SMOOTHED REDUCTION OF FRACTURE MECHANICS SOLUTIONS TO 1D CRACKED MODELS

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Abstract: Simulating the behaviour of structural members with defects and cracks is an important task in various engineering fields. In fracture mechanics the primary objective is to determine the extreme stress field near the crack that may produce unstable crack growth. In the field of structural health monitoring, the variation of measurable structure’s responses due to the existence of cracks is of interest. These responses may refer to static or dynamic strains and displacements, eigenfrequencies and/or eigenmodes etc. A new representation model for cracked bars based on the introduction of a stiffness function derived from the analytical solution of a cracked plate is presented herein. The new model, as shown in the test examples considered in the manuscript, captures with satisfactory accuracy effects that are revealed in the 2 or 3 dimensional numerical simulation like the strain field variation near a crack, while preserving simplicity and computational efficiency.

Keywords: Crack model; cracked bar structural element; stiffness function; finite element analysis; structural health monitoring; fracture mechanics.

1. INTRODUCTION

The main task when modelling cracks into a fracture mechanics’ framework, is to evaluate the stress field near crack tip, in order to assess the possible unstable crack growth. In the context of structural damage identification where the main objective is to localize and quantify possible damages, modelling cracks mainly implies stiffness degradation, since such an approach produces measurable changes in structural response. Depending on the type of the examined structural system, different models (usually based on the finite element method) are utilized; cracks are usually simulated either as a springs (e.g. [1-3]), either by introducing local stiffness degradations expressed as modulus of elasticity reductions (see for example in [4]), or related to overall finite element stiffness e.g. [5-8] and many others. Damage identification is formulated as an inverse problem in the model parameter domain, in the usual formalism it is decomposed in various stages [9]. Stage one deals with the determination of the existence of damage. The rest of the stages deal with the identification of the

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geometric location, type and quantification of the potential damages (cracks). The most common practice, is the adaptation of finite element model updating schemes. Cracks and damages are represented as an overall stiffness reduction corresponding to the damaged structural member/ finite element.

The main motivation of the presented work is to develop a method that models cracks that is not dependent on the size of the FEM discretisation while at the same time exploit the fact that the effect of the presence of the crack in the stain field may not always be absolutely local. The work presented here is composed of three parts. In part one, the equivalence of the spring models and the models representing damage as a stiffness modulus reduction are examined by introducing an appropriate stiffness function. In part two, based on the strain energy as a function of the distance from the crack tip built on the celebrated solution of Inglis for stresses on an elliptical hole \[10\] and the concept of energy release of fracture mechanics, the appropriate form of the introduced stiffness function is derived. In the third part, numerical validation of the proposed method for modelling cracks and its possible use in damage identification is demonstrated.

2. CRACK MODELS - ANALYSIS OF DAMAGES PRESENTED AS STIFFNESS MODULI REDUCTION

In the case of one dimensional structural members, two methods are usually invoked in order to describe crack(s). The first one is based on the idea to use an equivalent spring at the location of the crack and its stiffness is defined by fracture mechanics principles. This modelling method has the advantage to achieve the necessary “jump” in the structural member’s displacement field around the crack and to introduce localized damages. The second method is based on the idea to describe damage (or crack) as stiffness reduction of the appropriate finite element \[11\]. In order to analyse the two methods and develop an alternative one, an appropriate stiffness function needs to be introduced. Such a function for the case of an element of length $2\delta$ is depicted in Figure 1. $EA$ or $EI$ is the bar or beam element stiffness, where $E$ is the modulus of elasticity, $A$ is the area of the structural element cross section, $I$ is the cross section moment of inertia, $s$ is the percentage of the remaining stiffness in the damaged element and $y_0$ is the element’s middle point (in the longitudinal direction).

