Entropy of fully packed hard rigid rods on $d$-dimensional hyper-cubic lattices

Deepak Dhar$^{1,*}$ and R. Rajesh$^{2,3,†}$

$^1$Indian Institute of Science Education and Research, Dr. Homi Bhabha Road, Pashan, Pune 411008, India
$^2$The Institute of Mathematical Sciences, C.I.T. Campus, Taramani, Chennai 600113, India
$^3$Homi Bhabha National Institute, Training School Complex, Anushakti Nagar, Mumbai 400094, India

(Dated: December 15, 2020)

We determine the asymptotic behavior of the entropy of configurations of straight rigid rods of size $k \times 1$ and $1 \times k$ that fully cover a finite $L \times M$ rectangular portion of square lattice. We show that full coverage is possible only if at least one of $L$ and $M$ is a multiple of $k$, and that all allowed configurations can be reached from a standard configuration of all rods being parallel, using only basic flip moves that replace a $k \times k$ square of parallel horizontal rods by vertical rods, and vice versa. In the limit of large $k$, we show that the entropy per site $S_2(k)$ tends to $Ak^{-2}\ln k$, with $A = 1$. We argue that this large-$k$ behavior of entropy per site is super-universal and continues to hold on all $d$-dimensional hyper-cubic lattices, with $d \geq 2$.

I. INTRODUCTION

Systems of particles with only hard core interactions between them have been studied as prototypical models for phase transitions in equilibrium statistical mechanics as well as for understanding aspects of non-equilibrium statistical mechanics. In equilibrium statistical mechanics, hard sphere systems serve as minimal models of solid to fluid transition in molecular solids [1–3], and in colloidal crystals [4]. Dimer models are equivalent to the Ising model, and anisotropic particles can effectively model different phases and phase transitions in liquid crystals [5–10]. In non-equilibrium statistical mechanics, hard core models like symmetric or asymmetric exclusion processes provide basic models for driven systems and jamming in granular systems [11–13].

Lattice models of hard-core models have been of particular interest, as they are analytically more tractable. The phases of assemblies of particles of many different shapes have been studied. Examples include squares [14–19], triangles [20], hexagons [21], long rods [22–24], rectangles [25–27], Y-shaped molecules [28–30], tetraminoes [31], lattice models for spheres where the first $k$ nearest neighbors are excluded [32–37], cubes [38], plates [39], etc. An analytical exact solution has been possible only for the case of hard hexagons so far [21]. Phase transitions have also been studied in mixtures of different shapes, for example squares and dimers [17, 40], rods of different lengths [41, 42], polydispersed spheres [43], etc. For square-dimers, it was shown that the critical exponents of the order-disorder transition depend continuously on the relative concentration of the components. Despite a long history, many basic questions about these systems remain open; for example, for a given shape of particles, what are the possible ordered phases, and in which sequence will they appear on increasing the density?

The lattice model of linear $k \times 1$ hard rods ($k$-mers) has attracted a lot of interest, starting with the pioneering work of Onsager [44] who showed that a system of thin, long cylinders in three dimensional continuum undergo a phase transition from a disordered phase to an orientationally ordered nematic phase. On a $d$-dimensional hyper-cubic lattice, rods can only orient in one of the $d$ directions. It was realized in Ref. [23], based on Monte Carlo simulations and high density expansions, that nematic order is present at intermediate densities for large enough $k$, and that the lattice model at high densities must undergo a second disordering transition at a critical density $1 - \rho_c \approx A/k^2$ for large $k$, when the nematic order is lost. Usual Monte Carlo techniques with local moves are rather inefficient in sampling states at high-density due to high rates of rejection of moves due to jamming, but recently-introduced strip-update Monte Carlo technique has made it possible to reach densities within a few percent of maximum packing density [45, 46]. Using these techniques, it is found that on the square lattice, for $k < 6$, there is no phase transition, but for $k > 6$, as we increase density, there are three phases: the low-density disordered phase, intermediate-density nematic phase, and the high-density phase in which there is no long ranged positional or orientational order [46]. The existence of the transition may be rigorously proved [47]. The first phase transition belongs to the Ising [48–50] or three-state Potts universality classes [48, 49, 51] depending on whether the rods are on a square or triangular/honeycomb lattice. The nature of the second transition is not so clear. There is some indication of the high density phase having power law correlations [46] with the second transition being in the non-Ising universality class [46, 52], while the exact solution of soft repulsive rods on a tree-like lattice [53] suggests otherwise. More recently, the transitions in two dimensions have been studied using measures such as entanglement entropy, mutability, Shannon entropy and data compression [52, 54, 55].

In three dimensions, there is no phase transition for
For $k \leq 4$, the system undergoes phase transitions from disordered to nematic to a layered disordered phase as density is increased. In the layered disordered phase, the system breaks up into very weakly interacting two dimensional planes within which the rods are disordered. For $4 < k < 7$, there is no nematic phase, and a single phase transition from a disordered to a layered disordered phase [56, 57].

In this paper, we focus on the fully packed limit of linear rods on the square lattice and give heuristic arguments to extend the results to higher dimensions. In particular, we focus on the entropy per site $S_d(k)$. For the case of dimers ($k = 2$) - the only case that is exactly solvable [58–61] - the entropy per site for square lattice is $S_2(2) = G/\pi = 0.29156 \ldots$, where $G$ is the Catalan’s constant [58]. On square lattice, the orientation-orientation correlation of two dimers separated by a distance $r$ decays as a power law $r^{-3/2}$ for large $r$ [62], while on a triangular lattices, these correlations are short-ranged [63]. A review of the method of solution of dimer problems on planar lattices may be found in Ref. [64]. In three dimensions, there is a class of lattices (not cubic lattice) for which an exact solution can be found [65], while for the cubic lattice, the orientation-orientation correlations decay as a power law [6]. The entropy of fully packed trimer ($k = 3$) tilings on square lattice have also been studied [66]. By numerically diagonalising the transfer matrices for strips, the entropy per site was found to be $S_2(3) = 0.15852 \pm 0.000015$. Much less is known for higher values of $k$. It is known that the tilings admit a vector height field representation [67].

For larger values of $k$, Gagunashvili and Priezzhev obtained an upper bound for the entropy on the square lattice: $S_2(k) \leq k^{-2} \ln \gamma k$, where $\gamma = \exp(4G/\pi)/2$, with $G$ being the Catalan’s constant [68]. It is clear that the full-packing constraint induces strong correlations in the orientations of rods, and one would generally expect orientation-orientation correlations to decrease with distance as a power-law. There is Monte Carlo evidence that the high density phase of long rods in three dimensions shows two-dimensional layering [56], and this suggests that, in the fully packed limit, configurations in higher dimensions may be dominated by layered two-dimensional configurations.

