WZW action in odd dimensional gauge theories

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Abstract

It is shown that Wess-Zumino-Witten (WZW) type actions can be constructed in odd dimensional space-times using Wilson line or Wilson loop. WZW action constructed using Wilson line gives anomalous gauge variations and the WZW action constructed using Wilson loop gives anomalous chiral transformation. We show that pure gauge theory including Yang-Mills action, Chern-Simons action and the WZW action can be defined in odd dimensional space-times with even dimensional boundaries. Examples in 3D and 5D are given. We emphasize that this offers a way to generalize gauge theory in odd dimensions. The WZW action constructed using Wilson line can not be considered as action localized on boundary space-times since it can give anomalous gauge transformations on separated boundaries. We try to show that such WZW action can be obtained in the effective theory when making localized chiral fermions decouple.

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1 Introduction

Wess-Zumino-Witten action [1] as originally constructed for low energy mesons is known to produce the prediction of chiral anomaly at the level of Nambu-Goldstone boson. It is constructed in the non-linear sigma model using the field $\Sigma = e^{2\pi i/F}$ where $\pi = \sum_a T^a \pi^a$ is the meson field. Under a left-right transformation $\Sigma' = U_L \Sigma U_R^{-1}$ the action is not invariant and gives anomalous transformation. WZW action achieved beautiful success in its electromagnetic version which fixes the strength of processes like $\pi^0 \rightarrow 2\gamma$ and $K^+K^- \rightarrow 3\pi$.

WZW action is of broader interests in quantum field theories. The availability of such action offers an alternative way to construct anomaly-free gauge theories. The canonical way to make the chiral gauge theory anomaly-free is to arrange the fermion content in such a way that anomaly contributions of individual fermions cancel in the sum. This is exactly what happened in the Standard Model. The alternative way using the WZW action states that one can take fermion content which has non-vanishing gauge anomaly. The gauge anomalies in the fermionic sector and WZW part can be arranged to cancel.

The canonical way and the alternative to build anomaly-free theory can be connected by studying the decoupling limit of heavy fermions in a gauge theory with anomaly-free fermionic content. Heavy fermions in the theory which decouple can be arranged to give non-zero gauge anomalies. The consistency of the gauge theory in the effective theory is guaranteed with the appearance of the WZW action. For example some fermions, say $\Psi$'s, get massive from the following term.

$$\lambda (\Psi_L \Phi \Psi_R + \text{h.c.})$$
As long as \( \lambda \) is large enough one can integrate out the heavy degrees of freedom \( \Psi \)'s. The effective action will be the action of the field \( \Phi \), the gauge field and the light fermions \( \psi \)'s. The effective action of \( \Phi \) and the gauge field has to reproduce the gauge anomaly of \( \Psi \)'s and cancels the gauge anomaly contributed by light fermions \( \psi \)'s. So the effective theory is still anomaly-free.

Gauge anomalies and chiral anomalies studied in literature can be classified into anomalies of LA form and VA form. Consider a theory with Lagrangian

\[
\mathcal{L} = \bar{\psi} i \gamma^\mu (\partial_\mu - i V_\mu - i \gamma_5 A_\mu) \psi,
\]

\[
= \bar{\psi}_L i \gamma^\mu (\partial_\mu - i A_{L\mu}) \psi_L + \bar{\psi}_R i \gamma^\mu (\partial_\mu - i A_{R\mu}) \psi_R,
\]

where

\[
V_\mu = \frac{1}{2}(A_{R\mu} + A_{L\mu}), \quad A_\mu = \frac{1}{2}(A_{R\mu} - A_{L\mu}), \quad \psi_{L,R} = \frac{1 \mp \gamma_5}{2} \psi.
\]

\( V_\mu = \sum_a T^a V^a_\mu \) is the vector field and \( A_\mu = \sum_a T^a A^a_\mu \) is the axial-vector field. The coupling constants are absorbed into \( V_\mu \) and \( A_\mu \). \( T^a \) is the generator of the gauge group \( G \). \( V_\mu \) and \( A_\mu \) which can be gauge fields or auxiliary fields are taken as external fields in computing one-loop anomalies. The LR form of the anomaly is

\[
D^\mu J^a_{L\mu} = G^a_L(A_L) = \frac{1}{24\pi^2} \varepsilon^{\mu
\rho\sigma\tau} T^a \{ [T^a \partial_\mu (A_{L\nu} \partial_\rho A_{L\sigma}) - \frac{i}{2} A_{L\nu} A_{L\rho} A_{L\sigma}] \}, \tag{1}
\]

\[
D^\mu J^a_{R\mu} = G^a_R(A_R) = -\frac{1}{24\pi^2} \varepsilon^{\mu
\rho\sigma\tau} T^a \{ [T^a \partial_\mu (A_{R\nu} \partial_\rho A_{R\sigma}) - \frac{i}{2} A_{R\nu} A_{R\rho} A_{R\sigma}] \}, \tag{2}
\]

where \( \varepsilon^{\mu
\rho\sigma\tau} \) is the anti-symmetric tensor with \( \varepsilon^{0123} = 1 \), and

\[
J^a_{L\mu} = \bar{\psi}_L T^a \gamma_\mu \psi_L, \quad J^a_{R\mu} = \bar{\psi}_R T^a \gamma_\mu \psi_R. \tag{3}
\]

(1) and (2) are called LR form because in computing it the loop momentum is labeled in such a way that the anomaly takes the left-right symmetric form. Gauge anomaly of this form is also called consistent gauge anomaly. It is known that there is no unique way to label the loop momentum and anomaly can be shifted between vector current and axial-vector current [2]. The loop momentum can also be labeled in such a way that the vector current is covariantly conserved and then anomaly is completely shifted to the axial-vector current, that is the VA form of the anomaly [3]

\[
D^\mu J^5_{a\mu} = G^a_A(V,A)
\]

\[
= -\frac{1}{4\pi^2} \varepsilon^{\mu
\rho\sigma\tau} T^a \{ \frac{1}{4} V_{\mu\nu} V_{\rho\sigma} + \frac{1}{12} A_{\mu\nu} A_{\rho\sigma} - \frac{8}{3} A_{\mu\nu} A_{\rho\sigma} A_{\sigma} \}, \tag{4}
\]
where
\[ J^{5a}_\mu = \bar{\psi} T^a \gamma_\mu \gamma^5 \psi, \]  
\[ V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - i[V_\mu, V_\nu] - i[A_\mu, A_\nu], \]  
\[ A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[V_\mu, A_\nu] - i[A_\mu, V_\nu]. \]

In a theory with both left and right gauged symmetries it is natural to take the LR form of the anomaly. On the other hand, if only the vector part of the symmetry is gauged it is more natural to shift the anomaly to be in the axial-vector current and take the vector current conserved. This form of anomaly when applied to QED coincides with the original chiral anomaly obtained in [4]. The WZW action which produces anomalies at the level of Nambu-Goldstone boson can also be built to produce anomalies of LR form or VA form [2].

In 5D models it is interesting to notice that a Wilson line along the extra space-like dimension
\[ W(x^\mu) = \mathcal{P} e^{i \int_0^{\pi R} dy A_4(x^\mu, y)}, \]
transforms as bifundamental, i.e. \( W' = U(y = 0)WU^{-1}(y = \pi R) \). \( A_4 \) is the gauge field of the fourth space-like dimension. Since \( U(y = 0) \) and \( U(y = \pi R) \) are gauge transformations at different points in the extra dimension, they can be considered as independent gauge transformations from 4D point of view. Then \( W \) is similar to the \( \Sigma = e^{2\pi i/F_4} \) field in the sigma model. As will be seen in the following we can build WZW type action using the Wilson line or the Wilson loop. The WZW action constructed using Wilson line give gauge anomalies (of LR form). The WZW action constructed using Wilson loop is gauge invariant and gives anomalous variation under chiral transformation (of VA form). It will be shown that it is possible to construct pure gauge theories with the WZW action which gives anomalous gauge transformation. The theory is defined on odd dimensional space-times (3D and 5D) and it consists of the pure Yang-Mills, Chern-Simons and the WZW action constructed using the Wilson line. The theory is made gauge invariant by requiring the anomalous gauge variations of the Chern-Simons action and the WZW action have the same magnitude and the opposite sign and hence cancel on the boundary space-times.

