On free 4D Abelian 2-form and anomalous 2D Abelian 1-form gauge theories

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Abstract: We demonstrate a few striking similarities and some glaring differences between (i) the free four (3 + 1)-dimensional (4D) Abelian 2-form gauge theory, and (ii) the anomalous two (1 + 1)-dimensional (2D) Abelian 1-form gauge theory, within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism. We demonstrate that the Lagrangian densities of the above two theories transform in a similar fashion under a set of symmetry transformations even though they are endowed with a drastically different variety of constraint structures. Taking the help of our understanding of the 4D Abelian 2-form gauge theory, we prove that the gauge invariant version of the anomalous 2D Abelian 1-form gauge theory is a new field-theoretic model for the Hodge theory where all the de Rham cohomological operators of differential geometry find their physical realizations in the language of proper symmetry transformations. The corresponding conserved charges obey an algebra that is reminiscent of the algebra of the cohomological operators. We briefly comment on the consistency of the 2D anomalous 1-form gauge theory in the language of restrictions on the harmonic state of the (anti-) BRST and (anti-) co-BRST invariant version of the above 2D theory.

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1 Introduction

The Becchi-Rouet-Stora-Tyutin (BRST) formalism is one of the most elegant and intuitive methods that is required for the covariant canonical quantization of any arbitrary $p$-form ($p = 1, 2, 3, ...$) gauge and/or reparameterization invariant theories that are endowed with the first-class constraints in the language of Dirac’s prescription for the classification scheme [1,2]. In this formalism, the unitarity and “quantum” gauge (i.e. BRST) invariance are respected together [3-5] at any arbitrary order of perturbative computation for a given physical process that is allowed by the above type of theories.

In recent years, the Abelian 2-form ($B^{(2)} = \frac{1}{2} (dx^\mu \wedge dx^\nu) B_{\mu\nu}$) gauge theory with antisymmetric ($B_{\mu\nu} = -B_{\nu\mu}$) potential $B_{\mu\nu}$ [6,7] has become quite popular because of its relevance in the context of (super) gravity and (super) string theories [8-10]. Its existence is crucial for the emergence of non-commutativity in the realm of string theory [11]. Furthermore, it has been shown that this gauge theory provides a field-theoretic model for the quasi-topological field theory [12] and the Hodge theory [13-15]. This gauge theory, endowed with the first class constraints [16], has also been discussed in the framework of BRST formalism [17-19]. The (anti-) BRST symmetry transformations for this theory, however, have been shown to be anticommuting only up to a $U(1)$ vector gauge transformation (see, e.g. [13]).

We have applied the superfield formalism to the 4D Abelian 2-form (and 3-form) gauge theories in our recent endeavor [20]. One of the key outcomes of our work in [20] is that the nilpotent (anti-) BRST symmetry transformations must be absolutely anticommuting because they are identified with the translational generators along the Grassmannian directions of the (4,2)-dimensional supermanifold on which the 4D theory is generalized. This aspect has been obtained because of the existence of a Curci-Ferrari (CF) type restriction [21] that emerges due to the application of superfield approach to the above 2-form gauge theory. As is well-known, the original CF condition [21] was invoked to ensure the anticommutativity of (anti-) BRST symmetry transformations in the context of non-Abelian 1-form gauge theory in 4D.

We have been able to capture the CF type restriction in the Lagrangian formulation and have shown explicitly the existence of the absolutely anticommuting (anti-) BRST transformations for the free 4D Abelian 2-form gauge theory [22]. Added to this, we have been able to demonstrate the connection of the CF type restriction to the concepts of gerbs that have become fairly relevant in the context of string theories. In our present investigation, we shall exploit the mathematical beauty of the coupled Lagrangian densities [22] and show their relevance in the context of anomalous 2D Abelian 1-form gauge theory [23-25] for the specific set of symmetry considerations.
It is interesting to point out that we have proposed, in our earlier works [26,14], an alternative set of Lagrangian densities for the 4D Abelian 2-form gauge theories which are more economical than the ones proposed in [22]. However, in our present endeavor, it is the Lagrangian densities of [22] that have the features that are reminiscent of the specific properties associated with the anomalous 2D Abelian gauge theory [25]. To be precise, as it turns out, under the ordinary $U(1)$ gauge transformations, the Lagrangian density of the bosonized version of the 2D anomalous gauge theory transforms to a total spacetime derivative plus a term which is nothing but the off-shoot of the Euler-Lagrange equations of motion, derived from the very same Lagrangian density. Exactly the same feature appears for the basic Lagrangian densities of the 4D Abelian 2-form gauge theory [22] under a specific set of symmetry transformations within the BRST approach (see, Appendices B and C below).

The central theme of our present investigation is to establish an underlying mathematical similarity between the free 4D Abelian 2-form gauge theory and the anomalous 2D Abelian 1-form gauge theory. For this purpose, we focus on the (anti-) BRST invariant Lagrangian densities, proposed in our earlier work [22], where a Lagrange multiplier vector field has been incorporated to obtain, in a single step, the CF type restriction that is required for the absolute anticommutativity of the nilpotent (anti-) BRST symmetries. In addition, a consistent transformation on this multiplier field ensures the perfect symmetry invariance of the coupled Lagrangian densities of the theory. The other set of Lagrangian densities, that are proposed in our earlier works [14,26], play no meaningful role in our present endeavor.

We demonstrate that, under the nilpotent (anti-) BRST and (anti-) co-BRST symmetry transformations, the basic Lagrangian densities of the 4D Abelian 2-form gauge theory transform to a total spacetime derivative plus a term which turns out to be the equation of motion for the same Lagrangian densities. This feature is exactly same as the one we encounter in the case of the bosonized version of the 2D anomalous Abelian 1-form gauge theory. To be precise, the Lagrangian density of the latter theory transforms exactly as the former theory under the (dual-) gauge, (anti-) BRST and (anti-) co-BRST symmetry transformations (cf. Sec. 3 below). Furthermore, we demonstrate that perfectly (anti-) BRST and (anti-) co-BRST invariant version of the free 4D Abelian 2-form and gauge invariant version of the 2D anomalous Abelian 1-form theories are the cute field theoretical models for the Hodge theory where symmetry considerations play an important role. We compare and contrast these theories in Sec. 4 and pin-point explicitly the high degree of similarities and decisive features of differences between them.

In our present endeavor, for the first time, we demonstrate the existence of the dual-gauge and dual-BRST symmetry transformations for the gauge
invariant version [29] of the anomalous 2D Abelian gauge theory. This 2D
gauge invariant and bosonized version of the chiral Schwinger model (CSM),
to the best of our knowledge, is proven to be a field-theoretic model for Hodge
theory for the first time. The physical state of the theory is chosen to be
the most symmetric (i.e. harmonic) state of any arbitrarily Hodge decom-
posed state (of the total quantum Hilbert space of states). The physicality
criteria on this state with the BRST and co-BRST charges demonstrate that
the anomalous 2D Abelian gauge theory is a consistent theory because the
physical (harmonic) state is annihilated by the individual terms (and their
time derivatives) of the expression that appears in the anomalous behavior
[23-25] of the 2D theory (see, Sec. 3 below for details).

Our present investigation is essential on the following counts. First and
foremost, it is always very important to explore a web of mathematical
and/or theoretical relationships between two different and distinct theories.
Our present paper does provide some mathematical similarities between 4D
Abelian 2-form gauge theory and the anomalous 2D Abelian 1-form gauge
theory. Second, we propose a set of different looking Lagrangian densities for
the Abelian 2-form gauge theory where the beauty of the mathematical prop-
erties of the (anti-) BRST and (anti-) co-BRST symmetries are exploited in
an elegant manner. These Lagrangian densities are different from our earlier
Lagrangian densities [22,26,14]. Both the above sets, however, have their
own importance and individuality. Third, we provide a new field theoretical
model for the Hodge theory in 2D which is inspired by our understanding
of the 4D Abelian 2-form gauge theory. The new field-theoretic model hap-
pens to be the gauge invariant version of the anomalous 2D gauge theory.
Four, the physicality condition on the harmonic state proves the consistency
of the anomalous 2D Abelian theory because the anomaly term and its time
derivative annihilate the physical (harmonic) state. Finally, we discuss, the
similarities and differences between the above two theories. These observa-
tions might turn out to be useful in our main goal of studying the higher
$p$-form ($p \geq 3$) gauge theories within the framework of BRST formalism.

Our present paper is organized as follows. Our second section is dedicated
to the description of the symmetry properties of the free 4D Abelian 2-form
gauge theory. This study, ultimately, enables us to prove that the present
theory is a field-theoretic model for Hodge theory. In Sec. 3, we discuss, in
detail, some of the key features associated with the gauge invariant version
of the anomalous 2D Abelian 1-form gauge theory. The subject matter of
our Sec. 4 concerns itself with the discussion of the striking similarities and
glaring differences between the above two theories. Finally, in our Sec. 5, we
summarize our key results, discuss a bit about some subtle issues present in
our endeavor and point out a few future directions for further investigations.
Our Appendix A provides a synopsis of the (dual-) gauge transformations that exist for the 4D Abelian 2-form gauge theory. In Appendices B and C, we discuss about the derivation of the coupled Lagrangian densities of this theory that respect nilpotent and absolutely anticommuting (anti-) BRST and (anti-) co-BRST symmetries together.

2 Free 4D Abelian 2-form gauge theory: symmetries

In this section, we first discuss the absolutely anticommuting (anti-) BRST and (anti-) co-BRST symmetry transformations in subsection 2.1. Our subsection 2.2 is devoted to the discussion of a bosonic symmetry transformation. In subsection 2.3, we discuss the discrete and ghost scale symmetry transformations. Finally, our subsection 2.4 deals with the algebraic structure obeyed by the symmetry operators.

2.1 Absolutely anticommuting (anti-) BRST and (anti-) co-BRST symmetries: Lagrangian formulation

The coupled Lagrangian densities, that respect the nilpotent and absolutely anticommuting (anti-) BRST as well as (anti-) co-BRST symmetry transformations together, are\[\] (see, Appendices B and C below for more details)

\[
\mathcal{L}_{(B, \bar{B})}^{(L, M)} = \frac{1}{2} \partial_{\mu} \phi_{2} \partial^{\mu} \phi_{2} - \frac{1}{2} B^{\mu} \epsilon_{\mu \nu \kappa \lambda} \partial_{\nu} B_{\kappa \lambda} + \frac{1}{2} (B \cdot B + \bar{B} \cdot B - \frac{1}{2} \partial_{\mu} \phi_{1} \partial^{\mu} \phi_{1} + \partial_{\mu} \bar{B}_{\mu} \partial^{\mu} \beta + \frac{1}{2} (\partial_{\mu} \bar{C}_{\nu} - \partial_{\nu} \bar{C}_{\mu})(\partial^{\mu} C^{\nu} + \partial_{\mu} \bar{C} + \rho) + \frac{1}{2} (\partial_{\mu} \bar{C} + \rho) \lambda \lambda + L^{\mu}(B_{\mu} - \bar{B}_{\mu} - \partial_{\mu} \phi_{1}) + M^{\mu}(B_{\mu} - \bar{B}_{\mu} - \partial_{\mu} \phi_{2}),
\]

(1)

\[
\mathcal{L}_{(B, \bar{B})}^{(L, M)} = \frac{1}{2} \partial_{\mu} \phi_{2} \partial^{\mu} \phi_{2} - \frac{1}{2} \bar{B}^{\mu} \epsilon_{\mu \nu \kappa \lambda} \partial_{\nu} B_{\kappa \lambda} + \frac{1}{2} (B \cdot B + \bar{B} \cdot B - \frac{1}{2} \partial_{\mu} \phi_{1} \partial^{\mu} \phi_{1} + \partial_{\mu} \bar{B}_{\mu} \partial^{\mu} \beta + \frac{1}{2} (\partial_{\mu} \bar{C}_{\nu} - \partial_{\nu} \bar{C}_{\mu})(\partial^{\mu} C^{\nu} + \partial_{\mu} \bar{C} + \rho) + \frac{1}{2} (\partial_{\mu} \bar{C} + \rho) \lambda \lambda + L^{\mu}(B_{\mu} - \bar{B}_{\mu} - \partial_{\mu} \phi_{1}) + M^{\mu}(B_{\mu} - \bar{B}_{\mu} - \partial_{\mu} \phi_{2}),
\]

(2)

\(^{1}\)We adopt here the notations such that Greek indices \(\mu, \nu, \kappa, \ldots = 0, 1, 2, 3\) stand for the spacetime directions of the 4D flat Minkowski manifold with a metric that possesses signatures (+1, -1, -1, -1) and the 4D Levi-Civita tensor \(\epsilon_{\mu \nu \kappa \lambda}\) is taken with convention \(\epsilon_{0123} = +1\). We also follow \(B \cdot \bar{B} = B_{\mu} \bar{B}_{\mu} \equiv B_{0} \bar{B}_{0} - B_{i} \bar{B}_{i}\), where Latin indices \(i, j, k, \ldots = 1, 2, 3\) correspond to the space directions only.
where \( L_\mu \) and \( M_\mu \) are the Lorentz vector Lagrange multiplier fields and \( B_\mu, \bar{B}_\mu \) are the Nakanishi-Lautrup type auxiliary Lorentz vector fields. The above vector fields are bosonic in nature. The Lorentz vector fermionic \((C_\mu^2 = \bar{C}_\mu^2 = 0, \ C_\mu C_\nu + C_\nu C_\mu = 0, \ C_\mu \bar{C}_\nu + \bar{C}_\nu C_\mu = 0, \ etc.)\) (anti-) ghost fields \((\bar{C}_\mu)C_\mu\) as well as the Lorentz scalar bosonic (anti-) ghost fields \((\bar{\beta})\beta\) are needed for the validity of unitarity (at any arbitrary order of the perturbative calculations). The auxiliary ghost fields \( \rho \) and \( \lambda \) are fermionic (i.e. \( \rho \lambda + \lambda \rho = 0, \ \rho^2 = \lambda^2 = 0 \)) in nature and massless (\( \Box \phi_1 = 0, \Box \phi_2 = 0 \)) scalar fields \( \phi_1 \) and \( \phi_2 \) are required for the stage-one reducibility in the theory (see, e.g. [14] for more discussions). It is to be noted that the totally antisymmetric curvature tensor \( H_{\mu
u\eta} = \partial_\mu B_{\nu\eta} + \partial_\nu B_{\eta\mu} + \partial_\eta B_{\mu\nu} \) is hidden in the above Lagrangian density in a subtle manner through \( \varepsilon_{\mu\nu\eta\kappa} \partial^\nu B^{\eta\kappa} = (1/3)\varepsilon_{\mu\nu\eta\kappa} H^{\nu\eta\kappa} \).

