New Hamiltonian formalism and Lagrangian representations for integrable hydrodynamic type systems.

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Abstract

New Hamiltonian formalism based on the theory of conjugate curvilinear coordinate nets is established. All formulas are “mirrored” to corresponding formulas in Hamiltonian formalism constructed by B.A. Dubrovin and S.P. Novikov (in a flat case) and E.V. Ferapontov (in a non-flat case). In the “mirrored-flat” case Lagrangian formulation is found. Multi-Hamiltonian examples are presented. In particular Egorov’s case, generalizations of local Nutku–Olver’s Hamiltonian structure and corresponding Sheftel–Teshukov’s recursion operator are presented. An infinite number of local Hamiltonian structures of all odd orders is found.

In honour of Yavuz Nutku

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1 Introduction

The modern theory of integrable hydrodynamic type systems — i.e. quasilinear systems of first-order PDE’s (see e.g. [50], [51])

\[
\begin{pmatrix}
  u_1^t \\
  \vdots \\
  u_n^t
\end{pmatrix} = \begin{pmatrix}
  v_1^1(u) & \cdots & v_1^n(u) \\
  \vdots & \ddots & \vdots \\
  v_n^1(u) & \cdots & v_n^n(u)
\end{pmatrix} \begin{pmatrix}
  u_1^x \\
  \vdots \\
  u_n^x
\end{pmatrix},
\]  

(1)

where \( u^i = u^i(x,t), i = 1, \ldots, n \) are the field variables (unknown functions), \( x \) and \( t \) are one-dimensional space and time variables — started with the paper [14] where a general Hamiltonian formalism for such systems was proposed. Physical examples of (1) with a “natural” Hamiltonian structure are numerous (see e.g. [3], [15], [30], [31]) for physical examples of systems with Hamiltonian structures originating from infinite-dimensional Lie algebras naturally related to the respective systems; such Hamiltonian structures are a particular case of the more general Hamiltonian structure described by B.A. Dubrovin and S.P. Novikov. Their generalization allows us to cast into the Hamiltonian form another vast class of hydrodynamic type systems (1), derived from integrable nonlinear equations via the so called “averaging” procedure, which gives a system (usually of type (1)) describing slowly varying (“modulated”) quasiperiodic solutions (see e.g. [9], [14], [18], [24], [33], [35], [56], [57]). These Whitham equations possess the Hamiltonian structure of the Dubrovin–Novikov type. Some particular cases of this Hamiltonian structure for averaged equations were studied in [29].

This method may be applied to a wide range of “averaged” integrable equations which possess alongside with the Hamiltonian structure of the Dubrovin–Novikov type the remarkable property of diagonalizability: after a suitable change of the field variables \( r^i(u) \) in (1) one obtains the hydrodynamic type system

\[
\dot{r}_i^i = v^i(r)r_{x}^i, \quad i = 1, 2, ..., N.
\]  

(2)

(no summation over repeated indices hereafter!) with diagonal matrix \( v_i^i(r) \delta_{ij} \). These new variables \( r^i \) are called Riemann invariants (in the Whitham theory they are branch points of corresponding Riemann surfaces, see e.g. [24], [48]). For \( n = 2 \) every hyperbolic system (1) may be diagonalized; for \( n > 2 \) this is not true in general, a special criterion [28] shall be applied to check diagonalizability of (1).
On the other hand the methods of [56] (see also [57]) lead to an unexpected link between the theory of Hamiltonian diagonalizable hydrodynamic type systems and a classical object of local differential geometry intensively studied in the end of 19th and beginning of 20th century — orthogonal curvilinear coordinates in the flat Euclidean space $\mathbb{R}^n$. Practically all “physical” entities (conservation laws, symmetries etc.) have their counterparts in the theory of orthogonal curvilinear coordinates; amazingly enough the basic formulas relating conservation laws, symmetries and the Hamiltonian structure of integrable systems of hydrodynamic type may be found in [10]! Further investigation discovered a deep relation between other types of integrable nonlinear systems studied in the modern theory of integrable physical equations and classical problems studied in [5, 6, 10, 11, 12, 17, 27] and other papers at the beginning of previous century.

In this paper we expose this remarkable link as well as some recently discovered deeper relations between the classical differential geometry flourished at the end of the 19th century and the modern theory of integrable nonlinear PDEs appearing in different applications to mathematical physics.

The theory of semi-Hamiltonian hydrodynamic type systems is a most developed part of integrable PDE systems (see [14], [56], [57]). A one of such reasons is that this theory is based on classical differential geometry, and more precisely, on the theory of conjugate curvilinear coordinate nets (see [10]). For instance, the local Hamiltonian formalism associated with the differential-geometric Poisson brackets of the first order is connected with theory of orthogonal curvilinear coordinate nets (see [13], [32], [56], [57], [59]).

This paper is devoted to an alternative construction based on so-called anti-flatness property. An existence and, moreover, an explicit construction of Hamiltonian structures play important role in integrability of hydrodynamic type systems, because a general solution of semi-Hamiltonian hydrodynamic type system depends on general solution of a linear problems describing conservation laws and commuting flows, which are related by the Hamiltonian structure.

In general case the problem is following: it is necessary to describe all possible Hamiltonian formalisms associated with integrable hydrodynamic type systems or with conjugate curvilinear coordinate nets. The local Hamiltonian formalism determined by a differential operator of the first order is constructed in ([14] (see also [56], [57]). The nonlocal Hamiltonian formalism of the first order associated with a flat normal bundle is constructed in [19]. This paper is devoted to the nonlocal Hamiltonian formalism associated with the local Lagrangian representation of corresponding hydrodynamic type systems. Suppose a given hydrodynamic type system possesses the local Hamiltonian structure of the Dubrovin–Novikov type (see [14]) and the above mentioned Lagrangian representation, simultaneously. It means, that this hydrodynamic type system has local Hamiltonian and local symplectic structures. A most amazing consequence is that the corresponding hydrodynamic type system has infinitely many local Hamiltonian structures of all odd orders (as well as infinitely many local symplectic structures and corresponding local Lagrangian representations).

This paper is organized as follows. In Section 2, the relationship between local Hamiltonian formalism of the Dubrovin–Novikov type and orthogonal curvilinear coordinate nets is briefly described. In Section 3, the first “puzzle” hidden in the paper [42] written by Yavuz Nutku and Peter Olver is formulated. In Section 4, integrable hydrodynamic
type systems determined by a special local Lagrangian representation are found. The existence of such Lagrangians is based on the concept “anti-flatness”, which is established in this paper. In Section 5, simultaneously, hydrodynamic type systems associated with a local Hamiltonian structure and with above mentioned local Lagrangian representation are considered. Local Hamiltonian structures of all odd orders are described. In Section 6, hydrodynamic type systems possessing anti-flat multi-Hamiltonian structures are discussed. In the Section 7, the above construction is restricted on the symmetric case. 

Statement: suppose the Egorov hydrodynamic type system has just one known local Hamiltonian structure of the Dubrovin–Novikov type (or vice versa has just one known local Lagrangian representation), then infinitely many other local Hamiltonian structures of all odd orders can be constructed. All of them can be expressed via corresponding solutions of the WDVV equation (see [13], [34], [38], [46]). In Section 8, the second “puzzle” hidden in the paper [42] written by Yavuz Nutku and Peter Olver is formulated. A natural interpretation via local Hamiltonian structures of the Dubrovin–Novikov type is given. In Section 9, a generalization of above construction on a nonlocal case associated with surfaces with a flat normal bundle is briefly discussed. In Section 10, the above local Lagrangian formulation is extended on higher order commuting flows of hydrodynamic type systems.

2 Flat case

The theory of conjugate curvilinear coordinate nets (see [10]) is based on nonlinear PDE system

$$\partial_i \beta_{jk} = \beta_{ji} \beta_{ik}, \quad i \neq j \neq k,$$

where rotation coefficients $\beta_{ik}$ are functions of $N$ independent variables $r^k$ (which are called principal curvature coordinates, see [54]). A general solution of this system is parameterized by $N(N-1)/2$ functions of two variables. The above nonlinear PDE system is a consequence of the compatibility conditions $\partial_i (\partial_k H_j) = \partial_k (\partial_i H_j), \partial_i (\partial_k \psi_j) = \partial_k (\partial_i \psi_j), \quad i \neq j \neq k,$ where Lame coefficients $H_j$ and adjoint Lame coefficients $\psi_j$ are solutions of a linear PDE system and an adjoint linear PDE system, respectively

$$\partial_i H_k = \beta_{ik} H_i, \quad \partial_k \psi_i = \beta_{ik} \psi_k, \quad i \neq k,$$

which have general solutions parameterized by $N$ arbitrary functions of a single variable for any given rotation coefficients $\beta_{ik}$ satisfying (3). In general case a relationship between these linear PDE systems is unknown.

A link connecting conjugate curvilinear coordinate nets and integrable hydrodynamic type systems is following.

- Let $\bar{H}_i$ and $\tilde{H}_i$ be two particular solutions of the first linear PDE system (4).
- Let field variables $r^k$ be functions of two independent variables $x$ and $t$.
- Let construct a hydrodynamic type system written via Riemann invariants $r^k$

$$r^i_t = \frac{\bar{H}_i}{\tilde{H}_i} r^i_x.$$
• Let field variables $r^k$ be functions of three independent variables $x$, $t$ and $\tau$, simultaneously. Then the above hydrodynamic type system possesses a commuting flow (i.e. $(r^i_\tau)_t = (r^i_t)_\tau$)

$$r^i_\tau = \frac{H^i}{H^i_x} r^i, \quad (6)$$

parameterized by $N$ arbitrary functions of a single variable (see (4) and [56], [57]).

• Let introduce the function $h$ such that

$$\partial_i h = \psi_i H_i. \quad (7)$$

Then the above hydrodynamic type system has a conservation law (written in the potential form)

$$d\xi = h dx + g dt + f d\tau,$$

where

$$\partial_i g = \psi_i \tilde{H}_i, \quad \partial_i f = \psi_i H_i.$$

Thus, these hydrodynamic type systems possess infinitely many conservation laws parameterized by $N$ arbitrary functions of a single variable.

• A general solution of the hydrodynamic type system (4) is given (in an implicit form) by the generalized hodograph method (see [56])

$$x \tilde{H}_i + t \tilde{H}_i = H_i.$$

The first problem in the theory of hydrodynamic type systems (5) is a description of solutions of the nonlinear PDE system (3), the second problem is a description of solutions of the linear PDE systems with variable coefficients (4). These problems are very complicated in general. By this reason, conjugate curvilinear coordinate nets can be separated on some sub-classes due to some criterion (features, characteristics etc...), then corresponding conjugate curvilinear coordinate nets can be integrated within corresponding sub-classes.

The approach developed by B.A. Dubrovin, S.P. Novikov (see [14]) and S.P. Tsarev (see [56]) is based on a local Hamiltonian formalism for hydrodynamic type systems (5) connected with the theory of orthogonal curvilinear coordinate nets (see [56], [57]), which is a sub-class of the theory of conjugate curvilinear coordinate nets.