In the following analysis, which in the context of the present work without loss of generality is focused on truss structures, the expression of displacement field of a bar element as a function of $s$ will be derived. The case of localized damages as implemented with the spring model will be extracted as a special case of the distributed case (i.e. the stiffness reduction model) and both modelling methods will be directly related with $s$. The bar element stiffness as a function of the abscissa value $y$ along the element is defined as follows:

$$K(y) = K_0 - Q(y), \text{ where } K_0 = EA$$

where

$$Q(y) = (1-s)K_0 \left( H \left( y - (y_o - \delta) \right) - H \left( x - (y_o + \delta) \right) \right)$$

\[2\]
and $H(y)$ is the Heaviside step function. The bar element under equilibrium is governed by the following differential equation:

$$\frac{d}{dy} \left[ K(y) \frac{du}{dy} \right] = 0$$

(3)

$$K(y) \frac{du}{dy} = K(y) \varepsilon = F$$

(4)

where $u(y)$ is the bar element displacement field, $F$ is the axial force developed at the bar element and $\varepsilon$ is the standard engineering strain. Integrating both sides of Eq. (4) results into the following expression:

$$K_o[u]^y_{y_a} - [Qu]^y_{y_a} + \int_{y_a}^{y} (1-s)uK_o\left( \delta(y-(y_o-\delta)) - \delta(y-(y_o+\delta)) \right)dy = F[y]^y_{y_a}$$

(5)

where $\delta(y)$ is the Dirac delta function. Three cases are defined with respect to the abscissa value $y$ along the element:

- for $y \leq y_o - \delta$
  $$u(y) = \frac{F}{K_o} (y - y_a) + u(y_a)$$

- for $y_o - \delta < y \leq y_o + \delta$
  $$K_o[u]^y_{y_a} - (1-s)K_o \mu(y) + (1-s)K_o u(y_o - \delta) = F[y]^y_{y_a}$$

$$u(y) = \frac{F}{sK_o} \left( y - (y_o - \delta) \right) + \frac{F}{K_o} \left( (y_o - \delta) - y_a \right) + u(y_a)$$

(6)

- for $y > y_o + \delta$
  $$K_o[u]^y_{y_a} + (1-s)K_o \left( u(y_o - \delta) - u(y_o + \delta) \right) = F[y]^y_{y_a}$$

$$u(y) = \frac{F}{K_o} \left( y - y_a - 2\delta \right) + \frac{2\delta F}{sK_o} + u(y_a)$$

Although the displacement field $u(y)$ is described by a continuous function, its derivatives are not:

- for $y < y_o - \delta$ or $y > y_o + \delta$
  $$\frac{du}{dy} = \frac{F}{K_o}$$

(7)

- for $y_o - \delta < y < y_o + \delta$
  $$\frac{du}{dy} = \frac{F}{sK_o}$$

As $\delta$ tends to 0 the following observations need to be addressed: (i) for a finite value of $s$ the derivative is continuous everywhere except at $y_o$ where it takes a finite value. However this does not influence $u(y)$ as it is observed from Eq. (6), therefore no change in the displacement field is noticed and consequently no noticeable stiffness reduction of the larger component related to the undamaged case. (ii) Modelling cracks using springs with stiffness derived from fracture mechanics, actually produces a “jump” in the displacement field around the crack. If $\Delta u = u(y_o^+) - u(y_o^-)$ is the jump of the
displacement field around the crack, in order to be consistent with this formulation the following equation needs to be satisfied at $y_0$:

$$\frac{sK_\Delta u}{2\delta} = F, \delta \to 0$$

(8)

Eq. (8) is valid only if $s$ is proportional to $\delta$, i.e.:

$$s = w\delta$$

(9)

where $w$ is a constant of finite value. Therefore:

$$\Delta u = \frac{2F}{wK_0}$$

for $y > y_0$

$$u(y) = \frac{F}{K_0}(y - y_a) + \frac{2F}{wK_0} + u(y_a)$$

(10)

At this part of our analysis, it is possible correlate the first and the second modelling methods, since it is observed that:

$$C_f = \frac{K_0w}{2} = \frac{EAw}{2}$$

(11)

where $C_f$ is the stiffness of the spring adopted in the first modeling method.

Closing the analysis of the two methods an observation regarding the first crack model can be inferred, as the jump in the displacement field $\Delta u$ is of finite value the stiffness function value ($s$) needs to be equal to zero at the location of the crack. Furthermore, the first model does not predict any variation in the strain field near the crack. The second modelling scheme is unsatisfactory since the length of the crack influence depends on the FE discretization. Therefore, both modeling schemes yield unrealistic behavior of the cracked structural members.

