INFINITELY MANY ALGEBRAS DERIVED EQUIVALENT TO A BLOCK

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1. Introduction

Let $k$ be an algebraically closed field of characteristic $p > 0$, let $D$ be a finite $p$-group. Broué’s Abelian Defect Conjecture [Bro] predicts that if $D$ is abelian and $G$ is a finite group with $D \leq G$, and $B$ is a block algebra of $kG$ with defect group $D$, then $B$ is derived equivalent to its Brauer correspondent, a block algebra of $N_G(D)$ which also has defect group $D$.

Donovan’s Conjecture predicts that there are, up to Morita equivalence, only finitely many different block algebras of finite groups with defect group $D$.

Both of these conjectures are still open, although progress has been made on special cases of both. It is natural to ask (and a number of people have asked) if they are related in the following way: if it were true that there are only finitely many Morita equivalence classes of algebras derived equivalent to each block of defect $D$, then Broué’s Conjecture would imply Donovan’s Conjecture for abelian $D$.

In some small cases, this is true. If $D$ is cyclic, every block with defect $D$ is Morita equivalent to a Brauer tree algebra, and it is known that the only algebras derived equivalent to a Brauer tree algebra are also Morita equivalent to Brauer tree algebras for trees with the same number of edges and same multiplicity of the exceptional vertex, so there are only finitely many possibilities. If $D$ is a Klein 4-group, then it is also known that there are only a very small number of Morita classes of algebras derived equivalent to any block with defect group $D$ (and in fact all of these occur as blocks of group algebras.

However, for larger $D$ it seems that this is very rarely true. In fact, we believe that the cases mentioned above are probably the only cases where this is true.

In this paper, we give a method of showing by a fairly simple calculation that a given block algebra has infinitely many Morita equivalence classes of algebras derived equivalent to it, and show that the method applies to several blocks with small abelian defect group.

In fact, our method produces a sequence of algebras with unbounded Cartan invariants (which is how we can detect that there are infinitely many Morita equivalence classes. A weaker version of Donovan’s Conjecture states that the blocks with a given defect group $D$ have bounded Cartan invariants, and this means that even this weaker version wouldn’t follow in a straightforward way from Broué’s Conjecture.

2. The main theorem

We fix a field $k$, and a finite-dimensional algebra $A$ over $k$.

By a “module” for such an algebra, we shall always mean a right module unless specified otherwise.
If $X$ is an object of an additive category, then by $\text{add}(X)$ we mean the category of direct summands of finite direct sums of copies of $X$.

Let $A$ be a $k$-algebra. By $\text{mod } (A)$ we denote the category of finitely generated $A$-modules, and by $P_A$ the category of finitely generated $A$-modules. So, if $A A$ is the regular $A$-module, considered as an object of $\text{mod } (A)$, then $P_A$ is just $\text{add}(A A)$. We also identify $\text{mod } (A)$ in the usual way with the full subcategory of $D^b(A)$ consisting of complexes concentrated in degree zero.

By $K^b(P_A)$ we denote the homotopy category of bounded complexes over $P_A$, and by $D^b(A)$ the bounded derived category of $A$-modules. We regard $K^b(P_A)$ as a full subcategory of $D^b(A)$ in the usual way.

Let $\{S_i : i \in I\}$ be a set of representatives of the isomorphism classes of simple $A$-modules, and for each $i \in I$ let $P_i$ be a projective cover of $S_i$.

Recall [Ri1] that a tilting complex for $A$ is an object $T$ of $K^b(P_A)$ such that

$$\text{Hom}_{K^b(P_A)}(T, T[n]) = 0$$

if $n \neq 0$ and such that $\text{add}(T)$ generates $K^b(P_A)$ as a triangulated category, and that a $k$-algebra $B$ is derived equivalent to $A$ if and only if $B \cong \text{End}_{K^b(P_A)}(T)$ for $T$ some tilting complex for $A$.

The following construction of a tilting complex for a symmetric $k$-algebra was described in [Ri2] and [Oku], but since neither of these has been published (at least, not in English), we shall give details here.

First, recall [Ri4 Corollary 3.2] the following duality for finite-dimensional symmetric $k$-algebras.

