DETECTING COHOMOLOGY FOR LIE SUPERALGEBRAS

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Abstract. In this paper we use invariant theory to develop the notion of cohomological detection for Type I classical Lie superalgebras. In particular we show that the cohomology with coefficients in an arbitrary module can be detected on smaller subalgebras. These results are used later to affirmatively answer questions, which were originally posed in [BKN1] and [BaKN], about realizing support varieties for Lie superalgebras via rank varieties constructed for the smaller detecting subalgebras.

1. Introduction

1.1. For finite groups there are well known local-global principles which enable the study of their representation theory and cohomology via that of proper subgroups. For example if $G$ is a finite group, $k$ is field of characteristic $p > 0$, and $P$ is a $p$-Sylow subgroup of $G$, then the restriction map induces an embedding $\text{res} : \text{Ext}^\bullet_k(M, N) \hookrightarrow \text{Ext}^\bullet_k(M, N)$ where $kG$ (resp. $kP$) is the group algebra of $G$ (resp. $P$). We therefore say that the cohomology is “detected by” the $p$-Sylow subgroups. Another collection of subgroups which detect the cohomology is the set $E$ of elementary abelian $p$-subgroups. Here the restriction map induces an inseparable isogeny ($F$-isomorphism):

$$H^\bullet(G, k) \rightarrow \lim_{E \in E} H^\bullet(E, k).$$

Moreover, a cohomology class $\zeta \in \text{Ext}^\bullet_k(M, M)$ is nilpotent if and only if $\text{res}(\zeta)$ is nilpotent in $\text{Ext}^\bullet_k(M, M)$ for every $E \in E$.

Such cohomological “detection theorems” may be used to deduce properties of support varieties of finite groups. Let $M$ be a finite-dimensional module for $kG$ and write $V_k(M)$ (resp. $V_k(M)$) for its support variety over $kG$ (resp. $kP$). The restriction map induces a morphism of algebraic varieties $\text{res}^* : V_k(M) \rightarrow V_k(M)$ which is finite to one. Moreover,

$$V_k(M) = \bigcup_{E \in E} \text{res}^*(V_k(E)).$$

Detectability on small subgroups is a rather special feature of modular group algebras. For other finite-dimensional commutative Hopf algebras, like the restricted enveloping algebra of a restricted Lie algebra, one can define cohomology and support varieties, but cohomology is rarely detected on a finite set of smaller (proper) subalgebras.

1.2. We shall be concerned in this work with the representation theory of finite-dimensional classical Lie superalgebras $g = g_0 \oplus g_1$ over the complex numbers, which has strong analogies with the finite group case. Boe, Kujawa, and Nakano [BKN1] recently initiated the study of local-global principles in the setting of Lie superalgebras. Using natural properties of the action of the reductive group $G_0$ (where Lie $G_0 = g_0$) on $g_1$, they proved the existence of two types of detecting subalgebras, $f = f_0 \oplus f_1$ and $e = e_0 \oplus e_1$. These subalgebras were used to study representations in the category $F_{(g, g_0)}$ of finite-dimensional $g$-modules which are semisimple over $g_0$. The present work is a continuation of that program.

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In this situation, the restriction maps induce isomorphisms:

\[ H^*(\mathfrak{g}, \mathfrak{g}_0, \mathbb{C}) \cong H^*(\mathfrak{f}, \mathfrak{f}_0, \mathbb{C})^{N/N_0} \cong H^*(\mathfrak{e}, \mathfrak{e}_0, \mathbb{C})^W \]

where \( N/N_0 \) is a reductive group and \( W = W(\mathfrak{e}) \) is a finite pseudoreflection group. These relative cohomology rings may be identified with the invariant ring \( S^*(\mathfrak{g}_1^*)^{G_0} \), where \( S^* \) denotes the symmetric algebra, and so are finitely generated. This property was used to construct support varieties for \( M \) in \( \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)} \). The restriction maps in cohomology induce embeddings of support varieties:

\[ V_{(\mathfrak{e}, \mathfrak{e}_0)}(M)/W \leftrightarrow V_{(\mathfrak{f}, \mathfrak{f}_0)}(M)/(N/N_0) \leftrightarrow V_{(\mathfrak{g}, \mathfrak{g}_0)}(M). \]

It was suspected that these embeddings are in fact isomorphisms, but when these varieties were introduced there was no reasonable theory of cohomological detection for arbitrary modules in \( \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)} \).

An analogous theory has been developed for the Lie superalgebras \( W(n) \) and \( S(n) \) of Cartan type in \( \text{Bal} \) [Ba] and \( \text{BaKN} \). These Lie superalgebras are \( \mathbb{Z} \)-graded and detecting subalgebras were constructed using the reductive group corresponding to the zero component of the Lie superalgebra.

1.3. The main goal of this paper is to develop the remarkable theory of cohomological detection for arbitrary Type I classical Lie superalgebras and the Lie superalgebras of Cartan type, \( W(n) \) and \( S(n) \). The class of classical Lie superalgebras under consideration includes the general linear Lie superalgebra \( \mathfrak{g} = \mathfrak{gl}(m|n) \).

In Section 2, we review the fundamental definitions of classical Lie superalgebras and the constructions of the detecting subalgebras \( \mathfrak{f} \) and \( \mathfrak{e} \). We also indicate how detecting subalgebras are constructed for \( W(n) \) and \( S(n) \). In the following section, we prove that for Type I classical Lie superalgebras, the restriction map

\[ \text{res} : H^n(\mathfrak{g}, \mathfrak{g}_0, M) \rightarrow H^n(\mathfrak{f}, \mathfrak{f}_0, M) \]

is injective for all \( n \geq 0 \) and \( M \) in \( \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)} \). The same arguments may be used to verify cohomological embedding results for \( W(n) \) and \( S(n) \).

In Section 4, we show by example that (1.3.1) does not hold when \( \mathfrak{f} \) is replaced by \( \mathfrak{e} \). However, one can describe a relationship between support varieties of \( \mathfrak{f} \) and \( \mathfrak{e} \) by using an auxiliary sub Lie superalgebra \( \tilde{\mathfrak{f}} \). These cohomological embedding results are then applied to the theory of support varieties, and used to prove that the embeddings given in (1.2.1) are indeed isomorphisms of varieties. One consequence of this result is the concrete realization of the support variety \( V_{(\mathfrak{g}, \mathfrak{g}_0)}(M) \) as a quotient of a rank variety over \( \mathfrak{e}_1 \) by the finite (pseudo) reflection group \( W = W(\mathfrak{e}) \) (cf. Theorem 5.2.1(a)). Our results indicate the importance of the subalgebras \( \mathfrak{f} \) and \( \mathfrak{e} \) for the theory of classical Lie superalgebras. Finally, in Section 5, we apply our results to show that these support varieties can be viewed as support data as defined by Balmer [Bal]. We also indicate how the support theory fits into the classical combinatorial notion of atypicality as defined by Kac, Wakimoto and Serganova.

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2. Detecting subalgebras

2.1. Notation: We will use and summarize the conventions developed in [BKN1] [BKN2] [BKN3]. For more details we refer the reader to [BKN1] Section 2.

Throughout this paper, let \( \mathfrak{g} \) be a Lie superalgebra over the complex numbers \( \mathbb{C} \). In particular, \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) is a \( \mathbb{Z}_2 \)-graded vector space with a supercommutator \( \{ \cdot, \cdot \} : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \). A finite dimensional Lie superalgebra \( \mathfrak{g} \) is called classical if there is a connected reductive algebraic group \( G_0 \) such that \( \text{Lie}(G_0) = \mathfrak{g}_0 \), and the action of \( G_0 \) on \( \mathfrak{g}_1 \) differentiates to the adjoint action of \( \mathfrak{g}_0 \) on
Let $U(\mathfrak{g})$ be the universal enveloping superalgebra of $\mathfrak{g}$. We will be interested in supermodules which are $\mathbb{Z}_2$-graded left $U(\mathfrak{g})$-modules. If $M$ and $N$ are $\mathfrak{g}$-supermodules one can use the antipode and coproduct of $U(\mathfrak{g})$ to define a $\mathfrak{g}$-supermodule structure on the dual $M^*$ and the tensor product $M \otimes N$. For the remainder of the paper the term $\mathfrak{g}$-module will mean a $\mathfrak{g}$-supermodule. In order to apply homological algebra techniques, we will restrict ourselves to the underlying even category, consisting of $\mathfrak{g}$-modules with the degree preserving morphisms. In this paper we will study homological properties of the category $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$ which is the full subcategory of finite dimensional $\mathfrak{g}$-modules which are finitely semisimple over $\mathfrak{g}_0$ (a $\mathfrak{g}_0$-module is finitely semisimple if it decomposes into a direct sum of finite dimensional simple $\mathfrak{g}_0$-modules).

