Lattice QCD Impact on Determination of the CKM Matrix

Steven Gottlieb
Indiana University, Bloomington IN 47405, USA,
sg@indiana.edu,
WWW home page: http://physics.indiana.edu/~sg/

Abstract. We review many lattice QCD calculations that impact the precise determination of the CKM matrix. We focus on decay constants and semileptonic form factors of both light ($\pi$ and $K$) and heavy-light ($D_{(s)}$ and $B_{(s)}$) mesons. Implication of $A_b$ form factors will be shown. When combined with experimental results for branching fractions and differential decay rates, the above calculations strongly constrain the first two rows of the CKM matrix. We discuss a long standing difference between $|V_{ub}|$ and $|V_{cb}|$ as determined from exclusive or inclusive decays.

Keywords: lattice QCD, CKM matrix, leptonic decays, semileptonic decays

1 Introduction

Lattice QCD contributes strongly to understanding the CKM matrix and the search for beyond the Standard Model (SM) physics. To begin, I would like to relate a little about my background. I am a member of the Flavour Lattice Averaging Group (FLAG) and participate in the $D$ and $B$ semileptonic decays working group. However, this is not a FLAG approved talk. The most recent FLAG review [1] dates from 2016, and I will concentrate on more recent plots. The closing date for papers to appear in the next review is a couple of months after FPCP 2018, so the next FLAG plots and averages are not yet available.

For more than a decade, my own research has been in the context of the MILC collaboration and the Fermilab Lattice/MILC Collaborations (FNAL/MILC). Much of our work is directed toward more precisely determining the CKM matrix, and looking for discrepancies that would indicate physics beyond the SM, I will liberally use plots from my own collaborations when they contain results not yet reviewed by FLAG.

I also happen to be a member of the local organizing committee for Lattice 2018, which takes place the week after FPCP 2018. I can assure you that topics discussed here are very active areas within the lattice QCD community. For Lattice 2018, there were 37 abstracts in the weak matrix element category. Of them, five dealt with decay constants, 12 with $K$, $D$, or $B$ meson semileptonic decay, and seven with nucleon or nuclear matrix elements.
1.1 CKM matrix

In expression (1), CKM matrix elements are in bold and below each of them are one or two processes that can be used to determine that element. Below the last row, the matrix elements represent the $B(s)\bar{B}(s)$ mixing phenomena that depend on $V_{td}$ and $V_{ts}$ through loop diagrams.

$$
\begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
\pi \rightarrow l\nu & K \rightarrow l\nu & B \rightarrow l\nu \\
V_{cd} & V_{cs} & V_{cb} \\
D \rightarrow \pi l\nu & D \rightarrow K l\nu & B \rightarrow D^0 l\nu \\
V_{td} & V_{ts} & V_{tb} \\
\langle \overline{B}_d|\overline{B}_d \rangle & \langle \overline{B}_s|\overline{B}_s \rangle 
\end{pmatrix}
$$

(1)

The CKM matrix is unitary, so each row and column is a complex unit vector and each row (or column) is orthogonal to the other two. Violation of unitarity would be evidence for non-SM physics. Since different decays can depend on the same CKM matrix element, if the value of the matrix element inferred from the different decays do not agree, that would be evidence for new physics.

Consider the branching fraction $\mathcal{B}$ for leptonic decay of a $D$ or $D_s$ meson.

$$
\mathcal{B}(D_{(s)} \rightarrow \ell\nu) = \frac{G_F^2 |V_{cq}|^2 \tau_{D_{(s)}}}{8\pi f_{D_{(s)}}^2 m_{D_{(s)}}^2} \left(1 - \frac{m_{\ell}^2}{m_{D_{(s)}}^2}\right)^2
$$

(2)

where $V_{cq}$ is the (unknown) CKM matrix element with $q = d$, or $s$, $f_{D_{(s)}}$ is the decay constant of the meson that we calculate using lattice QCD, and the other factors on the RHS are well known. For semileptonic decays, the LHS would be a differential decay rate, and the RHS would involve a CKM matrix element and form factors describing the transition matrix element between the initial and final state hadrons induced by the weak current responsible for the decay. In both cases, the experimental measurement and hadronic input from lattice QCD allow determination of the CKM matrix element.

