Reconstructing Embedded Graphs from Persistence Diagrams

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Abstract

The persistence diagram (PD) is an increasingly popular topological descriptor. By encoding the size and prominence of topological features at varying scales, the PD provides important geometric and topological information about a space. Recent work has shown that particular sets of PDs can differentiate between different shapes. This trait is desirable because it provides a method of representing complex shapes using finite sets of descriptors. The problem of choosing such a set of representative PDs and then using them to uniquely determine the shape is referred to as reconstruction. In this paper, we present an algorithm for reconstructing embedded graphs in $\mathbb{R}^d$ (plane graphs in $\mathbb{R}^2$) with $n$ vertices from $n^2 - n + d + 1$ directional PDs. Lastly, we empirically validate the correctness and time-complexity of our algorithm in $\mathbb{R}^2$ on randomly generated plane graphs using our implementation, and explain the numerical limitations of implementing our algorithm.

1 Introduction

Topological data analysis (TDA) provides a set of promising tools to help analyze data in fields as varied as materials science, transcriptomics, and neuroscience [18 22 26]. The wide applicability is due to the fact that many forms of data can be modeled as graphs or simplicial complexes, two widely-studied types of topological spaces. Topological spaces are described in terms of their invariants—such as the homotopy type or homology classes. Persistent homology considers the evolution of the homology groups in a filtered topological space.

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Motivation  The problem of manifold and stratified space learning is an active research area in computational mathematics. For example, Chambers et al. use persistent homology in a stratified space setting [5], describing an algorithm to identify shapes that simplify a noisier shape, and then confirm that the given simplification still satisfies the desired topological properties. Zheng et al. also address a problem in space learning, and study the 3D reconstruction of plant roots from multiple 2D images [28], using persistent homology to ensure the resulting 3D root model is connected. Another reconstruction problem involves reconstructing road networks. Three common approaches to solving these problems involve Point Clustering, Incremental Track Insertion, and Intersection Linking [1]. Ge, Safa, Belkin, and Wang develop a point clustering algorithm using Reeb graphs to extract the skeleton graph of a road from point-cloud data [16]. The original embedding can be reconstructed using a principal curve algorithm [21]. Karagiorgou and Pfois give an algorithm to reconstruct a road network from vehicle trajectory GPS data by identifying intersections with clustering, then using vehicle trajectories to connect them [20]. Ahmed et al. provide an incremental track insertion algorithm to reconstruct road networks from point cloud data [2]. The reconstruction is done incrementally, using a variant of the Fréchet distance to partially match input trajectories to the reconstructed graph. Ahmed, Karagiorgou, Pfois, and Wenk describe all these methods in [1]. Finally, Dey, Wang, and Wang use persistent homology to reconstruct embedded graphs. This research has also been applied to input trajectory data [12]. Dey et al. use persistence to guide the Morse cancellation of critical simplices. We see from these applications the necessity for reconstruction algorithms, and in particular the necessity for reconstruction algorithms of graphs since much of the research involving reconstruction of road networks involves reconstructing graphs.

Our Contribution  In this work, we focus on graphs embedded in \( \mathbb{R}^d \) (plane graphs in \( \mathbb{R}^2 \)) and use directional APDs to reconstruct a graph. In particular, our main contributions are an upper bound on the number of APDs required for reconstructing embedded graphs in \( \mathbb{R}^d \), a polynomial-time algorithm for reconstructing plane graphs, and the first deterministic reconstruction algorithm for embedded graphs in arbitrary dimension.
The current paper is an extension of conference proceedings from CCCG 2018. We extend the proceedings paper in the following ways: (1) we revise proofs for clarity; (2) we extend our algorithms for graph reconstruction to $\mathbb{R}^d$; (3) we expand our literature review to include a discussion of recent results; (4) we publicly release code for our algorithm\textsuperscript{1}; and (5) we provide an experimental section to demonstrate the implementation.

2 Preliminaries

We begin by summarizing the necessary background information, but refer the reader to [4] for a more comprehensive overview of computational topology.

Plane Graphs Our main object of study is plane graphs with straight-line embeddings (referred to simply as plane graphs throughout this paper). A plane graph is a set of vertices and a set of straight line connections between pairs of vertices called edges (denoted by $V$ and $E$ respectively), such that no two edges in the embedding cross. We will frequently denote $|V| = n$ as the number of vertices in $G$. Throughout this paper, we make assumptions about the positioning of vertices in graphs.

Assumption 1 (General Position). Let $G$ be a graph embedded in $\mathbb{R}^d$ with vertices $V$. We assume that for all $x, y \in V$ where $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$, $x_i \neq y_i$ for any $i \in \{1, \ldots, d\}$. Furthermore, we assume that no $(d + 1)$ vertices are coplanar.

Height Filtration Let $G$ be a plane graph. Consider a direction $s$ in the unit sphere $S^{d-1}$ in $\mathbb{R}^d$; we define the lower-star filtration with respect to direction $s$ in two steps. First, let $h_s : G \rightarrow \mathbb{R}$ be defined for a simplex $\sigma \subseteq G$ by $h_s(\sigma) = \max_{v \in \sigma} v \cdot s$, where $v \cdot s$ is the inner (dot) product and measures height of $v$ in the direction of $s$, since $s$ is a unit vector. Thus, the height $h_s(\sigma)$ is the maximum height of all vertices in $\sigma$. Then, for each $t \in \mathbb{R}$, the subcomplex $G_t := h^{-1}_s([-\infty, t])$ is composed of all simplices that lie entirely below or at height $t$, with respect to direction $s$. Notice $G_t \subseteq G_t$ for all $r \leq t$ and $G_t = G_t$ if no vertex has height in the interval $[r, t]$. The sequence of all such subcomplexes, indexed by $\mathbb{R}$, is the height filtration with respect to $s$, denoted $F_s := F_s(G)$. Notice that the complex changes a finite number of times in this filtration. We note that the lower-star filtration is the discrete analog to a lower-levelset filtration, where the complex is intersected with a raising closed half-plane.

Axis Aligned Directions When considering a standard unit basis vector of $\mathbb{R}^d$ in dimension $i \in \{1, \ldots, d\}$, we will write $e_i$ to denote the direction.

\textsuperscript{1}The code is available in a git repo hosted on GitHub: https://github.com/compTAG/reconstruction
**Augmented Persistence Diagrams** The (directional) persistence diagram for a simplicial complex \( K \) is a summary of the homology groups \( H_\ast(K_t) \) as the height parameter \( t \) ranges from \(-\infty \) to \( \infty \): in particular, the persistence diagram is a set of birth-death pairs \((b_i, d_i)\), each with a corresponding dimension \( k_i \). Each pair represents an interval \([b_i, d_i)\) corresponding to a generator of the \( k_i \)-th homology group. In the specific application of plane graphs, a birth event may occur either when the height filtration discovers a new vertex \((k_i = 0)\), representing a new component, or when a one-cycle appears \((k_i = 1)\). Zero-dimensional deaths correspond to connected components merging. One-dimensional deaths occur when two cycles merge together, which includes the case where a cycle is filled in. By definition [14], all points in the diagonal \( y = x \) are also included with infinite multiplicity. However, in what follows, we use the multi-set of diagonal points that are explicitly computed, as in the algorithm given in Chapter VII of [14]. We refer to persistence diagrams containing only these explicitly computed diagonal points, rather than all points on the diagonal, as **augmented persistence diagrams** (APDs). We denote the space of all APDs by \( \mathcal{D} \).

For a direction \( s \in S^1 \), let the **directional augmented persistence diagram** \( D_s(F_s(K)) \) be the set of birth-death pairs for the \( i \)-th homology group from the height filtration \( F_s(K) \). As with the height filtration, we simplify notation and define \( D_s(s) := D_s(F_s(K)) \) when the complex is clear from context. If \( i \) is not specified, we write \( D(F_s(K)) \) to indicate the set of birth death pairs for homology groups in all dimensions from the height filtration \( F_s(K) \). We denote \( \beta_i \) to be the \( i \)-th Betti number, i.e., the rank of the \( i \)-th homology group. In general, the complexity of computing a persistence diagram is matrix multiplication time, with respect to the number \( n \) of simplicies in the filtration: that is, the complexity is \( \Theta(n^\omega) \), where \( \omega \) corresponds to the smallest known exponent for matrix multiplication time. In some cases (e.g., for computing \( D_0(F) \) or \( D_{d-1}(F) \) when \( F \) is a height filtration in \( \mathbb{R}^d \)), the computation time is \( \Theta(n\alpha(n)) \), where \( \alpha \) is the inverse Ackermann function.

In what follows, for the unknown complex \( K \), we assume that we have an **oracle** \( \mathcal{O} \) that can take a direction \( s \in S^{d-1} \) and a value \( i \in \mathbb{Z} \), and return the diagram \( D_i(s) \). We define \( T_G \) such that \( \Theta(T_G) \) is the time complexity for \( \mathcal{O} \) to return this diagram. Notice that \( T_G = \Omega(|D_i(s)|) \), where \(|D_i(s)|\) denotes the number of off-diagonal points in \( D_i(s) \).

Next, we state a lemma relating birth-death pairs in APDs to the simplices in \( K \). We omit the proof, but refer the reader to [14] pp. 120–121 of §V.4 for more details.

