Dynamic feedback stabilisation of edge states in channel flow

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The transition to turbulence in many wall-bounded parallel shear flows is connected with the stable manifold of specific nonlinear invariant solutions of the Navier-Stokes equations, which separates the turbulent and laminar basins of attraction. The properties of such solutions, or edge states, are thus of interest for transitional flows. In pipe flow, Willis et al. (2017) have shown that edge states can be stabilised by an adjustment of the Reynolds number in response to an energy-based observable. Here, we construct a similar pressure-based feedback strategy for channel flow and study its effect on the stable and unstable directions of edge states in different periodic cells. Even though the original instability can be removed, new instabilities can emerge as the feedback procedure affects not only the unstable but also the stable directions. We quantify these adverse effects and construct a control procedure that leaves the stable directions unaffected. This new strategy successfully stabilises travelling wave solutions on the edge.

Key words: transition, control, Poiseuille flow, channel flow

1. Introduction

The transition to turbulence in many wall-bounded parallel shear flows such as pipe or channel flow, is connected to the presence of a lower-dimensional manifold in state space, the edge of chaos (Itano & Toh 2001; Skufca et al. 2006; Eckhardt et al. 2007), which distinguishes between initial conditions resulting in laminar or turbulent flow. Relative attractors on the edge manifold, so-called edge states, have thus at least one unstable direction transversal to it (Duguet et al. 2008), such that the dynamics will not remain confined to the edge. The latter makes the determination of these exact coherent structures (ECS) on the edge manifold difficult. Numerical methods such as direct shooting (Itano & Toh 2001; Schneider et al. 2007) or edge tracking (Skufca et al. 2006) or adjoint methods (Farazmand 2016) are available, but they are either not guaranteed to converge or costly due to slow convergence and high computational effort. However, in most cases the edge states are part of an unstable lower branch of exact nonlinear solutions of the Navier-Stokes equations (or, ECS), that appear in a saddle-node bifurcation. This suggests that low-dimensional feedback stabilisation methods could be used to remove the effect of the unstable directions, such that the edge state is stabilised. In pipe flow, a simple feedback control strategy, where the Reynolds number is adjusted in response to an observable connected with deviations from laminar flow, indeed stabilises

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the dynamics to remain on the edge \cite{Willisetal2017}. Forward integration of the controlled system converged to previously unknown edge states in form of travelling waves, for more complicated edge states such as relative periodic orbits or chaotic solutions, the controlled simulations converged to objects in the vicinity of ECS of the uncontrolled system.

Exact coherent structures and their stability properties are not only of interest to studies concerned with transitional flows. There is ample evidence supporting the concept whereby the turbulent region of the state space of a wall-bounded, parallel shear flow consists of many unstable ECS \cite{Nagata1990, Hofetal2004, Hofetal2005, Duguetetal2008b, Duguetetal2008a, Eckhardtetal2007, Kawaharaetal2012, Cvitanovic2013, Willisetal2016, Budanuretal2017, Reetzetal2019}, with turbulence corresponding to a state-space trajectory travelling along the ECS’ stable and unstable manifolds resulting in frequent close passes to different ECS. Exact solutions of the Navier-Stokes equations can differ considerably in their global and local properties, such as e.g. drag, mean profile or turbulence intensity. The application of a feedback control procedure can be a useful strategy to avoid states with undesirable properties such as high drag by altering their stability properties, thus preventing state-space trajectories to remain close to certain ECS or to confine the dynamics to certain phase-space volumes. A dynamic feedback procedure based on adjustments of the Richardson number succeeded in temporal stabilisation of otherwise transient turbulent spots and stripes in stratified plane Couette flow \cite{Tayloretal2016}.

Here, we focus on feedback strategies in channel flow. We investigate the effect of a pressure-based feedback control on the stable and unstable directions in order to stabilise states on the edge, and we construct a feedback procedure that acts on the unstable direction only. We begin with theoretical considerations on linear feedback control in sec. \ref{sec:2} where the procedure is explained and its effect is illustrated in low-dimensional examples. In sec. \ref{sec:3} we use the general formalism outlined in sec. \ref{sec:2} to develop a pressure-based control strategy for pressure-driven shear flows such as channel flow. Before applying the control procedure to edge states in direct numerical simulations of channel flow, we summarise the numerical details and describe the target states in sec. \ref{sec:4}. Section \ref{sec:5} contains the main investigation of the stabilisation of edge states including the effect of the feedback control on the stable directions and the construction of a feedback strategy that acts only on the unstable directions. We summarise our results in sec. \ref{sec:6}.

\section{Stabilisation and control} \label{sec:2}

Consider a system with two variables, a positive observable $A$ and a control variable $R$. We assume that the uncontrolled system has stationary solutions that appear in a saddle-node bifurcation at $(A^*, R^*)$ with an unstable lower branch $A_{LB}(R)$. The aim is to stabilise an operating point $(A_0, R_0)$ on the lower branch. Without loss of generality we further assume that the uncontrolled dynamics is such that the observable grows if it exceeds the lower branch value $A_{LB}(R)$,

$$\dot{A} = \lambda (A - A_{LB}(R)),$$ \hspace{1cm} (2.1)

with $\lambda > 0$ being the Lyapunov exponent, which we assume to be independent, or a slowly varying function, of $R$. To control and avoid the exponential instability, the control variable must be repeatedly adjusted such that the ensuing dynamics of the system results in convergence to the operating point, for example through an interaction procedure where the lower branch is crossed at each adjustment of the control variable.
Feedback control of edge states

For the uncontrolled dynamics as in eq. (2.1), this can be achieved by adjusting the control variable according to

$$\dot{R} = -\gamma (R - R_0) - \gamma \mu (A - A_0) ,$$

(2.2)

where $A_0 = A(R_0) = A_{LB}(R_0)$ is the value of the observable at the reference point and $\gamma > 0$ and $\mu > 0$ are adjustable parameters. The signs are for the cases that $A_{LB}(R)$ decreases with $R$, i.e.

