OCCUPANCY FRACTION, FRACTIONAL COLOURING, AND TRIANGLE FRACTION

EWAN DAVIES, RÉMI DE JOANNIS DE VERCLOS, ROSS J. KANG, AND FRANC ¸OIS PIROT

Abstract. Given $\varepsilon > 0$, there exists $f_0$ such that, if $f_0 \leq f \leq \Delta^2 + 1$, then for any graph $G$ on $n$ vertices of maximum degree $\Delta$ in which the neighbourhood of every vertex in $G$ spans at most $\Delta^2/f$ edges,

(i) an independent set of $G$ drawn uniformly at random has at least $(1/2 - \varepsilon)(n/\Delta) \log f$ vertices in expectation, and

(ii) the fractional chromatic number of $G$ is at most $(2 + \varepsilon)\Delta/\log f$.

These bounds cannot in general be improved by more than a factor 2 asymptotically. One may view these as stronger versions of results of Ajtai, Komlós and Szemerédi (1981) and Shearer (1983). The proofs use a tight analysis of the hard-core model.

1. Introduction

By a deletion argument, Ajtai, Komlós and Szemerédi [2] noted a more general statement as corollary to their seminal bound on the independence number of triangle-free graphs. There is some $C > 0$ and some $f_0$ such that, in any graph on $n$ vertices of maximum degree $\Delta$ with at most $\Delta^2 n/f$ triangles, where $f_0 < f < \Delta$, there is an independent set of size at least $C(n/\Delta) \log f$. In other words, an upper bound on the fraction of triangles in the graph yields a corresponding lower bound on independence number. Somewhat later, Alon, Krivelevich and Sudakov [3] proved a stronger version of this in terms of an upper bound on the chromatic number. Recently, using a sophisticated “stochastic local search” framework, Achlioptas, Iliopoulos and Sinclair [1] tightened the result of [3], corresponding to a constant $C$ above of around $1/4$ in general. In fact, shortly after the work in [2], using a sharper bootstrapping from the triangle-free case, Shearer [12] had improved the above statement on independence number essentially as follows.

2010 Mathematics Subject Classification. Primary 05C35, 05D10; Secondary 05C15.

Key words and phrases. Independent sets, fractional colouring, hard-core model.

(E. Davies) The research leading to these results has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement No 339109.

(R. de Joannis de Verclos, R. J. Kang) Supported by a Vidi grant (639.032.614) of the Netherlands Organisation for Scientific Research (NWO).

1They also obtained an asymptotic estimate around $1/2$ for $f$ very close to $\Delta^2 + 1$.

2The results in [2, 12] are in terms of given average degree.
Theorem 1 (Shearer [12]). Given $\varepsilon > 0$, there exists $f_0$ such that, if $f_0 \leq f \leq \Delta^2/\varepsilon^2$, then in any graph on $n$ vertices of maximum degree $\Delta$ with at most $\Delta^2 n/\varepsilon^2$ triangles, there is an independent set of size at least $(1/2 - \varepsilon)(n/\Delta) \log f$.

The case $f = \Delta^2 - o(1)$ as $\Delta \to \infty$ includes the triangle-free case and yields the best to date asymptotic lower bound on the off-diagonal Ramsey numbers. The asymptotic factor $1/2$ cannot be improved above 1, due to random regular graphs; see Section 5 for more details on sharpness.

Our main contribution is to give two stronger forms of Theorem 1, one on occupancy fraction (see Theorem 5 below), the other on fractional chromatic number, combining for the result promised in the abstract. We show how either easily implies Theorem 1 in Section 2.

Theorem 2. Given $\varepsilon > 0$, there exists $f_0$ such that, if $f_0 \leq f \leq \Delta^2 + 1$, then for any graph $G$ on $n$ vertices of maximum degree $\Delta$ in which the neighbourhood of every vertex in $G$ spans at most $\Delta^2 f$ edges,

(i) an independent set of $G$ drawn uniformly at random has at least $(1/2 - \varepsilon)(n/\Delta) \log f$ vertices in expectation, and

(ii) the fractional chromatic number of $G$ is at most $(2 + \varepsilon) \Delta/\log f$.

We prove Theorem 2 by an analysis of the hard-core model. In Section 5, we give some indication that our application of this analysis is essentially tight. The same method was used for similar results specific to triangle-free graphs [8, 7]; to an extent, the present work generalises that earlier work.

