Abstract

We give an elementary explicit construction of cell decomposition of the moduli space of projective structures on a two dimensional surface, analogous to the decomposition of Penner/Strebel for moduli space of complex structures. The relations between projective structures and $PGL(2, \mathbb{C})$ flat connections are also described.
The moduli space of projective structures is in a sense a phase space of conformal field theories. After the well known paper of Kontsevich \[3\] the role of the Penner/Strebel construction for cell decomposition of moduli spaces of complex structures \[1, 2\] was realized in understanding relations between the old approach to string theory via conformal field theories and the matrix models of nonperturbative gravity (see also \[4\]). Briefly the Penner/Strebel construction is as follows. Let $\Sigma$ be a two dimensional real surface of genus $g$ with $n$ punctures, $\mathcal{M}_{g,n}$ be the moduli space of complex structures on $\Sigma$. The Penner/Strebel construction gives an isomorphism for $g + n \geq 3$, $n > 0$:

$$\mathcal{M}_{g,n} \times \mathbb{R}^n_+ \rightarrow \mathcal{M}_{g,n}^{\text{comb}},$$

where $\mathcal{M}_{g,n}^{\text{comb}}$ is the space of fat graphs, homotopically equivalent to $\Sigma_0$ with real positive numbers assigned to each edge. (Henceforth we denote by $\Sigma_0$ the surface $\Sigma$ with removed punctures.) This construction provides us with cell decomposition of $\mathcal{M}_{g,n} \times \mathbb{R}^n_+$ with real coordinates for each cell.

Here we shall consider a slightly different moduli space: the moduli space of projective structures. This space is in a sense the phase space for conformal field theories, inasmuch as the moduli space of complex structures is the configuration space for these theories. It turns out, that for our case the construction for cell decomposition gives complex coordinates on the moduli space of projective structures and can be described by means of rather elementary mathematical tools. We also consider the relations between the moduli spaces of projective structures and of flat $PGL(2, \mathbb{C})$-connections, which are known to be very similar. Our approach allows to show that really the moduli space of projective structures is indeed a blown up covering of the moduli space of flat $PGL(2, \mathbb{C})$-connections. As a by-product we get a parameterization of Fuchsian groups.

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1 Generalities on projective structures.

Let us describe for completeness the notion of projective structure on Riemann surface. The definition of projective structure is analogous to that of a complex structure. We have only to replace holomorphic functions to Möbius ones as follows.

A complex structure on a surface is defined if there is a full set of coordinate patches with complex coordinates $z_\alpha$ and holomorphic transition functions between them:

$$z_\alpha = \phi_{\alpha\beta}(z_\beta), \quad \bar{\partial}\phi_{\alpha\beta} = 0$$

A projective structure on a surface is defined if there is a full set of coordinate patches with complex coordinates $z_\alpha$ and Möbius transition functions between them:

$$z_\alpha = \frac{a_{\alpha\beta}z_\beta + b_{\alpha\beta}}{c_{\alpha\beta}z_\beta + d_{\alpha\beta}}$$
A function on a surface equipped with a projective structure (a projective surface) is called projective if it can be represented as a Möbius function of projective coordinates. A mapping between two projective surfaces is called projective if it sends projective functions into projective ones, or equivalently if this mapping is given by Möbius functions in projective coordinates.

Projective structure on a surface defines a complex structure on it. Conversely, for each complex structure there exists at least one projective structure which defines it. Indeed, let \( D \) be a unit disk in the complex plane and \( \Sigma_0 \) be the surface \( \Sigma \) with removed punctures; and let \( p : D \to \Sigma_0 \), be the canonical projection of the universal covering of \( \Sigma_0 \), which by the Poincaré uniformization theorem can be identified with the unit disk \( D \). Let us take a set of pull-downs of standard projective coordinates on \( D \) as a set of coordinates on \( \Sigma_0 \). The transition functions between such coordinates are given by Möbius functions thus defining a projective structure on \( \Sigma_0 \). We shall call it the Poincaré projective structure. In other words the Poincaré projective structure is the unique one such that the mapping \( p \) is projective w.r.t. it. The Poincaré projective structure is unambiguously defined by the complex structure of the surface.

