On cyclic branched coverings of prime knots

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Abstract

We prove that a prime knot $K$ is not determined by its $p$-fold cyclic branched cover for at most two odd primes $p$. Moreover, we show that for a given odd prime $p$, the $p$-fold cyclic branched cover of a prime knot $K$ is the $p$-fold cyclic branched cover of at most one more knot $K'$ non equivalent to $K$. To prove the main theorem, a result concerning the symmetries of knots is also obtained. This latter result can be interpreted as a characterisation of the trivial knot.

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1 Introduction

Two knots $K$ and $K'$ are equivalent if there is a homeomorphism of $S^3$ sending $K$ to $K'$. Given a knot $K \subset S^3$ and an integer $p \geq 2$ one can construct the (total space of the) $p$-fold cyclic cover $M_p(K)$ of $S^3$ branched along $K$: it is a fundamental object in knot theory. There are non-prime knots all of whose cyclic branched covers are homeomorphic. This is no longer true for prime knots: S. Kojima [K] proved that for each prime knot $K \subset S^3$ there is an integer $n_K \geq 2$ such that two prime knots $K$ and $K'$ are equivalent if their $p$-fold cyclic branched covers are homeomorphic for some $p > \max(n_K, n_K')$.

There are many examples of prime knots in $S^3$ which are not equivalent but share homeomorphic $p$-fold cyclic branched covers due to C. Giller [G], C. Livingston [L], Y. Nakanishi [N], M. Sakuma [SaI]. Moreover there is no universal bound for $n_K$.

The main goal of this article is to study the relationship between prime knots and their cyclic branched covers when the number of sheets is an odd prime number.

Definition 1. Let $K \subset S^3$ be a prime knot. A knot $K' \subset S^3$ which is not equivalent to $K$ and which has the same $p$-fold cyclic branched cover as $K$ is called a $p$-twin of $K$. 
There are examples of prime knots, even hyperbolic knots (e.g. Montesinos knots) with an arbitrarily large number of non-equivalent 2-twins. In contrast, for an odd prime number $p$, the number of $p$-twins is very restricted, according to our main result:

**Theorem 1.** Let $K \subset S^3$ be a prime knot. Then:

(i) There are at most two odd prime numbers $p$ for which $K$ admits a $p$-twin.
(ii) For a given odd prime number $p$, $K$ admits at most one $p$-twin.
(iii) Suppose that a prime knot $K$ admits the same knot $K'$ as a $p$-twin and a $q$-twin for two distinct odd prime numbers $p$ and $q$. Then $K$ has two commuting rotational symmetries of order $p$ and $q$ with trivial quotients.

A rotational symmetry of order $p$ of a knot $K \subset S^3$ is an orientation preserving periodic diffeomorphism $\psi$ of the pair $(S^3, K)$ with period $p$ and non-empty fixed-point set disjoint from $K$. We say that the rotational symmetry $\psi$ has trivial quotient if $K/\psi$ is the trivial knot.

For hyperbolic knots Theorem 1 is in fact a consequence of B. Zimmermann's result in [Z1] whose proof uses the orbifold theorem and the Sylow theory for finite groups.

The result in Theorem 1 is sharp: for any pair of coprime integers $p > q > 2$ B. Zimmermann has constructed examples of prime hyperbolic knots with the same $p$ -fold and $q$-fold branched coverings [Z2].

The second named author [P2] has proved that a hyperbolic knot is determined by three cyclic branched covers of pairwise distinct orders. The following, straightforward corollary of Theorem 1 shows that a stronger conclusion holds for arbitrary prime knots when we focus on branched coverings with odd prime orders.

**Corollary 1.** A prime knot is determined by three cyclic branched covers of pairwise distinct odd prime orders. More specifically, for every knot $K$ there is at least one integer $p_K \in \{3, 5, 7\}$ such that $K$ is determined by its $p_K$-cyclic branched cover.

Another straightforward consequence of Theorem 1 is:

**Corollary 2.** Let $K = K_1 \sharp ... \sharp K_t$ and $K' = K'_1 \sharp ... \sharp K'_t$ be two composite knots with the same cyclic branched covers of orders $p_j$, $j = 1, 2, 3$, for three fixed, pairwise distinct, odd prime numbers. Then, after a reordering, the (non-oriented) knots $K_i$ and $K'_i$ are equivalent for all $i = 1, ..., t$.

Part (ii) of Theorem 1 states that for a given odd prime number $p$ a closed, orientable 3-manifold can be the $p$-fold cyclic branched cover of at most two non-equivalent knots in $S^3$. In [BPZ] it has been shown that an integer homology sphere which is a $n$-fold cyclic branched cover of $S^3$ for four distinct odd prime numbers $n$ is in fact $S^3$. By putting together these two results we get the following corollary:

**Corollary 3.** Let $M$ be an irreducible integer homology 3-sphere. Then: there are at most three distinct knots in $S^3$ having $M$ as cyclic branched cover of odd prime order.
Our main task will be to prove Theorem 1 for a satellite knot: that is a knot whose exterior $S^3\setminus \mu(K)$ has a non trivial Jaco-Shalen-Johannson decomposition $JS$, $JS$ (in the sequel we use $JSJ$-decomposition for short). Otherwise the knot is called simple: in this case, due to Thurston’s hyperbolization theorem $\text{(12)}$, its exterior is either hyperbolic, and the proof follows already from the works in $\text{(12)}$ and $\text{(Z1)}$, or it is a torus knot and a simple combinatorial argument applies.

The proof of Theorem 1 for satellite knots relies on the study of the partial symmetries of the exterior $E(K)$ of $K$ induced by the covering transformations associated to the twins of $K$ and on the localization of their axes of fixed points in the components of the $JSJ$-decomposition of $E(K)$. In particular the proof uses the following result about rotational symmetries of prime knots which is of interest in its own right.

**Theorem 2.** Let $K$ be a knot in $S^3$ admitting three rotational symmetries with trivial quotients and whose orders are three pairwise distinct numbers $> 2$. Then $K$ is the trivial knot.

Since the trivial knot admits a rotational symmetry with trivial quotient of order $p$ for each integer $p \geq 2$, the above Theorem 2 can be interpreted as a characterisation of the trivial knot, i.e. a knot is trivial if and only if it admits three rotational symmetries of pairwise distinct orders $> 2$ and trivial quotients.

## 2 Rotational symmetries of knots

A rotational symmetry of order $p$ of a knot $K \subset S^3$ is an orientation preserving, periodic diffeomorphism $\psi$ of the pair $(S^3, K)$ of order $p$ and non-empty fixed-point set disjoint from $K$. We say that the rotational symmetry $\psi$ has trivial quotient if $K/\psi$ is the trivial knot.

**Remark 1.** Let $K$ be a knot and let $\psi$ be a rotational symmetry of $K$ of order $p$. The symmetry $\psi$ lifts to a periodic diffeomorphism $\tilde{\psi}$ of the $p$-fold branched cover $M_p(K)$ of order $p$ and non-empty fixed-point set, which commutes with the covering transformation $h$ of $K$ acting on $M_p(K)$. Then the symmetry $\tilde{\psi}$ has trivial quotient if and only if $(M, \text{Fix}(\tilde{\psi}))/ < \tilde{\psi} > \cong (S^3, K')$. Moreover in this case $K$ and $K'$ have a common quotient link with two trivial components (see $\text{(Z1)}$).

In particular a symmetry of a knot $K$ induced by the covering transformation associated to a $p$-twin $K'$ of $K$ is a $p$-rotational symmetry with trivial quotient. This follows from the fact that the two commuting deck transformations associated to the two twins induce on $M_p(K)$ a $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$-cover of $S^3$ branched over a link with two unknotted components.

The main result of this section is the following theorem whose assertion (i) is Theorem 2.

**Theorem 3.** Let $K$ be a knot in $S^3$.

(i) Assume that $K$ admits three rotational symmetries with trivial quotients and whose orders are three pairwise distinct numbers $> 2$. Then $K$ is the trivial knot.
Assume that $K$ admits two rotational symmetries $\psi$ and $\varphi$ with trivial quotients and of distinct orders $> 2$. Then the fixed-point sets $\text{Fix}(\psi)$ and $\text{Fix}(\varphi)$ sit in the $\text{JSJ}$-component of $E(K)$ which contains $\partial E(K)$.

We prove first a weaker version of Theorem 2 that we shall use in the remaining of this section (see also [P2, Scholium]).

**Proposition 1.** Let $K$ be a knot in $S^3$ admitting three commuting rotational symmetries of orders $p > q > r \geq 2$. If the symmetries of order $q$ and $r$ have trivial quotients, then $K$ is the trivial knot.

**Proof.** Denote by $\varphi$, $\psi$ and $\rho$ the three symmetries. If two of them -say $\varphi$, $\psi$- have the same axis, then by hypothesis the one with smaller order -say $\psi$- must have trivial quotient, i.e. $K/\psi$ is the trivial knot. Since the three symmetries commute, $\varphi$ induces a rotational symmetry of $K/\psi$ which is non trivial for the order of $\varphi$ is larger than that of $\psi$. The axis $A$ of this induced symmetry is the image of $\text{Fix}(\psi)$ in the quotient by the action of $\psi$. In particular $K/\psi$ and $A$ form a Hopf link and $K$ is the trivial knot: this follows from the equivariant Dehn lemma, see [H]. We can thus assume that the axes are pairwise disjoint.

Note that even if $r = 2$, since the symmetries commute, the symmetry of order 2 cannot act as a strong inversion on the axes of the other two symmetries. In this case we would have that the axis of $\rho$, which is a trivial knot, admits two commuting rotational symmetries, $\varphi$ and $\psi$, with distinct axes, which is impossible: this follows, for instance, from the fact (see [EL, Thm 5.2]) that one can find a fibration of the complement of the trivial knot which is equivariant with respect to the two symmetries.

The proof of Theorem 2 is based on a series of Lemmata.

The first result concerns the structure of the $\text{JSJ}$-decomposition of the $p$-fold cyclic branched cover $M$ of a prime knot $K \subset S^3$. Let $h$ be the covering transformation, then the quotient space $M/\langle h \rangle$ has a natural orbifold structure, denoted by $O_p(K)$, with underlying space $S^3$ and singular locus $K$ with local group a cyclic group of order $p$ (cf. [BMP, Chap. 2]). According to Bonahon-Siebenmann [BS] and the orbifold theorem [BoP], [CHK], such an orbifold admits a characteristic collection of toric 2-suborbifolds, which split $O_p(K)$ into geometric suborbifolds. Moreover this characteristic collection of toric 2-suborbifolds lifts to the $\text{JSJ}$-collection of tori for $M$. It follows that for $p > 2$ the Bonahon-Siebenmann characteristic collection of toric 2-suborbifolds coincides with the $\text{JSJ}$-collection of tori for the exterior $E(K) = S^3 \setminus \mathcal{U}(K)$ of $K$.

**Lemma 1.** Let $p > 2$ be an integer and let $M$ be the $p$-fold cyclic branched cover of a prime knot $K$ in the $3$-sphere. Then:

(a) The dual graph associated to the $\text{JSJ}$-decomposition of $M$ is a tree.

