\[ \times \left( \frac{\int_B r_{m'} \psi_{r_{i_0}} \Delta_{DB}}{\int_B r_{m'} \psi_{r_{i_0}} \Delta_{DB}} - \rho_{r_{i_0}}(c(\xi, t)) \right) \]

\[ ((\xi, t) \in M \times [0, T_1)). \]

Next we shall calculate \( \frac{\partial c}{\partial t} \). Denote by \( D_t \) the closed domain surrounded by \( P \) and \( M_t \), and \( \tilde{D} \) the maximal domain in \( T_{\perp} B \) containing the 0-section such that \( \exp^{|\tilde{D}} \) is a diffeomorphism into \( M \). From \( c_t(\xi) = (\pi \circ (\exp^{|\tilde{D}})^{-1})(f_t(\xi)) \), we have

\[ \frac{\partial c}{\partial t}(\xi, t) = (\pi \circ (\exp^{|\tilde{D}})^{-1})_* \left( \frac{df_t(\xi)}{dt} \right) \]

\[ = (\tilde{H}_t - H_{(\xi, t)})(\pi \circ (\exp^{|\tilde{D}})^{-1})_* (N_{(\xi, t)}). \]

On the other hand, we have

\[ N_{(\xi, t)} = \frac{P_{\gamma_{w(\xi, t)}}(w^1(\xi, t) - (D_{w(\xi, t)})^{-1}(\text{grad } r_{i_0})_{c(\xi, t)}) \}

\[ \sqrt{1 + \|\left( \frac{D_{w(\xi, t)}}{D_{w(\xi, t)}} \right)^{-1}(\text{grad } r_{i_0})_{c(\xi, t)}\|^2} \]

\[ \times \cos(\sqrt{k_0} b \tilde{\gamma}(\xi, t)) + \|\text{grad } r_{i_0}(\xi, t)\|^2 \]

by Lemma 3.2 in [Ko1]. Let \( s \mapsto J(s) \ (0 \leq s < \infty) \) be the Jacobi field along \( \gamma_{w^1(\xi, t)} \) with \( J(0) = (\text{grad } r_{i_0})_{c(\xi, t)} \) and \( J'(0) = 0 \). Then we have

\[ (\pi \circ (\exp^{|\tilde{D}})^{-1})_* (J(s)) = J(0) = (\text{grad } r_{i_0})_{c(\xi, t)}. \]
On the other hand, according to (3.1), \(J(s)\) is described as
\[
J(s) = P_{\gamma_{w_{\xi,t}}(\xi,t)}(D^{\gamma_{w_{\xi,t}}(\xi,t)}(\text{grad } r_t)_{c(\xi,t)}) = \cos(\sqrt{k_0}b_0)s)P_{\gamma_{w_{\xi,t}}(\xi,t)}(\text{grad } r_t)_{c(\xi,t)},
\]
where we use \(\tau^{-1}_{c(\xi,t)}(\text{grad } r_t)_{c(\xi,t)} \in \mathfrak{p}_{k_0}\beta\). Therefore we obtain
\[
(\pi \circ (\exp^{|\hat{\gamma}})^{-1})_s(P_{\gamma_{w_{\xi,t}}(\xi,t)}(\text{grad } r_t)_{c(\xi,t)}) = \frac{1}{\cos(\sqrt{k_0}b_0)}(\text{grad } r_t)_{c(\xi,t)}.
\]

Also, it is clear that \((\pi \circ (\exp^{|\hat{\gamma}})^{-1})_s(P_{\gamma_{w(\xi,t)}}(w^1(\xi,t)) = 0\). Therefore we obtain
\[
(\pi \circ (\exp^{|\hat{\gamma}})^{-1})_s(N_{\xi,t}) = -\frac{\text{grad } r_t)_{c(\xi,t)}}{\cos(\sqrt{k_0}b_0)(\xi,t))\sqrt{\cos^2(\sqrt{k_0}b_0)(\xi,t)) + ||(\text{grad } r_t)_{c(\xi,t)}||^2}.
\]

From (2.23), (3.5), (4.8) and (4.10), we can derive
\[
\frac{\partial c}{\partial t}(\xi,t) = \rho_{r_t}(c(\xi,t)) = \frac{\int_B r^0 c_{r_t} d\nu}{\int_B r^0 c_{r_t} d\nu}(\text{grad } r_t)_{c(\xi,t)}.
\]

Next we shall calculate the Laplacian \(\Delta_t \hat{\gamma}_t\) with respect to the metric \(g_t\) on \(M\) induced by \(f_t\). Let \(\mathcal{H}_t^t\) be the horizontal distribution on \(t_{r_t}(B)\) and \(\mathcal{H}_t^t\) be the distribution \(M\) with \(f_t(\mathcal{H}_t^t) = \exp^{|\hat{\gamma}}(\mathcal{H}_t^t)\). Also, let \(\mathcal{V}_t^t\) be the vertical distribution on \(t_{r_t}(B)\). Note that \(f_t(\mathcal{V}_t^t) = \exp^{|\hat{\gamma}}(\mathcal{V}_t^t)\). For \(X \in T_{c(\xi,t)}B\), denote by \(\tilde{X}_{w_{\xi,t}}(\xi)\) the natural lift of \(X\) to \(w_t(\xi) \in t_{r_t}(B)\) and let \(\tilde{X}_t^\xi\) be the element of \(T_\xi M\) with \(f_t(\tilde{X}_t^\xi) = \exp^{|\hat{\gamma}}(\tilde{X}_{w_{\xi,t}}(\xi))\). Note that \(\{\tilde{X}_t^\xi | X \in T_{c(\xi,t)}B\} = \mathcal{H}_t^t\). Denote by \(\nabla\) the Riemannian connection of \(g_t\). Fix \(x_0 \in B\) and \(\xi_0 \in S_{x_0}^1 B\). Let \((e_1 \cdots e_m)\) be an orthonormal tangent frame of \(B\) at \(c_{\xi_0}(x_0) = c_{\xi}(\xi_0)\) and \(\gamma_t\) the geodesic in \(B\) with \(\gamma_t'(0) = e_1\), where we take \(e_1\) as \(e_1 = (\text{grad } r_t)_{c_{\xi_0}(x_0)} \in \mathbb{R}^n\) in the case of \(k_0\) \(\neq 0\). Since \(\tau_{c(\xi,t)}^{-1}(e_1) \in \mathfrak{p}_{k_0}\beta\), we may assume that \(\tau_{c(\xi,t)}^{-1}(e_1) \in \mathfrak{p}_{k_i}\beta\) for some \(k_i \in \mathcal{K}\) \((i = 1, \cdots, m^H)\). Note that \(k_1 = k_0\). Also, let \(\gamma_t^i\) be the natural lift of \(\gamma_t\) to \(M_t\) starting from \(f_t(\xi_0)\). Note that \(\gamma_t^i(s) = f_t(\gamma_t^i(s)) = \bigl(\gamma_t^i(s)_{f^{-1}_t(\gamma_t(s))}\bigr)\). Set
\[
(E_t^i)_s := \frac{\gamma_t^i(s)_{f^{-1}(\gamma_t^i(s))}}{||\gamma_t^i(s)_{f^{-1}(\gamma_t^i(s))}||_t} (i = 1, \cdots, m^H).
\]
orthonormal base of the horizontal subspace $\mathcal{H}^*_{\xi_0}$ with respect to $(g_t)_{\xi_0}$ because we take $e_1$ as above.

![Diagram](image)

**Figure 3.**

Also, let $\{E^t_{m^H+1}, \ldots, E^t_n\}$ be a local orthonormal frame field (with respect to $g_t$) of the vertical distribution $\mathcal{V}$ around $\xi_0$. Since $\tilde{\gamma}^t_i$ is a pregeodesic in $M_t$, we have

$$\nabla^t_{E^t_i} E^t_i = 0 \quad (i = 1, \ldots, m^H).$$  

(4.12)

Also, since $\tilde{r}_t$ is constant along $M_t \cap F^1_x$ for each $x \in B$, we have

$$E^t_i \tilde{r}_t = 0 \quad (i = m^H + 1, \ldots, n)$$

(4.13)

From (4.12) and (4.13), we have

$$\Delta_t \tilde{r}_t \xi_0 = \sum_{i=1}^n \nabla^t_{E^t_i(0)} \nabla^t_{E^t_i(0)} \tilde{r}_t = \sum_{i=1}^{m^H} (E^t_i(0)(E^t_i \tilde{r}_t)).$$

(4.14)

Also we have $(\Delta r_t)_{f_t(\xi_0)} = \sum_{i=1}^{m^H} e_i(\gamma'_i r_t)$. By using (2.21), we can show

$$\langle E^t_i \rangle(0)(E^t_i \tilde{r}_t) = \frac{e_i(\gamma'_i r_t)}{\|\gamma'_i(0)\|^2} = \frac{e_i(\gamma'_i r_t)}{\cos^2(\sqrt{k_i c_t(\xi_0)}) + (e_i r_t)^2}.$$