2 First Row: Light Quarks

We start with decays of $\pi$ and $K$ mesons. As implied by the subsequent discussion generalized to other mesons, we need $f_\pi$ and $f_K$ to describe the leptonic decays and a form factor to describe the kaon semileptonic decay. The two decay constants can each be calculated; however, $f_\pi$ is often used to set the lattice scale, so the ratio $f_K/f_\pi$ which has the advantage of smaller systematic errors is a key quantity. From experiment (2) it is known that

$$
\left| \frac{V_{us}}{V_{ud}} \right| \frac{f_{K^\pm}}{f_{\pi^\pm}} = 0.2760(4)
$$

(3)
Thus, knowledge of the decay constant ratio, allows us to determine the ratio of the first two elements of the CKM matrix. The decay constant ratio has recently been updated by FNAL/MILC \cite{3}. The result is $f_{K^+}/f_{\pi^+} = 1.1950(^{+15}_{-22})$ which may be compared with the FLAG 2016 \cite{1} $N_f = 2 + 1 + 1$ average $1.193(3)$. $N_f$ is the number of dynamical sea quarks in the calculation. Figure 1 summarizes the most relevant calculations including those with $N_f = 2 + 1$.

For kaon semileptonic decay, $p_K = p_\pi + q_\ell + q_\nu$ by energy-momentum conservation. The relevant variable for the form factors is $q^2$ with $q = q_\ell + q_\nu$ the 4-momentum of the leptons. One could, in principle, determine the shape of the vector form factor $f_+(q^2)$ to predict the shape of the differential cross section. However, it is a bit easier to just calculate $f_+(0)$ using lattice QCD and take the experimental measurement \cite{4} that determines $|V_{us}|f_+(0) = 0.21654(41)$. The latest result for the form factor is \cite{5}

$$f_+(0) = 0.9696(15)_{\text{stat}}(11)_{\text{sys}} = 0.9696(19),$$

separating statistical and systematic errors before combining in quadrature. The total theoretical error is 0.19%, the same size as the experimental error. Figure 2 taken from \cite{5} shows the result just quoted (denoted “This work”) along with FLAG 2016 \cite{1} averages in black and the results included in the averages as green squares. Blue circles come from non-lattice QCD calculations. See \cite{5} for details. Using the previously quoted experimental value \cite{4}, we find

$$|V_{us}| = 0.22333(43)f_+(0)(42)_{\text{exp}} = 0.22333(60).$$

We consider the implications of these results on first row unitarity in Fig. 3. Since $|V_{ub}|$ is so small, we can neglect it. The vertical band labeled $0^+ \to 0^+$ comes from analysis of superallowed nuclear $\beta$-decays \cite{6} and is independent

**Fig. 1.** Comparison of calculations of decay constant ratio $f_{K^+}/f_{\pi^+}$ with $N_f = 2 + 1 + 1$ and $2 + 1$ sea quark flavors. From \cite{5}. 

\[ T \]
of lattice QCD input. The diagonal band labeled $K_{12}$ comes from the ratio of decay constants $f_K/f_\pi$, and the horizontal band labeled $K_{13}$ comes from the kaon semileptonic form factor. The diagonal band, the vertical band, and the unitarity curve nicely intersect. However, there is tension with unitary when we look at the kaon semileptonic decay. The small blue ellipse uses the result in Eq. 5 and the value of $|V_{ud}|$ from [6], and we see that it does not intersect the unitarity curve. We find that

