ENTROPY-BOUNDED SOLUTIONS TO THE COMPRESSIBLE NAVIER-STOKES EQUATIONS: WITH FAR FIELD VACUUM

JINKAI LI AND ZHOUPING XIN

Abstract. The entropy is one of the fundamental states of a fluid and, in the viscous case, the equation that it satisfies is highly singular in the region close to the vacuum. In spite of its importance in the gas dynamics, the mathematical analyses on the behavior of the entropy near the vacuum region, were rarely carried out; in particular, in the presence of vacuum, either at the far field or at some isolated interior points, it was unknown if the entropy remains its boundedness. The results obtained in this paper indicate that the ideal gases retain their uniform boundedness of the entropy, locally or globally in time, if the vacuum occurs at the far field only and the density decays slowly enough at the far field. Precisely, we consider the Cauchy problem to the one-dimensional full compressible Navier-Stokes equations without heat conduction, and establish the local and global existence and uniqueness of entropy-bounded solutions, in the presence of vacuum at the far field only. It is also shown that, different from the case that with compactly supported initial density, the compressible Navier-Stokes equations, with slowly decaying initial density, can propagate the regularities in the inhomogeneous Sobolev spaces.

1. Introduction

Let $\rho$, $u$, and $\theta$ be the density, velocity, and temperature of a fluid, and denote $t$ and $x$ as the time and spatial variables. Then, the full compressible Navier-Stokes equations read as

$$\partial_t \rho + \text{div} (\rho u) = 0,$$
$$\partial_t (\rho u) + \text{div} (\rho u \otimes u) = \text{div} T + \rho f,$$
$$\partial_t (\rho E) + \text{div} (\rho u E) + \text{div} q = \text{div} (T u) + \rho Q,$$

where $E = \frac{|u|^2}{2} + e$ is the specific total energy, $e = e(\rho, \theta)$ is the specific internal energy, $T$ is the stress tensor, $q$ is the internal energy flux directly related to the transfer of heat, $f$ is the external force, and $Q$ is the external heat source.

By $(1.1)-(1.3)$, one can obtain the following equation for $e$:

$$\partial_t (\rho e) + \text{div} (\rho u e) + p \text{div} u + \text{div} q = \mathbb{S} : \nabla u + \rho Q.$$
The stress tensor $\mathbf{T}$ is given by

$$
\mathbf{T} = \mathbf{S} - p \mathbf{I}, \quad \mathbf{S} = 2\mu \nabla \mathbf{u} + \lambda \text{div} \mathbf{u} \mathbf{I}, \quad \nabla \mathbf{u} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T),
$$

where $\mathbf{I}$ is the $3 \times 3$ identity matrix, $p$ is the pressure, and $\mu$ and $\lambda$ are viscosity coefficients, satisfying $\mu > 0$ and $2\mu + 3\lambda \geq 0$. In this paper, we consider the ideal gases, and state equations are

$$
p = R\rho \theta, \quad e = c_v \theta,
$$

for two positive constants $R$ and $c_v$. Then, it follows from (1.4) that

$$
c_v [\partial_t (\rho \theta) + \text{div}(\rho u \theta)] + p \text{div} u + \text{div} q = \mathbf{S} : \nabla \mathbf{u} + \rho Q. \quad (1.5)
$$

Recalling the state equations for $p$ and $e$, by the Gibb’s equation $\theta D s = D e + p D (\frac{1}{\rho})$, where $s$ is the state of the entropy, one has the following relationship between $p$ and $s$:

$$
p = Ae^{\frac{\gamma}{\gamma - 1}} \rho^\gamma,
$$

for some positive constant $A$, where $\gamma - 1 = \frac{R}{c_v}$. Thanks to this, and using the state equations for $p$ and $e$ again, one can derive from (1.1), (1.2), and (1.5) the following equation for the entropy $s$:

$$
\partial_t (\rho s) + \text{div}(\rho su) + \text{div} \left(\frac{q}{\theta}\right) = \frac{1}{\theta} \left(\mathbf{S} : \nabla \mathbf{u} - \frac{q \cdot \nabla \theta}{\theta}\right) + \rho \frac{Q}{\theta}.
$$

For the internal energy flux $q$, by the Fourier’s law of heat conduction, we assume that $q = -\kappa \nabla \theta$, where $\kappa \geq 0$ is the heat conduction coefficient.

There are extensive literatures on the mathematical analyses of the compressible Navier-Stokes equations. In the absence of vacuum, that is the density is bounded from below by some positive constant, the local well-posedness results were proved by Nash [32], Itaya [15], Vol’pert–Hudjaev [38], Tani [35], Valli [36], and Lukaszewicz [27]. The first global well-posedness result was established by Kazhikhov–Shelukhin [20], where they proved the global well-posedness of strong solutions of the initial boundary value problem to the one-dimensional compressible Navier-Stokes equations, for arbitrary $H^1$ initial data, and the corresponding result for the Cauchy problem was later proved by Kazhikhov [19]; global well-posedness of weak solutions to the one-dimensional compressible Navier-Stokes equations was proved by Zlotnik–Amosov [42, 43] and by Chen–Hoff–Trivisa [1] for the initial boundary value problems, and by Jiang–Zlotnik [18] for the Cauchy problem. Large time behavior of solutions to the one dimensional compressible Navier-Stokes equations with large initial data was recently proved by Li–Liang [23]. For the multi-dimensional case, the global wellposedness of strong solutions were established only for small perturbed initial data around some non-vacuum equilibrium or for spherically symmetric large initial data, see Matsumura–Nishida [28–31], Ponce [33], Valli–Zajaczkowski [37], Deckelnick [7], Jiang [16], Hoff [11], Kobayashi–Shibata [21], Danchin [6], and Chikami–Danchin [2].

One of the major differences between one dimensional case from the multi-dimensional
one is that if no vacuum is contained initially, then no vacuum will form later on in finite time, for the one dimensional compressible Navier-Stokes equations, as shown by Hoff-Smoller \cite{12}, while the similar result remains unknown for the multi-dimensional case.

In the presence of vacuum, that is the density may vanish on some set, or tends to zero at the far field, the breakthrough was made by Lions \cite{25, 26}, where he proved the global existence of weak solutions to the isentropic compressible Navier-Stokes equations, with adiabatic constant $\gamma \geq \frac{9}{5}$; the requirement on $\gamma$ was later relaxed by Feireisl–Novotný–Petzeltová \cite{3} to $\gamma > \frac{3}{2}$, and further by Jiang–Zhang \cite{17} to $\gamma > 1$ but only for the axisymmetric solutions. For the full compressible Navier-Stokes equations, global existence of the variational weak solutions was proved by Feireisl \cite{9, 10}; however, due to the assumptions on the constitutive equations made in \cite{9, 10}, the ideal gases were not included there. Local well-posedness of strong solutions, in the presence of vacuum, was proved first for the isentropic case by Salvi–Stráskraba \cite{34}, Cho–Choe–Kim \cite{3}, and Cho–Kim \cite{4}, and later for the polytropic case by Cho–Kim \cite{5}. It should be noticed that, in \cite{3, 3, 34}, the solutions were established in the homogeneous Sobolev spaces, that is, it is $\sqrt{\rho u}$ rather than $u$ itself that has the $L^\infty(0, T; L^2)$ regularity. Generally, one can not expect that the strong solutions to the compressible Navier-Stokes equations lie in the inhomogeneous Sobolev spaces, if the initial density has compact support. Actually, it was proved recently by Li–Wang–Xin \cite{22} that: neither isentropic nor the full compressible Navier-Stokes equations on $\mathbb{R}$, with $\kappa = 0$ for the full case, has any solution $(\rho, u, \theta)$ in the inhomogeneous Sobolev spaces $C^1([0, T]; H^m(\mathbb{R}))$, with $m > 2$, if $\rho_0$ is compactly supported and some appropriate conditions on the initial data are satisfied; the $N$-dimensional full compressible Navier-Stokes equations, with positive heat conduction, have no solution $(\rho, u, \theta)$, with finite entropy, in the inhomogeneous Sobolev spaces $C^1([0, T]; H^m(\mathbb{R}^N))$, with $m > \left[\frac{N}{2}\right] + 2$, if $\rho_0$ is compactly supported. Global existence of strong and classical solutions to the compressible Navier-Stokes equations, in the presence of initial vacuum, was first proved by Huang–Li–Xin \cite{14}, where they established the global well-posedness of strong and classical solutions, with small initial basic energy, to the three-dimensional isentropic compressible Navier-Stokes equations, see Li–Xin \cite{24} for further developments. However, due to the finite in time blow-up results by Xin \cite{40} and Xin–Yan \cite{41}, one can not expect the global well-posedness of classical solutions, in either inhomogeneous or homogeneous Sobolev spaces, to the full compressible Navier-Stokes equations in the presence of vacuum. In particular, it was proved in \cite{41} that, for the full compressible Navier-Stokes equations, if initially there is an isolated mass group surrounded by the vacuum region, then for the case $\kappa = 0$, any classical solution must blow-up in finite time, and for the case $\kappa > 0$, any classical solutions, with finite entropy in the vacuum region, must blow-up in finite time. Global existence of strong solutions to the heat conducting full compressible Navier-Stokes equations were obtained by Huang–Li \cite{13} for the case that with non-vacuum
far field, and by Wen–Zhu [39] for the case that with vacuum far field. The spaces of
the solutions obtained in [13, 39] can not exclude the possibility that the entropy is
infinite somewhere in the vacuum region, even if it is initially finite; in fact, due to
the results in [41], the corresponding entropy in [13, 39] must be infinite somewhere
in the vacuum region, if initially there is an isolated mass group surrounded by the
vacuum region.

Recalling the state equations for the ideal gases, the entropy can be expressed in
terms of the density and temperature as

\[ s = c_v \left( \log \frac{R}{A} + \log \theta - (\gamma - 1) \log \rho \right), \]

from which one can see that the entropy may develop singularities or even is not well
defined in the vacuum region and, consequently, it is impossible to obtain the desired
regularities of \( s \) merely from those of \( \theta \) and \( \rho \), in the presence of vacuum. Therefore,
though the vacuums are allowed for the solutions established in [5, 13, 39] by choosing
\((\rho, u, \theta)\) as the unknowns, no regularities of the entropy \( s \) can be implied in the vacuum
region there and, due to the result in [41], the entropy of the solutions obtained in
[13, 39] must be infinite in the vacuum region. To the best of our knowledge, in the
existing literatures, there were no such results that provided the uniform lower or
upper bounds of the entropy near the vacuum.

As stated in the previous paragraph, since the entropy can not be even defined at
the places where the density vanishes, it may be unreasonable to study the entropy for
the full compressible Navier-Stokes equations of the ideal gases, if the vacuum region
is an open set; however, when the vacuum occurs only at some isolated interior points
or at the far field and if, moreover, the entropy behaves well when the fluid tends
to these vacuum points or to the far field, it is still possible to define the entropy
there. Therefore, a natural question is what kind of behavior of the entropy, at the
vacuum far field or near the isolated interior vacuum points, can be preserved by the
ideal gases, when the flow evolves. The aim of this paper is to give some answers to
this question and, in particular, as indicated in our main results, the ideal gases can
preserve their boundedness of the entropy, locally or globally in time, if the vacuum
happens at the far field only.

Another question that we want to address in this paper is: under what kind of
assumptions on the initial density, beyond the the case that the initial density is
uniformly away from the vacuum, the compressible Navier-Stokes equations admit
solutions in the inhomogeneous Sobolev spaces. On one hand, recalling the result
in [22], for the case that the initial density has a compact support, the compressible
Navier-Stokes equations are ill-posed in the inhomogeneous Sobolev spaces; on
the other hand, for the case that the initial density is uniformly away from the vac-
uum, the compressible Navier-Stokes equations are well-posed in the inhomogeneous
Sobolev spaces. Comparing these two cases, by understanding the case that with
compact support as having supper fast decay at the far field, it is natural to ask if
the fast decay of the density can cause the ill-posedness of the compressible Navier-Stokes equations in the inhomogeneous spaces, or if the compressible Navier-Stokes equations will be well-posed in the inhomogeneous spaces when the initial density decays slowly at the far field. We will show in this paper that if the initial density decays slower than $\frac{K_0}{|x|^2}$, for some positive constant $K_0$, at the far field, then the compressible Navier-Stokes equations are indeed well-posed in the inhomogeneous Sobolev spaces, where $K_0$ is an arbitrary positive constant. Note that this is consistent with the well-posedness result for the compressible Navier-Stokes equations in the inhomogeneous Sobolev spaces in the absence of vacuum.

In this paper, we consider the one dimensional case, and assume that there are no external forces and heating source, i.e. $f \equiv Q \equiv 0$, and that there is no heat conduction in the fluids, that is $\kappa = 0$, while the muti-dimensional case and the cases that with heat conduction will be studied in the further works. Under these assumptions, the system considered in this paper is the following one-dimensional compressible Navier-Stokes equations:

\begin{align}
\rho_t + (\rho u)_x &= 0, \\
\rho (u_t + uu_x) - \mu u_{xx} + p_x &= 0, \\
c_v [(\rho \theta)_t + (\rho u \theta)_x] + p u_x &= \mu (u_x)^2.
\end{align}

Recalling the state equation $p = R \rho \theta$ and $c_v = \frac{R}{\gamma - 1}$, equation (1.8) can be rewritten equivalently as an equation for the pressure $p$, that is

\begin{equation}
 p_t + u p_x + \gamma u_x p = \mu (\gamma - 1) (u_x)^2.
\end{equation}

As will be seen later, it is more convenient to use (1.9), instead of (1.8), to state and prove the results, in other words, we will use the pressure, instead of the temperature, as one of the unknowns, throughout this paper; however, it should be mentioned that, as we consider the case that the vacuum happens only at the far field, (1.9) is equivalent to (1.8).

We will consider the Cauchy problem and, therefore, complement system system (1.6), (1.7), and (1.9), with the following initial condition

\begin{equation}
(\rho, u, p)|_{t=0} = (\rho_0, u_0, p_0).
\end{equation}

Before stating the main results, we first clarify some necessary notations being used throughout this paper. For $1 \leq q \leq \infty$ and positive integer $m$, we use $L^q = L^q(\mathbb{R})$ and $W^{1,q} = W^{m,q}(\mathbb{R})$ to denote the standard Lebesgue and Sobolev spaces, respectively, and in the case that $q = 2$, we use $H^m$ instead of $W^{m,2}$. For simplicity, we also use the notations $L^q$ and $H^m$ to denote the $N$ product spaces $(L^q)^N$ and $(H^m)^N$, respectively. We always use $\|u\|_q$ to denote the $L^q$ norm of $u$. For shortening the expressions, we sometimes use $\|(f_1, f_2, \cdots, f_n)\|_X$ to denote the sum $\sum_{i=1}^N \|f_i\|_X$ or its equivalent norm $\left(\sum_{i=1}^N \|f_i\|_X^2\right)^{\frac{1}{2}}$. 


We have the following two theorems on the local and global well-posedness of solutions to system (1.6), (1.7), and (1.9), subject to (1.10):

**Theorem 1.1 (Local well-posedness).** Assume that

\[
\inf_{y \in (-R, R)} \rho_0(y) > 0, \quad \forall R \in (0, \infty), \quad \left(\rho_0, \sqrt{\rho_0 u_0}, u_0, p_0, \frac{p'_0}{\sqrt{\rho_0}}\right) \in L^2, \tag{1.11}
\]

and denote \( F_0 := \mu u_0' - p_0 \).

Then, the following two hold:

(i) There is a unique local solution \((\rho, u, p)\) to system (1.6)–(1.7) and (1.9), subject to (1.10), satisfying

\[
\rho - \rho_0 \in C([0, T]; L^2), \quad \rho_x \in L^\infty(0, T; L^2), \quad \rho_t \in L^\infty(0, T; L^2((-r, r))),
\]

\[
\sqrt{\rho} u \in C([0, T]; L^2), \quad u_x \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1), \quad \sqrt{\rho} u_t \in L^2(0, T; L^2),
\]

\[
p - p_0 \in C([0, T]; L^2), \quad p_x \in L^\infty(0, T; L^2), \quad p_t \in L^4(0, T; L^2((-r, r))),
\]

for all \( r \in (0, \infty) \), where \( T \) is a positive constant depending only on \( \gamma, \mu, \|\rho_0\|_{\infty}, \|u_0\|_2, \|p_0\|_2 \), and \( \|p_0\|_\infty \).

(ii) Assume in addition that

\[
\left|\left(\frac{1}{\sqrt{\rho_0}}\right)'(y)\right| \leq \frac{K_0}{2}, \quad \forall y \in \mathbb{R}, \quad \rho_0^{\frac{d}{2}} F_0 \in L^2, \tag{1.12}
\]

for two positive constants \( \delta \) and \( K_0 \). Then, \((\rho, u, p)\) has the additional regularities

\[
\begin{cases}
  u \in L^\infty(0, T; H^1), & \text{if } u_0 \in H^1 \text{ and } \delta \geq 1, \\
  \theta \in L^\infty(0, T; H^1), & \text{if } \theta_0 \in H^1 \text{ and } \delta \geq 1, \\
  s \in L^\infty(0, T; L^\infty), & \text{if } s_0 \in L^\infty \text{ and } \delta \geq \gamma,
\end{cases} \tag{1.13}
\]

where \( \theta := \frac{p}{\rho} \) and \( s := c_v \log \left( \frac{\rho}{\rho_0} \right) \), respectively, are the temperature and entropy.

**Theorem 1.2 (Global well-posedness).** Assume that (1.11) holds, and that both \( \rho_0 \) and \( p_0 \) are in \( L^1 \). Then, there is a unique global solution \((\rho, u, p)\) to system (1.6)–(1.7) and (1.9), subject to (1.10), satisfying the regularities stated in (i) of Theorem 1.1, for any \( T \in (0, \infty) \). Moreover, if assume in addition that (1.12) holds, then (1.13) holds for any \( T \in (0, \infty) \).

Theorem 1.1 and Theorem 1.2 respectively, are the corollaries of the more accurate results Theorem 2.1 and Theorem 2.2 being stated in the next section in the Lagrangian coordinates. Therefore, in the rest of this paper, we focus on studying the compressible Navier-Stokes equations in the Lagrangian coordinates.

