On degree-colorings of multigraphs

Mark K. Goldberg
Department of Computer Science,
Rensselaer Polytechnic Institute
Troy, NY, 12180.
goldbm4@rpi.edu

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Abstract

A notion of degree-coloring is introduced; it captures some, but not all properties of
standard edge-coloring. We conjecture that the smallest number of colors needed for degree-
coloring of a multigraph \( G \) [the degree-coloring index \( \tau(G) \)] equals \( \max\{\Delta, \omega\} \), where \( \Delta \) and
\( \omega \) are the maximum vertex degree in \( G \) and the multigraph density, respectively. We prove
that the conjecture holds iff \( \tau(G) \) is a monotone function on the set of multigraphs.

1 Introduction.

The chromatic index \( \chi'(G) \) of a multigraph \( G(V, E) \) is the minimal number of colors
(positive integers) that can be assigned to the edges of \( G \) so that no two adjacent edges
receive the same color. Clearly, \( \Delta(G) \leq \chi'(G) \), where \( \Delta(G) \) is the maximal vertex degree
in \( G \). The famous result by Vizing [10] establishes \( \chi = \chi'(G) \leq \Delta(G) + p(G) \), where
\( p(G) \) is the maximal number of parallel edges in \( G \). For graphs, in particular, \( \Delta(G) \leq \chi'(G) \leq \Delta(G) + 1 \). The problem of computing the exact value of the chromatic index was
proved by Holyer [5] to be NP-hard even for cubic graphs. It is suspected that for every
multigraph with \( \chi'(G) > \Delta(G) + 1 \), its chromatic index is determined by the parameter
\( \omega(G) \), called the multigraph density:

\[
\omega(G) = \max_{H \subseteq G} \left[ \frac{e(H)}{\left\lfloor v(H)/2 \right\rfloor} \right],
\]

where \( H \) is a sub-multigraph of \( G \), and \( v(H) \) (resp. \( e(H) \)) denotes the number of vertices
(resp. edges) in \( H \). It is easy to prove that \( \omega(G) \leq \chi'(G) \) for every multigraph \( G \).
Seymour in [7] and Stahl in [8] proved the equality \( \max(\Delta(G), \omega(G)) = \chi^*(G) \), where
\( \chi^*(G) \) is the the fractional chromatic index of \( G \), known to be polynomially computable
(see [6]).

The following variation of the multigraph density idea was considered in [3]. Let \( \pi(F) \)
denote the size of a maximum matching composed of the edges in a set \( F \subseteq E \). Denote
\[ \omega^*(G) = \max_{F \subseteq E} \left\lceil \frac{|F|}{\pi(F)} \right\rceil. \text{ Then, it is easy to see that} \]

\[ \omega(G) \leq \omega^*(G) \leq \chi'(G). \]

It turns out (see [3]) that \( \omega^*(G) = \max(\Delta(G), \omega(G)) \).

Conjectures connecting \( \chi'(G), \omega(G), \) and \( \Delta(G) \) were independently proposed by Goldberg ([1]) and Seymour ([7]) more than 30 years ago ([8], [9]). Currently, the strongest variation of the conjecture ([2]) is as follows:

**Conjecture 1** If \( \Delta(G) \neq \omega(G) \), then \( \chi'(G) = \max(\Delta(G), \omega(G)) \), else \( \chi'(G) \leq \Delta(G) + 1 \).

Every edge-coloring with colors 1, 2, \ldots, \( c \) yields an assignment \( \mu : V \to 2^{[1,c]} \), where for every \( x \in V \), \( \mu(x) \) denotes the set of colors used on the edges incident to \( x \). Given \( S \subseteq V \) and \( i \in [1,c] \), the set of vertices \( x \in S \) such that \( i \in \mu(x) \) is denoted \( S^{(i)}(\mu) \). It is easy to prove that the assignment \( \mu \) originated by an edge-coloring using colors 1, \ldots, \( c \) satisfies the following three conditions:

**Degree condition:** \( \forall x \in V(G), |\mu(x)| = \deg_G(x) \);

**Cover condition:** \( \forall S \subseteq V, |E(S)| \leq \sum_{i=1}^{c} \left\lfloor \frac{|S^{(i)}(\mu)|}{2} \right\rfloor \);

**Matching condition:** \( \forall i \in [1,c] \), the submultigraph induced on \( V^{(i)} \) either has a perfect matching, or is empty.

**Definition 1** An assignment \( \mu : V(G) \to 2^{[1,c]} \) satisfying the degree and the cover conditions is called a degree-coloring.