These Lagrangian densities, respecting maximum number of symmetries, are completely new for the 4D Abelian 2-form gauge theories which have totally different appearance than the ones proposed in [14,20,22,26]. It can be checked that the Lagrangian densities (1) and (2) respect the following off-shell nilpotent \((s_{ab}^2 = 0)\) and absolutely anticommuting \((s_b s_{ab} + s_{ab} s_b = 0)\) (anti-) BRST transformations \((s_{(a)b})\)

\[
\begin{align*}
 s_b B_\mu &= - (\partial_\mu C_\nu - \partial_\nu C_\mu), \quad s_b C_\mu = - \partial_\mu \beta, \quad s_b \bar{C}_\mu = - B_\mu, \\
 s_b \phi_1 &= \lambda, \quad s_b \bar{\beta} = - \rho, \quad s_b \bar{B}_\mu = - \partial_\mu \lambda, \quad s_b L_\mu = - \partial_\mu \lambda, \\
 s_b[\rho, \lambda, \beta, \phi_2, B_\mu, \bar{B}_\mu, M_\mu, H_{\mu\nu\eta\kappa}] &= 0, \quad (3)
\end{align*}
\]

\[
\begin{align*}
 s_{ab} B_\mu &= - (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu), \quad s_{ab} C_\mu = - \partial_\mu \bar{\beta}, \quad s_{ab} \bar{C}_\mu = + \bar{B}_\mu, \\
 s_{ab} \phi_1 &= \rho, \quad s_{ab} \beta = - \lambda, \quad s_{ab} B_\mu = \partial_\mu \rho, \quad s_{ab} L_\mu = - \partial_\mu \rho, \\
 s_{ab}[\rho, \lambda, \beta, \bar{\beta}, B_\mu, \bar{B}_\mu, \bar{B}_\mu, M_\mu, H_{\mu\nu\eta\kappa}] &= 0, \quad (4)
\end{align*}
\]

because of the fact that the following explicit transformations are valid:

\[
\begin{align*}
 s_b L_{(B,\bar{B})}^{(L,M)} &= - \partial_\mu \left[ (\partial^\nu C^\nu - \partial^\nu \bar{C}^\nu) B_\nu + \lambda B^\mu + \rho \partial^\mu \bar{\beta} \right], \quad (5)
\end{align*}
\]

\[
\begin{align*}
 s_{ab} L_{(B,\bar{B})}^{(L,M)} &= - \partial_\mu \left[ (\partial^\nu \bar{C}^\nu - \partial^\nu C^\nu) \bar{B}_\nu - \rho \bar{B}^\mu + \lambda \partial^\mu \bar{\beta} \right]. \quad (6)
\end{align*}
\]

Under the BRST and anti-BRST symmetry transformations, the curvature tensor \( H_{\mu\nu\eta} \) and the massless scalar field \( \phi_2 \) remain invariant. Thus, the total kinetic term, owing its origin to the exterior derivative, remains invariant under the (anti-) BRST symmetry transformations. It is, therefore, concluded that the (anti-) BRST symmetries are the analogue of the exterior derivative. For more discussion on this issue, we refer the reader to our earlier work [14].
It is to be remarked that the absolute anticommutativity of the (anti-) BRST symmetry transformations imply that only one of them would be really the analogue of the exterior derivative (see, equations (28), (69) below).

In a similar fashion, it can be seen that the following off-shell nilpotent \((s^2_{a(d)} = 0)\) and absolutely anticommuting \((s_d s_{ad} + s_{ad} s_d = 0)\) (anti-) co-BRST symmetry transformations \((s_{(a)d})\)

\[
\begin{align*}
s_d B_{\mu\nu} &= -\varepsilon_{\mu\nu\rho\sigma} \partial^\rho C^\sigma, \quad s_d C_{\mu} = -\partial_\mu \bar{\beta}, \quad s_d \bar{C}_{\mu} = -B_{\mu}, \\
s_d \phi_2 &= -\rho, \quad s_d \beta = -\lambda, \quad s_d \bar{B}_{\mu} = \partial_\mu \rho, \quad s_d M_\mu = -\partial_\mu \rho, \\
s_{ad} B_{\mu\nu} &= -\varepsilon_{\mu\nu\rho\sigma} \partial^\rho C^\sigma, \quad s_{ad} C_{\mu} = \partial_\mu \bar{\beta}, \quad s_{ad} \bar{C}_{\mu} = \bar{B}_{\mu}, \\
s_{ad} \phi_2 &= -\lambda, \quad s_{ad} \bar{B}_{\mu} = -\partial_\mu \lambda, \quad s_{ad} M_\mu = -\partial_\mu \lambda, \\
s_{ad} [\rho, \beta, \bar{B}_{\mu}, B_{\mu}, B_{\mu}, (\partial^\nu B_{\nu\mu}), L_\mu] = 0, 
\end{align*}
\tag{7}
\]

leave the Lagrangian densities quasi-invariant because

\[
\begin{align*}
s_d \mathcal{L}^{(L,M)}_{(B,B)} &= \partial_{\mu} \left[ (\partial^\nu \bar{C}^\nu - \partial^\nu \bar{C}^\nu) B_{\nu} - \rho B^\mu + \lambda \partial^\mu \bar{\beta} \right], \\
s_{ad} \mathcal{L}^{(L,M)}_{(B,B)} &= \partial_{\mu} \left[ (\partial^\nu C^\nu - \partial^\nu C^\nu) B_{\nu} + \lambda \bar{B}^\mu + \rho \partial^\mu \beta \right]. 
\end{align*}
\tag{9, 10}
\]

It is evident that the gauge-fixing term \((\partial^\nu B_{\nu\mu})\), owing its origin to the co-exterior derivative, and the field \(\phi_1\) remain invariant under the nilpotent (anti-) co-BRST symmetry transformations. It can be explicitly checked that \(\delta B^{(2)} = -* d * B^{(2)} \equiv (\partial^\nu B_{\nu\mu}) dx^\mu\) where \(\delta = -* d *\) is the co-exterior derivative and \(*\) is the Hodge duality operation. In fact, the nomenclature of (anti-) co-BRST symmetry transformations owes its origin to the co-exterior derivative (see, e.g. [14]). Thus, the (anti-) co-BRST symmetry transformations are the analogue of the co-exterior derivative of differential geometry. The absolute anticommutativity of the (anti-) co-BRST symmetry transformations, however, imply that only one (of these two transformations) would be identified with the co-exterior derivative (see, equations (28), (69) below).

### 2.2 Anticommutator of fermionic symmetries: a bosonic symmetry

Our present theory is endowed with a set of four fermionic type \((s^2_{(a)b} = 0, s^2_{(a)d} = 0)\) symmetry transformations \(s_{(a)b}\) and \(s_{(a)d}\). It can be explicitly checked that the following operator equations are true, namely;

\[
\begin{align*}
\{s_b, s_{ab}\} &= 0, \quad \{s_b, s_{ad}\} = 0, \quad \{s_d, s_{ab}\} = 0, \quad \{s_d, s_{ad}\} = 0, 
\end{align*}
\tag{11}
\]
when they are applied on any arbitrary field of the theory. Furthermore, we have to impose the field equations $B_\mu - \bar{B}_\mu - \partial_\mu \phi_1 = 0$ and $B_\mu - \bar{B}_\mu - \partial_\mu \phi_2 = 0$ for the validity of (11) which emerge from the Lagrangian densities (1) and/or (2) as equations of motion with respect to $L_\mu$ and $M_\mu$.

The operator $s_\omega = \{s_b, s_d\}$ is a bosonic type symmetry transformation. The following infinitesimal version of this transformation

$$s_\omega B_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + \varepsilon_{\mu \nu \rho \kappa} \partial^\rho B^\kappa,$$
$$s_\omega \left[ \rho, \lambda, \phi_1, \phi_2, \beta, \bar{\beta}, B_\mu, \bar{B}_\mu, B_\mu, L_\mu, M_\mu \right] = 0, \quad s_\omega \tilde{C}_\mu = \partial_\mu \rho, \quad (12)$$

leaves the Lagrangian density $L^{(L,M)}_{(B,B)}$ quasi-invariant because

$$s_\omega L^{(L,M)}_{(B,B)} = \partial_\mu \left[ B^\mu (\partial \cdot B) - B^\mu (\partial \cdot \bar{B}) - B^\kappa \partial^\mu B_\kappa + B^\kappa \partial^\mu \bar{B}_\kappa - \lambda \delta^\mu \rho + (\partial^\mu \lambda) \rho \right]. \quad (13)$$

Thus, transformations (12) are the symmetry transformation for our present theory because the action corresponding to the Lagrangian density (1) remains invariant under (12). These transformations are the analogue of the Laplacian operator and are same as in our earlier work [14].

The anticommutators of the fermionic transformations $s_{ad}$ and $s_{ab}$ leads to the derivation of an infinitesimal version of a bosonic symmetry transformations ($s_\omega$) as given below

$$s_\omega B_{\mu \nu} = - (\partial_\mu \bar{B}_\nu - \partial_\nu \bar{B}_\mu + \varepsilon_{\mu \nu \rho \kappa} \partial^\rho \bar{B}^\kappa), \quad s_\omega C_\mu = - \partial_\mu \lambda,$$
$$s_\omega \left[ \rho, \lambda, \phi_1, \phi_2, \beta, \bar{\beta}, B_\mu, \bar{B}_\mu, B_\mu, \bar{B}_\mu, L_\mu, M_\mu \right] = 0, \quad s_\omega \tilde{C}_\mu = - \partial_\mu \rho. \quad (14)$$

It is straightforward to check that $s_\omega + s_\bar{\omega} = 0$ on the constrained submainfold defined by the field equation $B_\mu - \bar{B}_\mu - \partial_\mu \phi_2 = 0$. Thus, we conclude that the bosonic transformations $s_\omega$ are not independent bosonic symmetry transformations $vis-a-vis$ transformations $s_\bar{\omega}$. In other words, we have the operator relationship $\{s_b, s_d\} = s_\omega \equiv - \{s_{ad}, s_{ab}\}$.

2.3 Ghost and discrete symmetries: ramifications

In the Lagrangian densities $L^{(L,M)}_{(B,B)}$ and $L^{(L,M)}_{(\bar{B},\bar{B})}$, the fields $\phi_1, \phi_2, B_{\mu \nu}, B_\mu, \bar{B}_\mu, B_\mu, \bar{B}_\mu, L_\mu, M_\mu$ have ghost number equal to zero and the (anti-) ghost fields $(\bar{\beta})\beta, (\bar{C}_\mu)C_\mu$ and $(\rho)\lambda$ have ghost number equal to $(\mp 2), (\mp 1)$ and $(\pm 1)$, respectively. The ghost part of the Lagrangian densities respect the following
infinitesimal transformations \((s_g)\) [14]

\[
\begin{align*}
  s_g \beta &= +2 \Sigma \beta, \quad s_g \bar{\beta} = -2 \Sigma \bar{\beta}, \\
  s_g C_\mu &= +\Sigma C_\mu, \\
  s_g \bar{C}_\mu &= -\Sigma \bar{C}_\mu, \\
  s_g \rho &= -\Sigma \rho, \quad s_g \lambda = +\Sigma \lambda,
\end{align*}
\]

(15)

where \(\Sigma\) is a global scale parameter. In the above, the numerical factors \((\pm 2)\) and \((\pm 1)\) denote the corresponding ghost number of the ghost field(s).

It is evident that the fields, having ghost number equal to zero, do not transform at all under the ghost transformations. Thus, we have the following infinitesimal ghost transformations \((s_g)\) for all such fields, namely;

\[
\begin{align*}
  s_g \Psi &= 0, \\
  \Psi &= B_{\mu\nu}, \phi_1, \phi_2, B_\mu, \bar{B}_\mu, B_\mu, \bar{B}_\mu, L_\mu, M_\mu.
\end{align*}
\]

(16)

We observe that the above Lagrangian densities (1) and (2) remain invariant under the transformations \((s_g)\) because \(s_g L^{(L,M)}_{(B,B)} = 0\) and \(s_g L^{(L,M)}_{(B,B)} = 0\).

The Lagrangian densities (1) and (2) also respect the following discrete symmetry transformations

\[
\begin{align*}
  B_{\mu\nu} &\rightarrow \mp \frac{i}{2} \varepsilon_{\mu\nu\rho\sigma} B^{\rho\sigma}, \\
  C_\mu &\rightarrow \pm i \bar{C}_\mu, \quad \bar{C}_\mu &\rightarrow \pm i C_\mu, \\
  \beta &\rightarrow \pm i \bar{\beta}, \quad \bar{\beta} &\rightarrow \mp i \beta, \\
  \phi_1 &\rightarrow \pm i \phi_2, \quad \phi_2 &\rightarrow \mp i \phi_1, \\
  \rho &\rightarrow \mp i \lambda, \quad \lambda &\rightarrow \pm i \rho, \\
  L_\mu &\rightarrow \mp i M_\mu, \quad M_\mu &\rightarrow \pm i L_\mu, \\
  B_\mu &\rightarrow \pm i \bar{B}_\mu, \quad B_\mu &\rightarrow \mp i B_\mu, \\
  \bar{B}_\mu &\rightarrow \pm i \bar{B}_\mu, \quad \bar{B}_\mu &\rightarrow \mp i B_\mu.
\end{align*}
\]

(17)

The above symmetry transformations play very important role in establishing a connection between the symmetries on the one hand and some key concepts of the differential geometry on the other. For instance, these discrete symmetry transformations are the analogue of the Hodge duality \((\ast)\) operation of differential geometry. Under two successive operations of the transformations (17), it is interesting to point out that the following relationships are true on the generic fields of the theory (see, e.g. [27] for details)

\[
\begin{align*}
  \ast (\ast B) &= +B, \quad B = B_{\mu\nu}, B_\mu, \bar{B}_\mu, B_\mu, \bar{B}_\mu, \phi_1, \phi_2, L_\mu, M_\mu, \beta, \bar{\beta}, \\
  \ast (\ast F) &= -F, \quad F = C_\mu, \bar{C}_\mu, \rho, \lambda,
\end{align*}
\]

(18)

where \((\ast)\) corresponds to the discrete symmetry transformations (17).

Thus, we note that the fermionic and bosonic fields of the theory transform in a different manner under a couple of successive operations of the discrete transformations. This observation plays an important role in the following operator relationship (with \(s^2_{(a)b} = 0, s^2_{(a)d} = 0\)):

\[
\begin{align*}
  s_{(a)d} &= \pm \ast s_{(a)b} \ast,
\end{align*}
\]

(19)
where ± signs, in the above, are decided by such signs in (18) and \(s_{(a)b}\) and \(s_{(a)d}\) are the symmetry transformations (3), (4), (7) and (8). It is evident that the above relationship is the analogue of the relationship between the cohomological operators \(\delta\) and \(d\) (i.e. \(\delta = \pm d\) with \(\delta^2 = d^2 = 0\)).