(b) The fixed-point set of the group of deck transformations is entirely contained in one geometric piece of the decomposition.

**Proof.**

(a) Note, first of all, that $M$ is irreducible since $K$ is prime. Hence the Bonahon-Siebenmann decomposition of the orbifold $O_p(K)$ lifts to the $\text{JSJ}$-collection.
for $M$ since $p > 2$. Moreover, the graph dual to the Bonahon-Siebenmann decomposition of the orbifold $\mathcal{O}_p(K)$, which lifts to the JSJ-decomposition for $M$, is a tree. Cutting along a torus of former decomposition and considering the component $C$ which does not contain $K$ one gets the complement of a knot in $S^3$. The lemma follows now from the fact that each connected component of a cyclic branched cover of $C$ has a unique boundary component. 

(b) Note that the group of deck transformations preserves the JSJ-collection of tori. If $p > 2$, the fixed-point set of this group does not meet any torus of the JSJ-decomposition, because each JSJ-torus is separating and the fixed point set is connected. Since the fixed point set is connected, it is entirely contained in one geometric piece of the JSJ-decomposition.

Remark 2. Note that the conclusion of the first part of the lemma holds also for covers of order 2. For covers of prime order this property follows also from the fact that $M_p(K)$ is a $\mathbb{Z}/p\mathbb{Z}$-homology sphere (see [Go]).

Lemma 2. If a knot $K \subset S^3$ has a rotational symmetry with trivial quotient, then $K$ is prime.

Proof. M. Sakuma [Sa2, Thm 4] showed that the only possible rotational symmetries of a composite knot must either permute cyclically its prime summands, or act as a symmetry of one prime summand while permuting the remaining ones. In particular the quotient knot cannot be trivial.

The following is a key lemma for the proofs of Theorems 1 and 3.

Lemma 3. Let $K$ be a knot admitting a rotational symmetry $\psi$ of order $p > 2$ and consider the JSJ-decomposition of its exterior $E(K) = S^3 \setminus \mathcal{U}(K)$.

(i) $T$ is a torus of the decomposition which does not separate $\partial E(K)$ from $\text{Fix}(\psi)$ if and only if the orbit $\psi T$ has $p$ elements.

(ii) Under the assumption that $\psi$ has trivial quotient, each torus which separates $\partial E(K)$ from $\text{Fix}(\psi)$ corresponds to a prime companion of $K$ on which $\psi$ acts with trivial quotient.

Proof. Let $T$ be a torus of the JSJ-decomposition of $E(K)$ considered as a torus inside $S^3$: $T$ separates the 3-sphere into a solid torus containing $K$ and the exterior of a non trivial knot $K_T$ which is a companion of $K$. Note that, since the order of the symmetry $\psi$ is $> 2$, its axis cannot meet $T$. Assume that the axis $\text{Fix}(\psi)$ of the symmetry is contained in the solid torus.

If the orbit of $T$ under $\psi$ does not contain $p$ elements, then a non-trivial power of $\psi$ leaves $T$ invariant, and thus it also leaves the solid torus and the knot exterior invariant. The restriction of this power of $\psi$ to the solid torus acts as a rotation of order $m > 1$ around its core and leaves invariant each meridian. This non-trivial power of $\psi$ would then be a rotational symmetry about the non trivial knot $K_T$ which is absurd because of the proof of the Smith’s conjecture (see [MB]).

For the reverse implication, it suffices to observe that the geometric pieces of the decomposition containing $\partial E(K)$ and $\text{Fix}(\psi)$ must be invariant by $\psi$. 

5
and so must be the unique geodesic segment joining the corresponding vertices in the tree dual to the decomposition.

For the second part of the Lemma, note that $K_T/\psi$ is a companion of $K/\psi$, which is trivial by hypothesis. In particular $K_T/\psi$ is also trivial and thus, by Lemma 2 must be prime.

The following lemma gives a weaker version of assertion (ii) of Theorem 3 under a commutativity hypothesis:

Lemma 4. Let $K$ be a prime knot admitting two commuting rotational symmetries $\psi$ and $\varphi$ of orders $p, q > 2$. Then:

(i) The fixed-point sets of $\psi$ and $\varphi$ are contained in the same geometric component of the JSJ-decomposition for $E(K)$;

(ii) If $\psi$ has trivial quotient and $p \neq q$, the fixed-point sets of $\psi$ and $\varphi$ sit in the component which contains $\partial E(K)$.

Proof.

Part (i) Let $v_\psi$ (respectively $v_\varphi$) the vertex of the graph $\Gamma_K$ dual to the JSJ-decomposition of $E(K)$ corresponding to the geometric component containing $Fix(\psi)$ (respectively $Fix(\varphi)$). Since the two rotational symmetries commute, $\psi$ (respectively $\varphi$) must leave $Fix(\varphi)$ (respectively $Fix(\psi)$) invariant, and so the geodesic segment of $\Gamma_K$ joining $v_\psi$ to $v_\varphi$ must be fixed by the induced actions of $\psi$ and $\varphi$ on $\Gamma_K$. If this segment contains an edge $e$, the corresponding JSJ-torus $T$ in $E(K)$ cannot separate both $Fix(\psi)$ and $Fix(\varphi)$ from $\partial E(K)$. This would contradict part (i) of Lemma 3.

Part (ii) Let $M$ be the $p$-fold cyclic branched cover of $K$ and let $h$ be the associated covering transformation. According to Remark 4 the lift $\tilde{\psi}$ of $\psi$ to $M$ is the deck transformation of a cyclic cover of $S^3$ branched along a knot $K'$. Note that both $\tilde{\psi}$ and $\tilde{\varphi}$ (the lift of $\varphi$ to $M$) commute on $M$ with the covering transformation $h$. In particular $\tilde{\varphi}$ and $h$ induce commuting rotational symmetries of $K'$ with order $q$ and $p$ respectively. According to part (i), $Fix(h)$ and $Fix(\tilde{\psi})$ belong to the same piece of the JSJ-decomposition of $M$. Since $Fix(h)$ maps to $K$ and $p \neq q$, $Fix(\varphi)$ sits in the JSJ-piece of $E(K)$ which contains $\partial E(K)$ and the conclusion follows since $Fix(\psi)$ belongs to the same JSJ-piece as $Fix(\varphi)$.

Lemma 5. Let $K$ be a knot admitting a rotational symmetry $\psi$ with trivial quotient and of order $p > 2$. Let $M$ be the $p$-fold cyclic branched cover of $K$ and denote by $\pi : M \longrightarrow (S^3, K)$ the associated branched cover. Let $T$ be a torus in the JSJ-collection of tori of $E(K)$.

(i) The torus $T$ is left invariant by $\psi$ if and only if $\pi^{-1}(T)$ is connected.

(ii) If $\pi^{-1}(T)$ is connected, then the companion $K_T$ of $K$ corresponding to $T$ is prime and the winding number of $T$ with respect to $K$ is prime with $p$, so in particular it is not zero.

(iii) The torus $T$ is not left invariant by $\psi$ if and only if $\pi^{-1}(T)$ has $p$ components.
Part (i). According to Remark 1, the $p$-fold cyclic branched cover $M$ of $K$ admits two commuting diffeomorphisms of order $p$, $h$ and $h' = \tilde{\psi}$, such that: $(M, Fix(h))/\langle h \rangle \cong (S^3, K')$ on which $h'$ induces a $p$-rotational symmetry $\psi$ with trivial quotient, and $(M, Fix(h'))/\langle h' \rangle \cong (S^3, K)$ on which $h$ induces a $p$-rotational symmetry $\psi'$ with trivial quotient. The preimage $\pi^{-1}(T) = \tilde{T}$ is connected if and only if it corresponds to a torus $\tilde{T}$ of the JSJ-decomposition of $M$ which is left invariant by $h$. If, by contradiction, $\psi$ does not leave $T$ invariant, then the $h'$-orbit of $\tilde{T}$ consists of $m > 1$ elements. Cutting $M$ along these $m$ separating tori, one gets $m + 1$ connected components.

Claim 1. Both $Fix(h)$ and $Fix(h')$ must be contained in the same connected component.

Proof. The diffeomorphism $h'$ cyclically permutes the $m$ connected components which do not contain $Fix(h')$. Since $h$ and $h'$ commute, $h$ leaves invariant each of these $m$ components and it acts in the same way on each of them (that is, the restrictions of $h$ to each component are conjugate). Since the set $Fix(h)$ is connected, the claim follows.

The $m$ components permuted by $h'$ project to a connected submanifold of the exterior $\mathcal{E}(K')$ of the knot $K'$ with connected boundary the image $T'$ of $T$. This submanifold is invariant by the action of $\psi'$ but does not contain $Fix(\psi')$. This contradicts Lemma 3(i). To conclude the proof of Lemma 5(i), it suffices to observe that $h$ and $h'$ play symmetric roles.

Part (ii) The first part of assertion (ii) is a straightforward consequence of assertion (i) and of Lemma 1. The second part follows from the fact that for $\pi^{-1}(T)$ to be connected, the winding number of $T$ and $p$ must be coprime.

Part (iii) is a consequence of the proof of part (i) of Lemma 5 and of the fact that $h$ and $h'$ play symmetric roles.

Proof of Theorem 3. The proof is achieved in three steps.

Step 1. Theorem 3 is true under the assumption that the rotational symmetries commute pairwise.

In this case, assertion (i) is the statement of Proposition 1. Assertion (ii) follows from Lemma 1

Step 2. Theorem 3 is true under the assumption that every companion of $K$ is prime (i.e. $K$ is totally prime) and has non vanishing winding number (i.e. $K$ is pedigreed).

Assume that we are in the hypotheses of Theorem 3. Then Lemma 2 assures that $K$ is a prime knot. If $K$ is also totally prime and pedigreed then M. Sakuma [Sa2, Thm 4 and Lemma 2.3] proved that, up to conjugacy, the rotational symmetries belong either to a finite cyclic subgroup or to an $S^1$-action in $Diff^{+,+}(S^3, K)$. Thus after conjugacy, step 1 applies. For part (ii) note that the distances of the fixed point set of the symmetries to the vertex containing $\partial E(K)$ in the JSJ-graph $\Gamma_K$ do not change by conjugacy.

Step 3. Reduction of the proof to step 2.
If $K$ is not totally prime or pedigreed, then it is non-trivial. We shall construct a non trivial, totally prime and pedigreed knot verifying the hypothesis of Theorem 3. Assertion (i) then follows by contradiction. For Assertion (ii) we need to verify that the construction does not change the distance of the pieces containing the axes of rotations to the root containing $\partial E(K)$. Roughly speaking we consider the $JSJ$-tori closest to $\partial E(K)$ and corresponding either to non-prime or to winding number zero companions. Then we cut $E(K)$ along these tori and keep the component $W$ containing $\partial E(K)$ and suitably Dehn-fill $W$ along these tori to get the exterior of a non-trivial knot $\hat{K}$ in $S^3$, which verifies Sakuma’s property.

