Supercoset CFT’s for String Theories on Non-compact Special Holonomy Manifolds

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Abstract

We study aspects of superstring vacua of non-compact special holonomy manifolds with conical singularities constructed systematically using soluble \( \mathcal{N} = 1 \) superconformal field theories (SCFT’s). It is known that Einstein homogeneous spaces \( G/H \) generate Ricci flat manifolds with special holonomies on their cones \( \simeq \mathbb{R}_+ \times G/H \), when they are endowed with appropriate geometrical structures, namely, the Sasaki-Einstein, tri-Sasakian, nearly Kähler, and weak \( G_2 \) structures for \( SU(n), Sp(n), G_2, \) and \( Spin(7) \) holonomies, respectively. Motivated by this fact, we consider the string vacua of the type: \( \mathbb{R}^{d-1,1} \times (\mathcal{N} = 1 \text{ Liouville}) \times (\mathcal{N} = 1 \text{ supercoset CFT on } G/H) \) where we use the affine Lie algebras of \( G \) and \( H \) in order to capture the geometry associated to an Einstein homogeneous space \( G/H \). Remarkably, we find the same number of spacetime and worldsheet SUSY’s in our “CFT cone” construction as expected from the analysis of geometrical cones over \( G/H \) in many examples. We also present an analysis on the possible Liouville potential terms (cosmological constant type operators) which provide the marginal deformations resolving the conical singularities.
1 Introduction

String theories/M-theory on special holonomy manifolds developing conical singularities are of great importance by several reasons: Firstly, they exhibit the non-perturbative quantum effects, such as gauge symmetry enhancements, due to the appearance of light solitonic states [1–3]. Secondly, they are expected to provide frameworks to discuss the holographic dualities with the local or non-local interacting theories defined on the asymptotic boundaries [4, 5], which naturally generalize the AdS/CFT-correspondence [6].

It is known that an arbitrary \((m-1)\)-dimensional Einstein space \(X_{m-1}\) possesses a Ricci flat metric on its \(k\)-dimensional cone \(C(X_{m-1})\) of the form

\[
ds^2 = dr^2 + r^2 ds^2_{X_{m-1}},
\]

where \(r\) is the radial coordinate, and the special holonomies on \(C(X_{m-1})\) originate from the “weak special holonomies” on \(X_{m-1}\) [7–10]. To be more precise, the \(SU(n)\), \(Sp(n)\), \(G_2\) and \(Spin(7)\) holonomies on the cone \(C(X_{m-1})\) are in one to one correspondence with the Sasaki-Einstein \((m = 2n)\) [11], tri-Sasakian \((m = 4n)\) [12,13], nearly Kähler \((m = 7)\) [14] and weak \(G_2\) \((m = 8)\) structures [15] on \(X_{m-1}\), respectively as proved in [16]1 (See the table 1.). This fact is very useful to systematically construct special holonomy manifolds with conical singularities, because the Einstein homogeneous spaces \(X_{m-1} = G/H\) endowed with these geometrical structures are well understood since the old days of Kaluza-Klein supergravity (SUGRA) [17–23] (and [24] for a review) as well as from the mathematical literature mentioned above [11–16] and [25].

On the other hand, there also exists extensive literature on the worldsheet approaches to these conical backgrounds in string theory. Early literature for the conifold and \(K3\)-singularity is [26] and [27]. More recent studies are given in [28–31] for the \(SU(n)\)-holonomies, emphasizing the role of \(\mathcal{N} = 2\) Liouville theory [32] and the holographically dual descriptions based on the (wrapped) NS5-brane geometry (see also [33]). All of these cases possess the worldsheet \(\mathcal{N} = 2\) superconformal symmetry and have been discussed in [34–37] from the viewpoints of the “non-compact extensions” of Gepner models [38]. There are also several related results [39–42] from the spacetime view points, but with the RR-flux at infinity, and also a work based on the Hybrid formalism [43] is given in [44].

While these constructions in the case of the \(\mathcal{N} = 2\) supersymmetry have been rather successful, it is difficult to construct string vacua on the conical backgrounds with \(Spin(7)\) and \(G_2\) holonomies, which possess at most the \(\mathcal{N} = 1\) worldsheet SUSY. Partial attempts to construct the string vacua of \(Spin(7)\) and \(G_2\) holonomies with conical singularities have been given in [45, 46]. General structure of string theory on manifolds with \(Spin(7)\) and \(G_2\) was discussed in [47, 48] in particular from the point of view of the existence

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1 A good review for physicists on these mathematical concepts is found in the paper [8].
of extended chiral algebras. Structure of these chiral algebras has also been discussed in [49–51]. There are also several results [52] for the CFT constructions of compact $G_2$ and $\text{Spin}(7)$ manifolds based on the geometrical method of Joyce [53].

The main purpose of this paper is to give a systematic way of constructing special holonomy manifolds with conical singularities based on the solvable $\mathcal{N} = 1$ SCFT’s, which may be regarded as a natural generalization of the construction in the $\mathcal{N} = 2$ category mentioned above. Our strategy is quite simple: We formally replace an Einstein homogeneous space $X = G/H$ by an $\mathcal{N} = 1$ supercoset CFT $\mathcal{M} = (G \times \text{SO}(n))/H$, $(n = \dim G - \dim H)$ based on the affine Lie algebra of $G$ and $H$. $\text{SO}(n)$ stands for $n$ free fermions. We then add the $\mathcal{N} = 1$ Liouville sector in place of the radial degrees of freedom. We may call our construction as the “CFT cone” as opposed to the original geometrical cone construction. Of course, one should keep in mind that the coset CFT (WZW model) of $G/H$ is not identical to the non-linear $\sigma$-model with the target manifold $G/H$ because of the presence of NS $B$ field in WZW models. Nevertheless, as we will see in the following, taking the supercoset CFT associated with the Einstein homogeneous space provides a very good anzats for the superstring vacua of special holonomy. We completely classify these coset constructions (at least for the cosets $G/H$ with compact simple groups $G$), which include the models found in [45,46] as well as many of the vacua in the $\mathcal{N} = 2$ category presented in [26–31]. Among other things, we will find that our CFT cone approach leads to the right amount of worldsheet and spacetime SUSY’s as expected from geometrical grounds in many examples.

This paper is organized as follows: In section 2 we clarify the properties of conformal blocks characterizing the special holonomies in our CFT cone ansatzs. We also demonstrate the explicit constructions of spacetime supercharges. In section 3, which is the main part of this paper, we exhibit the classification of our string vacua with diverse spacetime dimensions, and observe how we obtain the special holonomies which coincides (or differs) with those of the geometrical constructions of Ricci flat cones. In section 4 we analyse the spectrum of possible cosmological constant type operators, which will provide the marginal deformations resolving the conical singularities in backgrounds. Section 5 is devoted to discussions on open problems.
Table 1: We summarize the relation between the special holonomies on the cone $C(X_{m-1})$ and the geometrical structures on the Einstein spaces $X_{m-1}$.

| dim | name | holonomy | Killing spinor | ½ spacetime SUSY | base of the cone |
|-----|------|----------|----------------|-----------------|------------------|
| 4   | hyper Kähler | $SU(2)$ | (2,0)          | 16              | tri-Sasakian     |
| 6   | Calabi-Yau   | $SU(3)$ | (1,1)          | 8               | Sasaki-Einstein  |
| 7   | $G_2$        | $G_2$   | 1              | 4               | nearly Kähler    |
| 8   | hyper Kähler | $Sp(2)$ | (3,0)          | 6               | tri-Sasakian     |
| 8   | Calabi-Yau   | $SU(4)$ | (2,0)          | 4               | Sasaki-Einstein  |
| 8   | $Spin(7)$    | $Spin(7)$ | (1,0)      | 2               | weak $G_2$       |

2 Superstring Vacua of Non-compact Special Holonomy Manifolds as the ‘Cone over SCFT’s’

2.1 General Set Up and Aspects of Spacetime SUSY

In this paper we shall search for supersymmetric string vacua with the form

$$R^{d-1,1} \times \mathbb{R}_\phi \times \mathcal{M},$$

where $\mathbb{R}_\phi$ denotes the $\mathcal{N} = 1$ Liouville theory ($\mathcal{N} = 1$ linear dilaton SCFT) with the background charge $Q_\phi$ and $\mathcal{M}$ is a rational $\mathcal{N} = 1$ SCFT with a central charge $c_M$.

We would like to identify this conformal system as a string vacuum of singular special holonomy manifold realized as the “cone over $\mathcal{M}$” in the decoupling limit $g_s \to 0$. By assumption we have a linear dilaton background $\Phi(\phi) = -\frac{Q_\phi}{2} \phi$. The weak coupling region $\phi \sim +\infty$ corresponds to the holographic boundary $R^{d-1,1}$ and the opposite region $\phi \sim -\infty$ is located around the singularity (“tip of the cone”).

The criticality condition is written as

$$\frac{3}{2}(d-2) + \left(\frac{3}{2} + 3Q_\phi^2\right) + c_M = 12,$$

or equivalently,

$$\frac{Q_\phi^2}{8} = \frac{9 - d}{16} - \frac{c_M}{24}.$$  \hspace{1cm} (2.2)

(2.3)

To build up the consistent string vacua the modular invariance is an important criterion. Let us first discuss the values of Liouville momentum which enters into the modular invariant partition function. We recall that the conformal dimension of a Liouville exponential $e^{\gamma \phi}$ is given by

$$h(e^{\gamma \phi}) = -\frac{1}{2} \gamma^2 - \frac{1}{2} Q_\phi \gamma = -\frac{1}{2}(\gamma + \frac{Q_\phi}{2})^2 + \frac{Q_\phi^2}{8}.$$  \hspace{1cm} (2.4)
It is known that the Liouville momentum in the partition function takes values in the “principal continuous series” (at least in the semi-classical analysis) \[54\]

\[\gamma = -\frac{Q_\phi}{2} + ip, \quad p \in \mathbb{R}\] \hspace{1cm} (2.5)

and thus

\[h(e^{\gamma \phi}) = \frac{1}{2}p^2 + \frac{Q_\phi^2}{8} \geq \frac{Q_\phi^2}{8}.\] \hspace{1cm} (2.6)

Liouville exponential with momentum in the principal range represents a plane wave propagating in spacetime and thus is a delta-function normalizable state. We note the presence of a “mass” gap \(Q_\phi^2/8\) in the Liouville spectrum. Therefore, conformal blocks are expanded only in terms of the “massive representations” of the extended superconformal algebra which characterizes each special holonomy. The list of massive representations are given in appendix B.

We here summarize the aspects of spacetime SUSY in (2.1) for various spacetime dimensions for later convenience.

- **SU\((n)\)-holonomy**: 

  This case corresponds to \(d = 10 - 2n\), and the worldsheet SUSY is required to be enhanced to \(\mathcal{N} = 2\). More explicitly, the criterion for the SU\((n)\)-holonomy is to ask whether it is possible to rewrite the theory as\(^2\)

\[\mathbb{R}_\phi \times \mathcal{M} \cong [\mathbb{R}_\phi \times U(1)] \times \frac{\mathcal{M}}{U(1)},\] \hspace{1cm} (2.7)

where \(\mathbb{R}_\phi \times U(1)\) stands for the \(\mathcal{N} = 2\) Liouville theory and the coset \(\mathcal{M}/U(1)\) describes an \(\mathcal{N} = 2\) SCFT. In the case when \(\mathcal{M}\) is an \(\mathcal{N} = 1\) coset, \(\mathcal{M}/U(1)\) should be a Kazama-Suzuki model \[56\] associated to a Kähler homogeneous space.

When making use of this criterion (2.7), the most crucial point is as follows: The \(\mathcal{N} = 2\) Liouville theory includes a compact boson \(Y\) with the radius equal to \(Q_\phi\) whose conformal blocks are written in terms of the theta functions of the level \(n^2 \cdot 2/Q_\phi^2\), where \(n\) is an integer, as discussed in \[34, 36\]. Ambiguity \(n^2\) of the level

\(^2\)In this paper we often use the concise expressions such as

\[\mathcal{M} \cong \frac{\mathcal{M}}{H_k} \times H_k,\]

where \(H_k\) denotes the conformal theory of the level \(k\) \(H\)-current algebra. Precisely speaking, if \(H\) is abelian, the R.H.S must be interpreted as an orbifoldization with respect to the \(H\)-charge as in the Gepner model \[38\]. If \(H\) is semi-simple, the R.H.S strictly means the “projected tensor product” discussed in \[55\].
originates from the choice of possible momentum lattice of compact boson as shown by the theta function identity (A.6).

We have a natural geometric interpretation of (2.7): The Calabi-Yau cones are build up over Sasaki-Einstein spaces, which have the $U(1)$ fibration over the Kähler-Einstein manifolds

$$ \mathcal{M} \xrightarrow{U(1)} \mathcal{M}/U(1) . \quad (2.8) $$

The standard GSO projection with respect to the total $U(1)_R$-charge of $\mathcal{N} = 2$ SCA leads to the spacetime SUSY as in Gepner models. The explicit construction of supercharges is given in [29] and we can check the cancellation of conformal blocks directly.

The characteristic features of the conformal blocks for the $SU(n)$-holonomy in each case $n = 2, 3, 4$ is summarized as follows;

(i) $SU(2)$-holonomy: In this case the conformal blocks are expanded by the massive characters of $\mathcal{N} = 4$ superconformal algebra with $c = 6$ (level 1), which have the forms as $q^{h-\check{\theta}_3^2/\eta^3}$ in the NS sector [59]. The SUSY cancellation is expressed by the familiar Jacobi’s abstruse identity

$$ \left( \frac{\theta_3}{\eta} \right)^4 - \left( \frac{\theta_4}{\eta} \right)^4 - \left( \frac{\theta_2}{\eta} \right)^4 \equiv 0 , \quad (2.9) $$

and we have 16 unbroken supercharges in 6-dimensional spacetime.

(ii) $SU(3)$-holonomy: In this case the conformal blocks are expanded by the massive characters of “$c = 9$ extended superconformal algebra” [60,61], which have the form $q^{h-\check{\theta}_3\check{\theta}_3/\eta^3}$ in the NS sector [60]. The SUSY cancellation is expressed as the following identities

$$ \left[ \left( \frac{\theta_3}{\eta} \right)^2 - \left( \frac{\theta_1}{\eta} \right)^2 \right] \frac{\Theta_{0,1}}{\eta} - \left( \frac{\theta_2}{\eta} \right)^2 \frac{\Theta_{1,1}}{\eta} \equiv 0 , $$

$$ \left[ \left( \frac{\theta_3}{\eta} \right)^2 + \left( \frac{\theta_1}{\eta} \right)^2 \right] \frac{\Theta_{1,1}}{\eta} - \left( \frac{\theta_2}{\eta} \right)^2 \frac{\Theta_{0,1}}{\eta} \equiv 0 , \quad (2.10) $$

and we have 8 supercharges in 4-dimensional spacetime.

(iii) $SU(4)$-holonomy: In this case the conformal blocks are expanded by the massive characters of “$c = 12$ extended superconformal algebra”, which take the form $q^{h-\check{\theta}_3^2/\eta^3}$ in the NS sector [62]. The SUSY cancellation is
ensured by the identities
\[
\frac{\theta_3 \Theta_{0,0} (\tau)}{\eta} - \frac{\theta_4 \widetilde{\Theta}_{0,0} (\tau)}{\eta} - \frac{\theta_2 \Theta_{\frac{1}{2},0} (\tau)}{\eta} \equiv 0, \\
\frac{\theta_3 \Theta_{1,\frac{1}{2}} (\tau)}{\eta} + \frac{\theta_4 \widetilde{\Theta}_{1,\frac{1}{2}} (\tau)}{\eta} - \frac{\theta_2 \Theta_{\frac{3}{2},\frac{1}{2}} (\tau)}{\eta} \equiv 0, \quad (2.11)
\]
and we have 4 supercharges in 2-dimensional spacetime.

- **Sp(n)-holonomy:**

These superstring vacua correspond to \( d = 10 - 4n \) and are described by the \( \mathcal{N} = 4 \) SCFT of \( c = 6n \) (level \( n \)). In the case when \( \mathcal{M} \) is defined as an \( \mathcal{N} = 1 \) supercoset CFT, the condition for the \( \mathcal{N} = 4 \) enhancement of worldsheet SUSY has been studied in [57]. We have the (small) \( \mathcal{N} = 4 \) worldsheet SUSY, if and only if the following rewriting is possible:
\[
\mathbb{R}_\phi \times \mathcal{M} \cong [\mathbb{R}_\phi \times SU(2)] \times \frac{\mathcal{M}}{SU(2)}, \quad (2.12)
\]
where the \( \mathcal{N} = 1 \) coset \( \mathcal{M}/SU(2) \) is associated to a Wolf space (quaternionic symmetric space) [58].

In the similar manner as (2.8), the tri-Sasakian homogeneous spaces are known to have \( SU(2) \)-fibrations over Wolf spaces;
\[
\mathcal{M} \xrightarrow{SU(2)} \mathcal{M}/SU(2), \quad (2.13)
\]
which yields the geometrical interpretation of (2.12).

Concerning the structure of conformal blocks, the case of \( Sp(1) (\cong SU(2)) \)-holonomy was already illustrated above. In the case of \( Sp(2) \)-holonomy, on the other hand, the conformal blocks should be expanded by the massive characters of \( \mathcal{N} = 4 \) SCA with \( c = 12 \), which are written [59] as
\[
\frac{q^{h_{-\ast}}}{\eta} \left( \frac{\theta_3}{\eta} \right)^2 \chi_{SU(2)}^{\ast} \equiv \frac{q^{h_{-\ast}}}{\eta} \left( \frac{\theta_3}{\eta} \right)^2 \frac{\Theta_{\ast,1}}{\eta}. \quad (2.14)
\]
The SUSY cancellation is expressed by the identity (2.10) again. However, it turns out that we obtain the \( \mathcal{N} = (3,3) \) (or \( \mathcal{N} = (6,0) \)) spacetime SUSY, that is, there exist only 6 supercharges in 2-dimensional spacetime.
• $G_2$-holonomy:

We set $d = 3$, and the spacetime SUSY requires the condition

$$U(1)_{3/2} \subset \mathcal{M}, \quad (2.15)$$

where $U(1)_{3/2}$ denotes a $c = 1$ CFT whose conformal blocks are described by characters $\Theta_{*,3/2}(\tau)/\eta(\tau)$. Precisely this relation means that the conformal blocks of the $\mathcal{M}$-sector are expanded in terms of characters $\Theta_{*,3/2}(\tau)/\eta(\tau)$ and the branching coefficients are not hit by the GSO projection. The SUSY cancellation is realized again as (2.11). If one takes account of the Liouville fermion, we have a relation [51]

$$SO(1) \times U(1)_{3/2} \cong \frac{SO(6)_1}{SU(3)_1} \times SO(1)$$

$$\cong \frac{SO(7)_1}{(G_2)_1} \times \frac{(G_2)_1}{SU(3)_1}$$

$$\cong \text{tri-critical Ising} \times 3\text{-state Potts}, \quad (2.16)$$

Thus our condition is consistent with the criterion for the $G_2$-holonomy given in [47].