In the following, we determine the asymptotic behavior of the entropy of the fully packed configurations in the limit of large rod lengths $k$: first in two dimensions, and then generalized to higher dimensions. The number of coverings depends strongly on the boundary conditions imposed. We will consider configurations of a finite $L \times M$ rectangular portion of square lattice fully covered by rectangles of size $k \times 1$ or $1 \times k$. Equivalently, we can consider this a lattice model, with all sites covered using straight rigid rods of length $k$. We will call this open boundary conditions. We prove that full coverage in the open boundary case is possible only if at least one of $L$ and $M$ is a multiple of $k$. All the allowed configurations for this case can be reached from the standard configuration of all horizontal rods, using only basic flip moves that flip a $k \times k$ square of parallel horizontal rods by vertical rods, and vice versa. Using rigorous upper and lower bound estimates, we show that $S_2(k)$ varies $A k^{-2} \ln k$, for large $k$, with $A = 1$. We give a heuristic argument that this large-$k$ behavior of entropy per site is ‘super-universal’, and continues to hold on $d$-dimensional hypercubical lattices for all $d > 2$

\[
\lim_{k \to \infty} \frac{k^2 S_d(k)}{\ln k} = 1,
\]

independent of $d$.

The remainder of the paper is organized as follows. In Sec. II, we define the problem precisely. We derive some basic properties of the fully packed phase by showing that an $L \times M$ rectangle can be completely covered by $k$-mers, only if at least one of $L$ or $M$ is a multiple of $k$ and that all full packing configurations on an open $L \times M$ rectangle can be obtained from the standard configuration of all horizontal rods by a combination of basic flip moves. In Sec. III, we obtain lower bounds for entropy by solving exactly for the entropy of rods on semi-infinite strips $k \times \infty$ and $2k \times \infty$. In Sec. IV, we obtain an upper bound for the entropy by considering truncated generated functions. The lower and upper bounds are combined in Sec. V to obtain Eq. (1). In Sec. VI, we use a perturbation theory estimate to argue that this result should also hold for $d$-dimensional hypercubical lattices with $d > 2$.}

### II. PRELIMINARIES

We consider tilings of a $L \times M$ rectangle, with $L, M$ positive integers, by $k \times 1$ and $1 \times k$ rectangles ($k$-mers). Each $k$-mer can only be in one of two orientations: horizontal or vertical. An example is shown in Fig. 1 for the case $k = 3$. Equivalently, we can consider this a lattice model, with all sites covered using straight rigid rods of length $k$. Let $N(L, M)$ be the number of such tilings.

#### A. Divisibility of $L, M$ by $k$

We first show that $N(L, M)$ is non-zero, if and only if at least one of $L$ and $M$ is divisible by $k$. The ‘if’ part is trivial. For the other part, clearly $LM$ has to be a multiple of $k$, for full coverage. We now argue that in this case, at least one of $L$ and $M$ has to be a multiple of $k$.

Assign one of the $k$ colors, called here 0, 1, 2, \ldots, $(k-1)$ to each of the squares of the lattice, with square $(x, y)$ given color $q = (x-y) \mod k$. The coloring of the squares for the case $k = 3$ is shown in Fig. 2. Then each $k$-mer covers exactly one square of each color. Let $L = k \ell + \alpha, M = k m + \beta$, with $0 < \alpha, \beta \leq k - 1$. Divide the rectangle into three smaller rectangles of sizes $k \ell \times \alpha$, $\alpha \times km$ and $\alpha \times \beta$, as shown in Fig. 3. Then, clearly the
FIG. 1. A tiling of the Euclidean plane by $k$-mers, with $k = 3$. Only a part of the tiling is shown here, and some $k$-mers do not fully fall in the region shown.

FIG. 2. Assigning colors to $1 \times 1$ squares for the case $k = 3$.

rectangles of size $k\ell \times M$, and $\alpha \times km$ can be covered by $k$-mers, implying that the number of squares of different colors in these two rectangles are equal. However, the small rectangle of size $\alpha \times \beta$ has $\min(\alpha, \beta)$ squares of same color along the diagonal. To cover them would require $\min(\alpha, \beta)$ rods, with total area $k \min(\alpha, \beta)$. Equating this to the total area $\alpha \beta$, we obtain $k = \max(\alpha, \beta)$. This contradicts the assumption that $\alpha, \beta < k$. Hence, the rectangle can not be fully covered by $k$-mers, unless either $L$ or $M$ is divisible by $k$.

For simplicity of presentation, in the following, we shall assume that both $L$ and $M$ are multiples of $k$.

**B. Ergodicity of the flip moves**

In this subsection, we show that all configurations of rods can be reached from any configurations by just using the flip move (defined below).

We define the standard tiling configuration of $k\ell \times M$ rectangle by $k$-mers as one using only horizontal $k$-mers. A basic flip move is defined as replacing a $k \times k$ square filled with vertical $k$-mers by one with horizontal $k$-mers, and vice versa, as illustrated in Fig. 4.

A combination of two flip moves defines a ‘slide’ move, where a vertical $k$-mer next to a $k \times k$ flippable square exchanges position (see Fig. 5), and the vertical $k$-mer will be said to slide across the flippable square.

We now argue that that any full tiling of $k\ell \times M$ rectangle by $k$-mers may be reached from the standard configuration by using only the basic flip and slide moves.

**Proof:** Look at the lowest row. If it consists of only horizontal $k$-mers, then we ignore this row, and the problem reduces to one with a smaller $M$. Else, it would have $\ell' = \ell - \Delta$ horizontal $k$-mers, and $k\Delta$ vertical $k$-mers. In Fig. 6, we have shown an example of a 4-mer tiling of a $48 \times 12$ rectangle, where $\ell = 12$, $\Delta = 1$. We move to the left any $k \times k$ block of horizontal flippable rods we find between these $k\Delta$ vertical $k$-mers, using the slide move, and make the vertical rods closer to each other. If now there is any block of consecutive vertical $k$-mers, we can flip these to horizontal, and reduce the problem to one with fewer number on vertical $k$-mers.

**FIG. 3.** Dividing an $(k\ell + \alpha) \times (km + \beta)$ rectangle, where $0 < \alpha, \beta \leq k - 1$, into smaller rectangles.

**FIG. 4.** The basic flip moves consists of replacing a small $k \times k$ square in the configuration covered by $k$ horizontal $k$-mers, by vertical $k$-mers, and vice versa.

**FIG. 5.** The slide move consists of transposing a rod and an adjacent flippable square, i.e. sliding the rod across the square, may be thought of as a combination of two flip moves.
FIG. 6. A tiling of a 48 × 12 rectangle with rods of length $k = 4$, where only the rods in the bottom row are shown. The vertical rods split the rectangles into smaller rectangles, and aids in finding a block of flippable $k$-mers (see text for details).

If there is no such horizontal flippable block of rods, we look at the bottom row. Let us say that it has segments of $i_1, i_2, ..., i_s$ horizontal rods, interspersed with vertical rods. [In Fig. 6, there are 4 segments, with $i_1 = 1$, $i_2 = 3$, $i_3 = 4$, $i_4 = 3$.] Clearly, these are bordered by vertical $k$-mers at the ends, unless the segment itself is at the end of the rectangle. Then we look at the sub-rectangles of sizes $i_1k \times M, i_2k \times M, ...$ made up of these segments and bounded by vertical boundaries. In the example shown in Fig. 6, these rectangles are shown with orange boundaries.

We now argue that there will be a flippable $k \times k$ block within each of these small rectangles. This is clear if the width of the rectangle is exactly $k$. Then the sites just above can only be covered by a horizontal rod, or $k$ vertical rods. In the latter case, it forms a vertical flippable rectangle. If not, then eventually, we will have $k$ horizontal rods just above each other, and form a horizontal flippable rectangle.