The rest of this paper is arranged as follows: in Section 2 we reformulate the system in the Lagrangian coordinates, and state our main results; in Section 3, we consider the system in the absence of vacuum, and carry out some a priori estimates, which are independent of the positive lower bound of the density; the proof of Theorem 2.1 is given in Section 4 while that of Theorems 2.2 is given in the last section.
2. Reformulation in Lagrangian coordinates and main results

Let \( y \) be the Lagrangian coordinate, and define the coordinate transform between the Lagrangian coordinate \( y \) and the Euler coordinate \( x \) as

\[
x = \eta(y, t),
\]

where \( \eta(y, t) \) is the flow map determined by \( u \), that is,

\[
\begin{cases}
\partial_t \eta(y, t) = u(\eta(y, t), t), \\
\eta(y, 0) = y.
\end{cases}
\]

Denote by \( \varrho, v, \) and \( \pi \) the density, velocity, and pressure, respectively, in the Lagrangian coordinate, that is we define

\[
\begin{align*}
\varrho(y, t) &= \rho(\eta(y, t), t), \\
v(y, t) &= u(\eta(y, t), t), \\
\pi(y, t) &= p(\eta(y, t), t).
\end{align*}
\]

Recalling the definition of \( \eta(y, t) \), by straightforward calculations, one can check that

\[
(u_x, p_x) = \left( \frac{v_y}{\eta_y}, \frac{\pi_y}{\eta_y} \right), \quad u_{xx} = \frac{1}{\eta_y} \left( \frac{v_y}{\eta_y} \right)_y,
\]

\[
\begin{align*}
\varrho_t + u \varrho_x &= \varrho_t, \\
u_t + uu_x &= v_t, \\
p_t + up_x &= \pi_t.
\end{align*}
\]

Define the function \( J = J(y, t) \) as

\[
J(y, t) = \eta_y(y, t),
\]

then it follows that

\[
J_t = v_y.
\]

Thanks to the above, system (1.6), (1.7), and (1.9) can be rewritten in the Lagrangian coordinate as

\[
\begin{align*}
J_t &= v_y, \\
\varrho_t + \frac{v_y}{J} \varrho &= 0, \\
\varrho v_t - \frac{\mu}{J} \left( \frac{v_y}{J} \right)_y + \frac{\pi_y}{J} &= 0, \\
\pi_t + \gamma \frac{v_y}{J} \pi &= (\gamma - 1) \mu \left( \frac{v_y}{J} \right)^2.
\end{align*}
\]

One can further reduce the above system to a simpler version. In fact, due to (2.1) and (2.2), it holds that

\[
(J \varrho)_t = J_t \varrho + J \varrho_t = v_y \varrho - J \frac{v_y^2}{J} \varrho = 0,
\]

from which, by setting \( \varrho|_{t=0} = \varrho_0 \) and noticing that \( J|_{t=0} = 1 \), we have

\[
J \varrho = \varrho_0.
\]
Therefore, one can drop (2.2) from system (2.1)–(2.4), and rewrite (2.3) as
\[
\rho_0 \frac{\partial v}{\partial t} - \mu \frac{\partial}{\partial y} \left( \frac{\nu y}{J} \right)_y + \pi_y = 0.
\]

In summary, we only need to consider the following system
\[
J_t = v_y, \quad (2.5)
\]
\[
\rho_0 \frac{\partial v}{\partial t} - \mu \frac{\partial}{\partial y} \left( \frac{\nu y}{J} \right)_y = 0, \quad (2.6)
\]
\[
\pi_t + \gamma \frac{\nu y}{J} \pi = (\gamma - 1) \mu \left( \frac{\nu y}{J} \right)^2. \quad (2.7)
\]

As will be shown later, the effective viscous flux \( G \) defined as
\[
G := \mu \frac{\nu y}{J} - \pi.
\]
plays a crucial role in proving the global existence of solutions to system (2.5)–(2.7). By straightforward calculations, it follows from (2.6) and (2.7) that
\[
G_t - \mu \frac{\partial}{\partial y} \left( \frac{G_y}{\rho_0} \right)_y = -\gamma \frac{\nu y}{J} G. \quad (2.8)
\]

We will consider the Cauchy problem and, thus, complement system (2.5)–(2.7) with the initial condition
\[
(J, v, \pi)|_{t=0} = (J_0, v_0, \pi_0), \quad (2.9)
\]
where \( J_0 \) has uniform positive lower and upper bounds.

It should be pointed out that, by the definition of \( J \), the initial \( J_0 \) should be identically one; however, for the aim of extending a local solution \( (J, v, \pi) \) to be a global one, we need the local existence of solutions to system (2.5)–(2.7), with initial \( J_0 \) not being identically one. Therefore, in this paper, when studying the local solutions, the initial \( J_0 \) is allowed to be not identically one, but when studying the global solutions, we always assume that \( J_0 \) is identically one.

Local and global strong solution to system (2.5)–(2.7), subject to (2.9), are defined in the following two definitions.

**Definition 2.1.** Given a positive time \( T \in (0, \infty) \). A triple \( (J, v, \pi) \) is called a strong solution to system (2.5)–(2.7), subject to (2.9), on \( \mathbb{R} \times (0, T) \), if it has the properties

\[
\inf_{y \in \mathbb{R}, t \in (0, T)} J(y,t) > 0, \quad \pi \geq 0 \ \text{on} \ \mathbb{R} \times (0, T),
\]
\[
J - J_0 \in C([0, T]; L^2), \quad \frac{J}{\sqrt{\rho_0}} \in L^\infty(0, T; L^2), \quad J_t \in L^\infty(0, T; L^2),
\]
\[
\sqrt{\rho_0} v \in C([0, T]; L^2), \quad v_y \in L^\infty(0, T; L^2), \quad \left( \sqrt{\rho_0} v_t, \frac{v_y}{\sqrt{\rho_0}} \right) \in L^2(0, T; L^2),
\]
\[
\pi \in C([0, T]; L^2), \quad \frac{\pi}{\sqrt{\rho_0}} \in L^\infty(0, T; L^2), \quad \pi_t \in L^4(0, T; L^2),
\]
satisfies equations (2.5)–(2.7), a.e. in \( \mathbb{R} \times (0, T) \), and fulfills the initial condition (2.9).
Definition 2.2. A triple \((J, v, π)\) is called a global strong solution to system (2.5)–(2.7), subject to (2.9), if it is a strong solution to the same system on \(\mathbb{R} \times (0, T)\), for any positive time \(T \in (0, \infty)\).

The main results of this paper are the following two theorems concerning the local and global existence of strong solutions to system (2.5)–(2.7), subject to (2.9).

Theorem 2.1 (Local well-posedness). Assume that

\[
\inf_{y \in (-R, R)} \varrho_0(y) > 0, \quad \forall R \in (0, \infty), \quad \varrho_0 \leq \bar{\varrho} \text{ on } \mathbb{R},
\]

\[
\left( \sqrt{\varrho_0 v_0}, v'_0, \pi_0, \frac{\pi'_0}{\sqrt{\varrho_0}} \right) \in L^2, \quad \pi_0 \geq 0 \text{ on } \mathbb{R},
\]

\[
0 < J_0 \leq J \leq \bar{J} < \infty \text{ on } \mathbb{R}, \quad J'_0 \sqrt{\varrho_0} \in L^2,
\]

for positive constants \(\bar{\varrho}, J_0\), and \(\bar{J}\), and denote \(G_0 := \mu v'_0 - \pi_0\).

The following two hold:

(i) There is a positive time \(T\) depending only on \(\gamma, \mu, \bar{\varrho}, J, J'_0, \|v'_0\|_2, \|\pi_0\|_2\), and \(\|\pi_0\|_\infty\), such that system (2.5)–(2.7), subject to the initial condition (2.9), has a unique strong solution \((J, v, \pi)\), on \(\mathbb{R} \times (0, T)\).

(ii) Assume in addition that

\[
\left| \left( \frac{1}{\sqrt{\varrho_0}} \right)'(y) \right| \leq \frac{K_0}{2}, \quad \forall y \in \mathbb{R}, \quad \varrho_0^{-\frac{\delta}{2}} G_0 \in L^2,
\]

for two positive constants \(\delta\) and \(K_0\).

Then, \((J, v, \pi)\) has the additional regularities

\[
\varrho_0^{-\frac{\delta}{2}} G \in L^\infty(0, T; L^2) \cap L^4(0, T; L^\infty), \quad \varrho_0^{-\frac{\delta+1}{2}} G_y \in L^2(0, T; L^2),
\]

where \(G := \mu v'_0 - \pi\) is the effective viscous flux, and

\[
\begin{cases}
  v \in L^\infty(0, T; H^1), & \text{if } v_0 \in H^1 \text{ and } \delta \geq 1,
  \\
  \vartheta \in L^\infty(0, T; H^1), & \text{if } \vartheta_0 \in H^1 \text{ and } \frac{J'_0}{\varrho_0} \in L^2, \text{ and } \delta \geq 1,
  \\
  s \in L^\infty(0, T; L^\infty), & \text{if } s_0 \in L^\infty \text{ and } \delta \geq \gamma,
\end{cases}
\]

where \(\vartheta := \frac{\pi}{R_0}\) and \(s := c_v \log \left( \frac{\pi}{A_0 \vartheta^2} \right)\), respectively, are the corresponding temperature and entropy, with \(\varrho := \frac{\pi}{R_0}\) being the density, which satisfies equation (2.2), and \(\vartheta_0 := \frac{\pi_0}{R_0 \varrho_0}\) and \(s_0 := c_v \log \left( \frac{\pi_0}{A_0 \varrho_0^2} \right)\), respectively are the initial temperature and entropy.
Remark 2.1. Basically, the condition $\left|\frac{(\frac{1}{\sqrt{\rho_0}})'}{\sqrt{\rho_0}}\right| \leq \frac{K_0}{2}$ or equivalently $|\mathcal{G}_0| \leq K_0^{3/2}$ on $\mathbb{R}$ means that $\rho_0$ decays no faster than $\frac{K_0}{y^2}$ at the far field: if choosing $\rho_0(y) = \frac{K_0}{\langle y \rangle \ell_\rho}$, $0 < K_0 < \infty$, $0 \leq \ell_\rho < \infty$, where $\langle y \rangle = (1 + y^2)^{1/2}$, then $\left|\frac{(\frac{1}{\sqrt{\rho_0}})'}{\sqrt{\rho_0}}\right| \leq \frac{K_0}{2}$ on $\mathbb{R} \iff \ell_\rho \leq 2$.

Remark 2.2. Choose

$$\rho_0(y) = \frac{K_0}{\langle y \rangle \ell_\rho}, \quad J_0 \equiv 1, \quad v_0 \in C^\infty_c, \quad \pi_0 = Ae^{cv_0} \tilde{\gamma},$$

where $K_0$ and $\ell_\rho$ are positive numbers.

(i) If $\frac{1}{2(\gamma-1)} < \ell_\rho \leq 2$, then $(\rho_0, v_0, \pi_0)$ satisfies conditions (H1), (H2), and (H3), with $\delta = 1$. Therefore, by Theorem 2.1, there is a unique local strong solution $(J, v, \pi)$, with $v$ being in the inhomogeneous Sobolev space $L^\infty(0, T; H^1)$; if moreover that $\gamma > \frac{5}{4}$ and $\frac{1}{2(\gamma-1)} < \ell_\rho \leq 2$, then $\vartheta_0 \in H^1$, and, consequently, the temperature $\vartheta$ also lies in the inhomogeneous Sobolev space $L^\infty(0, T; H^1)$. Note that this does not contradict to the ill-posedness results for the compressible Navier-Stokes equations in [22], as the initial density there was assumed to be compactly supported.

(ii) If $\frac{1}{\gamma} < \ell_\rho \leq 2$, then $(\rho_0, v_0, \pi_0)$ satisfies conditions (H1), (H2), (H3), with $\delta = \gamma$, and $s_0 \equiv 1$. Therefore, by Theorem 2.1, there is a unique local strong solution $(J, v, \pi)$, and the corresponding entropy $s$ is uniformly bounded on $\mathbb{R} \times (0, T)$. To the best of our knowledge, this is the first time that the boundedness of the entropy is achieved, in the presence of vacuum at the far field, for the compressible Navier-Stokes equations.

(iii) Combining (i) with (ii), if $\gamma > \frac{5}{4}$ and $\max\left\{\frac{1}{\gamma}, \frac{1}{2(\gamma-1)}\right\} < \ell_\rho \leq 2$, there is a unique local strong solution $(J, v, \pi)$, with the properties that the corresponding entropy is uniformly bounded, and the velocity and the corresponding temperature lie in the inhomogeneous space $L^\infty(0, T; H^1)$.

Remark 2.3. (i) The assumption $\inf_{y \in (-R, R)} \rho_0(y) > 0$, for all $R \in (0, \infty)$, is used for the boundedness of the entropy and the regularities of the velocity and temperature in the inhomogeneous Sobolev spaces, but it is not needed for the local well-posedness (in the homogeneous Sobolev spaces).

(ii) The compressible Navier-Stokes equations propagate the regularities in the homogeneous Sobolev spaces, see [3, 5], but not in the inhomogeneous Sobolev spaces (in particular, the $L^2$ regularity of $v$ can not be propagated), see [22], if the initial density has a compact support. While (ii) of Theorem 2.1 shows that, if the initial density decays slowly to the vacuum at the far field, then the regularities in the homogeneous
Sobolev spaces, in particular, the $L^2$ regularity of $v$, can be also propagated by the compressible Navier-Stokes equations.

(iii) The result in (ii) of Theorem 2.1 also indicates that the uniform boundedness of the entropy can be propagated by the compressible Navier-Stokes equations, if the initial density decays slowly to the vacuum at the far field.

Remark 2.4. By the definition of strong solutions, we have the regularities

$$\sqrt{\varrho_0} v \in L^\infty(0,T;L^2), \quad v_y \in L^2(0,T;H^1),$$

which implies $v \in L^2(0,T;\text{Lip})$. Define the Euler coordinate as

$$x = \eta(y,t), \quad \partial_t \eta(y,t) = v(y,t), \quad \eta(y,0) = y.$$

Noticing that

$$\partial_{ty} \eta y = v_y = \partial_t J, \quad \eta y (y,0) = 1 = J(y,0),$$

we have

$$\eta y \equiv J \text{ on } \mathbb{R} \times (0,T).$$

Recalling that $J$ has uniform positive lower and upper bounds on $\mathbb{R} \times (0,T)$, for any fixed $t \in (0,T)$, $\eta$ is reversible in $y$. Therefore, one can define the density $\rho$, velocity $u$, and pressure $p$, in the Euler coordinate as

$$\rho(x,t) = \varrho(y,t), \quad u(x,t) = v(y,t), \quad p(x,t) = \pi(y,t),$$

where

$$\varrho(y,t) := \frac{\varrho_0(y)}{\sqrt{\varrho_0} \eta(y,t)}.$$ We can check that $(\rho, u, p)$ has appropriate regularities, in particular $u \in L^1(0,T;\text{Lip})$, and it is a solution to system (1.6), (1.7), and (1.9), subject to the initial data $$(\rho_0, v_0, \pi_0);$$ while the uniqueness in the Euler coordinate can be proven by transforming it to the Lagrangian coordinate, as $u \in L^1(0,T;\text{Lip})$, and apply the uniqueness result stated in Proposition 4.1.

Theorem 2.2 (Global well-posedness). Assume that (H1)–(H2) hold, and that

$$\varrho_0 \in L^1, \quad \pi_0 \in L^1, \quad \varrho_0(y) \geq \frac{A_0}{(1+|y|)^2}, \quad \forall y \in \mathbb{R},$$

for some positive constant $A_0$.

The following two hold:

(i) There is a unique global strong solution $(J, v, \pi)$ to system (2.5)–(2.7), subject to the initial condition $(J, v, \pi)|_{t=0} = (1, v_0, \pi_0)$. Moreover, we have the following

$$\int_{\mathbb{R}} \left(\frac{1}{2} \varrho_0(y)v^2(y,t) + \frac{1}{\gamma - 1} J(y,t)\pi(y,t)\right)dy = \mathcal{E}_0,$$

$$\int_{\mathbb{R}} \varrho_0(y)v(y,t)dy = m_0, \quad \inf_{y \in \mathbb{R}} J(y,t) \geq c_0,$$

for any $t \in [0,\infty)$, where

$$\mathcal{E}_0 := \int_{\mathbb{R}} \left(\frac{\varrho_0 v_0^2}{2} + \frac{\pi_0}{\gamma - 1}\right)dy, \quad m_0 = \int_{\mathbb{R}} \varrho_0 v_0 dy, \quad c_0 = e^{-\frac{2\sqrt{2}}{\mu} \sqrt{\mathcal{E}_0} \varrho_0},$$

(ii) Assume further that (H3) holds, for two positive constants $\delta$ and $K_0$. Then, (2.10) and (2.11) hold for any $T \in (0,\infty)$. 

Remark 2.5. If the initial data has more regularities, then the corresponding solution \((\rho, v, \pi)\) in Theorem 2.2 can be classical ones and, consequently, we obtain the global existence of classical solutions to the compressible Navier-Stokes equations without heat conduction, in the presence of far field vacuum. To the best of our acknowledge, this is the first result on the global existence of strong solutions to the compressible Navier-Stokes equations without heat conduction, for arbitrary large initial data, in the presence of far field vacuum. Note that this global existence result does not contradict to the finite time blow-up results in [41], as the assumption that having initial isolated mass group there is excluded in our case.

Remark 2.6. The following assumption in (H4)

\[
\rho_0(y) \geq \frac{A_0}{(1 + |y|)^2}, \quad \forall y \in \mathbb{R}
\]

can be removed. In fact, noticing that, essentially, the role that this assumption played in the proof of Theorem 2.2 is to justify some integration by parts of some integrals defined on the whole line, so that one can get the basic energy inequality and the estimates on \(G\), see Proposition 5.1, Proposition 5.4, and Proposition 5.6. Alternatively, to get the desired basic energy inequality and the estimates on \(G\), one can approximate the Cauchy problem by a sequence of initial-boundary value problems, while for the initial-boundary value problems, the integration by parts to the corresponding integrals, defined on the finite intervals, holds without the above assumption.

Remark 2.7. Choose

\[
\rho_0(y) = \frac{K_\rho}{\langle y \rangle^{\ell_\rho}}, \quad J_0 \equiv 1, \quad v_0 \in C^\infty_c, \quad \pi_0 = A e^{\frac{1}{c_v} \vartheta_0},
\]

where \(K_\rho\) and \(\ell_\rho\) are positive numbers.

(i) If \(1 < \ell_\rho \leq 2\), then \((\rho_0, v_0, \pi_0)\) satisfies assumptions (H1), (H2), (H3), with \(\delta = 1\), and (H4). Therefore, by Theorem 2.2, there is a unique global strong solution \((J, v, \pi)\), with \(v\) being in the inhomogeneous Sobolev space \(L^{\infty}(0, T; H^1)\); if moreover that \(\gamma > \frac{5}{4}\) and \(\max \left\{1, \frac{1}{2(\gamma - 1)} \right\} < \ell_\rho \leq 2\), then \(\vartheta_0 \in H^1\), and, consequently, the temperature \(\vartheta\) also lies in the inhomogeneous Sobolev space \(L^{\infty}(0, T; H^1)\).

(ii) If \(\max \left\{1, \frac{1}{\gamma} \right\} < \ell_\rho \leq 2\), then \((\rho_0, v_0, \pi_0)\) satisfies conditions (H1), (H2), (H3), with \(\delta = \gamma\), (H4), and \(s_0 \equiv 1\). Therefore, by Theorem 2.2, there is a unique strong solution \((J, v, \pi)\), and the corresponding entropy \(s\) is uniformly bounded on \(\mathbb{R} \times (0, T)\).

