Inter-Brane Potential and the Decay of a non-BPS-D-brane to Closed Strings

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Abstract

We calculate the potential for $D_p - \overline{D}_p$ pair and show that the coincident $D_p - \overline{D}_p$ system has $(11 - p)$ tachyonic modes, with $(9 - p)$ of them due to radiative corrections. We propose that the decay width of an unstable non-BPS-$D_p$-brane to closed strings is given by the imaginary part of the one-loop contribution to the effective potential of the open string tachyon mode.

I. INTRODUCTION

The study of non-BPS brane systems are important in string theory. Some of the dynamics involved play crucial roles in the inflationary scenario in the brane world [1,2]. We shall start by calculating the $D_p - \overline{D}_p$ potential per unit world volume $V(y)$, which is complex. We first examine the potential $V(\theta, y)$ between two $D_p$-branes at an angle $\theta$ and separation $y$. (The advantage of studying $V(\theta, y)$ first is clear: the underlying physics is much easier to keep track, since the open string spectrum, in particular the tachyon mode, depends on both $\theta$ and $y$. For $\theta = \pi$, $V(\pi, y) = V(y)$ is the potential between the $D_p - \overline{D}_p$ pair.) The lightest open string mode has a mass that depends on $\theta$ and $y$. It is tachyonic for small $y$. The real part of the open string one-loop contribution simply yields the closed string exchange potential of the $D_p - \overline{D}_p$ pair separated at a distance $y$. The large $y$ behavior is simply Coulombic. However, at small $y$, this Coulombic behavior is truncated. Because the supersymmetry breaking is soft, a mass-level “supersymmetric” organization of the open string modes [1] yields a finite effective potential, as shown in Fig 1. For $V(y)$, the open string spectrum level crossing and the unbounded growth in the “soft” supersymmetry breaking for massive modes is qualitatively different from the small $\theta$ case [1]. We shall see that the scalar modes $y$ (there are $9 - p$ of them) are also tachyonic when the $D_p$-brane is on top of the $\overline{D}_p$-brane. Their tachyonic property is due to quantum effects (radiative correction), a feature first seen in Ref [1] for small $\theta$.

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FIG. 1. The potential $V(y)$ as a function of the separation $y$ for the $Dp - \overline{Dp}$-brane pair for $p=4$, where $\alpha' = 1$. The dashed curve is the imaginary part of $V(y)$. The thick line is the real part of $V(y)$. The Coulombic potential (the thin red curve) is shown for comparison.

In the open string one-loop channel, since only the tachyon mode contribution has an imaginary part at the one-loop level, the evaluation of $\text{Im} V(y)$ is completely field theoretic \cite{3}. A simple generalization from 4 spacetime dimensions to $(p + 1)$ dimensions yields the imaginary part of $V(y)$, as shown in Fig. 1.

\begin{equation}
\text{Im}V(y) = \frac{\pi}{\Gamma((p + 3)/2)} \left( \frac{m_{\text{tachyon}}^2}{4\pi} \right)^{(p+1)/2}
\end{equation}

where ($\alpha'$ is the open string Regge slope),

\begin{equation}
\alpha' m_{\text{tachyon}}^2 = \frac{y^2}{4\pi^2\alpha'} - \frac{1}{2}
\end{equation}

Its physical meaning has been discussed extensively \cite{3–5}.

In the dual channel, that is, the closed string channel, we should obtain exactly the same result. For large separation $y$, one obtains the well-known Coulombic potential \cite{6}, due to the NS-NS and RR exchanges, which correspond to the attractive gravitational and RR forces. As we go to short distances, the massive closed string modes that are Yukawa-suppressed start contributing to the potential $V(y)$. As $y \to 0$, we see that (naively) the potential diverges. This apparent divergence appears when the Hagedorn-like degeneracy overcomes the Yukawa suppression. In fact, this happens precisely when the lightest open string mode becomes tachyonic. A regularization (that is, an analytic continuation of the integral) renders the result finite, but with an imaginary part, as expected, precisely reproducing the result (1) obtained in the open string channel. This is clearly related to the decay of the $Dp - \overline{Dp}$-brane pair. When applied to a non-BPS $Dp$-brane, this same regularization approach yields a finite $\text{Im} < Dp | \Delta | Dp >$.

The decay of an unstable non-BPS D-brane to closed strings has been studied extensively \cite{7–9}. Adapting the above calculation to this case, we consider $< Dp | \Delta | Dp >$ where $\Delta$ is
the closed string propagator and the sum over the closed string spectrum is implied. This
is equal to $V(0)/2$, which is finite with an imaginary part. (The factor of $1/2$ is because
the open string spectrum of the non-BPS $Dp$-brane is half of those that stretched between
the $Dp - \overline{D}p$-brane pair at $y = 0$.) Using optical theorem (i.e., perturbative unitarity), as
shown in Fig. 2, we interpret this as the decay width $\Gamma$ of the non-BPS $Dp$-brane to on-shell
bosonic closed string modes:

$$\Gamma = V_p \, \text{Im} \, V(0) \quad (3)$$

where $V_p$ is the $Dp$-brane world volume. As expected, $\Gamma$ is 0th order in the string coupling.
Here, the finite imaginary part appears due to the Hagedorn-like degeneracy of the massive
closed string modes, and the analytic continuation moves the closed string modes from
off-shell to on-shell. The decay first goes to very massive, non-relativistic on-shell bosonic
closed string modes [7,8], with transverse momentum very small compared to the mass $m$
($k_\perp/m \sim 1/\sqrt{m}$). These non-relativistic massive modes then decay to relativistic light
closed string modes, both bosonic and fermionic.

