A semi-local trace identity and the Riemann hypothesis for function fields

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Abstract The asymptotic trace formula of Connes is restated in a semi-local form, thus showing that the difficulties in proving it directly do not lie in the change of topology when transgressing from finitely many to infinitely many places.

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Introduction

The Riemann hypothesis for Hecke $L$-series “mit Größencharakteren” over function fields has been proved by Weil \[8\]. In \[1\], Bombieri gave a simplified proof that is based on the Riemann-Roch theorem for curves over finite fields. This Riemann-Roch theorem, however, can be rephrased in terms of adeles and proved by means of Fourier analysis. On the other hand, as a consequence of the explicit formulae, the Riemann hypothesis can be reformulated as the positivity of the Weil distribution on the ideles. Thus it should be possible to give a proof of the Riemann hypothesis via the Weil positivity that is based entirely on Fourier analysis on the adeles and ideles. In this spirit, A. Connes \[2\] gave an asymptotic trace identity that is equivalent to the Riemann hypothesis. He managed to give direct proves for analogous formulae in the local and the semi-local case, but the global formula can as yet only be proved as a consequence of the Riemann hypothesis. At this point it looks as if the difficulty lies in the change of topology when transgressing from finitely many to infinitely many places. In this note we restate the trace formula as a semi-local trace identity, in which the global situation makes no appearance at all. The semi-local trace identity thus becomes equivalent to the Riemann hypothesis for function fields.

1 Connes’ theorem

In this section we fix notations and recall Connes’ result. Let $k$ be a global field of positive characteristic $p$. Then $k$ is the function field of some curve defined over a finite field. Let $q = p^m$ be the number of elements of the field of constants in $k$. Let $V$ be the set of valuations or places of $k$. For each $v \in V$ let $k_v$ be the completion of $k$ at $v$ and let $\mathcal{O}_v$ be the ring of integers of the local field $k_v$, i.e. $\mathcal{O}_v$ consists of all $x \in k_v$ which satisfy $v(x) \geq 0$. For each $v \in V$ fix a uniformizer at $p$, i.e. an element $\pi_v$ of $\mathcal{O}_v$ such that $v(\pi_v) = 1$. Let $A$ be the adele ring of $k$, i.e. the subset of the infinite product $\prod_{v \in V} k_v$ consisting of all elements $(x_v)_v$ with $x_v \in \mathcal{O}_v$ for all but finitely many $v$. We say that $A$ is the restricted product of the $k_v$ and write this as $A = \prod_v k_v$. For any subset $S$ of $V$ let $A_S = \prod_{v \in S} k_v$ and $A^S = \prod_{v \notin S} k_v$, then $A = A_S \times A^S$. The ring $A$ is a locally compact ring and $k$ embeds diagonally as a discrete subring that is cocompact as additive group.
For $x \in k_v$, let $|x|_v$ be its modulus, i.e. the unique positive real number such that for any measurable subset $A$ of $k_v$ we have $\mu(xA) = |x|_v \mu(A)$, where $\mu$ is any additive Haar measure on $k_v$. It then turns out that $|x|_v = q_v^{-w(x)}$, where $q_v$ is the number of elements of the residue class field of $k_v$.

The group of ideles, i.e. the multiplicative group $A^\times$ of invertible elements of $A$ is the restricted product of the $k_v^\times$ with respect to their compact subgroups $O_v^\times$. In this way the ideles form a locally compact group whose topology differs from that inherited from the adeles. The group $k^\times$ embeds diagonally as a discrete subgroup of $A^\times$. Let the absolute value on $A^\times$ be defined as $|.| = \prod_v |x_v|_v$, which does make sense since almost all factors are one. Then this coincides with the modulus of $x$ for any given additive Haar measure on $A$. Let $A^1$ be the set of all $a \in A^\times$ such that $|a| = 1$, then $k^\times$ forms a cocompact subgroup of $A^1$.

The image of $|.| : A^\times \to \mathbb{R}$ equals $q_0^Z$, where $q_0$ is some positive power of $q$. Let $\pi_v$ be an element of $A^\times$ with $|\pi_v| = q_0^{-1}$.

For any subset $S$ of $V$ let $O_S = \prod_{v \in S} O_v$. For $S = V$ the ring $O_V$ is a compact subring of $A$ and $O_V^\times$ is a compact subgroup of $A^\times$. There is a finite set $E \subset A^1$ such that $A^1 = k^\times E O_V^\times$ and the intersection $k^\times \cap O_V^\times$ is the group of nonzero constants in $k$, i.e. $\mathbb{F}_q^\times$.

Let $S(A)$ be the Schwartz-Bruhat space of $A$, i.e. the space of all locally constant functions on $A$ with compact support. Any $f \in S(A)$ is a finite sum of functions of the form $f = \prod_v f_v$, where $f_v$ is locally constant and of compact support on $k_v$ and $f(x) = \prod_v f_v(x_v)$; further for all but finitely many $v$ the function $f_v$ then coincides with $1_{O_v}$, the characteristic function of $O_v \subset k_v$. Fix a nontrivial additive character $\psi$ on $A$ which is trivial on $k$. Note that then the lattice $k$ in $A$ becomes self-dual $\mathbb{F}_q$, i.e., for $x \in A$ we have

$$\psi(x \gamma) = 1 \quad \text{for every } \gamma \in k \iff x \in k.$$ 

Every additive character on $A$ decomposes into a product

$$\psi = \prod_v \psi_v,$$

where $\psi_v$ is a character of $k_v$. For $v \in V$ let $n(v)$ denote the order of $\psi_v$, i.e., $n(v)$ is the greatest integer $k$ such that $\psi_v$ is trivial on $\pi_v^{-k} O_v$. Then $n(v) = 0$.
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for all but finitely many $v$ (Cor. 1.IV in [3]). Let $F_v$ denote the $O_v$-module $\pi_v^{-n(v)}O_v$. Then $F_v = O_v$ for all but finitely many $v$.

For any set $S$ of places let $\psi_S = \prod_{v \in S} \psi_v$. Then $\psi_S$ is a nontrivial additive character on $A_S$. Fix a Haar measure $dx$ on $A_S$ by the condition that it is self-dual with respect to $\psi_S$. To explain this, let the Fourier transform on $S(A_S)$ be defined by

$$\hat{f}(x) = \int_A f(y)\psi(xy)dy.$$

Then the measure is normalized so that $\hat{f}(x) = f(-x)$. More explicitly this means that $dx = \prod_{v \in S} dx_v$ with $dx_v$ giving the set $O_v$ the volume $q_v^{-n(v)/2}$.

