We review some recent developments in the theory of rotating atomic gases. These studies have thrown light on the process of nucleation of vortices in regimes where mean-field methods are inadequate. In our review, we shall describe and compare quantum vortex nucleation of a dilute ultracold bosonic gas trapped in three different configurations: a one-dimensional ring lattice, a one-dimensional ring superlattice and a two-dimensional asymmetric harmonic trap. In all of them, there is a critical rotation frequency, at which the particles in the ground state exhibit strong quantum correlations. However, the entanglement properties vary significantly from case to case. We explain these differences by characterizing the intermediate states that participate in the vortex nucleation process. Finally, we show that noise correlations are sensitive to these differences. These new studies have, therefore, shown how novel quantum states may be produced and probed in future experiments with rotating neutral atom systems.

Keywords: Bose-Einstein condensates; optical lattices; vortices; entanglement

1. Introduction

Vortex nucleation is a topic at the heart of the nature of superfluids and their intrinsic quantum character. Superfluid flow is a direct manifestation of quantum mechanics at the macroscopic level and is stable only below a critical velocity. Above the critical velocity, the generation of phonons, rotons (in $^4$He) or vortices can lead to a breakdown of superfluidity (Ihas et al. 1992). The microscopic nature of these processes cannot, however, be studied in superfluid liquid helium. There has been a resurgence in the study of vortices and their production in the new class of superfluids produced using ultracold atoms. Indeed, ultracold atoms offer a unique opportunity to investigate topological excitations, as recent experiments

*Author for correspondence (andreas.nunnenkamp@yale.edu).

One contribution to the 2010 Anniversary Series ‘A collection of reviews celebrating the Royal Society’s 350th Anniversary’.
have demonstrated with both bosonic and fermionic atoms (Madison et al. 2000; Haljan et al. 2001; Raman et al. 2001; Zwierlein et al. 2005). The production of vortex arrays has, in fact, been crucial in demonstrating the presence of a superfluid order parameter in the ultracold atom systems.

Aspects of vortex nucleation in ultracold atoms have been analysed in various theoretical studies based on a mean-field treatment (Feder et al. 1999; Sinha & Castin 2001; Kasamatsu et al. 2003; Lobo et al. 2004). More recently, attempts have been made to approach situations where the neutral atom vortices require methods beyond mean-field techniques. This should be the case, for example, in rapidly rotating clouds where the neutral atom vortices are analogous to those studied in the quantum Hall effect. An overview of the experimental field has been given by Stock et al. (2005), while Cooper (2009) reviews the different regimes theoretically.

In recent studies, the strongly correlated nature of the ground state around the region for quantum nucleation has been elucidated. The importance of such studies is at least twofold: firstly, production of novel quantum states and, secondly, investigating the role of strong correlation in the nucleation process. Two different physical scenarios were considered. In one of them, the superfluid ultracold atoms were trapped in an optical ring lattice that can be rotated to introduce angular momentum into the system (Hallwood et al. 2006, 2007; Rey et al. 2007; Nunnenkamp & Rey 2008; Nunnenkamp et al. 2008); in the other, the atoms were confined in a rotating asymmetric trap under conditions equivalent to having charged particles in a magnetic field (Dagnino et al. 2009a,b).

In the case of the rotating ring lattice, the authors were most concerned with the production of Schrödinger cat states, i.e. macroscopic superpositions of states with and without a vortex. They were inspired by the similarities of this system with a superconducting quantum interference device (SQUID) that exhibits macroscopic tunnelling between states of opposite current flow (Rouse et al. 1995). As discussed in Leggett (2002), these macroscopic superposition states are important for testing the limits of validity of quantum mechanics and can be used to achieve quantum-limited measurements in precision spectroscopy (Leibfried et al. 2004; Cappellaro et al. 2005; Lee & Khitrin 2005), which is important for ultra-precise gyroscopes. In Hallwood et al. (2006, 2007), Rey et al. (2007), Nunnenkamp et al. (2008) and Nunnenkamp & Rey (2008), the goal was to find the optimal conditions for cat state production. In the case of the vortex nucleation in the lowest Landau level (LLL), Dagnino et al. (2009a,b) demonstrated that the mean-field picture breaks down close to the nucleation point. The nature of the correlated states, while highly entangled, was clearly distinct from those produced in the rotating ring lattice.

In the present work, we show how this difference comes about by studying the nature of the intermediate states, i.e. the routes that connect states with and without vortices. By constructing effective Hamiltonians, we clarify the competing role of interactions and trap or lattice asymmetry in each of these cases and show that the entangled nature of the ground state and the strong correlations involved in the nucleation process can be seen in quantum noise correlations. We believe that rotating atomic gases offer significant new opportunities to study strongly correlated atomic systems. They may also have applications to areas where such states can be used in quantum information science and precision measurement.
2. Vortex nucleation in the rotating ring lattice

In this section, we investigate the effects of rotation on ultracold bosons confined to one-dimensional ring lattices and superlattices. This is an attractive system for study as it can be produced in the laboratory and isolates important issues of the underlying physics of the nucleation process. We find that, at commensurate filling, there exists a critical rotation frequency, at which the ground state of the weakly interacting gas is fragmented into a macroscopic superposition of different quasi-momentum states (Hallwood et al. 2006, 2007; Rey et al. 2007; Nunnenkamp & Rey 2008; Nunnenkamp et al. 2008). We note that Watanabe & Pethick (2007) and Danshita & Polkovnikov (2009) have studied related aspects in a similar system.