$$\frac{dA_{LB}(R)}{dR} \bigg|_{R_0} = -\alpha ,$$

(2.3)

with $\alpha > 0$. With $r = R - R_0$ and $a = A - A_0$ we can write

$$A - A_{LB}(R) = A - A_{LB}(R_0 + r) \approx a + \alpha r ,$$

(2.4)

so that eqs. (2.1) and (2.2) become

$$\frac{da}{dt} \begin{pmatrix} a \\ r \end{pmatrix} = \begin{pmatrix} \lambda & \lambda \alpha \\ -\gamma \mu & -\gamma \end{pmatrix} \begin{pmatrix} a \\ r \end{pmatrix} .$$

(2.5)

For the operating point $(A_0, R_0)$ to be stable, the matrix on the right-hand side of eq. (2.5) must have eigenvalues with negative real parts. The conditions for such eigenvalues are that the trace of the matrix, as the sum of the eigenvalues, has to be negative, and the determinant, the product of the eigenvalues, has to be positive. With the trace

$$\text{Tr} = \lambda - \gamma ,$$

(2.6)

and the determinant

$$\det = -\lambda \gamma + \lambda \gamma \alpha \mu ,$$

(2.7)

the conditions for stability become

$$\gamma > \lambda$$

(2.8)

$$\alpha \mu > 1 .$$

(2.9)

The conditions are such that the adjustment in $R$ (related to the parameter $\gamma$) has to be faster than the escape (as measured by $\lambda$). Similarly, the amplitude of the change in the control variable with the deviation in the observable has to be larger than the inverse of $\alpha$, so that the changes in $R$ outrun the changes in $A$. For what follows it will be useful to visualise the stability condition (2.9) geometrically: Since $\alpha$ is the slope of the tangent to the lower branch at $(A_0, R_0)$, the inequality (2.9) results in a line through $(A_0, R_0)$ with a slope $1/\mu < \alpha$ which is shallower than that of the control line, i.e. the tangent at the operating point. The feedback control procedure applied by Willis et al. (2017) corresponds in this context to an immediate adjustment of $R$, i.e. to $\gamma \to \infty$.

2.1. One-dimensional linear model

Before applying the feedback control to a high-dimensional dynamical system such as channel flow, we consider the dynamics of the linearised one-dimensional system given by eq. (2.5), with Lyapunov exponent $\lambda = 0.01$, and lower branch slope $\alpha = 1.5 \times 10^{-5}$. These values correspond to measurements of $\alpha$ and $\lambda$ for an edge state in DNS of channel flow, which will be discussed in further detail in sec. 4. Figure 2 presents phase portraits of this system for $\gamma = 1$ and two different values of the control strength $\mu$, i.e. $\mu = 2 \times 10^5$ and $\mu = 2.4 \times 10^7$. The tangent line as indicated in orange (light grey) has a steeper slope than the control line (blue/dark grey) in both cases, as required by eq. (2.9), and both lines cross at the operating point. The time evolution follows the green/grey curve,
beginning at the red/grey points located in the top right quadrants of the two panels, and it ends at the operating point. That is, in both cases the operating point has been stabilised.

In both cases the instability has been removed, leading to eigenvalues of the matrix in eq. (2.5) that have negative real parts. The eigenvalues do not only yield information on the stability of an equilibrium in the controlled system, they also determine the dynamic relaxation process. For real eigenvalues we expect monotonic exponential relaxation, while complex eigenvalues with non-zero imaginary part lead to an oscillatory approach to the stabilised equilibrium. In the present linear one-dimensional (1D) model system, the eigenvalues are real for $\mu = 2 \times 10^5$ and complex for $\mu = 2.4 \times 10^7$, and the relaxation towards the equilibrium does indeed proceed differently for the two values of the control strength. For $\mu = 2 \times 10^5$ the relaxation proceeds monotonically along the control line as shown in the left panel of fig. 2, while $\mu = 2.4 \times 10^7$ results in oscillatory relaxation as shown in the right panel of fig. 2. The latter is reminiscent of the schematic behaviour illustrated in fig. 1.

### 2.2. Effect on the stable directions

Equilibria in higher-dimensional systems can have several stable and unstable directions. Even if we assume that only one direction is unstable, as is generally the case for edge states in canonical wall-bounded parallel shear flows, the 1D control procedure may not only have the desired influence on the unstable direction, it may also couple to the stable directions. This effect is known in control theory, where its mitigation is essential in the design of successful controllers (Barbagallo et al. 2009). In order to illustrate what
Figure 2. Stabilisation of the linear model system given by eq. (2.5). The (linear) lower branch is indicated by the dashed line, it crosses the control line (solid black) at the operating point. The time evolution of the system follows the red (grey) curve starting at the blue (dark grey) square. Left: monotonic relaxation for $\mu = 2 \times 10^5$ corresponding to negative real eigenvalues. Right: oscillatory relaxation for $\mu = 2.4 \times 10^7$ corresponding to complex eigenvalues with negative real parts.

