A COMMON FORMULA FOR CERTAIN GENERALIZED HANKEL TRANSFORMS

MARIO GARCÍA ARMAS

Abstract. We study the generalized Hankel transform of the family of sequences satisfying the recurrence relation \(a_{n+1} = (\alpha + \frac{\beta}{n+\gamma})a_n\). We apply the obtained formula to several particular important sequences. Incidentally, we find a connection between some well known formulas that had previously arisen in literature in dissimilar settings. Additionally, given a non-zero sequence satisfying the above recurrence, we evaluate the Hankel transform of the sequence of its reciprocals.

1. Introduction

We recall some terminology from the theory of Hankel matrices. Given a sequence \((a_n)_{n=0}^{\infty}\), we consider the doubly-indexed sequence of Hankel matrices \(H_n^{(k)}\), \(n = 1, 2, \ldots; k = 0, 1, \ldots\), defined by

\[
H_n^{(k)} = \begin{pmatrix}
    a_k & a_{k+1} & a_{k+2} & \cdots & a_{k+n-1} \\
    a_{k+1} & a_{k+2} & a_{k+3} & \cdots & a_{k+n} \\
    a_{k+2} & a_{k+3} & a_{k+4} & \cdots & a_{k+n+1} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{k+n-1} & a_{k+n} & a_{k+n+1} & \cdots & a_{k+2n-2}
\end{pmatrix}.
\]

The (generalized) Hankel transform of the sequence \((a_n)_{n=0}^{\infty}\) is the doubly-indexed sequence of determinants \(d_n^{(k)} = \det H_n^{(k)}\) (for a similar treatment, see [12, 20]). It is important to mention that several authors refer to the Hankel transform only as the sequence \(d_n = \det H_n^{(0)}\) (see, for example, [3, 17]).

The theory of Hankel matrices have beautiful connections with many areas of mathematics, physics and computer science (see, for example, [3, 12, 21, 21]). Although Hankel determinants associated to sequences had been previously studied (see, for example, [3, 7, 9, 10, 11, 14]), the term Hankel transform was introduced and first studied by J. W. Layman in [17]. Several later studies of Hankel transforms of sequences have appeared in literature (see, for example, [3, 7, 9, 10, 11, 14]).

In the evaluation of Hankel determinants, several techniques have proved to be useful. For an extended set of tools, as well as a significant bibliography, please refer to [15] Sec. 2.7, [16] Sec. 5.4 and [21].

In this paper, we study the generalized Hankel transform of a sequence \((a_n)_{n=0}^{\infty}\) satisfying

\[
a_{n+1} = \left(\alpha + \frac{\beta}{n+\gamma}\right)a_n, \quad \forall n \geq 0,
\]

Date: May 1, 2009.

1991 Mathematics Subject Classification. Primary 11C20, 15A15; Secondary 11B37, 05A10.

Key words and phrases. determinant evaluation, Hankel transform, Hankel matrix.
for some complex numbers $\alpha, \beta, \gamma$.

Moreover, we study particular choices of these values, for which important sequences are obtained. Some of these Hankel transforms have been previously studied in quite independent settings. Consequently, one of our main goals is to illustrate the hidden connection between them.

In addition, we note that the sequence of reciprocals of any non-zero sequence satisfying (1.2) is also defined by a similar recurrence relation for some parameters depending on $\alpha, \beta, \gamma$ (see Equation (3.27)) and consequently, we evaluate the Hankel transform of the sequence of reciprocals.

It is worth mentioning that in general, it cannot be obtained a closed product form evaluation for Hankel transforms of sequences satisfying the recurrence

$$a_{n+1} = \frac{p(n)}{q(n)} a_n,$$

where $p$ and $q$ are arbitrary polynomials with $\max\{\deg p, \deg q\} > 1$. The non-existence of product evaluations can be proved by means of the procedure described in [10], i.e., computing the first terms of the transforms associated to integer (or rational) sequences and factor them into primes. The presence of extremely large primes in the factorizations shows that there cannot be a product form evaluation of the transforms. For example, the Hankel transform of the sequence $a_n = (3n+1)$ does not have a product form evaluation (see [10]) and however, it is defined by the recurrence

$$a_{n+1} = \frac{3(3n+2)(3n+4)}{2(n+1)(2n+3)} a_n.$$

Analogously, it can be proved for example, that the Hankel transforms of the sequences defined by the recurrence relations

$$a_{n+1} = \frac{n^2+1}{n+1} a_n$$

and

$$a_{n+1} = \frac{n+1}{n^2+1} a_n$$

cannot be evaluated in closed product form.

