Analytic solution for relativistic transverse flow at the softest point

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Abstract: We obtain an extension of Bjorken’s 1+1 dimensional scaling relativistic flow solution to relativistic transverse velocities with cylindrical symmetry in 1+3 dimension at constant, homogeneous pressure (vanishing sound velocity). This can be the situation during a first order phase transition converting quark matter into hadron matter in relativistic heavy ion collisions.

Hydrodynamics often allow for nonrelativistic scaling solutions. Relativistic flow, however, seems to be an exception: besides Bjorken’s 1+1 dimensional ansatz and the spherically symmetric relativistic expansion, no analytical solution is known [1].

In this paper we present an extension of Bjorken’s ansatz [2] for longitudinally and transversally relativistic flow patterns with cylindrical symmetry in 1+3 dimensions. This is an analytical solution of the flow equations of a perfect fluid for physical situations when the sound velocity is zero, \( c_s^2 = \frac{dp}{d\epsilon} = 0 \), with energy density \( \epsilon \) and pressure \( p(\epsilon) \).

In particular this happens during a first order phase transition, the pressure is constant while the energy density changes (in heavy ion collisions increases and drops again). This should, in principle, be signalled by a vanishing sound velocity. A remnant of this effect in finite size, finite time transitions might be a softest point of the equation of state, where \( c_s^2 \) is minimal. In fact, this has been suggested as a signal of phase transition by Shuryak [3], and investigated numerically in several recent works [4, 5].

In the light of this research, the presentation of an analytical solution including relativistic transverse flow is worthwhile. For a general equation of state \( p(\epsilon) \) the analytical solution given in this paper cannot be extended, only a perturbative expansion in terms of mild \( (v \ll 1) \) transverse velocities can be established. Such an approximation has been recently presented in [6]. Nonrelativistic analytic solution has been also given several times, with respect to heavy ions see [7, 8, 9].

The 1+1 dimensional Bjorken flow four-velocity is a normalized, timelike vector. It is natural to choose this as the first of our comoving frame basis vectors (vierbein). The

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three further, spacelike vectors will be constructed orthogonal to this, separating the two transverse directions. This basis fits excellently to a cylindrical symmetry and to longitudinally extreme relativistic flow.

\[
e^0_\mu = \left( \frac{t}{\tau}, \frac{z}{\tau}, 0, 0 \right), \quad e^1_\mu = \left( \frac{z}{\tau}, \frac{t}{\tau}, 0, 0 \right),
\]

\[
e^2_\mu = \left( 0, 0, \frac{x}{r}, \frac{y}{r} \right), \quad e^3_\mu = \left( 0, 0, -\frac{y}{r}, \frac{x}{r} \right),
\]

(1)

with \( t \) time coordinate and \( z \) longitudinal (beam-along) coordinate, \( x \) and \( y \) transverse, cartesian coordinates. The cylindrical radius is given by, \( r = \sqrt{x^2 + y^2} \), and \( \tau \) is the longitudinal proper time: \( \tau = \sqrt{t^2 - z^2} \).

This way the basis can be re-written in terms of a hyperbolic angle (coordinate-rapidity) \( \eta \) and a polar angle \( \phi \)

\[
e^0_\mu = (\cosh \eta, \sinh \eta, 0, 0), \quad e^1_\mu = (\sinh \eta, \cosh \eta, 0, 0),
\]

\[
e^2_\mu = (0, 0, \cos \phi, \sin \phi), \quad e^3_\mu = (0, 0, -\sin \phi, \cos \phi).
\]

(2)

This basis is orthonormal, \( e^a_\mu \cdot e^{\mu,b} = g^{ab} \), and satisfies the differential relations:

\[
d e^0_\mu = e^1_\mu d\eta, \quad de^1_\mu = e^0_\mu d\eta, \quad de^2_\mu = e^3_\mu d\phi, \quad de^3_\mu = -e^2_\mu d\phi.
\]

(3)

The space-time coordinate differential and the partial derivative are in our basis given by,

\[
dx_\mu = e^0_\mu d\tau + e^1_\mu \tau d\eta + e^2_\mu dr + e^3_\mu r d\phi,
\]

\[
\partial_\mu = e^0_\mu \frac{\partial}{\partial \tau} - e^1_\mu \frac{1}{\tau} \frac{\partial}{\partial \eta} - e^2_\mu \frac{\partial}{\partial r} - e^3_\mu \frac{1}{r} \frac{\partial}{\partial \phi}.
\]

