Unbounded Weighted Composition Operators in $L^2$-Spaces

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Abstract. Basic properties of unbounded weighted composition operators are investigated. A description of the polar decomposition and a characterization of quasinormality in this context are provided. Criteria for subnormality of unbounded weighted composition operators written in terms of measurable families of probability measures are established.

1. PRELIMINARIES

1.1. Introduction. This paper is devoted to the study of unbounded weighted composition operators in $L^2$-spaces over $\sigma$-finite spaces (abbreviated to “weighted composition operators”). This class of operators includes the classes of composition and multiplication operators. What is more, it includes the class of weighted shifts on directed trees which has been a subject of intensive research in recent years. The questions of seminormality, k-expansivity and complete hyperexpansivity of unbounded composition operators were investigated in [13, 21]. A more systematic study of unbounded composition operators has recently been undertaken in [10] and continued in papers [7, 11, 12] (the last of the three provides new criteria for subnormality of unbounded composition operators). The class of weighted shifts on directed trees has been introduced in [22] and studied in [8, 9, 22, 23, 24, 26].

The literature on unbounded weighted composition operators (even on composition ones) is meagre. To the best of our knowledge, only one paper dealt with unbounded weighted composition operators (cf. [13, Section 6]). It contains characterizations of hyponormality and cohyponormality in this context (under some additional technical assumptions).

The notion of a bounded subnormal operator was introduced by Halmos in 1950 (cf. [19]). He also gave its first characterization. It was successively simplified by Bram [5], Embry [16] and Lambert [27]. None of them remains valid for unbounded operators (see [15] and [33, 34, 35] for foundations of the theory of bounded and unbounded subnormal operators). The only known general characterizations of subnormality of unbounded operators refer to semispectral measures.
or elementary spectral measures (cf. [4, 18, 37]). The other known general criteria for subnormality (with the exception of [38]), require the operator in question to have an invariant domain (cf. [34, 36, 14, 1]). In this paper we give criteria for subnormality of densely defined weighted composition operators in $L^2$-spaces without assuming the invariance of domains. These criteria are written in terms of measurable families of probability measures which satisfy the consistency condition (cf. Theorem 29). We also study the relationship between Radon-Nikodym derivatives attached to the operator in question and quasi-moments of the family mentioned above (cf. Theorem 27). This is related to the research undertaken in [12].

The present paper is an outcome of the first stage of our investigations on unbounded weighted composition operators.

1.2. Prerequisites. We write $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ for the sets of integers, real numbers and complex numbers, respectively. We denote by $\mathbb{N}$, $\mathbb{Z}_+$ and $\mathbb{R}_+$ the sets of positive integers, nonnegative integers and nonnegative real numbers, respectively. Set $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$. In what follows, we adhere to the convention that $0 \cdot \infty = \infty \cdot 0 = 0$ and $\frac{1}{0} = \infty$.

Put $\Delta \Delta' = (\Delta \setminus \Delta') \cup (\Delta' \setminus \Delta)$ for subsets $\Delta$ and $\Delta'$ of a set $X$. Given subsets $\Delta_n$ of $X$, $n \in \mathbb{N}$, we write $\Delta_n \not

\not \Delta$ as $n \to \infty$ if $\Delta_n \subset \Delta_{n+1}$ for every $n \in \mathbb{N}$ and $\Delta = \bigcup_{n=1}^{\infty} \Delta_n$. The characteristic function of a subset $\Delta$ of $X$ is denoted by $\chi_{\Delta}$. If $f$ is a $C$-valued or $\overline{\mathbb{R}}_+$-valued function on a set $X$, then we put $\{f = 0\} = \{x \in X : f(x) = 0\}$, $\{f \neq 0\} = \{x \in X : f(x) \neq 0\}$ and $\{f > 0\} = \{x \in X : f(x) \in \mathbb{R}_+\} \cap \{f \neq 0\}$. All measures considered in this paper are assumed to be positive. Given two measures $\mu$ and $\nu$ on the same $\sigma$-algebra, we write $\mu \ll \nu$ if $\mu$ is absolutely continuous with respect to $\nu$; if this is the case, then $\frac{d\mu}{d\nu}$ stands for the Radon-Nikodym derivative of $\mu$ with respect to $\nu$ (provided it exists). The $\sigma$-algebra of all Borel sets of a topological space $Z$ is denoted by $\mathcal{B}(Z)$. Given $t \in \mathbb{R}_+$, we write $\delta_t$ for the Borel probability measure on $\mathbb{R}_+$ concentrated at $t$.

The following auxiliary lemma is a direct consequence of [28, Proposition I-6-1] and [2, Theorem 1.3.10].

**Lemma 1.** Let $\mathcal{P}$ be a semi-algebra of subsets of a set $X$ and $\mu_1, \mu_2$ be measures on $\sigma(\mathcal{P})$ such that $\mu_1(\Delta) = \mu_2(\Delta)$ for all $\Delta \in \mathcal{P}$. Suppose there exists a sequence $\{\Delta_n\}_{n=1}^{\infty} \subseteq \mathcal{P}$ such that $\Delta_n \not \Delta X$ as $n \to \infty$ and $\mu_1(\Delta_k) < \infty$ for every $k \in \mathbb{N}$. Then $\mu_1 = \mu_2$.

The proof of the following useful fact is left to the reader.

**Lemma 2.** If $(X, \mathcal{A}, \mu)$ is a $\sigma$-finite measure space and $f, g$ are $\mathcal{A}$-measurable complex functions on $X$ such that $\int_\Delta |f| \, d\mu < \infty$, $\int_\Delta |g| \, d\mu < \infty$ and $\int_\Delta f \, d\mu = \int_\Delta g \, d\mu$ for every $\Delta \subseteq \mathcal{A}$ such that $\mu(\Delta) < \infty$, then $f = g$ a.e. $[\mu]$.

In what follows we write $\int_0^{\infty}$ in place of $\int_{\mathbb{R}_+}$. Using [32, Theorem 1.39(a)], we obtain a characterization of Borel probability measures on $\mathbb{R}_+$ whose first moments vanish.

**Lemma 3.** If $\mu$ is a Borel probability measure on $\mathbb{R}_+$, then the following conditions are equivalent:

(i) $\mu = \delta_0$,

(ii) $\int_0^{\infty} t \, d\mu(t) = 0$, 

(iii) $\int_0^\infty t^k \, d\mu(t) = 0$ for some $k \in \mathbb{N}$ (equivalently, for every $k \in \mathbb{N}$).

Let $A$ be an operator in a complex Hilbert space $\mathcal{H}$ (all operators considered in this paper are linear). Denote by $\mathcal{D}(A)$, $\mathcal{N}(A)$, $\mathcal{R}(A)$ and $A^*$ the domain, the kernel, the range and the adjoint of $A$ (in case it exists) respectively. We write $\| \cdot \|_A$ for the graph norm of $A$, i.e., $\|f\|^2_A = \|f\|^2 + \|Af\|^2$ for $f \in \mathcal{D}(A)$. Given two operators $A$ and $B$ in $\mathcal{H}$, we write $A \subseteq B$ if $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and $Af = Bf$ for all $f \in \mathcal{D}(A)$. A densely defined operator $N$ in $\mathcal{H}$ is said to be normal if $N$ is closed and $N^*N = NN^*$ (or equivalently if and only if $\mathcal{D}(N) = \mathcal{D}(N^*)$ and $\|Nf\| = \|N^*f\|$ for all $f \in \mathcal{D}(N)$, see [3]). A closed densely defined operator $Q$ in $\mathcal{H}$ is said to be quasinormal if $U|Q| \subseteq |Q|U$, where $|Q|$ is the modulus of $Q$ and $Q = U|Q|$ is the polar decomposition of $Q$ (cf. [6, 34]). As shown in [25, Theorem 3.1],

A closed densely defined operator $Q$ in $\mathcal{H}$ is quasinormal if and only if $Q|Q|^2 = |Q|^2Q$. (1)

We say that a densely defined operator $S$ in $\mathcal{H}$ is subnormal if there exist a complex Hilbert space $\mathcal{K}$ and a normal operator $N$ in $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding), $\mathcal{D}(S) \subseteq \mathcal{D}(N)$ and $Sf = Nf$ for all $f \in \mathcal{D}(S)$. A densely defined operator $A$ in $\mathcal{H}$ is called hyponormal if $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $\|A^*f\| \leq \|Af\|$ for all $f \in \mathcal{D}(A)$. It is well-known that quasinormal operators are subnormal (see [6, Theorem 1] and [34, Theorem 2]) and subnormal operators are hyponormal (cf. [30, Lemma 2.8]), but none of these implications can be reversed in general (this can be seen by considering unilateral weighted shifts). In what follows $B(\mathcal{H})$ stands for the $C^*$-algebra of all bounded operators in $\mathcal{H}$ whose domains are equal to $\mathcal{H}$.

We write $I = I_\mathcal{H}$ for the identity operator on $\mathcal{H}$.

The following lemma turns out to be useful.

**Lemma 4.** Let $A$ and $B$ be positive selfadjoint operators in $\mathcal{H}$ such that $\mathcal{D}(A) = \mathcal{D}(B)$ and $\|Af\| = \|Bf\|$ for every $f \in \mathcal{D}(A)$. Then $A = B$.

**Proof.** Note that $\overline{\mathcal{R}(A)} = \mathcal{N}(A)^\perp = \mathcal{N}(B)^\perp = \overline{\mathcal{R}(B)}$. It follows from our assumptions that there exists a unique unitary operator $\hat{U} \in B(\overline{\mathcal{R}(A)})$ such that $\hat{U}A = B$. Then $U := \hat{U} \oplus I_{\mathcal{N}(A)}$ is unitary and $UA = B$. Hence $B^2 = B^*B = A^*U^*UA = A^2$, which, by uniqueness of square roots, implies that $A = B$. \hfill $\Box$

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\mathcal{B} \subseteq \mathcal{A}$ be a $\sigma$-algebra. We say that $\mathcal{B}$ is relatively $\mu$-complete if $\mathcal{A}_0 \subseteq \mathcal{B}$, where $\mathcal{A}_0 = \{\Delta \in \mathcal{A} : \mu(\Delta) = 0\}$ (cf. [32, Chapter 8]). The smallest relatively $\mu$-complete $\sigma$-algebra containing $\mathcal{B}$, denoted by $\mathcal{B}^\mu$, is equal to the $\sigma$-algebra generated by $\mathcal{B} \cup \mathcal{A}_0$. Moreover, we have

$$\mathcal{B}^\mu = \{\Delta \in \mathcal{A} : \exists \Delta' \in \mathcal{B} : \mu(\Delta \triangle \Delta') = 0\}. \quad (2)$$

The $\mathcal{B}^\mu$-measurable functions are described in [32, Lemma 1, p. 169].

