On local energy decay for large solutions of the Zakharov-Kuznetsov equation

Argenis J. Mendez, Claudio Muñoz, Felipe Poblete, and Juan C. Pozo

Count for Mathematical Modeling, Universidad de Chile, Santiago, Chile; bCNRS and Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático (UMI 2807 CNRS), Universidad de Chile, Santiago, Chile; cInstituto de Ciencias Físicas y Matemáticas, Facultad de Ciencias, Universidad Austral de Chile, Valdivia, Chile; dDepartamento de Matemáticas, Facultad de Ciencias, Universidad de Chile, Santiago, Chile

ABSTRACT

We consider the Zakharov-Kuznetsov (ZK) equation posed in \( \mathbb{R}^d \), with \( d = 2 \) and 3. Both equations are globally well-posed in \( L^2(\mathbb{R}^d) \). In this article, we prove local energy decay of global solutions: if \( u(t) \) is a solution to ZK with data in \( L^2(\mathbb{R}^d) \), then

\[
\liminf_{t \to \infty} \int_{\Omega(t)} u^2(x, t) \, dx = 0,
\]

for suitable regions of space \( \Omega(t) \subseteq \mathbb{R}^d \) around the origin, growing unbounded in time, not containing the soliton region. We also prove local decay for \( H^1(\mathbb{R}^d) \) solutions. As a byproduct, our results extend decay properties for KdV and quartic KdV equations proved by Gustavo Ponce and the second author. Sequential rates of decay and other strong decay results are also provided as well.

ARTICLE HISTORY

Received 20 July 2020
Accepted 17 December 2020

KEYWORDS

Zakharov-Kuznetsov; decay; scattering

1991 MATHEMATICS SUBJECT CLASSIFICATION

Primary: 35Q53; Secondary: 35Q05

1. Introduction and main results

In this article, we consider the Zakharov-Kuznetsov equation (ZK),

\[
\partial_t u + \partial_x \Delta u + u \partial_x u = 0, 
\]

(ZK)

where \( u = u(x, t) \in \mathbb{R}, t \in \mathbb{R} \) and \( x \in \mathbb{R}^d \), with \( d = 2, 3 \). In Physics, (ZK) arises as an asymptotic model of wave propagation in a magnetized plasma, see e.g. [1, 2]. It was originally proposed by Zakharov and Kuznetsov [3] for \( d = 3 \), see also [4] for a formal derivation. Additionally, Eq. (ZK) is a natural multi-dimensional generalization of the Korteweg-de Vries (KdV) equation (which is the case \( d = 1 \)). In particular, it has solitary wave solutions, or solitons (see Section 1.1).

Very recently it was proved that the two and three dimensional ZK equations are globally well-posed in \( L^2 \) and \( H^1 \) [5, 6]. This global behavior can be seen as a consequence of the fact that (ZK) enjoys conservation of mass and energy:
In this article, our main goal is to study the asymptotic behavior of these global ZK solutions, under minimal assumptions (essentially, data only in $L^2(\mathbb{R}^d)$ or $H^1(\mathbb{R}^d)$). In particular, we shall describe the dynamics in local regions of space where solitons are absent. No scattering seems to be available for 2D and 3D (ZK), not even in the small data case, except if the nonlinearity is sufficiently large [7].

First, we consider the two dimensional case. We always assume $t \geq 2$. Let $\Omega(t)$ denote the following rectangular box (see Figure 1)

$$\Omega(t) := \{(x,y) \in \mathbb{R}^2 \mid |x| < t^b \wedge |y| < t^{br}\}, \quad \frac{1}{3} < r < 3, \quad 0 < b < \frac{2}{3 + r}. \quad (1.1)$$

(Note that $b < \frac{3}{5}$ and $br < 1$.) Under rough data assumptions, we show decay along a sequence of times.

**Theorem 1.2** ($L^2$-decay in 2d). *Suppose that $u_0 \in L^2(\mathbb{R}^2)$ and let $u = u(x,y,t)$ be the bounded in time solution to 2D (ZK) such that $u \in C(\mathbb{R} : L^2(\mathbb{R}^2))$. Then*

$$\liminf_{t \to \infty} \int_{\Omega(t)} u^2(x,y,t) \, dx \, dy = 0. \quad (1.3)$$

Moreover, there exist constant $C_0 > 0$ and an increasing sequence of times $t_n \to +\infty$ such that

$$\int_{\Omega(t_n)} u^2(x,y,t) \, dx \, dy \leq \frac{C_0}{\ln^2(t_n)}. \quad (1.4)$$

The previous result holds for arbitrarily large data in $L^2$, despite the fact that 2D ZK is scattering critical (the standard scattering trick is $u\partial_x u \sim \frac{1}{i} u$, see Faminskii [8] for
required linear decay estimates). We also present in (1.4) a mild decay rate valid along a sequence of times growing to infinity. No $L^\infty$ decay seems reasonable here because the $H^s$ regularity needed is at least $s > 1$. Since $\Omega(t)$ grows with time, it contains any compact region in $\mathbb{R}^2$, but it does not contain the soliton region $x \sim t$. However, combining (1.3) with the asymptotic stability of the 2D ZK soliton proved in [9] (see also Section 1.1), a better description of the soliton dynamics is obtained. In that sense, Theorem 1.2 and the results below can be understood as one step forward the validity of the soliton resolution conjecture for (ZK). We also state in Lemmas 3.4 and 3.5 some interesting consequences of (1.3), which we believe are of independent interest.

**Remark 1.1.** Theorem 1.2 can be extended to the non-centered case with some minor modifications. Indeed, (1.3) still holds if $\Omega(t)$ is given by the expression

$$
\Omega(t) := \{(x,y) \in \mathbb{R}^2 \mid |x| < t^m \wedge |y| < t^n, \quad 1/3 < r < 3, \quad 0 < b < 2/3 + r, \quad 0 \leq m < 1 - 1/2 b(1 + r), \quad 0 \leq n < 1 - 1/2 b(3 - r) \}.
$$

Note that when $r \approx 3$, one has the maximum value for $n$, which is $\approx 1$. At the same time, $br \approx 1$, which makes sense with the fact that one cannot go further proving decay in the $y$ variable, not more than the maximum value of the centered case. However, by making the rectangle smaller if necessary, one can go further in the $x$ variable: take $b$ small; since $m \approx 1$, one can reach the soliton limit $x \sim t$, but the size of the decay window must be small. Clearly one improves the region of decay obtained in the centered case, which is $b < 3^5/3$; see (1.1). See Section 3.4 for the proofs.

**Remark 1.2.** The area of the region $\Omega(t)$ is not preserved with respect to variations of the parameters $b$ and $r$. The supremum value of the area is obtained in the limit $r = 3$ and $b = 1/3$, which is $t^4$.

Obtaining the remaining lim sup property is left here as an open question, even in the small data case. The related problem about the maximum size of $\Omega(t)$ is very relevant here, since the global $L^2$ norm is always conserved, and positive for nontrivial solutions. One could guess that for $\Omega(t)$ large enough,

$$
\limsup_{t \to \infty} \int_{\Omega(t)} u^2(x,y,t) \, dx \, dy > 0,
$$

and therefore a smaller $\Omega(t)$ than in our results is probably needed. In this direction, if $u(t)$ has better decay properties, such as being in $L^\infty([0,\infty), L^1(\mathbb{R}^2))$, then the zero lim sup part can be recovered following [10]. However, having such strong decay is extremely far from being known in the ZK case, except if the solution is a soliton.

Coming back to (1.3), some key results in the dispersive PDE literature have been established primarily via a sequence of times. We mention the work by Duyckaerts et al. [11] for the proof of the soliton resolution conjecture in the focusing, energy critical wave equation. Unlike the wave equation, our problem is energy subcritical in nature, and of infinite speed of propagation. In particular, an infinite number of solitons could emerge from large $L^2$ data (see [12] for the existence of multi-solitons). The
conclusion stated in (1.3) was used by Tao [13] as a condition to prove a weak form of soliton resolution for cubic focusing NLS in 3D under bounded finite $H^1$ norm. Note that the problem considered by Tao is mass supercritical, but energy subcritical, and these restrictions are key whenever scattering is treated. The $L^2$ subcritical condition on (ZK), and more importantly the scattering critical condition, make things definitely more subtle. See also [14, 15] for early but fundamental results involving sequential in time convergence.

Recovering the decay of the gradient of $u$ in the case of $H^1$ data requires a slight modification of the parameters in (1.1).

**Theorem 1.6** ($H^1$-decay in 2d). Suppose additionally that $u_0 \in H^1(\mathbb{R}^2)$, and let $u = u(x,y,t)$ be the solution to 2D (ZK) such that $u \in C(\mathbb{R} : H^1(\mathbb{R}^2))$. If now $1 < r < 3$ in (1.1), one has

$$\liminf_{t \to \infty} \int_{\Omega(t)} \left( u^2 + |\nabla u|^2 \right) \, dx \, dy = 0.$$ 

A similar decay rate as in (1.4) also holds in this case.

One could guess that by employing the Grünrock-Herr’s dilation/rotation trick [16] on the ZK variables, the extra condition $r > 1$ may be lifted, but as of today it is not clear to us that such an improvement is possible.

The techniques required for the proof of Theorems 1.2 and 1.6 are reminiscent of the works by Ponce and the second author [10, 17] in the case of 1D KdV and Benjamin-Ono equations (see also [18, 19] for applications to other 1D models). Here we deal with the ZK model, which contains additional difficulties because of the higher dimension. Additionally, in this article, we lift the $L^1$ conditions posed in [10, 17] and consider data only in the energy space $L^2$ or $H^1$. The rates of decay that we obtain (see e.g. (1.4)) are clearly weaker than the ones obtained by assuming much more regularity and decay on the initial data, but in the vastly energy space, it is hard to think about a possible universal rate of decay. Finally, the proof works equally for quadratic and quartic KdV in 1D as well, with some minor modifications, giving relative improvements to the results stated in [10] (the $L^\infty_t L^1_x$ condition on the solution is lifted, at the expense of only having liminf in the decay property).

Indeed, consider quadratic and quartic KdV equations

$$\partial_t u + \partial_x (\partial_x^2 u + u^p) = 0, \quad p = 2, 4, \quad u = u(x,t) \in \mathbb{R}, \quad t, x \in \mathbb{R}. \quad (1.7)$$

The IVP for these problems is very well-known, global solutions are known for $L^2$ and $H^1$ data, see [20] for instance. Define

$$\Omega(t) := \{ x \in \mathbb{R} : |x \pm t^n| < t^b \}, \quad 0 < b < \frac{p}{2p - 1}, \quad 0 \leq n < 1 - \frac{b}{2}. \quad (1.8)$$

(The $\pm$ signs are considered at the same time.) For this set, we have the following large data sequential decay.

**Theorem 1.9** (Decay in gKdV). Suppose that $u_0 \in L^2(\mathbb{R})$ if $p = 2$, and $u_0 \in H^1(\mathbb{R})$ if $p = 4$. Let $u = u(x,t)$ be the solution to (1.7). Then
\[
\liminf_{t \to \infty} \int_{\Omega(t)} u^p(x, t) \, dx = 0. 
\]

(1.10)

A similar sequential rate of decay as in (1.4) can be obtained as well. Note that by making \( b \) smaller if necessary, one can almost reach the soliton region \( x \sim t \). From the proof itself, if the data is in \( H^1 \), (1.10) also holds for nonintegrable perturbations of KdV, of the form \( u^2 + o(u^2) \), following [10]. The proof for cubic KdV (\( p = 3 \), the so-called mKdV) does not work for obvious reasons: there exist periodic-in-time solutions around zero, spatially localized in the Schwartz class, called breathers, which do not decay. See [21] and references therein for more details on that important case. Finally, for \( p = 4 \), thanks to the Cauchy-Schwarz inequality and (1.10), there is decay of the \( L^2 \) norm along sequential times, on any fixed compact set of space. See [10] and references therein for a detailed description of the gKdV Cauchy problem and the corresponding scattering results.

Let us explain in more detail the idea of proof in Theorems 1.2 and 1.6. Given an \( L^2 \) solution \( u(x, y, t) \) of (ZK), we introduce an \( L^1 \) virial-type functional of the form\(^1\)

\[
\Xi(t) = \frac{1}{\eta(t)} \int_{\mathbb{R}^2} u(x, y, t) \psi \left( \frac{x}{\lambda_1(t)} \right) \phi \left( \frac{x}{\lambda_2^2(t)} \right) \phi \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy, \quad q > 1, 
\]

(1.11)
in the spirit of Bona-Souganidis-Strauss [22] and Martel-Merle [23]. Here \( \psi \) denotes a smooth increasing and bounded function (e.g. \( \tanh \)), and \( \phi \) is a very localized function (assume \( \phi = \text{sech} \)). The time-dependent parameters \( \lambda_1(t), \lambda_2(t) \) and \( \eta(t) \) are key for the proof, and will be chosen following special requirements, related to the structure of the spatial region \( \Omega(t) \) described in (1.1), among other not less important conditions. There are some differences between \( \Xi(t) \) here and the same functional introduced in [10], the most important being the double localization in \( x, y \) via the function \( \phi \), and the introduction of the compensation function \( \eta(t) \) (first introduced in [17]). These procedures make possible to give a meaning for \( \Xi(t) \) even for \( L^2 \) data, but introduce plenty of new error terms that must be controlled with care. This is done by using appropriate choices for \( q > 1 \) and \( \lambda_2(t) \) in terms of \( \eta(t) \) and \( \lambda_1(t) \). Once it is proved that \( \Xi(t) \) makes sense, we show that the local \( L^2 \) integral bound

\[
\int_{\{t \gg 1\}} \frac{1}{t \ln t} \left( \int_{\Omega(t)} u^2(x, y, t) \, dx \, dy \right) dt \leq C_0 < \infty,
\]

is valid no matter the size of \( u \). This last bound is essentially the statement in Theorem 1.1, and is proved analyzing the dynamics of \( \Xi(t) \) in the long time regime, in the same spirit as previous work by Martel and Merle [23], and more recent works [10, 17, 24]. Theorem 1.2 is not different in nature, but follows the more standard use of Kato smoothing estimates, and the previously proved Theorem 1.1. The more restrictive restriction \( r > 1 \) appears from some interactions between mixed derivatives that require stronger control than other less complicated terms.

The techniques used to prove Theorems 1.2 and 1.6 are sufficiently versatile to provide, as far as we understand, a first proof of decay in the ZK 3D case. Consider now the region (see Figure 2)

\(^1\)Recall that \( \int_{\mathbb{R}^2} u \, dx \, dy \) is formally conserved, and scattering critical for the (ZK) scaling \( \lambda^2 u(\lambda x, \lambda y, \lambda^2 t) \).

$$\Omega(t) := \{ (x,y,z) \in \mathbb{R}^3 \mid |x| < t^b, |y| < t^{br_1}, |z| < t^{br_2} \},$$

$$b > 0, \quad r_1, r_2 > 1, \quad r_1 + r_2 < 3, \quad r_1 + 1 < 3r_2, \quad r_2 + 1 < 3r_1, \quad b < \frac{2}{3 + r_1 + r_2}. \quad (1.12)$$

See Lemma 5.1 for a detailed geometric description of this set. Two particularly important cases are $r_1 \approx r_2 \approx 1$, for which $b \approx \frac{2}{3}, br_1 \approx \frac{2}{3}$ and $br_2 \approx \frac{2}{3}$; and $r_1 \approx 1, r_2 \approx 2$ (and symmetric case), for which $b \approx \frac{1}{3}, br_1 \approx \frac{1}{3}$ and $br_2 \approx \frac{2}{3}$. For this region we immediately go to the $H^1$ case, proving the following result.

**Theorem 1.13** (Local decay in the 3D case). Suppose $u_0 \in H^1(\mathbb{R}^3)$, and let $u = u(x,y,z,t)$ be the solution to (ZK) in 3D such that $u \in C(\mathbb{R} : H^1(\mathbb{R}^3))$. Then

$$\liminf_{t \to \infty} \int_{\Omega(t)} (u^2 + |\nabla u|^2)(x,y,z,t) \, dx \, dy \, dz = 0.$$  

A similar decay rate as in (1.4) also holds in this case.

**Remark 1.3.** If only $L^2$ decay is considered, the conditions $r_1, r_2 > 1$ in (1.12) can be extended to $r_1, r_2 > \frac{1}{2}$.

**Remark 1.4.** The volume of the region $\Omega(t)$ is not preserved with respect to variations of the parameters $b$ and $r_1$ and $r_2$. The supremum value of the area is obtained in the limit $r_1 = r_2 = \frac{3}{2}$ and $b = \frac{1}{3}$, which is $\frac{3}{2}$.

The proof of Theorem 1.13 is similar to proof for the 2D case, with care needed to take into account the bigger number of error terms appearing in the dynamics, as well as some new technical estimates for cubic terms. However, the key of the argument is contained in the 2D case.

Is it possible to obtain strong decay with data only in the energy space? This question is not easy at all, essentially because the dynamics in the large data case may be extremely complex. However, one can show strong $L^2$ decay for $H^1$ data in some particular regions of the space, characterized for being too far from the previously considered regions, and the soliton region. Recall that $x = (x,y) = (x_1,x_2)$ in the 2D case, and...
$x = (x, y, z) = (x_1, x_2, x_3)$ in the 3D one. For any $p \geq 1, \epsilon > 0$ and $t \geq 2$, consider the region (see Figure 3)

$$\Omega(t) := \left\{ x \in \mathbb{R}^d \mid |x_j| \sim t^p \ln^{1+t} t \right\}, \quad j = 1, \ldots, d.$$ 

Recall that $p \geq 1$ is arbitrary. Here $a \sim b$ means that there exist $C_0, c_0 > 0$ independent of $a$ and $b$ such that $c_0 b \leq a \leq C_0 b$. Our last result is the following strong decay property.

