MOD-TWO COHOMOLOGY OF SYMMETRIC GROUPS AS A HOPF RING

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1. Introduction

We determine the mod-two cohomology of the disjoint union $BS_\bullet = \bigsqcup_n BS_n$ as a Hopf ring, shedding considerable light on the cup product structure in the cohomology of individual symmetric groups.

Definition 1.1. A Hopf ring is a ring object in the category of cocommutative coalgebras. Explicitly, a Hopf ring is a vector space $V$ with two multiplications and one comultiplication $(\cdot, \cdot, \Delta)$ such that each multiplication forms a bialgebra with the comultiplication and such that the three satisfy the distributivity relation

$$\alpha \cdot (\beta \circ \gamma) = \sum_{\Delta \alpha = \sum a' \otimes a''} (a' \cdot \beta) \circ (a'' \cdot \gamma).$$

On the cohomology of $BS_\bullet$ the second product $\cdot$ is cup product, which is zero for classes supported on disjoint components. The first product $\circ$ is the relatively new transfer product first studied by Strickland and Turner [21], (see Definition 3.1). It is akin to the “induction product” in the representation theory of symmetric groups, which dates back to Young and has been in standard use [9, 22]. The coproduct $\Delta$ on cohomology is dual to the standard Pontrjagin product on the homology of $BS_\bullet$. This Hopf ring structure was used by Strickland in [20] to calculate the Morava $E$-theory of symmetric groups.

Though Hopf rings were introduced by Milgram to study the homology of the sphere spectrum [11] and thus of symmetric groups [4], the Hopf ring structure we study does not fit into the standard framework. In particular it exists in cohomology rather than homology. See [21] for a lucid, complete discussion of the relationships between all of these structures. But like in calculations such as that of Ravenel and Wilson [17], we find this Hopf ring presentation to be quite efficient, given by a simple list of generators and relations.

Theorem 1.2. As a Hopf ring, $H^\ast(BS_\bullet; \mathbb{F}_2)$ is generated by classes $\gamma_{\ell,n} \in H^{n(2^\ell-1)}(BS_{n2^\ell})$, along with unit classes on each component. The coproduct of $\gamma_{\ell,n}$ is given by

$$\Delta \gamma_{\ell,n} = \sum_{i+j=n} \gamma_{\ell,i} \otimes \gamma_{\ell,j}.$$ 

Relations between transfer products of these generators are given by

$$\gamma_{\ell,n} \circ \gamma_{\ell,m} = \binom{n+m}{n} \gamma_{\ell,n+m}.$$ 

Relations between cup products of generators are that cup products of generators on different components are zero.
Since there are no relations between cup products of generators on a given component, all of the relations in the cohomology of symmetric groups follow from the distributivity of cup product over transfer product. Building on this presentation we give an additive basis, which is fairly immediate, and an explicit presentation of the multiplication rules for both products using that basis. The rule for cup product is more complicated but accessible, akin to the expression of multiplication of symmetric polynomials in terms of the additive basis of symmetrized monomials.

After giving an explicit additive basis presentation, we connect with previous work which built on invariant theory. We identify various quotient maps as restrictions in cohomology to elementary abelian subgroups. We revisit some of Feshbach’s calculations \cite{7} and reproduce his cup-product generators in terms of our Hopf ring generators. Some of Feshbach’s techniques can be adapted to the Hopf ring setting and in some places streamlined, but we defer such results for a full treatment at all primes. Ultimately, we find that the Hopf ring presentation of all components is more straightforward, while the cup ring structure for a single symmetric group is still complicated.

We also give our own invariant-theoretic presentation.

**Definition 1.3.** Consider the ring of polynomials $\mathbb{F}_2[x_A]$, where $A \subseteq \mathbb{m} = \{1, \ldots, m\}$. We call $x_{A'}$ a translate of $x_A$ if they are disjoint and of the same cardinality. A collection of translates is to be mutually disjoint.

Call a monomial $\prod x_{A_i}$ proper if whenever some $A_i$ and $A_j$ intersect, one is contained in the other, say $A_i \subset A_j$, and $A_j$ is the union of translates of $A_i$.

**Theorem 1.4.** The cohomology of symmetric groups $H^\ast(B\mathbb{S}_\bullet; \mathbb{F}_2)$ is isomorphic to the quotient of $\bigoplus \mathbb{F}_2[x_A | A \subseteq \mathbb{m}]$ by the additive submodule consisting of symmetrizations of monomials which are not proper.

The cohomology of symmetric groups is a classical topic, dating back to Steenrod’s \cite{19} and Adem’s \cite{2} studies of them in the context of cohomology operations. We heavily rely on Nakaoka’s seminal work \cite{13} which in particular determined the homology groups and cup coproduct structure on individual components. More explicit treatment of the cup product structure on cohomology was later given at the prime two partially by Hu‘ng \cite{8} and more definitively by Feshbach \cite{7}, using restriction to elementary abelian subgroups and invariant theory in different ways. Note that while Feshbach’s generators for cup ring structure are accessible, the relations are given recursively. The Hopf ring structure gives not only a compact, immediate description of all components at once but seems to extend to other primes, to other configuration spaces, and to related spaces.

We proceed with algebra first and develop Hopf ring structures on rings of symmetric invariants. Then we define the Hopf ring structure on the cohomology of symmetric groups and quickly obtain our descriptions, using primitivity of the transfer coproduct. We defer proof of this keystone primitivity result, a proof which is geometric at the moment, until the algebra is worked out and relationships with previous approaches have been discussed. At the end of the paper we show that Stiefel-Whitney classes for the standard representations can be used as Hopf ring generators, forging another tie between the categories of finite sets and vector spaces.
2. Hopf rings and symmetric sequences

The connection between the cohomology of symmetric groups and invariant theory has a distinguished history. Though invariant theory is a classical subject, to our knowledge the development of Hopf ring structures is a new tool in their study.

Definition 2.1. Let $A$ be an algebra which is flat over a ground ring $R$ (which is suppressed from notation). Let $\mu_{m,n} : A^\otimes m \otimes A^\otimes n \to A^\otimes m+n$ denote the standard isomorphism, and let $\Delta_{m,n}$ denote its inverse.

Let $A^S = \bigoplus_n (A^\otimes n)^{S_n}$, which we call the symmetric invariants of $A$. Define a coproduct $\Delta$ to be the sum of restrictions of $\Delta_{m,n}$. Define a product $\circ : (A^\otimes m)^{S_m} \otimes (A^\otimes n)^{S_n} \to (A^\otimes m+n)^{S_{m+n}}$ as the symmetrization of $\mu_{m,n}$ over $S_{m+n}/(S_m \times S_n)$.

Proposition 2.2. The symmetric invariants of $A$, namely $A^S$, with the product $\circ$, its standard product (which is zero for elements from different summands), and the coproduct $\Delta$ forms a Hopf ring.

Remark 2.3. This construction can be generalized in significant ways. First, the rings $A^\otimes n$ can be replaced by more general rings with $S_n$ action and analogues of maps $\mu$ and $\Delta$. More generally, they could be replaced by schemes, obtaining Hopf rings through regular functions or perhaps some sort of cohomology. Also, instead of symmetric groups other sequences of groups with inclusions $G_n \times G_m \to G_{m+n}$, in particular linear groups over finite fields, can be used. As we’ll see below, these invariant Hopf rings shed new light on some classical constructions. We content ourselves here with the minimum needed to treat cohomology of symmetric groups, leaving full generalization to further work.

Proof. The fact that the standard product and $\Delta$ form a bialgebra follows from the fact that the $\Delta_{m,n}$ are ring homomorphisms.
That $\circ$ and $\Delta$ form a bialgebra is also possible to establish for all of $A^\otimes m \otimes A^\otimes n$, not just the $S_m \times S_n$-invariants. First we consider $a_1 \otimes \cdots \otimes a_m \circ b_1 \otimes \cdots \otimes b_n$, which by definition is $\Delta_{m,n}$ of the symmetrization over $S_{m+n}$ of $\tau = a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_m$. We choose this symmetrization to be given by shuffles. Next we consider only the $\Delta_{m',n'}$-summand of $\Delta$, which takes the first $m'$ and last $n'$ tensor factors of a given tensor. The result of applying this to one of the shuffles at hand will be a shuffle of $a_1, \ldots, a_i$ and $b_1, \ldots, b_j$ tensored with a shuffle of $a_{i+1}, \ldots, a_m$ with $b_{j+1}, \ldots, b_n$. But these pairs of smaller shuffles are exactly what is obtained if one first applies $\Delta_{i,m-i} \otimes \Delta_{j,n-j}$ to $\tau$ and then $\circ$-multiplies, establishing the result.

Finally, for distributivity we start with $a \in (A^\otimes m+n)^{S_{m+n}}$ and $b$ and $c$ in $A^\otimes m$ and $A^\otimes n$ respectively. Then $a \cdot (b \circ c)$ is the product of $a$ with the symmetrization by shuffles of $b \otimes c$. But since $a$ is already symmetric this is equal to the symmetrization of $a \cdot (\mu_{m,n} b \otimes c) = \mu_{m,n} (\Delta_{m,n}(a) \cdot b \otimes c)$. Since there are no other terms in the coproduct of $a$ which non-trivially multiply $b \otimes c$, we get that $a \cdot (b \circ c) = \sum \Delta_a = a' \otimes a'' \cdot b \otimes a'' \circ c$.

In sum, what we have proven is that both $(\cdot, \Delta)$ and $(\circ, \Delta)$ define bialgebra structures on all of $\oplus_n A^\otimes n$, which then restrict to invariants. Moreover, distributivity of $\cdot \circ$ holds when multiplying something which is $S_n$ invariant, which means that when passing to such invariants we obtain a Hopf ring.

**Example 2.4.** The symmetric invariants $k[x]^S$ consists of the classical rings of symmetric polynomials over $k$.

The second product in the Hopf ring structure is the standard product of symmetric polynomials, defined to be zero if the number of variables differs. The coproduct is the standard “de-coupling” of two sets of variables, so for example

$$\Delta_{2,1}(x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2) = (x_1^2 x_2 x_1 x_2^2) \otimes x_1 + x_1 x_2 \otimes x_1^2.$$

The first product $f \circ g$ reindexes the variables of $g$, multiplies that by $f$, and then symmetrizes with respect to $S_{n+m}/S_n \times S_m$, as can be done with shuffles. A reasonable name for this product would be the shuffle product. For example

$$(x_1^2 x_2 + x_1 x_2^2) \circ x_1 = (x_1^2 x_2 + x_1 x_2^2) x_3 + (x_1 x_2^3 + x_1 x_3^2) x_2 + (x_2^2 x_3 + x_2 x_3^2) x_1 = 2\text{Sym}(x_1^2 x_2 x_3),$$

where $\text{Sym}$ denotes the standard $S_n$-symmetrization.

