On two dimensional coupled bosons and fermions

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Abstract

We study complex bosons and fermions coupled through a generalized Yukawa type coupling in the large-$N_c$ limit following ideas of Rajeev [Int. Jour. Mod. Phys. A 9 (1994) 5583]. We study a linear approximation to this model. We show that in this approximation we do not have boson-antiboson and fermion-antifermion bound states occurring together. There is a possibility of having only fermion-antifermion bound states. We support this claim by finding distributional solutions with energies lower than the two mass threshold in the fermion sector. This has important implications from the point of view of scattering theory. We discuss some aspects of the scattering above the two mass threshold of boson pairs and fermion pairs. We also briefly present a gauged version of the same model and write down the linearized equations of motion.

1 Introduction

Quantum field theories are both fundamental and challenging. Despite the fact that our description of the world of elementary particles is based on quantum field theory, we still do not have a complete understanding of interacting field theories, especially their bound state structure. For this reason it is interesting to study simple examples where one can actually make more progress. There are various such models in two dimensions and they have been a valuable source for new ideas and testing ground for many years (see [1] for a comprehensive selection of topics).
In this article we study a two dimensional model which could be another possible toy model for understanding Yukawa coupled field theories in four dimensions. The physically important one is the gauged Yukawa theory, as we know from the present day version of the standart model. In this work we will also study some aspects of the gauge coupled interacting bosons and fermions. We will actually present the linearized equations of the gauged version at the end of our paper, but our main emphasis is to understand the simpler case without the gauge potentials (of course it is not so clear if this is really a simpler theory). A more complicated version of the model we discuss is investigated in [2] using path integral techniques. The results of the present paper are somewhat different since we follow a Hamiltonian approach (and we are not taking the most complicated possible version).

Let us comment on some fundamental work in the literature that we are aware of: the literature on Yukawa theory is vast, we will mention only a few of them, a rigorous construction of two dimensional model is given in the papers [3, 4], the construction of the probability measure within the Euclidean formalism is done in [1]. The most recent rigorous analysis was given in [1] by following a renormalization group type idea essentially inspired from [7]. It will be interesting to attempt such a rigorous approach for the model we discuss below. The standart Yukawa coupling in two dimensions in the light-cone approach is discussed in [8, 9]. A further analysis using the Tamm-Dancoff approximation in the light-cone is pursued in [10]. A very interesting discussion of the Yukawa model in four dimensions is presented in [11], clearly a four dimensional model has many more interesting features. A more recent analysis of the fermion bound states of the same model is discussed in [12]. The equivalence between the light-cone and covariant perturbation theory is analysed in [13]. (Good reviews of field theories in the light cone are given in [14] and renormalization of light cone Hamiltonians in [15]).

In this work, we introduce a color degree of freedom for the purpose of reaching a Hartree-Fock type approximation. Following the ideas suggested by Rajeev in [16] we reformulate the problem in terms of color invariant bilinears (see also [17] for some similar ideas). The details of this refomulation are explained in our previous work [18] within the context of gauge theories, therefore in this work we will use the results of the cited article directly. In some sense the present article is a natural continuation. The reader should consult [18] for more information on the geometry of the resulting classical phase space. We also recommend the lectures notes of Rajeev on two dimensional QCD [19]. In [16], it is shown that the 1+1 dimensional QCD in the large-$N_c$ limit can be reformulated as a classical field theory with an infinite dimensional phase space, which is identified to be the restricted Grassmanian. The study of two dimensional QCD in the large-$N_c$ limit is given by ’t Hooft in his well-known paper [20], its generalization to scalar fields is done in [21] and in [22] using Hamiltonian methods. The two dimensional combined fermions and bosons QCD is given by Aoki in [23, 24] and also discussed by Cavicchi [2]. Within Rajeev’s approach one can reach the same results by using a linear approximation to the full theory. One can further study baryons by using a variational ansatz which does not correspond to small fluctuations of the fields, therefore cannot be seen by the linear approximation. Here we obtain the general Hamiltonian in the large-$N_c$ limit for gauge coupled bosons and fermions which are also interacting through a generalized Yukawa type interaction. The meson equations are given
for the linearized theory. Our presentation is incomplete since we do not study beyond the linear approximation and we plan to come to a more detailed analysis in the future.

We note that in the simpler model we discuss one can actually solve the integral equations, ending up with some eigenvalue or scattering solutions. These equations require a simple renormalization to be meaningful (which at the end amounts to defining the singular integrals as the Hadamard principal value). We warn the reader that the form of the resulting Hamiltonian suggests that a physically more relevant approximation in the non-gauged models could be given by a variational ansatz. This is due to the fact that the interactions in the linear approximation are all multiplied by the fermion mass. For heavy fermions we expect that the linear approximation gives valuable results, but for example in the case of massless fermions all the information is contained in the higher order terms, which cannot be accessed by the method we use.

2 Coupling between Complex Bosons and Fermions

We start with the action of our model with two Yukawa type couplings,

$$S = \int d^2x \left( i\bar{\psi}^\alpha \gamma^\mu \partial_\mu \psi^\alpha - \bar{\psi}^\alpha (\mu_1 Y \phi^{*\beta} \phi^\alpha + \mu_2 Y \phi^{*\lambda} \phi^\delta \delta^\beta + m_F \phi^{*\alpha} \phi^\alpha) + \partial^\mu \phi^{*\alpha} \partial_\mu \phi^\alpha - m^2_{B0} \phi^{*\alpha} \phi^\alpha - \frac{\lambda^2_{B0}}{4} (\phi^{*\alpha} \phi^\alpha)^2 \right).$$

Here $\alpha$ refers to a common flavor index and it goes from 1 to $N_c$ (we continue to write it as color symmetry, since at the end we will also talk about the gauged model). It is more natural to keep $\mu_1 = \mu_2$ when there are no gauge fields, since we will use the color degrees as a way of reaching a mean-field description, but we will keep this more general form. We rewrite the action in the light-cone coordinate system, and we choose $x^+$ as our “time” coordinate,

$$S = \int dx^+ dx^- \left( i\sqrt{2} \bar{\psi}^\alpha_L \partial_- \psi^\alpha_L + i\sqrt{2} \bar{\psi}^\alpha_R \partial_+ \psi^\alpha_R + \phi^{*\alpha}(-2\partial_-)\partial_+ \phi^\alpha - m^2_{B0} \phi^{*\alpha} \phi^\alpha \right) - (\bar{\psi}^\alpha_L \psi^\alpha_R + \bar{\psi}^\alpha_R \psi^\alpha_L) (\mu_1 Y \phi^{*\beta} \phi^\alpha + \mu_2 Y \phi^{*\lambda} \phi^\delta \delta^\beta + m_F \delta^\beta) - \frac{\lambda^2_{B0}}{4} (\phi^{*\alpha} \phi^\alpha)^2 \right).$$

There are many good introductions to the light-cone field theory, we refer the reader to [25, 26, 27]. We note that the lefthanded components of the fermion field are nondynamical, therefore we will remove $\psi^\alpha_L$ and its complex conjugate through the equations of motion in the quantum theory (we refer to [18] for our conventions in quantizing this theory, below we summarize the results),

$$\hat{\psi}^\alpha_L = \frac{\sqrt{2}}{2i\partial_-} [\mu_1 Y : \hat{\phi}^\alpha \bar{\hat{\psi}}^{\dagger\beta} \hat{\psi}^-_{R\beta} : + \mu_2 Y : \hat{\phi}^{\dagger\beta} \hat{\phi}^\beta \hat{\psi}^\alpha_R : + m_F \hat{\psi}^\alpha_R].$$

We do not really need to worry about the normal ordering in the first factor, since in the large-$N_c$ limit these corrections will be of smaller order, we wrote it to emphasize that the reduction should be performed at the quantum level. At the second term we have a normal ordering for the color contracted bosons only. The second thing we notice from the light-cone action is that we are already in the Hamiltonian formalism. Therefore we can read off the
Hamiltonian directly from the action when we insert the solution of the nondynamical field back into the action. Thus we arrive at the following Hamiltonian, (from now on we write \( \psi \) for \( \psi_R \) since this is the only fermionic field we have),

\[
\hat{H} = \int dx^- \left( \frac{\sqrt{2} m_F^2}{2} : \psi^{\dagger \alpha} \frac{1}{i\partial_-} \psi_\alpha + m_{B_0}^2 : \phi^{\dagger \alpha} \phi_\alpha : + \frac{\lambda_{B_0}^2}{4} (:\phi^{\dagger \alpha} \phi_\alpha :)^2 \right) \\
+ \frac{\sqrt{2}}{2} \frac{1}{\mu_Y m_F} [\psi^{\dagger \alpha} \frac{1}{i\partial_-} \phi^\beta \phi^\dagger \psi^\beta + \psi^\dagger \phi^\beta \phi^\dagger \psi^\beta \frac{1}{i\partial_-} \psi_\alpha ] + \frac{\sqrt{2}}{2} \frac{1}{\mu_Y} \psi^{\dagger \beta} \phi^\sigma \phi^\dagger \psi^\beta \frac{1}{i\partial_-} \psi_\alpha \\
+ \frac{\sqrt{2}}{2} \mu_Y^2 \left[ \psi^{\dagger \alpha} \frac{1}{i\partial_-} \phi^\beta \phi^\dagger \psi^\beta : + \phi^\dagger \phi^\dagger \phi^\beta \frac{1}{i\partial_-} \phi_\lambda \psi^\beta \right] \\
+ \frac{\sqrt{2}}{2} \mu_Y^2 \left[ \psi^{\dagger \alpha} \frac{1}{i\partial_-} \phi^\beta \phi^\dagger \psi^\beta : + \phi^\dagger \phi^\dagger \phi^\beta \frac{1}{i\partial_-} \phi_\lambda \psi^\beta \right] \\
+ \frac{\sqrt{2}}{2} \mu_Y^2 \left[ \phi^\dagger \phi^\dagger \phi^\beta \phi^\dagger \psi^\beta : + \phi^{\dagger \beta} \phi^\dagger \phi_\lambda \psi^\beta \right] \right). \tag{3}
\]

This Hamiltonian as it stands is not normal ordered, to define it properly we need to normal order the color singlet products of bosons in the sixth term and the products of fermions in the last three terms. All these terms except one will give some divergences which can be cancelled by redefinitions of \( m_{B_0} \) and \( \lambda_{B_0}^2 \) in the original Hamiltonian as we will see. One of them cannot be removed by the original terms in the action and we will add a counter term which cannot be put into the original action. This Hamiltonian could be a better two dimensional representative of the four dimensional Yukawa theory, since in four dimensions phi-four coupling is necessary to renormalize the Yukawa interaction.