**Lemma 1** (Adding a Simplex). Let \( K \) be a simplicial complex and \( \sigma \subseteq K \) be a \( k \)-simplex such that \( \partial(\sigma) \subseteq K \). Then, the addition of \( \sigma \) to \( K \) will either increase \( \beta_k \) by one or decrease \( \beta_{k-1} \) by one.

Thus, we form a bijection between simplices of \( K \) and birth-death events in an APD. If \( G \) is any graph, then the maximum number of edges in \( G \) is \( n(n-1)/2 \), and so \(|E| = O(n^2)\). In the case when \( G \) is a plane graph, \(|E| = O(n)\) due the planarity of \( G \). Furthermore, an APD will have at least \( n \) points from
the vertices in $G$ corresponding to births in the zero-dimensional diagram. These observations give us the following corollary on the size of APDs for graphs.

**Corollary 2 (Size of Augmented Persistence Diagrams).** Let $G$ be an embedded graph and $n$ be the number of vertices in $G$. Then an augmented persistence diagram will have $O(n^2)$ birth-death pairs. In the case when $G$ is a plane graph, the augmented persistence diagram will have $\Theta(n)$ birth-death pairs.

## 3 Related Work

Let $K$ be a simplicial complex embedded in $\mathbb{R}^d$, for some $d \geq 2$. Turner et al. introduced the **persistent homology transform** $PHT_K : S^{d-1} \to \mathcal{D}$, defined by $PHT_K(s) = D(F_s(K))$; see [27]. Intuitively, the $PHT$ considers the persistent homology of a simplicial complex using filtrations induced from every direction in $S^{d-1}$. The **Euler characteristic transform** (ECT) defined by $ECT_K : S^{d-1} \to \mathbb{Z}$ is a similar function that maps each direction in $S^{d-1}$ to an Euler characteristic curve that tracks the Euler characteristic of each subcomplex induced by a height filtration. Turner et al. show that both of these functions are injective when the vertices of $K$ are in general position and $\dim(K) < d \leq 3$. Recently, variations of these functions have attracted interest in other research domains and researchers are realizing the potential of persistent homology as an effective data descriptor. For example, in [9] the **smooth Euler characteristic transform** (SECT) is introduced as a method of determining clinical outcomes using MRIs from patients with glioblastoma multiforme (GBM). Furthermore, a recent survey by Oudot and Solomon explores the current state of inverse problems in topological persistence as a potential tool for producing explainable data descriptors [25]. The authors suggest new research directions and potential applications in the field of machine learning using approaches such as the PHT and ECT. These experiments and extensions offer insight into new applications and future work for topological summary statistics and demonstrate effectiveness in new research domains, suggesting further exploration of these tools.

In order to leverage the injectivity for shape comparison, two approaches can be taken: (1) provide an algorithm that reconstructs the shape from a subset of the directions; (2) show that $PHT_K$ has a finite representation. In the current paper, we take the first approach and show that not only is $PHT_K$ injective for graphs embedded in Euclidean space, but we can select a finite set of directions $P \subset S^{d-1}$ that will allow us to reconstruct the original complex from the directional augmented persistence diagrams from directions in $P$. In particular, we prove that a quadratic number of directions (with respect to the number of vertices) is sufficient to reconstruct a graph, given an oracle that can compute directional APDs. One method to tackle the second approach is by observing that diagrams only provide new information when a transposition in the ordering of the filtration occurs. Furthermore, changes in the filtration bound changes in the distance between persistence diagrams [7]. For example,
consider \( d = 3 \) and a finite geometric simplicial complex \( K \). Since transpositions can only happen when two vertices are swapped in the filtration, the set of directions in \( S^2 \) for which two vertices occur at the same height in the height filtration is finite. In other words, there are two (most likely unequal) ‘hemispheres’ of \( S^2 \) for which the vertices occur at different heights. For each pair of vertices, we have two different hemispheres that maintain the ordering of these vertices relative to one another in the filtration. We can consider these hemispheres for all \( \binom{n}{2} \) pairs of vertices and consider the regions \( R \) for which the ordering is stable. For each region \( R \subset S^2 \), the persistence diagram continuously varies and no transpositions (or ‘knees’) are witnessed \([8]\). This alternate approach was investigated independently by \([10]\).

In an independent investigation \([17]\), Ghrist, Levanger, and Mai further explore the persistent homology transform (and other invertible transforms), providing an alternate proof of injectivity. The authors show that \( PHT_k \) can be converted into a Radon integral transform, which is shown to be invertible. However, in contrast to our work, the main result of \([17]\) is a proof of theoretical injectivity and does not describe any reconstruction methods that could be implemented in practice.

In an arXiv preprint \([10]\), Curry et al. proved a finite bound on the number of directions necessary to reconstruct an unknown geometric simplicial complex \( K \) in \( \mathbb{R}^d \) from Euler characteristic curves, i.e., using a finite subset of curves from \( ECT_K \). First, the embedding for the vertices of \( K \) are determined using an assumed lower bound \( \delta \) on the local geometry around any given vertex. The vertices are discovered by generating topological summaries (i.e., Euler characteristic curves or persistence diagrams) by choosing directions in \( S^{d-1} \) based on \( \delta \) to ensure that the embedding of each vertex is identified. Then, the \( (d-1) \)-dimensional sphere is stratified using hyperplanes that intersect pairs of vertices. Directions from each stratum are sampled to generate a set of Euler characteristic curves. Theorem 7.14 of \([10]\) shows that this set of Euler characteristic curves uniquely identifies the embedding of \( K \) with a finite number of Euler characteristic curves (or persistence diagrams). However, these bounds are dependent on assumptions about the curvature of the complex and are exponential in the dimension.

It should be noted that persistence diagrams encode at least the same information encoded by Euler characteristic curves, so this finite bound also applies to persistence diagrams. However, extending methods that use persistence diagrams for reconstruction to methods that use Euler characteristic curves for reconstruction introduces its own set of challenges, which are explored in \([15]\). In particular, certain arrangements of vertices are identified that are reconstructible by PDs generated by a finite set of directions that are not reconstructible by the ECCs generated from the same set of directions. Our work differs from these approaches by providing an explicit reconstruction algorithm, with complexity analysis and implementation.
4 Vertex Reconstruction

Next, we present an algorithm for recovering the locations of vertices of an embedded graph. We begin with a plane graph $G$, where we are able to use three augmented directional augmented persistence diagrams. We then extend this method for any embedded graph in $\mathbb{R}^d$, using $d + 1$ directional augmented persistence diagrams.

4.1 Vertex Reconstruction for Plane Graphs

Intuitively, for each direction, we identify the lines on which the vertices of $G$ must lie. We show how to choose specific directions so that we can identify all vertex locations by searching for points in the plane where three lines intersect. We call these lines filtration lines:

**Definition 3** (Filtration Hyperplanes and Filtration Lines). Given a direction $s \in S^{d-1}$ and a height $h \in \mathbb{R}$, the filtration hyperplane at height $h$ is the $(d-1)$-dimensional hyperplane, denoted $\ell(s, h)$, through $h \ast s$ and perpendicular to direction $s$, where $\ast$ denotes scalar multiplication. Given a finite set of vertices $V \subset \mathbb{R}^d$, the filtration hyperplanes of $V$ are the set of hyperplanes $L(s, V) := \{\ell(s, h_s(v))\}_{v \in V}$.

In the special case when $d = 2$, we refer to filtration hyperplanes as filtration lines.

Notice that all hyperplanes in $L(s, V)$ are parallel, and that $\ell(s, h) = \ell(-s, -h)$. Intuitively, if $v \in V$ where $V$ is the vertex set of some plane graph $G$, then the line $\ell(s, h_s(v))$ occurs at the height where the filtration $\mathcal{F}_s(G)$ includes $v$ for the first time. If the height is known but the complex is not, then we know that $v$ must be contained on the line $\ell(s, h_s(v))$. By Lemma 3, the births in the zero-dimensional augmented persistence diagram are in one-to-one correspondence with the vertices of the plane graph $G$. Thus, we can construct $L(s, V)$ from a directional diagram in $\Theta(n)$ time by iterating through the points in the zero-dimensional augmented persistence diagram. Using filtration lines, we show a correspondence between intersections of three sets of filtration lines and the vertices in $G$. In what follows, given a direction $s_i \in S^1$ and a point $p \in \mathbb{R}^2$, define $\ell_i(p) := \ell(s_i, h_{s_i}(p))$ as a way to simplify notation.

**Lemma 4** (Vertex Existence). Let $G = (V, E)$ be a plane graph and let $n = |V|$. Let $s_1, s_2 \in S^1$ be linearly independent and further suppose that $L(s_1, V)$ and $L(s_2, V)$ each contain $n$ lines. Let $A$ be the $n^2$ intersection points between lines in $L(s_1, V)$ and in $L(s_2, V)$. Let $s_3 \in S^1$ such that each $v \in A$ has a unique height in direction $s_3$. Then, for all $\ell \in L(s_3, V)$, $A \cap \ell \neq \emptyset$, i.e., each $\ell$ shares an intersection point with filtration lines from $s_1$ and $s_2$.

