Nonfactorizable Contributions to Hadronic Matrix Elements by Use of Twist Wave Functions

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Abstract

A general approach for evaluating the nonfactorizable contributions to the hadronic matrix elements is described in terms of the higher twist wave functions of the mesons. An example is illustrated for the $\bar{B}^0 \rightarrow \pi^+ \pi^-$ decays.

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The factorization approximation has conventionally been used to describe the two-body nonleptonic decays of D and B mesons. However, it has been observed [1] that the naive factorization approach to nonleptonic heavy meson decays is incompatible with experimental data. The available data for D-meson decays seems to support the so-called rule of discarding \(1/N_c\) corrections. This rule has been formulated in ref. [2], which is in contrast to the standard prescription that keeps the \(1/N_c\) term. The dynamical explanation of this approximate cancellation of \(1/N_c\) terms in the D decays has been presented in ref. [3]. From such an explanation, it appears that discarding \(1/N_c\) terms may not necessarily be a universal rule. The recent experimental data on B-meson decays suggests that discarding \(1/N_c\) terms is not a universal rule for all channels. To understand these problems, one needs to see what approximation is being made in the naive factorization approach and how to systematically improve upon the approach. It will be seen that the issues concern the nonfactorizable contributions of the operator of two color-octet currents.

The leading nonfactorizable contributions to hadronic matrix element arise from the soft gluon interactions. It turns out, such soft gluon contributions to hadronic weak decays of mesons can be further factorized in terms of twist wave functions (TWF) of the mesons. We shall describe in this note a general approach for their evaluation.

As an illustration, let us consider an effective QCD-corrected weak Hamiltonian with two familiar operators \(O_1\) and \(O_2\).

\[
H_{\text{eff}} \sim c_1(\mu)O_1 + c_2(\mu)O_2
\]

where \(c_i(\mu)\) are the Wilson coefficient functions and \(O_i\) the effective four quark operators. For \(B \rightarrow \pi \pi\) decays, \(O_i\) are given by

\[
O_1 = (\bar{u}b)_{V-A}(\bar{d}u)_{V-A}, \\
O_2 = (\bar{u}_\alpha b_\beta)_{V-A}(\bar{d}_\beta u_\alpha)_{V-A}
\]

In the factorization approximation, the hadronic matrix element is factorized into the product of two matrix elements of singlet currents, which are evaluated in terms of decay constants and formfactors, and characterized by the parameters \(a_1^{(o)}\) and \(a_2^{(o)}\). These parameters are related to the Wilson coefficients \(c_i\)
This naive factorization fails to describe the nonleptonic charm meson decays. The approximation neglects the contributions from the product of two color-octet currents,

$$a_1^{(o)} = c_1 + \frac{1}{N_c}c_2, \quad a_2^{(o)} = c_2 + \frac{1}{N_c}c_1.$$  \hspace{1cm} (3)

Where \( t^a \) (\( a = 1, \cdots , 8 \)) are SU(3)\(_c\) generators, and the color indices in the operators \( O_i^c \) have been omitted. It becomes manifest that in the naive factorization approach at this level the contribution to the \( B \to \pi^+\pi^-\) decay from the operators \( O_2^c \) vanishes due to the mismatching of color.

Using the algebraic relation \( t^a t^b = \frac{1}{2N_c} \delta_{ab} + \frac{1}{2} (f^{abc} + id^{abc}) t^c \), one then obtains the following identities

$$O_i \equiv \frac{1}{N_c} O_j + O_i^c$$  \hspace{1cm} (5)

where \((i,j)\) represents the pair \((1,2)\) and \( N_c \) is the number of colors. The leading nonzero contribution of the operator \( O_2^c \) to the \( B \to \pi^+\pi^- \) decay arises from the soft gluon emission which recovers the color matching of the octet currents to the hadron. The resulting new effective operators due to the soft gluon interaction can be written as a product of two color-singlet currents again. The leading nonzero contribution is given by

$$A_{O_2} = -i \int dx < \pi^+(p_+)\pi^-(p_-)|T (j(x)O_2^c(0))|B(p) >$$  \hspace{1cm} (6)

where \( j(x) = \bar{d}(x)A(x)d(x) \) with \( A(x) \equiv g_s A(x)_{\mu} t^\mu \gamma^\mu \) (other quark flavors contribute to \( j(x) \), and their effects can be similarly calculated). Contracting the down quark pair, and using the algebraic relation of the SU(3) generators \( t^a t^b = \frac{1}{N_c} \delta_{ab} + \frac{1}{2} (f^{abc} + id^{abc}) t^c \), one then obtains