If the width is greater than $k$, and the row just above is not made of all horizontal rods, then it will be made up of a number of horizontal segments, separated by vertical rods. And we can repeat the argument with this smaller set. This process can not continue for ever, as the total width is finite, and the width decreases at each step.

Thus, we will be able to find a flippable $k \times k$ box at each stage, and eventually, the number of vertical rods becomes zero, and the standard tiling of all horizontal $k$-mers is reached. Since all moves are reversible, and any valid configuration of full-packing can be changed to standard configuration, we can go from any full packing configuration on the rectangle to any other using only flip moves.

III. LOWER BOUND FOR ENTROPY FOR LARGE $k$

It is easy to see that at full packing, there is a finite entropy per site. We can divide the lattice into $k \times k$ squares. There are $LM/k^2$ such squares, and each can be tiled in two ways, independent of the others. Then the total number of such tilings is $2^{LM/k^2}$ (see Fig. 7). Of course, more complicated tilings are possible, as shown in Fig. 1, and the above only provides a lower bound. We define entropy per site

$$S_2(k) = \lim_{L,M \to \infty} \frac{\ln N(L,M)}{LM}.$$ (2)

Then, $S_2(k) \geq k^{-2} \ln 2$.

A. Entropy of strips $k \times \infty$

We can easily obtain a better bound on $S_2(k)$. Break the $L \times kM$ lattice in $M$ strips of width $k$ each. Denoting $N(L, k)$ by $F_L$, it is clear that

$$N(L, kM) \geq [F_L]^M.$$ (3)

It is straightforward to see that $F_L$'s satisfy the recursion relation

$$F_L = F_{L-1} + F_{L-k}.$$ (4)

This implies that $F_L$ increases as $\lambda^L$ where $\lambda$ is the largest root of the equation

$$\lambda^k = \lambda^{k-1} + 1.$$ (5)

Then, it is easily seen that in the limit of large $k$,

$$\lambda = 1 + \frac{A(k)}{k} + O(k^{-2}),$$ (6)

where $A(k)$ is the solution of the equation

$$A(k) \exp[A(k)] = k.$$ (7)

The function $A(k)$ is called the Lambert function (usually denoted as $W(k)$) [69], and it is easy to see that for large $k$,

$$W(k) \approx \ln \left( \frac{k}{\ln k} \right).$$ (8)
with corrections that only grow slower than \( \ln(\ln k) \). Thus we obtain
\[
\lambda = 1 + \frac{1}{k} \ln \left( \frac{k}{\ln k} \right) + \text{higher order terms.} \tag{9}
\]
Thus, the entropy per site is bounded from below by \((\ln \lambda)/k\) and this gives the leading behavior:
\[
\frac{k^2 S_2(k)}{\ln k} \geq 1, \quad k \gg 1. \tag{10}
\]

B. Entropy of strips \( 2k \times \infty \)

In this subsection, we describe the exact calculation of the entropy of tilings of the semi-infinite \( 2k \times \infty \) stripe with \( k \)-mers, where the \( x \)-coordinate is \( \geq 1 \), and \( y \)-coordinate lies in the range \([1, 2k]\). We define the generating function \( \Omega_{2k}(x) \) as the sum over all covering of rectangles of size \( r \times 2k \), summed over all positive integer values of \( r \), where the weight of a covering with \( n \) tiles is \( x^n \). Then, we have
\[
\Omega_{2k}(x) = \sum_{r=0}^{\infty} N(r, 2k)x^{2r}. \tag{11}
\]

We also define a partial covering of the strip with rods to the left of some reference line \( x = s > 0 \), so that no site with \( x \)-coordinate less than \( s \) is left uncovered (see Fig. 8), and all rods must cover at least one site with \( x \)-coordinate less than \( s \). Clearly, all rods that do not lie completely to the left of \( x = s \) must be horizontal. A partial covering is a rectangular covering if no site with \( x \)-coordinate larger than \( s \) is covered.

A partial covering may be characterized by its right boundary \( \{h_y\} \), for \( y = 1 \) to \( 2k \), where \( h_y \) specifies how many sites to the right of the reference line \( x = s \) are covered in the row with ordinate \( y \). We will choose \( s \) to be as large as possible, so that at least one of the \( h_y \)'s has to be zero, and \( h_y \leq k - 1 \), for all \( y \). For example, the boundary of the configuration shown in Fig. 8 is specified by \( \{0, 0, 3, 3, 3, 3, 0, 0\} \). In a more compact notation, denoted as \( \{0^23^40^2\} \).

It is easy to see that for a partial covering of the \( 2k \times L \) stripe, the only allowed height configurations are \( \{0^{2k}\} \), \( \{h^k0^k\}, \{0^k0^k\}, \{h^k0^k0^k\} \), with \( h \) and \( j \) taking values from 1 to \( k - 1 \).

We define the generating functions \( \psi(\{h_y\}) \) as the generating function of all possible ways of completing a partial tilings with a given height profile \( \{h_y\} \), where the completed covering is rectangular, and the weight of tiling in which we add \( n \) extra rods is \( x^n \). Therefore, for example,
\[
\psi(\{0^{2k}\}) = \Omega_{2k}(x), \tag{12}
\]
\[
\psi(\{0^k1^k\}) = x + x^3 + \ldots. \tag{13}
\]

Consider a particular height configuration \( \{h_y\} \). We can write recursion equations for the corresponding generating function \( \psi(\{h_y\}) \), by considering all possible ways of filling the column of sites immediately to the right of the reference line by \( k \)-mers, such that the right edge of the full tiling is vertical, and no sites are left uncovered.

For example, it is easily seen that (see Fig. 9)
\[
\Psi(\{0^{2k}\}) = 1 + (x^2 + x^{2k})\Psi(\{0^{2k}\})
+2x^{k+1}\Psi(\{0^k(k-1)^k\})
+\sum_{j=1}^{k-1} x^{k+1}\Psi(\{(k-1)^j0^k(k-1)^{k-j}\}). \tag{14}
\]

The different terms in this equation correspond to the cases where the next column is left empty, or filled by two vertical rods, or by \( 2k \) horizontal rods, or by first \( j \) horizontal rods, then a vertical rod, then \( k - j \) horizontal rods.

Writing such generating functions for all possible boundaries, we obtain a set of inhomogeneous linear equations in approximately \( 2k^2 \) variables. This may be written as a transfer matrix of dimension \( 2k^2 \times 2k^2 \). However, using the symmetries of the problem, this number can be considerably reduced.

We note that the recursion equations for the generating function \( \Psi(\{h^j0^k0^k\}) \) is
\[
\Psi(\{h^j0^k0^k\}) = x\Psi(\{(h - 1)^j0^k(h - 1)^{k-j}\})
+x^k\Psi(\{0^k(h - h)0^k\}), \quad 1 \leq h < k. \tag{15}
\]

FIG. 8. A partial filling of the strip by rods for the case \( k = 4 \).
The vertical red line shows the reference line. The boundary of the configuration is specified by the projection to the right of the reference line; in this case \( \{0, 0, 3, 3, 3, 3, 0, 0\} \), or in a more compact notation, denoted as \( \{0^23^40^2\} \).

