A note on local antimagic chromatic number of lexicographic product graphs

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Abstract

Let $G = (V, E)$ be a connected simple graph. A bijection $f : E \rightarrow \{1, 2, \ldots, |E|\}$ is called a local antimagic labeling of $G$ if $f^+(u) \neq f^+(v)$ holds for any two adjacent vertices $u$ and $v$, where $f^+(u) = \sum_{e \in E(u)} f(e)$ and $E(u)$ is the set of edges incident to $u$. A graph $G$ is called local antimagic if $G$ admits at least a local antimagic labeling. The local antimagic chromatic number, denoted $\chi_{la}(G)$, is the minimum number of induced colors taken over local antimagic labelings of $G$. Let $G$ and $H$ be two disjoint graphs. The graph $G[H]$ is obtained by the lexicographic product of $G$ and $H$. In this paper, we obtain sufficient conditions for $\chi_{la}(G[H]) \leq \chi_{la}(G)\chi_{la}(H)$. Consequently, we give examples of $G$ and $H$ such that $\chi_{la}(G[H]) = \chi(G)\chi(H)$, where $\chi(G)$ is the chromatic number of $G$. We conjecture that (i) there are infinitely many graphs $G$ and $H$ such that $\chi_{la}(G[H]) = \chi_{la}(G)\chi_{la}(H) = \chi(G)\chi(H)$, and (ii) for $k \geq 1$, $\chi_{la}(G[H]) = \chi(G)\chi(H)$ if and only if $\chi(G)\chi(H) = 2\chi(H) + \lfloor \frac{\chi(H)}{2k+1} \rfloor$, where $2k+1$ is the length of a shortest odd cycle in $G$.

Keywords: lexicographic product; regular; local antimagic chromatic number.

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1 Introduction

Let $G = (V, E)$ be a connected simple graph of order $p$ and size $q$. A bijection $f : E \rightarrow \{1, 2, \ldots, q\}$ is called a local antimagic labeling of $G$ if $f^+(u) \neq f^+(v)$ holds for any two adjacent vertices $u$ and $v$, where $f^+(u) = \sum_{e \in E(u)} f(e)$ and $E(u)$ is the set of edges incident to $u$. Clearly, a local antimagic labeling induces a proper coloring of $G$. The function $f$ is called a local antimagic $t$-coloring of $G$ if $f$ induces a proper $t$-coloring of $G$, and we say $c(f) = t$. The local antimagic chromatic number of $G$, denoted by $\chi_{la}(G)$, is the minimum number of $c(f)$, where $f$ takes over all local antimagic labelings of $G$. Interested readers may refer to [5, 6, 10] for results related to local antimagic chromatic numbers of graphs.

Let $G$ and $H$ be two disjoint graphs. The lexicographic product $G[H]$ of graphs $G$ and $H$ is a graph such that its vertex set is the cartesian product $V(G) \times V(H)$, and any two vertices $(u, u')$ and $(v, v')$ are adjacent in $G[H]$ if and only if either $uv \in E(G)$ or $u = v$ and $u'v' \in E(H)$. In [9], the first two authors studied the exact value of $\chi_{la}(G[O_n])$, where $O_n$ is a null graph of order $n \geq 2$. Motivated by the above result, we investigate the sharp upper bound of $\chi_{la}(G[H])$ for any two disjoint non-null graphs $G$ and $H$ in this paper. We present the sufficient conditions for $\chi_{la}(G[H]) \leq \chi_{la}(G)\chi_{la}(H)$ holding. Further, we conjecture that (i) there are infinitely

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many graphs $G$ and $H$ with $\chi_{la}(G[H]) = \chi_{la}(G)\chi_{la}(H) = \chi(G)\chi(H)$, where $\chi(G)$ is the chromatic number of $G$; and (ii) for any positive integer $k$, $\chi_{la}(G[H]) = \chi(G)\chi(H)$ if and only if $\chi(G)\chi(H) = 2\chi(H) + \lceil \frac{\chi(H)}{k} \rceil$, where $2k + 1$ is the length of the shortest odd cycle in $G$. We refer to [3] for all undefined notation.

2 Bounds of $\chi_{la}(G[H])$

Before presenting our main results, we introduce some necessary notation and known results which will be used in this section.

Let $[a, b] = \{n \in \mathbb{Z} \mid a \leq n \leq b\}$ and $S \subseteq \mathbb{Z}$. Let $S^-$ and $S^+$ be a decreasing sequence and an increasing sequence of $S$, respectively.