For any set $S$ of places with at least two elements define the ring

$$k_S = \{x \in k^\times : |x|_v \leq 1 \text{ for } v \notin S\}$$

and let $k_S^\times$ be the units of this ring, then

$$k_S^\times = \{x \in k^\times : |x|_v = 1 \text{ for } v \notin S\}.$$

Then $k_S^\times$ is a discrete subgroup of $A_S^\times$, contained in $A_S^1$ and such that the quotient $k_S^\times \backslash A_S^1$ is compact.

For the multiplicative Haar measures we adopt Weil’s normalization as follows: Suppose that $G$ is a locally compact group and $g \mapsto |g|$ a nontrivial proper continuous homomorphism to $\mathbb{R}_+^\times$. Then there is a unique Haar measure on $G$ such that

$$\text{vol}(\{g \in G \mid 1 \leq |g| \leq \Lambda\}) \sim \log \Lambda,$$

as $\Lambda \to \infty$. We will fix this Haar measure on the groups $k_v^\times$ for any place $v$ and for $k_S^\times \backslash A_S^\times$, where $S$ is any set of places with at least two elements.

We will denote these Haar measures by $d^x$. In this normalization we get $\text{vol}(O_v^\times) = \log q_v$, and $\text{vol}(k_S^\times \backslash A_S^1) = \log q_S$, where $q_S > 1$ is the generator of the group of absolute values $|x|$, where $x \in A_S^\times$. On $A_S^\times$ we install the measure given by

$$\int_{A_S^\times} \varphi(x) d^x x = \int_{k_S^\times \backslash A_S^\times} \sum_{\gamma \in k_S^\times} \varphi(\gamma x) d^x x.$$
Note that if \( f \in S(A) \) is a product, say, \( f = \prod_v f_v \), then \( \hat{f} = \prod_v \hat{f}_v \), where on the right hand side one takes the local Fourier transforms. Further for a place \( v \) and \( z \in k_v^\times \) let \( 1_{z\mathcal{O}_v} \) be the characteristic function of the set \( z\mathcal{O}_v \).

Then the local transform satisfies \( \hat{1_{z\mathcal{O}_v}} = |z|q_v^{-\frac{m(v)}{2}}1_{z^{-1}F_v} \).

Let \( S_0(A) = \{ f \in S(A) | f(0) = 0 = \hat{f}(0) \} \). For \( f \in S_0(A) \) let

\[
E(f)(x) = |x|^{1/2} \sum_{\gamma \in k^\times} f(x\gamma).
\]

Then \( E(f) \) is a function on \( A^\times \), invariant under \( k^\times \), and A. Connes proved in \( [2] \) that \( E(f) \) is an element of \( L^2(k^\times \backslash A^\times) \). Moreover, as a consequence of Theorem 1 of \( [2] \) it follows that the image of \( E \) forms a dense subspace of \( L^2(k^\times \backslash A^\times) \).

Let \( \Lambda > 0 \) be in the value group \( qZ^0 \), and let \( \tilde{Q}_{\Lambda,0} \) be the subspace of \( S_0(A) \) consisting of all functions \( f \) with \( f(x) = 0 = \hat{f}(x) \) for all \( x \) with \( |x| > \Lambda \). Let \( Q_{\Lambda,0} \) be the closure in \( L^2(k^\times \backslash A^\times) \) of \( E(\tilde{Q}_{\Lambda,0}) \). We denote the orthogonal projection onto the space \( Q_{\Lambda,0} \) by the same symbol.

Let \( h \) be a Schwartz-Bruhat function on \( k^\times \backslash A^\times \), i.e. the function \( h \) is locally constant and of compact support. We write \( h \in S(k^\times \backslash A^\times) \). Given \( h \), define an operator \( U(h) \) on \( L^2(k^\times \backslash A^\times) \) by

\[
U(h)\varphi(x) = \int_{k^\times \backslash A^\times} |a|^{-\frac{1}{2}} h(a) \varphi(xa) d^\times a.
\]

Let \( \Lambda > 1 \) be in the value group \( qZ^0 \), and let

\[
\log' \Lambda = \text{vol}\{a \in k^\times \backslash A^\times | \Lambda^{-1} \leq |a| \leq \Lambda \}.
\]

Further let for any \( v \in V: h \mapsto \int' \frac{h(u)}{|u-1|} d^\times u \) be the distribution on \( k_v^\times \) that agrees with \( \frac{d^\times u}{|u-1|} \) for \( u \neq 1 \) and whose Fourier transform vanishes at 1. Another way to characterize this distribution is to say it is the unique distribution that agrees with \( \frac{d^\times u}{|u-1|} \) for \( u \neq 1 \) and that sends \( h = 1_{\mathcal{O}_v^\times} \) to zero.

Let \( \hat{h} : k^\times \backslash A^\times \rightarrow \mathbb{C} \) be the Fourier transform of \( h \). In particular write

\[
\hat{h}(0) = \int_{k^\times \backslash A^\times} h(x) d^\times x,
\]
and
\[ \hat{h}(1) = \int_{k^\times \setminus \mathbb{A}^\times} h(x)|x|^{-1} \, d^\times x, \]

The next theorem has been proved by A. Connes in \[2\].

**Theorem 1.1** The following are equivalent:

(a) For any \( h \in S(k^\times \setminus \mathbb{A}^\times) \) we have, as \( \Lambda \to \infty \), the asymptotic expansion:
\[ \text{tr} (Q_{\Lambda,0} U(h)) = 2h(1) \log \Lambda - \hat{h}(0) - \hat{h}(1) + \sum_{\nu \notin V} \int_{k^\nu} \frac{h(u)}{|u - 1|} d^\times u + o(1). \]

(b) All \( L \)-functions with Größencharacters on \( k \) satisfy the Riemann hypothesis.

In the light of the fact that (b) is known to be true one has to read this theorem as follows: (a) is implied by (b) and an independent proof of (a) gives an independent proof of (b).

Note that Connes gives a slightly different version of the trace formula. It is, however, easy to see that the two versions are equivalent.

### 2 The semi-local trace identity

Let \( S \subset V \) be a finite set of places that is supposed to be *large enough*. This means that \( S \) satisfies the following conditions:

- \( |S| \geq 2. \)
- The image of the absolute value \( |\cdot|_S : \mathbb{A}_S^\times \to \mathbb{R}^\times \) is the full value group of \( \mathbb{Q}_S^\times \).
- \( S \) contains all places \( v \) for that the order \( n(v) \) of the character \( \psi \) in nonzero.
• $S$ is so large, that $c_S = \prod_{v \in S} q_v^{1+n(v)} \geq 1$.

