Gravity Localization and Mass Hierarchy on Pure Geometric Thin Branes

Ke Yang, Yu-Xiao Liu, Yuan Zhong, Shao-Wen Wei

Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, People’s Republic of China

E-mail: yangke09@lzu.edu.cn, liuyx@lzu.edu.cn, zhongy2009@lzu.edu.cn, weishaow06@lzu.edu.cn

Abstract: We consider a simple toy model with flat thin branes embedded in a 5-dimensional Weyl integrable manifold, where the Weyl scalar provides the material to constitute the brane configurations. As we work with an $S^1/Z_2$ orbifold extra dimension, thin brane solutions which have two flat branes located at the two boundaries of the orbifold are found. Especially, we focus our attention on one of the solutions, whose warp factor is decreasing along the extra dimension like RS1 model, while its massless graviton is localized on the brane located at the boundary $z_b$ but not at the origin one. Further, the mass hierarchy problem is discussed based on this solution. Interestingly, we find that the spacing of mass spectrum in this scenario is very tiny, but the light gravitons cannot be seen individually in colliders for their weak enough interaction with matter on the visible brane.

Keywords: Brane World, Field Theories in Higher Dimensions, Weyl Geometry.

*Corresponding author.
1. Introduction

Motivated by string/M theory, the brane world scenario has been increasing interest during recent years (see [1, 2, 3, 4, 5] for introduction). In this scenario, our world (including Standard Model matter fields) is trapped in a four-dimensional submanifold (called brane) embedded in a fundamental multi-dimensional space-time (called bulk), while the gravity still propagates in the whole space-time. This scenario provides a mechanism that one could possibly solve some disturbing problems of high-energy physics, such as the hierarchy problem (the problem of why the electroweak scale $M_{EW} \approx 1\text{TeV}$ is so different from the Planck scale $M_{Pl} \approx 10^{16}\text{TeV}$) and the cosmological constant problem [6, 7, 8, 9, 10, 11]. A landmark theory in brane world scenario is the Randall-Sundrum (RS) model put forward in 1999 [8, 9]. In RS1 model [8], only one compact extra dimension with the topology $S^1/Z_2$ and two 3-branes located at boundaries are required. The one called visible brane is that our world lives on and the other called invisible brane is that spin-2 gravitons localize on. The hierarchy problem is solved in this model by introducing an exponential warp factor to warp the extra dimension and further this warped extra dimension generates an exponential hierarchy to decrease the Planck scale to weak scale on the visible brane. For micro-size of extra dimension, the gravity is indeed effective four-dimensional from a macro-scale point of view. When the radius of $S^1$ approaches infinite large, the RS1 model transforms to RS2 model [9]. Even the extra dimension is noncompact now, nevertheless, effective Newtonian gravity still can be recovered on the brane in this model. Since then a lot of brane world models have been proposed and the corresponding researches such as the mechanisms for localization of SM matter fields on the brane have been widely discussed, see e.g. [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

As the mainstay of gravitational theory, most investigations of brane worlds are based on Einstein’s relativity. However, there are still many other alternative gravitational theories which may present us with some novel and interesting properties of the universe,
such as scalar-tensor gravity \[26\], f(R) gravity \[27, 28\], Born-Infeld gravity \[29, 30\] and Horava-Lifshitz gravity \[31, 32\]. In this paper, we are interested in the brane world model based on the integrable Weyl geometry. The Weyl geometry was proposed by Weyl in 1918 \[33, 34\] as a generalization of Riemann geometry to attempt to unify gravity with electromagnetism. In Riemann geometry the covariant derivative is compatible with the metric, i.e., $\nabla_K g_{MN} = 0$, while this is not hold in Weyl geometry and now the assumption is given by \[35, 36\]

$\nabla_K g_{MN} = \omega_K g_{MN}$, \hspace{1cm} (1.1)

where $\omega_K$ is a “gauge” vector field on the Weyl manifold specified by the pair $(g_{MN}, \omega_K)$. Now consider an infinitesimal parallel transport $dx^K$, with the definition \[(1.1)\], the length $L = g_{MN}l^M l^N$ of a vector $l^K$ is changed by

$dL = L \omega_K dx^K$. \hspace{1cm} (1.2)

So this means Weyl geometry allows for possible variations in the length of vectors during parallel transport.

Nevertheless, Einstein firstly pointed out that the Weyl’s original theory is inadequate as a physical acceptable one since atomic clocks would depend on their past history, known as the “second clock effect”. In order to overcome this defect, i.e., hold the synchronization of clocks traveling alone different paths between space-time points $A$ and $B$, one just has to impose that circuit integral of eq. \[(1.2)\] vanishes for an arbitrary closed path containing $A$ and $B$, i.e., $\oint dL = 0$. With Stoke’s theorem this condition leads to the result that $\omega_K = \omega_K$. It means that the vector field is a gradient of a scalar denoted as $\omega$ here. This particular type of Weyl geometry is called a Weyl integrable manifold. Further with \[(1.1)\], the Weylian affine connection is expressed as

$\Gamma^P_{MN} = \{P_{MN}\} - \frac{1}{2} \left( \omega_{,M} g^P_N + \omega_{,N} g^P_M - g_{MN} \omega_P \right)$, \hspace{1cm} (1.3)

where $\{P_{MN}\}$ represents the Riemannian Christoffel symbol. Since the scalar field $\omega$ enters into the definition of the affine connection of the Weyl manifold, the Weyl scalar is actually a geometrical field.

Weyl geometry has been received more attention on the study of gravitation and cosmology \[35, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47\]. Background scalar fields with self-interaction potential are crucial for generating smooth thick brane configurations \[12\], while a geometric scalar is naturally provided by Weyl integrable manifold, thus the authors in \[11, 48, 49, 50, 51\] considered various thick brane models in Weyl geometry. The technique applied to achieve thick brane solutions in these papers is that one can map the Weyl manifold to the Riemann one via a conformal transformation and can solve the structure in Riemann geometry. Moreover, the corresponding problems of localizing matter fields were considered in references \[52, 53\]. In this paper, we investigate flat Minkowski 3-branes embedded in a vacuum of 5-dimensional Weyl integrable space-time, and the theory naturally reduces to three cases to allow us to find solutions. As we work with an $S^1/Z_2$ orbifold extra dimension, thin brane solutions are existed, while here the thin brane
configurations are not created by delta functions introduced by hand in the action like RS1 model [8], but the Weyl scalar and $S^1/Z_2$ extra dimension. Furthermore, the hierarchy problem is discussed based on the solution of case 2, which is found to be an interesting scenario and can provide us some interesting features in physical phenomenology.

The paper is organized as follows: In section 2, we propose our simple model and solve the theory directly in Weyl frame. In section 3, gravitational fluctuations are considered. In section 4, we simply discuss the physical implications based on the solution of case 2. Finally, brief conclusions and discussion are presented.