Another example of projective structure can be given in an analogous way by Schottky uniformization mapping. We call it the Schottky projective structure. This projective structure is unambiguously defined by the complex structure and the Schottky uniformization data: choice of maximal set of nonintersecting loops on \( \Sigma \).

The third example of the projective structure on a surface can be given by representing the surface as a ramified covering of the Riemann sphere. This projective structure (which we shall call covering projective structure) is well defined outside the ramification points.

The notion of projective structure can be also defined in local terms:

**Definition 1** Projective connection \( T \) on a surface \( \Sigma_0 \) is a holomorphic section of a one dimensional complex bundle on \( \Sigma_0 \) defined by the transition functions:

\[
T_\alpha(z_\alpha) = T_\beta(z_\beta) \left( \frac{dz_\beta}{dz_\alpha} \right)^2 + \frac{1}{2} S(z_\beta, z_\alpha),
\]

where \( S(z_\alpha, z_\beta) \) is the Shwarzian derivative

\[
S(z_\alpha, z_\beta) = \frac{\left( \frac{d^3 z_\alpha}{dz_\beta^3} \right)}{\left( \frac{dz_\alpha}{dz_\beta} \right)} - \frac{3}{2} \left( \frac{d^2 z_\alpha}{dz_\beta^2} \right)^2 / \left( \frac{dz_\alpha}{dz_\beta} \right)^2
\]

**Proposition 1** The set of projective structures on a surface compatible with a given complex structure is in a bijective correspondence with projective connections on the surface.

**Proof.** Let \( T \) be a holomorphic projective connection on the surface \( \Sigma_0 \). Consider a ratio of two linearly independent holomorphic solutions of the differential equation

\[
\partial^2 f + Tf = 0.
\]
One can easily check that such ratios can be taken as a set of projective coordinates i.e. that one such ratio is a Möbius function of another one. Conversely, let $u_\alpha$ be a full set of projective coordinates on a surface. Then the expression

$$T_\alpha(z) = S(u_\alpha, z)$$  (7)

correctly defines a holomorphic projective connection on the surface. □

**Corollary.** The moduli space of projective structures is an affine bundle over the moduli space of complex structures $\mathcal{M}_{g,n}$. The fiber of this bundle over a given complex structure is an affine space over the space of holomorphic quadratic differentials.

Indeed, the difference of two projective connections is a quadratic differential and a sum of a projective connection and a quadratic differential is a projective connection. The existence of at least one projective connection for each complex structure proves the corollary.

Note, that the Poincaré projective connection gives us a section of this bundle over $\mathcal{M}_{g,n}$ and thus we are able to consider this bundle as a vector bundle. However this section is not holomorphic. The Shottky projective structure gives us a holomorphic, but multivalued section.

The space of projective structures compatible with a given complex structure is an infinite dimensional space provided that the surface $\Sigma$ has at least one puncture, since a projective connection $T$ can have arbitrary singularities at the punctures. Call a projective structure regular at a puncture $p$ if the corresponding projective connection has at $p$ a pole of order two or less:

$$T(z) = \frac{a}{z^2} + \frac{b}{z} + \text{reg. terms},$$  (8)

where $z$ is a coordinate at a neighborhood of $p$ ($z(p) = 0$).

Note that a projective structure regular at $p$ corresponds to a regular at $p$ differential equation (8) regular at $p$ in the sense of Fuchs theory [5].

## 2 Fat graphs and projective surfaces.

Let us denote by $\mathcal{MP}_{g,n}$ the space of regular projective structures on a surface of genus $g$ with $n$ punctures, (we call a projective structure regular if it is regular at each puncture of the surface); and let $\mathcal{MP}_{g,n}^{\text{comb}}$ be the space of threevalent fat graphs with positive imaginary part complex numbers assigned to its edges.

**Proposition 2** An open dense subset of $\mathcal{MP}_{g,n}$ is isomorphic to $\mathcal{MP}_{g,n}^{\text{comb}}$.

(A rigoristically inclined reader can eliminate the word dense from the formulation, and consider the present formulation as a conjecture.)

