Abstract To identify the $3\alpha$ BEC state with the excess neutron, we have investigated the monopole strength of the excited states of $^{13}\text{C}$ by using the theoretical framework of the real-time evolution method. The calculations have revealed several candidates of the Hoyle-analog states in highly excited region.

1 Introduction

In this decade, it has been intensively discussed that the Hoyle state (the $0^+_2$ state of $^{12}\text{C}$) is a dilute gas-like $\alpha$ cluster state [1–4] and regarded as a BEC of three $\alpha$ particles [5,6]. The idea of the $\alpha$ particle condensate has been extended to the other excited states of $^{12}\text{C}$. Namely, the $2^+$ state at 10.03 MeV [7,8] and the $4^+$ state at 13.3 MeV [9] are expected as the rotational excitation mode of the Hoyle state and assigned as the “Hoyle band” [10,11]. More recently, the $0^+_3$ state at 10.3 MeV has been suggested as the “breathing mode” of the Hoyle state [11–14].

The Hoyle-analog states in neighboring nuclei have also been the important subjects. In particular, $^{13}\text{C}$ is an interesting system as it is composed of three $\alpha$ particles (bosons) and an excess neutron (fermion). It is a fascinating question how the $3\alpha$ BEC is disturbed by the addition of a neutron as an impurity. It has been also studied in other fields that quantum dynamics of impurities within a bath of bosons has led to many important exotic effects in solid-state, plasma, and atomic physics [15,16]. For example, in recent years, researchers have observed the formation of quasi-particles with the possibility of exploring impurity physics in the presence of dipolar interactions [17]. Coupling two distinct atomic species could help us to explore new quantum behaviors of ultracold gases. In order to answer the question on the side of nuclear physics, several experimental and theoretical studies have been conducted [18–22], but the conclusion still remains controversial.

Recently a theoretical model named the real-time evolution method (REM) [23,24] has been proposed as a powerful approach to the $\alpha$ clustering of light nuclei. It was demonstrated that the model precisely describes the
Hoyle state (3α BEC). Furthermore, the model has also been applied to the low-lying states \(^{13}\text{C}\) to investigate the underlying symmetry of the nuclear shape \([25]\). Therefore, it is natural to extend the discussion to the highly excited states of \(^{13}\text{C}\) to search for the Hoyle-analog state (3α BEC plus an excess neutron). In this work, we report the structure of the highly excited states of \(^{13}\text{C}\) calculated by REM. We focus on the excited 1/2\(^{-}\) states and their monopole strengths as a signature of the BEC formation.

2 Theoretical Framework

The Hamiltonian and the theoretical framework used in this work are essentially same with those in Ref. \([25]\). The Hamiltonian is given as,

\[
\hat{H} = \sum_{i=1}^{13} \hat{t}_i + \sum_{i<j}^{13} \hat{v}_N(r_{ij}) + \sum_{i<j}^{13} \hat{v}_C(r_{ij}) - \hat{t}_{cm},
\]

where \(\hat{t}_i\) and \(\hat{t}_{cm}\) are the kinetic energies of nucleon and that of the center-of-mass, respectively. The nucleon-nucleon interaction \(\hat{v}_N\) is composed of the Volkov No. 2 force \([26]\) with the exchange parameters \(W = 0.4, B = H = 0.125\) and \(M = 0.6\), and the spin-orbit part of the G3RS force \([27]\) with the strength of \(u_{ls} = 1000\) MeV.

The basis wave function of REM is the Brink-Bloch wave function \([28]\) which consists of three \(\alpha\) clusters with (0s)\(^4\) configuration coupled with an excess neutron,

\[
\Phi(Z_1, \ldots, Z_4) = A \left\{ \Phi(Z_1)\Phi(Z_2)\Phi(Z_3)\phi(r, Z_4)\chi_n \right\}, \quad (2)
\]

\[
\Phi(Z) = A \left\{ \phi(r_1, Z)\chi_p \cdots \phi(r_4, Z)\chi_n \right\}, \quad (3)
\]

\[
\phi(r, Z) = (2\nu/\pi)^{3/4} \exp \left\{ -\nu (r - Z)^2 \right\}, \quad (4)
\]

where \(\Phi(Z_1)\ldots\Phi(Z_3)\) describe the \(\alpha\) clusters. The excess neutron is located at \(Z_4\) and its spin is fixed to up. The size parameter \(\nu\) is fixed to 0.235 fm\(^{-2}\) to reproduce the radius of \(\alpha\) particle. We calculate the real-time evolution of \(Z_1, \ldots, Z_4\) by solving the equation-of-motion, and obtain a set of the basis wave functions,

