In this paper we consider features of graviton scattering in Matrix theory compactified on a 2-torus. The features which interest us can only be determined by nonperturbative effects in the corresponding 2+1 dimensional super Yang Mills theory. We show that the superconformal symmetry of strongly coupled Super Yang Mills Theory in 2 + 1 dimensions almost determines low energy, large impact parameter ten dimensional graviton scattering at zero longitudinal momentum in the Matrix model of IIB string theory. We then show that amplitudes involving arbitrary transverse momentum transfer are governed by instanton processes similar to the Polchinski Poulit process. Finally we consider the influence of instantons on a conjectured nonrenormalization theorem. This theorem is violated by instanton processes. Far from being a problem, this fact is seen to be crucial to the consistency of the IIB interpretation. We suggest that the $SO(8)$ invariance of strongly coupled SYM theory may lead to a proof of eleven dimensional Lorentz invariance.
1. Introduction

The emergence of spatial dimensions in string theory often occurs through mechanisms which seem extremely bizarre from any conventional viewpoint. Two notable examples are the emergence of the 11th direction of M-theory in strongly coupled IIA string theory and the 10th direction of type \( \text{II}B \) theory which materializes when M-theory is compactified on a 2-torus of vanishing area [1]. In both cases a discrete quantum number is identified which then plays the role of momentum for the new direction. In the first case the discrete quantum number is the number of D0-branes. In Matrix theory this becomes \( N \), the rank of the \( U(N) \) gauge group. As discussed in [2], \( N \) is identified with the longitudinal momentum of the light cone description. In the small 2-torus case the 2-brane wrapping number is the conjugate momentum to the new direction which we call \( Y \). This quantum number also has significance in Matrix theory. Matrix theory on a 2-torus is described by 2+1 dimensional \( U(N) \) super Yang Mills theory with 16 supersymmetries [2, 3]. The 2-brane wrapping number becomes the magnetic flux through the torus [4 – 7].

In both of the above cases the symmetry which rotates these new directions into the other directions is highly non-manifest. In the first case the symmetry is the difficult “angular conditions” which rotate longitudinal into transverse directions. In the second case they are the transverse rotations which rotate the new Aspinwall-Schwarz direction into the other 7 transverse directions. In each case there is evidence for the exactness of these symmetries in appropriate limits; the large \( N \) limit in one case and the large membrane wrapping number in the other. These limits generally involve letting the discrete quantum number tend to infinity while the corresponding quantum of energy tends to zero. In the second case we allow the torus to shrink to zero area while increasing the integer valued magnetic flux.

Evidence for the restoration of Lorentz invariance in the large \( N \) limit of Matrix theory was given in [2] where it was shown that low energy graviton-graviton scattering with zero longitudinal momentum transfer has exactly the form of single graviton exchange. However, to establish the full invariance, it is necessary to gain control over processes in which longitudinal momentum is exchanged. Progress has been made on this problem by Polchinski and Pouliot [8], who studied processes in which longitudinal momentum is transferred between infinite membrane configurations of Matrix theory. Here the basic process of unit momentum transfer was found to correspond to an instanton in a 2+1 dimensional gauge theory describing the membranes. The instanton amplitude exactly agrees with the expected behavior.
from single graviton exchange. Further progress was made by Dorey, Khoze and Mattis [9], who were able to study the sum over instantons, allowing any number of momentum units to be exchanged. At the moment it has not yet been possible to study longitudinal momentum exchange between types of configurations other than these extended membranes. For example one would also like to exchange longitudinal momentum between gravitons.

In the case of the vanishing torus and type $IIB$ theory two arguments have been given for rotational invariance [6, 7, 10]. Electric-magnetic duality of 3+1 dimensional super Yang Mills theory was used to indirectly prove that the manifest $SO(7)$ invariance of the super Yang Mills theory is enhanced to $SO(8)$ in the limit of vanishing torus area. An alternate argument is based on the fact that the vanishing torus limit should be described by a 2+1 dimensional fixed point theory. Then the superconformal invariance requires the $SO(8)$ symmetry. We will see in the next section that the superconformal invariance also determines a great deal about how graviton scattering amplitudes depends on impact parameter.

In this paper we will be interested in scattering amplitudes of gravitons in which $Y$ momentum is exchanged. For graviton-graviton scattering amplitudes involving no exchange of either longitudinal momentum or momentum in the $Y$ direction the method of [2] shows that the scattering is described by single graviton exchange in Matrix theory. However, to demonstrate the rotational invariance which rotates $Y$ into the other noncompact directions it is necessary to exchange $P_Y$. We shall see in what follows that this process is again governed by the same 2+1 dimensional instanton amplitudes as in [8].