• Fix a finite set $E \subset A^1$ such that $A^1 = E k^\times O^\times _V$. Then for each $e \in E$, the set $S$ contains all places $v$ with $e_v \notin O^\times _v$.

If we embed $A_S^\times$ into $A^\times$ by $x \mapsto (x,1,\ldots)$ and $S$ is large enough, then the set $E$ can be chosen to be contained in $A_S^\times$. We will tacitly assume this. We will further assume that $E$ is a set of representatives for $A^\times / k^\times O^\times$. Then it also is a set of representatives for $A_S^\times / k_S^\times O_S^\times$.

Recall that $k_S^\times$ is a discrete subgroup of $A_S^\times$. For $f \in S(A_S)$ let

$$E_S(f)(x) = |x|_{S}^{\frac{1}{2}} \sum_{\gamma \in k_S^\times} f(\gamma x).$$

Then $E_S(f)$ lies in $L^2(k_S^\times \backslash A_S^\times)$, as is shown in [2]. Further, let $k^0_S = k_S \setminus \{0\}$ and define

$$\tilde{E}_S(f)(x) = |x|_{S}^{\frac{1}{2}} \sum_{\gamma \in k^0_S} f(\gamma x).$$

We define $\psi_S = \prod_{v \in S} \psi_v$ as a character of $A_S$ and the Fourier transform on $S(A_S)$ by $\hat{f}(x) = \int_{k_S} f(y) \psi(xy) dy$. Let $S_0(A_S)$ be the space of all $f \in S(A)$ with $f(0) = 0 = \hat{f}(0)$.

**Proposition 2.1** For any $f_S \in S_0(A_S)$ the function $\tilde{E}_S(f_S)$ lies in the space $L^2(k_S^\times \backslash A_S^\times)$. Furthermore, we have $\tilde{E}_S(f)(x) = \tilde{E}_S(\hat{f})(x^{-1})$.

**Proof:** Let $T = V \setminus S$. We extend a given $f_S \in S(A_S)$ to $f \in S(A)$ by $f = f_S \otimes 1_{O_T}$. A. Connes has shown [2] that $E(f) \in L^2(k^\times \backslash A^\times)$.

**Lemma 2.2** The map

$$\alpha : k_S^\times \backslash A_S^\times \to k^\times \backslash A^\times / O_T^\times$$

$$k_S^\times a \mapsto k^\times a O_T^\times$$

is a homeomorphism, equivariant under the action of $A_S^\times$ and Haar-measure preserving, where on $k^\times \backslash A^\times / O_T^\times$ we install the quotient measure given by the normalized measure (i.e. $\text{vol}(O_T^\times) = 1$) on $O_T^\times$. 
Proof: The map $\alpha$ is well defined since $k^x_S a = k^x_S a'$ implies $k^x a = k^x a'$. It is clearly continuous and $A^\infty_S$-equivariant. It is injective since for $a, a' \in A^\infty_S$ embedded into $A^\infty$ by filling up with ones, the equation $k^x a \mathcal{O}^x_T = k^x a' \mathcal{O}^x_T$ implies that there are $x \in k^x$ and $y \in \mathcal{O}^x_T$ such that $a = xa'y$. Considering the places in $v \in T$ gives $1 = x_v y_v$, which implies $x_v \in \mathcal{O}^x_v$ and thus $x \in k^x$. The map $\alpha$ finally is surjective since $S$ is large enough and $A^\infty = k^x \mathcal{E} \mathcal{O}^x_T \pi_S^{\infty}$, where $\pi_S \in A^\infty_S$ is a uniformizer, i.e., $|\pi_S| = q_0^{-1}$. The map $\alpha$ preserves Haar measures as a consequence of our normalizations. Finally, since $\mathcal{O}^x_T$ is open in $A^\infty_T$ it follows that the map $\alpha$ is open, i.e. the inverse map is continuous.

Q.E.D.

As a consequence of the lemma, $\alpha$ induces a pullback isomorphism

$$\alpha^*: L^2(k^x \backslash A^\infty / \mathcal{O}^x_T) \rightarrow L^2(k^x_S \backslash A^\infty_S).$$

Now let $x \in A_S$ and write $(x, 1)$ for the element of $A$ that coincides with $x$ in $S$ and is equal to 1 everywhere else. We get $f((x, 1)) = f_S(x)$, and $E f(x, 1) = E_S(f)(x)$. The function $f$ is by construction invariant under the multiplication by $\mathcal{O}^x_T$, hence lies in the space of invariants $L^2(k^x \backslash A^\infty)^{\mathcal{O}^x_T} \simeq L^2(k^x \backslash A^\infty / \mathcal{O}^x_T)$, and we conclude that $E_S(f_S) = \alpha^* E(F)$. The second assertion of the proposition follows from Lemma 2 in Appendix I of [3].

Q.E.D.

Let $\Lambda > 0$ and $\tilde{Q}_{S, \Lambda, 0}$ be the subspace of $S_0(A_S)$ consisting of all $f \in S_0(A_S)$ such that $f(x) = 0 = \hat{f}(x)$ whenever $|x| > \Lambda$. Let $Q_{S, \Lambda, 0}$ be the closure in $L^2(k^x_S \backslash A^\infty_S)$ of the space $E(\tilde{Q}_{S, \Lambda, 0})$. Likewise, let $\tilde{Q}_{S, \Lambda, 0}$ be the closure in $L^2(k^x_S \backslash A^\infty_S)$ of the space $E_S(\tilde{Q}_{S, \Lambda, 0})$. We also write $Q_{S, \Lambda, 0}$ and $\tilde{Q}_{S, \Lambda, 0}$ for the orthogonal projections.

