Study of Thermodynamic Quantities in Generalized Gravity Theories

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In this work, we have studied the thermodynamic quantities like temperature of the universe, heat capacity and squared speed of sound in generalized gravity theories like Brans-Dicke, Hořava-Lifshitz and \( f(R) \) gravities. We have considered the universe filled with dark matter and dark energy. Also we have considered the equation of state parameters for open, closed and flat models. We have observed that in all cases the equation of state behaves like quintessence. The temperature and heat capacity of the universe are found to decrease with the expansion of the universe in all cases. In Brans-Dicke and \( f(R) \) gravity theories the squared speed of sound is found to exhibit increasing behavior for open, closed and flat models and in Hořava-Lifshitz gravity theory it is found to exhibit decreasing behavior for open and closed models with the evolution of the universe. However, for flat universe, the squared speed of sound remains constant in Hořava-Lifshitz gravity.

I. INTRODUCTION

Recently, it has become well known that the universe has not only undergone the period of early-time accelerated expansion (inflation), but also is currently in the so-called late-time accelerating epoch (dark energy era). The unified description of inflation and dark energy is achieved by modifying the gravitational action at the very early Universe as well as at the very late times [1, 2]. A number of viable modified gravity theories has been suggested [3, 4, 5, 6, 7]. In reference [8], the connection between modified gravity and M-string theory was indicated. The modified gravity gives the qualitative answers to the number of fundamental questions about dark energy. Indeed, the origin of dark energy may be explained by some sub-leading gravitational terms which become relevant with the decrease of the curvature (at late times). Moreover, there are many proposals to consider the gravitational terms relevant at high curvature (perhaps, due to quantum gravity effects) as the source of the early-time inflation. Hence, there appears the possibility to unify and to explain both: the inflation and late-time acceleration as the modified gravity effects [8]. Reviews on modified gravity are available in the references like [9] and [10]. Among the recent attempts to construct a consistent theory of quantum gravity, much attention has been paid to the quite remarkable Hořava-Lifshitz quantum gravity [11]. An extensively studied generalization of general relativity involves modifying the Einstein-Hilbert Lagrangian in the simplest possible way, replacing \( R - 2\Lambda \) by a more general function \( f(R) \) [10, 11, 12]. Recently the modified Hořava-Lifshitz \( f(R) \) gravity has been proposed in ref.[1]. Discussions on Hořava-Lifshitz gravity have been made in references [13, 14]. The basic idea of Hořava-Lifshitz gravity is to modify the UV behavior of the general theory so that the theory is perturbatively renormalizable [14]. However this modification is only possible on condition when we abandon Lorentz symmetry in the high energy regime [14]. In reference [15], the very interesting physical implications of Hořava-Lifshitz gravity are summarized as: (i) the novel solution subclasses, (ii) the gravitational wave production, the perturbation spectrum, (iii) the matter bounce, (iv) the dark energy phenomenology, (iv) the astrophysical phenomenology, and (v) the observational constraints on the theory. Recently, scalar-tensor theories have received renewed interest. The Brans-Dicke theory [16] is the simplest example of a scalar-tensor theory of gravity. In Brans-Dicke theory, Newton’s constant becomes a function of space and time, and a new parameter \( \omega \) is introduced. General relativity is recovered in the limit \( \omega \rightarrow \infty \) [17]. Interacting dark energy [18, 19] and holographic dark energy [20, 21, 22, 23, 24] models have been considered in Brans-Dicke theory. Brans-Dicke scalar field as chameleon field has been considered in the references [25] and [26].

A profound connection between gravity and thermodynamics was first established by Jacobson [27], who first

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showed that the Einstein gravity can be derived from the first law of thermodynamics in the Rindler spacetime. Thermodynamic aspects of the cosmological horizons have been reviewed in [28] and [29]. Investigating the generalized second law (GSL) of thermodynamics in gravity has gained immense interest in recent years. A plethora of papers have studied the thermodynamics in Einstein gravity theory [30, 31, 32, 33, 34, 35]. As the modified theory of gravity was argued to be a possible candidate to explain the accelerated expansion of our universe by various authors [36, 37, 38], thus it is interesting to examine the GSL in the extended gravity theories [39, 40, 41, 42]. Thermodynamics has been studied in the brane world scenario [43, 44, 45, 46], Hořava-Lifshitz gravity [47, 48, 49], Brans-Dicke gravity [50, 51, 52] and in $f(R)$ gravity [53, 54, 55]. Extending the study of [15], two of the authors of the present paper, examined the validity of the GSL in various cosmological horizons of a universe governed by the Hořava-Lifshitz gravity and the GSL was proved to be valid in different horizons [56].

In the present work, we have studied the thermodynamic quantities of the universe in generalized gravity theories like Brans-Dicke, Hořava-Lifshitz and $f(R)$ gravities. Instead of investigating the validity of the laws of thermodynamics, we have tried to investigate how the thermodynamic quantities like heat capacity ($C_v$), temperature $T$ and squared speed of sound $v_s^2$ behave during the evolution of the universe governed by the said gravity theories. In addition to this, the equation of state parameters have also been studied for all of the said gravity theories. Organization of the rest of the paper is as follows: In section II, we have discussed the thermodynamic quantities. In sections IIIA, IIIB and IIIC we have discussed the thermodynamic quantities under Brans-Dicke, Hořava-Lifshitz and $f(R)$ gravity theories respectively. Finally, in section IV, we have discussed the results.