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 - 1 = -0.00104(27)_{V_{us}}(41)_{V_{ud}}$$  

which is $2.1\sigma$ from zero. The large blue ellipse does not rely on $V_{ud}$ from $\beta$-decay and only uses results from pion leptonic decay and kaon leptonic and semileptonic decay. There is a clear tension with unitarity. In this case, we have

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 - 1 = -0.0151(38)_{f^+}(0.35)_{f^+} + 0.36\exp(27)_{EM}$$  

which is $2.2\sigma$ different from zero. The tension between leptonic and semileptonic determination of $|V_{us}|$ and $|V_{ud}|$ can also be seen in the FLAG 2016 Fig. 7 summary plot for $|V_{us}|$ and $|V_{ud}|$ in which semileptonic results are triangles and leptonic results are squares. The tension is most noticeable for the $N_f = 2+1+1$ calculations where the precision is higher.

### 3 Second Row of CKM Matrix

We now turn to the decay of charm mesons to determine $|V_{cd}|$ and $|V_{cs}|$. In 2005, the initial $N_f = 2+1$ calculations of decay constants were done with an accuracy of roughly 10%. The latest results for $f_{D^+}$ and $f_{D_s}$ now have errors < 0.3%. We have [3]

$$f_{D^+} = 212.7(0.6)\text{MeV}, \quad f_{D_s} = 249.9(0.4)\text{MeV}.$$  

Fig. 2. Comparison of calculations of the kaon form factor $f^K(q^2 = 0)$ with $N_f = 2+1$, or $2+1$ sea quark flavors, and non-lattice QCD calculations [5].
Fig. 3. Constraints in the $|V_{ud}|$–$|V_{us}|$ plane from leptonic ($Kl_2$) and semileptonic ($Kl_3$) decays. In addition, we have the result of nuclear $\beta$-decay on $|V_{ud}|$, unitarity, and a wide horizontal band from $|V_{cd}|$ combined with unitarity. (See [5].)

Prior to that calculation, the FLAG [1] and Particle Data Group (PDG) [2] average values had errors of roughly 1–3 MeV. Figure 4 summarizes the best recent calculations [3].

We use experimental results from the PDG [2] to determine the CKM matrix elements from the lattice QCD decay constants. They have $f_{D}|V_{cd}| = 45.91(1.05)$ MeV and $f_{Ds}|V_{cs}| = 250.9(4.0)$ MeV. The experimental error is 1.6–2.3%. For the CKM matrix element, we have

$$|V_{cd}|_{SM, f_{D}} = 0.2152(5)f_{D}(49)_{\text{expa}}(6)_{\text{EM}}, \quad (9)$$

$$|V_{cs}|_{SM, f_{Ds}} = 1.001(2)f_{Ds}(16)_{\text{expa}}(3)_{\text{EM}}, \quad (10)$$

where the errors are from lattice decay constant, experiment, and a structure dependent electromagnetic correction. The experimental errors are dominant.

Turning to the charm meson semileptonic decays, the FLAG 2016 form factor average is based on HPQCD results from 2010 [7] and 2011 [8]. There are new results from ETM Collaboration [9] and JLQCD [10]. In addition, FNAL/MILC is completing an analysis that should soon have errors smaller than those of HPQCD. I have taken a rough average of the three results above, even though they mix results with $N_f = 2 + 1$ and $2 + 1 + 1$. I find $f^{D\to\pi}(0) = 0.637(20)$ and $f^{D\to K}(0) = 0.745(15)$. These values may be a little aggressive. The FLAG 2016 values are 0.666(29) and 0.747(19), respectively. Using the HFLAV 2016 values [11] $f^{D\to\pi}_{+}|V_{cd}| = 0.1426(20)$ and $f^{D\to K}_{+}|V_{cs}| = 0.7226(34)$, we obtain $|V_{cd}| = 0.2239(76)$ and $|V_{cs}| = 0.970(20)$ corresponding to errors of 3.4% and 2.1%. In each case, the error is dominated by the error in the lattice form factor input. I updated the experimental input after FPCP 2018, so the values here are different from those in my slides.
Fig. 4. Comparison of recent calculations of $f_{D^+}$ and $f_{D_s}$ with $N + f = 2 + 1 + 1$ and $2 + 1$. From [3].