Let $h \in S(k^x_S \backslash A^\infty_S)$, i.e. $h$ is locally constant and of compact support. Define the operator $U(h)$ on $L^2(k^x_S \backslash A^\infty_S)$ by

$$U(h) \phi(x) = \int_{k^x_S \backslash A^\infty_S} |y|^{-\frac{1}{2}} h(y) \phi(xy) d^x y.$$ 

Let $\hat{h} : k_S^x \backslash A^\infty_S \rightarrow \mathbb{C}$ be the Fourier transform of $h$. In particular write $\hat{h}(0) = \int_{k^x_S \backslash A^\infty_S} h(x) d^x x$, and $\hat{h}(1) = \int_{k^x_S \backslash A^\infty_S} h(x) |x|^{-1} d^x x$. Let $Q_{S, \Lambda}$ be the subspace of $S(A_S)$ consisting of all $f \in S(A_S)$ with $f(x) = 0 = \hat{f}(x)$ whenever $|x| > \Lambda$. Let $Q_{S, \Lambda}$ be the closure in $L^2(k^x_S \backslash A^\infty_S)$ of the space $E_S(Q_{S, \Lambda})$. 


Theorem 2.3 (Semi-local trace identity) The following assertions are equivalent.

(a) If \( h \) is supported in \( \{ q_0^{-r} \leq |x| \leq q_0^{r} \} \) and \( S \) contains all places \( v \) with \( q_v \leq q_0^{r} \), then, as \( \Lambda \to \infty \) we have

\[
\text{tr} Q_{S,\Lambda,0} U(h) = \text{tr} \tilde{Q}_{S,\Lambda,0} U(h) + o(1).
\]

(b) Connes’ global trace formula.

(c) The Riemann hypothesis for all \( L \)-functions with Größencharacters.

Since Connes showed that (b) is equivalent to (c) it suffices to show that (a) is equivalent to (b). The proof of this theorem will be given in section 5.

3 A variant of the semi-local trace formula

Let \( S \) be a finite set of places that is large enough (see section 2). Let \( h \in S(k_S^{\times} \backslash \mathbb{A}_S^{\times}) \).

Theorem 3.1 As \( \lambda \to \infty \), we have

\[
\text{tr} Q_{S,\Lambda,0} U(h) = 2h(1) \log' \Lambda - \hat{h}(0) - \hat{h}(1) + \sum_{v \in S} \int_{k_v^{\times}}' \frac{h(u)}{|u - 1|} d^{\times} u + o(1).
\]

Proof: In [2] it is shown that

\[
\text{tr} (Q_{S,\Lambda} U(h)) = 2h(1) \log' \Lambda + \sum_{v \in S} \int_{k_v^{\times}} \frac{h(u)}{|u - 1|} d^{\times} u + o(1),
\]

as \( \Lambda \to \infty \). Actually, Connes shows a slightly different assertion in [2], namely, instead of one projection \( Q_{S,\Lambda} \) there is a product of two projections \( \tilde{P}_\Lambda P_\Lambda \). But similar to the global case it is easy to see that, in the absence of infinite places, Connes’ statement is equivalent to the above.
Let $z = \prod_{v \in S} z_v$ be in $\mathbb{A}_S^\times$. Then, locally
\[
\widehat{1_{z \mathcal{O}^\times_S}} = |z_v| q_v^{-\mathcal{O}_S} \left( 1_{z_v^{-1} F_v} - \frac{1}{q_v} 1_{\pi_v^{-1} z_v^{-1} F_v} \right).
\]
So we get
\[
\widehat{1_{z \mathcal{O}^\times_S}} = |z| \prod_{v \in S} q_v^{-n(v)} \left( 1_{z_v^{-1} F_v} - \frac{1}{q_v} 1_{\pi_v^{-1} z_v^{-1} F_v} \right).
\]
Let $c_S = \prod_{v \in S} q_v^{1+n(v)}$. Since $S$ is large enough we infer that $c_S \geq 1$. Then for $a \in \mathbb{A}_S^\times$ with $\frac{c_S}{\Lambda} \leq |a| \leq \Lambda$, the function $f = 1_a \mathcal{O}^\times_S$ lies in $\tilde{Q}_{S, \Lambda}$ and $E_S(f)(x) = (q-1)|x|^{\frac{1}{2}} 1_{a \mathcal{O}^\times_S k_S^x}(x)$. For $\alpha < \beta$ in the value group $\mathfrak{q}_0^\mathbb{Z}$, let $A(\alpha, \beta)$ be a set of representatives of $\{x \in \mathbb{A}_S \mid \alpha \leq |x| \leq \beta\}/\mathcal{O}_S^x k_S^x$. Note that $A(\alpha, \beta)$ is a finite set. Let $f_{1, \Lambda} = \frac{1}{q-1} \sum_{a \in A(\alpha, \beta)} 1_a \mathcal{O}^\times_S$. Then $f_{1, \Lambda} \in \tilde{Q}_{S, \Lambda}$ and $E_S(f_{1, \Lambda})(x) = |x|^\frac{1}{2} 1_{\{\mathfrak{q}_0^\mathbb{Z} \leq |x| \leq \Lambda\}}$. Let $f_{0, \Lambda}$ be the Fourier transform of $f_{1, \Lambda}$.

**Lemma 3.2** Suppose that $h$ is supported in $\{|x| \leq 1\}$. Then
\[
Q_{S, \Lambda} U(h) E_S(f_{1, \Lambda}) \equiv \frac{c_S - \Lambda q_0}{c_S - \Lambda q_0} h(0) E_S(f_{1, \Lambda}) \mod (Q_{S, \Lambda, 0}).
\]

**Proof:** Note first that the assertion only depends on the $\mathcal{O}_S^\times$-invariant projection $h_{\mathcal{O}_S^\times}(x) = \frac{c_S}{\text{vol}(\mathcal{O}_S^\times)} \int_{\mathcal{O}_S^\times} h(xy) dx$ of $h$. So we may assume that $h$ is $\mathcal{O}_S^\times$-invariant. Then it follows that $h$ is a finite linear combination of functions of the form $1_{b \mathcal{O}_S^\times k_S^x}$, for $b \in \mathbb{A}_S^\times$. We may thus assume that $h = (q-1) 1_{b \mathcal{O}_S^\times k_S^x}$ with $|b| \leq 1$. Then $U(h) E_S(f_{1, \Lambda}) = E_S(R(g)f_{1, \Lambda})$, where $g = 1_{b \mathcal{O}_S^\times}$, and $R(g)f_{1, \Lambda}(x) = \int_{b \mathcal{O}_S^\times} f_{1, \Lambda}(xy) dx y = \text{vol}(\mathcal{O}_S^\times) f_{1, \Lambda}(xb)$. So that
\[
U(h) E_S(f_{1, \Lambda}) = \text{vol}(\mathcal{O}_S^\times) E_S \left( \frac{1}{q-1} \sum_{a \in A(\mathfrak{q}_0^\mathbb{Z}, \Lambda)} 1_{a \mathcal{O}_S^\times} \right).
\]
Since $|b| \leq 1$, we get that $Q_{S, \Lambda} U(h) E_S(f_{1, \Lambda})$ equals
\[
\text{vol}(\mathcal{O}_S^\times) E_S \left( \frac{1}{q-1} \sum_{a \in A(\mathfrak{q}_0^\mathbb{Z}, \Lambda)} 1_{a \mathcal{O}_S^\times} \right).
\]
For \( f \in \mathcal{S}(\mathbb{A}_S) \) let \( \hat{\ell}(f) = \hat{f}(0) = \int_{\mathbb{A}_S} f(x) \, dx \). Then

\[
\hat{\ell}\left(1_{a\mathcal{O}_S^x}\right) = \int_{\mathbb{A}_S} 1_{a\mathcal{O}_S^x}(x) \, dx = |a| \int_{\mathcal{O}_S^x} dx.
\]