Therefore, it is unlikely that general closed products form evaluations of Hankel transforms of sequences satisfying (1.3) could be obtained outside from the linear case (i.e., $\deg p = \deg q = 1$), which is the one studied in our paper.

2. Basic Computations

Throughout this paper, we consider a finite product $\prod_{s=a}^b c_s = 1$ when $b < a$.

Let $(a_n)_{n=0}^\infty$ be a sequence satisfying (1.2). For simplicity, we may assume that $a_0 = 1$. It is quite obvious that the results we derive in this section can be straightforwardly extended to the general case (see Remark 2.4 below).

Note that for every $m \geq n \geq 0$ we can write

$$a_m = a_n \prod_{s=n+1}^m \frac{\alpha(s+\gamma-1)+\beta}{s+\gamma-1} = a_n \prod_{s=1}^{m-n} \frac{\alpha(s+n+\gamma-1)+\beta}{s+n+\gamma-1}. \quad (2.1)$$

Let us consider the Hankel matrix $H_n^{(k)} = (m_{ij})_{1 \leq i, j \leq n}$, with the notation of (1.1). Note that we have $m_{ij} = a_{i+j+k-2}$ and consequently, from (2.1) we obtain
the relation

\[ m_{ij} = a_{j+k-1} \prod_{s=1}^{i-1} \left( \frac{\alpha(s+j+k+\gamma-2)+\beta}{s+j+k+\gamma-2} \right). \] (2.2)

We now obtain a first tentative expression for \( d_n^{(k)} \).

Lemma 2.1.

\[ d_n^{(k)} = a_k a_{k+1} \ldots a_{k+n-1} \prod_{i=1}^{n-1} \frac{n! \left[ \alpha(i-1)-\beta \right]^{n-i}}{\prod_{i=1}^{n-1} \left[ i+k+\gamma-1 \right]^{i} \left[ 2n-i+k+\gamma-2 \right]^{i}}. \] (2.3)

Proof. This formula could be derived, after some work, from the generalization of the Cauchy determinant described in [6, Thm. 4.3].

Alternatively, a different (and more elementary) approach will be outlined. Let \( M_n(a, b, c) \) denote the \( n \times n \) matrix with entries

\[ m'_{ij} = \prod_{s=1}^{i-1} \frac{a(s+j)+b}{s+j+c}, \quad i, j = 1, 2, \ldots, n; \] (2.4)

where \( a, b, c \) are complex numbers. Then, it can be proved that the following recurrence relation holds:

\[ \det M_n(a, b, c) = \frac{(n-1)! (ac-b)^{n-1}}{\prod_{i=1}^{n-1} (i+1+c)(i+2+c)} \det M_{n-1}(a, a+b, c+2). \] (2.5)

This can be achieved after the following sequence of steps:

- Subtracting the \( (k-1) \)-st column of \( M_n(a, b, c) \) from the \( k \)-th one for \( k = n, n-1, \ldots, 2 \), in that order.
- Expanding by the minors of the first row.
- Taking common factors from rows and columns out of the determinant of the lower right \( (n-1) \times (n-1) \) block (These common factors form the fraction on the left of the RHS of Equation (2.5)).
- Noticing that the resulting matrix is \( M_{n-1}(a, a+b, c+2) \).

As a consequence of Equation (2.5), we obtain the formula

\[ \det M_n(a, b, c) = \prod_{i=1}^{n-1} \frac{i! \left[ a(i-1)+ac-b \right]^{n-i}}{\prod_{i=1}^{n-1} \left[ i+1+c \right]^{i} \left[ 2n-i+c \right]^{i}}. \] (2.6)

The proposition follows easily from Equation (2.6) after the substitution

\[ (a, b, c) := (\alpha, \alpha \gamma + \alpha k - 2\alpha + \beta, \gamma + k - 2). \] (2.7)

Let us find a nicer expression for \( d_n^{(k)} \). We make the following observation:
Remark 2.2. It is easily seen that \( d_n^{(k)} = 0 \) implies \( d_n^{(k+1)} = 0 \). In fact, the numerator of the fraction on the right of (2.3) does not depend on \( k \) and on the other hand, if \( a_i = 0 \) for some \( i \in \{k, \ldots, k+n-1\} \), then Equation (1.2) yields \( a_j = 0 \) for all \( j \geq i \).