FIG. 9. Recursion equation for \( \Psi(\{0^{2k}\}) \). The jagged boundary at the right end indicates sum over all possible configurations on the right.
This equation has no \( j \)-dependence. Hence, we may expect that \( \Psi(\{h^0 h^k h^{k-j}\}) \) is independent of \( j \). It can be checked that this ansatz is consistent with the remaining recursion equations. Similarly, we find that \( \Psi(\{0^h h^0 k^{-j}\}) \) is also independent of \( j \). With this simplification, the number of independent variables reduces to approximately 2k.

The remaining recursion equations are easily written down, we obtain for all \( 1 \leq h \leq k - 1 \),

\[
\Psi(\{0^h h^0 k^{-j}\}) = x^h \Psi(\{(k - h)^0 h^k (k - h)^{k-j}\}),
\]

and

\[
\Psi(\{0^k h^k\}) = x \Psi(\{(h - 1)^0 h^k (k - h)^{k}\}).
\]

Substituting for \( \Psi(\{0^k h^k\}) \) from Eq. (15) from Eq. (16), we obtain

\[
\Psi(\{h^0 h^k h^{k-j}\}) = \frac{x}{(1 - x^{2k})^h} \Psi(\{0^2 k\}), 1 \leq h, j \leq k - 1.
\]

Substituting for \( \Psi(\{(k - 1) h^0 h^k (k - 1)^{-j}\}) \) in Eq. (14) from Eq. (19) and simplifying, we obtain

\[
[1 - x^2 - x^{2k} - (k - 1)x^{2k}(1 - x^{2k})^{-k+1}] \Psi(\{0^2 k\})
= 1 + 2x^{k+1} \Psi(\{0^k (k - 1)^{-k}\}).
\]

To close the equations, we have to determine \( \Psi(\{0^k h^k\}) \) in terms of \( \Psi(\{0^2 k\}) \). The values of \( \Psi(\{0^k h^k\}) \) for one value of \( h \) are related by Eq. (17) to arguments \( (h - 1) \) and to \( (k - h) \). This seems complicated, but it is easily checked that the ansatz

\[
\Psi(\{0^k h^k\}) = Ca^h + D\alpha^{-h}, \quad 0 \leq h \leq k - 1.
\]

satisfies Eq. (17), so long as

\[
(1 - \frac{x}{\alpha})C = x^k \alpha^{-k} D,
\]

\[
(1 - x\alpha)D = x^k \alpha^k C.
\]

Eliminating \( C/D \) from Eq (22), we obtain

\[
(1 - \alpha x)(1 - \frac{x}{\alpha}) = x^{2k}.
\]

This is a quadratic equation in \( \alpha \), and determines \( \alpha \) for any given value of \( x \). Explicitly, we obtain

\[
\alpha^{\pm 1} = \frac{1}{2} + x^2 - x^{2k} \pm \sqrt{[(1 - x)^2 - x^{2k}][(1 + x)^2 - x^{2k}]}.
\]

Using Eq. (22), we can express \( C \) and \( D \) in terms of a single variable \( C' \):

\[
C = \kappa \alpha^{-k/2} \sqrt{1 - x/\alpha},
\]

\[
D = \kappa \alpha^{k/2} \sqrt{1 - x/\alpha}.
\]

The actual values of \( C \) and \( D \) can be determined from the boundary condition at \( h = 0 \):

\[
C + D = \Psi(\{0^2 k\}).
\]

We obtain

\[
\kappa = \Psi(\{0^2 k\})\alpha^{-k/2} \frac{1}{\sqrt{1 - x/\alpha}}.
\]

Substituting for \( \kappa \) in Eqs. (25) and (26), we obtain \( C \) and \( D \) in terms of \( x \):

\[
C = \frac{1 - x\alpha}{1 - x\alpha + x^{k+1}} \Psi(\{0^2 k\}),
\]

\[
D = \frac{\alpha^{k+1}}{1 - x\alpha + x^{k+1}} \Psi(\{0^2 k\}).
\]

Finally, substituting the value of \( C \) and \( D \) in Eq. (21), we obtain

\[
\psi(\{0^k (k - 1)^{-k}\}) = \frac{(1 - x\alpha)\alpha^{-k+1} + x^{k+1} \alpha}{1 - x\alpha + x^{k+1} \alpha} \Psi(\{0^2 k\}).
\]

Equation (31) may be simplified by substituting for \( x \) from Eq. (23):

\[
\Psi(\{0^k (k - 1)^{-k}\}) = \frac{\alpha^{k/2 - 1} \sqrt{1 - x\alpha + x^{-k+1} \sqrt{1 - x/\alpha}}}{\alpha^{-k/2} \sqrt{1 - x\alpha + \alpha^{k/2} \sqrt{1 - x/\alpha}}}
\]

Note the explicit symmetry of the expression under the exchange of \( \alpha \leftrightarrow 1/\alpha \).

Putting the expressions for \( \alpha \), \( \Psi(\{0^k (k - 1)^{-k}\}) \) in Eq. (20), we obtain an explicit expression for \( \Psi(\{0^2 k\}) \) of the form

\[
\Psi(\{0^2 k\}) = \frac{1}{E(x)},
\]

where the denominator \( E(x) \) equals

\[
E(x) = 1 - x^2 - x^{2k} - \frac{(k - 1)x^{2k}}{(1 - x^{2k})^{k-1}} - 2x^{k+1} \left[ \frac{\alpha^{k/2 - 1} \sqrt{1 - x\alpha + \alpha^{-k+1} \sqrt{1 - x/\alpha}}}{\alpha^{-k/2} \sqrt{1 - x\alpha + \alpha^{k/2} \sqrt{1 - x/\alpha}}} \right]
\]

The entropy is given by \(-k^{-1} \ln x^*, \) where \( x^* \) is the singularity of \( \Psi(\{0^2 k\}) \) that is closest to the origin. We will show below that asymptotic behavior of entropy for strips of width \( 2k \) is the same as that of strips of width \( k \). The explicit values of the entropies for strips \( k \times \infty \) and \( 2k \times \infty \) for \( k \) up to 21 are given in Table I, and compared with the asymptotic result \(-k^{-2} \ln k \) in Eq. (1).

We now determine the leading singularity \( x^* \) of \( \Psi(\{0^2 k\}) \) in the limit \( k \gg 1 \). To do so, consider the denominator \( E(x) \). It has a square root singularity at \( x_c \) when the discriminant in Eq. (24) equals zero. It is easy to see that \( x_c \) satisfies the equation

\[
1 - x_c - x_c^k = 0,
\]
TABLE I. Entropy $S$ for full packing of rods of length $k$ on strips $k \times \infty$ and $2k \times \infty$, compared with the leading asymptotic result $k^{-2} \ln k$ in Eq. (1).