• $Spin(7)$-holonomy:

We set $d = 2$, and the existence of spacetime SUSY requires the condition

$$\text{tri-critical Ising} \subset \mathcal{M}, \quad (2.17)$$

which precisely means that the conformal blocks in the $\mathcal{M}$-sector can be decomposed by the $\mathcal{N} = 1$ characters of the tri-critical Ising model (the first model in the $\mathcal{N} = 1$ minimal series with $c = 7/10$). The SUSY cancellation is realized as the following identities $^3$:

$$\sqrt{\frac{\theta_3}{\eta}} \chi^{\text{tri NS}}_0 - \sqrt{\frac{\theta_4}{\eta}} \chi^{\text{tri NS}}_1 \equiv 0$$

$$\sqrt{\frac{\theta_3}{\eta}} \chi^{\text{tri NS}}_1 + \sqrt{\frac{\theta_4}{\eta}} \chi^{\text{tri NS}}_0 \equiv 0.$$

$^3$In the convention here, the $\mathcal{N} = 1$ characters of tri-critical Ising are written in terms of the $\mathcal{N} = 0$ ones as follows:

$$\chi^{\text{tri NS}}_0 = \chi_0 + \chi_{3/2}, \quad \chi^{\text{tri NS}}_1 = \chi_0 - \chi_{3/2}, \quad \chi^{\text{tri NS}}_{1/10} = \chi_{1/10} + \chi_{3/10}, \quad \chi^{\text{tri NS}}_{1/10} = \chi_{1/10} - \chi_{3/10},$$

$$\chi^{\text{tri R}}_{7/16} = 2\chi^{\text{tri}}_{7/16}, \quad \chi^{\text{tri R}}_{3/80} = 2\chi^{\text{tri}}_{3/80}.$$
Table 2: We summarize the algebraic structures of worldsheet theories describing manifolds with special holonomies. The “structure for the massive reps.” is the algebraic structure which is manifest in the characters of massive representations.

### 2.2 Constructions of Spacetime Supercharges

Although we have just seen the characteristic features of conformal blocks in the string vacua (2.1), it may still be helpful to construct explicitly the spacetime supercharges for these vacua. We shall only consider the left-movers for simplicity. It is straightforward to combine the left and right movers so as to be consistent with the GSO conditions for the type IIA or type IIB string theories.

1. **Supercharges for the $SU(n)$-holonomies**:

   Construction of supercharges in these cases is quite standard and presented in [29]. We prepare the spin fields $S_{\alpha}^{(\pm)}$ for the Minkowski spacetime $\mathbb{R}^{d-1,1}$ with the conformal weight $h = d/16$, where $\alpha = 1, 2, \ldots, 2^\lceil \frac{d-2}{2} \rceil$, and $\pm$ denotes the chirality. We also introduce the spin fields for the sector of internal space described by the $N=2$ SCFT

   \[ \mathbb{R}_{\phi} \times \mathcal{M} \cong [\mathbb{R}_{\phi} \times U(1)] \times \frac{\mathcal{M}}{U(1)}. \]

   The total $U(1)_R$ current $J_R$ is bosonized as

   \[ J_R = i\partial H, \quad H(z)H(0) \sim -n \ln z, \]

   and the relevant spin fields are defined as

   \[ S_{\pm} = e^{\pm \frac{i}{2}H}, \]

   which have the conformal weight $h = n/8$. 

2
The GSO projection gives the chiral supercharges for \( d = 2,6 \), and non-chiral supercharges for \( d = 4; \)

\[
Q_+^\alpha = \oint S_0^{\alpha(+)} S^\pm e^{-\frac{\varphi}{2}},
\]

\[
Q_-^\alpha = \oint S_0^{\alpha(-)} S^\mp e^{-\frac{\varphi}{2}},
\]

where \( \varphi \) denotes the standard bosonized superghost.

2. Supercharges for the Spin(7)-holonomy

The Spin(7)-holonomy admits one Killing spinor with a definite chirality. We thus assume the spin-field in \( \mathbb{R}^{1,1} \) should have a definite chirality, say, \( S_0^{(+)} \). \(^4\) In order to define the spin fields for the internal space, we treat the Liouville and \( \mathcal{M} \) sectors separately. Since the \( \mathcal{N} = 1 \) Liouville sector includes a free fermion, we have the doubly degenerate spin fields \( \sigma^\phi_\pm \), which have the conformal weight \( h = 1/16 \) and satisfy the OPEs

\[
\psi^\phi(z)\sigma^\phi_\pm(0) \sim \frac{\pm i}{\sqrt{2}z^{1/2}}\sigma^\phi_\mp, \quad \sigma^\phi_\pm(z)\sigma^\phi_\pm(0) \sim \frac{1}{z^{1/8}}, \quad \sigma^\phi_+(z)\sigma^\phi_- (0) \sim \frac{z^{3/8}}{\sqrt{2}}\psi^\phi(0).
\]

On the other hand, the assumption (2.17) implies the existence of spin fields from tricritical Ising model with \( h = 7/16 \). They are again doubly degenerate and we denote them as \( \sigma^M_\pm \).

The candidate supercharges are now written as \( \sim \oint S_0^{(+)} \sigma^\phi_\sigma^M e^{-\varphi/2} \). To achieve the correct construction we must take account of the BRST invariance, especially the condition \( G_0(\equiv G^\phi_0 + G^M_0) = 0 \). The following relation is helpful for this purpose;

\[
G^M(z)\sigma^M_\pm(0) \sim \frac{1}{z^{3/2}}\sqrt{\frac{7}{16} - \frac{c_M}{24}}\sigma^M_\pm(0)
\]

\[
= \frac{1}{z^{3/2}}\frac{Q_\phi}{2\sqrt{2}}\sigma^M_\pm(0),
\]

where the first line is deduced from \((G^M_0)^2 = L^M_0 - \frac{c_M}{24}\), and the second line is due to (2.3). Now, it is not difficult to find out the following BRST invariant combinations;

\[
Q = \oint S_0^{(+)} \left( \sigma^\phi_+\sigma^M_\pm + \sigma^\phi_-\sigma^M_- \right) e^{-\varphi/2}, \quad Q' = \oint S_0^{(+)} \left( \sigma^\phi_+\sigma^M_+ + \sigma^\phi_-\sigma^M_- \right) e^{-\varphi/2}.
\]

\(^4\)Without this assumption, we would have twice as many supercharges as those expected from supergravity. In fact, we further construct the BRST invariant supercharge \( \tilde{Q} = \oint S_0^{(-)} \left( \sigma^\phi_+\sigma^M_\pm + \sigma^\phi_-\sigma^M_\mp \right) e^{-\varphi/2} \) which is mutually local with \( Q \) (\( Q' \)). This disagreement is probably an artifact originating from the fact that we are now working on the singular background without Liouville potential terms, while the 2-dimensional supergravity should correspond to low energy effective theory on smooth backgrounds.
They are mutually non-local and we must take only one of them, which amounts to process of GSO projection. We have thus obtained the desired supercharge.

3. Supercharges for the $G_2$-holonomy

Construction of supercharges for the $G_2$-holonomy is almost parallel. The condition (2.15) ensures the existence of the doubly degenerate spin fields $\sigma^M_\pm$ with $h = 3/8$, and the BRST invariant supercharges are found out to be

$$Q^\alpha = \oint S^\alpha_0 \left( \sigma^\phi_+ \sigma^M_+ + \sigma^\phi_- \sigma^M_- \right) e^{-\varphi/2}, \quad Q'^\alpha = \oint S^\alpha_0 \left( \sigma^\phi_- \sigma^M_+ + \sigma^\phi_+ \sigma^M_- \right) e^{-\varphi/2},$$

(2.27)

where $S^\alpha_0 (\alpha = 1, 2)$ is the 3-dimensional spin field with the conformal weight $h = 3/16$. $Q^\alpha$ and $Q'^\alpha$ are again mutually non-local, and we have to take either one by the GSO projection.

4. Supercharges for the $Sp(2)$-holonomy

The last case is the most non-trivial. The $Sp(2)$-holonomy admits three independent Killing spinors with the same chirality. So, we assume the longitudinal part of spin fields is $S^0_0^{(+)}$, as in the case of $Spin(7)$-holonomy.

To define the spin fields in the internal space, we first recall the $SU(2)_2$-current algebra $\{K^a(z)\}$ describing the $SU(2)_R$-symmetry in the $\mathcal{N} = 4$ superconformal theory, and the total $U(1)_R$ current of the $\mathcal{N} = 2$ SCFT is identified as $2K^3(z)$. Moreover, since all conformal blocks are expanded into massive characters (2.14), the $SU(2)_2$ current algebra is also embedded in $SU(2)_1 \times SO(4)_1$. In particular, we have the following decomposition of the total $U(1)_R$-current;

$$2K^3(z) = i\partial H_1(z) + i\partial H_2(z) + \sqrt{2}i\partial H_3(z) ,$$

(2.28)

$$H_i(z)H_j(0) \sim -\delta_{ij} \ln z .$$

(2.29)

Here the compact bosons $H_1, H_2$ have radii equal to 1, and describe 1+3=4 free fermions corresponding to the superpartners of the Liouville mode and the $SU(2)$-factor in (2.12) (the fermions along the fiber of (2.13)). $H_3$ has the radius equal to $\sqrt{2}$ (self-dual radius) and generates the $SU(2)_1$-current algebra. We now introduce the following spin fields (up to cocycle factors)

$$S^{\epsilon_1 \epsilon_2 \epsilon_3} = e^{i\frac{1}{4}H_1 + i\frac{3}{4}H_2 + i\frac{\sqrt{2}}{4}H_3} , \quad (\epsilon_i = \pm 1).$$

(2.30)

Candidate supercharges are now (linear combinations of) $\oint S^0_0^{(+)} S^{\epsilon_1 \epsilon_2 \epsilon_3} e^{-\varphi/2}$ which have 8 independent components. Again the non-trivial condition for the BRST invariance is
the constraint \( G_0 \equiv G_0^M + G_0^\phi = 0 \), and one finds the following solutions

\[
Q^\pm = \oint S_0^{(+) S^{\pm \pm}} e^{-\varphi/2},
\]

\[
Q^3 = \frac{1}{2} \oint S_0^{(+) (S^{++} + S^{--})} e^{-\varphi/2},
\]

(2.31)

and also

\[
Q' = \frac{1}{2} \oint S_0^{(+) (S^{+-} - S^{-+})} e^{-\varphi/2}.
\]

(2.32)

As is easily shown, \( Q^+ \), \( Q^- \) are the same supercharges as those of \( SU(4) \)-holonomy (2.22). Moreover, \( Q^a(a = \pm, 3) \) composes a triplet of \( SU(2)_R \) and \( Q' \) is a singlet. In particular, \( \{Q^a\} \) are identified as the spin 1 primary fields of the current algebra \( K^a(z)(a = \pm, 3) \). \( Q^a \) and \( Q' \) are mutually non-local, and we should choose either one by the GSO projection. We take the triplet \( Q^a \) to recover the string vacua of \( Sp(2) \)-holonomy.

Finally let us make a comment. It is well-known [7–10] that the Ricci flat background \( \mathbb{R}^{d-1,1} \times C(X_{9-d}) \) is converted into \( AdS_{d+1} \times X_{9-d} \) by letting \((d-1)\)-branes “fill” the Minkowski spacetime \( \mathbb{R}^{d-1,1} \), and taking the near horizon limit. We here denote the \((9-d)\)-dim. Einstein space as \( X_{9-d} \) and its Ricci flat cone as \( C(X_{9-d}) \). Even though the worldsheet approach to string theory on such backgrounds is usually very difficult due to the RR-flux, the \( d = 2 \) cases are tractable. Namely, we can fill the NS1 branes and interpolate the background \( \mathbb{R}^{1,1} \times C(X_7) \) to \( AdS_3 \times X_7 \).

The analogous relation in our study of “cone over SCFT” could be depicted as

\[
\mathbb{R}^{1,1} \times \mathbb{R}_\phi \times \mathcal{M} \overset{+\text{NS1}}{\longrightarrow} AdS_3 \times \mathcal{M}.
\]

(2.33)

In the R.H.S, the \( AdS_3 \) sector is described by the \( SL(2; \mathbb{R}) \) super WZW model with the (bosonic) level \( k \) which is determined by the relation \( Q_\phi^2 = \frac{2}{k+2} \). This relation (2.33) has already been pointed out in [29] in the cases of \( SU(4) \)-holonomy. There it is also discussed that the spacetime SUSY algebra should be enhanced to the \( \mathcal{N} = 2 \) superconformal algebra (only for the left or right movers) acting on the boundary of \( AdS_3 \), which is explicitly constructed in [63] along the same line as [64]. The similar observations are also possible for the \( Spin(7) \) and \( Sp(2) \)-holonomy cases:

(i) \( Spin(7) \)-holonomy : The easiest way to move from the L.H.S to R.H.S in (2.33) is by making the formal replacements; \( \psi^0 \rightarrow i\Psi^2 \), \( \psi^1 \rightarrow \Psi^1 \), and \( \psi^\phi \rightarrow i\Psi^3 \), where \( \psi^0 \), \( \psi^1 \), \( \psi^\phi \) are the free fermions along the longitudinal and Liouville directions and \( \Psi^A \) denotes the fermionic coordinates in the \( \mathcal{N} = 1 \ SL(2; \mathbb{R}) \)-WZW model. The structure of spin fields are quite similar. The essential difference lies in the
BRST charges, especially in the definitions of superconformal currents \( G(z) \): \( G(z) \) of \( SL(2; \mathbb{R}) \) super WZW model includes a cubic fermionic term \( \sim \Psi^1 \Psi^2 \Psi^3 \), while that of \( \mathcal{N} = 1 \) Liouville theory \( (\times \mathbb{R}^{1,1}) \) only includes a linear term \( \sim Q \phi \partial \psi \).

With these preparations, we can construct twice as many supercharges for the \( AdS_3 \times M \) backgrounds as compared with (2.26):

\[
G_{+1/2} = \oint S_0^{(+)} \left( \sigma^3_+ \sigma^M_+ + \sigma^3_- \sigma^M_- \right) e^{-\psi/2},
\]

\[
G_{-1/2} = \oint S_0^{(-)} \left( \sigma^3_+ \sigma^M_+ - \sigma^3_- \sigma^M_- \right) e^{-\psi/2},
\]

(2.34)

where we denote the spin fields associated \( \Psi^1, \Psi^2 \) as \( S_0^{(+)} \), and \( \sigma^3_\pm \) is defined similarly as in (2.24) with respect to \( \Psi^3 \). They are naturally identified as the “zero-modes” of the spacetime \( \mathcal{N} = 1 \) superconformal algebra. It is also not difficult to construct the full superconformal current oscillators based on the Wakimoto free field representation along the line of [64].

(ii) \( Sp(2) \)-holonomy : For the \( Sp(2) \)-holonomy the argument is almost parallel. In the background \( AdS_3 \times M \) we can obtain twice as many supercharges

\[
G^+_{+1/2} = \oint S_0^{(+)} S^{++} e^{-\psi/2},
\]

\[
G^3_{+1/2} = \frac{1}{2} \oint S_0^{(+)} \left( S^{++} + S^{--} \right) e^{-\psi/2},
\]

\[
G^\pm_{-1/2} = \oint S_0^{(-)} S^{\pm\mp} e^{-\psi/2},
\]

\[
G^3_{-1/2} = \frac{1}{2} \oint S_0^{(-)} \left( S^{++} + S^{--} \right) e^{-\psi/2}.
\]

(2.35)

They are again enhanced to the full generators of \( \mathcal{N} = 3 \) superconformal algebra [65], where the superconformal currents behave as a triplet of \( SU(2)_R \) symmetry, and coincide with those constructed in [66] (up to normalizations and the convention of spin fields). In addition, we will show in the next section that the \( Sp(2) \)-holonomy can be achieved only for \( M = SO(5)/SO(3), \ SU(3)/U(1) \), if we assume supercoset CFT’s for the \( M \)-sector. This fact is consistent with the observation given in [66].
3 Supercoset CFT’s for Superstring Vacua of Special Holonomies

3.1 Preliminary: Some Notes on Supercoset CFT’s

In order to fix our discussions we shall adopt the coset construction for the $\mathcal{M}$-sector from now on. The $\mathcal{N}=1$ supercoset CFT’s are easily constructed by using the super affine Kac-Moody algebras. They have the general form

$$\mathcal{M} = \frac{G_k \times SO(D)_1}{H}, \quad \dim G/H = D, \quad (3.1)$$

with a Lie group $G$ and its subgroup $H$. $SO(D)_1$ stands for the current algebra generated by $D$ free fermions. The condition $\dim G/H = D$ ensures the $\mathcal{N}=1$ superconformal symmetry. We especially concentrate on the cases of $D=9-d$ for the superstring vacua of the type $\mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times \mathcal{M}$ (namely, for the $10-d$ dimensional internal space). The reason why we do so is as follows: The coset spaces $G/H$ are generically endowed with Einstein metrics and were studied extensively in old days of Kaluza-Klein super gravity [17–24]. The cones over these spaces are known to possess Ricci flat metrics [7–10], leading to supersymmetric string vacua, when the coset spaces possess the geometrical structures listed in the table 1. In a formal sense the string vacua (2.1) we are studying are the CFT versions of the Ricci flat cones. We point out that $c_M \leq 3D/2$ holds in general (equality holds at $k=\infty$). Thus we always have the real background charge $Q_\phi$. (Recall the condition (2.3).)

The level of the current algebra of $H$ is slightly non-trivial to determine and depends on the embedding of $H$ in general. We assume the decomposition $H = H_0 \times H_1 \times \cdots \times H_r$, where $H_0$ is the abelian part and $H_i$ ($i=1,\ldots,r$) are simple parts. The level $k_i$ of each simple factor $H_i$ is defined in the standard manner;

$$J_{(i)}^a(z)J_{(i)}^b(0) \sim \frac{k_i(t_i^a \cdot t_i^b)}{z^2} + \frac{i f_i^{ab}}{z} J_{(i)}^c(0) \quad (\text{for } i \neq 0),$$

$$J_{(0)}^a(z)J_{(0)}^b(0) \sim \frac{k_0(t_0^a \cdot t_0^b)}{z^2} \quad (\text{for } H_0). \quad (3.2)$$

where $\{t_i^a\}$ is a basis of the Lie algebra of $H_i$ and $J_{(i)}^a(z) \equiv (J_{(i)}^a(z), t_i^a)$, $J_{(0)}^a(z) \equiv (J_{(0)}^a(z), t_0^a)$. The Killing metrics ($\ , \ )$ for $G$ and ($\ , \ )$, for the $H_i$ sectors are canonically normalized as $(\theta, \theta) = 2$, $(\theta_i, \theta_i) = 2$, where $\theta, \theta_i$ are the highest roots of $G, H_i$, respectively. Notice that the inner products ($\ , \ )$ and ($\ , \ )$, have different normalizations in general. $(\theta_i, \theta_i)$ is not necessarily equal to 2 and depends on the choice of embedding of $H_i$. Now, the levels $k_i$ are determined by comparing the Schwinger terms of the super affine Lie algebras of $G$.
and $H_i$:

$$k_0 = k + g^*$$

$$k_i = \frac{2}{(\theta_i, \theta_i)}(k + g^*) - h_i^* \quad (i = 1, \ldots, r) \quad (3.3)$$

where $g^*$, $h_i^*$ denote the dual Coxeter numbers of $G$, $H_i$ respectively. Care is needed when $G$ is non-simply laced and some $H_i$'s are embedded along its short roots. The central charges of these coset CFT's are evaluated as

$$c_M = \frac{k \dim G}{k + g^*} + \frac{1}{2} D - \sum_{i=0}^{r} k_i \dim H_i$$

We also remark that the sub-root lattice describing the charge spectrum of $H_0$-sector must be specified in order to define the coset model uniquely. The conformal blocks of this sector are written in terms of the theta functions associated to the charge lattice $\Gamma \subset \sqrt{k + g^*}Q$, where $Q$ is the root lattice of $G$. They are explicitly written as

$$F_{\lambda}^{H_0}(\tau) = \frac{\Theta_{\lambda}^{(\Gamma)}(\tau)}{\eta(\tau)^L}, \quad (\dim H_0 = L)$$

$$\Theta_{\lambda}^{(\Gamma)}(\tau) = \sum_{\alpha \in \lambda + \Gamma} q^{\frac{1}{2}(\alpha, \alpha)}, \quad (\lambda \in \Gamma^*) \quad (3.5)$$

For example, suppose that the charge lattice is given by

$$\Gamma = \mathbb{Z}\nu_1 + \cdots + \mathbb{Z}\nu_L, \quad (L = \dim H_0),$$

$$(\nu_a, \nu_b) = 0 \quad (\forall a \neq b) \quad (\nu_a, \nu_a) = 2l_a(k + g^*), \quad (3.6)$$

then we have the decomposition of the theta functions

$$\Theta_{\lambda}^{(\Gamma)}(\tau) = \prod_{a=1}^{L} \Theta_{m_a l_a(k + g^*)}(\tau), \quad \lambda = \sum_{a=1}^{L} m_a \nu_a^*, \quad (3.7)$$

where $\nu_a^*$ are the dual bases of $\Gamma^*$ such that $(\nu_a, \nu_b^*) = \delta_{ab}$. In this paper we shall adopt the convention that “$U(1)_k$” means the $c = 1$ conformal theory composed of the conformal blocks of the forms $\Theta_{\lambda, k}(\tau)/\eta(\tau)$. Namely, we here find

$$(H_0)_{k + g^*} \cong U(1)_{l_1(k + g^*)} \times \cdots \times U(1)_{l_L(k + g^*)} \quad (3.8)$$

We will later face non-trivial examples in which the different choice of charge lattice leads to inequivalent string vacua.