3. **STIFFNESS RELAXATION CRACK MODEL BASED ON FRACTURE ENERGY AT A DISTANCE FORM THE CRACK TIP**

Significant work has been carried out aiming to develop finite element models for simulating cracked structural members. The equivalent model is often developed by invoking stress intensity factor (SIF).

For cracks of mode I the equation for SIF is given by the following expression [12]:

$$K_I = R\sqrt{\pi a_c} F_c \left(\frac{a_c}{b}\right)$$

(12)

where $R$ is the uniform stress, $a_c$ is the crack length (or half-length for a central crack), $F_c$ is the correction function considering the finite dimensions and $b$ is the width (or half width in case of a central crack) of the bar structural member. The correction function for the case of bar structural members with prismatic cross-section and a central crack is given by:

$$F_c(a_c/b) = \left[1 - 0.025\left(\frac{a_c}{b}\right)^2 + 0.06\left(\frac{a_c}{b}\right)^4\right] \sqrt{\sec\left(\frac{\pi a_c}{2b}\right)}$$

(13)

while the strain energy due to a central crack is given by [11]:
\[ U_{sle} = \frac{1}{E} \int_0^a K_i^2 t da, \]  
(14)

where \( E \) is the modulus of elasticity and \( t \) is the structural member thickness.

### 3.1 Crack Energy Function

As indicated by fracture mechanics principles, stress (and strain) concentration is observed near cracks. In this part of the study, a new stiffness relaxation model will be derived using the redistribution of stresses (based on an energetic approach) in a finite distance from the crack. The mathematical tools invoked are the solution of a plate with a central elliptic hole which limits to a crack as described in [10] along with a crack energy expression similar to that derived by Griffith in [13] but different in the sense that in the current study the expression of the crack energy is formulated as a function of the distance from the crack tip. Similar to the prodromal studies [10, 13] curvilinear coordinates are employed. In particular, \( \alpha \) and \( \beta \) define a family of confocal ellipses for the case that \( \alpha \) is constant and the orthogonal family of hyperbolae for the opposite case where \( \beta \) is constant. The \( x \) axis of the orthogonal system defined at the center of the ellipses coincides with the major axes of the ellipses while axis \( y \) coincides with the minor ones. The Cartesian coordinates are connected to the curvilinear ones by the relation:

\[
\begin{align*}
    x &= c \cosh \alpha \cos \beta \\
    y &= c \sinh \alpha \sin \beta
\end{align*}
\]  
(15)

The stresses and displacements of the system described above are denoted as: \( R_{\alpha\alpha} \) and \( u_\alpha \) denote the tensile stress and displacement respectively along the normal to \( \alpha = \text{constant} \); \( R_{\beta\beta} \) and \( u_\beta \) denote the tensile stress and displacement respectively along the normal to \( \beta = \text{constant} \); \( S_{\alpha\beta} \) is the shear stress in the same directions.

The expressions derived for the stress and displacement fields refer to a plate having infinite dimensions with an elliptic hole at the center when \( \alpha = \alpha_0 \). The stress and displacement fields for the case of a uniform stress \( R \) developed at large (infinite) distance from the center of the hole are expressed as follows:

\[
\begin{align*}
    R_{\alpha\alpha} &= R \frac{\sinh 2\alpha (\cosh 2\alpha - \cosh 2\alpha_0)}{(\cosh 2\alpha - \cos 2\beta)^2} \\
    R_{\beta\beta} &= R \frac{\sinh 2\alpha (\cosh 2\alpha + \cosh 2\alpha_0 - 2\cos 2\beta)}{(\cosh 2\alpha - \cos 2\beta)^2} \\
    S_{\alpha\beta} &= R \frac{\sin 2\beta (\cosh 2\alpha - \cosh 2\alpha_0)}{(\cosh 2\alpha - \cos 2\beta)^2} \\
    \frac{u_\alpha}{h} &= \frac{c^2 R}{8\mu} \left[ (p - 1) \cosh 2\alpha - (p + 1) \cos 2\beta + 2 \cosh 2\alpha_0 \right] \\
    \frac{u_\beta}{h} &= 0
\end{align*}
\]  
(16)
where \( c \) is the half length of the focal line, \( E \) is the modulus of elasticity, \( \mu \) is the shear modulus \( \mu = E/(1+\sigma) \), \( p \) is equal to \((3-4\sigma)\) for the case of plain strain and \((3-\sigma)/(1+\sigma)\) for plain stress cases, \( \sigma \) is the Poisson ratio and \( h \) is the modulus of transformation given by the following expression:

\[
h = \frac{2}{c^2 \left( \cosh 2\alpha - \cos 2\beta \right)}
\]  