At this time, we do not know whether this Hopf ring structure on the collection of all symmetric functions has been considered.

Let $1_k$ denote the unit function on $k$ variables and $\sigma_n(k)$ the $n$th symmetric function in $k$ variables. Because $\sigma_n(k) = \sigma_n(n) \circ 1_{k-n}$, as a Hopf ring this collection of symmetric functions is generated by the $1_k$ and $\sigma_n = \sigma_n(n)$. The $\sigma_n \circ$-multiply according to the rule $\sigma_n \circ \sigma_m = \binom{n+m}{n} \sigma_{n+m}$, a divided powers algebra. Thus over the rationals only $\sigma_1$ is required to be a Hopf ring, while over $\mathbb{F}_p$ one needs all $\sigma_{np}$. Note that because of “periodicity” of binomial coefficients modulo $p$, the Hopf sub-rings generated by classes $\sigma_{np}$, for fixed $i$ (or equivalently, the quotient Hopf rings obtained by setting other symmetric polynomials to zero) are isomorphic to the full Hopf ring of symmetric functions. This isomorphism accounts for some “self-similarity” in the cohomology of symmetric groups.

The Hopf ring monomials in the $\sigma_n$ correspond to symmetrized monomials in the $x_i$. That is,

$$\sigma_n^{p_1} \circ \sigma_{n^2}^{p_2} \circ \cdots \circ \sigma_{n^k}^{p_k} \circ 1_j = \text{Sym}(x_1^{p_1} \cdots x_{n^1}^{p_1} x_{n^1+1}^{p_2} \cdots x_{n^1+n_2}^{p_2} \cdots x_{n_1+n_2}^{p_2} \cdots x_{n_1+n_2+\cdots+n_k}^{p_k}),$$
where \( \text{Sym} \) denotes symmetrization (note that any Hopf ring monomial has the form given by the left-hand side, since the generators \( \sigma_n \) have different numbers of variables and thus are zero when multiplied together under the first, standard product). The Hopf ring structure gives rise to a method to multiply symmetrized monomials. This approach is fairly indifferent to the classical theorem that such rings with a fixed number of variables form a polynomial algebra.

More generally, we consider \( A = k[x(1), \ldots, x(m)] \) in which case the ring of symmetric functions are symmetric polynomials in \( m \) collections of variables. Explicitly, we take polynomials in variables \( x(i) \) with \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \) which are invariant under permutation of the subscripts (alone). We then have the following.

**Proposition 2.5.** The symmetric invariants of \( A = k[x(1), \ldots, x(m)] \) is the Hopf ring generated over \( k \) by unit elements and \( \sigma(\ell)_n = x(\ell)_1 \cdot x(\ell)_2 \cdot \cdots \cdot x(\ell)_n \), for \( 1 \leq \ell \leq m \). The coproduct is given by

\[
\Delta \sigma(\ell)_n = \sum_{i+j=n} \sigma(\ell)_i \otimes \sigma(\ell)_j.
\]

The \( \circledast \)-products are given by

\[
\sigma(\ell)_i \circledast \sigma(\ell)_j = \binom{i+j}{i} \sigma(\ell)_{i+j},
\]

while \( \odot \)-products between classes with different \( \ell \) are free. In this presentation, the standard product is determined by Hopf ring distributivity and the fact that the collection of \( \sigma(\ell)_n \) for all \( \ell \) with fixed \( n \) form a polynomial ring.

**Proof.** That these symmetric functions have coproducts, \( \circledast \)-products and ordinary products as stated is straightforward. The fact that these basic symmetric function in each collection of variables are Hopf ring generators follows from the fact that their associated Hopf monomial basis coincides with the symmetrized monomial basis for the symmetric polynomials. The case of one set of variables is given in Equation (2) above. In general, we explicitly have that

\[
\begin{align*}
\bigcirc \prod_{1 \leq i \leq k} \sigma(\ell)_n_{1}^{p_{1,\ell}} &= \\
\text{Sym} \left( \prod_{\ell} x(\ell)_{1}^{p_{1,\ell}} \cdots x(\ell)_{n_{1}}^{p_{1,\ell}} \cdot \prod_{\ell} x(\ell)_{1+n_{1}}^{p_{2,\ell}} \cdots x(\ell)_{n_{1}+n_{2}}^{p_{2,\ell}} \cdots \cdot \prod_{\ell} x(\ell)_{1+\sum_{i<k} n_{i}}^{p_{k,\ell}} \cdots x(\ell)_{\sum n_{i}}^{p_{k,\ell}} \right).
\end{align*}
\]

Using the symmetric group action on subscripts to gather repeated exponents, any monomial in the \( x(\ell)_i \) can be put in the form of the right-hand-side.

This Hopf ring structure on symmetric functions is straightforward, closely tied to using symmetric monomials as an additive basis, and notationally convenient. Understanding only the standard product structure on its own with a fixed number of variables is complicated.

**Example 2.6.** Consider two sets of two variables, whose invariant functions have an additive basis of \( \sigma(1)_1^p \sigma(2)_1^q \circ \sigma(1)_1^r \sigma(2)_1^s \), with either \( p \neq q \) or \( r \neq s \), and \( \sigma(1)_2^p \sigma(2)_2^r \), which we can multiply in a straightforward manner.

Understanding this ring in terms of generators and relations is already involved. Over \( \mathbb{F}_2 \) the generators are: \( \sigma(1)_1 \circ 1 = x(1)_1 + x(1)_2; \sigma(1)_2 = x(1)_1 x(1)_2; \sigma(2)_1 \circ 1 = x(2)_1 + x(2)_2; \)
\[\sigma(2)_2 = x(2)_1 x(2)_2; \text{ and } \sigma(1)_1 \sigma(2)_1 \odot 1 = x(1)_1 x(2)_1 + x(1)_2 x(2)_2. \]

There is a fourth-degree relation, namely
\[
0 = (\sigma(1)_1 \sigma(2)_1 \odot 1)^2 + (\sigma(1)_1 \sigma(2)_1 \odot 1)(\sigma(1)_1 \odot 1)(\sigma(2)_1 \odot 1) + \\
\sigma(1)_2 (\sigma(2)_1 \odot 1) + (\sigma(1)_1 \odot 1) \sigma(2)_2.
\]

Indeed, the classical theorem that symmetric functions in one set of variables form a polynomial algebra is an anomaly, as even the simplest cases of multiple sets of variables are quite involved. The structure of such rings over \( \mathbb{F}_2 \) is the computational heart of Feshbach’s work on symmetric groups [7]. To our knowledge, even generators of these rings have not been computed over \( \mathbb{F}_p \) with \( p \) odd. Our Hopf ring presentation gives a manageable description with two products which might be helpful for many applications, and can help illuminate the analysis of the first product alone, as we see in work in progress.

3. Definition of the transfer product in the cohomology of unordered configuration spaces

The classifying space for symmetric groups is often modeled by unordered configuration spaces, which are a natural context to define the second product in our Hopf ring structure. Let \( \text{Conf}_n(X) = \{(x_1, \ldots, x_n) \in X^\times^n | x_i \neq x_j \text{ if } i \neq j \} \). Let \( \overline{\text{Conf}}_n(X) = \text{Conf}_n(X)/S_n \), where \( S_n \) acts on \( \text{Conf}_n(X) \) by permuting indices.

Definition 3.1. Consider the following maps
\[
\begin{array}{ccc}
\overline{\text{Conf}}_{m,n}(X) & \xrightarrow{p} & \overline{\text{Conf}}_n(X) \times \overline{\text{Conf}}_m(X) \\
\downarrow f & & \downarrow \\
\overline{\text{Conf}}_{m+n}(X).
\end{array}
\]

Here \( \overline{\text{Conf}}_{m,n}(X) \) is the space of \( m + n \) points in \( X \), \( m \) of which have label 0 and \( n \) of which have label 1. The map \( f \) forgets labels, and it is a covering map with \( \binom{n+m}{n} \) sheets. The map \( p \) is the product of maps which remember each of the two groups of points separately.

Define the transfer product \( \odot \) as the the composite \( \tau_f \circ p^* \), where \( \tau_f \) denotes the transfer map associated to \( f \) on cohomology.

When \( X = \mathbb{R}^\infty \) so that \( \overline{\text{Conf}}_n(X) \simeq B\mathcal{S}_n \), this product was previously studied by Strickland and Turner [21]. In this case the map \( p \) is a homotopy equivalence, so this composite is essentially the transfer map itself. Moreover, in this case the map \( f \) is homotopic to the map defining the product on \( B\mathcal{S}_n \). Recall that by either applying the classifying space functor to the standard inclusions \( S_n \times S_m \hookrightarrow S_{n+m} \) or by taking unions of unordered configurations in the \( \overline{\text{Conf}}_n(\mathbb{R}^\infty) \) model, we get a product on \( B\mathcal{S}_n \) which passes to a commutative product \( * \) on its homology. Its dual \( \Delta \) defines a cocommutative coalgebra structure on cohomology.

We paraphrase Theorem 3.2 of [21] to get the following.

Theorem 3.2. The transfer product \( \odot \) along with the cup product \( \cdot \) and the coproduct \( \Delta \) on cohomology dual to the standard product \( * \) on homology of \( B\mathcal{S}_n \) define a Hopf ring structure on \( H^*(B\mathcal{S}_n) \).
Remark 3.3. Strickland and Turner [21] actually show that the generalized cohomology of symmetric groups for any ring theory forms a Hopf ring. The $K$-theory of symmetric groups is a completion of the representation ring, by the Atiyah-Segal theorem [3], which is then a Hopf ring under induction product, tensor product and restriction coproduct. The bialgebra on the representation ring itself given by induction product and restriction coproduct alone was developed and computed by Zelevinsky [22]. We are currently determining what the Hopf ring structure with tensor product (also known as Kronecker product) can say about this classical object.

The cohomology (and representation theory) of various types of linear groups over finite fields also form Hopf rings, as do some rings of invariants under these groups such as Dickson algebras, as do the cohomology of some symmetric products. In work in progress, we are showing in both cases that those structures are compatible with that on rings of symmetric functions as in Proposition 2.5 and attempting to extend calculations.

Note that while the group completion of $BS_\bullet$ is the infinite loop space which represents the sphere spectum, this Hopf ring structure is not the standard one which arises on homology of $E_\infty$-ring spaces, since it is in particular a structure on cohomology and not homology.

We will require an alternate view of the coproduct dual to the transfer product.