**Proof.** Assume, for contradiction, that there exists $\ell \in L(s_3, V)$ such that $A \cap \ell = \emptyset$. Since $\ell \in L(s_3, V)$, there is some vertex $v \in V$ such that $\ell = \ell_3(v)$. 

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Figure 1: A vertex set $V$ of size four, with three sets of filtration lines. Here, notice that $e_1 \in S_1$ and $e_2 \in S_1$ are linearly independent, and the third direction $s$ satisfies the assumptions of Lemma 4. The lines of $L(e_1, V)$ are the blue vertical lines, $L(e_2, V)$ are the black horizontal lines, and $L(s, V)$ are the pink diagonal lines. The three-way intersection points (one from each set of filtration lines) is in one-to-one correspondence to the vertices in $V$. Lines 4 through 7 in Algorithm 1 provide details for finding $s_3$ using the width of the vertical lines (marked $w$) and the minimum height difference between horizontal lines (marked $h$).

and $v$ lies on $\ell$. However, $v$ is in $\ell_1(v) \cap \ell_2(v) \subset A$, contradicting the hypothesis that $A \cap \ell = \emptyset$.

If we generate vertical lines, $L_V = L(e_1, V)$, and horizontal lines, $L_H = L(e_2, V)$, for our first two directions, then only a finite number of directions in $S_1$ have been eliminated for the choice of $s_3$. In the next lemma, we choose a specific third direction by considering a bounding region defined by the largest distance between any two lines in $L_V$ and smallest distance between any two consecutive lines in $L_H$. Then, we pick the third direction so that if one of the corresponding lines intersects the bottom left corner of this region then it will also intersect the along the right edge of the region. In Figure 1 the third direction was computed using this procedure with the region having a width that is the length between the left most and right most vertical lines, and height that is the length between the top two horizontal lines. Next, we give a more precise description of the vertex localization procedure.

**Lemma 5 (Vertex Localization).** Let $L_H$ and $L_V$ be $n$ horizontal and $n$ vertical lines, respectively. Let $w$ (and $h$) be the largest (and smallest) distance between two lines of $L_V$ (and $L_H$, respectively). Let $B$ be the smallest axis-aligned bounding region containing the intersections of lines in $L_H \cup L_V$. Let $s = (w, h/2)/||w, h/2||$, i.e., a unit vector oriented towards the point $(w, h/2)$. Any line parallel to $s$ can intersect at most one line of $L_H$ in $B$. 

Proof. Note that, by definition, $s$ is a vector in the direction that is at a slightly smaller angle than the diagonal of the region with width $w$ and height $h$. Assume, by contradiction, that a line parallel to $s$ can intersect two lines of $L_H$ within $B$. Specifically, let $\ell_1, \ell_2 \in L_H$ and let $\ell_s$ be a line parallel to $s$ such that the points $\ell_1 \cap \ell_s = (x_i, y_i)$ for $i = \{1, 2\}$ are the two such intersection points within $B$. Since the lines of $L_H$ are horizontal and by the definition of $h$, we observe that $|y_1 - y_2| \geq h$. Let $w' = |x_1 - x_2|$, and observe $w' \leq w$. Since the slope of $\ell_s$ is $\frac{h}{w'}$, we have $|y_1 - y_2| < h$, which is a contradiction. \hfill \Box

We conclude the discussion of plane graph reconstruction with an algorithm to determine the coordinates of the vertices of the original graph in $\mathbb{R}^2$, using only three height filtrations.

**Theorem 6 (Vertex Reconstruction).** Let $G$ be a plane graph. We can compute the coordinates of all $n$ vertices of $G$ using three directional augmented persistence diagrams in $\Theta(n \log n + T_G)$ time, where $\Theta(T_G)$ is the time complexity of computing a single directional augmented persistence diagram for $G$.

**Proof.** We proceed with a constructive proof that is presented as an algorithm in Algorithm 1. Let $O$ be an oracle that can take a direction $s \in S^1$ and returns the zero-dimensional directional APD for the unknown plane graph $G$ in direction $s$ in $\Theta(T_G)$ time.

We start with requesting two directional augmented persistence diagrams from the oracle, $D_0(e_1)$ and $D_0(e_2)$. Note that, by our general position assumption, no two vertices of $G$ share an $x$- or $y$-coordinate. By Corollary 2 the sets $L(e_1, V)$ and $L(e_2, V)$ (which we do not explicitly construct) each contain $n$ distinct lines. Let $L_1 = \{h_1, h_2, \ldots, h_n\}$ be the resulting set of heights of lines in $L(e_1, V)$, in increasing order. Likewise, let $L_2 = \{h'_1, h'_2, \ldots, h'_n\}$ be the ordered set of heights of lines in $L(e_1, V)$, also in increasing order. We explicitly construct and sort these two sets $L_1$ and $L_2$: the birth times in the persistence diagrams correspond to heights of the filtration lines in $L(e_1, V)$.

Let $A$ be the set of $n^2$ intersection points between the lines in $L(e_1, V)$ and in $L(e_2, V)$. Exactly $n$ of these points correspond to vertices of $V$. The next step is to identify a third direction $s$ such that each line in $L(s, V)$ intersects with only one point in $A$, which we will use in order to distinguish which $n$ intersection points correspond to vertices in $V$.

Let $w = h_n - h_1$ and let $h$ be the minimum of $\{h'_i - h'_{i-1}\}_{i=2}^n$. In words, $w$ is the difference between the maximum and minimum heights of lines in $L(e_1, V)$ and $h$ is the minimum height difference between consecutive lines in $L(e_2, V)$; see Figure 1. Note that we can compute $w$ in $\Theta(1)$ time from $L_1$ and $h$ in $\Theta(n)$ time from $L_2$. Let $B$ be the smallest axis-aligned bounding region containing the intersection points $A$, and let $s$ be a unit vector perpendicular to the vector $[w, h]$. We request the set $D_0(s)$ from our oracle $O$. As before, the heights of the lines in $L(s, V)$ are the birth times of points in $D_0(s)$. We save this set of heights as $L_s$ in $\Theta(n)$ time, and sort $L_s$ in $\Theta(n \log n)$ time.

Finally, by Lemma 5 any line in $L(s, V)$ intersects no more than one line of $L(e_2, V)$ within $B$. Furthermore, by Lemma 4 each line in $L(s, V)$ inte-
sects \( A \). Thus, there are exactly \( n \) intersection points of \( L(s, V) \) with the set \( A \), locating the \( n \) vertices in \( G \). We compute these intersections by intersecting the \( i \)-th line of \( L(e_2, V) \) with the \( i \)-th line of \( L(s, V) \) in \( \Theta(n) \) time.

In total, this algorithm, summarized in Algorithm 1, uses three directional diagrams, two requested from the oracle in Line 1 and one requested in Line 8. These two lines take \( \Theta(T_G) \) time each, Lines 1 and 8 take \( \Theta(n \log n) \) time each, and the for loop in Lines 11 through 15 takes \( \Theta(n) \) time. All other lines are linear or constant, with respect to \( n \). Thus, the total time complexity is \( \Theta(n \log n + T_G) \).

**Algorithm 1** Reconstruct Vertices

**Input:** Oracle \( \mathcal{O} \) for an unknown graph \( G = (V, E) \subset \mathbb{R}^2 \).

**Output:** Set of vertex locations.

1. Consult \( \mathcal{O} \) to obtain diagrams \( D_0(e_1) \) and \( D_0(e_2) \)
2. \( L_1 \leftarrow \) birth times from points in \( D_0(e_1) \)
3. \( L_2 \leftarrow \) birth times from points in \( D_0(e_2) \)
4. Sort \( L_1 \) and \( L_2 \) in increasing order
5. \( w \leftarrow \) maximum minus minimum in \( L_1 \)
6. \( h \leftarrow \) minimum gap between two consecutive values in \( L_2 \)
7. \( s \leftarrow \) a unit vector perpendicular to the vector \([w, h/2]\)
8. Consult \( \mathcal{O} \) to obtain diagram \( D_0(s) \)
9. \( L_s \leftarrow \) birth times from points in \( D_0(s) \)
10. Sort \( L_s \) in increasing order
11. for \( i = 1, 2, \ldots, n \) do
12. \( \ell_i \leftarrow \) horizontal line with the \( i \)-th element of \( L_2 \) as the \( y \)-coordinate
13. \( \ell'_i \leftarrow \) line perpendicular to \( s \) at height equal to the \( i \)-th element of \( L_s \)
14. \( v_i = \ell_i \cap \ell'_i \)
15. end for
16. return \( \{v_i\}_{i=1}^n \)

### 4.2 Vertex Reconstruction in \( \mathbb{R}^d \)

The vertex reconstruction algorithm of the previous subsection generalizes to higher dimensions. In \( \mathbb{R}^d \), a filtration line becomes a filtration hyperplane, a \((d-1)\)-dimensional hyperplane that goes through one of the vertices in the vertex set (and is perpendicular to a given direction). Similar to filtration lines, filtration hyperplanes generated by a fixed direction are parallel and are in a 1-1 correspondence with the vertices (for almost all directions).

