First and Second-Order Energy Eigenvalues of One-Dimensional Quantum Harmonic and Anharmonic Oscillator with Linear, Quadratic, Cubic and Polynomial Perturbation Potential

B. I. Madububa¹, N. I. Achuko¹, JP. C. Mbagwu², C. I. Jonas³ and J.O. Ozuomba⁴

¹Department of Physics, Federal University of Technology Owerri, P.M.B. 1526, Owerri, Imo State, Nigeria
²Department of Physics and Astronomy, University of Kansas, Lawrence, KS, 66045, USA
³Department of Mathematics, Federal University of Technology Owerri, P.M.B. 1526, Owerri, Imo State, Nigeria
⁴Department of Physics, Faculty of Physical Science, Imo State University, Owerri, Nigeria

This work is aimed at obtaining the energy eigenvalues for one-dimensional quantum harmonic and anharmonic oscillators perturbed by linear, quadratic, cubic and polynomial potentials. To obtain the solutions of the energy eigenvalues, we employed the time-independent perturbation theory to calculate the first and the second-order energy correction, which we used to obtain the complete generalised energy eigenvalues of the quantum harmonic oscillators with linear, quadratic, cubic and polynomial perturbation potential of the same unperturbed Hamiltonian ($H$).

**Keywords:** Quantum Harmonic and Anharmonic Oscillator; Perturbation Theory; Eigenvalue; Perturbation potential; Polynomial perturbation potential

I. INTRODUCTION

The quantum harmonic and anharmonic oscillators have important applications in all areas of physics. Specifically, a harmonic oscillator is a model that has an important analogy when describing physical systems (Halil, 2018; Habtamu, 2019). Harmonic oscillator eigenvalue problems can be solved analytically (Mojtaba & Davood, 2002); but in cases where the exact solution of a problem cannot be found, it is more appreciable to use approximation methods such as perturbation theory, variational method or Wentzel, Kramers and Brillouin (WKB). The perturbation theory approach has been adopted in several ways to obtain the energy eigenvalues of the harmonic oscillator (David, 2016; Nouredine et al., 2009).

Most quantum mechanical problems are solved by harmonic oscillator analogy with appropriate boundary conditions. Eigenvalue perturbation theory was first used by Lord Rayleigh in acoustics and Schrodinger in his fundamental series introduced the quantum theory in the 20th century (Mbagwu et al., 2020). In remembrance of their contributions, the series is called Rayleigh–Schrodinger perturbation theory but the mathematical foundations were only adopted by Rellich a few years ago (Barry, 1991; Rellich, 1937). In recent years, immense contributions have been made to develop mathematical methods for solving eigenvalues and eigenfunctions of the quantum harmonic and anharmonic oscillators such as the SU (2) group method (SGM) (Kunihiro, 1993). However, perturbation theory is a method used in obtaining an approximate solution of an exact harmonic oscillator problem (Sergei, 2006). Thus, numerical and analytical procedures can be used to calculate these perturbed harmonic oscillators with theoretical contributions (Louisell, 1973; Peidaee et al., 2007).

The anharmonic oscillator is one of the key models in solving problems in physics (Bender & Wu, 1969; Bhaumik & Dutta, 1975; Hioe et al., 1978). It can be solved using either the analytical method or approximation method (Hsue & Chern, 1984; Chahjilany et al., 1991; Bacus et al., 1995). The
anharmonic quantum oscillator with quartic potential has been solved using a different approach (McRae & Vrscay, 1997; Ivanov, 1998; Ferrándiz & Guardiola, 1993). Quantum anharmonic oscillator with sextic, octic, decatic and the generalised nth term of the potentials has equally been solved recently (Skała et. al., 1999; Sharma & Fiase, 2000; Speliotopoulos, 2000). A lot of studies have been done on an anharmonic oscillator with only one perturbation term (Jafarpour et. al., 2003; Vinett & Čížek, 1991; Matamala & Maldonado, 2003). The goal of this work is to calculate the first and second-order energy correction and obtained the generalised energy eigenvalues for the quantum harmonic and anharmonic oscillator with linear, quadratic, cubic and polynomial potential by using the perturbation method whose Hamiltonian as in (Ariel & Philippe, 2018) is given by:

\[ \hat{H}_0 = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 \]  

where \( x \) is the position operator, \( m \) is the mass of the particle, \( p \) is the linear momentum operator and \( \omega \) is the angular frequency.

II. ONE DIMENSIONAL QUANTUM HARMONIC OSCILLATOR WITH LINEAR PERTURBATION POTENTIAL

Considering a particle with a linear perturbation Potential given as;

\[ \hat{H}' = \lambda_1 x \]  

where \( \lambda_1 \) is the perturbation coefficient. Hence, the total Hamiltonian then becomes;

\[ \hat{H} = \hat{H}_0 + \hat{H}' \]  

such that;

\[ \hat{H} = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 + \lambda_1 x \]  

At this point, we seek to determine the correction of first-order energy and second-order energy by using first order and second order energy shift formulas.

