NON-GROUP GRADINGS ON SIMPLE LIE ALGEBRAS

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Abstract. A set grading on the split simple Lie algebra of type $D_{13}$, that cannot be realized as a group-grading, is constructed by splitting the set of positive roots into a disjoint union of pairs of orthogonal roots, following a pattern provided by the lines of the projective plane over $GF(3)$. This answers in the negative [EK13, Question 1.11].

Similar non-group gradings are obtained for types $D_n$ with $n \equiv 1 \pmod{12}$, by substituting the lines in the projective plane by blocks of suitable Steiner systems.

1. Introduction

The aim of this paper is the construction of a set grading on a simple Lie algebra that cannot be realized as a group grading, thus answering in the negative [EK13, Question 1.11].

The systematic study of gradings on Lie algebras was initiated in 1989 by Patrera and Zassenhaus [PZ89], although particular gradings have been used from the beginning of Lie theory.

A set grading on a Lie algebra $L$ over a field $F$ is a decomposition into a direct sum of vector subspaces: $\Gamma : L = \bigoplus_{s \in S} L_s$, where $L_s \neq 0$ for any $s$ in the grading set $S$, such that for any $s_1, s_2 \in S$, either $[L_{s_1}, L_{s_2}] = 0$ or there exists an element $s_3 \in S$ such that $[L_{s_1}, L_{s_2}] \subseteq L_{s_3}$. The subspaces $L_s$ are called the homogeneous components of $\Gamma$.

Two such gradings $\Gamma : L = \bigoplus_{s \in S} L_s$ and $\Gamma' : L' = \bigoplus_{s' \in S'} L'_s$ are said to be equivalent if there is an isomorphism of Lie algebras $\varphi : L \to L'$ such that for any $s \in S$ there is an $s' \in S'$ such that $\varphi(L_s) = L'_{s'}$.

On the other hand, given a (semi)group $G$, a $G$-grading on $L$ is a decomposition as above $\Gamma : L = \bigoplus_{g \in G} L_g$, such that $[L_{g_1}, L_{g_2}] \subseteq L_{g_1g_2}$ for all $g_1, g_2 \in G$. The support of $\Gamma$ is the subset $\text{Supp}(\Gamma) := \{ g \in G \mid L_g \neq 0 \}$.

Given a set grading $\Gamma : L = \bigoplus_{s \in S} L_s$, it is said that $\Gamma$ can be realized as a (semi)group grading, if there exists a (semi)group $G$ and a one-to-one map $\iota : S \hookrightarrow G$ such that the subspaces $L_{\iota(s)} := L_s$, and $L_g := 0$ if $g \notin \iota(S)$, form a $G$-grading of $L$.

Given the set grading $\Gamma : L = \bigoplus_{s \in S} L_s$, its universal group is the group defined by generators (the elements of $S$) and relations as follows:

$U(\Gamma) := \langle S \mid s_1s_2 = s_3 \text{ for all } s_1, s_2, s_3 \in S \text{ with } 0 \neq [L_{s_1}, L_{s_2}] \subseteq L_{s_3} \rangle$.

There is a natural map $\iota : S \to U(\Gamma)$ taking $s$ to its coset modulo the relations, and $\Gamma$ can be realized as a group grading if and only if $\iota$ is one-to-one.

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The diagonal group of $\Gamma$ is the subgroup $\text{Diag}(\Gamma)$ of the group $\text{Aut}(L)$ of automorphisms of $L$, consisting of those automorphisms that act by multiplication by a scalar on each homogeneous component. Any $\varphi \in \text{Diag}(\Gamma)$ gives a map $\chi : S \to F^\times$ by the equation $\varphi|_L \cdot s = \chi(s) \cdot \text{id}$. This map induces a group homomorphism (a character with values in $F$), denoted by the same symbol, $\chi : U(\Gamma) \to F^\times$. And conversely, any character $\chi$ determines a unique element in $\text{Diag}(\Gamma)$. This shows that the diagonal group is isomorphic to the group of characters with values in $F$ of the universal group. If $F$ is algebraically closed of characteristic $0$ and $U(\Gamma)$ is abelian, characters separate points, and hence if $\Gamma$ is a group grading, the homogeneous components are the common eigenspaces for the action of $\text{Diag}(\Gamma)$.

For the basic results on gradings, the reader is referred to [EK13, Chapter 1].

In the seminal work [PZ89], it was erroneously asserted that any set grading on a finite-dimensional Lie algebra can be realized as a semigroup grading [PZ89, Theorem 1.(d)]. Counterexamples were given in [Eld06] and [Eld09]. In particular, in [Eld09] there appears a non-semigroup grading on the semisimple Lie algebra $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. On the other hand, Cristina Draper proved that if $G$ is a semigroup and $\Gamma : L = \bigoplus_{g \in G} L_g$ is a $G$-grading on a simple Lie algebra $L$, then $\text{Supp}(\Gamma)$ generates an abelian subgroup of $G$ so, in particular, $\Gamma$ can be realized as a group grading. (See [EK13] Proposition 1.12.) As a direct consequence, the universal group of any set grading on a finite-dimensional simple Lie algebra is always abelian.

There appeared naturally the following question [EK13, Question 1.11]:

Can any set grading on a finite-dimensional simple Lie algebra over $\mathbb{C}$ be realized as a group grading?

The main goal of this paper is to answer this question in the negative.

The author is indebted to Alice Fialowski, who during the “Workshop on Non-Associative Algebras and Applications”, held in Lancaster University in 2018, showed him a $(\mathbb{Z}/2)^n$-grading on the orthogonal Lie algebra $L = \mathfrak{so}(2n)$ over $\mathbb{C}$, constructed by Andriy Panasyuk, with some nice properties: one of the homogeneous component $L_g$, with $g \neq e$, is a Cartan subalgebra, while all the other nonzero homogeneous components are toral two-dimensional subalgebras. With hindsight, this grading turns out to be the one in Example 2.9. This is an example of what Hesselink [Hes82] called pure gradings. Some properties of these pure gradings, useful for our purposes, are developed in Section 2.

In an attempt to understand this grading on $\mathfrak{so}(2n)$ and to define similar gradings, the author used the 13 lines in the projective plane over the field of three elements to split the set of positive roots of $\mathfrak{so}(26)$ (say, over $\mathbb{C}$) into the disjoint union of pairs of orthogonal roots, thus obtaining the grading in (3.3). Checking that this is indeed a set grading is not obvious and relies on a pure group grading on $\mathfrak{so}(8)$ in Example 2.8. The next step was to check whether it is a group grading, but surprisingly it turned out not to be the case (Theorem 3.2). Moreover, the result can be extended to the classical split simple Lie algebra of type $D_{13}$ over any arbitrary field of characteristic not two (Theorem 3.3). All this appears in Section 3.