---

In the case $H_i = SO(3)$, one must use the formula

$$k_i = \frac{1}{(\theta_i, \theta_i)}(k + g^*) - 1$$

instead of (3.3). This fact is due to the equivalence $SO(3)_k \cong SU(2)_k$ as an affine Lie algebra.
3.2 Coset Constructions of $d = 6$ Superstring Vacua

We start with the study of $d = 6$ string vacua. The spacetime SUSY corresponds to the $SU(2)$-holonomy in this case. Assuming the $\mathcal{N} = 1$ supercoset CFT $\mathcal{M} = G/H$ with the condition that $G$ is compact, simple and $\dim G/H = 3$, the possible example is $G = SU(2)$, $H = \{\text{id}\}$. We also study the case $G = SO(4)$, $H = SO(3)$ as a particular example of a semi-simple $G$.

The first example, $G = SU(2)$, $H = \{\text{id}\}$, is the familiar CHS $\sigma$-model describing the NS 5-brane \[67\];

$$
\mathbb{R}_\phi \times \psi^\phi \times \mathcal{M} = \mathbb{R}_\phi \times \psi^\phi \times SU(2)_k \times SO(3)_1
$$

$$
\cong \left[ \mathbb{R}_\phi \times \psi^\phi \times U(1)_{k+2} \times SO(1)_1 \right] \times \frac{SU(2)_k \times SO(2)_1}{U(1)_{k+2}}. \quad (3.9)
$$

Liouville field, together with the WZW model $SU(2)_k$, describes the configuration of the throat region $\mathbb{R}_+ \times S^3$ transverse to the NS 5-brane. In the last line the part described by $[\cdots]$ is the $\mathcal{N} = 2$ Liouville theory with $Q_\phi = \sqrt{2/(k + 2)}$ and the remaining part is the $\mathcal{N} = 2$ minimal model of the level $k$. In [27] this system is interpreted as describing the string theory compactified on an ALE space with the $A_{k+1}$-type singularity (The $D$ and $E$-type singularities are naturally incorporated as the modular data of $SU(2)_k$ sector [28].) Thus (3.9) represents the well-known T-duality between NS 5-brane and ALE space [27, 68].

In the second example, $G = SO(4)$, $H = SO(3)$, we have two possibilities of $SO(3)$-embedding. The first choice is the embedding into one of the $SO(3)$’s of $SO(4) \simeq SO(3) \times SO(3)$, which again reduces to the CHS $\sigma$-model because of the relation

$$
\mathcal{M} = \frac{SO(4)_k \times SO(3)_1}{SO(3)_{k/2}}
$$

$$
\cong SU(2)_k \times SO(3)_1. \quad (3.10)
$$

The second choice is the diagonal embedding in $SO(3) \times SO(3)$, which leads to

$$
\mathcal{M} = \frac{SO(4)_k \times SO(3)_1}{SO(3)_{k+1}}. \quad (3.11)
$$

This case corresponds to vacua with no spacetime SUSY.

In addition, we can consider the following generalization of (3.11)

$$
\mathcal{M} = \frac{SU(2)_{k_1} \times SU(2)_{k_2} \times SO(3)_1}{SU(2)_{k_1 + k_2 + 2}}. \quad (3.12)
$$

These $\mathcal{N} = 1$ diagonal coset theories have been studied intensively in [50]. It is obvious that (3.12) cannot provide any $d = 6$ supersymmetric vacuum. However, we will later
show that the $d = 3$ and $d = 2$ supersymmetric vacua of this form exist under the suitable restrictions of the levels of current algebras.

### 3.3 Coset Constructions of $d = 4$ Superstring Vacua

In this case the supersymmetric string vacua correspond to $SU(3)$-holonomy. The criterion for spacetime SUSY is thus whether the worldsheet SUSY is enhanced to $N = 2$.

We assume $\dim G/H = 5$. If $G$ is simple, we only have the possibilities

$$G/H = SO(6)/SO(5), \ \text{SU}(3)/SU(2).$$

We also study the case $G/H = (SU(2) \times SU(2))/U(1)$ as a particular example of a semi-simple $G$.

1. $G/H = SO(6)/SO(5)$

   In this case the $N = 1$ supercoset CFT is given as

   $$\mathcal{M} = \frac{SO(6)_k \times SO(5)}{SO(5)_{k+1}}.$$  \hspace{1cm} (3.13)

   The worldsheet SUSY for the total system $\mathbb{R}_\phi \times \psi^\phi \times \mathcal{M}$ is not enhanced since the coset $\mathcal{M}/U(1)$ is not well-defined in this case. Therefore, this model corresponds to a string vacuum with no spacetime SUSY.

2. $G/H = SU(3)/SU(2)$

   We have two possibilities of the embedding of $SU(2)$.

   (i) $SU(2)$ embedded as the “isospin subgroup”:

   The easier embedding is of course into the usual isospin subgroup of $SU(3)$. In this case, we can show that the worldsheet SUSY is enhanced to $N = 2$ because of the equivalence

   $$\mathbb{R}_\phi \times \psi^\phi \times \mathcal{M} \cong [\mathbb{R}_\phi \times \psi^\phi \times U(1)_{3(k+3)} \times SO(1)_1] \times \frac{SU(3)_k \times SO(4)}{SU(2)_{k+1} \times U(1)_{3(k+3)}},$$

   \hspace{1cm} (3.15)

   as we already mentioned in (2.7). The part $\cdots$ is interpreted as the $N = 2$ Liouville. In fact, the criticality condition (2.3) gives us $Q^2_\phi = 6/(k+3)$, resulting in $3^2 \cdot 2/Q^2_\phi = 3(k+3)$. Thus, the $U(1)_{3(k+3)}$ piece is identified as the compact boson of $N = 2$ Liouville theory. The remaining coset CFT is the Kazama-Suzuki model
for $\mathbb{CP}_2$. In this way we have confirmed that this string vacuum corresponds to a non-compact CY$_3$ manifold.

Notice that the coset $SU(3)/SU(2)$ is isomorphic with $S^5$, which is an elementary example of Sasaki-Einstein manifold with the $U(1)$-fibration

$$\begin{align*}
S^5 \overset{U(1)}{\longrightarrow} \mathbb{CP}_2.
\end{align*}$$

(3.16)

This is the geometrical interpretation of (3.15).

**(ii) $SU(2)$ embedded as the maximal subgroup**

Another embedding is defined by identifying the simple root of $SU(2)$ with $\theta/2$, where $\theta \equiv \alpha_1 + \alpha_2$ is the highest root of $SU(3)$. In practice, this is given by restricting the canonical action of $SU(3)$ on complex 3-vectors to that of $SO(3)(\simeq SU(2))$ on real 3-vectors. By this embedding, the adjoint representation of $SU(3)$ is decomposed as $8 \rightarrow 3 + 5$, and thus it is “maximal” (see [22] for the detail). The coset space is again isomorphic with $S^5$.

Since $(\theta/2)^2 = 1/2$ holds, we find that the relevant supercoset CFT should have the following form due to the formula (3.3)

$$\begin{align*}
\mathcal{M} = \frac{SU(3)_k \times SO(5)_1}{SU(2)_{4k+10}}.
\end{align*}$$

(3.17)

It may be helpful to confirm that this coset CFT is really well-defined. To this aim it will be enough to check the existence of $SO(5)_1/SU(2)_{10}$. In fact, this coset CFT corresponds to the maximal embedding $SU(2) \subset SO(5)$ which we later consider. Another explanation is given as follows: We have a conformal embedding $SU(2)_{10} \subset SO(5)_1$ ($c = 5/2$ for both of $SO(5)_1$, $SU(2)_{10}$) based on the $E_6$-type modular invariance of $SU(2)_{10}$ [70]. Namely, the corresponding partition function is given by

$$\begin{align*}
Z = \left| \chi^{(10)}_0 + \chi^{(10)}_6 \right|^2 + \left| \chi^{(10)}_3 + \chi^{(10)}_7 \right|^2 + \left| \chi^{(10)}_4 + \chi^{(10)}_{10} \right|^2,
\end{align*}$$

(3.18)

where we denote the $SU(2)_{10}$ character of spin $\ell/2$ as $\chi^{(k)}_\ell$, and the following character relations are known (see [69], for example);

$$\begin{align*}
\chi^{SO(5)_1}_{\text{basic}}(\tau) &= \chi^{(10)}_0(\tau) + \chi^{(10)}_6(\tau), \quad \chi^{SO(5)_1}_{\text{spinor}}(\tau) = \chi^{(10)}_3(\tau) + \chi^{(10)}_7(\tau),
\chi^{SO(5)_1}_{\text{vector}}(\tau) &= \chi^{(10)}_4(\tau) + \chi^{(10)}_{10}(\tau).
\end{align*}$$

(3.19)

We have no extra $U(1)$ symmetry in (3.17) since the $SU(2)$ is a maximal subgroup, and thus the worldsheet SUSY cannot be enhanced. Therefore, the corresponding string vacua are not supersymmetric.
3. \( G/H = (SU(2) \times SU(2))/U(1) \)

Lastly, let us consider the most non-trivial example

\[
\mathcal{M} = \frac{SU(2)_{k_1} \times SU(2)_{k_2} \times SO(5)}{U(1)}
\]

(3.20)

As was already illustrated, the charge lattice of \( U(1) \) must have the following form because of the worldsheet SUSY

\[
\Gamma = \mathbb{Z}\left(p\sqrt{k_1 + 2\alpha} + q\sqrt{k_2 + 2\beta}\right)
\]

(3.21)

where \( \alpha, \beta \) denote the simple roots of the two \( SU(2) \) factors and \( p, q \in \mathbb{Z}_{\geq 0} \) \( ((p, q) \neq (0, 0)) \). We can reexpress \( \mathcal{M} \) as

\[
\mathbb{R}_\phi \times \psi^\phi \times \mathcal{M} \cong \mathbb{R}_\phi \times \psi^\phi \times \frac{SU(2)_{k_1} \times SO(2)}{U(1)_{k_1+2}} \times \frac{SU(2)_{k_2} \times SO(2)}{U(1)_{k_2+2}}
\]

\[
\times \frac{U(1)_{k_1+2} \times U(1)_{k_2+2}}{U(1)_{p^2(k_1+2)+q^2(k_2+2)}} \times SO(1)
\]

\[
\cong \left[ \mathbb{R}_\phi \times \psi^\phi \times U(1)_{(k_1+2)(k_2+2)} \{p^2(k_1+2)+q^2(k_2+2)\} \times SO(1) \right] \times \mathcal{M}_{k_1} \times \mathcal{M}_{k_2}
\]

(3.22)

where \( \mathcal{M}_k \) denotes the \( \mathcal{N} = 2 \) minimal model with level \( k \) \( (c = \frac{3k}{k+2}) \). In the last line, we made use of the product formula of theta functions. The \( \mathcal{N} = 2 \) worldsheet SUSY requires that the part \( \cdots \) becomes the \( \mathcal{N} = 2 \) Liouville theory. The criticality condition (2.3) gives us

\[
Q^2_\phi = \frac{2(k_1 + k_2 + 4)}{(k_1 + 2)(k_2 + 2)}
\]

(3.23)

and hence if and only if \( p = q \) holds, the correct factor \( U(1)_{(k_1+2)(k_2+2)(k_1+k_2+4)} \) is obtained. In conclusion, only in the cases \( p = q \), we have the \( \mathcal{N} = 2 \) worldsheet SUSY, leading to the \( SU(3) \)-holonomy.

This type string vacua were studied in [36], and especially the simple cases of \( k_1 = k \), \( k_2 = 0 \) correspond to the \( CY_3 \) singularity of \( A_{k+1} \)-type \([29, 34, 71]\) \( (D, E \text{-types are also described by the modular data}) \).

We also make a comment on a geometric interpretation. The Einstein homogeneous space \( T^{p,q} = (SU(2) \times SU(2))/U(1) \) is defined by the \( U(1) \)-action

\[
e^{i\theta} \mapsto \left(e^{i\theta \frac{2\pi}{p}}, e^{i\theta \frac{2\pi}{q}}\right)
\]

(3.24)
with relatively prime integers $p, q$. It is well-known that only the case $p = q = 1$ allows the spacetime SUSY [23]. More precisely, $T^{p,q}$ becomes a Sasaki-Einstein space only for $p = q = 1$. The cone over $T^{1,1}$ is the well-known conifold. The parameters $p, q$ precisely correspond to those introduced in our discussions (at least for the cases $k_1 = k_2$). Thus the condition for the presence of spacetime SUSY in our construction is in agreement with geometrical considerations.

Such a correspondence was partly suggested in [71] and also discussed in [72] in relation to the gauged WZW model of the GMM type [73].

Additionally we note that $T^{1,1}$ is isomorphic with the Stiefel manifold $V_2(\mathbb{R}^4)$ and have the $U(1)$-fibration

$$T^{1,1} \xrightarrow{U(1)} \mathbb{C}P_1 \times \mathbb{C}P_1 , \quad (3.25)$$

which gives the geometric interpretation of (3.22) for the case $p = q = 1$.

### 3.4 Coset Constructions of $d = 3$ Superstring Vacua

Since we are now assuming $\dim G/H = 6$, the condition for the spacetime SUSY (2.15) reduces to the simple criterion given in [46]: The spacetime SUSY exists if and only if it holds that

$$\mathcal{M} = \frac{G_k \times SO(6)_1}{H} \cong \frac{G_k \times SU(3)_1}{H} \times \frac{SO(6)_1}{SU(3)_1}$$

$$\cong \frac{G_k \times SU(3)_1}{H} \times U(1)_{3/2} , \quad (3.26)$$

as a well-defined coset CFT. Here we used the equivalence

$$SO(6)_1/SU(3)_1 \cong U(1)_{3/2} . \quad (3.27)$$

We first focus on the examples when $G$ is a compact simple group. The possible coset spaces are listed as follows;

$$G/H = SO(7)/SO(6) , \ SU(4)/(SU(3) \times U(1)) , \ G_2/SU(3) ,$$

$$SO(5)/(SO(3) \times U(1)) , \ SU(3)/(U(1) \times U(1)) . \quad (3.28)$$

We also pick up the following example of a semi-simple $G$

$$G/H = (SU(2) \times SU(2) \times SU(2))/SU(2) . \quad (3.29)$$
1. \( G/H = SO(7)/SO(6), \ SU(4)/(SU(3) \times U(1)) \)

The corresponding supercoset CFT's are defined by

\[
\mathcal{M} = \frac{SO(7)_k \times SO(6)_1}{SO(6)_{k+1}}, \quad \mathcal{M} = \frac{SU(4)_k \times SO(6)_1}{SU(3)_{k+1} \times U(1)_{6(k+4)}}. \tag{3.30}
\]

Clearly, the denominator groups are “too big” to allow the rearrangement (3.26). We thus find no vacua with spacetime SUSY.

2. \( G/H = G_2/SU(3) \)

In this case we find

\[
\mathcal{M} = \frac{(G_2)_k \times SO(6)_1}{SU(3)_{k+1}} \cong \frac{(G_2)_k \times SU(3)_1}{SU(3)_{k+1}} \times U(1)_{3/2}. \tag{3.31}
\]

We thus obtain a supersymmetric vacuum corresponding to a non-compact \( G_2 \) holonomy manifold. This model has been studied in [46].

This coset space is isomorphic with \( S^6 \) and it is a typical example of a nearly Kähler but non-Kähler manifold [14]. Our CFT result is in accordance with this fact.

3. \( G/H = SO(5)/(SO(3) \times U(1)) \)

We have two possibilities of the embedding of \( SO(3) \times U(1) \); one is to embed \( SO(3) \) along the long root of \( SO(5) \), and another is along its short root. \( U(1) \) has to be embedded in the direction orthogonal to \( SO(3) \) in each case.

(i) \( SO(3) \) **embedded along a long root**:

Suppose \( SO(3) \) is embedded along a long root of \( SO(5) \). The coset space is topologically isomorphic with \( S^7/U(1) \cong \mathbb{CP}^3 \) and can be endowed with a (non-HSS) parabolic structure.\(^6\) Hence \( \mathcal{M} \) is a (non-HSS) Kazama-Suzuki model. Based on

\(^6\)The most familiar coset realization of \( \mathbb{CP}^3 \) is of course \( SU(4)/(SU(3) \times U(1)) \), which is a hermitian symmetric space (HSS). The non-HSS Kähler coset \( SO(5)/(SO(3) \times U(1)) \) gives an inequivalent parabolic decomposition

\[
g = h + m_+ + m_-, \quad [h, m_\pm] \subset m_\pm, \quad [m_\pm, m_\pm] \subset m_\pm,
\]

with \( m_+, m_- \) being non-abelian.
the formulas (3.3) we obtain

\[ M = \frac{SO(5)_k \times SO(6)}{SO(3)} \times U(1)_{k+3} \]

\[ \cong \frac{SO(5)_k \times SO(6)}{SU(2)_{k+1} \times U(1)_{k+3}} \]

\[ \cong \frac{SO(5)_k \times SU(3)}{SU(2)_{k+1} \times U(1)_{k+3}} \times U(1)_{3/2}. \] (3.32)

We thus obtain superstring vacua of non-compact $G_2$-holonomy manifolds.

The coset space defined here can be equipped with an Einstein metric that is $SO(5)$ invariant, but not compatible with a definite complex structure. This is an example of a nearly Kähler manifold and gives rise to the well-known solution with the $G_2$-holonomy metric on its cone [74,75] (see also [5]). Our result seems consistent with this fact.