Theorem 2(ii) and the results in [1] hint at their common strengthening.

Conjecture 3. Given $\varepsilon > 0$, there exists $f_0$ such that, if $f_0 \leq f \leq \Delta^2 + 1$, then any graph of maximum degree $\Delta$ in which the neighbourhood of every vertex spans at most $\Delta^2 f$ edges has (list) chromatic number at most $(2 + \varepsilon) \Delta/\log f$.

In Section 6, motivated by quantitative Ramsey theory, we briefly discuss a more basic problem setting in terms of bounded triangle fraction.

1.1. Notation and preliminaries. We write $\mathcal{I}(G)$ for the set of independent sets (including the empty set) of a graph $G$.

Given $\lambda > 0$, the hard-core model on $G$ at fugacity $\lambda$ is a probability distribution on $\mathcal{I}(G)$, where each $I \in \mathcal{I}(G)$ occurs with probability proportional to $\lambda^{|I|}$. Writing $I$ for the random independent set, we have

$$\Pr(I = I) = \frac{\lambda^{|I|}}{Z_G(\lambda)},$$

where the normalising term in the denominator is the partition function (or independence polynomial) $Z_G(\lambda) = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}$. The occupancy fraction is $\mathbb{E}[|I|]/|V(G)|$. Note that this is a lower bound on the proportion of vertices in a largest independent set of $G$. 

If $\mu$ denotes the standard Lesbegue measure on $\mathbb{R}$, then a fractional colouring of $G$ is an assignment $w : \mathcal{I}(G) \to 2^\mathbb{R}$ of pairwise disjoint measurable subsets of $\mathbb{R}$ to independent sets such that $\sum_{I \in \mathcal{I}(G), I \ni v} \mu(w(I)) \geq 1$ for all $v \in V(G)$. The total weight of the fractional colouring is $\sum_{I \in \mathcal{I}(G)} \mu(w(I))$. The fractional chromatic number $\chi_f(G)$ of $G$ is the minimum total weight in a fractional colouring of $G$. Note that the reciprocal is a lower bound on the proportion of vertices in a largest independent set of $G$.

We make use of a special case of a lemma from our earlier work [7].

Lemma 4 ([7], cf. also [11]). Let $G$ be a graph of maximum degree $\Delta$, and $\alpha, \beta > 0$ be positive reals. Suppose that for every induced subgraph $H \subseteq G$, there is a probability distribution on $\mathcal{I}(H)$ such that, writing $I_H$ for the random independent set from this distribution, for each $v \in V(H)$ we have

$$\alpha P(v \in I_H) + \beta \mathbb{E}|N_H(v) \cap I_H| \geq 1.$$  

Then $\chi_f(G) \leq \alpha + \beta \Delta$.

The function $W : [-1/e, \infty) \to [-1, \infty)$ is the inverse of $z \mapsto z e^z$, also known as the Lambert $W$ function. It is monotonic and satisfies $W(x) = \log x - \log \log x + o(1)$ and $W((1 + o(1))x) = W(x) + o(1)$ as $x \to \infty$, and $e^{-W(y)} = W(y)/y$ for all $y$.

2. The main result

We next discuss our main result, Theorem 2, in slightly deeper context. We in fact show a sharp, general lower bound on occupancy fraction for graphs of bounded local triangle fraction, to which Theorem 2(i) is corollary.

Theorem 5. Suppose $f, \Delta, \lambda$ satisfy that $f \leq \Delta^2 + 1$ and, as $f \to \infty$, that 

$$\Delta \log(1 + \lambda) = \omega(1) \quad \text{and} \quad \frac{2(\Delta \log(1 + \lambda))^2}{fW(\Delta \log(1 + \lambda))} = o(1).$$

In any graph $G$ of maximum degree $\Delta$ in which the neighbourhood of every vertex spans at most $\Delta^2/f$ edges, writing $I$ for the random independent set from the hard-core model on $G$ at fugacity $\lambda$, the occupancy fraction satisfies

$$\frac{1}{|V(G)|} \mathbb{E}|I| \geq (1 + o(1)) \frac{\lambda W(\Delta \log(1 + \lambda))}{1 + \lambda \Delta \log(1 + \lambda)}.$$  

