Is mathematics able to give insight into current questions in finance, economics and politics?

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Abstract

Democrats in the US say that taxes can be used to “grease the wheels” of the economy and create wealth enough to recover taxes and thereby increase employment; the Republicans say that taxation discourages investment and so increases unemployment. These arguments cannot both be correct, but both arguments seem meritorious. Faced with this paradox, one might hope that a rigorous mathematical approach might help determine which is the truth.

We attempt to do this, and we reach a firm conclusion with a reasonable model which, as it turns out, favors the democratic or socialistic side by showing that if money is lent to a profitable company, there is always a way to provide enough additional profit to pay the cost of the loan and still provide the profits obtainable in the absence of the loan, at least on average. More precisely, we show that for every profitable company the additional funds can be used to increase the mean discounted profits and these additional profits can be used to pay for the expected discounted cost of the loans to the company.

This would appear to settle the issue, at least for a mathematician, that truth lies on the socialist side. However, there is a subtlety: if the company is free to take its profits in its own way so as to maximize its time discounted total dividends, then the statement becomes false in the sense that there are some profitable companies where, if the company can set its own rule for taking dividends, the additional profit may not be enough to pay the expected discounted loan cost and provide the old profits too. Thus interference in the companies internal decision making is required to obtain the advantage promised by the democratic claim.

This weakens the conclusion of the democrats if the greater society is not in favor of taking the freedom away from companies to use their own plan for profit taking. Perhaps the only conclusion is that mathematics is unable to give insight unless the rules for interference in the internal affairs of the companies is made clearer. On the other hand, this in itself is insightful. Moreover, society might feel that it is correct for a company’s profit taking to be subjected to government interference if it takes a government loan.
The strategy needed to obtain sufficient wealth to pay for the mean discounted loans necessarily involves government control of how and when the company takes its profits. The answer to the question of the title of this paper then comes down to deciding whether the society is willing to give up a little freedom of the company to set its profit-taking the way it wants to, and accept some government interference. This decision is not included in our mathematical model, and it would appear not to be easily included in any mathematical model. This decision seems to belong to social or political science rather than to mathematics.

Statement of results We show, rigorously, within a reasonable mathematical model, that the claim that government providing of additional funds to a company produces enough additional profit to cover the cost of providing the money and also increases the profits to the firm, at least in an average sense. That is, we prove rigorously that the increased profits are always sufficient to cover the cost of the loan, for any profitable company, though sometimes it is close. This surprised one of us, who is a registered Republican, and who is convinced that, since the model, though simple, is quite reasonable, one needs to look more closely to see where the flaw lies. He thinks there is a flaw and that it is that the company must follow closely to the optimum policy in order to increase its profits enough to cover the cost of the loan. A reasonable alternate policy (which gives higher profits to the firm) fails to achieve cost-coverage in many cases; would the government be wise enough to grease the right companies? Since it has no “self-interest”, maybe not. The other one of us, a registered Democrat, attributes the conclusion to a subtlety rather a flaw, a semantic difference - the company has an obligation to follow the socially optimal policy (especially since it has taken the government money).

The model We follow [1] and suppose that a company can be characterized by 4 parameters, \((x, \mu, \sigma, r)\), the present worth, \(x\), the profit rate, \(\mu\), the riskiness, \(\sigma\), and the prevailing interest rate, \(r\). It is presumed that the objective of the firm is to maximize discounted total profit, and also that the company chooses its policy optimally.

Radner and Shepp [1] found the company’s optimal policy and showed among other results, that the company that follows the optimal policy goes bankrupt in a finite time w.p. 1. At first this was surprising but later it became almost confirmational of the Dutta-Radner model, along with other consequences of the model, in particular that companies pay dividends at a certain threshold of their net worth.

In the new model, the government gives the company a line of credit, allowing it to borrow at a certain maximum rate, until bankruptcy. One might hope that there are some companies (four-tuples) for which the expected additional profit from having the line of credit is greater than the expected cost of the loan until bankruptcy. Indeed, we prove that for every company it holds that the increased profit covers the loan, provided the company takes profits (dividends) in an optimal way to maximize its presumed objective. This is the main
result of the paper.

Dutta and Radner \[2\] posed the following problem and Radner and Shepp \[1\] solved it.\[1\] The new model will be a slight variant of \[2\], which we now review for ease of the reader; \[2\] assume the model

\[
dX(t) = \mu dt + \sigma dW(t) - dZ(t), \quad t \geq 0; \quad X(0) = x
\]

where \(X(t)\) is the fortune of the company at time \(t\) and \(Z(t)\) is the total profit (dividends) (which subtracts from the fortune) taken up to time \(t\), so \(Z(t) \uparrow\) (increases) and \(W\) is Brownian motion, representing uncertainty and risk. We will later relax the assumption that \(\frac{dZ}{dt} \geq 0\) in order to discuss greasing the wheels.