Straightforward checking of the assignment presented in the Figure below shows that the assignment is a degree-coloring of the multicycle \( C \). However, it is not originated by any edge-coloring of \( C \), since the submultigraph of \( C \) induced on \( V^{(6)} \) has no perfect matching.

Let \( \tau(G) \) denote the smallest integer \( c \) for which a degree-coloring of \( G \) exists. It is easy to prove

**Lemma 1** \( \max(\Delta(G), \omega(G)) \leq \tau(G) \leq \chi'(G) \).

**Conjecture 2** (the \( \tau \)-conjecture): For every multigraph \( G \), \( \tau(G) = \max(\Delta(G), \omega(G)) \).
A real-valued function $\kappa(G)$ defined on the set of multigraphs is called **monotone** if for any multigraph $G$ and any submultigraph $H \subseteq G$, $\kappa(H) \leq \kappa(G)$. Clearly, $\Delta(G)$ and $\omega(G)$ are monotone functions.

**Conjecture 3** *The degree-coloring index $\tau(G)$ is a monotone function on multigraphs.*

It is easy to see that Conjecture 2 implies Conjecture 3. We prove in this paper that the reverse is also true: the monotonicity of $\tau(G)$ implies conjecture 2.

We use the standard graph-theoretical terminology which can be found in [11].

2 **Monotonicity of $\tau(G)$ and the $\tau$-conjecture.**

It is easy to construct a $\tau(G)$-degree-coloring for a regular multigraph $G$ with $\omega(G) \leq \Delta(G)$.

**Lemma 2** *If $G$ is a $\Delta$-regular multigraph, and $\omega(G) \leq \Delta$, then $\tau(G) = \Delta$.***

**Proof.** From the definition, $\tau(G) \geq \Delta$. Consider the following assignment:

$$\forall x \in V(G), \mu(x) = \{1, 2, \ldots, \Delta\}.$$

Given $S \subseteq V(G), \forall i \in [1, \Delta], S^{(i)}(\mu) = S$. Thus,

$$\sum_{i=1}^{\Delta} \left\lfloor \frac{|S^{(i)}(\mu)|}{2} \right\rfloor = \left\lfloor \frac{|S|}{2} \right\rfloor \Delta.$$

Since $\omega(G) \leq \Delta$, for any $S \subseteq V$, $\left\lfloor \frac{|S|}{2} \right\rfloor \Delta \geq \left\lfloor \frac{|S|}{2} \right\rfloor \omega(G) \geq |E(S)|$ implying $\tau(G) = \Delta$.  

Constructing a degree-coloring for a non-regular multigraph can be done via operation Regularization which, for every multigraph $G$, creates a regular multigraph $R(G)$ containing $G$ as an induced sub-multigraph.

**Regularization:** If a multigraph $G$ is regular and $\omega(G) \leq \Delta(G)$, then $R(G) = G$; else

1. generate a disjoint isomorphic copy $G' = (V', E')$ of $G(V, E)$ with an isomorphic mapping $f : V \rightarrow V'$ from $G$ onto $G'$;
2. let $V(R(G)) = V \cup V'$ and initialize $E(R(G))$ by setting $E(R(G)) = E(G) \cup E(G')$;
3. $\forall x \in V$, add $\max(\Delta(G), \omega(G)) - \deg(x)$ new edges $xf(x)$ to $E(R(G))$.

**Lemma 3** *$\forall G, \omega(G) \leq \omega(R(G)) \leq \max(\omega(G), \Delta(G))$ and $\Delta(R(G)) = \max(\Delta(G), \omega(G))$.***
Proof. If \( G = R(G) \), the lemma is obvious. Let \( G \neq R(G) \). Denote \( \Delta = \Delta(G) \), \( \omega = \omega(G) \), and \( \rho = \max(\Delta, \omega) \). Obviously, \( \Delta(R(G)) = \rho \) and \( \omega \leq \omega(R(G)) \).

To prove \( \omega(R(G)) \leq \rho \), denote \( R = R(G) \), \( V(R) = V_1 \cup V_2 \), where \( V_1 = V(G) \) and \( V_2 = V(G') \). Let \( f \) be an isomorphic mapping from \( V_1 \) onto \( V_2 \). Given \( S \subseteq V(R) \), let \( S_1 = S \cap V_1 \), \( S_2 = S \cap V_2 \), \( S' = S_1 \cap f^{-1}(S_2) \), and \( S'' = S_2 \cap f(S_1) \). Note that \( |S'| = |S''| \) and \( |E(S')| = |E(S'')| \).