We can test second row unitarity using a variety of determinations of $|V_{cd}|$ and $|V_{cs}|$. In this case, $|V_{cb}| \approx 0.0414(8)$ contributes about 0.0017(6) to the unitarity sum. We consider in Table 1 the latest result using leptonic decay constants from [3], the FLAG 2016 $N_f = 2 + 1 + 1$ result, the latest ETMC semileptonic result [12], and my rough average of semileptonic results. We find a slight (1.5σ) tension from the leptonic decay determination and none from the semileptonic. The semileptonic error is dominated by experimental error and the semileptonic by theory. However, both will improve.

Table 1. Tests of second row unitarity from various determinations of $|V_{cd}|$ and $|V_{cs}|$.

| $|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2 - 1$ input | $0.049(2)|V_{cd}|(32)|V_{cs}|(0)|V_{cb}|$ FNAL/MILC leptonic [3] |
|---------------------------------------------|--------------------------------------------------|
| 0.049(2)|V_{cd}|(32)|V_{cs}|(0)|V_{cb}| FNAL/MILC leptonic [3] |
| -0.004(64) | ETMC semileptonic [12] |
| 0.005(53) | my semileptonic average |

4 Decays of Hadrons with $b$ Quarks

Decays of hadrons containing $b$ quarks have been studied in order to determine $|V_{ub}|$ and $|V_{cb}|$. Mesonic decays have been extensively studied by a number of groups. Recently, Meinel and his collaborators have been looking at several decays of baryons with $b$ or $c$ quarks [14]. Rare decays involving flavor changing neutral currents (FCNC) are a good place to look for new physics as FCNC processes vanish at the tree level. These processes also may involve third row CKM...
matrix elements and provide an alternative to $B_{(s)}$ meson mixing for determining $|V_{td}|$ and $|V_{ts}|$. Meson mixing is covered by FLAG.

Reference [3] provides the best values for $B_{(s)}$ meson decay constants. (See Fig. 5). Errors are < 1.3MeV or 0.7%. There is good agreement with earlier calculations that have errors as small as 5–7 MeV. To exploit these results to get $|V_{ub}|$, we await precise results from Belle II for $B \to \tau \nu$, as the difference between BaBar and Belle is large [2].

Turning to bottom hadron semileptonic and rare decays, there are many possible channels. For $|V_{ub}|$ these include $B \to \pi \ell \nu$, $B_s \to K \ell \nu$, $B_s \to K^* \ell \nu$, and $A_b \to p \ell \nu$. For $|V_{cb}|$, we might study $B \to D \ell \nu$, $B \to D^* \ell \nu$, $B_s \to D_1^{(*)} \ell \nu$, and $A_b \to A_\ell \nu$. We can test lepton universality as $\ell$ can be $e$, $\mu$, or $\tau$. There are also interesting rare decays such as $B^0 \to \mu^+ \mu^-$, $B_s \to \mu^+ \mu^-$, $B_s \to \phi \ell \nu$, and $B \to K \ell^+ \ell^-$. Unfortunately, there is not enough time to cover all of these decays. Let’s consider the long standing difference between CKM matrix elements as determined in exclusive and inclusive decay measurements. In 2015, the form factors needed for $B \to \pi \ell \nu$ were updated [13] and the resulting value of $|V_{ub}| = 3.72(16) \times 10^{-3}$, was in somewhat better agreement with the inclusive value as seen in Fig. 6(L). The figure shows several different determinations of $|V_{ub}|$ including one based on $A_b$ decay (triangle) [14]. The inclusive result is plotted as a diamond, unitarity as a circle. The exclusive value of $|V_{ub}|$ is in good agreement with unitarity, but the inclusive one is not. As mentioned above, the leptonic decay $B \to \tau \nu$ could shed light on $|V_{ub}|$, but we’ll need to wait for Belle II results for that.