Let us recall the quantization of this system in the light-cone coordinates, the Fourier mode expansions read,

\[
\phi_\alpha(x^-) = \int a_\alpha(p) e^{-ipx^-} \frac{[dp]}{(2|p|)^{1/2}} , \quad \psi_{L\alpha}(x^-) = \int \chi_\alpha(p) e^{-ipx^-} \frac{[dp]}{2|p|}.
\]

The normalization factors are chosen to reproduce the correct classical limits. To precisely define these expansions, we assume that the momenta range between \( (-\infty, -\epsilon_0) \) and \( [\epsilon_0, \infty) \), at the end of our calculations we set \( \epsilon_0 \to 0 \). This is physically meaningful due to charge conjugation invariance, and amounts to the principal value prescription. (see [18, 19] for details)

\[
[\chi_\alpha(p), \chi^{\dagger \beta}(q)]_+ = \delta^\beta_\alpha \delta[p - q] , \quad [a_\alpha(p), a^{\dagger \beta}(q)] = \text{sgn}(p) \delta^\beta_\alpha \delta[p - q]. \tag{4}
\]

One defines a Fock vacuum state \( |0\rangle \) by conditions,

\[
a_\alpha(p)|0\rangle = \chi_\alpha(p)|0\rangle = 0 \text{ for } p > 0 \quad a^{\dagger \alpha}(p)|0\rangle = \chi^{\dagger \alpha}(p)|0\rangle = 0 \text{ for } p < 0. \tag{5}
\]

(recall that we are assuming that there is an infinitesimal hole around \( p = 0 \) to be taken to zero at the end of the calculations). The corresponding normal orderings are defined in [18]. It is useful to keep in mind the normal ordering rules of the bilinears,

\[
:a^{\dagger \alpha}(p)a_\beta(q) := a^{\dagger \alpha}(p)a_\beta(q) - \frac{1}{2} \delta^\alpha_\beta (1 - \text{sgn}(p)) \delta[p - q]. \tag{6}
\]
and
\[ \chi^\alpha(p)\chi_\beta(q) := \chi^\alpha(p)\chi_\beta(q) \chi^\alpha_\beta(p) \delta[p-q]. \] (7)

We provide some of the details of the reorganization of the Hamiltonian into normal ordered bilinears in the appendix. We formulate the theory in terms of the color invariant bilinears following the idea proposed by Rajeev [10] and use our results in [13]. For the convenience of the reader we recollect some of the essential points: to define the large-\(N_c\) limit we introduce,

\[ \hat{M}(p,q) = \frac{2}{N_c} : \chi^\alpha(p)\chi_\alpha(q) : \]

\[ \hat{N}(p,q) = \frac{2}{N_c} : a^\alpha(p)a_\alpha(q) : \] (8)

and their odd counterparts,

\[ \hat{Q}(p,q) = \frac{2}{N_c} \chi^\alpha(p)a_\alpha(q), \quad \hat{\bar{Q}}(r,s) = \frac{2}{N_c} a^\alpha(r)\chi_\alpha(s) \] (9)

In the large-\(N_c\) limit these bilinears become classical variables [28], and we postulate the super Poisson brackets satisfied by these variables, which defines the kinematics of our theory:

\[ \{M(p,q),M(r,s)\} = -2i[M(p,s)\delta[q-r]-M(r,q)\delta[p-s]] \]

\[ \{N(p,q),N(r,s)\} = -2i[N(p,s)\delta[q-r]N(r,q)\delta[p-s]] \]

\[ \{Q(p,q),\bar{Q}(r,s)\} = -2i[N(p,s)\delta[q-r]N(r,q)\delta[p-s]] \]

\[ \{M(p,q),Q(r,s)\} = -2i\delta[p-s]Q(p,s) \]

\[ \{N(p,q),\bar{Q}(r,s)\} = 2i\alpha[p-s]Q(r,q) \]

\[ \{M(p,q),\bar{Q}(r,s)\} = 2i\alpha[p-s]\bar{Q}(r,q) \]

\[ \{N(p,q),\bar{Q}(r,s)\} = -2i\alpha[q-r]Q(p,s). \] (10)

These classical variables now satisfy,

\[ M(p,q) = M^*(q,p) \quad N(p,q) = N^*(q,p) \quad \hat{Q}(p,q) = Q^*(q,p). \] (11)

There are also constraints satisfied by these variables when we restrict ourselves to the subspace of the color invariant states. For our problem, this is an approximation, since there is no reason to expect that all the physical states are color singlets. In fact we will see that there are scattering states of our linearized equations. We write explicitly the constraints for the basic variables,

\[ (M+\epsilon)^2 + Q\epsilon Q^\dagger = 1 \]

\[ \epsilon Q^\dagger M + \epsilon Q^\dagger \epsilon + \epsilon N\epsilon Q^\dagger + Q^\dagger = 0 \]

\[ MQ + \epsilon Q + \epsilon Q N + Q\epsilon = 0 \]

\[ (\epsilon N + \epsilon)^2 + \epsilon Q^\dagger Q = 1. \] (12)
Above we use an operator notation, \( \epsilon(p, q) = -\text{sgn}(p)\delta[p-q] \), and \((AB)(p, q) = \int dsA(p, s)B(s, q)\). The phase space of the resulting restricted theory is shown to be a super-Grassmannian in [18], with its natural symplectic sturcture generalizing the results in [16].

We can reexpress our Hamiltonian in terms of the above mentioned basic variables. After a somewhat long but straightforward computation, the large-\(N\) limit Hamiltonian becomes,

\[
H = H_0 + H_Y,
\]

where,

\[
H_0 = \frac{1}{4} m_F^2 \int [dp] M(p, p) + \frac{1}{4} m_{BR}^2 \int [dp] N(p, p),
\]

\[
H_Y = \int \frac{[dpdqdsdt]}{\sqrt{|sq|}} \delta[p - q + s - t] \left( \frac{1}{16} \mu_1 m_F \left[ \frac{1}{p} + \frac{1}{t} \right] + \kappa \frac{1}{s - t} \right) Q(p, q)Q(s, t)
\]

\[
- \frac{\mu_2 \mu_1}{16} \int \frac{[dpdqdsdt]}{\sqrt{|st|}} \delta[p - q + s - t] \left[ \frac{1}{p} + \frac{1}{q} \right] M(p, q)N(s, t)
\]

\[
+ \frac{1}{64} \int \frac{[dpdqdk]}{\sqrt{|pqkl|}} \lambda_R^2 \delta[p - q + k - l]N(p, q)N(k, l)
\]

\[
- \int \frac{[dpdqdsdtdk]}{\sqrt{|qskl|}} \delta[p - q + s - t + k - l] \left[ \frac{\mu_1 \mu_2}{64} \left( \frac{1}{k - l - t} + \frac{1}{s - t - q} \right) + \frac{\mu_2 Y}{64} \left( \frac{1}{s - t - l} \right) \right] Q(p, q)Q(s, t)N(k, l)
\]

\[
+ \frac{\mu_2^2 Y}{64} \int \frac{[dpdqdsdtdk]}{\sqrt{|pqkl|}} \delta[p - q + k - l + s - t] \frac{1}{t + l - k} N(p, q)N(k, l)M(s, t).
\]

Note that we have rescaled the coupling constants as \( \mu_1 N_c \mapsto \mu_1, \mu_2 N_c \mapsto \mu_2 \) and \( \lambda_R^2 N_c \mapsto \lambda_R^2 \).

As we discussed in the appendix, there are two possible renormalizations. In the first case we allow for the non-local counter terms and remove the divergent parts finding a local Hamiltonian–thus \( \kappa \) and \( \lambda_{RB} \) are just constants. If we only allow for the local counter terms then we find,

\[
\kappa = \kappa_R(\mu_R) - \frac{\mu_2^2 Y}{64 \pi} \ln \left| \frac{s - t}{\mu_R} \right|
\]

\[
\lambda_R^2 = \lambda_{RB}^2(\mu_R) + \frac{\mu_2^2 Y}{\pi} \ln \left| \frac{k - l}{\mu_R} \right|
\]

We introduce a renormalization scale \( \mu_R \), and assume that the renormalized values of the couplings, \( \kappa_R(\mu_R), \lambda_{RB}^2(\mu_R) \) vary with the scale \( \mu_R \) such that the Hamiltonian does not really depend on this scale. This means we should impose,

\[
\kappa_R(\mu_R) = \kappa_R(\tilde{\mu}_R) - \frac{\mu_2^2 Y}{64 \pi} \ln \left| \frac{\mu_R}{\tilde{\mu}_R} \right|
\]
\[
\lambda_{RB}^2(\mu_R) = \lambda_{RB}^2(\bar{\mu}_R) + \frac{\mu_{2Y}^2}{\pi} \ln \left| \frac{\mu_R}{\bar{\mu}_R} \right|.
\]  

(16)

The sign of the new coupling $\kappa_R$ should not be fixed since it does not exist in the original action, and it has dimensions of mass square.

For the rest of this work we will take the simpler Hamiltonian, that is we assume that all the renormalized couplings are ordinary numbers. From a more conservative point of view the local counter terms should be the general class of models we should investigate. We hope to return to a more detailed analysis in the future.

This Hamiltonian along with the Poisson brackets and the constraint define our model completely. As it stands this is a complicated system. We plan to study a variational approach to this model in a future work. We will study a linear approximation to this model in the next section.

### 3 The linear approximation

We assume that all the basic variables deviate from the vacuum by small amounts, therefore we keep everything to first order. This means the linearization of the constraint and the linearization of the equations of motion. The constraint implies that

\[
M(u, v) = 0 \quad N(u, v) = 0 \quad Q(u, v) = 0 \text{ if } uv > 0.
\]

The equations of motion for $u > 0, v < 0$, found from

\[
\frac{\partial O(u, v)}{\partial x^+} = \{O(u, v), H\},
\]

(18)

where $O$ refers to any one of our variables, could be linearized. Let us write these linearized equations of motion for all the variables,

\[
\frac{\partial M(u, v)}{\partial x^+} = i \frac{m_F^2}{2} \left[ \frac{1}{u} - \frac{1}{v} \right] M(u, v) - i \frac{\mu_{2Y} m_F}{4} \left[ \frac{1}{u} + \frac{1}{v} \right] \int \frac{N(s, s - (u - v))}{\sqrt{|s((u - v) - s)|}} \frac{ds}{s},
\]

(19)

\[
\frac{\partial N(k, l)}{\partial x^+} = i \frac{m_{BR}^2}{2} \left[ \frac{1}{k} - \frac{1}{l} \right] N(k, l) - i \frac{\mu_{2Y} m_F}{4 \sqrt{|kl|}} \int \frac{N(s, s - (k - l)) M(s, s - (k - l))}{\sqrt{|s((u - v) - s)|}} \frac{ds}{s},
\]

(20)

and

\[
\frac{\partial Q(u, v)}{\partial x^+} = i \frac{m_F^2}{2} \left[ \frac{1}{u} - \frac{1}{v} \right] Q(u, v) - \kappa \frac{4i}{u - v} \int \frac{Q(p, p - (u - v))}{\sqrt{|v(p - (u - v))|}} \frac{dp}{p} + \frac{\mu_{2Y} m_F}{4} \int \frac{Q(p, p - (u - v))}{\sqrt{|(p - (u - v))v|}} \frac{dp}{p}.
\]

(21)
The equation of motion for $\bar{Q}$ can be found by complex conjugation and does not carry new information. We note that the equations of motion for $M$ and $N$ are coupled, but the equations of motion for $Q$ within the linear approximation is decoupled from the rest.