Let $(X, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space. We shall abbreviate the expressions “almost everywhere with respect to $\mu$” and “for $\mu$-almost every $x$” to “a.e. $[\mu]$” and “for $\mu$-a.e. $x$”, respectively. We call a map $\phi : X \to X$ a transformation of $X$ and write $\phi^{-1}(\mathcal{A}) = \{\phi^{-1}(\Delta) : \Delta \in \mathcal{A}\}$. For $n \in \mathbb{N}$, we denote by $\phi^n$ the $n$-fold composition of $\phi$ with itself. We write $\phi^0$ for the identity transformation $\text{id}_X$ of $X$. Let $\phi^n(\Delta) = (\phi^n)^{-1}(\Delta)$ for $\Delta \in \mathcal{A}$ and $n \in \mathbb{N}$. A transformation $\phi$ of $X$ is said to be $\mathcal{A}$-measurable if $\phi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$. Clearly, if $\phi$ is $\mathcal{A}$-measurable, then so is $\phi^n$ for every $n \in \mathbb{N}$. Let $\phi$ be an $\mathcal{A}$-measurable transformation of $X$. If
It is easily seen that the measures \( \mu^w, \mu_w : \mathcal{A} \to \mathbb{R}_+ \) are \( \sigma \)-finite and mutually absolutely continuous. Moreover, if \( u : X \to \mathbb{C} \) is \( \mathcal{A} \)-measurable and \( u = w \) a.e. \( [\mu] \), then \( \mu(\{u \neq 0\} \triangle \{w \neq 0\}) = 0 \), \( \mu^w = \mu^u \) and \( \mu_u = \mu_w \). It is worth noting that a property \( \mathcal{P} \) of points of the set \( X \) holds a.e. \( [\mu_w] \) if and only if \( \mathcal{P} \) holds a.e. \( [\mu] \) on \( \{w \neq 0\} \).

By the Radon-Nikodym theorem (cf. [2, Theorem 2.2.1]), if \( \mu_w \circ \phi^{-1} \ll \mu \), then there exists a unique (up to a.e. \( [\mu] \) equivalence) \( \mathcal{A} \)-measurable function \( h_{\phi,w} : X \to \mathbb{R}_+ \) such that

\[
\mu_w \circ \phi^{-1}(\Delta) = \int_{\Delta} h_{\phi,w} \, d\mu, \quad \Delta \in \mathcal{A}.
\]

It follows from [2, Theorem 1.6.12] and [32, Theorem 1.29] that for every \( \mathcal{A} \)-measurable function \( f : X \to \mathbb{R}_+ \) (or for every \( \mathcal{A} \)-measurable function \( f : X \to \mathbb{C} \) such that \( f \circ \phi \in L^1(\mu_w) \)),

\[
\int_X f \circ \phi \, d\mu_w = \int_X f \, h_{\phi,w} \, d\mu.
\]

Note that the set \( \{h_{\phi,w} > 0\} \), which often appears in this paper, is determined up to a.e. \( [\mu] \) equivalence.

The following assumption will be used frequently.

The triplet \( (X, \mathcal{A}, \mu) \) is a \( \sigma \)-finite measure space, \( w \) is an \( \mathcal{A} \)-measurable complex function on \( X \) and \( \phi \) is an \( \mathcal{A} \)-measurable transformation of \( X \) such that \( \mu_w \circ \phi^{-1} \ll \mu \).

**Lemma 5.** Suppose (5) holds and \( f \) and \( g \) are \( \mathcal{A} \)-measurable \( \mathbb{R}_+ \)-valued or \( \mathbb{C} \)-valued functions on \( X \). Then

\[
f \circ \phi = g \circ \phi \text{ a.e. } [\mu_w], \text{ if and only if } \chi_{\{h_{\phi,w} > 0\}} \cdot f = \chi_{\{h_{\phi,w} > 0\}} \cdot g \text{ a.e. } [\mu].
\]

**Proof.** This follows from the equality \( \mu_w(\phi^{-1}(Y)) = \int_Y h_{\phi,w} \, d\mu \), where \( Y = \{x \in X : f(x) \neq g(x)\} \). \( \square \)

The following lemma is written in flavour of [20, 17, 10, 12].

**Lemma 6.** Suppose (5) holds. Then the following assertions are valid:

(i) \( h_{\phi,w} \circ \phi > 0 \) a.e. \( [\mu_w] \),

(ii) if \( h_{\phi,w} < \infty \) a.e. \( [\mu] \), then for every \( \mathcal{A} \)-measurable \( f : X \to \mathbb{R}_+ \),

\[
\int_X f \circ \phi \, d\mu_w = \int_{\{h_{\phi,w} > 0\}} f \, d\mu.
\]

**Proof.** (i) This follows from the equalities

\[
\mu_w(\{h_{\phi,w} = 0\}) = \int_X \chi_{\{h_{\phi,w} = 0\}} \circ \phi \, d\mu_w = \int_{\{h_{\phi,w} = 0\}} h_{\phi,w} \, d\mu = 0.
\]
(ii) Employing (i), we get
\[
\int_X \frac{f \circ \phi}{h_{\phi,w} \circ \phi} \, d\mu_w = \int_{\{h_{\phi,w} \circ \phi > 0\}} \frac{f \circ \phi}{h_{\phi,w} \circ \phi} \, d\mu_w
\]
\[
= \int_X \chi_{\{h_{\phi,w} > 0\}} \circ \phi \cdot \frac{f \circ \phi}{h_{\phi,w} \circ \phi} \, d\mu_w \quad \text{by (4)}
\]
which completes the proof. \(\square\)

1.3. Weighted composition operators. Let \((X, \mathcal{A}, \mu)\) be a \(\sigma\)-finite measure space, \(w\) be an \(\mathcal{A}\)-measurable complex function on \(X\) and \(\phi\) be an \(\mathcal{A}\)-measurable transformation of \(X\). Denote by \(L^2(\mu)\) the Hilbert space of all square summable (with respect to \(\mu\)) \(\mathcal{A}\)-measurable complex functions on \(X\). Define a weighted composition operator \(C_{\phi,w}: L^2(\mu) \supseteq \mathcal{D}(C_{\phi,w}) \to L^2(\mu)\) by
\[
\mathcal{D}(C_{\phi,w}) = \{f \in L^2(\mu): w \cdot (f \circ \phi) \in L^2(\mu)\},
\]
\[
C_{\phi,w}f = w \cdot (f \circ \phi), \quad f \in \mathcal{D}(C_{\phi,w}).
\]
We call \(\phi\) and \(w\) the symbol and the weight of \(C_{\phi,w}\) respectively. Of course, such operator may not be well-defined. The circumstances under which the definition of \(C_{\phi,w}\) is correct are provided below. In addition, it is shown that the weighted composition operator \(C_{\phi,w}\) does not depend on a change of the weight and the symbol on a set of measure zero.

PROPOSITION 7. Let \((X, \mathcal{A}, \mu)\) be a \(\sigma\)-finite measure space, \(w\) be an \(\mathcal{A}\)-measurable complex function on \(X\) and \(\phi\) be an \(\mathcal{A}\)-measurable transformation of \(X\). Then the following conditions are equivalent:

(i) \(C_{\phi,w}\) is well-defined,
(ii) \(\mu^w \circ \phi^{-1} \ll \mu ,\)
(iii) \(\mu_w \circ \phi^{-1} \ll \mu .\)

Moreover, if \(C_{\phi,w}\) is well-defined, \(u: X \to \mathbb{C}\) and \(\psi: X \to X\) are \(\mathcal{A}\)-measurable, \(u = w\ a.e.\ [\mu]\) and \(\psi = \phi\ a.e.\ [\mu],\) then \(C_{\psi,u}\) is well-defined, \(C_{\psi,u} = C_{\phi,w},\)
\[
\mu_u = \mu_w, \quad h_{\psi,u} = h_{\phi,w} \ a.e.\ [\mu] \quad \text{and} \quad (\psi^{-1}(\mathcal{A}))^{\mu_u} = (\phi^{-1}(\mathcal{A}))^{\mu_w}.
\]

PROOF. First note that for every function \(f: X \to \mathbb{C},\)
\[
\{w \cdot (f \circ \phi) \neq 0\} = \phi^{-1}(\{f \neq 0\}) \cap \{w \neq 0\}.
\]
(7)

(i)\(\Rightarrow\)(ii) Take \(\Delta \in \mathcal{A}\) such that \(\mu(\Delta) = 0.\) Then \(\chi_{\Delta} = 0\ a.e.\ [\mu],\) and thus, by (i), \(w \cdot (\chi_{\Delta} \circ \phi) = 0\ a.e.\ [\mu].\) Using (7) with \(f = \chi_{\Delta},\) we get \(\mu^w(\phi^{-1}(\Delta)) = 0.\)

(ii)\(\Rightarrow\)(i) If \(f: X \to \mathbb{C}\) is \(\mathcal{A}\)-measurable and \(f = 0\ a.e.\ [\mu],\) then \(\mu(\{f \neq 0\}) = 0,\) which, by (7) and (ii), gives \(w \cdot (f \circ \phi) = 0\ a.e.\ [\mu].\) Hence \(C_{\phi,w}\) is well-defined.

(i)\(\Leftrightarrow\)(iii) Note that the measures \(\mu^w\) and \(\mu_{\phi}\) are mutually absolutely continuous and apply the equivalence (i)\(\Leftrightarrow\)(ii).

To justify the “moreover” part, note that \(\mu_w = \mu_u,\) and \(\mu_w \circ \phi^{-1} = \mu_u \circ \psi^{-1}\) because
\[
\mu_w(\psi^{-1}(\Delta)) \Delta \phi^{-1}(\Delta) = \int_X |\chi_{\Delta} \circ \psi - \chi_{\Delta} \circ \phi|^2 \, d\mu_w = 0, \quad \Delta \in \mathcal{A}.
\]
This together with (2) completes the proof. \(\square\)

Below, we discuss a particular instance of a weighted composition operator.
Remark 8. Let \((X, \mathcal{A}, \mu)\) is a \(\sigma\)-finite measure space, \(Y\) be a nonempty subset of \(X\) and \(\psi : Y \to X\) be an \(\mathcal{A}\)-measurable mapping (i.e., \(\psi^{-1}(\Delta) \in \mathcal{A}\) for every \(\Delta \in \mathcal{A}\)). Call the operator \(C_\psi : L^2(\mu) \supseteq \mathcal{D}(C_\psi) \to L^2(\mu)\) given by

\[
\mathcal{D}(C_\psi) = \left\{ f \in L^2(\mu) : \int_Y |f \circ \psi|^2 \, d\mu < \infty \right\},
\]

\[
(C_\psi f)(x) = \begin{cases} f(\psi(x)) & \text{if } x \in Y, \\ 0 & \text{if } x \in X \setminus Y, \end{cases}
\]
a partial composition operator (note that \(Y = \psi^{-1}(X) \in \mathcal{A}\)). Arguing as in [29, p. 38], one can show that \(C_\psi\) is well-defined if and only if \(\mu \circ \psi^{-1} \ll \mu\), where \(\mu \circ \psi^{-1}(\Delta) = \mu(\psi^{-1}(\Delta))\) for \(\Delta \in \mathcal{A}\). Set \(w = \chi_Y\) and take any \(\mathcal{A}\)-measurable transformation \(\phi\) of \(X\) which extends \(\psi\). Since \(\mu(\psi^{-1}(\Delta)) = \mu^w(\phi^{-1}(\Delta))\) for \(\Delta \in \mathcal{A}\), we deduce that \(C_\psi\) is well-defined if and only if \(C_{\phi,w}\) is well-defined. If this is the case, then \(C_\psi = C_{\phi,w}\).

