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Abstract: We study categorial properties of the operadic twisting functor Tw. In particular, we show that Tw is a comonad. Coalgebras of this comonad are operads for which a natural notion of twisting by Maurer-Cartan elements exists. We give a large class of examples, including the classical cases of the Lie, associative and Gerstenhaber operads, and their infinity-counterparts Lie$_\infty$, As$_\infty$, Ger$_\infty$. We also show that Tw is well behaved with respect to the homotopy theory of operads. As an application we show that every solution of Deligne’s conjecture is homotopic to a solution that is compatible with twisting.

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OPERADIC TWISTING – WITH AN APPLICATION TO DELIGNE’S CONJECTURE

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Abstract. We study the categorial properties of the operadic twisting functor Tw. It is shown that Tw is a comonad. The coalgebras of this comonad are operads that are “natively twistable”. As an application we show that every solution of Deligne’s conjecture is homotopic to a solution that is compatible with twisting.

1. Introduction

It is well known that some algebraic objects allow for twisting by Maurer-Cartan elements. For example, let \( g \) be a differential graded Lie algebra. Let \( m \in g \) be a Maurer-Cartan element, i.e., a degree 1 element satisfying the Maurer-Cartan equation

\[
dm + \frac{1}{2} [m, m] = 0.
\]

Then the twisted Lie algebra \( g^m \) is the graded space \( g \) with differential \( d + [m, \cdot] \) and bracket \( [\cdot, \cdot] \). In a similar manner the notion of Maurer-Cartan element and twisting can be defined for associative and Gerstenhaber algebras, and their homotopy versions, i.e., Lie\(_\infty\), Assoc\(_\infty\) and Ger\(_\infty\) algebras. In this paper we study classes of algebraic objects which can be twisted by Maurer-Cartan elements, using the operadic twisting functor Tw introduced in [12]. Concretely, we show in Theorem 4.7 that Tw is naturally equipped with the structure of a comonad. This comonad furthermore has the following properties:

- The coalgebras of the comonad Tw are (in particular) operads whose algebras can be twisted by Maurer-Cartan elements (see section 4.5).
- A map \( O \to O' \) of operads induces a functor from the category of \( O' \)-algebras to the category of \( O \)-algebras. If \( O \to O' \) gives rise to a map of Tw-coalgebras, then this functor is compatible with the operation of twisting by Maurer-Cartan elements (see Theorem 4.11).
- The functor Tw preserves quasi-isomorphisms between operads (see Theorem 4.14).

Here we postpone the precise statements to later sections, since they require terminology introduced in the definition of Tw. We apply these concepts to study solutions of (our version of) the Deligne conjecture. By this we mean, abusing notation, a quasi-isomorphism between the homotopy Gerstenhaber and the Braces operad.

\[
F : \text{Ger}_\infty \to \text{Br}
\]

Both operads are Tw coalgebras, i.e., their algebras can be twisted by Maurer-Cartan elements. In general \( F \) does not need to be a morphism of Tw coalgebras.

Theorem 1.1. Suppose \( F : \text{Ger}_\infty \to \text{Br} \) is a quasi-isomorphism of operads. Then there is a homotopic quasi-isomorphism of operads \( F' : \text{Ger}_\infty \to \text{Br} \) that is also a morphism of Tw coalgebras.

The morphism \( F' \) is given by explicit formulas, see below. For example the Braces operad naturally acts on the Hochschild cochain complex \( C(A) \) of any algebra \( A \), and by the map \( F \) we obtain a Ger\(_\infty\) structure on \( C(A) \).

\[
F_A : \text{Ger}_\infty \to \text{End}(C(A))
\]

Suppose \( m \) is a Maurer-Cartan element in \( C(A) \), i.e., \( \mu + m \) is another associative product on \( A \), with \( \mu \) denoting the original product. There are two ways to produce a Ger\(_\infty\) structure on \( C(A^m) \), where \( A^m \) is the twisted algebra with product \( \mu + m \). First, since \( A^m \) is an associative algebra, one may form \( F_A^m \) as before. Secondly, one may twist \( F_A \) by \( m \) to obtain \( F_A^m \). The following result is a consequence of Theorem 1.1

Corollary 1.2. The maps \( F_A^m \) and \( F_A^m \) are homotopic.
1.1. Structure of the paper. In section 2 we fix our notations and recall some facts about operads. Section 3 contains a description of the twisting functor. Its categorical properties are studied in section 4. Sections 5, 6, and 7 treat the braces operad and show how to recover it (essentially) as the twisted version of some operad $\mathcal{BT}$. Finally the proof of Theorem 1.1 is given in section 8.

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2. Preliminaries

The underlying field $K$ has characteristic zero. The notation $S_n$ is reserved for the symmetric group on $n$ letters. The subset of $(p_1, p_2, \ldots, p_k)$-shuffles in $S_n$, $n = p_1 + p_2 + \ldots p_k$ is denoted by $\text{Sh}_{p_1, p_2, \ldots, p_k}$. The underlying symmetric monoidal category $\mathcal{C}$ is either the category $\text{grVect}_K$ of $\mathbb{Z}$-graded $K$-vector spaces or the category $\text{Ch}_K$ of unbounded cochain complexes of $K$-vector spaces. For a homogeneous vector $v$ in a cochain complex, $|v|$ denotes the degree of $v$. Furthermore, we denote by $s$ (resp. $s^{-1}$) the operation of suspension (resp. desuspension) on $\text{grVect}_K$ or $\text{Ch}_K$.

The notation $\text{Lie}$ (resp. $\text{Com}$, $\text{Ger}$) is reserved for the operad in $\text{Ch}_K$ governing Lie algebras (resp. commutative algebras without unit, Gerstenhaber algebras without unit). Dually, the notation $\text{coLie}$ (resp. $\text{coCom}$) is reserved for the cooperad in $\text{Ch}_K$ governing Lie coalgebras (resp. commutative coalgebras without counit).

For an operad $\mathcal{O}$ in $\text{Ch}_K$ we denote by $\Lambda \mathcal{O}$ the operad with the spaces:

$$\Lambda \mathcal{O}(n) = s^{1-n} \mathcal{O}(n) \otimes \text{sgn}_n,$$

where $\text{sgn}_n$ denotes the sign representation of $S_n$. For example, an algebra over $\Lambda \text{Lie}$ in $\text{Ch}_K$ is a cochain complex $V$ equipped with the binary operation:

$$\{,\} : V \otimes V \to V$$

of degree $-1$ satisfying the identities:

$$\{v_1, v_2\} = (-1)^{|v_1||v_2|}\{v_2, v_1\}$$

$$\{\{v_1, v_2\}, v_3\} + (-1)^{|v_1||v_2|+|v_3|}\{\{v_2, v_3\}, v_1\} + (-1)^{|v_1||v_2|}\{\{v_1, v_3\}, v_2\} = 0,$$

where $v_1, v_2, v_3$ are homogeneous vectors in $V$.

$\text{Ger}^\vee$ denotes the Koszul dual cooperad [3, 5, 6] for $\text{Ger}$. It is known [7] that

$$\text{Ger}^\vee = \Lambda^2 \text{Ger}^\ast,$$

where $\text{Ger}^\ast$ is obtained from the operad $\text{Ger}$ by taking the linear dual. In other words, algebras over the linear dual $(\text{Ger}^\vee)^\ast$ are very much like Gerstenhaber algebras except that the bracket carries degree 1 and the multiplication carries degree 2.

2.1. Trees. In this paper all trees are rooted and the root vertex has always valency 1. (Such trees are sometimes called planted). The remaining vertices of valency 1 are called leaves. A vertex is called internal if it is neither a root nor a leaf. We always orient trees in the direction towards the root. Thus every internal vertex has at least 1 incoming edge and exactly 1 outgoing edge. An edge adjacent to a leaf is called external. We allow a degenerate tree, that is a tree with exactly two vertices (the root vertex and a leaf) connected by a single edge.

Let us recall that for every planar tree $t$ the set $V(t)$ of its vertices is equipped with a natural linear order. To define this linear order on $V(t)$ we introduce the function

$$\mathcal{N} : V(t) \to V(t).$$

To a vertex $v \in V(t) \backslash V_{\text{root}}(t)$ the function $\mathcal{N}$ assigns the next vertex along the (unique) path connecting $v$ to the root vertex. Furthermore $\mathcal{N}$ sends the root vertex to the root vertex.

Let $v_1, v_2$ be two distinct vertices of $t$. If $v_1$ lies on the path which connects $v_2$ to the root vertex then we declare that

$$v_1 < v_2.$$ 

Similarly, if $v_2$ lies on the path which connects $v_1$ to the root vertex then we declare that

$$v_2 < v_1.$$ 

If neither of the above options realize then there exist numbers $k_1$ and $k_2$ such that

$$\mathcal{N}^{k_1}(v_1) = \mathcal{N}^{k_2}(v_2)$$

(2.4)
but

\[ \mathcal{N}^{k_1-1}(v_1) \neq \mathcal{N}^{k_2-1}(v_2). \]

Since the tree \( t \) is planar the set of \( \mathcal{N}^{-1}(\mathcal{N}^{k_1}(v_1)) \) is equipped with a linear order. Furthermore, since both vertices \( \mathcal{N}^{k_1-1}(v_1) \) and \( \mathcal{N}^{k_2-1}(v_2) \) belong to the set \( \mathcal{N}^{-1}(\mathcal{N}^{k_1}(v_1)) \), we may compare them with respect to this order.

We declare that, if \( \mathcal{N}^{k_1-1}(v_1) < \mathcal{N}^{k_2-1}(v_2) \), then

\[ v_1 < v_2. \]

Otherwise we set \( v_2 < v_1 \).

It is not hard to see that the resulting relation \( < \) on \( V(t) \) is indeed a linear order. Keeping this order in mind, we can say things like “the first vertex”, “the second vertex”, and “the \( i \)-th vertex” of a planar tree \( t \). In fact, the first vertex of a tree is always its root vertex.

We have an obvious bijection between the set of edges \( E(t) \) of a tree \( t \) and the subset of vertices:

\( V(t) \setminus \{ \text{root vertex} \} \).

This bijection assigns to a vertex \( v \) in \( V(t) \setminus \{ \text{root vertex} \} \) its outgoing edge. Thus the canonical linear order on the set \( V(t) \setminus \{ \text{root vertex} \} \) gives us a natural linear order on the set of edges \( E(t) \). For our purposes we also extend the linear orders on the sets \( V(t) \setminus \{ \text{root vertex} \} \) and \( E(t) \) to the disjoint union

\( V(t) \setminus \{ \text{root vertex} \} \sqcup E(t) \)

by declaring that a vertex is bigger than its outgoing edge. For example, the root edge is the minimal element in the set \( V(t) \setminus \{ \text{root vertex} \} \sqcup E(t) \).

Let us recall the notion of the height of a vertex (DO WE NEED IT?)

**Definition 2.1.** For every tree \( t \) we have the obvious function

\( \text{ht} : V(t) \to \{0,1,2,3\ldots\} \)

from the set of vertices to the set of non-negative integers. The function assigns to a vertex \( v \) the length of the path from this vertex to the root vertex. We call \( \text{ht}(v) \) the *height* of a vertex \( v \).

2.1.1. *Groupoid of labelled planar trees.* Let \( n \) be a non-negative integer. An \( n \)-labeled planar tree \( t \) is a planar tree equipped with an injective map

\( l : \{1,2,\ldots,n\} \to L(t) \)

from the set \( \{1,2,\ldots,n\} \) to the set \( L(t) \) of leaves of \( t \). Although the set \( L(t) \) has a natural linear order we do not require that the map \( l \) is monotonous.

The set \( L(t) \) of leaves of an \( n \)-labeled planar tree \( t \) splits into the disjoint union of the image \( l(\{1,2,\ldots,n\}) \) and its complement. We call leaves in the image of \( l \) *labelled*.

A vertex \( x \) of an \( n \)-labeled planar tree \( t \) is called *nodal* if it is neither a root vertex, nor a labelled leaf. We denote by \( V_{\text{nod}}(t) \) the set of all nodal vertices of \( t \). Keeping in mind the canonical linear order on the set of all vertices of \( t \) we can say things like “the first nodal vertex”, “the second nodal vertex”, and “the \( i \)-th nodal vertex”.

**Example 2.2.** An example of a 4-labelled planar tree is depicted on figure 2.1. On figures we use small white circles for nodal vertices and small black circles for labelled leaves and the root vertex.

![Fig. 2.1. A 4-labelled planar tree](image)
For our purposes we need to upgrade the set of \( n \)-labelled planar trees to the groupoid \( \text{Tree}(n) \). Objects of \( \text{Tree}(n) \) are \( n \)-labelled planar trees and morphisms are non-planar isomorphism of the corresponding (non-planar) trees compatible with labeling. The groupoid \( \text{Tree}(n) \) is equipped with an obvious left action of the symmetric group \( S_n \).

There is exactly one labelled planar tree with no nodal vertices at all. This is the degenerate tree with a single edge connecting the root vertex with a single leaf labelled by 1. For our purposes it is convenient to discard this tree. However, we would still like to keep the degenerate tree depicted on figure 2.7.

The notation \( \text{Tree}_2(n) \) is reserved for the full sub-groupoid of \( \text{Tree}(n) \) whose objects are \( n \)-labelled planar trees with exactly 2 nodal vertices. It is not hard to see that every object in \( \text{Tree}_2(n) \) has at most \( n + 1 \) leaves. Furthermore, isomorphism classes of \( \text{Tree}_2(n) \) are in bijection with the union
\[
\bigsqcup_{0 \leq p \leq n} \text{Sh}_{p,n-p}
\]
where \( \text{Sh}_{p,n-p} \) denotes the set of \((p, n-p)\)-shuffles in \( S_n \). The bijection assigns to a \((p, n-p)\)-shuffle \( \tau \) the isomorphism class of the planar tree depicted on figure 2.2.

\[
\tau(1) \quad \tau(p) \quad \tau(p+1) \quad \tau(n)
\]

Fig. 2.2. Here \( \tau \) is a \((p, n-p)\)-shuffle

2.1.2. Insertions of trees. Let \( \bar{t} \) be an \( n \)-labelled planar tree. If the \( i \)-th nodal vertex of \( \bar{t} \) has \( m_i \) incoming edges then for every \( m_i \)-labelled planar tree \( t \) we can define the insertion \( \cdot_i \) of the tree \( t \) into the \( i \)-th nodal vertex of \( \bar{t} \). The resulting planar tree \( \bar{t} \cdot_i t \) is also \( n \)-labelled.

If \( m_i = 0 \) then \( \bar{t} \cdot_i t \) is obtained via identifying the root edge of \( t \) with edge originating at the \( i \)-th nodal vertex.

If \( m_i > 0 \) then the tree \( \bar{t} \cdot_i t \) is built following these steps:

- First, we denote by \( E_i(\bar{t}) \) the set of edges terminating at the \( i \)-th nodal vertex of \( \bar{t} \). Since \( \bar{t} \) is planar, the set \( E_i(\bar{t}) \) comes with a linear order;
- second, we erase the \( i \)-th nodal vertex of \( \bar{t} \);
- third, we identify the root edge of \( t \) with the edge of \( \bar{t} \) which originates at the \( i \)-th nodal vertex;
- finally, we identify the edges of \( t \) adjacent to labelled leaves with the edges in the set \( E_i(\bar{t}) \) following this rule: the external edge with label \( j \) gets identified with the \( j \)-th edge in the set \( E_i(\bar{t}) \). In doing this, we keep the same planar structure on \( t \), so, in general, branches of \( \bar{t} \) move around.

Example 2.3. Let \( \bar{t} \) be the 4-labelled planar tree depicted on figure 2.1 and \( t \) be the 3-labelled planar tree depicted on figure 2.3. Then the insertion \( \bar{t} \cdot_1 t \) of \( t \) into the first nodal vertex of \( \bar{t} \) is shown on figure 2.4.

2.2. Operads, pseudo-operads, and their dual versions.
2.2.1. Collections. By a collection we mean the sequence \( \{ P(n) \}_{n \geq 0} \) of objects of the underlying symmetric monoidal category \( \mathcal{C} \) such that for each \( n \), the object \( P(n) \) is equipped with a left action of the symmetric group \( S_n \).

Given a collection \( P \) we form covariant functors for \( n \geq 0 \)

\[
P_n : \text{Tree}(n) \to \mathcal{C}.
\]

To an \( n \)-labelled planar tree \( t \) the functor \( P_n \) assigns the object

\[
P_n(t) = \bigotimes_{x \in V_{\text{nod}}(t)} P(m(x)),
\]

where \( V_{\text{nod}}(t) \) is the set of all nodal vertices of \( t \), the notation \( m(x) \) is reserved for the number of edges terminating at the vertex \( x \), and the order of the factors in the right hand side of the equation agrees with the natural order on the set \( V_{\text{nod}}(t) \).

To define the functor \( P_n \) on the level of morphisms we use the actions of the symmetric groups and the structure of the symmetric monoidal category \( \mathcal{C} \) in the obvious way.

**Example 2.4.** Let \( t_1 \) (resp. \( t_2 \)) be a 3-labelled planar tree depicted on figure 2.5 (resp. figure 2.6). There is a unique morphism \( \lambda \) from \( t_1 \) to \( t_2 \) in \( \text{Tree}(3) \). For these trees we have

\[
P_2(t_1) = P(2) \otimes P(3) \otimes P(0) \otimes P(0),
\]

\[
P_2(t_2) = P(2) \otimes P(0) \otimes P(3) \otimes P(0),
\]

and the morphism

\[
P_2(\lambda) : P(2) \otimes P(3) \otimes P(0) \otimes P(0) \to P(2) \otimes P(0) \otimes P(3) \otimes P(0)
\]

is the composition

\[
P_2 = \beta \circ (\sigma_{12} \otimes \sigma_{13} \otimes 1 \otimes 1),
\]

where \( \sigma_{12} \) (resp. \( \sigma_{13} \)) is the corresponding transposition in \( S_2 \) (resp. in \( S_3 \)) and \( \beta \) is the braiding

\[
\beta : (P(3) \otimes P(0)) \otimes P(0) \to P(0) \otimes (P(3) \otimes P(0)).
\]

2.2.2. Pseudo-operads and operads. We now recall that a pseudo-operad is a collection \( \{ P(n) \}_{n \geq 0} \) equipped with multiplication maps

\[
\mu_t : P_n(t) \to P(n)
\]

for all \( n \)-labelled trees \( t \) and for all \( n \geq 0 \). These multiplications should satisfy the axioms which we list below.

