Towards classification of quasi-local symmetries of evolution equations

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Abstract

We develop efficient group-theoretical approach to the problem of classification of evolution equations that admit non-local transformation groups (quasi-local symmetries), i.e., groups involving integrals of the dependent variable. We classify realizations of two- and three-dimensional Lie algebras leading to equations admitting quasi-local symmetries. Finally, we generalize the approach in question for the case of an arbitrary system of evolution equations with two independent variables.

1 Introduction.

Consider the general evolution equation in one spatial dimension

\[ u_t = F(t, x, u, u_1, u_2, \ldots, u_n), \quad n \geq 2, \tag{1} \]

where \( u = u(t, x) \) is a real-valued function of two real variables \( t, x \), \( u_i = \partial^i u / \partial x^i \), \( i = 1, 2, \ldots, n \), and \( F \) is an arbitrary smooth real-valued function.

The most general Lie transformation group leaving differential equation (1) invariant has the form (see, e.g., [1, 2])

\[ t' = T(t, \bar{\theta}), \quad x' = X(t, x, u, \bar{\theta}), \quad u = U(t, x, u, \bar{\theta}). \tag{2} \]

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Here $T, X, U$ are smooth real-valued functions satisfying the non-singularity condition $\frac{D(T,X,U)}{D(t,x,u)} \neq 0$ in some open domain of $\mathbb{R}^3$ and $\vec{\theta} = (\theta_1, \theta_2, \ldots, \theta_r) \in \mathbb{R}^r$ is the vector of group parameters.

If a transformation of the space of variables $t, x, u$ changes the specific form of (1) leaving invariant its differential structure, then we arrive at the concept of equivalence group. More precisely, if the (locally) invertible change of variables,

$$ t \rightarrow t' = T(t, x, u), \quad x \rightarrow x' = X(t, x, u), \quad u \rightarrow u' = U(t, x, u) $$

maps Eq.(1) into a possibly different $n$-th order evolution equation

$$ u'_{t'} = G(t', x', u', u'_1, u'_2, \ldots, u'_n), $$

then this change of variables is called equivalence transformation. The set of all possible equivalence transformations forms a diffeomorphism group, $E$, and is called the equivalence group of Eq.(1). Clearly, if we require that $G \equiv F$ then the equivalence group reduces to the symmetry group of Eq.(1). Consequently, Lie symmetry group of a given equation is a subgroup of its equivalence group.

Let partial differential equation (1) be invariant under Lie transformation group (2). What would happen to this Lie symmetry if we perform a transformation from the equivalence group of the equation under study? Evidently, transformation group (2) after being rewritten in the 'new' variables $t', x', u'$ becomes Lie symmetry of the transformed equation.

Now suppose that we allow for the more general equivalence transformation group

$$ t \rightarrow t' = T(t, x, u, \bar{v}), \quad x \rightarrow x' = X(t, x, u, \bar{v}), \quad u \rightarrow u' = U(t, x, u, \bar{v}), $$

where $\bar{v} = (u_1, u_2, \ldots, u_p, \partial^{-1}u, \partial^{-2}u, \ldots, \partial^{-s}u)$ with $\partial^{-1}u = \int u(t, x)dx$ and $\partial^{-k-1} = \partial^{-1}\partial^{-k}$. Saying it another way, we allow for an equivalence transformation to include derivatives and integrals of the dependent variable $u$. If such a transformation still preserves the differential structure of evolution equation (1), what would happen to Lie symmetries of the latter? The answer is, 'it depends'. In some cases, Lie symmetry transforms into another Lie symmetry. However, it could happen that some Lie symmetries would 'disappear' after performing equivalence transformation, meaning that they cannot be found within the framework of the infinitesimal Lie method. The
reason is that the transformation rule for the variables $t, x, u$ might contain derivatives and integrals of $u$, which are beyond reach of the standard Lie method. One needs to apply the generalized Lie \[3\]-\[5\] or non-Lie \[6\] approaches to be able to handle those symmetries.

To single out Lie symmetries, which after non-local equivalence transformation of an equation under study turn into non-Lie symmetries Ibragimov et al. \[7\] introduced the term 'quasi-local symmetry', which we use throughout the paper. Independently, the concept of quasi-local symmetry has been suggested in \[8\].

In the present paper we suggest a simple regular method for deriving quasi-local symmetries (QLS) of evolution equations. Note that the basic idea of the method has been suggested in our paper \[9\] and some non-trivial examples of second-order evolution equations with QLS are given in \[2\].

Next, we demonstrate how to apply the method developed to arbitrary systems of evolution equations with two independent variables.

2 Method description.

The most general Lie transformation group admitted by evolution equation \[1\] is of the form \[2\]. The infinitesimal operator, $Q$, of this group reads as

\[Q = \tau(t) \partial + \xi(t, x, u) \partial_x + \eta(t, x, u) \partial_u. \tag{3}\]

Provided, $\tau \equiv 0$ there is a transformation,

\[t \to \bar{t} = t, \quad x \to \bar{x} = X(t, x, u), \quad u \to \bar{u} = U(t, x, u), \tag{4}\]

that reduces $Q$ to the canonical form $\partial_u$ (we drop the bars). Evolution equation \[1\] now becomes

\[u_t = f(t, x, u_1, u_2, \ldots, u_n). \tag{5}\]

Note that the right-hand side of Eq.\[5\] does not depend explicitly on $u$.

Differentiating \[5\] with respect to $x$ yields

\[u_{tx} = \frac{\partial f}{\partial x} + \sum_{i=1}^{n} \frac{\partial f}{\partial u_i} u_{i+1}. \]

Making the change of variables

\[\bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = u_x \tag{6}\]
and dropping the bars we finally get

\[ u_t = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} + \sum_{i=2}^{n} \frac{\partial f}{\partial u_{i-1}} u_i, \]

(7)

where \( f = f(t, x, u, u_1, \ldots, u_{n-1}) \).

Thus non-local transformation (6) preserves the differential structure of the class of evolution equations (5).