Now we describe the graph norm of \(C_{\phi,w}\).

**Proposition 9.** If (5) holds, then

(i) \(\mathcal{D}(C_{\phi,w}) = L^2((1 + h_{\phi,w}) \, d\mu)\),

(ii) \(\|f\|_{C_{\phi,w}}^2 \leq \int_X |f|^2 (1 + h_{\phi,w}) \, d\mu\) for \(f \in \mathcal{D}(C_{\phi,w})\),

(iii) \(C_{\phi,w}\) is closed.

**Proof.** If \(f : X \to \mathbb{C}\) is \(\mathcal{A}\)-measurable, then

\[
\int_X |f \circ \phi|^2 |w|^2 \, d\mu = \int_X |f \circ \phi|^2 \, d\mu_w \overset{(4)}{=} \int_X |f|^2 h_{\phi,w} \, d\mu,
\]

which implies (i) and (ii), and consequently (iii). \(\square\)

For later use we single out the following fact whose proof is left to the reader.

If (5) holds and \(h_{\phi,w} < \infty\) a.e. \([\mu]\), then there exists a sequence \(\{X_n\}_{n=1}^{\infty} \subseteq \mathcal{A}\) such that \(\mu(X_n) < \infty\) and \(h_{\phi,w} \leq n\) a.e. \([\mu]\) on \(X_n\) for every \(n \in \mathbb{N}\), and \(X_n \nearrow X\) as \(n \to \infty\).

The question of dense definiteness of \(C_{\phi,w}\) is answered below.

**Proposition 10.** If (5) holds, then the following conditions are equivalent:

(i) \(C_{\phi,w}\) is densely defined,

(ii) \(h_{\phi,w} < \infty\) a.e. \([\mu]\),

(iii) \(\mu_w \circ \phi^{-1}\) is \(\sigma\)-finite,

(iv) \(\mu_w |_{\phi^{-1}(\mathcal{A})}\) is \(\sigma\)-finite.

**Proof.** (i)\(\Rightarrow\)(ii) Set \(Y = \{h_{\phi,w} = \infty\}\). Clearly, by Proposition 9(i), \(f|_Y = 0\) a.e. \([\mu]\) for every \(f \in \mathcal{D}(C_{\phi,w})\). This and (i) imply that \(f|_Y = 0\) a.e. \([\mu]\) for every \(f \in L^2(\mu)\). By the \(\sigma\)-finiteness of \(\mu\), we have \(\mu(Y) = 0\).

(ii)\(\Rightarrow\)(iii) Let \(\{X_n\}_{n=1}^{\infty}\) be as in (8). Then

\[
\mu_w \circ \phi^{-1}(X_n) \overset{(3)}{=} \int_{X_n} h_{\phi,w} \, d\mu \leq n\mu(X_n) < \infty, \quad n \in \mathbb{N}.
\]

This yields (iii).

(iii)\(\Rightarrow\)(iv) Evident.

(iv)\(\Rightarrow\)(i) Let \(\{X_n\}_{n=1}^{\infty} \subseteq \mathcal{A}\) be a sequence such that \(\phi^{-1}(X_n) \nearrow X\) as \(n \to \infty\) and \(\mu_w(\phi^{-1}(X_k)) < \infty\) for every \(k \in \mathbb{N}\). Without loss of generality we can assume
that $X_n \nearrow X_\infty := \bigcup_{n=1}^\infty X_k$ as $n \to \infty$. It follows from (3) that $h_{\phi,w} < \infty$ a.e. $[\mu]$ on $X_k$ for every $k \in \mathbb{N}$. This implies that $h_{\phi,w} < \infty$ a.e. $[\mu]$ on $X_\infty$. Since $\phi^{-1}(X_n) \nearrow \phi^{-1}(X_\infty) = X$ as $n \to \infty$, we get $\phi^{-1}(X \setminus X_\infty) = \emptyset$. By (3), we have $\int_{X \setminus X_\infty} h_{\phi,w} \, d\mu = 0$, which yields $h_{\phi,w} = 0$ a.e. $[\mu]$ on $X \setminus X_\infty$. Hence, $h_{\phi,w} < \infty$ a.e. $[\mu]$. Applying Proposition 9(i) and \cite{10, Lemma 12.1}, we obtain (i).

**Caution.** To simplify terminology throughout the rest of the paper, in saying that “$C_{\phi,w}$ is densely defined”, we tacitly assume that $C_{\phi,w}$ is well-defined.

Our next aim is to adapt \cite{10, Proposition 6.2} to the present context. Before doing this, let us note that, by (4), the kernel of $C_{\phi,w}$ can be described as follows.

**Lemma 11.** Suppose (5) holds. Then $N(C_{\phi,w}) = \chi_{\{h_{\phi,w} = 0\}} L^2(\mu)$. Moreover, $N(C_{\phi,w}) = \{0\}$ if and only if $h_{\phi,w} > 0$ a.e. $[\mu]$.

As opposed to the case of composition operators, a subnormal weighted composition operator may not be injective (because multiplication operators are particular instances of weighted composition operators). However, the part of the kernel of such operator that is related to the support of the weight must be trivial.

**Proposition 12.** Suppose (5) holds. Then the following conditions are equivalent:

1. $\chi_{\{w \neq 0\}} N(C_{\phi,w}) \neq \{0\}$,
2. $\mu(\{h_{\phi,w} = 0\} \cap \{w \neq 0\}) = 0$,
3. $h_{\phi,w} > 0$ a.e. $[\mu_w]$,
4. $\chi_{\{h_{\phi,w} = 0\}} = \chi_{\{h_{\phi,w} = 0\}} \circ \phi$ a.e. $[\mu_w]$.

Moreover, if $C_{\phi,w}$ is densely defined, then any of the above conditions is equivalent to the following one:

5. $\chi_{\{w \neq 0\}} N(C_{\phi,w}) \subseteq N(C_{\phi,w}^*)$.

**Proof.** (i)$\Rightarrow$(ii) This follows from the $\sigma$-finiteness of $\mu$ and Lemma 11.

(ii)$\Rightarrow$(iii) Clear.

(iii)$\Rightarrow$(iv) Since, by Lemma 6(i), $\chi_{\{h_{\phi,w} = 0\}} \circ \phi = 0$ a.e. $[\mu_w]$, we are done.

Now we assume that $C_{\phi}$ is densely defined.

(i)$\Rightarrow$(v) Evident.

(v)$\Rightarrow$(iii) By Proposition 10, $h_{\phi,w} < \infty$ a.e. $[\mu]$. Hence, there exists a sequence $\{X_n\}_{n=1}^\infty \subseteq \mathscr{A}$ such that $X_n \nearrow X$ as $n \to \infty$ and $\mu(X_k) < \infty$, $h_{\phi,w} \leq k$ a.e. $[\mu]$ on $X_k$ and $|w| \leq k$ a.e. $[\mu]$ on $X_k$ for every $k \in \mathbb{N}$. Set $Y_n = X_n \cap \{h_{\phi,w} = 0\}$ for $n \in \mathbb{N}$. It follows from Proposition 9(i) that $\{w \cdot \chi_{Y_n}\}_{n=1}^\infty \subseteq \chi_{\{w \neq 0\}} \mathcal{D}(C_{\phi,w})$ and $\{\chi_{X_n}\}_{n=1}^\infty \subseteq \mathcal{D}(C_{\phi,w})$. This implies that

$$
\|C_{\phi,w}(w \cdot \chi_{Y_n})\|^2 = \int_X |w \circ \phi|^2 \cdot \chi_{Y_n} \circ \phi \, d\mu_w \overset{(4)}{=} \int_X |w|^2 \chi_{Y_n} h_{\phi,w} \, d\mu = 0, \quad n \in \mathbb{N},
$$

and thus by our assumptions $\{w \cdot \chi_{Y_n}\}_{n=1}^\infty \subseteq N(C_{\phi,w}^*)$. As a consequence, we have

$$
0 = \langle w \cdot \chi_{Y_n}, C_{\phi,w} \chi_{X_n} \rangle = \int_X \chi_{X_n} \circ \phi \cdot \chi_{Y_n} \, d\mu_w = \mu_w(Y_n \cap \phi^{-1}(X_n)), \quad n \in \mathbb{N}.
$$
By continuity of measures, we conclude that \( \mu_w(\{h_{\phi,w} = 0\}) = 0 \), which gives (iii). This completes the proof. \( \square \)

Since \( N(A) \subseteq N(A^*) \) for every hyponormal operator \( A \), we get the following.

**Corollary 13.** If (5) holds and \( C_{\phi,w} \) is hyponormal, then \( h_{\phi,w} > 0 \) a.e. \([\mu_w] \).

It is worth mentioning that, in view of Proposition 12, a characterization of injectivity of composition operators given in [10, Proposition 6.2] remains valid for weighted composition operators \( C_{\phi,w} \) for which \( w \neq 0 \) a.e. \([\mu] \) (because then the measures \( \mu_w \) and \( \mu \) are mutually absolutely continuous).

### 1.4. Conditional expectation

In this section we discuss basic properties of the conditional expectation \( E_{\phi,w} \) which plays a crucial role in our considerations. Suppose (5) holds and \( h_{\phi,w} < \infty \) a.e. \([\mu] \). By Proposition 10, the measure \( \mu_w|_{\phi^{-1}(\mathcal{A})} \) is \( \sigma \)-finite. It follows from the Radon-Nikodym theorem that for every \( \mathcal{A} \)-measurable function \( f : X \to \mathbb{R}_+ \) there exists a unique (up to a.e. \([\mu_w] \) equivalence) \( \phi^{-1}(\mathcal{A}) \)-measurable function \( E_{\phi,w}(f) : X \to \mathbb{R}_+ \) such that

\[
\int_{\phi^{-1}(\Delta)} f \, d\mu_w = \int_{\phi^{-1}(\Delta)} E_{\phi,w}(f) \, d\mu_w, \quad \Delta \in \mathcal{A}.
\]

We call \( E_{\phi,w}(f) \) the conditional expectation of \( f \) with respect to \( \phi^{-1}(\mathcal{A}) \) (cf. [31]).

By applying the standard approximation procedure, we deduce that for all \( \mathcal{A} \)-measurable functions \( f, g : X \to \mathbb{R}_+ \), we have

\[
\int_X g \circ \phi \cdot f \, d\mu_w = \int_X g \circ \phi \cdot E_{\phi,w}(f) \, d\mu_w.
\]

As shown in [10, Appendix B], \( E_{\phi,w}(f) \in L^2(\mu_w) \) for every nonnegative \( f \in L^2(\mu_w) \).

Hence, by linearity we can define \( E_{\phi,w} \) on \( L^2(\mu_w) \) so that

if \( f \in L^2(\mu_w) \), then \( E_{\phi,w}(f) \in L^2(\mu_w) \), \( E_{\phi,w}(f) = E_{\phi,w}(\overline{f}) \) and (10) holds for every \( \mathcal{A} \)-measurable complex function \( g \) on \( X \) such that \( g \circ \phi \in L^2(\mu_w) \).