Dutta and Radner assumed the company chooses \(Z\) optimally to maximize the expected value of time-discounted profit without knowing the future of \(W\), i.e., the market conditions; they asked to find the company’s true worth, defined by:

\[
V(x) = V(x; \mu, \sigma, r) = \sup_{\substack{Z: dZ/dt \geq 0 \\tau_0 \leq \infty}} \mathbb{E}_x \left[ \int_0^{\tau_0} e^{-rt} dZ(t) \right],
\]

where \(r > 0\) is the discounting parameter, and \(\tau_0 \leq \infty\) is the time that the company goes bankrupt. Note that \(V(x) \geq x\) since one strategy for taking profit is to take the fortune immediately (“take the money and run”); bankruptcy occurs at time \(t = 0^+\). We will see that the company which follows the provably optimal policy does go bankrupt in a finite time, w.p. 1. We were at first surprised by this, but now it seems almost a (qualitative) confirmation of the model because companies do indeed go bankrupt. They try to maximize profits and so they operate “on the edge of bankruptcy”.

Here is the proof of the Dutta-Radner-Shepp theorem: note that \(X\) depends upon \(Z\) and \(Z\) also depends upon \(X\), which is somewhat paradoxical and seems to require an existence proof that there is a pair \(X, Z\) with this relationship, for every possible strategy, \(Z\), of profit taking. The standard way around this difficulty is to take the supremum over all \(\Omega\)-space realizations of the pair of processes, \(X, Z\).

The convex programming approach is to choose a function \(\bar{V}(x)\), for which if we define the process, \(Y(t)\), on an \(\Omega\) space where \(X, Z\) are both defined, by (with \(X = XZ\)),

\[
Y(t) = \bar{V}(X(t)) e^{-rt} + \int_0^t e^{-ru} dZ(u), \quad t \geq 0,
\]

\[1\] The publication of \[2\] was delayed.
then \( Y \) should be a supermartingale for any choice of \( X, Z \), because the second term represents the discounted profit already taken, and the first term the remaining profit in the firm’s future, also discounted. The duality argument of linear programming now shows that if \( Y \) is always a supermartingale then \( \bar{V} \) is an upper bound on \( V(x) \), i.e., \( V(x) \leq \bar{V}(x) \), because:

\[
\mathbb{E}_x \left[ \int_0^{\tau_0} e^{-rt} dZ(t) \right] = \mathbb{E}_x [Y(\tau_0)] \leq \mathbb{E}_x [Y(0)] = \bar{V}(x),
\]

and since this holds for every \( Z \), we get that \( V(x) \leq \bar{V}(x) \).

Ito calculus says that the condition that \( Y \) is a supermartingale is that the drift of \( Y \) should always be negative, i.e.,

\[
\mathbb{E} [dY \mid \mathcal{F}_t] = \left( -r\bar{V}(x) + \mu \bar{V}'(x) + \frac{\sigma^2}{2} \bar{V}''(x) + \frac{dZ}{dt} (1 - \bar{V}'(x)) \right) e^{-rt} dt \leq 0.
\]

One guesses that \( \frac{dZ}{dt} = 0 \) or \( \infty \) according as \( x = X(t) < a \) or \( > a \) for some \( a \). The supremum is attained by the \( \bar{V}(x) \) given by

\[
\bar{V}(x) = \begin{cases} 
  e^{\gamma_+ x} - e^{\gamma_- x} & 0 \leq x \leq a \\
  \frac{e^{\gamma_+ a} - e^{\gamma_- a}}{\gamma_+ e^{\gamma_+ a} - \gamma_- e^{\gamma_- a}} & a \leq x < \infty 
\end{cases}
\]

Since \( \bar{V}''(x) \equiv 0 \) for \( x > a \), the smooth fit heuristic says that \( \bar{V}''(a^-) = 0 \) which determines the profit-taking threshold, \( a = a(\mu, \sigma, r) \) as

\[
a = \frac{\log \left( \frac{\gamma_+}{\gamma_-} \right)^2}{\gamma_+ - \gamma_-}
\]

where the roots of the indicial equation are \( \gamma_{\pm} \), are

\[
\gamma_{\pm} = \frac{-\mu \pm \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}
\]
Moreover, since $\bar{V}(x)$ is achieved by an allowable strategy, namely take profit at infinite rate whenever $X(t) \geq a$ and otherwise forego taking profit, we have $\bar{V}(x) \leq V(x)$. So then