II. GENERAL DESCRIPTION OF THERMODYNAMIC QUANTITIES

The Einstein field equations for homogeneous, isotropic FRW universe are given by [57] (choosing $c = 1$)

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho$$  \hspace{1cm} (1)

and

$$\dot{H} - \frac{k}{a^2} = -4\pi G (\rho + p)$$  \hspace{1cm} (2)

where $H (= \frac{\dot{a}}{a})$ is the Hubble parameter and $k = 0, -1, +1$ denote the curvature index for flat, open and closed universe respectively. Here, $\rho$ and $p$ denote the energy density and pressure of the universe. The energy momentum tensor $T^\mu_\nu$ is conserved by virtue of the Bianchi identities, leading to the continuity equation [57]

$$\dot{\rho} + 3H (\rho + p) = 0$$  \hspace{1cm} (3)

where $p$ is the isotropic pressure and $\rho$ is the energy density of the fluid defined by

$$\rho = \frac{U}{V}$$  \hspace{1cm} (4)

Here, $U$ is the internal energy and $V$ is the volume of the universe.

We consider the FRW universe treated as a thermodynamical system. Then from Gibb’s equation of thermodynamics, we have [33]

$$TdS = d(\rho V) + pdV = d((\rho + p)V) - V dp$$  \hspace{1cm} (5)

where $S$ is the entropy, $T$ is the temperature and $V$ is the volume of the universe. The integrability condition of thermodynamic system is given by [58]
\[ \frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T} \]  

which leads to the relation between pressure, energy density and temperature as

\[ dp = \frac{\rho + p}{T}dT \]  

(7)

From (5) and (7), we get

\[ dS = d\left( \frac{(\rho + p)V}{T} \right) \]  

and integrating, we can obtain the expression of the entropy as (except for an additive constant)

\[ S = \frac{(\rho + p)V}{T} \]  

(9)

However, for adiabatic process entropy is constant and consequently, the equation (5) becomes

\[ d[(\rho + p)] = Vdp \]  

(10)

Relation (9) can also be obtained using (7) into (10). Hence for adiabatic process equation (9) may be considered as the temperature defining equation as

\[ T = \frac{(\rho + p)V}{S} \]  

(11)

The square speed of sound and heat capacity are defined by

\[ v_s^2 = \frac{\partial p}{\partial \rho} \]  

and

\[ C_V = V\frac{\partial \rho}{\partial T} \]  

(when entropy \( S \) is constant = \( S_0 \), say)

(13)

These thermodynamic quantities would be investigated for their evolution with the expansion of the universe in the subsequent sections.

III. THERMODYNAMIC QUANTITIES IN GENERALIZED GRAVITY THEORIES

A. Brans-Dicke Theory

The Jordan-Fierz-Brans-Dicke theory (heretofore, we will call it Brans-Dicke (BD) theory for simplicity) is the simplest example of a scalar-tensor theory of gravity. A brief introduction of the BD theory has been presented in the previous section. The Lagrangian density for the Brans-Dicke theory is

\[ \mathcal{L} = \sqrt{-g} \left[ -\phi R + \frac{\omega}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \mathcal{L}_m \right] \]  

(14)
where \( \phi \) is the Brans-Dicke field, and \( \mathcal{L}_m \) is the Lagrangian density for the matter fields. The self-interacting BD theory is described by the Jordan-Brans-Dicke (JBD) action (choosing \( c = 1 \)) as:

\[
S = \int \frac{d^4x \sqrt{-g}}{16\pi} \left[ \phi R - \frac{\omega(\phi)}{\phi} \phi^{\alpha} \phi_{\alpha} - V(\phi) + 16\pi \mathcal{L}_m \right]
\]

where \( V(\phi) \) is the self-interacting potential for the BD scalar field \( \phi \) and \( \omega(\phi) \) is modified version of the BD coupling parameter which is a function of \( \phi \). In this theory \( \frac{1}{\phi} \) plays the role of the gravitational constant \( G \).

This action also matches with the low energy string theory action for \( \omega = -1 \). The matter content of the Universe is composed of matter fluid, so the energy-momentum tensor is given by

\[
T^m_{\mu\nu} = (\rho + p) u^\mu u^\nu + p g_{\mu\nu}
\]

where \( u^\mu \) is the four velocity vector of the matter fluid satisfying \( u^\mu u_\mu = -1 \) and \( \rho, p \) are respectively energy density and isotropic pressure.

From the Lagrangian density we obtain the field equations

\[
G_{\mu\nu} = 8\pi T^m_{\mu\nu} + \frac{\omega(\phi)}{\phi^2} \left[ \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\alpha} \phi_{,\alpha} \right] + \frac{1}{\phi} \left[ \phi_{,\mu;\nu} - g_{\mu\nu} \Box \phi \right] - \frac{V(\phi)}{2\phi} g_{\mu\nu}
\]

and

\[
\Box \phi = \frac{8\pi T}{3 + 2\omega(\phi)} - \frac{1}{3 + 2\omega(\phi)} \left[ 2V(\phi) - \phi \frac{dV(\phi)}{d\phi} \right] - \frac{d\omega(\phi)}{d\phi} \left( \phi_{,\mu} \phi_{,\mu} \right)
\]

where \( T = T^m_{\mu\nu} g^{\mu\nu} \). Equation (17) can also be written as

\[
G_{\mu\nu} = 8\pi \tilde{T}_{\mu\nu} = \frac{8\pi}{\phi} \left( T^m_{\mu\nu} + \frac{1}{8\pi} T^\phi_{\mu\nu} \right)
\]

where \( \tilde{T}_{\mu\nu} \) can be treated as effective energy momentum tensor. The line element for Friedman-Robertson-Walker space-time is given by

\[
ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right]
\]

where, \( a(t) \) is the scale factor and \( k = 0, -1, +1 \) is the curvature index describe the flat, open and closed model of the universe.

