POWERS OF SUMS AND THEIR HOMOLOGICAL INVARIANTS

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Dedicated to the memory of Diana Taylor (1941–2016)

Abstract. Let \( R \) and \( S \) be standard graded algebras over a field \( k \), and \( I \subseteq R \) and \( J \subseteq S \) homogeneous ideals. Denote by \( P \) the sum of the extensions of \( I \) and \( J \) to \( R \otimes_k S \). We investigate several important homological invariants of powers of \( P \) based on the information about \( I \) and \( J \), with focus on finding the exact formulas for these invariants. Our investigation exploits certain Tor vanishing property of natural inclusion maps between consecutive powers of \( I \) and \( J \). As a consequence, we provide fairly complete information about the depth and regularity of powers of \( P \) given that \( R \) and \( S \) are polynomial rings and either \( \text{char } k = 0 \) or \( I \) and \( J \) are generated by monomials.

1. Introduction

Let \( R \) be a standard graded algebra over a field \( k \), i.e. \( R \) is a noetherian \( \mathbb{N} \)-graded ring with \( R_0 = k \) and \( R \) is generated as an algebra over \( k \) by elements of degree 1. Let \( \mathfrak{m} \) be the unique graded maximal ideal of \( R \) and \( I \subseteq \mathfrak{m} \) a homogeneous ideal of \( R \). Studying the asymptotic behavior of the algebraic invariants associated to \( I \) has been an important problem and has attracted much attention of commutative algebraists and algebraic geometers. A classical result states that the Hilbert–Samuel function associated to any \( \mathfrak{m} \)-primary ideal is eventually a polynomial function. Another well–known result due to Brodmann [6] establishes, for \( s \gg 0 \), that depth \( I^s \) is a constant depending only on \( I \).

In another direction, by results due to Cutkosky, Herzog, Kodiyalam, N.V. Trung and Wang [16, 38, 49], the Castelnuovo–Mumford regularity \( \text{reg } I^s \) is a linear function of \( s \) for \( s \gg 0 \). By definition, for a finitely generated graded \( R \)-module \( M \), its Castelnuovo–Mumford regularity is \( \text{reg } M = \sup \{ i + j : H^i_m(M)_j \neq 0 \} \) where \( H^i_m(M) \) denotes the \( i \)th local cohomology of \( M \) with support in \( \mathfrak{m} \). For more information on asymptotic properties of powers of ideals, see, e.g., [5, 9, 10, 11, 20, 21, 27, 39, 46] and their references.

In this paper, we study several homological invariants of powers of sums of ideals. In detail, let \( R \) and \( S \) be standard graded \( k \)-algebras, \( I \) and \( J \) non-zero, proper homogeneous ideals of \( R \) and \( S \), respectively. Denote by \( P \) the sum \( I + J \subseteq T = R \otimes_k S \), where \( I \) and \( J \) are regarded as ideals of \( T \). To avoid lengthy phrases later on, we call \( P \) the mixed sum of \( I \) and \( J \). We consider the problem of characterizing (asymptotically) the homological invariants of powers of \( P \), including the projective dimension, regularity, in terms of the data of \( I \) and \( J \). Besides the
theory of asymptotic homological algebras of powers of ideals, another source of our motivation comes from recent work of Hâ, N.V. Trung and T.N. Trung [25] on the depth and regularity of powers of $P$ in the case $R$ and $S$ are polynomial rings. While influenced by the last paper, the method of this paper is conceptually more transparent and yields more precise results under reasonable assumptions.

Our method makes substantial use of Betti splittings, first introduced by Francisco, Hâ and Van Tuyl [23] in their study of free resolutions of monomial ideals. One of the main findings of this paper is that Betti splittings are ubiquitous and most relevant to study all the powers of mixed sums. The decomposition of $P$ as $I + J$ is probably the easiest example of a Betti splitting: letting $\beta_i(I) = \dim_k \operatorname{Tor}_i^R(k, I)$ be the $i$th Betti number of $I$, then the formula $\beta_i(P) = \beta_i(I) + \beta_i(J) + \beta_{i-1}(I \cap J)$ holds for any $i \geq 0$. These equations define Betti splittings in general and allow us to give a fairly complete understanding of the depth and regularity of the powers of $P$. We can prove, rather surprisingly, that the inequalities for depth $T/P^s$ and reg $T/P^s$ in [25, Theorem 2.4] are both equalities under certain conditions.

**Theorem 1.1.** Let $R$ and $S$ be polynomial rings over $k$. Suppose that either $\operatorname{char} k = 0$ or $I$ and $J$ are monomial ideals. Then for any $s \geq 1$, there are equalities

(i) $\operatorname{depth} T/P^s = \min_{i \in [1, s-1], j \in [1, s]} \{ \operatorname{depth} R/I^{s-i} + \operatorname{depth} S/J^j + 1, \operatorname{depth} R/I^{s-j+1} + \operatorname{depth} S/J^j \}$,

(ii) $\operatorname{reg} T/P^s = \max_{i \in [1, s-1], j \in [1, s]} \{ \operatorname{reg} R/I^{s-i} + \operatorname{reg} S/J^j + 1, \operatorname{reg} R/I^{s-j+1} + \operatorname{reg} S/J^j \}$.

Theorem 1.1 is surprising since normally we only expect to get hold of information about large enough powers of an ideal. Prior to this result, the authors of [25] expressed in page 820 the view that “...there are no general formulae for the depth and regularity of $T/P^s$ (even when $R$ and $S$ are polynomial rings). Thanks to Theorem 1.1, we can also provide an upper bound for the stability index of depth of powers of $P$, and study when $P$ has a constant depth function (Section 5.3). Our results generalize or strengthen previous work of Hâ–Trung–Trung [25], Herzog–Vladoiu [34]. We expect Theorem 1.1 to hold in positive characteristics as well (but by Example 5.3 it does not hold over all non-polynomial base rings).

We do not restrict our considerations of mixed sums to the setting of polynomial base rings. To study Betti splittings of higher powers of $P$, we introduce in Section 4 the following notion.

**Definition 1.2.** We say $I$ is an ideal of small type if for all $s \geq 1$ and all $i \geq 0$, the map $\operatorname{Tor}_i^R(k, P^s) \to \operatorname{Tor}_i^R(k, P^{s-1})$ induced by the natural inclusion $I^s \to I^{s-1}$ is zero.

One of the main results of Section 4 is Theorem 4.10: if $I$ and $J$ are of small type then the powers of $P$ admit natural Betti splittings. While simple, Theorem 4.10 is enough to produce the formulas of Theorem 1.1 since if $R$ and $S$ are polynomial rings then $I$ and $J$ are of small type if either $\operatorname{char} k = 0$ (Ahangari Maleki [1, Theorem 2.5(i)]) or $I$ and $J$ are monomial ideals (Theorem 4.5).

Another goal of this paper is to study the linearity defect of powers of mixed sums. The linearity defect, introduced by Herzog and Iyengar [32] measures the failure of minimal free resolutions to be linear (see Section 2 for more details). From
the theory of componentwise linear ideals initiated by Herzog and Hibi [26], linearity defect is a natural invariant: over a polynomial ring, componentwise linear ideals are exactly ideals with linearity defect zero (see [45, Section 3.2]). The linearity defect, as can be expected, behaves well asymptotically. Denote by $\text{ld}_R M$ the linearity defect of a finitely generated graded $R$-module $M$. We prove in [42, Theorem 1.1] that if $R$ is a polynomial ring, then the sequence $(\text{ld}_R I^s)_{s \geq 1}$ is eventually constant. To understand the meaning of the limit $\lim_{s \to \infty} \text{ld}_R I^s$, one should naturally look for some computations of the asymptotic linearity defect. This is the original motivation of this paper which led to the discovery of Betti splittings of powers of mixed sums. The main results of Section 6 yield the following computational statement.

**Theorem 1.3** (Corollary 6.7). Let $(R, m)$ and $(S, n)$ be polynomial rings over $k$, $(0) \neq I \subseteq m^2$ and $(0) \neq J \subseteq n^2$ homogeneous ideals. Suppose that one of the following conditions holds:

(i) $\text{char} k = 0$ and $I$ and $J$ are non-trivial powers of some homogeneous ideals of $R$ and $S$, resp.

(ii) $I$ and $J$ are non-trivial powers of some monomial ideals of $R$ and $S$, resp.

(iii) All the powers of $I$ and $J$ are componentwise linear.

Then for all $s \gg 0$, we have an equality

$$\text{ld}_T P^s = \max \left\{ \lim_{i \to \infty} \text{ld}_R I^i + \max_{j \geq 1} \text{ld}_S J^j + 1, \max_{i \geq 1} \text{ld}_R I^i + \lim_{j \to \infty} \text{ld}_S J^j + 1 \right\}.$$ 

We believe that Theorem 1.3 is still true without the extra assumptions (i), (ii) and (iii).

The paper is organized as follows. Section 2 is devoted to the necessary background and useful facts. In Section 3, we introduce Betti splittings and the closely related notion of Tor-vanishing morphisms between modules. Our study of powers of mixed sums rests on the fact that several homological invariants behave well with respect to Betti splittings and Tor-vanishing morphisms. In Section 4, we study ideals of small type and the subclass of ideals of doubly small type. Theorem 1.3 is available thanks to the formula for $\text{ld}_T P^s$ when $I$ and $J$ are of doubly small type (Theorem 6.1(ii)). The first main result of Section 4 is Theorem 4.5 establishing that any proper monomial ideal of a polynomial ring is of small type. The second main result of this section is Theorem 4.10 which, among other things, establishes Betti splittings for powers of mixed sums whose summands are of small type. We study in Sections 5 and 6 the homological invariants, including projective dimension, regularity and linearity defect, of powers of mixed sums. The main focus in both sections is to provide exact formulas for these invariants. We also compute the asymptotic values of these invariants whenever possible. Some applications to the theory of ideals with constant depth functions and the index of depth stability are given at the end of Section 5.

Remark. Some materials of this paper originally belong to a preprint titled “Linearity defects of powers are eventually constant” [41]. After further thoughts, we split the last preprint into three parts. The first part establishes the asymptotic constancy of the linearity defect of powers [42]. This paper is the second part, which contains Section 4 of [41] but goes significantly beyond that. A third part which contains the results in the last section of [41] is in preparation.
2. Background

We begin with some basic notions and facts that will be used later. Standard knowledge of commutative algebra may be found in [8], [17]. For the theory of free resolutions, we refer to [4] and [44].

2.1. Linearity defect. Let \((R, m, k)\) be a noetherian ring which is one of the following:

(i) a local ring with the maximal ideal \(m\) and the residue field \(k = R/m\),

(ii) a standard graded algebra over a field \(k\) with the graded maximal ideal \(m\).

We usually omit \(k\) and write simply \((R, m)\). Let \(M\) be a finitely generated \(R\)-module. If \(R\) is a graded algebra, then the \(R\)-modules that we will study are assumed to be graded, and various structures concerning them, e.g. their minimal free resolutions are also taken in the category of graded \(R\)-modules and degree preserving homomorphisms.

Let \(F\) be the minimal free resolution of \(M\):

\[ F : \cdots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0. \]

For each \(i \geq 0\), the minimality of \(F\) gives rise to the following subcomplex of \(F\):

\[ F^i F : \cdots \longrightarrow F_{i+1} \longrightarrow F_i \longrightarrow mF_{i-1} \longrightarrow \cdots \longrightarrow m^{i-j}F_j \longrightarrow \cdots . \]

Following Herzog and Iyengar [32], we define the so-called linear part of \(F\) by the formula

\[ \text{lin}^R F := \bigoplus_{i=0}^{\infty} \frac{\mathcal{F}^i F}{\mathcal{F}^{i+1} F}. \]

Prior to the work of Herzog and Iyengar, the linear part was also studied by Eisenbud et al [18] in the graded setting. Observe that \((\text{lin}^R F)_i = (\text{gr}_m F_i)(-i)\) for every \(i \geq 0\), where \(\text{gr}_m M = \bigoplus_{i \geq 0} m^i M/(m^{i+1} M)\). Note that \(\text{lin}^R F\) is a complex of graded modules over \(\text{gr}_m R\). The construction of \(\text{lin}^R F\) has a simple interpretation in the graded case. For each \(i \geq 1\), apply the following rule to all entries in the matrix representing the map \(F_i \longrightarrow F_{i-1}\): keep it if it is a linear form, and replace it by 0 otherwise. Then the resulting complex is \(\text{lin}^R F\).

Following [32], the linearity defect of \(M\) is

\[ \text{ld}_R M := \sup\{ i : H_i(\text{lin}^R F) \neq 0 \}. \]

If \(M \cong 0\), we set \(\text{ld}_R M = 0\).

Most results on the linearity defect of the present paper are built upon the following theorem.

Theorem 2.1 (Sèga, [47, Theorem 2.2]). Let \(d \geq 0\) be an integer. The following statements are equivalent:

(i) \(\text{ld}_R M \leq d\);

(ii) The natural morphism \(\text{Tor}_i^R(R/m^{q+1}, M) \longrightarrow \text{Tor}_i^R(R/m^q, M)\) is zero for every \(i > d\) and every \(q \geq 0\).

If a module has linearity defect 0, then as in [32], we also say that it is a Koszul module. Note that by our convention, the trivial module \((0)\) is a Koszul module. The ring \(R\) is called Koszul if \(\text{ld}_R k = 0\). In the graded case, \(R\) is said to be a Koszul algebra if \(\text{reg}_R k = 0\). These two notions are compatible in the graded case by [32,
Proposition 1.13]. Moreover by [32, Remark 1.10], a noetherian local ring \((R, \mathfrak{m})\) is Koszul if and only if \(\text{gr}_R \mathfrak{m}\) is a Koszul algebra.

2.2. Regularity. Let \(R\) be a standard graded \(k\)-algebra, \(M\) a finitely generated graded \(R\)-module. The number \(\beta_{i,j}(M) = \dim_k \text{Tor}^R_i(k, M)_j\) is called the \((i, j)\) graded Betti number of \(M\) and \(\beta_i(M) = \dim_k \text{Tor}^R_i(k, M)\) its \(i\)th Betti number. We define the regularity of \(M\) over \(R\) as follows

\[
\text{reg}_R M = \sup\{j - i : \beta_{i,j}(M) \neq 0\}.
\]

If \(\text{ld}_R M < \infty\) then by [2, Proposition 3.5], there is an equality

\[
\text{reg}_R M = \sup_{0 \leq i \leq \text{ld}_R M} \{j - i : \text{Tor}^R_i(k, M)_j \neq 0\}.
\]

In particular, if \(M\) is a Koszul module then \(\text{reg}_R M\) equals the maximal degree of a minimal homogeneous generator of \(M\).

Recall that \(M\) is said to have a \(d\)-linear resolution (where \(d \in \mathbb{Z}\)), if for all \(i \in \mathbb{Z}\) and all \(j \neq d\), it holds that \(\text{Tor}^R_i(k, M)_{i+j} = 0\). If \(M\) has a \(d\)-linear resolution, then necessarily \(M\) is generated in degree \(d\) and \(\text{reg}_R M = d\).