(ii) $SO(3)$ embedded along a short root:

If $SO(3)$ is embedded along a short root, the coset space is the Grassmannian $G_2(\mathbb{R}^5)$, which is an HSS. We find from (3.3)

\[ M = \frac{SO(5)_k \times SU(3)}{SU(3)} \times U(1)_{3/2}. \] (3.33)

In this case we cannot rewrite it as in (3.26). The vacua have no SUSY.

4. $G/H = SU(3)/(U(1) \times U(1))$

We must specify the charge lattice $\Gamma$ associated to the $U(1) \times U(1)$ action to define the supercoset CFT $M$, although the corresponding coset space is always isomorphic with the flag manifold $F_{1,2}(\mathbb{C}^3)$ irrespective of the choice of $\Gamma$. Let $Q$ be the root lattice of $SU(3)$. According to (3.3), $\Gamma$ must be a sublattice of $\sqrt{k+3}Q$, where $k$ is the level of $SU(3)$, in order to maintain the worldsheet SUSY. On the other hand, the criterion for spacetime SUSY (3.26) requires that the coset

\[ \frac{SU(3)_k \times SU(3)}{U(1) \times U(1)} \] (3.34)

is well-defined, which implies the condition $\Gamma \subset \sqrt{k+3}Q \oplus Q$. Especially, we cannot adopt the simplest choice $\Gamma = \sqrt{k+3}Q$ for a generic $k$.

The charge lattices $\Gamma$ which admit both worldsheet and spacetime SUSY are constructed as follows: Let $\alpha_i, \beta_i (i = 1, 2)$ be the simple roots corresponding to the current algebras $SU(3)_k, SU(3)_1$ respectively, normalized as $\alpha_i^2 = \beta_i^2 = 2, (\alpha_1, \alpha_2) = (\beta_1, \beta_2) = -1,
\((\alpha_i, \beta_j) = 0\). We set
\[
\nu_1 = \sqrt{k\alpha_1 + \beta_1 + 2\beta_2} ,
\]
\[
\nu_2 = \sqrt{k(\alpha_1 + 2\alpha_2) + 3\beta_1} ,
\]
and define the lattice
\[
\Lambda = \mathbb{Z}\nu_1 + \mathbb{Z}\nu_2 .
\]
They satisfy
\[
(\nu_1, \nu_1) = 2(k + 3) , \quad (\nu_2, \nu_2) = 6(k + 3) , \quad (\nu_1, \nu_2) = 0 .
\]
By definition \(\Lambda\) is a sublattice of \(\sqrt{k}Q \oplus Q\), and by means of the identification
\[
\sqrt{k + 3} \gamma_1 = \nu_1 , \quad \sqrt{k + 3} (\gamma_1 + 2\gamma_2) = \nu_2 ,
\]
where \(\gamma_1, \gamma_2\) denote the simple roots corresponding to the total current \(SU(3)_{k+3}\), we can also regard \(\Lambda\) as a sublattice of \(\sqrt{k + 3}Q\). More precise argument is given as follows:
Recall that the total Cartan current of \(SU(3)_{k+3}\) is decomposed as \([56, 76]\)
\[
(t, H(z)) = (t, \hat{H}(z)) + \sum_{\alpha \in \Delta_+} (\alpha, t) : \psi^\alpha \psi^{-\alpha} : ,
\]
where \(\hat{H}\) is the bosonic current and \(\psi^\alpha\) are the free fermions along the coset direction. \((\psi^\alpha(z)\psi^{\alpha'}(0) \sim \delta_{\alpha+\alpha',0}/z). \Delta_+\) denotes the set of positive roots. On the other hand, the embedding \(SU(3)_1 \subset SO(6)_1\) is given by defining the \(SU(3)_1\) current \(K(z)\) from the 6 free fermions \(\chi_i, \chi_i^* (i = 1, 2, 3), (\chi_i^*(z)\chi_j(0) \sim \delta_{ij}/z, \chi_i(z)\chi_j(0) \sim \chi_i^*(z)\chi_j^*(0) \sim 0)\);
\[
(E_{ii} - E_{jj}, K(z)) = : \chi_i^*\chi_i : - : \chi_j^*\chi_j : , \quad (E_{ij}, K(z)) = \chi_i^*\chi_j \quad (i \neq j) ,
\]
where we set \((E_{ij})_{ab} = \delta_{ia}\delta_{jb}\). Identifying the coset fermions \(\psi^\alpha\) and \(\chi_i, \chi_i^*\) by the relation
\[
\chi_1 = \psi^{-\theta} , \quad \chi_1^* = \psi^\theta ,
\]
\[
\chi_2 = \psi^{\alpha_2} , \quad \chi_2^* = \psi^{-\alpha_2} ,
\]
\[
\chi_3 = \psi^{\alpha_1} , \quad \chi_3^* = \psi^{-\alpha_1} ,
\]
we find that
\[
(\lambda_3, H) = (\lambda_3, \hat{H}) + (:\chi_1^*\chi_1 : + : \chi_2^*\chi_2 : - 2 : \chi_3^*\chi_3 :) ,
\]
\[
(\lambda_8, H) = (\lambda_8, \hat{H}) + \sqrt{3}(\chi_1^*\chi_1 : - : \chi_2^*\chi_2 :) ,
\]
where $\lambda_3$, $\lambda_8$ are the Gell-Mann matrices corresponding to $\alpha_1, \frac{1}{\sqrt{3}}(\alpha_1 + 2\alpha_2)$;

$$
\lambda_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.
$$

These relations justify the identification (3.35), (3.38).

We should also take account of the symmetry by the Weyl group $W$. Therefore, the charge lattice $\Gamma$ gives SUSY, if (and only if) it is a sublattice of $w \cdot \Lambda$ with some Weyl reflection $w \in W$. Especially, in the simplest case $\Gamma = \Lambda$, we obtain

$$
\mathcal{M} = \frac{SU(3)_k \times SO(6)_1}{U(1)_{k+3} \times U(1)_{3(k+3)}}
$$

$$
\cong \frac{SU(3)_k \times SU(3)_1}{U(1)_{k+3} \times U(1)_{3(k+3)}} \times U(1)_{3/2}.
$$

(3.44)

One can directly check that the coset part in the last line is actually well-defined because of the relation $SU(3)_1 \sim U(1)_1 \times U(1)_3 \sim U(1)_9 \times U(1)_3$. In conclusion, we have obtained an infinite family of supersymmetric vacua of $G_2$-holonomy corresponding to the charge lattice $\Gamma$ mentioned above.

We have a comment: The flag manifold $F_{1,2}(\mathbb{C}^3)$ has the standard Kähler-Einstein metric, which is not $SU(3)$-invariant. However, this space is known to have a second (non-Kähler) Einstein metric that is $SU(3)$ invariant and compatible with a nearly Kähler structure [14] (see also [5]). There exists a well-known solution of $G_2$-holonomy metric on this cone [74, 75]. Our CFT result again seems to be consistent with the classical geometry. However, while the different choice of $\Gamma$ leads to the same homogeneous space $F_{1,2}(\mathbb{C}^3)$ classically, we obtain inequivalent string vacua depending on the choice of $\Gamma$ in our coset CFT construction.

We further consider examples of a non-simple $G$, motivated by the example of a nearly Kähler space $S^3 \times S^3$.

5. $G/H = (SU(2) \times SU(2) \times SU(2))/SU(2)$

We have three ways of $SU(2)$ embedding. In all the three cases the coset space is topologically isomorphic with $S^3 \times S^3$.

(i) $SU(2)$ embedded only in an $SU(2)$-factor :

This case is very easy, since one of the $SU(2)$-factors in the numerator is canceled.

The relevant supercoset reduces to

$$
\mathcal{M} = SU(2)_{k_1} \times SU(2)_{k_2} \times SO(6)_1.
$$

(3.45)
This obviously gives rise to supersymmetric vacua with 16 supercharges.

If we further make an $S^1$ compactification, this model will be converted by (2.33) into the background $AdS_3 \times S^3 \times S^3 \times S^1$ discussed in [77], which realizes the large $\mathcal{N} = 4$ superconformal symmetry on the boundary of $AdS_3$.

(ii) $SU(2)$ embedded in two $SU(2)$-factors:

In this case the relevant supercoset becomes

$$\mathcal{M} = \frac{SU(2)_k \times SU(2)_k \times SO(3)}{SU(2)} \times SU(2)_k \times SO(3) . \quad (3.46)$$

It leads to non-supersymmetric string vacua.

(iii) $SU(2)$ embedded in all of the $SU(2)$-factors:

This third case is the most interesting. The relevant supercoset is defined as

$$\mathcal{M} = \frac{SU(2)_k \times SU(2)_k \times SU(2)_k \times SU(3)}{SU(2)} \times U(1/2) . \quad (3.47)$$

In the same way as before the existence of spacetime SUSY is confirmed by the equivalence

$$\mathcal{M} \cong \frac{SU(2)_k \times SU(2)_k \times SU(2)_k \times SU(3)}{SU(2)} \times U(1) . \quad (3.48)$$

Recall the conformal embedding $SU(2)_4 \subset SU(3)_1$ is associated with the $D_4$-type modular invariant. Namely, we have the character relations

$$\chi^{SU(3)}_{\text{fund}} (\tau) = \chi^{SU(3)} (\tau), \quad \chi^{SU(3)}_{\text{fund}} (\tau) = \chi^{SU(3)} (\tau), \quad (3.49)$$

where $\chi^{(k)}$ denotes the $SU(2)_k$ character of spin $\ell/2$. Accordingly, the coset part in (3.48) is really well-defined.

This type string vacua of $G_2$-holonomy are regarded as natural generalizations of those given in [45]. In fact, the special cases of $k_1 = k$, $k_2 = k_3 = 0$ reduces to

$$\mathcal{R}_\phi \times \psi^\phi \times \frac{SU(2)_k \times SO(6)}{SU(2)} \cong \mathcal{R}_\phi \times \psi^\phi \times \frac{SU(2)_k \times SU(2)}{SU(2)} \times \frac{SU(2)_{k+2} \times SU(2)}{SU(2)}$$

$$\cong \mathcal{R}_\phi \times \psi^\phi \times M^{N=1}_{k+2} \times M^{N=1}_{k+4} , \quad (3.50)$$
where $\mathcal{M}_m^{N=1}$ denotes the $m$-th $\mathcal{N} = 1$ minimal model ($c = \frac{3}{2} - \frac{12}{m(m+2)}$). These are precisely the models constructed in [45].

We further make a comment: As mentioned in [5], the trivial round sphere metric on $S^3 \times S^3$ does not generate the $G_2$-holonomy on its cone. While, the second Einstein metric based on the identification $S^3 \times S^3 \cong (SU(2))^3/SU(2)$ leads to a $G_2$-holonomy on its cone and there exists a well-known $G_2$ metric [74, 75]. It seems plausible to relate the former case with the SCFT (3.45) and the latter with (3.47), although the precise interpretation of this relation is yet unclear.

To complete our classification of the $d = 3$ string vacua, let us also consider the diagonal coset (3.12). Although dim $G/H \neq 6$, we obtain the supersymmetric vacua of $G_2$-holonomy, by restricting the levels of current algebras. In fact, if we set $k_2 = 2$ (or $k_1 = 2$), the model again reduces to the one just considered (3.50). It is easy to see that any other choices of levels do not lead to supersymmetric vacua.

### 3.5 Coset Constructions of $d = 2$ Superstring Vacua

The $d = 2$ cases are the most involved because we have three possibilities of supersymmetric string vacua (except for the case of trivial flat space), that is, the $Sp(2)$, $SU(4)$ and $Spin(7)$-holonomies, each of which corresponds to a tri-Sasakian, Sasaki-Einstein, and weak $G_2$ base spaces, respectively.

The 7-dimensional Einstein homogeneous spaces are completely classified in [22]. We first focus on the cases of simple $G$ listed as

\[
G/H = SO(8)/SO(7), \; SO(7)/G_2, \; SU(4)/SU(3), \; SU(3)/U(1), \; SO(5)/SO(3).
\]  

We further discuss a few cases with a non-simple $G$; $G/H = (SU(3) \times SU(2))/(SU(2) \times U(1)), \; G/H = (SU(2) \times SU(2) \times SU(2))/(U(1) \times U(1))$, which are also found in the list of [22] (see also [24]). The other example is again (3.12). Although dim $G/H \neq 7$ here, it will turn out that the $Spin(7)$-holonomy vacua are also obtained under a suitable restriction of levels, as in the $d = 3$ case.

The SUSY condition (2.17) now reduces to a criterion [46];

\[
\mathcal{M} = \frac{G_k \times SO(7)}{H} \cong \frac{G_k \times (G_2)}{H} \times \text{tri-critical Ising},
\]  

25
by using the identification

$$SO(7)_1/(G_2)_1 \cong \text{tri-critical Ising} ,$$

which was first pointed out in [47].

1. \( G/H = SO(8)/SO(7) \):
   We have no spacetime SUSY in this case, because the denominator group is too large to make the rearrangement (3.52) possible.

2. \( G/H = SO(7)/G_2 \):
   This model gives the superstring vacua of \( Spin(7) \)-holonomy studied in [46]. The condition (3.52) is easily checked as follows;

$$\mathcal{M} = \frac{SO(7)_k \times SO(7)_1}{(G_2)_{k+1}} \cong \frac{SO(7)_k \times (G_2)_1}{(G_2)_{k+1}} \times \frac{SO(7)_1}{(G_2)_1}$$

$$\cong \frac{SO(7)_k \times (G_2)_1}{(G_2)_{k+1}} \times \text{tri-critical Ising} .$$

(3.54)

The trivial case \( k = 0 \) (i.e. \( \mathcal{M} = \text{tri-critical Ising} \)) corresponds to the model discussed in [45].

3. \( G/H = SU(4)/SU(3) \):
   We can similarly prove that the condition (3.52) is satisfied here. However, based on (2.7), we can further show that the worldsheet SUSY is enhanced to \( \mathcal{N} = 2 \);

$$\mathcal{R}_\phi \times \psi^\phi \times \mathcal{M} = \mathcal{R}_\phi \times \psi^\phi \times \frac{SU(4)_k \times SO(7)_1}{SU(3)_{k+1}}$$

$$\cong \left[ \mathcal{R}_\phi \times \psi^\phi \times U(1)_{6(k+4)} \times SO(1)_1 \right] \times \frac{SU(4)_k \times SO(6)_1}{SU(3)_{k+1} \times U(1)_{6(k+4)}} \quad \text{ (3.55)}$$

The part \([\cdots]\) is interpreted as the \( \mathcal{N} = 2 \) Liouville theory. The criticality condition (2.3) gives \( Q_\phi = 12/(k + 4) \), and hence \( U(1)_{6(k+4)} \) describes precisely the compact boson of \( \mathcal{N} = 2 \) Liouville theory. The remaining coset CFT is the Kazama-Suzuki model for \( \mathbb{CP}_3 \). Therefore, these string vacua correspond to non-compact \( CY_4 \) manifolds.

4. \( G/H = SO(5)/SO(3) \):
   This example is quite amazing. We find that all of the three possible holonomies \( Sp(2) \), \( SU(4) \), and \( Spin(7) \) are realized.
(i) \(SO(3)\) embedded along a long root:

Suppose \(SO(3)\) is embedded along a long root of \(SO(5)\), in other words, embedded in one of the \(SO(3)\)'s of the \(SO(3) \times SO(3) (\cong SO(4))\) subgroup of \(SO(5)\). This coset space is isomorphic with \(S^7\), and as is obvious by construction, we have a remaining \(SO(3) (\cong SU(2))\) symmetry. This space is an elementary example of the tri-Sasakian manifold and can be regarded as an \(SU(2)\)-bundle over a Wolf space:

\[
\mathcal{M} = \frac{SO(5)}{SO(3)} \cong S^7 \overset{SU(2)}\rightarrow \mathcal{M}/SU(2) \cong S^4. \tag{3.56}
\]

This is nothing but the familiar (quaternionic) Hopf fibration.

In terms of the coset CFT, we obtain

\[
\mathbb{R}_\phi \times \psi^\phi \times \mathcal{M} = \mathbb{R}_\phi \times \psi^\phi \times \frac{SO(5)_k \times SO(7)_1}{SO(3)} \cong \mathbb{R}_\phi \times \psi^\phi \times \frac{SO(5)_k \times SO(7)_1}{SU(2)_{k+1}} \cong \left[ \mathbb{R}_\phi \times \psi^\phi \times SU(2)_{k+1} \times SO(3)_1 \right] \times \frac{SO(5)_k \times SO(4)_1}{SU(2)_{k+1} \times SU(2)_{k+1}}.
\tag{3.57}
\]

The criticality condition (2.3) yields \(Q^2_\phi = 8/(k + 3)\). Since the supercoset part of the last line is associated to a Wolf space, the worldsheet SUSY should be enhanced to \(\mathcal{N} = 4\), as we already discussed. In this way, we have obtained the superstring vacua corresponding to the \(Sp(2)\)-holonomy. The SUSY cancellation reduces to the identity (2.10).

(ii) \(SO(3)\) embedded along a short root:

Suppose \(SO(3)\) is embedded along a short root, in other words, embedded diagonally into the \(SO(3) \times SO(3)\) subgroup. This is found to be the “canonical embedding” so that the vector representation of \(SO(5)\) is decomposed as \(5 \rightarrow 3 + 1 + 1\). Hence the coset space is isomorphic with the Stiefel manifold \(V_2(\mathbb{R}^5)\), which has the canonical \(U(1) (\cong SO(2))\)-fibration over the Grassmannian \(G_2(\mathbb{R}^5)\) (an example of HSS);

\[
\mathcal{M} = \frac{SO(5)}{SO(3)} \cong V_2(\mathbb{R}^5) \overset{U(1)}\rightarrow \mathcal{M}/U(1) \cong G_2(\mathbb{R}^5). \tag{3.58}
\]

This is a typical example of the Sasaki-Einstein homogeneous space. In terms of
the coset CFT we thus find that

$$\mathbb{R}_g \times \psi^g \times \mathcal{M} = \mathbb{R}_g \times \psi^g \times \frac{SO(5)_k \times SO(7)_1}{SO(3)_{k+2}}$$

$$\cong [\mathbb{R}_g \times \psi^g \times U(1)_{2(k+3)} \times SO(1)_1] \times \frac{SO(5)_k \times SO(6)_1}{SO(3)_{k+2} \times U(1)_{2(k+3)}}. (3.59)$$

We here obtain $Q^2_\phi = 9/(k + 3)$, resulting in $3^2 \cdot 2/Q^2_\phi = 2(k + 3)$. Therefore, $U(1)_{2(k+3)}$-factor exactly reproduces the $\mathcal{N} = 2$ Liouville theory. The remaining coset CFT is the Kazama-Suzuki model associated to $G_2(\mathbb{R}^5)$. In this way we find that the total system has $\mathcal{N} = 2$ worldsheet SUSY and $SU(4)$-holonomy.