More precisely, let $\Gamma_K$ be the tree dual to the $JSJ$-decomposition of $E(K)$ and let $\Gamma_0$ be its maximal (connected) subtree with the following properties:

- $\Gamma_0$ contains the vertex $v_\partial$ corresponding to the geometric piece whose boundary contains $\partial E(K)$. Note that the geometric piece of the decomposition corresponding to $v_\partial$ cannot be a composing space for $K$ is prime;
- no vertex of $\Gamma_0$ corresponds to a composing space (i.e. a space homeomorphic to a product $S^1 \times B$ where $B$ is an $n$-punctured disc with $n \geq 2$);
- no edge of $\Gamma_0$ corresponds to a torus whose meridian has linking number $0$ with $K$.

Denote by $X(\Gamma_0)$ the submanifold of $E(K)$ corresponding to $\Gamma_0$.

The following claim describes certain properties of $X(\Gamma_0)$ with respect to a rotational symmetry $\psi$ of $(S^3,K)$.

**Claim 2.** Let $\psi$ be a rotational symmetry of $(S^3,K)$ with order $p > 2$ and trivial quotient. Then:

(i) The fixed-point set of $\psi$ is contained in $X(\Gamma_0)$.

(ii) The tree $\Gamma_0$ is invariant by the automorphism of $\Gamma_K$ induced by $\psi$ and the submanifold $X(\Gamma_0)$ is invariant by $\psi$.

**Proof.**

**Assertion (i).** Let $\gamma$ be the unique geodesic segment in $\Gamma_K$ which joins the vertex $v_\partial$ to the vertex corresponding to the geometric piece containing $\partial E(K)$ (see Lemma 1; note that here we use $p > 2$). According to assertion (ii) of Lemma 3 no vertex along $\gamma_i$ can be a composing space. Since the linking number of $K$ and $Fix(\psi)$ must be coprime with $p$, no torus corresponding to an edge of $\gamma$ can have winding number $0$ (see Lemma 5).

**Assertion (ii).** This is just a consequence of the maximality of $\Gamma_0$ and the fact that elements of the group $\langle \psi \rangle$ generated by $\psi$ must preserve the $JSJ$-decomposition of $E(K)$ and the winding numbers of the $JSJ$-tori, as well as send composing spaces to composing spaces.

Let $\pi : M_p(K) \longrightarrow (S^3,K)$ be the $p$-fold cyclic branched cover. Let $T$ be a torus of the $JSJ$-collection of tori for $E(K)$. Denote by $E_T$ the manifold obtained as follows: cut $E(K)$ along $T$ and choose the connected component whose boundary consists only of $T$. Note that $E_T$ is the exterior of the companion $K_T$ of $K$ corresponding to $T$. 
Claim 3. Let \( T \) be a torus of \( \partial X(\Gamma_0) \setminus \partial E(K) \). The preimage \( \pi^{-1}(T) \) consists of \( p \) components, each bounding a copy of \( E_T \) in \( M_p(K) \). In particular, there is a well-defined meridian-longitude system \((\mu_T, \lambda_T)\) on each boundary component of \( X(\Gamma_0) \), different from \( \partial E(K) \), which is preserved by taking the \( p \)-fold cyclic branched covers.

Proof. According to Lemma 5, the preimage of \( T \) is either connected or consists of \( p \) components. If the preimage of \( T \) were connected, the tree \( \Gamma_0 \) would not be maximal according to Lemma 5(ii). The remaining part of the Claim is then easy. \( \Box \)

We wish now to perform Dehn fillings on the boundary of \( X(\Gamma_0) \) in order to obtain a totally prime and pedigreed knot admitting pairwise distinct rotational symmetries with trivial quotients. On each component \( T \) of \( \partial X(\Gamma_0) \setminus \partial E(K) \) we fix the curve \( \alpha_n = \lambda_T + n\mu_T \).

Claim 4. For all but finitely many \( n \in \mathbb{Z} \), the Dehn filling of each component \( T \) of \( \partial X(\Gamma_0) \setminus \partial E(K) \) along the curve \( \alpha_n \) produces the exterior of a non-trivial, prime and pedigreed knot \( \hat{K} \) in \( S^3 \).

Proof. Note that by the choice of surgery curves the resulting manifold \( \hat{X}(\Gamma_0) \) is the exterior of a knot \( \hat{K} \) in the 3-sphere, i.e. \( \hat{X}(\Gamma_0) \subset S^3 \), and thus is irreducible. We distinguish two cases:

1. The JSJ-component \( X_T \) of \( X(\Gamma_0) \) adjacent to \( T \) is Seifert fibred. Then, by the choice of \( \Gamma_0 \), \( X_T \) is a cable space (i.e. the exterior of a \((a,b)\)-torus knot in the solid torus bounded by \( T \) in \( S^3 \)). Moreover the fiber \( f \) of the Seifert fibration of \( X(\Gamma_0) \) is homologous to \( a\mu_T + b\lambda_T \) on \( T \) and the intersection number \( |\Delta(f, \mu_T)| = b > 1 \). The intersection number of the filling curve \( \alpha_n \) with the fiber \( f \) is then \( |\Delta(f, \alpha_n)| = |na-b| \) and is \( > 1 \) for all but finitely many \( n \in \mathbb{Z} \).

In this case the resulting manifold \( X_T(\alpha_n) \) is the exterior of a non trivial torus knot which is prime and pedigreed [CGLS].

2. The JSJ-component \( X_T \) of \( X(\Gamma_0) \) adjacent to \( T \) is hyperbolic. By Thurston’s hyperbolic Dehn filling theorem [11] Chap. 5 (see also [BoPAppendix B]) for all but finitely many \( n \in \mathbb{Z} \) the Dehn filling of each component \( T \subset \partial X_T \cap (\partial X(\Gamma_0) \setminus \partial E(K)) \) along the curve \( \alpha_n \) produces a hyperbolic manifold \( X_T(\alpha_n) \) with finite volume.

Therefore for all but finitely many \( n \)'s \( \in \mathbb{Z} \) the Dehn filling of each component \( T \subset \partial X(\Gamma_0) \setminus \partial E(K) \) along the curve \( \alpha_n \) produces a \( \partial \)-irreducible 3-manifold \( \hat{X}(\Gamma_0) \subset S^3 \) such that each Seifert piece of its JSJ-decomposition is either a Seifert piece of \( X(\Gamma_0) \) or a non-trivial torus knot exterior. Hence it corresponds to the exterior of a non-trivial knot \( \hat{K} \subset S^3 \) which is totally prime. It is also pedigreed by the choice of \( \Gamma_0 \). \( \Box \)

Let \( \psi \) a rotational symmetry of \((S^3, K)\) with order \( p > 2 \). Then the restriction \( \psi|_{(X, \alpha_n)} \), given by Claim 2 extends to \( \hat{X}(\Gamma_0) \), giving a \( p \)-rotational symmetry \( \hat{\psi} \) of the non-trivial, totally prime and pedigreed knot \((S^3, \hat{K})\). In order to be able to apply step 2 to the knot \( \hat{K} \) and the induced rotational symmetries, we still need to check that the rotational symmetry \( \hat{\psi} \) has trivial quotient when \( \psi \) has trivial quotient. This is the aim of the following:
Claim 5. If the knot $K/\psi$ is trivial, then the knot $\hat{K}/\hat{\psi}$ is trivial.

Proof. Let $\pi: M_p(K) \rightarrow (S^3, K)$ be the $p$-fold cyclic branched cover. Let $h$ be the deck transformation of this cover and $h'$ the lift of $\psi$. According to Remark II $h'$ is the deck transformation for the $p$-fold cyclic cover of the 3-sphere branched along a knot $K'$. Note that, by Claim $M_p(K) \setminus \pi^{-1}(X(\Gamma_0) \cup U(K))$ is a disjoint union of $p$ copies of $E(K) \setminus X(\Gamma_0)$. It follows that the $p$-fold cyclic branched cover $M_p(K)$ of $\hat{K}$ is the manifold obtained by a $(\lambda T + n\mu T)$-Dehn filling on all the boundary components of $\pi^{-1}(X(\Gamma_0) \cup U(K))$. The choice of the surgery shows that both $h$ and $h'$ extend to diffeomorphisms $\hat{h}$ and $\hat{h}'$ of order $p$ of $M_p(\hat{K})$. By construction we have that $M_p(\hat{K})/\hat{h} \cong S^3$. In the same way $M_p(\hat{K})/\hat{h}' \cong S^3$ by cutting off a copy of $E(K) \setminus X(\Gamma_0)$ and Dehn filling along $\partial X(\Gamma_0)$. The choice of the surgery curve assures that the resulting manifold is again $S^3$ and the conclusion follows from Remark II. 

From the non-trivial prime knot $K$, we have thus constructed a non-trivial, totally prime and pedigreed knot $\hat{K}$ which has the property that every rotational symmetry $\psi$ of $K$ with trivial quotient and order > 2 induces a rotational symmetry $\hat{\psi}$ of $\hat{K}$ with trivial quotient and the same order. Moreover, by the choice of the Dehn filling curve in the construction of $\hat{K}$, the vertex containing $Fix(\hat{\psi})$ remains at the same distance from the vertex containing $\partial E(K)$ in the JSJ-tree $\Gamma_{\hat{K}}$ as the vertex containing $Fix(\psi)$ from the vertex containing $\partial E(K)$ in the JSJ-tree $\Gamma_K$. Then the conclusion is a consequence of step 2. 

3 Twins of a prime knot

In this section we prove Theorem I. If $K$ is trivial, the theorem is a consequence of the proof of Smith’s conjecture (see [MB]). We shall thus assume in the remaining of this section that $K$ is non trivial and $p$ is an odd prime number.

Let $M$ be the common $p$-fold cyclic branched cover of two prime knots $K$ and $K'$ in $S^3$. Let $h$ and $h'$ be the deck transformations for the coverings of $K$ and $K'$ respectively. By the orbifold theorem [BoP], see also [CHK] one can assume that $h$ and $h'$ act geometrically on the geometric pieces of the JSJ-decomposition of $M$, i.e. by isometries on the hyperbolic pieces and respecting the fibration on the Seifert fibred ones.

The following lemma describes the Seifert fibred pieces of the JSJ-decomposition of the $p$-fold branched cyclic cover $M$ (see also [Ja] and [K, Lemma 2]).

Lemma 6. Let $p$ be an odd prime integer and let $M$ be the $p$-fold cyclic branched cover of $S^3$ branched along a prime, satellite knot $K$. If $V$ is a Seifert piece in the JSJ-decomposition for $M$. Then the base $B$ of $V$ can be:

1. A disc with 2, $p$ or $p+1$ singular fibres;
2. A disc with 1 hole, i.e. an annulus, with 1 or $p$ singular fibres;
3. A disc with $p-1$ holes and 1 singular fibre;
4. A disc with $p$ holes and 1 singular fibre;
5. A disc with \( n \) holes, \( n \geq 2 \).

