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Reconstruction and higher-dimensional geometry

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Abstract

Tutte proved that, if two graphs, both with more than two vertices, have the same collection of vertex-deleted subgraphs, then the determinants of the two corresponding adjacency matrices are the same. In this paper, we give a geometric proof of Tutte’s theorem using vectors and angles. We further study the lowest eigenspaces of these adjacency matrices.

Keywords: Vertex-deleted subgraphs; Reconstruction; Hypomorphism; Angles; Higher-dimensional geometry

1. Introduction

Given the graph \( G = \{V, E\} \), let \( G_i \) be the graph obtained by deleting the \( i \)th vertex \( v_i \). Fix \( n \geq 3 \) from now on. Let \( G \) and \( H \) be two graphs of \( n \) vertices. The main conjecture in reconstruction theory, states that if \( G_i \) is isomorphic to \( H_i \) for every \( i \), then \( G \) and \( H \) are isomorphic (up to a reordering of \( V \)). This conjecture is also known as the Ulam’s conjecture.

The reconstruction conjecture can be formulated in purely algebraic terms. Consider two \( n \times n \) real symmetric matrices \( A \) and \( B \). Let \( A_i \) and \( B_i \) be the matrices obtaining by deleting the \( i \)th row and \( i \)th column of \( A \) and \( B \), respectively.

Definition 1. Let \( \sigma_i \) be an \( n - 1 \) by \( n - 1 \) permutation matrix. Let \( A \) and \( B \) be two \( n \times n \) real symmetric matrices. We say that \( A \) and \( B \) are hypomorphic if there exists a set of \( n - 1 \times n - 1 \) permutation matrices

\[ \{\sigma_1, \sigma_2, \ldots, \sigma_n\}, \]

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such that \( B_i = \sigma_i A_i \sigma_i^t \) for every \( i \). Put \( \Sigma = \{ \sigma_1, \sigma_2, \ldots, \sigma_n \} \). We write \( B = \Sigma(A) \). \( \Sigma \) is called a hypomorphism.

The algebraic version of the reconstruction conjecture can be stated as follows.

**Conjecture 1.** Let \( A \) and \( B \) be two \( n \times n \) symmetric matrices. If there exists a hypomorphism \( \Sigma \) such that \( B = \Sigma(A) \), then there exists a \( n \times n \) permutation matrix \( \tau \) such that \( B = \tau A \tau^t \).

We start by fixing some notations. If \( M \) is a symmetric real matrix, then the eigenvalues of \( M \) are real. We write \( \text{eigen}(M) = (\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)) \).

If \( \alpha \) is an eigenvalue of \( M \), we denote the corresponding eigenspace by \( \text{eigen}_\alpha(M) \). Let \( 1_n \) be the \( n \)-dimensional row vector \((1, 1, \ldots, 1)\). We may drop the subscript \( n \) if it is implicit. Put \( J = 1^t 1 \).

If \( A \) and \( B \) are hypomorphic, so are \( A + tJ \) and \( B + tJ \).

**Theorem 1** (Tutte). Let \( B \) and \( A \) be two real \( n \times n \) symmetric matrices. If \( B \) and \( A \) are hypomorphic then \( \det(B - \lambda I + tJ) = \det(A - \lambda I + tJ) \) for all \( t, \lambda \in \mathbb{R} \).

In this paper, we will study the geometry related to Conjecture 1. Our main result can be stated as follows.

**Theorem 2** (Main theorem). Let \( B \) and \( A \) be two real \( n \times n \) symmetric matrices. Let \( \Sigma \) be a hypomorphism such that \( B = \Sigma(A) \). Let \( t \) be a real number. Then there exists an open interval \( T \) such that for \( t \in T \) we have

1. \( \lambda_n(A + tJ) = \lambda_n(B + tJ) \);
2. \( \text{eigen}_{\lambda_n}(A + tJ) \) and \( \text{eigen}_{\lambda_n}(B + tJ) \) are both one-dimensional;
3. \( \text{eigen}_{\lambda_n}(A + tJ) = \text{eigen}_{\lambda_n}(B + tJ) \).