### 2.4 Conserved currents and charges: Noether theorem

According to Noether’s theorem, the continuous symmetry transformations \(s_{(a)b}, s_{(a)d}, s_g, s_\omega\) would lead to the derivation of the conserved currents as

\[
J^\mu_{(b)} = (\partial_\nu \tilde{C}^\nu - \partial_\nu C^\nu) \partial_\nu \beta - \varepsilon^{\mu\nu\rho\kappa}(\partial_\nu C_{\rho\kappa})B_\nu - \rho \partial^\mu \phi_1 - \lambda L^\mu, \quad (20)
\]

\[
J^\mu_{(ab)} = -\rho L^\mu - \rho \partial^\mu \phi_1 - \lambda \partial^\mu \tilde{\beta} - (\partial_\nu \tilde{C}^\nu - \partial_\nu C^\nu)(\partial_\nu \tilde{\beta}) - \varepsilon^{\mu\nu\rho\kappa}(\partial_\nu C_{\rho\kappa})\bar{B}_\nu - (\partial_\nu \tilde{C}^\nu - \partial_\nu C^\nu)\bar{B}_\nu, \quad (21)
\]

\[
J^\mu_{(d)} = (\partial_\nu \tilde{C}^\nu - \partial_\nu C^\nu)B_\nu - \varepsilon^{\mu\nu\rho\kappa}(\partial_\nu C_{\rho\kappa})B_\nu - \rho \partial^\mu \phi_2 + \rho M^\mu - \lambda \partial^\mu \tilde{\beta} - (\partial_\nu \tilde{C}^\nu - \partial_\nu C^\nu)(\partial_\nu \tilde{\beta}), \quad (22)
\]

\[
J^\mu_{(ad)} = (\partial_\nu \tilde{C}^\nu - \partial_\nu C^\nu)\bar{B}_\nu - \varepsilon^{\mu\nu\rho\kappa}(\partial_\nu C_{\rho\kappa})\bar{B}_\nu - \rho \partial^\mu \phi_2 + \lambda M^\mu + \rho \partial^\mu \tilde{\beta} - (\partial_\nu \tilde{C}^\nu - \partial_\nu C^\nu)(\partial_\nu \tilde{\beta}), \quad (23)
\]

\[
J^\mu_{(g)} = 2\beta \partial^\mu \tilde{\beta} - 2\tilde{\beta} \partial^\mu \beta + (\partial_\nu \tilde{C}^\nu - \partial_\nu C^\nu)\tilde{C}_\nu + (\partial_\nu \tilde{C}^\nu - \partial_\nu C^\nu)C_\nu + C^\mu \rho - \tilde{C}^\mu \lambda, \quad (24)
\]

\[
J^\mu_{(\omega)} = \varepsilon^{\mu\nu\rho\kappa}(\partial_\nu B_\kappa + (\partial_\nu B_\kappa)B_\nu) + \partial_\nu \left[ B^\mu B^\nu - B^\mu B^\nu \right] + (\partial_\nu \tilde{C}^\nu - \partial_\nu C^\nu)(\partial_\nu \lambda) - (\partial_\nu \tilde{C}^\nu - \partial_\nu C^\nu)(\partial_\nu \rho). \quad (25)
\]

It can be checked that \(\partial_\mu J^\mu_{(i)} = 0\) (i.e. \(i = b, ab, d, ad, g, \omega\)) if we use the equations of motion derived from \(\mathcal{L}_{(B,B)}^{(L,M)}\) and \(\mathcal{L}_{(B,B)}^{(L,M)}\). For instance, the equations of motion from \(\mathcal{L}_{(B,B)}^{(L,M)}\) are

\[
\varepsilon^{\mu\nu\rho\kappa}\partial_\nu B_\kappa + (\partial_\mu B^\nu - \partial_\nu B^\mu) = 0, \quad \varepsilon^{\mu\nu\rho\kappa}\partial_\nu B_\kappa - (\partial_\mu B^\nu - \partial_\nu B^\mu) = 0,
\]

\[
B_\mu = \frac{1}{2}(\partial_\mu \phi_1 - \partial_\nu B_\nu), \quad \bar{B}_\mu = -\frac{1}{2}(\partial_\mu \phi_1 + \partial_\nu B_\nu), \quad \partial \cdot B = 0, \quad \partial \cdot B = 0,
\]

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The above operator algebra is analogous to the algebra obeyed by the de Rham cohomological operators of differential geometry\(^2\) [30-32].

We have, ultimately, the following interpretations for our continuous and discrete symmetry transformations

\[ B_\mu = \frac{1}{2}(\partial_\mu \phi_2 - \frac{1}{2} \epsilon_{\mu
u\rho\sigma} \partial^\nu B^{\rho\sigma}), \quad \bar{B}_\mu = -\frac{1}{2}(\partial_\mu \phi_2 + \frac{1}{2} \epsilon_{\mu
u\rho\sigma} \partial^\nu B^{\rho\sigma}), \]

\[ \partial \cdot B = 0, \quad \partial \cdot \bar{B} = 0, \quad B_\mu - \bar{B}_\mu - \partial_\mu \phi_1 = 0, \quad B_\mu - \bar{B}_\mu - \partial_\mu \phi_2 = 0, \]

\[ L^\mu = \bar{B}^\mu, \quad M^\mu = -\bar{B}^\mu, \quad \partial \cdot L = 0, \quad \partial \cdot M = 0, \quad \Box C^\mu = \frac{1}{2} \partial^\mu (\partial \cdot C) \equiv \partial^\mu \lambda, \]

\[ \Box \beta = 0, \quad \Box \bar{\beta} = 0, \quad \Box \rho = 0, \quad \Box \lambda = 0, \quad \Box \phi_1 = 0, \quad \Box \phi_2 = 0. \quad (26) \]

For the Lagrangian density \( L^{(L,M)}_{(\bar{B}, \bar{B})} \), however, the equations of motion are same as the above except the following additional relationships

\[ \epsilon^\mu\nu\rho\sigma \partial_\eta \bar{B}_\kappa + (\partial^\mu \bar{B}^\nu - \partial^\nu \bar{B}^\mu) = 0, \quad L^\mu = -B^\mu, \]

\[ \epsilon^\mu\nu\rho\sigma \partial_\eta B_\kappa - (\partial^\mu \bar{B}^\nu - \partial^\nu \bar{B}^\mu) = 0, \quad M^\mu = B^\mu. \quad (27) \]

It is interesting to point out that the expressions for the conserved currents in (20)-(25) look somewhat different from our earlier work [14]. If we exploit the equations of motion \( L^\mu = \bar{B}^\mu, \quad M^\mu = -\bar{B}^\mu \) and the CF type restriction \( B_\mu - \bar{B}_\mu - \partial_\mu \phi_1 = 0 \), however, we find that the expression for \( J^\mu_{(b)} \) and \( J^\mu_{(ad)} \) become exactly same as in [14].

Similar is the situation with \( J^\mu_{(ab)} \) and \( J^\mu_{(ad)} \) if we use \( L^\mu = -B^\mu, \quad M^\mu = +B^\mu \) and \( B_\mu - \bar{B}_\mu - \partial_\mu \phi_2 = 0 \). Thus, we conclude that the charges (derived from these conserved currents) would be same as in [14] and their algebraic structure would be exactly identical to the ones obtained in [14]. Thus, the Lagrangian densities (1) and (2) provide a field theoretic model for the Hodge theory because the following operator algebra is satisfied, namely;

\[ s^2 (a)_b = 0, \quad s^2 (a)_{ad} = 0, \quad \{s_b, s_{ad}\} = \{s_d, s_{ab}\} = 0, \]

\[ s_\omega = \{s_b, s_d\} \equiv -\{s_{ab}, s_{ad}\}, \quad [s_\omega, s_r] = 0, \quad (r = b, ab, d, ad, g), \]

\[ [s_g, s_b] = +s_b, \quad [s_g, s_d] = -s_d, \quad [s_g, s_{ad}] = +s_{ad}, \quad [s_g, s_{ab}] = -s_{ab}. \quad (28) \]

The above operator algebra is analogous to the algebra obeyed by the de Rham cohomological operators of differential geometry\(^2\) [30-32].

We have, ultimately, the following interpretations for our continuous and discrete symmetry transformations

\(^2\) It is worthwhile to mention that on a compact manifold without a boundary, there are three cohomological operators \( d, \delta, \Delta \) of differential geometry. These are christened as the exterior derivative, co-exterior derivative and Laplacian operator, respectively. They follow the algebra \( d^2 = \delta^2 = 0, \Delta = (d+\delta)^2, [\Delta, d] = 0, [\Delta, \delta] = 0 \) where \( \delta = *d* \) on a 4D spacetime manifold. The \((*)\), in the above, corresponds to the Hodge duality operation.
(i) only one of the nilpotent and absolutely anticommuting (anti-) BRST symmetry transformations is the analogue of the nilpotent \((d^2 = 0)\) exterior derivative \(d\) of differential geometry,

(ii) only one of the nilpotent and absolutely anticommuting (anti-) co-\(\text{BRST}\) symmetry transformations are the analogue of the nilpotent \((\delta^2 = 0)\) co-exterior derivative \(\delta\) of differential geometry,

(iii) the anticommutator (i.e. \(\{s_b, s_d\} \equiv -\{s_{ad}, s_{ab}\}\)) of the two fermionic \((s^2_{(a)b} = 0, s^2_{(a)d} = 0)\) transformations leads to the definition of a bosonic symmetry transformation which is the analogue of Laplacian operator, and

(iv) the discrete symmetry transformations \((17)\) and ensuing equation \((18)\) provide us the analogue of the relationship between co-exterior derivative \((\delta)\) and exterior derivative \((d)\) (i.e. \(\delta = \pm \ast d\ast\)).

To sum up, we have the following mappings: 
\[ (s_b, s_{ad}) \rightarrow d, \] 
\[ (s_d, s_{ab}) \rightarrow \delta, \] 
\[ \{s_b, s_d\} = -\{s_{ad}, s_{ab}\} \rightarrow \Delta \text{ (see, also [14] for details).} \]

3 Anomalous 2D Abelian 1-form gauge theory

We discuss here the gauge and dual-gauge transformations and corresponding BRST and dual-BRST transformations for the Lagrangian density of the anomalous 2D Abelian 1-form theory and its gauge invariant version.

3.1 Gauge and dual-gauge transformations: a synopsis

Let us begin with the following effective Lagrangian density of the bosonized version of the anomalous 2D Abelian 1-form gauge theory\(^3\) [23-25]

\[
\mathcal{L}_{\text{eff}} = -\frac{1}{4} F^{\mu
u} F_{\mu\nu} + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{2} a e^2 A_{\mu} A^{\mu} + e (g^{\mu\nu} - \varepsilon^{\mu\nu}) \partial_{\mu} \phi A_{\nu},
\]

\[
\equiv \frac{1}{2} E^2 + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{2} a e^2 A_{\mu} A^{\mu} + e (g^{\mu\nu} - \varepsilon^{\mu\nu}) \partial_{\mu} \phi A_{\nu},
\]  

(29)

where the 1-form \((A^{(1)} = dx^\mu A_\mu)\) defines the gauge potential \(A_\mu\) and the 2-form \(dA^{(1)} = F^{(2)} = \frac{1}{2!}(dx^\mu \wedge dx^\nu)F_{\mu\nu}\) (with \(d = dx^\mu \partial_\mu, d^2 = 0\) as exterior derivative) leads to the definition of the curvature tensor \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\).

Here ‘\(a\)’ is the parameter that shows the ambiguity in the regularization of

\(^3\)We adopt here the convention such that Minkowski metric \((g^{\mu\nu})\) is with signature \((+1, -1)\) and the antisymmetric Levi-Civita tensor \(\varepsilon_{\mu\nu}\) is with \(\varepsilon_{01} = +1 = -\varepsilon^{01}, \varepsilon_{\mu\nu} \varepsilon^{\mu\lambda} = -\delta^\lambda_\nu, \) etc. In 2D spacetime, the field strength tensor \(F_{\mu\nu}\) has only electric field \((E)\) as its existing component and the mass dimension of \(A_\mu\) as well as \(\phi\) is zero (i.e. \([A_\mu] = [\phi] = 0\)) and that of the electric charge \(e\) is one (i.e. \([e] = [M]\)). Here the Greek indices \(\mu, \nu... = 0, 1\) and Latin indices \(i, j... = 1.\) Thus, we have \(\Box = \partial_0^2 - \partial_1^2\) and \((\partial \cdot A) = \partial_0 A_0 - \partial_1 A_1.\)
the fermion determinant when the fermionic chiral Schwinger model (CSM) is bosonized in terms of the scalar field $\phi$ and the derivative on it.

It is straightforward to note that under the following infinitesimal gauge transformations (with gauge parameter $\chi(x)$) (see, e.g. \cite{25})

$$\delta_g A_\mu = -\partial_\mu \chi(x), \quad \delta_g \phi = +e \chi(x),$$
$$\delta_g E \equiv \delta_g F_{\mu\nu} = 0, \quad \delta_g (\partial \cdot A) = -\Box \chi,$$

(30)

the Lagrangian density (29) transforms as

$$\delta_g \mathcal{L}_{\text{eff}} = -\partial_\mu [e^2 \varepsilon^{\mu\nu} \chi A_\nu + e^2 (a - 1) A^\mu \chi - e \varepsilon^{\mu\nu} \phi \partial_\nu \chi]$$
$$+ \ e^2 \chi [(a - 1)(\partial \cdot A) + \varepsilon^{\mu\nu} \partial_\mu A_\nu].$$

(31)

It can be easily seen that the curvature $F_{\mu\nu}$, owing its origin to the exterior derivative $d = dx^\mu \partial_\mu$ (with $d^2 = 0$), remains invariant under the gauge transformations (30). Furthermore, the following Euler-Lagrange equations of motion, derived from the Lagrangian density (29), namely;

$$\Box \phi + e(g^{\mu\nu} - \varepsilon^{\mu\nu}) \partial_\mu A_\nu = 0, \quad \partial_\mu F^{\mu\nu} + ae^2 A^\nu + e(g^{\nu\eta} + \varepsilon^{\nu\eta}) \partial_\eta \phi = 0,$$

(32)

imply the following relationship (for $e \neq 0$)

$$(a - 1) \partial_\mu A^\mu + \varepsilon^{\mu\nu} \partial_\mu A_\nu = 0.$$  

(33)

Thus, it is clear that, even though the 2D CSM is anomalous (i.e. endowed with the second-class constraints \cite{24}) it respects usual gauge symmetry if the equations of motion (32) are imposed. The relationship in (33) has also been shown to be true by exploiting the Hamiltonian formalism where the Hamiltonian density is shown to commute with the second-class constraints of the 2D anomalous gauge theory \cite{24}. At the moment, we do not know the key reason(s) behind the existence of a symmetry like (31) because the theory is endowed with only second-class constraints and, therefore, there should not exist any gauge type symmetry.

It is very interesting to check that, under the following dual-gauge transformations (with an infinitesimal parameter $\Sigma(x)$):

$$\delta_{dg} A_\mu = -\varepsilon_{\mu\nu} \partial^\nu \Sigma, \quad \delta_{dg} \phi = -\epsilon \Sigma, \quad \delta_{dg} E = \Box \Sigma, \quad \delta_{dg} (\partial \cdot A) = 0,$$

(34)

the Lagrangian density (29) transforms as follows

$$\delta_{dg} \mathcal{L}_{\text{eff}} = \partial_\mu \left[e^2 (a + 1) \varepsilon^{\mu\nu} A_\nu \Sigma - e^2 A^\mu \Sigma \right]$$
$$- e \varepsilon^{\mu\nu} \phi \partial_\nu \Sigma + E \partial^\mu \Sigma - (\partial^\mu E) \Sigma$$
$$+ e^2 \Sigma \left[\frac{\Box E}{e^2} + (\partial \cdot A) - (a + 1) \varepsilon^{\mu\nu} \partial_\mu A_\nu \right].$$

(35)
We christen these transformations as the dual-gauge transformations because it is the gauge-fixing term $\partial_\mu A^\mu \equiv (\partial \cdot A)$, owing its origin to the dual-exterior derivative, that remains invariant under (34). In explicit terms, it can be checked that $\delta A^{(1)} = - * d * (dx^\mu A_\mu) = (\partial \cdot A)$. The equations of motion (32) imply that the following relationship is true, namely;

$$\Box E_{e^2} + (\partial \cdot A) - (a + 1) \varepsilon^{\mu\nu} \partial_\mu A_\nu = 0,$$

(36)

where $\Box = \partial_0^2 - \partial_1^2$ is the d’Alembertian operator in 2D. Thus, we note that the Lagrangian density (29) of the 2D anomalous gauge theory respects

(i) the infinitesimal local gauge symmetry transformations (30), and

(ii) the infinitesimal dual-gauge symmetry transformations (34),

if we impose the equations of motion (32) (and their off-shoots (33), (36)). This observation is exactly same as the ones, we have encountered, in the context of the Abelian 2-form gauge theory (see, Appendix A).