![FIG. 2. The vertical dashed line on the left side indicates taking the imaginary part of the
$< Dp |$ closed strings $|Dp >$ amplitude in the decay of a non-BPS $Dp$ brane to closed string modes.
The right side is $|f(Dp \to \text{closed string})|^2$.](image)

Intuitively, this is quite reasonable, since the imaginary part of the effective potential
should be related to some decay width. The above decay width is obtained perturbatively
so it is expected to go to perturbative modes, i.e., closed and/or open string modes. In the
decay of a non-BPS $Dp$-brane, there is no brane left after the decay, so no open string modes
can be present. Since only closed string modes are present at the end of the decay, the decay
width $\Gamma$ should be a measure of the decay to closed string modes. (Lower dimensional branes
are solitonic, which are non-perturbative, so their production during the decay should not
be included in $\Gamma$.)

Although $\text{Im} \, V(y)$ is clearly related to the decay of the $Dp - \overline{D}p$ pair, it is harder to
interpret $-2V_p \text{Im} \, V(y)$ as its decay to closed strings. However, since closed string modes
are the only perturbative modes left after the annihilation of the $Dp - \overline{D}p$ pair, its value
may provide an estimate of the actual decay width to closed string modes. This result has
applications to brane inflation [2,1].
We shall first obtain the \( Dp - Dp \) potential in the open string channel by extrapolating the potential for branes-at-angle. Next we see how the same finite complex \( Dp - Dp \) potential emerges in the closed string channel. We then relate its imaginary part at zero separation to the decay of a non-BPS-\( Dp \)-brane to closed strings.

II. \( DP - DP \) POTENTIAL

Let us consider the potential \( V(y) \) per unit volume between a parallel \( Dp - Dp \) pair separated by a distance \( y \), where the \( Dp \)-branes are BPS with respect to each other. Let us write \( V(y) = 2\tau_p + V(y) \), where the \( \tau_p \) is \( Dp \)-brane tension. We shall consider \( p \leq 7 \). In the closed string channel, \( V(y) \) is given by [6,11]

\[
V(y) = -\int_0^\infty \frac{ds}{s} \left( \frac{16\pi^3\alpha'}{s} \right)^{(p+1)/2} e^{-\frac{y^2}{2\pi s}} \left( \frac{2\pi}{s} \right)^4 16 \prod_{m=1}^{\infty} \frac{(1 + w^m)^8}{(1 - w^m)^8}
\]

where \( w = e^{-\frac{y}{\pi \alpha'}} \). At large \( y \), the above integral is dominated by large \( s \) (which corresponds to long cylinder), so

\[
V(y) \approx -\frac{\kappa^2 \tau_p^2}{\pi^{(9-p)/2}} \Gamma((7 - p)/2) \frac{1}{y^{7-p}}
\]

where \( \kappa^2 = 8\pi G_{10} \). For \( p < 7 \), \( V(y) \) vanishes as \( y \to \infty \). \( V(y) \sim \ln(y) \) for \( p = 7 \). This is simply the attractive NS-NS (gravitational) plus RR interaction between the branes.

At short distances (small \( s \)), this expression naively diverges, although \( V(y) \) is expected to approach a finite value even for \( y = 0 \). Let us first calculate \( V(y) \) in the open string channel, which yields a finite \( V(y) \) for all \( y \). Then we come back and show how to obtain the same result in the closed string channel.

A. \( Dp - Dp \) Potential In the Open String Channel

Using the Poisson resummation formula, the above \( V(y) \) for the \( Dp - Dp \) system may be rewritten as the open string one-loop amplitude:

\[
V(y) = -\int_0^\infty \frac{dt}{t} (8\pi^2\alpha')^{-(p+1)/2} e^{-y^2/2\pi\alpha'} Z(\pi, t)
\]

\[
Z(\pi, t) = \frac{1}{q^{1/2}} \prod_{m=1}^{\infty} \frac{(1 - q^{m-1/2})^8}{(1 - q^m)^8}
\]

\[
= \frac{1}{q^{1/2}} - 8 + 36q^{1/2} - 128q + 402q^{3/2} - 1152q^2 + 3064q^{5/2} - 7680q^3
\]

\[
+ 18351q^{7/2} - 42112q^4 + 93300q^{9/2} - 200448q^5 + ...
\]

where \( q = e^{-2\pi t} \). This sum is over the open string modes. For large \( y \), this sum converges rapidly to give \( V(y) \). Notice that as \( y \) decreases, the mass of the lightest open string mode is given in (2), which becomes tachyonic for small values of \( y \). In the presence of the tachyon, the above sum apparently becomes ill-defined, that is, the integral diverges. The tachyonic mode contribution to \( V(y) \) is
\[ V(y) = - \int_0^\infty \frac{dt}{t} \left( 8\pi^2 \alpha' t \right)^{-(p+1)/2} \exp \left( -2\pi t \left( \frac{y^2}{4\pi^2 \alpha'} - rac{1}{2} \right) \right) \tag{7} \]

which is divergent. This integral can be regularized by analytic continuation [4].

To see how the divergence can be regularized using analytic continuation, consider a slightly different integral:

\[ \int_0^\infty \frac{dx}{x} x^{-\alpha} e^{(a+\i\epsilon)x} = (a \exp[-\i(\pi - \epsilon)])^\alpha \int_0^\infty \frac{dx}{x} x^{-\alpha} e^{-x} \]

This integral has a branch cut along the negative \( x \) axis. The \( \i\epsilon \) prescription tells us to integrate under the branch cut. The integral is now finite. The divergence has been removed in favor of an imaginary part which does not depend on the value of \( \delta \). The divergence is not an infinity of the theory, but rather an indication of the amplitude becoming complex.