(4)

We consider ideal fluids (“dry water”), where the energy momentum tensor is given by

\[
T_{\mu\nu} = (\epsilon + p) u_\mu u_\nu - p g_{\mu\nu},
\]

(5)

and the equation of state is given in the form of \( p(\epsilon) \). The ansatz for an almost boost invariant flow with some transverse, cylindrically symmetric component is then given by

\[
u_\mu = \gamma \left( e^0_\mu + v e^2_\mu \right),
\]

(6)

using the Lorentz factor \( \gamma = \left( 1 - v^2 \right)^{-1/2} \). This four-velocity is normalized to one:

\[
u_\mu \nu^\mu = \gamma^2 - \gamma^2 v^2 = 1.
\]

(7)

For the four divergence of the flow we obtain

\[
\partial_\mu \nu^\mu = \frac{\partial}{\partial \tau} \gamma + \frac{\partial}{\partial r} (\gamma v) + \gamma \left( \frac{1}{\tau} + \frac{v}{r} \right).
\]

(8)
The co-moving flow derivative is

\[ u_\mu \partial^\mu = \gamma \left( \frac{\partial}{\partial \tau} + v \frac{\partial}{\partial r} \right). \]  

(9)

In order to formulate the Euler equation describing a possible acceleration or deceleration of the flow, one also needs a projection orthogonal to \( u_\mu \). We use

\[ \nabla_\lambda = (g_{\lambda \mu} - u_\lambda u_\mu) \partial^\mu, \]  

(10)

which is given by

\[ \nabla_\lambda = -\gamma^2 \left( e^2_\lambda + \epsilon_\lambda^0 \right) \left( \frac{\partial}{\partial r} + v \frac{\partial}{\partial \tau} \right) - e^1_\lambda \frac{1}{\tau} \frac{\partial}{\partial \eta} - e^3_\lambda \frac{1}{r} \frac{\partial}{\partial \phi}. \]  

(11)

The relativistic equation of local energy and momentum conservation, \( \partial_\mu T^{\mu \nu} = 0 \), can then be projected to a component parallel to \( u_\mu \) and components orthogonal to that. Denoting the co-moving derivative (9) by \( \gamma \) times an overdot, (i.e. \( u_\mu \partial^\mu = \gamma \dot{f} \) for any \( f \)), we arrive at

\[ \gamma \dot{\epsilon} + w \partial_\mu u^\mu = 0, \]
\[ w \gamma \dot{u}_\lambda - \nabla_\lambda p = 0. \]  

(12)

Here we introduced the enthalpy density \( w = \epsilon + p \). The first equation of (12) is the local form of the \( dE + pdV = 0 \) adiabatic flow condition; in fact this is equivalent to the conservation of the entropy flow in a one-component matter. The second equation resembles the familiar form of the nonrelativistic Euler equation.

Summarizing we have altogether four independent equations,

\[ \left( \frac{\partial}{\partial \tau} + v \frac{\partial}{\partial r} \right) \epsilon + wD = 0, \]
\[ \gamma^2 w \left( \frac{\partial}{\partial r} + v \frac{\partial}{\partial \tau} \right) v + \left( \frac{\partial}{\partial r} + v \frac{\partial}{\partial \tau} \right) p = 0, \]
\[ \frac{1}{\tau} \frac{\partial}{\partial \eta} p = 0, \]
\[ \frac{1}{r} \frac{\partial}{\partial \phi} p = 0. \]  

(13)

Here the four divergence (8) without the Lorentz factor \( \gamma \) is denoted by \( D = \frac{1}{\gamma} \partial_\mu u^\mu \), which can be expanded to

\[ D = \gamma^2 \left( \frac{\partial}{\partial r} + v \frac{\partial}{\partial \tau} \right) v + \frac{1}{\tau} + \frac{v}{r}. \]  

(14)

Let us restrict ourselves now to \( p = p_0 = \) constant situations. In this case \( c^2_s = 0 \) and \( w = \epsilon + p_0 \). From (13) we arrive at two independent equations only, the first and the second
one. Since derivatives of the pressure vanish, we obtain an equation for the transverse flow component alone:

$$\left( \frac{\partial}{\partial \tau} + v \frac{\partial}{\partial r} \right) v = 0. \quad (15)$$