Let us define some notations. The size of the two cycles of the torus are $L_1, L_2$. The longitudinal direction of the infinite momentum description of M-theory will be denoted by $x^{11}$. Following [2] we take $x^{11}$ to be compact with radius $R$. The large $N$ limit effectively decompactifies $x^{11}$. The 9 transverse directions are $X^i$ with $i = 1, \ldots, 9$. The two compact directions of the 2-torus are $X^1, X^2$. This leaves 7 manifest transverse spatial directions. Finally the newly emergent transverse direction is called $Y$. We can also denote this direction by $X^{10}$ but we prefer a notation which emphasizes the asymmetry of the M-theoretic origin of $Y$. Finally the term “longitudinal” momentum always refers to the component of momentum along the longitudinal axis of the light cone frame.

To conclude this introduction we will make some general remarks about scattering in Matrix theory. The definition of scattering amplitudes in Matrix theory involves path inte-
grals with fixed boundary conditions on the moduli space in the asymptotic past and future. As in the LSZ formalism of field theory, it is only necessary to choose boundary conditions which have a finite overlap with some stable state of the model. Scattering amplitudes are extracted from these path integrals by dividing by a wave function renormalization factor which extracts the overlap between the state defined by the asymptotic boundary conditions, and the true scattering states of the system.

Now consider the scattering of two graviton states of the toroidally compactified matrix model, with zero longitudinal momentum exchange and very large transverse separation. Even though we do not know the wave function of these states for generic values of the IIB string coupling it is extremely plausible that this can be encoded in an effective Lagrangian for the coordinate describing the separation between their centers of mass. On the moduli space, this coordinate is the zero mode of a field $\Delta^a$, $a = 1 \ldots 8$, which is the difference between the coefficients of the unit matrices in the two blocks representing the particles.

Supersymmetry guarantees that any correction to the free motion on the moduli space must contain at least four derivatives (or various powers of fermions). Thus, we may expect a Lagrangian of the schematic form

$$\delta \mathcal{L} = F_{abcd}(\Delta^a) \partial \Delta^a \partial \Delta^b \partial \Delta^c \partial \Delta^d$$

Calculating low energy scattering amplitudes between gravitons requires computing the functions $F_{abcd}$.

---

* See [11] for a more detailed discussion.

† In local field theory this is a theorem and can be proven by dispersion relations. In the present context one may be suspicious of it because of the growth of wave functions with $N$ described in [2]. In particular, Steve Shenker has reminded us that in the stringy regime, $N$ (which controls the world sheet cutoff) must always be taken large enough for the perturbative string wave functions to overlap, in order to reproduce long range gravitational interactions. However, in the presence of maximal SUSY, it appears that one can obtain correct results without taking the large $N$ limit.
2. Instantons and Momentum Transfer in IIB Theory

The 2+1 dimensional super Yang Mills theory describing IIB theory lives on the torus dual to $L_1, L_2$ with sides $\Sigma_1, \Sigma_2$ [12].

$$\Sigma_i = (2\pi)^2 \frac{l_{11}^3}{L_i R} \quad (2.1)$$

where $l_{11}$ is the 11 dimensional Planck length and $R$ is the compactification radius of the longitudinal direction [2].

The field content is seven scalars, $\phi$, a vector $A$ and fermions $\psi$. The Yang Mills coupling constant $g$ is given by [12]

$$g^2 = (2\pi)^2 \frac{R}{L_1 L_2} \quad (2.2)$$

The fields $\phi$ are proportional to the 7 uncompactified transverse coordinates $X^3, X^4, ..., X^9$. The precise connection is

$$\phi = \frac{g X}{\sqrt{R \Sigma_1 \Sigma_2}} \quad (2.3)$$

The $U(1)$ magnetic flux through the torus $\Sigma_{1,2}$ is quantized in integer units. The energy of a single unit of flux is given by

$$E_M = \frac{(2\pi)^2}{2Ng^2\Sigma_1 \Sigma_2} \quad (2.4)$$

It can be reexpressed in terms of the $L_i$, $R$ and $l_{11}$

$$E_M = \frac{R}{2N} \frac{L_1^2 L_2^2}{(2\pi)^4 l_{11}^6} = \frac{1}{2p_{11}^2} \frac{L_1^2 L_2^2}{(2\pi)^4 l_{11}^6} \quad (2.5)$$