• Let establish a link of the first order between solutions of the linear and adjoint linear PDE systems (4)

$$H_i = \partial_i \psi_i + \sum_{m \neq i} \beta_{mi} \psi_m. \quad (8)$$

• Then the flatness condition is given by (see [10])

$$\partial_i \beta_{ik} + \partial_k \beta_{ki} + \sum_{m \neq i,k} \beta_{mi} \beta_{mk} = 0, \quad i \neq k \quad (9)$$
which can be obtained by the substitution (8) in (4). Thus, the nonlinear PDE system (3) and (9) is a consequence of compatibility conditions of the linear PDE system (4) and (8). It means that orthogonal curvilinear coordinate nets are described by the nonlinear PDE system (3) and (9).

- If l.h.s. of (8) is vanished, then \( N \) particular solutions \( \psi_{(\gamma)i} \) are determined by the linear ODE systems (for each fixed index \( i \); see (4) and (8))

\[
\partial_k \psi_{(\gamma)k} = \beta_{ki} \psi_{(\gamma)i}, \quad i \neq k, \quad \partial_k \psi_{(\gamma)i} + \sum_{m \neq i} \beta_{mi} \psi_{(\gamma)m} = 0.
\]

- These linear ODE systems have \( N(N + 1)/2 \) constraints ("first integrals")

\[
\bar{g}_{\beta\gamma} = \sum \bar{\psi}_{(\beta)m} \bar{\psi}_{(\gamma)m} = \text{const}.
\]

This is non-degenerate matrix. Then \( \bar{\psi}_{(\beta)} = \bar{g}^{\beta\gamma} \bar{\psi}_{(\gamma)i} \).

- Let \( a^\gamma \) be flat coordinates, i.e. \( \partial_i a^\gamma = \bar{\psi}_{(\gamma)H_i} \) (see (7)) and

\[
\frac{\partial r^i}{\partial a^\gamma} = \frac{\bar{\psi}_{(\gamma)H_i}}{\bar{H}_i}.
\]

Then (5) under a point transformation \( r^i = r^i(a) \) can be written in the conservative form

\[
a^\beta_i = \bar{g}^{\beta\gamma} D_x \frac{\partial \bar{h}}{\partial a^\gamma},
\]

where \( \bar{h}(a) \) is a Hamiltonian density, the momentum density \( \bar{h}(a) \) is a quadratic expression with respect to flat coordinates \( a^\gamma \)

\[
\bar{h} = \frac{1}{2} \bar{g}_{\beta\gamma} a^\beta a^\gamma.
\]

It means (see [49]), that the above hydrodynamic type system has two extra conservation laws (of the momentum and of the energy, respectively) associated with the above local Hamiltonian structure

\[
\partial_i \bar{h} = D_x \left( a^\beta \frac{\partial \bar{h}}{\partial a^\beta} - \bar{h} \right), \quad \partial_i \bar{h} = D_x \left( \bar{g}^{\beta\gamma} \frac{\partial \bar{h}}{\partial a^\beta} \frac{\partial \bar{h}}{\partial a^\gamma} \right).
\]

Then the above conservation law densities \( \bar{h} \) and \( \tilde{h} \) can be expressed via Riemann invariants \( r^h \) (in quadratures)

\[
\frac{\partial \bar{h}}{\partial \bar{h}} = \bar{\psi}_{(\gamma)H_i}, \quad \partial_i \left( a^\beta \frac{\partial \bar{h}}{\partial a^\beta} - \bar{h} \right) = \bar{\psi}_{(\gamma)H_i}, \quad \partial_i \frac{\partial \bar{h}}{\partial a^\beta} = \bar{\psi}_{(\beta)\bar{H}_i},
\]

\[
\frac{\partial \tilde{h}}{\partial \tilde{h}} = \tilde{\psi}_{(\gamma)\bar{H}_i}, \quad \partial_i \left( \bar{g}^{\beta\gamma} \frac{\partial \tilde{h}}{\partial a^\beta} \frac{\partial \tilde{h}}{\partial a^\gamma} \right) = \tilde{\psi}_{(\gamma)\bar{H}_i}, \quad \partial_i \frac{\partial \tilde{h}}{\partial a^\beta} = \tilde{\psi}_{(\beta)\bar{H}_i}.
\]
where (see (8))

\[
\begin{align*}
\bar{H}_i &= \partial_i \bar{\psi}_i + \sum_{m \neq i} \beta_{mi} \bar{\psi}_m, \\
\tilde{H}_i &= \partial_i \tilde{\psi}_i + \sum_{m \neq i} \beta_{mi} \tilde{\psi}_m.
\end{align*}
\]

(14)

Corollary: The momentum density (see the first above equation) is given by (cf. (13))

\[
\bar{h} = \frac{1}{2} \sum \bar{\psi}_m^2.
\]

(15)

- It means, that the local Poisson bracket of the Dubrovin–Novikov type in flat coordinates is given by

\[
\{a^\beta(x), a^\gamma(x')\} = \tilde{g}^{\beta \gamma}(x-x');
\]

in Riemann invariants

\[
\{r^i(x), r^j(x')\} = \hat{A}^{ij}(x-x'),
\]

where

\[
\hat{A}^{ii} = \frac{1}{H_i} \partial_x \frac{1}{H_i}, \quad \hat{A}^{ik}|_{k \neq i} = \frac{1}{H_i H_k} (\beta_{ki} r^i_x - \beta_{ik} r^k_x).
\]

(16)

- Thus, an arbitrary commuting flow (6) also can be written in the same local canonical Hamiltonian form

\[
a^\beta = \tilde{g}^{\beta \gamma} D_x \frac{\partial h}{\partial a^\gamma},
\]

where

\[
\partial_i \left( a^\beta \frac{\partial h}{\partial a^\beta} - h \right) = \bar{\psi}_i H_i, \quad \partial_i h = \psi_i H_i, \quad \partial_i \frac{\partial h}{\partial a^\beta} = \tilde{\psi}_i(\beta) H_i, \quad \partial_i \left( \tilde{g}^{\beta \gamma} \frac{\partial h}{\partial a^\gamma} \frac{\partial h}{\partial a^\beta} \right) = \psi_i H_i,
\]

and solutions \(\psi_i\) and \(H_i\) of linear PDE systems (4) are related by (8).

- The relationship of the first order between the linear problems (4) is given by (8). Taking into account another identity

\[
\frac{\partial h}{\partial a^\beta} = \sum \bar{\psi}_m(\beta) \frac{\partial h}{\partial a^\beta},
\]

(17)

an inverse (nonlocal) relationship between the linear problems (4) is given by

\[
\psi_i = \tilde{\psi}_i(\beta) \frac{\partial h}{\partial a^\beta}.
\]

(18)

Indeed (see (8)),

\[
H_i = \partial_i \left( \tilde{\psi}_i(\beta) \frac{\partial h}{\partial a^\beta} + \sum_{m \neq i} \beta_{mi} \tilde{\psi}_m(\beta) \frac{\partial h}{\partial a^\beta} \right) \equiv \tilde{\psi}_i(\beta) \frac{\partial h}{\partial a^\beta} \iff \partial_i \frac{\partial h}{\partial a^\beta} = \tilde{\psi}_i(\beta) H_i.
\]
3 First Nutku–Olver’s “puzzle”

The ideal gas dynamic system (see e.g. [41], [42])

\[ \rho_t = (\rho u)_x, \quad u_t = \left( \frac{u^2}{2} + \rho f''(\rho) - f'(\rho) \right)_x \]  

has only one local Hamiltonian structure of the Dubrovin–Novikov type

\[
\left( \begin{array}{c} \rho \\ u \end{array} \right)_t = \left( \begin{array}{cc} 0 & D_x \\ D_x & 0 \end{array} \right) \left( \begin{array}{c} \frac{\delta H_4}{\delta \rho} \\ \frac{\delta H_4}{\delta u} \end{array} \right),
\]

where the Hamiltonian and other lower functionals are given by

\[ H_4 = \int \left[ \frac{\rho u^2}{2} + \rho f'(\rho) - 2 f(\rho) \right] dx, \quad H_3 = \int \rho udx, \quad H_2 = \int \rho dx, \quad H_1 = \int u dx. \]

However, Ya. Nutku and P. Olver (see [42]) found an absolutely new local Hamiltonian structure determined by the third order purely differential operator

\[ \hat{B} = \hat{R}^2 \left( \begin{array}{cc} 0 & D_x \\ D_x & 0 \end{array} \right), \]

where the recursion operator \( \hat{R} \) is a purely differential operator of the first order

\[ \hat{R} = D_x(W_x)^{-1} \]

and the matrix \( W \) is given by

\[ W = \left( \begin{array}{cc} u & \rho \\ f''(\rho) & u \end{array} \right). \]

This paper is devoted to an investigation of this phenomenon. Indeed, this local Hamiltonian structure of the third order has clear pure differential-geometric meaning. Moreover, a corresponding differential-geometric construction is generalized on \( N \) component hydrodynamic type systems.

So, we have the set of questions:

- How to explain the ORIGIN of this local Hamiltonian structure?
- How to generalize this local Hamiltonian structure on \( N \) component case?
- How to recognize that a given hydrodynamic type system possesses such a local Hamiltonian structure?
- How to construct such a local Hamiltonian structure?
- How many such local Hamiltonian structures a given hydrodynamic type system can possess?
Is it any *relationship* between these local Hamiltonian structures and local Hamiltonian structures of the Dubrovin–Novikov type?

The key idea used for opening this puzzle is hidden in the local Lagrangian representation for this hydrodynamic type system (see [43])

\[ S = \int \left[ \frac{1}{2} \rho \frac{u_t}{u_x} - \frac{u_x \rho_t}{u_x^2 - f''(\rho) \rho_x^2 - \rho} \right] dxdt. \] (23)

4 Local Lagrangian representations

The Lagrangian

\[ S = \int [g_k(r, r_x, r_xx, ...) r_i^k - h(r, r_x, r_xx, ...)] dxdt \]

determines the Euler–Lagrange equations

\[ \dot{M}_{ik} r^k_t = \frac{\delta H}{\delta r^i} \]

in the Hamiltonian form (see [40])

\[ r^i_t = \dot{K}_{ij} \frac{\delta H}{\delta r^j}, \]

where the Hamiltonian \( H = \int h(r, r_x, r_xx, ...) dx \), and the Hamiltonian operator \( \dot{K}_{ij} \) is an inverse operator to the symplectic operator

\[ \dot{M}_{ij} = \frac{\partial g_i}{\partial r^j} D_x^n - (-1)^n D_x^n \frac{\partial g_j}{\partial r^i}. \] (24)

Let us restrict our further consideration on the case

\[ S = \int [g_k(r, r_x) r_i^k - h(r, r_x, r_xx, ...)] dxdt. \]

Then corresponding Euler–Lagrange equations can be written in the form

\[ \dot{M}_{ik} r^k_t = \frac{\delta H}{\delta r^i}, \]

where a corresponding symplectic operator is given by (see (24))

\[ \dot{M}_{ik} = \frac{\partial g_k}{\partial r^i} - \frac{\partial g_i}{\partial r^k} - \left( \frac{\partial g_i}{\partial r^k} D_x + D_x \left( \frac{\partial g_k}{\partial r^i} \right) \right). \]