A. First Order Energy Shift

\[ E_n^1 = \langle \psi_n^0 | H' | \psi_n^0 \rangle \]  

where \( H' \) is the perturbed Hamiltonian of the system and \( \psi_n^0 \) is the unperturbed wave function of the system. Hence, substituting Equation (2) into Equation (5) that is;

\[ E_n^1 = \lambda_1 \langle \psi_n^0 | x | \psi_n^0 \rangle \]  

If \( \psi_n^0 = n \) in which \( n \) is the \( n^{\text{th}} \) eigenfunction. Therefore, we have that;

\[ E_n^1 = \lambda_1 \langle n | x | n \rangle \]  

For the harmonic oscillator the position function \( x \) is expressed in terms of ladder operators coined from Dirac Operator Technique which is;

\[ x = \sqrt{\frac{\hbar}{2m\omega}} [a^\dagger + a] \]  

where \( a^\dagger \) is the step-up (creation) operator and \( a \) is the step-down (annihilation) operator. Note: If they don’t have the same number of step-up or step-down operators the function will not be part of the state because they will annihilate each other and hence alter the state (Mbagwu et al., 2021).

Thus, substituting Equation (8) into Equation (7), we then have;

\[ E_n^1 = \lambda_1 \sqrt{\frac{\hbar}{2m\omega}} \]  

Applying the operator properties that is;

\[ a^\dagger | n \rangle \sqrt{n+1} | n+1 \rangle \]  

\[ a | n \rangle \sqrt{n} | n-1 \rangle \]  

Substituting Equation (10a) and (10b) into Equation (9) so we have;

\[ E_n^1 = \lambda_1 \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n+1} | n+1 \rangle + \sqrt{n} | n-1 \rangle) \]
From the orthogonality condition of ket and bra of;

\[ (m \mid n) = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \]

The ket and bra function in Equation (11) are not equal, thus we can generalise that;

\[ E_n^1 = 0 \]  \hspace{1cm} (12)

Therefore Equation (12) is the first-order energy correction to the eigenvalue \( E_n \) of a one-dimensional quantum harmonic oscillator with a linear perturbation potential.

**B. Second-Order Energy Shift**

Here we seek to obtain the second-order energy correction, using the eigenvalue fundamental equation of the second-order perturbation theory which is given as (Habtamu, 2019).

\[ E_n^2 = \sum_{n \neq m} \left| \langle \psi_m^0 \mid H' \mid \psi_n^0 \rangle \right|^2 \]  \hspace{1cm} (13)

\( \psi_m^0 \) and \( \psi_n^0 \) in Equation (13) represents the eigenstates for \( m^{th} \) and \( n^{th} \) ingers, where \( E_n^0 \) and \( E_m^0 \) are the ground state energy of a harmonic oscillator for \( m^{th} \) and \( n^{th} \) ingers and \( H' \) is the perturbation Hamiltonian.

Solving the numerator of the function in Equation (13) separately which is:

\[ \langle \psi_m^0 \mid H' \mid \psi_n^0 \rangle \]  \hspace{1cm} (14)

Hence substituting Equation (2) into Equation (14) we have that;

\[ \langle \psi_m^0 \mid H' \mid \psi_n^0 \rangle = \lambda_1 \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n+1} - m) \]  \hspace{1cm} (15)

where \( \psi_m^0 = m \) and \( \psi_n^0 = n \); in which \( m \) and \( n \) are the \( m^{th} \) and \( n^{th} \) eigenfunction. Therefore, substituting Equation (8) into Equation (15), hence we have;

\[ \langle \psi_m^0 \mid H' \mid \psi_n^0 \rangle = \lambda_1 \sqrt{\frac{\hbar}{2m\omega}} (m + \frac{1}{2}) \]  \hspace{1cm} (16)

Applying the operator properties in Equation (10a) and (10b) into Equation (16), thus having that;

\[ \langle \psi_m^0 \mid H' \mid \psi_n^0 \rangle = \lambda_1 \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n+1} - m) \]  \hspace{1cm} (17)

Applying the Kronecker delta function, it is important to note from Equation (17) that;

\[ \langle m \mid n + 1 \rangle = \delta_{m,n+1} \]  \hspace{1cm} (18a)

\[ \langle m \mid n - 1 \rangle = \delta_{m,n-1} \]  \hspace{1cm} (18b)

Therefore, substituting Equation (18a) and (18b) into Equation (17), thus we have;

\[ \langle \psi_m^0 \mid H' \mid \psi_n^0 \rangle = \lambda_1 \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n+1} \delta_{m,n+1} + \sqrt{n} \delta_{m,n-1}) \]  \hspace{1cm} (19)