So we will start with this one and make an ansatz as in [16, 19]. Let us assume that the solution can be written as $Q(u, v; x^+) = c_Q(x)e^{iP_+x^+}$, where $x = u/P_-, P_- = u - v$ and define an invariant mass $\Lambda_Q^2 = 2P_-P_+$. (Strictly speaking we could take the solution of the form $c_Q(x)f_Q(P_-)e^{iP_+x^+}$, the arbitrary function $f_Q$ factors out in the equations). Then we find,

$$\Lambda_Q^2 c_Q(x) = \left[ \frac{m_F^2}{x} + \frac{m_{BR}^2}{1 - x} \right] c_Q(x) - 8\kappa \int_0^1 \frac{dy}{\sqrt{(1 - y)(1 - x)}} c_Q(y) + \frac{1}{4}\mu_1 Y m_F \int_0^1 \frac{dy}{\sqrt{(1 - y)(1 - x)}} \frac{1}{x + y} c_Q(y).$$

This innocent looking equation actually requires a renormalization, as we will see shortly.

Let

$$\int_0^1 \frac{dy}{\sqrt{(1 - y)}} c_Q(y) = A \quad \int_0^1 \frac{dy}{y\sqrt{(1 - y)}} c_Q(y) = B. \tag{22}$$

Then solve for $c_Q(x)$,

$$c_Q(x) = \sqrt{(1 - x)} \left( \frac{aA + bB) + bA}{\Lambda_Q^2 x(1 - x) - m_F^2(1 - x) - m_{B}^2 x} \right). \tag{23}$$

where

$$a = -8\kappa, \quad b = \frac{1}{4}\mu_1 Y m_F. \tag{24}$$

A straightforward solution will actually produce a divergence, the integration defining $B$ is divergent. To find a finite result we need a renormalization prescription. Let us assume that the phase is given by $e^{(P_+ + \delta P_+(\epsilon_0)x^+)}$, where $\delta P_+(\epsilon_0)$ denotes a divergent phase of the solution that we remove from the equations, and $\epsilon_0$ denotes a low momentum cut-off. The time derivative will drop a factor of $\delta P_+(\epsilon_0)$, and multiplying by $2P_-$ we denote it as $\delta \Lambda_Q(\epsilon_0)$ and rewrite the same equation as,

$$\left[ \Lambda_Q^2 + \delta \Lambda_Q(\epsilon_0) - \frac{m_F^2}{x} \right] c_Q(x) = \frac{aA}{\sqrt{1 - x}} + \frac{bA}{x\sqrt{1 - x}} + \frac{bB(\epsilon_0)}{\sqrt{1 - x}}. \tag{25}$$

Since the divergent part comes from the $B$ term we expect that $\delta \Lambda_Q(\epsilon_0)c_Q(x)$ can be taken as a counterterm on the other side of the equality with the leading form $-\frac{\alpha_c(\epsilon_0)}{\sqrt{1 - x}}$. The unknown function now is given by the same formula with a shifted coefficient of $x$,

$$c_Q(x) = \sqrt{(1 - x)} \left( \frac{(aA + bB(\epsilon_0) - \alpha_c(\epsilon_0)) + bA}{\Lambda_Q^2 x(1 - x) - m_F^2(1 - x) - m_{B}^2 x} \right). \tag{26}$$
Let us insert this back into (22) and find the constants $A, B(\epsilon_0)$. After some algebra, we reach,

$$A = -F_2 \frac{aA + bB(\epsilon_0) - \alpha_c(\epsilon_0)}{2\Lambda_Q^2} - \left[\left(\frac{aA + bB(\epsilon_0) - \alpha_c(\epsilon_0)}{2\Lambda_Q^2}\right)(\Lambda_Q^2 - m_{RB}^2 + m_F^2) + bA\right] F_1(\Lambda_Q),$$

$$B(\epsilon_0) = \frac{bA}{m_F^2} \ln(\epsilon_0) + \frac{bA}{2m_F^2} F_2 - \left[\left(\frac{bA}{2m_F^2}\right)(\Lambda_Q^2 - m_{RB}^2 + m_F^2) + (aA + bB(\epsilon_0) - \alpha_c(\epsilon_0))\right] F_1(\Lambda_Q),$$

where,

$$F_2 = \int_0^1 dx \frac{2\Lambda_Q^2 x - (\Lambda_Q^2 - m_{RB}^2 + m_F^2) x + m_F^2}{\Lambda_Q^2 x^2 - (\Lambda_Q^2 - m_{RB}^2 + m_F^2) x + m_F^2} = \ln \left[\frac{m_{RB}^2}{m_F^2}\right],$$

$$F_1(\Lambda_Q) = \int_0^1 dx \frac{\Lambda_Q^2 x^2 - (\Lambda_Q^2 - m_{RB}^2 + m_F^2) x + m_F^2}{\Lambda_Q^2 x^2 - (\Lambda_Q^2 - m_{RB}^2 + m_F^2) x + m_F^2}.$$

If we are looking for a bound state of a boson and a fermion, this requires,

$$|m_{RB} - m_F| < \Lambda_Q < m_{RB} + m_F,$$

then the last integral gives,

$$F_1(\Lambda_Q) = \frac{2}{u} \left(\arctan \left[\frac{\Lambda_Q^2 + m_{BR}^2 - m_F^2}{u}\right] - \arctan \left[\frac{m_{BR}^2 - m_F^2 - \Lambda_Q^2}{u}\right]\right),$$

where,

$$u = \sqrt{(\Lambda_Q^2 - (m_{RB} - m_F)^2)((m_{RB} + m_F)^2 - \Lambda_Q^2)}.$$

Let us impose the two conditions,

$$B_* = B(\epsilon_0) - \frac{bA}{m_F^2} \ln(\epsilon_0), \quad bB(\epsilon_0) - \alpha_c(\epsilon_0) = bB_*,$$

then we see that if we set

$$\alpha_c(\epsilon_0) = \frac{b^2 A}{m_F^2} \ln(\epsilon_0),$$

we can take $\epsilon_0 \to 0^+$ limit, and keep $B_*, A$ finite. The renormalized equations become,

$$A = -F_2 \frac{aA + bB_*}{2\Lambda_Q^2} - \left[\left(\frac{aA + bB_*}{2\Lambda_Q^2}\right)(\Lambda_Q^2 - m_{RB}^2 + m_F^2) + bA\right] F_1(\Lambda_Q),$$

$$B_* = \frac{bA}{2m_F^2} F_2 - \left[\left(\frac{bA}{2m_F^2}\right)(\Lambda_Q^2 - m_{RB}^2 + m_F^2) + (aA + bB_*)\right] F_1(\Lambda_Q).$$

If we solve for the ratio $A/B_*$, after some algebra, this gives us a consistency condition for the excitation energy $\Lambda_Q^2$,

$$\left[ F_2 - s(\Lambda_Q) F_1(\Lambda_Q) - \frac{2am_F^2}{b} F_1(\Lambda_Q) \right] [F_2 + s(\Lambda_Q) F_1(\Lambda_Q)] =$$
\[- [1 + b F_1(\Lambda_Q)] [F_2 + (s(\Lambda_Q) + \frac{2b\Lambda^2_Q}{a}) F_1(\Lambda_Q) + \frac{2\Lambda^2_Q}{a} \frac{2m^2 a}{b^2}], \quad (37)\]

or equivalently,
\[
\left[ \ln \frac{m^2_{RB}}{m^2_F} - s(\Lambda_Q) F_1(\Lambda_Q) + \frac{64\kappa m_F}{\mu_Y} F_1(\Lambda_Q) \right] \left[ \ln \frac{m^2_{RB}}{m^2_F} + s(\Lambda_Q) F_1(\Lambda_Q) \right] = \\
\left[ 1 + \frac{1}{4} \mu_Y m_F F_1(\Lambda_Q) \right] \left[ \ln \frac{m^2_{RB}}{m^2_F} + s(\Lambda_Q) - \frac{\mu_Y m_F \Lambda^2_Q}{16\kappa} \right] F_1(\Lambda_Q) + \frac{\Lambda^2_Q}{4\kappa} \frac{128\kappa}{\mu^2_Y}. \quad (38)\]

where \( s(\Lambda_Q) = \Lambda^2_Q - m^2_{RB} + m^2_F \). This equation is written in terms of dimensionless ratios of the variables, and it should be investigated numerically under the conditions we have stated before for \( \Lambda_Q \). Instead of numerically solving these equations we will look at one extreme case, when \( \Lambda_Q \approx m_{BR} + m_F \) (weak coupling). (It is interesting to investigate the opposite limit \( \Lambda_Q \approx |m_F - m_{BR}| \), but the result is not so simple to interpret). It is better to use a different variable to study such limiting cases, we define \( \Delta \), via \( \Lambda_Q = m^2_{BR} + m^2_F + 2m_{BR}m_F \Delta \). Note that we have \(-1 < \Delta < 1\). Our function \( F_1(\Lambda_Q) \) becomes,
\[
F_1(\Delta) = \frac{1}{m_F m_{BR}} \frac{1}{\sqrt{1 - \Delta^2}} \left( \arctan \left( \frac{\omega + \Delta}{\sqrt{1 - \Delta^2}} \right) + \arctan \left( \frac{1}{\omega + \Delta} \right) \right). \quad (39)\]

Here \( \omega = m_{BR}/m_F \) and if we take \( \Delta \to 1^+ \), that means \( \Lambda_Q \to (m_{BR} + m_F)^{-1} \) and \( \Delta \to -1^+ \) corresponds to \( \Lambda_Q \to |m_F - m_{BR}|^+ \). There is nothing subtle about the first limit, keeping everything to first order gives us,
\[
\Delta = 1 - \left[ 1 - \left( 2 - \frac{\mu_Y}{\kappa} (m_F + m_{BR}) \right) \right]^{-1} \frac{\mu_Y}{m_F + m_{BR}} \ln \frac{m^2_{RB}}{m^2_F} \right]^{-2} \frac{\pi^2 \mu^2_Y}{m^2_B}. \quad (40)\]

We assumed all the way \( \Delta \approx 1 \), this could be consistent if for example we choose the coupling constant \( \mu_Y \) such that \( \mu_Y << m_B \) when the ratio \( m_F/m_{BR} \) is not too large and if \( \kappa < 0 \). There are other possibilities, but this simple one shows that there are solutions with the expected behaviour. If we assume \( \Delta \to -1^+ \), the function \( F_1(\Delta) \to 1/m_F m_{BR} \), and the calculations are more complex. If we set \( \Delta = -1 + \delta^2 \), we have \( F_1(-1 + \delta^2) \approx \frac{1}{m_F m_{BR}} (1 - \frac{3}{2} \delta^2) \). This can be used to study the opposite limit, but due to its algebraic complexity we leave it out, and only state that for various possible cases to be consistent, we find that \( m_F >> m_{BR} \) is a necessary condition.