Let differential equation (5) admit \( r \)-parameter Lie transformation group (2) with \( \vec{\theta} = (\theta_1, \ldots, \theta_r) \) and \( r \geq 2 \). To obtain the symmetry group of Eq. (5) we need to transform (2) according to (6). To this end we compute the first prolongation of formulas (2) and derive the transformation rule for the first derivative of \( u \)

\[
\frac{\partial u'}{\partial x'} = \frac{U_v u_x + U_x}{X_v u_x + X_x}.
\]

So symmetry group (2) now reads as

\[
t' = T(t, \vec{\theta}), \quad x' = X(t, x, v, \vec{\theta}), \quad u' = \frac{U_v u_x + U_x}{X_v u_x + X_x}
\]

(8)

with \( v = \partial^{-1} u \) and \( U = (t, x, v, \vec{\theta}) \). Consequently, if the right-hand sides of (8) depend explicitly on the non-local variable \( v \), then transformation group (8) is a quasi-local symmetry of Eq. (7).

Evidently, transformations (8) include variable \( v \) if and only if

\[
X_v \neq 0 \quad \text{or} \quad \frac{\partial}{\partial v} \left( \frac{U_v u_x + U_x}{X_v u_x + X_x} \right) \neq 0
\]

or, equivalently (since all the functions involved are real-valued),

\[
(X_v)^2 + (U_{vv}X_v - U_v X_{vv})^2 + (U_x X_v - U_v X_{xv})^2 + (U_{vv} X_x + U_{xv} X_v - U_v X_{vv} - U_v X_{xv})^2 \neq 0.
\]

(9)

If \( X_v \neq 0 \), then the above inequality holds true. If \( X_v \) does vanish identically, then (9) reduces to \( U_{vv}^2 \neq 0 \). It is straightforward to express the above constraints in terms of the coefficients of the corresponding infinitesimal operator of group (2). As a result, we get the following assertion.
Theorem 1 Equation (4) can be reduced to evolution equation (7) having QLS if it admits Lie symmetry, whose infinitesimal generator satisfies one of the inequalities

\[ \frac{\partial \xi}{\partial u} \neq 0, \quad (10) \]
\[ \frac{\partial \xi}{\partial u} = 0, \quad \left( \frac{\partial^2 \eta}{\partial u \partial x} \right)^2 + \left( \frac{\partial^2 \eta}{\partial u \partial u} \right)^2 \neq 0. \quad (11) \]

Now we can formulate an algorithm for constructing evolution equations of the form (1) admitting QLS.

1. We compute the maximal Lie symmetry group \( \mathcal{S} \) of differential equation (1).
2. We classify inequivalent one-parameter subgroups of \( \mathcal{S} \) and select subgroups \( \mathcal{S}_1, \ldots, \mathcal{S}_p \) whose infinitesimal operators are of the form \( Q = \xi(t, x, u) \partial_x + \eta(t, x, u) \partial_u \).
3. For each subgroup, \( \mathcal{S}_i \), we construct a change of variables (4) reducing the corresponding infinitesimal operator \( Q \) to the canonical operator \( \partial \bar{u} \), which leads to evolution equations of the form (5).
4. Since the invariance group, \( \bar{\mathcal{S}} \), admitted by (5) is isomorphic to \( \mathcal{S} \), we can utilize the results of subgroup classification of \( \mathcal{S} \). For each of the one-parameter subgroups of \( \bar{\mathcal{S}} \) we check whether its infinitesimal generators satisfies one of conditions (10), (11) of Theorem 1. This yields the list of evolution equations that can be reduced to those having QLS.
5. Performing non-local change of variables (6) yields evolution equations (7) admitting quasi-local symmetries (8).

Full implementation of the above approach will be the topic of our future publications. Here we restrict our considerations to classifying realizations of two- and three-parameter Lie transformation groups leading to evolution equations (1) that admit QLS.

Hereafter we suggest that evolution equation (1) admits a Lie symmetry \( Q = \xi(t, x, u) \partial_x + \eta(t, x, u) \partial_u \) and therefore can be reduced to the form (5). Differential equation (5) is guaranteed to admit at least a one-parameter
symmetry group, which is generated by the operator $\partial_u$. What we are going to do is to describe all realizations of two- and three-dimensional Lie algebras, which

- are Lie symmetry algebras of equations of the form (5), and,

- have coefficients satisfying one of the inequalities (10), (11) from Theorem 1.

With these realizations in hand, the problem of describing equations having QLS reduces to a straightforward application of the infinitesimal Lie method [4, 10, 11], which boils down to integrating Euler-Lagrange system for calculating differential invariants of the corresponding Lie algebras of first-order differential operators.

Let us remind that the most general symmetry generator admitted by (5) is of the form (3), while the most general equivalence group admitted by Eq.(11) reads as (see, e.g., [16])

$$
\bar{t} = T(t), \quad \bar{x} = X(t, x, u), \quad \bar{u} = U(t, x, u),
$$

(12)

where $T, X, U$ are arbitrary smooth real-valued functions.

We can always choose basis operators of Lie symmetry algebra of Eq.(1) so that

$$
e_0 = \partial_u, \quad e_i = \tau_i(t)\partial_t + \xi_i(t, x, u)\partial_x + \eta_i(t, x, u)\partial_u,
$$

where $i = 1, \ldots, r$. Note that by the Magadeev theorem [12] the maximal possible value for $r$ is $n + 4$, $n$ being the order of evolution equation (5), provided (5) is not locally equivalent to a linear equation. In particular, for the second-order evolution equation we have $r \leq 6$. By the definition of Lie algebra there are constant $r \times r$ matrix $C$ and constant $r$-component vector $\vec{c}$ such that

$$
[e_0, e_i] = \sum_{j=1}^{r} C_{ij} e_j + c_i e_0, \quad i = 1, \ldots, r.
$$

(13)

Here $[Q_1, Q_2] \equiv Q_1 Q_2 - Q_2 Q_1$.