Hence putting Equation (19) into Equation (13), Equation (20) becomes;

\[ E_n^2 = \sum_{n \neq m} \left| \lambda_1 \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n+1} \delta_{m,n+1} + \sqrt{n} \delta_{m,n-1}) \right|^2 \]  \hspace{1cm} (20)

here \( E_n^0 \) and \( E_m^0 \) are the ground state energy of a harmonic oscillator given by;

\[ E_n^0 = \left( n + \frac{1}{2} \right) \hbar \omega \]  \hspace{1cm} (21a)

\[ E_m^0 = \left( m + \frac{1}{2} \right) \hbar \omega \]  \hspace{1cm} (21b)

\[ E_n^2 = \frac{\lambda_1^2}{2m\omega^2} \sum_{n \neq m} \left| \frac{\sqrt{n+1} \delta_{m,n+1} + \sqrt{n} \delta_{m,n-1}}{n - m} \right|^2 \]  \hspace{1cm} (22)

If \( m = n + 1; \) \( m = n - 1 \) we obtain that;

\[ E_n^2 = \frac{\lambda_1^2}{2m\omega^2} \]  \hspace{1cm} (23)

Therefore Equation (23) is the second-order energy correction to the eigenvalue \( E_n \) of a one-dimensional
quantum harmonic oscillator with a linear perturbation potential. Thus, to get the complete eigenvalue of the system. The general expression of energy function due to \( H' \) for any perturbation is given by:

\[
E_n = E_n^0 + E_n^1 + E_n^2
\]  

(24)

Therefore, inserting Equation (21a), (12) and (23) into Equation (24) to obtain the general expression of the energy correction. That is;

\[
E_n = \left( n + \frac{1}{2} \right) \hbar \omega + \frac{\lambda^2}{2m\omega^2}
\]  

(25)

where \( n \) is a positive integer \((n = 0, 1, 2, 3 \ldots)\). Therefore Equation (25) gives the exact correction to the energy and also the complete eigenvalues for the first and second approximation of one-dimensional quantum harmonic oscillator perturbed by a linear potential.

III. ONE DIMENSIONAL QUANTUM HARMONIC OSCILLATOR WITH QUADRATIC PERTURBATION POTENTIAL

Now consider a quantum harmonic oscillator with unperturbed Hamiltonian \((H_0)\), which is already stated in Equation (1). Suppose that the system is perturbed with a quadratic potential so that;

\[
H' = \lambda_x x^2
\]  

(26)

Equation (26) is added to the unperturbed Hamiltonian \((H_0)\), in Equation (3). Hence we have that;

\[
H = H_0 + H' = \frac{p^2}{2m} + \lambda_x x^2
\]  

(27)

At this point we seek to determine the correction of the first and second-order energy by applying the first and second-order energy shift formulas.

A. First Order Energy Shift

Since the formula for the first-order energy correction in Equation (5) is; \( \psi_n^1 = (\psi_n^0 \mid H' \mid \psi_n^0) \). Thus, putting Equation (26) into Equation (5), we have that;

\[
E_n^1 = \lambda_x (\psi_n^0 \mid x^2 \mid \psi_n^0)
\]  

(28)

If \( \psi_n^0 = n \) in which \( n \) is the \( n \)th eigenfunction then;

\[
E_n^1 = \lambda_x (n \mid x^2 \mid n)
\]  

(29)

Applying the ladder operators stated in Equation (8) that is;

\[
x = \sqrt{\frac{\hbar}{2m\omega}} [a^\dagger + a] \text{ but for } x^2 \text{ we have that}
\]

\[
x^2 = \frac{\hbar}{2m\omega} [a^\dagger + a]^2
\]  

(30)

Hence inserting Equation (30) into Equation (29), we now have that;

\[
E_n^1 = \lambda_x \left( \frac{\hbar}{2m\omega} \right) (n \mid [a^\dagger + a]^2 \mid n)
\]  

(31)

Expanding the ladder operator in Equation (31) we then have that;

\[
E_n^1 = \frac{\lambda_x \hbar}{2m\omega} (n \mid a^\dagger a^\dagger + aa + a^\dagger a + aa \mid n)
\]  

(32)

Applying the operator properties in Equation (10a) and (10b) we now have that;

\[
E_n^1 = \frac{\lambda_x \hbar}{2m\omega} \left( \sqrt{(n + 1)(n + 2)} \right) (n \mid n + 2)
\]

\[
+ \sqrt{(n - 1)n} (n \mid n - 2 + 2n
\]

\[
+ 1(n \mid n)
\]  

(33)

Applying the orthogonality condition, we have;

\[
E_n^1 = \frac{\lambda_x \hbar}{m\omega} \left( n + \frac{1}{2} \right)
\]  

(34)

Therefore Equation (34) is the first-order energy correction to the eigenvalue \( E_n \) of a one-dimensional quantum harmonic oscillator.