The result of the analytic continuation is:

\[ \text{Im} \left( \int_0^\infty \frac{dx}{x} x^{-\alpha} e^{ax} \right) = \frac{\pi \Gamma((p+3)/2)}{\Gamma(1+\alpha)} \alpha^\alpha \tag{8} \]

for positive \( a \). Using (8), this gives, for \( y^2 < 2\pi^2 \alpha' \), the imaginary part of \( V(y) \):

\[ \text{Im} V(y) = \frac{\pi}{\Gamma((p+3)/2)} \left( \frac{|m_{\text{tachyon}}^2|}{4\pi} \right)^{(p+1)/2} \tag{9} \]

This \( \text{Im} V(y) \) is shown in Fig. 1. Since only the tachyon mode contributes to the imaginary part of \( V(y) \), we can also evaluate \( \text{Im} V(y) \) using the standard quantum field theory method. Its contribution can be calculated via the Coleman-Weinberg effective potential [3] where the one loop vacuum-to-vacuum amplitude for a point particle of mass \( m \) in \( (p+1) \) dimensions is given by:

\[ Z(m^2) = iV_{p+1} \int_0^\infty dl \int \frac{dp^{p+1}k}{(2\pi)^{p+1}} e^{-(k^2+m^2)l/2} = \frac{iV_{p+1}}{(2\pi)^{p+1}} \int_0^\infty dl e^{-m^2l/2} \tag{10} \]

Inserting the open string tachyon mass \( \alpha' m^2 < 0 \) we see that the above integral diverges. After a proper regularization using analytic continuation, we get the imaginary part of the energy density \( \text{Im}(E) \) as (9).

To calculate the real part of \( V(y) \) one has to include the whole tower of open string modes. The result depends on how the oscillating terms are grouped. The particular way of grouping the terms should be dictated by the soft supersymmetry breaking, as suggested by Garcia-Bellido, Rabadan and Zamora [1]. They applied it to the branes-at-small-angle case, where \( \theta = 0 \) corresponds to two parallel BPD \( Dp \)-branes while \( \theta = \pi \) corresponds to the \( Dp - \overline{Dp} \)-brane pair. For the \( Dp - \overline{Dp} \) system, supersymmetry breaking becomes large and level-crossings happen. We show that, despite these, the convergence remains intact.

When the two branes are parallel there is no potential between them because of the supersymmetry. Each mass level contains a set of the supermultiplets. The contribution to the potential \( V(y) \) from the open string bosons is exactly cancelled by the contribution from the open string fermions, mass level by mass level. We keep this mass level grouping. The way we identify the grouping as the angle \( \theta \) between the branes increases from zero to \( \pi \) is by following the spectral flow, that is, the splitting of open string modes at each level due to the soft supersymmetric breaking. This splitting leads to unequal contributions from the bosons and the fermions, resulting in a finite potential between the brane and the antibrane.
B. Branes-at-an-Angle Potential

Let us first review the branes-at-an-angle case. For simplicity, let us consider the case of two $D_4$-branes at an angle $\theta$.

The potential for two $D_4$-branes at an angle $\theta$ in the open string one-loop channel is given by,

$$V(y, \theta) = -V_4 \int_{0}^{\infty} \frac{dt}{t} (8\pi^2 \alpha'^t)^{-2} e^{-\frac{y^2}{\pi \alpha'}} Z(\theta, t),$$

$$Z(\theta, t) = \frac{\Theta_{11}^4\left(i\theta t/2\pi, it\right)}{i\Theta_{11}(i\theta t/\pi, it)\eta^9(it)}$$

(11)

For $\theta = 0$, the $D_p$-brane pair is supersymmetric and $Z(\theta = 0, t) = 0$.

![Figure 3: The spectral flow of the lowest two level open string modes as a function of the angle $\theta$ between the branes for $y = 0$. $\theta = 0$ corresponds to the 2 parallel BPS $D_p$-branes (left) while $\theta = \pi$ corresponds to the $D_p - \overline{D_p}$ case (right). The red (solid) lines show the NS states, the blue (dashed) lines show the R states. The number written above a solid/dashed line is the number of states represented by the line. The numbers in boldface to the right of the diagram show the states at the given mass levels for the $\theta = \pi$ case. + is for bosons and − is for fermions.](image)

The splitting of states is shown in the figure 3. To calculate the splitting one observes that the NS sector zero point energy has an angle dependence given by $\epsilon_{NS} = -1/2 + \theta/2\pi$. The R sector zero point energy remains unaffected by the angle between the branes. Considering $D_4$ branes in the light cone gauge, we take the branes to make an angle $\theta$ in the $7 - 8$ plane, then the creation operators in these two dimensions are given by $b_{7,8}^{r+\theta/\pi}$, where $r = 1/2, 3/2, \ldots$, and for the rest of the six dimensions the creation operators are $b_{-r}$, where
\[ i = 1, 2, ..., 6. \] Similarly the creation operators in the \( R \) sector are given by \( d_{7-n+\theta/2\pi}^{7,8} \) for the 7 and the 8 dimension and by \( d_{-n}^i \) for the other six dimensions, where \( n = 0, 1, 2, \ldots \). Note that since we have only six zero modes for the \( R \) sector for nonzero \( \theta \), therefore, the fermions will be in 4-dimensional representation of the Dirac algebra. The above consideration suggest the following grouping of the terms in the partition function:

\[
Z(t) = \frac{\Theta_{11}(it/2, it)}{e^{\pi t \eta(it)^{1/2}}} = \frac{(q^{1/2} - 1)^4}{q^{1/2}} \prod_{n=1}^{\infty} \frac{(1 - q^{m+1/2})}{(1 - q^{m-1/2})^4} (1 - q^m)^8
\]

(12)

The long distance behavior is determined by the \( t \to 0 \) limit of the partition function. In this limit, \( Z(\theta, t) \to 4t^3 \sin^2(\theta/2) \tan(\theta/2) \), and the potential becomes:

\[
V(y, \theta) = -\frac{\sin^2(\theta/2) \tan(\theta/2)}{8\pi^3 \alpha' y^2}
\]

(13)

The short distance behavior is determined by the \( t \to \infty \) limit. The open string spectrum splits into copies of the (broken) supermultiplet. The SUSY breaking is due to the “expectation value” \( \theta \), which is a soft SUSY breaking, so that, for any \( y \),

\[
\sum_i (-1)^F m_{2n} = 0, \quad n = 1, 2, 3
\]