\[
\frac{i\hbar}{\partial t} \sum_{j,\sigma} C_{ij\sigma} \frac{dZ_{j\sigma}}{dt} = \frac{\partial H}{\partial Z_{j\sigma}} \hat{H}, \quad \mathcal{H} = \frac{\langle \Phi|\hat{H}|\Phi \rangle}{\langle \Phi|\Phi \rangle}, \quad C_{ij\sigma} = \frac{\partial^2}{\partial Z_{j\sigma} \partial Z_{i\sigma}} \ln(\langle \Phi|\Phi \rangle).
\]

Then, we apply the generator coordinate method (GCM) superposing the wave functions,

\[
\Psi^{1/2}_{M} = \sum_{iK} \hat{P}_{MK}^{1/2} f_{iK} \Phi(Z_1(t_i), \ldots, Z_4(t_i)), \quad (6)
\]

where \(\hat{P}^{1/2}_{MK}\) is the parity and the angular momentum projector and the real-time \(t_i\) is discretized. The amplitude \(f_{iK}\) and eigen-energy are determined by solving the Hill-Wheeler equation \([29]\).

From thus-obtained wave functions, we calculate the monopole transition matrix elements as a signature of the Hoyle-analog state,

\[
M_{E0,\text{IS0}} = \left\langle \Psi^{1/2}_{M} | \mathcal{M}_{E0,\text{IS0}} | \Psi^{1/2}_{M} \right\rangle (g.s.), \quad (7)
\]

where \(\Psi^{1/2}_{M}\) (g.s.) and \(\Psi^{1/2}_{M}\) (ex.) denote the wave functions of the ground state and excited 1/2\(^{-}\) states, respectively. \(\mathcal{M}_{E0,\text{IS0}}\) is either of the electric (E0) or isoscalar (IS0) monopole operator,

\[
\mathcal{M}_{E0} = e^2 \sum_{i=1}^{A} r_i^2 \frac{1 + \tau_i^z}{2}, \quad \mathcal{M}_{IS0} = \sum_{i=1}^{A} r_i^2. \quad (8)
\]
The Isoscalar Monopole Strength of $^{13}\text{C}$

Fig. 1 Calculated and observed partial level scheme of $^{13}\text{C}$. The energy is measured relative to the $^{12}\text{C} + n$ threshold.

Table 1 The properties of the ground and excited $1/2^-$ states; excitation energies; matter, neutron and proton root-mean-square radii; isoscalar and electric monopole transition matrices. The energies, radii, isoscalar and electric transition matrices are given in the unit of MeV, fm, fm$^2$ and e fm$^2$, respectively. The observed data [30] are also given.

| REM | Exp. |
|-----|------|
| $E_x$ | $\sqrt{\langle r_m^2 \rangle}$ | $\sqrt{\langle r_n^2 \rangle}$ | $\sqrt{\langle r_p^2 \rangle}$ | $M_{1SO}$ | $M_{E0}$ | $E_x$ | $M_{1SO}$ |
| g.s. | 0.0 | 2.44 | 2.49 | 2.39 | – | – | 0.0 | – |
| ex. | 14.1 | 3.14 | 3.56 | 2.57 | 6.0 | 2.2 | 8.86 | 6.1 |
| | 14.6 | 3.03 | 3.34 | 2.61 | 3.7 | 1.1 | 11.08 | 4.2 |
| | 15.1 | 3.00 | 3.27 | 2.63 | 12.5 | 5.6 | 12.50 | 4.9 |

3 Results and Discussions

Figure 1 shows the calculated and observed spectra of the $1/2^\pm$, $3/2^\pm$ and $5/2^\pm$ states, which is essentially same with that discussed in our previous work [25].

Although the calculation yields the correct order of the ground band ($1/2^-$, $3/2^-$ and $5/2^-$ states), it overestimates the moment-of-inertia. Furthermore, it does not yield some of the observed negative-parity states lying between the $^{12}\text{C} + n$ and $^{9}\text{Be} + \alpha$ thresholds. These discrepancies are due to the inability of the model in describing the distortion the cluster structure as we have assumed $3\alpha + n$ cluster structure. On the contrary, the calculation describes the positive-parity spectrum relatively better than the negative-parity one. This indicates that the positive parity states have more pronounced $3\alpha + n$ cluster structure than the low-lying negative-parity states.