(iii) Combining (i) with (ii), if \(\gamma > \frac{5}{4}\) and \(\max \left\{1, \frac{1}{\gamma}, \frac{1}{2(\gamma - 1)} \right\} < \ell_\rho \leq 2\), then there is a unique global strong solution \((J, v, \pi)\), with the properties that the corresponding entropy is uniformly bounded, and the velocity and the corresponding temperature lie in the inhomogeneous space \(L^{\infty}(0, T; H^1)\).
Remark 2.8. Same as in Remark 2.4, one can obtain the corresponding global existence of solutions in the Euler coordinates to the compressible Navier-Stokes equations (1.6), (1.7), and (1.9), subject to the initial data \((\bar{\rho}_0, v_0, \pi_0)\).

3. LOCAL EXISTENCE IN THE ABSENCE OF VACUUM

In this section, we study system (2.5)–(2.7), subject to (2.9), in the absence of vacuum, that is, the density \(\bar{\rho}_0\) is assumed to have a positive lower bound. We focus on those a priori estimates of the solutions which are independent of the positive lower bound of the density \(\bar{\rho}_0\).

Then the following local existence result holds:

**Proposition 3.1.** Given a function \(\bar{\rho}_0\) satisfying \(\bar{\rho}_0 \leq \rho_0 \leq \bar{\rho}\) on \(\mathbb{R}\), for two positive constants \(\bar{\rho}_0\) and \(\rho_0\). Assume that the initial data \((J_0, v_0, \pi_0)\) satisfies \(J_0 \leq J_0 \leq \bar{J}, v_0 \in H^1, 0 \leq \pi_0 \in H^1\), for two positive constants \(\rho_0\) and \(\bar{\rho}\).

Then, there is a unique local strong solution \((J, v, \pi)\) to system (2.5)–(2.7), subject to the initial condition (2.9), on \(\mathbb{R} \times (0, T)\), satisfying

\[
\frac{3}{4} J \leq J \leq \frac{5}{4} \bar{J}, \quad \pi \geq 0, \quad \text{on } \mathbb{R} \times [0, T],
\]

\[
J - J_0 \in C([0, T]; H^1), \quad J_t \in L^\infty(0, T; L^2),
\]

\[
v \in C([0, T]; H^1) \cap L^2(0, T; H^2), \quad v_t \in L^2(0, T; L^2),
\]

\[
\pi \in C([0, T]; H^1), \quad \pi_t \in L^2(0, T; L^2),
\]

where \(T = T(\mu, \gamma, \bar{\rho}, \rho_0, \ell_0)\), with \(\ell_0 = \frac{1}{2} + \bar{J} + \|J_0\|_2 + \|\pi_0\|_{H^1}\), and the existence time \(T\) viewing as a function of \(\ell_0\) is continuous in \(\ell_0 \in (0, \infty)\).

**Proof.** Let \(T\) be a small positive time to be determined by the quantity \(\ell_0\). Given a velocity \(v \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2)\), define \(J\) and \(\pi\), successively, as the unique solutions to the following two ordinary differential equations:

\[
J_t = v_y,
\]

and

\[
\pi_t + \gamma \frac{v_y}{J} \pi = (\gamma - 1)\mu \left( \frac{v_y}{J} \right)^2,
\]

with initial data \(J|_{t=0} = J_0\) and \(\pi|_{t=0} = \pi_0\), respectively. Then, define \(V\) as the unique solution to the following uniform parabolic equation

\[
V_t - \mu \frac{\mu}{\rho_0} \left( \frac{V_y}{J} \right)_y = -\frac{\pi_y}{\rho_0},
\]

subject to the initial data \(V|_{t=0} = v_0\). Define a solution mapping \(\mathcal{M} : v \mapsto V\), with \(V\) defined as above. By standard energy estimates, and choosing \(T = T(\mu, \gamma, \bar{\rho}, \rho_0, \ell_0)\) small enough, one can show that \(\mathcal{M}\) is a contracting mapping on the space \(X = \)
\(L^\infty(0, T; H^1) \cap L^2(0, T; H^2)\) and, thus, there is a unique fixed point, denoted by \(v\), to \(\mathcal{M}\) in \(X\). Then \((J, v, \pi)\), with \((J, \pi)\) defined in the way as stated above, is a desired solution to system \((2.5)-(2.7)\), subject to \((2.9)\), on \(\mathbb{R} \times (0, T)\). Since the proof is lengthy but standard, the details are omitted here. \(\square\)

By Proposition 3.1, there is a positive time \(T_1\), such that system \((2.5)-(2.7)\), subject to \((2.9)\), has a unique solution \((J, v, \pi)\), on the time interval \((0, T_1)\), satisfying

\[
\begin{cases}
\frac{3}{4}J \leq J \leq \frac{5}{4}J, & \pi \geq 0, \quad \text{on } \mathbb{R} \times [0, T_1], \\
J - J_0 \in C([0, T_1]; H^1), & J_t \in L^\infty(0, T_1; L^2), \\
v \in C([0, T_1]; H^1) \cap L^2(0, T_1; H^2), & v_t \in L^2(0, T_1; L^2), \\
\pi \in C([0, T_1]; H^1), & \pi_t \in L^2(0, T_1; L^2),
\end{cases}
\]

where \(T_1\) is a positive constant depending only on \(\mu, \gamma, \varrho, \bar{\varrho}\), and \(\frac{1}{4} + J + \|J_0\|_2 + \|(v_0, \pi_0)\|_{H^1}\). Starting from the time \(T_1\), noticing that \((J, v, \pi)|_{t=T_1}\) satisfies the conditions on the initial data stated in Proposition 3.1, one can extend the solution \((J, v, \pi)\) forward in time to another time \(T_2 = T_1 + t_1\), for some positive time

\[t_1 = t_1(\mu, \gamma, \varrho, \bar{\varrho}, \ell(T_1)),\]

where, for simplicity of notations, we have denoted

\[
\ell(t) := \left(\inf_{y \in \mathbb{R}} J \right)^{-1} + \|J\|_\infty + \|J_0\|_2 + \|v\|_{H^1} + \|\pi\|_{H^1}(t), \tag{3.1}
\]

such that \((J, v, \pi)\) is the unique solution to system \((2.5)-(2.7)\), subject to \((2.9)\), on the time interval \((0, T_2)\), and that it enjoys the same regularities as above in the time interval \((0, T_2)\), and \((\frac{3}{4})^2 J \leq J \leq (\frac{5}{4})^2 J\) on \(\mathbb{R} \times [0, T_2]\). Continuing this procedure, one obtains two sequences of positive numbers \(\{t_j\}_{j=1}^\infty\) and \(\{T_j\}_{j=1}^\infty\), with

\[t_j = t_j(\mu, \gamma, \varrho, \bar{\varrho}, \ell(T_j)),\]

and \(T_{j+1} = T_j + t_j\), such that the solution \((J, v, \pi)\) can be extended to time intervals \((0, T_j)\), satisfying

\[
\begin{cases}
(\frac{3}{4})^j J \leq J \leq (\frac{5}{4})^j J, & \pi \geq 0, \quad \text{on } \mathbb{R} \times [0, T_j], \\
J - J_0 \in C([0, T_j]; H^1), & J_t \in L^\infty(0, T_j; L^2), \\
v \in C([0, T_j]; H^1) \cap L^2(0, T_j; H^2), & v_t \in L^2(0, T_j; L^2), \\
\pi \in C([0, T_j]; H^1), & \pi_t \in L^2(0, T_j; L^2),
\end{cases}
\]

for \(j = 1, 2, \ldots\). Set the maximal existing time \(T_\infty\) as

\[T_\infty = T_1 + \sum_{j=1}^\infty t_j.\]
Then, the solution \((J, v, \pi)\) can be extended to the time interval \((0, T_\infty)\), such that

\[
0 < \inf_{y \in \mathbb{R}, t \in [0, T]} J(y, t) \leq \sup_{y \in \mathbb{R}, t \in [0, T]} J(y, t) < \infty, \\
\pi \geq 0 \quad \text{on } \mathbb{R} \times [0, T], \\
J - J_0 \in C([0, T]; H^1), \\
v \in C([0, T]; H^1) \cap L^2(0, T; H^2), \\
\pi \in C([0, T]; H^1), \\
\|v\|_{L^2(0, T; L^2)} < \infty,
\]

for any \(T \in (0, T_\infty)\). Moreover, if \(T_\infty < \infty\), it must have \(\lim_{j \to \infty} t_j = 0\) and, consequently, one has

\[
\lim_{j \to \infty} \ell(T_j) = \infty,
\]

where \(\ell(t)\) is defined by \((3.1)\); otherwise, if \((3.3)\) is not true, then \(\ell(T_j)\) is uniformly bounded and, thus, by Proposition 3.1, \(t_j = t_j(\mu, \gamma, \bar{\rho}, \ell(T_j))\), \(j = 1, 2, \ldots\), have a uniform positive lower bound, contradicting to the fact that \(\lim_{j \to \infty} t_j = 0\).

Thanks to the statements in the above paragraph, in the rest of this section, we always assume, without any further mention, that \((J, v, \pi)\) is the unique solution to system \((2.5) - (2.7)\), subject to \((2.9)\), and that it has been extended, in the same way as above, to the maximal existing time interval \((0, T_\infty)\), where the maximal time \(T_\infty\) is constructed in the same way as above.

To obtain the a priori estimates on \((J, v, \pi)\), we define a positive time

\[
T_s := \sup \left\{ T \in (0, T_\infty) \left| \frac{J}{2} \leq J \leq 2J \right. \text{ on } \mathbb{R} \times [0, T] \right\}.
\]

We start with the following estimate on \(G\):

**Proposition 3.2.** There is a positive constant \(t_*^1 = t_*^1(\gamma, \mu, \bar{\rho}, J, \|J_0G_0\|_2)\), such that

\[
\sup_{0 \leq t \leq T_*^1} \|\sqrt{J}G\|^2_2 + \mu \int_0^{T_*^1} \frac{C_y}{\sqrt{G_0}} \|G_y\|^2 dt \leq 3(1 + \|J_0G_0\|^2_2),
\]

\[
\int_0^{T_*^1} \|G\|^4 dt \leq C(\mu, \bar{\rho}, J, \|\sqrt{J_0G_0}\|_2),
\]

where \(G_0 := \mu \nu_0 - \pi_0\), \(T_*^1 := \min\{1, t_*^1, T_s\}\), and \(T_s\) is defined by \((3.4)\).

**Proof.** Multiplying \((2.8)\) by \(JG\), and integrating the resultant over \(\mathbb{R}\), one gets by integration by parts that

\[
\int_{\mathbb{R}} JGG_t dy + \mu \int_{\mathbb{R}} \frac{(G_y)^2}{\nu_0} dy = -\gamma \int_{\mathbb{R}} v_y G^2 dy.
\]

Then \((2.5)\) shows

\[
\int_{\mathbb{R}} JGG_t dy = \frac{1}{2} \left( \frac{d}{dt} \int_{\mathbb{R}} JG^2 dy - \int_{\mathbb{R}} v_y G^2 dy \right),
\]
which, together with (3.5), yields
\[
\frac{1}{2} \frac{d}{dt} \| \sqrt{J} G \|_2^2 + \mu \left\| \frac{G_y}{\sqrt{\theta_0}} \right\|_2^2 = \left( \frac{1}{2} - \gamma \right) \int_\mathbb{R} v_y G^2 dy.
\] (3.6)

It follows from \( v_y = \frac{\mu}{2} (G + \pi) \) that
\[
\left( \frac{1}{2} - \gamma \right) \int_\mathbb{R} v_y G^2 dy = \frac{1 - 2\gamma}{2\mu} \int_\mathbb{R} J(G + \pi) G^2 dy \leq \frac{1 - 2\gamma}{2\mu} \int_\mathbb{R} JG^3 dy \leq \frac{\gamma}{\mu} \| G \|_\infty \| \sqrt{J} G \|_2^2;
\]
where we have used \( \pi \geq 0 \) and \( \gamma > 1 \). Therefore, it follows from (3.6) that
\[
\frac{1}{2} \frac{d}{dt} \| \sqrt{J} G \|_2^2 + \mu \left\| \frac{G_y}{\sqrt{\theta_0}} \right\|_2^2 \leq \frac{\gamma}{\mu} \| G \|_\infty \| \sqrt{J} G \|_2^2.
\] (3.7)

By the Gagliardo-Nirenberg inequality \( \| f \|_{L^\infty(\mathbb{R})} \leq C \| f \|_{L^2(\mathbb{R})} \| f' \|_{L^2(\mathbb{R})} \), one has
\[
\| G \|_\infty \leq C \| G \|_2^\frac{1}{2} \| G_y \|_2^\frac{1}{2} \leq C(\bar{\rho}, \mathcal{J}) \| \sqrt{J} G \|_2^\frac{1}{2} \left\| \frac{G_y}{\sqrt{\theta_0}} \right\|_2^\frac{1}{2}. \] (3.8)

Combining (3.7) and (3.8), one obtains from the Young inequality that
\[
\frac{1}{2} \frac{d}{dt} \| \sqrt{J} G \|_2^2 + \mu \left\| \frac{G_y}{\sqrt{\theta_0}} \right\|_2^2 \leq C(\gamma, \mu, \bar{\rho}, \mathcal{J}) \| \sqrt{J} G \|_2^2 \left\| \frac{G_y}{\sqrt{\theta_0}} \right\|_2^\frac{1}{2}
\]
\[
\leq \frac{\mu}{2} \left\| \frac{G_y}{\sqrt{\theta_0}} \right\|_2^2 + C(\gamma, \mu, \bar{\rho}, \mathcal{J})(1 + \| \sqrt{J} G \|_2^2)^2
\]
and, thus,
\[
\frac{d}{dt}(1 + \| \sqrt{J} G \|_2^2) + \mu \left\| \frac{G_y}{\sqrt{\theta_0}} \right\|_2^2 \leq C_1(\gamma, \mu, \bar{\rho}, \mathcal{J})(1 + \| \sqrt{J} G \|_2^2)^2. \] (3.9)

Solving (3.9) yields
\[
-(1 + \| \sqrt{J} G \|_2^2)^{-1}(t) \leq -(1 + \| \sqrt{J}_0 G_0 \|_2^2)^{-1} + C_1(\gamma, \mu, \bar{\rho}, \mathcal{J}) t
\]
\[
\leq -\frac{1}{2}(1 + \| \sqrt{J}_0 G_0 \|_2^2)^{-1},
\]
for any \( t \in [0, T^*_s) \), where
\[
T^*_s := \min\{1, t^*_s, T_s\}, \quad t^*_s := \frac{1}{2(1 + \| \sqrt{J}_0 G_0 \|_2^2) C_1(\gamma, \mu, \bar{\rho}, \mathcal{J})}.
\] (3.10)

Therefore, we have
\[
\sup_{0 \leq t < T^*_1} (1 + \| \sqrt{J} G \|_2^2) \leq 2(1 + \| \sqrt{J}_0 G_0 \|_2^2),
\]
and further from (3.9) that
\[
\sup_{0 \leq t < T_1^*} \| \sqrt{JG} \|_2^2 + \mu \int_0^{T_1^*} \left\| \frac{G_y}{\sqrt{\rho_0}} \right\|_2^2 \, dt \leq 3(1 + \| \sqrt{J_0G_0} \|_2^2).
\]
The estimate \( \int_0^{T_1^*} \|G\|_4^4 \, dt \leq C(\mu, \bar{\rho}, \bar{J}, \|G_0\|_2) \) follows from the above estimate and (3.8). The proof is complete. \( \square \)

Based on Proposition 3.2, we can derive the following estimates on \((J, v, \pi)\).

**Proposition 3.3.** (i) Let \( T_{1*} \) be as in Proposition 3.2. Then, it holds that
\[
\sup_{0 \leq t \leq T_{1*}} \left\| \left( \pi, \frac{\pi_y}{\sqrt{\rho_0}}, J - J_0, J_t, \frac{J_y}{\sqrt{\rho_0}}, v_y \right) \right\|_2 \leq C,
\]
\[
\int_0^{T_{1*}} \left( \|\pi_t\|_2^4 + \left\| \left( \sqrt{\rho_0}v_t, \frac{v_y}{\sqrt{\rho_0}} \right) \right\|_2^2 \right) \, dt \leq C,
\]
and
\[
\sup_{0 \leq t \leq T_{1*}} \left\| \sqrt{\rho_0}v \right\|_{L^2((-R,R))} \leq \|v_0\|_{L^2((-R,R))} + C,
\]
for a positive constant \( C \) depending only on \( \gamma, \mu, \bar{\rho}, \bar{J}, \|J_0\|_2, \|G_0\|_2, \|\pi_0\|_2, \) and \( \left\| \frac{\pi_0}{\sqrt{\rho_0}} \right\|_2 \), but independent of \( \bar{\rho} \).

(ii) There is a positive constant \( t_{2*} \) depending only on \( \gamma, \mu, \bar{\rho}, \bar{J}, \|G_0\|_2, \) and \( \|\pi_0\|_\infty \), but independent of \( \bar{\rho} \), such that
\[
\frac{3}{4}J \leq J(y, t) \leq \frac{5}{4}J, \quad \text{on } \mathbb{R} \times [0, T_{2*}),
\]
where \( T_{2*} := \min\{T_{1*}, t_{2*}\} = \min\{1, t_{1*}, t_{2*}, T_s\} \), with \( T_s \) defined by (3.4).