(a) Hamiltonian

We consider a system of $N$ ultracold bosons with mass $M$ confined in a one-dimensional ring lattice of $L$ sites with lattice constant $d$. Optical ring lattices can be created with Laguerre–Gaussian (LG) laser beams, as proposed by Amico et al. (2005) and realized by Franke-Arnold et al. (2007). LG beams can be derived from ordinary Gaussian beams, e.g. by means of computer-generated phase holograms (Chavez-Cerda et al. 2002). A conceptionally different approach to arbitrary two-dimensional potentials are spatial light modulators that have recently been used for dynamical manipulation of Bose–Einstein condensates (Boyer et al. 2006). The ring is rotated in its plane (about the $z$-axis) with angular velocity $U$.

By subtracting the rotation energy \( \hat{H}_{\text{rot}} = \int dx \hat{F}^\dagger U \hat{L}_z \hat{F} \), we transform to the rotating frame, where the potential $V(x)$ is time-independent and the many-body Hamiltonian is given by (Bhat et al. 2006; Rey et al. 2007)

\[
\hat{H} = \int dx \hat{F}^\dagger \left[ -\frac{\hbar^2}{2M} \nabla^2 + V(x) + \frac{4\pi \hbar^2 a_s}{2M} \hat{F}^\dagger \hat{F} - U \hat{L}_z \right] \hat{F}. \tag{2.1}
\]

In this expression, $a_s$ is the $s$-wave scattering length, $V(x)$ the lattice potential, $\hat{L}_z$ the angular momentum and $x$ the three-dimensional spatial coordinate vector. $\hat{F}(x)^\dagger$ and $\hat{F}(x)$ are bosonic creation and annihilation field operators.

We assume that the lattice potential $V(x)$ confines the motion along the $z$-axis as well as the radial direction in the $x$–$y$ plane so strongly that only the motion of the atoms along the ring has to be taken into account. In addition, we assume that the lattice is deep enough to restrict tunnelling to nearest-neighbour sites and that the band gap is larger than the rotational energy. These assumptions imply that the bosonic field operator $\hat{F}$ can be expanded in Wannier orbitals confined to the first band $\hat{F}(x) = \sum_j \hat{a}_j W'_j(x)$ (Jaksch et al. 1998). Here, $W'_j(x)$ are the Wannier orbitals in the rotating frame and $\hat{a}_j$ the bosonic annihilation operator of a particle at site $j$. We recall that the Hamiltonian of a neutral particle in a frame rotating at frequency $\Omega$ around the $z$-axis, $\hat{H} = \hat{p}^2/(2M) - \Omega \hat{L}_z$, is equivalent to the Hamiltonian of a charged particle in a magnetic field along the $z$-axis, $\hat{H} = (\hat{p} - \hat{A})^2/(2M)$ with the effective vector potential, $\hat{A}(x) = M\Omega(\hat{z} \times \hat{x})$. This implies that we can first calculate...
the Wannier orbitals of the stationary lattice $W_j(x)$ and then account for the presence of the effective vector potential $A(x)$ via the gauge transformation $W'_j(x) = \exp\left((-i/h) \int_{x_j}^x A(x') \, dx'\right) W_j(x)$.  

In terms of these quantities and up to on-site diagonal terms, the many-body Hamiltonian can be written as (Bhat et al. 2006; Rey et al. 2007)

$$
\hat{H} = -\sum_{j=1}^L (J_j e^{i\theta} \hat{a}_{j+1}^\dagger \hat{a}_j + \text{H.c.}) + \frac{U}{2} \sum_{j=1}^L \hat{n}_j (\hat{n}_j - 1). 
$$

In this expression, $\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j$ is the number operator at site $j$, $\theta$ is the effective phase twist (Peierls 1933) induced by the gauge field, $\theta \equiv \int_{x_j}^{x_{j+1}} A(x') \, dx' = (M \Omega L d^2 / h)$, $J_j$ is the hopping energy between nearest-neighbour sites $j$ and $j + 1$: $J_j \equiv \int dx W_j^* [-(h^2 / 2M) \nabla^2 + V(x)] W_{j+1}$, and $U$ the on-site interaction energy: $U \equiv (4\pi a_s h^2 / M) \int dx |W_j|^4$.

(b) Superposition states in rotating ring lattices

We start our discussion with the case of a uniform ring lattice, i.e. $J_j = J$ for all $j$. To understand the effect of rotation on the atoms in the ring lattice, we write the many-body Hamiltonian (2.2) in terms of the quasi-momentum operators $\hat{b}_q = (1 / \sqrt{L}) \sum_{j=1}^L \hat{a}_j e^{-2\pi i qj / L}$, where $2\pi q / dL$ is the quasi-momentum and $q = 0, \ldots, L - 1$ an integer. In this basis, the Hamiltonian (2.2) has the form (Hallwood et al. 2006; Rey et al. 2007)

$$
\hat{H} = \hat{H}_{sp} + \hat{H}_{\text{int}} = \sum_{q=0}^{L-1} E_q \hat{b}_q^\dagger \hat{b}_q + \frac{U}{2L} \sum_{q,s,l=0}^{L-1} \hat{b}_q^\dagger \hat{b}_s^\dagger \hat{b}_l \hat{b}_{[q+s-l] \text{mod } L},
$$

where $E_q = -2J \cos(2\pi q / L - \theta)$ are the single-particle energies, and the modulus is taken because, in collision processes, the quasi-momentum is conserved up to an integer multiple of the reciprocal lattice vector $2\pi / d$, i.e. modulo Umklapp processes.