This may be viewed as generalising [8, Thm. 3]. By monotonicity of the occupancy fraction in $\lambda$ (see e.g. [8, Prop. 1]), and the fact that a uniform choice from $\mathcal{I}(G)$ is a hard-core distribution with $\lambda = 1$, Theorem 2(1) follows from Theorem 5 with $\lambda = f^{1/(2+\epsilon/2)}/\Delta$. Theorem 5 is asymptotically optimal. More specifically, in [8] it was shown how the analysis of [4] yields that, for any fugacity $\lambda = o(1)$ in the range allowed in Theorem 5, the random $\Delta$-regular graph (conditioned to be triangle-free) with high probability has occupancy fraction asymptotically equal to the bound in Theorem 5. In Section 5, we show our methods break down for $\lambda$ outside this range, so that new ideas are needed for any improvement in the bound for larger $\lambda$. 
Moreover, the asymptotic bounds of Theorems 1 and 2 cannot be improved, for any valid choice of \( f \) as a function of \( \Delta \), by more than a factor of between 2 and 4. This limits the hypothetical range of \( \lambda \) in Theorem 5.

This follows by considering largest independent sets in a random regular construction or in a suitable blow-up of that construction [12]; see Section 5.

Observe that Theorem 2(ii) trivially fails with a global, rather than local, triangle fraction condition by the presence of a \((\Delta + 1)\)-vertex clique as a subgraph. So Theorems 1 and 2 may appear incompatible, since the former has a global condition, while the latter has a local one. Nevertheless, either assertion in Theorem 2 is indeed (strictly) stronger.

Proof of Theorem 1. Without loss of generality we may assume that \( \varepsilon > 0 \) is small enough so that

\[
(1/2 - \varepsilon^2)(1 - 3\varepsilon^2)(1 - \varepsilon^2) \geq 1/2 - \varepsilon.
\]

Let \( G \) be a graph on \( n \) vertices of maximum degree \( \Delta \) with at most \( \Delta^2 n / f \) triangles. Call \( v \in V(G) \) bad if the number of triangles of \( G \) that contain \( v \) is greater than \( \varepsilon^2 \Delta^2 / f \). Let \( B \) be the set of all bad vertices. Note that \( 3\Delta^2 n / f > |B|\varepsilon^2 \Delta^2 / f \) and so \( |B| < 3\varepsilon^2 n \). Let \( H \) be the subgraph of \( G \) induced by the subset \( V(G) \setminus B \). Then \( H \) is a graph of maximum degree \( \Delta \) on at least \((1 - 3\varepsilon^2)n\) vertices such that the neighbourhood of any vertex spans at most \( \Delta^2 / (\varepsilon^2 f) \) edges. Provided we take \( f \) large enough, either of (i) and (ii) in Theorem 2 implies that \( H \), and thus \( G \), contains an independent set of size

\[
(1/2 - \varepsilon^2)(1 - 3\varepsilon^2)n/\Delta \log(\varepsilon^2 f) \geq (1/2 - \varepsilon^2)(1 - 3\varepsilon^2)(1 - \varepsilon^2)n/\Delta \log f \\
\geq (1/2 - \varepsilon)n/\Delta \log f,
\]

where on the first line we used that \( \varepsilon^2 f \geq f^{1 - \varepsilon^2} \) for \( f \) large enough. \( \square \)

3. An analysis of the hard-core model

A crucial ingredient in the proofs is an occupancy guarantee from the hard-core model, which we establish in Lemma 7 below. This refines an analysis given in [5]. Given \( G \), \( I \in \mathcal{I}(G) \), and \( v \in V(G) \), let us call a neighbour \( u \in N(v) \) of \( v \) externally uncovered by \( I \) if \( u \notin N(I \setminus N(v)) \).

Lemma 6. Let \( G \) be a graph and \( \lambda > 0 \). Let \( I \) be an independent set drawn from the hard-core model at fugacity \( \lambda \) on \( G \).