**Proof.** (i) Equation (2.7) can be rewritten in terms of \( G \) as
\[
\pi_t + \frac{1}{\mu} \left( \pi + \frac{2 - \gamma}{2}G \right)^2 = \frac{\gamma^2}{4\mu}G^2,
\]
from which, one obtains
\[
0 \leq \pi(y, t) \leq \pi_0(y) + \frac{\gamma^2}{4\mu} \int_0^t G^2(y, \tau) \, d\tau.
\]
Thanks to the above, it follows from Proposition 3.2 and the Hölder inequality that
\[
\sup_{0 \leq t \leq T_2} \|\pi\|_2 \leq \|\pi_0\|_2 + \frac{\gamma^2}{4\mu} \int_0^{T_2} \|G\|_\infty \|G\|_2 \, dt \leq \|\pi_0\|_2 + C \left( \gamma, \mu, \bar{\rho}, \bar{J}, \|G_0\|_2 \right)
\]
Hence, it follows from (3.11) and Proposition 3.2 that

\[ \int_{0}^{T_{1}} \| \pi_{t} \|_{2}^{4} dt \leq C \int_{0}^{T_{1}} (\| G \|_{4}^{4} + \| \pi \|_{\infty}^{4}) (\| G \|_{2}^{4} + \| \pi \|_{2}^{4}) dt \]

\[ \leq C(\gamma, \mu, \bar{p}, J, \bar{J}, \| G_{0} \|_{2}, \| \pi_{0} \|_{2}, \| \pi_{0} \|_{\infty}). \]  

(3.14)

Differentiating equation (3.11) with respect to \( y \) yields

\[ \pi_{yt} + \frac{2}{\mu}(\pi + \frac{2}{2} - \gamma G) \left( \pi_{y} + \frac{2}{2} - \gamma G_{y} \right) = \frac{\gamma^{2}}{2\mu} G G_{y}. \]

Multiplying the above equation by \( \pi_{yt} \) and integrating over \( \mathbb{R} \), it follows from the Hölder and Cauchy inequalities that

\[ \frac{d}{dt} \left( \frac{\pi_{yt}}{\sqrt{20}} \right) \leq C(\gamma, \mu)(\| \pi \|_{\infty} + \| G \|_{\infty}) \left( \left\| \frac{\pi_{yt}}{\sqrt{\varrho_{0}}} \right\|_{2} + \left\| \frac{G_{y}}{\sqrt{\varrho_{0}}} \right\|_{2} \right) \left( \left\| \frac{\pi_{y}}{\sqrt{\varrho_{0}}} \right\|_{2} \right) \]

\[ \leq C(\gamma, \mu) \left( \left\| \frac{G_{y}}{\sqrt{\varrho_{0}}} \right\|_{2}^{2} + (1 + \| \pi \|_{\infty}^{2} + \| G \|_{\infty}^{2}) \left( \left\| \frac{\pi_{y}}{\sqrt{\varrho_{0}}} \right\|_{2}^{2} \right) \right), \]

from which, by Proposition 3.2 and (3.13), it follows from the Gronwall inequality that

\[ \sup_{0 \leq t \leq T_{1}^{*}} \left\| \frac{\pi_{yt}}{\sqrt{20}} \right\|_{2} \leq C(\gamma, \mu, \bar{p}, J, \bar{J}, \| G_{0} \|_{2}, \| \pi_{0} \|_{\infty}, \left\| \frac{\pi_{yt}}{\sqrt{20}} \right\|_{2}). \]  

(3.15)

Recalling the definition of \( G \), one can rewrite the equation for \( J \) as

\[ J_{t} = \frac{J}{\mu}(G + \pi), \]

(3.16)

from which, we deduce

\[ \| J - J_{0} \|_{2} + \| J_{t} \|_{2} = \left\| \int_{0}^{t} J_{t} d\tau \right\|_{2} + \| J_{t} \|_{2} \]

\[ \leq \frac{2\bar{J}}{\mu} \left( \int_{0}^{t} (\| G \|_{2} + \| \pi \|_{2}) d\tau + \| G \|_{2} + \| \pi \|_{2} \right), \]

and, thus, it follows from Proposition 3.2 and (3.12) that

\[ \sup_{0 \leq t \leq T_{1}^{*}} (\| J - J_{0} \|_{2} + \| J_{t} \|_{2}) \leq C(\gamma, \mu, \bar{p}, J, \bar{J}, \| G_{0} \|_{2}, \| \pi_{0} \|_{2}). \]  

(3.17)

Solving the ordinary differential equation (3.16) yields

\[ J(y, t) = \exp \left\{ \frac{1}{\mu} \int_{0}^{t} (G + \pi) d\tau \right\} J_{0}(y) \]
and, thus,

\[ J_y = \left( \frac{1}{\mu} \int_0^t (G_y + \pi_y) d\tau J_0 + J'_0 \right) \exp \left\{ \frac{1}{\mu} \int_0^t (G + \pi) d\tau \right\}, \]

from which, applying Propositions \ref{3.2} and using (3.13) and (3.15), one obtains

\[
\sup_{0 \leq t \leq T_1} \left\| J_y \right\|_\infty \leq \left( \frac{J}{\mu} \int_0^{T_1} \left\| \left( \frac{G_y}{\sqrt{\theta_0}}, \frac{\pi_y}{\sqrt{\theta_0}} \right) \right\|_2 dt + \left\| J'_0 \right\|_2 \right) \times \exp \left\{ \frac{1}{\mu} \int_0^{T_1} \left\| (G, \pi) \right\|_\infty dt \right\} \leq C, \tag{3.18} \]

for a positive constant \( C \) depending only on \( \gamma, \mu, \bar{\theta}, \bar{J}, \| \bar{J} \|_2, \| G_0 \|_2, \| \pi_0 \|_2 \), and \( \| \frac{\pi_y}{\sqrt{\theta_0}} \|_2 \), but independent of \( \bar{\theta} \).

Recalling the definition of \( G \), and noticing that \( \bar{\theta}_0 v_t = G_y \), one gets from Proposition \ref{3.2} and (3.12) that

\[
\sup_{0 \leq t \leq T_1} \left\| v_y \right\|_2 \leq \sup_{0 \leq t \leq T_1} \left\| \frac{J}{\mu} (G + \pi) \right\|_2 \leq C(\gamma, \mu, \bar{\theta}, \bar{J}, \| G_0 \|_2, \| \pi_0 \|_2) \tag{3.19} \]

and

\[
\int_0^{T_1} \left\| \sqrt{\bar{\theta}_0} v_t \right\|_2^2 dt = \int_0^{T_1} \left\| \frac{G_y}{\sqrt{\theta_0}} \right\|_2^2 dt \leq C(\mu, \bar{\theta}, \bar{J}, \| G_0 \|_2). \tag{3.20} \]

Therefore, it holds that

\[
\sup_{0 \leq t \leq T_1} \left\| \sqrt{\bar{\theta}_0} v \right\|_{L^2((-R, R))} = \sup_{0 \leq t \leq T_1} \left\| \sqrt{\bar{\theta}_0} v_0 + \int_0^t \sqrt{\bar{\theta}_0} v_t d\tau \right\|_{L^2((-R, R))} \leq \left\| \sqrt{\bar{\theta}_0} v_0 \right\|_{L^2((-R, R))} + \int_0^{T_1} \left\| \sqrt{\bar{\theta}_0} v_t \right\|_2^2 d\tau \leq \left\| \sqrt{\bar{\theta}_0} v_0 \right\|_{L^2((-R, R))} + C(\mu, \bar{\theta}, \bar{J}, \| G_0 \|_2), \]

for any \( 0 < R \leq \infty \). Noticing that

\[
v_{yy} = \left[ \frac{J}{\mu} (G + \pi) \right]_y = \frac{J_y}{\mu} (G + \pi) + \frac{J}{\mu} (G_y + \pi_y)\]

it follows from Propositions \ref{3.2}, (3.13), (3.15), (3.18), and the Hölder inequality that

\[
\int_0^{T_0} \left\| \frac{v_{yy}}{\sqrt{\theta_0}} \right\|_2^2 dt \leq C \int_0^{T_0} \left( \left\| \left( \frac{G_y}{\sqrt{\theta_0}}, \frac{\pi_y}{\sqrt{\theta_0}} \right) \right\|_2^2 + \left\| (G, \pi) \right\|_\infty^2 \left\| \frac{J_y}{\sqrt{\theta_0}} \right\|_2^2 \right) dt \leq C, \tag{3.21} \]

for a positive constant \( C \) depending only on \( \gamma, \mu, \bar{\theta}, \bar{J}, \| G_0 \|_2, \| \pi_0 \|_2 \), and \( \left\| \frac{\pi_y}{\sqrt{\theta_0}} \right\|_2 \), but independent of \( \bar{\theta} \).
(ii) Due to Proposition 3.2 and (3.13), one gets from the Hölder inequality that
\[
\frac{1}{\mu} \int_0^t \|J(G + \pi)\|_\infty d\tau \leq \frac{2\bar{J}}{\mu} \int_0^t \|G\|_\infty + \|\pi\|_\infty d\tau \\
\leq Ct + \frac{2}{\mu} \left( \int_0^t \|G\|_\infty^4 d\tau \right)^{\frac{1}{4}} t^\frac{3}{4} \leq C_2 t^\frac{3}{4} \leq \frac{J}{4},
\]
for any \( t \in [0, T_*^2) \),

where
\[ T_*^2 := \min\{ T_*^1, t_*^2 \} = \min\{ t_*^1, t_*^2, T_* \}, \]
for a positive constant \( C_2 \) depending only on \( \gamma, \mu, \tilde{g}, \bar{J}, \|G_0\|_2 \), and \( \|\pi_0\|_\infty \), but independent of \( \rho \). Consequently, it follows from (3.16) that
\[
|J - J_0| = \left| \int_0^t J \, d\tau \right| = \left| \int_0^t \frac{J}{\mu} (G + \pi) \, d\tau \right| \leq \frac{1}{\mu} \int_0^t \|J(G + \pi)\|_\infty d\tau \leq \frac{J}{4},
\]
which implies
\[
\frac{3}{4} J \leq J_0 - \frac{J}{4} \leq J \leq J_0 + \frac{J}{4} \leq \bar{J} + \frac{J}{4} = \frac{5}{4} \bar{J} \quad \text{on } \mathbb{R} \times [0, T_*^2),
\]
which proves (ii). \( \square \)

Thanks to the estimates stated in Proposition 3.3, one can evaluate the lower bound of the time \( T_s \) as stated in the following proposition:

**Proposition 3.4.** Let \( T_s \) be defined by (3.4), \( t_*^1 \) and \( T_*^1 \) be the constants stated in Proposition 3.2, and \( t_*^2 \) and \( T_*^2 \) the constants in Proposition 3.3. Then, we have \( T_s > T_*^2 \) and, consequently, \( T_*^1 > T_*^2 \geq \min\{1, t_*^1, t_*^2\} \).

**Proof.** Assume, by contradiction, that \( T_s < T_*^2 \). Recall that \( T_*^2 = \min\{1, t_*^1, t_*^2, T_s\} \), which gives \( T_*^2 \leq T_s \) and, therefore, \( T_s = T_*^2 \).

If \( T_s < T_\infty \), then \( T_*^2 = T_s < T_\infty \). Recalling that (3.2) holds for any \( T < T_\infty \), we have \((J - J_0, v, \pi) \in C([0, T_\infty); H^1(\mathbb{R}))\), which, by the embedding \( H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \), we have \( J \in C([0, T_\infty); L^\infty(\mathbb{R})) \). Thanks to this and by (ii) of Proposition 3.3 there is a positive time \( T_* \in (T_*^2, T_\infty) \), such that \( \frac{1}{2} J \leq J \leq \frac{3}{2} \bar{J} \) on \( \mathbb{R} \times [0, T_*] \). By the definition of \( T_s \), then \( T_s \leq T_* = T_*^2 \), which contradicts to \( T_s \in (T_*^2, T_\infty) \).

If \( T_s = T_\infty \), then \( T_s = T_*^2 = T_\infty \). By Proposition 3.3 and noticing that \( T_*^2 \leq T_*^1 \), we have
\[
\sup_{0 \leq t < T_*^2} (\|J_y\|_2 + \|\sqrt{\rho_0} v\|_2 + \|v_y\|_2 + \|\pi\|_{H^1}) < \infty
\]
and \( \frac{3}{4} J \leq J \leq \frac{5}{4} \bar{J} \) on \( \mathbb{R} \times [0, T_*^2) \). Therefore, recalling that \( \rho_0 \geq \rho > 0 \), we have \( \lim_{t \to T_\infty} \ell(t) = \lim_{t \to T_*^2} \ell(t) < \infty \), which contradicts to (3.3).
Combining the statements of the above two paragraphs yields \( T_s > T_s^2 \). By the aid of this, and recalling the definition of \( T_s^2 = \min\{1, t_s^1, t_s^2, T_s\} \), we have \( T_s^2 = \min\{1, t_s^1, t_s^2\} \). This proves the conclusion. \( \square \)

Then Propositions 3.1, 3.4 give the following:

**Corollary 3.1.** Given a function \( \varrho_0 \) satisfying \( \varrho \leq \varrho_0 \leq \bar{\varrho} \) on \( \mathbb{R} \), for two positive constants \( \underline{\varrho} \) and \( \bar{\varrho} \). Assume that the initial data \((\bar{J}_0, v_0, \pi_0)\) satisfies

\[
\underline{J} \leq J_0 \leq \bar{J} \text{ on } \mathbb{R}, \quad J_y^0 \in L^2, \quad v_0 \in L^1_{\text{loc}}, \quad v_y^0 \in L^2, \quad 0 \leq \pi_0 \in H^1,
\]

for two positive constants \( \underline{J} \) and \( \bar{J} \).

Then, there is a positive time \( T_0 \) depending only on \( \gamma, \mu, \bar{\varrho}, \underline{J}, \bar{J}, \|v_0\|_2, \|\pi_0\|_2 \), and \( \|\pi_0\|_\infty \), but independent of \( \varrho \), such that system \((2.5)-(2.7)\), subject to \((2.9)\), has a unique solution \((J, v, \pi)\) on \( \mathbb{R} \times [0, T_0] \), satisfying

\[
\pi \geq 0, \quad \frac{3}{4} \underline{J} \leq J \leq \frac{5}{4} \bar{J}, \quad \text{on } \mathbb{R} \times [0, T_0],
\]

\[
\sup_{0 \leq t \leq T_0} \left\| \left( \frac{\pi_y}{\sqrt{\varrho_0}}, J - J_0, J_t, \frac{J_y}{\sqrt{\varrho_0}} ; v_y \right) \right\|_2 \leq C,
\]

\[
\int_0^{T_0} \left( \|\pi_t\|_2^4 + \left\| \left( \sqrt{\varrho_0} v_y, \frac{v_y}{\sqrt{\varrho_0}} \right) \right\|_2^2 \right) \, dt \leq C,
\]

\[
\sup_{0 \leq t \leq T_0} \|\sqrt{\varrho_0} v\|_{L^2((-R,R))} \leq \|\sqrt{\varrho_0} v_0\|_{L^2((-R,R))} + C,
\]

for any \( 0 < R \leq \infty \), where \( C \) is a positive constant depending only on \( \gamma, \mu, \bar{\varrho}, \underline{J}, \bar{J}, \|v_0\|_2, \|v_y^0\|_2, \|\pi_0\|_2 \), and \( \|\varrho_0\|_2 \), but independent of \( \varrho \).

**Proof.** The H"older inequality yields

\[
|v_0(y)| = \left| v_0(0) + \int_0^y v_0'(z) \, dz \right| \leq |v_0(0)| + \|v_0'||2 \sqrt{|y|}, \quad \forall y \in \mathbb{R}. \tag{3.22}
\]

Choose a function \( 0 \leq \phi \in C^\infty_c((-2,2)) \), with \( \phi \equiv 1 \) on \((-1,1)\), \( 0 \leq \phi \leq 1 \) on \((-2,2)\), and \( |\phi'| \leq 2 \) on \( \mathbb{R} \). For any positive integer \( n \), we set \( \phi_n(\cdot) = \phi\left(\frac{\cdot}{n}\right) \) and \( v_{0n} = v_0 \phi_n \). Thanks to (3.22), noticing that \( \text{supp} \phi_n \subseteq (-2n, -n) \cup (n, 2n) \), and that

\[
v'_{0n} = v_0' \phi_n + v_0 \phi'_n = v_0' \phi_n + \frac{v_0}{n} \phi'\left(\frac{\cdot}{n}\right),
\]

one has

\[
\|v'_{0n}\|_2 \leq \|v'_0\|_2 + \frac{1}{n} \left\| v_0 \phi'\left(\frac{\cdot}{n}\right) \right\|_2
\]

\[
= \|v'_0\|_2 + \frac{1}{n} \left( \int_{-n < |y| < 2n} |v_0|^2 \left| \phi'\left(\frac{y}{n}\right) \right|^2 \, dy \right)^{\frac{1}{2}}
\]
Due to (3.23), it holds that for any $n$ for any $T \in [3.1–3.4]$, there is a positive time $\|v\|_2 \leq \frac{1}{n} \left[ \int_{n < |y| < 2n} (|v_0(0)| + \sqrt{|y|} \|v'_0\|_2) \, dy \right]^{\frac{1}{2}}$ 
\[ \leq \|v'_0\|_2 + \frac{2}{n} \left( |v_0(0)| + \sqrt{2n} \|v'_0\|_2 \right) \sqrt{2n} \] 
\[ = 5 \|v'_0\|_2 + \frac{2\sqrt{2}}{\sqrt{n}} |v_0(0)| \leq 5 \|v'_0\|_2 + 1, \quad (3.23) \]
for any $n \geq 8|v_0(0)|^2$.

Consider system (2.5)–(2.7), subject to the initial condition 
\[ (J, v, \pi)|_{t=0} = (J_0, v_0, \pi_0). \quad (3.24) \]
Due to (3.23), it holds that 
\[ \|G_{0n}\|_2 \leq \mu \|v'_{0n}\|_2 + \|\pi_0\|_2 \leq \mu (5 \|v'_0\|_2 + 1) + \|\pi_0\|_2, \]
for any $n \geq 8|v_0(0)|^2$, where $G_{0n} := \mu \nu v'_{0n} - \pi_0$. Thanks to this and Propositions 3.1–3.4, there is a positive time $T_0$ depending only on $\gamma, \mu, \tilde{\beta}, \tilde{J}, \|v'_0\|_2, \|\pi_0\|_2$, and $\|\pi_0\|_\infty$, but independent of $n$, such that system (2.5)–(2.7), subject to (3.24), has a unique solution $(J_n, v_n, \pi_n)$, on $\mathbb{R} \times (0, T_0)$, satisfying 
\[ \pi_n \geq 0, \quad \frac{3}{4} \tilde{J} \leq J_n \leq \frac{5}{4} \tilde{J}, \quad \text{on } \mathbb{R} \times [0, T_0], \]
\[ \sup_{0 \leq t \leq T_0} \left\| \left( \pi_n, \frac{\partial_y \pi_n}{\sqrt{\nu_0}}, J_n - J_0, \partial_t J_n, \frac{\partial_y J_n}{\sqrt{\nu_0}}, \partial_y v_n \right) \right\|_2 \leq C, \]
\[ \int_0^{T_0} \left( \|\partial_t \pi_n\|_2 + \left\| \left( \sqrt{\nu_0} \partial_y v_n, \frac{\partial^2_y v_n}{\sqrt{\nu_0}} \right) \right\|_2 \right)^2 \, dt \leq C, \]
\[ \sup_{0 \leq t \leq T_0} \left\| \sqrt{\nu_0} v_n \right\|_{L^2((-R, R))} \leq \left\| \sqrt{\nu_0} v_0 \right\|_{L^2((-R, R))} + C, \]
for any $n \geq 8|v_0(0)|^2$, and for any $0 < R \leq \infty$, where $C$ is a positive constant depending only on $\gamma, \mu, \tilde{\beta}, \tilde{J}, \left\| \frac{f}{\sqrt{\nu_0}} \right\|_2, \|v'_0\|_2, \|\pi_0\|_2$, and $\left\| \frac{\pi'_0}{\sqrt{\nu_0}} \right\|_2$, but independent of $n$ and $n \geq 8|v_0(0)|^2$.

With the above a priori estimates in hand, one can apply the Banach-Alaoglu theorem, use Cantor’s diagonal arguments, apply the Aubin-Lions lemma, and make use of the weakly lower semi-continuity of the norms, to show that there is a subsequence, still denoted by $(J_n, v_n, \pi_n)$, and a triple $(J, v, \pi)$, which satisfies the same a priori estimates as above, such that $(J_n, v_n, \pi_n)$ converges, weakly or weak-* in appropriate spaces, to $(J, v, \pi)$, and $(J, v, \pi)$ is a solution to system (2.5)–(2.7), subject to (2.9). Since the proof is very similar to that of (i) of Theorem 2.11 in the next section, we omit the details here. The uniqueness is guaranteed by Proposition 4.4 in the next section. \qed
As the end of this section, we give some more estimates on $G$ stated in the next proposition, which will be the key to obtain the boundedness of the entropy.