| $k$ | $S_{k \times \infty}$ | $S_{2k \times \infty}$ | $k^{-2} \ln k$ |
|-----|-----------------------|------------------------|-----------------|
| $2^2$ | $2.400659 \times 10^{-1}$ | $2.609982 \times 10^{-1}$ | $1.732868 \times 10^{-1}$ |
| $2^3$ | $2.608540 \times 10^{-2}$ | $2.929916 \times 10^{-2}$ | $3.249127 \times 10^{-2}$ |
| $2^4$ | $2.503880 \times 10^{-3}$ | $2.797511 \times 10^{-3}$ | $3.384508 \times 10^{-3}$ |
| $2^5$ | $2.190142 \times 10^{-4}$ | $2.429123 \times 10^{-4}$ | $2.961444 \times 10^{-4}$ |
| $2^6$ | $1.791444 \times 10^{-5}$ | $1.975853 \times 10^{-5}$ | $2.379732 \times 10^{-5}$ |
| $2^7$ | $1.396705 \times 10^{-6}$ | $1.532938 \times 10^{-6}$ | $1.817851 \times 10^{-6}$ |
| $2^8$ | $1.051617 \times 10^{-7}$ | $1.148760 \times 10^{-7}$ | $1.342731 \times 10^{-7}$ |
| $2^9$ | $7.14252 \times 10^{-9}$ | $8.388809 \times 10^{-9}$ | $9.683154 \times 10^{-9}$ |
| $2^{10}$ | $5.546713 \times 10^{-10}$ | $6.061340 \times 10^{-10}$ | $6.858901 \times 10^{-10}$ |
| $2^{11}$ | $3.925774 \times 10^{-11}$ | $4.234237 \times 10^{-11}$ | $4.791144 \times 10^{-11}$ |
| $2^{12}$ | $2.743428 \times 10^{-12}$ | $2.948320 \times 10^{-12}$ | $3.396672 \times 10^{-12}$ |
| $2^{13}$ | $1.897264 \times 10^{-13}$ | $2.032237 \times 10^{-13}$ | $2.655549 \times 10^{-13}$ |
| $2^{14}$ | $1.300702 \times 10^{-14}$ | $1.389037 \times 10^{-14}$ | $1.539969 \times 10^{-14}$ |
| $2^{15}$ | $8.851686 \times 10^{-16}$ | $9.426769 \times 10^{-16}$ | $1.038890 \times 10^{-15}$ |
| $2^{16}$ | $5.895929 \times 10^{-17}$ | $6.358717 \times 10^{-17}$ | $6.974028 \times 10^{-17}$ |
| $2^{17}$ | $4.025907 \times 10^{-18}$ | $4.266698 \times 10^{-18}$ | $4.659372 \times 10^{-18}$ |

The leading behavior of the singularity is identical to that obtained for strips of width $k$ [see Eq. (9) with $\lambda = 1/x$], and we thus obtain the same asymptotic lower bound for entropy as obtained for strips of width $k$ [see Eq. (10)].

IV. UPPER BOUND FOR ENTROPY FOR LARGE $k$

Gagunashvili and Priezzhev obtained an upper bound for $N(L, M)$ [68]. They considered a subset of sites of the square lattice whose coordinates are multiple of $k$, and assumed that we are given the configuration of $k$-mers that cover these sites. Then, they proved that there is at most one way to cover the remaining sites with $k$-mers. Then, the number of coverings allowed is bounded from above by the number of ways the subset of sites can be covered by $k$-mers. But each of these can be covered in at most $2k^2$ ways. Since there are at most $N/k^2$ such sites, they obtain

$$N(L, M) \leq (2k)^{LM/k^2}. \quad (42)$$

This implies that $S_2(k) \leq \lfloor \ln(2k)^2/k^2 \rfloor$.

In fact, Gagunashvili and Priezzhev proved a stronger upper bound which for large $k$ is $S_2(k) \leq k^{-2} \ln(\gamma k)$, where $\gamma = \exp(4G/\pi)/2$, with $G$ being the Catalan’s constant. Numerically, $\gamma \approx 1.605$. However, the weaker bound is adequate for our purpose here, and combined with Eq. (10) implies that

$$\lim_{k \to \infty} \frac{k^2 S_2(k)}{\ln k} = 1. \quad (43)$$

We will now provide an alternate proof of this proposition.

We define the generating function

$$\Omega_M(x) = \sum_{L=0}^{\infty} N(L, M) x^{LM/k}, \quad (44)$$

with $N(0, M)$ defined to be 1, by convention. Then, clearly,

$$\Omega_1(x) = 1 + x + x^2 + x^3 + \ldots, \quad (45)$$
$$\Omega_2(x) = 1 + x^2 + x^4 + x^6 + \ldots, \quad k > 2. \quad (46)$$

$\Omega_M(x)$ is sum of weights of all configurations of rods on a semi-infinite strip of width $M$, with the weight of a configuration of $n$ rods being $x^n$. We have the further constraint that all rods must lie fully in a rectangular region, with both left and right edge vertical, and no uncovered regions within the rectangle. As a slightly more complicated example, it is easily seen that

$$\Omega_k(x) = \frac{1}{1 - x - x^k}. \quad (47)$$

In determining admissible tilings, a useful concept is of concatenation. This is illustrated in Fig. 10. Given
two tilings of rectangles of sizes $L_1 \times M$ and $L_2 \times M$, we construct a tiling of a rectangle of size $(L_1 + L_2) \times M$ by just putting the smaller rectangles adjacent to each other.

A tiling is said to be vertically indecomposable, if it cannot be expressed as a concatenation of two admissible tilings. For a vertically decomposable tiling, there is a vertical line that divides the rectangle into two smaller tilings, such that no rod crosses the vertical line.

We now define $R_M(x)$ as the sum of weights of all vertically indecomposable tilings of rectangles of width $M$. Clearly, $R_M(x)$ is a series in powers of $x$, with all coefficients as non-negative integers. Then, we have

$$\Omega_M(x) = \frac{1}{1 - R_M(x)}.$$  

Let the radius of convergence of the power series of $\Omega_M(x)$ be $x^*_M$. Then,

$$R_M(x^*_M) = 1.$$  

Since $R_M(x)$ is a series of positive coefficients, we may truncate the series at any order, and obtain an upper bound estimate of $x^*_M$ and hence of $x^*$, the limit of $x^*_M$, for large $M$. This, in turn, will provide an upper bound for the entropy.

We will take $M$ to be a multiple of $k$, as these give the best bounds. The simplest case is $M = k$. In this case, $R_k(x)$ is a finite polynomial, and we have

$$R_k(x) = x + x^k.$$  

We see that this is consistent with Eq. (47).

Now, let us consider the more complicated case $M = 2k$. In this case, $R_{2k}(x)$ is not a finite polynomial. But it has an interesting structure: the lowest order term is $x^2$, corresponding to a configuration of two vertical $k$-mers. But, then terms of order $x^4, x^6, \ldots$ are all zero, as the corresponding tilings are decomposable. The first non-zero term is of order $x^{2k}$, which corresponds to configurations consisting of a plaquette of $k$ aligned vertical rods and $k$ horizontal rods tiling the remaining area, and another of $2k$ horizontal $k$-mers. With a small amount of brute force enumeration, it is easily seen that the plaquette can be placed in $(k + 1)$ ways to give

$$R_{2k}(x) = x^2 + (k + 2)x^{2k} + O(x^{2k+2}).$$  

If we truncate the equation at order $x^{2k}$, we obtain an upper bound estimate for $x^*$. It turns out that for large $k$, the terms that have been dropped make only a negligible contribution to $R_{2k}(x)$ at $x = x^*$. We will verify this claim later. First, we solve the truncated equation for $x^{2k}$:

$$x^2 + (k + 2)x^{2k} = 1.$$  

Writing $x = \exp(-B/k)$, we see that $(1 - x^2) \approx 2B/k$, if $k$ is large, to leading order in $k$, $B$ satisfies the equation

$$2B \exp(2B) \approx k(k + 2),$$  

which has the solution $B \approx (1/2)W(k(k + 2))$ [W(x) being the Lambert function], which for large $k$ has the leading behavior

$$B \approx \ln \left(\sqrt{\frac{k(k + 2)}{2 \ln k}}\right) \approx \ln \left(\frac{k}{\sqrt{2 \ln k}}\right).$$  

This is a bit larger than the estimate using strips of width $k$, which gave $A \approx \ln(\frac{k}{\ln k})$ [see Eq. (8)], but for large $k$, the leading behavior remains $(\ln k)/k$, with the difference showing only in the sub-leading correction of order $(\ln \ln k)/k$.