First, we require that the standard corolla \( q_n \) (see figures 2.7 and 2.8) acts via identity

\[
\mu_{q_0} = \text{id}_{P(n)}.
\]
Second, we require that the operation $\mu_t$ is $S_n$-equivariant
\begin{equation}
\mu_{\sigma(t)} = \sigma \circ \mu_t.
\end{equation}

Third, for every morphism $\lambda : t \to t'$ in $\text{Tree}(n)$ we have
\begin{equation}
\mu_{t'} \circ P_n(\lambda) = \mu_t.
\end{equation}

Finally, we need to formulate the associativity axiom for multiplications (2.10). For this purpose we consider the following quadruple $(\tilde{t}, i, m, t)$ where $\tilde{t}$ is an $n$-labelled planar tree with $k$ nodal vertices, $1 \leq i \leq k$, $m_i$ is the number of edges terminating at the $i$-th nodal vertex of $\tilde{t}$, and $t$ is an $m_i$-labelled planar tree.

The associativity axioms states that for each such quadruple $(\tilde{t}, i, m, t)$ we have
\begin{equation}
\mu_{\tilde{t}} \circ (\text{id} \otimes \cdots \otimes \text{id} \otimes \mu_{\tilde{t}} \otimes \text{id} \otimes \cdots \otimes \text{id}) \circ \beta_{\tilde{t},i,m_i,t} = \mu_{\tilde{t},t}
\end{equation}
where $\tilde{t} \circ t$ is the $n$-labelled planar tree obtained by inserting $t$ into the $i$-th vertex of $\tilde{t}$ and $\beta_{\tilde{t},i,m_i,t}$ is the isomorphism in $\mathcal{C}$ which is responsible for "putting tensor factors in the correct order".

Given integers $n \geq 1$, $k \geq 0$, $1 \leq i \leq n$ and a permutation $\sigma \in S_{n+k-1}$ we can form the $(n+k-1)$-labelled planar tree $t_{\sigma}^{n,k,i}$ shown on figure 2.9

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.9.pdf}
\caption{The $(n+k-1)$-labelled planar tree $t_{\sigma}^{n,k,i}$}
\end{figure}

It is often convenient to use a different notation for the multiplication map $\mu_{t_{\sigma}^{n,k,i}}$ corresponding to the tree $t_{\sigma}^{n,k,i}$. More precisely, for a vector $v \in P(n)$ and $w \in P(k)$ of a pseudo-operad $P$ we will use this notation
\begin{equation}
v(\sigma(1), \ldots, \sigma(i-1), w(\sigma(i), \ldots, \sigma(i+k-1)), \sigma(i+k), \ldots, \sigma(n+k-1)) := \mu_{t_{\sigma}^{n,k,i}}(v, w).
\end{equation}

Recall that, for $\sigma = \text{id} \in S_{n+k-1}$, the multiplications
\begin{equation}
\mu_{t_{\text{id}}^{n,k,i}} : P(n) \otimes P(k) \to P(n+k-1)
\end{equation}
are called the elementary insertion and often denoted by $o_i$. Namely, for $v \in P(n)$ and $w \in P(k)$ we have
\begin{equation}
v \circ_i w := \mu_{t_{\text{id}}^{n,k,i}}(v, w).
\end{equation}

It is not hard to see that a pseudo-operad structure on a collection $P$ is uniquely determined by elementary insertions (2.16). All the remaining multiplications (2.10) can be expressed in terms of (2.16) using the axioms of pseudo-operad.

An equivalent definition of pseudo-operad in terms of elementary insertions (2.16) is given in [9] Definition 17. In loc. cit. pseudo-operad is called non-unital Markl’s operad.

Let us now recall that an operad is a pseudo-operad $P$ with unit, that is a map
\begin{equation}
u : K \to P(1)
\end{equation}
for which the compositions
\begin{equation}
P(n) \cong P(n) \otimes K \xrightarrow{id \otimes \nu} P(n) \otimes P(1) \xrightarrow{\circ_1} P(n)
\end{equation}
\begin{equation}
P(n) \cong K \otimes P(n) \xrightarrow{\nu \otimes id} P(1) \otimes P(n) \xrightarrow{\circ_1} P(n)
\end{equation}
coincide with the identity map on $P(n)$.

\footnote{Numbers $n$ and $k$ are suppressed from the notation.}
Morphisms of pseudo-operads and operads are defined in the obvious way.

2.2.3. Pseudo-cooperads and cooperads. Reversing the arrows in the definition of pseudo-operad we get the definition of a pseudo-cooperad. More precisely, a pseudo-cooperad is a collection $Q$ equipped with comultiplication maps
\[(2.19) \Delta_t : Q(n) \to Q_n(t),\]
which satisfy a similar list of axioms.

As well as for pseudo-operads, we have
\[(2.20) \Delta_{Q_n} = \text{id}_{Q(n)},\]
where $Q_n$ is the standard corolla shown on figures 2.7 and 2.8.

We also require that the operations (2.19) are $S_n$-equivariant
\[(2.21) \Delta_{\sigma(t)} \circ \sigma = \Delta_t.\]

For every morphism $\lambda : t \to t'$ in Tree$(n)$ we have
\[(2.22) \Delta_{t'} = Q_n(\lambda) \circ \Delta_t.\]

Finally, to formulate the coassociativity axiom for (2.19), we consider the following quadruple $(\tilde{t}, i, m_i, t)$ where $\tilde{t}$ is an $n$-labelled planar tree with $k$ nodal vertices, $1 \leq i \leq k$, $m_i$ is the number of edges terminating at the $i$-th nodal vertex of $\tilde{t}$, and $t$ is an $m_i$-labelled planar tree.

The coassociativity axioms states that for each such quadruple $(\tilde{t}, i, m_i, t)$ we have
\[(2.23) (\text{id} \otimes \cdots \otimes \text{id} \otimes \Delta_{\text{i-th spot}} \otimes \text{id} \otimes \cdots \otimes \text{id}) \circ \Delta_{\tilde{t}} = \Delta_{\tilde{t}_0, t} \circ \beta_{\tilde{t}, i, m_i, t},\]

where $\tilde{t}_0, t$ is the $n$-labelled planar tree obtained by inserting $t$ into the $i$-th nodal vertex of $\tilde{t}$ and $\beta_{\tilde{t}, i, m_i, t}$ is the isomorphism in $\mathcal{C}$ which is responsible for "putting tensor factors in the correct order".

As well as for pseudo-operads, a pseudo-cooperad structure on a collection $Q$ is uniquely determined by the comultiplications:
\[(2.24) \Delta_i := D_{t_n,k,i} : Q(n + k - 1) \to Q(n) \otimes Q(k),\]

where $\{t_{n,k,i}\}_{\sigma \in S_{n+k-1}}$ is the family of labelled planar trees defined on figure 2.9.

The comultiplications (2.24) are called elementary co-insertions.

We now recall that a cooperad is a pseudo-cooperad $Q$ with counit, that is a map
\[(2.25) u^* : Q(1) \to K\]
for which the compositions
\[(2.26) Q(n) \xrightarrow{\Delta_i} Q(n) \otimes Q(1) \xrightarrow{\text{id} \otimes u^*} Q(n) \otimes K \cong Q(n)\]
\[(Q(n) \xrightarrow{\Delta_1} Q(1) \otimes Q(n) \xrightarrow{u^* \otimes \text{id}} K \otimes Q(n) \cong Q(n))\]

coincide with the identity map on $Q(n)$.

Morphisms of pseudo-cooperads and cooperads are defined in the obvious way.

2.2.4. Augmented operads and coaugmented cooperads. Let us observe that the collection
\[(2.27) *(n) = \begin{cases} K & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}\]
is equipped with the unique structure of an operad and a unique structure of a cooperad. In fact, $*$ is the initial object in the category of operads and $*$ is the terminal object in the category of cooperads.

We say that an operad $\mathcal{O}$ is augmented if $\mathcal{O}$ is equipped with an operad morphism
\[(2.28) \varepsilon : \mathcal{O} \to *.\]

Similarly, a cooperad $\mathcal{C}$ is called coaugmented if $\mathcal{C}$ is equipped with a cooperad morphism
\[(2.29) \varepsilon' : * \to \mathcal{C}.\]
It is not hard to see that a kernel of an augmentation (2.28) (resp. cokernel of a coaugmentation (2.29)) is naturally a pseudo-operad (resp. pseudo-cooperad).

This assignment can be easily upgraded to a functor. Furthermore, according to M. Markl [9, Proposition 21] this functor gives us an equivalence between the category of augmented operads and the category of pseudo-operads. Dually, we have an equivalence between the category of coaugmented cooperads and the category of pseudo-cooperads.

In this paper we denote by $O_\circ$ (resp. $C_\circ$) the kernel (resp. the cokernel) of augmentation (resp. coaugmentation) of an augmented operad $O$ (resp. a coaugmented cooperad $C$).

2.3. Convolution Lie algebra and its properties. Let $O$ (resp. $C$) be a pseudo-operad (resp. pseudo-cooperad) in the category $\mathcal{Ch}_K$ of unbounded cochain complexes of $K$-vector spaces.

We consider the following cochain complex
\begin{equation}
\text{Conv}(C, O) = \prod_{n \geq 0} \text{Hom}_{S_n}(C(n), O(n)).
\end{equation}
with the binary operation $\bullet$ defined by the formula
\begin{equation}
f \bullet g(X) = \sum_{z \in \text{Isom}_2(n)} \mu_z(f \otimes g \Delta_z(X)),
\end{equation}
where $t_z$ is any representative of the isomorphism class $z \in \text{Isom}_2(n)$. The axioms of pseudo-operad (resp. pseudo-cooperad) imply that the right hand side of (2.31) does not depend on the choice of representatives $t_z$.

It follows directly from the definition that the operation $\bullet$ is compatible with the differential on $\text{Conv}(C, O)$ coming from $C$ and $O$. Furthermore, we claim that

**Proposition 2.5.** The bracket
\[
[f, g] = -(f \bullet g - (-1)^{|f||g|}g \bullet f)
\]

satisfies the Jacobi identity.

**Proof.** For the proof we refer the reader to [1] and [11]. □

The construction of the Lie algebra $\text{Conv}(C, O)$ is justified by the following statement:

**Theorem 2.6.** Let $C$ be a coaugmented cooperad in the category $\mathcal{Ch}_K$ with $C_\circ$ being the cokernel of coaugmentation. For every augmented operad $O$ in $\mathcal{Ch}_K$ there is bijection between the set of operad maps $\text{Cobar}(C) \to O$ and MC elements of the Lie algebra $\text{Conv}(C_\circ, O)$.

**Proof.** A proof can be found in [1]. In fact this statement is a particular case of something from [11]. □

2.4. Intrinsic derivations of an operad. Let $P$ be an operad in $\mathcal{Ch}_K$. Then the operation $\circ_1$ equips $P(1)$ with a structure of an associative algebra. We consider $P(1)$ as a Lie algebra with the Lie bracket being the commutator.

We claim that

**Proposition 2.7.** The formula
\begin{equation}
\delta_b(a) = b \circ_1 a - (-1)^{|a||b|} \sum_{i=1}^n a \circ_i b
\end{equation}

with
\[b \in P(1), \quad \text{and} \quad a \in P(n)\]
defines an operadic derivation of $P$ for every $b \in P(1)$.
Proof. Let $a_1 \in P(n_1)$ and $a_2 \in P(n_2)$. Then for every $b \in P(1)$ and $1 \leq j \leq n_1$ we have

$$
\delta_b(a_1 \circ_j a_2) = b \circ_1 (a_1 \circ_j a_2) - (-1)^{|a_1|+|a_2|} \sum_{i=1}^{n_1+n_2-1} a_1 \circ_j (a_2 \circ_i b) =
$$

$$(b \circ_1 a_1) \circ_j a_2 - (-1)^{|a_1|} \sum_{i \neq j} a_1 \circ_i b \circ_j a_2 =
$$

$$
= (b \circ_1 a_1) \circ_j a_2 - (-1)^{|a_1|} \sum_{i=1}^{n_1} a_1 \circ_i b \circ_j a_2 + (-1)^{|a_1|+|a_2|} \sum_{i=1}^{n_2} a_1 \circ_j (a_2 \circ_i b) =
$$

$$
= (b \circ_1 a_1) \circ_j a_2 - (-1)^{|a_1|} \sum_{i=1}^{n_1} a_1 \circ_i b \circ_j a_2 + (-1)^{|a_1|+|a_2|} \sum_{i=1}^{n_2} a_1 \circ_j (a_2 \circ_i b) =
$$

$$
= \delta_b(a_1) \circ_j a_2 + (-1)^{|a_1|} a_1 \circ_j \delta_b(a_2). 
$$

Hence $\delta_b$ is indeed an operadic derivation of $P$.

The identity $[\delta_{b_1}, \delta_{b_2}] = \delta_{[b_1, b_2]}$ follows from a similar direct computation. \qed

3. Twisting of operads

Let $O$ be an operad in the category $Ch_k$ equipped with a map

$$(3.1) \quad \hat{\varphi} : \Lambda \text{Lie}_\infty \to O.$$ 

Let $V$ be an algebra over $O$. Using the map $\hat{\varphi}$, we equip $V$ with an $\Lambda \text{Lie}_\infty$-structure. If we assume, in addition, that $V$ is equipped with a complete descending filtration

$$(3.2) \quad V \supset F_1 V \supset F_2 V \supset F_3 V \supset \ldots, \quad V = \lim_k V / F_k V$$

and the $O$-algebra structure on $V$ is compatible with this filtration then we may define MC elements of $V$ as degree 2 elements $\alpha \in F_1 V$ satisfying the equation

$$(3.3) \quad \partial(\alpha) + \sum_{n \geq 2} \frac{1}{n!} \{\alpha, \alpha, \ldots, \alpha\}_n = 0$$

where $\partial$ is the differential on $V$ and $\{\ldots, \ldots, \}$ are operations of the $\Lambda \text{Lie}_\infty$-structure on $V$. Given such a MC element $\alpha$ we can twist the differential on $V$ and insert $\alpha$ into various $O$-operations on $V$. This way we get a new algebra structure on $V$. It turns out that this new algebra structure is governed by an operad $\text{Tw } O$ which is built from the pair $(O, \hat{\varphi})$. This section is devoted to the construction of $\text{Tw } O$.

First, we recall that, since $\Lambda \text{Lie}_\infty = \text{Cobar}(\Lambda^2 \text{coCom})$, the morphism (3.1) is determined by a MC element

$$\varphi \in \text{Conv}(\Lambda^2 \text{coCom}_n, O).$$

The $n$-the space of $\Lambda^2 \text{coCom}_n$ is the trivial $S_n$-modules placed in degree $2 - 2n$:

$$\Lambda^2 \text{coCom}(n) = s^{2-2n} k.$$ 

So we have

$$\text{Conv}(\Lambda^2 \text{coCom}_n, O) = \prod_{n \geq 2} \text{Hom}_{S_n}(s^{2-2n} k, O(n)) = \prod_{n \geq 2} s^{2n-2}(O(n))^S_n.$$

For our purposes we will need to extend the DG Lie algebra $\text{Conv}(\Lambda^2 \text{coCom}_n, O)$ to

$$(3.5) \quad \mathcal{L}_O = \text{Conv}(\Lambda^2 \text{coCom}, O) = \prod_{n \geq 1} \text{Hom}_{S_n}(s^{2-2n} k, O(n)).$$
It is clear that
\[ \mathcal{L}_\mathcal{O} = \prod_{n \geq 1} s^{2n-2}(\mathcal{O}(n))^{S_n}. \]

For \( n, r \geq 1 \) we realize the group \( S_r \) as the following subgroup of \( S_{n+r} \)
\begin{equation}
S_r \cong \{ \sigma \in S_{n+r} \mid \sigma(i) = i, \ \forall i > r \}. \tag{3.6} \end{equation}
In other words, the group \( S_r \) may be viewed as subgroup of \( S_{n+r} \) permuting only the first \( r \) letters. We set \( S_0 \) to be the trivial group. Using this embedding of \( S_r \) into \( S_{n+r} \), we introduce the following collection \( \tilde{\mathcal{T}}w \mathcal{O}(n) \)
\begin{equation}
\tilde{\mathcal{T}}w \mathcal{O}(n) = \prod_{r \geq 0} \text{Hom}_{S_r}(s^{-2r}, \mathcal{O}(r+n)). \tag{3.7} \end{equation}

It is clear that
\[ \tilde{\mathcal{T}}w \mathcal{O}(n) = \prod_{r \geq 0} s^{2r}(\mathcal{O}(r+n))^{S_r}. \]

To define an operad structure on (3.7) we denote by \( 1_r \) the generator \( 1_r \in s^{-2r} \). Then the identity element \( u \) in \( \tilde{\mathcal{T}}w \mathcal{O}(1) \) is given by
\begin{equation}
u(1_r) = \begin{cases} u_\mathcal{O} & \text{if } r = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3.8} \end{equation}
where \( u_\mathcal{O} \in \mathcal{O}(1) \) is the identity element for the operad \( \mathcal{O} \). Next, for \( f \in \tilde{\mathcal{T}}w \mathcal{O}(n) \) and \( g \in \tilde{\mathcal{T}}w \mathcal{O}(m) \), we define the \( i \)-th elementary insertion \( \circ_i \) for \( \tilde{\mathcal{T}}w \mathcal{O} \) by
\begin{equation}
f \circ_i g(1_r) = \sum_{p=0}^r \sum_{\sigma \in S_{p,r-p}} \mu_{\sigma,i}(f(1_p) \otimes g(1_{r-p})). \tag{3.9} \end{equation}
where the tree \( t_{\sigma,i} \) is depicted on figure 3.1.

**Fig. 3.1.** Here \( \sigma \) is a \((p,r-p)\)-shuffle

Sometimes it is convenient to use a different but equivalent definition of the \( i \)-th elementary insertion \( \circ_i \) for \( \tilde{\mathcal{T}}w \mathcal{O} \). This definition is given by the formula
\begin{equation}
f \circ_i g(1_r) = \sum_{p=0}^r \sum_{\sigma \in S_{p,r-p}} \sigma((f(1_p) \circ g(1_{r-p}))). \tag{3.10} \end{equation}
To see that the element \( f \circ_i g(1_r) \in \mathcal{O}(r+n+m-1) \) is \( S_r \)-invariant one simply needs to use the fact that every element \( \tau \in S_r \) can be uniquely presented as the composition \( \tau_h \circ \tau_{p,r-p} \), where \( \tau_h \) is a \((p,r-p)\)-shuffle and \( \tau_{p,r-p} \in S_p \times S_{r-p} \).