Now we turn to the highly excited states, in particular, the $1/2^-$ states. They can have enhanced monopole transition strength which is regarded as a signature of the enhanced clustering. Indeed, an experiment has been conducted to measure the monopole transition strength and it reported three $1/2^-$ states as candidates of the pronounced $3\alpha + n$ clustering. Table 1 summarizes the properties of the ground state and the excited $1/2^-$ states. It is notable that all of the calculated $1/2^-$ states have considerably enhanced monopole strengths comparable with the Weisskopf estimate (1WU = 5.9 e fm$^2$), and show reasonable agreement with the data reported by an experiment [30]. Because of the enhanced monopole strength, they are regarded as the candidate of the Hoyle-analog states. In fact, their matter radii are much larger than the ground state showing their dilute density. However, the proton radii are not as large as that of the Hoyle state which is calculated as 3.7 fm by using the same Hamiltonian [2]. This implies the reduction of the $3\alpha$ BEC size due to the attraction between the $\alpha$ cluster and the excess neutron. It is also noted that the present calculation overestimates the observed excitation energies of these states, that tells us a need for improvement in our Hamiltonian and model wave functions.

To elucidate the internal structure of the ground and excited $1/2^-$ states, we have calculated the overlap between the $1/2^-$ states and Brink-Bloch wave functions, which is defined as,

$$|\langle \psi_n^{1/2^-} | \rho_{MK}^{1/2^-} \Phi(Z_1(t_i), ..., Z_4(t_i)) \rangle|^2,$$ (9)
Fig. 2 Density distributions of the Brink-Bloch wave functions which have maximum overlap with the $1/2^-_{1-4}$ states. Solid lines and color plots show the density distributions of $3\alpha$ particles and the excess neutron, respectively. Each panel shows the densities in $xy$ and $yz$ planes. Numbers in the panels show the maximum value of the overlap.

where $\Psi_n^{1/2^-}$ is the wave function of the $1/2^-_{1-4}$ states, while $\Phi(Z_1(t_i), ..., Z_4(t_i))$ is the basis wave function. The ground state maximally overlaps with the Brink-Bloch wave function shown in Fig. 1a. Note that the overlap is as large as 0.83, and hence, the ground state may be reasonably approximated by this wave function which has relatively compact spatial distribution. On the contrary, the excited $1/2^-_{1-4}$ states only have the maximum overlaps ranging 0.25–0.36 with the wave functions in panels (b)–(d). This clearly indicates that the excited states cannot be approximated by a single Brink-Bloch wave function as they have non-localized cluster configurations. Indeed, these excited states also overlap with various Brink-Bloch wave functions with different cluster configurations indicating that they do not have definite nuclear shape nor underlying spatial symmetry. This feature is common with the Hoyle state and reasonably in accordance with the dilute gas-like nature. Thus, within the scope of our study, these three excited states are good candidates of the Hoyle-analog state. The main difference from $^{12}$C is that there are three BEC candidates of $^{13}$C in both the experimental and theoretical results. We also note that our conclusion is contradictory to previous theoretical studies [20,22] which concluded that the BEC in $^{13}$C is not the $1/2^-_{1-4}$ states but the $1/2^+$ states. To identify the BEC state precisely, we need to investigate the spectroscopic factors and occupation probabilities, and such calculation is ongoing.

4 Summary

We have investigated the highly excited states of $^{13}$C and their monopole transition strengths to identify the Hoyle-analog state. We have obtained three candidates which have enhanced monopole transition strengths from the ground state. The calculated monopole strength are as large as those reported by an experiment, although the calculation overestimated the excitation energies. The calculation also showed that the radii of these candidates are considerably smaller than that of the Hoyle state. This implies that the extra neutron causes the shrinkage of $3\alpha$ BEC due to its interaction with $\alpha$ particles.