the consequences of such a coupling can be, we consider a two-dimensional (2D) extension of the controller 1D-model given in linearised form in eq. (2.5)

$$\frac{d}{dt} \begin{pmatrix} r \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -\gamma & -\gamma \mu_1 & -\gamma \mu_2 \\ \lambda_1 \alpha_1 & \lambda_1 & 0 \\ -\lambda_2 \alpha_2 & 0 & -\lambda_2 \end{pmatrix} \begin{pmatrix} r \\ a_1 \\ a_2 \end{pmatrix}, \tag{2.10}$$

where $a_1$ corresponds to the unstable and $a_2$ to the stable direction with $\lambda_1 > 0$ and $\lambda_2 > 0$. The dynamics are coupled to the control procedure through $\mu_1$ and $\mu_2$, respectively. For simplicity, we assume that the stable and unstable directions decouple. For $a_2$ and $\lambda_1$ as in fig. 2 we randomly choose $a_1$ and $\lambda_1 > 0$ and calculate the number of eigenvalues of the matrix on the right-hand side of eq. (2.10) that have a positive real part as a function of $\mu_1$ and $\mu_2$. An example of the results obtained from such a calculation is shown in fig. 3. If the control is weakly coupled to the dynamical system, we find one eigenvalue with positive real part, as expected for a system with one stable and one unstable direction. Increasing $\mu_1$ for small $\mu_2$ eventually stabilises the operating point, which can also be expected from the results in the 1D case. However, we find a large part of parameter space where one or two eigenvalues have a positive real part, hence the control is not able to stabilise the operating point if it overlaps significantly with the stable direction.

In summary, the success of the control strategy in higher-dimensional systems depends on how the dynamics along the stable directions couple to the control. Stabilisation of the operating point then requires a control strategy that acts on a hyperplane orthogonal to all stable directions. Such a strategy can be constructed in numerical simulations only, and we will come back to this point in sec. 5.2. In practice, the control is more likely to destabilise stable directions with a small negative real part, which suggests that it may be sufficient to design the control to be orthogonal to the least stable directions in order to achieve stabilisation. Similar procedures are indeed sometimes applied in control theory in the context of model reduction (Åkervik et al. 2007) and will be successful provided the chosen modes are observable and controllable (Barbagallo et al. 2009).
3. A pressure-based control strategy

Having introduced a general one-dimensional feedback control strategy and discussed its properties in low-dimensional model systems, we now turn to its application to wall-bounded shear flows, whose dynamics is governed by the incompressible Navier-Stokes equations

$$\partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + \frac{1}{\rho} f, \quad (3.1)$$

$$\nabla \cdot u = 0, \quad (3.2)$$

where $u$ is the velocity field, $p$ the pressure divided by the constant density $\rho$, $\nu$ the kinematic viscosity and $f$ a force that drives the flow, e.g. a constant pressure gradient as in pipe or channel flow. The implementation of the feedback procedure introduced in sec. 2 requires a choice of observable and control variable. Here, care must be taken in the non-dimensionalisation of eq. (3.1), as the choice of control variable may result in the usual characteristic scales becoming time-dependent. Furthermore, eq. (3.1) must be supplemented with an auxiliary equation that describes the time evolution of the control variable as a function of the observable. The feedback loop is then closed by coupling the control variable to eq. (3.1).

In principle, there are two conceptual choices for the control variable, one that results in a modulation of the flow and one that results in an adjustment of $f$. Since eq. (3.1) is usually made dimensionless using a characteristic length scale $h$ and a reference velocity $U_0$, the choice of control variable must be such that $U_0$ and $h$ remain time-independent. Otherwise the dimensionless form of eq. (3.1) is not applicable any longer because the time-derivative does not commute with the now time-dependent reference velocity $U_0(t)$. This occurs if the feedback is implemented through a modulation of the flow. Therefore, we focus here on the second possibility, that of an adjustment of $f$ in response to a control variable. Assuming that $f(t)$ fluctuates around a reference state $f_0$, the velocity scale $U_0$ that is associated with that particular value of the force is used to rescale eq. (3.1). Specifically, the forcing is made dimensionless in units of $h/U_0^2$, and variations in the
force can be measured in the same units. In what follows, $U_0$ is the laminar centerline velocity and $h$ the half-height of the channel.

For pressure-driven pipe or channel flow, $f$ can be identified with a time-dependent streamwise pressure gradient $dP/dx(t)e_x$ that fluctuates around a reference value $(dP/dx)_0 e_x$. The controlled system in non-dimensionalised form then reads

$$\partial_t u + u \cdot \nabla u + \nabla p - \frac{1}{Re} \Delta u + \left( \frac{dP}{dx} \right)_0 e_x = -\frac{dP}{dx}(t)e_x,$$

(3.3)

$$\nabla \cdot u = 0,$$

(3.4)

$$\dot{R} = -\gamma (R - R_0) - \gamma \mu (A - A_0),$$

(3.5)

$$\frac{dP}{dx}(t) = -\frac{2}{R_0} \frac{R(t)}{R_0} R_0,$$

(3.6)

with $A$ being an observable, $R$ the control variable with $(R_0, A_0)$ defining the operating point. The last equation, which implements the feedback, is based on $R$ representing a Reynolds number such that for $R = R_0 = Re$ the reference pressure gradient is recovered. The time-dependent Reynolds number that is used by [Willis et al.](2017) cannot be realised with a change in the pressure gradient or similar, since that would give a different velocity scale, as discussed above. As it stands, a modulation in Reynolds number can only be obtained as a consequence of variations in viscosity, which is difficult to achieve in experiments.

In order to stabilise the operating point, the control must overlap with the expanding directions of the operating point’s tangent space. Since the linear operator representing the linearised Navier-Stokes dynamics close to the operating point is non-normal, its eigenvectors are not orthogonal. That is, it is in principle possible to stabilise an operating point with a one-dimensional control procedure, provided that all unstable directions overlap. Here, the proposed feedback control acts in the streamwise direction only and it is translationally invariant in both streamwise and spanwise directions. That is, it can only stabilise unstable directions that have streamwise component with a non-zero streamwise and spanwise mean.