2. The Model

We start with a simple 5-dimensional pure geometric action on a Weyl integrable manifold $M_5^W$,

$$S_5^W = \frac{1}{2\kappa} \int_{M_5^W} d^5x \sqrt{|g|} f(\omega) R,$$

(2.1)

where $\kappa \equiv 8\pi G_5$, which will be set to 1 for simplifying the notation, $G_5$ is the gravitational constant, and $f(\omega)$ is an arbitrary function of the Weyl scalar $\omega(z)$ which only refers to the extra dimension $z$. In this frame the Weylian Ricci tensor reads

$$R_{MN} = \nabla_M \nabla_N f - \frac{1}{2} g_{MN} \nabla_K \nabla_K f.$$

(2.2)

where the semicolon is the covariant derivative $\nabla$ with respect to the Christoffel symbol. Thus the Weylian action (2.1) can be rewritten as

$$S_5^W = \frac{1}{2} \int_{M_5^W} d^5x \sqrt{|g|} \left[ f(\omega) \beta - 3f(\omega) \nabla_K \omega_K \right],$$

(2.3)

where $f_\omega(\omega)$ denotes the derivative with respect to the scalar $\omega$. Then from the action (2.3), the equations of motion are

$$f \hat{G}_{MN} = (3f + 4f_\omega) \left( \omega_{M,N} - \frac{1}{2} g_{MN} \omega_{K,K} \right) + \left( \nabla_M \nabla_N f - g_{MN} \nabla_K \nabla_K f \right),$$

(2.4a)

$$f_\omega \beta = - (3f + 4f_\omega) \omega_{K,K} - 2 (3f + 4f_\omega) \nabla_K \nabla_K \omega.$$  

(2.4b)

Further, we can write the field equation (2.4a) in the form of Einstein equations with an effective stress-energy tensor composed of all Weyl scalar terms moved to the right-hand side. This approach has been proved to be useful in practice in scalar-tensor gravity and $f(R)$ gravity [28], namely,

$$\hat{G}_{MN} = \hat{T}_{MN},$$

(2.5)
where $\hat{G}_{MN}$ is the effective Einstein tensor and $\hat{T}_{MN}$ is the effective energy-momentum tensor given by

$$
\hat{T}_{MN} = (3 + 4 \frac{f_{\omega}}{f}) \left[ \omega_{,M} \omega_{,N} - \frac{1}{2} g_{MN} \omega^{R} \omega_{,R} \right] + \frac{f_{\omega}}{f} \left[ \hat{\nabla}_{M} \hat{\nabla}_{N} \omega - g_{MN} \hat{\nabla}^{K} \hat{\nabla}_{K} \omega \right] \\
+ \frac{f_{\omega}}{f} \left[ \omega_{,M} \omega_{,N} - g_{MN} \omega^{R} \omega_{,R} \right].
$$

(2.6)

Furthermore, the null-null component of effective energy-momentum tensor in (2.6) is defined as the effective energy density $\hat{\rho}$.

As the ideal proposed in RS1 model [8], we consider the embedding of 3-branes, which preserve 4-dimensional Poincaré invariance, in a 5-dimensional Weyl space-time with an $S^1/Z_2$ orbifold extra dimension. The ansatz for the most general metric satisfying these properties is given by

$$
ds_5^2 = a^2(y) \eta_{\mu\nu} dx^\mu dx^\nu + dy^2,
$$

(2.7)

where $a(y)$ is the warp factor and $y \in [-y_b, y_b]$ denotes the physical coordinate of the extra dimension. The orbifold is compact and its physical size is $[0, y_b]$. However, if the boundary $y_b \to \infty$, the topology of extra dimension $S^1/Z_2 \to R^1/Z_2$. Therefore, the extra dimension will be noncompact anymore in this situation. After a coordinate transformation, $dy = a(y(z))dz$, one introduces a conformal metric which is useful for discussing the gravitational perturbations:

$$
ds_5^2 = a^2(z) (\eta_{\mu\nu} dx^\mu dx^\nu + dz^2),
$$

(2.8)

here $z$ denotes the conformal coordinate of the extra dimension with $z \in [-z_b, z_b]$. This conformal coordinate still preserves $Z_2$-symmetry. Therefore, with this conformal metric the related Ricci scalar and Einstein tensor are given by

$$
\hat{R} = -8 \frac{a''}{a} - 4 \frac{a'^2}{a^3},
$$

(2.9a)

$$
\hat{G}_{\mu\nu} = 3 \eta_{\mu\nu} \frac{a''}{a},
$$

(2.9b)

$$
\hat{G}_{55} = 6 \frac{a'^2}{a^2},
$$

(2.9c)

where the prime denotes the derivative with respect to the extra dimension $z$.

Thus the equations of motion (2.4) can be explicitly expressed as

$$
f_{\omega} \omega'' + (f_{\omega\omega} + 2f_{\omega} + \frac{3}{2} f) \omega'^2 + 2f_{\omega} \frac{a'}{a} \omega' + 3f \frac{a''}{a} = 0,(2.10a)
$$

$$
(\omega' - 2 \frac{a'}{a}) \left[ (4f_{\omega} + 3f) \omega' + 6f \frac{a'}{a} \right] = 0,(2.10b)
$$

$$
(4f_{\omega} + 3f) \omega'' + (2f_{\omega\omega} + \frac{3}{2} f_{\omega}) \omega'^2 + 3(4f_{\omega} + 3f) \frac{a'}{a} \omega' - 2f_{\omega} (2 \frac{a''}{a} + \frac{a'^2}{a^2}) = 0.(2.10c)
$$

On the other hand, the Weyl action (1.1) is invariant under the Weyl rescaling,

$$
\bar{g}_{MN} = e^{-\phi} g_{MN}, \quad \bar{\omega} = \omega - \phi,
$$

(2.11)
where $\phi$ is a smooth function defined on $M_W$. And further, it is easy to check that the Weyl action \((2.1)\) is invariant under this rescaling if and only if $f(\omega) = e^{-\frac{3}{2} \omega}$. However, when this invariance is broken, the Weyl scalar field transforms into an physical observable matter degree of freedom to generate the brane configuration \([11]\). Therefore, we simply choose $f(\omega) = e^{k \omega}$ with $k$ a constant in following discussion. Especially when $k \neq -3/2$ the invariance will be broken.

It is interesting to note that via a particular conformal transformation $\hat{g}_{MN} = e^{-\omega}g_{MN}$, the rescaling \((2.1)\) gives $\hat{\omega} = 0$. Thus the condition \((1.1)\) becomes
\[
\nabla_K \hat{g}_{MN} = 0.
\]
Therefore, the Weylian affine connection changes to Christoffel symbol $\Gamma^P_{MN} \rightarrow \{^P_{MN}\}$, and the Weyl structure maps into the Riemann one. Now under this conformal transformation, the Weylian action \((2.1)\) maps into a Riemannian one as
\[
S_{R}^{5} = \frac{1}{2} \int_{M_5} R_5 \sqrt{|\hat{g}|} e^{(k+\frac{3}{2}) \omega} \hat{R}.
\]
Especially, if $k = -3/2$ \([40, 41]\), this action is just the standard Einstein-Hilbert action in vacuum.

