INTERPOLATION BETWEEN $L_0(M, \tau)$ AND $L_\infty(M, \tau)$

J. HUANG AND F. SUKOCHEV

Abstract. Let $M$ be a semifinite von Neumann algebra with a faithful semifinite normal trace $\tau$. We show that the symmetrically $\Delta$-normed operator space $E(M, \tau)$ corresponding to an arbitrary symmetrically $\Delta$-normed function space $E(0, \infty)$ is an interpolation space between $L_0(M, \tau)$ and $\mathcal{M}$, which is in contrast with the classical result that there exist symmetric operator spaces $E(M, \tau)$ which are not interpolation spaces between $L_1(M, \tau)$ and $\mathcal{M}$. Besides, we show that the $K$-functional of every $X \in L_0(M, \tau)$ coincides with the $K$-functional of its generalized singular value function $\mu(X)$. Several applications are given, e.g., it is shown that the pair $(L_0(M, \tau), \mathcal{M})$ is $K$-monotone when $\mathcal{M}$ is a non-atomic finite factor.

1. Introduction

Recall the Calkin correspondence between symmetrically $\Delta$-normed operator spaces and symmetrically $\Delta$-normed function spaces introduced in [17]. Let $E(0, \infty)$ be an arbitrary symmetrically $\Delta$-normed function space equipped with a $\Delta$-norm $\| \cdot \|_E$ (see Section 2) and let $M$ be an arbitrary semifinite von Neumann algebra equipped with a faithful normal semifinite trace $\tau$. Then,

$$E(M, \tau) := \{ X \in S(M, \tau) : \mu(X) \in E(0, \infty) \}, \quad \| X \|_{E(M, \tau)} := \| \mu(X) \|_E$$

is a symmetrically $\Delta$-normed operator space, where $S(M, \tau)$ is the space of all $\tau$-measurable operators affiliated with $M$. Moreover, if $E(0, \infty)$ is complete, then $E(M, \tau)$ is also complete (see [17, Theorem 3.8], see also [22, 34] for the Banach case and the quasi-Banach case). For brevity, we omit below the term “operator” and refer just to symmetrically $\Delta$-normed spaces.

Let $A(M, \tau)$ and $B(M, \tau)$ be two symmetrically $\Delta$-normed spaces. A symmetrically $\Delta$-normed space $E(M, \tau)$ is said to be intermediate for $A(M, \tau)$ and $B(M, \tau)$ if the continuous embeddings

$$A(M, \tau) \cap B(M, \tau) \subset E(M, \tau) \subset A(M, \tau) + B(M, \tau)$$

hold. Let $E(M, \tau)$ be a symmetrically $\Delta$-normed space intermediate between $A(M, \tau)$ and $B(M, \tau)$. If every linear operator on $A(M, \tau) + B(M, \tau)$ which is bounded from $A(M, \tau)$ to $A(M, \tau)$ and $B(M, \tau)$ to $B(M, \tau)$ is also a bounded operator from $E(M, \tau)$ to $E(M, \tau)$, then $E(M, \tau)$ is called an interpolation space between the spaces $A(M, \tau)$ and $B(M, \tau)$.

Interpolation function spaces have been widely investigated (see e.g. [1, 4, 18, 23, 26, 31]) since Mityagin [28] and Calderón [5] gave characterizations of the class of all interpolation spaces with respect to $(L_1(0, \infty), L_\infty(0, \infty))$ (see also [9, 10] for results...
in the noncommutative setting). Among several real interpolation methods, the K-method of interpolation linked to the so-called K-functional is very important (we refer [2][26][27][36] for applications of K-functionals in different areas). Calculating the K-functionals for a given couple of spaces is very important in the K-method [25]. In [18], the K-functionals for the couple \( L_0(0,\infty) \) and \( L_\infty(0,\infty) \) are obtained. We give a description of the K-functional of every element \( X \in (L_0 + L_\infty)(\mathcal{M}, \tau) \) in terms of singular value function as well as in terms of its distribution function, showing that the K-functional of \( X \) coincides with the K-functional of its generalized singular value function \( \mu(X) \) (see Section 2, which extends [18, Proposition 3]).

It is well-known (see e.g. [3][28], see also [9][12][23]) that the operator space \( E(\mathcal{M}, \tau) \) corresponding to a fully symmetric (see e.g Section 5) function space \( E(0,\infty) \) is an interpolation space between \( L_1(\mathcal{M}, \tau) \) and \( \mathcal{M} \). In particular, there exist symmetric normed spaces which are not interpolation spaces between \( L_1(\mathcal{M}, \tau) \) and \( L_\infty(\mathcal{M}, \tau) \) (see [23] Chapter II, § 4.2 and § 5.7]). However, in this paper, it is shown that if we consider \( L_0(\mathcal{M}, \tau) \) (the set of all \( \tau \)-measurable operators having finite-trace support) instead of \( L_1(\mathcal{M}, \tau) \), then the operator space \( E(\mathcal{M}, \tau) \) corresponding to an arbitrary symmetrically \( \Delta \)-normed function space \( E(0,\infty) \) is necessarily an interpolation space between \( L_0(\mathcal{M}, \tau) \) and \( \mathcal{M} \), which is a non-commutative version of results in [18] (see also [2]).

As an application of the previous result, we describe the orbits and K-orbits for an arbitrary \( A \in S(\mathcal{M}, \tau) \) in the last section. It is shown that the unit balls of K-orbits do not coincide with the unit balls of orbits in the pair \( (L_0(\mathcal{M}, \tau), \mathcal{M}) \), which generalises [2] Theorem 4. In [2], it is asserted that the commutative pair \( (L_0(0,\infty), L_\infty(0,\infty)) \) is not K-monotone, that is, K-orbits do not necessarily coincide with orbits in the pair \( (L_0(0,\infty), L_\infty(0,\infty)) \). However, it is known that this assertion is incorrect and \( (L_0(0,\infty), L_\infty(0,\infty)) \) is indeed K-monotone (see e.g. Section 0). A non-commutative version of this result is established, that is, the pair \( (L_0(\mathcal{M}, \tau), \mathcal{M}) \) is K-monotone in the setting when \( \mathcal{M} \) is a non-atomic finite factor. We would like to thank Professor Astashkin for providing us with the proof for the commutative pair \( (L_0(0,\infty), L_\infty(0,\infty)) \).

2. Preliminaries

2.1. Generalized singular value functions.

In what follows, \( \mathcal{H} \) is a Hilbert space and \( B(\mathcal{H}) \) is the *-algebra of all bounded linear operators on \( \mathcal{H} \), and \( 1 \) is the identity operator on \( \mathcal{H} \). Let \( \mathcal{M} \) be a von Neumann algebra on \( \mathcal{H} \). For details on von Neumann algebra theory, the reader is referred to e.g. [7], [15], [20] or [37]. General facts concerning measurable operators may be found in [30], [32] (see also [38] Chapter IX and the forthcoming book [12]). For convenience of the reader, some of the basic definitions are recalled.

