Entangled bases with fixed Schmidt number

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Abstract
An entangled basis with a fixed Schmidt number \(k\) (EB\(k\)) is a set of orthonormal basis states with the same Schmidt number \(k\) in a product Hilbert space \(\mathbb{C}^d \otimes \mathbb{C}^{d'}\). It is a generalization of both the product basis and the maximally entangled basis. We show here that for any \(k \leq \min\{d, d'\}\), EB\(k\) exists in \(\mathbb{C}^d \otimes \mathbb{C}^{d'}\) for any \(d\) and \(d'\). Consequently, general methods of constructing SEB\(k\) (EB\(k\) with the same Schmidt coefficients) and EB\(k\) (but not SEB\(k\)) are proposed. Moreover, we extend the concept of EB\(k\) to multipartite cases and find out that the multipartite EB\(k\) can be constructed similarly.

Keywords: entangled basis, Schmidt number, pure state

1. Introduction

Entanglement is a fundamental feature of quantum physics, and it is also proven to be a central resource in quantum information and quantum computation [1, 2]. Consequently, characterizing entanglement is one of the fundamental problems in this field. One indispensable approach is to analyze the basis of the state space, such as the unextendible product basis (UPB) [3], the unextendible maximally entangled basis (UMEB) [4], and the unextendible entangled basis with Schmidt number \(k\) (UEB\(k\)) [5], etc.

Bipartite bases (complete or incomplete) in \(\mathbb{C}^d \otimes \mathbb{C}^{d'}\) are a fundamental problem in both quantum physics and mathematics. The unextendible product bases is a set of orthogonal product vectors whose complementary space does not contain product states, which can be used for constructing bound entangled states [6]. A UMEB is a set of orthonormal maximally entangled pure states in a two-qudit system consisting of fewer than \(d^2\) members which have no other maximally entangled vectors orthogonal to all of them. It is shown that there is no UMEB in the two-qubit system, a six-member UMEB exists in \(\mathbb{C}^3 \otimes \mathbb{C}^3\) and a 12-member
UMEB exists in $\mathbb{C}^d \otimes \mathbb{C}^d$ [4]. Later, Chen and Fei proved in [7] that a set of $d^2$-member UMEB exists in $\mathbb{C}^d \otimes \mathbb{C}^d$ ($d' < d < d'$) and questioned the existence of UMEBs in the case of $d' \geq 2d$. Recently, in [8], the authors proved that there might be two sets of UMEBs in any bipartite system, and an explicit construction of UMEBs is put forward. Some properties of UMEBs are given in [9].

One of the most crucial quantities associated with the bipartite pure state is the Schmidt number, which can be used to characterize and quantify the degree of bipartite entanglement for pure states directly [10, 11]. In [5], we introduced the concept of unextendible entangled basis with Schmidt number $k$ ($2 \leq k < \min\{d, d'\}$) (UEBk) and proposed a general way of constructing a UEBk with arbitrary $d$ and $d'$. Consequently, it is shown that there are at least $k - r$ (here $r = d \mod k$, or $r = d' \mod k$) different UEBk when $d$ or $d'$ is not the multiple of $k$, while there are at least $2(k - 1)$ UEBk when both $d$ and $d'$ are the multiples of $k$. UEBk can be considered as a generalization of both UPB and UMEB from different directions. In general, a UEBk in a $d \times d'$-dimensional composite Hilbert space is not a complete basis, as it contains less than $dd'$ states; in order to have a complete basis, one needs to include states that have Schmidt numbers different from $k$.

Instead of investigating an unextendible basis that is actually an incomplete basis, here we study a complete basis that contains a complete set of basis states with the same Schmidt number. In [12, 13], it is shown that an MEB (maximally entangled basis) exists in $\mathbb{C}^d \otimes \mathbb{C}^d$ for any $d$. We remark that the product basis (PB) always exists in $\mathbb{C}^d \otimes \mathbb{C}^d$ for arbitrary $d$ and $d'$ and it is a set of pure states whose Schmidt numbers are 1. Furthermore, the MEB is a set of pure states whose Schmidt numbers are $d$. Then, a related question arises: is there an entangled basis with any fixed Schmidt number in $\mathbb{C}^d \otimes \mathbb{C}^d$? In this paper, we show that such bases exist in both $\mathbb{C}^d \otimes \mathbb{C}^d$ and $\mathbb{R}^d \otimes \mathbb{R}^d$ for any $d$ and $d'$, and we provide methods to construct them. We also show that our methods can be directly extended to the multipartite case. Such bases could be useful when one studies projective measurements onto bases states with a fixed Schmidt number.

The material in this paper is arranged as follows. In section 2, we introduce the concepts of EBk, SEBk and MEB. In addition the relation between EBk and the rank-$k$ Hilbert–Schmidt bases of the associated matrix space is illustrated. Section 3 contains methods of constructing EBk whenever $dd'$ is a multiple of $k$. In section 4, we discuss the case when $dd'$ is not a multiple of $k$ by analyzing the Hilbert–Schmidt basis with fixed rank $k$ in the space of coefficient matrices. Both EBk and SEBk are discussed. The multipartite case is discussed in section 5. Finally, we conclude in section 6.

2. Definition and preliminary

Recall that the Schmidt number of a pure state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$, denoted by $S_r(|\psi\rangle)$, is defined as the length of the Schmidt decomposition [14]: if $|\psi\rangle = \sum_{k=1}^{m} \lambda_k |e_k \rangle |e_k \rangle$ is its Schmidt decomposition, then $S_r(|\psi\rangle) = m$. It is clear that $S_r(|\psi\rangle) = \text{rank}(\rho_1) = \text{rank}(\rho_2)$, where $\rho_i$ denotes the reduced state of the $i$th part. A state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ is called a maximally entangled state if $S_r(|\psi\rangle) = d$ and $\lambda_1 = \lambda_2 = \cdots = \lambda_d$. Hereafter, we always assume that $d \leq d'$ for simplicity.

Definition 1. An orthonormal basis $\{|\phi_i\rangle\}$ in $\mathbb{C}^d \otimes \mathbb{C}^d$ is called an entangled basis with Schmidt number $k$ (EBk) if $S_r(|\phi_i\rangle) = k \geq 2$ for any $i$. Particularly, it is called a special entangled basis with Schmidt number $k$ (SEBk) if it is an EBk with all the Schmidt
coefficients of $|\phi_i\rangle$ is equal to $1/\sqrt{2}$, and it is called a maximally entangled basis (MEB) if $|\phi_i\rangle$ is maximally entangled for any $i$.