Suppose now that we have \( d_n^{(j)} \neq 0 \) for some \( j \geq 0 \). Using Equations (2.1) and (2.3) together, we compute the ratio

\[
\frac{d_n^{(j+1)}}{d_n^{(j)}} = \frac{a_{j+n}}{a_j} \times \frac{\prod_{i=1}^{n-1} \left[ i + j + \gamma - 1 \right]^i \left[ 2n - i + j + \gamma - 2 \right]^i}{\prod_{i=1}^{n-1} \left[ i + j + \gamma \right]^i \left[ 2n - i + j + \gamma - 1 \right]^i} \\
= \frac{a_{j+n}}{a_j} \times \frac{\prod_{i=1}^{n} (i + j + \gamma - 1)}{\prod_{i=1}^{n} (i + j + \gamma + n - 1)} \\
= \frac{\prod_{i=1}^{n} (\alpha (i + j + \gamma - 1) + \beta)}{\prod_{i=1}^{n} (i + j + \gamma + n - 2)}.
\]

(2.8)

Joining the last result with Remark 2.2, we conclude that the relation

\[
d_n^{(j+1)} = d_n^{(j)} \prod_{i=1}^{n} \frac{\alpha (i + j + \gamma - 1) + \beta}{i + j + \gamma + n - 2}
\]

(2.9)

is always valid and consequently, we obtain the formula

\[
d_n^{(k)} = d_n^{(0)} \prod_{j=0}^{k-1} \prod_{i=1}^{n} \frac{\alpha (i + j + \gamma - 1) + \beta}{i + j + \gamma + n - 2}.
\]

(2.10)

Now, let us focus on \( d_n^{(0)} \). Following Equation (2.3), we can write

\[
d_n^{(0)} = a_0 a_1 \cdots a_{n-1} \frac{\prod_{i=1}^{n-1} \left[ \alpha (i) - \beta \right]^{n-i}}{\prod_{i=1}^{n} \left[ i + \gamma - 1 \right]^i \left[ 2n - i + \gamma - 2 \right]^i}.
\]

(2.11)
It is easy to see that, similarly to Remark 2.2, $d_n^{(0)} = 0$ implies $d_{n+1}^{(0)} = 0$. Again, if we have $d_j^{(0)} \neq 0$ for some $j \geq 1$, we compute the ratio

$$
\frac{d_{j+1}^{(0)}}{d_j^{(0)}} = a_j \prod_{i=1}^{j} \frac{i[\alpha(i+1) - \beta]}{i+\gamma - 1} \times \frac{\prod_{i=1}^{j} [i + \gamma - 1]^i [2j - i + \gamma - 2]^i}{\prod_{i=1}^{j} [i + \gamma - 1]^i [2j - i + \gamma]^i}.
$$

It is easy to see that, similarly to Remark 2.2, $d_n^{(0)} = 0$ implies $d_{n+1}^{(0)} = 0$. Again, if we have $d_j^{(0)} \neq 0$ for some $j \geq 1$, we compute the ratio

$$
\frac{d_{j+1}^{(0)}}{d_j^{(0)}} = a_j \prod_{i=1}^{j} \frac{i[\alpha(i+1) - \beta]}{i+\gamma - 1} \times \frac{\prod_{i=1}^{j} [i + \gamma - 1]^i [2j - i + \gamma - 2]^i}{\prod_{i=1}^{j} [i + \gamma - 1]^i [2j - i + \gamma]^i}.
$$

Hence, we can conclude that the relation

$$
d_{j+1}^{(0)} = d_j^{(0)} \prod_{i=1}^{j} \frac{i[\alpha(i+1) - \beta]}{i+\gamma - 1} \frac{\prod_{i=1}^{j} [i + \gamma - 1]^i [2j - i + \gamma - 2]^i}{\prod_{i=1}^{j} [i + \gamma - 1]^i [2j - i + \gamma]^i}
$$

is always valid and therefore, from the obvious $d_0^{(0)} = a_0 = 1$ we obtain

$$
d_n^{(0)} = \prod_{1 \leq i \leq j \leq n-1} \frac{i[\alpha(i+1) - \beta]}{i+\gamma - 1} \frac{\prod_{i=1}^{j} [i + \gamma - 1]^i [2j - i + \gamma - 2]^i}{\prod_{i=1}^{j} [i + \gamma - 1]^i [2j - i + \gamma]^i}.
$$

We summarize the obtained results in the following theorem.