A linear operator \( X : \mathfrak{D}(X) \to \mathcal{H} \), where the domain \( \mathfrak{D}(X) \) of \( X \) is a linear subspace of \( \mathcal{H} \), is said to be affiliated with \( \mathcal{M} \) if \( YX \subseteq XY \) for all \( Y \in \mathcal{M}' \), where \( \mathcal{M}' \) is the commutant of \( \mathcal{M} \). A linear operator \( X : \mathfrak{D}(X) \to \mathcal{H} \) is termed measurable with respect to \( \mathcal{M} \) if \( X \) is closed, densely defined, affiliated with \( \mathcal{M} \) and there exists a sequence \( \{P_n\}_{n=1}^\infty \) in the logic of all projections of \( \mathcal{M} \), \( \mathcal{P}(\mathcal{M}) \), such that \( P_n \uparrow 1 \), \( P_n(\mathcal{H}) \subseteq \mathfrak{D}(X) \) and \( 1 - P_n \) is a finite projection (with respect to \( \mathcal{M} \)) for all \( n \). It should be noted that the condition \( P_n(\mathcal{H}) \subseteq \mathfrak{D}(X) \) implies that \( X P_n \in \mathcal{M} \). The collection of all measurable operators with respect to \( \mathcal{M} \) is
denoted by $S(\mathcal{M})$, which is a unital $*$-algebra with respect to strong sums and products (denoted simply by $X + Y$ and $XY$ for all $X, Y \in S(\mathcal{M})$).

Let $X$ be a self-adjoint operator affiliated with $\mathcal{M}$. We denote its spectral measure by $\{E^X\}$. It is well known that if $X$ is a closed operator affiliated with $\mathcal{M}$ with the polar decomposition $X = U|X|$, then $U \in \mathcal{M}$ and $E \in \mathcal{M}$ for all projections $E \in \{E^{|X|}\}$. Moreover, $X \in S(\mathcal{M})$ if and only if $X$ is closed, densely defined, affiliated with $\mathcal{M}$ and $E^{|X|}(\lambda, \infty)$ is a finite projection for some $\lambda > 0$. It follows immediately that in the case when $\mathcal{M}$ is a von Neumann algebra of type $III$ or a type $I$ factor, we have $S(\mathcal{M}) = \mathcal{M}$. For type $II$ von Neumann algebras, this is no longer true. From now on, let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace $\tau$.

An operator $X \in S(\mathcal{M})$ is called $\tau$-measurable if there exists a sequence $\{P_n\}_{n=1}^\infty$ in $P(\mathcal{M})$ such that $P_n \uparrow 1$, $P_n(\mathcal{H}) \subseteq \mathfrak{D}(X)$ and $\tau(1 - P_n) < \infty$ for all $n$. The collection of all $\tau$-measurable operators is a unital $*$-subalgebra $S(\mathcal{M})$ denoted by $S(\mathcal{M}, \tau)$. It is well known that a linear operator $X$ belongs to $S(\mathcal{M}, \tau)$ if and only if $X \in S(\mathcal{M})$ and there exists $\lambda > 0$ such that $\tau(E^{|X|}(\lambda, \infty)) < \infty$. Alternatively, an unbounded operator $X$ affiliated with $\mathcal{M}$ is $\tau$-measurable (see [14]) if and only if

$$\tau\left(E^{|X|}\left(\frac{1}{n}, \infty\right)\right) = o(1), \quad n \to \infty.$$

**Definition 2.1.** Let a semifinite von Neumann algebra $\mathcal{M}$ be equipped with a faithful normal semi-finite trace $\tau$ and let $X \in S(\mathcal{M}, \tau)$. The generalized singular value function $\mu(X) : t \to \mu(t; X)$ of the operator $X$ is defined by setting

$$\mu(s; X) = \inf\{|XP| : P = P^* \in \mathcal{M} \text{ is a projection, } \tau(1 - P) \leq s\}.$$

An equivalent definition in terms of the distribution function of the operator $X$ is the following. For every self-adjoint operator $X \in S(\mathcal{M}, \tau)$, setting

$$d_X(t) = \tau(E^X(t, \infty)), \quad t > 0,$$

we have (see e.g. [14])

$$\mu(t; X) = \inf\{s \geq 0 : d_X(s) \leq t\}.$$  \hspace{1cm} (1)

Consider the algebra $\mathcal{M} = L^\infty(0, \infty)$ of all Lebesgue measurable essentially bounded functions on $(0, \infty)$. Algebra $\mathcal{M}$ can be seen as an abelian von Neumann algebra acting via multiplication on the Hilbert space $\mathcal{H} = L^2(0, \infty)$, with the trace given by integration with respect to Lebesgue measure $m$. It is easy to see that the algebra of all $\tau$-measurable operators affiliated with $\mathcal{M}$ can be identified with the subalgebra $S(0, \infty)$ of the algebra of Lebesgue measurable functions which consists of all functions $x$ such that $m(\{|x| > s\})$ is finite for some $s > 0$. It should also be pointed out that the generalized singular value function $\mu(x)$ is precisely the decreasing rearrangement $\mu(x)$ of the function $x$ (see e.g. [23]) defined by

$$\mu(t; x) = \inf\{s \geq 0 : m(\{|x| \geq s\}) \leq t\}.$$

The two-sided ideal $\mathcal{F}(\mathcal{M}, \tau)$ in $\mathcal{M}$ consisting of all elements of $\tau$-finite rank is defined by setting

$$\mathcal{F}(\mathcal{M}, \tau) = \{X \in \mathcal{M} : \tau(\nu(X)) < \infty\} = \{X \in \mathcal{M} : \tau(s(X)) < \infty\}.$$
For convenience of the reader we also recall the definition of the measure topology \( t_\tau \) on the algebra \( S(M, \tau) \). For every \( \varepsilon, \delta > 0 \), we define the set

\[
V(\varepsilon, \delta) = \{ X \in S(M, \tau) : \exists P \in P(M) \text{ such that } \| X(1 - P) \| \leq \varepsilon, \tau(P) \leq \delta \}.
\]

The topology generated by the sets \( V(\varepsilon, \delta) \), \( \varepsilon, \delta > 0 \), is called the measure topology \( t_\tau \) on \( S(M, \tau) \) \([12, 14, 30]\). It is well known that the algebra \( S(M, \tau) \) equipped with the measure topology is a complete metrizable topological algebra \([30]\) (see also \([29]\)). A sequence \( \{ X_n \}_{n=1}^\infty \subset S(M, \tau) \) converges to zero with respect to measure topology \( t_\tau \) if and only if \( \tau(X^n | (\varepsilon, \infty)) \to 0 \) as \( n \to \infty \) for all \( \varepsilon > 0 \) \([11, 12]\).

2.2. \( \Delta \)-normed spaces.

For convenience of the reader, we recall the definition of \( \Delta \)-norm. Let \( \Omega \) be a linear space over the field \( \mathbb{C} \). A function \( \| \cdot \| \) from \( \Omega \) to \( \mathbb{R} \) is a \( \Delta \)-norm, if for all \( x, y \in \Omega \) the following properties hold:

1. \( \| x \| \geq 0, \| x \| = 0 \Leftrightarrow x = 0; \)
2. \( \| \alpha x \| \leq \| x \| \) for all \( \alpha \in \mathbb{C}, \| x \| \leq 1; \)
3. \( \lim_{n \to 0} \| \alpha x \| = 0; \)
4. \( \| x + y \| \leq C_\Omega \cdot (\| x \| + \| y \|) \) for a constant \( C_\Omega \geq 1 \) independent of \( x, y \).