Let \( f \in \tilde{\mathcal{T}}w \mathcal{O}(n) \), \( g \in \tilde{\mathcal{T}}w \mathcal{O}(m) \), \( h \in \tilde{\mathcal{T}}w \mathcal{O}(k) \), \( 1 \leq i \leq n \), and \( 1 \leq j \leq m \). To check the identity
\begin{equation}
f \circ_i (g \circ_j h) = (f \circ_i g) \circ_{j+i-1} h \tag{3.11} \end{equation}
we observe that
\[
f \circ_i (g \circ_j h)(1_r) = \sum_{p=0}^{r} \sum_{\sigma \in Sh_p, r-p} \mu_{t, \sigma, i} (f(1_p) \otimes (g \circ_j h)(1_{r-p}))
\]
\[
= \sum_{p_1+p_2+p_3=r} \sum_{\sigma \in Sh_{p_1, p_2+p_3}} \sum_{\sigma' \in Sh_{p_2, p_3}} \mu_{t, \sigma, i} \circ (1 \otimes \mu_{t, \sigma, j}) (f(1_{p_1}) \otimes g(1_{p_2}) \otimes h(1_{p_3}))
\]
\[
= \sum_{p_1+p_2+p_3=r} \sum_{\tau \in Sh_{p_1, p_2, p_3}} \mu_{t, \tau, i, j} (f(1_{p_1}) \otimes g(1_{p_2}) \otimes h(1_{p_3}))
\]
where the tree \(t_{\tau, i, j}\) is depicted on figure 3.2. Similar calculations show that

\[
(f \circ_i g) \circ_{j+i-1} h = \sum_{p_1+p_2+p_3=r} \sum_{\sigma \in Sh_{p_1, p_2+p_3}} \mu_{t, \sigma, i, j} (f(1_{p_1}) \otimes g(1_{p_2}) \otimes h(1_{p_3}))
\]

with \(t_{\sigma, i, j}\) being the tree depicted on figure 3.2.

We leave the verification of the remaining axioms of the operad structure for the reader.

Our next goal is to define an auxiliary action of \(O\) on the operad \(Tw\). For a vector \(f \in \widetilde{Tw} O(n)\) the action of \(v \in L_{O}\) on \(f\) is defined by the formula

\[
(3.12) \quad v \cdot f(1_r) = -(-1)^{|v||f|} \sum_{p=1}^{r} \sum_{\sigma \in Sh_{p, r-p}} \mu_{t, \sigma, p, r-p} (f(1_{r-p+1}) \otimes v(1_p)),
\]

where the tree \(t_{\sigma, p, r-p}\) is depicted on figure 3.3.

We claim that

**Proposition 3.1.** Formula (3.12) defines an action of \(L_{O}\) on the operad \(Tw\).

**Proof.** A simple degree bookkeeping show that the degree of \(v \cdot f\) is \(|v| + |f|\).

Then we need to check that for two homogeneous vectors \(v, w \in L_{O}\) we have

\[
(3.13) \quad [v, w] \cdot f(1_r) = (v \cdot (w \cdot f))(1_r) - (-1)^{|v||w|} (w \cdot (v \cdot f))(1_r)
\]
\[ (v \cdot (w \cdot f))(1_r) - (-1)^{|v||w|}(w \cdot (v \cdot f))(1_r) = \]
\[ (-1)^{|f|(|v|+|w|)+|v||w|} \sum_{p \geq 1, q \geq 0} \sum_{\tau \in Sh_{p,q,r-p-q}} \mu^{t^{p,q}_r}(f(1_{r-p-q+1}) \otimes w(1_{q+1}) \otimes v(1_p)) \]
\[ + (-1)^{|f|(|v|+|w|)+|v||w|} \sum_{p,q \geq 1} \sum_{\tau \in Sh_{p,q,r-p-q}} \mu^{\tilde{t}^{p,q}_r}(f(1_{r-p-q+2}) \otimes w(1_{q}) \otimes v(1_p)) \]
\[ - (-1)^{|v||w|}(v \leftrightarrow w), \]
where the trees \( t^{p,q}_r \) and \( \tilde{t}^{p,q}_r \) are depicted on figures 3.4 and 3.5, respectively.

Thus equation (3.13) follows. It remains to check that the operation \( f \mapsto v \cdot f \) is an operadic derivation and we leave this step as an exercise for the reader. \( \square \)

3.1. The action of \( \mathcal{L}_O \) on \( \widetilde{Tw} O \). Let us view \( \widetilde{Tw} O(1) \) as the Lie algebra with the bracket being commutator.

We have a obvious degree zero map
\[ \kappa : \mathcal{L}_O \rightarrow \widetilde{Tw} O(1) \]
defined by the formula
\[ \kappa(v)(1_r) = v(1_{r+1}) \]
\[ \Theta(v) = v + \kappa(v) \]

defines a Lie algebra homomorphism
\[ \Theta : \mathcal{L}_O \to \mathcal{L}_O \ltimes \widetilde{\text{Tw}} \mathcal{O}(1) \]

**Proof.** First, let us prove that for every pair of homogeneous vectors \( v, w \in \mathcal{L}_O \) we have
\[ \kappa([v, w]) = [\kappa(v), \kappa(w)] + v \cdot \kappa(w) - (-1)^{|v||w|} w \cdot \kappa(v). \]

Indeed, unfolding the definition of \( \kappa \) we get
\[ \kappa([v, w])(1_r) = \sum_{p=1}^{r} \sum_{\tau \in \text{Sh}_{p, r-p}} v_{r-p+2} (w_p(\tau(1), \ldots, \tau(p)), \tau(p+1), \ldots, \tau(r), r+1) \]
\[ + \sum_{p=0}^{r} \sum_{\tau \in \text{Sh}_{p, r-p}} v_{r-p+1} (w_{p+1}(\tau(1), \ldots, \tau(p), r+1), \tau(p+1), \ldots, \tau(r)) \]
\[ - (-1)^{|v||w|} (v \leftrightarrow w), \]
where \( v_t = v(1_t) \) and \( w_t = w(1_t) \). The first sum in (3.18) equals
\[ - (-1)^{|v||w|} (\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 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3.2. The operad $\Tw \mathcal{O}$. Algebras over $\Tw \mathcal{O}$. To an operad $\mathcal{O}$ in $\Ch\mathbb{K}$ with a chosen morphism $\hat{\varphi}$ (3.1) we assign another operad $\Tw \mathcal{O}$. As an operad in the category $\gr\Vect_{\mathbb{K}}$, $\Tw \mathcal{O} = \widetilde{\mathcal{O}}$. The differential $\partial^{\Tw}$ on $\Tw \mathcal{O}$ is given by equation

$$\partial^{\Tw} = \partial \mathcal{O} + \varphi \cdot + \delta_{\kappa}(\varphi)$$

(3.20)

where $\partial \mathcal{O}$ is the differential on $\widetilde{\mathcal{O}}$ coming from the one on $\mathcal{O}$ and $\varphi$ is the MC element in $\mathcal{L}_\mathcal{O}$ corresponding to a morphism $\hat{\varphi}$ (3.1) from the operad $\mathcal{L}_{\mathbb{K} \text{Lie}_\infty}$ to $\mathcal{O}$.

Corollaries 3.3 and 3.4 imply that $\partial^{\Tw}$ is indeed a differential on $\Tw \mathcal{O}$.

Let us now assume that $V$ is an algebra over $\mathcal{O}$ equipped with a complete decreasing filtration (3.2). We also assume that the $\mathcal{O}$-algebra structure on $V$ is compatible with this filtration.

Given a MC element $\alpha \in F_1 V$ the formula

$$\partial^{\alpha} (v) = \partial (v) + \sum_{r=1}^{\infty} \frac{1}{r!} \varphi(1_{r+1})(\alpha, \ldots, \alpha, v)$$

(3.21)

defines a new (twisted) differential on $V$. We will denote by $V^{\alpha}$ the cochain complex $V$ with this new differential. In this setting we have the following theorem

**Theorem 3.5.** If $V^{\alpha}$ is the cochain complex obtained from $V$ via twisting the differential by the Maurer-Cartan element $\alpha$ then the formula

$$f(v_1, \ldots, v_n) = \sum_{r=0}^{\infty} \frac{1}{r!} f(1_r)(\alpha, \ldots, \alpha, v_1, \ldots, v_n)$$

(3.22)

for $f \in \Tw \mathcal{O}(n), \quad v_i \in V$

defines a $\Tw \mathcal{O}$-algebra structure on $V^{\alpha}$.

**Proof.** The desired equation

$$(-1)^{|g|(|v_1| + \cdots + |v_{n-1}|)} f(v_1, \ldots, v_{i-1}, g(v_i, \ldots, v_{i+k-1}), v_{i+k}, \ldots, v_{n+k-1}) - f \circ g(v_1, \ldots, v_{n+k-1})$$

(3.23)

is a straightforward consequence of the definition (see eq. (3.9)). Next, we need to show that

$$\partial^{\Tw}(f)(v_1, \ldots, v_n) = \partial^{\alpha} f(v_1, \ldots, v_n) - (-1)^{|f|} \sum_{i=1}^{n} (-1)^{|v_i| + \cdots + |v_{i-1}|} f(v_1, \ldots, v_{i-1}, \partial^{\alpha}(v_i), v_{i+1}, \ldots, v_n)$$

(3.24)

The right hand side of (3.24) can be rewritten as

R.H.S. of (3.24) =

$$\sum_{p \geq 0} \frac{1}{p!} \partial f_p(\alpha, \ldots, \alpha, v_1, \ldots, v_n) + \sum_{p \geq 0, q \geq 1} \frac{1}{p!q!} \varphi_q(\alpha, \ldots, \alpha, f_p(\alpha, \ldots, \alpha, v_1, \ldots, v_n))$$

$$- (-1)^{|f|} \sum_{i=1}^{n} \frac{(-1)^{|v_1| + \cdots + |v_{i-1}|}}{p!} f_p(\alpha, \ldots, \alpha, v_1, \ldots, v_{i-1}, \partial(v_i), v_{i+1}, \ldots, v_n)$$

$$- (-1)^{|f|} \sum_{i=1}^{n} \sum_{p \geq 0, q \geq 1} \frac{(-1)^{|v_1| + \cdots + |v_{i-1}|}}{p!q!} f_p(\alpha, \ldots, \alpha, v_1, \ldots, v_{i-1}, \partial \varphi_q(\alpha, \ldots, \alpha, v_i), v_{i+1}, \ldots, v_n)$$

where $f_p = f(1_p)$ and $\varphi_q = \varphi(1_q)$. Using the symmetry of $f_p$ (resp. $\varphi_q$) with respect to the action of $S_p$ (resp. $S_q$) we can rewrite this. Let us now add to and subtract from the right hand side of (3.24) the sum

$$- (-1)^{|f|} \sum_{p \geq 0} \frac{1}{p!} f_{p+1}(\partial \alpha, \ldots, \alpha, v_1, \ldots, v_n).$$

We get

R.H.S. of (3.24) =

$$\sum_{p \geq 0} \frac{1}{p!} \partial f_p(\alpha, \ldots, \alpha, v_1, \ldots, v_n) - (-1)^{|f|} \sum_{p \geq 0} \frac{1}{p!} f_{p+1}(\partial \alpha, \ldots, \alpha, v_1, \ldots, v_n)$$
we have

Hence, using the simple combinatorial formula

following two categories:

The construction from Theorem 3.5 induces an isomorphism of categories between the fol-

Since $V$ is equipped with a complete descending filtration, the endomorphism operad $\text{End}_V$ also carries a complete descending filtration. So does the operad $\text{Tw} \mathcal{O}$. Namely,

(3.25) $F_k \text{Tw} \mathcal{O}(n) = \{ f \in \text{Tw} \mathcal{O}(n) \mid f(1_r) = 0 \quad \forall \ r < k \}$

It is clear that this defines a complete descending filtration compatible with the operad structure.

**Theorem 3.6.** The construction from Theorem 3.5 induces an isomorphism of categories between the following two categories:

1. The category of pairs $(V, \alpha)$, where $V$ is a $\mathcal{O}$ algebra with a complete descending filtration as in (3.2) and $\alpha \in V$ is a Maurer-Cartan element. The morphism between pairs $(V, \alpha)$ and $(V, \alpha')$ are morphisms of (filtered) $\mathcal{O}$-algebras $f : V \to V'$ such that $f$ sends $\alpha$ to $\alpha'$.

2. The category of $\text{Tw} \mathcal{O}$ algebras $V$ with a complete descending filtration as in (3.2). (The $\text{Tw} \mathcal{O}$ is required to be compatible with the filtrations on $\text{Tw} \mathcal{O}$ and $V$.)
Proof. The construction of Theorem 3.5 yields a functor from the first category to the second. Let us define a functor in the reverse direction and show that they are inverse to each other. Using the identity element \( u_\mathcal{O} \in \mathcal{O}(1) \) we produce a degree 2 element \( u^o \in F_1 \text{Tw} \mathcal{O}(0) \):

\[
(3.26) \quad u^o(1_r) = \begin{cases} 
  u_\mathcal{O} & \text{if } r = 1 \\
  0 & \text{otherwise}
\end{cases}
\]

It is easy to see that \( \text{Tw} \mathcal{O} \) is generated (as operad of complete topological vector spaces) by \( \mathcal{O} \) and \( u^o \). Concretely, any element of \( \text{Tw} \mathcal{O} \) can be written as a convergent infinite series each of whose terms is a composition of some element of \( \mathcal{O} \) with some copies of \( u^o \).

Let \( V \) be a \( \text{Tw} \mathcal{O} \)-algebra and let \( \alpha \in F_1 V \) be the image of \( u^o \) in \( V \). The functor we want to construct sends the representation \( V \) to the pair \((V', \alpha)\). Here \( V' \) is the same as \( V \) as a graded vector space, but the differential is \( d_V - \kappa(\varphi) \cdot \). Here \( d_V \) is the differential on \( V \), \( \kappa \) is as in (3.15) and \( \varphi \) is the MC element in \( L_\mathcal{O} \) corresponding to a morphism \( \hat{\varphi} \) from the operad \( \Lambda\text{Lie}_\infty \) to \( \mathcal{O} \). The notation \( \kappa(\varphi) \cdot \) denotes the action of the unary element \( \kappa(\varphi) \in \text{Tw} \mathcal{O}(1) \) on \( V \). We have to show that \( V' \) is indeed an \( \mathcal{O} \)-algebra and that \( \alpha \) is a Maurer-Cartan element in \( V' \). First note that \( V \) is, as a filtered graded vector space an algebra over \( \mathcal{O} \), as graded operad, because \( \mathcal{O} \subset \text{Tw} \mathcal{O} \) is a graded sub-operad. However, it is not a differential graded sub-operad. Instead we have the equation

\[
\partial \text{Tw} \iota(o) - \iota(\partial^\mathcal{O} o) = \delta \kappa(\varphi) o
\]

where \( \iota : \mathcal{O} \to \text{Tw} \mathcal{O} \) is the inclusion, \( o \in \mathcal{O} \) is some element and the remaining notation is as in (3.20). Note that by the preceding equation the action of \( \mathcal{O} \) on \( V' \) is compatible with the differentials. Finally let us check the Maurer-Cartan equation for \( \alpha \). First note that by the formula for the differential in \( \text{Tw} \mathcal{O} \), i.e. (3.20), \( \alpha \) satisfies the equation

\[
d_V \alpha = \sum_n \frac{1}{n!} \varphi_n(\alpha, \cdots, \alpha) + \kappa(\varphi) \cdot \alpha.
\]

In particular, this means that \( \alpha \) satisfies the Maurer-Cartan equation in \( V' \) (sign??). The construction is natural in \( V \) and hence we obtain a functor from the second category in the theorem to the first.

Next we claim that the functor just constructed and the one from Theorem 3.5 are inverse to each other. In one direction, take \((V', \alpha)\) as constructed above and form a new representation of \( \text{Tw} \mathcal{O} \) by the method of Theorem 3.5. It is clear from the definitions that the dg vector space with twisted differential is \( V \) again, i.e., \((V')^o = V\). Furthermore the \( \mathcal{O} \) action (as graded operad action on a graded space) agrees with the one on \( V \). By construction the element \( u^o \) acts as \( \alpha \). But since \( \text{Tw} \mathcal{O} \) is generated by \( \mathcal{O} \) and \( u^o \), the new representation is the same we started with. In the other direction, we leave the (straightforward) verification to the reader.

\[\square\]

4. Categorical properties of twisting

The goal of this section is to show that the operation \( \text{Tw} \) defines a co-monad on the under-category \( \Lambda\text{Lie}_\infty \downarrow \text{Operads} \), where \( \text{Operads} \) is the category of dg operads. Recall that the under-category \( \Lambda\text{Lie}_\infty \downarrow \text{Operads} \) is the category of arrows \( \Lambda\text{Lie}_\infty \to \mathcal{O} \) in \( \text{Operads} \), with morphisms the commutative diagrams

\[
\begin{tikzcd}
\Lambda\text{Lie}_\infty \ar[rd] & \\
\mathcal{O} & \mathcal{O}' \ar[ru]
\end{tikzcd}
\]
4.1. **Twisting as a functor.** Consider arrows $\mathcal{A}\mathsf{Lie}_\infty \to \mathcal{O} \xrightarrow{f} \mathcal{O}'$ in the category $\textit{Operads}$. Twisting the operads $\mathcal{O}$, $\mathcal{O}'$, we obtain new operads

$$\text{Tw } \mathcal{O}(n) = \prod_{r \geq 0} \text{Hom}_{S_r}(s^{-2r}K, \mathcal{O}(r+n))$$

$$\text{Tw } \mathcal{O}'(n) = \prod_{r \geq 0} \text{Hom}_{S_r}(s^{-2r}K, \mathcal{O}'(r+n))$$

By composing morphisms on the right with (components of) the map $f: \mathcal{O} \to \mathcal{O}'$ we obtain a map (of $\Sigma$-modules)

$$\text{Tw } f: \text{Tw } \mathcal{O} \to \text{Tw } \mathcal{O}'$$

**Lemma 4.1.** The above map $\text{Tw } f: \text{Tw } \mathcal{O} \to \text{Tw } \mathcal{O}'$ is a map of dg operads. Furthermore, if

$$\mathcal{A}\mathsf{Lie}_\infty \to \mathcal{O} \xrightarrow{f} \mathcal{O}' \xrightarrow{g} \mathcal{O}''$$

are maps of dg operads, then $\text{Tw } (f \circ g) = (\text{Tw } f) \circ (\text{Tw } g)$.

**Proof.** It is clear since we only used natural operations (i.e., the operad structure) in defining $\text{Tw } \mathcal{O}$.

By applying this result to $\mathcal{A}\mathsf{Lie}_\infty \to \mathcal{A}\mathsf{Lie}_\infty \to \mathcal{O}$ we obtain a morphism

$$\text{Tw } \mathcal{A}\mathsf{Lie}_\infty \to \text{Tw } \mathcal{O}$$

**Lemma 4.2.** There is a morphism of operads

$$\mathcal{A}\mathsf{Lie}_\infty \to \text{Tw } \mathcal{A}\mathsf{Lie}_\infty$$

such that

$$\mathcal{A}\mathsf{Lie}_\infty(n) \ni l_n \mapsto \sum_{r \geq 0} (1_r \mapsto l_{n+r}) \in \text{Tw } \mathcal{A}\mathsf{Lie}_\infty(n)$$

for $n = 1, 2, \ldots$. Here $l_n$ is the $n$-ary generator of $\mathcal{A}\mathsf{Lie}_\infty$.