If we choose

\[ g_k(r, r_x) = \frac{\dot{H}_k^2(r)}{2r_x^k}, \]
then corresponding components of the above symplectic operator are given by (cf. (16))

\[ \dot{M}_{ii} = \frac{\bar{H}_i}{r_x^i} D_x \frac{\bar{H}_i}{r_x^i}, \quad \dot{M}_{ik}|_{k \neq i} = \frac{\bar{H}_i}{r_x^i} \frac{\beta_{ik}}{r_x^k} - \frac{\beta_{ki}}{r_x^k}, \] (25)

where

\[ \beta_{ik} = \frac{\partial_i \bar{H}_k}{\bar{H}_i}, \quad k \neq i. \] (26)

**Theorem 1**: The Hamiltonian operator

\[ \dot{K}^{ij} = \bar{\varepsilon}^{\alpha\beta} w^i_{(\alpha)} r_x^i D_x^{-1} w^j_{(\beta)} r_x^j \] (27)

is an inverse operator to the above symplectic operator \( \dot{M}_{ik} \) iff

1. \( \beta_{ik} \) are rotation coefficients of a corresponding conjugate curvilinear coordinate net determined by the Bianchi–Darboux–Lame system (3) and by the “anti-flatness” condition

\[ \partial_i \beta_{ki} + \partial_k \beta_{ik} + \sum_{m \neq i} \beta_{im} \beta_{km} = 0, \quad i \neq k; \] (28)

2. affinors \( w^i_{(\alpha)} \) determine \( N \) commuting hydrodynamic type systems (structural flows)

\[ r_t^i = w^i_{(\alpha)} r_x^i, \] (29)

where \( w^i_{(\alpha)} = \bar{H}_{(\alpha)i}/\bar{H}_i \), and \( \bar{H}_{(\alpha)i} \) are solutions of the linear ODE systems

\[ \partial_i \bar{H}_{(\alpha)k} = \beta_{ik} \bar{H}_{(\alpha)i}, \quad i \neq k, \quad \partial_i \bar{H}_{(\alpha)i} + \sum_{m \neq i} \beta_{im} \bar{H}_{(\alpha)m} = 0; \]

3. \( \bar{\varepsilon}^{\alpha\beta} \) is a constant non-degenerate symmetric matrix such that

\[ \bar{\varepsilon}^{\alpha\beta} = \sum \bar{H}^{(\alpha)}_m \bar{H}^{(\beta)}_m, \quad \bar{H}^{(\alpha)}_i = \varepsilon^{\alpha\beta} \bar{H}^{(\beta)i}. \] (30)

**Proof**: can be obtained by the direct verification

\[ \dot{K}^{is} \dot{M}_{sj} = \delta^i_j, \quad \dot{M}_{is} \dot{K}^{sj} = \delta^i_j. \]

**Corollary**: The Lagrangian

\[ S = \int \left[ \frac{1}{2} \sum \bar{H}^2_k(r) \frac{r_x^k}{r_x^i} - h(r) \right] dxdt \] (31)

determines an integrable hydrodynamic type system (6), where (cf. (18))

\[ H_i = \bar{H}^{(\beta)}_i q_{\beta}, \quad q_{\beta} = \sum \bar{H}^{(\beta)m}_i H_m, \quad \partial_i q_{\beta} = \psi_i \bar{H}^{(\beta)i}. \] (32)

It means that the Hamiltonian density \( h \) (see (7)) is determined by the linear system

\[ \partial_i \psi_k = \beta_{ki} \psi_i, \quad i \neq k, \quad \psi_i = \partial_i H_i + \sum_{m \neq i} \beta_{im} H_m, \] (33)
where \( \partial_i h = \psi_i \bar{H}_i \); the momentum density (cf. (15))

\[
\bar{h} = \frac{1}{2} \sum \bar{H}_m^2
\]  

(34)
can be obtained by the replacement \( \psi_i \rightarrow \bar{\psi}_i, \bar{H}_i \rightarrow \bar{H}_i \) in the above system, where \( \partial_i \bar{h} = \bar{\psi}_i \bar{H}_i \). If \( \tau \rightarrow t, \psi_i \rightarrow \tilde{\psi}_i, H_i \rightarrow \tilde{H}_i \) in the above system (cf. (14)), the above Lagrangian determines the hydrodynamic type system (5), where \( \partial_i \bar{h} = \psi_i \bar{H}_i \).

**Remark:** Since the Lagrangian density

\[
L = L(r, r_x, r_t)
\]
determines an energy-momentum tensor, whose components are conservation laws of the energy and the momentum, respectively

\[
\partial_t \left( L - r_t^k \frac{\partial L}{\partial r_t^k} \right) = \partial_x \left( r_x^k \frac{\partial L}{\partial r_x^k} \right), \quad \partial_t \left( r_t^k \frac{\partial L}{\partial r_x^k} \right) = \partial_x \left( L - r_x^k \frac{\partial L}{\partial r_t^k} \right),
\]

then the hydrodynamic type system (6) determined by the Lagrangian (31) possesses these conservation laws too, which are given by (see (34))

\[
\partial_t h = \frac{1}{2} \partial_x \left( \sum \bar{H}_m^2 \right), \quad \frac{1}{2} \partial_t \left( \sum \bar{H}_m^2 \right) = \partial_x \left( \sum \bar{H}_m H_m - h \right).
\]

The first example of integrable hydrodynamic type systems associated with the above Lagrangian representation was found in \([43]\). Indeed, under the point transformation (from physical variables to the Riemann invariants)

\[
dr^\pm = du \pm \sqrt{f^m(r)} dr
\]

the Lagrangian (23) reduces to (31), where the Lame coefficients are given by

\[
\bar{H}_m^2 = \pm \frac{1}{2 \sqrt{f^m(r)}},
\]

and the ideal gas dynamics (19) reduces to the diagonal form

\[
r_t^\pm = \left( \frac{r^+ + r^-}{2} \pm \varphi(r^+ - r^-) \right) r_x^\pm,
\]

where the function \( \varphi(z) \) is given in the implicit form

\[
\varphi(z) = \rho \sqrt{f^m(r)}, \quad dz = 2 \sqrt{f^m(r)} dr.
\]

Taking \( N \) arbitrary solutions \( \psi^{(\beta)} \) of the adjoint linear problem (4), the hydrodynamic type system (5) can be re-written in the Hamiltonian form (see (27) and cf. (12))

\[
a_t^\alpha = \bar{\varepsilon}^{\beta \delta} (w_\beta^\gamma)_x D_x^{-1} (w_\delta^\gamma)_x \frac{\partial h}{\partial a^\gamma},
\]

(35)

where conservation law densities and corresponding fluxes of structural flows (29)

\[
a_t^\alpha = (w_\beta^\alpha (a))_x
\]

(36)

are determined by their derivatives

\[
\partial_i a^\beta = \psi^{(\beta)}_i \bar{H}_i, \quad \partial_i w^\alpha_\gamma = \psi^{(\alpha)}_i \bar{H}_{(\gamma)i}.
\]
5 \textit{Mixed} bi-Hamiltonian structure

Let us consider an arbitrary semi-Hamiltonian hydrodynamic type system (2). It means, that characteristic velocities $v^i(r)$ satisfy the integrability condition (see [56])

$$
\frac{\partial_j}{v^k - v^i} = \frac{\partial_k}{v^j - v^i}, \quad i \neq j \neq k.
$$

Following S. Tsarev (see [56]) the Lame coefficients $\tilde{H}_k$ are given by

$$
\partial_k \ln \tilde{H}_i = \frac{\partial_k v^i}{v^k - v^i}, \quad i \neq k.
$$

(37)

Following G. Darboux (see [10]) rotation coefficients $\beta_{ik}$ are given by (26). Suppose the rotation coefficients $\beta_{ik}$ associated with a given hydrodynamic type system satisfy the flatness (9) and anti-flatness (28) conditions, simultaneously. It means that this hydrodynamic type system has two Hamiltonian structures

$$
r^i_t = \{r^i, H_1\}_1 = \{r^i, H_2\}_2,
$$

where components of the local (Dubrovin–Novikov) Poisson structure

$$
\{r^i, r^j\}_1 = \hat{A}^{ij} \delta(x - x')
$$

are given by (16)

$$
\hat{A}^{ii} = \frac{1}{\tilde{H}_i} D_x \frac{1}{\tilde{H}_i}, \quad \hat{A}^{ik}|_{i \neq k} = \frac{1}{\tilde{H}_i \tilde{H}_k} (\beta_{ki} r^i_x - \beta_{ik} r^k_x)
$$

and components of the local symplectic structure

$$
\hat{M}_{jk}\{r^k, r^i\}_2 = \delta^i_j \delta(x - x')
$$

are given by (25)

$$
\hat{M}_{ii} = \frac{\tilde{H}_i}{r^i_x} D_x \frac{\tilde{H}_i}{r^i_x}, \quad \hat{M}_{ik}|_{i \neq k} = \tilde{H}_i \tilde{H}_k \left( \frac{\beta_{ik}}{r^k_x} - \frac{\beta_{ki}}{r^i_x} \right).
$$

Thus, the recursion operator of the second order is a product of local Hamiltonian and symplectic operators of the first orders

$$
\hat{R}^i_t = \hat{A}^{ij} \hat{M}_{jk},
$$

and the above hydrodynamic type system has infinitely many local Hamiltonian structures of all odd orders

$$
r^i_t = \hat{A}^{ij} \frac{\delta H_1}{\delta r^j} = \hat{A}^{ij} \hat{M}_{jk} \hat{A}^{ks} \frac{\delta H_0}{\delta r^s} = \hat{A}^{ij} \hat{M}_{jk} \hat{A}^{ks} \hat{M}_{sn} \hat{A}^{nm} \frac{\delta H_{-1}}{\delta r^m} = \ldots
$$

(38)

and infinitely many local symplectic structures of all odd orders

$$
\hat{M}_{ij} r^i_t = \frac{\delta H_2}{\delta r^i}, \quad \hat{M}_{ij} \hat{A}^{jk} \hat{M}_{kn} r^n_t = \frac{\delta H_3}{\delta r^i}, \quad \hat{M}_{ij} \hat{A}^{jk} \hat{M}_{kn} \hat{A}^{ns} \hat{M}_{sm} r^m_t = \frac{\delta H_4}{\delta r^i}, \ldots
$$
Theorem 2: The nonlinear PDE system
\[
\partial_i \beta_{jk} = \beta_{ji} \beta_{ik}, \quad i \neq j \neq k, \\
\partial_i \beta_{ik} + \partial_k \beta_{ki} + \sum_{m \neq i} \beta_{mi} \beta_{mk} = 0, \quad \partial_i \beta_{ki} + \partial_k \beta_{ik} + \sum_{m \neq i} \beta_{im} \beta_{km} = 0, \quad i \neq k
\]  
(39)
is an integrable system, which is a consequence of compatibility conditions following from the over-determined linear PDE system
\[
\partial_i H_k = \beta_{ik} H_i, \quad i \neq k, \quad \lambda H_i = \partial_i \left( \partial_i H_i + \sum_{n \neq i} \beta_{mi} H_n \right) + \sum_{m \neq i} \beta_{im} \left( \partial_m H_m + \sum_{n \neq m} \beta_{mn} H_n \right), \\
\partial_k \psi_i = \beta_{ik} \psi_k, \quad i \neq k, \quad \lambda \psi_i = \partial_i \left( \partial_i \psi_i + \sum_{n \neq i} \beta_{ni} \psi_n \right) + \sum_{m \neq i} \beta_{in} \left( \partial_m \psi_m + \sum_{n \neq m} \beta_{nm} \psi_n \right).
\]