Definition 3.4. Define the map

$$\phi_n : \overline{\text{Conf}}_{m+n}(X) \rightarrow SP^{(m+n)}\left(\overline{\text{Conf}}_m(X) \times \overline{\text{Conf}}_n(X)\right),$$

where $SP$ denotes the symmetric product functor, by taking an unordered configuration of $m + n$ points and dividing it into pair of unordered configurations, of $m$ and $n$ points, in all possible ways. The composition of $\phi_n$ with the standard inclusion of symmetric products,

$$\overline{\text{Conf}}_{m+n}(X) \xrightarrow{\phi_n} SP^{(m+n)}\left(\overline{\text{Conf}}_m(X) \times \overline{\text{Conf}}_n(X)\right) \hookrightarrow SP^\infty\left(\overline{\text{Conf}}_m(X) \times \overline{\text{Conf}}_n(X)\right),$$

extends canonically to

$$\phi'_n : SP^\infty(\overline{\text{Conf}}_n(X)) \rightarrow SP^\infty\left((\overline{\text{Conf}}_k(X) \times \overline{\text{Conf}}_{n-k}(X))\right).$$

Taking the product of these maps over $n$, applying homotopy and using the Dold-Thom theorem yields a map $\Gamma$ on homology.

Proposition 3.5. The map $\Gamma$ coincides with the coproduct $\Delta_\circ$ dual to the transfer product.

Because the fundamental geometry underlying the transfer product is that of taking a configuration and partitioning it into two configurations in all possible ways, we sometimes call it the partition product. Indeed, partitioning is part of the geometry as seen through Poincaré duality. The usual cup product of course corresponds to intersection of Poincaré duals, which means taking the the locus of configurations of $n$ points which satisfy the conditions defining the two cocycles in question. The locus defining the transfer product is similar, but we instead require that some $k$ points satisfy the first condition and then the complementary $n - k$ points satisfy the latter condition.

4. Review of homology of symmetric groups

We now focus on calculations with $\mathbb{F}_2$ coefficients. We start with the well-known structure of the Kudo-Araki-Dyer-Lashof algebra and the closely-related homology of symmetric groups. We recollect standard facts as a way to set notation.
Definition 4.1. Let $\mathcal{K}$ be the associative algebra over $\mathbb{F}_2$, with product $\circ$, generated by $q_0, q_1, \ldots$ with the following relations,

\[(\text{Adem})\: \text{For} \: m > n, \: q_m \circ q_n = \sum_i \left( \frac{i - n - 1}{2i - m - n} \right) q_{m+2n-2i} \circ q_i.\]

Given a sequence $I = i_1, \ldots, i_k$ of non-negative integers, let $q_I = q_{i_1} \circ \cdots \circ q_{i_k}$. Using the Adem relations, $\mathcal{K}$ is spanned by $q_I$ whose entries are non-decreasing. We call such an $I$ admissible. If such an $I$ has no zeros we call it strongly admissible.

Following [5], we call $\mathcal{K}$ the Kudo-Araki algebra, to distinguish it from a closely related presentation usually called the Dyer-Lashof algebra which has been more commonly used, for example in [6]. The algebra $\mathcal{K}$ is one of the main characters in algebraic topology because it acts on the homology of any infinite loop space, or more generally any $E_\infty$-space (see I.1 of [6]).

Definition 4.2. An action of $\mathcal{K}$ on a graded algebra $A$ with product denoted $\ast$ and grading denoted $\deg$ is a map from $\mathcal{K} \otimes A \to A$, typically written using operational notation, with the following properties:

- (Action) $(q_i \circ q_j)(a) = q_i(q_j(a))$.
- (Grading) $\deg q_i(a) = 2\deg a + i$.
- (Squaring) $q_0(a) = a^2$.
- (Vanish) $q_i(1) = 0$ for $i > 0$.
- (Cartan) For any $a, b$, we have $q_n(a \ast b) = \sum_{i+j=n} q_i(a) \ast q_j(b)$.

If $\mathcal{K}$ acts on $A$ we call $A$ a $\mathcal{K}$-algebra.

We denote powers in such an algebra $A$ by $a \ast \cdots \ast a = a^m$.

As is standard, there is a free $\mathcal{K}$-algebra functor, left adjoint to the forgetful functor from $\mathcal{K}$-algebras to vector spaces. With this, we can give the simplest reformulation of Nakaoka’s seminal result.

Theorem 4.3 ([13]). $H_*(BS_\bullet)$, with its standard product $\ast$, is isomorphic to the free $\mathcal{K}$-algebra generated by $H_0(BS_1)$. Thus, as a ring under $\ast$ it is isomorphic to the polynomial algebra generated by the nonzero class $i \in H_0(BS_1)$ and $q_I(i) \in H_*(BS_{2^k})$ for $I$ strongly admissible.

The second statement, which is essentially Nakaoka’s formulation, follows straightforwardly from the first statement. We will often abuse notation and refer to $q_I(i)$ as simply $q_I \in H_{|I|}BS_{2^k}$ with $|I| = i_1 + 2i_2 + \cdots + 2^{k-1}i_k$.

For our applications, we require geometric representatives for this homology.

Definition 4.4. Given $I = i_1, i_2, \ldots, i_k$ inductively define manifolds $\text{Orb}_I$ and maps $Q_I : \text{Orb}_I \to \overline{\text{Conf}}_{2^k}(\mathbb{R}^d)$ where $d > i_\ell$ for all $\ell$ as follows.

- If $I$ is empty $\text{Orb}_I$ is a point. Otherwise, $\text{Orb}_I = S^{i_1} \times \mathbb{Z}/2 (\text{Orb}_{I'} \times \text{Orb}_{I'})$, the quotient of $\mathbb{Z}/2$ acting antipodally on $S^{i_1}$ and by permuting the two factors of $\text{Orb}_{I'}$, where $I' = i_2, \ldots, i_k$.
- Let $\varepsilon = \frac{1}{4}$. If $I$ is empty, $Q_I$ sends $\text{Orb}_I$ to $0 = \mathbb{R}^0$. Otherwise, $Q_I(v \times \mathbb{Z}/2 (o_1, o_2))$ is given by $(v + \varepsilon Q_{I'}(o_1)) \cup (-v + \varepsilon Q_{I'}(o_2))$. Here we consider $v \in S^{i_1}$ to be a unit vector in $\mathbb{R}^{i_1+1}$. The configuration $v + \varepsilon Q_{I'}(o_1)$ is the configuration obtained by scaling each point in $Q_{I'}(o_1)$ by $\varepsilon$ and then adding $v$, perhaps after either the configuration or $v$ is included (canonically) into the larger of the two Euclidean spaces in which they are defined.
Theorem 4.5. The class $q_I$ in Theorem 4.3 is equal to $(Q_I)_*\text{[Orb}_I] \in H^*_*(\text{Conf}_{2k}(\mathbb{R}^\infty))$, where $[\text{Orb}_I]$ is the fundamental class of $\text{Orb}_I$.

On these $q_I$, the coproduct dual to the cup product is classically known, and thus it is determined on the entire homology of symmetric groups because of the bialgebra structure.

Definition 4.6. Define a coproduct $\Delta$ on $H^*_*(B\mathbb{S}_\bullet)$ by extending the formula for $I$ admissible

\[
\Delta(q_I) = \sum_{J+K=I} q_J \otimes q_K,
\]

where when $I = i_1, \ldots, i_n$ we have that $J$ and $K$ range over partitions of the same length such that for each $\ell, j_\ell + k_\ell = i_\ell$.

This coproduct is more complicated than it seems at first look. Even when starting with an admissible $I$, the sum above is over all possible $J$ and $K$. Thus to get an expression in the standard basis, as needed for example to apply the coproduct again, one must apply Adem and $q_0$ relations. The ones which get used most often are the relations $q_{2n+1}q_0 = 0$ and $q_{2n}q_0 = q_0q_n$.

Theorem 4.7 (See for example I.2 of [6]). Under the isomorphism of Theorem 4.3, the diagonal map on $B\mathbb{S}_\bullet$ induces the map $\Delta$ on homology.

On the other hand, one of our main results is that the coproduct dual to the transfer product is as simple as possible.

Theorem 4.8. The transfer product is linearly dual to the primitive coproduct on the Kudo-Araki-Dyer-Lashof algebra. That is, $\Delta_<(q_I) = q_I \otimes 1 + 1 \otimes q_I$, where $1$ is the non-zero class in $H_0(B\mathbb{S}_0)$.

We prove this theorem in Section 9. Presently we use it to quickly determine the cohomology of symmetric groups as a Hopf ring.

5. Hopf ring structure through generators and relations

The primitivity of the transfer coproduct coupled with some classical theorems immediately leads to algebraic presentations of $H^*(B\mathbb{S}_\bullet)$.

Recall from Theorem 7.15 of Milnor and Moore’s standard reference [12] that a Hopf algebra which is polynomial and primitively generated has a linear dual that is exterior, generated by linear duals (in the monomial basis) to generators raised to powers of two. If $\alpha \in H^*(B\mathbb{S}_\bullet)$ is a monomial in the $q_I$ we let $\alpha^\vee \in H^*(B\mathbb{S}_\bullet)$ denote the cohomology class which evaluates to one on $\alpha$ and is zero on all other monomials. Theorem 4.8 implies the following.
Corollary 5.1. Under the transfer product $\odot$ alone, the cohomology of $BS_\bullet$ is exterior, generated by $(q_{1}^{\ast 2k})^{\lor}$ for $I$ strongly admissible, or equivalently by $q_{I}^{\lor}$ for $I$ admissible.

It is straightforward to incorporate the cup product structure, through another classical theorem. As in I.3 of [6], let $R[n]$ be the span of the $q_{n}$ of length $n$, a submodule of $H_{*}(BS_{2^{n}})$. Note that in [6] the authors prefer to work with the Dyer-Lashof algebra generated by $Q^{I}$ rather than the Kudo-Araki algebra generated by $q_{I}$, which differ by re-indexing.

Definition 5.2. Let $q_{\ell 1}$ denote $q_{1,\ldots,\ell}$ and similarly let $q_{k-0,\ell 1}$ denote $q_{0,\ldots,0,1,\ldots,1} = q_{1,\ldots,1}^{* 2k}$ in $H_{2^{k}(2^{\ell}-1)}(BS_{2^{k+\ell}})$, where there are $k$ zeros and $\ell$ ones.

We let $\gamma_{\ell,n}$ denote the linear dual to $(q_{\ell 1})^{* n}$ in the monomial basis, so that in particular $\gamma_{\ell,2^{k}}$ is $q_{k-0,\ell 1}^{\lor}$.

Theorem 5.3 (Theorem I.3.7 of [6]). The linear dual of $R[n]$, which is an algebra under cup product, is a polynomial algebra generated by the classes $\gamma_{\ell,2^{k}}$ with $k + \ell = n$.