We try to show that such kind pure gauge theory can arise as an effective theory of a gauge invariant theory with localized fermions. Chiral fermions \( \Psi^{L,R} \) charged under gauge group are localized on different boundary branes. Hence the gauge anomalies are localized on the branes. The theory is made gauge invariant by including the Chern-Simons term in the bulk and requiring that its anomalous gauge variations cancel those localized on the boundary space-times. Chiral fermions localized on boundary space-times can couple to the Wilson line \( W \) which links the two boundaries with interaction
\[ m(\bar{\Psi}_L W \Psi_R + h.c.) \]
One can integrate out the fermion \( \Psi^{L,R} \) by sending \( m \to \infty \). As is required by the consistency of the theory, the effective theory after integrating out fermion should also be anomaly-free. This is obtained by the appearance of the WZW action built of \( W \) in the effective theory.

We motivate that we can generalize the gauge theory in odd dimensional space-time using the WZW action constructed using Wilson line. The Wilson line used in the theory links
the boundary branes. The WZW action constructed like this gives non-local interaction in odd dimensional space-time and can not be considered as action localized on even dimensional boundaries. The present work is inspired by a series of recent works [11, 12] by C. Hill who constructed 5D models with localized chiral fermions, hence localized gauge anomalies, on separate branes. In section 2 we illustrate the point of our paper with a 3D example. We construct the WZW action in 3D space-time with boundaries using Wilson line and show that it gives anomalous gauge transformations on boundaries which are gauge anomalies of the consistent form (LR form). We construct a pure gauge theory in 3D which includes Yang-Mills action, Chern-Simons and WZW action. In section 3 we give example in 5D which is a bit more complicated. We study how such WZW action arises from integrating out heavy chiral fermions localized on different branes. In section 4 we construct WZW action using Wilson loop. The action constructed is gauge invariant but gives anomalous chiral transformation (of VA form). We also try to obtain this action from decoupling heavy fermions localized on boundaries. We comment and summarize in section 5.

2 WZW action in 3D gauge theory

We begin with the simple example in 3 dimension. Consider 3D flat spacetime $\Sigma = M_2 \times [0, \pi R]$ with coordinates $x^M = (x^\mu, x^2 = y)$ ($\mu = 0, 1$). $M_2$ is two dimensional Minkowski space-time. There are two 2D boundary branes L and R at $y = 0$ and $y = \pi R$ separately. Gauge field $A_M$ of gauge group $G$ propagates in the 3D space-time. Gauge group on boundaries $G_L$ and $G_R$ are determined by the boundary conditions and can be smaller than $G$. For simplicity we assume $G_L = G_R = G$. We introduce gauge fields on the boundaries which are obtained from reducing $A_M$ to the boundaries:

$$A_{L\mu} = A_\mu(x^\mu, y = 0), \quad A_{R\mu} = A_\mu(x^\mu, y = \pi R).$$

(8)

$A_M = \sum a T^a A_M^a$, similarly for $A_{L,R\mu}$. $A_M$ and $A_{L,R\mu}$ are defined as having dimension $[M]$. We then introduce $U_L$ and $U_R$,

$$U_L(x^\mu) = U(x^\mu, y = 0), \quad U_R(x^\mu) = U(x^\mu, y = \pi R).$$

(9)

So the gauge transformation

$$A_M^I(x^\mu, y) = U(x^\mu, y) A_M(x^\mu, y) U^{-1}(x^\mu, y) + iU(x^\mu, y) \partial_M U^{-1}(x^\mu, y),$$

(10)

when reduced to L and R boundary branes are written as

$$A_{L\mu}^I = U_L(x^\mu) A_{L\mu} U_L^{-1}(x^\mu) + iU_L(x^\mu) \partial_\mu U_L^{-1}(x^\mu),$$

(11)

$$A_{R\mu}^I = U_R(x^\mu) A_{R\mu} U_R^{-1}(x^\mu) + iU_R(x^\mu) \partial_\mu U_R^{-1}(x^\mu).$$

(12)

A Wilson line linking two boundaries is defined as

$$W(x^\mu) = \mathcal{P} e^{i \int_0^{\pi R} dy A_2(x^\mu, y)},$$

(13)
where $\mathcal{P}$ is the path-ordering operator and $A_2$ is the gauge field along the compact space-like dimension. Under the gauge transformation (10), the Wilson line transforms as
\[
W'(x^\mu) = U(x^\mu, 0) W(x^\mu) U^{-1}(x^\mu, \pi R) = U_L(x^\mu) W(x^\mu) U_R^{-1}(x^\mu).
\] (14)

We also introduce $W_y(x^\mu, y)$
\[
W_y(x^\mu, y) = \mathcal{P} e^{i \int_0^y dy' A_2(x^\mu, y')}.
\] (15)

$W_y(x^\mu, y)$ satisfies
\[
W_y(x^\mu, y = 0) = 1, \quad W_y(x^\mu, y = \pi R) = W(x^\mu).
\] (16)

For gauge transformed $W'$ one can also introduce $W'_y$. Condition (16) is also satisfied for gauge transformed $W'_y$. We have
\[
W'_y(x^\mu, y) = U(x^\mu, 0) W_y(x^\mu, y) U^{-1}(x^\mu, y).
\] (17)

We note that $W_y(x^\mu, y = 0) = 1$ and the configuration of $W_y$ can be taken as a mapping of $\Sigma_3$ to the space of gauge group $G$. $\Sigma_3$ is $\Sigma_3$ with the boundary at $y = 0$ shrinking to a point. So $\Sigma_3$ has a single boundary at $y = \pi R$: $\partial \Sigma_3 = \mathcal{M}_2$.

An anomalous action can be constructed as
\[
\Gamma_{WZW} = \frac{1}{12\pi} \int_{\Sigma_3} d^3x \, \varepsilon^{RST} \, Tr[(\partial_R W_y) W_y^{-1}(\partial_S W_y) W_y^{-1}(\partial_T W_y) W_y^{-1}] + \frac{i}{4\pi} \int_{\mathcal{M}_2} d^2x \, \varepsilon^{\mu\nu} \, Tr[A_{L\mu} W_{L\nu} + A_{R\mu} W_{R\nu} - i A_{R\mu} W^{-1} A_{L\nu} W],
\] (18)

where $R, S, T$ run over $0, 1, 2$, $\varepsilon^{012} = \varepsilon^{01} = 1$ and
\[
W_{L\mu} = (\partial_\mu W) W^{-1}, \quad W_{R\mu} = W^{-1}(\partial_\mu W).
\] (19)

(18) is formally of the 2D WZW action. However its interpretation and physical content is quite different. $W_y$ is not an auxiliary extension of $W$ to the third auxiliary dimension and the 3D integration is not in an auxiliary space-time either. Furthermore this action defined using Wilson line gives non-local interaction for $A_L$ and $A_R$ at different branes and can not be interpreted as action localized on boundaries. One can see that last term in (18) mixes gauge fields on two boundaries $A_L$ and $A_R$ via the link field $W$. One can also make the point clear by studying the gauge transformation properties of action (18). Using (10), (17), (11) and (12), action (18) transforms under an infinitesimal transformation as
\[
\delta \Gamma_{WZW} = -\frac{1}{4\pi} \int_{\mathcal{M}_2} \varepsilon^{\mu\nu} \, Tr[\epsilon_L \, \partial_\mu A_{L\nu}] + \frac{1}{4\pi} \int_{\mathcal{M}_2} \varepsilon^{\mu\nu} \, Tr[\epsilon_R \, \partial_\mu A_{R\nu}],
\] (20)

where $\epsilon_{L,R}$ is given by $U = e^{i\epsilon}$ which approaches unity at infinity and
\[
U_{L,R} = e^{i\epsilon_{L,R}}, \quad \epsilon_L(x^\mu) = \epsilon(x^\mu, y = 0), \quad \epsilon_R(x^\mu) = \epsilon(x^\mu, y = \pi R).
\] (21)
Figure 1: Pure gauge theory in odd dimensional (3D or 5D) space-time. Gauge fields on L and R branes are induced by gauge field in bulk: $A_{L\mu}(x^\mu) = A_\mu(x^\mu, y = 0)$ and $A_{R\mu}(x^\mu) = A_\mu(x^\mu, y = \pi R)$. 

