ABSTRACT

Long lived modes of elliptical galaxies can exist à la van Kampen. Specific systems may possess long lived oscillations which Landau damp on time scales longer than a Hubble time. Some physical processes such as a close encounter, tidal forces from a cluster or an orbiting satellite could preferentially excite a coherent mode. These may relate to the observed faint structure in elliptical galaxies such as shells and ripples. Their detection in projected phase space would ultimately provide a detailed probe of the underlying potential.

I give an overview of linear perturbations to stationary solutions of the Vlasov equation, including a discretized Hermite polynomial expansion which explicitly demonstrates completeness and orthogonality of solutions. Some exact solutions are shown, which implies the feasibility of such a procedure and suggest future fully numerical studies.

Subject headings: galaxies: internal motions, galaxies: structure, stars: stellar dynamics

1. Introduction

Elliptical galaxies provide a clean laboratory to study the physics of self-gravitating fluids. We would like to study the possibility of applying normal mode analysis to complement the observational data. The conventional wisdom is that most excitations Landau damp or phase mix at rates comparable to the dynamical time, making it impossible to observe the equivalent of p-modes in stars. Exceptions to this rule have been studied by previous authors. Mathur (1990) has shown that not all modes decay by proving the existence of discrete eigenvalues that allow stable oscillations. Weinberg (1991) numerically studied these modes in the vertical structure of the disk. A different approach was shown in (Weinberg 1993) where he numerically studied modes which decay on time scales long compared to the dynamical time. In this article I wish to extend this search by examining qualitative relationships between various modes in linear theory.
I follow the van Kampen approach to construct these modes. Since Case (1959) has shown van Kampen’s method to be equivalent to Landau’s treatment, we can use the qualitative features of van Kampen modes to infer properties of Landau damping. In the Jeans analysis, for example, the normal van Kampen modes are singular (Binney and Tremaine 1987, appendix 5A). This not only makes them impossible to excite, but also violates the Landau damping assumptions, where the distribution must be differentiable for $v$. Conversely, I will suggest how non-singular modes might be constructed, and show explicit examples in some one dimensional toy systems. A differentiable van Kampen mode implies the existence of a mode which does not Landau damp. If the system is furthermore proven to be stable, we know that the Landau integration contour must contain a pole on the real axis, without actually having to find it.

In this paper I first give a general treatment of linear perturbations to review the notation and physical interpretation. Next I will assume spherical symmetry to study the modes found in simple stationary systems. While they are too simple to be applicable to real astrophysical systems, they demonstrate how non-singular van Kampen modes might exist. Following that, I will construct a framework based on matrix expansions, which could be used to study modes numerically. This method is applied to the example of the Jeans’ instability. We then construct an explicitly self-adjoint complete representation of the linear Vlasov operator in the presence of discontinuous functions. By showing how an unbounded operator which is not explicitly self-adjoint can possess a complete set of orthogonal eigenfunctions we find a resolution to Case’s puzzle. The argument also lends strength to the feasibility of a general mode decomposition. The paper concludes with some speculation on possible astrophysical implications and suggestions for future work.

2. Formulation

Qualitatively, one proceeds as follows: we consider a stable stationary system (such as an elliptical galaxy), and look for periodic perturbations. This is accomplished by first constructing a small perturbation consisting of a smooth periodic pattern in the background potential, i.e. a periodic Hamiltonian flow. To linear order, this pattern does not interact with itself. It does, however, effect the background fluid. Having assumed the system stable, we know that no resonant growth can result. We need only study the response of the background, and we are done. If we are unfortunate, the background response may cancel the perturbation exactly, and such an example is given in the next section. Or the background response to a periodic perturbation may be chaotic, for example if the potential is not integrable. In general, however, we have some non-trivial background response, as constructed explicitly in section [1] for the case of the Jean’s instability.
To phrase this mathematically, we are interested in the solution to Vlasov’s equation

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left[ v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right] = 0 \quad (1)$$

$$\nabla^2 \Phi = 4 \pi G \rho, \quad \rho(\vec{x}) = \int f d^3 v, \quad \vec{x} = (x_1, x_2, x_3), \quad f = f(\vec{x}, \vec{v}; t). \quad (2)$$

The problem under consideration consists of a stationary background solution $f_b(\vec{x}, \vec{v})$ and a periodic perturbation $f_p(\vec{x}, \vec{v})$ such that $f = f_b + \epsilon e^{i\omega t} f_p$, where $\epsilon$ is a small number. The sufficient condition for the perturbation equation to be valid is that

$$\frac{\partial \Phi_p}{\partial x_i} \ll \frac{\partial \Phi_b}{\partial x_i} \quad (3)$$

at all points $\vec{x}$ for $i = 1, 2, 3$, which is weaker than requiring $\epsilon f_p < f_b$ at all points in phase space. The latter condition only becomes necessary when we consider negative perturbations, but positive perturbations are allowed even when singular in velocity space.