(4.15)
Let \((t, s) \mapsto \xi_{t,s}^i\) be the smooth curve in \(M\) such that \(f_t(\xi_{t,s}^i) = \gamma_{t}^i(s)\). By using (4.9), we can show
\[
\gamma_{t}^i r_t = -\sqrt{1 + \|(D_{w(\xi_{t,s}^i,t)})^{-1}(\text{grad } r_t)\|_{c(\xi_{t,s}^i,t)}}^2\times \langle P_{\gamma_{w(\xi_{t,s}^i,t)}[0,1]}(D_{w(\xi_{t,s}^i,t)}(\gamma_{t}^i)), N \rangle.
\]
Furthermore, since \(B\) is reflective, we can show
\[
e_i(\gamma_t r_t) = -\left\langle \left(\frac{\nabla}{ds}\bigg|_{s=0} D_{w(\xi_{t,s}^i,t)} \right)(e_i), D_{w(\xi_0,t)}(\text{grad } r_t)_{c(\xi_0,t)} \right\rangle
\]
\[-\left\langle D_{w(\xi_0,t)}^co e_i, \left(\frac{\nabla}{ds}\bigg|_{s=0} D_{w(\xi_{t,s}^i,t)} \right)(\text{grad } r_t)_{c(\xi_0,t)} \right\rangle.
\]
For \(w_1, w_2 \in T_p \overline{M}\), we define a linear transformation \(Q_{w_1, w_2}\) by
\[
Q_{w_1, w_2} := \tau_p \circ \left( \sum_{k=0}^{\infty} \frac{1}{(2k)!} \sum_{j=1}^{2k} (\text{ad}(w_1))^{j-1} \circ \text{ad}(w_2) \circ (\text{ad}(w_1)^{2k-j}) \right) \circ \tau_p^{-1}.
\]
Then we have
\[
\frac{\nabla}{ds}\bigg|_{s=0} D_{w(\xi_{t,s}^i,t)} = Q_{w(\xi_0,t), \xi_{t,s}^i,t}.
\]
On the other hand, since \(s \mapsto \frac{w(\xi_{t,s}^i,t)}{||w(\xi_{t,s}^i,t)||}\) is parallel with respect to the normal connection of \(f_t\) and \(F\) is reflective, it is parallel with respect to \(\nabla\). Hence we have
\[
\frac{\nabla}{ds}\bigg|_{s=0} w(\xi_{t,s}^i,t) = \frac{e_i r_t}{r_t(c(\xi_0,t))} w(\xi_0,t).
\]
From this relation and (4.18), we can derive
\[
\frac{\nabla}{ds}\bigg|_{s=0} D_{w(\xi_{t,s}^i,t)}^co = \frac{e_i r_t}{r(\xi_0,t)} (D_{w(\xi_0,t)}^si \circ \text{ad}(w(\xi_0,t))^2).
\]
From this relation, (4.17) and the arbitrariness of \(\xi_0\), we can derive
\[
e_i(\gamma_t r_t) = \frac{2(e_i r_t)}{r(\xi_0,t)} ((D_{w(\xi,t)}^co \circ D_{w(\xi,t)}^si \circ \text{ad}(w(\xi,t))^2)(\text{grad } r_t)_{c(\xi,t)}, e_i)
\]
\[
= \left\{ \begin{array}{ll}
-\sqrt{e_k} b \sin(2\sqrt{e_k} b r(\xi,t)) || (\text{grad } r_t)_{c(\xi,t)} ||^2 & (i = 1) \\
0 & (i = 2, \ldots, m^H)
\end{array} \right.
\]
$((\xi,t) \in M \times [0, T_1))$. From (4.14), (4.15), (4.19) and the arbitrariness of $\xi_0$, we obtain

$$
(\triangle t \widehat{r}_t)(\xi) = - \frac{||\text{grad } r_t c(\xi, t)||^2 \sqrt{\varepsilon k_0 b \sin(2 \sqrt{\varepsilon k_0 b t}(\xi, t))}}{\cos^2(\sqrt{\varepsilon k_0 b t}(\xi, t)) + ||\text{grad } r_t c(\xi, t)||^2}
$$

$((\xi, t) \in M \times [0, T_1))$. From (4.7) and (4.20), the following evolution equation is derived.

**Lemma 4.1.** The radius functions $r_t$'s satisfies the following evolution equation:

$$
\frac{\partial \widehat{r}_t}{\partial t}(\xi, t) - (\triangle t \widehat{r}_t)(\xi) = \cos(\sqrt{\varepsilon k_0 b t}(\xi, t))
$$

$$
\sqrt{\cos^2(\sqrt{\varepsilon k_0 b t}(\xi, t)) + ||\text{grad } r_t c(\xi, t)||^2}
$$

$$
\times \left( \int_F \! \! F r_t^m v \rho_v \psi_r d\nu_F \right) - \rho_r c(\xi, t))
$$

$$
+ \frac{||\text{grad } r_t c(\xi, t)||^2 \sqrt{\varepsilon k_0 b \sin(2 \sqrt{\varepsilon k_0 b t}(\xi, t))}}{\cos^2(\sqrt{\varepsilon k_0 b t}(\xi, t)) + ||\text{grad } r_t c(\xi, t)||^2}
$$

$((\xi, t) \in M \times [0, T_1)).$

Denote by $\text{grad}_t \widehat{r}_t$ the gradient vector field of $\widehat{r}_t$ with respect to $g_t$. Also, the following relation holds between $\text{grad } r_t$ and $\text{grad}_t \widehat{r}_t$.

**Lemma 4.2.** The gradient vector $(\text{grad } r_t)_c(\xi, t)$ is described as

$$
(\text{grad } r_t)_c(\xi, t) = \frac{1}{\cos(\sqrt{\varepsilon k_0 b t}(\xi, t))} \{ (P^{-1}_{g_{w(\xi, t)}} \circ f_{\text{t}})(\text{grad}_t \widehat{r}_t)_c(\xi, t)
$$

$$
- \frac{||\text{grad } r_t c(\xi, t)||^2}{\widehat{r}(\xi, t)} w(\xi, t) \}. \}
$$

Also, the squared norm $||\text{grad } r_t c(\xi, t)||^2$ is described as

$$
||\text{grad } r_t c(\xi, t)||^2 = \frac{1}{2} \left\{ - \cos^2(\sqrt{\varepsilon k_0 b t}(\xi, t)) \right\}
$$

$$
+ \sqrt{\cos^4(\sqrt{\varepsilon k_0 b t}(\xi, t)) + 4||\text{grad}_t \widehat{r}_t||^2_c},
$$

where $|| \cdot ||_t$ is the norm of $(\cdot)$ with respect to $g_t$. 

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Proof. Denote by \( \tilde{\gamma}(w(\xi,t)) \) the natural lift of \( (\cdot) \) to \( w(\xi,t) \in f_t(M) = \exp^*(t_r(B)) \). It is clear that

\[
 f_t((\text{grad}_t \tilde{r}_t)_{\xi}) = \exp^*((\text{grad}_t c(\xi,t))_{w(\xi,t)}). 
\]

Hence it follows from (2.21) that

\[
 f_t((\text{grad}_t \tilde{r}_t)_{\xi}) = \text{pr}_{\tilde{W}} \left( D_w^o(\text{grad}_t c(\xi,t)) + \frac{||\text{grad}_t c(\xi,t)||^2}{\tilde{r}(\xi,t)} w(\xi,t) \right) 
\]

From this relation, we can derive the first relation in this lemma. Furthermore, by noticing \( \tilde{g}(\text{grad}_t c(\xi,t), w(\xi,t)) = 0 \), we can derive the second relation in this lemma from the first relation. q.e.d.