**Lemma 7** (Generalized Vertex Existence). Let \( G = (V, E) \) be a straight-line embedded graph in \( \mathbb{R}^d \). Let \( s_1, s_2, \ldots, s_d \) be linearly independent directions in \( \mathbb{S}^{d-1} \) and further suppose that \( L(s_i, V) \) contain \( n \) filtration hyperplanes for each \( i \in \{1, 2, \ldots, d\} \). Choosing one hyperplane in each set \( L(s_i, V) \), the intersection of these hyperplanes is a point. Let \( A \) denote the \( n^d \) such intersection points.
Let $s_{d+1} \in \mathbb{S}^{d-1}$ such that $s_{d+1} \neq \pm s_i$ for any $i \in \{1,2,\ldots,d\}$, and $L(s_i, V)$ contains $n$ filtration hyperplanes. Then, for all $\ell \in L(s_{d+1}, V)$, $A \cap \ell \neq \emptyset$, i.e., each $\ell$ shares an intersection point with filtration lines from $s_1, \ldots, s_d$.

Proof. Assume, for contradiction, that there exists $\ell \in L(s_{d+1}, V)$ such that $A \cap \ell = \emptyset$. Since $\ell \in L(s_{d+1}, V)$, there exists a vertex $v \in V$ such that $\ell = \ell_{d+1}(v)$ and $v$ lies on $\ell$. However, $v$ is in $\ell_1(v) \cap \ldots \cap \ell_d(v) \subset A$, contradicting the hypothesis $A \cap \ell = \emptyset$. $\square$

Just as in the case of plane graphs, we can now describe a method for locating all vertices. The following lemma is a higher-dimensional analogue of Lemma 5.

**Lemma 8** (Generalized Vertex Localization). Let $G = (V, E)$ be a straight-line embedded graph in $\mathbb{R}^d$, with $n = |V|$. Choosing one hyperplane in each set $L(e_i, V)$ for $1 \leq i \leq d$, the intersection of these hyperplanes is a point. Let $A$ denote the $n^d$ such intersection points. Then, we can find a direction $s$ such that each hyperplane in $L(s, V)$ intersects at most one of the $n^d$ points in $A$ in $\Theta(dn \log n)$ time.

Proof. Let $L_i = \{h^i_1, h^i_2, \ldots, h^i_n\}$ be the ordered set of heights of lines in $L(e_i, V)$.

Let $w_i = h^i_n - h^i_1$ be the largest distance between any two hyperplanes in $L(e_i, V)$, and let $w = \max_{1 \leq i \leq d} w_i$. Let $h_i = \{h^i_j - h^i_{j-1}\}_{j=2}^n$ be the smallest height difference between any two (adjacent) hyperplanes in $L(e_i, V)$, and let $h = \frac{1}{2} \min_{1 \leq i \leq d} h_i$. We describe next how to choose a direction perpendicular to the hyperplane that intersects the origin and each $we_i + \langle 0, \ldots, 0, \frac{h}{d-1} \rangle$. Let our hyperplane $H$ be defined by the rows in

$$
H = \begin{bmatrix}
0 & 0 & \ldots & 0 & 0 \\
\frac{h}{d-1} & \frac{h}{d-1} & \ldots & \frac{h}{d-1} \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & \frac{h}{d-1}
\end{bmatrix}
$$

Then, we choose a vector orthogonal to the hyperplane by solving $Hs = 0$. We note that there are two solutions, but without loss of generality, we choose $s = \left[-\frac{1}{w}, \ldots, -\frac{1}{w}, \frac{d-1}{w} \right]^T$ to solve the equation.

We now show that $s$ satisfies the claim that each hyperplane in $L(s, V)$ intersects at most one of the $n^d$ points in $A$ in $\Theta(dn \log n)$ time. Let $p = (p_1, p_2, \ldots, p_d)$ and $q = (q_1, q_2, \ldots, q_d)$ be points in $A$ with $p \neq q$, and assume, for contradiction, that they lie on the same hyperplane in $L(s, V)$. Then, $s \cdot p = s \cdot q$. By the definition of dot product, we have the following equation:

$$
\frac{d-1}{h} (p_d - q_d) - \sum_{i=1}^{d-1} \frac{1}{w} (p_i - q_i) = 0
$$
Since \( \frac{d-1}{h} \) and \( w \) are positive numbers, we can rearrange this equality to obtain:

\[
|p_d - q_d| = \frac{h}{w(d-1)} \left| \sum_{i=1}^{d-1} (p_i - q_i) \right|.
\] (1)

Recall that \( h = \frac{1}{2} \min_{1 \leq i \leq d} h_i \leq \frac{1}{2} h_d \). Therefore, we know that \( |p_d - q_d| \geq 2h \). Applying Equation (1) to this inequality, we obtain:

\[
2h \leq \frac{h}{w(d-1)} \left| \sum_{i=1}^{d-1} (p_i - q_i) \right|
\]

\[
\leq \frac{h}{w(d-1)} (d-1) \max_{1 \leq i \leq d-1} |(p_i - q_i)|
\]

\[
\leq \frac{h}{w} \max_{1 \leq i \leq d-1} w_i
\]

Thus, we have \( w < \max_{1 \leq i \leq d} w_i \), which is a contradiction to \( w = \max_{1 \leq i \leq d} w_i \).

We analyze the complexity of computing \( w \) and \( h \). For each direction \( e_i \), we first sort the heights of \( L(e_i, V) \), which takes \( \Theta(n \log n) \) time. Then, to compute \( w_i \), the sorted set is constant time (as it is the maximum value minus the minimum value of the heights), and computing \( h_i \) is \( \Theta(n) \) time. Computing \( w \) and \( h \) from the sets \( \{w_i\} \) and \( \{h_i\} \) is \( \Theta(n) \) time. Thus, the bottleneck is sorting in each direction, which makes the total runtime \( \Theta(dn \log n) \).

We proceed with a constructive proof, generalizing the constructive proof from Theorem 6 and Algorithm 1. Let \( O \) be an oracle that takes a direction \( s \in S^{d-1} \) and returns the \( D_0(s) \) in \( \Theta(T_G) \) time.

For \( i \in \{1, \ldots, d\} \), we use this oracle to obtain \( D_0(e_i) \). Note that, by Assumption 1 (General Position) and Corollary 2, for each of these directions, we have exactly \( n \) distinct filtration hyperplanes, in one-to-one correspondence with the vertices. Note that, for a given direction \( e_i \), we store the filtration hyperplanes as a list of the vertex heights. Choosing one hyperplane in each direction yields \( d \) pairwise orthogonal hyperplanes; their intersection is a point in \( \mathbb{R}^d \) and this point is a potential vertex location. In total, we have \( n^d \) potential vertex locations, of which only \( n \) are actual vertices. We denote this set of \( n^d \)
potential vertex locations by $A$. By Corollary 2, $D_0(s)$ has at least $n$ points. Thus, computing these lists of vertex heights takes $\Theta(T_G)$ time per dimension to account for computing and listing the points of the APD.

Let $s$ be chosen as in Lemma 8 in time $\Theta(d \log n)$. By Lemma 8, each line $\ell_s(v)$ intersects at most one point in $A$ for each $v \in V$. Furthermore, by Lemma 7, each hyperplane in $\mathbb{L}(s,V)$ intersects $A$. Thus, there are exactly $n$ distinct intersections between $\mathbb{L}(s,V)$ and $A$, in one-to-one correspondence with the $n$ vertices.

Then, to identify vertex locations in $\mathbb{R}^d$, we employ the following brute force algorithm. We check each element $v \in A$ for intersections with any hyperplane $\ell \in \mathbb{L}(s,V)$. Since $|A| = n^d$ and $|\mathbb{L}(s,V)| = n$, we have $n^{d+1}$ checks that we must perform, with each check taking $\Theta(d)$ time. Thus, the total time complexity of calculating $V$ from the $d+1$ sets of filtration hyperplanes is $\Theta(d n^{d+1})$ and no additional augmented persistence diagrams are computed.

In total, this algorithm uses $d+1$ directional diagrams. The time complexity of constructing the $d+1$ sets of filtration hyperplanes is $\Theta(d \log n + d T_G)$, and an additional $\Theta(d n^{d+1})$ time to compute the actual vertex locations. Thus, the total time complexity is $\Theta(d n^{d+1} + d T_G)$.

5 Edge Reconstruction

Given the vertices constructed in Section 4, we describe how to reconstruct the edges in an embedded graph using $n^2 - n$ augmented persistence diagrams. The key to determining whether an edge exists or not is counting the degree of a vertex for edges in the half plane “below” the vertex with respect to a given direction. We begin with a method for reconstructing plane graphs, and then extend our method to embedded graphs in $\mathbb{R}^d$.

5.1 Edge Reconstruction for Plane Graphs

We first define necessary terms, and then describe our algorithm for constructing edges.

**Definition 10** (Indegree of Vertex). Let $G$ be an straight-line embedded graph in $\mathbb{R}^d$ with vertex set $V$. Then, for every vertex $v \in V$ and every direction $s \in S^{d-1}$, we define:

$$\text{INDEG}(v, s) = |\{(v, v') \in E \mid s \cdot v' \leq s \cdot v\}|.$$  

Thus, the indegree of $v$ is the number of edges incident to $v$ that lie below $v$, with respect to direction $s$; see Figure 2.