We are considering the universe filled with dark energy (with energy density \( \rho_D \)) and dark matter (with energy density \( \rho_m \)). As we are not considering interacting situation, the conservation equations are separately satisfied for dark matter and dark energy. Thus

\[
\dot{\rho}_D + 3H(\rho_D + p_D) = 0
\]

and

\[
\dot{\rho}_m + 3H(1 + w_m)\rho_m = 0
\]

Solving the conservation equation for dark matter, we get the density and pressure of dark matter as \( \rho_m = \rho_{m_0}(1 + z)^{3(1 + w_m)} \) and \( p_m = \rho_{m_0}(1 + w_m)(1 + z)^{3(1 + w_m)} \). Defining \( \rho_1 = \frac{\rho_m}{\phi} \) and \( p_1 = \frac{p_m}{\phi} \) the Einstein’s field equations can be written as
Fig. 1 shows the EOS parameter $w_{total} = \frac{p_1 + p_D}{\rho_1 + \rho_D}$ for $k = -1$ (the red line), $k = 1$ (the green line) and $k = 0$ (the blue line) for Brans-Dicke model where dark energy and dark matter satisfy the conservation equation separately. We have taken $\alpha = 3$, $B = 0.1$, $\rho_{m0} = 0.23$, $w_m = 0.003$.

$$H^2 + \frac{k}{a^2} = \frac{8\pi}{3}(\rho_1 + \rho_D)$$

(23)

$$\dot{H} - \frac{k}{a^2} = -4\pi(\rho_1 + p_1 + \rho_D + p_D)$$

(24)

where

$$\rho_D = \frac{\omega}{16\pi}\frac{\dot{\phi}^2}{\phi^2} - \frac{3}{8\pi}H\frac{\dot{\phi}}{\phi} + \frac{V(\phi)}{16\pi\phi}$$

(25)

and

$$p_D = \frac{\omega}{16\pi}\frac{\dot{\phi}^2}{\phi^2} + \frac{H}{4\pi}\frac{\dot{\phi}}{\phi} + \frac{1}{8\pi}\frac{\ddot{\phi}}{\phi} - \frac{V(\phi)}{16\pi\phi}$$

(26)

To find the thermal quantities we use the choices of $\phi$ and $V$ as

$$\phi = \phi_0 a^\alpha, \quad V = V_0 a^{-\frac{3(1+w)}{\alpha}}$$

(27)

Using EOS for dark energy $p_D = w\rho_D$ and the solution of (21) $\rho_D = \rho_D_0 a^{-3(1+w)}$ in the field equations we get

$$H^2 = kAa^{-2} + Ba^{-\frac{2\alpha(1+w)\alpha-1}{2\alpha}} + Ca^{-\alpha-3(1+w)}$$

(28)

$$\dot{H} = kA_1a^{-2} + B_1a^{-\frac{2\alpha(1+w)\alpha-1}{2\alpha}} + C_1a^{-\alpha-3(1+w)}$$

(29)
Fig. 2 shows the heat capacity $C_v$ for $k = -1$ (the thick line), $k = 1$ (the dotted line) and $k = 0$ (the broken line) for Brans-Dicke model where dark energy and dark matter satisfy the conservation equation separately. We have taken $\alpha = 3$, $B = 0.1$, $\rho_{m0} = 0.23$, $w_m = 0.003$.

Fig. 3 shows the temperature $T$ with evolution of the universe for $k = -1$ (the red line), $k = 1$ (the green line) and $k = 0$ (the blue line) for Brans-Dicke model. We have taken $\alpha = 3$, $B = 0.1$, $\rho_{m0} = 0.23$, $w_m = 0.003$.

Fig. 4 shows the squared speed of sound $v_s^2$ with evolution of the universe for $k = -1$ (the red line), $k = 1$ (the green line) and $k = 0$ (the blue line) for Brans-Dicke model. We have taken $\alpha = 3$, $B = 0.1$, $\rho_{m0} = 0.23$, $w_m = 0.003$.

where,

\[
\begin{align*}
A &= \frac{2}{\alpha (1 + \omega) \alpha (1 + \omega) - (2 + \alpha)}; \quad C = - \frac{16\pi (1 + \omega) \rho_D}{\phi_0} \\
A_1 &= \frac{2 + \alpha (1 - \alpha (1 + \omega))}{2 + \alpha}; \quad B_1 = \frac{\alpha (1 - \alpha (1 + \omega)) B}{2 + \alpha}; \quad C_1 = \frac{\alpha (1 - \alpha (1 + \omega)) \phi_0 C - 8\pi (1 + \omega) \rho_D}{\phi_0} \\
A_2 &= 6\phi_0 + AB_2/B; \quad B_2 = 6\rho_D(1 + \alpha - \omega \alpha^2/6)B; \quad C_2 = B_2 C/B - 16\pi \rho_D \\
\end{align*}
\]

Equation of state parameter $w_{total} = \frac{p_1 + p_D}{\rho_1 + \rho_D}$ is computed for flat, open as well as closed universes and are plotted against redshift $z$ in figure 1 and it is found that in all of the above cases the EOS parameters are staying above $-1$, which indicates quintessence-like behavior. The behaviour of the EOS parameters further indicate that the energy density is increasing with evolution of the universe irrespective of it curvature.