The category $\mathcal{F} := \mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$ has enough injective (and projective) modules. In fact, $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$ is a Frobenius category (i.e., where injectivity is equivalent to projectivity) \cite{BKN3}. Given $M, N$ in $\mathcal{F}$, let $\text{Ext}^d_{\mathcal{F}}(M, N)$ be the degree $d$ extensions between $N$ and $M$. In practice, there is a concrete realization for these extension groups via the relative Lie superalgebra cohomology for the pair $(\mathfrak{g}, \mathfrak{g}_0)$:

$$\text{Ext}^d_{\mathcal{F}}(M, N) \cong H^d(\mathfrak{g}, \mathfrak{g}_0; M^* \otimes N).$$

The relative cohomology can be computed using an explicit complex (cf. \cite{BKN1}, Section 2.3). Moreover, the cohomology ring

$$R := H^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) = S^\bullet(\mathfrak{g}_1^+)_{G_0}.$$

Since $G_0$ is reductive it follows that $R$ is finitely generated.

### 2.2 Classical Lie superalgebras:

In this section we review the construction of the two families of (cohomological) detecting subalgebras for classical Lie superalgebras defined in \cite{BKN1} Section 3.4 using the invariant theory of $G_0$ on $\mathfrak{g}_1$.

First we consider the case when $G_0$ has a stable action on $\mathfrak{g}_1$ (cf. \cite{BKN1} Section 3.2). That is, there is an open dense subset of $\mathfrak{g}_1$ consisting of semisimple points. Recall that a point $x \in \mathfrak{g}_1$ is called semisimple if the orbit $G_0 \cdot x$ is closed in $\mathfrak{g}_1$.

Let $x_0$ be a generic point in $\mathfrak{g}_1$; that is, $x_0$ is semisimple and regular, in the sense that its stabilizer has minimal dimension. Let $H = \text{Stab}_{G_0} x_0$ and $N := N_{G_0}(H)$. In order to construct a detecting subalgebra, we let $\mathfrak{f}_1 = \mathfrak{g}_1^H$, $\mathfrak{f}_0 = \text{Lie } N$, and set

$$\mathfrak{f} = \mathfrak{f}_0 \oplus \mathfrak{f}_1.$$

Then $\mathfrak{f}$ is a classical Lie superalgebra and a sub Lie superalgebra of $\mathfrak{g}$. The stability of the action of $G_0$ on $\mathfrak{g}_1$ implies the following properties.

1. **The restriction homomorphism $S(\mathfrak{g}_1^N) \to S(\mathfrak{f}_1^N)$ induces an isomorphism**

   $$\text{res} : H^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \to H^\bullet(\mathfrak{f}, \mathfrak{f}_0; \mathbb{C})_{N/N_0}.$$

   Here $N_0$ is the connected component of the identity in $N$.

2. **The set $G_0 \cdot \mathfrak{f}_1$ is dense in $\mathfrak{g}_1$.**

Next we recall the notion of polar action introduced by Dadok and Kac \cite{DK}. Let $v \in \mathfrak{g}_1$ be a semisimple element, and set

$$\mathfrak{c}_v = \{ x \in \mathfrak{g}_1 \mid \mathfrak{g}_0 \cdot x \subseteq \mathfrak{g}_0 \cdot v \},$$

where $\mathfrak{g}_0$ is the Lie algebra of $G_0$. The action of $G_0$ on $\mathfrak{g}_1$ is called polar if for some semisimple element $v \in \mathfrak{g}_1$ we have $\dim \mathfrak{c}_v = \dim S(\mathfrak{g}_1^+)_{G_0}$, where $\dim S(\mathfrak{g}_1^+)_{G_0}$ is the Krull dimension of this ring. The vector space $\mathfrak{c}_v$ is called a Cartan subspace; let $\mathfrak{c}_1$ denote a fixed choice of a Cartan subspace.

If the action of $G_0$ on $\mathfrak{g}_1$ is both stable and polar, we can further assume

$$x_0 \in \mathfrak{c}_{x_0} = \mathfrak{c}_1 \subseteq \mathfrak{f}_1,$$
where \( x_0 \) and \( f_1 \) are as above. Furthermore, the Cartan subspace is unique up to conjugation by \( G_0 \) (cf. \[DK\] Theorem 2.3). Set \( \mathfrak{e}_0 = \text{Lie}(H) \). Then the detecting subalgebra \( \mathfrak{e} \) is the classical Lie sub-superalgebra of \( \mathfrak{g} \) defined by:

\[
\mathfrak{e} = \mathfrak{e}_0 \oplus \mathfrak{e}_1.
\]

Assume \( \mathfrak{g} \) is a classical Lie superalgebra where the action of \( G_0 \) is both stable and polar on \( \mathfrak{g}_1 \). Then by \[BKN1\] Theorem 3.3.1 we have the following two facts.

1. The restriction homomorphism \( S(\mathfrak{g}_1^\ast) \to S(\mathfrak{e}_1^\ast) \) induces an isomorphism

\[
\text{res} : \mathbb{H}^\ast(\mathfrak{g}, \mathfrak{g}_0, \mathbb{C}) \to \mathbb{H}^\ast(\mathfrak{e}, \mathfrak{e}_0, \mathbb{C})^W,
\]

where \( W = W(\mathfrak{g}) \) is a finite pseudoreflection group. In particular, \( R \) is a polynomial algebra.

2. The set \( G_0 \cdot \mathfrak{e}_1 \) is dense in \( \mathfrak{g}_1 \).

For the Lie superalgebras \( \mathfrak{g} = W(n) \) and \( S(n) \) detecting families (analogous to the \( f \)'s) were also constructed using stable actions. We will describe a basis for these subalgebras below.

### 2.3. Type I Lie superalgebras:

A Lie superalgebra is said to be of Type I if it admits a \( \mathbb{Z} \)-grading \( \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+ \) concentrated in degrees \(-1, 0, \) and \( 1 \) with \( \mathfrak{g}_0 = \mathfrak{g}_0 \) and \( \mathfrak{g}_1 = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \). Otherwise, \( \mathfrak{g} \) is of Type II. Examples of Type I Lie superalgebras include: \( \mathfrak{gl}(m|n) \) and simple Lie superalgebras of types \( A(m, n), C(n) \) and \( P(n) \).

The simple modules for \( \mathfrak{g} \), a Type I classical Lie superalgebra, can be constructed in the following way. Let \( t \) be a Cartan subalgebra of \( \mathfrak{g}_0 \) and \( X^+_0 \subseteq t^\ast \) be the set of dominant integral weights (with respect to a fixed Borel subalgebra). For \( \lambda \in X^+_0 \), let \( L_0(\lambda) \) be the simple finite dimensional \( \mathfrak{g}_0 \)-module of highest weight \( \lambda \). Set

\[
\mathfrak{g}^+ = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \quad \text{and} \quad \mathfrak{g}^- = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}.
\]

Since \( \mathfrak{g} \) is a Type I Lie superalgebra \( \mathfrak{g}_{\pm 1} \) is an abelian ideal of \( \mathfrak{g}^\pm \). We can therefore view \( L_0(\lambda) \) as a simple \( \mathfrak{g}^\pm \)-module via inflation.

For each \( \lambda \in X^+_0 \), we construct the Kac module and the dual Kac module by using the tensor product and the Hom-space in the following way:

\[
K(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}^+)} L_0(\lambda) \quad \text{and} \quad K^-(\lambda) = \text{Hom}_{U(\mathfrak{g}^-)} (U(\mathfrak{g}), L_0(\lambda)).
\]

The module \( K(\lambda) \) has a unique maximal submodule. The head of \( K(\lambda) \) is the simple finite dimensional \( \mathfrak{g} \)-module \( L(\lambda) \). Then \( \{ L(\lambda) : \lambda \in X^+_0 \} \) is a complete set of non-isomorphic simple modules in \( F_{(\mathfrak{g}_0)} \). Let \( P(\lambda) \) (resp. \( I(\lambda) \)) denote the projective cover (resp. injective hull) in \( F_{(\mathfrak{g}_0)} \) for the simple \( \mathfrak{g} \)-module \( L(\lambda) \). These are all finite-dimensional. Moreover, the projective covers admit filtrations with sections being Kac modules and the injective hulls have filtrations whose sections are dual Kac modules. These filtrations also respect the ordering on weights and thus \( F_{(\mathfrak{g}_0)} \) is a highest weight category (cf. \[BKN3\] Section 3) as defined in \[CPS\].