Given a directional augmented persistence diagram, we prove that we can determine the indegree of a vertex with respect to that direction:

**Lemma 11** (Indegree from Diagram). Let $G = (V, E)$ be a straight-line embedded graph in $\mathbb{R}^d$. Let $s \in S^{d-1}$ such that no two vertices have the same height
with respect to \( s \) (and thus \(|L(s, V)| = n\)). Let \( D_0(s) \) and \( D_1(s) \) be the zero- and one-dimensional augmented persistence diagrams resulting from the height filtration \( F_s(G) \). Then, for all \( v \in V \),

\[
\text{Indeg}(v, s) = |\{(x, y) \in D_0(s) \mid y = v \cdot s\}| + |\{(x, y) \in D_1(s) \mid x = v \cdot s\}|.
\]

Furthermore, if \( n = |V| \) and \( d = 2 \) then \( \text{Indeg}(v, s) \) can be computed in \( \Theta(n) \) time. If \( d > 2 \), then \( \text{Indeg}(v, s) \) can be computed in \( O(n^2) \) time.

Proof. Let \( v, v' \in V \) such that \( s \cdot v' < s \cdot v \), i.e., the vertex \( v' \) is lower in direction \( s \) than \( v \). Let \( e = (v, v') \in E \). Then, by Lemma \( \text{I} \) we have two cases to consider when \( e \) is added to \( F_s \):

Case 1: \( e \) joins two disconnected components. If \( e \) connects two previously disconnected components, then \( e \) is associated with a death in \( D_0(s) \) at height \( s \cdot v \). Moreover, since all deaths in \( D_0(s) \) are associated with adding an edge, we know that the set of all edges that fall into this case with \( v \) as the top endpoint is \( A = \{(x, y) \in D_0(s) \mid y = v \cdot s\} \).

Case 2: \( e \) creates a one-cycle. In this case, \( e \) is associated with a birth in \( D_0(s) \) at height \( s \cdot v \). Thus, we have that \( B = \{(x, y) \in D_1(s) \mid x = v \cdot s\} \) is the set of edges that fall into this case with \( v \) as the top endpoint.

The union \( A \cup B \) is the set of all edges ending at \( v \) with respect to \( s \), hence \( \text{Indeg}(v, s) = |A \cup B| \). Furthermore, by Corollary \( \text{II} \) if \( d = 2 \), then \( G \) is a plane graph and each of \( D_0(s) \) and \( D_1(s) \) have \( \Theta(n) \) points, so we count and sum the points joining two disconnected components or creating one-cycles at height \( v \) in time \( \Theta(n) \) time. If \( d > 2 \), then each of \( D_0(s) \) and \( D_1(s) \) have \( O(n^2) \) points, so we count and sum the points joining two disconnected components or creating one-cycles at height \( v \) in time \( O(n^2) \) time.

In order to decide whether an edge \((v, v')\) exists between two vertices, we look at the degree of \( v \) as seen by two close directions such that \( v' \) is the only vertex in what we call a **wedge at** \( v \):
Definition 12 (Wedge). Let \( v \in V \), and choose \( s_1, s_2 \in S^{d-1} \). Then, a wedge at \( v \) is the symmetric difference between the half planes below \( v \) in directions \( s_1 \) and \( s_2 \). In the special case when \( d = 2 \), we refer to the wedge as a bow tie.

Because we assume that no three vertices in our plane graph are collinear, for each pair of vertices \( v, v' \in V \), we can always find a bow tie centered at \( v \) that contains the vertex \( v' \) and no other vertex in \( V \); see Figure 3. We use bow tie regions to determine if there exists an edge between \( v \) and \( v' \). In the next lemma, we show how to decide if the edge \((v, v')\) exists in our plane graph.

![Figure 3: Bow tie \( B \) at \( v \), denoted by the shaded area. \( B \) contains exactly one vertex, \( v' \), so the only potential edge in \( B \) is \((v, v')\). In order to determine if there exists an edge between \( v \) and \( v' \), we compute \( \text{Indeg}(v, s_1) \) and \( \text{Indeg}(v, s_2) \), i.e., the number of edges incident to \( v \) in the solid and dashed arcs, respectively. An edge exists between \( v \) and \( v' \) if and only if \(|\text{Indeg}(v, s_1) - \text{Indeg}(v, s_2)| = 1\).

Lemma 13 (Edge Existence). Let \( G = (V, E) \) be a straight-line embedded graph in \( \mathbb{R}^d \). Let \( v, v' \in V \). Let \( s_1, s_2 \in S^{d-1} \) such that the wedge \( B \) at \( v \) defined by \( s_1 \) and \( s_2 \) satisfies: \( B \cap V \setminus \{v\} = v' \). Then,
\[
|\text{Indeg}(v, s_1) - \text{Indeg}(v, s_2)| = 1 \iff (v, v') \in E.
\]

Proof. Since edges in \( G \) are straight lines, any edge incident to \( v \) will either fall in the wedge region \( B \) or will be on the same side (above or below) of both hyperplanes. Let \( A \) be the set of edges that are incident to \( v \) and below both hyperplanes; that is, \( A = \{(v, w) \in E \mid s_i \cdot w < s_i \cdot v\} \). Furthermore, suppose we split the wedge into the two infinite cones. Let \( B_1 \) be the set of edges in one cone and \( B_2 \) be the set of edges in the other cone. We note that \(|B_1| - |B_2|\) is equal to one if there is an edge \((v, v') \in E \) with \( v' \in B_1 \) or \( v' \in B_2 \) and zero otherwise. Then, by definition of indegree,
\[
|\text{Indeg}(v, s_1) - \text{Indeg}(v, s_2)| = |A| + |B_1| - |A| - |B_2|
= |B_1| - |B_2|
= |B \cap E|,
\]
which equals one iff \((v, v') \in E\). Then \(|\text{Indeg}(v, s_1) - \text{Indeg}(v, s_2)| = 1 \iff (v, v') \in E\), as required. \( \square \)
Next, we prove that we can find the embedding of the edges in the original graph using $\Theta(n^2)$ directional augmented persistence diagrams. See \ref{A} for an example of walking through the reconstruction.

**Theorem 14 (Edge Reconstruction).** Let $G = (V, E)$ be a plane graph. If $V$ is known, then we can compute $E$ using $n^2 - n$ directional augmented persistence diagrams in $\Theta(n^2 T_G)$ time, where $\Theta(T_G)$ is the time complexity of computing a single diagram.

**Proof.** We prove this theorem constructively, and summarize the construction in Algorithm 2. In the algorithm, we first preprocess in order to find a global bow tie half-angle. Then we iterate through each pair of vertices and test to see if the edge exists. This edge test is done in two steps: first create a bow tie that isolates the potential edge, then apply Lemma 13 to determine if the edge exists or not by comparing the indegrees of $v$ with respect to the two directions defining the bow tie.

**Preprocessing (Lines 1–7 of Algorithm 2).** We initialize a set $E$ of edges to be the empty set in Line 1. Next, we compute an angle that is sufficiently small to be used to construct bow ties for every edge. For each vertex $v \in V$, we consider the cyclic ordering of the points in $V \setminus \{v\}$ around $v$; let $c[v]$ denote this ordered list of vertices. By Lemmas 1 and 2 of \cite{23}, we compute $c[v]$ for all $v \in V$ in $\Theta(n^2)$ total time. Once we have these cyclic orderings, we compute all $n$ lines through $v$ and $v_i \in V \setminus \{v\}$ and can compute a cyclic ordering of all such lines through $v$ in $\Theta(n)$ per vertex. (The step of obtaining the cyclic ordering of lines given the cyclic ordering of vertices is similar to the merge step of merge sort). Given two adjacent lines through $v$, consider the angle between these lines; see the angles labeled $\theta$ in Figure 4. For each vertex $v$, the minimum of such angles, denoted $\theta(v)$, is computed in Line 5 in $\Theta(n)$ time. Finally, we define $\theta = \frac{1}{2} \min_{v \in V} \theta(v)$ in Line 7. The value $\theta$ will be used to compute bow ties in the edge test. The runtime for this preprocessing is $\Theta(n^2)$ and requires no augmented persistence diagrams.

**Edge Test (Lines 9–17).** Let $O$ be an oracle that takes a direction $s$ in $S^1$ and returns the zeroth- and first-dimensional directional APDs for the unknown plane graph $G$ in direction $s$ in time $\Theta(T_G)$. Let $v, v' \in V$ such that $v \neq v'$. We now provide the two steps necessary to test if $(v, v') \in E$ using only two diagrams.

