SHARP THRESHOLDS OF BLOW-UP AND GLOBAL EXISTENCE FOR THE COUPLED NONLINEAR SCHRÖDINGER SYSTEM

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Abstract. In this paper, we establish two new types of invariant sets for the coupled nonlinear Schrödinger system on \( \mathbb{R}^n \), and derive two sharp thresholds of blow-up and global existence for its solution. Some analogous results for the nonlinear Schrödinger system posed on the hyperbolic space \( \mathbb{H}^n \) and on the standard 2-sphere \( S^2 \) are also presented. Our arguments and constructions are improvements of some previous works on this direction. At the end, we give some heuristic analysis about the strong instability of the solitary waves.

Keywords: Coupled Schrödinger system, Sharp thresholds.

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1. Introduction

In this paper, we establish two new types of invariant sets for the \( N \)-coupled nonlinear Schrödinger system on \( \mathbb{R}^n \) given by

\[
\begin{aligned}
- i \partial_t \phi_j &= \Delta \psi_j + \mu_j |\phi_j|^{p-1} \phi_j + \sum_{i \neq j} \beta_{ij} |\phi_i|^{(p+1)/2} |\phi_j|^{(p-3)/2} \phi_j, \\
\phi_j(t,x) &= \phi_j(t,x) \in \mathbb{C}, \quad x \in \mathbb{R}^n, \quad t > 0, \quad j = 1, \ldots, N, \\
\phi_j(0,x) &= \phi_{0j}(x), \quad \phi_{0j} : \mathbb{R}^n \to \mathbb{C},
\end{aligned}
\]

where \( 1 \leq p < 1 + 4/(n-2)^+ \) (we use the convention: \( 4/(n-2)^+ = \infty \) when \( n = 1,2 \), and \( (n-2)^+ = n-2 \) when \( n \geq 3 \)), \( \mu_j > 0 \)'s are positive constants and \( \beta_{ij} \)'s are coupling constants subjected to \( \beta_{ij} = \beta_{ji} \). Based on our new invariant sets, we then derive two sharp thresholds of blow-up and global existence for the solutions. We point out that our results have no restriction on the dimension \( n \), which plays an important role in the previous related studies \[4\]. We also give the sharp thresholds when \( \mathbb{H}^n \) is considered on the hyperbolic space \( \mathbb{H}^n \) and on the standard 2-sphere \( S^2 \). These results rely heavily on the geometric structure of the manifolds and behave very differently from the ones considered on \( \mathbb{R}^n \).

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At the end, we give some heuristic analysis about the strong instability of the solitary waves.

The system \((1)\) has applications in many physical problems, especially in nonlinear optics. Physically, the solution \(\phi_j\) denotes the \(j^{th}\) component of the beam in Kerr-like photo-refractive media (cf. \([1]\)). The positive constant \(\mu_j\) is for self-focusing in the \(j^{th}\) component of the beam. The coupling constant \(\beta_{ij}\) is the interaction between the \(i^{th}\) and the \(j^{th}\) component of the beam. We refer to \([5]\) for more precision on the meaning of the constants. When the spatial dimension \(n \leq 3\), there are many analytical and numerical results on the system. We shall quote the recent works \([11, 15, 16, 17, 18, 22]\), where a comprehensive list of references on this subject can be found. However, there are few works describing the blow-up phenomena of the solution. Hereafter, we focus on the blow-up analysis for the system \((1)\) when \(\beta_{ij} = \beta_{ji}\). For notational simplicity, we write \(\Phi_0 = (\phi_{01}, ..., \phi_{0N})\) as the initial data and \(\Phi = (\phi_1, ..., \phi_N)\) as the solution. We denote

\[
\|\Phi\|_p := \left( \sum_{j=1}^{N} \int_{\mathbb{R}^n} |\phi_j|^p \right)^{1/p}
\]

for \(1 \leq p < \infty\) and

\[
\|\nabla \Phi\|_2 := \left( \sum_{j=1}^{N} \int_{\mathbb{R}^n} |\nabla \phi_j|^2 \right)^{1/2}.
\]

We define the testing functional

\[
P(\Phi) := \sum_{j=1}^{N} \mu_j \int_{\mathbb{R}^n} |\phi_j|^{p+1} + \sum_{i,j=1}^{N} \beta_{ij} \int_{\mathbb{R}^n} |\phi_i|^{(p+1)/2} |\phi_j|^{(p+1)/2}.
\]

The local existence theorem for the single Schrödinger equation in \(H^1(\mathbb{R}^n)\) (see \([10, 12]\)) still holds true for the Schrödinger system \((1)\). In fact, by solving the equivalent integral system

\[
\phi_j = e^{it \Delta} \phi_{0j} + i \mu_j \int_0^t e^{i(t-s) \Delta} |\phi_j|^{p-1} \phi_j(s) ds
\]

\[
+ i \sum_{i \neq j} \beta_{ij} \int_0^t e^{i(t-s) \Delta} |\phi_i|^{(p+1)/2} |\phi_j|^{(p-3)/2} \phi_j(s) ds
\]

in the space

\[
(H^1(\mathbb{R}^n))^N = H^1(\mathbb{R}^n) \times ... \times H^1(\mathbb{R}^n)
\]

with a standard Picard iteration method as in \([10, 12]\), one gets easily the following proposition.
Proposition 1. (Local Existence) Assume that $1 \leq p < 1 + 4/(n - 2)^+$. Then for any $\Phi_0 \in (H^1(\mathbb{R}^n))^N$, there exists a $T > 0$ and a unique solution $\Phi \in C([0, T), (H^1(\mathbb{R}^n))^N)$ such that either $T = \infty$ or else $T < \infty$ and $\|\nabla \Phi\|_2 \to \infty$ as $t \to T$.

When $\beta_{ij} = \beta_{ji}$, the system (1) admits the mass and the energy conservation laws in the space $(H^1(\mathbb{R}^n))^N$, which are stated in (2) and (3) below.

