THRESHOLD SOLUTIONS FOR THE FOCUSING GENERALIZED HARTREE EQUATIONS

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Abstract. We study the global behavior of solutions to the focusing generalized Hartree equation with \( H^1 \) data at mass-energy threshold in the inter-range case. In the earlier works of Arora-Roudenko [2], the behavior of solutions below the mass-energy threshold was classified. In this paper, we first exhibit three special solutions: \( e^{it}Q, Q^\pm \), where \( Q^\pm \) exponentially approach to the \( e^{it}Q \) in the positive time direction, \( Q^+ \) blows up and \( Q^- \) scatters in the negative time direction. Then we classify solutions at this threshold, showing that they behave exactly as the above three special solutions up to symmetries, or scatter or blow up in both time directions. The argument relies on the uniqueness and non-degeneracy of ground state, which we regard as an assumption for the general case.

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1. Introduction

In this paper, we consider the Cauchy problem

\[
\begin{cases}
    i\partial_t u + \Delta u + (|\cdot|^{-(N-\gamma)} * |u|^p) |u|^{p-2} u = 0, \ (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\
    u(0, x) = u_0(x) \in H^1(\mathbb{R}^N),
\end{cases}
\]

where \( \gamma \in (0, N) \) and \( p \geq 2 \).
The equation (1.1) is a generalization of the standard Hartree equation with $p = 2$,

$$i\partial_t u + \Delta u + \left( | \cdot |^{-(N-\gamma)} \ast |u|^2 \right) u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

which can be considered as a classical limit of a field equation describing a quantum mechanical non-relativistic many-boson system interacting through a two-body potential $V(x) = \frac{1}{|x|^{N-\gamma}}$, see [14]. How it arises as an effective evolution equation in the mean-field limit of many-body quantum systems can be traced to [15]. Lieb & Yau [28] mentioned it in a context of developing theory for stellar collapse, and in particular, in the boson particles setting. A special case of the convolution term $\frac{1}{|x|}$ in $\mathbb{R}^3$ is referred as the Coulomb potential, which goes back to the work of Lieb [26] and has been intensively studied since then, see reviews [12] and [13].

The equation (1.1) can be written as the Schrödinger-Poisson system of the form

$$\begin{cases}
    i\partial_t u + \Delta u + V|u|^{p-2}u = 0, \\
    (-\Delta)\frac{\partial}{\partial t} V = (N-2)|S|^{N-1}|u|^p.
\end{cases}$$

This can be thought of as an electrostatic version of the Maxwell-Schrödinger system, describing the interaction between the electromagnetic field and the wave function related to a quantum nonrelativistic charged particle, see [27] for examples.

The Cauchy problem (1.1) is locally wellposed in $H^1(\mathbb{R}^N)$ for

$$\begin{cases}
    2 \leq p \leq 1 + \frac{2}{N-2}, & \text{if } N \geq 3, \\
    2 \leq p < \infty, & \text{if } N = 1, 2,
\end{cases}$$

see [2] for details. We denote the forward lifespan by $[0, T_+]$ and the backward by $(T_-, 0]$. If $T_+(u) < +\infty$, then $\|u(t)\|_{H^1} \to \infty$, and it is said that the solution blows up in finite time. And the same as $T_-(u) > -\infty$.

When in its lifespan, the solutions to (1.1) satisfy mass, energy and momentum conservations:

$$M[u(t)] \triangleq \int |u(x, t)|^2 \, dx = M[u_0],$$
$$E[u(t)] \triangleq \frac{1}{2} \int |\nabla u(x, t)|^2 \, dx - \frac{1}{2p} \int \left( | \cdot |^{-(N-\gamma)} \ast |u(\cdot, t)|^p \right) (x)|u(x, t)|^p \, dx = E[u_0],$$
$$P[u](t) = 3 \int \bar{u}(x, t) \nabla u(x, t) \, dx = P[u_0].$$

The equation (1.1) has several invariances: if $u(x, t)$ is a solution to (1.1), then

- by invariance under scaling, so is $\lambda^{\frac{\gamma+2}{2(p-1)}} u(\lambda x, \lambda^2 t), \lambda > 0$;
- by invariance under spatial translation, so is $u(x + x_0, t), x_0 \in \mathbb{R}^N$;
- by invariance under phase rotation, so is $e^{i\theta_0} u, \theta_0 \in \mathbb{R}$;
- by invariance under Galilean transformation, so is $e^{ix \cdot \xi_0 - it |\xi_0|^2} u(x - 2\xi_0 t, t), \xi_0 \in \mathbb{R}^N$;
- by invariance under time translation, so is $u(x, t + t_0), t_0 \in \mathbb{R}$;
- by invariance under time reversal symmetry, so is $u(x, -t)$.

By scaling invariance,

$$\|\lambda^{\frac{\gamma+2}{2(p-1)}} u_0(\lambda x)\|_{\dot{H}^{s_c}(\mathbb{R}^N)} = \|u_0\|_{\dot{H}^{s_c}(\mathbb{R}^N)},$$

where

$$s_c = \frac{N}{2} - \frac{\gamma + 2}{2(p-1)},$$

(1.2)
and here we call the problem (1.1) is $\dot{H}^{s_c}$-critical. In particular, when $s_c \in (0,1)$, we say the problem (1.1) is in the inter-range case.

Furthermore, as shown in [2, Section 4], the minimization problem

$$
\inf_{0 \neq u \in H^1(\mathbb{R}^N)} \frac{\|u\|_2^{(N+\gamma)-(N-2)p} \|\nabla u\|_2^{Np-(N+\gamma)}}{\int_{\mathbb{R}^N} (|\cdot|^{-(N-\gamma)} * |u|^p) |u|^p dx}
$$

(1.3)

has a minimizer $Q$, and we call it the ground state of problem (1.3). As shown in [2, Section 4], such minimizer $Q$ is a strictly positive, radially symmetric and decreasing function. Moreover, after scaling if necessary, $Q$ satisfies the following Euler-Lagrange equation

$$
-\Delta Q + Q - \left(|\cdot|^{-(N-\gamma)} * Q^p\right) Q^{p-1} = 0.
$$

(1.4)

As for the uniqueness and nondegeneracy of the ground state, where the nondegeneracy means that

$$
\ker L_+ = \text{span}\{\partial_{x_1} Q, ..., \partial_{x_N} Q\} \text{ and } L_+ \text{ is defined in (3.2),}
$$

it is an intricate issue for general $(\gamma, p, N)$ and is still an open question at least to the author’s knowledge. As for the uniqueness of ground state to (1.4) for $(\gamma, p, N) = (2, 2, 3)$, it dates back to [26], later it was extended to dimension $N = 4$ by Krieger, Lenzmann and Raphaël in [20], and Arora and Roudenko [2] generalized to $3 \leq N \leq 5$ with $(\gamma, p) = (2, 2)$. In addition, Lenzmann dealt with the non-degeneracy of the ground state when $(\gamma, p, N) = (2, 2, 3)$ in [22]. With the satisfying result for $(\gamma, p) = (2, 2)$ as a preliminary, there are also some related results for $(\gamma, p)$ sufficiently closed to $(2, 2)$ by the perturbation method. As for $(\gamma, p, N) = (2, 2 + \varepsilon, 3)$ with $0 < \varepsilon \ll 1$, Xiang [39] proved the uniqueness and the nondegeneracy of the ground state. In recent work by Li [24], he extended the results in [39] and proved the uniqueness and nondegeneracy of ground state for $(\gamma, p)$ close to $(2, 2)$ when $N \in \{3, 4, 5\}$. Furthermore, when $N \geq 3$, Seok [36] checked the validity of the uniqueness and non-degeneracy of the ground state for $p \in \left(2, \frac{2N}{N-2}\right)$ and $\gamma$ sufficiently close to 0 or $p \in \left[2, \frac{2N}{N-2}\right)$ and $\gamma$ sufficiently close to $N$.

Next, as in [2], Arora and Roudenko consider the global behavior of solutions to (1.1) below the mass-energy threshold. Precisely, if we define

- **renormalized mass-energy**: $\mathcal{M}E[u] = \frac{M[u]^\theta E[u]}{M[Q]^\theta E[Q]}$,

- **renormalized gradient**: $\mathcal{G}[u] = \frac{\|u\|_2^\theta \|\nabla u\|_2}{\|Q\|_2^\theta \|\nabla Q\|_2}$,

- **renormalized momentum**: $\mathcal{P}[u] = \frac{\|u\|_2^{\theta - 1} P[u]}{\|Q\|_2^{\theta} \|\nabla Q\|_2}$,

where $\theta = \frac{1-s_c}{s_c}$, then

**Theorem 1.1** ([2], classification of solutions to (1.1) below mass-energy threshold). Assume $\gamma \in (0, N)$, $p \geq 2$ and $s_c \in (0,1)$. Let $u_0 \in H^1(\mathbb{R}^N)$ with $P[u_0] = 0$ and let $u(t)$ be the corresponding solution to (1.1) with the maximal time interval of existence $(T, T_+)$. Suppose that $\mathcal{M}E[u_0] < 1$.

1. If $\mathcal{G}[u_0] < 1$, then the solution exists globally in time with $\mathcal{G}[u(t)] < 1$ for all $t \in \mathbb{R}$, and $u$ scatters in $H^1(\mathbb{R}^N)$;
2. If $\mathcal{G}[u_0] > 1$, then $\mathcal{G}[u(t)] > 1$, for all $t \in (T, T_+)$. Moreover, if $u_0$ is of finite variance or $u_0 \in L^2(\mathbb{R}^N)$ is radial, then the solution blows up in finite time. If $u_0$ is of infinite variance and nonradial, then either the solution blows up in finite time or there exists a time sequence $t_n \to \infty$ (or $t_n \to -\infty$) such that $\|\nabla u(t_n)\|_{L^2(\mathbb{R}^N)} \to \infty$. 


When it comes to the dynamics of threshold solutions, Duyckaerts and Merle dealt with the energy-critical NLW and NLS in [8] and [9] respectively for \( N \in \{3, 4, 5\} \), with high-dimensional generalizations in [23] and [6] respectively. Moreover, Miao, Wu and Xu considered the energy-critical Hartree equations in [29]. As for the inter-range case, Duyckaerts and Roudenko solved the classification of threshold solutions to cubic NLS in \( \mathbb{R}^3 \) in [10], and later Campos, Farah and Roudenko extended the results for NLS to all the inter-range cases in [6].

The dynamics above the mass-energy threshold is mostly open. As for NLS, [35] constructed a stable manifold near the soliton family with an improvement to optimal topology in [3], and [32] clarified the global dynamics of radial solutions slightly above the threshold. When it is above the threshold, after [18] gave two blow-up criteria, [11] showed a scattering versus blow-up dichotomy for finite variance solutions. Moreover, the reader can refer to [4], [19], [21] and to [31], [33], [34] for the dynamics of NLW and NLKG respectively.

In this paper, we mainly consider the threshold solutions to (1.1) for \( s_c \in (0, 1) \) and \( p \geq 2 \). The argument needs the uniqueness of ground state, and we regard it as an assumption for general case:

**Assumption 1.2.** *(uniqueness of the ground state)* Assume \( \gamma \in (0, N), p \geq 2 \) and \( s_c \in (0, 1) \), then the minimizer of (1.3) is unique up to symmetries.

In addition, the argument in this paper needs the spectral properties of linearized operator \( L \) and its coercivity, requiring us figure out the null space of \( L \), which we also have to put as an assumption for general case.

**Assumption 1.3.** *(nondegeneracy of the ground state)* As for \( (\gamma, p, N) \) shown above, the kernel of linearized operator \( L_+ \) is

\[
\ker L_+ = \text{span}\left\{ \partial_{x_1} Q, \partial_{x_2} Q, \ldots, \partial_{x_N} Q \right\},
\]

(1.5)

where \( L_+ \) is defined in (3.2).

Next, we clarify the restrictions on parameters \( (\gamma, p, N) \). First, the inter-range case means that \( s_c \in (0, 1), p \geq 2 \) and \( 0 < \gamma < N \implies 2 < Np - N - \gamma < 2p, p \geq 2 \) and \( \gamma \in (0, N) \). (1.6)

However, out of some technical reasons, when it comes to the threshold solution with initial data satisfying \( u_0 \in L^2_{\text{rad}} \) and \( \mathcal{Q}[u_0] > 1 \) in subsection 5.2, we have to add more restrictions on the parameters \( (\gamma, p, N) \), i.e.

\[
Np \leq 3N + \gamma \text{ and } Np \leq 2N - p + 6, \quad N \geq 2.
\]

(1.7)

**Remark 1.4.** The restriction (1.7) has only to be imposed in the case of classification of threshold solutions with initial data satisfying \( \mathcal{Q}[u_0] > 1 \) and \( u_0 \in L^2_{\text{rad}} \), see Theorem 1.6 for details.

Next, we establish the existence of special solutions to (1.1) at the mass-energy threshold

\[
\mathcal{M}[u] = 1.
\]

(1.8)

**Theorem 1.5.** Under the Assumption 1.2 and Assumption 1.3, there exist two radial solutions \( Q^+(t, x) \) and \( Q^-(t, x) \) in \( H^1(\mathbb{R}^N) \) such that

a. \( M[Q^+] = M[Q^-] = M[Q] \), \( E[Q^+] = E[Q^-] = E[Q] \). \([0, +\infty) \) is in the domain of the lifespan of \( Q^\pm \) and

\[
\|Q^\pm(t) - e^{it\mathcal{Q}}\|_{H^1(\mathbb{R}^N)} \leq Ce^{-\varepsilon_0 t}, \quad \forall t \geq 0,
\]

where \( \varepsilon_0 \) is the unique positive eigenvalue of linearized operator \( L \) defined in (3.1).

b. \( \|\nabla Q^+_0\|_2 < \|\nabla Q^-\|_2 \), \( Q^- \) is globally defined and scatters in negative time,

c. \( \|\nabla Q^+_0\|_2 > \|\nabla Q^-\|_2 \), and the negative time of existence of \( Q^+ \) is finite.
Then we classify all solutions to (1.1) at the mass-energy critical threshold with zero momentum as follows:

**Theorem 1.6.** Under the Assumption 1.2 and Assumption 1.3, let u be a solution to (1.1) satisfying (1.8) and $P[u] = 0$.

a. If $G[u_0] < 1$, then either $u$ scatters or $u = Q^-$ up to symmetries.

b. If $G[u_0] = 1$, then $u = e^{it}Q$ up to the symmetries.

c. If $G[u_0] > 1$, and $u_0$ is of finite variance, i.e. $|x|u_0 \in L^2$, then either the lifespan of $u$ is finite or $u = Q^+$ up to the symmetries.

The symmetries cited above include all symmetries except for Galilean transformation.

As for the threshold solutions to (1.1) with nonzero momentum, we take the Galilean transformation into account. Precisely, let $\xi_0 = -\frac{P[u]}{M[u]}$, we get a solution $v$ to (1.1) with zero momentum, which is the minimal energy solution among all Galilean transformations of $u$, and $v$ satisfies

$$M[v] = M[u], \quad E[v] = E[u] - \frac{1}{2} \frac{P[u]^2}{M[u]}, \quad \|\nabla v_0\|^2 - \frac{P[u_0]^2}{M[u_0]} = 1.$$

Applying Theorem 1.6 to $v$,

**Theorem 1.7.** Under the same conditions as what in Theorem 1.6, let $u$ be a solution to (1.1) with $P[u] \neq 0$ and

$$\mathcal{M}E[u_0] = \frac{N(p-1) - \gamma}{N(p-1) - \gamma - 2} \frac{P[u_0]^2}{M[u_0]} \leq 1.$$

a. If $G[v_0]^2 = G[u_0]^2 - P[u_0]^2 < 1$, then either $u$ scatters or $u = Q^-$ up to the symmetries.

b. If $G[v_0]^2 = G[u_0]^2 - P[u_0]^2 = 1$, then $u = e^{it}Q$ up to the symmetries.

c. If $G[u_0] > 1$, and $u_0$ is of finite variance, i.e. $|x|u_0 \in L^2$, then either the lifespan of $u$ is finite or $u = Q^+$ up to the symmetries.

The symmetries cited above include all symmetries.

When dimension $N = 5$, if $(\gamma, p)$ is sufficiently close to $(2, 2)$, then [24] checks the uniqueness and nondegeneracy of the ground state. In addition, $(\gamma, p)$ also satisfies the restriction (1.7). Consequently we can clarify the threshold solutions when $(\gamma, p)$ is sufficiently close to $(2, 2)$ when $N = 5$.

**Corollary 1.8.** If $N = 5$, $p \geq 2$ and $(\gamma, p)$ is close to $(2, 2)$, then the results in Theorem 1.6 and Theorem 1.7 hold.

There are several difficulties in dealing with threshold solutions to the generalized Hartree equation. The major one is the nonlocal nature of the nonlinear term. Unlike the power type nonlinear term $|u|^{p-1}u$, the behavior of $(|x|^{-\gamma} - |x|^{-\gamma}) |u|^p |u|^{p-2}u$ at one point is determined by the global behavior of $u$, which motivates us to impose additional restrictions on parameters $(\gamma, p, N)$ out of technical reason when dealing with $L^2_{rad}$ initial data.

Another problem is to deal with the fractional, low power of the parameter $p$. The fractional power makes us unable to expand the nonlinear term directly, always with high order remaining terms. Even worse, if the parameter $p \in (2, 3)$, then the mean value theorem is invalid when dealing with the remaining terms, and we need to use the fact that $f(x) = |x|^{-\alpha}$ is Hölder continuous of $\alpha$ order when $\alpha \in (0, 1)$. Moreover, when constructing the special solution, instead of relying on $H^s$ estimates as in [10], we choose $\|\langle \nabla \rangle \cdot \|_{L^2}$ estimates. However, when we are in the case for
$p \in (2, 3)$, the power of difference term is too low to use the contraction method, which requires to combine $S(H^{s_c})$ norm.

Moreover, the previous papers only mention the exponential decay ground state $Q$ and its regularity. We know little about the properties of high order derivatives of ground state $Q$, and we even do not have the a priori upper bound of $|\partial^a Q|$, let alone $Q \in S$, so it seems not rigorous to simply follow the idea from [6, Corollary 3.8]. Instead, we try to use the classical elliptic comparison theory and iteration argument to overcome it.

Finally, it is worth mentioning that the choice of orthogonal condition $\int \Delta Q h_1 = 0$ used in [10] is not suitable for our setting, then we try to use other better orthogonal conditions and find that $\int (|\cdot|^{-((N-\gamma)\ast Q^p)} Q^{p-1} h_1 = 0$ just meets our demand, which is analogous to the orthogonal condition used in [6] and [9].

The paper is organized as follows:

In section 2, we introduce the Strichartz pairs widely used in this paper and present some basic preliminaries for the later discussion.

In section 3, we consider the linearized equation and explore the spectral properties of the linearized operator and the coercivity of the linearized energy.

In section 4, we discuss the modulation stability near the ground state solution. Here we identify the spatial and phase parameters which control the variations from $e^{it}Q$ when the entire variation is small in $H^1$ norm.

In section 5 and section 6, we study the solutions with initial data from Theorem 1.6 part (a) and (c) respectively. Our main goal is to obtain the exponential convergence to $e^{it}Q$ in positive time direction.

In section 7, we improve the rate of exponential convergence. After we construct special solutions $Q^\pm$ stated in Theorem 1.5 by a fixed point argument, we prove the rigidity, showing the solutions discussed in section 5 and section 6 are equal to $Q^\pm$ respectively.

In Appendix A, we are devoted to the properties of the ground state and the eigenfunctions $Y_\pm$ of $L$ with eigenvalues $\pm e_0$, especially the exponential decay of any order derivatives of ground state $Q$. Finally, we give some useful estimates on the nonlinear term $R(h)$ defined in (3.4) in Appendix B.

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2. Preliminaries

2.1. Strichartz estimates and admissible pairs. In this section, we will introduce the admissible Strichartz pairs and recall the corresponding Strichartz estimates.

**Definition 2.1.** For $s > 0$, the pair $(q, r)$ is $\dot{H}^s$ admissible if

$$\frac{2}{q} + \frac{N}{r} = \frac{N}{2} - s, \quad 2 \leq q, r \leq \infty, \quad \text{and} \quad (q, r, N) \neq (2, \infty, 2).$$

(2.1)

If $s = 0$, we say that the pair $(q, r)$ is $L^2$ admissible.
In order to control the constants uniformly in Strichartz estimates, we restrict the range for the pair \((q, r)\),

\[
\begin{cases}
\left(\frac{2}{1-s}\right)^+ \leq q \leq \infty, & \frac{2N}{N-2s} \leq r \leq \left(\frac{2N}{N-2}\right)^-, \text{ if } N \geq 3, \\
\left(\frac{2}{1-s}\right)^+ \leq q \leq \infty, & \frac{2}{1-s} \leq r \leq \left(\frac{2}{1-s}\right)^+, \text{ if } N = 2, \\
\frac{2}{1-2s} \leq q \leq \infty, & \frac{2}{1-2s} \leq r \leq \infty, \text{ if } N = 1.
\end{cases}
\] (2.2)

Then we define \(S(\dot{H}^s)\) norm by

\[
\|u\|_{S(\dot{H}^s)} = \sup \left\{ \|u\|_{L^q_t L^r_x} : (q, r) \text{ satisfies (2.1) and (2.2)} \right\}.
\]

Moreover, to define the corresponding dual Strichartz norm, we should set the following restrictions:

\[
\begin{cases}
\left(\frac{2}{1+s}\right)^+ \leq q \leq \left(\frac{1}{s}\right)^-, & \left(\frac{2N}{N-2s}\right)^+ \leq r \leq \left(\frac{2N}{N-2}\right)^-, \text{ if } N \geq 3, \\
\left(\frac{2}{1+s}\right)^+ \leq q \leq \left(\frac{1}{s}\right)^-, & \left(\frac{2}{1+s}\right)^+ \leq r \leq \left(\frac{2}{1+s}\right)^+, \text{ if } N = 2, \\
\frac{2}{1+s} \leq q \leq \left(\frac{1}{s}\right)^-, & \left(\frac{2}{1+s}\right)^+ \leq r \leq \infty, \text{ if } N = 1,
\end{cases}
\] (2.3)

then we define the dual Strichartz norm as below:

\[
\|u\|_{S'(H^{-s})} = \inf \left\{ \|u\|_{L^{q'}_t L^{r'}_x} : (q, r) \text{ satisfies (2.1) and (2.3)} \right\}.
\]

**Definition 2.2.** In the later discussion in this paper, for given \(N, p, \gamma\) and hence a fixed \(s_c \in (0, 1)\), we select specific Strichartz pairs as follows:

- **\(L^2\)-admissible pair:** \(S(L^2) = L^{q_1}_t L^{r_1}_x \cap L^{q_2}_t L^{r_2}_x\), where

  \[
  (q_1, r_1) = \left(\frac{2p}{1+s_c(p-1)}, \frac{2Np}{N+\gamma}\right) \quad \text{and} \quad (q_2, r_2) = \left(\frac{2p}{1-s_c}, \frac{2Np}{N+\gamma+2s_c p}\right);
  \]

- **\(L^2\) dual admissible pair:** \(S'(L^2) = L^{q'_1}_t L^{r'_1}_x\), where \((q'_1, r'_1) = \left(\frac{2p}{2p-1-s_c(p-1)}, \frac{2Np}{2Np-N-\gamma}\right)\);

- **\(\dot{H}^{s_c}\)-admissible pair:** \(S(\dot{H}^{s_c}) = L^{q'_2}_t L^{r'_2}_x\), where \((q'_2, r'_2) = \left(\frac{2p}{1-s_c}, \frac{2Np}{N+\gamma}\right)\);

- **\(\dot{H}^{-s_c}\) dual admissible pair:** \(S'(\dot{H}^{-s_c}) = L^{q''_1}_t L^{r''_1}_x\), where \((q''_1, r''_1) = \left(\frac{2p}{(2p-1)(1-s_c)}, \frac{2Np}{2Np-N-\gamma}\right)\).

By the assumption on \((N, p, \gamma)\), all the Strichartz pairs shown above satisfy (2.2) and (2.3) respectively.