The first step is to construct bow ties (Lines 9–12 of Algorithm 2). Let $s$ be a unit vector perpendicular to vector $v' - v$, and let $s_1, s_2$ be the two unit vectors that form angles $\pm \theta$ with $s$. Note that $v$ and $v'$ are at the same height in direction $s$, but different heights in direction $s_1$ and $s_2$ (and, in fact, their order changes between directions $s_1$ and $s_2$). We consult the oracle $O$ to obtain the APDs $D(s_1)$ and $D(s_2)$. Note that we need the zeroth- and first-dimensional diagrams only, and these are the only non-trivial diagrams for a graph. Let $\mathcal{B}$ be

---

2Note that the na"ive approach would be to sort about each vertex independently, which would take $\Theta(n^2 \log n)$ time, but the results of \cite{23} improve this to $\Theta(n^2)$. The lemmas in \cite{23} use big-O notation, but the presented algorithm is actually asymptotically tight.
Figure 4: Ordering of all vertices about \( v \). Lines are drawn through all vertices and then angles are computed between all adjacent pairs of lines. The smallest angle is denoted as \( \theta(v) \). Here, \( \theta(v) = \theta_2 \).

the bow tie between \( \ell(s_1, h_{s_1}(v)) \) and \( \ell(s_2, h_{s_2}(v)) \). Note that, by construction, \( B \) contains exactly one point from \( V \), namely \( v' \). This first step of the edge test takes \( \Theta(T_G) \) time and will use two augmented persistence diagrams.

The second step of the edge test is to compute indegrees of \( v \) in order to determine if there exists an edge between \( v \) and \( v' \) (Lines 13–17 of Algorithm 2). By Lemma 11, we compute the indegrees \( \text{Indeg}(v, s_1) \) and \( \text{Indeg}(v, s_2) \) from \( D(s_1) \) and \( D(s_2) \), respectively, in \( \Theta(n) \) time; see Lines 13 and 14. Then, using Lemma 13, we determine whether the edge \((v, v')\) is in \( E \) by checking if \(|\text{Indeg}(v, s_1) - \text{Indeg}(v, s_2)| = 1\). If this inequality holds, the edge exists; if not, the edge does not; see Lines 15–17. The bottleneck of the edge test is the \( \Theta(n) \) indegree computation, the second step of the edge test takes \( \Theta(n) \) time. We do not compute additional diagrams in this step.

We apply the edge test for all \( \binom{n}{2} = \frac{1}{2}(n^2 - n) \) distinct pairs in \( V \). For all pairs, the complexity of the edge test uses \( n^2 - n \) persistence diagrams and takes \( \Theta(n^2 T_G + n^3) \) time. Observing that the time to compute a diagram \( D \) is \( \Omega(|D|) \) and using Corollary 2 we observe that \( T_G \) is \( \Omega(n) \). As a result, we can simplify \( \Theta(n^2 T_G + n^3) \) to \( \Theta(n^2 T_G) \). Thus, the runtime of Algorithm 2 is \( \Theta(n^2 T_G) \) (\( \Theta(n^2 T_G) \) for preprocessing and \( \Theta(n^3) \) for the edge tests).

Putting together Theorem 6 and Theorem 14 leads us to our primary result:

**Theorem 15 (Plane Graph Reconstruction).** Let \( G = (V, E) \) be a plane graph with \( n \) vertices embedded in \( \mathbb{R}^2 \). Algorithm 1 and Algorithm 2 calculate the vertex locations and edges using \( n^2 - n + 3 \) different directional augmented persistence diagrams in \( \Theta(n^2 T_G) \) time, where \( \Theta(T_G) \) is the time complexity of computing a single diagram.

**Proof.** By Theorem 6, Algorithm 1 reconstructs the vertices \( V \) using three APDs in \( \Theta(n \log n + T_G) \) time. By Theorem 14, Algorithm 2 reconstructs the edges \( E \) with \( n^2 - n \) directional augmented persistence diagrams in \( \Theta(n^2 T_G) \) time. Thus, we can reconstruct all vertex locations and edges of \( G \) using \( n^2 - n + 3 \) augmented persistence diagrams in \( \Theta(n^2 T_G) \) time. \( \square \)
Algorithm 2: Reconstruct Edges

**Input:** Oracle $\mathcal{O}$ for an unknown graph $G = (V, E) \subset \mathbb{R}^2$; the vertex set $V$

**Output:** the edge set $E$

1: $E \leftarrow \emptyset$
2: $c \leftarrow$ use Algorithm 23 to compute cyclic orderings for all vertices in $V$ stored as lists, indexed by $v \in V$
3: **for** $v \in V$ **do**
4: \[ \ell[v] \leftarrow \text{cyclic ordering of lines though } v, \text{ computed from } c[v] \]
5: \[ \theta(v) \leftarrow \text{minimum angle between any two lines in } \ell[v] \]
6: **end for**
7: \[ \theta = \frac{1}{2} \min_{v \in V} \theta(v) \]
8: **for** $(v, v') \in V \times V, v \neq v'$ **do**
9: \[ s \leftarrow \text{unit vector perpendicular to } (v' - v) \]
10: \[ s_1 \leftarrow s \text{ rotated by } \theta \]
11: \[ s_2 \leftarrow s \text{ rotated by } -\theta \]
12: Consult $\mathcal{O}$ to obtain diagrams $D(s_1)$ and $D(s_2)$
13: Compute $\text{INDEG}(v, s_1)$ from $D(s_1)$
14: Compute $\text{INDEG}(v, s_2)$ from $D(s_2)$
15: **if** $|\text{INDEG}(v, s_1) - \text{INDEG}(v, s_2)| = 1$ **then**
16: \[ \text{Add } (v, v') \text{ to } E \]
17: **end if**
18: **end for**
19: **return** $E$
5.2 Edge Reconstruction in $\mathbb{R}^d$

We can also reconstruct edges of graphs embedded in higher dimensions. We can form a higher-dimensional version of the bow tie, referred to as a wedge, which is the symmetric difference of two $(d - 1)$-dimensional hyperplanes.

**Theorem 16** (Higher-dimensional Edge Reconstruction). Let $G = (V, E)$ be a straight-line embedded graph in $\mathbb{R}^d$ for some $d > 1$. If $V$ is known, then we can compute $E$ using $n^2 - n$ directional augmented persistence diagrams in $O(n^2 T_G + n^4)$ time, where $\Theta(T_G)$ is the time complexity of computing a single diagram.

**Proof.** Let $\mathbb{P}_{e_1, e_2}$ be the subspace spanned by $e_1$ and $e_2$. Let $\pi: \mathbb{R}^d \to \mathbb{R}^2$ be defined by $\pi(x_1, x_2, \ldots, x_d) = (x_1, x_2)$; in other words, $\pi$ is the projection to $\mathbb{P}_{e_1, e_2}$. Let $V_s = \pi(V)$. We follow Lines 1–7 of Algorithm 2 to compute an angle $\theta$ for the vertex set $\theta$.

Let $v, v' \in V$. Using Lines 8–11 of Algorithm 2, we define $s_1$ and $s_2$ such that the lines $\ell_1(\pi(v))$ and $\ell_2(\pi(v))$, which are perpendicular to $s_1$ and $s_2$ that go through $\pi(v)$, define a bow tie at $\pi(v)$ that isolates the edge $(\pi(v), \pi(v'))$. This bow tie extends to a wedge in $\mathbb{R}^d$ by replacing the lines with hyperplanes that intersect $\mathbb{P}_{e_1, e_2}$ orthogonally; specifically, the line $a_1x_1 + a_2x_2 = h$ in $\mathbb{P}_{e_1, e_2}$ corresponds to the $(d - 1)$-dimensional hyperplane, $a_1x_1 + a_2x_2 + 0x_3 + \ldots + 0x_d = h$ in $\mathbb{R}^d$. Let $s^d_1$ and $s^d_2$ be directions in $\mathbb{R}^{d-1}$ that define this wedge. Notice that $v'$ is the only vertex in this wedge; for this reason, we say that the wedge isolates the edge $(v, v')$.

We then compute the indegrees of $v$ with respect to $s^d_1$ and $s^d_2$, just as we did in the two-dimensional case in Lines 13 and 14 of Algorithm 2. However, we note that since our dimension may be greater than two, Lemma 11 states that this step takes $O(n^2)$ time for each indegree computation. We perform indegree checks $\binom{n}{2}$ pairs of vertices. Finally, by Lemma 13, we test for an edge by determining if the difference between the indegrees is one or not using the same technique as Lines 15–16 of Algorithm 2.

The runtime for computing the indegree for all $\binom{n}{2}$ pairs of vertices, and reconstructing the edges, is $O(n^2T_G + n^4)$. The total number of diagrams necessary is $n^2 - n$, two diagrams for each pair of vertices. \hfill $\square$

Putting together Theorem 9 and Theorem 16, we obtain a graph reconstruction algorithm for graphs embedded with straight-line edges in $\mathbb{R}^d$.

**Theorem 17** (Generalized Graph Reconstruction). If $G = (V, E)$ is a straight-line embedded graph in $\mathbb{R}^d$ for $d > 1$, then $V, E$, and the embedding of $G$ can be determined using augmented persistence diagrams from $n^2 - n + d + 1$ different directions in time $O(dn^{d+1} + n^4 + (d + n^2)T_G)$, where $\Theta(T_G)$ is the time complexity of computing a single diagram.
6 Experiments

In this section, we empirically validate both the correctness of our algorithm and the runtime of various steps of the algorithm. Furthermore, we explore the runtime of checking for the presence of edges for a fixed vertex set as the number of edges varies. We conclude our experiments with an analysis of the probability of having ‘bad input;' that is, input with small angles. Timings were taken on a MacBook Pro with a 2.3GHz Intel Core i5 processor, 8GB RAM, running macOS Mojave 10.14.6. Our implementations were written in C++, compiled with the LLVM toolchain version 10.0.1. Our code is publicly available and experiments were run using the reconstruction library at git commit hash d9fd61d.[1]

Implementation To complement the theoretical results of this paper, we have written the reconstruction code in C++. In the implementation, an oracle is given a direction and returns an augmented persistence diagram. We use the Dionysus 2.0 library [24] for computing the augmented persistence diagram. We store the computed birth-death pairs in two lists, one for zero-dimension points (representing connected components) and one for one-dimensional points (representing loops).