**Mass ($L^2$ norm):**

\begin{equation}
M(\Phi) := \|\Phi\|_2 = M(\Phi_0);
\end{equation}

**Energy:**

\begin{equation}
E(\Phi) := \frac{1}{2}\|\nabla \Phi\|_2^2 - \frac{1}{p + 1}\mathcal{P}(\Phi) = E(\Phi_0).
\end{equation}

Furthermore, let $\rho > 0$ be a $C^4$ real function on $\mathbb{R}^n$ (independent of $t$), and then for

\begin{equation}
J(t) := \sum_{j=1}^{N} \int_{\mathbb{R}^n} \rho(x)|\phi_j(t, x)|^2;
\end{equation}

we have

\begin{equation}
J'(t) = 2\mathfrak{R} \sum_{j=1}^{N} \int_{\mathbb{R}^n} (\nabla \phi_j \cdot \nabla \rho) \bar{\phi}_j
\end{equation}

and

\begin{equation}
J''(t) = 4\sum_{j=1}^{N} \int_{\mathbb{R}^n} D^2 \rho(\nabla \phi_j, \nabla \bar{\phi}_j) - \sum_{j=1}^{N} \int_{\mathbb{R}^n} (\Delta^2 \rho)|\phi_j|^2
\end{equation}

\begin{equation}
- \frac{2p-1}{p+1} \sum_{j=1}^{N} \mathcal{P}_j \int_{\mathbb{R}^n} (\Delta \rho)|\phi_j|^{p+1}
\end{equation}

\begin{equation}
- \frac{2p-1}{p+1} \sum_{i,j=1}^{N} \beta_{ij} \int_{\mathbb{R}^n} (\Delta \rho)|\phi_i|^{(p+1)/2}|\phi_j|^{(p+1)/2}
\end{equation}

under the assumption $\beta_{ij} = \beta_{ji}$. Especially, if we choose $\rho(x) = |x|^2$ (see [13] and [21]), we then get that

\begin{equation}
J'(t) = 4\mathfrak{R} \sum_{j=1}^{N} \int_{\mathbb{R}^n} (\nabla \phi_j \cdot x) \bar{\phi}_j
\end{equation}

and

\begin{equation}
J''(t) = 16Q(\Phi),
\end{equation}
where
\begin{equation}
Q(\Phi) := \frac{1}{2} \|\nabla \Phi\|_2^2 - \frac{n(p-1)}{4(p+1)} P(\Phi).
\end{equation}

Applying the classical energy argument, one has for \( p < 1 + 4/n \), the solution of (1) exists globally. In fact, assuming \( |E(\Phi_0)| < \infty \) and thanks to the Gagliardo-Nirenberg inequality on \( \mathbb{R}^n \), we find from the energy conservation law that
\[
\|\nabla \Phi\|_2^2 \leq 2E(\Phi_0) + C\|\nabla \Phi\|_2^{n(p-1)/2}\|\Phi\|_2^{p+1-n(p-1)/2}.
\]
Clearly an uniform bound on \( \|\nabla \Phi\|_2 \) results, provided \( p < 1 + 4/n \), and accordingly the solution exists globally. For \( p \geq 1 + 4/n \), blow-up of the solution may occur. In fact, if there exists a constant \( \delta < 0 \) such that \( Q(\Phi) \leq \delta < 0 \) or \( Q(\Phi) < 0 \) and \( J'(0) \leq 0 \) simultaneously, it’s obvious from the facts \( J''(t) \leq 16\delta < 0 \) or \( J'(0) \leq 0 \) and \( J''(t) < 0 \) that the solution blows up in finite time.

In the case \( p \geq 1 + 4/n \), the sharp thresholds of blow-up and global existence become very interesting. For the single Schrödinger equation, the sharp thresholds of blow-up and global existence have been extensively studied (see the related works \[4, 19, 23, 25\]). Our present work in this paper is to derive two types of sharp thresholds for the system (1). To our knowledge, these are the first results in this direction for the system (1), which seem new even for the single nonlinear Schrödinger equation on \( \mathbb{R}^n \). See Theorems 2, 3 below.

Recall that we have defined the \( C^0 \) functionals \( M(u), E(u) \) and \( Q(u) \) for \( u = (u_1, ..., u_N) \in (H^1(\mathbb{R}^n))^N \) in (2), (3) and (6).

**Theorem 2.** *(Sharp Threshold I)* Assume that \( 1 + 4/n \leq p < 1 + 4/(n-2)^+ \). The constrained variational problem
\[
d_I := \inf_{\{u \in (H^1(\mathbb{R}^n))^N \setminus \{0\}; \ G(u)=0\}} \frac{1}{2}\|\nabla u\|_2^2
\]
with
\[
G(u) = (M(u))^{p+1-n(p-1)/2} - \frac{1}{p+1} P(\Phi)
\]
satisfies \( d_I > 0 \). Besides, assume the initial data \( \Phi_0 \in (H^1(\mathbb{R}^n))^N \) satisfies
\[
(M(\Phi_0))^{p+1-n(p-1)/2} + E(\Phi_0) < d_I.
\]
We have:
(A). If \( G(\Phi_0) > 0 \), then the solution exists globally;
(B). If \( G(\Phi_0) < 0 \), \(|x|\Phi_0(x) \in (L^2(\mathbb{R}^n))^N\), and
\[
3 \sum_{j=1}^{N} \int_{\mathbb{R}^n} (\nabla \phi_{0j} \cdot x) \bar{\phi}_{0j} \leq 0
\]
when \( p > 1 + 4/n \), then the solution blows up in finite time.

**Theorem 3.** (Sharp Threshold II) Assume that \( 1 + 4/n < p < 1 + 4/(n-2)^+ \). Let \( \gamma > 0 \) be any fixed constant. The constrained variational problem
\[
d_{II} := d_{II}(\gamma) = \inf_{\{u \in (H^1(\mathbb{R}^n))^N \setminus \{0\}; \ Q(u)=0\}} (M(u)^\gamma + E(u))
\]
satisfies \( d_{II} > 0 \). Besides, assume the initial data \( \Phi_0 \in (H^1(\mathbb{R}^n))^N \) satisfies
\[
(M(\Phi_0))^\gamma + E(\Phi_0) < d_{II},
\]
then we have
(A). If \( Q(\Phi_0) > 0 \), the solution exists globally;
(B). If \( Q(\Phi_0) < 0 \) and \(|x|\Phi_0(x) \in (L^2(\mathbb{R}^n))^N\), the solution blows up in finite time.

As corollaries, we invoke the sharp thresholds to obtain small data criterions for the global existence of (1). We get the following two results.

**Corollary 4.** (Small Data Criterion I) Assume that \( 1 + 4/n \leq p < 1 + 4/(n-2)^+ \). Then if the initial data \( \Phi_0 \in (H^1(\mathbb{R}^n))^N \) satisfies
\[
\frac{1}{2} \|\nabla \Phi_0\|_2^2 + (M(\Phi_0))^{p+1-n(p-1)/2} < d_I,
\]
the solution of (1) exists globally.

**Corollary 5.** (Small Data Criterion II) Assume that \( 1 + 4/n < p < 1 + 4/(n-2)^+ \). Then if the initial data \( \Phi_0 \in (H^1(\mathbb{R}^n))^N \) satisfies
\[
\frac{1}{2} \|\nabla \Phi_0\|_2^2 + (M(\Phi_0))^\gamma < d_{II},
\]
the solution of (1) exists globally.