$$\bar{V}(x) \leq V(x) \leq \bar{V}(x)$$

Thus the formula for the mean discounted total profit of the firm and therefore its true worth is

$$V(x, \mu, \sigma, r) = \begin{cases} \frac{e^{\gamma^+ a} - e^{\gamma^- x}}{\gamma^+ e^{\gamma^+ a} - \gamma^- e^{\gamma^- a}}, & 0 \leq x \leq a \\ V(a) + x - a; & x \geq a \end{cases}$$

where $a = a(\mu, \sigma, r) = \frac{\log(\frac{\gamma^-}{\gamma^+})^2}{\gamma^+ - \gamma^-}$

A graph of $V(x)$ for a particular choice of the parameters, $\mu, \sigma, r$, is given in Figure 1. We point out for later use that $V(a) = \frac{\mu}{r}$, because from the above formula, dividing top and bottom by $e^{\gamma^- a}$, and using the value

$$a = \frac{\log(\frac{\gamma^-}{\gamma^+})^2}{\gamma^+ - \gamma^-}$$

we verify immediately that

$$V(a) = \frac{e^{\gamma^+ a} - e^{\gamma^- a}}{\gamma^+ e^{\gamma^+ a} - \gamma^- e^{\gamma^- a}} = \frac{\gamma^- + \gamma^-}{\gamma^+ \gamma^-} = \frac{\mu}{r}$$

We should point out that if $\mu \leq 0$, then $V(x, \mu, \sigma, r) \equiv x$, i.e. it’s optimal to take the money and run. However, if $\mu > 0$, the company is profitable, and then $a(\mu, \sigma, r) > 0$; it can be seen that the threshold $a$ goes to zero as $\mu \downarrow 0$ or as $\mu \uparrow \infty$. Note that the optimal solution is of the form that the company takes profit only when it is “rich”, i.e., when $X(t) \geq a$. This is proved to be optimal.

So this completely and explicitly rigorously solves the problem that Dutta and Radner posed; [1] did a bit more: they allowed for different pairs $\mu_i, \sigma_i, i = 1, \ldots, n$, and they found which pair to follow at any given value of $X(t) = x$. This question was also part of the Dutta-Radner problem; the solution gave rise to some surprising results, for example that if the company is nearly bankrupt then it should be very conservative and use the $(\mu_i, \sigma_i)$ pair with the smallest $\sigma_i$ which seems to be paradoxical to many economists, see the work of Sheth et al [3].
The new problem: greasing the wheels  For simplicity in discussing the present question we will limit the company to only one corporate direction, \( n = 1 \). We will also assume that \( \mu > 0 \), i.e., the company is profitable.

To use the model to study the effect of taxation, we consider the “greasing the wheels” argument of the Democrats. Suppose the government subsidizes the firm to put it in a better position. Instead of \( \mu \) suppose the government (or perhaps a private investor like Warren Buffett) increases the \( \mu \) to \( \mu^* \). This might be done by a direct subsidy or by infrastructure improvements. To keep it simple, we don’t consider taxes at all, which would raise irrelevant issues, but instead we assume a loan coming to the firm. Let us assume that the government chooses certain firms to subsidize - we will see that it is not a difficult choice by the government to find a company where the increased profits will pay for the loan, at least in an expected value sense; in fact, it turns out that it will work for any profitable company.\(^2\)

We should point out that it is possible that the firm will go bankrupt before even issuing profits, but the expected total discounted profit will cover the expected discounted cost of the loan for every profitable firm. This seems to imply that the Democrats are correct.

We assume the company is supplied any amount less than or equal to \((\mu^* - \mu)dt\) in each interval \( dt \), if it chooses to take this money, where \( \mu^* \) is a new parameter with \( \mu^* > \mu \). The firm then could choose an operating level, \( a^* \), at which to take profits. It may choose the Radner-Shepp optimal level \( a^* = a(\mu^*) \) which goes with \( \mu^* \) in place of \( \mu \), but if it chooses this “socially-undesirable” or “greedy” strategy then there are firms for which the expected additional profit does not cover the expected loan cost. However, it can do better, at least in a socialistic sense as we will show and so much better that it will always provide enough money to cover both the loan cost and the old profit.

To find a socially acceptable policy under borrowing, where the socially acceptable goal of recovering the loan and the original profit (on average) we consider another problem completely, the following variant of the Dutta-Radner problem: Suppose the firm’s worth at time \( t \) is again \( X(t) \), which satisfies the sde:

\[
\frac{dX(t)}{dt} = \mu dt + \sigma dW(t) - dZ(t), X(0) = x
\]

but now, we allow borrowing,

\[
\frac{dZ}{dt} \geq -(\mu^* - \mu)
\]

instead of \( \frac{dZ}{dt} \geq 0 \), as in the Dutta-Radner formulation. This means the firm can borrow at a limited rate, or, take negative dividends.