B. Second-Order Energy Shift

Here we seek to obtain the second-order energy correction using the eigenvalue fundamental equation of the second-order perturbation theory which is already stated in Equation
Expanding the ladder operator in Equation (36) hence we have that;

\[
\langle \psi_m^0 \mid H' \mid \psi_n^0 \rangle = \frac{\lambda_2 h}{2m \omega} \left( m \mid a^\dagger a + aa^\dagger + aa \mid n \right) \tag{37}
\]

Applying the operator properties in Equation (10a) and (10b) into Equation (32), thus having that;

\[
(m \mid H' \mid n) = \frac{\lambda_2 h}{2m \omega} \left( \sqrt{(n+1)(n+2)} \delta_{m,n+2} + \sqrt{(n-1)n} \delta_{m,n-2} + (2n+1) \delta_{m,n} \right) \tag{38}
\]

Inserting Equation (38) into Equation (13) then we have that;

\[
E_n^2 = \sum_{m \neq n} \frac{\left( \frac{\lambda_2 h}{2m \omega} \left( \sqrt{(n+1)(n+2)} \delta_{m,n+2} + \sqrt{(n-1)n} \delta_{m,n-2} + (2n+1) \delta_{m,n} \right) \right)^2}{E_n^0 - E_m^0} \tag{39}
\]

From Equation (39), the first term contributes when \( m = n + 2 \), the second term contributes when \( m = n - 2 \), but the third term contributes only when \( m = n \), which is excluded in our relation. Hence substituting Equation (21a) into Equation (39), we get;

\[
E_n^2 = \frac{\lambda_2^2 h}{4m^2 \omega^2} \left[ - \frac{1}{2} (n+1)(n+2) + \frac{1}{2} n(n-1) \right] \tag{40}
\]

Solving Equation (40) further we finally obtain that;

\[
E_n^2 = - \frac{\lambda_2^2 h}{2m^2 \omega^2} \left( n + \frac{1}{2} \right) \tag{41}
\]

Therefore Equation (41) is the second-order energy correction to the eigenvalue \( E_n \) of a one-dimensional quantum harmonic oscillator with a quadratic perturbation potential. Hence, inserting Equation (21a), (34) and (41) into Equation (24) to get the complete eigenvalues of the system. The general expression of energy function due to \( H' \) for any perturbation now becomes;

\[
E_n = \hbar \omega \left( n + \frac{1}{2} \right) + \frac{\lambda_2 h}{m \omega} \left( n + \frac{1}{2} \right) - \frac{\lambda_2^2 h}{2m^2 \omega^2} \left( n + \frac{1}{2} \right) \tag{42}
\]

Equation (42) can be solved further to obtain that;

\[
E_n = \hbar \omega \left( n + \frac{1}{2} \right) \left[ 1 + \frac{\lambda_2}{m \omega} + \frac{\lambda_2^2}{2m^2 \omega^2} \right] \tag{43}
\]

where \( n \) is a positive integer i.e. \( n = 0, 1, 2, 3 \ldots \). Therefore Equation (43) gives the exact correction to the energy and also the complete eigenvalues for the first and second approximation of one-dimensional quantum harmonic oscillator perturbed by a quadratic potential.

**IV. ONE DIMENSIONAL QUANTUM HARMONIC OSCILLATOR WITH CUBIC PERTURBATION POTENTIAL**

This time we will consider a cubic potential perturbing on the same unperturbed Hamiltonian\( (H_0) \). The cubic perturbation potential is given as;

\[
H' = \lambda_3 x^3 \tag{44}
\]

Inserting Equation (44) and Equation (1) into Equation (3) so that the total Hamiltonian become;

\[
H = \frac{\hbar^2}{2m} + \frac{1}{2} m \omega^2 x^2 + \lambda_3 x^3 \tag{45}
\]
To solve for the correction of the first-order energy and second-order energy by using the first order and second order energy shift formula, we adopted the method of perturbation theory.

**A. First Order Energy Shift**

Adopting Equation (5) which is $E_n^1 = \langle \psi_n^0 | H' | \psi_n^0 \rangle$. Thus, putting Equation (44) into the equation we have that;

$$E_n^1 = \lambda_3 (\psi_n^0 | x^3 | \psi_n^0)$$

(46)

If $\psi_n^0 = n$ in which n is the nth eigenfunction, therefore;

$$E_n^1 = \lambda_3 (n | x^3 | n)$$

(47)

Applying the ladder operators stated in Equation (8) that is;

$$x = \sqrt{\frac{\hbar}{2m_o}} [a \dagger + a]$$

but for $x^3$ we have that

$$x^3 = (\frac{\hbar}{2m_o})^\frac{3}{2} [a \dagger + a]^3$$

(48)

Hence inserting Equation (48) into Equation (47), we now have that;

$$E_n^1 = \lambda_3 (\frac{\hbar}{2m_o})^\frac{3}{2} (n | [a \dagger + a]^3 | n)$$

(49)

Expanding the ladder operator in Equation (49) we then have that;

$$E_n^1 = \lambda_3 (\frac{\hbar}{2m_o})^\frac{3}{2} (n | a \dagger a \dagger a \dagger + aaa \dagger + a \dagger a \dagger a + a \dagger a \dagger + a \dagger + a a + a a | n)$$

(50)

Applying again the orthogonality condition; Equation (51) reduces to;

$$E_n^1 = 0$$

(52)

Therefore Equation (52) is the first-order energy correction to the eigenvalue $E_n$ of a one-dimensional quantum harmonic oscillator with a cubic perturbation potential.