**Remark 6.** Notice that the first type of thresholds deals with \( p \geq 1 + 4/n \) while the second type only deals with \( p > 1 + 4/n \).

Both for the physical and mathematical reasons, in the last five years, many authors paid much attention to the Cauchy problem of the Schrödinger equation posed on an arbitrary Riemannian manifold \((M, g)\) with \( \Delta_g \) being the associated Laplace-Beltrami operator (where \( \Delta_g u = u'' \) on the real line \( \mathbb{R} \)). See the recent papers [2, 3, 6, 7, 8, 9] and
the references therein. In the setting of \((H^1(\mathcal{M}))^N\), the conservation laws of mass \(2\) and energy \(3\) hold true for \((\mathcal{M}, g)\) with \(\int_{\mathbb{R}^n}\) replaced by \(\int_{\mathcal{M}}\) (the volume integration on \(\mathcal{M}\)). The virial identities \(4\) and \(5\) are also valid with \(\rho\) being a \(C^4\) function on \(\mathcal{M}\) and \(\nabla_{\mathcal{M}}\) being the associated gradient operator \((20)\).

For \((\mathcal{M}, g)\) on \(\mathbb{H}^n\) and on \(S^2\), a similar local existence result as Proposition 1 still holds when we replace \((H^1(\mathbb{R}^n))^N\) in Proposition 1 by \((H^1(\mathbb{H}^n))^N\) and \((H^1(S^2))^N\) respectively. The readers can consult \([3, 6, 7]\) for more related discussions about the single Schrödinger equation on \(\mathbb{H}^n\) and \(S^n\). The reason why we restrict ourselves on \(S^2\) instead of \(S^n\) is that when \(n \geq 3\) the global wellposedness and the blow-up phenomena seem more delicate than the case \(n = 1, 2\). For \(n \geq 3\), some negative results of wellposedness on \(S^n\) attributed to N. Burq, P. Gérard, and N. Tzvetkov, which are in strong contrast with the case \(\mathbb{R}^n, \mathbb{H}^n\) and \(S^2\), can be found in \([6, 7, 9]\) (see also \([2]\)). Our results for the Schrödinger system \((1)\) on \(\mathbb{H}^n\) read as follows. We emphasize that on \(\mathbb{H}^n\) we have to make a difference dealing with the radial case and the nonradial case, due to the nonvanishing curvature of the manifold.

**Theorem 7.** (Sharp Threshold I on \(\mathbb{H}^n\): Radial Case) Assume that \(1 + 4/n < p < 1 + 4/(n - 2)^+\). The constrained variational problem

\[
\mathcal{J} := \inf_{\{u \in (H^1(\mathbb{H}^n))^N \backslash \{0\} : G(u) = 0\}} \frac{1}{2} \left\| \nabla_{\mathbb{H}^n} u \right\|^2_2
\]

with

\[
G(u) = (M(u))^{p+1-n(p-1)/2} - \frac{1}{p+1} \mathcal{P}(\Phi)
\]

satisfies \(\mathcal{J} > 0\). Besides, assume the initial data \(\Phi_0 \in (H^1(\mathbb{H}^n))^N\) is radial and satisfies

\[
(M(\Phi_0))^{p+1-n(p-1)/2} + E(\Phi_0) < \mathcal{J}.
\]

Then we have:

(A). If \(G(\Phi_0) > 0\), the solution exists globally;

(B). If \(G(\Phi_0) < 0\), \(|x| \Phi_0(x) \in (L^2(\mathbb{H}^n))^N\), and

\[
\exists \sum_{j=1}^N \int_{\mathbb{H}^n} (\nabla_{\mathbb{H}^n} \phi_{0j} \cdot \nabla_{\mathbb{H}^n} \rho) \bar{\phi}_{0j} \leq 0
\]

when \(p > 1 + 4/n\), the solution blows up in finite time.

Here \(\rho = r^2\), where \(r = r(x)\) is the geodesic distance from \(x \in \mathbb{H}^n\) to the origin \(O \in \mathbb{H}^n\).
Theorem 7 doesn’t work for the nonradial case. However, the second type of thresholds on $\mathbb{H}^n$ (see Theorem 8) holds for the nonradial case fortunately. To state it, we need the following definition

$$Q^*(\Phi) := \frac{1}{2}\|\nabla_{\mathbb{H}^n}\Phi\|^2_2 - \frac{(n-1)(p-1)}{4(p+1)}P(\Phi).$$

**Theorem 8. (Sharp Threshold II on $\mathbb{H}^n$)**

**Radial Case:**
Assume that $1 + 4/n < p < 1 + 4/(n-2)^+$. The constrained variational problem

$$d_{\mathbb{H}^n II} := \inf_{\{u \in (H^1(\mathbb{H}^n))^N \setminus \{0\}; \ Q(u)=0\}} (M(u))^\gamma + E(u)$$

with $\gamma > 0$ being an arbitrary constant satisfies $d_{\mathbb{H}^n II} > 0$. Besides, assume the initial data $\Phi_0 \in (H^1(\mathbb{H}^n))^N$ is radial and satisfies

$$(M(\Phi_0))^\gamma + E(\Phi_0) < d_{\mathbb{H}^n II},$$

then we have

(A). If $Q(\Phi_0) > 0$, the solution exists globally;

(B). If $Q(\Phi_0) < 0$ and $|x|\Phi_0(x) \in (L^2(\mathbb{H}^n))^N$, the solution blows up in finite time.

**Nonradial Case:**
Assume $n \geq 2$ and $1 + 4/(n-1) < p < 1 + 4/(n-2)^+$. The constrained variational problem

$$d_{\mathbb{H}^n II}^* := \inf_{\{u \in (H^1(\mathbb{H}^n))^N \setminus \{0\}; \ Q^*(u)=0\}} (M(u))^\gamma + E(u)$$

with $\gamma > 0$ being an arbitrary constant satisfies $d_{\mathbb{H}^n II}^* > 0$. Besides, assume the initial data $\Phi_0 \in (H^1(\mathbb{H}^n))^N$ satisfies

$$(M(\Phi_0))^\gamma + E(\Phi_0) < d_{\mathbb{H}^n II}^*,$$

then we have

(A). If $Q(\Phi_0) > 0$, the solution exists globally;

(B). If $Q(\Phi_0) < 0$ and $|x|\Phi_0(x) \in (L^2(\mathbb{H}^n))^N$, the solution blows up in finite time.

Corresponding to Theorems 7, 8, we have the small data criterions below.

