Relativistic entanglement of Bell states with general momentum

Young Hoon Moon\textsuperscript{1,2}, Doyeol Ahn\textsuperscript{1,2,*}, and Sung Woo Hwang\textsuperscript{1,3†}

\textsuperscript{1}Institute of Quantum Information Processing and Systems, University of Seoul, Seoul, 130-743, Korea

\textsuperscript{2}Department of Electrical and Computer Engineering, University of Seoul, Seoul, 130-743, Korea

\textsuperscript{3}Department of Electronic Engineering, Korea University, Seoul, 136-701, Korea

Abstract

In this paper, the Lorentz transformation of the entangled Bell states with momentum, not necessarily orthogonal to the boost direction, and spin, is studied. We extended quantum correlations and Bell’s inequality to the relativistic regime by considering normalized relativistic observables. It is shown that quantum information, along the perpendicular direction to the boost, is eventually lost and Bell’s inequality is not always violated for entangled states in special relativity. This could impose restrictions to certain quantum information processing such as quantum cryptography using massive particles.

*\textsuperscript{e-mail:}dahn@uoscc.uos.ac.kr

†\textsuperscript{e-mail:}swhwang@korea.ac.kr
Relativistic quantum information processing is of growing interest, not only for the logical completeness but also the new features such as the physical bounds on the information transfers, processing and the errors provided by the full relativistic treatment [1]–[11]. It would be also interesting to study quantum correlations and Bell’s inequality in different Lorentz frames. Violation of Bell’s inequality is perhaps the most drastic feature distinguishing the quantum theory from the classical physics [12]. Bell’s proof that there are states of two-quantum-particle systems that do not satisfy the Bell’s inequality derived from Einstein’s assumptions [13] of the principle of local causes has changed our traditional viewpoint of Nature quite significantly. Specifically, it was shown that all the non-product states or otherwise known as the entangled states always violate the Bell inequality when special relativity is not taken into account [14]. So it would be an interesting question to ask if above mentioned condition changes if one considers special relativity.

Under the Lorentz transformation, the Hilbert space vectors representing the quantum states undergo the unitary transformations [15]. On the other hand, the Pauli matrices are not Lorentz covariant, so there are needs to find relativistically invariant operators corresponding to the spin in order to investigate the Bell’s inequality within the special relativity [16]. Sometime ago, Fleming [17] showed that covariant spin-vector operator which reduces to the ordinary spin operator in the non-relativistic limit, can be derived from the Pauli-Lubanski pseudo vector and Czachor [2] showed that the degree of violation of the Bell’s inequality depends on the velocity of the pair of spin−1/2 particles with respect to the laboratory. Unitary transformation corresponding the Lorentz boost of the quantum states was not considered, in those works.

In the previous work [6], we calculated the Bell observables for entangled states in the rest frame with both momentum vector and spin in the z-direction, seen by the observer moving in the x-direction, and showed that the entangled states do not always violate the Bell’s inequality when the boost speed approaches the speed of light. This paper is
a direct continuation of a preceding one [6](I). In this paper, we study the case of the general momentum not necessarily in perpendicular to the boost direction as described in Figures 1 and 2 and derived transformation rules for the entangled states. We also calculated the average of the Bell observable for the momentum-conserved entangled Bell states for spin−\(\frac{1}{2}\) particles and show that Bell’s inequality is not always violated for the case of general momentum in special relativity. It is also shown that quantum information, along the perpendicular direction to the boost, is eventually lost. This could impose restrictions to certain quantum information processing such as quantum cryptography using massive particles. Unless both sender and receiver measures along the boost direction, there will be information loss.
FIGURES

**FIG. 1.** The case of momentum vector in the $x$-$z$ plane, $\vec{p} = p(\sin \theta, 0, \cos \theta)$ and the boost $\Lambda$ in the $x$-direction.

**FIG. 2.** The case of momentum vector out of plane, $\vec{p} = p(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ and the boost in the $x$-direction.

**II. RELATIVISTIC ENTANGLEMENTS**

A multi-particle state vector is denoted by

$$\Psi_{p_1 \sigma_1; p_2 \sigma_2; \ldots} = a^+(\vec{p}_1, \sigma_1) a^+(\vec{p}_2, \sigma_2) \ldots \Psi_0,$$

(1)
where \( p_i \) labels the four-momentum, \( \sigma_i \) is the spin \( z \) component, \( a^+(\vec{p}_i, \sigma_i) \) is the creation operator which adds a particle with momentum \( \vec{p}_i \) and spin \( \sigma_i \), and \( \Psi_0 \) is the Lorentz invariant vacuum state. The Lorentz transformation \( \Lambda \) induces unitary transformation on vectors in the Hilbert space

\[
\Psi \rightarrow U(\Lambda)\Psi
\]

and the operators \( U \) satisfies the composition rule

\[
U(\bar{\Lambda})U(\Lambda) = U(\bar{\Lambda}\Lambda),
\]

while the creation operator has the following transformation rule

\[
U(\Lambda)a^+(\vec{p}, \sigma)U(\Lambda)^{-1} = \sqrt{(\Lambda p)^0} \sum_\sigma D^{(j)}(W(\Lambda, p))a^+(\vec{p}_\Lambda, \sigma).
\]

Here, \( W(\Lambda, p) \) is the Wigner’s little group element given by

\[
W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p),
\]

with \( D^{(j)}(W) \) the representation of \( W \) for spin \( j \), \( p^\mu = (\vec{p}, p^0) \), \( (\Lambda p)^\mu = (\vec{p}_\Lambda, (\Lambda p)^0) \) with \( \mu = 1, 2, 3, 0 \) and \( L(p) \) is the Lorentz transformation such that

\[
p^\mu = L^\mu_\nu k^\nu
\]

where \( k^\nu = (0, 0, 0, m) \) is the four-momentum taken in the particle’s rest frame. One can also use the conventional ket-notation to represent the quantum states as

\[
\Psi_{p,\sigma} = a^+(\vec{p}, \sigma)\Psi_0 \\
= |\vec{p}, \sigma\rangle \\
= |\vec{p}\rangle \otimes |\sigma\rangle.
\]

The Wigner representation of the Lorentz group for the spin-\( \frac{1}{2} \) becomes:

\[
D^{(1/2)}(W(\Lambda, p)) \\
= \frac{1}{[(p^0 + m)((\Lambda p)^0 + m)]^{1/2}} \left\{ (p^0 + m) \cosh \frac{\alpha}{2} + (\vec{p} \cdot \hat{e}) \sinh \frac{\alpha}{2} - i \sinh \frac{\alpha}{2} \vec{\sigma} \cdot (\vec{p} \times \hat{e}) \right\} \\
= \cos \frac{\Omega_p}{2} + i \sin \frac{\Omega_p}{2} (\vec{\sigma} \cdot \hat{n}),
\]
where cosh $\delta$ transformed Bell states are given by

$$0 \Psi_0 = \frac{\cosh \frac{\alpha}{2} \cosh \delta + \sinh \frac{\alpha}{2} \sinh \frac{\delta}{2} (\hat{\epsilon} \cdot \hat{p})}{\left[ \frac{1}{2} + \frac{1}{2} \cosh \alpha \cosh \delta + \frac{1}{2} \sinh \alpha \sinh \delta (\hat{\epsilon} \cdot \hat{p}) \right]^{1/2}},$$

(9)

and

$$\sin \frac{\Omega}{2} \hat{n} = \frac{\sinh \frac{\alpha}{2} \sinh \frac{\delta}{2} (\hat{\epsilon} \times \hat{p})}{\left[ \frac{1}{2} + \frac{1}{2} \cosh \alpha \cosh \delta + \frac{1}{2} \sinh \alpha \sinh \delta (\hat{\epsilon} \cdot \hat{p}) \right]^{1/2}},$$

(10)

where $\cosh \delta = \frac{\alpha}{m}$. We note that the eq. (8) indicates the Lorentz group can be represented by the pure rotation by axis $\hat{n} = \hat{\epsilon} \times \hat{p}$ for the two-component spinor.

We define the momentum-conserved entangled Bell stats for spin-$\frac{1}{2}$ particles in the rest frame as follows:

$$\Psi_{00} = \frac{1}{\sqrt{2}} \left\{ a^+ (\vec{p}, \frac{1}{2}) a^+ (-\vec{p}, \frac{1}{2}) + a^+ (\vec{p}, -\frac{1}{2}) a^+ (-\vec{p}, -\frac{1}{2}) \right\} \Psi_0,$$

(11a)

$$\Psi_{01} = \frac{1}{\sqrt{2}} \left\{ a^+ (\vec{p}, \frac{1}{2}) a^+ (-\vec{p}, -\frac{1}{2}) - a^+ (\vec{p}, -\frac{1}{2}) a^+ (-\vec{p}, \frac{1}{2}) \right\} \Psi_0,$$

(11b)

$$\Psi_{10} = \frac{1}{\sqrt{2}} \left\{ a^+ (\vec{p}, \frac{1}{2}) a^+ (-\vec{p}, \frac{1}{2}) + a^+ (\vec{p}, -\frac{1}{2}) a^+ (-\vec{p}, \frac{1}{2}) \right\} \Psi_0,$$

(11c)

$$\Psi_{11} = \frac{1}{\sqrt{2}} \left\{ a^+ (\vec{p}, \frac{1}{2}) a^+ (-\vec{p}, -\frac{1}{2}) - a^+ (\vec{p}, -\frac{1}{2}) a^+ (-\vec{p}, -\frac{1}{2}) \right\} \Psi_0,$$

(11d)

where $\Psi_0$ is the Lorentz invariant vacuum state.