4. Datasets and numerical details

Direct numerical simulations (DNS) of channel flow have been carried out using the pseudospectral open-source code channelflow2.0 ([Gibson] 2014; [Gibson et al.](2019)). The code numerically solves the incompressible Navier-Stokes equations (3.1) in a rectangular domain with periodic boundary conditions in streamwise and spanwise ($x, z$) directions, and no-slip boundary conditions in the wall-normal ($y$) direction. The spatial discretisation is obtained through Fourier expansions in $x$- and $z$-directions using $N_x$ and $N_z$ collocation points, respectively, and a Chebyshev expansion in the $y$-direction on $N_y$ points. Aliasing errors in the periodic directions are removed by 2/3-Galerkin truncation ([Orszag](1971)). A third-order semi-implicit Adams-Bashforth scheme is used for the temporal discretisation. The code has been adapted to run the controlled simulations as the core dynamical system in order to make use of the methods for numerical stability analysis provided in channelflow2.0.

As discussed in sec. 3, a control strategy based on a streamwise-averaged quantity will not be able to remove strictly periodic instabilities in the streamwise direction, which are more likely to develop in streamwise elongated domains. For this reason all simulations in this study are carried out in a short computational domains of size $L_x/h \times L_y/h \times L_z/h =$
| id       | $L_x/h$ | $L_y/h$ | $L_z/h$ | $N_x$ | $N_y$ | $N_z$ | control type | observable | $Re_0$ | $\mu$ | $\|\delta u\|_2$ |
|----------|---------|---------|---------|-------|-------|-------|--------------|------------|--------|-------|----------------|
| TW1-A1   | 2$\pi$  | 2       | 2$\pi$  | 32    | 49    | 48    | $dP/dx$     | $L_2$-norm | 1395   | $2 \times 10^5$ | 0.003   |
| TW1-A2   | 2$\pi$  | 2       | 2$\pi$  | 32    | 49    | 48    | $dP/dx$     | $L_2$-norm | 1395   | $6 \times 10^5$ | 0.003   |
| TW1-A3   | 2$\pi$  | 2       | 2$\pi$  | 32    | 49    | 48    | $dP/dx$     | $L_2$-norm | 1395   | $10^6$          | 0.003   |
| TW1-A-stab | 2$\pi$ | 2       | 2$\pi$  | 32    | 49    | 48    | $dP/dx$     | $L_2$-norm | 1395   | $0 - 10^6$       | $10^{-6}$|
| TW1-B1   | 2$\pi$  | 2       | 2$\pi$  | 32    | 49    | 48    | $dP/dx$     | $C_f$      | 1395   | $2 \times 10^5$ | 0.003   |
| TW1-B2   | 2$\pi$  | 2       | 2$\pi$  | 32    | 49    | 48    | $dP/dx$     | $C_f$      | 1395   | $10^6$          | 0.003   |
| TW1-B3   | 2$\pi$  | 2       | 2$\pi$  | 32    | 49    | 48    | $dP/dx$     | $C_f$      | 1395   | $3 \times 10^6$ | 0.003   |
| TW1-C1   | 2$\pi$  | 2       | 2$\pi$  | 32    | 49    | 48    | $e_n^*$     | $L_2$-norm | 1395   | $10^6$          | 0.003   |
| TW1-C2   | 2$\pi$  | 2       | 2$\pi$  | 32    | 49    | 48    | $e_n^*$     | $L_2$-norm | 1395   | $10^7$          | 0.003   |
| TW1-D1   | 2$\pi$  | 2       | 2$\pi$  | 32    | 49    | 48    | $e_n^*$     | $Ecf$      | 1395   | $10^9$          | 0.003   |
| TW1-D2   | 2$\pi$  | 2       | 2$\pi$  | 32    | 49    | 48    | $e_n^*$     | $Ecf$      | 1395   | $10^{10}$       | 0.003   |
| TW-sym-A1 | 2$\pi$ | $\pi$  | 48     | 65    | 48    | 48    | $dP/dx$     | $L_2$-norm | 1010   | $10^4$          | 0.01    |
| TW-sym-A2 | 2$\pi$ | $\pi$  | 48     | 65    | 48    | 48    | $dP/dx$     | $L_2$-norm | 1010   | $5 \times 10^4$ | 0.01    |
| TW-sym-A3 | 2$\pi$ | $\pi$  | 48     | 65    | 48    | 48    | $dP/dx$     | $L_2$-norm | 1010   | $10^5$          | 0.01    |
| TW-sym-B1 | 2$\pi$ | $\pi$  | 48     | 65    | 48    | 48    | $dP/dx$     | $Ecf$      | 1010   | $3 \times 10^5$ | 0.01    |
| TW-sym-B2 | 2$\pi$ | $\pi$  | 48     | 65    | 48    | 48    | $dP/dx$     | $Ecf$      | 1010   | $5 \times 10^5$ | 0.01    |
| TW-sym-B3 | 2$\pi$ | $\pi$  | 48     | 65    | 48    | 48    | $dP/dx$     | $Ecf$      | 1010   | $6 \times 10^5$ | 0.01    |
| TW-sym-C1 | 2$\pi$ | $\pi$  | 48     | 65    | 48    | 48    | $e_n^*$     | $L_2$-norm | 1010   | $3 \times 10^5$ | 0.01    |
| TW-sym-C2 | 2$\pi$ | $\pi$  | 48     | 65    | 48    | 48    | $e_n^*$     | $L_2$-norm | 1010   | $3.5 \times 10^5$ | 0.01    |
| TW-sym-C3 | 2$\pi$ | $\pi$  | 48     | 65    | 48    | 48    | $e_n^*$     | $L_2$-norm | 1010   | $4.75 \times 10^5$ | 0.01    |
| TW-sym-C4 | 2$\pi$ | $\pi$  | 48     | 65    | 48    | 48    | $e_n^*$     | $L_2$-norm | 1010   | $5.25 \times 10^5$ | 0.01    |
| TW-sym-C5 | 2$\pi$ | $\pi$  | 48     | 65    | 48    | 48    | $e_n^*$     | $L_2$-norm | 1010   | $6 \times 10^5$ | 0.01    |