System of equations (13) is the starting point of our classification algorithm. First of all, let us note that by re-arranging the basis of the Lie algebra, $e_{\mu} \to \sum_{\nu=0}^{r} c_{\mu\nu} e_\nu$, $\nu = 0, 1, \ldots, r$, we can always reduce the constant matrix $C$ to the canonical form. Consequently, without any loss of generality we may assume that the matrix $C$ is in the canonical form.
Computing commutators in the left-hand sides of (13) and equating the coefficients of linearly-independent operators $\partial_t, \partial_x, \partial_u$ we get the following system of partial differential equations:

\[
\begin{align*}
\frac{\partial \tilde{\xi}}{\partial u} &= C\xi, \\
\frac{\partial \tilde{\eta}}{\partial u} &= C\eta + \tilde{c}, \\
C\tau &= 0,
\end{align*}
\]

(14)

where $\tilde{\xi} = (\xi_1, \ldots, \xi_r), \tilde{\eta} = (\eta_1, \ldots, \eta_r), \tilde{\tau} = (\tau_1, \ldots, \tau_r)$. After integrating differential equations (14) we need to ensure that the operators $e_1, \ldots, e_r$ do form a basis of Lie algebra and satisfy the additional set of commutation relations,

\[
[e_i, e_j] = \sum_{k=1}^{r} c_{ij}^k e_k, \quad k = 1, \ldots, r.
\]

Next, we simplify the form of operators $e_1, \ldots, e_r$ using the suitable equivalence transformations from the group $E$. As a final step, we verify that at least one of the coefficients of one the operators $e_1, \ldots, e_r$ satisfy either (10) or (11).

3 Realizations of two-dimensional QLS algebras

For the case when $r = 1$, system (14) reduces to a pair of non-coupled differential equations

\[
\begin{align*}
\xi_u &= \lambda \xi, \\
\eta_u &= \lambda \eta + c, \\
\lambda \tau &= 0,
\end{align*}
\]

(15)

where $\lambda, c$ are constants.

While integrating (15) we need to differentiate between the two cases $\lambda \neq 0$ and $\lambda = 0$.

**Case 1.** $\lambda \neq 0$. The general solution of (15) has the form

\[
\begin{align*}
\tau(t) &= 0, \\
\xi(t, x, u) &= W_1(t, x) \exp(\lambda u), \\
\eta(t, x, u) &= W_2(t, x) \exp(\lambda u) - c\lambda^{-1}.
\end{align*}
\]

where $W_1, W_2$ are arbitrary smooth functions. So that the two-dimensional Lie algebra $\langle e_0, e_1 \rangle$ read as

\[
e_0 = \partial_u, \quad e_1 = W_1(t, x) \exp(\lambda u) \partial_x + (W_2(t, x) \exp(\lambda u) - c\lambda^{-1}) \partial_u.
\]
As $\lambda \neq 0$ we can always re-scale $u$, i.e., make a transformation $u \rightarrow ku$, $k = \text{const}$, in order to get $\lambda = 1$. Next taking as new $e_1$ the linear combination $c\lambda^{-1}e_0 + e_1$ we get rid of the term in $e_1$ which is proportional to $c$, namely,

$$e_0 = \partial_u, \quad e_1 = W_1(t,x) \exp(u)\partial_x + W_2(t,x) \exp(u)\partial_u.$$ 

It is not difficult to verify that the most general subgroup of the equivalence group $\mathcal{E}$ not altering the form of equation (5) and operator $\partial_u$ is given by the formulas

$$\tilde{t} = T(t), \quad \tilde{x} = X(t,x), \quad \tilde{u} = u + U(t,x),$$

(16)

where $T, X, U$ are arbitrary smooth functions.

Since the functions $W_1$ and $W_2$ do not vanish simultaneously, we have three possible subcases, (1) $W_1 \neq 0, W_2 \neq 0$, (2) $W_1 \neq 0, W_2 = 0$, and $W_1 = 0, W_2 \neq 0$.

**Case 1.1.** $W_1 \neq 0, W_2 \neq 0$. Applying (16) with $T = t$ we reduce the operator $e_1$ to the form

$$e_1 = \epsilon_1 \exp(u)\partial_x + \epsilon_2 \exp(u)\partial_u,$$

where $\epsilon_1 = \pm 1, \epsilon_2 = \pm 1$. Combining equivalence transformation $t \rightarrow t$, $x \rightarrow -x$, $u \rightarrow u$ and re-scaling $e_1 \rightarrow -e_1$ we get the final form of the basis elements $e_0, e_1$

$$e_0 = \partial_u, \quad e_1 = \exp(u)(\partial_x + \partial_u).$$

Since $\xi_u = \exp(u) \neq 0$, the condition (10) of Theorem 1 holds true and the evolution equation invariant under the above algebra is equivalent to a quasi-linear evolution equation that admits QLS.

**Case 1.2.** $W_1 \neq 0, W_2 \neq 0$. Applying transformation (15) with $T(t) = t, U(t,x) = 0, k = 1$ we reduce the operator $e_1$ to the form $e_1 = \exp(u)\partial_x$. Again, the coefficient $\xi = \exp(u)$ obeys condition (10) of Theorem 1 and, therefore, it gives rise to the two-dimensional Lie algebra

$$e_0 = \partial_u, \quad e_1 = \exp(u)\partial_x,$$

that leads to an evolution equation admitting QLS.

**Case 1.3.** $W_1 = 0, W_2 \neq 0$. Applying transformation (15) with $T(t) = t, X(t,x) = x$ we reduce the operator $e_1$ to the form $e_1 = \pm \exp(u)\partial_u$. Rescaling, if necessary, the operator $e_1$ to $-e_1$ we can make sure that $e_1$ reads as $\exp(u)\partial_u$ and finally get

$$e_0 = \partial_u, \quad e_2 = \exp(u)\partial_u.$$
Since the coefficient $\eta = \exp(u)$ obeys condition (11) of Theorem 1, an evolution equation invariant under the above algebra is equivalent to a partial differential equation of the form (7) which has QLS.