We note that we have made some changes from the graph reconstruction algorithm presented in the previous sections. In particular, instead of using [23] to compute a cyclic ordering of all vertices in $\Theta(n^2)$, in Algorithm 2 Line 2, we implemented a naïve approach of ordering around each vertex independently in $O(n^2 \log n)$.

Data Our experimental data is a set of random plane graphs embedded in $\mathbb{R}^2$. For $n \in \mathbb{Z}^+$, we randomly generate $n$ points from the uniform distributions over the unit square. Next, we compute the Delaunay triangulation using the SciPy library [19] in Python. This random triangulation is similar to those of [4, 6, 11], but uses a binomial point process with a uniform density rather than a Poisson point process. Finally, we arrive at our random graph by setting a percentage $\alpha$ of edges $E$ from the triangulation to keep and delete edges until $\alpha \cdot |E|$ remain. The result is a random subgraph of a Delaunay triangulation.

Timing For each experiment about timing, we subtract out the time spent computing persistence diagrams as to capture only the time spent in the algorithm. In addition, we run each experiment five times and report the average of the runs.

6.1 Experimental Runtimes

Our first experiment validates the theoretical runtime of the plane graph reconstruction algorithm. We fix the percent of edges at $\alpha = 10\%$ and vary the
number of vertices $n \in \{10, 20, \ldots, 80\}$. For each value of $n$, we reconstruct ten random graphs as described in Paragraph 6. Data. We track the time spent in vertex and edge reconstruction and record the timings as described in Paragraph 6. Timing.

In Figure 5a and Figure 5b, we plot the time for reconstructing vertices and edges versus $n$, respectively. Times are measured in milliseconds and each mark is the mean for five runs on each random graph. In Figure 5a we fit $t = \beta_0 + \beta_1 n \log n$ to the experimental data, which had a RMSE of 0.0019 ms. The figure agrees with Theorem 6, which showed that the vertex reconstruction runtime is $O(n \log n)$ (ignoring the time for computing persistence diagrams). Figure 5b must be interpreted with a little more care. Recall that Theorem 14 showed that the edge reconstruction runtime is $\Theta(n^2 T_G)$. However, the runtime of the loop in Lines 8–18 of Algorithm 2 is $O(n^3)$, and in the experiments, we subtract the time for computing persistence diagrams. Thus, we expect the time to be upper-bounded by $n^3$. We fit the curve $t = \beta_0 + \beta_1 n^3$ to the experimental data, which had a RMSE of 0.45 ms. The figure agrees with Theorem 14.

![Figure 5: Experimental times for reconstructing plane graphs.](image)

(a) Vertex Reconstruction  
(b) Edge Reconstruction

Figure 5: Experimental times for reconstructing plane graphs. For each value of $n$, we record the vertex and edge reconstruction times for 10 random graphs and compare to the theoretical runtimes for vertex reconstruction (Theorem 6) and edge reconstruction (Theorem 14). The experimental timings agree with the theoretical runtimes.

### 6.2 Effect of Edges on Runtime

Our second experiment investigates the effect of edge density on our algorithm. Note that while our runtimes are expressed in the number of vertices, investigating the edges adds an additional insight. As in the previous section, each graph is generated as described in Paragraph 6. Data and we track the time spent in the vertex and edge reconstruction as described in Paragraph 6. Timing. Specifically, for a fixed Delaunay triangulation on a vertex set of size $n$, we vary the percent
of edges $\alpha$ from the Delaunay triangulation. We use $\alpha \in \{10, 20, \ldots, 100\}$ and $n \in \{10, 25, 50, 100\}$. In Figure 6a and Figure 6b, each value of $n$ is represented by a different line in the plot.

In the first part of this experiment, we investigate the effect of edge density on vertex reconstruction times. In Figure 6b, we see a constant relationship in the average runtime per vertex as a function of the percent of edges. To confirm, we fit the curve $t = \beta_0$, with RMSE 218.5 ns for $n = 10$, 62.4 ns for $n = 25$, 33.4 ns for $n = 50$, and 18.2 ns for $n = 100$. Thus, as predicted by the analysis of Theorem 6, the runtime of the vertex reconstruction algorithm is independent of the number of edges.

In the second part of this experiment, we focus on the reconstruction time of edges in the graph. In Figure 6b, we see a linear relationship in the average runtime per edge as a function of the percent of edges. To confirm, we fit the curve $t = \beta_0 + \beta_1 n$, with RMSE 108.3 ns for $n = 10$, 72.9 ns for $n = 25$, 61.9 ns for $n = 50$, and 33.4 ns for $n = 100$. The linear growth is somewhat unexpected as the number of edges does not appear in the analysis of Theorem 14. But, recall that the edge reconstruction algorithm computes indegree. By Lemma 11, the computation takes time proportional to the size of the augmented persistence diagrams (in dimensions zero and one combined). Moreover, the size of the one dimensional diagram grows linearly with respect to the number of edges. Thus,
when $\alpha$ is small, as would be the case in sparse graphs, we see fewer cycles, which makes $D_1(\cdot)$ smaller. This results in a small improvement in runtime.

6.3 Minimum Angle

Very small angles cause numeric issues, which is a problem that we encountered in the above experiments. In order to mitigate the issues, our code has an assertion that bow tie half-angles are at least $10^{-6}$ radians. We observed that many of the experiments with over 50 vertices were failing this assertion. In addition, a bounded angle assumption was used in [10]. Thus, we investigate the probability of encountering small angles, both with a back-of-the-envelope calculation and empirical observations.

Back-of-the-Envelope Calculation

Let $a, b,$ and $c$ be three points sampled i.i.d. from the uniform distribution on the unit square. We consider $\angle abc$. Without loss of generality, we can translate and rotate the points such that $b$ is at the origin, and $c$ is on the positive $x$-axis. Then, consider the line $ba$. If $a$ was randomly chosen, then any angle it makes with the positive $x$-axis ($ba$) is equally likely, so

$$P(\angle abc < 10^{-6}) = \frac{2 \cdot 10^{-6}}{2\pi} = \frac{10^{-6}}{\pi} \approx 3.18 \times 10^{-7}.$$ 

Furthermore, let $S_n$ be a set of $n$ (different) points sampled i.i.d. from the uniform distribution on the unit square. Notice that we can make $\binom{n}{3}$ = $n(n-1)(n-2)$ angles defined by the points in $S_n$. Let $A_n$ be the event that there exists an angle less than $10^{-6}$ in $S_n$. Assuming independence of angles in $S$, we have:

$$P(A_n) = 1 - \left(1 - \frac{10^{-6}}{\pi}\right)^{n(n-1)(n-2)}.$$ 

We observe that $P(A_n) > 5\%$ when $n \geq 56$. In the above runtime experiments, it is not surprising that we see the assertion failing. In this calculation, we assume that all angles are independent, so $P(A_n)$ is an overestimation of the true probability. We conjecture:

Conjecture 18 (Probability of Encountering Small Angles). Let $G = (V, E)$ be a randomly generated plane graph embedded in $\mathbb{R}^2$. If $|V| > 55$, then with probability at least $5\%$, the minimum angle as described in Theorem [14] is less than $10^{-6}$ radians.

Our final experiment is an investigation of the minimum angle of random point sets. This is motivated by understanding the frequency of small angles, and the extent to which our assertion and the assumption of [10] may or may not be limiting in practice.
Empirical Observations  This experiment illustrates how the bounded angle assumption is actually quite limiting, and that small angles appear with high probability as the number of vertices increases.

We generated random sets of vertices of size $n \in \{10, 20, \ldots, 350\}$, and measured the minimum angle between all triples of vertices. In Figure 7, we show the box plots of the minimum angles for 1000 random graphs for each $n$. In Figure 8, we show the probability of a graph having a minimum angle less than $10^{-6}$ radians as predicted by the theoretical model and by the experimental data. We notice at $n = 20$, we begin to see minimum angles of $10^{-6}$ radians appear in our random graphs. At $n = 70$, the number of random graphs with a minimum angle of $10^{-6}$ radians or less becomes even more substantial, with 8% of the angles less than $10^{-6}$ experimentally. This experiment suggests that for plane graphs with $n \geq 50$, our algorithm may encounter numerical errors due to small angles. And, as $n$ grows into the hundreds, the probability gets close to one. For the range of $n$ values we tested experimentally, we see—as we expected—that the theoretical back-of-the-envelope calculation upper bounds the experimental probability of finding a small angle. As we mentioned above, the differences in Figure 8 are due to the fact that the predicted probability is done assuming independence of angle. Further investigations of minimum
angles encountered in benchmark datasets such as MPEG7 and MNIST are also
being performed [13].