5 Uniformly boundedness of the radius function and the average mean curvature

We use the notations in Introduction and Sections 1-4. Also, denote by \( g_t \) the metric induced from \( \tilde{g} \) by \( f_t \) and \( \nabla^t \) the Levi-Civita connection of \( g_t \). In this section, we shall show that the radius function \( \tilde{r}_t \) and the average mean curvature \( \overline{H}_t \) are uniformly bounded. Denote by \( r_{\text{cut}} \) the cut radius of the symmetric space \( F_{\perp} \), where we note that \( r_{\text{cut}} \) is independent of the choice of \( x \in F \). Also, denote by \( \pi_F^\perp \) the bundle projection of the normal bundle \( T^\perp F \) of \( F \). Set

\[
 \tilde{W} := \{ \exp^*(\xi) \mid \xi \in T^\perp F \text{ s.t. } ||\xi|| < r_{\text{cut}} \}
\]

and let \( \text{pr}_F \) be the submersion of \( W \) onto \( F \) defined by \( \text{pr}_F(\exp^*(\xi)) := \pi_F^\perp(\xi) \), where \( \xi \in \tilde{W} \). Let \( \tilde{r} : \overline{M} \to \mathbb{R} \) be the distance function from \( F \), where we note that \( \tilde{r}(\exp^*(\xi)) = ||\xi|| \) holds for \( \xi \in \tilde{W} \). Define a function \( \tilde{\psi} \) over \([0,r_{\text{cut}}]\) by

\[
 \tilde{\psi}(s) := \left( \prod_{k=0}^{2} \left( \frac{\sin(\sqrt{kbs})}{\sqrt{kbs}} \right)^{m_k^\perp} \right) \left( \prod_{k \in \mathbb{K} \cup \{0\}} \cos^{m_k^\perp}(\sqrt{kbs}) \right). 
\]

From (2.21) and (3.2), we can give the following explicit descriptions of the volume element \( dv_{\overline{M}} \) of \( \overline{M} \) and the volume \( \text{Vol}(D_t) \).
Proposition 5.1. (i) The volume element \( dv_{\bigwedge} \) is given by

\[
(dv_{\bigwedge})_p = (\overline{\psi} \circ \overline{r})(p) \left( \left( (\exp^* |_{\tilde{S}(pr_F(p), \overline{r}(p))})^{-1} \right) \ast dv_{\tilde{S}(pr_F(p), \overline{r}(p))} \right)_p \\
\wedge (d\overline{r})_p \wedge (pr_F^* dv_F)_p \quad (p \in W)
\]

(ii) The volume \( \text{Vol}(D_i) \) is given by

\[
\text{Vol}(D_i) = v_m v \int_{p \in B} \left( \int_0^{r_i(p)} s^m \overline{\psi}(s) ds \right) dv_F.
\]

Denote by \( \exp_F^x \) the exponential map of \( F \) at \( x \) and \( \tilde{S}'(x, a) \) the hypersphere of radius \( a \) in \( T_x F \) centered the origin. Since \( B \) is star-shaped with respect to \( x_0 \), \( B \) is described as

\[
B = \{ \exp_F^x(zX) \mid X \in \tilde{S}'(x_0, 1), 0 \leq z \leq r^B(X) \}
\]

for some positive function \( r^B \) over \( \tilde{S}'(x_0, 1) \). Set \( r^B_{\text{min}} := \min_{X \in \tilde{S}'(x_0, 1)} r^B(X) \) and \( r^B_{\text{max}} := \max_{X \in \tilde{S}'(x_0, 1)} r^B(X) \). Since \( F \) is a symmetric space, we can define the operators corresponding to \( D_{w^o}^F \) and \( D_{w}^{si} \) \( (w \in TM) \) for each \( X \in TF \). Denote by \( (DF)^{si}_{X} \) and \( (DF)^{si}_{X} \) the operators. Define a function \( \psi^F_{\text{min}} \) (resp. \( \psi^F_{\text{max}} \)) over \( \mathbb{R} \) by

\[
\psi^F_{\text{min}}(z) := \min_{X \in \tilde{S}'(x_0, 1)} \det((DF)^{si}_{X}) \quad (\text{resp. } \psi^F_{\text{max}}(z) := \max_{X \in \tilde{S}'(x_0, 1)} \det((DF)^{si}_{X})).
\]

Then we can describe the volume element \( dv_B \) of \( B \) and can estimate the volume \( \text{Vol}(B) \) as follows.

Proposition 5.2. (i) The volume element \( dv_B \) is given by

\[
(dv_B)_{\exp^F_{x_0}(zX)} = \det((DF)^{si}_{zX}) \left( \left( (\exp^F_{x_0} | \tilde{S}'(x_0, z))^{-1} \right) \ast dv_{\tilde{S}'(x_0, z)} \right)_{\exp^F_{x_0}(zX)} \\
\wedge (d\overline{z})_{\exp^F_{x_0}(zX)}
\]

for \( \exp^F_{x_0}(zX) \in B \), where \( \overline{z} \) is the distance function from \( x_0 \) in \( F \).

(ii) The volume \( \text{Vol}(B) \) is estimated as

\[
v_m u_{-1} \int_0^{r^B_{\text{min}}} z^{mH-1} \psi^B_{\text{min}}(z) dz \leq \text{Vol}(B) \leq v_m u_{-1} \int_0^{r^B_{\text{max}}} z^{mH-1} \psi^B_{\text{max}}(z) dz.
\]

Proof. In similar to (3.2), we have

\[
(\exp^F_{x_0})_*(Y) = P_{zX} ((DF)^{si}_{zX}(Y)),
\]
where $X \in \overline{S}'(x_0, 1)$ and $Y \in T_x \overline{S}'(x_0, z)$. From this relation, (5.4) follows directly. The estimate (5.5) follows directly from (5.3). 

q.e.d.

Define a function $\delta : [0, r_{\text{cut}}] \to \mathbb{R}$ by $\delta(s) := \int_0^s s^{m-1} \psi(s) ds$. It is clear that $\delta$ is increasing. Set

$$b_t := \delta^{-1} \left( \frac{\text{Vol}(D_t)}{v_m \text{Vol}(B)} \right).$$

Denote by $(r_t)_{\text{max}}$ (resp. $(r_t)_{\text{min}}$) the maximum (resp. the minimum) of $r_t$. Then we have

$$\delta(b_t) v_m \text{Vol}(B) = \text{Vol}(D_t) = v_m \int_{B} \left( \int_{0}^{r_t(p)} s^{m-1} \psi(s) ds \right) dv_F \geq v_m \text{Vol}(B) \delta((r_t)_{\text{min}})$$

and hence $\delta(b_t) \geq \delta((r_t)_{\text{min}})$, that is, $b_t \geq (r_t)_{\text{min}}$. By using Proposition 3.2, we have

$$\text{Vol}(M_t) = v_m \int_{B} r_t^{m-1} \psi_{r_t} dv_F \geq v_m \int_{B} r_t^{m-1} (\psi \circ r_t) dv_F \geq v_m v_m^{\mu-1} \int_{(r_t)_{\text{min}}}^{(r_t)_{\text{max}}} s^{m-1} \psi(s) ds \geq v_m v_m^{\mu-1} (\delta((r_t)_{\text{max}}) - \delta(b_t)),$$

where we use $\psi_{r_t}(x) \geq \overline{\psi}(r_t(x))$ ($x \in B$) and $\delta(b_t) \geq \delta((r_t)_{\text{min}})$. On the other hand, since $\text{Vol}(D_t)$ preserves invariantly along the volume-preserving mean curvature flow and $\text{Vol}(M_t)$ is decreasing along the flow, we have $\text{Vol}(D_t) = \text{Vol}(D_0)$ and $\text{Vol}(M_t) \leq \text{Vol}(M_0)$. Hence we have

$$(r_t)_{\text{max}} \leq \delta^{-1} \left( \frac{\text{Vol}(D_0)}{v_m \text{Vol}(B)} + \frac{\text{Vol}(M_0)}{v_m v_m^{\mu-1}} \right).$$

That is, we obtain

$$(5.8) \sup_{(x,t) \in B \times [0,T]} r_t(x) < \delta^{-1} \left( \frac{\text{Vol}(D_0)}{v_m \text{Vol}(B)} + \frac{\text{Vol}(M_0)}{v_m v_m^{\mu-1}} \right).$$

Thus we obtain the following result.

**Proposition 5.3.** The family $\{r_t\}_{t \in [0,T]}$ of the radius functions is uniformly bounded. In more detail, we have $\sup_{t \in [0,T]} \max_{B} r_t < r_{\text{cut}}$. 

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For uniformly boundedness of the average mean curvatures $|H|$, we have the following result.

**Proposition 5.4.** If $0 < a_1 \leq \tilde{r}_t(x) \leq a_2 < r_{\text{cut}}$ holds for all $(x,t) \in M \times [0,T_0]$ ($T_0 < T$), then $\max_{t \in [0,T_0]} |H_t| \leq C(a_1,a_2)$ holds for some constant $C(a_1,a_2)$ depending only on $a_1$ and $a_2$.

**Proof.** According to (3.6), we have

$$H = \frac{\int_B r^m \rho_t \psi_t dv_B}{\int_B r^m \psi_t dv_B}.$$  

Also, according to (2.23), the function $r^m \rho_t \psi_t$ is described as

$$r^m \rho_t \psi_t = \Delta(\eta(r,||\text{grad } r||^2)) + r^m \psi_t \tilde{\eta}_r$$

for some function $\eta$ over $\mathbb{R} \times [0,\infty)$ and some function $\tilde{\eta}_r$ over $B$ depending only on $r$. Furthermore, we can show that $|\tilde{\eta}_r| \leq C(a_1,a_2)$ for some positive constant $C(a_1,a_2)$ depending only on $a_1$ and $a_2$ if $a_1 \leq r \leq a_2$, where $a_1$ and $a_2$ are constants. Hence, by using the divergence theorem, we can derive

$$\int_B r^m \rho_t \psi_t dv_B \leq \int_B \Delta(\eta(r,||\text{grad } r||^2)) dv_B + C(a_1,a_2) \int_B r^m \psi_t dv_B$$

$$= - \int_{\partial B} d(\eta(r,||\text{grad } r||^2))(N_B)dv_B + C(a_1,a_2) \int_B r^m \psi_t dv_B.$$  

Since $\text{grad } r_t = 0$ and $\nabla \text{grad } r_t = 0$ hold along $\partial B$ for all $t \in [0,T)$ by the assumption, we see that $N_B r_t = N_B (||\text{grad } r_t||^2) = 0$ holds for all $t \in [0,T)$. Hence we have

$$\int_B r^m \rho_t \psi_t dv_B \leq C(a_1,a_2) \int_B r^m \psi_t dv_B.$$  

Therefore we obtain $\max_{t \in [0,T_0]} |H_t| \leq C(a_1,a_2)$.

q.e.d.

