Boundary conditions and consistency of effective theories

Alicja Siwek

IPHC, Université de Strasbourg
IP, Wrocław University of Technology

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Outline

Motivation

Quantum Mechanics
  Linear spaces with indefinite norm
  Free particle dynamics
  Time reversal transformation

Reflection Positivity
  Model with higher order time derivatives
  Lattice regularization
  Positivity of transfer matrix
  Boundary conditions
  Conclusion
Effective theories:
⇒ elimination of degrees of freedom - heavy particles

Consequences

- long range correlations
  ⇒ higher order derivative terms in the effective action
- low energy - truncation of the gradient expansion

Two issues:
⇒ specification of states - boundary conditions
⇒ unitarity of the effective theory
Scalar field governed by the action

\[ S[\phi] = \int dx \left[ \phi \left( \sum_{n=0}^{n_d} c_n \Box^n \right) \phi(x) - V(\phi(x)) \right] \]

- time reversal invariant model
- coefficients \( c_n \) and potential \( V(\phi) \) real and \((-1)^{n_d} c_{n_d} > 0\)

Free propagator in momentum space

\[ D(p) = \left( \sum_{n=0}^{n_d} (-1)^n c_n (p^2)^n \right)^{-1} = \sum_{j=1}^{n_d} \frac{Z_j}{p^2 - m_j^2} \]

\( \Rightarrow \) at least one negative \( Z \) factor
- negative norm states
Linear Space with Indefinite Norm

Linear space $H$ with non-definite metric

1. $\langle u|v \rangle = \langle v|u \rangle^*$
2. $\langle u|(a|v \rangle + b|w \rangle \rangle = a \langle u|v \rangle + b \langle u|w \rangle$
3. $H = H_+ + H_-$ where $H_\pm = \{|u\rangle|\langle u|u\rangle \geq 0\}$ and $\langle H_+|H_- \rangle = 0$
4. $|u\rangle = |u_+\rangle + |u_-\rangle$, $\langle u_\pm|u_\pm \rangle \geq 0$

- basis $\{|n\rangle\}$, non-definite metric $\eta_{mn} = \langle m|n \rangle$ where $\eta^\dagger = \eta$
- matrix elements $A_{jk}$ of an operator $A$ defined by $\langle m|A|n \rangle = \sum_k \eta_{mk}A_{kn}$
Self-adjoint and Skew-adjoint Operators

⇒ adjoint $\bar{A}$ and Hermitian adjoint $A^\dagger$

$$\langle u | \bar{A} | v \rangle = \langle v | A | u \rangle^*$$

so $\bar{A} = \eta^{-1} A^\dagger \eta \neq A^\dagger$

▶ Condition : $\bar{A} = \sigma_A A$

⇒ valid for self-adjoint operators, $\sigma_A = +1$, and skew-adjoint operators, $\sigma_A = -1$.

▶ Eigenvectors : $A |\lambda\rangle = \lambda |\lambda\rangle$, $A |\rho\rangle = \rho |\rho\rangle$

⇒ relation for the spectrum

$$(\lambda - \sigma_A \rho^*) \langle \rho | \lambda \rangle = 0$$
Free Particle Dynamics

Canonical pair of operators $\hat{q}_\sigma$ and $\hat{p}_\sigma$

$\Rightarrow$ either self- or skew-adjoint
$\Rightarrow$ commutation relation $[\hat{q}_\sigma, \hat{p}_\sigma] = i$

Real spectrum

$\Rightarrow \eta(q, q') = \delta(q - \sigma q')$ and $\eta(p, p') = \delta(p - \sigma p')$

Closing relations in coordinate and momentum space

$1 = \int dq |\sigma q\rangle \langle q| = \int dp |\sigma p\rangle \langle p|$

Hamiltonian of harmonic oscillator

$\hat{H}_\sigma = \frac{\sigma}{2}(\hat{p}_\sigma^2 + \hat{q}_\sigma^2) = \sigma \bar{a}_\sigma a_\sigma$

where operator $a_\sigma = (\hat{q}_\sigma + i\hat{p}_\sigma)/\sqrt{2}$ and $[a_\sigma, \bar{a}_\sigma] = \sigma$
Bounded Hamiltonian Conditions

