Oscillating factorial cumulants in counting statistics are due to interactions

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Abstract. We discuss our recent theoretical proposal to detect interactions in the electron transport through a nano-scale system by measuring high-order factorial cumulants of the full counting statistics. Our proposal is based on theoretical studies which have demonstrated that the zeros of the generating function for the full counting statistics are always real and negative for non-interacting electrons in a two-terminal scattering setup. As we have shown, this implies that the factorial cumulants do not oscillate as functions of any system parameter. Interactions, however, can cause the zeros to move away from the negative real axis into the complex plane. This transition is clearly visible in the factorial cumulants which start oscillating. We illustrate our findings with a model of transport through a two-level Coulomb blockade quantum dot, which we analyze both for finite times and in the long-time limit, and we discuss possible experimental implementations to test our predictions.

1. Introduction

Full counting statistics (FCS) concerns the stochastic transport of electrons in nano-scale systems [1, 2, 3]. FCS is expected to provide more information about the physical processes taking place inside a sub-micron electronic device compared to what is available from measurements of the mean current or the current noise alone. The theory of FCS already has a history that extends over nearly two decades and recent years have moreover witnessed the first measurements of the distribution of transferred charge beyond the first two cumulants of the current (the mean current and the current noise): the third cumulant (the skewness) of the charge transfer distribution has been measured in tunnel junctions [4] and in quantum point contacts [5, 6], and more recently the detection of cumulants of even higher orders in quantum dots [7, 8, 9, 10, 11] and in avalanche diodes [12] has also been experimentally demonstrated. Remarkably, time-dependent cumulants of up to the 15th order were recently measured in a quantum dot [10, 11]. A wealth of statistical data has thereby become available and the question of how to extract physical information from the statistical distributions is now both important and interesting.

In this contribution we discuss our recent theoretical work on the detection of interactions in FCS using factorial cumulants [13]. A recent work has shown that the high-order cumulants in general grow factorially with the cumulant order and moreover oscillate as functions of basically any system parameter including the observation time [10]. Mathematically, this is due to singularities of the cumulant generating function in the complex plane [10, 14, 15]. At first glance, this generic behavior, which is expected for a large variety of systems, might seem to imply that the high-order statistics actually contain very limited information about the underlying physical processes in a given nano-scale system. However, as we have shown, the
so-called factorial cumulants are in fact very useful to extract physical information about the charge transfer process \[13\]. Briefly, the factorial cumulants do not oscillate for non-interacting electrons in a two-terminal scattering problem. In contrast, if a factorial cumulant oscillates as a function of some parameter it must be due to interactions among the electrons passing through the conductor.

Our finding relies on a deeper mathematical analysis of the generating function for the counting statistics by Abanov and Ivanov who showed that its zeros must always be real and negative for non-interacting electrons in a two-terminal configuration \[16, 17\]; the multi-terminal problem is analyzed in Ref. \[18\]. Only in the presence of interactions the zeros may move away from the negative real axis into the complex plane. As we have shown, this transition is clearly visible in the factorial cumulants, which start oscillating when the zeros of the generating function become nonreal. Importantly, factorial cumulants are measurable in Coulomb blockade quantum dots using currently available experimental techniques \[7, 8, 9, 10, 11\], thus enabling an experimental test of our predictions.

In the following we give a short introduction to FCS, including the basic definitions of moments and cumulants as well as their factorial counterparts. We discuss why the factorial cumulants do not oscillate for non-interacting electrons in a two-terminal scattering problem. As an illustrative example, we then consider the FCS of transport through an interacting two-level quantum dot for which the factorial cumulants clearly oscillate due to interactions. In our recent work we only analyzed the FCS for finite times \[13\], but here we explicitly show that our findings also hold in the long-time limit. Finally, we present our concluding remarks.

2. Full counting statistics

The central object in FCS is the probability distribution \(P(n, t)\) of having transferred \(n\) charges through a nano-scale system during the time span \([0, t]\) together with the corresponding generating function

\[
G(z, t) = \sum_n P(n, t) z^n. \tag{1}
\]

The generating function provides us with a convenient mathematical representation of \(P(n, t)\) and it allows us for example to define the moments and cumulants of \(n\). The moment generating function is

\[
M(z, t) = G(e^z, t) = \sum_n P(n, t) e^{nz}, \tag{2}
\]

from which the moments follow as derivatives with respect to \(z\) evaluated at \(z = 0\),

\[
\langle n^m \rangle(t) = \frac{\partial^m}{\partial z^m} M(z, t) \big|_{z \to 0} = \sum_n n^m P(n, t). \tag{3}
\]

The cumulant generating function is

\[
S(z, t) = \ln [M(z, t)] = \ln [G(e^z, t)] \tag{4}
\]

and the cumulants are similarly defined as the derivatives

\[
\langle \langle n^m \rangle \rangle(t) = \frac{\partial^m}{\partial z^m} S(z, t) \big|_{z \to 0}. \tag{5}
\]