### 6 Proof of Theorem A

We use the notations in Introduction and Sections 1-5. In this section, we shall prove Theorem A. Let $T_1$ be as in Section 4. Define a function $\tilde{u}_t : M \to \mathbb{R}$ ($t \in [0,T)$) by

$$\tilde{u}_t(\xi) := \tilde{g}(N(\xi,t), P_{\gamma_{w(\xi,t)}}|_{[0,1]}(w^1(\xi,t))) \quad (\xi \in M)$$

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and a map $\tilde{\nu}_t : M \to \mathbb{R}$ by $\tilde{\nu}_t := \frac{1}{\tilde{u}_t} (0 \leq t < T_1)$. Define a map $\tilde{u} : M \times [0, T) \to \mathbb{R}$ by $\tilde{u}(\xi, t) := \tilde{u}_t(\xi)$ ($((\xi, t) \in M \times [0, T))$ and a map $\tilde{v} : M \times [0, T) \to \mathbb{R}$ by $\tilde{v}(\xi, t) := \tilde{v}_t(\xi)$ $((\xi, t) \in M \times [0, T))$. Define a function $u_t$ (resp. $\tilde{v}_t$) over $B$ by $\tilde{u}_t(x) := \tilde{u}_t(\xi)$ ($x \in B$) (resp. $\tilde{v}_t(x) := \tilde{v}_t(\xi)$ ($x \in B$), where $\xi$ is an arbitrary element of $M \cap S_x^1 B$. It is clear that these functions are well-defined. Set $u_t := \tilde{u}_t \circ \tilde{c}_t^{-1}$ and $v_t := \tilde{v}_t \circ \tilde{c}_t^{-1}$. We have only to show $\inf_{(x,t) \in B \times [0,T)} u(x, t) > 0$, that is, $\sup_{(x,t) \in B \times [0,T)} v(x, t) < \infty$. In the sequel, assume that $t < T_1$, where $T_1$ is as in Assumption of Section 4. From (4.9), we have

$$
\tilde{u}_t(\xi) = \frac{\cos(\sqrt{\varepsilon k_0 b R}(\xi, t))}{\sqrt{\cos^2(\sqrt{\varepsilon k_0 b R}(\xi, t)) + ||(\grad r_t)_{c(\xi,t)}||^2}}.
$$

It is easy to show that the outward unit normal vector field $N_t$ satisfies the following evolution equation:

$$
\frac{\partial N}{\partial t} = f_t*(\grad H_t).
$$

For simplicity, we set $\tilde{w}^1(\xi, t) := P_{\gamma_{w(\xi, t)}|[0,1]}([w^1(\xi, t)])$. We calculate $\tilde{w}^1(\xi, t)$. Define a map $\delta : [0, T_1) \times \mathbb{R} \to \mathcal{M}$ by

$$
\delta(t, s) := \exp_{c(\xi,t)}(sw^1(\xi, t)).
$$

Then we have $\left.\frac{\partial \delta}{\partial s}\right|_{s = \tilde{r}(\xi, t)} = \tilde{w}^1(\xi, t)$. For a fixed $t \in [0, T_1)$, $\tilde{Y}_t : s \mapsto \left(\frac{\partial \delta}{\partial t}\right)(t, s)$ is the Jacobi field along the geodesic $\gamma_{w^1(\xi, t)}$. Since $\tilde{Y}_t(0) = \left(\frac{\partial c}{\partial t}\right)(\xi, t)$ and

$$
\tilde{Y}_t'(0) = \left(\nabla^{\delta}_{\frac{\partial \delta}{\partial t}}\right)_{s=0} = \left(\nabla^{\delta}_{\frac{\partial \delta}{\partial s}}\right)_{s=0} = \nabla^{\delta}_{\frac{\partial \delta}{\partial s}} w^1(\xi, t) = \nabla^{\perp_B}_{\tilde{Y}_t(0)} w^1(\xi, t)
$$

by the reflectivity of $F$, it follows from (3.1) that

$$
\tilde{Y}_t(s) = P_{\gamma_{w^1(\xi, t)}|[0, s]} \left(D_{sw^1(\xi, t)}^{co}(\frac{\partial c}{\partial t})(\xi, t) + sD_{sw^1(\xi, t)}^{si}(\nabla^{\perp_B}_{\tilde{Y}_t(0)} w^1(\xi, t))\right),
$$

where $\nabla^{\perp_B}$ is the normal connection of $B$. This implies together with (4.11) that

$$
\tilde{Y}_t(s) = - \frac{(\Pi_t - H(\xi, t)) \cos(\sqrt{\varepsilon k_0 b R})}{\tilde{v}(\xi, t) \cos^2(\sqrt{\varepsilon k_0 b R}(\xi, t))} P_{\gamma_{w^1(\xi, t)}|[0, s]}((\grad r_t)_{c(\xi,t)}) + sP_{\gamma_{w^1(\xi, t)}|[0, s]}(D_{sw^1(\xi, t)}^{si}(\nabla^{\perp_B}_{\tilde{Y}_t(0)} w^1(\xi, t))).
$$
In particular, we have
\[
\tilde{Y}_t(\tilde{r}(\xi, t)) = -\frac{H_t - H(\xi, t)}{\tilde{v}(\xi, t) \cos(\sqrt{\varepsilon k_0 b^2}(\xi, t))} P_{\gamma_{w(\xi,t)}}((\text{grad}_t r_t)_{\text{c}(\xi,t)}) \\
+ \tilde{r}(\xi, t) P_{\gamma_{w(\xi,t)}}[D^\text{co}_{w(\xi,t)}((\nabla_{\text{Y}^0})^B w^1(\cdot, t)))].
\]

Also, we have
\[
\left(\frac{\partial \hat{w}^1}{\partial t}\right)(\xi, t) = \left(\nabla^t_{\delta} \frac{\partial \delta}{\partial s}\right)_{s = \tilde{r}(\xi, t)} = \nabla^t_{\delta} Y_t \\
= \frac{(H_t - H(\xi, t))\sqrt{\varepsilon k_0 b^2}(\xi, t))}{\tilde{v}(\xi, t) \cos^2(\sqrt{\varepsilon k_0 b^2}(\xi, t))} P_{\gamma_{w(\xi,t)}}((\text{grad}_t r_t)_{\text{c}(\xi,t)}) \\
+ P_{\gamma_{w(\xi,t)}}[D^\text{co}_{w(\xi,t)}((\nabla_{\text{Y}^0})^B w^1(\cdot, t)))].
\]

Hence we obtain
\[
\hat{g} \left( \left( N(\xi, t), \left(\frac{\partial \hat{w}^1}{\partial t}\right)(\xi, t) \right) \right) = -\frac{(H_t - H(\xi, t))\sqrt{\varepsilon k_0 b^2}(\xi, t))}{\tilde{v}(\xi, t) \cos^2(\sqrt{\varepsilon k_0 b^2}(\xi, t))} \left(\frac{\partial \hat{w}^1}{\partial t}\right)(\xi, t)^2 \right)^2. \]

From (6.2) and (6.6), we obtain
\[
\frac{\partial \hat{u}}{\partial t} = \hat{g}(f_s(\text{grad} H_t), \hat{w}^1) \\
- \frac{(H_t - H(\xi, t))\sqrt{\varepsilon k_0 b^2}(\xi, t))}{\tilde{v}(\xi, t) \cos^2(\sqrt{\varepsilon k_0 b^2}(\xi, t))} \left(\frac{\partial \hat{w}^1}{\partial t}\right)(\xi, t)^2 \right)^2.
\]