Following Herzog and Hibi [26], \(M\) is said to be \(\text{componentwise linear}\) if for each \(d\), the submodule generated by homogeneous elements of degree \(d\) in \(M\) has a \(d\)-linear resolution. If \(R\) is a Koszul algebra, then we have the following implications, with the first one being strict in general:

\[
M \text{ has a linear resolution} \implies M \text{ is componentwise linear} \implies M \text{ is Koszul}.
\]

The equivalence is due to Römer [45, Theorem 3.2.8]. Throughout, we use the term “Koszul module” instead of “componentwise linear module” to streamline the exposition.

The \(\text{Castelnuovo–Mumford regularity}\) of \(M\) measures the degree of vanishing of its local cohomology modules: \(\text{reg} M = \sup\{i + j : H^i_m(M)_j \neq 0\}\), where \(H^i_m(M)\) denotes the \(i\)th local cohomology module of \(M\).

2.3. Elementary lemmas. The identities in the next lemma extend [35, Lemma 1.1] and [25, Proposition 3.2]. They are the basis of most discussions about mixed sums in this paper. Note that the proof of [25, Proposition 3.2] is defective since the authors assume following false claim: Let \(M_1, \ldots, M_p, N\) are submodules of an ambient module \(M_0\) over a ring \(R\) (where \(p \geq 1\)) such that \(M_i \cap M_j \subseteq N\) for all \(i \neq j\). Denote \(M = M_1 + \cdots + M_p\). Then the module \((M + N)/N\) admits a direct sum decomposition \((M + N)/N = \bigoplus_{i=1}^p (M_i + N)/N\).

**Lemma 2.2.** Let \(R, S\) be affine \(k\)-algebras. Let \(I, J\) be ideals of \(R, S\), resp., and \(P = I + J \subset T = R \otimes_k S\). Then for all \(p, q, r, s \geq 0\), there are identities:

\[
\begin{align*}
I^{p+r} J^q & \cap I^r J^{s+q} = I^{p+r} P^q \cap I^r J^{s+q} = I^{p+r} J^{s+q}, \\
\frac{I^r P^s}{I^{r+i} P^{s+i}} & \cong \bigoplus_{i=0}^s \left( \frac{I^{r+i}}{I^{r+i+1}} \otimes_k \frac{J^{s-i}}{J^{s-i+1}} \right).
\end{align*}
\]

**Proof.** (2.1) First we treat the case \(p = s = 1, r = q = 0\). Since \(k\) is a field, we have the following identities

\[
I \cap J = (I \otimes_k S) \cap (R \otimes_k J) = (I \cap R) \otimes_k (S \cap J) = I \otimes_k J = IJ.
\]
Generally for all ideals $I$ and $J$, we have

$$I^{r+i}, J^{s-i} \subseteq I^{r+i} P^q \cap I^{r+i} J^{s-i} \subseteq I^{r+i} P^q \cap I^{r+i} J^{s-i} = I^{r+i} J^{s-i}. \tag{2.5}$$

The last equality in the display follows from the case treated above. Hence all the inclusions are in fact equalities.

(2.2) We have $I^r P^s = \sum_{i=0}^{s} I^{r+i} J^{s-i}$. We claim that for $0 \leq i \leq s$, the following inclusion holds

$$I^{r+i} J^{s-i} \cap \left( \sum_{0 \leq j \leq s, j \neq i} I^{r+j} J^{s-j} + I^r P^{s+1} \right) \subseteq I^{r+i} P^{s+1}. \tag{2.3}$$

Note that $P^{s+1} \subseteq I^{r+i} + J^{s-i+1}$, so $I^r P^{s+1} \subseteq I^{r+i+1} + J^{s-i+1}$. Moreover $J^{s-j} \subseteq J^{s-i+1}$, $I^{r+i} \subseteq I^{r+i+1}$ if $j < i < t$. Hence

$$I^{r+i} J^{s-i} \cap \left( \sum_{0 \leq j \leq s, j \neq i} I^{r+j} J^{s-j} + I^r P^{s+1} \right) = I^{r+i} J^{s-i} \cap \left( \sum_{j<i} I^{r+j} J^{s-j} + \sum_{j>i} I^{r+j} J^{s-j} + I^r P^{s+1} \right) \subseteq I^{r+i} J^{s-i} \cap \left( I^{r+i+1} + J^{s-i+1} \right). \tag{2.4}$$

Generally for all ideals $I_1, I_2, I_3$ in a common ring, we have

$$I_1 \cap (I_2 + I_3) \subseteq (I_2 \cap (I_1 + I_3)) + (I_3 \cap (I_1 + I_2)).$$

Take $I_1 = I^{r+i} J^{s-i}, I_2 = I^{r+i+1}, I_3 = J^{s-i+1}$ in $T$. Then as $I_1 + I_3 \subseteq J^{s-i}, I_1 + I_2 \subseteq I^{r+i}$, the last display implies the inclusion in the following chain

$$I^{r+i} J^{s-i} \cap (I^{r+i+1} + J^{s-i+1}) \subseteq I^{r+i+1} \cap J^{s-i} + J^{s-i+1} \cap I^{r+i} \subseteq I^{r+i+1} J^{s-i} + I^{r+i} J^{s-i+1}. \tag{2.5}$$

The equality holds because of (2.1). Note that the last chain also yields

$$I^{r+i} J^{s-i} \cap I^r P^{s+1} = I^{r+i+1} J^{s-i} + I^{r+i} J^{s-i+1}. \tag{2.6}$$

Indeed, the right-hand side is trivially contained in the left-hand one. For the reverse inclusion, we only need to recall that $I^r P^{s+1} \subseteq I^{r+i+1} + J^{s-i+1}$. The chain (2.5) takes care of the rest.

The combination of (2.4) with (2.5) and (2.6) yields the desired inclusion (2.3).

So we have the first equality in the following chain

$$\frac{I^r P^s}{I^r P^{s+1}} = \bigoplus_{i=0}^{s} \frac{I^{r+i} J^{s-i} + I^r P^{s+1}}{I^r P^{s+1}} \cong \bigoplus_{i=0}^{s} \frac{I^{r+i} J^{s-i}}{I^{r+i+1} J^{s-i+1} + I^{r+i} J^{s-i+1}} \cong \bigoplus_{i=0}^{s} \left( \frac{I^{r+i} J^{s-i}}{I^{r+i+1} J^{s-i+1}} \otimes_k J^{s-i+1} \right).$$

The second equality is (2.6), the last isomorphism is standard. This finishes the proof. \qed
Lemma 2.3. Let $R, S$ be standard graded $k$-algebras, and $M, N$ be non-zero finitely generated graded modules over $R, S$, resp. Then denoting $T = R \otimes_k S$, there are equalities
\[
\begin{align*}
\text{pd}_T(M \otimes_k N) &= \text{pd}_R M + \text{pd}_S N, \\
\text{reg}_T(M \otimes_k N) &= \text{reg}_R M + \text{reg}_S N, \\
\text{ld}_T(M \otimes_k N) &= \text{ld}_R M + \text{ld}_S N.
\end{align*}
\]

Proof. Let $F, G$ be the minimal graded free resolutions of $M, N$ over $R, S$, resp. Then $F \otimes_k G$ is a minimal graded free resolution of $M \otimes_k N$ over $T$. A simple accounting then yields the first two equalities.

We immediately have $\text{lin}^T(F \otimes_k G) = \text{lin}^R F \otimes_k \text{lin}^S G$. Hence using the Künneth's formula, we get $\text{ld}_T(M \otimes_k N) = \text{ld}_R M + \text{ld}_S N$. \hfill \Box

Recall that a map of noetherian local rings $\theta : (R, m) \longrightarrow (S, n)$ is called an algebra retract if there exists a local homomorphism $\phi : S \rightarrow R$ such that the composition $\phi \circ \theta$ is the identity of $R$. In such a situation, we call the map $\phi$ the retraction map of $\theta$.

Lemma 2.4. Let $\theta : (R, m) \rightarrow (S, n)$ be an algebra retract of noetherian local rings with the retraction map $\phi : S \rightarrow R$. Let $I \subseteq m$ be an ideal of $R$. Let $J \subseteq n$ be an ideal containing $\theta(1)S$ such that $\phi(J)R = I$. Then there are inequalities
\[
\text{ld}_R(R/I) \leq \text{ld}_S(S/J),
\]
\[
\text{ld}_R I \leq \text{ld}_S J.
\]

Proof. The hypothesis implies that there is an induced algebra retract $R/I \rightarrow S/J$. For each $i \geq 0, q \geq 0$, there is a commutative diagram of $R$-modules
\[
\begin{array}{ccc}
\text{Tor}^R_i(R/m^q+1, R/I) & \xrightarrow{\iota^q+1} & \text{Tor}^S_i(S/n^q+1, S/J) & \xrightarrow{\varphi^q} & \text{Tor}^R_i(R/m^q+1, R/I) \\
\mu^q_R & \downarrow & \mu^q_S & & \\
\text{Tor}^R_i(R/m^q, R/I) & \xrightarrow{\iota^q} & \text{Tor}^S_i(S/n^q, S/J) & \xrightarrow{\varphi^q} & \text{Tor}^R_i(R/m^q, R/I)
\end{array}
\]

By functoriality, the composition of the horizontal maps on the second row is the identity map of $\text{Tor}^R_i(R/m^q, R/I)$. From this, we deduce that $\iota^q$ is injective.

Take $i > \text{ld}_S(S/J)$, then from Theorem 2.1, $\mu^q_S$ is trivial for all $q \geq 0$. Since $\iota^q$ is injective, we also have $\mu^q_R$ is trivial for all $q \geq 0$. Using Theorem 2.1, this implies that $\text{ld}_R(R/I) \leq \text{ld}_R(S/J)$. The remaining inequality is immediate. \hfill \Box

3. Maps of Tor and algebraic invariants

This section makes the preparation for Section 4. We will introduce the notion of Tor-vanishing and doubly Tor-vanishing homomorphisms. Morphisms of both type are well-suited to study the linearity defect but the latter yields more precise information. We also recall some results from [40], [43], which will be used frequently later.

3.1. Tor-vanishing morphisms. Let $(R, m)$ be a noetherian local ring. Let $\phi : M \rightarrow P$ be a morphism of finitely generated $R$-modules. We say that $\phi$ is Tor-vanishing if for all $i \geq 0$, we have $\text{Tor}^R_i(k, \phi) = 0$. We say that $\phi$ is doubly Tor-vanishing if there exist (possibly non-minimal) free resolutions $F$ and $G$ of $M$ and $P$, resp., and a lifting $\varphi : F \rightarrow G$ of $\phi$ such that $\varphi(F) \subseteq m^2G$. 

We use the same terminology for the setting of standard graded algebras over a field and graded modules. Clearly, if $\phi$ is doubly Tor-vanishing then it is Tor-vanishing, but not vice versa: For example, take $R = k[[x]]$ and let $\phi$ be the map $R \xrightarrow{x} R$.

**Remark 3.1.** Let $F, G$ be the minimal free resolutions of $M, P$, resp.

(1) The following statements are equivalent:

(i) $\phi$ is Tor-vanishing;

(ii) There exists a lifting $\varphi : F \to G$ of $\phi$ such that $\varphi(F) \subseteq mG$;

(iii) For any lifting $\varphi : F \to G$ of $\phi$, we have $\varphi(F) \subseteq mG$.

(2) The following statements are equivalent:

(i) $\phi$ is doubly Tor-vanishing;

(ii) There exists a lifting $\varphi : F \to G$ of $\phi$ such that $\varphi(F) \subseteq m^2G$;

(iii) For any lifting $\varphi : F \to G$ of $\phi$, we have $\varphi(F) \subseteq m^2G$.

Compositing Tor-vanishing morphisms produce doubly Tor-vanishing ones.

**Lemma 3.2.** Let $M \xrightarrow{\phi} P$ and $P \xrightarrow{\psi} Q$ be Tor-vanishing morphisms. Then $\psi \circ \phi$ is doubly Tor-vanishing.

**Proof.** Let $F, G, H$ be the minimal free resolutions of $M, P, Q$, resp. Since $\phi$ is Tor-vanishing, it admits a lifting $\varphi : F \to G$ such that $\varphi(F) \subseteq mG$. Similarly, there exists a lifting $\Psi : G \to H$ of $\psi$ such that $\Psi(G) \subseteq mH$. The composition $\Psi \circ \varphi : F \to H$ yields a lifting of $\psi \circ \phi$ such that $(\Psi \circ \varphi)(F) \subseteq m^2H$. □

The following result illustrates the utility of (doubly) Tor-vanishing morphisms.

**Lemma 3.3.** Let $0 \to M \xrightarrow{\phi} P \to N \to 0$ be an exact sequence of non-zero finitely generated $R$-modules.

(i) Assume that $\phi$ is Tor-vanishing. Then

$$\text{pd}_R N = \max\{\text{pd}_R P, \text{pd}_R M + 1\}.$$  

If moreover $R$ is a standard graded algebra and $M, P, N$ are graded modules, then

$$\text{reg}_R N = \max\{\text{reg}_R P, \text{reg}_R M - 1\}.$$  

(ii) Assume that $\phi$ is doubly Tor-vanishing. Then

$$\text{ld}_R N = \max\{\text{ld}_R P, \text{ld}_R M + 1\}.$$  

**Proof.** (i) The long exact sequence of Tor yields $\beta_i(N) = \beta_i(P) + \beta_{i-1}(M)$ for all $i \geq 0$. Hence the equality for the projective dimension follows. Similar arguments for the regularity. (ii) follows by using Remark 3.1(2) and [42, Lemma 3.5]. □

### 3.2. Betti splittings

Betti splittings were first introduced by Francisco, Hà, and Van Tuyl [23] for monomial ideals. A motivation for this notion comes from work of Eliahou and Kervaire [22]. Let $(R, m)$ be a noetherian local ring (or a standard graded $k$-algebra) and $P, I, J \neq (0)$ be proper (homogeneous) ideals of $R$ such that $P = I + J$.

**Definition 3.4.** The decomposition of $P$ as $I + J$ is called a Betti splitting if for all $i \geq 0$, the following equality of Betti numbers holds: $\beta_i(P) = \beta_i(I) + \beta_i(J) + \beta_{i-1}(I \cap J)$.
The lemma below is a straightforward generalization of [23, Proposition 2.1] and
admits the same proof. We will frequently use the characterization (ii) of Betti
splittings in the next sections.

**Lemma 3.5.** The following are equivalent:

(i) The decomposition $P = I + J$ is a Betti splitting;
(ii) The morphisms $I \cap J \to I$ and $I \cap J \to J$ are Tor-vanishing;
(iii) The mapping cone construction for the map $I \cap J \to I \oplus J$ yields a minimal
free resolution of $P$.

Most results of this paper are motivated by the next simple observation, pre-
sentated in [43, Example 4.7].

**Example 3.6.** Let $R, S$ be standard graded $k$-algebras and $I, J$ be non-zero, proper
homogeneous ideals of $R, S$, resp. Let $P = I + J \subseteq T = R \otimes_k S$, then the
decomposition $P = I + J$ is a Betti splitting.

The following result signifies the utility of Betti splittings. The case when $R$ is
a polynomial ring is well-known [23, Corollary 2.2].