Lemma 2.1 ([7] Lemma 2.2). For positive integers $q$ and $p$, let $S_p(a) = [p(n-1) + 1, pa]$, $1 \leq a \leq q$. Then,

(i) \( S_p(a) \mid 1 \leq a \leq q \) is a partition of \([1, pq]\);

(ii) when $a < b$, every term of $S_p(a)$ is less than that of $S_p(b)$;

(iii) for each $1 \leq i \leq p$, the sum of the $i$-th term of $S_p^+(a)$ and that of $S_p^-(b)$ is independent of $i$, where $1 \leq a, b \leq q$;

(iv) for any positive integer $k$ and each $1 \leq i \leq p$, $\sum_{l=1}^{k} (i$-th term of $S_p^+(a_l)) + \sum_{l=1}^{k} (i$-th term of $S_p^-(b_l))$ is independent of $i$, where $1 \leq a_l, b_l \leq q$.

Note that the proof of Lemma 2.1 in [7] shows that the sum of $i$-term of $S_p^+(a)$ and that of $S_p^-(b)$ is $p(a + b - 1) + 1$. According to the definitions of $S_p^+(a)$ and $S_p^-(b)$, we shall write the sequence $S_p^+(a)$ and $S_p^-(a)$ as column vectors in this paper. Now we are ready to present our first main result.

Theorem 2.2. Suppose $H$ admits a local antimagic $t$-coloring $f$ that satisfies the following conditions:

(a) for each vertex, the number of even incident edge labels equals the number of odd incident edge labels under $f$;

(b) when $f^+(u) = f^+(v)$, $\deg(u) = \deg(v)$;

(c) when $f^+(u) \neq f^+(v)$, $p f^+(u) + \frac{1}{2} \deg(u)(p - 1) \neq p f^+(v) - \frac{1}{2} \deg(v)(p - 1)$ holds for a fixed integer $p$.

Then $\chi_{la}(pH) \leq t$.

Proof. Let $V(H) = \{x_1, \ldots, x_n\}$ and $L$ be the labeling matrix of $H$ according to $f$ (for definition of labeling matrix, please see [11]). Now we define a guide matrix $M$ by adding a ‘+’ sign to all odd entries and a ‘−’ sign to all even entries in $L$. The concept of guide matrix was introduced in [7].

We define $p$ matrices $L_1, \ldots, L_p$ as follows. For each $1 \leq i \leq p$, the $(j, k)$-entry of $L_i$ is the $i$-th term of $S_p^+(a)$ (resp. $S_p^-(a)$) if the corresponding $(j, k)$-entry of $M$ is $+a$ (resp. $-a$), where $1 \leq a \leq |E(H)|$.

From the condition (a), for each row of $L$, the number of odd entries equals that of even entries. Thus, let $a_1, \ldots, a_s$ denote the odd numerical entries of the $j$-th row of $L$ and $b_1, \ldots, b_s$ denote the even numerical entries of the $j$-th row of $L$, where $s$ is a positive integer. Now,

$$r_j(L_i) = \sum_{l=1}^{s} [i$-th term of $S_p^+(a_l)] + \sum_{l=1}^{s} [i$-th term of $S_p^-(b_l)]$$.
By Lemma \(2.1\) (iv), \(r_j(L_i)\) is constant for a fixed \(j\). Actually, it is \(p \sum_{l=1}^{s} (a_l + b_l) - ps + s = pr_j(L) - k(p - 1) = pf^+(x_j) - \frac{1}{2} \deg(x_j)(p - 1)\). By conditions (a) and (b), the diagonal block matrix

\[
\begin{pmatrix}
L_1 & \star & \cdots & \star \\
\star & L_2 & \cdots & \star \\
\vdots & \vdots & \ddots & \vdots \\
\star & \star & \cdots & L_p
\end{pmatrix}
\]

is a local antimagic labeling of \(pH\). Thus \(\chi_{la}(pH) \leq t\). \(\square\)

It is known that \(\chi_{la}(K_{1,2n}) = 2n + 1\) and \(\chi_{la}(mK_{1,2n}) = 2nm + 1\) \([2\ Corollary 3]\). Clearly, the upper bound stated in Theorem \(2.2\) is not sharp. From Theorem \(2.2\) we obtain the following result immediately.

**Corollary 2.3.** If \(H\) is an \(r\)-regular graph \((r \geq 2)\) with a local antimagic \(t\)-coloring \(f\) satisfying the condition (a) of Theorem \(2.2\) then \(\chi_{la}(pH) \leq t\) holds for any positive integer \(p\).