Proof: follows from the above construction.
Indeed, suppose some particular solution \( H_i^{(0)} \) of the first linear system (4) is given. Then corresponding solution \( \psi_i^{(0)} \) of the adjoint linear system (4) can be obtained from (33)
\[
\psi_i^{(0)} = \partial_i H_i^{(0)} + \sum_{n \neq i} \beta_{ni} H_n^{(0)}.
\]
The new particular solution \( H_i^{(1)} \) is given by (8)
\[
H_i^{(1)} = \partial_i \psi_i^{(0)} + \sum_{m \neq i} \beta_{mi} \psi_m^{(0)}.
\]
Thus, an iteration of the above substitutions
\[
H_i^{(k)} = \partial_i \psi_i^{(k+1)} + \sum_{m \neq i} \beta_{mi} \psi_m^{(k+1)}, \quad \psi_i^{(k+1)} = \partial_i H_i^{(k+1)} + \sum_{n \neq i} \beta_{ni} H_n^{(k+1)}, \quad k = 0, 1, 2, \ldots \tag{40}
\]
leads to semi-infinite series of solutions of linear problems (4). It means, that corresponding commuting flows and conservation law densities can be found too
\[
r_{ik} = \frac{H_i^{(k)}}{H_i} r_{ik}, \quad dh^{(k)} = \sum \psi_i^{(k)} H_i dr^i.
\]
Since the iterations (40) are invertible (see (18) and (32)), another semi-infinite series can be found in quadratures
\[
H_i^{(-k-1)} = H_i^{(\beta)} q_\beta^{(-k)}, \quad \partial_\beta q_\beta^{(-k)} = \psi_i^{(-k)} H_i^{(\beta)}, \quad \psi_i^{(-k)} = \bar{\psi}^{(\beta)} \partial h^{(-k)} \partial_\alpha, \quad \partial_i \frac{\partial h^{(-k)}}{\partial_\alpha} = \bar{\psi}^{(\beta)} H_i^{(-k)}.
\]
Let structural flows (36) be written via flat coordinates of the first local Hamiltonian structure of the Dubrovin–Novikov type (see (12))
\[
a_{r^\alpha}^{\alpha'} = g^{\alpha \beta} D_x \frac{\partial h_r}{\partial \alpha}, \tag{13}
\]
13
where
\[ \partial_i a^\beta = \bar{\psi}_i^\beta \bar{H}_i, \quad \partial_i w_\gamma^\alpha = \bar{\psi}_i^{(\alpha)} \bar{H}(\gamma)_i, \quad \bar{H}(\gamma)_i = \partial_i \bar{\psi}_{(\gamma)i} + \sum_{m \neq i} \beta_m w_{(\gamma)m}, \quad \partial_i h_\gamma = \bar{\psi}_{(\gamma)i} \bar{H}_i. \]

Then an integrable hierarchy of hydrodynamic type systems can be written in the bi-Hamiltonian form
\[
\partial_t a^\alpha_i = \bar{g}^\alpha_\beta D_x \frac{\partial h^{(k)}}{\partial a^\beta_i} = \bar{\varepsilon}^\beta_\gamma (w_\beta^\alpha)_x D_x^{-1} (w_\gamma^\beta)_x \frac{\partial h^{(k+1)}}{\partial a^\gamma}, \quad (41)
\]

**Remark:** More general compatible bi-Hamiltonian structure recently was investigated in [38]. However, the nonlocal Hamiltonian operator (27) can be effectively constructed for any given flat metric namely in the above (flat–anti-flat) case. Moreover, just in this mixed case a corresponding recursion operator is local.

Indeed, since
\[ \bar{\varepsilon}^\gamma_\kappa (U^{\eta_\beta} \bar{g}^{\beta_\gamma}) D_x (U^{\xi_\mu} \bar{g}_{\mu\sigma}) a^{\sigma}_i = \frac{\partial h^{(k+1)}}{\partial a^\gamma}, \]

where
\[ U^{\alpha_\gamma} (w_{\gamma_\beta})_x = \delta^{\alpha}_\beta, \quad (w_{\gamma_\beta})_x U^{\beta_\alpha} = \delta^{\alpha}_\gamma, \quad U^{\alpha_\gamma} = \sum \bar{\psi}_m^{(\gamma)} \bar{H}_m^{(\alpha)} \frac{\partial m}{x}, \quad \partial_i w_{\gamma_\beta} = \bar{\psi}_{(\gamma)i} \bar{H}(\beta)_i, \]

then an integrable hierarchy of hydrodynamic type systems possesses the local Hamiltonian structure of the third order given by
\[
a^{\alpha}_i = \bar{\varepsilon}^\gamma_\kappa D_x U^{\gamma_\alpha} D_x U^{\xi_\sigma} D_x \frac{\partial h^{(k-1)}}{\partial a^{\sigma}}, \quad (42)
\]

**Remark:** In N component case, an inverse matrix \( U^{\beta_\gamma}(a, a_x) \) is a very complicated expression with respect to the first order derivatives. By this reason, the Riemann invariants \( r^k \) (see (38)) are most appropriate coordinates (for instance, in the Whitham theory for an arbitrary genus higher than 0), except the case when the Riemann invariants cannot be found explicitly, for instance for a dispersionless limit of integrable dispersive systems (see e.g. [48]).

**Remark:** One of very powerful approaches in an integrability of the nonlinear PDE system (3) was presented in [8]. Two linear PDE systems (4) should be related by the linear ODE system (with respect to each fixed index \( i \))
\[ H_i = [c_i \partial_i^n + \ldots + c_n(r)] \psi_i. \]

If \( n = 1 \), this is exactly already well-known flat case (8). The case considered in [8] can be restricted on a compatible couple of the above transformations
\[ H^{(k+1)}_i = [c_i(0) \partial_i^n + \ldots + c_n(r)] \psi^{(k)}_i, \quad \psi^{(k)}_i = [\bar{c}_i(0) \partial_i^n + \ldots + \bar{c}_n(r)] H^{(k)}_i \]

generalizing the case considered in this Section.

**Remark:** The above transformation of the second order (see (39) and (40))
\[ H^{(k+1)}_i = (\partial_i^2 + \ldots) H^{(k)}_i \]
is a first example of more general transformations

\[ H_i^{(k+1)} = (\partial^k_i + \ldots)H_i^{(k)} \]

never studied except \( n = 1 \) (see below, Section 7).

**Remark:** Thus, the first “puzzle” hidden in [42] is a consequence of the existence of a mixed bi-Hamiltonian structure discussed in this section. It means, that the sub-class of so-called “separable” Hamiltonians for two component hydrodynamic type systems introduced in [42] has pure differential-geometric meaning via the language of conjugate curvilinear coordinate nets in \( N \) component case.

**Remark:** The nonlinear PDE system (39) is well-known in classical differential geometry (see e.g. [5]). See also recent investigation in [54], where (39) is called \( O(2N)/O(N) \times O(N) \)-system (see also the generalized wave equation in [53]). This nonlinear PDE system (39) is a natural generalization of the Egorov case (see below Section 7): if rotation coefficients \( \beta_{ik} \) are symmetric, then (39) reduces to well-known system describing Egorov flat metrics.

### 6 Multi-Hamiltonian structures

A comparison nonlinear and linear PDE systems ((3), (4), (8) and (9)) describing orthogonal curvilinear coordinate nets \( \beta_{ik} \) with corresponding nonlinear and linear PDE systems ((28) and (33)) describing anti-orthogonal curvilinear coordinate nets \( \tilde{\beta}_{ik} \) leads to the simple identity

\[ \tilde{\beta}_{ik} = \beta_{ki}. \]

**Definition:** Corresponding curvilinear coordinate nets are said to be **mirrored**.

Since a description of orthogonal curvilinear coordinate nets is equivalent to a description of anti-orthogonal curvilinear coordinate nets, then, in fact, all known theorems in the theory of orthogonal curvilinear coordinate nets can be simply reformulated for anti-orthogonal case.

**Interpretation:** Any integrable (semi-Hamiltonian) hydrodynamic type system has an infinite set of conservation laws and commuting flows.

Let us write \( N \) arbitrary conservation laws for \( N - 1 \) arbitrary nontrivial commuting flows in the potential form

\[
\begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_N
\end{pmatrix} =
\begin{pmatrix}
h_{11} & h_{12} & \ldots & h_{1N} \\
h_{21} & h_{22} & \ldots & h_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
h_{N1} & h_{N2} & \ldots & h_{NN}
\end{pmatrix}
\begin{pmatrix}
t_1 \\
t_2 \\
\vdots \\
t_N
\end{pmatrix}.
\]

(44)

It means, that \( N - 1 \) commuting flows

\[ r_{ik}^i = \frac{H_{(k)}i}{H_{(1)i}} r_{ik}^i, \quad k = 2, 3, \ldots, N \]

possess \( N \) conservation laws

\[ \partial_{t_k} h_{i1} = \partial_{t_i} h_{ik}, \quad i = 1, 2, \ldots, N, \]
where
\[ \partial_j h_{ik} = \psi_{(i)j} H_{(k)j}. \]

Let us construct another set of \( N \) conservation laws and \( N - 1 \) commuting flows determined by the transposed matrix

\[
\begin{pmatrix}
  z_1 \\
  z_2 \\
  \vdots \\
  z_N
\end{pmatrix}
= \begin{pmatrix}
  h_{11} & h_{21} & \cdots & h_{N1} \\
  h_{12} & h_{22} & \cdots & h_{N2} \\
  \vdots & \vdots & \ddots & \vdots \\
  h_{1N} & h_{2N} & \cdots & h_{NN}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_N
\end{pmatrix}. \tag{46}
\]

It means, that \( N - 1 \) commuting flows

\[
r_{x^k}^i = \frac{\psi_{(k)i}}{\psi_{(1)i}} r_{x^i}^i, \quad k = 2, 3, ..., N
\]

possess \( N \) conservation laws

\[
\partial_{x^k} h_{1i} = \partial_{x^i} h_{ki}, \quad i = 1, 2, ..., N.
\]

Corresponding rotation coefficients are given by (see (43))

\[
\tilde{\beta}_{ik} = \frac{\partial_i \tilde{H}_{(1)k}}{\tilde{H}_{(1)i}} \equiv \frac{\partial_i \psi_{(1)k}}{\psi_{(1)i}} = \beta_{ki},
\]

because the new Lame coefficients \( \tilde{H}_{(1)i} \equiv \psi_{(1)i} \) (cf. (45) and (47)).