Table 1. Simulation parameters and observables. The Reynolds number is $Re_0 = U_0 h / \nu$, where $U_0$ is the laminar centerline velocity, $h = L_y / 2$ half the domain height, $\nu$ the kinematic viscosity, $\mu$ the control strength as in Eq. (5.5) and $\delta u$ the perturbation about the respective operating point. The adjustment rate in Eq. (3.5) is $\gamma = 1$ in all cases. The control type $dP/dx$ refers to the pressure-based control given in Eqs. (3.3)-(3.6), while that labelled $e_n^*$ refers to the control along the dual vector of the unstable direction implemented according to Eqs. (3.5), (3.6) and (3.9). The number of Fourier modes in x and z-directions, $N_x$ and $N_z$, contain the dealiased modes.

$2\pi \times 2 \times 2\pi$ and $L_x/h \times L_y/h \times L_z/h = 2\pi \times 2 \times \pi$. Further details of all simulations are summarised in Table 1.

### 4.1. Operating points: Edges state in channel flow

The edge states we wish to stabilise are travelling waves with one unstable direction, which have been obtained by means of edge tracking. They differ in their spatial localisation. The first one, TW1, has been calculated at $Re_0 = 1394$ in a domain of size $L_x/h \times L_y/h \times L_z/h = 2\pi \times 2 \times 2\pi$ [Zammert & Eckhardt 2014]. It is localised in the spanwise direction, with two low-speed streaks accompanied by four vortices and is mirror-symmetric about the midplane. The second one, TW-sym, has been obtained by a Newton-Krylov search at $Re_0 = 1010$ from an ECS in the domain $L_x/h \times L_y/h \times L_z/h = 2\pi \times 2 \times \pi$ that had originally been calculated with constant flow rate [Zammert & Eckhardt 2015]. It consists of two high-speed streaks, four low-speed streaks and eight vortices and is not localised in the spanwise direction. Visualisations of the streamwise-averaged structures are presented in Fig. 4 where the colour-coding represents the streamwise velocity component and the superimposed arrows the cross-flow.

Calculations of TW-sym are carried out in a subspace that enforces mirror-symmetry.
about the midplane \((y = 0)\) and in spanwise direction about the plane \(z = \pi/2\)

\[
\begin{align*}
\mathbf{s}_y \left( \left( u, v, w \right)^t(x, y, z) \right) &= \left( u, -v, w \right)^t(x, -y, z), \\
\mathbf{s}_z \left( \left( u, v, w \right)^t(x, y, z) \right) &= \left( u, v, -w \right)^t(x, y, -z),
\end{align*}
\]

where the superscript denotes the transpose. Invariant solutions obtained in symmetry-reduced subspaces are also solutions with respect to the unrestricted dynamics, where the number of unstable directions is usually higher (Duguet \textit{et al.} 2008b; Kreilos & Eckhardt 2012; Avila \textit{et al.} 2013). In this context it thus is of interest to assess the effect of symmetry-reduced calculations on feedback stabilisation. For this reason, we also carried out controlled simulations of TW1 within its symmetry-reduced subspace.

As mentioned before, the pressure-based feedback control introduced in sec. 3 can only stabilise unstable directions with non-zero streamwise average, as it is translationally invariant in the streamwise direction. Stability analyses including calculations of the eigenvectors of the Jacobian at the respective operating points were carried out for the free and the controlled dynamics through Arnoldi iteration. In both cases, the unstable directions indeed have non-zero streamwise mean, as can be seen in fig. 5.

### 5. Stabilisation

Figure 6 presents phase-space trajectories of the controlled system for perturbations about TW1 and TW-sym with \(\|u\|_2\), the friction factor \(C_f = 2\tau_w/(\rho U_0^2)\), where \(\tau_w\) is the shear stress at the bottom wall, and the cross-flow energy

\[
E_{cf}(t) = \frac{1}{L_x L_y L_z} \int_0^{L_x} \int_{-L_y/2}^{L_y/2} \int_0^{L_z} dxdydz \left( v^2(x, y, z, t) + w^2(x, y, z, t) \right),
\]

as functions of the control parameter \(R\), i.e. series TW1-A, TW1-B, TW-sym-A and TW-sym-B in table 1. The top panels correspond to results for series TW1-A (left) and TW1-B (right), and the bottom row for TW-sym-A (left) and TW-sym-B. All panels contain datasets from simulations carried out with different values of the control strength \(\mu\) indicated by the colour gradient, where darker colours correspond to higher values of \(\mu\) and hence stronger control. The corresponding control lines, which must intersect at the operating point, are shown in black. As can be seen from the data shown in the two panels, the feedback control results in phase-space trajectories where the perturbed edge state is driven towards the operating point for all observables. In case of TW1-A, the trajectories resemble those from the model system discussed in sec. 3 and shown in fig. 2. For the friction factor (TW1-B), the trajectories first approach intermediate states on the control line and subsequently follow the control line towards the operating point. For TW-sym-A the trajectories show large excursions and eventually return to the operating point, while for TW-sym-B the dynamics evolves along the control lines. We note that the operating point was not approached in our attempts to control the dynamics of TW1 with respect to \(E_{cf}\) or that of TW-sym with respect to \(C_f\).