**Theorem 2.4.** Let \(G\) be a graph of order \(p\) admitting a local antimagic \(\chi_{la}(G)\)-coloring \(g\) and \(H\) be a graph of order \(n\) admitting a local antimagic \(\chi_{la}(H)\)-coloring \(h\). Suppose \(h\) satisfies the following conditions:

(i) For each vertex, the number of even incident edge labels equals the number of odd incident edge labels under \(h\);

(ii) when \(h^+(u) = h^+(v)\), \(\deg_H(u) = \deg_H(v)\);

(iii) when \(h^+(u) \neq h^+(v)\), \(ph^+(u) - \frac{1}{2} \deg_H(u)(p - 1) \neq ph^+(v) - \frac{1}{2} \deg_H(v)(p - 1)\).

Moreover, \(g\) satisfies the following conditions:

(iv) when \(g^+(u) = g^+(v)\), \(\deg_G(u) = \deg_G(v)\), and

(v) when \(g^+(u) \neq g^+(v)\), \(g^+(u)n^3 - \frac{(n^3 - n)\deg_G(u)}{2} \neq g^+(v)n^3 - \frac{(n^3 - n)\deg_G(v)}{2}\).

Then \(\chi_{la}(G[H]) \leq \chi_{la}(G)\chi_{la}(H)\).

**Proof.** Let \(q(G)\) and \(q(H)\) denote the sizes of \(G\) and \(H\) respectively. Clearly, \(G[H]\) is a graph of order \(pn\) and size \(pq(H) + q(G)n^2\). Suppose that \(\{u_1, \ldots, u_p\}\) and \(\{x_1, \ldots, x_n\}\) are the vertex lists of \(G\) and \(H\) respectively. According to these vertex lists, we define that \(A_G\) and \(A_H\) are the adjacency matrices of \(G\) and \(H\) respectively. Thus the adjacency matrix of \(G[H]\) can be expressed as

\[A_G \otimes J_n + I_p \otimes A_H,\]

where \(J_n\) is an \(n \times n\) matrix whose entries are all 1, \(I_p\) is an identity matrix of order \(p\), and \(A_G \otimes J_n\) is the Kronecker product of \(A_G\) and \(J_n\). Note that \(A_G \otimes J_n\) is the adjacency matrix of \(G[O_n]\) and \(I_p \otimes A_H\) is the adjacency matrix of \(O_p[H]\), where \(O_n\) and \(O_p\) are null graphs of orders \(n\) and \(p\). Therefore, the diagonal blocks of \(A_G \otimes J_n\) are zero matrices and only the diagonal blocks of \(I_p \otimes A_H\) are non-zero matrices.

Now we shall label the edges of \(O_p[H]\) and \(G[O_n]\) separately. According to the definition, \(O_p[H] \cong pH\). Since \(H \in \mathcal{H}\), by Theorem \(2.2\) \(pH\) has a local antimagic \(\chi_{la}(H)\)-coloring, say \(\phi\), by using integers in \([1, pq(H)]\) such that for each vertex \((u_i, x_j)\) in \(O_p[H]\), \(\phi^+(u_i, x_j)\) is independent of \(i\), where \(1 \leq i \leq p\). The labeling matrix of \(\phi\) is denoted by \(\mathcal{M}_1\).

Next we shall label \(G[O_n]\) by integers in \([1, q(G)n^2]\). This labeling was constructed in the proof of \(2.1\) Theorem 2.1. For completeness, we list the outline of the construction.

Let \(M_g\) be the labeling matrix of \(G\) corresponding to \(g\). Suppose \(\Omega\) is a magic square of order \(n\). Let \(\Omega_i = \Omega + (i - 1)n^2J_n\), where \(1 \leq i \leq q(G)\) and \(\psi_0\) be the labeling of \(G[O_n]\) such that its labeling matrix \(\mathcal{M}\) is defined by replacing each entry of \(M_G\) with an \(n \times n\) matrix as follows:

3
(1) replace \( \ast \) by \( \star \) which is an \( n \times n \) matrix whose entries are \( \ast \);

(2) replace \( i \) by \( \Omega_i \), if \( i \) lies in the upper triangular part of \( M_i \);

(3) replace \( i \) by \( \Omega_i^T \), if \( i \) lies in the lower triangular part of \( M_i \), where \( \Omega_i^T \) is the transpose of \( \Omega_i \).