(iii) $SO(3)$ embedded as a maximal subgroup of $SO(5)$:

The third case is the most non-trivial. Consider the embedding of $SO(3)$ along $2\alpha_1 + 3\alpha_2$, where $\alpha_1$, $\alpha_2$ are the long and short roots of $SO(5)$ ($\alpha_1^2 = 2$, $\alpha_2^2 = 1$, $\alpha_1 \cdot \alpha_2 = -1$). The simple root of $SO(3)$ is defined as the projection of the highest root of $SO(5)$ and thus identified as $\theta' \equiv \frac{1}{7}(2\alpha_1 + 3\alpha_2)$. In this embedding the adjoint representation of $SO(5)$ is decomposed as $10 \rightarrow 3 + 7$, which means it is a maximal embedding (see [22] for the detail). Since we have $(\theta', \theta') = 1/5$, the relevant supercoset should be

$$\mathcal{M} = \frac{SO(5)_k \times SO(7)_1}{SO(3)_{5k+14}}. (3.60)$$

Since we have no remaining symmetry of $SU(2)$ or $U(1)$ in this coset, it is obvious that the worldsheet SUSY cannot be enhanced. We thus obtain at most 2 supercharges in spacetime corresponding to the $Spin(7)$-holonomy. The criterion for the spacetime SUSY (3.52) is now expressed as

$$\mathcal{M} \cong \frac{SO(5)_k \times (G_2)_1}{SO(3)_{5k+14}} \times \frac{SO(7)_1}{(G_2)_1}$$

$$\cong \frac{SO(5)_k \times (G_2)_1}{SU(2)_{10k+28}} \times \text{tri-critical Ising}, (3.61)$$

and we require that the coset CFT in the last line should be well-defined. Especially, we must ask whether we can consistently define $(G_2)_1/SU(2)_{28}$. We first note that $(G_2)_1$ and $SU(2)_{28}$ have the equal central charge $c = 14/5$. So, $(G_2)_1/SU(2)_{28}$ would be a topological coset CFT, if it is well-defined. It is actually known that the
conformal embedding $SU(2)_{28} \subset (G_2)_1$ exists (see for example, [69])\footnote{The explicit embedding of $SU(2) \subset G_2$ here is given as follows: The simple root of $SU(2)$ is identified with $\theta' = \frac{1}{14}(\alpha_1 + 6\alpha_2)$, where $\alpha_1$, $\alpha_2$ denote the long and short roots of $G_2$ ($\alpha_1^2 = 2$, $\alpha_2^2 = 2/3$, $\alpha_1 \cdot \alpha_2 = -1$). By this embedding, the adjoint representation of $G_2$ is decomposed as $14 \to 3 + 11$. Hence, this is also the maximal embedding. The square length of $\theta'$ is equal 1/14, which is compatible with the existence of coset CFT $(G_2)_1/SU(2)_{28}$.}. More precise relation is as follows: $SU(2)_{28}$ is known to have the $E_8$-type modular invariant [70], in which the partition function is given as

$$Z = \left| \chi_0^{(28)} + \chi_{10}^{(28)} + \chi_{18}^{(28)} \right|^2 + \left| \chi_6^{(28)} + \chi_{12}^{(28)} + \chi_{16}^{(28)} + \chi_{22}^{(28)} \right|^2 . (3.62)$$

This partition function is in fact equivalent to the diagonal modular invariant of $(G_2)_1$ due to the character relations;

$$\chi_{\text{basic}}^{(G_2)_1}(\tau) = \left( \chi_0^{(28)} + \chi_{10}^{(28)} + \chi_{18}^{(28)} \right)(\tau) ,$$

$$\chi_{\text{fund}}^{(G_2)_1}(\tau) = \left( \chi_6^{(28)} + \chi_{12}^{(28)} + \chi_{16}^{(28)} + \chi_{22}^{(28)} \right)(\tau) . \quad (3.63)$$

Accordingly, the rewriting (3.61) is actually well-defined. We have achieved the string vacua of $Spin(7)$-holonomy.

5. $G/H = SU(3)/U(1)$:

To define the supercoset CFT, we have to specify the $U(1)$-embedding, which is characterized by the 1-dimensional charge lattice $\Gamma \subset \sqrt{k + 3}Q$, where $Q = Z\alpha_1 + Z\alpha_2$ is the root lattice of $SU(3)$. $((\alpha_1, \alpha_i) = 2, (\alpha_1, \alpha_2) = -1.)$

The spacetime SUSY requires that the following rewriting should be possible; \footnote{This is a slightly stronger condition than (3.52) and leads to twice as many supercharges as compared to the $Spin(7)$-holonomy case (i.e. the same number of supercharges as the $SU(4)$ and $G_2$-holonomy). However, it is easy to show that (3.52) inevitably reduces to (3.64) in this case.}

$$\mathcal{M} = \frac{SU(3)_k \times SO(7)_1}{U(1)} \cong \frac{SU(3)_k \times SU(3)_1}{U(1)} \times SO(1)_1 \times U(1)_{3/2} . \quad (3.64)$$

As we discussed in the $d = 3$ analysis, this rewriting is possible if and only if $\Gamma$ is generated by an element $\mu_1$ of the lattice $w \cdot \Lambda$, where $\Lambda$ is defined in (3.36) and $w$ is a Weyl reflection. However, we can show that $\sqrt{k + 3}Q = W \cdot \Lambda$. So, we have the spacetime SUSY for an arbitrary choice of $\mu_1$.

On the other hand, the $N = 4$ enhancement of worldsheet SUSY occurs in the special cases $\mu_1 = m(w \cdot \nu_2) (\equiv m(w \cdot (\alpha_1 + 2\alpha_2)))$, where $m$ is an arbitrary integer and $w$ is a
Weyl reflection. In fact, (only) in that case we can find an $SU(2)$ symmetry along the transverse direction $\mu_2 = w \cdot \nu_1 \equiv w \cdot \alpha_1$, resulting in the equivalence

$$\mathbb{R}_\phi \times \psi^\phi \times \mathcal{M} \cong \left[ \mathbb{R}_\phi \times \psi^\phi \times SU(2)_{k+1} \times SO(3) \right] \times \frac{SU(3)_k \times SO(4)}{U(1)_{3(k+3)} \times SU(2)_{k+1}} \quad (3.65)$$

The coset part is associated to $\mathbb{C}P_2$, which possesses the structures of both the HSS and Wolf space. Hence the worldsheet SUSY is enhanced to $\mathcal{N} = 4$.

As a consistency check, we can also check the $\mathcal{N} = 2$ structure on the worldsheet. The following rewriting is also possible;

$$\mathbb{R}_\phi \times \psi^\phi \times \mathcal{M} \cong \left[ \mathbb{R}_\phi \times \psi^\phi \times U(1)_{k+3} \times SO(1) \right] \times \frac{SU(3)_k \times SO(6)}{U(1)_{3(k+3)} \times U(1)_{k+3}} \quad (3.66)$$

The part $[\cdots]$ describes the $\mathcal{N} = 2$ Liouville since $Q_\phi^2 = 8/(k + 3)$, and the coset part is the Kazama-Suzuki model considered in the $d = 3$ analysis.

It is also obvious that, if we cannot write $\mu_1$ as the form $\mu_1 = m(w \cdot \nu_2)$, the worldsheet SUSY cannot be enhanced for generic values of $k$.

In summary, we have shown that

(i) If $\Gamma$ has the form $\Gamma = \mathbb{Z}(mw \cdot \nu_2)$, with some Weyl reflection $w$ and an integer $m$, we have the $\mathcal{N} = 4$ worldsheet SUSY, and the string vacua corresponds to an $Sp(2)$-holonomy.

(ii) If $\Gamma$ does not, we have the supersymmetric vacua with 4 supercharges, but the worldsheet SUSY is at most $\mathcal{N} = 1$. In this case, the coset CFT $\mathcal{M}$ has a remaining $U(1)$-symmetry and seems to correspond to a compactification on a space of the form $S^1 \times G_2$-manifold. However, the $U(1)$-charge is not independent of the quantum numbers in the remaining sector and this is some kind of an orbifold space.

Let us make a few comments: The coset space $SU(3)/U(1)$ is known under the name Aloff-Wallach space [25] and written as $N(m, \ell) \quad ^9$ with the $U(1)$-action

$$e^{i\theta} \mapsto \begin{pmatrix} e^{im\theta} & 0 & 0 \\ 0 & e^{i\ell\theta} & 0 \\ 0 & 0 & e^{-i(m+\ell)\theta} \end{pmatrix} \quad (3.67)$$

[^9]: In many literature it is also denoted as $N^{pqr}$ [19], where $N^{pqr} = \frac{SU(3) \times U(1)}{U(1) \times U(1)}$ with the integer parameters $p, q, r$ characterizing the remaining $U(1)$ symmetry as

$$Z = p \frac{i\sqrt{3}}{2}\lambda_8 + q \frac{i}{2}\lambda_3 + riY .$$

($\lambda_3, \lambda_8$ are the Gell-Mann matrices (3.43) and $Y$ is the generator of $U(1)$ in the numerator.) In particular, $N(1, 1)$ is equal to $N^{010}$.
where $m$, $\ell$ are relatively prime integers. They are not diffeomorphic for different parameters $m$, $\ell$ (unless we have a Weyl reflection connecting them).

It is known that the spaces $N(m, \ell)$ can be endowed with two types of Einstein metrics \[19,20\], denoted as $N(m, \ell)_{I}$ and $N(m, \ell)_{II}$ (the “squashed” one) in \[24\]. The generic cases of $(m, \ell) \neq (1, 1)$ become weak $G_2$ manifolds irrespective of the choice of Einstein metrics. On the other hand, $N(1, 1)_I$ is known to be tri-Sasakian (and also Sasaki-Einstein) \[12,19\], while $N(1, 1)_{II}$ has the weak $G_2$ holonomy \[12,20\]. The choice of $m$, $\ell$ is in one-to-one correspondence with the charge lattice $\Gamma$ introduced above. The first case (i) corresponds to $N(1, 1)$ and leads to the $\mathcal{N} = 4$ worldsheet SUSY, while the second case (ii) corresponds to the cases $(m, \ell) \neq (1, 1)$ and at most the $\mathcal{N} = 1$ worldsheet SUSY is allowed. In this sense our algebraic construction agrees with the geometrical analysis. The amount of worldsheet SUSY is exactly as expected. Among other things, $N(1, 1)_I$ has the $SU(2)$-fibration characteristic of the tri-Sasakian homogeneous space:

$$N(1, 1)_I \xrightarrow{SU(2)} \mathbb{CP}_2,$$

which corresponds to \(3.65\). It also has the $U(1)$-fibration for the Sasaki-Einstein space:

$$N(1, 1)_I \xrightarrow{U(1)} F_{1,2}(\mathbb{C}^3),$$

where we should regard the flag manifold $F_{1,2}(\mathbb{C}^3)$ as a Kähler-Einstein space. This of course corresponds to \(3.66\).

We will later discuss the CFT interpretation of the squashed geometry of $N(1, 1)_{II}$.

We further analyse a few examples with a non-simple group $G$.

6. $G/H = (SU(3) \times SU(2))/(SU(2) \times U(1))$

We have various possibilities of embedding of $SU(2) \times U(1)$ as listed in \[22\].

(i) $SU(2)$ embedded as the isospin subgroup:

The relevant coset SCFT should have the form

$$\mathcal{M} = \frac{SU(3)_{k_1} \times SU(2)_{k_2} \times SO(7)}{SU(2)_{k_1+1} \times U(1)}.$$ \[3.70\]

To define the model completely, we still have to fix the $U(1)$ embedding. Choosing the isospin $SU(2)$ subgroup along the simple root $\alpha_1$ (simple roots of $SU(3)$: $\alpha_1$, $\alpha_2$ with $\alpha_1^2 = \alpha_2^2 = 2$, $\alpha_1 \cdot \alpha_2 = -1$, as usual), let us introduce the following charge lattice for $U(1)$-action:

$$\Gamma = \mathbb{Z} \left( q\sqrt{k_1+3(\alpha_1+2\alpha_2)} - p'\sqrt{k_2+2\beta} \right),$$ \[3.71\]
where $\beta$ is the simple root of the $SU(2)$ factor ($\beta^2 = 2$). The $U(1)$ action generated by $\Gamma$ obviously commutes with $SU(2)_{\text{isospin}}$ for arbitrary $q, p'$. The theta function associated to this charge lattice yields the $U(1)$ factor $U(1)_{3(k_1+3)q^2+(k_2+2)p'^2}$ in (3.70), and we obtain

$$
\mathcal{M} \cong \frac{SU(3)_{k_1} \times SO(4)_1}{SU(2)_{k_1+1} \times U(1)_{3(k_1+3)}} \times \frac{SU(2)_{k_2} \times SO(2)_1}{U(1)_{k_2+2}} \\
\times SO(1)_1 \times U(1)_{3(k_1+3)(k_2+2)} \left\{3q^2(k_1+3)+p'^2(k_2+2)\right\},
$$

(3.72)

using the relation

$$
\frac{U(1)_{3(k_1+3)} \times U(1)_{k_2+2}}{U(1)_{3(k_1+3)q^2+(k_2+2)p'^2}} \cong U(1)_{3(k_1+3)(k_2+2)} \left\{3q^2(k_1+3)+p'^2(k_2+2)\right\},
$$

(3.73)

of the product formula of theta functions. The first coset part in (3.72) is the Kazama-Suzuki model for $\mathbb{C}P_2$ and the second coset is the $\mathcal{N} = 2$ minimal model of level $k_2$. On the other hand, the criticality condition (2.3) leads to

$$
Q^2_\phi = \frac{2(k_1 + 3k_2 + 9)}{(k_1 + 3)(k_2 + 2)}.
$$

(3.74)

We thus need the factor $U(1)_{3(k_1+3)(k_2+2)(k_1+3k_2+9)}$ in (3.72) to get the $\mathcal{N} = 2$ Liouville sector. Therefore, the worldsheet SUSY is enhanced to $\mathcal{N} = 2$, if and only if $p' = 3q$ holds, which yields a string vacuum of $SU(4)$-holonomy.

In the case of $p' \neq 3q$, we have at most $\mathcal{N} = 1$ worldsheet SUSY. Then, the spacetime SUSY requires that the following rewriting should be possible (as in the analysis of $SU(3)/U(1)$);

$$
\mathcal{M} \cong \frac{SU(3)_{k_1} \times SU(2)_{k_2} \times SU(3)_1}{SU(2)_{k_1+1} \times U(1)_{3q^2(k_1+3)+p'^2(k_2+9)}} \times SO(1)_1 \times U(1)_{3/2}.
$$

(3.75)

For generic values of $k_1, k_2$, this is possible only for $p' = 3q$, which goes back to the $\mathcal{N} = 2$ case already discussed. In this way, we conclude that this type string vacua are supersymmetric if and only if $p' = 3q$ holds, and in those cases we obtain the $SU(4)$-holonomy.

We have a comment in connection to known results in Kaluza-Klein SUGRA: This case corresponds to the homogeneous space with $SU(3) \times SU(2) \times U(1)$ isometry first studied in [17];

$$
M^{pq} = \frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1) \times U(1)},
$$

(3.76)

where $SU(2)$ is embedded as the isospin subgroup of $SU(3)$ and the integer parameters $p, q, r$ characterize the remaining $U(1)$-symmetry as

$$
Z = p\frac{i\sqrt{3}}{2}\lambda_8 + q\frac{i}{2}\sigma_3 + riY,
$$

(3.77)
where $\lambda_8$ is the Gell-Mann matrix (the generator transverse to the $SU(2)_{\text{isospin}}$) (3.43), $\sigma_3$ is the Pauli matrix and $Y$ is the generator of the $U(1)$-factor in the numerator. Our coset CFT $\mathcal{M}$ with the charge lattice $\Gamma$ is naturally associated to the space $M^{pq0}$ under the identification $p' = 3p$ (at least for the cases $k_1 + 3 = k_2 + 2$). It is known [21] that every coset space $M^{pqr}$ can be equipped with Einstein metrics and allows the spacetime SUSY only in the case $p = q$. $M^{110}$ is a regular Sasaki-Einstein space and $M^{ppr}$ is regarded as an orbifold of it. The $U(1)$-fibration for the Sasaki-Einstein space $M^{110}$ is written as

$$M^{110} \xrightarrow{U(1)} \mathbb{CP}_2 \times \mathbb{CP}_1,$$

which corresponds to (3.72) for $q = 1, p' = 3$. These aspects fit nicely with our algebraic construction.

(ii) $SU(2)$ embedded in the explicit $SU(2)$ factor:

Obviously, this reduces to the case of $SU(3)/U(1)$ we studied previously.

(iii) $SU(2)$ embedded in both of $SU(3)$ and the explicit $SU(2)$ factor:

In this case the only possibility is the diagonal embedding $SU(2) \subset SU(2)_{\text{isospin}} \times SU(2) \subset SU(3) \times SU(2)$. The $U(1)$-factor must necessarily be embedded along the $\lambda_8 \propto \alpha_1 + 2\alpha_2$ direction. The corresponding coset SCFT is defined as

$$\mathcal{M} = \frac{SU(3)_{k_1} \times SU(2)_{k_2} \times SO(7)_1}{SU(2)_{k_1+k_2+3} \times U(1)_{3(k_1+3)}}.$$ (3.79)

We can rewrite it as

$$\mathcal{M} \approx \frac{SU(3)_{k_1} \times SU(2)_{k_2} \times (G_2)_1}{SU(2)_{k_1+k_2+3} \times U(1)_{3(k_1+3)}} \times \text{tri-critical Ising}.$$ (3.80)

(Recall that $(G_2)_1 \sim SU(2)_3 \times U(1)_1 \sim SU(2)_3 \times U(1)_9$.) Therefore, we have obtained a string vacuum with $Spin(7)$-holonomy.

The coset space considered here is topologically isomorphic with $N(1,1)$, and this string vacuum is supposed to correspond to the squashed geometry of $N(1,1)$ mentioned before. Under the limit $k_2 \to \infty$, the $SU(2)$-factors in (3.79) are canceled out, and we recover the $N(1,1)$ coset SCFT $\mathcal{M} = \frac{SU(3)_k \times SO(7)_1}{U(1)_{3(k+3)}}$. 

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(iv) \( SU(2) \) embedded as a maximal subgroup of \( SU(3) \):

In this case \( U(1) \) has to be embedded only in the \( SU(2) \) factor. As in (3.17), the relevant supercoset CFT is written as

\[
\mathcal{M} = \frac{SU(3)_{k_1} \times SU(2)_{k_2} \times SO(7)_1}{SU(2)_{4k_1+10} \times U(1)_{k_2+2}} \quad (3.81)
\]

The second line corresponds to the fact that this coset space is topologically isomorphic with \( S^5 \times S^2 \) (see [22]), and no remaining \( U(1) \) symmetry exists. As is easily shown, we have no spacetime SUSY in this case. It is again consistent with the known results of Kaluza-Klein SUGRA [22].