(i) For every \( v \in V(G) \), writing \( F_v \) for the subgraph of \( G \) induced by the neighbours of \( v \) externally uncovered by \( I \),

\[
\mathbb{P}(v \in I) \geq \frac{\lambda}{1 + \lambda}(1 + \lambda)^{-\mathbb{E}|F_v|}.
\]

(ii) Moreover,

\[
\mathbb{E}|I| \geq \frac{\lambda}{1 + \lambda}|V(G)|(1 + \lambda)^{-\frac{2|E(G)|}{|V(G)|}}.
\]
Proof. The first part follows from two applications of the spatial Markov property of the hard-core model. First, we have

\[ \mathbb{P}(v \in I) = \frac{\lambda}{1 + \lambda} \mathbb{P}(I \cap N(v) = \emptyset), \]

because conditioned on a value \( I \setminus \{v\} = J \) such that \( J \cap N(v) = \emptyset \) there are two realisations of \( I \), namely \( J \) and \( J \cup \{v\} \), giving

\[ \mathbb{P}(v \in I \mid J) = \frac{\lambda|J| + 1}{\lambda|J| + \lambda|J| + 1} = \frac{\lambda}{1 + \lambda}, \]

and conditioned on \( I \setminus \{v\} = J \) such that \( J \cap N(v) \neq \emptyset \), \( v \) cannot be in \( I \).

Second, the spatial Markov property gives that \( I \cap N(v) \) is a random independent set drawn from the hard-core model on \( F_v \). Then \( I \cap N(v) = \emptyset \) if and only if this random independent set in \( F_v \) is empty. It follows that

\[ \mathbb{P}(I \cap N(v) = \emptyset) = \frac{1}{Z_{F_v}(\lambda)} \geq \mathbb{E}(1 + \lambda)^{-|V(F_v)|} \geq (1 + \lambda)^{-\mathbb{E}|V(F_v)|}, \]

since the graph on \( |V(F_v)| \) vertices with largest partition function is the graph with no edges, and by convexity. This completes the proof of (i).

By the fact that \( |V(F_v)| \leq \deg(v) \) we also have for all \( v \in V(G) \) that

\[ \mathbb{P}(v \in I) \geq \frac{\lambda}{1 + \lambda} (1 + \lambda)^{-\deg(v)}. \]

Then (ii) follows by convexity:

\[ \mathbb{E}|I| = \sum_{v \in V(G)} \mathbb{P}(v \in I) \geq \frac{\lambda}{1 + \lambda} |V(G)| \sum_{v \in V(G)} \mathbb{E}[(1 + \lambda)^{-\deg(v)}] \geq \frac{\lambda}{1 + \lambda} |V(G)|(1 + \lambda)^{-\frac{\mathbb{E}|V(F_v)|}{|V(G)|}}. \]

Lemma 7. Let \( G \) be a graph of maximum degree \( \Delta \) in which the neighbourhood of every vertex in \( G \) spans at most \( \Delta^2/f \) edges and \( \lambda, \alpha, \beta > 0 \). Let \( I \) be an independent set drawn from the hard-core model at fugacity \( \lambda \) on \( G \). Then we have, for every \( v \in V(G) \),

(1) \[ \alpha \mathbb{P}(v \in I) + \beta \mathbb{E}|N(v) \cap I| \geq \frac{\lambda}{1 + \lambda} \min_{z \geq 0} \left( \alpha(1 + \lambda)^{-z} + \beta z(1 + \lambda)^{-\frac{2z^2}{\Delta}} \right). \]

Moreover,

(2) \[ \frac{1}{|V(G)|} \mathbb{E}|I| \geq \min_{z \in \mathbb{R}^+} \left\{ \frac{\lambda}{1 + \lambda} (1 + \lambda)^{-z}, \frac{\lambda}{1 + \lambda} \frac{z}{\Delta} (1 + \lambda)^{-\frac{2z^2}{\Delta}} \right\}. \]

Proof. Write \( F_v \) for the graph induced by the neighbours of \( v \) externally uncovered by \( I \) and \( z_v = \mathbb{E}|V(F_v)| \). By Lemma (i) we have

\[ \mathbb{P}(v \in I) \geq \frac{\lambda}{1 + \lambda} (1 + \lambda)^{-z_v}. \]
For the other term, we apply Lemma 6(ii) to the graph \( F_v \), for which by assumption \( \frac{2|E(F_v)|}{|V(F_v)|} \leq \frac{2\Delta^2}{f|V(F_v)|} \). If \( J \) is an independent set drawn from the hard-core model at fugacity \( \lambda \) on \( F_v \), then by convexity