**Proof.** It suffices to observe that \( V \) projects to a Seifert fibred piece \( V' \) of the Bonahon-Siebenmann decomposition for the orbifold \( \mathcal{O}_p(K) \). There are four possible cases:

(a) \( V' \) contains \( K \): \( V' \) is topologically a non trivially fibred solid torus and \( K \) is a regular fibre of the fibration, i.e. a torus knot \( K(a,b) \), since it cannot be the core of the fibred solid torus. The knot \( K \) lifts to a singular fibre of order \( p \) if \( p \) does not divide \( ab \) and to a regular fibre otherwise. The core of the solid torus is a singular fibre of order -say- \( a \). It lifts to a regular fibre if \( a = p \), a singular fibre of order \( a/(a,p) \) if \( p \) does not divide \( b \), or to \( p \) singular fibres of order \( a \) if \( p \) divides \( b \). Thus \( V \) has \( p \) boundary components if \( p \) divides \( a \) and 1 otherwise. An Euler characteristic calculation shows that \( B \) is either a disc with 2 or \( p \) singular fibres, or a disc with \( p - 1 \) holes and with at most 1 singular fibre.

(b) \( V' \) is the complement of a torus knot \( K(a,b) \) in \( S^3 \). In this case, \( V \) is either a copy of \( V' \), and \( B \) is a disc with 2 singular fibres or \( V \) is a true \( p \)-fold cover of \( V' \). In this case \( V \) has exactly one boundary component. Reasoning as in case (a), we see that the two singular fibres of \( V' \) must lift to either 2 singular fibres, or 1 regular fibre and \( p \) singular fibres or 1 singular fibre and \( p \) singular fibres. In particular \( B \) is a disc with 2, \( p \) or \( p + 1 \) singular fibres.

(c) \( V' \) is the complement of a torus knot \( K(a,b) \) in a solid torus, i.e. a cable space, and its base is an annulus with 1 singular fibre. Reasoning as in (b) we find that \( B \) can be a disc with 1 hole and 1 or \( p \) singular fibres or a disc with \( p \) holes and at most 1 singular fibre.

(d) \( V' \) is a composing space with at least 3 boundary components and thus so is \( V \). More precisely, note that either \( V' \) lifts to \( p \) disjoint copies of itself, or \( V \) and \( V' \) are homeomorphic and \( V' \) is obtained by quotienting \( V \) via the \( p \)-translation along the \( S^1 \) fibre. In this case \( B \) is a disc with at least 2 holes.

This analysis ends the proof of Lemma 6. \( \square \)

**Proposition 2.** Let \( M \) be the common \( p \)-fold cyclic branched cover of two prime knots \( K \) and \( K' \) in \( S^3 \), \( p \) an odd prime number, and let \( h \) be the deck transformation for the covering of \( K \). Let \( \Gamma \) be the tree dual to the JSJ-decomposition of \( M \). The deck transformation \( h' \) for the covering of \( K' \) can be chosen (up to conjugacy) in such a way that:

(i) There exists a subtree \( \Gamma_f \) of \( \Gamma \) on which the actions induced by \( h \) and \( h' \) are trivial;

(ii) The vertices of \( \Gamma \) corresponding to the geometric pieces of the decomposition which contain \( \text{Fix}(h) \) and \( \text{Fix}(h') \) belong to \( \Gamma_f \);

(iii) Let \( M_f \) the submanifold of \( M \) corresponding to \( \Gamma_f \). The restrictions of \( h \) and \( h' \) to \( M_f \) commute.

**Proof.** The proof relies on the study of the actions of the two covering transformations \( h \) and \( h' \) on the JSJ-decomposition of the common \( p \)-fold cyclic branched covering \( M \). Since \( \Gamma \) is finite, the group generated by the tree automorphisms induced by \( h \) and \( h' \) is finite as well. Standard theory of group actions on trees assures that a finite group acting on a tree without inversion...
must have a global fixed point and that its fixed-point set is connected. Thus part (i) of the proposition follows, using the fact that \( h \) and \( h' \) have odd orders.

Choose now \( h' \), up to conjugacy in \( \Diff^+(M) \), in such a way that \( \Gamma_f \) is maximal. We want to show that, in this case, \( M_f \) contains \( \text{Fix}(h) \) and \( \text{Fix}(h') \).

Assume by contradiction that the vertex \( v_h \) of \( \Gamma \) corresponding to the geometric piece containing \( \text{Fix}(h) \), whose existence is ensured by Lemma 1, does not belong to \( \Gamma_f \). Let \( \gamma_h \) the unique geodesic path in \( \Gamma \) connecting \( v_h \) to \( \Gamma_f \). Let \( e_h \) the edge in \( \gamma_h \) adjacent to \( \Gamma_f \) and denote by \( T \) the corresponding torus of the JSJ-collection of tori for \( M \). Let \( U \) be the connected component of \( M \setminus T \) which contains \( \text{Fix}(h) \). Consider the \( \langle h, h' \rangle \)-orbit of \( U \). This orbit is the disjoint union of \( h \) (and \( h' \)) orbits of \( U \).

Claim 6. The orbit \( \langle h, h' \rangle U \) must contain an \( h \)-orbit, different from \( \{U\} \) and containing a unique element.

Proof. Otherwise all the \( h \)-orbits in \( \langle h, h' \rangle U \) different from \( \{U\} \) would have \( p \) elements, since \( p \) is prime. In particular, the cardinality of \( \langle h, h' \rangle U \) would be of the form \( kp + 1 \). This implies that at least one of the \( h' \)-orbits in \( \langle h, h' \rangle U \) must contain one single element \( U' \). Up to conjugacy with an element of \( \langle h, h' \rangle \) (whose induced action on \( \Gamma_f \) is trivial), we can assume that \( U = U' \), contradicting the hypothesis that \( h' \) was chosen up to conjugacy in such a way that \( \Gamma_f \) is maximal.

Let \( U' \neq U \) the element of \( \langle h, h' \rangle U \) such that \( h(U') = U' \). Note that \( U \) and \( U' \) are homeomorphic since they belong to the same \( \langle h, h' \rangle \)-orbit.

Claim 7. \( U \) is homeomorphic to the exterior \( E(K) \) of a knot \( K \subset S^3 \) admitting a free symmetry of order \( p \).

Proof. The first part of the Claim follows from the fact that, by maximality of \( \Gamma_f \), \( h' \) cannot leave \( U \) invariant, so must freely permute \( p \) copies of \( U \) belonging to \( \langle h, h' \rangle U \). Thus \( U \) must appear as a union of geometric pieces of the JSJ-splitting of \( E(K') \). The second part follows from the fact that \( h \) must act freely on \( U' \) which is homeomorphic to \( U \).

Remark 3. Note that the quotient of \( U \) by the action of its free symmetry of order \( p \) is also a knot exterior because \( h \) acts freely on \( U' \) and \( U' \) must project to a union of geometric pieces of the JSJ-splitting of \( E(K) \).

Claim 8. \( U \) admits a rotational symmetry of order \( p \) whose quotient \( U/\langle h \rangle \) is topologically a solid torus.

Proof. The quotient \( U/\langle h \rangle \) is obtained by cutting \( S^3 \) along an essential torus in \( E(K) \). Since \( K \subset U/\langle h \rangle \), it must be a solid torus.

It follows from Claim 6 and Lemma 2 that the knot \( K \) is prime. Moreover, according to Claims 7 and 8, \( K \) admits a rotational symmetry and a free symmetry, both of order \( p \). This is however impossible because M. Sakuma [Sa2, Thm. 3] showed that a prime knot can only have one symmetry of odd order up to conjugacy. This contradiction proves part (ii) of Proposition 2.
To prove part (iii) we shall consider two cases, according to the structure of $\Gamma_f$.

**Case (a):** $\Gamma_f$ contains an edge. Choose an edge in $\Gamma_f$ and let $T$ be the corresponding torus in the JSJ-collection of tori for $M$. Let $V$ be a geometric piece of the JSJ-decomposition of $M$ adjacent to $T$. Then Lemma 7 below together with a simple induction argument show that $h'$ can be chosen (up to conjugacy) in such a way that its restriction to $M_f$ commutes with the restriction of $h$.

**Lemma 7.** If the covering transformations $h$ and $h'$ preserve a JSJ-torus $T$ of $M$ then, up to conjugacy in $\text{Diff}^+(M)$, $h$ and $h'$ commute on the union of the geometric components of the JSJ-decomposition adjacent to $T$.

**Proof.** First we show that $h$ and $h'$ commute on each geometric component adjacent to $T$. Since $h$ and $h'$ preserve the orientation of $M$, we deduce that $h(V) = V$ and $h'(V) = V$, and that $h$ and $h'$ act geometrically on the geometric piece $V$. A product structure on $T$ can always be induced by the geometric structure on $V$: either by considering the induced Seifert fibration on $T$ if $V$ is Seifert fibred, or by identifying $T$ with a section of a cusp in the complete hyperbolic manifold $V$. Since $h$ and $h'$ are isometries of order $p$, for such a product structure on $T$ they act as (rational) translations, i.e. their action on $T = S^1 \times S^1$ is of the form $(\zeta_1, \zeta_2) \mapsto (e^{2\pi r_1/p} \zeta_1, e^{2\pi r_2/p} \zeta_2)$, where $p$ and at least one between $r_1$ and $r_2$ are coprime. Thus $h$ and $h'$ commute on $T$.

If $V$ is hyperbolic, we have just seen that $h$ and $h'$ are two isometries of $V$ which commute on the cusp corresponding to $T$. Thus they must commute on $V$.

If $V$ is Seifert fibred, then the Seifert fibration is unique up to isotopy, and $h$ and $h'$ preserve this fibration.

**Remark 4.** Note that the quotient of $V$ by a fiber-preserving diffeomorphism of finite order $h$ only depends on the combinatorial behaviour of $h$, i.e. its translation action along the fibre and the induced permutation on cone points and boundary components of the base. In particular, the conjugacy class of $h$ only depends on these combinatorial data. Note moreover that two geometric symmetries having the same combinatorial data are conjugate via a diffeomorphism isotopic to the identity.

Since the translation along the fibres commutes with every fiber-preserving diffeomorphism of $V$, it suffices to see whether $h$ and $h'$ commute, up to a conjugation of $h'$, on the base $B$ of $V$. It is enough then to consider the possible actions of order $p$ on the possible bases. According to Lemma 6 the possible actions of $h$ and $h'$ are described below:

1. If $B$ is a disc with 2 singular fibres, or an annulus with 1 singular fibre, or a disc with $n$ holes, $n \neq p$, or a disc with $p - 1$ holes and 1 singular fibre, then the action on $B$ is necessarily trivial and there is nothing to prove. Note that, according to the proof of Lemma 6 if $B$ is a disc with $p - 1$ holes with one singular fibre, no boundary torus is left invariant, so this possibility in fact does not occur.
2. If $B$ is a disc with $p$ holes and 1 singular fibre or a disc with $p + 1$ singular fibres, then the only possible action is a rotation about a singular fibre cyclically permuting the holes or the remaining singular fibres.

3. If $B$ is a disc with $p$ singular fibres then the action must be a rotation about a regular fibre which cyclically exchanges the singular fibres.

4. If $B$ is an annulus with $p$ singular fibres the action must be a free rotation cyclically exchanging the singular fibres. Note that in the three latter cases the action can never be trivial on the base.