**Theorem 2.3.** Let $(a_n)_{n=0}^{\infty}$ be a sequence such that $a_0 = 1$ and satisfies Equation (1.2). Then, its generalized Hankel transform is given by

$$
d_n^{(k)} = \prod_{1 \leq i \leq j \leq n-1} \frac{i[\alpha(i+1) - \beta]}{i+\gamma - 1} \frac{\prod_{i=1}^{j} [i + \gamma - 1]^i [2j - i + \gamma - 2]^i}{\prod_{i=1}^{j} [i + \gamma - 1]^i [2j - i + \gamma]^i} \times \prod_{j=0}^{k-1} \prod_{i=1}^{n} \frac{\alpha(i+j+\gamma-1) + \beta}{i+j+\gamma+n-2}.
$$

**Remark 2.4.** If we drop the restriction $a_0 = 1$, then it is easily seen that the expression for $d_n^{(k)}$ will be multiplied by $a_0^n$.

### 3. Applications to Particular Sequences

We now consider several applications of Theorem 2.3 to particular important sequences. In some cases, we show how Formula (2.15) applies to derive some
well known closed product form evaluations, which have been obtained before using several independent methods. We emphasize that empty products are always considered to be 1.

**Application 1.** We start with the sequence \( a_n = 1/(n+1) \), whose associated matrices \( H_n^{(k)} \) are (generalized) Hilbert matrices. This sequence satisfies (1.2) for \( \alpha = 1, \beta = -1, \gamma = 2 \) and thus, its Hankel transform is given by

\[
d_n^{(k)} = \prod_{1 \leq i \leq j \leq n-1} \frac{i^3}{(i+1)(i+j)(i+j+1)} \times \prod_{j=0}^{k-1} \prod_{i=1}^{n} \frac{i+j}{i+j+n}.
\] (3.1)

After some elementary transformations, we obtain the identities

\[
\prod_{1 \leq i \leq j \leq n-1} \frac{i^3}{(i+1)(i+j)(i+j+1)} = \frac{n-1}{\prod_{i,j=1} (i+j-1)},
\]

\[
\prod_{j=0}^{k-1} \prod_{i=1}^{n} \frac{i+j}{i+j+n} = \prod_{i,j=1}^{n} \frac{i+j-1}{i+j+k-1},
\]

which allow us to write \( d_n^{(k)} \) in the more familiar form

\[
d_n^{(k)} = \frac{n-1}{\prod_{i,j=1}^{n} (i+j+k-1)} \frac{\prod_{i=1}^{n-1} (i!)^2}{\prod_{i,j=1}^{n} (i+j+k-1)}.
\] (3.2)

This is a very well known formula and can be proved by several methods (for some historical remarks, see [18, vol. III, pp. 311]). For example, it can be easily derived from *Cauchy’s double alternant* (see [15, Eq. 2.7]). An extensive literature exists on Hilbert matrices and their generalizations. For an interesting compilation of results about Hilbert matrices, please refer to [4].

More generally, it is easy to see that the Hankel transform of the sequence \( a_n = 1/(\lambda n + \mu) \) is given by

\[
d_n^{(k)} = \frac{1}{\mu^n} \prod_{1 \leq i \leq j \leq n-1} \frac{i^2[i-1+\kappa]}{(i+\kappa)(i+j+\kappa-1)(i+j+\kappa)} \times \prod_{j=0}^{k-1} \prod_{i=1}^{n} \frac{i+j+\kappa-1}{i+j+n+\kappa-1}.
\] (3.3)
A COMMON FORMULA FOR HANKEL TRANSFORMS

where \( \kappa = \mu / \lambda \) (see Remark 2.4). Again, using the identities

\[
\prod_{1 \leq i \leq j \leq n-1} \frac{i^2 [i - 1 + \kappa]}{[i + \kappa] [i + j + \kappa - 1][i + j + \kappa]} = \kappa^n \prod_{i,j=1}^{n} \frac{(i!)^2}{(i + j + \kappa - 2)} \]

we obtain the more general (and well known) formula

\[
d_n^{(k)} = \frac{1}{\lambda^n} \times \frac{n-1}{\prod_{i,j=1}^{n} (i + j + k + \kappa - 2)}, \quad (3.4)
\]

which could also be derived, for example, by means of the partial fractions method (cf. [6, Thm. 4.1]).