For an observer in another reference frame $S'$ described by an arbitrary boost $\Lambda$, the transformed Bell states are given by

$$\Psi_{ij} \rightarrow U(\Lambda) \Psi_{ij}.$$  

(12)

For example, from equations (4) and (11a), $U(\Lambda) \Psi_{00}$ becomes

$$U(\Lambda) \Psi_{00} = \frac{1}{\sqrt{2}} \left\{ U(\Lambda) a^+ (\vec{p}, \frac{1}{2}) U^{-1}(\Lambda) U(\Lambda) a^+ (-\vec{p}, \frac{1}{2}) U^{-1}(\Lambda) \right\} \Psi_0$$

$$+ U(\Lambda) a^+ (\vec{p}, -\frac{1}{2}) U^{-1}(\Lambda) U(\Lambda) a^+ (-\vec{p}, -\frac{1}{2}) U^{-1}(\Lambda) \Psi_0$$

$$= \frac{1}{\sqrt{2}} \sum_{\sigma, \sigma'} \left\{ \sqrt{\frac{(\Lambda p)^0}{p^0}} D^{(\frac{1}{2})}_{\sigma'} (W(\Lambda, p)) \sqrt{\frac{(\Lambda P p)^0}{(P p)^0}} D^{(\frac{1}{2})}_{\sigma} (W(\Lambda, P p)) a^+ (\vec{p}_\Lambda, \sigma) a^+ (-\vec{p}_\Lambda, \sigma') \right\} \Psi_0$$

$$+ \sqrt{\frac{(\Lambda p)^0}{p^0}} D^{(\frac{1}{2})}_{\sigma'} (W(\Lambda, p)) \sqrt{\frac{(\Lambda P p)^0}{(P p)^0}} D^{(\frac{1}{2})}_{\sigma} (W(\Lambda, P p)) a^+ (\vec{p}_\Lambda, \sigma) a^+ (-\vec{p}_\Lambda, \sigma') \} \Psi_0$$

(13)
and so on.

A: The momentum and the boost vectors in the same plane.

We assume that \( \vec{p} \) is in the \( x-z \) plane, \( \vec{p} = (p \sin \theta, 0, p \cos \theta) \) and the boost \( \Lambda \) is in \( x \)-direction. In this case, we have

\[
\cos \frac{\Omega \pm \vec{p}}{2} = \frac{\cosh \frac{\alpha}{2} \cosh \frac{\delta}{2} \pm \sinh \frac{\alpha}{2} \sinh \frac{\delta}{2} \sin \theta}{\frac{1}{2} + \frac{1}{2} \cosh \alpha \cosh \delta \pm \frac{1}{2} \sinh \alpha \sinh \delta \sin \theta}^{1/2};
\]

\[
\sin \frac{\Omega \pm \vec{p}}{2} \hat{n}_\pm = \frac{\mp \hat{y} \sin \frac{\alpha}{2} \sin \frac{\delta}{2} \cos \theta}{\frac{1}{2} + \frac{1}{2} \cosh \alpha \cosh \delta \pm \frac{1}{2} \sinh \alpha \sinh \delta \sin \theta}^{1/2};
\]

and

\[
\mathcal{D}^{1/2}(W(\Lambda, p)) = \cos \frac{\Omega \pm \vec{p}}{2} - i \sigma_y \sin \frac{\Omega \pm \vec{p}}{2}
\]

\[
= \begin{pmatrix}
\cos \frac{\Omega \pm \vec{p}}{2} - \sin \frac{\Omega \pm \vec{p}}{2} \\
\sin \frac{\Omega \pm \vec{p}}{2} \cos \frac{\Omega \pm \vec{p}}{2}
\end{pmatrix},
\]

\[
\mathcal{D}^{1/2}(W(\Lambda, \mathcal{P}p)) = \cos \frac{\Omega \mp \vec{p}}{2} + i \sigma_y \sin \frac{\Omega \mp \vec{p}}{2}
\]

\[
= \begin{pmatrix}
\cos \frac{\Omega \mp \vec{p}}{2} \sin \frac{\Omega \mp \vec{p}}{2} \\
- \sin \frac{\Omega \mp \vec{p}}{2} \cos \frac{\Omega \mp \vec{p}}{2}
\end{pmatrix},
\]

where \( \hat{n}_\pm = \mp \hat{y} \).

Then from equations (16),(17) and (13), we obtain

\[
U(\Lambda)\Psi_{00} = \frac{(\Lambda p)^0}{p^0} \cos \frac{\Omega \pm \vec{p}}{2} + \frac{1}{\sqrt{2}} \{ a^+(\vec{p}_\Lambda, \frac{1}{2}) a^+(-\vec{p}_\Lambda, \frac{1}{2}) + a^+(\vec{p}_\Lambda, -\frac{1}{2}) a^+(-\vec{p}_\Lambda, -\frac{1}{2}) \} \Psi_0
\]

\[
- \frac{(\Lambda p)^0}{p^0} \sin \frac{\Omega \pm \vec{p}}{2} + \frac{1}{\sqrt{2}} \{ a^+(\vec{p}_\Lambda, \frac{1}{2}) a^+(-\vec{p}_\Lambda, -\frac{1}{2}) - a^+(\vec{p}_\Lambda, -\frac{1}{2}) a^+(-\vec{p}_\Lambda, \frac{1}{2}) \} \Psi_0
\]

\[
= \frac{(\Lambda p)^0}{p^0} |\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \{ \cos \frac{\Omega \pm \vec{p}}{2} + \frac{1}{\sqrt{2}} (|\frac{1}{2}, \frac{1}{2}\rangle + |\frac{1}{2}, -\frac{1}{2}\rangle) \}
\]

\[
- \frac{(\Lambda p)^0}{p^0} |\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \{ \sin \frac{\Omega \pm \vec{p}}{2} + \frac{1}{\sqrt{2}} (|\frac{1}{2}, -\frac{1}{2}\rangle - |\frac{1}{2}, \frac{1}{2}\rangle) \}
\]

\[
= \frac{(\Lambda p)^0}{p^0} \{ \cos \frac{\Omega \pm \vec{p}}{2} \Psi_{00}' - \sin \frac{\Omega \pm \vec{p}}{2} \Psi_{11}' \},
\]

where \( \Psi_{ij}' \) is the Bell states in the moving frame \( S' \) whose momenta are transformed as \( \vec{p} \rightarrow \vec{p}_\Lambda, -\vec{p} \rightarrow -\vec{p}_\Lambda \).
Likewise, we have

\[
U(\Lambda)\Psi_{01} = \frac{(\Lambda p)^0}{p^0} \cos \frac{\Omega p - \Omega p'}{2} \frac{1}{\sqrt{2}} \{a^+(\vec{p}_\Lambda, \frac{1}{2})a^+(-\vec{p}_\Lambda, \frac{1}{2}) - a^+(\vec{p}_\Lambda, -\frac{1}{2})a^+(-\vec{p}_\Lambda, -\frac{1}{2})\} \Psi_0 \\
+ \frac{(\Lambda p)^0}{p^0} \sin \frac{\Omega p - \Omega p'}{2} \frac{1}{\sqrt{2}} \{a^+(\vec{p}_\Lambda, \frac{1}{2})a^+(-\vec{p}_\Lambda, -\frac{1}{2}) + a^+(\vec{p}_\Lambda, -\frac{1}{2})a^+(-\vec{p}_\Lambda, \frac{1}{2})\} \Psi_0 \\
= \frac{(\Lambda p)^0}{p^0} |\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \{\cos \frac{\Omega p - \Omega p'}{2} \frac{1}{\sqrt{2}}((\frac{1}{2}, \frac{1}{2}) - | - \frac{1}{2}, -\frac{1}{2})\} \\
+ \frac{(\Lambda p)^0}{p^0} |\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \{\sin \frac{\Omega p - \Omega p'}{2} \frac{1}{\sqrt{2}}((\frac{1}{2}, -\frac{1}{2}) + | - \frac{1}{2}, \frac{1}{2})\} \\
= \frac{(\Lambda p)^0}{p^0} \{\cos \frac{\Omega p - \Omega p'}{2} \Psi_{01} + \sin \frac{\Omega p - \Omega p'}{2} \Psi_{10}\},
\]

and

\[
U(\Lambda)\Psi_{10} = \frac{(\Lambda p)^0}{p^0} \cos \frac{\Omega p - \Omega p'}{2} \frac{1}{\sqrt{2}} \{a^+(\vec{p}_\Lambda, \frac{1}{2})a^+(-\vec{p}_\Lambda, -\frac{1}{2}) + a^+(\vec{p}_\Lambda, -\frac{1}{2})a^+(-\vec{p}_\Lambda, \frac{1}{2})\} \Psi_0 \\
- \frac{(\Lambda p)^0}{p^0} \sin \frac{\Omega p - \Omega p'}{2} \frac{1}{\sqrt{2}} \{a^+(\vec{p}_\Lambda, \frac{1}{2})a^+(-\vec{p}_\Lambda, \frac{1}{2}) - a^+(\vec{p}_\Lambda, -\frac{1}{2})a^+(-\vec{p}_\Lambda, -\frac{1}{2})\} \Psi_0 \\
= \frac{(\Lambda p)^0}{p^0} |\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \{\cos \frac{\Omega p - \Omega p'}{2} \frac{1}{\sqrt{2}}((\frac{1}{2}, \frac{1}{2}) + | - \frac{1}{2}, -\frac{1}{2})\} \\
- \frac{(\Lambda p)^0}{p^0} |\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \{\sin \frac{\Omega p - \Omega p'}{2} \frac{1}{\sqrt{2}}((\frac{1}{2}, -\frac{1}{2}) - | - \frac{1}{2}, \frac{1}{2})\} \\
= \frac{(\Lambda p)^0}{p^0} \{\cos \frac{\Omega p - \Omega p'}{2} \Psi_{10} - \sin \frac{\Omega p - \Omega p'}{2} \Psi_{01}\},
\]