The simulations shown in fig. 6 reached close vicinity of the operating point after very short simulation times (around 20 time units for norm-controlled simulations around 50 time units for friction-controlled simulations) of both TW1 and TW-sym. However, if the controlled system is evolved for very long times, the trajectories leave the operating point again. This is demonstrated by the time evolution of \(\|u\|_2\) shown in red (light grey) and \(C_f\) shown in blue (dark grey) in the left panel of fig. 7 for the operating point TW1. A deviation of \(\|u\|_2\) from the reference value is visible after about 1000 time units, while \(C_f\) appears to remain constant. The right panel of fig. 7 presents the time evolution
of the cross-flow energy with $v$ and $w$ being the wall-normal and spanwise components of $\mathbf{u} = (u, v, w)$. The control is unable to prevent the dynamics from escaping from the operating point towards the laminar fixed point. Interestingly, this happens on a much shorter timescale compared to the departure of the control observables from their target values. Similar observations can be made for the dynamics of TW1 controlled
Figure 5. Visualisation of the unstable eigenmodes of TW1 (top) and TW-sym (bottom) showing the deviation of the streamwise average of the streamwise velocity component, $\langle u \rangle_x$ from the laminar profile. The crossflow ($v, w$) is indicated by the arrows.

with respect to $C_f$, for TW-sym and for controlled simulations of TW1 carried out in its symmetry-reduced subspace (not shown).

The results shown in fig. 7 suggest the presence of a residual instability in the controlled simulations. According to the discussion in sec. 2.2, an instability in the controlled system could result from the control being too weak to completely remove the original instability,
from the control being orthogonal to the unstable direction as would be the case for strictly periodic instabilities, or from an undesired destabilising effect of the control on the stable directions. The first possibility can be ruled out by an exhaustive parameter scan. The second possibility does not apply either, as the unstable directions have non-zero streamwise mean as discussed in sec. 4.1 and thus overlap with the control. In what follows we therefore investigate in detail how the control alters the tangent space structure of the chosen invariant solutions.

5.1. Effect of the control on stable and unstable directions

In order to quantify the effect of the control on the tangent space of the invariant solutions investigated here, stability analyses of TW1 with respect to the coupled system consisting of DNS and feedback control as in eqs. (3.3)-(3.6) have been carried out, see series TW1-A-stab listed in table [1]. Figure 8 shows the eigenvalues of the Jacobian at TW1 for the free dynamics and for the $L_2$-norm controlled system for different values of the control strength $\mu$. As can be seen, the free dynamics is such that TW1 has one unstable direction as expected for an edge state. For low values of $\mu$ the corresponding single positive real eigenvalue decreases with increasing $\mu$. At the same time, a pair of complex conjugate eigenvalues with negative real part move closer to the line where the latter vanishes. Eventually, their real part becomes positive indicating the presence of new unstable directions. For a small set of parameters, both old and new unstable directions are present. That is, even though the original unstable direction is removed...
feedback control of edge states

Figure 7. Time evolution of the control observables (left) and the cross-flow energy (right) for the controlled simulations with target state TW1. Dynamics controlled with respect to the $L_2$-norm with $\mu = 10^6$ and with respect to $C_f$ with $\mu = 3 \times 10^6$ are shown in red (grey) and blue (dark grey), respectively.

Figure 8. Spectrum of the Jacobian at TW1 for the combined system DNS with feedback control according to eqs. 3.3-3.6 as a function of the control strength $\mu$. The thick black dots correspond to the uncontrolled system and the decreasing color gradient indicates increasing values of $\mu$.

For large enough $\mu$, the control indeed destabilises stable directions of the uncontrolled system. Willis et al. (2017) also calculated eigenvalues and Floquet exponents for their successfully stabilised invariant solutions. In both cases there are stable eigenvalues whose real parts move closer to zero in the controlled system, see fig. 3a of Willis et al. (2017) for the spectrum of a travelling wave, and fig. 4c for the Floquet exponents of a stabilised periodic orbit. In summary, a simple one-dimensional feedback control can have adverse effects on the stable directions, whereby the real parts of the stable eigenvalues tend to zero and may even become positive as shown here. This precludes the application of the pressure-based feedback control to the search for new invariant solutions in channel flow following...
the procedure proposed by Willis et al. (2017) for pipe flow, as without any information about eventual overlaps between the control and the stable directions it is difficult to know a-priori if such a feedback-induced instability indeed occurs. Hence a black-box application of such feedback strategies without good knowledge of the coefficients is not guaranteed to work.

Before returning to this point in more detail in the following section, we briefly discuss the experimental applicability of our results in terms of turbulence control. Although the feedback control does not stabilise the operating point, it is able to find global target observables connected with the streamwise component of the flow, which do not require knowledge of all velocity components and are thus easier accessible experimentally. This suggests that the proposed pressure-based feedback control can be used to confine turbulent dynamics to a region of phase space close to the edge state and to prevent large fluctuations in kinetic energy or drag. Preliminary results for a wall-suction-based feedback control for plane Couette flow (Linkmann & Eckhardt 2019) show that this is indeed the case.

5.2. Revised control strategy

The destabilising effect of the control given by eqs. (3.3)-(3.6) calls for a new control strategy, which acts on the unstable direction only. In what follows we construct a control that acts on a hyperplane orthogonal to the stable subspace of the ECS’s tangent space and hence cannot couple and destabilise the contracting directions. Similar approaches are used in controlling linear, infinite-horizon problems. There, the optimal control strategy is of feedback type and proceeds by projection of the state vector onto its unstable eigenspace and an appropriate choice of coupling coefficients such that domain of the linear operator representing the controlled system has only stable eigenmodes (Anderson & Moore 1990; Burl 1999). We consider a general n-dimensional dynamical system

$$\dot{\xi} = F(\xi) ,$$

where $F$ is a differentiable function that governs the time-evolution of $\xi$. In the present application $\xi$ represents the Galerkin-truncated velocity field and $F$ the time evolution given by the appropriately truncated version of eq. (3.1) in terms of a finite number of coupled ordinary differential equations. Let $\xi_0$ correspond to the operating point, then the linearised dynamics close to $\xi_0$ are given by