(14)

where \( i \) runs over the spectrum in each “softly broken” supermultiplet (corresponding to a \( N = 4 \) supermultiplet). The lightest open string mode has mass

\[
\alpha' m_{\text{tachyon}}^2 = \frac{y^2}{4\pi^2 \alpha'} - \frac{\theta}{2\pi}
\]

(15)

In Fig 3, the open string modes are grouped into integer mass levels at \( \theta = 0 \), each of which contains a set of (broken) supermultiplets. Following the spectral flow and keeping this grouping for non-zero \( \theta \), \( Z(\theta, t) \) can be written as:

\[
Z(\theta, t) = \frac{(z - 1)^4}{z} (\sum_{n=0}^{\infty} z^{2n}) \prod_{m=1}^{\infty} \frac{(1 - q^m z)^4}{(1 - q^m z^{-1})} \frac{(1 - q^m z)^4}{(1 - q^m z^2)(1 - q^m z^{-2})}
\]

\[
= \frac{(z - 1)^4}{z} (\sum_{n=0}^{\infty} z^{2n}) [1 + \frac{(z - 1)^4}{z^2} q + \frac{(z - 1)^4}{z^4} q^2 + \ldots]
\]

(16)

where \( q = e^{-2\pi \alpha'} \) and \( z = e^{-\theta t} \). Here, for each mass level, the sum of the Landau levels in \( \sum z^{2n} \). After the integration over \( t \), we obtain

\[
V_{1\text{-loop}} = -\frac{1}{(8\pi \alpha')^2} \int_0^{\infty} \frac{dt}{t^3} \sum_i (-1)^F e^{-2\pi \alpha' m_i^2} \]

\[
= \frac{1}{32 \pi^2} \sum_i (-1)^F m_i^4 \ln(2\pi \alpha' m_i^2)
\]

(17)

Notice that \( V_{1\text{-loop}} \) is finite, since the quadratic divergence is absent for each multiplet.

Now let us go back to the \( Dp-D\bar{p} \) case. The above expression diverges in this limit. This divergence is well understood: it is due to the appearance of an additional dimension in
the overlapping world volume of the $Dp$-branes. To remove this divergence, we make the substitution:

$$\Theta_{11}(i\theta t/\pi, it) \to iL(8\pi^2\alpha' t)^{-1/2}e^{-\pi t}\eta(it)^{-3}$$

This substitution gives us the expected partition function for the brane-antibrane case:

$$Z(\theta, t) \to Z(\pi, t) = (q)^{1/2}\frac{\Theta_{11}(it/2, it)^4}{\eta(it)^{12}}$$

$$= \frac{1}{(q)^{1/2}} \prod_{m=1}^{\infty} \frac{(1 - q^{-m-1/2})^8}{(1 - q^m)^8}$$

This gives the $Dp$-$\overline{Dp}$ potential $V(y)$ in the open string channel (6).

To be specific, let us now focus on the $p = 4$ case. The long distance behavior is found by taking the $t \to 0$ limit of the partition function, with $Z(\pi, t) \to 16t^4$, we have

$$V(y, \pi) = -2.56 \times 10^{-4} \left(\frac{2\pi \alpha'}{y^2}\right)^{3/2}$$

To see the short distance behavior, retain the mutiplet grouping structure at $\theta = \pi$, we write the partition function as a function of $q$ and $z = \sqrt{q}$:

$$Z(\pi, t) = \frac{(z - 1)^4}{z} \prod_{m=1}^{\infty} \frac{(1 - q^m z)^4(1 - q^m z^{-1})^4}{(1 - q^m)^8}$$

$$= \frac{(z - 1)^4}{z} \left[1 - 4 \left(\frac{z - 1}{z}\right)q + 6 \left(\frac{z - 1}{z}\right)^2(1 - 4z + z^2)q^2 + ...\right]$$

We use this to evaluate the following integral which we shall need:

$$V_{1-\text{loop}} = \frac{-1}{(8\pi \alpha')^{5/2}} \int_{0}^{\infty} \frac{dt}{t^{7/2}} \sum_{i} (-1)^{F_i} e^{-2\pi \alpha' t m_i^2}$$

$$= \frac{1}{(8\pi \alpha')^{5/2}} \sum_{i} (-1)^{F_i} \frac{8\Gamma(1/2)}{15} (2\pi \alpha' m_i^2)^{5/2}$$

where we have used the soft SUSY breaking condition to evaluate this integral. Now we can write down the potential terms for different string levels (taking the relevant power of $q$ from the above expansion:

Order $q^0$ in $V(y, \pi)$:

$$\frac{8\Gamma(1/2)}{15(4\pi)^{5/2}} [(y^2 - \theta')^{5/2} - 4(y^2)^{5/2} + 6(y^2 + \theta')^{5/2} - 4(y^2 + 2\theta')^{5/2} + (y^2 + 3\theta')^{5/2}]$$

Order $q^1$ in $V(y, \pi)$:

$$\frac{8\Gamma(1/2)}{15(4\pi)^{5/2}} [-4(1 + y^2 - 2\theta')^{5/2} + 24(1 + y^2 - \theta')^{5/2} - 60(1 + y^2)^{5/2} + 80(1 + y^2 + \theta')^{5/2} - 60(1 + y^2 + 2\theta')^{5/2} + 24(1 + y^2 + 3\theta')^{5/2} - 4(1 + y^2 + 4\theta')^{5/2}]$$
Order $q^2$ in $V(y, \pi)$:

$$
6 \times 8 \frac{\Gamma(1/2)}{15(4\pi)^{5/2}} \left[ (2 + y^2 - 3\theta')^{5/2} - 10(2 + y^2 - 2\theta')^{5/2} + 40(2 + y^2 - \theta')^{5/2} \right]
$$

$$
-86(2 + y^2)^{5/2} + 110(2 + y^2 + \theta')^{5/2} - 86(2 + y^2 + 2\theta')^{5/2} + 40(2 + y^2 + 3\theta')^{5/2}
$$

$$
-10(2 + y^2 + 4\theta')^{5/2} + (2 + y^2 + 5\theta')^{5/2}
$$

(25)

and so on, where $y'$ stands for $y/(2\pi\alpha')$ and $\theta'$ stands for $\theta/(2\pi\alpha')$. This yields the potential in Fig. 1. The imaginary part comes only from the lightest open string mode when it becomes tachyonic. The real part converges quite rapidly, we need to keep only up to $q^3$ to get within 1% accuracy.