Indeed, it is simple to see that $q_{0,\ldots,0,1,\ldots,1}$ are primitive under $\Delta$, because $q_{1} \odot q_{0} = 0$. It is a straightforward induction to show that there are no other primitives, and then a counting argument to show that there are no relations among the $q_{k-0,\ell 1}^{\lor}$. Note however that because of the Adem relations in the Kudo-Araki-Dyer-Lashof algebra, the pairing between $q_{I}$ and polynomials in $\gamma_{\ell,2^{k}} = (q_{0,\ell 1})^{\lor}$ is complicated. For example, $\gamma_{1,2}^{3} = (q_{0,1}^{\lor})^{3} = q_{0,3}^{\lor} + q_{2,2}^{\lor}$, in part since $\Delta q_{2,2}$ includes a term $q_{2,0} \odot q_{0,2} = q_{0,1} \odot q_{0,2}$.

Theorem 5.3 gives us the last input we need to understand the cohomology of symmetric groups as a Hopf ring, since we see that under $\odot$ alone the generators are polynomials in the $\gamma_{\ell,2^{k}}$.

Theorem 5.4. As a Hopf ring, $H^{*}(BS_{\bullet})$ is generated by the classes $\gamma_{\ell,2^{k}}$.

The transfer product is exterior. The $\gamma_{\ell,2^{k}}$ with $\ell + k = n$ form a polynomial ring, and this along with Hopf ring distributivity determine the cup product structure.

This theorem is immediate from the fact that the transfer product generators $q_{I}^{\lor}$ of Corollary 5.1 are polynomials in $\gamma_{\ell,2^{k}}$.

Corollary 5.5. Any collection of classes $\{\alpha_{\ell,2^{k}}\}$ such that $\alpha_{\ell,2^{k}} \in H^{2^{k}(2^{\ell}-1)}(BS_{2^{k+\ell}})$ pairs non-trivially with $q_{k-0,\ell 1}$ constitutes a generating set for $H^{*}(BS_{\bullet})$ as a Hopf ring.

In order to fully understand the Hopf ring structure, and in particular be able to apply the distributivity relation, we need to compute the coproducts of the $\gamma_{\ell,2^{k}}$. More generally we consider $\gamma_{\ell,n}$, which by definition is $(q_{\ell 1})^{* n}$. We thus have that its coproduct is given as follows.

Proposition 5.6. The coproduct of $\gamma_{\ell,n}$ is given by

$$\Delta \gamma_{\ell,n} = \sum_{i+j=n} \gamma_{\ell,i} \otimes \gamma_{\ell,j}. $$

Taking the binary expansion $j = \sum 2^{k_{i}}$ with $k_{i}$ distinct, we have that $q_{\ell,1}^{* j} = \prod q_{\ell,1}^{* 2^{k_{i}}}$. By Theorem 5.3 and linear duality it follows that

$$\gamma_{\ell,j} = \bigodot_{j=\sum 2^{k_{i}}} \gamma_{\ell,2^{k_{i}}}. $$

Because the transfer product is exterior, we obtain the following.
Proposition 5.7. The transfer products of classes $\gamma_{\ell,n}$ are given by

$$\gamma_{\ell,n} \otimes \gamma_{\ell,m} = \binom{n+m}{n} \gamma_{\ell,n+m},$$

while transfer products between other classes have no relations.

Finally, the collection of all $\gamma_{\ell,n}$ such that $n2^\ell = m$ for a fixed $m$ are on the same component, so we need to understand their structure under cup product.

Proposition 5.8. The classes $\{\gamma_{\ell,n}\}$ such that $n2^\ell = m$ generate a polynomial subalgebra of $H^*(BS_m)$.

Proof. We start with $m = 2^p$ which is covered by Theorem 5.3, which says that the classes $\gamma_{\ell,2^k}$ with $k + \ell = n$ form a polynomial subring of $H^*(BS_{2^p})$. We can then use induction on the number of ones in the binomial expansion of $m$ and Hopf ring distributivity to establish the general case. $\square$

Collecting Theorem 5.4 and Propositions 5.6, 5.7 and 5.8 yields a proof of Theorem 1.2, our first presentation of the cohomology of symmetric groups as a Hopf ring.

6. Presentation of product structures through additive bases

We can use our knowledge of the Hopf ring structure on $H^*(BS_m)$ to explicitly understand the cup and transfer product structures through additive bases. We set the notational conventions that cup product has priority over transfer product, so that $a \cdot b \otimes c$ means $(a \cdot b) \otimes c$, and that exponents always refer to repeated application of cup product (an easy choice, since transfer product is exterior). Let $1_m$ denote the unit for cup product on component $m$.

We warm up with some calculations on the first even components (since the $F_2$-cohomology of $BS_{2k+1}$ is isomorphic to that of $BS_{2k}$). As $BS_2 \simeq \mathbb{R}P^\infty$ its cohomology is a polynomial ring generated by $\gamma_{1,1}$. 
For $H^*(BS_4)$ the Hopf ring monomial basis consists of classes $\gamma_{1,1}^i \circ \gamma_{1,1}^j \in H^{i+j}(BS_4)$ (which are zero if $i = j$) along with polynomials in $\gamma_{1,2} \in H^2(BS_4)$ and $\gamma_{2,1} \in H^3(BS_4)$. Using Hopf ring distributivity, we have that

$$(\gamma_{1,1}^i \circ \gamma_{1,1}^j) \cdot (\gamma_{1,1}^k \circ \gamma_{1,1}^\ell) = \gamma_{1,1}^{i+k} \circ \gamma_{1,1}^{j+\ell} + \gamma_{1,1}^{i+\ell} \circ \gamma_{1,1}^{j+k},$$

where we note one of these terms could be zero, if either $i + k = j + \ell$ or if $i + \ell = j + k$. In order to compute some products with $\gamma_{1,2}$ we have to compute its coproduct, which by Proposition 5.6 is equal to $\gamma_{1,2} \otimes 1_2 + \gamma_{1,1} \otimes \gamma_{1,1} + \gamma_{1,2} \otimes 1_2$. Using the distributivity relation,

$$\gamma_{1,2} \cdot (\gamma_{1,1}^n \circ \gamma_{1,1}^m) = (\gamma_{1,2} \cdot \gamma_{1,1}^n) \circ (1_0 \cdot \gamma_{1,1}^m) + (\gamma_{1,1} \cdot \gamma_{1,1}^n) \circ (\gamma_{1,2} \cdot \gamma_{1,1}^m) = \gamma_{1,1}^{n+1} \circ \gamma_{1,1}^{m+1}.$$

In general, most terms arising from the Hopf ring distributivity relation are zero because they involve multiplication of classes supported on different components. The last basic products to compute for $BS_4$ are $\gamma_{2,1} \cdot (\gamma_{1,1}^n \circ \gamma_{1,1}^m)$, which are zero because the coproduct of $\gamma_{2,1}$ is just $\gamma_{2,1} \otimes 1_0 + 1_0 \otimes \gamma_{2,1}$. Applying distributivity repeatedly we get that if $k \neq 0$

$$\gamma_{1,2}^p \gamma_{2,1}^q \cdot (\gamma_{1,1}^k \circ \gamma_{1,1}^\ell) = \begin{cases} \gamma_{1,1}^{k+p} \circ \gamma_{1,1}^{\ell+p} & \text{if} \ q = 0, \\ 0 & \text{if} \ q \neq 0, \end{cases}$$

which completes an understanding of how to multiply elements of our additive basis. The case of $BS_4$ is one of the very few in which it is simpler to understand the cup multiplicative structure in terms of ring generators and relations. From the multiplicative rules just given, it is a straightforward exercise to deduce that $\gamma_{1,1} \circ 1_2$, $\gamma_{1,2}$ and $\gamma_{2,1}$ generate the cohomology on this component, with the lone relation being $(\gamma_{1,1} \circ 1_2) \cdot \gamma_{2,1} = 0$. Similarly, a good exercise for the reader is to write down an additive basis for $H^*(BS_6)$, determine its multiplication rules, and then show that it is generated by $\gamma_{1,1} \circ 1_4$, $\gamma_{1,2} \circ 1_2$, $\gamma_{2,1} \circ 1_2$ and $\gamma_{1,2}^2 \circ \gamma_{1,1} \circ 1_2$, with the relation that $\gamma_{2,1} \cdot (\gamma_{1,1}^2 \circ \gamma_{1,1} \circ 1_2) = 0$.

In general, presentations in terms of generators and relations are quite complicated. We instead understand cup and transfer products explicitly in terms of a canonical additive basis.

**Definition 6.1.** A Hopf ring monomial in classes $x_i$ is one of the form $f_1 \circ f_2 \circ \cdots \circ f_k$, where each $f_j$ is a monomial under the first product in the $x_i$.

Because of the distributivity condition, any class in a sub-Hopf ring generated by some collection of classes is a sum of Hopf ring monomials of those classes.

**Definition 6.2.** A gathered monomial in the cohomology of symmetric groups is a Hopf ring monomial in the generators $\gamma_{\ell,n}$ where such $n$ are maximal or equivalently the number of transfer products which appear is minimal.

For example, $\gamma_{1,4} \gamma_{2,2}^3 \circ \gamma_{1,2} \gamma_{2,1}^3 = \gamma_{1,6} \gamma_{2,3}^3$. Gathered monomials such as the latter in which no transfer products appear are building blocks for general gathered monomials.

**Definition 6.3.** A gathered block is a monomial of the form $\prod_{i} \gamma_{\ell_i,n_i}^{d_i}$, where the product is the cup product. Its profile is defined to be the collection of pairs $(\ell_i, d_i)$.

Non-trivial gathered blocks must have all of the numbers $2^\ell n_i$ equal, and we call this number the length. We assume that the factors are ordered from smallest to largest $n_i$ (or largest to smallest $\ell_i$), and then note that $n_i = 2^{d_i - \ell_i} n_1$. 


Since the gathered monomials form a canonical additive basis for the cohomology, it would be interesting to understand their pairing with Nakaoka’s monomial basis for homology. Gathered blocks in just the $\gamma_{\ell,2^k}$ are the fundamental case, which as mentioned after Theorem 5.3 are interesting.

Definition 6.4. In the notation above, let $n_1 = \sum_{j=1}^{k} m_j$ be a partition of $n_1$. A partition of a gathered block into $k$ is defined by the set consisting of the $k$ blocks $\prod_i \gamma_{\ell_i,2^i-1} \cdots \gamma_{\ell_m,d_i}$. We allow for some $m_j$ to be zero, in which case the corresponding elements of the partition will be $1_0$.

A splitting of a gathered monomial $f_1 \odot \cdots \odot f_k$ into two is a pair of gathered monomials $f'_1 \odot \cdots \odot f'_k$ and $f''_1 \odot \cdots \odot f''_k$ where each $\{f'_i, f''_i\}$ is a partition of $f_i$ into two (which could be trivial - that is, of the form $\{1_0, f_i\}$).