It follows that there is a \( \lambda \in \mathbb{C} \) with \( Q_{S,A}U(h)E_S(f_{1,\Lambda}) - \lambda E_D(f_{1,\Lambda}) \in Q_{S,A,0} \) and this \( \lambda \) is

\[
\lambda = \text{vol}(\mathcal{O}_S^x) \frac{\sum_{a \in A(\frac{\pi_S}{\alpha}, \Lambda)} |a|}{\sum_{a \in A(\frac{\pi_S}{\alpha}, \Lambda)} |a|}.
\]

For \( \alpha, \beta \) in the value group we can choose \( A(\alpha, \beta) = \bigcup_{\alpha \leq q_0^k \leq \beta} \pi_S^{-k} \mathcal{E} \), where \( \pi_S \in \mathbb{A}_S^\times \) is an element with \( |\pi_S| = q_0^{-1} \). Let \( \Lambda = q_0^k, c_S = q_0^{k_0} \), and \( v(b) \geq 0 \) the valuation of \( b \). Then

\[
\lambda = \text{vol}(\mathcal{O}_S^x) \frac{|\mathcal{E}| \sum_{j=0}^{k_0 - k - v(b)} q_0^j}{|\mathcal{E}| \sum_{j=0}^{k_0 - k} q_0^j} \quad = \quad \text{vol}(\mathcal{O}_S^x) \frac{\sum_{j=0}^{2k_0 - k - v(b)} q_0^j}{\sum_{j=0}^{k_0 - k} q_0^j} \quad = \quad \text{vol}(\mathcal{O}_S^x) q_0^{v(b)} \frac{1 - q_0^{-2k_0 - v(b) + 1}}{1 - q_0^{-2k_0 + 1}} = \text{vol}(\mathcal{O}_S^x) \frac{1}{|b|} \frac{c_S - \Lambda^2 q_0 |b|}{c_S - \Lambda^2 q_0}.
\]

Since \( \text{vol}(\mathcal{O}_S^x k_S^x / k_S^x) = \frac{\text{vol}(\mathcal{O}_S^x)}{q - 1} \), we get \( \hat{h}(0) = \text{vol}(\mathcal{O}_S^x) \). The lemma follows.

Q.E.D.

Now write \( f_{1,\Lambda} = e_{1,\Lambda} + q_{1,\Lambda} \), where \( q_{1,\Lambda} \in Q_{S,A,0} \) and \( e_{1,\Lambda} \) is orthogonal to \( Q_{S,A,0} \). Likewise write \( f_0 = e_0 + q_0 \). Let \( l(E_S(f)) = f(0) \) for \( E_S(f) \in Q^C_{S,A} \). Note that \( l \) is well defined and that \( \ker(l) = Q^C_{S,A,0} \oplus \mathbb{C} e_{1,\Lambda} \). Likewise let \( \tilde{l}(E_S(f)) = \hat{f}(0) \) and note that \( \ker(\tilde{l}) = Q_{S,A,0} \oplus \mathbb{C} e_{0,\Lambda} \). The operator \( Q_{S,A}U(h) \) preserves \( \ker(l) \) if \( \text{supph} \subset \{|x| \leq 1\} \), and it preserves \( \ker(\tilde{l}) \) if \( \text{supph} \subset \{|x| \geq 1\} \). Let \( \tilde{f}_{1,\Lambda} = \frac{f_{1,\Lambda}}{\|E_S(f_{1,\Lambda})\|} \).

**Lemma 3.3** If \( h \) is supported in \( \{|x| \geq 1\} \), then, as \( \Lambda \to \infty \),

\[
Q_{S,A}U(h)E_S(\tilde{f}_{1,\Lambda}) = \hat{h}(1)E_S(\tilde{f}_{1,\Lambda}) + \varphi + o(1),
\]

where \( \varphi \in Q_{S,A,0} \).
Proof: Without loss of generality we can assume that $h = (q - 1)1_{bO_S^x}$ for some $|b| \geq 1$. Let

$$f_{1,\Lambda}^1 = \frac{1}{q - 1} \sum_{a \in A(\frac{cS}{a}, \Lambda)} 1_{aO_S^x},$$

and let

$$\tilde{f}_{1,\Lambda}^1 = \frac{f_{1,\Lambda}^1}{\|E_S(f_{1,\Lambda})\|}, \tilde{e}_{1,\Lambda} = \frac{e_{1,\Lambda}}{\|E_S(f_{1,\Lambda})\|}, \tilde{q}_{1,\Lambda} = \frac{q_{1,\Lambda}}{\|E_S(f_{1,\Lambda})\|}. $$

Then $\|E_S(f_{1,\Lambda} - \tilde{f}_{1,\Lambda}^1)\|$ tends to zero as $\Lambda \to \infty$. Similar to the last proof we get

$$U(h)E_S(f_{1,\Lambda}^1) = \text{vol}(O_S^x)E_S \left( \frac{1}{q - 1} \sum_{a \in A(\frac{cS}{a}, \Lambda)} 1_{aO_S^x} \right),$$

and this also equals $Q_{S,\Lambda}U(h)E_S(f_{1,\Lambda}^1)$. Repeating the argument of the last lemma we get $Q_{S,\Lambda}U(h)E_S(f_{1,\Lambda}^1) \equiv \lambda E_S(f_{1,\Lambda}^1)$, with

$$\lambda = \frac{\text{vol}(O_S^x)\sum_{a \in A(\frac{cS}{a}, \Lambda)} |a|}{\sum_{a \in A(\frac{cS}{a}, \Lambda)} |a|} = \frac{\text{vol}(O_S^x)\sum_{a \in A(\frac{cS}{a}, \Lambda)} |ab|}{\sum_{a \in A(\frac{cS}{a}, \Lambda)} |ab|} = \frac{1}{|b|} \text{vol}(O_S^x).$$

Since $\hat{h}(1) = \frac{1}{|b|} \text{vol}(O_S^x)$, the claim follows. Q.E.D.