Following Hallwood et al. (2006) and Rey et al. (2007), we show in figure 1 the single-particle spectrum as a function of the phase twist $\theta$. In the absence of rotation, i.e. $\theta = 0$, the state with zero quasi-momentum $|q = 0\rangle$ is the single-particle ground state. In the rotating system, the ground state depends on the phase twist $\theta$. Writing $\theta = (2\pi / L) m + (\Delta \theta / L)$ with $m$ an integer and $0 \leq \Delta \theta < 2\pi$, the ground state is the quasi-momentum state $|q = m\rangle$ for $0 \leq \Delta \theta < \pi$ and $|q = m + 1\rangle$ for $\pi < \Delta \theta < 2\pi$.

In the absence of interactions $U = 0$, the ground state of a bosonic many-body system is the state with all $N$ bosons occupying the lowest energy single-particle state. At rotation frequencies corresponding to phase twists with $\Delta \theta = \pi$, the ground state of the single-particle Hamiltonian $\hat{H}_{sp}$ is twofold degenerate, i.e. $E_m = E_{m+1}$, so that there is an $N + 1$-dimensional degenerate subspace at the $N$-particle level (figure 2). A convenient basis for this subspace are the Fock
states $|n, N - n\rangle$ with $0 \leq n \leq N$, where $n$ particles are in the quasi-momentum state $|q = m\rangle$ with energy $E_m$ and $N - n$ particles in the quasi-momentum state $|q = m + 1\rangle$, with energy $E_{m+1}$, respectively.

For weak interactions, i.e. $U \ll J$, we can use first-order perturbation theory to account for the effect of interactions. It predicts that the energies of the states $|n, N - n\rangle$ are

$$E_n^{(1)} = nE_m + (N - n)E_{m+1} + \frac{U}{2L} [N(N - 1) + 2n(N - n)]. \quad (2.4)$$

We see from this expression that the degeneracy is lifted and the states of lowest energy are $|N, 0\rangle$ and $|0, N\rangle$. These two states are still degenerate and higher order coupling is needed to break the degeneracy (figure 3).

At this point, it is important whether the number of atoms $N$ is commensurate or incommensurate with the number of lattice sites $L$. While there are many different paths that couple the states $|N, 0\rangle$ and $|0, N\rangle$ in the commensurate case, in the incommensurate case there is no coupling between these two states and they thus remain degenerate at all orders of perturbation theory. To understand this fact, let us consider the total quasi-momentum operator $\hat{K} = (2\pi/L)\sum_{q=0}^{L-1} q \hat{b}_q^\dagger \hat{b}_q \mod L$. Since the many-body Hamiltonian (2.3) commutes with the quasi-momentum operator $\hat{K}$, i.e. $[\hat{H}, \hat{K}] = 0$, the Hamiltonian has block diagonal form if the quasi-momentum Fock states are ordered according to the
Figure 2. Many-particle spectrum versus phase twist (or rotation frequency). In the rotating ring lattice with \( L = N = 4 \) and \( t = 0.7J \) the \((N + 1)\)-fold degeneracy of the non-interacting system \((a)\) is lifted to an anti-crossing in the interacting system \((b)\) \cite{Hallwood2006}. In the LLL system with \( N = 6 \) particles, the \( N \)-fold degeneracy of the interacting isotropic \((g = 1 \text{ and } A = 0)\) case \((c)\) is lifted in the asymmetric \((g = 1 \text{ and } A = 0.03)\) case \((d)\) \cite{Dagnino2009b}.

In the commensurate case, we can construct an effective \( 2 \times 2 \) Hamiltonian by projecting the many-body Hamiltonian \((2.3)\) onto the subspace spanned by the states \(|N, 0⟩\) and \(|0, N⟩\)

\[
\hat{H}_{2\times 2} = \begin{pmatrix} E_0^{(1)} & \Delta \\ \Delta & E_N^{(1)} \end{pmatrix},
\]

where the coupling \( \Delta \) is given in the perturbation theory by

\[
\Delta = \sum \left( \frac{H_0, H_{ij} \ldots H_{pN}}{(E_0^{(1)} - \varepsilon_i)(E_0^{(1)} - \varepsilon_j) \ldots (E_0^{(1)} - \varepsilon_p)} \right) \propto \left( \frac{U}{2L} \right)^{\frac{N}{L} - 1}.
\]