The last Section 4 is devoted to obtain similar non-group gradings on the classical split simple Lie algebra of type $D_{2n}$ with $n \equiv 1 \pmod{12}$, by substituting the lines in the projective plane $\mathbb{P}^2(F_3)$ by the blocks of Steiner systems of type $S(2,4,n)$.

Throughout the work, all the algebras considered will be assumed to be defined over a field $F$ and to be finite-dimensional. Up to Theorem 3.2, the ground field $F$ will be assumed to be algebraically closed of characteristic $0$. 
2. Pure gradings on semisimple Lie algebras

Let $\mathcal{L}$ be a semisimple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0. Following [Hes82], a grading $\Gamma$ by the abelian group $G$ is said to be pure if there exists a Cartan subalgebra $\mathfrak{H}$ of $\mathcal{L}$ and an element $g \in G$, $g \neq e$, such that $\mathfrak{H}$ is contained in $L_g$. (Here $e$ denotes the neutral element of $G$.)

This section follows the ideas in [Hes82].

Let $\Phi$ be the set of roots relative to $\mathfrak{H}$, so that $\mathcal{L} = H \oplus \left( \bigoplus_{\alpha \in \Phi} \mathcal{L}_\alpha \right)$ is the root space decomposition of $\mathcal{L}$, and fix a system of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ relative to the Cartan subalgebra $\mathfrak{H}$. Denote by $\Phi^+$ the set of positive roots relative to $\Delta$.

As in [Hes82, §4], consider the torus $T := \{ \varphi \in \text{Aut}(\mathcal{L}) \mid \varphi|_\mathfrak{H} = \text{id} \}$ and its 2-torsion part: $T_2 := \{ \varphi \in T \mid \varphi^2 = \text{id} \}$. Consider too the larger subgroup $T_e := \{ \varphi \in \text{Aut}(\mathcal{L}) \mid \varphi|_\mathfrak{H} = \pm \text{id} \}$.

**Lemma 2.1.** Let $\sigma$ be an automorphism of $\mathcal{L}$ with $\sigma|_\mathfrak{H} = -\text{id}$. Then the following conditions hold:

1. The order of $\sigma$ is 2: $\sigma^2 = \text{id}$, $\sigma$ permutes $\mathcal{L}_\alpha$ and $\mathcal{L}_{-\alpha}$ for all $\alpha \in \Phi$, and a system of generators $e_i \in \mathcal{L}_{\alpha_i}$, $f_i \in \mathcal{L}_{-\alpha_i}$, $i = 1, \ldots, n$, can be chosen such that $[e_i, f_i] = h_i$, $\sigma(e_i) = -f_i$, and $\sigma(f_i) = -e_i$, where $h_i \in \mathfrak{H}$ satisfies $\alpha_i(h_i) = 2$.
2. The subgroup $T_e$ is the semidirect product of the torus $T$ and the cyclic group of order 2 generated by $\sigma$.
3. The centralizer of $\sigma$ in $T$ is $T_2$.

**Proof.** This follows from [Hes82, Lemma 4.3]. We include a proof for completeness. From $\sigma|_\mathfrak{H} = -\text{id}$ it follows at once that $\sigma$ permutes $\mathcal{L}_\alpha$ and $\mathcal{L}_{-\alpha}$ for all $\alpha \in \Phi$. For any $\alpha \in \Phi^+$, choose nonzero elements $e_\alpha \in \mathcal{L}_\alpha$ and $f_\alpha \in \mathcal{L}_{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha$ with $\alpha(h_\alpha) = 2$. Then we have $\sigma(h_\alpha) = -h_\alpha$, $\sigma(e_\alpha) = \mu f_\alpha$, and $\sigma(f_\alpha) = \mu^{-1} e_\alpha$, for some $0 \neq \mu \in \mathbb{F}$. Let $\eta \in \mathbb{F}$ be a square root of $-\mu^{-1}$. Then we get $\sigma(\eta e_\alpha) = -\eta^{-1} f_\alpha$ and $\sigma(\eta^{-1} f_\alpha) = -\eta e_\alpha$. This completes the proof of the first part.

For any $\varphi \in T_e$, either $\varphi|_\mathfrak{H} = \text{id}$ and hence we have $\varphi \in T$, or $\varphi|_\mathfrak{H} = -\text{id}$ and hence $\sigma \varphi \in T$. Therefore, $T_e$ equals $T \cup T \sigma$. Besides, $T$ is the kernel of the natural homomorphism $T_e \rightarrow \{ \pm 1 \}$, so it is a normal subgroup of $T_e$. This proves (2).

For any $\varphi \in T$, there are nonzero scalars $\mu_i \in \mathbb{F}$ such that $\varphi(e_i) = \mu_i e_i$ and $\varphi(f_i) = \mu_i^{-1} f_i$ for all $i = 1, \ldots, n$. Then we get

\[\sigma \varphi(e_i) = \mu_i \sigma(e_i) = -\mu_i e_i,\]
\[\varphi(\sigma(e_i)) = -\varphi(f_i) = -\mu_i^{-1} f_i,\]

and hence, if $\varphi$ and $\sigma$ commute, we get $\mu_i = \mu_i^{-1}$, so that $\mu_i^2 = 1$ for all $i$ and, therefore, $\varphi^2 = \text{id}$. \hfill \Box

The automorphism $\sigma$ in item (1) above play a key role in the proof of the existence of Chevalley bases. (See [Hum72, §25.2].)

**Proposition 2.2.** Let $\mathcal{L}$ be a semisimple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0. Let $G$ be an abelian group and let $\Gamma$ be a pure $G$-grading of $\mathcal{L}$. Let $g \in G \setminus \{e\}$ such that $\mathcal{L}_g$ contains a Cartan subalgebra $\mathfrak{H}$ of $\mathcal{L}$. With $T$ and $T_2$ as before, there is an automorphism $\sigma \in \text{Aut}(\mathcal{L})$ with $\sigma|_\mathfrak{H} = -\text{id}$, such that $\text{Diag}(\Gamma)$ is the cartesian product of $\text{Diag}(\Gamma) \cap T_2$ and the subgroup generated by $\sigma$:

$$\text{Diag}(\Gamma) = (\text{Diag}(\Gamma) \cap T_2) \times \langle \sigma \rangle.$$  \hfill (2.1)