5. If $B$ is a disc with $n$ holes then two situations can arise: either the action is trivial on the base (case (d) in the proof of Lemma 6 note that in case (a), when $n = p - 1$, all boundary components must be cyclically permutated), or $n = p$ and the action is a rotation about a regular fibre which cyclically permutes the $p$ holes (see part (c) of Lemma 6).

We shall now show that, if both $h$ and $h'$ induce non trivial actions on the base of $V$, then, up to conjugacy, $h$ and $h'$ can be chosen so that their actions on $B$ coincide. Note that for $h$ and $h'$ to commute it suffices that the action of $h'$ on $B$ coincides with the action of some power of $h$, however this stronger version will be needed in the proof of Corollary 10.

First of all remark that, if $B$ is a disc with $p + 1$ singular fibres (case 2) and $h$ and $h'$ leave invariant distinct singular fibres, then all the singular fibres must have the same order (in fact, must have the same invariants). This means that, after conjugating $h'$ by a homeomorphism of $V$ which is either an isotopy exchanging two regular fibres or a Dehn twist along an incompressible torus exchanging two singular fibres, one can assume that, in cases 2 and 3, $h$ and $h'$ leave set-wise invariant the same fibre. Note that this homeomorphism is isotopic to the identity on $\partial V$ and thus extends to $M$. In fact, using Lemma 6 one can show that the fibres cannot all have the same order.

Since the actions of $h$ and $h'$ consist in permuting exactly $p$ holes or singular fibres, it suffices to conjugate $h'$ via a homeomorphism of $V$ (which is a composition of Dehn twists along incompressible tori) in such a way as to exchange the order of the holes or singular fibres so that $h'$ and $h$ cyclically permute them in the same order. Note that in the case of singular fibres this product of Dehn twists is isotopic to the identity on $\partial V$ and thus extends to $M$. In the case of holes, the product of Dehn twists extends to $M$ since it induces the identity on the fundamental groups of the tori of $\partial V$ and the connected components of $M \setminus V$ adjacent to boundary tori different from $T$ are necessarily homeomorphic.

Once the two diffeomorphisms $h$ and $h'$ commute on the two geometric pieces adjacent to $T$, the commutation can be extended on a product neighborhood of $T$, since the two finite abelian groups generated by the restrictions of $h$ and $h'$ on each side of $T$ have the same action on $T$. Indeed, the slope of the translation induced by $h'$ on $T$ has been left unchanged by the conjugation.

\[\square\]

**Remark 5.** Note that in case 1 of the proof of the above Lemma, the actions of $h$ and $h'$ must coincide after taking a power, i.e. $h$ and $h'$ generate the same cyclic group. This is not necessarily true in the remaining cases, even if $h$ and $h'$ induce the same action on $B$. Indeed, they can induce different translations
along the fibres. Nevertheless, in both cases, to assure that the actions of $h$ and $h'$ coincide on $V$, it suffices to check that they coincide on $T$.

**Case (b):** $\Gamma_f$ is a single vertex. Let $V = M_f$ be the geometric piece corresponding to the unique vertex of $\Gamma_f$. If $V = M$, then the result is already known. We can thus assume that $V \neq M$. According to part (ii) of Proposition 2, we can assume that the fixed-point sets of $h$ and $h'$ are contained in $V$. If $V$ is Seifert fibred then, case (a) of the proof of Lemma 6 shows that the base $B$ of $V$ is either a disc with 2 or $p + 1$ singular fibres, or a disc with $p - 1$ holes and with 1 or 2 singular fibres. In the first case the boundary torus of $V$ is preserved by $h$ and $h'$ and the assertion follows from Lemma 4. In the second case the action on the base is necessarily a rotation fixing two points (either the unique singular fibre and a regular one, or the two singular fibres) and cyclically permuting the $p$ boundary components. Then conjugating $h'$ by a product of Dehn twists along incompressible tori, which extends to $M$ as in the proof of Lemma 7, leads to the desired conclusion.

The case where $V$ is hyperbolic is due to B. Zimmermann [Z1]. We give the argument for completeness. Since $V$ is hyperbolic, we consider the group $\mathcal{I}_V$ of isometries of $V$ induced by diffeomorphisms of $M$ which leave $V$ invariant. Let $\mathcal{S}$ be the $p$-Sylow subgroup of $\mathcal{I}_V$. Up to conjugacy, we can assume that both $h = h|_V$ and $h' = h'|_V$ belong to $\mathcal{S}$. If the groups $\langle h \rangle$ and $\langle h' \rangle$ generated by $h$ and $h'$ are conjugate, we can assume that $h = h'$ and we are done. So we assume that $\langle h \rangle$ and $\langle h' \rangle$ are not conjugate. Then it suffices to prove that $h'$ normalises $\langle h \rangle$ because each element normalising $\langle h \rangle$ must leave invariant $\text{Fix}(h)$ and the subgroup of $\mathcal{I}_V$ which leaves invariant a simple closed geodesic, like $\text{Fix}(h)$, must be a finite subgroup of $\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z})$. In particular, elements of odd order must commute. Assuming that $\langle h \rangle$ and $\langle h' \rangle$ are not conjugate, we have that $\langle h \rangle \subseteq \mathcal{S}$ and, by [St] Ch. 2, 1.5], either $\langle h \rangle$ is normal in $\mathcal{S}$ and we have reached the desired conclusion, or there exist an element $h = ghg^{-1}$, conjugate to $h$ in $\mathcal{S}$, which normalises $\langle h \rangle$ and such that $\langle h \rangle \cap \langle h \rangle = \{1\}$.

We want to show that $h'$ normalises $\langle h \rangle$. Assume, by contradiction that $h'$ is not contained in $\langle h, \bar{h} \rangle = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. Then this group is smaller than $\mathcal{S}$ and again we are able to find a new cyclic group $H$ of order $p$ whose intersection with $\langle h, \bar{h} \rangle$ is reduced to the identity and which normalises $\langle h, \bar{h} \rangle$. Since the order of $H$ is an odd prime number and since $\langle h \rangle$ and $\langle h \rangle$ are the only subgroups of $\langle h, \bar{h} \rangle$ which fix point-wise a geodesic by [MZ], Proposition 4], $H$ would commute with $\langle h, \bar{h} \rangle$ which is a contradiction to the structure of a group leaving a geodesic invariant. This final contradiction shows that, up to conjugacy, the subgroups $\langle h \rangle$ and $\langle h' \rangle$ either commute or coincide on $V$. This finishes the proof of Proposition 2.

The following proposition shows that a prime knot $K$ having a $p$-twin either admits a rotational symmetry of order $p$, or a well-specified submanifold $E_p(K)$ built up of geometric pieces of the JSJ-decomposition of $E(K)$ admits a symmetry of order $p$ with non-empty fixed-point set.

**Definition 2.** Let $K$ be a prime knot in $S^3$. For each odd prime number $p$ we define $E_p(K)$ to be the connected submanifold of $E(K)$ containing $\partial E(K)$ and such that $\partial E_p(K) \setminus \partial E(K)$ is the union of the JSJ-tori of $E(K)$ with winding number $p$ which are closest to $\partial E(K)$.
Proposition 3. Let $K$ be a prime knot and let $p$ be an odd prime number. Then for any $p$-twin $K'$, the deck transformation of the branched cover $M \rightarrow (S^3, K')$ induces on $E_p(K)$ a symmetry of order $p$, with non-empty fixed-point set and which extends to $U(K)$.

**Proof.** First we show that the deck transformation of the branched cover $M \rightarrow (S^3, K')$ associated to a $p$-twin of $K$ induces on $E_p(K)$ a symmetry of order $p$.

Let $K'$ be a $p$-twin of $K$. Let $h$ and $h'$ be the deck transformations on $M$ for the $p$-fold cyclic branched covers of $K$ and $K'$. We shall start by understanding the behaviour of $h$ and $h'$ on $M$. We have seen in Proposition 2 that $h$ and $h'$ can be chosen to commute on the submanifold $M_\ell$ of $M$ corresponding to the maximal subtree of $\Gamma$ on which both $h$ and $h'$ induce a trivial action. Let $\Gamma_\ell$ the maximal $(h, h')$-invariant subtree of $\Gamma$ containing $\Gamma_\ell$, such that, up to conjugacy, $h$ and $h'$ can be chosen to commute on the corresponding submanifold $M_\ell$ of $M$.

If $M_\ell = M$ then after conjugation $h'$ commutes with $h$ on $M$, but is distinct from $h$ because the knots $K$ and $K'$ are not equivalent. Hence it induces a rotational symmetry of order $p$ of the pair $(S^3, K)$ and we are done.

So we consider now the case where $\partial M_\ell$ is not empty. It is sufficient to show that $E_p(K) \subset M_\ell/\langle h \rangle$: then the symmetry of order $p$ induced by $h'$ on $M_\ell/\langle h \rangle$ must preserve $E_p(K)$ since each JSJ-torus of $E(K)$ can only be mapped to another torus of the family with the same winding number and the same distance from $\partial E(K)$. First we show:

Claim 9. Let $T$ be a connected component of $\partial M_\ell$. The $h$-orbit of $T$ consists of $p$ elements which are permuted in the same way by $h$ and $h'$.

**Proof.** Let $T$ be a torus in $\partial M_\ell$ and let $U$ be the connected component of $M \setminus M_\ell$ adjacent to $T$. Because of Lemma 7 $T$ cannot be preserved by both $h$ and $h'$ for else $M_\ell$ would not be maximal. Without loss of generality, we can assume that either:

(a) $h(T) \neq T$ and $h'(T) \neq T$;

or

(b) $h(T) = T$ but $h'(T) \neq T$; in this case since $h$ and $h'$ commute on $M_\ell$, we have that $h(h'^\alpha(U)) = h'^\alpha(U)$. Then part (ii) of Proposition 2 implies that $h$ acts freely on $h'^\alpha(U)$ for each $\alpha = 0, ..., p - 1$.

In case (a), the orbit of $T$ by the action of the group $\langle h, h' \rangle$ consists of $p$ or $p^2$ elements which bound on one side $M_\ell$ and on the other side a manifold homeomorphic to $U$. If the orbit consist of $p$ elements, since $h$ and $h'$ commute on $M_\ell$, up to choosing a different generator in $\langle h' \rangle$ we can assume that $h$ and $h'$ permute the elements of the orbit in the same way. Indeed, we have $h'h(T) = hh'(T) = h(h'^\alpha(T)) = h'^\alpha(h(T))$.

If the orbit consist of $p^2$ elements, $U$ is a is a knot exterior and there is a well-defined longitude-meridian system on each component of the $\langle h, h' \rangle$-orbit of $T$. In particular, there is a unique way to glue a copy of $U$ along the projection of $T$ in $M_\ell/\langle h, h' \rangle$. This implies that $h$ and $h'$ commute up to conjugacy on $M_\ell \cup \langle h, h' \rangle U$, contradicting the maximality of $M_\ell$. Note also that in this latter
case the stabiliser of each component of \(\langle h, h'\rangle U\) is reduced to the identity which clearly extends to \(\langle h, h'\rangle U\).