$$\dot{\delta \xi} = J_F \delta \xi ,$$

where $J_F = J_F(\xi_0)$ is the Jacobian of $F$ at $\xi_0$. The tangent space at $\xi_0$ is then spanned by the right eigenvectors $\{v_i\}_{i=1,...,n}$ of $J_F$. Since $J_F$ is in general non-normal, the $\{v_i\}$ are not mutually orthogonal, i.e. $(v_i, v_j) \neq \delta_{ij}$, with $(\cdot, \cdot)$ being an inner product on the tangent space at $\xi_0$. Hence a control procedure that overlaps with the unstable directions may also overlap with the stable ones. However, the dual basis $\{v^*_i\}$ satisfies the desired bi-orthogonality constraints by definition $(v_i, v^*_j) = \delta_{ij}$. If we have $k < n$ unstable directions, $v_1, ..., v_k$, a control that is constructed as a linear combination of the duals $v^*_1, ..., v^*_k$ will be orthogonal to all stable directions. More specifically, the purpose of a feedback control $f(\xi)$ is to stabilise $\xi_0$, that is, to ensure that all eigenvalues of $J_F + J_f$ have negative or zero real parts. In what follows we assume that $J_F$ has one expanding direction $v_u$. If we construct $f$ to act along $v^*_u$ such that the controlled dynamical system is given by

$$\dot{\xi} = F(\xi) + f(\xi) = F(\xi) + \kappa(\xi)v^*_u ,$$
where $\kappa$ is a function of $\xi$ implementing the feedback, then the controlled linearised system is
\[
\delta \dot{\xi} = (J_F + v_u \otimes \nabla \kappa) \delta  \xi ,
\]
where $\nabla \kappa$ denotes the gradient of $\kappa$ at $\xi_0$. Since the dimension of the tangent space at any point equals that of the underlying manifold, we can expand $\delta \xi$ at any point in time in terms of the basis $v_i$,
\[
\delta \xi(t) = \sum_i a_i(t) v_i ,
\]
where $a_i$ are time-dependent coefficients. Equation (5.5) becomes
\[
\sum_i \dot{a}_i(t) v_i = (J_F + v_u \otimes \nabla \kappa) \sum_i a_i(t) v_i = \sum_i \left( \lambda_i + \sum_j k_j v_u \otimes v_j \right) a_i(t) v_i
\]
\[
= \sum_i \lambda_i a_i(t) v_i + \sum_{i,j} a_i(t) k_j (v_j^*, v_i) v_u
\]
\[
= \sum_i \lambda_i a_i(t) v_i + \sum_{i,j} a_i(t) k_j \delta_{ij} v_u
\]
\[
= \sum_{i \neq u} \lambda_i a_i(t) v_i + \left( \lambda_u a_u(t) + \sum_i k_i a_i(t) \right) v_u ,
\]
where $\lambda_i$ are the eigenvalues of $J_F$ and $k_j = (\nabla \kappa^*, v_j)$. By taking the inner product of both sides of this equation with $v_u^*$ it can be seen that $\xi_0$ is stabilised if $k_i = 0$ for $i \neq u$ and if
\[
\lambda_u + k_u \leq 0 ,
\]
that is, the gradient of the feedback function $\kappa$ at the operating point must be colinear with the unstable direction. In the present example of channel flow, $\kappa$ is determined by the choice of observable. An observable that is quadratic in the velocity field will result in $\nabla \kappa$ being colinear with the operating point, at least in approximation. If the latter then has a significant overlap with the unstable direction, the choice of observable may work well. Close inspection of the unstable direction can yield further information, for example if the instability is mostly in span- or wall-normal directions, the cross-flow energy is a good observable.

For time-independent operating points, i.e. equilibria of eq. (5.2), with one unstable eigenmode, the implementation of such a control procedure results in replacing the unit vector $e_x$ on the right-hand side of eqs. (3.3) with the dual of the solution’s unstable eigenmode. The generalisation to more unstable directions is straightforward. For travelling wave or periodic solutions, the implementation is slightly more complicated as the time-dependence of the target state has to be accounted for. For a wave travelling in streamwise direction with speed $c$ the revised control strategy is given by eqs. (3.4)-(3.5), with (3.3) replaced by
\[
\partial_t u + u \cdot \nabla u + \nabla p - \frac{1}{Re} \Delta u + \left( \frac{dP}{dx} \right)_0 e_x = \frac{2R(t)}{R_0^2} \sigma_c(t)(e_u^*) .
\]
where $\sigma_c$ is the shift operator in streamwise direction
\[
\sigma_c(t) : u(x, y, z) \mapsto u(x + ct, y, z) .
\]
Shifts in spanwise direction can be accounted for analogously.
Figure 9 shows phase-space trajectories with respect to the $L_2$-norm and the cross-flow energy and time-series of the latter obtained with the control implemented according to eq. (5.9) and eqs. (3.4)-(3.6) for TW1, i.e. series TW1-C and TW1-D summarised in table 1. The panels in the left column correspond to DNSs controlled with respect to $L_2$-norm and the panels in the right column to controlled runs with respect to the cross-flow energy. As can be seen from the phase space trajectories in the top-row panels, all controlled simulations approach the targeted values of the chosen observables, as has been the case for the streamwise-invariant control discussed in the beginning of sec. 5. The time evolution of the cross-flow energy (bottom-row panels) indicates that the controlled system now also approaches the actual operating point and stays in its vicinity for around 200 time units for the $L_2$-norm control and for over 300 time units for cross-flow control.