\(^2\)Solyndra was a bad choice, but any firm with \( \mu > 0 \), and any choice of the level of the loan will be shown to produce enough increased profits that there will be a way to levy taxes to cover (on average) the cost of the loan.
The problem is now to maximize the mean discounted profit and to find the new “social” value of the firm,

\[ \hat{V}(x, \mu, \sigma, r, \mu^*) = \sup_{\{ \frac{dZ}{dt} \geq -(\mu^* - \mu) \}} \mathbb{E}_x \left[ \int_0^{\tau_0} e^{-rt} dZ(t) \right] \]

where the sup is over all choices of \( Z \), nonanticipating, with derivative \( \frac{dZ}{dt} \) now allowed to be negative, but it must be at least as big as \( -(\mu^* - \mu) \). Note that negative increments of \( Z \) count negatively in computing the profit earned by the company. Nevertheless, the company will earn more profit than in the Radner case since it has more choices of strategy, \( Z \), that is, the Radner-Shepp value, \( V_{R-S} \), satisfies,

\[ V_{R-S}(x, \mu, \sigma, r) \leq \hat{V}(x, \mu, \sigma, r, \mu^*) \]

Note also that the value, \( \hat{V}(x, \mu, \sigma, r, \mu^*) \) can be written in the form, for the optimal choice of \( Z \),

\[ \hat{V}(x, \mu, \sigma, r, \mu^*) = \mathbb{E}_x \left[ \int_0^{\tau_0} e^{-rt} dZ_+(t) \right] - \mathbb{E}_x \left[ \int_0^{\tau_0} e^{-rt} dZ_-(t) \right] \]

where \( Z_+(t) \) is the total dividends issued up to time \( t \), which may be taxed, and \( Z_-(t) \) is the total loans issued up to time \( t \), where \( Z_\pm \) are part of the optimal strategy.

It is clear that \( \hat{V}(x) \geq V(x) \) for any choice of the parameters, \( x, \mu, \sigma, r, \mu^* > \mu \), because the criterion allows for a larger set of strategies, \( Z \), now \( \frac{dZ}{dt} \geq -(\mu^* - \mu) \), whereas in the first model we required \( \frac{dZ}{dt} \geq 0 \). Thus in a sense we have chosen the model with an em built-in advantage for the socialist side. On the other hand what is wrong with this assumption is that society regularly objects to outsize profits taken by executives of money losing companies, which is close to what is under discussion in our society and it is not clear that mathematics can shed light on public opinion of this issue. It might be stated that a company that wants the loan (under which it makes more money) might be willing to agree to conditions on how and when it takes its profits. It appears credible and appears to justify the investment of the money provided. We will find the optimal strategy under the loan: we will show that the company should always take the loan at the maximum rate, and it should again issue profits at a threshold which is always lower than the threshold for the company with profit parameter, \( \mu^* \). Note that the optimal threshold for \( \mu \) may be above or below the threshold for the larger profit parameter \( \mu^* \). It is only when the company chooses its threshold optimally to maximize expected (discounted) profit minus cost that the threshold decreases. This makes it appear that the company is simply issuing part of the loan as profit, which is true in a sense.
Calculation of the optimal expected profit minus loan cost  Note that we will use

\[ Z(t) = Z_+(t) - Z_-(t) \]

where \( Z_+(t), t \geq 0 \) is a nondecreasing, nonanticipating control, representing the total profit (dividend) taken up to time \( t \geq 0 \), and \( Z_-(t), t \geq 0 \) is a nondecreasing, nonanticipating control representing the total money borrowed by the firm up to time \( t \geq 0 \). There is an additional restriction on \( Z_- \), that

\[ \frac{dZ_-(t)}{dt} \leq c = \mu^* - \mu. \]

If the firm employs the two controls, \( Z_\pm \), then the worth of the company until time \( \tau^X_0 \), when bankruptcy occurs, follows

\[ dX(t) = \mu dt + \sigma dW(t) + dZ_-(t) - dZ_+(t), 0 \leq t \leq \tau^X_0 \]

since \( dZ_+(t) \) decreases \( X(t) \), while \( dZ_-(t) \) increases \( X(t) \). Note that both \( Z_\pm \) are increasing processes.