Furthermore, it should be noted that we have taken the limit $a << 1$ so that $1/(a - 1) \sim -(1 + a)$. Throughout the whole body of our text, we shall stick to this assumption (i.e. $a << 1$). It is elementary, then, to check that, in this limit, we have $(\partial \cdot A) - (a + 1) \varepsilon^{\mu\nu} \partial_\mu A_\nu = 0$ that emerges from (33) and, furthermore, we also have $\Box E = 0$ which is valid only for $a << 1$. More discussions on the choice of this region of parameter space is given in Sec. 5.

Before we wrap up this subsection, we would like to mention, in passing, that the (dual-) gauge transformations for the 4D Abelian 2-form gauge theory has been discussed in [28] where we have obtained a specific set of restrictions on the infinitesimal local (dual-) gauge parameters for the (dual-) gauge invariance in the theory. These restrictions, however, can be converted into the product of the above local parameters and equations of motion corresponding to the specific fields of the theory. Thus, we claim that there is one-to-one correspondence between the above mentioned theories as far as the symmetry properties are concerned (see, Appendix A).

The similarities between the two theories motivate us to look for the existence of the (anti-) BRST and (anti-) co-BRST symmetries for the 2D theory. This is what precisely we do in our forthcoming subsections.

3.2 BRST and anti-BRST transformations: gauge invariant CSM

Corresponding to the gauge transformations (30), we have the off-shell nilpotent ($s^2_{(a)b} = 0$) and absolutely anticommuting ($s_b s_{ab} + s_{ab} s_b = 0$) (anti-) BRST symmetry transformations $s_{(a)b}$

$$s_b A_\mu = - \partial_\mu C, \quad s_b C = 0, \quad s_b \bar{C} = ib, \quad s_b b = 0, \quad s_b \phi = eC,$$
\[ s_b E = s_b F_{\mu \nu} = 0, \quad s_b (\partial \cdot A) = -\Box C, \quad s_b \theta = -e^2 C, \]
\[ s_{ab} A_\mu = -\partial_\mu \bar{C}, \quad s_{ab} \bar{C} = 0, \quad s_{ab} b = 0, \quad s_{ab} \phi = e \bar{C}, \]
\[ s_{ab} E = s_{ab} F_{\mu \nu} = 0, \quad s_{ab} (\partial \cdot A) = -\Box \bar{C}, \quad s_{ab} \theta = -e^2 \bar{C}, \quad (37) \]

under which, the following Lagrangian density (with an additional field \( \theta \))
\[ L_b = \frac{1}{2} E^2 + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} a e^2 A_\mu A^\mu + e(g^{\mu \nu} - \varepsilon^{\mu \nu}) \partial_\mu \phi A_\nu \]
\[ + \theta \left[ (a - 1)(\partial \cdot A) + \varepsilon^{\mu \nu} \partial_\mu A_\nu \right] + \frac{(a - 1)}{2e^2} \partial_\mu \theta \partial^\mu \theta \]
\[ + b(\partial \cdot A) + \frac{1}{2} b^2 + i \partial_\mu \bar{C} \partial^\mu C, \quad (38) \]

remains quasi-invariant because it changes to a total spacetime derivative as
\[ s_b L_b = \partial_\mu \left[ (1 - a)e^2 CA^\mu - e^2 \varepsilon^{\mu \nu} CA_\nu + e\varepsilon^{\mu \nu} \phi \partial_\nu C \right] + (1 - a) \theta \partial^\mu C - b \partial^\mu C, \quad (39) \]
\[ s_{ab} L_b = \partial_\mu \left[ (1 - a)e^2 \bar{C} A^\mu - e^2 \varepsilon^{\mu \nu} \bar{C} A_\nu + e\varepsilon^{\mu \nu} \phi \partial_\nu \bar{C} \right] + (1 - a) \theta \partial^\mu \bar{C} - b \partial^\mu \bar{C}. \quad (40) \]

It will be noted that the constrained relationship (33), which was derived in two steps from the Lagrangian density (29), is now derived in one step because the equation of motion with respect to \( \theta \), namely;
\[ \frac{(a - 1)}{e^2} \Box \theta = (a - 1) (\partial \cdot A) + \varepsilon^{\mu \nu} \partial_\mu A_\nu, \quad (41) \]
produces it if we set the limit \( \theta \to 0 \). Furthermore, it is worth emphasizing that the mass dimensions of \([\theta] = [M], [b] = [M], [C] = 0, [\bar{C}] = 0 \), etc., ensure the appropriate mass dimension of the Lagrangian density (38).

In the above, the \( b \) field is the Nakashii-Lautrup auxiliary field and \((\bar{C})C\) are the fermionic \( C^2 = \bar{C}^2 = 0, CC + \bar{C}C = 0 \) (anti-) ghost fields that are needed for the unitarity in the theory. In the Lagrangian density (38), it is clear that the gauge-fixing and Faddeev-Popov terms can be expressed as
\[ s_b s_{ab} \left( -\frac{i}{2} A_\mu A^\mu + \frac{1}{2} C \bar{C} \right) = b(\partial \cdot A) + \frac{1}{2} b^2 + i \partial_\mu \bar{C} \partial^\mu C. \quad (42) \]

If we do not incorporate the terms that contain \( \theta \) fields in (38), then, under the (anti-) BRST transformations, the Lagrangian density would transform
As

\[
\begin{align*}
    s_b L_{(b)}^{(\theta \to 0)} &= -\partial_{\mu}[e^2 \varepsilon^{\mu\nu} CA_{\nu} + e^2 (a-1) A^\mu C - e \varepsilon^{\mu\nu} \phi \partial_{\nu} C + b \partial^\mu C] \\
    + e^2 C[(a-1)(\partial \cdot A) + \varepsilon^{\mu\nu} \partial_{\mu} A_{\nu}], \\
    s_{ab} L_{(b)}^{(\theta \to 0)} &= -\partial_{\mu}[e^2 \varepsilon^{\mu\nu} \bar{C} A_{\nu} + e^2 (a-1) \bar{A}^\mu \bar{C} - e \varepsilon^{\mu\nu} \phi \partial_{\nu} \bar{C} + b \partial^\mu \bar{C}] \\
    + e^2 \bar{C}[(a-1)(\partial \cdot \bar{A}) + \varepsilon^{\mu\nu} \partial_{\mu} A_{\nu}].
\end{align*}
\]

This observation is exactly same as the one we have encountered in the context of the 4D Abelian 2-form gauge theory (Appendix B) where the Lagrangian density transforms to a total spacetime derivative plus a term that is found to be zero on-shell. Thus, to obtain a perfect BRST symmetry transformation, it is essential to add these additional terms containing \( \theta \).

There are a few points that have to be emphasized at this stage. First, the \( \theta \) field here is not a Lagrange multiplier field because it possesses a kinetic term and, therefore, is a dynamical field. Second, the motivation for adding the term \( (a-1)(\partial \cdot A) + \varepsilon^{\mu\nu} \partial_{\mu} A_{\nu} \) has come from the fact that (33) is the equation of motion from our starting Lagrangian density (29). We have incorporated this term in the Lagrangian density following our understanding of the Abelian 2-form gauge theory where we have incorporated the terms \( L^\mu(B_\mu - \bar{B}_\mu - \partial_\mu \phi_1) \) and \( M^\mu(B_\mu - \bar{B}_\mu - \partial_\mu \phi_2) \) in the Lagrangian densities (cf. (1), (2)). Finally, it can be seen that under the (anti-) BRST symmetry transformations, the equation (33) transforms as

\[
    s_{b}[(a-1)(\partial \cdot A) + \varepsilon^{\mu\nu} \partial_{\mu} A_{\nu}] = -(a-1)\Box C,
\]

and

\[
    s_{ab}[(a-1)(\partial \cdot A) + \varepsilon^{\mu\nu} \partial_{\mu} A_{\nu}] = -(a-1)\Box \bar{C}.
\]

This is why, a kinetic term for the field \( \theta \) has to be added in the Lagrangian density to achieve the perfect (anti-) BRST symmetries. Thus, we conclude that there is a perfect conceptual analogy between our present 2D theory and the 4D Abelian 2-form gauge theory (see, Sec. 2 and Appendices B, C). In the latter case, it was found that the equations of motion \( B_\mu - \bar{B}_\mu - \partial_\mu \phi_1 = 0 \) and \( \bar{B}_\mu - B_\mu - \partial_\mu \phi_2 = 0 \) were invariant under the nilpotent (anti-) BRST and (anti-) co-BRST symmetry transformations (i.e. \( s_{(a)b}[B_\mu - \bar{B}_\mu - \partial_\mu \phi_1] = 0 \) and \( s_{(a)d}[B_\mu - \bar{B}_\mu - \partial_\mu \phi_2] = 0 \). This is why there was no need to add a kinetic term for the Lagrange multiplier field \( L_\mu \) as well as \( M_\mu \).

Before we close up this subsection, it is worth noting that the Lagrangian density of [29], that results in from the gauge-invariant generating functional of the bosonized version of the CSM in 2D, is same as the one quoted in (38)
modulo some constant factors and the gauge-fixing and Faddeev-Popov ghost terms. However, the logic behind the derivation of the Lagrangian density (38) is totally different and it has emerged out from our understanding of the derivation of the Lagrangian densities (1) and (2) in the context of 4D free Abelian 2-form gauge theory (see, Sec. 2 and Appendices B and C). It is worthwhile to mention that the inclusion of the $\theta$ field has, in fact, rendered the second-class constraints of the original bosonized version of the CSM to the first-class constraints [29]. This is why, there is existence of a perfect (anti-) BRST symmetry invariance (cf. (37), (39) and (40)) in the theory.

3.3 (Anti-) dual BRST symmetry transformations: a discussion

The BRST invariant Lagrangian density $L_b$ is also endowed with the off-shell nilpotent $(s^2_{a,d}) = 0$ (anti-) dual BRST symmetry transformations $s_{(a,d)}$. For this purpose, we linearize the kinetic term ($-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}E^2$) by invoking an auxiliary field $\bar{b}$ in the following fashion

$$L_{(b,d)} = \bar{b}E - \frac{1}{2}\bar{b}^2 + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2}ae^2 A_\mu A^\mu + e(g^{\mu\nu} - \varepsilon^{\mu\nu})\partial_\mu \phi A_\nu,$$

$$+ \theta \left[(a - 1)(\partial \cdot A) + \varepsilon^{\mu\nu} \partial_\mu A_\nu\right] + \frac{(a - 1)}{2\varepsilon^2} \partial_\mu \theta \partial^\mu \theta,$$

$$+ b(\partial \cdot A) + \frac{1}{2}b^2 + i\partial_\mu \bar{C} \partial^\mu C. \quad (46)$$

It can be checked that the following nilpotent $(s^2_{(a,d)} = 0)$ and absolutely anticommuting $(s_d s_{ad} + s_{ad} s_d = 0)$ (anti-) dual BRST transformations $s_{(a,d)}$

$$s_d A_\mu = -\varepsilon_{\mu\nu} \partial^\nu \bar{C}, \quad s_d \bar{C} = 0, \quad s_d C = \bar{b}, \quad s_d \bar{b} = 0,$$

$$s_d \phi = -\bar{C}, \quad s_d E = \Box \bar{C}, \quad s_d (\partial \cdot A) = 0, \quad s_d b = 0,$$

$$s_d \theta = -\frac{e^2 \bar{C}}{(a - 1)} \cong e^2 \bar{C}(1 + a), \quad (47)$$

$$s_{ad} A_\mu = -\varepsilon_{\mu\nu} \partial^\nu C, \quad s_{ad} C = 0, \quad s_{ad} \bar{C} = -\bar{b}, \quad s_{ad} \bar{b} = 0,$$

$$s_{ad} \phi = -e C, \quad s_{ad} E = \Box C, \quad s_{ad} (\partial \cdot A) = 0, \quad s_{ad} b = 0,$$

$$s_{ad} \theta = -\frac{e^2 C}{(a - 1)} \cong e^2 C(1 + a), \quad (48)$$

leave the Lagrangian density (46) quasi-invariant because

$$s_d L_{(b,d)} = \partial_\mu \left[\bar{b} \partial^\mu \bar{C} - e \varepsilon^{\mu\nu} \phi \partial_\nu \bar{C} - \theta \partial^\mu \bar{C}\right],$$

$$- e^2 A^\mu \bar{C} + (a + 1)e^2 C \varepsilon^{\mu\nu} A_\nu \right], \quad (49)$$
Thus, the action corresponding to the above Lagrangian density \( L_{(b,d)} \) remains invariant under the (anti-) dual-BRST symmetry transformations \( s_{(a)d} \).

It is an interesting point to note that if \( \theta \) terms are not incorporated in the Lagrangian density \( L_{(b,d)} \), the latter would transform, under the (anti-) co-BRST symmetry transformations, as

\[
\begin{align*}
  s_{d}L_{(b,d)}^{(\theta=0)} &= \partial_{\mu} \left[ e^{2}(a + 1)\epsilon^{\mu\nu}A_{\nu}\bar{C} - e^{2}A^{\mu}\bar{C} - e\epsilon^{\mu\nu}\phi\partial_{\nu}\bar{C} + \partial_{\mu}\bar{C} \right] \\
  + &\ e^{2}\bar{C} \left[ (\partial \cdot A) - (a + 1)\epsilon^{\mu\nu}\partial_{\mu}A_{\nu} \right],
\end{align*}
\]

\[
\begin{align*}
  s_{ad}L_{(b,d)}^{(\theta=0)} &= \partial_{\mu} \left[ e^{2}(a + 1)\epsilon^{\mu\nu}A_{\nu}C - e^{2}A^{\mu}C - e\epsilon^{\mu\nu}\phi\partial_{\nu}C + \partial_{\mu}C \right] \\
  + &\ e^{2}C \left[ (\partial \cdot A) - (a + 1)\epsilon^{\mu\nu}\partial_{\mu}A_{\nu} \right],
\end{align*}
\]

which are the analogues of (43) where we have taken \((a - 1)^{-1} \sim -(1 + a)\) because of the fact that \(a \ll 1\). It is worth pointing out that the nature of transformations in the above is exactly same as the one, we have encountered in the context of Abelian 2-form gauge theory (see, Appendix C below).

The analogue of the equation (42) can be written in terms of the (anti-) dual-BRST symmetry transformations \((s_{(a)d})\) as

\[
\begin{align*}
  s_{d}s_{ad} \left( -\frac{i}{2}A_{\mu}A^{\mu} + \frac{1}{2}C\bar{C} \right) &= \bar{b}E - \frac{1}{2}\bar{b}^{2} + i\partial_{\mu}\bar{C}\partial^{\mu}C.
\end{align*}
\]

Thus, we note that the kinetic term for the gauge field \(A_{\mu}\) and the Faddeev-Popov ghost terms can be written in the exact-form with the help of the (anti-) co-BRST symmetry transformations. In the above form, the (anti-) co-BRST invariance of the Lagrangian density \(L_{(b,d)}\) becomes quite simple because of the nilpotency (i.e. \(s_{(a)d}^{2} = 0\), anticommutativity (i.e. \(s_{d}s_{ad} + s_{ad}s_{d} = 0\)) and the invariance of the gauge-fixing term (i.e. \(s_{a}b(\partial \cdot A) = 0, s_{(a)d}b = 0\)).