For each vertex \( (u_i, x_j) \in V(G[O_n]) \), the row sum of \( \mathcal{M} \) corresponding to the vertex \( (u_i, x_j) \) is \( \psi_0^+(u_i, x_j) = g^+(u_i)M_i \), which is independent of \( j \). By condition (i), \( \psi_0 \) is a local antimagic labeling of \( G(O_n) \). According to condition (v), there are at most \( \chi(G) \) distinct row sums of \( \mathcal{M} \). Let \( \mathcal{M}_2 \) be the matrix obtained from \( \mathcal{M} \) by adding all numerical entries with \( pq(H) \) and \( \psi \) be the corresponding labeling. Then, \( \psi^+(u_i, x_j) = \psi_0^+(u_i, x_j) + n pq(H) \), which is independent of \( j \).

Therefore, \( \mathcal{M}_1 + \mathcal{M}_2 \) is a labeling matrix that corresponds to a local antimagic labeling of \( G[H] \), where \( \ast \) is treated as 0. Hence \( \chi_{la}(G[H]) \leq \chi_{la}(G)\chi_{la}(H) \).

The following is an example of Theorem 2.4.

**Example 2.1.** Let \( G \) be the one point union of two 4-cycles and \( H \) be the one point union of two 3-cycles. Figure 1 show the local antimagic 3-colorings of \( G \) and \( H \).

![Figure 1: Local antimagic 3-colorings of graphs G and H](image)

Note that \( \chi_{la}(G) = \chi_{la}(H) = 3 \). It is easy to check that the above local antimagic 3-colorings of \( G \) and \( H \) satisfy the conditions of Theorem 2.4 respectively. Then, the labeling matrices of \( G \) and \( H \) are shown below:

\[
M_G = \begin{pmatrix}
8 & 1 \\
2 & 7 \\
3 & 6 \\
1 & 4 \\
\end{pmatrix}, \quad M_H = \begin{pmatrix}
6 & 1 \\
5 & 2 \\
3 & 4 \\
\end{pmatrix}.
\]

Let
\[
L_1 = \begin{pmatrix}
42 & 1 \\
29 & 14 \\
2 & 28 \\
1 & 25 \\
\end{pmatrix}, \quad L_2 = \begin{pmatrix}
41 & 2 \\
30 & 13 \\
2 & 27 \\
1 & 23 \\
\end{pmatrix}, \quad L_3 = \begin{pmatrix}
40 & 3 \\
31 & 12 \\
3 & 24 \\
1 & 17 \\
\end{pmatrix},
\]
\[
L_4 = \begin{pmatrix}
39 & 4 \\
32 & 11 \\
3 & 25 \\
4 & 18 \\
\end{pmatrix}, \quad L_5 = \begin{pmatrix}
38 & 5 \\
33 & 10 \\
3 & 24 \\
4 & 19 \\
\end{pmatrix}, \quad L_6 = \begin{pmatrix}
37 & 6 \\
34 & 9 \\
3 & 23 \\
4 & 20 \\
\end{pmatrix},
\]
\[
L_7 = \begin{pmatrix}
36 & 7 \\
35 & 8 \\
3 & 21 \\
7 & 22 \\
\end{pmatrix}.
\]
Obviously, for each 1 ≤ i ≤ 7, the row sums of $L_i$ are 43, 43, 57, 57, 58 respectively. Let $\Omega$ be a magic square of order 5 with row sum 65 and $\Omega_i = \Omega + 25(i - 1)J_5$, where 1 ≤ i ≤ 8. For each 1 ≤ i ≤ 8, let $\Psi_i = \Omega_i + 42J_5$. Then, the labeling matrix of $G[H]$ is

$$
\begin{pmatrix}
L_1 & \star & \star & \star & \star & \Psi_8 & \star & \Psi_1 \\
\star & L_2 & \star & \star & \star & \Psi_2 & \star & \Psi_7 \\
\star & \star & L_3 & \star & \star & \star & \Psi_6 & \Psi_3 \\
\star & \star & \star & L_4 & \star & \star & \Psi_4 & \Psi_5 \\
\Psi_5^T & \Psi_2^T & \star & \star & \star & \star & \Psi_7^T & \Psi_1^T \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\Psi_5^T & \Psi_2^T & \Psi_6^T & \Psi_4^T & \Psi_7^T & \Psi_1^T & \Psi_3^T & \star & \star & L_7
\end{pmatrix}
$$

By calculating the row sums of the above matrix, we obtain that the distinct row sums are 1468, 1482, 1483, 1593, 1607, 1608, 2643, 2657, 2658. Thus, $\chi_{la}(G[H]) \leq 9$.