As for the Strichartz pairs selected above, by Sobolev embedding, we have

**Lemma 2.3.** \(\forall f \in S \left(I, \langle \nabla \rangle L^2 \right), \text{ then } f \in S \left(I, \dot{H}^{s_c} \right) \) and

\[
\|f\|_{S(\dot{H}^{s_c})} = \|f\|_{L^{q_2}_t L^{r_2}_x} \lesssim \|\nabla^{s_c} f\|_{L^{q_1}_t L^{r_1}_x} \lesssim \|\langle \nabla \rangle f\|_{L^{q_2}_t L^{r_2}_x} \lesssim \|\langle \nabla \rangle f\|_{S(L^2)};
\] (2.4)

**Lemma 2.4** (Kato-Strichartz estimates). \(\forall f \in S \left(I, \langle \nabla \rangle L^2 \right), \text{ then}

\[
\left\| \int_{s>t} e^{i(t-s)\Delta} f(s) ds \right\|_{S(I, L^2)} \lesssim \|f\|_{S(I, \dot{H}^{s_c})};
\]

\[
\left\| \int_{s>t} e^{i(t-s)\Delta} f(s) ds \right\|_{S(I, \dot{H}^{s_c})} \lesssim \|f\|_{S(I, \dot{H}^{-s_c})}.
\]
2.2. Gradient separation. In this subsection, we want to give a rough classification of solutions to (1.1) through the gradient. Precisely, by the same argument as what in [10, Lemma 2.2],

**Lemma 2.5.** Under the Assumption 1.2, we consider the solutions to (1.1) with initial data \( u_0 \) satisfying (1.8) and \( P[u_0] = 0 \).

(a) If \( \|u_0\|_2^2 \|\nabla u_0\|_2 = \|Q\|_2^2 \|\nabla Q\|_2 \), then \( u = e^{it}Q \) up to symmetries.

(b) If \( \|u_0\|_2^2 \|\nabla u_0\|_2 < \|Q\|_2^2 \|\nabla Q\|_2 \), then \( u \) is globally defined and

\[
\|u\|_2^2 \|\nabla u(t)\|_2 < \|Q\|_2^2 \|\nabla Q\|_2, \quad \forall t \in \mathbb{R}.
\]

(c) If \( \|u_0\|_2^2 \|\nabla u_0\|_2 > \|Q\|_2^2 \|\nabla Q\|_2 \), then

\[
\|u\|_2^2 \|\nabla u(t)\|_2 > \|Q\|_2^2 \|\nabla Q\|_2, \quad \text{for all } t \text{ belonging to the lifespan of } u.
\]

2.3. Qualitative rigidity. In this part, we give a criterion of the closeness of solution to (1.1) to the soliton family:

**Proposition 2.6.** Under the Assumption 1.2, there exists a function \( \varepsilon(\rho) \) defined for small \( \rho > 0 \) such that \( \lim_{\rho \to 0} \varepsilon(\rho) = 0 \) and \( \forall u \in H^1 \) such that

\[
\begin{align*}
&\left| \|u\|_2^2 - \|Q\|_2^2 \right| + \left| \|\nabla u\|_2^2 - \|\nabla Q\|_2^2 \right| \\
&+ \left| \int \left( |\cdot|^{(N-\gamma)} \ast |u|^p \right) |u|^p dx - \int \left( |\cdot|^{-(N-\gamma)} \ast Q^p \right) Q^p dx \right| \leq \rho,
\end{align*}
\]

there exists \( (\theta_0, x_0) \in \mathbb{R} \setminus 2\pi \mathbb{Z} \times \mathbb{R}^N \) such that

\[ \|u - e^{i\theta_0}Q(\cdot - x_0)\|_{H^1} \leq \varepsilon(\rho). \] (2.5)

In particular, for solution to (1.1) satisfying

\[ M[u] = M[Q], \quad E[u] = E[Q], \] (2.6)

if we let

\[ \delta(t) = \left| \|\nabla u(t)\|_2^2 - \|\nabla Q\|_2^2 \right|, \] (2.7)

and assume \( \delta(t) \leq \rho \), then there exists \( (\theta_0, x_0) \) such that (2.5) holds.

**Proof.** If not, then there exists a sequence \( \{u_n\}_{n=1}^\infty \subset H^1(\mathbb{R}^N) \) such that

\[
\begin{align*}
&\left| \|u_n\|_2^2 - \|Q\|_2^2 \right| + \left| \|\nabla u_n\|_2^2 - \|\nabla Q\|_2^2 \right| \\
&+ \left| \int \left( |\cdot|^{(N-\gamma)} \ast |u_n|^p \right) |u_n|^p dx - \int \left( |\cdot|^{-(N-\gamma)} \ast Q^p \right) Q^p dx \right| \to 0,
\end{align*}
\]

but

\[
\inf_{(\theta_0, x_0) \in \mathbb{R} \times \mathbb{R}^N} \|u_n - e^{i\theta_0}Q(\cdot - x_0)\|_{H^1} \geq \varepsilon_0.
\]

By linear decomposition for \( \{u_n\}_{n=1}^\infty \) as used in [17, Proposition 3.1], after extracting a subsequence, there exist a family of linear profiles \( \{U_j\}_{j=1}^\infty \subset H^1 \) and \( \{x_n^j\} \) such that

(1) For every \( k \neq j \), \( |x_n^k - x_n^j| \to \infty \), \( n \to \infty \);

(2) For every \( l \geq 1 \) and every \( x \in \mathbb{R}^N \), \( u_n(x) = \sum_{j=1}^l U_j^*(x - x_n^j) + r_n(x) \) with

\[
\lim_{l \to \infty} \limsup_{n \to \infty} \|r_n\|_{L^s(\mathbb{R}^N)} = 0, \quad \forall s \in (2, 2^*), \quad \text{where } 2^* = \begin{cases} \frac{2N}{N-2}, & \text{if } N \geq 3, \\ +\infty, & \text{if } N = 1, 2. \end{cases} \] (2.9)
(3). Orthogonality. For any fixed \( l \in \mathbb{N} \),
\[
\|u_n\|_2^2 = \sum_{j=1}^{l} \|U_j\|_2^2 + \|r_n\|_2^2 + o_n(1),
\]
(2.10)
\[
\|\nabla u_n\|_2^2 = \sum_{j=1}^{l} \|\nabla U_j\|_2^2 + \|\nabla r_n\|_2^2 + o_n(1).
\]
(2.11)
By (2.10) and (2.11), we have
\[
\|Q\|_2^2 = \lim_{n \to \infty} \|u_n\|_2^2 \geq \sum_{j=1}^{l} \|U_j\|_2^2, \quad \|\nabla Q\|_2^2 = \lim_{n \to \infty} \|\nabla u_n\|_2^2 \geq \sum_{j=1}^{l} \|\nabla U_j\|_2^2.
\]
(2.12)
Moreover, after extracting a subsequence if necessary, by (2.9) and the fact that \( |x_n^k - x_n^j| \to \infty \) as \( n \to \infty \), we have
\[
\lim_{l \to \infty} \limsup_{n \to \infty} \int |r_n^l(x)|^p r_n^l(y)^p \frac{1}{|x - y|^{N-\gamma}} \, dx \, dy = 0,
\]
and hence
\[
\lim_{n \to \infty} \int \int |u_n(x)|^p |u_n(y)|^p \frac{1}{|x - y|^{N-\gamma}} \, dx \, dy = \lim_{l \to \infty} \int \int |U_j(x)|^p |U_j(y)|^p \frac{1}{|x - y|^{N-\gamma}} \, dx \, dy.
\]
From Proposition A.1, note that
\[
\sum_{j=1}^{l} \int \int \frac{|U_j(x)|^p |U_j(y)|^p}{|x - y|^{N-\gamma}} \, dx \, dy \leq C_{GN} \sum_{j=1}^{l} \|U_j\|_2^{N+\gamma-(N-2)p} \|\nabla U_j\|_2^{Np-(N+\gamma)}
\]
we obtain that
\[
C_{GN}^{-1} \leq \liminf_{l \to \infty} \frac{\sum_{j=1}^{l} \|U_j\|_2^{N+\gamma-(N-2)p} \|\nabla U_j\|_2^{Np-(N+\gamma)}}{\sum_{j=1}^{l} \int \int \frac{|U_j(x)|^p |U_j(y)|^p}{|x - y|^{N-\gamma}} \, dx \, dy}
\]
\[
\leq \liminf_{l \to \infty} \frac{\left( \sum_{j=1}^{l} \|U_j\|_2^{N+\gamma-(N-2)p} \right)^{N+\gamma-(N-2)p} \left( \sum_{j=1}^{l} \|\nabla U_j\|_2^{Np-(N+\gamma)} \right)^{Np-(N+\gamma)}}{\sum_{j=1}^{l} \int \int \frac{|U_j(x)|^p |U_j(y)|^p}{|x - y|^{N-\gamma}} \, dx \, dy}
\]
\[
\leq \liminf_{l \to \infty} \frac{\|Q\|_2^{N+\gamma-(N-2)p} \|\nabla Q\|_2^{Np-(N+\gamma)}}{\sum_{j=1}^{l} \int \int \frac{|U_j(x)|^p |U_j(y)|^p}{|x - y|^{N-\gamma}} \, dx \, dy}
\]
\[
= \frac{\|Q\|_2^{N+\gamma-(N-2)p} \|\nabla Q\|_2^{Np-(N+\gamma)}}{\int \int \frac{|Q(x)|^p |Q(y)|^p}{|x - y|^{N-\gamma}} \, dx \, dy} = C_{GN}^{-1},
\]
(2.13)
thus the inequalities are all equalities. In particular, from (2.13), there is only one nonzero profile. Without loss of generality, we may assume it is \( U^1 \), then
\[
u_n(x) = U^1(x - x_n^1) + r_n^1(x)
\]
and therefore
\[
\frac{\|U^1\|_2^{N+\gamma-(N-2)p} \|\nabla U^1\|_2^{Np-(N+\gamma)}}{\int \int \frac{|U^1(x)|^p |U^1(y)|^p}{|x - y|^{N-\gamma}} \, dx \, dy} = C_{GN}^{-1} = \frac{\|Q\|_2^{N+\gamma-(N-2)p} \|\nabla Q\|_2^{Np-(N+\gamma)}}{\int \int \frac{|Q(x)|^p |Q(y)|^p}{|x - y|^{N-\gamma}} \, dx \, dy}.\]
If we have the uniqueness of ground state, i.e., Assumption 1.2, then there exists \((\mu_2, x_2, \lambda_2) \in \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{C}\) such that
\[
U^1(x) = \lambda_2 Q(\mu_2 x + x_2),
\]
where \(\mu_2 = 1\) and \(|\lambda_2| = 1\) by (2.8), thus
\[
U^1(x) = e^{ib_2} Q(x + x_2) \Rightarrow u_n(x + x_n^1) = e^{ib_2} Q(x + x_2) + r_n^1(x + x_n^1).
\]
Together with (2.10) and (2.11), we obtain that \(|r_n^1|_{H^1} = o_n(1)\), which means that
\[
\inf_{(\theta_0, x_0) \in \mathbb{R} \times \mathbb{R}^N} \|u_n - e^{ib_0} Q(\cdot - x_0)\|_{H^1} \leq \|r_n^1\|_{H^1} = o_n(1),
\]
a contradiction!

\[\Box\]

3. The linearized equation and properties of linearized operator

3.1. The linearized equation. Consider the solution \(u\) to (1.1) around \(e^{it} Q\) and write \(u\) as
\[
u(x, t) = e^{it} (Q(x) + h(t, x)).
\]
If we write \(h = h_1 + ih_2\), then \(h = \left(\begin{array}{c} h_1 \\ h_2 \end{array}\right)\) satisfies
\[
\partial_t h + \mathcal{L} h = iR(h), \quad \mathcal{L} = \left(\begin{array}{cc} 0 & -L_- \\ L_+ & 0 \end{array}\right),
\]
where
\[
L_+ = -\Delta + 1 - p \left( | \cdot |^{-(N-\gamma)} (Q^{p-1}) \right) Q^{p-1} - (p - 1) \left( | \cdot |^{-(N-\gamma)} * Q^p \right) Q^{p-2},
\]
\[
L_- = -\Delta + 1 - \left( | \cdot |^{-(N-\gamma)} * Q^p \right) Q^{p-2},
\]
and
\[
R(h) = \left( | \cdot |^{-(N-\gamma)} * |Q + h|^p \right) |Q + h|^{p-2}(Q + h)
- \left( | \cdot |^{-(N-\gamma)} * Q^p \right) \left( Q^{p-1} + \frac{p - 2}{2} Q^{p-2} \tilde{h} + \frac{p}{2} Q^{p-2} \bar{h} \right)
- \left( | \cdot |^{-(N-\gamma)} * \left( \frac{p}{2} Q^{p-1} \tilde{h} + \frac{p}{2} Q^{p-1} \bar{h} \right) \right) Q^{p-1}.
\]

3.2. The spectral properties of the linearized operator. In this subsection, we explore the spectral properties of the linearized operator. Before we present the main result, we give the following auxiliary lemma about the non-negativity of linearized energy \(\Phi\) on a subspace of \(H^1(\mathbb{R}^N)\) with co-dimension 1, where \(\Phi\) is defined by
\[
\Phi(h) = \frac{1}{2} \int (L_+ h_1) h_1 dx + \frac{1}{2} \int (L_- h_2) h_2 dx.
\]

Lemma 3.1 (non-negativity of the linearized energy). For any function \(h \in H^1\) satisfying
\[
\left\langle \left( | \cdot |^{-(N-\gamma)} * |Q|^p \right) Q^{p-1}, h_1 \right\rangle = 0,
\]
we have \(\Phi(h) \geq 0\).
Proof. Similar as the process in [10, A.1], since \( Z[u] \) attains its infimum at \( Q \),
\[
I[u] \triangleq \frac{\|u\|_{L^2}^{N+\gamma-(N-2)p} \|\nabla u\|_{L^2}^{N-(N+\gamma)}}{\|Q\|_{L^2}^{N+\gamma-(N-2)p} \|\nabla Q\|_{L^2}^{N-(N+\gamma)}} - \frac{\int (|\cdot|^{-(N-\gamma)} * |u|^p) |u|^p dx}{\int (|\cdot|^{-N} * Q^p) Q^p dx} \geq 0, \quad \forall u \in H^1(\mathbb{R}^N).
\]
For any function \( h \in H^1(\mathbb{R}^N) \), the function \( \lambda \mapsto I[Q + \lambda h] \) with domain \( \mathbb{R} \) attains its minimum at \( \lambda = 0 \), which implies that
\[
\frac{d^2}{d\lambda^2} I[Q + \lambda h] \geq 0, \tag{3.7}
\]
Next, we expand \( \|Q + \lambda h\|_{L^2}^{N\gamma-(N-2)p}, \|\nabla Q + \lambda \nabla h\|_{L^2}^{N-(N+\gamma)} \) and \( \int (|\cdot|^{-N} * |Q + \lambda h|_p^p) |Q + \lambda h|_p^p dx \) with respect to \( \lambda \) up to order 2 respectively and then compute the expansion of \( I[Q + \lambda h] \) in \( \lambda \) of order 2. Then by \( \int \nabla Q \cdot \nabla h_1 = - \int Q h_1 \), which simply follows from (1.4) and (3.6),
\[
\frac{N_p - (N + \gamma)}{2 \|\nabla Q\|_2^2} \Phi(h) - \frac{(N_p - (N + \gamma))(p - 2)}{2 \|\nabla Q\|_2^2} \int (|\cdot|^{-(N-\gamma)} |Q^p| Q^{p-2}) |h_2|_2^2 dx - \frac{2p(N_p - (N + \gamma))}{N + \gamma - (N - 2)p} \left( \frac{\int \nabla Q \cdot \nabla h_1}{\|\nabla Q\|_2^2} \right)^2 \geq 0,
\]
thus it implies \( \Phi(h) \geq 0 \) by (1.6).

Remark 3.2. If we choose the orthogonal condition \( \int \Delta Q h_1 dx = 0 \) as in [10], in order to get the non-negativity of linearized operator, some additional restriction on parameters \( (N, p, \gamma) \) should be imposed. By contrast, if we choose \( \int (|\cdot|^{-\gamma} * Q^p) Q^{p-1} h_1 dx = 0 \), we can get the optimal range, i.e. (1.6).

Next, we obtain the spectral properties of linearized operator \( \mathcal{L} \) as follows.

Proposition 3.3. Under the Assumption 1.3, let \( \sigma(\mathcal{L}) \) be the spectrum of the linearized operator \( \mathcal{L} \) defined on \( L^2(\mathbb{R}^N) \) \( \times L^2(\mathbb{R}^N) \) and let \( \sigma_{ess}(\mathcal{L}) \) be its essential spectrum. Then
\[
\sigma(\mathcal{L}) = \{ \xi \in \mathbb{R} : \xi \geq 0 \}, \quad \sigma_{ess}(\mathcal{L}) \cap \mathbb{R} = \{ -e_0, 0, e_0 \} \text{ for some } e_0 > 0.
\]
Furthermore, \( e_0, -e_0 \) are the only pair of eigenvalues of \( \mathcal{L} \) on \( \mathbb{R} \setminus \{ 0 \} \) with eigenfunctions \( \mathcal{Y}_+ \in \mathcal{S} \) and \( \mathcal{Y}_- \in \mathcal{S} \) satisfying \( \mathcal{Y}_+ = \overline{\mathcal{Y}}_- \). And if we let \( \mathcal{Y}_1 = \mathbb{R}\mathcal{Y}_+ = \mathbb{R}\mathcal{Y}_- \) and \( \mathcal{Y}_2 = \mathbb{R}\mathcal{Y}_+ - \mathbb{R}\mathcal{Y}_- \), then
\[
L_+ \mathcal{Y}_1 = e_0 \mathcal{Y}_2 \quad \text{and} \quad L_- \mathcal{Y}_2 = -e_0 \mathcal{Y}_1.
\]
Moreover, as for the null space of \( \mathcal{L} \), we have
\[
\ker \mathcal{L} = \text{span}\{ i Q, \partial x_1 Q, ..., \partial x_N Q \}.
\]

Proof. As for the spectral properties of \( \mathcal{L} \), it suffices to consider the self-adjoint operator \( \mathcal{P} = L_+^{\frac{1}{2}} L_+ L_+^{\frac{1}{2}} \) with domain \( H^4(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \). Since it can be checked that \( \mathcal{P} \) is a relatively compact perturbation of \( (-\Delta + 1)^2 \), by Weyl’s theorem (see [16, Theorem 14.6]),
\[
\sigma_{ess}(\mathcal{P}) = [0, +\infty).
\]
Moreover, by min-max principle, following the similar argument as in [29, Lemma C.1] or in [6, Lemma 3.2], \( \mathcal{P} = L_+^{\frac{1}{2}} L_+ L_+^{\frac{1}{2}} \) has a negative eigenvalue \( -e_0^2 \) with associated eigenfunction \( g \). Defining \( \mathcal{Y}_1 = L_+^\frac{1}{2} g, \mathcal{Y}_2 = \frac{1}{e_0} L_+ \mathcal{Y}_1 \) and \( \mathcal{Y}_\pm = \mathcal{Y}_1 \pm i \mathcal{Y}_2 \), we obtain that \( \mathcal{L} \mathcal{Y}_\pm = \pm e_0 \mathcal{Y}_\pm \).

The simplicity of eigenvalue \( -e_0^2 \) of \( \mathcal{P} \) and the uniqueness of negative eigenvalue of \( \mathcal{P} \) follow from the non-negativity of \( L_+ \) acting on \( \left\{ (|\cdot|^{-N_\gamma} * Q^p) Q^{p-1} \right\}^\perp \), which simply follows from
Lemma 3.1. As for the null space of $\mathcal{L}$, it is Assumption 1.3 and the fact that $\ker L_- = \text{span}\{Q\}$ (see [37, Theorem XIII. 48]).

To prove $\mathcal{Y}_\pm \in \mathcal{S}$, it suffices to check $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathcal{S}$. First, by the bootstrap argument, we know that $\mathcal{Y}_1, \mathcal{Y}_2 \in H^\infty$, which further implies that $\mathcal{Y}_1, \mathcal{Y}_2 \in (\cap_{k \geq 1} W^{k, \infty}) \cap C^\infty$ by Sobolev embedding. Note that $\mathcal{Y}_1$ satisfies that

$$[(1 - \Delta)^2 + \epsilon_0^2] \mathcal{Y}_1 = F(x),$$

where

$$F(x) = \left( | \cdot |^{-2(N-\gamma) + p^\circ} - (\Delta + 1) \mathcal{Y}_1 \right)$$

and

$$+ (\Delta - 1) \left[ p \left( | \cdot |^{-2(N-\gamma)} \mathcal{Y}_1 \right) \right] Q^{p-1} + (p-1) \left( | \cdot |^{-2(N-\gamma)} \mathcal{Y}_1 \right) Q^{2-p} \mathcal{Y}_1$$

satisfies $|F(x)| \lesssim \langle x \rangle^{-\alpha}$ for some $\alpha > 0$ by the exponential decay of $\partial^\beta Q$ and the boundness of $\partial^\beta \mathcal{Y}_1$ for any $\beta \in \mathbb{Z}_0^N$.