**Case 2.** $\lambda = 0$. System (15) is readily integrated to yield

$$T(t) = W_0(t), \quad \xi = W_1(t, x), \quad \eta = W_2(t, x) + cu.$$  

Here $W_0, W_1, W_2$ are arbitrary smooth functions. Checking the conditions of Theorem 1 we see that neither of them can be satisfied by the coefficients of the operator $e_1$. Consequently, this case yields no equations admitting QLS.

Below we give the full list of $E$-inequivalent realizations of two-dimensional Lie algebras spanned by the operators $e_0 = \partial_u$, $e_1 = T(t)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u$ satisfying the conditions of Theorem 1

$$A^1_2 : \langle \partial_u, \exp(u)(\partial_x + \partial_u) \rangle,$$

$$A^2_2 : \langle \partial_u, \exp(u)\partial_x \rangle,$$

$$A^3_2 : \langle \partial_u, \exp(u)\partial_u \rangle.$$

Evolution equations (5) invariant under the above algebras are reduced to differential equations that admit QES. The corresponding QLS are obtained by re-writing the transformation groups generated by the operators $e_1$ in terms of new (non-local) variables $t, x, u$ and $\partial^{-1}u$.

Consider, for example, the algebra $A^2_2 = \langle \partial_u, \exp(u)\partial_x \rangle$. Applying the standard infinitesimal Lie algorithm [10] we obtain the determining equations for the function $f$

$$-u_xf - f_x + u_x^2f_{ux} + (u_x^3 + 3u_xu_{xx})f_{uxx} = 0.$$  

The general solution of the above equation reads as

$$f(t, x, u_x, u_{xx}) = u_x \hat{f}(\omega_0, \omega_1, \omega_2),$$

where $\hat{f}$ is an arbitrary smooth function and $\omega_0 = t, \omega_1 = (xu_x - 1)u_x^{-1}, \omega_2 = (u_{xx} + u_x^2)u_x^{-3}$ are absolute invariants of the transformation group generated by the operators $\partial_u$ and $\exp(u)\partial_x$. Consequently, the evolution equation invariant under the algebra $A^2_2$ is of the form

$$u_t = u_x \hat{f}(\omega_0, \omega_1, \omega_2).$$
Differentiating the above equation with respect to $x$ and replacing $u_\times$ with $u$ according to (6) we arrive at the evolution equation

$$u_t = u_\times \tilde{f} + \frac{u_\times + u^2}{u^2} \tilde{f}_\omega_1 + \frac{u u_\times x - 3(u^2 + 1)u_\times x}{u^4} \tilde{f}_\omega_2$$

with $\omega_0 = t$, $\omega_1 = x - u^{-1}$, $\omega_2 = (u_\times + u^2)u^{-3}$. This differential equation admits the following quasi-local symmetry group

$$t' = t, \quad x' = x + \theta \exp(v), \quad u' = \frac{u}{1 + \theta u \exp(v)}.$$

where $\theta$ is a group parameter and $v = \partial^{-1}u$.

### 4 Realizations of three-dimensional QLS algebras

Consider system of partial differential equations (14) with $r = 2$. The constant $2 \times 2$ matrix $C$ has been reduced to the canonical real Jordan form. There are three inequivalent cases that need to be considered separately, namely, when eigenvalues

1. are complex conjugate,
2. are real, and the matrix $C$ is diagonal matrix,
3. are real and the matrix $C$ is the $2 \times 2$ canonical Jordan box

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

We consider in detail the class of realizations of three-dimensional Lie algebras obtained for the case when $C$ has two complex eigenvalues $\lambda_1, \lambda_2$. For the remaining two classes we present the final results only.

#### 4.1 Case of diagonal canonical form with complex eigenvalues

As the characteristic equation of the real matrix $C$ is real, the eigenvalues have to satisfy the additional constraint $\lambda_1^2 = \lambda_2$. Consequently, if we define
\[ \lambda = \frac{\lambda_1 + \lambda_2}{2} \] and \[ a = \frac{(\lambda_1 - \lambda_2)}{(2i)} \], then the general solution of (14) can be represented in the form

\[ \begin{align*}
\tau_1 &= 0, \quad \tau_2 = 0, \\
\xi_1 &= (W_1(t, x) \cos(au) + W_2(t, x) \sin(au)) \exp(\lambda u), \\
\xi_2 &= (W_2(t, x) \cos(au) - W_1(t, x) \sin(au)) \exp(\lambda u), \\
\eta_1 &= (W_3(t, x) \cos(au) + W_4(t, x) \sin(au)) \exp(\lambda u) + b_1, \\
\eta_2 &= (W_4(t, x) \cos(au) - W_3(t, x) \sin(au)) \exp(\lambda u) + b_2.
\end{align*} \] (17)

Here \( W_1, W_2, W_3, W_4 \) are arbitrary smooth real-valued functions, \( b_1, b_2 \) are real constants. Hence, the most general form of the basis operators \( e_1, e_2 \) is

\[ e_1 = (W_1(t, x) \cos(au) + W_2(t, x) \sin(au)) \exp(\lambda u) \partial_x + \left( (W_3(t, x) \cos(au) + W_4(t, x) \sin(au)) \exp(\lambda u) + b_1 \right) \partial_u, \]

\[ e_2 = (W_2(t, x) \cos(au) - W_1(t, x) \sin(au)) \exp(\lambda u) \partial_x + \left( (W_4(t, x) \cos(au) - W_3(t, x) \sin(au)) \exp(\lambda u) + b_2 \right) \partial_u. \]

Note that \( a \neq 0 \), otherwise, \( \lambda_1, \lambda_2 \) are not complex. By re-scaling the variable \( u \) we can make \( a \) equal to 1. Next, applying to the operators \( e_1, e_2 \) an equivalence transformation (15) with \( T(t) = t, X(t, x) = x \) we can get rid of the function \( W_2 \). With these remarks the operators \( e_1, e_2 \) take the form

\[ e_1 = \left( (W_3(t, x) \cos u + W_4(t, x) \sin u) \exp(\lambda u) + b_1 \right) \partial_u + W_1(t, x) \cos u \exp(\lambda u) \partial_x, \]

\[ e_2 = \left( (W_4(t, x) \cos u - W_3(t, x) \sin u) \exp(\lambda u) + b_2 \right) \partial_u - W_1(t, x) \sin u \exp(\lambda u) \partial_x. \] (18)

In what follows we need to distinguish between the cases, \( \lambda \neq 0 \) and \( \lambda = 0 \).