7. \( G/H = (SU(2) \times SU(2) \times SU(2))/(U(1) \times U(1)) \):

This model may regarded as a natural generalization of the \( d = 4 \) vacuum \( (SU(2) \times SU(2))/U(1) \) and also the CHS \( \sigma \)-model. The supercoset has the form

\[
\mathcal{M} = \frac{SU(3)_{k_1} \times SU(2)_{k_2} \times SU(2)_{k_3} \times SO(7)_1}{U(1) \times U(1)} \quad (3.82)
\]

Similarly to the \( d = 4 \) case, we try to rewrite as

\[
\mathbb{R}_\phi \times \psi^\phi \times \mathcal{M} \sim \left[ \mathbb{R}_\phi \times \psi^\phi \times U(1) \times SO(1)_1 \right] \times \mathcal{M}_{k_1} \times \mathcal{M}_{k_2} \times \mathcal{M}_{k_3} \quad ,
\]

where \( \mathcal{M}_k \equiv \frac{SU(2)_k \times SO(2)}{U(1)_{k+2}} \) denotes the \( N = 2 \) minimal model again. Since we obtain

\[
Q^2_\phi = \frac{2(N_2N_3 + N_3N_1 + N_1N_2)}{N_1N_2N_3} \quad , \quad (N_i \equiv k_i + 2) \quad ,
\]

from the criticality condition (2.3), the criterion for the part \([\cdots]\) to become the \( N = 2 \) Liouville is whether we can factorize \( U(1)N_1N_2N_3(N_2N_3 + N_3N_1 + N_1N_2) \). Namely, we want to derive a relation

\[
\frac{U(1)_{N_1} \times U(1)_{N_2} \times U(1)_{N_3}}{U(1) \times U(1)} \equiv U(1)_{N_1N_2N_3(N_2N_3 + N_3N_1 + N_1N_2)} \quad ,
\]

by a suitable choice of the charge lattice \( \Gamma \) of \( U(1) \times U(1) \). To describe it explicitly, let \( \alpha_i \) \((i = 1, 2, 3)\) be the simple roots of each \( SU(2) \) factors, normalized as \( (\alpha_i, \alpha_j) = 2\delta_{ij} \). The two dimensional lattice \( \Gamma \) must be defined as a sublattice of \( \mathbb{Z}\sqrt{N_1\alpha_1} + \mathbb{Z}\sqrt{N_2\alpha_2} + \mathbb{Z}\sqrt{N_3\alpha_2} \). Introducing integer parameters \( p, q, r \), we set

\[
\nu_1 = q\sqrt{N_1}\alpha_1 - p\sqrt{N_2}\alpha_2 \quad ,
\]

\[
\nu_2 = prN_2N_3\sqrt{N_1}\alpha_1 + qrN_3N_1\sqrt{N_2}\alpha_2 - N_3(p^2N_2 + q^2N_1)\sqrt{N_3}\alpha_2 \quad ,
\]

\[
\nu_3 = pN_2N_3\sqrt{N_1}\alpha_1 + qN_3N_1\sqrt{N_2}\alpha_2 + rN_1N_2\sqrt{N_3}\alpha_3 \quad .
\]

(3.86)
Then we find that they are orthogonal to each other, and also,

\[(\nu_3, \nu_3) = 2N_1N_2N_3(p^2N_2N_3 + q^2N_3N_1 + r^2N_1N_2) .\]  (3.87)

If we choose \(\Gamma\) as (a sublattice of) \(\mathbb{Z}\nu_1 + \mathbb{Z}\nu_2\), we obtain the theta function identity such as

\[
\Theta^* \cdot N_1(\tau) \Theta^* \cdot N_2(\tau) \Theta^* \cdot N_3(\tau) = \sum \Theta^*(\Gamma) \Theta^* \cdot N_1N_2N_3(p^2N_2N_3 + q^2N_3N_1 + r^2N_1N_2) .
\]  (3.88)

Accordingly, if and only if \(p = q = r\) holds, the wanted relation (3.85) is obtained and hence the worldsheet SUSY enhances to \(\mathcal{N} = 2\). We find the string vacuum with \(SU(4)\)-holonomy in this case. It is also not difficult to see that the spacetime SUSY cannot exist in other cases as in the previous analysis. The special cases of \(k_1 = k, k_2 = k_3 = 0\) correspond to the CY_4 with the \(A_{k+1}\)-type singularity studied in [29, 34, 71].

We also make a comment in connection to the Kaluza-Klein SUGRA: Consider the Einstein homogeneous space \([18]\]

\[Q(p, q, r) = \frac{SU(2) \times SU(2) \times SU(2)}{U(1) \times U(1)} ,\]  (3.89)

where \(p, q, r\) parameterize the remaining \(U(1)\)-symmetry as before. It is known that we have the spacetime SUSY only in the cases \(p = q = r\) and the \(Q(p, q, r)\) space becomes a Sasaki-Einstein space in this case. For the cases of \(N_1 = N_2 = N_3\), the above choice of charge lattice \(\Gamma\) precisely reproduces the coset space \(Q(p, q, r)\) (the vector \(\nu_3\) describes the remaining \(U(1)\) symmetry), and thus the SUSY condition coincides precisely. Moreover, the \(U(1)\)-fibration

\[Q(p, q, r) \xrightarrow{U(1)} \mathbb{C}P_1 \times \mathbb{C}P_1 \times \mathbb{C}P_1\]  (3.90)

is naturally related with (3.83). Our CFT analysis again is consistent with that of Kaluza-Klein SUGRA.

8. \(G/H = (SU(2) \times SU(2))/SU(2) :\)

Finally we present an example with \(\dim G/H \neq 7\). Consider again the \(\mathcal{N} = 1\) diagonal coset (3.12) and set \(k_1 = k, k_2 = 1;\)

\[
\mathcal{M} = \frac{SU(2)_k \times SU(2)_1 \times SO(3)_1}{SU(2)_{k+3}} .\]  (3.91)

Since \(\dim G/H = 3\) rather than \(\dim G/H = 7\), we use a different relation to present the spacetime SUSY;

\[
\frac{SU(2)_1 \times SU(2)_2}{SU(2)_3} \cong \text{tri-critical Ising} .\]  (3.92)
We then obtain

\[ \mathcal{M} \cong \frac{SU(2)_k \times SU(2)_3}{SU(2)_{k+3}} \times \frac{SU(2)_1 \times SU(2)_2}{SU(2)_3} \]

\[ \cong \frac{SU(2)_k \times SU(2)_3}{SU(2)_{k+3}} \times \text{tri-critical Ising}, \]

which satisfies the SUSY condition (2.17). In this way we obtain superstring vacua of \( Spin(7) \)-holonomy manifolds. Simplest case \( k = 0 \) again reduces to the model given in [45]. Since \( \dim G/H \neq 7 \), it seems difficult to relate these string vacua with the solutions of SUGRA.

Before closing this section we want to make a few comments on the relation between our CFT and the classical geometry of special holonomy manifolds based on the cone construction.

1. The comparison between the geometrical cones and our “CFT cones” leads to an obvious disagreement when \( G/H \) is isomorphic with a round sphere \( S^{9-d} \). For the \( d = 2 \) string vacua, for example, all the cosets \( G/H = SO(8)/SO(7), SO(7)/G_2, SU(4)/SU(3), SO(5)/SO(3) \) (the case when \( SO(3) \) is embedded along a long root) are found to be topologically and metrically isomorphic with \( S^7 \). The cone over \( S^7 \) is of course the flat space \( \mathbb{R}^8 \), and hence allows the maximal spacetime SUSY. On the other hand, the supercoset CFT’s based on them are really inequivalent with each other, and yield string vacua with less spacetime SUSY. As we discussed above, \( SO(8)/SO(7) \) leads to non-SUSY vacua, and \( SO(7)/G_2, SU(4)/SU(3), SO(5)/SO(3) \) provide \( Spin(7), SU(4), Sp(2) \) holonomies, respectively.

2. It is known that every 7-dim. tri-Sasakian manifold allows the second “squashed” Einstein metric that provides a weak \( G_2 \) holonomy [13, 15] (see also [8, 12]). A way to present the squashing procedure is to replace the original tri-Sasakian coset \( G/H \) by \( \frac{G \times SU(2)}{H \times SU(2)} \), which is topologically isomorphic with \( G/H \).

For the case of \( N(1,1) = SU(3)/U(1) \), the “squashed supercoset CFT” is defined in (3.79). The similar construction is also possible for the tri-Sasakian coset \( SO(5)/SO(3) \) (3.57), and could be identified as the “squashed \( S^7 \)” (often denoted as \( J^7 \)). Namely, we deform (3.57) as

\[ \mathcal{M} = \frac{SO(5)_{k_1} \times SU(2)_{k_2} \times SO(7)_1}{SO(3)_{k_1+k_2+1} \times SU(2)_{k_1+k_2+3}}, \]

(3.94)
where the $SU(2)$ in the denominator is embedded diagonally into $SU(2) \times SU(2)$, in which the first $SU(2)$-factor is the remaining one of $SO(5)/SO(3)$ and the second is the explicit $SU(2)$-factor. We can easily show that it leads to string vacua with $Spin(7)$-holonomy in the similar manner as in the $N(1,1)$ case, and the original tri-Sasakian coset (3.57) is recovered under the limit $k_2 \to \infty$.

The analogous relation is also found in the $d = 3$ example $S^3 \times S^3$. The coset CFT (3.47) may be regarded as the squashed version of (3.45). We again recover the unsquashed one (3.45) in the limit $k_3 \to \infty$.

4 Marginal Deformations: Spectrum of Cosmological Constant Operators

4.1 Cosmological Constant Operators Preserving Special Holonomy

Since the linear dilation CFT is singular as a worldsheet theory, we should introduce the “cosmological constant operators” (Liouville potential terms) in order to eliminate its singular behavior at the tip of the cone (Liouville exponential prevents the field $\phi$ going out to $-\infty$). In this section we consider cosmological constant terms which preserve the spacetime supersymmetry and thus act as marginal perturbations in various models discussed in previous sections, focusing in particular on the $G_2$ and $Spin(7)$ holonomy cases.

As in the old days of two-dimensional gravity [78], the cosmological constant operator is defined as the most relevant primary field of the “matter sector” multiplied by the Liouville exponential. Here one might worry about the “$\hat{c} = 1$ ($c = 3/2$) barrier”, since the conformal matter $\mathbb{R}^{d-1,1} \times \mathcal{M}$ has the central charge bigger than $3/2$ for any unitary $\mathcal{M}$. In fact if one considers an identity operator (of matter sector) multiplied by a Liouville exponential, one finds the trouble of a complex Liouville exponent. This difficulty is avoided in our case by the requirement of GSO projection for vacua with unbroken spacetime SUSY. We should define the cosmological constant term for the most relevant primary operator allowed by the GSO condition, and we can show that the Liouville exponential is then always real as we shall see below. On the other hand in the case of broken spacetime SUSY, it seems difficult to define a suitable cosmological operator that resolves the singularity, since the most relevant operator becomes tachyonic and has a complex Liouville exponential.
The general form of the marginal perturbation operator is written as follows;

\[
\left[ G_{-\frac{1}{2}}, \left[ G_{-\frac{1}{2}}, e^{\gamma \phi} \mathcal{O}_{\mathcal{M}}^{(\text{NS})} \right] \right], \tag{4.1}
\]

where \( \mathcal{O}_{\mathcal{M}}^{(\text{NS})} \) denotes an NS primary field in the \( \mathcal{M} \) sector. Here one cannot choose the identity operator \( \mathcal{O}_{\mathcal{M}}^{(\text{NS})} = \text{id} \) since the operator (4.1) then becomes mutually non-local with respect to the spacetime SUSY operator (violates GSO condition). The BRST invariance requires the on-shell condition

\[
h(\mathcal{O}_{\mathcal{M}}^{(\text{NS})}) + h(e^{\gamma \phi}) \equiv h(\mathcal{O}_{\mathcal{M}}^{(\text{NS})}) - \frac{1}{2} \gamma^2 - \frac{1}{2} Q_\phi \gamma = \frac{1}{2}, \tag{4.2}
\]

under which (4.1) manifestly preserves the worldsheet \( \mathcal{N} = 1 \) supersymmetry. To analyse the spectrum of operators \( \mathcal{O}_{\mathcal{M}}^{(\text{NS})} \) it is convenient to consider their supersymmetric partners \( \mathcal{O}_{\mathcal{M}}^{(\text{R})} \) in the Ramond sector, which have the conformal weight

\[
h(\mathcal{O}_{\mathcal{M}}^{(\text{R})}) = h(\mathcal{O}_{\mathcal{M}}^{(\text{NS})}) + \frac{1 - d}{16}. \tag{4.3}
\]

This relation follows from the spacetime supersymmetry: Ramond states in the partition functions are dressed by the spin fields of the Minkowski space and the Liouville fermion and possess the same conformal weights as the NS states. Dimensions of the spin fields add up to \( \frac{(d-1)}{16} \) which accounts for the factor in (4.3).

The above relation may also be derived from the structure of our coset theories: NS and R states in the coset theory have the general form,

\[
\mathcal{O}_{\mathcal{M}}^{(\text{NS})} = \Phi_{\Lambda,s=2,\lambda} \left[ (G \times SO(9 - d))/H \right],
\]

\[
\mathcal{O}_{\mathcal{M}}^{(\text{R})} = \Phi_{\Lambda,s=1(-1),\lambda} \left[ (G \times SO(9 - d))/H \right], \tag{4.4}
\]

where \( \Phi_{\Lambda,s,\lambda} \left[ (G \times SO(9 - d))/H \right] \) denotes a primary state in the coset \( (G \times SO(9 - d))/H \) defined by the highest weights \( \Lambda, s \) and \( \lambda \) of the affine Lie algebras \( G, SO(9 - d) \) and \( H \), respectively. \( s = 0, 2, 1, -1 \) stand for the basic, vector, spinor and cospinor representations of \( SO(9 - d) \). Note that the weights \( \Lambda, \lambda \) of NS and R states are the same for supersymmetric partners and thus the difference in their dimensions come from that of the representations of \( SO(9 - d) \). Difference of spinor and vector dimensions \( (9 - d)/16 - 1/2 = (1 - d)/16 \) accounts again for the RHS of (4.3). (In case a basic representation \( s = 0 \) of the current algebra \( SO(9 - d) \) is used in the NS state, there is an additional factor \( +1/2 \) in the RHS of (4.3).)

The unitarity of \( \mathcal{M} \) sector requires the inequality \( h(\mathcal{O}_{\mathcal{M}}^{(\text{R})}) \geq c_\mathcal{M}/24 \), and thus we can easily show that the Liouville exponent \( \gamma \) is always real with the help of the condition (2.3).
Let us now fix the value of the exponent $\gamma$ in the marginal perturbation operators (4.1). From the old days of two-dimensional gravity it is known that Liouville exponentials have different characteristics depending on whether $\gamma > -\frac{Q}{2}$ or $\gamma < -\frac{Q}{2}$ [29, 31, 54] (see also [39, 40]).

- $\gamma > -\frac{Q}{2}$: The operator $e^{\gamma \phi}$ is called non-normalizable, since the corresponding wave function exponentially diverges at the asymptotic region $\phi \rightarrow +\infty$. It is interpreted as a coupling constant of the dual field theory, since the fluctuation has a divergent kinetic energy. In the context of two dimensional gravity it is also identified as the local scaling operators.

- $\gamma < -\frac{Q}{2}$: The operator $e^{\gamma \phi}$ is called normalizable. The wave function is peaked around the singular region $\phi \rightarrow -\infty$ as opposed to the above case. It is interpreted as a VEV of the dynamical fields of the dual theory (the modulus of vacuum), since the fluctuation has a finite kinetic energy.

Now we propose to choose the critical value for $\gamma$

$$\gamma = -\frac{Q}{2} \ , \quad (4.5)$$

for our marginal operators (4.1). It corresponds to the maximal value of conformal weight $h(e^{-\frac{Q}{2} \phi}) = \frac{Q^2}{8}$ of the Liouville exponential with real $\gamma$ and corresponds also to the minimum value in the continuous spectrum of delta-function normalizable states $\gamma = -\frac{Q}{2} + ip$ ($p \in \mathbb{R}$). This situation is quite reminiscent of that of the $c = 1$ conformal matter coupled to two dimensional gravity, or equivalently, the critical string on the background of two dimensional black hole [79].

It turns out that the condition $\gamma = -\frac{Q}{2}$ reduces the problem of finding marginal perturbations (4.1) to that of Ramond ground states in the theory $\mathcal{M}$. In fact when we take account of the $\mathcal{N} = 1$ Liouville degrees of freedom and set $\gamma = -\frac{Q}{2}$, the Ramond sector operator is given by

$$O^{(R)}(R) = \sigma^{\phi} e^{-\frac{Q}{2} \phi} O^{(R)}(\mathcal{M}) \ , \quad (4.6)$$

where $\sigma^{\phi}$ denotes the spin field associated with the Liouville fermion. If we recall the criticality condition (2.2)

$$\frac{3}{2}(d-2) + \frac{3}{2} + 3Q^2 + c_\mathcal{M} = 12 \ , \quad (4.7)$$

and divide the formula by 24, we find

$$\frac{1}{16} + \frac{Q^2}{8} + \frac{c_\mathcal{M}}{24} = \frac{10 - d}{16} . \quad (4.8)$$
A state with $h = (10 - d)/16$ is in fact the Ramond ground state for the internal space with $10 - d$ dimensions. Thus the Ramond ground state of the system $(\mathcal{N} = 1 \text{ Liouville}) \times \mathcal{M}$ is constructed from the Ramond ground state $h(\mathcal{O}_\mathcal{M}^{(R)}) = c_{\mathcal{M}}/24$ of the theory $\mathcal{M}$.

We thus have an one-to-one correspondence between the marginal perturbation and Ramond ground state as

$$O^{(\text{NS})} = e^{-\frac{Q \phi}{2}} O^{(\text{NS})}_\mathcal{M} \iff O^{(\text{R})} = \sigma_\phi e^{-\frac{Q \phi}{2}} O^{(\text{R})}_\mathcal{M}.$$ (4.9)

Here $O^{(\text{NS})}_\mathcal{M}$ and $O^{(\text{R})}_\mathcal{M}$ are related as (4.4). If one uses (4.3), one finds $h(O^{(\text{NS})}) = 1/2$. Such a correspondence between Ramond ground states and marginal operators was first pointed out in [47]. We will show below that in fact these operators $O^{(\text{NS})}$ in the NS sector are marginal perturbations preserving the special holonomies. We also note that $O^{(\text{NS})}_\mathcal{M}$ is the most relevant primary field since the Liouville exponential $e^{-\frac{Q \phi}{2}}$ has the maximum dimension.

Our remaining task is to confirm that the cosmological constant operators defined here are really marginal deformations preserving the spacetime SUSY. To this aim let us recall the discussions given in [47]. First of all, the SCFT characterizing $G_2$ holonomy contains the tri-critical Ising model, and the energy momentum tensor is decomposed as

$$T = T^{\text{tri}} + T^r,$$

$$T^{\text{tri}}(z) T^r(w) \sim 0,$$

where $T^{\text{tri}}$ and $T^r$ satisfy the Virasoro algebra with central charge $7/10$ and $49/5$. We express the conformal weights as $(h^{\text{tri}}, h^r)$ for $T^{\text{tri}}$ and $T^r$ respectively. We only treat here operators with the same right moving quantum numbers with the left moving ones, and focus only on the left movers.

As shown in [47], the deformation (4.1) preserves the spacetime supersymmetry and exactly marginal, if and only if $e^{-\frac{Q \phi}{2}} O^{(\text{NS})}_\mathcal{M}$ is a primary of the type $(h^{\text{tri}}, h^r) = \left(\frac{1}{10}, \frac{7}{5}\right)$. Its corresponding Ramond sector operator

$$O_\pm \phi e^{-\frac{Q \phi}{2}} O^{(\text{R})}_\mathcal{M},$$ (4.10)

(generated by the “spectral flow operator” $\left(\frac{7}{16}, 0\right)$) must then be an operator of the type $\left(\frac{3}{80}, \frac{2}{5}\right)$. The operator (4.10) is doubly degenerate because of the spin field $\sigma_\phi$, and we must take a suitable linear combination compatible with the GSO projection, namely, the mutual locality with spacetime supercharges.