In [4], N. Ćižek and S. Klavžar gave the lower bound of chromatic number of the lexicographic product as follows.

**Corollary 2.5 ([4, Corollary 3]).** Let $G$ be a nonbipartite graph. Then for any graph $H$, $\chi(G[H]) \geq 2\chi(H) + \lceil \frac{\chi(H)}{k} \rceil$, where $k \geq 1$ and $2k + 1$ is the length of a shortest odd cycle in $G$.

Combining Theorem 2.4 and Corollary 2.5, we obtain the following results.

**Corollary 2.6.** Suppose $G$ and $H$ are graphs satisfying the conditions listed in Theorem 2.4. If the length of a shortest odd cycle in $G$ is $2k + 1$, then $2\chi(H) + \lceil \frac{\chi(H)}{k} \rceil \leq \chi(G[H]) \leq \chi_{la}(G[H]) \leq \chi_{la}(G)\chi_{la}(H)$. In particular, if $C_3$ is a subgraph of $G$, then $3\chi(H) \leq \chi_{la}(G[H]) \leq \chi_{la}(G)\chi_{la}(H)$.

**Proof.** $\chi(G[H]) \leq \chi_{la}(G[H])$ is trivial. The lower bound follows from Corollary 2.5 and the upper bound follows from Theorem 2.4. \qed

**Corollary 2.7.** Let $G$ and $H$ be regular graphs and $H$ admit a local antimagic $\chi_{la}(H)$-coloring $h$. Suppose for each vertex of $H$, the number of even incident edge labels equals the number of odd incident edge labels under $h$. If the length of a shortest odd cycle in $G$ is $2k + 1$, then $2\chi(H) + \lceil \frac{\chi(H)}{k} \rceil \leq \chi(G[H]) \leq \chi_{la}(G[H]) \leq \chi_{la}(G)\chi_{la}(H)$. In particular, if $C_3$ is a subgraph of $G$, then $3\chi(H) \leq \chi_{la}(G[H]) \leq \chi_{la}(G)\chi_{la}(H)$.

By applying Corollary 2.7, we can obtain $\chi_{la}(G[H])$ for some graphs $G$ and $H$. An example is shown in Example 2.2.

**Example 2.2.** Let $G = C_3 \times K_2$ and $H$ be the octahedral graph. Figure 2 presents their local antimagic 3-colorings which are shown in [8].

![Figure 2: Local antimagic 3-colorings of graphs G and H](image-url)
It is easy to verify that $G$ and $H$ satisfy the conditions of Corollary 2.7, which implies that $3\chi(H) \leq \chi_{la}(G[H]) \leq \chi_{la}(G)\chi_{la}(H)$. Since $\chi_{la}(G) = \chi_{la}(H) = 3$, $\chi_{la}(G[H]) = 9$. 

In [8, Theorem 3.3], the first two authors proved that $\chi_{la}(C_{2m} \vee O_{2n}) = 3$ for $m \geq 2, n \geq 1$, where $C_{2m} \vee O_{2n}$ is the join of graphs $C_{2m}$ and $O_{2n}$. In the following, we give another local antimagic 3-coloring of $C_{2m} \vee O_{2n}$ that satisfies the conditions (i) and (ii) of Theorem 2.4.

**Theorem 2.8.** For $m \geq 2$ and $n \geq 1$, there is a local antimagic 3-coloring of $C_{2m} \vee O_{2n}$ satisfying conditions (i) and (ii) of Theorem 2.4.

**Proof.** Let $V(C_{2m}) = \{u_i \mid 1 \leq i \leq 2m\}$ and $V(O_{2n}) = \{v_j \mid 1 \leq j \leq 2n\}$. We separate $C_{2m} \vee O_{2n}$ into two edge-disjoint graphs, $C_{2m}$ and $O_{2m} \vee O_{2n}$, where $V(O_{2m}) = V(C_{2m})$.

Firstly, define a labeling for $C_{2m}$. Let $f: V(C_{2m}) \to [1, 2m]$ such that $f(u_i u_{i+1}) = i$, where $1 \leq i \leq 2m$ and $u_{2m+1} = u_1$.

Thus, $f^+(u_1) = 2m + 1$, $f^+(u_i) = 2i - 1$ for $2 \leq i \leq 2m$.

Next, we define a labeling $g$ for $O_{2m} \vee O_{2n} \cong K_{2m, 2n}$. The labeling matrix of $g$ is $\begin{pmatrix} \star & B \\ B^T & \star \end{pmatrix}$ under the vertex lists $V(O_{2m}) = \{u_1, u_3, \ldots, u_{2m-1}, u_2, u_4, \ldots, u_{2m}\}$ and $V(O_{2n}) = \{v_1, v_2, \ldots, v_{2n}\}$. So we only need to fill the integers in $[2m + 1, 2m + 4m]$ into the matrix $B$.