Next, we calculate the integral kernel $G_-(x, y) = (-\Delta + 1 - i\epsilon_0)^{-1}(x, y)$. Note that

$$\left| \int_{\mathbb{R}_N} e^{2\pi i x \cdot \xi} \left( 4\pi^2 |\xi|^2 + 1 - i\epsilon_0 \right)^{-1} d\xi \right| = \left| \int_{\mathbb{R}_N} e^{2\pi i x \cdot \xi} \left( \int_0^\infty e^{-4\pi^2 |\xi|^2 + 1 - i\epsilon_0} \frac{d\delta}{d\delta} \right) d\xi \right|$$

$$\leq (4\pi)^{-N/2} \int_0^\infty e^{-\frac{|x|^2}{4\delta}} e^{-\delta^{-N/2}} d\delta,$$

together with the estimate that

$$e^{-\frac{|x|^2}{4\delta}} e^{-\delta} \leq \min \left\{ e^{-\frac{1}{2\delta}}, e^{-|x|} \right\} \leq e^{-\frac{1}{2\delta} - \frac{1}{2\delta} e^{-\frac{|x|}{2}}}$$

for $|x| \geq 1$,

we have

$$|G_-(x)| \lesssim e^{-\frac{|x|^2}{4\delta}} e^{-\delta^{-N/2}} d\delta \lesssim e^{-\frac{|x|^2}{2}}, \quad |x| \geq 1.$$
Remark 3.4. By the analogous process of proving $\mathcal{Y}_\pm \in \mathcal{S}(\mathbb{R}^N)$,
\begin{equation}
(\mathcal{L} - (k + 1)\epsilon_0)^{-1} \mathcal{S}(\mathbb{R}^N) \subset \mathcal{S}(\mathbb{R}^N), \quad \forall k \geq 1.
\end{equation}

3.3. Coercivity of linearized energy. As for the linearized energy $\Phi$ defined in (3.5), we denote the bilinear symmetric form associated to $\Phi$ by $B(f, g)$, i.e.
\begin{equation}
B(g, h) = \frac{1}{2} \int (L+g_1)h_1 dx + \frac{1}{2} \int (L-g_2)h_2 dx.
\end{equation}

Next, we want to obtain the coercivity of $\Phi$. Before we present the result, except for (3.6), we give some other orthogonality relations:
\begin{equation}
\int (\partial_{x_1}Q)h_1 = ... = \int (\partial_{x_N}Q)h_1 = \int Qh_2,
\end{equation}

where $\mathcal{Y}_\pm$ are defined in Proposition 3.3. We define $G_\perp$ to be the set of all $h \in H^1(\mathbb{R}^N)$ satisfying (3.6) and (3.11), and $G_\perp'$ to be the set of all $h \in H^1(\mathbb{R}^N)$ satisfying (3.11) and (3.12), then

**Proposition 3.5** (coercivity of $\Phi$). Under the Assumption 1.3, there exists constant $c > 0$ such that
\begin{equation}
\Phi(h) \geq c\|h\|^2_{H^1}, \quad \forall h \in G_\perp \cup G_\perp'.
\end{equation}

**Proof.** Step 1. Coercivity of $\Phi$ on $G_\perp$. Note that $\Phi(h) = \Phi_1(h_1) + \Phi_2(h_2)$, where
\begin{align*}
\Phi_1(h_1) &= \frac{1}{2} \int (L+h_1) h_1, \quad \Phi_2(h_2) = \frac{1}{2} \int (L-h_2) h_2.
\end{align*}

W want to prove the coercivity of $\Phi_1$ and $\Phi_2$ respectively. As for the coercivity of $\Phi_1$, we first claim that
\begin{equation}
\Phi_1(h_1) \geq c\|h_1\|^2_{H^1}, \quad \forall h_1 \text{ satisfying } \int (\partial_{x_j}Q)h_1 = \int (|\cdot|^{-(N-\gamma)} * Q^p)Q^{p-1}h_1 = 0, \forall 1 \leq j \leq N.
\end{equation}

If not, we then assume that there exists a sequence of real-valued functions $\{f_n\}_n \subset H^1(\mathbb{R}^N)$ such that
\begin{equation}
\lim_{n \to \infty} \Phi_1(f_n) = 0, \quad \|f_n\|_2 = 1 \text{ and } \int (\partial_{x_j}Q) f_n = \int (|\cdot|^{-(N-\gamma)} * Q^p)Q^{p-1}f_n = 0, \quad \forall 1 \leq j \leq N.
\end{equation}

Note that
\begin{align}
\Phi_1(f_n) &= \frac{1}{2} \int |\nabla f_n|^2 + \frac{1}{2} \int \|f_n\|^2 - \frac{p}{2} \int \left(|\cdot|^{-(N-\gamma)} * (Q^{p-1}f_n)\right)(Q^{p-1}f_n) \\
&\quad - \frac{p-1}{2} \int \left(|\cdot|^{-(N-\gamma)} * Q^p\right)Q^{p-2}|f_n|^2 = o_n(1),
\end{align}

by Hardy-Littlewood-Sobolev inequality and exponential decay of $\partial^\alpha Q, \forall \alpha \in \mathbb{Z}^N_{\geq 0}$ (see Lemma A.5) we have $\int |\nabla f_n|^2 \leq C\|f_n\|^2 < \infty$, which means that $\{f_n\}$ is bounded in $H^1$, hence there exists a subsequence such that
\begin{equation}
f_n \rightharpoonup f_\ast \text{ in } H^1(\mathbb{R}^N), \text{ for some } f_\ast \in H^1(\mathbb{R}^N).
\end{equation}

Moreover, by compactness argument,
\begin{equation}
\int \left(|\cdot|^{-(N-\gamma)} * Q^p\right)Q^{p-2}|f_n|^2 \rightharpoonup \int \left(|\cdot|^{-(N-\gamma)} * Q^p\right)Q^{p-2}|f_\ast|^2, \quad n \to \infty
\end{equation}
and
\[ \int \left( | \cdot |^{-(N-\gamma)} \ast (Q^{p-1}f_n) \right) (Q^{p-1}f_n) \to \int \left( | \cdot |^{-(N-\gamma)} \ast (Q^{p-1}f_\ast) \right) (Q^{p-1}f_\ast), \ n \to \infty. \] (3.17)

Then by semi-lower continuity of weak convergence [5, Proposition 3.5 (iii)],
\[ 0 = \liminf_{n \to \infty} \Phi_1(f_n) \geq \Phi(f_\ast) \geq 0, \]
which implies that \( \Phi_1(f_\ast) = 0 \) with
\[ \int (\partial_{x_j} Q) f_\ast = \int (| \cdot |^{-(N-\gamma)} \ast Q^p) Q^{p-1} f_\ast = 0, \ \forall 1 \leq j \leq N. \]

Consequently, by (3.15), (3.16), (3.17) and the assumption that \( \|f_n\|_2 = 1 \),
\[ \frac{p}{2} \int \left( | \cdot |^{-(N-\gamma)} \ast (Q^{p-1}f_\ast) \right) (Q^{p-1}f_\ast) + \frac{p-1}{2} \int \left( | \cdot |^{-(N-\gamma)} \ast Q^p \right) Q^{p-2} f_\ast^2 \geq \frac{1}{2}, \]
then \( 0 \neq f_\ast \in H^1(\mathbb{R}^N) \) is a minimizer of the following variational problem
\[ \inf_{f \in \mathcal{K}} \Phi_1(f) = \inf_{f \in \mathcal{K}} \frac{1}{2} \int (L_+ f) f \, dx, \]
where
\[ \mathcal{K} \triangleq \left\{ f \in H^1(\mathbb{R}^N) : 0 \neq f \text{ is real, } \|f\|_2 = 1, \ f \perp \left( | \cdot |^{-(N-\gamma)} \ast Q^p \right) Q^{p-1}, \ f \perp \partial_{x_j} Q, \ 1 \leq j \leq N \right\}, \]
so there exists \( \{\lambda_j\}_{0 \leq j \leq N+1} \) such that
\[ L_+ f_\ast = \lambda_0 \left( | \cdot |^{-(N-\gamma)} \ast Q^p \right) Q^{p-1} + \sum_{j=1}^{N} \lambda_j \partial_{x_j} Q + \lambda_{N+1} f_\ast. \]

Taking inner product with \( f_\ast \) and \( \partial_{x_j} Q \) in \( L^2(\mathbb{R}^N) \) on both sides respectively, we can easily get
that \( \lambda_j = 0, \ \forall 1 \leq j \leq N+1 \) and then
\[ L_+ f_\ast = \lambda_0 \left( | \cdot |^{-(N-\gamma)} \ast Q^p \right) Q^{p-1}. \]

Note that
\[ L_+ Q = -(2p-2) \left( | \cdot |^{-(N-\gamma)} \ast Q^p \right) Q^{p-1}, \]
by Assumption 1.3 again,
\[ f_\ast = -\frac{\lambda_0}{2(p-1)} Q + \sum_{j=1}^{N} \mu_j \left( \partial_{x_j} Q \right) \]
for some \( \{\mu_j\}_{1 \leq j \leq N} \). By orthogonal conditions and the fact that \( \partial_{x_j} Q \perp Q \) for any \( 1 \leq j \leq N \), \( \mu_j = 0, \ \forall 1 \leq j \leq N \), then \( f_\ast = -\frac{\lambda_0}{2(p-1)} Q \) and
\[ \Phi_1(f_\ast) = \frac{1}{2} \int (L_+ f_\ast) f_\ast \, dx = -\frac{\lambda_0^2}{4(p-1)} \int \left( | \cdot |^{-(N-\gamma)} \ast Q^p \right) Q^p \, dx = 0, \]
which implies that \( \lambda_0 = 0 \) and thus \( f_\ast = 0 \), a contradiction! Then the coercivity of \( \Phi_1 \) in the sense of \( H^1(\mathbb{R}^N) \) follows from a simple interpolation of (3.14) and
\[ \Phi_1(h_1) \geq \int \frac{1}{2} |\nabla h_1| \, dx - C \|h_1\|^2_{L^2(\mathbb{R}^N)}. \]
By the same argument as above, it is easy to get the coercivity of $\Phi_2$, then we have proved the coercivity of $\Phi$ on $G_\perp$.

Step 2. Coercivity of $\Phi$ on $G'_\perp$. First, we prove that

$$\Phi(h) > 0, \quad \forall h \in G'_\perp \setminus \{0\}.$$  \hfill (3.18)

If not, then there exists $f \in G'_\perp \setminus \{0\}$ such that

$$0 \geq \Phi(f) = B(f, f) = \frac{1}{2} \int (L_+ f_1) f_1 + \frac{1}{2} \int (L_- f_2) f_2.$$  

We define a space

$$E \triangleq \text{span}\{\partial_{x_1} Q, \ldots, \partial_{x_N} Q, iQ, \mathcal{H}_+, f\}.$$  

Then $\forall h \in E$, which we may assume $h = \sum_{j=1}^N \lambda_j \partial_{x_j} Q + \lambda_{N+1} iQ + \lambda_{N+2} \mathcal{H}_+ + \lambda_{N+3} f$, $\lambda_j \in \mathbb{R}$, $\forall 1 \leq j \leq N + 3$,

$$\Phi(h) = B(h, h) = \lambda_{N+3}^2 B(f, f) = \lambda_{N+3}^2 \Phi(f) \leq 0.$$  

Furthermore, we claim that $\dim_E E = N + 3$.

Step 2. Coercivity of $\Phi$ on $G'_\perp$. First, we prove that

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Furthermore, we claim that $\dim_E E = N + 3$.

If it is true, then it contradicts to the fact that $\Phi$ is positive on subspace $G_{\perp}$ with codimension $N + 2$, hence it remains us to check the validity of the claim. Assume that there exist $\{\lambda_j\}_{j=1}^{N+3} \subset \mathbb{R}^{N+3}$ such that

$$\sum_{j=1}^N \lambda_j \partial_{x_j} Q + \lambda_{N+1} iQ + \lambda_{N+2} \mathcal{H}_+ + \lambda_{N+3} f = 0,$$

then $\lambda_{N+2} = 0$ simply follows from the facts that

$$0 = B \left( \sum_{j=1}^N \lambda_j \partial_{x_j} Q + \lambda_{N+1} iQ + \lambda_{N+2} \mathcal{H}_+ + \lambda_{N+3} f, \mathcal{H}_- \right) = \lambda_{N+2} B(\mathcal{H}_+, \mathcal{H}_-) = 0.$$  

Since $\partial_{x_j} Q, iQ, f$ are orthogonal to each other in $L^2$, it is easy to check that $\lambda_j = 0, \forall i \neq N + 2$. Then $\dim_E E = N + 3$ and we have proved (3.18).

As for the coercivity of $\Phi$ on $G'_\perp$, we assume that there exists a sequence $\{h_n\} \subset G'_\perp$ such that

$$\Phi(h_n) \to 0, \quad \text{and} \quad \|h_n\|_2^2 = 1,$$

then we can also extract a subsequence $\{h_n\}$ such that $h_n \rightharpoonup h^*$ in $H^1$ sense. By the same argument as in Step 1, $h^*$ also satisfies $h^* \in G'_\perp$, $h^* \neq 0$ and $\Phi(h^*) = 0$, which contradicts to (3.18). Consequently,

$$\Phi(h) \geq C \|h\|_2^2, \quad \forall h \in G'_\perp.$$  \hfill (3.19)

Then the coercivity of $\Phi$ on $G'_\perp$ immediately follows from the interpolation between (3.19) and

$$\Phi(h) \geq \frac{1}{2} \|h\|_{H^1}^2 - C \|h\|_{L^2}^2, \quad \forall h \in H^1.$$

\[\square\]

4. Modulation

For $u$ solution to (1.1) with

$$M[u] = M[Q], \quad E[u] = E[Q],$$

by Proposition 2.6, if $\delta(t)$ is sufficiently small, then there exists $(\tilde{\sigma}, \tilde{X})$ such that $\|e^{-i\tilde{\sigma}} u(\cdot + \tilde{X}) - Q\|_{H^1} \leq \varepsilon(\delta(t))$. And we further have
Lemma 4.1. Under the Assumption 1.2, for any solution \( u \) satisfying \( M[u] = M[Q] \), \( E[u] = E[Q] \), there exist \( \delta_0 > 0 \) such that there exists \((\sigma, X) \in \mathbb{R}^{2\pi \mathbb{Z} \times \mathbb{R}^N} \) such that \( v = e^{-i\sigma}u(\cdot + X) \) satisfies \[ \|v - Q\|_{H^1} \leq \varepsilon \] and

\[ \exists \int Qv = 0, \quad \Re \int (\partial_x Q)v = 0, \quad k = 1, \ldots, N, \]  

where \( \varepsilon \) is defined in Proposition 2.6. Moreover, the parameters defined above are unique and the mapping \( u \mapsto (\sigma, X) \) is \( C^1 \).

Proof. Note by Proposition 2.6, there exists \( \tilde{\sigma}, \tilde{X} \) such that

\[ \left\| e^{-i\tilde{\sigma}}u(\cdot + \tilde{X}) - Q \right\|_{H^1(\mathbb{R}^N)} \leq \varepsilon(\delta), \quad \text{if} \ \delta(t) < \delta_0 < 1. \]

Without loss of generality, we assume \( u \) is close to \( Q \) in \( H^1 \), and if not, we can replace \( u \) by \( \tilde{u} = e^{-i\tilde{\sigma}}u(\cdot + \tilde{X}) \). Next, we define a mapping \( J : \mathbb{R} \setminus 2\pi \mathbb{Z} \times \mathbb{R}^N \times H^1 \rightarrow \mathbb{R}^{N+1} \) as

\[ J(\sigma, X, u) = \begin{pmatrix} J_1(\sigma, X, u) \\ J_2(\sigma, X, u) \end{pmatrix} = \begin{pmatrix} \Re \int e^{-i\sigma}u(x + X)Q^*(x)dx \\ \Im \int e^{-i\sigma}u(x + X)Q^*(x)dx \end{pmatrix}, \]

where \( J(0, 0, Q) = 0 \) and the corresponding Jacobian matrix at \( (0, 0, Q) \) is invertible. By the implicit function theorem, there exists \( \varepsilon_0, \eta_0 > 0 \) such that

\[ \|u - Q\|_{H^1} < \varepsilon_0 \quad \Rightarrow \quad \exists! \ (\sigma, X), \ |\sigma| + |X| \leq \eta_0 \text{ and } J(\sigma, X, u) = 0. \]

The uniqueness of \((\sigma, X)\) and the regularity of the mapping \( u \mapsto (\sigma, X) \) follow from the implicit function theorem and the regularity of solution to (1.1) in \( H^1 \) sense.

Let \( u \) be the solution to (1.1) satisfying \( M[u] = M[Q] \) and \( E[u] = E[Q] \), and let \( D_{\delta_0} \) be the open set of all times in the lifespan of \( u \) such that \( \delta(t) < \delta_0 \). On \( D_{\delta_0} \), there exist \( C^1 \) functions \( \sigma(t), X(t) \) such that \( e^{-i\sigma(t)}u(\cdot + X(t)) \) is close to \( Q \) and the orthogonal property (4.1) holds. Next, we want to get more information about the behavior of such modulation parameters when it is in \( D_{\delta_0} \). Here we tend to work with the parameters \( X(t) \) and \( \theta(t) = \sigma(t) - t \). Precisely, we rewrite

\[ e^{-it - i\theta(t)}u(x + X(t), t) = (1 + \alpha(t))Q(x) + h(t, x), \quad \forall t \in D_{\delta_0}, \]  

where \( \alpha(t) \in \mathbb{R} \) is a continuous function such that \( \Re \langle h, (|\cdot|^{-\gamma} * Q^p)^{p-1} \rangle = 0. \) Then by Proposition 3.5, \( h \in G_{\perp} \) and thus

\[ \Phi(h) \simeq \|h\|_{H^1(\mathbb{R}^N)}. \]

Furthermore, the behavior of these modulation parameters is as follows:

Lemma 4.2. Under the Assumption 1.2 and Assumption 1.3, let \( u \) be the solution to (1.1) satisfying \( M[u] = M[Q] \), \( E[u] = E[Q] \), then taking a smaller \( \delta_0 \) if necessary, the following estimates hold for \( t \in D_{\delta_0} \):

\[ |\alpha(t)| \simeq \left| \int Qh_1(t)dx \right| \simeq \|h(t)\|_{H^1} \simeq \delta(t). \]  

Proof. Let \( \widetilde{\delta}(t) \triangleq |\alpha(t)| + \delta(t) + \|h(t)\|_{H^1} \). Considering the variation of \( u \mapsto M[u] \) around the ground state \( Q \),

\[ M[u] = M[Q + (\alpha Q + h)] = M[Q] + 2\Re \langle Q, \alpha Q + h \rangle + O(\alpha^2 + \|h\|_{H^1}^2), \]

then

\[ |\alpha(t)| = \frac{1}{\|Q\|^2} \left| \int Qh_1 dx \right| + O(\widetilde{\delta}^2). \]  

(4.5)
Moreover, if we consider the variation of $u \mapsto \|\nabla u\|_2^2$ near the ground state $Q$, then
\[
\|\nabla u\|_2^2 = \left\| \nabla (Q + (\alpha Q + h)) \right\|_2^2 = \|\nabla Q\|_2^2 + 2\alpha \|\nabla Q\|_2^2 + 2 \int \nabla Q \cdot \nabla h_1 dx + O(\alpha^2 + \|h\|_{H^1}^2).
\]
Together with orthogonal condition $\int (|\cdot|^{-N-\gamma} * \inta)^p \, Q^{p-1} h_1 dx = 0$, we obtain that
\[
\delta(u(t)) = \left| \|\nabla u(t)\|_2^2 - \|\nabla Q\|_2^2 \right|
= 2\alpha \|\nabla Q\|_2^2 + 2 \int Q h_1 dx + O(\tilde{\delta}^2)
= 2 \left( \|\nabla Q\|_2^2 + \|Q\|_2^2 \right) |\alpha(t)| + O(\delta^2), \tag{4.6}
\]
As for the variation of energy $E[u]$ around the ground state $Q$, if we let $v = \alpha Q + h$,
\[
E[u] = E[Q + v]
= \frac{1}{2} \int |\nabla Q + \nabla v|^2 dx - \frac{1}{2p} \int \left( |\cdot|^{-N-\gamma} * |Q + v|^p \right) |Q + v|^p dx
= E[Q] - \langle Q, v_1 \rangle
\]
\[
+ \frac{1}{2} \left\langle -\Delta v_1 - p \left( |\cdot|^{-N-\gamma} * Q^{p-1} v_1 \right) Q^{p-1} - (p - 1) \left( |\cdot|^{-N-\gamma} * Q^p \right) Q^{p-2} v_1, v_1 \right\rangle
+ \frac{1}{2} \left\langle -\Delta v_2 - \left( |\cdot|^{-N-\gamma} * Q^p \right) Q^{p-2} v_2, v_2 \right\rangle + \text{h.o.t.}
= E[Q] + \Phi(v) + \text{h.o.t.,}
\]
where the last equality follows from the assumption $M[Q + v] = M[Q]$. Moreover, by Lemma B.1, the h.o.t. satisfies
\[
\text{h.o.t.} = \begin{cases} 
O(\|v\|_{H^1}^3), & \text{if } p = 2 \text{ or } p \geq 3, \\
O(\|v\|_{H^1}^p), & \text{if } p \in (2, 3),
\end{cases}
\]
and without loss of generality, we denote $h.o.t. = O(\|v\|_{H^1}^s) = O(\tilde{\delta}^s)$ for some $s > 2$. Then by the orthogonal condition $\int (|\cdot|^{-N-\gamma} * \inta)^p \, Q^{p-1} h_1 dx = 0$, we obtain that $B(Q, h) = 0$ and
\[
O(\tilde{\delta}^s) = \Phi(\alpha Q + h) = \alpha^2 \Phi(Q) + \Phi(h) + 2\alpha B(Q, h) = \alpha^2 \Phi(Q) + \Phi(h),
\]
so together with (4.3) and (A.4),
\[
\|h\|_{H^1} \simeq \Phi(h)^{\frac{1}{2}} \simeq |\alpha| + O(\tilde{\delta}^s). \tag{4.7}
\]
Combining (4.5), (4.6) and (4.7), $\tilde{\delta}(t) \lesssim \delta(t) + O(\tilde{\delta}^\kappa(t))$ for some $\kappa > 1$, then by continuity argument, $\tilde{\delta}(t) \lesssim \delta(t)$, $\forall t \in D_{\delta_0}$, $\delta_0 \ll 1$. Then using (4.5), (4.6) and (4.7) again, we immediately get (4.4). \(\square\)

When it comes to the evolution of derivatives of modulation parameters,

**Lemma 4.3** (Bounds on the time-derivatives). Under the conditions in Lemma 4.2,
\[
|\dot{\alpha}| + |\dot{X}| + |\dot{\theta}| = O(\delta), \quad \forall t \in D_{\delta_0}. \tag{4.8}
\]

**Proof.** Let $\delta^*(t) = \delta(t) + |\dot{\alpha}(t)| + |\dot{X}(t)| + |\dot{\theta}(t)|$. By (1.1) and (4.2),
\[
i \partial_t h + \Delta h + i \dot{\alpha} Q - i \dot{X} \cdot \nabla Q - \dot{\theta} Q = O(\delta + \delta^*) \text{ in } L^2. \tag{4.9}
\]
Since \( h \) satisfies (3.6) and (3.11), multiplying (4.9) by \( Q \), integrating on \( \mathbb{R}^N \) and then taking the real part, we get
\[
-\dot{\theta}\|Q\|^2 = \mathcal{O}(\delta + \delta^*) \Rightarrow |\dot{\theta}| = \mathcal{O}(\delta + \delta^*).
\]
If we multiply (4.9) by \( \partial_{x_j} Q \) and \( (|\cdot|^{(N-\gamma)} * Q^p) Q^{p-1} \) respectively, integrate on \( \mathbb{R}^N \) and take the imaginary part, then
\[
|\dot{X}_j| = \mathcal{O}(\delta + \delta^*) \quad \text{and} \quad |\dot{\alpha}| = \mathcal{O}(\delta + \delta^*).
\]
Consequently,
\[
\delta^* = \mathcal{O}(\delta + \delta^*), \quad \forall t \in D_{\delta_0},
\]
then it concludes the desired result for \( \delta_0 < 1 \). \( \square \)

We end this section with the following lemma, which will be used in the next two sections.

**Lemma 4.4.** Under the Assumption 1.2 and Assumption 1.3, let \( u \) be the solution to (1.1) satisfying \( M[u] = M[Q], E[u] = E[Q], \) and assume \( u \) is defined on \([0, +\infty)\) and that there exists \( c, C > 0 \) such that
\[
\int_{t}^{\infty} \delta(s)ds \leq Ce^{-ct}, \forall t \geq 0,
\]
then there exists \( (\theta_0, x_0) \in \mathbb{R}\backslash 2\pi\mathbb{Z} \times \mathbb{R}^N \) such that
\[
\|u - e^{it\theta_0} e^{itQ(-x_0)}\|_{H^1(\mathbb{R}^N)} \leq Ce^{-ct}.
\]

**Proof.** By assumption (4.10), it can be easily to see that there must exist \( \{t_n\}_{n \in \mathbb{N}} \), such that \( \delta(t_n) \to 0 \). With the help of convergence to 0 of \( \delta(t) \) up to a subsequence, we first need to check \( \lim_{t \to \infty} \delta(t) = 0 \). If not, there exists \( \{t'_n\}_{n \in \mathbb{N}} \), such that
\[
t'_n \to \infty, \quad \text{and} \quad \delta(t'_n) \geq \varepsilon_1 > 0
\]
for some \( \varepsilon_1 > 0 \). Then we can adjust the value of \( \{t'_n\} \) and extract subsequences from \( \{t_n\} \) and \( \{t'_n\} \) with the properties below:
\[
t_n < t'_n, \quad \forall n \in \mathbb{N}, \quad \delta(t_n) = \varepsilon_1, \quad \delta(t) < \varepsilon_1, \quad \forall t \in [t_n, t'_n).
\]
Hence
\[
|\alpha(t_n) - \alpha(t'_n)| \leq \int_{t_n}^{t'_n} |\dot{\alpha}(t)| dt \leq Ce^{-ct} \to 0, \quad \Rightarrow \quad \alpha(t'_n) \to 0, \quad n \to \infty,
\]
which leads a contradiction.