### 2.4. General Linear Superalgebra:

The prototypical example of a Type I classical Lie superalgebra admitting both a stable and polar action of \( G_0 \) on \( \mathfrak{g}_1 \) is \( \mathfrak{g} = \mathfrak{gl}(m|n) \), which as a vector space is the set of \( m+n \) matrices. As basis one may take the matrix units \( E_{i,j} \) where \( 1 \leq i, j \leq m+n \). The degree zero component \( \mathfrak{g}_0 \) is the span of \( E_{i,j} \) where \( 1 \leq i, j \leq m \) or \( m+1 \leq i, j \leq m+n \). As a Lie algebra \( \mathfrak{g}_0 \cong \mathfrak{gl}(m) \times \mathfrak{gl}(n) \), and the corresponding reductive group is \( G_0 = \text{GL}(m) \times \text{GL}(n) \). Note that \( G_0 \) acts on \( \mathfrak{g}_1 \) via the adjoint representation. A basis for \( \mathfrak{g}_1 \) is given by the \( E_{i,j} \) such that \( m+1 \leq i \leq m+n \) and \( 1 \leq j \leq n \) or \( 1 \leq i \leq m \) and \( m+1 \leq j \leq m+n \).

Observe that \( \mathfrak{g}_1 = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1 \) where \( \mathfrak{g}_{-1} \) (resp. \( \mathfrak{g}_1 \)) consists of the lower triangular matrices (resp. upper triangular matrices) in \( \mathfrak{g}_1 \). The action of \( G_0 \) on \( \mathfrak{g}_{-1} \) is given by \((A, B).X = BXA^{-1}\) so the orbits are the matrices of a given rank in \( \mathfrak{g}_{-1} \). By results from elementary linear algebra, \( G_0 \cdot \mathfrak{f}_1 \) and \( G_0 \cdot \mathfrak{e}_1 \) are dense in \( \mathfrak{g}_1 \).

For simplicity of exposition, let us assume that \( m = n = r \). With an appropriate choice of \( x_0 \) the detecting subalgebras have the following descriptions. The detecting subalgebra \( \mathfrak{f} = f_0 \oplus f_1 \)
where \( f_1 \) is the span of \( \{ E_{i,i+r} : i = 1, 2, \ldots, r \} \cup \{ E_{i+r,i} : i = 1, 2, \ldots, r \} \) and \( f_0 \) is the span of \( \{ f_1, f_1^0 \} \). Here \( H \cong T^r \) where \( T \) is a one-dimensional torus, and \( N \cong W \times T^r \) where \( W = \Sigma_r \ltimes (\mathbb{Z}_2)^r \) (hyperoctahedral group). The detecting subalgebra \( \mathfrak{e} = e_0 \oplus e_1 \) where \( e_1 \) is the span of \( \{ E_{i,i+r} + E_{i+r,i} : i = 1, 2, \ldots, r \} \) and \( e_0 \) is the span of \( \{ e_1, e_1^0 \} \). Constructions of detecting subalgebras for other classical Lie superalgebras are explicitly described in [BKN1, Section 8].

### 2.5. The Witt algebra \( W(n) \) and \( S(n) \):

Let \( n \geq 2 \), and \( \Lambda^*(V) \) be the exterior algebra of the vector space \( V = \mathbb{C}^n \). The Lie superalgebra \( W(n) \) is the set of all superderivations of \( \Lambda^*(V) \), and the superalgebra structure is provided via the supercommutator bracket. The Lie superalgebra \( W(n) \) inherits a \( \mathbb{Z} \)-grading,

\[
\mathfrak{g} := W(n) = W(n)_{-1} \oplus W(n)_0 \oplus \cdots \oplus W(n)_{n-1},
\]

from \( \Lambda^*(V) \) by letting \( \mathfrak{g}_k := W(n)_k \) be the superderivations which increase the degree of a homogeneous element by \( k \). Furthermore, the \( \mathbb{Z}_2 \)-grading on \( W(n) \) is obtained from the \( \mathbb{Z} \)-grading by taking \( W(n)_0 = \oplus_k W(n)_2k \) and \( W(n)_1 = \oplus_k W(n)_{2k+1} \).

One can give an explicit basis for \( W(n) \) in the following way. We have

\[
W(n) \cong \Lambda(n) \otimes V^*.
\]

We can fix an ordered basis \( \{ \xi_1, \ldots, \xi_n \} \) for \( V \). For each ordered subset \( I = \{ i_1, \ldots, i_s \} \) of \( N = \{ 1, \ldots, n \} \) with \( 1 \leq i_1 < i_2 < \cdots < i_s \), let \( \xi_I = \xi_{i_1} \xi_{i_2} \cdots \xi_{i_s} \). The set of all such \( \xi_I \) forms a basis for \( \Lambda(n) \). If \( 1 \leq i \leq n \) let \( \partial_i \) be the element of \( W(n) \) given by partial differentiation: i.e., \( \partial_i(\xi_j) = \delta_{ij} \). Write \( \xi_I \partial_i \) for \( \xi_I \otimes \partial_i \). Then an explicit basis for \( W(n) \) is given by \( \xi_I \partial_i \) where \( I \) runs over all ordered subsets of \( N \) and \( i = 1, 2, \ldots, n \). Observe that \( W(n)_k \) is spanned by the basis elements \( \xi_I \partial_i \) with \( |I| = k + 1 \). In particular, \( \mathfrak{g}_0 \) is spanned by the elements \( \xi_i \partial_j \), from which one easily sees that \( \mathfrak{g}_0 = \mathfrak{gl}(n) \).

Let \( \mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)} \) be the category of \( \mathfrak{g} \)-supermodules which are completely reducible over \( \mathfrak{g}_0 \). In [BaKN Section 5.5], a detecting subalgebra was constructed as follows. There exists a generic point \( x_0 \in \mathfrak{g}_{-1} \oplus \mathfrak{gl} \) for the action of \( G_0 = GL(n) \). Set \( H = \text{Stab}_{G_0} x_0 = T_{n-1} \), an \( (n-1) \)-dimensional torus. The normalizer \( N = N_{G_0}(H) \) is \( \Sigma_{n-1} \ltimes T \) where \( \Sigma_{n-1} \) is the symmetric group of degree \( n-1 \) and \( T \) is the torus of \( G_0 \) consisting of diagonal matrices. Set \( f_1 = (\mathfrak{g}_{-1} \oplus \mathfrak{gl})^H \). Then \( f_1 \) is the span of the vectors \( \{ \partial_i, \xi_i \xi_j \partial_i : i = 2, 3, \ldots, n \} \). Set \( f_0 = \text{Lie} (N) \). Then \( f = f_0 \oplus f_1 \) detects cohomology (as in the classical stable case), that is the restriction map:

\[
\text{res} : H^\bullet(\mathfrak{g}, \mathfrak{g}_0, \mathbb{C}) \rightarrow H^\bullet(f, f_0, \mathbb{C})^N
\]

is an isomorphism.

In [BaKN] a support variety theory was developed for finite dimensional modules in \( \mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)} \) (and \( \mathcal{C}_{(f,f_0)} \)), it was shown that for such a module \( M \) there exists an injection of algebraic varieties:

\[
(\text{2.5.1}) \quad V_{(f,f_0)}(M)/\Sigma_{n-1} \hookrightarrow V_{(\mathfrak{g}, \mathfrak{g}_0)}(M).
\]

In Theorem [5.1.2] we will prove that this is indeed an isomorphism of varieties.

A similar development of cohomology and support varieties has been investigated by Bagci for the Lie superalgebra of Cartan type \( S(n) \). Detecting subalgebras analogous to \( f \) have been constructed. Details are omitted here and left to the interested reader (see [Ba]).

### 3. Cohomological Embedding

#### 3.1. The Stable Case:

The goal in this section is to prove that the relative \( (\mathfrak{g}, \mathfrak{g}_0) \) cohomology for \( M \in \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)} \) embeds in the relative cohomology for \( (f, f_0) \). Our first result uses a dimension shifting argument to reduce to looking at cohomology in degree one.