In this expression, \( \bar{n} = N/L \) is the number density, the \( H_{ij} \) are transition matrix elements introduced by the interaction Hamiltonian \( \hat{H}_{\text{int}} \), and \( \varepsilon_i \) are either given by \( E_n^{(1)} \) in equation \((2.4)\) or non-interacting many-body eigen-energies depending...
Figure 3. Coupling path. The degeneracies of the many-particle spectrum are lifted by the interactions \( U \) in the case of the ring lattice and superlattice (a, b), respectively, or by the asymmetric potential \( V \) in the case of the LLL system (c). While in the ring lattice the degenerate states are coupled via non-resonant states (Hallwood et al. 2007), i.e. states separated by an energy gap \( J \), both in the superlattice and LLL cases interactions or asymmetry couple the degenerate states within themselves. In the superlattice case, however, there are diagonal couplings that introduce small energy gaps within the resonant manifold making the \(|N, 0\rangle\) and \(|0, N\rangle\) the lowest lying states (Numnenkamp & Rey 2008). These considerations explain why, in the lattice cases, the ground state is a macroscopic superposition of two quasi-momentum states, while in the LLL system it is a more complicated superposition state.

upon whether or not the intermediate states are in the \(|n, N - n\rangle\) manifold. The factor \( \tilde{n}(L - 1) \) corresponds to the minimum number of collision processes necessary to couple the states \(|N, 0\rangle\) and \(|0, N\rangle\) and the sum is taken over all possible coupling paths. In the case of the ring lattice, the coupling is exclusively though states outside the degenerate manifold (figure 3), and the coupling \( A \) decreases exponentially with increasing number of particles \( N \).

At the critical phase twist \( \Delta \theta = \pi \), we have \( E_0^{(1)} = E_N^{(1)} \) and, owing to the non-zero value of the coupling \( A \), the symmetric and anti-symmetric superpositions \(|\pm\rangle\) become the ground and first excited state separated by an energy gap \( 2A \)

\[
|\pm\rangle = \frac{|N, 0\rangle \pm |0, N\rangle}{\sqrt{2}}. \tag{2.7}
\]

In figure 4, we plot the overlap with the states \(|N, 0\rangle\) and \(|0, N\rangle\). Below and above the critical phase twist, the bosons form the non-rotating condensate \(|N, 0\rangle\) and the vortex state \(|0, N\rangle\), respectively, while at resonance the macroscopic superposition state (2.7) occurs. These Schrödinger cat states are central to high-precision spectroscopy, amplified quantum detection and measurement...
(Leibfried et al. 2004; Cappellaro et al. 2005; Lee & Khitrin 2005), where they improve the resolution by a factor of $\sqrt{N}$ with respect to the classical shot noise limit.

(c) Superposition states in rotating ring superlattices

In Nunnenkamp et al. (2008), we further explored whether the situation can be improved by introducing a lattice modulation and, instead, considered a ring superlattice, where $J_j = J$ for $j$ even and $J_j = t$ for $j$ odd. Owing to the superlattice potential, the quasi-momentum states $|q\rangle$ and $|q + L/2\rangle$ are coupled and the single-particle Hamiltonian is no longer diagonal in the quasi-momentum basis

$$\hat{H}_{sp} = \sum_{q=0}^{L/2-1} \left( \hat{b}_q^\dagger \hat{c}_q^\dagger \hat{c}_{q+L/2}^\dagger \hat{b}_{q+L/2}^\dagger \right) \begin{pmatrix} -(J + t) \cos \phi & -i(J - t) \sin \phi \\ +i(J - t) \sin \phi & +(J + t) \cos \phi \end{pmatrix} \begin{pmatrix} \hat{b}_q \\ \hat{b}_{q+L/2} \end{pmatrix}, \quad (2.8)$$

with $\phi = \theta - (2\pi q/L)$. We can diagonalize the single-particle Hamiltonian (2.8) via a unitary basis transformation $(\hat{c}_q, \hat{c}_{q+L/2}) = M_U(\hat{b}_q, \hat{b}_{q+L/2})$ and obtain

$$\hat{H}_{sp} = \sum_{q=0}^{L/2-1} \left( \hat{c}_q^\dagger \hat{c}_q \right) \begin{pmatrix} E^-_q & 0 \\ 0 & E^+_q \end{pmatrix} \begin{pmatrix} \hat{c}_q \\ \hat{c}_{q+L/2} \end{pmatrix}, \quad (2.9)$$

where the single-particle energies are given by $E^\pm_q = \pm \sqrt{J^2 + t^2 + 2Jt \cos(2\theta - \frac{4\pi q}{L})}$.

In the uniform ring, $t/J = 1$, the eigenstates of the single-particle Hamiltonian $\hat{H}_{sp}$ are quasi-momentum states. At certain phase twists $\theta$, they are doubly degenerate. For example, for $L = 4$ sites the quasi-momentum states $|q = 1\rangle$ and $|q = -1\rangle$ are degenerate at $\theta = 0$, whereas at $\theta = \pi/4$ the states $|q = 0\rangle$ and $|q = 1\rangle$
as well as \(|q = 2\) and \(|q = -1\) are degenerate. Reducing the symmetry of the ring by choosing \(t \neq J\), the quasi-momentum states that differ by \(L/2\) quasi-momentum units are coupled by the single-particle Hamiltonian (2.8), so that the quasi-momentum states \(|q = 1\) and \(|q = -1\) hybridize and the degeneracy at \(\theta = 0\) is lifted, i.e. \(E_1^+ - E_1^- = 2(J - t)\).