(17)

Worth mentioning that the expressions of Eq. (16) limits to those derived for a plate with a central crack when \( \alpha_0 = 0 \). The strain energy \( U \) of the material per unit thickness of the plate with in an ellipse \( \alpha \) is:

\[
U = \frac{1}{2} \int_0^{2\pi} R_{\alpha\alpha} \frac{u_{\alpha}}{h} d\beta + \frac{1}{2} \int_0^{2\pi} S_{\alpha\beta} \frac{u_{\beta}}{h} d\beta
\]

(18)

In order to derive the expression for the excess energy due to the crack \( CE \) the following steps are implemented; the expressions of Eq. (16) are substituted into Eq. (18), the resulting expression is integrated and the energy terms that are due to the cavity \( \alpha_0 \) when \( \alpha_0 \) limits to 0 are preserved (more details can be found in the Appendix of the current study). The expression for the excess energy due to the crack \( CE \) that is derived, is given by:

\[
CE = \frac{\pi(1+\sigma)R^2c^2}{16E} \left[ A_1 \left( -(3-p)e^{-4a} + 4e^{-2a} - 4e^{-2a} + (3-p)e^{4a} \right) + \\
+ A_2 \left( -2(1+p)e^{-2a} + 2(1+p)e^{2a} \right) \right]
\]

(19)

\[
A_1 = \frac{\cosh 2\alpha}{(\cosh 2\alpha - 1)^{3/2} (\cosh 2\alpha + 1)^{3/2}}
\]

\[
A_2 = \frac{1}{(\cosh 2\alpha - 1)^{3/2} (\cosh 2\alpha + 1)^{3/2}}
\]

When \( a \) tends to infinity, Eq. (19) tends to the following vale:

\[
CE_{a\to\infty} = \frac{\pi(3-p)R^2c^2}{8\mu}
\]

(20)

That defines the well-known crack energy as derived by Griffith in [13]. In order to find a minimum of the crack energy of Eq. (19) its derivative defined as follows:

\[
\frac{\partial CE}{\partial a} = -\frac{\pi(1+\sigma)R^2c^2}{16E} \left[ 128\sinh(\alpha)^2 + 384\sinh(\alpha)^4 - 256\sinh(\alpha)^8 \right]
\]

\[
\left( \frac{\sinh(\alpha)^2}{(2\sinh(\alpha)^2)} \right)^{5/2} \left( \frac{\sinh(\alpha)^2 + 2}{\sinh(\alpha)^2 + 2} \right)^{3/2}
\]

(21)

is set to zero and a minimum is achieved for the following parameters’ values:

\[
a = \arcsinh \left[ \frac{2^{0.5} \sqrt{3^{0.5} + 1}}{2} \right]
\]

\[
y_m = e^{0.5} \frac{3^{0.5} + 1}{2} \sin(\beta)
\]

\[
y_m\beta = \frac{1.1688c}{2}
\]

(22)
The corresponding value of the minimum crack energy defined as follows:

$$CE_{\text{min}} \approx \frac{\pi (2.732 - p) R^2 c^2}{8 \mu}$$  \hspace{1cm} (23)

### 3.2 Stiffness Relaxation Model

Since the range of interest for the distribution of the crack energy is defined for \( y \geq y_m \) and aiming to develop a simple representation of the stiffness function, the following expression is considered as a very good approximation of the energy distribution:

$$E_a = \Theta \arctan(By)$$  \hspace{1cm} (24)