### 3.4 Bosonic symmetry: anticommutator of fermionic symmetries

In the context of the gauge invariant version of the 2D anomalous gauge theory, we have established the existence of four nilpotent (fermionic) symmetries (i.e. \(s_{(a)b}, s_{(a)d}\)). The following infinitesimal version of the bosonic (i.e. \(s_{\omega} = \{s_{b}, s_{a}\}\)) transformations \(s_{\omega}\)

\[
\begin{align*}
  s_{\omega}A_{\mu} &= -i(\epsilon_{\mu\nu}(\partial_{\nu}b + \partial_{\nu}\bar{b}) = -i\epsilon^{2}\left( \frac{b}{a - 1} + \bar{b} \right), \\
  s_{\omega}\theta &= -i\epsilon^{2}\left( \frac{b}{a - 1} + \bar{b} \right), \\
  s_{\omega}E &= i\square b, \\
  s_{\omega}\phi &= +i\epsilon(\bar{b} - b), \\
  s_{\omega}(\partial \cdot A) &= -i\square \bar{b}, \\
  s_{\omega}[C, \bar{C}, b, \bar{b}] &= 0.
\end{align*}
\]

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is the symmetry transformation in the theory because the Lagrangian density (46) transforms, under the above infinitesimal transformations, as

\[ s_\omega \mathcal{L}_{(b,d)} = \partial_\mu X^\mu, \]

\[ X^\mu = i(\bar{b}\partial^\mu b - b\partial^\mu \bar{b}) - i \theta \partial^\mu \left[b + (a - 1)\bar{b}\right] \]

\[ - i e^2 b A^\mu - i e^2 \varepsilon^\mu\nu \left[\bar{b} A_\nu + \frac{1}{(a - 1)} b A_\nu\right] \]

\[ - i e^2 (a - 1)\bar{b} A^\mu + i e \varepsilon^\mu\nu \phi \partial_\nu (\bar{b} - b). \] (54)

As a result, the action of the theory remains invariant under (53).

It can be explicitly checked that the anticommutators \( \{s_d, s_{ad}\} = 0, \{s_b, s_{ab}\} = 0, \{s_d, s_{ab}\} = 0 \). Thus, the remaining anticommutator \( \{s_{ad}, s_{ab}\} = s_\omega \) produces a bosonic symmetry transformation \( s_\omega \) which is not independent of \( s_{\bar{\omega}} \). It can be explicitly checked that

\[ (s_\omega + s_{\bar{\omega}}) \Omega = 0, \] (55)

where \( \Omega \) is an arbitrary generic field of the theory. This establishes the fact that \( s_{\bar{\omega}} = -s_\omega \). In other words, we have \( s_\omega = \{s_b, s_d\} \equiv -\{s_{ad}, s_{ab}\} \).

### 3.5 Ghost and discrete symmetries: outcomes

The bosonic fields \( A_\mu, \theta, \phi, b, \bar{b} \) of the theory have ghost number equal to zero whereas the fermionic fields \( C \) and \( \bar{C} \) have ghost number equal to \( \pm 1 \). Thus, we have the following ghost scale transformations

\[ A_\mu \rightarrow A_\mu, \quad \theta \rightarrow \theta, \quad \phi \rightarrow \phi, \quad b \rightarrow b, \]

\[ \bar{b} \rightarrow \bar{b}, \quad C \rightarrow e^{\pm \Lambda} C, \quad \bar{C} \rightarrow e^{\pm \Lambda} \bar{C}, \] (56)

under which the Lagrangian density \( \mathcal{L}_{(b,d)} \) remains invariant. The numbers \( \pm 1 \) in the exponentials of \( C \) and \( \bar{C} \) transformations correspond to the ghost numbers and \( \Lambda \) is a global infinitesimal scale parameter.

It can be, furthermore, checked that under the following discrete symmetry transformations

\[ a \rightarrow -a, \quad e \rightarrow \pm i e, \quad C \rightarrow \pm i \bar{C}, \quad \bar{C} \rightarrow \pm i C; \]

\[ b \rightarrow \pm i \bar{b}, \quad \bar{b} \rightarrow \pm i b, \quad A_\mu \rightarrow \pm i \varepsilon_{\mu\nu} A^\nu, \]

\[ \phi \rightarrow \phi, \quad \theta \rightarrow \pm \frac{i\theta}{(a + 1)} \cong \mp i\theta(a - 1), \] (57)
the Lagrangian density \( L_{(b,d)} \) remains invariant. Equivalently, under another discrete symmetry transformations

\[
\begin{align*}
a &\rightarrow -a, \quad e \rightarrow \pm ie, \quad C \rightarrow \pm i\bar{C}, \quad \bar{C} \rightarrow \pm iC, \\
b &\rightarrow \pm ib, \quad \bar{b} \rightarrow \pm ib, \quad A_\mu \rightarrow A_\mu, \quad \partial_\mu \rightarrow \pm i\varepsilon_\mu\nu\partial^\nu, \\
\phi &\rightarrow \phi, \quad \theta \rightarrow \pm \frac{i\theta}{a+1} \approx \mp i\theta(a-1),
\end{align*}
\]

(58)

the Lagrangian density \( L_{(b,d)} \) remains unchanged. The above discrete symmetry transformations (57) and (58) play very important role as it can be clearly seen that the following relationships are true, namely;

\[
s_{(a)b} \Omega = \pm \ast \, s_{(a)b} \ast \Omega, \quad \Omega = A_\mu, \phi, b, \bar{b}, C, \bar{C}, \theta, E, \quad (\partial \cdot A).
\]

(59)

where \( \Omega \) is an arbitrary generic field of the theory and the nilpotent transformations \( s_{(a)b} \) and \( s_{(a)d} \) are explicitly illustrated in (37), (47) and (48).

For the 2D theory, it can be also checked that the reverse relationship \( s_{(a)b} \Omega = \pm \ast \, s_{(a)d} \ast \Omega \) is also true (see, e.g. [27] for details).

In the above, the \((\ast)\) symbol corresponds to the discrete symmetry transformations (57) and/or (58) and the signs \((\pm)\) are dictated by such signs that appear in the two successive operations of \((\ast)\) as given below

\[
\ast \, [\ast \, (\Omega_1)] = \pm \Omega_1, \quad \Omega_1 = \phi, \\
\ast \, [\ast \, (\Omega_2)] = - \Omega_2, \quad \Omega_2 = A_\mu, b, \bar{b}, C, \bar{C}, \theta, (\partial \cdot A), E.
\]

(60)

It can be explicitly checked that, with \((\Omega = \Omega_1, \Omega_2)\), we have [27]

\[
\ast \, [\ast \, (\Omega_1)] = + \Omega_1, \quad \Omega_1 = \phi, \\
\ast \, [\ast \, (\Omega_2)] = - \Omega_2, \quad \Omega_2 = A_\mu, b, \bar{b}, C, \bar{C}, \theta, (\partial \cdot A), E.
\]

(61)

The above relations are true with respect to the discrete transformations (57). A bit different relation emerges with the transformations in (58) where we find \( \Omega_1 = \phi, A_\mu \) and \( \Omega_2 = b, \bar{b}, C, \bar{C}, \theta, (\partial \cdot A), E \). The relationship in (59) is the analogue of the relationship between the co-exterior derivative \((\delta)\) and the exterior derivative \(d\) (i.e. \( \delta = \pm \ast \, d \ast \)). Thus, we note that the analogue of the Hodge duality \((\ast)\) operation is the discrete symmetry transformations (57) and/or (58) for the 2D Abelian 1-form gauge theory.

3.6 Conserved charges and algebra: impacts

The continuous symmetry transformations, according to Noether’s theorem, lead to the conserved currents. These conserved currents, corresponding to the symmetry transformations \( s_{a(b)}, s_{a(d)}, s_\omega \) and ghost transformations are:

\[
J_\mu^{(b)} = ae^2CA^\mu + F^{\mu\nu}(\partial_\nu C) - b\partial_\mu C - (a - 1)C(\partial_\mu \theta) + eC(\partial_\mu \phi) - \theta\varepsilon^{\mu\nu}(\partial_\nu C) - e\varepsilon^{\mu\nu}\phi(\partial_\nu C),
\]

(62)
\[ J^\mu_{(d)} = e\varepsilon^{\mu\nu}(\partial_\nu \bar{C}) - ae^2\bar{C}\varepsilon^{\mu\nu}A_\nu - e\bar{C}(\partial^\mu \phi) + \bar{b}\partial^\mu \bar{C} - \bar{C}(\partial_\mu \theta) - \theta(a - 1)e\varepsilon^{\mu\nu}\partial_\nu \bar{C} - b\varepsilon^{\mu\nu}(\partial_\nu \bar{C}), \tag{63} \]

\[ J^\mu_{(\omega)} = i(e\bar{b} - b)\partial^\mu \phi + ie^2\bar{b}A^\mu + i\bar{b}\varepsilon^{\mu\nu}\partial_\nu \bar{b} - i\theta\varepsilon^{\mu\nu}\partial_\nu \bar{b} \]

\[ - i\bar{b}\varepsilon^{\mu\nu}\partial_\nu b + i\bar{b}\varepsilon^{\mu\nu}\phi\partial_\nu (b - \bar{b}) - i\left[ b + (a - 1)\bar{b} \right] \partial^\mu \theta \]

\[ - i(a - 1)\theta\varepsilon^{\mu\nu}\partial_\nu b - i\bar{e}^2\bar{b}\varepsilon^{\mu\nu}A_\nu, \tag{64} \]

\[ J^\mu_{(g)} = -i(C\partial^\mu \bar{C} + \bar{C}\partial^\mu C), \tag{65} \]

where \( J^\mu_{(g)} \) is the ghost Noether current. It will be noted that the expressions for \( J^\mu_{(ab)} \) and \( J^\mu_{(ad)} \) can be obtained from \( J^\mu_{(b)} \) and \( J^\mu_{(d)} \) by the replacements: \( C \rightarrow \bar{C} \) and \( C \rightarrow C \). The above currents are conserved because it can be checked that \( \partial_\mu J^\mu_{(i)} = 0 \) for \( i = b, ab, d, ad, \omega, g \). For this proof, however, the following equations of motion, emerging from \( \mathcal{L}_{(b,d)} \), have to be used:

\[ \varepsilon^{\mu\nu}\partial_\nu \bar{b} + (a - 1)\partial^\mu \theta - \varepsilon^{\mu\nu}\partial_\nu b + \partial^\mu \theta - ae^2A^\mu - e(g^{\mu\nu} + \varepsilon^{\mu\nu})\partial_\nu \phi = 0, \]

\[ \Box \theta = \frac{e^2}{(a - 1)}[(a - 1)(\partial \cdot A) + \varepsilon^{\mu\nu}\partial_\mu A_\nu], \quad b = - (\partial \cdot A), \quad \bar{b} = E, \]

\[ \Box \phi + e(g^{\mu\nu} - \varepsilon^{\mu\nu})\partial_\mu A_\nu = 0, \quad \Box C = \Box \bar{C} = 0, \quad \Box b = \Box \bar{b} = 0. \tag{66} \]

It is worth mentioning that we have always taken \( a < < 1 \) and consequently \( 1/(a - 1) \sim -(1 + a) \) has been used throughout the whole body of the text.

Using the equations of motion (66), it is straightforward to obtain the expression for the conserved charges \( Q_i = \int J^\mu_{(i)} dx \) \((i = b, ab, d, ad, \omega, g)\) as

\[ Q_b = \int dx \left[ \bar{b} \bar{C} - b \bar{C} \right], \quad Q_{ab} = \int dx \left[ \bar{b} C - b \bar{C} \right], \]

\[ Q_d = \int dx \left[ \bar{b} \bar{C} - b \bar{C} \right], \quad Q_{ad} = \int dx \left[ \bar{b} \bar{C} - b \bar{C} \right], \]

\[ Q_\omega = -i \int dx \left[ \bar{b} \bar{b} - b \bar{b} \right], \quad Q_g = -i \int dx \left[ C \bar{C} + \bar{C} \bar{C} \right]. \tag{67} \]

These conserved charges obey an algebra that is reminiscent of the algebra of the cohomological operators. These are succinctly expressed as

\[ Q^2_{a(b)} = 0, \quad Q^2_{a(d)} = 0, \quad [Q_\omega, Q_r] = 0, \quad (r = b, ab, d, ad, g), \]

\[ \{Q_b, Q_{ab}\} = 0, \quad \{Q_d, Q_{ad}\} = 0, \quad \{Q_b, Q_{ad}\} = 0, \]

\[ Q_\omega = \{Q_d, Q_b\} = -\{Q_{ad}, Q_{ab}\}, \quad \{Q_d, Q_{ab}\} = 0, \]

\[ i[Q_g, Q_b] = +Q_b, \quad i[Q_g, Q_{ab}] = -Q_{ab}, \]

\[ i[Q_g, Q_d] = -Q_d, \quad i[Q_g, Q_{ad}] = +Q_{ad}. \tag{68} \]
Thus, we note that there exists a two-to-one mapping between the conserved charges on the one hand and the de Rham cohomological operators on the other. This statement can be captured in the following set of equations:

\[
\begin{align*}
(Q_b, Q_{ad}) & \rightarrow d, \\
(Q_{ab}, Q_d) & \rightarrow \delta, \\
Q_\omega & = \{Q_b, Q_d\} = -\{Q_{ab}, Q_{ad}\} \rightarrow \Delta.
\end{align*}
\]  

(69)

It is clear, therefore, that the symmetries and conserved charges are the realizations of the de Rham cohomological operators. The physical reasons behind the mapping in (69) are exactly the same as the ones we have discussed in the context of the free 4D Abelian 2-form gauge theory (cf. Sec. 2).

If a state \(|\psi >_n\), in the quantum Hilbert space, has the ghost number equal to \(n\) (i.e. \(iQ_g|\psi > = n|\psi >\)), the following relationships turn out to be true if we exploit the algebraic relations (68), namely:

\[
\begin{align*}
iQ_gQ_b & |\psi >_n = (n + 1)Q_b|\psi >_n, \\
iQ_gQ_d & |\psi >_n = (n - 1)Q_d|\psi >_n, \\
iQ_gQ_{ab} & |\psi >_n = (n - 1)Q_{ab}|\psi >_n, \\
iQ_gQ_{ad} & |\psi >_n = (n + 1)Q_{ad}|\psi >_n, \\
iQ_gQ_\omega & |\psi >_n = nQ_\omega|\psi >_n.
\end{align*}
\]  

(70)

Thus, the ghost numbers of the states \(Q_b|\psi >_n, Q_d|\psi >_n\) and \(Q_\omega|\psi >_n\) are \((n + 1), (n - 1)\) and \(n\), respectively. This observation is the analogue of the basic facts connected with the differential geometry where the degree of an \(n\)-form \((f_n)\) increases by one, decreases by one and remains intact due to the operations of the exterior, dual-exterior and the Laplacian operator, respectively. That is to say, in the mathematical terms, we have: \(df_n \sim f_{n+1}\), \(\delta f_n \sim f_{n-1}\) and \(\Delta f_n \sim f_n\), respectively.