Now, the term of order $x^{2k+2}$ in $R_{2k}$ is only $2Bx^{2k+2}$, and its contribution to sum is smaller than that of the term of order $x^{2k}$ by a factor $(1/k)$. At higher orders, the term of order $x^{3k}$ has a coefficient of order $k^3$. Using the fact that $x^k$ is of order $1/k$, the net contribution of this term decreases as $1/k^3$. This also does not change the leading order $k$-dependence of $x^*$.

The above results, obtained by truncating the series for $R_{2k}(x)$ are consistent with the exact result given in Sec. III B.

A similar argument works for other values of $R_{mk}(x)$. We will only sketch the arguments here. The series expansion for $R_{mk}(x)$ in powers of $x$ is of the form

$$R_{mk}(x) = x^m + C_m x^{mk} + \text{higher powers of } x^m.$$  

Here $C_m$ is an $m$-dependent coefficient. The leading contribution to this term comes from configurations depicted in Fig. 11, consisting of of $(m-1)$ flippable vertical squares, interspersed with $k$ horizontal rods. The number of such configurations is \binom{k+m-1}{m-1}. Thus, keeping only the first two non-zero terms in the expansion for $R_{m}(x)$, we write

$$x^m + \binom{k+m-1}{m-1}x^{km} = 1.$$  

FIG. 10. A schematic diagram illustrating the procedure of concatenation. Here $T_1$ and $T_2$ are two admissible tilings, and the tiling $T_2T_1T_1$ is obtained by putting the tilings end to end in the specified order.
Solving this equation, we see that its smallest positive root has the leading $k$-dependence given by

$$x_m^* = \exp\left[ -\frac{1}{k} \ln \frac{k}{(m! \ln k)^{1/m}} \right], \quad k \gg m. \quad (57)$$

Thus, we obtain the same upper bound for large $k$ as in Eq. (42), and we conclude that

$$\frac{k^2 S_2(k)}{\ln k} \leq 1, \quad k \gg 1. \quad (58)$$

V. MAIN RESULT

We now combine the lower bound obtained for entropy in Eq. (10) and the upper bound obtained for entropy in Eq. (58). Since these two bounds are the same, we conclude that the entropy for fully packed rods on a square lattice has the asymptotic behavior as given in Eq. (1), thus proving our main result.

We now look at how the bounds converge to the asymptotic result. The entropies on the strips $k \times \infty$ and $2k \times \infty$ provide lower bounds for the entropy on infinite lattices. Ref. [68] gives an upper bound for large $k$ as $S_2(k) \leq k^{-2} \ln(\gamma k)$, where $\gamma \approx 1.605$. Since the leading form is the same for both the upper bounds as well as lower bounds, we divide it out by considering $k^2 S(k)/\ln k$, which converges to 1 for large $k$. The strips provide lower bounds while $\ln \gamma$ provides an upper bound for this quantity. These bounds are shown in Fig. 12.

VI. EXTENSION TO HIGHER DIMENSIONS

We now present a heuristic argument that extends the above result to higher dimensions $d > 2$. For a system of $k$-mers on a $d$-dimensional hypercubical lattice, with $d > 2$, the expectation is that the fully packed state shows spontaneous symmetry breaking, and the system breaks up into a system of parallel 2-dimensional layers, with almost all the rods lying within a layer, and very weak correlations between the rods in different layers. In this way, one obtains the full packing constraint satisfied within a 2-dimensional layer, and different layers can be occupied independently, leading to a large entropy. The existence of layering in the high-density phase has been seen in simulations for rods of length larger than or equal to five in $d = 3$ [56].

We first consider the case $d = 3$. The argument is easily extended to higher $d$.

For the simple cubic lattice, we consider different activities $w_1, w_2, w_3$ for rods oriented along the $x$, $y$, and $z$-directions. Let the corresponding partition function for an $L \times L \times L$ lattice be denoted by $\Omega_L(w_1, w_2, w_3)$.

We will consider a perturbation expansion of this in powers of $w_3$. We start with the case $w_1 = w_2 = w$, and $w_3 = 0$. The the grand partition function of a $L \times L \times L$ cuboid $\Omega_L(w, w, 0)$ can be written as product of 2-dimensional partition functions

$$\Omega_L(w, w, 0) = \left[ \Omega_{2dL}(w) \right]^L, \quad (59)$$

where $\Omega_{2dL}(w)$ is the partition function of a full packing of $L \times L \times 1$ layer by $k$-mers. We write $\Omega_L(w_1, w_2, w_3)$ as a perturbation expansion in $w_3$ about $w_3 = 0$. It is easy to see that this expansion is of the form

$$\Omega_L(w, w, w_3) = \Omega_L(w, w, 0) \left[ 1 + L^d A_1 w_3^k + O(w_3^{2k}) \right], \quad (60)$$

where $A_1 = \frac{\langle h \rangle \ln \gamma}{\ln k}$ and $\langle h \rangle$ is the average entropy of a strip of width 1.
where the first nontrivial term is proportional to \( w_3^k \), and the coefficient \( A_1 \) is determined in terms of the number of configurations of \( k \) z-type rods (to be also called vertical rods), with the rest of the rods being of the x- and y-type. These vertical rods will have to be in the same vertical slab of height \( k \). Let the x- and y-coordinates of the lowest point of these vertical rods \( \{ \alpha_i, \beta_i \} \), \( i = 1 \) to \( k \). Let the number of possible coverings of rods in one plane, given unoccupied sites \( \{ x_i, y_i \} \), be \( N(\{ x_i, y_i \}) \).

Then the number of coverings of the cuboid is proportional to \( [N(\{ x_i, y_i \})]^k \), and the relative weight of this term will be \( [N(\{ \alpha_i, \beta_i \})]/\Omega_{2d,L} \). We then sum over different \( \{ \alpha_i, \beta_i \} \). We expect \( N_{\alpha_i,\beta_i}/\Omega_{2d,L} \) to be less than 1, and to decrease as a power law of the distances between \( \{ \alpha_i, \beta_i \} \), but with a power large enough so that

\[
\sum_{\{ \alpha_i, \beta_i \}} \left[ \frac{N(\{ \alpha_i, \beta_i \})}{\Omega_{2d,L}} \right]^k < \infty, \tag{61}
\]

where the primed sum is over \( i = 2, 3, \ldots, k \), with \( \{ \alpha_i, \beta_i \} \) fixed. Then the sum over \( \alpha_1 \) and \( \beta_1 \) gives a factor proportional to \( L^3 \).