Let $M$ be a guide matrix as follows:

$$
\begin{pmatrix}
-2 & -3 & -2 & -1 & 2 & -2n + 1 & -2n + 4 & -2n + 2 & -2n + 3 & -5 \\
2n + 1 & 2n & (2n + 1) & (2n - 1) & (2n) & (2n - 2) & (2n - 3) & \cdots & (2n - 3) & 2n - 2
\end{pmatrix}
$$

We replace each entry of $M$ to a column vector according to the rules:

1. replace $-a$ to $2S_m^m(a) - J_{m,1}$; replace $+a$ to $2S_m^m(a) - J_{m,1}$, where $J_{m,1}$ is an $m \times 1$ matrix with all entries 1;
2. replace $-a$ to $2S_m^m(a)$; replace $+a$ to $2S_m^m(a)$.

Let $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ be the resulting matrix, where $B_1$ and $B_2$ are $m \times 2n$ matrices. The row sums of $B_1$ in column matrix is

$$
(2S_m^m(2) - J_{m,1}) + (2S_m^m(3) - J_{m,1}) + 2S_m^m(2) + 2S_m^m(2n - 1) + \sum_{i=1}^{n-2} 2S_m^m(2i + 1) + \sum_{j=2}^{n-1} 2S_m^m(2j + 1) - J_{m,1}
$$

$$
= 4S_m^m(2n - 1) + 2[S_m^m(2) + S_m^m(2)] + 2 \sum_{i=1}^{n-2} [S_m^m(2i + 1) + S_m^m(2i + 1)] - nJ_{m,1}
$$

$$
= 4S_m^m(2n - 1) + 2(3m + 1)J_{m,1} + 2 \sum_{i=1}^{n-2} [m(4i + 1) + 1]J_{m,1} - nJ_{m,1}
$$

$$
= 4S_m^m(2n - 1) + [4mn^2 - 10mn + 10m + n - 2]J_{m,1} = A_1.
$$

Clearly, the entries of the column matrix $A_1$ form a decreasing sequence with common difference 4. Now the first column of $B_1$ is the vector $2S_m^m(2) - J_{m,1}$. We shift each entry of this vector downward to 1 and move the last entry of this vector to the top, i.e., add the entries by 2 except the $(1, 1)$-entry and subtract the $(1, 1)$-entry by 2$(m - 1)$. Let this new matrix be $B_1'$. Now, the first column of $B_1'$ has entries $2m + 1, 4m - 1, 4m - 3, \ldots, 2m + 3$ so that the second entry up to the last entry of the first column of $B_1'$ form a decreasing sequence with common difference 2 and the difference between the first entry and second entry is 2 - 2m.

Similarly the row sums of $B_2$ in column matrix is
\[
4S_m^-(2n + 1) + 4S_m^-(2n) + 4S_m^+(4) + 2 \sum_{i=3}^{n-1} [S_m^+(2i) + S_m^-(2i)] - nJ_{m,1}
\]
\[
= 4S_m^-(2n + 1) + (4m(2n + 3) + 1)J_{m,1} + [2m(n - 3)(2n + 3) + 2(n - 3)]J_{m,1} - nJ_{m,1}
\]
\[
= 4S_m^-(2n + 1) + [4mn^2 + 2mn - 6m + n - 2]J_{m,1} = A_2.
\]

It is clear that the entries of the column matrix \(A_2\) form a decreasing sequence with common difference 4.

Combining the labelings \(f\) and \(g\), we have a labeling \(\phi\) for the whole graph \(C_{2m} \cup O_{2n}\). One may check that \(\phi^+(u_{2j-1}) = f^+(u_{2j-1}) + r_j(B_1) = 4mn^2 - 2mn + 6m + n + 1\) for each \(1 \leq j \leq m\); and \(\phi^+(u_{2i}) = f^+(u_{2i}) + r_i(B_2) = 4mn^2 + 10mn - 2m + n + 1\) for each \(1 \leq i \leq m\). Hence \(\phi^+(u_{2i}) > \phi^+(u_{2j-1})\) for \(1 \leq i, j \leq m\).