**Case 1.** \( \lambda \neq 0 \). Performing, if necessary, transformation (15) with \( T(t) = t, U(t, x) = u \) we can always make non-vanishing identically function \( W \) equal to 1. Next, taking as \( e_1 \) and \( e_2 \) the linear combinations \( e_1 - b_1 e_0 \) and \( e_2 - b_2 e_0 \) we can get rid of parameters \( b_1, b_2 \).

Now we need to ensure that the operators \( e_0, e_1, e_2 \) do form a realization of a Lie algebra. To this end we have to verify that the relation

\[ [e_1, e_2] = \alpha e_1 + \beta e_2 + \gamma e_0 \] (19)
with some real \( \alpha, \beta, \gamma \) holds true. Calculating the commutators and equating the coefficients of linearly-independent operators \( \partial_t, \partial_x, \partial_u \) we get the system of differential equations for \( W_3, W_4 \). Its general solution is given by the formulas

\[
W_3 = \lambda (\lambda^2 + 1)^{-1} (x - p(t))^{-1}, \quad W_4 = -(\lambda^2 + 1)^{-1} (x + p(t))^{-1}.
\]

Here \( p(t) \) is an arbitrary smooth function. Making the equivalence transformation (15) with \( T(t) = t, X(t, x) = p(t), U(t, x) = 0 \) we eliminate the function \( p(t) \) and arrive at the following realization of a three-dimensional Lie algebra

\[
\begin{align*}
\mathcal{E}_0 &= \partial_u, \\
\mathcal{E}_1 &= \exp(\lambda u) \cos u \partial_x + (\lambda^2 + 1)^{-1} (\lambda \cos u - \sin u) x^{-1} \exp(\lambda u) \partial_u, \\
\mathcal{E}_2 &= -\exp(\lambda u) \sin u \partial_x - (\lambda^2 + 1)^{-1} (\cos u - \lambda \sin u) x^{-1} \exp(\lambda u) \partial_u,
\end{align*}
\]

Evidently, the coefficients of \( \mathcal{E}_1, \mathcal{E}_2 \) satisfy condition (10) of Theorem 1 and, consequently, evolution equation invariant under the symmetry algebra \( \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2 \) can be reduced to the one having QLS.

Case 2. \( \lambda = 0 \). Operators (18) take the form

\[
\begin{align*}
\mathcal{E}_1 &= \left( (W_3(t, x) \cos u + W_4(t, x) \sin u) + b_1 \right) \partial_u + W_1(t, x) \cos u \partial_x, \\
\mathcal{E}_2 &= \left( (W_4(t, x) \cos u - W_3(t, x) \sin u) + b_2 \right) \partial_u - W_1(t, x) \sin u \partial_x.
\end{align*}
\]

Replacing \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) with the linear combinations \( \mathcal{E}_1 - b_1 \mathcal{E}_0 \) and \( \mathcal{E}_2 - b_2 \mathcal{E}_0 \) eliminates the parameters \( b_1, b_2 \). So that we can assume that \( b_1 = 0, b_2 = 0 \) without any loss of generality.

Case 2.1. \( W_1 \neq 0 \). Utilizing equivalence transformation (15) with \( T = t, U = 0 \) we can make \( W_1 \) equal to 1. After simple algebra we prove that for the operators \( \mathcal{E}_1, \mathcal{E}_2 \) to satisfy the remaining commutation relation (19) the functions \( W_3, W_4 \) have to take one of the following forms:

\[
\begin{align*}
W_3 &= 0, \quad W_4 = \mu \tan(\mu x), \\
W_3 &= 0, \quad W_4 = -\mu \tanh(\mu x), \\
W_3 &= 0, \quad W_4 = x^{-1},
\end{align*}
\]

where \( \mu \) is an arbitrary real parameter. Inserting the above expressions into
the corresponding formulas for \( e_1, e_2 \) we finally get

\[
\begin{align*}
e_0 &= \partial_u, \\
e_1 &= \cos u \partial_x + \mu \tan(\mu x) \sin u \partial_u, \\
e_2 &= -\sin u \partial_x + \mu \tan(\mu x) \cos u \partial_u; \\
e_0 &= \partial_u, \\
e_1 &= \cos u \partial_x - \mu \tanh(\mu x) \sin u \partial_u, \\
e_2 &= -\sin u \partial_x + \mu \tanh(\mu x) \cos u \partial_u; \\
e_0 &= \partial_u, \\
e_1 &= \cos u \partial_x - x^{-1} \sin u \partial_u, \\
e_2 &= -\sin u \partial_x - x^{-1} \cos u \partial_u.
\end{align*}
\]

**Case 2.2.** \( W_1 = 0 \). In this case using transformation (15) with \( T = t, X = x \) we can eliminate \( W_4 \). Inserting the corresponding expressions for \( e_1, e_2 \) into (19) and solving the obtained equations within the equivalence relation \( \mathcal{E} \) yields

\[
\begin{align*}
e_0 &= \partial_u, \\
e_1 &= \cos u \partial_u, \\
e_2 &= \sin u \partial_u.
\end{align*}
\]

Note that all realizations of three-dimensional Lie algebras obtained under Cases 2.1, 2.2 satisfy the conditions of Theorem 1. Consequently, evolution equations invariant with respect to the above algebras can be transformed into equations admitting QLS.