Fix $x_0 \in B$, $\xi_0 \in S^1_{x_0}B$ and $t_0 \in [0, T_1)$. Let $(e_1 \cdots e_{m^H})$, $\gamma_i$, $\tilde{\gamma}_i$, $(E_i^t)$, $\xi_{i,s}^t$ ($i = 1, \cdots, m^H$) and $\{E_i^{t_{m^H+1}}, \cdots, E_i^{t_k}\}$ be as in Section 4. Define a map $\delta_i : [0, T_1) \times (-\varepsilon, \varepsilon) \rightarrow M$ ($i = 1, \cdots, m^H$) by
\[
\delta_i(t, s) := \exp_{\gamma_i(t)}(tw(\xi_{i,s}, t_0)),
\]
where $\varepsilon$ is a small positive number. Set $Y_i^{s_0} := \frac{\partial \delta_i}{\partial s}$. Since $Y_i^{s_0}$ is the Jacobi field along $\gamma_{w(\xi_{i,s}^{t,s}, t_0)}$ with $Y_i^{s_0}(0) = \gamma_i^t(s_0)$ and $(Y_i^{s_0})^t(0) = (e_i r_t)w^1(\xi_{i,s}^{t,s}, t_0)$, it is described as
\[
Y_i^{s_0}(t) = P_{\gamma_{w(\xi_{i,s}^{t,s}, t_0)}}(D_{tw(\xi_{i,s}^{t,s}, t_0)}^{co}(\gamma_i^t(s_0)) + tD_{tw(\xi_{i,s}^{t,s}, t_0)}^{si}(e_i r_t)w^1(\xi_{i,s}^{t,s}, t_0))) \\
= P_{\gamma_{w(\xi_{i,s}^{t,s}, t_0)}}(D_{tw(\xi_{i,s}^{t,s}, t_0)}^{co}(\gamma_i^t(s_0)) + t(e_i r_t)w^1(\xi_{i,s}^{t,s}, t_0))
\]
Hence we have

\[
\nabla^{f_{t_0}}_{(E_{t_0}^{s_0})_{t_0}} \tilde{w}_{t_0}^1 = (Y_{s_0}^t)'(1) \\
= P_{\gamma_{w(\xi_{t_0},t_0))}^0} (D_{\frac{\partial}{\partial s}} \circ \text{ad}(w(\xi_{t_0},t_0)))^2 e_1 + (e_{i(t_0)}) \tilde{w}_{(\xi_{i(t_0),s_0},t_0)},
\]

where \(\nabla^{f_{t_0}}\) is the pull-back connection of \(\nabla\) by \(f_{t_0}\). Let \(\alpha_j : (-\varepsilon, \varepsilon) \to M\) \((j = mH + 1, \cdots, n)\) be the integral curve of \(E_{t_0}^{s_0}\) with \(\alpha_j(0) = \xi_0\), where \(\varepsilon\) is a small positive number. Define a map \(\tilde{\gamma}_j : [0, T_1) \times (-\varepsilon, \varepsilon) \to \overline{M}\) \((j = mH + 1, \cdots, n)\) by

\[
\tilde{\gamma}_j(t, s) := \exp_{x_0}(t(f_{t_0} \circ \alpha_j)(s)).
\]

Set \(\tilde{Y}_{s_0}^t := \frac{\partial \tilde{\gamma}_j}{\partial s} \bigg|_{s=s_0}\). Since \(\tilde{Y}_{s_0}^t\) is the Jacobi field along \(\gamma_{w(\alpha_j(s),t_0)}\) with \(\tilde{Y}_{s_0}^t(0) = 0\) and \((\tilde{Y}_{s_0}^t)'(0) = (\exp_{x_0})^{-1}(f_{t_0}^*(E_{\alpha_j(s_0)}^{s_0}), t_0))\), it is described as

\[
\tilde{Y}_{s_0}^t(t) = P_{\gamma_{w(\alpha_j(s),t_0))}^0} \left(D_{\frac{\partial}{\partial s}} \circ \text{ad}(w(\alpha_j(s),t_0))^{-1}(f_{t_0}^*(E_{\alpha_j(s_0)}^{s_0}))) \right).
\]

For simplicity, we denote \((E_{t_0}^{s_0})_{\alpha_j(s)}\) by \((E_{t_0}^{s_0})_{\alpha_j}\). Hence we have

\[
\nabla^{f_{t_0}}_{(E_{t_0}^{s_0})_{s_0}} \tilde{w}_{t_0}^1 = (\tilde{Y}_{s_0}^t)'(1) \\
= P_{\gamma_{w(\alpha_j(s),t_0))}^0} \left(D_{\frac{\partial}{\partial s}} \circ \text{ad}(w(\alpha_j(s),t_0))^{-1}(f_{t_0}^*(E_{\alpha_j(s_0)}^{s_0}))) \right) = (E_{t_0}^{s_0})_{s_0}.
\]

Denote by \(\triangle \tilde{u_t}\) be the rough Laplacian of \(\tilde{u_t}\) with respect to \(g_t\). On the other hand, we have

\[
(\triangle_t \tilde{u}_{t_0})(\xi_0) = \sum_{i=1}^{n} \nabla^{f_{t_0}}_{(E_{t_0}^{s_0})_0} \nabla^{f_{t_0}}_{(E_{t_0}^{s_0})_0} g(N_{t_0}, \tilde{w}_{t_0}^1) \\
= \sum_{i=1}^{mH} \left( \frac{\partial}{\partial s} \bigg|_{s=0} (E_{t_0}^{s_0})_s (g(N_{t_0}, \tilde{w}_{t_0}^1)) \right) \\
= \tilde{g}(f_{t_0}^*(\text{grad} H_{t_0} \xi_0), \tilde{w}_{t_0}^1(\xi_0, t_0)) - ||A_t||^2(\xi_0) \tilde{u}_{t_0}(\xi_0) + \text{Tr}(A_t|_{\gamma_{w(\xi_0),t_0)}) \\
+ \sum_{i=1}^{mH} \tilde{g}(f_{t_0}^*(A_t((\bar{e}_{i_0})^{s_0}_0), \nabla^{f_{t_0}}_{(E_{i_0}^{s_0})_0} \tilde{w}_{(e_{i0}^{s_0})_0}^1),)
\]

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where we use \( \tilde{g}(N_{t_0}, \nabla f_{t_0}^{(E_j, t_0)} \hat{w}_{t_0}) = 0 \) \((i = 1, \cdots, n)\) and \( \nabla f_{t_0}^{(E_j, t_0)} \hat{w}_{t_0} = (E_j f_{t_0})_0 \) \((j = m + 1, \cdots, n)\). From (6.7) and (6.10), we have

\[
\frac{\partial \tilde{u}}{\partial t}_{t=t_0} = \triangle_{t_0} \tilde{u}_{t_0} + \|A_{t_0}\|^2 \tilde{u}_{t_0} - \text{Tr}(A_{t_0}\nu_{w(\xi_0, t_0)})
\]

\[
(6.11)
\]

\[
- \sum_{i=1}^2 \tilde{g}(f_{t_0}^{(A_{t_0}((\tilde{e}^i)_{\xi_0}), \nabla f_{t_0}^{(E_j, t_0)} \hat{w}_{t_0}))}
\]

\[
(\varepsilon_{k,t}-H_{(\xi,t)})\sqrt{\varepsilon k b \sin(\varepsilon k b \tilde{r}(\xi, t))}||\text{grad } r_{t_c(\xi, t)}||^2.
\]

From (2.24) and (6.1), we have

\[
(6.12)
\]

\[
\text{Tr}(A_{t_0}\nu_{w(\xi_0, t_0)}) = \tilde{u}(\xi_0, t_0) \sum_{k=0}^{2} \frac{m_k \sqrt{\varepsilon k b}}{\tan(\sqrt{\varepsilon k b \tilde{r}(\xi_0, t_0))}}.
\]

Fix \((\xi_0, t_0) \in M \times [0, T)\). From (2.5) and \(r_{t_c(\xi, t_0)}^{-1} e_i \in \mathbb{P}_{k_i, \beta}\), we have

\[
(6.13)
\]

\[
- \tilde{u}(\xi_0, t_0) \sqrt{\varepsilon k b} \tan(\sqrt{\varepsilon k b} \tilde{r}(\xi_0, t_0)) f_{t_0}^{(e_i, \nu_{w(\xi_0, t_0)})}
\]

\[
+ \tilde{u}(\xi_0, t_0)^3(e_i r_{t_0}) \sqrt{\varepsilon k b} \tan(\sqrt{\varepsilon k b} \tilde{r}(\xi_0, t_0))
\]

\[
\times f_{t_0}^{(e_i r_{t_0})}((D_{\zeta_{w(\xi_0, t_0)}}^{-1})^{-2}(\text{grad } r_{t_c(\xi_0, t_0)}))_{\xi_0}
\]

\[
- \tilde{u}(\xi_0, t_0)^2(P_{\zeta_{w(\xi_0, t_0)}} \circ r_{t_c(\xi_0, t_0)})(Z_{e_i, w(\xi_0, t_0)}(1))T.
\]