In order to prove the proposition we shall explicitly construct the mappings $\mathcal{MP}_{g,n} \to \mathcal{MP}_{g,n}^{\text{comb}}$ and $\mathcal{MP}_{g,n}^{\text{comb}} \to \mathcal{MP}_{g,n}$ and then show that the former mapping is inverse to the latter.
2.1 Fat graphs from projective structures

Let us first describe the mapping \( \mathcal{MP}_{g,n} \rightarrow \mathcal{MP}_{g,n}^{\text{comb}} \). Let a projective disk on \( \Sigma \) be a mapping \( u : D \rightarrow \Sigma_0 \) of the open standard unit disk equipped with the standard projective structure into the surface \( \Sigma_0 \) considered up to the action of the group \( PGL(2, R) \) of authomorphisms of \( D \). Let \( \mathcal{D}_\Sigma \) be the set of all projective disks on \( \Sigma \). Define a partial order on \( \mathcal{D}_\Sigma \) by taking \( u_1 \geq u_2 \) \((u_1, u_2 \in \mathcal{D}_\Sigma)\) if there exists a commutative diagram of projective mappings:

Now consider the set \( \mathcal{D}_\Sigma^{\text{max}} \subset \mathcal{D}_\Sigma \) of maximal disks w.r.t. this ordering. We shall say that a disk \( u \in \mathcal{D}_\Sigma \) leans on a puncture \( p \) if \( p \) belongs to the closure of image of \( u \) in \( \Sigma \). In this situation the puncture \( p \) define a discrete set of points \( \overline{\pi}^{-1}(p) \) on the unit circle \( \partial D \). (Here \( \overline{u} \) is the extension of \( u \) on the closed unit disk.) Note that it is not necessary that this set consists of one element. We shall say that a disc \( u \) leans on \( p \) with multiplicity \( \# \overline{\pi}^{-1}(p) \). In these terms an evident proposition describes the set \( \mathcal{D}_\Sigma^{\text{max}} \) of maximal disks:

**Proposition 3** The set \( \mathcal{D}_\Sigma^{\text{max}} \) is topologically isomorphic to a graph with canonical fat graph structure. The vertices of the graph correspond to projective disks leaning on the punctures at least three times.

One can easily see that the set of disks leaning on two given punctures (or leaning on one given puncture twice) is a one dimensional manifold (fig 1).
The set of disks close to a disk leaning on \( n \) punctures is isomorphic to a neighborhood of an \( n \)-valent vertex of a graph (fig 2).

![Fig. 2](image)

(11)

The cyclic order of the set \( \cup_p \pi^{-1}(p) \) on the unit circle induces the cyclic order of ends of edges incident to the corresponding vertex of this graph. □

Now let us provide \( D^\text{max}_\Sigma \) with an additional structure – complex numbers on edges – which, as it will be demonstrated in the next section, contains all the information about the isomorphism class of projective structure on \( \Sigma \).

Let \( u_1 \) and \( u_2 \) be two projective disks which correspond to the beginning and to the end of an oriented edge \( \alpha \) respectively. Let \( z_\alpha \) be a (multivalued) projective function on \( \Sigma \) such that (i) it is equal to \(-1, 0, \infty\) at the points the disk \( u_1 \) leans on, (ii) its imaginary part is negative within the disk, (iii) The disk \( u_2 \) leans on points \( z_\alpha = 0 \) and \( z_\alpha = \infty \). These conditions define the function \( z_\alpha \) unambiguously. (In order to avoid multivaluedness one can consider here the disks, projective functions, e.t.c. on the universal covering \( \tilde{\Sigma}_0 \).) Let now \( x_\alpha = \ln z_\alpha \) be a branch of logarithm taking positive real values. Let \( Z_\alpha \) be the value of \( x_\alpha \) at the point the disk \( u_2 \) leans on (other than the points \( z_\alpha = 0 \) and \( z_\alpha = \infty \))(cf. fig. 3 A,B).

![Fig. 3A](image)

(12)
This point is evidently outside the disk $u_1$ and therefore $\text{Im} Z_\alpha > 0$. Assign this complex number $Z_\alpha$ with positive imaginary part to the edge $\alpha$. The complex number $Z_\alpha$ unambiguously determines a configuration of two disks leaning on four punctures. One can straightforwardly check that the definition of the number $Z_\alpha$ is correct:

**Proposition 4** The value of $Z_\alpha$ is independent on the orientation of the edge $\alpha$. The coordinate $x_\alpha$ changes to

$$x_\alpha \mapsto Z_\alpha - i\pi - x_\alpha$$

under change of orientation of $\alpha$.