To first order in $\epsilon$ the perturbation satisfies the linear equation

$$\mathbf{L} f_p = \omega f_p, \quad \mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2,$$

$$\mathbf{L}_1 = i \sum_{i=1}^{3} \left[ v_i \frac{\partial}{\partial x_i} - \frac{\partial \Phi_b}{\partial x_i} \frac{\partial}{\partial v_i} \right],$$

$$\mathbf{L}_2 = -4i \pi G \sum_{i=1}^{3} \frac{\partial f_b}{\partial v_i} \frac{\partial}{\partial x_i} \nabla^{-2} \int d^3 v. \quad (4)$$

This decomposition allows a simple physical interpretation. Solutions to $\mathbf{L}_1$ describe the motion of the perturbation, and $\mathbf{L}_2$ describes the response of the background system. Mathur (1990) showed that for any $\omega$ which is not an eigenvalue of $\mathbf{L}_1$, $\mathbf{L}$ can inherit a discrete eigenvalue from $\mathbf{L}_2$. In this paper, however, we will consider the eigenvalues of $\mathbf{L}_1$. Any eigenvalue of $\mathbf{L}_1$ corresponds to an eigenvalue of $\mathbf{L}$ since the velocity space structure of $\mathbf{L}_2 f_p$ depends solely on the background distribution. To see this, let $\mathbf{L}_1 f_\alpha = \omega f_\alpha$. We can construct the response $f_\beta$ such that the total perturbation $f_p = f_\alpha + f_\beta$. It satisfies the inhomogeneous equation

$$(\mathbf{L} - \omega) f_\beta = \sum_{i=1}^{3} \frac{\partial f_b}{\partial v_i} \frac{\partial \Phi_\alpha}{\partial x_i} \quad (5)$$

Equation (5) may be singular. We thus can not simply invert $\mathbf{L} - \omega$. Instead one must establish that at least one solution $f_\beta$ exists. This can be seen by going back to the linearized Vlasov equation (4), where we use the full time dependence to solve a linear hyperbolic problem in a periodically driven field $e^{i\omega t} \Phi_\alpha$:

$$\dot{f}_\gamma - i \mathbf{L} f_\gamma = e^{i\omega t} \sum_{i=1}^{3} \frac{\partial f_b}{\partial v_i} \frac{\partial \Phi_\alpha}{\partial x_i} \quad (6)$$
Start with the initial value \( f_\gamma(t = 0) = 0 \), and we have a unique solution to the initial value problem. Wait for a time \( T \), and Fourier transform the result, yielding \( f_\gamma(T, \omega t) \). Repeat this for larger values of \( T \), taking the limit as \( T \to \infty \). Since we are working with a stable system, there can be no growing modes with negative imaginary eigenvalues. Any damped modes with negative eigenvalues we simply discard. We take only the resonant frequency piece, and transform back to get \( f_\delta = f_\gamma(T = \infty, \omega t = \omega)e^{i\omega t} \). This will satisfy (3). To see this, take any interval \([t_0, t_1 = t_0 + \omega] \) with \( t_0 > 0 \), and apply a discrete Fourier transform on that interval to equation (3). The RHS contains only one frequency component \( \omega \). Therefore the LHS operator \( L_d \) applied to the solution \( f_\delta = L_df_\gamma \) will contain only that same frequency component. Since \( L_d \) does not depend on \( t \), it can only map functions to zero, but not generate new frequencies. Thus \( f_\delta \) solves (3), and \( f_\beta = f_\delta e^{-i\omega t} \).