The following lemma is patterned on [13, page 325].

**Proposition 14.** Suppose (5) holds. Assume that \( f \) is an \( \mathcal{A} \)-measurable \( \mathbb{R}_+ \)-valued (respectively, \( \mathbb{C} \)-valued) function on \( X \). Then there exists an \( \mathcal{A} \)-measurable \( \mathbb{R}_+ \)-valued (respectively, \( \mathbb{C} \)-valued) function \( g \) on \( X \) such that \( f \circ \phi = g \circ \phi \) a.e. \([\mu_w] \) and \( g = 0 \) a.e. \([\mu] \) on \( \{h_{\phi,w} = 0\} \). Moreover, such \( g \) is unique up to a.e. \([\mu] \) equivalence and is given by \( g = f \cdot \chi_{\{h_{\phi,w} > 0\}} \).

**Proof.** By Lemma 6(i), \( \chi_{\{h_{\phi,w} > 0\}} \circ \phi = 1 \) a.e. \([\mu_w] \). This implies that

\[
(f \cdot \chi_{\{h_{\phi,w} > 0\}}) \circ \phi = (f \circ \phi) \cdot \chi_{\{h_{\phi,w} > 0\}} \circ \phi = f \circ \phi \text{ a.e. } [\mu_w].
\]

The uniqueness statement follows from Lemma 5. \( \square \)

Assume that (5) holds and \( h_{\phi,w} < \infty \) a.e. \([\mu] \). If \( f : X \to \mathbb{R}_+ \) is an \( \mathcal{A} \)-measurable function (respectively, \( f \in L^2(\mu_w) \)), then, by the well-known description of \( \phi^{-1}(\mathcal{A}) \)-measurable functions and Proposition 14, \( E_{\phi,w}(f) = g \circ \phi \) a.e. \([\mu_w] \) with some \( \mathcal{A} \)-measurable \( \mathbb{R}_+ \)-valued (respectively, \( \mathbb{C} \)-valued) function \( g \) on \( X \) such that
g = g \cdot \chi_{\{h_{\phi,w} > 0\}} \text{ a.e. } [\mu]. \text{ Set } E_{\phi,w}(f) \circ \phi^{-1} = g \text{ a.e. } [\mu]. \text{ By Proposition 14, this definition is correct, and by Lemma 5 the following equality holds}
\begin{equation}
(E_{\phi,w}(f) \circ \phi^{-1}) \circ \phi = E_{\phi,w}(f) \text{ a.e. } [\mu_{w}|_{\phi^{-1}(A)}].
\end{equation}
(Of course, the expression “a.e. [\mu_{w}|_{\phi^{-1}(A)}]” in (12) can be replaced by “a.e. [\mu_{w}].”)

It is worth noticing that
\begin{equation}
\int_{\phi^{-1}(A)} E_{\phi,w}(f) \, d\mu_{w} = \int_{\phi^{-1}(A)} f \, d\mu_{w} \leq \int_{\phi^{-1}(A)} c \, d\mu_{w}, \quad \Delta \in \mathcal{A},
\end{equation}
and using the fact that \(\mu_{w}|_{\phi^{-1}(A)}\) is \(\sigma\)-finite (cf. Proposition 10), we get \(E_{\phi,w}(f) \leq c \text{ a.e. } [\mu_{w}]\). To prove the remaining part of the conclusion, note that by (4) we have
\begin{equation}
\int_{\Delta} E_{\phi,w}(f) \circ \phi^{-1} \cdot h_{\phi,w} \, d\mu = \int_{\phi^{-1}(A)} E_{\phi,w}(f) \, d\mu_{w}
\leq \int_{\phi^{-1}(A)} c \, d\mu_{w} = \int_{\Delta} c \cdot h_{\phi,w} \, d\mu, \quad \Delta \in \mathcal{A}.
\end{equation}
Since \(\mu \text{ is } \sigma\)-finite, \(h_{\phi,w} < \infty \text{ a.e. } [\mu] \) and \(E_{\phi,w}(f) \circ \phi^{-1} = 0 \text{ a.e. } [\mu] \) on \(\{h_{\phi,w} = 0\}\), the proof is complete. \(\square\)

Note that, in view of (13), Proposition 15 is no longer true if “\(\leq\)” is replaced by “\(=\)” (or by “\(\geq\)”).

The Radon-Nikodym derivative \(\frac{d\mu_{w}|_{\phi^{-1}}}{d\mu} \) can be expressed in terms of \(h_{\phi,w}\).

**Proposition 16.** If (5) holds and \(h_{\phi,w} < \infty \text{ a.e. } [\mu] \), then \(\mu_{w} \circ \phi^{-1} \ll \mu \) and
\begin{equation}
\frac{d\mu_{w} \circ \phi^{-1}}{d\mu} = h_{\phi,w} \cdot E_{\phi,w}(\chi_{\{w \neq 0\}} \cdot \frac{1}{|w|^{2}}) \circ \phi^{-1} \text{ a.e. } [\mu].
\end{equation}

**Proof.** Since
\begin{equation}
\mu_{w}(\phi^{-1}(\Delta)) = \int_{\phi^{-1}(A)} \chi_{\{w \neq 0\}} \cdot \frac{1}{|w|^{2}} \, d\mu_{w} = \int_{\phi^{-1}(\Delta)} E_{\phi,w}(\chi_{\{w \neq 0\}} \cdot \frac{1}{|w|^{2}}) \, d\mu_{w}
\end{equation}
\(\text{(4) & (12)}\)
\begin{equation}
\int_{\Delta} h_{\phi,w} \cdot E_{\phi,w}(\chi_{\{w \neq 0\}} \cdot \frac{1}{|w|^{2}}) \circ \phi^{-1} \, d\mu, \quad \Delta \in \mathcal{A},
\end{equation}
the proof is complete. \(\square\)

**1.5. Adjoint and polar decomposition.** An unexplicit description of the adjoint of a weighted composition operator has been given in [13, Lemma 6.4].

Below, we provide another one which is complete and written in terms of the conditional expectation \(E_{\phi,w}\) different than that used therein.

We single out the following useful fact.

If \((X, \mathcal{A}, \mu)\) is a measure space and \(w: X \to \mathbb{C}\) is \(\mathcal{A}\)-measurable, then the mapping \(L^{2}(\mu) \ni f \mapsto f_{w} := \chi_{\{w \neq 0\}} \cdot \frac{f}{w} \in L^{2}(\mu_{w})\) is a well-defined linear contraction. \(\text{(14)}\)
Proposition 17. Suppose that (5) holds and $C_{\phi,w}$ is densely defined. Then
\begin{equation}
\mathcal{D}(C_{\phi,w}^*) = \left\{ f \in L^2(\mu) : h_{\phi,w} \cdot E_{\phi,w}(f_w) \circ \phi^{-1} \in L^2(\mu) \right\}, \tag{15}
\end{equation}

\begin{equation}
C_{\phi,w}^*(f) = h_{\phi,w} \cdot E_{\phi,w}(f_w) \circ \phi^{-1}, \quad f \in \mathcal{D}(C_{\phi,w}^*). \tag{16}
\end{equation}

Proof. It follows from (14) that $E_{\phi,w}(f_w) \in L^2(\mu_w)$ for every $f \in L^2(\mu)$. In turn, if $g \in \mathcal{D}(C_{\phi,w})$, then by (4) and Proposition 9(i), we get $g \circ \phi \in L^2(\mu_w)$. This, (4) and (12) yield
\begin{equation}
\langle C_{\phi,w} g, f \rangle = \int_X g \circ \phi \cdot \overline{f_w} \ d\mu_w \overset{(11)}{=} \int_X g \circ \phi \cdot \overline{E_{\phi,w}(f_w)} \ d\mu_w
= \int_X g \cdot h_{\phi,w} \cdot \overline{E_{\phi,w}(f_w)} \circ \phi^{-1} \ d\mu_w, \quad g \in \mathcal{D}(C_{\phi,w}), \ f \in L^2(\mu). \tag{17}
\end{equation}

Denote by $\mathcal{E}$ the right-hand side of (15). Clearly, if $f \in \mathcal{E}$, then, by (17), $f \in \mathcal{D}(C_{\phi,w}^*)$ and (16) holds. It suffices to prove that if $f \in \mathcal{D}(C_{\phi,w}^*)$, then $\xi := h_{\phi,w} \cdot E_{\phi,w}(f_w) \circ \phi^{-1} \in L^2(\mu)$. By (17), $g \cdot \xi \in L^1(\mu)$ and $\int_X g \cdot \xi \ d\mu = \int_X g \cdot \hat{\xi} \ d\mu$ for every $g \in \mathcal{D}(C_{\phi,w})$, where $\hat{\xi} := C_{\phi,w}^* f \in L^2(\mu)$. Let $\{X_n\}_{n=1}^\infty$ be as in (8). Considering $g = \chi_{\Delta \cap X_n}$, $\Delta \in \mathcal{F}$, and applying Lemma 2 we get $\xi = \tilde{\xi}$ a.e. $[\mu]$ on $X_n$ for every $n \in \mathbb{N}$. Hence $\xi = \tilde{\xi}$ a.e. $[\mu]$, which completes the proof. \hfill \Box

Recall the definition of the multiplication operator. Given an $\mathcal{F}$-measurable function $u : X \to \mathbb{C}$, we denote by $M_u$ the operator of multiplication by $u$ in $L^2(\mu)$ defined by
\begin{align*}
\mathcal{D}(M_u) &= \{ f \in L^2(\mu) : u \cdot f \in L^2(\mu) \},
M_u f &= u \cdot f, \quad f \in \mathcal{D}(M_u).
\end{align*}

The operator $M_u$ is a normal operator (cf. [3, Section 7.2].

The polar decompositions of $C_{\phi,w}$ and $C_{\phi,w}^*$ can be described as follows.

Proposition 18. Suppose (5) holds and $C_{\phi,w}$ is densely defined. Let $C_{\phi,w} = U |C_{\phi,w}|$ be the polar decomposition of $C_{\phi,w}$. Then
\begin{enumerate}
\item[(i)] $|C_{\phi,w}| = M_{h_{\phi,w}^{1/2}},$
\item[(ii)] $U = C_{\phi,\tilde{w}}$, where $\tilde{w} : X \to \mathbb{C}$ is an $\mathcal{F}$-measurable function such that\footnote{Because of Lemma 6(i), the right-hand side of the equality in (18) is well-defined a.e. $[\mu]$.}
\begin{equation}
\tilde{w} = w \cdot \frac{1}{(h_{\phi,w} \circ \phi)^{1/2}} \quad \text{a.e. } [\mu], \tag{18}
\end{equation}
\item[(iii)] $U^* f = h_{\phi,w}^{1/2} \cdot E_{\phi,w}(f_w) \circ \phi^{-1}$ for $f \in L^2(\mu)$, where $f_w$ is as in (14),
\item[(iv)] the modulus $|C_{\phi,w}^*|$ is given by
\begin{align}
\mathcal{D}(|C_{\phi,w}^*|) &= \{ f \in L^2(\mu) : w \cdot (h_{\phi,w} \circ \phi)^{1/2} \cdot E_{\phi,w}(f_w) \in L^2(\mu) \},
|C_{\phi,w}^*| f &= w \cdot (h_{\phi,w} \circ \phi)^{1/2} \cdot E_{\phi,w}(f_w), \quad f \in \mathcal{D}(|C_{\phi,w}^*|). \tag{19}
\end{align}
\end{enumerate}
PROOF. First note that, by Proposition 7, $h_{\phi,w} < \infty$ a.e. [$\mu$].