As of 2015, the situation for $|V_{cb}|$ is depicted in Fig. 6(R). At that time, exclusive decay processes $B \to D^* \ell \nu$ and $B \to D \ell \nu$ were both being studied. The experimental error was larger in the $D$ channel (3.9%) whereas it was just 1.4% for the $D^*$ channel. In the figure, $w$ is an alternate kinematic variable equivalent to $q^2$. The form factor at $w = 1$ (zero recoil) can be calculated using lattice
QCD; however, it is difficult to get the corresponding experimental value as the differential decay rate vanishes there, so it is necessary to fit the experimental results as a function of $w$. For the $D$ channel, the theoretical form factors were available for a range of $w$. The notation HFAG '14 in the figure indicates that the experimental input for $w = 1$ came from the fit of the Heavy Flavor Averaging Group (now HFLAV). In 2016, Bigi and Gambino [16] used updated Belle data for $B \to D\ell\nu$ and the BGL parameterization [17] to obtain $|V_{cb}| = (40.49 \pm 0.97) \times 10^{-3}$. In 2017, Bigi, Gambino, Schacht [18]; and Grinstein and Kobach [19] examined new Belle data [20] for $B \to D^*\ell\nu$ and found a 10% difference when changing between CLN [21] and BGL parameterizations of the experimental data. Using CLN, they found $(38.2 \pm 1.5) \times 10^{-3}$, and for BGL, they found $(41.7 \pm 2.0) \times 10^{-3}$. The PDG inclusive value for $|V_{cb}| = (42.2 \pm 0.8) \times 10^{-3}$. We see that for $B \to D^*$, exclusive and inclusive $|V_{cb}|$ values are totally compatible, and for $B \to D$, the difference between inclusive and exclusive determinations is only 1.36σ. Thus, for $|V_{cb}|$, the issue is largely resolved (at least) until the errors can be reduced.

5 Conclusions and Outlook

There has been very significant progress using lattice QCD to calculate hadronic matrix elements needed for precise evaluation of SM contributions to numerous decay processes. This theoretical input is essential to determine the CKM matrix. A number of quantities can now be calculated to sub-percent accuracy. The interplay between theory and experiment will continue to yield increasingly stringent tests of the SM. In semileptonic decays, we see some tension with unitarity in the first and second rows. In the first row, we see slightly $> 2\sigma$ tension with unitarity from semileptonic K decay. There is some tension between leptonic and semileptonic determinations of $|V_{ud}|$ and $|V_{us}|$. The tests of unitarity in the second row are not as stringent. The difference between exclusive and inclusive determination of $|V_{cb}|$ may be due to how the experimental data had been fit; however, for $|V_{ub}|$ a difference remains. Although I had hoped to cover some of the other recent observations that are in tension with the SM predictions,
there was not enough time. Future results from Belle II, BES III, and LHCb, combined with increasingly precise calculations from lattice QCD, will provide more critical tests of the SM and opportunities to find evidence of new physics.

Acknowledgments

I thank the FPCP organizers for their wonderful hospitality and a stimulating conference. I gratefully acknowledge my colleagues in the Fermilab Lattice and MILC Collaborations for wonderful working relationships and friendships. I also thank FLAG members who contribute countless hours to making lattice QCD results more easily available to the wider community. This work was supported by the US DOE grant [de-sc0010120].