One of the decisive features of the present 2D model of the Hodge theory is that, under the discrete symmetry transformations (57) and/or (58), we have the following relationships:

\[
\begin{align*}
*Q_b & = + Q_d, \quad *Q_d = + Q_b, \quad *Q_\omega = + Q_\omega, \\
*Q_g & = - Q_g, \quad *Q_{ab} = + Q_{ad}, \quad *Q_{ad} = + Q_{ab}.
\end{align*}
\]  

(71)

This feature is distinctly different [14] from the 4D Abelian 2-form gauge theory where \(*Q_b = +Q_d, *Q_d = -Q_b, *Q_\omega = -Q_\omega, *Q_{ad} = -Q_{ab}, *Q_{ab} = +Q_{ad}, *Q_g = -Q_g\). This difference is connected with the dimensionality of the two different theories [27]. It is interesting to point out that the total algebra (68) remains invariant under the \((*)\) operation corresponding to the discrete symmetry transformations listed in (57) and/or (58).
3.7 Physical state as a harmonic state: consequences

It is worth pointing out that, consistent with the algebraic structures in (68), (69) and (70), one can write an arbitrary state $| \psi >_n$ (with ghost number $n$) in the quantum Hilbert space, as the following sum

$$| \psi >_n = | h >_n + Q_b | \chi >_{(n-1)} + Q_d | \xi >_{(n+1)}$$

$$\equiv | h >_n + Q_{ad} | \chi >_{(n-1)} + Q_{ab} | \xi >_{(n+1)},$$

(72)

where, in the first line, the state $Q_b | \chi >_{(n-1)}$ is a BRST exact state, the state $Q_d | \xi >_{(n+1)}$ is the BRST co-exact state and $| h >_n$ is the harmonic state. A similar kind of statement can be made for the second line. The above equation is the analogue of the Hodge decomposition theorem (HDT) [30-32] which states that any arbitrary $n$-form $f_n$, on a compact manifold without a boundary, can be uniquely written as the sum of a harmonic form $h_n$ with $(\Delta h_n = 0, \partial h_n = 0, \delta h_n = 0)$, an exact form $(d e_{n-1})$ and a co-exact form $(\delta c_{n+1})$. Mathematically, this statement can be expressed as

$$f_n = h_n + d e_{n-1} + \delta c_{n+1}.$$  

(73)

Due to the two-to-one mapping (cf. (69)), however, the HDT can be expressed in two different ways in the quantum Hilbert space of states. Taking the help of mapping in (69), we have captured this statement in (72).

The most symmetric state, in the quantum Hilbert space of the states, is the harmonic state $| h >_n$ in (72) which is annihilated by $Q_{(a)b}, Q_{(a)d}$ and $Q_{\omega}$. We choose this state as the physical state of the theory (i.e. $| h >_n \equiv | \text{phys} >$). This immediately implies that

$$Q_{\omega} | \text{phys} > = 0, \quad Q_{(a)b} | \text{phys} > = 0, \quad Q_{(a)d} | \text{phys} > = 0. \quad (74)$$

It will be noted that all the above restrictions are consistent with one-another. The latter two relations, in the above, produce the following restrictions on the physical state (that are different from the ghost states), namely;

$$b | \text{phys} > = 0, \quad \dot{b} | \text{phys} > = 0, \quad \ddot{b} | \text{phys} > = 0,$$  

(75)

so that the physical state could become symmetric with respect to the nilpotent and conserved (anti-) BRST and (anti-) co-BRST charges.

It is evident from the equations of motion (66) that the above restriction in (75) imply the following restrictions on the physical state

$$(\partial \cdot A) | \text{phys} > = 0, \quad \partial_0 (\partial \cdot A) | \text{phys} > = 0,$$

$$E | \text{phys} > = 0, \quad \dot{E} | \text{phys} > = 0.$$  

(76)
Thus, we notice that the anomalous behavior, that appears in the r.h.s. of the conservation law $\partial_\mu J^\mu \sim [(a-1)(\partial \cdot A) + \varepsilon^{\mu\nu}\partial_\mu A_\nu]$ (see, e.g. [24,25]), is trivially zero because of the physicality condition. Here $J^\mu$ is defined through $\partial_\nu F^\nu_\mu = J^\mu$ (which is also equivalent to $\varepsilon^{\mu\nu}\partial_\nu \bar{b} = -J^\mu$ because of the equations of motion and the observation that $F^\mu_\nu = \varepsilon^\mu_\nu \bar{b}$). The above statement is true because this conservation law is valid in the quantum Hilbert space as

$$<\text{phys}| \partial_\mu J^\mu |\text{phys}> \sim <\text{phys}| [(a-1)(\partial \cdot A) + \varepsilon^{\mu\nu}\partial_\mu A_\nu] |\text{phys}>. \quad (77)$$

However, as we have seen that $Q^{(a)b}|\text{phys}>=0 \Rightarrow (\partial \cdot A)|\text{phys}>=0, \partial_\nu (\partial \cdot A)|\text{phys}>=0$ and $Q^{(a)d}|\text{phys}>=0 \Rightarrow E|\text{phys}>=0, \dot{E} |\text{phys}>=0$, it is clear that the individual terms of the anomalous expression (and their time derivatives, too) annihilate the physical state of the theory.

It should be mentioned that the above statements are valid in the limit $\theta \to 0$ which corresponds to the true anomalous 2D Abelian 1-form gauge theory. On the face value, the $\theta$-dependent terms do not appear in the expressions for $Q^{(a)b}$ and $Q^{(a)d}$. However, they turn up in the expressions for the time derivatives of $(\partial \cdot A)$ and $E = -\varepsilon^{\mu\nu}\partial_\mu A_\nu$ due to the dynamical equations of motion listed in (66). Thus, we conclude that the anomalous 2D Abelian 1-form gauge theory is a consistent theory because of the physicality conditions on the harmonic state with the (anti-) BRST and (anti-) co-BRST charges (which are conserved and nilpotent of order two).

4. Similarities and differences: a bird’s-eye view

The two theories, under discussion, are completely different theories in different dimensions of spacetime. Thus, there are bound to be too many differences. However, the interesting and amazing aspects of these theories are that they have some common points of similarities. We point out here some striking similarities and key conceptual differences between these theories. In particular, we concentrate more on the common features of similarity and focus only on the conceptual issues as far as the differences are concerned.

The first and foremost aspect of similarity is the nature of the transformations of the Lagrangian densities under the (anti-) BRST and (anti-) co-BRST symmetry transformations. It can be seen from equations (92), (93), (31) and (43) that, under the nilpotent and absolutely anticommuting (anti-) BRST symmetry transformations, the Lagrangian densities of the two theories transform to a total spacetime derivative plus a term that is proportional to one of the equations of motion (see, Appendix B for (92) and (93)). In exactly similar fashion, from equations (105), (106), (35) and (51), it can be noted that the Lagrangian densities of the two theories behave in exactly
the same manner under the nilpotent and absolutely anticommuting (antii-)co-BRST symmetry transformations (see, Appendix C for (105) and (106)).

The second feature that draws our attention is that, for the existence of the perfect symmetry invariance, we incorporate a couple of terms (e.g. $L^\mu (B_\mu - \bar{B}_\mu - \partial_\mu \phi_1)$, $M^\mu (B_\mu - \bar{B}_\mu - \partial_\mu \phi_2)$ in the Lagrangian densities, through the Lagrange multiplier fields, in the case of the free 4D Abelian 2-form gauge theory (cf. (1),(2)). The above logic of the 4D Abelian 2-form theory, with a bit of modification, also works in the case of anomalous 2D Abelian 1-form gauge theory. In fact, to begin with, we add a term proportional to the equation of motion (i.e. $\theta((a - 1)(\partial \cdot A) + \varepsilon^{\mu\nu} \partial_\mu A_\nu]$ with $\theta$ as a Lagrange multiplier field) in the Lagrangian density of the original theory. However, this turns out to be insufficient for our purpose. The above features are totally different from our understanding of the 4D (non-)Abelian 1-form gauge theories where there is absolutely no need of any kind of multiplier fields (see, e.g., [3,4] for details).

Despite our logic being same for both the theories, a bit of difference crops up because of the following reasons. It is straightforward to note that the field equations $B_\mu - \bar{B}_\mu - \partial_\mu \phi_1$ and $B_\mu - \bar{B}_\mu - \partial_\mu \phi_2$ remain off-shell invariant under the nilpotent (anti-)BRST and (anti-)co-BRST transformations. The above statement can be mathematically expressed as

$$s\,(a)\,(b)\, [B_\mu - \bar{B}_\mu - \partial_\mu \phi_1] = 0, \quad s\,(a)\,(b)\, [B_\mu - \bar{B}_\mu - \partial_\mu \phi_2] = 0,$$

$$s\,(a)\,(d)\, [B_\mu - \bar{B}_\mu - \partial_\mu \phi_1] = 0, \quad s\,(a)\,(d)\, [B_\mu - \bar{B}_\mu - \partial_\mu \phi_2] = 0,$$

where $s\,(a)\,(b)$ and $s\,(a)\,(d)$ are given in (3), (4), (7) and (8). The same does not hold good with the field equation $(a - 1)(\partial \cdot A) + \varepsilon^{\mu\nu} \partial_\mu A_\nu$ in the context of anomalous 2D Abelian 1-form theory. This statement, besides (44) and (45), can be mathematically stated as

$$s\,(d)\,[(a - 1)(\partial \cdot A) + \varepsilon^{\mu\nu} \partial_\mu A_\nu] = -\Box \bar{C},$$

$$s\,(a)\,(d)\,[(a - 1)(\partial \cdot A) + \varepsilon^{\mu\nu} \partial_\mu A_\nu] = -\Box C.$$  

Thus, we note that, in the context of anomalous 2D Abelian theory, the equation of motion $(a - 1)(\partial \cdot A) + \varepsilon^{\mu\nu} \partial_\mu A_\nu$ remains invariant under the (anti-) BRST and (anti-) co-BRST symmetry transformations only on the on-shell (i.e. $\Box C = \Box \bar{C} = 0$) for $a \neq 1$ (cf. (44),(45),(79)).

This is the reason that a “kinetic” piece, corresponding to the field $\theta$, has to be incorporated in the Lagrangian density for the perfect symmetry invariance in the context of anomalous 2D Abelian 1-form gauge theory (cf. (46)). However, such addition makes the $\theta$-field as a dynamical (propagating) field in the theory. It is to be emphasized, at this juncture, that the logic
behind the derivation of the Lagrangian densities (1), (2) and (46) for the 4D Abelian 2-form and anomalous 2D Abelian 1-form theories is the same. Thus, there is a striking similarity between these two theories. It should be re-emphasized that the above features are completely different from our understanding of the 4D (non-)Abelian 1-form gauge theories where there is no need to incorporate any kind of CF type restriction explicitly in the (anti-) BRST invariant Lagrangian density of the above theories [3,4].

The third point of similarity between the two theories is the observation that the modified Lagrangian densities (cf. (1),(2),(46)) of the two theories are endowed with continuous symmetry transformations and discrete symmetry transformations which render them to be a field theoretic-model for the Hodge theory. Of course, the original anomalous 2D Abelian 1-form theory is described by the Lagrangian density that is a limiting case of the Lagrangian density (46) when \( \theta \to 0 \). However, the point to be noted is that both the theories, in some sense, are the modified versions of the basic theories (as far as the true philosophy of BRST formalism is concerned).

At the conceptual level, we enumerate here a few key differences between the two theories. Both the theories are drastically different in the sense that the free 4D Abelian 2-form gauge theory is endowed with the first-class constraints (see, e.g. [16]) but the original anomalous 2D Abelian 1-form gauge theory possesses only second-class constraints [24] in the language of Dirac’s prescription for classification scheme. Furthermore, they exist in different dimensions of the spacetime. They are rendered to be the models for the Hodge theory through symmetry considerations. However, the methods to achieve the perfect symmetries, in both the theories, are different in the sense that the former needs only the Lagrange multipliers fields but the latter requires the “kinetic” piece for the “Lagrange multiplier” field as well.

The CF type restrictions, in the context of the 4D 2-form theory, play double roles because, not only they accomplish the anticommutativity of the (anti-) BRST and (anti-) co-BRST symmetries, but they also render the theory to possess the maximum number of perfect symmetries. The role of the equation of motion \((a - 1)(\partial \cdot A) + \epsilon^{\mu\nu} \partial_\mu A_\nu = 0\), on the other hand, is totally different in the context of anomalous 2D Abelian 1-form theory. Whereas the CF type restrictions (i.e. \(B_\mu - \bar{B}_\mu - \partial_\mu \phi_1 = 0\) and \(\mathcal{B}_\mu - \bar{\mathcal{B}}_\mu - \partial_\mu \phi_2 = 0\)) are derived directly from the Lagrangian densities (1) and (2), the condition \((a - 1)(\partial \cdot A) + \epsilon^{\mu\nu} \partial_\mu A_\nu = 0\) emerges from (66) as the limiting case when \( \theta \to 0 \). Furthermore, in the proof of consistency of the anomalous 2D Abelian 1-form gauge theory the latter condition plays an important role (i.e. \(\partial_\mu J^\mu \sim (a - 1)(\partial \cdot A) + \epsilon^{\mu\nu} \partial_\mu A_\nu\)). We have briefly commented about it through the physicality condition with the conserved and nilpotent (anti-) BRST and (anti-) co-BRST charges (see, Subsec. 3.7, for details).
5. Summary and discussion

In our present investigation, we have demonstrated the similarity of the coupled Lagrangian densities of the free 4D Abelian 2-form gauge theory [22] with the Lagrangian density of the anomalous 2D Abelian gauge theory under a specific set of symmetry transformations. To be precise, we have established that the basic Lagrangian densities\(^4\) of the 4D Abelian 2-form gauge theory transform, under the (anti-) BRST and (anti-) co-BRS T symmetry transformations, to a total spacetime derivative plus a term that is zero on the equations of motion that are derived from the coupled Lagrangian densities (cf. (92)-(95), (105)-(108) in Append. B and C). This feature is exactly the same as the nature of transformations in the context of the anomalous 2D Abelian 1-form gauge theory under the (dual-) gauge, (anti-) BRST and (anti-) co-BRST symmetry transformations (cf. (31),(43),(51)).

It is to be noted that, only in the context of the 4D Abelian 2-form gauge theory, the extra pieces (e.g. \(L^\mu (B_\mu - \bar{B}_\mu - \partial_\mu \phi_1)\) and \(M^\mu (B_\mu - \bar{B}_\mu - \partial_\mu \phi_2)\)) have to be incorporated in the Lagrangian densities (cf. (1),(2)) for the perfect symmetry invariance. On the contrary, such kind of extra pieces are absolutely not required for the symmetry invariance in the context of 4D (non-)Abelian 1-form gauge theories (see, e.g. [3,4]). In fact, the analogue of (42), in the case of 4D (non-)Abelian 1-form gauge theories, is good enough for the perfect symmetry invariance. It has been claimed in our earlier work [22] that the CF type restrictions, in the context of the free 4D Abelian 2-form gauge theory, have deep connection with the concept of gerbes and they would always appear in the context of higher-form (\(p \geq 2\)) gauge theories. In our recent works [20], the above claim has been shown to be true in the case of the free 4D Abelian 3-form gauge theory.

To obtain the perfect symmetry invariance, we have introduced a pair of Lagrange multiplier fields (i.e. \(L^\mu\) and \(M^\mu\)) for the Abelian 2-form gauge theory. A noteworthy point is that, the “kinetic terms” for these multiplier fields, are not required for the perfect symmetry invariance in the theory. This is due to the fact that the CF type restrictions (i.e. equations of motion) remain absolutely invariant under the (anti-) BRST and (anti-) co-BRST symmetry transformations (cf. (78)). We follow the same trick in the context of anomalous 2D Abelian theory and introduce a Lagrange multiplier field \(\theta\). However, the constraint conditions (i.e. equations of motion) are not found

\(^4\)We call the Lagrangian densities (91) as basic because these are similar to the Lagrangian densities of 4D (non-)Abelian 1-form gauge theories (having no interaction with matter fields) [3,4]. In the latter theories there is no need of any Lagrange multiplier fields.
to be absolutely invariant under the (anti-) BRST as well as (anti-) co-BRST symmetry transformations. Rather, they are found to be invariant only on the on-shell conditions $\Box C = \Box \bar{C} = 0$ (see, Sec. 4 for details).