The couple \((\Omega, \| \cdot \|)\) is called a \( \Delta \)-normed space. We note that the definition of a \( \Delta \)-norm given above is the same with that given in \([21]\). It is well-known that every \( \Delta \)-normed space \((\Omega, \| \cdot \|)\) is metrizable and conversely every metrizable space can be equipped with a \( \Delta \)-norm \([21]\). Note that properties (2) and (4) of a \( \Delta \)-norm imply that for any \( \alpha \in \mathbb{C} \), there exists a constant \( M \) such that \( \| \alpha x \| \leq M \| x \| , x \in \Omega \), in particular, if \( \| x_n \| \to 0 \), \( \{ x_n \}_{n=1}^\infty \subset \Omega \), then \( \| \alpha x_n \| \to 0 \).

Let \( E(0, \infty) \) be a space of real-valued Lebesgue measurable functions on \((0, \infty)\) (with identification \( m \)-a.e.), equipped with a \( \Delta \)-norm \( \| \cdot \|_E \). The space \( E(0, \infty) \) is said to be absolutely solid if \( x \in E(0, \infty) \) and \( |y| \leq |x|, y \in S(0, \infty) \) implies that \( y \in E(0, \infty) \) and \( \| y \|_E \leq \| x \|_E \). An absolutely solid space \( E(0, \infty) \subseteq S(0, \infty) \) is said to be symmetric if for every \( x \in E(0, \infty) \) and every \( y \in S(0, \infty) \), the assumption \( \mu(y) = \mu(x) \) implies that \( y \in E(0, \infty) \) and \( \| y \|_E = \| x \|_E \) (see e.g. \([23]\)).

We now come to the definition of the main object of this paper.

**Definition 2.2.** Let a semifinite von Neumann algebra \( M \) be equipped with a faithful normal semi-finite trace \( \tau \). Let \( E \) be a linear subset in \( S(M, \tau) \) equipped with a \( \Delta \)-norm \( \| \cdot \|_E \). We say that \( E \) is a symmetrically \( \Delta \)-normed operator space if \( X \in E \) and every \( Y \in S(M, \tau) \) the assumption \( \mu(Y) \leq \mu(X) \) implies that \( Y \in E \) and \( \| Y \|_E \leq \| X \|_E \).

One should note that a symmetrically \( \Delta \)-normed space \( E(M, \tau) \) does not necessarily satisfy

\[
\| AXB \|_E \leq \| A \|_\infty \| B \|_\infty \| X \|_E, A, B \in M, X \in E(M, \tau).
\]

It is clear that in the special case, when \( M = L_\infty(0,1) \), or \( M = L_\infty(0, \infty) \), or \( M = L_1 \), the definition of symmetrically \( \Delta \)-normed operator spaces coincides with definition of the symmetric function (or sequence) spaces. In the case, when \( M = B(H) \) and \( \tau \) is a standard trace \( \text{Tr} \), we shall call a symmetrically \( \Delta \)-normed operator space introduced in Definition \([2.2]\) a symmetrically \( \Delta \)-normed operator ideal (for the symmetrically normed ideals we refer to \([15, 16, 33]\)).

As mentioned before, the operator space \( E(M, \tau) \) defined by

\[
E(M, \tau) := \{ X \in S(M, \tau) : \mu(X) \in E(0, \infty) \}, \| X \|_{E(M, \tau)} := \| \mu(X) \|_E
\]
is a complete symmetrically \( \Delta \)-normed operator space whenever the symmetrically
\( \Delta \)-normed function space \( E(0, \infty) \) equipped with a \( \Delta \)-norm \( \| \cdot \|_E \) is complete \[17\].

3. \((L_0 + L_\infty)(\mathcal{M}, \tau) = S(\mathcal{M}, \tau)\)

By \( L_0(0, \infty) \) we denote the space of all measurable functions on \( (0, \infty) \) whose
support has finite measure. This space is endowed with the group-norm \[18\]

\[
\| f \|_{L_0} = m(\text{supp}(f)),
\]

where \( \text{supp}(f) = \{ t \in (0, \infty) : f(t) \neq 0 \} \). It is clearly that the corresponding
operator space \( L_0(\mathcal{M}, \tau) \) is the subspace of \( S(\mathcal{M}, \tau) \) which consists of all operators
\( X \) such that \( \tau(s(X)) < \infty \). It is easy to see that \((L_0 + L_\infty)(\mathcal{M}, \tau)\) coincides with
\( S(\mathcal{M}, \tau) \). For the sake of completeness, we present a brief proof below.

**Proposition 3.1.** \((L_0 + L_\infty)(\mathcal{M}, \tau) = S(\mathcal{M}, \tau)\).

**Proof.** Since \((L_0 + L_\infty)(\mathcal{M}, \tau) \subset S(\mathcal{M}, \tau)\), it suffices to prove \( S(\mathcal{M}, \tau) \subset (L_0 + L_\infty)(\mathcal{M}, \tau)\). For any operator \( X \in S(\mathcal{M}, \tau) \), there exists \( \lambda > 0 \) such that \( t := \tau(\mathcal{E}(X)(\lambda)) < \infty \). By \[12\] Chapter III, Eq. (4), we have

\[
d_{\mu(X)}(\lambda) = \tau(\mathcal{E}(X)(\lambda)) = t.
\]

This together with \[18\] Proposition 3 implies that \( \mu(X) \in (L_0 + L_\infty)(0, \infty) \). Thus,
\( X \in (L_0 + L_\infty)(\mathcal{M}, \tau) \). \( \square \)

In \[18\] Proposition 3, the description of the \( K \)-functional (which is a \( \Delta \)-norm
on \( S(0, \infty) \) ) \( K_u(f) = \inf \{ \| g \|_0 + u \| h \|_\infty : f = g + h, g \in L_0(0, \infty), h \in L_\infty(0, \infty), u > 0 \} \),
of any \( f \in S(0, \infty) \) is given in terms of its distribution function and its
singular value function. That is, for every \( f \in S(0, \infty) \), we have

\[
K_u(f) = \inf_{s > 0} [su + d_{\mu(f)}(s)] = \inf_{t > 0} [t + u\mu(t; f)].
\]

Similarly, for every \( X \in S(\mathcal{M}, \tau) \), the \( K \)-functional is defined by

\[
K_u(X) := \inf \{ \| G \|_0 + u \| H \|_\infty : X = G + H, G \in L_0(\mathcal{M}, \tau), H \in L_\infty(\mathcal{M}, \tau), u > 0 \}.
\]

In particular, we define a \( \Delta \)-norm (see e.g. Remark \[8\]) on \( S(\mathcal{M}, \tau) \) by

\[
\| X \|_S := K_1(X) = \inf \{ \| G \|_0 + \| H \|_\infty : X = G + H, G \in L_0(\mathcal{M}, \tau), H \in L_\infty(\mathcal{M}, \tau) \}
\]

for any \( X \in S(\mathcal{M}, \tau) \). The following result complements an earlier result from \[9\]
for the pair \((L_1(\mathcal{M}, \tau), L_\infty(\mathcal{M}, \tau))\).

