Thermalization and localization of an oscillating Bose-Einstein condensate in a disordered trap

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We numerically simulate an oscillating Bose-Einstein condensate in a disordered trap [Phys. Rev. A 82, 033603 (2010)] and the results are in good agreement with the experiment. It allows us to verify that total energy and particle number are conserved in this quantum system. The disorder acts as a medium, which results in a relaxation from nonequilibrium to equilibrium, i.e., thermalization. An algebraic localization is realized when the system approaches the equilibrium, and if the system falls into the regime when the healing length of the condensate exceeds the correlation length of the disorder, exponential Anderson localization is to be observed.

Anderson localization (AL) had been a long-studied phenomenon in electronic systems [1]. When transporting in an environment with random disorder, waves of electrons get localized after multiple scattering with the disorder. Recently there is a resurgence of studies of AL in a variety of systems such as the photonic crystals [2, 3], the ultrasound in 3D elastic networks [4], the quantum chaotic systems [5], and the cold atoms [6, 7]. The experiment [6] of cold atoms was done by expanding Bose condensate in a weak random potential in which the healing length (ξ) of the condensate exceeds the correlation length (σD) of the disorder, ξ > σD. In this regime, the particle interaction is relatively unimportant to which ξ → ∞ corresponds to a noninteracting limit. The experiment [6] confirmed that the localized condensate exhibits an exponential density profile in a one-dimensional geometry, in agreement with the theory of Sanchez-Palencia et al. [8].

In this Letter, we show that an oscillating condensate in a disordered trap, such as the experiment done by Dries et al. [5], can also exhibit AL when it comes to equilibrium and if it falls into the regime ξ > σD. Using exactly the same parameters of the experiment reported in Fig. 2 of Ref. [5] for ξ < σD, we perform a numerical simulation based on the Gross-Pitavskii (GP) approach in the presence of a spatially random disorder potential. The results are in good agreement with the experiment as far as the overall spatial dynamics is concerned. It allows us to verify that when time passes a relaxation time t∗ (discussed later), the system enters an algebraical localized state [8]. This motivates to carry out another simulation with the same parameters except by reducing σD to make ξ > σD. In this case, exponential AL is eventually observed.

Another important factor in such a system is that it provides a simple framework to investigate the long-standing question on how an “isolated” many-body quantum system, without coupling to the reservoir, can relax to a steady state that seems to be in thermodynamic equilibrium, i.e., thermalization [10–12]. Our results show that total energy and particle number of the system are conserved and temporal entropy reveals that a relaxation process from nonequilibrium to equilibrium does exhibit. Random disorder plays the role of a medium (or transistor) which results in the exchange of partial kinetic energy with partial potential energy. When t ≫ t∗, the system reaches a thermodynamic equilibrium in which both kinetic and potential energies come to a constant. There is no dissipation of the total energy through any kind of friction. It becomes evident that the thermalization to equilibrium is accompanied by the localization in the system.

To make a direct comparison with the experiment reported in Fig. 2 of Ref. [5], we consider a 1D Bose gas with a repulsive contact interaction that is trapped in a harmonic potential V mono(z) = mω 2z 2 /2. In the dilute and ultracold condition, the condensate wave function ψ(z, t) is governed by the GP equation in the presence of a real spatially random disordered potential V_{dis}(z),

\[ i\hbar \partial_t \psi = \left( -\frac{\hbar^2}{2m} \partial_z^2 + V_{\text{mono}}(z) + V_{\text{dis}}(z) + N g |\psi|^2 - \mu \right) \psi. \]  \hspace{1cm} (1)

Here N is the total number of atoms, g is the coupling constant of contact interaction, μ is the chemical potential, and ψ is normalized to one. \( \int |\psi|^2 dz = 1 \). The healing length at the center of the condensate is defined as \( \xi = \hbar / \sqrt{2m\mu} \).

The disorder correlation length, σD, is defined by fitting the autocorrelation function \( V_{\text{dis}}(z)V_{\text{dis}}(z + \Delta z) = V_D^2 \exp(-2\Delta z^2 / \sigma_D^2) \) with V_D the strength of V_{dis}(z) [13].