A similar statement holds for the highest eigenspaces.

Since the sets of majors of \( A + tJ \) and of \( B + tJ \) are the same, for every \( t \in T \) and \( \lambda \in \mathbb{R} \),

\[
\det(A + tJ - \lambda I) - \det(B + tJ - \lambda I) = \det(A + tJ) - \det(B + tJ). \tag{1}
\]

If \( t \in T \), by taking \( \lambda = \lambda_n(A + tJ) \), we obtain

\[
\det(A + tJ) - \det(B + tJ) = \det(A + tJ - \lambda I) - \det(B + tJ - \lambda I) = 0.
\]

Since the above statement is true for \( t \in T \), \( \det(A + tJ) = \det(B + tJ) \) for every \( t \). By Eq. (1), we obtain \( \det(B - \lambda I + tJ) = \det(A - \lambda I + tJ) \) for all \( t, \lambda \in \mathbb{R} \). This is Tutte’s theorem, which was proved using rank polynomials and Hamiltonian circuits. I should also mention that Kocay [1] found a simpler way to deduce the reconstructibility of characteristic polynomials.

Here is the content of this paper. We begin by presenting a positive semidefinite matrix \( A + \lambda I \) by \( n \) vectors in \( \mathbb{R}^n \). We then interpret the reconstruction conjecture as a generalization of a congruence theorem in Euclidean geometry. Next we study the presentations of \( A + \lambda I \) under the perturbation by \( tJ \). We define a norm of angles in higher dimensions and establish a comparison theorem. Our comparison theorem then forces hypomorphic matrices to have the same lowest eigenvalue and eigenvector.
2. Notations

Unless stated otherwise,

(1) all linear spaces in this paper will be finite-dimensional real Euclidean spaces;
(2) all linear subspaces will be equipped with the induced Euclidean metric;
(3) all vectors will be column vectors;
(4) vectors are sometimes regarded as points in \( \mathbb{R}^n \).

Let \( U = \{ u_1, u_2, \ldots, u_m \} \) be an ordered set of \( m \) vectors in \( \mathbb{R}^n \). \( U \) is also interpreted as a \( n \times m \) matrix.

(1) Let \( \text{conv} \ U \) be the convex hull spanned by \( U \), namely,

\[
\left\{ \sum_{i=1}^{m} \alpha_i u_i \left| \alpha_i \geq 0, \sum_{i=1}^{m} \alpha_i = 1 \right. \right\}.
\]

(2) Let \( \text{aff} \ U \) be the affine space spanned by \( U \), namely,

\[
\left\{ \sum_{i=1}^{m} \alpha_i u_i \left| \sum_{i=1}^{m} \alpha_i = 1 \right. \right\}.
\]

(3) Let \( \text{span} \ U \) be the linear span of \( U \), namely,

\[
\left\{ \sum_{i=1}^{m} \alpha_i u_i \left| \alpha_i \in \mathbb{R} \right. \right\}.
\]

Then \( \text{conv} \ U \subset \text{aff} \ U \subset \text{span} \ U \).

Let \( A \) be a matrix. We denote the \((i,j)\)th entry of \( A \) by \( a_{ij} \). We denote the transpose of \( A \) by \( A^t \). Let \( \mathbb{R}^+ \) be the set of vectors with only positive coordinates.

3. Geometric interpretation

Fix a standard Euclidean space \((\mathbb{R}^n, (,))\).

**Definition 2.** Let \( A \) be a symmetric positive semidefinite real matrix. An ordered set of vectors \( V = \{ v_1, v_2, \ldots, v_n \} \) is said to be a presentation of \( A \) if and only if \( (v_i, v_j) = a_{ij} \).