**B. Second-Order Energy Shift**

At this point, we seek to obtain the second-order energy correction using the eigenvalue fundamental equation of the second-order perturbation theory as stated in Equation (13).

$$\langle \psi_m^0 | H' | \psi_n^0 \rangle = \lambda_3 (m | x^3 | n)$$

(53)

Therefore, substituting Equation (48) into Equation (53) hence we have that;

$$\langle m | H' | n \rangle = \lambda_3 (\frac{\hbar}{2m_o})^\frac{3}{2} (m | [a \dagger + a]^3 | n)$$

(54)

From the expression obtained in Equation (53) if we substitute the ket function Equation (51) can be rewritten as;

$$\langle m | H' | n \rangle = \lambda_3 (\frac{\hbar}{2m_o})^\frac{3}{2} \left( \sqrt{(n + 1)(n + 2)(n + 3)} \delta_{m,n+3} + (n + 1)\sqrt{n} \delta_{m,n-1} + (n + 2)\sqrt{n + 1} \delta_{m,n+1} + (n \sqrt{n + 1}) \delta_{m,n+3} + (n - 1)\sqrt{n} \delta_{m,n-1} \right)$$

(55)

Contributions of these terms in Equation (53) lies when $m = n + 3, m = n - 1, m = n + 1, m = n - 3$; hence it will be more appreciable to solve these relations separately to obtain an appropriate function thus;

**Case I:** $m = n + 3$

$$| \langle n+3 | H' | n \rangle |^2 = | \lambda_3 (\frac{\hbar}{2m_o})^\frac{3}{2} \sqrt{(n + 1)(n + 2)(n + 3)} |^2$$

$$= \lambda_3^2 (\frac{\hbar}{2m_o})^\frac{3}{2} (n + 1)(n + 2)n + 3$$

(56)
Case II: \( m = n - 1 \). Since it has three terms in the relation, hence we will obtain the single relation by adding the terms up so therefore we have that;

\[
|\langle n - 1 | H' | n \rangle|^2 = |\lambda_3 \left( \frac{h}{2m\omega} \right)^3 n + 1 + \sqrt{n} + n - 1\sqrt{n} + \frac{n}{\sqrt{n}}|^2 = \lambda_3^2 \left( \frac{h}{2m\omega} \right)^3 9n^3
\]

(57)

Case III: \( m = n + 1 \). Since it has three terms in the relation, hence we will obtain the single relation by adding the terms up so therefore we have that;

\[
|\langle n + 1 | H' | n \rangle|^2 = |\lambda_3 \left( \frac{h}{2m\omega} \right)^3 n + 1 + \sqrt{n} + 1 + n + 2\sqrt{n} + n + 1|^2 = \lambda_3^2 \left( \frac{h}{2m\omega} \right)^3 9(n + 1)^3
\]

(58)

Case IV: \( m = n - 3 \)

\[
|\langle n - 3 | H' | n \rangle|^2 = |\lambda_3 \left( \frac{h}{2m\omega} \right)^3 \sqrt{(n-2)(n-1)n}|^2 = \lambda_3^2 \left( \frac{h}{2m\omega} \right)^3 (n-2)(n-1)n
\]

(59)

Hence combining the contributions from all these terms obtained from Equation (56) to Equation (59) substituting them into Equation (13) we then have;

\[
E_n^2 = \frac{\lambda_3^4 \left( \frac{h}{2m\omega} \right)^6 ((n+1)(n+2)(n+3)+(n-2)(n-1)n+9(n+1)^2+9n^2)}{E_n^2 + E_n^2 n^2}
\]

(60)

Substituting Equation (21a) and (21b) into Equation (60) we obtain;

\[
E_n^2 = \lambda_3^2 \left( \frac{h}{2m\omega} \right)^3 \frac{1}{\hbar\omega} \left[ -\frac{1}{3} (n+1)(n+2)(n+3) + \frac{1}{3} (n-2)(n-1)n - 9(n+1)^3 + 9n^3 \right]
\]

(61)

With little effort in algebra we can solve further to obtain;

\[
E_n^2 = \frac{1}{8} \lambda_3^2 \frac{h^2}{m^2\omega^4} \left[ 30 \left( n+\frac{1}{2} \right)^2 + \frac{7}{2} \right]
\]

(62)

Therefore Equation (62) is the correction to the eigenvalue \( E_n \) of a one-dimensional quantum harmonic oscillator with a cubic perturbation potential. Thus, inserting Equation (21a), (52) and (62) into Equation (24) to get the complete eigenvalue of the system. The general expression of energy function due to \( H' \) for any perturbation now becomes;

\[
E_n = \hbar\omega \left( n + \frac{1}{2} \right) - \frac{1}{8} \frac{\lambda_k h^2}{m^2\omega^4} \left[ 30 \left( n + \frac{1}{2} \right)^2 + \frac{7}{2} \right]
\]

(63)

where \( n \) is a positive integer i.e. \( n = 0, 1, 2, 3 \ldots \). Therefore Equation (63) gives the exact correction to the energy and also the complete eigenvalues for the first and second approximation of one-dimensional quantum harmonic oscillator perturbed by a cubic potential.