In the experiment, the condensate is released at a position off the center of the harmonic trap that results in the subsequent oscillations. For numerical convenience, we take an alternative scheme such that the condensate is released at the trap center but with an initial velocity. To obtain an initial wave function with a velocity ν_0, we apply the Galilean transformation: \( \psi = \varphi \exp(i m \nu_0 z) \) and in the absence of disorder, the corresponding GP equation for the residual wave function \( \varphi \) is

\[ i\hbar \partial_t \varphi = \left[ \frac{1}{2m} \left( \frac{\hbar}{i} \partial_z - m\nu_0 \right)^2 + V_{\text{mono}}(z) + N g |\varphi|^2 - \mu \right] \varphi. \]  \hspace{1cm} (2)
Long-term imaginary-time evolution of Eq. (2) gives $\varphi$ which in turn gives the initial wave function $\psi$. A cutoff wavevector $k_c$ corresponding to the shortest length scale or the largest $k$ scale in association with the healing length $\xi$ is naturally introduced in the simulation which gives the best results. In natural units: $\hbar = m = \omega = 1$, experimental parameters are $\mu = 200$, $v_0 = 37.5$, $V_D = 50.9$, and $\sigma_D = 0.25$ that correspond to the regime $\xi < \sigma_D$.

Fig. 1(a) shows the spatial and temporal results of the oscillating condensate simulation. We plot, at various times, the calculated norm of the spatial condensate wave function in $z$-direction. For comparison, black dots correspond to experimentally measured temporal center-of-mass coordinates (reported in Fig. 2 of Ref. [9]). Surprisingly it gives a very good agreement between the simulation and the experiment. Close examinations of the results are shown in Fig. 1(b) and (c) for $t < t_c \approx 4s$ when the system is still out of equilibrium and for $t > t_c$ (approaching the equilibrium in accompany of localization).
From Fig. 2, the condition with equation (1), the following condition is satisfied at equilibrium. For the current system described by the GP virial theorem is satisfied when the system approaches the equilibrium. Moreover, as clearly seen in Fig. 2, the energy exchange rate or the transportation of the condensate is significantly reduced when $t > t_c$. One can also verify whether the virial theorem is satisfied when the system approaches the equilibrium. For the current system described by the GP equation (1), the following condition

$$2E_{\text{kin}} - 2E_{\text{pot}} + dE_{\text{int}} = 0 \tag{3}$$

with $d$ the dimension should be satisfied at equilibrium \[\text{[17]}\]. From Fig. 2 the condition $E_{\text{pot}} \simeq E_{\text{kin}} + E_{\text{int}}/2$ is indeed satisfied with $d = 1$.

Owing to the randomness nature of the wave function, it is particularly useful to study the corresponding wave action spectrum $n_k(t)$ in the context of wave turbulence \[\text{[18]}\]. When expressing the condensate wave function $\psi(z,t)$ in terms of Madelung transformation, $\psi(z,t) = \sqrt{\rho(z,t)} \exp [i \varphi(z,t)]$ with $\rho$ and $\varphi$ the density and phase, the hydrodynamic kinetic-energy density is $K = (m/2) |\nabla \varphi|^2$ with $\mathbf{u} \equiv (h/m) \partial \varphi$ the velocity. To study the scaling laws, one applies the sum rule:

$$K(t) = \int k_0 K(k,t) dk,$$

where $K(k,t)$ is the kinetic-energy spectrum and $k_0$ is the cutoff wavevector mentioned earlier. The corresponding wave action spectrum is then given by $n_k(t) = k^{-2} \tilde{K}(k,t)$ and one can define the entropy $S(t)$ in association with $n_k$ \[\text{[14,22]}\],