**Lemma 2.1.** Let $A$ be a finite-dimensional symmetric $k$-algebra, and $X, Y$ objects of $K^b(P_A)$. Then $\text{Hom}_{K^b(P_A)}(X, Y)$ is dual to $\text{Hom}_{K^b(P_A)}(X, Y)$, and hence (applying this with $Y[n]$ in place of $Y$), $\text{Hom}_{K^b(P_A)}(X, Y[n])$ is dual to $\text{Hom}_{K^b(P_A)}(Y, X[-n])$.

Let $I_0$ be a subset of the set $I$ indexing the simple $A$-modules, and let $i \in I$.

If $i \in I_0$, let $T_i$ be the object

$$\cdots \rightarrow 0 \rightarrow P_i \rightarrow 0 \rightarrow 0 \rightarrow \cdots$$

of $K^b(P_A)$, where $P_i$ is in degree 1.

If $i \in I \setminus I_0$, then let $T_i$ be the object

$$\cdots 0 \rightarrow R_i \rightarrow P_i \rightarrow 0$$

of $K^b(P_A)$, where $P_i$ is in degree zero, and $R_i \rightarrow P_i \rightarrow M_i \rightarrow 0$ is the minimal projective presentation of the largest quotient $M_i$ of $P_i$ with no composition factors isomorphic to elements of $\{S_j : j \in I_0\}$ (or equivalently, $R_i$ is the projective cover of the largest submodule of $P_i$ whose simple quotients are all isomorphic to elements of $\{S_j : j \in I_0\}$). Thus, in particular, $R_i$ is a direct sum of indecomposable projective modules from $\{P_j : j \in I_0\}$ and every map $P_j \rightarrow P_i$, with $j \in I_0$, factors through $R_i \rightarrow P_i$.

**Proposition 2.2.** If $A$ is a finite-dimensional symmetric $k$-algebra, then

$$T = \bigoplus_{i \in I} T_i$$

is a tilting complex for $A$. 
Let\( B = \text{End}_{K^h(P_A)}(T) \), so that there is an equivalence of derived categories between \( F : D^b(A) \to D^b(B) \) sending the objects \( T_i \) to the indecomposable projective \( B \)-modules. In this way the isomorphism classes of indecomposable projective \( B \)-modules are naturally indexed by \( I \), with the projective indexed by \( i \in I_0 \) being \( F(P_i)[1] \).

Since the class of finite-dimensional symmetric \( k \)-algebras is closed under derived equivalence \cite[Corollary 5.3]{Rin} we can iterate this construction, using the same subset \( I_0 \subset I \), obtaining after \( t \) iterations a tilting complex

\[
T^{(t)} = \bigoplus_{i \in I} T_i^{(t)}
\]

for \( A \), whose indecomposable summands are \( T_i^{(t)} = P_i[t] \) for \( i \in I_0 \) and

\[
T_i^{(t)} = \cdots \to 0 \to R_i^{(t-1)} \to R_i^{(t-2)} \to \cdots \to R_i^{(0)} \to P_i \to 0 \to \ldots
\]

(with \( R_i \) in degree zero) for \( i \in I \setminus I_0 \), where \( R_i^{(0)} \to P_i \) is the map \( R_i \to P_i \) defined above and for \( k > 0 \), \( R_i^{(k)} \) is the projective cover of the largest submodule of the kernel of the next differential whose simple quotients are all isomorphic to elements of \( \{S_i : i \in I_0\} \).

Thus, if we set

\[
B^{(t)} = \text{End}_{K^h(P_A)}(T^{(t)}),
\]

we obtain a sequence \( B = B^{(0)}, B^{(1)}, \ldots \) of algebras that are derived equivalent to \( A \). For \( i \in I \), let \( P_i^{(t)} \) be the indecomposable projective \( B^{(t)} \)-module corresponding to \( T_i^{(t)} \) under the derived equivalence between \( B^{(t)} \) and \( A \).

We shall give a technique that allows us to prove by a simple calculation that the sequence \( B^{(0)}, B^{(1)}, \ldots \) of algebras very often contains infinitely many algebras from different Morita equivalence classes, even when \( A \) is a block of a finite group algebra, and therefore, in such cases, if Donovan’s Conjecture is true, almost all the algebras in the sequence are not Morita equivalent to blocks of finite group algebras.