where

\[
\cos \frac{\Omega p + \Omega p'}{2} = \cos \frac{\Omega p'}{2} \cos \frac{\Omega p}{2} - \sin \frac{\Omega p'}{2} \sin \frac{\Omega p}{2} \\
= \frac{(\cosh \frac{\alpha}{2} \cosh \frac{\delta}{2})^2 - (\sinh \frac{\alpha}{2} \sinh \frac{\delta}{2})^2}{[(\frac{1}{2} + \frac{1}{2} \cosh \alpha \cosh \delta)^2 - (\frac{1}{2} \sinh \alpha \sinh \delta)^2]^\frac{1}{2}},
\]
\[ \cos \frac{\Omega_{\vec{p}} - \Omega_{\vec{p}'}}{2} = \cos \frac{\Omega_{\vec{p}}}{2} \cos \frac{\Omega_{\vec{p}'} + \Omega_{\vec{p}'}}{2} + \sin \frac{\Omega_{\vec{p}}}{2} \sin \frac{\Omega_{\vec{p}'} + \Omega_{\vec{p}'}}{2} \]

\[ = \frac{(\cosh \frac{\alpha}{2} \cosh \delta \theta')^2 + (\sinh \frac{\alpha}{2} \sinh \delta \theta')^2 \cos 2\theta}{[(\frac{1}{2} + \frac{1}{2} \cosh \alpha \cosh \delta)^2 - (\frac{1}{2} \sinh \alpha \sinh \delta \sin \theta)^2]^{1/2}}, \quad (19b) \]

\[ \sin \frac{\Omega_{\vec{p}} + \Omega_{\vec{p}'}}{2} = \sin \frac{\Omega_{\vec{p}}}{2} \cos \frac{\Omega_{\vec{p}'} + \Omega_{\vec{p}'}}{2} + \cos \frac{\Omega_{\vec{p}}}{2} \sin \frac{\Omega_{\vec{p}'} + \Omega_{\vec{p}'}}{2} \]

\[ = \frac{2 \cosh \frac{\alpha}{2} \delta \theta' \sinh \frac{\alpha}{2} \sinh \delta \theta \cos \theta}{[(\frac{1}{2} + \frac{1}{2} \cosh \alpha \cosh \delta)^2 - (\frac{1}{2} \sinh \alpha \sinh \delta \sin \theta)^2]^{1/2}}, \quad (19c) \]

\[ \sin \frac{\Omega_{\vec{p}} - \Omega_{\vec{p}'}}{2} = \sin \frac{\Omega_{\vec{p}}}{2} \cos \frac{\Omega_{\vec{p}'} - \Omega_{\vec{p}'}'}{2} - \cos \frac{\Omega_{\vec{p}}}{2} \sin \frac{\Omega_{\vec{p}'} - \Omega_{\vec{p}'}'}{2} \]

\[ = \frac{-(\sinh \frac{\alpha}{2} \sinh \delta \theta')^2 \sin 2\theta}{[(\frac{1}{2} + \frac{1}{2} \cosh \alpha \cosh \delta)^2 - (\frac{1}{2} \sinh \alpha \sinh \delta \sin \theta)^2]^{1/2}}. \quad (19d) \]

**B: The case of the momentum and the boost vectors not in the same plane.**

We consider the general case of momentum vector out of plane, \( \vec{p} = (p \sin \theta \cos \phi, p \sin \theta \sin \phi, p \cos \theta) \) and the boost in the \( x \)-direction.

In this case, we have

\[ \cos \frac{\Omega_{\vec{p}}}{2} = \cosh \frac{\alpha}{2} \cosh \frac{\delta \theta}{2} \pm \sinh \frac{\alpha}{2} \sinh \frac{\delta \theta}{2} \sin \theta \cos \phi \]

\[ \sin \frac{\Omega_{\vec{p}}}{2} \hat{n}_\pm = \frac{r \hat{n}_+ \sinh \frac{\alpha}{2} \sinh \delta \theta}{\frac{1}{2} + \frac{1}{2} \cosh \alpha \cosh \delta \pm \frac{1}{2} \sinh \alpha \sinh \delta \sin \theta \cos \phi} \]

and

\[ D^{1/2}(W(\Lambda, p)) = \cos \frac{\Omega_{\vec{p}}}{2} + i \sin \frac{\Omega_{\vec{p}}}{2} \hat{\sigma} \cdot (-\hat{\gamma} \cos \eta + \hat{\zeta} \sin \eta) \]

\[ = \left( \begin{array}{cc} \cos \frac{\Omega_{\vec{p}}}{2} + i \sin \frac{\Omega_{\vec{p}}}{2} \sin \eta & -\sin \frac{\Omega_{\vec{p}}}{2} \cos \eta \\ \sin \frac{\Omega_{\vec{p}}}{2} \cos \eta & \cos \frac{\Omega_{\vec{p}}}{2} - i \sin \frac{\Omega_{\vec{p}}}{2} \sin \eta \end{array} \right), \quad (22) \]

\[ D^{1/2}(W(\Lambda, \mathcal{P} p)) = \cos \frac{\Omega_{\vec{p}'}}{2} - i \sin \frac{\Omega_{\vec{p}'}}{2} \hat{\sigma} \cdot (-\hat{\gamma} \cos \eta + \hat{\zeta} \sin \eta) \]

\[ = \left( \begin{array}{cc} \cos \frac{\Omega_{\vec{p}'}}{2} - i \sin \frac{\Omega_{\vec{p}'}}{2} \sin \eta & \sin \frac{\Omega_{\vec{p}'}}{2} \cos \eta \\ -\sin \frac{\Omega_{\vec{p}'}}{2} \cos \eta & \cos \frac{\Omega_{\vec{p}'}}{2} + i \sin \frac{\Omega_{\vec{p}'}}{2} \sin \eta \end{array} \right) \quad (23) \]

where \( \hat{n}_\pm = \pm (-\hat{\gamma} \cos \eta + \hat{\zeta} \sin \eta), \cos \eta = \frac{\cos \theta}{r}, \sin \eta = \frac{\sin \theta \sin \phi}{r}, r = \sqrt{\sin^2 \theta \sin^2 \phi + \cos^2 \theta}. \)
Let $\bar{\Omega} = \frac{\Omega_{\bar{p}} + \Omega_{-\bar{p}}}{2}$, $\Delta \bar{\Omega} = \frac{\Omega_{\bar{p}} - \Omega_{-\bar{p}}}{2}$.

Then from equations (22), (23) and (13), we obtain

$$U(\Lambda)\Psi_{00} = \frac{(\Lambda p)^0}{p^0}((\cos \bar{\Omega}_p \cos^2 \eta + \cos \Delta \Omega_p \sin^2 \eta)\frac{1}{\sqrt{2}}\{a^+(\bar{p}_\Lambda, \frac{1}{2})a^+(-\bar{p}_\Lambda, \frac{1}{2})$$

$$+ a^+(\bar{p}_\Lambda, -\frac{1}{2})a^+(-\bar{p}_\Lambda, -\frac{1}{2})\} \Psi_0$$

$$- \frac{(\Lambda p)^0}{p^0} \sin \bar{\Omega}_p \cos \eta \frac{1}{\sqrt{2}}\{a^+(\bar{p}_\Lambda, \frac{1}{2})a^+(-\bar{p}_\Lambda, -\frac{1}{2}) - a^+(\bar{p}_\Lambda, -\frac{1}{2})a^+(-\bar{p}_\Lambda, \frac{1}{2})\} \Psi_0$$

$$+ i \frac{(\Lambda p)^0}{p^0} \sin \Delta \Omega_p \sin \eta \frac{1}{\sqrt{2}}\{a^+(\bar{p}_\Lambda, \frac{1}{2})a^+(-\bar{p}_\Lambda, -\frac{1}{2}) - a^+(\bar{p}_\Lambda, -\frac{1}{2})a^+(-\bar{p}_\Lambda, \frac{1}{2})\} \Psi_0$$

$$- i \frac{(\Lambda p)^0}{p^0} (-\cos \bar{\Omega}_p + \cos \Delta \Omega_p) \sin \eta \cos \eta \frac{1}{\sqrt{2}}\{a^+(\bar{p}_\Lambda, \frac{1}{2})a^+(-\bar{p}_\Lambda, -\frac{1}{2})$$

$$+ a^+(\bar{p}_\Lambda, -\frac{1}{2})a^+(-\bar{p}_\Lambda, \frac{1}{2})\} \Psi_0$$

$$= \frac{(\Lambda p)^0}{p^0}\{((\cos \bar{\Omega}_p \cos^2 \eta + \cos \Delta \Omega_p \sin^2 \eta)\Psi_{00} - \sin \bar{\Omega}_p \cos \eta \Psi_{11}$$

$$+ i \sin \Delta \Omega_p \sin \eta \Psi_{01} - i (-\cos \bar{\Omega}_p + \cos \Delta \Omega_p) \sin \eta \cos \eta \Psi_{10}\},$$