### 2.2 A surface with projective structure from a fat graph

Consider now the an inverse procedure i.e. how to restore a surface starting from a graph with complex numbers assigned to the edges. It turns out that all the procedure from the above section can be performed in the reversed direction. Moreover it turns out that one is able to build a surface out of a graph with weaker condition on the numbers on edges, than that of positivity of the imaginary part.

**Proposition 5** For any fat graph with positive imaginary part complex numbers assigned to the edges there exists a surface such that the above described construction reproduces this graph.

**Proof** Assign a strip $-i\pi < \text{Im} x_\alpha < \text{Im} Z_\alpha$ to each oriented edge of the graph and define transition functions between the strips:

$$x_\alpha = Z_\alpha + \ln(e^{x_\beta} + 1)$$

$$x_{\alpha^\vee} = Z_\alpha - i\pi - x_\alpha$$

where $\alpha^\vee$ is an edge obtained from $\alpha$ by orientation changing. Here the orientations of edges are chosen in such a way that the end $v$ of $\alpha$ coincides with the beginning of $\beta$. 

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and the edge $\alpha$ is next after the edge $\beta$ in counterclockwise direction w.r.t. $v$. (fig 4).

The described set of strips and transition functions between them defines the required surface provided that it is smooth. (For arbitrary set of patches and transition functions between them the corresponding space may be neither smooth manifold nor even Hausdorff topological space.) Remind, that formally the surface is defined by the set of strips and transition functions as a set of equivalence classes of the points of disjoint union of strips; two points $a$ and $b$ are taken to be equivalent if there exist a chain of points $a = z_0, \ldots, z_n = b$ and of transition functions $f_1, \ldots, f_n$ such that $z_{i+1} = f_i(z_i)$. The surface defined by such rules is evidently smooth if (but not only if) each equivalence class consists of finite number of points. We now prove, that this condition holds in our case.

**Lemma.** Let $\alpha, \beta$ be two subsequent edges of the graph (subsequent means that the beginning of $\beta$ coincides with the end of $\alpha$). Then

$$\Im x_\alpha > \Im Z_\alpha + \Im x_\beta$$

**Proof of the lemma:** For $\alpha$ next after $\beta$ in counterclockwise direction it follows directly from (15) and from the fact that the transition function is defined only for $-i\pi < x_\alpha < 0$. For $\beta$ next to $\alpha$ combining (15) and (14) we get the analog of (15) for this case:

$$x_\alpha = Z_\alpha - \ln(e^{-x_\beta} + 1)$$

(19)

One can easily check, that (18) holds for this case also. □

Now consider a sequence of equivalent points. Without loss of generality it is sufficient to consider only such sequences for which $f_i \neq f_{i+1}^{-1}$ for all $i$ and let $\alpha_1, \ldots, \alpha_n$ be the corresponding sequence of edges forming a path on the graph and oriented along this path. Applying (18) to the pairs of subsequent edges of this path we get:

$$\Im z_0 \geq \sum_{i=1}^{n} \Im Z_{\alpha_i} + \Im z_n$$

(20)

which can be valid only for a finite sequence of edges. □

Note, that the proof of this theorem shows that a surface can be defined starting from a graph with weaker condition on the numbers assigned to edges:

$$\sum_{\alpha \in \gamma} \Im z_\alpha \geq 0$$

(21)
Where \( \gamma \) is any closed path on the graph without returns. (Without returns means that it does not go along an edge forth and immediately back).

Two different graphs with numbers satisfying (21) can correspond to the same projective structure (which never holds for graphs with positive imaginary part numbers). In particular holds the following

**Proposition 6** The graphs with numbers on the edges connected by the operation of flipping an edge (fig. 5) correspond to the same surface.

\[
\begin{array}{c}
\text{Z} \\
\text{Z}
\end{array}
\]  

![Fig. 5](image.png)

The condition (21) allows us to extend the functions \( Z_\alpha \) from the domain of \( MP_\Sigma \) described by a given graph and the proposition (6) gives us the transition functions between the coordinates on \( MP_\Sigma \) which correspond to different graphs.