Proposition 6.5. The coproduct of a gathered monomial is given by

$$\Delta f_1 \odot \cdots \odot f_k = \sum f'_1 \odot \cdots \odot f'_k \odot f''_1 \odot \cdots \odot f''_k,$$

where the sum is over all splittings of the monomial into two.

Proof. To establish the special case of gathered blocks - that is, having only one $f$ - we use the Hopf algebra compatibility of cup product and Pontryagin coproduct. The coproduct $\Delta$ of any $\gamma_{\ell_i,n_i,d_i}$ will correspond to partitions of $n_i$ into two. But only for the partitions of $n_1$ will there be corresponding partitions for all $n_i$ which yield non-trivial classes when cupped together. The resulting products correspond to the partitions of $f$ into two.

The general case follows from the Hopf algebra compatibility of partition product and Pontryagin coproduct. Because the monomial is gathered, no terms in the coproducts of $f_i$ can be equal, so we obtain no trivial transfer products when such terms collected. □

Definition 6.6. A partition of a gathered monomial in $H^*(BS_m)$ is a partition of each of its gathered blocks. The associated component partition is the partition of $m$ given by the components of the classes in the partition.

We define refinement of a partition of a gathered monomial in the obvious way, reflected faithfully by the refinement structure of the associated component partitions.

A matching $\mu$ between partitions of two gathered monomials is an isomorphism of their respective component partitions. We say that one matching refines another if that isomorphism commutes with inclusions of components under refinement.

For any gathered monomial in $H^*(BS_m)$ there is a canonical partition of $m$ defined by the components of its constituent gathered block monomials. The associated component partition of a monomial partition is a refinement of this canonical partition.

We are now ready to describe product structures in terms of our additive basis of gathered monomials.

Theorem 6.7. The transfer product of two gathered monomials is defined as the gathering of the Hopf ring monomial whose set of blocks is the union of their gathered blocks. The product is zero if the two monomials contain gathered blocks with the same profile whose lengths share some non-zero digit of their binary expansion.

If $x$ and $y$ are two gathered monomials, we let $M_{x,y}$ denote the set of matchings between any of their partitions which are not a refinement of some other matching. The cup product of $x$ and
$y$ is the sum

$$\sum_{\mu \in M_{x,y}} \left( \bigotimes_{b,b' \text{ matched by } \mu} \beta_{\mu} b \cdot b' \right),$$

where $\beta_{\mu}$ is zero if there are two products $b \cdot b'$ which result in blocks with the same profile and whose widths have binary expansion which share a non-zero digit.

**Proof.** We use gathered blocks, whose multiplication is polynomial by Theorem 1.2 as a base case for an induction on the total number of blocks in $x$ and $y$. View $x$ as a non-trivial transfer product of $x'$ and $x''$ which preserves blocks, so that $x'$ and $x''$ each has fewer blocks than $x$. The key is to see that each matching in $M_{x,y}$ coincides with some (arbitrary) partition of $y$ into $\{y', y''\}$ along with matchings of partitions of those pieces with partitions of $x'$ and $x''$. From this observation, the induction follows, with the coefficient $\beta_{\mu}$ accounting for when such a process yields a partition product of some monomial in the $\gamma_{\ell, 2^k}$ with itself. \qed

7. Topology and the Invariant Theoretic Presentation

Compare the presentation for the cohomology of symmetric groups as a Hopf ring, as given in Theorem 1.2, with the Hopf ring presentations of rings of symmetric functions, as given in Example 2.4 and Proposition 2.5. Seeing classes which behave similarly, we obtain some immediate identifications of split quotient rings of the cohomology of symmetric groups.

**Definition 7.1.** Define the level-$\ell$ quotient Hopf ring of the cohomology of symmetric groups, denoted $P_\ell$, to be the quotient Hopf ring obtained by setting all $\gamma_{\ell', n}$ for $\ell' \neq \ell$ equal to zero. It is isomorphic to the sub-Hopf ring generated by the classes $\gamma_{\ell, n}$. Let $P_\ell[m]$ be the sub-module of $P_\ell$ supported on $BS_m$, which is an algebra under cup product.

**Proposition 7.2.** The level-$\ell$ Hopf ring $P_\ell$ is isomorphic to the Hopf ring of classical symmetric polynomials. Thus $P_\ell[m]$ is a polynomial ring for any $m$.

The proof is an immediate comparison of their two presentations. We originally proved the second part directly from the Hopf ring presentation of $P_\ell[m]$, before realizing that we were mimicking the proof that symmetric functions form a polynomial algebra.

This identification has the following significant generalization.

**Definition 7.3.** Define the scale of $\gamma_{\ell, n}$ to be the product $\ell \cdot |n|_2$, where $|n|_2$ is the 2-adic valuation of $n$ (that is, the largest power of two which divides $n$). Define the scale-$k$ quotient Hopf ring of the cohomology of symmetric groups, denoted $Q_k$, to be the quotient Hopf ring obtained by setting all $\gamma_{\ell, n}$ with either scale less than $k$ or with $\ell > k$ to zero. It is isomorphic to the sub-Hopf ring generated by $\gamma_{\ell, n}$ with scale greater than or equal to $k$ and $\ell \leq k$.

**Proposition 7.4.** The scale-$k$ Hopf ring $Q_k$ is isomorphic to the Hopf ring of symmetric polynomials in $k$ sets of variables.

Once again, the proof is by a direct comparison, a proof made possible by the Hopf ring approach. The canonical isomorphism between them sends $\gamma_{\ell, n}$ with $\ell < k$ and scale greater than $k$ to the symmetric polynomial $\sigma(\ell)_m$ with $m = \frac{n}{2^{k-\ell}}$.

Our goals in the rest of this section are twofold. First we develop the standard topology which underlies these isomorphisms. Then, we move from these identifications of local invariant-theoretic sub/quotient rings to the global invariant-theoretic description of Theorem 1.4.
The predominant approach to the cohomology of symmetric groups has been through restricting cohomology to that of elementary abelian subgroups. For the following, we refer to Chapters 3 and 4 in [1]. We let $V_n$ denote the subgroup of $(\mathbb{Z}/2)^n \subset S_{2^n}$ defined by having $(\mathbb{Z}/2)^n$ act on itself. If we view this action as given by linear translations on the $\mathbb{F}_2$-vector space $\mathbb{F}_2^n$, then we can see that the normalizer of this subgroup is isomorphic to all affine transformations of $(\mathbb{F}_2)^n$. The Weyl group is thus $GL_n(\mathbb{F}_2)$, which acts as expected on the cohomology of $V_n$. The invariants $\mathbb{F}_2[x_1, \ldots, x_n]^{GL_n(\mathbb{F}_2)}$ are known as Dickson algebras, which are polynomial on generators $d_{k,\ell}$ in dimensions $2^k(2^\ell - 1)$ where $k + \ell = n$. (As mentioned earlier, these Dickson algebras together form a Hopf ring, which we are currently investigating.)

Since we base our work so squarely on Nakaoka’s homology calculation, our analysis of elementary abelian subgroups involves homology as well as cohomology.

**Lemma 7.5.** The image of the homology of $BV_n$ in that of $BS_{2^n}$ is exactly the span of the $q_I$ for $I$ admissible of length $n$.

**Proof.** The inclusion of $V_n$ into $S_{2^n}$ factors through the $n$-fold iterated wreath product of $\mathbb{Z}/2$ with itself, that is $\mathbb{Z}/2 \wr (\mathbb{Z}/2 \wr (\cdots (\mathbb{Z}/2 \wr \mathbb{Z}/2) \cdots ))$. But a well-known alternate definition of the Dyer-Lashof operations $q_I$ is through the homology of the inclusion of wreath products $\mathbb{Z}/2 \wr S_n \to S_{2^n}$. Inductively, the image of this iterated wreath product is given by length-$n$ Kudo-Araki-Dyer-Lashof classes, so the image of $V_n$ is contained in the span of such operations.

To see that the image of $V_n$ yields all such classes we compare ranks using the dual map in cohomology. The image in cohomology of the inclusion of $V_n$ in $S_{2^n}$ is all of the Dickson invariants $\mathbb{F}_2[x_1, \ldots, x_n]^{GL_n(\mathbb{F}_2)} \cong \mathbb{F}_2[d_{k,\ell}]$. This is most readily seen through the fact that the standard representation of $S_{2^n}$ through permutation matrices gives rise to a vector bundle which when pulled back to $BV_n$ splits as the sum of all possible line bundles. So the total Stiefel-Whitney class of this standard bundle in the cohomology of $BS_{2^n}$ maps to $\prod_{y \in H^1(BV_n)}(1 + y)$, where $y$ ranges over linear combinations of the $x_i$. But classical invariant theory identifies $\sum d_{k,\ell}$ with the product of all $1 + \lambda$ where $\lambda$ varies over all linear functions in the $x_i$. So these Stiefel-Whitney classes map exactly to the Dickson generators (or to zero).

By Madsen’s calculation, recounted in Theorem 5.3, the dual linear dual to the span of the $q_I$ length $n$ is a polynomial algebra in classes of dimension $2^k(2^\ell - 1)$ with $k + \ell = n$. Since the image of the cohomology of $BS_{2^n}$ in that of $BV_n$ has the same rank as this polynomial algebra, and thus as the span of $q_I$ of length $n$, the image in homology must be all of this span. □

Because the only classes in the Weyl-invariant cohomology of $BV_n$ in degrees $2^k(2^\ell - 1)$ are the Dickson classes, and the map in homology sends a generator in that degree to $q_{k,0,\ell,1}$, we have the following.

**Corollary 7.6.** The restriction of $\gamma_{\ell,2k}$ with $k + \ell = n$ to the elementary abelian subgroup $V_n$ is the Dickson class $d_{k,\ell}$.

Milgram, following Quillen [15], [16], showed that $H^*(BS_n)$ injects in the direct sum of the cohomology of elementary abelian subgroups (Quillen showed that this map has kernel consisting of nilpotent classes for any finite group). This lemma gives an alternate proof for that theorem through the following refinement.

**Corollary 7.7.** The image of the elementary abelian subgroup $\prod_k V_{k_j}$ in the homology of any symmetric group which contains it (that is, of order $\sum 2^{k_j}$ or greater) is the span of products
\[ \prod q_I, \text{ where } I \text{ is of length } k. \] Thus, the map from the homology of all elementary abelian subgroups to the homology of symmetric groups is surjective.

We now give a topological interpretation of Proposition 7.4.

**Theorem 7.8.** The map from \( H^\ast(BS) \) to its image in the cohomology of \( \coprod_m BV_k^m \) coincides with the quotient map defining the scale-\( k \) quotient ring \( Q_k \).

**Proof.** By Corollary 7.7, the image in homology of \( \coprod_m BV_k^m \) is the submodule of products of \( q_I \) of length \( k \). By Theorem 4.7 and Theorem 4.8, it is closed under the coproducts dual to cup and transfer product. Thus the image of this map of classifying spaces in cohomology, linear dual to this image of homology, is a quotient of the cohomology of symmetric groups as a Hopf ring.