**Proposition 3.5.** In addition to the assumptions in Corollary 3.1, we assume that

$$\left| \left( \frac{1}{\sqrt{\rho_0}} \right)'(y) \right| \leq \frac{K_0}{2}, \forall y \in \mathbb{R},$$

for some positive constant $K_0$. Let $T_0$ be the positive constant in Corollary 3.1 and $(J, v, \pi)$ the solution stated in Corollary 3.1.

Then, for any $\delta \in (0, \infty)$, there is a positive constant $C$ depending only on $\gamma, \mu, \bar{\rho}, J, K_0, \|J\|_2, \|v\|_2, \|\pi\|_2, \|\frac{\rho_0'}{\sqrt{\rho_0}}\|_2$, but independent of $\rho_0$, such that

$$\sup_{0 \leq t \leq T_0} \left\| \frac{G}{\rho_0^{\frac{\delta}{4}}} \right\|_2^2 + \int_0^{T_0} \left\| \frac{G_y}{\rho_0^{\frac{\delta}{4}+1}} \right\|_2^2 \, dt \leq C.$$

Proof. Multiplying (2.8) by $\frac{JG}{\rho_0}$ and integrating over $\mathbb{R}$, one gets from integration by parts that

$$\int_{\mathbb{R}} JG \frac{G}{\rho_0} \, dy + \mu \int_{\mathbb{R}} \frac{G_y}{\rho_0} \left( \frac{G}{\rho_0^\delta} \right)_y \, dy = -\gamma \int_{\mathbb{R}} v_y \frac{G^2}{\rho_0^\delta} \, dy. \quad (3.25)$$

Direct calculations yields

$$\int_{\mathbb{R}} \frac{G_y}{\rho_0} \left( \frac{G}{\rho_0^\delta} \right)_y \, dy = \int_{\mathbb{R}} \frac{G_y}{\rho_0} \left( \frac{G_y}{\rho_0^\delta} - \delta \frac{\rho_0'}{\rho_0} \frac{G}{\rho_0^\delta} \right) \, dy$$

$$= \left\| \frac{G_y}{\rho_0^\delta} \right\|_2^2 - \delta \int_{\mathbb{R}} \frac{\rho_0'}{\rho_0^\delta+1} G G_y \, dy. \quad (3.26)$$

It follows from (2.5) that

$$\int_{\mathbb{R}} JG \frac{G}{\rho_0^\delta} \, dy = \frac{1}{2} \left( \frac{d}{dt} \int_{\mathbb{R}} JG^2 \, dy - \int_{\mathbb{R}} J_1 G^2 \, dy \right)$$

$$= \frac{1}{2} \left( \frac{d}{dt} \left\| J \frac{G}{\rho_0^\delta} \right\|_2^2 - \int_{\mathbb{R}} v_y \frac{G^2}{\rho_0^\delta} \, dy \right). \quad (3.27)$$

Plugging (3.26) and (3.27) into (3.25) yields

$$\frac{1}{2} \frac{d}{dt} \left\| \frac{J}{\rho_0^\delta} G \right\|_2^2 + \mu \left\| \frac{G_y}{\rho_0^\delta+1} \right\|_2^2 = \left( \frac{1}{2} - \gamma \right) \int_{\mathbb{R}} v_y \frac{G^2}{\rho_0^\delta} \, dy + \delta \mu \int_{\mathbb{R}} \frac{\rho_0'}{\rho_0^\delta+1} G G_y \, dy. \quad (3.28)$$
Due to the assumption \( \left| \left( \frac{1}{\sqrt{\varrho_0}} \right) \right| \leq \frac{K_0}{2} \), or equivalently \( |\varrho_0| \leq K_0 \frac{3}{2} \), it follows from the Cauchy inequality that
\[
\int_{\mathbb{R}} \frac{\varrho_0^2 G G_y y}{\frac{\varrho_0^2}{\varrho_0^2 + 1}} dy \leq K_0 \int_{\mathbb{R}} \frac{G_y}{\varrho_0^2 + 1} \left| \frac{G_y}{\varrho_0^2 + 1} \right| dy \leq \frac{1}{2\delta} \left\| \frac{G_y}{\varrho_0^2 + 1} \right\|_2^2 + \frac{\delta K_0^2}{2} \left\| \frac{G}{\frac{3}{2}} \right\|_2^2.
\] (3.29)

Using the definition of \( G \) leads to
\[
\left( \frac{1}{2} - \gamma \right) \int_{\mathbb{R}} v_y^2 \frac{G^2}{\varrho_0^2} dy = \left( \frac{1}{2} - \gamma \right) \int_{\mathbb{R}} \frac{J}{\mu} (G + \pi) \frac{G^2}{\varrho_0^2} dy 
\leq \frac{1 - 2\gamma}{2\mu} \int_{\mathbb{R}} \frac{J G^3}{\varrho_0^2} dy \leq \frac{\gamma}{\mu} \| G \|_\infty \left( \left\| \frac{J}{\varrho_0^2} \right\|_2^2 + 1 \right),
\] (3.30)

here, we have used the fact that \( \gamma > 1 \) and \( \pi \geq 0 \). Plugging (3.29) and (3.30) into (3.28) yields
\[
\frac{d}{dt} \left( \left\| \sqrt{\frac{J}{\varrho_0^2}} G \right\|_2^2 + \frac{\mu}{\varrho_0^2 + 1} \left\| \frac{G_y}{\varrho_0^2 + 1} \right\|_2^2 \right) \leq C(\gamma, \mu, \delta, K_0)(1 + \| G \|_\infty) \left( \left\| \sqrt{\frac{J}{\varrho_0^2}} G \right\|_2^2 + 1 \right),
\]
from which, by Corollary 3.1 and the Gronwall inequality, we have
\[
\sup_{0 \leq t \leq T_0} \left\| \frac{G}{\varrho_0^2} \right\|_2^2 + \int_0^{T_0} \left\| \frac{G_y}{\varrho_0^2 + 1} \right\|_2^2 dt \leq C,
\]
for a positive constant \( C \) depending only on \( \gamma, \mu, \varrho_0, J, K_0, \| \frac{J}{\varrho_0^2} \|_2, \| v_0' \|_2, \| \pi_0 \|_2, \| \frac{\varrho_0}{\sqrt{\varrho_0}} \|_2 \), and \( \| \frac{\varrho_0^2}{\varrho_0} G_0 \|_2^2 \) but independent of \( \varrho_0 \) \( \square \)

4. LOCAL EXISTENCE IN THE PRESENCE OF FAR FIELD VACUUM

In this section, we prove the local existence and uniqueness of strong solutions to system (2.5)–(2.7), subject to (2.9), in the presence of far field vacuum, in other words and, thus, prove Theorem 2.1.

We start with the uniqueness of the solutions.

**Proposition 4.1.** Given a function \( \varrho_0 \) satisfying \( \inf_{y \in (-R, R)} \varrho_0(y) > 0 \), for any \( R \in (0, \infty) \), and \( \varrho_0 \leq \bar{\varrho} \) on \( \mathbb{R} \), for a positive constant \( \bar{\varrho} \). Let \((J_1, v_1, \pi_1)\) and \((J_2, v_2, \pi_2)\) be two solutions to system (2.5)–(2.7), subject to the same initial data, on \( \mathbb{R} \times (0, T) \), satisfying \( c_0 \leq J_i \leq C_0 \) on \( \mathbb{R} \times (0, T) \), for two positive numbers \( c_0 \) and \( C_0 \), and

\[
\pi_i \in L^2(0, T; L^\infty), \quad (\partial_t J_i, \partial_t \pi_i) \in L^1_{\mathrm{loc}}(\mathbb{R} \times [0, T]),
\]

\[
(\sqrt{\varrho_0 v}, \partial_y J_i, \partial_y \pi_i) \in L^\infty(0, T; L^2), \quad (\sqrt{\varrho_0} \partial_t v_i, \partial_y v_i, \partial_y^2 v_i) \in L^2(0, T; L^2),
\]

for \( i = 1, 2 \). Then \((J_1, v_1, \pi_1) \equiv (J_2, v_2, \pi_2)\) on \( \mathbb{R} \times [0, T] \).
\textbf{Proof.} Define \((J, v, \pi)\) as
\[
J = J_1 - J_2, \quad v = v_1 - v_2, \quad \pi = \pi_1 - \pi_2.
\]
Then, \((J, v, \pi)\) satisfies
\[
\begin{align*}
\partial_t J &= \partial_y v, \\
\rho_0 \partial_t v - \mu \partial_y \left( \frac{\partial_y v}{J_1} \right) + \partial_y \left( \pi + \frac{\alpha}{J_1} J \right) &= 0, \\
\partial_t \pi + \gamma \beta \pi &= \chi (\partial_y v - \alpha J),
\end{align*}
\]
where \(\alpha = \alpha(y, t), \beta = \beta(y, t),\) and \(\chi = \chi(y, t)\) are given functions as follows
\[
\alpha(y, t) = \frac{\partial_y v_2}{J_2}, \quad \beta(y, t) = \frac{\partial_y v_1}{J_1}, \quad \chi(y, t) = \left[ (\gamma - 1) \mu \left( \frac{\partial_y v_1}{J_1^2} + \frac{\partial_y v_2}{J_1 J_2} \right) - \gamma \frac{\pi_2}{J_1} \right].
\]
Due to the regularities of \((J_i, v_i, \pi_i), i = 1, 2,\) it holds that
\[
(\alpha, \beta, \chi) \in L^2(0, T; L^2), \quad (\alpha, \beta, \chi) \in L^2(0, T; L^\infty).
\]
Choose a function \(\eta \in C^\infty_c((-2, 2)),\) with \(\eta \equiv 1\) on \((-1, 1),\) and \(0 \leq \eta \leq 1\) on \((-2, 2)\). For each \(r \geq 1,\) we set \(\eta_r(y) = \eta(\frac{y}{r}),\) for \(y \in \mathbb{R}.\) Multiplying equations (4.1), (4.2), and (4.3), respectively, by \(J\eta_r^2, v\eta_r^2,\) and \(\pi \eta_r^2,\) summing the resultants up, and integrating over \(\mathbb{R},\) one gets from integration by parts that
\[
\frac{1}{2} \frac{d}{dt} \int_\mathbb{R} (J^2 + \rho_0 v^2 + \pi^2) \eta_r^2 dy + \mu \int_\mathbb{R} \left( \frac{\partial_y v}{J_1} \right)^2 \eta_r^2 dy
\]
\[
= \int_\mathbb{R} \left[ \partial_y v J + \left( \pi + \frac{\alpha}{J_1} J \right) \partial_y v + \chi (\partial_y v - \alpha J) \pi - \gamma \beta \pi^2 \right] \eta_r^2 dy
\]
\[
+ 2 \int_\mathbb{R} \left( \pi + \frac{\alpha}{J_1} J - \mu \frac{\partial_y v}{J_1} \right) v \eta_r \eta'_r dy.
\]
Note that
\[
\int_\mathbb{R} \left[ \partial_y v J + \left( \pi + \frac{\alpha}{J_1} J \right) \partial_y v + \chi (\partial_y v - \alpha J) \pi - \gamma \beta \pi^2 \right] \eta_r^2 dy
\]
\[
\leq \frac{\mu}{2} \int_\mathbb{R} \left( \frac{v_y}{J_1} \right)^2 \eta_r^2 dy + C \int_\mathbb{R} (J^2 + \pi^2 + \alpha^2 J^2 + \chi^2 \pi^2 + |\beta| \pi^2) \eta_r^2 dy
\]
\[
\leq \frac{\mu}{2} \int_\mathbb{R} \left( \frac{v_y}{J_1} \right)^2 \eta_r^2 dy + C(1 + \|(\alpha, \beta, \chi)\|_\infty^2) \int_\mathbb{R} (J^2 + \pi^2) \eta_r^2 dy,
\]
so
\[
\frac{d}{dt} \int_\mathbb{R} (J^2 + \rho_0 v^2 + \pi^2) \eta_r^2 dy \leq C(1 + \|(\alpha, \beta, \chi)\|_\infty^2) \int_\mathbb{R} (J^2 + \pi^2) \eta_r^2 dy
\]
\[
+ C \int_\mathbb{R} (|\pi| + |\alpha| |J| + |\partial_y v|) v \eta_r \eta'_r dy.
\]
It follows from this, the Gronwall inequality, and (4.4) that
\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}} (J^2 + \varrho_0 \nu^2 + \pi^2) \eta^2_r dy \leq C \int_0^T \int_{\mathbb{R}} (|\pi| + |\alpha||J| + |\partial_y v||v||\eta^r|| dy dt =: Q_r.
\]
In order to prove the conclusion, it suffices to show that \(Q_r\) tends to zero as \(r \to \infty\).

Note that
\[
v(y, t) = v(z, t) + \int_z^y \partial_y v(y', t) dy', \quad 0 < z < 1 < y < \infty.
\]
Integrating the above identity with respect to \(z\) over the interval \((0, 1)\), and denoting \(D := \sup_{y \in (-1, 1)} \frac{1}{\varrho_0(y)}\), one obtains by the H"older inequality that
\[
|v(y, t)| \leq \left| \int_0^1 v(z, t) dz + \int_0^y \partial_y v(y', t) dy' dz \right| \leq \int_0^1 |v| dz + \int_0^y |\partial_y v| dy' \leq \frac{1}{\sqrt{D}} \int_0^1 |\sqrt{\varrho_0}v| dz + \int_0^y |\partial_y v| dy' \leq \frac{1}{\sqrt{D}} \|\sqrt{\varrho_0}v\|_2 + \sqrt{y} \|\partial_y v\|_2 \leq C \sqrt{y}, \quad \forall y \geq 1.
\]

In the same way, one has \(|v(y, t)| \leq C \sqrt{|y|}, \text{ for any } y \leq -1 \text{ and, thus,}
\]
\[
|v(y, t)| \leq C \sqrt{|y|}, \quad \forall y \in (-\infty, -1] \cup [1, \infty).
\] (4.5)

Due to (4.1), one has
\[
\sup_{0 \leq t \leq T} \|J\|_2 \leq \int_0^T \|\partial_y v\|_2 d\tau < \infty,
\]
thanks to which, by solving (4.3) as
\[
\pi = e^{-\gamma} \int_0^T \beta d\tau \int_0^t e^{\gamma} \int_0^T \delta d\tau' \chi(\partial_y v - \alpha J) d\tau,
\]
and recalling (4.4), one obtains from the Cauchy inequality that
\[
\sup_{0 \leq t \leq T} \|\pi\|_2 \leq e^{\gamma} \int_0^T \|\beta\|_\infty d\tau \int_0^T \|\chi\|_\infty ((\|\partial_y v\|_2 + \|\alpha\|_\infty \|J\|_2)) d\tau \\
\leq C \int_0^T (\|\chi\|_\infty^2 + \|\partial_y v\|_2^2 + \|\alpha\|_\infty^2 + 1) d\tau < \infty. \quad (4.6)
\]

Thanks to (4.5) and (4.6), noticing that \(\text{supp } \eta^r \subseteq (-2r, -r) \cup (r, 2r)\), and \(\|\eta^r\| \leq \frac{C}{r}\),
one obtains by the H"older inequality that
\[
Q_r = C \int_0^T \int_{\mathbb{R}} (|\pi| + |\alpha||J| + |\partial_y v||v||\eta^r|| dy dt
\]
for any $r \geq 1$, which, together with the fact that $\pi, \alpha, \partial_y v \in L^2(\mathbb{R} \times (0, T))$, implies $Q_r \to 0$, as $r \to \infty$.

The proof is complete. \hfill \Box

We are now ready to give the proof of Theorem 2.1.

**Proof of Theorem 2.1.** (i) Since the uniqueness is a direct corollary of Proposition 4.1, it remains to prove the existence. For any positive number $\varepsilon \in (0, 1)$, set $\varrho_0(y) = \varrho_0(y) + \varepsilon$, for $y \in \mathbb{R}$. It is clear that $\varepsilon \leq \varrho_0(y) \leq \bar{\varrho} + 1$, for $y \in \mathbb{R}$. Consider the following approximate system of (2.5)–(2.7):

$$
\begin{cases}
J_t = v_y, \\
\varrho_0 v_t - \mu \left( \frac{v_y}{f} \right)_y + \pi_y = 0, \\
\pi_t + \gamma \frac{\varrho_0}{f} \pi = (\gamma - 1) \mu \left( \frac{v_y}{f} \right)^2.
\end{cases}
$$

(4.7)

By Corollary 3.1 there is a positive constant $T$ depending only on $\gamma, \mu, \bar{\varrho}, J, \bar{J}, ||v'||_2, ||\pi_0||_2$, and $||\pi_0||_\infty$, but independent of $\varepsilon$, such that system (4.7), subject to (2.9), has a unique solution $(J_\varepsilon, v_\varepsilon, \pi_\varepsilon)$, satisfying

$$
\pi_\varepsilon \geq 0, \quad \frac{3}{4} L \leq J_\varepsilon \leq \frac{5}{4} \bar{J}, \quad \text{on } \mathbb{R} \times [0, T],
$$

(4.8)

$$
\sup_{0 \leq t \leq T} \left\| \left( J_\varepsilon - J_0, \frac{\partial_0 J_\varepsilon}{\varrho_0}, \frac{\partial_0 v_\varepsilon}{\varrho_0}, \frac{\partial_0 \pi_\varepsilon}{\varrho_0} \right) \right\|_2^2 \leq C,
$$

(4.9)

$$
\sup_{0 \leq t \leq T} \| \sqrt{\varrho_0} v_\varepsilon \|_{L^2((-R, R))} \leq \| \sqrt{\varrho_0} v_0 \|_{L^2((-R, R))} + C,
$$

(4.10)

for any $0 < R < \infty$, and

$$
\int_0^T \left( \left\| \left( \frac{\partial_y^2 v_\varepsilon}{\varrho_0}, \sqrt{\varrho_0} \partial_y v_\varepsilon \right) \right\|^2 + \left\| \partial_t \pi_\varepsilon \right\|^2 \right) dt \leq C,
$$

(4.11)

for a positive constant $C$ independent of $\varepsilon$.