Summing up we present the full list of realizations of three-dimensional Lie algebras, obtained for the case when \( 2 \times 2 \) matrix \( C \) in (14) has two complex eigenvalues.

\[
\begin{align*}
A_1^3 & : \langle \partial_u, \exp(\mu u) \cos u \partial_x + (\mu^2 + 1)^{-1}(\mu \cos u - \sin u) x^{-1} \exp(\mu u) \partial_u, \\
A_2^3 & : \langle \partial_u, \cos u \partial_x + \mu \tan(\mu x) \sin u \partial_u, -\sin u \partial_x + \mu \tan(\mu x) \cos u \partial_u, \\
A_3^3 & : \langle \partial_u, \cos u \partial_x - \mu \tanh(\mu x) \sin u \partial_u, -\sin u \partial_x - \mu \tanh(\mu x) \cos u \partial_u, \\
A_4^3 & : \langle \partial_u, \cos u \partial_x - x^{-1} \sin u \partial_u, -\sin u \partial_x - x^{-1} \cos u \partial_u, \\
A_5^3 & : \langle \partial_u, \cos u \partial_u, \sin u \partial_u \rangle.
\end{align*}
\]

Here \( \mu \) is an arbitrary real constant.

Evolution equations (15) invariant under the algebras \( A_1^3, \ldots, A_5^3 \) can be reduced to equations admitting QLS.
4.2 Case of diagonal canonical form with real eigenvalues

Without loss of generality we may assume that the matrix $C$ from (14) has been reduced to the diagonal matrix \[
\begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2 
\end{pmatrix}
\]. Since $\lambda_1, \lambda_2$ do not vanish simultaneously we may assume that $\lambda_1 \ne 0$. Re-scaling $u$ we make $\lambda_1$ equal to 1. With this choice of $C$, system (14) takes the form

\[
T_1 = 0, \quad \lambda_2 T_2 = 0,
\]
\[
\xi_{1u} = \xi_1, \quad \xi_{2u} = \lambda_2 \xi_2,
\]
\[
\eta_{1u} = \eta_1 + c_1, \quad \eta_{2u} = \lambda_2 \eta_2 + c_2.
\]

Integrating the above system within the equivalence relation $\mathcal{E}$, inserting the result into the remaining commutation relation (13) and solving the equations obtained we arrive at the following realizations of three-dimensional Lie algebras:

\begin{align*}
A_3^6 & : \langle \partial_u, \exp(u)\partial_x, (x^2 + \sigma(t)) \exp(-u)\partial_x + 2x \exp(-u)\partial_u \rangle, \\
A_3^7 & : \langle \partial_u, \exp(u)\partial_x + \epsilon \exp(u)\partial_u, (\sigma(t) \exp(x) \pm \exp(2x) + \mu) \exp(-u)\partial_x \\
& \quad +(\pm \exp(2x) - \mu)\partial_u \rangle, \\
A_3^8 & : \langle \partial_u, \exp(u)\partial_x + \exp(u)\partial_u, (\sigma(t) \exp(-\mu x) \pm \exp((1 - \mu)x)) \\
& \quad \times \exp(\mu u)\partial_x \pm \exp((1 - \mu) x) \exp(\mu u)\partial_u \rangle, \\
A_3^9 & : \langle \partial_u, \exp(u)\partial_x, \sigma(t)x \exp(\mu u)\partial_x + \sigma(t) \exp(\mu u)\partial_u \rangle, \\
A_3^{10} & : \langle \partial_u, \exp(u)\partial_x, \sigma(t) \exp(\mu u)\partial_u \rangle, \\
A_3^{11} & : \langle \partial_u, \exp(u)\partial_x + \partial_u, (\sigma(t) + \exp(x))\partial_x + \exp(x)\partial_u + \epsilon \partial_t \rangle, \\
A_3^{12} & : \langle \partial_u, \exp(u)\partial_x + \partial_u, \sigma(t)\partial_x + \epsilon \partial_t \rangle, \\
A_3^{13} & : \langle \partial_u, \exp(u)\partial_x, \sigma(t)(x\partial_x + \partial_u) + \epsilon \partial_t \rangle, \\
A_3^{14} & : \langle \partial_u, \exp(u)\partial_x, \sigma(t)(\partial_x + \partial_u) + \epsilon \partial_t \rangle, \\
A_3^{15} & : \langle \partial_u, \exp(u)\partial_u, \sigma(t)\partial_u + \epsilon \partial_t \rangle, \\
A_3^{16} & : \langle \partial_u, \exp(u)\partial_u, \partial_x + \sigma(t)\partial_u \rangle, \\
A_3^{17} & : \langle \partial_u, \exp(u)(\partial_x + \partial_u), \exp(u)(\sigma_1(t) \exp(-x) + \sigma_2(t))\partial_x \\
& \quad + \sigma_1(t) \exp(u)\partial_u \rangle, \\
A_3^{18} & : \langle \partial_u, \exp(u)\partial_x, \exp(u)(\sigma_1(t)x + \sigma_2(t))\partial_x + \sigma_1(t) \exp(u)\partial_u \rangle,
\end{align*}
Here $\sigma(t), \sigma_1(t), \sigma_2(t)$ are arbitrary real-valued smooth functions, $\mu$ is an arbitrary real parameter, and $\epsilon = 0, 1$.

By construction the coefficients of algebras $A_3^6 - A_3^{20}$ satisfy the conditions of Theorem 1. Consequently, evolution equations (13) invariant under these algebras can be reduced to ones having QLS.