V. ONE DIMENSIONAL QUANTUM ANHARMONIC OSCILLATOR WITH POLYNOMIAL PERTURBATION POTENTIAL

This time we will consider a polynomial potential perturbing on the same unperturbed Hamiltonian \( (H_o) \) as stated in Equation (1). Polynomial perturbation potential for integral values of \( k \), where \( k \), the order of the polynomial, is a positive integer i.e. \( k = 0, 1, 2, 3 \ldots \). as given in (Barry S, 1991);

\[
H' = \lambda_k x^k
\]

(64)

Equation (64) is the generalised perturbation. Hence, inserting it and Equation (1) into Equation (3) so that the total Hamiltonian will now become;

\[
H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 + \lambda_k x^k
\]

(65)

Equation (65) is the correction to the first-order energy and second-order energy by using the first order and second order energy shift formula, we adopted the method of perturbation theory.

A. First Order Energy Shift

Adopting Equation (5) which is \( E_n^1 = \langle \psi_n^0 | H' | \psi_n^0 \rangle \). Thus, putting Equation (63) into the equation we have that;

\[
E_n^1 = \lambda_k \langle \psi_n^0 | x^k | \psi_n^0 \rangle
\]

(66)

If \( \psi_n^0 = n \) in which \( n \) is the \( n \)th eigenfunction, therefore;
\[ E_n^1 = \lambda_k (n \mid x^k \mid n) \] (67)

Applying the ladder operators stated in Equation (8) that is;

\[ x = \sqrt{\frac{\hbar}{2m\omega}} [a^\dagger + a] \] but for \( x^k \) we have that

\[ x^k = \left( \frac{\hbar}{2m\omega} \right)^k [a^\dagger + a]^k \] (68)

Inserting Equation (68) into Equation (67), we now have that;

\[ E_n^1 = \lambda_k \left( \frac{\hbar}{2m\omega} \right)^k (n \mid [a^\dagger + a]^k \mid n) \] (69)

Solving the ladder operator in Equation (69) which is a binomial expansion of the power \( k \) we then have that;

\[ [a^\dagger + a]^k = a^k + kC_1 a^k \omega^{-1} a^{-1} + \ldots + kC_1 a^r a^{k-r} + \ldots \]

\[ + a^{r+k} \] (70)

For any integral values of \( k \) as given in (Pathak, 2000). Is expressed as;

\[ [a^\dagger + a]^k = \sum_{r=0,2,4,\ldots}^{k} t_r C_r (a^\dagger + a)^{k-r} \] (71)

\[ t_r = \frac{(r-1)!}{2^{(r-1)} \left( \frac{r}{2} - 1 \right)!} \] for \( r \geq 4 \) and \( t_0 = t_2 = 1 \) (72)

\[ E_n^1 = \lambda_k \left( \frac{\hbar}{2m\omega} \right)^k (n \mid \sum_{r=0,2,4,\ldots}^{k} t_r C_r (a^\dagger + a)^{k-r} \mid n) \] (73)

\[ = \lambda_k \left( \frac{\hbar}{2m\omega} \right)^k (n \mid \sum_{r=0,2,4,\ldots}^{k} t_r C_r e^{-k} a^{k-r} \mid n) \] (74)

\[ = \lambda_k \left( \frac{\hbar}{2m\omega} \right)^k \sum_{r=0,2,4,\ldots}^{k} t_r C_r e^{-k} a^{k-r} \left( \frac{k-r}{2} \right)! \] (75)

Equation (74) is the total correction to the energy eigenvalues for the first approximation of a one-dimensional quantum anharmonic oscillator perturbed by a polynomial potential. It involves summations that are more easily to evaluate.

\[ E_n^2 = \lambda^2 \left( \frac{\hbar}{2m\omega} \right)^k \sum_n \sum_m \left| \frac{\sum_{r=0,2,4,\ldots}^{k} t_r C_r (a^\dagger + a)^{k-r} \mid n} {E_m - E_n} \right|^2 \] (76)

Equation (78) can be rewritten by substituting Equation (21a) \& (21b).

\[ E_n^2 = \lambda^2 \left( \frac{\hbar}{2m\omega} \right)^k \sum_n \sum_m \left| \frac{\sum_{r=0,2,4,\ldots}^{k} t_r C_r (a^\dagger + a)^{k-r} \mid n} {E_m - E_n} \right|^2 \] (77)

B. Second-Order Energy Shift

At this point, we seek to obtain the second-order energy correction using the eigenvalue fundamental equation of the second-order perturbation theory as stated in Equation (13). Inserting Equation (64) into the numerator function of Equation (13). Thus, we then have that;

\[ \langle \psi_m^0 \mid H' \mid \psi_n^0 \rangle = \lambda_k (m \mid x^k \mid n) \] (78)

Therefore, substituting Equation (68) into Equation (75) we have that;

\[ \langle m \mid H' \mid n \rangle = \lambda_k \left( \frac{\hbar}{2m\omega} \right)^k (m \mid [a^\dagger + a]^k \mid n) \] (79)