**Theorem 1.14.** Suppose $u_0 \in H^1(\mathbb{R}^d)$, $d = 2, 3$ and let $u = u(x, t)$ be the solution to (ZK) such that $u \in C(\mathbb{R} : H^1(\mathbb{R}^d))$. Then,

$$\lim_{t \to \infty} \int_{\Omega_j(t)} u^2(x, t) dx = 0, \quad j = 1, \ldots, d. \quad (1.15)$$

To prove this result we employ a new virial estimate first introduced in [25], which is extended here to the $d$-dimensional case (in particular, the transversal directions $y$ and $z$ work very well in terms of decay properties). The fact that we obtain the strong limit here is due to an improved virial estimate that profits of a key sign condition and not only forced decay estimates; however, it only works for long distances.

### 1.1. A brief description of the ZK literature

In this section, we briefly describe key previous results for (ZK) posed in the $\mathbb{R}^d$ setting. Originally derived by Zakharov and Kuznetsov [3], the mathematical study of the ZK equation has attracted the attention of many authors in past years. Unlike KdV, ZK is not integrable. It was rigorously derived from the Euler-Poisson system with magnetic field as a long-wave and small-amplitude limit, see [26, Section 10.3.2.6].

Faminskii [8] showed local well-posedness (LWP) in $H^s(\mathbb{R}^2)$ for $s \geq 1$. After him, many researchers have contributed to the low regularity LWP theory. We mention the works of Linares and Pastor [27], Molinet and Pilod [28], and Grünrock and Herr [16],...
who showed LWP at regularity $s > \frac{1}{2}$. Very recently, Kinoshita [5] has proved local-wellposedness for $s > -1/4$. This is best possible range. See also [29] for the the proof of LWP in the case of a 2D modified ZK equation. Uniqueness results vs. spatial decay have been recently proved in [30], and propagation of regularity along regions of space has been considered in [31].

Concerning the 3D case, Linares and Saut [32], Molinet and Pilod [28], and Ribaud-Vento [33] proved local and global well-posedness (GWP) in $H^s(\mathbb{R}^3)$ for $s > 1$. Herr and Kinoshita [6] showed LWP for $s > -\frac{1}{2}$, and GWP in the energy space. Moreover, Herr and Kinoshita have proved that LWP holds in $H^s(\mathbb{R}^d)$, with $d \geq 3$ and $s > \frac{d-4}{2}$. This last information, and the fact that (ZK) is $L^2$-critical in dimension 4 (possibly having blow-up solutions as well), has stopped us to get decay results in 4D.

**Solitons.** Similar to the one dimensional KdV equation, (ZK) possesses soliton solutions of the form

$$u(x,t) = Q_c(x - ct, x'), \quad c > 0, \quad x' \in \mathbb{R}^{d-1}.$$ 

Here $Q_c = cQ(\sqrt{c}x)$ and $Q$ is the $H^1(\mathbb{R}^d)$ radial solution of the elliptic PDE

$$\Delta Q - Q + Q^2 = 0, \quad Q > 0, \quad d \leq 5.$$ 

Unlike KdV, no explicit formula is known for ZK solitons. However, for any $R > 0$,

$$\int_{|x-c| \leq R, \ |x'| \leq R} Q_c^2(x - ct, x') dx \geq c^{2-d} c_0(R) > 0,$$

revealing that Theorem 1.2 cannot hold in the vicinity of solitons. However, Theorem 1.2 is still valid in $\Omega(t)$ even if the initial data contains infinitely many solitons adding finite $L^2$ norm. Anne de Bouard [34] showed that $L^2$ subcritical ZK solitons are orbitally stable in $H^1$, and supercritical ones are unstable. The asymptotic stability of the solution has been proven in [9] in the 2D case, and recently in [35] in 3D. Both works are non-trivial extensions of the foundational works by Martel and Merle [36, 37] concerning the one dimensional KdV case. Well-decoupled multi-solitons were proved stable in 2D, see [9]. The modified ZK equation (cubic nonlinearity in (ZK)) is $L^2$ critical, and recently finite or infinite time blow up solutions were constructed around the solitary wave [38], in close relation with a similar result obtained by Merle [39] for the $L^2$-critical, quintic generalized KdV. Finally, see the recent work [12] for the construction of multi-soliton like solutions for 2D and 3D ZK.

### 1.2. Organization of this article

This article is organized as follows. Section 2 contains basic tools needed for the proofs in remaining Sections. Sections 3 and 4 contain the proofs of Theorems 1.2 and 1.6, respectively. Section 5 is devoted to the proof of Theorem 1.13, and Section 6 deals with the proof of Theorem 1.14. Finally, Theorem 1.9 is proved in Section 7.
2. Preliminaries

The purpose of this section is to gather all the necessary auxiliary results that will be needed in forthcoming sections. We start by describing the weighted functions used to define our local norms.

2.1. Weighted functions

Let \( \phi \) be a smooth even and positive function such that

(i) \( \phi' \leq 0 \) on \( \mathbb{R}^+ = [0, \infty) \),
(ii) \( \phi|_{[0,1]} = 1 \), \( \phi(x) = e^{-x} \) on \([2, \infty)\), \( e^{-x} \leq \phi(x) \leq 3e^{-x} \) on \( \mathbb{R}^+ \).
(iii) The derivatives of \( \phi \) satisfy:

\[
|\phi'(x)| \leq c\phi(x) \quad \text{and} \quad |\phi''(x)| \leq c\phi(x),
\]

for some positive constant \( c \).

Let \( w(x) = \int_0^x \phi(s) \, ds \). Then \( w \) is an odd function such that \( w(x) = x \) on \([-1, 1] \) and \( |w(x)| \leq 3 \).

For \( \sigma \) a parameter, we set

\[
\psi_\sigma(x) = \sigma \psi\left(\frac{x}{\sigma}\right) \quad \text{so that} \quad \psi'_\sigma(x) = \phi\left(\frac{x}{\sigma}\right) =: \phi_\sigma(x) \tag{2.1}
\]

and

\[
\psi_\sigma(x) = x \quad \text{on} \quad [-\sigma, \sigma],
\]

\[
|\psi_\sigma(x)| \leq 3\sigma, \quad e^{-\frac{|x|}{\sigma}} \leq \phi_\sigma(x) \leq 3e^{-\frac{|x|}{\sigma}} \quad \text{on} \quad \mathbb{R}. \tag{2.2}
\]

Also, as part of our analysis we require to define functions \( \lambda_1, \lambda_2 \) and \( \eta \) that will be described later in a more detailed manner, so that for the moment we will assume that such functions are smooth enough for \( t \gg 1 \).\(^2\)

2.2. Compactly supported weights

In this paragraph we consider weights needed for the proof of Theorem 1.14, see Section 6 for more details. We will consider the following a function \( \chi \in C^\infty(\mathbb{R}) \) such that:

\[
0 \leq \chi \leq 1,
\]

\[
\chi(x) = \begin{cases} 
1 & x \leq -1 \\
0 & x \geq 0, 
\end{cases} \tag{2.3}
\]

with \( \text{supp}(\chi) \subset (-\infty, 0] \) and \( \chi'(x) \leq 0 \) for all \( x \in \mathbb{R} \). Also, for \( x \in [-3/4, -1/4] \), the function \( \chi' \) satisfies the inequality

\[
-\chi'(x) \geq c_0 1_{[-3/4, -1/4]}(x) \quad \text{for all} \quad x \in \mathbb{R}, \tag{2.4}
\]

where \( c_0 \) is a universal, positive constant.

\(^2\)Through all the document we will use the notation \( t \gg 1 \) that for us mean \( t \geq 10 \).
3. $L^2$ decay in 2D. Proof of Theorem 1.2

Recall the region $\Omega(t)$ introduced in (1.1). The purpose of this section is to first show the following auxiliary result.

**Lemma 3.1.** Assume that $u_0 \in L^2(\mathbb{R}^2)$. Let $u \in (C \cap L^\infty)(\mathbb{R} : L^2(\mathbb{R}^2))$ be the corresponding unique solution of (ZK) with initial data $u(t = 0) = u_0$. Then, there exists a constant $C_0 > 0$ such that

$$
\int_{\{t > 1\}} \frac{1}{t \ln t} \left( \int_{\Omega(t)} u^2(x, y, t) \, dx \, dy \right) \, dt \leq C_0 < \infty. \quad (3.1)
$$

The proof of (3.1) is the key element to conclude Theorem 1.2, but its proof is technical; it requires the introduction of the modified virial functional (1.11).

Assuming Lemma 3.1, we can easily prove Theorem 1.2, following the lines in [17].

3.1. End of proof of Theorem 1.1

Since the function $\frac{1}{t \ln t} \not\in L^1(\{t \gg 1\})$ we can ensure that there exist a sequence of positive time $(t_n) \uparrow \infty$ as $n$ goes to infinity, such that

$$
\lim_{n \to \infty} \int_{\Omega(t_n)} u^2(x, y, t_n) \, dx \, dy = 0.
$$

The convergence of this sequence shows that 0 is an accumulation point, the least one because of nonnegativity. This proves the liminf and concludes the proof of Theorem 1.2.

Now, we prove (1.4). It requires a simple result, Lemma 3.4, which is stated and proved below. One of the conclusions of Lemma 3.4 is that for each $t \gg 1$

$$
\int_t^\infty \frac{1}{s \ln s} \left( \int_{\Omega(s)} u^2(x, y, s) \, dx \, dy \right) \, ds \leq C_0 \int_t^\infty \frac{ds}{s \ln^{1+1/b}(s)}. \quad (3.2)
$$

For $t = t_0 \gg 1$ in (3.2), there exists $t_1 \geq t_0$ such that

$$
\frac{1}{t_1 \ln t_1} \left( \int_{\Omega(t_1)} u^2(x, y, t_1) \, dx \, dy \right) \leq \frac{C_0}{t_1 \ln^{1+1/b}(t_1)}.
$$

Otherwise we arrive to a contradiction with (3.2). Let $t^*_1 = t_1 + 1$, by a similar argument as above we obtain $t_2 \geq t^*_1 > t_1$ satisfying

$$
\frac{1}{t_2 \ln t_2} \left( \int_{\Omega(t_2)} u^2(x, y, t_2) \, dx \, dy \right) \leq \frac{C_0}{t_2 \ln^{1+1/b}(t_2)}.
$$

Recursively, one can ensure the existence of a sequence of positive times $(t_n) \uparrow \infty$ as $n$ goes to infinity, satisfying (1.4).

The rest of this section will be devoted to the proof of Lemma 3.1.
3.2. Setting

Recall the weighted functions $\psi_\sigma$ and $\phi_\sigma$ defined in (2.1), the parameters $(b, r)$ in (1.1), and $\delta_1, \delta_2 > 0$.

In the next for each $b, r > 0$ satisfying (1.1) we denote by $q \in (1, 2)$ a number such that

$$b \leq \frac{2}{2 + q + r} < \frac{2}{3 + r}, \quad \frac{1}{3} < r < 3.$$  \hfill (3.3)

For $u$ a solution of the Zakharov-Kuznetsov equation (ZK), we set the functional

$$\Xi(t) := \frac{1}{\eta(t)} \int_{\mathbb{R}^2} u(x, y, t) \psi_\sigma \left( \frac{x}{\hat{\lambda}_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\hat{\lambda}_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\hat{\lambda}_2(t)} \right) \ dy, \quad (3.4)$$

where $\hat{\lambda}_1, \hat{\lambda}_2$ and $\eta$ are functions depending on $t$. We consider

$$\hat{\lambda}_1(t) = \frac{t^b}{\ln t} \quad \text{and} \quad \eta(t) = t^p \ln^2 t,$$  \hfill (3.5)

where $p$ is a positive constant satisfying

$$p + b = 1.$$  \hfill (3.6)

We also consider

$$\hat{\lambda}_2(t) = \hat{\lambda}_1^r(t) \quad \text{where} \quad r > 0.$$  \hfill (3.7)

Then,

$$\frac{\hat{\lambda}_1'(t)}{\hat{\lambda}_1(t)} \sim \frac{\eta'(t)}{\eta(t)} \sim \frac{1}{t} \quad \text{for} \quad t \gg 1.$$  \hfill (3.8)

Also,

$$\hat{\lambda}_1'(t) = \frac{1}{t^{1-b} \ln t} \left( \frac{b \ln t - 1}{\ln t} \right) \quad \text{and} \quad \hat{\lambda}_1(t) \eta(t) = t \ln t.$$  \hfill (3.9)

**Lemma 3.2.** For $u \in L^2(\mathbb{R}^2)$, the functional $\Xi$ is well defined and bounded in time.

**Proof.** By Cauchy-Schwarz inequality we obtain

$$|\Xi(t)| \leq \frac{1}{\eta(t)} \| u(t) \|_{L_{x,y}} \| \psi_\sigma \|_{L_\infty^2} \| \phi_{\delta_1} \left( \frac{\cdot}{\hat{\lambda}_1^q(t)} \right) \phi_{\delta_2} \left( \frac{\cdot}{\hat{\lambda}_2^r(t)} \right) \|_{L_{x,y}^2},$$

$$= \left( \frac{\hat{\lambda}_1^q(t) \hat{\lambda}_2^r(t)}{\eta(t)} \right)^{1/2} \| u_0 \|_{L_{x,y}^2} \| \psi_\sigma \|_{L_\infty^2} \| \phi_{\delta_2} \|_{L_x^2} \| \phi_{\delta_1} \|_{L_x^2}$$

$$\approx \delta_1, \delta_2, \sigma \frac{1}{(\ln(t))^{(2+q+r)/2}} \frac{1}{t^{(2-b(2+q+r))/2}}.$$  \hfill (3.10)

Since (3.3) is satisfied we have

$$\sup_{t \gg 1} |\Xi(t)| \leq C_0 < \infty,$$

which finishes the proof. \hfill \Box
3.3. Dynamics for $\Xi(t)$

In what follows, we compute and estimate the dynamics of $\Xi(t)$ in the long time regime.

**Lemma 3.3.** For any $t \geq 10$, one has the bound
\[
\frac{1}{\lambda_1(t)\eta(t)} \int_{\mathbb{R}^2} u^2 \psi'(x, \frac{x}{\lambda_1(t)}) \phi_{\delta_1} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy \leq \frac{d\Xi}{dt}(t) + \Xi_{int}(t),
\]
where $\Xi_{int}(t)$ are terms that belong to $L^1(\{ t \gg 1 \})$.

Assuming this estimate, (3.5), (3.6) and Lemma 3.2 imply Lemma 3.1, after noticing that the set $\Omega(t)$ defined in (1.1) is nothing but a set where
\[
\frac{1}{\lambda_1(t)\eta(t)} \int_{\Omega(t)} u^2 \, dxdy \leq \frac{1}{\lambda_1(t)\eta(t)} \int_{\mathbb{R}^2} u^2 \psi'(x, \frac{x}{\lambda_1(t)}) \phi_{\delta_1} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy,
\]
provided $q > 1$ is chosen sufficiently close to 1 in (3.3), and the log terms in (3.5) are discarded after making the parameter $b$ slightly smaller if necessary. The rest of the section will be devoted to the proof of Lemma 3.3.

**Proof of Lemma 3.3.** We have
\[
\frac{d}{dt} \Xi(t) = -\frac{\eta'(t)}{\eta^2(t)} \int_{\mathbb{R}^2} u \psi'(x, \frac{x}{\lambda_1(t)}) \phi_{\delta_1} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy - \frac{\eta'(t)}{\eta^2(t)} \int_{\Omega(t)} u \psi'(x, \frac{x}{\lambda_1(t)}) \phi_{\delta_1} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy - \Xi_{1}(t) + \Xi_{2}(t).
\]
First, we bound $\Xi_{2}$, that in virtue of (3.10) the same analysis applied there yields
\[
|\Xi_{2}(t)| \leq \left| \int_{\mathbb{R}^2} u \psi'(x, \frac{x}{\lambda_1(t)}) \phi_{\delta_1} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy \right| \leq \frac{1}{t \eta(t)} \left( \lambda_1^q(t) \lambda_2(t) \right)^{1/2} \eta'(t) \frac{1}{\eta^2(t)}.
\]
From (3.3) we have $b \leq \frac{2}{q + r + 2}$ then $2 - b(1 + \frac{q + r}{2}) \geq 1$. Thus, $\Xi_{2} \in L^1(\{ t \gg 1 \})$.