Assume we are in case (b). Consider the restriction of \(h\) and \(h_\alpha = h^{-\alpha}hh^{\alpha}\) to \(U\). Since \(h\) and \(h'\) commute on \(M_c\), \(h\) and \(h_\alpha\) coincide on \(T\). Let \(V\) be the geometric piece of the JSJ-decomposition for \(M\) adjacent to \(T\) and contained in \(U\). Using Lemma 9 we see that \(h\) and \(h_\alpha\) commute on \(V\) and thus coincide on it, because they coincide on \(T\). Thus \(h\) and \(h'\) commute on \(M_c \cup_{\alpha=0}^{p-1} h^{\alpha}(V)\), and again we reach a contradiction to the maximality of \(M_c\).

We can thus assume to be in case (a) and that the \(\langle h, h'\rangle\)-orbit of \(T\) has \(p\) elements.

\textbf{Claim 10.} \textit{Each torus in the boundary of \(M_c/\langle h\rangle\) has winding number \(p\) with respect to \(K\).}

\textbf{Proof.} Since a boundary component \(T\) of \(M_c/\langle h\rangle\) lifts to \(p\) boundary components of \(M_c\), the winding number of \(T\) with respect to \(K\) must be a multiple of \(p\). We shall now reason by induction on the number \(n\) of boundary components of \(M_c/\langle h\rangle\). If \(n = 0\) there is nothing to prove.

If \(n = 1\) the quotient spaces \(M_c/\langle h\rangle\) and \(M_c/\langle h'\rangle\) are solid tori, i.e. the exterior of a trivial knot which can be identified with a meridian of each solid torus. Note that the winding number of \(T\) is precisely the linking number of \(K\) with such a meridian. Note, moreover, that the spaces \(M_c/\langle h\rangle\) and \(M_c/\langle h'\rangle\) have a common quotient \(O\) which is obtained by quotienting \(M_c/\langle h\rangle\) (respectively \(M_c/\langle h'\rangle\)) via the the symmetry \(\psi\) (respectively \(\psi'\)) of order \(p\) and with non-empty fixed-point set, induced by \(h'\) (respectively \(h\)). Since \(\psi'\) preserves \(\partial(M_c/\langle h'\rangle)\) and has non-empty fixed-point set, \(\text{Fix}(\psi')\) and the meridian of \(\partial(M_c/\langle h'\rangle)\) must form a Hopf link, in particular, their linking number is 1. The image of \(\text{Fix}(\psi')\) and of the meridian of \(\partial(M_c/\langle h'\rangle)\) form again a Hopf link in \(O = (M_c/\langle h'\rangle)/\psi\). By lifting them up to \(M_c/\langle h\rangle\) we see that the meridian lifts to a meridian and the image of \(\text{Fix}(\psi')\) lifts to \(K\) which thus have linking number \(p\). Hence the property is proved in this case.

If \(n > 1\), we shall perform trivial Dehn surgery on \(n - 1\) boundary components of \(M_c/\langle h\rangle\). Note that such a surgery does not change the winding number of the remaining boundary components (for the boundary components are unlinked), that the symmetry of order \(p\) of \(M_c/\langle h\rangle\) extends to the resulting solid torus, and that the surgery can be lifted on \(M_c\) in such a way that the quotient of the resulting manifold by the action of the diffeomorphism induced by \(h'\) is again a solid torus. This last property follows from the fact that each connected component of \((E(K) \setminus \langle M_c/h\rangle)\) is the exterior of a knot which lifts in \(M\) to \(p\) diffeomorphic copies. These \(p\) copies of the knot exterior are permuted by \(h'\) and a copy appears in the JSJ-decomposition of \(E(K')\). This means that on each boundary component there is a well-defined meridian-longitude system which is preserved by \(h\) and \(h'\) and by passing to the quotient. The claim follows now from case \(n = 1\).

Now Claims 9 and 10 imply that \(E_p(K)\) is a submanifold of \(M_c/\langle h\rangle \cap E(K)\).

Note, moreover, that the fixed-point set of the induced symmetry is contained in \(M_f/\langle h\rangle \subset M_c/\langle h\rangle\). In particular, each torus of the JSJ-family separating such fixed-point set from \(K\) lifts to a single torus of the JSJ-family for
$M$ and its winding number cannot be a multiple of $p$. We can thus conclude that the fixed-point set of the symmetry induced by $h'$ is contained in $E_p(K)$.

This finishes the proof of Proposition $\blacksquare$

Remark 6. Note that $M_c/h \cap E(K)$ can be larger than $E_p(K)$ for there might be tori of the JSJ-collection for $M$ which have an $\langle h, h' \rangle$-orbit containing $p^2$ elements and which project to tori with winding number $p$. Note also that $E_p(K)$ coincides with $E(K)$ if there are no JSJ-tori in $E(K)$ with winding number $p$.

Remark 7. The deck transformations $h$ and $h'$ cannot commute on the submanifolds $U$ of $M$ corresponding to branches of $\Gamma$ whose $h$- and $h'$-orbits coincide and consist of $p$ elements, if $h$ and $h'$ are different; that is, the stabiliser $h'h^{-1}$ is a finite order diffeomorphism of $U$ if and only if it is trivial. To see this, assume that there is a unique orbit of this type and assume by contradiction that $h$ and $h'$ commute on $M$ and are distinct. The diffeomorphism $h'$ would induce a non-trivial symmetry of $E(K)$ of order $p$ and non-empty fixed-point set which fixes set-wise the projection of $U$ and acts freely on it. This contradicts the first part of Lemma $\blacksquare$. If there are $n > 1$ such orbits an equivariant Dehn surgery argument on $n-1$ components leads again to a contradiction.

Here is a straightforward corollary of Proposition $\blacksquare$ which generalises a result proved by B. Zimmermann $[Z1]$ for hyperbolic knots.

Corollary 4. Let $K$ be a prime knot and let $p$ be an odd prime number. If $K$ has no companion of winding number $p$ and has a $p$-twin, then $K$ admits a rotational symmetry of order $p$ with trivial quotient. $\blacksquare$

So far we have proved that if a prime knot $K$ has a $p$-twin either $E(K)$ admits a $p$-rotational symmetry or a well-specified submanifold $E_p(K)$ of $E(K)$ admits a symmetry of order $p$ with non-empty fixed-point set. We shall say that the $p$-twin induces a symmetry, respectively a partial symmetry, of $K$.

Proposition 4. Let $K$ be a prime knot. Assume that $K$ has a $p$-twin and a $q$-twin for two distinct odd prime numbers.

(i) At least one twin, say the $q$-twin, induces a $q$-rotational symmetry $\psi_q$ of $K$. Moreover:

(ii) If the $p$-twin induces a partial $p$-symmetry of $K$, then $\partial E_p(K) \setminus \partial E(K)$ is a JSJ-torus which separates the fixed point set $\text{Fix}(\psi_q)$ from $\partial E(K)$.

First we study some properties of partial symmetries induced by $p$-twins for an odd prime number $p$.

Lemma 8. Let $K$ be a prime knot and let $\psi$ be the partial symmetry of order $p$ induced on $E_p(K)$ by a $p$-twin. Let $T$ be a torus of the JSJ-collection of $E_p(K)$ which is not in the boundary. Then $T$ does not separate $\partial E(K)$ from $\text{Fix}(\psi)$ if and only if its $\psi$-orbit has $p$ elements. Moreover, this is the case if and only if the lift of $T$ to the $p$-fold cyclic branched cover of $K$ has $p$ elements.
Proof. It suffices to perform ψ-equivariant Dehn fillings on the boundary components ∂E_p(K) \ ∂E(K) of E_p(K) in such a way that the resulting manifold is a knot exterior E(\hat{K}) and that the graph dual to the JSJ-decomposition of E(\hat{K}) remains unchanged after filling (see the proof of Theorem \textbf{3}). Part (i) of Lemma \textbf{3} then applies to the resulting knot \hat{K} and the induced rotational symmetry. To apply Lemma \textbf{5} it suffices to note that, as in the proof of Claim \textbf{10} the fillings can be chosen in such a way that the induced fillings on the quotient E_p(K)/⟨ψ⟩ give also a solid torus (see Remark \textbf{4}).

Remark 8. In particular, case (b) of the proof of Claim \textbf{4} cannot happen for a torus T in the situation of Lemma \textbf{5}.

Lemma 9. Let K be a prime knot and let ψ be the partial symmetry of order p induced on E_p(K) by a p-twin. Let T ⊂ ∂E_p(K) \ ∂E(K) be a torus which is ψ-invariant. Let e_T be the corresponding edge in the tree dual to the JSJ-decomposition of E_p(K). Let v_K and v_ψ be the vertices corresponding to the geometric pieces containing ∂E(K) and Fix(ψ) respectively. Then v_ψ belongs to the unique geodesic joining v_K to e_T in this JSJ-tree.

Proof. If we cut S^3 along a torus of the JSJ-collection of E_p(K), the connected component which does not contain K is a knot exterior and is thus contained in a ball in S^3. If the conclusion of the Lemma were false, then we could find two tori of the JSJ-decomposition of E_p(K) contained in two disjoint balls, one torus separating Fix(ψ) from K and the other coinciding with T or separating it from K. In particular the linking number of Fix(ψ) and a meridian of the solid torus bounded by T (i.e. the winding number of T with respect to Fix(ψ)) would be zero. This is impossible since ψ leaves set-wise invariant T.

Remark 9. Lemma \textbf{9} has two interesting consequences. Since h and h' play symmetric roles, we deduce that Fix(ψ) and ∂E(K) must belong to the same geometric piece of the JSJ-decomposition of E_p(K). This follows from the fact that, in E_p(K') \ U(K'), Fix(ψ) maps to K', K maps to Fix(ψ)', and T maps to a ψ'-invariant torus. Moreover, each invariant boundary torus T is adjacent to the geometric component containing Fix(ψ) and K, else, we would get a contradiction to Lemma \textbf{5}.

Proof of Proposition \textbf{4}(i). We argue by contradiction, assuming that there are a p-twin and a q-twin of K which induce only partial symmetries of E(K) for two distinct odd prime numbers p and q. Then ∂E_p(K) and ∂E_q(K) are not empty. Moreover, we must have E(K) \ E_p(K) ⊂ E_q(K) since the winding number along nested tori is multiplicative and thus the winding number of any JSJ-torus contained in E(K) \ E_p(K) must be of the form kp and cannot be q. In particular ∂E_p(K) \ ∂E(K) ⊂ int(E_q(K)).

Let T ∈ ∂E_p(K) \ ∂E(K) be a torus and let ψ be the q-symmetry with non-empty fixed-point set induced on E_q(K) by the q-twin. Since the winding number of T is p, its lift to the q-fold cyclic branched cover of K is connected.
According to part (i) of Lemma 3 and to Lemmata 5 and 8, $T$ must separate $\partial E(K)$ from $\text{Fix}(\psi)$. Since $\text{Fix}(\psi)$ is connected, we see that so must be $\partial E_p(K) \setminus \partial E(K) = T$. The final contradiction is then reached by applying Remark 3.