With the help of (4.4) and (4.8), we claim that there exists \( X_\infty \) and \( \theta_\infty \) such that
\[
\delta(t) + |\alpha(t)| + \|h(t)\|_{H^1} + |X(t) - X_\infty| + |\theta(t) - \theta_\infty| \leq Ce^{-ct}, \forall t \geq 0.
\]
In fact, by Lemma 4.3 and Lemma 4.4 and together with (4.10),
\[
\delta(t) \simeq \|h(t)\|_{H^1} \simeq |\alpha(t)| = \left| \int_t^{\infty} \dot{\alpha}(s)ds \right| \lesssim \int_t^{\infty} \delta(s)ds \leq Ce^{-ct}, \quad \forall t \geq 0,
\]
then as a by-product,
\[
|\dot{X}(t)| + |\dot{\theta}(t)| \leq Ce^{-ct},
\]
which yields that there exist \( X_\infty \) and \( \theta_\infty \) such that
\[
|X(t) - X_\infty| + |\theta(t) - \theta_\infty| \lesssim \int_t^{\infty} |\dot{X}(s)| + |\dot{\theta}(s)|ds \lesssim \int_t^{\infty} e^{-cs}ds \lesssim Ce^{-ct},
\]
and it solves (4.11). Lemma 4.4 then immediately follows from (4.11) and the decomposition (4.2). \( \square \)
5. Convergence to $Q$ for $\|\nabla u_0\|_2^{\frac{1}{p-\gamma}} \|u_0\|_2 > \|\nabla Q\|_2^{\frac{1}{p-\gamma}} \|Q\|_2$

**Theorem 5.1.** Under the same conditions as what in Theorem 1.6, let $u$ be the solution to (1.1) satisfying

$$M[u] = M[Q], \quad E[u] = E[Q], \quad \|\nabla u_0\|_2 > \|\nabla Q\|_2,$$

and $u$ globally exist in positive time, and assume

$$\begin{cases} u_0 \text{ is of finite variance, i.e. } |x|u_0 \in L^2, \\ \text{or } u_0 \in L^2_{\text{rad}}(\mathbb{R}^N) \text{ and } (N, p, \gamma) \text{ satisfies (1.7) in addition,} \end{cases}$$

then there exists $\theta_0 \in \mathbb{R}\setminus 2\pi\mathbb{Z}, x_0 \in \mathbb{R}^N, c, C > 0$ such that

$$\|u - e^{it+i\theta_0}Q(\cdot - x_0)\|_{H^1(\mathbb{R}^N)} \leq Ce^{-ct}.$$  

Moreover, the lifespan of $u$ in negative time direction is finite.

5.1. **Finite variance solutions.** In this section, we are devoted to the proof of the Theorem 5.1 when $u_0$ is of finite variance. Here we introduce virial quantity

$$y(t) \triangleq \int |x|^2 |u(x, t)|^2 dx,$$

then

$$\dot{y} = \Re \int_{\mathbb{R}^N} |x|^2 u \bar{u} dx = -2\Re \int_{\mathbb{R}^N} |x|^2 (\Delta u) \bar{u} dx = 4\Re \int_{\mathbb{R}^N} x \cdot \nabla u \bar{u} dx,$$

and

$$\dot{y}(t) = 16(s_c(p-1)+1)E[u] - 8s_c(p-1)\|\nabla u\|_2^2.$$  

By the assumption that

$$E[u] = E[u_0] = E[Q]$$

and Pohozaev identity (A.4), $\dot{y}(t)$ can be written as

$$\dot{y}(t) = 16(s_c(p-1)+1)E[Q] - 8s_c(p-1)\|\nabla u\|_2^2$$

$$= 8s_c(p-1) (\|\nabla Q\|_2^2 - \|\nabla u\|_2^2) = -8s_c(p-1)\delta(t) < 0,$$  

where $\delta(t) = \left|\|\nabla u\|_2^2 - \|\nabla Q\|_2^2\right|$.  

Then we devise the argument into the following three steps with the previous preliminary.

**Step 1.** Claim:

$$\dot{y}(t) > 0, \forall t \geq 0.$$  

Here we use the convexity argument. If not, then $\exists t_0 \geq 0$ such that $\dot{y}(t_0) \leq 0$. Then by (5.4), there exists $t_1 > t_0$, such that $\forall t \geq t_1$, $\dot{y}(t) \leq \dot{y}(t_1) < \dot{y}(t_0) \leq 0$, hence $y(t) \to -\infty$ as $t \to \infty$, a contradiction!

**Step 2.** Claim: If $\varphi \in C^1(\mathbb{R}^N)$, $f \in H^1(\mathbb{R}^N)$ satisfying $\|f\|_2 = \|Q\|_2$ and $E[f] = E[Q]$, moreover, assume that $\int |\nabla \varphi|^2 |f|^2 dx < \infty$, then

$$\Re \int (\nabla \varphi \cdot \nabla f) \bar{f} \leq C \delta^2(f) \int |\nabla \varphi|^2 |f|^2.$$  

Since $Q$ is the minimizer of minimizing problem

$$\inf_{0 \neq f \in H^1(\mathbb{R}^N)} \frac{\|\nabla f\|_2^{Np-(N+\gamma)} \|f\|_2^{N+\gamma-(N-2)p}}{\int (|\cdot|^{-(N-\gamma)} * |f|^p) |f|^p dx},$$
we have
\[ \left( \int |\cdot|^{-N}\, |f|^p \, |f|^p \, dx \right) \| \nabla Q \|_{L^2}^{Np-(N+\gamma)} \leq \left( \int |\cdot|^{-N}\, |f|^p \, Q^p \, dx \right) \| \nabla f \|_{L^2}^{Np-(N+\gamma)} \]
for any \( f \in H^1(\mathbb{R}^N) \). The inequality above is also valid for \( e^{i\lambda \varphi} f, \forall \lambda \in \mathbb{R} \), hence
\[ \left( \int |\nabla \varphi|^2 |f|^2 \, dx \right) \lambda^2 - 2 \left( \Re \int (\nabla \phi \cdot \nabla f) \, \bar{f} \right) \lambda \]
\[ + \| \nabla f \|_{L^2}^2 - \frac{\| \nabla Q \|_{L^2}^2 \left( \int |\cdot|^{-N}\, |f|^p \, |f|^p \, dx \right)_{\frac{Np-(N+\gamma)}{2}}}{\left( \int |\cdot|^{-N}\, |f|^p \, Q^p \, dx \right)_{\frac{Np-(N+\gamma)}{2}}} \geq 0, \quad \forall \lambda \in \mathbb{R}, \]
which means that
\[ \left( \Re \int (\nabla \phi \cdot \nabla f) \, \bar{f} \right)^2 \leq \left( \int |\nabla \varphi|^2 |f|^2 \, dx \right) \left( \| \nabla f \|_{L^2}^2 - \frac{\| \nabla Q \|_{L^2}^2 \left( \int |\cdot|^{-N}\, |f|^p \, |f|^p \, dx \right)_{\frac{Np-(N+\gamma)}{2}}}{\left( \int |\cdot|^{-N}\, |f|^p \, Q^p \, dx \right)_{\frac{Np-(N+\gamma)}{2}}} \right). \]

By the assumption that \( E[f] = E[Q] \) and Pohozaev identity (A.4),
\[ \| \nabla f \|_{L^2}^2 - \frac{\| \nabla Q \|_{L^2}^2 \left( \int |\cdot|^{-N}\, |f|^p \, |f|^p \, dx \right)_{\frac{Np-(N+\gamma)}{2}}}{\left( \int |\cdot|^{-N}\, |f|^p \, Q^p \, dx \right)_{\frac{Np-(N+\gamma)}{2}}} \]
\[ = \| \nabla f \|_{L^2}^2 - \| \nabla Q \|_{L^2}^2 \left( 1 + \frac{p\delta(t)}{\int |\cdot|^{-N}\, |f|^p \, Q^p \, dx} \right)_{\frac{Np-(N+\gamma)}{2}} \]
\[ \leq \| \nabla f \|_{L^2}^2 - \| \nabla Q \|_{L^2}^2 - \frac{2p}{Np-(N+\gamma)} \int |\cdot|^{-N}\, |f|^p \, Q^p \, dx \, \delta(t) + C\delta(t)^2 \]
\[ = \left( 1 - \frac{2p}{Np-(N+\gamma)} \int |\cdot|^{-N}\, |f|^p \, Q^p \, dx \right) \delta(t) + C\delta(t)^2 = C\delta(t)^2, \]
the claim then immediately follows from (5.7) and (5.8).

**Step 3.** Complete the proof. Let \( \varphi(x) = |x|^2 \), then by (5.3), (5.4) and (5.6),
\[ (\dot{y})^2 \leq C(\bar{y})^2 y \quad \Rightarrow \quad \frac{\dot{y}}{\sqrt{y}} \leq -C\bar{y}. \]

By (5.5),
\[ \sqrt{\bar{y}(t)} - \sqrt{\bar{y}(0)} = \int_0^t \frac{\dot{y}(s)}{\sqrt{\bar{y}(s)}} \, ds \leq -C (\bar{y}(t) - \bar{y}(0)) \leq C\bar{y}(0) \]
and \( y(t) \) is uniformly bounded for \( t \geq 0 \). Therefore,
\[ \dot{y} \leq -c\bar{y}, \quad \Rightarrow \quad \frac{d}{dt} (e^{ct} \bar{y}) \leq 0, \quad \Rightarrow \quad \bar{y}(t) \leq Ce^{-ct}, \quad \forall t \geq 0, \]
for some \( C, c > 0 \). Then by (5.4),
\[ \int_0^\infty \delta(s)ds = -\frac{1}{8s_c(p-1)} \int_0^\infty \dot{y}(s)ds \leq e^{-ct}, \quad \forall t \geq 0. \]

Then we immediately complete this part by Lemma 4.4.
5.2. Radial solutions. In this subsection, we consider the case for \( u_0 \in L^2_{\text{rad}}(\mathbb{R}^N) \), and our goal is to prove \( u_0 \) is of finite variance. Here we consider the localized virial quantity

\[
y_R(t) = \int_{\mathbb{R}^N} \varphi_R(x)|u(x,t)|^2 dx = \int_{\mathbb{R}^N} R^2 \varphi \left( \frac{x}{R} \right) |u(x,t)|^2 dx,
\]

where \( \varphi \) is a \( C^\infty \) positive radial function on \( \mathbb{R}^N \) such that

\[
\varphi(x) = \begin{cases} |x|^2, & \text{if } |x| \leq 1, \\
0, & \text{if } |x| \geq 2,
\end{cases} \quad \varphi(r) \geq 0, \quad \varphi''(r) \leq 0,
\]

then after some careful calculation, we obtain that

\[
y_R(t) = 23 \int_{\mathbb{R}^N} \nabla \varphi_R(x) \cdot \nabla u \dd ud x = 2R \int_{\mathbb{R}^N} \nabla \varphi \left( \frac{x}{R} \right) \cdot \nabla u \dd ud x,
\]

and

\[
y_R(t) = -8s_c(p-1)\delta(t) + A_R(u(t)),
\]

where

\[
A_R(u(t)) = 4 \int_{\mathbb{R}^N} \left( \varphi'' \left( \frac{x}{R} \right) - 2 \right) |\nabla u|^2 dx - \int_{\mathbb{R}^N} (\Delta^2 \varphi_R) |u|^2 dx \\
+ \left( \frac{2}{p} - 2 \right) \int_{\mathbb{R}^N} (\Delta \varphi_R - 2N) \left( |\cdot|^{-\gamma(N-\gamma)} * |u|^p \right) |u|^p dx \\
- \frac{2}{p} \int_{\mathbb{R}^N \times \mathbb{R}^N} u(x)p|^p|u(y)|^p |\nabla y - \left( (\nabla \varphi_R(x) - 2x) - (\nabla \varphi_R(y) - 2y) \right)| dx dy.
\]

Next, we claim that there exists \( R_0 > 0 \), such that

\[
y_R(t) \leq -4s_c(p-1)\delta(t), \quad \forall R \geq R_0,
\]

and it suffices to check

\[
|A_R(u(t))| \leq 4s_c(p-1)\delta(t), \quad \forall R \geq R_0
\]

for some \( R_0 > 0 \). Here we divide the proof into following two cases: \( Case 1. \ t \in D_{\delta_1} \) for some \( 0 < \delta_1 \leq \delta_0 \ll 1 \). Here we try to make full use of modulation. By (4.2) and the symmetry of \( u \),

\[
e^{-it}u = (1 + \alpha)e^{it}Q + e^{it}h = e^{it}Q + e^{it}(\alpha Q + h).
\]

We denote \( v = \alpha Q + v \), then

\[
A_R(u) = A_R(e^{-it}u) = A_R(e^{it}Q + e^{it}v) = A_R(Q + v) - A_R(Q),
\]

where the last equality follows from the fact that

\[
y_R(t) = 2R \int e^{-it}Q \nabla \varphi \left( \frac{x}{R} \right) \cdot e^{it}Q dx = 0 \quad \text{and} \quad \delta(u(t)) = 0 \quad \text{if} \quad u(t) = e^{it}Q.
\]

Furthermore, note that when \( |x| \leq R \), \( \nabla \varphi_R(x) = R \nabla \varphi \left( \frac{x}{R} \right) = 2x \), the function \( (\nabla \varphi_R(x) - 2x) - (\nabla \varphi_R(y) - 2y) \) is supported on

\[
\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |x| \geq R \text{ or } |y| \geq R \}.
\]

In addition, by mean-value theorem,

\[
\frac{\left| \nabla_y \left( \frac{(2x - \nabla \varphi_R(x)) - (2y - \nabla \varphi_R(y))}{x - y} \right) \right|}{|x - y|^{N-\gamma}} \lesssim \left| \frac{-2N + \Delta \varphi_R(y)}{|x - y|^{N-\gamma}} \right| + \left| \frac{x - y}{|x - y|^{N-\gamma}} \right| \lesssim \frac{1}{|x - y|^{N-\gamma}}.
\]

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Then by Lemma 4.2, Lemma A.5 and Lemma B.1,
\[
A_R(u(t)) = \left| A_R(Q + v) - A_R(Q) \right|
\lesssim \left| \int_{|x| \geq R} |\nabla Q + \nabla v|^2 - |\nabla Q|^2 \, dx \right| + \left| \int_{|x| \geq R} |Q + v|^2 - Q^2 \, dx \right|
+ \left| \left( Q(x) + v(x) \right) |Q(y) + v(y)| - \frac{|Q(x)|^p |Q(y)|^p}{|x - y|^{N-\gamma}} \right| \, dxdy
\lesssim e^{-cRt} + \delta^2(t) + \delta^{2p}(t),
\] (5.16)
for some \( c > 0 \), which shows that there exists \( R_1 \gg 1 \) and \( 0 < \delta_1 \ll 1 \) such that \( |A_R(u(t))| \leq 4s_c(p-1)\delta_1 \), \( \forall R \geq R_1 \), and \( \forall t \in D_{\delta_1} \).

Case 2. \( t \notin D_{\delta_1} \). Since we have required that \( \varphi''(r) \leq 2 \),
\[
\int \left( \varphi'' \left( \frac{x}{R} \right) - 2 \right) |\nabla u|^2 \, dx \leq 0.
\] (5.17)
Moreover,
\[
\left| \int_{\mathbb{R}^3} (\Delta^2 \varphi_R) |u|^2 \, dx \right| = \left| \frac{1}{R^2} \int \left( \Delta^2 \varphi_R \left( \frac{x}{R} \right) \right) u(t,x)^2 \, dx \right|
\leq \frac{C}{R^2} \|u\|_2^2 = \frac{C}{R^2} \|Q\|_2^2 \leq 2s_c(p-1)\delta_1 \leq 2s_c(p-1)\delta(t)
\] (5.18)
for \( R \geq R_2 = \sqrt{\frac{C\|Q\|_2^2}{2s_c(p-1)\delta_1}} \).

It remains us to estimate the last two terms of \( A_R(u(t)) \) defined in (5.12). By (5.14) and (5.15), they can both be bounded by \( \left\| (| \cdot |^{-(N-\gamma)} * |u|^p) |u|^p \right\|_{L^1(|x| \geq R)} \). Our aim is to prove
\[
\left\| (| \cdot |^{-(N-\gamma)} * |u|^p) |u|^p \right\|_{L^1(|x| \geq R)} \lesssim R^{-\alpha} \|u\|_{L^2}^{2p-\alpha} \|\nabla u\|_2^\alpha
\] (5.19)
for some \( \beta > 0 \) and \( \alpha \in [0,2] \). If we check the validity of (5.19), then by mass conservation of \( u \) and Young’s inequality,
\[
\left\| (| \cdot |^{-(N-\gamma)} * |u|^p) |u|^p \right\|_{L^1(|x| \geq R)} \lesssim R^{-\beta} \delta(t).
\] (5.20)

Remark 5.2. If we have (5.20), not only can we get a decay of coefficient with respect to \( R \), but also we get that the behavior of nonlinear term can be control by \( \delta(t) \) when \( t \notin D_{\delta_1} \). However, if we wish \( \delta(t) \) to dominate the behavior of nonlinear term, we are motivated to restrict \( \alpha \in [0,2] \) in (5.19), which, by further calculation, is why we should add the additional restrictions onto the parameters \( (\gamma, N, p) \) as shown in (1.7).

We then continue the argument. To prove (5.19), we divide the integral above into the following two parts:
\[
\left\| (| \cdot |^{-(N-\gamma)} * |u|^p) |u|^p \right\|_{L^1(|x| \geq R)} \leq \left\| (| \cdot |^{-(N-\gamma)} * \left( \chi_{\{|x| \geq \frac{1}{2}R\}} |u|^p \right)) |u|^p \right\|_{L^1(|x| \geq R)}
+ \left\| (| \cdot |^{-(N-\gamma)} * \left( \chi_{\{|x| \leq \frac{1}{2}R\}} |u|^p \right)) |u|^p \right\|_{L^1(|x| \geq R)}
\triangleq I + II.
\]
As for $I$, by Hölder inequality and Hardy-Littlewood-Sobolev inequality,
\[
I \lesssim \left\| \cdot \right\|^{2(pNp)}_{L^2} \left( \frac{\partial_{\gamma}N}{x} \right) \|u\|^{2pNp}_{L^2} \left( \|x\| \right) \|u\|^{2\frac{Np}{N-\gamma}}_{L^2} \left( \|x\| \right)
\]
\[
\lesssim \left\| u \right\|^{2pNp}_{L^2} \left( \|x\| \right) \|u\|^{2\frac{Np}{N-\gamma}}_{L^2} \left( \|x\| \right)
\]
\[
\lesssim \frac{1}{R^{\frac{(N-1)(Np-1)}{N}}} \left\| u \right\|_{2}^{\frac{Np-N-\gamma}{N}} \left\| \nabla u \right\|_{2}^{\frac{Np-N-\gamma}{N}}.
\]

Note we cannot get any decay of $R$ when $N = 1$, we are motivated to restrict the parameters to
\[
N \geq 2, \quad \frac{Np-N-\gamma}{N} \in [0, 2]. \tag{5.21}
\]

As for $II$, noting that $|x-y| \sim |x|$ when $(x, y) \in \{(x, y) : |x| \geq R, |y| \leq \frac{R}{2}\},$
\[
II \approx \int_{\{x| \geq R\} \cap \{|y| \leq \frac{R}{2}\}} \frac{|u(x)|^p |u(y)|^p}{|x|^{-\gamma}} \ dx \ dy = \left( \int_{|y| \leq \frac{R}{2}} |u(y)|^p \ dy \right) \left( \int_{|x| \geq R} \frac{|u(x)|^p}{|x|^{-\gamma}} \ dx \right).
\]

If $s \in \left( \frac{N}{N-\gamma}, +\infty \right)$, then
\[
II \lesssim \left\| |x|^{-\gamma} \right\|_{L^s(|x| \geq R)} \left\| |u|^p \right\|_{L^s(|x| \geq R)} \left\| |u|^p \right\|_{L^{\frac{p}{p}}(|x| \leq \frac{R}{2})} \lesssim R^{-(N-\gamma)+\frac{N}{s}} \left\| u \right\|_{L^{s,p}(|x| \geq R)} \left\| u \right\|_{L^{p}(|x| \leq \frac{R}{2})},
\]
where
\[
\left\| u \right\|_{L^{p}(|x| \leq \frac{R}{2})} \lesssim \left\| u \right\|_{L^2}^{\frac{2Np-2N}{2}} \left\| \nabla u \right\|_{L^2}^{\frac{Np-N}{2}} \tag{5.22}
\]
by Hölder inequality and Sobolev embedding. Moreover, as for $\left\| u \right\|_{L^{s,p}(|x| \geq R)}$, since the integral regime is away from the origin, except for following the argument in (5.22), there is another method to handle it, which is by using interpolation between $L^2$ norm and $L^\infty$ norm together with Strauss lemma. Compared with the former one, not only can we get a decay of $R$, but also can get less power of $\left\| \nabla u \right\|_{L^2}$ followed by the latter one. Thus we follow the latter way, then
\[
\left\| u \right\|_{L^{s,p}(|x| \geq R)} = \left( \int_{|x| \geq R} |u(x)|^{s,p} \ dx \right)^{\frac{1}{s,p}}
\]
\[
\leq \left( \int_{|x| \geq R} |u(x)|^{2} \ dx \right)^{\frac{1}{s}} \left\| u \right\|_{L^{\infty}(|x| \geq R)}^{\frac{1}{s}(s,p-2)} \tag{by interpolation}
\]
\[
\lesssim R^{-\frac{N-1}{2}(p-\frac{1}{s})} \left\| u \right\|_{L^2}^{\frac{2}{s}} \left\| \nabla u \right\|_{L^2}^{\frac{Np-N-\gamma}{2}} \tag{by Strauss lemma}
\]
Consequently, $II$ is bounded by
\[
II \lesssim R^{-\frac{N-1}{2}(p-\frac{1}{s})-(N-\gamma)+\frac{N}{s}} \left\| u \right\|_{L^2}^{\frac{2}{s}+\frac{Np-N-\gamma}{2}} \left\| \nabla u \right\|_{L^2}^{\frac{Np-N-\gamma}{2}},
\]
where the power of $\left\| \nabla u \right\|_{L^2}$ satisfies
\[
\frac{p}{2} - \frac{1}{s} + \frac{Np}{2} - N = \frac{1}{s} + \left( \frac{p}{2} - 1 \right) (N+1) \in \left[ \left( \frac{p}{2} - 1 \right) (N+1), \frac{N-\gamma}{N} + \left( \frac{p}{2} - 1 \right) (N+1) \right], \tag{5.23}
\]
and the minimum is obtained at $s = \infty$. In order to get (5.19), by (5.21) and (5.23), we then impose
\[
N \geq 2, \quad \frac{Np-N-\gamma}{N} \in [0, 2] \text{ and } \frac{p}{2} - \frac{1}{s} + \frac{Np}{2} - N \in [0, 2]. \tag{5.24}
\]
Since we hope to get range of \((\gamma, N, p)\) as wide as possible, we choose \(s = \infty\), then at this time (5.24) is definitely (1.7).

Under the restriction (1.7), combining with (5.17), (5.18) and (5.20), we can choose \(R \geq R_0 \gg 1\) such that
\[
|A_R(u(t))| \leq 4s_c(p - 1)\delta(t), \quad \forall t \notin D_{\delta_1},
\]
which implies (5.13). As a simple by-product, \(\dot{y}_R(t) > 0\) and \(\dot{y}_R(t)\) is monotonically decreasing, which respectively imply that
\[
y_R(0) = \int R^2 \varphi \left( \frac{x}{R} \right) |u_0|^2 dx \leq y_R(t), \forall t \geq 0, \forall R \geq R_0,
\]
and there exists \(A \geq 0\) such that \(\dot{y}_R(t) \downarrow A\) as \(t \to \infty\). The latter fact ensures
\[
\int_0^\infty \delta(t) dt \leq \int_0^\infty |\dot{y}_R(t)| dt = \left| \int_0^\infty \dot{y}_R(t) dt \right| = \left| \dot{y}_R(\infty) - \dot{y}_R(0) \right| < \infty,
\]
so there exists a subsequence \(\{t_n\}\) such that \(t_n \to \infty\) and \(\delta(t_n) \to 0\). Since \(M[u] = M[Q], E[u] = E[Q]\), by Proposition 2.6, there exists \(\theta_0\) such that
\[
\|u(t_n) - e^{\theta_0 Q}\|_{H^1(\mathbb{R}^N)} \to 0, \quad n \to \infty.
\]
Taking \(t = t_n\) in (5.25) and then letting \(n \to \infty\), we have
\[
\int R^2 \varphi \left( \frac{x}{R} \right) |u_0|^2 dx \leq \int R^2 \varphi \left( \frac{x}{R} \right) Q^2 dx, \quad \forall R \geq R_0,
\]
thus by Fatou’s Lemma,
\[
\int |x|^2 |u_0(x)|^2 dx \leq \liminf_{R \to \infty} \int R^2 \varphi \left( \frac{x}{R} \right) |u_0(x)|^2 dx
\]
\[
\leq \liminf_{R \to \infty} \int R^2 \varphi \left( \frac{x}{R} \right) Q^2 dx < \infty,
\]
which implies the finite variance of \(u_0\), then we finish the proof.