Unlike $\Xi_{2}$, to bound $\Xi_{1}$ it is required to take into consideration the dispersive part associated to the ZK equation, as well as, the non-linear interaction. More precisely, we shall decompose such term as follows:
\[ \Xi_1(t) = \frac{1}{\eta(t)} \int_{\mathbb{R}^2} \partial_t u \psi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy 
- \frac{\lambda_1'(t)}{\lambda_1(t)} \frac{\eta(t)}{u} \int_{\mathbb{R}^2} u \psi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy 
- q \frac{\lambda_1'(t)}{\lambda_1(t) \eta(t)} \int_{\mathbb{R}^2} \psi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_1(t)} \right) \phi'_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy 
- \frac{\lambda_2'(t)}{\lambda_2(t) \eta(t)} \int_{\mathbb{R}^2} \psi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy \]

= \Xi_{1,1}(t) + \Xi_{1,2}(t) + \Xi_{1,3}(t) + \Xi_{1,4}(t). \tag{3.14} \]

Concerning to \( \Xi_{1,1} \) we have by (ZK) and integration by parts
\[ \Xi_{1,1}(t) = - \frac{1}{\eta(t)} \int_{\mathbb{R}^2} \partial_x \left( \Delta u + \frac{u^2}{2} \right) \psi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy 
+ \frac{1}{\eta(t) \lambda_1'(t)} \int_{\mathbb{R}^2} \Delta \psi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy 
+ \frac{1}{\eta(t)^2 \lambda_1'(t)} \int_{\mathbb{R}^2} u^2 \psi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy 
+ \frac{1}{\eta(t)^2 \lambda_1'(t)} \int_{\mathbb{R}^2} u^2 \psi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy \]

= \Xi_{1,1,1}(t) + \Xi_{1,1,2}(t) + \Xi_{1,1,3}(t) + \Xi_{1,1,4}(t). \tag{3.15} \]

For \( \Xi_{1,1,1} \) we have after combining integration by parts
\[ \Xi_{1,1,1}(t) = \frac{1}{\eta(t) \lambda_1'(t)} \int_{\mathbb{R}^2} \psi_{\sigma}'' \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy 
+ \frac{2}{\eta(t) \lambda_1'(t)^2} \int_{\mathbb{R}^2} \psi_{\sigma}'' \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy 
+ \frac{1}{\eta(t) \lambda_1'(t)^2} \int_{\mathbb{R}^2} \psi_{\sigma}'' \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy 
+ \frac{1}{\eta(t) \lambda_1(t) \lambda_2'(t)} \int_{\mathbb{R}^2} \psi_{\sigma}'' \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy. \]

First, we bound each term using Cauchy-Schwarz inequality, as follows:
\[|\Xi_{1,1}(t)| \leq \left(\frac{\lambda_1(t)\lambda_2(t)}{\eta(t)\lambda_1(t)}\right)^{1/2}\|u_0\|_{L^2} \\|\psi''\|_{L^2} \\|\phi_{\delta_1}\|_{L^\infty} \\|\phi_{\delta_2}\|_{L^2} \]
\[+ \left(\frac{\lambda_1(t)\lambda_2(t)}{\eta(t)\lambda_1(t)}\right)^{1/2}\|u_0\|_{L^2} \\|\psi''\|_{L^2} \\|\phi_{\delta_1}\|_{L^\infty} \\|\phi_{\delta_2}\|_{L^2} \]
\[+ \left(\frac{\lambda_1(t)\lambda_2(t)}{\eta(t)\lambda_1(t)}\right)^{1/2}\|u_0\|_{L^2} \\|\psi''\|_{L^2} \\|\phi_{\delta_1}\|_{L^\infty} \\|\phi_{\delta_2}\|_{L^2} \]
\[+ \left(\frac{\lambda_1(t)\lambda_2(t)}{\eta(t)\lambda_1(t)}\right)^{1/2}\|u_0\|_{L^2} \\|\psi''\|_{L^2} \\|\phi_{\delta_1}\|_{L^\infty} \\|\phi_{\delta_2}\|_{L^2}. \]

Consequently,
\[|\Xi_{1,1}(t)| \leq \frac{1}{\eta(t)\lambda_1(t)^{5/2-2/r}} + \frac{1}{\eta(t)\lambda_1(t)^{3/2+q-r/2}} \]
\[+ \frac{1}{\eta(t)\lambda_1(t)^{1/2+2q-r/2}} + \frac{1}{\eta(t)\lambda_1(t)^{1/2+3r/2}} \]
\[= \frac{1}{t^{1+b(\frac{5}{2})}} \ln^{r-2-1/2}(t) + \frac{1}{t^{1+b(1/2+q-r/2)}} \ln^{1/2-q+r/2}(t) \]
\[+ \frac{1}{t^{1+b(-1/2+2q-r/2)}} \ln^{3/2-2q+r/2}(t) + \frac{1}{t^{1+b(-1/2+3r/2)}} \ln^{3/2-3r/2}(t). \]

We claim \(\Xi_{1,1,1} \in L^1(\{t \gg 1\})\). Indeed, from (3.3) we have \(4q - 1 > 1 + 2q > 3 > r > \frac{1}{3}\). Thus
\[3 > r, \quad 1 + 2q > r, \quad 4q - 1 > r, \quad r > \frac{1}{3}, \]
or equivalently
\[\frac{5}{2} - \frac{r}{2} > 1, \quad \frac{1}{2} + q - \frac{r}{2} > 0, \quad -\frac{1}{2} + 2q - \frac{r}{2} > 0, \quad -\frac{1}{2} + \frac{3}{2}r > 0. \]

Next, applying integration by parts,
\[\Xi_{1,1,2}(t) = \frac{1}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}^2} \Delta u \psi_\sigma \left(\frac{x}{\lambda_1(t)}\right) \phi_{\delta_1} \left(\frac{x}{\lambda_2(t)}\right) \psi_\sigma \left(\frac{y}{\lambda_2(t)}\right) \, dx \, dy \]
\[= \frac{1}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}^2} \psi''_\sigma \left(\frac{x}{\lambda_1(t)}\right) \phi_{\delta_1} \left(\frac{x}{\lambda_2(t)}\right) \psi_\sigma \left(\frac{y}{\lambda_2(t)}\right) \, dx \, dy \]
\[+ \frac{2}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}^2} \psi'_\sigma \left(\frac{x}{\lambda_1(t)}\right) \phi_{\delta_1} \left(\frac{x}{\lambda_2(t)}\right) \phi_{\delta_2} \left(\frac{y}{\lambda_2(t)}\right) \, dx \, dy \]
\[+ \frac{1}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}^2} \psi'_\sigma \left(\frac{x}{\lambda_1(t)}\right) \phi_{\delta_1} \left(\frac{x}{\lambda_2(t)}\right) \phi_{\delta_2} \left(\frac{y}{\lambda_2(t)}\right) \, dx \, dy \]
\[+ \frac{1}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}^2} \psi'_\sigma \left(\frac{x}{\lambda_1(t)}\right) \phi_{\delta_1} \left(\frac{x}{\lambda_2(t)}\right) \phi_{\delta_2} \left(\frac{y}{\lambda_2(t)}\right) \, dx \, dy, \]
and the Cauchy-Schwarz inequality yields

\[
|\Xi_{1,1,2}(t)| \\
\leq \frac{(\lambda_1(t) \lambda_2(t))^{1/2}}{\eta(t) \lambda_1^{1/2 + q/2}(t)} \|u_0\|_{L^2} \|\psi''\|_{L^2} \|\phi_{\delta_1}\|_{L^2} \|\phi_{\delta_2}\|_{L^2} \\
+ \frac{(\lambda_1(t) \lambda_2(t))^{1/2}}{\eta(t) \lambda_1^{1/2 + 2q/2}(t)} \|u_0\|_{L^2} \|\phi''\|_{L^2} \|\phi_{\delta_1}\|_{L^2} \|\phi_{\delta_2}\|_{L^2} \\
+ \frac{(\lambda_1(t) \lambda_2(t))^{1/2}}{\eta(t) \lambda_1^{3q/2}(t)} \|u_0\|_{L^2} \|\phi''\|_{L^2} \|\phi_{\delta_1}\|_{L^2} \|\phi_{\delta_2}\|_{L^2} \\
+ \frac{(\lambda_1(t) \lambda_2(t))^{1/2}}{\eta(t) \lambda_1^{3q/2} \lambda_2^2(t)} \|u_0\|_{L^2} \|\phi''\|_{L^2} \|\phi_{\delta_1}\|_{L^2} \|\phi_{\delta_2}\|_{L^2}.
\]

Hence,

\[
|\Xi_{1,1,2}(t)| \leq \frac{1}{\eta(t) \lambda_1^{3/2 + q/2}(t)} + \frac{1}{\eta(t) \lambda_1^{1/2 + 2q/2}(t)} \\
+ \frac{1}{\eta(t) \lambda_1^{5q/2 + 3r}(t)} + \frac{1}{\eta(t) \lambda_1^{2q + 3r/2}(t)} \\
= \frac{1}{t^{1 + b(1/2 + q/2 + r/2)} \ln^{1/2 + q/2 + r/2}(t)} + \frac{1}{t^{1 + b(-1/2 + 2q + r/2)} \ln^{3/2 - 2q + r/2}(t)} \\
+ \frac{1}{t^{1 + b(-1 + 5q/2 + 3r/2)} \ln^{2q + 3r/2}(t)} + \frac{1}{t^{1 + b(-1 + q/2 + 3r/2)} \ln^{2q - 3r/2}(t)}.
\]

(3.17)

From (3.3), we have \(5q - 2 > 4q - 1 > 1 + 2q > 3 > r > 1/3 > (2 - q)/3.\) Thus,

\[
1 + 2q > r, \quad 4q - 1 > r, \\
5q - 2 > r, \quad r > \frac{1}{3}(2 - q),
\]
or equivalently

\[
\frac{1}{2} + \frac{q - r}{2} > 0, \quad -\frac{1}{2} + \frac{2q - r}{2} > 0, \\
-1 + \frac{5}{2} q - \frac{r}{2} > 0, \quad -1 + \frac{q}{2} + \frac{3}{2} r > 0.
\]

Hence \(\Xi_{1,1,2} \in L^1(\{t \gg 1\}).\)

We emphasize that the term \(\Xi_{1,1,3}\) in (3.15)

\[
\Xi_{1,1,3}(t) = \frac{1}{2\eta(t) \lambda_1(t)} \int_{\mathbb{R}^2} u^2 |\psi''(\frac{x}{\lambda_1(t)}\phi_{\delta_1}(\frac{x}{\lambda_2(t)}\phi_{\delta_2}(\frac{y}{\lambda_2(t)}) dx dy,
\]

(3.18)

is the term to be estimated after integrating in time, leading to the L.H.S. in (3.11). Therefore, it will remain unchanged nearly until the end of the proof.
The term $\Xi_{1, 1, 4}$ in (3.15) satisfies the following estimate

$$\left| \Xi_{1, 1, 4}(t) \right| \leq \frac{1}{2\eta(t)\lambda_1(t)} \left| \int_{\mathbb{R}^2} u^2 \psi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_1^q(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy \right|$$

$$\leq \frac{1}{2\eta(t)\lambda_1^q(t)} \left\| \psi_{\sigma} \right\|_{L^\infty} \left\| \phi_{\delta_1} \right\|_{L^\infty} \left\| \phi_{\delta_2} \right\|_{L^\infty}$$

(3.19)

Since (3.3) are satisfied, we obtain $q > 1$ (note that $b > 0$ is needed here). Thus $\Xi_{1, 1, 4} \in L^1(\{t \gg 1\})$. This last estimate ends the study of the term $\Xi_{1, 1}$ in (3.14).

Now, we focus our attention in the remaining terms in (3.14). First, by means of Young’s inequality, we have for $\epsilon > 0,

$$\left| \Xi_{1, 2}(t) \right| \leq \frac{1}{4\epsilon} \left| \frac{\lambda_1'(t)}{\lambda_1(t)\eta(t)} \right| \int_{\mathbb{R}^2} u^2 \psi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_1^q(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy$$

$$+ \epsilon \left| \frac{\lambda_1'(t)}{\lambda_1(t)\eta(t)} \right| \int_{\mathbb{R}^2} \psi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_1^q(t)} \right)^2 \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy$$

$$= \frac{1}{4\epsilon} \left| \frac{\lambda_1'(t)}{\lambda_1(t)\eta(t)} \right| \left\| \cdot \right\|_{L^2} \left\| \phi_{\delta_1} \right\|_{L^\infty} \left\| \phi_{\delta_2} \right\|_{L^\infty},$$

so that, taking $\epsilon = \lambda_1'(t) > 0$ for $t \gg 1$; it is clear that

$$\left| \Xi_{1, 2}(t) \right| \leq \frac{1}{2\lambda_1(t)\eta(t)} \int_{\mathbb{R}^2} u^2 \psi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_1^q(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy$$

$$+ \left( \frac{\lambda_1'(t)}{\lambda_1(t)} \right)^2 \frac{\lambda_2(t)}{\eta(t)(\lambda_1(t))^2} \left\| \cdot \right\|_{L^2} \left\| \phi_{\delta_1} \right\|_{L^\infty} \left\| \phi_{\delta_2} \right\|_{L^\infty}$$

$$=: \frac{1}{2} \Xi_{1, 1, 3}(t) + \Xi_{1, 2}(t).$$

Note that the first term in the R.H.S. is the quantity to be estimated (see (3.18)), unlike the remaining term $\Xi_{1, 2}$ which satisfies

$$0 \leq \Xi_{1, 2}(t) \leq \left( \frac{\lambda_1'(t)}{\lambda_1(t)} \right)^2 \frac{\lambda_2(t)}{\eta(t)(\lambda_1(t))^2} \leq \frac{1}{t^2} \frac{1}{\eta(t)(\lambda_1(t))^2} = \frac{1}{t^{3-b(3+r)}}.$$

(3.20)

The term $\Xi_{1, 2}$ belongs in $L^1(\{t \gg 1\})$, since (3.3) implies that $b < \frac{2}{r+3}$ or equivalent $3 - b(3 + r) > 1$. This ends the estimate of $\Xi_{1, 2}(t)$. 

COMMUNICATIONS IN PARTIAL DIFFERENTIAL EQUATIONS
Now we consider the term \( \Xi_{1,3}(t) \).

\[
\begin{align*}
|\Xi_{1,3}(t)| &= q \left| \frac{\lambda_1'(t)}{\lambda_1(t) \eta(t)} \int_{\mathbb{R}^2} u \psi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \left( \frac{x}{\lambda_1^2(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \ dx \ dy \right| \\
&\leq q \left| \frac{\lambda_1'(t) \lambda_2^{q/2}(t) \lambda_2^{1/2}(t)}{\lambda_1(t) \eta(t)} \right| \|u_0\|_{L_2} \|\phi_{\delta_1}\|_{L_2^2} \|\phi_{\delta_2}\|_{L_2^2} \\
&\leq \frac{1}{t^{2-b(1+q/2+r/2)} \ln^{2+q/2+r/2}(t)}.
\end{align*}
\]

By (3.3) we have \( b \leq \frac{2}{2+q+r} \), consequently \( 2 - b \left( 1 + \frac{q}{2} + \frac{r}{2} \right) \geq 1 \). Thus, \( \Xi_{1,3} \in L^1(\{t \gg 1\}) \).

As before,

\[
\begin{align*}
|\Xi_{1,4}(t)| &\leq \left| \frac{\lambda_2'(t)}{\lambda_2(t) \eta(t)} \int_{\mathbb{R}^2} u \psi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \left( \frac{x}{\lambda_1^2(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \ dx \ dy \\
&\leq \left| \frac{\lambda_1'(t) \lambda_2^{q/2}(t) \lambda_2^{1/2}(t)}{\lambda_2(t) \eta(t)} \right| \|u_0\|_{L_2} \|\phi_{\delta_1}\|_{L_2^2} \|\phi_{\delta_2}\|_{L_2^2} \\
&\leq \frac{1}{t^{2-b(1+q/2+r/2)} \ln^{2+q/2+r/2}(t)}.
\end{align*}
\]

Hence we obtain that \( \Xi_{1,4} \in L^1(\{t \gg 1\}) \).

Now, combining (3.12), (3.14), (3.15) and (3.20) we have

\[
0 \leq \Xi_{1,1,3}(t) = \frac{d\Xi_{1,1,3}(t)}{dt} - \Xi_{1,1,1}(t) - \Xi_{1,1,2}(t) - \Xi_{1,1,4}(t) \\
- \Xi_{1,2}(t) - \Xi_{1,3}(t) - \Xi_{1,4}(t) - \Xi_2(t) \\
\leq \frac{d\Xi_{1,1,3}(t)}{dt} - \Xi_{1,1,1}(t) - \Xi_{1,1,2}(t) - \Xi_{1,1,4}(t) \\
+ \frac{1}{2} \Xi_{1,1,3}(t) + \Xi_{1,2}(t) - \Xi_{1,3}(t) - \Xi_{1,4}(t) - \Xi_2(t).
\]

Thus,

\[
\begin{align*}
\frac{1}{2} \Xi_{1,1,3}(t) &\leq \frac{d\Xi_{1,1,3}(t)}{dt} - \Xi_2(t) - \Xi_{1,1,1}(t) - \Xi_{1,1,2}(t) - \Xi_{1,1,4}(t) \\
&\quad + \Xi_{1,2}(t) - \Xi_{1,3}(t) - \Xi_{1,4}(t). \\
\end{align*}
\]

Note that all the terms on the right above, including \( \frac{d\Xi_{1,1,3}(t)}{dt} \), lie in \( L^1(\{t \gg 1\}) \), by (3.13), (3.16), (3.17) and (3.19)–(3.22). In consequence, we conclude (3.11).