Now with $f(\omega) = e^{k \omega}$, the above eq. \((2.1)\) can be simply expressed as
\[
k \omega'' + (k^2 + 2k + \frac{3}{2}) \omega'^2 + 2k \frac{a'}{a} \omega' + 3 \frac{a''}{a} = 0, \quad \text{(2.12a)}
\]
\[
(k' - 2 \frac{a'}{a}) \left[ (4k + 3) \omega' + 6 \frac{a'}{a} \right] = 0, \quad \text{(2.12b)}
\]
\[
(4k + 3) \omega'' + (2k^2 + \frac{3}{2} k) \omega'^2 + 3(4k + 3) \frac{a'}{a} \omega' - 2k \left( \frac{a''}{a} + \frac{a'^2}{a^2} \right) = 0. \quad \text{(2.12c)}
\]
From eq. \((2.12)\), we easily get
\[
\frac{a'}{a} = \frac{1}{2} \omega', \quad \text{(2.13)}
or
\[
\frac{a'}{a} = - \frac{4k + 3}{6} \omega'. \quad \text{(2.14)}
\]
Note that when $k = -3/2$, eqs. \((2.13)\) and \((2.13)\) are just equivalent. Then substituting \((2.13)\) into eqs. \((2.12a)\) and \((2.12c)\), one finds that the two equations are identical to each other and can be rewritten as
\[
(2k + 3) \left[ \frac{a''}{a} + (2k + 2) \frac{a'^2}{a^2} \right] = 0. \quad \text{(2.15)}
\]
Thus, when $k = -3/2$, this constraint is vanishing, while $k \neq -3/2$, the constraint equation $\frac{a''}{a} + (2k + 2) \frac{a'^2}{a^2} = 0$ is needed. So with \((2.13)\), we can divide eq. \((2.12)\) into two independent cases:

Case 1:
\[
k = -\frac{3}{2}, \quad \text{(2.16a)}
\]
\[
\omega' = 2 \frac{a'}{a}. \quad \text{(2.16b)}
\]
Case 2:

\[ k \neq -\frac{3}{2}, \quad (2.17a) \]

\[ \omega' = 2 \frac{a'}{a}, \quad (2.17b) \]

\[ \frac{a''}{a} + (2k + 2) \frac{a'^2}{a^2} = 0. \quad (2.17c) \]

On the other hand, with the relation (2.14), eqs. (2.12a) and (2.12c) can be rewritten respectively as

\[ (2k + 3) \left[ \omega'' - \left( k + \frac{3}{2} \right) \omega'^2 \right] = 0, \quad (2.18) \]

\[ (2k + 3) \left[ \omega'' - \left( k + \frac{3}{2} \right) \omega'^2 \right] (4k + 3) = 0. \quad (2.19) \]

When \( k = -3/4 \), eq. (2.19) is satisfied automatically. In this case the constraints (2.14) and (2.18) can be rewritten as

\[ \frac{a'}{a} = 0, \quad (2.20) \]

\[ \omega'' - \frac{3}{4} \omega'^2 = 0. \quad (2.21) \]

However, it is clear that eq. (2.20) just gives rise to the warp factor \( a(z) \) a constant, so we do not take an interest in this trivial case. While if \( k \neq -3/4 \), eqs. (2.18) and (2.19) are just equivalent. As we have noted that the case \( k = -3/2 \) is just the same to the previous case 1, therefore, with the equation (2.14) yields another independent case, namely,

Case 3:

\[ k \neq -\frac{3}{4}, \quad (2.22a) \]

\[ \frac{a'}{a} = -\frac{4k + 3}{6} \omega', \quad (2.22b) \]

\[ \omega'' - \left( k + \frac{3}{2} \right) \omega'^2 = 0. \quad (2.22c) \]

Since we have divided the equations of motion into three cases, next we will solve them to find solutions of this theory.

**Case 1.** In this case, eq. (2.16) shows that there is just one constraint on the warp factor \( a(z) \) and the scalar \( \omega(z) \), which is given by equation (2.16b) as

\[ \omega = 2 \ln a, \quad (2.23) \]

where we have set the integral parameter as zero to fix \( \omega(0) = 0 \). Thus we can chose the warp factor freely if it preserves \( Z_2 \)-symmetry. However, as we have discussed above, the action is invariant under the Weyl rescaling for \( k = -3/2 \), thus the scalar is indeed a hidden degree of freedom and cannot be fixed dynamically in this case. Furthermore, as we can
see in next section, the gravitational phenomenology referring to (2.23) are just like the large extra dimensions scenario (known as ADD model) with one extra dimension \([6, 7]\), but a \(Z_2\)-symmetry is appended. So we are not interested in this case in this paper.

**Case 2.** Form eq. (2.17c), the warp factor is read as
\[
a(z) = C_1[(2k + 3)z - C_2]^{\frac{1}{2k+3}},
\]
(2.24)
where \(k \neq -\frac{3}{2}\). We set \(C_1^{2k+3}C_2 = -1\) to fix \(a(0) = 1\) and define the parameter \(p = C_1^{2k+3} > 0\). Therefore, after a redefinition of parameters, the warp factor is rewritten as a concise form
\[
a(z) = [1 + (2k + 3)pz]^{\frac{1}{2k+3}}.
\]
(2.25)
However, in order to satisfy the \(Z_2\)-symmetric condition and, furthermore, make sure that the null signal takes an infinite amount of time to travel from \(z_b\) to \(z = 0\) when \(z_b \to \infty\), as suggested in RS model \([8, 9]\), we rewrite the warp factor as the form
\[
a(z) = [1 - (2k + 3)p|z|]^{\frac{1}{2k+3}},
\]
(2.26)
where \(|z|\) is the absolute value of \(z\), and the parameter \(k < -\frac{3}{2}\). As shown in Fig. 1(a), the first order derivative of this warp factor is not continuous at the boundaries \(z = 0\) and \(z = z_b\), so it actually suggests a thin brane solution. Then substitute (2.26) into eq. (2.17b), the Weyl scalar is read as
\[
\omega(z) = C_3 + \frac{2}{2k + 3} \ln[1 - (2k + 3)p|z|].
\]
(2.27)
Set \(C_3 = 0\) to fix \(\omega(0) = 0\). As shown in Fig. 1(b), \(\omega'(z)\) is also not continuous at boundaries. Thus in this case the solution is given by

**Figure 1:** The shapes of the warp factor \(a(z)\), scalar \(\omega(z)\) and energy density \(\hat{\rho}(z)\) in case 2. The parameters are set to be \(p = 1\), \(k = -2\) and \(z_b = 4\).

\[
f(\omega) = e^{k\omega}, \quad (k < -3/2)
\]
(2.28a)
\[
a(z) = [1 - (2k + 3)p|z|]^{\frac{1}{2k+3}},
\]
(2.28b)
\[
\omega(z) = \frac{2}{2k + 3} \ln[1 - (2k + 3)p|z|].
\]
(2.28c)
Since $\omega'(z)$ is not continuous at boundaries, form eq. (2.6), there are delta functions in the energy density at these boundaries. Thus in this case the effective energy density is found to be

$$\hat{\rho}(z) = \frac{6(1+k)p^2}{1-(3+2k)p|z|^2} - \frac{4kp}{1-(3+2k)p|z|} [\delta(z) - \delta(z-z_b)]. \quad (2.29)$$

Therefore, this solution suggests two thin 3-branes localized at $z = 0$ and $z = z_b$, respectively. The shape of this effective energy density is plotted in Fig. 1(c). This brane configuration is similar to the RS1 model, however, as we will see in next section, the gravitational fluctuations are quite different from that in RS1 model, hence in this case we call the brane located at the origin as a visible brane, which is assumed as the one our world living on, while the other located at $z_b$ is an invisible brane.