Regarding \( v_i \) as column vectors and \( V \) as a \( n \times n \) matrix, \( V \) is a presentation of \( A \) if and only if \( V^t V = A \). Every positive semidefinite real matrix \( A \) has a presentation. In addition, the presentation \( V \) is unique up to a left multiplication by an orthogonal matrix.

**Definition 3.** Let \( S \) and \( T \) be two sets of vectors in \( \mathbb{R}^n \). \( S \) and \( T \) are said to be congruent if there exists an orthogonal linear transformation in \( \mathbb{R}^n \) that maps \( S \) onto \( T \).

So \( A = \sigma B \sigma^t \) for some permutation \( \sigma \) if and only if \( A \) and \( B \) are presented by two congruent subsets in \( \mathbb{R}^n \).

Now consider two hypomorphic matrices \( B = \Sigma(A) \). Observe that \( B + \lambda I = \Sigma(A + \lambda I) \). Without loss of generality, assume \( A \) and \( B \) are both positive semidefinite. Let \( U \) and \( V \) be their
presentations, respectively. Since $B_i = \sigma_i A_i \sigma_i^t$, $U - \{u_i\}$ is congruent to $V - \{v_i\}$. Then the reconstruction conjecture can be stated as follows.

**Conjecture 2** (Geometric reconstruction). Let

$$S = \{u_1, u_2, \ldots, u_n\}$$

and

$$T = \{v_1, v_2, \ldots, v_n\}$$

be two finite sets of vectors in $\mathbb{R}^m$. Assume that $S - \{u_i\}$ is congruent to $T - \{v_i\}$ for every $i$. Then $S$ and $T$ are congruent.

Generically, $m = n$.

**Definition 4.** We say that $U = \{u_i\}_1^n$ is in good position if the point 0 is in the interior of the convex hull of $U$ and the convex hull of $U$ is of dimension $n - 1$.

**Lemma 1.** Let $A$ be a symmetric positive semidefinite matrix. The following are equivalent.

1. $A$ has a presentation in good position.
2. Every presentation of $A$ is in good position.
3. $\text{rank}(A) = n - 1$ and $\text{eigen}_0(A) = \mathbb{R} \alpha$ for some $\alpha \in (\mathbb{R}^n)^n$.

**Proof.** Since $A$ is symmetric positive semidefinite, $A$ has a presentation. Let $U$ be a presentation of $A$.

If $U$ is in good position, then every presentation obtained from an orthogonal linear transformation is also in good position. Since a presentation is unique up to an orthogonal linear transformation, $(1) \iff (2)$.

Suppose $U$ is in good position. Then $\text{rank}(U) = n - 1$. So $\text{rank}(A) = n - 1$. Since 0 is in the interior of the convex hull of $U$, there exists $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^t$ such that

$$0 = \sum_{i=1}^{n} \alpha_i u_i, \quad \sum_{i=1}^{n} \alpha_i = 1, \quad \alpha_i > 0 \forall i.$$ 

Since $\text{rank}(U) = n - 1$, $\alpha$ is unique. Now $U\alpha = 0$ implies

$$A\alpha = U^t U \alpha = U^t 0 = 0.$$ 

Since $\text{rank}(A) = \text{rank}(U) = n - 1$, $\text{eigen}_0(A) = \mathbb{R} \alpha$. So $(2) \implies (3)$.

Conversely, suppose $\text{rank}(A) = n - 1$ and $\text{eigen}_0(A) = \mathbb{R} \alpha$ with $\alpha \in (\mathbb{R}^n)^n$. Then $\sum_i \alpha_i u_i = 0$ and the linear span $\text{span} U$ is of dimension $n - 1$. Thus, 0 is in $\text{conv} U$. It follows that $\text{aff} U = \text{span} U$. So $\dim(\text{conv} U) = \dim(\text{aff} U) = \dim(\text{span} U) = n - 1$. So $(3) \implies (1)$. 