The total excess energy due to the crack is defined according to Eq. (14) while its spatial distribution is determined with the aid of Eqs. (20) to (23). Therefore, \( \Theta \) is computed by the following relation:

$$E_a(y \to \infty) = \frac{1}{2} U_{\text{SIF}}$$

$$\Theta = \frac{U_{\text{SIF}}}{\pi}$$  \hspace{1cm} (25)

and \( B \) is computed by the relation:

$$E_a(y_m) = \frac{CE_{\text{crack}} U_{\text{SIF}}}{CE_{a \to \infty} \frac{2}{2(3 - p)}} U_{\text{SIF}}$$

$$B = \frac{1}{1.1688c} \tan \left[ \frac{2.732 - p}{2(3 - p) \Theta} U_{\text{SIF}} \right]$$

$$= \frac{1}{1.1688c} \tan \left[ \frac{2.732 - p}{2(3 - p) \pi} \right]$$  \hspace{1cm} (26)

The total strain energy of a bar structural member and the excess energy due to the crack are given by:

$$W = \int \frac{R^2 A}{E s(y)} dy$$

$$= \int \frac{R^2 A}{E} \left[ 1 + \frac{1 - s}{s} \right] dy$$

$$W_{\text{crack}} = \int \frac{R^2 A}{E} \left[ \frac{1 - s}{s} \right] dy$$  \hspace{1cm} (27)

where \( A \) is the bar’s cross section area; therefore, equating Eqs. (24) with (27) and then differentiating the resulting equation the following expression is obtained:

$$\frac{\partial W_{\text{crack}}}{\partial y} = \frac{t}{\partial} \frac{\partial E_a}{\partial y}$$  \hspace{1cm} (28)

the stiffness function for the 1D bar structural member model is obtained:
\[
\begin{align*}
\frac{1}{2Bt} & \left[ \bar{a}_c F_c^2 \left( \frac{a_c}{b} \right) \right]_0^\bar{a}_c \frac{da_c}{(1+(By)^2)A} \\
\end{align*}
\]

(29)

where \(a_c\) is the last expression stands for the crack half-length.

Figures 2 depicts the distribution of the excess energy due crack \(CE\) computed using the expression of Eq. (19) (scaled to \(U_{si}\)) and the crack energy distribution model given by Eq. (24) and Figure 3 shows the corresponding stiffness function of Eq. (29); both for a steel bar structural member with \(E=200\,\text{GPa}, \sigma=0.265, R=100\,\text{MPa}, 2b=0.4\,\text{m}, t=0.4\,\text{m}\) and half crack length \(a_c=0.13\,\text{m}\), (the results are achieved assuming plane stress conditions).

4 NUMERICAL STUDY

Aiming to present the efficiency of the proposed model of the cracks two levels of comparison are employed herein: (i) in the first one, the new model is compared against two usually applied approaches for modelling cracks in the case of 1D simulations, (ii) in the second one, the new method is integrated into a damage identification approach and is compared against a 3D implementation.

4.1 Cracked Bar Finite Element Implementing the Proposed Stiffness Function

In order to derive the stiffness coefficients \(k_{ij}\) of the stiffness matrix for the proposed cracked bar finite element of length \(L\), first Eq. (4) is integrated and \(K(y)\) is substituted with the term \(sEA\), where \(s\) is given by Eq. (29). Then differentiating twice the external work as follows:

\[
k_{ij} = \frac{\partial^2 W_{el}}{\partial u_i \partial u_j}
\]

(30)

the stiffness coefficients \(k_{ij}\) of the stiffness matrix are obtained:

\[
K = \frac{EA}{L+C \left[ \arctan(Bx) \right]_y^{y_r}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\]

(31)

where \(y_1, y_2\) and \(y_r\) are the coordinates along the bar structural element of the start-end edge nodes of the finite element and that of the location of the crack, respectively.

4.2 Comparison with Existing Models for 1D Simulations of Cracked Structural Elements

In the first level of comparison employed in the current study, the proposed modelling approach is compared against two highly used modelling practices for the case of 1D simulations of cracked
structural members. The three models are compared with reference to the normal strain and displacement profiles along the length of the structural element achieved. For this case an indicative test case is used corresponding to a steel structural element of length $L=2m$, of rectangular cross-section $(2b \times t)$ $2b=0.4m$, $t=0.4m$, subjected to a tensile force $F=10MN$ (applied uniformly to one edge of the bar, while the other edge is fixed). The material properties considered are modulus of elasticity $E=200GPa$, poison ration $\sigma=0.3$ while a crack located at the half-length of the structural element is considered with half-length equal to $a_c=0.15m$.