Case of $\sigma = +1$ (for $\sigma = -1$ $a \leftrightarrow \bar{a}$)

- operators $b = a_+, \bar{b} = \bar{a}_+$
- eigenstate of self-adjoint operator $\bar{b}b : \bar{b}b|\lambda\rangle = \lambda|\lambda\rangle$
- double infinite series of states
  $\cdots, b^2|\lambda\rangle, b|\lambda\rangle, |\lambda\rangle, \bar{b}|\lambda\rangle, \bar{b}^2|\lambda\rangle, \cdots$
  with corresponding eigenvalues
  $\cdots, \lambda - 2, \lambda - 1, \lambda, \lambda + 1, \lambda + 2, \cdots$ of $\bar{b}b$

Bounded Hamiltonian

Series limited on the left or on the right $\rightarrow \lambda$ integer

$\langle \lambda|\bar{b}b|\lambda\rangle = \lambda\langle \lambda|\lambda\rangle$ and $\langle \lambda|b\bar{b}|\lambda\rangle = (\lambda + 1)\langle \lambda|\lambda\rangle$

$\Rightarrow$ either $\lambda \geq 0$ or $\lambda \leq -1$

$\Rightarrow$ either $\text{sign}(\langle \lambda + 1|\lambda + 1\rangle) = \text{sign}(\langle \lambda|\lambda\rangle)$

or $\text{sign}(\langle \lambda - 1|\lambda - 1\rangle) = -\text{sign}(\langle \lambda|\lambda\rangle)$
Time Reversal Properties of Operators

Time reversal $\Theta$ (way to trace negative norm states)

$\sigma = +1$, $\Theta : \hat{q} \rightarrow \hat{q}$, $\hat{p} \rightarrow -\hat{p}$, $c \rightarrow c^*$

$\hat{q}$-type operators

- operators with well-defined time reversal parity: $\Theta A = \tau_A A$, with $\tau_A = \pm 1$

- Heisenberg representation $\Theta : A(t) \rightarrow \bar{A}(-t)$

$\Rightarrow$ for the time derivative $\tau_0 A = -\tau_A$

$\Rightarrow$ self- or skew-adjoint operators $\Theta : A \rightarrow \bar{A} = \sigma_A A$

Time parity and the signature of the linear space $\Rightarrow$

$\tau_A = \sigma_A$
Yang-Mills-Higgs Model in Euclidean Space-time

Model with higher order derivatives
\[ S[\phi, \phi^\dagger, A] = \int d^d x \left[ K(D) - \phi^\dagger L(D^2) D^2 \phi + V(\phi^\dagger \phi) \right] \]

- \( D_\mu = \partial_\mu - iA_\mu \) - covariant derivative
- \( A_\mu = A^a_\mu \tau^a \) - gauge field
- \( \tau^a \) - generators of the gauge group
- \( K, L \) - bounded from below, polynomials of \( D \) of order at most \( 2n_d \) and \( 2n_d - 2 \), respectively
- \( V(\phi^\dagger \phi) \) - scalar field potential
Appearance of higher order time derivatives

⇒ Solution in classical systems:

- introduction of new coordinates for each higher order derivative except the last one
  
  \[ A_{j\mu}(x) = D_0^j A_\mu(x) \text{ and } \phi_j(x) = \partial_0^j \phi(x) \]
  
  for \( j = 0, \ldots, n_d - 1 \)

- time reversal parity of the coordinates
  
  \[ A_{j\mu}(x) : \tau = (-1)^{j+\delta_{\mu,0}} \text{ and } \phi_j(x) : \tau = (-1)^j \]

- applicable also in case of quantum fields by means of path integral for \( A_{j\mu}(x) \) and \( \phi_j(x) \)
Theory on the Lattice

Aim

Finding fields with Green functions satisfying Wightman’s axioms in real time

- reflection positivity $\Rightarrow$ lattice regularization -
  - positivity of the transfer matrix in imaginary time