Due to the a priori estimates (4.9)–(4.11), by the Banach-Alaoglu theorem, and using Cantor’s diagonal argument, there is a subsequence, still denoted by $(J_\varepsilon, v_\varepsilon, \pi_\varepsilon)$, and a triple $(J, v, \pi)$, such that

$$
J_\varepsilon - J_0 \to J - J_0, \quad \text{weak-* in } L^\infty(0, T; H^1),
$$

(4.12)
\[ \partial_t J_\varepsilon \to J_t, \quad \text{weak-* in } L^\infty(0, T; L^2), \quad (4.13) \]
\[ v_\varepsilon \to v, \quad \text{weakly in } L^2(0, T; H^2((-R, R))), \quad (4.14) \]
\[ \partial_t v_\varepsilon \to v_t, \quad \text{weakly in } L^2(0, T; L^2((-R, R))), \quad (4.15) \]
\[ \partial_y v_\varepsilon \to v_y, \quad \text{weak-* in } L^\infty(0, T; L^2) \text{ and weakly in } L^2(0, T; H^1), \quad (4.16) \]
\[ \pi_\varepsilon \to \pi, \quad \text{weak-* in } L^\infty(0, T; H^1), \quad (4.17) \]
\[ \partial_t \pi_\varepsilon \to \pi_t, \quad \text{weakly in } L^4(0, T; L^2), \quad (4.18) \]

for any \( R \in (0, \infty), \) and

\[ J - J_0 \in L^\infty(0, T; H^1), \quad J_t \in L^\infty(0, T; L^2), \quad (4.19) \]
\[ v_y \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1), \quad (4.20) \]
\[ \pi \in L^\infty(0, T; H^1), \quad \pi_t \in L^4(0, T; L^2). \quad (4.21) \]

It remains to prove that \((J, v, \pi)\) is a strong solution to system (2.5)–(2.7), subject to (2.9), on \( \mathbb{R} \times (0, T). \) One can verify that \((J, v, \pi)\) has the regularities stated in Definition 2.1. Other desired regularities of \((J, v, \pi), \) beyond those in (4.19)–(4.21), are verified as follows. First, thanks to (4.19), (4.21), and

\[ Y := \{ f | f \in L^\infty(0, T; L^2), f' \in L^1(0, T; L^2) \} \hookrightarrow C([0, T]; L^2), \]

it is clear that \((J - J_0, \pi) \in C([0, T]; L^2). \) Next, noticing that \( \sqrt{\partial_0 \varepsilon} \to \sqrt{\partial_0} \) and \( \frac{1}{\sqrt{\partial_0 \varepsilon}} \to \frac{1}{\sqrt{\partial_0}}, \) in \( L^\infty((-R, R)), \) for any \( R \in (0, \infty), \) one gets from (4.12), (4.14)–(4.15), and (4.17) that

\[ \left( \frac{\partial_y J_\varepsilon}{\sqrt{\partial_0 \varepsilon}}, \frac{\partial_y \pi_\varepsilon}{\sqrt{\partial_0 \varepsilon}} \right) \to \left( \frac{J_y}{\sqrt{\partial_0}}, \frac{\pi_y}{\sqrt{\partial_0}} \right), \quad \text{weak-* in } L^\infty(0, T; L^2((-R, R))), \]
\[ \left( \frac{\partial^2 v_\varepsilon}{\sqrt{\partial_0 \varepsilon}}, \sqrt{\partial_0 \varepsilon} \partial_t v_\varepsilon \right) \to \left( \frac{v_{yy}}{\sqrt{\partial_0}}, \sqrt{\partial_0} \partial_t v \right), \quad \text{weakly in } L^2((-R, R) \times (0, T)), \]

for any \( R \in (0, \infty). \) Consequently, it follows from the weakly lower semi-continuity of the norms, (4.19), and (4.11) that

\[ \left\| \left( \frac{J_y}{\sqrt{\partial_0}}, \frac{\pi_y}{\sqrt{\partial_0}} \right) \right\|_{L^\infty(0, T; L^2((-R, R)))} \leq \lim_{\varepsilon \to 0} \left\| \left( \frac{\partial_y J_\varepsilon}{\sqrt{\partial_0 \varepsilon}}, \frac{\partial_y \pi_\varepsilon}{\sqrt{\partial_0 \varepsilon}} \right) \right\|_{L^\infty(0, T; L^2((-R, R)))} \leq C, \]
\[ \left\| \left( \frac{v_{yy}}{\sqrt{\partial_0}}, \sqrt{\partial_0} \partial_t v \right) \right\|_{L^2(0, T; L^2((-R, R)))} \leq \lim_{\varepsilon \to 0} \left\| \left( \frac{\partial^2 v_\varepsilon}{\sqrt{\partial_0 \varepsilon}}, \sqrt{\partial_0 \varepsilon} \partial_t v_\varepsilon \right) \right\|_{L^2(0, T; L^2((-R, R)))} \leq C, \]

for a positive constant \( C \) independent of \( R. \) Therefore,

\[ \left( \frac{J_y}{\sqrt{\partial_0}}, \frac{\pi_y}{\sqrt{\partial_0}} \right) \in L^\infty(0, T; L^2), \quad \left( \sqrt{\partial_0} v_t, \frac{v_{yy}}{\sqrt{\partial_0}} \right) \in L^2(0, T; L^2). \]

And finally, since \( \sqrt{\partial_0} v_t \in L^2(0, T; L^2), \) then \( \sqrt{\partial_0} v \in C([0, T]; L^2). \) Combining all the regularities obtained in the above, we can see that \((J, v, \pi)\) meet the required regularities in Definition 2.1.
Next, we show that \( \pi \geq 0 \), \( J \) has a uniform positive lower bound on \( \mathbb{R} \times (0, T) \), and that \((J, v, \pi)\) fulfills the initial condition \((2.9)\). Thanks to \((4.12)-(4.15), (4.17)-(4.18)\), the Aubin-Lions compactness lemma, and Cantor’s diagonal argument again, there is a subsequence, still denoted by \((J_\varepsilon, v_\varepsilon, \pi_\varepsilon)\), such that

\[
J_\varepsilon \to J, \quad \text{in } C([0, T]; L^2((-R, R))), \quad (4.22)
\]

\[
v_\varepsilon \to v, \quad \text{in } C([0, T]; L^2((-R, R))) \cap L^2(0, T; H^1((-R, R))), \quad (4.23)
\]

\[
\pi_\varepsilon \to \pi, \quad \text{in } C([0, T]; L^2((-R, R))), \quad (4.24)
\]

for any \( R \in (0, \infty) \). It follows from \((4.22), (4.24)\), and \((4.8)\) that \( \pi \geq 0 \) and \( \frac{3}{4} J \leq J \leq \frac{5}{4} J \) on \( \mathbb{R} \times [0, T] \). Moreover, \((4.22)-(4.24)\) guarantees that \((J, v, \pi)\) fulfills the initial condition \((2.9)\).

And finally, we prove that \((J, v, \pi)\) satisfies equations \((2.5)-(2.7)\). Thanks to \((4.12)-(4.18)\) and \((4.22)-(4.24)\), by taking \( \varepsilon \to 0^+ \) to system \((2.7)\), one can see that \((J, v, \pi)\) satisfies equations \((2.5)-(2.7)\). Therefore, \((J, v, \pi)\) is a strong solution to system \((2.5)-(2.7)\), subject to \((2.9)\), on \( \mathbb{R} \times (0, T) \), which proves (i).

(ii) We first prove the regularities of \( G \), i.e., \((2.10)\). Let \( \varrho_0, (v_\varepsilon, J_\varepsilon, \pi_\varepsilon) \), and \( T \) be the same as in (i). Then, \( \frac{3}{4} J \leq J_\varepsilon \leq \frac{5}{4} J \) on \( \mathbb{R} \times (0, T) \), and \((4.12)-(4.18)\) and \((4.22)-(4.24)\) hold. It is clear that

\[
\left| \left( \frac{1}{\sqrt{\varrho_\varepsilon}} \right)' \right| = \frac{1}{2} \left| \frac{\varrho_\varepsilon'}{\varrho_\varepsilon} \right| = 1 \left| \frac{\varrho_\varepsilon'}{\varrho_\varepsilon} \right| = \frac{1}{2} \left| \frac{\varrho_\varepsilon'}{\varrho_\varepsilon} \right| = \left( \frac{1}{\sqrt{\varrho}} \right)' \leq \frac{K_0}{2}, \quad \forall y \in \mathbb{R}. 
\]

Therefore, one can apply Proposition \(3.5\) to get

\[
\sup_{0 \leq t \leq T} \left\| \frac{G_\varepsilon}{\varrho_\varepsilon} \right\|_2^2 + \int_0^T \left\| \frac{\partial_y G_\varepsilon}{\varrho_\varepsilon} \right\|_2^2 \, dt \leq C, \quad (4.25)
\]

for a positive constant \( C \) independent of \( \varepsilon \), where \( G_\varepsilon := \mu \frac{\varrho_0 v_\varepsilon}{J_\varepsilon} - \pi_\varepsilon \).

Recalling \( \frac{3}{4} J \leq J_\varepsilon \leq \frac{5}{4} J \), and using \((4.12), (4.16)-(4.17), (4.12)-(4.23)\), one can show that

\[
G_\varepsilon \to G, \quad \text{weak-* in } L^\infty(0, T; L^2((-R, R))),
\]

\[
\partial_y G_\varepsilon \to G_y, \quad \text{weakly in } L^2(0, T; L^2((-R, R))),
\]

for any \( R \in (0, \infty) \). Therefore, noticing that \( \frac{1}{\varrho_\varepsilon} \to \frac{1}{\varrho_0} \) in \( L^\infty((-R, R)) \), for any \( R \in (0, \infty) \), it is easily to verify that

\[
\frac{G_\varepsilon}{\varrho_\varepsilon} \to \frac{G}{\varrho_0}, \quad \text{weak-* in } L^\infty(0, T; L^2((-R, R))),
\]

\[
\frac{\partial_y G_\varepsilon}{\varrho_\varepsilon} \to \frac{G_y}{\varrho_0}, \quad \text{weakly in } L^2(0, T; L^2((-R, R))),
\]

for any \( R \in (0, \infty) \).
Due to (4.26), (4.27), and the weakly lower semi-continuity of the norms, it follows from (4.25) that $\frac{G}{\varrho_0}$ and $\frac{G_y}{\varrho_0^{\frac{3}{2}}}$, respectively, are bounded in $L^\infty(0, T; L^2(\langle -r, r \rangle))$ and $L^2(0, T; L^2(\langle -r, r \rangle))$, uniformly in $r \in (0, \infty)$, and, consequently, it holds that

$$\frac{G}{\varrho_0^3} \in L^\infty(0, T; L^2), \quad \frac{G_y}{\varrho_0^{\frac{3}{2}}} \in L^2(0, T; L^2).$$

Thanks to these regularities of $G$, it follows from the Gagliardo-Nirenberg inequality $\|f\|_{L^\infty(\mathbb{R})} \leq C \|f\|_{L^2(\mathbb{R})}^{\frac{2}{3}} \|f'\|_{L^2(\mathbb{R})}^{\frac{1}{3}}$, for $f \in H^1(\mathbb{R})$, and the assumption $|\left(\frac{1}{\sqrt{\varrho_0}}\right)'| \leq \frac{K_0}{2}$, or equivalently $|\varrho_0'| \leq K_0\varrho_0^{\frac{3}{2}}$, that

$$\int_0^T \left\| \frac{G}{\varrho_0^3} \right\|_\infty^4 dt \leq C \int_0^T \left\| \frac{G}{\varrho_0^3} \right\|_2^2 \left( \left\| \frac{G_y}{\varrho_0^3} \right\|_2^2 \right) dt$$

$$= C \int_0^{T_0} \left\| \frac{G}{\varrho_0^3} \right\|_2^2 \left( \left\| \frac{G_y}{\varrho_0^3} \right\|_2^2 - \frac{\delta \varrho_0'}{2 \varrho_0^{\frac{3}{2}}} \right) dt$$

$$\leq C \int_0^T \left\| \frac{G}{\varrho_0^3} \right\|_2^2 \left( \left\| \frac{G_y}{\varrho_0^{\frac{3}{2}}} \right\|_2^2 \right) + \delta^2 K_0^2 \left\| \frac{G}{\varrho_0^{\frac{3}{2}}} \right\|_2^2 dt$$

$$\leq C \int_0^T \left\| \frac{G}{\varrho_0^3} \right\|_2^2 \left( \left\| \frac{G_y}{\varrho_0^{\frac{3}{2}}} \right\|_2^2 + \left\| \frac{G}{\varrho_0^{\frac{3}{2}}} \right\|_2^2 \right) dt < \infty$$

and, thus, $\frac{G}{\varrho_0^3} \in L^4(0, T; L^\infty)$.

Next, we show the regularity that $v \in L^\infty(0, T; H^1)$, under the assumption that in this case $\delta \geq 1$ and $v_0 \in H^1$. Noticing that $\frac{G_x}{\varrho_0} \in L^2(0, T; L^2)$ and $\varrho_0 v_t = G_y$, it is straightforward that $v_t \in L^2(0, T; L^2)$ and, consequently, recalling $v_0 \in L^2$, one obtains $v \in L^\infty(0, T; L^2)$, and further $v \in L^\infty(0, T; H^1)$.

We verify the regularity $\vartheta \in L^\infty(0, T; H^1)$, under the assumption that $\delta \geq 1$ and $\vartheta_0 \in H^1$, as follows. In this case, one has

$$\frac{G}{\sqrt{\varrho_0}} \in L^\infty(0, T; L^2) \cap L^4(0, T; L^\infty), \quad \frac{G_y}{\varrho_0} \in L^2(0, T; L^2)$$

and, by the assumption $|\left(\frac{1}{\sqrt{\varrho_0}}\right)'| \leq \frac{K_0}{2}$, or equivalently $|\varrho_0'| \leq K_0\varrho_0^{\frac{3}{2}}$, one can verify that $\varrho_0^{\frac{3}{2}} \in L^2$.

Recalling that

$$\pi_t + \frac{1}{\mu} \left( \pi + \frac{2 - \gamma}{2} G \right)^2 = \frac{\gamma^2}{4\mu} G^2,$$

(4.28)
one obtains
\[
\sup_{0 \leq t \leq T} \left\| \frac{\pi}{\varrho_0} \right\|_2 \leq \left\| \frac{\pi_0}{\varrho_0} \right\|_2 + \frac{\gamma^2}{4\mu} \int_0^T \left\| \frac{G}{\sqrt{\varrho_0}} \right\|_2 \left\| \frac{G}{\sqrt{\varrho_0}} \right\|_2 \, dt < \infty
\]
and, thus, \( \frac{\pi}{\varrho_0} \in L^\infty(0, T; L^2) \). Therefore, recalling \( J \in L^\infty(0, T; L^\infty) \), we have
\[
\vartheta := \frac{\pi}{R \varrho} = \frac{J \pi}{R \varrho_0} \in L^\infty(0, T; L^2), \quad \text{where} \quad \varrho := \frac{\varrho_0}{J}.
\]
Differentiating (4.28) in \( y \), multiplying the resultant by \( \frac{\pi}{\varrho_0} \), and integrating over \( \mathbb{R} \), one gets from the Hölder and Cauchy inequalities, and (3.13) that
\[
\frac{d}{dt} \left\| \frac{\pi_y}{\varrho_0} \right\|^2_2 \leq C(\gamma; \mu) \left( \left\| \pi \right\|_\infty + \left\| G \right\|_\infty \right) \left( \left\| \frac{\pi_y}{\varrho_0} \right\|_2 + \left\| \frac{G_y}{\varrho_0} \right\|_2 \right) \left\| \frac{\pi_y}{\varrho_0} \right\|_2 \leq C(\gamma; \mu) \left( \left\| \frac{G_y}{\varrho_0} \right\|^2_2 + \left( 1 + \left\| \pi \right\|^2_\infty + \left\| G \right\|^2_\infty \right) \left\| \frac{\pi_y}{\varrho_0} \right\|^2_2 \right),
\]
from which, by the Gronwall inequality, we have \( \sup_{0 \leq t \leq T} \left\| \frac{\pi_y}{\varrho_0} \right\|^2_2 < \infty \) and, thus, \( \frac{\pi_y}{\varrho_0} \in L^\infty(0, T; L^2) \).

Rewrite equation (2.3) in terms of \( G \) as \( J_t = \frac{J}{\pi}(G + \pi) \), from which, one obtains
\[
J_y(t, y) = \left( \frac{1}{\mu} \int_0^t (G_y + \pi_y) \, dJ_0 + J_0' \right) \tau \exp \left\{ \frac{1}{\mu} \int_0^t (G + \pi) \, d\tau \right\}.
\]
Hence,
\[
\sup_{0 \leq t \leq T} \left\| \frac{J_y}{\varrho_0} \right\|_2 \leq \left( \frac{J}{\mu} \int_0^T \left\| \frac{G_y}{\varrho_0} \right\|_2 + \left\| J_0' \right\|_2 \right) e^{\frac{1}{\mu} \int_0^t \left\| (G, \pi) \right\|_\infty \, dt} < \infty,
\]
that is \( \frac{J_y}{\varrho_0} \in L^\infty(0, T; L^2) \).

Thanks to the regularities \( \left( \frac{\pi}{\varrho_0}, \frac{\pi_y}{\varrho_0}, \frac{J_y}{\varrho_0} \right) \in L^\infty(0, T; L^2) \), noticing that \( (J_t, \pi) \in L^\infty(0, T; L^\infty) \), and recalling the assumption \( \left| \left( \frac{\pi}{\varrho_0} \right)' \right| \leq \frac{K_0}{2} \), or equivalently \( |\varrho_0| \leq K_0 \varrho_0^3 \), we have
\[
\vartheta_y = \frac{1}{R} \left( \frac{J \pi_y}{\varrho_0} + \frac{J_y \pi}{\varrho_0} - \frac{\varrho_0 J \pi}{\varrho_0^3} \right) \in L^\infty(0, T; L^2).
\]

It remains to prove that \( s \in L^\infty(0, T; L^\infty) \), under the assumption that \( s_0 \in L^\infty \) and \( \delta \geq \gamma \). To this end, by the definition of \( s \), it suffice to verify that \( \frac{\pi}{\varrho_0} = \frac{J \pi}{\varrho_0} \)
has uniform positive lower and upper bounds on \( \mathbb{R} \times (0, T) \). Since \( \delta \geq \gamma \), it follows from (2.10) that \( \frac{G}{\varrho_0^2} \in L^4(0, T; L^\infty) \). To show the boundedness from the above of \( \frac{J \pi}{\varrho_0^3} \),
Proposition 5.1. recalling that $J \in \left[\frac{3}{4}, \frac{5}{4}\right]$ on $\mathbb{R} \times [0, T]$, we need only to verify that of $\frac{\pi_\nu}{\varrho_0}$. (4.28) implies that

$$\sup_{0 \leq t \leq T} \left\| \frac{\pi_\nu}{\varrho_0} \right\|_\infty \leq \left\| \frac{\pi_0}{\varrho_0} \right\|_\infty + \frac{\gamma^2}{4\mu} \int_0^T \left\| \frac{G_\gamma}{\varrho_0} \right\|_\infty^2 \, dt < \infty$$

and, thus, $\frac{\pi_\nu}{\varrho_0}$ has a uniform upper bound on $\mathbb{R} \times (0, T)$. Concerning the uniform positive lower bound, by (2.5) and (2.7), one has

$$(J_\gamma \pi)_t = (\gamma - 1)\mu J_\gamma^{-2}(v_y)^2 \geq 0$$

and, thus, $J_\gamma(y, t)\pi(y, t) = J_\gamma(0)(y)\pi_0(y) = \pi_0(y)$, which leads to

$$\inf_{y \in \mathbb{R}, t \in [0, T]} J_\gamma(y, t)\pi(y, t) \geq \inf_{y \in \mathbb{R}} \frac{\pi_0(y)}{\varrho_0^2(y)} > 0.$$ 

Hence, $\frac{\pi_\nu}{\varrho_0}$ has a uniform positive lower bound on $\mathbb{R} \times (0, T)$. $\square$

5. Global existence in the presence of far field vacuum

This section is devoted to proving the global existence of strong solutions to system (2.5)–(2.7), subject to (2.9), which proves Theorem 2.2. Throughout this section, it is always set $J_0 \equiv 1$.