**Proposition 3.2.** For every \( X \in S(\mathcal{M}, \tau) \), we have

\[
K_u(X) = K_u(\mu(X)), u > 0.
\]

In particular, \( \| X \|_S = \inf_{s > 0} [s + d_{\mu(X)}(s)] = \inf_{t > 0} [t + \mu(t; X)] \). Moreover, \( S(\mathcal{M}, \tau) \)
is a complete \( \Delta \)-normed space with respect to the \( \Delta \)-norm \( K_u \) for every \( u > 0 \).
Proof. Firstly, for every $X \in S(M, \tau)$, we have
\[
K_u(X) = \inf \{\|G\|_0 + u\|H\|_\infty : X = G + H, G \in L_0(M, \tau), H \in L_\infty(M, \tau)\}
\leq \inf_{s > 0} \{\|XE^{|X|}(s, \infty)\|_0 + u\|XE^{|X|}(0, s)\|_\infty\}
\leq \inf_{s > 0} \{d_{\mu(XE^{|X|}(s, \infty))}(0) + us\}
= \inf_{s > 0} \{d_{\mu(XE^{|X|}(s, \infty))}(0) + us\} \quad \text{(by [12] Chapter III, Eq. (4))}
= \inf_{s > 0} \{d_{\mu(X)}(0) + us\}
\leq \inf_{s > 0} \{d_{\mu(X)}(s) + us\}
= K_u(\mu(X)).
\]
Conversely, for every $G \in L_0(M, \tau)$ with $t := \|G\|_0$ and $X - G \in L_\infty(M, \tau)$, by [12] Chapter III, Proposition 2.20, we have
\[
t + u\mu(t; X)
= t + u \cdot \inf \{\|X - Y\|_\infty : Y \in L_0(M, \tau), \|Y\|_0 \leq t, X - Y \in M\}
\leq \|G\|_0 + u\|X - G\|_\infty.
\]
Hence, we obtain
\[
\inf_{t > 0} [t + u\mu(t; X)] \leq \inf_{G \in L_0(M, \tau)} \inf_{X - G \in L_\infty(M, \tau)} (\|G\|_0 + u\|X - G\|_\infty) = K_u(X).
\]
The fact that $S(0, \infty)$ is complete with respect to $K_u$ together with [17] Theorem 3.8 implies the completeness of $S(M, \tau)$ with respect to $K_u$. 

Remark 3.3. Recall that $d_{\mu(X)}(s) = d_{|X|}(s)$ (see e.g. [12] Chapter III, Eq. (4)). For every $X \in S(M, \tau)$, the $K$-functional can be also defined by the formula
\[
K_u(\mu(X)) = \inf_{s > 0} [su + d_{|X|}(s)], \ u > 0.
\]
Remark 3.4. It is still unknown whether the Calkin correspondence preserves the constant $C_E$ for an arbitrary symmetrically $\Delta$-normed function space $E(0, \infty)$ (see [17], [24]). However, it is well-known that $(S(0, \infty), K_u(\cdot))$, $u > 0$, is an $F$-space (i.e., a complete $\Delta$-normed space with $C_E = 1$) and Proposition 3.2 implies that for every $u > 0$, $K_u(\mu(\cdot))$ is not only a $\Delta$-norm but also an $F$-norm on $S(M, \tau)$. Indeed, for every $X, Y \in S(M, \tau)$, by Proposition 3.2 we have
\[
K_u(\mu(X)) = K_u(X) + K_u(Y)
= \inf \{\|X_1\|_0 + u\|X_2\|_\infty : X = X_1 + X_2, X_1 \in L_0(M, \tau), X_2 \in L_\infty(M, \tau)\}
+ \inf \{\|Y_1\|_0 + u\|Y_2\|_\infty : Y = Y_1 + Y_2, Y_1 \in L_0(M, \tau), Y_2 \in L_\infty(M, \tau)\}
= \inf \{\|X_1\|_0 + \|Y_1\|_0 + u\|X_2\|_\infty + u\|Y_2\|_\infty : \}
\geq \inf \{\|X_1 + Y_1\|_0 + u\|X_2 + Y_2\|_\infty : X = X_1 + X_2, Y = Y_1 + Y_2, X_1, Y_1 \in L_0(M, \tau), X_2, Y_2 \in L_\infty(M, \tau)\}
\geq \inf \{\|Z_1\|_0 + u\|Z_2\|_\infty : X + Y = Z_1 + Z_2, Z_1 \in L_0(M, \tau), Z_2 \in L_\infty(M, \tau)\}
= K_u(X + Y) = K_u(\mu(X + Y)).
where we used the fact that \( \|X\|_0 + \|Y\|_0 = \tau(\text{supp}(X)) + \tau(\text{supp}(Y)) \geq \tau(\text{supp}(X + Y)) = \|X + Y\|_0 \) for every \( X, Y \in S(M, \tau) \).

4. An embedding theorem

It is well-known that for every symmetrically normed function space \( E(0, \infty) \), the corresponding operator space \( E(M, \tau) \) is symmetrically normed \([22]\) and is an intermediate space for the noncommutative pair \((L_1(M, \tau), M)\) \([13, 12]\). In this section, we prove an analogue for the \( \Delta \)-normed case, that is, every operator space \( E(M, \tau) \) corresponding to a \( \Delta \)-normed function space \( E(0, \infty) \) is an intermediate space for the noncommutative pair \((L_0(M, \tau), M)\).

Before we proceed to the proof of the embedding theorem, we show that the topology given by \( \| \cdot \|_S \) is equivalent with the measure topology.

**Proposition 4.1.** Let \( \{X_n\} \) be a sequence in \( S(M, \tau) \). Then, \( \|X_n\|_S \to 0 \) if and only if \( X_n \to_\tau 0 \).

**Proof.** By \([17]\) Lemma 2.4, it suffices to show that \( \|X_n\|_S \to 0 \) whenever \( X_n \to_\tau 0 \). By \([12]\) Chapter II, Proposition 5.7, we have \( \tau(E[\chi_{(\varepsilon, \infty)}]) \to_\tau 0 \) for every \( \varepsilon > 0 \). By Proposition 3.2, we have \( \|X_n\|_S \leq \varepsilon + \tau(E[\chi_{(\varepsilon, \infty)}]) \), which completes the proof. \( \square \)

Notice that the two-sided ideal \( \mathcal{F}(\tau) \) in \( M \) coincides with \((L_0 \cap L_{\infty})(M, \tau)\). For every \( X \in \mathcal{F}(\tau) \), we define the group-norm \( \|X\|_{\mathcal{F}} \) by
\[
\|X\|_{\mathcal{F}} := \max\{\|X\|_0, \|X\|_{\infty}\}.
\]

The following embedding theorem is the main result of this section, which extends \([18]\) Theorem 1 to the non-commutative case.

**Theorem 4.2.** If \( E(0, \infty) \) is a nontrivial symmetrically \( \Delta \)-normed function space, then
\[
\mathcal{F}(\tau) \subset E(M, \tau) \subset S(M, \tau).
\]

Moreover, the embeddings are continuous. That is, \( E(M, \tau) \) is an intermediate space between \( L_0(M, \tau) \) and \( M \).

**Proof.** Since \( E(0, \infty) \) is not empty, there is a non-zero element \( x_0 \in E(0, \infty) \). Then, there is a scalar \( t > 0 \) such that \( \mu(t; x_0) > 0 \). It is clear that \( \mu(t; x_0) \chi_{(0, t]} \leq \mu(x_0) \), which implies that \( \chi_{(0, t]} \in E(0, \infty) \). Since \( m(\chi_{(0, t]}(\cdot)) = m(\chi_{(t, 2t]}(\cdot)) = \cdots = m(\chi_{(nt-1, nt]}(\cdot)) \) and \( E(0, \infty) \) is a linear space, it follows that \( \chi_{(0, nt]} \in E(0, \infty) \). Hence, \( \mathcal{F}(0, \infty) \subset E(0, \infty) \).

