Simulation of 2D Acoustic Wave Propagation with Absorbing Boundary Condition

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Abstract. In this work we design an algorithm to simulate the 2D acoustic wave (P wave) propagation in a medium with different regional velocities. We use the Generalized Conjugate Residual (GCR) algorithm to solve the trapezoidal time integration, where for a given time step of 0.01 seconds, the error is approximately 8% with respect to the reference. We further speed up the simulation by a factor of 900 by implementing a model reduction scheme using Proper Orthogonal Decomposition (POD). Lastly, to suppress the artificial reflections formed when the wave bounces off the fixed boundaries, we created two virtual absorbing layers by tapering the solution of nodes close to the corresponding boundaries after each time iteration. However, we note that the successful implementations of the model reduction and absorbing boundary condition are not compatible with each other and provide explanations for why and how to combine them in future work.

1. Introduction
Earthquakes generate seismic energy waves that propagate the earth’s interior and get recorded by geophones when they reach the surface, as shown in Fig. 1. The seismic recordings are then analysed to extract geological information about the Earth. The simulation of seismic wave propagation is widely used in the seismology community for various purposes. Sometimes, the synthetic data generated by simulation is treated as the theoretical replacement of real data to be tested with novel imaging processing techniques. For instance, techniques like the curvelet transform and slant stacking are used on the seismic data to analyse the Earth’s mantle [1,2]; however, the real data are usually contaminated with ambient noise and false instrumental response. Thus, synthetic data is first used before any work on the real data. Since different earthquakes generate different seismic data, seismologists wish to generate simulations for different scenarios as quickly and as accurately as possible. To cater to these demands, we focus on speeding up the simulation and mimicking the real-world scenarios by adopting model reduction and absorbing boundary conditions. However, we simplify the Earth’s geometry to a rectangular 2D grid due to the limited scope of this work.[3]
2. Problem formulation

Our work starts from the 1D wave equation:

\[
\frac{1}{v_p^2} \frac{d^2 q}{dt^2} = \frac{\partial^2 q}{\partial r^2}
\]  \hspace{1cm} (1)

where, \( q \equiv \text{particle displacement} \), \( v_p \equiv P - \text{wave velocity} \), \( r \equiv \text{spatial distance} \). We take \( \frac{d^2 q}{dt^2} \) equals \( \frac{dv}{dt} \) construct the state vector, and formulate the wave equation as:
where $u \equiv \text{excitation}$. The function is synthesized into a linear system, where $A$ is made of the finite difference discretization matrix $E$ and identity matrix $I$:

$$\frac{dx(t)}{dt} = Ax(t) + bu(t)$$ (3)

$$A = \begin{bmatrix} 0 & 0 \\ \frac{v_p^2}{\Delta r^2} E & I \end{bmatrix}$$ (4)

We setup the grid as a $m \times m$ network, where $m$ is the number of nodes.

- The state of our system is a vector of length $m^2$ and each node includes displacement $q$ and particle velocity $v$:

$$x = [q_1, ..., q_N, v_1, ..., v_N]^T$$ (5)

- The parameters of our components include:

$$p = [v_p, \Delta r, R, \Delta t]^T$$ (6)

where $v_p$ is the P-wave velocity, $\Delta r$ is the spatial increment, $R$ is the model distance, $s$ is the location of excitation, and $\Delta t$ is the time increment.

- The excitation of the system is:

$$u(t) = \cos\left(\frac{2\pi nt}{T}\right), t \in [0, t_0]$$ (7)

- Thus, the final set of equations are written explicitly as:

$$f_i(x, p, u) = \frac{dq_i}{dt} = v_i, \; i \in [1, N]$$ (8)

$$f_{N+1}(x, p, u) = \frac{dv_1}{dt} = v_p^2 \frac{-2q_1 + q_2}{\Delta r^2} + v_p^2 \frac{u_1}{\Delta r^2}$$ (9)

$$f_{N+i}(x, p, u) = \frac{dv_i}{dt} = v_p^2 \frac{q_{i-1} - 2q_i + q_{i+1}}{\Delta r^2} + v_p^2 \frac{u_i}{2\Delta r^2}$$ (10)

$$f_{N+N}(x, p, u) = \frac{dv_N}{dt} = v_p^2 \frac{q_{N-1} - 2q_N + q_{N+1}}{\Delta r^2} + v_p^2 \frac{u_N}{\Delta r^2}$$ (11)

- The output $y$ is the displacement of surface nodes as a function of both distance $i \times \Delta r$ and time $t$, which is known as the synthetic seismic recording at the surface.

$$y(t) = C^T x(t) = [q_1(t), q_2(t), ..., q_m(t)]^T$$ (12)

### 3. Fundamental numerical methods

#### 3.1. Model parameters

In this study, $\Delta r = 200m$, $\Delta t = 0.01s$. $v_p^{top} = 6600m/s$ for the top half block and $v_p^{bot} = 9900m/s$ for the bottom half block. The total horizontal and vertical distances are 18000m and 9000m respectively, which yield a total number of 4186 simulating points. The acoustic wave velocity $v_p$ is the physical property assigned to each node as constant. The scope of this study is to simulate the first 5 seconds of the wave propagation.