**Lemma 3.7.** Let $R$ be a standard graded $k$-algebra and $P$ a homogeneous ideal
with a Betti splitting $P = I + J$. Then there are equalities

\[
\text{pd}_R P = \max\{\text{pd}_R I, \text{pd}_R J, \text{pd}_R (I \cap J) + 1\},
\]

\[
\text{reg}_R P = \max\{\text{reg}_R I, \text{reg}_R J, \text{reg}_R (I \cap J) - 1\}.
\]

**Proof.** Looking at the long exact sequence of $\text{Tor}_R^R(k, -)$, we see that the second
equality is always true and the first one is true if $I \cap J \neq (0)$. If $I \cap J = (0)$, then
$P = I \oplus J$ so $\text{pd}_R P = \max\{\text{pd}_R I, \text{pd}_R J\}$. If $\max\{\text{pd}_R I, \text{pd}_R J\} \geq 1$ then the
first equality is again true. If $\text{pd}_R I = \text{pd}_R J = 0$ then $I$ and $J$ are free $R$-modules.
Hence there are non-zero divisors $x \in I, y \in J$. But then $0 \neq xy \in I \cap J$, a
contradiction. This finishes the proof of the first equality. \qed

### 3.3. Exact sequence estimates

Let $(R, \mathfrak{m})$ be a noetherian local ring and $0 \to M \xrightarrow{\phi} P \xrightarrow{\lambda} N \to 0$ be an exact sequence of finitely generated $R$-modules. From Šega’s Theorem 2.1, it follows that the vanishing of $\text{Tor}_R^R(k, \phi), \text{Tor}_R^R(k, \lambda)$ and the
connecting map $\text{Tor}_R^R(k, N) \to \text{Tor}_R^R(k, M)$ are useful for comparing the numbers
$\text{ld}_R M, \text{ld}_R P$ and $\text{ld}_R N$. For example, we have

**Proposition 3.8 ([43, Proposition 4.3]).** With the notations as above, assume that
$\phi$ is Tor-vanishing. Then there are inequalities

\[
\text{ld}_R N \leq \max\{\text{ld}_R P, \text{ld}_R M + 1\},
\]

\[
\text{ld}_R P \leq \max\{\text{ld}_R M, \text{ld}_R N\},
\]

\[
\text{ld}_R M \leq \max\{\text{ld}_R P, \text{ld}_R N - 1\}.
\]

Using Lemma 3.5 and Proposition 3.8 for the short exact sequence $0 \to I \cap J \to I + J \to P \to 0$, we get

**Proposition 3.9 ([43, Theorem 4.9]).** Let $P = I + J$ be a Betti splitting of non-zero
proper ideals of $R$. Then there are inequalities

\[
\text{ld}_R P \leq \max\{\text{ld}_R I, \text{ld}_R J, \text{ld}_R (I \cap J) + 1\},
\]

\[
\max\{\text{ld}_R I, \text{ld}_R J\} \leq \max\{\text{ld}_R (I \cap J), \text{ld}_R P\},
\]

\[
\text{ld}_R (I \cap J) \leq \max\{\text{ld}_R I, \text{ld}_R J, \text{ld}_R P - 1\}.
\]
The following result will be invoked several times. Note that, compared with the original statements in [40], below we additionally allow trivial modules. No contradiction arises in doing so because of the convention that the trivial module is Koszul.

**Theorem 3.10** (Nguyen [40, Theorem 3.5 and its proof]). Let $0 \rightarrow M \xrightarrow{\phi} P \rightarrow N \rightarrow 0$ be a short exact sequence of finitely generated $R$-modules such that:

(i) $P$ is a Koszul module,

(ii) $M \subseteq \mathfrak{m}P$.

Then $\phi$ is Tor-vanishing, and $\text{ld}_R N - 1 \leq \text{ld}_R M \leq \max\{0, \text{ld}_R N - 1\}$. Furthermore, $\text{ld}_R N = 0$ if and only if $\text{ld}_R M = 0$ and $M \cap \mathfrak{m}^{s+1}P = \mathfrak{m}^s M$ for all $s \geq 0$.

### 3.4. Invariants of mixed sums

Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be standard graded $k$-algebras, $I \subseteq \mathfrak{m}$ and $J \subseteq \mathfrak{n}$ be non-zero homogeneous ideals of $R$ and $S$, resp. In the sequel, modules over $R$ or $S$ are identified with their extensions to $T$ (via the obvious faithfully flat maps). For simplicity, we will call $P = I + J$ the *mixed sum* of $I$ and $J$. More generally, if $R_1, \ldots, R_c$ are standard graded $k$-algebras (where $c \geq 2$) and $I_i \subseteq R_i$ is a homogeneous ideal for $1 \leq i \leq c$, we call $I_1 + \cdots + I_c \subseteq R_1 \otimes_k \cdots \otimes_k R_c$ the mixed sum of $I_1, \ldots, I_c$.

Part (i) of the following result is folkloric and is a consequence of Example 3.6 and Lemma 3.7.

**Proposition 3.11.** The following statements hold.

(i) There are equalities

\[
\begin{align*}
\text{pd}_T P &= \text{pd}_R I + \text{pd}_S J + 1, \\
\text{reg}_T P &= \text{reg}_R I + \text{reg}_S J - 1.
\end{align*}
\]

(ii) If $R/I$ is a Koszul $R$-module then $\text{ld}_T P = \text{ld}_S J$. If $\text{ld}_R(R/I)$ and $\text{ld}_S(S/J)$ are $\geq 1$ then $\text{ld}_T P = \text{ld}_R I + \text{ld}_S J + 1$.

**Proof.** (i) The proof was mentioned above.

(ii) Applying Lemma 2.3 for the modules $R/I$ and $S/J$, we have $\text{ld}_T(T/P) = \text{ld}_R(R/I) + \text{ld}_S(S/J)$. This implies the first part of (ii). For the second part, note that $\text{ld}_R(R/I) \geq 1$, hence $\text{ld}_R I = \text{ld}_R(R/I) - 1$. Similar equalities hold for $J$ and $P$, and the conclusion follows. \qed

**Remark 3.12.** If $R$ is a Koszul algebra, then by Subsection 2.2, $R/I$ is a Koszul module $\iff$ $R/I$ has a linear resolution over $R$ $\iff$ $I$ has a $1$-linear resolution over $R$.

In any case, if $I$ is not generated by linear forms then $\text{ld}_R(R/I) \geq 1$. Hence the hypotheses of Proposition 3.11(ii) are satisfied if $I \subseteq \mathfrak{m}^2, J \subseteq \mathfrak{n}^2$. Later on, we will have different kinds of results about $\text{ld}_T P^s, s \geq 2$ according to whether $I$ is generated by linear forms or $I \subseteq \mathfrak{m}^2$ (compare for example Corollary 6.7 and Theorem 6.8).

### 4. Ideals of (doubly) small type

Let $(R, \mathfrak{m})$ be a noetherian local ring (or a standard graded $k$-algebra), and $I$ a proper (homogeneous) ideal. We say that $I$ is of small type (of doubly small type) if for all $r \geq 1$, the natural map $I^r \rightarrow I^{r-1}$ is Tor-vanishing (doubly Tor-vanishing,
resph.). Ideals which are not of small type abound: Let \( R \) be a non-Koszul algebra. Then by [32, Proposition 1.13], \( \text{ld}_R \mathfrak{m} = \infty \), hence by Theorem 2.1, \( \mathfrak{m} \) is not of small type. Moreover, even over some Koszul complete intersections of codimension 2 ideals not of small type exist, e.g. consider the ideal \( (x + y) \) of \( k[x, y, z]/(x^2, yz) \) in Example 6.2. In this section, we provide diverse classes of ideals of small or doubly small type. An important result in the current section is Theorem 4.10 relating ideals of small type and Betti splittings of powers of their mixed sums. This will be useful to studying invariants of powers of mixed sums in Sections 5 and 6.

4.1. Differential criteria. In this subsection, let \( R = k[x_1, \ldots, x_m] \) (where \( m \geq 0 \)) be a polynomial ring, \( \mathfrak{m} \) its graded maximal ideal, and \( (0) \neq I \subseteq \mathfrak{m} \) a homogeneous ideal. We denote by \( \partial(I) \) the ideal generated by elements of the form \( \partial f/\partial x_i \), where \( f \in I, i = 1, \ldots, m \). In the proof of Proposition 4.8, we will make use of the following criterion for detecting Tor-vanishing homomorphisms.

**Lemma 4.1** (Ahangari Maleki [1, Theorem 2.5 and its proof]). Assume that \( \text{char} \ k = 0 \).

(i) Let \( I_1 \) and \( I_2 \) be homogeneous ideals of \( R \) such that \( \partial(I_1) \subseteq I_2 \). Then \( I_1 \subseteq \mathfrak{m}I_2 \) and the map \( I_1 \rightarrow I_2 \) is Tor-vanishing.

(ii) In particular, any homogeneous ideal \( I \) of \( R \) is of small type.

The only new statement in Lemma 4.1 is the inclusion \( I_1 \subseteq \mathfrak{m}I_2 \), which results from Euler’s identity for homogeneous polynomials.

We wish to prove that every proper monomial ideal of a polynomial ring is of small type. For this, we record the following simple but very useful lemma. It is an application of the Taylor’s resolution [48]; see [28, Section 7.1]. Below, for a set \( G' \) of polynomials in \( R \), denote \( \text{lcm} G' = \text{lcm}(x : x \in G') \). If \( I \) is a monomial ideal, let \( G(I) \) denote the set of its minimal monomial generators.

**Lemma 4.2** (Eliahou and Kervaire [22, Proof of Proposition 3.1]). Let \( (0) \neq I_1 \subseteq I_2 \) be monomial ideals of \( R \). Assume that there exists a function \( \phi : G(I_1) \rightarrow G(I_2) \) with the following property:

(LCM) For any non-empty subset \( G' \) of \( G(I_1) \), \( \text{lcm} G' \) belongs to \( (\text{lcm} \phi(G'))\mathfrak{m} \).

Then the inclusion map \( I_1 \rightarrow I_2 \) is Tor-vanishing.

For \( I \) being a monomial ideal of \( R \), we denote by \( \partial^*(I) \) the ideal generated by elements of the form \( f/\partial x_i \), where \( f \) is a minimal monomial generator of \( I \) and \( x_i \) is a variable dividing \( f \). The following lemma is straightforward so we leave the detailed proof to the interested reader.

**Lemma 4.3.** Let \( I_1, I_2, L \) be monomial ideals of \( R \) where \( L \subseteq I_1 \). Then the following statements hold:

(i) \( \partial^*(I_1) \) is a monomial ideal and \( \partial(I_1) \subseteq \partial^*(I_1) \).

(ii) \( I_1 \subseteq \mathfrak{m}\partial^*(I_1) \).

(iii) Let \( g \geq 1 \) be maximal such that there exists a variable \( x_i \) of \( R \) with the property that \( x_i^g \) divides an element of \( G(I_1) \). If \( \text{char} \ k = 0 \) or \( \text{char} \ k > g \) then \( \partial^*(I_1) = \partial(I_1) \). In particular, if \( I_1 \) is squarefree then \( \partial^*(I_1) = \partial(I_1) \).

(iv) \( \partial^*(L) \subseteq \partial^*(I_1) \).

(v) \( \partial^*(I_1I_2) = \partial^*(I_1)I_2 + I_1\partial^*(I_2) \).

(vi) \( \partial^*(I^s) = \partial^*(I)I^{s-1} \) for all \( s \geq 1 \).
We have the following criterion for maps between sets of minimal generators of monomial ideals to have the (LCM) property.

**Proposition 4.4.** Let $I_1, I_2$ be non-zero proper monomial ideals of $R$ such that $\partial^*(I_1) \subseteq I_2$. Then $I_1 \subseteq \mathfrak{m}I_2$ and there exists a map $\phi : \mathcal{G}(I_1) \to \mathcal{G}(I_2)$ having the property (LCM) of Lemma 4.2.

**Proof.** From Lemma 4.3(ii), we have $I_1 \subseteq \mathfrak{m}\partial^*(I_1) \subseteq \mathfrak{m}I_2$.

For the remaining statement, we use induction on the number of generators of $I_1$. For a monomial $f \in R$, denote by $\supp(f)$ the set of variables dividing $f$. If $I_1$ is a principal ideal $(f)$, then $\partial^*(I_1) = (f/x_i : x_i \in \supp(f))$. By the hypothesis, $f/x_i \in I_2$ for some $x_i \in \supp(f)$, so there exists $g \in \mathcal{G}(I_2)$ dividing $f/x_i$. We define $\phi$ by $\phi(f) = g$, then clearly $f \in gm$.

Assume that $|\mathcal{G}(I_1)| \geq 2$. Let $x$ be a variable which divides a minimal monomial generator of $I_1$. We can write in a unique way $I_1 = xK + L$, where $L$ is generated by the elements of $\mathcal{G}(I_1)$ which are not divisible by $x$ and $xK$ is generated by the remaining elements.

Observe that $\mathcal{G}(xK) \cap \mathcal{G}(L) = \emptyset$, and $\mathcal{G}(I_1) = \mathcal{G}(xK) \cup \mathcal{G}(L)$. Furthermore $K \subseteq \partial^*(I_1) \subseteq I_2$.

Firstly, assume that $L \neq (0)$. Since $|\mathcal{G}(L)| < |\mathcal{G}(I_1)|$ and $\partial^*(L) \subseteq \partial^*(I_1) \subseteq I_2$ thanks to Lemma 4.3(iv), by induction hypothesis, there exists a function $\psi : \mathcal{G}(L) \to \mathcal{G}(I_2)$ which has the property (LCM).

We define $\phi : \mathcal{G}(I_1) \to \mathcal{G}(I_2)$ as follows: if $y \in \mathcal{G}(xK)$, since $K \subseteq I_2$, for a choice of monomials $f \in R, g \in \mathcal{G}(I_2)$ such that $y = xfg$, we let $\phi(y) = g$. If $y \in \mathcal{G}(L)$ then we set $\phi(y) = \psi(y)$. We verify that $\phi$ has the property (LCM).

Consider a set $G' = \{y_1, \ldots, y_r, z_1, \ldots, z_s\} \subseteq \mathcal{G}(I_1)$ where $y_i \in \mathcal{G}(xK)$ and $z_j \in \mathcal{G}(L)$. We write $y_i = xfg_i$ where $g_i = \phi(y_i) \in \mathcal{G}(I_2)$. If $r = 0$ then $\lcm G' = \lcm \{z_1, \ldots, z_s\}$ is strictly divisible by $\lcm \phi(G') = \lcm \psi(G')$ by the choice of $\psi$. If $r \geq 1$, denote $g = \lcm(\phi(y_1), \ldots, \phi(y_r)) = \lcm(g_1, \ldots, g_r)$. Denote $b = \lcm(z_1, \ldots, z_s), b' = \lcm(z_1, \ldots, z_s)$ then $b'$ divides $b$. Now $\lcm \phi(G') = \lcm(g, b')$ and $\lcm G'$ is divisible by $\lcm(xg_1, \ldots, xg_r, z_1, \ldots, z_s) = \lcm(xg, b) = x \lcm(g, b)$, where the second equality holds since $\gcd(x, b) = 1$. Since $b'$ divides $b$, we conclude that $\lcm \phi(G')$ strictly divides $\lcm G'$, as desired.

It remains to consider the case $L = 0$. In this case $I_1 = xK$, so $K \subseteq \partial^*(I_1) \subseteq I_2$.