This equation is valid even for relativistic transverse flow, it is a quasilinear partial differential equation. A general solution can be obtained by the factorizing ansatz $v = a(\tau) \cdot b(r)$. The above eq.(15) leads to

$$\frac{a'(\tau)}{a^2(\tau)} + b'(r) = 0. \quad (16)$$

Here both terms are constant, balancing each other to zero. The solution, which is regular at the cylindrical axis $r = 0$ is given by

$$v = \frac{\alpha r}{1 + \alpha \tau}. \quad (17)$$

The first equation of the system (13) is the cooling equation. It reduces to

$$\left( \frac{\partial}{\partial \tau} + v \frac{\partial}{\partial r} \right) \epsilon + (\epsilon + p_0) D = 0, \quad (18)$$

with $D$ given by eq.(14). Utilizing the solution (17) for $v(\tau, r)$ we obtain

$$D = \frac{1}{\tau} + \frac{2\alpha}{1 + \alpha \tau}, \quad (19)$$

which can also be written as a comoving derivative,

$$D = \left( \frac{\partial}{\partial \tau} + v \frac{\partial}{\partial r} \right) \log (E(\tau)), \quad (20)$$

with

$$E(\tau) = \tau(1 + \alpha \tau)^2. \quad (21)$$

Using $D$ in this form the cooling equation (18) can be rewritten as

$$\left( \frac{\partial}{\partial \tau} + v \frac{\partial}{\partial r} \right) \log (E(\tau)(\epsilon + p_0)) = 0. \quad (22)$$

A particular solution of this equation is, when the quantity on which $\left( \frac{\partial}{\partial \tau} + v \frac{\partial}{\partial r} \right)$ operates is constant,

$$\epsilon + p_0 = \frac{\text{const}}{\tau(1 + \alpha \tau)^2} \quad (23)$$

This solution interpolates between the one-dimensional cooling law $\epsilon \propto 1/\tau$ and a three-dimensional one for large longitudinal proper time, $\epsilon \propto 1/\tau^3$.

There is, however, a more general solution to the “cooling law”. First inserting the analytic solution (17) for $v$ in the co-moving derivative operation we get,

$$\left( \frac{\partial}{\partial \tau} + \frac{\alpha}{1 + \alpha \tau} \frac{\partial}{\partial r} \right) (\log w + \log E(\tau)) = 0. \quad (24)$$
Multiplying this by \((1 + \alpha \tau) / \alpha\) we obtain

\[
\left( \frac{\partial}{\partial \log(1 + \alpha \tau)} + \frac{\partial}{\partial \log r} \right) (\log w + \log E(\tau)) = 0.
\]

This equation is of type

\[
\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) F(x, y) = 0
\]

whose solution is simply a function of \(x - y\), \(F(x, y)_{\text{solution}} = F(x - y)\). Henceforth we obtain

\[
\log w + \log E(\tau) = F(\log r - \log(1 + \alpha \tau)).
\]

In this result one easily recognizes the analytic form of the transverse flow velocity \(v\) \([17]\) after collecting the terms in the argument of the general function \(F\) inside one logarithm. Finally we arrive at

\[
\log w + \log E(\tau) = F(v/\alpha) = \log f(v)
\]

as an analytic solution of the cooling law with relativistic cylindrical transverse flow:

\[
\epsilon(r, \tau) = -p_0 + \frac{f(v)}{\tau(1 + \alpha \tau)^2},
\]

\[
v = \frac{\alpha}{1 + \alpha \tau} r.
\]

The unknown function of one variable \(f(v)\) can be used to match the initial profile of the radial distribution of the energy density at \(\tau = \tau_0\),

\[
\epsilon(r, \tau_0) = -p_0 + \frac{1}{\tau_0(1 + \alpha \tau_0)^2} f \left( \frac{\alpha r}{1 + \alpha \tau_0} \right).
\]