### 3.4. The non-centered case

Here we explain how Theorem 1.1, can be extended to the non-centered case, as explained in Remark 1.1. The proof is simple and require some minor modifications of the proof in the centered case. First of all, we consider this time the functional (compare with (3.4))
Lemma 3.4. Let $b$ described in (3.3). Then for some $C_0 > 0$ we have

$$\int_t^{\infty} \frac{1}{s \ln s} \left( \int_{\Omega(s)} u^2(x, y, s) \, dx \, dy \right) \, ds \leq C_0 \int_t^{\infty} \frac{1}{s \ln^{1+1/b}(s)} \, ds, \quad t \gg 1,$$

(3.24)
and
\[
\int_a^b \frac{1}{s \ln s} \left( \int_{\Omega(s)} u^2(x, y, s) \, dx \, dy \right) \, ds \leq \frac{C_0}{\ln^{1+1/b}(a)} + C_0 \int_a^b \frac{1}{s \ln^{1+1/b}(s)} \, ds \quad a, b \gg 1. \tag{3.25}
\]

**Proof.** We adopt the same notation given in the proof of Lemma 3.2. After comparing the terms in (3.23) we set \(q = \frac{2}{b} - r - 2\), for \(0 < b < \frac{3}{r+2}\). From (3.13), (3.16), (3.17) and (3.19)–(3.22) we have
\[
|\Xi_{1,1}(t)|, \quad |\Xi_{1,2}(t)|, \quad |\Xi_{1,4}(t)| \leq \frac{1}{t \ln^{1+1/b}(t)} \quad \text{and} \quad |\Xi_2(t)|, \quad |\Xi_{1,3}(t)| \leq \frac{1}{t \ln^{2+1/b}(t)} = \frac{1}{t \ln^{1+1/b}(t)}.
\]
Consequently by (3.23) we have
\[
\Xi_{1,1,3}(t) \leq \frac{d\Xi}{dt}(t) + \frac{C_1}{t \ln^{1+1/b}(t)}, \quad t \gg 1,
\]
for some \(C_1 > 0\). Integrating on time and taking into consideration (3.10) we get for \(t \gg 1\)
\[
\int_t^\infty \Xi_{1,1,3}(s) \, ds \leq \int_t^\infty \frac{d\Xi}{ds}(s) \, ds + C_1 \int_t^\infty \frac{1}{s \ln^{1+1/b}(s)} \, ds
\]
\[
\leq -\Xi(t) + C_1 \int_t^\infty \frac{1}{s \ln^{1+1/b}(s)} \, ds
\]
\[
\leq \frac{C_2}{\ln^{1+1/b}(t)} + C_1 \int_t^\infty \frac{1}{s \ln^{1+1/b}(s)} \, ds
\]
\[
\leq \frac{C_2}{1+1/b} \int_t^\infty \frac{1}{s^{2+1/b}(s)} \, ds + C_1 \int_t^\infty \frac{1}{s \ln^{1+1/b}(s)} \, ds \leq C_0 \int_t^\infty \frac{1}{s \ln^{1+1/b}(s)} \, ds,
\]
where \(C_0 = \frac{C_2}{1+1/b} + C_1\) and \(C_2 = ||u_0||_{L^2_{\xi,y}} ||\psi_\eta||_{L^\infty_{\xi,y}} ||\phi_{\xi_0}||_{L^2_{\xi}} ||\phi_{\xi_1}||_{L^2_{\xi}}\). Thus, (3.24) is satisfied. Finally by a similar argument we conclude (3.25).

**Lemma 3.5.** Let \(\epsilon > 0\) and \(E_\epsilon := \{ s > 10 : \int_{\Omega(s)} u^2(x, y, s) \, dx \, dy > \epsilon \}\) and \(b\) described in (3.3). Then the following conditions are satisfied:

1. \(E_\epsilon = \bigsqcup_{n \in \mathbb{N}} (a_n, b_n)\) (disjoint union).
2. If \(a_n \gg 1\) then we have
\[
b_n < a_n \left( \exp \left( \frac{2C_0}{\ln^{1/b}(a_n)} \right) \right). \tag{3.26}
\]

**Proof.** Since \(E_\epsilon\) is open on \(\mathbb{R}\) we have that \(E_\epsilon = \bigsqcup_{n \in \mathbb{N}} I_n\), where \(I_n\) is an open interval for each \(n \in \mathbb{N}\). We note that \(I_n\) can not be unbounded by (3.24). Hence item (1) is satisfied. Now, we prove item (2). First we note from (3.3) that \(b < \frac{3}{r}\). Let \(F(s) := \int_{\Omega(s)} u^2(x, y, s) \, dx \, dy\) and \(b_n > a_n \gg 1\). Since \(F(s) > \epsilon\) for \(s \in (a_n, b_n)\) and (3.25) is satisfied we obtain
\[
\int_{a_n}^{b_n} \frac{\epsilon}{s \ln(s)} \, ds < \int_{a_n}^{b_n} \frac{F(s)}{s \ln(s)} \, ds \leq \frac{C_0}{\ln^{1+1/b}(a_n)} + C_0 \int_{a_n}^{b_n} \frac{1}{s \ln^{1+1/b}(s)} \, ds,
\]
then we have
\[ e \left( \ln \left( \ln (b_n) \right) - \ln \left( \ln (a_n) \right) \right) < \frac{C_0}{\ln^{1+1/b}(a_n)} + \frac{C_0}{1/b} \left( \frac{1}{\ln^{1/b}(a_n)} - \frac{1}{\ln^{1/b}(b_n)} \right). \]

The last inequality implies
\[ e \left( \ln \left( \ln (b_n) \right) \right) < e \ln \left( \ln (a_n) \right) + \frac{C_0}{\ln^{1/b}(a_n)} + \frac{C_0}{1/b} \ln^{1/b}(a_n) \]
\[ < e \ln \left( \ln (a_n) \right) + \frac{C_0}{\ln^{1/b}(a_n)} + \frac{C_0}{1/b}, \]

consequently (3.26) is satisfied.

\[ \square \]

**Remark 3.1** (An explicit construction of the times \( t_n \)). Item (2) of the Lemma 3.5 is useful to specify the times of a sequence \((t_n)_n \uparrow \infty\) as \( n \) goes to infinity such that
\[ \lim_{n \to \infty} \int_{\Omega(t_n)} u^2(x, y, t_n) \, dx \, dy = 0. \]

In fact let \( F(s) := \int_{\Omega(s)} u^2(x, y, s) \, dx \, dy, t_0 \gg 1 \) and \( \epsilon_1 = F(t_0)/2 \). From (3.26) we infer \( F(t_1) \leq \epsilon_1 \), if \( t_1 = t_0 \exp \left( \frac{2C_0}{\ln^{1/b}(a_0)} \right) \). Similar as above for \( \epsilon_2 = F(t_1)/2 \) we have \( F(t_1) \leq \epsilon_2 \) if \( t_2 = t_1 \exp \left( \frac{2C_0}{\ln^{1/b}(t_1)} \right) \). Recursively the sequence of time \((t_n)\) described by
\[ t_{n+1} = t_n \exp \left( \frac{2C_0}{\ln^{1/b}(t_n)} \right), \]

is such that
\[ \int_{\Omega(t_n)} u^2(x, y, t_n) \, dx \, dy \leq F(t_0)/2^n \to 0, \]
as \( n \) goes to infinity.

**4. Asymptotic behavior of solutions in \( H^1(\mathbb{R}^2) \)**

The purpose of this section is to prove Theorem 1.6. We follow similar ideas as in previous section, with two additional ingredients. First, we use Theorem 1.2, more precisely, Lemma 3.1. Second, we use some technical estimates for nonlinear terms first introduced by Kenig and Martel [40] in the case of the Benjamin-Ono equation.

Recall the region \( \Omega(t) \) introduced in (1.1), with the additional assumption \( 1 < r < 3 \).

Theorem 1.6 follows from the following result.

**Lemma 4.1.** Assume now that \( u_0 \in H^1(\mathbb{R}^2) \). Let \( u \in (C \cap L^\infty) (\mathbb{R} : H^1(\mathbb{R}^2)) \) be the corresponding unique solution of (ZK) with initial data \( u(t = 0) = u_0 \). Then, there exists a constant \( c > 0 \) such that
\[ \int_{\{t > 1\}} \frac{1}{t \ln t} \left( \int_{\Omega(t)} |\nabla u|^2(x, y, t) \, dx \, dy \right) \, dt \leq c < \infty. \]  
(4.1)
Recall $\psi$ and $\phi$ defined in Section 2.1. In what follows, $\sigma'$ is a positive constant to be determined later. To prove (4.1), we consider the functional

$$Q(t) := \frac{1}{\eta(t)} \int_{\mathbb{R}^2} u^2(x, y, t) \psi(x (\lambda_1(t)), y (\lambda_2(t))) \, dx \, dy,$$

that is clearly well defined for solutions of the IVP (ZK).

Next, we compute:

$$\frac{d}{dt} Q(t) = \frac{2}{\eta(t)} \int_{\mathbb{R}^2} u \partial_t u \psi(x (\lambda_1(t)), y (\lambda_2(t))) \, dx \, dy$$

$$- \frac{\lambda_2'(t)}{\lambda_2(t) \eta(t)} \int_{\mathbb{R}^2} u^2 \left( x \phi_{\lambda_1(t)} \phi_{\lambda_2(t)} \right) \, dx \, dy$$

$$- \frac{\lambda_1'(t)}{\lambda_1(t) \eta(t)} \int_{\mathbb{R}^2} u^2 \phi_{\lambda_1(t)} \phi_{\lambda_2(t)} \, dx \, dy$$

$$- \frac{\eta'(t)}{\eta^2(t)} \int_{\mathbb{R}^2} u^2 \phi_{\lambda_1(t)} \phi_{\lambda_2(t)} \, dx \, dy.$$ 

Now we bound each of the terms above. In the first place we have, after applying integration by parts, that

$$A_1(t) = \frac{2}{\eta(t)} \int_{\mathbb{R}^2} \partial_x u \left( \Delta u + \frac{u^2}{2} \right) \psi(x (\lambda_1(t)), y (\lambda_2(t))) \, dx \, dy$$

$$+ \frac{2}{\lambda_1(t) \eta(t)} \int_{\mathbb{R}^2} u \left( \Delta u + \frac{u^2}{2} \right) \phi_{\lambda_1(t)} \phi_{\lambda_2(t)} \, dx \, dy$$

$$= A_{1,1}(t) + A_{1,2}(t).$$

In this sense, we have that

$$A_{1,1}(t) = - \frac{1}{\lambda_1(t) \eta(t)} \int_{\mathbb{R}^2} (\partial_x u)^2 \phi_{\lambda_1(t)} \phi_{\lambda_2(t)} \, dx \, dy$$

$$+ \frac{1}{\lambda_1(t) \eta(t)} \int_{\mathbb{R}^2} (\partial_t u)^2 \phi_{\lambda_1(t)} \phi_{\lambda_2(t)} \, dx \, dy$$

$$- \frac{2}{\lambda_2(t) \eta(t)} \int_{\mathbb{R}^2} \partial_x u \partial_x u \psi(x (\lambda_1(t)), y (\lambda_2(t))) \, dx \, dy$$

$$- \frac{1}{3 \eta(t) \lambda_1(t)} \int_{\mathbb{R}^2} u^3 \phi_{\lambda_1(t)} \phi_{\lambda_2(t)} \, dx \, dy$$

$$= A_{1,1,1}(t) + A_{1,1,2}(t) + A_{1,1,3}(t) + A_{1,1,4}(t).$$
The terms $A_{1,1,1}$ and $A_{1,1,2}$ are essentially some portion of the quantities appearing in (4.1). The others come from $A_{1,2}(t)$. Instead, the terms $A_{1,1,3}$ and $A_{1,1,4}$ need to be estimated.

First, we handle $A_{1,1,3}$. We obtain

$$
\int_{\{t \geq 1\}} |A_{1,1,3}(t)| \, dt \leq 3 \|u\|_{L^2_t H^1_x}^2 \int_{\{t \geq 1\}} \eta(t) \lambda_2(t) < \infty,
$$

whenever $r > 1$ (see (3.5) and (3.7)). This is the extra condition needed for the proof of Theorem 1.6.

Next, we focus our attention into $A_{1,1,4}(t)$. Here we consider a smooth cutoff function $\chi : \mathbb{R} \to \mathbb{R}$ such that

$$
\chi \equiv 1 \text{ on } [0,1], \quad 0 \leq \chi \leq 1 \quad \text{and} \quad \chi \equiv 0 \text{ on } (-\infty, -1] \cup [2, \infty).
$$

For $n \in \mathbb{Z}$, we set $\chi_n(x) := \chi(x - n)$, such that $\chi_n \equiv 1$ in $[n, n + 1]$. Similarly, we define for $m \in \mathbb{Z}$ the function $\chi_m(y) := \chi(y - m)$.

First, notice that by the Gagliardo-Nirenberg-Sobolev inequality

$$
\int_{\mathbb{R}^2} |u|^3 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy
$$

$$
= \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \int_{m}^{m+1} \int_{n}^{n+1} |u|^3 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy
$$

$$
\leq \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \|u\chi_n\|_{L^3_{xy}} \left( \max_{x \in [n, n+1]} \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \right) \left( \max_{y \in [m, m+1]} \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \right)
$$

$$
\leq \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \|\nabla (u\chi_n\chi_m)\|_{L^2_{xy}} \|u\chi_n\chi_m\|_{L^2_{xy}} \left( \max_{x \in [n, n+1]} \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \right) \left( \max_{y \in [m, m+1]} \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \right).
$$

Nevertheless, by hypothesis

$$
\|\nabla (u\chi_n\chi_m)\|_{L^2_{xy}} = \|\chi_n\chi_m \nabla u + u \nabla (\chi_n\chi_m)\|_{L^2_{xy}} \leq \|u(t)\|_{H^1(\mathbb{R}^1)} \leq c \|u\|_{L^2_t H^1_x} < \infty.
$$

Therefore, we obtain the simpler bound

$$
\int_{\mathbb{R}^2} |u|^3 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy
$$

$$
\leq \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \|u\chi_n\chi_m\|_{L^2_{xy}}^2 \left( \max_{x \in [n, n+1]} \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \right) \left( \max_{y \in [m, m+1]} \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \right)
$$

(4.6)

Also, from (2.2),

$$
e^{-\|x\|_{\alpha_1(t)}} \leq \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \leq 3e^{-\|x\|_{\alpha_1(t)}}, \quad e^{-\|x\|_{\alpha_2(t)}} \leq \phi_{\delta_2} \left( \frac{y}{\lambda_2(t)} \right) \leq 3e^{-\|x\|_{\alpha_2(t)}}.
$$

Then, we consider the cases described below.

- **Case $x > 0$**: In this case we have that

$$
\max_{x \in [n, n+1]} \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \leq 3 \max_{x \in [n, n+1]} e^{-\|x\|_{\alpha_1(t)}} = 3e^{-\|x\|_{\alpha_1(t)}}.
$$

(4.7)
Additionally, the minimum value at the same interval is given by
\[ \min_{x \in [n, n+1]} \phi_\sigma'(\frac{x}{\lambda_1(t)}) \geq \min_{x \in [n, n+1]} e^{-\frac{\left| x \right|}{\lambda_1(t)}} = e^{-\frac{\left| x \right|}{\lambda_1(t)}} e^{-\frac{\left| x \right|}{\lambda_1(t)}}, \tag{4.8} \]
so that, after combining both inequalities above we get
\[ \max_{x \in [n, n+1]} \phi_\sigma'(\frac{x}{\lambda_1(t)}) \leq 3e^{\frac{x}{\lambda_1(t)}} \min_{x \in [n, n+1]} \phi_\sigma'(\frac{x}{\lambda_1(t)}), \quad \text{whenever } x > 0. \tag{4.9} \]

- Case \( x \leq 0 \): By parity, we obtain a similar bound as before.

Finally, combining both cases and using that \( t \gg 1 \), we get that for a universal constant \( C \),
\[ \max_{x \in [n, n+1]} \phi_\sigma'(\frac{x}{\lambda_1(t)}) \leq C \min_{x \in [n, n+1]} \phi_\sigma'(\frac{x}{\lambda_1(t)}), \quad \text{for all } x \in \mathbb{R}. \tag{4.10} \]

A similar analysis yields
\[ \max_{y \in [m, m+1]} \phi_{\delta_1}'\left(\frac{y}{\lambda_2(t)}\right) \leq C \min_{y \in [m, m+1]} \phi_{\delta_1}'\left(\frac{y}{\lambda_2(t)}\right), \quad \text{for all } y \in \mathbb{R}. \]

Next, we incorporate the two last estimates above into the original one (4.6). Combined with the Monotone Convergence Theorem, we get
\[
\int_{\mathbb{R}^2} |u|^3 \phi_\sigma'(\frac{x}{\lambda_1(t)}) \phi_{\delta_1}'\left(\frac{y}{\lambda_2(t)}\right) \, dx \, dy \\
\leq \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \|u\chi_m \chi_n\|^2_{L_\eta^2} \left( \max_{x \in [n, n+1]} \phi_\sigma'(\frac{x}{\lambda_1(t)}) \right) \left( \max_{y \in [m, m+1]} \phi_{\delta_1}'\left(\frac{y}{\lambda_2(t)}\right)\right) \\
\leq \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \left( \int_{\mathbb{R}^2} u^2 \chi_m \chi_n \phi_\sigma'(\frac{x}{\lambda_1(t)}) \phi_{\delta_1}'\left(\frac{y}{\lambda_2(t)}\right) \, dx \, dy \right) \\
\leq \int_{\mathbb{R}^2} u^2 \phi_\sigma'(\frac{x}{\lambda_1(t)}) \phi_{\delta_1}'\left(\frac{y}{\lambda_2(t)}\right) \, dx \, dy.
\]
(Note that the implicit constants in the inequalities above does not depend on the variable \( m \) nor \( n \).)