### 4.3 Case of non-diagonal canonical form

For the case under consideration, the matrix $C$ from (14) is of the form

\[
\begin{pmatrix}
\lambda & 1 \\
0 & \lambda
\end{pmatrix}
\]

System of partial differential equations (13) now reads as

\[
\lambda T_1 + T_2 = 0, \quad \lambda T_2 = 0,
\]

\[
\xi_1 u = \lambda \xi_1 + \xi_2, \quad \xi_2 u = \lambda \xi_2,
\]

\[
\eta_1 u = \lambda \eta_1 + \eta_2 + c_1, \quad \eta_2 u = \lambda \eta_2 + c_2.
\]

Integrating the above system within the equivalence relation $\mathcal{E}$, inserting the result into (19) and solving the relations obtained we obtain eleven realizations of three-dimensional Lie algebras:

- $A_3^{21}$ : $\langle \partial_u, u \exp(u)\partial_x + x^{-1}(u + 1) \exp(u)\partial_u, \exp(u)\partial_x + x^{-1} \exp(u)\partial_u \rangle$,
- $A_3^{22}$ : $\langle \partial_u, u \partial_x + (\mu u^2 + \sigma(t) \exp(4\mu x))\partial_u, \partial_x + 2\mu \partial_u \rangle$,
- $A_3^{23}$ : $\langle \partial_u, u \partial_x + (\mu x + \nu u + \sigma_1(t))\partial_u + \sigma_2(t)\partial_x, \partial_x \rangle$,
- $A_3^{24}$ : $\langle \partial_u, \partial_x + \mu u \tan(\mu x)\partial_u, \tan(\mu x)\partial_u \rangle$,
- $A_3^{25}$ : $\langle \partial_u, \partial_x - \mu u \tanh(\mu x)\partial_u, \tanh(\mu x)\partial_u \rangle$,
- $A_3^{26}$ : $\langle \partial_u, \partial_x - x^{-1} u \partial_u, x^{-1} \partial_u \rangle$,
- $A_3^{27}$ : $\langle \partial_u, (u^2 + x)\partial_u, u \partial_u \rangle$,
- $A_3^{28}$ : $\langle \partial_u, (u^2 + t)\partial_u, u \partial_u \rangle$,
- $A_3^{29}$ : $\langle \partial_u, u(\nu + 2\mu \tan(\mu t + \alpha x))\partial_u + 2\partial_t, (\nu + 2\mu \tan(\mu t + \alpha x))\partial_u \rangle$,
- $A_3^{30}$ : $\langle \partial_u, u(\nu + 2\mu \tanh(-\mu t + \alpha x))\partial_u + 2\partial_t, (\nu + 2\mu \times \tanh(-\mu t + \alpha x))\partial_u \rangle$,
- $A_3^{31}$ : $\langle \partial_u, (\mu \nu x - \mu t - 2)(t - \nu x)^{-1} u \partial_u + 2t \partial_t, (\mu \nu x - \mu t - 2) \times (t - \nu x)^{-1} u \partial_u \rangle$.  

15
Here $\sigma(t), \sigma_1(t), \sigma_2(t)$ are arbitrary real-valued smooth functions, $\alpha, \mu, \nu$ are arbitrary real parameters.

Evolution equations (5) admitting symmetry algebras $A_3^{21}, \ldots, A_3^{31}$ can be reduced to equations having QLS.

5 Some generalizations

The technique developed in the previous sections naturally expands to cover general systems of evolution equations

$$\vec{u}_t = \vec{F}(t, x, \vec{u}, \vec{u}_1, \ldots, \vec{u}_n), \quad n \geq 2.$$  \hfill (20)

Here $\vec{F} = (F^1, \ldots, F^m)$ is an arbitrary $m$-component real-valued smooth function, $\vec{u} = (u^1, \ldots, u^m) \in \mathbb{R}^m$, and $\vec{u}_i = (\partial^i \vec{u})/(\partial x^i), i = 1, \ldots, n$.

Suppose that system of evolution equations (20) admits $m$-parameter Abelian symmetry group which leaves the temporal variable, $t$, invariant. The infinitesimal generators of this group have to be of the form

$$Q_i = \xi_i(t, x, \vec{u})\partial_x + \sum_{j=1}^{m} \eta_{ij}(t, x, \vec{u})\partial_{u^j}, \quad i = 1, \ldots, n$$  \hfill (21)

and what is more, the rank of the matrix composed of the coefficients of differential operators $\partial_x, \partial_{u^1}, \ldots, \partial_{u^m}$ equals to $m$. Given these conditions, there exists a change of variables

$$\tilde{t} = t, \quad \tilde{x} = X(t, x, \vec{u}), \quad \tilde{u} = \tilde{U}(t, x, \vec{u})$$  \hfill (22)

that reduces operators (21) to canonical ones $\tilde{Q}_i = \partial_{\tilde{u}^i}, i = 1, \ldots, m$ (see, e.g., [10]). Consequently system of evolution equations (20) takes the form

$$\vec{u}_t = \vec{f}(t, x, \vec{u}_1, \ldots, \vec{u}_n), \quad n \geq 2.$$  \hfill (23)

Note that we dropped the bars.

Now we can apply the same trick we utilized for the case of a single evolution equation. Namely, we differentiate (23) with respect to $x$ and map $\vec{u} \rightarrow \vec{u}_x$ thus getting

$$\vec{u}_t = \frac{\partial \vec{f}}{\partial x} + \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial \vec{f}}{\partial u^j_{i-1}} u^j_i$$  \hfill (24)
with \( u_0^j \equiv u^j \).

Now if the system of evolution equations (23) admits the Lie transformation group
\[
t' = T(t, \theta), \quad x' = X(t, x, \vec{u}, \theta), \quad \vec{u}' = \vec{U}(t, x, \vec{u}, \theta),
\]
where \( \theta \in \mathbb{R} \) is a group parameter, then the transformed system of equations
(25) admits the group
\[
t' = T(t, \theta), \quad x' = X(t, x, \vec{v}, \theta), \quad \vec{u}' = \vec{U}(t, x, \vec{v}, \theta)
\]
with \( v^i = \partial^{-1} u^i \equiv \int u^i dx \) and \( \vec{U} = \vec{U}(t, x, \vec{v}, \theta) \). Consequently, provided either of relations
\[
\partial X / \partial v^j \neq 0, \quad \frac{\partial}{\partial v^j} \left( \frac{\vec{U}_x + \sum_{i=1}^m \vec{U}_{v^i} u^i}{X_x + \sum_{i=1}^m X_{v^i} u^i} \right) \neq 0
\]
holds for some \( j, \ 1 \leq j \leq m \), system of evolution equations admits QLS (26).