To numerically approximate (3), one can discretize the operators by expanding all functions in a given basis (Pen and Jiang 1992, Pen 1992). If we expand \( f_\beta \) as a discrete sum along the lines of section 4., we obtain a sequence of approximations \( f_\beta^i \), which will converge to \( f_\beta \). For each order in the expansion, the differential and integral operators are finite and discrete, so we can solve (3). We use the Gram-Schmidt orthogonalization to invert \( L - \omega \), and require orthogonality to \( f_\alpha \) when we encounter the matrix singularity. Specific examples of exact and series solutions will follow below.

We now only need to consider \( L_1 \). Its solutions describe the motion of an ensemble of particles in a static field. A single periodic orbit corresponds to a localized distribution (\( \delta \)-function) in phase space, which is a periodic solution, and thus a discrete sum of eigenstates. In the case of integrable systems, one can describe all eigenstates in terms of the action-angle coordinates. In general it is possible to have periodic patterns, whether or not individual orbits are periodic. The pattern need not have the same period as its constituent orbits. To illustrate this point, consider a single orbit in an integrable system with period \( T \). A single particle corresponds to a distribution function with pattern period \( T \), but \( n \) particles equispaced along the angle variable with have period \( T/n \). If we fill the orbit with a continuum of particles, the pattern will be stationary. Now consider a second orbit with a different period. If the ratio of the two periods is a rational number, we can construct a pattern with period equal to any linear sum of the two constituent orbit periods. Given a set with a continuum parameter of orbital periods, we can, by judicious choice of orbits and phase angles, construct a pattern with any period we wish. In astrophysical situations, such patterns might be observable. We will now invert the procedure and try to construct patterns directly from the Eulerian description of the distribution function.

To linear order in phase space, the Liouville theorem assures that these solutions do not diffuse or damp since dynamical friction is a higher order non-linear phenomenon. From (3) it follows that a constant mass density perturbation will have more linear behavior if it is smeared out in space. One would thus expect that long wavelength perturbations are dynamically longer lived, which will make their detection easier observations with limited spatial resolution.
3. Spherical Potentials

In general, the eigenmodes of a system must be computed numerically. For a few systems, however, it is possible to find exact or series expressions of some modes. I will show some examples to illustrate the effects in simple models. They are radial toy models with non-singular van-Kampen modes, which implies the existence of initial conditions which do not Landau damp. While they do not have a clear correspondence to three dimensional real galaxies, they are easier to calculate and may provide some hints about the behaviour of realistic systems.

Consider simple power law models, where the background distribution has a power law dependence on the radial coordinate $\rho \propto r^n$. The simplest is $n = 0$ which is realized in the Einstein sphere, see for example (Mikahilovskii 1972). The Einstein sphere is a stable distribution describing a constant density sphere where all particles are on tangential (non-radial) orbits, and the distribution function is isotropic in the tangent plane. All perturbation orbits which do not leave the sphere are periodic, and thus all modes have the same eigenvalue. This exemplifies the existence of an isolated eigenvalue in $L_1$.

For $n = -1$, the potential gradient $\nabla \Phi_b = F$ is constant. Now consider purely radial perturbation orbits. The linear Vlasov equation (4) becomes

$$-iL_1 f_p = v_r \frac{\partial f_p}{\partial r} - F \frac{\partial f_p}{\partial v_r} = i\omega f_p.$$  

Laplace transforming $f_p$ with respect to $r$, such that $f_p = \int f_k e^{-kr} dk$ allows us to express the radial eigenmode as

$$f_k = \exp(-\frac{k^2 v^2}{2F} + i\frac{\omega v}{F} - k|r|).$$

This solution is certainly smooth everywhere except at $r = 0$. We thus need to modify the background potential to allow a smooth transition. The $1/r$ density has rapidly divergent mass, so one must limit the power law at some radius $R$, which provides a characteristic scale for $k$. By using positive and negative perturbations of the lower harmonics, one can easily construct non-trivial waves and modes for standing radial density waves.