On the other hand, according to (2.21), we have

\[
(6.14)
\]

\[
f_{t_0}^{(X_{\xi_0})} = \cos(\sqrt{\varepsilon k b} \tilde{r}(\xi_0, t_0)) P_{\zeta_{w(\xi_0, t_0)}}(X) + (X r_{t_0}) \hat{w}_{t_0}(\xi_0, t_0)
\]

for any \(X \in T_{c(\xi, t_0)}B \cap r_{t_c(\xi, t_0)} \mathbb{P}_{k, \beta}\). By using (6.13) and (6.14), we can show

\[
(6.15)
\]

\[
f_{t_0}^{(A_{t_0}((\tilde{e}^i)_{\xi_0}))}
\]

\[
= - \tilde{u}(\xi_0, t_0) \sqrt{\varepsilon k b} \tan(\sqrt{\varepsilon k b} \tilde{r}(\xi_0, t_0)) P_{\zeta_{w(\xi_0, t_0)}}(e_i)
\]

\[
+ \tilde{u}(\xi_0, t_0)^3(e_i r_{t_0}) \sqrt{\varepsilon k b} \tan(\sqrt{\varepsilon k b} \tilde{r}(\xi_0, t_0))
\]

\[
\cos(\sqrt{\varepsilon k b} \tilde{r}(\xi_0, t_0))
\]

\[
\times P_{\zeta_{w(\xi_0, t_0)}}((\text{grad } r_{t_c(\xi_0, t_0)})_{\xi_0})
\]

\[
- \tilde{u}(\xi_0, t_0)^2(P_{\zeta_{w(\xi_0, t_0)}} \circ r_{t_c(\xi_0, t_0)})(Z_{e_i, w(\xi_0, t_0)}(1))T
\]

\[
- \tilde{u}(\xi_0, t_0) \sqrt{\varepsilon k b} \tan(\sqrt{\varepsilon k b} \tilde{r}(\xi_0, t_0)) (e_i r_{t_0}) \hat{w}_{t_0}(\xi_0, t_0)
\]

\[
\tilde{u}(\xi_0, t_0)^3 \sqrt{\varepsilon k b} \tan(\sqrt{\varepsilon k b} \tilde{r}(\xi_0, t_0)) (e_i r_{t_0}) ||\text{grad } r_{t_0}||^2 \hat{w}_{t_0}(\xi_0, t_0).
\]
On the other hand, from (6.8), we have
\[
(6.16)
\nabla f_{t_0}^{i_0} \tilde{w}_{t_0}^1 = \sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0) \sin(\sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0)) P_{\gamma_{t_0}^{i_0}}(e_i) + (e_\varepsilon r_{t_0}) \tilde{w}_{t_0}^1(\xi_0, t_0).
\]

From (6.15) and (6.16), we obtain
\[
(6.17)
\nabla (f_{t_0}^i, (A_{t_0}^{i_0}((\xi_0, t_0)), \nabla_{\xi_0}^{i_0} \tilde{w}_{t_0}^1)) \\
= -\tilde{u}(\xi_0, t_0) \cos(\sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0)) + \tilde{u}(\xi_0, t_0), t_0)^3 \sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0) + (e_\varepsilon r_{t_0})^2 \\
\times (\sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0) \tan(\sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0)) - 1) \\
- \tilde{u}(\xi_0, t_0)^2 \sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0) \sin(\sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0)) \\
\times \sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0) \tan(\sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0)) - 1) \\
\times \frac{\cos(\sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0))}{\cos(\sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0))} \\
\times ((\tau_{c(\xi_0, t_0)}(Z_{e_i w(\xi_0, t_0)}(1), e_i)) \\
\quad \times ((\tau_{c(\xi_0, t_0)}(Z_{e_i w(\xi_0, t_0)}(1))) r_{t_0}).
\]

Also, from (2.19), we have
\[
(6.18)
Z_{e_i w(\xi_0, t_0)}(1) \\
\equiv \cos(\sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0)) \tau_{c(\xi_0, t_0)}^{-1}\left(\nabla_{\xi_0}^{i_0} \left(2(\gamma_{t_0}^{i_0}) - (\gamma_{t_0}^{i_0})^2(\gamma_{t_0}^{i_0})\right) \right) \\
\quad + \left(1 - \frac{1}{\cos(\sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0))} + \tan(\sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0)) \right) \\
\quad \times \frac{2\varepsilon_0 r_{t_0}}{\tilde{r}(\xi_0, t_0)} \tau_{c(\xi_0, t_0)}^{-1}\left(\gamma_{t_0}^{i_0} \left(\gamma_{t_0}^{i_0}\right)^2(\gamma_{t_0}^{i_0}) \right) (\text{mod} \, p^\perp).
\]

Hence we have
\[
\nabla (\tau_{c(\xi_0, t_0)}(Z_{e_i w(\xi_0, t_0)}(1), e_i)) \\
\quad \equiv \nabla (\nabla_{\xi_0}^{i_0} \left(2(\gamma_{t_0}^{i_0}) - (\gamma_{t_0}^{i_0})^2(\gamma_{t_0}^{i_0})\right) \right) \\
\quad + \left(1 - \frac{1}{\cos(\sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0))} + \tan(\sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0)) \right) 2\varepsilon_0 r_{t_0}^2 \tilde{r}(\xi_0, t_0).
\]

Also, we have
\[
\nabla_{\xi_0}^{i_0} \left(2(\gamma_{t_0}^{i_0}) - (\gamma_{t_0}^{i_0})^2(\gamma_{t_0}^{i_0})\right) \\
\quad \equiv \nabla_{\xi_0}^{i_0} \left(\frac{1}{\cos^2(\sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0))} (\gamma_{t_0}^{i_0})^2(\gamma_{t_0}^{i_0})\right) \\
\quad \equiv 2\sin(\sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0)) \cos(\sqrt{\varepsilon k_i b_{t_0}}(\xi_0, t_0)) \nabla_{e_i} \gamma_{t_0}^{i_0}(\xi_0, t_0).
\]
From these relations, we can derive
\[
\bar{g}(\gamma_c(\xi_0, t_0), (Z_{e_i}, w(\xi_0, t_0))((1)), e_i) \\
= \left( \frac{2 \sin(\sqrt{\varepsilon k_0 b \tilde{r}(\xi_0, t_0)})}{\cos^2(\sqrt{\varepsilon k_0 b \tilde{r}(\xi_0, t_0)}) + 2(1 - \frac{\cos(\sqrt{\varepsilon k_0 b \tilde{r}(\xi_0, t_0)})}{\cos(\sqrt{\varepsilon k_0 b \tilde{r}(\xi_0, t_0)}) + \tan(\sqrt{\varepsilon k_0 b \tilde{r}(\xi_0, t_0)})})} \right) \tilde{r}(\xi_0, t_0) \\
\times \|\text{grad} \\ t_0 \| ^2 \delta_{i1} + \frac{\bar{g}(\nabla e_i, \text{grad} r_{t_0}, e_i)}{\cos(\sqrt{\varepsilon k_0 b \tilde{r}(\xi_0, t_0)})}
\]
where \( \delta_{i1} \) denotes the Kronecker’s delta. By using these relations, we obtain the following evolution equations for \( \hat{u}_t \) and \( \hat{v}_t \).

**Lemma 6.1.**

(i) The function \( \hat{u}_t \) satisfies the following evolution equation:
\[
\frac{\partial \hat{u}_t}{\partial t}(\xi, t) - (\Delta \hat{u}_t)(\xi) = \sum_{j=0}^{6} \eta_j(\hat{r}(\xi, t))\hat{u}(\xi, t)^j + \eta_{\Pi}(\hat{r}(\xi, t))\Pi_t \\
+ g_t((\text{grad}_i \hat{u}_t)\xi, X(\xi, t))
\]

((\xi, t) \in M \times [0, T_1]), where \( \eta_j \ (j = 0, \cdots, 6) \) and \( \eta_{\Pi} \) are some \( C^\omega \)-functions over [0, \( r_{\text{cut}} \]), and \( X(\xi, t) \) is some tangent vector of \( M \) at \( \xi \).

(ii) The function \( \hat{v}_t \) satisfies the following evolution equation:
\[
\frac{\partial \hat{v}_t}{\partial t}(\xi, t) - (\Delta \hat{v}_t)(\xi) = \sum_{j=-4}^{2} \tilde{\eta}_j(\hat{r}(\xi, t))\hat{v}(\xi, t)^j + \tilde{\eta}_{\Pi}(\hat{r}(\xi, t))\Pi_t \\
+ g_t((\text{grad}_i \hat{v}_t)\xi, \tilde{X}(\xi, t))
\]

((\xi, t) \in M \times [0, T_1]), where \( \tilde{\eta}_j \ (j = -4, \cdots, 2) \) and \( \tilde{\eta}_{\Pi} \) are some \( C^\omega \)-functions over [0, \( r_{\text{cut}} \]), and \( \tilde{X}(\xi, t) \) is some tangent vector of \( M \) at \( \xi \).