Recall that \( \gamma_{\ell,n} = (q_{\ell+1,n})^\vee \), so that all \( \gamma_{\ell,n} \) with either scale less than \( k \) or \( \ell > k \) will evaluate to zero on the image of homology. An elementary counting argument shows that this ideal, the quotient by which defines \( Q_k \), is as large as possible so that the restriction of the cohomology of symmetric groups to these elementary abelian subgroups is exactly \( Q_k \).

We now restate and prove Theorem 1.4, giving a global invariant theoretic description of the cohomology of symmetric groups.

**Theorem 7.9.** The cohomology of symmetric groups \( H^\ast(BS; \mathbb{F}_2) \) is isomorphic to the quotient of \( \bigoplus_m \mathbb{F}_2[x_A | A \subset m]^{S_m} \) by the additive submodule consisting of symmetrizations of monomials which are not proper.

**Proof.** We begin with the abstract Hopf ring description of \( H^\ast(BS; \mathbb{F}_2) \) given in Theorem 1.2, and show that it is isomorphic to the quotient stated.

Given some \( A = \{1, \cdots, 2^k\} \subset m \) define its \( i \)th translate \( \tau_i A \) to be \( \{1+i2^k, \cdots, (i+1)2^k\} \). We start to define a map between \( H^\ast(BS; \mathbb{F}_2) \) and this ring of invariants by sending \( \gamma_{\ell,n} \) to \( \prod_{i=1}^n x_{\tau_i A} \) where \( A = \{1, \cdots, 2^k\} \). More generally, the gathered block \( \prod \gamma_{\ell_i,n_i} \) maps to the product where the first \( n_i \) translates of \( x_{1,\cdots,2^{i}} \) are raised to the \( \delta_i \)th power. Such products are proper monomials. The transfer products of gathered blocks go to symmetrized products of such monomials, after reindexing so that the subscripts corresponding to different gathered blocks are distinct. Just as we indicated for symmetric functions in Proposition 2.5, with patience we can see that all proper monomials can, after reindexing, be put in this form.

As we have mentioned, the standard approach to the cohomology of symmetric groups has long been through invariant theory, starting with Milgram’s theorem that this cohomology is given by the inverse limit of the Weyl invariant cohomology of elementary abelian subgroups. Theorem 1.4 is motivated by this approach, and we could also prove it using this framework and calculations of Dickson invariants. Since we have found the abstract description of this Hopf ring to be more useful and more closely tied to the well-understood homology, we have adopted that as our basic approach.

8. **Cup product generators after Feshbach**

Feshbach gives in [7] a complete minimal set of ring generators for \( H^\ast(BS_m; \mathbb{F}_2) \) along with relations which are minimal but not entirely explicit. Using some results from the previous section, we can express his generators in terms of our Hopf ring generators. The combinatorics of even the generating set is somewhat involved.
A minimal generating set under cup product for $H^*(BS_{12})$.

**Definition 8.1.** A level-$n$ Dickson partition of $p$ is an equality $p = \sum_{k<n} t_k (2^k (2^{n-k} - 1))$, where at least one of the positive integers $t_k$ is odd. Consistent with [7], we denote such a partition $\Lambda(n; t)$ or just $\Lambda$.

**Definition 8.2.** Let $\Lambda$ be a level-$n$ Dickson partition. Let $2^\ell$ be the largest power of two which occurs twice in the dyadic expansion of the $t_k$'s ($\ell = -\infty$ if there is no such overlap), and then define $\mu(\Lambda)$ to be $2^\ell + \sum 2^d$, where the sum is over all powers of two greater than $2^\ell$ which occur in the dyadic expansion of some $t_k$. In particular, $\mu(\Lambda)$ is just the sum of the $t_k$ if all powers of two in the dyadic expansion of the $t_k$ are distinct.

The maxwidth of $\Lambda$, denoted $w(\Lambda)$, is defined as $2^n \mu(\Lambda)$.

One of the main results of [7] is the following.

**Theorem 8.3.** There is a minimal generating set of the cohomology of $BS_m$ where the generators $v_\Lambda$ in dimension $p$ are indexed by Dickson partitions $\Lambda$ of $p$ with maxwidth less than $m$.

These generators must be expressible in terms of our Hopf ring generators.

**Theorem 8.4.** The generator $v_\Lambda$ can be taken to be equal to $\prod_{k+\ell = n} \gamma_{\ell,2^k} \otimes 1_{m-2^k}$. Moreover, one obtains a generating set by replacing $\gamma_{\ell,2^k}$'s by any classes which pair non-trivially with $q_{k,0,\ell,1}$.

**Proof of Theorem 8.4.** First, we must recall Feshbach’s characterization of his generators. Recall that the elementary abelian 2-subgroups $W_\pi$ of $S_m$ up to conjugacy are naturally indexed by partitions $\pi$ of $m$ as a sum of powers of two. If $\Lambda$ is a Dickson partition, we let $r_\Lambda$ be the smallest $r$ such that $t_r \neq 0$. We say that a partition $\pi$ of $m$ is $\mu$-subordinate to a level-$n$ Dickson partition $\Lambda$ if it contains a partition $\mu$ of the form $2^n = \sum_{j<r} s_j 2^{n-j}$. The generator $v_\Lambda$ is characterized by its restriction to $W_\pi$ being the sum $\sum_\mu v_{\mu,\pi}(\Lambda)$, where $\mu$ ranges over sub-partitions for which $\pi$ is $\mu$-subordinate to $\Lambda$ of $\pi$ and $v_{\mu,\pi}$ is the symmetrization of

$$1 \otimes 1 \otimes \cdots \otimes_j \left( \prod_i d_{n-j,i-j}^{t_i} \right) \otimes s_j \otimes 1 \otimes \cdots \otimes 1,$$
where $d_{n-j,i-j}$ is the appropriate Dickson polynomial.

The $\prod_{k+\ell=n} \gamma_{\ell,2k} t^k \circ 1_{m-2^k}$ has these same restrictions. The basic case of $m = 2^n$ is covered by Corollary 7.6, which says that the $\gamma_{\ell,2k}$ map to Dickson invariants $d_{k,\ell}$ in the cohomology of $V_n$. The general case follows from the fact that the inclusion $W_\pi = V_{n_1} \times \cdots \times V_{n_q}$ in $S_m$ factors through the inclusion of $S_{2^n_1} \times \cdots \times S_{2^n_q}$, which defines the product on homology. We can thus use the coproduct formula, namely Proposition 5.6 and the fact that the transfer coproduct is primitive to see that $\prod_{k+\ell=n} \gamma_{\ell,2k} t^k \circ 1_{m-2^k}$ maps to a class in the cohomology of $S_{2^n_1} \times \cdots \times S_{2^n_q}$ which in turn maps to the sum of $v_{\mu,\pi}$ as stated. □

In work in progress, we are investigating alternate generating sets built from transfer products alone (no cup products) of the $\gamma_{\ell,n}$, which might yield more tractable relations.

9. Proof of primitivity of the transfer coproduct

In this section we prove Theorem 4.8, which we restate here for convenience.

**Theorem 9.1.** The transfer product is linearly dual to the primitive coproduct on the Kudo-Araki-Dyer-Lashof algebra. That is, $\Delta \circ (q_I) = q_I \otimes 1 + 1 \otimes q_I$, where 1 is the non-zero class in $H_0(BS_0)$.

Our proof is geometric, proceeding through a dimension-counting argument which requires some “compactification technology” to get started, technology for which we refer to [IS].

**Definition 9.2.** Let $AS_{n,d}$ be the subspace of $\prod_{1 \leq i \neq j \leq n} S^{d-1}$ of collections of vectors $(v_{12}, v_{13}, \ldots)$ with $v_{ij} = -v_{ji}$. Let $\Pi : \text{Conf}_n(\mathbb{R}^d) \to AS_{n,d}$ send $(x_1, \ldots, x_n)$ to the point with $v_{ij} = \frac{x_j - x_i}{|x_j - x_i|}$. Let $\text{Conf}_n([\mathbb{R}^d])$ be the closure of the image of $\Pi$.

The following gathers some results from [IS], namely Corollary 4.5, Theorem 5.10 and Theorem 4.10, along with the fact that equivariant homotopy equivalences between compact manifolds with free actions by finite groups pass to homotopy equivalences of their quotients.

**Theorem 9.3.** Consider the following commutative diagram, where each vertical map is the quotient by the standard $S_n$ action. The top horizontal map $\Pi'$ is essentially $\Pi$, with domain changed to the closure of its image, and $\bar{\Pi}'$ is its passage to quotients.

$$
\begin{array}{ccc}
\text{Conf}_n(\mathbb{R}^d) & \xrightarrow{\Pi'} & \text{Conf}_n([\mathbb{R}^d]) \\
\downarrow & & \downarrow \\
\overline{\text{Conf}}_n(\mathbb{R}^d) & \xrightarrow{\bar{\Pi}'} & \overline{\text{Conf}}_n([\mathbb{R}^d]).
\end{array}
$$

The maps $\Pi'$ and $\bar{\Pi}'$ are homotopy equivalences.

The basic map in our second definition of the transfer (co)product passes as expected to these compactifications.

**Proposition 9.4.** The map $\phi_n : \overline{\text{Conf}}_{m+n}(\mathbb{R}^d) \to SP^{m+n}((\overline{\text{Conf}}_m(\mathbb{R}^d) \times \overline{\text{Conf}}_n(\mathbb{R}^d)))$ extends to a map $\phi_{(n)} : \overline{\text{Conf}}_{m+n}([\mathbb{R}^d]) \to SP^{m+n}((\overline{\text{Conf}}_m([\mathbb{R}^d]) \times \overline{\text{Conf}}_n([\mathbb{R}^d])))$. 

Sketch of proof. First define
\[ \hat{\phi}_n : AS_{m+n,d} \to \prod_{\alpha, \beta = n} AS_{\#\alpha, d} \times AS_{\#\beta, d} \]
by, for each \( \alpha \prod \beta \), projecting onto the collection of \( x_{ij} \) with \( i \) and \( j \) in both \( \alpha \) or both in \( \beta \). We may show that \( \hat{\phi}_n \) restricted to \( \text{Conf}_n([\mathbb{R}^d]) \) maps to the product of \( \overline{\text{Conf}}_m([\mathbb{R}^d]) \times \overline{\text{Conf}}_n([\mathbb{R}^d]) \), by showing that its image satisfies the conditions of Theorem 5.14 of [18]. Call the resulting restriction \( \hat{\phi}_{(n)} \). We can then pass from \( \hat{\phi}_{(n)} \) to a unique map \( \phi_{(n)} \) on quotient spaces. Finally, it is straightforward to show that \( \phi_{(n)} \) agrees with \( \hat{\phi}_n \) on the image of \( \Pi \).