In particular, $\text{Diag}(\Gamma)$ is a finite 2-elementary group.
Proof. Without loss of generality, we may assume that $G$ is the universal group $U(\Gamma)$. Then the group $\text{Diag}(\Gamma)$ can be identified with the group of characters of $G$. Let $\chi$ be a character of $G$ with $\chi(g) \neq 1$. The automorphism $\sigma$ of $\text{Diag}(\Gamma)$ defined by $\sigma(x) = \chi(h)x$ for any $h \in G$ and $x \in L_h$ satisfies that its restriction to $\mathcal{H}$ is $\chi(\sigma)\text{id}$. This forces $\chi(g)^{-1}\alpha$ to be a root for any root $\alpha$ relative to $\mathcal{H}$. As the only scalar multiples of a root $\alpha$ are $\pm\alpha$, we get $\chi(g) = -1$, and hence $\sigma|_{\mathcal{H}} = -\text{id}$. This also shows that $\chi(g) = \pm 1$ for any character $\chi$ of $G$, and hence $\text{Diag}(\Gamma)$ is contained in $T_\sigma$. We conclude that $\text{Diag}(\Gamma)$ is contained in the centralizer of $\sigma$ in $\mathcal{T}_\sigma$, which is $T_2 \times \langle \sigma \rangle$ by Lemma 2.1.(3). The result follows. \hfill \Box

From now on, fix a Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}$ and an automorphism $\sigma \in \text{Aut}(\mathcal{L})$ such that $\sigma|_{\mathcal{H}} = -\text{id}$. Note that $\sigma$ is unique up to conjugation because of Lemma 2.1.(1). Denote by $Q$ the root lattice $Q := \mathbb{Z}\Phi = \mathbb{Z}\Delta$ (notation as above). The torus $T$ is naturally isomorphic to the group of characters of $Q$, and this isomorphism restricts to a group isomorphism $T_2 \simeq \text{Hom}(Q/2Q, \{\pm 1\})$. Any $\chi \in \text{Hom}(Q/2Q, \{\pm 1\})$ corresponds to the automorphism $\tau_\chi$ whose restriction to $\mathcal{L}_\alpha$ is $\chi(\alpha + 2Q)\text{id}$ for all $\alpha \in \Phi$. In particular, any element of $T_2$ acts a $\pm 1$ on the two-dimensional subspace $\mathcal{L}_\alpha + \mathcal{L}_{-\alpha}$ for any $\alpha \in \Phi^\circ$.

This gives a bijection:

$$\{\text{subgroups of } T_2\} \rightarrow \{\text{subgroups of } Q \text{ containing } 2Q\}$$

$$S \quad \mapsto \quad E \quad \text{such that } E/2Q = \bigcap_{\tau_\chi \in S} \ker \chi. \quad (2.2)$$

In the reverse direction, a subgroup $E$ with $2Q \leq E \leq Q$ corresponds to the subgroup $T_E := \{\tau_\chi \mid \chi(E/2Q) = 1\}$.

For any positive root $\alpha \in \Phi^+$, pick a nonzero element $x_\alpha \in \mathcal{L}_\alpha$. Let $E$ be a subgroup with $2Q \leq E \leq Q$ and denote by $\overline{Q}_E$ the 2-elementary group $Q/E \times \mathbb{Z}/2$. Define the $\overline{Q}_E$-grading $\Gamma_E$ on $\mathcal{L}$ as follows:

$$\mathcal{L}_{(q+E,0)} = \bigoplus_{\alpha \in \Phi^+ \cap (q+E)} \mathbb{F}(x_\alpha + \sigma(x_\alpha)),$$

$$\mathcal{L}_{(q+E,1)} = \begin{cases} \mathcal{H} \oplus \bigoplus_{\alpha \in \Phi^+ \cap (q+E)} \mathbb{F}(x_\alpha - \sigma(x_\alpha)) & \text{if } q + E \neq E, \\ \mathcal{H} \cap \bigoplus_{\alpha \in \Phi^+ \cap (q+E)} \mathbb{F}(x_\alpha - \sigma(x_\alpha)) & \text{if } q + E = E. \end{cases} \quad (2.3)$$

Remark 2.3. Note that the homogeneous spaces of $\Gamma_E$ are the common eigenspaces for the action of the group $T_E \times \langle \sigma \rangle$.

The extreme cases are $\Gamma_Q$, which is a grading by $\mathbb{Z}/2$ with $\mathcal{L}_0 = \{x \in \mathcal{L} \mid \sigma(x) = x\}$ and $\mathcal{L}_1 = \{x \in \mathcal{L} \mid \sigma(x) = -x\}$, and $\Gamma_{2Q}$, which is a grading by $Q/2Q \times \mathbb{Z}/2 \simeq (\mathbb{Z}/2)^{\alpha_{2Q}}$.

For any $E$ as above, define the subgroup $E^\circ$ as follows:

$$E^\circ := 2Q + \mathbb{Z}(\Phi^+ \cap E) + \mathbb{Z}\{\alpha - \beta \mid \alpha, \beta \in \Phi^+ \text{ and } \alpha - \beta \in E\}. \quad (2.4)$$

Note that $2Q \leq E^\circ \leq E \leq Q$.

Remark 2.4. Another way to define $E^\circ$ is as the subgroup of $E$ generated by $2Q$, by the elements $\alpha \in \Phi^+$ such that $x_\alpha - \sigma(x_\alpha)$ is in the homogeneous component of $\Gamma_E$ that contains $\mathcal{H}$, and by the elements $\alpha - \beta$ for $\alpha, \beta \in \Phi^+$ such that $x_\alpha - \sigma(x_\alpha)$ and $x_\beta - \sigma(x_\beta)$ are in the same homogeneous component of $\Gamma_E$.

**Proposition 2.5.** Under the conditions above, the gradings $\Gamma_E$ and $\Gamma_{E^\circ}$ are equivalent and $\text{Diag}(\Gamma_E) = \text{Diag}(\Gamma_{E^\circ})$ equals $T_{E^\circ} \times \langle \sigma \rangle$.

**Proof.** The homogeneous components of $\Gamma_E$ and $\Gamma_{E^\circ}$ coincide, so they are trivially equivalent.
Proposition 2.2 gives $\text{Diag}(\Gamma_E) = (\text{Diag}(\Gamma_E) \cap T_2) \times \langle \sigma \rangle$. For any character $\chi \in \text{Hom}(Q/2Q, \{\pm 1\})$, let $\tau_\chi$ be the associated element in $T_2$. For any $\alpha \in \Phi^+$, $\tau_\chi(x_\alpha \pm \sigma(x_\alpha)) = \chi(\alpha + 2Q)(x_\alpha \pm \sigma(x_\alpha))$. Then $\tau_\chi$ belongs to $\text{Diag}(\Gamma_E)$ if and only if $\chi(\alpha + 2Q) = 1$ for any $\alpha \in \Phi^+ \cap E$ (because $\tau_\chi|_{\mathcal{H}} = \text{id}$), and $\chi(\alpha + 2Q) = \chi(\beta + 2Q)$ for any $\alpha, \beta \in \Phi^+$ with $\alpha - \beta \in E$. In other words, $\tau_\chi$ lies in $\text{Diag}(\Gamma_E)$ if and only if $E^\circ/2Q$ is contained in $\ker \chi$, if and only if $\tau_\chi$ lies in $T_{E^\circ}$. \hfill \square

Let us show now that $E^\circ$ may be strictly contained in $E$.