Suppose further that $h$ is supported in $\{|x| \geq 1\}$. Then $Q_{S,\Lambda}U(h)$ preserves $\ker(\tilde{l}) = Q_{S,\Lambda,0} \oplus \mathbb{C}\tilde{e}_{1,\Lambda}$, and we get

$$Q_{S,\Lambda}U(h)\tilde{e}_{1,\Lambda} \equiv Q_{S,\Lambda}U(h)E_S(\tilde{f}_{1,\Lambda}) \mod \ker(\tilde{l})$$

$$\equiv \hat{h}(1)E_S(\tilde{f}_{1,\Lambda}) + o(1) \mod \ker(\tilde{l})$$

$$\equiv \hat{h}(1)\tilde{e}_{1,\Lambda} + o(1) \mod \ker(\tilde{l}).$$

Next we show that $\tilde{q}_{1,\Lambda}$ tends to zero as $\lambda \to \infty$. We use the fact that for any $\varphi \in Q_{S,\Lambda,0}$ the integral $\int \varphi(x)|x|^\frac{d^\Lambda}{2}d^\Lambda x$ vanishes, to compute

$$\langle \tilde{q}_{1,\Lambda}, \tilde{q}_{1,\Lambda} \rangle = \langle \tilde{q}_{1,\Lambda}, E_S(\tilde{f}_{1,\Lambda}) \rangle = \int_{|x| \geq \frac{cS}{\Lambda}} \tilde{q}_{1,\Lambda}(x)|x|^\frac{d^\Lambda}{2}d^\Lambda x$$

$$= -\int_{|x| < \frac{cS}{\Lambda}} \tilde{q}_{1,\Lambda}(x)|x|^\frac{d^\Lambda}{2}d^\Lambda x.$$
We infer that
\[ \| \tilde{q}_{1,\Lambda} \|^2 \leq \sqrt{\int_{|x| < \frac{c\Lambda}{\pi}} |\tilde{q}_{1,\Lambda}(x)|^2 d^x x} \int_{|x| < \frac{c\Lambda}{\pi}} |x|^2 d^x x. \]
The right hand side tends to zero as \( \Lambda \to \infty \). Therefore \( \tilde{q}_{1,\Lambda} \to 0 \). Using Lemma 3.2 we get for \( \text{supp} h \subset \{|x| \leq 1\} \),
\[
Q_{S,\Lambda} U(h) \tilde{e}_{1,\Lambda} = Q_{S,\Lambda} U(h) E_S(\tilde{f}_{1,\Lambda}) + o(1)
\equiv \hat{h}(0) E_S(\tilde{f}_{1,\Lambda}) + o(1) \mod Q_{S,\Lambda,0}
= \hat{h}(0) \tilde{e}_{1,\Lambda} + o(1).
\]
The Fourier transform turns \( \ker(\hat{l}) = Q_{S,\Lambda,0} \oplus \mathbb{C} \tilde{e}_{0,\Lambda} \) into \( \ker(l) = Q_{S,\Lambda,0} \oplus \mathbb{C} \tilde{e}_{1,\Lambda} \). For \( \varphi \in L^2(k_S^x \setminus \mathbb{A}_S^x) \) we have \( \hat{U}(\varphi) = U(\hat{h}) \hat{\varphi} \), where \( \hat{h}(x) = |x|h(x^{-1}) \).
Since \( \hat{h}(0) = \hat{h}(1) \) and \( \hat{h}(1) = \hat{h}(0) \), we get for \( \text{supp}(h) \subset \{|x| \geq 1\} \),
\[
Q_{S,\Lambda} U(h) \tilde{e}_{0,\Lambda} \equiv \hat{h}(1) \tilde{e}_{0,\Lambda} + o(1) \mod ker(l).
\]
Likewise, we get for \( \text{supp}(h) \subset \{|x| \geq 1\} \),
\[
Q_{S,\Lambda} U(h) \tilde{e}_{0,\Lambda} \equiv \hat{h}(0) \tilde{e}_{0,\Lambda} + o(1) \mod ker(l).
\]
We have proved the following lemma, which also implies the theorem.

**Lemma 3.4** As \( \Lambda \to \infty \), we have
\[
\text{tr} (Q_{S,\Lambda} - Q_{S,\Lambda,0}) U(h) = \hat{h}(0) + \hat{h}(1) + o(1).
\]

**Q.E.D.**

### 4 The localization

Let \( h \in S(k^x \setminus \mathbb{A}^x) \). There is a function \( g \in S(\mathbb{A}^x) \) such that \( h(x) = \sum_{\gamma \in k^x} g(\gamma x) \). There is a finite set of places \( S \) that is large enough (sec. 2) such that \( g = g_S g^S \), where \( g_S \in S(\mathbb{A}_S^x) \) and \( g^S = \prod_{v \notin S} 1_{O_v^x} \). For \( x \in \mathbb{A}_S^x \) set
\[
h_S(x) = \sum_{\gamma \in k_S^x} g_S(\gamma x).
\]
Theorem 4.1 (Localization) For $\Lambda > 1$ we have
\[
\text{tr } Q_{\Lambda,0} U(h) = \text{tr } \tilde{Q}_{S\Lambda,0} U(h_S).
\]

**Proof:** For $\varphi \in L^2(k^x \backslash A^x)$ we get $U(h)\varphi = R(g)\varphi$, where
\[
R(g)\varphi(x) = \int_{A^x} g(y)|y|^{-\frac{1}{2}} \varphi(xy)dy.
\]
Then $R(g) = R(g_S)R(g^S)$, where, for $\varphi \in L^2(k^x \backslash A^x)$ we have
\[
R(g_S)\varphi(x) = \int_{A^x_S} |a|^{-\frac{1}{2}} g_S(a)\varphi(xa)da
\]
and
\[
R(g^S)\varphi(x) = \int_{A^{S,x}} |a|^{-\frac{1}{2}} g^S(a)\varphi(xa)da,
\]
where the measure in the latter integral is the quotient of the normalized measures on $A^x$ and $A^x_S$. Let $T = V \backslash S$. With this measure the set $O^x_T \subset A^{S,x}$ has volume 1.