Define $\phi$ in the same way as above, we get the desired conclusion. □

An immediate consequence of Proposition 4.4 is the following.

**Theorem 4.5.** Any proper monomial ideal of $R$ is of small type.

**Proof of Theorem 4.5.** Let $I \subseteq \mathfrak{m}$ be a monomial ideal. There is nothing to do if $I = (0)$, so we assume that $I \neq (0)$. By Lemma 4.2, it suffices to prove that for any $s \geq 1$, there exists a map $\mathcal{G}(I^s) \to \mathcal{G}(I^{s-1})$ with the (LCM) property. Thanks to Proposition 4.4, we only need to check that $\partial^*(I^s) \subseteq I^{s-1}$, but this follows from Lemma 4.3(vi). □

4.2. **A catalog.** We provide in this section more classes of ideals which are of small or doubly small type. For some classes of ideals satisfying the hypothesis of the next result, see, e.g., [7], [29], [30].
Proposition 4.6. Assume that all the powers of $I$ are Koszul. Then:

(i) $I$ is of small type.
(ii) If moreover $R$ is a Koszul algebra and $I \subseteq \mathfrak{m}^2$, then $I$ is of doubly small type.

Proof. Take $r \geq 1$. Since $I^r \subseteq \mathfrak{m}^{r-1}$ and $I^{r-1}$ is Koszul, the map $I^r \to I^{r-1}$ is Tor-vanishing, thanks to Theorem 3.10.

Now assume further that $R$ is a Koszul algebra and $I \subseteq \mathfrak{m}^2$. For $r \geq 1$, consider the chain $I^r \subseteq \mathfrak{m}^{r-1} \subseteq I^{r-1}$. Since $I^{r-1}$ is Koszul and $R$ is a Koszul algebra, we obtain by [40, Corollary 3.8] that $I^{r-1}$ is Koszul. From the inclusion $I^r \subseteq \mathfrak{m}^2 I^{r-1}$ and Theorem 3.10, the map $I^r \to \mathfrak{m} I^{r-1}$ is Tor-vanishing. As seen above, the map $\mathfrak{m} I^{r-1} \to I^{r-1}$ is Tor-vanishing as well, so Lemma 3.2 implies that $I^r \to I^{r-1}$ is doubly Tor-vanishing for all $r \geq 1$. In other words, $I$ is of doubly small type. \Box

Proposition 4.7. Let $(R, \mathfrak{m})$ be a noetherian local ring and $I$ a proper ideal generated by a regular sequence. Then:

(i) The ideal $I$ is of small type.
(ii) If moreover $I$ is contained in $\mathfrak{m}^2$ then it is of doubly small type.

Proof. (i) We can use the Eagon–Northcott resolution to directly construct a suitable lifting of the map $I^r \to I^{r-1}$. Here is an alternative argument. By assumption $I = (f_1, \ldots, f_p)$ where $p \geq 1$ and $f_1, \ldots, f_p$ is a regular sequence of elements in $\mathfrak{m}$. We use induction on $p \geq 1$ and $r \geq 1$ that there exists a lifting on the level of minimal free resolutions of the map $I^r \to I^{r-1}$ which induces the zero map after tensoring with $k$. If $p = 1$ or $r = 1$ then the conclusion is clear. Assume that $p \geq 2$ and $r \geq 2$. Denote $K = (f_2, \ldots, f_p)$ and $f = f_1$. We have $I^r = fI^{r-1} + K^r$. Since $f$ is $R/K^r$-regular (see [8, Page 6]), we have $fI^{r-1} \cap K^r = fK^r$.

Let $A, B$ be the minimal free resolution of $K^r, I^{r-1}$, resp. Using the induction hypothesis, the map $K^r \to I^{r-1}$ is Tor-vanishing, hence so is $K^r \to I^{r-1}$. Since $f$ is $R/K^r$-regular, a lifting of the map $fK^r \to K^r$ is given by $A \xrightarrow{\lambda} A$. Therefore a minimal free resolution of $I^r$ is obtained from the mapping cone construction for the map $fK^r \to fI^{r-1} \oplus K^r$.

Let $\varepsilon$ be a lifting of the map $K^r \to I^{r-1}$ such that $\varepsilon \otimes_R k = 0$. We have the following lifting diagram.

Specifically, a lifting $\Lambda : A \to B \oplus A$ of the map $fK^r \to fI^{r-1} \oplus K^r$ is given by

A lifting $\Phi : B \oplus A \to B$ of the map $fI^{r-1} \oplus K^r \to I^{r-1}$ is given by

$B \oplus A \ni (y, z) \mapsto fy - \varepsilon(z) \in B$. 

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The composition $\Phi \circ \Lambda$ is zero, hence we can extend $\Phi$ to a map from the mapping cone of $\Lambda$ to $B$ by setting it to be zero on $A$. The extended map is a lifting of the map $I^r \to I^{r-1}$ which is zero after tensoring with $k$. Hence $I^r \to I^{r-1}$ is Tor-vanishing. The induction and hence the proof is finished.

(ii) Argue similarly as for (i). \qed

Assume that $R = k[x_1, \ldots, x_m]$ is a polynomial ring over $k$ (where $m \geq 0$) and $(0) \neq I \subseteq R$ is a proper homogeneous ideal. Following Herzog and Huneke [31], if char $k = 0$ we say that a homogeneous (but possibly non-monomial) ideal $I$ is strongly Golod if $\partial(I)^2 \subseteq I$. Independently of char $k$, we say that an ideal $I$ is strongly Golod if $I$ is a monomial ideal and $\partial^*(I)^2 \subseteq I$. The two notions are compatible when char $k = 0$: $I$ is strongly Golod if and only if $I$ is strongly Golod in the sense of Herzog and Huneke. Below, let $\hat{I} = \bigcup_{s \geq 1}(I : m^s)$ be the saturation of $I$. Let $\overline{I}$, the integral closure of $I$, be the set of $x \in R$ which satisfies a relation $x^n + a_1x^{n-1} + \cdots + a_n = 0$ where $n \geq 1, a_i \in I$ for $1 \leq i \leq n$. The integral closure is an ideal containing $I$, and if $I$ is a monomial ideal then so is $\overline{I}$: In fact, by Proposition 1.4.2 and the discussion preceding it in [36], $\overline{I}$ is generated by monomials $f \in R$ such that $f^r \in I^r$ for some $r \geq 1$.

For each $s \geq 1$, let $I^{(s)} = \bigcap_{P \in \text{Min}(I)} I^sP \cap R$ be the $s$-th symbolic power of $I$, where $\text{Min}(I)$ denotes the set of minimal primes of $I$.

**Proposition 4.8.** Assume that either char $k = 0$ or $I$ is a monomial ideal.

(i) If $I$ is strongly Golod or strongly Golod then it is of doubly small type.

(ii) For all $s \geq 2$, the ideals $I^s, \overline{I^s}, I^{(s)}$ are of doubly small type.

(iii) If $I$ is monomial then for all $s \geq 2$, $\overline{I^s}$ is of doubly small type.

First, we establish the following lemma, which is inspired by [31, Theorem 2.3 and Proposition 3.1]. The proof of [31, Proposition 3.1] in page 96–97 contains several problems/mistakes, e.g., in p. 97, line 5, the best one can say is $(w/x_k)^r \in I^rI^{(b-a)/2}$. In p. 97, line 7, the equality $r = d + b$ is generally false, one can only say $r \leq d + b$. The crucial claim in the proof is nevertheless correct. We salvage the above problems and mistakes in the proof of part (i) of Lemma 4.9.

**Lemma 4.9.** Let $I$ be a monomial ideal.

(i) If $I$ is strongly Golod, then so is $\overline{I}$.

(ii) Let $I$ be strongly Golod, and $L$ a monomial ideal such that $I : L = I : L^2$.
Then $I : L$ is also strongly Golod.

(iii) For all $s \geq 2$, $\overline{I^s}$ is strongly Golod.

(iv) For all $s \geq 2$, $\overline{I^s}, I^{(s)}$ are strongly Golod.

**Proof.** (i) Let $f$ be a monomial such that $f^r \in I^r$ for some $r \geq 1$ and $x_i \in \text{supp}(f)$.
We claim that $(f/x_i)^r \in I^{[r/2]}$.
Write $f^r = m_1 \cdots m_r$ where for each $1 \leq j \leq r$, $m_j$ is a monomial in $I$. For $1 \leq j \leq r$, let $d_j$ be maximal so that $x_i^{d_j}$ divides $m_j$. By permuting the indices, we can assume that $d_1 = \cdots = d_a = 0$ and $1 \leq d_{a+1} \leq \cdots \leq d_r$. Clearly $x_i^{d_j} f^r$, hence $d_{a+1} + \cdots + d_r \geq r$. If $a \geq \lfloor r/2 \rfloor$, then

$$(f/x_i)^r = (m_1 \cdots m_a) \left((m_{a+1} \cdots m_r)/x_i^r\right) \in I^a \subseteq I^{[r/2]}.$$  

Consider the case $a < \lfloor r/2 \rfloor$.

**Observation:** It holds that $d_{r-a+1} + \cdots + d_r \geq 2a$. 

Indeed, assume that \( d_{r-a+1} + \cdots + d_r < 2a \), then since \( d_{r-a+1} \leq \cdots \leq d_r \), we get \( d_{r-a+1} < 2 \). In particular, \( d_{a+1} = \cdots = d_r = 1 \). This yields \( d_{a+1} + \cdots + d_r = r - 2a + (d_{r-a+1} + \cdots + d_r) < r \), which is a contradiction. Hence the observation is true.

Since \( I \) is strongly Golod and \( x_i \) divides \( m_{a+1}, \ldots, m_{r-a} \), we have an inclusion \( (m_{a+1}/x_i) \cdots (m_{r-a}/x_i) \in I^{[(r-2a)/2]} \). Together with the observation, we get the inclusion in the following chain
\[
\left( \frac{f}{x_i} \right)^r = \left( m_1 \cdots m_a \right) \left( \frac{m_{a+1}}{x_i} \cdots \frac{m_{r-a}}{x_i} \right) \frac{m_{r-a+1} \cdots m_r}{x_i^{2a}} \in I^a \left[ \frac{x_i}{x_i^{2a}} \right] = I^i.
\]

This finishes the proof of the claim.

Now take \( f, g \in \mathcal{I} \) and \( x_i \in \text{supp}(f), x_j \in \text{supp}(g) \). There exist \( r, s \geq 1 \) such that \( f^r \in \mathcal{I}^r, g^s \in \mathcal{I}^s \). Note that \( f^{2rs}, g^{2rs} \in \mathcal{I}^{2rs} \), hence by the above claim we obtain \( (fg/\langle x_ix_j \rangle)^{2rs} = (f/x_i)^{2rs}(g/x_j)^{2rs} \in \mathcal{I}^{rs} \mathcal{I}^{rs} = \mathcal{I}^{2rs} \). Hence \( (f/x_i)(g/x_j) \in \mathcal{I} \).

This implies that \( \mathcal{I} \) is *strongly Golod.

(ii) Take \( f, g \in I : L \) and \( x_i \in \text{supp}(f), x_j \in \text{supp}(g) \). Take any \( h_1, h_2 \in L \). Then \( fh_1, gh_2 \in I \), so \( (f/x_i)h_1, (g/x_j)h_2 \in \partial^*(I) \). In particular \( (f/x_i)(g/x_j)h_1h_2 \in \partial^*(I)^2 \subseteq I \). The last chain together with the hypothesis yields \( (f/x_i)(g/x_j) \in I : L^2 = I : L \). This implies that \( \partial^*(I : L)^2 \subseteq I : L \), namely \( I : L \) is *strongly Golod.

(iii) By Lemma 4.3(vi), \( \partial^*(I^s) = \partial^*(I) I^{s-1} \). Hence \( \partial^*(I^s)^2 \subseteq (I^{s-1})^2 \subseteq I^s \), proving that \( I^s \) is *strongly Golod.

(iv) By (i) and (iii), we have that \( \mathcal{I}^* \) is *strongly Golod.

Let \( J \) be the intersection of all the associated, non-minimal prime ideals of \( I^s \). Then \( I^{(s)} = \cup_{i \geq 1} (I^s : J^i) \). Since \( R \) is noetherian, for large enough \( i \), \( I^{(s)} = I^s : J^i \). Set \( L = J^i \), then \( I^{(s)} = I^s : L = I^s : L^2 \). Hence by (ii) and (iii), \( I^{(s)} \) is *strongly Golod.

Similarly from (ii) and (iii), we deduce that \( \mathcal{I}^* \) is *strongly Golod. The proof is concluded.

Proof of Proposition 4.8. (i) First, consider the case \( \text{char } k = 0 \). Let \( P = \partial(I) I^r \), then \( I^{r+1} \subseteq P \subseteq I^r \) by Euler’s identity for homogeneous polynomials. By Lemma 3.2, it suffices to show that the inclusion maps \( I^{r+1} \subseteq P \) and \( P \subseteq I^r \) are Tor-vanishing. For this, we wish to apply Lemma 4.1, so we claim that \( \partial(I^{r+1}) \subseteq P \) and \( \partial(P) \subseteq I^r \). The first inclusion is clear. Now \( \partial(P) \subseteq \partial(\partial(I)) I^r + \partial(I)^2 I^{r-1} \subseteq I^r \), where the second inclusion follows from the hypothesis.

The case \( I \) is *strongly Golod is proved similarly, where \( \partial^*(I) \) and Proposition 4.4 are used in places of \( \partial(I) \) and Lemma 4.1.

(ii) If \( \text{char } k = 0 \), by [31, Theorem 2.3], for any \( s \geq 2 \), the ideals \( I^s, \mathcal{I}^s, I^{(s)} \) are strongly Golod. If \( I \) is monomial, by Lemma 4.9, the same ideals are *strongly Golod. Together with part (i) we get that such ideals are of doubly small type.

(iii) This follows by combining Lemma 4.9(iv) and part (i).

4.3. Powers of mixed sums. The original purpose of introducing ideals of small type is contained in part (i) of the following result. The purpose of introducing ideals of doubly small type is contained in Theorem 6.1(ii).
Theorem 4.10. Let \((R, m)\) and \((S, n)\) be standard graded algebras over \(k\). Let \(I \subseteq R, J \subseteq S\) be non-zero proper homogeneous ideals. Denote \(T = R \otimes_k S\) and \(P = I + J \subseteq T\).

(i) Assume that \(I\) and \(J\) are of small type. Then for all \(r \geq 0\) and all \(s \geq 1\), we have a Betti splitting \(I^r P^s = I^{r+1}P^{s-1} + I^r J^s\). Furthermore, \(P\) is also of small type.

(ii) Assume that \(I\) and \(J\) are of doubly small type. Then for all \(r \geq 0\) and all \(s \geq 1\), the maps \(I^{r+1}J^s \to I^r J^s\) and \(I^{r+1}J^s \to I^{r+1}P^{s-1}\) are doubly Tor-vanishing, and the ideal \(P\) is of doubly small type.