**Corollary 9. (Small Data Criterion I on $\mathbb{H}^n$)** Assume that $1 + 4/n \leq p < 1 + 4/(n-2)^+$. Then if the initial data $\Phi_0 \in (H^1(\mathbb{H}^n))^N$ no matter radial or not satisfies

$$\frac{1}{2}\|\nabla_{\mathbb{H}^n}\Phi_0\|_2^2 + (M(\Phi_0))^{p+1-n(p-1)/2} < d_{\mathbb{H}^n I},$$

the solution of (7) exists globally.
Corollary 10. (Small Data Criterion II on $\mathbb{H}^n$) Assume that $1 + 4/n < p < 1 + 4/(n - 2)$. Then if the initial data $\Phi_0 \in (H^1(\mathbb{H}^n))^N$ no matter radial or not satisfies
\[
\frac{1}{2} \|\nabla_{\mathbb{H}^n} \Phi_0\|_2^2 + (M(\Phi_0))^\gamma < d_{\mathbb{H}^n,II},
\]
the solution of (1) exists globally.

We are now in position to state the sharp threshold for the Schrödinger system (1) posed on $S^2$. From the viewpoint of geometry, the compactness of $S^2$ results in the difference between the Sobolev embedding on $S^2$ and the ones on $\mathbb{R}^n$ and $\mathbb{H}^n$. To display the same spirit as in the analysis on $\mathbb{R}^n$ and $S^n$, we prefer to work on the function space
\[
\Lambda := \{u \in (H^1(S^2))^N \setminus \{0\}; u \text{ is antisymmetric about the equator}\},
\]
and we define
\[
Q^{**}(\Phi) := \frac{1}{2} \|\nabla_{S^2} \phi\|_2^2 - \frac{p - 1}{4(p + 1)} P(\Phi).
\]
Our results are as below.

Theorem 11. (Sharp Threshold on $S^2$) Assume that $5 < p < \infty$. Let $\gamma > 0$ be an arbitrary constant. The constrained variational problem
\[
d_{S^2} := \inf \{u \in \Lambda; Q^{**}(u) = 0\} (M(u))^\gamma + E(u)
\]
satisfies $d_{S^2} > 0$. Besides, assume the initial data $\Phi_0 \in \Lambda$ satisfies
\[
(M(\Phi_0))^\gamma + E(\Phi_0) < d_{S^2},
\]
then we have
(A) If $Q^{**}(\Phi_0) > 0$, the solution exists globally;
(B) If $Q^{**}(\Phi_0) < 0$, the solution blows up in finite time.

Corollary 12. (Small Data Criterion on $S^2$) Assume that $5 < p < \infty$. Then if the initial data $\Phi_0 \in \Lambda$ satisfies
\[
\frac{1}{2} \|\nabla_{S^2} \Phi_0\|_2^2 + (M(\Phi_0))^\gamma < d_{S^2},
\]
the solution of (1) exists globally.

The rest of our paper is organized as follows. In section 2, we prepare some abstract analysis for the invariant sets. In section 3, we give the proofs of Theorems 2, 3 and Corollaries 4, 5. In section 4, we give the proofs of Theorems 7, 8 and Corollaries 9, 10. In section 5, we give the proofs of Theorem 11 and Corollary 12. At the end, we give some heuristic analysis about the strong instability of the solitary waves in Section 6.
2. Some abstract analysis

In this section, we will establish the invariant sets for (1) via the three $C^0$ functionals $M(u)$, $E(u)$ and $Q(u)$. Our analysis can be formed as the following proposition. Let $\mathbb{M} = \mathbb{R}^n$, $\mathbb{H}^n$ or $\mathbb{S}^2$.

**Proposition 13.** (Invariant Sets) Let $F(u)$ and $G(u)$ be two $C^0$ functionals on $(H^1(\mathbb{M}))^N$, and $f(x,y)$ be a $C^0$ function on $\mathbb{R}^2$. Suppose that the cross-constrained minimization problem

$$d := \inf_{\{u \in (H^1(\mathbb{M}))^N \setminus \{0\}; \ G(u) = 0\}} F(u)$$

satisfies $d > 0$. If in addition

(7) \[ G(u) = 0 \Rightarrow F(u) \leq f(M(u), E(u)), \]

then the sets

$$K_+ = \{u \in (H^1(\mathbb{M}))^N; \ G(u) > 0, \ f(M(u), E(u)) < d\}$$

and

$$K_- = \{u \in (H^1(\mathbb{M}))^N; \ G(u) < 0, \ f(M(u), E(u)) < d\}$$

are all invariant sets of the Schrödinger system (1) on $\mathbb{M}$.

**Proof.** Assume $\Phi_0 \in K_+$, that is, $G(\Phi_0) > 0$ and $f(M(\Phi_0), E(\Phi_0)) < d$. Noticing that $M(\Phi)$ and $E(\Phi)$ are conservation quantities for (1), we have

$$f(M(\Phi), E(\Phi)) = f(M(\Phi_0), E(\Phi_0)) < d.$$ \(\quad\) We now show that $G(\Phi) > 0$. Otherwise, from the continuity, there were a $t^* \in (0, T)$ such that $G(\Phi(t^*)) = 0$ and $\Phi(t^*) \neq 0$. We infer from (7) that

$$F(\Phi(t^*)) \leq f(M(\Phi(t^*)), E(\Phi(t^*))) < d,$$

which is a contradiction with the minimization of $d$. Thus we get that $G(\Phi) > 0$ and therefore $\Phi \in K_+$.

By the same argument, we have $K_-$ is also invariant under the flow generated by (1). \(\square\)

**Remark 14.** The idea of this proposition goes back to H. Berestycki and T. Cazenave [4]. However, they restricted themselves only to the case

$$f(M, E) = M + E.$$ \(\quad\) As a consequence, they obtained the invariant sets for the Schrödinger equation only on $\mathbb{R}^2$. The reader will see below that we introduce

(8) \[ f(M, E) = M^\gamma + E \]
to enlarge the invariant sets of the Schrödinger equation on \( \mathbb{R}^2 \) to the Schrödinger system \([1]\) on \( \mathbb{R}^n \) for all \( n \geq 1 \) and on some other Riemannian manifolds. The power \( \gamma > 0 \) in \([8]\) relies heavily on the Gagliardo-Nirenberg inequality.

Suppose we already get that

\[
K_+ = \{ u \in (H^1(M))^N; \ G(u) > 0, \ f(M(u), E(u)) < d \}
\]

and

\[
K_- = \{ u \in (H^1(M))^N; \ G(u) < 0, \ f(M(u), E(u)) < d \}
\]

are invariant sets of \([1]\). Moreover, if we can show that there exist two constants \( M, \delta \) such that

\[
\Phi_0 \in K_+ \Rightarrow \| \nabla M \Phi \|_2 \leq M < \infty
\]

and

\[
\Phi_0 \in K_- \Rightarrow J''(t) \leq \delta < 0 \text{ or } J''(0) \leq 0 \text{ simultaneously}
\]

then we arrive at the conclusion that \( \Phi_0 \in K_+ \) implies the solution exists globally and \( \Phi_0 \in K_- \) implies that the solution blows up in finite time. In this sense, under the assumption \( f(M(\Phi_0), E(\Phi_0)) < d \), we say \( G(\Phi_0) = 0 \) is a sharp threshold of blow-up and global existence.