Following the same strategy, we look for stationary solutions for the coupled equations: we start with the following ansatz for \( M, N \): \( M(u, v) = \xi_M(x) e^{ip_x x} \) and \( N(k, l) = \xi_N(y) e^{ip_y y} \), where \( x = u/P_M, P_M = u - v \) and similarly for \( N \). If we now substitute these into the coupled equations of motion we see that the oscillations in time cancel out (since we select the same \( P_\pm \)) and we end up with,
\[
\Lambda^2_M \xi_M(x) = m^2_F \left[ \frac{1}{x} + \frac{1}{1 - x} \right] \xi_M(x) - \frac{\mu^2_Y m_F}{4\pi} \left[ \frac{1}{x} - \frac{1}{1 - x} \right] \int_0^1 \frac{\xi_N(y)}{\sqrt{y(1 - y)}} dy, \quad (41)\]

\[
\Lambda^2_N \xi_N(x) = m^2_{BR} \left[ \frac{1}{x} + \frac{1}{1 - x} \right] \xi_N(x) - \frac{\mu^2_Y m_F}{4\pi \sqrt{(1 - x)x}} \int_0^1 dy \left( \frac{1}{y} - \frac{1}{1 - y} \right) \xi_M(y)
\]

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\[ + \frac{\lambda_R^2}{8\pi} \int_0^1 \frac{dy}{\sqrt{x(1-x)y(1-y)}} \xi_N(y), \]  

(42)

where we set \( \Lambda_M^2 = 2P_MP_+ \), and \( \Lambda_N^2 = 2P_NP_+ \) for the invariant masses of the excitations of \( M \) and \( N \) respectively. We notice that the desired decoupling of the total momentum variables would not have happened in the above equations if we had used the more general Hamiltonians.

Before we plunge into the standard way to solve these equations we will talk about an interesting possibility, if we admit that the distributional solutions are also proper solutions of these equations.

Let us write the equation for \( M \), if we call

\[ D = \int_0^1 dy \frac{\xi_N(y)}{\sqrt{y(1-y)}}, \]  

(43)

then we can rewrite it as,

\[ \Lambda_M^2 \xi_M(x) = m_F^2 \left[ \frac{1 - dD}{x} + \frac{1 + dD}{1 - x} \right] \xi_M(x), \quad \text{where } d = \frac{\mu_{2Y}}{4\pi m_F}. \]  

(44)

For the solution we have in mind we need to impose

\[ dD < 1, \]  

(45)

otherwise the energy will be unbounded from below. Let us assume that the last two terms in the \( \xi_N \) equation cancel against each other. This condition implies that

\[ \frac{\mu_{2Y} m_F}{4\pi} \int_0^1 dy \left( \frac{1}{y} - \frac{1}{1-y} \right) \xi_M(y) = \frac{\lambda_R^2}{8\pi} \int_0^1 \frac{dy}{\sqrt{y(1-y)}} \xi_N(y), \]  

(46)

or equivalently,

\[ \frac{\mu_{2Y} m_F}{4\pi} \int_0^1 dy \left( \frac{1}{y} - \frac{1}{1-y} \right) \xi_M(y) = \frac{\lambda_R^2}{8\pi} D. \]  

(47)

Thus we have to consistently choose everything to satisfy these conditions. The equation for \( M \) and \( N \) can be solved by using \( \xi_M(x) = \delta(x - x_F) \) and \( \xi_N(x) = \delta(x - x_B) \). What should we take as \( x_F \) and \( x_B \)? One way is to minimize the excitation energy for fermions, and then fix the bosonic parameter to have the cancellation. The eigenvalue for \( M \) becomes, after the minimizing choice is made,

\[ \Lambda_M^2 = m_F^2 \left[ (1 + dD)^{1/2} + (1 - dD)^{1/2} \right]^2 < 4m_F^2. \]  

(48)

The last inequality is interesting since it implies that the fermions actually form a bound state at this energy. Now the condition we should have for the cancellation reads,

\[ \frac{(1 + dD)^{1/2} - (1 - dD)^{1/2}}{(1 + dD)^{1/2} + (1 - dD)^{1/2}} = fD, \]  

(49)
where
\[ f = \frac{\lambda_R^2}{2\mu_{2Y} m_F}. \]  

From here we can solve for \( D \),
\[ D^2 = \frac{1}{d^2} - \frac{4}{f^2}. \]  

We should have \( D^2 > 0 \), this puts a condition on our couplings and the fermion mass. But there is a stronger condition, once we have the solution for the value of \( D \), we can find the parameter \( x_B \) to choose for the bosons from the definition of \( D \),
\[ D = \int_0^1 \frac{dy}{\sqrt{y(1-y)}} \xi_N(y) = \frac{1}{\sqrt{x_B(1-x_B)}}. \]

It is possible to find \( x_B \) if \( D \geq 2 \), since the minimum of the function on the right is 2, thus we need \( D^2 \geq 4 \). This implies a condition on our couplings,
\[ \frac{1}{\pi} \left( \frac{\mu_{2Y}^2}{m_F^2} \right) \left[ 1 - \frac{\mu_{2Y}^2}{4\pi^2 m_F^2} \right]^{-1/2} < \frac{\lambda_R^2}{m_F^2}, \]
where we used dimensionless variables to express this inequality.

We have the other condition about \( D \), which says \( dD < 1 \). This is actually satisfied by our solution, so we need
\[ 2 \leq D < 1/d. \]

From these we have a condition on the strength of the Yukawa coupling constant,
\[ \mu_{2Y} < 2\pi m_F. \]

Now we can go back and find the actual value of the fermion bound state and the mass of the boson pair. The boson pair mass is simply given by
\[ \Lambda_N^2 = \frac{m_{RB}^2}{x_B(1-x_B)} = m_{RB}^2 D^2 = m_{RB}^2 \left[ \frac{1}{d^2} - \frac{4}{f^2} \right] = 16 m_{RB}^2 \left[ \frac{\pi^2 m_F^2}{\mu_{2Y}^2} - \frac{\mu_{2Y}^2 m_F^2}{\lambda_R^4} \right] > 4 m_{RB}^2. \]

Similarly we have for the fermion pair,
\[ \Lambda_M^2 = m_F^2 \left[ \left( 1 + \left[ 1 - \frac{4d^2}{f^2} \right]^{1/2} \right)^{1/2} \right] \left( 1 - \left[ 1 - \frac{4d^2}{f^2} \right]^{1/2} \right)^{1/2} \]
\[ = m_F^2 \left[ \left( 1 + \left[ 1 - \frac{\mu_{2Y}^4}{\pi^2 \lambda_R^4} \right]^{1/2} \right)^{1/2} \right] \left( 1 - \left[ 1 - \frac{\mu_{2Y}^4}{\pi^2 \lambda_R^4} \right]^{1/2} \right)^{1/2} \]
\[ = m_F^2 \left[ \left( 1 + \left[ 1 - \frac{\mu_{2Y}^4}{\pi^2 \lambda_R^4} \right]^{1/2} \right)^{1/2} \right] \left( 1 - \left[ 1 - \frac{\mu_{2Y}^4}{\pi^2 \lambda_R^4} \right]^{1/2} \right)^{1/2} \]
\[ = m_F^2 \left[ \left( 1 + \left[ 1 - \frac{\mu_{2Y}^4}{\pi^2 \lambda_R^4} \right]^{1/2} \right)^{1/2} \right] \left( 1 - \left[ 1 - \frac{\mu_{2Y}^4}{\pi^2 \lambda_R^4} \right]^{1/2} \right)^{1/2} \]
\[ = \left( m_F^2 \left[ \left( 1 + \left[ 1 - \frac{\mu_{2Y}^4}{\pi^2 \lambda_R^4} \right]^{1/2} \right)^{1/2} \right] \left( 1 - \left[ 1 - \frac{\mu_{2Y}^4}{\pi^2 \lambda_R^4} \right]^{1/2} \right)^{1/2} \right)^2 \]
\[ \approx 2 m_F^2 \left( 1 + \frac{2d^2}{f^2} \right) \approx 2 m_F^2 \left[ 1 + \frac{1}{\pi} \frac{\mu_{2Y}^4}{\lambda_R^4} \right]. \]

We note that the above solution is quite interesting: we assume that the relative strength of the Yukawa coupling is small, i.e. \( \mu_{2Y} << \lambda_R \), and expand the square roots,
\[ \Lambda_M^2 \approx 2 m_F^2 \left( 1 + \frac{2d^2}{f^2} \right) \approx 2 m_F^2 \left[ 1 + \frac{1}{\pi} \frac{\mu_{2Y}^4}{\lambda_R^4} \right]. \]
Furthermore the boson pair becomes in this approximation,

$$\Lambda_N^2 \approx 16\pi^2 \left( \frac{\mu_{2Y}}{m_F} \right)^2 m_{BR}^2 > 16m_{BR}^2. \quad (58)$$

If we further assume that $\mu_{2Y} << m_F$ this implies that the boson pair mass becomes very large.

Actually, there is a whole range of solutions with $\xi_M(x) = \delta(x-x_F)$ and $\xi_N(x) = \delta(x-x_B)$. We are free to choose one of them, say $x_F$ then the other one will be determined by the same consistency relation as above. We see that the boson pair excitation will always be bigger than $2m_{BR}$, since the minimum is given by $x_B = 1/2$. For the fermion pair we choose the consistent $x_F$'s such that the mass is less than the two mass threshold. Let us briefly present our findings using the same notation as above.