**Case 3.** From eq. (2.22), the scalar and warp factor are given by

$$\omega(z) = \frac{2}{3+2k} \ln[1 + (3+2k)pz], \quad (2.30)$$
$$a(z) = [1 + (3+2k)pz]^{-\frac{3+4k}{3+6k}}. \quad (2.31)$$

We have set the parameters to fix $\omega(0) = 0$, $a(0) = 1$ and $k \neq -3/2$, $p > 0$. This solution is still not a $Z_2$-symmetric one, therefore like the case 2, we redefine the solution to achieve a $Z_2$-symmetric thin brane model. In this case, a little more complicated than case 2, we have two acceptable forms according to different value intervals of the parameter $k$:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{plot.pdf}
\caption{The shapes of the warp factor $a(z)$ and scalar $\omega(z)$ in case 3 with the parameter $k < -3/2$. The parameters are set to be $p = 1$, $k = -2$ and $z_b = 4$.}
\end{figure}

$$f(\omega) = e^{k\omega}, \quad (k < -3/2), \quad (2.32a)$$
$$a(z) = [1 - (3+2k)p|z|]^{-\frac{3+4k}{3+6k}}, \quad (2.32b)$$
$$\omega(z) = \frac{2}{3+2k} \ln[1 - (3+2k)p|z|]. \quad (2.32c)$$
Figure 3: The shapes of the warp factor $a(|z|)$ and scalar $\omega(|z|)$ in case 3 with the parameter $k > -3/4$. The parameters are set to be $p = 1$, $k = 1$ and $z_b = 4$.

\[
f(\omega) = e^{k\omega}, \quad (k > -3/4) \tag{2.33a}
\]
\[
a(z) = \left[1 + (3 + 2k)p|z|\right]^{\frac{3+4k}{3+4k}}, \quad (2.33b)
\]
\[
\omega(z) = \frac{2}{3+2k}\ln[1 + (3 + 2k)p|z|]. \quad (2.33c)
\]

The shapes of warp factor and scalar in each solution are plotted in Fig. 2 and Fig. 3 respectively. With these two solutions, the effective energy density can be expressed as

Figure 4: The shapes of the energy densities $\hat{\rho}(z)$ in case 3. The left one refers to $k = -2$ and right one $k = 1$. The parameters are set to be $p = 1$ and $z_b = 4$.

\[
\hat{\rho}(z) = -\frac{2(-9 + 3k + 8k^2)p^2}{3[1 - (3 + 2k)p|z|]^2} - \frac{4kp}{1 - (3 + 2k)p|z|}\left[\delta(z) - \delta(z - z_b)\right], \quad (k < -3/2), \quad (2.34)
\]
\[
\hat{\rho}(z) = -\frac{2(-9 + 3k + 8k^2)p^2}{3[1 + (3 + 2k)p|z|]^2} + \frac{4kp}{1 + (3 + 2k)p|z|}\left[\delta(z) - \delta(z - z_b)\right], \quad (k > -3/4). \quad (2.35)
\]

The effective energy densities are plotted in Fig. 4. Differing from the previous case, here since the brane configuration and gravitational fluctuations are similar to those in RS1
model, thus we call the brane located at $z_b$ as a visible brane where our word lives on and the other located at the origin is an invisible one.

Furthermore, via a coordinate transformation $dy = a(z)dz$, the solutions with respect to the non-conformal metric (2.7) can be achieved. Now the warp factors $a(z(y))$ and Weyl scalars $\omega(z(y))$ are given by

$$a(y) = e^{-p|y|}, \quad \omega(z) = -2p|y|, \quad (k = -2), \quad \text{(2.36)}$$

$$a(y) = \left[1 - 2(2 + k)p|y|\right]^{\frac{1}{4(2+k)}},$$

$$\omega(z) = \frac{1}{2 + k} \ln \left[1 - 2(2 + 2k)p|y|\right], \quad (k < -\frac{3}{2}, \quad k \neq -2) \quad \text{(2.37)}$$

in case 2, and

$$a(y) = e^{-3p|y|}, \quad \omega(z) = -2p|y|, \quad (k = -3), \quad \text{(2.38)}$$

$$a(y) = \left[1 - \frac{2}{3}(3 + k)p|y|\right]^{\frac{3 + 4k}{3 + 6k}},$$

$$\omega(y) = \frac{3}{3 + k} \ln \left[1 - \frac{2}{3}(3 + k)p|y|\right], \quad (k < -\frac{3}{2}, \quad k \neq -3), \quad \text{(2.39)}$$

$$a(y) = \left[1 + \frac{2}{3}(3 + k)p|y|\right]^{\frac{3 + 4k}{3 + 6k}},$$

$$\omega(y) = \frac{3}{3 + k} \ln \left[1 - \frac{2}{3}(3 + k)p|y|\right], \quad (k > -\frac{3}{4}) \quad \text{(2.40)}$$

in case 3. What interesting is that the extra dimension $y$ will have horizons after the coordinate transformation in case 2 with parameter $-2 < k < -\frac{3}{2}$ and in case 3 with $-3 < k < -\frac{3}{2}$, and the horizons are located at $y = \frac{1}{2(2+k)p}$ and $y = \frac{3}{2(2+k)p}$, respectively. Therefore the boundary of extra dimension should satisfy $y_b < \frac{1}{2(2+k)p}$ for the former case and $y_b < \frac{3}{2(3+k)p}$ for the later. We note that the curvature scalar of the Weyl bulk is the Weylian Ricci scalar $R$, and with the solutions given in previous cases it can be explicitly expressed as: $R = 0$ for case 2, $R = -\frac{24}{3}(3 + 2k)^2p^2(1 - (3 + 2k)p|z|)^{\frac{12 + 4k}{9 + 6k}}$ for case 3 with $k < -3/2$, and $R = -\frac{24}{3}(3 + 2k)^2p^2(1 + (3 + 2k)p|z|)^{\frac{12 + 4k}{9 + 6k}}$ for case 3 with $k > -3/4$.

Due to the parameter $p$ with a mass dimension referring to the bulk curvature, therefore schematically, we set $p$ as the order of the 5-dimensional fundamental Planck scale $M_5$ in following discussions.