**Example 2.6.** Let $\mathcal{L}$ be the orthogonal Lie algebra $\mathfrak{so}(V, b)$ of a vector space $V$ of dimension 12, endowed with a nondegenerate symmetric bilinear form $b$. Pick a basis $\{u_1, \ldots, u_6, v_1, \ldots, v_6\}$ of $V$ with $b(u_i, u_j) = b(v_i, v_j)$ and $b(u_i, v_j) = \delta_{ij}$ (Kronecker delta) for all $i, j$. The diagonal elements of $\mathfrak{so}(V, b)$ relative to this basis form a Cartan subalgebra $\mathcal{H}$, and the weights of $V$ relative to $\mathcal{H}$ are $\pm \varepsilon_1, \ldots, \varepsilon_6$, where $h.u_i = \varepsilon_i(h)u_i$, $h.v_i = -\varepsilon_i(h)v_i$ for all $h \in \mathcal{H}$ and $i = 1, \ldots, 6$. The set of roots is $\Phi = \{\pm \varepsilon_i \pm \varepsilon_j \mid i \neq j\}$. Up to a scalar, the nondegenerate bilinear form on $\mathcal{H}^*$ induced by the Killing form is given by $\langle \varepsilon_i | \varepsilon_j \rangle = \delta_{ij}$ (Kronecker delta).

As a system of simple roots take $\Delta = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_4 - \varepsilon_5, \varepsilon_5 - \varepsilon_6, \varepsilon_6 \}$. The highest weight of one of the half-spin modules is $\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_6)$. Denote by $\Lambda'$ the set of weights of this half spin module: $\Lambda' = \{\frac{1}{2}(\pm \varepsilon_1 \pm \cdots \pm \varepsilon_6) \mid \text{even number of } + \text{ signs}\}$. Write $W' = Z\Lambda'$. Then $E = 2W'$ satisfies $Q \leq E = 2Q + Z(\varepsilon_1 + \cdots + \varepsilon_6) \leq Q$.

It follows at once that for any $r < s$ and $(i, j) \neq (r, s)$, the element $\frac{1}{2}(\varepsilon_i + \varepsilon_j - \varepsilon_r - \varepsilon_s)$ does not belong to $Q$, nor to $\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_6) + Q$. Hence we get that for any two different positive roots $\alpha, \beta \in \Phi^+$, we get $\alpha - \beta \notin E$. It is also clear that $\alpha$ does not belong to $E$ for any $\alpha \in \Phi^+$, so $\mathfrak{g}_0$ equals $Q = 2Q$.

**Remark 2.7.** A word of caution is in order here. The notion of equivalence in $\text{Hom}(Q/2Q, \{\pm 1\})$ is more restrictive. It turns out that $\Gamma_E$ and $\Gamma_E^\circ$ are not equivalent in this more restrictive sense if $E^\circ$ is contained properly in $E$, as in Example 2.3, even though their homogeneous components coincide, so they are trivially equivalent in our sense.

The situation is different for $D_4$, and this will be crucial in our examples of non-group gradings in the next sections.

**Example 2.8.** Let $\mathcal{L}$ be the orthogonal Lie algebra $\mathfrak{so}(V, b)$ of a vector space $V$ of dimension 8, endowed with a nondegenerate symmetric bilinear form $b$. As in Example 2.3 pick a basis $\{u_1, \ldots, u_4, v_1, \ldots, v_4\}$ of $V$ with $b(u_i, u_j) = b(v_i, v_j)$ and $b(u_i, v_j) = \delta_{ij}$ for all $i, j$, and consider the corresponding Cartan subalgebra of diagonal elements relative to this basis, and the system of simple roots $\Delta = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_4\}$. Pick the highest weight $\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$ of one of the half-spin modules, and let $\Lambda'$ be its set of weights. Finally, write $W' = Z\Lambda'$ and consider the subgroup $E = 2W'$ of $2Q + Z(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$, whose index $|E : 2Q| = 2$.

For any $1 \leq r < s \leq 4$, let $\bar{r} < \bar{s}$ be such that $\{r, s, \bar{r}, \bar{s}\} = \{1, 2, 3, 4\}$. Then the positive roots $\varepsilon_r + \varepsilon_s$ and $\varepsilon_r + \varepsilon_s$ satisfy that its difference belongs to $E$, and the same happens with $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = (\varepsilon_r + \varepsilon_s) - (\varepsilon_r + \varepsilon_s) + 2(\varepsilon_r + \varepsilon_s)$. Therefore in this case we get $E = E^\circ$.

Proposition 2.3 shows that the diagonal group of the grading $\Gamma_E$ is $T_E \times \langle \sigma \rangle$, which is isomorphic to $(\mathbb{Z}/2)^4$. Our last example is Panayuk’s example mentioned in Section 4.
Example 2.9. Let \( \mathcal{L} \) be the orthogonal Lie algebra \( \mathfrak{so}(V, b) \) of a vector space \( V \) of dimension \( 2n \), endowed with a nondegenerate symmetric bilinear form \( b \). Pick a basis \( \{ u_1, \ldots, u_n, v_1, \ldots, v_n \} \) of \( V \) with \( b(u_i, u_j) = 0 = b(v_i, v_j) \) and \( b(u_i, v_j) = \delta_{ij} \) for all \( i, j \) as in Example 2.6 and use the notation there. Let \( W = \mathbb{Z} \epsilon_1 \oplus \cdots \oplus \mathbb{Z} \epsilon_n \) be the \( \mathbb{Z} \)-span of the weights of the natural module \( V \). Note that \( W = Q + \mathbb{Z} \epsilon_1 \) and \( 2 \epsilon_1 = (\epsilon_1 - \epsilon_2) + (\epsilon_1 + \epsilon_2) \) belongs to \( Q \). Hence the index \([ W : Q ]\) is 2. Consider the subgroup \( E = 2W \), which lies between \( 2Q \) and \( Q \), and the associated grading \( \Gamma_{2W} \) over \( Q/(2W) \times \mathbb{Z}/2 \cong (\mathbb{Z}/2)^n \) in \((3.3)\).