Figures 4 to 6 depict the profiles of the normal strains obtained for the steel beam. In particular, Figure 4 illustrates the normal strains $\varepsilon_{yy}$ (defined at the cross section) derived when implementing the first modelling method where a linear spring is used at the location of the crack. The FE discretization used for the implementation of the first method is composed by 20 1D beam elements and one spring. Figure 5 illustrates the normal strains derived when implementing the second modelling method applying the appropriate cracked finite element at the location of the crack. The FE discretization used for the implementation of the second method is composed by 21 1D beam elements. Finally, Figure 6 illustrates the normal strains profile obtained when implementing the proposed modelling method. The FE discretization used for the implementation of the third method is composed by 20 1D beam elements whose stiffness matrices were derived using Eq. (31). Comparing the three methods it can be seen that first and second ones result into unrealistic profiles of the normal strain, exhibiting no variation.

Additionally, in order to further examine the differences between the three modelling approaches, the displacement profiles along the bar structural element were also obtained using the three crack representation methods. The displacements’ profiles obtained are depicted in Figures 7, 8 and 9 for the three modelling methods, respectively. Similar to the normal strain profiles, those obtained for the first and second method develop displacement profiles with sharp discontinuities, while the proposed one displace a smoother one.

### 4.3 Application to Damage Identification

In this part of the numerical study, it is presented how the performance of damage identification procedures based on measured static strains can be improved with the aid of the proposed 1D crack model. For this purpose, the same steel structural member is used (i.e. length $L=2m$, rectangular cross-section, $2b=0.4m$, $t=0.4m$, also subjected to the same tensile force $F=10MN$, applied uniformly to one edge of the bar, while the other edge is fixed, with $E=200GPa$ and $\sigma=0.3$). A structural damage is also considered at the half-length of the structural element, having though smaller value; the crack implemented has half-length equal to $a_c=0.10m$.

In this part of the comparative study, the proposed 1D modelling approach of cracked structural elements is compared against a more detailed numerical model. Thus, in order to define the basis of
comparison, the structural element was numerically simulated by means of the FE method using a fine 3D discretization. In particular, a numerical model composed by 15,116 3D solid finite elements is used and the numerical analysis was performed with ANSYS R17.0 software. The type solid element used is the SOLID186, which is a higher order 3-D 20-node solid element that exhibits quadratic displacement behavior. In order to identify possible damages in the 3D numerical simulation strain gauges are implemented along the longitudinal direction of the structural element. In particular, strain gauges were placed every 0.3m at perpendicular faces and they were simulated using appropriate zero stiffness elements (see Figures 10 and 11). Strain gauges were simulated with the use of SHELL181 shell elements with zero stiffness.

The proposed crack representation refers to 1-dimensional modeling of a cracked structural element; thus, its strain and displacement fields refer to “averaged” values. Therefore, an appropriate definition of the “averaged” values needs to be adopted able to translate 2D fields obtained from the measurements into 1D ones, which is not a trivial task. For the test case examined herein, without loss of generality, only central cracks are assumed to be present. Furthermore, strain gauges are assumed to be in proximity to the crack in order to capture the “averaged 1D strain” labeled as $\varepsilon_{yy,i}$ by applying the following relation:

$$\varepsilon_{yy,i} \approx \frac{\varepsilon_{\text{max},i}(2b-a_c) + \varepsilon_{\text{min},i}a_c}{2b}$$

where $\varepsilon_{\text{max},i}$ and $\varepsilon_{\text{min},i}$ are the maximum and minimum strains measured for the $i^{th}$ cross section (see Table 1). The “averaged 1D strains” are derived assuming a linear approximation as described in Figure 12 for the strains in the range $[\varepsilon_{\text{min},i}, \varepsilon_{\text{max},i}]$ when the values of $x$ is defined in the range $0 \leq x \leq a_c$ while remaining constant equal to $\varepsilon_{\text{max},i}$ for the values of $x$ defined in the range $a_c \leq x \leq b$, where $x$ denotes the distance from the center of the cross section along a direction parallel to the crack. The averaged strains are then computed by the proposed method as follows:

$$\varepsilon_{\text{myy},i} = \frac{F}{sEA}$$

where $s$ is computed using Eq. (29). For each cross section different values for the strains suggest the existence of a crack in the vicinity. The value of $F$ can be computed from the cross sections where the values of the strains are equal (and small) which suggests uniform distributions for stresses and strains.