Clearly, the column sum of \((B'_1 \ B'_2)\) is \((4mn + 4m + 1)m\). So \(\phi^+(v_i) = (4mn + 4m + 1)m\).

\[
\phi^+(u_{2j-1}) - \phi^+(v_i) = 4mn^2 - 4m^2n - 4m^2 - 2mn + 5m + n + 1
\]
\[
= 4mn(m - m - 1) - 4m^2 + 2mn + 5m + n + 1. \tag{2.1}
\]

If \(n \geq m + 2\), then \(\phi^+(u_{2i}) - \phi^+(v_i) \geq \phi^+(u_{2j-1}) - \phi^+(v_i) \geq 4mn - 4m^2 + 2mn + 5m + n + 1 > 0\).

\[
\phi^+(v_i) - \phi^+(u_{2i}) = 4m^2n - 4mn^2 + 4m^2 - 10mn + 3m - n - 1
\]
\[
= 4mn(m - n - 2) + 4m^2 - 2mn + 3m - n - 1. \tag{2.2}
\]

If \(m \geq n + 2\), then \(\phi^+(v_i) - \phi^+(u_{2j-1}) > \phi^+(v_i) - \phi^+(u_{2i}) > 0\).

1) If \(n = m + 1\), then \(\phi^+(u_{2j-1}) - \phi^+(v_i) = -2m^2 + 8m + 2 \neq 0\) (since the discriminant is not a prefect square) and \(\phi^+(u_{2i}) - \phi^+(v_i) = 10m^2 + 12m + 2 > 0\).

2) If \(n = n\), then \(\phi^+(u_{2j-1}) - \phi^+(v_i) = -6m^2 + 6m + 1 > 0\) and \(\phi^+(u_{2i}) - \phi^+(v_i) = 6m^2 - 2m + 1 > 0\).

3) If \(n = m - 1\), then \(\phi^+(u_{2j-1}) - \phi^+(v_i) = -10m^2 + 12m < 0\), but \(\phi^+(u_{2i}) - \phi^+(v_i) = 2m^2 - 8m \neq 0\) when \(m \\neq 4\). So, for \(n = m - 1 = 3\), we have to find another labeling for \(C_8 \cup O_6\).

Label the edges of \(C_8\) by 1, 3, 2, 5, 4, 7, 6 in natural order. Let this labeling be \(f\). So the induced vertex of \(u_1, u_2, \ldots, u_8\) are 7, 9, 11, 5, 7, 9, 11, 13.

We start from a \(8 \times 6\) magic rectangle \(\Omega\) (shown below). Add each entry by 8 and swap some entries within the same column (indicated in italic). We have

\[
\Omega = \begin{pmatrix}
1 & 44 & 9 & 36 & 29 & 28 \\
2 & 43 & 10 & 35 & 30 & 27 \\
3 & 42 & 11 & 34 & 31 & 26 \\
4 & 41 & 12 & 33 & 32 & 25 \\
45 & 8 & 37 & 16 & 17 & 24 \\
46 & 7 & 38 & 15 & 18 & 23 \\
47 & 6 & 39 & 14 & 19 & 22 \\
48 & 5 & 40 & 13 & 20 & 21 \\
\end{pmatrix} \quad \rightarrow \quad \begin{pmatrix}
7 & 9 & 52 & 17 & 44 & 37 & 36 \\
11 & 10 & 51 & 18 & 43 & 38 & 31 \\
7 & 11 & 50 & 19 & 42 & 39 & 34 \\
11 & 12 & 49 & 20 & 41 & 40 & 29 \\
9 & 53 & 16 & 45 & 24 & 27 & 32 \\
9 & 54 & 15 & 46 & 23 & 26 & 33 \\
13 & 55 & 14 & 47 & 22 & 25 & 30 \\
5 & 56 & 13 & 48 & 21 & 28 & 35 \\
\end{pmatrix} = 202
\]

This matrix forms a labeling matrix of a labeling \(g\) of \(K_{8,6}\) under the vertex list \(\{u_1, u_3, u_5, u_7, u_2, u_6, u_8, u_4\}\) of \(C_8\). The column in front of the matrix is the corresponding induced vertex labels under \(f\) on \(C_8\), and the column behind of the matrix is the induced vertex labels of the labeling \(\phi\) for \(C_8 \cup O_6\). Thus \(\phi^+(u_{2i-1}) = 202, \phi^+(u_{2i}) = 206\) and \(\phi^+(v_j) = 260\) for \(1 \leq i \leq 4\) and \(1 \leq j \leq 6\).