#### 3.2. ODE integration method

We compared the Forward Euler (FE) and Trapezoidal approximation methods, and found that Trapezoidal is preferred. During the FE progression, the system diverges easily because of the
accumulated errors, which resemble introducing fake energy into the system. Thus, the largest feasible \( \Delta t \) for FE is \( 10^{-6} \), which is computationally expensive. Trapezoidal on the other hand, increases \( \Delta t \) to \( 10^{-2} \).

The structure of algorithm is shown in Fig. 3. We use GCR iteration inside Newton method to solve for each time interval. Note that, the system is completely linear and does not require 6 Newton method, but it is retained for the future implementation of nonlinear terms, such as energy loss. In our case, GCR takes about 3.3 min to finish, compared to around 23 min for LU decomposition.

![Figure 3. Fundamental ODE structure.](image)

4. The technical challenge

We completed two technical challenges: model order reduction using the Proper Orthogonal Decomposition (POD) method and absorbing boundary conditions.

4.1. POD model order reduction

In seismology, only nodes at the surface matter because that is where geophones are located. Therefore, model reduction is desired to speed up the simulation by preserving only the input-output relationship between the excitation and surface nodes. With POD, singular value decomposition (SVD) is applied to matrix \( W \), constructed with a sufficient number of \( x \)'s generated by the original model (501 in this case). Projection matrix \( V_q \) is constructed by taking the singular vectors corresponding to \( q \) largest singular values, in our case \( q = 140 \). The model can then be reduced with \( \hat{A} \), \( \hat{b} \) and \( \hat{c} \) and can quickly and effectively handle different excitations as shown in Figure 4.

![Figure 4. POD: Singular values of SVD of matrix \( W \).](image)

**Algorithm 1** Trapezoidal Time Integration Method

| Require: | initial condition \( x(t_s) = x^{init} \) |
|---|---|
| Ensure: | \( x \) is the solution of \( \frac{dx}{dt} = f(x, p, u) \) |
function TRAPEZOIDAL($f(\cdot), x^{\text{init}}, p, u(\cdot), t_s, t_f \Delta t$)

\[ x^0 \leftarrow x^{\text{init}} \]
\[ t_l \leftarrow t_s \]
\[ l \leftarrow 1 \]

repeat \hspace{1cm} \triangleright Time iteration, index $l$

\[ k \leftarrow 0 \]
\[ \hat{x}^{l,k} \leftarrow \hat{x}^{l-1} \]
\[ \gamma \leftarrow \hat{x}^{l-1} + \frac{\Delta t}{2} f \left( \hat{x}^{l-1}, u(t_{l-1}) \right) \]

repeat \hspace{1cm} \triangleright Newton iteration, index $k$

\[ r_0 \leftarrow -\left[ \hat{x}^{l-1} - \frac{\Delta t}{2} f \left( \hat{x}^{l,k}, u(t_l) \right) - \gamma \right] \]
\[ j \leftarrow 0 \]

repeat \hspace{1cm} \triangleright GCR iteration, index $j$

\[ p_j \leftarrow r_j \]
\[ \text{compute image of residual } (r_j): A p_j = r_j - \frac{\Delta t}{2} \left( f \left( \hat{x}^{l,k} + \varepsilon r_j, u(t_l) \right) - f \left( \hat{x}^{l,k}, u(t_l) \right) \right) \]
\[ \alpha_j \leftarrow \langle r_j, A p_j \rangle \]
\[ \Delta x^{l,k+1}_j \leftarrow \Delta x^{l,k}_j + \alpha_j p_j \]
\[ r_{j+1} \leftarrow r_j + \alpha_j A p_j \]
\[ j \leftarrow j + 1 \]

until \[ \|r_j\| \text{ small} \]
\[ \hat{x}^{l,k+1} \leftarrow \hat{x}^{l,k} + \Delta x^{l,k+1} \]
\[ k \leftarrow k + 1 \]

until \[ \|\Delta x^{l,k}\|, \|\hat{x}^{l,k} - \frac{\Delta t}{2} f \left( \hat{x}^{l,k}, u(t_l) \right) - \gamma \| \text{ small} \]
\[ \hat{x}^l \leftarrow \hat{x}^{l,k} \]
\[ t_{l+1} \leftarrow t_l + \Delta t \]
\[ l \leftarrow l + 1 \]

until \[ t_l \geq t_f - t_s \]

return \[ [x^0 \ x^1 \ \ldots \ x^l] \]

Algorithm 2 POD model order reduction

Require: matrix $W = [\hat{x}^1 \ \hat{x}^2 \ \ldots \ \hat{x}^m]$ sampled from $x(t_i)$
Ensure: $m$ at least 200-300

function POD($A, b, c, W, q$)

compute $V_L$ through $\text{SVD}(W)$
\[ V_q \leftarrow \text{column 1 to } q \text{ of } V_L \]
\[ \hat{A} \leftarrow V_q^T A V_q \]
\[ \hat{b} \leftarrow V_q^T b \]
\[ \hat{c} \leftarrow V_q^T c \]

return $\hat{A}, \hat{b}, \hat{c}$

4.2. Absorbing boundary condition

The Earth rarely possesses vertical geological features, so when waves hit the vertical boundaries of our 2D grid, they should pass through instead of being reflected. To address this, we taper the displacement of 10% of nodes on both left and right sides. Those tapered nodes form the absorbing layers. It is
implemented by multiplying a tapering matrix $T$ to state $x(t_i)$ at the end of each time iteration, shown in Fig.5.