**5.3. Complete the proof of Theorem 5.1.** It remains for us to check that the lifespan of \(u\) in negative time direction is finite. First, from the proof of (5.5),

**Lemma 5.3.** Let \(u\) be a solution to (1.1) satisfying (5.1) and \(T_+(u) = +\infty\) with initial data \(u_0\) of finite variance, then
\[
\exists \int x \cdot \nabla u(x, t) \bar{u}(x, t) dx = \frac{1}{4} \dot{y}(t) > 0
\]
for all \(t\) in the interval of existence of \(u\).

As for \(u\) given in Theorem 5.1, we assume \(T_-(u) = -\infty\). Since the equation (1.1) satisfies the time reversal symmetry, we can see that \(v(x, t) = \bar{u}(x, -t)\) is also the solution to (1.1) on \(\mathbb{R}_+\) and \(v\) satisfies the condition in Lemma 5.3, which implies that
\[
0 < \exists \int x \cdot \nabla v(x, t) \bar{v}(x, t) dx = -\exists \int x \cdot \nabla u(x, -t) \bar{u}(x, -t) dx, \quad \forall t > 0,
\]
but it contradicts to the fact that
\[
\exists \int x \cdot u(x, t) \bar{u}(x, t) dx > 0, \quad \forall t \in (T_-(u), T_+(u)).
\]
Hence the negative time of existence of \(u\) is finite.
6. Convergence to $Q$ for $\|\nabla u_0\|_2^{\frac{1+\varepsilon}{2}} \|u_0\|_2 < \|\nabla Q\|_2^{\frac{1+\varepsilon}{2}} \|Q\|_2$

**Theorem 6.1.** Under the same conditions as what in Theorem 1.6, let $u$ be the solution to (1.1) satisfying
\[
M[u] = M[Q], \quad E[u] = E[Q], \quad \|\nabla u\|_2 < \|\nabla Q\|_2, \quad (6.1)
\]
and assume it does not scatter in positive time. Then there exists $(\theta_0, x_0) \in \mathbb{R} \setminus 2\pi \mathbb{Z} \times \mathbb{R}^N, c, C > 0$ such that
\[
\|u - e^{it+i\theta_0}Q(\cdot-x_0)\|_{H^1(\mathbb{R}^N)} \leq Ce^{-ct}. \quad (6.5)
\]

First of all, we present a result about concentration compactness, which is essential to the later discussion in this section.

**Lemma 6.2** (Concentration compactness). Let $u$ be a solution to (1.1) satisfying the conditions in Theorem 6.1. There exists a continuous function $x(t)$ such that
\[
K \triangleq \{u(x + x(t), t) : t \in [0, +\infty)\} \quad (6.2)
\]
is pre-compact in $H^1(\mathbb{R}^N)$.

**Proof.** The proof is essentially the same as [10, Lemma 6.1]. □

Repeating the analogous argument to what in [10, Section 6.1], the continuous function $x(t)$ defined in Lemma 6.2 can be further modified such that not only does it retain the property in Lemma 6.2, but also it satisfies
\[
x(t) = X(t), \quad \forall t \in D_{\delta_0}, \quad (6.3)
\]
where $X(t)$ is defined in (4.2).

Moreover, we can obtain more informations about the behavior of mass center $x(t)$ as follows:

**Lemma 6.3.** As $u$ given in Theorem 6.1. Let $x(t)$ defined above, then
(i)
\[
P[u] = 3 \int \bar{u} \nabla u dx = 0, \quad (6.4)
\]
and thus
\[
\lim_{t \to +\infty} \frac{x(t)}{t} = 0. \quad (6.5)
\]
(ii) there exists a constant $C > 0$ such that
\[
|x(t) - x(s)| \leq C, \quad \forall s, t \geq 0 \text{ with } |s - t| \leq 2. \quad (6.6)
\]

**Proof.** (i) The proof is almost the same as [7, Proposition 4.1, Lemma 5.1].
(ii) The result simply follows from Lemma 6.2, the continuous dependence of the flow onto the initial data and the continuity of the flow. □

Next, we want to study the behavior of $\delta(t)$ at infinity. First we have the vanishing of $\delta(t)$ at infinity in the mean sense.

**Lemma 6.4** (Convergence in mean). Let $u$ be a solution of (1.1) satisfying (6.1), then
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \delta(t) dt = 0. \quad (6.7)
\]
Proof. Here we use the localized virial quantity (5.9). We have already known that \( \dot{y}_R(t) \) satisfies (5.10) and \( \ddot{y}_R(t) \) satisfies (5.11) with
\[
A_R(u(t)) = 4R \sum_{i,j} \int_{\mathbb{R}^N} \frac{\partial^2 \varphi_R}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial \bar{u}}{\partial x_j} \, dx + \sum_{i=1}^N \int_{\mathbb{R}^N} \left( \frac{\partial^2 \varphi_R}{\partial x_i^2} - 2 \right) |\frac{\partial u}{\partial x_i}|^2 \, dx - \int_{\mathbb{R}^N} (\Delta^2 \varphi_R) |u|^2 \, dx
= \frac{2}{p} \int_{\mathbb{R}^N} (\Delta \varphi_R - 2N) (|\cdot|^{-3} * |u|^2) |u|^2 \, dx
- \frac{2}{p} \int_{\mathbb{R}^N \times \mathbb{R}^N} |u(x)|^p |u(y)|^p \nabla_y \cdot \left( \frac{\nabla \varphi_R(x) - 2x}{|x-y|^{N-\gamma}} \right) \, dx \, dy.
\]
Note that if \( |x| \leq R \),
\[
\frac{\partial^2 \varphi_R}{\partial x_i \partial x_j} (x) = (\Delta^2 \varphi_R)(x) = \frac{\partial^2 \varphi_R}{\partial x_i^2} (x) - 2 = 2N - \Delta \varphi_R(x) = 0, \quad 2x - \nabla \varphi_R(x) = 0,
\]
it means
\[
|A_R(u(t))| \lesssim \int_{|x| \geq R} |\nabla u|^2 + \frac{1}{R^2} |u|^2 + \left( |\cdot|^{-(N-\gamma)} * |u|^p \right) |u|^p \, dx.
\]
With the compactness of \( K, \forall \varepsilon > 0, \exists R_0(\varepsilon) > 0 \) sufficiently large, such that
\[
\int_{|x-x(t)| \geq R_0(\varepsilon)} |\nabla u|^2 + |u|^2 + \left( |\cdot|^{-(N-\gamma)} * |u|^p \right) |u|^p \, dx < \varepsilon.
\]
By (6.5), for \( \varepsilon > 0 \) given above, \( \exists t_0(\varepsilon) > 0 \) such that
\[
\left| \frac{x(t)}{t} \right| < \varepsilon, \text{ i.e. } |x(t)| \leq \varepsilon t, \quad \forall t \geq t_0(\varepsilon).
\]
Note that
\[
\left| \frac{1}{T} \int_{t_0(\varepsilon)}^T (8s_c(p-1)\delta(t) + A_R(u(t))) \, dt \right| = \left| \frac{1}{T} \int_{t_0(\varepsilon)}^T \ddot{y}_R(t) \, dt \right| \leq CR \cdot \frac{T}{T},
\]
where the last inequality is followed by
\[
|\ddot{y}_R(t)| = 2R \int \nabla \varphi \left( \frac{x}{R} \right) \cdot \nabla \bar{u} \, dx \leq R \|\nabla \varphi\|_\infty \|\nabla u\|_2 \|u\|_2 \leq CR \text{ uniformly in } t > 0.
\]
Now we have to choose an appropriate \( R \) to ensure the smallness of \( A_R(u(t)) \) on \([t_0(\varepsilon), T]\) and here we let \( R = R_0(\varepsilon) + \varepsilon T + 1 \), then by (6.9), (6.10) and (6.11),
\[
|A_R(u(t))| \lesssim \int_{|x| \geq R} |\nabla u|^2 + \frac{1}{R^2} |u|^2 + \left( |\cdot|^{-(N-\gamma)} * |u|^p \right) |u|^p \, dx
\leq \int_{|x-x(t)| \geq R_0(\varepsilon)} |\nabla u|^2 + |u|^2 + \left( |\cdot|^{-(N-\gamma)} * |u|^p \right) |u|^p \, dx < \varepsilon.
\]
Together with (6.12) and (6.13),
\[
\frac{1}{T} \int_0^T \delta(t) \, dt = \frac{1}{T} \int_0^{t_0(\varepsilon)} \delta(t) \, dt + \frac{1}{T} \int_{t_0(\varepsilon)}^T \delta(t) \, dt \leq \frac{1}{T} \int_0^{t_0(\varepsilon)} \delta(t) \, dt + \frac{R_0(\varepsilon) + 1}{T} + \varepsilon.
\]
Letting \( T \to \infty \) on both sides, (6.7) immediately follows from the arbitrariness of \( \varepsilon > 0 \). \( \square \)

As a corollary of Lemma 6.4, \( \delta(t) \) vanishes at infinity up to a subsequence.

**Corollary 6.5.** Let \( u \) be the a solution of (1.1) satisfying (6.1), then there exists a time sequence \( \{t_n\}_{n=1}^\infty \) tending to \( +\infty \), such that \( \delta(t_n) \to 0 \).

Moreover, the behavior of \( \delta(t) \) can be controlled by the mass center \( x(t) \) in the following sense:
Lemma 6.6 (Virial-type estimates on $\delta$). Under the Assumption 1.2 and Assumption 1.3, there exists a constant $C > 0$ such that if $0 \leq \sigma < \tau$

$$
\int_{\sigma}^{\tau} \delta(t)dt \leq C \left(1 + \sup_{\sigma \leq t \leq \tau} |x(t)|\right) \left(\delta(\sigma) + \delta(\tau)\right).
$$

(6.14)

Proof. As for the localized virial quantity $y_R$ as defined in (5.9), we wish $\delta$ to control the behavior of $\dot{y}_R$, requiring us to estimate $A_R(u(t))$. We divide the argument into the following two cases, one for $\delta(t) \leq \delta_0$ and another one for $\delta(t) > \delta_0$, while the former can be dealt by modulation and the latter relies on the concentration compactness. Precisely, as for the former case, by (4.2), Lemma 4.2, Lemma A.5, Lemma B.1 and the fact that for any fixed $\theta_0, X_0$, $A_R(e^{i\theta_0}e^{i\tau}Q(\cdot + X_0)) = 0$ for any $R, t$, we can repeat the same argument as what in (5.16) to get

$$
|A_R(u(t))| = \left|A_R(u) - A_R \left(e^{i(t+\theta(t))}Q(x - X(t))\right)\right| \\
\lesssim e^{-C\delta_0}\delta(t) + \delta(t)^2 + \delta(t)^{2p}, \quad \text{for any } R \geq R_0 + |X(t)|.
$$

Then we choose $R_0 > 0$ sufficiently large and $\delta_0 \ll 1$, such that

$$\forall R \geq R_0 + |X(t)| = R_0 + |x(t)| \Rightarrow |A_R(u(t))| \leq 4s_c(p - 1)\delta(t), \forall t \in D_{\delta_0}.$$

As for the latter case, if $\delta(t) \geq \delta_0$, then by (6.8), (6.9) and (6.10),

$$\begin{align*}
|A_R(u(t))| &\leq C \int_{|x| \geq R} \left|\nabla u\right|^2 + |u|^2 + \left|\nabla \cdot (N - \gamma) * |u|^p\right| |u|^p \, dx \\
&\leq C \int_{|x - x(t)| \geq R - |x(t)|} \left|\nabla u\right|^2 + |u|^2 + \left|\nabla \cdot (N - \gamma) * |u|^p\right| |u|^p \, dx < C\varepsilon, \quad \forall R \geq |x(t)| + R_0(\varepsilon),
\end{align*}$$

where $R_0(\varepsilon)$ is the constant chosen in (6.10). Therefore, if we let $\varepsilon = \frac{4s_c(p - 1)\delta_0}{C} > 0$, then there exists $R_1 = R(\varepsilon) > 0$ such that

$$\forall R \geq R_1 + |x(t)| \Rightarrow |A_R(u(t))| \leq C\varepsilon = 4s_c(p - 1)\delta_0 \leq 4s_c(p - 1)\delta(t), \forall t \notin D_{\delta_0}.$$

Consequently, if we let $R_2 = \max\{R_0, R_1\}$, then

$$|A_R(u(t))| \leq 4s_c(p - 1)\delta(t), \quad \forall R \geq R_2(1 + |x(t)|), \quad \forall t \geq 0.$$

In particular, if we let $R = R_2(1 + \sup_{\sigma \leq t \leq \tau} |x(t)|)$, then

$$\dot{y}_R(t) = 8s_c(p - 1)\delta(t) + A_R(u(t)) \geq 4s_c(p - 1)\delta(t), \quad \forall t \in [\sigma, \tau],$$

which indicates that

$$\int_{\sigma}^{\tau} \delta(t)dt \lesssim \int_{\sigma}^{\tau} \dot{y}_R(t)dt = \dot{y}_R(\tau) - \dot{y}_R(\sigma).$$

It remains for us to estimate $\dot{y}_R(t) = 2R^3 \int_{\mathbb{R}^N} \nabla \varphi \left(\frac{x}{R}\right) \cdot \nabla u\bar{v}dx$. If $\delta(t) \geq \delta_0$, then

$$|\dot{y}_R(t)| \leq 2R \|\nabla \varphi\|_{\infty} \|\nabla u\|_2 \|\bar{v}\|_2 \leq CR \|Q\|_2 \left(\delta(t) + \|\nabla Q\|_2^2\right)^{\frac{1}{4}} \leq CR\delta(t).$$

If $t \in D_{\delta_0}$, then by (4.2),

$$\begin{align*}
\dot{y}_R(t) &= 2R^3 \int_{\mathbb{R}^N} \nabla \varphi \left(\frac{x + X(t)}{R}\right) \cdot \nabla Q\bar{v}dx + 2R^3 \int_{\mathbb{R}^N} \nabla \varphi \left(\frac{x + X(t)}{R}\right) \cdot \nabla vQdx \\
&\quad + 2R^3 \int_{\mathbb{R}^N} \nabla \varphi \left(\frac{x + X(t)}{R}\right) \cdot \nabla v\bar{v}dx, \quad \text{where } v = \alpha Q + h,
\end{align*}$$

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which yields $|\dot{y}_R(t)| \leq CR(\delta(t) + \delta(t)^2) \leq CR\delta(t)$ by Lemma 4.2. Consequently, combining the argument above, we immediately get
\[
\int_{\sigma}^{\tau} \delta(t) dt \leq R(\delta(\sigma) + \delta(\tau)) = R_2 \left( 1 + \sup_{\sigma \leq t \leq \tau} |x(t)| \right) (\delta(\sigma) + \delta(\tau)).
\]

\[\square\]

**Lemma 6.7** (Control of the variations of $x(t)$). **Under the Assumption 1.2 and Assumption 1.3, there exists a constant $C > 0$ such that**
\[
\forall \sigma, \tau > 0 \text{ with } \sigma + 1 \leq \tau, \quad |x(\tau) - x(\sigma)| \leq C \int_{\sigma}^{\tau} \delta(t) dt.
\]

**Proof.** First we claim that there exists a constant $\delta_2 > 0$ such that $\forall \tau \geq 0$,
\[
\inf_{t \in [\tau, \tau + 2]} \delta(t) \geq \delta_2 \quad \text{or} \quad \sup_{t \in [\tau, \tau + 2]} \delta(t) < \delta_0.
\]
If not, there exist two sequences $\{t_n\}, \{t'_n\}$ such that
\[
\delta(t_n) \to 0, \quad \delta(t'_n) \geq \delta_0, \quad |t_n - t'_n| < 2.
\]
After extracting a subsequence, we have $t'_n - t_n \to t_0 \in [-2, 2]$. By the compactness of $K$, there exists $v_1 \in H^1$ such that $u(\cdot + x(t_n), t_n) \to v_1$ in $H^1(\mathbb{R}^N)$, then together with the fact that $\delta(t_n) \to 0$,
\[
M[u] = M[v_1] = M[Q], \quad E[u] = E[v_1] = E[Q], \quad \|\nabla v_1\|_{L^2} = \|\nabla Q\|_{L^2},
\]
which indicates that $v_1 = e^{i\theta_0}Q(\cdot - x_0)$ for some $(\theta_0, x_0) \in \mathbb{R} \setminus 2\pi \mathbb{Z} \times \mathbb{R}^N$ by Lemma 2.5. Moreover, it is easy to see that $e^{it}e^{i\theta_0}Q(\cdot - x_0)$ is the solution to (1.1) with initial data $v_1$. By the continuous dependence of solution to (1.1) onto the initial data,
\[
u(\cdot + x(t_n), t_n + t_0) \to e^{it_0}e^{i\theta_0}Q(\cdot - x_0).
\]
Thus together with the continuity of the flow,
\[
u(\cdot + x(t_n), t'_n) = u(\cdot + x(t_n), t_n + t'_n - t_n) \to e^{it_0}e^{i\theta_0}Q(\cdot - x_0).
\]
Since $|t_n - t'_n| \leq 2$, $|x(t_n) - x(t'_n)| \leq C < \infty$ uniformly by (6.6), so there exists $x_1$ such that $x(t_n) - x(t'_n) \to x_1$ and
\[
u(\cdot + x(t'_n), t'_n) = u(\cdot + x(t'_n) - x(t_n), t'_n) \to e^{it_0}e^{i\theta_0}Q(\cdot - x_0 - x_1)
\]
in $H^1$.
Therefore,
\[
\delta(t'_n) = \|\nabla u(t'_n)\|_{L^2}^2 - \|\nabla Q\|_{L^2}^2 \to 0,
\]
which contradicts to our assumption $\delta(t'_n) \geq \delta_0, \forall n \in \mathbb{N}$ and therefore we have proved (6.16).

Next, we claim that (6.15) holds when $\tau \leq \sigma + 2$. In fact, if $\inf_{t \in [\tau, \tau + 2]} \delta(t) \geq \delta_2$, then by (6.6),
\[
\int_{\sigma}^{\tau} \delta(t) dt \geq \left( \inf_{t \in [\tau, \tau + 2]} \delta(t) \right) (\tau - \sigma) \geq \delta_2 (\tau - \sigma) \geq \delta_2 \geq |x(\tau) - x(\sigma)|.
\]
If $\sup_{t \in [\tau, \tau + 2]} \delta(t) < \delta_0$, then by Lemma 4.3 and Corollary 6.3,
\[
|x(\tau) - x(\sigma)| = |X(\tau) - X(\sigma)| \leq \int_{\sigma}^{\tau} |\dot{X}(t)| dt \lesssim \int_{\sigma}^{\tau} \delta(t) dt.
\]
When it comes to the general case for $\tau \geq \sigma + 2$, we can divide $[\sigma, \tau]$ into intervals of length at least 1 and at most 2 and then stick the inequalities together to get (6.15) without the assumption $\tau \leq \sigma + 2$. \[\square\]
Proof of Theorem 6.1. By Corollary 6.5, there exists \( \{t_n\}_{n \geq 1} \) such that
\[
\delta(t_n) \to 0, \quad t_n \to \infty \quad \text{and} \quad 1 + t_n \leq t_{n+1}. \tag{6.17}
\]
Then for any fixed \( N > 0, \forall n \geq N + 1, \)
\[|x(t) - x(t_N)| \leq C \int_{t_N}^{t} \delta(s)ds \leq C \int_{t_N}^{t_n} \delta(t)dt \leq C (\delta(t_n) + \delta(t_N)) \left(1 + \sup_{t \in [t_N, t_n]} |x(t)| \right), \quad \forall t \in [t_N, t_n],
\]
which implies that
\[
\sup_{t \in [t_N, t_n]} |x(t)| \leq |x(t_N)| + C (\delta(t_n) + \delta(t_N)) \left(1 + \sup_{t \in [t_N, t_n]} |x(t)| \right).
\]
By the assumption on \( \{t_n\}_{n \geq 1} \) as shown in (6.17), we can choose \( N \geq 1 \) sufficiently large such that \( \forall n \geq N + 1, C (\delta(t_n) + \delta(t_N)) \leq \frac{1}{2} \), then
\[
\sup_{t \in [t_N, t_n]} |x(t)| \leq C|x(t_N)| + \frac{1}{2} < \infty, \quad \forall n \geq N + 1.
\]
By the arbitrariness of \( n \geq N + 1 \) and the continuity of \( x(t) \) on \( \mathbb{R}^+ \), \( x(t) \) is uniformly bounded on \( [0, +\infty) \) and thus by Lemma 6.6,
\[
\int_{\sigma}^{t_n} \delta(t)dt \lesssim \delta(\sigma) + \delta(t_n)
\]
uniformly in \( n \). Letting \( n \to \infty \),
\[
\int_{\sigma}^{\infty} \delta(t)dt \lesssim \delta(\sigma),
\]
which ensures \( \int_{\sigma}^{\infty} \delta(t)dt \leq Ce^{-ct} \) by Gronwall’s inequality. Then Theorem 6.1 simply follows from Lemma 4.4. \( \square \)

7. Uniqueness and completing the main theorem

7.1. Exponential Strichartz estimates for solutions to linearized equation. Before we proceed with the argument, we remark that the case for \( p = 2 \) is easy to deal with because we can expand the nonlinear term directly. As for \( p > 2 \), the question becomes tougher since we encounter fractional power, in which case we can not expand nonlinear term directly anymore, and it requires us to analyze more carefully, especially for \( p \in (2, 3) \), which causes the low regularity of nonlinear term. Thus in the later discussion, we mainly focus on \( p > 2 \). As for \( p = 2 \), we just mention the underlying difficulties in later discussion.