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References

1. E. Uegaki, Y. Abe, S. Okabe, H. Tanaka, Prog. Theor. Phys. **59**, 1031 (1978)
2. M. Kamimura, Nucl. Phys. A **351**, 456 (1981)
3. Y. Kanada-En’yo, Phys. Rev. Lett. 81, 5291 (1998)
4. M. Chernykh, H. Feldmeier, T. Neff, P. von Neumann-Cosel, A. Richter, Phys. Rev. Lett. 98, 032501 (2007)
5. A. Tohsaki, H. Horiiuchi, P. Schuck, G. Röpke, Phys. Rev. Lett. 87, 192501 (2001)
6. P. Schuck, Y. Funaki, H. Horiiuchi, G. Röpke, A. Tohsaki, T. Yamada, Phys. Scr. 91, 123001 (2016)
7. M. Freer, H. Fujita, Z. Buthelezi, J. Carter, R.W. Fearick, S.V. Förtsch, R. Neveling, S.M. Perez, P. Papka, F.D. Smit, J.A. Swartz, I. Usman, Phys. Rev. C 80, 041303 (2009)
8. M. Itoh, H. Akimune, M. Fujiwara, U. Garg, N. Hashimoto, T. Kawabata, K. Kawase, S. Kishi, T. Murakami, K. Nakanishi, Y. Nakatsugawa, B.K. Nayak, S. Okumura, H. Sakaguchi, H. Takeda, S. Terashima, M. Uchida, Y. Yasuda, M. Yosoi, J. Zenhiro, Phys. Rev. C 84, 054308 (2011)
9. M. Freer, S. Almaraz-Calderon, A. Aprahamian, N.I. Ashwood, M. Barr, B. Bucher, P. Copp, M. Couder, N. Curtis, X. Fang, F. Jung, S. Lesher, W. Lu, J.D. Malcolm, A. Roberts, W.P. Tan, C. Wheldon, V.A. Ziman, Phys. Rev. C 83, 034314 (2011)
10. M. Freer, H. Fynbo, Prog. Part. Nucl. Phys. 78, 1 (2014)
11. Y. Funaki, Phys. Rev. C 92, 021302 (2015)
12. C. Kurokawa, K. Kato, Phys. Rev. C 71, 021301 (2005)
13. S.I. Ohtsubo, Y. Fukushima, M. Kamimura, E. Hiyama, Prog. Theor. Exp. Phys. 2013, 73 (2013)
14. B. Zhou, A. Tohsaki, H. Horiiuchi, Z. Ren, Phys. Rev. C 94, 044319 (2016)
15. F.M. Cucchietti, E. Timmermans, Phys. Rev. Lett. 96, 210401 (2006)
16. J. Tempere, W. Casteels, M.K. Oberthaler, S. Knoop, E. Timmermans, J.T. Devreese, Phys. Rev. B 80, 184504 (2009)
17. A. Trautmann, P. Ilzöfiér, G. Durastante, C. Politi, M. Sohmen, M.J. Mark, F. Ferlaino, Phys. Rev. Lett. 121, 213601 (2018)
18. N. Furutachi, M. Kimura, Phys. Rev. C 83, 021303 (2011)
19. C. Wheldon, N.I. Ashwood, M. Barr, N. Curtis, M. Freer, T. Kokalova, J.D. Malcolm, V.A. Ziman, T. Faestermann, H.F. Wirth, R. Hertenberger, R. Lutter, Phys. Rev. C 86, 044328 (2012)
20. T. Yamada, Y. Funaki, Phys. Rev. C 92, 034326 (2015)
21. J.P. Ebran, E. Khan, T. Nikšić, D. Vretenar, J. Phys. G 44, 103001 (2017)
22. Y. Chiba, M. Kimura, Phys. Rev. C 101, 024317 (2020)
23. R. Imai, T. Tada, M. Kimura, Phys. Rev. C 99, 064327 (2019)
24. B. Zhou, M. Kimura, Q. Zhao, S.H. Shin, Eur. Phys. J. A 56, 298 (2020)
25. S. Shin, B. Zhou, M. Kimura, Phys. Rev. C 103, 054313 (2021)
26. A.B. Volkov, Nucl. Phys. 74, 33 (1965)
27. N. Yamaguchi, T. Kasahara, S. Nagata, Y. Akashi, Prog. Theor. Phys. 62, 1018 (1979)
28. D.M. Brink, Proc. Int. School of Physics Enrico Fermi, Course 36, Varenna (Academic Press, New York, 1966)
29. D.L. Hill, J.A. Wheeler, Phys. Rev. 89, 1102 (1953)
30. T. Kawabata, Y. Sasamoto, Y. Maeda, S. Sakaguchi, Y. Shimizu, K. Suda, T. Uesaka, M. Fujiwara, H. Hashimoto, K. Hatanaka, K. Kawase, H. Matsubara, K. Nakanishi, Y. Tameshige, A. Tamiil, K. Itoh, M. Itoh, H.P. Yoshida, Y. Kanada-En’yo, M. Uchida, Int. J. Mod. Phys. E 17, 2071 (2008)

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