Using compactifications, we can give a more canonical representative for the homology classes \( q_I \).

Definition 9.5. Consider the unique binary tree \( T_k \) with \( k \) levels, thus having \( 2^k \) leaves. Thus there are \( 2^\ell \) vertices which are \( \ell \) edges away from the root for \( 0 \leq j \leq k - 1 \), and each vertex has two incoming edges. If \( I = i_1, \ldots, i_k \) let \( T_I \) be the labeled tree with underlying tree \( T_k \) and with labels on internal vertices given by \( i_\ell \) as the label of each vertex of level \( \ell \); notationally, \( \lambda(v) = i_\ell \).

Redefine \( \text{Orb}_I \) as the quotient of \( \prod_{v \in \text{Vert} T_I} S^{\lambda(v)} \) by the action of the group of automorphisms of \( T_I \). This automorphism group is an iterated wreath product of \( \mathbb{Z}/2 \)'s. An automorphism acts by permuting factors corresponding to the vertices, and by the antipodal map on \( S^{\lambda(v)} \) if it switches left and right incoming edges of \( v \).

Let \( T_I \) be \( T_I \) with a labeling of its leaves by \( 1, \ldots, 2^k \). Given leaf labels \( i \) and \( j \) let \( v(ij) \) be their “join”, that is the vertex of greatest height under leaves \( i \) and \( j \). Define the map
\[ \hat{Q}_{(I)} : \prod_{v \in \text{Vert} T_I} S^{\lambda(v)} \to AS_{n,d} \]
by sending \( \prod_{v \in \text{Vert} T_I} x_v \) to \( \prod_{1 \leq i \neq j \leq k} y_{ij} \) with \( y_{ij} = \pm x_{v(ij)} \) where the sign is +1 if \( i < j \) and -1 if \( i > j \).

Proposition 9.6. \( \hat{Q}_{(I)} \) has image in \( \text{Conf}_n([\mathbb{R}^d]) \subset AS_{n,d} \).

- Considered as a map to \( \text{Conf}_n([\mathbb{R}^d]) \) the map \( \hat{Q}_{(I)} \) covers a well-defined map \( Q_{(I)} : \text{Orb}_I \to \overline{\text{Conf}}_n([\mathbb{R}^d]) \).
- The maps \( \Pi \circ Q_{(I)} \) and \( Q_{\{I\}} \) are homotopic, and thus send the fundamental class of \( \text{Orb}_I \) to the same homology class, which we identify to be \( q_I \) in the homology of \( BS_{2^k} \).

Sketch of proof. The first fact is a straightforward check that the image of \( \hat{Q}_{(I)} \) satisfies the conditions of Definition 5.11 of [18] for being in \( \text{Conf}_n([\mathbb{R}^d]) \). The second statement is also straightforward just from the definitions, essentially seeing that \( \hat{Q}_{(I)} \) composed with the projection from ordered to unordered configuration spaces is independent of the choice of labeling of the leaves of \( T_I \). The third statement is the most involved, as one has to check the continuity of the homotopy defined by having “\( \varepsilon \) go to zero” in Definition 4.1.

With this more canonical model for \( q_I \) we can prove our primitivity result.

Proof of Theorem 4.8 Consider partitions of the leaves of \( T_I \) into two subsets \( b, w \) which we call black and white, of respective cardinality \( i \) and \( 2^k - i \). Let \( S_{k,i} \) denote the set of isomorphism
classes of such partitions. For example $S_{2,2}$ contains two elements which we call $\alpha$ and $\beta$, for which in $\alpha$ the two black leaves lie over the same vertex of level one and in $\beta$ they do not.

Given a partition $\alpha = (b, w)$ define the tree $T^{b(\alpha)}_I$ (or sometimes just $T^{b}_I$) as obtained by taking the minimal sub-tree containing the leaves labelled $b$ and then removing any bivalent vertices – that is, those with only one incoming and one outgoing edge – and identifying their adjacent edges. Define $T^{w(\alpha)}_I$, or just $T^{w}_I$ when $\alpha$ is understood, similarly. For example for $\alpha$ as above, $T^{b(\alpha)}_I$ has a single internal vertex, corresponding to the vertex of level one under the black leaves, while $T^{w(\alpha)}_I$ has one vertex corresponding to the one under the white leaves. On the other hand $T^{w(\beta)}_I$ and $T^{b(\beta)}_I$ are trees with a single pair of edges emanating from a vertex which corresponds to the root of the original tree.

We let $T^{b}_I$ and $T^{w}_I$ inherit the vertex labels of $T_I$. We may apply Definition 9.5 and Proposition 9.6 to $T^{b}_I$ and $T^{w}_I$, and the trees $\hat{T}^{b}_I$ and $\hat{T}^{w}_I$ obtained by adding labels, noting that their automorphism group is no longer necessarily a simple iterated wreath product of $\mathbb{Z}/2$. By doing so we obtain manifolds $\text{Orb}^{b}_I$ and $\text{Orb}^{w}_I$ and maps $Q^{b}_I$ and $Q^{w}_I$ respectively from those manifolds to $\text{Conf}_I([\mathbb{R}^d])$ and $\text{Conf}_{2k-i}([\mathbb{R}^d])$ respectively.

We construct a commutative diagram as follows,

$$
\begin{array}{ccc}
\text{Orb}_I & \xrightarrow{Q_I} & \text{Conf}_{2k}([\mathbb{R}^d]) \\
\varphi_I & & \phi_{(i)} \\
\prod_{\alpha \in S_{k,i}} SP(\#\alpha) \left( \text{Orb}^{b(\alpha)}_I \times \text{Orb}^{w(\alpha)}_I \right) & \xrightarrow{SP(Q^{b(\alpha)}_I \times Q^{w(\alpha)}_I)} & SP(\mathbb{R}^k) \left( \text{Conf}_I([\mathbb{R}^d]) \times \text{Conf}_{2k-i}([\mathbb{R}^d]) \right).
\end{array}
$$

Here $\#\alpha$ is the number of partitions of leaves of $T_I$ which are isomorphic to $\alpha$ (which is equal to the order of the automorphisms of $T_I$ modulo those which leave $\alpha$ invariant). The map $\varphi_I$ is defined as a symmetrized product of quotient maps (recall that $\text{Orb}^{b}_I$ and $\text{Orb}^{w}_I$ are quotients of $\text{Orb}_I$). The existence of this factorization follows from seeing that in the image of $\phi_{(i)} \circ Q_I$ only the factors labelled by vertices and $T^{b(\alpha)}_I$ and $T^{w(\alpha)}_I$ appear in the composite.

As in Definition 4.1, extend this diagram to a diagram of multiplicative maps between (products of) infinite symmetric products. If we apply homotopy groups and then use the Dold-Thom theorem we obtain the diagram

$$
\begin{array}{ccc}
H_*(\text{Orb}_I) & \xrightarrow{Q_I} & H_*(\text{Conf}_{2k}([\mathbb{R}^d])) \\
\oplus_{\alpha \in S_{k,i}} H_*(\text{Orb}^{b(\alpha)}_I) \otimes H_*(\text{Orb}^{w(\alpha)}_I) & \xrightarrow{\otimes Q^{b(\alpha)}_I \otimes Q^{w(\alpha)}_I} & H_*(\text{Conf}_I([\mathbb{R}^d])) \otimes H_*(\text{Conf}_{2k-i}([\mathbb{R}^d])).
\end{array}
$$

To complete the proof, we show that the dimension of $\text{Orb}^{b(\alpha)}_I \times \text{Orb}^{w(\alpha)}_I$ is strictly less than that of $\text{Orb}_I$ for proper $\alpha$, that is for $0 < i < 2^k$. The commutativity of the diagram above then implies that $\Gamma_i$, the relevant summand of $\Gamma$, applied to $q_I = Q_I|_{\text{Orb}_I}$ factors through a trivial homology group.
The dimensions of $\text{Orb}_I$, $\text{Orb}_I^{b(\alpha)}$ and $\text{Orb}_I^{w(\alpha)}$ are given by the sum of the labels of their vertices, so we want to show that
\[
\sum_{v \in T_I} \lambda(v) > \sum_{v^b \in T^b_I} \lambda(v^b) + \sum_{v^w \in T^w_I} \lambda(v^w).
\]
First note that because a binary tree with $k$ leaves has $k - 1$ vertices, the left-hand side of Equation 3 has $2^k - 1$ terms and the right-hand side has $i - 1 + (2^k - i) - 1 = 2^k - 2$ terms. We claim that if a vertex $v_r$ is repeated on the right-hand side of Equation 3 (that is, some vertex is common to the black and white trees), then some vertex above $v_r$ will not contribute to the sum. Indeed, apply the previous observation to the subtree of $T_I$ which lies above $v_r$, which necessarily inherits a non-trivial partition since there are both black and white leaves above both edges of $v_r$. Also, no vertices of highest level are repeated. Because $I$ is admissible, this means that the greatest possible value for the right-hand side of Equation 3 is that for which only the root vertex term is missing, establishing that the dimension of $\text{Orb}_I$ is greater by at least $i_1$ than the dimension of $\text{Orb}_I^{b(\alpha)} \times \text{Orb}_I^{w(\alpha)}$.

We end discussion of primitivity with a “non-result.” One view of the cohomology of $BS_m$ is as a quotient of that of $BS_\infty$, as first established by Steenrod. We originally searched for a Hopf ring structure there, but the primitivity of $\Delta_\otimes$ as a coproduct with respect to the product $*$ shows that the transfer product is a strictly unstable phenomenon. We see for example that for $t^n$ the non-zero class in $H_0(BS_m)$ the coproduct is given by $\Delta_\otimes(t^n) = \sum_{i+j=n} \binom{n}{i} t^i \otimes t^j$. The standard inclusion map $BS_m \to BS_\infty \times \mathbb{Z}$, which is a “group completion” map, gives rise to inverting $t$ on homology. There is no good way to extend the coproduct formula for $t^n$ for negative $n$. The prevalence of viewing the homology and cohomology of symmetric groups through the limit perhaps explains the fact that the transfer product was not exploited until recently.

10. Stiefel-Whitney generators

We now give an alternate, more geometric presentation of this Hopf ring which uses Stiefel-Whitney classes. Though it has a more complicated coproduct formula, such a presentation should be useful for geometric applications. It also could give a better presentation for understanding Steenrod operations by building on the Wu formula once it is known how Steenrod operations behave on transfer products. Finally, the appearance of Stiefel-Whitney classes does forge another significant link between the categories of finite sets and finite-dimensional vector spaces, showing that the cohomology of automorphisms of finite sets is in this two-product sense generated by that of automorphisms of vector spaces.