It follows that
\[
R(g^S) = \frac{1}{\text{vol}(O^x_T)} \int_{A^x_S} |a|^{-\frac{1}{2}} 1_{O^x_T}(a)\varphi(xa)d^xA = \frac{1}{\text{vol}(O^x_T)} \int_{O^x_T} \varphi(xa)d^xA.
\]
Thus it emerges that $R(g^S)$ coincides with $Pr_T$, the orthogonal projection onto the space $L^2(k^x \backslash A^x)^{O^x_T}$ of $O^x_T$-invariants. On $L^2(k^x \backslash A^x)$ we now have two orthogonal projections, $Q_{\Lambda}$ and $R(g^S)$, which commute with each other. So we get
\[
\text{tr } Q_{\Lambda,0} U(h) = \text{tr } Q_{\Lambda,0} R(g) = \text{tr } \left( Q_{\Lambda,0} R(g) \mid L^2(k^x \backslash A^x)^{O^x_T} \right)
\]
\[
= \text{tr } \left( \alpha^* Q_{\Lambda,0} R(g_S)(\alpha^*)^{-1} \mid L^2(k^x_S \backslash A^x_S) \right).
\]
It is easy to see that $R(g_S)$ commutes with $\alpha^*$.

**Lemma 4.2** We have the identity of projections $\alpha^* Q_{\Lambda,0}(\alpha^*)^{-1} = \tilde{Q}_{S\Lambda,0}$. 
Proof: Let $Q^\mathcal{O}_{\Lambda,0}^{\mathcal{T}}$ be the space of $\mathcal{O}_{\Lambda,0}^{\mathcal{T}}$-invariants in $Q_{\Lambda,0}$. The lemma will follow from the identity of vector spaces $\alpha^* \left( Q^\mathcal{O}_{\Lambda,0}^{\mathcal{T}} \right) = \tilde{Q}_{S,\Lambda,0}$. For this let $f \in \tilde{Q}_{\Lambda,0}$ and suppose that $E(f)$ is $\mathcal{O}_{\Lambda,0}^{\mathcal{T}}$-invariant. We may then assume that $f$ itself is $\mathcal{O}_{\Lambda,0}^{\mathcal{T}}$-invariant. Then $f$ can be written as a finite sum $f = \sum_j f_j$, where each $f_j$ lies in $\tilde{Q}_{\Lambda}$ and is a product $f_j = \prod_v f_{j,v}$. We may assume that if $v \notin S$, then $f_{j,v} = 1_{\mathcal{O}_v}$ for some $b \in k_v^\times$. All but finitely many of the $b$’s can be chosen to be 1. Since $S$ is large enough, there is, for each $j$, a $\gamma_j \in k^\times$ such that with $\gamma_j(x) = f_j(\gamma_j x)$ we have $f_{j,v}^{\gamma_j} = 1_{\mathcal{O}_v}$ for every $v \notin S$. Let $f_1 = \sum_j f_j^{\gamma_j}$. We claim that $f_1$ lies in $\tilde{Q}_{\Lambda,0}$ again. By $f_j \in \tilde{Q}_{\Lambda}$ and $|\gamma_j| = 1$ we get that $f_j^{\gamma_j} \in \tilde{Q}_{\Lambda}$. So we only have to show that $f_1(0) = 0 = \hat{f}_1(0)$.

For the first, recall that

$$f_1(0) = \sum_j f_j^{\gamma_j}(0) = \sum_j f_j(0) = f(0) = 0.$$  

For the second recall

$$\hat{f}_1(0) = \sum_j \hat{f}_j^{\gamma_j}(0) = \sum_j \hat{f}_j^{\gamma_j^{-1}}(0) = \sum_j \hat{f}_j(0) = \hat{f}(0) = 0.$$  

Finally,

$$E(f_1) = \sum_j E(f_j^{\gamma_j}) = \sum_j E(f_j) = E(f),$$

and this shows that $E(\tilde{Q}_{\Lambda,0})^{\mathcal{O}_{\Lambda,0}^{\mathcal{T}}} = E(\tilde{Q}_{\Lambda,0}^{\mathcal{T}})$, where $\tilde{Q}_{\Lambda,0}^{\mathcal{T}}$ is the space of all $f \in \tilde{Q}_{\Lambda,0}$ that can be written as a product $f = f_S \left( \prod_{v \notin S} 1_{\mathcal{O}_v} \right)$ for some $f_S \in \tilde{Q}_{S,\Lambda,0}$. For such a function $f$ and $x \in A_S^\times$ we get

$$E(f)(x) = \sum_{\gamma \in k^\times} f(\gamma x) = \sum_{\gamma \in k^\times} f_S(\gamma x) = \hat{E}_S(f_S)(x).$$

Since all $f_S$ in $\tilde{Q}_{S,\Lambda,0}$ can occur, the lemma follows and this implies the theorem. Q.E.D.

5 Proof of the main theorem

We will now prove Theorem 2.3. We use the notation of the previous section.
A SEMILOCAL TRACE IDENTITY...

By Theorem 4.1 Connes’ trace formula is equivalent to
\[ \text{tr} \bar{Q}_{S,0} U(h) = 2h(1) \log' \Lambda - \hat{h}(0) - \hat{h}(1) + \sum_{v \notin V} \int_{k_v} \frac{h(u)}{|u-1|} d\times u + o(1). \]

By Theorem 3.1, on the other hand, we know that
\[ \text{tr} (Q_{S,0} U(h_S)) = 2h_S(1) \log_S' \Lambda - \hat{h}_S(0) - \hat{h}_S(1) + \sum_{v \in S} \int_{k_v} \frac{h_S(u)}{|u-1|} d\times u + o(1). \]

So the main theorem will follow from identifying the right hand sides of both of these formulae.

**Lemma 5.1** We have

(i) \( h(x) = h_S(x) \) if \( x \in \mathbb{A}_S^\times \),

(ii) \( \hat{h}(0) = \hat{h}_S(0) \) and \( \hat{h}(1) = \hat{h}_S(1) \),

(iii) \( \log' \Lambda = \log' \Lambda \).