(24a)

where $\Psi_{ij}$ is the Bell states in the moving frame $S'$ whose momenta are transformed as $\bar{p} \rightarrow \bar{p}_\Lambda, -\bar{p} \rightarrow -\bar{p}_\Lambda$. Likewise, we have

$$U(\Lambda)\Psi_{01} = \frac{(\Lambda p)^0}{p^0}\cos \Delta \Omega_p \frac{1}{\sqrt{2}}\{a^+(\bar{p}_\Lambda, \frac{1}{2})a^+(-\bar{p}_\Lambda, \frac{1}{2}) - a^+(\bar{p}_\Lambda, -\frac{1}{2})a^+(-\bar{p}_\Lambda, -\frac{1}{2})\} \Psi_0$$

$$+ \frac{(\Lambda p)^0}{p^0} \sin \Delta \Omega_p \cos \eta \frac{1}{\sqrt{2}}\{a^+(\bar{p}_\Lambda, \frac{1}{2})a^+(-\bar{p}_\Lambda, -\frac{1}{2}) + a^+(\bar{p}_\Lambda, -\frac{1}{2})a^+(-\bar{p}_\Lambda, \frac{1}{2})\} \Psi_0$$

$$+ i \frac{(\Lambda p)^0}{p^0} \sin \Delta \Omega_p \sin \eta \frac{1}{\sqrt{2}}\{a^+(\bar{p}_\Lambda, \frac{1}{2})a^+(-\bar{p}_\Lambda, -\frac{1}{2}) + a^+(\bar{p}_\Lambda, -\frac{1}{2})a^+(-\bar{p}_\Lambda, \frac{1}{2})\} \Psi_0$$

$$= \frac{(\Lambda p)^0}{p^0}\{\cos \Delta \Omega_p \frac{1}{\sqrt{2}}\{(\frac{1}{2}, \frac{1}{2}) - | - \frac{1}{2}, \frac{1}{2})\}\}$$

$$+ \frac{(\Lambda p)^0}{p^0}\{\sin \Delta \Omega_p \cos \eta \frac{1}{\sqrt{2}}\{(\frac{1}{2}, \frac{1}{2}) + | - \frac{1}{2}, \frac{1}{2})\}\}$$

10
$$+i \frac{(\Lambda p)^0}{p^0} |\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \{\sin \Delta \Omega_\rho \sin \eta \frac{1}{\sqrt{2}} ((\frac{1}{2}, \frac{1}{2}) + | -\frac{1}{2}, -\frac{1}{2}\rangle)\}$$

$$= \frac{(\Lambda p)^0}{p^0} \{\cos \Delta \Omega_\rho \Psi'_{10} + \sin \Delta \Omega_\rho \cos \eta \Psi'_{10} + i \sin \Delta \Omega_\rho \sin \eta \Psi'_{00}\}, \quad (24b)$$

$$U(\Lambda)\Psi_{10} = \frac{(\Lambda p)^0}{p^0} \{\cos \frac{1}{\sqrt{2}} ((\frac{1}{2}, \frac{1}{2}) + | -\frac{1}{2}, -\frac{1}{2}\rangle)\}$$

$$= \frac{(\Lambda p)^0}{p^0} |\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \{\cos \Delta \Omega_\rho \sin \eta \frac{1}{\sqrt{2}} ((\frac{1}{2}, \frac{1}{2}) - | -\frac{1}{2}, -\frac{1}{2}\rangle)\}$$

$$-i \frac{(\Lambda p)^0}{p^0} \{\cos \Delta \Omega_\rho \cos \eta \frac{1}{\sqrt{2}} ((\frac{1}{2}, \frac{1}{2}) + | -\frac{1}{2}, -\frac{1}{2}\rangle)\}$$

$$+i \frac{(\Lambda p)^0}{p^0} \{\cos \Delta \Omega_\rho \cos \eta \sin \eta \frac{1}{\sqrt{2}} ((\frac{1}{2}, \frac{1}{2}) + | -\frac{1}{2}, -\frac{1}{2}\rangle)\}$$

$$= \frac{(\Lambda p)^0}{p^0} \{\cos \frac{1}{\sqrt{2}} ((\frac{1}{2}, \frac{1}{2}) - | -\frac{1}{2}, -\frac{1}{2}\rangle)\}$$

$$-i \frac{(\Lambda p)^0}{p^0} \{\cos \frac{1}{\sqrt{2}} ((\frac{1}{2}, \frac{1}{2}) + | -\frac{1}{2}, -\frac{1}{2}\rangle)\}$$

$$+i \sin \Omega_\rho \sin \eta \Psi'_{11} - i(\cos \Omega_\rho - \cos \Delta \Omega_\rho) \cos \eta \sin \eta \Psi'_{01}\}, \quad (24c)$$

and

$$U(\Lambda)\Psi_{11} = \frac{(\Lambda p)^0}{p^0} \cos \Omega_\rho \frac{1}{\sqrt{2}} \{a^+(\vec{p}_\Lambda, \frac{1}{2})a^+(-\vec{p}_\Lambda, -\frac{1}{2}) - a^+(\vec{p}_\Lambda, -\frac{1}{2})a^+(-\vec{p}_\Lambda, \frac{1}{2})\} \Psi_0$$

$$+ \frac{(\Lambda p)^0}{p^0} \sin \Omega_\rho \frac{1}{\sqrt{2}} \{a^+(\vec{p}_\Lambda, \frac{1}{2})a^+(-\vec{p}_\Lambda, -\frac{1}{2}) + a^+(\vec{p}_\Lambda, -\frac{1}{2})a^+(-\vec{p}_\Lambda, \frac{1}{2})\} \Psi_0$$

$$+ i \frac{(\Lambda p)^0}{p^0} \sin \Omega_\rho \sin \eta \frac{1}{\sqrt{2}} \{a^+(\vec{p}_\Lambda, \frac{1}{2})a^+(-\vec{p}_\Lambda, -\frac{1}{2}) + a^+(\vec{p}_\Lambda, -\frac{1}{2})a^+(-\vec{p}_\Lambda, \frac{1}{2})\} \Psi_0$$

$$= \frac{(\Lambda p)^0}{p^0} |\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \{\sin \Omega_\rho \frac{1}{\sqrt{2}} ((\frac{1}{2}, \frac{1}{2}) - | -\frac{1}{2}, -\frac{1}{2}\rangle)\}$$

$$+ \frac{(\Lambda p)^0}{p^0} |\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \{\sin \Omega_\rho \frac{1}{\sqrt{2}} ((\frac{1}{2}, \frac{1}{2}) + | -\frac{1}{2}, -\frac{1}{2}\rangle)\}$$
\[ i \frac{(\Lambda p)^0}{p^0} |\vec{p}_\Lambda, -\vec{p}_\Lambda \rangle \otimes \{\sin \Omega \vec{p} \sin \eta \frac{1}{\sqrt{2}}(|\frac{1}{2}, -\frac{1}{2}) + | -\frac{1}{2}, \frac{1}{2})\} \]
\[ = \frac{(\Lambda p)^0}{p^0} \{\cos \Omega \vec{p} \Psi_{11}' + \sin \Omega \vec{p} \cos \eta \Psi_{00}' + i \sin \Omega \vec{p} \sin \eta \Psi_{10}'\}. \] (24d)

If we regard \( \Psi_{ij}' \) as Bell states in the moving frame \( S' \), then to an observer in \( S' \), the effects of the Lorentz transformation on entangled Bell states among themselves should appear as rotations of Bell states in the frame \( S' \).

**III. BELL’S INEQUALITY.**

We are now ready to check whether the Lorentz transformed Bell states always violate the Bell’s inequality in special relativity.

One of the most essential features of quantum mechanics distinguished from classical physics is that the expectation value, or the quantum correlation of the measurement of the observables \( \vec{a}_1 \cdot \vec{\sigma}_1 \) and \( \vec{a}_2 \cdot \vec{\sigma}_2 \) for two-particle system, where \( \vec{\sigma}_1 \) and \( \vec{\sigma}_2 \) are the Pauli spin matrices pertaining to the two particles and \( \vec{a}_1 \) and \( \vec{a}_2 \) are unit vectors, given by [18]

\[ \langle \vec{a}_1 \cdot \vec{\sigma}_1 \vec{a}_2 \cdot \vec{\sigma}_2 \rangle = -\vec{a}_1 \cdot \vec{a}_2 \] (25)

for the singlet state, is always stronger than the classical correlations. Original Bell’s inequality was derived for any physical system with dichotomic observables, whose values are \( \pm 1 \). Since any Hermitian operator defines an observables, one could extend the Bell’s inequality to the relativistic regime for any normalized relativistic observables.