References

1. S. Aoki et al., Eur. Phys. J. C 77, no. 2, 112 (2017) doi:10.1140/epjc/s10052-016-4509-7 [arXiv:1607.00299 [hep-lat]].
2. C. Patrignani et al. [Particle Data Group], Chin. Phys. C 40, no. 10, 100001 (2016). doi:10.1088/1674-1137/40/10/100001.
3. A. Bazavov et al., Phys. Rev. D 98, no. 7, 074512 (2018) doi:10.1103/PhysRevD.98.074512 [arXiv:1712.09262 [hep-lat]].
4. M. Moulsan, PoS CKM 2016, 033 (2017) doi:10.22323/1.291.0033 [arXiv:1704.04104 [hep-ex]].
5. A. Bazavov et al. [Fermilab Lattice and MILC Collaborations], arXiv:1809.02827 [hep-lat].
6. J. C. Hardy and I. S. Towner, arXiv:1807.01146 [nucl-ex].
7. H. Na, C. T. H. Davies, E. Follana, G. P. Lepage and J. Shigemitsu, Phys. Rev. D 82, 114506 (2010) doi:10.1103/PhysRevD.82.114506 [arXiv:1008.4562 [hep-lat]].
8. H. Na, C. T. H. Davies, E. Follana, J. Koponen, G. P. Lepage and J. Shigemitsu, Phys. Rev. D 84, 114505 (2011) doi:10.1103/PhysRevD.84.114505 [arXiv:1109.1501 [hep-lat]].
9. V. Lubicz et al. [ETM Collaboration], Phys. Rev. D 96, no. 5, 054514 (2017) doi:10.1103/PhysRevD.96.054514 [arXiv:1706.03017 [hep-lat]].
10. T. Kaneko et al. [JLQCD Collaboration], EPJ Web Conf. 175, 13007 (2018) doi:10.1051/epjconf/201817513007 [arXiv:1711.11235 [hep-lat]].
11. Y. Amhis et al. [HFLAV Collaboration], Eur. Phys. J. C 77, no. 12, 895 (2017) doi:10.1140/epjc/s10052-017-5058-4 [arXiv:1612.07233 [hep-ex]].
12. L. Riggio, G. Salerno and S. Simula, Eur. Phys. J. C 78, no. 6, 501 (2018) doi:10.1140/epjc/s10052-018-5943-5 [arXiv:1706.03657 [hep-lat]].
13. J. A. Bailey et al. [Fermilab Lattice and MILC Collaborations], Phys. Rev. D 92, no. 1, 014024 (2015) doi:10.1103/PhysRevD.92.014024 [arXiv:1503.07839 [hep-lat]].
14. W. Detmold, C. Lehner and S. Meinel, Phys. Rev. D 92, no. 3, 034503 (2015) doi:10.1103/PhysRevD.92.034503 [arXiv:1503.01421 [hep-lat]].
15. J. A. Bailey et al. [MILC Collaboration], Phys. Rev. D 92, no. 3, 034503 (2015) doi:10.1103/PhysRevD.92.034503 [arXiv:1503.07237 [hep-lat]].
16. D. Bigi and P. Gambino, Phys. Rev. D 94, no. 9, 094008 (2016) doi:10.1103/PhysRevD.94.094008 [arXiv:1606.08030 [hep-ph]].
17. C. G. Boyd, B. Grinstein and R. F. Lebed, Phys. Rev. D 56, 6895 (1997) doi:10.1103/PhysRevD.56.6895 [hep-ph/9705252].
18. D. Bigi, P. Gambino and S. Schacht, Phys. Lett. B 769, 441 (2017) doi:10.1016/j.physletb.2017.04.022 [arXiv:1703.06124 [hep-ph]].
19. B. Grinstein and A. Kobach, Phys. Lett. B 771, 359 (2017) doi:10.1016/j.physletb.2017.05.078 [arXiv:1703.08170 [hep-ph]].
20. A. Abdesselam et al. [Belle Collaboration], arXiv:1702.01521 [hep-ex].
21. I. Caprini, L. Lellouch and M. Neubert, Nucl. Phys. B 530, 153 (1998) doi:10.1016/S0550-3213(98)00350-2 [hep-ph/9712417].