Let us search for a solution of the equation

$$m_F^2 \left[ \frac{1 - dD}{x} + \frac{1 + dD}{1 - x} \right] \xi_M(x) = \Lambda_M^2 \xi_M(x) \quad \text{with} \quad \Lambda_M^2 < 4m_F^2. \quad (59)$$

We assume again that $dD < 1$, then the equality is satisfied if we set $\xi_M(x) = \delta(x-x_F)$, $x_F = 1/2(1 - \alpha dD)$ for $0 < \alpha < 1$. We require the same delta function solution for $\xi_N$, this means we should solve for the equation,

$$\frac{1}{x_F} - \frac{1}{1 - x_F} = f D. \quad (60)$$

If we solve for $D$ now, we find

$$D^2 = \left[ 1 - \frac{4\alpha d}{f} \right] \frac{1}{\alpha^2 d^2}. \quad (61)$$

This implies that the first factor in the big parenthesis should be positive. This is true if $\lambda_R^2 > \frac{2}{\pi} \alpha \mu_{2Y}$. Again to have a solution for $x_B$ we need $4 \leq D^2$, this means

$$4\alpha^2 d^2 + \frac{4\alpha d}{f} - 1 \geq 0, \quad (62)$$

which can be satisfied if the quadratic equation for $d$ has real roots and we choose $d$ in between–assuming alpha is chosen. This implies for $\lambda_R$ an inequality,

$$\frac{2\alpha}{\pi} \left( \frac{\mu_{2Y}}{m_F} \right)^2 \left[ 1 - \left( \frac{\alpha \mu_{2Y}}{2\pi m_F} \right)^2 \right]^{-1} < \frac{\lambda_R^2}{m_F^2}. \quad (63)$$

This means that we should choose $\lambda_R$ above a certain value, and this condition is stronger than the first one we found above. A uniform bound for various values of $\alpha$ can be chosen,

$$\frac{2}{\pi} \left( \frac{\mu_{2Y}}{m_F} \right)^2 \left[ 1 - \left( \frac{\mu_{2Y}}{2\pi m_F} \right)^2 \right]^{-1} < \frac{\lambda_R^2}{m_F^2}. \quad (64)$$

Incidentally this requires $\mu_{2Y} < 2\pi m_F$. We still have $dD < 1$ to satisfy. If we simply use this condition assuming $\alpha$ as given we arrive at the positivity of a quadratic expression in $\alpha$:

$$\alpha^2 + \frac{2\alpha \mu_{2Y}^2}{\pi \lambda_R^2} - 1 > 0. \quad (65)$$
Notice that the range of allowed $\alpha$ will be bigger if we take $\mu_{2Y}/\lambda_R$ ratio as large as possible. If we use the uniform lower bound for $\lambda_R$ we will find the largest region, and if we denote the deviation from this value by a multiplicative factor $k > 1$, we can insert this ratio into the quadratic expression,

$$\alpha^2 + \frac{1}{k}[1 - \left(\frac{\mu_{2Y}}{2\pi m_F}\right)^2] \alpha - 1 > 0. \quad (66)$$

The positivity is guaranteed if we choose $\alpha$ outside of the region between the two roots. The choice consistent with $0 < \alpha < 1$ gives us,

$$\frac{1}{2k} \left(\sqrt{\left[1 - \frac{\mu_{2Y}^2}{4\pi^2 m_F^2}\right]^2 + 4k^2 - \left[1 - \frac{\mu_{2Y}^2}{4\pi^2 m_F^2}\right]}\right) < \alpha < 1. \quad (67)$$

The lower bound is a decreasing function of $k$, so the stronger relative values of $\lambda_R$ will have a smaller domain of $\alpha$’s. Since for these choices we have a continuous range of $\alpha$’s, the spectrum of the problem is rather different. Such distributional solutions are not eigenvalues but they typically refer to the continuous part of the spectrum. In some sense these are still scattering states. This means we cannot use the free parts of the original Hamiltonian to study the scattering theory below the two mass thresholds. Above these values we will see that the scattering theory can be studied by conventional methods. Perhaps below this we need to use the minimum value of the spectrum to define new effective pair mass for the fermionic sector. We are not able to resolve this issue at the moment.

If we now go back to the standard approach, again as in the case of boson-fermion pair, we will need to renormalize our equations, by assuming a divergent common phase $\delta P_+(\epsilon_0)$, $2i\xi_M e^{i(\delta P_+(\epsilon_0) + P_+)}x^+$, and similarly for $N$. The derivative will bring terms of the form $\delta \Lambda_N(\epsilon_0) = 2P_N \delta P_+(\epsilon_0)$ and $\delta \Lambda_M(\epsilon_0) = 2P_M \delta P_+(\epsilon_0)$. An inspection of the resulting equations show that, we have the leading behaviour

$$\delta \Lambda_M(\epsilon_0) \xi_M(x) \sim \frac{\alpha_c(\epsilon_0)(1 - 2x)}{x(1 - x)}, \quad \delta \Lambda_N(\epsilon_0) \xi_N(x) \sim \frac{\beta_c(\epsilon_0)}{\sqrt{x(1 - x)}}. \quad (68)$$

In general, $\beta_c(\epsilon_0) = \sigma \alpha_c(\epsilon_0)$, as we will see the precise value of $\sigma$ is not important. Now we can solve the unknown functions,

$$\xi_M(x) = \frac{(1 - 2x)(aA(\epsilon_0) - \alpha_c(\epsilon_0))}{\Lambda_M^2 x(1 - x) - m_F^2},$$

$$\xi_N(x) = \sqrt{x(1 - x)} \frac{2aB(\epsilon_0) + bA(\epsilon_0) - \sigma \alpha_c(\epsilon_0)}{\Lambda_N^2 x(1 - x) - m_{BR}^2}, \quad (69)$$

where

$$A(\epsilon_0) = \int_{\epsilon_0}^1 dy \frac{\xi_N(y)}{\sqrt{y(1 - y)}}, \quad B(\epsilon_0) = \int_{\epsilon_0}^1 dy \frac{\xi_M(y)}{y}, \quad (70)$$

and

$$a = -\frac{\mu_{2Y} m_F}{4\pi}, \quad b = \frac{\lambda_R^2}{8\pi}. \quad (71)$$
If we introduce
\[ F(\Lambda, m) = \mathcal{P} \int_0^1 \frac{dy}{\Lambda^2 y(1 - y) - m^2}, \] (72)
we find from the defining conditions of \( A(\epsilon_0), B(\epsilon_0) \),
\[ B(\epsilon_0) = \left( aA(\epsilon_0) - \alpha_c(\epsilon_0) \right) \frac{1}{m_F^2} \ln \epsilon_0 + \left( aA(\epsilon_0) - \alpha_c(\epsilon_0) \right) \left( \frac{\Lambda^2}{2m_F^2} - 2 \right) F(\Lambda_M, m_F) \]
\[ A(\epsilon_0) = \left[ 2aB(\epsilon_0) + bA(\epsilon_0) - \sigma \alpha_c(\epsilon_0) \right] F(\Lambda_N, m_{RB}). \] (73)
We now define \( aA_\star = aA(\epsilon_0) - \alpha_c(\epsilon_0) \), and \( B_\star = B(\epsilon_0) - \frac{a}{m_F^2} A_\star \ln \epsilon_0 \), and insert these back into our equations,
\[ B_\star = A_\star a \left( \frac{\Lambda^2}{2m_F^2} - 2 \right) F(\Lambda_M, m_F) \]
\[ A_\star = \left[ 2aB_\star + bA_\star + \frac{2a^2}{m_F^2} A_\star \ln \epsilon_0 + \left[ \frac{b}{a} - \sigma - \frac{1}{aF(\Lambda_N, m_{RB})} \right] \alpha_c(\epsilon_0) \right] F(\Lambda_N, m_{RB}). \] (74)
If we set
\[ \alpha_c(\epsilon_0) = - \left( \frac{b}{a} - \sigma - \frac{1}{aF(\Lambda_N, m_{RB})} \right)^{-1} \frac{2a^2}{m_F^2} A_\star \ln \epsilon_0, \] (75)
we can take \( \epsilon_0 \to 0^+ \) limit while keeping \( A_\star, B_\star \) finite. These will be our renormalized equations,
\[ B_\star = aA_\star \left( \frac{\Lambda^2}{2m_F^2} - 2 \right) F(\Lambda_M, m_F) \quad A_\star = [2aB_\star + bA_\star] F(\Lambda_N, m_{BR}). \] (76)
Incidentally we note that this physical prescription implies that the proper way we should define these integral equations is to use the Hadamard finite value (see the similar issue in [19] and [29] for a recent discussion of the renormalization and distribution theory).
Let us assume that we are looking for bound state solutions, the principal value integral in \( F(\Lambda, m) \) then becomes an ordinary integral. Now we have two different expression for the ratio \( B_\star/A_\star \), which give us the desired eigenvalues when we require square integrable solutions. If we assume that both of the eigenvalues are bound states we find,
\[ \left[ \frac{\mu^2}{4\pi^2} \lambda^2 - 4m^2 \right] \arctan \left[ \frac{\Lambda_M}{\sqrt{4m^2 - \lambda^2}} \right] + \frac{\lambda^2}{8\pi} \arctan \left[ \frac{\Lambda_N}{\sqrt{4m^2 - \lambda^2}} \right] = -1. \] (77)
Since the left hand-side is positive and the right one is negative this has no solution! if we demand both boson-boson and fermion-fermion pairs to form bound states, there is no solution. But we should not be alarmed, what we can do is to demand a resonance for the bosonic sector, then,
\[ \left[ \frac{\mu^2}{4\pi^2} \lambda^2 - 4m^2 \right] \arctan \left[ \frac{\Lambda_M}{\sqrt{4m^2 - \lambda^2}} \right] + \frac{\lambda^2}{8\pi} \ln \frac{\Lambda_N + \sqrt{\lambda^2 - 4m^2}}{\Lambda_N - \sqrt{\lambda^2 - 4m^2}} = 1, \] (78)
\[ B_\star = aA_\star \left( \frac{\Lambda^2}{2m_F^2} - 2 \right) F(\Lambda_M, m_F) \quad A_\star = [2aB_\star + bA_\star] F(\Lambda_N, m_{BR}). \] (76)
and this has solutions in general. The resonance case seems to be special to 1+1 dimensions, the other possibility is to require bosons to have scattering states and fermions to be bound. This can be studied by analyzing the pole structure of the analytic continuation of the scattering amplitudes that will be worked out below. Due to the algebraic complexity of the resulting formulae, we will not be able to answer it in this work.