Such families of hydrodynamic type systems are said to be mirrored. The aforementioned relationship between these hydrodynamic type systems (44) and (46) is well known in the projective differential geometry as the “duality condition” (see [1] and other references therein).

One of the most interesting questions of the classical differential geometry which has appeared at studying of semi-Hamiltonian systems of hydrodynamical type is the description of the surfaces admitting not trivial deformations with preservation of principal directions and principal curvatures. Then the number of essential parameters on which such deformations depend, is actually equal to number of various local Hamiltonian structures of a corresponding hydrodynamical type system [20]. Such local Hamiltonian structures are determined by differential-geometrical Poisson brackets of the first order (see [14]). In the same paper a bi-Hamiltonian structure of the system of averaged one-phase solutions of Korteweg de Vries equation has been considered. Later multi-Hamiltonian structures of systems of hydrodynamical type were studied in [44], [49].

Example [22]: The Lame coefficients (37) for the hydrodynamic type system

\[
r^i_i = \left( \sum r^m + 2 r^i \right) r^i_x, \quad i = 1, 2, ..., N \tag{48}
\]

are given by

\[
\tilde{H}_i = \mu^{-1/2}_i \prod_{m \neq i} (r^i - r^m)^{1/2},
\]
where \( \mu_i(r^i) \) are arbitrary functions. The rotation coefficients (26) are given by

\[
\beta_{ik} = \frac{\mu_i^{1/2}(r^i)}{\mu_k^{1/2}(r^k)} \frac{\bar{H}^0_i}{\bar{H}^0_k} r^i - r^k,
\]

where we use the notation

\[
\bar{H}^0_i = \prod_{m \neq i} (r^i - r^m)^{1/2}.
\]

The substitution these rotation coefficients in the flatness condition (9) yields the following functional-differential system (see [56])

\[
\mu_i \partial_i \beta_{ik}^0 + \frac{1}{2} \mu_i' \beta_{ik}^0 + \mu_k \partial_k \beta_{ki}^0 + \frac{1}{2} \mu_k' \beta_{ki}^0 + \sum_{m \neq i, k} \mu_m \beta_{mi}^0 \beta_{mk}^0 = 0,
\]

where we use the notation \( \beta_{ik}^0 = \partial_i \bar{H}^0_k / \bar{H}^0_i \).

**Lemma [22]:** The above functional-differential equation has a polynomial solution parameterized by \( N + 1 \) arbitrary constants

\[
\mu_i = C_0 (r^i)^N + C_1 (r^i)^{N-1} + C_2 (r^i)^{N-2} + \ldots + C_{N-2} (r^i)^2 + C_{N-1} r^i + C_N.
\]

**Remark [44]:** These Riemann invariants \( r^k \) are nothing but elliptic coordinates (see [10]).

**Theorem 3 [22]:** Semi-Hamiltonian hydrodynamic type system cannot possess more than \( N + 1 \) local Hamiltonian structures of the Dubrovin–Novikov type.

Thus, the above hydrodynamic type system (48) has a maximum of local Hamiltonian structures of the Dubrovin–Novikov type.

**Remark:** Bi-Hamiltonian structures of the Dubrovin–Novikov type are described in the set of publications (see [13], [21], [39], [47]). Semi-Hamiltonian hydrodynamic type systems connected with multi-orthogonal curvilinear coordinate nets, where the number of corresponding local Hamiltonian structures of the Dubrovin–Novikov type is less than \( N + 1 \) is not described yet.

Since orthogonal and anti-orthogonal cases are dual to each other, then following theorem is valid.

**Theorem 4:** Semi-Hamiltonian hydrodynamic type system cannot possess more than \( N + 1 \) anti-flat nonlocal Hamiltonian structures (see (27), (28)).

**Example:** The hydrodynamic type system

\[
r_i^j = \left( \sum r^m - 2r^i \right) r_x^i, \quad i = 1, 2, \ldots, N
\]

has \( N + 1 \) local Lagrangian representations (see (31))

\[
S_n = \int \left[ \frac{1}{2} \sum_{m \neq k} \frac{(r^k)^{n-1}}{(r^k - r^m)} \frac{r_x^k}{r_x^m} - h_n(r) \right] dx dt, \quad n = 1, 2, \ldots, N + 1.
\]
Indeed, the Lame coefficients are given by (cf. (49))
\[ H_i^0 = \prod_{m \neq i} (r^i - r^m)^{-1/2}. \]

It means, that corresponding rotation coefficients \( \tilde{\beta}_{ik} \) are mirrored (see (43)) to the rotation coefficients \( \beta_{ik} \) associated with elliptic coordinates (see (48)).

7 The Egorov flat hydrodynamic type systems

Let us consider the semi-Hamiltonian hydrodynamic type system (2) possessing the local Hamiltonian structure of the Dubrovin–Novikov type (12) and the couple of conservation laws
\[ a_t = b_x, \quad b_t = c_x. \tag{52} \]

**Definition:** The above Hamiltonian hydrodynamic type system is said to be Egorov flat.

**Theorem 5:** The Egorov flat hydrodynamic type system has infinitely many local Hamiltonian structures of all odd orders.

**Proof:** If the semi-Hamiltonian hydrodynamic type system (2) has the couple of conservation laws (52), then (see [49]) \( a \) is a potential of the Egorov metric, i.e.
\[ \partial_i a = \tilde{H}_i^2, \quad \partial_i b = \tilde{H}_i\tilde{H}_i, \quad \partial_i c = \tilde{H}_i^2 \]
and rotation coefficients \( \beta_{ik} \) are symmetric (see [16]). It means, that both linear PDE systems (4) coincide to each other (thus, we can identify \( \psi_k \) and \( H_k \) below). Also the flatness (9) and anti-flatness (28) conditions coincide and reduce to
\[ \delta \beta_{ik} = 0, \tag{53} \]
where \( \delta = \sum \partial/\partial r^m \) is a shift operator. Thus, if the Egorov hydrodynamic type system has a local Hamiltonian structure of the Dubrovin–Novikov type, then another nonlocal Hamiltonian structure associated with a local Lagrangian representation exists automatically; and vice versa.

The main statement of this section: If rotation coefficients \( \beta_{ik} \) are symmetric and depend on differences of Riemann invariants (see (53)) only, then corresponding hydrodynamic type system (2) has an infinite set of local Hamiltonian structures of all odd orders.

**Remark:** In general case, an infinite set of Hamiltonian structures can be constructed starting with a couple of given Hamiltonian structures (one of them must be invertible) due to Magri’s theorem (see [36]). In the Egorov case, we need just one local Hamiltonian structure of the Dubrovin–Novikov type or one local Lagrangian representation (31).

Indeed, the first transformation (8) (associated with the flatness condition (9)) reduces to (see [56], [57])
\[ H_i^{(k)} = \delta H_i^{(k+1)}; \tag{54} \]
the second transformation (33) (associated with the anti-flatness condition (28)) reduces to (cf. (40))

\[ H_i^{(k-1)} = \delta H_i^{(k)}. \]

Thus, the transformation of the second order (40) factorizes in a product of the above transformations of the first orders. The well-known theorem (in fact written by L. Bianchi in [5]) is a consequence of such a factorization:

**Theorem 6**: The nonlinear PDE system (Bianchi–Lame–Darboux–Egorov) system (3), (53) is integrable. The corresponding linear PDE system (which is equivalent to \( N \) linear ODE systems for each fixed index \( i \)) is given by

\[ \partial_i H_k = \beta_{ik} H_i, \quad i \neq k; \quad \lambda H_i = \delta H_i. \quad (55) \]

**Proof**: can be obtained by a computation of the compatibility conditions \( \partial_i(\partial_j H_k) = \partial_j(\partial_i H_k) \) for \( i \neq j \neq k \) and \( \partial_i(\partial_k H_k) = \delta_k(\partial_i H_k) \) for \( i \neq k \).

Let us consider the consistency of the linear problem (4) with the symmetry operator \( \hat{S} = \sum s_k(r^k) \partial/\partial r^k \). Under the scaling \( dR^k(r^k) = dr^k/s_k(r^k) \) this symmetry operator \( \hat{S} \) is equivalent to the shift operator \( \delta \).

**Theorem 7**: The first linear problem (4) is compatible with the symmetry operator \( \hat{S} \); i.e. the compatibility conditions \( \partial_i(\partial_j H_k) = \partial_j(\partial_i H_k) \) for \( i \neq j \neq k \) and \( \partial_i(\partial_k H_k) = \delta_k(\partial_i H_k) \) for \( i \neq k \) are valid, where

\[ \lambda H_i = (\hat{S} + c_i(r^i)) H_i, \]

if and only if

\[ \hat{S} \beta_{ik} = (c^i - c^k - s_i^k) \beta_{ik}. \]

The second linear problem (4) is compatible with the symmetry operator \( \hat{S} \); i.e. the compatibility conditions \( \partial_i(\partial_j \psi_k) = \partial_j(\partial_i \psi_k) \) for \( i \neq j \neq k \) and \( \partial_i(\partial_k \psi_k) = \delta_k(\partial_i \psi_k) \) for \( i \neq k \) are valid, where

\[ \lambda \psi_i = (\hat{S} + c_i(r^i)) \psi_i, \]

if and only if

\[ \hat{S} \beta_{ik} = (c^k - c^i - s_k^i) \beta_{ik}. \]

**Proof**: can be obtained by a straightforward computation.

In the symmetric case (\( \beta_{ik} = \beta_{ki} \)) the above formulas reduce to

\[ \lambda H_i = (\hat{S} + s_i^i(r^i)/2) H_i, \quad \hat{S} \beta_{ik} + (s_i^i + s_k^i) \beta_{ik}/2 = 0. \quad (56) \]

**Theorem 8** [44], [49]: The semi-Hamiltonian Egorov hydrodynamic type system (2) has a local Hamiltonian structure of the Dubrovin–Novikov type iff the symmetry restrictions (56) are fulfilled (and vice versa).

**Proof**: Indeed, if the Egorov hydrodynamic type system has a local Hamiltonian structure of the Dubrovin–Novikov type, then the flatness condition (see (50))

\[ s_i \partial_i \beta_{ik} + \frac{1}{2} s_i^i \beta_{ik} + s_k \partial_k \beta_{ki} + \frac{1}{2} s_k^k \beta_{ki} + \sum_{m \neq i,k} s_m \beta_{mi} \beta_{mk} = 0 \quad (57) \]
reduces to (56).

Thus, if the functional-differential system (56) has \( M \) independent solutions \( (M < N+2) \), then a corresponding Egorov hydrodynamic type system has also \( M \) local Hamiltonian structures of the Dubrovin–Novikov type as well as \( M \) nonlocal Hamiltonian structures associated with local Lagrangian representations (31).