PB always exists in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ for any $d$ and $d'$. It is clear that SEB reduces to PB (MEB) when $k = 1$ ($k = d$). That is, SEB is a generalization of both PB and MEB. MEB is equal to SEBd when an EBd is not necessarily an MEB. By definition, it is clear that $\{|\phi_i\rangle\rangle\}$ is an MEB if and only if $\{|U_i \otimes U_i |\phi_i\rangle\rangle\}$ is an EB, where $U_{iab}$ is any unitary operator on $\mathbb{C}^{dd}$. In addition, it is worth noting that the EB is far different from the UEB introduced in [5] since an EB is always a complete basis while a UEB is just a special non-completed set of orthonormal states in the associated Hilbert space, namely, a UEB is not a basis in the strict sense.

We assume that $\{|\psi_i\rangle\rangle_{i=0}^{dd-1}$ is a set of pure states in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$. Let 

$$|\psi_i\rangle = \sum_{k,l} a_{kl}^{(i)} |k\rangle |l\rangle,$$

where $|k\rangle$ and $|l\rangle$ are the standard computational bases of the first $\mathbb{C}^d$ and the second $\mathbb{C}^{d'}$, respectively. We write 

$$A_i = \left[ a_{kl}^{(i)} \right],$$

then $A_i$ is a $d$ by $d'$ matrix, $S_r(|\psi_i\rangle) = \text{rank}(A_i)$ and $\langle \psi_i | \psi_j \rangle = \text{Tr}(A_i^\dagger A_j)$. Let $\mathcal{M}_{ddx^{d'}}$ be the space of all $d$ by $d'$ complex matrices. Then $\mathcal{M}_{ddx^{d'}}$ is a Hilbert space with the inner product defined by $\langle A|B\rangle = \text{Tr}(A^\dagger B)$ for any $A, B \in \mathcal{M}_{ddx^{d'}}$. It turns out that $\{A_i; \text{rank}(A_i) = k\}$ is a Hilbert–Schmidt basis of the space of $\mathcal{M}_{ddx^{d'}}$ if and only if $\{|\psi_i\rangle\rangle_{i=0}^{dd-1}$ is an EB in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$. For simplicity, we call $\{A_i; \text{rank}(A_i) = k, \text{Tr}(A_i^\dagger A_j) = \delta_{ij}\}$ a rank-$k$ basis in the following. That is, there is a one-to-one relation between the EB $\{|\psi_i\rangle\rangle\}$ and the rank-$k$ basis $\{A_i\}$:

$$|\psi_i\rangle \leftrightarrow A_i, \quad \{|\psi_i\rangle\rangle\} \leftrightarrow \{A_i\}. \quad (1)$$

Therefore, the EB problem is equivalent to the rank-$k$ basis of the associated matrix space. The important role of equation (1) will be seen in section 4 for the case when $dd'$ is not a multiple of $k$, although the case when $dd'$ is a multiple of $k$ in the next section can be worked out without using equation (1).

### 3. $dd'$ is a multiple of $k$

This section is divided into three cases. We first discuss the case $k = d = d'$, then consider the case $k = d < d'$, and finally study the case $k < d \leq d'$. The methods of constructing EB can be induced from the structure of the MEB.

#### 3.1. $k = d = d'$

For the two-qubit case, the Bell basis states, $(1/\sqrt{2})(|0\rangle |0\rangle + |1\rangle |1\rangle)$, $(1/\sqrt{2})(|0\rangle |0\rangle - |1\rangle |1\rangle)$, $(1/\sqrt{2})(|0\rangle |1\rangle + |1\rangle |0\rangle)$, and $(1/\sqrt{2})(|0\rangle |1\rangle - |1\rangle |0\rangle)$ form an MEB. In general, for the case of $d \otimes d$, $d \geq 2$, let

$$|\Omega_{0,0}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle |i\rangle$$
and
\[ \hat{W}_{m,n} |l\rangle = \xi^{m(i-n)} |i-n\rangle, \]  
where \( \hat{W}_{m,n} \)'s are the Weyl operators, \( \xi = e^{2\pi i} \) and \( |i-n\rangle \equiv |i-n+d\rangle \) (here, the roman letter \( i \) denotes the imaginary unit, \( i - n + d \) means \( i - n + d \mod d \)). Then the actions of the Weyl operators produce an MEB \([12, 13]\):
\[ |\Omega_{m,n}\rangle = (\hat{W}_{m,n} \otimes 1) |\Omega_{0,0}\rangle, \]
where \( 0 \leq m, n \leq d - 1 \). In fact, equation (2) can be simplified as
\[ \hat{W}_{m,n} |l\rangle = \xi^{m} |i-n\rangle. \]  
Then
\[ |\Omega_{m,n}\rangle = (\hat{W}_{m,n} \otimes 1) |\Omega_{0,0}\rangle \]
with \( 0 \leq m, n \leq d - 1 \) also form an MEB in \( \mathbb{C}^d \otimes \mathbb{C}^d \). Symmetrically, both \( (1 \otimes \hat{W}_{m,n}) |\Omega_{0,0}\rangle \) and \( (1 \otimes \hat{W}_{m,n}) |\Omega_{0,0}\rangle \) induce an MEB as well.