Let \( \{X_n\}_n \subseteq \mathcal{F}(\tau) \) be a sequence such that \( \|X_n\|_{\mathcal{F}} \to 0 \). For every \( 0 < \varepsilon < 1 \), we can find an \( N \) such that for every \( n \geq N \), we have \( \|X_n\|_{\mathcal{F}} \leq \varepsilon \), that is,
\[
\text{supp}(\mu(X_n)) \leq \varepsilon \quad \text{and} \quad \|X_n\|_{\infty} \leq \varepsilon.
\]

Hence, \( \mu(X_n) \leq \varepsilon \chi_{(0, t]} \leq \varepsilon \chi_{(0, 1]} \) and therefore \( \|X_n\|_E \leq \varepsilon \chi_{(0, 1]} \|X\|_E \). By the continuity of \( \Delta \)-norm \( \| \cdot \|_E \), we obtain that \( \|X_n\|_E \to_\tau 0 \).

Lemma 2.4 in \([17]\) together with Proposition 4.1 implies that \( E(M, \tau) \) is continuously embedded into \( S(M, \tau) \). \( \square \)

The set of all self-adjoint elements in \( E(M, \tau) \) is denoted by \( E_b(M, \tau) \). Then, \([12]\) Chapter II, Proposition 6.1 together with Proposition 4.1 and Theorem 4.2 implies the following results immediately.
Corollary 4.3. Let $E(0, \infty)$ be a symmetrically $\Delta$-normed function space. The following statements hold.

1. The positive cone $E(M, \tau)^+$ is closed in $E(M, \tau)$ with respect to $\| \cdot \|_E$.
2. If $\{X_n\}_{n=1}^{\infty}$ is a sequence in $E(M, \tau)$ and $X, Y \in E_h(M, \tau)$ are such that $\|X_n - X\|_E \to 0$ and $X_n \leq Y$ for all $n$, then $X \leq Y$.
3. If $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence in $E_h(M, \tau)$ and $X \in E_h(M, \tau)$ with $\|X_n - X\|_E \to 0$, then $X_n \uparrow X$.

5. Interpolation in the pair $(L_0(M, \tau), M)$

Introduce the dilation operator $\sigma_s$ on $S(0, \infty)$, $s > 0$, by setting

$$(\sigma_s(x))(t) = x\left(\frac{t}{s}\right), \quad t > 0.$$ 

It is well-known that $\mu(X + Y) \leq \sigma_2(\mu(X) + \mu(Y))$, $X, Y \in S(M, \tau)$ [24]. We note also that $\sigma_2x \in E(0, \infty)$ with

$$(3) \quad \|\sigma_2x\|_E \leq (2C_E)^k\|x\|_E$$

for all $x \in E(0, \infty)$ and $k \in \mathbb{N}$ (see e.g. [23], see also [18]).

Recall that $(L_0 + L_\infty)(M, \tau) = S(M, \tau)$ (see Proposition 3.1). Let $T : S(M, \tau) \to S(M, \tau)$ be a homomorphism, i.e.,

$T(X + Y) = TX + TY$ and $T(-X) = -TX$

for any $X, Y \in S(M, \tau)$. Let $E(0, \infty)$ be a $\Delta$-normed function space. A homomorphism $T : E(M, \tau) \to E(M, \tau)$ is called continuous if for any given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\|X\|_E < \delta(\varepsilon)$ implies that $\|TX\|_E < \varepsilon$ [21].

A homomorphism is called bounded if

$$\|T\|_{E \to E} = \sup_{x \neq 0} \frac{\|TX\|_E}{\|X\|_E} < \infty.$$ 

The homomorphism $T$ is said to be bounded on the pair $(L_0(M, \tau), M)$ if $T$ is a bounded mapping from $L_0(M, \tau)$ into $L_0(M, \tau)$ and from $M$ into $M$.

Theorem 5.1. Let $E(0, \infty)$ be a symmetrically $\Delta$-normed function space and $T : S(M, \tau) \to S(M, \tau)$ be a homomorphism which is bounded on $(L_0(M, \tau), M)$ with

$$\|TX\|_0 \leq M_0\|X\|_0, \quad \forall X \in L_0(M, \tau),$$

$$\|TX\|_\infty \leq M_1\|X\|_\infty, \quad \forall X \in M$$

for some constants $M_0, M_1 > 0$. Then, $T$ maps $E(M, \tau)$ into itself and

$$(4) \quad \mu(M_0; TX) \leq \mu(t; M_1X), \quad X \in E(M, \tau).$$

Proof. For any $P \in \mathcal{P}(M)$ with $\tau(1 - P) < t$, we have $\|TX(1 - P)\|_0 \leq M_0\|X(1 - P)\|_0 \leq M_0t$. By [21] Theorem 2.3.13, for every $t > 0$, we have

$$\mu(M_0t; TX) = \inf\{\|TX - B\|_\infty : B \in S(M, \tau), \|B\|_0 \leq M_0t\}$$

$$\leq \|TX - TX(1 - P)\|_\infty = \|XP\|_\infty \leq M_1\|XP\|_\infty.$$ 

By Definition 2.21, we have

$$\mu(M_0t; TX) \leq M_1\mu(t; X).$$

This implies that $\sigma_{1/M_0}(\mu(TX)) \in E(0, \infty)$ and therefore, by appealing to [23], we conclude that $\mu(TX) = \sigma_{M_0}(\sigma_{1/M_0}(\mu(TX))) \in E(0, \infty)$. □
If \(X, Y \in S(M, \tau)\), then \(X\) is said to be submajorized by \(Y\), denoted by \(X \prec\prec Y\), if
\[
\int_0^t \mu(s; X) ds \leq \int_0^t \mu(s; Y) ds, \quad t \geq 0.
\]

A linear subspace \(E\) of \(S(M, \tau)\) equipped with a complete norm \(\| \cdot \|_E\), is called fully symmetric space (of \(\tau\)-measurable operators) if \(X \in S(M, \tau), Y \in E\) and \(X \prec\prec Y\) imply that \(X \in E\) and \(\|X\|_E \leq \|Y\|_E\) [11][12][24].

For a symmetric normed function space \(E(0, \infty)\), by [12] Theorem 10.13 (see also [9] and [23]), \(E(M, \tau)\) is an interpolation space between \(L_1(M, \tau)\) and \(M\) if \(E(0, \infty)\) is fully symmetric. In particular, one can find symmetric normed spaces \(E(M, \tau)\) which are not interpolation spaces between \(L_1(M, \tau)\) and \(L_\infty(M, \tau)\) [23] Chapter II, § 5.7. However, for an arbitrary symmetrically \(\Delta\)-normed function space \(E(0, \infty)\), \(E(M, \tau)\) is, in fact, an interpolation space between \(L_0(M, \tau)\) and \(M\).

**Corollary 5.2.** Let \(E(0, \infty)\) be a symmetrically \(\Delta\)-normed function space and \(T : S(M, \tau) \to S(M, \tau)\) be a homomorphism which is bounded on \((L_0(M, \tau), \mathcal{M})\) with
\[
\|TX\|_0 \leq M_0\|X\|_0, \quad \forall X \in L_0(M, \tau),
\]
\[
\|TX\|_\infty \leq M_1\|X\|_\infty, \quad \forall X \in \mathcal{M}
\]
for some constants \(M_0, M_1 > 0\). Then, \(T\) is a bounded homomorphism from \(E(M, \tau)\) into itself. In particular, \(\sup_{X \in S(M, \tau)} \frac{\|TX\|_\infty}{\|X\|_\infty} < \infty\).