From (2.5) it is seen that the energy gap tends to zero as the torus $L_{1,2}$ shrinks. This is properly interpreted as the decompactification of $Y$. By matching energy scales it was shown in [12] that the size of the $Y$ circle $L_Y$, satisfies

$$L_Y L_1 L_2 = (2\pi)^3 l_{11}^3 \quad (2.6)$$

The $Y$ component of momentum $P_Y$ is related to the integer valued magnetic flux quantum number $n$ by

$$P_Y = 2\pi \frac{n}{L_Y} \quad (2.7)$$

We will consider the scattering of a pair of gravitons in the IIB theory. For simplicity we
take the longitudinal momenta of the gravitons to be equal. This condition may be relaxed without introducing any essential complication. The configuration is described in Matrix theory by considering block diagonal matrices composed of equal size blocks \( \frac{N}{2} \times \frac{N}{2} \) blocks. The center of masses of the two blocks are well separated in the transverse 7 dimensional space. The separation 7-vector has length \( \rho \) which corresponds to a symmetry breaking field

\[
|\phi| = \rho \frac{g}{\sqrt{R\Sigma_1\Sigma_2}}
\]

This corresponds to spontaneously breaking the \( U(N) \) gauge group to \( U(\frac{N}{2}) \times U(\frac{N}{2}) \). We will assume that all momentum invariants are small and that the impact parameter is large.

The spatial momentum transfer in the scattering process is described by a nine-vector \( Q \) with components in the seven directions \( X \), the longitudinal direction \( x_{11} \) and the \( Y \) direction. For processes with \( Q_{11} = Q_Y = 0 \), the amplitude is given by the same methods as used in [2]. Recall that when the \( U(N) \) symmetry is broken, the strings connecting the two blocks become massive. Integrating out these degrees of freedom in the one loop approximation gives a graviton-graviton scattering amplitude which agrees with single graviton exchange. The thing we wish to emphasize is that the amplitude in this case is computed by a purely perturbative method in the super Yang Mills theory.

We will continue to consider vanishing \( Q_{11} \) but relax the condition \( Q_Y = 0 \). As we shall see such processes are nonperturbative in the Yang Mills coupling \( g \). With no loss of generality we may suppose that the two initial gravitons have equal and opposite transverse momentum including the component \( P_Y \). The momenta \( P_Y \) of the two gravitons are described as follows. When the gauge group is broken to \( U(\frac{N}{2}) \times U(\frac{N}{2}) \). There are two Abelian \( U(1) \) magnetic fluxes \( n_1, n_2 \) associated with the two blocks. Choosing \( n_1 = -n_2 = n \), the initial \( Y \) momenta are \( \pm \frac{n}{L_Y} \). Now consider a process in which \( n_1 \rightarrow n_1 + 1 \) and \( n_2 \rightarrow n_2 - 1 \) or equivalently \( n \rightarrow n + 1 \). This corresponds to a momentum transfer with \( Q_Y = \frac{2\pi}{L_Y} \). We may also think of it in gauge theory language. Consider an \( SU(2) \) subgroup of \( U(N) \) between a zero brane in each group. For example the \( SU(2) \) which mixes the entries labeled 1 and \( \frac{N}{2} + 1 \). Its generators are Pauli matrices \( \tau \) acting in the \( 2 \times 2 \) subspace \( (1, \frac{N}{2} + 1) \). This group is broken to the \( U(1) \) generated by \( \tau_3 \). The transition which exchanges \( Q_Y = \frac{2\pi}{L_Y} \) may be thought of as changing the magnetic flux of this \( U(1) \) subgroup by one quantum.
A process in which $SU(2)$ magnetic flux changes by an integer flux quantum is topologically nontrivial and requires a nonvanishing instanton number. The process is almost mathematically identical to the process studied in [8]. There are however some differences:

1. Polchinski and Pouliot were considering the scattering of infinite 2-branes which exchange longitudinal momentum. The 2+1 dimensional field theory describing the branes lives on an infinite plane. In the present case the 2+1 dimensional field theory lives on the torus dual to the space-time torus.

2. In [8] the gauge group corresponding to the two 2-branes was $U(2)$. In our case the system carries $N$ units of longitudinal momentum (not to be confused with $P_Y$) and is described by the gauge group $U(N)$.

However the difference between the two cases does not lead to substantial differences in the calculation. First, in the $IIB$ limit in which $L_Y \to \infty$ the field theory torus tends to infinite size while the instantons of interest remain small. Second, the instantons live in $SU(2)$ subspaces. Integrating over the orientation of the instantons only affects the results by multiplying the $n$ instanton result by an $N$ dependent factor $C_N$.