**Application 2.** Let \( a_n = 2/(n^2 + 3n + 2) \) be the sequence of the reciprocals of triangular numbers shifted by one. It satisfies (1.2) for \( \alpha = 1, \beta = -2, \gamma = 3 \). Thus, its Hankel transform is given by

\[
d_n^{(k)} = \prod_{1 \leq i \leq j \leq n-1} \frac{i^2 [i + 1]}{[i + 2] [i + j + 1][i + j + 2]} \times \prod_{j=0}^{k-1} \prod_{i=1}^{n} \frac{i + j}{i + j + n + 1}. \quad (3.5)
\]

By considering the identities

\[
\prod_{1 \leq i \leq j \leq n-1} \frac{i^2 [i + 1]}{[i + 2] [i + j + 1][i + j + 2]} = 2^n \prod_{i,j=1}^{n} \frac{(i!)^2}{(i + j)},
\]

\[
\prod_{j=0}^{k-1} \prod_{i=1}^{n} \frac{i + j}{i + j + n + 1} = \left( \frac{n + k}{n} \right)^{-1} \prod_{i,j=1}^{n} \frac{i + j}{i + j + k},
\]

we obtain the simpler formula

\[
d_n^{(k)} = 2^n \left( \frac{n + k}{n} \right)^{-1} \times \frac{n-1}{\prod_{i,j=1}^{n} (i + j + k)}, \quad (3.6)
\]

We could have easily evaluated the Hankel transform of the slightly more general sequence \( a_n = \mu (n + \lambda)^{-1}(n + \lambda + 1)^{-1} \) by the same method. However, note that the quadratic polynomials in the denominators are carefully chosen such that the sequence satisfy (1.2) for some \( \alpha, \beta, \gamma \). In fact, if we consider the apparently similar sequence \( a_n = (n+1)^{-2} \), we note that its Hankel transform cannot even be evaluated
in closed product form! This can be proven by the techniques from [10] (see also Introduction) by noting for example the factorizations

\[ d^{(0)}_3 = 647 \cdot 2^8 \cdot 3^6 \cdot 5^2, \]

\[ d^{(0)}_5 = 179 \cdot 179357 \cdot 2^{20} \cdot 3^6 \cdot 5^{10} \cdot 7^5, \]

\[ d^{(0)}_7 = 23 \cdot 1280587616051046200369 \cdot 2^{36} \cdot 3^{22} \cdot 5^{10} \cdot 7^{14} \cdot 11^9 \cdot 13^2. \]

The amazingly large primes in the numerators confirm our claim.

**Application 3.** Consider the sequence \( a_n = 1/n! \). It satisfies (1.2) for \( \alpha = 0, \beta = 1, \gamma = 1 \) and therefore, we have

\[ d^{(k)}_n = (-1)^{\binom{n+k}{2}} \prod_{1 \leq i \leq j \leq n-1} \frac{1}{[i+j-1][i+j]} \times \prod_{i=0}^{k-1} \prod_{j=0}^{n-1} \frac{1}{i+j+n-1}. \]

(3.10)

Alternatively, the following identities hold:

\[ \prod_{1 \leq i \leq j \leq n-1} \frac{1}{[i+j-1][i+j]} = \prod_{i=0}^{n-1} \frac{i!}{(i+n-1)!}, \]

(3.11)

\[ \prod_{j=0}^{n-1} \prod_{i=1}^{k-1} \frac{1}{i+j+n-1} = \prod_{i=0}^{n-1} \frac{(i+n-1)!}{(i+k+n-1)!}. \]

(3.12)

Hence, we obtain

\[ d^{(k)}_n = (-1)^{\binom{n+k}{2}} \prod_{i=0}^{n-1} \frac{i!}{(i+k+n-1)!}, \]

(3.13)

which recovers the formula from [2, Thm. 1.3].