3.2. \( k = \frac{d}{d'} \)

We begin with the simple case of \( \otimes 2 \otimes 3 \). It is clear that
\[
\begin{align*}
|\phi_1\rangle &= \frac{1}{\sqrt{2}} (|0\rangle|0\rangle + |1\rangle|1\rangle), \\
|\phi_2\rangle &= \frac{1}{\sqrt{2}} (|0\rangle|0\rangle - |1\rangle|1\rangle), \\
|\phi_3\rangle &= \frac{1}{\sqrt{2}} (|0\rangle|1\rangle + |1\rangle|2\rangle), \\
|\phi_4\rangle &= \frac{1}{\sqrt{2}} (|0\rangle|1\rangle - |1\rangle|2\rangle), \\
|\phi_5\rangle &= \frac{1}{\sqrt{2}} (|0\rangle|2\rangle + |1\rangle|0\rangle), \\
|\phi_6\rangle &= \frac{1}{\sqrt{2}} (|0\rangle|2\rangle - |1\rangle|0\rangle)
\end{align*}
\]
is an MEB in \( \mathbb{C}^2 \otimes \mathbb{C}^3 \). In general, in a \( d \otimes d' \) \( (d < d') \) system, let
\[ |\Omega_{0,0}\rangle = \frac{1}{\sqrt{d'}} \sum_{i=0}^{d-1} |i\rangle|i\rangle \]
and
\[ \hat{W}_{m,n} |i\rangle = \xi^{m} |(i-n)'\rangle, \]
where \( \xi = e^{2\pi i} \) and \( |i-n\rangle \equiv |i-n+d'\rangle \) (here \( i - n + d' \) means \( i - n + d' \mod d' \)). Then
\[ |\Omega_{m,n}\rangle = (1 \otimes \hat{W}_{m,n}) |\Omega_{0,0}\rangle \]
with \( 0 \leq m \leq d - 1 \) and \( 0 \leq n \leq d' - 1 \) induce an MEB in \( \mathbb{C}^d \otimes \mathbb{C}^{d'} \).

That is, SEBd always exists in \( \mathbb{C}^d \otimes \mathbb{C}^{d'} \) with \( d \leq d' \). We can also show that there is an EBd but not an SEBd in \( \mathbb{C}^d \otimes \mathbb{C}^{d'} \). Let
then \( O_d \) is a \( d \) by \( d \) orthogonal matrix and \( U_d = iO_d \) is a unitary matrix. Replacing the coefficients in equations (5) and (7) by the entries in the columns (or rows) of the \( U_d \) or \( O_d \) respectively, one can verify the existence of \( EB_d \) in any \( d \otimes d' \) system.

In fact, any \( d \) by \( d \) isometric matrix \( X = [x_{ij}] \) without zero entries can induce an \( EB_d \) since we can replace the coefficients in equations (5) and (7) by the entries in the columns of any \( d \) by \( d \) isometric matrix without zero entries. That is, we replace the coefficient \( \xi^{m_i} \) of \( \xi^{mi} |i \rangle \langle n - i \rangle \) by \( x_{m_i+1,m+1} \). Here, an \( m \) by \( n \) matrix \( A \) is an isometric matrix if \( A^H A = I_n \), \( I_n \) is the \( n \) by \( n \) identity matrix. Since there are infinitely many isometric matrices, we can construct infinitely many \( EB_d \)s. Also note that there are many ways of constructing \( MEB \) since we can replace the coefficients \( \xi^{m_i} \) by any other ones that guarantee the orthogonality.

### 3.3. \( k < d \leq d' \)

We now consider the \( EB_k \) in a \( d \otimes d' \) system with \( dd' \) a multiple of \( k \), and \( k < d \leq d' \). For clarity, we give an example of \( EB_3 \) in \( \mathbb{C}^4 \otimes \mathbb{C}^6 \). Let \( \xi = e^{\frac{2\pi}{3}} \), it is obvious that the following states constitute an \( EB_3 \) in a \( 4 \otimes 6 \) system.

\[
\begin{align*}
|\phi_0\rangle &= \frac{1}{\sqrt{3}} (|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle), \\
|\phi_1\rangle &= \frac{1}{\sqrt{3}} (|0\rangle|0\rangle + \xi^1 |1\rangle|1\rangle + \xi^2 |2\rangle|2\rangle), \\
|\phi_2\rangle &= \frac{1}{\sqrt{3}} (|0\rangle|0\rangle + \xi^2 |1\rangle|1\rangle + \xi^4 |2\rangle|2\rangle), \\
|\phi_3\rangle &= \frac{1}{\sqrt{3}} (|3\rangle|3\rangle + |0\rangle|1\rangle + |1\rangle|2\rangle), \\
|\phi_4\rangle &= \frac{1}{\sqrt{3}} (|3\rangle|3\rangle + \xi |0\rangle|1\rangle + \xi^2 |1\rangle|2\rangle), \\
|\phi_5\rangle &= \frac{1}{\sqrt{3}} (|3\rangle|3\rangle + \xi^2 |0\rangle|1\rangle + \xi^4 |1\rangle|2\rangle), \\
|\phi_6\rangle &= \frac{1}{\sqrt{3}} (|2\rangle|3\rangle + |3\rangle|4\rangle + |0\rangle|1\rangle), \\
|\phi_7\rangle &= \frac{1}{\sqrt{3}} (|2\rangle|3\rangle + \xi |3\rangle|4\rangle + \xi^2 |0\rangle|1\rangle), \\
|\phi_8\rangle &= \frac{1}{\sqrt{3}} (|2\rangle|3\rangle + \xi^2 |3\rangle|4\rangle + \xi^4 |0\rangle|1\rangle), \\
|\phi_9\rangle &= \frac{1}{\sqrt{3}} (|1\rangle|3\rangle + |2\rangle|4\rangle + |3\rangle|5\rangle).
\end{align*}
\]
\[ \psi_{10} = \frac{1}{\sqrt{3}} \left( |1\rangle|3\rangle + \xi |2\rangle|4\rangle + \xi^2 |3\rangle|5\rangle \right), \]
\[ \psi_{11} = \frac{1}{\sqrt{3}} \left( |1\rangle|3\rangle + \xi^2 |2\rangle|4\rangle + \xi |3\rangle|5\rangle \right), \]
\[ \psi_{12} = \frac{1}{\sqrt{3}} \left( |0\rangle|3\rangle + |1\rangle|4\rangle + |2\rangle|5\rangle \right), \]
\[ \psi_{13} = \frac{1}{\sqrt{3}} \left( |0\rangle|3\rangle + \xi |1\rangle|4\rangle + \xi^2 |2\rangle|5\rangle \right), \]
\[ \psi_{14} = \frac{1}{\sqrt{3}} \left( |0\rangle|3\rangle + \xi^2 |1\rangle|4\rangle + \xi |2\rangle|5\rangle \right), \]
\[ \psi_{15} = \frac{1}{\sqrt{3}} \left( |1\rangle|0\rangle + |0\rangle|4\rangle + |1\rangle|5\rangle \right), \]
\[ \psi_{16} = \frac{1}{\sqrt{3}} \left( |3\rangle|0\rangle + \xi |0\rangle|4\rangle + \xi^2 |1\rangle|5\rangle \right), \]
\[ \psi_{17} = \frac{1}{\sqrt{3}} \left( |3\rangle|0\rangle + \xi^2 |0\rangle|4\rangle + \xi |1\rangle|5\rangle \right), \]
\[ \psi_{18} = \frac{1}{\sqrt{3}} \left( |2\rangle|0\rangle + |3\rangle|1\rangle + |0\rangle|5\rangle \right), \]
\[ \psi_{19} = \frac{1}{\sqrt{3}} \left( |2\rangle|0\rangle + \xi |3\rangle|1\rangle + \xi^2 |0\rangle|5\rangle \right), \]
\[ \psi_{20} = \frac{1}{\sqrt{3}} \left( |2\rangle|0\rangle + \xi^2 |3\rangle|1\rangle + \xi |0\rangle|5\rangle \right), \]
\[ \psi_{21} = \frac{1}{\sqrt{3}} \left( |1\rangle|0\rangle + |2\rangle|1\rangle + |3\rangle|2\rangle \right), \]
\[ \psi_{22} = \frac{1}{\sqrt{3}} \left( |1\rangle|0\rangle + \xi |2\rangle|1\rangle + \xi^2 |3\rangle|2\rangle \right), \]
\[ \psi_{23} = \frac{1}{\sqrt{3}} \left( |1\rangle|0\rangle + \xi^2 |2\rangle|1\rangle + \xi |3\rangle|2\rangle \right). \]