**Lemma 2.** Let $U$ be a presentation of $A$. Suppose that $U$ is in good position. Let $\alpha_i$ be the volume of the convex hull of $\{0, u_1, u_2, \ldots, \hat{u}_i, \ldots, u_n\}$. Then

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^t$$

is a lowest eigenvector.
The proof can be found in many places. For the sake of completeness, I will give a proof using the language of exterior product.

**Proof.** Choosing an orthonormal basis properly, we may assume that every $u_i \in \mathbb{R}^{n-1}$. $U$ becomes a $(n-1) \times n$ matrix. Let $x_1, x_2, \ldots, x_{n-1}$ be the row vectors of $U$. Consider the exterior product

$$x_1 \wedge x_2 \wedge \cdots \wedge x_{n-1}.$$ 

Let $\beta_i$ be the $i$th coordinate in terms of the standard basis

$$\left\{(-1)^{i-1} e_1 \wedge e_2 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_n \mid i \in [1, n] \right\}.$$ 

Put $\beta = (\beta_1, \beta_2, \ldots, \beta_n)^t$. Notice that $x_i \wedge (x_1 \wedge x_2 \wedge \cdots \wedge x_{n-1}) = 0$ for $1 \leq i \leq n-1$. Therefore, $(x_i, \beta) = 0$ for every $i$. So $U\beta = 0$. It follows that $\sum_{i=1}^n \beta_i u_i = 0$. Since 0 is in the convex hull of $\{u_i\}_{i=1}^n$, $\beta_i$ must be either all negative or all positive. Clearly,

$$|\beta_i| = |u_1 \wedge \cdots \wedge \hat{u}_i \wedge \cdots \wedge u_n| = (n-1)!a_i.$$ 

Therefore, we have $U\alpha = 0$. Then $A\alpha = U^t U \alpha = 0$. $\alpha$ is a lowest eigenvector. \hfill \Box

**Theorem 3.** Suppose that $B = \Sigma(A)$. Suppose that $A$ and $B$ have presentations in good position. Then $eigen_0(A) = eigen_0(B) \cong \mathbb{R}$.

**Proof.** Let $U$ and $V$ be presentations of $A$ and $B$, respectively. Then $U$ and $V$ are in good position. Notice that the volume of the convex hull of

$$\{0, u_1, u_2, \ldots, \hat{u}_i, \ldots, u_n\}$$

equals the volume of the convex hull of

$$\{0, v_1, v_2, \ldots, \hat{v}_i, \ldots, v_n\}.$$ 

By Lemmas 2 and 1, $eigen_0(A) = eigen_0(B) \cong \mathbb{R}$. So the lowest eigenspace of $A$ is equal to the lowest eigenspace of $B$. \hfill \Box

### 4. Perturbation by $J$

Recall that $J = 1_n^t 1_n$. We know that $B = \Sigma(A)$ if and only if $B + tJ = \Sigma(A + tJ)$. Let us see how presentations of $A + tJ$ depend on $t$. Let $A$ be a positive definite matrix. Let $U = \{u_i\}_{i=1}^n$ be a presentation of $A$.

Let $aff U$ be the affine space spanned by $U$. Then $\{u_i\}$ are affinely independent. Let $u_0$ be the orthogonal projection of the origin onto $aff U$. Then $(u_0, u_i - u_0) = 0$ for every $i$. We obtain

$$U^t u_0 = \|u_0\|^2 1.$$ 

It follows that $u_0 = \|u_0\|^2 (U^t)^{-1} 1$. Consequently,

$$\|u_0\|^2 = (u_0, u_0) = \|u_0\|^4 1^t U^{-1} (U^t)^{-1} 1 = \|u_0\|^4 1^t A^{-1} 1.$$ 