**Proof.** By simple calculations, we have
\[
\|(\text{grad} r_t)_{c(\xi, t)}\|^2 = \cos^2(\sqrt{\varepsilon k_0 b \tilde{r}(\xi, t)}) \left( \frac{1}{u(\xi, t)^2} - 1 \right)
\]
and
\[
g_t((\text{grad}_i \hat{u}_t)\xi, Y) = -\frac{\hat{u}(\xi, t)^3}{\cos^2(\sqrt{\varepsilon k_0 b \tilde{r}(\xi, t)})} \{ \bar{g}((\nabla e_i, Y) \text{grad} r_t, (\text{grad}_i \hat{u}_t)_{c(\xi, t)})) \\
+ \sqrt{\varepsilon k_0 b} \tan(\sqrt{\varepsilon k_0 b \tilde{r}(\xi, t)}) ||(\text{grad}_i \hat{u}_t)_{c(\xi, t)}||^2 \cdot g_t((\text{grad}_i \hat{r}_t)\xi, Y) \}
\]
for any tangent vector $Y$ of $M$ at $\xi$. The relation (6.20) follows directly from (6.11), (6.12), (6.17), (6.18), (6.19) and these relations. The relation (6.21) follows from (6.20) and the following relations:

$$\frac{\partial \hat{v}}{\partial t} = -\frac{1}{u_t^2} \frac{\partial \hat{u}}{\partial t}, \quad \text{grad} \hat{v}_t = -\frac{1}{u_t^2} \text{grad} \hat{u}_t \quad \text{and} \quad \Delta_t \hat{v}_t = -\frac{1}{u_t^2} \Delta_t \hat{u}_t + \frac{2}{u_t^2} \|\text{grad} \hat{u}_t\|^2.$$

q.e.d.

From Lemmas 4.1 and 6.1, we can derive the following evolution equation.

**Lemma 6.2.** Set $\hat{\phi} := e^{C\hat{r}}\hat{v}$, where $C$ is a fixed positive constant. This function $\hat{\phi}$ satisfies the following evolution equation:

$$\frac{\partial \hat{\phi}}{\partial t}(\xi,t) - (\Delta_t \hat{\phi}_t)(\xi) = \sum_{j=-4}^2 \bar{\eta}_j(\hat{r}(\xi,t))\hat{\phi}(\xi,t)^j + \bar{\eta}_{\Pi}(\hat{r}(\xi,t))\Pi_t$$

$$+ g_t((\text{grad}_t \hat{\phi}_t)_\xi, \bar{X}(\xi,t))$$

$$((\xi,t) \in M \times [0,T_1]), \quad \text{where} \quad \bar{\eta}_j \quad (j = -4, \cdots, 2) \quad \text{and} \quad \bar{\eta}_{\Pi} \quad \text{are some} \quad C^\omega \text{-functions over} \quad [0,r_{\text{cut}}), \quad \text{and} \quad \bar{X}(\xi,t) \quad \text{is some tangent vector of} \quad M \quad \text{at} \quad \xi.$$

**Proof.** By simple calculations, we have

$$e^{-C\hat{r}(\xi,t)} \left( \frac{\partial \hat{\phi}}{\partial t}(\xi,t) - (\Delta_t \hat{\phi}_t)(\xi) \right) = C\hat{v}(\xi,t) \left( \frac{\partial \hat{\phi}}{\partial \hat{r}}(\xi,t) - (\Delta_t \hat{\phi}_t)(\xi) \right) + \left( \frac{\partial \hat{v}}{\partial t}(\xi,t) - (\Delta_t \hat{v}_t)(\xi) \right)$$

$$- C^2 \| (\text{grad}_t \hat{r}_t)_\xi \|^2 \hat{v}(\xi,t) - 2C g_t((\text{grad}_t \hat{r}_t)_\xi, (\text{grad}_t \hat{v}_t)_\xi)$$

and

$$g_t((\text{grad}_t \hat{r}_t)_\xi, (\text{grad}_t \hat{v}_t)_\xi) = \frac{1}{C\hat{v}(\xi,t)} g_t((\text{grad}_t \hat{\phi}_t)_\xi, (\text{grad}_t \hat{v}_t)_\xi) \left( \frac{\partial \hat{r}}{\partial \hat{r}}(\xi,t) - (\Delta_t \hat{r}_t)(\xi) \right)$$

$$= \frac{1}{C\hat{v}(\xi,t)} \| (\text{grad}_t \hat{v}_t)_\xi \|^2.$$

According to (4.21), we have

$$\frac{\partial \hat{\phi}}{\partial t}(\xi,t) - (\Delta_t \hat{\phi}_t)(\xi) = \sum_{j=-4}^2 \bar{\eta}_j(\hat{r}(\xi,t))\hat{\phi}(\xi,t)^j + \bar{\eta}_{\Pi}(\hat{r}(\xi,t))\Pi_t$$

$$+ g_t((\text{grad}_t \hat{\phi}_t)_\xi, \bar{X}(\xi,t)).$$

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where \( \hat{\eta}_j \) \((j = -4, \cdots, 2)\) and \( \hat{\eta}_H \) are some \( C^\omega \)-functions over \([0, r_{\text{cut}}]\), and \( \hat{\phi}(\xi, t) \) is some tangent vector of \( M \) at \( \xi \). According to (6.21), \( \frac{\partial \hat{P}}{\partial t}(\xi, t) = (\Delta_t \hat{\nu}_t)(\xi) \) also is described similarly. These descriptions together with (6.23) and (6.24) derive the desired relation. q.e.d.

By using Propositions 5.3, 5.4 and Lemma 6.2, we shall prove Theorem A.

**Proof of Theorem A.** According to Proposition 5.3, we have \( \sup_{t \in [0, T_1]} \max_M \hat{r}_t < r_{\text{cut}} \).

By using this fact, we can derive

\[
\sup_{t \in [0, T_1]} \max_{\xi \in M} \hat{\eta}_j(\hat{r}(\xi, t)) < \infty \quad (j = -4, \cdots, 2)
\]

and

\[
\sup_{t \in [0, T_1]} \max_{\xi \in M} \hat{\eta}_H(\hat{r}(\xi, t)) < \infty.
\]

Also, we have

\[
\sup_{t \in [0, T_1]} \max_{\xi \in M} \hat{\eta}_1(\hat{r}(\xi, t)) < 0
\]

and

\[
\sup_{t \in [0, T_1]} \max_{\xi \in M} \hat{\eta}_2(\hat{r}(\xi, t)) \leq 0
\]

by choosing \( C \) suitably. Set

\[
a_r := \sup_{t \in [0, T_1]} \max_M \hat{r}_t,
\]

\[
C_j := \sup_{t \in [0, T_1]} \max_{\xi \in M} \hat{\eta}_j(\hat{r}(\xi, t)) < \infty \quad (j = -4, \cdots, 2),
\]

\[
C_H := \sup_{t \in [0, T_1]} \max_{\xi \in M} \hat{\eta}_H(\hat{r}(\xi, t))
\]

and

\[
C'_1 := - \sup_{t \in [0, T_1]} \max_{\xi \in M} \hat{\eta}_1(\hat{r}(\xi, t)).
\]

Take any \( t_0 \in [0, T) \). Set

\[
a(t_0) := \min_{t \in [0, t_0]} \min_{\xi \in M} \hat{r}(\xi, t).
\]

According to Proposition 5.4, we have

\[
\max_{t \in [0, t_0]} |\hat{P}| \leq \hat{C}(a(t_0), a_r)
\]
for some positive constant $\hat{C}(a(t_0), a_r)$ depending only on $a(t_0)$ and $a_r$. Suppose that $T_1 < t_0$. Since $\hat{v}(\xi, t) \geq 1$ and hence $\hat{\phi}(\xi, t)^{-1} \leq 1$ ($0 \leq t < T_1$), it follows from (6.22) that

$$
\frac{\partial \hat{\phi}}{\partial t}(\xi, t) - (\nabla_t \hat{\phi})(\xi) \leq -C_1' \hat{\phi}(\xi, t) + \sum_{j=-4}^0 C_j + C_H \hat{C}(a(t_0), a_r) + g_t((\nabla_t \hat{\phi}_t)_{\xi}, \bar{X}(\xi, t)) \quad (0 \leq t < T_1).
$$

For simplicity, set

$$
\tilde{C}(a(t_0), a_r) := \sum_{j=-4}^0 C_j + C_H \hat{C}(a(t_0), a_r).
$$

By the maximum principle, we can derive

$$
\sup_{t \in [0, T_1]} \max_{\xi \in M} \hat{\phi}(\xi, t) \leq \max \left\{ \max_{\xi \in M} \hat{\phi}_0(\xi), \frac{\tilde{C}(a(t_0), a_r)}{C_1'} \right\}.
$$

For simplicity, set

$$
\overline{C}(t_0) := \max \left\{ \max_{\xi \in M} \hat{\phi}_0(\xi), \frac{\tilde{C}(a(t_0), a_r)}{C_1'} \right\}.
$$

From $\hat{v}_t \leq \hat{\phi}_t$, we have

$$
\sup_{t \in [0, T_1]} \max_{\xi \in M} \hat{v}(\xi, t) \leq \overline{C}(t_0)
$$

and hence

$$
\inf_{t \in [0, T_1]} \min_{\xi \in M} \hat{u}(\xi, t) \geq \overline{C}(t_0)^{-1}.
$$

By the continuity, we have

$$
\inf_{t \in [0, T_1 + \varepsilon]} \min_{\xi \in M} \hat{u}(\xi, t) > 0
$$

for a sufficiently small positive number $\varepsilon$. This contradicts the definition of $T_1$. Hence we obtain $T_1 \geq t_0$. Furthermore, from the arbitrariness of $t_0$, we obtain $T_1 = T$. q.e.d.
7 Examples

Polars and meridians of symmetric spaces of compact type are reflective. If \( F \subset M \) is either a Helgason sphere (which is a meridian) or the corresponding polar of rank one symmetric spaces of compact type in Table 2, then \( R(\cdot, \xi)|_{T_x F} \) is the constant-multiple of the identity transformation of \( T_x F \) for any \( x \in F \) and for any \( \xi \in T_x^\perp F \). Also, if \( F \subset M \) is one of reflective submanifolds of a rank one symmetric spaces of non-compact type as in Table 3 (which are given as an orbit of the dual action of a Hermann action having a Helgason sphere or the corresponding polar in Table 2 as an orbit), then \( R(\cdot, \xi)|_{T_x F} \) is the constant-multiple of the identity transformation of \( T_x F \) for any \( \xi \in T_x^\perp F \), that is, the condition (i) in Theorem A holds for any \( r_t \).