(20) is of the form of the consistent gauge anomaly (of LR form) in two dimension [5]. We note that $A_L$ and $A_R$ are gauge fields on two boundaries. Under gauge transformation the action gives anomalous variations on two boundaries. This can not be achieved by a single WZW action localized on boundary.

A pure gauge theory can be defined in $\Sigma_3$ using (18) together with Chern-Simons action, as shown in Fig. 1. The action of the theory is

$$\Gamma = \Gamma_{YM} + \Gamma_{CS} + \Gamma_{WZW}. \quad (22)$$

$\Gamma_{YM}$ is the Yang-Mills kinetic action which is itself gauge invariant and will not elaborated in the following. The Chern-Simons action is

$$\Gamma_{CS} = -\frac{1}{4\pi} \int_{\Sigma_3} d^3x \varepsilon^{RST} Tr[A_R \partial_S A_T - \frac{2i}{3} A_R A_S A_T]. \quad (23)$$

Under infinitesimal gauge transformation given in (21), (10), (11) and (12), action $\Gamma_{CS}$ transforms as

$$\delta \Gamma_{CS} = -\frac{1}{4\pi} \int_{\Sigma_3} d^3x \varepsilon^{RST} \partial_R Tr[\varepsilon \partial_S A_T]$$

$$= -\frac{1}{4\pi} \int_{M_2} d^2x \varepsilon^\mu\nu Tr[\varepsilon_R \partial_\mu A_{R\nu}] + \frac{1}{4\pi} \int_{M_2} d^2x \varepsilon^\mu\nu Tr[\varepsilon_L \partial_\mu A_{L\nu}], \quad (24)$$

where the integration over total divergence in $\Sigma_3$ is reduced to the integration on boundaries in $M_2$. It is clear that (24) and (20) cancel in (22), i.e. gauge invariance is achieved in (22): $\delta \Gamma = 0$. (22) defines a generalized gauge theory in 3D space-time with boundaries. In this theory gauge anomalies in Chern-Simons part and the WZW part are canceled on the boundary space-times.
3 WZW action in 5D gauge theory

In this section we present a 5D example which is a bit more complicated. Consider a 5D flat space-time $\Sigma_5 = \mathcal{M}_4 \times [0, \pi R]$ with coordinate $x^M = (x^\mu, x^4 = y)$ ($\mu = 0, 1, 2, 3$). $\mathcal{M}_4$ is 4 dimensional Minkowski space-time. Gauge fields of gauge group $G$ are propagating in the bulk. There are two boundary branes in $\Sigma_5$, namely brane $L$ at $y = 0$ and brane $R$ at $y = \pi R$. The gauge groups $G_L$ and $G_R$ on the boundary branes $L$ and $R$ are determined by the boundary conditions and can be smaller than the gauge group $G$ in the bulk. For simplicity we assume $G_L = G_R = G$. We introduce gauge fields on the boundary branes $L$ and $R$ as $A_{L\mu}$ and $A_{R\mu}$, defined using Eq. (8) with $\mu = 0, 1, 2, 3$. Eq. (9), (10), (11) and (12) give the gauge transformations of $A_M$ and $A_{L,R\mu}$ with $M = 0, 1, 2, 3, 4$ and $\mu = 0, 1, 2, 3$.

We also introduce Wilson line $W(x^\mu)$ and $W_y(x^\mu, y)$

$$W(x^\mu) = \mathcal{P} e^{i \int_0^y dy A_4(x^\mu,y)}, \quad W_y(x^\mu, y) = \mathcal{P} e^{i \int_0^y dy' A_4(x^{\mu,y'})},$$

where $\mathcal{P}$ is the path-ordering operator. Under gauge transformation (10) they transform as

$$W'(x^\mu) = U_L(x^\mu) W(x^\mu) U_R^{-1}(x^\mu),$$

$$W'_y(x^\mu, y) = U(x^\mu, 0) W_y(x^\mu, y) U^{-1}(x^\mu, y).$$

$W_y$ satisfies the condition

$$W_y(x^\mu, y = 0) = 1, \quad W_y(x^\mu, y = \pi R) = W(x^\mu).$$

Since $W_y(x^\mu, y = 0) = 1$ the configuration of $W_y(x^\mu, y)$ can be considered as a mapping of $\Sigma_5$ to space of the gauge group $G$ where $\Sigma_5$ is $\Sigma_5$ with the boundary at $y = 0$ shrinking to a point. So $\Sigma_5$ has a single boundary: $\mathcal{M}_4 = \partial \Sigma_5$.

3.1 A pure gauge theory with WZW action

WZW action is defined as

$$\Gamma_{WZW} = \frac{-i}{240 \pi^2} \int_{\Sigma_5} d^5x \mathcal{T} R \left[ e^{\mathcal{L}^{MNRST}} \frac{\partial W_y}{\partial x^M} W_{y^{-1}} - \frac{\partial W_{x^N}}{\partial x^M} W_{y^{-1}} \frac{\partial W_{x^N}}{\partial x^M} W_{y^{-1}} - \frac{\partial W_{y}}{\partial x^T} W_{y^{-1}} \frac{\partial W_{y}}{\partial x^T} W_{y^{-1}} \right]$$

$$+ \frac{i}{4 \pi^2} \int_{\mathcal{M}_4} d^4x \, e^{\mu_{\nu\rho\sigma}} \mathcal{T} R \left[ \frac{1}{2} (W_{L\mu} A_{R\nu} W_{R\rho} A_{L\sigma} - W_{L\mu} A_{L\nu} W_{L\rho} A_{L\sigma}) + W_{L\mu} (A_{L\nu} \partial_{\rho} A_{L\sigma} + (\partial_{\nu} A_{L\rho}) A_{L\sigma} - i A_{L\nu} A_{L\rho} A_{L\sigma} - i W_{L\nu} W_{L\rho} A_{L\sigma}) + (L \rightarrow R) \right]$$

$$+ \frac{1}{4 \pi^2} \int_{\mathcal{M}_4} d^4x \, e^{\mu_{\nu\rho\sigma}} \mathcal{T} R \left[ i A_{L\mu} W A_{R\nu} W^{-1} W_{L\rho} W_{L\sigma} - i A_{R\mu} W^{-1} A_{L\nu} W W_{R\rho} W_{R\sigma} + (A_{R\mu} \partial_{\nu} A_{R\rho}) A_{R\sigma} - i A_{R\mu} A_{R\nu} A_{R\rho} W^{-1} A_{L\sigma} W - (L \leftrightarrow R, W^{-1} \leftrightarrow W) \right.$$

$$\left. + A_{R\mu} W^{-1} A_{L\nu} W A_{R\rho} W_{R\sigma} + A_{L\mu} W A_{R\nu} W^{-1} A_{L\rho} W_{L\sigma} - \frac{i}{2} A_{R\mu} W^{-1} A_{L\nu} W A_{R\rho} W^{-1} A_{L\sigma} W + \frac{i}{2} (\partial_{\mu} A_{R\nu})(\partial_{\rho} W^{-1}) A_{L\sigma} W - (L \leftrightarrow R, W^{-1} \leftrightarrow W) \right].$$

(29)
where \( M, N, R, S, T \) run over 0, 1, 2, 3, 4, \( \varepsilon^{01234} = \varepsilon^{0123} = 1 \) (\( \varepsilon_{01234} = -\varepsilon_{0123} = 1 \)) and

\[
W_{L\mu} = \frac{\partial W}{\partial x^\mu} W^{-1}, \quad W_{R\mu} = W^{-1} \frac{\partial W}{\partial x^\mu},
\]  
(30)

\( M, N, R, S, T \) run over 0, 1, 2, 3, 4. (29) is of the form of the 4D WZW action [6, 7, 8, 9, 10]. The interpretation and the physics content are however quite different. (29) defines non-local interactions of gauge fields on two boundaries, \( A_L \) and \( A_R \), via the link field \( W \) and can not be understood as action localized on the boundaries. Further discussions on this action closely follow the discussions on (18).