### 3 Projective structures and flat \( PGL(2, \mathbb{C}) \) connections.

Projective structures on a Riemann surface \( \Sigma \) are closely related to flat connections on a \( PGL(2, \mathbb{C}) \)-bundle on \( \Sigma \) (cf. \[7\], where these relations was discussed in terms of smooth \( SL(2, \mathbb{C}) \) flat connections). Here we define a mapping

\[
MP_\Sigma \rightarrow A_\Sigma(PGL(2, \mathbb{C}))
\]  

(23)

where \( A_\Sigma(PGL(2, \mathbb{C})) \) is the moduli space of flat structures of a \( PGL(2, \mathbb{C}) \)-bundle or in other words the space of flat \( sl(2, \mathbb{C}) \)-connections on \( \Sigma \) modulo gauge transformations. Then we describe this mapping in terms of the coordinates \( \{Z_\alpha\} \) on \( MP_\Sigma \) and graph connections and also give in these terms a construction for the inverse (multivalued) mapping.

#### 3.1 Flat \( PGL(2, \mathbb{C}) \) connections from projective structures.

Let \( \Sigma \) be a surface equipped with a projective structure and \( \{z_\alpha\} \) be a full set of projective coordinates on it. For each two of these coordinates \( z_\alpha \) and \( z_\beta \) with non-trivial intersection of their definition domains one can define an element of the group...
$PGL(2, \mathbb{C}) = GL(2, \mathbb{C})/\mathbb{C}^* = SL(2, \mathbb{C})/\{\pm 1\} = (\text{group of Möbius transformations.})$ represented by a matrix (24)

$$g_{\alpha, \beta} = \begin{pmatrix} a_{\alpha, \beta} & b_{\alpha, \beta} \\ c_{\alpha, \beta} & d_{\alpha, \beta} \end{pmatrix}$$

(Strictly speaking this matrix is defined unambiguously only for topologically trivial system of coordinate patches $\{z_\alpha\}$. In general the matrix is defined if we have fixed also the connected component of the intersection of definition domains of $z_\alpha$ and $z_\beta$.) It is evident that this system of matrices satisfies the cocycle condition i.e. that for any three coordinates $z_\alpha, z_\beta, z_\gamma$ with intersecting definition domains one has

$$g_{\alpha, \beta} g_{\beta, \gamma} g_{\gamma, \alpha} = 1 \in PGL(2, \mathbb{C})$$

and thus this system of elements of $PGL(2, \mathbb{C})$ can be taken as a set of transition functions of a $PGL(2, \mathbb{C})$-bundle with canonical locally flat connection. (Flat sections w.r.t. this connection are those given by constant functions in the trivialization fixed by the transition functions.)

The isomorphism class of a flat connection (the gauge equivalence class of a flat connection) is fixed if one know the monodromy operators along a sufficient number of paths. For example one can describe such class by fixing monodromy operators along edges of a graph drawn on the surface and homotopically equivalent to it with, may be, some additional holes. The assignment of group elements to oriented edges of a graph (in such a way that if one changes the orientation of an edge the corresponding group element changes to its inverse) is called graph connection. Now we describe a $PGL(2, \mathbb{C})$ graph connection which corresponds to a given projective structure i.e. describe a procedure which makes a graph connection on a graph starting from a projective structure, defined by some (may be another) graph $\Gamma$ with complex numbers on edges. For our purpose it is convenient to define the required graph connection on a graph obtained from $\Gamma$ by blowing up vertices: The blown up graph $\tilde{\Gamma}$ is the graph $\Gamma$ with $k$-valent vertices replaced by $k$-vertex polygons (fig. 6A).

Orient the new edges in counterclockwise direction w.r.t the interior of the polygons
and assign the matrices $g_\alpha = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \in PGL(2, \mathbb{C})$ to them and $\begin{pmatrix} e^{Z_\alpha} & -1 \\ e^{Z_\alpha} + 1 & 1 \end{pmatrix} \in PGL(2, \mathbb{C})$ to the old edges \{\alpha\} (their orientations are inessential because $g_\alpha = g_\alpha^{-1}$).

**Proposition 7** The above described graph connection (fig. 6A) on $\tilde{\Gamma}$ corresponds to the projective structure defined by the graph $\Gamma$ and the numbers \{$Z_\alpha$\}.