The damage identification procedure is formulated as the unconstrained optimization problem of the following expression:

$$\arg\min_{a_c, a_y} \left[ \sum_{i=1}^{K} (\varepsilon_{yy,i}^m - \varepsilon_{yy,i})^2 \right]$$

(34)

For the solution of the problem of Eq. (34) a well-known metaheuristic algorithm is used, in particular the genetic algorithm (GA) is employed. This should not been considered as an implication related to the efficiency of other algorithms, any algorithm able to solve the problem of Eq. (34) can be
considered for the solution of the optimization problem based on user’s experience. The formulation described by Eq. (34) can also be employed for the case of multiple cracks. The values of \(a_c\) (half-length of crack) and \(y_r\) (crack position) are presented in Table 2 as it can be seen both position and length of the crack are identified with acceptable accuracy. The values obtained for \(\varepsilon_{yy}^m\) and \(\varepsilon_{yy}^d\) are depicted in Figure 13; as it can be observed the strain field obtained by the FE analysis compared to that obtained using the proposed modeling method are also in good agreement. It should be noted that it is not possible to perform a similar identification procedure implementing any other 1D crack model, since they are not able to reproduce the strain field variation due to the presence of a crack.

5 CONCLUSIONS

A new model for representation of cracked bars is presented in the current study. The new model was developed by introducing an appropriate stiffness function derived from the analytical solution of a cracked plate. As it was observed through the numerical tests performed, the new model captures with satisfactory accuracy effects that are revealed only in 2 or 3 dimensional numerical simulation of the strain field variation near the crack, while preserving simplicity and computational efficiency.

For comparative reasons the proposed modelling approach was compared with existing crack models for static analysis. In addition, it was also implemented in the framework of damage identification using measured static strains; worth mentioning that such a task was not able to be performed with the existing 1D crack models. In future work, it is intended to study the efficiency of the proposed new model in case of dynamic problems like the calculation of eigenfrequencies and eigenmodes while its efficiency will also be examined when integrated in case of damage identification problems using vibration data.

ACKNOWLEDGEMENTS

This research has been supported by the OptArch project: “Optimization Driven Architectural Design of Structures” (No: 689983) belonging to the Marie Sklodowska-Curie Actions (MSCA) Research and Innovation Staff Exchange (RISE) H2020-MSCA-RISE-2015.

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APPENDIX

Integrating the expression of Eq. (18) gives the total strain energy within an ellipse $a$:

\[
U = \int_{0}^{2\pi} \frac{R^2c^2}{64e^{6a}E(cosh 2a - cos 2\beta)^2} \left[ (1 - p)(1 + \sigma) + e^{4a}(1 - p)(1 + \sigma) - e^{8a}(1 - p)(1 + \sigma) - e^{12a}(1 - p)(1 + \sigma) + 
+ \cos 2\beta \left( 2e^{2a}(1 + p)(1 + \sigma) - 2e^{10a}(1 + p)(1 + \sigma) \right) - 
- 2e^{2a} \cosh 2a_0(3 - p)(1 + \sigma) + 8e^{4a} \cosh^2 2a_0(1 + \sigma) - 
- 8e^{6a} \cosh^2 2a_0(1 + \sigma) + 2(3 - p)(1 + s) \cosh 2a_0e^{10a} - 
- 4 \cos 2\beta \cosh 2a_0e^{4a}(1 + p)(1 + \sigma) + 
+ 4 \cos 2\beta \cosh 2a_0e^{8a}(1 + p)(1 + \sigma) \right] d\beta 
\]

(A1)

The terms $cosh2a_0$ are of interest here as they express the energy related to the elliptical hole therefore the energy due to the hole is:

\[
U_1 = \int_{0}^{2\pi} \frac{R^2c^2}{64e^{6a}E(cosh 2a - cos 2\beta)^2} \left[ I_1 \left( -2e^{2a} \cosh 2a_0(3 - p)(1 + \sigma) + 8e^{4a} \cosh^2 2a_0(1 + \sigma) - 
- 8e^{6a} \cosh^2 2a_0(1 + \sigma) + 2(3 - p)(1 + s) \cosh 2a_0e^{10a} - 
- 4 \cos 2\beta \cosh 2a_0e^{4a}(1 + p)(1 + \sigma) + 
+ 4 \cos 2\beta \cosh 2a_0e^{8a}(1 + p)(1 + \sigma) \right) \right] d\beta 
\]