**Examples:** The ideal gas dynamics (19) is the Egorov hydrodynamic type system. Indeed, the Egorov pair of conservation laws (52) is given by

\[
\rho_t = (\rho u)_x, \quad (\rho u)_t = (\rho u^2 + \rho^2 f''(\rho) - 2\rho f'(\rho) + 2f(\rho))_x.
\]

Just one symmetry operator \( \delta = \partial/\partial u \) is connected with sole local Hamiltonian structure of the Dubrovin–Novikov type and with sole nonlocal Hamiltonian structure associated with a local Lagrangian representation (31). Thus, only one infinite series of local Hamiltonian structures of all odd orders is connected with the ideal gas dynamics (19).

**Whitham equations** (see [44]). The averaged \( N \) phase solution of the Sinh-Gordon equation is the Egorov hydrodynamic type system possessing sole local Hamiltonian structure of the Dubrovin–Novikov type. The averaged \( N \) phase solution of the Korteweg de Vries equation is the Egorov hydrodynamic type systems possessing two local Hamiltonian structures of the Dubrovin–Novikov type. The averaged \( N \) phase solution of the nonlinear Schrödinger equation is the Egorov hydrodynamic type system possessing three local Hamiltonian structures of the Dubrovin–Novikov type.

Thus, the first Egorov hydrodynamic type system possesses sole infinite series of local Hamiltonian structures of all odd orders; the second Egorov hydrodynamic type system possesses two infinite series of local Hamiltonian structures of all odd orders; the third Egorov hydrodynamic type system possesses three infinite series of local Hamiltonian structures of all odd orders.

**Exceptional** linear-degenerate hydrodynamic type system

\[
r^i_t = \left( \sum \varepsilon_m r^m - \varepsilon_i \sum r^m \right) r^i_x, \quad i = 1, 2, ..., N.
\]

Since the Lame coefficients \( H_i = 1/\Sigma r^m \) (see (37)), the potential of the Egorov metric \( a = -1/\Sigma r^m \) and the rotation coefficients \( \beta_{ik} = -1/\Sigma r^m \), then the functional-differential system (57) has \( N \) independent solutions. Indeed, the substitution rotation coefficients in (56) leads to the following identity

\[
\sum s_k(r^k) = \frac{1}{2}(s^i_k + s^i_k) \sum r^m.
\]

This system has the solution \( s_k(r^k) = r^k \) and \( N - 1 \) parametric solution \( s_k = \text{const} \), where \( \Sigma s_k = 0 \). Thus, the above hydrodynamic type system possesses \( N \) infinite series of local Hamiltonian structures of all odd orders. \( N \) parametric local Lagrangian representation is given by

\[
S = \int \left[ \sum s_k r^k + s_k r^k \frac{r^k}{\sum r^m} - h(r) \right] dxdt.
\]

Let us restrict our consideration on \( N - 1 \) commuting Egorov flat hydrodynamic type systems written via flat coordinates \( a^\beta \), whose first conservation law is given by (see [45])

\[
\partial_{t^\gamma} a_1 = \partial_{t^1} a_{\gamma},
\]
where \( a^\beta = \tilde{g}^{\beta\gamma} a_\gamma \) (see (12)).

**Theorem 9 [13]:** These set of hydrodynamic type systems

\[
\partial_\gamma a_\beta = \partial_1 \frac{\partial^2 F}{\partial a^\gamma \partial a^\beta}
\]  

is determined by a solution of the WDVV equation

\[
\frac{\partial^3 F}{\partial a^\alpha \partial a^\beta \partial a^\gamma} \tilde{g}^{\gamma\eta} \frac{\partial^3 F}{\partial a^\eta \partial a^\delta \partial a^\mu} = \frac{\partial^3 F}{\partial a^\alpha \partial a^\gamma \partial a^\beta} \tilde{g}^{\gamma\eta} \frac{\partial^3 F}{\partial a^\eta \partial a^\delta \partial a^\mu}.
\]

**Corollary:** Conservation law densities satisfy the linear PDE system

\[
\tilde{g}^{\gamma\eta} \frac{\partial^2 h}{\partial a^\alpha \partial a^\beta} \frac{\partial^3 F}{\partial a^\gamma \partial a^\delta \partial a^\mu} = \tilde{g}^{\gamma\eta} \frac{\partial^2 h}{\partial a^\mu \partial a^\gamma} \frac{\partial^3 F}{\partial a^\delta \partial a^\beta}.
\]

Taking into account (this is a consequence from (58) when \( \beta = 1 \))

\[
\tilde{g}_{\alpha\beta} = \frac{\partial^3 F}{\partial a^1 \partial a^\alpha \partial a^\beta}
\]

the above linear PDE system reduces (for \( \beta = 1 \)) to

\[
\tilde{g}^{\gamma\eta} \frac{\partial^2 h}{\partial a^1 \partial a^\gamma} \frac{\partial^3 F}{\partial a^\delta \partial a^\mu} = \frac{\partial^2 h}{\partial a^\mu \partial a^\gamma} \frac{\partial^3 F}{\partial a^\delta}.
\]

Since this linear PDE system is consistent with the symmetry (shift) operator \( \delta = \partial / \partial a^1 \), then this formula reduces to iterations (in quadratures) of all conservation law densities (see [13])

\[
\frac{\partial^2 h_{\beta}^{(k+1)}}{\partial a^\mu \partial a^\delta} = \tilde{g}^{\gamma\eta} \frac{\partial h_{\beta}^{(k)}}{\partial a^\gamma} \frac{\partial^3 F}{\partial a^\mu \partial a^\delta \partial a^\alpha}, \quad k = 0, 1, 2, ..., \quad p = 1, 2, ..., N,
\]

where first \( N \) conservation law densities \( h_{\beta}^{(0)} \) are (for instance) flat coordinates \( a^\beta \). Thus, the generating function of conservation law densities \( h(a, \lambda) \) is determined by the linear PDE system

\[
\lambda \frac{\partial^2 h}{\partial a^\mu \partial a^\delta} = \tilde{g}^{\gamma\eta} \frac{\partial h}{\partial a^\gamma} \frac{\partial^3 F}{\partial a^\mu \partial a^\delta \partial a^\alpha},
\]  

(59)

whose compatibility conditions (see [13]) lead to the aforementioned WDVV equation.

**Theorem 10:** The above linear PDE system is equivalent to (55).

**Proof:** A differentiation (58) with respect to Riemann invariants (45) leads to the identity (see [13])

\[
\frac{\partial^3 F}{\partial a^\alpha \partial a^\beta \partial a^\gamma} = \sum \frac{\bar{H}_{(\alpha)m} \bar{H}_{(\beta)m} \bar{H}_{(\gamma)m}}{\bar{H}_{(1)m}}.
\]

Expressing derivatives of \( h \) with respect to flat coordinates via derivatives of \( h \) with respect to Riemann invariants (see (7), (17) and (11), where we must identify \( \tilde{\psi}_{(\gamma)i} \equiv \bar{H}_{(\gamma)i} \)), (59) reduces to (55).
In this case the bi-Hamiltonian structure (41) reduces to (see [38])

\[ a_{ik}^\alpha = \bar{g}^{\alpha \beta} D_\beta \frac{\partial h(k)}{\partial a^\alpha} = \bar{g}^{\beta \gamma} \bar{g}^{\alpha \eta} \gamma^\eta \left( \frac{\partial^2 F}{\partial a^\alpha \partial a^\beta} \right) D_x^{-1} \left( \frac{\partial^2 F}{\partial a^\beta \partial a^\mu} \right) \frac{\partial h(k+1)}{\partial a^\gamma}, \]

because the structural flows (29) (see also (36)) are given by (58) and (cf. (10) with (30)) \( \bar{g}^{\alpha \beta} \equiv \bar{\varepsilon}^{\alpha \beta}. \) A corresponding local Hamiltonian structure of the third order (42) reduces to (cf. (20))

\[ a_{ik}^\alpha = \bar{g}_{\beta \gamma} D_\gamma U^\alpha \beta U^\gamma \sigma D_x \frac{\partial h(k-1)}{\partial a^\sigma}, \]

where

\[ U^\alpha \beta \left( \frac{\partial^2 F}{\partial a^\beta \partial a^\gamma} \right)_x = \delta^\alpha_\gamma, \quad \left( \frac{\partial^2 F}{\partial a^\alpha \partial a^\beta} \right)_x U^\beta \gamma = \delta^\beta_\alpha, \quad U^\alpha \beta = \sum \frac{\bar{H}^{\alpha \beta}_{m}}{r^m_x}, \quad \partial_{\beta} \frac{\partial^2 F}{\partial a^\beta \partial a^\gamma} = \bar{H}^{\beta}_{(\gamma)i}. \]

Thus, the above local Hamiltonian structure of the third order is \( N \) component generalization of (20).

**Remark:** Ya. Nutku and P. Olver explained the origin of the second order recursion operator for the ideal gas dynamics (19) as a product of two Sheftel–Teshukov first order recursion operators in [42]. E.V. Ferapontov has proved (mentioned in [56], see also [25]) that the existence of the Sheftel–Teshukov recursion operator (see [52], [55]) is equivalent to the existence of the symmetry operator \( \delta \) (see (54)) in appropriate Riemann invariants. Indeed, the eigenfunction problem (59) can be written in the form

\[ \lambda \frac{\partial h}{\partial a^\alpha} = \bar{g}^{\beta \gamma} D_\gamma D_x^{-1} \left( \frac{\partial^2 F}{\partial a^\alpha \partial a^\beta} \right) \frac{\partial h}{\partial a^\gamma}. \]

Thus, the recursion operator (translating the gradient \( \partial h(k) / \partial a^\alpha \) to the gradient \( \partial h(k+1) / \partial a^\alpha \)) is given by

\[ \hat{Q}^\beta_\alpha = \bar{g}^{\beta \gamma} D_\gamma D_x^{-1} \left( \frac{\partial^2 F}{\partial a^\alpha \partial a^\gamma} \right)_x; \]

the inverse recursion operator

\[ \hat{G}^\alpha_\beta = U^{\alpha \gamma} \bar{g}_{\gamma \beta} D_x \]

is connected with the Sheftel–Teshukov recursion operator (translating corresponding commuting flows) written via flat coordinates (cf. (21))

\[ \hat{R}^\alpha_\beta = \bar{g}^{\alpha \gamma} D_\gamma \hat{G}^\sigma_\gamma D_x^{-1} g_{\sigma \beta} = D_x (\bar{g}_{\beta \sigma} U^{\sigma \alpha}) \]

(60)

Thus, this observation (made by Ya. Nutku and P. Olver) is valid for \( N \) component Egorov Hamiltonian hydrodynamic type systems (cf. (42)).

**Example:** Let us consider the couple of commuting flows

\[ a_t = b_x, \quad a_y = c_x, \]

\[ b_t = \partial_x (c + z_{bb}), \quad b_y = \partial_x z_{ab}, \]

\[ c_t = \partial_x z_{ab}, \quad c_y = \partial_x z_{aa}, \]

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where the function \( z(a, b) \) satisfies the associativity equation (see [13])

\[
z_{aaa} = z_{abb} - z_{aab}z_{bbb}.
\]

These Egorov hydrodynamic type systems have only one local Hamiltonian structure

\[
a_t = D_x \frac{\partial h_1}{\partial c}, \quad b_t = D_x \frac{\partial h_1}{\partial b}, \quad c_t = D_x \frac{\partial h_1}{\partial a},
\]

\[
a_y = D_x \frac{\partial h_2}{\partial c}, \quad b_y = D_x \frac{\partial h_2}{\partial b}, \quad c_y = D_x \frac{\partial h_2}{\partial a}
\]

of the Dubrovin–Novikov type in general case, where \( h_1 = bc + z_b, \ h_2 = c^2/2 + z_a \).