**Proof of Proposition 4(ii).** This is a consequence of the proof of part (i): note that in the proof $\psi$ may be a global or partial symmetry.

We are now in a position to prove Theorem 1.

**Proof of part (i) of Theorem 1.** We argue by contradiction, assuming that $K$ admits twins for three distinct, odd prime numbers $p, q, r$. Under this assumption, it follows that $K$ is a non-trivial knot.

If the three twins induce rotational symmetries of the knot $K$, then part (i) of Theorem 3 gives a contradiction.

Therefore part (i) of Proposition 4 implies that twins of orders, say $q$ and $r$, induce rotational symmetries $\psi_q$ and $\psi_r$ of $K$ having order $q$ and $r$ respectively, while a $p$-twin induces only a partial rotational symmetry of $E(K)$ of order $p$.

Then part (ii) of Proposition 4 shows that $\partial E_p(K) \setminus \partial E(K)$ is a JSJ-torus in $E(K)$ which separates $\partial E(K)$ from both $\text{Fix}(\psi_q)$ and $\text{Fix}(\psi_r)$. This contradicts part (ii) of Theorem 3 which states that $\text{Fix}(\psi_q)$ and $\text{Fix}(\psi_r)$ must sit in the JSJ-component containing $\partial E(K)$.

**Proof of part (ii) of Theorem 1.** Let $K$ be a prime knot and let $p$ be an odd prime number. We assume that $K$ has at least two non-equivalent $p$-twins $K_1$ and $K_2$ and look for a contradiction.

If both $\psi_1$ and $\psi_2$ are rotational symmetries of order $p$ of $K$, then by M. Sakuma [Sa2, Thm. 3] they are conjugate since $K$ is prime. This would contradict the hypothesis that the knots $K_1$ and $K_2$ are not equivalent.

Assume now that at least one symmetry, say $\psi_1$ is partial. Then $\psi_1$ and $\psi_2$ are rotational symmetries of order $p$ of the submanifold $E_p(K) \subset E(K)$. Let $X_0$ be the geometric piece of the JSJ-decomposition of $E(K)$ containing $\partial E(K)$. Then $\psi_1$ (respectively $\psi_2$) generates a finite cyclic subgroup $G_1$ (respectively $G_2$) of the group $\text{Diff}^{++}(X_0, \partial E(K))$ of diffeomorphisms of the pair $(X_0, \partial E(K))$ which preserve the orientations of $X_0$ and of $\partial E(K)$. Moreover, one can assume that $G_1$ and $G_2$ act geometrically on $X_0$.

If $X_0$ admits a hyperbolic structure, it is a consequence of the proof of the Smith conjecture (see for example [Sa2 Lemma 2.2]) that the subgroup of $\text{Diff}^{++}(X_0, \partial E(K))$ consisting of restrictions of isometries of $X_0$ is finite cyclic. Hence $G_1 = G_2$ and up to taking a power $\psi_1 = \psi_2$ on $X_0$.

If $X_0$ is Seifert fibred, then it must be a cable space, since $K$ is prime. The uniqueness of the Seifert fibration and the fact that the basis of the Seifert fibration has no symmetry of finite order imply that the cyclic groups $G_1$ and $G_2$ belong to the circle action $S^1 \subset \text{Diff}^{++}(X_0, \partial E(K))$ inducing the Seifert fibration of $X_0$, see [Sa2 Lemma 2.3]. Since $G_1$ and $G_2$ have the same prime order, up to taking a power $\psi_1 = \psi_2$ on $X_0$.

Let $h_1$ and $h_2$ be the deck transformations on $M$ associated to the $p$-fold cyclic coverings branched along $K_1$ and $K_2$, and which induce $\psi_1$ and $\psi_2$. Then by taking a suitable powers, $h_1$ and $h_2$ coincide up to conjugacy on the geometric piece $X_0$ of the JSJ-decomposition of $M$ containing the preimage of
$K$. The following lemma shows that they will coincide on $M$, contradicting our hypothesis.

Lemma 10. If the covering transformations $h$ and $h'$ preserve a JSJ-piece or a JSJ-torus of $M$ and coincide on it, then they can be chosen, up to conjugacy, to coincide everywhere.

Proof. This is a consequence of the proofs of Propositions 2 and 3. We shall start by showing that we can always assume that there is a piece $V$ of the JSJ-decomposition on which $h$ and $h'$ coincide. To this purpose, assume that $h$ and $h'$ coincide only on a JSJ-torus $T$. According to Lemma 7 and Remark 8, $h$ and $h'$ coincide on the geometric pieces of the decomposition adjacent to $T$, which are also invariant. Consider now the maximal subtree $\Gamma_1$ of $\Gamma$ such that the restrictions of $h$ and $h'$ to the corresponding submanifold $M_1$ of $M$ coincide, up to conjugacy, and such that $V \subset M_1$. Let $S$ be a JSJ-torus for $M$ in the boundary of $M_1$. Since $h$ and $h'$ coincide on $M_1$, the $h$-orbit and the $h'$-orbit of $S$ coincide as well and consist of either one single element \{S\} or $p$ elements \{S, h(S) = h'(S), ..., h^{p-1}(S) = h'^{p-1}(S)\}. In the former case, according to Lemma 4, $\Gamma_1$ would not be maximal. In the latter case, we are precisely in the situation described in part (a) of Claim 9. Once more, $\Gamma_1$ is not maximal because one can impose that $h$ and $h'$ act in the same way on the $p$ connected components with connected boundary obtained by cutting $M$ along the $\langle h, h' \rangle$-orbit of $S$ (see Remark 7). This contradiction shows that $M = M_1$ and the lemma is proved.

Proof of part (iii) of Theorem 1. First we analyse the case of a knot admitting two twins, one of which induces a partial symmetry.

Proposition 5. Let $K$ be a prime knot admitting a $p$-twin $K'$ and a $q$-twin $K''$ for two distinct odd prime numbers $p$ and $q$. If $K'$ induces a partial symmetry of $K$ then $K'$ and $K''$ are not equivalent.

Proof. By part (ii) of Proposition 4, $E_p(K)$ has a unique boundary component which separates $\partial E(K)$ from the fixed-point set of the $q$-rotational symmetry $\psi$ induced by $K''$. By cutting $S^3$ along $T = \partial E_p(K)$ we obtain a solid torus $V = E_p(K) \cup U(K)$ containing $K$, and a knot exterior $E_T$. $K$ admits a $q$-rotational symmetry $\psi$ induced by $K''$ which preserves this decomposition and induces a $q$-rotational symmetry with trivial quotient (see Lemma 8) on $E_T$ and a free $q$-symmetry $\bar{\psi}$ on $V$. The covering transformation for the knot $K'$ induces a $p$-symmetry $\varphi$ of $V$ with non-empty fixed-point set.

Assume now by contradiction that $K' = K''$. Since $K'$ induces a partial symmetry of $K$ and vice versa, $S^3$ admits a decomposition into two pieces: $V' = E_p(K') \cup U(K')$ and $E_T$. On the other hand, since $K''$ induces a genuine $q$-rotational symmetry of $K$, $K''$ admits a $q$-rotational symmetry $\psi''$ induced by $K$ which preserves the aforementioned decomposition and induces a $q$-rotational symmetry with trivial quotient on $E_T$. Using the fact that $E_T$ is the exterior of a prime knot (see Lemma 8) and M. Sakuma’s result [Sa2] Thm. 3, we see that the two $q$-rotational symmetries with trivial quotient induced by $\psi$ and $\psi''$ on $E_T$ act in the same way. Let now $E_0$ be the smallest knot exterior of the JSJ-decomposition of $E_T$ on which $\psi = \psi''$ induces a $q$-rotational symmetry
with trivial quotient (this is obtained by cutting $E_T$ along the torus of the \textit{JSJ}-decomposition closest to $\text{Fix}(\psi)$ -respectively $\text{Fix}(\psi'')$- and separating it from $T$. Consider now the lift, denoted by $(X,K)$, to $(S^3,K')$ of $(E_0,\text{Fix}(\psi))/\psi$. We claim that $(X,K) = (V',K')$. Indeed, $X$ contains $K'' = K'$ by construction, and its boundary is the unique torus of the \textit{JSJ}-decomposition which is left invariant by the $q$-rotational symmetry of $K''$-by construction again- and which is closest to $K'$ (compare Remark 9). Since $E_0/\psi = E_0/\psi''$, and a solid torus has a unique $p$-fold cyclic cover, we deduce that $(V',K') = (X,K) = (V,K)$. In particular, the deck transformations for $K$ and $K'$ on their common $p$-fold cyclic branched cover can be chosen to coincide on the lift of $V = V'$. Lemma implies that $K = K'$ contradicting the fact that $K'$ is a $p$-twin.

Let $K'$ be a $p$-twin and a $q$-twin of $K$ for two distinct odd prime numbers $p$ and $q$. Proposition implies that $K'$ induces two rotational symmetries $\psi_p$ and $\psi_q$ of $K$ with trivial quotients and orders $p$ and $q$. Part (ii) of Theorem shows that the fixed-point sets $\text{Fix}(\psi_p)$ and $\text{Fix}(\psi_q)$ lie in the $\text{JSJ}$-component of $E(K)$ which contains $\partial E(K)$. Then the proof of part (iii) of Theorem follows from the following:

\textbf{Lemma 11.} Let $K$ be a prime knot admitting two rotational symmetries $\psi$ and $\varphi$ of odd prime orders $p > q$. If the fixed-point sets of $\psi$ and $\varphi$ lie in the component which contains $\partial E(K)$, then the two symmetries commute up to conjugacy.

\textbf{Proof.} Reasoning as in the proof of part (ii) of Theorem one can show that $\psi$ and $\varphi$ commute on the component which contains $\partial E(K)$. Since all other components are freely permuted according to part (i) of Lemma the conclusion follows as in the proof of part (a) of Claim.

\textbf{Proof of Corollary 2.} First of all note that, because of the uniqueness of the Milnor-Kneser decomposition of the covers of $K$ and $K'$, the number of prime summands of $K$ and $K'$ is the same. After ditching components of $K$ and $K'$ that appear in both decompositions in equal number, we can assume that $K_i$ is not equivalent to $K'_j$, for all $i, \ell = 1,\ldots,t$. If $K$ and $K'$ have three common cyclic branched covers of odd prime orders, we deduce that for each $i = 1,\ldots,t$, $K_i$ is not determined by its $p_j$-fold cyclic branched cover, $j = 1,2,3$, for it is also the $p_j$-fold cyclic branched cover of some $K'_{i_j}$ not equivalent to $K_i$. Hence $K_i$ would have twins for three distinct odd prime orders which is impossible by Theorem.

\section{Examples}

Examples of prime knots admitting a $p$-twin which induces a global rotational symmetry of order $p$ were first constructed by Y. Nakanishi and M. Sakuma. They considered a prime link with two trivial components whose linking number is 1. By taking the $p$-fold cyclic cover of $S^3$ branched along the first (respectively the second) component of the link one gets again $S^3$ and the second (respectively first) component lifts to a prime knot. The two knots thus constructed have the same $p$-fold cyclic branched cover by construction (see...
also Remark 1, moreover, by computing their Alexander polynomial they were shown to be distinct.