3. The proofs of Theorems 2, 3 and Corollaries 4, 5

This section is devoted to the proofs of Theorems 2, 3 and Corollaries 4, 5. Let’s recall the Gagliardo-Nirenberg inequality \([24]\) for \( 1 \leq p < 1 + 4/(n - 2)^+ \):

\[
\| \phi \|_{p+1} \leq C \| \nabla \phi \|_2^{(p-1)/2} \| \phi \|_2^{1-n(p-1)/2}, \quad \forall \ \phi \in H^1(\mathbb{R}^n).
\]

The proof of Theorem 2

Step 1. We claim that the constrained variational problem in Theorem 2 satisfies \( d_I > 0 \). For \( u \in (H^1(\mathbb{R}^n))^N \setminus \{0\} \) subjected to \( G(u) = 0 \), it follows from \([9]\) that

\[
(M(u))^{p+1-n(p-1)/2} = \frac{1}{p+1} \mathcal{P}(\Phi) \leq C \| \nabla u \|_2^{(p-1)/2} (M(u))^{p+1-n(p-1)/2},
\]

which indicates \( d_I > 0 \).

Step 2. Choosing \( F(u) = \frac{1}{2} \| \nabla u \|_2^2 \) and \( f(M, E) = M^{p+1-n(p-1)/2} + E \) in Proposition 13, we see that

\[
K_+ = \{ u \in (H^1(\mathbb{R}^n))^N; \ G(u) > 0, \ (M(u))^{p+1-n(p-1)/2} + E(u) < d_I \}
\]
and

\[ K_\pm = \{ u \in (H^1(\mathbb{R}^n))^N ; \ G(u) < 0, \ (M(u))^{p+1-n(p-1)/2} + E(u) < d_I \} \]

are invariant sets of (1).

Step 3. Assume that \( \Phi_0 \) satisfies \( G(\Phi_0) > 0 \). Then from step 2, we have \( G(\Phi) > 0 \) and \( (M(\Phi))^{p+1-n(p-1)/2} + E(\Phi) < d_I \), which imply

\[
\frac{1}{2} \| \nabla \Phi \|_2^2 < d_I,
\]

and consequently the solution exists globally.

Step 4. Assume that \( \Phi_0 \) satisfies \( G(\Phi_0) < 0 \). From step 2, we have \( G(\Phi) < 0 \) and \( (M(\Phi))^{p+1-n(p-1)/2} + E(\Phi) < d_I \).

Case (i): \( p = 1 + 4/n \). In this case, \( Q(\Phi) = E(\Phi) = E(\Phi_0) \). From \( G(\Phi_0) < 0 \) we get that there exists a \( \lambda \in (0, 1) \) such that

\[
G(\lambda \Phi_0) = 0, \text{ that is, } (M(\Phi_0))^{p-1} = \frac{\lambda^2}{p+1} P(\Phi).
\]

Then it follows from the minimization of \( d_I \) that

\[
\frac{1}{2} \| \nabla (\lambda \Phi_0) \|_2^2 \geq d_I > (M(\Phi_0))^{p-1} + E(\Phi_0).
\]

Inserting (10) into (11) yields

\[
\frac{\lambda^2}{2} \| \nabla \Phi_0 \|_2^2 \geq d_I > (M(\Phi_0))^{p-1} + E(\Phi_0) = \frac{1}{2} \| \nabla \Phi_0 \|_2^2 + \frac{\lambda^2 - 1}{p+1} P(\Phi),
\]

that is, \((1 - \lambda^2)E(\Phi_0) < 0\). Thus we have \( J''(t) = 16 E(\Phi_0) < 0 \) and therefore the solution blows up in finite time.

Case (ii). \( p > 1 + 4/n \). In this case, for any fixed \( t \in (0, T) \), there exists a \( \lambda \in (0, 1) \) such that \( G(\lambda \Phi) = 0 \), that is,

\[
(M(\Phi))^{p+1-n(p-1)/2} = \frac{\lambda^{n(p-1)/2}}{p+1} P(\Phi).
\]

Then it follows from the minimization of \( d_I \) that

\[
\frac{1}{2} \| \nabla (\lambda \Phi) \|_2^2 \geq d_I > (M(\Phi))^{p+1-n(p-1)/2} + E(\Phi).
\]

Inserting (12) into (13) yields

\[
\frac{\lambda^2}{2} \| \nabla \Phi \|_2^2 \geq d_I > (M(\Phi))^{p+1-n(p-1)/2} + E(\Phi) = \frac{1}{2} \| \nabla \Phi_0 \|_2^2 + \frac{\lambda^{n(p-1)/2} - 1}{p+1} P(\Phi),
\]
that is,
\[ \frac{1}{2} \|\nabla \Phi\|^2 \leq \frac{\lambda^{-n(p-1)/2} - 1}{1 - \lambda^2} (M(\Phi))^{p+1-n(p-1)/2}. \]

We infer from the above inequality that
\[
J''(t) = Q(\Phi) = \frac{1}{2} \|\nabla \Phi\|^2 - \frac{n(p-1)}{4} \lambda^{-n(p-1)/2} (M(\Phi))^{p+1-n(p-1)/2} \\
\leq h(\lambda) (M(\Phi))^{p+1-n(p-1)/2} < 0,
\]
with the fact
\[
h(\lambda) = \frac{\lambda^{-n(p-1)/2}}{1 - \lambda^2} (1 - \lambda^n(p-1)/2 - \frac{n}{4}(p-1)(1 - \lambda^2)) < 0, \quad \forall \lambda \in (0, 1)
\]
used in the last step.

Thus we get that \(J'(0) \leq 0\) and \(J''(t) < 0\), which suggest that the solution blows up in finite time. The proof of Theorem 2 is concluded.

The proof of Corollary 4.
From the assumption
\[
\frac{1}{2} \|\nabla \Phi_0\|^2 + (M(\Phi_0))^{p+1-n(p-1)/2} < d_I,
\]
it’s obvious that
\[
(M(\Phi_0))^{p+1-n(p-1)/2} + E(\Phi_0) < d_I.
\]
In view of Theorem 2 we only have to check that \(G(\Phi_0) > 0\). If else, one would have \(G(\Phi_0) \leq 0\). Due to the minimization of \(d_I\), \(G(\Phi_0) \neq 0\).