These commuting flows can be written in the potential form

\[
d \begin{pmatrix} \Omega_x \\
\Omega_t \\
\Omega_y \end{pmatrix} = \begin{pmatrix} a & b & c \\
b & c + z_{bb} & z_{ab} \\
c & z_{ab} & z_{aa} \end{pmatrix} d \begin{pmatrix} x \\
t \\
y \end{pmatrix}, \quad (61)
\]

where

\[
d\Omega = \Omega_x dx + \Omega_t dt + \Omega_y dy.
\]

Then a new local Hamiltonian operator of the third order (cf. (20))

\[
\hat{B} = \hat{R}^2 \begin{pmatrix} 0 & 0 & D_x \\
0 & D_x & 0 \\
D_x & 0 & 0 \end{pmatrix},
\]

where the recursion operator \( \hat{R} \) is a purely differential operator of the first order (cf. (21))

\[
\hat{R} = D_x(W_x)^{-1}
\]

according to (60), and the matrix \( W \) is given by (cf. (61))

\[
W = \begin{pmatrix} c & b & a \\
z_{ab} & c + z_{bb} & b \\
z_{aa} & z_{ab} & c \end{pmatrix}.
\]

8 Second Nutku–Olver’s “puzzle”

Let us consider a particular case of the ideal gas dynamics (19) determined by the special choice \( f(\rho) = \rho^\gamma/\gamma(\gamma - 1)(\gamma - 2) \). The polytropic gas dynamics (see [41])

\[
\rho_t = (\rho u)_x, \quad u_t = \left( \frac{u^2}{2} + \frac{\rho^{\gamma-1}}{\gamma - 1} \right)_x
\]

has three local Hamiltonian structures of the Dubrovin–Novikov type

\[
\begin{pmatrix} \rho \\
u \end{pmatrix}_t = \begin{pmatrix} \hat{A}_{k}^{11} & \hat{A}_{k}^{12} \\
\hat{A}_{k}^{21} & \hat{A}_{k}^{22} \end{pmatrix} \begin{pmatrix} \delta H_{5-k}^{1}\partial \rho \\
\delta H_{5-k}^{1}\partial u \end{pmatrix}, \quad k = 1, 2, 3,
\]

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where the Hamiltonians are
\[
\begin{align*}
\hat{H}_4 &= \int \left[ \frac{\rho u^2}{2} + \frac{\rho}{\gamma(\gamma - 1)} \right] dx, & \hat{H}_3 &= \int \rho u dx, & \hat{H}_2 &= \int \rho dx, & \hat{H}_1 &= \int u dx.
\end{align*}
\]

Corresponding components \(\hat{A}^{ij}_k\) of the above local Hamiltonian operators are given by
\[
\hat{A}_1 = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix},
\]
\[
\hat{A}_2 = \begin{pmatrix} \rho D_x + D_x \rho & (\gamma - 2) D_x u + u D_x \\ D_x u + (\gamma - 2) u D_x & \rho \gamma^{-2} D_x + D_x \rho \gamma^{-2} \end{pmatrix},
\]
\[
\hat{A}_3 = \begin{pmatrix} u \rho D_x + D_x u \rho & D_x \left[ \frac{1}{2}(\gamma - 2)u^2 + \frac{1}{\gamma - 1}\rho^{-1} \right] + \left[ \frac{1}{2}u^2 + \frac{1}{\gamma - 1}\rho^{-1} \right] D_x \\ D_x \left[ \frac{1}{2}u^2 + \frac{1}{\gamma - 1}\rho^{-1} \right] + \left[ \frac{1}{2}(\gamma - 2)u^2 + \frac{1}{\gamma - 1}\rho^{-1} \right] D_x & u \rho \gamma^{-2} D_x + D_x u \rho \gamma^{-2} \end{pmatrix}.
\]

Then the first recursion operator
\[
\hat{R}_1 = \begin{pmatrix} (\gamma - 1) u + (\gamma - 2)u_x D_x^{-1} & 2\rho + \rho_x D_x^{-1} \\ 2\rho \gamma^{-2} + (\gamma - 2) \rho \gamma^{-3} \rho_x D_x^{-1} & (\gamma - 1)u + u_x D_x^{-1} \end{pmatrix}
\]
is a ratio of first two local Hamiltonian structures, i.e. \(\hat{A}_2 = \hat{R}_1 \hat{A}_1\). However \(\hat{A}_3 \neq \hat{R}_1 \hat{A}_2\). It means that
\[
\hat{R}_1 \hat{A}_2 = 2(\gamma - 1) \begin{pmatrix} 2u \rho D_x + (u \rho)_x & \frac{1}{2}(\gamma - 1)u^2 + 2\rho \gamma^{-1} D_x \\ \frac{1}{2}(\gamma - 1)u^2 + 2\rho \gamma^{-1} D_x & 2u \rho \gamma^{-2} D_x + (u \rho \gamma^{-2})_x \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(\gamma - 1)u^2 + 2\rho \gamma^{-1} D_x \\ 2u \rho \gamma^{-2} D_x + (u \rho \gamma^{-2})_x \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(\gamma - 1)u^2 + 2\rho \gamma^{-1} D_x \\ 2u \rho \gamma^{-2} D_x + (u \rho \gamma^{-2})_x \end{pmatrix}
\]
is a nonlocal Hamiltonian operator of the Ferapontov type (see [19]). A comparison of the above nonlocal Hamiltonian operator \(\hat{R}_1 \hat{A}_2\) and the third local Hamiltonian operator \(\hat{A}_3\) leads to the purely nonlocal Hamiltonian operator (27)
\[
2\frac{\gamma - 1}{\gamma - 2} \hat{A}_3 - \frac{1}{\gamma - 2} \hat{R}_1 \hat{A}_2 = \begin{pmatrix} \frac{u}{\rho \gamma^{-2}} & \rho \\ \rho \gamma^{-2} & u \end{pmatrix} D_x^{-1} \begin{pmatrix} \frac{u}{\rho \gamma^{-2}} & \rho \\ \rho \gamma^{-2} & u \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
where the matrix \(W\) is given by the particular choice \(f(\rho) = \rho^\gamma / \gamma(\gamma - 1)(\gamma - 2)\) in (22).
Taking into account the identity (see (27), (41), (42))

\[ \hat{B} = \hat{A}_1 \hat{K}^{-1} \hat{A}_1, \]

where \( \hat{B} \) is a local Hamiltonian operator of the third order (20) and \( \hat{K} \) is aforementioned nonlocal Hamiltonian operator connected with the local Lagrangian representation (23), we obtain an interesting algebraic relationship

\[ \hat{B} = \hat{A}_1 \left( \frac{2\gamma - 1}{\gamma - 2} \hat{A}_3 - \frac{1}{\gamma - 2} \hat{A}_2 \hat{A}_1^{-1} \hat{A}_2 \right)^{-1} \hat{A}_1 \]  \hspace{1cm} (62)

for local Hamiltonian operators found by Ya. Nutku and P. Olver in [42].

The above relationship

\[ \hat{K} = 2 \frac{\gamma - 1}{\gamma - 2} \hat{A}_3 - \frac{1}{\gamma - 2} \hat{A}_2 \hat{A}_1^{-1} \hat{A}_2 \]

has a simple differential-geometric interpretation (see [19]). Any semi-Hamiltonian hydrodynamic type system has an infinite series of nonlocal Hamiltonian structures of the Ferapontov type. These Hamiltonian structures are associated with surfaces with a flat normal bundle. A reconstruction of corresponding surfaces with a flat normal bundle is a very complicated problem (see [7]). Let \( g_{ii} \) be a diagonal metric of the first Hamiltonian structure (i.e. \( \hat{A}_1 \) is written via Riemann invariants \( r^k \), see (16)), then \( g_{ii}^{(2)} = \hat{H}_i^2 / r^i \) is a metric of the second Hamiltonian structure (i.e. \( \hat{A}_2 \)), and \( g_{ii}^{(3)} = \hat{H}_i^2 / (r^i)^2 \) is a metric of the third Hamiltonian structure (i.e. \( \hat{A}_3 \) and \( \hat{A}_2 \hat{A}_1^{-1} \hat{A}_2 \)).

Thus, the third metric is associated with two different submanifolds. It means that corresponding the two-dimensional Riemann space with a flat normal bundle is embedded into a four dimensional Riemann space in the second (nonlocal) case.

**Remark:** Ya. Nutku and P. Olver have found one local Hamiltonian operator of the third order. However, the polytropic gas dynamics has three local Hamiltonian structures and three local Lagrangian representations (31) too, because this Egorov hydrodynamic type systems possesses three symmetry operators (the shift operator \( \delta = \Sigma \partial / \partial r^m \), the scaling operator \( \tilde{S} = \Sigma \partial / \partial r^m \), the projective operator \( \tilde{P} = \Sigma (r^m)^2 \partial / \partial r^m \), see (56) and [2]), where each of them is connected with one local Hamiltonian structure and with one local Lagrangian representation, simultaneously. Thus, the polytropic gas dynamics has three local Hamiltonian structures of the third order. Corresponding flat coordinates are found in [43]. Then two other similar algebraic relationships (see (62)) can be found.

### 9 Generalizations

Theory of orthogonal curvilinear coordinate nets associated with a flat metric can be extended on surfaces with a flat normal bundle [19]. In such a case two linear PDE systems (4) are connected by the nonlocal transformation of the first order (cf. (8))

\[ H_i = \partial_i \psi_i + \sum_{m \neq i} \beta_{mi} \psi_m + \varepsilon_{\alpha \beta} H_i^{(\alpha)} p^{(\beta)}, \quad \partial_i p^{(\beta)} = \psi_i H_i^{(\beta)}, \]
where $\varepsilon_{\alpha\beta}$ is a constant symmetric non-degenerate matrix, $H_i^{(\alpha)}$ are particular solutions of (4), the flatness condition replaces by the more general Gauss-Codazzi equation

$$\partial_i \beta_{ik} + \partial_k \beta_{ki} + \sum_{m \neq i,k} \beta_{mi} \beta_{mk} = \varepsilon_{\alpha\beta} H_i^{(\alpha)} H_k^{(\beta)}.$$ 

Thus, an obvious generalization of an anti-flatness condition and a corresponding nonlocal transformation of the first order between two linear PDE systems (4) can be given by

$$\psi_i = \partial_i H_i + \sum_{m \neq i} \beta_{mi} H_m + \varepsilon_{\alpha\beta} \psi_i^{(\alpha)} \tilde{p}^{(\beta)}, \quad \partial_i \tilde{p}^{(\beta)} = \psi_i^{(\beta)} H_i,$$

where

$$\partial_i \beta_{ki} + \partial_k \beta_{ik} + \sum_{m \neq i,k} \beta_{im} \beta_{km} = \tilde{\varepsilon}_{\alpha\beta} \psi_i^{(\alpha)} \psi_k^{(\beta)}.$$ 

**Example:** The hydrodynamic type system (51) has $N+1$ local symplectic structures $\hat{M}_{ik}^{(n)}$ related by the recursion operator (cf. [22])

$$\hat{R} = \begin{pmatrix} r^1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & r^2 & 0 \\ & & \ddots & \hat{r} \\ 0 & \cdots & \cdots & r^N \end{pmatrix} - \frac{1}{2} \begin{pmatrix} r^1_x & r^1_x & \cdots & r^1_x \\ r^2_x & r^2_x & \cdots & r^2_x \\ \vdots & \vdots & \ddots & \vdots \\ r^N_x & r^N_x & \cdots & r^N_x \end{pmatrix} D^{-1}.$$ 

Indeed, it is easy to verify that $\hat{M}_{ik}^{(2)} = \hat{M}_{im}^{(1)} \hat{R}_m, \hat{M}_{ik}^{(3)} = \hat{M}_{im}^{(2)} \hat{R}_m, \ldots, \hat{M}_{ik}^{(N+1)} = \hat{M}_{im}^{(N)} \hat{R}_m$ are local. However, $\hat{M}_{ik}^{(N+2)} = \hat{M}_{im}^{(N+1)} \hat{R}_m$ is no longer local. Then one can check that corresponding relationships between linear PDE systems (4) associated with higher symplectic structures $\hat{M}_{ik}^{(N+1+n)}$ are given by (63).