Under infinitesimal transformation \( U(x^\mu, y) \) which approaches unity at infinity,

\[
U = e^{i\epsilon}, \quad U_L = U(y = 0) = e^{i\epsilon_L}, \quad U_R = U(y = \pi R) = e^{i\epsilon_R},
\]  
(31)

\( \epsilon_L(x^\mu) = \epsilon(x^\mu, y = 0) \), \( \epsilon_R(x^\mu) = \epsilon(x^\mu, y = \pi R) \),

we obtain

\[
\delta \Gamma_{WZW} = -\frac{1}{24\pi^2} \int_{M_4} d^4 x \, \omega^1_4(A_L, \epsilon_L) + \frac{1}{24\pi^2} \int_{M_4} d^4 x \, \omega^1_4(A_R, \epsilon_R),
\]  
(33)

where \( \omega^1_4 \) is

\[
\omega^1_4(B_\mu(x^\mu), \varepsilon(x^\mu)) = Tr[\varepsilon^{\mu\nu\rho\sigma} \varepsilon(x^\mu) \partial_\mu (B_\nu \partial_\rho B_\sigma - \frac{i}{2} B_\nu B_\rho B_\sigma)].
\]  
(34)

(33) takes the form of the 4D gauge anomaly (LR form) [6, 7, 8, 9]. Its interpretation is that under the gauge transformation (10) and (27) in the bulk the action (29) gives anomalous gauge transformations on two separated boundaries. This can not be achieved by a single WZW action localized on boundaries.

A pure gauge theory can be defined on \( \Sigma_5 \) using (29) and the Chern-Simons action, as illustrated in Fig. 1. The action of the theory is

\[
\Gamma = \Gamma_{YM} + \Gamma_{CS} + \Gamma_{WZW}.
\]  
(35)

The pure Yang-Mills action \( \Gamma_{YM} \) is itself gauge invariant. The Chern-Simons action to be included is given as

\[
\Gamma_{CS}(A_M) = -\frac{1}{24\pi^2} \int_{\Sigma_5} d^5 x \, \omega_5(A_M, F_{MN}),
\]  
(36)

where

\[
\omega_5(A_M, F_{MN}) = \varepsilon^{MNRST} Tr[\frac{1}{4} A_M F_{NR} F_{ST} + \frac{i}{4} A_M A_N A_R F_{ST} - \frac{1}{10} A_M A_N A_R A_S A_T].
\]  
(37)

It can be checked that under infinitesimal transformation (31) and (32), action (36) gives a total derivative in the integrand and the integration \( \Sigma_5 \) is then reduced to the boundary [15]:

\[
\delta \Gamma_{CS} = \frac{1}{48\pi^2} \int_{\Sigma_5} d^5 x \, \varepsilon^{MNRST} \partial_M Tr[(\partial_N \epsilon)(A_R \partial_S A_T + (\partial_R A_S) A_T - i A_R A_S A_T)]
\]  

\[= -\frac{1}{24\pi^2} \int_{M_4} d^4 x \, \omega^1_4(A_{R\mu}, \epsilon_R) + \frac{1}{24\pi^2} \int_{M_4} d^4 x \, \omega^1_4(A_{L\mu}, \epsilon_L),
\]  
(38)
where $\varepsilon^{4\mu\rho\sigma} = \varepsilon^{\mu\rho\sigma}$ and integration by part on $\mathcal{M}_4$ has been used.

Using (33) and (38) one can see $\delta \Gamma = 0$. The gauge invariance of the theory is achieved by making the anomalous gauge variations of Chern-Simons part and the WZW part cancel on the boundary space-times.

### 3.2 WZW action from decoupling fermion

In this subsection we study a gauge theory with chiral fermions localized on boundaries and Chern-Simons action in the 5D bulk. We try to obtain the WZW action described in the last subsection in the effective theory when making the localized chiral fermions heavy and decouple. We work again in $\Sigma_5$. We introduce $\psi_L$ on brane L and $\psi_R$ on brane R. They are charged under the gauge group $G_L$ and $G_R$, namely coupled to $A_L$ and $A_R$ separately. The gauge invariance of the theory is achieved by making the anomalous gauge variations of bulk part, i.e. of Chern-Simons part at the classical level, cancel the anomalous gauge variations given by the boundary fermions which are at the quantum level.

This cancellation is possible by noting that the chiral gauge theories on the 4D boundaries are known to be anomalous, i.e. the currents are not covariantly conserved on two boundaries. The non-conservation is understood at the functional level as arising from non-invariance of the fermionic measure under the left-right transformation [13, 14]. Under infinitesimal transformation (31) and (32)

$$\psi'_L = U_L \psi_L, \quad \psi'_R = U_R \psi_R,$$

where $U_{L,R}$ is defined in (9), the functional measure changes

$$D\psi'_L D\bar{\psi}'_L = D\psi_L D\bar{\psi}_L e^{-i \int d^4x \epsilon^a_L G^a_L(A_L)}, \quad D\psi'_R D\bar{\psi}'_R = D\psi_R D\bar{\psi}_R e^{-i \int d^4x \epsilon^a_R G^a_R(A_R)},$$

where $\epsilon_{L,R} = \sum_a T^a \epsilon^a_{L,R}$. $G_L(A_L)$ and $G_R(A_R)$ are given in (1) and (2).

We consider the action

$$\Gamma = \Gamma_{YM} + \Gamma_{CS} + \Gamma_{eff},$$

(41)

$\Gamma_{YM}$ is the pure Yang-Mills action which is gauge invariant itself and will not be elaborated in the following. $\Gamma_{CS}$ is the Chern-Simons action given in (36). It is not gauge invariant and gives the anomalous gauge variations on the boundary branes given in (38). We then check $\Gamma_{eff}$ and show how the gauge invariance is achieved in (41). $\Gamma_{eff} = \Gamma_{eff}(A_L, A_R, W)$ is the action quantum corrected by boundary fermions $\psi_{L,R}$ and is given at the functional level as

$$e^{i\Gamma_{eff}(A_L, A_R, W)} = \int D\psi_L D\bar{\psi}_L D\psi_R D\bar{\psi}_R e^{i\Gamma_{\psi}(\psi_L, \psi_R, W, A_L, A_R)},$$

(42)

where

$$\Gamma_{\psi} = \int d^4x \left[ \bar{\psi}_L i \gamma^\mu (\partial_\mu - i A_L) \psi_L + \bar{\psi}_R i \gamma^\mu (\partial_\mu - i A_R) \psi_R - m(\bar{\psi}_L W \psi_R + h.c.) \right].$$

(43)

$m$ can be complex in general. We have taken $m$ as real for simplicity. $W$ is the Wilson line in (25). Notice that a non-local term $\bar{\psi}_L W \psi_R$ is included in the Lagrangian. Under gauge
transformation given in (11), (12), (26) and (39) $\mathcal{L}_\psi$ is gauge invariant but the functional measure is not as shown in (40). For an infinitesimal transformation (31) and (32) we get

$$\delta \Gamma_{\text{eff}} = \Gamma_{\text{eff}}(A'_L, A'_R, W') - \Gamma_{\text{eff}}(A_L, A_R, W)$$

$$= - \int d^4x \left[ \epsilon^a_L G^a_L(A_L) + \epsilon^a_R G^a_R(A_R) \right]$$

$$= - \frac{1}{24\pi^2} \int d^4x \, \omega^1_L(A_L, \epsilon_L) + \frac{1}{24\pi^2} \int d^4x \, \omega^1_R(A_R, \epsilon_R). \quad (44)$$

Putting this into (41) and using (38) one sees that

$$\Gamma(A'_M, \psi'_L, \psi'_R) = \Gamma(A_M, \psi_L, \psi_R).$$

The anomalous variations cancel in the action under consideration and the theory is gauge invariant.