$$A_{O_2^c} = \frac{1}{N_c} \int dx \int \frac{dq}{4\pi^2} \frac{e^{-iqx}}{q^2-m_d^2} < \pi^+(p_+)\pi^-(p_-)|J^{\nu\mu}(x,q)J^G_{\nu\mu}|B(p) >$$  \hspace{1cm} (7)

with

$$J^{\nu\mu} = \bar{d}(x)\gamma^\nu(q^\rho \gamma_\rho + m_d)\gamma^\mu(1-\gamma_5)u(0),$$

$$J^G_{\nu\mu} = \int_0^1 d\alpha x^\rho \bar{u}(0)G_{\rho\nu}(\alpha x)\gamma_\mu(1-\gamma_5)b(0).$$  \hspace{1cm} (8)
where we have neglected the terms given by two color-octet currents since their nonzero contributions involve the next-to-leading order contributions of the coupling constant $g_s$.

At this level, one can again use the factorization approach to evaluate the hadronic matrix element, and obtain

$$A_G = \frac{1}{N_c} \int dx \int \frac{dq}{(4\pi)^4} \frac{e^{-iqx}}{q^2 - m_d^2} \cdot <\pi^- (p_-) | J^\mu(x, q)| 0 > <\pi^+ (p_+) | J^G_\mu B(p) >$$

As the currents $J_{\mu\nu}(x, q)$ and $J^G_{\mu\nu}$ are nonlocal, their hadronic matrix elements will be evaluated by so-called twist wave functions

$$T^{\mu\nu}(p_-, q, x) = <\pi^- (p_-) | J^\mu(x, q)| 0 > = i f_\pi \int_0^1 d\alpha e^{i\alpha x - p_-} \cdot \left\{ (p_-\mu q^\nu + p_-^\mu q^\nu - q \cdot p_- g^{\mu\nu} - i\varepsilon^{\mu\nu\rho\sigma} q_\rho p_-^\sigma) \phi_A(\alpha) \right\}$$

where $\phi_A, \phi_P$ and $\phi_T$ are pion twist wave functions defined as follows

$$<\pi^- (p_-) | \bar{u}(x) i\gamma_5 d(0)| 0 > = \frac{f_\pi m_\pi^2}{m_u + m_d} \int_0^1 d\alpha e^{i\alpha x - p_-} \phi_P(\alpha)$$

$$<\pi^- (p_-) | \bar{u}(x) \gamma_{\mu\nu} \gamma_5 d(0)| 0 > = -i f_\pi m_\pi^2 \int_0^1 d\alpha e^{i\alpha x - p_-} \phi_T(\alpha)$$

The twist wave functions $\phi_P$, $\phi_T$ and $\phi_A$ satisfy the normalization conditions

$$\int_0^1 d\alpha \phi_A(\alpha) = \int_0^1 d\alpha \phi_P(\alpha) = \int_0^1 d\alpha \phi_T(\alpha) = 1$$

These wave functions can be obtained from QCD sum rules. The results depend on the renormalization group scale. For $\mu^2 = m_B^2 - m_0^2 = 5$ GeV$^2$, 3
they are given by \[ \phi_A(\alpha, \mu) = 19.32\alpha(1 - \alpha)[(2\alpha - 1)^2 + 0.11], \phi_P(\alpha, \mu) = \frac{3}{4}[1.078 + 0.766(2\alpha - 1)^2], \phi_T(\alpha, \mu) = \alpha(1 - \alpha). \]

To find the matrix element \(< \pi^+(p_+)|J_{\nu\mu}^G|B(p) >\), let us consider the correlator

\[
T_{\nu\mu}^G(p_+, p, x) = i \int dy e^{-ipy} < \pi^+(p_+)|J_{\nu\mu}^G \bar{b}(y)i\gamma_5d(y)|0 >
\]

\[
= \frac{m^2_fB}{m_B} \frac{1}{m_B^2 - p^2} < \pi^+(p_+)|J_{\nu\mu}^G|B(p) > + \cdots
\]

where \(< B(p)|\bar{b}(y)i\gamma_5d(y)|0 > = (m^2_fB/m_B)e^{-ipy}.\) The ellipses denote the excited and continuum states which are usually taken into account by the dispersion integral with the effective threshold. The single soft gluon contribution is given by