The optimization criterion is that of Dutta-Radner,

\[ \hat{V}(x) = \sup_{Z_+, Z_-} E_x \left[ \int_0^{\tau^X_0} e^{-rt} [dZ_+(t) - dZ_-(t)] \right] \]

We note that

\[ \hat{V}(x, \mu, \mu^*, \sigma, r) = V_+(x, \mu, \mu^*, \sigma, r) - C(x, \mu, \mu^*, \sigma, r) \]

where \( Z_+ \) are the positive dividend payouts to the firm, and \( C \) is the cost of the loans (discounted), i.e.,

\[ V_+(x) = E \left[ \int_0^{\tau^0_0} e^{-rt} dZ_+(t) \right], \quad C(x) = E \left[ \int_0^{\tau^0_0} e^{-rt} dZ_-(t) \right] \]

It turns out that the optimal dividend payout control, \( dZ_+ \) is not a differentiable process, but increases in local time at a different threshold, \( \hat{A} \), while our additional restriction on the borrowing process forces \( dZ_-(t) \) to be differentiable, and under optimal control, we show that
the firm borrows at the maximum possible rate, $c = \mu^* - \mu$, all the time. Taking dividends in local time appears complicated but the thesis\cite{4} shows that if dividends are required to be lump-sum payments rather than “dribs and drabs”, nearly the same total profit values are obtained. We see that

$$V_{R-S}(x, \mu, \sigma, r) + C(x) \leq V_+(x, \mu, \sigma, r, \mu^*)$$

because this inequality is just a rearrangement of the terms in $V \leq \hat{V}$, which we have already seen. It follows that $V_{R-S}(x) \leq V_+(x)$ since $C(x) \geq 0$.

**Remark** The Dutta-Radner choice of additive rather than multiplicative Brownian motion is an old and well-discussed issue, see \cite{1}, p. 1390, where it is shown that inside a Samuelson-Black-Scholes multiplicative model the company either becomes infinitely rich ($\mu > r$) or $V(x) \equiv x$ (take-the-money-and-run), i.e., the multiplicative model would not represent observed real-world behavior.

We state the new form of the Dutta-Radner-Shepp theorem:

The solution to the new socialistic optimization problem, $\hat{V}(x, \mu, \sigma, r, \mu^*)$, when borrowing at rate $c = \mu^* - \mu$ is allowed, is the solution to the new differential equation,

$$-r\hat{V}(x) + \mu^*\hat{V}'(x) + \frac{\sigma^2}{2}\hat{V}''(x) = \mu^* - \mu, \quad x \in [0, \hat{a}]$$

and for $x > a^*$, $V''(x) \equiv 1$. Here, $\hat{a}$ is again determined by the smooth fit condition,

$$\hat{V}(0) = 0, \quad \hat{V}'(a^*) = 1, \quad \hat{V}''(\hat{a}) = 0$$

The proof is similar to that of the Radner-Shepp theorem; one merely rearranges the terms to exploit the new inequality that

$$\frac{dZ}{dt} \geq -(\mu^* - \mu) \quad \text{instead of} \quad \frac{dZ}{dt} \geq 0$$

so that $e^{-rt}dt$ times
\[-r \hat{V}(x) + \mu^* \hat{V}'(x) + \frac{\sigma^2}{2} \hat{V}''(x) + (1 - \hat{V}^*(x)) \left( \frac{dZ}{dt} + \mu^* - \mu \right) \leq \mu^* - \mu \]

and this gives rise to \( \mu^* \) on the left instead of \( \mu \), and the right side being \( \mu^* - \mu \) instead of zero. Since \( \frac{dZ}{dt} \) may be any number \( \geq -(\mu^* - \mu) \) the second term is nonpositive if and only if \( \hat{V}'(x) \geq 1 \). Further, in order to get the least upper bound we guess that we should make the first term zero and we get an ode. The solution to the ode is explicit in terms of exponentials involving now \( e^{\gamma \pm x} \), which now go with \( \mu^* \), i.e., just as before,

\[ \gamma^*_\pm = -\mu^* \pm \sqrt{(\mu^*)^2 + 2r\sigma^2} \]

quite analogous to \( \gamma_\pm \) in the simpler Dutta-Radner problem, as well as a particular solution to the now non-homogeneous ode.