Clearly, $\|u_0\|^2 = \frac{1}{1^t A^{-1} 1}$. We obtain the following lemma.
Lemma 3. Let $A$ be a positive definite matrix. Let $U = \{u_i\}_{i=1}^{n}$ be a presentation of $A$. Let $u_0$ be the orthogonal projection of the origin onto $\text{aff} U$. Then $\|u_0\|^2 = \frac{1}{\text{tr}A^{-1}}$ and
\[
u_0 = \frac{1}{\text{tr}A^{-1}}(U^t)^{-1}1.
\]
Consider $\{u_i - su_0\}_{i=1}^{n}$. Notice that
\[
(u_i - su_0, u_j - su_0) = (u_i - u_0 + (1-s)u_0, u_j - u_0 + (1-s)u_0)
= (u_i - u_0, u_j - u_0) + (1-s)^2(u_0, u_0).
\]
Taking $s = 0$, we have
\[
(u_i, u_j) = (u_i - u_0, u_j - u_0) + (u_0, u_0).
\]
Therefore
\[
(u_i - su_0, u_j - su_0) = (u_i, u_j) - (u_0, u_0) + (1-s)^2(u_0, u_0)
= (u_i, u_j) + (s^2 - 2s)\|u_0\|^2.
\]
We see clearly that $A + (s^2 - 2s)\|u_0\|^2 J$ is presented by $\{u_i - su_0\}_{i=1}^{n}$. Observe that
\[
\text{span}(u_1 - su_0, u_2 - su_0, \ldots, u_n - su_0)
\]
is of dimension $n$ for all $s \neq 1$. So $A + (s^2 - 2s)\|u_0\|^2 J$ is positive definite for all $s \neq 1$. If $s = 1$, we see that $A - \|u_0\|^2 J$ is presented by $\{u_i - u_0\}_{i=1}^{n}$ whose linear span is of dimension $n - 1$. We obtain the following lemma.

Lemma 4. Let $A$ be a symmetric positive definite matrix. Let $U$ be a presentation of $A$. Let $u_0$ be the orthogonal projection of the origin onto $\text{aff} U$. Then $\{u_i - su_0\}_{i=1}^{n} \{u_i - u_0\}_{i=1}^{n}$ is a presentation of $A + (s^2 - 2s)\|u_0\|^2 J$. Let \( t = (s^2 - 2s)\|u_0\|^2 \). Then $A + t J$ is positive definite for all $t > -\|u_0\|^2$ and positive semidefinite for $t = -\|u_0\|^2$.

Notice that
\[
u_0 = \frac{1}{\text{tr}A^{-1}}(U^t)^{-1}1 = \frac{1}{\text{tr}A^{-1}}U(U^{-1}(U^t)^{-1})1 = \frac{1}{\text{tr}A^{-1}}UA^{-1}1.
\]

Theorem 4. Let $A$ be a symmetric positive definite matrix. Let $U$ be a presentation of $A$. Let $u_0$ be the orthogonal projection of the origin onto $\text{aff} U$. Then $u_0 = \frac{1}{\text{tr}A^{-1}}UA^{-1}1$ and the following are equivalent.

1. $A - \|u_0\|^2 J$ has a presentation in good position;
2. $u_0$ is in the interior of $\text{conv} U$;
3. $A^{-1}1 \in \mathbb{R}^n$.

Corollary 1. Let $A$ be a real symmetric matrix. There exists $\lambda_0$ such that for every $\lambda \geq \lambda_0$ there exists a real number $t$ such that $A + \lambda I + t J$ has a presentation in good position.