Now we try to show that $\Gamma_{\text{eff}}$ contains the WZW action as a part and the WZW action given in the last subsection appears in the effective theory when decoupling the heavy fermions localized on the boundary branes as $m \to \infty$. We change the variables $\psi_{L,R}$ to $\chi_{L,R}$ and diagonalize the fermion mass matrix using finite transformation $g_{L,R}$

$$\chi_{L,R} = g_{L,R} \psi_{L,R}, \quad W = g_L W g_R^{-1} = 1,$$

$$\mathcal{L}_\chi = \bar{\chi} i \gamma^\mu (\partial_\mu - i A^q_{L,R}) \chi - m \bar{\chi} \chi + (L \leftrightarrow R). \quad (45)$$

Computing the Jacobian when changing $\psi$ to $\chi$ with $\psi = g^{-1} \chi$ one obtains

$$\Gamma_{\text{eff}}(A_L, A_R, \Sigma) = \Gamma_\chi(A^q_L, A^q_R) + \Gamma_L(A^q_L, g_L) + \Gamma_R(A^q_R, g_R), \quad (46)$$

where $\Gamma_\chi$ is the effective action computed in the base of $\chi$, and

$$\Gamma_L(A^q_L, g_L(s)) = \Gamma_A(A^q_L, g_L(s)), \quad \Gamma_R(A^q_R, g_R(s)) = -\Gamma_A(A^q_R, g_R(s)), \quad (47)$$

where

$$\Gamma_A(A^q, g(s)) = \frac{i}{24\pi^2} \int_{D_5} ds d^4x \, \omega^1(A^q(s), g(s) \partial_s g^{-1}(s)), \quad (48)$$

$$A^q_\mu = g(x) A^q g^{-1}(x) + ig(x) \partial^q g^{-1}(x). \quad (49)$$

$D_5 = M_4 \times [0, 1]$ is an auxiliary extension of $M_4$. (47) is the opposite of the Jacobian changing variable $\chi$ to $\psi$ using $\chi = g \psi$. $g_{L,R}(s)(s \in [0, 1])$ interpolates $g_{L,R}(s = 0) = 1$ and $g_{L,R}(s = 1) = g_{L,R}$. A computation of (48) is done in Appendix A and an explicit formula of $\Gamma_A$ is given.

We can check the transformation properties of the effective action as follows. There is a transformation leaving $\mathcal{L}_\chi$ invariant, i.e.

$$\chi \to V \chi, \quad A^q_L \to (A^q_L)^V = A^{V^q_L}, \quad A^q_R \to (A^q_R)^V = A^{V^q_R}. \quad (50)$$
This is a vector-like transformation. The left-right transformation $W' = U_L W R^{-1}$ is manifested on $g_L$ and $g_R$ as

$$g'_L = g_L U_L^{-1} \quad g'_R = g_R U_R^{-1} ,$$

which leaves $g'_L W' g'_R^{-1} = 1$ invariant. We note that the left-right transformation has not effect on $A^{q_L}_L$ and $A^{q_R}_R$. It can be checked that under the left-right transformation

$$(A'_L)^{q_L}_L = (A^{U_L}_L)^{q_L}_L = A^{q_L}_L ,$$

same for $A^{q_R}_R$. That means $L$ is not affected by the left-right transformation.

In total $g_{L,R}$ can be transformed by $V$ and $U_{L,R}$, that is

$$g_{L,R} \rightarrow g'_{L,R} = V g_{L,R} U_{L,R}^{-1} .$$

Look at $\Gamma$ for example. What we need to compute for $g'_L$ and $A'_L$ is

$$\Gamma_L((A^{U_L}_L)^{q_L}_L, g'_L) = \frac{i}{24 \pi^2} \int d^4 x \omega^1(A^{U_L}_L)^{q_L}_L(s), g'_L(s) \partial_s g^{-1}_L(s)) ,$$

where $g'_L(s)$ interpolates $g'_L(s = 0) = 1$ and $g'_L(s = 1) = g'_L$. The integration is over gauge configurations from $A^{U_L}_L$ to $(A^{U_L}_L)^{q_L}_L = A^{q_L}_L U_L = A^{q_L}_L$ in which (52) is used. Compared to $\Gamma(A^{q_L}_L, g_L)$, the difference is in the integration from gauge configurations $A_L$ to $A^{U_L}_L$ and $A^{q_L}_L$ to $A^{q_L}_L$. For infinitesimal transformation (31), (32) and $V = e^{\epsilon \psi V}$ we have

$$\delta \Gamma_L = \Gamma_L((A^{U_L}_L)^{q_L}_L, g'_L) - \Gamma_L(A^{q_L}_L, g_L)$$

$$= \frac{1}{24 \pi^2} \int d^4 x \omega^1(A^{q_L}_L, \epsilon_V) - \frac{1}{24 \pi^2} \int d^4 x \omega^1(A_L, \epsilon_L) ,$$

(53)

Similarly we can get transformation properties for $\Gamma$ with a difference on the sign. We see that (53) produces exactly what we expect in (44) for the left-right transformation. The extra $V$ transformation corresponds to the freedom to reparametrize $g_L$ and $g_R$, that is $W = (V g_L)^{-1} V g_R$. The total effective action must be the same for different parametrization, that is $\Gamma_{eff}$ is invariant under $V$ transformation.

So $\Gamma$ has to give the right transformation under $V$, i.e.

$$\delta \Gamma = (A^{q_L}_L, A_R^{q_R}) = -\frac{1}{24 \pi^2} \int d^4 x \omega^1(A^{q_L}_L, \epsilon_V) + \frac{1}{24 \pi^2} \int d^4 x \omega^1(A^{q_R}_R, \epsilon_V) ,$$

and this cancels variation in $\Gamma$ and $\Gamma$. A counterterm, $\Gamma_B$, is known to give the correct transformation properties [9], that is

$$\Gamma_B(B_L, B_R) = \frac{1}{48 \pi^2} \int d^4 x \varepsilon_{\mu \nu \rho \sigma} Tr \left[ \frac{1}{2} (F_{\mu \nu}^{BL} + F_{\mu \nu}^{BR})(B_{R \rho} B_{L \sigma} - B_{L \rho} B_{R \sigma}) + i(B_{R \mu} B_{R \nu} B_{L \sigma} - B_{L \mu} B_{L \nu} B_{R \sigma}) - \frac{i}{2} B_{R \mu} B_{L \nu} B_{R \rho} B_{L \sigma} \right] ,$$

(54)
where
\[ F_{\mu \nu}^{BL} = \partial_\mu B_{L\nu} - \partial_\nu B_{L\mu} - i [B_{L\mu}, B_{L\nu}], \quad F_{\mu \nu}^{RR} = \partial_\mu B_{R\nu} - \partial_\nu B_{R\mu} - i [B_{R\mu}, B_{R\nu}]. \] (55)

\( \Gamma \chi \) must be obtained as *
\[ \Gamma_\chi(A_L^{gl}, A_R^{gr}) = \Gamma_B(A_L^{gl}, A_R^{gr}) + \text{polynomials invariant under} \ V, \] (56)

Armed with this observation we can write the anomalous effective action \( \Gamma_{WZW} \) as
\[ \Gamma_{WZW}(A_L, A_R, W) = \frac{1}{2} [\Gamma_B(A_L^{W^{-1}}, A_R) + \Gamma_B(A_L, A_R^{W}) + \Gamma_L(A_L^{W^{-1}}, W^{-1}) + \Gamma_R(A_R^{W}, W)]. \] (57)

This is obtained as the mean of the actions in two cases of \( g_L = W^{-1}, g_R = 1 \) and \( g_L = 1, g_R = W \). (57) reproduces (29) when changing the integration over \( s \) to integration over \( y \) and taking \( g_L(s) = W_y^{-1}(s \times \pi R), g_R(s) = 1 \) and \( g_L(s) = 1, g_R(s) = W_y(s \times \pi R) \) in two cases.