5. Results

Fig. 6 is a snapshot of results in time, and it shows a clear reflection at the depth of 4500 m, where an intermediate interface is introduced by setting the top and bottom blocks with different p-wave velocities. This is a good sign suggesting that our model is working properly.

(a) before POD
(b) after POD

Figure 7. Results of original (left) & reduced model (right)
Fig. 7 shows a comparison between the original model and the reduced model. Both look very similar, and specifically the $L^2$ error between the two is only 0.0124%. However, the reduced model only takes 0.22s to run, while the original one takes about 25 min, a speed up factor of 900.

Fig. 8 shows a comparison between the model with and without absorbing boundary condition. It is obvious to see, on the synthetic 2D seismic recording at the surface in the left panel, the straight-line features are being effectively suppressed, which are introduced by the side reflections. Also, in the right panel, the simulation result with absorbing boundary condition (bottom) no longer contains as much side-reflection energy as the one without absorbing boundary condition (top).

6. Technical discussion

In Fig. 8, the reciprocal of the slope where the hyperbola converges is 6000 m/s for the top and 9000 m/s for the bottom, which is close to the true P-wave velocities assigned to the top and bottom blocks ($v_{p,\text{top}} = 6600 \text{m/s}, v_{p,\text{bottom}} = 9900 \text{m/s}$). The velocity approximations together with the half-depth reflection suggest that our model is working properly, and the results are valid. Although our reduced model makes the simulation 900 times faster than the original model, one can easily speed it up more. In this study, we pick 140 largest singular values, and achieve a low error of 0.0124%, which may not be necessary in other applications. Thus, choosing fewer singular values by making $q$ smaller is feasible to further speed up the simulation. Since we only take 10% of nodes from both sides to be the absorbing layer, the result still experiences faint side reflections, shown in 9. If one increases the size of the layer, the performance of suppression will be better, but it will become more computationally expensive. Implementing the absorbing boundary condition using tapering window has its limit. The tapering process after each time iteration is manipulating matrix $A$ in GCR, but the model reduction requires $A$ to be unchanged throughout the simulation. Thus, our implementation of absorbing boundary condition is not compatible with the POD model reduction. The address it, more advanced implementation of absorbing boundary condition is needed. For instance, Neumann boundary condition is working directly on matrix $A$, but it requires one to consider first order spatial derivative, which adds another layer of constrains to the model [4-7].
7. Conclusions
Though this work addresses a physical problem without direct technical concerns, we note several limitations.

- Seismic waves contain energy loss through the conversion from kinetic to thermal energy. We used a constant loss term - 10% reduction along the diagonal of nodal matrix $E$ – to approximate this process. However, in reality, the attenuation should be non-linear terms. Thus, any amplitude analysis should not be performed with this model.
- The current model is tapered manually with sharp suppression in 10% of the nodes at both sides. Reflections still exist from those borders. Our method is not yet perfect given the incompatibility with other techniques. Future work is needed to provide better performance.
- The bottom boundary is an artificial boundary, and the reflection produced is not geologically meaningful. In application, the property of this interface depends on the difference of density across materials from both sides.
- $\Delta t = 0.01 \text{s}$ for Trapezoidal integration introduces of 8% with respect to the true solution. If more accuracy is needed, one may need to make $\Delta t$ smaller, but the computation will be more demanding.
- The harmonic solutions of the wave equation are sinusoidal function of both time and space domain. In this work the wave equation is simulated numerically, and we are sampling from the real solution spatially and temporally. Thus, governed by the Nyquist frequency rule, the sampling frequency of both time and space have to be greater than twice the frequency of the excitation $u$ in time and space.

8. Conclusions
In this study, we design a modelling algorithm to quickly and accurately simulate 2D acoustic wave propagation. To solve the time integration of 2D wave equation, we use the Trapezoidal method, which allows us to set a large time step while maintaining a reasonable level of error. We adopt the POD model reduction technique to speed up the simulation by a factor of 900. Also, we implement absorbing boundary conditions to reduce the artificial reflections from vertical boundaries. The simulation results can be used in seismological research to quickly assess the quality of real data. Moreover, they can be treated as constructive a-priori information for inverse problems. It is also worth mentioning that our implementation of absorbing boundary condition is not compatible with POD model reduction. Thus, more work needs to be done in the future to combine the two.

9. References
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