**Application 4.** Let \( a_n = (n+1)^{-1} \binom{2n}{n} \) be the sequence of Catalan numbers. It satisfies (1.2) for \( \alpha = 4, \beta = -6 \) and \( \gamma = 2 \). Therefore, its Hankel transform is given by

\[ d^{(k)}_n = \prod_{1 \leq i \leq j \leq n-1} \frac{i[4i-2][4i+2]}{[i+1][i+j][i+j+1]} \times \prod_{i=0}^{k-1} \prod_{j=0}^{n-1} \frac{4(i+j) - 2}{i+j+n}. \]

(3.14)

Now, the left product reduces to 1 and the right product can be rewritten as

\[ \prod_{1 \leq i \leq j \leq k-1} \frac{i+j+2n}{i+j} \]

(3.15)

for \( k \geq 0 \), which is obvious from the identity

\[ \prod_{i=1}^{j} \frac{4(i+j) - 2}{i+j+n} = \prod_{i=1}^{j} \frac{i+j+2n}{i+j}, \quad j \geq 0. \]

(3.16)

Accordingly, we obtain the well known formula

\[ d^{(k)}_n = \prod_{1 \leq i \leq j \leq k-1} \frac{i+j+2n}{i+j}, \]

(3.17)

which was primarily found by Desainte-Catherine and Viennot (8), who also gave a combinatorial interpretation for this transform (see also [14] for further generalizations). It is also proved in [20], by means of the Dodgson’s condensation method.
Hence, its Hankel transform is given by (\ref{eq:transform}) and discussed in \cite{16} Thm. 33, with some additional remarks. The cases \( k = 0,1 \) and 2 are also studied by Aigner in [1], where the author describes a beautiful generalization of Catalan numbers (called Catalan-like numbers) inspired by the property \( d_n^{(0)} = d_n^{(1)} = 1. \)

**Application 5.** Let \( a_n = \binom{2n}{n} \) be the sequence of even central binomial coefficients. It is immediate to see that \((a_n)\) satisfies (\ref{eq:identities}) for \( \alpha = 4, \beta = -2 \) and \( \gamma = 1. \) Hence, its Hankel transform is given by

\[
d_n^{(k)} = \prod_{1 \leq i \leq j \leq n-1} \frac{[4i-2]^2}{i + j - 1} \frac{1}{i + j} \times \prod_{j=0}^{k-1} \prod_{i=1}^{n} \frac{4(i+j) - 2}{i + j + n - 1}. \quad (3.18)
\]

The left product is readily seen to be \( 2^{n-1} \). As for the right product, it can be rewritten as

\[
2 \times \prod_{1 \leq i \leq j \leq k-1} \frac{i + j - 1 + 2n}{i + j - 1} \quad (3.19)
\]

for \( k \geq 1. \) This can be deduced directly from the identities

\[
\prod_{i=1}^{n} \frac{4(i+j)-2}{i+j+n-1} = \prod_{i=1}^{j} \frac{i + j - 1 + 2n}{i + j - 1}, \quad j \geq 1, \quad (3.20)
\]

\[
\prod_{i=1}^{n} \frac{4i-2}{i+n-1} = 2. \quad (3.21)
\]

Thus, we are able to obtain the formula

\[
d_n^{(k)} = \begin{cases} 
2^{n-1}, & \text{if } k = 0; \\
2^n \times \prod_{1 \leq i \leq j \leq k-1} \frac{i + j - 1 + 2n}{i + j - 1}, & \text{if } k \geq 1.
\end{cases} \quad (3.22)
\]

Taking into account that \( \binom{2m}{m} = 2^{2m-1} \), we recover the formula for the Hankel transform of odd central binomial coefficients from \cite{21} Eq. 1.5 (see also \cite{12} Eq. 13).

It is worth mentioning that, given the identity \( \binom{2n}{n} = (-1)^n 2^n \binom{-1/2}{n} \), the Hankel transform could also be computed using \cite{15} Eq. 3.12. For interesting connections of this Hankel transform with combinatorics and algebra, see \cite{1} and \cite{12} respectively.

**Application 6.** Let \( a_n = \binom{\lambda}{n} \), where \( \lambda \in \mathbb{C}. \) It satisfies (\ref{eq:identities}) for \( \alpha = -1, \beta = \lambda + 1 \) and \( \gamma = 1; \) consequently, its Hankel transform is given by

\[
d_n^{(k)} = (-1)^{nk} \prod_{1 \leq i \leq j \leq n} \frac{(i-\lambda-1)(i+\lambda)}{i+j-1} \times \prod_{j=0}^{k-1} \prod_{i=1}^{n} \frac{i + j - \lambda - 1}{i + j + n - 1}. \quad (3.23)
\]

This evaluation, as well as the one from the next example, could be also derived after some work from \cite{15} Thm. 26, where the author describes a powerful set of tools for evaluating determinants of matrices with entries given by binomial coefficients. Some additional results in binomial determinant evaluations can be found in \cite{5,6}. It is also worth mentioning \cite{13}, where the authors describe interesting combinatorial interpretations of a wide class of binomial determinants.