The zero eigenvalues of isothermal spheres (or any other power law system) can also be solved. This scenario is attractive since many astrophysical objects have such a dependence. Let $\sigma_b$ be the velocity dispersion of the background material. Consider an isothermal Maxwellian perturbation with some other velocity dispersion $\sigma_p$, as might describe an elliptical galaxy embedded in some halo. From $L_1 f_a = 0$ we obtain the well known result $\rho_a = r^n, \ n = -2\sigma_b/\sigma_p$. The background (halo) response $\rho_b$ to the imbedding is

$$\rho_b = -\frac{2}{n^2 + 5n + 8}\rho_a,$$

and the net density fluctuation is $\rho_p = \rho_a + \rho_b$. In the case that $\sigma_b = \sigma_p$, one would obtain $n = -2$. The background response is equal and opposite to the perturbation and exactly cancels it. In this
case, no net perturbation actually happened, except for relabeling $b$ particles to belong to $p$. But if we solve the perturbation equation exactly, $\partial_r r^4 L f_p = 0$ becomes

$$\frac{\partial}{\partial r} (r^4 \frac{\partial \rho_p}{\partial r}) + 2 r^3 \frac{\partial \rho_p}{\partial r} + 2 r^2 \rho_p = 0$$

which implies $n = (\frac{-5 \pm \sqrt{-7}}{2})$, and we have a nontrivial solution. This eigenmode can be obtained from (3) in the limit as $\rho_r \rightarrow \infty$.

4. Jeans Instability

The Jeans instability is well understood with known analytic van Kampen modes. It is thus a good testbed for general analysis. The spatial translation invariance allows an exact series solution in terms of an infinite sequence of discrete matrix operators on successively refined basis functions.

Consider a homogeneous isotropic Maxwellian background fluid whose distribution function is given as $f_b = \rho_b \exp(-v^2/2\sigma^2)/\sqrt{2\pi}$. Then apply the Jeans swindle and set the potential $\Phi$ of this component to zero. Let the Fourier mode $f_p$ be periodic in space with wave number $k$ in one dimension (say $x$) and constant along the other two ($y, z$) so the perturbation equation reads

$$\dot{f}_p + ik v f_p - \frac{4i\pi G \rho_b v}{\sqrt{2\pi \sigma^3 k}} e^{-v^2/s^2} \int dv f_p = 0$$

where $s \equiv \sqrt{2}\sigma$. Now $L = kv - (k_J^2/\sqrt{2\pi \sigma k})v \exp(-v^2/s^2) \int dv$ with $k_J^2 \equiv 4\pi G \rho_b / \sigma^2$ being the Jeans wavenumber. Equation (11) can be solved as

$$\dot{f}_p = iL f_p,$$

$$f_p(t) = e^{itL} f_p(0).$$

All we need to do is find the eigenvalues of $L$ and project the initial condition $f(0)$ onto the eigenstates to obtain the complete solution.

The simplest case is the free equation, where $G = 0$ so we have no gravity. Then the eigenmodes of $L$ are $f_{v_0} = \delta(v - v_0)$, Dirac $\delta$-functions which move with frequency $\omega = kv_0$. The next simplest case are stationary solutions, i.e. distributions corresponding to zero eigenvalues, $Lf_p = 0$. Let $f_p = \exp(ikx)(a\delta(v) + bm(v))$ where $m(v) = \exp(-v^2/s^2)/s\sqrt{\pi}$ is a Maxwellian distribution similar to $f_b$. Equation (11) becomes

$$\left(\frac{k}{k_J}\right)^2 - 1 = \frac{a}{b}$$

and we have solved the static perturbation equation as being the sum of the Gaussian and a $\delta$ function. The relevant limits are:
\( k \to \infty \): for very short wavelength perturbations, \( b \to 0 \) and our solutions are \( \delta \)-functions, which solve the free equations as expected.

\( k \to k_J \): at the Jeans’ length, the solution is a plain Gaussian without any delta function, which is a static perturbation.

\( k \to \sqrt{2}k_J \): this is the equality point, where half the density is in a \( \delta \)-function and half is in the Gaussian, so \( a = b \).

\( k \to 0 \): it certainly seems curious that we have static solutions with wavelengths much longer than the Jeans’ length. Note that \( a \to -b \), which means that \( \rho_p \to 0 \) and the net density fluctuation tends to zero. All the structure is in velocity space, implying that we can always have coherent stable perturbations, even though they contain less and less mass.

We can now consider the complete spectral decomposition of \( L \). From the static and free solutions we are led to the ansatz \( f_p = \exp(ikx)(\mu\delta(v - v_0) + f_h) \) and expand in Hermite polynomials

\[
\begin{equation}
  f_h = \sum_{i=0}^{\infty} c_i N_i H_i(\nu) e^{-\nu^2}
\end{equation}
\]

where \( \nu \equiv v/s \), and we get \( sc_0 = \rho_h \). The \( H_i = 1, x, x^2 - 1, \ldots \) are defined using the conventions of Gradshteyn and Ryzhik (1980), and the normalization \( N_i = (2^n n! \sqrt{\pi})^{-1/2} \). The completeness of this expansion for \( L^1 \) Lebesque integrable functions is shown in (Keener). Note that projection (14) is onto a skew basis, where the inverse projection occurs through plain Hermite polynomials \( c_i = N_i \int dv H_i f_h \) without an exponential weighting.