Let \( h \) satisfy \( u = e^{it} (Q + h) \), then \( h \) satisfies the following linearized equation
\[
i \partial_t h + \Delta h - h + V(h) + R(h) = 0, \tag{7.1}
\]
where \( V(h) \) is the linear term defined by
\[
V(h) = p \left( | \cdot |^{-(N-\gamma)} \ast (Q^{p-1}h) \right) Q^{p-1}
+ \left( | \cdot |^{-(N-\gamma)} \ast Q^p \right) \left( \left( \frac{P}{2} - 1 \right) Q^{p-2}h + \frac{P}{2} Q^{p-2}h \right), \tag{7.2}
\]
and \( R(h) \) is as in (3.4). We remark here that the notation \( h \) in this section is a little confusing with (4.2), but this section is not involved in the decomposition (4.2), so we no longer use other notations in this section to distinguish them anymore.
For any time interval $I \subset \mathbb{R}$, together with Lemma B.3, Lemma B.4 and Remark B.5,

\[
\| \langle \nabla \rangle R(h) \|_{S'(I, L^2)} \lesssim \begin{cases} 
|I|^{\alpha_1} \| \langle \nabla \rangle h \|^2_{S(I, L^2)} & + |I|^{\alpha_2} \| \langle \nabla \rangle h \|_{S(I, L^2)}^{p-1} + \| \langle \nabla \rangle h \|_{S(I, L^2)}^{2p-1}, \\
|I|^{\beta} \| \langle \nabla \rangle h \|_{S(I, L^2)}^{p-1} & + \| \langle \nabla \rangle h \|_{S(I, L^2)}^{2p-1},
\end{cases}
\]

for some $\alpha_1, \alpha_2, \beta > 0$. As for the linear term $V(h)$, by Lemma B.2, we obtain that

\[
\| \langle \nabla \rangle V(h) \|_{S'(I, L^2)} \lesssim |I|^{\gamma} \| \langle \nabla \rangle h \|_{S(I, L^2)}
\]

for some $\gamma > 0$. Then by Lemma 2.4, (7.3) and (7.4),

\[
\| \langle \nabla \rangle h \|_{S(I, L^2)} \leq \left\| \langle \nabla \rangle \left[ e^{i(t'-t)(\Delta-1)} h(t) - i \int_t^{t'} e^{i(t'-s)(\Delta-1)} (-V(h) - R(h))(s) ds \right] \right\|_{S(I, L^2)}
\]

\[
\lesssim \| h(t) \|_{H^1} + \| \langle \nabla \rangle V(h) \|_{S'(I, L^2)} + \| \langle \nabla \rangle R(h) \|_{S'(I, L^2)}
\]

\[
\leq \| h(t) \|_{H^1} + \tau_0^\gamma \| \langle \nabla \rangle h \|_{S(I, L^2)}
\]

\[
+ \begin{cases} 
\tau_0^{\alpha_1} \| \langle \nabla \rangle h \|^2_{S(I, L^2)} & + \tau_0^{\alpha_2} \| \langle \nabla \rangle h \|_{S(I, L^2)}^{p-1} + \| \langle \nabla \rangle h \|_{S(I, L^2)}^{2p-1}, \\
\tau_0^{\beta} \| \langle \nabla \rangle h \|_{S(I, L^2)}^{p-1} & + \| \langle \nabla \rangle h \|_{S(I, L^2)}^{2p-1},
\end{cases}
\]

\[
\| \langle \nabla \rangle h \|_{S(I, L^2)} \leq \sum_{n=0}^{\infty} \| \langle \nabla \rangle h \|_{S(t+n\tau_0, t+(n+1)\tau_0, L^2)} \leq \sum_{n=0}^{\infty} e^{-c(t+n\tau_0)} \lesssim e^{-ct}.
\]

Consequently, we have the following exponential Strichartz estimate:

**Lemma 7.1** (Exponential Strichartz estimate). For $h$ the solution to (7.1), if $\| h(t) \|_{H^1} \lesssim e^{-ct}$, $\forall t \geq T$ for some $c > 0$, then

\[
\| \langle \nabla \rangle h \|_{S((t, \infty), L^2)} \lesssim e^{-ct}, \quad \forall t \geq T.
\]

**Remark 7.2.** We can extract the argument above to conclude the result as follows: let $h$ satisfy

\[
\partial_t h + \mathcal{L}h = R \text{ with } \| h(t) \|_{H^1} \lesssim e^{-ct} \text{ and } \| \langle \nabla \rangle R \|_{S'((t, \infty), L^2)} \lesssim e^{-c't} \text{ for some } c' \geq c > 0, \text{ then }
\]

\[
\| \langle \nabla \rangle h \|_{S((t, +\infty), L^2)} \lesssim e^{-c't}.
\]

**7.2. Improving the rate of decay.**

**Lemma 7.3.** Under the Assumption 1.2 and Assumption 1.3, consider the solution $h$ to

\[
\partial_t h + \mathcal{L}h = R, \quad (x, t) \in \mathbb{R}^N \times (t_0, +\infty),
\]

with

\[
\| h(t) \|_{H^1} \lesssim e^{-c_0 t}, \quad \| \langle \nabla \rangle R \|_{S'((t, \infty), L^2)} \lesssim e^{-c_1 t}, \quad \text{and } c_0 < c_1.
\]

- If $c_0 < c_1 \leq c_0$ or $c_0 < c_0 < c_1$, then $\| h(t) \|_{H^1} + \| \langle \nabla \rangle h \|_{S((t, +\infty), L^2)} \lesssim e^{-c_1 t}$. 
- If $c_0 \leq e_0 < c_1$, then there exists $\mathcal{A} \in \mathbb{R}$ such that

\[
\| h(t) - Ae^{-c_0 t} \mathcal{Y}_+ \|_{H^1(\mathbb{R}^N)} + \| \langle \nabla \rangle (h(t) - Ae^{-c_0 t} \mathcal{Y}_+) \|_{S((t, +\infty), L^2)} \lesssim e^{-c_1 t}.
\]
Proof. By Remark 7.2, it suffices to consider the behavior of \( \|h(t)\|_{H^1} \). First, we decompose \( h \) into

\[
h(t) = \alpha_+(t)\mathcal{Y}_+ + \alpha_-(t)\mathcal{Y}_- + \sum_{j=0}^{N} \beta_j(t)Q_j + h_\perp(t), \quad h_\perp \in G'_\perp,
\]

(7.7)

where \( Q_0 = \frac{iq}{\|q\|_2} \), \( Q_j = \frac{\partial_q Q_j}{\|\partial_q Q_j\|_2} \), \( 1 \leq j \leq N \), and out of simplicity, we can also normalize the eigenfunctions \( \mathcal{Y}_\pm \) such that \( B(\mathcal{Y}_+^\perp, \mathcal{Y}_-) = 1 \). Then

\[
\begin{align*}
\alpha_+(t) &= B(h(t), \mathcal{Y}_+), \\
\alpha_-(t) &= B(h(t), \mathcal{Y}_+), \\
\beta_j(t) &= \langle h(t), Q_j \rangle - \alpha_+(t) \langle \mathcal{Y}_+, Q_j \rangle - \alpha_-(t) \langle \mathcal{Y}_-, Q_j \rangle .
\end{align*}
\]

(7.8) (7.9)

The estimates on \( \alpha_\pm \). First, by Proposition 3.3, we have the decay of \( \alpha_\pm(t) \) at infinity,

\[
|\alpha_\pm| = |B(h, \mathcal{Y}_\pm)| \lesssim \left| \frac{1}{2} \int (L_+ h_1)\mathcal{Y}_1 dx \right| + \left| \frac{1}{2} \int (L_- h_2)\mathcal{Y}_2 dx \right| \lesssim \|h(t)\|_{L^2} \lesssim e^{-\epsilon_0 t}.
\]

(7.10)

Next, we want to improve the decay of \( \alpha_\pm \). As for \( \alpha_+ \), noting that \( \alpha_+ \) satisfies the following differential equation

\[
\dot{\alpha}_+(t) = \partial_t B(h, \mathcal{Y}_-) = B(\partial_t h, \mathcal{Y}_-) = B(-\mathcal{L} h + R, \mathcal{Y}_-) = -e_0 \alpha_+ + B(R, \mathcal{Y}_-),
\]

we get

\[
\partial_t \left( e^{\epsilon_0 t} \alpha_+ \right) = e^{\epsilon_0 t} B(R, \mathcal{Y}_-),
\]

(7.11)

which means that

\[
e^{\epsilon_0 t} \alpha_+(t) = \alpha_+(0) + \int_0^t e^{\epsilon_0 s} B(R, \mathcal{Y}_-) ds.
\]

If \( \epsilon_0 \geq \epsilon_1 > \epsilon_0 \), then by using the estimate that

\[
\int_n^{n+1} e^{\epsilon_0 s} \|B(R, \mathcal{Y}_-)\| |ds| \lesssim e^{\epsilon_0 n} \|R\|_{L^2} \lesssim e^{(\epsilon_0 - \epsilon_1) n},
\]

(7.12)

we conclude that

\[
\left| \int_0^t e^{\epsilon_0 s} B(R, \mathcal{Y}_-) ds \right| \lesssim \sum_{n=0}^{[t]} \int_n^{n+1} e^{\epsilon_0 s} \|B(R, \mathcal{Y}_-)\| ds \lesssim \sum_{n=0}^{[t]} e^{(\epsilon_0 - \epsilon_1) n} \lesssim \begin{cases} e^{(\epsilon_0 - \epsilon_1) t}, & \text{if } \epsilon_0 > \epsilon_1, \\ t, & \text{if } \epsilon_0 = \epsilon_1, \end{cases}
\]

and thus \( |\alpha_+(t)| \lesssim e^{-\epsilon_1 t} \).

If \( \epsilon_0 < \epsilon_0 < \epsilon_1 \), then (7.10) implies that \( e^{\epsilon_0 t} \alpha_+(t) \rightarrow 0 \) as \( t \rightarrow \infty \) and thus by (7.11) we have

\[
e^{\epsilon_0 t} \alpha_+(t) = \int_t^{\infty} e^{\epsilon_0 s} B(R, \mathcal{Y}_-) ds.
\]

By the same argument as what in (7.12),

\[
|e^{\epsilon_0 t} \alpha_+(t)| \lesssim \sum_{n=[t]}^{\infty} \left| \int_n^{n+1} e^{\epsilon_0 s} B(R, \mathcal{Y}_-) ds \right| \lesssim \sum_{n=[t]}^{\infty} e^{(\epsilon_0 - \epsilon_1) n} \lesssim e^{(\epsilon_0 - \epsilon_1) t},
\]

which means that \( |\alpha_+(t)| \lesssim e^{-\epsilon_1 t} \).

If \( \epsilon_0 \leq \epsilon_0 < \epsilon_1 \), then

\[
A = \alpha_+(0) + \int_0^{\infty} e^{\epsilon_0 s} B(R, \mathcal{Y}_-) ds \in \mathbb{R}
\]

by

\[
\int_0^{\infty} e^{\epsilon_0 s} B(R, \mathcal{Y}_-) ds \leq \sum_{n=0}^{\infty} \int_n^{n+1} e^{\epsilon_0 s} B(R, \mathcal{Y}_-) ds \lesssim \sum_{n=0}^{\infty} e^{(\epsilon_0 - \epsilon_1) n} < \infty.
\]
As for $A$ given above, we then obtain that

$$|e^{\epsilon_0 t} \alpha_+(t) - A| = \left| \int_t^\infty e^{\epsilon_0 s} B(R, \mathcal{Y}_-) ds \right| \lesssim \sum_{n=[t]}^\infty e^{(\epsilon_0 - c_1)n} \lesssim e^{(\epsilon_0 - c_1)t}.$$  

Hence $\alpha_+(t)$ satisfies

$$\begin{cases}
|\alpha_+(t)| \lesssim e^{-c_1 t}, & \text{if } c_0 < c_1 \leq \epsilon_0 \text{ or } \epsilon_0 < c_0 < c_1, \\
|\alpha_+(t) - A e^{-\epsilon_0 t}| \lesssim e^{-c_1 t}, & \text{if } c_0 \leq \epsilon_0 < c_1.
\end{cases} \quad (7.13)$$

Similarly, as for $\alpha_-$,

$$\partial_t \left( e^{-\epsilon_0 t} \alpha_- \right) = e^{-\epsilon_0 t} B(R, \mathcal{Y}_+) \Rightarrow e^{-\epsilon_0 t} \alpha_-(t) = \int_t^\infty e^{-\epsilon_0 s} B(R, \mathcal{Y}_+) ds.$$  

By the same argument as what for $\alpha_+$ when $\epsilon_0 < c_0 < c_1$,

$$|\alpha_-(t)| \lesssim e^{\epsilon_0 t} \sum_{n=[t]}^{n+1} \left| \int_n^{n+1} e^{-\epsilon_0 s} B(R, \mathcal{Y}_+) ds \right| \lesssim e^{\epsilon_0 t} \sum_{n=[t]}^\infty e^{-(\epsilon_0 + c_1)n} \lesssim e^{-c_1 t}. \quad (7.14)$$

The estimates on the remaining terms and completing the proof of the case for $A = 0$. First, since $B(Lf, g) = -B(f, Lg)$, we have $B(Lh, h) = 0$, which further implies that

$$\partial_t \Phi(h) = 2B(\partial_t h, h) = 2B(-\mathcal{L} h + R, h) = 2B(R, h).$$

Moreover, by the assumption $\|h(t)\|_{H^1} \lesssim e^{-c_0 t}$ and Remark 7.2, we obtain that

$$\int_n^{n+1} B(h, R) ds = \frac{1}{2} \int_n^{n+1} \int_{\mathbb{R}^N} (L_+ h_1) R_1 dx ds + \frac{1}{2} \int_n^{n+1} \int_{\mathbb{R}^N} (L_- h_2) R_2 dx ds \lesssim \| (\nabla) h \|_{S([n, +\infty), L^2)} \| (\nabla) R \|_{S'([n, +\infty), L^2)} \lesssim e^{-c_0 t}$$

which then implies that

$$|\Phi(h)| = \left| \int_t^\infty 2B(R, h) ds \right| \lesssim e^{-(\epsilon_0 + c_1)t}.$$  

Furthermore, by decomposition of $h$ in (7.7),

$$\Phi(h) = B(h, h) = 2\alpha_+ \alpha_- + B(h_\perp, h_\perp),$$

by (4.3) and (7.10),

$$\|h_\perp(t)\|_{H^1}^2 \simeq B(h_\perp, h_\perp) = \Phi(h) - 2\alpha_+ \alpha_- \lesssim e^{-(\epsilon_0 + c_1)t} + e^{-2c_1 t} \lesssim e^{-(\epsilon_0 + c_1)t}$$

for large $t$, i.e.

$$\|h_\perp(t)\|_{H^1} \lesssim e^{-\frac{\epsilon_0 + c_1}{2}t}, \quad \forall t \geq t_0 \text{ for some } t_0 > 0. \quad (7.15)$$

Besides, by (7.9),

$$\dot{\beta}_j(t) = \left< h(t), Q_j \right> - \dot{\alpha}_+(t) \left< \mathcal{Y}_+, Q_j \right> - \dot{\alpha}_-(t) \left< \mathcal{Y}_-, Q_j \right>$$

$$= (-\mathcal{L} h + R - \dot{\alpha}_+ \mathcal{Y}_+ - \dot{\alpha}_- \mathcal{Y}_-, Q_j)$$

$$= (-\dot{\alpha}_+ + \epsilon_0 \alpha_+) \left< \mathcal{Y}_+, Q_j \right> + (-\dot{\alpha}_- + \epsilon_0 \alpha_-) \left< \mathcal{Y}_-, Q_j \right> - \left< \mathcal{L} h, Q_j \right> + \left< R, Q_j \right>,$$
then together with the $\beta_j(t) \to 0$ as $t \to \infty$,

$$|\beta_j(t)| = \left| \int_t^\infty \dot{\beta}_j(s) ds \right| \lesssim \int_t^\infty |B(R, Y_\pm)| + \|h_\bot(s)\|_{H^1} + \|R, Q_j\| ds \lesssim \sum_{n=0}^\infty ||\langle \nabla \rangle R||_{L^2((t+n, t+n+1), L^2)} + \|h_\bot\|_{L^\infty H^1((t+n, t+n+1)}$$

$$\lesssim \sum_{n=0}^\infty \left( e^{-c_1(t+n)} + e^{-\frac{\alpha_0+c_1}{2}(t+n)} \right) \lesssim e^{-\frac{\alpha_0+c_1}{2} t}, \forall t \geq t_0. \quad (7.16)$$

Combining (7.7), (7.13), (7.14), (7.15) and (7.16), we obtain that

$$\|h(t)\|_{H^1} \lesssim e^{-\frac{\alpha_0+c_1}{2} t}, \forall t \geq t_0.$$ 

Taking $c_0 \to \frac{\alpha_0+c_1}{2}$ and $c_1 \to c_1$ and repeating the argument again and again, we then improve the decay to $\|h(t)\|_{H^1} \lesssim e^{-c_1 t}, \forall t \geq t_0.$

*Complete the proof.* It remains for us to check the case for $A \neq 0$, i.e. when $c_0 \leq e_0 < c_1$. In fact, we notice that the function

$$\tilde{h}(t) = h(t) - Ae^{-c_0 t} Y_+$$

satisfies the equation

$$\tilde{\partial}_t \tilde{h} + \tilde{L} \tilde{h} = R$$

with $\|\tilde{h}(t)\|_{H^1(R^N)} \lesssim e^{-c_0 t}$, and $|\tilde{\alpha}_\bot(t)| = B(\tilde{h}(t), Y_-) \lesssim e^{-c_1 t}$ by (7.13). Repeating the previous argument on $\tilde{h}$, we immediately get the desired conclusion. \qed

### 7.3. Existence of special solutions.

**Proposition 7.4.** Under the Assumption 1.3. Let $A \in \mathbb{R}$, then there exists a sequence $(Z_j)_{j \geq 1} \subset S$ such that $\mathcal{V}^A_k \triangleq \sum_{j=1}^k e^{-j\alpha_0 t} Z_j^A$ satisfies

$$\partial_t \mathcal{V}^A_k + L \mathcal{V}^A_k = iR(\mathcal{V}^A_k) + O(e^{-(k+1)\alpha_0 t}) \text{ in } S(R^N). \quad (7.17)$$

**Remark 7.5.** Let $U^A_k = e^{it}(Q + \mathcal{V}^A_k)$, then by the proposition above, $U^A_k$ satisfies

$$i\partial_t U^A_k + \Delta U^A_k + \left( |\cdot|^{-\gamma - (N-\gamma)} \ast |U^A_k|^p \right) |U^A_k|^{p-2} U^A_k = O(e^{-(k+1)\alpha_0 t}) \text{ in } S(R^N). \quad (7.18)$$

**Proof of Proposition 7.4.** As for $p > 2$, we follow the proof of [6, Proposition 4.1] and [9, Lemma 6.1] . As for $J(z), K(z)$ defined in (B.1) and (B.2), they are both real-analytic on the disk $\{ z \in \mathbb{C} ||z| < 1 \}$ and satisfy

$$J(z) = \sum_{j_1+j_2 \geq 2} a_{j_1, j_2} z^{j_1} z^{j_2}, \quad K(z) = \sum_{k_1+k_2 \geq 2} b_{k_1, k_2} z^{k_1} z^{k_2}$$
with normal convergence of the series and all its derivatives for $|z| \leq \frac{1}{2}$. Thus if $|\omega| \equiv |Q^{-1} \epsilon| \leq \frac{1}{2}$, we obtain that

$$R(h) = \left( \left| \cdot \right|^{- (N - \gamma)} \cdot \left( Q^p \sum_{j_1 + j_2 \geq 2} a_{j_1 j_2} \omega^{j_1 \omega j_2} \right) \right) Q^{p-1} \left( \sum_{k_1 + k_2 \geq 2} b_{k_1 k_2} \omega^{k_1 \omega k_2} + \left( 1 + \frac{p}{2} \omega + \frac{p}{2} \right) \right)

+ \left( \left| \cdot \right|^{- (N - \gamma)} \cdot \left( Q^p \left( 1 + \frac{p}{2} \omega + \frac{p}{2} \right) \right) \right) Q^{p-1} \left( \sum_{k_1 + k_2 \geq 2} b_{k_1 k_2} \omega^{k_1 \omega k_2} \right)

+ \left( \left| \cdot \right|^{- (N - \gamma)} \cdot \left( Q^p \left( \frac{p}{2} \omega + \frac{p}{2} \right) \right) \right) Q^{p-1} \left( \frac{p}{2} \omega + \frac{p}{2} \omega \right)

= \sum_{j_1 + j_2 + k_1 + k_2 \geq 2} c_{j_1 j_2 k_1 k_2} \left( \left| \cdot \right|^{- (N - \gamma)} \cdot \left( Q^p \omega^{j_1 \omega j_2} \right) \right) Q^{p-1} \omega^{k_1 \omega k_2},$$  

(7.19)

where $R(h)$ is defined in (3.4). Denote $\epsilon_k = \partial_t \mathcal{V}_k^A + \mathcal{L} \mathcal{V}_k^A - iR(\mathcal{V}_k^A)$. As for $k = 1$, let $\mathcal{Z}_1^A = A \mathcal{V}_+^A$, then $\mathcal{V}_1^A = e^{-\epsilon_0 t} \mathcal{Z}_1^A$ and $\omega = Q^{-1} \mathcal{V}_1^A = e^{-\epsilon_0 t} Q^{-1} \mathcal{V}_+^A$. Since the decay rate of $\mathcal{V}_+^A$ is faster than $Q$ (see Lemma A.5), $R(\mathcal{V}_1^A)$ can be expanded as in (7.19) for large $t$, then

$$|\epsilon_1| = \left| \partial_t \mathcal{V}_1^A + \mathcal{L} \mathcal{V}_1^A - iR(\mathcal{V}_1^A) \right| = |iR(\mathcal{V}_1^A)| \leq e^{-2\epsilon_0 t}, \quad \text{for } t \geq t_0 \gg 1.$$

Similarly, by Lemma A.5 again, it can be also easy to check that

$$|x|^\alpha |\partial^\beta \epsilon_1| \leq e^{-2\epsilon_0 t}, \quad \forall \alpha, \beta \in \mathbb{Z}^N, \forall t \geq t_0(\alpha, \beta)$$

and

$$\|Q^{-1} e^{j x \cdot |\partial^\alpha \mathcal{Z}_1^A|} \|_{L^\infty} < \infty, \quad \forall \alpha \in \mathbb{Z}^N_{\geq 0} \text{ for some } 0 < \eta \ll 1.$$

As for general case, we use induction. In fact, we assume that $\mathcal{Z}_1^A, \mathcal{Z}_2^A, ..., \mathcal{Z}_k^A \in \mathcal{S}(\mathbb{R}^N)$ have been constructed and satisfy

$$\|Q^{-1} e^{j x \cdot |\partial^\alpha \mathcal{Z}_j|} \|_{L^\infty} < \infty, \quad \forall j \leq k, \forall \alpha \in \mathbb{Z}^N_{\geq 0},$$

(7.20)

then $\epsilon_k = \partial_t \mathcal{V}_k^A + \mathcal{L} \mathcal{V}_k^A - iR(\mathcal{V}_k^A) = O(e^{-(k+1)\epsilon_0 t})$ in $\mathcal{S}(\mathbb{R}^N)$ and $|Q^{-1} \mathcal{V}_k^A| \leq \frac{1}{2}$ for large $t$. Hence by (7.19) again,

$$-iR(\mathcal{V}_k^A) = \sum_{j=1}^{k+1} e^{-j \epsilon_0 t} F_j + O(e^{-(k+2)\epsilon_0 t}) \text{ in } \mathcal{S}(\mathbb{R}^N), \forall t > t_0 \gg 1,$$

for some $F_j \in \mathcal{S}(\mathbb{R}^N), \forall 1 \leq j \leq k + 1$. Furthermore, by induction assumption (7.20), $F_j$ also satisfies

$$\|Q^{-1} e^{j x \cdot |\partial^\beta F_j|} \|_{L^\infty} < \infty, \quad \forall \beta \in \mathbb{Z}^N_{\geq 0}, \quad \forall 1 \leq j \leq k + 1.$$  

(7.21)

Thus

$$\epsilon_k = \sum_{j=1}^{k} e^{j \epsilon_0 t} \left( F_j + (\mathcal{L} - j \epsilon_0) \mathcal{Z}_j^A \right) + e^{-(k+1)\epsilon_0 t} F_{k+1} + O(e^{-(k+2)\epsilon_0 t}) = O \left( e^{-(k+1)\epsilon_0 t} \right),$$

which then implies that

$$\epsilon_k = e^{-(k+1)\epsilon_0 t} F_{k+1} + O(e^{-(k+2)\epsilon_0 t}).$$
Let $Z_{k+1}^A \triangleq -(\mathcal{L} - (k + 1)e_0)^{-1} F_{k+1}$, then by (3.9), (7.21) and Lemma A.5, $Z_{k+1}^A \in S(\mathbb{R}^N)$ and $Z_j^A$ satisfies the induction assumption (7.20) for $j \leq k + 1$. Consequently,

$$
\varepsilon_{k+1} = -i \left( R(V_{k+1}^A) - R(V_k^A) \right) + \varepsilon_k + e^{-(k+1)e_0t} (\mathcal{L} - (k + 1)e_0) Z_{k+1}^A
$$

$$
= -i \left( R(V_{k+1}^A) - R(V_k^A) \right) + O\left(e^{-(k+2)e_0t}\right).
$$

Using (7.19) again for large $t$,

$$
R(V_{k+1}^A) - R(V_k^A) = \sum_{j_1+j_2+k_1+k_2 \geq 2} c_{j_1j_2k_1k_2} \left( |\cdot|^{-(N-\gamma)} \ast \left( Q^p Y_{k+1}^{j_1} \bar{Y}_{k+1}^{j_2} \right) \right) Q^{p-1} Y_{k+1}^{k_1} \bar{Y}_{k+1}^{k_2}
$$

$$
- \sum_{j_1+j_2+k_1+k_2 \geq 2} c_{j_1j_2k_1k_2} \left( |\cdot|^{-(N-\gamma)} \ast \left( Q^p Y_{k}^{j_1} \bar{Y}_{k}^{j_2} \right) \right) Q^{p-1} Y_{k}^{k_1} \bar{Y}_{k}^{k_2}
$$

$$
= O\left(e^{-(k+2)e_0t}\right),
$$

we then conclude that $\varepsilon_{k+1} = O(e^{-(k+2)e_0t})$ in $S(\mathbb{R}^N)$, and the construction is completed for $p > 2$.