**Application 7.** Let \( a_n = \binom{n+\lambda}{m} \), where \( m \) is a nonnegative integer and \( \lambda \) is either an integer with \( \lambda \geq m \) or \( \lambda \in \mathbb{C}\setminus\mathbb{Z}. \) Note that \( a_0 = \binom{\lambda}{m} \) and \((a_n)\) satisfies...
the relation
\[ a_{n+1} = \left(1 + \frac{m}{n + \lambda - m + 1}\right) a_n, \quad \forall n \geq 0. \quad (3.24) \]

Thus, by applying Theorem 2.3 together with Remark 2.4, we easily see that the Hankel transform of \((a_n)\) is given by
\[
d^{(k)}_n = \binom{\lambda}{m}^n \prod_{1 \leq i \leq j \leq n-1} \frac{i \gamma}{i + \lambda - m} \frac{[i + j - m - 1]}{[i + j + \lambda - m - 1]} \times \prod_{j=0}^{k-1} \prod_{i=1}^{n} \frac{i + j + \lambda - m - 1}{i + j + n + \lambda - m - 1}.
\quad (3.25)\]

Then, it is straightforward to note that \(d^{(k)}_n = 0\) if (and only if) \(n \geq m + 2\). Another interesting consequence is that in the limiting case \(n = m + 1\), we have
\[
d^{(k)}_{m+1} = (-1)^{\left\lfloor \frac{n+1}{2} \right\rfloor},
\quad (3.26)\]
which can be proved without difficulty after some manipulations (note that the second product reduces to 1).

3.1. Hankel Transforms of Reciprocal Sequences. We finish this section by noting the following beautiful property: if \((a_n)\) is a non-zero sequence satisfying Equation (1.2) with \(\alpha \neq 0\), then it is obvious that the reciprocal sequence \((a^{-1}_n)\) satisfies the relation
\[
\frac{1}{a_{n+1}} = \frac{n + \gamma}{\alpha n + \alpha \gamma + \beta} \frac{1}{a_n} = \frac{1}{\alpha} - \frac{\beta}{\alpha} \frac{1}{a_n}.
\quad (3.27)\]

**Corollary 3.1.** Let \((a_n)_{n=0}^{\infty}\) be a non-zero sequence such that \(a_0 = 1\) and satisfies Equation (1.2) for some \(\alpha \neq 0\). Then, the generalized Hankel transform \(d^{(k)}_n\) of \((a^{-1}_n)_{n=0}^{\infty}\) is given by
\[
\prod_{1 \leq i \leq j \leq n-1} \frac{i \gamma}{\alpha(i + \gamma - 1)} \frac{[\alpha(i - 1) + \beta]}{[\alpha(i + j + \gamma - 2) + \beta]} \frac{[\alpha(i + j + \gamma - 1) + \beta]}{[\alpha(i + j + \gamma + n - 2) + \beta]} \times \prod_{j=0}^{k-1} \prod_{i=1}^{n} \frac{i + j + \gamma - 1}{\alpha(i + j + \gamma + n - 2) + \beta}.
\quad (3.28)\]

**Proof.** The result follows from Equation (3.27) and Theorem 2.3. \(\square\)

**Remark 3.2.** The formula remains valid in the case \(\alpha = 0\); an easy way to see this is by making \(\alpha \to 0\) in Equation (3.28).

As an immediate consequence of Corollary 3.1 we evaluate some non-trivial generalized Hankel transforms:
A COMMON FORMULA FOR HANKEL TRANSFORMS

• Let \( a_n = (n + 1) \binom{2n}{n}^{-1} \) be the sequence of the reciprocals of Catalan numbers. Then, its Hankel transform is given by

\[
d^{(k)}_n = \frac{1}{2^{n(n+k-1)}} \prod_{1 \leq i \leq j \leq n-1} \frac{i(i+1)(2i-5)}{(2i-1)(2(i+j)-3)(2(i+j)-1)} \times \prod_{j=0}^{k-1} \prod_{i=1}^{n} \frac{i+j+1}{2(i+j+n)-3}.
\]

\[ (3.29) \]