**Proposition 3.1.1.** Let \( \mathfrak{g} \) be a classical Lie superalgebra which is stable. Suppose that \( \text{res} : H^1(\mathfrak{g}, \mathfrak{g}_0, M) \rightarrow H^1(f, f_0, M) \) is an injective map for every \( M \in \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)} \). Then \( \text{res} : H^n(\mathfrak{g}, \mathfrak{g}_0, M) \rightarrow H^n(f, f_0, M) \) is an injective map for all \( n \geq 0 \) and \( M \in \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)} \).
Proof. For \( n = 0 \) the statement of the theorem is clear because \( H^0(g, g_0, M) = M^g \) (fixed points under \( g \)) and \( H^0(f, f_0, M) = M^f \). By assumption the result holds for \( n = 1 \). Now assume by induction the result holds for \( n < t \), and consider the short exact sequence
\[
0 \to M \to I \to \Omega^{-1}(M) \to 0
\]
where \( I \) is the injective hull of \( M \) in \( \mathcal{F}_n(g, g_0) \). Note that \( I \) is also injective in \( \mathcal{F}_n(f, f_0) \). Now applying the long exact sequence to the short exact sequence above and using these facts, we have the following commutative diagram:
\[
\begin{array}{ccc}
H^{t-1}(g, g_0, \Omega^{-1}(M)) & \xrightarrow{\text{res}} & H^{t-1}(f, f_0, \Omega^{-1}(M)) \\
\downarrow & & \downarrow \\
H^t(g, g_0, M) & \xrightarrow{\text{res}} & H^t(f, f_0, M)
\end{array}
\]
where the vertical maps are isomorphisms. The top horizontal map is injective by induction. Therefore, the bottom res map is also injective. \( \square \)

3.2. Let \( g = g_{-1} \oplus g_0 \oplus g_1 \) be a Type I Lie superalgebra. The detecting subalgebra \( f \) has a triangular decomposition which is compatible with the \( \mathbb{Z} \)-grading: \( f = f_{-1} \oplus f_0 \oplus f_1 \) with \( f_{\pm 1} = g_{H}^{\pm} \). Note that \( f^{\pm} = f_0 \oplus f_{\pm 1} \) and \( f^{\pm} = f \cap g^{\pm} \).

In order to analyze the question of embedding of cohomology of Type I Lie superalgebras we first investigate the case \((g^{\pm}, g_0)\) and \((f^{\pm}, f_0)\).

**Theorem 3.2.1.** Let \( g \) be a classical Type I Lie superalgebra which is stable. Then for all \( M \) in \( \mathcal{F}_{(g, g_0)} \) and \( n \neq 0 \), the restriction map
\[
H^n(g^{\pm}, g_0, M) \to H^n(f^{\pm}, f_0, M)
\]
is injective.

**Proof.** Without loss of generality we can consider the case \( g^{+} = g_0 \oplus g_1 \). Since \( g_1 \) is an ideal of \( g^{+} \), one has the Lyndon-Hochschild-Serre (LHS) spectral sequence for the pair \((g_1, \{0\})\) in \((g^{+}, g_0)\) (cf. proof of [BKN3, Theorem 3.3.1]):
\[
\tilde{E}^{i,j}_{2} = H^i(g_0, g_0, H^j(g_1, M)) \Rightarrow H^{i+j}(g^{+}, g_0, M).
\]
This spectral sequence collapses because modules are completely reducible over \( g_0 \) and yields:
\[
H^n(g^{+}, g_0, M) \cong H^n(g_1, M)^{g_0}
\]
for all \( n \geq 0 \). Similarly, for \( n \geq 0 \)
\[
H^n(f^{+}, f_0, M) \cong H^n(f_1, M)^{f_0}.
\]

Next observe that \( f_1 \) is an abelian Lie superideal in \( g_1 \). Consequently, we have another LHS spectral sequence:
\[
E^{i,j}_{2} = H^i(g_1/f_1, H^j(f_1, M)) \Rightarrow H^{i+j}(g_1, M).
\]
This gives rise to an exact sequence (i.e., the first three terms of the standard five term exact sequence):
\[
0 \to H^1(g_1/f_1, M^{f_1}) \to H^1(g_1, M) \to H^1(f_1, M)^{g_1/f_1} \to \ldots
\]
Under the restriction map we have
\[
\text{res} : H^1(g_1, M)^{g_0} \to H^1(f_1, M)^{f_0}.
\]
In order to prove the theorem it suffices by Proposition 3.1.1 (3.2.1), and (3.2.2) to demonstrate that the restriction map above is injective.
Consider Hom\(_{G_0}(g_1, M) \cong [(g_1)^* \otimes M]^{G_0}\) as a subspace of \((g_1)^* \otimes M\). The restriction of the original map \(\beta\) to this subspace \(\beta : \text{Hom}_{G_0}(g_1, M) \rightarrow [(f_1)^* \otimes M]^{f_0}\) is given by \(\beta(\psi) = \psi |_{f_1}\). Now \(\beta\) is an injective map. This can be seen as follows. If \(\psi |_{f_1} = 0\) then \(\psi(\cdot) = 0\). The fact that \(\psi\) is \(G_0\)-invariant shows that \(\psi(G_0 \cdot f_1)\). Finally using the density of \(G_0 \cdot f_1\) in \(g_1\) implies that \(\psi = 0\).

This means that \(\text{Im} \alpha \cap [(g_1)^* \otimes M]^{G_0} = 0\). The first map in (3.2.3) is induced by \(\alpha\), thus restricting the second map to \(H^1(g_1, M)^{G_0}\) yields an embedding \(H^1(g_1, M)^{G_0} \hookrightarrow H^1(f_1, M)^{g_1/f_1}\). Note that the representative cocycles in \(H^1(g_1, M)^{G_0}\) can be chosen to be in \([(g_1)^* \otimes M]^{G_0}\) because the fixed point functor \((-)^{G_0}\) is exact.

3.3. We now combine information from both sides of the triangular decomposition of \(g\) and \(f\) to prove that the relative cohomology for \((g, g_0)\) embeds in the relative cohomology for \((f, f_0)\).

**Theorem 3.3.1.** Let \(g\) be a classical Type I Lie superalgebra which is stable. Then for all \(M \in \mathcal{F}(g, g_0)\) and \(n \neq 0\) the restriction map

\[
H^n(g, g_0, M) \rightarrow H^n(f, f_0, M)
\]

is injective.

**Proof.** For an explicit definition of relative Lie superalgebra cohomology via a cochain complex we refer the reader to [BKN1, Section 2.3]. By Proposition 3.1.1 it suffices to verify the case \(n = 1\). Consider cochain differentials whose images are respectively used to define \(H^1(f^+, f_0, M), H^1(g^+, g_0, M), H^1(f, f_0, M), \) and \(H^1(g, g_0, M)\):

\[
d_{f^\pm} : M^{f_0} \rightarrow [(f_1)^* \otimes M]^{f_0};
\]

\[
d_{g^\pm} : M^{g_0} \rightarrow [(g_1)^* \otimes M]^{g_0};
\]

\[
d_f : M^{f_0} \rightarrow [(f_1)^* \otimes M]^{f_0};
\]

\[
d_g : M^{g_0} \rightarrow [(g_1)^* \otimes M]^{g_0}.
\]

In Theorem 3.2.1 we proved that the restriction map embeds \(H^1(g^+, g_0, M)\) into \(H^1(f^+, f_0, M)\). At the cochain level, we have the following commutative diagram:

\[
\begin{array}{ccc}
M^{g_0} & \longrightarrow & M^{f_0} \\
\downarrow d_{g^\pm} & & \downarrow d_{f^\pm} \\
[(g_1)^* \otimes M]^{g_0} & \rightarrow & [(f_1)^* \otimes M]^{f_0}
\end{array}
\]

where \(\sigma_{\pm}\) is the map obtained by restriction of functions. The embedding of \(H^1(g^+, g_0, M)\) into \(H^1(f^+, f_0, M)\) implies that

\[
\sigma_{\pm}^{-1}(\text{Im} d_{f^\pm}) = \text{Im} d_{g^\pm}
\]

or

\[
\text{Im} d_{f^\pm} = \sigma_{\pm}(\text{Im} d_{g^\pm}).
\]