The intersection \( \Phi^+ \cap 2W \) is empty, and the only pairs of positive roots that belong to the same coset modulo \( 2W \) are \( \epsilon_i - \epsilon_j \) and \( \epsilon_i + \epsilon_j \) for \( 1 \leq i < j \leq n \). It follows that \( E^0 = E \) and that the homogeneous component \( \mathcal{L}_{(E, 1)} \) coincides with the Cartan subalgebra, while all the other nonzero homogeneous components have dimension two.

Remark 2.10. The same properties are valid for the grading in Example 2.8. But for \( n = 4 \), the natural and half-spin modules are related by triality, so the grading in Example 2.8 is equivalent to the one in Example 2.9 for \( n = 4 \). However, the specific \( E \) in Example 2.8 will be crucial in the next section (proof of Proposition 3.1).

3. A non-group grading on \( D_{13} \)

We may number the points in the projective plane \( \mathbb{P}^2(\mathbb{F}_3) \) over the field \( \mathbb{F}_3 \) of three elements from 1 to 13, so that the lines in \( \mathbb{P}^2(\mathbb{F}_3) \) are the ones consisting of the points:

\[
\begin{align*}
\{1, 2, 3, 4\} &\quad \{2, 5, 8, 11\} &\quad \{3, 5, 9, 13\} &\quad \{4, 5, 10, 12\} \\
\{1, 5, 6, 7\} &\quad \{2, 6, 9, 12\} &\quad \{3, 6, 10, 11\} &\quad \{4, 6, 8, 13\} \\
\{1, 8, 9, 10\} &\quad \{2, 7, 10, 13\} &\quad \{3, 7, 8, 12\} &\quad \{4, 7, 9, 11\}
\end{align*}
\]

(3.1)

(The reader can check that any two points lie in a unique line, and any two lines intersect in a unique point.) Denote by \( \mathcal{L} \) this set of lines.

As in Examples 2.6 and 2.8, let \( \mathcal{L} \) be the orthogonal Lie algebra \( \mathfrak{so}(V, b) \) of a vector space \( V \) of dimension 26, endowed with a nondegenerate symmetric bilinear form \( b \). Pick a basis \( \{ u_1, \ldots, u_{13}, v_1, \ldots, v_{13} \} \) of \( V \) with \( b(u_i, u_j) = 0 = b(v_i, v_j) \) and \( b(u_i, v_j) = \delta_{ij} \) for all \( i, j \). The diagonal elements of \( \mathfrak{so}(V, b) \) relative to this basis form a Cartan subalgebra \( \mathcal{H} \), and the weights of \( V \) relative to \( \mathcal{H} \) are \( \pm \epsilon_1, \ldots, \pm \epsilon_{13} \), where \( h.u_i = \epsilon_i(h)u_i, h.v_1 = -\epsilon_i(h)v_1 \) for all \( h \in \mathcal{H} \) and \( i = 1, \ldots, 13 \). The set of roots is \( \Phi = \{ \epsilon_i \pm \epsilon_j \mid i \neq j \} \). As a system of simple roots take \( \Delta = \{ \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \ldots, \epsilon_{12} - \epsilon_{13}, \epsilon_{12} + \epsilon_{13} \} \).

Fix an automorphism \( \sigma \) of \( \mathcal{L} \) with \( \sigma|_{\mathcal{H}} = -\text{id} \), and for any positive root \( \alpha \in \Phi^+ \) pick a nonzero element \( x_\alpha \in \mathcal{L}_\alpha \).

For any line \( \ell = \{ i, j, k, l \} \in \mathcal{L} \), with \( i < j < k < l \) consider the set \( \mathcal{P}_\ell \) whose elements are the following six subsets of pairs of orthogonal positive roots:

\[
\begin{align*}
\{ \epsilon_i + \epsilon_j, \epsilon_k + \epsilon_l \}, &\quad \{ \epsilon_i + \epsilon_k, \epsilon_j + \epsilon_l \}, &\quad \{ \epsilon_i + \epsilon_l, \epsilon_j + \epsilon_k \} \\
\{ \epsilon_i - \epsilon_j, \epsilon_k - \epsilon_l \}, &\quad \{ \epsilon_i - \epsilon_k, \epsilon_j - \epsilon_l \}, &\quad \{ \epsilon_i - \epsilon_l, \epsilon_j - \epsilon_k \}.
\end{align*}
\]

(3.2)

Denote by \( \mathcal{P} \) the union \( \mathcal{P} = \bigcup_{\ell \in \mathcal{L}} \mathcal{P}_\ell \). Its elements are subsets consisting of a pair of orthogonal positive roots. Hence \( \mathcal{P} \) contains \( 13 \times 6 = 78 \) elements, and each
positive root appears in exactly one of the pairs in \( \Psi \). Then \( \mathcal{L} \) decomposes as

\[
\mathcal{L} = \mathcal{H} \oplus \left( \bigoplus_{(\alpha, \beta) \in \Psi} (\mathcal{F}(x_\alpha + \sigma(x_\alpha)) + \mathcal{F}(x_\beta + \sigma(x_\beta))) \right)
\]

\[
\oplus \left( \bigoplus_{(\alpha, \beta) \in \Psi} (\mathcal{F}(x_\alpha - \sigma(x_\alpha)) + \mathcal{F}(x_\beta - \sigma(x_\beta))) \right).
\]

Hence \( \mathcal{L} \) decomposes into the direct sum of the Cartan subalgebra and the direct sum of \( 13 \times 6 \times 2 = 156 \) two-dimensional abelian subalgebras. The reader may recall here Example 2.9 where a group-grading with these properties is highlighted.

**Proposition 3.1.** The decomposition in Equation (3.3) is a set grading of \( \mathcal{L} \).

**Proof.** For any \( h \in \mathcal{H} \) and \( \alpha \in \Phi^+ \), \([h, x_\alpha \pm \sigma(x_\alpha)] = \alpha(h)(x_\alpha \mp \sigma(x_\alpha)) \). Also, for any two orthogonal positive roots \( \alpha, \beta \), \([x_\alpha \pm \sigma(x_\alpha), x_\beta \pm \sigma(x_\beta)] = 0 \), because \( \alpha \pm \beta \) is not a root. We must check that the bracket of two subspaces of the form \( \mathcal{F}(x_\beta \pm \sigma(x_\beta)) \), with \( \{\alpha, \beta\} \in \Psi \), is contained in another subspace of this form. So let us take two such elements \( \{\alpha, \beta\} \in \Psi_{\ell_1} \) and \( \{\alpha', \beta'\} \in \Psi_{\ell_2} \).