The solution for \( \hat{V}(x) = \hat{V}(x, \mu, \sigma, r, \mu^*) \) is

\[ \hat{V}(x) = A_+(e^{\gamma^*_+ x} - 1) + A_-(e^{\gamma^*_- x} - 1), \quad x \in [0, \hat{a}] \]

where we need \( A_+ + A_- = \frac{\mu^* - \mu}{r} \) to include the particular solution. The boundary conditions determine \( A_\pm \) and \( \hat{a} \),

\[ A_+ = \frac{-\gamma^*_- e^{-\gamma^*_+ \hat{a}}}{\gamma^*_+(\gamma^*_+ - \gamma^*_-)}, \quad A_- = \frac{\gamma^*_+ e^{-\gamma^*_- \hat{a}}}{\gamma^*_-(\gamma^*_+ - \gamma^*_-)} \]

The form of the optimal profit taking involves a barrier-threshold, \( \hat{a} \), similar to the barrier \( a = a(\mu) \) in the Radner-Shen case for \( \mu \), or for \( a^* = a(\mu^*) \). Note that \( \hat{a} \) is now only \textit{implicitly} defined uniquely by smooth-fit, since \( A_\pm \) and \( \hat{a} \), are determined from the ode and the smooth-fit conditions,

\[ \hat{V}'(\hat{a}) = 1, \quad \hat{V}''(\hat{a}) = 0 \]

Finally, \( \hat{a} \) is determined as the root of the equation,

\[ \frac{(-\gamma^*_-) e^{-\gamma^*_+ \hat{a}}}{(\gamma^*_+)(\gamma^*_+ - \gamma^*_-)} - \frac{(\gamma^*_+) e^{-\gamma^*_- \hat{a}}}{(-\gamma^*_-)(\gamma^*_+ - \gamma^*_-)} - \frac{\mu^* - \mu}{r} = 0 \]
Call the left side \( f(\hat{a}) \), and verify that \( f \) is decreasing with a negative second derivative in \( \hat{a} \), and \( f(0) = \frac{\mu}{r} > 0 \), if \( \mu > 0 \), and \( f(\infty) = -\infty \), so that there is one and only one root, \( \hat{a} \). We will next prove that it is always true that the new threshold where dividends are paid is less than the old Radner-Shepp threshold, \( a \), i.e., \( \hat{a} \leq a \) always holds. To prove this, we showed above that \( V(a) = \frac{\mu}{r} \) and now we note also that \( \hat{V}(\hat{a}) = \frac{\mu^* - \mu}{r} \) because

\[
\hat{V}(\hat{a}) = A_+ e^{\gamma_+ \hat{a}} + A_- e^{\gamma_- \hat{a}} - \frac{\mu^* - \mu}{r} = \frac{-\gamma_-}{\gamma_+} + \frac{\gamma_+}{\gamma_-} - \frac{\mu^* - \mu}{r} = \frac{\mu}{r}
\]

But for any \( x \), \( \hat{V}(x) \geq V(x) \), because the space of strategies for \( \hat{V} \) includes all non-anticipating \( Z \)'s with

\[
\frac{dZ}{dt} \geq -(\mu^* - \mu)
\]

while, in the Radner-Shepp problem, \( \frac{dZ}{dt} \geq 0 \), and \( \mu^* \geq \mu \).

But for \( x \) sufficiently large (\( x > \max(a, \hat{a}) \)), we have both

\[
V(x) = V(a) + x - a \quad \text{and} \quad V(x) \leq \hat{V}(x) = \hat{V}(\hat{a}) + x - \hat{a}
\]

holding. Cancelling the \( x \)'s and the common values of \( V(a) \) and \( \hat{V}(\hat{a}) \) proves that \( \hat{a} \leq a \). qed.

This gives the formula for \( \hat{V}(x, \mu, \sigma, r, \mu^*) \), the increased value of the firm which has access to the loan \( (\mu^* - \mu)dt \) in each time interval of solvency. See Figure 2 for a graph of \( \hat{V}(x, \mu, \sigma, r, \mu^*) \) for the same values of the parameters, \( \mu, \sigma, r \), used to in Figure 1 to graph \( V(x, \mu, \sigma, r) \), where \( \mu^* \) was not involved. they are quite similar.

To calculate the cost of the loan for the optimal “greedy” strategy for the firm, as well as for the “socially acceptable strategy”. we need to calculate for a fixed threshold, say \( a \), at which there is a reflecting barrier, the time discounted mean cost of the loan until bankruptcy, \( C \), where

\[
C = C(x, \mu, \mu^*, a, \sigma, r) = (\mu^* - \mu)E_x \left[ \int_0^{\tau_0(a)} e^{-rt} dt \right] = (\mu^* - \mu) \frac{1 - g(x)}{r}
\]
where $\tau_0(a)$ is the first time to hit zero if there is a reflecting barrier at $a$, where $a$ can be $a(\mu)$, $a(\mu^*)$, or $a(\hat{a})$, and $g(x) = e^{x}e^{-r\tau_0(a)}$.