Proof. Instead, consider $I + sA$ with $s = \frac{1}{\lambda}$. $I + sA$ is related to $A + \lambda I$ by a constant multiplication:
\[
\lambda(I + sA) = \lambda I + A.
\]
Let \( s_0 = \frac{1}{\|A\|+1} \) where \( \|A\| \) denote the operator norm. Suppose that \( 0 \leq s \leq s_0 \). Then \( I + sA \) is positive definite. For \( s = 0 \), \( (I + sA)^{-1}1 \in \mathbb{R}^n \). Since \( s \rightarrow (I + sA)^{-1}1 \)
is continuous on \((0, s_0)\), there exists an \( s_1 \in (0, s_0) \) such that \( (I + sA)^{-1}1 \in \mathbb{R}^n \) for every \( s \in (0, s_1) \). So for every \( \lambda \in \left[\frac{1}{s_1}, \infty\right) \), \( (A + \lambda I)^{-1}1 = \lambda^{-1}(sA + I)^{-1}1 \in \mathbb{R}^n \). Let \( \lambda_0 = \frac{1}{s_1} \). So for every \( \lambda \geq \lambda_0 \), \( (A + \lambda I)^{-1}1 \in \mathbb{R}^n \). By Theorem 4, for every \( \lambda \geq \lambda_0 \) there exists a \( t \) such that \( A + \lambda I + tJ \) has a presentation in good position. \( \square \)

5. Higher-dimensional angle and comparison theorem

**Definition 5.** Let \( U = \{u_1, u_2, \ldots, u_n\} \) be a subset in \( \mathbb{R}^n \). \( \mathbb{R}^n \) may be contained in some other Euclidean space. Let \( u \) be a point in \( \mathbb{R}^n \). The angle \( \angle(u, U) \) is defined to be the region

\[
\left\{ \sum_{i=1}^{n} \alpha_i (u_i - u) \mid \alpha_i \geq 0 \right\}.
\]

Two angles are congruent if there exists an isometry that maps one angle to the other. Let \( B \) be the unit ball in \( \mathbb{R}^n \). The norm of \( \angle(u, U) \) is defined to be the volume of \( \angle(u, U) \cap B \), denote it by \( |\angle(u, U)| \).

Let me make a few remarks.

1. Firstly, if two angles are congruent, their norms are the same. But, unlike the 2-dimensional case, if the norms of two angles are the same, these two angles may not be congruent.
2. Secondly, if \( \{u_i - u\}_1^n \) are linearly dependent, then \( |\angle(u, U)| = 0 \). If \( u \) happens to be in aff \( U \), then \( |\angle(u, U)| = 0 \).
3. According to our definition, \( |\angle(u, U)| \) is always less than half of the volume of \( B \).
4. More generally, one can allow \( \{\alpha_i\}_1^n \) to be in a collection of other sign patterns which correspond to quadrants in two-dimensional case. Then the norm of an angle can be greater than half of the volume of \( B \).

**Lemma 5.** If \( \angle(u, U) \subseteq \angle(u, V) \), then \( |\angle(u, U)| \leq |\angle(u, V)| \). If \( |\angle(u, U)| > 0 \) and \( \angle(u, U) \) is a proper subset of \( \angle(u, V) \) then \( |\angle(u, U)| < |\angle(u, V)| \).

**Theorem 5 (Comparison theorem).** Let \( \angle(u, U) \) be an angle and \( |\angle(u, U)| \neq 0 \). Suppose that \( v \) is contained in the interior of the convex hull of \( \{u\} \cup U \). Then \( |\angle(u, U)| < |\angle(v, U)| \).

**Proof.** Without loss of generality, assume \( u = 0 \). Suppose \( |\angle(0, U)| > 0 \). Let \( U = \{u_1, u_2, \ldots, u_n\} \). Then \( U \) is linearly independent. Since \( v \) is in the interior of \( \text{conv}(0, U) \), \( v \) can be written as

\[
\sum_{i=1}^{n} \alpha_i u_i
\]

with \( \alpha_i \in \mathbb{R}_+^n \) and \( \sum_{i}^{n} \alpha_i < 1 \).
Let $U' = \{u_i - v\}_1^n$. It suffices to prove that $\angle(0, U)$ is a proper subset of $\angle(0, U')$. Let $x$ be a point in $\angle(0, U)$ with $x \neq 0$. Then $x = \sum_i x_i u_i$ for some $x_i \geq 0$ with $\sum_i x_i > 0$. Define for each $i$