There are two different possibilities:

- \( \ell_1 \cap \ell_2 \) consists of a single point: \( \ell_1 = \{i, j, k, l\}, \ell_2 = \{i, p, q, r\} \) for distinct \( i, j, k, l, p, q, r \). This shows that, up to reordering, \( \{\alpha|\alpha'\} \neq 0 \), but \( \{\alpha|\beta'\} = 0 = \{\alpha'|\beta\} \). This is because, up to reordering, the roots involved are of the form \( \alpha = \pm \varepsilon_i \pm \varepsilon_j = \pm \varepsilon_k \pm \varepsilon_l, \alpha' = \pm \varepsilon_k \pm \varepsilon_l, \beta = \pm \varepsilon_q \pm \varepsilon_r \). Since \( \{\alpha|\alpha'\} \neq 0 \), neither \( \alpha + \alpha' \) or \( \alpha - \alpha' \) is a root (but not both), while \( \alpha \pm \beta' \) and \( \alpha' \pm \beta \) are not roots. It follows that the bracket of \( \mathcal{F}(x_\alpha \pm \sigma(x_\alpha)) + \mathcal{F}(x_\beta \pm \sigma(x_\beta)) \) with \( \mathcal{F}(x_{\alpha'} \pm \sigma(x_{\alpha'})) + \mathcal{F}(x_{\beta'} \pm \sigma(x_{\beta'})) \) equals \( \mathcal{F}[x_\alpha \pm \sigma(x_\alpha), x_{\alpha'} \pm \sigma(x_{\alpha'})] \), which equals \( \mathcal{F}[x_{\alpha \pm \alpha'} \pm \sigma(x_{\alpha \pm \alpha'})] \), depending on whether \( \alpha + \alpha' \) or \( \alpha - \alpha' \) is a root.

- \( \ell_1 = \ell_2 \). Without loss of generality we may assume that \( \ell = \{1, 2, 3, 4\} \). Then the subalgebra generated by \( x_\alpha \) and \( \sigma(x_\alpha) \) for \( \alpha \) in the six subsets associated to \( \ell \) generate a subalgebra of \( \mathcal{L} \) isomorphic to \( D_4 \). The decomposition induced on this subalgebra from the decomposition in (3.3) is exactly the group-grading \( \Gamma_E \) in Example 2.8 and hence the bracket of \( \mathcal{F}(x_\alpha \pm \sigma(x_\alpha)) + \mathcal{F}(x_\beta \pm \sigma(x_\beta)) \) with \( \mathcal{F}(x_{\alpha'} \pm \sigma(x_{\alpha'})) + \mathcal{F}(x_{\beta'} \pm \sigma(x_{\beta'})) \) is contained in another ‘homogeneous component’ in (3.3).

We conclude that the decomposition in (3.3) is indeed a set grading on \( \mathcal{L} \). \( \square \)

Denote the grading given by (3.3) by \( \Gamma \). Our aim now is to show that \( \Gamma \) cannot be realized as a group-grading. We will do it by computing its diagonal group.

To begin with, note that the automorphism \( \sigma \) belongs to \( \text{Diag}(\Gamma) \), and that any element of \( \text{Diag}(\Gamma) \) must act as a scalar on \( \mathcal{H} \). The arguments in the proof of Proposition 2.11 work for \( \Gamma \) and hence Equation (2.11) holds: \( \text{Diag}(\Gamma) = (\text{Diag}(\Gamma) \cap T_2) \times \sigma \).

For any \( \chi \in \text{Hom}(Q/2Q, \{\pm 1\}) \), let \( \tau_\chi \) be the corresponding element in \( T_2 \) (recall Equation (2.22)). Then, as in the proof of Proposition 2.2 \( \tau_\chi \) is in \( \text{Diag}(\Gamma) \) if and only if \( \chi(\alpha+2Q) = \chi(\beta+2Q) \) for all \( \{\alpha, \beta\} \in \Psi \), if and only if \( \{\alpha - \beta + 2Q | \{\alpha, \beta\} \in \Psi\} \) is contained in \( \ker \chi \). Therefore we get \( \text{Diag}(\Gamma) = T_E \times \langle \sigma \rangle \) for \( E = 2Q + \mathbb{Z}\{\alpha - \beta | \{\alpha, \beta\} \in \Psi\} \).

In particular, for \( \ell = \{i, j, k, l\} \in \mathcal{L} \), with \( 1 \leq i < j < k < l \leq 13 \), the sum \( \varepsilon_i + \varepsilon_j + \varepsilon_k + \varepsilon_l = (\varepsilon_1 + \varepsilon_2) - (\varepsilon_k + \varepsilon_l) + 2(\varepsilon_k + \varepsilon_l) \) lies in \( E \). Considering the lines in the first column of (3.1), and adding the elements \( \varepsilon_1, \varepsilon_j, \varepsilon_k, \varepsilon_l \) for these lines, we check that \( E \) contains the elements \( 4\varepsilon_1 + \sum_{i=2}^{13} \varepsilon_i \). As \( 4\varepsilon_1 = 2((\varepsilon_1 - \varepsilon_2) + (\varepsilon_1 + \varepsilon_2)) \) lies in \( 2Q \leq E \), it follows that \( \sum_{i \neq j} \varepsilon_i \) lies in \( E \). In the same vein \( \sum_{i \neq j} \varepsilon_i \) lies in \( E \), and subtracting these two elements, we see that \( \varepsilon_1 - \varepsilon_2 \) lies in \( E \). The
same argument shows that all the roots of the form $\varepsilon_i - \varepsilon_j$ lie in $E$. But then we get $Q = Z\Delta \subseteq E + Z(\varepsilon_{12} + \varepsilon_{13}) \subseteq Q$, so $Q = E + Z(\varepsilon_{12} + \varepsilon_{13})$. Also we have $2(\varepsilon_{12} + \varepsilon_{13}) \in 2Q \subseteq E$, and hence the index of $E$ in $Q$ is at most 2, and $\text{Diag}(\Gamma) = T_E \times \langle \sigma \rangle$ is isomorphic to either $C_2$ or $C_2 \times C_2$. Actually, the splitting $V = (\mathbb{F}u_1 + \cdots + \mathbb{F}u_6) \oplus (\mathbb{F}v_1 + \cdots + \mathbb{F}v_6)$ gives a $Z/2$-grading on $L$, where the even (respectively odd) part consists of the elements in $L$ that preserve (respectively swap) the subspaces $\mathbb{F}u_1 + \cdots + \mathbb{F}u_6$ and $\mathbb{F}v_1 + \cdots + \mathbb{F}v_6$. The order two automorphism $\tau$ that fixes the even part and is $-\text{id}$ on the odd part lies in $T$. It fixes the root spaces $L_{\varepsilon_i - \varepsilon_j}$ and is $-\text{id}$ on the root spaces $L_{\varepsilon_i + \varepsilon_j}$. It follows that $\tau$ is in $\text{Diag}(\Gamma) \cap T_2$, and we conclude that the diagonal group of $\Gamma$ is exactly $\text{Diag}(\Gamma) = \langle \tau, \sigma \rangle$.