\[
\mathbb{E}|N(v) \cap I| = \mathbb{E}|J| \geq \frac{\lambda}{1 + \lambda} \mathbb{E} \left[ |V(F_v)| (1 + \lambda)^{-\frac{2\Delta^2}{f|V(F_v)|}} \right] \\
\geq \frac{\lambda}{1 + \lambda} z_v (1 + \lambda)^{-\frac{2\Delta^2}{f z_v}},
\]

so (1) follows. For (2), by above we may bound \( \mathbb{E}|I| \) in two distinct ways:

\[
\mathbb{E}|I| \geq \sum_{v \in V(G)} \sum_{u \in N(v)} \mathbb{P}(u \in I) \geq \frac{1}{\Delta} \sum_{v \in V(G)} \mathbb{E}|N(v) \cap I| \\
\geq \left| V(G) \right| \frac{\lambda}{1 + \lambda} z(1 + \lambda)^{-\frac{2\Delta^2}{f z}} \quad \text{and} \\
\mathbb{E}|I| \geq \left| V(G) \right| \frac{\lambda}{1 + \lambda} (1 + \lambda)^{-z},
\]

where \( z \) is the expected number of externally uncovered neighbours of a uniformly random vertex. Now (2) follows.

\section{The Proofs}

\textbf{Proof of Theorem 5.} We optimise (2) in Lemma 7. As the first argument of the maximisation in (2) is increasing in \( z \) while the second is decreasing, the minimum occurs where these two arguments are equal: at \( z \in \mathbb{R}^+ \) satisfying

\[
(1 + \lambda)^{-z} = \frac{z}{\Delta} (1 + \lambda)^{-\frac{2\Delta^2}{f z}}.
\]

Writing \( \Lambda = \Delta \log(1 + \lambda) \) and \( y = z \log(1 + \lambda) \), we have

\[
ye^y = \Lambda e^{\frac{2\Delta^2}{f y}} \geq \Lambda
\]

for any (positive) values of the parameters. This means that \( W(\Lambda) \leq y \), since \( W \) is monotonic. Substituting, it then implies

\[
ye^y = \Lambda e^{\frac{2\Delta^2}{W(\Lambda)}} \leq \Lambda e^{\frac{2\Delta^2}{f W(\Lambda)}},
\]

whence, in terms of the original parameters,

\[
W(\Delta \log(1 + \lambda)) \leq z \log(1 + \lambda) \leq W(\Delta \log(1 + \lambda) e^{\frac{2(\Delta \log(1 + \lambda))^2}{W(\Delta \log(1 + \lambda))}}).
\]

By the assumptions,

\[
z \log(1 + \lambda) \leq W((1 + o(1))\Delta \log(1 + \lambda)) = W(\Delta \log(1 + \lambda)) + o(1).
\]

Substituting this into the first argument in (2), we obtain, as \( f \to \infty \),

\[
\frac{1}{|V(G)|} \mathbb{E}|I| \geq \frac{\lambda}{1 + \lambda} e^{-W(\Delta \log(1 + \lambda)) + o(1)} = (1 + o(1)) \frac{\lambda W(\Delta \log(1 + \lambda))}{(1 + \lambda) \Delta \log(1 + \lambda)}. \qed
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Proof of Theorem 2 (ii). Supposing that we have chosen $\alpha$, $\beta$, and $\lambda$, write

$$g(x) = \frac{\lambda}{1 + \lambda} \left( \alpha (1 + \lambda)^{-x} + \beta x (1 + \lambda)^{-2\Delta^2 f x} \right).$$

By (1) in Lemma 7 and Lemma 4, we have $\chi_f(G) \leq \alpha + \beta \Delta$, provided $g(x) \geq 1$ for all $x \geq 0$. It is easy to verify that with $\alpha, \beta, \lambda > 0$ the function $g$ is strictly convex, so the minimum of $g(x)$ occurs when $g'(x) = 0$, or

$$(1 + \lambda)^{-x} = \frac{\beta}{\alpha} \left( \frac{1}{\log(1 + \lambda)} + \frac{2\Delta^2 f x}{2}\right) (1 + \lambda)^{-2\Delta^2 f x}.$$  