The Lemma reflects the fact that $\mathcal{L}_\infty$ algebras can be twisted by Maurer-Cartan elements.

**Proof of Lemma 4.2.** Since the operad $\mathcal{A}\mathsf{Lie}_\infty$ is quasi-free it suffices to check that the map above is compatible with the differential. The differential on the generators $l_n$ of $\mathcal{A}\mathsf{Lie}_\infty$ takes the form

$$dl_n = -\sum_{p=2}^{n-1} \sum_{\sigma \in S_{p}(p,n-p)} \sigma l_{n-p+1} \circ_1 l_p$$

where the sum is over all $(p, n-p)$-shuffle permutations. Under the map $\mathcal{A}\mathsf{Lie}_\infty \to \text{Tw } \mathcal{A}\mathsf{Lie}_\infty$ the right hand side is mapped to

$$-\sum_{p=2}^{n-1} \sum_{\sigma \in S_{p}(p,n-p)} \sigma (1_r \mapsto l_{n-p+1+r}) \circ_1 (1_r' \mapsto l_{p+r})$$

On the other hand, the differential of $(1_r \mapsto l_{n+r})$ in $\text{Tw } \mathcal{A}\mathsf{Lie}_\infty$ is defined by equation (3.20). Unravelling the definitions the individual parts read (using notation as in (3.20))

$$\partial^\mathcal{O}(1_r \mapsto l_{n+r}) = -\sum_{r'+r''=r} \sum_{p=0}^{n} \sum_{\sigma \in S_{p}(p,n-p)} \sigma (1_r' \mapsto l_{n-p+1+r'}) \circ_1 (1_r'' \mapsto l_{p+r''})$$

$$\phi \cdot (1_r \mapsto l_{n+r}) = \sum_{r'+r''=r} (1_r' \mapsto l_{n+1+r'}) \circ_1 (1_r'' \mapsto l_{r''})$$

$$\delta_{\sigma}(\phi)(1_r \mapsto l_{n+r}) = \sum_{r'+r''=r} \sum_{p=1,n} (1_r' \mapsto l_{n-p+1+r'}) \circ_1 (1_r'' \mapsto l_{p+r''})$$

One sees that the latter two terms cancel the terms $p = 0, 1, n$ in the sum and hence together yield the desired result.
Using this lemma, we obtain a morphism
\[ \Lambda \text{Lie}_\infty \to \text{Tw } O \]
by composition. From lemma 4.1 it is then evident that the following diagram commutes:
\[ \Lambda \text{Lie}_\infty \xrightarrow{\text{Tw } f} \text{Tw } O \]
\[ \text{Tw } f \] \[ \text{Tw } O \]
\[ \text{Tw } O' \]
Here \( f, O, O' \) are as above. Summarizing, we obtain the following result:

**Corollary 4.3.** The operation \( \text{Tw } \) defines an endofunctor on the undercategory \( \Lambda \text{Lie}_\infty \downarrow \text{Operads} \).

### 4.2. The natural projection.

Let \( \Lambda \text{Lie}_\infty \to O \) be an arrow in \( \text{Operads} \). There is the natural map
\[ \eta_O : \text{Tw } O \to O \]
projecting the product
\[ \text{Tw } O(n) = \prod_{r \geq 0} \text{Hom}_{S_r}(s^{-2r}K, O(r + n)) \]
to its first \((r = 0)\) factor. It is easy to see that this map is a map of dg operads.

**Lemma 4.4.** The maps \( \eta_O \) assemble to form a natural transformation \( \eta : \text{Tw } \Rightarrow \text{id} \), where \( \text{id} \) is the identity functor on \( \Lambda \text{Lie}_\infty \downarrow \text{Operads} \).

**Proof.** We have to show that for all arrows \( \Lambda \text{Lie}_\infty \to O \to O' \) the following diagram commutes.
\[ \text{Tw } O \xrightarrow{\text{Tw } f} \text{Tw } O' \]
\[ \eta_O \]
\[ O \]
\[ f \]
\[ O' \]
This is obvious. \( \square \)

### 4.3. \text{Tw} as a co-monad.

Let again \( \Lambda \text{Lie}_\infty \to O \) be an arrow in \( \text{Operads} \). Consider
\[ \text{Tw } \text{Tw } O = \prod_{r, s \geq 0} \text{Hom}_{S_r}(s^{-2r}K, \text{Hom}_{S_s}(s^{-2r}K, O(r + s + n))) \cong \prod_{r, s \geq 0} \text{Hom}_{S_r \times S_s}(s^{-2r-2s}K, O(r + s + n)). \]
There are natural inclusions
\[ \text{Hom}_{S_r \times S_s}(s^{-2r-2s}K, O(r + s + n)) \to \text{Hom}_{S_r \times S_s}(s^{-2r-2s}K, O(r + s + n)) \]
and they assemble to form a map
\[ \mu_O : \text{Tw } O \to \text{Tw } \text{Tw } O. \]

**Lemma 4.5.** The map \( \mu_O \) is a map of operads and the diagram
\[ \Lambda \text{Lie}_\infty \]
\[ \text{Tw } O \xrightarrow{\mu_O} \text{Tw } \text{Tw } O \]
\[ \text{Tw } O \]
\[ \text{Tw } \text{Tw } O \]
commutes.

**Proof.** Let us consider commutativity of the diagram first. The right hand arrow sends the generator \( l_n \) to
\[ (\sum_{r, s \geq 0} (1_{n+r+s} \mapsto L_{n+r+s}) ) \in \text{Tw } \text{Tw } O \]
where \( L_n \) is the image of \( l_n \) in \( O \). The left hand arrow sends the generator \( l_n \) to
\[ \sum_{r \geq 0} (1_{n+r} \mapsto L_{n+r}) \in \text{Tw } O. \]
The bottom map sends this element to

\[(4.2) \quad \sum_{r \geq 0} \sum_{j=0}^{r} (1_{n+r} \mapsto L_{n+j+(r-j)}) \in \text{Tw } \mathcal{O}.
\]

Here the notation shall indicate that \((1_{n+r} \mapsto L_{n+j+(r-j)})\) should be understood as an element of

\[\text{Hom}_{S_j \times S_{r-j}}(\mathbb{S}^{-2r K}, \mathcal{O}(r + n)) \subset \text{Tw } \mathcal{O}.
\]

Comparing (4.1) and (4.2) we see that the two sums agree.

Next let us show that \(\mu_{\mathcal{O}}\) is a map of operads. It is simpler to show this statement on algebras. So let \(A\) be a \(\text{Tw } \mathcal{O}\) algebra. We have to show that the pullback of the \(\text{Tw } \mathcal{O}\) algebra structure via the map \(\mu_{\mathcal{O}}\) makes it into a \(\text{Tw } \mathcal{O}\) algebra. We may assume that \(A\) arises by the construction of Theorem 3.5 (applied twice). Hence we are given (i) an \(\mathcal{O}\) algebra structure on \(A\), (ii) a Maurer-Cartan element \(m \in A\) and (iii) another Maurer-Cartan element \(m'\) in the twisted (by \(m\)) \(\Lambda\) Lie \(\infty\) algebra \(A^m\).

\[\mu_{\mathcal{O}}(f(v_1, \ldots, v_n)) = \sum_{r=0}^{\infty} \frac{1}{r!} f(1_r)(m + m', \ldots, m + m', v_1, \ldots, v_n).
\]

We know by Theorem 3.5 that these formulas define a \(\text{Tw } \mathcal{O}\) algebra structure if \(m + m'\) is a Maurer-Cartan element. But it is, since \(m'\) is a Maurer-Cartan element in the twisted (by \(m\)) \(\Lambda\) Lie \(\infty\) algebra \(A^m\).

\[\square\]

**Lemma 4.6.** The maps \(\mu_{\mathcal{O}}\) from above assemble to form a natural transformation \(\mu : \text{Tw } \Rightarrow \text{Tw } \circ \text{Tw}\).

**Proof.** We have to show that for all arrows \(f : \mathcal{O} \rightarrow \mathcal{O}'\) (respecting the maps from \(\Lambda\) Lie \(\infty\)) the following diagram commutes.

\[
\begin{array}{ccc}
\text{Tw } \mathcal{O} & \xrightarrow{\text{Tw } f} & \text{Tw } \mathcal{O}' \\
\mu_{\mathcal{O}} \downarrow & & \downarrow \mu_{\mathcal{O}'} \\
\text{Tw Tw } \mathcal{O} & \xrightarrow{\text{Tw } \mu_{\mathcal{O}}} & \text{Tw Tw } \mathcal{O}'
\end{array}
\]

Unravelling the definitions this is amounts to saying that the following diagrams commute

\[
\begin{array}{ccc}
\text{Hom}_{S_{r+s}}(\mathbb{S}^{-2r-2s K}, \mathcal{O}(r + s + n)) & \xrightarrow{f_{\mathcal{O}}} & \text{Hom}_{S_{r+s}}(\mathbb{S}^{-2r-2s K}, \mathcal{O}'(r + s + n)) \\
\downarrow & & \downarrow \\
\text{Hom}_{S_r \times S_s}(\mathbb{S}^{-2r-2s K}, \mathcal{O}(r + s + n)) & \xrightarrow{f_{\mathcal{O}}} & \text{Hom}_{S_r \times S_s}(\mathbb{S}^{-2r-2s K}, \mathcal{O}'(r + s + n))
\end{array}
\]

for all \(r, s, n\). This is clear. \[\square\]

**Theorem 4.7.** The functor \(\text{Tw}\) together with the natural transformations \(\eta, \mu\) from above, is a co-monad on the under-category \(\Lambda\) Lie \(\infty\) ↓ Operads.

**Proof.** We have to verify the defining relations for a co-monad. The two co-unit relations boil down to the statement that for any operad \(\mathcal{O}\) the compositions

\[
\begin{array}{ccc}
\text{Tw } \mathcal{O} & \xrightarrow{\mu_{\mathcal{O}}} & \text{Tw } \mathcal{O} \\
\eta_{\mathcal{O}} \downarrow & & \downarrow \eta_{\mathcal{O}} \\
\text{Tw } \mathcal{O} & \xrightarrow{\text{Tw } \eta_{\mathcal{O}}} & \text{Tw } \mathcal{O}
\end{array}
\]

are the identity maps on \(\mathcal{O}\). This statement follows immediately from the definitions. Next consider the co-associativity axiom. In our case it boils down to the statement that for any operad \(\mathcal{O}\) the diagram

\[
\begin{array}{ccc}
\text{Tw } \mathcal{O} & \xrightarrow{\mu_{\mathcal{O}}} & \text{Tw } \mathcal{O} \\
\mu_{\mathcal{O}} \downarrow & & \downarrow \mu_{\text{Tw } \mathcal{O}} \\
\text{Tw Tw } \mathcal{O} & \xrightarrow{\text{Tw } \mu_{\mathcal{O}}} & \text{Tw Tw } \mathcal{O}
\end{array}
\]

are the identity maps on \(\mathcal{O}\). This statement follows immediately from the definitions.
commutes. Unravelling the definitions, we have to show that the following diagram commutes

\[
\begin{array}{ccc}
\text{Hom}_{S_{r+s+t}}(s^{-2r-2s-2t}\mathbb{K}, \mathcal{O}(r + s + t + n)) & \longrightarrow & \text{Hom}_{S_{r+s+t}}(s^{-2r-2s-2t}\mathbb{K}, \mathcal{O}(r + s + t + n)) \\
\downarrow & & \downarrow \\
\text{Hom}_{S_r \times S_{s+t}}(s^{-2r-2s-2t}\mathbb{K}, \mathcal{O}(r + s + t + n)) & \longrightarrow & \text{Hom}_{S_r \times S_{s+t}}(s^{-2r-2s-2t}\mathbb{K}, \mathcal{O}(r + s + t + n))
\end{array}
\]

for all \( r, s, t, n \). This is again clear. \( \square \)

4.4. Tw-co-algebras. Let us next consider the co-algebras of the co-monad Tw. These are arrows of operads \( \Lambda \text{Lie}_\infty \rightarrow \mathcal{O} \) together with an operad map

\[ c: \mathcal{O} \rightarrow \text{Tw} \mathcal{O} \]

such that the following axioms hold:

- The following diagram commutes:

\[
\begin{array}{ccc}
\Lambda \text{Lie}_\infty & \longrightarrow & \text{Tw} \mathcal{O} \\
\downarrow c & & \downarrow \\
\mathcal{O} & \longrightarrow & \text{Tw} \mathcal{O}
\end{array}
\]

- The composition

\[ \mathcal{O} \xrightarrow{\eta} \text{Tw} \mathcal{O} \xrightarrow{\eta} \mathcal{O} \]

is the identity.

- The following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{O} & \longrightarrow & \text{Tw} \mathcal{O} \\
\downarrow c & & \downarrow \mu_{\mathcal{O}} \\
\text{Tw} \mathcal{O} & \longrightarrow & \text{Tw} \text{Tw} \mathcal{O}
\end{array}
\]

Let \( A \) be an \( \mathcal{O} \)-algebra with Maurer-Cartan element \( m \in A \). Then the existence of the morphism \( c \) above means that \( A \) can be twisted by \( m \) not just to a \( \text{Tw} \mathcal{O} \)-algebra, but to an \( \mathcal{O} \)-algebra. Call this \( \mathcal{O} \)-algebra \( A^m \). The three additional axioms translate into the following properties of this operation of twisting.

- Considered as an \( \Lambda \text{Lie}_\infty \) algebras, \( A^m \) is obtained by twisting the \( \Lambda \text{Lie}_\infty \) algebra \( A \) by the Maurer-Cartan element \( m \).
- Twisting by the zero Maurer-Cartan element \( m = 0 \) does not change the algebra, i.e., \( A^0 = A \).
- Twisting \( A \) by a Maurer-Cartan element \( m \in A \), and then again by a Maurer-Cartan element \( m' \in A^m \) is the same as twisting by the Maurer-Cartan element \( m + m' \in A \). I.e.,

\[
(A^m)^{m'} = A^{m+m'}.
\]

We will sometimes be a bit sloppy notationwise and call the operad \( \mathcal{O} \) a Tw-coalgebra, even if in fact not the operad, but the arrow \( \Lambda \text{Lie}_\infty \rightarrow \mathcal{O} \) is the Tw-coalgebra.

Example 4.8. The operads \( \Lambda \text{Lie}, \text{Ger} \) and their cofibrant resolutions \( \Lambda \text{Lie}_\infty, \text{Ger}_\infty \) (with the natural maps from \( \Lambda \text{Lie}_\infty \)) are Tw-coalgebras. For \( \Lambda \text{Lie} \) and \( \text{Ger} \) the coalgebra structures are the inclusions

\[
\Lambda \text{Lie} \cong \text{Hom}(\mathbb{K}, \Lambda \text{Lie}(0 + n)) \subset \text{Tw} \Lambda \text{Lie}
\]

\[
\text{Ger} \cong \text{Hom}(\mathbb{K}, \text{Ger}(0 + n)) \subset \text{Tw} \text{Ger}.
\]

For \( \Lambda \text{Lie}_\infty \) the coalgebra structure is given by the formula of Lemma 4.2. For \( \text{Ger}_\infty \) the formula is similar.

Example 4.9. If \( \Lambda \text{Lie}_\infty \rightarrow \mathcal{O} \) is an arrow, then \( \text{Tw} \mathcal{O} \) is a Tw-coalgebra.
4.5. **Algebras over Tw-coalgebras.** To any operad $\mathcal{O}$ we can assign the category $\text{Alg}_{\mathcal{O}}$. This assignment is natural in the sense that $\text{Alg}$ defines a contravariant functor from the category $\text{Operads}$ to the quasi-category of categories. In particular, given an operad map $\mathcal{O} \to \mathcal{O}'$ we obtain a functor $\text{Alg}_{\mathcal{O}} \to \text{Alg}_{\mathcal{O}}$. In this section we want to make precise the following statement:

“For a map of Tw-coalgebras $\mathcal{O} \to \mathcal{O}'$ the induced functor $\text{Alg}_{\mathcal{O}} \to \text{Alg}_{\mathcal{O}}$ is compatible with the operation of twisting of algebras by Maurer-Cartan elements.”

Note that for $(\mathcal{O}, c)$ a Tw-coalgebra, the composition $c \circ \eta$ defines an endomorphism of $\text{Tw} \mathcal{O}$. For a morphism $F : (\mathcal{O}, c) \to (\mathcal{O}', c')$ of Tw-coalgebras, of course,

$$\text{Tw} F \circ (c \circ \eta) = (c' \circ \eta) \circ \text{Tw} F.$$ Using the functoriality of $\text{Alg}$ above, $c \circ \eta$ defines an endofunctor $\text{ATw}_{\mathcal{O}} := \text{Alg}(c \circ \eta)$ of $\text{Alg}_{\text{Tw} \mathcal{O}}$.

**Remark 4.10.** By Theorem 3.6 the sub-category $\text{Alg}_{\text{Tw} \mathcal{O}}$ formed by algebras with a compatible complete descending filtration is equivalent to the category of pairs $(A, m)$ where $A$ is an $\mathcal{O}$-algebra and $m \in A$ is a Maurer-Cartan element. The functor $\text{ATw}_{\mathcal{O}}$ maps $(A, m) \mapsto (A^m, 0) =: \text{ATw}_{\mathcal{O}}(A, m)$.

So the functor $\text{ATw}_{\mathcal{O}}$ encodes the operation of twisting of $\mathcal{O}$-algebras by Maurer-Cartan elements.

The following result is almost a tautology.

**Theorem 4.11.** For $F : \mathcal{O} \to \mathcal{O}'$ a map of Tw-coalgebras, the induced functor $\text{Alg}(F) : \text{Alg}_{\mathcal{O}'} \to \text{Alg}_{\mathcal{O}}$ is compatible with twisting in the sense that $\text{Alg}(\text{Tw} F)$ intertwines the functors $\text{ATw}_{\mathcal{O}}$ and $\text{ATw}_{\mathcal{O}'}$, encoding the operation of twisting on the level of algebras.

4.6. **Twisting and homotopy theory.**

**Definition 4.12.** An arrow $\Lambda\text{Lie}_\infty \to \mathcal{O}$ is called a homotopy fixed point of $\text{Tw}$ if the counit $\eta_{\mathcal{O}} : \text{Tw} \mathcal{O} \to \mathcal{O}$ is a quasi-isomorphism. If the map from $\Lambda\text{Lie}_\infty$ is clear from the context, we will say (abusing notation) that $\mathcal{O}$ is a homotopy fixed point of $\text{Tw}$.