In [Z1 Thm 3 and Cor. 1] B. Zimmerman showed that if a hyperbolic knot has a $p$-twin, for $p \geq 3$, then the $p$-twin induces a global symmetry and the two knots are thus obtained by Y. Nakanishi and M. Sakuma’s construction where the quotient link is hyperbolic and admits no symmetry which exchanges its two components.

As a matter of fact, the links considered by Y. Nakanishi and M. Sakuma are in fact hyperbolic and so are the resulting twins if $p \geq 3$, according to the orbifold theorem [BoP], see also [CHK]. Note that, when $p = 2$, the situation, even in the case of hyperbolic knots, is much more complex and there are several ways to construct 2-twins of a given knot. In this section we shall see how one can construct, for each given odd prime $p$, two prime, non simple, knots which are $p$-twins, and such that the symmetries they induce are not global.

4.1 Knots admitting a $p$-twin inducing only a partial symmetry

Assume we are given a hyperbolic link $L = L_1 \cup ... \cup L_{\nu+2}$, with $\nu + 2 \geq 3$ components, satisfying the following requirements:

Property *

1. The sublink $L_3 \cup ... \cup L_{\nu+2}$ is the trivial link;
2. For each $i = 1, 2$ and $j = 3, ..., \nu + 2$, the sublink $L_i \cup L_j$ is a Hopf link;
3. $\text{lk}(L_1, L_2)$ is prime with $p$;
4. No symmetry of $L$ exchanges $L_1$ and $L_2$.

We shall consider the orbifold $\mathcal{O} = (S^3, (L_1 \cup L_2)_p) \setminus U(L_3 \cup ... \cup L_{\nu+2})$ which is the 3-sphere with singular set of order $p$ the (sub)link $L_1 \cup L_2$ and an open tubular neighbourhood of the (sub)link $L_3 \cup ... \cup L_{\nu+2}$ removed. $\mathcal{O}$ is hyperbolic if $p \geq 3$, and will represent the quotient of $\mathcal{O}_p = E_p(K) \cup U(K)$ and $\mathcal{O}_{p}' = E_{p'}(K') \cup U(K')$ via the action of the partial $p$-symmetries. Indeed, to obtain $\mathcal{O}_p$ (respectively $\mathcal{O}_{p}'$) take the $p$-fold cyclic orbifold cover of $(S^3, (L_1 \cup L_2)_p) \setminus U(L_3 \cup ... \cup L_{\nu+2})$ which desingularises $L_2$ (respectively $L_1$). Observe that one can fix a longitude-meridian system on each boundary component of $\mathcal{O}$, induced by those of $L_i$, $i = 3, \ldots, \nu + 2$. Note that, because of condition 4 of Property *, the two orbifolds $\mathcal{O}_p$ and $\mathcal{O}_{p}'$ with the fixed peripheral systems are distinct.

Remark that $\mathcal{O}_p$ and $\mathcal{O}_{p}'$ can be obtained by the orbifold covers, analogous to those described above, of $(S^3, (L_1 \cup L_2)_p)$ (which are topologically $S^3$) by
removing open regular neighbourhoods of the lifts of the components \( L_3 \cup \ldots \cup L_{\nu+2} \). Note that these components lift to trivial components whose linking number with the lift of \( L_i \), \( i = 1, 2 \), is precisely \( p \), because of condition 2, and which form again a trivial link.

For each \( j = 3, \ldots, \nu+2 \), choose a knot exterior \( E(K_j) \) to be glued along the \( j \)-th boundary component of \( O_p \) and \( O'_p \) in such a way that a fixed longitude-meridian system on \( E(K_j) \) is identified with the lift of the longitude-meridian system on the \( j \)-th boundary component of \( O \). The underlying spaces of the orbifolds \( O_p \cup_{j=3}^{\nu+2} E(K_j) \) and \( O'_p \cup_{j=3}^{\nu+2} E(K_j) \) are topologically \( S^3 \) and it is easy to see that their singular sets are connected (see condition 3). The resulting knots have the same \( p \)-fold cyclic branched cover, however, since \( O_p \) and \( O'_p \) are distinct, they are not equivalent.

\[ \text{Remark 10.} \] Observe that we have just shown that the number of connected components of \( \partial E_p(K) \setminus \partial E(K) \), which is precisely \( \nu \), can be arbitrarily large. Note also that if \( \nu \geq 2 \), according to Proposition 4, the knot \( K \) has no \( q \)-twins for \( q \neq p \) odd prime.

We shall now prove that links with Property \( * \) exist. Notice that for \( \nu = 1 \) links satisfying all the requirements where constructed by Zimmermann in [Z2], see also [P1].

Consider the link given in Figure 1 for \( \nu = 3 \) (the generalization for arbitrary \( \nu \geq 1 \) is obvious). Most conditions are readily checked just by looking at the figure, and we only need to show that \( L \) is hyperbolic and has no symmetries which exchange \( L_1 \) and \( L_2 \). To this purpose, we shall describe the Bonahon-Siebenmann decomposition of the orbifold \((S^3, (L)_2)\), where all components have

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**Figure 1:** The link \( L \) and its Bonahon-Siebenmann decomposition.
as local group. The decomposition consists of one single hyperbolic piece (see Figure 1) and \( \nu + 1 \) (respectively 1) Seifert fibred pieces if \( \nu \geq 2 \) (respectively \( \nu = 1 \)). Since the Seifert fibred pieces contain no incompressible torus, the hyperbolicity of \( L \) follows.

Note now that every symmetry of \( L \) must leave invariant the unique hyperbolic piece of the decomposition. This piece is obtained by quotienting the hyperbolic knot 10\( _{155} \) via its full symmetry group \( \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \) and thus has no symmetries (for more details see [P1]), so we conclude that the components \( L_1 \) and \( L_2 \) are non exchangeable.

4.2 Knots admitting a \( p \)-twin inducing a partial symmetry and a \( q \)-twin inducing a global symmetry

Let \( K \) be a hyperbolic knot admitting a \( p \)-twin and a \( q \)-twin; the twins of \( K \) induce global symmetries, so that \( K \) admits a \( p \)- and a \( q \)-rotational symmetry with trivial quotient (see [Z2], where a method to construct hyperbolic knots with two twins is described). Remove a tubular neighbourhood of the axis of the symmetry of order \( q \) (note that the two symmetries have disjoint axes), and use the resulting solid torus \( V \) to perform Dehn surgery on the exterior \( E \) of the \((2, q)\)-torus knot. Denote by \( K \) the image of \( K \) after surgery. We require that:

1. The resulting manifold is \( S^3 \);
2. The \( q \)-rotational symmetry of \( E \) and the restriction of the \( q \)-rotational symmetry of \( K \) to \( V \) give a global \( q \)-rotational symmetry of \( K \);
3. The \( q \)-rotational symmetry of \( K \) has trivial quotient.

Note that the last requirement can be met by choosing appropriately the longitude when satellising, as illustrated in Figure 2. We claim that \( K \) admits a \( q \)-twin, \( K'' \), and a \( p \)-twin, \( K' \). \( K'' \) is obtained by the standard method described in Remark 1. Note that \( K \neq K'' \), for the roots of the JSJ-decompositions of the exteriors of \( K \) and \( K'' \) are hyperbolic and Seifert fibred respectively. To construct \( K' \), consider the \( p \)-twin \( K' \) of \( K \) and let \( V' \) be the solid torus obtained by removing the axis of the \( q \)-rotational symmetry of \( K' \). Note that \( V \) and \( V' \) have a common quotient obtained by taking the space of orbits of the \( p \)-rotational symmetries, however \( V \) and \( V' \) are different orbifolds by construction.

Fix a longitude-meridian system on \( V \) (the one used for the surgery); by first quotienting and then lifting it, get a longitude-meridian system on \( V' \) that must be used to perform surgery along a copy of \( E \). The image of \( K' \) after the surgery will be \( K' \). Note that, when taking the \( p \)-fold cyclic branched covers of \( K \) and \( K' \), the hyperbolic orbifolds \( V \) and \( V' \) lift to the same manifold by construction, while the Seifert fibred part lifts, in both cases, to \( p \) copies of \( E \). Again by construction, the gluings are compatible and the two covers coincide. It is also evident that \( K' \) can only induce a partial symmetry of \( K \), and the claim is proved.

Remark 11. Note that according to Proposition 2 the \( p \)-twins and \( q \)-twins obtained in this construction cannot be equivalent.
5 Homology spheres as cyclic branched covers

By the proof of the Smith conjecture Corollary 3 is true for the 3-sphere $S^3$. So from now on we assume that the integral homology sphere $M$ is not homeomorphic to $S^3$. Then by [BPZ, Thm1], $M$ can be a $p_i$-fold cyclic branched cover of $S^3$ for at most three pairwise distinct odd prime numbers $p_i$. Moreover if $M$ is irreducible and is the $p_i$-fold cyclic branched cover of $S^3$ for three pairwise distinct odd prime numbers $p_i$, then the proof of [BPZ Corollary 1.(i)] shows that for each prime $p_i$, $M$ is the $p_i$-fold cyclic branched cover of precisely one knot. Since a knot admits at most one $p$-twin for an odd prime integer $p$, we need only to consider the case when the irreducible integral homology sphere $M$ is the branched cover of $S^3$ for precisely two distinct odd primes, say $p$ and $q$. Moreover [BPZ Corollary 1.(ii)] shows that $M$ has a non trivial JSJ-decomposition.

Looking for a contradiction, we can assume that, for each prime, $M$ is the branched covering of two distinct knots with covering transformations $\psi$, $\psi'$ of order $p$ and $\varphi$, $\varphi'$ of order $q$.

If each rotation of order $p$ commutes with each rotation of order $q$ up to conjugacy, then the contradiction follows from the following claim which is an easy consequence of Sakuma’s result [Sa2 Thm. 3] (see [BPZ Claim 8]).

Claim 11. Let $n \geq 3$ be a fixed odd integer. Let $\rho$ be a rotation with trivial quotient of an irreducible manifold $M$. All the rotations of $M$ of order $n$ which commute with $\rho$ are conjugate in $\text{Diff}(M)$ into the same cyclic group of order $n$.

Otherwise, consider the subgroup $G = \langle \psi, \psi', \varphi, \varphi' \rangle$ of diffeomorphisms of
According to the proof of [BPZ, Proposition 4], each rotation of order \( p \) commutes with each rotation of order \( q \) up to conjugacy, unless the induced action of \( G \) on the dual tree of the JSJ-decomposition for \( M \) fixes precisely one vertex corresponding to a hyperbolic piece \( V \) of the decomposition and \( \{p,q\} = \{3,5\} \). In this case, one deduces as in the proof of [BPZ, Corollary 1.(ii)] that the restrictions of \( \psi \) and \( \psi' \) (respectively \( \varphi \) and \( \varphi' \)) coincide up to conjugacy on \( V \). Then the desired contradiction follows from Lemma 10 which implies that \( \psi \) and \( \psi' \) (respectively \( \varphi \) and \( \varphi' \)) coincide up to conjugacy on \( M \).

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