We study the scattering states when we are beyond the two mass threshold. Let us make a digression for the moment and study a simpler problem, the lambda-phi-four coupling. It is easy to see from our equations for $N(u, v)$ that the same ansatz for the solution leads to

$$\Lambda^2 \xi(x) = \frac{m^2}{x(1-x)} \xi(x) + \frac{\lambda_B^2}{8\pi} \frac{1}{\sqrt{x(1-x)}} \int_0^1 \frac{\xi(y)}{\sqrt{y(1-y)}}. \quad (79)$$

We assume that the operator on the right is acting on $L^2([0,1])$, with vanishing at the end points boundary conditions. The free part,

$$H_0 = \frac{m^2}{x(1-x)} \quad (80)$$

is an unbounded operator with a continuous spectrum $[4m^2, \infty)$. This is easy to understand by studying a particle and antiparticle in the center of momentum frame.

If the added term is not a “too strong” perturbation then the absolutely continuous part of the spectrum of the full operator on the right is the same as the spectrum of the free part and we can study the scattering states using the free part (there are various conditions we can state so that “too strong” becomes a precise statement, we recommend [30] for a thorough mathematical discussion of these issues, and [31] with more physical emphasis). The type of problem we study is analyzed in a recent valuable book by Albeverio and Kurasov [32]. The interaction term is called a rank one perturbation. The Hamiltonian,

$$H = \frac{m^2}{x(1-x)} + \frac{\lambda_B^2}{8\pi} \frac{1}{\sqrt{x(1-x)}} \int_0^1 dy \frac{1}{\sqrt{y(1-y)}}, \quad (81)$$

where everything acts on functions in $L^2([0,1])$, can be written as

$$H = H_0 + \frac{\lambda_B^2}{8\pi} |f><f|, \quad (82)$$

with $<x|f> = f(x) = \frac{1}{\sqrt{x(1-x)}}$. If such a perturbation is relatively form bounded then the scattering states are given by the scattering states of the free part. To verify this condition it is enough to show that the added term satisfies

$$\|f\|_{-1} = \|\frac{1}{(|H_0| + 1)^{1/2}} f\| < \infty, \quad (83)$$

where $\|\cdot\|$ denotes the usual $L^2$ norm. It is now simple to check that,

$$\|f\|_{-1}^2 < \int_0^1 dx \left( \frac{m^2}{x(1-x)} \right)^{-1} \left( \frac{1}{\sqrt{x(1-x)}} \right)^2 < \infty. \quad (84)$$
In fact in the above problem we can find the resolvent of our integral operator (see [32]):

\[(R_H(Z)f)(x) = [(H - Z)^{-1}f](x)\]

\[= \left(\frac{m^2}{x(1-x)} - Z\right)^{-1}f(x) + \frac{\lambda B}{8\pi} \tilde{A}(Z) \left[\frac{1}{Zx(1-x) - m^2} \int_0^1 \frac{\sqrt{y(1-y)}}{Zy(1-y) - m^2} dy f(y)\right],\]

for \(Z\) outside of the spectrum (and complex in general), and here we use the analytic continuation of \(\tilde{A}(\lambda)\) to complex numbers and its explicit form is given below. The knowledge of the resolvent gives everything about the operator, for example we can find the spectral density function \(\rho(\Lambda)\) (which heuristically corresponds to the “eigenfunction” expansion) by using the well-known Stone’s identity,

\[\rho(\Lambda) = \frac{1}{\pi i \epsilon_0} \lim_{\epsilon_0 \to 0^+} \left(\frac{1}{H - \Lambda + i\epsilon_0} - \frac{1}{H - \Lambda - i\epsilon_0}\right).\]  

(85)

In [32] the scattering theory of finite rank perturbations has been worked out by using rigorous methods. We will only content with the result that the scattering theory makes sense beyond the bound state thresholds for both particles, and we can find the resolvents explicitly. For simplicity of our presentation we study the scattering theory by the standart methods in physics, and only find the wave operators. One can see that for a given value of \(\Lambda^2 = \frac{m^2}{\lambda(1-\lambda)}\) we have two roots,

\[\lambda_\pm = \frac{1}{2} \pm \left[\frac{1}{4} - \frac{m^2}{\Lambda^2}\right]^{1/2}.\]  

(86)

In physics we typically think of two particles approaching and then scattering off to infinity. An inspection of the kinematics of a particle and an antiparticle pair in the center of momentum frame reveals that \(\lambda_+\) corresponds to the particle moving in the positive \(x^1\) direction (which we may take as “incoming” states), and \(\lambda_-\) corresponds to the particle moving in the negative \(x^1\) direction. From a physical point of view, the scattering data should give us the information about transmission and reflection of the pair. An equivalent description would be to find the wave operator, \(\Omega\) which maps (in general the projection to the absolutely continuous part of the spectrum of the original operator) the Hilbert space to the scattering states (if we take the absolutely continuous part of the spectrum and use the spectral projections corresponding to these values) of the interacting Hamiltonian:

\[\xi = \Omega f.\]  

(87)

(In physics one typically uses \(\Omega_+\) which takes the wave functions at time zero and evolve them to positive infinity, this requires the \(+i\epsilon\) prescriptions in the integrals, we will see that for our problem it is more suitable to define the principal value one, this is why we use \(\Omega\)).

To find the scattering amplitudes we rewrite the eigenvalue equation in Lippmann-Schwinger form,

\[\xi_\lambda(x) = \delta(x - \lambda) + \frac{\lambda B}{8\pi} \left[\Lambda^2 - \frac{m^2}{x(1-x)}\right]^{-1} \frac{1}{\sqrt{x(1-x)}} \int_0^1 dy \frac{\xi_\lambda(y)}{\sqrt{y(1-y)}}.\]  

(88)
We formally use the eigenvalue \( \delta_\lambda(x) = \delta(x - \lambda) \) of the free Hamiltonian. We can now solve for \( A(\lambda) = \int_0^1 \frac{\delta_\lambda(y)}{\sqrt{y(1-y)}} \),

\[
A(\lambda) = \left[ 1 + \frac{\lambda^2}{4\pi \Lambda \sqrt{(\Lambda^2 - 4m^2)}} \ln \left\{ \frac{\Lambda - \sqrt{\Lambda^2 - 4m^2}}{\Lambda + \sqrt{\Lambda^2 - 4m^2}} \right\} \right]^{-1} \frac{1}{\sqrt{\lambda(1 - \lambda)}},
\]

(89)

where \( \Lambda^2 = \frac{m^2}{\lambda(1 - \lambda)} \). Thus the wave operator acting on the (formal) eigenfunctions of the free Hamiltonian can be written as

\[
\xi_\lambda(x) = (\Omega \delta_\lambda)(x) = \int_0^1 dy \left( \delta(x - y) + \frac{\lambda^2}{8\pi} \mathcal{P} \left[ \frac{m^2}{y(1-y)} - \frac{m^2}{x(1-x)} \right]^{-1} \sqrt{x(1-x)y(1-y)} \right) \delta(y - \lambda),
\]

with \( \tilde{A}(\lambda) = \sqrt{\lambda(1 - \lambda)} A(\lambda) \). The left side of the expression gives us the distributional kernel of the wave operator. (If we are interested in incoing pair we could restrict ourselves to \( \lambda_+ \) values). We can expand an arbitrary vector into a series of the form \( f = \int_0^1 d\lambda f(\lambda) \delta_\lambda(x) \), and we find

\[
\xi(x) = [\Omega \left( \int_0^1 f(\lambda) \delta_\lambda \right)](x) = \int_0^1 d\lambda \Omega(x, \lambda) f(\lambda),
\]

(90)

and this makes sense in general. Note that this result is exact (within the linearized large-\( N_c \) limit) and we have a complete characterization of the set of scattering states once \( A(\lambda) \) is given.

We will study the scattering states of the coupled equations beyond the bound state thresholds, that is when the energies are larger than both \( 2m_F \) and \( 2m_{BR} \). The discussion preceding these indicates that the free part and the interacting coupled equations may not have the same scattering states (see the distributional solution we present). We follow the same idea as in the above problem and restrict ourselves to the heuristic Lippmann-Schwinger type approach. One can also find the resolvent exactly and verify the formulae below by more careful analysis.

We solve for the scattering amplitude by using a renormalized Lipmann-Schwinger equation: thus we have for the scattering,

\[
\xi_M(x; \Lambda) = \delta(x - \lambda_M) + \mathcal{P} \frac{1}{\Lambda^2 x(1-x) - m_F^2} (1 - 2x) a A_*(\Lambda)
\]

\[
\xi_N(x; \Lambda) = \delta(x - \lambda_N) + \mathcal{P} \frac{1}{\Lambda_N^2 x(1-x) - m_{BR}^2} \sqrt{x(1-x)} (aB_*(\Lambda) + bA_*(\Lambda)),
\]

where \( A_*(\Lambda), B_*(\Lambda) \) satisfy,

\[
A_*(\Lambda) = \frac{1}{\sqrt{\lambda_N(1 - \lambda_N)}} + (aB_*(\Lambda) + A_*(\Lambda)) F(\Lambda, m_{BR})
\]

\[
B_*(\Lambda) = \frac{1}{\lambda_M} - \frac{1}{1 - \lambda_M} + aA_*(\Lambda) \left( \frac{\Lambda^2}{m_F^2} - 4 \right) F(\Lambda, m_F),
\]

18
and $F(\Lambda, m)$ is the same function as before and we choose $\text{Max}(4m_B^2, 4m_F^2) < \Lambda^2$, with
$\Lambda^2 = \frac{m_B^2}{\lambda_M(1-\lambda_M)} = \frac{m_B^2}{\lambda_N(1-\lambda_N)}$. We can use $\lambda_M$ as the only parameter and call it simply $\lambda$. If we solve for the scattering amplitudes,

$$A_s(\lambda) = \left[ 1 - \left( \frac{\lambda R^2}{4\pi m_F^2} + \frac{\mu_2^2 Y}{4\pi m_F^2} \sqrt{(1 - 4\lambda(1 - \lambda))} \ln \left| \frac{1 - \sqrt{1 - 4\lambda(1 - \lambda)}}{1 + \sqrt{1 - 4\lambda(1 - \lambda)}} \right| \right] \right.$$

$$\times \frac{\lambda(1 - \lambda)}{\sqrt{1 - 4\left(\frac{m_B^2}{m_F^2}\right)\lambda(1 - \lambda)}} \ln \left| \frac{1 - \sqrt{1 - 4\left(\frac{m_B^2}{m_F^2}\right)\lambda(1 - \lambda)}}{1 + \sqrt{1 - 4\left(\frac{m_B^2}{m_F^2}\right)\lambda(1 - \lambda)}} \right|^{-1}$$

$$\times \left[ \frac{m_F}{m_B} \frac{1}{\sqrt{1 - \lambda}} - \frac{\mu_2 Y}{2\lambda m_F^{3/2}} \frac{\lambda(1 - \lambda)}{\sqrt{1 - 4\left(\frac{m_B^2}{m_F^2}\right)\lambda(1 - \lambda)}} \ln \left| \frac{1 - \sqrt{1 - 4\left(\frac{m_B^2}{m_F^2}\right)\lambda(1 - \lambda)}}{1 + \sqrt{1 - 4\left(\frac{m_B^2}{m_F^2}\right)\lambda(1 - \lambda)}} \right| \right] A_s(\lambda).$$

Note that this result is written in terms of dimensionless variables. We can read off $B_s(\lambda)$ as well,

$$B_s(\lambda) = \frac{1 - 2\lambda}{\lambda(1 - \lambda)} - \frac{\mu_2 Y}{2\pi m_F^{3/2}} \sqrt{1 - 4\lambda(1 - \lambda)} \ln \left| \frac{1 - \sqrt{1 - 4\lambda(1 - \lambda)}}{1 + \sqrt{1 - 4\lambda(1 - \lambda)}} \right| A_s(\lambda). \tag{91}$$

The reader can check that the above results actually reduce to the phi-four theory results we have found if we set $\mu_2 Y = 0$.