If \(G(\Phi_0) < 0\), there exists a \(\lambda \in (0, 1)\) such that \(G(\lambda \Phi_0) = 0\) and consequently we have
\[
\frac{1}{2} \|\nabla (\lambda \Phi_0)\|^2 \geq d_I,
\]
which is contradictory with
\[
\frac{1}{2} \|\nabla \Phi_0\|^2 < d_I.
\]

The proof of Theorem 3.
Step 1. The constrained variational problem in Theorem 3 satisfies \(d_{II} > 0\). We argue by contradiction. Suppose there exists a sequence
\( u_k \in (H^1(\mathbb{R}^n))^N \setminus \{0\} \) satisfying \( Q(u_k) = 0 \) and \( (M(u_k))^\gamma + E(u_k) \to 0 \) as \( k \to 0 \). By \( Q(u_k) = 0 \) we get that

\[
(M(u_k))^\gamma + E(u_k) = (M(u_k))^\gamma + \frac{n(p-1) - 4}{2n(p-1)}\|\nabla u_k\|_2^2 \to 0,
\]

which indicates that \( M(u_k) \to 0 \) and \( \|\nabla u_k\|_2 \to 0 \). On the other hand, by the Gagliardo-Nirenberg inequality, we get from \( G(u_k) = 0 \) that

\[
\frac{1}{2}\|\nabla u_k\|_2^2 = \frac{n(p-1)}{4(p+1)} P(\Phi) \leq C\|\nabla u_k\|_2^{n(p-1)/2}(M(u_k))^{(p+1)-n(p-1)/2},
\]

that is

\[
\|\nabla u_k\|_2^{n(p-1)/2-2} (M(u_k))^{(p+1)-n(p-1)/2} \geq \frac{1}{2C} > 0,
\]

which contradicts with \( M(u_k) \to 0 \) and \( \|\nabla u_k\|_2 \to 0 \).

Step 2. Choosing \( F(u) = (M(u))^\gamma + E(u) \) and \( f(M,E) = M^\gamma + E \) in Proposition 13 we have that

\[
K_+ = \{ u \in (H^1(\mathbb{R}^n))^N; \ Q(u) > 0, \ (M(u))^\gamma + E(u) < d_{II} \}
\]

and

\[
K_- = \{ u \in (H^1(\mathbb{R}^n))^N; \ Q(u) < 0, \ (M(u))^\gamma + E(u) < d_{II} \}
\]

are invariant sets of \([11]\).

Step 3. Assume that \( \Phi_0 \) satisfies \( Q(\Phi_0) > 0 \). Then from step 2, we have \( Q(\Phi) > 0 \) and \( (M(\Phi))^\gamma + E(\Phi) < d_{II} \), which imply

\[
\frac{n(p-1) - 4}{2n(p-1)}\|\nabla \Phi\|_2^2 < d_{II},
\]

and consequently the solution exists globally.

Step 4. Assume that \( \Phi_0 \) satisfies \( Q(\Phi_0) < 0 \). From step 2, we have \( Q(\Phi) < 0 \) and \( (M(\Phi))^\gamma + E(\Phi) < d_{II} \). We assert that

\[
Q(\Phi) \leq (M(\Phi_0))^\gamma + E(\Phi_0) - d_{II} < 0,
\]

following which the solution blows up in finite time.

In actuality, the fact \( Q(\Phi) < 0 \) yields a \( \lambda \in (0, 1) \) such that \( Q(\lambda \Phi) = 0 \) and accordingly \( (M(\lambda \Phi))^\gamma + E(\lambda \Phi) \geq d_{II} \). Moreover, \( Q(\Phi) < 0 \).
implies that $P(\Phi) > 0$. Next, we do computation to achieve
\[
(M(\Phi_0))^{\gamma} + E(\Phi_0) - d_{II} \\
\geq [(M(\Phi))^{\gamma} + E(\Phi)] - [(M(\lambda\Phi))^{\gamma} + E(\lambda\Phi)] \\
= (1 - \lambda^\gamma)(M(\Phi))^{\gamma} + \frac{1}{2}(1 - \lambda^2)\|\nabla \Phi\|_2^2 - \frac{1 - \lambda^{p+1}}{p + 1}P(\Phi) \\
\geq \frac{1}{2}(1 - \lambda^2)\|\nabla \Phi\|_2^2 - \frac{n(p-1)(1 - \lambda^{p+1})}{4(p + 1)}P(\Phi) \\
= Q(\Phi) - Q(\lambda\Phi) = Q(\Phi).
\]
This concludes the proof of Theorem 3.

The proof of Corollary 5.

From the assumption
\[
\frac{1}{2} \|\nabla \Phi_0\|_2^2 + (M(\Phi_0))^{\gamma} < d_{II},
\]
it’s obvious that
\[
(M(\Phi_0))^{\gamma} + E(\Phi_0) < d_{II}.
\]
In view of Theorem 3, we only have to check that $Q(\Phi_0) > 0$. If else, one would have $Q(\Phi_0) \leq 0$. Due to the minimization of $d_{II}$, $Q(\Phi_0) \neq 0$. If $Q(\Phi_0) < 0$, there exists a $\lambda \in (0, 1)$ such that $Q(\lambda\Phi_0) = 0$ and consequently we have
\[
(M(\lambda\Phi_0))^{\gamma} + E(\lambda\Phi_0) \geq d_{II} \\
\Rightarrow \frac{\lambda^2}{2}\|\nabla \Phi_0\|_2^2 + \lambda^{\gamma}(M(\Phi_0))^{\gamma} \geq d_{II},
\]
which is contradictory with
\[
\frac{1}{2} \|\nabla \Phi_0\|_2^2 + (M(\Phi_0))^{\gamma} < d_{II}.
\]

4. The proofs of Theorems 7, 8 and Corollaries 9, 10

In this section, we focus on the Schrödinger system on $\mathbb{H}^n$. The Sobolev inequality on the hyperbolic space (see [14]) writes as
\[
\|\phi\|_{2n/(n-2)} \leq K_n\|\nabla_{\mathbb{H}^n}\phi\|_2 - \omega_n^{-2/n}\|\phi\|_2, \ \forall \ \phi \in H^1(\mathbb{H}^n),
\]
where $K_n$ is the best constant for the Sobolev embedding on $\mathbb{R}^n$, and $\omega_n$ is the volume of the sphere $\mathbb{S}^n$. By interpolation between the $L^2$
and the $L^{2n/(n-2)}$ norms, we get the Gagliardo-Nirenberg inequality for functions on $H^1(\mathbb{H}^n)$ for $1 \leq p < 1 + 4/(n-2)$:

$$\|\phi\|_{p+1}^{p+1} \leq C\|\nabla_{\mathbb{H}^n}\phi\|_2^{n(p-1)/2}\|\phi\|_2^{p+1-n(p-1)/2}, \ \forall \phi \in H^1(\mathbb{H}^n).$$

Let’s firstly consider the radial case.