## 4 WZW action and anomalous chiral symmetry

In this section we construct WZW action which is gauge invariant but gives anomalous chiral transformation. This happens in the case chiral symmetry is not gauged. Consider a toy model on \( M_4 \times I \) with coordinates \( x^M = (x^\mu, x^4 = y) \). The space-like extra dimension \( I \) is compact and flat. It can be a cycle \( I = S^1 \) or orbifold like \( I = S^1/Z_2 \), etc. Gauge fields \( B_M(x^\mu, y) \) of gauge group \( G \) propagate in the bulk. Depending on the boundary condition, the gauge group on the brane at \( y = 0, H \) which is \( G \) restricted at \( y = 0 \), can be smaller or equal to the gauge group in the bulk: \( H \subseteq G \). We explain in detail the toy model as follows.

We define the periodicity of the compact space \( I \) using the equivalence class
\[ y + 2\pi R \sim y. \] (58)

Gauge fields are equivalent up to gauge transformation:
\[ B_M(x^\mu, y + 2\pi R) = \Omega B_M(x^\mu, y) \Omega^{-1} + i \Omega \partial_M \Omega^{-1}. \] (59)

Up to a global phase \( \Omega = \Omega(x^\mu, y) \) is an element of \( G \). Under a local gauge transformation \( U(x^\mu, y), B_M \) and \( \Omega \) are transformed according to
\[ B'_M(x^\mu, y) = U(x^\mu, y) B_M(x^\mu, y) U^{-1}(x^\mu, y) + i U(x^\mu, y) \partial_M U^{-1}(x^\mu, y); \] (60)
\[ \Omega'(x^\mu, y) = U(x^\mu, y + 2\pi R) \Omega(x^\mu, y) U^\dagger(x^\mu, y). \] (61)

We define a Wilson loop \( W(x^\mu) \) as
\[ W(x^\mu) = \text{P} \text{e}^{i \int_0^{2\pi R} dy \ B_4(x^\mu, y) \times \Omega(x^\mu, 0)}, \] (62)

*To author’s knowledge \( \Gamma_B \) known as the Bardeen’s counterterm is not yet obtained in computation of the effective action when decoupling heavy chiral fermions in 4D chiral gauge theories. The detailed computation is out of the scope of this paper and a computation will be presented in a further publication.
where $\mathcal{P}$ is the path-ordering operator. Using (60) and (61), $W(x^\mu)$ can be shown to be gauge covariant:

\[
W' = Pe^{i\int_0^{2\pi R} dy B'_4} \times \Omega'(x^\mu, 0)
\]

\[
= U(x^\mu, 0) Pe^{i\int_0^{2\pi R} dy B_4} U^\dagger(x^\mu, 2\pi R) \times \Omega'(x^\mu, 0)
\]

\[
= U(x^\mu, 0) W U^\dagger(x^\mu, 0). \tag{63}
\]

We introduce the gauge fields $V_\mu$ and $A_\mu$ on the brane at $y = 0$ as

\[
V_\mu(x^\mu) = B_\mu(x^\mu, y = 0), \quad A_\mu = 0. \tag{64}
\]

For convenience of later discussion we introduce $A_L$ and $A_R$

\[
A_R = V_\mu + A_\mu, \quad A_L = V_\mu - A_\mu, \tag{65}
\]

so that $V_{\mu\nu}$ and $A_{\mu\nu}$ defined in (6) and (7) are

\[
V_{\mu\nu} = \frac{1}{2}(F^R_{\mu\nu} + F^L_{\mu\nu}), \quad A_{\mu\nu} = \frac{1}{2}(F^R_{\mu\nu} - F^L_{\mu\nu}). \tag{66}
\]

$F^L_{\mu\nu} = \partial_\mu A_{L,R\nu} - \partial_\nu A_{L,R\mu} - i[A_{L,R\mu}, A_{L,R\nu}]$ are the field strengths for $A_{L,R}$. On the brane at $y = 0$ the gauge transformation $U(x^\mu, y)$ is reduced to be $U(x^\mu)$, i.e.

\[
U(x^\mu) = U(x^\mu, y = 0), \quad W' = U(x^\mu) W U^{-1}(x^\mu), \tag{67}
\]

\[
A'_L,R_{\mu} = U(x^\mu) A_{L,R\mu} U^{-1}(x^\mu) + iU(x^\mu)\partial_\mu U^{-1}(x^\mu). \tag{68}
\]

We introduce $W(s)$ for $s \in [0, 1]$ with condition

\[
W(s = 0) = 1, \quad W(s = 1) = W. \tag{69}
\]

$W(s)$ interpolates unit matrix and $W$. We can construct the WZW action as

\[
\Gamma_{WZW} = \frac{i}{8\pi^2} \int_0^1 ds \int_{\mathcal{M}_4} d^4x \omega^2_4(V^{W(s)}, A^{W(s)}, W(s)), \tag{70}
\]

where

\[
\omega^2_4(V^{W(s)}, A^{W(s)}, W(s)) = \varepsilon^{\mu\nu\sigma\tau} Tr[W^{-1}(s) \frac{\partial W(s)}{\partial s} (\frac{1}{4} V^{W(s)}_{\mu\nu} V^{W(s)}_{\rho\sigma} + \frac{1}{12} A^{W(s)}_{\mu\nu} A^{W(s)}_{\rho\sigma})
\]

\[
+ \frac{2i}{3} (V^{W(s)}_{\mu\nu} A^{W(s)}_{\rho} A^{W(s)}_{\sigma} + A^{W(s)}_{\mu} V^{W(s)}_{\nu\rho} A^{W(s)}_{\sigma} + A^{W(s)}_{\mu} A^{W(s)}_{\nu} V^{W(s)}_{\rho\sigma})
\]

\[
- \frac{8}{3} A^{W(s)}_{\mu} A^{W(s)}_{\nu} A^{W(s)}_{\rho} A^{W(s)}_{\sigma}], \tag{71}
\]
and
\[ V^{W(s)}_{\mu\nu} = \frac{1}{2} [F^R_{\mu\nu} + W^{-1}(s)F^L_{\mu\nu}W(s)], \quad A^{W(s)}_{\mu\nu} = \frac{1}{2} [F^R_{\mu\nu} - W^{-1}(s)F^L_{\mu\nu}W(s)], \] (72)
\[ A^{W(s)}_{\mu} = \frac{1}{2} [A_{R\mu} - W^{-1}(s)A_{L\mu}W(s) - iW^{-1}(s)\partial_\mu W(s)]. \] (73)

It's easy to check the transformation property of (70). Under the gauge transformation given in (67) and (68) we have

\[ W'(s) = U(x^\mu)W(s)U^{-1}(x^\mu), \quad A^{W'(s)}_{\mu} = U(x^\mu)A^{W(s)}_{\mu}U^{-1}(x^\mu), \] (74)
\[ V^{W'(s)}_{\mu\nu} = U(x^\mu)V^{W(s)}_{\mu\nu}U^{-1}(x^\mu), \quad A^{W'(s)}_{\mu\nu} = U(x^\mu)A^{W(s)}_{\mu\nu}U^{-1}(x^\mu). \] (75)

\( W'(s) \) satisfies the condition (69) as can be easily seen. So it's clear that (70) and (71) are invariant under the vector-like gauge transformation.