We replace $p$ and $\rho$ by $(p_1 + p_D)$ and $(\rho_1 + \rho_D)$ in equation (11) we get the temperature $T$, which is used in equation (13) to get the heat capacity $C_v$. Heat capacity $C_v$ is computed for open, closed and flat universes and are plotted in figure 2. This figure shows that for all of the three universes, the heat capacity is decreasing with increase in the redshift. This means that the heat capacity is increasing with evolution of the universe
Fig. 5 shows the behavior of the total equation of state parameter $w_{\text{total}}$ with evolution of the universe for $k = -1$ (the red line), $k = 1$ (the green line) and $k = 0$ (the blue line) for Hořava-Lifshitz gravity. We have taken $\lambda = 1.2$, $\mu = 1.03$, $w_m = 0.03$ and $\rho_{\text{m0}} = 0.23$.

irrespective of its curvature. Also, we present the temperature $T$ against redshift $z$ for all of the curvatures. We find that the temperature is decreasing with evolution of the universe.

### B. Hořava-Lifshitz Gravity

Thermodynamics in cosmology has been extensively studied either in Einstein’s theory of gravity or in modified theories of gravity. In this section, we shall generalize such studies to the Hořava-Lifshitz (HL) Cosmology. An exhaustive review of HL cosmology is available in [59]. We briefly review the scenario where the cosmological evolution is governed by HL gravity. The dynamical variables are the lapse and shift functions, $N$ and $N_i$, respectively, and the spatial metric $g_{ij}$. In terms of these fields the full metric is written as [60]

$$ds^2 = -N^2dt^2 + g_{ij}(dx^i + N_i dt)(dx^j + N_j dt)$$

(31)

where indices are raised and lowered using $g_{ij}$. The scaling transformation of the coordinates reads: $t \rightarrow l^3 t$ and $x^i \rightarrow lx^i$.

The action of the HL gravity is given by [60]

$$I = dt \int d^3x \left( \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_m \right)$$

$$\mathcal{L}_0 = \sqrt{g} N \left[ \frac{\kappa^2}{2}(K_{ij}K^{ij} - \lambda K^2) + \frac{\mu^2}{8(1-3\lambda)}(\Lambda R - 3\Lambda^2) \right]$$

$$\mathcal{L}_1 = \sqrt{g} N \left[ \frac{\kappa^2 \mu^2(1-4\lambda)}{32(1-3\lambda)} R^2 - \frac{\kappa^2}{2} \left( C_{ij} - \frac{\mu^2}{2} R_{ij} \right) (C^{ij} - \frac{\mu^2}{2} R^{ij}) \right]$$

(32)

where, $\kappa^2$, $\lambda$, $\mu$, $\omega$ and $\Lambda$ are constant parameters, and $C_{ij}$ is Cotton tensor (conserved and traceless, vanishing for conformally flat metrics). The first two terms in $\mathcal{L}_0$ are the kinetic terms, others in $(\mathcal{L}_0 + \mathcal{L}_1)$ give the potential of the theory in the so-called “detailed-balance” form, and $\mathcal{L}_m$ stands for the Lagrangian of other matter field. Comparing the action to that of the general relativity, one can see that the speed of light and the cosmological Newtons constant are
Fig. 6 shows the behavior of temperature $T$ with evolution of the universe for $k = -1$ (the red line), $k = 1$ (the green line) and $k = 0$ (the blue line) for Hořava-Lifshitz gravity. We have taken $\lambda = 1.2$, $\mu = 1.03$, $w_m = 0.03$ and $\rho_{tot} = 0.23$. Fig. 7 shows the behavior of heat capacity $C_v$ with evolution of the universe for $k = -1$ (the red line), $k = 1$ (the green line) and $k = 0$ (the blue line) for Hořava-Lifshitz gravity. We have taken $\lambda = 1.2$, $\mu = 1.03$, $w_m = 0.03$ and $\rho_{tot} = 0.23$.

Fig. 8 shows the squared speed of sound $v_s^2$ with evolution of the universe for $k = -1$ (the red line), $k = 1$ (the green line) and $k = 0$ (the blue line) for Hořava-Lifshitz gravity. We have taken $\lambda = 1.2$, $\mu = 1.03$, $w_m = 0.03$ and $\rho_{tot} = 0.23$.

\[ c = \frac{\kappa^2 \mu}{4} \sqrt{\frac{\Lambda}{1 - 3\lambda}}, \quad G_c = \frac{\kappa^2 c}{16\pi(3\lambda - 1)} \]  \hspace{1cm} (33)

It may be noted that when $\lambda = 1$, $\mathcal{L}_0$ reduces to the usual Lagrangian of Einsteins general relativity. Thus, when $\lambda = 1$, the general relativity is approximately recovered at large distances.