Next note that \(\text{Im} d_{f^\pm} \cong M^{f_0}/\text{Ker} d_{f^\pm}\) and \(\text{Im} d_{g^\pm} \cong M^{g_0}/\text{Ker} d_{g^\pm}\). Since \(G_0 \cdot f_1\) is dense in \(g_1\) the map obtained by restriction of functions

\[
\sigma : [(g_1)^* \otimes M]^{g_0} \hookrightarrow [(f_1)^* \otimes M]^{f_0}
\]

is injective. In order to prove that the induced map in cohomology from \(H^1(g, g_0, M) \rightarrow H^1(f, f_0, M)\) is injective, it suffices (using reasoning similar to that above) to prove that \(\sigma(\text{Im} d_g) = \text{Im} d_f\).
From the definition of the differential note that
\[(3.3.4) \quad K_\delta := \text{Ker } d_{\delta} = \text{Ker } d_{\delta}^- \cap \text{Ker } d_{\delta}^+,
\]
\[(3.3.5) \quad K_\bar{\delta} := \text{Ker } \bar{d}_{\delta} = \text{Ker } d_{\delta}^- \cap \text{Ker } d_{\delta}^+.
\]
Because \(\text{Ker } d_{\delta}^\pm = M^{\delta^\pm}\) and \(\text{Ker } \bar{d}_{\delta}^\pm = M^{\bar{\delta}^\pm}\). We have the following commutative diagram
\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Ker } d_{\delta}^-/K_\delta & \longrightarrow & M^{\delta^-}/K_\delta & \longrightarrow & M^{\delta^-}/\text{Ker } d_{\delta}^+ & \longrightarrow & 0 \\
\sigma & & \downarrow \sigma & & \downarrow \sigma & & \downarrow \\
0 & \longrightarrow & \text{Ker } d_{\bar{\delta}}^-/K_{\bar{\delta}} & \longrightarrow & M^{\bar{\delta}^-}/K_{\bar{\delta}} & \longrightarrow & M^{\bar{\delta}^-}/\text{Ker } d_{\bar{\delta}}^+ & \longrightarrow & 0
\end{array}
\]
\[(3.3.6) \quad (\sigma \downarrow \sigma \downarrow \sigma).
\]
From our analysis above the map \(\sigma_+\) is an isomorphism. Therefore, in order to prove the theorem it suffices to show that
\[
\sigma : \text{Ker } d_{\delta}^+/K_\delta \rightarrow \text{Ker } d_{\bar{\delta}}^+/K_{\bar{\delta}}
\]
is an isomorphism because the five lemma would imply that the map in the middle is an isomorphism. This would show that \(\sigma(\text{Im } d_{\delta}) = \text{Im } d_{\delta}^-\).

By the second isomorphism theorem we have the following isomorphisms:
\[(3.3.7) \quad \text{Ker } d_{\delta}^+/K_{\bar{\delta}} \cong (\text{Ker } d_{\delta}^+ + \text{Ker } d_{\delta}^-)/\text{Ker } d_{\delta}^-,
\]
\[(3.3.8) \quad \text{Ker } d_{\bar{\delta}}^+/K_{\bar{\delta}} \cong (\text{Ker } d_{\bar{\delta}}^+ + \text{Ker } d_{\bar{\delta}}^-)/\text{Ker } d_{\bar{\delta}}^-.
\]
Now observe that from the relationship between \((\mathfrak{g}^-, \mathfrak{g}_0)\) and \(((\mathfrak{f}^-, \mathfrak{f}_0)\), we have a commutative diagram
\[
\begin{array}{ccccccc}
(\text{Ker } d_{\delta}^+ + \text{Ker } d_{\delta}^-)/\text{Ker } d_{\delta}^- & \longrightarrow & M^{\delta^-}/\text{Ker } d_{\delta}^+ & \longrightarrow & 0 \\
\sigma_- & & \downarrow \sigma_- & & \downarrow \\
(\text{Ker } d_{\bar{\delta}}^+ + \text{Ker } d_{\bar{\delta}}^-)/\text{Ker } d_{\bar{\delta}}^- & \longrightarrow & M^{\bar{\delta}^-}/\text{Ker } d_{\bar{\delta}}^+ & \longrightarrow & 0
\end{array}
\]
\[(3.3.9) \quad (\sigma_- \downarrow \sigma_- \downarrow \downarrow).
\]
The horizontal maps are embeddings and the rightmost vertical map is an isomorphism (using the fact that \(\text{Im } d_{\delta}^- = \sigma_-(\text{Im } d_{\delta}^-))\). Therefore, the map
\[
\sigma_- : (\text{Ker } d_{\delta}^+ + \text{Ker } d_{\delta}^-)/\text{Ker } d_{\delta}^- \rightarrow (\text{Ker } d_{\delta}^+ + \text{Ker } d_{\delta}^-)/\text{Ker } d_{\delta}^-
\]
is injective. In order to finish the proof we need to show that \(\sigma_-\) is surjective.

Suppose that \(y + \text{Ker } d_{\delta}^- \in (\text{Ker } d_{\delta}^+ + \text{Ker } d_{\delta}^-)/\text{Ker } d_{\delta}^-\) with \(y = y_- + y_+\) where \(y_\pm \in \text{Ker } d_{\delta}^\pm\). From the isomorphism given by \(\sigma_-\) above we have \(g(y_\pm + \text{Ker } d_{\delta}^-) = y_\pm + \text{Ker } d_{\delta}^-\) for all \(g \in G_0\). Moreover, \(d_{\delta}^- (y_\pm) = 0\) so in particular \(f_1.y_+ = 0\) and \(f_- . y_- = 0\). Now \(0 = g.(f.y_\pm) = (g.f)(g^{-1}.y_\pm) = (g.f).y_\pm\) for all \(g \in G_0\) and \(f \in f_{\pm 1}\). Since \(G_0 \cdot f_{\pm 1}\) is dense in \(g_{\pm 1}\) it follows that \(x.y_\pm = 0\) for all \(x \in g_{\pm 1}\), thus \(y_\pm \in \text{Ker } d_{\delta}^\pm\).

\[\square\]

3.4. Let \(\mathfrak{h}\) be a classical Lie subsuperalgebra of \(\mathfrak{g}\) with the property that
\[(3.4.1) \quad \text{res : } H^n(\mathfrak{g}, \mathfrak{g}_0, \mathcal{M}) \leftrightarrow H^n(\mathfrak{h}, \mathfrak{h}_0, \mathcal{M})
\]
is an injective map for all \(n \geq 0\) and \(\mathcal{M} \in \mathcal{F}(\mathfrak{g}_0)\).

Let \(\mathcal{M}\) be a module in \(\mathcal{F}(\mathfrak{g}_0)\) which is projective when considered as a module in \(\mathcal{F}(\mathfrak{h}, \mathfrak{h}_0)\). By \((3.4.1)\), we have
\[
\text{res : } H^1(\mathfrak{g}, \mathfrak{g}_0, \mathcal{M} \otimes S^*) \rightarrow H^1(\mathfrak{h}, \mathfrak{h}_0, \mathcal{M} \otimes S^*)
\]
for all simple modules \(\mathcal{S}\) in \(\mathcal{F}(\mathfrak{g}_0)\). By the projectivity of \(\mathcal{M}\) in \(\mathcal{F}(\mathfrak{h}, \mathfrak{h}_0)\), we further have \(H^1(\mathfrak{h}, \mathfrak{h}_0, \mathcal{M} \otimes \mathcal{S}^*) = 0\). Consequently,
\[
0 = H^1(\mathfrak{g}, \mathfrak{g}_0, \mathcal{M} \otimes S^*) \cong \text{Ext}^1_{\mathcal{F}(\mathfrak{g}_0)}(\mathcal{S}, \mathcal{M})
\]
for all simple modules \(\mathcal{S}\) in \(\mathcal{F}(\mathfrak{g}_0)\), whence \(\mathcal{M}\) is projective in \(\mathcal{F}(\mathfrak{g}_0)\). For Type I classical Lie superalgebras when \(\mathfrak{h} = \mathfrak{f}\) this fact about projectivity was earlier deduced using geometric methods.
involving support varieties (cf. [BKN3, Theorem 3.5.1, Theorem 3.7.1]). Theorem 3.3.1 can be viewed as a strong generalization of this projectivity result between modules in \( \mathcal{F}(g_0, f_0) \) and \( \mathcal{F}(f, i_0) \).