As preparations, several a priori estimates are stated in the next propositions. We start with the basic energy identity of a strong solution to system (2.5)–(2.7), subject to (2.9).

**Proposition 5.1.** Given a positive time $T$, and let $(J, v, \pi)$ be a strong solution to system (2.5)–(2.7), subject to (2.9), on $\mathbb{R} \times (0, T)$, with $(\varrho_0, v_0, \pi_0)$ satisfying (H1), (H2), and (H4). Then, it holds that

$$\int_\mathbb{R} \left( \frac{\varrho_0 v^2}{2} + \frac{J_\pi}{\gamma - 1} \right) \, dy = \int_\mathbb{R} \left( \frac{\varrho_0 v_0^2}{2} + \frac{\pi_0}{\gamma - 1} \right) \, dy, \quad t \in [0, T].$$

**Proof.** Take a nonnegative function $\eta \in C_0^\infty((-2, 2))$ such that $\eta \equiv 1$ on $[-1, 1]$ and $0 \leq \eta \leq 1$ on $(-2, 2)$. For each $r \in (0, \infty)$, we define a function $\eta_r$ as $\eta_r(\cdot) = \eta(\frac{\cdot}{r})$. Multiplying (2.6) and (2.7), respectively, by $v_\eta^2$ and $J_\eta^2$, and integrating the resultants over $\mathbb{R}$, one gets from integration by parts that

$$\frac{1}{2} \frac{d}{dt} \int_\mathbb{R} \varrho_0 v^2 \eta_r^2 \, dy + \mu \int_\mathbb{R} \frac{v_\eta^2}{J} \, dy = \int_\mathbb{R} \pi v_\eta^2 \eta_r^2 \, dy + 2 \int_\mathbb{R} (\pi v - \frac{v_\nu v}{J}) \eta_r \eta_r^2 \, dy. \quad (5.1)$$

and

$$\int_\mathbb{R} J_\pi v_\eta^2 \eta_r^2 \, dy + \gamma \int_\mathbb{R} \pi v_\eta^2 \eta_r^2 \, dy = \frac{J_\gamma}{\gamma - 1} \mu \int_\mathbb{R} \frac{v_\eta^2}{J} \eta_r \, dy. \quad (5.2)$$

(2.5) implies that

$$\int_\mathbb{R} J_\pi v_\eta^2 \eta_r^2 \, dy = \frac{d}{dt} \int_\mathbb{R} J_\pi \eta_r^2 \, dy - \int_\mathbb{R} J_\pi \pi \eta_r^2 \, dy = \frac{d}{dt} \int_\mathbb{R} J_\pi \eta_r^2 \, dy - \int_\mathbb{R} v_\eta \pi \eta_r^2 \, dy.$$
which, plugged in to (5.2) yields,
\[
\frac{d}{dt} \int_{\mathbb{R}} J \pi \eta_r^2 dy + (\gamma - 1) \int_{\mathbb{R}} v_y \pi \eta_r^2 dy = (\gamma - 1) \mu \int_{\mathbb{R}} \frac{v_y^2 \eta_r^2}{J} dy.
\]
This, together with (5.1), yields
\[
\frac{d}{dt} \int_{\mathbb{R}} \left( \frac{\rho_0 v^2}{2} + \frac{J \pi}{\gamma - 1} \right) \eta_r^2 dy = 2 \int_{\mathbb{R}} \left( \pi - \mu \frac{v_y}{J} \right) v_y \eta_r \eta_r' dy,
\]
which implies
\[
\int_{\mathbb{R}} \left( \frac{\rho_0 v^2}{2} + \frac{J \pi}{\gamma - 1} \right) \eta_r^2 dy = 2 \int_0^t \int_{\mathbb{R}} \left( \pi - \mu \frac{v_y}{J} \right) v_y \eta_r \eta_r' dy d\tau
\]
\[
+ \int_{\mathbb{R}} \left( \frac{\rho_0 v_0^2}{2} + \frac{\pi_0}{\gamma - 1} \right) \eta_r^2 dy,
\]
(5.3)
for any \( t \in [0, T] \). Noticing that \( |\eta_r'| \leq \frac{||\eta'||_{\infty}}{r} \), by the assumption \( \rho_0(y) \geq \frac{A_0}{(1 + |y|)^2} \), one deduces
\[
|\eta_r'(y)| \leq \frac{||\eta'||_{\infty}}{r} \leq \frac{2||\eta'||_{\infty}}{|y|} \leq \frac{4||\eta'||_{\infty}}{|y| + 1} \leq \frac{4||\eta'||_{\infty}}{\sqrt{A_0}} \sqrt{\rho_0(y)}, \quad \forall \ 1 \leq r < |y| < 2r,
\]
from which, noticing that \( \text{supp} \ \eta_r' \subseteq (-2r, r) \cup (r, 2r) \), we obtains
\[
|\eta_r'(y)| \leq M_0 \sqrt{\rho_0(y)}, \quad \forall y \in \mathbb{R}, \quad \text{where} \quad M_0 = \frac{4||\eta'||_{\infty}}{\sqrt{A_0}},
\]
(5.4)
for any \( r \geq 1 \). Therefore, denoting \( \delta_T = \inf_{y \in \mathbb{R}, t \in [0, T]} J(y, t) \), and noticing that \( \sqrt{\rho_0 v \pi}, \sqrt{\rho_0 v y} \in L^1(\mathbb{R} \times (0, T)) \), we deduce
\[
\left| \int_0^t \int_{\mathbb{R}} \left( \pi - \mu \frac{v_y}{J} \right) v_y \eta_r \eta_r' dy d\tau \right| = \int_0^t \int_{r<|y|<2r} \left( \pi - \mu \frac{v_y}{J} \right) v_y \eta_r \eta_r' dy d\tau
\]
\[
\leq M_0 \int_0^t \int_{r<|y|<2r} \left( |\pi| + \frac{\mu}{\delta_T} |v_y| \right) |\sqrt{\rho_0 v}| dy d\tau \to 0, \quad \text{as} \quad r \to \infty,
\]
for any \( t \in [0, T] \). Thanks to this, and taking \( r \to \infty \) in (5.3), the conclusion follows.  \( \square \)

The next proposition yields the uniform positive lower bound of \( J \).

**Proposition 5.2.** Under the same assumptions as in Proposition 5.1, it holds that
\[
\inf_{y \in \mathbb{R}} J(y, t) \geq e^{-\frac{2v_T}{\mu} \sqrt{E_0}} \sqrt{\rho_0}, \quad t \in [0, T],
\]
where \( E_0 := \int_{\mathbb{R}} \left( \frac{\rho_0 v_0^2}{2} + \frac{\pi_0}{\gamma - 1} \right) dy \).
Proof. Due to (2.3), one rewrite (2.6) as
\[ \varrho_0 v_t - \mu (\log J)_{yt} + \pi_y = 0. \]
Thus,
\[ \varrho_0 v - \mu (\log J)_y + \int_0^t \pi_y d\tau = \varrho_0 v_0. \]
Integrating the above identity over the interval \((z_0, z)\) yields
\[ J(z, t) = J(z_0, t) \exp \left\{ -\frac{1}{\mu} \int_{z_0}^z \varrho_0 (v - v_0) dz \right\} \exp \left\{ -\frac{1}{\mu} \int_0^t (\pi - \pi(z_0, \tau)) d\tau \right\}, \tag{5.5} \]
for any \(z, z_0 \in \mathbb{R}\) and \(t \in (0, T_\infty)\). By Proposition 5.1, it follows from the Hölder inequality that
\[ \left| \int_{z_0}^z \varrho_0 (v - v_0) dy \right| \leq \| \varrho_0 \|_1 (\| \sqrt{\varrho_0 v} \|_2 + \| \sqrt{\varrho_0 v_0} \|_2) \leq 2 \sqrt{2E_0} \| \varrho_0 \|_1. \]
It then follows from this, \(\pi \geq 0\), and (5.5) that
\[ J(z, t) \geq \exp \left\{ -2\sqrt{2} \frac{\sqrt{E_0} \| \varrho_0 \|_1}{\mu} \right\} \exp \left\{ -\frac{1}{\mu} \int_0^t \pi(z_0, \tau) d\tau \right\} J(z_0, t), \tag{5.6} \]
for any \(z, z_0 \in \mathbb{R}\) and \(t \in (0, T_\infty)\).

Since \(v_y \in L^2(0, T; H^1)\), it is clear that, for any fixed \(t \in [0, T]\), \(\int_0^t v_y d\tau \in H^1\). Thanks to this, and using the fact that \(f(y) \to 0\), as \(y \to \infty\), for any \(f \in H^1\), one obtains from equation (2.3) that
\[ J(y, t) = 1 + \int_0^t v_y(y, \tau) d\tau \to 1, \quad \text{as } y \to \infty, \tag{5.7} \]
for any \(t \in [0, T]\). By the Sobolev embedding inequality, it follows from \(v_y \in L^2(0, T; H^1)\) that \(v_y \in L^2(0, T; L^\infty)\); therefore, recalling \(v_y \in L^\infty(0, T; L^2)\), it follows from the Hölder inequality that
\[ (v_y)^2 \in L^2(0, T; L^2), \quad v_y v_{yy} \in L^1(0, T; L^2), \]
which imply \(\int_0^t (v_y)^2 d\tau \in H^1\), for any fixed \(t \in [0, T]\). Thanks to this, and using again the fact that \(f(y) \to 0\), as \(y \to \infty\), for any \(f \in H^1\), one has
\[ \lim_{y \to \infty} \int_0^t |v_y|(y, \tau) d\tau \leq \sqrt{t} \left( \lim_{y \to \infty} \int_0^t (v_y)^2(y, \tau) d\tau \right)^{\frac{1}{2}} = 0, \]
for any \(t \in [0, T]\). Therefore, denoting \(\delta_T = \inf_{y \in \mathbb{R}, t \in (0, T)} J(y, t)\), and solving equation (2.7), we deduce
\[ \pi(y, t) = \exp \left\{ -\gamma \int_0^t \frac{v_y}{J} d\tau \right\} \left( \pi_0(y) + (\gamma - 1) \mu \int_0^t e^{\gamma \int_0^{\tau'} \frac{v_y}{J} d\tau'} (v_y)^2 d\tau' \right) \]
\[
\frac{d}{dt} \int_{\mathbb{R}} \varrho_0 v \eta_r \, dy = \int_{\mathbb{R}} \left( \pi - \mu \frac{v_y}{J} \right) \eta'_r \, dy, 
\]
which implies that
\[
\int_{\mathbb{R}} \varrho_0 v \eta_r \, dy = \int_{\mathbb{R}} \varrho_0 v_0 \eta_r \, dy + \int_{0}^{t} \int_{\mathbb{R}} \left( \pi - \mu \frac{v_y}{J} \right) \eta'_r \, dy \, d\tau. 
\] (5.9)

We claim that
\[
\lim_{r \to \infty} \int_{0}^{t} \int_{r<|y|<2r} \left( \pi - \mu \frac{v_y}{J} \right) \eta'_r \, dy \, d\tau = 0, 
\] (5.10)
thanks to which, noticing that \( \sqrt{\varrho_0} \pi \in L^1(\mathbb{R} \times (0, T)) \) and \( \sqrt{\varrho_0} v_y \in L^1(\mathbb{R} \times (0, T)) \), one obtains the desired conclusion by taking the limit \( r \to \infty \) in (5.9).

It remains to verify (5.10). To this end, recalling \( \text{supp} \eta_r \subseteq (-2r, -r) \cup (r, 2r) \) and (5.4), by Proposition 5.2 and noticing that \( \sqrt{\varrho_0} \pi \in L^1(\mathbb{R} \times (0, T)) \) and \( \sqrt{\varrho_0} v_y \in L^1(\mathbb{R} \times (0, T)) \), one can get
\[
\left| \int_{0}^{t} \int_{\mathbb{R}} \pi \eta'_r \, dy \, d\tau \right| = \left| \int_{0}^{t} \int_{r<|y|<2r} \pi \eta'_r \, dy \, d\tau \right|
\leq M_0 \int_{0}^{t} \int_{r<|y|<2r} \sqrt{\varrho_0} \pi \, dy \, d\tau \to 0, \quad \text{as } r \to \infty, 
\]
and
\[
\left| \int_{0}^{t} \int_{\mathbb{R}} v_y \eta'_r \, dy \, d\tau \right| = \left| \int_{0}^{t} \int_{r<|y|<2r} v_y \eta'_r \, dy \, d\tau \right|
\leq M_0 \int_{0}^{t} \int_{r<|y|<2r} \sqrt{\varrho_0} |v_y| \, dy \, d\tau \to 0, \quad \text{as } r \to \infty, 
\]
where \( c_0 = e^{-2\sqrt{\frac{\pi}{\sqrt{2}}} \sqrt{\|\varrho_0\|_1}} \). Therefore, (5.10) holds and, consequently, the conclusion follows. \( \square \)

The following proposition on the global in time a priori estimates on the effective viscous flux \( G \) is the key for proving the global existence of strong solutions.

**Proposition 5.4.** Under the same assumptions as in Proposition 5.1, it holds that

\[
\sup_{0 \leq t \leq T} \|G\|_2^2 + \int_0^T \left\| \frac{G_y}{\sqrt{\varrho_0}} \right\|^2 dt + \left( \int_0^T \|G\|_\infty^4 dt \right)^\frac{1}{2} \leq C e^{CT} \|G_0\|_2^2,
\]

for a positive constant \( C \) depending only on \( \gamma, \mu, \bar{\varrho}, E_0 \), and \( \|\varrho_0\|_1 \).

**Proof.** Let \( \eta \) and \( \eta_r \) be the same functions as in the proof of Proposition 5.1. Note that \( G \) satisfies (2.8). Multiplying (2.8) by \( JG\eta_r^2 \), and integrating the resultant over \( \mathbb{R} \), one gets by integration by parts that

\[
\int_\mathbb{R} JGG_\eta_r^2 dy + \mu \int_\mathbb{R} \left( \frac{G_y}{\varrho_0} \right)^2 \eta_r^2 dy = -\gamma \int_\mathbb{R} v_yG^2 \eta_r^2 dy - 2\mu \int_\mathbb{R} \frac{G_y}{\varrho_0} G\eta_r \eta_r' dy.
\] (5.11)

On the other hand, (2.5) implies that

\[
\int_\mathbb{R} JGG_\eta_r^2 dy = \frac{1}{2} \left( \frac{d}{dt} \int_\mathbb{R} JG^2 \eta_r^2 dy - \int_\mathbb{R} v_yG^2 \eta_r^2 dy \right).
\]

Plugging the above into (5.11) and integrating by parts show that

\[
\frac{1}{2} \frac{d}{dt} \int_\mathbb{R} JG^2 \eta_r^2 dy + \mu \int_\mathbb{R} \left( \frac{G_y}{\varrho_0} \right)^2 \eta_r^2 dy = \left( \frac{1}{2} - \gamma \right) \int_\mathbb{R} v_yG^2 \eta_r^2 dy - 2\mu \int_\mathbb{R} \frac{G_y}{\varrho_0} G\eta_r \eta_r' dy
\]

\[
= (2\gamma - 1) \int_\mathbb{R} vGGy \eta_r^2 dy + \int_\mathbb{R} [(2\gamma - 1)vG^2 - 2\mu \frac{G_y}{\varrho_0} G] \eta_r \eta_r' dy,
\]

which implies that

\[
\int_\mathbb{R} JG^2 \eta_r^2 dy + 2\mu \int_0^t \int_\mathbb{R} \left( \frac{G_y}{\varrho_0} \right)^2 \eta_r^2 dy \, d\tau
\]

\[
= 2 \int_0^t \int_\mathbb{R} \left( (2\gamma - 1)vG^2 - 2\mu \frac{G_y}{\varrho_0} G \right) \eta_r \eta_r' d\tau + (4\gamma - 4) \int_0^t \int_\mathbb{R} vGGy \eta_r^2 dy \, d\tau + \int_\mathbb{R} G_0^2 \eta_r^2 dy,
\] (5.12)

for any \( t \in [0, T] \).

We are going to show that

\[
\lim_{r \to \infty} \int_0^t \int_\mathbb{R} vGGy \eta_r^2 dy \, d\tau = \int_0^t \int_\mathbb{R} vGGy dy \, d\tau,
\] (5.13)

\[
\lim_{r \to \infty} \int_0^t \int_\mathbb{R} vG^2 \eta_r \eta_r' dy \, d\tau = \lim_{r \to \infty} \int_0^t \int_\mathbb{R} \frac{G_y}{\varrho_0} G\eta_r \eta_r' dy \, d\tau = 0,
\] (5.14)
for any $t \in [0, T]$. To verify (5.13), it suffices to show $vGG_y \in L^1(\mathbb{R} \times (0, T))$. By the regularities of $(J, v, \pi)$, one can check that

$$G \in L^\infty(0, T; L^2), \quad \frac{G_y}{\sqrt{\varrho_0}} \in L^2(0, T; L^2),$$

and further, by the Sobolev embedding, that $G \in L^2(0, T; L^\infty)$. As a result, by the Hölder inequality, we have $vGG_y = \sqrt{\varrho_0}vG \frac{G_y}{\sqrt{\varrho_0}} \in L^1(0, T; L^1)$ and, thus, (5.13) holds.

Observing that $\sqrt{\varrho_0}vG^2 \in L^1(0, T; L^1)$ and $\frac{G_yG}{\sqrt{\varrho_0}} \in L^1(0, T; L^1)$, and recalling $\text{supp } \eta_r \subset (-2r, r) \cup (r, 2r)$ and (5.4), we have

$$\left| \int_0^T \int_\mathbb{R} vG^2 \eta_r \eta'_r dyd\tau \right| = \left| \int_0^T \int_{|y|<2r} vG^2 \eta_r \eta'_r dyd\tau \right| \leq M_0 \int_0^T \int_{|y|<2r} \sqrt{\varrho_0}|v|G^2 dyd\tau \to 0, \quad \text{as } r \to \infty,$$

and

$$\left| \int_0^T \int_\mathbb{R} \frac{G_y}{\varrho_0} G \eta_r \eta'_r dyd\tau \right| = \left| \int_0^T \int_{|y|<2r} \frac{G_y}{\varrho_0} G \eta_r \eta'_r dyd\tau \right| \leq M_0 \int_0^T \int_{|y|<2r} \frac{|G_y||G|}{\sqrt{\varrho_0}} dyd\tau \to 0, \quad \text{as } r \to \infty,$$

for any $t \in [0, T]$. Therefore, (5.14) holds.