This is a complicated problem, for which we do not know the exact closed form expression. However, we note that each term decreases exponentially with \( k \) for large \( k \) since we expect \( N_{\alpha_i,\beta_i}/\Omega_{2d,L} < 1 \). We note that \( N(\{ \alpha_i, \beta_i \}) \) would be expected to be largest, when the holes are near each other. In fact, the closest they can be be is in a continuous single line of \( k \) points, which may be created by removing a single rod from the 2d-covering. In this case, the contribution of the term is \( [1/(2k)]^{k-1} \).

If we sum to all orders, the dominant contribution will be expected to be of the same form. We thus conclude that for large \( k \),

\[
S_3(k) = S_2(k) + \text{terms of order } k^{-k}. \tag{62}
\]

Then \( \Omega_{L}(w_3) \), as a function of \( w_3 \), and expand in powers of \( w_3^k \), order by order, each term in the perturbation series will give an exponentially small contribution in the large- \( k \) limit.

The proof is immediately extended to higher \( d \), and we obtain Eq. (1).

VII. CONCLUDING REMARKS

In this paper, we studied the tiling of a finite rectangular part of the plane by rectangles of size \( k \times 1 \) and \( 1 \times k \). We showed that in order to get a full coverage, one of the sides of the rectangle to be covered should a multiple of \( k \). We also studied the structure of the tilings of rectangles, and showed that all tilings can be obtained from each other by a sequence of basic flip move that exchanges a small \( k \times k \) small square made of \( k \) parallel vertical rectangles into horizontal ones, and vice versa. We also showed that in the thermodynamic limit, \( S_d(k) \) the entropy per site for \( k \)-mers on a \( d \)-dimensional hypercubical lattice covered by straight rods of length \( k \), for all \( d \geq 2 \) satisfies Eq. (1).

The fact that this limit is independent of dimension deserves some comment. In general, we would expect the coefficients of logarithms encountered in the study of critical phenomena to be ‘universal’, because by definition, they do not change under a change of length scale in a renormalization transformation. But, such coefficients are in general not dimension independent. In fact, here, \( S_d \) has a multiplying factor \( k^{-2} \), which indicates that the relevant quantity is the number of allowed configurations per unit square of length \( k \) (which is the natural length scale in the problem). This number is proportional to \( k \), and the entropy is proportional to \( \ln k \). The fact that this is independent of dimension is only reflecting the fact that for large \( k \), the problem essentially reduces a two-dimensional problem, because of spontaneous symmetry breaking, and most of the configurations at full packing are the ones where the system breaks up into disjoint two-dimensional layers. Consequently, for large \( k \), the leading behavior of entropy in higher dimensions is same as the two-dimensional case.

VIII. ACKNOWLEDGMENTS

DD’s work was partially supported by the grant DST-SR-S2/JCB-24/2005 of the Government of India.

[1] B. J. Alder and T. E. Wainwright, Phase transition for a hard sphere system, J. Chem. Phys. 27. 1208 (1957).
[2] B. J. Alder and T. E. Wainwright, Phase transition in elastic disks, Phys. Rev. 127. 359 (1962).
[3] E. G. Noya, C. Vega, and E. de Miguel, Determination of the melting point of hard spheres from direct coexistence simulation methods, J. Chem. Phys. 128. 154507 (2008).
[4] P. N. Pusey and W. van Megen, Phase behaviour of concentrated suspensions of nearly hard colloidal spheres, Nature 320, 340 (1986).
[5] M. E. Fisher, On the dimer solution of planar ising models, J. Math. Phys. 7. 1776 (1966).
[6] D. A. Huse, W. Krauth, R. Moessner, and S. L. Sondhi, Coulomb and liquid dimer models in three dimensions, Phys. Rev. Lett. 91. 167004 (2003).
[7] R. Moessner and S. L. Sondhi, Ising and dimer models in two and three dimensions, Phys. Rev. B 68, 054405 (2003).
[8] P. G. de Gennes and J. Prost, The physics of liquid crystals, Vol. 83 (Oxford university press, 1995).
D. Frenkel, Structure of hard-core models for liquid crystals, J. Phys. Chem. 92, 3280 (1988).

C. Care and D. Cleaver, Computer simulation of liquid crystals, Rep. Prog. Phys. 68, 2665 (2005).

A. Donev, S. Torquato, F. H. Stillinger, and R. Connelly, Jamming in hard sphere and disk packings, J. Appl. Phys. 95, 989 (2004).

T. M. Liggett, Interacting particle systems, Vol. 276 (Springer Science & Business Media, 2012).

K. Mallick, The exclusion process: A paradigm for non-equilibrium behaviour, Physica A 418, 17 (2015).

A. Bellemans and R. K. Nigam, Phase transitions in the hard-square lattice gas, Phys. Rev. Lett. 16, 1038 (1966).

A. Bellemans and R. K. Nigam, Phase transitions in two-dimensional lattice gases of hard-square molecules, J. Chem. Phys. 46, 2922 (1967).

K. Ramola and D. Dhar, High-activity perturbation expansion for the hard square lattice gas, Phys. Rev. E 86, 031135 (2012).

K. Ramola, K. Damle, and D. Dhar, Columnar order and askin-teller criticality in mixtures of hard squares and dimers, Phys. Rev. Lett. 114, 190601 (2015).

T. Nath, D. Dhar, and R. Rajesh, Stability of columnar order in assemblies of hard rectangles or squares, Eur. Phys. Lett. 114, 10003 (2016).

D. Mandal, T. Nath, and R. Rajesh, Estimating the critical parameters of the hard square lattice gas model, J. Stat. Mech. 2017, 043201 (2017).

A. Verberkmoes and B. Nienhuis, Triangular trimers on the triangular lattice: An exact solution, Phys. Rev. Lett. 83, 3986 (1999).

R. J. Baxter, Hard hexagons: exact solution, J. Phys. A 13, L61 (1980).

P. J. Flory, Phase equilibria in solutions of rod-like particles, Proc. Roy. Soc. A 234, 73 (1956).

A. Ghosh and D. Dhar, On the orientational ordering of long rods on a lattice, Eur. Phys. Lett. 78, 20003 (2007).

D. Dhar, R. Rajesh, and J. F. Stilck, Hard rigid rods on a bethe-like lattice, Phys. Rev. E 84, 011140 (2011).

J. Kundu and R. Rajesh, Phase transitions in a system of hard rectangles on the square lattice, Phys. Rev. E 89, 052124 (2014).

J. Kundu and R. Rajesh, Phase transitions in systems of hard rectangles with non-integer aspect ratio, Eur. Phys. J. B 88, 133 (2015).

J. Kundu and R. Rajesh, Asymptotic behavior of the isotropic-nematic and nematic-columnar phase boundaries for the system of hard rectangles on a square lattice, Phys. Rev. E 91, 012105 (2015).

P. Szabelski, W. Rżysko, T. Pańczyk, E. Ghijsens, K. Tahara, Y. Tobe, and S. De Feyter, Self-assembly of molecular tripods in two dimensions: structure and thermodynamics from computer simulations, RSC Adv. 3, 25159 (2013).