3.5. **The Polar Case:** In this section we will consider the example when \( g = g(1\mid 1) \) and demonstrate that the analogue Theorem 3.3.1 does not hold when the detecting subalgebra \( f \) is replaced by \( e \). The simple modules in the principal block of \( \mathcal{F}(g_0, f_0) \) are labelled by \( L(\lambda) \) where \( \lambda \) denotes the highest weight \( (\lambda - \lambda) \). Consider the two-dimensional dual Kac module \( K^-(\lambda) \). The module \( K^-(\lambda) \) remains indecomposable when restricted to \( e \), and the only 2-dimensional indecomposable \( e \)-modules are the projective indecomposable modules in the category \( \mathcal{F}(e, e_0) \). Consequently, \( K^-(\lambda) \) is a projective module in \( \mathcal{F}(e, e_0) \), thus \( \text{H}^n(e, e_0, K^-(\lambda)) = 0 \) for \( n > 0 \). It is well known that the Kac modules in the principal block are not projective in \( \mathcal{F}(g_0, f_0) \), thus by the argument in Section 3.4, the property (3.4.1) cannot hold for \( h = e \).

We also see that the restriction map is not injective by direct computation. By Frobenius reciprocity

\[
\text{H}^n(g, g_0, K^-(\lambda)) \cong \text{H}^n(g^-, g_0, \lambda) \cong [\text{H}^n(g^-, g_0) \otimes \lambda]^G_0 \cong [S^n((g^-)^\ast) \otimes \lambda]^G_0.
\]

In this instance \( G_0 \) a torus (specifically, the set of invertible diagonal matrices). The subspace \( g^- \) is one-dimensional and is spanned by a weight vector having weight \((-1\mid 1)\). Consequently,

\[
\text{H}^n(g, g_0, K^-(\lambda)) \cong \begin{cases} \mathbb{C} & \lambda = -n \\ 0 & \text{otherwise}. \end{cases}
\]

In particular, we have \( \text{H}^1(g, g_0, K^-(1\mid 1)) \cong \mathbb{C} \), and \( \text{H}^1(e, e_0, K^-(1\mid 1)) = 0 \). Therefore,

\[
\text{res} : \text{H}^1(g, g_0, K^-(1\mid 1)) \to \text{H}^1(e, e_0, K^-(1\mid 1))
\]

is not an injective map. However,

\[
\text{res} : \text{H}^n(g, g_0, K^-(1\mid 1)) \to \text{H}^n(e, e_0, K^-(1\mid 1))
\]

is an injective map for \( n \geq 2 \).

It was shown in [BKN1] that when \( \dim e_1 = 1 \) the restriction map is injective for \( n \) sufficiently large. An interesting problem would be to determine whether this occurs for arbitrary \( e \).

3.6. **Types \( W(n) \) and \( S(n) \):** The techniques used to prove Theorem 3.2.1 and 3.3.1 can be used with the triangular decomposition given by the \( \mathbb{Z} \)-grading for \( W(n) \) and \( S(n) \) to prove the following detection theorem. Let \( f \) be as in Section 2.5.

**Theorem 3.6.1.** Let \( g = W(n) \) or \( S(n) \). Then for all \( M \) in \( C(g_0, f_0) \) and \( n \neq 0 \). The restriction map

\[
\text{H}^n(g, g_0, M) \to \text{H}^n(f, f_0, M).
\]

is injective.

4. **Support Varieties**

4.1. We first recall the definition of the support variety of a finite dimensional \( g \)-supermodule \( M \) (cf. [BKN1 Section 6.1]). Let \( g \) be a classical Lie superalgebra, \( R := \text{H}^\ast(g, g_0; \mathbb{C}) \), and \( M_1, M_2 \) be \( F := \mathcal{F}(g_0, f_0) \). According to [BKN1] Theorem 2.5.3, \( \text{Ext}_F^\ast(M_1, M_2) \) is a finitely generated \( R \)-module. Set \( J_{(g, f_0)}(M_1, M_2) = \text{Ann}_R(\text{Ext}_F^\ast(M_1, M_2)) \) (i.e., the annihilator ideal of this module). The **relative support variety of the pair** \( (M, N) \) is

\[
(4.1.1) \quad \mathcal{V}_{(g, f_0)}(M, N) = \text{MaxSpec}(R/J_{(g, f_0)}(M, N))
\]

In the case when \( M = M_1 = M_2 \), set \( J_{(g, f_0)}(M) = J_{(g, f_0)}(M, M) \), and

\[
\mathcal{V}_{(g, f_0)}(M) := \mathcal{V}_{(g, f_0)}(M, M).
\]
The variety $\mathcal{V}(\mathfrak{g}, \mathfrak{g}_0)(M)$ is called the support variety of $M$. In this situation, $J_{(\mathfrak{g}, \mathfrak{g}_0)}(M) = \text{Ann}_R \text{Id}$ where $\text{Id}$ is the identity morphism in $\text{Ext}^1_R(M, M)$.

4.2. We will now compare support varieties for the classical Lie superalgebras $\mathfrak{g}$, $\mathfrak{f}$, and $\mathfrak{e}$. Assume that $\mathfrak{g}$ is both stable and polar. Without the assumption that $\mathfrak{g}$ is polar, the statements concerning cohomology and support varieties for $\mathfrak{g}$ and $\mathfrak{f}$ remain true.

First there are natural maps of rings given by restriction:

$$\text{res} : H^*(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \rightarrow H^*(\mathfrak{f}, \mathfrak{f}_0; \mathbb{C}) \rightarrow H^*(\mathfrak{e}, \mathfrak{e}_0; \mathbb{C}).$$

which induce isomorphisms

$$\text{res} : H^*(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \rightarrow H^*(\mathfrak{f}, \mathfrak{f}_0; \mathbb{C})^{N_0/N_0} \rightarrow H^*(\mathfrak{e}, \mathfrak{e}_0; \mathbb{C})^W.$$

The map on cohomology above induces morphisms of varieties:

$$\text{res}^* : \mathcal{V}(\mathfrak{e}, \mathfrak{e}_0)(\mathbb{C}) \rightarrow \mathcal{V}(\mathfrak{f}, \mathfrak{f}_0)(\mathbb{C}) \rightarrow \mathcal{V}(\mathfrak{g}, \mathfrak{g}_0)(\mathbb{C})$$

and isomorphisms (by passing to quotient spaces)

$$\text{res}^* : \mathcal{V}(\mathfrak{e}, \mathfrak{e}_0)(\mathbb{C})/W \rightarrow \mathcal{V}(\mathfrak{f}, \mathfrak{f}_0)(\mathbb{C})/(N_0/N_0) \rightarrow \mathcal{V}(\mathfrak{g}, \mathfrak{g}_0)(\mathbb{C}).$$

Let $M$ be a finite dimensional $\mathfrak{g}$-module. Then $\text{res}^*$ induces maps between support varieties:

$$\mathcal{V}(\mathfrak{e}, \mathfrak{e}_0)(M) \rightarrow \mathcal{V}(\mathfrak{f}, \mathfrak{f}_0)(M) \rightarrow \mathcal{V}(\mathfrak{g}, \mathfrak{g}_0)(M).$$

Since $M$ is $\mathfrak{g}_0$-module, the first two varieties are stable under the action of $W$ and $N_0/N_0$, respectively. Consequently, we obtain the following induced maps of varieties:

$$\mathcal{V}(\mathfrak{e}, \mathfrak{e}_0)(M)/W \hookrightarrow \mathcal{V}(\mathfrak{f}, \mathfrak{f}_0)(M)/(N_0/N_0) \rightarrow \mathcal{V}(\mathfrak{g}, \mathfrak{g}_0)(M).$$

These maps are embeddings because if $x \in R$ annihilates the identity in $H^0(\mathfrak{g}, \mathfrak{g}_0, M^* \otimes M)$ then it must annihilate the identity elements in $H^0(\mathfrak{f}, \mathfrak{f}_0, M^* \otimes M)$ and $H^0(\mathfrak{e}, \mathfrak{e}_0, M^* \otimes M)$.