It is easy to see that $g$ is determined by the requirements that

$$g'(a) = 0, \quad g(0) = 1,$$

and that $g(X(t))e^{-rt}$ is a martingale

so the answer is

$$g(x) = \frac{e^{\gamma_+(x-a)}}{e^{\gamma_+(a)}} = \frac{e^{\gamma_-(x-a)}}{e^{\gamma_-(a)}}$$

The program in the appendix of the paper uses this formula to calculate $C$.

**Conclusions** The Democratic policy of greasing the wheels is sometimes presented as “risk-sharing”. It is tempting to believe that risk-sharing can indeed bring benefit to all, as does the promise of socialism or communism. Indeed, if it were true that government loans to some perhaps well-chosen firms would increase the rate of income to the point that the increased profit would pay for itself in expectation then, as the Democrats have been arguing, this extra money could be used to supplement taxes and the extra capital above the cost of the loans could be used in many ways, including creating jobs.

Unfortunately, if the company uses the Radner-Shepp policy with $\mu^*$ in place of $\mu$ this will lead to choosing a dividend threshold which is not $\hat{a}$, the one for the socially acceptable policy of maximizing $\hat{V}(x)$. The profit obtained will be $V(x, \mu, \sigma, r)$ with $\mu$ replaced by $\mu^*$. It is true that this value is greater than $V(x, \mu, \sigma, r)$, but it is not clear that it is larger than $V(x, \mu, \sigma, r) + C(x, \mu, \mu^*, a^*, \sigma, r)$, where this $C = C(x, \mu^*, \mu, \sigma, r)$ is the cost until bankruptcy of financing this greedy policy. A program, given in the Appendix shows that in a large (26%) percent of random choices for $\mu, \mu^*,$ and $x$, with $\sigma = 1, r = .1$, the increased profit is not enough to cover its cost, but verifies what we have proven, that $\hat{V}(x, \mu^*, \mu, \sigma, r) > V(x, \mu, \sigma, r)$ for all values of the parameters, except for a few cases where the two values are very close. This is due to numerical roundoff. We chose the threshold, $\hat{a}$, inherent in $\hat{V}$ to maximize the profit minus cost rather than just the profit, but a greedy company might simply go with the optimal strategy since it has little self-interest in minimizing the cost to the government. The government might then be “forced” to step in to ensure that the “correct” dividend policy is utilized, or it might try to recognize which companies would be in the “good class” where the increased profits will pay for the loan. In fact, we ourselves do not know how to tell which companies are good in this sense even if we assume that the toy model is an exact description.
Acknowledgement  We are grateful to Cun-Hui Zhang for useful discussions and ideas, and to Alain Kornhauser, who, upon hearing the earlier erroneous conclusion, made it clear that he suspected that there was an error somewhere and properly but politely gave the speaker a hard time. We are also grateful to J. Michael Harrison, for pointing out that an earlier version of the paper, in which we arrived at the opposite conclusion double-counted the cost of the loan.

References

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[3] “Risk-taking, financial distress, and innovation”, (A.Sheth, L.Shepp, O.Palmon), Academy of Business Journal: Special Issue on the Global Debt Crisis, 2:5-18, 2011.

[4] PhD thesis, Rutgers.
Program for numerical calculations.

#include <math.h>
#include <stdio.h>

main()
{
    /* Note that mu1 = \mu^* in the paper, gp, gm = \gamma_\pm, gp1, gm1 = \gamma^*_\pm, C is the cost, Ap, Am = A_\pm
    The first 5 lines below "for" compute a, the next 3 compute a2 = the threshold
    for the Radner-Shepp threshold with \mu replaced by \mu^*. Starting with the
    line b1 = -eps, and continuing through the while and for loops calculate al
    which is a^* in the paper. Then V, V1, V2 etc. are computed, where V is
    Radner-Shepp value for \mu, and V2 for \mu^*. V1 is the new value, called
    V^* in the paper. Similarly for g and C */

    int i, j, k, N = 1000;
    double mu, mu1, sig, sig2, r, gm, gp, gm1, gp1, C, C1, C2, eps = .0001;
    double x, x1, b, b0, b1, a, a1, a2, V, V1, V2, g, g1, g2, f, F0, F1;
    double big = exp(31.*log(2.)), Ap, Am;
    sig = 1; r = .1;