As we are considering dark energy with dark matter the conservation equations are given by (21) and (22). The field equations are

\[ H^2 + \frac{k}{a^2} = \frac{8\pi G_c}{3}(\rho_m + \rho_D) \]  \hspace{1cm} (34)
and

$$\dot{H} + \frac{3}{2} H^2 + \frac{k}{2a^2} = -4\pi G_c (p_m + p_D) \quad (35)$$

where

$$\rho_D = \frac{3\kappa^2 \mu^2 k^2}{8(3\lambda - 1)a^4} + \frac{3\kappa^2 \mu^2 \Lambda^2}{8(3\lambda - 1)} \equiv \frac{1}{16\pi G_c} \left( \frac{3k^2}{\Lambda a^4} + 3\Lambda \right) \quad (36)$$

and

$$p_D = \frac{\kappa^2 \mu^2 k^2}{8(3\lambda - 1)a^4} - \frac{3\kappa^2 \mu^2 \Lambda^2}{8(3\lambda - 1)} \equiv \frac{1}{16\pi G_c} \left( \frac{k^2}{\Lambda a^4} - 3\Lambda \right) \quad (37)$$

Using the solution for the conservation equation for dark matter given in (22) we get the total energy density as a function of redshift \((z = \frac{1}{a} - 1)\) as

$$\rho(z) = \rho_m + \rho_D = \rho_{m0} (1 + z)^{3(1+w_m)} + \frac{1}{16\pi G_c} \left( \frac{3k^2(1+z)^4}{\Lambda} + 3\Lambda \right) \quad (38)$$

Similarly, the total pressure as a function of redshift \(z\) is

$$p(z) = p_m + p_D = \rho_{m0} w_m (1 + z)^{3(1+w_m)} + \frac{1}{16\pi G_c} \left( \frac{k^2(1+z)^4}{\Lambda} - 3\Lambda \right) \quad (39)$$

The squared speed of sound is given as a function of \(z\) by

$$v_s^2(z) = \frac{k^2(1+z)\kappa^2 \mu^2 + 6(1+z)^3w_m(-1 + 3\lambda)w_m \rho_{m0}(1 + w_m)}{3\{k^2(1+z)\kappa^2 \mu^2 + 2(1+z)^3w_m(-1 + 3\lambda)(1 + w_m)\rho_{m0}\}} \quad (40)$$

and the heat capacity becomes

$$C_v(z) = \frac{3S_0 \{k^2(1+z)\kappa^2 \mu^2 + 6(1+z)^3w_m(-1+3\lambda)(1 + w_m)\rho_{m0}\}}{k^2(1+z)\kappa^2 \mu^2 + 6(1+z)^3w_m(-1 + 3\lambda)w_m \rho_{m0}(1 + w_m)} \quad (41)$$

The thermodynamic quantities expressed above are now plotted against redshift to see their behavior with the evolution of the universe. In figure 5, where we have plotted the equation of state parameter for the Horava-Lifshitz gravity, we see that the behavior is like Brans-Dicke theory. It is staying above \(-1\). However, at lower redshifts the equation of state parameter is tending to \(-1\). However, it never crosses \(-1\). Like Brans-Dicke, this behavior remains the same for flat, open and closed universes. In figure 6 we find that in the case of flat, open and closed universes, the temperature \(T\) is decreasing with the evolution of the universe. From figure 6 we see that the heat capacity \(C_v\) is increasing as we are approaching towards the lower redshifts. From figure 7 we understand that for open and closed universes, the squared speed of sound \(v_s^2\) decreases with the evolution of the universe. However, for flat universe, the \(v_s^2\) remains constant throughout the evolution of the universe.

C. \(f(R)\) Gravity

Motivated by astrophysical data which indicate that the expansion of the universe is accelerating, the modified theory of gravity (or \(f(R)\) gravity) which can explain the present acceleration without introducing dark energy, has received intense attention. Extensive review of \(f(R)\) gravity is available in [61]. The action of \(f(R)\) gravity is given by [62]

$$S = \int d^4x \sqrt{-g} \left[ \frac{f(R)}{2\kappa^2} + \mathcal{L}_{matter} \right] \quad (42)$$
where $g$ is the determinant of the metric tensor $g_{\mu \nu}$, $\mathcal{L}_{\text{matter}}$ is the matter Lagrangian and $\kappa^2 = 8\pi G$. The $f(R)$ is a non-linear function of the Ricci curvature $R$ that incorporates corrections to the Einstein-Hilbert action which is instead described by a linear function $f(R)$. The gravitational field equations in this theory are

\begin{equation}
H^2 + \frac{k}{a^2} = \frac{\kappa^2}{3f'(R)}(\rho + \rho_c) \tag{43}
\end{equation}

\begin{equation}
\dot{H} - \frac{k}{a^2} = -\frac{\kappa^2}{2f'(R)}(\rho + p + \rho_c + p_c) \tag{44}
\end{equation}

where $\rho_c$ and $p_c$ can be regarded as the energy density and pressure generated due to the difference of $f(R)$ gravity from general relativity given by [61] (choosing $G = 1$)

\begin{equation}
\rho_c = \frac{1}{8\pi f'} \left[ \frac{f - Rf'}{2} - 3Hf'' \right] \tag{45}
\end{equation}

\begin{equation}
p_c = \frac{1}{8\pi f'} \left[ \frac{f - Rf'}{2} + f''R + f'''R^2 + 6f''R \right] \tag{46}
\end{equation}

where, the scalar tensor $R = -6\left(\dot{H} + 2H^2 + \frac{k}{a^2}\right)$.