(A2)

Integrals $I_1$ and $I_2$ present a singularity at $\pi/2$ and $3\pi/2$ therefore they are appropriately computed:

\[
F_i(\beta) = \int_{0}^{2\pi} \frac{1}{(T - \cos 2\beta)^2} d\beta 
= Tarc tan \left( \frac{\tan(\beta)(2T + 2)}{2(T - 1)^{1/2}(T + 1)^{1/2}} \right) \frac{1}{(T - 1)^{1/3}(T + 1)^{1/3}} + 
+ \tan \beta \frac{T - 1)(T + 1) \times \tan^2 \beta + T - 1 \right) 
\]

(A3a)

\[
I_1 = F_i(\beta \rightarrow \frac{\pi}{2}) - F_i(0) + F_i(\beta \rightarrow \frac{3\pi}{2}) - F_i(\beta \rightarrow \frac{\pi}{2}) + F_i(2\pi) - F_i(\beta \rightarrow \frac{3\pi}{2}) = 4F_i(\beta \rightarrow \frac{\pi}{2}) = 2\pi A_i 
\]
\[
F_z(\beta) = \int \frac{\cos 2\beta}{(T - \cos 2\beta)^2} \, d\beta \\
= \arctan\left(\frac{\tan(\beta)(2T + 2)}{2(T - 1)^{1/2}(T + 1)^{1/2}}\right) \frac{1}{(T - 1)^{3/2}(T + 1)^{3/2}} + \\
\frac{T \tan \beta}{(T - 1)(T + 1)((T + 1)^* \tan^2 \beta + T - 1)}
\]

\[
I_2 = F_z(\beta \to \frac{\pi}{2}^-) - F_z(0) + F_z(\beta \to \frac{3\pi}{2}^-) - F_z(\beta \to \frac{\pi}{2}^+) + F_z(2\pi) - F_z(\beta \to \frac{3\pi}{2}^+) \\
= 4F_z(\beta \to \frac{\pi}{2}^-) \\
= 2\pi A_2
\]

Substituting \(\cosh \alpha = 1\) when the elliptical hole limits to a crack results to Eq. (19) for the crack energy.
FIGURES

Figure 1: Stiffness function in the longitudinal direction of 1D elements

Figure 2: Crack energy distribution
Figure 3: Stiffness function (the origin $y=0$ coincides with the crack position)

Figure 4: Normal strains, first modelling method
Figure 5: Normal strains, second modelling method

Figure 6: Normal strains, proposed modelling method
Figure 7: Displacements, first modelling method

Figure 8: Displacements, second modelling method
Figure 9: Displacements, proposed modelling method

Figure 10: FEM model of the cracked bar
Figure 11: Normal strain $\varepsilon_{yy}$ results obtained from FEM

Figure 12: Calculation of approximate “averaged” strain $\varepsilon_{yy}$ from measurements

Figure 13: Strain distributions $\varepsilon_{yyi}$ and $\varepsilon_{myyi}$ from FEM and the proposed method respectively
TABLES

**Table 1:** Strains obtained from the FEM analysis

| Distance of the cross section from the start of the bar (m) | “Measured” strain at Face 1 (m/m) | “Measured” strain at Face 2 (m/m) |
|----------------------------------------------------------|----------------------------------|----------------------------------|
| 0.2                                                      | 3.030E-04                        | 3.035E-04                        |
| 0.5                                                      | 3.117E-04                        | 3.118E-04                        |
| 0.8                                                      | 3.996E-04                        | 2.100E-04                        |
| 1.1                                                      | 4.582E-04                        | 1.083E-04                        |
| 1.4                                                      | 3.149E-04                        | 3.053E-04                        |
| 1.7                                                      | 3.119E-04                        | 3.130E-04                        |

**Table 2:** Damage identification results

| FEM simulation | Proposed method |
|---------------|-----------------|
| Crack half length (a_c) | 0.1m | 0.12m |
| Crack position (y_r) | 1m | 0.97m |