$$y_i = x_i + \frac{\sum x_i}{1 - \sum \alpha_i}.$$ 

The reader can easily verify that $\sum y_i(u_i - v) = x$. Observe that $y_i > x_i \geq 0$. So $\angle(0, U)$ is a proper subset of $\angle(0, U')$. It follows that

$$|\angle(0, U)| < |\angle(0, U')| = |\angle(v, U)|. \quad \Box$$

**Theorem 6.** Let $U = \{u_1, u_2, \ldots, u_n\} \subset \mathbb{R}^m$ for some $m \geq n$. Suppose that $|\angle(u, U)| > 0$. Suppose that the orthogonal projection of $u$ onto aff $U$ is in the interior of conv $U$. Let $v$ be a vector such that $(u - v, u - u_i) = 0$ for every $i$. If $u \neq v$ then $|\angle(u, U)| > |\angle(v, U)|$.

**Proof.** Without loss of generality, assume that $u = 0$. Then $U$ is linearly independent and $v \perp u_i$ for every $i$. Let $u_0$ be the orthogonal projection of $u$ onto aff $U$. By our assumption, $u_0$ is in the interior of conv $U$ and $\|u_0\| \neq 0$. Let

$$v' = \left(1 - \sqrt{\frac{\|v\|^2}{\|u_0\|^2} + 1}\right) u_0.$$ 

Then

$$\|v' - u_0\|^2 = \left(\frac{\|v\|^2}{\|u_0\|^2} + 1\right)\|u_0\|^2 = \|v\|^2 + \|u_0\|^2.$$ 

Notice that $v \perp u_i$ and $u_0 \perp u_i - u_0$. We obtain

$$(u_i - v', u_j - v') = (u_i - u_0 + \sqrt{\frac{\|v\|^2}{\|u_0\|^2} + 1} u_0, u_j - u_0 + \sqrt{\frac{\|v\|^2}{\|u_0\|^2} + 1} u_0)$$

$$= (u_i - u_0, u_j - u_0) + \left(\frac{\|v\|^2}{\|u_0\|^2} + 1\right)\|u_0\|^2$$

$$= (u_i - u_0, u_j - u_0) + (u_0, u_0) + (v, v)$$

$$= (u_i, u_j) + (v, v)$$

$$= (u_i - v, u_j - v). \quad (2)$$

Hence $\angle(v, U) \cong \angle(v', U)$. Notice that $1 - \sqrt{\frac{\|v\|^2}{\|u_0\|^2} + 1} < 0$. So the origin sits between $v'$ and $u_0$ which is in the interior of conv $U$. Therefore, $0$ is in the interior of $\angle(v', U)$. By the comparison theorem, $|\angle(v', U)| > |\angle(0, U)|$. Consequently, $|\angle(v, U)| > |\angle(0, U)|$. \quad \Box

**Theorem 7.** Suppose that $B = \Sigma(A)$. Let $\lambda_0$ be as in Corollary 1 for both $B$ and $A$. Fix $\lambda > \lambda_0$. Let $t_1$ and $t_2$ be two real numbers such that $A + \lambda I + t_1 J$ and $B + \lambda I + t_2 J$ have presentations in good position. Then $t_1 = t_2$.

**Proof.** We prove by contradiction. Without loss of generality, suppose that $t_1 > t_2$. Let $U$ be a presentation of $A + \lambda I + t_1 J$. Then $U$ is in good position. So $0$ is in the interior of conv $U$. Let $V$ be a representation of $B + \lambda I + t_2 J$. Then $V$ is in good position. So $0$ is in the interior of conv $V$
and \( \dim(\text{span } V) = n - 1 \). Let \( v_0 \perp \text{span } V \) and \( \|v_0\|^2 = t_1 - t_2 \). Let \( V' = \{v_i + v_0\} \). Clearly, \( V' \) is a presentation of \( B + \lambda I + t_1 J \).