In the $\text{Spin}(7)$ holonomy cases almost the same argument works. The energy momentum tensor can be decomposed into the Ising part and the rest, and we write the conformal weights as $(h^{\text{Isi}}, h^r)$. If and only if $e^{-\frac{Q \phi}{2}} O^{(\text{NS})}_\mathcal{M}$ is a primary of the type $(h^{\text{Isi}}, h^r) = \left(\frac{1}{16}, \frac{7}{16}\right)$, the deformation (4.1) is an exactly marginal operator preserving $\text{Spin}(7)$-holonomy. Moreover, such an NS primary corresponds to a Ramond state of the dimension $\left(\frac{1}{16}, \frac{7}{16}\right)$. 40
Now, let us confirm that these are indeed the cases for our cosmological constant operators (4.1). In the $G_2$ holonomy case, the Ramond ground state operator must be either $(\frac{3}{80}, \frac{2}{5})$ or $(\frac{7}{16}, 0)$. In our construction $T^{\text{tri}}$ is made up of sectors $U(1)_{3/2} \times \psi^\phi$ and does not contain the Liouville field $\phi$. We can hence conclude that $h^r \neq 0$ for the operator (4.10). Consequently the operator (4.10) must be of the $(\frac{3}{80}, \frac{2}{5})$ type.

In the $\text{Spin}(7)$ holonomy case, the problem is a little more subtle. There are three types of Ramond ground states: $(0, \frac{1}{2}), (\frac{1}{16}, 0), (\frac{1}{16}, \frac{7}{16})$. Since $h^r \neq 0$ holds by the same reason as $G_2$ holonomy case, the remaining possibilities are $(0, \frac{1}{2})$ or $(\frac{1}{16}, \frac{7}{16})$. Actually, the doubly degenerate operators (4.10) can be either of these types. However, it is possible to show that the operator $(\frac{1}{16}, \frac{7}{16})$, which has a mutually non-local OPE with the spectral flow operator $(\frac{1}{2}, 0)$, survives after the GSO projection.

In conclusion, the problem of enumerating marginal perturbations is reduced to the classification of Ramond ground states of the conformal theory $\mathcal{M}$ in the case of holonomies $G_2$ and $\text{Spin}(7)$. It turns out that this also amounts to enumerating conformal blocks $F_2^{(\tau)}(h = 1/2; \tau)$ defined in appendix B, appearing in the modular invariant partition functions. Each function in the NS sector $F_2^{(\tau)}(h = 1/2; \tau)$ is one-to-one correspondence with the operator $e^{-\frac{Q^\phi}{2} \phi} \mathcal{O}_M^{(\text{NS})}$ considered above.

We finally make a comment on the vacua with $N = 2$ worldsheet SUSY; 

$$\mathbb{R}_\phi \times \mathcal{M} \cong (\mathbb{R}_\phi \times S^1_\phi) \times \mathcal{M}/U(1),$$

(4.11)

where $\mathcal{M}/U(1)$ is assumed to be an $\mathcal{N} = 2$ SCFT and $\mathbb{R}_\phi \times S^1_\phi$ denotes the $\mathcal{N} = 2$ Liouville theory. It is not difficult to show that the cosmological constant operator (4.1) does not preserve the worldsheet $\mathcal{N} = 2$ SUSY. The easiest way to see it is to observe that the integrality of the $U(1)_R$-charge fails for the operator of the type (4.1). Thus we now need to shift the exponent $\gamma$ away from $-Q_\phi/2$. Typical operators which preserve $\mathcal{N} = 2$ SUSY are now written as:

$$\left[G_{-\frac{1}{2}}, \left[\bar{G}_{-\frac{1}{2}}, \mathcal{O}_M^{(\text{NS})} \exp \gamma (\phi + iY)\right]\right] + (\text{c.c.}),$$

$$-\frac{Q_\phi}{2} \gamma + h(\mathcal{O}_M^{(\text{NS})}) = \frac{1}{2},$$

(4.12)

where $\mathcal{O}_M^{(\text{NS})}$ denotes a (cc) chiral primary field in the $\mathcal{M}/U(1)$ sector. Again for the cases $\gamma > -Q_\phi/2$, (4.12) is non-normalizable and corresponds to a coupling constant, while the operators $\gamma < -Q_\phi/2$ describe the normalizable moduli that resolve the singularity. The most relevant primary is of course $\mathcal{O}_M^{(\text{NS})} = \text{id}$, which corresponds to $\gamma = -1/Q_\phi$ and the well-known Liouville potential for the $\mathcal{N} = 2$ Liouville theory. In the case of singular Calabi-Yau $n$-folds, all of these operators (4.12) are normalizable for $n = 2$ while non-normalizable for $n = 4$. In the $CY_3$ case, one “half” of them are normalizable and the remaining are non-normalizable. See for the detail [29, 31, 39, 40, 71].
4.2 Ramond Ground States in the $\mathcal{N} = 1$ Supercoset CFT’s

As we have shown, finding the possible marginal deformations preserving $\text{Spin}(7)$ or $G_2$ holonomies amounts to the classification of Ramond ground states in the supercoset theory $\mathcal{M}$. We start by summarizing how to analyse this problem. It is quite reminiscent of the analysis on the chiral rings in the Kazama-Suzuki models [76].

Let us recall the structure of $\mathcal{N} = 1$ coset (3.1) and again assume $H = H_0 \times H_1 \times \cdots \times H_r$, where $H_0$ denotes the abelian part and $H_i$ are simple parts. For each $H_i$ ($i \neq 0$), the conformal dimension of the highest weight state $\lambda^{(i)}$ is evaluated as

$$h(\lambda^{(i)}) = \frac{(\lambda^{(i)}, \lambda^{(i)} + 2\rho^{(i)})_i}{2(k_i + h_i^*)} = \frac{|\lambda^{(i)} + \rho^{(i)}|_i^2}{2(k_i + h_i^*)} + \frac{c_{H_i}}{24} - \frac{\dim H_i}{24},$$

where $\rho^{(i)}$ denotes the Weyl vector of $H^{(i)}$ and $k_i$ is defined in (3.3). The norm $| \cdot |_i$ is associated with the inner product $(\cdot, \cdot)_i$. We here made use of the well-known Freudenthal-de Vries strange formula: $(\rho^{(i)}, \rho^{(i)})_i = h_i^* \dim H_i / 12$, and $c_{H_i} \equiv \frac{k_i \dim H_i}{k_i + h_i^*}$ denotes the central charge of $(H_i)_{k_i}$. This formula also applies to the abelian part if we set $\rho^{(0)} = 0$. We thus find that the conformal dimension of the highest weight state $(\Lambda, s = 1, \{\lambda^{(i)}\})$ of the Ramond sector becomes

$$h(\Lambda, s = 1, \{\lambda^{(i)}\}) = \frac{(\Lambda, \Lambda + 2\rho_G)}{2(k + g^*)} + \frac{D}{16} - \sum_{i=0}^{r} \frac{(\lambda^{(i)}, \lambda^{(i)} + 2\rho^{(i)})_i}{2(k_i + h_i^*)},$$

$$= \frac{|\Lambda + \rho_G|^2}{2(k + g^*)} - \sum_{i=0}^{r} \frac{|\lambda^{(i)} + \rho^{(i)}|^2}{2(k + g^*)} + \frac{c_{\mathcal{M}}}{24},$$

where $\rho_G$ is the Weyl vector of $G$, $D \equiv \dim G/H$, and $c_{\mathcal{M}}$ is given in (3.4). This relation is valid only when the representation $\{\lambda^{(i)}\}$ is included in the representation $\Lambda \times ((\co)spinor)$ as embedding of finite dimensional Lie algebra $G \times SO(D) \supset H$. Therefore we find the relation for the Ramond ground state

$$|\Lambda + \rho_G|^2 - |\lambda + \rho_H|^2 = 0,$$

In this equation, we define $\lambda = \sum_i \lambda^{(i)}$ and $\rho_H = \sum_i \rho^{(i)}$. Formula (4.16) has been known in $\mathcal{N} = 2$ coset theories [76] and is now generalized to the case of $\mathcal{N} = 1$ coset theories with spacetime supersymmetry.
Note that the condition (4.16) does not depend on the level \( k \) of the affine algebra. The level \( k \) enters only through the restriction on the possible set of representations \( \Lambda \). Because of the unitarity, the \( \Lambda = \sum_i \Lambda_i \omega_i \) must satisfy the following relations.

\[
\Lambda_i \in \mathbb{Z}, \quad \Lambda_i \geq 0, \quad (\theta, \Lambda) \leq k .
\] (4.17)

Only the finite number of \( \Lambda \)'s satisfy these relations.

In the next subsection we explicitly analyse the spectra of Ramond ground states. We treat all the cases of simple group \( G \), and also the example \( \mathcal{M} = SU(2)^3/SU(2) \) of \( G_2 \) holonomy (3.47).

### 4.3 Spin(7) Holonomy Cases

We first study the Ramond ground states in the Spin(7) holonomy cases. The wanted states are labeled by \((\Lambda, s = 1, \lambda)\), where \( \Lambda \) is the integrable highest weight of \( G_k \) and \( \lambda \) is the integrable highest weight of \( H \). We also use the label \( \Lambda = \sum_i \Lambda_i \omega_i \) and \( \lambda = \sum_i \lambda_i \omega_i' \), where \( \omega_i \) and \( \omega_i' \) are the fundamental weights of \( G \) and \( H \) respectively.

1. \( SO(7)/G_2 \): The relevant supercoset is

\[
\mathcal{M} = \frac{SO(7)_k \times SO(7)}{(G_2)_{k+1}} .
\] (4.18)

Let us denote the highest weight of \( SO(7)_k, SO(7), (G_2)_{k+1} \) by \( \Lambda = \sum_{j=1}^3 \Lambda_j \omega_j, s = 0, 1, 2, -1, \lambda = \sum_{j=1}^3 \lambda_j \omega_j' \), respectively. In this case, the Ramond ground states satisfy

\[
\Lambda_3 = 2\Lambda_1 + 1 , \quad \lambda_1 = \Lambda_2 , \quad \lambda_2 = \Lambda_1 + \Lambda_3 + 1 ,
\] (4.19)

in addition to the relations (4.17). The simplest example of the Ramond ground states is \( \Lambda_1 = 1, \Lambda_2 = \Lambda_3 = 0 \), which means \( \Lambda \) is the highest weight of spinor representation of \( SO(7) \) and \( \lambda \) is the highest weight of 27 dimensional representation of \( G_2 \).

2. \( SU(3)/U(1) \): The relevant supercoset is written as

\[
\mathcal{M} = \frac{SU(3)_k \times SO(7)}{U(1)} ,
\] (4.20)

and the level of \( U(1) \) is determined when we fix the embedding. Consider the \( N(m, \ell) \) type coset where \( U(1) \) lies along the direction \( \nu = (m - \ell) \omega_1 + (m + 2\ell) \omega_2 \). We set \( m \) and \( \ell \) relatively prime, and \( (m - \ell) \geq 0, (m + 2\ell) \geq 0 \). If we use
a canonically normalized $U(1)$ charge $p$ (which means that the dimension of an exponential operator becomes $p^2/2$), the Ramond ground state is obtained if it satisfies either of the following two conditions

(a) $(m - \ell)(\Lambda_2 + 1) = (m + 2\ell)(\Lambda_1 + 1)$, \hspace{1cm} $p = \frac{m(\Lambda_1 + 1) + (m + \ell)(\Lambda_2 + 1)}{\sqrt{2(k+3)(m^2 + \ell^2 - m\ell)}}$. \hspace{1cm} (4.21)

(b) $(m - \ell)(\Lambda_1 + 1) = (m + 2\ell)(\Lambda_2 + 1)$, \hspace{1cm} $p = -\frac{m(\Lambda_2 + 1) + (m + \ell)(\Lambda_1 + 1)}{\sqrt{2(k+3)(m^2 + \ell^2 - m\ell)}}$. \hspace{1cm} (4.22)

As we already mentioned, we have the $\mathcal{N} = 4$ enhanced SUSY for the special case of $m = \ell = 1$, and the $\mathcal{N} = 1$ cosmological terms (4.1) cannot be applied in this case. We would like to further discuss this point elsewhere.

3. $SO(5)/SO(3)_{\text{max}}$: The relevant supercoset is expressed as (3.60):

$$\mathcal{M} = \frac{SO(5)_k \times SO(7)_1}{SU(2)_{10k+28}}.$$ \hspace{1cm} (4.23)

If we denote by $(\ell+1)$ the the dimension of the representation of $SU(2)$, the Ramond ground state satisfies the relation

$$\Lambda_2 = 2\Lambda_1 + 1, \quad \ell = 2\Lambda_1 + 5\Lambda_2 + 6.$$ \hspace{1cm} (4.24)

The number of the Ramond ground state is evaluated as

$$\left[ \frac{k+1}{3} \right],$$

where $[\ast]$ denotes the Gauss symbol.

The simplest example of the Ramond ground state is $\Lambda_1 = 0, \Lambda_2 = 1$, which means $\Lambda$ is the highest weight of spinor representation of $SO(5)$ and $\lambda$ is the highest weight of 12 dimensional representation of $SU(2)$.

### 4.4 $G_2$ Holonomy Cases

We next analyse the Ramond ground states for the $G_2$ holonomy case. In these cases, the coset fermions $SO(6)_1$ has two representations in Ramond sector: spinor and cospinor. We express these representation by $s = \pm 1$ respectively.
1. $G_2/SU(3)$: The relevant supercoset is given as

\[ \mathcal{M} = \frac{(G_2)_k \times SO(6)_1}{SU(3)_{k+1}}. \]  

(4.25)

A Ramond ground state is obtained if either one of the following conditions is satisfied

(a) \( s = -1, \quad \lambda_1 = \Lambda_1, \quad \lambda_2 = \Lambda_1 + \Lambda_2 + 1 \).

(4.26)

(b) \( s = 1, \quad \lambda_2 = \Lambda_1, \quad \lambda_1 = \Lambda_1 + \Lambda_2 + 1 \).

(4.27)

The number of the Ramond ground states is evaluated as follows;

\[ \frac{1}{2}(k + 2)(k + 3) - \left[ \frac{k + 2}{2} \right]. \]

(4.28)

The simplest Ramond ground states come from the basic representation of $G_2$, and represented as $(\Lambda = 0, s = -1, \lambda_1 = 0, \lambda_2 = 1)$ and $(\Lambda = 0, s = +1, \lambda_1 = 1, \lambda_2 = 0)$.

2. $SO(5)/(SU(2) \times U(1))$: The relevant coset corresponds to the (non-HSS) $\mathbb{CP}^3$ (3.32);

\[ \mathcal{M} = \frac{SO(5)_k \times SO(6)_1}{SU(2)_{k+1} \times U(1)_{k+3}}. \]

(4.29)

We denote by $(\ell + 1)$ the dimension of the representation of $SU(2)$, and by $m$ the charge of $U(1)$ normalized so that the conformal dimension of its vertex operator is given by $\frac{m^2}{4(k+3)}$. In this notation, a Ramond ground state is obtained if it satisfies one of the following four conditions

(a) \( s = -1, \quad \ell = \Lambda_1, \quad m = \Lambda_1 + \Lambda_2 + 2 \).

(4.30)

(b) \( s = +1, \quad \ell = \Lambda_1 + \Lambda_2 + 1, \quad m = \Lambda_1 + 1 \).

(4.31)

(c) \( s = -1, \quad \ell = \Lambda_1 + \Lambda_2 + 1, \quad m = -\Lambda_1 - 1 \).

(4.32)

(d) \( s = +1, \quad \ell = \Lambda_1, \quad m = -\Lambda_1 - \Lambda_2 - 2 \).

(4.33)

As a result, there are four Ramond ground states for each representation $\Lambda$ of $SO(5)$. However there are the field identifications originating from the outer automorphism $\mathbb{Z}_2$;

\[ (\Lambda_1, \Lambda_2, s, \ell, m) \cong (k - \Lambda_1 - \Lambda_2, \Lambda_2, s + 2, k - \ell, m + (k + 3)). \]

(4.34)
Hence, the number of Ramond ground states is given by

\[(k + 1)(k + 2).\]  

(4.35)

Simplest ground states are comes from the basic representation of \(SO(5)\): \((\Lambda = 0, s = 1, \ell = 1, m = 1)\) and \((\Lambda = 0, s = 1, \ell = 0, m = -2)\).

3. \(SU(3)/U(1)^2\) : The relevant coset is characterized by the charge lattice \(\Gamma\) of \(U(1)^2\). We here only consider the simplest case \(\Gamma = \mathbb{Z}\nu_1 + \mathbb{Z}\nu_2\), where \(\nu_1\) and \(\nu_2\) are defined in (3.35), leading to

\[\mathcal{M} = \frac{SU(3)_k \times SO(6)_1}{U(1)_{k+3} \times U(1)_{3(k+3)}}.\]  

(4.36)

We label the \(U(1)\) charge by \(\lambda = \lambda_1\omega_1 + \lambda_2\omega_2\) normalized so that the dimension of the vertex operator is given by \(\frac{(\lambda, \lambda)}{2(k+3)}\). The Ramond ground states satisfy the condition

\[w(\Lambda + \rho) = \lambda, \quad s = -\epsilon(w),\]  

(4.37)

for an element \(w\) of the \(SU(3)\) Weyl group, where \(\epsilon(w)\) is the signature of \(w\). We here note a slightly non-trivial point. The charge lattice \(\Gamma\) here is not invariant under the outer automorphism \(\mathbb{Z}_3\) which usually implies the necessity of field identification. We thus should consider the spectrum without field identification contrary to the standard Kazama-Suzuki coset \(\Gamma = \sqrt{k+3}Q\). Consequently, the number of the Ramond ground states becomes

\[3(k + 1)(k + 2).\]  

(4.38)

The simplest Ramond ground states are expressed as \((\Lambda = 0, s = -1, \lambda_1 = 1, \lambda_2 = 1)\) and its Weyl transforms.

4. \(SU(2)^3/SU(2)\) : The supercoset is expressed as (3.47);

\[\mathcal{M} = \frac{SU(2)_{k_1} \times SU(2)_{k_2} \times SU(2)_{k_3} \times SO(6)_1}{SU(2)_{k_1+k_2+k_3+4}}.\]  

(4.39)

Let us denote the dimension of representation by \(\ell_j + 1, \ (j = 1, 2, 3)\) for each \(SU(2)\) factor in the numerator, and the one in the denominator by \(\ell_4 + 1\) One obtains a Ramond ground state if

\[\frac{\ell_1 + 1}{k_1 + 2} = \frac{\ell_2 + 1}{k_2 + 2} = \frac{\ell_3 + 1}{k_3 + 2},\]  

\[\ell_4 = \ell_1 + \ell_2 + \ell_3 + 3, \quad s = \pm 1.\]  

(4.40)

are satisfied. In this case, by denoting the greatest common divisor of \(\{k_j + 2\}\) as \(p\), the number of Ramond ground states becomes equal to \((p - 1)\), when we take a suitable field identification into account.
5 Discussions

In this paper we have investigated aspects of superstring vacua of the type:

\[ \mathbb{R}^{d-1,1} \times (\mathcal{N} = 1 \text{ Liouville}) \times (\mathcal{N} = 1 \text{ supercoset CFT on } G/H) , \]

motivated by an analogy with the geometrical cone constructions of special holonomy manifolds over the Einstein homogeneous spaces \( G/H \). We made an almost exhaustive analysis and obtained results which led in most cases to the same amount of supersymmetries as expected from the geometrical approach.