Clearly all labels are used. So \(\phi\) is a local antimagic 3-coloring for \(C_{2m} \cup O_{2n}\). Moreover, the number of even incident edge labels equals the number of odd incident edge labels for each vertex. Hence \(\phi\) satisfies condition (i) and (ii) of Theorem 2.4.
**Corollary 2.9.** If $G = C_3 \times K_2$ and $H = C_{2m} \lor O_{2n}$, $m \geq 2$, $n \geq 1$, then $\chi_{la}(G[H]) = 9$.

**Proof.** Keep all notation defined in the proof of Theorem 2.3. Now $\deg_H(u_i) = 2n + 2$, $\deg_H(v_i) = 2m$ and $p = 6$. By Theorems 2.4 and 2.8, it suffices to check condition (iii) of Theorem 2.4, i.e., $6[\phi^+(u_1) - \phi^+(v_1)] - 5(n + 1 - m) \neq 0$ and $6[\phi^+(v_1) - \phi^+(u_2)] - 5(m - n - 1) \neq 0$.

By (2.1), we have
\[
6[4mn(n - m - 1) - 4m^2 + 2mn + 5m + n + 1] - 5(n + 1 - m)
= -24m^2 + 24mn^2 - 24m^2n - 12mn + 35n + m + 1
= 24mn(n - m) - 24m(m - 1) - 12mn + 11m + n + 1. \tag{2.3}
\]

Clearly (2.3) is a perfect square. When $n \geq m + 2$, (2.3) $\geq 36mn - 24m^2 + 35m + n + 1 > 0$. When $n = m + 1$, (2.3) $= 12mn - 24m^2 + 35m + n + 1 = -12m^2 + 48m + 2 \neq 0$ since the discriminant is 2400 which is not a perfect square.

By (2.2), we have
\[
6[4mn(m - n - 2) + 4m^2 - 2mn + 3m - n - 1] - 5(m - n - 1)
= 4m^2 + 24m^2n - 24mn^2 - 60mn + 13m - n - 1
= 24mn(m - n) - 12m(m - n) + 12m^2 + 13m - n - 1. \tag{2.4}
\]

Clearly (2.4) $> 0$ for $m \geq n + 2$. When $m \leq n$, (2.4) $\leq -48mn + 12m + 13m - n - 1 = 12m(m - 4n + 1)m - n - 1 < 0$. When $m = n + 1$, then $H$ is regular so condition (iii) holds. The proof is complete.

**Example 2.3.** Let $V(C_6) = \{u_1, u_3, u_5, u_2, u_4, u_6\}$ and $V(O_8) = \{v_j \mid 1 \leq j \leq 8\}$ be the vertex lists of $C_6$ and $O_8$. According to the proof of Theorem 2.4, we label the edges of $C_6$ by 1 to 6 in natural order. So the induced vertex labels are 7, 3, 5, 7, 9, 11. Then

| $G$ | 7 | 17 | 8 | 42 | 14 | 41 | 26 | 29 | 191 |
|-----|---|----|---|----|----|----|----|----|-----|
| 5   | 11 | 15 | 10| 40 | 16 | 39 | 28 | 27 | 191 |
| 9   | 9  | 13 | 12| 38 | 18 | 37 | 30 | 25 | 191 |
| 3   | 54 | 48 | 53| 19 | 47 | 20 | 35 | 32 | 311 |
| 7   | 52 | 46 | 51| 21 | 45 | 22 | 33 | 34 | 311 |
| 11  | 50 | 44 | 49| 23 | 43 | 24 | 31 | 36 | 311 |

The column in front of the matrix is the corresponding induced vertex labels under $\chi_{la}$, and the column behind of the matrix is the induced vertex labels of the labeling $\phi$ for $C_6 \lor O_8$. One may check that the column sum of the matrix is 183, which is $\phi^+(v_j)$ for all $j$.

Corollary 2.9 shows that there exist infinitely many graphs $H$ such that $\chi_{la}(G[H]) = \chi_{la}(G)\chi_{la}(H) = \chi(G)\chi(H)$. We end this note with the following conjectures.

**Conjecture 2.1.** There exist infinitely many graphs $G$ and $H$ respectively such that $\chi_{la}(G[H]) = \chi_{la}(G)\chi_{la}(H) = \chi(G)\chi(H)$.

**Conjecture 2.2.** For $k \geq 1$ and graph $G$ and $H$, $\chi_{la}(G[H]) = \chi(G)\chi(H)$ if and only if $\chi(G)\chi(H) = 2\chi(H) + \left\lceil \frac{\chi(H)}{2k} \right\rceil$, where $2k + 1$ is the length of a shortest odd cycle in $G$.

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