(i) By the well-known properties of multiplication operators, $M_{h_{\phi,w}}^{1/2}$ is positive and selfadjoint. So is the operator $|C_{\phi,w}|$. By Proposition 9(i), $D(|C_{\phi,w}|) = D(M_{h_{\phi,w}}^{1/2})$ and thus

$$
\| |C_{\phi,w}| f \| = \| C_{\phi,w} f \| = \int_X |f|^2 \circ \phi \ d\mu_w = \int_X |f|^2 h_{\phi,w} \ d\mu = \| M_{h_{\phi,w}} f \|
$$

for every $f \in D(|C_{\phi,w}|)$. It follows from (i) that $|C_{\phi,w}| = M_{h_{\phi,w}}^{1/2}$.

(ii) By Lemma 6(ii), we have

$$
\int_X \left| \frac{f \circ \phi}{(h_{\phi,w} \circ \phi)^{1/2}} \right|^2 \ d\mu = \int_{\{h_{\phi,w} > 0\}} |f|^2 \ d\mu, \quad f \in L^2(\mu),
$$

which implies that the operator $C_{\phi,w}$ is well-defined and $C_{\phi,w} \in B(L^2(\mu))$. According to (21) and Lemma 11, we see that $N(C_{\phi,w}) = N(C_{\phi,w}) = \chi_{\{h_{\phi,w} > 0\}} L^2(\mu)$ and $C_{\phi,w} L^2(\mu) \cap N(C_{\phi,w})$ is an isometry. This means that $C_{\phi,w}$ is a partial isometry. It follows from (i) that $C_{\phi,w} = C_{\phi,w} |C_{\phi,w}|$. By the uniqueness statement in the polar decomposition theorem, $U = C_{\phi,w} \circ \phi$, which yields (ii).

(iii) Clearly, $d\mu_{\tilde{w}} = \frac{1}{h_{\phi,w} \circ \phi} \ d\mu_w$, which means that the measures $\mu_{\tilde{w}}$ and $\mu_w$ are mutually absolutely continuous and thus $\mu_{\tilde{w}} \circ \phi^{-1} \ll \mu$. By Lemma 6(ii), we have

$$
(\mu_{\tilde{w}} \circ \phi^{-1})(\Delta) = \int_{\Delta} \frac{\chi_{\Delta} \circ \phi}{(h_{\phi,w} \circ \phi)^{1/2}} \ d\mu_w = \int_{\Delta} \chi_{\{h_{\phi,w} > 0\}} \ d\mu, \quad \Delta \in \mathcal{A},
$$

which implies that $h_{\phi,w} = \chi_{\{h_{\phi,w} > 0\}}$ a.e. [$\mu$].

Now we show that

$$
E_{\phi,w}(f_{\tilde{w}}) = h_{\phi,w}^{1/2} \circ \phi \cdot E_{\phi,w}(f_w) \ \text{a.e.} \ [\mu_{\tilde{w}}], \quad f \in L^2(\mu).
$$

For this define $q_{\Delta} = \chi_{\{h_{\phi,w} > 0\}} \cdot \frac{h_{\phi,w}}{h_{\phi,w}^{1/2}} \ \text{a.e.} \ [\mu]$ for $\Delta \in \mathcal{A}$. Then, by Lemma 6(i) and Lemma 5, we have

$$
q_{\Delta} \circ \phi = \frac{\chi_{\Delta} \circ \phi}{(h_{\phi,w} \circ \phi)^{1/2}} \ \text{a.e.} \ [\mu_w], \quad \Delta \in \mathcal{A}.
$$

Take $f \in L^2(\mu)$. Let $\Delta \in \mathcal{A}$ be such that $\mu_{\tilde{w}}(\phi^{-1}(\Delta)) < \infty$. Then

$$
\int_X |q_{\Delta} \circ \phi|^2 \ d\mu_w = \int_{\{h_{\phi,w} > 0\}} \chi_{\Delta} \ d\mu_w = \mu_{\tilde{w}}(\phi^{-1}(\Delta)) < \infty.
$$

This combined with (11), (14) and (24) yields

$$
\int_{\phi^{-1}(\Delta)} E_{\phi,w}(f_{\tilde{w}}) \ d\mu_{\tilde{w}} = \int_X q_{\Delta} \circ \phi \cdot f_w \ d\mu_w
$$

and

$$
= \int_X q_{\Delta} \circ \phi \cdot E_{\phi,w}(f_w) \ d\mu_w
$$

$$
= \int_{\phi^{-1}(\Delta)} h_{\phi,w}^{1/2} \circ \phi \cdot E_{\phi,w}(f_w) \ d\mu_{\tilde{w}}.
$$

Applying Lemma 2 to the measure $\mu_{\tilde{w}}(\phi^{-1}(\mathcal{A}))$ gives (23).

Since the measures $\mu_{\tilde{w}}$ and $\mu_w$ are mutually absolutely continuous, we infer from (23) and Proposition 14 that

$$
E_{\phi,w}(f_w) \circ \phi^{-1} = h_{\phi,w}^{1/2} \cdot E_{\phi,w}(f_w) \circ \phi^{-1} \ \text{a.e.} \ [\mu], \quad f \in L^2(\mu).
$$
This together with Proposition 17, applied to $C_{\phi,w}$, yields (iii).

(iv) It follows from \cite[Exercise 7.26(b)]{cite} that $|C_{\phi,w}^*| = C_{\phi,w}U^*$. In view of Lemma 9, $f \in L^2(\mu)$ belongs to $\mathcal{D}(|C_{\phi,w}^*|)$ if and only if $U^*f \in L^2(h_{\phi,w}d\mu)$. Since, by (iii), (4) and (12), the following equalities hold

$$\int_X |U^*f|^2 \cdot h_{\phi,w} \, d\mu = \int_X h_{\phi,w}^2 \cdot |E_{\phi,w}(f_w) \circ \phi^{-1}|^2 \, d\mu = \int_X h_{\phi,w} \circ \phi \cdot |E_{\phi,w}(f_w)|^2 \, d\mu_w, \quad f \in L^2(\mu),$$

we get (19). The formula (20) follows from the equality $|C_{\phi,w}^*| = C_{\phi,w}U^*$, the condition (iii) and the equality (12). This completes the proof.

\[\square\]

Remark 19. Regarding Proposition 18, note that $E_{\phi,w}(f) = E_{\phi,w}(f) \text{ a.e. } [\mu_w]$ for every $\mathcal{A}$-measurable function $f : X \to \mathbb{R}^+$. Indeed, this is because

$$\int_{\phi^{-1}(\Delta)} f \, d\mu_w \overset{(24)}{=} \int_X (q_\Delta \circ \phi)^2 \cdot f \, d\mu_w = \int_X (q_\Delta \circ \phi)^2 \cdot E_{\phi,w}(f) \, d\mu_w \overset{(24)}{=} \int_{\phi^{-1}(\Delta)} E_{\phi,w}(f) \, d\mu_w, \quad \Delta \in \mathcal{A}.$$ 

1.6. Quasinormality. Below we characterize quasinormal weighted composition operators.

Theorem 20. If (5) holds and $C_{\phi,w}$ is densely defined, then $C_{\phi,w}$ is quasinormal if and only if $h_{\phi,w} \circ \phi = h_{\phi,w} \text{ a.e. } [\mu_w]$.

Proof. It follows from Proposition 18(i) that $|C_{\phi,w}|^2 = M_{h_{\phi,w}}$. We claim that

$$\mathcal{D}(C_{\phi,w}|C_{\phi,w}|^2) = L^2((1 + h_{\phi,w}^3) \, d\mu).$$

Indeed, if $f \in \mathcal{D}(C_{\phi,w}|C_{\phi,w}|^2)$, then $f \in \mathcal{D}(|C_{\phi,w}|^2) = L^2((1 + h_{\phi,w}^3) \, d\mu)$ and, by Lemma 9(i),

$$\int_X |f|^2 h_{\phi,w}^3 \, d\mu = \int_X |M_{h_{\phi,w}} f|^2 h_{\phi,w} \, d\mu < \infty,$$

which yields $f \in L^2((1 + h_{\phi,w}^3 + h_{\phi,w}^3) \, d\mu) = L^2((1 + h_{\phi,w}^3) \, d\mu)$. Reversing the above reasoning proves (25).

Suppose $C_{\phi,w}$ is quasinormal. By Proposition 10, $h_{\phi,w} < \infty \text{ a.e. } [\mu]$. Let $\{X_n\}_{n=1}^\infty$ be as in (8). In view of (25), $\{\chi_{X_n}\}_{n=1}^\infty \subseteq \mathcal{D}(C_{\phi,w}|C_{\phi,w}|^2)$. Therefore

$$w \cdot (h_{\phi,w} \circ \phi) \cdot (\chi_{X_n} \circ \phi) = C_{\phi,w} |C_{\phi,w}|^2 \chi_{X_n} \overset{(1)}{=} |C_{\phi,w}|^2 C_{\phi,w} \chi_{X_n} = w \cdot h_{\phi,w} \circ \phi \cdot (\chi_{X_n} \circ \phi) \text{ a.e. } [\mu], \quad n \in \mathbb{N}. \quad (26)$$

Since $\phi^{-1}(X_n) \not\subseteq X$ as $n \to \infty$, we see that $w \cdot (h_{\phi,w} \circ \phi) = w \cdot h_{\phi,w} \text{ a.e. } [\mu]$, or equivalently that $h_{\phi,w} \circ \phi = h_{\phi,w} \text{ a.e. } [\mu_w]$.