For practical purposes it is often more convenient to describe flat connections in terms of another graph $\tilde{\tilde{\Gamma}}$ — the graph $\Gamma$ with blown up edges and vertices. This graph can be obtained from the graph $\Gamma$ by replacing its edges by rectangulars (fig. 6B).

Let us call the *long* edges the edges parallel to the edges of the original graph $\Gamma$ and *short* edges the other ones. Orient the edges in counterclockwise direction w.r.t. the interior of the rectangulars and assign the matrices $\begin{pmatrix} e^{Z_\alpha} \\ e^{Z_\alpha} \\ 0 \\ 1 \end{pmatrix}$ to the long edges and $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ to the short ones. Then the obtained graph connection on $\tilde{\tilde{\Gamma}}$ defines an element of $\mathcal{A}_\Sigma(\text{PGL}(2, \mathbb{C}))$ which corresponds to the graph $\Gamma$ and the set of numbers \{$Z_\alpha$\} on edges.

### 3.2 Projective structures from flat $\text{PGL}(2, \mathbb{C})$ connections.

Now consider the problem, what is the preimage of a flat connection. In order to solve it we give an explicit construction of all graphs with numbers corresponding to a given graph and to a given $\text{PGL}(2, \mathbb{C})$-graph connection on it.

Let $\Gamma$ be a threevalent graph homotopically equivalent to $\Sigma$ and $g_\alpha$ be a graph connection (i.e. group elements assigned to the edges). Consider a monodromy operator $g_{v,f}$ around a face $f$ starting and ending at a vertex $v$ (i.e. a path ordered product of group elements assigned to the sites of the face $f$). Choose an eigenspace $l_{v,1}$ for each $g_{v,f}$ in such a way, that the monodromy operator around $f$ sends $l_{v_1,f}$ to $l_{v_2,f}$ for any corners $v_1$ and $v_2$ of $f$ (i.e. one has two possible choices for each face). Thus for each vertex we have three eigenspaces — one for each face this vertex is a corner of. Now consider an
edge α of the graph. The monodromy operator along the edge sends two of the three eigenspaces assigned to one end of the edge onto two eigenspaces assigned to another one. The image of the third eigenspace together with the eigenspaces assigned to this end form a quadruple of one dimensional subspaces in \( \mathbb{C}^2 \) (or they can be thought of as a quadruple of points in \( \mathbb{C}P^1 \) which we denote as \( P_{-1}, P_0, P_\infty \) and \( P \)). Such quadruple is known to have one \( GL(2, \mathbb{C}) \)-invariant — the double ratio:

\[
z = -\frac{(P_0 - P) (P_\infty - P_{-1})}{(P_\infty - P) (P_0 - P_{-1})}
\]

Let \( Z_\alpha \) be any positive imaginary part value of \( \ln z \). Thus we have assigned a positive imaginary part complex number to all edges of the graph.

**Proposition 8** The \( PGL(2, \mathbb{C}) \)-flat structure which corresponds to the projective structure determined by the constructed graph with complex numbers coincides with that we have started from and all sets of numbers assigned to edges can be obtained in this way.

The proof can be given by a direct verification.

This construction shows that the inverse image of a \( PGL(2, \mathbb{C}) \)-flat structure is a set the elements of which are numerated by different ways to choosing the graph, the eigenspaces an the branches of logarithm. At a generic point where the monodromy operators around all faces are diagonalizable and not equal to unity this set is discrete and thus the mapping (23) is a covering (with infinite number of sheets). For the points, where the monodromy operator around at least one face is not diagonalizable this covering is not locally trivial. For the points where there exist faces the monodromy operators around which equal to unity the inverse image is not discrete.

Therefore we have described the mappings of the diagram

\[
\begin{array}{c}
\mathcal{M}P_\Sigma \\
\downarrow 2.1 \\
\mathcal{M}P_{\Sigma \text{comb}} \\
\downarrow 3.2 \\
\mathcal{A}_\Sigma(PGL(2, \mathbb{C})) \\
\downarrow 3.1
\end{array}
\]

and proved its commutativity. (The numbers indicate section where the corresponding mapping was considered.)