**Example 4.13.** The operads $\Lambda\text{Lie}$, $\Lambda\text{Lie}_\infty$, $\text{Ger}$, $\text{Ger}_\infty$ are homotopy fixed points of $\text{Tw}$. For $\Lambda\text{Lie}$ and $\text{Ger}$ this is shown in Appendix D (The proof is essentially a copy of one from [12]). For $\Lambda\text{Lie}_\infty$ and $\text{Ger}_\infty$ this follows from the following proposition.

**Theorem 4.14.** Let $\Lambda\text{Lie}_\infty \to \mathcal{O} \xrightarrow{F} \mathcal{O}'$ be morphisms with $F$ being a quasi-isomorphism. Then the map $\text{Tw}(F) : \text{Tw} \mathcal{O} \to \text{Tw} \mathcal{O}'$ is also a quasi-isomorphism.

To show this proposition we will need the following lemma.

**Lemma 4.15.** Let $C$ be a complex with a bounded above complete filtration. I.e.,

$$C = F^0 \supseteq F^1 \supseteq \cdots$$

and $C = \varprojlim C/F^p$. Suppose further that the associated graded $\text{gr} C$ is acyclic. Then $C$ is acyclic.

**Proof.** Suppose $c \in C$ is some cocycle. Suppose $c \in F^j$. Then by acyclicity of the associated graded there is a $b_j \in F^j$ such that $c - db_j \in F^{j+1}$. Proceed in this manner to define $b_{j+1}, b_{j+2}$ etc. Then set $b := \sum b_j$. The sum converges since $C$ is complete. Again by completeness

$$c - db = \lim_{N \to \infty} c - \sum_{j \leq N} db_j = 0.$$ Hence $c$ is exact. \qed
We want to show that $\text{Tw} \mathcal{O}(n) \to \text{Tw} \mathcal{O}'(n)$ is a quasi-isomorphism for every $n = 0, 1, 2, \ldots$. This equivalent to saying that the mapping cone

$$C := \text{Tw} \mathcal{O}(n) \oplus \text{Tw} \mathcal{O}'(n)[1]$$

is acyclic for every $n$. There is a natural complete filtration $C = F^0 \supset F^1 \supset \cdots$ such that

$$F^p = \prod_{r \geq p} \text{Hom}_{S_n}(s^{-2r}K, \mathcal{O}(r + n)) \oplus \prod_{r \geq p+1} \text{Hom}_{S_n}(s^{-2r}K, \mathcal{O}'(r + n))[1].$$

The associated graded is $\prod_p F^p / F^{p+1}$. Note that

$$F^p / F^{p+1} \cong \text{Hom}_{S_n}(s^{-2p}K, \mathcal{O}(p + n) \oplus \mathcal{O}'(p + n))[1].$$

The complex $\mathcal{O}(r + n) \oplus \mathcal{O}'(r + n)[1]$ is the mapping cone of $\mathcal{O}(p + n) \to \mathcal{O}'(p + n)$ and hence acyclic by assumption. Since taking invariants with respect to a finite group action commutes with taking cohomology, we conclude that the associated graded is acyclic as well. By the previous lemma the statement of the proposition follows.

**Example 4.16.** There is a quasi-isomorphism of operads $\text{Ger}_\infty \to \text{Br}$. It hence follows from the theorem and the above example that $\text{Br}$ is a homotopy fixed point of $\text{Tw}$.

### 4.7. The functor $\text{Tw}$ has “many” homotopy fixed points.

Let $(\mathcal{O}, i)$ be a pair of a dg operad $\mathcal{O}$ and a morphism (of dg operads)

$$i : \Lambda\text{Lie} \to \mathcal{O}.$$

Recall that, since $\Lambda\text{Lie}$ is generated by the vector $\{a_1, a_2\} \in \Lambda\text{Lie}(2)$, the map (4.3) is uniquely determined by the vector

$$\beta = i(\{a_1, a_2\}) \in \mathcal{O}(2).$$

Since the map $i$ induces a $\Lambda\text{Lie}$-algebra structure on any $\mathcal{O}$-algebra $V$, we can ask the following question: for which pairs $(\mathcal{O}, i)$ the adjoint action $\{v, \}$ on $V$ is a derivation of the $\mathcal{O}$-algebra structure for all $v \in V$?

The following proposition describes a large class of dg operads which satisfy this property:

**Proposition 4.17.** Let $(\mathcal{O}, i)$ be a pair of a dg operad $\mathcal{O}$ and a morphism (of dg operads) (4.3). If the vector $\beta \in \mathcal{O}(2)$ (4.4) satisfies identity

$$\beta \circ_2 \gamma = \sum_{i=1}^{n} c_{1,i} (\gamma \circ_i \beta) \quad \forall \gamma \in \mathcal{O}(n)$$

then for every $\mathcal{O}$-algebra $V$, the adjoint action $\{v, \}$ is a derivation of the $\mathcal{O}$-algebra structure on $V$ for every $v \in V$.

**Proof.** The proof is straightforward.

**Definition 4.18.** We say that the pair $(\mathcal{O}, i)$ satisfies the Leibniz condition if the vector $\beta \in \mathcal{O}(2)$ (4.4) satisfies (4.5).

**Example 4.19.** Operads $\Lambda\text{Lie}$, $\text{Ger}$ are examples of operads which satisfy the Leibniz property. Operads $\text{BV}$ and $\text{BT}$ do not satisfy this property.

We claim that

**Theorem 4.20.** If a pair $(\mathcal{O}, i)$ satisfies the Leibniz condition then $\mathcal{O}$ is a $\text{Tw}$-coalgebra and is a homotopy fixed point of $\text{Tw}$.

**Proof.** It is not hard to see that $\mathcal{O}$ is a coalgebra over the comonad $\text{Tw}$. Indeed, identity (4.5) implies that the canonical embedding

$$\text{emb}_\mathcal{O} : \mathcal{O} \to \text{Tw} \mathcal{O}$$

(4.6)

$$\left(\text{emb}_\mathcal{O}(v)\right)(1_r) = \begin{cases} v & \text{if } r = 0, \\ 0 & \text{otherwise} \end{cases}$$

is a derivation of the $\mathcal{O}$-algebra structure on $\text{Tw} \mathcal{O}$ for every $\mathcal{O}$-algebra $V$. Since $\text{Tw} \mathcal{O}$ is a coalgebra over the comonad $\text{Tw}$ and satisfies the Leibniz condition, it hence follows from the theorem and the above example that $\text{Br}$ is a homotopy fixed point of $\text{Tw}$.
is compatible with the differentials $\partial_{\text{Tw}}$ and $\partial_{\mathcal{O}}$.

Let us consider the free $\mathcal{O}$-algebra generated by $n$ dummy variables $a_1, a_2, \ldots, a_n$ of degree zero and one dummy variable $a$ of degree 2. Let $\mathcal{O}(a, a_1, a_2, \ldots, a_n)$ be the free $\mathcal{O}$-algebra in variables $a, a_1, a_2, \ldots, a_n$ and let

$$\mathcal{O}_n' \subset \mathcal{O}(a, a_1, a_2, \ldots, a_n)$$

be the subspace spanned by monomials in which each variable from the set $\{a_1, a_2, \ldots, a_n\}$ appears exactly once and variable $a$ appears $r$-times.

For example, if $v \in \mathcal{O}(2 + 3)$ then $(v; a_1, a, a_2, a_3)$ is a vector in $\mathcal{O}_n'$.

It is obvious that

$$\mathcal{O}_n' \cong S^{2r}(\mathcal{O}(r + n))^S.$$

and hence

$$\text{Tw} \mathcal{O}(n) \cong \prod_{r=0}^{\infty} \mathcal{O}_n'.$$

Under this identification the differential $\partial_{\text{Tw}}$ takes this form

$$\partial_{\text{Tw}}(v) = \partial_{\mathcal{O}} v + \{a, v\} - v \in \mathcal{O}_n'.$$

AA

**TECHNICAL LEMMA.**

Let us consider the free $\Lambda\text{Lie}$-algebra

$$\Lambda\text{Lie}(a, a_1, a_2, \ldots, a_n)$$

generated by $n$ dummy variables $a_1, a_2, \ldots, a_n$ of degree zero and one dummy variable $a$ of degree 2.

Let us denote by $\delta$ the degree 1 derivation of (4.11) defined by the formulas:

$$\delta(a) = \frac{1}{2} \{a, a\}, \quad \delta(a_i) = 0 \quad \forall \ 1 \leq i \leq n.$$

Due to the Jacobi identity, we have $\delta^2 = 0$. Hence, $\delta$ is a differential on (4.11).

Let us introduce two subspace

$$\Lambda\text{Lie}'(a_1, a_2, \ldots, a_n) \subset \Lambda\text{Lie}(a, a_1, a_2, \ldots, a_n)$$

and

$$\Lambda\text{Lie}'(a, a_1, a_2, \ldots, a_n) \subset \Lambda\text{Lie}(a, a_1, a_2, \ldots, a_n).$$

Here $\Lambda\text{Lie}'(a_1, a_2, \ldots, a_n)$ is spanned by by $\Lambda\text{Lie}$-monomials in $\Lambda\text{Lie}(a, a_1, a_2, \ldots, a_n)$ which involve each variable from the set $\{a_1, a_2, \ldots, a_n\}$ at most once, and $\Lambda\text{Lie}'(a, a_2, \ldots, a_n)$ is spanned by $\Lambda\text{Lie}$-monomials in $\Lambda\text{Lie}'(a, a_1, a_2, \ldots, a_n)$ which do not involve the variable $a$ at all.

It is clear that both subspaces (4.13) and (4.14) are subcomplexes of (4.11). Moreover, the restriction of the differential $\delta$ to (4.13) is zero.

We claim that

**Lemma 4.21. The embedding**

$$\text{emb} : \Lambda\text{Lie}'(a_1, a_2, \ldots, a_n) \hookrightarrow \Lambda\text{Lie}'(a, a_1, a_2, \ldots, a_n)$$

is quasi-isomorphism of cochain complexes. In other words, for every cocycle $c \in \Lambda\text{Lie}'(a_1, a_2, \ldots, a_n)$, there exists a vector $c_1 \in \Lambda\text{Lie}'(a, a_1, a_2, \ldots, a_n)$ such that

$$c - \delta(c_1) \in \Lambda\text{Lie}'(a_1, a_2, \ldots, a_n).$$
Proof. We consider a non-empty ordered subset \( \{ i_1 < i_2 < \cdots < i_k \} \) of \( \{1, 2, \ldots, n \} \) and denote by
\[
\mathcal{L}_{i_1 \ldots i_k} (a, a_1, \ldots, a_n)
\]
the subcomplex of \( \mathcal{L}_{i_1} (a, a_1, \ldots, a_n) \) which is spanned by \( \mathcal{L}_{i_1} \)-monomials in \( \mathcal{L}_{i_1} (a, a_1, \ldots, a_n) \) involving each variable in the set \( \{ a_1, \ldots, a_n \} \) exactly once.

It is clear that \( \mathcal{L}_{i_1 \ldots i_k} (a, a_1, \ldots, a_n) \) splits into the direct sum of subcomplexes:
\[
\mathcal{L}_{i_1, \ldots, i_k} (a, a_1, \ldots, a_n) = \mathbb{K} \langle a, \{ a \} \rangle \oplus \bigoplus_{\{i_1 < i_2 < \cdots < i_k \}} \mathcal{L}_{i_1 \ldots i_k} (a, a_1, \ldots, a_n),
\]
where the summation runs over all non-empty ordered subsets \( \{ i_1 < i_2 < \cdots < i_k \} \) of \( \{1, 2, \ldots, n \} \).

It is not hard to see that the subcomplex \( \mathcal{L}_{i_1, \ldots, i_k} (a, a_1, \ldots, a_n) \) is acyclic. Thus our goal is to show that every cocycle in \( \mathcal{L}_{i_1, \ldots, i_k} (a, a_1, \ldots, a_n) \) is cohomologous to cocycle in the intersection
\[
\mathcal{L}_{i_1, \ldots, i_k} (a, a_1, \ldots, a_n) \cap \mathcal{L}_{i_1 \ldots i_k} (a_1, a_2, \ldots, a_n).
\]

To prove this fact we consider the tensor algebra
\[
T(\mathbb{K} (s^{-1} a, s^{-1} a_1, s^{-1} a_2, \ldots, s^{-1} a_1 a))
\]
in the variables \( s^{-1} a, s^{-1} a_1, s^{-1} a_2, \ldots, s^{-1} a_{1k-1} \) and denote by
\[
T'(s^{-1} a, s^{-1} a_1, s^{-1} a_2, \ldots, s^{-1} a_{1k-1})
\]
the subspace of (4.18) which is spanned by monomials involving each variable from the set \( \{ s^{-1} a_1, s^{-1} a_2, \ldots, s^{-1} a_{1k-1} \} \) exactly once.

It is not hard to see that the formula
\[
\nu(x_{i_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_n}) = \{ s x_{i_1}, \{ s x_{j_2}, \{ \ldots s x_{j_n}, a_{i_1} \} \} \ldots \}
\]
defines an isomorphism of the graded vector spaces
\[
\nu : T'(s^{-1} a, s^{-1} a_1, s^{-1} a_2, \ldots, s^{-1} a_{1k-1}) \xrightarrow{\sim} \mathcal{L}_{i_1, \ldots, i_k} (a, a_1, \ldots, a_n).
\]

Let us denote by \( \delta_T \) a degree 1 derivation of the tensor algebra (4.18) defined by the equations
\[
\delta_T(s^{-1} a_{i_1}) = 0, \quad \delta_T(s^{-1} a) = s^{-1} a \otimes s^{-1} a.
\]
It is not hard to see that \( (\delta_T)^2 = 0 \). Thus, \( \delta_T \) is a differential on the tensor algebra (4.18).

The subspace (4.19) is obviously a subcomplex of (4.18). Furthermore, using the following consequence of Jacobi identity
\[
\{a, \{a, X\}\} = -\frac{1}{2} \{\{a, a\}, X\}, \quad \forall X \in \mathcal{L}_{i_1, \ldots, i_k} (a, a_1, \ldots, a_n),
\]
it is easy to show that
\[
\delta \circ \nu = \nu \circ \delta_T.
\]

Thus \( \nu \) is an isomorphism from the cochain complex
\[
(T'(s^{-1} a, s^{-1} a_1, s^{-1} a_2, \ldots, s^{-1} a_{1k-1}), \delta_T)
\]
to the cochain complex
\[
(\mathcal{L}_{i_1, \ldots, i_k} (a, a_1, \ldots, a_n), \delta).
\]

To compute cohomology of the cochain complex
\[
(T(\mathbb{K} (s^{-1} a, s^{-1} a_1, s^{-1} a_2, \ldots, s^{-1} a_{1k-1})), \delta_T)
\]
we observe that the truncated tensor algebra
\[
T_{s^{-1} a} := T(\mathbb{K} (s^{-1} a))
\]
form an acyclic subcomplex of (4.22).

We also observe that the cochain complex (4.22) splits into the direct sum of subcomplexes
\[
T(\mathbb{K} (s^{-1} a, s^{-1} a_1, s^{-1} a_2, \ldots, s^{-1} a_{1k-1})) = T(\mathbb{K} (s^{-1} a_1, s^{-1} a_2, \ldots, s^{-1} a_{1k-1})) \oplus \bigoplus_{m \geq 2, p_1, \ldots, p_m} V_{s^{-1} a}^p \otimes T_{s^{-1} a} \otimes V_{s^{-1} a}^p \otimes \cdots \otimes V_{s^{-1} a}^p \otimes T_{s^{-1} a} \otimes V_{s^{-1} a}^p,
\]
where $V_{a*}$ is the cochain complex

$$V_{a*} := \mathbb{K}(s^{-1} a_1, s^{-1} a_{i_2}, \ldots, s^{-1} a_{i_k})$$

with the zero differential and the summation runs over all combinations $(p_1, \ldots, p_m)$ of integers satisfying the conditions

$$p_1, p_m \geq 0, \quad p_2, \ldots, p_{m-1} \geq 1.$$ 

By Künneth’s theorem all the subcomplexes

$$V_{a*}^{\otimes p_1} \otimes T_{s^{-1} a} \otimes V_{a*}^{\otimes p_2} \otimes T_{s^{-1} a} \otimes \cdots \otimes V_{a*}^{\otimes p_m} \otimes T_{s^{-1} a} \otimes V_{a*}^{\otimes p_n}$$

are acyclic. Hence for every cocycle $c$ in (4.22) there exists a vector $c_1$ in (4.22) such that

$$c - \delta_T(c_1) \in T(\mathbb{K}(s^{-1} a_1, s^{-1} a_{i_2}, \ldots, s^{-1} a_{i_k})).$$

Combining this observation with the fact that the subcomplex (4.19) is a direct summand in (4.22), we conclude that, for every cocycle $c$ in (4.19) there exists a vector $c_1$ in (4.19) such that

$$c - \delta_T(c_1) \in T'(s^{-1} a, s^{-1} a_1, s^{-1} a_{i_2}, \ldots, s^{-1} a_{i_k-1}) \cap T(\mathbb{K}(s^{-1} a_1, s^{-1} a_{i_2}, \ldots, s^{-1} a_{i_k})).$$

Since the map $\nu$ (4.20) is an isomorphism from the cochain complex (4.19) with the differential $\delta_T$ to the cochain complex (4.16) with the differential $\delta$, we deduce that every cocycle in (4.16) is cohomologous to a unique cocycle in the intersection

$$\Lambda \text{Lie}''(a, a_1, \ldots, a_n) \cap \Lambda \text{Lie}'(a_1, \ldots, a_n).$$

Therefore every cocycle in $\Lambda \text{Lie}'(a, a_1, \ldots, a_n)$ is cohomologous to a unique cocycle in the subcomplex

$$\Lambda \text{Lie}'(a_1, \ldots, a_n).$$

The lemma is proved. \hfill \Box

5. The operad of brace trees $\mathcal{BT}$

In this section we introduce the operad $\mathcal{BT}$ of brace trees. This is an operad in $\text{grVect}_\mathbb{K}$. This operad receives a natural map from $\Lambda \text{Lie}$ and the operad $\mathcal{Br}$ of braces is very closely related to $\text{Tw} \mathcal{BT}$.

To define the space $\mathcal{BT}(n)$ we introduce an auxiliary set $\mathcal{T}(n)$. An element of this set is a planar tree $T$ equipped with a bijection between $\{1, 2, \ldots, n\}$ and the set

$$V(T) \setminus \{\text{root vertex}\}$$

of non-root vertices. For $n = 0$ the set $\mathcal{T}(n)$ is empty.

We call elements of $\mathcal{T}(n)$ brace trees. Examples of brace trees are shown on figures 5.1 and 5.2 on figures, 5.1. A brace tree $T' \in \mathcal{T}(6)$  

\begin{center}
Fig. 5.1. A brace tree $T' \in \mathcal{T}(6)$
\end{center}

\begin{center}
Fig. 5.2. The brace tree $T_{id} \in \mathcal{T}(1)$
\end{center}

non-root vertices are depicted by white circles with the corresponding numbers inscribed.