Let us finally discuss some properties of this analytic solution. First one would like to be convinced that the Bjorken scaling limit is contained in eq.(29). (The 1+1 dimensional solution was presented by Gyulassy and Matsui in 1984 \([10]\).) This is indeed the case, for \(v \to 0\) one obtains \(\alpha = 0\) and

\[
\epsilon(r, \tau) = -p_0 + \frac{f(0)}{\tau},
\]

leading to

\[
(\epsilon(\tau) + p_0) = (\epsilon_0 + p_0) \frac{\tau_0}{\tau}.
\]

As an example of the \(p = \text{const.}\) situation let us consider an oversimplified system: massless quark-gluon plasma described by the bag equation of state,

\[
\epsilon_Q = \sigma T^4 + B, \quad p_Q = \frac{1}{3} \sigma T^4 - B,
\]

keeps a Gibbs equilibrium with a light, relativistic pion gas, described by

\[
\epsilon_H = h T^4, \quad p_H = \frac{1}{3} h T^4.
\]
During a first order phase transition, for simplicity assumed to take place in the total volume, the pressure is constant and homogeneous (Gibbs criterion). Actually from this requirement \( p_Q = p_H \) one usually obtains the transition temperature due to

\[
\frac{1}{3}(\sigma - h)T_c^4 = B. \tag{35}
\]

The energy density is that of a mixture, containing \( x \) part quark matter and \((1 - x)\) part hadron matter,

\[
\epsilon = x\epsilon_Q(T_c) + (1 - x)\epsilon_H(T_c). \tag{36}
\]

This partition \( x \) is what changes according to the cooling law, respectively its solution (29). Utilizing eqs. (33, 34) and (35) we arrive at

\[
\epsilon = 4Bx(\tau) + \epsilon_H = -p_H + (\epsilon_Q + p_Q)\frac{T_0}{\tau}. \tag{37}
\]

Assuming at \( \tau = \tau_0 \) pure quark matter \( x(\tau_0) = 1 \), the time of the total conversion \( \tau_1 \) when \( x(\tau_1) = 0 \) is given by

\[
\tau_1 = \frac{\epsilon_Q + p_Q}{\epsilon_H + p_H} \tau_0 = \frac{\sigma}{h} \tau_0. \tag{38}
\]

It is determined by the relative number of degrees of freedom in the two phases in this simple scenario.

The same discussion is somewhat more complex using the analytic solution with a cylindrical flow. At the beginning of the phase transition \( x(\tau_0) = 1 \) let be a transverse velocity,

\[
v(\tau_0, r) = v_0 \frac{r}{R_0}. \tag{39}
\]

The analytic solution (17) leads to

\[
v(\tau, r) = \frac{v_0}{R_0 + v_0(\tau - \tau_0)}r \tag{40}
\]

featuring a slowing down of the transverse flow. From the initial condition for the transverse flow we also obtained the parameter \( \alpha \)

\[
\alpha = \frac{v_0}{R_0 - v_0\tau_0}. \tag{41}
\]

Since the four-flow is given by

\[
u_\mu = \gamma \left( e^0_\mu + v e^2_\mu \right)
\]

a slowing down of the transverse component implies a slowing down of the original Bjorken component as well (\( \gamma \) is decreasing to one, when \( v \) is decreasing to 0).

The cooling of the energy density can be obtained now by fitting the profile at \( \tau = \tau_0 \) to \( \epsilon(r, \tau_0) = \epsilon(r) \),

\[
\epsilon(r, \tau) + p_0 = \frac{\tau_0 R_0^2}{\tau R^2} \left( \epsilon \left( \frac{R_0}{R}r \right) + p_0 \right) \tag{42}
\]

with

\[
R(\tau) = R_0 + v_0(\tau - \tau_0). \tag{43}
\]
This solution follows the Bjorken cooling law at the beginning $\tau \approx \tau_0$,

$$\epsilon(r, \tau) + p_0 \approx \frac{\tau_0}{\tau} (\epsilon(r) + p_0), \quad (44)$$

but for long times $\tau \gg \tau_0$ turns over to a three-dimensional scaling

$$\epsilon(r, \tau) + p_0 \approx \frac{\tau_0 R_0^2}{v_0^3} \left( \frac{R_0}{v_0 \tau} r \right) + p_0. \quad (45)$$

Here for large times, the driving force besides the constant pressure will be $\epsilon(0)$, the original energy density at the axis.