Equation (71) can be substituted into Equation (76) to obtain that;

\[ \langle m \mid H' \mid n \rangle = \lambda_k \left( \frac{\hbar}{2m\omega} \right)^k \sum_{r=0,2,4,\ldots}^{k} t_r C_r (a^\dagger + a)^{k-r} \mid n) \] (80)

As functions of \( t_r \) as stated by (Pathak, 200). Thus, we can have the total correction to the energy eigenvalues for the second approximation of one-dimensional quantum anharmonic oscillator perturbed by a polynomial potential by substituting Equation (77) into (13) as;

\[ E_n^2 = \lambda^2 \left( \frac{\hbar}{2m\omega} \right)^k \sum_n \sum_m \left| \frac{\sum_{r=0,2,4,\ldots}^{k} t_r C_r (a^\dagger + a)^{k-r} \mid n} {E_m - E_n} \right|^2 \] (78)

Solving further we have;

\[ E_n^2 = \lambda^2 \left( \frac{\hbar}{2m\omega} \right)^k \sum_n \sum_m \left| \frac{\sum_{r=0,2,4,\ldots}^{k} t_r C_r (a^\dagger + a)^{k-r} \mid n} {E_m - E_n} \right|^2 \] (79)

Therefore, Equation (80) is the second-order energy correction to the eigenvalue \( E_n \) of a one-dimensional quantum anharmonic oscillator with a polynomial perturbation potential. Thus, inserting Equation (21a), (75)
and (80) into Equation (24), we get the complete eigenvalue of the system. The general expression of energy function due to $H'$ for any perturbation now becomes;

$$E_n = \hbar \omega \left( n + \frac{1}{2} \right) +$$

$$\lambda_k \left( \frac{\hbar}{2m\omega} \right)^k \sum_{i=0,2,4,\ldots} t_r^k C_r \frac{C_{kr-1}}{r} \left( \frac{k-\ell}{2} \right) ! +$$

$$A^2 \left( \frac{\hbar}{2m\omega} \right)^k \sum_{n,m} n^m \left| \left( m \right) \sum_{i=0,2,4,\ldots} t_r^k C_r \left( a^\dagger a^{-1} \right)^n \right| n^2 \right| \right|^2$$

where $n$, $m$ and $k$ are positive integers i.e., $(n,m,k = 0,1,2,3 \ldots )$. Therefore Equation (82) gives the exact correction to the energy and also the complete eigenvalues for the first and second approximation of one-dimensional quantum anharmonic oscillator perturbed by a polynomial potential.

VI. CONCLUSION

In this work, we have reported the solutions of one-dimensional quantum harmonic and anharmonic oscillator perturbed by a linear, quadratic, cubic and polynomial potential using the time-independent perturbation theory method. We presented the correction to first and second-order energy using the first and second-order energy shift equations. From the analysis, we observed that the first-order energy eigenvalues for linear and cubic potential are zero, but non-zero for a quadratic potential. The general expression of the energy eigenvalues due to $H'$ for the linear, quadratic, cubic and polynomial potential was obtained, and the values of the energy function depend on the values of $n$ and the order of polynomial $k$. The solutions could find applications in nonlinear deterministic equations encountered in quantum field theory and the qubit when quantum degrees of freedom in a potential well is either bound, free or scattered. More so, the solutions of the quantum anharmonic oscillator could be used to describe as well as predict the time evolution of quasiclassical systems employed in modelling equations describing stochastic processes such as probability distributions of stock price return which dynamics of its movement is considered an analogue of the motion of a quantum particle.

VII. ACKNOWLEDGEMENTS

The authors wish to give their sincere gratitude to the referee for his kind comments which improved the manuscript.

VIII. REFERENCES

Ariel, E & Philippe, L 2018, 'First and second-order relativistic corrections to the two and higher dimensional isotropic harmonic oscillator obeying the spinless Salpeter equation Ariel Edery and Philippe Laporte', J. Phys. Commun, vol. 2, no. 2, pp. 025024-33 5607.

Bacus, B, Meurice, Y & Soemadi A 1995, 'Precise determination of the energy levels of the anharmonic oscillator from the quantisation of the angle variable', Journal of Physics A: Mathematical and General, vol. 28, no. 14, pp. 381-386.

Barry, S 1991, 'Fifty years of eigenvalues perturbation theory', Bulletin (New series) of the American Mathematical Society, vol. 24, no. 2, pp. 303-319.

Bender, CM & Wu, TM 1969, 'Anharmonic oscillator', Physical Review Journal, vol. 184, no. 5, pp. 1231-1260.

Bhaumik, K & Dutta-Roy, B 1975, 'The classical nonlinear oscillator and the coherent state', Journal of Mathematical Physics, vol. 16, no. 5, pp. 1131-1155.

Chahjiny, SC, Letov, DA & Malney, VN 1991, 'Calculation of energy eigenvalues for the quantum anharmonic oscillator with a polynomial potential’, Journal of Physics A: Mathematical and General, vol. 35, no. 1, pp. 87-92.