In summary, we have proved that
\[
|A_{1,1,4}(t)| \leq \frac{1}{\eta(t) \lambda_1(t)} \int_{\mathbb{R}^2} |u|^3 \phi_\sigma'(\frac{x}{\lambda_1(t)}) \phi_{\delta_1}'\left(\frac{y}{\lambda_2(t)}\right) \, dx \, dy \\
\leq \frac{1}{\eta(t) \lambda_1(t)} \int_{\mathbb{R}^2} u^2 \phi_\sigma'(\frac{x}{\lambda_1(t)}) \phi_{\delta_1}'\left(\frac{y}{\lambda_2(t)}\right) \, dx \, dy.
\]
Nevertheless, by (3.11), we have for \( \sigma' > 0 \) satisfying
\[
\frac{1}{\sigma} + \frac{1}{\delta_1} \leq \frac{1}{\sigma'},
\]
one has...
\[
\int_{\{t \geq 1\}} \left( \frac{1}{\eta(t) \lambda_1(t)} \int_{\mathbb{R}^2} u^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy \right) \, dt < \infty,
\]

which clearly implies that \( A_{1,1,4} \in L^1(\{t \gg 1\}) \). Summarizing, \( A_{1,1} \) defined in (4.3) and (4.4) satisfies

\[
A_{1,1}(t) = A_{1,1,1}(t) + A_{1,1,2}(t) + A_{\text{int}}(t), \quad A_{\text{int}}(t) \in L^1(\{t \gg 1\}).
\]

Next, recall \( A_{1,2}(t) \) from (4.3). We have

\[
A_{1,2}(t) = \frac{2}{\lambda_1(t) \eta(t)} \int_{\mathbb{R}^2} (\partial_y u)^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy
\]

\[
- \frac{2}{\lambda_1(t) \eta(t)} \int_{\mathbb{R}^2} (\partial_x u)^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy
\]

\[
+ \frac{2}{\lambda_1(t) \eta(t) \lambda_2(t)} \int_{\mathbb{R}^2} u^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy
\]

\[
= A_{1,2,1}(t) + A_{1,2,2}(t) + A_{1,2,3}(t) + A_{1,2,4}(t) + A_{1,2,5}(t).
\]

We only will be focus on the terms \( A_{1,2,2} \) and \( A_{1,2,4} \), since \( A_{1,2,1} \) is the quantity to be estimated after integrating in time in (4.1), the same as \( A_{1,2,3} \). Note additionally that the bad sign term \( (\partial_y u)^2 \) in (4.4) is solved by adding the term \( A_{1,2,3} \). Finally, for \( A_{1,2,5} \) we described above how to obtain upper bounds that implies \( A_{1,2,5} \in L^1(\{t \gg 1\}) \), and we omit its proof here.

Thus, since from (3.6) \( 3b + p > 1 \),

\[
|A_{1,2,2}(t)| \leq \|u_0\|_{L_2} \frac{1}{\lambda_1^2(t) \eta(t)} \in L^1(\{t \gg 1\}).
\]

For \( A_{1,2,4} \) we have that \( p + b + 2br > 1 \) and

\[
|A_{1,2,4}(t)| \leq \|u_0\|_{L_2} \frac{1}{\eta(t) \lambda_1(t) \lambda_2^2(t)} \in L^1(\{t \gg 1\}).
\]

We partially conclude from (4.2) and the previous computations that

\[
\frac{d}{dt} Q(t) = -\frac{3}{\lambda_1(t) \eta(t)} \int_{\mathbb{R}^2} (\partial_y u)^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy
\]

\[
- \frac{1}{\lambda_1(t) \eta(t)} \int_{\mathbb{R}^2} (\partial_x u)^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_2(t)} \right) \, dx \, dy
\]

\[
+ A_{\text{int}}(t) + A_2(t) + A_3(t) + A_4(t),
\]

with \( A_{\text{int}}(t) \in L^1(\{t \gg 1\}) \). Finally, we consider the remainders terms in (4.2). First,

\[
|A_2(t)| \leq \|u_0\|_{L_2} \frac{\lambda_2(t)}{\eta(t) \lambda_2(t)} \in L^1(\{t \gg 1\}).
\]
since $p > 0$. The term $A_3(t)$ is completely similar. Finally,

$$|A_4(t)| \leq \eta(t)^2 \frac{\eta'(t)}{\eta(t)} \in L^1\{t \gg 1\},$$

since $p > 0$. Gathering these estimates in (4.11), we conclude (4.1) in the same vein as in the proof of Theorem 1.2.

5. The 3D case. Proof of Theorem 1.4

The proof in the 3D case follow similar lines as the one in 2D, but it is clearly more cumbersome.

5.1. $L^1$ virial and $L^2$ local decay

As we did in the previous case, we start by defining the functional that will provide the mass behavior associated with the solutions to (ZK) in the case $d = 3$. More precisely, we set

$$\Xi(t) = \frac{1}{\eta(t)} \int_{\mathbb{R}^3} u(x, y, z, t)\psi_\epsilon \left( \frac{x}{\hat{\lambda}_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\hat{\lambda}_2(t)} \right) \phi_{\delta_2} \left( \frac{y}{\hat{\lambda}_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\hat{\lambda}_4(t)} \right) \ dx \ dy \ dz, \quad (5.1)$$

where

$$\hat{\lambda}_1(t) = \frac{t^{p_1}}{\ln q_1(t)}, \quad \hat{\lambda}_2(t) = t^{p_2}, \quad \hat{\lambda}_3(t) = t^{p_3}, \quad \text{and} \quad \hat{\lambda}_4(t) = t^{p_4}, \quad (5.2)$$

with $p_1, p_2, p_3, p_4 > 0$ and $q_1 > 0$ parameters to be determined. Also, we consider

$$\eta(t) = t^{r_1} \ln t. \quad (5.3)$$

We will consider $p_1, r_1$ and $q_1, r_2$ satisfying the following conditions:

$$r_1 = 1 - p_1, \quad p_1, r_1 > 0, \quad r_2 = 1 + q_1, \quad q_1 > 0. \quad (5.4)$$

The parameters $p_1, p_2, p_3$ and $p_4$, are chosen in such a way to satisfy the following array of conditions

$$p_1, p_2, p_3, p_4 > 0, \quad p_1 < 1, \quad (5.5)$$

$$0 < 2p_1 + p_2 + p_3 + p_4 < 2, \quad (5.6)$$

$$p_2 > p_1, \quad (5.7)$$

$$p_3 > p_1, \quad (5.8)$$

$$p_4 > p_1, \quad (5.9)$$

$$p_1 > \frac{1}{3}(p_3 + p_4), \quad (5.10)$$

$$\frac{1}{2}p_1 + p_2 > \frac{1}{2}(p_3 + p_4), \quad (5.11)$$

$$p_2 > \frac{1}{4}(p_1 + p_3 + p_4), \quad (5.12)$$
\[ p_3 > \frac{1}{3}(p_1 + p_4), \]  
(5.13)  
\[ p_4 > \frac{1}{3}(p_1 + p_3), \]  
(5.14)  
\[ p_2 > \frac{1}{5}(2p_1 + p_3 + p_4), \]  
(5.15)  
\[ p_2 + 3p_3 > p_4 + 2p_1, \]  
(5.16)  
\[ p_2 + 3p_4 > p_3 + 2p_1, \]  
(5.17)  

and
\[ 3p_1 + (p_3 + p_4) < 2. \]  
(5.18)  

Recall (5.2). Define now
\[ \mathcal{P} := \{ (p_1, p_2, p_3, p_4) \in (0, \infty)^4 : (5.5) - (5.18) \text{ are satisfied} \}, \]

and
\[ \Omega(t) := \{ (x, y, z) \in \mathbb{R}^3 : |x| \leq \lambda_1(t), \ |y| \leq \lambda_3(t), \ |z| \leq \lambda_4(t), \ (p_1, p_2, p_3, p_4) \in \mathcal{P} \}. \]

This set is precisely the set (1.9) stated in Theorem 1.4, but with plenty of redundant conditions inside. Before continuing, we need some simplifications in conditions (5.4)–(5.18), that will lead to the simpler definition of \( \Omega(t) \) in (1.9).

**Lemma 5.1.** \( \mathcal{P} \) is nonempty. Moreover, it can be reduced to the following simpler set of conditions
\[ \mathcal{P} = \{ (p_1, p_2, p_3, p_4) \in (0, \infty)^4 : (5.5) - (5.13), (5.10) \text{ and (5.14) are satisfied.} \}. \]

In particular, the conditions \( p_1 < \frac{1}{2} \) and \( p_2, p_3, p_4 > p_1 \) must be satisfied.

Using this result, it is easy to describe the set \( \Omega(t) \) in (1.12), probably taking \( b \) slightly smaller if needed. Just redefine \( p_1 = b \), \( p_3 = br_1 \) and \( p_4 = br_2 \). The parameter \( p_2 \) can be written as \( p_1 + \epsilon_0 = b + \epsilon_0 \), any \( \epsilon_0 > 0 \), and it is a free parameter, leading to the last, nonlinear condition in (1.12). Also, the two conditions (5.8)–(5.9) are not needed for proving \( L^2 \) decay, only for proving decay of the \( \dot{H}^1 \) norm.

**Proof of Lemma 5.1.** For the proof we consider the following set
\[ \mathcal{T} := \left\{ (p, r, q, q) \in \mathbb{R}^4 \mid 0 < p < r < \frac{1}{3}, \ p < q < \frac{3p}{2} \right\}. \]  
(5.19)

We claim that the set \( \mathcal{T} \) is not empty. In fact, for \( \epsilon > 0 \), the point whose coordinates are given by \( (p, r, q, q) = (\frac{1}{3} - 2\epsilon, \frac{1}{6} - \epsilon, \frac{1}{3}, \frac{1}{3}) \in \mathcal{T} \), whenever \( \epsilon < 1/18 \).

Additionally, each point \( (p, r, q, q) \in \mathcal{T} \) satisfies the following chain of inequalities
\[ p - \frac{r}{2} < \frac{p}{2} < p < q < \frac{3p}{2} < \frac{p}{2} + r < 2r - \frac{p}{2} < \frac{5r}{2} - p < 1 - \frac{r}{2} - p < 1 - \frac{3p}{2}. \]  
(5.20)

In fact it can be verified that any point \( (p, r, q, q) \) satisfying (5.20) also satisfy the conditions (5.5)–(5.18), that are precisely the conditions that define to \( \mathcal{P} \). In summary, we have proved that \( \mathcal{T} \subset \mathcal{P} \).
On the other hand, after defining
\[ \tilde{p}_2 := \frac{p_2}{p_1}, \quad \tilde{p}_3 := \frac{p_3}{p_1}, \quad \tilde{p}_4 := \frac{p_4}{p_1}, \]
Eqs. (5.10), (5.13) and (5.14) become
\[ 1 > \frac{1}{3} (\tilde{p}_3 + \tilde{p}_4), \quad \tilde{p}_3 > \frac{1}{3} (1 + \tilde{p}_4), \quad \tilde{p}_4 > \frac{1}{3} (1 + \tilde{p}_3). \tag{5.21} \]
The solution to this system of inequalities corresponds to the interior of the triangle of vertices \( \left( \frac{1}{2}, \frac{1}{2} \right), (2, 1) \) and \((1, 2)\) in the \( \tilde{p}_3 - \tilde{p}_4 \) plane (see Figure 4). The extrema of the function \( a := \tilde{p}_3 + \tilde{p}_4 \) in this set satisfies
\[ 1 < a = \tilde{p}_3 + \tilde{p}_4 < 3. \]
Also, it is not difficult to check
\[ \tilde{p}_4 - 3\tilde{p}_3 < -1, \quad \tilde{p}_3 - 3\tilde{p}_4 < -1. \tag{5.22} \]
Now, Eqs. (5.7), (5.11), (5.12) and (5.15) reduce to
\[ \tilde{p}_2 > 1, \quad \tilde{p}_2 > \frac{1}{2} (a - 1), \quad \tilde{p}_2 > \frac{1}{4} (1 + a), \quad \tilde{p}_2 > \frac{1}{5} (2 + a). \tag{5.23} \]
A quick checking (see Figure 4) reveals that the condition \( \tilde{p}_2 > 1 \) is the most restrictive one. Also, from (5.16) and (5.17) one has
\[ \tilde{p}_2 > \tilde{p}_4 - 3\tilde{p}_3 + 2, \quad \tilde{p}_2 > \tilde{p}_3 - 3\tilde{p}_4 + 2. \]
From the condition \( \tilde{p}_2 > 1 \) and (5.22) we have that the two last conditions are redundant.
Now we consider the conditions (5.6) and (5.18). Written in terms of normalized variables, one has
\[ 0 < 2 + \tilde{p}_2 + a < \frac{2}{\tilde{p}_1}, \quad 3 + a < \frac{2}{\tilde{p}_1}. \]
Since \( a > 0 \) and \( \tilde{p}_2 > 1 \), always \( 2 + \tilde{p}_2 + a > 3 + a \). Therefore, the last condition is redundant. Finally, (5.5) and (5.6) imply that \( p_1 \) must be below \( \frac{1}{2} \).

There are two remaining conditions to be considered. These are (5.8) and (5.9), which become \( \tilde{p}_3 > 1 \) and \( \tilde{p}_4 > 1 \). The representation of these conditions can be found in Figure 4, left panel.

This ends the proof of the lemma. \( \square \)

Now, we continue with the estimate of \( \Xi(t) \). 

CLAIM: Under (5.4) and (5.6), the functional (5.1) is well-defined.

Proof. Assume (5.4) and (5.6). Since \( u \in L^2 \) and the mass is conserved, it is clear that

\[
|\Xi(t)| \leq \|u_0\|_{L^2} \frac{\left( \lambda_2(t) \lambda_3(t) \lambda_4(t) \right)^{1/2}}{\eta(t)}.
\]

Therefore, from (5.2) and (5.3),

\[
\sup_{t \geq 1} |\Xi(t)| < \infty,
\]

and the claim is proved. \( \square \)

5.1.1. Mass behavior

Now we compute the evolution of \( \Xi(t) \). We follow similar estimates as the one performed to prove Lemma 3.3. First of all,

\[
\frac{d}{dt} \Xi(t) = \frac{1}{\eta(t)} \int_{\mathbb{R}^2} \partial_t u \psi_+ \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) dx dy dz
\]

\[
- \frac{\eta'(t)}{\eta^2(t)} \int_{\mathbb{R}^2} u \psi_+ \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) dx dy dz
\]

\[
+ \frac{1}{\eta(t)} \int_{\mathbb{R}^2} u \partial_t \left( \psi_+ \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \right) dx dy dz
\]

\[
= \Xi_1(t) + \Xi_2(t) + \Xi_3(t).
\]

(5.24)

First, we bound \( \Xi_1 \). Using (ZK) in the 3D case, we get after applying integration by parts

\[
\Xi_1(t)
\]

\[
= \frac{1}{\eta(t)} \int_{\mathbb{R}^3} \left( \Delta u + \frac{u^2}{2} \right) \partial_x \left( \psi_+ \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) dx dy dz
\]

\[
= \frac{1}{\eta(t) \lambda_1(t)} \int_{\mathbb{R}^3} \left( \Delta u + \frac{u^2}{2} \right) \phi_+ \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) dx dy dz
\]

\[
+ \frac{1}{\eta(t) \lambda_2(t)} \int_{\mathbb{R}^3} \left( \Delta u + \frac{u^2}{2} \right) \psi_+ \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1}' \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) dx dy dz
\]

\[
= \Xi_{1,1}(t) + \Xi_{1,2}(t).
\]

(5.25)
The term $\Xi_{1,1}$ will be treated as follows:

$$
\Xi_{1,1}(t) = 
\frac{1}{\eta(t) \lambda_1(t)} \int_{\mathbb{R}^3} \Delta \mu \phi_\sigma \left( \frac{x}{\lambda_1(t)} \right) \phi_\delta_1 \left( \frac{x}{\lambda_2(t)} \right) \phi_\delta_2 \left( \frac{y}{\lambda_3(t)} \right) \phi_\delta_3 \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz 
$$

(5.26)

$$
= \Xi_{1,1,1}(t) + \Xi_{1,1,2}(t).
$$

The term $\Xi_{1,1,2}(t)$ is precisely the term that we want to estimate, and we save it. For $\Xi_{1,1,1}$, using integration by parts, we obtain

$$
\Xi_{1,1,1}(t) = 
\frac{1}{\eta(t) \lambda_1(t)} \int_{\mathbb{R}^3} \Delta \mu \phi_\sigma \left( \frac{x}{\lambda_1(t)} \right) \phi_\delta_1 \left( \frac{x}{\lambda_2(t)} \right) \phi_\delta_2 \left( \frac{y}{\lambda_3(t)} \right) \phi_\delta_3 \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz 
$$

$$
= \Xi_{1,1,1,1}(t) + \Xi_{1,1,1,2}(t) + \Xi_{1,1,1,3}(t) + \Xi_{1,1,1,4}(t) + \Xi_{1,1,1,5}(t).
$$

Recall that $p_2 > p_1$ from (5.7). Therefore, $\Xi_{1,1,1,1}$ is bounded as

$$
|\Xi_{1,1,1,1}(t)| \lesssim \frac{(\lambda_3(t) \lambda_4(t))^{1/2}}{\eta(t) (\lambda_1(t))^{5/2}} \in L^1(\{t \gg 1\}),
$$

since $p_1 > \frac{1}{4} (p_3 + p_4)$ in (5.10). Next,

$$
|\Xi_{1,1,1,2}(t)| \lesssim \frac{(\lambda_3(t) \lambda_4(t))^{1/2}}{\eta(t) (\lambda_1(t))^{3/2} \lambda_2(t)} \in L^1(\{t \gg 1\}),
$$

whenever \( \frac{p_1}{2} + p_2 > \frac{p_3 + p_4}{2} \) in (5.11).