By Theorem 6, for every \( i \),
\[
\left| \angle(0, V \setminus \{v_i\}) \right| > \left| \angle(-v_0, V \setminus \{v_i\}) \right| = \left| \angle(0, V' \setminus \{v_i + v_0\}) \right|.
\]
Since \( B + \lambda I + t_1 J = \Sigma(A + \lambda I + t_1 J) \), \( V' \setminus \{v_i + v_0\} \) is congruent to \( U \setminus \{u_i\} \) for every \( i \).
Therefore \( \left| \angle(0, V' \setminus \{v_i + v_0\}) \right| = \left| \angle(0, U \setminus \{u_i\}) \right| \). Since 0 is in the interiors of the convex hulls of \( U \) and of \( V' \), we have
\[
\text{Vol}(B) = \sum_{i=1}^{n} \left| \angle(0, V \setminus \{v_i\}) \right| > \sum_{i=1}^{n} \left| \angle(0, V' \setminus \{v_i + v_0\}) \right| = \sum_{i=1}^{n} \left| \angle(0, U \setminus \{u_i\}) \right| = \text{Vol}(B).
\]
This is a contradiction. Therefore, \( t_1 = t_2 \). \( \square \)

6. Proof of the main theorem

Suppose \( B = \Sigma(A) \). Suppose \( \lambda_0 \) satisfies Corollary 1 for both \( A \) and \( B \). So for every \( \lambda \geq \lambda_0 \) there exist real numbers \( t_1 \) and \( t_2 \) such that \( A + \lambda I + t_1 J \) has a presentation in good position and \( B + \lambda I + t_2 J \) has a presentation in good position. By Theorem 7, \( t_1 = t_2 \). Because of the dependence on \( \lambda \), put \( t(\lambda) = t_1 = t_2 \). By Theorem 3,
\[
eigen_0(A + \lambda I + t(\lambda)J) = \text{eigen}_0(B + \lambda I + t(\lambda)J) \cong \mathbb{R}.
\]
Since 0 is the lowest eigenvalue of \( A + \lambda I + t(\lambda)J \) and \( B + \lambda I + t(\lambda)J \), \( \lambda \) is the lowest eigenvalue of \( A + t(\lambda)J \) and \( B + t(\lambda)J \). In addition,
\[
eigen_{-\lambda}(A + t(\lambda)J) = \text{eigen}_{-\lambda}(B + t(\lambda)J) \cong \mathbb{R}.
\]
Now it suffices to show that \( t([\lambda_0, \infty)) \) covers a nonempty open interval.

By Lemmas 4 and 3,
\[
t(\lambda) = -\|u_0\|^2 = -\frac{1}{1' (A + \lambda I) \cdot 1}.
\]
So \( t(\lambda) \) is a rational function. Clearly, \( t([\lambda_0, \infty)) \) contains a nonempty open interval \( T \). For \( t \in T \), we have \( \lambda_n(A + tJ) = \kappa_n(B + tJ) \) and \( \text{eigen}_{\lambda_n}(A + tJ) = \text{eigen}_{\lambda_n}(B + tJ) \cong \mathbb{R} \). This finishes the proof of Theorem 2.

Tutte’s proof involves certain polynomials associated with a graph. It is algebraic in nature. The main instrument in our proof is the comparison theorem. Presumably, there is a connection between the geometry in this paper and the polynomials defined in Tutte’s paper. In particular, given \( n \) unit vectors \( u_1, u_2, \ldots, u_n \), can we compute the function \( |\angle(0, U)| \) explicitly in terms of \( U'U = A \)? This question turns out to be hard to answer. The norm \( |\angle(0, U)| \) as a function of \( A \) may be closely related to the functions studied in Tutte’s paper [2].

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References

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