**The proofs of Theorem 7 and the radial case of Theorem 8.**

If the initial data $\Phi_0$ is radial about the origin $O \in \mathbb{H}^n$, by the symmetry of the system (1) we see easily the solution $\Phi$ is also radial. We take $\rho = r^2$, where $r = r(x)$ is the geodesic distance from $x \in \mathbb{H}^n$ to $O \in \mathbb{H}^n$. By the noteworthy estimates (see [3] for details)

$$\begin{cases}
D^2\rho(\nabla_{\mathbb{H}^n}\phi_j, \nabla_{\mathbb{H}^n}\bar{\phi}_j) \leq 2|\nabla_{\mathbb{H}^n}\phi_j|^2, \\
\Delta_{\mathbb{H}^n}\rho > 0, \\
\Delta_{\mathbb{H}^n}\rho \geq 2n,
\end{cases}$$

we indicate from (5) that

$$J''(t) \leq 16Q(\Phi)$$

with $Q(\Phi)$ defined as in (6). Then the proofs of Theorem 7, the radial case of Theorem 8 proceed exactly the same as the ones of Theorems 2, 3.

Now we turn to the nonradial case.

**The proof of the nonradial case of Theorem 8.**

When $\Phi$ is nonradial, the crucial estimate

$$D^2\rho(\nabla_{\mathbb{H}^n}\phi_j, \nabla_{\mathbb{H}^n}\bar{\phi}_j) \leq 2|\nabla_{\mathbb{H}^n}\phi_j|^2$$

doesn’t hold. We choose another positive radial function

$$\rho(r) = \int_0^r \left( \int_0^s \sinh^{n-1}\tau d\tau \right) \left( \sinh^{n-1}s \right)^{-1} ds,$$

which satisfies (see [20] for details)

$$\begin{cases}
D^2\rho(\nabla_{\mathbb{H}^n}\phi_j, \nabla_{\mathbb{H}^n}\bar{\phi}_j) \leq \frac{1}{n-1}|\nabla_{\mathbb{H}^n}\phi_j|^2, \\
\Delta_{\mathbb{H}^n}\rho = 1.
\end{cases}$$

Then from (5) we obtain that

$$J''(t) \leq \frac{8}{n-1}Q^*(\Phi)$$

Following the proof of Theorem 8 with the modification that $Q(\Phi)$ is substituted by $Q^*(\Phi)$ and $p > 1 + 4/n$ is substituted by $p > 1 + 4/(n-1)$, we easily arrive at the conclusions of Theorem 8.

**The proof of Corollaries 9, 10.**
The idea to prove Corollary 9 is the same as the proof of Corollary 4. In fact, in view of Proposition 13 in section 2, we see that

\[ K_+ = \{ \Phi \in (H^1(\mathbb{H}^n))^N; G(\Phi) > 0, (M(\Phi))^{p+1-n(p-1)/2} + E(\Phi) < d_{\mathbb{H}^n} \} \]

is an invariant set under the flow generated by the Schrödinger system (1) on \( \mathbb{H}^n \). Once \( \Phi \in K_+ \), it follows that

\[ \frac{1}{2} \| \nabla_{\mathbb{H}^n} \Phi \| < d_{\mathbb{H}^n}, \]

which yields the global existence of the solution \( \Phi \). We check as exactly as we did in the proof of Corollary 5 that

\[ \Phi_0 \in K_+ \]

and subsequently the proof of Corollary 9 is concluded. The proof of Corollary 10 proceeds along the way of the proof of Corollary 5 similarly, and the details are omitted.

\[ \square \]

5. THE PROOF OF THEOREM 11 AND COROLLARY 12

In this section, we complete the proofs of Theorem 11 and Corollary 12. The Sobolev embedding has its analogue on \( \mathbb{S}^n \). See the following proposition, which is taken from [14].

**Proposition 15.** Assume \( 2 \leq p \leq 2n/(n-2) \) when \( n \geq 3 \) and \( 2 \leq p < \infty \) when \( n = 2 \). Then for any \( \phi \in H^1(\mathbb{S}^n) \), there holds

\[ (\int_{\mathbb{S}^n} |\phi|^p)^{2/p} \leq \frac{p-2}{n\omega_n^{1-2/p}} \int_{\mathbb{S}^n} |\nabla_{\mathbb{S}^n} \phi|^2 + \frac{1}{\omega_n^{1-2/p}} \int_{\mathbb{S}^n} |\phi|^2. \]

Hereafter we concentrate on \( \mathbb{S}^2 \) and work on the space

\[ \Lambda := \{ u \in (H^1(\mathbb{S}^2))^N \setminus \{0\}; \ u \text{ is antisymmetric about the equator} \}. \]

We have the following estimate.

**Proposition 16.** For any function \( \phi \in H^1(\mathbb{S}^n) \) which is antisymmetric about the equator, there holds

\[ \| \phi \|_2 \leq 4 \| \nabla_{\mathbb{S}^2} \phi \|_2. \]

Before proving this proposition, we list some facts which will be used in the sequel. In the paper of the same authors [20], we introduce the positive function

\[ \rho(r) = \begin{cases} -2 \log \cos(r/2), & 0 < r \leq \pi/2, \\ 0, & r = 0. \end{cases} \]

We cut off the sphere \( \mathbb{S}^2 \) along the equator into two hemispheres \( \mathbb{S}^2_+ \) and \( \mathbb{S}^2_- \), which contains the north pole \( N \in \mathbb{S}^2 \) and the south pole \( S \in \mathbb{S}^2 \).
respectively. If we regard $r = r(x)$ as the sphere distance between the point $x \in S^2$ to $N \in S^2$ or to $S \in S^2$, then $\rho = \rho(r)$ is a $C^4$ function defined on $S^2_+$ radial about $N$ or defined on $S^2_-$ radial about $S$. We denote it by $\rho_+$ and $\rho_-$ respectively. An remarkable property of $\rho_+$ and $\rho_-$ is that

$$\Delta_{S^2} \rho_{\pm} = 1.$$  

Furthermore, we have (see [20] for details)

$$\left\{ \begin{array}{ll} |\nabla_{S^2} \rho_{\pm}| & \leq 1, \\ D^2 \rho_{\pm}(\nabla_{S^2} \phi, \nabla_{S^2} \bar{\phi}) & \leq |\nabla_{S^2} \phi|^2, \quad \forall \phi \in H^1(S^2_\pm). \end{array} \right.$$  

We now prove Proposition [16].