The estimates for $\Xi_{1,1,1,3}(t), \Xi_{1,1,1,4}(t), \text{ and } \Xi_{1,1,1,5}(t)$ are very similar in nature. For $\Xi_{1,1,1,3}$ we have that

$$
|\Xi_{1,1,1,3}(t)| \lesssim \frac{(\lambda_3(t) \lambda_4(t))^{1/2}}{\eta(t) (\lambda_1(t))^{1/2} (\lambda_2(t))^{3/2}} \in L^1(\{t \gg 1\}),
$$

since $p_2 > \frac{1}{4} (p_1 + p_3 + p_4)$ in (5.12). Next, for $\Xi_{1,1,1,4}$ we have that

$$
|\Xi_{1,1,1,4}(t)| \lesssim \frac{(\lambda_4(t))^{1/2}}{\eta(t) (\lambda_1(t))^{1/2} (\lambda_3(t))^{3/2}} \in L^1(\{t \gg 1\}),
$$
since $p_3 > \frac{1}{3}(p_1 + p_4)$ in (5.13). Finally, we focus our attention on $\Xi_{1,1,1,5}$, to get the bound
\[
|\Xi_{1,1,1,5}(t)| \lesssim \frac{(\lambda_3(t))^{1/2}}{\eta(t)(\lambda_1(t))^{1/2}(\lambda_4(t))^{3/2}} \in L^1\{t \gg 1\},
\]
valid for $p_4 > \frac{1}{3}(p_1 + p_3)$, thanks to (5.14). We conclude that
\[
|\Xi_{1,1,1}(t)| \lesssim \frac{1}{t^{1+\kappa_0}(t)} \in L^1\{t \gg 1\},
\]
for some fixed positive constants $\kappa_0$ and $\epsilon_0$ depending on the parameters $p_i$, $q_i$, $i = 1, 2, 3, 4$.

Concerning the term $\Xi_{1,2}(t)$ from (5.25), we first set
\[
\Xi_{1,2}(t) = \frac{1}{\eta(t)\lambda_2(t)} \int_{\mathbb{R}^3} \Delta u \psi_\sigma \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_1} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz
\]
\[
+ \frac{1}{2\eta(t)\lambda_2(t)} \int_{\mathbb{R}^3} u^2 \psi_\sigma \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_1} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz.
\]
Thus, after applying integration by parts,
\[
\Xi_{1,2,1}(t) = \frac{1}{\eta(t)(\lambda_2(t))^2} \int_{\mathbb{R}^3} u \phi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz
\]
\[
+ \frac{2}{\eta(t)(\lambda_2(t))^2} \int_{\mathbb{R}^3} u \phi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_1} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz
\]
\[
+ \frac{1}{\eta(t)(\lambda_2(t))^3} \int_{\mathbb{R}^3} u \psi_\sigma \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_1} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz
\]
\[
+ \frac{1}{\eta(t)(\lambda_2(t))^3} \int_{\mathbb{R}^3} u \psi_\sigma \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_1} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz
\]
\[
+ \frac{1}{\eta(t)(\lambda_2(t))^3} \int_{\mathbb{R}^3} u \psi_\sigma \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_1} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz
\]
\[
= \Xi_{1,2,1,1}(t) + \Xi_{1,2,1,2}(t) + \Xi_{1,2,1,3}(t) + + \Xi_{1,2,1,4}(t) + + \Xi_{1,2,1,5}(t).
\]

In the first place,
\[
|\Xi_{1,2,1,1}(t)| \lesssim \frac{(\lambda_3(t)\lambda_4(t))^{1/2}}{\eta(t)(\lambda_1(t))^{1/2}(\lambda_2(t))^{3/2}} \in L^1\{t \gg 1\},
\]
for $\frac{p_1}{2} + p_2 > \frac{p_3 + p_4}{2}$ as in (5.11). Next,
\[
|\Xi_{1,2,1,2}(t)| \lesssim \frac{(\lambda_3(t)\lambda_4(t))^{1/2}}{\eta(t)(\lambda_1(t))^{1/2}(\lambda_2(t))^{3/2}} \in L^1\{t \gg 1\},
\]
since $p_2 > \frac{p_1 + p_1 + p_4}{4}$ from (5.12). Similarly,

$$|\Xi_{1,2,1,3}(t)| \leq \frac{(\lambda_3(t)\lambda_4(t))^{1/2}}{\eta(t)(\lambda_2(t))^{3/2}} \in L^1(\{t \gg 1\}),$$

for $p_2 > \frac{1}{3}(2p_1 + p_3 + p_4)$, see (5.15). Again,

$$|\Xi_{1,2,1,4}(t)| \leq \frac{(\lambda_4(t))^{1/2}}{\eta(t)(\lambda_2(t))^{1/2}(\lambda_3(t))^{3/2}} \in L^1(\{t \gg 1\}),$$

for $p_2 + 3p_3 > p_4 + 2p_1$ as in (5.16). We conclude with

$$|\Xi_{1,2,1,5}(t)| \leq \frac{(\lambda_3(t))^{1/2}}{\eta(t)(\lambda_2(t))^{1/2}(\lambda_4(t))^{3/2}} \in L^1(\{t \gg 1\}),$$

for $p_2 + 3p_4 > p_3 + 2p_1$ as in (5.17). We conclude that $\Xi_{1,2,1}(t)$ satisfies

$$\Xi_{1,2,1}(t) \in L^1(\{t \gg 1\}).$$

Finally, from (5.27),

$$|\Xi_{1,2,2}(t)| = \left| \frac{1}{2\eta(t)\lambda_2(t)} \int_{\mathbb{R}^3} u^2 \psi_\sigma \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx dy dz \right| \leq \frac{1}{\eta(t)\lambda_2(t)} \in L^1(\{t \gg 1\}),$$

whenever $p_2 > p_1$, which is just (5.7).

Second, we bound $\Xi_2(t)$ from (5.24). Following a similar analysis as in the 2D case (3.10),

$$|\Xi_2(t)| \leq \left| \frac{\eta'(t)}{\eta^2(t)} \int_{\mathbb{R}^3} u \psi_\sigma \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx dy \right| \leq_{\sigma, \delta_1, \delta_2} \|u_0\|_{L^2} (\lambda_2(t)\lambda_3(t)\lambda_4(t))^{1/2}|\eta'(t)| \eta^2(t) \in L^1(\{t \gg 1\}),$$

which is valid if $2p_1 + p_2 + p_3 + p_4 < 2$, that is, (5.6). Next, from (5.24),

$$\Xi_3(t)$$

$$= -\frac{\lambda'_1(t)}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}^3} u \psi_\sigma \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_2} \left( \frac{x}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx dy dz$$

$$- \frac{\lambda'_2(t)}{\eta(t)\lambda_2(t)} \int_{\mathbb{R}^3} u \psi_\sigma \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx dy dz$$

$$- \frac{\lambda'_3(t)}{\eta(t)\lambda_3(t)} \int_{\mathbb{R}^3} u \psi_\sigma \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx dy dz$$

$$- \frac{\lambda'_4(t)}{\eta(t)\lambda_4(t)} \int_{\mathbb{R}^3} u \psi_\sigma \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx dy dz$$

$$= \Xi_{3,1}(t) + \Xi_{3,2}(t) + \Xi_{3,3}(t) + \Xi_{3,4}(t).$$
Concerning the term $\Xi_{3,1}$, we have for all $\epsilon > 0$,

$$|\Xi_{3,1}(t)| \leq \frac{\lambda'_1(t)}{4\eta(t) \lambda_1(t)} \left| \int_{\mathbb{R}^3} u^2 \phi_\sigma \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) dx dy dz \right|$$

$$+ \frac{\epsilon \lambda'_1(t)}{\eta(t) \lambda_1(t)} \left| \int_{\mathbb{R}^3} \phi_\sigma \left( \frac{x}{\lambda_1(t)} \right) \left( \frac{x}{\lambda_2(t)} \right)^2 \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_2} \left( \frac{z}{\lambda_4(t)} \right) dx dy dz \right|.$$  

Hence we get, after choosing $\epsilon = |\lambda'_1(t)| > 0$,

$$|\Xi_{3,1}(t)| \leq \frac{1}{4\eta(t) \lambda_1(t)} \left| \int_{\mathbb{R}^3} u^2 \phi_\sigma \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) dx dy dz \right|$$

$$+ \frac{(\lambda'_1(t))^2 \lambda_3(t) \lambda_4(t)}{\eta(t)}.$$  

Note that the first term in the R.H.S. above is half the quantity $\Xi_{1,1,2}(t)$ to be estimated, see (5.26). Therefore, that term is absorbed properly. Instead, the second quantity satisfies

$$\frac{(\lambda'_1(t))^2 \lambda_3(t) \lambda_4(t)}{\eta(t)} \in L^1(\{t \gg 1\}),$$

whenever $3p_1 + (p_3 + p_4) < 2$, giving by (5.18). Next, we have

$$|\Xi_{3,2}(t)| \leq \left| \frac{\lambda'_2(t) \lambda_3(t) \lambda_4(t)}{\eta(t) (\lambda_2(t))^{1/2}} \right| \in L^1(\{t \gg 1\}),$$

for $2p_1 + p_2 + p_3 + p_4 < 2$, which is (5.6). Next, for $\Xi_{3,3}$ we obtain the bound

$$|\Xi_{3,3}(t)| \leq \left| \frac{\lambda'_3(t) \lambda_2(t) \lambda_4(t)}{\eta(t) (\lambda_3(t))^{1/2}} \right| \in L^1(\{t \gg 1\}),$$

since $2p_1 + p_2 + p_3 + p_4 < 2$ again. Finally, we bound $\Xi_{3,4}$ as follows:

$$|\Xi_{3,4}(t)| \leq \frac{(\lambda'_1(t) \lambda_2(t) \lambda_3(t))^{1/2}}{\eta(t) (\lambda_4(t))^{1/2}} \in L^1(\{t \gg 1\}),$$

once again by (5.6). We conclude

$$\frac{1}{4\eta(t) \lambda_1(t)} \left| \int_{\mathbb{R}^3} \frac{u^2}{2} \phi_\sigma \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{x}{\lambda_2(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) dx dy dz \right|$$

$$= \Xi_{1,1,2}(t)$$

$$\leq 2 \frac{d\Xi(t)}{dt} + \Xi_{aux}(t),$$

where $\Xi_{aux}(t) \in L^1(\{t \gg 1\})$, with

$$|\Xi_{aux}(t)| \leq \frac{1}{t^{1+\kappa_0} \log^{\epsilon_0}(t)}, \quad \kappa_0, \epsilon_0 > 0.$$  

After this, we conclude essentially in the same form as in the 2D case.
5.2. L^2 virial and H^1 local decay

We follow the lines of Section 4, devoted this time to the 3D case. Some additional estimates are needed, and some care will be put in some particular parts of the proof.

As in the 2D case, we consider the functional

$$Q(t) := \frac{1}{\eta(t)} \int_{\mathbb{R}^3} u^2(x, y, z, t) \phi_{\sigma'_x} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\sigma_y} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\sigma_z} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz,$$

that is clearly well defined for solutions of the IVP (ZK) with \( d = 3 \). Moreover, it is a bounded in time functional.

In what follows, we consider the evolution of \( Q(t) \):

$$\frac{d}{dt} Q(t) = \frac{2}{\eta(t)} \int_{\mathbb{R}^3} u \partial_t u \sigma_x \left( \frac{x}{\lambda_1(t)} \right) \phi_{\sigma_x} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\sigma_z} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz$$

$$- \frac{\lambda_2'(t)}{\lambda_2(t) \eta(t)} \int_{\mathbb{R}^3} u^2 \psi_x \left( \frac{x}{\lambda_1(t)} \right) \phi_{\sigma_x} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\sigma_x} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\sigma_z} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz$$

$$- \frac{\lambda_4'(t)}{\lambda_4(t) \eta(t)} \int_{\mathbb{R}^3} u^2 \psi_x \left( \frac{x}{\lambda_1(t)} \right) \phi_{\sigma_x} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\sigma_x} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\sigma_z} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz$$

$$- \frac{\eta'(t)}{\eta^2(t)} \int_{\mathbb{R}^3} u^2 \psi_x \left( \frac{x}{\lambda_1(t)} \right) \phi_{\sigma_x} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\sigma_x} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\sigma_z} \left( \frac{z}{\lambda_4(t)} \right) \phi_{\sigma_z} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz.$$

(5.30)

Compared with the 2D case in (4.2), the term \( A_4(t) \) is a new contribution. First, \( A_1(t) \)

$$= \frac{2}{\eta(t)} \int_{\mathbb{R}^3} \partial_x u \Delta u \sigma_x \left( \frac{x}{\lambda_1(t)} \right) \phi_{\sigma_x} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\sigma_z} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz$$

$$+ \frac{1}{\eta(t)} \int_{\mathbb{R}^3} \partial_x u^2 \sigma_x \left( \frac{x}{\lambda_1(t)} \right) \phi_{\sigma_x} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\sigma_z} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz$$

$$+ \frac{2}{\eta(t) \lambda_1(t)} \int_{\mathbb{R}^3} u \left( \Delta u + \frac{u^2}{2} \right) \phi_{\sigma_x} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\sigma_x} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\sigma_z} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz$$

$$= A_{1,1}(t) + A_{1,2}(t) + A_{1,3}(t).$$

The term \( A_{1,2}(t) \) will produce a local cubic term which will be the most difficult one to be controlled. For the moment, we concentrate ourselves in the term \( A_{1,1}(t) \). We have
We have $A_{1,1,5}(t)$ as the new contribution in 3D, but the reminders also need some care because one does not obtain the same conditions on the parameters as in the previous subsection.

From this expression we will only focus on estimate $A_{1,1,2}$ and $A_{1,1,5}$, since $A_{1,1,1}, A_{1,1,3}$ and $A_{1,1,4}$ are part of the quantities to be estimated. So that,

$$|A_{1,1,2}(t)| \leq \|u\|_{H^2/w}^2 \frac{1}{\eta(t) \lambda_3(t)} \in L^1(\{t \gg 1\}),$$

whenever $p_3 + r_1 = p_3 + 1 - p_1 > 1$, valid thanks to (5.8). Also,

$$|A_{1,1,5}(t)| \leq \|u\|_{H^3/w}^2 \frac{1}{\eta(t) \lambda_4(t)} \in L^1(\{t \gg 1\}),$$

whenever $p_4 + 1 - p_1 > 1$, which is (5.9). Next,

$$A_{1,3}(t) = -\frac{2}{\eta(t) \lambda_1(t)} \int (\partial_x u)^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz$$

$$+ \frac{1}{\eta(t) \lambda_3(t)} \int u^2 \phi_{\sigma''} \left( \frac{x}{\lambda_3(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz$$

$$- \frac{2}{\eta(t) \lambda_1(t) \lambda_3(t)} \int (\partial_x u)^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz$$

$$+ \frac{1}{\eta(t) \lambda_1(t) \lambda_3^2(t)} \int u^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz$$

$$- \frac{2}{\eta(t) \lambda_1(t) \lambda_3^2(t)} \int (\partial_x u)^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz$$

$$+ \frac{1}{\eta(t) \lambda_1(t) \lambda_3^2(t)} \int u^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz$$

$$= A_{1,3,1}(t) + A_{1,3,2}(t) + A_{1,3,3}(t) + A_{1,3,4}(t) + A_{1,3,5}(t) + A_{1,3,6}(t) + A_{1,3,7}(t).$$

We will provide upper bounds for the terms $A_{1,3,2}, A_{1,3,4},$ and $A_{1,3,6}$ since the terms $A_{1,3,1}, A_{1,3,3}$ and $A_{1,3,5}$ are part of the quantities to be estimated. The term $A_{1,3,7}$ will be described below since it represents the most harder term to estimate at this step.
First, similar to the 2D case,

\[ |A_{1,3,2}(t)| \leq \frac{1}{\eta(t)\lambda_1(t)} \in L^1(\{t \gg 1\}), \]

since \( p_1 > 0 \). Second,

\[ |A_{1,3,4}(t)| \leq \frac{1}{\eta(t)\lambda_1(t)\lambda_3(t)} \in L^1(\{t \gg 1\}), \]

since \( p_3 > 0 \). Third,

\[ |A_{1,3,6}(t)| \leq \frac{1}{\eta(t)\lambda_1(t)\lambda_4(t)} \in L^1(\{t \gg 1\}), \]

since \( p_4 > 0 \). Only the terms \( A_{1,2} \) and \( A_{1,3,7} \) remain to be bounded, but both are similar in nature and it is only necessary to estimate one of them.

Now, we come back to (5.30). Concerning \( A_2, A_3, A_4 \) and \( A_5 \), we have in the first place the following bounds: for \( A_2 \) it verifies that

\[ |A_2(t)| \leq \frac{\lambda_3(t)}{\eta(t)\lambda_3(t)} \leq \frac{1}{\eta(t)} \in L^1(\{t \gg 1\}), \]

since \( r_1 > 0 \). Also, for \( A_3 \) a quite similar results holds, that is,

\[ |A_3(t)| \leq \frac{\lambda_3(t)}{\eta(t)\lambda_3(t)} \leq \frac{1}{\eta(t)\ln \gamma t} \in L^1(\{t \gg 1\}), \]

since \( r_1 > 0 \). Next, for \( A_4 \) and \( A_5 \), the bound is exactly the same, so we skip it. This ends the estimation of all terms, except \( A_{1,2} \) and \( A_{1,3,7} \) which contain cubic powers.