With the same spirit in mind, in general, we let
\[ |\psi_i\rangle = \begin{cases} |i\rangle\bar{\psi}', & 0 \leq i < d, \\ |\psi\rangle(t \oplus r)', & i = td + r, 0 \leq r < d, \end{cases} \]
where \( t \oplus r \) means \( t + r \mod d' \). If \( dd' = sk, 2 \leq k < d \), then
\[ |\tilde{\psi}_{m,n}\rangle = \frac{1}{\sqrt{k}} \sum_{i=0}^{k-1} \xi^{ml} |\psi\rangle_{nk+i} \] (8)
constitute an EBk, where \( \xi = e^{2\pi i}, 0 \leq m \leq k - 1, 0 \leq n \leq s - 1 \).

The EBk above is also an SEBk. In order to construct an EBk that is not an SEBk, one can replace the coefficients in equation (8) by the entries of any \( k \) by \( k \) isometric matrix \( X = [x_{ij}] \) without zero entries, namely, we can replace the coefficient \( \xi^{ml} \) in equation (8) by \( x_{(i+m+1)} \). From the discussion in subsections 3.1 and 3.2 we know that we can construct infinitely many such EBks that are not SEBks, and we also know that there are other methods of constructing SEBks.
4. \(dd'\) is not a multiple of \(k\)

In this section, we assume that \(dd'\) is not a multiple of \(k\). We discuss an EB\(k\) that is not an SEB\(k\) firstly and then deal with SEB\(k\). We begin with the case \(k = 2\). If \(dd'\) is not a multiple of 2, then the Hilbert space \(M_{dd'}\) is a direct sum of three subspaces which are equivalent to \(M_{2p \times 3}, M_{2q \times 2}\) and \(M_{3 \times 3}\) respectively, with \(p, q \geq 1\). For example, the space of \(M_{3 \times 5}\) is a direct sum of \(M_{3 \times 2}\) and \(M_{3 \times 3}\):

\[
\begin{pmatrix}
* & * & * & * & * \\
* & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
* & * & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \oplus \begin{pmatrix}
0 & 0 & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

The space of \(M_{5 \times 5}\) is a direct sum of \(M_{5 \times 2}, M_{1 \times 3}\) and \(M_{3 \times 3}\):

\[
\begin{pmatrix}
* & * & * & * & * \\
* & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
* & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \oplus \begin{pmatrix}
0 & 0 & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Since rank-2 bases always exist in \(M_{2p \times 3}\) and \(M_{1 \times 2q}\), we only need to discuss \(M_{3 \times 3}\). Observe that

\[
\begin{pmatrix}
* & * & * \\
* & * & * \\
* & * & * \\
\end{pmatrix} = \begin{pmatrix}
0 & * & * \\
0 & * & * \\
0 & * & * \\
\end{pmatrix} \oplus \begin{pmatrix}
* & 0 & 0 \\
* & 0 & 0 \\
* & 0 & 0 \\
\end{pmatrix},
\]

we thus only need to check that

\[
\mathcal{L}_{2,1} = \left\{ \begin{pmatrix}
* & 0 \\
0 & * \\
0 & *
\end{pmatrix} \right\}
\]

have rank-2 bases since \(\begin{pmatrix}
0 & * & * \\
* & 0 & * \\
* & 0 & *
\end{pmatrix}\) also have rank-2 bases (it is indeed a 2 \(\otimes\) 3 case). (Hereafter, we denote by \(\mathcal{L}_{k,s}\) with \(1 \leq s \leq k - 1\) the linear space of all \(k + s\) by \(k\) matrices with the following properties: (i) the first \(k\) rows form a \(k\) by \(k\) diagonal matrix and (ii) the last \(s\) rows form an \(s\) by \(k\) matrix with all entries in the first \(k - 1\) columns at zero.) Similarly, for \(k = 3\), the Hilbert space \(M_{dx3d'}\) is a direct sum of three subspaces which are equivalent to \(M_{dx3p}, M_{3pq(3 + t)}\) and \(M_{(3 + s)x(3 + t)}\) respectively, where \(p, q \geq 1, 1 \leq s, t \leq 2\). We thus only need to show that

\[
\mathcal{L}_{3,1} = \left\{ \begin{pmatrix}
* & 0 & 0 \\
0 & * & 0 \\
0 & 0 & *
\end{pmatrix} \right\} \quad \text{and} \quad \mathcal{L}_{3,2} = \left\{ \begin{pmatrix}
* & 0 & 0 \\
0 & * & 0 \\
0 & 0 & * \\
\end{pmatrix} \right\}
\]
have rank-3 bases. In general, for any \(k\), if \(d = sk + r\) and \(d' = s'k + r'\), then the Hilbert space \(\mathcal{M}_{d \times d'}\) is a direct sum of three subspaces which are equivalent to \(\mathcal{M}_{d \times (s' - 1)k}\), \(\mathcal{M}_{(s' - 1)k \times (k + r')}\) and \(\mathcal{M}_{(k + r') \times (k + r')}\) respectively. We only need to consider rank-\(k\) bases in the following spaces (assume that \(EB_l(l < k)\) exists)