• Let \( a_n = \binom{2n}{n}^{-1} \) be the sequence of the reciprocals of even central binomial coefficients. Then, its Hankel transform is given by

\[
d^{(k)}_n = \frac{1}{2^{n(n+k-1)}} \prod_{1 \leq i \leq j \leq n-1} \frac{i^2(2i-3)}{(2i-1)(2(i+j)-3)(2(i+j)-1)} \times \prod_{j=0}^{k-1} \prod_{i=1}^{n} \frac{i+j+1}{2(i+j+n)-3}.
\]

\[ (3.30) \]

• Let \( a_n = \binom{\lambda n}{n}^{-1} \), with \( \lambda \in \mathbb{C}\setminus\mathbb{N} \). Then, its Hankel transform is given by

\[
d^{(k)}_n = (-1)^n (-\lambda)^{n+k} \prod_{1 \leq i \leq j \leq n-1} \frac{i^2(i-\lambda-2)}{(i-\lambda-1)(i+j-\lambda-2)(i+j-\lambda-1)} \times \prod_{j=0}^{k-1} \prod_{i=1}^{n} \frac{i+j+n-\lambda-2}{i+j+n-\lambda-2}.
\]

\[ (3.31) \]

Acknowledgments. The author would like to thank Prof. B. A. Sethuraman for introducing him to the subject and for so many fruitful comments and discussions.

References

[1] M. Aigner, Catalan–like numbers and determinants, J. Combin. Theory Ser. A 87 (1999), 33–51.
[2] R. Bacher, Determinants of matrices related to the Pascal triangle, J. Théor. Nombres Bordeaux 14 (2002), 19–41.
[3] M. Chamberland and C. French, Generalized Catalan numbers and generalized Hankel transformations, J. Integer Seq. 10 (2007), Article 07.1.1.
[4] M. Choi, Tricks or treats with the Hilbert matrix, Amer. Math. Monthly 80 (1983), 301–312.
[5] W. Chu, Binomial convolutions and determinant identities, Discrete Math. 204 (1999), 129–153.
[6] W. Chu and L. V. di Claudio, Binomial determinant evaluations, Ann. Comb. 9 (2005), 363–377.
[7] A. Cvetković, P. Rajković and M. Ivković, Catalan numbers, the Hankel transform, and Fibonacci numbers, J. Integer Seq. 5 (2002), Article 02.1.3.
[8] M. Desainte-Catherine and X. G. Viennot, Enumeration of certain Young tableau with bounded height, Combinatoire Énumérative (Montreal 1985), Lect. Notes in Math. 1234 (1986), 58–67.
[9] Ö. Egecioglu, T. Redmond and C. Ryavec, From a polynomial Riemann hypothesis to alternating sign matrices, Electron. J. Combin. 8 R36 (2001).
[10] Ö. Egecioglu, T. Redmond and C. Ryavec, Almost product evaluation of Hankel determinants, Electron. J. Combin. 15(1) R6 (2008).
[11] C. French, Transformations preserving the Hankel transform, J. Integer Seq. 10 (2007), Article 07.7.3.
[12] M. Garcia Armas and B. A. Sethuraman, A note on the Hankel transform of the central binomial coefficients, J. Integer Seq. 11 (2008), Article 08.5.8.
[13] I. M. Gessel and X. G. Viennot, Binomial determinants, paths, and hook length formulae, Adv. Math. 58 (1985), 300–321.
[14] I. M. Gessel and X. G. Viennot, Determinants, paths, and plane partitions, preprint (1989).
[15] C. Krattenthaler, Advanced determinant calculus, Sém. Lothar. Combin. 42 (1999), Article B42q.
[16] C. Krattenthaler, Advanced determinant calculus: a complement, Linear Algebra Appl. 411 (2005), 68–166.
[17] J. W. Layman, The Hankel transform and some of its properties, J. Integer Seq. 4 (2001), Article 01.1.5.
[18] T. Muir, The Theory of Determinants in the Historical Order of Development, 4 vols., MacMillan, 1906–1923.
[19] M. Z. Spivey and L. L. Steil, The \( k \)-binomial transform and the Hankel transform, J. Integer Seq. 9 (2006), Article 06.1.1.
[20] U. Tamm, Some aspects of Hankel matrices in coding theory and combinatorics, Electron. J. Combin. 8(1) A1 (2001).
[21] R. Vein and A. Dale, Determinants and Their Applications in Mathematical Physics, Springer, 1991.

Departamento de Matemática, Universidad de La Habana, San Lázaro y L, La Habana, CP 10400, Cuba
E-mail address: marioga@matcom.uh.cu