It is already known [16] that neither the rest frame spin \( \vec{\sigma} \) nor the Dirac Spin operator \( \vec{\Sigma} \) which is associated with the spin of a moving particle as seen by a stationary observer can not be the relativistic spin operator. Another plausible candidate is the Pauli-Lubanski pseudovector \( W^\mu \) which itself is a Casimir operator satisfying \( W^\mu W_\mu = m^2 s(s + 1) \), where \( m \) and \( s \) are the mass and spin of the particle, respectively, and \( W^\mu = (p^0 (\vec{e} \cdot \vec{s}) \vec{e} + mc(\vec{s} - (\vec{e} \cdot \vec{s}) \vec{e}), p^0 \vec{v} \cdot \vec{s}/c^2) \) for the observer in the moving frame with boost velocity \( \vec{v} \) [1,17,19]. Here \( \vec{s} \) is the spin vector in the rest frame, \( \vec{e} \) is the unit vector in the Lorentz boost direction, and \( \beta = v/c \), the ratio of the boost speed and the speed of light.
In non-relativistic case, the measurement of the spin in the direction of the unit vector direction $\vec{a}$ is represented by the observable $\vec{a} \cdot \vec{s}$ and if we extend this definition of the observable to the relativistic case as $\vec{a} \cdot \vec{s}_\Lambda$, then

$$\vec{a} \cdot \vec{s}_\Lambda = \left[ \sqrt{1 - \beta^2 (\vec{a} - \vec{e}(\vec{a} \cdot \vec{e})) + \vec{e}(\vec{a} \cdot \vec{e})} \right] \cdot \vec{s}$$

Here we postulate the relativistic spin as [1,6]

$$\vec{s}_\Lambda = \frac{mc}{p^0} \vec{s} + (1 - \frac{mc}{p^0})(\vec{e} \cdot \vec{s})\vec{e} = \sqrt{1 - \beta^2 (\vec{s} - \vec{e}(\vec{s} \cdot \vec{e})) + \vec{e}(\vec{s} \cdot \vec{e})} = \frac{\vec{W}}{p^0},$$

and the normalized relativistic spin observable is given by [1,6]

$$\hat{a} = \frac{\left[ \sqrt{1 - \beta^2 (\vec{a} - \vec{e}(\vec{a} \cdot \vec{e})) + \vec{e}(\vec{a} \cdot \vec{e})} \right]}{\sqrt{1 + \beta^2 (\vec{e} \cdot \vec{a})^2 - 1}} \cdot \sigma,$$

where we normalized the relativistic spin observable by the absolute value of its eigenvalue. Here $\vec{a}$ and $\vec{s}_\Lambda$ are unit direction vector and relativistic spin operator seen by the moving observer. We can give more clear physical meaning of Eqs. (27) and (28) by invoking the principle of the special relativity. If we note $\vec{a}_\Lambda$ as the Lorentz transformation (now seen in the rest frame) of direction vector (of the moving frame), then from eqs. (27) and (28), we obtain

$$\frac{\vec{a}_\Lambda \cdot \vec{s}}{|\lambda(\vec{a}_\Lambda \cdot \vec{s})|} = \frac{\vec{a} \cdot \vec{s}_\Lambda}{|\lambda(\vec{a} \cdot \vec{s}_\Lambda)|},$$

which is consistent with the principle of the special relativity which tells the physics doesn’t change across the frame. As a result we can interpret $\hat{a}$ as correct normalized relativistic observable for the observer in the moving frame. Here $\lambda(\hat{O})$ denotes the eigenvalue of an operator $\hat{O}$.

It is straightforward to calculate the classical correlation $\langle \hat{a}\hat{b} \rangle_{\text{classical}}$ when the moving observer is receding (approaching) from (to) the rest frame with the speed of light and is given by

$$\langle \hat{a}\hat{b} \rangle_{\text{classical}} = \frac{\vec{a} \cdot \vec{e}}{|\vec{a} \cdot \vec{e}|} \cdot \frac{\vec{b} \cdot \vec{e}}{|\vec{b} \cdot \vec{e}|} = \pm 1,$$
and it should be noted that the information in the perpendicular direction to the unit boost vector $\vec{e}$ is lost as both spins are tilted toward the boost axis as a result of the Lorentz transformation.

Normalized relativistic spin observables $\hat{a}, \hat{b}$ are given by [6]

$$\hat{a} = \frac{(\sqrt{1 - \beta^2} \hat{a}_\perp + \hat{a}_\parallel) \cdot \vec{\sigma}}{\sqrt{1 + \beta^2[(\hat{e} \cdot \hat{a})^2 - 1]}}$$

(31)

and

$$\hat{b} = \frac{(\sqrt{1 - \beta^2} \hat{b}_\perp + \hat{b}_\parallel) \cdot \vec{\sigma}}{\sqrt{1 + \beta^2[(\hat{e} \cdot \hat{b})^2 - 1]}}.$$  

(32)

where the subscript $\perp$ and $\parallel$ denote the components of $\hat{a}$ (or $\hat{b}$) which are perpendicular and parallel to the boost direction, respectively. Moreover, $|\hat{a}| = |\hat{b}| = 1$.

A: The momentum and the boost vectors in the same plane.

Case I: $\Psi_{00} \to U(\Lambda)\Psi_{00}$

From eq. (18a), we have

$$U(\Lambda)\Psi_{00} = \frac{(\Lambda p)^0}{p^0} |\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \left[ \frac{1}{\sqrt{2}} \cos \frac{\Omega_{\rho} + \Omega_{-\rho}}{2} (|\frac{1}{2}, \frac{1}{2}\rangle + | -\frac{1}{2}, -\frac{1}{2}\rangle) - \frac{1}{\sqrt{2}} \sin \frac{\Omega_{\rho} + \Omega_{-\rho}}{2} (|\frac{1}{2}, -\frac{1}{2}\rangle - | -\frac{1}{2}, \frac{1}{2}\rangle) \right].$$

(33)

Then, after some mathematical manipulations, we get

$$\hat{a} \otimes \hat{b} |\frac{1}{2}, \frac{1}{2}\rangle = \frac{1}{\sqrt{[1 + \beta^2(a_x^2 - 1)][1 + \beta^2(b_x^2 - 1)]}} \{(1 - \beta^2)a_z b_z |\frac{1}{2}, \frac{1}{2}\rangle$$

$$+ (1 - \beta^2)a_x (b_x + i b_y \sqrt{1 - \beta^2}) |\frac{1}{2}, -\frac{1}{2}\rangle$$

$$+ (1 - \beta^2)b_z (a_x + i a_y \sqrt{1 - \beta^2}) | -\frac{1}{2}, \frac{1}{2}\rangle$$

$$+ (a_x + i a_y \sqrt{1 - \beta^2})(b_x + i b_y \sqrt{1 - \beta^2}) | -\frac{1}{2}, -\frac{1}{2}\rangle \}.$$  

(34a)

$$\hat{a} \otimes \hat{b} | -\frac{1}{2}, -\frac{1}{2}\rangle = \frac{1}{\sqrt{[1 + \beta^2(a_x^2 - 1)][1 + \beta^2(b_x^2 - 1)]}} \{(a_x - i a_y \sqrt{1 - \beta^2})(b_x - i b_y \sqrt{1 - \beta^2}) |\frac{1}{2}, \frac{1}{2}\rangle$$

$$- (1 - \beta^2)b_z (a_x - i a_y \sqrt{1 - \beta^2}) |\frac{1}{2}, -\frac{1}{2}\rangle$$

$$- (a_x - i a_y \sqrt{1 - \beta^2})(b_x - i b_y \sqrt{1 - \beta^2}) | -\frac{1}{2}, \frac{1}{2}\rangle.$$  

(34b)
Then the Bell observable for the 4 relevant joint measurements becomes

\[
\hat{a} \otimes \hat{b} | \frac{1}{2}, -\frac{1}{2} \rangle = \frac{1}{\sqrt{[1 + \beta^2(a_x^2 - 1)][1 + \beta^2(b_y^2 - 1)]}} \{ \sqrt{1 - \beta^2} a_z (b_x - ib_y \sqrt{1 - \beta^2}) | \frac{1}{2}, \frac{1}{2} \rangle \\
- (1 - \beta^2) a_x b_z | \frac{1}{2}, \frac{1}{2} \rangle \\
+ (a_x + ia_y \sqrt{1 - \beta^2} (b_x - ib_y \sqrt{1 - \beta^2}) | \frac{1}{2}, -\frac{1}{2} \rangle \\
- \sqrt{1 - \beta^2} b_z (a_x + ia_y \sqrt{1 - \beta^2}) | -\frac{1}{2}, \frac{1}{2} \rangle \\
- (1 - \beta^2) a_y b_y - \sqrt{1 - \beta^2} (a_x b_x - b_x a_x) \sin(\Omega_{\vec{p}} + \Omega_{-\vec{p}}) \}
\]

for the boost in the \( x \)-direction. The calculation of \( \langle \hat{a} \otimes \hat{b} \rangle \) is straightforward and is given by

\[
\langle \hat{a} \otimes \hat{b} \rangle = \frac{1}{\sqrt{[1 + \beta^2(a_x^2 - 1)][1 + \beta^2(b_y^2 - 1)]}} \{ [a_x b_x + (1 - \beta^2) a_z b_z] \cos(\Omega_{\vec{p}} + \Omega_{-\vec{p}}) \\
- (1 - \beta^2) a_y b_y - \sqrt{1 - \beta^2} (a_x b_x - b_x a_x) \sin(\Omega_{\vec{p}} + \Omega_{-\vec{p}}) \}
\]

(35)

It is interesting to note that in the ultra-relativistic limit, \( \beta \to 1 \), equation (35) becomes

\[
\langle \hat{a} \otimes \hat{b} \rangle \to \frac{a_x}{|a_x|} \cdot \frac{b_x}{|b_x|} \cos(\Omega_{\vec{p}} + \Omega_{-\vec{p}}),
\]

(36)

implying that the joint measurements are not correlated at all. As a result, one might suspect that the entangled state satisfies the Bell’s inequality. We now consider the vectors

\[
\vec{a} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0), \vec{a'} = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0), \vec{b} = (0, 1, 0), \vec{b'} = (1, 0, 0) \]

which lead to the maximum violation of the Bell’s inequality in the non-relativistic domain, \( \Omega_{\vec{p}} = \Omega_{-\vec{p}} = 0 \) and \( \beta = 0 \). Then the Bell observable for the 4 relevant joint measurements becomes

\[
\langle \hat{a} \otimes \hat{b} \rangle + \langle \hat{a} \otimes \hat{b'} \rangle + \langle \hat{a'} \otimes \hat{b} \rangle - \langle \hat{a'} \otimes \hat{b'} \rangle = \frac{2}{\sqrt{2 - \beta^2}} (\sqrt{1 - \beta^2} + \cos(\Omega_{\vec{p}} + \Omega_{-\vec{p}})).
\]

(37)
In the ultra-relativistic limit where \( \beta = 1 \), the eq. (37) gives the maximum value of 2 satisfying the Bell’s inequality as expected.