**4 Examples and unsolved problems.**

**4.1 Poincaré projective structure.**

The construction from sect. 2.2 which makes a graph starting from the projective structure is not applicable for this case because the maximal disks w.r.t. the Poincaré projective structure pulled back on the universal covering of a curve are just the mappings onto
the universal covering and thus all maximal disks lean on all punctures infinitely many times. Nevertheless the inverse construction may give Poincaré projective structure:

**Proposition 9** Poincaré projective structure corresponds to graphs with real numbers assigned to edges (via construction of sect 2.3).

*Proof.* One can easily see that the coordinate patches for the case of real numbers assigned to edges are unit disks (in terms of coordinates $z_\alpha$) and the transition functions maps one disc onto another. In particular it means that the corresponding $PGL(2, \mathbb{C})$-bundle reduces to $PGL(2, \mathbb{R})$ one and that the procedure of gluing strips reduces to factorization of a single disk thus giving just the Poincaré uniformization mapping. Since the surface which can be constructed starting from a graph with positive imaginary parts numbers assigned to edges is smooth the same is true for graphs with real numbers on edges. ☐

Note, that the correspondence between surfaces with Poincaré projective structure and fat graphs with real numbers on edges is not one-to-one. In particular the graphs connected with each other as shown on fig. 5 correspond to the same surface. I seem to be probable, that all graphs corresponding the same surface can be obtained one from another by such operation.

Nevertheless this construction gives us at least local parameterization of the Poincaré projective structures and thus of the moduli space of complex structures. In these terms it is easy to find a Fuchsian group, corresponding to a given complex structure.

**Proposition 10** The complex surface corresponding to a given graph with real numbers on the edges is isomorphic to the quotient of a unit disc by the monodromy group of the flat connection shown on fig. 6 A,B.

This proposition follows immediately from prop. [3]. Note, that *a priori* it was not evident, that the Fuchsian group, given by this construction always corresponds to a smooth surface.

### 4.2 Covering projective structure.

Covering projective structures are in a sense the most simple ones, for which it is possible to construct the corresponding graphs in a very simple explicit way. The inverse operation – to restore the ramification points and the scheme of the covering starting from a graph – can also be done explicitly, provided the graph indeed corresponds to a covering projective structure. Here we give a simple proposition which allows to characterize such graphs.

**Proposition 11** Covering projective structure is characterized by the requirement that the corresponding $PGL(2, \mathbb{C})$-connection is trivial.
One can easily write down this condition in terms of the variables $Z_{\alpha}$ using the construction of sect. 3.1. In particular a projective structure at a neighborhood of a puncture is isomorphic to that at a neighborhood of a $k$-th order ramification point iff

$$\begin{align*}
Z_{\alpha_1} + \ldots + Z_{\alpha_n} &= 2\pi ik \\
e^{Z_{\alpha_1}} + e^{Z_{\alpha_2}} + \ldots + e^{Z_{\alpha_n}} &= 0
\end{align*}$$

(30)

where $\{\alpha_i\}$ is the sequence of sites of the face, corresponding to the given puncture. In particular $k = 1$ means that the projective structure at the puncture is nonsingular. For graphs which correspond to punctured spheres the equations (30) satisfied for all faces of the graph is enough for the corresponding projective structure to be a covering one. For surfaces with handles we have to impose additional conditions, which can be written down explicitly for each concrete graph.

### 4.3 Relations to Strebel construction

In this section we discuss the relation between our construction and that of Penner/Strebel which describes a one-to-one correspondence between the space of complex structures on a surface $\Sigma$ and the space of graphs with positive real numbers assigned to edges.

Let $\Gamma$ be a graph with positive real numbers $l_\alpha$ assigned to edges and let $\tilde{\Gamma}$ be the corresponding graph with blown up vertices. Assign the purely imaginary complex numbers $i(l_\alpha - \pi)$ to the edges which correspond to the edges $\alpha$ of the old graph and $i\pi$ to all other edges of $\tilde{\Gamma}$ (fig. 10).

A surface with projective structure which corresponds to this graph with numbers by the construction of sect. 2.1 (it is applicable here because the condition (21) is satisfied) has two kind of punctures: the punctures of the first kind correspond to the faces of $\Gamma$ and that of the second one — to the vertices of $\Gamma$. Extend the complex structure to the punctures of the second kind.

**Proposition 12** The complex surface obtained by such construction starting from a graph with positive real numbers on edges gives the complex surface isomorphic to that given by Strebel/Penner construction.