The behavior of the potential for short distances can be seen from Fig. 1. An interesting result first observed in Ref [1] is the emergence of new tachyonic modes in the $y$-directions. There is a dip in the potential close to the origin and the second derivative of the potential at the origin is negative. This yields a tachyonic mass for $y$ as a function of $\theta$.

$$
\alpha' m^2(y) = 3.36 \times 10^{-4} \theta/\pi \quad \theta \neq \pi
$$

$$
\alpha' m^2(y) = 4.51 \times 10^{-5} \quad \theta = \pi
$$

(26)

The appearance of the $y$ tachyons happens for generic $p$. In the open string classical limit, the brane separation $y$ are $(9 - p)$ moduli, so this tachyonic mass is a consequence of the one-loop open string contribution; that is, radiative corrections analogous to the Coleman-Weinberg mechanism. This feature first appears in the branes at small $\theta$ system [1]. We see that it persists for all values of $\theta > 0$. Since this $y$ tachyon mass is much smaller than the lightest open string tachyonic mode, we expect the latter to dominate the brane dynamics at short distances.

III. THE $Dp - \bar{Dp}$ POTENTIAL IN THE CLOSED STRING CHANNEL

The next step is to evaluate the same brane-antibrane potential $V(y)$ from the closed-string perspective. The $Dp$-brane is a solitonic object and it can emit/absorb closed string modes with arbitrary transverse momentum $k_\perp$.

$$
V(y) = -\sum_j \frac{1}{2\pi}(4\pi^2\alpha')^{1-p} \int_0^\infty ds \int \frac{d^{3-p}k_\perp}{(2\pi)^{3-p}} \exp \left( -s\alpha'(k^2_\perp + m_j^2)/4 + ik_\perp \cdot y \right)
$$

$$
= -\int_0^\infty \frac{ds}{s} \left( \frac{16\pi^3\alpha'}{s} \right)^{(p+1)/2} c^{-y^2/(s\alpha')} \left( \frac{2\pi}{s} \right)^4 Z(s)
$$

(27)

$$
Z(s) = 16 \prod_{m=1}^\infty \frac{(1 + w^m)^8}{(1 - w^m)^8} = \sum_0^\infty A(n) w^n
$$

(28)

where $n = \alpha'm^2/4$ and $w = e^{-s}$. This is (4), where

$$
A(n) = \frac{1}{2\pi i} \oint dw Z(w) \frac{1}{w^{n+1}} = \oint dz e^{nz} Z(s)
$$

(29)

(30)
We see that only bosonic closed string (NS-NS and RR) modes are included. Long distance behavior is governed by the light \((n = 0)\) closed string modes which dominate the s-channel contribution to the potential in the \(s \to \infty\) limit. The result is finite and goes as \(-y^{p-7}\).

Next, we consider the small \(y\) (small \(s\) and large \(n\)) behavior. Notice that \(A(n)\) grows monotonically with \(n\). So, unlike the open-string calculation, where the coefficients in the expansion for \(Z(s)\) oscillate in sign so that they can be grouped in a multiplet structure leading to a convergent sum, here the sum grows monotonically and will lead to a divergence if the degeneracy factor dominates the Yukawa suppressing factor \(\exp(-y^2/s \alpha')\). To get an idea of the behavior, we need the large \(n\) behavior of \(A(n)\). As \(s \to 0\) (see Appendix):

\[
Z(s) \approx \left(\frac{s}{2 \pi}\right)^4 \exp\left(\frac{2 \pi^2}{s}\right)
\]  
(31)  

and \(A(n)\) is obtained via the saddle-point approximation with the asymptotic form of \(Z(s)\) (31) in (30),

\[
A(n) \to (2n)^{-11/4} e^{(8 \pi^2 n)^{1/2}} \left(1 + O\left(\frac{1}{\sqrt{2n}}\right)\right)
\]  
(32)

Next, we perform the \(s\)-integration in \(V(y)\) to obtain, in the large \(n\) approximation:

\[
V(y) \sim -y^{p-7} \sum_n A(n) e^{-2 \sqrt{n \alpha'}} (1 + O(y/\sqrt{n \alpha'}) + ..)
\]  
(33)

\[
\sim -y^{p-7} \sum_n n^{-11/4} \exp\{4 \pi \sqrt{n} (1/\sqrt{2} - y/2 \pi \sqrt{\alpha'})\}
\]

the mass of the closed string mode being exchanged is \(2 \sqrt{n \alpha'}\) so that, for large values of \(y\), the Yukawa suppression factor \(\exp(-2 \sqrt{n y / \sqrt{\alpha'}})\) leads to a finite potential. However, for \(y \leq \sqrt{2 \alpha' \pi}\), the exponent blows up for large \(n\), so the potential \(V(y)\) apparently diverges. That is, the exponentially large (Hagedorn-like) number of massive closed string modes contributing to the potential overcomes the Yukawa suppression. Comparing with the open string tachyon mass formula (2), we see that the divergence appears exactly at the value of \(y\) where the lightest open string mode becomes tachyonic. To emphasize this point, let us go back to \(V(y)\) (4). In the large \(n\) approximation, we replace \(A(n)\) by its asymptotic value and replace the sum over \(n\) by an integration over a continuous \(u\) variable, where \(2n = u^2\) and \(s = 2\pi/t\):