**Proof.** By Theorem 5.1, for any \(X \in E(M, \tau)\), we have \(TX \in E(M, \tau)\) with
\[
\mu(M_0; TX) \leq M_1\mu(t; X).
\]
Let \(k \geq 0\) be an integer such that \(2^k \geq M_0\). Noticing that \(\|T X\|_0 \leq \|T X\|_0 \leq \|T X\|_0 \leq M_0\|X\|_0\).

By [30], we have
\[
\|\sigma_{2^k}(\mu(X))\|_E \leq \|\sigma_{2^k}(\mu(X))\|_E \leq (2C_E)^k\|\mu(X)\|_E
\]
for any \(X \in E(M, \tau)\).

Then, we get
\[
\|TX\|_E = \|\mu(TX)\|_E \leq (2C_E)^k\|\sigma_{1/M_0}(\mu(TX))\|_E \leq (2C_E)^k\|M_1\mu(X)\|_E
\]
\[
\leq (2C_E)^k\|([M_1]+1)\mu(X)\|_E \leq (2C_E)^k\sum_{i=1}^{[M_1]+1} C^k_i\|\mu(X)\|_E,
\]
where \([M_1]\) is the integer part of \(M_1\). The proof is complete.

The results in this section are applied to the study of orbits and \(K\)-orbits in the pair of \((L_0(M, \tau), \mathcal{M})\) in the next section.

**6. Orbits and \(K\)-Orbits**

For an element \(X \in S(M, \tau)\), the orbit \(\text{Orb}(X; L_0(M, \tau), \mathcal{M})\) of \(X\) is the set of all \(Y \in S(M, \tau)\) such that \(Y = TX\) for some homomorphism \(T\) which is bounded on the pair \((L_0(M, \tau), \mathcal{M})\). Furthermore, we define
\[
\|Y\|_{\text{Orb}} := \inf_{Y = TX} \|T\|_{S(M, \tau)}.
\]
where the infimum is taken over all bounded homomorphisms $T$ such that $TX = Y$ and $\|T\|_{S(M,\tau)} = \max\{\|T\|_{L_0\rightarrow L_0}, \|T\|_{L_\infty\rightarrow L_\infty}\}$.

By Theorem 5.1 we have the following proposition, which is an analogue of [2, Theorem 1].

**Proposition 6.1.** Let $X \in S(M,\tau)$. Then, for every $Y \in \text{Orb}(X; L_0(M,\tau), M)$, we have

$$\mu(t; Y) \leq \|Y\|_{\text{Orb}} \mu\left(\frac{t}{\|Y\|_{\text{Orb}}}; X\right), \ t > 0.$$  

**Proof.** Notice for every $\varepsilon > 0$, we can find a $T$ such that $Y = TX$ with $\|T\|_{S(M,\tau)} \leq \|Y\|_{\text{Orb}} + \varepsilon$. Then, by (4), we have

$$\mu(\|T\|_{S(M,\tau)}t; Y) = \mu(\|T\|_{S(M,\tau)}t; TX) \leq \|T\|_{S(M,\tau)} \mu(t; X).$$

Hence, $\mu(\|Y\|_{\text{Orb}} + \varepsilon)t; Y) \leq (\|Y\|_{\text{Orb}} + \varepsilon)\mu(t; X)$ for every $t > 0$. By the right-continuity of singular value functions, we have

$$\mu(\|Y\|_{\text{Orb}}t; Y) \leq \|Y\|_{\text{Orb}} \mu(t; X)$$

for every $t > 0$, which completes the proof. \Box

Let $(X_0, X_1)$ be a pair of symmetrically $\Delta$-normed spaces. The $K$-orbit of $A \in X_0 + X_1$ is defined by the set $K\text{O}(A; X_0, X_1)$ of all $X \in X_0 + X_1$ such that

$$\|X\|_{K\text{O}} := \sup_{t > 0} \frac{K(t, X; X_0, X_1)}{K(t, A; X_0, X_1)} < \infty,$$

where $K(t, Z; X_0, X_1) := \inf\{\|Z_0\|_{X_0} + t\|Z_1\|_{X_1} : Z = Z_0 + Z_1, Z_0 \in X_0, Z_1 \in X_1\}$.

A pair $(X_0, X_1)$ is called $K$-monotone if $K\text{O}(A; X_0, X_1) = \text{Orb}(A; X_0, X_1)$ for all $A \in X_0 + X_1$. The pair $(L_1(0, \infty), L_\infty(0, \infty))$ is a classical example of a $K$-monotone pair (see e.g. [2]). Moreover, the noncommutative pair $(L_1(M, \tau), L_\infty(M, \tau))$ is $K$-monotone (see [2], Proposition 2.5, Theorem 4.7).

It follows from the definition that the unit ball of $\text{Orb}(A; X_0, X_1)$ is a subset of the unit ball of $K\text{O}(A; X_0, X_1)$. Moreover, [9] Proposition 2.5 and Theorem 4.7 imply that the unit ball of $\text{Orb}(A; L_1(M, \tau), M)$ coincides with the unit ball of $K\text{O}(A; L_1(M, \tau), M)$. However, it is known that the reverse inclusion may fail for certain element in the pair $L_0(0, \infty) + L_\infty(0, \infty)$ [2]. One of the main results of this section is a non-commutative version of [2, Theorem 4].

By Proposition 3.2, the $K$-orbit $K\text{O}(A; L_0(M, \tau), M)$ of every $A \in S(M, \tau)$ is the set of all $X \in S(M, \tau)$ such that

$$\|X\|_{K\text{O}} := \sup_{t > 0} \frac{K_t(\mu(X))}{K_t(\mu(A))} < \infty.$$

**Theorem 6.2.** If $M$ is a non-trivial von Neumann algebra ($M \neq \mathbb{C}1$ and $M \neq 0$), then there exist $A, X \in S(M, \tau)$ such that

$$K_t(A) = K_t(X)$$

whereas $\mu(t; A) < \mu(t; X)$, $t \in E$, for some measurable set $E$, $m(E) > 0$. In particular, the unit ball of the $K\text{O}(A; L_0(M, \tau), M)$ does not coincide with the unit ball of $\text{Orb}(A; L_0(M, \tau), M)$.
Hence, we have
\[ L \]
That is, \[ \tau(P_1) > 0 \] and \[ \tau(P_2) > 0. \]
Let \( k_1, k_2 > 0 \) be such that
\[ \frac{\tau_2}{\tau_1 + \tau_2} \]
Define \[ X := k_1(P_1 + P_2). \]
Then, \( \mu(X) = k_1 \chi_{(0, \tau_1 + \tau_2)}. \) By \([2]\), we have
\[ K_i(X) = \min\{tk_1, \tau_1 + \tau_2\}. \]
Then, we have
\[ K_i(X) = \begin{cases} tk_1, & t < \frac{\tau_1 + \tau_2}{k_1} \\ \tau_1 + \tau_2, & t \geq \frac{\tau_1 + \tau_2}{k_1} \end{cases} \]
Define \[ A := k_1P_1 + k_2P_2. \]
Then, \( \mu(A) = k_1 \chi_{(0, \tau_1)} + k_2 \chi_{[\tau_1, \tau_1 + \tau_2]}. \) By \([2]\), we have
\[ K_i(A) = \min\{tk_1 + \kappa, \tau_1 + \kappa\}. \]
However, \([3]\) implies that there is no such a \( t \) such that \( \tau_1 + \kappa \leq \min\{tk_1 + \kappa\}. \) Hence, we have
\[ K_i(A) = \begin{cases} tk_1, & t < \frac{\tau_1 + \tau_2}{k_1} \\ \tau_1 + \tau_2, & t \geq \frac{\tau_1 + \tau_2}{k_1} \end{cases} \]
That is, \( K_i(A) = K_i(X). \) However, it is clear that
\[ \mu(t; A) < \mu(t; X), \quad \tau_1 \leq t < \tau_1 + \tau_2. \]
Assume that \( X \) lies in the unit ball of \( \text{Orb}(A; L_0(\mathcal{M}, \tau), \mathcal{M}). \) By \([1]\)
\[ \mu(t; X) \leq \frac{\|X\|_{\text{Orb}}}{\mu_{\text{Orb}}}(\frac{t}{\|X\|_{\text{Orb}}}; A) \leq \mu(t; A), \quad t > 0, \]
which is a contradiction with \([2]\). Hence, \( X \) lies in the unit ball of \( KO(A; L_0(\mathcal{M}, \tau), \mathcal{M}) \) but not in the unit ball of \( \text{Orb}(A; L_0(\mathcal{M}, \tau), \mathcal{M}). \) \( \square \)