The proof of the proposition can be done by direct comparison of our construction of a surface (sect 2.2) and the Strebel one [2].
4.4 Projective structures on the torus with one puncture.

Consider as an example the moduli space of projective structures on a torus with one puncture and with nonsingular behavior of the projective structure at this puncture. Such projective structure can be characterized by two parameters: the standard modular parameter $\tau$ of complex structure and a parameter $k$ for projective connection $T = k^2dz^2$, where $z$ is the standard coordinate on the torus $\mathbb{C}/\mathbb{Z}$ ($z \equiv z + m + n\tau$). The projective coordinates are given therefore by ratios of solutions of the equation (6), i.e. a general projective coordinate $u$ has the form

$$u = \frac{ae^{kz} + be^{-kz}}{ce^{kz} + de^{-kz}}, \quad \text{for } k \neq 0.$$  \hspace{1cm} (32)

$$u = \frac{az + b}{cz + d}, \quad \text{for } k = 0.$$  \hspace{1cm} (33)

For $k = 0$ the corresponding maximal disks are shown on fig. 9.

There are two inequivalent disks corresponding to two vertices of the graph and three ways to deform one into another, corresponding to the three edges of the graph. For $k \neq 0$ the disks are deformed (on the $z$-plane), but topologically the picture remains unchanged. To calculate the numbers on edges one has to take logarithms of double ratios of values of any projective coordinates at quadruples of punctures:

$$Z_1 = \ln \left( \frac{u(\tau) - u(1 + \tau)}{u(1) - u(1 + \tau)} \right) \left( \frac{u(1) - u(0)}{u(\tau) - u(0)} \right) = 2 \ln \left( -\frac{\text{sh}k\tau}{\text{sh}k} \right)$$  \hspace{1cm} (35)

$$Z_2 = \ln \left( \frac{-u(1) - u(1 - \tau)}{u(0) - u(1 - \tau)} \right) \left( \frac{u(0) - u(\tau)}{u(1) - u(\tau)} \right) = 2 \ln \left( -\frac{\text{sh}k\tau}{\text{sh}k(\tau - 1)} \right)$$  \hspace{1cm} (36)

$$Z_3 = \ln \left( \frac{-u(0) - u(\tau - 1)}{u(\tau) - u(\tau - 1)} \right) \left( \frac{u(\tau) - u(1)}{u(0) - u(1)} \right) = 2 \ln \left( -\frac{\text{sh}k(\tau - 1)}{\text{sh}k} \right)$$  \hspace{1cm} (37)
4.5 Projective structure on sphere with four nonsingular punctures.

The construction of a graph for a standard projective structure on a sphere with four punctures is illustrated in fig. 8.

![Fig. 8](image)

The procedure for this case as well as for all other coverings of the Riemann sphere reduces to calculating some double ratios of coordinates of the ramification points and nonsingular punctures.

4.6 Unsolved problems.

To conclude we discuss some yet unclear aspects of the above construction. First of all, we do not have a rigorous proof of the fact that our construction really describes the dense subset of $\mathcal{MP}_\Sigma$, though this conjecture seems to be very realistic. Another yet unclear related point is how to describe the transition functions between the maps of $\mathcal{MP}_\Sigma$, corresponding to different graphs. The proposition 6 shows how to do this in some cases, but for example, the question, what is glued to the component of the boundary of a cell corresponding to a graph with closed edge and real numbers on this edge remains unclear. The example 4.4 shows that when the projective structure tends to the Shottky one ($k \rightarrow i\pi$ in our example) some numbers $Z_\alpha$ tend to infinity. The question is, how to generalize the space $\mathcal{MP}_\Sigma^{comb}$ to be able to describe all projective structures.

Another group of problems are connected to the perspective of possible quantization of the space $\mathcal{MP}_\Sigma$. The space $\mathcal{MP}_\Sigma$ is a Poisson manifold and the first question is, how to express the Poisson structure in terms of the coordinates $Z_\alpha$. Another problem is how to describe the set of algebraic functions on $\mathcal{MP}_\Sigma$. The knowledge of such description is necessary for applying some algebraic quantization technique, like quantum groups e.t.c., analogously to what can be done for the space $\mathcal{A}_\Sigma$ (see [3]).

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