Proof. (i) By Lemma 2.2 we have \(I^{r+1}P^{s-1} \cap I^r J^s = I^{r+1}J^s\). By Lemma 3.5, it suffices to check that the maps \(I^{r+1}J^s \to I^r J^s\) and \(I^{r+1}J^s \to I^{r+1}P^{s-1}\) are Tor-vanishing. Since \(I\) is of small type, the map \(I^{r+1} \to I^r\) is Tor-vanishing. Tensoring over \(k\) with \(J^s\) is an exact functor, so the map \(I^{r+1}J^s \to I^r J^s\) is Tor-vanishing. The same holds for the map \(I^{r+1}J^s \to I^{r+1}J^{s-1}\), and hence also for \(I^{r+1}J^s \to I^{r+1}P^{s-1}\) which factors through the previous map.

We prove that for every \(s \geq 1\), the map \(I^r P^s \to I^r P^{s-1}\) is Tor-vanishing for all \(r \geq 0\). This clearly implies that \(P\) is of small type. Induct on \(s\).

Step 1: If \(s = 1\), using \(I^r P = I^{r+1} + I^r J\) and Lemma 2.2, we have the exactness of the first row in the following diagram

\[
\begin{array}{cccccc}
0 & \to & I^{r+1}J & \to & I^{r+1} \oplus I^r J & \to & I^r P & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & I^r & & & & \\
\end{array}
\]

Let \(i^r, j^r, h^r\) be the minimal graded free resolutions of \(i^r, j^r, p^r\) over \(R, S, T\), resp. Then minimal graded free resolutions over \(T\) of \(I^{r+1}J, I^r J^s\) and \(I^r\) are given by \(A = i^{r+1} \otimes_k l^r = i^r \otimes_k G, B = b \otimes \overline{T} = (i^{r+1} \otimes_k S) \oplus (i^r \otimes_k G)\) and \(C = e^r \otimes_k S\), resp.

Let \(\phi : i^{r+1} \to i^r\) be a lifting of the map \(i^{r+1} \to i^r\). Since \(I\) is of small type, we can choose \(\phi\) such that \(\phi \otimes_R k = 0\). Let \(\varepsilon : l^r \to S\) be a lifting of the map \(J \to S\) given by \(\varepsilon_i = 0\) for \(i \geq 1\). Choose bases for \(i^{r+1}, i^r, l^r\). We have a lifting diagram as follows.

\[
\begin{array}{ccc}
A & \xrightarrow{(id \otimes \varepsilon, \phi \otimes id)} & B \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
I^{r+1}J & \xrightarrow{a \otimes (a, a)} & I^{r+1} \oplus I^r J & \xrightarrow{(b, c) \mapsto b - c} & I^r \\
\end{array}
\]

In other words, a lifting of \(\Lambda : A \to B \oplus \overline{T}\) of the map \(I^{r+1}J \to I^{r+1} \oplus I^r J\) is given on the basis elements by

\[
A \ni x \otimes y \mapsto (x \otimes \varepsilon(y), \phi(x) \otimes y) \in B \oplus \overline{T}.
\]

A lifting \(\Pi : B \oplus \overline{T} \to C\) of the map \(I^{r+1} \oplus I^r J \to I^r\) is given on the basis elements by

\[
B \oplus \overline{T} \ni (z \otimes 1, u \otimes v) \mapsto \phi(z) \otimes 1 - u \otimes \varepsilon(v) \in C.
\]

In particular, the composition map \(A \to C\) is zero. Therefore we can extend \(\Pi : B \oplus \overline{T} \to C\) to a map on the mapping cone of \(\Lambda\) by setting \(\Pi|_A = 0\). By the choice of \(\phi\) and \(\varepsilon\), we have \(\Pi \otimes_T k = 0\). Hence \(I^r P \to I^r\) is Tor-vanishing.
Step 2: Now assume that \( s \geq 2 \). Thanks to the argument for the Betti splitting \( I^r P^s = I^{r+1} P^{s-1} + I^r J^s \) from above, we have a commutative diagram with exact rows.

\[
\begin{array}{cccccc}
0 & \rightarrow & I^{r+1} J^s & \rightarrow & I^{r+1} P^{s-1} \oplus I^r J^s & \rightarrow & I^r P^s & \rightarrow & 0 \\
0 & \rightarrow & I^{r+1} J^{s-1} & \rightarrow & I^{r+1} P^{s-2} \oplus I^r J^{s-1} & \rightarrow & I^r P^{s-1} & \rightarrow & 0
\end{array}
\]

Let \( 1^A, 1^B, \) and \( 1^C \) be the minimal graded free resolutions of \( I^{r+1} J^s, I^{r+1} P^{s-1}, \) and \( I^r J^s \), resp. Let \( 2^A, 2^B, \) and \( 2^C \) be the minimal graded free resolution of \( I^{r+1} J^{s-1}, I^{r+1} P^{s-2}, \) and \( I^r J^{s-1} \), resp. With the notations as above, we can choose \( 1^A = r+1F \otimes_k G, 1^C = rF \otimes_k G, 2^A = r+1F \otimes_k s^{-1}G \) and finally \( 2^C = rF \otimes_k s^{-1}G \).

Step 2a: We show that there exist liftings \( \Phi, \Pi, \Omega \) of \( \alpha, \beta, \gamma, \delta \) such that the following two maps \( 1^A \rightarrow 2^B \oplus 2^C \) are equal: \( \Omega \circ \Phi = \Pi \circ \Psi \).

Let \( p \) be the graded maximal ideal of \( T \). Since \( J \) is of small type, there is a lifting \( \Psi : 1^A \rightarrow 2^A \) of \( \gamma \) such that \( \Psi(1^A) \subseteq p(2^A) \). In more details, choose a lifting \( \eta : s^*G \rightarrow s^{-1}G \) of \( J^s \rightarrow J^{s-1} \) so that \( \eta_s \otimes_S k = 0 \) and choose \( \Psi = \text{id} \otimes_k \eta_s \).

We will define \( \Phi, \Omega, \Pi \) componentwisely; see the next two diagrams. Let \( \alpha_1 : I^{r+1} J^s \rightarrow I^{r+1} P^{s-1}, \alpha_2 : I^{r+1} J^s \rightarrow I^r J^s \) be the restrictions of \( \alpha \) to the first and second component of the image, resp. Similarly define \( \beta_1, \beta_2 \). Let \( \delta_1 : I^{r+1} P^{s-1} \rightarrow I^{r+1} P^{s-2}, \delta_2 : I^r J^s \rightarrow I^r J^{s-1} \) be the components of \( \delta \). By the induction hypothesis, \( \delta_1 \) is also Tor-vanishing, hence there is a lifting \( \Omega_1 : 1^B \rightarrow 2B \) of it such that \( \Omega_1(1^B) \subseteq p(2^B) \). Let \( \Delta \) be a lifting of the map \( \theta : I^{r+1} J^{s-1} \rightarrow I^{r+1} P^{s-1} \), then we have liftings \( \Phi_1 = \Delta \circ \Psi \) of \( \alpha_1 \) and \( \Pi_1 = \Omega_1 \circ \Delta \) of \( \beta_1 \). Moreover, \( \Pi_1 \circ \Psi = \Omega_1 \circ \Phi_1 \).

Since \( I \) is of small type, for each \( r \), we can choose a lifting \( \rho_r : rF \rightarrow r^{-1}F \) of \( I^r \rightarrow I^{r-1} \) such that \( \rho_r \otimes_R k = 0 \). Let \( \Phi_2 = \rho_{r+1} \otimes_k \text{id} : 1^A \rightarrow 1^C, \Omega_2 = \text{id} \otimes_k \eta_s : 1^C \rightarrow 2^C \) and \( \Pi_2 = \rho_{r+1} \otimes_k \text{id} : 2^A \rightarrow 2^C \), which are liftings of \( \alpha_2, \delta_2 \), and \( \beta_2 \), resp. It is immediate that \( \Pi_2 \circ \Psi = \Omega_2 \circ \Phi_2 \).
Define $\Phi$ by $\Phi_1$ and $\Phi_2$, $\Omega$ by $\Omega_1$ and $\Omega_2$, $\Pi = (\Pi_1, \Pi_2)$. Then one has $\Pi \circ \Psi = \Omega \circ \Phi$, as desired.

**Step 2b:** Let $1^D, 2^D$ be the mapping cones of $\alpha, \beta$, resp. By the construction of $\alpha$ and $\beta$, $1^D, 2^D$ are the minimal graded free resolutions of $I^rP^s, I^rP^{s-1}$, resp. The liftings $\Psi$ and $\Omega$ together with the commutativity relation $\Pi \circ \Psi = \Omega \circ \Phi$ yields a lifting $\chi : 1^D \to 2^D$ for $I^rP^s \to I^rP^{s-1}$. Since $\Psi \otimes_T k = 0$ and $\Omega \otimes_T k = 0$, we deduce that $\chi \otimes_T k = 0$. Hence $I^rP^s \to I^rP^{s-1}$ is Tor-vanishing, as desired. This finishes the induction and the proof of part (i).

(ii) can be proved similarly as (i). \qed

5. Projective dimension and regularity

From now on, unless stated otherwise, we fix the following notations: Let $(R, m)$ and $(S, n)$ be standard graded algebras over $k$. Let $I \subseteq R, J \subseteq S$ be non-zero proper homogeneous ideals. Denote $T = R \otimes_k S$ and let $P = I + J \subseteq T$ be the mixed sum of $I$ and $J$.

The main technical result of this section is Proposition 5.1. We will deduce easily from Proposition 5.1 the asymptotic formulas for the projective dimension and the (Castelnuovo–Mumford) regularity of $P^s$ for $s \gg 0$, recovering in part results of [25] (see Theorems 5.6 and 5.7).

5.1. Formulas.

**Proposition 5.1.** Assume that $I$ and $J$ are of small type. Assume further that $I^i, J^i \neq 0$ for all $i \geq 1$, e.g., $R$ and $S$ are reduced. Then for all $s \geq 1$, we have

\[
\begin{align*}
\text{pd}_T P^s &= \max_{i \in [1, s-1], j \in [1, s]} \{ \text{pd}_R I^{s-i} + \text{pd}_S J^i, \text{pd}_R I^{s-j+1} + \text{pd}_S J^j + 1 \}, \\
\text{reg}_T P^s &= \max_{i \in [1, s-1], j \in [1, s]} \{ \text{reg}_R I^{s-i} + \text{reg}_S J^i, \text{reg}_R I^{s-j+1} + \text{reg}_S J^j - 1 \}.
\end{align*}
\]

**Proof.** By the proof of Theorem 4.10, the decomposition $I^rP^s = I^{r+1}P^{s-1} + I^rJ^s$ is a Betti splitting and $I^{r+1}P^{s-1} \cap I^rJ^s = I^{r+1}J^s$. Hence by Lemma 3.7

\[
\text{pd}_T(I^rP^s) = \max\{\text{pd}_T(I^{r+1}P^{s-1}), \text{pd}_T(I^rJ^s), \text{pd}_T(I^{r+1}J^s) + 1\}.
\]

Together with the isomorphism $IJ \cong I \otimes_k J$ and Lemma 2.3, we conclude that

\[
\text{pd}_T(I^rP^s) = \max\{\text{pd}_T(I^{r+1}P^{s-1}), \text{pd}_R(I^r) + \text{pd}_S(J^s), \text{pd}_R(I^{r+1}) + \text{pd}_S(J^s) + 1\}.
\]

By induction on $s \geq 1$, we get

\[
\text{pd}_T(I^rP^s) = \max_{i \in [0, s], j \in [1, s]} \{ \text{pd}_R I^{r+s-i} + \text{pd}_S J^i, \text{pd}_R I^{r+s-j+1} + \text{pd}_S J^j + 1 \}.
\]

Setting $r = 0$, we reduce to

\[
\text{pd}_T P^s = \max_{i \in [0, s], j \in [1, s]} \{ \text{pd}_R I^{s-i} + \text{pd}_S J^j, \text{pd}_R I^{s-j+1} + \text{pd}_S J^j + 1 \}.
\]

The term corresponding to $i = s$ can be omitted since it is smaller than the term corresponding to $j = s$. Similarly we can omit the term corresponding to $i = 0$, since it is smaller than the term corresponding to $j = 1$. Hence we receive the desired formula for $\text{pd}_T P^s$. The formula for $\text{reg}_T P^s$ is proved similarly. \qed

**Example 5.2.** The following example shows that the equality for regularity in Proposition 5.1 does not hold without the condition $I$ and $J$ are of small type. We take a cue from the example in [12, Remark 2.7]. Take $R = k[a, b, c, d]/(a^2 - bc, ab, b^2, c^2, ad, bd, cd)$, $I = (a, d)$ and $S = k[x, y, z, t]/(x^2 - yz, xy, y^2, z^2, xt, yt, zt)$,
$J = (x,t)$. (By abuse of notations, we denote the residue class of $a$ in $R$ by $a$ itself, and so on.) We will show that the following hold:

(i) Both $I$ and $J$ are neither of small type nor nilpotent.
(ii) $\text{reg}_R I, \text{reg}_S J \geq 2$.
(iii) $\text{reg}_T P^2 = 3 < 4 \leq \max\{\text{reg}_R I + \text{reg}_S J, \text{reg}_R I^2 + \text{reg}_S J - 1, \text{reg}_R I + \text{reg}_S J^2 - 1\}$.

(i) Clearly $I^n = (d^n)$ for $n \geq 3$, hence $I$ is not nilpotent. By computation with Macaulay2 [24], $\beta_{1,3}(I) \neq 0$ while $\beta_{1,3}(I/I^2) = 0$, so the map $\text{Tor}_1^R(k,I) \to \text{Tor}_1^S(k, I/I^2)$ is not injective. This shows that $I$ is not of small type. By symmetry, $J$ is also neither nilpotent nor of small type.

(ii) As mentioned above, $\beta_{1,3}(I) \neq 0$, hence $\text{reg}_R I \geq 2$. Similarly $\text{reg}_S J \geq 2$.

(iii) Denote $U = P^2 + (d) = (d, t^2, ta, ax, a^2, x^2)$, $V = U + (t) = (d, t, ax, a^2, x^2)$. By direct computations, $P^2 : d = (a, b, c, d, x, t)$ and $U : t = (a, d, x, y, z, z)$. Hence there are exact sequences

$$0 \to P^2 \to P^2 + (d) = U \to \frac{(d)}{(d) \cap P^2} \cong \left( \frac{T}{(a,b,c,d,x,t)} \right) (-1) \to 0.$$ 

and

$$0 \to U \to U + (t) = V \to \frac{(t)}{(t) \cap U} \cong \left( \frac{T}{(a,d,x,y,z,t)} \right) (-1) \to 0.$$ 

By elementary arguments, for our purpose, it suffices to show that

(a) $\text{reg}_T(a, b, c, d, x, t) = 2$, $\text{reg}_T(a, d, x, y, z, t) = 2$,

(b) $\text{reg}_T V = 3$.