Consider next the chiral transformation

\[ W' = U^{-1}_A W U_A, \] (76)
\[ A'_{L\mu} = U^{-1}_A A_{L\mu} U_A + iU^{-1}_A \partial_\mu U_A, \quad A'_{R\mu} = U_A A_{R\mu} U_A^{-1} + iU_A \partial_\mu U_A^{-1}. \] (77)

\( W'(s) \), which interpolates the unit matrix and \( W' = U^{-1}_A W U_A^{-1} \), is then introduced into (70). If we write

\[ W'(s) = U_A^{-1}(x^\mu)\tilde{W}(s)U_A^{-1}(x^\mu), \] (78)
we then have \( \tilde{W}(s) \) which interpolates \( \tilde{W}(s = 0) = U_A^2(x^\mu) \) and \( \tilde{W}(s = 0) = W \). One can check

\[ A^{W'(s)}_{\mu} = U_A(x^\mu)A^{\tilde{W}(s)}_{\mu}U_A^{-1}(x^\mu) \]
\[ V^{W'(s)}_{\mu\nu} = U_A(x^\mu)V^{\tilde{W}(s)}_{\mu\nu}U_A^{-1}(x^\mu), \quad A^{W'(s)}_{\mu\nu} = U_A(x^\mu)A^{\tilde{W}(s)}_{\mu\nu}U_A^{-1}(x^\mu), \]

and the chirally transformed \( \omega^2_4 \) can be rewritten as

\[ \omega^2_4 (V^{W'(s)}, A^{W'(s)}, W'(s)) = \omega^2_4 (V^{\tilde{W}(s)}, A^{\tilde{W}(s)}, \tilde{W}(s)). \] (79)

Plugging it into (70) and comparing it with the un-transformed case the difference is the integration from \( W(s) = 1 \) to \( W(s) = U_A^2(x^\mu) \). For an infinitesimal transformation \( U_A = e^{iA} \) we have

\[ \delta \Gamma_{WZW} = -\epsilon_A^a G^a_A(V, A), \] (80)
where \( \epsilon_A = \sum_a T^a \epsilon_A^a \). This is exactly the anomaly in VA form.

The action (70) can also be obtained as arising from a decoupling fermion. We introduce fermions \( \psi = (\psi_L, \psi_R) \) which are localized on the brane at \( y = 0 \). They are transformed under the gauge transformation (68) as

\[ \psi'_{L,R}(x^\mu) = U(x^\mu)\psi_{L,R}(x^\mu). \] (81)
Consider the effective action $\Gamma_{eff}$

$$e^{i\Gamma_{eff}(A_L,A_R,W)} = \int D\psi_L D\bar{\psi}_L D\psi_R D\bar{\psi}_R \ e^{i\Gamma}. \quad (82)$$

$A_{L,R}$ and $W$ as introduced in (62) and (65) are considered as external fields and

$$\Gamma = \int d^4x \ [\bar{\psi}i\gamma^\mu(\partial_\mu - iT^a V^a_\mu)\psi - m(\bar{\psi}W\psi + h.c.)],$$

$$= \int d^4x \ [\bar{\psi}i\gamma^\mu(\partial_\mu - iA_L)\psi_L + \bar{\psi}_RI\gamma^\mu(\partial_\mu - iA_R)\psi_R - m(\bar{\psi}W\psi + h.c.)]. \quad (83)$$

In general $m$ is complex. We have chosen $m$ to be real for convenience. Action (83) is gauge invariant under the gauge transformation given in (67), (68) and (81).

Gauge couplings in (83) with gauge fields introduced in (64) and (65) take the form in which only the vector part is gauged. As discussed before, the anomaly takes the VA form for which the anomaly is completely shifted to the axial-vector current and the vector current is covariantly conserved. Anomaly of VA form can also be obtained in the functional level. It is achieved by carefully treating the Dirac operator in the evaluation of the Jacobian [14, 16]. Consequently, under a chiral transformation (76) and (77) supplemented by

$$\psi' = U_A \psi, \quad \bar{\psi}' = U_A^{-1} \bar{\psi}, \quad (84)$$

$\Gamma_{eff}$ changes by an anomalous term due to changing the measure of $\psi'$ to that of $\psi$. For an infinitesimal transformation $U_A = e^{iA}$ it gives

$$\delta \Gamma_{eff} = \Gamma_{eff}(A_L',A_R',W') - \Gamma_{eff}(A_L,A_R,W) = -i e_A^a G_A^a(V,A), \quad (85)$$

where $G_A^a(V,A)$ is given in (4) and $V_\mu$ and $A_\mu$ shown in (65).

Similar to the case in the last section we can obtain the anomalous action by doing a finite chiral transformation

$$\psi_R = \xi^{-1}\chi_R, \quad \psi_L = \xi\chi_L, \quad \text{with } W = \xi^2. \quad (86)$$

We introduce $\xi(s)$ and $W(s) = \xi^2(s)$ with condition

$$\xi(s = 0) = 1, \quad \xi(s = 1) = \xi,$$

$$W(s = 0) = 1, \quad W(s = 1) = W.$$

WZW action is obtained as a continuous integration of chiral transformation

$$\Gamma_{WZW} = i \int_0^1 ds \int_{M_4} d^4x \ Tr \left\{ \frac{1}{2}[\xi(s)\partial_s \xi^{-1}(s) - \xi^{-1}(s)\partial_s \xi(s)]G_A^a(V^{\xi(s)},A^{\xi(s)}) \right\}, \quad (87)$$

It is the opposite of the phase integration of changing $\chi_R = \xi\psi_R$ to $\psi_R$. $A^{\xi(s)}_\mu$ in (87) is the axial-vector part of $A^{\xi^{-1}(s)}_L$ and $A^{\xi(s)}_R$ computed using (49), that is

$$A^{\xi(s)}_\mu = \frac{i}{2}[\xi(s)A_R\xi^{-1}(s) + i\xi(s)\partial_\mu \xi^{-1}(s) - \xi^{-1}(s)\partial_\mu \xi(s) - i\xi^{-1}(s)\partial_\mu \xi(s)]$$

$$= \frac{i}{2}(\xi(s)[A_{R\mu} - W^{-1}(s)A_{L\mu}W(s) - iW^{-1}(s)\partial_\mu W(s)]\xi^{-1}(s)$$

$$= \xi(s)A^{W(s)}_\mu \xi^{-1}(s). \quad (88)$$
Similarly we have

\[ V^{\xi(s)}_{\mu\nu} = \xi(s)V^{W(s)}_{\mu\nu}\xi^{-1}(s), \quad A^{\xi(s)}_{\mu\nu} = \xi(s)A^{W(s)}_{\mu\nu}\xi^{-1}(s), \tag{89} \]

Noticing further that

\[ \xi(s)\partial_s\xi^{-1}(s) - \xi^{-1}(s)\partial_s\xi(s) = -\xi(s)W^{-1}(s)[\partial_sW(s)]\xi^{-1}(s), \]

it is then easy to rewrite \( \Gamma_{WZW} \) in (87) to be the form in (70).

5 Summary

Using examples in 3D and 5D we have shown that Wilson line or Wilson loop along the compact space-like dimension can be used to construct the WZW action. If \( W \) is a Wilson line which links the two boundary branes, \( W \) transforms as bifundamental under the gauge groups of the two boundaries. Gauge symmetries at two branes can be considered in 4D point of view as two independent symmetries though they are parts of the 5D gauge symmetry. Anomalous action constructed using \( W \) gives anomalous gauge variations (of LR form) on two boundary branes. On the other hand, the Wilson loop transforms under a single gauge symmetry of one brane. The anomalous action constructed using Wilson loop will then be invariant under vector-like gauge transformation. The WZW action gives the anomalous variations under the chiral transformation.

The WZW action constructed using Wilson line gives non-local interactions for gauge fields on separated boundary branes via the link field, i.e. the Wilson line. It should be understood in odd dimensional bulk, rather than in the even-dimensional boundary branes. We have used this action to generalize the gauge theory in 3D and 5D space-times with boundaries. The generalized gauge theory includes actions of pure Yang-Mills, Chern-Simons and WZW in the bulk. The gauge invariance of the theory is achieved by requiring the that the anomalous gauge variations of Chern-Simons and WZW actions cancel on the boundary space-times. We expect that this procedure can be generalized to odd dimensions larger than five. The action constructed is expected to be of the form of WZW action in even dimensions higher than four [18]. We have constructed action in flat space-time background. We expect procedure described in this paper can be generalized to curved space-time background. The generalization to include gravity is a problem and is worth further study.