![Figure 8: Probability of failure (encountering an angle less than 10^{-6}) from experimental data (solid line) and the theoretical model (dashed line). The theoretical model provides an upper bound for the experimental probability of finding a small angle.](image)

7 Discussion

In this paper, we address the problem of reconstructing graphs using persistence diagrams by making the following three contributions.

1. We provide an algorithm for reconstructing a plane graph with \( n \) vertices embedded in \( \mathbb{R}^2 \) using \( n^2 - n + 3 \) directional APDs in \( \Theta(n^2T_G) \) time.

2. We extend the algorithm for reconstructing plane graphs to reconstructing graphs embedded in \( \mathbb{R}^d \) for any \( d \in \mathbb{N} \). This algorithm uses \( n^2 - n + d + 1 \) directional APDs in \( O(dn^{d+1} + n^4 + (d + n^2)T_G) \) time.

3. We empirically validated the correctness and time complexity of our algorithm for reconstructing plane graphs by implementing the algorithm and testing the implementation on randomly generated plane graphs. A plane graph cannot be reconstructed on the machine we used for testing if the bow tie half-angle is less than 10^{-6} radians. We found that this numerical issue is likely to occur even when we only have tens of vertices.
The reconstruction presented in this paper suggests several avenues for future work, both theoretical and empirical. On the theoretical side, we conjecture that we can extend the work here to reconstruct geometric simplicial complexes using APDs, and are currently investigating this problem. However, the generalized indegree that counts $k$-simplices below a particular simplex in a given direction is non-trivial to calculate. We may also be interested in computing properties of an embedded graph, rather than fully reconstructing it. For example, we might want to classify vertices into connected components. Intuitively, determining such properties should require fewer persistence diagrams than required for a complete reconstruction. Another question is whether other topological descriptors can be used to fully reconstruct simplicial complexes. We explored this question using Euler characteristic curves in [15] and found that degree two vertices in graphs posed particular problems. Thus, we conjecture that generic plane graphs cannot be reconstructed from Euler characteristic curves using approaches similar to the ones described here.

The motivation for developing reconstruction algorithms is to find a representative set of persistence diagrams which can be used to compare simplicial complexes. If a general reconstruction algorithm is found, then an implementation and tests of the usefulness of representative persistence diagrams to differentiate between shapes would be an important next step.

Reconstruction problems using topological descriptors is a rich area of study and has gained attention from many researchers in applied topology. We hope that our research only continues to excite others to study problems in this area.

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A Demonstration of Plane Graph Reconstruction

We give an example of reconstructing a plane graph embedded in \( \mathbb{R}^2 \). Consider the complex, \( G \), given in Figure 9. The vertices of \( G \) are

\[
V = \{(-1, 2), (0, -1), (0.25, 0), (1, 1)\}
\]

and edges are given by the following pairs of vertices,

\[
E = \{((-1, 2), (0, -1)), ((0, -1), (0.25, 0)), ((0, -1), (1, 1)), ((0.25, 0), (1, 1))\}.
\]

**Vertex Reconstruction** We find vertex locations using the algorithm described in Section 4. Note, that in this example, \( n = 4 \). Using the persistence diagrams from height filtrations in directions \( e_1 = (1,0) \) and \( e_2 = (0,1) \), we construct the set of lines \( \mathbb{L}_{e_1}(V) = \{(-1, y), (0, y), (0.25, y), (1, y) \mid y \in \mathbb{R}\} \) and \( \mathbb{L}_{e_2}(V) = \{(x, 2), (x, -1), (x, 0), (x, 1) \mid x \in \mathbb{R}\} \) as shown in Figure 9c. The set \( \mathbb{L}_{e_1}(V) \cup \mathbb{L}_{e_2}(V) \) of \( 2n = 8 \) lines has \( n^2 = 16 \) possible locations for the vertices at the intersections in \( A \). We show these filtration lines and intersections in Figure 9b.

We compute the third direction, \( s \), using the algorithm outlined in Theorem 3. Recall, that we need to find the maximum width between two lines in \( \mathbb{L}_{e_1}(V) \), denoted by \( w \), and smallest height between two adjacent lines in \( \mathbb{L}_{e_2}(V) \), denoted by \( h \). In our example, \( w = 1 - (-1) = 2 \) and \( h = 2 - 1 = 1 \). Then, we use the rectangle of width, \( w \), and height, \( h \), to choose a direction \( s \). We pick \( s \) to be a unit vector perpendicular to \( [w, h/2] \). In particular, we compute \( s = (-0.243, 0.970) \in \mathbb{S}^1 \). Then, the four three-way intersections in \( \mathbb{L}_{e_1}(V) \cup \mathbb{L}_{e_2}(V) \cup \mathbb{L}(s, V) \) identify all Cartesian coordinates of the vertices in the graph.

**Edge Reconstruction** Next, we reconstruct all edges of \( G \) as described in Section 5. In order to do so, we first compute the bow tie half-angle denoted by \( \theta \). For each vertex \( v \in V \), we consider the cyclic ordering of the points in \( V \setminus \{v\} \) around \( v \) and compute \( \theta(v) \), which is the minimum angle between all adjacent pairs of lines through \( v \). Noting that the vertex set is \( V = \{(-1, 2), (0, -1), (0.25, 0), (1, 1)\} \), we find \( \theta(v) \) to be approximately 0.237, 0.219, 0.399, and 0.180 radians, respectively. Then, we fix \( \theta \) so that \( \theta = \min_v \theta(v)/2 = 0.09 \).

Now, for each of the \( n(n - 1)/2 \) pairs of vertices \( (v, v') \in V^2 \), we construct a bow tie \( B \) and then use this bow tie to determine whether an edge exists between the two vertices. We go through two examples: one for a pair of vertices that does have an edge between them, \( ((0.25, 0), (1, 1)) \), and one for a pair that does not, \( ((0.25, 0), (-1, 2)) \). First, consider the pair \( v = (0.25, 0) \) and \( v' = (1, 1) \). To construct the bow tie at \( v \) that contains \( v' \), we first find a unit vector perpendicular to the vector that points from \( v \) to \( v' \). Here, our unit
Figure 9: Example of vertex reconstruction from three directions, $e_2$, $e_1$ and $s$ with corresponding persistence diagrams built for height filtrations from these directions. The filtration lines are the dotted lines superimposed over the complex.
vector is \( s = (-0.8, 0.6) \). Now, we find \( s_1, s_2 \) such that the bow tie at \( v \) has half-angle \( \theta \) with \( s \). We choose \( s_1 = (-0.851, 0.526) \) and \( s_2 = (-0.743, 0.669) \). By Lemma 11, we use the persistence diagrams from these two directions to compute \( \text{INDEG}(v, s_1) \) and \( \text{INDEG}(v, s_2) \). We observe that \( D_0(s_1) \) contains exactly one birth-death pair \( (x, y) \) such that \( y = v \cdot s_1 \) and \( D_1(s_1) \) has one birth-death pair such that \( x = v \cdot s_1 \). Thus, \( \text{INDEG}(v, s_1) = 2 \). On the other hand, \( D_0(s_2) \) contains exactly one birth-death pair \( (x, y) \) such that \( y = v \cdot s_2 \), but \( D_1(s_2) \) contains no birth-death pair such that \( x = v \cdot s_2 \). So \( \text{INDEG}(v, s_2) = 1 \). Now, since \( |\text{INDEG}(v, s_1) - \text{INDEG}(v, s_2)| = 1 \), we know that \( (v, v') \in E \), by Lemma 13.

For the second edge example, consider the pair of vertices \( v = (0.25, 0) \) and \( v' = (-1, 2) \). Again, we construct the bow tie at \( v \) containing \( v' \), by finding a unit vector perpendicular to the vector, \( (v' - v) \). We choose this to be \( s = (0.848, 0.530) \). Then, the \( s_1 \) and \( s_2 \) that form angle \( \theta \) with \( s \) are \( s_1 = (0.892, 0.452) \) and \( s_2 = (0.797, 0.604) \). Again by Lemma 11, we examine the zero- and one-dimensional persistence diagrams from these two directions to compute the indegree from each direction for vertex \( v \). In \( D_0(s_1) \), we have one pair \( (x, y) \) that dies at \( y = v \cdot s_1 \), but in \( D_1(s_1) \), no pair is born at \( x = v \cdot s_1 \). So \( \text{INDEG}(v, s_1) = 1 \). We see the exact same for \( s_2 \), which means that \( |\text{INDEG}(v, s_1) - \text{INDEG}(v, s_2)| = 0 \). Since Lemma 13 tells us that we have an edge between \( v \) and \( v' \) only if the absolute value of the difference of indegrees is one, we know that there is no edge between vertices \( (0.25, 0) \) and \( (-1, 2) \).

In order to reconstruct all edges, we perform the same computations for all pairs of vertices. After doing this, we obtain the desired plane graph as shown in Figure 9.
Figure 10: Example of edge reconstruction for two edges. The first edge (top row) exists while the second edge (bottom row) does not. The bow tie is given on the left while the persistence diagrams $D_0(s_1)$ and $D_1(s_1)$ are given in the middle and the persistence diagrams $D_0(s_2)$ and $D_1(s_2)$ are given on the right. The dotted lines indicate $v \cdot s_1$ and $v \cdot s_2$ in diagrams for $s_1$ and $s_2$ respectively.