Case II: \( \Psi_{01} \to U(\Lambda)\Psi_{01} \)

From eq. (18b), we have

\[
U(\Lambda)\Psi_{01} = \frac{(\Lambda p)^0}{p^0} \left[ \vec{p}_\Lambda, -\vec{p}_\Lambda \right] \otimes \left[ \frac{1}{\sqrt{2}} \cos \frac{\Omega_{\vec{p}} - \Omega_{-\vec{p}}}{2} \left( | \frac{1}{2}, \frac{1}{2} \rangle - | -\frac{1}{2}, -\frac{1}{2} \rangle \right) + \frac{1}{\sqrt{2}} \sin \frac{\Omega_{\vec{p}} - \Omega_{-\vec{p}}}{2} \left( | \frac{1}{2}, -\frac{1}{2} \rangle + | -\frac{1}{2}, \frac{1}{2} \rangle \right) \right].
\]

(38)

From equations (34a) to (34d), we obtain

\[
\langle \hat{a} \otimes \hat{b} \rangle = \frac{1}{\sqrt{[1 + \beta^2(a_x^2 - 1)][1 + \beta^2(b_x^2 - 1)]}} \left\{ [-a_x b_x + (1 - \beta^2)a_z b_z] \cos(\Omega_{\vec{p}} - \Omega_{-\vec{p}}) \\
+ (1 - \beta^2)a_y b_y + \sqrt{1 - \beta^2(a_x b_x + b_z a_x)} \sin(\Omega_{\vec{p}} - \Omega_{-\vec{p}}) \right\}.
\]

(39)

Then, in the ultra-relativistic limit, \( \beta \to 1 \), we have

\[
\langle \hat{a} \otimes \hat{b} \rangle \to -\frac{a_x}{|a_x|} \cdot \frac{b_x}{|b_x|} \cos(\Omega_{\vec{p}} - \Omega_{-\vec{p}}),
\]

(40)

again, indicating the joint measurements, become uncorrelated in this limit. We consider the vectors \( \vec{a} = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), \vec{a}' = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0), \vec{b} = (0, 1, 0), \vec{b}' = (1, 0, 0) \) which lead to the maximum violation of the Bell’s inequality in the non-relativistic regime. Then the Bell observable for the 4 relevant joint measurements becomes

\[
\langle \hat{a} \otimes \hat{b} \rangle + \langle \hat{a} \otimes \hat{b}' \rangle + \langle \hat{a}' \otimes \hat{b} \rangle - \langle \hat{a}' \otimes \hat{b}' \rangle \\
= \frac{2}{\sqrt{2} - \beta^2} (\sqrt{1 - \beta^2} + \cos(\Omega_{\vec{p}} - \Omega_{-\vec{p}})),
\]

(41)

thus giving same maximum value as in the case I. It can also be shown that one can obtain the same value for the Bell observables given by eq. (41) for \( U(\Lambda)\Psi_{10} \) and \( U(\Lambda)\Psi_{11} \) implying eq. (41) is the universal result.

B: The momentum and the boost vectors not in the same plane.
Case I: $\Psi_{00} \rightarrow U(\Lambda)\Psi_{00}$

From eq. (24a), we have

$$U(\Lambda)\Psi_{00} = \frac{(\Delta p)^0}{\rho^0} |\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \{(\cos \bar{\Omega}_\rho \cos^2 \eta + \cos \Delta \Omega_\rho \sin^2 \eta) \frac{1}{\sqrt{2}} (|\frac{1}{2}, \frac{1}{2}\rangle + |-\frac{1}{2}, -\frac{1}{2}\rangle)$$

$$- \sin \bar{\Omega}_\rho \cos \eta \frac{1}{\sqrt{2}} (|\frac{1}{2}, -\frac{1}{2}\rangle - |\frac{1}{2}, \frac{1}{2}\rangle)$$

$$+ i \sin \Delta \Omega_\rho \sin \eta \frac{1}{\sqrt{2}} (|\frac{1}{2}, \frac{1}{2}\rangle - |\frac{1}{2}, -\frac{1}{2}\rangle)$$

$$- i(- \cos \bar{\Omega}_\rho + \cos \Delta \Omega_\rho) \sin \eta \cos \eta \frac{1}{\sqrt{2}} (|\frac{1}{2}, -\frac{1}{2}\rangle + \frac{1}{2} |\frac{1}{2}, \frac{1}{2}\rangle)) \}, \quad (42)$$

From equations (34a) to (34d), we obtain

$$\langle \hat{a} \otimes \hat{b} \rangle = \frac{1}{\sqrt{[1 + \beta^2(a_x^2 - 1)][1 + \beta^2(b_x^2 - 1)]}} \left\{ \frac{A_+ + A_-}{2} X^2 + \frac{C_+ + C_-}{2} Y^2 + \frac{E_+ + E_-}{2} Z^2 + \frac{G_+ + G_-}{2} W^2 - 2i \frac{B_+ + B_-}{2} XY + \frac{A_+ - A_-}{2i} XZ + \frac{B_+ - B_-}{2i} XY - \frac{D_+ - D_-}{2i} YZ + \frac{C_+ - C_-}{2i} YW + \frac{F_+ + F_-}{2} ZW \right\}. \quad (43)$$

where

$$X = \cos \bar{\Omega}_\rho \cos^2 \eta + \cos \Delta \Omega_\rho \sin^2 \eta, \quad (44a)$$

$$Y = \sin \bar{\Omega}_\rho \cos \eta, \quad (44b)$$

$$Z = \sin \Delta \Omega_\rho \sin \eta, \quad (44c)$$

$$W = - \cos \bar{\Omega}_\rho \cos \eta \sin \eta + \cos \Delta \Omega_\rho \sin \eta \cos \eta, \quad (44d)$$

$$\frac{A_+ + A_-}{2} = a_x b_x - (1 - \beta^2)a_y b_y + (1 - \beta^2)a_z b_z, \quad (44e)$$

$$\frac{C_+ + C_-}{2} = -a_x b_x - (1 - \beta^2)a_y b_y - (1 - \beta^2)a_z b_z, \quad (44f)$$

$$\frac{E_+ + E_-}{2} = -a_x b_x + (1 - \beta^2)a_y b_y + (1 - \beta^2)a_z b_z, \quad (44g)$$

$$\frac{G_+ + G_-}{2} = a_x b_x + (1 - \beta^2)a_y b_y - (1 - \beta^2)a_z b_z, \quad (44h)$$

$$\frac{B_+ + B_-}{2} = \frac{D_+ + D_-}{2} = \sqrt{1 - \beta^2(a_x b_x - b_z a_x)}, \quad (44i)$$

$$\frac{F_+ + F_-}{2} = \frac{H_+ + H_-}{2} = \sqrt{1 - \beta^2(a_z b_z + b_x a_x)}, \quad (44j)$$

$$\frac{A_+ - A_-}{2i} = \frac{E_+ - E_-}{2i} = \sqrt{1 - \beta^2(a_x b_y + b_y a_x)}, \quad (44k)$$
\[
\begin{align*}
\frac{C+ - C-}{2i} &= \frac{G+ - G-}{2i} = \sqrt{1 - \beta^2(a_x b_y - b_x a_y)}, \\
\frac{B+ - B-}{2i} &= \frac{H+ - H-}{2i} = (1 - \beta^2)(a_x b_y + b_x a_y), \\
\frac{D+ - D-}{2i} &= \frac{F+ - F-}{2i} = (1 - \beta^2)(a_x b_y - b_x a_y)
\end{align*}
\] (44l-m-n)