At \(\theta = \pi/4\), however, the degenerate quasi-momentum states are not coupled by the single-particle Hamiltonian (2.8), so that the degeneracy is also present in the non-uniform case. This remains true for arbitrary \(L\), i.e. \(E_0^- = E_{L/4}^-\) at \(\theta = \pi/4\), but for \(L \neq 4\) these states are not the ground states of the system. In figure 1 we plot the single-particle spectrum for \(L = 4\) sites as a function of the effective phase twist \(\theta\). It shows level crossings at \(\theta = \pi/4\) both for \(t = J\) as well as for \(t \neq J\). We will refer to \(\theta = \pi/4\) as the critical phase twist, since—as we will demonstrate below—weak on-site interactions lift the degeneracy at \(\theta = \pi/4\) and lead to the formation of strongly correlated states in the many-body system.

The crossing of two single-particle levels implies an \((N + 1)\)-fold degeneracy in the non-interacting many-body spectrum. In figure 2, we plot the many-body spectrum with \(L = N = 4\) and \(t/J = 0.7\) for \(U/J = 0\) and \(U/J = 0.5\) as a function of the phase twist \(\theta\) and find that interactions turn level crossings into avoided crossings.

In the weakly interacting regime, this effect can be understood by deriving an effective Hamiltonian within the \((N + 1)\)-dimensional degenerate subspace. A convenient basis for this subspace is spanned by the Fock states \(|n, N - n\rangle\) with \(0 \leq n \leq N\), where \(n\) particles are in the single-particle state of energy \(E_0^-\) and \(N - n\) particles in the one of energy \(E_{L/4}^-\), respectively. For weak interactions \(NU/L \ll 2\sqrt{J^2 + t^2}\), this subspace is the low-energy sector of the many-body problem for all phase twists \(\theta\) and tunnelling strength ratios \(t/J\). Starting from the interaction Hamiltonian (2.3), we restrict the Hilbert space to the relevant modes and keep only terms within the low-energy subspace. In this way, we obtain the effective Hamiltonian to first order in the on-site interaction strength \(U\)

\[
\hat{H}_{\text{eff}} = (E_0^- \hat{n}_0 + E_{L/4}^- \hat{n}_{L/4}) + \frac{U}{2L} (2\hat{n}_0 \hat{n}_{L/4} + N^2 - N) + \left(\frac{i\eta U}{2L} \hat{c}_0^\dagger \hat{c}_0^\dagger \hat{c}_{L/4} \hat{c}_{L/4} + \text{H.c.}\right),
\]

(2.10)

where \(\hat{n}_q = \hat{c}_q^\dagger \hat{c}_q\) are the number operators and the parameter \(\eta\) evaluated at \(\theta = \pi/4\) is given by \(\eta = (J^2 - t^2)/J^2 + t^2\). The first bracket of equation (2.10) contains the contributions from the single-particle Hamiltonian (2.9), whereas the terms in the second and third brackets arise from the on-site interaction. At the critical phase twist \(\theta = \pi/4\), the former are an unimportant zero-energy offset, whereas the terms in the second bracket shift the energies of the states in the subspace differently, e.g. they lead to an energy difference of \(U(N - 1)/L\) between the states \(|N, 0\rangle\) and \(|N - 1, 1\rangle\), while the states \(|n, N - n\rangle\) and \(|N - n, n\rangle\) remain pairwise degenerate. The terms in the third bracket are off-diagonal in the Fock basis of the subspace and describe two-particle scattering between the two single-particle modes.

Let us now determine the ground and first excited state for slightly non-uniform rings \(t/J \approx 1\), close to the critical phase twist \(\theta \approx \pi/4\). Since the terms in the second bracket of equation (2.10) increase the energy for all states in the

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subspace apart from $|N,0\rangle$ and $|0,N\rangle$ and the coupling between the states is weak (as the coupling $\eta$ is small in this limit), we project the effective Hamiltonian (2.10) onto the subspace spanned by these two nearly degenerate lowest energy states. As there is no direct coupling between $|N,0\rangle$ and $|0,N\rangle$, we calculate the total coupling through intermediate states using the perturbation theory. After eliminating the intermediate states, we obtain the following $2 \times 2$ Hamiltonian:

$$\hat{H}_{2\times2} = \begin{pmatrix} \Delta E/2 & \Delta \\ \Delta^* & -\Delta E/2 \end{pmatrix},$$ (2.11)

where $\Delta E$ is the energy difference between the states $|N,0\rangle$ and $|0,N\rangle$ caused by the detuning of the phase twist from resonance $\Delta \theta = \theta - \pi/4$, i.e. $\Delta E = N(E_{L/4}^- - E_0^-) \approx (4JtN \Delta \theta / \sqrt{J^2 + t^2})$, and $\Delta$ is the coupling between the states $|N,0\rangle$ and $|0,N\rangle$ because of the off-diagonal terms of the effective Hamiltonian (2.10). As the latter only directly couples the states $|n,N-n\rangle$ and $|n \pm 2,N-n \mp 2\rangle$, the first non-vanishing order is given by

$$\Delta = \frac{\langle N,0|\hat{H}_{\text{eff}}^{N/2}|0,N\rangle}{\prod_{j=1}^{N/2-1}(E_0^{(1)} - E_2^{(1)})} = \frac{U}{L} \cdot \left(\frac{i\eta}{2}\right)^{N/2} \cdot \frac{N!}{\prod_{j=1}^{N/2-1}(2j)^2},$$ (2.12)

with the interaction energy shift $E_n^{(1)} = (U/2L)(2n(N-n) + N^2 - N)$. Note that, in contrast to the case of the ring lattice, the perturbation couples states within the degenerate manifold (figure 3). This leads to a less severe but still exponential scaling of the gap $\Delta$ with the number of particles $N$ (Nunnenkamp et al. 2008).