First, set \( \mu = 0 \). Substituting (14) into (11), multiplying by \( N_i H_i \) and integrating over \( dv \), we have turned the continuum equation (11) into a discrete matrix equation, so writing the \( c_i \) as a column vector \( \vec{c} \), we obtain \( \dot{\vec{c}} = iL\vec{c} \) where \( L \) now becomes

\[
L = 2\pi^{1/4} \sigma k \left( V - \frac{k_J^2}{k^2} T \right),
\]

\[
T = \begin{pmatrix}
  0 & 0 & 0 & 0 & 0 \\
  1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
\]
The eigenvalues are easily read off from $L$. If $k > k_J$, we can symmetrize entries $L_{12}$ and $L_{21}$ to their geometric mean (Wilkinson 1965), yielding all real eigenvalues, implying that all eigenmodes are stable. The apparent dispersion becomes clear: a density perturbation $\delta c_0$ projects onto all eigenstates, which propagate at different velocities. Only the norm of $\vec{c}$ is conserved, and $c_0$ clearly decays, explaining the dispersion in density space. For $k = k_J$, the matrix becomes explicitly singular, with the Gaussian $c_0$ being a static eigenmode, as analyzed above. For $k < k_J$, we obtain two (and only two) imaginary eigenvalues, $\omega_1, 2 \approx \pm 2i \pi^{1/4} k \sigma \sqrt{k^2 / k_J^2 - 1}$. This result holds for small $k$ and is obtained as leading order term from the continued fraction expansion described below, by directly observing the singularity of the matrix. There are only two unstable eigenstates, one growing and one decaying.

Let us now construct eigenstates. The eigenstates of $V$ are $\delta$-functions, as is easily verified from the recurrence relation of Hermite polynomials. Unfortunately, convergence to such discontinuous functions is quite slow. We thus return to our ansatz and solve for the analytic part separately. Restoring $\mu$ into our calculation, we can solve for the eigenstate iteratively. Substitute $\mu \delta(v - v_0)$ back into equation (11). We then obtain

$$\left( L - \frac{kv_0}{\pi^{1/4}} I \right) \vec{c} = \frac{v_0 k^2}{k} \left( \begin{array}{c} 0 \\ \mu \\ 0 \\ \vdots \\ \end{array} \right)$$

(16)

which we can solve for $\vec{c}$, unless $k = 0$, which we had discussed earlier. Gaussian elimination is most easily applied from the bottom up, and we obtain for the top two coefficients

$$\lambda c_0 + c_1 = 0$$

(17)

$$\left( 1 - \frac{k_J^2}{k^2} \right) c_0 + \left( \lambda - C(\lambda^{-1}) \right) c_1 = \frac{k_J^2 v_0}{2\pi^{1/4} k^2 \sigma} \mu$$

(18)

Here $C(x)$ is defined from the continued fraction

$$C(x) = \frac{2x}{1 - \frac{3x}{1 + \frac{3x}{1 + \frac{3x}{1 + \cdots}}}}$$

(19)
while $\lambda = -v_0/2\sigma^1/4$. Knowing the first two elements, the remainder are simply given by the tridiagonal recurrence relation (13). This algorithm can be verified by starting with a finite matrix and iteratively refining the solution.

The same method can be applied to solve for the imaginary eigenvalue, where we simply require (18) to be singular, and interpret $\lambda$ as an eigenvalue. We thus have an algebraically implicit expression for the growth factor. Since the imaginary eigenvalue is unique up to sign, a growing ansatz should obtain the correct solution and the implicit solution described here equivalent to equation (5-31) in Binney and Tremaine (1987).

The qualitative features are easy to extract. As $k \to \infty$ or $v_0 \to \infty$, the analytic component vanishes, $\vec{c} \to 0$ and we recover the free non-interacting solution. As $k \to 0$ we obtain $-\lambda c_1 = c_0 = -\mu$ unless we move $v_0$ to $\infty$ at the same time, in which case arbitrary ratios of $c_0/c_1$ can be achieved.