The first three cumulants expressed in terms of the moments are shown in Table 1. For a Gauss distribution, only the first and second cumulants are nonzero, \(\langle \langle n^m \rangle \rangle = 0\) for \(m > 2\). Moreover, for large times \(t \to \infty\), the cumulants typically become linear in time such that \(\langle \langle n^m \rangle \rangle(t) \to \langle \langle l^m \rangle \rangle t\), where \(\langle \langle l^m \rangle \rangle\) are the (zero-frequency) cumulants of the current. Generally, the high-order cumulants grow factorially with the cumulant order and oscillate as functions of basically any system parameter as it has recently been shown \[10\].
The factorial cumulants can also be expressed in terms of the ordinary cumulants as
\[ s = \frac{1}{n!} \sum_{j=1}^{n} j \langle x^j \rangle. \]
which can be written in terms of the ordinary moments as \[22\]
The factorial moments are now
\[ m \quad \langle n^m \rangle_F(t) = \frac{\partial^m}{\partial z^m} M_F(z, t)|_{z=0}, \]
which can be written in terms of the ordinary moments as \[22\]
\[ m \quad \langle n^m \rangle_F = \langle n(n-1)(n-2) \cdots (n-m) \rangle = \sum_{j=1}^{m} s(m, j) \langle n^j \rangle, \]
where \( s(m, j) \) are the Stirling numbers of the first kind.\(^1\) The factorial cumulant generating function is
\[ S_F(z, t) = \ln \left[ M_F(z, t) \right] = \ln \left[ G(z + 1, t) \right] \]
whose derivatives at \( z = 0 \) deliver the factorial cumulants
\[ \langle n^m \rangle_F(t) = \frac{\partial^m}{\partial z^m} S_F(z, t)|_{z=0}. \]
The factorial cumulants can also be expressed in terms of the ordinary cumulants as
\[ \langle n^m \rangle_F = \langle n(n-1)(n-2) \cdots (n-m) \rangle = \sum_{j=1}^{m} s(m, j) \langle n^j \rangle. \]
The first three factorial cumulants given in terms of the ordinary moments are shown in Table 1. For a Poisson distribution, only the first factorial cumulant is nonzero, \( \langle n^m \rangle_F = 0 \) for \( m > 1 \). For large times \( t \to \infty \), the factorial cumulants typically also become linear in time such that \( \langle n^m \rangle_F(t) \to \langle I^m \rangle_F t \), where \( \langle I^m \rangle_F \) are the (zero-frequency) factorial cumulants of the current.

The factorial cumulants are useful to consider in the light of the findings by Abanov and Ivanov who showed that the generating function for non-interacting electrons can always be factorized as
\[ G(z, t) \xrightarrow{\text{generalized binomial}} z^{-Q} \prod_i (1 - p_i + p_i z), \]
\(^1\) The Stirling numbers of the first kind can be generated from the relation \( \ln (1 + x) = j! \sum_{m=j}^{\infty} \frac{s(m, j) x^m}{m!} \) \[22\].
Figure 1. Coulomb blockade quantum dot. The quantum dot is operated close to a charge degeneracy point where only a single (additional) electron at a time can occupy either the upper ($\varepsilon_+$) or the lower ($\varepsilon_-$) level. The bare couplings to the leads are denoted as $\Gamma_S$ and $\Gamma_D$, and $x = 1 - f(\varepsilon_-)$, where $f(\varepsilon_-)$ is the Fermi-Dirac distribution of the drain ($D$) evaluated at the energy of the lower level.

referred to as generalized binomial statistics. Here $p_i \in [0, 1]$ is the probability for a single quasi-particle transfer to occur, while the factor $z^{-Q}$ corresponds to a deterministic background charge transfer $Q = \sum_i p_i - \langle n \rangle \geq 0$ opposite to the positive direction of the current flow. For uni-directional transport $Q = 0$. This generating function has real and negative zeros at $z_i = 1 - 1/p_i \leq 0$.

In case of generalized binomial statistics we readily find for the factorial cumulants

$$\langle \langle n^m \rangle \rangle_F \text{ generalized binomial } (-1)^{m-1}(m-1)! \left[ \sum_i p_i^m - Q \right]. \quad (13)$$

This expression provides us with a direct test of whether or not a statistical distribution can be factorized into independent single particle events as described by Eq. (12). In particular, for uni-directional transport (where $Q = 0$), a factorial cumulant of a given order $m$ must have a fixed sign that is determined by the factor $(-1)^{m-1}$ and no oscillations can occur as some parameter is varied. We have tested that this prediction indeed holds for several different non-interacting systems, see e. g. Refs. [13, 23]. However, as we have also shown, once interactions cause the zeros of the generating function to become complex, the factorial cumulants start oscillating as some parameter is varied [13]. Below, we consider one such interacting model, where interactions cause oscillations of the factorial cumulants.