The following is a Koszul filtration for $R$ (see [15] for more details on Koszul filtrations): $\{(0), (c), (c, d), (c, b), (c, b, a), (c, b, a, d), \}$. Hence $R$ is a Koszul algebra by [15, Proposition 1.2], and by symmetry, the same is true for $S$. Letting $p = m + n$, we then obtain the following Koszul filtration for $T$:

$$\mathcal{F} = \{(0), (c), (c, d), (c, b), (c, b, a), (c, b, a, d, z), (c, b, a, d, t), (c, b, a, d, z, t), (c, b, a, d, z, y), (c, b, a, d, z, x), (c, b, a, d, x), (c, b, a, d, y), (c, b, a, d, x), (c, b, a, d, y), (c, b, a, d, x), (c, b, a, d, y), \}.$$ 

Enlarging the filtration $\mathcal{F}$ by adding the ideals $(d), (d, t), (d, z), (d, t, z), (d, z, y), (d, z, y, x), (d, z, y, x, t)$, we get a larger Koszul filtration of $T$. In particular, $\text{reg}_T T/(d,t) = 0$. By [13, Section 3.2, Proposition 3], this implies that $\text{reg}_T M = \text{reg}_T M$ for any finitely generated graded $T$-module $M$ with $(d,t)M = 0$. Replacing $R, S, T$ by $R' = R/(d), S' = S/(t), T' = T' \otimes_k S'$, we reduce (a) and (b) to the following statements

(c) $\text{reg}_{T'}(a, b, c, x) = 2$, $\text{reg}_{T'}(a, x, y, z) = 2$.

(d) $\text{reg}_{T'}(a, x)^2 = 3$.

(c) By symmetry, it suffices to prove the second equality, which is equivalent to

$$1 = \text{reg}_{T'}((R'/a) \otimes_k (S'/y, z)).$$ 

Using Lemma 2.3 and the Koszulness of $S'$, this reduces to $\text{reg}_{R'}(a) = 2$. Thanks to $(a) \cong (R'/b, ac)(-1)$, we will finish the proof of (c) by showing that $\text{reg}_{R'} (b, ac) = 2$. Note that $R'$ admits the following Koszul filtration $\{(0), (b), (b, a), (b, a, c)\}$. Moreover, $(0) : b = (b, a), (b) : (ac) = (b, a, c)$. Hence $(b, ac)$ is a Koszul module by [40, Proposition 5.11] and as $R'$ is a Koszul algebra, Subsection 2.2 yields $\text{reg}_{R'}(b, ac) = 2$. Therefore $\text{reg}_{T'}(a, x, y, z) = 2$, as desired.
(d) Denote \(V' = (a, x)^2, W = (a^2)\) then \(V' = W + (ax, x^2)\). Since \(W \cong (T'/(a, b, c))(−2)\), we have that \(\text{reg}_{T'} W = 2\). We show next that \(\text{reg}_{T'} (W + (ax)) = 3\). For this, look at the exact sequence

\[
0 \rightarrow W \rightarrow W + (ax) \rightarrow \left(\frac{T'}{W : ax}\right)(−2) \rightarrow 0.
\]

It remains to show that \(\text{reg}_{T'}(b, a, y, xz) = 2\). Arguing similarly as above, we obtain \(\text{reg}_{R}(b, a) = 1\) and \(\text{reg}_{S}(y, xz) = 2\), hence Proposition 3.11 yields \(\text{reg}_{T'}(b, a, y, xz) = 2\). The last exact sequence implies \(\text{reg}_{T'}(W + (ax)) \leq 3\) and the reverse inequality holds by direct inspection, so \(\text{reg}_{T'} (W + (ax)) = 3\).

Finally, since \(V' = W + (ax, x^2)\), to prove \(\text{reg}_{T'} V' = 3\), we look at

\[
0 \rightarrow W + (ax) \rightarrow V' \rightarrow \left(\frac{T'}{(W + (ax)) : x^2}\right)(−2) \rightarrow 0.
\]

By (c), the last term in the above exact sequence has regularity 3 over \(T'\), therefore \(\text{reg}_{T'} V' \leq 3\). The reverse inequality is true by direct inspection, hence \(\text{reg}_{T'} V' = 3\). The proof is completed.

Unfortunately, we could not find an example showing that the condition \(I\) and \(J\) are of small type is necessary for the equality of projective dimension in Proposition 5.1.

We are now able to deliver the

**Proof of Theorem 1.1.** Since \(\text{char } k = 0\) or \(I\) and \(J\) are monomial, by Lemma 4.1(ii) and Theorem 4.5, \(I\) and \(J\) are of small type. Using the Auslander–Buchsbaum’s formula, Proposition 5.1, and the equality \(\dim T = \dim R + \dim S\), we see that

\[
\text{depth } P^s = \min_{i \in [1, s-1], j \in [1, s]} \left\{\text{depth } I^{s-i} + \text{depth } J^i, \text{depth } I^{s-j+1} + \text{depth } J^j - 1\right\}.
\]

Since \(I, J, P \neq (0)\), we can use the equality \(\text{depth } R/I = \text{depth } I - 1\) to obtain (i).

As \(R\) is a polynomial ring, for any finitely generated graded \(R\)-module \(M\), \(\text{reg } M = \text{reg}_R M\) [19, Proposition, Page 89]. Arguing similarly as for the equality of depth, using the equality \(\text{reg } R/I = \text{reg } I - 1\) and Proposition 5.1, we get (ii).

**Example 5.3.** It is natural to ask whether Theorem 1.1 is still true if \(R\) and \(S\) are not polynomial rings. The answer is no, even if \(\text{char } k = 0\) and \(I^i, J^j \neq 0\) for all \(i \geq 1\). Consider \(R = \mathbb{Q}[a, b, c, d, e]/(a^2, ab, bc, d^2, de), I = (a - b, b + d), S = \mathbb{Q}[x, y, z]/(x^2, xz, yz)\) and \(J = (x + z)\). Computations with Macaulay2 [24] show that:

1. \(\text{depth } (T/P^2) = 1 > 0 = \text{depth } (R/I) + \text{depth } (S/J^2)\). Hence the equality (i) for depth of Theorem 1.1 does not hold.
2. \(\text{reg } (T/P^2) = 2 < 3 = \text{reg } (R/I) + \text{reg } (S/J) + 1\). Hence the equality (ii) for Castelnuovo–Mumford regularity of Theorem 1.1 does not hold.

On the other hands, both equalities of Proposition 5.1 are true, thanks to Proposition 5.4 below. Indeed, since \(J^n = ((x + z)^n)\) and \(x + z\) is a zero-divisor, \(\text{pd}_S J^n = \text{pd}_S J = \infty\) for all \(n \geq 1\). Furthermore, observe that \(0 : (x + z) = (x)\) and \(0 : (x + z)^n = 0 : z^n = (x, y)\) for \(n \geq 2\). Since the collection of ideals generated by residue classes of variables forms a Koszul filtration for \(S\), \(\text{reg}_S J^n = n\) for all \(n \geq 1\).
We may see from this example why Proposition 5.1, which deals with projective dimension and regularity over $T$, is a natural generalization of Theorem 1.1.

In some special cases, the formulas of Proposition 5.1 hold with the assumption that only one of $I$ and $J$ is of small type. The following result is a generalization of [25, Proposition 2.9]; the proof of the latter cannot be adapted to our situation.

**Proposition 5.4.** Assume that $I^i \neq 0$ for all $i \geq 1$.

(i) Assume that $J$ is of small type and $\text{pd}_S J^i = \text{pd}_S J$ for all $i \geq 1$. Then for all $s \geq 1$, there is an equality $\text{pd}_T P^s = \max_{i \in [1,s]} \{ \text{pd}_R I^i \} + \text{pd}_S J + 1$.

(ii) Assume that $\text{reg}_S J^i = i$ for all $i \geq 1$. Then for all $s \geq 1$, there is an equality $\text{reg}_T P^s = \max_{i \in [1,s]} \{ \text{reg}_R I^i - i \} + s$.

*Proof.* Following [3, Page 450], we say that a map of graded $R$-modules $M \xrightarrow{\phi} P$ is small if $\text{Tor}_i^R(k, \phi)$ is injective for all $i \geq 0$. First we claim that if $J$ is of small type then for all $r \geq 0, s \geq 1$, the inclusion map $I^{r+1}P^{s-1} \to I^r P^s$ is small. Consider the following diagram with exact rows and denote $U = (I^r J^s)/(I^{r+1} J^s) \cong (I^r/I^{r+1}) \otimes_k J^s$.

\[
\begin{array}{cccccc}
0 & \to & I^{r+1} J^s & \to & I^r J^s & \to & 0 \\
\downarrow{\gamma} & & \downarrow{\gamma} & & \downarrow{\gamma} & \\
0 & \to & I^{r+1} P^{s-1} & \to & I^r P^s & \to & 0
\end{array}
\]

The long exact sequence of homology induces the following commutative rectangle.

\[
\begin{array}{ccc}
\text{Tor}_{i+1}^T(k, U) & \to & \text{Tor}_i^T(k, I^{r+1} J^s) \\
\downarrow{=} & & \downarrow{=} \\
\text{Tor}_{i+1}^T(k, U) & \to & \text{Tor}_i^T(k, I^{r+1} P^{s-1})
\end{array}
\]

Since $J$ is of small type, similarly to the proof of Theorem 4.10, the map $I^{r+1} J^s \to I^{r+1} P^{s-1}$ is Tor-vanishing, hence the diagram yields that the connecting map on the second row is zero. In other words, $I^{r+1} P^{s-1} \to I^r P^s$ is small, as claimed.

(i) Since $J$ is of small type, by the claim, we deduce

\[
\text{pd}_T(I^r P^s) = \max \{ \text{pd}_T(I^{r+1} P^{s-1}), \text{pd}_T U \}
\]

\[
= \max \{ \text{pd}_T(I^{r+1} P^{s-1}), \text{pd}_R(I^r/I^{r+1}) + \text{pd}_S J^s \},
\]

\[
= \max \{ \text{pd}_T(I^{r+1} P^{s-1}), \text{pd}_R(I^r/I^{r+1}) + \text{pd}_S J \}.
\]

The second equality follows from Lemma 2.3. The last one follows from the hypothesis. Using induction on $s \geq 1$, we then get

\[
\text{pd}_T(I^r P^s) = \max_{i \in [1,s]} \{ \text{pd}_R(I^{r+i-1}/I^{r+i}) + \text{pd}_S J, \text{pd}_R I^{r+s} \}.
\]

Setting $r = 0$, we obtain \( \text{pd}_T P^s = \max_{i \in [1,s]} \{ \text{pd}_R(I^{i-1}/I^i) + \text{pd}_S J, \text{pd}_R I^s \} \). For the desired equality, it remains to show that

\[
\max_{i \in [1,s]} \{ \text{pd}_R(I^{i-1}/I^i), \text{pd}_R I^s - \text{pd}_S J \} = \max_{i \in [1,s]} \{ \text{pd}_R I^i + 1 \}.
\]

Using $\text{pd}_R(R/I) = \text{pd}_R I + 1$ and $\text{pd}_R(I^{i-1}/I^i) \leq \max \{ \text{pd}_R I^{i-1}, \text{pd}_R I^i + 1 \}$ for $2 \leq i \leq s$, we see that the left-hand side is \( \leq \) the right-hand side.
Conversely, using $pd_R I^i + 1 \leq \max\{pd_R I^{i-1} + 1, pd_R (I^{i-1}/I^i)\}$ for $2 \leq i \leq s$, we get the reverse inequality. This finishes the proof of (i).

(ii) By Proposition 4.6, $J$ is of small type. Take $r \geq 0$. By the claim from above, we again have the map $I^{r+1}P^{s-1} \to I^r P^s$ is small, so there is a chain

$$\text{reg}_T(I^r P^s) = \max\{\text{reg}_T(I^{r+1}P^{s-1}), \text{reg}_T U\} = \max\{\text{reg}_T(I^{r+1}P^{s-1}), \text{reg}_R(I^r/I^{r+1}) + \text{reg}_S J^s\},$$

$$= \max\{\text{reg}_T(I^{r+1}P^{s-1}), \text{reg}_R(I^r/I^{r+1}) + s\}.$$

By induction on $s \geq 1$, we obtain

$$\text{reg}_T(I^r P^s) = \max_{i \in [0, s-1]} \{\text{reg}_T I^{r+s}, \text{reg}_S (I^{r+i+1}/I^{r+1}) + s - i\}.$$

Setting $r = 0$, we have $\text{reg}_T P^s = \max_{i \in [0, s-1]} \{\text{reg}_T I^s, \text{reg}_R(I^i/I^{i+1}) + s - i\}$. For the desired conclusion, it remains to show

$$\max_{i \in [0, s-1]} \{\text{reg}_T I^{s} - s, \text{reg}_R(I^i/I^{i+1}) - i\} = \max_{i \in [1, s]} \{\text{reg}_R I^i - i\}.$$

This is standard: for $\leq$, we note that $\text{reg}_R(I^i/I^{i+1}) \leq \max\{\text{reg}_R I^i, \text{reg}_R I^{i+1} - 1\}$ for $0 \leq i \leq s - 1$. For $\geq$, we use the inequality max$\{\text{reg}_R I^{-1}, \text{reg}_R(I^{-1}/I^i) + 1\} \geq \text{reg}_R I^i$ for $2 \leq i \leq s$ and the equality $\text{reg}_R I^{-1} = \text{reg}_S(R/I)$. The proof is concluded. □

5.2. Asymptotes. Define the index of projective dimension stability of $I$ to be $\text{pstab}(I) = \min\{r \geq 1 : pd_R I^i = \lim_{j \to \infty} pd_R I^j\}$ for all $i \geq 1$. This is a well-defined finite number, e.g. by a result of Kodiyalam [37, Corollary 8].

**Theorem 5.5.** Assume that $I$ and $J$ are of small type and $I^i, J^i \not= 0$ for all $i \geq 1$. Then for all $s \geq \text{pstab}(I) + \text{pstab}(J)$, we have

$$pd_T P^s = \max \left\{ \lim_{i \to \infty} pd_R I^i + \max_{j \geq 1} pd_S J^j + 1, \max_{i \geq 1} pd_R I^i + \lim_{j \to \infty} pd_S J^j + 1 \right\}.$$

**Proof.** By Proposition 5.1 and its proof, we have

$$pd_T P^s = \max_{i \in [1, s], j \in [1, s]} \{pd_R I^{s-i} + pd_S J^j, pd_R I^{s-i+1} + pd_S J^j + 1\}.$$

First we determine max$\{pd_R I^{s-i} + pd_S J^j\}$. If $\text{pstab}(J) \leq i \leq s$, then $pd_S J^j = \lim_{j \to \infty} pd_S J^j$. For every such $i$, $0 \leq s - i \leq s - \text{pstab}(J)$ and $s - \text{pstab}(J) \geq \text{pstab}(I)$, hence

$$\max_{\text{pstab}(J) \leq i \leq s} pd_R I^{s-i} = \max_{i=0, \ldots, \text{pstab}(I)} pd_R I^i = \max_{i \geq 1} pd_R I^i.$$

So we obtain

$$\max_{i=\text{pstab}(J), \ldots, s} \{pd_R I^{s-i} + pd_S J^j\} = \max_{i \geq 1} \lim_{j \to \infty} pd_R I^i + \lim_{j \to \infty} pd_S J^j.$$

If $1 \leq i \leq \text{pstab}(J)$ then $s - i \geq \text{pstab}(I)$, hence $pd_R I^{s-i} = \lim_{i \to \infty} pd_R I^i$. Thus

$$\max_{i=1, \ldots, \text{pstab}(J)} \{pd_R I^{s-i} + pd_S J^j\} = \lim_{i \to \infty} pd_R I^i + \max_{j=1, \ldots, \text{pstab}(J)} pd_S J^j,$$

$$= \lim_{i \to \infty} pd_R I^i + \max_{j \geq 1} pd_S J^j.$$
Powers of Sums

Summing up

$$\max_{i=1,\ldots,s} \{pd_R I^{s-i} + pd_S J^i\} =$$

$$\max\left\{ \max_{i \geq 1} pd_R I^i + \lim_{j \to \infty} pd_S J^j, \lim_{i \to \infty} pd_R I^i + \max_{j \geq 1} pd_S J^j \right\}. $$

Similarly, we have

$$\max_{j=1,\ldots,s} \{pd_R I^{s-j+1} + pd_S J^j + 1\} =$$

$$\max\left\{ \max_{i \geq 1} pd_R I^i + \lim_{j \to \infty} pd_S J^j + 1, \lim_{i \to \infty} pd_R I^i + \max_{j \geq 1} pd_S J^j + 1 \right\}. $$

This yields the desired equality for $pd_T P^s$. \(\square\)

The index of depth stability of $I$, denoted by $dstab(I)$, is the least number $r$ such that $\text{depth } I^s = \lim_{s \to \infty} \text{depth } I^s$ for all $i \geq r$. The next theorem recovers partly [25, Theorem 4.6] with an easier argument.