Lattice

- fields: $\phi(n) = a\phi(x), \phi^\dagger(n) = a\phi^\dagger(x)$,
  $U_\mu(n) = U_{-\mu}^\dagger(n + \hat{\mu}) = e^{igaA_\mu(n)}$,
  $a$ - lattice spacing, $\hat{\mu}$ - unit vector in direction $\mu$

- gauge transformation: $\phi(n) \rightarrow \omega(n)\psi(n)$,
  $\phi^\dagger(n) \rightarrow \omega^\dagger(n)\psi^\dagger(n)$,
  $U_\mu(n) \rightarrow \omega(n + \hat{\mu})U_\mu(n)\omega^\dagger(n)$
Partition Function of the Model

Partition function

\[ Z = \int D[U] D[\phi^\dagger] D[\phi] e^{-S_L} \]

Lattice action

\[ S_L = \sum_n \sum_{\gamma'} a_{\gamma'} \text{tr} U_{\gamma'}(n) + \sum_n \phi^\dagger(n) \sum_{\gamma} U^\dagger_\gamma(n) \phi(n + \gamma) b_\gamma \]

\[ + \sum_n V(\phi^\dagger(n) \phi(n)) \]

\( \gamma', \gamma \) - closed and open paths respectively, up to \( n_d \)

\( U_{\gamma}(n) \) - the path ordered product of the link variables along this path

Action \( S_L \) real

\[ \Rightarrow \text{for each } \gamma, \Theta \gamma \text{ included in the sums with } a_{\Theta \gamma'} = a^*_{\gamma'}, \]

\[ b_{\Theta \gamma} = b^*_\gamma \]
Identification of Lattice Field Variables

Construction of $n_d$ lattice field variables

- regroupment of $n_d$ consecutive lattice sites in the time direction with their field variables into a single $n_d$-component field
  ⇒ a single blocked time slice

- fields:
  \[
  \phi_j(t, n) = \phi(n_d t + j, n), \\
  U_{j,\mu}(t, n) = U_\mu(n_d t + j, n)
  \]

with $t$ integer, $j = 1, \cdots, n_d$
Lattice Action in Terms of New Variables

Lattice action

\[ S_L = \sum_t [L_s(t) + L_{km}(t) + L_{kg}(t)] \]

where:

\[ L_s(t) = S_s[U(t), \phi^\dagger(t), \phi(t)] \]

\[ L_{kg}(t) = S_{kg}[U(t), U(t+1)] \]

\[ L_{km}(t) = \sum_{t,m,n} \phi^\dagger_j(t+1, m) \Delta_{j,k}(m, n; U(t), U(t+1)) \phi_k(t, n) \]

+ c.c.

Positivity of transfer matrix condition

For any functional \( \mathcal{F} \) of physical fields for positive \( t \)

\[ \langle 0 | \mathcal{F} \Theta[\mathcal{F}] | 0 \rangle \geq 0 \]
Time Reversal of Field Variables

Time reversal of the functional $\mathcal{F}$

$$\Theta \mathcal{F} [\phi, \phi^\dagger, U] = \mathcal{F} [\Theta \phi, \Theta \phi^\dagger, \Theta U]$$

with fields

$$\Theta \phi_j(n) = \phi_j^\dagger(\Theta n),$$

$$\Theta \phi_j^\dagger(n) = \phi_j(\Theta n),$$

$$\Theta U_{j,\mu}(n) = \begin{cases} 
U_{\Theta j,\mu}^\dagger(\Theta n) & \mu = 1, 2, 3, \\
U_{\Theta j-1,\mu}(\Theta n) & \mu = 0, j < n_d, \\
U_{j,\mu}(\Theta n - \hat{\Theta}) & \mu = 0, j = n_d,
\end{cases}$$

and space-time coordinate $n = (t, n)$

$$\Theta(t, n) = (-t, n)$$

$\Rightarrow$ site-inversion realization of time reversal $t \rightarrow -t$
Functionals with Well Defined Time Inversion
Parity

Functionals of the type
\[ \Theta \mathcal{F}[\phi(n), \phi^\dagger(n), U(n)] = \tau \mathcal{F}[\phi(\Theta n), \phi^\dagger(\Theta n), U(\Theta n)] \]