As we are considering both dark matter and dark energy, the Friedman equations take the form

\begin{equation}
H^2 + \frac{k}{a^2} = \frac{8\pi}{3} \rho_{\text{total}} \tag{47}
\end{equation}

\begin{equation}
\dot{H} - \frac{k}{a^2} = -4\pi(\rho_{\text{total}} + p_{\text{total}}) \tag{48}
\end{equation}

where,

\begin{equation}
\rho_{\text{total}} = \rho_1 + \rho_c, \quad p_{\text{total}} = p_1 + p_c \tag{49}
\end{equation}

with $\rho_1 = \frac{\rho_m}{a^3}$ and $p_1 = \frac{p_m}{a^4}$. As there is no interaction, like the previous two cases the dark energy and dark matter satisfy the conservation equation separately. Therefore, we have the density and pressure of dark matter as $\rho_m = \rho_{m0}(1 + z)^{3(1 + w_m)}$ and $p_m = \rho_{m0}(1 + w_m)(1 + z)^{3(1 + w_m)}$. In the present section, while considering the $f(R)$ gravity, we have illustrated with a solution,

\begin{equation}
f(R) = \beta R + \alpha R^m, \quad R = \frac{A}{a^n}, \quad m > 1, \quad \alpha > 0, \quad \beta > 0 \tag{50}
\end{equation}

Also we have chosen $n(m - 1) = 1$.

Using the above form of $f(R)$ in (44) and (45) we have computed the temperature $T$, squared speed of sound $v_s^2$ and heat capacity $C_v$ as functions of the redshift $z$ as follows:

\begin{equation}
\rho_1 = \frac{(1 + z)^{3(1 + w_m)}\rho_{m0}}{m(A(1 + z)^n)^{-1+m\alpha + \beta}} \tag{51}
\end{equation}

\begin{equation}
p_1 = \frac{w_m(1 + z)^{3(1 + w_m)}\rho_{m0}}{m(A(1 + z)^n)^{-1+m\alpha + \beta}} \tag{52}
\end{equation}
Fig. 9 shows the plot of heat capacity $C_v$ against redshift $z$ in $f(R)$ gravity. We see that $C_v$ is increasing with the evolution of the universe. Here $\alpha = 2.10132$, $\beta = 2.10101$, $w_m = 0.003$, $m = 2.3$, $\rho_m = 0.23$ and the red, green and blue lines correspond to $k = -1$, 1, 0 respectively.

Fig. 10 shows the plot of temperature $T$ against redshift $z$. We find that the temperature is decreasing with the evolution of the universe in $f(R)$ gravity $\alpha = 12.1$, $\beta = 10.1$, $w_m = 0.003$, $m = 12.3$, $\rho_m = 0.23$ and the red, green and blue lines correspond to $k = -1$, 1, 0 respectively.

Fig. 11 shows the plot of the squared speed of sound $v_s^2$ against redshift $z$ in $f(R)$ gravity. We find that $v_s^2$ is increasing with the evolution of the universe. Here, $\alpha = 12.1$, $\beta = 10.1$, $w_m = 0.003$, $m = 12.3$, $\rho_m = 0.23$ and the red, green and blue lines correspond to $k = -1$, 1, 0 respectively.

Using (51), (52), (53) and (54) we get temperature $T$, squared speed of sound $v_s^2$ and heat capacity $C_v$ as functions of redshift $z$ in the following forms

\[
T = \frac{\rho^1 + p^1 + \rho_c + p_c}{(1 + z)^3S_0}
\]
$$v^2 = \frac{\xi_1(z)}{\xi_2(z)}$$

where

$$\xi_1(z) = \left[16(m(A(1 + z)^n)^{-1+m}\alpha + \beta)^2 \times \right.$$  
$$\left. \frac{A_{n}(1+z)^{-1+4n}(-3A+6(-2+m)mn^2(1+n)^2(1+z)^4-n+6mn(1+z)^2-n(-5+n+z+zn))}{24\pi} \sqrt{C_1 - \frac{k}{(1+z)^2} + \frac{A_{n}(1+z)^{-4+n}}{4(-4+n)}} + \frac{A}{48\pi} \times \right.$$  
$$\left. \left(-6(-2+m)mn(-4+n)n^2(1+n)^2(1+z)^3+n + \frac{3mn(1+z)^2n(-5+n+z+zn)}{C_1 - \frac{k}{(1+z)^2} + \frac{A_{n}(1+z)^{-4+n}}{4(-4+n)}} - 6m(-2+n)n(1+z)^1+n(-5+n+z+zn) \times \right. \right.$$  
$$\left. \sqrt{C_1 - \frac{k}{(1+z)^2} + \frac{A_{n}(1+z)^{-4+n}}{4(-4+n)}} + \frac{A(1+z)^{-4+n}}{m(A(1+z)^n)^{1+m+\alpha+\beta}} \right) \alpha \times \right.$$  
$$\left( (-1+m)mn(A(1+z)^n)^m \left( 1 + \frac{2(-2+m)mn(1+z)^4-2n}{A^2} \left( 3C_1 - \frac{3k}{(1+z)^2} + \frac{A_{n}(1+z)^{-4+n}}{4(-4+n)} \right) \right) \alpha \times \right.$$  
$$\left. \left\{ \left( -1+m \right)(A(1+z)^n)^{-1+m}\alpha + m(A(1+z)^n)^{-1+m}\alpha + \beta \right\} + \right.$$  
$$\left. 2(-2+m)(-1+m)mn(1+z)^3-2n(A(1+z)^n)^m \left( -6(-2+n)C_1 + \frac{6k(4-5n+2n^2(1+z)^2-A_{n}(1+z)^n)}{(-4+n)(1+z)^3} \alpha(m(A(1+z)^n)^{-1+m}\alpha + \beta) \right) \right) \left( 16(-1+m)mn(1+z)^2+3wm(A(1+z)^n)^{-1+m}\alpha + \beta \right) \rho_{m0} \right) + 48(1+w_m)(1+z)^2+3wm(A(1+z)^n)^{-1+m}\alpha + \beta \rho_{m0} \right) \right)$$