**Proof:** For \( x \in \mathbb{A}_S^\times \) we have
\[ h(x) = \sum_{\gamma \in k^\times} g(\gamma x) = \sum_{\gamma \in k_S^\times} g_S(\gamma x) = h_S(x). \]

This proves (i). We further see that
\[ \hat{h}(0) = \int_{k^\times \setminus k_S^\times} h(x) d\times x = \int_{k_S^\times} g(x) d\times x = \int_{k_S^\times} g_S(x) d\times x = \hat{h}_S(0), \]
as well as \( \hat{h}(1) = \hat{h}_S(1) \). Finally note that
\[ \log' \Lambda = \int_{k_S^\times \setminus k_S^\times} \sum_{a \in A(1/\Lambda, \Lambda)} 1_{a O_S^\times k_S^\times}(x) d\times x = \frac{1}{q-1} \sum_{a \in A(1/\Lambda, \Lambda)} \int_{k_S^\times} 1_{a O_S^\times}(x) d\times x \]
\[ = \frac{\text{vol}(O_S^\times)}{q-1} \# A \left( \frac{1}{\Lambda}, \Lambda \right) = \log' \Lambda. \]

The last equation follows from the same computation with \( S \) replaced by \( V \) and the fact that \( A(1/\Lambda, \Lambda) \) also is a set of representatives of the set of all \( x \in \mathbb{A}^\times \) of absolute value between \( \frac{1}{\Lambda} \) and \( \Lambda \) modulo \( k^\times O_V^\times \). Q.E.D.
Lemma 5.2 Suppose $h$ is supported in $\{q^{-r} \leq |x| \leq q^{r}\}$. Then for every $v \in S$ we have
\[
\int_{k_0^S}^\prime \frac{h(u)}{|u-1|} d^x u = \int_{k_0^S}^\prime \frac{h(u)}{|u-1|} d^x u.
\]
For $v \notin S$ we have $\int_{k_0^S}^\prime \frac{h(u)}{|u-1|} d^x u = 0$.

**Proof:** The first assertion follows from (i) of the last lemma. For the second let $v \notin S$. Then
\[
\int_{k_0^S}^\prime \frac{h(u)}{|u-1|} d^x u = \sum_{\gamma \in k^\times} \int_{k_0^S}^\prime \frac{g(\gamma u)}{|u-1|} d^x u.
\]
Let $\gamma \in k^\times$. If $g(\gamma u) \neq 0$ for some $u \in k_0^S$, then it follows that $q_0^{-r} \leq |\gamma|_S \leq q_0^r$ and $|\gamma|_w = 1$ for every $w \notin S$, $w \neq v$. Therefore it follows $q_0^{r'} \leq |\gamma|_v \leq q_0^r$ which implies $|\gamma|_v = 1$ since $q_v > q_0$. Therefore $u \mapsto g(\gamma u)$ is a multiple of $1_{\phi^\times}$ and so
\[
\int_{k_0^S}^\prime \frac{g(\gamma u)}{|u-1|} d^x u = 0.
\]
Q.E.D.

### 6 Closing remarks

In this section we give a reformulation of the semi-local trace identity. Let $C$ be a compact open subgroup of $k_S^\times$, such that $h$ is invariant under translations by elements of $C$. Let $P_C$ be the orthogonal projections onto the space of $C$-invariants. Then $P_C = \frac{1}{\text{vol}(C)} \int_C d^x x$. This implies that $P_C$ leaves stable the spaces $Q_{S,\Lambda}$, $Q_{S,0}$ and $\tilde{Q}_{S,0}$. Further we have that $U(h) = P_C U(h) P_C$. Let $Q^C_{S,\Lambda}$, $\tilde{Q}^C_{S,0}$ and $Q^C_{S,0}$ be the subspaces of $C$-invariants. Every $\varphi \in \tilde{Q}^C_{S,0}$ is supported in $\{\frac{1}{\Lambda} \leq |x| \leq \Lambda\}$. Since the set $\{\frac{1}{\Lambda} \leq |x| \leq \Lambda\}/k_S^\times C$ is finite, it follows that $\tilde{Q}^C_{S,0}$ is finite dimensional. In particular the set $\tilde{Q}^C_{S,0}$ coincides with $\tilde{E}_S(\tilde{Q}^C_{S,0})$. Let $f \in \tilde{Q}_{S,0}$. Then $\tilde{E}_S(f)(x) = |x|^\frac{1}{2} \sum_{\gamma \in k_S^\times} f(\gamma x)$. Let $R = k_S^0/k_S^\times$. We use the same letter to indicate a set of representatives for the quotient $R$. Summing over $k_S^\times$ first and then over $R$ gives the sum expansion
\[
\tilde{E}_S(f)(x) = \sum_{r \in R} |r|^{-\frac{1}{2}} \tilde{E}_S(f)(rx).
\]
This sum converges pointwise, indeed the sum is locally finite, but it does not converge absolutely in $L^2(k_S^\times \backslash A_S^\times)$. In this way we get a canonical surjective map $T: E_S\left(\tilde{Q}_{S,\Lambda,0}^C\right) \to \bar{Q}_{S,\Lambda,0}^C$ defined by

$$T(E_S(f))(x) = \sum_{r \in R} |r|^{-\frac{1}{2}} E_S(f)(rx) = \bar{E}_S(f)(x).$$

It is not hard to see that there are coefficients $c_r \in \mathbb{R}$ such that for every $f \in \tilde{Q}_{S,\Lambda,0}^C$ we have

$$E_S(f)(x) = T'(\bar{E}_S(f))(x) = \sum_{r \in R} c_r \bar{E}_S(f)(rx),$$

where, again, the sum is locally finite. This implies that $E(\tilde{Q}_{S,\Lambda,0}^C)$ is finite dimensional and so coincides with $Q_{S,\Lambda,0}^C$. Further $T$ is a linear bijection with inverse $T^{-1} = T'$. If we extend $T$ to a bicontinuous linear bijection of $L^2(k_S^\times \backslash A_S^\times)$ that maps the orthogonal space of $Q_{S,\Lambda,0}$ to the orthogonal space of $\bar{Q}_{S,\Lambda,0}$, then we get

$$TQ_{S,\Lambda,0}T^{-1} = \bar{Q}_{S,\Lambda,0}.$$

So that the semi-local trace identity then becomes

$$\text{tr} Q_{S,\Lambda,0} U(h) = \text{tr} Q_{S,\Lambda,0} T U(h) T^{-1}. $$

Note that if $h$ is supported in the norm one elements, then $U(h)$ leaves invariant the space $Q_{S,\Lambda,0}$ and it commutes with $T$. So in this case the formula follows directly. It would be nice to find a direct prove of this identity in general.
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