We tried to show that WZW constructed using Wilson line or Wilson loop can be understood as arising from integrating out heavy fermion. In 5D examples, these fermions are localized on 4D boundaries. They can not couple to extra dimensional gauge field \( A_4 \) using \( \bar{\psi}\gamma_5\psi A_4\psi \) since there is no corresponding kinetic term \( \bar{\psi}\gamma_4\partial_4\psi \) on the boundary brane. The coupling using Wilson line or Wilson loop, such as

\[ m(\bar{\psi}_L W\psi_R + h.c.), \]

is perfectly allowed. This is a natural way for \( A_4 \) degrees of freedom to couple to localized fermion. Integrating out localized fermion as sending \( m \to \infty \) results in the WZW actions which manifest the gauge or global anomalies of fermions in the effective theory. Ref. [11]
one can write (90) as the Wilson line or the Wilson loop as $W$ than the single boundary in quantum Hall effect. Similar in the sigma model, one can write is similar to the setup of the quantum Hall effect, although we have two boundaries rather than the single boundary in quantum Hall effect. Similar in the sigma model, one can write the Wilson line or the Wilson loop as $W = e^{2i\pi/F}$ where $F = 1/(2\pi R)$ or $F = 1/(\pi R)$. The gauge field $A_4$ is then like meson fields. We note that KK modes that are not periodic around the extra dimension can contribute to the Wilson loop, e.g. the anti-periodic modes in some models. Then the meson fields $\tilde{\pi}$ as defined in $W = e^{2i\pi/F}$ can contain massive KK modes. In this case processes like $\tilde{\pi} \rightarrow 2\gamma$ is allowed if the gauge symmetry on the boundary brane includes that of electromagnetism. To understand the phenomenology of the WZW action in 5D, its KK mode expansion needs to be studied.

**Appendix A. Evaluation of phase integration**

The phase integration can be conveniently computed using differential forms. Consider the integration on $D_5 = M_4 \times [0, 1]$

$$\Gamma_A(A^g, g(s)) = \frac{i}{24\pi^2} \int_{D_5} ds d^4x \, Tr[g(s) \frac{\partial g^{-1}(s)}{\partial s} \varepsilon^{\mu\nu\rho\sigma} \partial_\mu (A_i^{g(s)} \partial_\nu A_\rho A_\sigma - \frac{i}{2} A_i^{g(s)} A_\nu A_\rho A_\sigma)](90)$$

Using

$$A_g = A_{i\mu} d\mu, \quad F_{g}\left(\mu, \nu, \sigma, \tau\right) = dA_{g} - iA^{2}_{g}, \quad \left(\partial_{\mu} A_{\nu}\right) = 0,$$

one can write (90) as

$$\Gamma_A(A^g, g(s)) = \frac{i}{24\pi^2} \int_{D_5} ds \, Tr[g(s) \frac{\partial g^{-1}(s)}{\partial s} d(A_{g}(s) dA_{g}) - \frac{i}{2} A_{g}(s) A_{g}(s) A_{g}(s)]]$$

$$= \frac{i}{24\pi^2} \int_{0}^{1} ds \, Tr[g(s) \frac{\partial g^{-1}(s)}{\partial s}(F_{g}^{2}(s)$$

$$+ \frac{i}{2}(A_{g}(s) F_{g} + F_{g}(s) A_{g} + F_{g}(s) A_{g}^{2}) - \frac{1}{2} A_{g}(s)^{4})]. \quad (92)$$

We extend the gauge fields $A$ to $\mathcal{A}$ with a fictitious dimension and fictitious gauge field $\mathcal{A}_s$

$$\mathcal{A}_\mu = A_\mu, \quad \mathcal{F}_{\mu\nu} = F_{\mu\nu}, \quad \mathcal{A}_s = 0, \quad (93)$$

$$\mathcal{A}^{g(s)} = g(s) \mathcal{A}_s g(s)^{-1} + ig(s) \partial_s g(s)^{-1} = ig(s) \partial_s g(s)^{-1}, \quad (94)$$

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_s - \partial_s \mathcal{A}_\mu - i[\mathcal{A}_\mu, \mathcal{A}_s] = 0, \quad (95)$$

$$\mathcal{F}^{g(s)} = g(s) \mathcal{F}_{\mu\nu} g(s)^{-1} = 0. \quad (96)$$
We can extend the 4-forms in (92) to 5-forms with \( \varepsilon^{0123} = 1 \) \( \varepsilon_{0123} = -1 \) and get

\[
\Gamma_A(A^g, g(s)) = \frac{1}{24\pi^2} \int_{D_5} \omega_5(A_{g(s)}, F_{g(s)}),
\]

(97)

where \( \omega_5(A_{g(s)}, F_{g(s)}) \) is the Chern-Simons 5-forms:

\[
\omega_5(A_{g(s)}, F_{g(s)}) = Tr[A_{g(s)}F_{g(s)}^2 + \frac{i}{2}A^3_{g(s)}F_{g(s)} - \frac{1}{10}A^5_{g(s)}].
\]

(98)

Compared to the last term in (92), a factor 1/5 arises in \( A^5_{g(s)} \) term due to the cyclic symmetry. \( \omega_5 \) is not a gauge invariant. One can write [15]

\[
\omega_5(A_{g(s)}, F_{g(s)}) = \omega_5(A + P, F) = \omega_5(A, F) + d\alpha_4(A, P) + \omega_5(P, 0),
\]

(99)

where \( P = i(dg(s)^{-1})g(s) \) is a 1-form and

\[
\alpha_4(A, P) = -Tr\left[\frac{1}{2}P(AdA + dAA) - \frac{i}{2}PA^3 + \frac{i}{2}P^3A + \frac{i}{4}PAPA\right].
\]

(100)

In this case \( \omega_5(A, F) = 0 \) because of \( A_s = 0 \). One can check (99) explicitly using \( dP = -iP^2 \).

 Putting (99) into (97) and integrate \( d\alpha_4 \) term over interval \([0, 1]\) one obtain

\[
i\Gamma_A(A^g, g(s)) = \frac{i}{24\pi^2} \int_{D_5} \omega_5(P, 0) + \frac{i}{24\pi^2} \int_{M_4} \alpha_4(A(s), P(s))\bigg|_{s=1}^{s=0}. \tag{101}
\]

Since \( P(s = 0) = 0 \) if reduced to \( M_4 \) we can get the action

\[
\Gamma_A(A^g, g(s)) = -\frac{i}{240\pi^2} \int_{D_5} ds \, d^4x \, Tr[\varepsilon^{PQRST}\frac{\partial g^{-1}}{\partial x^P}g\frac{\partial g^{-1}}{\partial x^Q}g\frac{\partial g^{-1}}{\partial x^R}g\frac{\partial g^{-1}}{\partial x^S}g\frac{\partial g^{-1}}{\partial x^T}g]
\]

\[
+ \frac{i}{48\pi^2} \int_{M_4} d^4x \, \varepsilon^{\mu\nu\rho\sigma} Tr[\frac{\partial g^{-1}}{\partial x^\mu}g(A_\mu A_\nu A_\sigma + \partial_\mu A_\rho A_\sigma - iA_\mu A_\rho A_\sigma)]
\]

\[
- \frac{i}{2\partial x^\mu}g\frac{\partial g^{-1}}{\partial x^\nu}g\frac{\partial g^{-1}}{\partial x^\rho}gA_\sigma - \frac{1}{2}\frac{\partial g^{-1}}{\partial x^\mu}gA_\nu \frac{\partial g^{-1}}{\partial x^\rho}gA_\sigma], \tag{102}
\]

where \( P, Q, R, S, T \) run over \( s, 0, 1, 2, 3 \) and \( \varepsilon^{\mu\nu\rho\sigma} = \varepsilon^{\mu\nu\rho\sigma} \) has been used.

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