![Diagram](image)

Then
\[
|E(S)| = |E(S_1)| + |E(S_2)| + \sum_{x \in S'} (\rho - \deg_G(x))
\]
\[
= |E(S_1 - S')| + |E(S_1 - S', S')| + |E(S')| + |E(S_2 - S'')| + |E(S_2 - S'', S'')| + |E(S'')| + |S'|\rho - \sum_{x \in S'} \deg_G(x).
\]

It is easy to check that
\[
|E(S_1 - S', S')| + |E(S')| + |E(S_2 - S'', S'')| + |E(S'')| =
|E(S_1 - S', S')| + |E(S_2 - S'', S'')| + 2|E(S')| \leq \sum_{x \in S'} \deg_G(x),
\]

which yields the following upper bound
\[
|E(S)| \leq |E(S_1 - S')| + |E(S_2 - S'')| + |S'|\rho
\]
\[
\leq \left( \frac{|S_1| - |S'|}{2} \right) \rho + \left( \frac{|S_2| - |S''|}{2} \right) \rho + |S'|\rho.
\]

To prove
\[
\left( \frac{|S_1| - |S'|}{2} \right) \rho + \left( \frac{|S_2| - |S''|}{2} \right) \rho + |S'|\rho \leq \left( \frac{|S_1| + |S_2|}{2} \right) \rho,
\]

note that it is straightforward if \( |S_1| + |S_2| \) is even. If \( |S_1| + |S_2| \) is odd, one out of two integers \( |S_1| - |S'| \) and \( |S_2| - |S''| \) is even and one is odd. Thus,
\[
\left( \frac{|S_1| - |S'|}{2} \right) \rho + \left( \frac{|S_2| - |S''|}{2} \right) \rho + |S'|\rho = \frac{|S_1|}{2} \rho + \frac{|S_2|}{2} \rho - \frac{1}{2} \rho.
\]
Since $|S_1| + |S_2|$ is odd,
\[
\left\lfloor \frac{|S_1| + |S_2|}{2} \right\rfloor \rho = \frac{|S_1| + |S_2|}{2} \rho - \frac{1}{2} \rho,
\]
which implies the result. 

**Theorem 1** If function $\tau(G)$ is monotone on the set of all multigraphs, then for any multigraph $G$,
\[
\tau(G) = \max\{\Delta(G), \omega(G)\}.
\]

**Proof.** By Lemma 1, $\max\{\Delta(G), \omega(G)\} \leq \tau(G)$. On the other hand, since $G \subseteq R(G)$, it follows from Lemma 2 that $\tau(G) \leq \tau(R(G)) = \max\{\Delta(G), \omega(G)\}$. 

**References**

[1] M. K. Goldberg, 1973, *On Multigraphs of Almost Maximal Chromatic Class*, Discret. Analiz, vol 23, pp. 3-7, In Russian

[2] M. K. Goldberg, 1984, *Edge coloring of multigraphs: Recoloring Technique* J. Graph Theory, vol 8, pp. 123-137.

[3] M. K. Goldberg, 2007, *Clusters in a multigraph with elevated density* The Electronic Journal of Combinatorics, vol 14(1), num R10.

[4] P. E. Haxell and H. A. Kierstead, 2015, *Edge coloring multigraphs without small dense subsets* Discrete Mathematics, vol 338, pp. 2502-2506.

[5] I.J. Holyer, 1981, *The NP-completeness of edge coloring* SIAM J. Comput., vol. 10, pp.718-720.

[6] E. R. Scheinerman and D. H. Ullman, 1997, *Fractional Graph Theory*, John Wiley & Sons, Inc.

[7] P. D. Seymour, 1979, *Some unsolved problems on one-factorizations of graphs*, Graph Theory and Related Topics, Academic Press”, Bondy and Murty, eds.

[8] S. Stahl, 1979, *Fractional edge colorings* Cahiers Center Etudes Rech. Oper., vol 21, pp. 127-131.

[9] Michael Steibitz, Diego Schide, Bjarne Toft, and Lene M. Favrholdt. 2012, *Graph Edge Coloring, WILEY Series in Discrete Mathematics and Optimization*

[10] V. G. Vizing. 1965, *Critical graphs with a given chromatic class* Discret. Analiz, volume 5, pp. 9-17, in Russian.

[11] D. B. West, 2003, *Introduction to Graph Theory* Prentice Hall, Upper Saddle River, NJ.