4. Model

We consider the model of a two-level Coulomb blockade quantum dot depicted in Fig. 1 [24]. The quantum dot is weakly coupled to source and drain electrodes and the system is operated close to a charge degeneracy point where only a single (additional) electron at a time can enter and leave the quantum dot from the electrodes. The quantum dot has two single-particle levels which are both below the chemical potential of the source electrode. However, only one of the levels at a time can be occupied due to the strong Coulomb interactions. The upper level is well above the chemical potential of the drain electrode, while the lower level is close to the chemical potential, determined by the parameter $x = 1 - f(\varepsilon_-)$ which depends both on the temperature $T$ and the applied bias voltage. Here, $f$ is the Fermi-Dirac distribution of the drain electrode.

Charge transport through the quantum dot can be described by a master equation of the form

$$\frac{d}{dt} \mathbf{p}(z) = \mathbf{M}(z)\mathbf{p}(z), \quad (14)$$

where $\mathbf{p}(z) = [p_0(z), p_-(z), p_+(z)]^T$ contains the probabilities for the system to be empty (0) or having the lower (−) or upper (+) level occupied, respectively. Here, counting of electrons is effected by the parameter $z$ which is conjugate to the number of electrons that have tunneled.
Figure 2. High-order factorial cumulants. We show from left to right the 13th to 15th factorial cumulants as functions of the parameter \( x \) at different times \( t \). We plot the logarithm \( \log \left( \frac{\langle n^m \rangle_F}{\langle n \rangle} \right) \), \( m = 13, 14, 15 \), for which dips occur as the factorial cumulants change sign. The colors of the curve segments correspond to the sign of the factorial cumulants (red is positive and blue is negative). Parameters are \( \Gamma_S = 1 \) and \( \Gamma_D = 6 \). Results are shown for the transient time \( t = 0.5/\Gamma_S \), long time \( t = 6/\Gamma_S \), and for infinite time \( t \to \infty \), where the ratio \( \frac{\langle n^m \rangle_F}{\langle n \rangle} \) is still well-defined and finite.

from the source electrode onto the quantum dot [25]. The corresponding rate matrix is

\[
M(z) = \begin{pmatrix}
-(1-x)\Gamma_D + 2\Gamma_S & x\Gamma_D & \Gamma_D \\
-x\Gamma_D & -x\Gamma_D & 0 \\
z\Gamma_S & z\Gamma_S + (1-x)\Gamma_D & -z\Gamma_D \\
\end{pmatrix},
\]

where \( \Gamma_S \) and \( \Gamma_D \) are the bare couplings to the source and drain electrodes, respectively. The parameter \( z \) multiplies the rate \( \Gamma_R \) in the off-diagonal elements of \( M(z) \), corresponding to charge transfer events across the source barrier. The generating function is then given as \( G(z,t) = p_0(z) + p_-(z) + p_+(z) \) [13].

5. Results
We now use the methods of Refs. [13] and [26, 27] to calculate the high-order factorial cumulants for finite times and in the long-time limit, respectively. In Fig. 2 we show the 13th to 15th factorial cumulants as functions of the parameter \( x \) at different times \( t \). The factorial cumulants vary over many orders of magnitude as a function of \( x \) and we therefore show the logarithm of the absolute value of the factorial cumulants. The colors of the curves then correspond to the sign of the factorial cumulants (red is positive and blue is negative). The factorial cumulants are normalized with respect to the mean of \( n \) such that the ratio \( \frac{\langle n^m \rangle_F}{\langle n \rangle} \) has a well-defined and finite value also in the long-time limit.
The figure clearly illustrates the oscillations of the high-order factorial cumulants as functions of the parameter $x$ (dips occur when the factorial cumulants cross zero and change sign). This holds both at finite times and in the long-time limit. The latter case was not considered explicitly in our work [13]. We stress that such oscillations cannot occur in a two-terminal scattering problem involving non-interacting electrons. Factorial cumulants are measurable using currently available counting techniques [7, 8, 9, 10, 11] and an experimental test of our predictions should therefore realistically be within reach. Experimentally it might be necessary to resolve the direction of the tunneling events. To this end, a double quantum dot with an asymmetrically coupled quantum point contact used for electron counting could be a promising candidate system [8].

6. Conclusions
We have discussed our recent theoretical proposal to detect interactions in the full counting statistics of charge transport through a nano-scale system by measuring high-order factorial cumulants. In the absence of interactions, the factorial cumulants do not oscillate. In contrast, oscillating factorial cumulants are a clear indication of interactions in the transport. This holds both at finite times and in the long-time limit. Factorial cumulants are measurable using available experimental techniques and we therefore believe that an experimental test of our prediction should be feasible. A more detailed account of our proposal can be found in Ref. [13].

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