**Proof.** Noticing that $\phi = 0$ on $\partial S^2_+ = \partial S^2_-$, we can use the technique of integration by parts to obtain that

$$\int_{S^2_+} |\phi|^2 = \int_{S^2_+} |\phi|^2 \Delta_{S^2} \rho_+ + \int_{S^2_-} |\phi|^2 \Delta_{S^2} \rho_-$$

$$= -\int_{S^2_+} \nabla_{S^2} |\phi|^2 \cdot \nabla_{S^2} \rho_+ - \int_{S^2_-} \nabla_{S^2} |\phi|^2 \cdot \nabla_{S^2} \rho_-$$

$$\leq 2 \int_{S^2_+} |\phi| |\nabla_{S^2} \phi| + 2 \int_{S^2_-} |\phi| |\nabla_{S^2} \phi|$$

$$\leq 2 \left( \int_{S^2} |\phi|^2 \right)^{1/2} \left( \int_{S^2} |\nabla_{S^2} \phi|^2 \right)^{1/2},$$

which gives the desired conclusion. 

Combining Propositions [15] and [16] we achieve for any $1 \leq p < \infty$, there exists a universal constant $C$ such that

$$\|\Phi\|_{p+1} \leq C \|\nabla_{S^2} \Phi\|_2^{p+1}, \quad \forall \Phi \in \Lambda,$$

which is a Sobolev type estimate. By virtue of (14), we argue as before to see that the constrained variational problem in Theorem [11] satisfies $d_{S^2} > 0$.

We define

$$J(t) = \int_{S^2_+} \rho_+ |\Phi|^2 + \int_{S^2_-} \rho_- |\Phi|^2.$$  

As in [20], we get that
\[ J''(t) \leq 4\left( \int_{S^2} \rho_+(\nabla_{S^2}\Phi, \nabla_{S^2}\Phi) + \int_{S^2} \rho_-(\nabla_{S^2}\Phi, \nabla_{S^2}\Phi) \right) - 2\frac{p-1}{p+1} \mathcal{P}(\Phi) \]
\[ \leq 4 \int_{S^2} |\nabla_{S^2}\Phi|^2 - 2\frac{p-1}{p+1} \mathcal{P}(\Phi) \]
\[ = 8Q^{**}(\Phi). \]

In view of (15), following the proof of Theorem 3 and Corollary 5 with
\[ p > 5 \] and \[ Q \] replaced by \[ Q^{**} \], we arrive at the conclusions of Theorem 11 and Corollary 12.

**Remark 17.** The sharp threshold of blow-up and global existence for the Schrödinger system (1) posed on \( S^2 \) with the initial data \( \Phi_0 \in (H^1(S^2))^N \setminus \Lambda \) leaves open.

6. **Remarks on instability of the solitary waves**

In this section, we are concerned with the strong instability of the solitary waves. We only consider the Schrödinger system (1) on \( \mathbb{R}^n \). For any \( \lambda_j > 0, j = 1, \ldots, N \), we define
\[ M_\lambda(\cdot) = \left( \sum_{j=1}^N \frac{\lambda_j}{2} \int_{\mathbb{R}^n} |\phi_j|^2 \right)^{1/2}. \]

Noticing that as a \( L^2 \) norm, \( M_\lambda(\cdot) \) is equivalent to \( M(\cdot) \), the conclusions of Theorem 3 still work with \( M \) replaced by \( M_\lambda \). Let \( \gamma = 2 \) in Theorem 3 and we are led to the variational minimizing problem
\[ d_{II} := \inf_{\{u \in (H^1(\mathbb{R}^n))^N \setminus \{0\}; Q(u) = 0\}} (M_\lambda(u))^2 + E(u). \]

We have proved that \( d_{II} > 0 \). In addition, we believe that under some reasonable assumptions, this minimization can be attained by some function \( w \in (H^1(\mathbb{R}^n))^N \setminus \{0\} \) subjected to an Euler-Lagrangian equation. Recently, there has been some literature on this topic, see [17, 22]. For our purpose, we make the following assumption.

**Assumption:** the minimization of (16) is attained by some function \( w \in (H^1(\mathbb{R}^n))^N \setminus \{0\} \), which satisfies
\[ \Delta w_j - \lambda_j w_j + \mu_j |w_j|^{p-1} w_j + \sum_{i \neq j} \beta_{ij} |w_i|^{(p+1)/2} |w_j|^{(p-3)/2} w_j \]
for \( j = 1, \ldots, N \).
It’s obvious that \( \phi_j(x,t) := e^{i\lambda_j t} w_j(x) \) is a solution to (1), which is called a ground solitary wave physically. Multiplying (17) by \( \bar{w}_j \) and integrating over \( \mathbb{R}^n \) by parts, we get that

\[
S(w) := \|\nabla w\|^2_2 + 2(M_\lambda(w))^2 - \mathcal{P}(w) = 0. \tag{18}
\]

Multiplying (17) by \( x \cdot \nabla \bar{w}_j \) and integrating over \( \mathbb{R}^n \) by parts, we get the Pohozaev identity

\[
\frac{n}{2} - 1)\|\nabla w\|^2_2 + n(M_\lambda(w))^2 - \frac{n}{p + 1}\mathcal{P}(w) = 0. \tag{19}
\]

Combining (18) and (19), we obtain

\[
Q(w) = 0.
\]

After these preliminaries, we prove the following instability theorem.

**Theorem 18.** Suppose that \( 1 + 4/n < p < 1 + 4/(n - 2)^+ \) and the above Assumption holds. Then for any \( \epsilon > 0 \), there exists a \( \Phi_0 \in (H^1(\mathbb{R}^n))^N \) with \( \|\Phi_0 - w\| < \epsilon \) such that the solution \( \Phi \) to the Schrödinger system (1) with initial data \( \Phi_0 \) blows up in finite time.

**Proof.** From \( S(w) = 0 \) and \( Q(w) = 0 \), we have

\[
S(kw) < 0, \quad Q(kw) < 0, \quad \forall \ k > 1.
\]

On the other hand, noticing that

\[
\frac{d}{dk} ((M_\lambda(kw))^2 + E(kw)) = \frac{1}{k}S(kw) < 0, \quad \forall \ k > 1,
\]

we obtain simultaneously for all \( k > 1 \) that

\[
\left\{
\begin{array}{l}
(M_\lambda(kw))^2 + E(kw) < d_{II}, \\
Q(kw) < 0,
\end{array}
\right.
\]

which suggest from Theorem 3 that the solution to (1) with initial data \( \Phi_0 = kw \) blows up in finite time. Then any \( \Phi_0 = kw \) with \( 1 < k < 1 + \epsilon \) is the desired one.

\( \square \)

Similar discussions can be made about the Schrödinger system (1) posed on \( \mathbb{H}^n \) and \( \mathbb{S}^2 \). However, due to the loss of the variational characterizations about the ground solitary solutions when (1) is considered on manifolds, we prefer not to go deep in this direction.
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