### 3. Gravitational Fluctuations

From the effective Einstein equations (2.3), we consider metric fluctuations in our model. However, as here we only care about the tensor fluctuations, which are thought to refer to spin-2 gravitons here, for simplicity, we use the gauges $g_{\mu 5} = 0$ and $g_{55} = 1$ to remove the vector and scalar fluctuations [1]. Therefore, the fluctuation metric is given by the form:

$$ds^2 = a^2(z)[(\eta_{\mu\nu} + h_{\mu\nu}(x, z))dx^\mu dx^\nu + dz^2], \quad \text{(3.1)}$$

where $a^2(z)h_{\mu\nu}$ represents tensor fluctuations in the background space-time. From this fluctuation metric, the first order components of Einstein equations (2.3) are expressed as
$\mu \nu$-component

$$\frac{1}{2} \eta_{\mu \nu} h'' + \left( \frac{3 a'}{2a} + \frac{1}{2} \frac{f_\omega}{f} \omega' \right) \eta_{\mu \nu} h' + \frac{1}{2} \eta_{\mu \nu} \Box^{(4)} h - \frac{1}{2} h''_{\mu \nu}$$

$$+ \frac{1}{2} \left( \partial_\mu \partial_\nu h'' + \partial_\mu \partial_\nu h'' + \partial_\mu \partial_\nu h'' - \partial_\mu \partial_\nu h'' \right) - \frac{1}{2} \eta_{\mu \nu} \partial_\rho \partial_\sigma h'' \rho \sigma - \left( \frac{3 a'}{2a} + \frac{1}{2} \frac{f_\omega}{f} \omega' \right) h''_{\mu \nu}$$

$$+ \frac{3 a}{2} \omega^2 + \frac{f_\omega}{f} \left( \omega'' + \frac{2 a'}{a} \omega' + 2 \omega'^2 \right) + \frac{f_\omega f_\omega}{f} \omega^2 + \frac{3 a'}{a} \omega_\mu h_\nu - \frac{1}{2} \Box^{(4)} h_{\mu \nu}$$

$$= \frac{f_\omega}{f} \left( \partial_\mu \partial_\nu \hat{\omega} - \eta_{\mu \nu} \Box^{(4)} \hat{\omega} \right) - \frac{f_\omega}{f} \eta_{\mu \nu} \hat{\omega}'' - \left[ 3 \omega' + \frac{f_\omega}{f} \left( \frac{2 a'}{a} + 4 \omega' \right) + \frac{2 f_\omega}{f} \omega' \right] \eta_{\mu \nu} \hat{\omega}'$$

$$- \left[ \left( \frac{f_\omega f_\omega}{f} - \frac{f_\omega^2}{f^2} \right) \left( \omega'' + \frac{2 a'}{a} \omega' + 2 \omega'^2 \right) + \left( \frac{f_\omega f_\omega}{f} - \frac{f_\omega^2}{f^2} \right) \omega^2 \right] \eta_{\mu \nu} \hat{\omega},$$

(3.2)

$\mu 5$-component

$$\frac{1}{2} \partial_\mu h''_{\mu} - \frac{1}{2} \partial_\mu h' = \frac{2 f_\omega}{f} \hat{\omega}_{\mu} + \left[ \left( 3 + \frac{4 f_\omega}{f} + \frac{f_\omega f_\omega}{f^2} \right) \omega' - \frac{A'}{A} \frac{f_\omega}{f} \right] \hat{\omega}_{\mu},$$

(3.3)

55-component

$$\frac{1}{2} \Box^{(4)} h + \left( \frac{3 A'}{2 A} + \frac{1}{2} \frac{f_\omega}{f} \omega' \right) h' - \frac{1}{2} \partial_\rho \partial_\sigma h''_{\rho \sigma}$$

$$= \left( \frac{f_\omega}{f} - \frac{f_\omega^2}{f^2} \right) \left( 2 \omega'^2 - 4 \frac{A'}{A} \omega' \right) \hat{\omega} - \frac{f_\omega}{f^2} \Box^{(4)} \hat{\omega} + \left[ 3 \omega' + \frac{4 f_\omega}{f} \left( \omega' - \frac{A'}{A} \right) \right] \hat{\omega},$$

(3.4)

where $\Box^{(4)} = \eta_{\mu \nu} \partial_\mu \partial_\nu, h''_{\mu} = \eta_{\mu \nu} h_{\mu \nu}$, and $h = \eta_{\mu \nu} h_{\mu \nu}$. And the first order component of scalar field equation (2.4b) is given by

$$\frac{f_\omega}{f} h'' - \left[ 3 \omega' + \frac{4 f_\omega}{f} \left( \omega' - \frac{A'}{A} \right) \right] h' + \frac{f_\omega}{f} \left( \Box^{(4)} h - \partial_\mu \partial_\nu h''_{\rho \sigma} \right)$$

$$= \left( 3 \frac{f_\omega}{f} + 4 \frac{f_\omega f_\omega}{f} \right) \omega'^2 \hat{\omega} + 2 \left( 3 \frac{f_\omega}{f} + 4 \frac{f_\omega f_\omega}{f} \right) \left( \omega'' \hat{\omega} + 3 \frac{A'}{A} \omega' \hat{\omega} + \omega' \hat{\omega}' \right)$$

$$+ 2 \left( 3 + \frac{4 f_\omega}{f} \right) \left( \omega'' + \frac{3 A'}{A} \omega' + \Box^{(4)} \hat{\omega} \right),$$

(3.5)

where $\hat{\omega}$ represents the fluctuation of Weyl scalar $\omega$.

Further, we consider the transverse-traceless (TT) components of $h_{\mu \nu}$ with notation $\bar{h}_{\mu \nu}$ [3, 4], which satisfy the TT condition,

$$\bar{h} = \partial_\mu \bar{h}''_{\mu} = 0.$$  

(3.6)

Inspection of above fluctuation equations reveals that eqs. (3.3), (3.4) and (3.5) are purely non-TT components and all TT components are involved in eq. (3.2). Thus we make use of the TT projection operator for symmetric tensor field, namely, $P_{\mu \nu \sigma \rho} = \Pi_{\mu \rho} \Pi_{\nu \sigma} - \frac{1}{3} \Pi_{\mu \nu} \Pi_{\sigma \rho}$, to obtain the TT components of eq. (3.2), where $\Pi_{\mu \nu} = \eta_{\mu \nu} - \partial_\mu \partial_\nu$. The TT projection operator has the properties that $P_{\mu \nu \sigma \rho} h_{\sigma \rho} = \bar{h}_{\mu \nu}, P_{\mu \nu \sigma \rho} \eta_{\sigma \rho} F = 0$, and
\( P_{\mu \nu} \sigma_{\rho \sigma} \partial_{\sigma} \partial_{\rho} F = 0 \), where \( F \) is any scalar function. Furthermore, with eq. (2.10a), we have a relation
\[
3 \frac{a''}{a} = - \left[ \frac{3}{2} \omega'^2 + \frac{f_\omega}{f} \left( \omega'' + 2 \frac{a'}{a} + 2 \omega'^2 \right) + \frac{f \omega \omega'}{f} \right]. \tag{3.7}
\]
Thus the TT components of equation (3.2) can be simplified as
\[
\bar{h}_{\mu \nu}^{''} + 3 \frac{a'}{a} \bar{h}_{\mu \nu}^{'} + \frac{f_\omega}{f} \omega \bar{h}_{\mu \nu}^{'} + \Box^{(4)} \bar{h}_{\mu \nu} = 0. \tag{3.8}
\]
Furthermore, we decompose \( \bar{h}_{\mu \nu} \) in the form
\[
\bar{h}_{\mu \nu}(x, z) = \varepsilon_{\mu \nu}(x) A \frac{3}{2}(z) \Psi(z), \tag{3.9}
\]
where \( A(z) = a(z) f \frac{1}{4}(\omega) \). And the four-dimensional mass \( m \) of a Kaluza-Klein (KK) excitation is defined as
\[
\Box^{(4)} \varepsilon_{\mu \nu}(x) = m^2 \varepsilon_{\mu \nu}(x), \tag{3.10}
\]
then a Schrödinger-like equation can be obtained from eq. (3.8)
\[
- \Psi''(z) + V(z) \Psi(z) = m^2 \Psi(z), \tag{3.11}
\]
where the effective potential \( V(z) \) is given by
\[
V(z) = \frac{3}{2} \frac{A''}{A} + \frac{3}{4} \frac{A'^2}{A^2}. \tag{3.12}
\]
Note that the Hamiltonian in eq. (3.11) can be factorized as the form
\[
H = \left( \frac{d}{dz} + \frac{3}{2} A' \right) \left( - \frac{d}{dz} + \frac{3}{2} A' \right). \tag{3.13}
\]
Thus supersymmetric quantum mechanics ensures that there are no normalizable modes with \( m^2 < 0 \), that means there are no unstable tachyonic excitations in this system. The spectrum of the eigenvalue \( m^2 \) in (3.11) parameterizes the spectrum of the observed 4-dimensional graviton masses. Especially when \( m = 0 \), eq. (3.10) gives us the 4-dimensional massless graviton. Obviously, setting \( m = 0 \) in eq. (3.11) one easily gets a zero mode wave-function
\[
\Psi_0(z) = \frac{A^\frac{3}{2}(z)}{N_0} = \frac{a^\frac{3}{2}(z) f^{\frac{1}{2}}(\omega)}{N_0}, \tag{3.14}
\]
where \( N_0 \) is a constant which can be fixed by normalization condition. Here we note that the Weyl scalar enters into the potential of Schrödinger equation and the zero mode wave-function, which plays a crucial role to decide the gravitational phenomenology in our model as we will see in the following discussions.