The $n$-th space $\mathcal{BT}(n)$ of $\mathcal{BT}$ consists of linear combinations of elements in $\mathcal{T}(n)$. The structure of a graded vector space on $\mathcal{BT}(n)$ is obtained by declaring that each non-root edge carries degree $-1$. In other words, for every $T \in \mathcal{T}(n)$ we have

$$|T| = 1 - |E(T)|,$$

where $E(T)$ is the set of all edges of $T$. A simple combinatorics shows that for every brace tree $T \in \mathcal{T}(n)$ we have

$$|T| = 1 - n.$$ 

Hence, the graded vector space $\mathcal{BT}(n)$ is concentrated in the single degree $1 - n$. 

Since $T(0)$ is empty, we have
\begin{equation}
(5.3) \quad \text{BT}(0) = \emptyset.
\end{equation}
Furthermore, since in $T(1)$ we have only element $T_{id}$ (see figure 5.2),
\begin{equation}
(5.4) \quad \text{BT}(1) = K.
\end{equation}

5.1. The operadic structure on $BT$. Let $T \in T(n)$, $T' \in T(k)$ and $1 \leq i \leq k$. Our goal is to define the output of the elementary insertion $T' \circ_i T \in \text{BT}(n + k - 1)$.

Let $v_i$ be the non-root vertex of $T'$ with label $i$. If $v_i$ is a leaf (i.e. $v_i$ does not have incoming edges) then the vector $T' \circ_i T \in \text{BT}(n + k - 1)$ is represented by a brace tree $T''$ which is obtained from $T'$ by erasing the vertex $v_i$ and gluing the brace tree $T$ via identifying the root edge of $T$ with the edge originating at $v_i$. After this operation we relabel elements of the set
\begin{equation}
V(T'') \setminus \{\text{root vertex}\} = (V(T) \setminus \{\text{root vertex}\}) \sqcup (V(T') \setminus \{v_i\})
\end{equation}
in the obvious way.

Let us now consider the case when $v_i$ has $q \geq 1$ incoming edges. Since $T'$ is a planar tree these $q$ incoming edges are ordered linearly. So we denote them by $e_1, e_2, \ldots, e_q$ keeping in mind that
\begin{equation}
(5.5) \quad e_1 < e_2 < \cdots < e_q.
\end{equation}

The desired vector $T' \circ_i T \in \text{BT}(n + k - 1)$ is represented by the sum
\begin{equation}
(5.6) \quad T' \circ_i T = \sum_{\alpha} (-1)^{f(\alpha)} T_{\alpha}
\end{equation}
where $T_{\alpha}$ is obtained from $T'$ and $T$ following these steps:

- first, we erase the vertex $v_i$ and glue the brace tree $T$ via identifying the root edge $T$ with the edge originating from $v_i$,
- second, we attach the edges $e_1, e_2, \ldots, e_q$ to vertices in the set
\begin{equation}
(5.7) \quad V(T) \setminus \{\text{root vertex}\}.
\end{equation}
- finally, we relabel elements of the set
\begin{equation}
V(T'') \setminus \{\text{root vertex}\} = (V(T) \setminus \{\text{root vertex}\}) \sqcup (V(T') \setminus \{v_i, \text{root vertex}\})
\end{equation}
in the obvious way.

Ways of connecting the edges $e_1, e_2, \ldots, e_q$ to vertices in the set $\{5.7\}$ should satisfy the following condition

**Condition 5.1.** The restriction of the total order on the set $E(T_{\alpha})$ of $T_{\alpha}$ to the subset $\{e_1, e_2, \ldots, e_q\}$ should coincide with the order $\{5.5\}$.

This condition can be reformulated in geometric terms as follows. If we choose a small tubular neighborhood of the tree $T$ (drawn on the plane) and walk along its boundary starting from the root vertex in the clockwise direction then we will cross the edges in this order: first, we will cross $e_1$, second, we will cross $e_2$, third, we will cross $e_3$, and so on.

The summation in $\{5.6\}$ goes over all ways $\alpha$ of connecting the edges $e_1, e_2, \ldots, e_q$ to vertices in the set $\{5.7\}$ satisfying Condition $\{5.1\}$

To define the sign $(-1)^{f(\alpha)}$ in $\{5.6\}$ we consider the set
\begin{equation}
(5.8) \quad E(T') \sqcup (E(T) \setminus \{\text{root edge}\})
\end{equation}
with the total order which agrees with the orders on $E(T')$ and $E(T) \setminus \{\text{root edge}\}$, and elements of $E(T')$ are declared to be smaller than elements of $E(T) \setminus \{\text{root edge}\}$.

Next we observe that the set $\{5.8\}$ is naturally isomorphic to the set $E(T_{\alpha})$ of edges of $T_{\alpha}$. On the other hand, the set $E(T_{\alpha})$ carries a possibly different total order coming from planar structure on $T_{\alpha}$.

So the factor $(-1)^{f(\alpha)}$ is the sign of the permutation which connects these total orders on the set
\begin{equation}
E(T_{\alpha}) \cong E(T') \sqcup (E(T) \setminus \{\text{root edge}\}).
\end{equation}

**Example 5.2.** Let $T'$ (resp. $T_{oo}$) be the brace tree depicted on figure 5.1 (resp. figure 5.3). The result of the insertion $T' \circ_2 T$ is the sum of brace trees shown on figure 5.4.
The symmetric group $S_n$ acts on $\mathcal{B}T(n)$ in the obvious way by rearranging the labels. It is not hard to see that operations (5.6) together with this action give us an operad structure on $\mathcal{B}T$ with the identity element represented by the brace tree $T_{id}$ depicted on figure 5.2.

6. THE OPERAD $\text{Tw}\mathcal{B}T$

The operad $\mathcal{B}T$ receives a natural map from $\Lambda\text{Lie}$

(6.1) $\varphi : \Lambda\text{Lie} \to \mathcal{B}T$.

Since the operad $\Lambda\text{Lie}$ is generated by the binary bracket operation $\{\cdot, \cdot\} \in \Lambda\text{Lie}(2)$, the map (6.1) is uniquely determined by its value $\varphi(\{\cdot, \cdot\})$, which equals

(6.2) $\varphi(\{\cdot, \cdot\}) = T_{oo} + \sigma_{12}T_{oo}$, 

where $T_{oo}$ is the brace tree depicted on figure 5.3 and $\sigma_{12}$ is the transposition in $S_2$.

The desired compatibility with the Jacobi relation

(6.3) $\varphi(\{\cdot, \cdot\}) \circ_1 \varphi(\{\cdot, \cdot\}) + \text{cyclic permutations} (1, 2, 3) = 0$

can be checked by a straightforward computation.

In this section we give an explicit description for the twisted operad $\text{Tw}\mathcal{B}T$ corresponding to the map (6.2).

As a graded vector space,

(6.4) $\text{Tw}\mathcal{B}T(n) = \prod_{r=0}^{\infty} s^{2r}(\mathcal{B}T(r + n))^{S_r}$.

Using the observation that for every $m$ the space $\mathcal{B}T(m)$ is concentrated in the single degree $1 - m$ we conclude that the subspace $\text{Tw}\mathcal{B}T^p(n)$ of degree $p$ vectors in $\text{Tw}\mathcal{B}T(n)$ is spanned by vectors of the form

(6.5) $\sum_{\sigma \in S_r} \sigma(T)$

where $T$ is an arbitrary brace tree in $T(r + n)$ and $r = p + n - 1$.

To represent vectors (6.5) it is convenient to extend the set $T(n)$ to another auxiliary set $T^{tw}(n)$. An element of $T^{tw}(n)$ is a planar tree $T$ equipped with the following data:
• a partition of the set \( V(T) \) of vertices
\[
V(T) = V_{\text{lab}}(T) \sqcup V_{\nu}(T) \sqcup V_{\text{root}}(T)
\]
into the singleton \( V_{\text{root}}(T) \) consisting of the root vertex, the set \( V_{\text{lab}}(T) \) consisting of \( n \) vertices, and the set \( V_{\nu}(T) \) consisting of vertices which we call neutral;
• a bijection between the set \( V_{\text{lab}}(T) \) and the set \( \{1, 2, \ldots, n\} \);

We also call elements of \( \mathcal{T}^{tw}(n) \) brace trees.

Figures 6.3, 6.1, 6.2 show examples of brace trees in \( \mathcal{T}^{tw}(2) \). Figures 6.3 and 6.4 show examples of a brace tree in \( \mathcal{T}^{tw}(1) \). On figures, neutral vertices of a brace tree in \( \mathcal{T}^{tw}(n) \) are depicted by black circles and vertices in \( V_{\text{lab}}(T) \) are depicted by white circles with the corresponding numbers inscribed.

We have the obvious bijection between brace trees in \( \mathcal{T}^{tw}(n) \) with \( r \) neutral vertices and linear combinations (6.5). This bijection assigns to a brace tree \( T' \) with \( r \) neutral vertices the linear combination (6.5) where \( T \) is obtained from \( T' \) by labeling the neutral vertices by 1, 2, \ldots, \( r \) in any possible way and shifting the labels for vertices in \( V_{\text{lab}}(T) \) by \( r \).

In virtue of this bijection, the \( n \)-th space \( \text{TwBT}(n) \) of \( \text{TwBT} \) is the space of (finite) linear combinations of brace trees in \( \mathcal{T}^{tw}(n) \). Furthermore, each brace tree \( T' \in \mathcal{T}^{tw}(n) \) carries the degree
\[
|T| = 2|V_{\nu}(T)| - |E(T)| + 1,
\]
where \( E(T) \) is the set of all edges of \( T \). In other words, each non-root edge carries degree \(-1\) and each neutral vertex carries degree \( 2 \).

Using this description of \( \text{TwBT} \) it is easy to get define the elementary insertions in terms of brace trees. Indeed, let \( T \in \mathcal{T}^{tw}(n) \), \( T' \in \mathcal{T}^{tw}(k) \), \( 1 \leq i \leq k \) and let \( v_i \) be the vertex in \( V_{\text{lab}}(T') \) with label \( i \). If \( v_i \) is a leaf (i.e. \( v_i \) does not have incoming edges) then the vector \( T' \circ_i T \in \text{TwBT}(n + k - 1) \) is represented by a tree \( T'' \) which is obtained from \( T' \) by erasing the vertex \( v_i \) and gluing the tree \( T \) via identifying the root edge of \( T \) with the edge originating at \( v_i \). After this operation we relabel elements of the set \( V_{\text{lab}}(T'') = V_{\text{lab}}(T) \sqcup (V_{\text{lab}}(T') \setminus \{v_i\}) \) in the obvious way.

Let us now consider the case when \( v_i \) has \( q \geq 1 \) incoming edges. Since the tree \( T' \) is planar these \( q \) incoming edges are ordered linearly. So we denote them by \( e_1, e_2, \ldots, e_q \) keeping in mind that
\[
e_1 < e_2 < \cdots < e_q.
\]

The desired vector \( T' \circ_i T \in \text{TwBT}(n + k - 1) \) is represented by the sum
\[
T' \circ_i T = \sum_{\alpha} (-1)^{f(\alpha)} T_{\alpha}
\]
where \( T_{\alpha} \) is obtained from \( T' \) and \( T \) following these steps:

• first, we erase the vertex \( v_i \in V_{\text{lab}}(T') \) and glue the tree \( T \) via identifying the root edge \( T \) with the edge originating from \( v_i \),
• second, we attach the edges \( e_1, e_2, \ldots, e_q \) to vertices in the set \( V_{\text{lab}}(T) \sqcup V_{\nu}(T) \)
• finally, we relabel elements of the set \( V_{\text{lab}}(T'') = V_{\text{lab}}(T) \sqcup (V_{\text{lab}}(T') \setminus \{v_i\}) \) in the obvious way.
Ways of connecting the edges \( e_1, e_2, \ldots, e_q \) to vertices in the set \( V_{lab}(T) \cup V_\nu(T) \) should obey the obvious analog of Condition 5.1.

**Condition 6.1.** The restriction of the total order on the set \( E(T_\alpha) \) of \( T_\alpha \) to the subset \( \{e_1, e_2, \ldots, e_q\} \) should coincide with the order \( (6.7) \).

The summation in \( (6.8) \) goes over all ways \( \alpha \) of connecting the edges \( e_1, e_2, \ldots, e_q \) to vertices in the set \( V_{lab}(T) \cup V_\nu(T) \) satisfying Condition 6.1.

The sing factor \( (-1)^{f(\alpha)} \) is defined in the same way as for elementary insertions in \( BT \).

**Example 6.2.** Let \( T_{<0} \) (resp. \( T_{<1} \)) be the brace tree in \( Tw^2(2) \) depicted on figure 5.3 (resp. figure 6.1). Then the vector \( T \circ_1 T_{<} \in Tw BT(3) \) equals to the sum shown on figure 6.5.

\[
T \circ_1 T_{<} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
3
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
2
\end{array}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
3
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
2
\end{array}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
2
\end{array}
\begin{array}{c}
3
\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
3
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
2
\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
2
\end{array}
\begin{array}{c}
3
\end{array}
\end{array}
\end{array}
\end{array}
\]

**Fig. 6.5.** The elementary insertion \( T \circ_1 T_{<} \)

It is clear that we can use the brace trees in \( Tw^2(0) \) to represent vectors in the Lie algebra

\[
(6.9) \quad L_{BT} = \operatorname{Conv}(\Lambda^2 \text{coCom}, BT) = \prod_{r=0}^{\infty} s^{2r-2} (BT(r))^S_r.
\]

However, the degree formula should be adjusted. A brace tree \( T \in Tw^2(0) \) representing a vector in \( L_{BT} \) carries the degree

\[
(6.10) \quad |T| = 2|V_\nu(T)| - |E(T)| - 1.
\]

For example, the vector \( T_{<} \) depicted on figure 6.6 is the MC element in \( L_{BT} \) corresponding to the map (6.1). This vector carries degree 1.

**6.1. The differential on** \( Tw BT \). Following the general procedure the differential \( \partial(T) \) of a brace tree \( T \in Tw BT(n) \) is given by the formula

\[
(6.11) \quad \partial(T) = T_{<} \cdot T + \kappa(T_{<}) \circ_1 T - (-1)^{|T|} \sum_{i=1}^{n} T \circ_1 k_i(T_{<}).
\]

Here \( T_{<} \cdot \) denotes the auxiliary action (3.12) of the Lie algebra \( L_{BT} \) on the operad \( Tw BT \) and

\[
(6.12) \quad \kappa(T_{<}) = T_{<1} + T_{1<},
\]

where \( T_{<1} \) and \( T_{1<} \) are the brace trees depicted on figures 6.3 and 6.4 respectively.

**Fig. 6.6.** The brace tree \( T_{<} \in Tw^2(0) \)
Unfolding the definition of the auxiliary action (3.12) and using (6.12) we get
\begin{equation}
\partial(T) = -(-1)^{|T|} \sum_{v \in V_c(T)} T_v \circ_{n+1} T_{\bullet \bullet} + T_{1 \bullet} \circ_1 T + T_{1 \bullet} \circ_1 T
\end{equation}

\begin{equation}
-(-1)^{|T|} \sum_{i=1}^{n} T \circ_i T_{1 \bullet} - (-1)^{|T|} \sum_{i=1}^{n} T \circ_i T_{\bullet 1}.
\end{equation}

where $T_v$ is a brace tree in $T^{tw}(n+1)$ which is obtained from $T$ by replacing the neutral vertex $v$ by a “white” vertex with label $n+1$. In other words, $T_{\bullet \bullet} \cdot T$ is the sum over insertions of the brace tree $T_{\bullet \bullet}$ (see fig. 6.6) into neutral vertices of $T$ with appropriate signs.

The expression $T_{1 \bullet} \circ_1 T$ is the sum of brace trees which are obtained from $T$ by attaching the single edge with a neutral vertex on one end to all non-root vertices of $T$. The brace tree $T_{1 \bullet} \circ_1 T$ is obtained from the tree $T$ by dividing the root edge into two parts and inserting a neutral vertex in the middle. The expression $T \circ_i T_{1 \bullet}$ (resp. the expression $T \circ_i T_{\bullet 1}$) is obtained from $T$ by inserting $T_{1 \bullet}$ (resp. $T_{\bullet 1}$) into the vertex of $T$ with label $i$.

**Example 6.3.** Let $T_{0 \bullet}, T_{1 \bullet}, T_{1 \bullet opp}, T_{\bullet 1}$ be the brace trees depicted on figures 5.3, 6.1, 6.2, and 6.3. Direct computations show that $T_{1 \bullet}$ and $T_{1 \bullet opp}$ are $\partial$-closed and $T$ is not $\partial$-closed but
\begin{equation}
\partial(T) = T_{1 \bullet opp} - T_{1 \bullet}.
\end{equation}

In other words, $T_{1 \bullet}$ is cohomologous to $T_{1 \bullet opp}$.

For the brace tree $T_{\bullet 1}$ we have
\begin{equation}
\partial(T_{\bullet 1}) = T_{\bullet 1}.
\end{equation}

where $T_{\bullet 1}$ is the brace tree depicted on figure 6.7.

The operad $T^{tw} BT$ acts naturally on the Hochschild cochain complex $C^\bullet(A)$ of any (flat) $A_\infty$-algebra $A$. This action is described in Appendix B.

### 7. The operad of braces $Br$ à la Kontsevich-Soibelman

In this section we define a suboperad $Br$ of $T^{tw} BT$. The operad $Br$ coincides with what M. Kontsevich and Y. Soibelman called the “minimal operad” in [8]. By slightly abusing notation we will call $Br$ the operad of braces and we will refer to algebras over $Br$ as brace algebras. At the end of this section we will show that the embedding $Br \hookrightarrow T^{tw} BT$ is a quasi-isomorphism of operads.

We will say that a brace tree $T \in T^{tw}(n)$ is admissible if $T$ satisfies the following condition:

**Condition 7.1.** Every neutral vertex of $T$ has at least 2 incoming edges.

Thus figures 5.3, 6.1, and 6.2 show examples of admissible brace trees while figures 6.3, 6.4, 6.6 show examples of inadmissible brace trees.

We define $Br(n)$ as the subspace of $T^{tw} BT(n)$ which is spanned by admissible brace trees. It is easy to see that, in $T^{tw}(0)$ there are no admissible brace trees. Hence,
\begin{equation}
Br(0) = 0.
\end{equation}

Furthermore, in $T^{tw}(1)$, there is exactly one admissible brace tree. This brace tree is depicted on figure 5.2.

\begin{equation}
Br(1) = K.
\end{equation}

We claim that...
Proposition 7.2. The sub-collection $\{ Br(n) \}_{n \geq 0}$ is a suboperad of $Tw BT$.