It is interesting to note that in the ultra-relativistic limit, \(\beta \to 1\), equation (43) becomes

\[
\langle \hat{a} \otimes \hat{b} \rangle \to \left. \frac{a_x}{|a_x|} \cdot \frac{b_x}{|b_x|} \right| (X^2 - Y^2 - Z^2 + W^2),
\] (45)

implying that the joint measurements are not correlated at all. As a result, one might suspect that the entangled state satisfies the Bell’s inequality. We now consider the vectors

\[
\vec{\alpha} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0), \quad \vec{\alpha}' = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0), \quad \vec{\beta} = (0, 1, 0), \quad \vec{\beta}' = (1, 0, 0)
\]

which lead to the maximum violation of the Bell’s inequality in the non-relativistic domain, \(\Omega_{\vec{\alpha}} = \Omega_{-\vec{\alpha}} = 0\) and \(\beta = 0\). Then the Bell observable for the 4 relevant joint measurements becomes

\[
\langle \hat{a} \otimes \hat{b} \rangle + \langle \hat{a} \otimes \hat{b}' \rangle + \langle \hat{a}' \otimes \hat{b} \rangle - \langle \hat{a}' \otimes \hat{b}' \rangle
\]

\[
= \frac{2}{\sqrt{2} - \beta^2} \{ (X^2 - Y^2 - Z^2 + W^2) + (X^2 + Y^2 - Z^2 - W^2) \sqrt{1 - \beta^2} \}.
\] (46)

In the ultra-relativistic limit where \(\beta = 1\), the eq. (46) gives the maximum value of 2 satisfying the Bell’s inequality as expected. (Appendix)

**Case II: \(\Psi_{01} \to U(\Lambda)\Psi_{01}\)**

From eq. (24b), we have

\[
U(\Lambda)\Psi_{01} = \frac{(\Lambda p)^0_0}{p^0}|\vec{p}_0, -\vec{p}_0\rangle \otimes \{ \cos \Delta \Omega_{\vec{p}} \frac{1}{\sqrt{2}} (|\frac{1}{2}, \frac{1}{2}\rangle - | -\frac{1}{2}, -\frac{1}{2}\rangle) \\
+ \sin \Delta \Omega_{\vec{p}} \cos \eta \frac{1}{\sqrt{2}} (|\frac{1}{2}, -\frac{1}{2}\rangle + | -\frac{1}{2}, \frac{1}{2}\rangle) \\
+ i \sin \Delta \Omega_{\vec{p}} \sin \eta \frac{1}{\sqrt{2}} (|\frac{1}{2}, \frac{1}{2}\rangle + | -\frac{1}{2}, -\frac{1}{2}\rangle) \}
\] (47)

From equations (34a) to (34d), we obtain

\[
\langle \hat{a} \otimes \hat{b} \rangle = \frac{1}{\sqrt{[1 + \beta^2(a_x^2 - 1)][1 + \beta^2(b_x^2 - 1)]}} \left\{ \frac{G+ + G-}{2} X'^2 + \frac{A+ + A_-}{2} Y'^2 \\
+ \frac{E_+ + E_-}{2} Z'^2 + \frac{F_+ + F_-}{2} X'Z' + \frac{A_+ - A_-}{2i} Y'Z' - \frac{B_+ - B_-}{2i} X'Y' \right\}.
\] (48)
where

\[ X' = \sin \Delta \Omega \vec{p} \cos \eta, \] (49a)

\[ Y' = \sin \Delta \Omega \vec{p} \sin \eta, \] (49b)

\[ Z' = \cos \Delta \Omega \vec{p}. \] (49c)

Then, in the ultra-relativistic limit, \( \beta \to 1 \), we have

\[ \langle \hat{a} \otimes \hat{b} \rangle \to -\frac{a_x}{|a_x|} \cdot \frac{b_x}{|b_x|} (-X'^2 - Y'^2 + Z'^2) = -\frac{a_x}{|a_x|} \cdot \frac{b_x}{|b_x|} \cos 2\Delta \Omega \vec{p}, \]

(50)

again, indicating the joint measurements, become uncorrelated in this limit. We consider the vectors \( \vec{a} = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), \vec{a}' = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), \vec{b} = (0, 1, 0), \vec{b}' = (1, 0, 0) \) which lead to the maximum violation of the Bell’s inequality in the non-relativistic regime. Then the Bell observable for the 4 relevant joint measurements becomes

\[
\langle \hat{a} \otimes \hat{b} \rangle + \langle \hat{a} \otimes \hat{b}' \rangle + \langle \hat{a}' \otimes \hat{b} \rangle - \langle \hat{a}' \otimes \hat{b}' \rangle \\
= \frac{2}{\sqrt{2} - \beta^2} \left\{ \left( -X'^2 - Y'^2 + Z'^2 \right) + \left( X'^2 - Y'^2 + Z'^2 \right) \sqrt{1 - \beta^2} \right\} \\
= \frac{2}{\sqrt{2} - \beta^2} \left\{ \cos 2\Delta \Omega \vec{p} + (\cos^2 \eta + \sin^2 \eta \cos 2\Delta \Omega \vec{p}) \sqrt{1 - \beta^2} \right\}
\] (51)

thus giving same maximum value as in the case I. It can also be shown that one can obtain the same value for the Bell observables given by eq. (51) for \( U(\Lambda) \Psi_{10} \) and \( U(\Lambda) \Psi_{11} \) implying eq. (51) is the universal result.

These agree with our previous results [1] which didn’t take into account the general momentum. It can also be shown that similar results would be obtained for the case [20] in which one observer is in the rest frame and the other observer is in the moving frame and do the joint measurements of spins. Now, one can see that the quantum correlation approaches to the classical correlation when the speed of the moving observer reaches the speed of light and in both cases, the information in the vertical direction to the boost axis is lost. This is somewhat analogous to the cases of \( \beta \)-decay of nuclei and high energy electrons and positrons emitted in the decay of muons for which emitted electrons and positrons are polarized such...
that their spins tend to lie in the same direction of the motion and their projections of the
spins in the direction of the motion became $\pm 1$ for the relativistic particles [21]. It should
be noted that if one simply rotates the spin directions instead of using the relativistic spin
observables, then the entanglement between the spins of the Bell states are not changed and
the results of the spin measurements would be exactly same as if they were done in the rest
frame thus give the maximum violation of the Bell inequality. It is interesting to note that
the entanglement is still remained though it is degraded, when Bell’s inequality is satisfied.
The most plausible reason for this is that the quantum correlations in the vertical direction
to the boost are lost and become classical. So we can also conclude that the Bell’s inequality
is not always violated for entangled state in special relativity.

IV. SUMMARY

In this work, we studied the Lorentz transformed entangled Bell states and the Bell
observables in the case of general momentum to investigate whether the Bell’s inequality is
always violated in special relativity. We have calculated the Bell observable for the joint 4
measurements and found that the results are universal for all entangled states:

$$c(\hat{a}, \hat{a}', \hat{b}, \hat{b}') = \langle \hat{a} \otimes \hat{b} \rangle + \langle \hat{a} \otimes \hat{b}' \rangle + \langle \hat{a}' \otimes \hat{b} \rangle - \langle \hat{a}' \otimes \hat{b}' \rangle$$

$$= \frac{2}{\sqrt{2-\beta^2}}(1 + \sqrt{1-\beta^2}),$$

where $\hat{a}, \hat{b}$ are the relativistic spin observables derived from the Pauli-Lubanski pseudo vec-
tor. It turn out that the Bell observable is a monotonically decreasing function of $\beta$ and
approaches the limit value of 2 when $\beta = 1$ indicating that the Bell’s inequality is not
always violated in the ultra-relativistic limit. It is also shown that quantum information,
along the perpendicular direction to the boost, is eventually lost and Bell’s inequality is
not always violated for entangled states in special relativity. This could impose restrictions
to certain quantum information processing such as quantum cryptography using massive
particles. Unless both sender and receiver measures along the boost direction, there will be
information loss.
Acknowledgements

This work was supported by the Korean Ministry of Science and Technology through the Creative Research Initiatives Program under Contract No. M1-0116-00-0008. We are also indebted to M. Czachor for valuable discussions.

APPENDIX: DERIVATION OF EQ.(39)

We have already known as follows, \( \cosh \alpha = \frac{1}{\sqrt{1-\beta^2}} \), \( \cosh \delta = \frac{\theta}{m} \) from (9) and (10), and defined as follows, \( \cos \eta = \frac{\cos \theta}{r} \), \( \sin \eta = \frac{\sin \theta \sin \phi}{r} \), \( r = \sqrt{\sin^2 \theta \sin^2 \phi + \cos^2 \theta} \), and \( \bar{\Omega} = \frac{\Omega_{\phi} + \Omega_{-\phi}}{2} \), \( \Delta \bar{\Omega} = \frac{\Omega_{\phi} - \Omega_{-\phi}}{2} \).