The ground state of the $2 \times 2$ Hamiltonian (2.11) is similar to the one obtained above in equation (2.7), i.e.

$$\frac{\alpha|N,0\rangle + i^{N/2}\beta|0,N\rangle}{\sqrt{2}}.$$ (2.13)

We see that to obtain a cat-like superposition, i.e. $\alpha/\beta \approx 1$, the energy difference must not dominate over the coupling $|\Delta E| \ll |2\Delta|$ (see Hallwood et al. (2007) and Nunnenkamp et al. (2008) for further details).

### 3. Vortex nucleation in the lowest Landau level

In this section, we discuss the effects of rotation on ultracold bosons confined to a two-dimensional harmonic potential. We first review the results of Dagnino et al. (2009a,b) in order to then compare and contrast them with our findings on rotating ring (super)lattices that we presented in the previous section.

(a) Hamiltonian

Following Dagnino et al. (2009a,b), we consider a system of $N$ ultracold bosons with mass $M$ confined to a two-dimensional symmetric harmonic potential $V_0$ and
rotating in the $x$--$y$ plane about the $z$-axis with angular velocity $\mathcal{Q}$, then as before the many-body Hamiltonian in the rotating frame is

$$\hat{H} = \int d\mathbf{x} \hat{\Phi}^\dagger \left[ -\frac{\hbar^2}{2M} \nabla^2 + V_0(\mathbf{x}) + \frac{4\pi \hbar^2 a_s}{2M} \hat{\Phi}^\dagger \hat{\Phi} - \hat{\Omega} \hat{L}_z \right] \hat{\Phi}. \quad (3.1)$$

If we assume that the potential $V_0$ confines the motion along the $z$-axis so strongly that only the transversal motion of the atoms in the $x$--$y$ plane needs to be considered, and that along the transverse direction the atoms feel a harmonic confinement $V_0(x, y) = M \omega^2 (x^2 + y^2)/2$ and the interaction energy is small compared with the Landau level splitting $\hbar(\omega + \mathcal{Q})$, the bosonic field operator $\hat{\Phi}$ can be expanded in the LLL basis, $\hat{\Phi} = \sum_m \hat{a}_m \varphi_m(x, y)$. Here, $\varphi_m(x, y)$ are the eigenfunctions of the single-particle angular momentum operator $\hat{L}_z$ with non-negative integer eigenvalue $m$, i.e. $\varphi_m(x, y) \propto (x + iy)^m e^{-(x^2 + y^2)/2\lambda^2}$ with the magnetic length $\lambda = \sqrt{\hbar/M \omega}$. In the course of the following discussion, an asymmetry of the trapping potential in the $x$--$y$ plane will be included by adding a single-particle potential, $V(x, y) = 2AM \omega^2 (x^2 - y^2)$, where $A$ is a measure of the asymmetry. For $A \ll 1$, the LLL remains a good basis set and the asymmetry can be treated as a perturbation. Within this approximation, the many-body Hamiltonian is (Dagnino et al. 2009a, b)

$$\hat{H} = \hat{H}_0 + \hat{U} + \hat{V}$$

$$= \hbar(\omega - \mathcal{Q}) \hat{L} + \frac{g}{4\pi \lambda^2} \sum_{ijkl} \frac{(k + l)! \delta_{i+j,k+l}}{2^{k+l} \sqrt{i!j!k!l!}} \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l$$

$$+ \frac{A}{2} \lambda^2 \sum_m \sqrt{m(m - 1)} \hat{a}_m^\dagger \hat{a}_{m-2} + \sqrt{(m + 1)(m + 2)} \hat{a}_m^\dagger \hat{a}_{m+2}. \quad (3.2)$$

Here, $\hat{H}_0$ is the unperturbed single-particle Hamiltonian proportional to the total angular momentum operator $\hat{L} = \sum_m m \hat{a}_m^\dagger \hat{a}_m$, $\hat{U}$ is the two-body interaction and $\hat{V}$ is the perturbation owing to the asymmetry in the single-particle potential. In the following we use $\lambda$, $\hbar \omega$ and $\omega$ as units of length, energy and frequency, respectively.

(b) Effective many-body Hamiltonian

Analogous to the previous section, we start our discussion with the single-particle spectrum $E_m = \hbar(\omega - \mathcal{Q}) m$. In figure 1, we plot it as a function of rotation frequency $\mathcal{Q}$. We see that all states in the LLL become degenerate at $\mathcal{Q} = \omega$, but, in contrast to the rotating ring lattice, there is no single-particle level crossing for $0 \leq \mathcal{Q} < \omega$.

Nonetheless, the spectrum of $N$ non-interacting bosons is highly degenerate for any rotation frequency $\mathcal{Q}$. This is a consequence of the fact that the single-particle Hamiltonian is not only diagonal in the angular momentum basis but proportional to the total angular momentum $L$, so that all many-body states with the same total angular momentum $L$ have the same energy—a degeneracy independent of the rotation rate $\mathcal{Q}$ and only present at the many-body level. Interactions will break this huge degeneracy, but, since the interaction Hamiltonian $\hat{U}$

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commutes with the total angular momentum operator $\hat{L}$, the interacting many-body Hamiltonian remains block diagonal, i.e. it only mixes states with the same total angular momentum $L$.