In fact, for the space of \(\mathcal{L}_{k,s}, 1 \leq s \leq k - 1\), we take

\[
\ell_{k,1} = \left\{ \begin{pmatrix} x_1 & 0 & \cdots & 0 & 0 \\ 0 & x_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & x_k \\ 0 & 0 & \cdots & 0 & x_{k+1} \end{pmatrix} \right\},
\]

\[
\ell_{k,2} = \left\{ \begin{pmatrix} x_1 & 0 & \cdots & 0 & 0 \\ 0 & x_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & x_k \\ 0 & 0 & \cdots & 0 & x_{k+1} \end{pmatrix} \right\},
\]

\[
\vdots
\]

\[
\ell_{k,k-1} = \left\{ \begin{pmatrix} x_1 & 0 & \cdots & 0 & 0 \\ 0 & x_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & x_k \\ 0 & 0 & \cdots & 0 & x_{k+1} \end{pmatrix} \right\}
\]

In fact, for the space of \(\mathcal{L}_{k,s}, 1 \leq s \leq k - 1\), we take

\[
x_{p}^{(i)} = \frac{1}{\sqrt{k + s}} \xi^{(p-1)(i-1)} \xi = e^{\frac{2\pi i}{k}},
\]

where \(1 \leq p, i \leq k + s\). That is, \(x_{p}^{(i)}\) are the coefficients of the \(i\)th element of an MEB in \((k + s) \otimes (k + s)\) system as in equation (5), \(i = 1, 2, ..., k + s\). Let

\[
A_{k+s}^{(i)} = \begin{pmatrix} x_{1}^{(i)} & 0 & \cdots & 0 & 0 \\ 0 & x_{2}^{(i)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & x_{k}^{(i)} \\ 0 & 0 & \cdots & 0 & x_{k+1}^{(i)} \end{pmatrix},
\]

then \(\{A_{k+s}^{(i)}\}\) is a rank-\(k\) basis of \(\mathcal{L}_{k,s}\). We thus prove that EB\(k\) exists in any bipartite system.
The coefficients in equation (9) can be replaced by the entries in the columns of any $k + s$ by $k + s$ isometric matrix without zero entries respectively. That is, there are infinitely many EBks in any $d \otimes d'$ system. Of course other types of EBk (but not SEBk) also exist. For example,

\[ |\psi_1\rangle = \frac{1}{2} |0\rangle |0\rangle' + \frac{\sqrt{3}}{2} |1\rangle |1\rangle', \]
\[ |\psi_2\rangle = \frac{\sqrt{3}}{2} |0\rangle |0\rangle' - \frac{1}{2} |1\rangle |1\rangle', \]
\[ |\psi_3\rangle = \frac{1}{\sqrt{2}} |0\rangle |1\rangle' + \frac{1}{\sqrt{2}} |1\rangle |0\rangle', \]
\[ |\psi_4\rangle = -\frac{1}{\sqrt{2}} |0\rangle |1\rangle' + \frac{1}{\sqrt{2}} |1\rangle |0\rangle', \]
\[ |\psi_5\rangle = \frac{1}{\sqrt{2}} |1\rangle |2\rangle' + \frac{1}{\sqrt{2}} |2\rangle |1\rangle', \]
\[ |\psi_6\rangle = \frac{1}{\sqrt{2}} |1\rangle |2\rangle' - \frac{1}{\sqrt{2}} |2\rangle |1\rangle', \]
\[ |\psi_7\rangle = \frac{1}{\sqrt{2}} |0\rangle |2\rangle' + \frac{1}{\sqrt{2}} |2\rangle |1\rangle', \]
\[ |\psi_8\rangle = \frac{1}{\sqrt{2}} |0\rangle |2\rangle' - \frac{1}{\sqrt{2}} |2\rangle |1\rangle' - \frac{\sqrt{2}}{\sqrt{3}} |2\rangle |2\rangle'. \] (11)

constitute an EB2 in $\mathbb{C}^3 \otimes \mathbb{C}^3$, but it is not an SEB2. We thus obtain the following result.

**Theorem 1.** For any $d, d'$ and $1 \leq k \leq d$, infinitely many EBks exist in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$.

Now we discuss SEBk. We begin with $k = 2$ and $d = d' = 3$. It is straightforward to show that for any rank-2 basis of $\mathcal{L}_{2,1}$, if it corresponds to three elements of an SEB2, then it must admit the following form

\[
\begin{pmatrix}
\frac{\sqrt{2}}{2} & 0 & e^{i\zeta_1} \\
0 & ae^{i\zeta_2} & 0 \\
0 & 0 & be^{i\zeta_3}
\end{pmatrix},
\begin{pmatrix}
\frac{\sqrt{2}}{2} & 0 & e^{i\zeta_4} \\
0 & ae^{i\zeta_5} & 0 \\
0 & 0 & be^{i\zeta_6}
\end{pmatrix}
\]

where $a > 0$, $b > 0$, $a^2 + b^2 = 1$, $x_i, y_i, z_i$ ($i = 1, 2$) are real numbers that should satisfy the following equalities

\[
\begin{cases}
x_1 - y_1 = x_2 - y_2 \pm \pi, \\
x_1 - z_1 = x_2 - z_2 \pm \pi, \\
y_1 - z_1 = y_2 - z_2 \pm \pi.
\end{cases}
\]

However, the above three equalities contradict each other. Therefore there is no SEB2 in $3 \otimes 3$ systems according to our scenario (maybe SEB2 can be constructed by other approaches). The space of $\mathcal{M}_{4 \times 4}$ can be decomposed as
It is obvious that \( \text{diag}(0,1,1,1) \), \( \text{diag}(1,0,-1,1) \), \( \text{diag}(1,1,0,-1) \) and \( \text{diag}(1,1,1,0) \) form a rank-3 basis of \( \{ \text{diag}(*,*,*,* \} \). Thus, SEB3 exists in any \( 4 \otimes d' \) systems or systems that can be divided into \( 3 \otimes p, 4 \otimes q, r \otimes 4 \) and \( s \otimes 3 \) systems with \( q, r \geq 4 \) and \( p, s \geq 3 \). We now can conclude the following.