David, JG 2016, 'Introduction to quantum mechanics', 2nd edn, ISBN-13: 978-1107179868.

Fernández, FM & Guardiola 1993, 'Accurate eigenvalues and eigenfunctions for quantum mechanical anharmonic oscillator', Journal of Physics A: Mathematical and General, vol. 26, no. 23, pp. 7169-7180.

Habtamu, D 2019, ‘Solution of harmonic oscillator perturbed by electric field’, Advances in Physics Theories and Applications, vol. 78, no. 1, pp. 1-4.
Halil, M 2018, ‘Energy levels of one-dimensional anharmonic oscillator via neural networks, Modern’, Physics Letter A, vol. 34, no. 12, pp. 1793-6632.

Hioe, FT, MacMillen, D & Montroll, EW 1978, ‘Quantum theory of anharmonic oscillators: energy levels of a single and a pair of coupled oscillators with quartic coupling’, Physics Report, vol. 43, no. 7, pp. 305-335.

Hsue, CS & Chern JL 1984, ‘Two-step approach to one-dimensional anharmonic oscillator’, Physical Review D, vol. 29, no. 4, pp. 643-647.

Ivanov, IA 1998, ‘Link between the strong-coupling and the weak-coupling asymptotic perturbation expansions for the quartic anharmonic oscillator’, Journal of Physics A: Mathematical and General, vol. 31, no. 33, pp. 6995-7003.

Jafarpour, M, Khalafi, G, Latifi AR & Ashrafpour, M 2003 ‘Classical and quantum sextic anharmonic oscillators: second-order solutions and the classical limit’, Nuovo Cimento, vol. 118, no.5, pp. 513-523.

Kunihiro, T 1993, ‘Renormalization – group resummation of a divergent series of the perturbation wave functions of the quantum anharmonic oscillator’, Physical Review D, vol. 57, no. 4, pp. 2035-2040.

Louissell, WH 1973, ‘Quantum statistical properties of radiation’, John Wiley and Sons, ISBN-13: 978-0471523659.

Matamala, AR & Maldonado, CR 2003, ‘A simple algebraic approach to a nonlinear quantum oscillator’, Physics Letter A, vol. 308, no. 5, pp. 319-322.

Mbagwu, JPC, Madububa, BI & Onwuemeka Ji 2020, ‘Article on basics of quantum theory’, International Journal of Scientific Research in Physics and Applied Sciences, vol. 8, no. 3, pp. 28-35.

Mbagwu JPC, Madububa, BI, Ozuomba, JO & Udoye MC, 2021, ‘Series solutions of mathematical problems of quantum mechanics’, Research Journal of Applied Sciences, vol. 16, no. 5, pp. 204-211.

McRae, SM & Vrscay 1997, ‘Perturbation theory and the classical limit of quantum mechanics’, Journal of Mathematical Physics, Vol. 38, no. 6, pp. 2899-2921.

Mojtaba, J & Davood A 2002, ‘Calculation of energy eigenvalues for the quantum anharmonic oscillator with a polynomial Potential’, Journal of Physics A: Mathematical and General, vol. 35, no. 2, pp. 87-92.

Nouredine, Z, Chichester, & Wiley 2009, ‘Quantum mechanics, concepts and application’, Contemporary Physics vol 52, no. 4, pp. 363-384.

Pathak, A 2000, ‘Generalized quantum anharmonic oscillator using an operator ordering approach’, Journal of Physics A: Mathematical and General, vol. 33, no. 31, pp. 5607-5613.

Peidae, P & Baghai-Wadji AR 2007, ‘On the calculation of linearly-perturbed harmonic oscillator’, Proceedings of ACES, Applied Computational Electromagnetic Society Conference, Verona, Italy, vol. 3, no. 4, pp. 485-489.

Rellich, F 1937, ‘Störungstheorie der spektralzerlegung’, Proceedings of the American Mathematical Society, vol. 113, no. 1, pp. 600-619.

Sergei, W 2006, ‘Perturbation theory for anharmonic oscillations’, GNU Free Documentation License version 1.2 or Later, The latex source code is a PDE file.

Sharma, LK & Fiase, JO 2000, ‘Non-perturbative energy expressions for the generalized anharmonic oscillator’, European Journal Physics, vol. 21, no. 2, pp. 167-174.

Sk’ala L, C & Zamastil, J 1999, ‘Large-order behaviour of the strong coupling perturbation expansion for anharmonic oscillator, oscillator using an operator ordering approach’, Journal of Physics A: Mathematical and General, vol. 32, no. 50, pp. 123-127.

Sk’ala L, C, Weniger, EJ & Zamastil, J 1999, ‘Large-order behaviour of the convergent perturbation theory for anharmonic oscillators’, Physical Review A, vol. 59 no.1, pp. 102-106.

Vinett, F & Čizek, J 1991, ‘Upper and lower bounds of the ground state energy of anharmonic oscillators using renormalized inner projection’, Journal Mathematical Physics, vol. 32, no. 12, pp. 3392-3.