\[
T_{\nu\mu}^G(p_+, p, x) = i \int dy \int dk \frac{e^{-i(p-k)y}}{(4\pi)^4(m^2_b - k^2)} \int_0^1 d\alpha_0 \alpha_0 x^\rho
\]

\[
\cdot < \pi^+(p_+)|\bar{u}(0)G_{\rho\nu}(\alpha_0\xi)\gamma_\mu\gamma_5(k^\rho\gamma_\sigma + m_b)\gamma_5d(y)|0 >
\]

which can be evaluated by introducing the pion wave functions with higher twist

\[
T_{\nu\mu}^G(p_+, p, x) = i \int dy \int dk \frac{e^{-i(p-k)y}}{(4\pi)^4(m^2_b - k^2)} \int_0^1 d\alpha_0 \alpha_0 \{ \]

\[
f_\pi m_\pi \int D\alpha_i \phi_{3\pi}(\alpha_i) e^{i\pi\alpha_0x}\]

\[
\cdot \{ (k_\nu\cdot p_+ - p_\nu\cdot k)p_{+\mu} - (g_{\nu\mu}\cdot p_+ - p_\nu\cdot x_\mu)k\cdot p_+ \}
\]

\[
+ f_\pi m_\pi \int D\alpha_i e^{i\pi\alpha_0x} \{ \phi_\perp(\alpha_i)(g_{\nu\mu}\cdot p_+ - p_\nu\cdot x_\mu)
\]

\[
+ (\phi_\parallel(\alpha_i) + \phi_\perp(\alpha_i))(p_{+\nu} y\cdot x - x\cdot p_+/ y\cdot p_+) \}
\]

\[
\}
\]

where \(\phi_{3\pi}\) and \(\phi_\parallel, \phi_\perp\) are the pion wave functions of twist 3 and 4, respectively. They are defined as \[ [4, 5]\]

\[
< \pi^+(q)|\bar{u}(0)G_{\rho\mu}\gamma_5d(y)|0 > = -i \int D\alpha_i \phi_{3\pi}(\alpha_i) e^{i\pi\alpha_0x}
\]

\[
\cdot f_\pi \frac{1}{2} \{ q_\mu(q_\rho g_{\nu\sigma} - q_\sigma g_{\rho\nu}) - q_\sigma(q_\rho g_{\nu\mu} - q_\nu g_{\rho\mu})\}
\]

\[
< \pi^+(q)|\bar{u}(0)G_{\rho\nu}\gamma_5d(y)|0 > = \int D\alpha_i \phi_\perp(\alpha_i) e^{i\pi\alpha_0x}
\]
\[
\begin{align*}
&\cdot f_\pi \delta^2_{4\pi} \cdot \left[ q_\nu (g_{\rho\mu} - \frac{y_\rho q_\mu}{y \cdot q}) - q_\mu (g_{\nu\mu} - \frac{y_\nu q_\mu}{y \cdot q}) \right] \\
&+ f_\pi \delta^2_{4\pi} q_\mu \frac{q_\nu y_\mu - q_\nu y_\rho}{y \cdot q} \int D\alpha_i \phi_\parallel (\alpha_i) e^{i\alpha (2\alpha_2 + \alpha_3 x)}
\end{align*}
\]

where \( D\alpha_1 = d\alpha_1 d\alpha_2 d\alpha_3 \delta(\alpha_1 + \alpha_2 + \alpha_3 - 1) \). The constants \( \delta_{3\pi} \) and \( \delta_{4\pi} \) are renormalization scale dependent and have the values \( [5] \) \( \delta_{3\pi} \approx 0.027 \text{GeV} \) and \( \delta_{4\pi} \approx 0.45 \text{GeV} \) at \( \mu \approx 1 \text{GeV} \). The results for \( \phi_{3\pi}, \phi_\perp \) and \( \phi_\parallel \) are given by:

\( \phi_{3\pi}(\alpha_i) = 360\alpha_i \alpha_2 \alpha_3^2 [1 + \omega_{1,0} \frac{1}{2} (7\alpha_3 - 3) + \omega_{2,0} (2 - 4\alpha_i \alpha_2 - 8\alpha_3 + 8\alpha_3^3) + \omega_{1,1} (3\alpha_i \alpha_2 - 2\alpha_3 + 3\alpha_3^2) + \cdots] \), \( \phi_\perp (\alpha_i) = 30(\alpha_1 - \alpha_2)\alpha_3^2 \left[ \frac{1}{3} + 2\varepsilon (1 - 2\alpha_3) \right] \), \( \phi_\parallel (\alpha_i) = 120\varepsilon (\alpha_1 - \alpha_2)\alpha_1 \alpha_2 \alpha_3 \). where \( \omega_{1,0} = -2.88 \), \( \omega_{2,0} = 10.5 \), \( \omega_{1,1} = 0 \) and \( \varepsilon \approx 0.5 \) at \( \mu \approx 1 \text{GeV} \).

The integral of eq. (18) for the variables \( y \) and \( k \) can be performed and is given by:

\[
T^G_{\nu \mu}(p_+, p, x) = i \int D\alpha_1 \frac{f_\pi}{m_b^2 - (p - \alpha_2 p_+)^2} \int_0^1 d\alpha_0 \alpha_0 e^{i\alpha_0 \alpha_3 p_+ \cdot x} \\
\{ \delta_{3\pi} \phi_{3\pi}(\alpha_i) (p_\nu x \cdot p_+ - p_+ x \cdot p) p_{+\mu} - (g_{\nu\mu} x \cdot p_+ - p_+ x \mu) \\
\cdot (p_+ - \alpha_2 p_+^2) \phi_{3\pi}(\alpha_i) - \delta^2_{4\pi} m_b \phi_\perp (\alpha_i) \} \\
+ \delta^2_{4\pi} m_b \left( \phi_\perp (\alpha_i) + \phi_\parallel (\alpha_i) \right) \int_0^\infty d\beta \frac{m_b^2 - (p - \alpha_2 p_+)^2}{m_b^2 - (p - (\alpha_2 + \beta) p_+)^2} \\
\cdot \left[ \frac{m_\pi^2}{m_d m_b} (g_{\nu\mu} x \cdot p_+ - p_+ x \mu) + \frac{2}{m_b^2 - (p - (\alpha_2 + \beta) p_+)^2} \\
\cdot (p_\nu x \cdot p_+ - p_+ x \cdot p) \left( p_{+\mu} + \frac{m_\pi^2}{m_d m_b} \alpha_2 ((\alpha_2 + \beta) p_{+\mu} + p_\mu) \right) \right] \}
\]

Here we have used the relation \( 1/p_+ \cdot y = -i \int_0^\infty d\beta e^{i\beta p_+ \cdot y} \). With the above results, we further consider:

\[
\frac{1}{N_c} \int dx \int \frac{dq}{(4\pi)^4 q^2 - m_d^2} T^{\nu\mu}(p_-, q, x) T^G_{\nu \mu}(p_+, p, x) = \frac{m_B^2 f_B}{m_b} \frac{1}{m_B^2 - p^2} A_\omega + \int_0^\infty \frac{\rho(m_\pi^2, s) ds}{s - p^2}
\]

where the dispersion integral with the effective threshold \( s_0 \) assembles the excited and continuum states. By substituting eqs.(10) and (24) to the left hand side of the above equation, it is not difficult, though tedious, to integrate out the variables \( x \) and \( q \). The result is found to be
\[ \frac{1}{N_c} \int dx \int \frac{dq}{(4\pi)^4} \frac{e^{-iqx}}{q^2 - m_d^2} T^{\nu\mu}(p_-, q, x) T^{\mu \nu}_0(p_+, p, x) \]

\[ = i \frac{1}{N_c} f_\pi^2 \int_0^1 d\alpha \int_0^1 d\alpha_0 \alpha_0 \int D\alpha_\hat{M} \frac{\phi_A(\alpha)}{m_d^2 - (p - \alpha_2p_+)^2} \{ \]

\[ 2[(\alpha - \alpha_0\alpha_3)^2m_\pi^2 + \alpha\alpha_0\alpha_3p^2 - m_d^2] \]

\[ \cdot (\delta_3\phi_{3\pi}(\alpha_i)p^2 - \delta_{4\pi}mb_\perp(\alpha_i) + \delta_{4\pi}m_b(\phi_\perp(\alpha_i) + \phi_\parallel(\alpha_i)) \]

\[ \cdot \int_0^\infty d\beta \frac{m_\pi^2 - (p - \alpha_2p_+)^2}{m_d^2 - (p - (\alpha_2 + \beta)p_+)^2} \]