Assume now that $h_{\phi,w} \circ \phi = h_{\phi,w} \text{ a.e. } [\mu_w]$. We claim that

$$\mathcal{D}(|C_{\phi,w}|^2 C_{\phi,w}) = L^2((1 + h_{\phi,w}^3) \, d\mu). \quad (27)$$
Indeed, if \( f \in \mathcal{D}((C_{\phi,w})^2C_{\phi,w}) \), then it follows from Proposition 18(i) that \( C_{\phi,w}f \in L^2((1 + h_{\phi,w}^2) \, d\mu) \) and
\[
\int_X |f|^2 \, d\mu = \int_X (h_{\phi,w}^2 \cdot |f|^2 \circ \phi) \, d\mu_w = \int_X h_{\phi,w}^2 |C_{\phi,w}f|^2 \, d\mu < \infty,
\]
which implies that \( f \in L^2((1 + h_{\phi,w}^2) \, d\mu) \). Reversing the above reasoning proves (27). Combining (25) and (27) shows that \( \mathcal{D}(C_{\phi,w}|C_{\phi,w}|^2) = \mathcal{D}((C_{\phi,w})^2C_{\phi,w}) \). An appropriate modification of (26) gives \( C_{\phi,w}|C_{\phi,w}|^2 = |C_{\phi,w}|^2C_{\phi,w} \). Applying (1) completes the proof. \( \square \)

2. Subnormality

2.1. General scheme. Let \((X, \mathcal{A})\) be a measurable space. A mapping \( P: X \times \mathcal{B}(\mathbb{R}_+) \to [0, 1] \) is called an \( \mathcal{A}\)-measurable family of probability measures if the set-function \( P(x, \cdot) \) is a probability measure for every \( x \in X \) and the function \( P(\cdot, \sigma) \) is \( \mathcal{A}\)-measurable for every \( \sigma \in \mathcal{B}(\mathbb{R}_+) \). Denote by \( \mathcal{A} \otimes \mathcal{B}(\mathbb{R}_+) \) the \( \sigma \)-algebra generated by the family
\[
\mathcal{A} \otimes \mathcal{B}(\mathbb{R}_+) := \{ \Delta \times \sigma: \Delta \in \mathcal{A}, \sigma \in \mathcal{B}(\mathbb{R}_+) \}.
\]
If \( \mu: \mathcal{A} \to \mathbb{R}_+ \) is a \( \sigma \)-finite measure, then, by [2, Theorem 2.6.2], there exists a unique \( \sigma \)-finite measure\(^2\) \( \rho \) on \( \mathcal{A} \otimes \mathcal{B}(\mathbb{R}_+) \) such that
\[
\rho(\Delta \times \sigma) = \int_{\Delta} P(x, \sigma) \mu(dx), \quad \Delta \in \mathcal{A}, \sigma \in \mathcal{B}(\mathbb{R}_+).
\]
(28)

Moreover, for every \( \mathcal{A} \otimes \mathcal{B}(\mathbb{R}_+)\)-measurable function \( f: X \times \mathbb{R}_+ \to \mathbb{R}_+ \),
\[
\text{the function } X \ni x \to \int_0^\infty f(x, t) P(x, dt) \in \mathbb{R}_+ \text{ is } \mathcal{A}\text{-measurable}
\]
(29)
and
\[
\int_{X \times \mathbb{R}_+} f \, d\rho = \int_X \int_0^\infty f(x, t) P(x, dt) \mu(dx).
\]
(30)

Let \( w: X \to \mathbb{C} \) be an \( \mathcal{A}\)-measurable function and \( \phi \) be an \( \mathcal{A}\)-measurable transformation of \( X \). Define the function \( W: X \times \mathbb{R}_+ \to \mathbb{C} \) and the transformation \( \Phi \) of \( X \times \mathbb{R}_+ \) by
\[
W(x, t) = w(x), \quad x \in X, t \in \mathbb{R}_+,
\]
(31)
\[
\Phi(x, t) = (\phi(x), t), \quad x \in X, t \in \mathbb{R}_+.
\]
(32)

It is easily seen that \( W \) and \( \Phi \) are \( \mathcal{A} \otimes \mathcal{B}(\mathbb{R}_+)\)-measurable. According to our convention, the measure \( \rho_W \) is defined as follows
\[
\rho_W(E) = \int_E |W|^2 \, d\rho = \int_X \int_0^\infty \chi_E(x, t) P(x, dt) \mu_w(x), \quad E \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}_+).
\]

\(^2\) Clearly the measure \( \rho \) depends on \( P \). Since we do not exploit this fact, we will not make this dependence explicit.
In what follows, we regard \( C_{\phi, W} \) as a weighted composition operator in \( L^2(\rho) \). There is a natural way of looking at \( L^2(\mu) \) as a subspace of \( L^2(\rho) \). Namely, by (30), the mapping \( U: L^2(\mu) \to L^2(\rho) \) given by
\[
(Uf)(x, t) = f(x), \quad x \in X, \ t \in \mathbb{R}_+, \ f \in L^2(\mu),
\]
is well-defined, linear and isometric. Moreover, if \( C_{\phi, W} \) is well-defined, then, combining Proposition 7, Lemma 21 and (30), we deduce that
\[
UC_{\phi, w} = C_{\phi, W} U.
\] (33)

In order to make the paper more readable, we single out the following assumption.

The triplet \((X, \mathcal{A}, \mu)\) is a \( \sigma \)-finite measure space, \( w \) is an \( \mathcal{A} \)-measurable complex function on \( X \), \( \phi \) is an \( \mathcal{A} \)-measurable transformation of \( X \) and \( P: X \times \mathcal{B}(\mathbb{R}_+) \to [0, 1] \) is an \( \mathcal{A} \)-measurable family of probability measures. The measure \( \rho \), the function \( W \) and the transformation \( \phi \) are determined by (28), (31) and (32), respectively.

We begin by proving a formula that connects \( h_{\phi, w} \) with \( \phi_{\phi, W} \) via \( E_{\phi, w} \).

**Lemma 21.** Suppose (34) holds and \( \rho_W \circ \phi^{-1} \ll \rho \). Then \( \mu_w \circ \phi^{-1} \ll \mu \). Moreover, if \( h_{\phi, w} < \infty \ a.e. \ [\mu] \), then \( \phi_{\phi, W} < \infty \ a.e. \ [\rho] \) and
\[
\begin{align*}
\left( E_{\phi, w}(P(\cdot, \sigma)) \circ \phi^{-1}\right)(x) & = \int_{\sigma} h_{\phi, W}(x, \sigma) P(x, d\sigma) \mu(dx) \\
& = \int_{X} \int_{\mathbb{R}_+} h_{\phi, W}(x, t) P(x, dt) \mu(dx) \quad \text{for } \mu-a.e. \ x, \ \sigma \in \mathcal{B}(\mathbb{R}_+).
\end{align*}
\] (35)

**Proof.** To prove that \( \mu_w \circ \phi^{-1} \ll \mu \), take \( \Delta \in \mathcal{A} \) such that \( \mu(\Delta) = 0 \). Then, by (28), \( \rho(\Delta \times \mathbb{R}_+) = 0 \). Hence, in view of (30), we have
\[
\mu_w \circ \phi^{-1}(\Delta) = \int_{\phi^{-1}(\Delta)} |w(x)|^2 \int_0^1 P(x, d\mu) d\rho = \int_{\phi^{-1}(\Delta \times \mathbb{R}_+)} |W|^2 d\rho = 0.
\]

Assume additionally that \( h_{\phi, w} < \infty \ a.e. \ [\mu] \). If \( \Delta \in \mathcal{A} \) and \( \sigma \in \mathcal{B}(\mathbb{R}_+) \), then
\[
\begin{align*}
\int_{\Delta} \int_{\mathbb{R}_+} h_{\phi, W}(x, t) P(x, dt) \mu(dx) & \overset{(30)}{=} \int_{\Delta \times \mathbb{R}_+} h_{\phi, W} \ d\rho \overset{(3)}{=} \int_{\phi^{-1}(\Delta \times \mathbb{R}_+)} |W|^2 d\rho \\
& \overset{(30)}{=} \int_{\phi^{-1}(\Delta)} |w(x)|^2 \int_{\sigma} P(x, dt) \mu(dx) \\
& = \int_{\phi^{-1}(\Delta)} P(x, \sigma) \mu_w(dx) \\
& = \int_{\phi^{-1}(\Delta)} E_{\phi, w}(P(\cdot, \sigma)) \ d\mu_w \\
& \overset{(1)}{=} \int_{\Delta} E_{\phi, w}(P(\cdot, \sigma)) \circ \phi^{-1} \cdot h_{\phi, w} \ d\mu,
\end{align*}
\] (36)

where \((1)\) follows from (4) and (12). Since \( \mu \) is \( \sigma \)-finite, (35) holds.

It suffices to show that \( h_{\phi, W} < \infty \ a.e. \ [\rho] \). Let \( \{X_n\}_{n=1}^\infty \) be as in (8). Then
\[
\begin{align*}
\int_{X \times \mathbb{R}_+} h_{\phi, W} \ d\rho & \overset{(30)}{=} \int_{X} \int_{\mathbb{R}_+} h_{\phi, W}(x, t) P(x, dt) \mu(dx) \\
& \overset{(35)}{=} \int_{X} \left( E_{\phi, w}(P(\cdot, \mathbb{R}_+)) \circ \phi^{-1}\right)(x) \cdot h_{\phi, w}(x) \mu(dx)
\end{align*}
\]
clear that \( P \) on \( X_n \times \mathbb{R}_+ \) for every \( n \in \mathbb{N} \), and thus \( h_{\phi, w} < \infty \) a.e. \([\rho]\). This completes the proof. \( \square \)

The consistency condition (CC) introduced below plays the crucial role in the present paper (see [12] for the case of composition operators).

**Lemma 22.** Suppose (34) holds, \( \mu_w \circ \phi^{-1} \ll \mu \), \( h_{\phi, w} < \infty \) a.e. \([\mu]\) and the following condition is satisfied:

\[
E_{\phi, w}(P(\cdot, \sigma))(x) = \frac{\int_{\sigma} t P(\phi(x), dt)}{h_{\phi, w}(\phi(x))} \text{ for } \mu_w \text{-a.e. } x \in X, \quad \sigma \in \mathcal{B}(\mathbb{R}_+). \tag{CC}
\]

Then \( \rho_W \circ \phi^{-1} \ll \rho \) and

\[
h_{\phi, W}(x, t) = \chi_{\{h_{\phi, w} > 0\}}(x) \cdot t \text{ for } \rho \text{-a.e. } (x, t) \in X \times \mathbb{R}_+.
\]

**Proof.** Arguing as in (36), we get

\[
\rho_W(\Phi^{-1}(\Delta \times \sigma)) = \int_{\Phi^{-1}(\Delta \times \sigma)} |W|^2 \, d\rho = \int_{\Phi^{-1}(\Delta)} E_{\phi, w}(P(\cdot, \sigma)) \, d\mu_w
\]

\[
\overset{(CC)}{=} \int_X \chi_{\Delta}(\phi(x)) \cdot \int_{\sigma} t P(\phi(x), dt) \, \mu_w(dx)
\]

\[
\overset{(6)}{=} \int_{\{h_{\phi, w} > 0\}} \chi_{\Delta}(x) \int_{\sigma} t P(x, dt) \, \mu(dx)
\]

\[
\overset{(30)}{=} \int_{\Delta \times \sigma} \chi_{\{h_{\phi, w} > 0\}}(x) \cdot t \, d\rho(x, t), \quad \Delta \in \mathcal{A}, \quad \sigma \in \mathcal{B}(\mathbb{R}_+). \tag{37}
\]

It is clear that \( \mathcal{P} := \mathcal{A} \otimes \mathcal{B}(\mathbb{R}_+) \) is a semi-algebra which generates \( \mathcal{A} \otimes \mathcal{B}(\mathbb{R}_+) \). Let \( \{X_n\}_{n=1}^{\infty} \) be as in (8). Then \( \{X_n \times \mathbb{R}_+\}_{n=1}^{\infty} \subseteq \mathcal{P} \) and

\[
\rho_W(\Phi^{-1}(X_n \times \mathbb{R}_+)) = \int_{\Phi^{-1}(X_n)} |w|^2 \, d\mu = \mu_w \circ \phi^{-1}(X_n)
\]

\[
= \int_{X_n} h_{\phi, w} \, d\mu \leq n \cdot \mu(X_n) < \infty, \quad n \in \mathbb{N}. \tag{38}
\]

Combining (37) and (38) with Lemma 1, we get

\[
\rho_W(\Phi^{-1}(E)) = \int_E \chi_{\{h_{\phi, w} > 0\}}(x) \cdot t \, d\rho(x, t), \quad E \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}_+).
\]

This completes the proof. \( \square \)

Now we provide some characterizations of the consistency condition (CC) that will be used later in this paper.