Finally we repeat the calculation in the simplified scenario assuming a first order phase transition between ideal quark gluon plasma and ideal pion gas. The general solution for the quark matter part, $x$ is given by

$$x(\tau, r) = -\frac{h}{\sigma - h} + \frac{\tau_0 R_0^2}{\tau R^2} \left( \frac{h}{\sigma - h} + x(\tau_0, R_0 r) \right). \quad (46)$$

Assuming a linearly decreasing transverse profile of the QGP part initially (at the beginning of the phase transition),

$$x(\tau_0, r) = 1 - r/R_1, \quad (47)$$

we obtain the transverse radius of the mixed phase as

$$r_b = \frac{R_1}{R_0} \left( \frac{1 - \frac{h}{\sigma - h} \left( \frac{\tau R^2}{\tau_0 R_0^2} - 1 \right)}{\sigma - h} \right). \quad (48)$$

In the simple case, when $R_0 = v_0 \tau_0$, the scaling transverse radius $R$ is proportional to the longitudinal proper time $\tau$: $R = v_0 \tau$. The transverse velocity field is also particularly simple: $v = r/\tau$. The above result simplifies to

$$r_b = \frac{R_1}{R_0} \left( \frac{\sigma - h}{\sigma - h} \right)^{1/3} \left( \frac{\sigma}{4h} \right)^{1/3} \left( \frac{\tau_0 R_0^2}{\tau R^2} - 1 \right). \quad (49)$$

This expression initially grows, achieves a maximum and then decreases towards zero. The maximum,

$$r_b^{\text{max}} = \frac{3}{4} \frac{R_1}{R_0} \left( \frac{\sigma}{4h} \right)^{1/3} \left( \frac{\sigma}{\sigma - h} \right)^{1/3}, \quad (50)$$

is achieved at time

$$\tau^{\text{max}} = \tau_0 \left( \frac{\sigma}{4h} \right)^{1/3}, \quad (51)$$

the $x = 0$ for the whole space (equivalent to $r_b = 0$) at

$$\tau_1 = \tau_0 \left( \frac{\sigma}{4h} \right)^{1/3}. \quad (52)$$

Realistic estimates use $\sigma = 37$ for the quark gluon plasma and $h = 3$ for the pion gas. This leads to $r_b^{\text{max}} \approx 1.14 R_1$, $\tau^{\text{max}} \approx 1.45 \tau_0$, and $\tau_1 \approx 2.31 \tau_0$. It is realistic to assume $\tau_0 \approx 5 \text{ fm/c}$ for a CERN SPS Pb+Pb experiment. The radial extension of the quark
matter can grow with about 14 per cent and then the conversion into hadrons eats it up, reaching the zero radius in about 11.5 fm/c. This is a fast hadronization even in this simple scenario.

For a smaller initial radial flow, realistically \( v_0 = 0.6 \) at the radial edge of the cylinder \( r = R_0 \), one obtains \( v_0 \tau_0 / R_0 = 0.428 \). Fig.1 shows the radial space-time evolution profiles of the mixture: the outmost curve corresponds to \( x = 0 \) (pure hadron matter) and the steps are 0.1. The innermost curve belonging to \( x = 1.0 \) shrinks to a point, since the phase conversion starts immediately at \( \tau = \tau_0 \). Here we assumed a linear initial profile \( x(r, \tau_0) = 1 - r / R_0 \).

With these parameters the mixed phase of \( R_0 = 7 \) fm longs for about \( \tau_0 \approx 17.5 \) fm/c.

In conclusion we have presented an analytical solution for relativistic transverse flow of a perfect fluid at the softest point. This can be realized during a first order phase transition, when the pressure is homogeneous and constant in a large volume. The phase conversion during this stage of expansion is given by this analytical solution which scales with the longitudinal proper time initially like the Bjorken flow \( 1 / \tau \), but eventually like a spherical flow \( 1 / \tau^3 \). For simple equations of state for the quark and hadronic matter side the estimated longitudinal proper time spent in the mixed phase turns out to be more than twice the time when the phase transition began.

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Figure 1: Contour lines of several QGP - hadronic mixtures (from 0 to 1 by 0.1 steps) in the $\tau/\tau_0 - r/R_0$ plane. The initial profile at $\tau = \tau_0$ was assumed to be linear giving zero percent quark matter exactly at $R_1 = R_0$. The initial transverse flow is $v_0 = 0.6$ at the edge $R = R_0$.

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