To circumvent the above difficulty, we have added a kinetic piece for the field $\theta$ to obtain the perfect symmetry invariance in the theory. As a consequence, the $\theta$-field becomes a propagating (dynamical) field and it behaves, no longer, as a Lagrange multiplier field. In fact, it is due to the presence of the $\theta$-terms that we have been able to show the existence of the (anti-) BRST and (anti-) co-BRST symmetry transformations for the modified version of the anomalous 2D Abelian 1-form gauge theory.

The existence of the dual-gauge and (anti-) co-BRST symmetry transformations is a completely new result as far as the modified version of the anomalous 2D Abelian 1-form theory is concerned. In fact, these symmetry transformations enable us to prove that the system, described by the Lagrangian density (46), provides a new field-theoretic model for the Hodge theory. In this context, it is pertinent to point out that, so far, we have been able to prove the following field-theoretic models for the Hodge theory:

(i) the free 2D (non-)Abelian 1-form gauge theories without any interaction with matter fields [33-35],

(ii) the interacting 2D $U(1)$ Abelian gauge theory with matter fields as Dirac fields [36,37], and

(iii) the free 4D Abelian 2-form gauge theory [13-15].

One of the key assumptions, in our present investigation, has been the choice of the ambiguity parameter $a$ to be in the region $a \ll 1$. In this context, it is to be pointed out that, in a very recent work [38], it has been demonstrated, with the help of the numerical computation, that $a = 1$ is an exceptional point in the theory. We have avoided this point by our choice $a \ll 1$ and have confined ourselves to the region of the parameter space where the modified version of the anomalous 2D theory respects maximum symmetries which render it to become a model for the Hodge theory.

At this juncture, it is worthwhile to mention that we have shown the existence of the dual-BRST symmetry transformations in the context of the 2D QED with Dirac fields [36,37]. The latter fields undergo an (anti-) BRST version of the chiral transformations corresponding to the (anti-) co-BRST symmetry transformations (i.e. $s_d A_\mu = -\varepsilon_{\mu\nu} \partial^\nu \bar{C}$, $s_{ad} A_\mu = -\varepsilon_{\mu\nu} \partial^\nu C$) on the $U(1)$ gauge field $A_\mu$ (which couples with the matter (Dirac) fields in a $U(1)$ gauge invariant fashion). In these works [36,37], there is no presence of any ambiguity parameter $a$ and, therefore, there is no restriction of any kind. The interesting point is that, even in this work, the expressions for the nilpotent (anti-) dual-BRST charges are same as in (67). As a consequence, the physicality condition on the harmonic (physical) state with these conserved and
nilpotent charges is: \( Q_{(a)\mu} |_{\text{phys}} = 0 \Rightarrow E |_{\text{phys}} = 0, \dot{E} |_{\text{phys}} = 0 \) where \( E \sim \varepsilon^{\mu\nu} F_{\mu\nu} \) is the anomaly term in 2D.

It is to be emphasized that the consistency of the anomalous 2D Abelian 1-form gauge theory is encoded in the physicality condition with the conserved and nilpotent (anti-) BRST and (anti-) co-BRST charges. The anomalous terms, which are on the r.h.s. of the conservation law \( \partial_{\mu} J^\mu \sim (a - 1)(\partial \cdot A) + \varepsilon^{\mu\nu} \partial_{\mu} A_{\nu} \) (see, e.g. [24],[25]), individually annihilate the harmonic (physical) state of the theory due to \( Q_{(a)b} |_{\text{phys}} = 0, Q_{(a)d} |_{\text{phys}} = 0 \). Furthermore, these restrictions remain invariant w.r.t. the time-evolution of the system because the physicality condition implies that \( (\partial \cdot A) |_{\text{phys}} = 0, \partial_{\mu}(\partial \cdot A) |_{\text{phys}} = 0 \) as well as \( (-\varepsilon^{\mu\nu} \partial_{\mu} A_{\nu} \equiv E) |_{\text{phys}} = 0, \dot{E} |_{\text{phys}} = 0 \).

The precise reasons behind the similarity between the anomalous 2D Abelian 1-form gauge theory and the free 4D Abelian 2-form gauge theory are not clear to us at the moment. This issue is an interesting problem for our future investigations. It would be nice to extend our present investigation to the 4D non-Abelian 2-form gauge theory and establish its hidden connection with the anomalous 2D non-Abelian gauge theory which has already been shown to be consistent and unitary [39]. To show that the above theories and their possible modified versions are the field-theoretic models for the Hodge theory, is a very challenging and demanding endeavor for us. We, at the moment, are actively involved with the above-mentioned issues and we hope to report about our results in our future publications [40].

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Appendix A: On (dual-) gauge transformations in 2-form theory

Let us begin with the following simple gauge-fixed Lagrangian density of the 4D Abelian 2-form gauge theory in the Feynman gauge

\[
\mathcal{L}_0 = \frac{1}{12} H_{\mu\nu\kappa} H^{\mu\nu\kappa} + \frac{1}{2} (\partial^\mu B_{\nu\mu}) (\partial_\eta B^{\eta\mu}),
\]  

where the totally antisymmetric curvature tensor \( H_{\mu\nu\kappa} = \partial_{\mu} B_{\nu\kappa} + \partial_{\nu} B_{\kappa\mu} + \partial_{\kappa} B_{\mu\nu} \) is derived from the 3-form \( H^{(3)} = dB^{(2)} = \frac{1}{3!}(dx^\mu \wedge dx^\nu \wedge dx^\kappa) H_{\mu\nu\kappa} \).

In the above, \( d = dx^\mu \partial_\mu \) (with \( d^2 = 0 \)) is the exterior derivative and the 2-form \( B^{(2)} = \frac{1}{2}(dx^\mu \wedge dx^\nu) B_{\mu\nu} \) defines the antisymmetric \( (B_{\mu\nu} = -B_{\nu\mu}) \) gauge
potential $B_{\mu\nu}$ of the present gauge theory. In a similar fashion, the gauge-fixing term is connected with the co-exterior derivative $\delta = - \ast d \ast$ because $\delta B^{(2)} = - \ast (d \ast B^{(2)}) = (\partial_\nu B^{\nu\mu}) dx_\mu$. Here the $(\ast)$ operation corresponds to the Hodge duality operation on the 4D Minkowski spacetime manifold. The following infinitesimal versions of the (dual-) gauge transformations

$$\delta_{dg} B_{\mu\nu} = - \varepsilon_{\mu\nu\eta\xi} \partial^\eta \xi \Sigma, \quad \delta_{g} B_{\mu\nu} = - (\partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu),$$

leave the gauge-fixing and the kinetic terms invariant, respectively, because

$$\delta_{dg} (\partial_\nu B^{\nu\mu}) = 0, \quad \delta_{g} (H_{\mu\nu\eta}) = 0.$$  

This invariance is the reason behind the above nomenclature associated with the symmetry transformations. Thus, the infinitesimal transformations $\delta_{dg}$, in the above, correspond to the (dual-) gauge transformations and $\Sigma_\mu$ and $\alpha_\mu$ are the corresponding infinitesimal parameters.

It can be readily checked that the gauge-fixed Lagrangian density $L_0$ transforms to a total spacetime derivative plus terms that are zero on the equation of motion $\Box B_{\mu\nu} = 0$. This statement can be captured by the following equations that represent the (dual-) gauge transformations, namely;

$$\delta_{dg} L_0 = - \frac{1}{2} \partial_\mu \left[ H^{\mu\nu\eta} \varepsilon_{\nu\xi\sigma} \partial^\xi \Sigma - \varepsilon^{\mu\nu\eta\rho} (\partial_\xi H_{\eta\sigma}) \Sigma \right]$$

$$- \frac{1}{2} \varepsilon_{\xi\nu\eta} (\partial_\xi \Box B^{\nu\eta}) \Sigma,$$  

$$\delta_{g} L_0 = - \partial_\mu \left[ (\partial^\nu B_{\nu\sigma}) (\partial^\sigma \alpha^\mu) - \partial^\sigma \partial^\nu (\partial^\nu B_{\nu\sigma}) + \alpha^\sigma \partial_\sigma (\partial_\nu B^{\nu\mu}) \right]$$

$$- (\partial_\nu \Box B_{\nu\mu}) \alpha^\mu.$$  

Thus, we note that the anomalous Abelian 1-form Lagrangian density (cf. Subsec. 3.1) and gauge-fixed version of the Lagrangian density of the Abelian 2-form gauge theory have a similarity as far as their properties under the (dual-) gauge transformations are concerned. We further point out that the above observations are the prelude to the existence of the (anti-) BRST and the (anti-) co-BRST symmetry transformations which we attempt below.

Appendix B: On (anti-) BRST invariant Lagrangian densities

We begin here with the basic (anti-) BRST invariant Lagrangian densities of the Abelian 2-form gauge theory in 4D [22]

$$L_B = \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + B^\mu (\partial^\nu B_{\nu\mu}) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1$$

$$+(\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu) (\partial^\nu C^\mu) + (\partial \cdot C - \lambda) \rho + (\partial \cdot \bar{C} + \rho) \lambda + \partial_\mu \beta \partial^\mu \beta,$$  

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\[ L_B = \frac{1}{12} H_{\mu\nu\kappa} H^{\mu\nu\kappa} + \bar{B}^{\mu} (\partial^\nu B_{\nu\mu}) + \frac{1}{2} (B \cdot B + \bar{B} \cdot B) + \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 \]
\[ + (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu) (\partial^{\mu} C^\nu) + (\partial \cdot C - \lambda) \rho + (\partial \cdot \bar{C} + \rho) \lambda + \partial_\mu \bar{\beta} \partial^{\mu} \beta. \quad (86) \]

The above basic Lagrangian densities \( L_B \) and \( \bar{L}_B \) are endowed with the gauge-fixing and Faddeev-Popov ghost terms as given below

\[ s_b s_{ab} \left[ 2\beta \bar{\beta} + \bar{C}_\mu C^\mu - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \right] = B^{\mu} (\partial^\nu B_{\nu\mu}) + B \cdot \bar{B} + \partial_\mu \bar{\beta} \partial^{\mu} \beta \]
\[ + (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu) (\partial^{\mu} C^\nu) + (\partial \cdot C - \lambda) \rho + (\partial \cdot \bar{C} + \rho) \lambda, \quad (87) \]
\[ - s_a s_b \left[ 2\beta \bar{\beta} + \bar{C}_\mu C^\mu - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \right] = \bar{B}^{\mu} (\partial^\nu B_{\nu\mu}) + B \cdot \bar{B} + \partial_\mu \bar{\beta} \partial^{\mu} \beta \]
\[ + (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu) (\partial^{\mu} C^\nu) + (\partial \cdot C - \lambda) \rho + (\partial \cdot \bar{C} + \rho) \lambda, \quad (88) \]

where the nilpotent \((s_{(a)b})^2 = 0\) and absolutely anticommuting \((s_b s_{ab} + s_{ab} s_b = 0)\) (anti-) BRST symmetry transformations \((s_{(a)b})\) are [22]

\[
\begin{align*}
& s_b B_{\mu\nu} = - (\partial_\mu C_\nu - \partial_\nu C_\mu), \quad s_b C_\mu = - \partial_\mu \beta, \quad s_b \bar{C}_\mu = - B_\mu, \\
& s_b \phi_1 = \lambda, \quad s_b \bar{\beta} = - \rho, \quad s_b \bar{B}_\mu = - \partial_\mu \lambda, \quad s_b [\rho, \lambda, B_\mu, \beta, H_{\mu\nu\kappa}] = 0, \quad (89) \\
& s_{ab} B_{\mu\nu} = - (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu), \quad s_{ab} \bar{C}_\mu = - \partial_\mu \bar{\beta}, \quad s_{ab} C_\mu = \bar{B}_\mu, \\
& s_{ab} \phi_1 = \rho, \quad s_{ab} \beta = - \lambda, \quad s_{ab} B_\mu = \partial_\mu \rho, \quad s_{ab} [\rho, \lambda, \bar{B}_\mu, \bar{\beta}, H_{\mu\nu\kappa}] = 0. \quad (90)
\end{align*}
\]

We have obtained \( L_B \) and \( \bar{L}_B \) (cf. (85) and (86)) by exploiting \( B \cdot \bar{B} = \frac{1}{2} (B \cdot B + \bar{B} \cdot B) \) because our present theory is defined on a constrained submanifold described by the constrained field equation \( B_\mu - \bar{B}_\mu = \partial_\mu \phi_1 \).

Furthermore, the absolute anticommutativity \((s_b s_{ab} + s_{ab} s_b = 0)\) of the above transformations is satisfied if and only if \( B_\mu - \bar{B}_\mu = \partial_\mu \phi_1 \). In particular, it can be checked that \( \{ s_b, s_{ab} \} B_{\mu\nu} \equiv (s_b s_{ab} + s_{ab} s_b) B_{\mu\nu} = 0 \) is true only if the above equation is precisely respected\(^5\).

It can be checked that the Lagrangian densities (85) and (86) can be expressed in term of the above (anti-) BRST symmetry transformations as

\[
\begin{align*}
L_B &= \frac{1}{12} H_{\mu\nu\kappa} H^{\mu\nu\kappa} + s_b s_{ab} \left[ 2\beta \bar{\beta} + \bar{C}_\mu C^\mu - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \right], \\
\bar{L}_B &= \frac{1}{12} H_{\mu\nu\kappa} H^{\mu\nu\kappa} - s_a s_b \left[ 2\beta \bar{\beta} + \bar{C}_\mu C^\mu - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \right]. \quad (91)
\end{align*}
\]

\(^5\)The (anti-) BRST invariant condition \( B_\mu - \bar{B}_\mu = \partial_\mu \phi_1 = 0 \) is obtained in our earlier work on the geometrical superfield approach to free 4D Abelian 2-form gauge theory [20]. This condition is the analogue of the Curci-Ferrari restriction [21] that appears in the context of the non-Abelian 1-form gauge theory. The above conditions ensure the \textit{absolute} anticommutativity of the off-shell nilpotent (anti-) BRST symmetries.
The BRST and anti-BRST invariance of (91) on the constrained surface $B_\mu - \bar{B}_\mu - \partial_\mu \phi_1 = 0$ becomes very clear and simple because of

(i) the nilpotency $(s^2_{ab} = 0)$ and absolute anticommutativity $(s_{ab} + s_{b} s_{ab} = 0)$ of the (anti-) BRST symmetry transformations, and

(ii) the (anti-) BRST invariance of the curvature term (i.e. $s_{(a)b} H_{\mu\nu\kappa} = 0$).