Theorem 2. If \( dd' \) is a multiple of \( k \), then infinitely many SEBks exist in \( \mathbb{C}^d \otimes \mathbb{C}^{d'} \).

Theorem 3. If \( dd' \) is not a multiple of \( k \), then \( \text{SEB}k \) exists in \( \mathbb{C}^d \otimes \mathbb{C}^{d'} \) if there exists a \( \mathbb{C}^{kr} \) isometric matrix \( X' \) such that each column

\[
(x_{ij}, x_{2j}, \ldots, x_{kj}, j \leq j, k)'
\]

(here, \( A' \) denotes the transpose of \( A \)) satisfies:

1. either there are only \( k \) nonzero entries and the modulus of them is \( \frac{1}{\sqrt{k}} \) or
2. \( |x_i| = \frac{1}{\sqrt{k}} \) when \( 1 \leq i \leq k - 1 \) and \( \sum_{j,k} |x_j|^2 = \frac{1}{k} \).

That is, the SEBk problem is reduced to the construction of the special isometric matrices. However, whether or not there exists \( \mathbb{C}^{kr} \) isometric matrix \( X' \) satisfying the condition (1) or (2) is unknown when \( k \geq 3 \). With the increase of dimensions \( d \) and \( d' \) and \( k \), the verification of the existence of SEBk becomes harder and harder.

5. Multipartite case

In this section, we consider the multipartite case. We first extend the concept of EBk to multipartite systems. In a product Hilbert space \( \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \ldots \otimes \mathbb{C}^{d_N} \) with \( N \geq 3 \), only specific pure states admit a Schmidt decomposition form [15, 16]. Now we discuss whether a basis can be constructed from such specific states for the multipartite case. Hereafter, we always assume with no loss of generality that \( d_1 \leq d_2 \leq \ldots \leq d_N \).

Definition 2. An orthonormal basis \( \{|\psi_i\rangle\} \) in \( \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \ldots \otimes \mathbb{C}^{d_N} \) is an \( N \)-partite EBk \( (1 \leq k \leq d_1) \) if

\[
|\psi_i\rangle = \sum_{j=0}^{k-1} \lambda_j^{(i)} |e_j^{(1)}\rangle |e_j^{(2)}\rangle \ldots |e_j^{(N)}\rangle
\]

for any \( i, 0 \leq i \leq d_1d_2\ldots d_N - 1 \), where \( \{|e_j^{(l)}\rangle\} \) is an orthonormal set of \( \mathbb{C}^{d_l} \), \( \lambda_j^{(i)} > 0 \), \( \sum_j (\lambda_j^{(i)})^2 = 1, 1 \leq l \leq N, N \geq 3 \). Particularly, it is an \( N \)-partite SEBk if it is an \( N \)-partite EBk with all the coefficients \( \lambda_j^{(i)} \)'s equal to \( \frac{1}{\sqrt{k}} \).

We begin with the tripartite case. Let \( \{|\psi_i\rangle\} \) be an EBk of \( \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \), and be written in the Schmidt decomposition form as \( |\psi_i\rangle = \sum_{j=0}^{k-1} \lambda_j^{(i)} |e_j\rangle |e_j\rangle \), of which the \( l \)-th product term
|$e_i\rangle|e'_i\rangle$ is denoted by $|\psi_i\rangle$ for convenience. We define

$$
|\psi_{ij}^{(2+1)}\rangle := \sum_{l=0}^{k-1} \lambda_j^{(l)} |\psi_i\rangle|j + l\rangle_3,
$$

(13)

where $j + l$ means $j + l \mod d_3$, $\{|j\rangle\}$ is an orthonormal bases of $C^{d_3}$, $0 \leq j \leq d_3 - 1$. Then $\{|\psi_{ij}^{(2+1)}\rangle\}$ with $0 \leq i \leq d_4d_2 - 1$ and $0 \leq j \leq d_3 - 1$ is a tripartite EB in $C^{d_4} \otimes C^{d_2} \otimes C^{d_3}$.

Generally, we denote by $|\psi\rangle = |\psi_1\rangle|\psi_2\rangle\cdots|\psi_m\rangle$ for convenience provided that $|\psi\rangle = \sum_{\lambda} \lambda |\psi\rangle$ with $\{|\psi\rangle\}$ is an orthonormal set of $C^d$, $1 \leq s \leq m$. If $\{|\psi_i\rangle\}$ is an $m$-partite EB of $C^{d_1} \otimes C^{d_2} \otimes \cdots \otimes C^{d_m}$, then

$$
|\psi_{ij}^{(m+1)}\rangle := \sum_{l=0}^{k-1} \lambda_j^{(l)} |\psi_i\rangle|j + l\rangle_{m+1}
$$

(14)

with $0 \leq j \leq d_{m+1} - 1$ form an $(m + 1)$-partite EB of $C^{d_1} \otimes C^{d_2} \otimes \cdots \otimes C^{d_{m+1}}$, where $\{|\psi_{ij}\rangle\}$ is an orthonormal basis of $C^{d_{m+1}}$, $j + l$ means $j + l \mod d_{m+1}$, $0 \leq l \leq d_1d_2 \cdots d_m - 1$. That is, an $(m + 1)$-partite EB can be obtained from an $m$-partite EB for any $m \geq 2$. Together with theorems 1–3, we thus get the following.

**Proposition 1.** For any $1 \leq k \leq d_1$, infinitely many $N$-partite EBks exist in $C^{d_1} \otimes C^{d_2} \otimes \cdots \otimes C^{d_N}$.

**Proposition 2.** If $d_1d_2 \cdots d_N$ is a multiple of $k$, then infinitely many $N$-partite SEBks exist in $C^{d_1} \otimes C^{d_2} \otimes \cdots \otimes C^{d_N}$.

**Proposition 3.** If $d_1d_2 \cdots d_N$ is not a multiple of $k$, then an $N$-partite SEBk exists in $C^{d_1} \otimes C^{d_2} \otimes \cdots \otimes C^{d_N}$ if the condition of theorem 3 holds with $\tilde{r} = \min \{r_i; 1 \leq i \leq N\}$, $d_i = s_jk + r_i$.