Alternatively, we can use some software, for instance SageMath, to compute $[Q : E]$. Let $W$ be the group generated by $\varepsilon_1, \ldots, \varepsilon_{13}$. (Note $[W : Q] = 2$.) Write in a SageMath cell the following:

```sage
A=matrix([[2,-2,0,0,0,0,0,0,0,0,0,0,0], [0,0,2,-2,0,0,0,0,0,0,0,0,0], [0,0,0,0,2,-2,0,0,0,0,0,0,0], [0,0,0,0,0,0,2,-2,0,0,0,0,0], [0,0,0,0,0,0,0,0,2,-2,0,0,0], [0,0,0,0,0,0,0,0,0,0,2,-2,0], [0,0,0,0,0,0,0,0,0,0,0,2,2], [1,1,1,1,0,0,0,0,0,0,0,0,0], [1,0,0,0,1,1,1,0,0,0,0,0,0], [1,0,0,0,0,0,0,1,1,1,0,0,0], [1,0,0,0,0,0,0,0,0,0,1,1,1], [0,1,0,0,1,0,0,1,0,0,1,0,0], [0,1,0,0,0,1,0,0,1,0,1,0,0], [0,0,1,0,1,0,0,0,1,0,0,1,0], [0,0,1,0,0,1,0,0,1,1,0,0,1], [0,0,0,1,1,1,0,0,0,0,1,0,0], [0,0,0,1,0,1,0,1,0,1,0,0,0], [0,0,0,1,0,0,1,0,1,0,0,0,0], [0,0,0,1,0,0,1,0,1,0,0,0,0]]);
A.elementary_divisors()
```

where the first six rows correspond to the coordinates of the elements of the basis $2\Delta$ of $2Q$ in the natural basis $\{\varepsilon_1, \ldots, \varepsilon_{13}\}$ of $W$. The last six rows are the coordinates of the elements $\varepsilon_i + \varepsilon_j + \varepsilon_k + \varepsilon_l$ for each line $\{i, j, k, l\} \subseteq \Sigma$.

The outcome of running the SageMath cell above is the following:

$\langle 1, 1, 1, 1, 1, 1, 1, 1, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \rangle$

giving the elementary divisors of the matrix $A$, and this shows that the index $[W : E]$ is 4, and hence the index $[Q : E]$ is 2.

We conclude that the universal group of $\Gamma$ is also isomorphic to $C_2 \times C_2$, because its group of characters is $\text{Diag}(\Gamma) \cong C_2 \times C_2$, but there are 157 nonzero homogeneous components, and hence the natural map $\iota : S \to U(\Gamma)$ cannot be one-to-one, where $S$ denotes the set of homogeneous components of $\Gamma$. We have proved the following result:

**Theorem 3.2.** The set grading $\Gamma$ of the simple Lie algebra of type $D_{13}$ in $\mathfrak{so}(V, b)$ cannot be realized as a group grading.

Actually, this grading makes sense over an arbitrary field $\mathbb{F}$ of characteristic not two. Indeed, let $\mathcal{L}$ be the orthogonal Lie algebra $\mathfrak{so}(V, b)$ of a vector space $V$ over $\mathbb{F}$ of dimension 26, endowed with a nondegenerate symmetric bilinear form $b$ of maximal Witt index. That is, $\mathcal{L}$ is the classical split simple Lie algebra of type $D_{13}$ over $\mathbb{F}$. Pick a basis $\{u_1, u_2, v_1, \ldots, v_{13}\}$ of $V$ with $b(u_i, u_j) = 0 = b(v_i, v_j)$ and $b(u_i, v_j) = \delta_{ij}$ for all $i, j$. As before, the diagonal elements of $\mathfrak{so}(V, b)$ relative to this basis form a Cartan subalgebra $\mathfrak{H}$, and the weights (elements of the dual $\mathfrak{H}^*$) of $V$ relative to $\mathfrak{H}$ are $\pm \varepsilon_1, \ldots, \pm \varepsilon_{13}$, where $h.u_i = \varepsilon_i(h)u_i$, $h.v_i = -\varepsilon_i(h)v_i$. 
for all $h \in H$ and $i = 1, \ldots, n$. Up to a scalar, the nondegenerate bilinear form on $\mathfrak{h}^*$ induced from the trace form on $V$ is given by $(\epsilon_i | \epsilon_j) = \delta_{ij}$ (Kronecker delta).

Identify any $x \in \mathcal{L}$ with its coordinate matrix relative to the basis above, which has the following block form
$$
\begin{pmatrix}
A & B \\
C & -A^t
\end{pmatrix},
$$
where the blocks are $13 \times 13$ matrices, with $B$ and $C$ symmetric: $B = B^t$, $C = C^t$.

The root space decomposition of $\mathcal{L}$ relative to $H$ is

$$
\mathcal{L} = H \oplus \bigoplus_{i \neq j} \mathcal{L}_{\epsilon_i - \epsilon_j} \oplus \bigoplus_{i < j} \mathcal{L}_{\epsilon_i + \epsilon_j} \oplus \bigoplus_{i < j} \mathcal{L}_{-\epsilon_i - \epsilon_j}
$$

(3.4)

where the root spaces are:

$$
\mathcal{L}_{\epsilon_i - \epsilon_j} = \mathbb{F} \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}, \quad \mathcal{L}_{\epsilon_i + \epsilon_j} = \mathbb{F} \begin{pmatrix} 0 & E_{ij} \\ 0 & 0 \end{pmatrix}, \quad \mathcal{L}_{-\epsilon_i - \epsilon_j} = \mathbb{F} \begin{pmatrix} 0 & 0 \\ E_{ij} & 0 \end{pmatrix},
$$

where $E_{ij}$ is the $13 \times 13$-matrix with 1 in the $(i, j)$-position and 0's elsewhere.

Let $W$ be the free abelian group with generators $\epsilon_1, \ldots, \epsilon_{13}$ (note the slightly different notation: $\epsilon_i$ for free generators, and $\epsilon_j$ for weights), and its index 2 subgroup $Q$ freely generated by $\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{12} - \epsilon_{13}, \epsilon_{12} + \epsilon_{13}$.