Let us first estimate the instanton amplitude for large values of the separation of the two gravitons (large Higgs field). The action of an $SU(2)$ instanton is given by

$$S = \frac{2\pi}{g^2} f$$  \hspace{1cm} (2.9)

where $f$ is the expectation value of $\phi$ in the broken symmetry state. Using (2.1), (2.2) and (2.3) we find

$$S = \frac{L_1 L_2 \rho}{2\pi^2 l_{11}^3}$$  \hspace{1cm} (2.10)

where $\rho$ represents the 7 dimensional distance between the gravitons. Finally, (2.6) gives

$$S = \frac{2\pi}{L_Y} \rho$$  \hspace{1cm} (2.11)

This means that the amplitude for the exchange of a single quantized unit of $P_Y$ is

$$A = \exp \left[ -\frac{2\pi}{L_Y} \rho \right].$$

It would be interesting to complete the instanton computation for gauge group $U(N)$ for any $N$ and finite volume and to perform the sum over instanton numbers along the lines
of [9]. This computation is not easy. However, as $L_Y \to \infty$ the field theory torus tends to infinite size and we may carry out our calculations in infinite space. In this limit the effective Lagrangian (1.1) (with zero modes replaced by local fields) describes a superconformal fixed point theory with scale invariance and $SO(8)$ symmetry [10]. We may use the scale invariance to constrain the functional form of the $F_{abcd}(r)$.

To do so we note that out along the flat directions the fields $X$ have scaling dimension $\frac{1}{2}$. This is determined by the form of the quadratic term in $\mathcal{L}$. The translation invariance of the system requires this term to be the usual canonical free field Lagrangian from which the dimensionality of $X$ follows. We then determine the dimensionality of $F_{abcd}(r)$ to be 3. This means that $F$ must be proportional to $r^{-6}$.

In fact, this scale invariance argument suggests that the full answer is not just proportional to $\frac{v^4}{r^6}$ but there can be three terms

$$A\frac{(v \cdot v)^2}{r^6} + B\frac{(v \cdot v)(r \cdot v)^2}{r^8} + C\frac{(r \cdot v)^4}{r^{10}}$$

where $A, B, C$ are coefficients which cannot be determined by using the $SO(8)$ and scale symmetries alone. It is likely that the ratios between them can be determined using supersymmetry.

3. Conclusion

When Matrix theory is compactified on a 2-torus a new direction of space which we called $Y$ emerges as the torus shrinks to zero size. We have seen that graviton scattering processes involving the exchange of the $Y$ component of momentum are nonperturbative instanton processes in the 2+1 dimensional super Yang Mills theory describing the compactified Matrix theory. Using the calculations of [8] and [9] we saw that the scattering amplitude for such processes agrees with the results of supergravity perturbation theory when the impact parameter is bigger than $L_Y$, the compactification scale of $Y$. However if one wants to study the theory for fixed impact parameter as $L_Y \to \infty$ one has to go beyond the leading order semiclassical instanton approximation. This may be done by appealing to the superconformal invariance of the fixed point theory describing the strongly coupled theory. The result derived in section 2 agrees with supergravity calculations at all length scales between $L_Y$.
and the 10 dimensional Planck scale. In particular, the amplitude is of the form $v^4/r^6$ and is invariant under the $SO(8)$ rotation group.

According to Polchinski and Pouliot [8] the same super Yang Mills theory which describes the toroidal compactification also describes the theory of 2-branes in the uncompactified theory. This theory may be used to describe longitudinal momentum transfer in the scattering of infinite 2-branes. One may wonder what the implications of the superconformal fixed point theory are for the membrane amplitudes. Let us suppose that we have a collection of any number $k$ of parallel membranes oriented in the $X^1, X^2$ plane. In the Polchinski, Pouliot description of Matrix theory this is described by $U(k)$ super Yang Mills theory. The membranes may not all be at rest in the same frame. In other words they may be relatively boosted along $x^{11}$ with respect to one another. This is described by turning on various magnetic fluxes associated with the unbroken $U(1)$ subgroups which survive when the branes are separated.

Polchinski and Pouliot find that the super Yang Mills theory becomes strongly coupled in the limit $R \to \infty$. Therefore when $x^{11}$ decompactifies the membrane interactions and scattering are described by the strongly coupled fixed point theory which is $SO(8)$ invariant. The $SO(8)$ group in this case is just the rotation group acting on the 8 spatial dimensions of the 11 dimensional theory which are transverse to the branes including $x^{11}$.