Proof. It is obvious that for admissible brace trees $T' \in T_{tw}(k)$ and $T \in T_{tw}(n)$

$$T' \circ_i T, \quad 1 \leq i \leq k$$

are linear combinations of admissible brace trees. Thus it remains to prove that for every brace tree $\partial T$ is a linear combination of admissible brace trees. Concretely, we have to show that in $\partial T$ there occur no trees with 1- or 2-valent neutral vertices. Let us consider first the 2-valent case. The vertex splitting produces $2k + 2$ terms with a 2-valent neutral vertex, 2 for each of the $k$ edges of $T$ and 2 for the root edge. Concretely the two terms for each edge are:

\[ \text{Here the left term comes from splitting of the lower vertex and the right term from splitting of the upper vertex. The gray vertices can be either external or neutral. The remainder of the tree is omitted. Note that the two terms shown cancel out for each edge, so the differential does not produce valence 2 neutral vertices.} \]

Next consider valence 1 neutral vertices. Again it can be seen that they come in pairs.

\[ \text{Here the left term comes from the splitting of the gray (i.e., white or black) vertex. The right term comes from the second term of the differential in (6.11). Again, checking the signs, these two terms cancel and no valence 1 neutral vertex is produced.} \]

\[ \square \]

7.1. A simpler description of the differential on $Br$. Let $T$ be an admissible brace tree in $T(n)$, $v$ be a vertex in $V_{lab}(T)$ carrying label $i$, and $\kappa(T_{\bullet \bullet})$ be the sum $T_{i+1} + T_{1\bullet}$ of inadmissible brace trees depicted on figures 6.3 and 6.4. We denote by $\partial_v(T)$ a vector in $Br(n)$ which is obtained from the sum

$$T \circ_i \kappa(T_{\bullet \bullet})$$

by omitting all non-admissible brace trees.

Let $v$ be a neutral vertex of $T$. To define $\partial_v(T)$ we change the color of vertex $v$ to white and assign to $v$ label $n + 1$. We denote this new brace tree by $T_v$ and obtain $\partial_v(T)$ from the sum

$$T_v \circ_{n+1} T_{\bullet \bullet}$$

via omitting all non-admissible brace trees.

Finally, we declare that the differential $\partial$ on $Br$ is the vector $\partial(T) \in Br(n)$ is represented by the sum

$$\partial(T) = \sum_{v \in V_{lab}(T) \cup V_{\nu}(T)} \pm \partial_v(T)$$

7.2. The inclusion $Br \hookrightarrow Tw BT$ is a quasi-isomorphism. The goal of this section is to prove the following theorem

Theorem 7.3. The operad map

$$Br \hookrightarrow Tw BT$$

is a quasi-isomorphism.
Proof. Let us put a filtration (of vectors spaces) on $\text{Tw } \mathcal{B} \mathcal{T}$ by the number of neutral vertices of valence 1 or 2.

$$\text{Tw } \mathcal{B} \mathcal{T} = \prod_{p \geq 0} \text{Tw } \mathcal{B} \mathcal{T}^{(p)}.$$ 

Here $\text{Tw } \mathcal{B} \mathcal{T}^{(p)}$ is spanned by trees with $p$ neutral vertices of valence $\leq 2$. In particular $\text{Br} = \text{Tw } \mathcal{B} \mathcal{T}^{(0)}$. The differential splits into two parts, $d = d_0 + d_1$, where $d_1 : \mathcal{B} \mathcal{T}^{(p)} \to \mathcal{B} \mathcal{T}^{(p+1)}$. By standard spectral sequence arguments, we will be done if we can show that the $d_1$ cohomology is $\text{Br}$,

$$H(\text{Tw } \mathcal{B} \mathcal{T}, d_1) = \text{Tw } \mathcal{B} \mathcal{T}^{(0)}.$$ 

For every twisted $\mathcal{B} \mathcal{T}$ tree $\Gamma$ let the core of $\Gamma$ be the tree $\Gamma'$ obtained by deleting all neutral valence 2 vertices and connecting their incident edges. Here is an example:

Since $d_1$ does not alter the core of trees, the complex $(\text{Tw } \mathcal{B} \mathcal{T}, d_1)$ splits into a direct product of subcomplexes, one for each possible core:

$$\text{Tw } \mathcal{B} \mathcal{T} \cong \prod_{\text{cores } \Gamma'} \text{Tw } \mathcal{B} \mathcal{T}_{\Gamma'}.$$ 

In turn, the subcomplexes $\text{Tw } \mathcal{B} \mathcal{T}_{\Gamma'}$ are tensor products of complexes, one for each edge in $\Gamma'$, including the root edge:

$$\text{Tw } \mathcal{B} \mathcal{T}_{\Gamma'} = \bigotimes_{\text{edges } e} V_e.$$ 

Here

$$V_e = \begin{cases} \mathbb{K} \xrightarrow{id} \mathbb{K} \to \mathbb{K} \xrightarrow{id} \mathbb{K} \to \cdots & \text{if } e \text{ connects to a valence 1 neutral vertex.} \\ \mathbb{K} \to \mathbb{K} \xrightarrow{id} \mathbb{K} \xrightarrow{0} \mathbb{K} \to \cdots & \text{otherwise.} \end{cases}$$ 

The various copies of $\mathbb{K}$ correspond to strings of valence 2 neutral vertices of various lengths:

As before, the gray vertices stand for either external or neutral vertices. The lower gray vertices may also be replaced by the root vertex. The rest of the tree is omitted, and just indicated by some stubs at the gray vertices. Clearly the cohomology of $V_e$ is trivial if $e$ connects to a valence 1 neutral vertex and $\mathbb{K}$ otherwise. The representative in the latter case is a single edge (i.e., a string of zero neutral vertices). This shows that the cohomology of $(\text{Tw } \mathcal{B} \mathcal{T}_{\Gamma'}, d_1)$ is zero if $\Gamma'$ contains any valence 1 internal vertices. Otherwise, the cohomology is one dimensional and a representative is the unique graph $\Gamma$ with core $\Gamma'$ that does not have valence 2 neutral vertices. This shows that $H(\text{Tw } \mathcal{B} \mathcal{T}, d_1) = \text{Tw } \mathcal{B} \mathcal{T}^{(0)}$ and we are done. \(\square\)
8. The proof of Theorem 1.1

Let us turn to the proof of the main Theorem. We will in fact deduce it from the following more general statement.

**Theorem 8.1.** Let $\Lambda\text{Lie}_\infty \to \mathcal{O}$, $\Lambda\text{Lie}_\infty \to \mathcal{O}'$ be to objects of the under-category $\Lambda\text{Lie}_\infty \downarrow \text{Operads}$. Let $\mathcal{O} \xrightarrow{\eta} \text{Tw} \mathcal{O}'$ be a morphism of dg operads, compatible with the maps from $\Lambda\text{Lie}_\infty$. If $(\mathcal{O}, c)$ is a $\text{Tw}$ coalgebra and $\mathcal{O}$ and $\text{Tw} \mathcal{O}'$ are homotopy fixed points of $\text{Tw}$, then the morphism

$$F' := (\text{Tw} (\eta_{\mathcal{O}'} \circ F)) \circ c: \mathcal{O} \to \text{Tw} \mathcal{O}'$$

is homotopic to $F$. In particular note that $F'$ is a morphism of $\text{Tw}$-coalgebras.

The Proposition implies that any morphism of operads between the homotopy $\text{Tw}$ fixed points $\mathcal{O}$ and $\text{Tw} \mathcal{O}'$ is homotopic to a morphism of operads respecting also the $\text{Tw}$ coalgebra structure.

**Proof.** Abbreviate $f := \eta_{\mathcal{O}'} \circ F$ and consider the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{O} & \xrightarrow{F} & \text{Tw} \mathcal{O}' \\
\downarrow{f} & & \downarrow{\eta_{\mathcal{O}'}} \\
\mathcal{O}' & & 
\end{array}
$$

Applying the functor $\text{Tw}$ we obtain the following commutative diagram:

$$
\begin{array}{ccc}
\text{Tw} \mathcal{O} & \xrightarrow{\text{Tw} F} & \text{Tw} \text{Tw} \mathcal{O}' \\
\downarrow{\text{Tw} f} & & \downarrow{\text{Tw} \eta_{\mathcal{O}'}} \\
\text{Tw} \mathcal{O}' & & 
\end{array}
$$

Finally consider the following diagram:

$$
\begin{array}{ccc}
\mathcal{O} & \xrightarrow{F} & \text{Tw} \mathcal{O}' \\
\downarrow{\eta_{\mathcal{O}}} & & \downarrow{\eta_{\text{Tw} \mathcal{O}'}} \\
\text{Tw} \mathcal{O} & \xrightarrow{\text{Tw} F} & \text{Tw} \text{Tw} \mathcal{O}' \\
\downarrow{?} & & \downarrow{\eta_{\text{Tw} \mathcal{O}'}} \\
\mathcal{O} & & 
\end{array}
$$

Our goal is to show that the triangle marked with "?" is homotopy commutative. We will do this by showing that all the other cells (including the big cell around the diagram) homotopy commute. We saw above that the upper right triangle commutes. Furthermore the big outer square commutes because $\eta$ is a natural transformation. The small cell on the left commutes since $\eta \circ c = id_{\mathcal{O}}$ by the definition of a $\text{Tw}$ coalgebra. Finally consider the small cell on the right. Note that by the defining property of the counit the following diagram commutes:

$$
\begin{array}{ccc}
\text{Tw} \mathcal{O} & \xrightarrow{\text{Tw} F} & \text{Tw} \text{Tw} \mathcal{O}' \\
\downarrow{?} & & \downarrow{\eta_{\text{Tw} \mathcal{O}'}} \\
\mathcal{O} & & 
\end{array}
$$

All arrows are quasi-isomorphisms since $\text{Tw} \mathcal{O}'$ is a homotopy fixed point of $\text{Tw}$. Hence the morphisms $\text{Tw} \eta_{\mathcal{O}'}$ and $\eta_{\text{Tw} \mathcal{O}'}$ are homotopic. This shows that the triangle ? in the previous diagram is homotopy commutative. This is the statement of the proposition. □

Now apply the Proposition to $\mathcal{O} = \text{Ger}_\infty$ and $\mathcal{O}' = \text{BT}$. $\text{Ger}_\infty$ is a $\text{Tw}$ coalgebra. Furthermore we already saw in the previous section that both $\text{Ger}_\infty$ and $\text{Tw} \text{BT}$ are homotopy fixed points of $\text{Tw}$. Hence the Proposition states that any morphism of operads $\text{Ger}_\infty \to \text{Tw} \text{BT}$ respecting the maps from $\Lambda\text{Lie}_\infty$ is
homotopic to a morphism of twisted coalgebras $F' : \text{Ger}_\infty \to \text{TwBT}$. By Theorem 4.11 it then follows that the induced functor $\text{Alg}(F')$ is compatible with twisting. This shows Theorem 1.1.

**APPENDIX A. THE OPERAD Ger**

A Gerstenhaber algebra is a graded vector space $V$ equipped with a commutative (and associative) product (without identity) and a degree $−1$ binary operation $\{ \}$ which satisfies the following relations:

(A.1) \[ \{ v_1, v_2 \} = (-1)^{|v_1||v_2|}\{ v_2, v_1 \}, \]

(A.2) \[ \{ v, v_1 v_2 \} = \{ v, v_1 \} v_2 + (-1)^{|v||v_1|+|v_1|}v_1 \{ v, v_2 \}, \]

(A.3) \[ \{ \{ v_1, v_2 \}, v_3 \} + (-1)^{|v_1||v_2|+|v_3|}\{ \{ v_2, v_3 \}, v_1 \} + (-1)^{|v_1||v_2|}\{ \{ v_3, v_1 \}, v_2 \} = 0. \]

To define spaces of the operad Ger governing Gerstenhaber algebras we introduce the free Gerstenhaber algebra $\text{Ger}_n$ in $n$ dummy variables $a_1, a_2, \ldots, a_n$ of degree 0. Next we set $\text{Ger}(0) = 0$ and $\text{Ger}(1) = \mathbb{K}$. And then we declare that, for $n \geq 2$, $\text{Ger}(n)$ is spanned by monomials of $\text{Ger}_n$ in which each dummy variable $a_i$ appears exactly once.

The symmetric group $S_n$ acts on $\text{Ger}(n)$ in the obvious way by permuting the dummy variables. It is also clear how to define elementary insertions.

**Example A.1.** Let us consider the monomials $u = \{ a_2, a_3 \}a_1\{ a_4, a_5 \} \in \text{Ger}(5)$ and $w = \{ a_1, a_2 \} \in \text{Ger}(2)$ and compute the insertions $u \circ_2 w$, $u \circ_4 w$ and $w \circ_1 u$. We get

\[ u \circ_2 w = -\{ \{ a_2, a_3 \}, a_4 \}a_1\{ a_5, a_6 \}, \quad u \circ_4 w = \{ a_2, a_3 \}a_1\{ \{ a_4, a_5 \}, a_6 \}, \]

\[ w \circ_1 u = \{ \{ a_2, a_3 \}, a_1\{ a_4, a_5 \}, a_6 \} = \{ a_6, \{ a_2, a_3 \}a_1\{ a_4, a_5 \} \} = \{ a_6, a_2, a_3 \}a_1\{ a_4, a_5 \} - \{ a_2, a_3 \}a_1\{ a_4, a_5 \}. \]

It is easy to see that Ger is generated by the monomials $a_1a_2, \{ a_1, a_2 \} \in \text{Ger}(2)$.

**APPENDIX B. ACTION OF THE OPERAD TwBT ON THE HOCHSCHILD COCHAIN COMPLEX OF AN $A_\infty$-ALGEBRA**

Let $A$ be a cochain complex. We form another cochain complex

(B.1) \[ C^\bullet(A) = \bigoplus_{m \geq 0} s^m \text{Hom}(A^{\otimes m}, A) \]

with the differential $\partial_A$ coming from $A$.

Let us show that (B.1) is a naturally an algebra over $\text{BT}$. To define an action $\text{BT}$ on (B.1) we observe that the collection

(B.2) \[ \{ \text{Hom}(A^{\otimes m}, A) \}_{m \geq 0} \]

is an naturally an operad, i.e. the endomorphism operad of $A$.

Using this observation, we will now construct an auxiliary map

(B.3) \[ g : \bigoplus_{N \geq 1} \left( \text{BT}(N) \otimes \left( C^\bullet(A) \otimes^N \right) \right)_{S_N} \to A \]

Given a brace tree $T \in \mathcal{T}(N)$ and $N$ homogeneous vectors

(B.4) \[ P_i \in s^{m_i} \text{Hom}(A^{\otimes m_i}, A), \quad 1 \leq i \leq N \]

we decorated non-root vertices of $T$ with vectors (B.4) following this rule: the vertex with label $i$ is decorated by the vector $P_i$.

We say that such a decoration is *admissible* if for every $1 \leq i \leq N$ the vertex with label $i$ has exactly $m_i$ incoming edges. Otherwise, we say that the decoration is *inadmissible*. In particular, if a decoration is admissible, then each leaf of $T$ is decorated by a vector in

\[ \text{Hom}(A^{\otimes 0}, A) = A. \]
Fig. B.1. A brace tree $T \in T(4)$

Fig. B.2. The brace corolla $T_k \in T(k+1)$

Since (B.2) is an operad, each brace tree $T \in T(N)$ with an admissible decoration gives us a vector in $A$. We denote this vector in $A$ by

$$m(T; P_1, P_2, \ldots, P_N).$$

For example, if $T$ the brace tree depicted on figure B.1, $P_1 \in \text{sHom}(A, A)$, $P_2 \in A$, $P_3 \in \text{s}^2\text{Hom}(A^{\otimes 2}, A)$, and $P_4 \in A$ then we have

$$m(T; P_1, P_2, P_3, P_4) = P_3(P_1(P_2), P_4).$$

So if $P_1, P_2, \ldots, P_N$ give us an inadmissible decoration of $T$ then we set

$$\varrho(T; P_1, \ldots, P_N) = 0.$$  \hspace{1cm} \text{(B.6)}$$

Otherwise, we declare that

$$\varrho(T; P_1, \ldots, P_N) = (-1)^{\varepsilon(T; P_1, \ldots, P_N)} m(T; P_1, \ldots, P_N)$$

where the sign factor $(-1)^{\varepsilon(T; P_1, \ldots, P_N)}$ comes from permutation on the set

$$\{E(T) \setminus \{\text{root edge}\}\} \sqcup \{P_1, \ldots, P_N\}$$

which we perform when we decorate the tree $T$ with vectors $P_1, P_2, \ldots, P_N$.

For example, $T$ is the brace tree depicted on figure B.1 and $P_1 \in \text{sHom}(A, A)$, $P_2 \in A$, $P_3 \in \text{s}^2\text{Hom}(A^{\otimes 2}, A)$, and $P_4 \in A$ then

$$\varepsilon(T; P_1, \ldots, P_N) = |P_3||P_1| + |P_2| + 3|P_3| + 2|P_1| + |P_2|.$$  \hspace{1cm} \text{(B.7)}$$

The sign factor $(-1)^{|P_3||P_1|+|P_2|}$ appears because we need to switch from the order $(P_1, P_2, P_3, P_4)$ to the order $(P_3, P_1, P_2, P_4)$; the sign factor $(-1)^{|P_3|}$ appears because $P_3$ “jumps over” three edges of brace tree $T$. Similarly, $P_1$ (resp. $P_2$) “jumps over” two edges (resp. one edge) of the brace tree $T$.

We can now define how a brace tree $T \in T(n)$ acts on $n$ homogeneous vectors

$$P_i \in \text{s}^{m_i}\text{Hom}(A^{\otimes m_i}, A), \quad 1 \leq i \leq n.$$  \hspace{1cm} \text{(B.8)}$$

For this purpose form the linear combinations ($k \geq 0$)

$$T_k \circ_1 T,$$

where $T_k$ is the brace corolla shown on figure B.2 and

$$k = \sum_{i=1}^{n} m_i + 1 - n.$$  \hspace{1cm} \text{(B.10)}$$

Next we set

$$T(P_1, \ldots, P_n; a_1, \ldots, a_k) = \varrho(T_k \circ_1 T, P_1, \ldots, P_n, a_1, \ldots, a_k),$$

where $a_1, \ldots, a_k$ are viewed as vectors in

$$\text{Hom}(A^{\otimes 0}, A).$$
Note that, the vectors $a_1, \ldots, a_k$ will decorate leaves (of brace trees in the linear combination $T_k \circ_1 T$) with labels $n+1, n+2, \ldots, n+k$. Moreover, if equation (B.10) were not satisfied then all decorations of $T_k \circ_1 T$ would be inadmissible.

Tedious but straightforward computations show that formula (B.11) indeed defines an action of the operad $\text{Br}$ on (B.1).