**Lemma 23.** Suppose (34) holds, \( \mu_w \circ \phi^{-1} \ll \mu \) and \( h_{\phi, w} < \infty \) a.e. \([\mu]\). Then the following conditions are equivalent:

(i) \( P \) satisfies (CC),

(ii) \( \left( E_{\phi, w}(P(\cdot, \sigma)) \circ \phi^{-1}\right)(x) \cdot h_{\phi, w}(x) = \chi_{\{h_{\phi, w} > 0\}}(x) \int_{\sigma} t P(x, dt) \) for \( \mu \)-a.e. \( x \in X \) and for every \( \sigma \in \mathcal{B}(\mathbb{R}_+) \),

(iii) \( \rho_W \circ \phi^{-1} \ll \rho \) and \( h_{\phi, W}(x, t) = \chi_{\{h_{\phi, w} > 0\}}(x) \cdot t \) for \( \rho \text{-a.e. } (x, t) \in X \times \mathbb{R}_+ \),

---

3 By Lemma 6 and (29), the right-hand side of the equality in (CC) is an \( \mathcal{A} \)-measurable function defined a.e. \([\mu]\).
(iv) $\rho_W \circ \phi^{-1} \ll \rho$ and $\int_\Delta h_{\phi,W}(\phi(x), t)P(\phi(x), dt) = \int_\sigma tP(\phi(x), dt)$ for $\mu_w$-a.e. $x \in X$ and for every $\sigma \in \mathcal{B}(\mathbb{R}_+)$.

**Proof.** (i)$\Rightarrow$(ii) This can be proved by using (12), Lemma 5 and Lemma 6(i).

(i)$\Rightarrow$(iii) Apply Lemma 22.

(iii)$\Rightarrow$(iv) Note that

$$\int_{\phi^{-1}(\Delta)} \int_\sigma h_{\phi,W}(\phi(x), t)P(\phi(x), dt)\mu_w(dx)$$

$$= \int_{\Delta} h_{\phi,w}(x) \int_\sigma h_{\phi,W}(\phi(x), t)P(x, dt)\mu(dx)$$

$$= \int_{\Delta \times \sigma} h_{\phi,w}(x)h_{\phi,W}(x, t)\,d\rho(x, t)$$

$$\equiv \int_{\Delta \times \sigma} h_{\phi,w}(x)\chi_{\{h_{\phi,w} \neq 0\}}(x) \cdot t\,d\rho(x, t)$$

$$= \int_{\Delta} h_{\phi,w}(x)\chi_{\{h_{\phi,w} \neq 0\}}(x) \int_\sigma tP(x, dt)\mu(dx)$$

$$= \int_{\phi^{-1}(\Delta)} \chi_{\{h_{\phi,w} \neq 0\}}(\phi(x)) \int_\sigma tP(\phi(x), dt)\mu_w(dx)$$

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$$\int_{\phi^{-1}(\Delta)} \int_\sigma tP(\phi(x), dt)\mu_w(dx), \quad \Delta \in \mathcal{F}, \, \sigma \in \mathcal{B}(\mathbb{R}_+).$$

Since, by Proposition 10, the measure $\mu_w|_{\phi^{-1}(\mathcal{F})}$ is $\sigma$-finite, we get (iv).

(iv)$\Rightarrow$(i) By Lemma 21, (35) holds. Composing both sides of (35) with $\phi$ and using Lemma 5, the equality (12) and (iv), we obtain (i). This completes the proof. □

The next lemma deals with a version of the consistency condition (CC).

**Lemma 24.** Suppose (34) holds, $\mu_w \circ \phi^{-1} \ll \mu$, $h_{\phi,w} < \infty$ a.e. $[\mu]$ and

$$(E_{\phi,w}(P(\cdot, \sigma)) \circ \phi^{-1} )(x) \cdot h_{\phi,w}(x)$$

$$= \int_\sigma tP(x, dt) \text{ for } \mu_w\text{-a.e. } x \in X, \quad \sigma \in \mathcal{B}(\mathbb{R}_+). \quad (39)$$

Then for every Borel function $f: \mathbb{R}_+ \to \mathbb{R}_+$,

$$\left(E_{\phi,w}\left(\int_0^\infty f(t)P(\cdot, dt)\right) \circ \phi^{-1}\right)(x) \cdot h_{\phi,w}(x)$$

$$= \int_0^\infty t \cdot f(t)P(x, dt) \text{ for } \mu_w\text{-a.e. } x \in X. \quad (40)$$

**Proof.** By Proposition 14, (40) holds for every simple Borel function $f: \mathbb{R}_+ \to \mathbb{R}_+$. Let $f: \mathbb{R}_+ \to \mathbb{R}_+$ be a Borel function. Take a sequence $\{s_n\}_{n=1}^\infty$ of simple Borel functions $s_n: \mathbb{R}_+ \to \mathbb{R}_+$ which is monotonically increasing and pointwise convergent to $f$. Then, by Lebesgue’s monotone convergence theorem, we have

$$\int_\Delta E_{\phi,w}\left(\int_0^\infty s_n(t)P(\cdot, dt)\right) \circ \phi^{-1} \cdot h_{\phi,w} \,d\mu_w$$

$$\equiv \int_{\phi^{-1}(\Delta)} E_{\phi,w}\left(\int_0^\infty s_n(t)P(\cdot, dt)\right) \cdot |w \circ \phi|^2 \,d\mu_w$$
Below, we provide recurrence formulas for the Radon-Nikodym derivatives $\mu$ way to the weighted composition operator $\varphi^w$. If $\varphi$ is densely defined, then

\[
\int_{0}^{\infty} f(t)P(x, dt)|w(\varphi(x))|^2\mu_{w}(dx)
\]

\[
\int_{0}^{\infty} s_n(t)P(x, dt)|w(\varphi(x))|^2\mu_{w}(dx)
\]

\[
\int_{0}^{\infty} t \cdot s_n(t)P(x, dt)\mu_{w}(dx) \xrightarrow{n \to \infty} \int_{0}^{\infty} t \cdot f(t)P(x, dt)\mu_{w}(dx),
\]

where (†) follows from (4) and (12). Since $\int_{0}^{\infty} t \cdot s_n(t)\mu_{w}(dx) \xrightarrow{n \to \infty} \int_{0}^{\infty} t \cdot f(t)\mu_{w}(dx)$, $\Delta \in \mathcal{A}$, and $\mu_{w}$ is $\sigma$-finite, the proof is complete. □

Applying the equivalence (i)$\Leftrightarrow$(ii) of Lemma 23, we obtain the following.

**Proposition 25.** Suppose (34) holds, $\mu_{w} \circ \varphi^{-1} \ll \mu$ and $h_{\varphi,w} < \infty$ a.e. $[\mu]$. Then the following two assertions are valid.

(i) If $P$ satisfies (39) and $(E_{\varphi,w}(P(\cdot, \sigma)) \circ \varphi^{-1})(x) \cdot h_{\varphi,w}(x) = \int_{\sigma} tP(x, dt)$ for $\mu$-a.e. $x \in \{h_{\varphi,w} > 0\} \cap \{w = 0\}$ and for every $\sigma \in \mathcal{B}(\mathbb{R}_+)$, then $P$ satisfies (CC).

(ii) If $P$ satisfies (CC) and $\int_{0}^{\infty} tP(x, dt) = 0$ for $\mu$-a.e. $x \in \{h_{\varphi,w} = 0\} \cap \{w \neq 0\}$, then $P$ satisfies (39).

Regarding the assertion (i) of Proposition 25, we note that if $(E_{\varphi,w}(P(\cdot, \sigma)) \circ \varphi^{-1})(x) \cdot h_{\varphi,w}(x) = \int_{\sigma} tP(x, dt)$ for $\mu$-a.e. $x \in \{h_{\varphi,w} > 0\} \cap \{w = 0\}$ and for every $\sigma \in \mathcal{B}(\mathbb{R}_+)$, then, by (13), $\int_{0}^{\infty} tP(x, dt) = 0$ for $\mu$-a.e. $x \in \{h_{\varphi,w} > 0\} \cap \{w = 0\}$ if and only if $\mu(\{h_{\varphi,w} > 0\} \cap \{w = 0\}) = 0$.

The $n$-th power of a weighted composition operator $C_{\varphi,w}$ is related in a natural way to the weighted composition operator $C_{\varphi^n,w_n}$ with explicitly given weight $w_n$. Below, we provide recurrence formulas for the Radon-Nikodym derivatives $h_{\varphi,w}$ attached to $C_{\varphi^n,w_n}$.

**Lemma 26.** Suppose (5) holds. Then $C_{\varphi,w}^{n}$ is well-defined and $C_{\varphi,w}^{n} \subseteq C_{\varphi^n,w_n}$ for every $n \in \mathbb{Z}_+$, where $w_0 = 1$ and $w_{n+1} = \prod_{j=0}^{n} w_0 \circ \varphi^j$ for $n \in \mathbb{Z}_+$. Moreover, if $C_{\varphi,w}$ is densely defined, then

\[
h_{\varphi,w}^{[n+1]} = E_{\varphi,w} \left( h_{\varphi,w}^{[n]} \circ \varphi^{-1} \cdot h_{\varphi,w} \right) a.e. [\mu], \quad n \in \mathbb{Z}_+,
\]

\[
h_{\varphi,w}^{[n+1]} \circ \varphi = E_{\varphi,w} \left( h_{\varphi,w}^{[n]} \cdot (h_{\varphi,w} \circ \varphi) \right) a.e. [\mu], \quad n \in \mathbb{Z}_+,
\]

where $h_{\varphi,w}^{[n]} := h_{\varphi,w}^{n}$ for $n \in \mathbb{Z}_+$.