Finally, we show how to handle the terms with a cubic power. We proceed as follows:

let \( \epsilon > 0 \) to be fixed later on. Write \( |u|^3 = (\epsilon^{-1/4}|u|^{5/2})(\epsilon^{1/4}|u|^{1/2}) \). Using Young’s inequality with \( p = 4/3 \) and \( p' = 4 \), it holds

\[
\int_{\mathbb{R}^3} |u|^3 \phi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \ dx \ dy \ dz \\
\leq \frac{3}{4\epsilon^{1/3}} \int_{\mathbb{R}^3} |u|^{10/3} \phi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \ dx \ dy \ dz \\
+ \frac{\epsilon}{4} \int_{\mathbb{R}^3} u^2 \phi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \ dx \ dy \ dz. \tag{5.31}
\]

From now on we will focus our attention to provide an adequate upper bound for the first term in the last line of (5.31). In this sense, following the same procedure as in (4.5),

\[
\int_{\mathbb{R}^3} |u|^3 \phi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \ dx \ dy \ dz \\
= \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3} \int_{m_1+1}^{m_1+1} \int_{m_2+1}^{m_2+1} \int_{m_3+1}^{m_3+1} |u|^3 \phi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \ dx \ dy \ dz \\
\leq \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3} \int_{\mathbb{R}^3} \left( |u|^{10} \lambda_{m_1}(x) \lambda_{m_2}(y) \lambda_{m_3}(z) \right)^{10} \phi_{\sigma} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \ dx \ dy \ dz \\
\leq \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3} \left\| u \lambda_{m_1} \lambda_{m_2} \lambda_{m_3} \right\|_{\bar{L}^{10}}^{10} \Lambda_{m_1} \Lambda_{m_2} \Lambda_{m_3},
\]
where
\[ \Lambda_{m_1} := \max_{x \in [m_1, m_1 + 1]} \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right), \quad \Lambda_{m_2} := \max_{y \in [m_2, m_2 + 1]} \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right), \]
and
\[ \Lambda_{m_3} := \max_{z \in [m_3, m_3 + 1]} \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right). \]

Now, by the Gagliardo-Nirenberg-Sobolev inequality, we get
\[ \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3} \left\| u x_{m_1} x_{m_2} x_{m_3} \right\|_{L^2_{xyz}}^2 \Lambda_{m_1} \Lambda_{m_2} \Lambda_{m_3} \]
\[ \leq \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3} c^2 \left\| \nabla (u x_{m_1} x_{m_2} x_{m_3}) \right\|_{L^2_{xyz}} \left\| u x_{m_1} x_{m_2} x_{m_3} \right\|_{L^2_{xyz}}^2 \Lambda_{m_1} \Lambda_{m_2} \Lambda_{m_3} \]
\[ \leq \left\| u_0 \right\|_{L^2_{xyz}}^2 \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3} c^2 \left( \left\| \nabla (u x_{m_1} x_{m_2} x_{m_3}) \right\|_{L^2_{xyz}}^2 + \left\| u x_{m_1} x_{m_2} x_{m_3} \right\|_{L^2_{xyz}}^2 \right) \Lambda_{m_1} \Lambda_{m_2} \Lambda_{m_3}, \tag{5.32} \]

note that \( c \) does not depend on \( m_1, m_2, m_3 \). Also,
\[ \left\| \nabla (u x_{m_1} x_{m_2} x_{m_3}) \right\|_{L^2_{xyz}} = \left\| x_{m_1} x_{m_2} x_{m_3} \nabla u + u \nabla (x_{m_1} x_{m_2} x_{m_3}) \right\|_{L^2_{xyz}} \]
\[ \leq \left\| x_{m_1} x_{m_2} x_{m_3} \nabla u \right\|_{L^2_{xyz}} + \left\| u \nabla (x_{m_1} x_{m_2} x_{m_3}) \right\|_{L^2_{xyz}}. \]

Additionally, an analysis similar to the employed in (4.7)–(4.10) allow us to obtain the following bounds:
\[ \Lambda_{m_1} \leq C \min_{x \in [m_1, m_1 + 1]} \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right), \quad \text{for all } x \in \mathbb{R}, \]
\[ \Lambda_{m_2} \leq C \min_{y \in [m_2, m_2 + 1]} \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right), \quad \text{for all } y \in \mathbb{R}, \]
and
\[ \Lambda_{m_3} \leq C \min_{z \in [m_3, m_3 + 1]} \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right), \quad \text{for all } z \in \mathbb{R}. \]

Next, we go back to (5.32). Incorporating the previous computations,
\[ \left\| u_0 \right\|_{L^2_{xyz}}^2 \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3} c^2 \left( \left\| \nabla (u x_{m_1} x_{m_2} x_{m_3}) \right\|_{L^2_{xyz}}^2 + \left\| u x_{m_1} x_{m_2} x_{m_3} \right\|_{L^2_{xyz}}^2 \right) \Lambda_{m_1} \Lambda_{m_2} \Lambda_{m_3} \]
\[ \leq 2 \left\| u_0 \right\|_{L^2_{xyz}}^2 \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3} c^2 \left( \left\| \nabla (u x_{m_1} x_{m_2} x_{m_3}) \right\|_{L^2_{xyz}}^2 + \left\| u x_{m_1} x_{m_2} x_{m_3} \right\|_{L^2_{xyz}}^2 \right) \Lambda_{m_1} \Lambda_{m_2} \Lambda_{m_3} \]
\[ \leq \tilde{c} \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3} \int_{\mathbb{R}^3} \left| \nabla u \right|^2 \lambda_{m_1, m_2, m_3}^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz \]
\[ + \tilde{c} \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3} \int_{\mathbb{R}^3} \left| \nabla (x_{m_1} x_{m_2} x_{m_3}) \right|^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz \]
\[ \leq \tilde{c} \int_{\mathbb{R}^3} \left| \nabla u \right|^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz \]
\[ + \tilde{c} \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3} \int_{\mathbb{R}^3} \left| \nabla (x_{m_1} x_{m_2} x_{m_3}) \right|^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx \, dy \, dz. \]
To estimate the remainder $L^2$ term above we consider a smooth function $\rho$ satisfying: 
\[ \rho \equiv 1 \text{ on } [-1, 2] \text{ and } \rho \equiv 0 \text{ on } (-\infty, -2] \cup [3, \infty). \] 
Then, for $m_1, m_2, m_3 \in \mathbb{Z}$, we set $\rho_{m_i}(x) := \rho(x - m_i), \rho_{m_i}(y) := \rho(y - m_2), \text{ and } \rho_{m_i}(z) := \rho(z - m_3)$.

Hence,
\[
\tilde{c} \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3} \int_{\mathbb{R}^3} |\nabla (\chi_{m_1, m_2, m_3})|^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx dy dz
\]
\[
= \tilde{c} \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3} \int_{\mathbb{R}^3} u^2 \left( \chi_{m_1, m_2, m_3} \right)^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx dy dz
\]
\[
+ \tilde{c} \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3} \int_{\mathbb{R}^3} u^2 \left( \chi_{m_1, m_2, m_3} \right)^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx dy dz
\]
\[
+ \tilde{c} \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3} \int_{\mathbb{R}^3} u^2 \left( \chi_{m_1, m_2, m_3} \right)^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx dy dz
\]
\[
\leq \tilde{c} \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3} \int_{\mathbb{R}^3} u^2 \left( \rho_{m_1, m_2, m_3} \right)^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx dy dz
\]
\[
+ \tilde{c} \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3} \int_{\mathbb{R}^3} u^2 \left( \chi_{m_1, m_2, m_3} \right)^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx dy dz
\]
\[
+ \tilde{c} \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3} \int_{\mathbb{R}^3} u^2 \left( \chi_{m_1, m_2, m_3} \right)^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx dy dz
\]
\[
\leq \tilde{c} \int_{\mathbb{R}^3} u^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx dy dz.
\]

Summarizing, for the cubic term we have proved that for all $\epsilon > 0$, the following inequality holds:
\[
\frac{1}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}^3} |u|^3 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx dy dz
\]
\[
\leq \frac{3\tilde{c}}{4\epsilon^{1/3}\eta(t)\lambda_1(t)} \int_{\mathbb{R}^3} |\nabla u|^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx dy dz
\]
\[
+ \frac{C\epsilon}{4\eta(t)\lambda_1(t)} \int_{\mathbb{R}^3} u^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx dy dz,
\]
where $C$ is a positive constant depending on $\tilde{c}$.

Note that, independent of the constant $\epsilon$ taken, the second term above satisfies:
\[
\frac{C\epsilon}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}^3} u^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \, dx dy dz \in L^1(\{t \gg 1\}),
\]
this is achieved after combining (2.2) and (5.29) with $\sigma' > 0$ satisfying:
Finally, after choosing $\epsilon > (3\tilde{c}/2)^3$, we obtain that

$$\int_{\{t\geq 1\}} \left( \frac{1}{\eta(t) \lambda_1(t)} \int_{\mathbb{R}^3} |\nabla u|^2 \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) \right) dx dy dz \right) dt < \infty.$$  

This result, together with (5.29), allow us to conclude Theorem 1.13, in the same form as in the 2D case. Therefore,

$$\lim_{t \to \infty} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)(x,y,z,t) \phi_{\sigma'} \left( \frac{x}{\lambda_1(t)} \right) \phi_{\delta_2} \left( \frac{y}{\lambda_3(t)} \right) \phi_{\delta_3} \left( \frac{z}{\lambda_4(t)} \right) dx dy dz = 0.$$  

6. Decay in far far regions. Proof of Theorem 1.5

This last section is devoted to the proof of Theorem 1.14. As stated in the Introduction, the proof differs from the other proofs in this article. In particular, we will need slightly different weighted functions, as stated in Section 2, Section 2.2. We will closely follow [25], with some key differences.

6.1. 2D case

Fix any $p \geq 1$ and $\epsilon > 0$. The proof consists of two independent decay estimates, one in a band of the form $|x| \sim t^p \log^{1+\epsilon} t$ and the other one for the band $|y| \sim t^p \log^{1+\epsilon} t$. Although both results are similar in nature, the proofs are slightly different, and some care is needed in both cases.

**Case** $|x| \sim t^p \log^{1+\epsilon} t$. Let $\chi$ be the cutoff function introduced in (2.3). Additionally, we will consider $\theta_1(t) := t^p \ln^{1+\epsilon} t$. So that, it is clear that

$$\frac{\theta_1'(t)}{\theta_1(t)} \sim \frac{1}{t} \quad \text{for} \quad t \gg 1.$$  

Note that unlike the previous analysis, the function $\theta_1^{-1} \in L^1(\{t \gg 1\})$. So that, it suggests that the proof will be obtained by exploiting properties of the weighted function $\chi$ as we will see below.

Firstly, we will estimate in the portion $x \sim -\theta_1(t)$. To estimate in the portion $x \sim \theta_1(t)$ the procedure follows by using an argument quite similar.

Formally, we get after multiplying the equation in (ZK) by $u \chi \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right)$ that

$$\left( u \partial_t u + u \partial_x \Delta u + u^2 \partial_x u \right) \chi \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) = 0. \quad (6.1)$$  

Then after integrating in space we obtain the following identity:
Finally, we gather the previous estimates, that combined with the fact that $\theta^{-1}_1 \in L^1(\{t \gg 1\})$ yield us to conclude that

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} u^2 \chi \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \, dx \, dy - \frac{1}{2} \int_{\mathbb{R}^2} u^2 \chi' \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \left( \frac{\theta'_1(t)}{\theta_1(t)} \right) \, dx \, dy \\
+ \frac{\theta'_1(t)}{2\theta_1(t)} \int_{\mathbb{R}^2} u^2 \chi' \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \, dx \, dy \\
+ \frac{3}{2\theta_1(t)} \int_{\mathbb{R}^2} (\partial_x u)^2 \chi' \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \, dx \, dy \\
- \frac{1}{2\theta_1^3(t)} \int_{\mathbb{R}^2} u^2 \chi'' \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \, dx \, dy \\
- \frac{1}{3\theta_1(t)} \int_{\mathbb{R}^2} u^3 \chi' \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \, dx \, dy = 0.
\]  

(6.2)

Since $\chi$ is monotone decreasing, then $\chi' \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \leq 0$, and in particular $\left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \chi' \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \geq 0$, so that, for $t \gg 1$, the terms $A_1(t)$ and $A_2(t)$ are positive.

Instead, to estimate the terms $A_3$ and $A_4$ the scenario is quite different due to $\theta^{-1}_1 \in L^1(\{t \gg 1\})$. More precisely, we obtain

$$|A_3(t)| \leq \|u\|_{L^\infty_x} \frac{1}{\theta_1(t)} \in L^1(\{t \gg 1\}).$$

Also,

$$|A_4(t)| \leq \|u\|_{L^\infty_x} \frac{1}{\theta_1(t)} \in L^1(\{t \gg 1\}).$$

Next,

$$|A_5(t)| \leq \|u\|_{L^2} \frac{1}{\theta_1^2(t)} \in L^1(\{t \gg 1\}).$$

To estimate $A_6$ we use the boundedness of the $H^1$ norm uniform in time and the fact that $\theta^{-1}_1 \in L^1(\{t \gg 1\})$; for the sake of brevity we omit the details here. In summary, we get

$$|A_6(t)| \leq \|u\|_{L^\infty_x} \frac{1}{\theta_1(t)} \in L^1(\{t \gg 1\}).$$

Finally, we gather the previous estimates, that combined with the fact that $\theta^{-1}_1 \in L^1(\{t \gg 1\})$ yield us to conclude that
\[ \int_{\{t \geq 1\}} \frac{1}{t} \left( \int_{\mathbb{R}^2} u^2 \left| \frac{x + \theta_1(t)}{\theta_1(t)} \right| \, dx \, dy + \int_{\mathbb{R}^2} u^2 \chi' \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \, dx \, dy \right) \, dt < \infty. \] (6.3)

Note that the weighted function \( \chi' \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \) in the first term in the L.H.S. above is supported on the region \(-2\theta_1(t) < x < -\theta_1(t), \) that is, \( x \sim -\theta_1(t). \)

The lack of integrability of the function \( t^{-1} \) for \( t \gg 1, \) implies that there exist a sequence of times for that the function in (6.3) converges to zero. More precisely, we ensure that there exist a sequence of positive times \((t_n)_n\), such that \( t_n \uparrow \infty \) as \( n \) goes to infinity and satisfying

\[ \lim_{n \to \infty} \int_{\Lambda(t_n)} u^2 \, dx \, dy = 0, \]

where \( \Lambda(t) := \{ (x, y) \in \mathbb{R}^2 \mid x \sim -\theta_1(t) \} \).

To prove the decay to zero in the right portion \( x \sim \theta_1(t) \) is enough to apply an argument similar to the one described above, but considering this time the weighted function \( \tilde{\chi}(x) := \chi(-x), \ x \in \mathbb{R}. \)

Finally, we conclude (1.15) in this case by performing again an estimate as (6.2) but with weighted function \( \tilde{\chi} \) nonnegative and supported in the interval \([-\frac{3}{4}, \frac{1}{4}], \) and using (2.4). The details are essentially in [25], and we skip them.

**Case** \(|y| \sim t^p \log^{1+\epsilon} t.\) This case is similar to the previous one, with some differences. We have from (6.1),

\[ (u \partial_t u + u \partial_x \Delta u + u^2 \partial_y u) \chi \left( \frac{y + \theta_1(t)}{\theta_1(t)} \right) = 0. \] (6.4)

Integrating in space and by parts, we get now

\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} u^2 \chi \left( \frac{y + \theta_1(t)}{\theta_1(t)} \right) \, dx \, dy - \frac{1}{2} \int_{\mathbb{R}^2} u^2 \chi' \left( \frac{y + \theta_1(t)}{\theta_1(t)} \right) \left( \frac{\theta'_1(t)}{\theta_1(t)} \right) \, dx \, dy \]

\[ + \frac{\theta'_1(t)}{2\theta_1(t)} \int_{\mathbb{R}^2} u^2 \chi' \left( \frac{y + \theta_1(t)}{\theta_1(t)} \right) \left( \frac{y + \theta_1(t)}{\theta_1(t)} \right) \, dx \, dy \]

\[ + \frac{1}{\theta_1(t)} \int_{\mathbb{R}^2} (\partial_x u \partial_y u) \chi \left( \frac{y + \theta_1(t)}{\theta_1(t)} \right) \, dx \, dy \]

\[ = 0. \] \( A_3(t) \)

The only new term here is \( A_3, \) for which we immediately have

\[ |A_3(t)| \leq \| u \|_{L^4(\mathbb{R}^2)} \, \frac{1}{\theta_1(t)} \in L^1(\{ t \gg 1 \}). \]

The rest of the proof is the same as in the previous case.
6.2. 3D case

Similar to the 2D case, we will describe how to obtain the decay to zero in the region \( x \sim -\theta_1(t) \) and \( y \sim -\theta_1(t) \). The decay in the remaining regions (in particular, in the \(|z| \sim \theta_1(t) \) can be obtained by using quite similar arguments.