As for $p = 2$, since we can’t get the parallel information about $Y_\pm$ as shown in Lemma A.5(ii) which is suitable for $p > 2$, it is not appropriate to simply follow the argument above to construct approximation solutions when $p = 2$. However, the even power $p = 2$ enables us to expand $|f + g|^2$ directly and we needn’t to use the expansion (7.19) anymore, which heavily relies on a priori assumption $|\omega| \triangleq |Q^{-1}h| \leq \frac{1}{2}$. Therefore, it is easier to handle the case for $p = 2$ and the proof is almost essentially the same as [29, Proposition 3.1] and [10, Proposition 3.4]. □

Next, we construct the special solution with threshold mass-energy by a fixed point argument.

**Proposition 7.6** (Construction of special solutions near the approximation solution). Under the Assumption 1.3, let $A \in \mathbb{R}$, there exists $k_0 > 0$ and $t_0 > 0$ such that $\forall k \geq k_0$, there exists a solution $U^A$ to (1.1) such that

$$
\| \langle \nabla \rangle (U^A - U^A_{l(k)}) \|_{S((t, +\infty); L^2)} \leq e^{-(k+\frac{1}{2})e_0t}, \quad \forall t \geq t_0,
$$

(7.22)

where $l(k) = \begin{cases} \max \left\{ k, \left( k + \frac{1}{2} \right) \frac{1}{p-2} \right\}, & p \neq 2, \\ k, & p = 2. \end{cases}$

Furthermore, $U^A$ is the unique solution to (1.1) satisfying (7.22) for large $t$. Finally, $U^A$ is independent of $k$ and satisfies

$$
\| U^A(t) - e^{itQ} - A e^{(i-\varepsilon_0)t} Y_+ \|_{H^1} \lesssim e^{-2e_0t}.
$$

(7.23)

**Proof.** Let $h$ be the solution to the linearized equation

$$
\partial_t h + \mathcal{L} h = i R(h),
$$

which is equivalent to the equation

$$
i \partial_t h + \Delta h - h + S(h) = 0, \quad \text{where } S(h) = V(h) + R(h) \text{ is defined in (7.1)}.
$$

By (7.17), the function $w \triangleq h - Y^A_{l(k)}$ satisfies the equation

$$
i \partial_t w + \Delta w - w + S(Y^A_{l(k)} + w) - S(Y^A_{l(k)}) = -\varepsilon_{l(k)} = O(e^{-(l(k)+1)e_0t}),
$$

and $w$ is given by the equation

$$
w(t) = M(w)(t),
$$

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where
\[ M(w)(t) = -i \int_t^{+\infty} e^{i(t-s)(\Delta-1)} \left[ S(V_{l(k)}^A + w) - S(V_{l(k)}^A) \right] s ds. \] (7.24)

Next, we begin to construct the special function for \( p \in (2, 3) \). As shown in Lemma B.3, if we only use \( S(I, \langle \nabla \rangle L^2) \) norm, we face a term \( \| \langle \nabla \rangle (f - g) \|^{p-2}_{S'(L^2)} \) with power \( p - 2 < 1 \), which makes contraction unable to work, and we turn to \( S(I, \dot{H}^{s_c}) \) norm for help. Precisely speaking, for \( k \geq 1 \) and \( t_k \) fixed later, we define the working space by
\[
B^k \triangleq \left\{ w \in X^k : \| w \|_{X^k} \leq 1 \right\},
\]
\[
X^k \triangleq \left\{ w : \langle \nabla \rangle w \in S([t_k, +\infty), L^2), w \in S([t_k, +\infty), \dot{H}^{s_c}) \text{ and } \| w \|_{X^k} < \infty, \right\}
\]
where \( \| w \|_{X^k} = \sup_{t \geq t_k} \| w \|_{S([t, +\infty), \dot{H}^{s_c})} e^{(k+\frac{1}{2}) \frac{1}{p-2} \epsilon_0 t} + \sup_{t \geq t_k} \| \langle \nabla \rangle w \|_{S([t, +\infty), L^2)} e^{(k+\frac{1}{2}) \frac{1}{p-2} \epsilon_0 t} \}
\]
equipped with the metric
\[ d_k(u, v) = \sup_{t \geq t_k} \| u - v \|_{S([t, +\infty), \dot{H}^{s_c})} e^{(k+\frac{1}{2}) \frac{1}{p-2} \epsilon_0 t}. \]

By the uniqueness of weak limitation, it is easy to see that \((B^k, d_k)\) is a complete metric space. We will show that \( M : B^k \to B^k \) and \( M \) is a contraction. For any fixed \( w \in B^k \), by Strichartz estimate, we can see that
\[
\| \langle \nabla \rangle M(w) \|_{S([t, +\infty), L^2)} \lesssim \| \langle \nabla \rangle (S(V_{l(k)}^A + w) - S(V_{l(k)}^A)) \|_{S'(([t, +\infty), L^2)} + \| \langle \nabla \rangle \varepsilon_{l(k)} \|_{S'(([t, +\infty), L^2)}
\]
\[
\lesssim \| \langle \nabla \rangle \left( R(V_{l(k)}^A + w) - R(V_{l(k)}^A) \right) \|_{S'(([t, +\infty), L^2)} + \| \langle \nabla \rangle V(w) \|_{S'([t, +\infty), L^2)}
\]
\[
+ \| \langle \nabla \rangle \varepsilon_{l(k)} \|_{S'(([t, +\infty), L^2)}.
\]

As for the estimate of the nonlinear term, by (B.5) and Remark B.5,
\[
\| \langle \nabla \rangle (R(V_{l(k)}^A + w) - R(V_{l(k)}^A)) \|_{S'([t, L^2])} \lesssim e^{-(k+\frac{3}{2}) \epsilon_0 n} \text{ for } I_n = [n, n+1],
\]
where the constant \( C > 0 \) is independent of \( n \in \mathbb{Z}^+ \). Sticking the intervals together, we obtain that
\[
\| \langle \nabla \rangle \left( R(V_{l(k)}^A + w) - R(V_{l(k)}^A) \right) \|_{S'([t, +\infty), L^2)}
\]
\[
\lesssim \sum_{n=0}^{\infty} \| \langle \nabla \rangle \left( R(V_{l(k)}^A + w) - R(V_{l(k)}^A) \right) \|_{S'([t+n, t+n+1], L^2)}
\]
\[
\lesssim \sum_{n=0}^{\infty} e^{-(k+\frac{3}{2}) \epsilon_0 (t+n)} \lesssim e^{-(k+\frac{3}{2}) \epsilon_0 t}. \] (7.25)

**Remark 7.7.** Before we continue proceeding with the proof, we emphasize that when handling
\[ \| \langle \nabla \rangle (R(V_{l(k)}^A + w) - R(V_{l(k)}^A)) \|_{S'([t, +\infty), L^2)} \text{, we face the term } \| w \|^{p-2}_{S(I_n, \dot{H}^{s_c})} \text{ again, whose low power is likely to bring us another obstacle. Precisely, if the decay of } \| w \|_{S([t, +\infty), \dot{H}^{s_c})} \text{ is just } e^{-(k+\frac{3}{2}) \epsilon_0 t}, \text{ then it is hard to get enough decay of } \| \langle \nabla \rangle (R(V_{l(k)}^A + w) - R(V_{l(k)}^A)) \|_{S'([t, +\infty), L^2)}, \text{ hence } \| w \|_{S([t, +\infty), \dot{H}^{s_c})} \text{ should have faster decay, and here we wish its decay rate to be } e^{-(k+\frac{3}{2}) \frac{1}{p-2} \epsilon_0 t} (\text{see definition of working space } X^k).
As for the estimate of the linear term $\| (\nabla) V(w) \|_{S^0([t, t+\tau_0), L^2]}$ by (7.4),
\[
\| (\nabla) V(w) \|_{S^0([t, t+\tau_0), L^2]} \leq C \tau_0^\gamma \| (\nabla) w \|_{S^0([t, t+\tau_0), L^2]} \leq C \tau_0^\gamma e^{-(k+\frac{1}{p})\varepsilon_0 t},
\]
where the constant $C > 0$ is independent of the choice of $\tau_0 > 0$, similarly, it yields
\[
\| (\nabla) V(w) \|_{S^0([t, t+\tau_0), L^2]} \leq \sum_{n=0}^{\infty} \| (\nabla) V(w) \|_{S^0([t+n\tau_0, t+n\tau_0+\tau_0), L^2]} \leq C \sum_{n=0}^{\infty} \tau_0^\gamma e^{-(k+\frac{1}{p})\varepsilon_0 (t+n\tau_0)} = \frac{C \tau_0^\gamma}{1 - e^{-(k+\frac{1}{p})\varepsilon_0 \tau_0}} e^{-(k+\frac{1}{p})\varepsilon_0 t}. \tag{7.26}
\]
Furthermore, as for the remaining term, by Proposition 7.4,
\[
\| (\nabla) \varepsilon_{l(k)} \|_{S(t, +\infty)} \lesssim_k e^{-(l(k)+1)\varepsilon_0 t}. \tag{7.27}
\]
Combining with (7.25), (7.26) and (7.27),
\[
e^{(k+\frac{1}{p})\varepsilon_0 t} \| (\nabla) M(w) \|_{S^0([t, t+\tau_0), L^2]} \leq C e^{\varepsilon_0 t} + \frac{C \tau_0^\gamma}{1 - e^{-(k+\frac{1}{p})\varepsilon_0 \tau_0}} + C_k e^{-\frac{1}{2} \varepsilon_0 t}.
\]
Choosing $\tau_0 \ll 1$ and then letting $k \geq 1$ and $t_k > 0$ sufficiently large, we can get
\[
e^{(k+\frac{1}{p})\varepsilon_0 t} \| (\nabla) M(w) \|_{S^0([t, t+\tau_0), L^2]} \leq \frac{1}{4}, \quad \forall w \in B^k, \forall t \geq t_k. \tag{7.28}
\]
Similarly, by (B.6), we can repeat the previous argument to check that
\[
\| R(V^A_{l(k)} + w) - R(V^A_{l(k)}) \|_{S^0([t, t+\tau_0), \dot{H}^{-\frac{s}{2}})} \lesssim e^{-(k+\frac{1}{p})\varepsilon_0 t} e^{-\varepsilon_0 t}, \tag{7.29}
\]
\[
\| V(w) \|_{S^0([t, t+\tau_0), \dot{H}^{-\frac{s}{2}})} \lesssim \frac{C \tau_0^\gamma}{1 - e^{-(k+\frac{1}{p})\varepsilon_0 \tau_0}} e^{-(k+\frac{1}{p})\varepsilon_0 t}. \tag{7.30}
\]
and
\[
\| (\nabla) \varepsilon_{l(k)} \|_{S(t, +\infty), \dot{H}^{-\frac{s}{2}}} \lesssim_k e^{-(l(k)+1)\varepsilon_0 t}, \tag{7.31}
\]
thus
\[
e^{(k+\frac{1}{p})\varepsilon_0 t} \| M(w) \|_{S^0([t, t+\tau_0), \dot{H}^{-\frac{s}{2}})} \leq C e^{\varepsilon_0 t} + \frac{C \tau_0^\gamma}{1 - e^{-(k+\frac{1}{p})\varepsilon_0 \tau_0}} + C_k e^{-\frac{1}{2} \varepsilon_0 t}.
\]
Similarly, after choosing $\tau_0 \ll 1$ and letting $k \geq 1$ and $t_k > 0$ sufficiently large,
\[
e^{(k+\frac{1}{p})\varepsilon_0 t} \frac{1}{4} \| M(w) \|_{S(t, +\infty), \dot{H}^{s}} \leq \frac{1}{4}, \quad \forall w \in B^k, \forall t \geq t_k. \tag{7.32}
\]
Therefore, together with (7.28) and (7.32), $\| M(w) \|_{\dot{X}^k} \leq \frac{1}{4} < 1$, $\forall h \in B^k$ for $k \gg 1$ and $t_k \gg 1$.

With the same argument as above, $M$ is also a contraction in $(B^k, d_k)$. Then we get a unique solution $w$ on $(t_k, +\infty)$, and $U^A \triangleq e^{it} \left( Q + V^A_{l(k)} + w \right)$ satisfies
\[
\| (\nabla) (U^A - U^A_{l(k)}) \|_{S(t, t+\tau_0), L^2]} \leq e^{-(k+\frac{1}{p})\varepsilon_0 t}, \quad \forall t \geq t_k
\]
and
\[
\| U^A - U^A_{l(k)} \|_{S^0([t, t+\tau_0), \dot{H}^{-\frac{s}{2}})} \leq e^{-(k+\frac{1}{p})\varepsilon_0 t}, \quad t \geq t_k.
\]
Moreover, by local well-posedness of (1.1), the uniqueness given by contraction and the embedding relationship
\[
X^{k'}(t_{k'} + \infty) \subset X^k(t_{k'} + \infty), \quad \forall k' \geq k,
\]
there exists \( k_0 \gg 1 \) and \( t_0 \gg 1 \) such that the solution \( U^A \) constructed above are coincident with each other for any \( k \geq k_0 \). By Duhamel formula (7.24) and Strichartz estimate
\[
\left\| \frac{U^A - e^{itQ} - e^{it\dot{\gamma}_{q(k)}(t)}}{H^1} \right\| = \left\| w(t) \right\|_{H^1} \lesssim e^{-(k+\frac{1}{2})e_{o}t}, \forall t \geq t_0.
\]
Then together with the fact that
\[
\left\| \gamma_{q(k)}(t) - Ae^{-e_{o}t}\gamma_+ \right\|_{H^1} \lesssim e^{-2e_{o}t},
\]
which follows from the construction of \( \gamma_{q(k)} \) in Proposition 7.4, we obtain (7.23) immediately.

As for \( p \geq 3 \) or \( p = 2 \), since we have more regularity for \( \| \cdot \|_p \) in this case, we do not need \( S(I, \dot{H}^{s_c}) \) anymore, and we can introduce a simpler working space as follows:
\[
B^k \triangleq \left\{ w \in X^k : \| w \|_{X^k} \leq 1 \right\},
\]
\[
X^k \triangleq \left\{ w : \langle \nabla \rangle w \in S([t_k, +\infty), L^2) : \| w \|_{X^k} = \sup_{t \geq t_k} \| (\nabla) w \|_{S([t, +\infty), L^2)} e^{(k+\frac{1}{2})e_{o} t} < \infty \right\},
\]
equipped with the metric
\[
d_k(u, v) = \| u - v \|_{X^k}.
\]
With almost the same argument as above, we can also get the desired special functions. \( \square \)

Remark 7.8. By Proposition 7.6 shown above, we can construct special solutions \( Q^\pm \) on \( \mathbb{R}^+ \) in Theorem 1.5. Precisely speaking, if we let \( A = \pm 1 \) in Proposition 7.6, then there exist solutions \( U^{\pm 1} \) on \([t_0, +\infty)\) such that
\[
\| U^{\pm 1} - e^{itQ} - e^{it\dot{\gamma}} \|_{H^1} \lesssim C e^{-2e_{o}t}, \quad \forall t \geq t_0,
\]
and \( U^\pm \) satisfy
\[
\| \nabla U^{\pm 1}(t) \|_2 = \| \nabla Q \|_2 + 2 e^{-e_{o}t} \int \nabla Q \cdot \nabla \gamma_1 + O(e^{-2e_{o}t}), \quad t \to \infty.
\]
Without loss of generality, we assume \( \int \nabla Q \cdot \nabla \gamma_1 > 0 \), then
\[
\| \nabla U^{\pm 1}(t) \|_2 > \| \nabla Q \|_2, \quad \text{and} \quad \| \nabla U^{-1}(t) \|_2 < \| \nabla Q \|_2, \quad \forall t \geq t_0 \text{ for } t_0 \gg 1.
\]
Letting
\[
Q^\pm(x, t) \triangleq e^{-it_0 U^{\pm 1}(x, t + t_0), t \geq 0}, \quad (7.33)
\]
we get two solutions satisfying
\[
M[Q^\pm] = M[Q], \quad E[Q^\pm] = E[Q], \quad \| \nabla Q^\pm(0) \|_2 > \| \nabla Q \|_2, \quad \| \nabla Q^{-}(0) \|_2 < \| \nabla Q \|_2,
\]
and
\[
\| Q^\pm - e^{itQ} \|_{H^1} \lesssim C e^{-e_{o}t}, \quad t \geq 0. \quad (7.34)
\]

Complete the proof of Theorem 1.5. As for \( Q^+ \) constructed in Remark 7.8, by Theorem 5.1 and (7.34), \( Q^+ \) satisfies the properties in Theorem 1.5. As for \( Q^- \), it remains to check that \( Q^- \) scatters in negative time direction. By contradiction, we assume that \( Q^- \) does not scatter in negative time, then \( t \to Q^-(x, t) \) globally exists in positive time but does not scatter in positive time. Repeating the argument of Theorem 6.1 onto \( Q^-(x, t) \), there exists a parameter \( x(t) \) for \( t \in \mathbb{R} \) such that
\[
\bar{K} = \{ Q^- \cdot + x(t), t \in \mathbb{R} \}
\]
has a compact closure in $H^1$, and $x(t)$ is bounded and $\delta(t) \to 0$, $t \to \pm\infty$. Then a simple adjustment of Lemma 6.6 yields
\[
\int_\sigma^T \delta(t) dt \leq C \left[ 1 + \sup_{\sigma \leq \tau} |x(t)| \right] \left( \delta(\sigma) + \delta(\tau) \right) \leq C \left( \delta(\sigma) + \delta(\tau) \right).
\]

Letting $\sigma \to -\infty$ and $\tau \to +\infty$, we get that $\int_{\mathbb{R}} \delta(t) dt = 0$, thus, $\delta(t) = 0$ for all $t$, which contradicts to the fact that $\|\nabla Q^{-1}(0)\|_2 < \|\nabla Q\|_2$.

\[\Box\]

### 7.4. Rigidity

Before we complete the Theorem 1.6, we still have to deal with a problem involved in uniqueness.

**Proposition 7.9 (Rigidity).** Under the Assumption 1.2 and Assumption 1.3, let $u$ be the solution to (1.1) on $[t_0, +\infty)$ such that $M[u] = M[Q], E[u] = E[Q]$ and
\[
\|u - e^{it}Q\|_{H^1} \leq Ce^{-ct}, \quad \forall t \geq t_0
\]
for some $c, C > 0$. Then there exists $A \in \mathbb{R}$ such that $u = U^A$, where $U^A$ is defined in Proposition 7.6.

**Proof.** Letting $u = e^{it}(Q + h)$, $h$ satisfies
\[\partial_t h + \mathcal{L}h = R(h)\]
and $\|h\|_{H^1(\mathbb{R}^N)} \leq Ce^{-ct}$, $\forall t \geq t_0$. By exponential Strichartz estimate shown in Lemma 7.1,
\[
\|h(t)\|_{H^1} + \| \langle \nabla \rangle h \|_{S((t, +\infty), L^2)} \leq Ce^{-ct}, \quad \forall t \geq t_0.
\]

**Step 1.** Improve the decay to
\[
\|h(t)\|_{H^1} + \| \langle \nabla \rangle h \|_{S((t, +\infty), L^2)} \leq Ce^{-c_0 t}, \quad \forall t \geq t_0.
\]

By (7.3),
\[
\| \langle \nabla \rangle R(h) \|_{S'((t, +\infty), L^2)} \lesssim \sum_{n=0}^\infty \| \langle \nabla \rangle R(h) \|_{S'((t+n, t+n+1), L^2)} \lesssim \begin{cases} e^{-(p-1)ct}, & p \in (2, 3), \\ e^{-2ct}, & p = 2 \text{ or } p \geq 3. \end{cases}
\]

Then $h$ and $R$ satisfy the conditions of Lemma 7.3 with
\[c_0 = c, \quad c_1 = \tau c, \quad \text{where} \quad \tau \triangleq \begin{cases} p - 1 > 1, & p \in (2, 3), \\ 2, & p = 2 \text{ or } p \geq 3. \end{cases}\]

If $\tau c > e_0$, then (7.36) is completed. If not, we get that
\[
\|h(t)\|_{H^1} + \| \langle \nabla \rangle h \|_{S((t, +\infty), L^2)} \leq Ce^{-\tau c t}, \quad \forall t \geq t_0,
\]
and then (7.36) follows from an iteration argument. And as a by-product, by Lemma 7.3 again, there exists $A \in \mathbb{R}$ such that
\[
\|h(t) - A^{-e^{-ct}Y} + \| \langle \nabla \rangle (h(t) - A^{-e^{-ct}Y}) \|_{S((t, +\infty), L^2)} \lesssim e^{-\tau c t}, \quad \forall t \geq t_0.
\]

**Step 2.** As for $U^A$ constructed in Proposition 7.6, where $A$ is as in Step 1, if $U^A \triangleq e^{it}(Q + h^A)$, then we claim
\[
\|h^A(t) - h(t)\|_{H^1} + \| \langle \nabla \rangle (h^A - h) \|_{S((t, +\infty), L^2)} \lesssim e^{-\gamma t}, \quad \forall t \geq t_0, \forall \gamma > 0.
\]

In fact, by Remark 7.2 and (7.23), $h^A$ satisfies
\[
\|h^A(t) - A^{-e^{-ct}Y} + \| \langle \nabla \rangle (h^A(t) - A^{-e^{-ct}Y}) \|_{S((t, +\infty), L^2)} \lesssim e^{-2\gamma_0 t}, \quad \forall t \geq t_0,
\]
then $h^A - h$ satisfies the equation
\[
\partial_t (h^A - h) + \mathcal{L} (h^A - h) = R(h^A) - R(h),
\]
and
\[
\|h^A - h\|_{H^1} + \left\| \langle \nabla \rangle (h^A - h) \right\|_{S((t, +\infty), L^2)} \lesssim e^{-\tau \epsilon_0^- t}, \quad \forall t \geq t_0.
\]
If $p \in (2, 3)$, then Lemma B.3 and Remark B.5,
\[
\| \langle \nabla \rangle (R(h^A) - R(h)) \|_{S((n,n+1), L^2)} \lesssim e^{-\lambda \epsilon_0^- n}
\]
for $\lambda = (p - 2)(p - 1) + 1 > p - 1 = \tau > 1$, which means
\[
\| \langle \nabla \rangle (R(h^A) - R(h)) \|_{S((t, +\infty), L^2)} \lesssim \sum_{n=0}^{\infty} \| \langle \nabla \rangle (R(h^A) - R(h)) \|_{S((t+n,n+n+1), L^2)} \lesssim \sum_{n=0}^{\infty} e^{-\lambda \epsilon_0^- (t+n)} \lesssim e^{-\lambda \epsilon_0^- t}, \quad \forall t \geq t_0,
\]
Using Lemma 7.3 with $c_0 = \tau \epsilon_0^- > \epsilon_0$ and $c_1 = \lambda \epsilon_0^-$, and repeating the process over and over again, we immediately get (7.37).

When it comes to $p = 2$ or $p > 3$, (7.37) also follows from the analogous argument.

**Step 3.** For any $k \in \mathbb{Z}^+$, because $\gamma$ is arbitrarily chosen, letting $\gamma = (k_0 + 1)\epsilon_0$ in (7.37), we get
\[
\left\| \langle \nabla \rangle (u - e^{it\mathcal{Q}^-} e^{i(t+k_0)\mathcal{Q}^+}) \right\|_{S((t, +\infty), L^2)} \lesssim e^{-(k_0 + \frac{1}{2})\epsilon_0^- t} \text{ for large } t,
\]
then $u = U^A$ follows from the uniqueness shown in Proposition 7.6.