Some nonlocal Hamiltonian structures of the Ferapontov type can be inverted. However, corresponding symplectic structures are nonlocal too (see [37]). Thus, a list of all possible local Hamiltonian structures associated with semi-Hamiltonian hydrodynamic type systems is exhausted in this paper.

**Remark:** The paper [1] particularly is devoted to a connection of Temple’s sub-class with a linear-degenerate sub-class of hydrodynamic type systems. For instance, the linear-degenerate hydrodynamic type system (see [23])

$$r^i_t = \left( \sum m^m - r^i \right) r^i_x$$

is mirrored to Temple’s hydrodynamic type system

$$r^i_y = \left( \sum m^m + r^i \right) r^i_x,$$

whose commuting flow is well-known electrophoresis hydrodynamic type system (see [4])

$$r^i_r = \left( r^i \prod m^m \right)^{-1} r^i_x.$$ 

The first hydrodynamic type system has an infinite series of nonlocal Hamiltonian structures parameterized by $N$ arbitrary functions of a single variable (see [23]); the second
hydrodynamic type system (as well as the third one) has another infinite series of nonlocal Hamiltonian structures parameterized by $N$ arbitrary functions of a single variable (see [19]). Thus, the construction presented in the Section 5 can be extended on a nonlocal (Ferapontov) case [19].

**Remark:** The mixed case (cf. (39))

\[
\partial_i \beta_{jk} = \beta_{ji} \beta_{ik}, \quad i \neq j \neq k,
\]

\[
\partial_i \beta_{ik} + \partial_k \beta_{ki} + \sum_{m \neq i,k} \beta_{mi} \beta_{mk} = \varepsilon_{\alpha \beta} H^{(\alpha)}_i H^{(\beta)}_k, \quad \partial_i \beta_{ik} + \partial_k \beta_{ik} + \sum_{m \neq i,k} \beta_{im} \beta_{km} = \tilde{\varepsilon}_{\alpha \beta} \psi^{(\alpha)}_i \psi^{(\beta)}_k
\]

was considered in [53] for special simple choice $\varepsilon_{\alpha \beta} = \delta_{\alpha \beta}$ and $\tilde{\varepsilon}_{\alpha \beta} = 0$. Such a case is called the generalized Sinh-Gordon equation (may be better to use another notation: the generalized Bonnet equation, see a corresponding comment, for instance, in [59]) associated with the symmetric spaces $SO(2N,1)/SO(N) \times SO(N,1)$ (see details in [53], [54]). Thus, the above nonlinear system is associated with an isometric immersion of $N$-dimensional submanifolds into a pseudo-Euclidean space.

### 10 Non-hydrodynamic type systems

Any semi-Hamiltonian hydrodynamic type system (5) has an infinite set of commuting hydrodynamic type systems (6) parameterized by $N$ arbitrary functions of a single variable (see [56], [57]). However, higher commuting flows

\[
u^i_y = f^i_1(u, u_x, u_{xx}), \quad \nu^i_z = f^i_2(u, u_x, u_{xx}, u_{xxx}), \ldots \tag{64}
\]

can be constructed in some cases (see [26], [52], [55]). If some hydrodynamic type system has the local Lagrangian representation (31), then its higher commuting flows (64) are determined by similar Lagrangians

\[
S = \int \left[ \frac{1}{2} \sum \bar{H}^2_k(r) \frac{r^k}{r_x} - h(r, r_x, r_{xx}, r_{xxx}, \ldots) \right] dx dt.
\]

Corresponding higher order conservation law densities also were investigated in [56], [58]).

**Theorem 11** [56]: A semi-Hamiltonian hydrodynamic type system has the conservation law density

\[
h(r, r_x) = \sum \bar{H}^2_k \frac{r^k}{r_x}
\]

if and only if the extra condition

\[
\sum \bar{H}^2_k \partial_k v^k = 0
\]

is fulfilled.

Taking into account (5), the above constraint reduces to

\[
\sum (H_k \partial_k \bar{H}_k - \bar{H}_k \partial_k H_k) = 0
\]
**Corollary:** If rotation coefficients $\beta_{ik}$ are symmetric and depend on differences of Riemann invariants (see (53)) only (the Egorov flat case), then the above constraint reduces to
\[
\sum \bar{H}_k \bar{\psi}_k = \sum \bar{H}_k \bar{\psi}_k,
\]
where (see (33)) $\bar{\psi}_k = \delta \bar{H}_k$ and $\bar{\psi}_k = \delta \bar{H}_k$.

**Lemma:** The set of commuting hydrodynamic type systems (58) possesses an infinite series of higher conservation laws and higher commuting flows.

**Proof:** The Lame coefficients $H_{(k)i}$ of the Egorov flat hydrodynamic type systems (58) (written in Riemann invariants (45)) depend on differences of Riemann invariants. Thus, the above constraint is fulfilled.

This infinite series can be constructed iteratively (see (38)). The first commuting flow of the third order is given by the above Hamiltonian $H_1 = \int h(r, r_x) dx$
\[
r^i_t = \hat{A}^{ij} \delta H_1 / \delta r^j.
\]
All higher commuting flows and higher conservation law densities can be obtained due to the second Hamiltonian structure (see (38))
\[
\frac{\delta H_2}{\delta r^i} = \hat{M}_{ij} r^j, \quad \frac{\delta H_3}{\delta r^i} = \hat{M}_{ij} \hat{A}^{jk} \hat{M}_{kn} r^n_t, \quad \frac{\delta H_4}{\delta r^i} = \hat{M}_{ij} \hat{A}^{jk} \hat{M}_{kn} \hat{A}^{ms} \hat{M}_{sm} r^m_t, \ldots
\]
Thus, this result also generalizes the corresponding construction presented in [42].

**Remark:** All these higher commuting flows (see (64)) are integrable “dispersive” systems with a rational dependence on first derivatives. They are integrable, because they possess infinitely many Hamiltonian structures, conservation laws and commuting flows. Their solutions should be considered elsewhere.

## 11 Conclusion and outlook

In this paper we were able to find answers on a set of important questions.

- Lagrangian formulation for nonlocal Hamiltonian structures of hydrodynamic type systems associated with the anti-flatness condition is presented.

- Existence of two Hamiltonian structures associated with flatness and anti-flatness conditions implies an infinite set of local Hamiltonian structures of odd higher orders.

- Description of multi-Hamiltonian structures associated with anti-flatness conditions is equivalent to the description of local multi-Hamiltonian structures of the Dubrovin–Novikov type.

- Existence of one local Hamiltonian structure for the Egorov hydrodynamic type system implies the existence of infinitely many local Hamiltonian structures of odd higher degrees.
- Non-local Hamiltonian structures associated with the anti-flatness condition are extended on an arbitrary “co-dimension”.

The problem of description of local Hamiltonian structures of odd orders was formulated by S.P. Tsarev (see the end of Dr. of Science Thesis). Below we formulate a most general conjecture, which should be investigated elsewhere.

The theory of local differential-geometric Poisson brackets

$$\{u^i(x), u^j(x')\} = A^{ij}(u, u_x, u_{xx}, \ldots) \delta(x - x'), \quad (65)$$

where

$$A^{ij}(u, u_x, u_{xx}, \ldots) = g^{ij}(u) D_x^2 + b^{ij}_k(u) u^k_x D_x^{n-1} + (c^{ij}_k(u) u^k_{xx} + c^{ij}_{km} u^k_x u^m_x) D_x^{n-2} + \ldots \quad (66)$$

starts from the first order (see [14])

$$\{u^i(x), u^j(x')\} = (g^{ij}(u) D_x + b^{ij}_k(u) u^k_x) \delta(x - x')$$

connected with hydrodynamic type systems

$$u^i_t = v^i_k(u) u^k_x, \quad k = 1, 2, \ldots, N.$$ 

Later these Poisson brackets were generalized on nonlocal case (see [19], [37])

$$\{u^i(x), u^j(x')\} = (A^{ij}(u, u_x, u_{xx}, \ldots)) + \sum \varepsilon^{\alpha\beta} w^{ij}_\alpha u^\alpha_x D_x^{-1} w^{ij}_\beta \delta(x - x'), \quad (67)$$

where functions $u^i(x, t)$ simultaneously satisfy commuting flows

$$u^i_{t^\alpha} = w^{ij}_{\alpha} (u, u_x, u_{xx}, \ldots).$$

Moreover, in general case the existence of such nonlocal Hamiltonian structure means integrability of corresponding PDE system.

Just the Poisson brackets of the first order (see [19])

$$\{u^i(x), u^j(x')\} = (g^{ij}(u) D_x + b^{ij}_k(u) u^k_x + \sum \varepsilon^{\alpha\beta} w^{ij}_\alpha u^\alpha_x D_x^{-1} w^{ij}_{\beta m} u^m_x) \delta(x - x') \quad (68)$$

are connected with hydrodynamic type systems.

This paper particularly deals with the local Poisson brackets (65), where the operator $A^{ij}(u, u_x, u_{xx}, \ldots)$ is given by (38) (cf. (66)).

The main conjecture of this paper is following: a most general Poisson bracket associated with an integrable hydrodynamic type system is given by (67), where the local part (cf. (66)) $A^{ij}(u, u_x, u_{xx}, \ldots)$ is determined by similar quasi-rational functions as in (38); the nonlocal part is exactly as in (68).

Remark: Aforementioned differential-geometric Poisson brackets are invariant under an arbitrary point transformation $\tilde{u}^i = \tilde{u}^i(u)$, while the new Poisson brackets (38) are not invariant. Quasi-rational function means rational with respect to higher derivatives.
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