For the sake of completeness we will also present the scattering solutions for $c_Q(x)$ variables. Below, we use the same shorthand symbols $A(\lambda)$ and $B_s(\lambda)$ as in the bound state equation for $c_Q$: The renormalized scattering equations become,

$$c_Q(x; \lambda) = \delta(x - \lambda) - \mathcal{P} \frac{1}{\Lambda^2 x^2 - (\Lambda^2 - m_B^2 + m_F^2)x + m_F^2} \sqrt{1 - x} [(aA(\lambda) + bB_s(\lambda))x + bA(\lambda)], \tag{92}$$

we should set

$$\Lambda^2 = \frac{m_F^2}{\lambda} + \frac{m_B^2}{1 - \lambda}, \tag{93}$$

not surprisingly $m_F + m_B^2 \leq \Lambda < \infty$, and $A(\lambda), B_s(\lambda)$ found from their definitions, when we simplify the result it becomes,

$$A(\lambda) = \left[ 1 - \frac{4\kappa}{\Lambda^2} F_2 + \left( F_2 \frac{\mu_Y^2}{64\Lambda^2} - \frac{\mu_Y^2}{64\Lambda^2} (\Lambda^2 - m_R^2 + m_F^2) F_1 + \frac{\mu_Y m_F^2 \kappa}{\Lambda^2} F_1 \right) \left( 1 + \frac{\mu_Y^2 m_F^2}{4 \Lambda^2} F_1 \right)^{-1} F_2 \right]^{-1}$$

$$\times \left[ \frac{1}{\sqrt{1 - \lambda}} \left[ 1 + \frac{\mu_Y^2 m_F^2}{32 \Lambda^2} \left( 1 + \frac{\mu_Y m_F^2}{4 \Lambda^2} F_1 \right)^{-1} F_2 \right] \right.$$ 

$$B_s(\lambda) = \left( 1 + \frac{\mu_Y m_F^2}{4 F_1} \right)^{-1} \left[ \frac{1}{\lambda \sqrt{1 - \lambda}} + \left( 8\kappa F_1 - \frac{\mu_Y^2}{8 m_F^2} (\Lambda^2 - m_R^2 + m_F^2) F_1 + \frac{\mu_Y}{8 m_F^2} F_2 \right) A(\lambda) \right].$$

where

$$F_2 = \ln \left| \frac{m_B^2}{m_F^2} \right|, \quad F_1 = \frac{1}{u} \ln \left| \frac{\Lambda^2 + m_B^2 - m_F^2 + u}{\Lambda^2 + m_B^2 - m_F^2 - u} \right|, \tag{94}$$
here \( u = \sqrt{(\Lambda^2 - (m_F - m_{BR})^2)(\Lambda^2 - (m_F + m_{BR})^2)} \).

These define the wave operators of our model, as discussed in the simpler model of phi-four coupling. We see that the results are fairly complex expressions. One should study various approximate forms of these equations, and a numerical investigation of the poles of the amplitudes should give information about the bound states.

Our present approach has one weakness, our results are nontrivial since the fermions have nonzero mass. This makes it sensitive to the sign of the coupling—treated them as positive, but the results are valid if we simply assume them to be negative. Physically more interesting case would be to study the massless fermions. If we set \( m_F = 0 \) all the interesting information we have is lost in the linear approximation, and we should go beyond this. This observation suggests that it is necessary to study some kind of variational approach to understand the system better. We plan to investigate this in the future.

4 Gauged model

We will assume in this part that the model is gauged by introducing \( SU(N_c) \) Lie algebra valued gauge potentials \( A_\mu \) and refer for our conventions to \([18]\). In the light-cone coordinates, and setting \( A_- = 0 \), the action becomes,

\[
S_Y = \int dx^+ dx^- \left[ -\frac{1}{2} Tr F_+ F^- + \sqrt{2} \psi^* \partial_- \psi L_0 + i \sqrt{2} \psi^*(\partial_+ + igA_+)^\beta_\alpha \psi_{R\beta} \right.
\]

\[
-2 \phi^{*\alpha} \partial_+ \phi_\alpha + ig(\partial_- \phi^{*\alpha} A_{+\alpha}^\beta \phi_\beta - \phi^{*\alpha} A_{+\alpha}^\beta \partial_- \phi_\beta) - m^2 B_0 \phi^{*\alpha} \phi_\alpha - \frac{\lambda^2 B_0}{4} (\phi^{*\alpha})^2
\]

\[
-(\psi^*_L \psi_{R\beta} + \psi^*_R \psi_{L\beta})(\mu_1 \phi^{*\beta} \phi_\alpha + \mu_2 \phi^{*\lambda} \phi_\delta \delta_\beta + m_F \delta_\alpha^\beta) \right].
\] (95)

The restriction to the color invariant states in the gauge theory is actually necessary to make the Hamiltonian finite. For this theory the large-\( N_c \) limit should be a better approximation. Furthermore one expects baryons in this theory, the geometry of the large-\( N_c \) phase space should be useful to find a variational ansatz (see \([19]\) for a nice discussion of these ideas). Following the same reduction process, the Hamiltonian becomes

\[
H = H_0 + H_Y + H_G,
\] (96)

where,

\[
H_0 = \frac{1}{4} \left( m_F^2 - \frac{g^2}{\pi} \right) \int \frac{|dp|}{p} M(p,p) + \frac{1}{4} \left( m_{BR}^2 - \frac{g^2}{\pi} \right) \int \frac{|dp|}{|p|} N(p,p),
\] (97)

\( H_Y \) is as given in equation (13), and the gauge contribution is exactly given in \([18]\),

\[
H_G = \frac{-g^2}{16} \int [dpdqdsdt] \left( \frac{1}{(p-t)^2 + \frac{1}{(q-s)^2}} \right) \delta[p+s-t-q] M(p,q) M(s,t)
\]

\[
+ \frac{g^2}{64} \int [dpdqdsdt] \left( \frac{1}{(p-t)^2 + \frac{1}{(q-s)^2}} \right) \delta[p+s-t-q] \frac{qt + ps + st + pq}{\sqrt{|pqst|}} N(p,q) N(s,t)
\]

\[
+ \frac{g^2}{8} \int [dpdqdsdt] \frac{q+s}{(q-s)^2} \frac{\delta[p+s-t-q]}{\sqrt{|qs|}} Q(p,q) Q(s,t),
\]
Above we rescaled our coupling constants by a factor of $N_c$ as before and $g^2 N_c \mapsto g^2$. Let us use exactly the same substitutions as before for the basic variables we have, and simplify the resulting equations into,

$$
\Lambda_M^2 \xi_M(x) = \left( m_F^2 - \frac{g^2}{\pi} \right) \left( \frac{1}{x} + \frac{1}{1-x} \right) \xi_M(x) - \frac{g^2}{\pi} \int_0^1 \frac{dy}{(y-x)^2} \xi_M(y) - \frac{\mu_2 m_F}{4\pi} \left[ \frac{1}{x} - \frac{1}{1-x} \right] \int_0^1 \frac{\xi_N(y)}{\sqrt{y(1-y)}} dy.
$$

and

$$
\Lambda_N^2 \xi_N(x) = \left( m_{RB}^2 - \frac{g^2}{\pi} \right) \left( \frac{1}{x} + \frac{1}{1-x} \right) \xi_N(x) - \frac{g^2}{4\pi} \int_0^1 \frac{dy}{(y-x)^2} \frac{(x+y)(2-x-y)}{\sqrt{x(1-x)y(1-y)}} \xi_N(y) - \frac{\mu_2 m_F}{8\pi} \int_0^1 \frac{dy}{\sqrt{x(1-x)y(1-y)}} \xi_M(y) + \frac{\lambda_R^2}{\sqrt{x(1-x)y(1-y)}}. \quad (98)
$$

Again we see that the equations for $\xi_M$ and $\xi_N$ are coupled and they should be solved together.

$$
\Lambda_Q^2 c_Q(x) = \left[ \left( m_F^2 - \frac{g^2}{\pi} \right) \frac{1}{x} + \left( m_{BR}^2 - \frac{g^2}{\pi} \right) \frac{1}{1-x} \right] c_Q(x) - \frac{g^2}{2\pi} \int_0^1 \frac{dy}{(y-x)^2} \frac{2-x-y}{\sqrt{(1-x)(1-y)}} c_Q(y) - 8\kappa \int_0^1 \frac{dy}{\sqrt{(1-y)(1-x)}} c_Q(y) + \frac{1}{4} \mu_1 y m_F \int_0^1 \frac{dy}{\sqrt{(1-y)(1-x)}} \left( \frac{1}{x} + \frac{1}{y} \right) c_Q(y). \quad (99)
$$

These singular integral equations can perhaps be investigated numerically. The linear approximation could be a better one for the gauged model, since the effect of the gauge interaction is to bring a singular operator. The non-gauged models require renormalizations, it is possible that the above equations will behave better due to the singular operators in them. Another important application is to find a variational ansatz for the baryonic solutions and have a linear expansion around these solutions. We plan to study these issues in more depth in the future.