As before, let $z$ satisfy (3). Then by choosing

$$\frac{\alpha}{\beta} = \frac{\Delta}{z} \left( \frac{1}{\log(1 + \lambda)} + \frac{2\Delta^2 f z}{f z} \right),$$

the minimum of $g$ occurs at $z$. Now the equations $g(z) = 1$ and (5) give us values of $\alpha$ and $\beta$ in terms of $\lambda, \Delta, f$. Using (3), this means

$$g(z) = \frac{\lambda}{1 + \lambda} (1 + \lambda)^{-z} (\alpha + \beta \Delta) = 1,$$

and hence by Lemma 4 we obtain

$$\chi_f(G) \leq \alpha + \beta \Delta = \frac{1 + \lambda}{\lambda} (1 + \lambda)^z.$$  

We take $\lambda = o(1)$ as $f \to \infty$ given by $\Delta \log(1 + \lambda) = f^{1/(2+\varepsilon/2)}$, and note the analysis of (3) gives in this case that $z \log(1 + \lambda) = W(\Delta \lambda) + o(1)$. Thus

$$\chi_f(G) \leq \frac{1 + \lambda}{\lambda} (1 + \lambda)^z = (1 + o(1)) (2 + \varepsilon/2) \frac{\Delta}{\log f} < (2 + \varepsilon) \frac{\Delta}{\log f},$$

provided $f_0$ is taken large enough. \hfill $\square$

5. Sharpness

5.1. Occupancy fraction. Since the occupancy fraction is increasing in $\lambda$, it might be intuitive that the lower bound on occupancy fraction that results from the proof of Theorem 5 is also increasing. This is true only up to a point, just as in [8]. Already for $\lambda$ slightly larger than admissible for Theorem 5 under mild assumptions, the method breaks down in the sense that the resulting lower bound is asymptotically smaller. In this case a novel analysis would be necessary; there is almost no slack in our treatment of (3).

Assume as $f \to \infty$ that $\lambda = o(1)$, $\Delta = f^{O(1)}$,

$$\lambda = \omega \left( \frac{\sqrt{f \log f}}{\Delta} \right)^3 \quad \text{and} \quad \lambda = o \left( \frac{f \log f}{\Delta} \right).$$

The choice of $\lambda$ used to obtain Theorem 2 is just shy of the above range. Substituting the extremes of the interval (4) into the second argument of (2),
we derive the following two expressions as \( f \to \infty \), the larger of which necessarily bounds the best guarantee to expect from our approach. First,

\[
\frac{\lambda}{1 + \lambda} \Delta \frac{z}{W(\Delta \log(1 + \lambda)))}
\]

\[
= (1 + o(1)) \log \left( \frac{\Delta}{\lambda} \right) \exp \left( - (1 + o(1)) \frac{2\Delta^2 \lambda^2}{f \log(\Delta \lambda)} \right)
\]

\[
\leq \Delta^{-1} \exp \left( - \omega \left( (\log f)^2 \right) \right) = o \left( \frac{\log(\Delta \lambda)}{\Delta} \right).
\]

Second, using the properties of \( W \) and the assumed bounds on \( \lambda \),

\[
\frac{\lambda}{1 + \lambda} \Delta \frac{z}{W(\Delta \log(1 + \lambda)))}
\]

\[
= (1 + o(1)) \frac{2\Delta^2 \lambda^2}{f \log(\Delta \lambda)} \exp \left( - \frac{2(\Delta \log(1 + \lambda))^2}{f W(e/W(\Delta \log(1 + \lambda))) + \log(\Delta \log(1 + \lambda)))} \right)
\]

\[
= (1 + o(1)) \frac{2\Delta^2 \lambda^2}{f \log(\Delta \lambda)} \exp \left( - \frac{W(\Delta \log(1 + \lambda))}{1 - (1 + o(1)) \frac{\log(\Delta \lambda)}{2(\Delta \lambda)^2}} \right)
\]

\[
= (1 + o(1)) \frac{2\Delta^2 \lambda^2}{f \log(\Delta \lambda)} \exp \left( - \frac{\log(\Delta \lambda)}{\Delta} \right)
\]

\[
= (1 + o(1)) \frac{2}{f} = o \left( \frac{\log(\Delta \lambda)}{\Delta} \right).
\]

5.2. Independence number and fractional chromatic number. By an analysis of the random regular graph \([10]\), there are triangle-free \( \Delta \)-regular graphs \( G_\Delta \) in which as \( \Delta \to \infty \) the largest independent set has size at most

\[
(2 + o(1)) \frac{|V(G_\Delta)|}{\Delta} \log \Delta.
\]