\[
V(y) \approx - \int_0^\infty \frac{dt}{t} \sum_n (8 \pi^2 \alpha' t)^{(p+1)/2} (2n)^{-11/4} \exp\{2 \pi \sqrt{2n - t} \cdot \frac{y^2}{2 \pi \alpha'} - \frac{2 \pi n}{t}\}
\]  
(34)

\[
\approx - \int_0^\infty \frac{dt}{t} \int_0^\infty \frac{du}{u} (8 \pi^2 \alpha' t)^{(p+1)/2} t^{7/2} \exp\{-\pi (u-t)^2 / t - 2 \pi \alpha' m_i^2 t\}
\]

where \(\alpha' m_i^2 = y^2 / 4 \pi^2 \alpha' - 1/2\) is precisely the mass squared of the lightest open string mode given in (2). The integral over \(u\) is gaussian-like so it is convergent, while the integral over \(t\) depends on \(y\). For small \(y\), \(m_i^2 < 0\) is tachyonic and the integral over \(t\) diverges.

Now we are ready to get \(\text{Im} \ V(y)\) in the closed string channel. Since the open string channel calculation of the potential is finite everywhere, we expect the same for the closed
string calculation of $V(y)$. The appearance of this divergence at small $y$ signals the appearance of an imaginary part of the potential. We may use Eq(8) to obtain the imaginary part of $V(y)$ in (34) when $m^2 < 0$. This gives a crude value of $Im V(y)$. It turns out that the correction (i.e., subleading) terms in $A(n)$ are important in the evaluation of $Im V(y)$, and the convergence seems slow.

In fact, it is easier to evaluate $Im V(y)$ simply by going back to $Z(s)$. Putting the asymptotic ($s \to 0$) form of $Z(s)$ (31) into $V(y)$ (4) and using (8), we obtain precisely the $Im V(y)$ given in (9). As we shall see, corrections to the $s \to 0$ form of $Z(s)$ do not contribute to $Im V(y)$. While the imaginary part seen in the open string channel is due to the single tachyonic mode, the imaginary part in the closed string channel is due to the cumulative effect of the Hagedorn-like degeneracy of the massive closed string states. These two effects are dual to each other. This point will become even more explicit below.

We still have to see how the real part of the potential $Re V(y)$ emerges in the closed string channel. This is carried out by an exact correspondence between the open string calculation and the closed string calculation. To demonstrate this exact correspondence, we shall start from the closed string channel and show that taking into account all orders of correction to the asymptotic expression for $Z(s)$ leads back to the open string channel expression. Furthermore, the asymptotic expression for $Z(s)$ (31) corresponds precisely to the open string tachyon mode. This shall also answer the question as to what happens to the supersymmetric grouping of the terms present in the open string partition function (that we make use of through the soft SUSY breaking condition to find a finite sum in the open string channel) when we go to the closed string channel.

We follow the calculation of the asymptotic form for $Z(s)$ making use of the Hardy-Ramanujan formula — Appendix— but this time by keeping track of the correction terms that would arise in the $s \neq 0$ region. The result is:

$$Z(s) = 16e^{2\pi^2/s}(\frac{s}{4\pi})^4 e^\chi$$

where $\chi$ contains the correction terms,

$$\chi = -De^{-2\pi^2/s} + \frac{D}{2}e^{-4\pi^2/s} - \frac{D}{3}e^{-6\pi^2/s} + \frac{5D}{4}e^{-8\pi^2/s} + ..... \quad (35)$$

Expanding the $e^\chi$ term for $D = 8$, we get:

$$Z(s) = (\frac{s}{2\pi})^4(e^{\frac{2\pi^2}{s}} - 8 + 36e^{-\frac{2\pi^2}{s}} - 128e^{-\frac{4\pi^2}{s}} + 402e^{-\frac{6\pi^2}{s}} - ...) \quad (36)$$

Substituting this into the closed string channel $V(y)$ (4), and making a change of variable $s \to \frac{2\pi}{t}$, we recover the potential $V(y)$ in the open string channel (6):

$$V(y) = -\int_0^\infty \frac{dt}{t} (8\pi^2\alpha' t)^{-5/2} e^{\frac{2\pi^2}{s}} (e^{\pi t} - 8 + 36e^{-\pi t} - 128e^{-2\pi t} + 402e^{-3\pi t} - ...) \quad (37)$$

One clearly sees that the imaginary part of the potential using the asymptotic form of $Z(s)$ in the closed string channel is exactly equivalent to the tachyon contribution in the open string channel. The massless and massive level contributions to the potential in the open string channel are exactly equal to the contribution coming from the corrections to
this asymptotic form. They do not contribute to $Im\ V(y)$. One could go on and start calculating the real part of the potential in the closed string channel by taking the term by term contribution coming from the subleading terms to $Z(s)$, grouping them as suggested by the soft SUSY breaking in the open string channel.

IV. DECAY OF A NON-BPS BRANE TO CLOSED STRINGS

The decay rate per unit world volume of a non BPS $Dp$-brane to closed strings can be written as the square of the amplitude $|f(Dp \rightarrow \text{closed string})|^2$. Via the optical theorem, we expect this to be given by $Im\ <Dp|\Delta|Dp>$, as shown in Fig. 2. $\Delta$ is the closed string propagator given by $\int_0^\infty ds\ e^{-s(L_0+\tilde{L}_0)/2}$, $L_0$ and $\tilde{L}_0$ being the Virasoro generators, and $(L_0-\tilde{L}_0)|Dp>=0$. Let us consider an unstable non-BPS-$Dp$-brane in Type II string theory. For a non-BPS-$Dp$-brane, both ends of an open string end in the same brane, so $y=0$. For the same reason, the open string spectrum in a non-BPS-$Dp$-brane has only one real tachyon mode, instead of a complex tachyon field, and it has only half of the spectrum of open strings that stretch between the branes in the $Dp\bar{D}p$ system at $y=0$. So the analysis of the one-loop open string effective potential in this case is identical to that for $V(0)/2$, and we obtain $Im\ V(0)/2$ as given in (9) with $\alpha'm_i^2=-1/2$. Performing the $s$-integration in (27) at $y=0$, we obtain

$$<Dp|\Delta|Dp> = \frac{V(0)}{2} = -\sum_j \int \frac{d^{9-p}k_\perp \pi}{(2\pi)^{9-p} 8 k_j^2 + m_j^2 + i\epsilon} (4\pi^2\alpha')^{3-p}$$