The root space decomposition (3.4) is a $Q$-grading, with $\mathcal{L}_{\pm \epsilon_i, \pm \epsilon_j} := \mathcal{L}_{\pm \epsilon_i, \pm \epsilon_j}$ for all $i, j$. The automorphism $\sigma : x \mapsto -x^t$ has order 2 and satisfies $\sigma(\mathcal{L}_\alpha) = \mathcal{L}_{-\alpha}$ for all roots $\alpha$. This automorphism $\sigma$ can be used to define the grading $\Gamma$ as in (3.3).

The only scalar multiples of a root $\alpha$ are $\pm \alpha$, hence the argument in the proof of Proposition 2.2 gives that any automorphism of $\mathcal{L}$ that acts as a scalar multiple of id on $\mathfrak{h}$ must act necessarily as id or $-\text{id}$. Now the arguments in Lemma 2.1 and Proposition 2.2 apply to show that $T_2$ is the centralizer of $\sigma$ in $T$ and that (2.1) holds here.

On the other hand, any $\varphi \in T_2$ is determined by its restriction to the root spaces $\mathcal{L}_{\epsilon_1 - \epsilon_2}, \ldots, \mathcal{L}_{\epsilon_{12} - \epsilon_{13}}, \mathcal{L}_{\epsilon_{12} + \epsilon_{13}}$, and this allows us to identify $T_2$ with the group $\text{Hom}(Q/2Q, \{\pm 1\})$.

Proposition 2.2 and its proof thus remain valid over an arbitrary field of characteristic not two and, therefore, we conclude that the diagonal group $\text{Diag}(\Gamma)$ is isomorphic to $C_2 \times C_2$. However, over arbitrary fields the diagonal group is, up to isomorphism, the group of characters of the universal grading group, but not conversely.

However, for any $\{\alpha, \beta\} \in \mathbb{P}$ consider the subalgebra $\mathfrak{H}\oplus (\mathbb{F}(x_\alpha + \sigma(x_\alpha)) + \mathbb{F}(x_\beta + \sigma(x_\beta)) + \mathbb{F}(x_\alpha - \sigma(x_\alpha)) + \mathbb{F}(x_\beta - \sigma(x_\beta)))$. It consists of three of the homogeneous components of $\Gamma$. The bracket of any two of these components is nonzero and lies in the other component, because $\alpha \pm \beta$ is not a root. This shows that all the generators of the universal group have order at most two and, therefore, the universal group is 2-elementary. But the characteristic of $\mathbb{F}$ being not two, the group of characters of a 2-elementary group is isomorphic to itself. Hence the universal group of $\Gamma$ is isomorphic to $C_2 \times C_2$, generated by $\sigma$ and by the order two automorphism $\tau$ that fixes the elements of the form
$$
\begin{pmatrix}
A & 0 \\
0 & -A^t
\end{pmatrix}
$$
and multiplies by $-1$ the elements of the form
$$
\begin{pmatrix}
0 & B \\
C & 0
\end{pmatrix}.
$$
We thus get the same contradiction leading to Theorem 5.2 which is then extended as follows.

**Theorem 3.3.** The set grading $\Gamma$ in (3.3) of the classical split simple Lie algebra of type $D_{13}$, over an arbitrary field of characteristic not two, cannot be realized as a group grading.
4. An infinite family of non-group gradings on orthogonal Lie algebras

A careful look at the non-group grading in Section 3 shows that a key point is the existence of the set of lines in $\mathcal{L}$. All the arguments in the previous section can be applied as long as $n > 4$ and there is a set $\mathcal{L}$ of subsets of 4 elements of $\{1, \ldots, n\}$, that we will call lines, such that any two points lie in a unique line. Any such $\mathcal{L}$ is called a 2-$(n, 4, 1)$ design, or a Steiner system of type $S(2, 4, n)$. (See e.g., the survey paper [RR10].)

Actually, all the arguments in the proof of Proposition 3.1 work, but a possibility must be added, as there are now three options for the intersection of two lines $\ell_1$ and $\ell_2$: either $\ell_1 \cap \ell_2 = \emptyset$, or $\ell_1 \cap \ell_2$ consists of a single point, or $\ell_1 = \ell_2$. The extra option: $\ell_1 \cap \ell_2 = \emptyset$, is dealt with easily, as in this case $\ell_1 = \{i, j, k, l\}$ and $\ell_2 = \{p, q, r, s\}$ for distinct $i, j, k, l, p, q, r, s$, and hence the subsets of roots $\{\alpha, \beta\}$, $\{\alpha, \beta\}$ satisfy that $\alpha, \beta, \alpha', \beta'$ are all orthogonal, and hence the bracket of $F(x_\alpha \pm \sigma(x_\alpha)) + F(x_\beta \pm \sigma(x_\beta))$ with $F(x_{\alpha'} \pm \sigma(x_{\alpha'})) + F(x_{\beta'} \pm \sigma(x_{\beta'}))$ is trivial.

Therefore, any Steiner system of type $S(2, 4, n)$ allows us to define a set grading on the simple Lie algebra of type $D_n$.

It turns out (see [Han61]) that a Steiner system of type $S(2, 4, n)$ exists if and only if $n \equiv 1$ or $4 \pmod{12}$. One direction is easy: if $\mathcal{L}$ is a Steiner system of type $S(2, 4, n)$, $n$ must be $\equiv 1 \pmod{3}$, as 1 lies in $\frac{n-1}{3}$ lines. On the other hand, if the system has $b$ lines, then necessarily $4b = n \frac{n-1}{3}$, so we have $n(n-1) \equiv 0 \pmod{12}$, and we get $n \equiv 1, 4 \pmod{12}$.

The lowest such $n$ is 13, where we get the Steiner system in (3.1). If $n$ is congruent to 1 modulo 12, and we take the corresponding set grading using (3.2) and (3.3), and compute $E$ as in the previous section, we check that $E$ contains the element $\frac{1}{4} \varepsilon_1 + \sum_{i \neq 1} \varepsilon_i$. But $\frac{1}{4}$ is a multiple of 4 so, as in the previous section, we see that $\sum_{i \neq 1} \varepsilon_i$ lies in $E$, and all the subsequent arguments work.

As a consequence, we obtain our last result:

**Theorem 4.1.** For any $n \equiv 1 \pmod{12}$, $n > 1$, there is a non-group grading on the classical split simple Lie algebra of type $D_n$ over an arbitrary field $F$ of characteristic $\neq 2$. One of the homogeneous components of this grading is a Cartan subalgebra, and all the other homogeneous components have dimension 2.

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