\[ \cdot (1 + \frac{m_\pi^2}{m_dm_b}\alpha_2(1 + \alpha_2 + \beta)) - \frac{m_\pi^2}{m_dm_b}) \]

\[ + \left( (\alpha - \alpha_0\alpha_3)^2m_\pi^2 + \alpha\alpha_0\alpha_3p^2 \right) \left( \delta_3\phi_{3\pi}(\alpha_i)p^2 - 2\delta_{4\pi}m_b(\phi_\perp(\alpha_i) + \phi_\parallel(\alpha_i)) \right) \]

\[ \cdot \int_0^\infty d\beta \frac{m_\pi^2 - (p - \alpha_2p_+)^2}{m_d^2 - (p - (\alpha_2 + \beta)p_+)^2} \]

\[ \cdot \left( (1 + \frac{m_\pi^2}{m_dm_b}\alpha_2(1 + \alpha_2 + \beta)) - \frac{m_\pi^2}{m_dm_b}) \right) \]

where we have neglected the terms of order \( m_\pi^2/m_B^2 \). The remaining integral may be carried out numerically (in fact, the integral for the variable \( \beta \) can be explicitly integrated out). Before doing so, one may use the QCD sum rule approach [3] by applying the Borel transformation in the variable \( p^2 \), i.e.,

\[ \hat{L}_{M^2} f(Q^2) = \lim_{Q^2, n \to \infty, (Q^2/n) = M^2} \frac{Q^{(n+1)}}{n!} \left( -\frac{d}{dQ^2} \right)^n f(Q^2) \equiv f(M^2) \quad (23) \]

to the left and right hand side of eq.(25) with use of the more explicit expression eq.(26) for the left hand side. \( M \) is the so-called Borel parameter. It can be seen that the Borel operator transforms the series in \( Q^{-2} \) into a series in \( M^{-2} \):
\[ \hat{L}_M \left( \frac{1}{Q^2} \right)^n = \frac{1}{(n-1)!} \left( \frac{1}{M^2} \right)^n \]  

(24)

From duality considerations, the dispersion integral on the right hand side of eq.(25) is cancelled against the corresponding part of the dispersion integral of the QCD result on the left hand side. One then reads off the needed amplitude \( A_{O_2} \). The procedure is quite standard following the usual QCD sum rule calculations\[6\]. The utility of the result is that it depends on twist wave functions that can be calculated from two point Greens’ functions. We have used \( B^0 \to \pi^+\pi^- \) as an example but the method applies equally well to other heavy to light channels. A detailed application will be discussed elsewhere.

In conclusion, we have shown that the nonfactorizable contributions to the hadronic matrix elements of the nonleptonic two body decays can be calculated directly from the QCD theory. It has been seen that the leading nonzero contributions from the single soft gluon emission can actually be factorized again in terms of the higher twist wave functions of the mesons. It is expected that the approach presented here will be helpful in the understanding hadronic decays of the heavy mesons, especially for the heavy to light meson decays. It shall provide a complement to the approaches presented in ref.\[7\] for the calculations of the \( B \to D\pi \) and \( B \to J/\psi K \) decays in which one of the final hadrons is heavy.

References

[1] M. Bauer, B. Stech and M. Wirbel, Z. Phys. C34 (1987) 103.

[2] A.S. Buras, J.M. Gerald and R. Ruckl, Nucl. Phys. B268 (1986) 16.

[3] B. Blok and M.A. Shifman, Nucl. Phys. B389 (1993) 534; B399 (1993) 441,459.

[4] V.L. Chernyak and A.R. Zhitnitsky, Nucl. Phys. B345 (1990) 137; Phys. Rep. 112 (1984) 173.

[5] V.M. Braun and I.E. Filyanov, Z. Phys. C48 (1990) 239.
[6] M. Shifman, A. Vainshtein and V. Zakharov, Nucl. Phys. B147 (1979) 385, 448.

[7] The nonfactorizable contribution to the decay $\bar{B}^0 \rightarrow D^0 \pi^0$ at the tree level was estimated by I. Halperin, (TECHNION-PHYS-94-16, (1994) within the light cone QCD sum rule method. The estimate for the nonfactorizable amplitude of the decay $B \rightarrow J/\psi K$ was given by A. Khodjamirian and R. Ruckl, MPI-PhT/95-19, LMU 05/95, 1995.