(38)

where the sum is over closed string modes. Recall that

$$A(Dp \rightarrow Dp) \simeq \sum_j f_j(Dp \rightarrow X_j)f_j(X_j \rightarrow \bar{D}p) \rightarrow \sum_j i\delta(k^2 - m_j^2)|f_j|^2$$

(39)

where, ignoring momenta $k_\parallel$ parallel to the brane, $k^2 = k_0^2 - k_\perp^2$. We see that $|f_j|$ is a constant independent of the closed string mode $j$, and that the imaginary part comes only from on-shell closed string modes.

Before analytic continuation, $V(0)/2$ is real and infinite, with $k_0 \sim 0$. After analytic continuation, $V(0)/2$ becomes finite with an imaginary part. A comparison of (38), (27) and (39) suggests that the analytic continuation moves the propagators on-shell, that is, $Im\ V(0)/2$ is due to the sum over the on-shell poles. Since the divergence comes from large $n$, it further suggests that most of $Im\ V(0)/2$ comes from massive on-mass-shell closed string modes. For the massive closed string states, the poles get so dense for large $n$ that they are well represented by a branch cut. Our prescription for regularizing the divergence and isolating the imaginary part is equivalent to applying the $i\epsilon$ prescription of the analytic continuation to all asymptotically massive closed string modes and moving them on-shell. So $\Gamma = V_0 Im\ V(0)$ should be interpreted as the decay width of a non-BPS $Dp$-brane into on-mass-shell closed string modes.

We may see this a little more clearly by separating the regularization procedure into 2 steps: first go on-shell and then perform the analytic continuation. Ignoring $k_\parallel$, we may restore the $k_0$ integral and rewrite (38) as
V(0) \sim \int \infty \, dn \int dk^9 - p \, dk_0 \delta(k^2 - 4n/\alpha') \theta(k_0) A(n) \sqrt{k_1^2 + 4n/\alpha'} \\
\rightarrow \int dk^9 - p \int_0^\infty \, dk_0 \frac{A(k^2)}{k_0} \tag{40}

where the sum over j is replaced by an integration over \( n = \alpha'n^2/4 \), with measure A(n). Carrying out the \( n \)-integration with the delta function keeps only the on-mass-shell states. Using the asymptotic form of A(n) (32), we find that A(k^2) \sim k^{-11/2} \exp(\sqrt{2\pi^2\alpha' k^2}). This integral is divergent. After the analytic continuation of this expression, the integral becomes finite and obtains an imaginary part. Presumably, the inclusion of subleading terms of A(n) will reproduce the correct Im V(0).

In Fig. 2, only bosonic closed string modes are involved. They are the NS-NS and RR modes. In (27) (with \( y = 0 \)), we see that \( k_\perp \) has a Gaussian distribution, with \( \langle k_\perp^2 \rangle = 2(9 - p)/\alpha' \). From (31) and (30), we have \( n \simeq 2\pi^2/s^2 \). Putting them together, we see that, for massive states,

\[
\frac{\langle k_\perp^2 \rangle}{m^2} \simeq \frac{9 - p}{2\pi \sqrt{2n}} \tag{41}
\]

so the momenta of the massive closed string modes transverse to the brane are negligible. Conservation of momenta tangential to the brane implies that they are negligible too. Since it is the large \( n \) behavior that brings in the imaginary part, the decay is to very massive non-relativistic closed string modes, as pointed out in Ref [7,8]. These massive closed string modes will then decay to light (both bosonic and fermionic) closed string modes, which are expected to be relativistic.

Apriori, the tachyon couples to other open string modes and the decay of a non BPS \( Dp \)-brane corresponds to the rolling of the tachyon [10], so one naively expects some energy to go to open string modes. However, no open string mode exists after the complete decay (and the disappearance) of the brane. This issue is resolved in Ref [12,13]: one expects the ends of an open string mode to have a flux tube (U(1)) between them, so that the flux tube together with the open string form a closed string. As a result, only closed string modes are produced.

The estimate of \( \Gamma \) is for the tachyon at or close to the top of the potential (at \( \sqrt{2\tau_p} \)). The time scale \( t_T \) of the tachyon rolling is around \( \sqrt{\alpha'} \). In the compactified case where the world volume \( V_p >> \alpha'^{p/2} \), \( t_T \) is comparable to or larger than the inverse of \( \Gamma \). In this case, the above estimate of \( \Gamma \) should be valid. For \( t_T \) much smaller than \( 1/\Gamma \), we may expect the decay to start with the tachyon rolling, which goes to tachyon matter [14], which then decays to relativistic closed strings. In this case, the above estimate may not be applicable.