Combinations of local fields satisfying:

\[ \phi_{\tau,j}(t, n) = \frac{1}{2}[\phi_j(t, n) + \tau \phi^\dagger_{\Theta j}(t, n)], \]
\[ \phi^\dagger_{\tau,j}(t, n) = \frac{1}{2}[\phi^\dagger_j(t, n) + \tau \phi_{\Theta j}(t, n)], \]
\[ U_{\tau,j,\mu}(t, n) = \frac{1}{2}[U_j,\mu(t, n) + \tau U^\dagger_{\Theta j,\mu}(t, n)], \quad \mu = 1, 2, 3 \]

with \( j = 1, \ldots, (n_d - 1)/2 \)
Gauge Invariance

Gauge fixing

- gauge transformation

\[ \omega(n_d t + j, n) = [U_0(n_d t + j - 1, n) \cdots U_0(n_d t + 1, n)]^\dagger \]

for \( 2 \leq j \leq n_d \) (on the original lattice, with simple time slices)

- cancellation of the time component of the gauge field within the block time slices and \( U_{j,0}(t, n) \rightarrow 1 \) for \( j < n_d \)
Proof of Positivity of Transfer Matrix

Lattice action split into three pieces

\[ S_L = S_0 + S_+ + S_- \]

Time reversal invariance of microscopic dynamics

\[ S_\pm[\Psi(t)] = \Theta[S_\mp[\Psi(t)]] = S_\mp[\Psi(\Theta t)] \]
\[ S_0[\Psi] = S_0[\Psi(\Theta t)] \]

Introducing \( \Psi = (U, \phi^\dagger, \phi) \)

\[ \langle 0 | \mathcal{F} \Theta \mathcal{F} | 0 \rangle = \int D[\Psi] e^{-S_0[\Psi]} e^{-S_+[\Psi]} \mathcal{F}[\Psi] e^{-S_-[\Psi]} \Theta[\mathcal{F}[\Psi]] \]
⇒ Time reversal invariance of the vacuum state $\Theta |0\rangle = |0\rangle$

$$\langle 0 | \mathcal{F} \Theta \mathcal{F} | 0 \rangle = \int D[\psi] e^{-S_0[\psi]} e^{-S_+[\psi]} \mathcal{F}[\psi] \Theta [e^{-S_+[\psi]} \mathcal{F}[\psi]]$$

$$= \int D_{t=0}[\psi] \int D_{t>0}[\psi] e^{-\frac{S_0}{2}} S_+ \mathcal{F} \Theta \mathcal{F} \int \Theta_{t>0}[\psi] e^{-\frac{S_0}{2}} S_+$$

Assuming $\tau_\mathcal{F}$ - time-reversal parity of $\mathcal{F}[\psi]$

$$\langle 0 | \mathcal{F} \Theta \mathcal{F} | 0 \rangle = \tau_\mathcal{F} \left( \int D_{t \geq 0}[\psi] e^{-\frac{1}{2} S_0[\psi]} - S_+[\psi] \mathcal{F}[\psi] \right)^2$$

For $\tau_\mathcal{F} = 1$

$$\langle 0 | \mathcal{F} \Theta \mathcal{F} | 0 \rangle \geq 0$$
Boundary Conditions

Positivity of the transfer matrix

⇒ equations valid for each trajectory in the path integral

Boundary conditions:
\[ \psi(t_f) = \tau \psi(t_i) \]

- periodic (antiperiodic) boundary conditions for time-reversal even (odd) variables
- generalized KMS conditions

Example:
\[ \phi_j(x) = \partial_0^j \phi(x), \quad \tau = (-1)^j \]
\[ \phi_j(t_f, x) = (-1)^j \phi_j(t_i, x) \]
Conclusion

- Truncation of the gradient expansion of the effective theory (after elimination of heavy particle modes) ⇒ question of unitarity
- Reflection positivity demonstrated for the Yang-Mills-Higgs model with higher order derivatives, within the Fock space span by local, time-reversal invariant functionals of the fields acting on the time-reversal invariant vacuum
- Restriction: generalized KMS boundary conditions
Thank you for your attention!