In \([2]\), it is asserted incorrectly that the \( \text{Orb}(A; L_0(0, \infty), L_\infty(0, \infty)) \neq KO(A; L_0(0, \infty), L_\infty(0, \infty)). \) The following proposition together \([2]\) \text{Theorem 1} explains why \( \text{Orb}(A; L_0(0, \infty), L_\infty(0, \infty)) = KO(A; L_0(0, \infty), L_\infty(0, \infty)), \) i.e., the pair \( (L_0(0, \infty), L_\infty(0, \infty)) \) is \( K \)-monotone. We would like to thank Professor Astashkin for providing the proof for the special case when \( \mathcal{M} = L_\infty(0, \infty). \)

**Proposition 6.3.** Let \( A, X \in S(\mathcal{M}, \tau). \) Then, the following statements are equivalent.

1. There exists \( C > 0 \) such that \( \mu(s; X) \leq C \mu(s; A) \) for every \( s > 0. \)
2. \( \sup_{t>0} \frac{K_i(X)}{K_i(A)} < \infty. \)
Proof. (i) For every $A, X$ satisfying condition (1), we have
\[
\sup_{t>0} \frac{K_t(X)}{K_t(A)} = \sup_{t>0} \frac{\inf_s \{s + t\mu(s; X)\}}{\inf_s \{s + t\mu(s; A)\}} \leq \sup_{t>0} \frac{\inf_s \{s + tC\mu(s/C; A)\}}{\inf_s \{s + t\mu(s; A)\}} = C < \infty,
\]
which proves the validity of condition (2).

(ii) Conversely, assume that
\[
\sup_{t>0} \frac{\inf_s \{s + t\mu(s; X)\}}{\inf_s \{s + t\mu(s; A)\}} \leq C
\]
for some $C > 0$. For every $Z \in S(M, \tau)$, we define
\[
M_t(Z) := \inf_s \{\max \{s, t\mu(s; Z)\}\}, \ t > 0.
\]
Clearly, we have
\[
M_t(Z) \leq K_t(Z) \leq 2M_t(Z), \ Z \in S(M, \tau), \ t > 0.
\]
Therefore,
\[
M_t(X) \leq 2CM_t(A), \ t > 0.
\]
Let $s \in (0, \infty)$ and let $t := \frac{s}{\mu(s, X)}$ (without loss of generality, we may assume that $\mu(s; X) > 0$). Notice that $s = t\mu(s; X) \leq t\mu(s - \Delta_1; X)$ and $s = t\mu(s; X) \leq s + \Delta_2$ for any $\Delta_1, \Delta_2 > 0$ with $s - \Delta_1 > 0$. We have,
\[
M_t(X) = s = t\mu(s; X).
\]
Let $s_1 := M_t(A)$. Then, we have
\[
t\mu(s_1^-; A)(:= t \lim_{k \uparrow s_1} \mu(k; A)) \geq s_1
\]
(again, we have $\max\{t\mu(s_1 - \varepsilon; A), s_1 - \varepsilon\} < s_1 = M_t(A)$ for some $\varepsilon > 0$, which is a contradiction to the definition of $M_t(A)$).

Since $s = M_t(X) \leq 2CM_t(A) = 2Cs_1$, it follows that
\[
\mu(s_1^-; A) \leq \mu((\frac{s}{2C})^-; A).
\]
Then, we obtain
\[
t\mu(s; X) \leq M_t(X) \leq 2CM_t(A) = 2Cs_1 \leq 2Ct\mu(s_1^-; A) \leq 2Ct\mu((\frac{s}{2C})^-; A),
\]
that is,
\[
\mu(s; X) \leq 2C\mu((\frac{s}{2C})^-; A).
\]
Since $s > 0$ is arbitrary taken, it follows that
\[
\mu(s; X) \leq 3C\mu((\frac{s}{3C})^-; A)
\]
for every $s > 0$, which completes the proof.

\[\square\]

By the above proposition and [2, Theorem 1], we obtain the following result immediately, which implies that the commutative pair $(L_0(0, \infty), L_\infty(0, \infty))$ is indeed $\mathcal{K}$-monotone.
COROLLARY 6.4. For every \( a \in S(0, \infty) \), we have

\[
\text{Orb}(a; L_0(0, \infty), L_\infty(0, \infty)) = \mathcal{KO}(a; L_0(0, \infty), L_\infty(0, \infty)).
\]

It is known that there cannot in general be a conditional expectation from \( S(\mathcal{M}, \tau) \) onto a subalgebra of \( S(\mathcal{M}, \tau) \) (see e.g. [13, Appendix B]), which is the main obstacle in extending [2] Theorem 1 to the non-commutative case. The following theorem is the last result of this section, giving a non-commutative version of [2] Theorem 1 in the setting of non-atomic finite factors by approaches which are completely different from those used in [2].

For the sake of convenience, we denote \( L_\infty(0, \tau(s(X))) \) by \( M_{\mu(A)} \). If \( \mathcal{M} \) is a non-atomic semifinite von Neumann algebra, then for every \( X \in S_0(\mathcal{M}, \tau) \), there exists a non-atomic commutative von Neumann subalgebra \( \mathcal{M}_{|X|} \) in \( s(|X|)M_s(|X|) \) and a trace-preserving \(*\)-isomorphism \( J \) from \( S(\mathcal{M}_{|X|}, \tau) \) onto the algebra \( S(M_{\mu(A)}, m) \) [6,8].

THEOREM 6.5. If \( \mathcal{M} \) is a non-atomic finite von Neumann factor with a faithful normal finite trace \( \tau \), then for every \( 0 \neq A \in S(\mathcal{M}, \tau) \), \( \text{Orb}(A; L_0(\mathcal{M}, \tau), L_\infty(\mathcal{M}, \tau)) \) is the set of all \( X \in S(\mathcal{M}, \tau) \) for which there exists \( C > 0 \) such that

\[
\mu(t; X) \leq C \mu\left(\frac{t}{t+C}; A\right), \quad t > 0.
\]

In particular, \( \text{Orb}(A; L_0(\mathcal{M}, \tau), L_\infty(\mathcal{M}, \tau)) = \mathcal{KO}(A; L_0(\mathcal{M}, \tau), L_\infty(\mathcal{M}, \tau)) \).

Proof. (i) It follows from Proposition 6.1 that for every \( X \in \text{Orb}(A; L_0(\mathcal{M}, \tau), L_\infty(\mathcal{M}, \tau)) \), there exists such a \( C > 0 \) satisfying (14). 