As usual our starting point is based on energy estimates. So that, a standard procedure allow us to obtain the identity

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} u^2 \chi \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \, dx \, dy \, dz - \frac{1}{2} \int_{\mathbb{R}^3} u^2 \chi' \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \frac{\theta_1'(t)}{\theta_1(t)} \, dx \, dy \, dz
\]

\[
+ \frac{\theta_1'(t)}{2 \theta_1(t)} \int_{\mathbb{R}^3} u^2 \chi' \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \, dx \, dy \, dz
\]

\[
+ \frac{3}{2 \theta_1(t)} \int_{\mathbb{R}^3} (\partial_x u)^2 \chi' \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \, dx \, dy \, dz
\]

\[
+ \frac{1}{2 \theta_1(t)} \int_{\mathbb{R}^3} (\partial_x u)^2 \chi' \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \, dx \, dy \, dz
\]

\[
+ \frac{1}{2 \theta_1(t)} \int_{\mathbb{R}^3} (\partial_x u)^2 \chi' \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \, dx \, dy \, dz
\]

\[- \frac{1}{2 \theta_1(t)} \int_{\mathbb{R}^3} u^2 \chi'' \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \, dx \, dy \, dz
\]

\[- \frac{1}{3 \theta_1(t)} \int_{\mathbb{R}^3} u^2 \chi' \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \, dx \, dy \, dz = 0.
\]

The constraints on \( \chi \) implies that for \( t \gg 1 \), the terms \( A_1 \) and \( A_2 \) are positive. Also, the terms \( A_3, A_4, \) and \( A_5 \) satisfy the bounds

\[
|A_3(t)| \leq \|u\|_{H^\infty} \frac{1}{\theta_1(t)} \in L^1(\{t \gg 1\}),
\]

\[
|A_4(t)| \leq \|u\|_{H^\infty} \frac{1}{\theta_1(t)} \in L^1(\{t \gg 1\}),
\]

and

\[
|A_5(t)| \leq \|u\|_{H^\infty} \frac{1}{\theta_1(t)} \in L^1(\{t \gg 1\}).
\]

For \( A_6 \) we have that

\[
|A_6(t)| \leq \|u_0\|_{L^2} \frac{1}{\theta_1(t)} \in L^1(\{t \gg 1\}).
\]

To estimate \( A_7 \) we can use the boundedness of the \( H^1 \) norm in time, exactly as in the 2D case. In summary, we get

\[
|A_7(t)| \leq \|u\|_{L^\infty} \frac{1}{\theta_1(t)} \in L^1(\{t \gg 1\}).
\]
Finally, after gathering the estimates in this step that combined with the fact that $\theta_1^{-1} \in L^1(\{t \gg 1\})$ yield us to conclude that

$$\int_{\{t \gg 1\}} \frac{1}{t} \left( -\int_{\mathbb{R}^3} u^2 \chi^2 \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \frac{dx}{\theta_1(t)} \right) \frac{dy}{\theta_1(t)} \frac{dz}{\theta_1(t)} + \int_{\mathbb{R}^3} u^2 \chi^2 \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \left( \frac{x + \theta_1(t)}{\theta_1(t)} \right) \frac{dx}{\theta_1(t)} \frac{dy}{\theta_1(t)} \frac{dz}{\theta_1(t)} dt < \infty.$$  

From this point, the rest of the proof is the same as in the 2D case.

Finally, the case $y \sim -\theta_1$ is obtained in similar terms starting from (6.4). We skip the details.

7. Proof of Theorem 1.3 in the gKdV case

In this section, our goal will be to provide a proof of the gKdV version of Theorem 1.1, namely Theorem 1.3. This new result complements [10].

Proof of Theorem 1.3. We sketch the proof, following the proof of Theorem 1.1. Using (3.4), we consider this time the functional

$$\Xi(t) := \frac{1}{\eta(t)} \int_{\mathbb{R}} u(x,t) \psi_\sigma \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{\tilde{x}}{\lambda_1^2(t)} \right) dx, \quad \tilde{x} := x - \rho(t),$$

for $\lambda_1$, $\rho$ and $\eta$ given by

$$\lambda_1(t) = \frac{t^b}{\ln t}, \quad \rho(t) = \pm t^n, \quad \text{and} \quad \eta(t) = t^m \ln^2 t, \quad m + b = 1, \quad b, m > 0,$$

$$0 < b \leq \min \left\{ \frac{p}{p + q(p-1)}, \frac{2}{2 + q}, \frac{2}{2p - 1} \right\}, \quad q > 1, \quad 0 \leq n \leq 1 - \frac{b}{2}. \quad (7.1)$$

Also, (3.8) and (3.9) are satisfied.

Here we have two cases. If $p = 2$, then $b < \frac{2}{3}$, by taking $q = 1 + \epsilon_0$, $\epsilon_0$ arbitrarily small. If $p = 4$, one has $\frac{p}{p + q(p-1)} < \frac{2}{2 + q}$ and we conclude $b < \frac{4}{7}$ performing the same trick as before. We conclude $b < \frac{p}{2p - 1}$ as the condition for the validity of Theorem 1.9, exactly as in (1.8).

Now, we estimate $\Xi$ and its derivative in time. First of all, by Cauchy-Schwarz inequality and (7.1) we obtain

$$\sup_{t \gg 1} |\Xi(t)| \leq \sup_{t \gg 1} \left( \frac{\lambda_1^2(t)}{\eta(t)} \right)^{1/2} \leq \sup_{t \gg 1} \frac{1}{t^{1-b/2} q \ln^{3/2} t} < \infty.$$  

In what follows, we compute and estimate the dynamics of $\Xi(t)$ in the long time regime. We will prove for $p = 2, 4$, and $C_0 > 0$,

$$\frac{1}{\lambda_1(t) \eta(t)} \int_{\mathbb{R}} u^p \psi_\sigma' \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{\tilde{x}}{\lambda_1^2(t)} \right) dx \leq C_0 \frac{\Xi}{dt} (t) + \Xi_{int}(t),$$

where $\Xi_{int}(t)$ are terms that belong to $L^1(\{t \gg 1\})$. Once this result is proved, the rest of the proof is direct. We have
\[
\frac{d}{dt} \Xi(t) = \frac{1}{\eta(t)} \int_{\mathbb{R}} \partial_t \left( u \phi(\frac{\tilde{x}}{\lambda_1(t)}) \phi(\frac{\tilde{x}}{\lambda_1^q(t)}) \right) dx - \frac{\eta'(t)}{\eta^2(t)} \int_{\mathbb{R}} u \phi(\frac{\tilde{x}}{\lambda_1(t)}) \phi(\frac{\tilde{x}}{\lambda_1^q(t)}) dx
\]
\[
= : \Xi_1(t) + \Xi_2(t).
\]

First, we bound \( \Xi_2 \). In virtue of (3.10) the same analysis applied there yields
\[
|\Xi_2(t)| \leq \left| \frac{\eta'(t)}{\eta^2(t)} \int_{\mathbb{R}} u \phi(\frac{\tilde{x}}{\lambda_1(t)}) \phi(\frac{\tilde{x}}{\lambda_1^q(t)}) dx \right| \leq \left( \frac{\lambda_1(t)^{q/2}}{t \eta(t)} \right) \frac{1}{t^{2-\frac{b}{2+q}}} \ln \frac{t^{2+q}}{t}.
\]
We need \( 2 - b - \frac{1}{2} b q \geq 1 \), which is \( b \leq \frac{2}{2+q} \). Since (7.1) hold, the last term integrates. Thus, \( \Xi_2 \in L^1(\{t \gg 1\}) \). Now,
\[
\Xi_1(t) = \frac{1}{\eta(t)} \int_{\mathbb{R}} \partial_t u \phi(\frac{\tilde{x}}{\lambda_1(t)}) \phi(\frac{\tilde{x}}{\lambda_1^q(t)}) dx
\]
\[
- \frac{\lambda_1'(t)}{\lambda_1(t) \eta(t)} \int_{\mathbb{R}} u \phi(\frac{\tilde{x}}{\lambda_1(t)}) \phi(\frac{\tilde{x}}{\lambda_1^q(t)}) dx
\]
\[
- \frac{q \lambda_1'(t)}{\lambda_1(t) \eta(t)} \int_{\mathbb{R}} u \phi(\frac{\tilde{x}}{\lambda_1(t)}) \phi(\frac{\tilde{x}}{\lambda_1^q(t)}) \phi'(\frac{\tilde{x}}{\lambda_1^q(t)}) dx
\]
\[
- \rho'(t) \int_{\mathbb{R}} u \phi(\frac{\tilde{x}}{\lambda_1(t)}) \phi(\frac{\tilde{x}}{\lambda_1^q(t)}) dx
\]
\[
- \rho'(t) \int_{\mathbb{R}} u \phi(\frac{\tilde{x}}{\lambda_1(t)}) \phi(\frac{\tilde{x}}{\lambda_1^q(t)}) dx
\]
\[
=: \Xi_{1,1}(t) + \Xi_{1,2}(t) + \Xi_{1,3}(t) + \Xi_{1,4}(t) + \Xi_{1,5}(t).
\]
Concerning to \( \Xi_{1,1} \) we have by (1.7) and integration by parts
\[
\Xi_{1,1}(t) = - \frac{1}{\eta(t)} \int_{\mathbb{R}} \partial_x \left( \partial_x^2 u + u^p \right) \phi(\frac{\tilde{x}}{\lambda_1(t)}) \phi(\frac{\tilde{x}}{\lambda_1^q(t)}) dx
\]
\[
= \frac{1}{\eta(t) \lambda_1(t)} \int_{\mathbb{R}} \partial_x^2 u \phi(\frac{\tilde{x}}{\lambda_1(t)}) \phi(\frac{\tilde{x}}{\lambda_1^q(t)}) dx
\]
\[
+ \frac{1}{\eta(t) \lambda_1^q(t)} \int_{\mathbb{R}} \partial_x^2 u \phi(\frac{\tilde{x}}{\lambda_1(t)}) \phi(\frac{\tilde{x}}{\lambda_1^q(t)}) dx
\]
\[
+ \frac{1}{\eta(t) \lambda_1^q(t)} \int_{\mathbb{R}} u \phi'(\frac{\tilde{x}}{\lambda_1(t)}) \phi(\frac{\tilde{x}}{\lambda_1^q(t)}) dx
\]
\[
+ \frac{1}{\eta(t) \lambda_1^q(t)} \int_{\mathbb{R}} u \phi'(\frac{\tilde{x}}{\lambda_1(t)}) \phi(\frac{\tilde{x}}{\lambda_1^q(t)}) dx
\]
\[
=: \Xi_{1,1,1}(t) + \Xi_{1,1,2}(t) + \Xi_{1,1,3}(t) + \Xi_{1,1,4}(t).
\]
For \( \Xi_{1,1,1} \) we have after combining integration by parts
First, we bound each term using Cauchy-Schwarz inequality, as follows: Since $q > 1$,

$$|\Xi_{1,1}(t)| \leq \frac{1}{\eta(t) \lambda_1^{3/2}(t)} + \frac{1}{\eta(t) \lambda_1^{3/2+q}(t)} + \frac{1}{\eta(t) \lambda_1^{1/2+2q}(t)} \leq \frac{1}{\eta(t) \lambda_1^{5/2}(t)},$$

which clearly integrates. Next, applying integration by parts,

$$\Xi_{1,1,2}(t) = \frac{1}{\eta(t) \lambda_1^{3/2}(t)} \int_{\mathbb{R}} \frac{\partial^2 u\psi_\sigma}{\partial x^2} \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{\tilde{x}}{\lambda_1^q(t)} \right) \, dx$$

$$= \frac{2}{\eta(t) \lambda_1^{3/2+q}(t)} \int_{\mathbb{R}} u\psi_\sigma'' \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \phi_{\delta_1}' \left( \frac{\tilde{x}}{\lambda_1^q(t)} \right) \, dx$$

$$+ \frac{1}{\eta(t) \lambda_1^{1/2+2q}(t)} \int_{\mathbb{R}} u\psi_\sigma' \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \phi_{\delta_1}'' \left( \frac{\tilde{x}}{\lambda_1^q(t)} \right) \, dx,$$

and the Cauchy-Schwarz inequality yields

$$|\Xi_{1,1,2}(t)| \leq \frac{1}{\eta(t) \lambda_1^{3/2+q}(t)} + \frac{1}{\eta(t) \lambda_1^{1/2+2q}(t)} + \frac{1}{\eta(t) \lambda_1^{5/2}(t)}.$$

Each term above integrates since $q > 1$. We emphasize that the term $\Xi_{1,1,3}$ in (7.3)

$$\Xi_{1,1,3}(t) = \frac{1}{2\eta(t) \lambda_1(t)} \int_{\mathbb{R}} u^p \psi_\sigma' \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{\tilde{x}}{\lambda_1^q(t)} \right) \, dx,$$

is the term to be estimated after integrating in time. Therefore, it will be taken until the end of the proof.

The term $\Xi_{1,1,4}$ in (7.3) satisfies the following estimate

$$|\Xi_{1,1,4}(t)| \leq \left| \frac{1}{2\eta(t) \lambda_1^q(t)} \int_{\mathbb{R}} u^p \psi_\sigma \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \phi_{\delta_1}' \left( \frac{\tilde{x}}{\lambda_1^q(t)} \right) \, dx \right| \leq \frac{1}{2\eta(t) \lambda_1^q(t)}.$$

Since $q > 1$, $\Xi_{1,1,4} \in L^1(\{ t \gg 1 \})$.

Now, we focus our attention in the remaining terms in (7.2). First, by means of Young’s inequality, we have for $\epsilon > 0$, $p = 2, 4$ and $p' = \frac{p}{p-1}$,
\[
|\mathbf{I}_{1,2}(t)| = \left| \frac{\lambda_1'(t)}{\lambda_1(t) \eta(t)} \int_{\mathbb{R}} u \psi_\sigma \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \left( \frac{\tilde{x}}{\lambda_1^2(t)} \right) \phi_{\delta_1} \left( \frac{\tilde{x}}{\lambda_1^3(t)} \right) \, dx \right|
\]

\[
\leq \frac{1}{4e^p} \left| \frac{\lambda_1'(t)}{\lambda_1(t) \eta(t)} \int_{\mathbb{R}} u \psi_\sigma \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \frac{\phi_{\delta_1}(\tilde{x})}{\lambda_1^3(t)} \, dx \right|
\]

\[
+c^p \left| \frac{\lambda_1'(t)}{\lambda_1(t) \eta(t)} \int_{\mathbb{R}} \psi_\sigma' \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \left( \frac{\tilde{x}}{\lambda_1^3(t)} \right) \, dx \right|
\]

\[
\leq \frac{1}{4e^p} \left| \frac{\lambda_1'(t)}{\lambda_1(t) \eta(t)} \int_{\mathbb{R}} u \psi_\sigma \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \phi_{\delta_1} \left( \frac{\tilde{x}}{\lambda_1^3(t)} \right) \, dx + c e^p \left| \frac{\lambda_1'(t)}{\eta(t)} \right|, \right.
\]

so that, taking \( e^p = \lambda_1'(t) > 0 \) for \( t \gg 1 \), it is clear that

\[
|\mathbf{I}_{1,2}(t)| \leq \frac{1}{4} |\mathbf{I}_{1,3}(t)| + |\mathbf{I}_{1,4}(t)|.
\]

Note that the first term in the R.H.S. is the quantity to be estimated. The remaining term \( \mathbf{I}_{1,2} \) integrates since \( b \leq \frac{p}{2(p-1)} \), see (7.1).

Now, we consider the term \( \mathbf{I}_{1,3}(t) \). Combining the properties attribute to \( \phi_{\delta_1} \), the fact that \( u \in L^2 \) for \( p = 2 \) and \( u \in L^4 \) for \( p = 4 \), and Young’s inequality we get for \( \theta(t) = t^{2m/p} \),

\[
|\mathbf{I}_{1,3}(t)| \leq \frac{1}{t \ln t} + \frac{1}{t^{2+\frac{1}{p-1}} - b(1+q+\frac{1}{p}) \ln (\ln t)}.
\]

which integrates if \( b \leq \frac{p}{p+q(p-1)} \), which is just (7.1). Thus, \( \mathbf{I}_{1,3} \in L^1(\{ t \gg 1 \}) \).

Now,

\[
|\mathbf{I}_{1,4}(t)| \leq \frac{1}{\eta(t) \lambda_1^{1/2}(t)} \leq \frac{1}{t^{2-(\frac{q-1}{p})} \ln \frac{1}{t}},
\]

which integrates since \( n \leq 1 - \frac{q}{p} \). Finally,
\[ |\Xi_{1.5}(t)| \leq \frac{|\rho'(t)|}{\eta(t)\eta(t)^{5/2}} \leq \frac{1}{t^{2 - \frac{b}{2} - n} \ln t} \in L^1(\{t \geq 1\}), \]

since \( n \leq 1 - \frac{b}{2} \).

Therefore, once again we conclude the integrability in time. The rest of the proof is the same as in (ZK), and we omit the details.

\[ \square \]

**Acknowledgments**

We thank Didier Pilod, Gustavo Ponce and Jean-Claude Saut for comments on a first version of this draft.

**Funding**

A. J. M. work was partially supported by CMM Conicyt PIA10.13039/100008674 AFB170001. C. M. work was funded in part by Chilean research grants ANID FONDECYT10.13039/501100002850 1191412, project France-Chile ECOS-Sud C18E06, MathAmSud EEQUADDII 20-MATH-04, and CMM Conicyt PIA10.13039/100008674 AFB170001. F. P. work is partially supported by ANID projects FONDECYT/Iniciación10.13039/501100002850 11181263 and FONDECYT/Regular10.13039/501100002850 1170466. J. C. P. work was partially supported by Chilean research grant FONDECYT10.13039/501100002850 1181084.

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