Smith & Wilkin (2000) have shown that the states $|\Phi_L\rangle$ with the wavefunctions $\Phi_L(x_1, y_1, \ldots, x_N, y_N) \propto \sum_{1 \leq i_1 \leq \cdots \leq N} (z_{i_1} - z_0) \cdots (z_{i_L} - z_0) \Phi_0(x_1, y_1, \ldots, x_N, y_N)$, (3.3)

where $z_j = x_j + iy_j$ are complex coordinates in the $x-y$ plane, $z_0 = \sum_j z_j/N$ is the centre of mass, and $\Phi_0 \propto e^{-\sum_j z_j^2/2}$, i.e. the non-rotating ground-state wave function, are exact eigenstates of the interaction Hamiltonian $\hat{U}$ with eigenvalue $E_L = gN/8\pi (2N - L - 2)$. (3.4)

They also present overwhelming numerical evidence that these are the ground states for $0 \leq L \leq N$ with $L \neq 1$. When expressed in the Fock basis $|n_0, n_1, \ldots\rangle$ with occupation numbers $n_m$ for the angular-momentum basis, these are seen to be complicated superposition states. For example, the many-body state with one unit of angular momentum per particle $|\Phi_N\rangle$ is not, in fact, the vortex state $|0, N, 0, \ldots\rangle$ with all particles occupying the single-particle wave function with $m = 1$, but rather the so-called yrast state whose occupation number of the first angular-momentum state is large $\langle n_1 \rangle \approx N$, but whose overlap with the vortex state is only about one-half (Bertsch & Papenbrock 1999).

Following Dagnino et al. (2009b), we plot in figure 2 the many-body spectrum as a function of rotation frequency $\Omega$. Since the single-particle and interaction energy of the states $|\Phi_L\rangle$ depend linearly on $L$, there is a rotation frequency $\Omega_c/\omega = 1 - (gN/8\pi)$ at which all many-body states $|\Phi_L\rangle$ are degenerate. Around this rotation frequency, the ground state rapidly changes: for $\Omega < \Omega_c$ the ground state is $|\Phi_0\rangle$, while above $\Omega > \Omega_c$ it is the yrast state $|\Phi_N\rangle$.

Turning to the effect of the perturbation $\hat{V}$ on the states involved, we plot in figure 1 the single-particle spectrum as a function of rotation frequency $\Omega$. We see that, apart from a single-particle crossing close to $\Omega \approx \omega$, the perturbation has little effect on the single-particle spectrum and it turns out that this level crossing is unimportant for the discussion that follows.

The many-body spectrum, on the other hand, is degenerate at $\Omega_c$, so that any additional perturbation can effectively mix the degenerate many-body states $|\Phi_L\rangle$. Like Dagnino et al. (2009b), we show in figure 2 the many-body spectrum for $A \neq 0$, where we find that the degeneracy is lifted into a set of anti-crossings.

At first glance, the situation seems to be similar to the case of the superlattice discussed in the previous section. In both cases, the $N+1$-fold (or $N$-fold) degeneracy in a set of many-body states is lifted by an external perturbation (interactions in the superlattice case and trap asymmetry in the LLL case) that couples the various many-body states among them. This is not the case in the uniform ring lattice, where coupling takes place via non-resonant states (figure 3).

However, there are two crucial differences between the two systems. First of all, the interaction Hamiltonian $\hat{U}$ does not simply couple the degenerate states but also shifts their energies differentially (figure 3). This is why we were able to construct

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an effective $2 \times 2$ Hamiltonian $\hat{H}_{2 \times 2}$ (2.11), which couples only the two lowest lying states, and found the ground state to be a macroscopic superposition state exhibiting an energy gap that exponentially decreases with increasing number of particles (owing to virtual couplings to the adiabatically eliminated states within the resonant manifold). In contrast, the perturbation $\hat{V}$ contains only off-diagonal matrix elements and all states within the degenerate manifold have to be treated on an equal footing. This view is corroborated in figure 4, where we, like Dagnino et al. (2009a), plot the overlap of the ground state with the states $|\Phi_L \rangle$ as a function of rotation frequency $\Omega$. We see that with increasing $\Omega$ the weight of the ground state in the various $|\Phi_L \rangle$ states shifts towards higher $L$ in steps of two units of angular momentum. At the nucleation point, Dagnino et al. (2009b) find that the state of the system is well described by a complicated superposition state

$$|\Psi_0 \rangle \propto |N, 0 \rangle + |N - 2, 2 \rangle + \cdots + |2, N - 2 \rangle + |0, N \rangle,$$  

(3.5)

where $|n, m \rangle$ is the state with $n$ and $m$ atoms in the two eigenfunctions of the single-particle density matrix $n^{(1)}(r, r') = \langle \psi_0 | \hat{\psi}^{\dagger}(r) \hat{\psi}(r') | \psi_0 \rangle$ with the largest eigenvalues, respectively. The nature of the intermediate states also leads to a different scaling of the energy gap with the number of particles. As is shown by Dagnino et al. (2009b), the energy gap within the subspace at the critical rotation frequency remains finite as the number of particles increases, as opposed to the exponential scaling of the energy gap for the macroscopic superposition states in the ring lattice. Note, however, that the energy difference between the ground state within the subspaces of even and odd $L$ decreases exponentially with the number of particles (Parke et al. 2008).