Let us start by giving another interpretation of this construction.
Let
\[ Q_0 = \bigoplus_{i \in I_0} P_i, \]
and \( E = \text{End}_A(Q_0). \) Then, considering \( Q_0 \) as an \( E-A \)-bimodule, the functor
\[ H = \text{Hom}_A(Q_0,-) : \text{mod} \ (A) \to \text{mod} \ (E) \]
restricts to an equivalence of categories
\[ \text{add}(Q_0) \to \text{P}_E. \]
Applying this functor to the complex \( T_i^{(t)} \) for \( i \in I \setminus I_0, \) we obtain a complex
\[ \cdots \to 0 \to H \left( R_i^{(t-1)} \right) \to H \left( R_i^{(t-2)} \right) \to \cdots \to H \left( R_i^{(0)} \right) \to H (P_i) \to 0 \to \cdots \]
where, since \( R_i^{(k)} \) is in \( \text{add}(Q_0) \) for each \( k, \) \( H \left( R_i^{(k)} \right) \) is a projective \( E \)-module, and the construction of the differentials in \( T_i^{(t)} \) translates into the fact that when we apply \( H \) we get a complex that, being acyclic except at \( H \left( R_i^{(t-1)} \right), \) is the truncation of a minimal projective \( E \)-module resolution of \( H (P_i). \)

Now, consider the Cartan invariants of \( B^{(t)}, \)
\[ c_{ij}^{(t)} = \dim_k \text{Hom}_{B^{(t)}} \left( P_i^{(t)}, P_j^{(t)} \right). \]
Then translating to the derived category of \( A, \)
\[ c_{ij}^{(t)} = \dim_k \text{Hom}_{K^b(P_A)} \left( T_i^{(t)}, T_j^{(t)} \right). \]
If we take \( i \in I_0 \) but \( j \in I \setminus I_0, \) so that \( T_i^{(t)} = P_i[t] \) and
\[ T_j^{(t)} = \cdots \to 0 \to R_j^{(t-1)} \to R_j^{(t-2)} \to \cdots \to R_j^{(0)} \to P_j \to 0 \to \cdots, \]
then there is a short exact sequence of complexes
\[ 0 \to T_j^{(t)} \to T_j^{(t+1)} \to R_j^{(t)}[t+1] \to 0, \]
and since \( \text{Hom}_{K^b(P_A)} \left( T_i^{(t)}, T_j^{(t)}[n] \right) = 0 \) for \( n \neq 0, \) \( \text{Hom}_{K^b(P_A)} \left( T_i^{(t)}, T_j^{(t+1)}[n] \right) = 0 \) for \( n \neq -1 \)
(since \( T_i^{(t+1)} = T_i^{(t)}[1] \)) and \( \text{Hom}_{K^b(P_A)} \left( T_i^{(t)}, R_j^{(t)}[n] \right) = 0 \) for \( n \neq t, \) the long exact sequence obtained by applying the cohomological functor \( \text{Hom}_{K^b(P_A)} \left( T_i^{(t)}, - \right) \) to this sequence yields a short exact sequence
\[ 0 \to \text{Hom}_{K^b(P_A)} \left( T_i^{(t+1)}, T_j^{(t+1)} \right) \to \text{Hom}_{K^b(P_A)} \left( T_i^{(t)}, T_j^{(t)}[t] \right) \to \text{Hom}_{K^b(P_A)} \left( T_i^{(t)}, T_j^{(t)} \right) \to 0 \]
and hence
\[ c_{ij}^{(t)} + c_{ij}^{(t+1)} = \dim_k \text{Hom}_A \left( P_i, R_j^{(t)} \right). \]
Therefore, if
\[ \left\{ \dim_k \text{Hom}_A \left( P_i, R_j^{(t)} \right) : 0 \leq t < \infty \right\} \]
is unbounded, then the set of Cartan invariants \( \{ c_i^j : 0 \leq t < \infty \} \) must also be unbounded, and hence the sequence of algebras \( B^{(0)}, B^{(1)}, \ldots \) must contain representatives of infinitely many Morita equivalence classes.

Since we have seen that the projective modules \( R_j^t \) correspond under an equivalence of categories to the terms in a minimal projective resolution of the \( E \)-module \( \text{Hom}_A(Q_0, P_j) \), where \( E = \text{End}_A(Q_0) \), and the modules \( P_i \) for \( i \in I_0 \) correspond to the indecomposable projective \( E \)-modules under the same equivalence of categories, the following theorem follows.