    for(i=0;i<N;i++) {
        mu = rand() / big; mu1 = mu + rand() / big; x = rand() / big;
        sig2 = sig * sig;
        gp = (-mu + sqrt(mu*mu + 2*r*sig2)) / sig2;
        gm = (-mu - sqrt(mu*mu + 2*r*sig2)) / sig2;
        a = (log((gm/gp)*(gm/gp))) / (gp-gm);
        gp1 = (-mu1 + sqrt(mu1*mu1+2*r*sig2))/sig2;
        gm1 = (-mu1 - sqrt(mu1*mu1+2*r*sig2))/sig2;
        a2 = (log((gm1/gp1)*(gm1/gp1)))/(gp1-gm1);
        b1 = -eps;
        f = F0 = mu/r;
        while(f > 0) {
            b1 += eps;
            F0 = f;
            f = ((-gm1*exp(-gp1*b1))/gp1+((gp1*exp(-gm1*b1))/gm1))/(gp1-gm1);
            f = f -((mu1-mu)/r);
            F1 = f;
            b0 = b1 - eps;
        }
        for(j=0;j<30;j++) {
            b = (b0+b1)/2;
            f = ((-gm1*exp(-gp1*b))/gp1+((gp1*exp(-gm1*b))/gm1))/(gp1-gm1);
            f = f -((mu1-mu)/r);
        }
    }
}
if(f >= 0) {
    b0 = b; F0 = f;
}
if(f < 0) {
    b1 = b; F1 = f;
}
a1 = b;
if(x < a) {
    V = (exp(x*gp)-exp(x*gm))/(gp*exp(a*gp)-gm*exp(a*gm));
    g = exp(gp*(x-a))/gp - exp(gm*(x-a))/gm;
    g = g/((exp(-gp*a)/gp) - (exp(-gm*a)/gm));
    C = (1-g)*(mu1-mu)/r;
}
if(x >= a) {
    V = (gp+gm)/(gp*gm) + (x-a);
    g = (1/gp) - (1/gm);
    g = g/((exp(-gp*a)/gp) - (exp(-gm*a)/gm));
    C = (1-g)*(mu1-mu)/r;
}
if(x < a2) {
    V2 = (exp(x*gp1)-exp(x*gm1))/(gp1*exp(a2*gp1)-gm1*exp(a2*gm1));
    g2 = exp(gp1*(x-a2))/gp1 - exp(gm1*(x-a2))/gm1;
    g2 = g2/((exp(-gp1*a2)/gp1) - (exp(-gm1*a2)/gm1));
    C2 = (1-g2)*(mu1-mu)/r;
}
if(x >= a2) {
    V2 = (gp1+gm1)/(gp1*gm1) + (x-a2);
    g2 = (1/gp1) - (1/gm1);
    g2 = g2/((exp(-gp1*a2)/gp1) - (exp(-gm1*a2)/gm1));
    C2 = (1-g2)*(mu1-mu)/r;
}
Ap = (-gm1*exp(-gp1*a1))/(gp1*(gp1-gm1));
Am = (gp1*exp(-gm1*a1))/(gm1*(gp1-gm1));
if(x < a1) {
    V1 = Ap*(exp(gp1*x) - 1) + Am*(exp(gm1*x) - 1);
    g1 = (exp(-gp1*(x-a1))/gp1) - (exp(-gm1*(x-a1))/gm1);
    g1 = g1/((exp(-gp1*a1)/gp1) - (exp(-gm1*a1)/gm1));
    C1 = (1-g1)*(mu1-mu)/r;
}
if(x >= a1) {
    V1 = (mu/r) + (x-a1);
}
printf("x,mu,mu1,a,a1,a2=%g,%g,%g,%g,%g\n",x,mu,mu1,a,a1,a2);
printf("x,mu,mu1,V,V1,V2=%g,%g,%g,%g,%g\n",x,mu,mu1,V,V1,V2);
printf("x,\mu,\mu_1,g_1,g_2=\%g,\%g,\%g,\%g,\%g
",x,\mu,\mu_1,g_1,g_2);
printf("x,\mu,\mu_1,C,C_1,C_2=\%g,\%g,\%g,\%g,\%g\n",x,\mu,\mu_1,C,C_1,C_2);
printf("A_p,A_m,V,V_2-C_2,V_1=\%g,\%g,\%g,\%g,\%g\n",A_p,A_m,V,V_2-C_2,V_1);
if(a_1 > a + .001) printf("Q1 a_1 \gt a = \%g,\%g\n",a_1,a);
if(V_2 - C_2 > V_1) printf("Q2 V_2 - C_2 > V_1 = \%g,\%g\n",V_2-C_2,V_1);
if(V_1 < V - .001) printf("Q3 V_1 \lt V = \%g,\%g\n",V_1,V);
if(V_2 - C_2 < V) printf("Q4 V_2 - C_2 < V\n");
}
Figure 1: Object by number of workers that found the object. The blue line represents the true number of objects, the quantity we’re ultimately trying to estimate.