$$S(t) = \int dk \ln [n_k(t)]. \tag{4}$$

Results of $S(t)$ are shown in Fig. 3. Major features are that $\Delta S(t) \equiv \lim_{\Delta t \to 0} [S(t + \Delta t) - S(t)] > 0$ when $t < t_c$ and $\Delta S(t) \to 0$ when $t > t_c$. It unambiguously identifies that thermalization is developed in the system. In earlier process, one also sees the Fermi-Pasta-Ulam-Tsingou (FPUT) recurrence effect that is consistent with the oscillation of the system. In the inset of Fig. 3, we show $n_k(t)$ at $t = 7.3s \gg t_c$. It follows the Rayleigh-Jeans spectrum, $n_k \sim k^{-2}$, which indicates that the energy spectrum $\tilde{K}$ is a constant, or equipartition in $k$ space. In other words, the system corresponds to a non-dissipative one with a detailed balance.

Here we discuss the relaxation time $t_c$. As studied by Bhongale et al. \[\text{[15]}\], by comparing the condensate center-of-mass speed $v$ to the sound speed $c \equiv \sqrt{\mu/m}$, the entire oscillation process can be divided into fast or supersonic ($v > c$) and slow or subsonic ($v < c$) regions. When $v \leq c$, the relatively slow motion of the (central) extended part does not affect much the distribution of the (long-tail) localized part. As a matter of fact, the relaxation time $t_c$ can be well defined as the time when the center-of-mass speed is equal to the sound speed ($v = c$). It has been identified for the experiment \[\text{[5]}\] that $t_c \approx 4s$ when $v = c$ and it is consistent with the value quoted according to the results of $S(t)$ in Fig. 3.

Finally we investigate in details whether a localization can be seen in the oscillation experiment. In a semi-log plot, Fig. 4(a) shows the eventual spatial density distributions at $t = 7.3s$. To smoothen the spikes arising from
randomness, the results are taken as the average over a period. As shown in Fig. 4(a), it is verified that for the current \( \xi < \sigma_D \) case the condensate is algebraically localized, in accordance with the previous theory [8]. As mentioned earlier, in the presence of random disorder, the condensate is actually separated into both localized and extended parts. In a free expansion experiment, the extended part will escape whereas the localized part will remain [6]. When a trapping potential is in place, the moving extended part is eventually stopped at the center of the trap, whereas the localized part comprises the long tails of the condensate. Fig. 4(a) shows a well-fitting curve \( |\psi(z)|^2 \sim |z|^{-2} \) at the range \( 20 < |z| < 70 \). The lower bound is determined by the initial size of the condensate, i.e., the Thomas-Fermi radius \( R_{\text{TF}} = \sqrt{2\mu} = 20 \), and the off-fitting data out of the range \( |z| > 70 \) is due to the trapping effect.

Can the AL be seen in a similar oscillation experiment? Here we perform another simulation for the same parameters except by reducing \( \sigma_D \) to 0.01 to make the regime \( \xi > \sigma_D \). In this case, the relaxation time is \( t_c \approx 2s \) only. As shown in Fig. 4(b), the results of the eventual density distributions at \( t = 3.7s \) are well fitted by an exponential one, \( |\psi(z)|^2 \sim \exp(-2\gamma_{\text{eff}} |z|) \) with \( \gamma_{\text{eff}} \) the Lyapunov exponent [6]. The fitting \( \gamma_{\text{eff}} \) is also in good agreement with the analytic one, \( \gamma_{\text{eff}} = (\pi/32\xi)(V_D/\mu)^2(\sigma_D/\xi)^2 \approx 0.025 \). Compared to the case \( \xi < \sigma_D \), higher percentage of atoms can be localized in the case \( \xi > \sigma_D \). Thus one sees that the fitting is as good as in the range \( 0 < |z| < 70 \). The off-fitting data out of the range is again due to the trapping effect.

In summary, we propose that Anderson localization can be observed in an oscillating condensate in a disordered trap when the system comes to an equilibrium and when the healing length of the condensate is exceeding the disorder correlation length. In addition, we show that in such an “isolated” system, the disorder plays the role as a medium and through it, the system undergoes a relaxation process from nonequilibrium to equilibrium. The occurrence of localization can thus be viewed as the development of thermalization in the system.

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