4.3. The Intermediate Subalgebra $\overline{\mathfrak{f}}$: We next define an intermediate subalgebra between $\mathfrak{e}$ and $\mathfrak{f}$ which will be useful for our purposes. Let $\overline{\mathfrak{f}}$ be defined as follows. Given $\mathfrak{f}$, let $\overline{\mathfrak{f}} := \mathfrak{f}_0 \oplus \mathfrak{f}_1$. From the proof of [BKN1] Theorem 4.1, we know that $\overline{\mathfrak{f}}$ is a Lie subsuperalgebra of $\mathfrak{f}$ and contains $\mathfrak{e}$ (in the case that we have a polar action). Moreover, $[\overline{\mathfrak{f}}_0, \overline{\mathfrak{f}}_1] = 0$. This implies that we have a rank variety description for $\mathcal{V}(\overline{\mathfrak{f}}, \overline{\mathfrak{f}}_0)(M)$ when $M \in \mathcal{F}(\overline{\mathfrak{f}}, \overline{\mathfrak{f}}_0)$.

The rank variety of $M$ is

$$\mathcal{V}^{\text{rank}}(M) = \{ x \in \overline{\mathfrak{f}}_1 \mid M \text{ is not projective as } U(\langle x \rangle)-\text{module} \} \cup \{0\}.$$

It was shown in [BKN1] Theorem 6.3.2 that there is an isomorphism

$$\mathcal{V}(\overline{\mathfrak{f}}, \overline{\mathfrak{f}}_0)(M) \cong \mathcal{V}^{\text{rank}}(M).$$

4.4. Comparing support varieties over $\mathfrak{f}$ and $\overline{\mathfrak{f}}$: In this section we compare support varieties for modules over $\overline{\mathfrak{f}}$ and $\mathfrak{f}$. Let $M \in \mathcal{F}(\mathfrak{f}, \mathfrak{f}_0)$. There is a map of varieties induced by the restriction map in cohomology:

$$\text{res}^* : \mathcal{V}(\overline{\mathfrak{f}}, \overline{\mathfrak{f}}_0)(M) \rightarrow \mathcal{V}(\mathfrak{f}, \mathfrak{f}_0)(M).$$

Using the fact that

$$H^*(\mathfrak{f}, \mathfrak{f}_0, \mathbb{C}) \cong \text{H}^*(\overline{\mathfrak{f}}, \overline{\mathfrak{f}}_0, \mathbb{C})^{N_0/H},$$

it is clear that the above map is the restriction of the orbit map:

$$\mathcal{V}(\overline{\mathfrak{f}}, \overline{\mathfrak{f}}_0)(\mathbb{C}) \twoheadrightarrow \mathcal{V}(\overline{\mathfrak{f}}, \overline{\mathfrak{f}}_0)(\mathbb{C})/(N_0/H) \twoheadrightarrow \mathcal{V}(\mathfrak{f}, \mathfrak{f}_0)(\mathbb{C}).$$

The following theorem demonstrates that support varieties in this context are natural with respect to taking quotients.
Therefore, res exists. Since the restriction map is injective, 0 = res(0). From this isomorphism, one sees that for all Q in \( F(\mathfrak{f}, \mathfrak{h}_0) \),

\[
\text{res} : H^* \otimes M \cong H^* \otimes M \subseteq H^* \otimes M.
\]

The ideal \( \text{res}^{-1}(J(\mathfrak{f}, \mathfrak{h}_0)(M)) \) defines the variety \( \text{res}^*(\mathcal{V}(\mathfrak{f}, \mathfrak{h}_0)(M)) \). We need to prove that

\[
\text{res}^{-1}(J(\mathfrak{f}, \mathfrak{h}_0)(M)) = J(\mathfrak{f}, \mathfrak{h}_0)(M).
\]

If \( x \in J(\mathfrak{f}, \mathfrak{h}_0)(M) \), then

\[
0 = \text{res}(x \cdot \text{id}_{\mathfrak{f}, \mathfrak{h}_0}) = \text{res}(x) \cdot \text{res} \cdot \text{id}_{\mathfrak{f}, \mathfrak{h}_0}.
\]

Therefore, \( \text{res}(x) \in J(\mathfrak{f}, \mathfrak{h}_0)(M) \), and so \( x \in \text{res}^{-1}(J(\mathfrak{f}, \mathfrak{h}_0)(M)) \). Conversely, if \( x \in \text{res}^{-1}(J(\mathfrak{f}, \mathfrak{h}_0)(M)) \), then

\[
0 = \text{res}(x \cdot \text{id}_{\mathfrak{f}, \mathfrak{h}_0}) = \text{res}(x) \cdot \text{res} \cdot \text{id}_{\mathfrak{f}, \mathfrak{h}_0}.
\]

Since the restriction map is injective, 0 = \( x \cdot \text{id}_{\mathfrak{f}, \mathfrak{h}_0} \) and so \( x \in J(\mathfrak{f}, \mathfrak{h}_0)(M) \).

Using the identification of \( \mathcal{V}(\mathfrak{f}, \mathfrak{h}_0)(M) \) as a rank variety, Theorem 4.4.1 provides a concrete realization of \( \mathcal{V}(\mathfrak{f}, \mathfrak{h}_0)(M) \). Since the tensor product theorem holds for rank varieties it follows that it also holds for support varieties in this context. The proof of [BaKN, Theorem 6.5.1] adapted to our setting yields the following result.

Corollary 4.4.1. Let \( M, N \) be modules in \( F(\mathfrak{f}, \mathfrak{h}_0) \). Then

(a) \( \mathcal{V}(\mathfrak{f}, \mathfrak{h}_0)(M) \cong \mathcal{V}^{\text{rank}}(M)/(N_0/H), \)

(b) \( \mathcal{V}(\mathfrak{f}, \mathfrak{h}_0)(M \otimes N) = \mathcal{V}(\mathfrak{f}, \mathfrak{h}_0)(M) \cap \mathcal{V}(\mathfrak{f}, \mathfrak{h}_0)(N) \).

4.5. Comparing support varieties over \( \mathfrak{f} \) and \( \mathfrak{c} \): In this section we compare the varieties for the detecting subalgebras \( \mathfrak{f} \) and \( \mathfrak{c} \). In general \( \mathfrak{f} \) does not have a simple rank variety description. This necessitates the use of the auxiliary algebra \( \mathfrak{f} \) to make the transition between \( \mathfrak{f} \) and \( \mathfrak{c} \).

Theorem 4.5.1. Let \( \mathfrak{g} \) be a classical Lie superalgebra which is stable and polar. If \( M \in \mathcal{F}(\mathfrak{f}, \mathfrak{h}_0) \) then we have the following isomorphism of varieties:

\[
\text{res}^* : \mathcal{V}(\mathfrak{c}, \mathfrak{h}_0)(M)/W \to \mathcal{V}(\mathfrak{f}, \mathfrak{h}_0)(M)/(N/N_0).
\]
Proof. Let \( M \in \mathcal{F}_{(f_0)} \). We have the following commutative diagram of varieties:

\[
\begin{array}{c}
\mathcal{V}_{(e,t_0)}(M) \xrightarrow{\text{res}^*} \mathcal{V}_{(\tilde{f_0})}(M) \xrightarrow{\text{res}^*} \mathcal{V}_{(f_0)}(M) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\mathcal{V}^{\text{rank}}_e(M) \xrightarrow{\beta} \mathcal{V}^{\text{rank}}_f(M)/(N_0/H) \\
\downarrow \quad \downarrow \\
\mathcal{V}^{\text{rank}}_f(C) \xrightarrow{\alpha} \mathcal{V}^{\text{rank}}_f(C)/(N_0/H)
\end{array}
\]

It suffices to show that the composition of the top (horizontal) maps

\[ \text{res}^*: \mathcal{V}_{(e,t_0)}(M) \to \mathcal{V}_{(f_0)}(M) \]

is surjective.

Let \( \sigma \) denote the middle (horizontal) composition of maps from \( \mathcal{V}^{\text{rank}}_e(M) \) to \( \mathcal{V}^{\text{rank}}_f(M)/(N_0/H) \). Since the first row of vertical maps are all isomorphisms, it suffices to prove that \( \sigma \) is surjective. Take \( y \in \mathcal{V}^{\text{rank}}_f(M)/(N_0/H) \). There exists \( x \in \mathcal{V}^{\text{rank}}_e(M) \) with \( \beta(x) = y \), and by naturality of rank varieties, we may consider \( x \) to be an element of \( \mathcal{V}^{\text{rank}}_f(C) \). Since the composition \( \beta \circ \alpha \) of the two lowest horizontal arrows is an isomorphism, there is an element \( z \in \mathcal{V}^{\text{rank}}_f(C) \) such that \( \beta(x) = \beta(z) \), where, since \( \alpha \) is an inclusion, we identify \( z \) as an element of \( \mathcal{V}^{\text{rank}}_f(C) \). But then \( z = gx \) for some \( g \in N_0/H \), and it follows that \( z \in \mathcal{V}^{\text{rank}}_e(M) \), and clearly \( \sigma(z) = y \). \( \square \)

4.6. In the case of \( \mathfrak{g} = W(n) \) or \( S(n) \) there is an auxiliary subalgebra \( \tilde{f} \) which is analogous to \( \tilde{f} \). In this setting we have the following result (cf. \[ \text{BaKN} \] Theorem 6.4.1, (6.3.4)).