Set of relations (27) is equivalent to the following system of inequalities
\[
\sum_{i=1}^m X_{v^i}^2 \neq 0,
\]
or
\[
\sum_{i=1}^m X_{v^i}^2 = 0, \quad \sum_{i,j=1}^m (U_{xv^i}^j)^2 + \sum_{i,j,k=1}^m (U_{v^i v^j}^k)^2 \neq 0.
\]

Note that all the functions involved are real-valued, so that vanishing of the sum of squares requires that every summand vanishes individually. Rewriting the obtained relations in terms of coefficients of the corresponding infinitesimal operators we arrive at the following assertion.

**Theorem 2** System of evolution equations (23) can be reduced to a system having QLS if it admits Lie symmetry whose infinitesimal operator
\[
Q = \tau(t) \partial_t + \xi(t, x, \vec{u}) \partial_x + \sum_{i=1}^m \eta_i(t, x, \vec{u}) \partial_{v^i}
\]
satisfies one of the inequalities
\[
\sum_{i=1}^m \xi_{v^i}^2 \neq 0,
\]
\[
\sum_{i=1}^{m} \xi_{vi}^2 = 0, \quad \sum_{i,j=1}^{m} (\eta_{ivj}^j)^2 + \sum_{i,j,k=1}^{m} (\eta_{ivij}^k)^2 \neq 0. \tag{29}
\]

Summing up we formulate the algorithm for classifying systems of evolution equations (20) admitting QLS.

1. We compute the maximal Lie symmetry group \(S\) of system of partial differential equations (20).

2. We classify inequivalent \(m\)-parameter Abelian subgroups \(S_1, \ldots, S_p\) of the group \(S\) and select subgroups whose infinitesimal operators are of the form (21).

3. For each subgroup \(S_i\) we construct change of variables (22) reducing commuting infinitesimal operators, \(Q_i\), to the canonical forms \(\partial_{\bar{u}_i}\), which leads to system of evolution equations (23).

4. Since the invariance group, \(\bar{S}\), admitted by (23) is isomorphic to \(S\), we can utilize the results of subgroup classification of \(S\). For each of the \(m\)-parameter Abelian subgroups of \(\bar{S}\) we check whether their infinitesimal generators satisfy one of conditions (28), (29) of Theorem 2. This yields the list of systems of evolution equations that can be reduced to those having QLS.

5. Performing the non-local change of variables \(u^i \rightarrow \bar{u}_i^i, i = 1, \ldots, m\) we obtain systems of evolution equations (24) admitting quasi-local symmetries (26).

We intend to devote a special publication to application of this algorithm to Schrödinger-type systems of partial differential equations. Here we present an example of Galilei-invariant nonlinear Schrödinger equation, which leads to the equation possessing QLS.

Consider nonlinear Schrödinger equation

\[
\begin{align*}
    i\psi_t &= \psi_{xx} + 2(x + i\alpha)^{-1}\psi_x - (i/2)(x + i\alpha) \\
    &\quad + F(2i\alpha(x + i\alpha)\psi_x - (x - i\alpha)(\psi - \psi^*)),
\end{align*}
\tag{30}
\]

where \(\psi = \psi_{RE}(t, x) + i\psi_{IM}(t, x), \psi^* = \psi_{RE}(t, x) - i\psi_{IM}(t, x), \alpha \neq 0\) is an arbitrary real constant and \(F\) is an arbitrary complex-valued function.
According to \[13\], Eq.(30) admits the Lie algebra of the Galilei group having the following basis operators:

\[
\begin{align*}
    e_1 &= \partial_t, \\
    e_2 &= \partial_\psi + \partial_{\psi^*}, \\
    e_3 &= (x + i\alpha)^{-1}\partial_\psi + (x - i\alpha)^{-1}\partial_{\psi^*}, \\
    e_4 &= \partial_x - (t + (x + i\alpha)^{-1}\psi)\partial_\psi - (t + (x - i\alpha)^{-1}\psi^*)\partial_{\psi^*}.
\end{align*}
\]

Operators \(e_2, e_3\) commute and the rank of the matrix of coefficients of operators \(\partial_t, \partial_x, \partial_\psi, \partial_{\psi^*}\) is equal to 2. Consequently, there is a change of variables that reduces \(e_2, e_3\) to canonical forms \(\partial_u, \partial_v\). Indeed, making the change of variables

\[
    u(t, x) = (1/2)(\psi + \psi^*), \quad v(t, x) = (2i\alpha)^{-1}(x^2 + \alpha^2)(\psi - \psi^*)
\]

transforms \(e_1, e_2\) to become \(e_1 = \partial_u, e_2 = \partial_v\). So we can apply the above algorithm to Eq.(30) transformed according to (31). As the coefficients of the transformed operator \(e_3\) satisfy (29), it leads to QLS of the system of evolution equations of the form (24).

6 Concluding remarks

In the present paper we develop the efficient approach to constructing evolution type partial differential equations which admit quasi-local symmetries. It is important to emphasize that the algorithm can be applied iteratively. Namely, if the transformed equation possesses Lie symmetries which satisfy conditions of Theorem 1, then it again can be transformed to a new evolution equation admitting QLS and so on. What is more, the equation obtained as the \(N\)th iteration of the algorithm admits QLS which involves non-local variables \(\partial^{-1}u, \partial^{-2}u, \ldots, \partial^{-N}u\).

It is also of great interest to explore the case of multi-component evolution equations. The most natural objects are the non-linear Schrödinger-type equations or systems of nonlinear reaction-diffusion equations.

There is a different approach to analyzing non-local symmetries for some special differential equations based on the notion of potential symmetries introduced by Bluman [14, 15]. It is of interest to explore the connection between two approaches, which would open a way to group classification of equations with potential symmetries.
Study of the above mentioned problems is in progress now and will be reported in our future publications.

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