In fact, we can confirm this fact as follows. In the cases where \( F \) is as in Tables 2 and 3, we have

\[ \mathcal{K} = \{1, 2\}, \ \ p_0 = a = b, \ \ p' = p_\beta \cap p' \quad \text{and} \quad p'^\perp = b + p_\beta \cap p'^\perp + p_{2\beta} \]

for any \( x \in F \) and any \( \xi \in T_x^\perp F \), where \( \mathcal{K}, \ p_\beta \) and \( p_{2\beta} \) are the quantities defined as in Section 2 for \( b = \text{Span}\{\tau_x^{-1} \xi\} \). That is, we have

\[ R((\text{grad } \psi)_x, \xi) = -\sqrt{\varepsilon} \beta (\tau_x^{-1} \xi)^2 (\text{grad } \psi)_x \]

for any positive function \( \psi \) over \( F \).

| \( M \) | \( F \) | \( F_x^\perp \) |
|------|-------|------------------|
| \( \mathbb{R}P_{n+1} \) | \( \mathbb{R}P^n \) (a Helgason sphere) | \( \mathbb{R}P^n \) (the corresponding polar) |
| \( \mathbb{C}P_{n+1} \) | \( \mathbb{C}P^n \) (a Helgason sphere) | \( \mathbb{C}P^n \) (the corresponding polar) |
| \( \mathbb{Q}P_{n+1} \) | \( \mathbb{Q}P^n \) (a Helgason sphere) | \( \mathbb{Q}P^n \) (the corresponding polar) |
| \( \mathbb{O}P^2 \) | \( \mathbb{O}P^n \) (a Helgason sphere) | \( \mathbb{O}P^n \) (the corresponding polar) |
| \( \mathbb{R}P_{n+1} \) | \( \mathbb{R}P^n \) (the corresponding polar) | \( \mathbb{R}S^1 \) (a Helgason sphere) |
| \( \mathbb{C}P_{n+1} \) | \( \mathbb{C}P^n \) (the corresponding polar) | \( \mathbb{C}S^2 \) (a Helgason sphere) |
| \( \mathbb{Q}P_{n+1} \) | \( \mathbb{Q}P^n \) (the corresponding polar) | \( \mathbb{Q}S^4 \) (a Helgason sphere) |
| \( \mathbb{O}P^2 \) | \( \mathbb{O}S^8 \) (the corresponding polar) | \( \mathbb{O}S^8 \) (a Helgason sphere) |

(\( \mathcal{M} \) : a rank one symmetric space of compact type)

Table 2.
If $F \subset M$ is one of meridians in irreducible rank two symmetric spaces of compact type in Table 4 and if $D$ is the corresponding distribution on $F$ as in Table 4, then $D_x$ is a common eigenspace of the family $\{R(\cdot, \xi)\xi \mid \xi \in T^\perp_x F\}$ for any $x \in F$. Hence the condition (i) in Theorem A holds for any $r_t$ with $(\text{grad } r_t)_x \in D_x (x \in B)$. In fact, we can confirm this fact as follows. First we consider the case of (1) - (4) in Tables 4 and 5. In these cases, we have $\mathcal{K} = \{1, 2\}$ and $\tau^{-1}_x(D_x) = p_{2\beta} \cap p'$ for any $x \in F$ and any $\xi \in T^\perp_x F$, where $\mathcal{K}$ and $p_{2\beta}$ are the quantities defined as in Section 2 for $b = \text{Span}\{\tau^{-1}_x\xi\}$. That is, we have

$$R((\text{grad } \psi)_x, \xi)\xi = -4\sqrt{\varepsilon^2 \beta (\tau^{-1}_x\xi)^2 (\text{grad } \psi)_x}$$

for any positive function $\psi$ over $F$ with $(\text{grad } \psi)_x \in D_x (x \in F)$. Next we consider the case of (5) in Tables 4 and 5. In these cases, $\mathcal{K} = \{1\}$ holds, for one of $D = TSp(1)'s$ in (5), $\tau^{-1}_x(D_x) = p_0 \cap p'$ $(x \in F, \xi \in T^\perp_x F)$ holds and, for another $D = TSp(1)'s$ in (5), $\tau^{-1}_x(D_x) = p_{\beta} \cap p'$ $(x \in F, \xi \in T^\perp_x F)$ holds, where $\mathcal{K}$, $p_0$ and $p_{\beta}$ are the quantities defined as in Section 2 for $b = \text{Span}\{\tau^{-1}_x\xi\}$. That is, in the first case, we have

$$R((\text{grad } \psi)_x, \xi)\xi = 0$$

for any positive function $\psi$ over $F$ with $(\text{grad } \psi)_x \in D_x (x \in F)$, and in the second case,

$$R((\text{grad } \psi)_x, \xi)\xi = -\sqrt{\varepsilon^2 \beta (\tau^{-1}_x\xi)^2 (\text{grad } \psi)_x}$$

$$(\mathcal{M} : \text{a rank one symmetric space of non-compact type})$$

Table 3.

| $M$   | $F$   | $F^\perp_x$ |
|-------|-------|-------------|
| $H^n$ | $H^1$ | $H^n$       |
| $\mathcal{C}H^{n+1}$ | $H^2$ | $\mathcal{C}H^n$ |
| $QH^{n+1}$ | $H^4$ | $QH^n$ |
| $\mathcal{O}H^2$ | $H^8$ | $H^8$ |
| $H^n$ | $H^n$ | $H^1$       |
| $\mathcal{C}H^{n+1}$ | $\mathcal{C}H^n$ | $H^2$ |
| $QH^{n+1}$ | $QH^n$ | $H^4$ |
| $\mathcal{O}H^2$ | $H^8$ | $H^8$ |


for any positive function \( \psi \) over \( F \) with \( (\text{grad} \, \psi)_x \in D_x \) \((x \in F)\).

\[
\begin{array}{|c|c|c|c|}
\hline
M & F & F \perp_x & D \\
\hline
(1) SU(3)/SO(3) & S^1 \cdot S^2 \text{ (meridian)} & \mathbb{R}P^2 \text{ (polar)} & TS^1 \\
(2) SU(6)/Sp(3) & S^1 \cdot S^5 \text{ (meridian)} & \mathbb{O}P^2 \text{ (polar)} & TS^1 \\
(3) SU(3) & S^1 \cdot S^3 \text{ (meridian)} & \mathbb{C}P^2 \text{ (polar)} & TS^1 \\
(4) E_6/F_4 & S^1 \cdot S^9 \text{ (meridian)} & \mathbb{O}P^2 \text{ (polar)} & TS^1 \\
(5) Sp(2) & Sp(1) \times Sp(1) \text{ (meridian)} & S^1 \text{ (polar)} & \text{One of } TS\text{Sp}(1)'s \\
\hline
\end{array}
\]

\((\overline{M} : \text{an irreducible rank two symmetric space of compact type})\)

Table 4.

\[
\begin{array}{|c|c|c|c|}
\hline
M & F & F \perp_x & D \\
\hline
(1) SL(3,\mathbb{R})/SO(3) & H^1 \times H^2 & H^2 & TH^1 \\
(2) SU^*(6)/Sp(3) & H^1 \times H^5 & \mathbb{O}H^2 & TH^1 \\
(3) SL(3,\mathbb{R}) & H^1 \times H^3 & \mathbb{C}H^2 & TH^1 \\
(4) E_6^{26}/F_4 & H^1 \times H^9 & \mathbb{O}H^2 & TH^1 \\
(5) Sp(2,\mathbb{C}) & Sp(1,\mathbb{C}) \times Sp(1,\mathbb{C}) & H^4 & \text{One of } TS\text{Sp}(1,\mathbb{C})'s \\
\hline
\end{array}
\]

\((\overline{M} : \text{an irreducible rank two symmetric space of non-compact type})\)

Table 5.

See [He] about the notations in these tables for example.

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Department of Mathematics, Faculty of Science
Tokyo University of Science, 1-3 Kagurazaka
Shinjuku-ku, Tokyo 162-8601 Japan
(koike@ma.kagu.tus.ac.jp)