We can compare this to the analytic expression for the van Kampen mode

$$f_{\rho}(v) = \frac{k^2}{(2\pi\sigma^2)^{3/2}k^2} \frac{kve^{-v^2/2}}{kv - \omega} + \left[1 - \frac{k^4}{k^2}W \omega \left(\frac{\omega}{k\sigma}\right)\right] \delta(v - \omega/k)$$

where

$$W \omega (Z) = \frac{1}{\sqrt{2\pi}} \varphi \int_{-\infty}^{\infty} x e^{-x^2/2} dx$$

and $\varphi$ denotes the Cauchy principal value for the integral. We verify correct convergence for $k > k_J$. The matrix result, however, is general, and correctly yields all the continuum and discrete eigenvalues and eigenvectors for all values of $k$.

This example illustrates the separation between the solutions of $L_1$ which are simply the $\delta$ functions, and the contribution from $L_2$ which contains the response. This approach can be numerically extended by expanding the distribution function at every point in space and solving the more complex set of equations for arbitrary potentials.

5. Discussion

Formally, $L$ appears self-adjoint, which unfortunately only holds for $C^1$ (differentiable) functions. But as we saw, the self-adjointness can also be elucidated for quite pathological function domains. In non-trivial background potentials, smoothness of eigenmodes should increase since $L_1$ contains a $\partial_{v_0}$, which would diverge strongly for singular functions, such as occurred in the Jeans analysis.

For general systems such as elliptical galaxies, there is no way to separate the distribution function into the product of a radial and a velocity piece, as we did in all the examples. This foils
any attempt to apply a Fourier or other integral transform to $r$ and $v$ separately to express the differential operators as algebraic ones. Already in the Jeans example, we saw that one cannot define dispersion measures for van Kampen modes. In the absence of new tools, the general problem is very difficult, which is to be expected of partial differential equations. Linearity allows systematic numerical studies as we will see below.

The assumption of being a small perturbation holds as long as the inequality (3) is satisfied, which is even possible in the presence of a divergent $\delta$-function. One need only make sure that the coefficient of that $\delta$-function is positive to prevent any possibility of causing negative densities in phase space.

Most elliptical galaxies are observed to possess some weak small scale structure, which should certainly be subject to perturbative modeling. If we can have a complete analytic or numerical understanding of the eigenmodes, these perturbations can teach us much about the detailed potential structure. A direct prediction of this analysis is that perturbations should posses a discrete symmetry: positive and negative density perturbations should occur with comparable frequencies, and display similar patterns. Linearity of (4) allows us to reverse the sign of the perturbation $f_p \rightarrow -f_p$, so we do not expect small perturbations to be skewed when averaged over many instantiations, i.e. $<\rho_p> = 0$. Asymmetry only arises as a non-linear effect for short wavelength or large mass perturbations.

These long-lived oscillations may help explain the presence of shells in elliptical galaxies. Quinn (1984) explained these patterns as transients that arise as an elliptical galaxy accretes a smaller system, and what we see is the track of a cannibalized dwarf galaxy, which forms shells through the phase wrapping process. Hernquist and Spergel (1992) suggested that these shells can also form through the merger of two equal mass spiral systems. The observed concentric structure of shells implies a large amount of dynamical friction and fast phase mixing, thus these shells are relatively short lived. Schweizer and Seitzer (1988) find that a large fraction of all ellipticals have shells, which implies a high accretion or merger rate with corresponding cosmological implications. From (3) we see that a sufficiently long wavelength perturbation does not exhibit dynamical friction and is subject to perturbative treatment. The diffuse remnant may continue to orbit for several dynamical times with little diffusion or damping. Furthermore, the periodic motion of particles in a shell can excite similar modes in the predator galaxy, which are also long lived. Therefore, a numerical simulation must necessarily take these excited modes into account in order to model such phenomena accurately.

Under certain assumptions, Case and others have proven that the spectrum of eigenvalues is continuous except for a finite number of discrete points. For an Antonov stable system we know that they cannot contain positive imaginary components. The matrix analysis suggests that complex eigenvectors come in complex conjugate pairs, so we expect all modes to be stable. Landau damping then only occurs through the loss of coherence in a superposition of states when projected onto the density axis. But any periodic process would preferentially excite coherent
modes, as would be the case in satellites orbiting about larger galaxies. These modes can survive even after the satellite is disrupted or accreted.