Due to (5.13) and (5.14), by taking $r \to \infty$, one obtains from (5.12) that

$$\int_\mathbb{R} JG^2 dy + 2\mu \int_0^T \int_\mathbb{R} \frac{(G_y)^2}{\varrho_0} dyd\tau = (4\gamma - 2) \int_0^T \int_\mathbb{R} vGG_y dyd\tau + \int_\mathbb{R} G_0^2 dy,$$

which and Proposition 5.2 give

$$\|G\|_2^2(t) + \int_0^t \left\| \frac{G_y}{\sqrt{\varrho_0}} \right\|_2^2 d\tau \leq C \left( \int_0^t \int_\mathbb{R} |v||G||G_y| dyd\tau + \|G_0\|_2^2 \right), \quad (5.15)$$

for any $t \in [0, T]$, where $C$ is a positive constant depending only on $\gamma, \mu, \mathcal{E}_0$, and $\|\varrho_0\|_1$. It follows from the Hölder, Yong, and Gagliardo-Nirenberg inequalities, and Proposition 5.1 that

$$C \int_\mathbb{R} |v||G||G_y| dy = C \int_\mathbb{R} \sqrt{\varrho_0}|v||G| \left\| \frac{G_y}{\sqrt{\varrho_0}} \right\| dy \leq C\|\sqrt{\varrho_0}v\|_2 \|G\|_{\infty} \left\| \frac{G_y}{\sqrt{\varrho_0}} \right\|_2 \leq C\|\sqrt{\varrho_0}v\|_2 \|G\|_2 \|G_y\|_{\frac{3}{2}} \left\| \frac{G_y}{\sqrt{\varrho_0}} \right\|_2 \leq C\|G\|_{\frac{3}{2}} \left\| \frac{G_y}{\sqrt{\varrho_0}} \right\|^2_{\frac{3}{2}} \leq \frac{1}{2} \left\| \frac{G_y}{\sqrt{\varrho_0}} \right\|^2_2 + C\|G\|_2^2.$$
for a positive constant $C$ depending only on $\gamma, \mu, \bar{\varrho}, E_0$, and $\|q_0\|_1$. Plugging the above estimate into (5.15) yields
\[
\|G\|_2^2(t) + \int_0^t \left( \left\| \frac{G_y}{\sqrt{q_0}} \right\|_2^2 \right) dt \leq C \left( \int_0^t \|G\|_2^2 \right),
\]
for any $t \in [0,T]$, for a positive constant $C$ depending only on $\gamma, \mu, \bar{\varrho}, E_0$, and $\|q_0\|_1$. Then the Gronwall inequality shows that
\[
\sup_{0 \leq t \leq T} \|G\|_2^2 + \int_0^T \left\| \frac{G_y}{\sqrt{q_0}} \right\|_2^2 dt \leq C e^{CT} \|G_0\|_2^2
\]
and, consequently, by the Gagliardo-Nirenberg inequality, one has
\[
\int_0^T \|G\|_4 dt \leq C \int_0^T \left( \left\| \frac{G_y}{\sqrt{q_0}} \right\|_2 \right) dt \leq C e^{CT} \|G_0\|_4.
\]
for a positive constant $C$ depending only on $\gamma, \mu, \bar{\varrho}, E_0$, and $\|q_0\|_1$. \hfill \Box

Based on Proposition 5.4 similar to Proposition 3.3 one can obtain the other a priori estimates stated in the next proposition.

**Proposition 5.5.** Under the same assumptions as in Proposition 5.7, it holds that
\[
\sup_{0 \leq t \leq T} \left\| \left( J - 1, \frac{J_y}{\sqrt{q_0}}, J_t, v_y, \pi, \frac{\pi_y}{\sqrt{q_0}} \right) \right\|_2 \leq C,
\]
and
\[
\int_0^T \left( \|\pi_t\|_2^2 + \left\| \left( \frac{v_y}{\sqrt{q_0}}, \sqrt{q_0} v_t \right) \right\|_2^2 \right) dt \leq C,
\]
for a positive constant $C$ depending only on $\gamma, \mu, \bar{\varrho}, E_0$, $\|q_0\|_1$, $\|\pi_0\|_2$, $\left\| \frac{\pi_y}{\sqrt{q_0}} \right\|_2$, and $T$, and the constant $C$, viewing as a function of $T$, is continuously increasing with respect to $T \in [0, \infty)$.

**Proof.** The proof is almost the same as that for Proposition 3.3 and the only difference is that the role played by Proposition 5.2 there is now played by Proposition 5.4 and we have to estimate the upper bound of $J$ on $[0,T]$, which is automatically guaranteed there by the assumption, before proving the estimates on $v_y$ and $\frac{v_y}{\sqrt{q_0}}$. Therefore, we only sketch the proof here.

Repeating the arguments as those for (3.12), (3.14), (3.15), (3.18), and (3.20), but applying Proposition 5.4 instead of applying Proposition 3.2 there, one obtains
\[
\sup_{0 \leq t \leq T} \left\| \left( \frac{J_y}{\sqrt{q_0}}, \pi, \frac{\pi_y}{\sqrt{q_0}} \right) \right\|_2 + \int_0^T \left( \|\pi_t\|_2^4 + \left\| \sqrt{q_0} v_t \right\|_2^2 \right) dt \leq C, \tag{5.16}
\]
here and throughout the proof of this proposition, $C$ is a positive constant depending only on $\gamma, \mu, \bar{\varrho}, E_0$, $\|q_0\|_1$, $\|\pi_0\|_2$, $\left\| \frac{\pi_y}{\sqrt{q_0}} \right\|_2$, and $T$, and it is continuously increasing with respect to $T \in [0, \infty)$. Thanks to this estimate and the Sobolev embedding...
inequality, we have $\sup_{0 \leq t \leq T} ||\pi||_\infty \leq C$. Noticing that $J(y, t) = \exp \left\{ \frac{1}{\mu} \int_0^t (G + \pi) d\tau \right\}$, it follows from Proposition 5.4 and (5.16) that $\sup_{0 \leq t \leq T} ||J||_\infty \leq C$. Thanks to (5.16) and $\sup_{0 \leq t \leq T} (||\pi||_\infty + ||J||_\infty) \leq C$, repeating the arguments as those for (3.17), (3.19), and (3.21), but applying Proposition 5.4 instead of applying Proposition 3.2 there, one can get

$$\sup_{0 \leq t \leq T} ||(J - 1, J_t, v_y)||_2^2 + \int_0^T \left| \frac{v_{yy}}{\sqrt{\rho_0}} \right|_2^2 dt \leq C.$$  

Thus, Proposition 5.5 is proved. □

The next proposition will be the key to show the global in time boundedness of the entropy.

**Proposition 5.6.** Let the assumption in Proposition 5.1 hold. Assume in addition that (H3) holds and $\frac{G}{\rho_0} \in L^2(0, T; L^2)$. Then, we have the following estimate:

$$\sup_{0 \leq t \leq T} \left| \frac{G}{\rho_0} \right|_2^2 + \int_0^T \left| \frac{G_y}{\xi_0 y^{1/2}} \right|_2^2 dt \leq C \left| \frac{G_0}{\rho_0} \right|_2^2,$$

for a positive constant $C$ depending only on $\gamma, \mu, \delta, K_0, \bar{\rho}, \mathcal{E}_0$, $||\rho_0||_1$, $||\pi_0||_2$, $||\frac{\pi_y}{\sqrt{\rho_0}}||_2$, and $T$, and the constant $C$, viewing as a function of $T$, is continuously increasing with respect to $T \in [0, \infty)$.

**Proof.** Let $\eta_r$ be the same function as in the proof of Proposition 5.1. Note that $G$ satisfies (2.8). Multiplying equation (2.8) by $\frac{JG\eta_r}{\rho_0}$, and integrating the resultant over $\mathbb{R}$, it follows from integration by parts that

$$\int_{\mathbb{R}} JG\eta_r^2 G_t dy + \mu \int_{\mathbb{R}} \frac{G_y}{\xi_0 y^{1/2}} \left( \frac{G\eta_r^2}{\rho_0} \right)_y dy = -\gamma \int_{\mathbb{R}} v_y G^2 \eta_r^2 dy. \quad (5.17)$$

Direct calculations yield

$$\int_{\mathbb{R}} \frac{G_y}{\xi_0} \left( \frac{G\eta_r^2}{\rho_0} \right)_y dy = \int_{\mathbb{R}} \frac{G_y}{\xi_0} \left[ \left( \frac{G_y}{\rho_0} - \frac{\delta}{\rho_0} G \right) \eta_r^2 + \frac{2}{\rho_0^2} \frac{\pi_y}{\sqrt{\rho_0}} \eta_r \right] dy. \quad (5.18)$$

Using equation (2.5), one can get

$$\int_{\mathbb{R}} JG\eta_r^2 G_t dy = \frac{1}{2} \left( \frac{d}{dt} \int_{\mathbb{R}} \frac{JG^2 \eta_r^2}{\rho_0} dy - \int_{\mathbb{R}} v_y G^2 \eta_r^2 dy \right). \quad (5.19)$$
Plugging (5.18) and (5.19) into (5.17), one can get from the assumption $\left| \left( \frac{1}{\sqrt{\varepsilon_0}} \right)' \right| \leq \frac{K_0}{2}$, or equivalently $|\theta_0'| \leq K_0 \theta_0^{3/2}$, and the Cauchy inequality that

$$\frac{1}{2} \int \frac{d}{dt} \frac{J G^2 \eta_r^2}{\theta_0^2} dy + \mu \int \frac{G^2 \eta_r^2}{\theta_0^{\delta + 1}} dy = \left( \frac{1}{2} - \gamma \right) \int v_y \frac{G^2 \eta_r^2}{\theta_0^2} dy + \mu \int \left( \frac{\delta \theta_0}{\theta_0^2} \frac{\eta_r^2}{\theta_0^2} - 2\eta_r \eta_r' \right) \frac{G^2 \eta_r^2}{\theta_0^{\delta + 1}} dy$$

$$\leq \gamma \int v_y \frac{G^2 \eta_r^2}{\theta_0^2} dy + \mu \int \left( \delta K_0 \sqrt{\theta_0^2} \eta_r^2 + 2\eta_r \eta_r' \right) \frac{G^2 \eta_r^2}{\theta_0^{\delta + 1}} dy$$

$$\leq \frac{\mu}{2} \int \frac{G^2 \eta_r^2}{\theta_0^{\delta + 1}} dy + C \int \left( |v_y| + 1 \right) \frac{G^2 \eta_r^2}{\theta_0^2} + \frac{G^2 \eta_r^2}{\theta_0^{\delta + 1}} |\eta_r'|^2 dy,$$

(5.20)

for a positive constant $C$ depending only on $\gamma, \mu, \delta, \text{ and } K_0$. It follows from $\text{supp } \eta_r \subseteq (-2r, r) \cup (r, 2r)$ and (5.19) that

$$\int \frac{G^2}{\theta_0^{\delta + 1}} |\eta_r'|^2 dy \leq M_0^2 \int_{r < |y| < 2r} \frac{G^2}{\theta_0^2} dy, \quad r \geq 1.$$

This, together with Proposition (5.2) and (5.20), implies that

$$\frac{d}{dt} \left\| \sqrt{\frac{J}{\theta_0}} G \eta_r \right\|_2^2 + \mu \left\| \frac{G_y \eta_r}{\theta_0^{\delta + 1}} \right\|_2^2$$

$$\leq C(|v_y|_\infty + 1) \left\| \sqrt{\frac{J}{\theta_0}} G \eta_r \right\|_2^2 + CM_0^2 \int_{r < |y| < 2r} \frac{G^2}{\theta_0^2} dy,$$

(5.21)

for $r \geq 1$, and for a positive constant $C$ depending only on $\gamma, \mu, \delta, K_0, \varepsilon_0$, and $\|\theta_0\|_1$. By Proposition (5.3) it follows from the Sobolev embedding inequality that $\int_0^T |v_y|_\infty dt \leq C$, for a positive constant $C$ depending only on $\gamma, \mu, \theta_0, \varepsilon_0, \|\theta_0\|_1, \|\pi_0\|_2, \|\eta_0\|_2$, and $T$, and $C$ is continuously increasing with respect to $T \in [0, \infty)$. Thanks to this, by applying the Gronwall inequality to (5.21), we obtain

$$\sup_{0 \leq t \leq T} \left\| \sqrt{\frac{J}{\theta_0}} G \eta_r \right\|_2^2 + \int_0^T \left\| \frac{G_y \eta_r}{\theta_0^{\delta + 1}} \right\|_2^2 dt \leq C \left( \left\| \frac{G_0}{\theta_0^2} \right\|_2^2 + M_0^2 \int_0^T \int_{r < |y| < 2r} \frac{G^2}{\theta_0^2} dy dt \right),$$

for $r \geq 1$, from which, noticing that the assumption $\frac{G_0}{\theta_0^2} \in L^2(0, T; L^2)$ implies that the last term of the right hand side of the above inequality tends to zero, as $r \to \infty$, we obtain by taking $r \to \infty$ that

$$\sup_{0 \leq t \leq T} \left\| \sqrt{\frac{J}{\theta_0}} G \right\|_2^2 + \int_0^T \left\| \frac{G_y}{\theta_0^{\delta + 1}} \right\|_2^2 \, dt \leq C \left\| \frac{G_0}{\theta_0^2} \right\|_2^2,$$
for a positive constant $C$ depending only on $\gamma, \mu, \bar{\rho}, K_0, \varrho, E_0, \|\varrho_0\|_1, \|\pi_0\|_2, \|\frac{\pi^2}{\sqrt{\varrho_0}}\|_2$, and $T$, and $C$ is continuously increasing with respect to $T \in [0, \infty)$. This, together with Proposition 5.2, yields the desired conclusion. □

We are now ready to prove Theorem 2.2.

**Proof of Theorem 2.2.** (i) By (i) of Theorem 2.1 there is a strong solution $(J, v, \pi)$ to system (2.5)–(2.7), subject to (2.9), on $\mathbb{R} \times (0, T_0)$, for some positive time $T_0$. Extend this local strong solution to the maximal time of existence $T_*$. Then, $(J, v, \pi)$ is a strong solution to system (2.5)–(2.7), subject to (2.9), one $\mathbb{R} \times (0, T)$, for any $T \in (0, T_*)$. Denote $E_0 = \int_\mathbb{R} \left( \frac{\varrho_0 v^2}{2} + \frac{J \pi}{\gamma - 1} \right) dy$. By Propositions 5.1–5.3 we have

$$
\left[ \int_\mathbb{R} \left( \frac{\varrho_0 v^2}{2} + \frac{J \pi}{\gamma - 1} \right) dy \right] (t) = E_0,
$$

$$
\inf_{y \in \mathbb{R}} J(y, t) \geq \exp \left\{ -\frac{2\sqrt{2}}{\mu} \sqrt{E_0} \|\varrho_0\|_1 \right\},
$$

$$
\left( \int_\mathbb{R} \varrho_0 v dy \right) (t) = \int_\mathbb{R} \varrho_0 v_0 dy,
$$

for any $t \in [0, T_*)$. In order to prove the conclusion, it suffices to show that $T_* = \infty$.

Assume, by contradiction, that $T_* < \infty$. By Proposition 5.5 we have

$$
\sup_{0 \leq t \leq T_*} \left\| \left( J - 1, \frac{J y}{\sqrt{\varrho_0}}, J_t, v_y, \pi, \frac{\pi y}{\sqrt{\varrho_0}} \right) \right\|_2 \leq C,
$$

$$
\int_0^T \left( \|\pi_t\|_2^4 + \left( \frac{v_{yy}}{\sqrt{\varrho_0}}, \sqrt{\varrho_0} v_t \right) \right)_2^2 dt \leq C,
$$

for any $T \in [0, T_*)$, where $C$ is a positive constant depending only on $\gamma, \mu, \bar{\rho}, E_0, \|\varrho_0\|_1, \|\pi_0\|_2, \|\frac{\pi^2}{\sqrt{\varrho_0}}\|_2$, and $T$, and this constant $C$, viewing as a function of $T$, is continuously increasing with respect to $T \in [0, \infty)$. Since $T_*$ is a finite positive number, we can see the positive constants in the above are actually independent of $T \in (0, T_*)$. Thanks to the above estimates, we have

$$
\inf_{y \in \mathbb{R}} J(y, T_1) > 0, \quad J(\cdot, T_1) \in L^\infty, \quad \left( \frac{\partial_y J}{\sqrt{\varrho_0}}, \sqrt{\varrho_0} v, \partial_y v, \pi, \frac{\partial_y \pi}{\sqrt{\varrho_0}} \right) (\cdot, T_1) \in L^2,
$$

and

$$
\inf_{y \in \mathbb{R}} J(y, T_1) + (\|J\|_\infty + \|G\|_2 + \|\pi\|_\infty) (T_1) \leq C_0,
$$

for a positive constant $C_0$ independent of $T_1 \in (0, T_*)$. Therefore, by Theorem 2.1 there is a positive time $t_0$, such that, starting from time $T_* - \frac{t_0}{2}$, one can extend the strong solution $(J, v, \pi)$ uniquely to another time $T_* - \frac{t_0}{2} + t_0 = T_* + \frac{t_0}{2} > T_*$,
which contradicts to the definition of $T_*$. Therefore, $T_* = \infty$ and, thus, one obtains a unique global strong solution to system (2.5)–(2.7), subject to (2.9).

(ii) We only prove that (2.10) holds for any finite $T \in (0, \infty)$, while the validity of (2.11) follows from (2.10) and (i), by exactly the same arguments as in the proof of Theorem 2.1. By (ii) of Theorem 2.1, there is a positive time $T_\ell$, such that (2.10) holds. Denote by $T_\ell$ the maximal time, such that (2.10) holds, for any $T \in (0, T_\ell)$. In order to verify that (2.10) holds for any finite $T \in (0, \infty)$, it suffices to show that $T_\ell = \infty$. Assume, by contradiction, that $T_\ell < \infty$. By Proposition 5.6, the following estimate holds

$$\sup_{0 \leq t \leq T} \left\| \frac{G}{\varrho_0^\gamma} \right\|_2^2 + \int_0^T \left\| \frac{G_y}{\varrho_0^\gamma} \right\|_2^2 \, dt \leq C \left( \frac{G_0}{\varrho_0^\gamma} \right),$$

for any $T \in (0, T_\ell)$, where $C$ is a positive constant depending only on $\gamma, \mu, \delta, K_0, \varrho, \mathcal{E}_0,$ $\|\varrho_0\|_1, ||\pi_0\|_2,$ $\|\sqrt{\varrho_0}\|_2$, and $T$, and this constant $C$, viewing as a function of $T$, is continuously increasing with respect to $T \in [0, \infty)$. Since $T_\ell$ is a positive finite number, the above constant $C$ is actually independent of $T \in (0, T_\ell)$. Due to this fact, one can see that

$$\frac{G(\cdot, T_\ell)}{\varrho_0^\gamma(\cdot)} \mid_{t = T_\ell} \in L^2.$$

With the aid of this, taking $T_\ell$ as the initial time, by Theorem 2.1, one can see that (2.10) holds for some other time $T'_\ell > T_\ell$, which contradicts to the definition of $T_\ell$. Therefore, one must have $T_\ell = \infty$, in other words, (2.10) holds for any finite time $T \in (0, \infty)$.

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(Jinkai Li) DEPARTMENT OF MATHEMATICS, THE CHINESE UNIVERSITY OF HONG KONG, HONG KONG, P.R.CHINA
E-mail address: jklimath@gmail.com

(Zhongping Xin) THE INSTITUTE OF MATHEMATICAL SCIENCES, THE CHINESE UNIVERSITY OF HONG KONG, HONG KONG, P.R.CHINA
E-mail address: zpxin@ims.cuhk.edu.hk