D. Ruth, R. Toral, D. Holz, J. Rickman, and J. Gunton, Impact of surface interactions on the phase behavior of y-shaped molecules, Thin Solid Films 597, 188 (2015).

D. Mandal, T. Nath, and R. Rajesh, Phase transitions in a system of hard y-shaped particles on the triangular lattice, Phys. Rev. E 97, 032131 (2018).

B. C. Barnes, D. W. Siderius, and L. D. Gelb, Structure, thermodynamics, and solubility in tetramino fluids, Langmuir 25, 6702 (2009).

A. Z. Panagiotopoulos, Thermodynamic properties of lattice hard-sphere models, J. Chem. Phys. 123, 104504 (2005).

H. C. M. Fernandes, J. J. Arenzon, and Y. Levin, Monte carlo simulations of two-dimensional hard core lattice gases, J. Chem. Phys. 126, 114508 (2007).

T. Nath and R. Rajesh, Multiple phase transitions in extended hard-core lattice gas models in two dimensions, Phys. Rev. E 90, 012120 (2014).

T. Nath and R. Rajesh, The high density phase of the k -nn hard core lattice gas model, J. Stat. Mech. 2016, 073203 (2016).

S. S. Akimenko, V. A. Gorbunov, A. V. Myslyavtsev, and P. V. Stishenko, Tensor renormalization group study of hard-disk models on a triangular lattice, Physical Review E 100, 022108 (2019).

F. C. Thewes and H. C. M. Fernandes, Phase transitions in hard-core lattice gases on the honeycomb lattice, Phys. Rev. E 101, 062138 (2020).

N. Vigneshwar, D. Mandal, K. Damle, D. Dhar, and R. Rajesh, Phase diagram of a hard cube on the cubic lattice, Phys. Rev. E 99, 052129 (2019).

M. Disertori, A. Giuliani, and I. Jauslin, Plate-nematic phase in three dimensions, Commun. Math. Phys. 373, 327 (2020).

D. Mandal and R. Rajesh, Columnar-disorder phase boundary in a mixture of hard squares and dimers, Phys. Rev. E 96, 012140 (2017).

J. Kundu, J. F. Stilck, and R. Rajesh, Phase diagram of a bidispersed hard-rod lattice gas in two dimensions, Eur. Phys. Lett. 112, 66002 (2016).

J. F. Stilck and R. Rajesh, Polydispersed rods on the square lattice, Phys. Rev. E 91, 012106 (2015).

N. T. Rodrigues and T. J. Oliveira, Three stable phases and thermodynamic anomaly in a binary mixture of hard particles, J. Chem. Phys. 151, 024504 (2019).

L. Onsager, The effects of shape on the interaction of colloidal particles, Ann. N. Y. Acad. Sci. 51, 627 (1949).

J. Kundu, R. Rajesh, D. Dhar, and J. F. Stilck, A monte carlo algorithm for studying phase transition in systems of hard rigid rods, AIP Conf. Proc. 1447, 113 (2012).

J. Kundu, R. Rajesh, D. Dhar, and J. F. Stilck, Nematic-disordered phase transition in systems of long rigid rods on two-dimensional lattices, Phys. Rev. E 87, 032103 (2013).

M. Disertori and A. Giuliani, The nematic phase of a system of long hard rods, Commun. Math. Phys. 323, 143 (2013).

D. A. Matoz-Fernandez, D. H. Linares, and A. J. Ramirez-Pastor, Determination of the critical exponents for the isotropic-nematic phase transition in a system of long rods on two-dimensional lattices: Universality of the transition, Eur. Phys. Lett. 82, 50007 (2008).

D. A. Matoz-Fernandez, D. H. Linares, and A. J. Ramirez-Pastor, Critical behavior of long straight rigid rods on two-dimensional lattices: Theory and monte carlo simulations, J. Chem. Phys. 128, 214902 (2008).

T. Fischer and R. L. C. Vink, Restricted orientation “liquid crystal” in two dimensions: Isotropic-nematic transition or liquid-gas one(?), Eur. Phys. Lett. 85, 56003 (2009).

D. Matoz-Fernandez, D. Linares, and A. Ramirez-Pastor, Critical behavior of long linear k-mers on honeycomb lattices, Physica A 387, 6513 (2008).
[52] C. Chatelain and A. Gendiar, Close packing of rectilinear polymers on a square lattice, Eur. Phys. J. B 93, 134 (2020).
[53] J. Kundu and R. Rajesh, Reentrant disordered phase in a system of repulsive rods on a bethe-like lattice, Phys. Rev. E 88, 012134 (2013).
[54] E. E. Vogel, G. Saravia, and A. J. Ramirez-Pastor, Phase transitions in a system of long rods on two-dimensional lattices by means of information theory, Phys. Rev. E 96, 062133 (2017).
[55] E. E. Vogel, G. Saravia, A. J. Ramirez-Pastor, and M. Pasinetti, Alternative characterization of the nematic transition in deposition of rods on two-dimensional lattices, Phys. Rev. E 101, 022104 (2020).
[56] N. Vigneshwar, D. Dhar, and R. Rajesh, Different phases of a system of hard rods on three dimensional cubic lattice, J. Stat. Mech. 2017, 113304 (2017).
[57] A. Gschwind, M. Klopotek, Y. Ai, and M. Oettel, Isotropic-nematic transition for hard rods on a three-dimensional cubic lattice, Phys. Rev. E 96, 012104 (2017).
[58] P. Kasteleyn, The statistics of dimers on a lattice, Physica 27, 1209 (1961).
[59] P. W. Kasteleyn, Dimer statistics and phase transitions, J. Math. Phys. 4, 287 (1963).
[60] H. N. V. Temperley and M. E. Fisher, Dimer problem in statistical mechanics-an exact result, Phil. Mag. 6, 1061 (1961).
[61] M. E. Fisher, Statistical mechanics of dimers on a plane lattice, Phys. Rev. 124, 1664 (1961).
[62] M. E. Fisher and J. Stephenson, Statistical mechanics of dimers on a plane lattice. ii. dimer correlations and monomers, Phys. Rev. 132, 1411 (1963).
[63] P. Fendley, R. Moessner, and S. L. Sondhi, Classical dimers on the triangular lattice, Phys. Rev. B 66, 214513 (2002).
[64] J. F. Nagle, C. S. Yokoi, and S. M. Bhattacharjee, Dimer models on anisotropic lattices, Phase transitions and critical phenomena 13, 235 (1989).
[65] D. Dhar and S. Chandra, Exact entropy of dimer coverings for a class of lattices in three or more dimensions, Phys. Rev. Lett. 100, 120602 (2008).
[66] A. Ghosh, D. Dhar, and J. L. Jacobsen, Random trimer tilings, Phys. Rev. E 75, 011115 (2007).
[67] R. Kenyon, Conformal invariance of domino tiling, Ann. Prob. , 759 (2000).
[68] N. D. Gagunashvili and V. B. Priezzhev, Close packing of rectilinear polymers on a square lattice, Theor Math Phys 39, 507–510 (1979).
[69] R. M. Corless, G. H. Gonnet, D. E. Hare, D. J. Jeffrey, and D. E. Knuth, On the Lambert-W function, Advances in Computational mathematics 5, 329 (1996).