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6 Appendix: Reduction of the Hamiltonian

In this appendix we will give some of the details of the reduction of the Hamiltonian to the desired color invariant products. Let us recall that the Hamiltonian is given in equation number (15). Let us start with the term,

$$\frac{\sqrt{2}}{2} \mu_{1Y}^2 \int dx^{-} \hat{\psi}^{i\alpha} \hat{\phi}_{\beta} \hat{\phi}^{+i\alpha} \frac{1}{i\partial_+} \hat{\phi}_\alpha \hat{\phi}^{\dagger i\lambda} \hat{\psi}_\lambda.$$  

(100)

When we write this in terms of the Fourier mode expansions, it becomes,

$$-\mu_{1Y}^2 \int \frac{dkdltpdqds}{8\sqrt{[stqk]}} \delta[k-l+t-s+q-p] \chi^{i\alpha}(p)a_\alpha(q) : a^{i\beta}(s)a_\beta(t) : a^{i\lambda}(k)\chi_\lambda(l)$$

$$-\mu_{1Y}^2 N_c \int \frac{dkdldqds}{16\sqrt{|qk|}} \chi^{i\alpha}(p)a_\alpha(q)a^{i\lambda}(k)\chi_\lambda(l)\delta[k-l+q-s] \left( \int [ds] \frac{\text{sgn}(s) - 1}{s(k-l-s)} \right).$$

Notice that the divergent integral is isolated and a principal value regularization calculation shows that

$$\mathcal{P} \int [ds] \frac{\text{sgn}(s) - 1}{s(k-l-s)} = \frac{1}{\pi(k-l)} \ln \frac{k-l}{\epsilon_R},$$  

(101)

where $\epsilon_R$ is an infrared cut-off. If we are only allowed to introduce local counter terms in the original action we should introduce a momentum scale $\mu_R$ so that we can separate the momentum dependent part and purely divergent part of these type expressions: (this point is somewhat difficult to decide for this particular term since it is not possible to write such a term in the original action)

$$\mathcal{P} \int [ds] \frac{\text{sgn}(s) - 1}{s(k-l-s)} = \frac{1}{\pi(k-l)} \ln \frac{k-l}{\epsilon_R} = \frac{1}{\pi(k-l)} \left( \ln \left| \frac{k-l}{\mu_R} \right| + \ln \left| \frac{\mu_R}{\epsilon_R} \right| \right).$$  

(102)

If we remove the divergent part by a counter term of the form

$$\left( \frac{\mu_{1Y}^2 N_c}{16\pi} \ln \frac{\mu_R}{\epsilon_R} + 8\kappa_R(\mu_R) \right) \int \frac{dpdqdkdl}{\sqrt{|qk|}} \frac{1}{k-l} \delta[p-q+k-l] \chi^{i\alpha}(p)a_\alpha(q)a^{i\beta}(k)\chi_\beta(l).$$  

(103)

the finite term comes out to be

$$\int \frac{dkdldqds}{\sqrt{|qk|}} \left[ 8\kappa_R(\mu_R) - \frac{\mu_{1Y}^2}{8\pi} \ln \left| \frac{k-l}{\mu_R} \right| \right] \frac{1}{q-s} \chi^{i\alpha}(k)a_\alpha(l)a^{i\beta}(q)\chi_\beta(s)\delta[k-l+q-s].$$  

(104)

If we require the theory not to have a dependence on the arbitrary scale we introduced, it is natural to demand that the residual coupling to vary under a change of scale according to

$$\kappa_R(\mu_R) = \kappa_R(\mu_R) - \frac{\mu_{1Y}^2}{64\pi} \ln \left| \frac{\mu_R}{\mu_R} \right|.$$

(105)

The reader may be alarmed by the nonlocal expression in the interaction, but if we actually go back to the position space, the inverse Fourier transform gives a term, up to some constants,

$$\int dxdy \left( a\text{sgn}(x-y) - \ln |\mu_R(x-y)|\text{sgn}(x-y) \right) \hat{\psi}^{i\alpha}(x) \hat{\phi}_\alpha(y) \hat{\phi}^{i\beta}(y) \hat{\psi}_\beta(x),$$  

(106)
which has a logarithmic correction to the sign function. (This behaves worse for the shortdistance than the coulomb potential $|x-y|$, but we should interpret $\text{sgn}(x-y) = 0$ if $x = y$, so there is no singularity at the short distance.) In the text we will only consider the cases where these non-local terms dropped, or removed by taking them as part of the counter terms in the action.

(Notice that this is the interaction one finds if we use a parity broken model as in [18]. The sign of the remaining interaction term is not determined since it is not in the original action it should be left as an arbitrary parameter).

As another example we will discuss the term,

$$\frac{\sqrt{2}}{2} m_F \mu_{2Y} \int dx^- \frac{1}{i \partial_+} \psi_+ \phi^\sigma \phi_\sigma,$$

(107)

a Fourier expansion and removing the vacuum expectation value gives us a term

$$-\frac{\mu_{2Y} m_F}{4} \int \frac{[dpdqdsdt]}{|st|} \frac{\delta[p-q+s-t]}{s-t-q} \chi^{ta}(p) \chi_\alpha(q) : a^{t\beta}(s)a_\beta(t) : + \mu_{2Y} N_c m_F \int \frac{[ds]}{|s|} : a^{t\beta}(s)a_\beta(s) : \int [dp] \frac{1-\text{sgn}(p)}{p},$$

and

$$\mathcal{P} \int [dp] \frac{1-\text{sgn}(p)}{p} = -2 \ln \left| \frac{\Lambda_R}{\epsilon_R} \right|,$$

(108)

where we have $\epsilon_R$ and $\Lambda_R$ as the infrared and ultraviolet cut-offs respectively. If we introduce a boson mass counter of the form

$$\frac{\mu_{2Y} N_c m_F}{4} \ln \left| \frac{\Lambda_R}{\epsilon_R} \right| \int \frac{[ds]}{|s|} : a^{t\beta}(s)a_\beta(s) :,$$

(109)

this term will be cancelled.

The other terms are also done in the same way, and the rest is to collect all terms to find the Hamiltonian in terms of large-$N_c$ bilinears.

References

[1] E. Abdalla, M. C. B. Abdalla and K. D. Rothe, Nonperturbative methods in two dimensional quantum field theory, World Scientific, UK (2001).

[2] M. Cavicchi, A bilocal field approach to the large-$N$ expansion of two dimensional (gauge) theories, Intr. Jour. Mod. Phys. A 10 (1995) 167.

[3] J. Glimm and A. Jaffe, The Yukawaz theory quantum field theory without cutoffs, Jour. Funct. Analysis 7 (1971) 323.

[4] R. Schrader, A Yukawa quantum field theory in two spacetime dimensions without cutoffs, Ann. Phys. 70 (1972) 412.
[5] E. Seiler, *Schwinger functions for the Yukawa model in two dimensions with space-time cutoff*, Comm. Math. Phys. 42 (1975) 163.

[6] A. Lesniewski, *Effective action for the Yukawa$_2$ quantum field theory* Comm. Math. Phys. 108 (1987) 437.

[7] K. Gawedzki and A. Kupiainen, *Gross-Neveu model through convergent perturbation expansions*, Comm. Math. Phys. 102 (1985) 1.

[8] H-C. Pauli and S. J. Brodsky, *Solving field theory in one space and one time dimension*, Phys. Rev. D 32, (1985) 1993.

[9] H-C. Pauli and S. J. Brodsky, *Discretized light-cone quantization: solution to a field theory in one space and one time dimension*, Phys. Rev. D 32, (1985) 2001.

[10] R. J. Perry and A. Harindranath, *Renormalization in the light-front Tamm-Dancoff approach to field theory*, Phys. Rev. D 43, (1991) 4051.

[11] S. Glazek, A. Haindranath, S. Pinsky, J. Shigemitsu, and K. Wilson, *Relativistic bound-state problem in the light-front Yukawa model*, Phys. Rev. D 47 (1993) 1599.

[12] M. Mangin-Brinet, J. Carbonell and V. A. Karmanov, *Relativistic bound states in the Yukawa model*, Phys. Rev. D 64 (2001) 125005.

[13] N. C. J. Schoonderwoerd and B. L. G. Baker, *Equivalence of renormalized covariant and light-front perturbation theory I and II*, Phys. Rev. D 57 (1998) 4965, Phys. Rev. D 58 (1998) 025013.

[14] S. Brodsky, H-C. Pauli and S. Pinsky, *QCD and other field theories on the light cone*, Phys. Report. 301 (1998) 299.

[15] R. J. Perry, *A renormalization group approach to Hamiltonian light-front field theory*, Annals Phys. 232 (1994) 116.

[16] S. G. Rajeev, *Quantum hadrondynamics in two dimensions*, Int. J. Mod. Phys.A 9 (1994) 5583.

[17] A. Dhar, G. Mandal, and S. R. Wadia, Nuc. Phys. B 436 (1994) 487.

[18] A. Konechny and O. T. Turgut, *Supergrassmannian and large N limit of quantum field theory with bosons and fermions*, Jour. Math. Phys. 43 (2002) 2988.

[19] S. G. Rajeev, *Derivation of Hadronic Structure Functions from QCD*, Conformal Field Theory, edited by Y. Nutku, C. Saclioglu, O. T. Turgut, Perseus Publishing, New York, 2000.

[20] G. ‘t Hooft, *A two dimensional model for mesons*, Nuc. Phys. B 75 (1974) 461.

[21] S. S. Shei and H-S. Tsao, *Scalar quantum Chromodynamics in two dimensions and the parton model*, Nucl. Phys. B 141 (1978) 445.

[22] T. N. Tomaras, *Scalar U(N) QCD in the large-N limit*, Nuc. Phys.B 163 (1980) 79.
[23] K. Aoki, *Boson-fermion bound states in two dimensional QCD*, Phys. Rev. D 49 (1994) 573.

[24] K. Aoki and T. Ichihara, *1+1 dimensional QCD with fundamental bosons and fermions*, Phys. Rev. D 52 (1995) 6453.

[25] T. Heinzl, *Light cone quantization: foundations and applications*, in *methods of quantization*, Proceedings of 39th Schlading winter school. [hep-th 0008096](http://arxiv.org/abs/hep-th/0008096).

[26] S-J. Chang, R. G. Root, and T-M. Yan, *Quantum field theories in infinite momentum frame. I. Quantization of scalar and Dirac fields*, Phys. Rev. D 7 (1973) 1133.

[27] A. Harindranath, *An introduction to light front dynamics for pedestrians*, Published in Light-Front Quantization and Non-Perturbative QCD, J.P. Vary and F. Woelz (eds.), International Institute of Theoretical and Applied Physics, ISU, Ames, IA 50011, U.S.A. ISBN: 1-891815-00-8.

[28] L. Yaffe, *Large-N limits as classical mechanics*, Rev. Mod. Phys. 54 (1982) 407.

[29] J. Gracia-Bondia, *Improved Epstein- Glaser renormalization in coordinate space I*, [hep-th 0202023](http://arxiv.org/abs/hep-th/0202023).

[30] M. Reed and B. Simon, *Methods of modern mathematical physics, vol III, scattering theory*, Academic Press, San Diego (1979).

[31] R. G. Newton, *Scattering theory of waves and particles* McGraw-Hill, USA (1966).

[32] S. Albeverio and P. Kurasov, *Singular perturbations of differential operators*, London Mathematical Society Lecture Series 271, Cambridge, United Kingdom (2000).