In other words, the key point of the multipartite case is in fact the same as the bipartite one. In addition, from our approach, the $N$-partite EBk can be constructed in many ways. For example, if $d_1d_2d_3$ is a multiple of $k$, one can either construct an EBk (or SEBk) of $C^{d_3} \otimes C^{d_3} \otimes \cdots \otimes C^{d_{m+1}}$, where $\{|\psi_{ij}\rangle\}$ is an orthonormal basis of $C^{d_{m+1}}$, $j + l$ means $j + l \mod d_{m+1}$, $0 \leq l \leq d_1d_2 \cdots d_m - 1$. That is, an $(m + 1)$-partite EB can be obtained from an $m$-partite EBk for any $m \geq 2$. Together with theorems 1–3, we thus get the following.

**Proposition 1.** For any $1 \leq k \leq d_1$, infinitely many $N$-partite EBks exist in $C^{d_1} \otimes C^{d_2} \otimes \cdots \otimes C^{d_N}$.

**Proposition 2.** If $d_1d_2 \cdots d_N$ is a multiple of $k$, then infinitely many $N$-partite SEBks exist in $C^{d_1} \otimes C^{d_2} \otimes \cdots \otimes C^{d_N}$.

**Proposition 3.** If $d_1d_2 \cdots d_N$ is not a multiple of $k$, then an $N$-partite SEBk exists in $C^{d_1} \otimes C^{d_2} \otimes \cdots \otimes C^{d_N}$ if the condition of theorem 3 holds with $\tilde{r} = \min \{r_i; 1 \leq i \leq N\}$, $d_i = s_jk + r_i$.

Finally, we give two examples for illustration. For the case $C^3 \otimes C^3 \otimes C^3$, it is easy to check that the eight 3-qubits GHZ states deduced from the four Bell states via relation equation (13) form a 3-partite SEB2 (or MEB), and a general 3-partite EB (that is not a SEB2) can be obtained through equation (13) from any EB in $C^2 \otimes C^2$. Namely, if $\{|\psi_i\rangle = a_i^{(0)}|\psi_i^{(0)}\rangle + a_i^{(1)}|\psi_i^{(1)}\rangle; 0 \leq i \leq 3, (a_i^{(0)})^2 + (a_i^{(1)})^2 = 1\}$ is an EB2 of a two-qubit system, then

$$
|\text{GHZ}^{(3)}\rangle = \sum_{j=0}^{k-1} \lambda_j |j\rangle |j\rangle |j\rangle,
$$

where $\{|j\rangle\}$ is an orthonormal set of $C^{d_3}$, $\lambda_j > 0$, $\sum_{j=0}^{k-1} \lambda_j^2 = 1, 1 \leq l \leq N$.
are a 3-partite EB2 in $C^2 \otimes C^2 \otimes C^2$ via relation equation (13).