(ii) Conversely, assume that \( A, X \in S(\mathcal{M}, \tau) \) satisfies (14).

Then, by [6, Lemma 1.3], there are isomorphisms \( J_A \) between \( S(M_{\mu(A)}, m) \) and \( S(\mathcal{M}_{|A|}, \tau) \) such that \( J_A \mu(A) = |A| \) and \( J_X \) between \( S(M_{\mu(A)}, m) \) and \( S(M_{\mu(A)}, \tau) \) such that \( J_X \mu(X) = |X| \).

Then, by (14), we have

\[
2C \mu\left(\frac{t}{2C}; A\right) - \mu(t; x) \geq C \mu\left(\frac{t}{2C}; A\right) + C \mu\left(\frac{t}{2C}; A\right) - \mu(t; x) \geq C \mu\left(\frac{\tau(s(X))}{2C}; A\right)
\]

for every \( t \in (0, \tau(s(X))) \). Without loss of generality, we can assume that \( C \) is an integer which is large enough such that

\[
2C \mu\left(\frac{t}{2C}; A\right) - \mu(t; x) \geq C \mu\left(\frac{\tau(s(X))}{2C}; A\right) \geq 1
\]

for every \( t \in (0, \tau(s(X))) \).

Let \( \varepsilon < \frac{1}{2C} \). Since \( J_A \mu(A) = |A| \), we can define \( A_n := [a_n, b_n], n \geq 0, \) by

\[
\chi_{A_n} = \chi_{a_n, b_n} = J_A^{-1}(E^{[A]}(n\varepsilon, (n+1)\varepsilon)).
\]

For every \( 0 \leq j \leq 2C - 1 \), we set \( [a_{nj}, b_{nj}] := [2Ca_n + j(b_n - a_n), 2Ca_n + (j + 1)(b_n - a_n)] \).

By [6, Lemma 1.3], for every \([a_{nj}, b_{nj}] \subset [0, \tau(s(X))]\), we have

\[
\tau(E^{[A]}(n\varepsilon, (n+1)\varepsilon)) = \tau(J_A \chi_{a_n, b_n}) = \tau(J_X \chi_{[a_{nj}, b_{nj}]})
\]

Since \( \mathcal{M} \) is a finite factor, due to (10), there exist partial isometries \( U_{nj} \in \mathcal{M} \) such that \( U_{nj}E^{[A]}(n\varepsilon, (n+1)\varepsilon) = J_A \chi_{a_n, b_n} \) and \( U_{nj}U_{nj} = J_X \chi_{[a_{nj}, b_{nj}]} \). If \( [a_{nj}, b_{nj}] \cap [0, \tau(s(X))] = \emptyset \), we define \( U_{nj} := 0 \). In the case when \( [a_{nj}, b_{nj}] \subset [0, \tau(s(X))] \) but \( [a_{nj}, b_{nj}] \cap [0, \tau(s(X))] \neq \emptyset \), we define \( U_{nj} \) as the partial isometry...
such that $U_{nj}U^*_{nj} = J_X \chi_{[\alpha_n, \alpha_n + \tau(s(X))] \leq E[|A|](n\varepsilon, (n + 1)\varepsilon]$ and $U_{nj}^*U_{nj} = J_X \chi_{\tau(s(X)))}$.

Denote $U_{j} := \sum_{n=0}^{\infty} U_{nj}, 0 \leq j \leq 2C - 1$ (note that $U_{kj}^*U_{lj} = 0$ and $U_{lj}U_{kj}^* = 0$ whenever $k \neq l$). Note that every $U_{j}$ is a partial isometry. Let

$$B_1 := \sum_{n=1}^{\infty} n\varepsilon E[|A|](n\varepsilon, (n + 1)\varepsilon] = \sum_{n=1}^{\infty} n\varepsilon J_X \chi_{[\alpha_n, \alpha_n]} \in S(M[|A|], \tau)$$

and

$$B_2 := \sum_{n,j} n\varepsilon J_X \chi_{[\alpha_n, \alpha_n]} \chi_{[0, \tau(s(X)))}) \in S(M[|X|], \tau).$$

Then, noting that $U_{n,j}^*E[|A|](n\varepsilon, (n + 1)\varepsilon]U_{n,j} = U_{n,j}^*J_X \chi_{[\alpha_n, \alpha_n]}U_{n,j} = J_X \chi_{[\alpha_n, \alpha_n]} \chi_{[0, \tau(s(X)))})$ for every $n, j$, we have

$$B_2 = \sum_{n,j} U_{n,j}^*n\varepsilon E[|A|](n\varepsilon, (n + 1)\varepsilon]U_{n,j} = \sum_{j=0}^{2C-1} U_{j}^*B_1U_{j}.$$

Since $\varepsilon < \frac{1}{2C}$, it follows that $0 \leq \mu(A) - \mu(B_1) < \frac{1}{2C}$ and therefore, by (15), we have

$$2C\mu(t; B_2) - \mu(t; X) = 2C\mu(t; \frac{1}{2C}; B_1) - \mu(t; X)$$

(17)

$$> 2C(\mu(t; \frac{1}{2C}; A) - \frac{1}{2C}) - \mu(t; X) \geq 0$$

for every $t \in [0, \tau(s(X)))$ (note that the $\mu(t; B_2) = \mu(t; \frac{1}{2C}; B_1)$ follows immediately from the definitions of $B_1$ and $B_2$).

Let $A_{\Delta} := \int \frac{\lambda/\varepsilon}{X} dE[|A|](\lambda)$ and $B_{\Delta} := J_X \mu(X)\mu(B_2)$. Here, $[\lambda/\varepsilon]$ is the integer part of $\lambda/\varepsilon$. Note that (17) implies that $B_{\Delta}$ is a bounded operator. Clearly, we have $A_{\Delta}|A| = B_1$ and $|X| = J_X \mu(X)\mu(B_2) = B_{\Delta}B_2$ (notice that $\sum_{n,j} n\varepsilon \chi_{[\alpha_n, \alpha_n]} \chi_{[0, \tau(s(X)))} = \mu(B_2)$). Let $U_{A}|A| = A$ and $U_{X}|X| = X$ be the polar decompositions. Define a homomorphism $T : S(M, \tau) \rightarrow S(M, \tau)$ by setting

$$TZ = U_X B_{\Delta}\left( \sum_{j=0}^{2C-1} U_j^*(A_{\Delta}U_j^*Z)U_j \right), \ Z \in S(M, \tau).$$

It is easy to verify that $TA = X$ and $T$ is a bounded homomorphism on the pair $(L_0(M, \tau), M)$ (one should note that operators of multiplication by $U_{j}$, $0 \leq j \leq 2C - 1$, $A_{\Delta}$ and $B_{\Delta}$ are bounded homomorphisms on the pair $(L_0(M, \tau), M)$ by Corollary 3).

The last statement follows immediately from Proposition 6.3.

**Remark 6.6.** The assumption that $M$ is a finite factor plays a crucial role in the above proof. The authors did not succeed in extending the result to the case for general semifinite von Neumann algebras.

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(Jinghao Huang) School of Mathematics and Statistics, University of New South Wales, Kensington, 2052, NSW, Australia E-mail: jinghao.huang@unsw.edu.au

(Fedor Sukochev) School of Mathematics and Statistics, University of New South Wales, Kensington, 2052, NSW, Australia E-mail: f.sukochev@unsw.edu.au