Landau’s approach using Laplace transforms is an equivalent method to solve the initial value problem. Stability here implies the lack of frequencies with positive imaginary components. In the Jeans’ argument, the damping occurs despite the existence of coherent van Kampen modes, since these are singular and thus violate the assumptions of the Landau analysis. In this paper we suggested that van Kampen modes need not in general be singular. This implies that there may exist poles in the inverse transform on the real axis, allowing undamped modes to exist.

The success of the series expansion for the Jeans instability suggests that this scheme is a practical algorithm to calculate normal modes in arbitrary potentials. A systematic search for observable modes is now possible. A large density of states near certain eigenvalues might lead to easily excitable modes.

A numerical eigenmode expansion of the full six dimensional system should be feasible if we choose the appropriate basis. Since our examples all have a basically Gaussian velocity space structure, together with spherical symmetry and power law radial dependence, the natural basis should involve Hermite functions in \( \tau \)-space, spherical harmonics for angular dependence and Bessel functions for the radial piece. One can envision a p-mode analysis of galaxies similar to their very successful application to stars. Observations of both the surface density and spatially resolved velocity perturbations can in principle supply us with a three-parameter data set, which we can compare to numerical calculations. This should enable us to determine the low order harmonics. Given infinite resolution, a three parameter observation allows us to infer the full three dimensional structure of the gravitational potential. From (3) one expects that long wavelength perturbations should be longest lived, since they are the most linear and less subject to dynamical friction. This simplifies life for both the observer and the theorist, since a moderate resolution will pin down the fundamental modes and low harmonics.

6. Conclusions

The picture presented in this article gives a simple physical interpretation of perturbations about collisionless systems in terms of eigenmodes. I have presented some examples of periodic and stable oscillations in one dimensional collisionless systems, and discussed general features of these modes, which one can try to find in elliptical galaxies.

We have explored a little of the structure of perturbations in velocity space, and saw that it lends itself to simple analysis when expanded in a suitable way. The analytic solutions are limited, but numerical studies can provide a detailed description of fundamental modes. An expansion in Hermite polynomials allows an explicit transformation of the perturbation equations into self-adjoint form for general \( L^1 \) integrable functions, which relaxes the standard
differentiability requirement. This argues that general stationary systems should also exhibit periodic nondispersive modes.

The linear perturbation solutions of the six dimensional phase space are vast, and we have seen but a tiny sampling of its rich solution space.

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REFERENCES

Binney and Tremaine 1987, *Galactic Dynamics*, Princeton University Press, Chapter 5.1 and appendix 5A.

Case, K.M. 1959, *Annals of Physics*, 7, 349.

Fridman, A.M., and Polyachenko, V.L., *Physics of Gravitating Systems*, New York: Springer Verlag.

Gradshteyn, I.S. and Ryzhik, I.M. 1980, *Table of Integrals, Series and Products*, Academic Press.

Habib, S., Kandrup, H.E., and Yip, P.F. 1986, ApJ, 309, 176.

Hernquist, L., and Spergel, D.N. 1992, ApJ, 399, L117.

Hernquist, L., and Quinn, P.J. 1988, ApJ, 331, 682.

Hernquist, L., and Quinn, P.J. 1989, ApJ, 342, 1.

Keener, J.P. 1988, *Principles of Applied Mathematics*, Addison-Wessley, p.301-303.

Louis, P.D., and Gerhard, O.E. 1988, MNRAS, 233, 337.

Mathur, S.D. 1990, MNRAS, 243, 529.

Mikahilovskii, A.B., Fridman, A.M., & Epel’baum, Ya.G. 1971, *Soviet Phys.–JETP*, 32, 878.

Pen, U. 1992, *Numerical Methods for Partial Differential Equations*, 7, 303-315.

Pen, U., and Jiang, T.F. 1992, *Phys. Rev. A*, 46, 4297.
Quinn, P.J. 1984, ApJ, 279, 596.

Schweizer, F., and Seitzer, P. 1988, ApJ, 328, 88.

Weinberg, M.D. 1991, ApJ, 373, 391.

Weinberg, M.D. 1993, submitted to ApJ.

Wilkinson, J.H. 1965, *The Algebraic Eigenvalue Problem*, Oxford, Clarendon Press, p.335-337.