For the case $C^3 \otimes C^3 \otimes C^3$,

\[
\begin{align*}
\psi_{0,0}^{(2+)} & = a_{0}^{(0)} |0\rangle |0\rangle |0\rangle + \frac{\sqrt{3}}{2} |1\rangle |1\rangle |1\rangle, \\
\psi_{0,1}^{(2+)} & = a_{0}^{(0)} |0\rangle |0\rangle |1\rangle + \frac{\sqrt{3}}{2} |1\rangle |1\rangle |2\rangle, \\
\psi_{0,2}^{(2+)} & = a_{0}^{(0)} |0\rangle |0\rangle |2\rangle + \frac{\sqrt{3}}{2} |1\rangle |1\rangle |0\rangle, \\
\psi_{1,0}^{(2+)} & = a_{1}^{(0)} |0\rangle |0\rangle |0\rangle - \frac{1}{2} |1\rangle |1\rangle |1\rangle, \\
\psi_{1,1}^{(2+)} & = a_{1}^{(0)} |0\rangle |0\rangle |1\rangle - \frac{1}{2} |1\rangle |1\rangle |2\rangle, \\
\psi_{1,2}^{(2+)} & = a_{1}^{(0)} |0\rangle |0\rangle |2\rangle - \frac{1}{2} |1\rangle |1\rangle |0\rangle, \\
\psi_{2,0}^{(2+)} & = a_{2}^{(0)} |0\rangle |1\rangle |0\rangle + \frac{1}{\sqrt{2}} |1\rangle |0\rangle |1\rangle, \\
\psi_{2,1}^{(2+)} & = a_{2}^{(0)} |0\rangle |1\rangle |1\rangle + \frac{1}{\sqrt{2}} |1\rangle |0\rangle |2\rangle, \\
\psi_{2,2}^{(2+)} & = a_{2}^{(0)} |0\rangle |1\rangle |2\rangle + \frac{1}{\sqrt{2}} |1\rangle |0\rangle |0\rangle, \\
\psi_{3,0}^{(2+)} & = a_{3}^{(0)} |0\rangle |1\rangle |0\rangle - \frac{1}{\sqrt{2}} |1\rangle |0\rangle |1\rangle, \\
\psi_{3,1}^{(2+)} & = a_{3}^{(0)} |0\rangle |1\rangle |1\rangle - \frac{1}{\sqrt{2}} |1\rangle |0\rangle |2\rangle, \\
\psi_{3,2}^{(2+)} & = a_{3}^{(0)} |0\rangle |1\rangle |2\rangle + \frac{1}{\sqrt{2}} |1\rangle |0\rangle |0\rangle, \\
\psi_{4,0}^{(2+)} & = a_{4}^{(0)} |0\rangle |1\rangle |0\rangle + \frac{1}{\sqrt{2}} |1\rangle |2\rangle |0\rangle + \frac{1}{\sqrt{2}} |2\rangle |0\rangle |1\rangle.
\end{align*}
\]
\[ |\psi_{4,0}^{(2+1)}\rangle = \frac{1}{\sqrt{2}} |1\rangle |2\rangle |1\rangle + \frac{1}{\sqrt{2}} |2\rangle |0\rangle |2\rangle, \]
\[ |\psi_{4,1}^{(2+1)}\rangle = \frac{1}{\sqrt{2}} |1\rangle |2\rangle |2\rangle + \frac{1}{\sqrt{2}} |2\rangle |0\rangle |0\rangle, \]
\[ |\psi_{5,0}^{(2+1)}\rangle = \frac{1}{\sqrt{2}} |1\rangle |2\rangle |0\rangle - \frac{1}{\sqrt{2}} |2\rangle |0\rangle |1\rangle, \]
\[ |\psi_{5,1}^{(2+1)}\rangle = \frac{1}{\sqrt{2}} |1\rangle |2\rangle |1\rangle - \frac{1}{\sqrt{2}} |2\rangle |0\rangle |2\rangle, \]
\[ |\psi_{5,2}^{(2+1)}\rangle = \frac{1}{\sqrt{2}} |1\rangle |2\rangle |2\rangle - \frac{1}{\sqrt{2}} |2\rangle |0\rangle |0\rangle, \]
\[ |\psi_{6,0}^{(2+1)}\rangle = \frac{1}{\sqrt{2}} |0\rangle |2\rangle |0\rangle + \frac{1}{\sqrt{2}} |2\rangle |1\rangle |1\rangle, \]
\[ |\psi_{6,1}^{(2+1)}\rangle = \frac{1}{\sqrt{2}} |0\rangle |2\rangle |1\rangle + \frac{1}{\sqrt{2}} |2\rangle |1\rangle |2\rangle, \]
\[ |\psi_{6,2}^{(2+1)}\rangle = \frac{1}{\sqrt{2}} |0\rangle |2\rangle |2\rangle + \frac{1}{\sqrt{2}} |2\rangle |1\rangle |0\rangle, \]
\[ |\psi_{7,0}^{(2+1)}\rangle = \frac{1}{\sqrt{3}} |0\rangle |2\rangle |0\rangle - \frac{1}{\sqrt{3}} |2\rangle |1\rangle |1\rangle + \frac{1}{\sqrt{3}} |2\rangle |2\rangle |1\rangle, \]
\[ |\psi_{7,1}^{(2+1)}\rangle = \frac{1}{\sqrt{3}} |0\rangle |2\rangle |1\rangle - \frac{1}{\sqrt{3}} |2\rangle |1\rangle |2\rangle + \frac{1}{\sqrt{3}} |2\rangle |2\rangle |2\rangle, \]
\[ |\psi_{7,2}^{(2+1)}\rangle = \frac{1}{\sqrt{3}} |0\rangle |2\rangle |2\rangle - \frac{1}{\sqrt{3}} |2\rangle |1\rangle |0\rangle + \frac{1}{\sqrt{3}} |2\rangle |2\rangle |0\rangle, \]
\[ |\psi_{8,0}^{(2+1)}\rangle = \frac{1}{\sqrt{3}} |0\rangle |2\rangle |0\rangle - \frac{1}{\sqrt{6}} |2\rangle |1\rangle |1\rangle - \sqrt{\frac{2}{3}} |2\rangle |2\rangle |1\rangle, \]
\[ |\psi_{8,1}^{(2+1)}\rangle = \frac{1}{\sqrt{6}} |0\rangle |2\rangle |1\rangle - \frac{1}{\sqrt{6}} |2\rangle |1\rangle |2\rangle - \sqrt{\frac{2}{3}} |2\rangle |2\rangle |2\rangle, \]
\[ |\psi_{8,2}^{(2+1)}\rangle = \frac{1}{\sqrt{6}} |0\rangle |2\rangle |2\rangle - \frac{1}{\sqrt{6}} |2\rangle |1\rangle |0\rangle - \sqrt{\frac{2}{3}} |2\rangle |2\rangle |0\rangle. \]

constitute an EB2 in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. $|\psi_{ij}^{(2+1)}\rangle$’s are in fact 3-qubits GHZ-like states. It is clear that $|\psi_{ij}^{(2+1)}\rangle$’s above are derived from equation (11) via relation equation (13). There is no SEB2 in a $3 \otimes 3$ system, so there is no SEB2 in a $3 \otimes 3 \otimes 3$ system either. An EB3 or SEB3 in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ can be obtained via relation equation (13) from any EB3 or SEB3 in $\mathbb{C}^3 \otimes \mathbb{C}^3$ respectively.

6. Conclusions

We have introduced the concept of an EBk and shown that an EBk exists in $\mathbb{C}^d \otimes \mathbb{C}^d$ for any $d$ and $d'$. In our discussion we have also proposed methods of constructing EBks by analyzing the structure of the matrix space of the coefficient matrices. We showed that the existence of EBks is equivalent to the existence of the special isometric matrices. Our result is a complement of the UB problem (here, UB contains UPB, UMEB and UEBk) to some extent. By now, the basis problems (namely, MEB, EBk, PB, UPB, UMEB and UEBk) are somewhat settled: almost any kind of basis (complete or incomplete) exists in any bipartite space
$C^d \otimes C^{d'}$. It is a pity that the SEBk problem is left open when $dd'$ is not a multiple of $k$ and the UMEB for $d'=d$ is known only for some special case.

Furthermore, our approaches can be used in the multipartite case and we thus have provided a complete characterization of EBk in both the bipartite case and the multipartite case. In either the bipartite case or multipartite case, the basis states of EBk (or $N$-partite EBk) are just the pure states that admit the Schmidt decomposition with the same length. Although only some special multipartite pure states have the Schmidt decomposition, the basis with such a special structure always exists. Our results provide a mathematical tool for studying projective measurements onto basis states that admit a Schmidt decomposition with the same length.

It is worth mentioning here that real space is different from complex space. From the arguments in sections 3 and 4, we conjecture that there is no SEB2 in the real space $\mathbb{R}^d \otimes \mathbb{R}^{d'}$ when $dd'$ is not a multiple of 2 (due to the fact that the real numbers with modulus 1 are only 1 and $-1$). Moreover, we conjecture that: (i) MEB exists only when $d = 2^s$ with $s \geq 0$; (ii) SEBk exists only when $k = 2^s$, $s \geq 0$, and $dd'$ is a multiple of $k$. In addition, from the orthogonal matrix $O_n$, we can conclude that an EBk that is not an SEBk exists in $\mathbb{R}^d \otimes \mathbb{R}^{d'}$ for any $d$ and $d'$ (of course there are other types of EBks). Similar results for the multipartite case can be obtained from the method in section 5.

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