In case 1, from eq. (2.23), we have \( A(z) = a f^{\frac{1}{2}} = 1 \), thus the effective potential is vanishing. Then the Schrödinger-like equation (3.11) gives that the zero mode \( \Psi_0 \) is a constant and the massive modes \( \Psi(z) \) are just plan wave with discrete eigenvalues \( m_n \propto \frac{n}{2z_0} \). Thus the gravitational phenomenology is similar to the ADD scenario with one extra
dimension \[6, 7\], but a \(Z_2\)-symmetry is appended. Therefore, we are not interested in this case here.

In the case 2, from eq. \(2.28\), up to a normalization constant, the gravitational zero mode and effective potential are expressed as

\[
\Psi_0(z) = [1 - (3 + 2k)p|z|]^{1/2},
\]
\[
V(z) = \frac{3(3+2k)p^2}{4[1-(3+2k)p|z|]^2} - \frac{(3+2k)p}{1-(3+2k)p|z|} \left[\delta(z) - \delta(z-z_b)\right].
\]

Since eq. \(3.11\) tells that in Weyl frame the gravitational modes are not just decided by the warp factor but also the geometrical Weyl scalar, thus as shown in Fig. 5, an interesting difference between our model and RS1 is that here the gravitational zero mode is localized on the brane at \(z_b\) but not on the one at origin, although the warp factor is also decrease towards to \(z_b\), as shown in figure \(1(a)\).

In the case 3, again up to normalization, from eq. \(2.32\) we have

\[
\Psi_0(z) = [1 - (3 + 2k)p|z|]^{-1/2}, \quad (k < -3/2)
\]
\[
V(z) = \frac{3(3+2k)p^2}{4[1-(3+2k)p|z|]^2} + \frac{(3+2k)p}{1-(3+2k)p|z|} \left[\delta(z) - \delta(z-z_b)\right],
\]

and

\[
\Psi_0(z) = [1 + (3 + 2k)p|z|]^{-1/2}, \quad (k > -3/4)
\]
\[
V(z) = \frac{3(3+2k)p^2}{4[1+(3+2k)p|z|]^2} - \frac{(3+2k)p}{1+(3+2k)p|z|} \left[\delta(z) - \delta(z-z_b)\right].
\]

As shown in Fig. 5 and Fig. 6, the shapes of the effective potentials and zero modes are similar with RS1 model and the gravitational zero modes are localized on the invisible brane at \(z = 0\).

Even these massless gravitational modes are normalizable in case 2 and case 3, nevertheless, because the scalar enters into the composition of massless gravitational wavefunction, these modes (3.15), (3.17) and (3.19) are not normalizable anymore when \(z_b \to \infty\).
Figure 6: The shapes of the zero mode $\Psi_0(z)$ and effective potential $V(z)$ in case 3 with the parameter $k < -3/2$. The parameters are set to be $p = 1$, $k = -2$ and $z_b = 4$.

Figure 7: The shapes of the zero mode $\Psi_0(z)$ and effective potential $V(z)$ in case 3 with the parameter $k > -3/4$. The parameters are set to be $p = 1$, $k = 1$ and $z_b = 4$.

While this is capable in RS1 model and further ensure it removing the visible brane to transform to RS2 model \cite{9}. Therefore, the need of compactifying the extra dimension is indeed crucial in our model. Furthermore, for the compact extra dimension all massive KK modes are bound states with discrete eigenvalues.

4. Physical Implications

Now with the explicitly expressions of effective potential, the Schrödinger equation (3.11) between the two boundaries could be rewritten as the form

$$-\Psi''(z) + \left(\alpha^2 - \frac{1}{4}\right) \frac{\beta^2}{(1 + \beta z)^2} \Psi(z) = m^2 \Psi(z),$$

where the parameters $\alpha$ and $\beta$ are $\alpha = 0$, $\beta = -(3 + 2k)p$ for the case 2, $\alpha = 1$, $\beta = -(3 + 2k)p$ for case 3 with $k < -3/2$ and $\alpha = 1$, $\beta = (3 + 2k)p$ for case 3 with $k > -3/4$. The general solution of this equation is given in terms of Bessel functions

$$\Psi_n(z) = (1 + \beta z)^{\frac{1}{2}} \left[ C_1 J_\alpha(m(z + 1/\beta)) + C_2 Y_\alpha(m(z + 1/\beta)) \right],$$
where $C_1$ and $C_2$ are $m$-dependent parameters. On the other hand, for working on an $S^1/Z_2$ orbifold, the appropriate boundary conditions could be chosen as $\partial_z \hat{h}_{\mu\nu}(x,z) = 0$ at the two boundary points to insure that gravitational wave-functions $\hat{h}_{\mu\nu}$ are $Z_2$-symmetric and $C^1$-smooth $\mathbb{R}$. Thus with the decomposition (3.9) we have

$$
\partial_z \Psi(z)|_{z=0} = \pm \beta z \Psi(z)|_{z=0}, \quad \partial_z \Psi(z)|_{z=\pm b} = \pm \frac{\beta}{2(1 + \beta z)} \Psi(z)|_{z=\pm b}, \quad (4.3)
$$

here the plus sign refers to the case 2 and minus sign to the case 3. With the first condition the wave function is

$$
\Psi_n(z) = \frac{(1 + \beta z)^{1/2}}{N_n} \left[ J_0(m(z + 1/\beta)) + \alpha_n Y_0(m(z + 1/\beta)) \right], \quad (4.4)
$$

where $N_n$ is a normalization factor and $\alpha_n = -J_{1-\alpha}(m/\beta)/Y_{1-\alpha}(m/\beta)$. Then with the second boundary condition, we have the graviton mass spectrum which is satisfied

$$
\frac{J_{1-\alpha}(m_n(z_b + 1/\beta))}{J_{1-\alpha}(m_n/\beta)} = \frac{Y_{1-\alpha}(m_n(z_b + 1/\beta))}{Y_{1-\alpha}(m_n/\beta)}. \quad (4.5)
$$