**Theorem 2.3.** Let \( A \) be a symmetric \( k \)-algebra, let \( \{ S_i : i \in I \} \) be a set of representatives of the isomorphism classes of simple modules, and for each \( i \in I \) let \( P_i \) be a projective cover of \( S_i \). Let \( I_0 \) be a subset of \( I \), \( Q_0 = \bigoplus_{i \in I_0} P_i \) and \( E = \text{End}_A(Q_0) \). If there is \( j \in I \setminus I_0 \) such that \( \text{Hom}_A(Q_0, P_j) \), considered as an \( E \)-module, has a minimal projective resolution whose terms have unbounded dimension, then there are infinitely many Morita equivalence classes of algebras derived equivalent to \( A \).

### 3. Applications

We shall apply Theorem 2.3 to some blocks of group algebras, but first let us explain why it doesn’t apply in the small cases where there are known to be only finitely many Morita equivalence classes of algebras derived equivalent to a certain block.

In the case of blocks \( A \) with cyclic defect group, then whatever subset \( I_0 \subset I \) we choose, the endomorphism algebra \( \text{End}_A(Q_0) \) is periodic, and so the minimal projective resolution of any \( E \)-module has bounded terms, so our theorem doesn’t apply.

Similarly, let us take \( A \) to be a block with defect group \( C_2 \times C_2 \): let us take \( A \) to be the group algebra of the alternating group \( A_4 \), although similar remarks apply to any block with the same defect group. There are three simple modules, so we have the choice of taking \( |I_0| = 1 \) or \( |I_0| = 2 \) (it is easy to see that if \( I_0 = \emptyset \) or \( I_0 = I \), the algebras \( B^{(t)} \) that we construct are all Morita equivalent to \( A \)). If \( |I_0| = 1 \), then \( E \) is isomorphic to the algebra \( k[x]/(x^2) \), which is periodic. If \( |I_0| = 2 \), then \( E \) is a Brauer tree algebra for the tree with two edges, and so again is periodic. So in neither case are there any \( E \)-modules with unbounded minimal projective resolution.

Another obvious case where our theorem certainly can’t be applied is to a block with only one simple module (so the only possibilities are \( I_0 = \emptyset \) or \( I_0 = I \)). Again, in this case, there are not infinitely many Morita classes of algebras derived equivalent to the block, since for a local algebra, the only tilting complexes are isomorphic to shifts of projective generators, and so all derived equivalent algebras are in fact Morita equivalent.

It may well be that the cases we have just described are the only examples of blocks with abelian defect group which are derived equivalent to only finitely many Morita equivalence classes of blocks, although there are some more small cases where Theorem 2.3 does not prove this.

Now let us examine some cases where the theorem does apply.

In characteristic 3, let \( D = C_3 \times C_3 \) be an elementary abelian group of rank 2.

First let \( G \) be the semidirect product \( D \rtimes C_2 \), where a generator of \( C_2 \) acts on \( D \) by inverting all elements, and take \( A = kG \).

Then \( A \) has two simple modules, obtained by inflating the two one-dimensional modules for \( kC_2 \). Denote these by \( S_k \) and \( S_t \). We’ll take \( I_0 = \{ k \} \), so that \( Q_0 = P_k \).
The projective modules $P_k$ and $P_\epsilon$ have Loewy series that are respectively

$$
\begin{array}{ccc}
S_k & S_k & S_k \\
S_k & S_k & S_k \\
S_k & S_k & S_k \\
\end{array}
$$

and

$$
\begin{array}{ccc}
S_\epsilon & S_\epsilon & S_\epsilon \\
S_\epsilon & S_\epsilon & S_\epsilon \\
S_\epsilon & S_\epsilon & S_\epsilon \\
\end{array}
$$

The endomorphism algebra $E$ of $P_k$ is a 5-dimensional commutative local algebra, with Loewy length 3, generated by three elements $x, y$ and $z$ subject to the relations $x^2 = xy = yz = z^2 = 0$ and $xz = y^2$. The set $\{1, x, y, z, y^2\}$ is a basis.

The $M = \text{Hom}_A(P_k, P_\epsilon)$ is a 4-dimensional indecomposable $E$-module, with Loewy length 2, generated by two elements $u$ and $v$, with $ux = uy^2 = vy^2 = vz = 0$. A basis is given by $\{u, v, uz, vx\}$. $M$ has two-dimensional head and two-dimensional socle.