**Theorem 5.6.** Assume that $R$ and $S$ are polynomial rings over $k$, and either $\text{char } k = 0$ or $I$ and $J$ are monomial ideals. Then for all $s \geq dstab(I) + dstab(J)$, we have

$$pd_T P^s = \max\left\{ \lim_{i \to \infty} pd_R I^i + \max_{j \geq 1} pd_S J^j + 1, \max_{i \geq 1} pd_R I^i + \lim_{j \to \infty} pd_S J^j + 1 \right\}. $$

**Proof.** Since $R, S, T$ are regular rings, by the Auslander–Buchsbaum’s formula, we have equalities $pstab(I) = dstab(I), pstab(J) = dstab(J)$. Moreover, by Lemma 4.1(ii) and Theorem 4.5, $I$ and $J$ are of small type. Hence the result follows from Theorem 5.5. \(\square\)

Assume that $R$ and $S$ are polynomial rings over $k$. By the result due of Kodiyalam [38] and Cutkosky–Herzog–N.V. Trung [16], we can define the regularity stabilization index of $I$, denoted $\text{reg} \text{stdab}(I)$, as follows

$$\min\{r \geq 1 : \text{there exist constants } a \text{ and } g \text{ such that } \text{reg } I^s = as + g \text{ for all } s \geq r\}. $$

Note that in [25], the notation $\text{lin}(I)$ was used in place of $\text{reg} \text{stdab}(I)$. The next result is a special case of [25, Proposition 5.7] but its proof is easier.

**Theorem 5.7.** Assume that $R$ and $S$ are polynomial rings over $k$. Assume in addition that either $\text{char } k = 0$ or $I$ and $J$ are monomial ideals. Suppose that $\text{reg } I^r = ar + g$ for all $r \geq \text{reg} \text{stdab}(I)$ and $\text{reg } J^s = bs + h$ for all $s \geq \text{reg} \text{stdab}(J)$, and moreover $a \geq b$. Denote $g^* = \max_{i=1,\ldots,\text{reg} \text{stdab}(I)}(\text{reg } I^i - bi)$ and $h^* = \max_{j=1,\ldots,\text{reg} \text{stdab}(J)}(\text{reg } J^j - aj)$. Then for all $s \geq \text{reg} \text{stdab}(I) + \text{reg} \text{stdab}(J)$, we have

$$\text{reg } P^s = \max\{a(s+1) + g + h^*, b(s+1) + g^* + h\} - 1. $$

**Sketch of proof.** By Lemma 4.1(ii) and Theorem 4.5, $I$ and $J$ are of small type. Hence applying Proposition 5.1 and its proof, we have

$$\text{reg } P^s = \max_{i \in [1,s], j \in [1,s]} \{\text{reg } I^{s-i} + \text{reg } J^j, \text{reg } I^{s-j+1} + \text{reg } J^j - 1\}. $$ (5.1)

First we show that

$$\max_{i=1,\ldots,s} \{\text{reg } I^{s-i} + \text{reg } J^j\} = \max\{as + g + h^*, bs + g^* + h\}. $$ (5.2)
The argument runs along the line of the proof of Theorem 5.5. Similarly to (5.2), we have
\[
\max_{i=1,\ldots,s} \{ \text{reg } I^{s-i+1} + \text{reg } J^i - 1 \} = \max \{ a(s+1) + g + h^* - 1, b(s+1) + g^* + h - 1 \}.
\] (5.3)
Combining Equalities (5.1), (5.2) and (5.3), we get the desired equality. \qed

5.3. Applications. In this subsection, assume that \( R \) and \( S \) are polynomial rings over \( k \). Recall that the depth function of \( I \) is given by \( r \mapsto \text{depth}(R/I^r) \) for \( r \geq 1 \). By Brodmann’s theorem [6], the sequence of values of any depth function is eventually constant. We say that \( I \) has a constant depth function if \( (\text{depth}(R/I^r))_{r \geq 1} \) is a constant sequence. One of the main applications of [25] is the following

**Proposition 5.8** (Hà, Trung and Trung, [25, Proposition 4.7]). Let \( I \) and \( J \) be squarefree monomial ideals of \( R \) and \( S \), resp. Then \( I + J \) has a constant depth function if and only if both \( I \) and \( J \) do.

For squarefree monomial ideals, the last result simplifies significantly the equivalence (i) \( \iff \) (ii) of [34, Theorem 1.1], which has to require that the Rees algebras of \( I \) and \( J \) are Cohen–Macaulay. Thanks to Theorem 1.1, we obtain the following improvement of (the “if” part of) Proposition 5.8.

**Corollary 5.9.** If either \( \text{char } k = 0 \), or \( I \) and \( J \) are (possibly non-squarefree) monomial ideals, then \( I + J \) has a constant depth function if \( I \) and \( J \) do. If \( I \) and \( J \) are squarefree monomial ideals, then conversely, if \( I + J \) has a constant depth function then both \( I \) and \( J \) do.

Our proof of the second part of this result is more straightforward than the original proof of [25, Proposition 4.7]. As noted in [25, Example 4.8], the second statement of Corollary 5.9 does not hold if \( I \) and \( J \) are non-squarefree monomial ideals, even in characteristic zero.

**Proof of Corollary 5.9.** The first statement is immediate from Theorem 1.1(i).

For the second statement, we use the inequality \( \text{depth } R/\sqrt{I} \geq \text{depth } R/L \) for \( L \) being a monomial ideal of \( R \) [33, Proof of Theorem 2.6]. In particular, since \( I \) and \( J \) are squarefree monomial ideals, \( \text{depth } R/I \geq \text{depth } R/I^s \) and \( \text{depth } S/J \geq \text{depth } S/J^s \) for all \( s \geq 1 \). By Theorem 1.1(i), \( \text{depth } T/P = \text{depth } R/I + \text{depth } S/J \) and for all \( n \geq 2 \),
\[
\text{depth } T/P^n = \min_{i \in [1,s-1], j \in [1,s]} \{ \text{depth } R/I^i + \text{depth } S/J^{s-i+1} + 1, \text{depth } R/I^i + \text{depth } S/J^{s-j} \}
\]
\[
\leq \min_{j \in [1,s]} \{ \text{depth } R/I^j + \text{depth } S/J^{s-j} \}
\]
\[
\leq \text{depth } R/I + \text{depth } S/J = \text{depth } T/P.
\]
Since \( P = I + J \) has a constant depth function, we conclude that \( \text{depth } I^s = \text{depth } I \) and \( \text{depth } J^s = \text{depth } J \) for all \( s \geq 1 \). The proof is finished. \qed

The authors of [25] could not provide an upper bound for the index of depth stability of \( P \) in terms of those of \( I \) and \( J \) (see page 821 of loc. cit.). With the assumptions of Theorem 1.1, we can easily provide a reasonable bound. Theorems 5.6 and 5.7 yield the following inequalities. The second of these inequalities provides a slightly better upper bound for \( \text{rstab}(I + J) \) than [25, Corollary 5.8].
Corollary 5.10. Assume that either \( \text{char } k = 0 \) or \( I \) and \( J \) are monomial ideals. Then \( \text{dstab}(I + J) \leq \text{dstab}(I) + \text{dstab}(J) \). If furthermore, in the notations of Theorem 5.7, we have \( a = b \), then \( \text{rstab}(I + J) \leq \text{rstab}(I) + \text{rstab}(J) \).

The following statement is possibly of independent interest.

Corollary 5.11. Assume that either \( \text{char } k = 0 \) or \( I \) is a monomial ideal. Then for all \( s \geq 1 \), the map \( \text{Tor}_R^i(k, I^s) \to \text{Tor}_R^i(k, I^{s-1}) \) is zero for every \( i \geq 0 \). Moreover, there are equalities

\[
\text{depth}(I^{s-1}/I^s) = \min\{\text{depth } I^{s-1}, \text{depth } I^s - 1\},
\]

\[
\text{reg}(I^{s-1}/I^s) = \max\{\text{reg } I^{s-1}, \text{reg } I^s - 1\}.
\]

Proof. The first statement follows from combining Lemma 4.1(ii) and Theorem 4.5. The equalities follow from Lemma 3.3 and the Auslander–Buchsbaum’s formula. \( \square \)

Remark 5.12. The authors of [25] expressed in page 832 their feeling that “In general, there seems to be no relationships between the stability indices of the functions depth \( R/I^n \) and depth \( I^{n-1}/I^n \)”. Corollary 5.11 renders this opinion disputable: the stability index of the function depth \( I^{s-1}/I^s \) is at most the stability index of the function depth \( R/I^s \), at least if \( \text{char } k = 0 \) or \( I \) is monomial.

6. LINEARITY DEFECT

In this section, we characterize the (asymptotic) linearity defect of powers of \( P \) in terms of the data of \( I \) and \( J \). The goal is to find formulas similar to those of Proposition 5.1 and Theorem 5.5. If \( I \) and \( J \) are of small type, using Betti splittings, we get similar upper bounds which in many situations are actual values for \( \text{ld}_T P^s \). In fact the equality occurs if \( I \) and \( J \) are of doubly small type. There is a lower bound \( \max\{\text{ld}_R I^s, \text{ld}_S J^s\} \leq \text{ld}_T P^s \) following from Lemma 2.4, but this is far from sharp even for the starting case \( s = 1 \), as seen from Proposition 3.11(ii).

6.1. Upper bounds and formulas. The first main result of this section is the following. Part (ii) of the next result is the motive for introducing ideals of doubly small type.

Theorem 6.1. Let \( I \) and \( J \) be ideals of small type. Assume that \( I^i \neq 0, J^i \neq 0 \) for all \( i \geq 1 \), e.g. \( R \) and \( S \) are reduced. Then the following statements hold:

(i) For all \( s \geq 1 \), we have an inequality

\[
\text{ld}_T P^s \leq \max_{i \in [1, s-1], j \in [1, s]} \{\text{ld}_R I^{s-i} + \text{ld}_S J^i, \text{ld}_R I^{s-j+1} + \text{ld}_S J^j + 1\}.
\]

(ii) Assume further that both \( I \) and \( J \) are of doubly small type. Then the inequality in (i) is an equality for all \( s \geq 1 \).

Proof. (i) By Theorem 4.10, the decomposition \( I^s P^s = I^{s+1} P^{s-1} + I^s J^s \) is a Betti splitting. Hence by Proposition 3.9 and Lemma 2.2, we obtain

\[
\text{ld}_T(I^s P^s) \leq \max\{\text{ld}_T(I^{s+1} P^{s-1}), \text{ld}_T(I^s J^s), \text{ld}_T(I^{s+1} J^s) + 1\}. \tag{6.1}
\]

Using Lemma 2.3, we then get

\[
\text{ld}_T(I^s P^s) \leq \max\{\text{ld}_T(I^{s+1} P^{s-1}), \text{ld}_R I^r + \text{ld}_S J^s, \text{ld}_R I^{s+1} + \text{ld}_S J^s + 1\}.
\]

Using induction on \( s \), we infer that

\[
\text{ld}_T(I^s P^s) \leq \max_{i \in [0, s], j \in [1, s]} \{\text{ld}_R I^{r+s-i} + \text{ld}_S J^i, \text{ld}_R I^{r+s-j+1} + \text{ld}_S J^j + 1\}. \tag{6.2}
\]
Setting $r = 0$ in (i), the conclusion is
\[ \text{ld}_R P^s \leq \max_{i \in [0, s], j \in [1, s]} \left\{ \text{ld}_R I^{s-1} + \text{ld}_R J^i, \text{ld}_R I^{r-j+1} + \text{ld}_R J^j + 1 \right\}. \]
The term on the right-hand side corresponding to $i = 0$ can be omitted, since it is smaller than the term corresponding to $j = 1$. Similarly, the term on the right-hand side corresponding to $i = s$ can be omitted, since it is smaller than the term corresponding to $j = s$. Hence the desired inequality follows.

(ii) Since $I$ and $J$ are of doubly small type, by Theorem 4.10, the maps $I^{r+1}J^s \to I^rJ^s$ and $I^{r+1}J^s \to I^{r+1}P^{s-1}$ are doubly Tor-vanishing. Consider the exact sequence $0 \to I^{r+1}J^s \to I^rJ^s \oplus I^{r+1}P^{s-1} \to I^rP^s \to 0$. As all the powers of $I$ and $J$ are non-trivial, we also have $I^rJ^j \cong I^r \otimes_k J^j \neq 0$ for all $i, j \geq 0$. Applying Lemma 3.3, we see that (6.1) is now an equality. Hence so is (6.2), giving the desired conclusion. \hfill \Box

**Example 6.2.** The following example shows that the condition $I$ and $J$ are of small type is necessary for the conclusion of Theorem 6.1. Consider the rings and ideals $R = k[a, b, c]/(a^2, bc), I = (a + b)$ and $S = k[x, y, z]/(x^2, yz), J = (x + y)$. We will show that

(i) $I$ and $J$ are neither of small type nor nilpotent,

(ii) $\text{ld}_R I = \text{ld}_S J = 1, \text{ld}_R I^2 = \text{ld}_S J^2 = 0$,

(iii) $\text{ld}_E P^2 \geq 4 > 2 = \max\{\text{ld}_R I + \text{ld}_S J, \text{ld}_R I^2 + \text{ld}_S J + 1, \text{ld}_R I + \text{ld}_S J^2 + 1\}$.

(i) We have $I^n = (a + b)b^{n-1}$ for $n \geq 2$, hence $I$ is not nilpotent. By direct inspection, $0 : (a + b) = (ac)$, hence $\beta_{1, 3}(I) \neq 0$. On the other hand, $(a + b)^2 : (a + b) = (a + b)$, hence $\beta_{1, 3}(I/I^2) = 0$. The map $\text{Tor}_1^R(k, I) \to \text{Tor}_1^R(k, I/I^2)$ being not injective, so $I$ is not of small type. By symmetry, $J$ is neither nilpotent nor of small type.