While these results seem satisfactory, we should also emphasize the obvious difference between our and geometrical approaches. In the non-linear \( \sigma \)-model on the geometrical cone over \( G/H \), the physics depends on the choice of its metric, and possibly on the global topology. On the other hand, in our algebraic approach based on the coset CFT's, the vacua are defined associated with the affine Lie algebras for \( G, H \) and further with the choice of embedding of \( \text{Lie}(H) \) into \( \text{Lie}(G) \). We do not use information on the metric structure and global topology of the manifold \( G/H \) explicitly. (It will be interesting to check if one obtains metrics used in the geometrical cone constructions when one computes the natural metric on homogeneous space making use of our embedding of \( H \) into \( G \) described in various examples). Actually we often encountered examples where the same manifold has several different coset realizations. They are of course equivalent in the sense of non-linear \( \sigma \)-model, but \emph{not} necessarily equivalent as the coset CFT's. The typical example is the case of round sphere. The geometrical cone over a round sphere is a trivial flat space and leads to a vacua with a maximal amount of SUSY, while we have several inequivalent coset CFT's based on a round sphere possessing different amount of supercharges.

Important questions for our analysis may as follows:

1. How can we identify our string vacua with the known solutions in supergravity theories?

2. How can we attach a physical meaning to the levels of current algebras in our construction?

3. What is the precise geometrical interpretation of the Ramond ground states discussed in section 4?

These problems are deeply related with each other and seem difficult in particular in the cases of \( G_2 \) and \( \text{Spin}(7) \) holonomies.

However, we now would like to point out some suggestive examples: For the \( d = 4 \) vacua with \( \mathcal{M} = (SU(2)_{k_1} \times SU(2)_{k_2})/U(1) \), we observed that the spacetime SUSY is
allowed only for the case of $U(1)$ embedding $p = q$ in (3.21) which corresponds to the Einstein homogeneous space $T^{1,1}$. Moreover, it is easy to see that the simplest case $k_1 = k_2 = 0$ is equivalent to the string theory on conifold as discussed in [26]. The brane construction of conifold was explored in [80] based on the ALE fibration and the well-known ALE-NS5 correspondence [27, 68]. It is realized as a system of two intersecting NS5-branes, both of which fill the 4-dimensional Minkowski spacetime. Because the level of WZW-model is translated into the brane charge of NS5, it is natural to suppose that the higher level models $\mathcal{M} = (SU(2)_{k_1} \times SU(2)_{k_2})/U(1)$ could be associated with a configuration of intersecting stacks of NS5-branes with brane charges $k_1$ and $k_2$. In fact, it is known [29, 33] that the vacua with $k_1 = k, k_2 = 0$ are understood as the ADE hierarchy of rational singularities, or the wrapped NS5 branes with charge $\approx k$.

On the other hand, the brane interpretation for the known supergravity solutions of $G_2$ (and $Spin(7)$) holonomy have been discussed in [5, 81]. In [5] the relevant brane configurations are identified as $D6$-brane wrapped around special Lagrangian cycles (“L-pictures”), which are again reinterpreted as the intersecting NS5-branes by some duality web [81]. The cases of intersecting stacks of NS5-branes still describe the vacua with the correct number of spacetime SUSY’s, although the explicit supergravity solutions are not available. It seems again plausible to assume that the charges of stacked NS5-branes are incorporated as the levels of WZW models in our construction.

Encouraged by these observations, we conjecture for the string vacua with $G_2$ and $Spin(7)$ holonomies:

1. Our coset CFT models for the nearly Kähler (weak $G_2$) homogeneous spaces $G_k/H$ should be identified as the geometrical $G_2$ ($Spin(7)$) cones over them in the special case of zero level $k = 0$.

2. The non-zero level models could be associated with the same configurations of intersecting NS5-branes as for the geometrical $G_2$ ($Spin(7)$) cones considered above, with the suitable numbers of stacked NS5-branes.

3. Marginal deformations discussed in section 4 correspond to various motions of intersecting NS5-branes preserving the spacetime SUSY.

In this paper we have assumed the special value of the Liouville momentum $\gamma = -Q_\phi/2$ and have constructed associated marginal perturbation operators. An obvious question is if one can construct other marginal operators with a different choice for the momentum. It seems that this is a problem whose answer depends to some extent on how one defines
the Liouville theory. As is well-known there are two branches for the Liouville momentum

\[
\begin{align*}
(i) \quad \text{"discrete series" :} & \quad \gamma \in \mathbb{R} \\
(ii) \quad \text{"principal continuous series" :} & \quad \gamma = -\frac{Q\phi}{2} + ip, \quad p \in \mathbb{R}
\end{align*}
\]

and our choice \(-Q\phi/2\) was special in the sense that with this value of \(\gamma\) the dimension of the Liouville exponential \(e^{\gamma\phi}\) is maximum for the series (i) and minimum for the series (ii). Relevant range of the “discrete series” here is \(\gamma < -Q\phi/2\) which provides operators peaked around the tip of the cone and possibly resolve its singular behavior.

When one takes the Liouville momentum from the “discrete series” \(\gamma < -Q\phi/2\), the dimension \(h(e^{\gamma\phi})\) could become negative and large as \(-\gamma\) becomes large. One may possibly construct marginal operators by pairing such an exponential with a field from the matter sector \(\mathcal{M}\) with a large positive dimension. However, such a possibility seems rather unphysical since the Liouville exponential with a large negative \(\gamma\) should effect strongly the physics around \(\phi \approx -\infty\) while matter fields of high dimensions should be largely irrelevant.

Unitarity of the Liouville theory is a difficult and subtle problem, however, it seems generally agreed that the theory becomes unitary when one restricts oneself to the sector of “principal continuous series”. If one takes this point of view, the value \(\gamma = -Q\phi/2\) appears to be the unique choice for marginal operators since this is the value for which the operator \(e^{\gamma\phi}\) is real and has the lowest dimension. In fact the central charge of the \(\mathcal{N} = 1\) Liouville system is given by \(c_L = 3/2 + 3Q^2\) and the operator \(\sigma^\phi e^{-Q\phi/2}\phi\) saturates its BPS bound \(c_L/24 = 1/16 + Q^2/8\).

Thus we further conjecture

4. Marginal operators we have constructed exhaust all possible marginal perturbations of the \(\mathcal{N} = 1\) coset model of string vacua for manifolds with special holonomy.

In order to discuss these conjectures it is quite important to establish the dictionary translating the cosmological constant operators into the geometrical data. To this aim, a possible future direction may be the boundary state analysis along the line similar to [71]. It may be also interesting and challenging to try to generalize the concept of chiral rings characteristic for the \(\mathcal{N} = 2\) string vacua [76]. It is quite suggestive that our cosmological constant operators are defined in one-to-one correspondence with the Ramond ground states for the “base” \(\mathcal{M}\), implying some cohomological structure behind the system. The approach based on the real Landau-Ginzburg theory may also be significant.
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Appendix A : Notations

Set $q := e^{2\pi i \tau}$, $y := e^{2\pi i z}$;

\[ \theta_1(\tau, z) = i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2/2} y^{n-1/2} \equiv 2 \sin(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 - yq^m)(1 - y^{-1}q^m) , \]
\[ \theta_2(\tau, z) = \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2} y^{n-1/2} \equiv 2 \cos(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 + yq^m)(1 + y^{-1}q^m) , \]
\[ \theta_3(\tau, z) = \sum_{n=-\infty}^{\infty} q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1 - q^m)(1 + yq^{-m})(1 + y^{-1}q^{-m}) , \]
\[ \theta_4(\tau, z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1 - q^m)(1 - yq^{-m})(1 - y^{-1}q^{-m}) . \]  
(A.1)

\[ \Theta_{m,k}(\tau, z) = \sum_{n=-\infty}^{\infty} q^{k(n+m\tau)^2} y^{k(n+m\tau)} , \]  
(A.2)

\[ \tilde{\Theta}_{m,k}(\tau, z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{k(n+\frac{\pi}{2\tau})^2} y^{k(n+\frac{\pi}{2\tau})} . \]  
(A.3)

We often use the abbreviations; $\theta_i \equiv \theta_i(\tau, 0)$ ($\theta_1 \equiv 0$), $\Theta_{m,k}(\tau) \equiv \Theta_{m,k}(\tau, 0)$, $\tilde{\Theta}_{m,k}(\tau) \equiv \tilde{\Theta}_{m,k}(\tau, 0)$. We also set

\[ \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) . \]  
(A.4)

The product formula of theta functions is useful for our analysis;

\[ \Theta_{m,k}(\tau, z)\Theta_{m',k'}(\tau, z') = \sum_{r \in \mathbb{Z}_{k+k'}} \Theta_{mk'-m'k+2kk',kk'(k+k')} (\tau, u) \Theta_{m+m'+2kr,k+k'} (\tau, v) , \]  
(A.5)

where we set $u = \frac{z - z'}{k + k'}$, $v = \frac{kz + k'z'}{k + k'}$.

The next formula is also useful

\[ \Theta_{m/p,k/p}(\tau, z) = \Theta_{m,k}(\tau/p, z/p) = \sum_{r \in \mathbb{Z}_p} \Theta_{m+2kr,pk}(\tau, z/p) , \]  
(A.6)
where $m, k$ are some real numbers and $p$ is an integer.

**Appendix B : Massive Characters of the Extended Chiral Algebras Associated to Special Holonomies**

Here we summarize the massive character formulas of the extended chiral algebras characterizing the special holonomy manifolds. For simplicity we shall only focus on the NS sector. The characters for other spin structures are obtained by making the half-integral spectral flows as follows;

\[
\begin{align*}
\text{Ch}^{(\tilde{\text{NS}})}(\ast; \tau, z) &= \text{Ch}^{(\text{NS})}(\ast; \tau, z + \frac{1}{2}), \\
\text{Ch}^{(R)}(\ast; \tau, z) &= q^{\frac{c}{12}} y^{-\frac{c}{6}} \text{Ch}^{(\text{NS})}(\ast; \tau, z + \frac{\tau}{2}), \\
\text{Ch}^{(\tilde{R})}(\ast; \tau, z) &= q^{\frac{c}{12}} y^{-\frac{c}{6}} \text{Ch}^{(\text{NS})}(\ast; \tau, z + \frac{\tau}{2} + \frac{1}{2}).
\end{align*}
\]

(B.1)

1. $Sp(k)$-holonomy ($k = 1, 2$) :

   This case corresponds to hyper Kähler manifolds of the real dimension $4k$. The relevant chiral algebra is the (small) $\mathcal{N} = 4$ superconformal algebra with the level $k$ ($c = 6k$), as is well-known. The massive representations are labeled by the conformal weight $h$ and the $SU(2)$ spin $\ell/2$, and the unitarity requires the constraints $h \geq \ell/2$. The character formulas are given in [59];

\[
\text{Ch}^{(\text{NS})}(h, \ell; \tau, z) = \frac{q^{\frac{h^2}{3(4k+1)}} \left( \frac{\theta_3(\tau, z)}{\eta(\tau)} \right)^2}{\chi^{(k)}_{\ell}(\tau, z)}, \quad (0 \leq \ell \leq k-1),
\]

(B.2)

where $\chi^{(k)}_{\ell}(\tau, z)$ denotes the character of $SU(2)_k$ with the spin $\ell/2$ ($0 \leq \ell \leq k$) representation;

\[
\chi^{(k)}_{\ell}(\tau, z) = \frac{\Theta_{\ell+1,k+2} - \Theta_{-\ell-1,k+2}}{\Theta_{1,2} - \Theta_{-1,2}}(\tau, z).
\]

(B.3)

We note that

\[
\begin{align*}
\chi^{(1)}_{0}(\tau, z) &= \frac{\Theta_{0,1}(\tau, 2z)}{\eta(\tau)} \\
\chi^{(1)}_{1}(\tau, z) &= \frac{\Theta_{1,1}(\tau, 2z)}{\eta(\tau)}.
\end{align*}
\]

(B.4)

2. $SU(n)$-holonomy :


This case corresponds to the Calabi-Yau $n$-fold ($\text{CY}_n$) compactification. The extended chiral algebras are defined by adding the integral spectral flow operator, which corresponds to the holomorphic $n$-form in $\text{CY}_n$, to the $\mathcal{N} = 2$ superconformal algebra with $c = 3n$. These conformal algebras are not Lie algebras but are W-algebras except for the $SU(2)$-holonomy case, which of course reduces to the above $Sp(1)$ case. For the $SU(3)$-holonomy the spectral flow operator generates the subsector described by $SO(6)/SU(3) \cong U(1)_{3/2}$, which is equivalent with the $\mathcal{N} = 2$ minimal model of level 1. The corresponding subsector for the $SU(4)$-holonomy is $SO(8)/SU(4) \cong U(1)_2$, which is equivalent with the conformal system of a complex fermion.

The massive representations are labeled by the conformal weight $h$ and the $U(1)_R$ charge $Q$ with the unitarity condition $h \geq Q/2$. The character formulas for $SU(3)$-holonomy are given in [60], and those for $SU(4)$-holonomy are given in [62].

- **$SU(3)$-holonomy**: We have two continuous series;

\[
\text{Ch}^{(\text{NS})}(h, Q = 0; \tau, z) = \frac{g^{h-\frac{1}{2}} \theta_3(\tau, z) \Theta_{0,1}(\tau, 2z)}{\eta(\tau)} \frac{\eta(\tau)}{\eta(\tau)}, \\
\text{Ch}^{(\text{NS})}(h, |Q| = 1; \tau, z) = \frac{g^{h-\frac{1}{2}} \theta_3(\tau, z) \Theta_{1,1}(\tau, 2z)}{\eta(\tau)} \frac{\eta(\tau)}{\eta(\tau)}.
\] (B.5)

The vacuum state is doubly degenerate for the second case ($Q = 1$ and $Q = -1$).

- **$SU(4)$-holonomy**: We have three continuous series;

\[
\text{Ch}^{(\text{NS})}(h, Q = 0; \tau, z) = \frac{g^{h-\frac{1}{2}} \theta_3(\tau, z) \Theta_{0,\frac{1}{2}}(\tau, 2z)}{\eta(\tau)} \frac{\eta(\tau)}{\eta(\tau)}, \\
\text{Ch}^{(\text{NS})}(h, Q = \pm 1; \tau, z) = \frac{g^{h-\frac{13}{24}} \theta_3(\tau, z) \Theta_{\pm 1,\frac{1}{2}}(\tau, 2z)}{\eta(\tau)} \frac{\eta(\tau)}{\eta(\tau)}.
\] (B.6)

3. **$G_2$ and $Spin(7)$-holonomies**

The chiral algebras associated to the $G_2$ and $Spin(7)$-holonomies are again the W-algebra like extensions of $\mathcal{N} = 1$ superconformal algebra explicitly defined in [47]\textsuperscript{10}. A characteristic feature of the $G_2$ extended algebra is the existence of the tri-critical Ising model ($\cong SO(7)/G_2$), and that for the $Spin(7)$ extended algebra is the Ising model ($\cong SO(8)/SO(7)$) as discussed in [47]. The unitary massive representations for these algebras are classified in [49, 50]: two continuous series with $h \geq 0$ and $h \geq 1/2$ exist for the each case. Unfortunately, the character formulas for these representations have

\textsuperscript{10}In some literature the $G_2$ extended algebra is denoted as $SW(3/2, 3/2, 2)$-algebra ($c = 21/2$) and the $Spin(7)$ extended algebra is denoted as $SW(3/2, 2)$-algebra ($c = 12$).
not been worked out. Nevertheless, based on the existence of spacetime SUSY as well as
the worldsheet \( \mathcal{N} = 1 \) superconformal symmetry, it seems plausible to propose that the
conformal blocks corresponding to the massive representations should be expanded into
the following functions:

- **Spin(7)**-holonomy : For the NS sector,

\[
F_{1}^{(\text{NS})}(h; \tau) = \frac{q^{h-\frac{49}{120}}}{\eta(\tau)} \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} \chi_{0,\text{tri,NS}}(\tau)
\]

\[
\equiv \frac{q^{h-\frac{49}{120}}}{\eta(\tau)} \left( \chi_{0,\text{Ising}}(\tau) \chi_{0}(\tau) + \chi_{1/2,\text{Ising}}(\tau) \chi_{1/2}(\tau) + \chi_{1/16,\text{Ising}}(\tau) \chi_{1/16}(\tau) \right),
\]

\((h \geq 0)\) (B.7)

\[
F_{2}^{(\text{NS})}(h; \tau) = \frac{q^{h-\frac{49}{120}}}{\eta(\tau)} \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} \chi_{1/10,\text{tri,NS}}(\tau)
\]

\[
\equiv \frac{q^{h-\frac{49}{120}}}{\eta(\tau)} \left( \chi_{0,\text{Ising}}(\tau) \chi_{3/5}(\tau) + \chi_{1/2,\text{Ising}}(\tau) \chi_{1/10}(\tau) + \chi_{1/16,\text{Ising}}(\tau) \chi_{3/80}(\tau) \right),
\]

\((h \geq 1/2)\) (B.8)

where we have written \( \chi_{h,\text{tri,NS}} \), \( \chi_{h,\text{tri}} \) for the \( \mathcal{N} = 1 \) and \( \mathcal{N} = 0 \) characters of the
tri-critical Ising model, and \( \chi_{h,\text{Ising}} \) for the Ising model. The second lines are consistent with the decompositions of massive representations of the Spin(7) extended
algebra with respect to the Virasoro modules of Ising model presented in [49]. This
fact suggests that the functions \( F_{i}^{(\text{NS})}(h; \tau) \) may in fact be the massive characters
although we have not been able to prove this.

- **G2**-holonomy : For the NS sector, the wanted functions are given as

\[
F_{1}^{(\text{NS})}(h; \tau) = \frac{q^{h-\frac{49}{120}}}{\eta(\tau)} \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} \eta(\tau)
\]

\[
\equiv \frac{q^{h-\frac{49}{120}}}{\eta(\tau)} \left( \chi_{0,\text{tri,NS}} \chi_{0,\text{Potts}} + \chi_{1/10,\text{tri,NS}} \chi_{2/5,\text{Potts}} \right)(\tau), \quad (h \geq 0)
\]

\[
F_{2}^{(\text{NS})}(h; \tau) = \frac{q^{h-\frac{49}{120}}}{\eta(\tau)} \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} \eta(\tau)
\]

\[
\equiv \frac{q^{h-\frac{49}{120}}}{\eta(\tau)} \left( \chi_{0,\text{tri,NS}} \chi_{2/3,\text{Potts}} + \chi_{1/10,\text{tri,NS}} \chi_{1/15,\text{Potts}} \right)(\tau), \quad (h \geq 1/2)\) (B.8)
where $\chi^{\text{tri. (NS)}}_h$ again means the $\mathcal{N} = 1$ character of tri-critical Ising and $\chi^\text{Potts}_h$ denotes the character of the 3-state Potts model ($\mathbb{Z}_3$ Parafermion) that is defined as the coset CFT $SU(2)_3/U(1)_3$. The second lines are easily derived using the equivalence

$$SO(1)_1 \times \frac{SO(6)_1}{SU(3)_1} \cong \frac{SO(7)_1}{(G_2)_1} \times \frac{(G_2)_1}{SU(3)_1}$$

$$\cong \text{tri-critical Ising} \times \text{3-state Potts}. \quad (B.9)$$

They are consistent with the structures of unitary massive representations given in [50].
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