The above statements can be corroborated by the following equations

\[ s_b L_B = -\partial_\mu \left[ (\partial^\mu C^\nu - \partial^\nu C^\mu) B_\nu + \lambda B^\mu + \rho \partial^\mu \beta \right] + (\partial^\mu \lambda) \left[ B_\mu - \bar{B}_\mu - \partial_\mu \phi_1 \right], \]  

\[ s_{ab} L_B = -\partial_\mu \left[ (\partial^\mu C^\nu - \partial^\nu C^\mu) B_\nu - \rho \bar{B}^\mu + \lambda \partial^\mu \bar{\beta} - \rho (\partial_\mu B^\nu) \right] + (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \partial_\mu \left[ B_\nu - \bar{B}_\nu - \partial_\nu \phi_1 \right] + (\partial^\mu \rho) \left[ B_\mu - \bar{B}_\mu - \partial_\mu \phi_1 \right], \]  

\[ s_{ab} L_{\bar{B}} = -\partial_\mu \left[ (\partial^\mu C^\nu - \partial^\nu C^\mu) \bar{B}_\nu + \rho \partial^\mu \beta + \lambda \partial_\mu B^\nu \right] - (\partial^\mu C^\nu - \partial^\nu C^\mu) \partial_\mu \left[ B_\nu - \bar{B}_\nu - \partial_\nu \phi_1 \right] + (\partial^\mu \lambda) \left[ B_\mu - \bar{B}_\mu - \partial_\mu \phi_1 \right]. \]  

Thus, it is clear that if we impose the constraint field equation $(B_\mu - \bar{B}_\mu - \partial_\mu \phi_1 = 0)$, we shall have the following:

(i) the absolute anticommutativity of the nilpotent (anti-) BRST symmetry transformations $s_{(a)b}$, and

(ii) the BRST and anti-BRST invariance of both the basic and equivalent Lagrangian densities $L_B$ and $L_{\bar{B}}$.

The constrained field equation $B_\mu - \bar{B}_\mu - \partial_\mu \phi_1 = 0$ is an (anti-) BRST invariant quantity (i.e. $s_{(a)b} [B_\mu - \bar{B}_\mu - \partial_\mu \phi_1] = 0$). As a side remark, it can be seen that the $(B \cdot \bar{B})$, present in (87) and (88), can also be expressed as

\[ B \cdot \bar{B} = B \cdot B - B^\mu \partial_\mu \phi_1, \quad B \cdot \bar{B} = \bar{B} \cdot \bar{B} + \bar{B}^\mu \partial_\mu \phi_1. \]  

In such a situation, the Euler-Lagrange equations of motion, that would emerge from $L_B$ and $L_{\bar{B}}$, are [14,26]

\[ B_\mu = -\frac{1}{2} (\partial^\nu B_{\nu\mu} - \partial_\mu \phi_1), \quad \bar{B}_\mu = -\frac{1}{2} (\partial^\nu B_{\nu\mu} + \partial_\mu \phi_1). \]  

These observations are important because we have seen a similar kind of symmetry structure in the case of anomalous 2D Abelian 1-form gauge theory in Sec. 3.
The above expressions would lead to the constrained field equation $B_\mu - \bar{B}_\mu - \partial_\mu \phi_1 = 0$. Thus, we obtain the CF type constrained field equation $(B_\mu - \bar{B}_\mu - \partial_\mu \phi_1 = 0)$ in two steps from the Lagrangian densities $\mathcal{L}_B$ and $\mathcal{L}_{\bar{B}}$ by exploiting the equations of motion and subtracting one from the other. It would be, however, very nice to obtain

(i) the above constrained field equation (i.e. $B_\mu - \bar{B}_\mu - \partial_\mu \phi_1 = 0$) in one step by exploiting the equation of motion, and

(ii) the perfect (anti-) BRST symmetry invariance of the Lagrangian densities $\mathcal{L}_B$ and $\mathcal{L}_{\bar{B}}$ without any imposition of $B_\mu - \bar{B}_\mu - \partial_\mu \phi_1 = 0$.

To this end in mind, we add a Lagrange multiplier field $(\mathcal{L}_\mu)$ in the Lagrangian densities in the following fashion (see, e.g. [22] for details).

\[
\mathcal{L}_{(L,B)} = \frac{1}{12} H_{\mu\nu\kappa} H^{\mu\nu\kappa} + B_\mu (\partial^\nu B^\nu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 \\
+ \partial_\mu \bar{B}_\nu \beta + (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu) (\partial^\mu C^\nu) + (\partial \cdot C - \lambda) \rho \\
+ (\partial \cdot \bar{C} + \rho) \lambda + L^\mu (B_\mu - \bar{B}_\mu - \partial_\mu \phi_1),
\]

(98)

\[
\mathcal{L}_{(L,\bar{B})} = \frac{1}{12} H_{\mu\nu\kappa} H^{\mu\nu\kappa} + \bar{B}_\mu (\partial^\nu B^\nu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 \\
+ \partial_\mu \bar{B}_\nu \beta + (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu) (\partial^\mu C^\nu) + (\partial \cdot C - \lambda) \rho \\
+ (\partial \cdot \bar{C} + \rho) \lambda + L^\mu (B_\mu - \bar{B}_\mu - \partial_\mu \phi_1).
\]

(99)

It is straightforward to check that the above Lagrangian densities remain quasi-invariant under the (anti-) BRST transformations if we take

\[
s_b L_\mu = -\partial_\mu \lambda, \quad s_{ab} L_\mu = -\partial_\mu \rho.
\]

(100)

The above transformations are consistent with the Euler-Lagrange equation of motion, nilpotency and absolute anticommutativity of the (anti-) BRST symmetry transformations. Furthermore, it can be checked that, under the symmetry transformations (89) and (90), the above Lagrangian densities transform to the total spacetime derivatives (as given in (5) and (6)) without any imposition of the restriction like $B_\mu - \bar{B}_\mu - \partial_\mu \phi_1 = 0$.

Appendix C: On (anti-) co-BRST invariant Lagrangian densities

The kinetic term $(\frac{1}{12} H_{\mu\nu\kappa} H^{\mu\nu\kappa})$ of the gauge field (cf. (85) and (86)) can be linearized by introducing a massless ($\Box \phi_2 = 0$) scalar field $\phi_2$ and the auxiliary fields $B_\mu$ and $\bar{B}_\mu$. The ensuing equivalent Lagrangian densities

\[
\mathcal{L}_{(B,\bar{B})} = \frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{1}{2} B^\mu \varepsilon_{\mu\nu\kappa} \partial^\nu B^{\nu\kappa} - \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B})
\]

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It is straightforward to check that the following are true, namely;

\[+ B^\mu (\partial^\nu B_{\nu \mu}) + \frac{1}{2} (B \cdot B + B \cdot \vec{B}) - \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \beta \partial^\mu \beta \]
\[+ (\partial_\mu \vec{C}_\nu - \partial_\nu \vec{C}_\mu)(\partial^\mu C^\nu) + (\partial \cdot C - \lambda) \rho + (\partial \cdot \vec{C} + \rho) \lambda, \quad (101)\]

\[L_{(B, \vec{B})} = \frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{1}{2} \vec{B}^\mu \varepsilon_{\mu \nu \rho \sigma} \partial^\nu B^\rho - \frac{1}{2} (B \cdot B + \vec{B} \cdot \vec{B})
\[+ \vec{B}^\mu (\partial^\nu B_{\nu \mu}) + \frac{1}{2} (B \cdot B + \vec{B} \cdot \vec{B}) - \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \beta \partial^\mu \beta \]
\[+ (\partial_\mu \vec{C}_\nu - \partial_\nu \vec{C}_\mu)(\partial^\mu C^\nu) + (\partial \cdot C - \lambda) \rho + (\partial \cdot \vec{C} + \rho) \lambda, \quad (102)\]

respect, in addition to the (anti-) BRST symmetry transformations (89) and (90)\(^7\) the following dual(co)-BRST and anti-dual(co)-BRST transformations

\[s_d B_{\mu \nu} = -\varepsilon_{\mu \nu \rho \sigma} \partial^\rho C^\sigma, \quad s_d \vec{C}_\mu = -\partial_\mu \beta, \quad s_d C_\mu = -B_\mu, \quad s_d \phi_2 = -\rho, \quad \]
\[s_d \beta = -\lambda, \quad s_d \vec{B}_\mu = \partial_\mu \rho, \quad s_d [\rho, \lambda, \beta, \phi_1, B_\mu, \vec{B}_\mu, (\partial^\nu B_{\nu \mu})] = 0, \quad (103)\]

\[s_{ad} B_{\mu \nu} = -\varepsilon_{\mu \nu \rho \sigma} \partial^\rho C^\sigma, \quad s_{ad} \vec{C}_\mu = \partial_\mu \beta, \quad s_{ad} C_\mu = \vec{B}_\mu, \quad s_{ad} \phi_2 = -\lambda, \quad s_{ad} \beta = \rho, \quad s_{ad} B_\mu = -\partial_\mu \lambda, \quad s_{ad} [\rho, \lambda, \beta, \phi_1, B_\mu, \vec{B}_\mu, (\partial^\nu B_{\nu \mu})] = 0. \quad (104)\]

It is straightforward to check that the following are true, namely;

\[s_d L_{(B, \vec{B})} = \partial_\mu \left[ (\partial^\mu \vec{C}_\mu) B_\nu - \partial^\nu \vec{C}_\mu B_\nu - \rho B^\mu + \lambda \partial^\mu \vec{B}_\mu \right]
\[+ (\partial^\nu \rho) [B_\mu - \vec{B}_\mu - \partial_\mu \phi_2], \quad (105)\]

\[s_{ad} L_{(B, \vec{B})} = \partial_\mu \left[ (\partial^\mu C^\nu - \partial^\nu \vec{C}_\mu) \vec{B}_\nu + \rho \partial^\mu \beta + \lambda \vec{B}_\mu \right]
\[+ (\partial^\mu \lambda) [B_\mu - \vec{B}_\mu - \partial_\mu \phi_2], \quad (106)\]

\[s_d L_{(\vec{B}, \vec{B})} = \partial_\mu \left[ (\partial^\mu C^\nu - \partial^\nu \vec{C}_\mu) B_\nu - \frac{\rho}{2} \varepsilon_{\mu \nu \rho \sigma} \partial_\nu B_{\rho \sigma} - \rho B^\mu - \lambda \partial^\mu \vec{B}_\mu \right]
\[+ (\vec{B}^\mu - \vec{B}^\mu - \partial^\mu \phi_2) (\partial_\mu \rho) + (\partial^\mu \vec{C}_\nu - \partial^\nu \vec{C}_\mu) \partial_\mu [B_\nu - \vec{B}_\nu - \partial_\nu \phi_2], \quad (107)\]

\[s_{ad} L_{(\vec{B}, \vec{B})} = \partial_\mu \left[ (\partial^\mu C^\nu - \partial^\nu \vec{C}_\mu) \vec{B}_\nu + \frac{\lambda}{2} \varepsilon_{\mu \nu \rho \sigma} \partial_\nu B_{\rho \sigma} + \rho \partial^\mu \beta + \lambda \vec{B}_\mu \right]
\[+ (\vec{B}^\mu - \vec{B}^\mu - \partial^\mu \phi_2) (\partial_\mu \lambda) + (\partial^\mu C^\nu - \partial^\nu \vec{C}_\mu) \partial_\mu [B_\nu - \vec{B}_\nu - \partial_\nu \phi_2]. \quad (108)\]

\(^7\)In fact, in addition to the transformations (89) and (90), we also need the transformations \(s_{(a)} \phi_2 = 0, s_{(a)} \vec{B}_\mu = 0, s_{(a)} \beta = 0\) for the perfect symmetry invariance of the Lagrangian densities of our present theory (cf. (105)–(108) below).
Thus, it is clear that, on the constraint surface $B_\mu - \bar{B}_\mu - \partial_\mu \phi_2 = 0$, the equivalent Lagrangian densities (101) and (102) are (anti-) co-BRST invariant. It is interesting to point out that similar kind of mathematical structure appears for the anomalous 2D gauge theory as well (see, Sec. 3).

Analogous to equation (91), the Lagrangian densities (101) and (102) can be expressed as the sum of the full gauge-fixing term and (anti-) co-BRST exact expressions as given below

$$
\mathcal{L}_{(B,\bar{B})} = \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) + B^\mu (\partial^\nu B_{\nu \mu}) - \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 + s_{ad} s_d \left[ 2 \beta \bar{\beta} + \bar{C}_\mu C^\mu - \frac{1}{4} B^\mu B_{\mu \nu} B_{\nu \mu} \right],
$$

$$
\mathcal{L}_{(B,\bar{B})} = \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) + \bar{B}^\mu (\partial^\nu B_{\nu \mu}) - \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 - s_{ad} s_d \left[ 2 \beta \bar{\beta} + \bar{C}_\mu C^\mu - \frac{1}{4} B^\mu B_{\mu \nu} B_{\nu \mu} \right].
$$

In this form, the (anti-) co-BRST invariance of the Lagrangian densities (101) and (102) becomes very simple (on the constrained submanifold defined by the field equation $B_\mu - \bar{B}_\mu - \partial_\mu \phi_2 = 0$) because of

(i) the nilpotency of the (anti-) co-BRST symmetry transformations, and

(ii) the invariance of the total gauge-fixing term (i.e. $s_{(a)\bar{a}} \left[ \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) + \bar{B}^\mu (\partial^\nu B_{\nu \mu}) - \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 \right] = 0, s_{(a)\bar{a}} \left[ \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) + B^\mu (\partial^\nu B_{\nu \mu}) - \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 \right] = 0$) under the (anti-) co-BRST transformations $s_{(a)\bar{a}}$.

The explicit expression, modulo some total spacetime derivative terms, for the following combination, namely:

$$
\begin{align*}
&\quad s_{ad} s_d \left[ 2 \beta \bar{\beta} + \bar{C}_\mu C^\mu - \frac{1}{4} B^\mu B_{\mu \nu} B_{\nu \mu} \right] = (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu) (\partial^\mu C^\nu) \\
&\quad + (\partial \cdot C - \lambda) \rho + (\partial \cdot \bar{C} + \rho) \lambda + \partial_\mu \bar{\beta} \partial^\mu \beta - B \cdot \bar{B} - \frac{1}{2} B^\nu \varepsilon_{\mu \nu \rho \sigma} \partial^\rho B^\sigma, \tag{110}
\end{align*}
$$

leads to the derivation of the Lagrangian density (101). In a similar fashion, the following relationship (modulo some total spacetime derivative terms):

$$
\begin{align*}
&\quad -s_{ad} s_d \left[ 2 \beta \bar{\beta} + \bar{C}_\mu C^\mu - \frac{1}{4} B^\mu B_{\mu \nu} B_{\nu \mu} \right] = (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu) (\partial^\mu C^\nu) \\
&\quad + (\partial \cdot C - \lambda) \rho + (\partial \cdot \bar{C} + \rho) \lambda + \partial_\mu \bar{\beta} \partial^\mu \beta - B \cdot \bar{B} - \frac{1}{2} B^\nu \varepsilon_{\mu \nu \rho \sigma} \partial^\rho B^\sigma, \tag{111}
\end{align*}
$$

leads to the derivation of (102) if we use

$$
B \cdot \bar{B} = \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - \frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2, \tag{112}
$$

that emerges due to the CF type of restriction $B_\mu - \bar{B}_\mu - \partial_\mu \phi_2 = 0$. It is worthwhile, once again, to point out that the Lagrangian densities (101)
and (102) are equivalent (on the constraint submanifold defined by the field equation $B^\mu - \tilde{B}^\mu - \partial_\mu \phi_2 = 0$) and both of them respect the (anti-) co-BRST symmetry transformations. Analogous to (98) and (99), we can also write (101) and (102) by incorporating $M^\mu (B^\mu - \tilde{B}^\mu - \partial_\mu \phi_2)$ (with $M^\mu$ as Lagrange multiplier field) which would respect the (anti-) co-BRST symmetries without any imposition. These issues have been taken into account in (1) and (2).

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