Now in the following, we just focus on the solution of case 2 (i.e., $\alpha = 0$), since this scenario is quite different from RS1 model with massless graviton localized on the brane at the origin, and it may provide us some new physical phenomenologies. Working in the limit $m_n/\beta \ll 1$ and $1 + \beta z_b \gg 1$, then $\alpha_n \approx \frac{m_n^2}{4z_b^2} \ll 1$, thus from (4.5), the spectrum is determined by $J_1(m_n(z_b + 1/\beta)) \approx 0$, namely,

$$
m_n = \frac{x_n}{z_b + 1/\beta}, \quad (4.6)
$$

where $x_n$ satisfies $J_1(x_n) = 0$, and $x_1 = 3.83$, $x_2 = 7.02$, $x_3 = 10.17, \ldots$. Thus the scale of mass splitting between the KK modes could be read off from this equation, as we could see in the following. Furthermore, from the normalization condition $\int_{-z_b}^{z_b} \Psi_n(z)\Psi_m(z)dz = \delta_{mn}$, the normalization factors are fixed by

$$
N_0^2 = 2z_b + \beta z_b^2, \quad N_n^2 \approx 2z_b + \beta z_b^2, \quad (n > 0). \quad (4.7)
$$

Thus for low excited states, we have

$$
\Psi_0(z) = \frac{(1 + \beta z)^{1/2}}{\sqrt{2z_b + \beta z_b^2}}, \quad \Psi_n(z) \approx \frac{(1 + \beta z)^{1/2}}{\sqrt{2z_b + \beta z_b^2}} J_0(m_n(z + 1/\beta)), \quad (n > 0). \quad (4.8)
$$

Further, with the decomposition (3.10), the normalized wave-functions are given by

$$
h_{\mu\nu}^0(x) = \frac{1}{\sqrt{2z_b + \beta z_b^2}} \varepsilon^0_{\mu\nu}(x), \quad h_{\mu\nu}^n(x,z) \approx \frac{J_0(m_n(z + 1/\beta))}{\sqrt{2z_b + \beta z_b^2}} \varepsilon^n_{\mu\nu}(x), \quad (n > 0). \quad (4.9)
$$

The interaction between gravitons and matter is achieved by including the action of SM matter fields with the action $S_m = \int d^5x \sqrt{g}L_m(x,z)$. Varying this action with respect to the metric, one has $S_{\text{int}} = \frac{1}{2} \int d^5x \sqrt{g}T^{MN}(x,z)\delta g_{MN} = \frac{\xi}{2} \int d^5x \sqrt{g}T^{MN}(x,z)h_{MN}(x,z)$,
where the factor $\xi = 2/M_s^{3/2}$ is chosen to give the 5-dimensional field $h_{\mu\nu}$ a correct dimension, namely, $h_{\mu\nu} \to \xi h_{\mu\nu}$. When the matter fields located on the visible brane, i.e., $T^{MN} = T^{\mu\nu}(x)\delta_\mu^M \delta_\nu^N \delta(z)$, it gives the usual form of the interaction Lagrangian in the 4-dimensional effective theory [34]

$$L_{\text{int}} = \frac{\xi}{2} \hat{T}^{\mu\nu}(x) h_{\mu\nu}(x, 0). \quad (4.10)$$

When $z = 0$, $J_0(m_0/\beta) \approx 1$, eq. (4.8) shows $\Psi_n(x, 0) \approx \Psi_0(x, 0)$, thus (4.9) is reduced to $h^n_{\mu\nu}(x, 0) = \epsilon^n_{\mu\nu}(x)/(2z_b + \beta z_b^2), (n \geq 0)$. Therefore eq. (4.10) gives

$$L_{\text{int}} = \frac{\xi}{2\sqrt{2z_b + \beta z_b^2}} \hat{T}^{\mu\nu}(x) \epsilon^n_{\mu\nu}(x) = \frac{\xi}{\sqrt{2z_b + \beta z_b^2}} \hat{T}^{\mu\nu}(x) \epsilon^n_{\mu\nu}(x), \quad (n \geq 0), \quad (4.11)$$

where $\xi = 1/(M_s^{3/2} \sqrt{2z_b + \beta z_b^2})$ is the effective coupling constant. It shows that the coupling of both the massless graviton and massive KK gravitons to matters are of the same order in our model, while in RS1 model the couplings of massless and massive gravitons to the matter fields on its visible brane are quite different: the coupling of massless mode is of order $1/M_{Pl}$ and the coupling of massive KK modes is of order $1/\text{TeV}$.

As is well known, RS1 model can provide an exponential twist mechanism to solve the hierarchy problem relying on the exponential warp factor and two-brane configuration. Thus we simply discuss the possibility of solving the mass hierarchy problem based on our simple model. Due to the embedding of branes in a Weyl bulk, some interesting features are expected to emerge. As in [8], including only the massless zero mode in the fluctuation metric (3.3)

$$d s_5^2 = a^2(z) [g_{\mu\nu}^{(4)}(x) dx^\mu dx^\nu + dz^2] = a^2(z) [(\eta_{\mu\nu} + h^0_{\mu\nu}(x)) dx^\mu dx^\nu + dz^2]. \quad (4.12)$$

This will provide the gravitational fields in our effective theory. Here we note that since we have assumed that the Weyl scalar depends only on the extra dimension, the condition (1.1) on these thin branes is $\tilde{\nabla}_\alpha \tilde{g}_{\mu\nu} = \omega_\alpha \tilde{g}_{\mu\nu} = 0$, and this implies that the connection $\Gamma^{(4)}_{\lambda\mu\nu}$ is just the Christoffel symbol constituted by the induced metric $\tilde{g}_{\mu\nu}(x) = a^2(z_0)g_{\mu\nu}^{(4)}(x)$ on the brane located at $z_0$. Thus the geometry of these thin branes is actually Riemannian.

Now calculating the contribution of the massless zero mode sector of the action (2.3) gives us the 4-dimensional effective gravitational theory

$$S_5^W \supset M_s^3 \int_{M_5^W} d^5x \sqrt{|g(\omega)| \hat{R}} \supset M_s^3 \int_{-z_b}^{z_b} dz a^3(z) f(\omega) \int_{M_4^W} d^4x \sqrt{|g^{(4)}| \hat{R}^{(4)}}, \quad (4.13)$$

where $M_s^{-3} = 16\pi G_5$, and $\hat{R}^{(4)}$ is the four-dimensional Riemannian Ricci scalar made out of $g_{\mu\nu}^{(4)} = \eta_{\mu\nu} + h^0_{\mu\nu}(x)$. Thus with the solution (2.28) in case 2, the 4-dimensional effective scale of gravitational interaction is read from above equation as

$$M_{Pl}^2 = M_s^3 \int_{-z_b}^{z_b} dz a^3(z) f(\omega) = M_s^3 (2z_b + \beta z_b^2), \quad (4.14)$$
where $M_{Pl}$ is our fundamental 4-dimensional Planck scale and $M_{Pl}^2 = 16\pi G_N$ with $G_N$ the Newton’s gravitational constant.