and

$$C_v = \frac{\hat{C}_1(z)}{\hat{C}_2(z)}$$

where

$$\hat{C}_1(z) = \frac{(-1+m)mn(1+z)^{-1+n}(A(1+z)^n)^{-1+m} \left\{ A^2 + 2(-2+m)mn(1+z)^4-2n \left( 3C_1 - \frac{3k}{(1+z)^2} + \frac{A_{n}(1+z)^{-4+n}}{4(-4+n)} \right) \right\} \alpha \times \right.$$  
$$\left. \left\{ \alpha \left\{ -1+m \right\}(A(1+z)^n)^{-1+m}\alpha + \beta \right\} + \right.$$  
$$\left. \left( 2(-2+m)mn(1+z)^4-2n \left( \frac{6k}{(1+z)^2} + A(1+z)^{-5+n} \right) + 2(-2+m)n(1+z)^3-2n \left( 3C_1 - \frac{3k}{(1+z)^2} + \frac{A_{n}(1+z)^{-4+n}}{4(-4+n)} \right) \right) \alpha \times \right.$$  
$$\left. \left\{ \left( 16(-1+m)mn(1+z)^2+3wm(A(1+z)^n)^{-1+m}\alpha + \beta \right) A_{m} \right\} \right) \left( 16(-1+m)mn(1+z)^2+3wm(A(1+z)^n)^{-1+m}\alpha + \beta \right) \left( \frac{3(1+w_m)(1+z)^2+3wm(A(1+z)^n)^{-1+m}\alpha + \beta}{m(A(1+z)^n)^{-1+m}\alpha + \beta} \right) \rho_{m0} \right) \right)$$

(56)
Fig. 12 shows the evolution of the equation of state parameter $w_{\text{total}}$ with the evolution of the universe in $f(R)$ gravity.

We find that for $k = -1, 1, 0$ the equation of state parameter $w_{\text{total}} > -1$. This indicates quintessence era.

Here, $\alpha = 10.32$, $\beta = 10.01$, $w_m = 0.003$, $m = 12.3$, $\rho_{\text{mol}} = 0.23$ and the red, green and blue lines correspond to $k = -1, 1, 0$ respectively.

and

$$\zeta(z) = -\frac{3}{S_0(1+z)^2} \left[ \begin{array}{c} A(1+z)^2n \left( -3A+6(-2+m)mn^2(1+n)^2(1+z)^{4-n}+6mn(1+z)^{2-n}(-5+n+z+zn) \right) \sqrt{C_1 - \frac{4}{A(1+z)^2} + \frac{A(1+z)^2-4n}{m(1+z)^{n+1}}} \right] +$$

$$\frac{(-1+m)(A(1+z)^n)^m}{48m\pi} \left( \begin{array}{c} A^2+2(-2+m)mn(1+z)^{3-2n} \left( 3C_1-\frac{4}{A(1+z)^2} + \frac{A(1+z)^2-4n}{m(1+z)^{n+1}} \right) \alpha + \frac{(1+z)^2(1+w_m)\rho_{\text{mol}}}{m(A(1+z)^n)^{n+1}m+\rho_{\text{mol}}} + \frac{w_m(A(1+z)^n)^{2n}(1+w_m)}{m(A(1+z)^n)^{n+1}m+\rho_{\text{mol}}} \right) \right] +$$

$$\frac{1}{S_0(1+z)^2} \left[ \begin{array}{c} A(1+z)^2n \left( -6(-2+m)mn(-4+n)n^2(1+n)^2(1+z)^{3-n} + 3mn(1+z)^{2-n}(-5+n+z+zn) \right) \sqrt{C_1 - \frac{k}{(1+z)^2} + \frac{A(1+z)^2-4n}{3(1+z)^{n+1}}} \right] +$$

$$\frac{6mn(1+z)^{1-n} \sqrt{C_1 - \frac{k}{(1+z)^2} + \frac{A(1+z)^2-4n}{3(1+z)^{n+1}}} \left( \begin{array}{c} A(1+z)^{2-2n} \left( -3A+6(-2+m)mn^2(1+n)^2(1+z)^{4-n}+6mn(1+z)^{2-n}(-5+n+z+zn) \right) \sqrt{C_1 - \frac{k}{(1+z)^2} + \frac{A(1+z)^2-4n}{3(1+z)^{n+1}}} \right) \right] +$$