**7.5. Complete the proof of Theorem 1.6.** Finally, we are devoted to the proof of Theorem 1.6.

As for $U^A$ constructed in Proposition 7.6, if $A \neq 0$, then $U^A = Q^+$ (if $A > 0$) or $Q^-$ (if $A < 0$) up to some symmetries. In fact, by the construction of $Q^\pm$,
\[
\left\| Q^\pm(t) - e^{it\mathcal{Q}^-} e^{-\epsilon_0 \epsilon_0^+ e^{i(t+k_0)\mathcal{Q}^+} \mathcal{Y}^+} \right\|_{H^1} = O(e^{-2\epsilon_0^- t}), \quad t \geq 0.
\]
Thus \(\forall t_1 \in \mathbb{R},\)
\[
e^{-it_1} Q^\pm(t + t_1, x) = e^{it\mathcal{Q}^-} e^{-\epsilon_0 e_0^+ e^{i(t+k_1)\mathcal{Q}^+} e^{-\epsilon_0^+} e^{i\mathcal{Y}^+} + O(e^{-2\epsilon_0^- t})} \text{ in } H^1.
\]
When $A > 0$, let $t_1 = -\epsilon_0^{-1} \log \epsilon$, then
\[
e^{-it_1} Q^+(t + t_1, x) = e^{it\mathcal{Q}^-} \epsilon e^{-\epsilon_0^+ e^{i(t+k_1)\mathcal{Q}^+} e^{-\epsilon_0^+} e^{i\mathcal{Y}^+} + O(e^{-2\epsilon_0^- t})} \text{ in } H^1.
\]
Consequently, by the rigidity shown in Proposition 7.9, there exists $\tilde{A} \in \mathbb{R}$ such that $e^{-it_1} Q^+(t + t_1, x) = U^A$, and by (7.23),
\[
e^{it\mathcal{Q}^-} \epsilon e^{-\epsilon_0^+ e^{i(t+k_1)\mathcal{Q}^+} e^{-\epsilon_0^+} e^{i\mathcal{Y}^+} + O(e^{-2\epsilon_0^- t})} = e^{-it_1} Q^+(t + t_1, x) = e^{it\mathcal{Q}^-} \epsilon e^{-\epsilon_0^+ e^{i(t+k_1)\mathcal{Q}^+} e^{-\epsilon_0^+} e^{i\mathcal{Y}^+} + O(e^{-2\epsilon_0^- t})} \text{ in } H^1.
\]
Comparing the first term with the last term, we obtain that $A = \tilde{A}$. Consequently, $U^A(t, x) = U^A(t, x) = e^{-it_1} Q^+(t + t_1, x)$. The case for $A < 0$ also follows from the same argument.

Let $u$ be a solution to (1.1) satisfying the conditions in Theorem 1.6, then $M[u]^{\frac{1}{1+\alpha}} E[u] = M[Q]^{\frac{1}{1+\alpha}} E[Q]$. Rescaling $u$ if necessary, we may assume
\[
M[u] = M[Q], \quad E[u] = E[Q].
\]
The classification of threshold solutions to (1.1) then immediately follows from Lemma 2.5, Theorem 5.1, Theorem 6.1, Proposition 7.9 and the argument above.
APPENDIX A. GROUND STATE

As for the existence of ground state, we can find it by minimizing a Weinstein-type functional. More precisely, for \( p \geq 2 \) and \( \gamma \in (0, N) \), by Hardy-Littlewood-Sobolev inequality, Hölder inequality and Sobolev embedding, one concludes that
\[
\int_{\mathbb{R}^N} \left( |\cdot|^{-(N-\gamma)} * |u|^p \right) |u|^p dx \leq C \|\nabla u\|_{L^2(\mathbb{R}^N)}^{Np-(N+\gamma)} \|u\|_{L^2(\mathbb{R}^N)}^{N+\gamma-(N-2)p}.
\] (A.1)

We want to determine the sharp constant for (A.1), which motivates us to minimize the following Weinstein-type functional
\[
Z[u] = \frac{\|u\|_2^{(N+\gamma)-(N-2)p} \|\nabla u\|_2^{Np-(N+\gamma)}}{\int (|\cdot|^{-(N-\gamma)} * |u|^p) |u|^p dx}.
\] (A.2)

By rearrangement inequalities [25, Theorem 3.4, Theorem 3.7] and Strauss Lemma, the infimum of (A.2) can be attained at a radial and strictly positive function. The readers can refer to Lemma 4.1 in [2] for details. And the result is stated as below:

**Proposition A.1** ([2], existence of ground state \( Q \).) *The minimizer of the problem*
\[
C_{GN}^{-1} \triangleq \inf_{0 \neq u \in H^1(\mathbb{R}^N)} Z[u]
\] (A.3)
*can be attained at a radial, positive function \( Q \). Moreover, after scaling if necessary, \( Q \) satisfies the equation (1.4).*

**Lemma A.2** (Pohozhaev identity). *As for \( Q \) given above, we have*
\[
\int \left( |\cdot|^{-(N-\gamma)} * Q^p \right) Q^p dx = \frac{2p}{\gamma+2p-N(p-1)}\|Q\|_2^2 = \frac{2p}{N(p-1)-\gamma}\|\nabla Q\|_2^2.
\] (A.4)

**Proof.** Multiplying \( x \cdot \nabla Q \) with both sides of
\[-\Delta Q + Q - \left( |\cdot|^{-(N-\gamma)} * Q^p \right) Q^{p-1} = 0,
\]
integrating on \( \mathbb{R}^N \), by some careful calculation, we have
\[-\frac{N-2}{2}\|\nabla Q\|_2^2 - \frac{N}{2}\|Q\|_2^2 + \frac{N+\gamma}{2p} \int \left( |\cdot|^{-(N-\gamma)} * Q^p \right) Q^p dx = 0.
\]
Moreover, if we multiply \( Q \) with both sides and integrate on \( \mathbb{R}^N \), then
\[
\|\nabla Q\|_2^2 + \|Q\|_2^2 - \int \left( |\cdot|^{-(N-\gamma)} * Q^p \right) Q^p dx = 0.
\]
Hence (A.4) immediately follows from the two equations above. \( \square \)

As for the behavior of ground state at infinity, V.Moroz and J. Van Schaftingen [30] shows the decay rate at infinity as follows:

**Proposition A.3** ([30], exponential decay of ground state). *Let \( N \in \mathbb{N}_* \), \( \gamma \in (0, N) \) and \( p \in (1, \infty) \). Assume that \( \frac{N-2}{N+\gamma} < \frac{1}{p} < \frac{N}{N+\gamma} \). Let \( Q \in W^{1,2}(\mathbb{R}^N) \) be a nonnegative ground state of
\[-\Delta Q + Q = \left( |\cdot|^{-(N-\gamma)} * |Q|^p \right) |Q|^{p-2}Q, \quad \text{in } \mathbb{R}^N.
\]
If \( p > 2 \),
\[
\lim_{|x| \to \infty} Q(x)|x|^\frac{N-1}{2} e^{|x|} \in (0, \infty);
\]
If $p = 2,$

$$\lim_{|x| \to \infty} Q(x)|x|^{-\frac{N+1}{p}} \exp \int_{|x|/2}^{|x|} \sqrt{1 - \frac{\mu^{N-\gamma}}{s^{N-\gamma}}} \, ds \in (0, \infty).$$

Next, we study the decay of any order of derivative of ground state $Q,$ and we need the following simple auxiliary lemma. Using the formula $\partial_r = \frac{x}{|x|} \cdot \nabla$ and induction method, we obtain the following estimate:

**Lemma A.4.** If $f \in W^{k,1}(\mathbb{R}^N)$ and $g \in L^1(\mathbb{R}^N)$ are two radial functions, then $(f * g)(x)$ are also radial, and

$$\left| \partial_r^k ((f * g)(x)) \right| \lesssim \sum_{j=1}^k \frac{(|D_j f| * |g|)(x)}{|x|^{k-j}}, \quad \forall x \in \mathbb{R}^N. \quad (A.5)$$

**Lemma A.5.** As for $Q$ given above. If $p > 2,$ then for any multi-index $\alpha \in \mathbb{Z}^N_{\geq 0},$ the following estimates hold:

(i) $\left\| Q^{-1} \partial^\alpha Q \right\|_{L^\infty} < \infty,$

(ii) $\left\| Q^{-1} e^{\eta |x|} \partial^\alpha Y_\pm \right\|_{L^\infty} < \infty$ for some $0 < \eta \ll 1,$

(iii) $\left\| Q^{-1} e^{\eta |x|} \partial^\alpha \left( (\mathcal{L} - \lambda)^{-1} f \right) \right\|_{L^\infty} < +\infty$ for every $\lambda \in \mathbb{R} \setminus \sigma(\mathcal{L})$ and every $f \in \mathcal{S}(\mathbb{R}^N)$ such that $\left\| Q^{-1} e^{\eta |x|} \partial^\beta f \right\|_{L^\infty} < +\infty$ for some $0 < \eta < \Re \sqrt{1 + \lambda^2}$ and any $\beta \in \mathbb{Z}^N_{\geq 0}.$

**Proof.** As for (i), we do not know whether $Q$ is in Schwartz class from the published papers, hence we can not use the method in [6, Corollary 3.8] directly, which heavily relies on the a priori upper bound of $\partial^\alpha Q.$ Instead, we use the comparison theorem together with the fact that $Q$ is radially symmetric. Taking gradient onto both sides of (1.4), since $Q$ is radial, we obtain that

$$\left( -\Delta + 1 + \frac{N - 1}{r^2} - (p - 1) \left( |\cdot|^{-(N-\gamma) * Q^p} \right) Q^{p-2} \right) (\partial_r Q) = \partial_r \left[ |\cdot|^{-((N-\gamma) * Q^p)} \right] Q^{p-1}.$$

Note that $Q$ is a positive, radial decreasing function on $\mathbb{R}^N,$ by an estimate on $\partial_r \left[ |\cdot|^{-((N-\gamma) * Q^p)} \right]$ (see [24, Proposition 2.2]), $\cdot^{-((N-\gamma) * Q^p)}$ is strictly radially decreasing, which means that

$$\left( -\Delta + 1 + \frac{N - 1}{r^2} - (p - 1) \left( |\cdot|^{-(N-\gamma) * Q^p} \right) Q^{p-2} \right) (\partial_r Q) \geq 0.$$

Observing that

$$\left( -\Delta + 1 + \frac{N - 1}{r^2} - (p - 1) \left( |\cdot|^{-(N-\gamma) * Q^p} \right) Q^{p-2} \right) (r^{-N} e^{-r}) \leq 0, \quad \forall r \geq R_0 \gg 1,$$

by the classical elliptic comparison theorem, we find that

$$-(\partial_r Q)(r) \geq \frac{r^{-N} e^{-r}}{R^{-N} e^{-R}} (\partial_r Q)(R), \quad \forall r \geq R \geq R_0,$$

which implies

$$Q(R) = \int_R^\infty (\partial_r Q)(s) \, ds \geq \int_R^\infty \frac{s^{-N} e^{-s}}{R^{-N} e^{-R}} (\partial_r Q)(R) \, ds \gtrsim (\partial_r Q)(R), \quad \forall R \geq R_0,$$

and it concludes (i) for $|\alpha| = 1.$ As for $|\alpha| = 2,$ it immediately follows from the equation

$$- \partial_r^2 Q - \frac{N - 1}{r} \partial_r Q + Q - \left( |\cdot|^{-((N-\gamma) * Q^p)} \right) Q^{p-1} = 0. \quad (A.6)$$
and (A.6) further implies that the higher order derivatives of \( Q \) can be controlled by the lower ones. By this intuition, as for more general case, we assume that \( |\partial_r^k Q(r)| \lesssim Q(r) \) holds for \( l \leq k + 1 \), and we will prove the validity of the estimate for \( l = k + 2 \). In fact, taking \( \partial_r^k \) onto both sides of (A.6),

\[
- \partial_r^{k+2} Q - \partial_r^k \left( \frac{N-1}{r} \partial_r Q \right) + \partial_r^k Q - \partial_r^k \left[ \left( | \cdot |^{-(N-\gamma)} * Q^p \right) Q^{p-1} \right] = 0, \tag{A.7}
\]

which, by induction assumption and (A.5), implies that

\[
\left| \partial_r^k \left( \frac{1}{r} \partial_r Q \right) \right| \lesssim \sum_{j=0}^{k} r^{-(k-j+1)} |\partial_r^{j+1} Q| \lesssim Q(r), \quad \forall \ r \geq 1
\]

and

\[
\left| \partial_r^k \left[ \left( | \cdot |^{-(N-\gamma)} * Q^p \right) Q^{p-1} \right] \right| = \left| \sum_{j=0}^{k} c_j \partial_r^j \left( | \cdot |^{-(N-\gamma)} * Q^p \right) \partial_r^{k-j} \left( Q^{p-1} \right) \right|
\]

\[
\lesssim \sum_{j=0}^{k} \sum_{i=1}^{j} \left| \frac{1}{|x|^{j-i}} \partial_r^{k-j} \left( Q^{p-1} \right) \right| \lesssim Q(r), \quad \forall \ r \geq 1,
\]

hence \( |\partial_r^{k+2} Q(r)| < Q(r), \forall \ r \geq 1 \). Then together with the fact that \( Q \in C^\infty \) (see [30, Theorem 3]) and \( Q \) is strictly positive, we obtain that \( |\partial_r^{k+2} Q| \lesssim Q \) on \( \mathbb{R}^N \). Thus we have completed the proof of (i).

As for (ii) and (iii), the proof is almost the same as [6, Corollary 3.8]. \( \square \)

**Remark A.6.** As for the decay of any order derivative of the ground state \( Q \) for \( p = 2 \), after repeating the same argument above, we can also get the desired conclusion, i.e. \( \| Q^{-1} \partial^\alpha Q \|_{L^\infty} < \infty, \forall \alpha \in \mathbb{Z}_{\geq 0}^N \). However, since \( \left| (1+|x|^{-\gamma}) Q^2 \right| \) only decays with rate \( |x|^{-\gamma} \) at infinity, we cannot follow the argument in [6] to extend (ii) and (iii) to the case for \( p = 2 \) temporarily.

**Appendix B. The estimates on nonlinear term \( R(h) \)**

**Lemma B.1.** If \( p > 2 \), the function

\[
J(z) = |1 + z|^p - \left( 1 + \frac{p}{2} z + \frac{p}{2} \bar{z} \right), \quad z \in \mathbb{C}
\]

satisfies the following properties:

(i) \( J(z) \in C^1(\mathbb{C}) \);
(ii) \( J_z(0) = J_{\bar{z}}(0) = 0 \);
(iii) \( |J_z(z_1) - J_z(z_2)| \lesssim (|z_1| + |z_2| + |z_1|^{p-1} + |z_2|^{p-1}) |z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{C} \);
(iv) \( |J_z(z_1) - J_z(z_2)|, |J_{\bar{z}}(z_1) - J_{\bar{z}}(z_2)| \lesssim (1 + |z_1|^{p-2} + |z_2|^{p-2}) |z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{C} \).

Similarly, the function

\[
K(z) = |1 + z|^{p-2}(1 + z) - \left( 1 + \frac{p}{2} z + \frac{p}{2} \bar{z} \right), \quad z \in \mathbb{C}
\]

satisfies the following properties:

(i) \( K(z) \in C^1(\mathbb{C}) \);
(ii) \( K_z(0) = K_{\bar{z}}(0) = 0 \);

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$$(iii) \ |K(z_1) - K(z_2)| \lesssim \begin{cases} \left( |z_1| + |z_2| + |z_1|^{p-2} + |z_2|^{p-2} \right) |z_1 - z_2|, & \text{if } p \geq 3, \\
\left( |z_1|^{p-2} + |z_2|^{p-2} \right) |z_1 - z_2|, & \text{if } p \in (2, 3), \\
\end{cases}$$

$$(iv) \ |K_\gamma(z_1) - K_\gamma(z_2)|, |K_{\beta'}(z_1) - K_{\beta'}(z_2)| \lesssim \begin{cases} \left( 1 + |z_1|^{p-3} + |z_2|^{p-3} \right) |z_1 - z_2|, & \text{if } p \geq 3, \\
|z_1 - z_2|^{p-2}, & \text{if } p \in (2, 3). \\
\end{cases}$$

**Proof.** The results are easily to be checked with the help of

$$J(z_1) - J(z_2) = \int_0^1 J_z(sz_1 + (1-s)z_2) (z_1 - z_2) + J_z(sz_1 + (1-s)z_2) \frac{(z_1 - z_2)}{dz} ds.$$ 

We omit the details here and just emphasize that when we consider the estimates on $K(z)$ and its derivative for $p \in (2, 3)$, the mean value theorem is invalid and we use an important fact that the function $z \mapsto |z|^a$ is Hölder continuous with order $\alpha$ for $\alpha \in (0, 1)$. \hfill \Box

**Lemma B.2.** For any $u_i \in S(I, \langle \nabla \rangle L^2)$, $1 \leq i \leq 5$, we have

$$\| \left( | \cdot |^{-(N-\gamma)} * (|u_1|^{p-2}u_2u_3) \right) |u_4|^{p-2}|u_5| \|_{S'(L^2)} \lesssim \min_{j=2,3,5} \left\{ \left\| u_1 \right\|_{S(H^{\infty})}^{p-2} \left\| u_4 \right\|_{S(L^2)} \left\| u_j \right\|_{S(L^2)} \prod_{i,k \in \{1,4,j\}} \left\| u_i \right\|_{S(H^{\infty})} \left\| u_k \right\|_{S(H^{\infty})} \right\}$$

(B.3)

and

$$\| (| \cdot |^{-(N-\gamma)} * (|u_1|^{p-2}u_2u_3)) |u_4|^{p-2}|u_5| \|_{S'(H^{-\infty})} \lesssim \| u_1 \|_{S(H^{\infty})}^{p-2} \| u_2 \|_{S(H^{\infty})} \| u_3 \|_{S(H^{\infty})} \| u_4 \|_{S(H^{\infty})}^{p-2} \| u_5 \|_{S(H^{\infty})},$$

(B.4)

**Proof.** As a matter of fact, by Hölder inequality and Hardy-Littlewood-Sobolev inequality and Definition 2.2, it is easily to check that

$$\| (| \cdot |^{-(N-\gamma)} * (|u_1|^{p-2}u_2u_3)) |u_4|^{p-2}|u_5| \|_{L^{q_1}L^{r_1}L^{q_2}L^{r_2}L^{q_3}L^{r_3}} \lesssim \min_{j=2,3,5} \left\{ \left\| u_1 \right\|_{L^{q_1}L^{r_1}L^{q_2}L^{r_2}L^{q_3}L^{r_3}} \left\| u_4 \right\|_{L^{q_1}L^{r_1}L^{q_2}L^{r_2}L^{q_3}L^{r_3}} \prod_{i,k \in \{1,4,j\}} \left\| u_i \right\|_{L^{q_1}L^{r_1}L^{q_2}L^{r_2}L^{q_3}L^{r_3}} \right\},$$

which is (B.3). Similarly, we can also get (B.4). \hfill \Box

With the help of Lemma B.1 and Lemma B.2, we are going to estimate the Strichartz norm of $\langle \nabla \rangle (R(f) - R(g))$.

**Lemma B.3.** Let $I$ be a bounded interval on $\mathbb{R}$. If $p \in (2, 3)$, then for any $f, g \in S(I, \langle \nabla \rangle L^2)$,

$$\| \langle \nabla \rangle (R(f) - R(g)) \|_{S'(I, L^2)} \lesssim \| (\langle \nabla \rangle (f - g)) \|_{S(I, L^2)} \left( \| f \|_{S(I, H^{\infty})}^{2p-2} + \| g \|_{S(I, H^{\infty})}^{2p-2} \right) \left( \| f \|_{S(I, H^{\infty})} + \| g \|_{S(I, H^{\infty})} \right) \left( \| f \|_{S(I, L^2)}^{p-2} + \| g \|_{S(I, L^2)}^{p-2} \right) \left( \| f \|_{S(I, L^2)} + \| g \|_{S(I, L^2)} \right) \left( \| f \|_{S(I, H^{\infty})}^p + \| g \|_{S(I, H^{\infty})}^p \right)$$

(B.5)
and
\[ \| R(f) - R(g) \|_{S(I, H^{s_c})} \leq \| f - g \|_{S(I, H^{s_c})} \left( \| f \|_{S(I, H^{s_c})}^{2p-2} + \| g \|_{S(I, H^{s_c})}^{2p-2} + \| Q \|_{S(I, H^{s_c})}^{2p-3} \left( \| f \|_{S(I, H^{s_c})} + \| g \|_{S(I, H^{s_c})} \right) \right). \] (B.6)

**Proof.** As a matter of fact, by (3.4), we obtain that
\[
R(f) - R(g) = \left( | \cdot |^{-(N-\gamma)} \right) \left( Q^p \left( J(Q^{-1}f) - J(Q^{-1}g) \right) \right) |Q + f|^{p-2} (Q + f) + \left( | \cdot |^{-(N-\gamma)} \right) (Q^p \left( Q^{-1}g \right)) \left( |Q + f|^{p-2} (Q + f) - |Q + g|^{p-2} (Q + g) \right) + \left( | \cdot |^{-(N-\gamma)} \right) \left( \frac{P_2}{2} Q^{p-1} (f - g) + \frac{P_2}{2} Q^{p-1} \left( \frac{f}{3} + \frac{g}{3} \right) \right) Q^{p-1} K(Q^{-1}f) + \left( | \cdot |^{-(N-\gamma)} \right) \left( \frac{P_2}{2} Q^{p-1} (f - g) + \frac{P_2}{2} Q^{p-1} \left( \frac{f}{3} + \frac{g}{3} \right) \right) \left( \frac{P_2}{2} Q^{p-2} f + \frac{P_2}{2} Q^{p-2} \left( \frac{f}{3} + \frac{g}{3} \right) \right),
\]
where \( J, K \) are two special functions defined in Lemma B.1, then we immediately get (B.6) and
\[
\| R(f) - R(g) \|_{S(I, L^2)} \leq \| f - g \|_{S(I, L^2)} \left( \| f \|_{S(I, H^{s_c})}^{2p-2} + \| g \|_{S(I, H^{s_c})}^{2p-2} + \| Q \|_{S(I, H^{s_c})}^{2p-3} \left( \| f \|_{S(I, H^{s_c})} + \| g \|_{S(I, H^{s_c})} \right) \right).
\]
When we take gradient onto \( R(f) - R(g) \), bearing in mind that any term with a gradient can only be bounded by its \( S(I, \langle \nabla \rangle L^2) \) norm, all the terms can be controlled by the first two terms on the right hand side of (B.5) except for the case when the gradient is left onto
\[ |Q + f|^{p-2} (Q + f) - |Q + g|^{p-2} (Q + g) \]
and \( Q^{p-1} \left( K(Q^{-1}f) - K(Q^{-1}g) \right) \), requiring us to add the third term on the right hand side of (B.5) to dominate their behaviors. \( \square \)

When \( p \geq 3 \), similarly we also obtain that
\[
\| \langle \nabla \rangle (R(f) - R(g)) \|_{S(I, L^2)} \leq \| \langle \nabla \rangle (f - g) \|_{S(I, L^2)} \left( \| f \|_{S(I, H^{s_c})}^{2p-2} + \| g \|_{S(I, H^{s_c})}^{2p-2} + \| Q \|_{S(I, H^{s_c})}^{2p-3} \left( \| f \|_{S(I, H^{s_c})} + \| g \|_{S(I, H^{s_c})} \right) \right) + \| f - g \|_{S(I, H^{s_c})} \left( \| \langle \nabla \rangle f \|_{S(I, L^2)} + \| \langle \nabla \rangle g \|_{S(I, L^2)} \right) \left( \| f \|_{S(I, H^{s_c})}^{2p-3} + \| g \|_{S(I, H^{s_c})}^{2p-3} + \| Q \|_{S(I, H^{s_c})}^{2p-3} \right).
\] (B.7)

**Remark B.5.** As widely used in Section 7, by (2.4), we can replace all the terms involving \( S(I, H^{s_c}) \) norm on the right hand side of (B.5),(B.6) and (B.7) with \( \| \langle \nabla \rangle \cdot \|_{S(I, L^2)} \).
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