\( Dp \)-branes in the bosonic string theory may also be considered:

\[
\langle Dp \rangle \Delta |Dp> = \frac{1}{4\pi(8\pi^2\alpha')} \int_0^\infty \, ds \, e^{s} \prod_{m=1}^{\infty} (1 - e^{-ms})^{-24} \tag{42}

\rightarrow \frac{1}{4\pi(8\pi^2\alpha')} \int_0^\infty \, ds \, e^{s} \, Z_B(s)
\]

For \( s \to 0 \), \( Z_B(s) \simeq (s/2\pi)^{12} \exp(4\pi^2/s) \) so the degeneracy \( D(n) \simeq 2^{-1/2}n^{-27/4} \exp(4\pi^2n) \). This yields for \( s = 2\pi/t \) and \( n = v^2 \) (ignoring powers of \( t \) and \( v \)),

##
\[ \langle Dp | \Delta | Dp \rangle \sim \int_0^\infty dt dv \exp(-2\pi(v-t)^2/t + 2\pi t) \]

Completely analogous to (34), the last term in the exponent is identified as \(-2\pi\alpha' m_t^2 t = 2\pi t\) with the tachyon mass \(\alpha' m_t^2 = -1\). This is the origin of the apparent divergence in the integration over \(t\). Using the \(s \to 0\) form of \(Z_B(s)\) in \(\langle Dp | \Delta | Dp \rangle\), we obtain, after analytic continuation, the expression (1) for \(2 \text{Im} \langle Dp | \Delta | Dp \rangle\), with \(p = 25\) and \(m_{\text{tachyon}}^2 = -1/\alpha'\). Following the earlier analysis, we see that subleading terms of \(Z_B(s)\) do not contribute to \(\text{Im} \langle Dp | \Delta | Dp \rangle\).

It is suggestive to take \(2V_p \text{Im} V(y)\) (9) to be the decay width of the \(Dp\overline{Dp}\)-brane pair to closed strings, since perturbatively, only closed strings are available after their annihilation. It will be important to understand this issue better. For branes at an angle \(\theta\), the branes recombine as \(y \to 0\). Perturbatively, we expect the decay to release energy to both open and closed strings.

It is interesting to compare this result to that in quantum field theory. In quantum field theory, the propagator of a single field at the tree level has only \(\delta\) function as its imaginary part. Here the imaginary part appears classically due to the Hagedorn-like degeneracy of the closed string spectrum. However, this is a perturbative quantum effect in the open string channel.

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V. APPENDIX : HARDY-RAMANUJAN FORMULA FOR THE CLOSED STRING LEVEL DEGENERACY

Let \(Z(s) = 16\hat{Z}(s)\) be given by:

\[ \hat{Z}(s) = \prod_{m=1}^{\infty} \frac{(1 + w^m)^D}{(1 - w^m)^D} = \sum_0^{\infty} \hat{A}(n)w^n \] (43)

where \(w = e^{-s}\). Using complex variables, we can write: \(A(n) = 16\hat{A}(n)\), where

\[ \hat{A}(n) = \frac{1}{2\pi i} \oint dw \frac{\hat{Z}(w)}{w^{n+1}} \] (44)

where the contour is a circle around the origin. To find the asymptotic form of \(\hat{A}(n)\), we write:

\[ \ln \hat{Z}(s) = DF(2s) - 2DF(s) \] (45)

where \(F(s)\) can be expressed in the Mellin representation:

\[ F(s) = \sum_{n=1}^{\infty} \ln(1 - e^{-ns}) = -\frac{1}{2\pi i} \int_{Re(z)=c} dz \Gamma(z)\zeta(1+z)\zeta(z)s^{-z}, \] (46)
where \( c > 1 \), and \( \zeta(z) \) is the Riemann zeta-function. The integrand has a first-order pole at \( z = 1 \) and a second-order pole at \( z = 0 \). Shifting the line of integration from \( \Re(z) = c > 1 \) to \( \Re(z) = c' \), \(-1 < c' < 0\), we arrive at:

\[
F(s) = -\frac{\pi^2}{6s} - \frac{1}{2} \ln\left(\frac{s}{2\pi}\right) - \frac{1}{2} \frac{s}{2\pi i} \int_{\Re(z)=c'} dz \Gamma(z) \zeta(1+z) \zeta(z) s^{-z}.
\]  
(47)

Letting \( z \to -z \) and using the identities:

\[
\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z) \zeta(1-z),
\]

\[
\Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin(\pi z)},
\]  
(48)

one obtains:

\[
F(s) = -\frac{\pi^2}{6s} - \frac{1}{2} \ln\left(\frac{s}{2\pi}\right) + \frac{s}{24} + F\left(\frac{4\pi^2}{s}\right)
\]  
(50)

In the limit \( s \to 0, F\left(\frac{4\pi^2}{s}\right) = 0 \), and one obtains the asymptotic behavior of \( \hat{Z}(s) \) as:

\[
\hat{Z}(s) \simeq \left(\frac{s}{4\pi}\right)^{D/4} e^{D\pi^2 s}/4s
\]  
(51)

Using this result and carrying out a saddle point evaluation of the contour integral, we get:

\[
\hat{A}(n) \simeq \left(\frac{D}{64}\right)^{(D+1)/4} n^{-(D+3)/4} e^{\sqrt{D\pi^2 n}} \left(1 - \frac{10}{\pi \sqrt{n}} + ...\right)
\]  
(52)

For general \( s \),

\[
\ln \hat{Z}(s) = \frac{D\pi^2}{4s} + \frac{D}{2} \ln\left(\frac{s}{4\pi}\right) - De^{-2\pi^2/s} + \frac{D}{2} e^{-4\pi^2/s} - \frac{D}{3} e^{-6\pi^2/s} + \frac{5D}{4} e^{-8\pi^2/s} + ....
\]  
(53)

This leads to the following asymptotic expansion for \( \hat{Z}(s) \):

\[
\hat{Z}(s) = \left(\frac{s}{4\pi}\right)^{D/4} e^{D\pi^2 s}[1 - De^{-2\pi^2/s} + \left(\frac{D^2}{2} + \frac{D}{2}\right) e^{-4\pi^2/s} - \left(\frac{D^3}{6} + \frac{D^2}{2} + \frac{4D}{3}\right) e^{-6\pi^2/s}
\]  
(54)

\[+ \left(\frac{D^4}{24} + \frac{D^3}{4} + \frac{35D^2}{24} + \frac{5D}{4}\right) e^{-8\pi^2/s} - ....].\]

A similar (and simpler) analysis can be carried out for the bosonic string.
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