On the other hand, consider a fundamental Higgs field on our visible brane, its action can be written as

$$S_H \supset \int d^4x \sqrt{|\tilde{g}|} \left[ \tilde{g}^{\mu\nu} D_\mu H^\dagger D_\nu H - \lambda (H^\dagger H - v_0^2)^2 \right],$$  \hspace{1cm} (4.15)$$

where $v_0$ is the vacuum expectation value (VEV) of Higgs scalar field and $\tilde{g}_{\mu\nu}(x)$ the induced metric on the visible brane. In case 2, the warp factor is $a(0) = 1$, thus it leads to the induced metric $\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}^{(4)}(x)$. Plugging this into the above action gives the effective action for the Higgs field

$$S_H \supset \int d^4x \sqrt{|g^{(4)}|} \left[ g^{(4)\mu\nu} D_\mu H^\dagger D_\nu H - \lambda (H^\dagger H - v_0^2)^2 \right].$$  \hspace{1cm} (4.16)$$

Here it shows that the Higgs field is just the usually 4-dimensional canonically normalized form on the brane, thus the VEV scale takes its physical value. Furthermore, since the Higgs VEV sets all the mass parameters, any effective physical mass $m_{\text{vis}}$ on this visible brane is identical to its mass parameter $m_*$ in the fundamental theory

$$m_{\text{vis}} = m_*.$$  \hspace{1cm} (4.17)$$

Eqs. (4.14) and (4.17) provide us a mechanism that can be used to solve the mass hierarchy problem. Thus if we set all the fundamental parameters $M_*, p, v_0$ to be of order of TeV scale in our theory, then from (4.14), we only require $\beta z_b \approx 10^{16}$ to provide a large twist of the two scale $\beta M_{Pl}^2 \approx 10^{32} M_*^2$.

On the other hand, in this case the mass spectrum given by eq. (4.6) is

$$m_n = x_n \beta \approx (\beta z_b)^{-1} x_n \beta \approx 10^{-4}\text{eV.}$$

Thus the spacing of the KK gravitons is quite small and it seems that the masses are tiny enough to allow energetics to produce these KK gravitons in colliders. Nevertheless, as is shown in eq. (4.8), both the massless mode and lower massive KK modes are suppressed on the visible brane and localized on the other invisible one. Thus the effective coupling constant in (4.11) is set to be $\tilde{\xi} = 1/(M_*^{3/2} \sqrt{2z_b + \beta z_b^2}) = 1/M_{Pl}$. It means that both massless and massive gravitons interact with our matter fields on the brane with 4-dimensional gravitational strength $1/M_{Pl}$, therefore these light KK gravitons can certainly not be seen individually. This is in contrast with that of RS1 model where the spacing of KK gravitons is of order of TeV scale, but similar to ADD model [6, 7] with two extra dimensions, where the spacing is also tiny and about $10^{-3}\text{eV}$.

Furthermore, the relation of non-conformal physical coordinate $y$ and conformal coordinate $z$ could be easily obtained via the coordinate transformation $dy = a(z)dz$, and it is expressed as

$$\beta z_b = e^{pyb} - 1, (k = -2),$$  \hspace{1cm} (4.18)$$

$$\beta z_b = [1 - 2(2 + k)pyb]^{\frac{1+k}{1+2k}} - 1, (k < -\frac{3}{2}, k \neq -2).$$  \hspace{1cm} (4.19)$$

Thus for $\beta z_b \approx 10^{16}$, these relations give us the physical size of extra dimension $y_b$. For $k = -2$, the warp factor $a(y) = e^{-py}$ is similar to that of RS1 model and provides an exponential
twist for extra dimension, thus the physical size is given by $y_b \approx 37/p \approx 10^{-15}$ cm. When $k < -2$, as the warp factor is not an exponential one, the twist provided for hierarchy is slow, thus the size of extra dimension needs to be set quite larger than the former case, for example, when $k = -3$, $y_b \approx 10^{10}/p$. And for $-2 < k < -3/2$, there is a horizon at the point $\frac{1}{2(2+k)p}$. Thus, in order to provide a large enough twist for the hierarchy problem, interestingly, the brane should be located near enough to the horizon, i.e., $y_b \approx \frac{1}{2(2+k)p}$.

5. Conclusions and Discussion

In this paper, we have considered a brane world model in Weyl integrable geometry. For the Weyl scalar depending only on the extra dimension, the geometry on the brane is still Riemannian. With the action (2.1) and metric ansatz (2.7), the theory naturally reduces to three cases to find brane solutions. In case 1, since the action is invariant under the Weyl rescaling, the Weyl scalar is a hidden degree of freedom. Therefore, only one constraint of the warp factor and Weyl scalar is got from the equations of motion. The gravitational phenomenology of this case is similar to the ADD scenario with an $S^1/Z_2$ extra dimension. Then in case 2 and case 3, we get thin brane solutions with two flat branes located at the boundaries. Since the Weyl scalar enters into the massless graviton wave-function in gravitational fluctuations (3.14), the massless gravitons in case 3 are still localized on the brane at $z_b$, while they are localized on the brane at origin in case 2. As gravitational zero modes are not normalizable when $z \to \infty$, compactifying the extra dimension is indeed necessary in our model.

With the existence of cosmology constant in 5-dimensional background, RS1 model supports an $AdS_5/Z_2$ bulk solution. And as a result of the fine tuning of brane tensions, two flat branes are constituted at boundaries of compact extra dimension. While in our simple toy model, without introducing the delta functions by hand, but by means of the Weyl scalar and $S^1/Z_2$ extra dimension, we constitute a two-brane configuration. The effective energy densities are infinite on the branes, such as $\hat{\rho}(0) = -4kp\delta(0)$ on the invisible brane in case 2. Thus only up to some constant factors, these effective densities equivalently play the roles of brane tensions (energy densities localized on the branes) as in RS1 model, and they actually compensate the effects produced by the bulk components of the effective energy densities and hence insure the existence of these 4-dimensional flat Minkowski branes.

As is well known, RS1 model can provide an exponential twist of hierarchy problem resorting to the exponential warp factor and two-brane configuration, thus finally we have simply discussed the possibility of solving the mass hierarchy problem based on the solution in case 2. Our world is assumed living on the brane located at the origin, and with all the fundamental parameters to be of order of TeV scale and $\beta z_b \approx 10^{16}$, a large hierarchy can be generated from the 4-dimensional Planck scale and fundamental weak scale. Moreover, the spacing of KK modes is found to be of order of $10^{-4}\text{eV}$ in this case, however, since the lower massive KK gravitons are suppressed on our visible brane and the strength of their interaction with our matter fields on the brane is the weak 4-dimensional gravitational one $1/M_{Pl}$, the light KK gravitons cannot be seen individually in colliders.
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