What does this say about Lorentz invariance of supergraviton scattering in 11 dimensions? To answer this we may use the fact that the poles in scattering amplitudes factorize. If for each external supergraviton in a process, we introduce a 2-brane to act as a source, the $SO(8)$ invariance of the membrane amplitude guarantees the corresponding invariance of the supergraviton scattering. This together with the manifest $SO(9)$ invariance of the Matrix theory of supergravitons should provide a basis for a proof of 11 dimensional Lorentz invariance.

We will conclude with some remarks about the nonperturbative breakdown of the non-renormalization theorem reported in [13]. Let us recall the argument in [2] for the necessity of such a theorem. In that reference a one loop matrix quantum mechanics calculation of the force between two gravitons in 11 dimensions was shown to exactly agree with supergravity at large distances and small transverse momentum. The amplitude had the form

$$A \sim N^2 \frac{v^4}{\rho^4}$$

(3.1)
where the factor $v^4$ is schematic for a quartic expression in the transverse velocities. As explained in [2], higher loop corrections, if they exist, will correct this by a factor of the form

$$1 + c_2 \frac{N}{\rho^3} + c_3 \left( \frac{N}{\rho^3} \right)^2 + c_4 \left( \frac{N}{\rho^3} \right)^3 + \cdots = F \left( \frac{N}{\rho^3} \right)$$

Eq. (3.2) represents the leading large $N$ behavior of the loop diagrams assuming no cancellation takes place. The amplitude will only agree with graviton exchange if $F \left( \frac{N}{\rho^3} \right) \to 1$ for $\rho \gg l_{11}$. For fixed $N$ (3.2) shows that this is so. However the correct limit is to fix $\rho$ and let $N \to \infty$. Evidently any nontrivial dependence of $F$ is dangerous in the large $N$ limit. This circumstance led to the conjecture that $F$ is not corrected beyond one loop. The conjecture has been confirmed at the two loop level [14]. Incidentally, it is obvious from what has been said that the nonrenormalization theorem is only really required for the leading large $N$ behavior, in other words for planar diagrams. The calculation in [14] does not test this issue because at the level of two loops the only graphs which contribute are planar.

Recently, a nonperturbative nonrenormalization theorem of this sort for 3+1 dimensional super Yang Mills theory was proven [13]. However, it was also shown that in 2+1 dimensions instantons violate any such theorem. In fact the instanton effects are exactly the ones we have been discussing in the previous section. The question is whether these effects are dangerous from the point of view of [2]. To answer this we note two things. First of all, for finite tori, the instanton effects are exponentially suppressed at large distance. Furthermore, unlike the perturbative corrections they do not depend on the ratio $N/\rho^3$. They are perfectly harmless for large $\rho$ as $N$ becomes large.

But as we have seen, the range of these effects grows as $L_Y \to \infty$. In this limit the long range behavior of the amplitude is indeed modified by the instanton effects. Far from being a problem, these effects are essential to describe the emergence of the new non-manifest $Y$ direction in $IIB$ theory.

Acknowledgments

We would like to thank Joe Polchinski and Steve Shenker for valuable conversations. W.F. would like to thank The Stanford Institute for Theoretical Physics for hospitality while this work was being done. The work of T.B. and N.S. was supported in part by the Department of Energy under Grant No. DE - FG02 - 96ER40959 and that of W.F. was
supported in part by the Robert A. Welch Foundation and by NSF Grant PHY-9511632. L.S. acknowledges the support of the NSF under Grant No. PHY - 9219345.

REFERENCES

1. P. Aspinwall, Talk Presented at ICTP Trieste Conf. on Physical and Mathematical Implications of Mirror Symmetry in String Theory, June 1995, hep-th/9508154. J. Schwarz, Phys. Lett. B367 (1996) 97, hep-th/9510080.

2. T. Banks, W. Fischler, S. Shenker and L. Susskind, hep-th/9610043.

3. W. Taylor, hep-th/9611042.

4. O.J. Ganor, S. Ramgoolam and W. Taylor, hep-th/9611202.

5. T. Banks, N. Seiberg and S. Shenker, Nucl. Phys. B490 (1997) 91, hep-th/9612157.

6. S. Sethi and L. Susskind, hep-th/9702101.

7. T. Banks and N. Seiberg, hep-th/9702187.

8. J. Polchinski and P. Pouliot, hep-th/9704029.

9. N. Dorey, V.V. Khoze and M.P. Mattis, hep-th/9704197.

10. N. Seiberg, hep-th/9705117.

11. T. Banks, Trieste Lectures 1997 to appear.

12. W. Fischler, E. Halyo, A. Rajaraman and L. Susskind, hep-th/9703102.

13. M. Dine and N. Seiberg, hep-th/9705057.

14. K. Becker and M. Becker, hep-th/9705091.