$$\frac{\rho_{\text{mol}}(A(1+z)^n)^m}{m(A(1+z)^n)^{n+1}m+\rho_{\text{mol}}} \left( A^2+2(-2+m)mn(1+z)^{3-2n} \left( 3C_1-\frac{k}{(1+z)^2} + \frac{A(1+z)^2-4n}{3(1+z)^{n+1}} \right) \right) \left( \begin{array}{c} \alpha \left( \frac{A^2+2(-2+m)mn(1+z)^{3-2n} \left( 3C_1-\frac{k}{(1+z)^2} + \frac{A(1+z)^2-4n}{3(1+z)^{n+1}} \right) \alpha}{m(A(1+z)^n)^{n+1}m+\rho_{\text{mol}}} \right) \right) +$$

$$\frac{(-1+m)(A(1+z)^n)^m}{8A^2(m(A(1+z)^n)^{n+1}m+\rho_{\text{mol}} + 3w_m(A(1+z)^n)^{2n} \rho_{\text{mol}})(1+w_m) \left( \begin{array}{c} A(1+z)^{2-2n} \left( -3A+6(-2+m)mn^2(1+n)^2(1+z)^{4-n}+6mn(1+z)^{2-n}(-5+n+z+zn) \right) \sqrt{C_1 - \frac{k}{(1+z)^2} + \frac{A(1+z)^2-4n}{3(1+z)^{n+1}}} \right) \left( \begin{array}{c} A^2+2(-2+m)mn(1+z)^{3-2n} \left( 3C_1-\frac{k}{(1+z)^2} + \frac{A(1+z)^2-4n}{3(1+z)^{n+1}} \right) \alpha \left( \frac{A^2+2(-2+m)mn(1+z)^{3-2n} \left( 3C_1-\frac{k}{(1+z)^2} + \frac{A(1+z)^2-4n}{3(1+z)^{n+1}} \right) \alpha}{m(A(1+z)^n)^{n+1}m+\rho_{\text{mol}}} \right) \right) \right] +$$

The thermodynamic quantities expressed above are now plotted against redshift $z$ to see their behavior with the evolution of the universe. We proper choice of the parameters we plot all of the quantities in figures 8, 9, and 10 respectively. In figure 8 we find the increasing behavior of the heat capacity with the evolution of the universe. This behavior remains the same irrespective of the curvature of the universe. From figure 9 we see that as the universe in evolving, the temperature $T$ is decreasing. Here also we get the same behavior for open, closed and flat universes. It may be interpreted that the temperature of the universe decreases as it expands under $f(R)$ gravity. In figure 8 we plot the squared speed of sound $v_s^2$. Like the temperature, $v_s^2$ is decreasing with the expansion of the universe under $f(R)$ gravity. The
choices of the parameters are mentioned in the figure captions. Behavior of the equation of state parameter $w_{total}$ is observed in figure 11. Throughout the evolution of the universe $w_{total} > -1$. This indicates quintessence like behavior of the equation of state parameter. Therefore, we see that in $f(R)$ gravity, where we are considering the coexistence of dark energy and dark matter with very small pressure without interaction, the equation of state parameter behaves like quintessence era. This holds true for flat, closed as well as open universes. However, it also discerned that $w_{total}$ is gradually increasing in the negative direction.

**IV. DISCUSSIONS**

In the present work, we have considered modified gravities as Brans-Dicke, Hořava-Lifshitz and $f(R)$ gravities. Various thermodynamic quantities like temperature, heat capacity and squared speed of sound have been investigated for all the gravity theories. In each case we have considered that the universe is filled with dark matter and dark energy which are not interacting. Prior to evaluating the thermodynamic quantities we have studied the behaviors of the equation of state parameters. In figure 1 we see that in the case of Brans-Dicke gravity theory, the equation of state parameter is staying above $-1$ throughout the evolution of the universe. This indicates that the equation of state parameter is behaving like quintessence in this case irrespective of the curvature of the universe. In figure 5, where we have plotted the equation of state parameter for the Hořava-Lifshitz gravity, we see that the behavior is like Brans-Dicke. It is staying above $-1$. However, at lower redshifts the equation of state parameter is tending to $-1$. However, it never crosses $-1$. Like Brans-Dicke, this behavior remains the same for flat, open and closed universes. Similar behavior of the equation of state parameter is discernible in $f(R)$ gravity also. The evolution of the equation of state parameter for $f(R)$ gravity has been presented in figure 12. From the figures 3, 6 and 10 we see that the temperature $T$ of the universe is decreasing with evolution of the universe in Brans-Dicke, Hořava-Lifshitz and $f(R)$ gravities respectively. This behavior remains the same in open, closed and flat universes. From figures 2, 7 and 9 we find that the heat capacity $C$ of the universe increases with the evolution of the universe. Moreover, in all of the cases $C$ remains at the positive level throughout the evolution of the universe. We have also investigated the squared speed of sound $v_s^2$ in all of the cases. From figure 4 we see that for closed universe $v_s^2$ starts decreasing from redshift $-0.2$. However, up to $z = -0.2$ it has gradually increased. In the cases of open and flat universes, $v_s^2$ has an increasing behavior throughout the evolution of the universe. From figure 8 we see that in Hořava-Lifshitz gravity $v_s^2$ has a decaying behavior throughout the evolution of the universe in the case of open and closed universes. However, for flat ($k = 0$) universe, the squared speed of sound remains constant throughout the evolution of the universe. From figure 11 we understand that $v_s^2$ increases throughout the evolution of the universe irrespective of the curvature of the universe.

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