**Proof.** Take $\Delta \in \mathcal{A}$ such that $\mu(\Delta) = 0$. Then by Proposition 7, we see that $w_n \cdot \chi_{(\varphi^n)^{-1}(\Delta)} = w_n \cdot (\chi_{\Delta} \circ \varphi^n) = 0$ a.e. $[\mu]$ for every $n \in \mathbb{Z}_+$, which means that $(\mu_{w_n} \circ (\varphi^n)^{-1})(\Delta) = 0$. Applying Proposition 7 again, we conclude that $C_{\varphi^n,w_n}$ is well-defined. The inclusion $C_{\varphi,w}^{n} \subseteq C_{\varphi^n,w_n}$ is easily seen to be true. By Lemma 5 and (12), the equality (42) follows from (41). To prove (41), note that $w_{n+1} = w_0 \circ \varphi \cdot w_n$ for $n \in \mathbb{Z}_+$. Hence, by (4) and (12), we have

\[
= \int_{0}^{\infty} s_n(t)P(x, dt)|w(\varphi(x))|^2\mu_{w}(dx) + \int_{0}^{\infty} f(t)P(x, dt)|w(\varphi(x))|^2\mu_{w}(dx)
\]

\[
\int_{0}^{\infty} t \cdot s_n(t)P(x, dt)\mu_{w}(dx) \xrightarrow{n \to \infty} \int_{0}^{\infty} t \cdot f(t)P(x, dt)\mu_{w}(dx),
\]

and $\mu_{w}$ is $\sigma$-finite, the proof is complete. □
\[
\mu_{w,n+1}((\phi^{n+1})^{-1}(\Delta)) = \int_X \chi_{\phi^{-1}(\Delta)} \circ \phi^n \cdot |w \circ \phi^n|^2 \, d\mu_{w,n} = \int_{\phi^{-1}(\Delta)} |w|^2 h^{[n]}(x) \, d\mu = \int_{\phi^{-1}(\Delta)} \mathbb{E}_{\phi,w}(h^{[n]}(x)) \, d\mu_w = \int_{\Delta} \mathbb{E}_{\phi,w}(h^{[n]}(x)) \circ \phi^{-1} \cdot h_{\phi,w} \, d\mu, \quad \Delta \in \mathcal{A}, \ n \in \mathbb{Z}.
\]
This completes the proof. \(\square\)

The result that follows will be used in the proof of Theorem 29. It clarifies the role played by the assumption “\(h_{\phi,w} > 0\) a.e. [\(\mu_w\)]” in this theorem.

**Theorem 27.** Suppose (34) holds, \(\mu_w \circ \phi^{-1} \ll \mu \) and \(h_{\phi,w} < \infty \) a.e. [\(\mu\)]. If \(P\) satisfies (CC), then the following conditions are equivalent:

(i) \(P\) satisfies (39),
(ii) \(h^{[n]}(x) = \int_t^\infty t^n P(x, dt)\) for every \(n \in \mathbb{Z}_+\) and for \(\mu_w\)-a.e. \(x \in X\),
(iii) \(\int_0^\infty t P(x, dt) = 0\) for \(\mu_w\)-a.e. \(x \in \{h_{\phi,w} = 0\}\),
(iv) \(P(x, \cdot) = \delta_0(\cdot)\) for \(\mu_w\)-a.e. \(x \in \{h_{\phi,w} = 0\}\),
(v) \(\rho \circ \Phi^{-1} \ll \rho\) and \(h_{\phi,w}(x,t) = t\) for \(\rho\)-a.e. \((x,t) \in X \times \mathbb{R}_+\),
(vi) \(\rho \circ \Phi^{-1} \ll \rho\) and \(h_{\phi,w} \circ \Phi = h_{\phi,w}\) \(\text{a.e.} [\rho]\),
(vii) \(h_{\phi,w} > 0\) \(\text{a.e.} [\mu_w]\).

**Proof.** (i) \(\Rightarrow\) (ii) We use induction on \(n\). The case of \(n = 0\) is obvious. The induction hypothesis is that \(h^{[n]} = H_n\) \(\text{a.e.} [\mu_w]\), where \(H_n(x) = \int_0^\infty t^n P(x, dt)\) for \(x \in X\) and \(n \in \mathbb{Z}_+\). By Lemma 24, (40) holds with \(f(t) = t^n\). This leads to

\[
H_{n+1}(x) \overset{(40)}{=} (\mathbb{E}_{\phi,w}(H_n) \circ \phi^{-1})(x) \cdot h_{\phi,w}(x) = (\mathbb{E}_{\phi,w}(h^{[n]}(x)) \circ \phi^{-1})(x) \cdot h_{\phi,w}(x) \overset{(41)}{=} h^{[n+1]}(x) \text{ for } \mu_w\text{-a.e. } x \in X, \quad (43)
\]
which completes the induction argument and gives (ii).

(ii) \(\Rightarrow\) (iii) Consider the equality in (ii) with \(n = 1\).

(iii) \(\Rightarrow\) (iv) Use Lemma 3.

(iii) \(\Rightarrow\) (v) It follows that

\[
\int_0^\infty t \cdot \chi_E(x,t)P(x, dt) = 0 \text{ for } \mu_w\text{-a.e. } x \in \{h_{\phi,w} = 0\}, \quad E \in \mathcal{A} \otimes \mathcal{B} (\mathbb{R}_+). \quad (44)
\]

Hence, by the implication (i) \(\Rightarrow\) (iii) of Lemma 23, \(\rho \circ \Phi^{-1} \ll \rho\) and

\[
\int_E h_{\phi,W} \, d\rho_W = \int_E |w(x)|^2 h_{\phi,W}(x, t) \, d\rho(x, t) = \int_X \chi_{\{h_{\phi,w} > 0\}}(x) \cdot t \cdot |w(x)|^2 \, d\rho(x, t) = \int_X \chi_{\{h_{\phi,w} > 0\}}(x) \int_0^\infty t \cdot \chi_E(x, t)P(x, dt)\mu_w(dx) \overset{(44)}{=} \int_X \int_0^\infty t \cdot \chi_E(x, t)P(x, dt)\mu_w(dx) = \int_E t \, d\rho_W(x, t), \quad E \in \mathcal{A} \otimes \mathcal{B} (\mathbb{R}_+).
\]

\(^4\) Note that all equalities in (43) except for the first one hold a.e. [\(\mu\)].
Since $\rho_w$ is $\sigma$-finite, (v) holds.

(v)⇒(vi) By the implication (i)⇒(iii) of Lemma 23 and Lemma 5, we have

$$
(h_{\phi,w} \circ \phi)(x,t) = \chi_{\{h_{\phi,w} > 0\}}(\phi(x)) \cdot t \quad \text{for } \rho_W\text{-a.e. } (x,t) \in X \times \mathbb{R}_+.
$$

Now we note that if $f,g: X \to \mathbb{R}_+$ are $\mathcal{A}$-measurable functions such that $f = g$ a.e. $[\mu_w]$, then $f(x) \cdot t = g(x) \cdot t$ for $\rho_W\text{-a.e. } (x,t) \in X \times \mathbb{R}_+$. Indeed, this is because

$$
\int_E f(x) \cdot t \, d\rho_W(x,t) = \int_X f(x) \int_0^{\infty} \chi_E(x,t) \cdot t \, P(x,dt) \mu_w(dx) = \int_E g(x) \cdot t \, d\rho_W(x,t), \quad E \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}_+).
$$

The above property combined with (45) and the fact that $\chi_{\{h_{\phi,w} > 0\}} \circ \phi = 1$ a.e. $[\mu_w]$ (cf. Proposition 6(i)) yields (vi).

(vi)⇒(vii) By Lemma 6(i) and Proposition 12, $\chi_{\{W \neq 0\}} \mathcal{N}(C_{\phi,w}) = \{0\}$. Take $f \in \chi_{\{W \neq 0\}} \mathcal{N}(C_{\phi,w})$. By Lemma 11, $f \in \mathcal{N}(C_{\phi,w})$ and thus, by (33), $Uf \in \mathcal{N}(C_{\phi,w})$. Since $\{W = 0\} = \{w = 0\} \times \mathbb{R}_+$ and $f = 0$ a.e. $[\mu]$ on $\{w = 0\}$, we deduce that

$$
\int_{\{W = 0\}} |Uf|^2 \, d\rho = \int_{\{w = 0\}} |f(x)|^2 \int_0^{\infty} P(x,dt) \mu(dx) = \int_{\{w = 0\}} |f(x)|^2 \mu(dx) = 0.
$$

As a consequence, $Uf \in \chi_{\{W \neq 0\}} \mathcal{N}(C_{\phi,w}) = \{0\}$. Since $U$ is injective, we get $f = 0$ a.e. $[\mu]$. This means that $\chi_{\{w \neq 0\}} \mathcal{N}(C_{\phi,w}) = \{0\}$. It follows from Proposition 12 that $h_{\phi,w} > 0$ a.e. $[\mu_w]$.

(vii)⇒(i) Apply the implication (i)⇒(ii) of Lemma 23.

\(\square\)

**Remark 28.** The implication (vi)⇒(vii) of Theorem 27 can be proved in a shorter (but more advanced) way by applying Theorem 20. Indeed, by Lemmas 10 and 21, and Theorem 20 the operator $C_{\phi,w}$ is quasinormal. In view of (33) and the fact that quasinormal operators are subnormal, $C_{\phi,w}$ is subnormal. As subnormal operators are hyponormal, an application of Corollary 13 yields $h_{\phi,w} > 0$ a.e. $[\mu_w]$.

Theorem 27 enables us to formulate a criterion for subnormality of unbounded weighted composition operators in $L^2$-spaces (see Theorem 29 below). Note that the assumption $h_{\phi,w} > 0$ a.e. $[\mu_w]$ that appears in Theorem 29 is not restrictive because it is always satisfied whenever $C_{\phi,w}$ is subnormal (cf. Corollary 13).

**Theorem 29.** Let $(X, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space, $w$ be an $\mathcal{A}$-measurable complex function on $X$ and $\phi$ be an $\mathcal{A}$-measurable transformation of $X$ such that $C_{\phi,w}$ is densely defined and $h_{\phi,w} > 0$ a.e. $[\mu_w]$. Suppose there exists an $\mathcal{A}$-measurable family of probability measures $P: X \times \mathcal{B}(\mathbb{R}_+) \to [0,1]$ that satisfies (CC). Then $C_{\phi,w}$ is subnormal, $C_{\phi,w}$ is its quasinormal extension (cf. (34)) and

$$
h_{\phi,w}^{[n]}(x) = \int_0^{\infty} t^n P(x,dt) \quad \text{for every } n \in \mathbb{Z}_+ \text{ and for } \mu_w\text{-a.e. } x \in X.
$$

**Proof.** By Propositions 7 and 10, the assumptions of Theorem 27 are satisfied. Hence (46) holds and, by Lemma 21 and Theorem 20, $C_{\phi,w}$ is quasinormal. Employing (33) completes the proof.

\(\square\)

**Remark 30.** It is worth pointing out that the above criterion can be applied to prove the normality of multiplication operators, which are particular instances of weighted composition operators. Indeed, then $\phi = \text{id}_X$, $\phi^{-1}(\mathcal{A}) = \mathcal{A}$, the
conditional expectation is the identity mapping, $\mu_w \circ \phi^{-1} = \mu_w \ll \mu$ and $h_{\phi,w} = |w|^2$ a.e. [\mu]. Hence $M_w = C_{\phi,w}$ is densely defined. Set $P(x,\sigma) = \chi_{\sigma}(|w(x)|^2)$ for $x \in X$ and $\sigma \in B(\mathbb{R}_+)$. Then, as easily seen, $P$ is an $\mathcal{A}$-measurable family of probability measures which satisfies (CC). By Theorem 29, $M_w$ is subnormal. It follows from Proposition 17 that $M_w^* = M_w$. As a consequence $M_w$ and $M_w^*$ are subnormal. Since subnormal operators are hyponormal, we conclude that $M_w$ is normal.

Acknowledgement. A substantial part of this paper was written while the first, the second and the fourth authors visited Kyungpook National University during the spring and the autumn of 2013. They wish to thank the faculty and the administration of this unit for their warm hospitality.

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