(ii) Since $R$ is Koszul and $\text{reg}_R I \geq 2, \text{ld}_R I \geq 1$. On the other hand, $0 : (a + b) = (ac)$ and as $R$ is defined by a quadratic monomial ideal, $0 : (ac) = (a, b)$ has a 1-linear resolution over $R$. This implies that $\text{ld}_R I = 1$.

Observe that $I^2 = (ab + b^2)$ and $0 : (ab + b^2) = (c)$ has a 1-linear resolution over $R$, so $\text{ld}_R I^2 = 0$. By symmetry, we also have $\text{ld}_S J = 1, \text{ld}_S J^2 = 0$.

(iii) By computations with Macaulay2 [24], $\beta_{3, 8}(P^2) \neq 0$ while $\beta_{3, i} = 0$ for $i \geq 7$. Let $F$ be the minimal graded free resolution of $P^2$ over $T$. Then in $\text{lin}^T F$, the map $(\text{lin}^T F)_4 \to (\text{lin}^T F)_3$ has non-trivial kernel. This implies that $\text{ld}_T P^2 \geq 4$, as desired.

The second main result of this section is an analog of Proposition 5.4(ii). We were benefited by the method of [25] in Step 2 of the proof.

**Theorem 6.3.** Assume that $J$ is of small type and $I$ is an ideal generated by linear forms of $R$ such that all the powers of $I$ are Koszul. Assume further that $I^i, J^j \neq 0$ for all $i \geq 1$. Then for all $s \geq 1$, we have $\text{ld}_T P^s = \max_{i \in [1, s]} \{ \text{ld}_S J^i \}$.

**Proof.** We prove more generally for all $r \geq 0, s \geq 1$ the equality $\text{ld}_T (I^rP^s) = \max_{i \in [1, s]} \{ \text{ld}_S J^i \}$.

**Step 1:** Consider the following exact sequence
\[ 0 \to I^{r+1}P^{s-1} \to I^rP^s \to I^rJ^s/(I^{r+1}J^s) \cong (I^r/I^{r+1}) \otimes_k J^s \to 0. \]
Since $J$ is of small type, by the claim in the proof of Proposition 5.4, the map $\text{Tor}_{i+1}^R(k, I^s/(I^{i+1} J^i)) \to \text{Tor}_i^R(k, I^{i+1} P^{s-1})$ is zero for all $i \geq 0$. By [40, Proposition 2.5(ii)], there is an inequality
\[
ld_T(I^s P^s) \leq \max \{ld_T(I^{i+1} P^{s-1}), ld_T((I^i/I^{i+1}) \otimes_k J^s)\}. \tag{6.3}
\]
Using Theorem 3.10, we prove that $I^i/I^{i+1}$ is a Koszul $R$-module. For this, we need to check that (a) $I^i$ and $I^{i+1}$ are Koszul, and (b) $I^{i+1} \cap m^{s+1} I^i = m^sI^{i+1}$ for all $s \geq 0$. The hypothesis guarantees (a), while (b) holds true by degree reason and the fact that $I$ is generated by linear forms. Hence by Theorem 3.10, $I^i/I^{i+1}$ is Koszul. Together with Lemma 2.3, we conclude that $\text{ld}_T((I^i/I^{i+1}) \otimes_k J^s) = \text{ld}_S J^s$. Hence from (6.3), we get the inequality $\text{ld}_T(I^s P^s) \leq \max\{\text{ld}_T(I^{i+1} P^{s-1}), \text{ld}_S J^s\}$. Using induction on $s \geq 1$ and the fact that the powers of $I$ are Koszul, we get $\text{ld}_T(I^s P^s) \leq \max_{i \in [1,s]} \{\text{ld}_S J^i\}$.

**Step 2:** It remains to show that $\text{ld}_T(I^s P^s) \geq \max_{i \in [1,s]} \{\text{ld}_S J^i\}$ for all $r \geq 0, s \geq 1$. There is nothing to do if the right-hand side is 0, hence we will assume its positivity.

By Lemmas 2.2 and 2.3, there is an equality
\[
\text{ld}_T(I^s P^s/I^r P^s) = \max_{i \in [0, s-1]} \{\text{ld}_R(I^{r+i}/I^{i+1}) + \text{ld}_S(J^{s-i-1}/J^{s-i})\}.
\]
As proved in (i), $I^i/I^{i+1}$ is Koszul for all $r$. So we get
\[
\text{ld}_T(I^s P^{s-1}/I^r P^s) = \max_{i \in [1,s]} \{\text{ld}_S(J^{i-1}/J^i)\}.
\]

**Claim:** Given $\max_{i \in [1,s]} \{\text{ld}_S J^i\} \geq 1$, it holds that
\[
\max_{i \in [1,s]} \{\text{ld}_S(J^{i-1}/J^i)\} = \max_{i \in [1,s]} \{\text{ld}_S J^i\} + 1. \tag{6.4}
\]

**Proof of the claim.** Since $J$ is of small type, Proposition 3.8 yields $\text{ld}_S(J^{i-1}/J^i) \leq \max\{\text{ld}_S J^{i-1}, \text{ld}_S J^i + 1\}$ for $1 \leq i \leq s$. These inequalities imply that the left-hand side of (6.4) is $\leq$ the right-hand side.

For the reverse inequality, again using Proposition 3.8, we have
\[
\text{ld}_S J^i \leq \max\{\text{ld}_S J^{i-1}, \text{ld}_S(J^{i-1}/J^i) - 1\}
\]
for $2 \leq i \leq s$. Hence it suffices to show that $\text{ld}_S J^{i+1} \leq$ the left-hand side of (6.4). If $\text{ld}_S(S/J) \geq 1$, we have $\text{ld}_S J + 1 = \text{ld}_S(S/J)$ and we are done. If $\text{ld}_S(S/J) = 0$ then also $\text{ld}_S J = 0$ and the desired inequality holds unless the left-hand side of (6.4) is zero. But if that happens then by induction on $i$ and Proposition 3.8, we have $\text{ld}_S J^i = 0$ for $1 \leq i \leq s$. This contradicts the assumption $\max_{i \in [1,s]} \{\text{ld}_S J^i\} \geq 1$. So $\text{ld}_S J + 1 \leq$ the left-hand side of (6.4) and the claim follows.

The previous discussions show that
\[
\text{ld}_T(I^s P^s/I^r P^s) = \max_{i \in [1,s]} \{\text{ld}_S J^i\} + 1.
\]

By the proof of Theorem 4.10, the map $I^r P^s \to I^s P^{s-1}$ is Tor-vanishing. Hence by Proposition 3.8, there is an inequality
\[
\text{ld}_T(I^s P^{s-1}/I^r P^s) \leq \max\{\text{ld}_T(I^r P^{s-1}), \text{ld}_T(I^s P^s) + 1\}.
\]

Putting everything together, we obtain
\[
\max_{i \in [1,s]} \{\text{ld}_S J^i\} + 1 \leq \max\{\text{ld}_T(I^r P^{s-1}), \text{ld}_T(I^s P^s) + 1\}.
\]
Recall that from Step 1, \( \text{ld}_T(I^P s^{s-1}) \leq \max_{i \in [1,s-1]} \{ \text{ld}_S J^i \} \), hence \( \text{ld}_T(I^P s) \geq \max_{i \in [1,s]} \{ \text{ld}_S J^i \} \), as desired. The proof is concluded. \( \square \)

Example 6.4. The conclusion of Theorem 6.3 does not hold if \( J \) is not of small type. Consider \( R = k[a], I = (a), S = k[x,y,z]/(x^2, yz) \) and \( J = (x + y) \). By Example 6.2, \( J \) is not of small type. We claim that in \( T = R \otimes_k S \), \( \text{ld}_T P^2 \geq 2 > 1 = \max\{ \text{ld}_S J, \text{ld}_S J^2 \} \). Indeed, by computations with Macaulay2 \([24]\), \( \beta_{2,5}(P^2) \neq 0 \) while \( \beta_{1,j}(P^2) = 0 \) for \( j \geq 4 \). Similarly to Example 6.2, \( \text{ld}_T P^2 \geq 2 \).

6.2. Asymptotes. Denote by

\[
\text{gl}\text{ld} R = \sup\{ \text{ld}_R M : M \text{ is a finitely generated graded } R\text{-module} \}
\]

the so-called *global linearity defect* of \( R \). For example, if \( R \) is a regular ring or more generally a quadratic hypersurface then \( \text{gl}\text{ld} R = \dim R \); see [14, Corollary 6.4]. Let the *index of linearity defect stability* of \( I, \text{lstab}(I) \), to be the least positive integer \( r \) such that for all \( i \geq r \), \( \text{ld}_R I^i \) is a constant depending only on \( I \). By [43, Theorem 1.1], if \( \text{gl}\text{ld} R < \infty \) then \( \text{lstab}(J) \) is well-defined and finite for every \( I \).

The third main result of Section 6 is

**Theorem 6.5.** Assume that \( R \) and \( S \) have finite global linearity defect and \( I^i \neq 0, J^i \neq 0 \) for all \( i \geq 1 \). Assume further that \( I \) and \( J \) are of doubly small type. Then for all \( s \geq \text{lstab}(I) + \text{lstab}(J) \), we have

\[
\text{ld}_T P^s = \max \left\{ \lim_{i \to \infty} \text{ld}_R I^i + \max_{j \geq 1} \text{ld}_S J^j + 1, \max \text{ld}_R I^i + \lim_{j \to \infty} \text{ld}_S J^j + 1 \right\}.
\]

**Proof.** The proof is parallel to that of Theorem 5.5, taking Theorem 6.1(iii) into account. \( \square \)

**Corollary 6.6.** With the notations and assumptions of Theorem 6.5, there is an inequality \( \text{lstab}(I + J) \leq \text{lstab}(I) + \text{lstab}(J) \).

We are able to compute \( \text{ld}_T P^s \), \( s \) large enough, in some fairly general cases.

**Corollary 6.7.** Let \( (R,m), (S,n) \) be polynomial rings over \( k \). Let \( (0) \neq I \subseteq m^2, (0) \neq J \subseteq n^2 \) be homogeneous ideals of \( R, S \), resp. Assume that one of the following conditions holds:

(i) \( \text{char } k = 0 \) and \( I = U^p \) and \( J = V^q \) where \( p, q \geq 2 \) and \( U, V \) are homogeneous ideals of \( R, S \), resp.

(ii) \( I = U^p \) and \( J = V^q \) where \( p, q \geq 2 \) and \( U, V \) are monomial ideals of \( R, S \), resp.

(iii) All of the powers of \( I \) and \( J \) are Koszul.

Then for all \( s \geq \text{lstab}(I) + \text{lstab}(J) \), there is an equality

\[
\text{ld}_T P^s = \max \left\{ \lim_{i \to \infty} \text{ld}_R I^i + \max_{j \geq 1} \text{ld}_S J^j + 1, \max \text{ld}_R I^i + \lim_{j \to \infty} \text{ld}_S J^j + 1 \right\}.
\]

**Proof.** This follows by combining Propositions 4.6, 4.8 and Theorem 6.5. \( \square \)

If additionally, one of \( I \) and \( J \) is generated by linear forms, we have a nicer asymptotic statement.

**Theorem 6.8.** Let \( R, S \) be polynomial rings over \( k \). Let \( I \) be generated by linear forms. Assume further that one of the following conditions holds:

(i) \( \text{char } k = 0 \).
(ii) $J$ is a monomial ideal of $S$.
(iii) All the powers of $J$ are Koszul.

Then for all $s \geq 1$, there is an equality $\text{ld}_{T} P^s = \max_{i \in [1,s]} \{ \text{ld}_{S} J^i \}$. Moreover, if $1 \leq p \leq \text{lstab}(J)$ be minimal such that $\text{ld}_{S} J^p = \max_{i \geq 1} \{ \text{ld}_{S} J^i \}$ then $\text{ld}_{T} P^s = \max_{i \geq 1} \{ \text{ld}_{S} J^i \}$ for all $s \geq p$ and $\text{lstab}(I + J) = p$.

**Proof.** By Lemma 4.1(ii), Theorem 4.5 and Proposition 4.6, $J$ is of small type. As all the powers of $I$ have a linear resolution, the result is immediate from Theorem 6.3. \qed

### 6.3. Ideals with Koszul powers.

**Corollary 6.9.** Assume that $R$ and $S$ are reduced and $I$ is generated by linear forms. The following statements are equivalent:

(i) All the powers of $I$ and $J$ are Koszul ideals;
(ii) All the powers of $P$ are Koszul ideals.

**Proof.** For (ii) $\Rightarrow$ (i): Take $s \geq 1$. Since $\text{ld}_{T} P^s = 0$, using Lemma 2.4, we get $\text{ld}_{R} I^s \leq \text{ld}_{T} P^s = 0$. Hence $J$ have Koszul powers. Similar arguments work for $J$.

For (i) $\Rightarrow$ (ii): Applying Proposition 4.6, we see that $J$ is of small type. By Theorem 6.3, we conclude the proof. \qed

The following result supplies computation of the linearity defect of powers for a non-trivial class of ideals.

**Theorem 6.10.** Let $(R_1, m_1), \ldots, (R_c, m_c)$ be Koszul, reduced $k$-algebras, where $c \geq 1$. For each $i = 1, \ldots, c$, let $(0) \neq I_i \subseteq m_i^2$ be a homogeneous ideal of $R_i$ such that all the powers of $I_i$ are Koszul. Denote $P = I_1 + I_2 + \cdots + I_c \subseteq T = R_1 \otimes_k \cdots \otimes_k R_c$ the mixed sum of $I_1, \ldots, I_c$. Then $\text{ld}_{T} P^s = c - 1$ for all $s \geq 1$.

**Proof.** We proceed by induction on $c \geq 1$. The case $c = 1$ is a tautology. Assume that $c \geq 2$.

By the hypothesis and Proposition 4.6, $I_i$ is of doubly small type for all $1 \leq i \leq c$. Denote $J = I_2 + \cdots + I_c \subseteq S = R_2 \otimes_k \cdots \otimes_k R_c$, then by Theorem 4.10(ii), $J$ is of doubly small type. By the induction hypothesis, $\text{ld}_{S} J^s = c - 2$ for all $s \geq 1$.

Note that all the powers of $J$ are non-zero, since all the powers of $I_2, \ldots, I_c$ are so. Hence applying Theorem 6.1(ii) to the sum $P = I_1 + J$, we get that for any $s \geq 1$,

$$\text{ld}_{T} P^s = \max_{i \in [1,s-1], j \in [1,s]} \{ \text{ld}_{R_i} I_1^{s-i} + \text{ld}_{S} J^i, \text{ld}_{R_i} I_1^{s-j-1} + \text{ld}_{S} J^j + 1 \} = c - 1.$$  

This concludes the induction and the proof. \qed

**Acknowledgments.** This project was completed in part thanks to our visits to the University of Nebraska – Lincoln in March 2015 and the Vietnam Institute for Advanced Study in Mathematics (VIASM) in April and May 2016. We are grateful to the highly efficient staff and the inspiring working environment at the VIASM, which made our stay an enjoyable experience. We are grateful to Ngô Viết Trung, Trần Nam Trung, and some useful suggestions related to this paper. Thanks are also due to Luchezar Avramov, Hailong Dao and Jeff Mermin for some useful discussions on the terminology. The first author was a Marie Curie fellow of the Istituto Nazionale di Alta Matematica.
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