From Eq.(20) and (21), we obtain

\[
\cos \bar{\Omega} = \cos \frac{\Omega_{\phi}}{2} \cos \frac{\Omega_{-\phi}}{2} - \sin \frac{\Omega_{\phi}}{2} \sin \frac{\Omega_{-\phi}}{2} = \frac{\cov^2 \frac{\alpha}{2} \cosh^2 \frac{\delta}{2} - \sinh^2 \frac{\alpha}{2} \sinh^2 \frac{\delta}{2}}{\left[ \left( \frac{1}{2} + \frac{1}{2} \cosh \alpha \cosh \delta \right)^2 - \left( \frac{1}{2} \sinh \alpha \sinh \delta \sin \theta \cos \phi \right)^2 \right]^2} \]

\[
= \frac{\left( \frac{1}{2} \cosh \alpha \cosh \delta + 1 \right)^2 - 4 \cosh^2 \frac{\alpha}{2} \cosh^2 \frac{\delta}{2} \left( 1 - r^2 \right)}{\left[ \left( \frac{1}{2} \coth \frac{\alpha}{2} \coth \frac{\delta}{2} - 1 \right)^2 + 4 \coth^2 \frac{\alpha}{2} \coth^2 \frac{\delta}{2} r^2 \right]^2} - \left( 1 - 2r^2 \right) \frac{1}{2r^2 t^2} \]

\[
= \frac{t - 1}{\left[ (t - 1)^2 + 4dr^2 \right]^2} \]

where \( t = \coth^2 \frac{\alpha}{2} \coth^2 \frac{\delta}{2} \), \( 1 \leq t \), and

\[
\cos \Delta \bar{\Omega} = \cos \frac{\Omega_{\phi}}{2} \cos \frac{\Omega_{-\phi}}{2} + \sin \frac{\Omega_{\phi}}{2} \sin \frac{\Omega_{-\phi}}{2} = \frac{\cov^2 \frac{\alpha}{2} \cosh^2 \frac{\delta}{2} - \sinh^2 \frac{\alpha}{2} \sinh^2 \frac{\delta}{2} \left( \sin^2 \theta \cos^2 \phi - \sin^2 \theta \sin^2 \phi \cos^2 \theta \right)}{\left[ \left( \frac{1}{2} + \frac{1}{2} \cosh \alpha \cosh \delta \right)^2 - \left( \frac{1}{2} \sinh \alpha \sinh \delta \sin \theta \cos \phi \right)^2 \right]^2} \]

\[
= \frac{\left( \frac{1}{2} \cosh \alpha \cosh \delta + 1 \right)^2 - (1 - 2r^2)}{\left[ \left( \frac{1}{2} \coth \frac{\alpha}{2} \coth \frac{\delta}{2} - 1 \right)^2 + 4 \coth^2 \frac{\alpha}{2} \coth^2 \frac{\delta}{2} \left( 1 - r^2 \right) \right]^2} \]
\[
\begin{align*}
= \frac{\coth^2 \frac{\alpha^2}{2} \coth^2 \frac{\delta^2}{2} - (1 - 2r^2)}{[(\coth^2 \frac{\alpha^2}{2} \coth^2 \frac{\delta^2}{2} + 1) - 4 \coth^2 \frac{\alpha^2}{2} \coth^2 \frac{\delta^2}{2}(1 - r^2)]^{\frac{1}{2}}} \\
= \frac{\coth^2 \frac{\alpha^2}{2} \coth^2 \frac{\delta^2}{2} - 1 + 2r^2}{[(\coth^2 \frac{\alpha^2}{2} \coth^2 \frac{\delta^2}{2} - 1) + 4 \coth^2 \frac{\alpha^2}{2} \coth^2 \frac{\delta^2}{2} r^2]^{\frac{1}{2}}} \\
= \frac{(t - 1) + 2r^2}{[(t - 1)^2 + 4tr^2]^{\frac{1}{2}}},
\end{align*}
\]

(A2)

From (A1) and (A2), we get

\[
X^2 - Y^2 - Z^2 + W^2 = \cos^2 \bar{\Omega}_p \cos^2 \eta + \cos^2 \Delta \Omega_p \sin^2 \eta - \sin^2 \bar{\Omega}_p \cos^2 \eta - \sin^2 \Delta \Omega_p \sin^2 \eta
\]

\[
= 2(\cos^2 \bar{\Omega}_p \cos^2 \eta + \cos^2 \Delta \Omega_p \sin^2 \eta) - 1
\]

\[
= 2 \frac{(t - 1)^2 \cos^2 \eta + (t - 1 + 2r^2)^2 \sin^2 \eta}{(t - 1)^2 + 4tr^2} - 1
\]

\[
= 1 - 8r^2 \cos^2 \eta \frac{t + (1 - r^2) \tan^2 \eta}{(t - 1)^2 + 4tr^2},
\]

(A3)

and

\[
X^2 + Y^2 - Z^2 - W^2 = \cos^2 \bar{\Omega}_p \cos^2 \eta \cos 2\eta - \cos^2 \Delta \Omega_p \sin^2 \eta \cos 2\eta + \sin^2 \bar{\Omega}_p \cos^2 \eta
\]

\[
- \sin^2 \Delta \Omega_p \sin^2 \eta + 4 \cos \bar{\Omega}_p \cos \Delta \Omega_p \sin^2 \eta \cos^2 \eta
\]

\[
= \cos 2\eta + (1 - \cos 2\eta)\{\cos^2 \Delta \Omega_p - \cos^2 \eta(\cos \bar{\Omega}_p - \cos \Delta \Omega_p)^2\}
\]

\[
= \cos 2\eta + (1 - \cos 2\eta)\left\{\frac{(t - 1 + 2r^2)^2 - \cos^2 \eta(-2r^2)^2}{(t - 1)^2 + 4tr^2}\right\}
\]

\[
= 1 - 8r^2 \sin^2 \eta \frac{1 - r^2 \sin^2 \eta}{(t - 1)^2 + 4tr^2}.
\]

(A4)

From (A3), we define

\[
f(t) = \frac{t + a}{(t - 1)^2 + 4r^2 t},
\]

(A5)

and then

\[
\frac{df(t)}{dt} = -\frac{(t - 1)(t + 2a + 1) + 4r^2a}{((t - 1)^2 + 4r^2 t)^2} < 0, \text{ for } t \geq 1, \forall \theta \text{ and } \forall \phi,
\]

(A6)

where \( a = (1 - r^2) \tan^2 \eta \geq 0. \)

From (A5) and (A6), we obtain

\[
0 = f(\infty) \leq f(t) \leq f(1) = \frac{1 + a}{4r^2}
\]

(A7)
and

\[ 1 - 8r^2 \cos^2 \eta f(1) = 1 - 8r^2 \cos^2 \eta \frac{1 + (1 - r^2) \tan^2 \eta}{4r^2} \]

\[ = 2 \sin^2 \theta \sin^2 \phi - 1, \quad (A8) \]

therefore

\[ 2 \sin^2 \theta \sin^2 \phi - 1 \leq X^2 - Y^2 - Z^2 + W^2 \leq 1. \quad (A9) \]

From (A4), we define

\[ g(t) = \frac{b}{(t - 1)^2 + 4r^2t}, \quad (A10) \]

and then

\[ \frac{dg(t)}{dt} = -2b \frac{(t - 1) + 2r^2}{(t - 1)^2 + 4r^2t} \leq 0, \text{ for } t \geq 1, \forall \theta \text{ and } \forall \phi, \quad (A11) \]

where \( b = 1 - r^2 \sin^2 \eta \geq 0. \)

From (A10) and (A11), we have

\[ 0 = g(\infty) \leq g(t) \leq g(1) = \frac{b}{4r^2} \quad (A12) \]

and

\[ 1 - 8r^2 \sin^2 \eta g(1) = 1 - 8r^2 \sin^2 \eta \frac{1 - r^2 \sin^2 \eta}{4r^2} \]

\[ = 1 - 2 \sin^2 \eta (1 - r^2 \sin^2 \eta) \]

\[ = \cos 2\eta + \frac{r^2}{2} (1 - \cos 2\eta)^2 \]

\[ \geq \cos 2\eta, \quad (A13) \]

therefore

\[ \cos 2\eta \leq X^2 + Y^2 - Z^2 - W^2 \leq 1. \quad (A14) \]
REFERENCES

[1] M. Czachor, Phys. Rev. A55, 77(1997).

[2] P. M. Alsing and G. J. Milburn, Quant. Inf. Comp. 2, 487 (2002).

[3] A. Peres, P. F. Scudo, and D. R. Terno, Phys. Rev. Lett. 88, 230402 (2002).

[4] R. M. Gingrich and C. Adami, Phys. Rev. Lett. 89, 270402 (2002).

[5] U. Yurtserver and J. P. Dowling, Phys. Rev. A 65, 052317 (2002).

[6] D. Ahn, H.-j. Lee, Y. H. Moon, and S. W. Hwang, Phys. Rev. A 67, 012103 (2003).

[7] D. Ahn, H.-j. Lee, and S. W. Hwang, Phys. Rev. A 67, 032309 (2003).

[8] A. Peres and D. R. Terno, Rev. Mod. Phys. 75, (2003), in press; e-print quant-ph/0212023.

[9] H. Terashima and M. Ueda, to be published in Quant. Inf. Comp; e-print quant-ph/0204138.

[10] H. Terashima and M. Ueda, to be published in Quant. Inf. Comp; e-print quant-ph/0211177.

[11] M. Czachor and M. Wilczewski, e-print quant-ph/0303077.

[12] J. S. Bell, Physics 1, 195 (1964).

[13] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).

[14] A. Gisin, Phys. Lett. A 154, 201 (1991).

[15] S. Weinberg, The Quantum Theory of Fields I (Cambridge University Press, New York, 1995).

[16] L. H. Ryder, Quantum Field Theory (Cambridge University Press, New York, 1986).

[17] G. N. Fleming, Phys. Rev. 137, B188 (1965).
[18] A. Peres, Quantum Theory: Concepts and Methods (Kluwer Academic Publishers, Dordrecht, 1998).

[19] N. N. Bogolubov, A. A. Logunov, and I. T. Todorov, Introduction to Axiomatic Quantum Field Theory (Benjamin, New York, 1975).

[20] D. Ahn, H.-j. Lee, and S. W. Hwang, e-print quant-ph/0207018.

[21] I. S. Hughes, Elementary Particles (Cambridge University Press, New York, 1972).