**Definition 10.1.** Let $w_{i,n} \in BS_n$ be the pull-back of the $i$th Stiefel-Whitney class through the classifying map of the standard representation $\rho_n : S_n \to O(n)$ given by permutation matrices.

**Remark 10.2.** Stiefel-Whitney classes in symmetric groups have Poincaré dual representatives which are simple to describe in the configuration space model of $BS_n$. First replace $\overline{\text{Conf}}_n(\mathbb{R}^\infty)$ by the homotopy equivalent subspace of configurations which are linearly independent. The tautological bundle over $BO(n)$ pulls back to the bundle $E_n$ whose fiber over some configuration $x$ is the vector space span $V_x$ of the points $x_i$ in $x = (x_1, \ldots, x_n)/\sim$. Recall that $w_i$ of any bundle is the Poincaré dual to the locus where a generic collection of $n - i + 1$ sections becomes linearly dependent. In this case, we may construct such sections by taking a standard basis $e_1, \ldots, e_{n-i+1}$
An intersection calculation, reflecting that $w_{2,4}$ pairs non-trivially with $q_1 * q_1$.

and projecting each one into $V_x$. Elementary linear algebra tells us that these projections will be dependent if and only if the projection of $V_x$ onto $\mathbb{R}^{n-i+1} = \text{Span}(e_1, \ldots, e_{n-i+1})$ is less than full rank.

That is, the Poincaré dual of $w_i$ is the collection of unordered configurations whose projection onto their first $n - i + 1$ coordinates is not of full rank. Thus for example $w_1$ records the linear dependence of a configuration of $n$ points in $\mathbb{R}^\infty$ when projected onto $\mathbb{R}^n$. If we replace the bundle $E_n$ by $\overline{E}_n$, defined by taking the span of the vectors $x_i - x_1$, then $w_{2,4}$ is Poincaré dual to the locus of configurations of four points in $\mathbb{R}^\infty$ which when projected onto $\mathbb{R}^2$ are collinear.

Through these Poincaré duals, we can explicitly see the pairings between Stiefel-Whitney classes (and their cup and partition products) and polynomials in $q_I$ by counting intersections as in the figure above.

**Definition 10.3.** Let $w(k, \ell) = w_{2k(2\ell-1), 2k+\ell}$.

We will use Corollary 5.5 to show that the Stiefel-Whitney classes $w(k, \ell)$ generate $H^*(BS\mathbb{S}_*)$ as a Hopf ring. The needed calculation is a special case of what is needed to understand the coproducts of these classes, so we treat the entire structure at once.

**Proposition 10.4.** Let $q$ be a monomial in the $q_I$. Then $w_{i,n}$ evaluated on $q$ is one if $q$ is a product of classes $q_{\ell,1}$ and $q_0$ and is zero otherwise.

**Proof.** To set notation $H_*(\coprod_n BO(n))$ is the free polynomial ring on classes $b_i$ in degree $i$, with $i \geq 0$, which are in the homology of $BO(1) = \mathbb{R}P^\infty$. Recall that $w_{i,n} \in H^*(BO(n))$ evaluates to one on $b_0^{n-i}b_1^i$ and to zero on all other classes. This calculation is easily established inductively using the fact that stably the coproduct of $w_i$ dual to the Pontrjagin product is $\sum w_{i-j} \otimes w_j$.

We use the fact that $\coprod_n B\rho_n$ is map of $E_\infty$-spaces (see for example II.7 of [6]) to see $q_I = q_I(\iota)$ maps to $q_I(b_0)$. Then we can use the structure of the homology of $\coprod_n BO(n)$ over the Kudo-Araki-Dyer-Lashof algebra, as computed by Kochman [10] and then Priddy [14], determined by

$$q_r(b_n) = \sum_i \binom{r+i-1}{i} b_{n-i}b_{r+n+i}. \tag{4}$$

Given that $w_{i,n}$ only pairs with $b_0^{n-i}b_1^i$ we call a monomial $m$, which is product of $b_j$’s, tainted if one of those $b_j$ has $j > 1$. The basic observation is that if $m$ is tainted then $q_k(m)$ is tainted for any $k$. Indeed, using the Cartan formula we see that $q_k(m)$ is sum of products of $q_k(b_{j_i})$. But using Equation 3 for the $j_i > 1$ the factor $q_{k_i}(b_{j_i})$ will be a sum of products of two $b$’s of total degree $2j_i + k_i \geq 4$, so at least one must have degree greater than one.
If in \( q_{i_1, \ldots, i_k} \) we have \( i_k > 1 \), then because \( q_{i_k}(b_0) = b_0 b_{i_k} \) is tainted, so will be every term in \( q_{i_1, \ldots, i_k}(b_0) \). Thus \( w_{i,n} \) must evaluate trivially on any monomial which is a product of at least one such \( q_1 \).

To see that on the other hand \( w_{i,n} \) does evaluate non-trivially on a monomial in the \( b_1, \ldots, 1 \), we calculate \( q_{1, \ldots, 1}(b_0) \). We get that \( q_1(b_0) = b_0 b_1 \), and then that

\[
q_{1,1}(b_0) = q_1(q_1(b_0)) = q_1(b_1 b_1) = q_1(b_0)q_0(b_1) + q_0(b_0)q_1(b_1) = b_0 b_1^3 + b_0^2 b_3 + b_0^3 b_1 b_2.
\]

In general, \( q_{1, \ldots, 1}(b_0) \) is equal to \( b_0 b_1^{2k-1} \) plus tainted monomials. Because \( B \rho_s \) is a map of rings, a product of such classes in degree \( i \) will equal \( b_0^{n-i} b_1^i \) plus tainted monomials, and thus be evaluated non-trivially by \( w_{i,n} \), completing the argument.

We now develop the combinatorics necessary to express the coproducts of Stiefel-Whitney classes.

**Definition 10.5.** A Dickson bi-partition of the pair \((k, \ell)\) is an equality of pairs of positive integers

\[
(2^k(2^\ell - 1), 2^{k+\ell}) = \sum_i \left(2^{k_i}(2^{\ell_i} - 1), 2^{k_i+\ell_i}\right).
\]

Here we allow the trivial one-term partition, and we allow \( k_i \) to be zero as well as \( \ell_i \) to be zero when the corresponding \( k_i \) is. We manipulate such a partition as a set \( p = \{ (k_i, \ell_i) \} \), and sometimes emphasize the numbers being partitioned by writing \( p = p(k, \ell) \).

We say one bi-partition refines another if it is obtained by substituting of some entry or entries by corresponding Dickson bi-partition(s).

For example, because \((24, 32) = (4, 8) + (6, 8) + (14, 16)\) we have the corresponding Dickson bi-partition \( p(3, 2) = \{ (2, 1), (1, 2), (1, 3) \} \). Because in turn of the equality \((4, 8) = (0, 2) + (1, 2) + (3, 4)\), we have that \( q = \{ (1, 0), (0, 1), (1, 1), (1, 2), (1, 3) \} \) refines \( p \).

**Definition 10.6.** Let \( \Pi_{k,\ell} \) denote the set consisting of Dickson bi-partitions \( p \) expressed as an ordered union of two smaller partitions \( p = p' \cup p'' \), each of which contains no repeated pairs of numbers. Define a partial order on \( \Pi_{k,\ell} \) by \( p' \cup p'' \leq q' \cup q'' \) if \( p' \) is a (possibly trivial) refinement of \( q' \) and \( p'' \) of \( q'' \).

Let \( \phi \) be the \( \mathbb{F}_2 \)-valued function on \( \Pi_{k,\ell} \) defined uniquely by

\[
\sum_{p' \cup p'' \leq q' \cup q''} \phi(q' \cup q'') = 1,
\]

for any \( p' \cup p'' \in \Pi_{k,\ell} \).

In other words, the function \( \phi \) is the inverse under convolution to the function which is one on all of \( \Pi_{k,\ell} \). Thus it could be determined by Möbius inversion, though we have not found that to be enlightening.

**Theorem 10.7.** As a Hopf ring, \( H^*(BS_\ast) \) is generated by Stiefel-Whitney classes \( w(k, \ell) \).

The transfer product is exterior, and there are no further relations other than the Hopf ring structure.
The coproduct is given by
\[
\Delta w(k, \ell) = \sum_{p' \cup p'' \in \Pi_k, \ell} \phi(p' \cup p'') \left( \bigotimes_{(k_i, \ell_i) \in p'} w(k_i, \ell_i) \right) \bigotimes_{(k_j, \ell_j) \in p''} w(k_j, \ell_j).
\]

Proof. That these Stiefel-Whitney classes generate is now an immediate application of Proposition 10.4 to verify the hypothesis of Corollary 5.5. The lack of further relations and the additive basis follow from Theorem 5.4 just as this Corollary 5.5 did.

The coproduct formula is verified by direct check using bialgebra structure. By Proposition 10.4, \( w(k, \ell) \) evaluated on some non-trivial product \( m \ast m' \) which is a monomial will be non-zero if and only if \( m \) and \( m' \) are products of \( (q_0)'s \) and \( q_1, \ldots, 1's \). Such products are in one-to-one correspondence with the set \( \Pi_{k, \ell} \). Looking at only \( m \), first we express each \( q^n_{1, \ldots, 1} \) uniquely as a product of \( q^n_{k, \ldots, 1} = q_0 \ldots q_{k-1} \) and then record the \( (k, \ell) \) which appear. For example, \( q_0 q_1 q_{1, 1} q_{1, 1, 1} = q_0 \ast q_1 \ast q_{1, 1} \ast q_{1, 1, 1} \) corresponds to \( \{(1, 0), (0, 1), (1, 1), (0, 2), (1, 3)\} \). Call this bijection \( \beta \) from the set of monomials in \( q_1, \ldots, 1 \) to Dickson bi-partitions.

Applying Proposition 10.4 we find that not only does \( m \otimes m' \) pair with \( \bigotimes_{(k_i, \ell_i) \in \beta(m)} w(k_i, \ell_i) \otimes \bigotimes_{(k_j, \ell_j) \in \beta(m')} w(k_j, \ell_j) \) but it also pairs with all similar products of Stiefel-Whitney classes over \( q' \cup q'' \) which are refined by \( \beta(m) \cup \beta(m') \). Thus, if we take the linear combination with coefficients given by \( \phi \), that sum will pair to one with \( m \otimes m' \).

By 10.2 we see that the cohomology of symmetric groups, when viewed through the model of unordered configurations in \( \mathbb{R}^\infty \) and applying Poincaré duality, is represented by subvarieties which are ultimately defined by quadratic equations controlling linear dependence. Originally the third author had conjectured that the cohomology would be Poincaré dual to subvarieties which, when lifted to the universal cover, would be defined by linear equations. While that is not implied by our results, and is probably not true, the basic intuition that cocycles defined through linear algebra should play a central role in the cohomology of symmetric groups is satisfyingly correct.

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