**Example B.1.** Let $T$ be the brace tree in $T(4)$ depicted on figure B.4 and

$$P_1 \in \mathfrak{s}^3 \text{Hom}(A \otimes A, A), \quad P_2 \in \text{shom}(A, A), \quad P_3 \in \mathfrak{s}^3 \text{Hom}(A^{\otimes 3}, A), \quad P_4 \in \text{Hom}(A^{\otimes 0}, A) = A.$$  

The vector $T(P_1, P_2, P_3, P_4)$ belongs to

$$\mathfrak{s}^3 \text{Hom}(A^{\otimes 3}, A)$$

The linear combination $T_3 \circ_1 T$ contains a lot of terms. On figure B.3 we list all brace trees (with signs) which get admissible decorations by the vectors

$$P_1, P_2, P_3, P_4, a_1, a_2, a_3$$

Thus we get

$$T(P_1, P_2, P_3, P_4)(a_1, a_2, a_3) = -(1)^{\varepsilon_1} P_3(a_1, P_1(a_2, P_2(a_3)), P_4)$$

$$-(1)^{\varepsilon_2} P_3(a_1, P_1(P_2(a_2), a_3), P_4) - (1)^{\varepsilon_3} P_3(P_1(a_1, P_2(a_2)), a_3, P_4),$$

where

$$\varepsilon_1 = \varepsilon(P_3, a_1, P_1, a_2, P_2, a_3, P_4) + 6|P_3| + 5|a_1| + 4|P_1| + 3|a_2| + 2|P_2| + |a_3|,$$

$$\varepsilon_2 = \varepsilon(P_3, a_1, P_1, P_2, a_2, a_3, P_4) + 6|P_3| + 5|a_1| + 4|P_1| + 3|P_2| + 2|a_2| + |a_3|,$$

$$\varepsilon_3 = \varepsilon(P_3, P_1, a_1, P_2, a_2, a_3, P_4) + 6|P_3| + 5|P_1| + 4|a_1| + 3|P_2| + 2|a_2| + |a_3|,$$

and $\varepsilon(\ldots)$ is the sign of the permutation of vectors $P_1, P_2, P_3, P_4, a_1, a_2, a_3$ from their standard position (B.13).

Since the operad $\text{Br}$ receives an operad map from $\Lambda \text{Lie}$, the cochain complex (B.1) is naturally a $\Lambda \text{Lie}$ algebra. The bracket is the famous Gerstenhaber bracket introduced in (4).

The cochain complex $C^\bullet(A)$ (B.1) is equipped with the obvious decreasing filtration:

$$F_q C^\bullet(A) = \left\{ P \in C^\bullet(A) \mid P(a_1, a_2, \ldots, a_k) = 0 \forall k \leq q \right\}.$$

It is not hard to see that this filtration is compatible with the action of $\text{Br}$.

WELL ... $C^\bullet(A)$ is not complete... OK..

Thus we may apply the general procedure of twisting described in Section 3 to the $\text{Br}$-algebra $C^\bullet(A)$. According to this procedure “NICE” $\text{Tw} \text{Br}$-algebra structures on $C^\bullet(A)$ are in bijection with MC elements $\alpha$ in the COMPLETION of $C^\bullet(A)$ of the form

$$\alpha = \sum_{k \geq 2} \alpha_k, \quad \alpha_k \in \mathfrak{s}^k \text{Hom}(A^{\otimes k}, A).$$

These are exactly flat $A_\infty$-structures on the cochain complex $A$. Furthermore, the cochain complex $C^\bullet(A)$ with the twisted differential is exactly the Hochschild cochain complex of the $A_\infty$-algebra $A$.

Thus we conclude that for every (flat) $A_\infty$-algebra $A$ its Hochschild cochain complex is equipped with a natural action of the operad $\text{Tw} \text{Br}$ and hence with a natural action of the operad $\text{Br}$. 

![Figure B.3](image-url)
Appendix C. A Direct Computation of the Cohomology of $\text{Br}$

In this section we give a direct proof of the following theorem.

**Theorem C.1.** For the operad $\text{Br}$ we have

$$H^\bullet(\text{Br}) = \text{Ger}.$$ 

The bracket operation $\{\cdot, \cdot\} \in \text{Ger}(2)$ is represented by the sum of trees $T + \sigma_{12}(T)$ and the product operation $\cdot \wedge \cdot \in \text{Ger}(2)$ is represented by

$$\frac{1}{2}(T_\cup + T_{\cup\text{opp}}),$$

where the trees $T$, $T_\cup$, and $T_{\cup\text{opp}}$ are depicted on figures 5.3, 6.1, and 6.2, respectively.

**Remark C.2.** In fact, mapping the generator of $\text{Lie}\{1\}$ to the element indicated in the Theorem, we obtain a map $\text{Lie}\{1\} \to \text{Br}$.

To see this, one has to check that the Jacobi identity holds in $\text{Br}$. The proof of the Theorem will show that this map is injective and that the image is given by the closed elements of $\text{Br}$ which do not contain neutral vertices.

**Proof.** We show that $H^\bullet(\text{Br}(n)) = \text{Ger}(n)$ by induction on $n$. For $n = 1$ there is nothing to show. So suppose we know that $H^\bullet(\text{Br}(j)) = \text{Ger}(j)$ for $j = 1, 2, \ldots, n - 1$ and let us attack the statement for $j = n$. We split

$$\text{Br}(n) = V_0 \oplus V_*$$

Here $V_*$ is the space spanned by trees whose lowest node (i.e., the node closest to the root) is neutral, while $V_0$ is the subspace spanned by trees whose lowest node is external. The arrows indicate the several parts of the differential. For example, $V_*$ is a subcomplex. By standard arguments (take a spectral sequence), $\delta_1$ induces a map, also called $\delta_1$ on $\delta_0$ cohomologies:

$$H(V_0, \delta_0) \xrightarrow{\delta_1} H(V_*, \delta_0)$$

Furthermore it holds that

$$H(\text{Br}(n)) = (\ker \delta_1) \oplus (\coker \delta_1).$$

We evaluate both parts in turn.

**Claim 1:** $\ker \delta_1 \cong \text{Lie}\{1\}(n)$. Furthermore this space can be identified with the image of the map $\text{Lie}(n) \to \text{Br}(n)$ described above.

We will (equivalently) show the dual statement. Let us compute $H(V_0^*, \delta_0^3)$. A representative of the class corresponding to the permutation $\sigma \in \Sigma_n$ is given by a string-like tree with vertices labelled according to the permutation, see figure C.1.

To see this statement we again use induction on $n$. For $n = 1$ the statement is clear. Otherwise split:

$$V_0^* = W_1 \oplus W_{\geq 2}$$

Here $W_1$ is spanned by trees in which the lowest node has exactly one child and $W_1$ is spanned by trees in which the lowest node has at least two children. It is easy to see that $\delta_1^3$ is surjective and its kernel is spanned by trees whose lowest node has an external node as child. The complex $(\ker \delta_1^3, \delta_0^3)$ is isomorphic to $(V_0^*, \delta_0^3)$, but for $n - 1$ external nodes. Hence using the induction hypothesis Claim 2 follows.

3Note that $\text{Br}(n)^*$ can be canonically identified with $\text{Br}(n)$ as graded space, just the differential is given by edge contractions instead of vertex splitting. So in particular the differential reduces the number of neutral vertices in a tree by one.
Let us return to showing Claim 1, or rather its dual statement. Note that $\text{Lie}\{1\}(n)^*$ is naturally identified as vector space with $K[\Sigma_n]/(\text{shuffles})$, where
\[(\text{shuffles}) = \text{span} \left\{ \sigma ( \sum_{\sigma' \in \text{sh}(p,q)} \pm \sigma')^{-1} \mid \sigma \in \Sigma_n, p, q \geq 1, p + q = n \right\}\]
is the subspace spanned by signed sums over shuffle permutations. The sum is over all $(p, q)$-shuffles. We want to show that the image of $H(V_\bullet^*, \delta_0^*)$ under $\delta_1^*$ is exactly the cohomology classes of (linear combinations of) string-like trees corresponding to shuffles. Of course, only trees with exactly one neutral vertex (namely the lowest one) can actually contribute, and all those trees are closed. Hence the following statement will show Claim 1.

**Claim 3:** The classes of trees in the image under $\delta_1^*$ of trees in $V_\bullet^*$ with exactly one neutral vertex are sums of string-like graphs corresponding to elements of the subspace (shuffles) $\subset K[\Sigma_n]$.

To prove the claim one has to compute the images and express them as linear combinations of representatives. Both steps are explicit, but tedious to describe in words. We refer to figure [C.3] for the “graphical proof". This shows the first statement of Claim 1. The second statement is a straightforward verification we leave to the reader. To show the theorem, it remains to show:

**Claim 4:** $\text{coker} \delta_1 \cong \text{Ger}(n)/\text{Lie}\{1\}(n)$.

Let us first compute $H(V_\bullet^*, \delta_0)$.

**Claim 5:** $H(V_\bullet^*, \delta_0) \cong \text{Ger}(n)/\text{Lie}\{1\}(n) \oplus (\wedge^2 \text{Lie}\{1\}(n))^\vee$. Furthermore the image of $\delta_1$ lands in the second summand.

Here the second summand deserves some comment. A basis is given by wedge products of Lie words in symbols $x_1, \ldots, x_n$ such that each $x_j$ occurs exactly once. One such element corresponds to a linear combination of trees as shown in figure [C.2]. If we believe Claim 5, Claim 4 is easy to show. We want to show that the image of $\delta_1$ is the second summand. But we know by the proof of Claim 1 that the image of $\delta_1$ is $n! - (n - 1)! = (n - 1)(n - 1)!$-dimensional. Hence by the second part of Claim 5 it suffices to compute the dimension of the second summand. The second summand is in itself isomorphic (as vector space) to $\text{Ger}(n)/\text{Lie}\{1\}(n)$ and hence it has dimension $n! - (n - 1)! = (n - 1)(n - 1)!$. Hence Claim 4 follows from Claim 5. To show Claim 5 consider a filtration of $V_\bullet$ by the number of children of the lowest node.

$$F^2 \subset F^3 \subset \cdots \subset F^n = V_\bullet.$$  

Here $F^p$ is spanned by trees whose lowest node has $\geq p$ children. Let us consider the spectral sequence associated to this filtration. The first differential, say $d_0$, splits vertices except the lowest vertex. The resulting complexes $(\text{gr} F^p, d_0)$ are sums of $p$-fold tensor products of complexes of the form $\text{Br}(j)$, one for each subtree of the lowest node. Here $j$ is the number of external vertices in the subtree. By our (first) induction hypothesis the cohomology of each $\text{Br}(j)$ is $\text{Ger}(j)$ (using $j \leq n - 1$). Hence

$$E^1 := H(\text{gr} F^p, d_1) \cong \bigoplus_{j_1, \ldots, j_p \geq 1} \text{Ger}(j_1) \otimes \cdots \otimes \text{Ger}(j_p).$$

The next differential, say $d_1$, splits the lowest node producing a neutral child node with two children. In $E^1$ this corresponds to taking the commutative product of two elements of the Gerstenhaber operad described by the two subtrees of the new neutral node. Recall that for a polynomial algebra $P$ the bar complex $B(P)$ has cohomology $\wedge P_{\text{gen}}[1]$ where $P_{\text{gen}}$ is the space spanned by the generators. Furthermore the truncated complex $B^{\geq 2}(P)$ has cohomology $P^{\geq 2} \oplus \wedge^2 P_{\text{gen}}[1]$. Here $P^{\geq 2}$ is the space of at least quadratic elements of $P$, i.e., polynomials $f$ such that $f(0) = f'(0) = 0$.

Now note that $\text{Ger} = \text{Com} \circ \text{Lie}\{1\}$. It can be thought of as a subspace of the polynomial algebra with space of generators $\text{Lie}\{1\}$. Furthermore $E^2$ is a subspace of the truncated bar complex of that polynomial algebra. Using the facts about bar complexes of polynomial algebras just recalled, one can check that

$$E^2 = H(E^1, d_1) = \text{Ger}(n)/\text{Lie}\{1\}(n) \oplus (\wedge^2 \text{Lie}\{1\}(n))[1]^\vee.$$ 

The spectral sequence abuts at this point. (All the classes here can be represented in $\text{Br}(n)$ by closed elements, hence all further differentials will vanish.)

(TODO: expand, draw pictures, gradings of vector spaces incorrect, check signs)
To each permutation one can define a certain “string-like” tree.

\[
[x_1, x_3] \wedge x_2 \wedge [x_4, x_5] \Leftrightarrow \begin{array}{c}
\begin{array}{c}
2 \quad 4 \\
1 \quad 3
\end{array} \\
\delta^* \\
\begin{array}{c}
4 \\
2 \\
3 \\
1
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
2 \quad 4 \\
1 \quad 3
\end{array} \\
\begin{array}{c}
4 \\
2 \\
\delta^* \\
\begin{array}{c}
4 \\
2 \\
3 \\
1
\end{array}
\end{array}
\end{array} + \ldots
\]

Illustration of how the differential \(\delta^*\) produces string-like graphs corresponding to shuffle permutations. Clockwise from top right: Some tree in \(V_*\). We compute its differential. This yields a sum of two trees. We want to represent them by string-like trees. Adding the boundary of the trees in the top right, we obtain to string-like trees and to others. Proceeding in the same way for the to others we obtain 6 string-like trees, corresponding to the 6 shuffles of symbols \((12)(34)\).

**Appendix D.** \(\Lambda\text{Lie}^\ast\) and \(\text{Ger}\) are homotopy \(\text{Tw}\) fixed points

The purpose of this section is to show a statement of example 4.13, namely that \(\Lambda\text{Lie}^\ast\) and \(\text{Ger}\) are homotopy \(\text{Tw}\) fixed points. The proof is essentially contained in Appendix A of [12]. Let us start with \(\Lambda\text{Lie}^\ast\). Elements of \(\Lambda\text{Lie}(n)\) are linear combinations of \(\Lambda\text{Lie}\)-words in \(n\) formal variables \(X_1, \ldots, X_n\), each occurring exactly once. Legal examples for \(n = 3\) are \([X_2, [X_1, X_3]]\). Elements of \(\text{Tw} \Lambda\text{Lie}(n)\) are infinite series of \(\Lambda\text{Lie}\) words.
where each \( \cdot \) stands for either \( M \) or some \( X_j \), \( j \in \{2, \ldots, n\} \). These basis elements can be identified with (associative) words in variables \( X_2, \ldots, X_N, M \), with each \( X_j \) occurring exactly once. For example

\[
[X_2, [X_3, [M, [M, X_1]]]] \leftrightarrow X_2X_3MM
\]

The differential acts by doubling \( M \)'s, e.g.

\[
\partial^{\text{Tw}} (X_2X_3MM) = -X_2X_3MMM + X_2X_3MMM = 0.
\]

It is clear that the cohomology of the resulting complex are those words without \( M \)'s. They correspond to the quotient \( \Lambda \text{Lie}(n) \) of \( \text{Tw} \Lambda \text{Lie}(n) \). Hence we have shown that the projection \( \text{Tw} \Lambda \text{Lie}(n) \to \Lambda \text{Lie}(n) \) is a quasi-isomorphism. In other words, \( \Lambda \text{Lie} \) is a homotopy fixed point of \( \text{Tw} \).

Next consider the operad \( \text{Ger} \). Elements of \( \text{Ger}(n) \) are those products of \( \Lambda \text{Lie} \)-words in the symmetric product \( S(\Lambda \text{Lie}(n)) \) that contain each formal variable \( X_1, \ldots, X_n \) exactly once. Similarly \( \text{Tw} \text{Ger}(n) \) can be seen as the subspace of the completed symmetric product space \( S(\text{Tw} \Lambda \text{Lie}(n)) \) spanned by products of \( \Lambda \text{Lie} \)-words in which each \( X_1, \ldots, X_n \) occurs exactly once. The same argument as in the Lie case then shows that the cohomology is spanned by products of \( \text{Lie} \) words without \( M \)'s. Hence the projection \( \text{Tw} \text{Ger}(n) \to \text{Ger}(n) \) is a quasi-isomorphism and \( \text{Ger} \) is a homotopy fixed point of \( \text{Tw} \).

**APPENDIX E. \( \Br \) is a homotopy \( \text{Tw} \) fixed point – combinatorial argument**

We showed in section ?? above that the operad \( \Br \) is a homotopy fixed point of \( \text{Tw} \), i. e., that \( \text{Tw} \Br \to \Br \) is a quasi-isomorphism. One may give an alternative proof of that fact by an elementary combinatorial argument, which we briefly sketch in this appendix.

The elements of \( \text{Tw} \Br \) can be seen as series in \( \Br \)-trees, for which some of the external vertices have been colored in, say, gray. The differential is schematically depicted in Figure [E.4]. The number of neutral ("black") vertices of \( \Br \)-trees yields a descending complete filtration

\[
\text{Tw} \Br = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \mathcal{F}^2 \supset \cdots
\]

where \( \mathcal{F}^p \) is composed of series in trees with \( \geq p \) neutral (black) vertices. Let us consider the associated graded \( \text{gr} \mathcal{F} \). Its differential misses those terms of Figure [E.4] that produce a black vertex. We claim that \( V_p := \mathcal{F}^p / \mathcal{F}^{p+1} \) is acyclic for \( p \geq 1 \). To show this we need to use additional notation. For a \( \Br \) tree, the first internal vertex is the one hit first when going around the tree in clockwise order, see Figure [E.1]. We filter \( V_p \) by the number of valence zero gray vertices attached to the very left of the first internal vertex, see Figure [E.2]. Taking a spectral sequence, the first differential increases that number by one, see Figure [E.3].

It is easy to see that the cohomology under this differential is zero. It follows that \( V_p \) is acyclic as claimed. Hence the projection \( \text{gr} \mathcal{F} \to \mathcal{F}^1 / \mathcal{F}^0 \) is a quasi-isomorphism. From this one can see that also the projection \( \text{Tw} \Br \to \text{Tw} \mathcal{B} \mathcal{T} \) is a quasi-isomorphism. But since \( \Br \to \text{Tw} \mathcal{B} \mathcal{T} \) is a quasi-isomorphism by section ?? we are done.
Fig. E.2. The filtration on $V_p$ we use comes from the number of valence zero gray vertices attached to the very left of the first internal vertex. In this example, that number is three.

Fig. E.3. The differential increases the number of valence zero gray vertices attached to the very left of the first internal vertex by at most one. The component that increases this number by exactly one is shown here. Note that if this number is odd, the differential acts as zero.

Fig. E.4. The differential on $g_1$ has two parts. The first (first row) is obtained by splitting off internal vertices from neutral, gray, or external vertices and comes from the differential on $Br$. The second part (second row) comes from the bracket with $\phi$ in the deformation complex. The very last term contains the terms depicted in Figure E.3.

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