So \( G_\Delta \) certifies Theorems 1 and 2 to be sharp up to an asymptotic factor 2 provided \( f = \Delta^2 - o(1) \) as \( \Delta \to \infty \). For smaller \( f \), let us for completeness reiterate an observation from \([12]\). For \( f < \Delta/2 \), let \( d = f - 1 \) and let \( bG_d \) be the graph obtained from \( G_d \) by substituting each vertex with a clique of size \( b = \lfloor \Delta/f \rfloor \). Then \( bG_d \) is regular of degree \( bf - 1 \leq \Delta \) such that each neighbourhood contains at most \( b^2 f/2 \leq \Delta^2/(2f) \) edges, and so \( bG_d \) has at most \( \Delta^2 |V(bG_d)|/f \) triangles. In \( G_d \) the largest independent set is of size

\[
(2 + o(1)) \frac{|V(G_d)|}{d} \log d = (2 + o(1)) \frac{|V(bG_d)|}{\Delta} \log f.
\]

The same is clearly true in \( bG_d \), and this is an asymptotic factor 4 greater than the lower bound in Theorem 1. Last, observe that for \( f \geq \Delta/2 \) and \( f \leq \Delta^2 - \Omega(1) \), \( G_\Delta \) certifies that Theorems 1 and 2 are at most an asymptotic factor 4 from extremal, and so this holds throughout the range of \( f \).
6. A more basic setting

Due to the close link with off-diagonal Ramsey numbers, we wonder, what occurs when we drop the degree bounding parameter? This may be variously interpreted. For example, over graphs on \( n \) vertices with at most \( n^3/f \) triangles, what is the best lower bound on the independence number?

One can deduce nearly the correct answer to an alternative, local version of this question: over graphs on \( n \) vertices such that each vertex \( v \) is contained in at most \( \deg(v)^2/f \) triangles, where \( 2 \leq f \leq (n-1)^2 + 1 \), what is the best lower bound on the independence number? By a comparison of Theorem [1] and Turán’s theorem (applied to a largest neighbourhood), as \( f \to \infty \) there must be an independent of size at least

\[
(1 + o(1)) \frac{f}{1 + \sqrt{1 + \frac{2f^2}{n \log \min\{f, n\}}}}.
\]

This expression is asymptotic to \( f/2 \) if \( f = o(\sqrt{n \log n}) \) and asymptotic to \( \sqrt{0.5n \log \min\{f, n\}} \) if \( f = \omega(\sqrt{n \log n}) \), so this in particular extends Shearer’s bound on off-diagonal Ramsey numbers to cover any \( f \geq n^{1-o(1)} \).

Over the range of \( f \) as a function of \( n \), (6) is asymptotically sharp up to some reasonably small constant factor by considering the final output of the triangle-free process [5, 9] or a blow-up of that graph by cliques.

We remark that the same argument as above yields a similar bound as in (6) for a more local version, where \( f = \deg(v)^a + 1 \) for some fixed \( a \in [0, 2] \).

By repeatedly extracting independent sets of the size guaranteed in (6) (cf. [6, Lem. 4.1]), it follows that as \( f \to \infty \) the chromatic number is at most

\[
(2 + o(1)) \left(1 + \sqrt{1 + \frac{2f^2}{n \log \min\{f, n\}}} \right) \frac{n}{f}.
\]

Is this the correct asymptotic order for the largest list chromatic number? Is the extra factor 2 unnecessary for the chromatic number? Even improving the bound only for the fractional chromatic number by a factor 2 would be very interesting. This generalises recent conjectures of Cames van Batenburg and a subset of the authors [6], for the triangle-free case \( f = (n-1)^2 + 1 \).

Acknowledgement. We are grateful to Matthew Jenssen, Will Perkins and Barnaby Roberts for helpful discussions, particularly in relation to the analysis in Section 3.

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Korteweg–De Vries Institute for Mathematics, University of Amsterdam, Netherlands.

E-mail address: maths@ewandavies.org

Department of Mathematics, Radboud University Nijmegen, Netherlands.

E-mail address: r.deverclos@math.ru.nl

Department of Mathematics, Radboud University Nijmegen, Netherlands.

E-mail address: ross.kang@gmail.com

Department of Mathematics, Radboud University Nijmegen, Netherlands and LORIA, Université de Lorraine, Nancy, France.

E-mail address: francois.pirot@loria.fr