**Theorem 4.6.1.** Let \( \mathfrak{g} = W(n) \) or \( S(n) \) and let \( M \) be a finite dimensional object in \( \mathcal{C}_{(f_0)} \). Then there exists a torus \( T \) such that

(a) \( \mathcal{V}_{(\tilde{f_0})}(M)/T \cong \text{res}^*(\mathcal{V}_{(\tilde{f_0})}(M)) = \mathcal{V}_{(f_0)}(M) \).

(b) \( \mathcal{V}_{(f_0)}(M)/T \cong \mathcal{V}_{f}^{\text{rank}}(M) \).

5. Applications

5.1. Isomorphism Theorems. In \[ \text{BKN1, BKN2} \] there was convincing theoretical and computational evidence of direct relationships between the support varieties for a classical Lie superalgebra \( \mathfrak{g} \) and those of its detecting subalgebras. The cohomological embedding theorem provided in Section 3 enables us to provide such a relation, which we now proceed to do.

**Theorem 5.1.1.** Let \( \mathfrak{g} \) be a classical Type I Lie superalgebra with \( \tilde{f} \) etc. as above, and let \( M \) be in the module category \( \mathcal{F}_{(\tilde{g},\tilde{f}_0)} \).

(a) If \( \mathfrak{g} \) is stable then the map on support varieties

\[ \text{res}^*: \mathcal{V}_{(\tilde{f}_0)}(M)/(N/N_0) \to \mathcal{V}_{(\tilde{g},\tilde{f}_0)}(M) \]

is an isomorphism.

(b) If \( \mathfrak{g} \) is stable and polar then the maps on support varieties

\[ \text{res}^*: \mathcal{V}_{(e,t_0)}(M)/W(\mathfrak{c}) \to \mathcal{V}_{(f_0)}(M)/(N/N_0) \to \mathcal{V}_{(\tilde{g},\tilde{f}_0)}(M) \]

are isomorphisms, where \( W = W(\mathfrak{c}) \) is the pseudoreflection group of (2.2.3).
Theorem 5.1.2. Let \( \mathfrak{g} \) be \( W(n) \) or \( S(n) \) and let \( M \) be a finite dimensional module in \( \mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)} \). The induced map of support varieties

\[
\text{res}^*: \mathcal{V}_{(\mathfrak{f}, \mathfrak{f}_0)}(M)/W \rightarrow \mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(M)
\]

is an isomorphism where \( W = \Sigma_{n-1} \) (resp. \( \Sigma_{n-2} \)) for \( W(n) \) (resp. \( S(n) \)).

Proof. This follows by using the analogue of Theorem 5.1.1 for \( \mathfrak{g} = W(n) \) or \( S(n) \) and applying the argument given above in the proof of Theorem 5.1.1(a).

\[ \square \]

5.2. Realizability and the Tensor Product Theorem: Theorem 5.1.1 allows us to provide a concrete realization for the variety \( \mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(M) \) when \( M \in \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)} \). The next theorem provides a rank variety description of \( \mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(M) \) when \( M \in \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)} \).

Theorem 5.2.1. Let \( \mathfrak{g} \) be a Type I classical Lie superalgebra which is both stable and polar, and let \( M_1, M_2 \) and \( M \) be in \( \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)} \). Then, writing \( W(\varepsilon) \) for the pseudoreflection group associated with \( \varepsilon \),

(a) \( \mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(M) \cong \mathcal{V}_{(\varepsilon, \varepsilon_0)}^{\text{rank}}(M)/W(\varepsilon) \);
(b) \( \mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(M_1 \otimes M_2) = \mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(M_1) \cap \mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(M_2) \).
(c) Let \( X \) be a conical subvariety of \( \mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(\mathbb{C}) \). Then there exists \( L \) in \( \mathcal{F} \) with \( X = \mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(L) \).
(d) If \( M \) is indecomposable then \( \text{Proj}(\mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(M)) \) is connected.

Proof. (a) This statement follows from the realization of \( \mathcal{V}_{(\varepsilon, \varepsilon_0)}(M) \) as a rank variety and Theorem 5.1.1(b). (b) This follows by using part (a) and the fact that the tensor product property for support varieties holds for \( \mathcal{V}_{(\varepsilon, \varepsilon_0)}^{\text{rank}}(-) \). Parts (c) and (d) are proved using the same arguments as those for support varieties of finite groups (cf. Car1, Car2).

We remark that since the stable module category of \( \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)} \) is a symmetric monoidal tensor category, one can consider the spectrum “\( \text{Spc(Stab} \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)}) \)” as in Bal. Our results show that the assignment \((-) \rightarrow \mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(-) \) satisfies the properties stated in Balmer’s paper for support datum.

Using Theorem 4.6.1 and the same arguments as in Theorem 5.2.1 we can similarly realize the support varieties for the Cartan type Lie superalgebras \( W(n) \) and \( S(n) \), and prove that they satisfy the tensor product property.

Theorem 5.2.2. Let \( \mathfrak{g} = W(n) \) or \( S(n) \) and let \( M_1, M_2, M \) be finite dimensional modules in \( \mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)} \).

(a) \( \mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(M) \cong \mathcal{V}_{(\mathfrak{f}, \mathfrak{f}_0)}^{\text{rank}}(M)/[W \ltimes T] \);
(b) \( \mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(M_1 \otimes M_2) = \mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(M_1) \cap \mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(M_2) \).
(c) Let \( X \) be a conical subvariety of \( \mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(\mathbb{C}) \). Then there exists \( L \) in \( \mathcal{F} \) with \( X = \mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(L) \).
(d) If \( M \) is indecomposable then \( \text{Proj}(\mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(M)) \) is connected.
5.3. Connections with Atypicality. Let $g$ be a basic classical Lie superalgebra with a non-degenerate invariant supersymmetric even bilinear form $(-,-)$. For a weight $\lambda \in t^*$, the *atypicality* of $\lambda$ is the maximal number of linearly independent, mutually orthogonal, positive isotropic roots $\alpha \in \Phi^+$ such that $(\lambda + \rho, \alpha) = 0$ where $\rho = \frac{1}{2}(\sum_{\alpha \in \Phi^+} \alpha - \sum_{\alpha \in \Phi^-} \alpha)$.

For a simple finite dimensional $g$-supermodule $L(\lambda)$ with highest weight $\lambda$, the atypicality of $L(\lambda)$ is defined to be $\text{atyp}(L(\lambda)) := \dim V_{(g,0)}(L(\lambda))$. In [BKN1] Conjecture 7.2.1, a conjecture was stated relating the atypicality of $L(\lambda)$ to the dimension of $V_{(\ell,\tau_0)}(L(\lambda))$. This conjecture was verified for $g = \mathfrak{gl}(m|n)$ in [BKN2]. In light of the results of the previous section, it seems reasonable to modify the “Atypicality Conjecture” to a statement which does not involve detecting subalgebras.

**Conjecture 5.3.1.** Let $g$ be a simple basic classical Lie superalgebra and let $L(\lambda)$ be a finite dimensional simple $g$-supermodule. Then

$$\text{atyp}(L(\lambda)) = \dim V_{(g,0)}(L(\lambda)).$$

In [BKN2] this modified version of the conjecture has been also verified, and the support varieties for the simple modules have been completely described. This leads one to believe that the atypicality for any classical Lie superalgebra $g$ should be defined for all modules $M$ in $\mathcal{F}_{(g,0)}$ (in a functorial way) as the dimension of $V_{(g,0)}(M)$.

For the Lie superalgebras of Cartan type, Serganova [Se] defined the notion of typical and atypical weights. In [BaKN] Theorem 7.3.1, it was shown that for typical weights the support varieties for simple $W(n)$-modules is $\{0\}$, and for atypical weights the support varieties are equal to $V_{(g,0)}(\mathbb{C})$. In this case it also makes sense to define the atypicality of a finite dimensional module in $\mathcal{C}_{(g,0)}$ as $\dim V_{(g,0)}(M)$.

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