The second difference between vortex nucleation in the two systems is the ground state above the critical rotation frequency. In the LLL system, it has a large overlap with the yrast state $|\Phi_N \rangle$. To compare with the superlattice system, we plot in figure 4 also the overlap between the ground state and the macroscopic occupied modes on both sides of the resonance, i.e. the condensate at rest $|N, 0, \ldots \rangle$ for $\Omega < \Omega_c$, and the vortex state $|0, N, \ldots \rangle$ for $\Omega > \Omega_c$. We see that the overlap with the corresponding states is always close to 1 for the superlattice, whereas in the LLL the overlap is only close to 1 for $\Omega < \Omega_c$. For $\Omega > \Omega_c$ is only about one-half. This is a consequence of the fact that, in the ring lattice away from the critical rotation frequency, the state $|0, N \rangle$ is the only low-lying state and therefore finite interactions lead to a small depletion of the superfluid ground state. In contrast, the non-interacting LLL system with one unit per particle is highly degenerate and interactions produce strongly correlated many-body ground-states like $|\Phi_N \rangle$.

### 4. Noise correlations

To further quantify the entanglement of the ground state at the critical frequency and to compare the different nature of the ground state in the ring lattice and LLL system, we calculate the noise correlation pattern, i.e.

$$\Delta(q, q') = \langle \hat{O}_q^{\dagger} \hat{O}_q \hat{O}_{q'}^{\dagger} \hat{O}_{q'} \rangle - \langle \hat{O}_q^{\dagger} \hat{O}_q \rangle \langle \hat{O}_{q'}^{\dagger} \hat{O}_{q'} \rangle.$$  

(4.1)
We will use quasi-momentum creation operators for the lattice case \( \hat{O}_q = \hat{b}_q \) and angular-momentum operators for the harmonically trapped system, \( \hat{O}_q = \hat{a}_q \).

Mintert et al. (2009) have argued that noise correlations are experimentally accessible quantities encoding information on entanglement. An alternative characterization of the entanglement properties of the ground state in the LLL system has been given by Liu et al. (2009), and Read & Cooper (2003) have proposed time-of-flight expansion to probe the vortex lattice in LLL systems.

For the lattice and superlattice cases, the noise interferogram shows the development of three sharp fringes, with positive and negative signs at the critical frequency. Their amplitudes are given by

\[
\Delta(0, 0) = \Delta(1, 1) = -\Delta(0, 1) = \frac{N^2}{2},
\]

heralding the development of a Schrödinger cat state at \( \Omega_c \). Away from \( \Omega_c \), the noise interference pattern disappears signaling the unentangled nature of the ground state and the macroscopic occupation of a single mode (Rey et al. 2007).

In striking contrast is the noise interferogram for the LLL system, which exhibits various sharp fringes that develop as the system is driven through the critical frequency. They signal the strong correlations and multi-mode nature of the ground state at \( \Omega_c \). Furthermore, the interferogram does not disappear for \( \Omega > \Omega_c \), behaviour that highlights the correlated nature of the yrast state \( |\Phi_N\rangle \). All these features are illustrated in figure 5, where we plot the non-zero \( \Delta(q, q') \) as a function of phase twist \( \theta \) and rotation frequency \( \Omega \), respectively.

Figure 5. Quasi-momentum (or angular-momentum) noise correlations. The two different strongly correlated ground states can be distinguished via noise correlations: in the first case (a) (Rey et al. 2007), they are non-zero only on resonance, while, in the latter (b), they are strongest on resonance but non-zero above resonance as well. This corroborates the fact that the atoms do not condense into one single-particle state but form a strongly correlated many-body state.
5. Conclusion

We have shown how a new generation of experiments can be used to examine the quantum nucleation process and its important link to entangled states of atoms. We have compared quantum vortex nucleation in rotating ring lattices and two-dimensional harmonic potentials and have shown how effective Hamiltonians can be used to describe the nearly degenerate ground-state manifold close to the rotation frequency at which the first vortex is nucleated. The degeneracy in the many-body spectrum is lifted by interactions in the case of the ring superlattice and by the asymmetry of the single-particle potential in the LLL system. In the first case, the interactions not only couple the states but also shift their energies differentially. This leads to macroscopic superpositions as low-lying states whose energy gap decreases exponentially with increasing number of particles. In contrast, in the latter case, all states contribute to the ground-state wave function and the energy gap within the subspace remains finite with increasing particle number. Finally, we showed that the two scenarios can be distinguished in noise correlation interferograms: while the ground state in a rotating ring lattice above the critical phase twist is the vortex state with vanishing noise correlations, interactions in the lowest Landau level produce the yrast state that has a non-trivial noise pattern.

A.N. thanks Steven M. Girvin for insightful discussions and careful reading of the manuscript. A.M.R. acknowledges support from NSF, NIST and the DARPA OLE programme, and A.N. acknowledges support from NSF DMR-0603369.

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