Since $E$ is a symmetric algebra, the syzygies $\Omega^s M$ will all be indecomposable with no projective summands, and so have Loewy length at most two, and the socle of $\Omega^{s+1} M$ will be isomorphic to the head of $\Omega^s M$ for every $s$. Denoting the dimension of the socle of $\Omega^s M$ by $a_s$ (so the dimension of its head is $a_{s+1}$), the projective cover of $\Omega^s M$ will be the direct sum of $a_{s+1}$ copies of the regular $E$-module. Since the dimension of $E$ is 5, the short exact sequence

$$0 \rightarrow \Omega^{s+1} M \rightarrow E^{a_{s+1}} \rightarrow \Omega^s M \rightarrow 0$$

gives a recurrence relation

$$a_{s+1} = 3a_s + 1.$$  
We have $a_0 = 2 = a_1$, and solving the recurrence relation we have

$$a_s = \left(1 - \frac{1}{\sqrt{5}}\right) \left(\frac{3 + \sqrt{5}}{2}\right)^s + \left(1 + \frac{1}{\sqrt{5}}\right) \left(\frac{3 - \sqrt{5}}{2}\right)^s,$$

and in particular $a_s$ grows exponentially, so Theorem 2.3 tells us that there are infinitely many Morita equivalence classes of algebras derived equivalent to $kG$.

However, if we take a different semidirect product $G = D \rtimes C_2$, where now a generator of $C_2$ acts trivially on one cyclic factor of $D = C_3 \times C_3$, but by inversion on the other, then the indecomposable projective $kG$-modules have Loewy series

$$
\begin{array}{ccc}
S_k & S_k & S_k \\
S_k & S_k & S_k \\
S_\epsilon & S_\epsilon & S_\epsilon \\
S_\epsilon & S_\epsilon & S_\epsilon \\
\end{array}
$$
and

\[
\begin{array}{ccc}
S_e & S_k & S_e \\
S_e & S_k & S_e \\
S_k & S_e & S_e \\
S_e & & \\
\end{array}
\]

This time the endomorphism algebra \( E = \text{End}_{kG}(P_k) \) is a six-dimensional commutative algebra, and \( M = \text{Hom}_{kG}(P_k, P_t) \) is a three-dimensional \( E \)-module with \( \Omega M \cong M \), so the minimal projective resolution of \( M \) is periodic, and our main theorem does not apply.

Of course, this doesn’t prove that there are only finitely many Morita equivalence classes of algebras that are derived equivalent to \( kG \), and we don’t know whether or not this is in fact the case.

4. Questions and concluding remarks

Our construction gives examples of families of infinitely many derived equivalent algebras with unbounded Cartan invariants. But there are well-known conjectures in modular representation theory (such as the weaker form of Donovan’s Conjecture, or some more precise conjectures) which predict a bound on the Cartan invariants of a block in terms of the defect group.

So an obvious question is whether, if such a bound were proved, we could then deduce Donovan’s Conjecture (at least for blocks with abelian defect group) from Broué’s Abelian Defect Group Conjecture. In other words:

**Question 1.** Are there infinite families of Morita equivalence classes of algebras, all derived equivalent to the same block algebra, but with bounded Cartan invariants.

It is known that there is only a countable set of algebras derived equivalent to a given algebra, so there can not be continuous families of derived equivalent algebras, at least over an uncountable field, but it does not seem obvious that there cannot be an infinite discrete set of derived equivalent, but not Morita equivalent, algebras with the same Cartan matrix. We don’t know any examples of this kind.

Another question that arises is that the construction proving our main theorem provides examples of infinite families of algebras derived equivalent to a given block, but if even the weaker form of Donovan’s Conjecture is true, only finitely many of them can be Morita equivalent to block algebras. However, they have most of the algebraic properties that they would have to have to be block algebras, by virtue of being derived equivalent to one, since many obvious properties are preserved by derived equivalence.

Some are ruled out by partial results in the direction of Donovan’s Conjecture, but to the best of our knowledge, all these results rely on the Classification of Finite Simple Groups, or at least, most of the force of that theorem.

**Question 2.** Is there any way to prove that any of the algebras we construct are not Morita equivalent to any block algebras, short of using the Classification of Finite Simple Groups or something close to that?

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