Results from controlled simulations targeting TW-sym, that is, series TW-sym-C in table 1 are presented in fig. 10. Here, stabilisation has been achieved using the $L_2$-norm as an observable, as can be seen from the phase-space trajectories of runs TW-sym-C3, TW-sym-C4 and TW-sym-C5 in the top left panel and the corresponding evolution of the cross-flow energy in the bottom left panel of the figure. Compared with the controlled dynamics targeting TW1 shown in fig. 9, the approach to the operating point is much slower, but the stabilisation is complete. For low values of the control strength $\mu$, i.e. for runs TW-sym-C1 and TW-sym-C2, the controlled dynamics gets trapped into new invariant tori, where the mean values of the $L_2$-norm and the cross-flow energy depend
Concerning the choice of observable, several observations can be made from a comparison of the visualisations of the ECS in fig. 4 and those of their respective unstable directions shown in fig. 5. For both structures we note that the cross-flow varies very little between the ECS and its unstable direction, while clear differences are visible in the streamwise velocity component at least for TW1. This suggests that the cross-flow energy should work better than the $L_2$-norm as a control observable for TW1, which is indeed the case. For TW-sym the $L_2$-norm worked well. Finally, we note that stabilisation through the revised feedback strategy could not be achieved using the friction factor $C_f$ as an observable. Since $C_f$ is linear in the velocity field, its gradient at the operating point is a constant vector and its dual hence not orthogonal to all stable directions.

Projections onto bi-orthogonal bases, stable and unstable eigenmodes used in the feedback strategy proposed here being only one example thereof, are used in controlling high-dimensional systems where the algorithm requires a reduction of the number of degrees of freedom to become viable (Antoulas et al. 2001; Lauga & Bewley 2003, 2004; Akervik et al. 2007; Ehrenstein & Gallaire 2008; Henningson & Akervik 2008; Barbagallo et al. 2009). There, a high-dimensional system is modelled by projection onto a lower dimensional subspace spanned by an appropriately chosen set of basis modes, and a control strategy for the reduced system is calculated. In order for this
control strategy to work on the full system, the subspace must, of course, include all unstable eigenmodes, but more importantly also the set of stable eigenmodes that are triggered by the control. [Ehrenstein & Gallaire (2008)] successfully stabilised an unstable flow by projection onto a subset of stable eigenmodes, however, this is not a strategy that works generically, and sometimes other bases such as proper orthogonal decomposition (POD) modes constitute a better choice ([Barbagallo et al. 2009]).

6. Conclusions

Here, we have devised two feedback control procedures designed to stabilise invariant solutions of the Navier-Stokes equations. We focussed thereby on edge states because of their relevance for the bypass transition to turbulence in parallel shear flows. The main result of the present paper is that stabilisation of edge states in plane Poiseuille flow can be achieved if the feedback control remains orthogonal to the contracting subspace of the target state’s tangent space. A simpler, pressure-based, and thus in principle experimentally viable, control strategy is shown to target set values of the $L_2^2$-norm, the cross-flow energy or the friction factor. As such, pressure-based dynamic feedback may be a useful tool to accelerate or prevent relaminarisation events.

The pressure-based control strategy was inspired by the work of [Willis et al. (2017)] on feedback stabilisation of edge states in pipe flow, where the viscosity was adjusted as a function of energy-type observables. In order to obtain a control procedure that, in principle, can be carried out experimentally, we proposed to adjust the pressure gradient instead of the viscosity as a function of either energy-type observables or the friction factor. Numerical simulations of the controlled system were carried out for two target edge states which are both travelling wave solutions and which differ in their respective degree of spanwise localisation. Even though the control resulted in the dynamics approaching the respective target values of the observables used in the control strategy, the actual edge states were not stabilised as the control procedure has a destabilising effect on some of the structures’ contracting directions.

This observation led to the construction of a more sophisticated feedback procedure whereby the control acts along the dual vector of the original unstable direction, that is, on a hyperplane orthogonal to all stable directions. Here, it was found that care must be taken in the choice of observable, because the latter results in different feedback terms whose gradients may or may not overlap with the stable directions. However, we found that for standard energy-type observables such as the $L_2^2$-norm or the energy of the transverse fluctuations, the target edge states were stabilised if not completely then at least for an extended time interval.

Closed-loop control strategies are commonly used in engineering and industrial applications, where the success of a given method must also be assessed in terms of its energy efficiency. In most cases, dynamics that do not correspond to an equilibrium or a periodic orbit of the uncontrolled system have to be enforced, a process which comes with an unavoidable energy expense. In contrast, the advantage of a stabilised invariant solution is that it maintains itself and does not require any external energy input to evolve. The expense of the control therefore results from the stabilisation only. Since the feedback strategies discussed here alter the stability of an exact solution to the Navier-Stokes equations, they can not only be used to stabilise an operating point, but also to further destabilise it, if so desired, or to confine the dynamics to a certain region in phase space. This may be useful in systems where there is an interest in avoiding certain flow states, e.g. those with enhanced drag. Such applications call for a generalisation of the proposed feedback from edge states with one unstable direction to ECS with
more unstable directions, which is a straightforward procedure for travelling waves. A generalisation to more complicated objects such as unstable relative periodic orbits or chaotic states will be more challenging, as these are not fixed points of the observables.

Recent results from numerical simulations of channel flow suggest that extreme fluctuations in the streamwise component of the wall-shear stress are less likely if the flow is maintained by prescription of a constant flow rate compared with forcing through a constant pressure drop or a fixed energy input \cite{Quadrio2016}, however, the differences concerned rare events. Dynamic feedback could be a possibility to avoid extreme fluctuations more effectively. \cite{FarazmandSapsis2019} showed that extreme events in 2D Kolmogorov flow can be avoided by dynamically regulating the dynamics of certain Fourier modes at the driving scale. Preliminary results show that a variant of the feedback strategy proposed here can be applied to damp transverse fluctuations in plane Couette flow through adjustable wall suction \cite{LinkmannEckhardt2019}. The latter calls for further investigation, which will be reported elsewhere in due course.

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