MIRROR DUALITY AND
STRING-THEORETIC HODGE NUMBERS

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Abstract
We prove in full generality the mirror duality conjecture for string-theoretic
Hodge numbers of Calabi-Yau complete intersections in Gorenstein toric Fano
varieties. The proof is based on properties of intersection cohomology.

1 Introduction
The first author has conjectured that the polar duality of reflexive polyhedra in-
duces the mirror involution for Calabi-Yau hypersurfaces in Gorenstein toric Fano
varieties [2]. The second author has proposed a more general duality which con-
jecturally induces the mirror involution for Calabi-Yau complete intersections in
Gorenstein toric Fano varieties [7]. The most general form of the combinatorial du-
ality which includes mirror constructions of physicists for rigid Calabi-Yau manifolds
was formulated by both authors in [4].

The main purpose of our paper is to show that all proposed combinatorial duali-
ties agree with the following Hodge-theoretic property of mirror symmetry predicted
by physicists:

If two smooth n-dimensional Calabi-Yau manifolds V and W form a mirror pair,
then their Hodge numbers satisfy the relation

\[ h^{p,q}(V) = h^{n-p,q}(W), \quad 0 \leq p, q \leq n. \]  (1)

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A verification of this property becomes rather non-trivial if we don’t make restrictions on the dimension $n$. The main difficulty is the necessity to work with singular Calabi-Yau varieties whose singularities in general don’t admit any crepant desingularization. This difficulty was the motivation for introduction of so called string-theoretic Hodge numbers $h^p_q \text{st}(V)$ for singular $V$ [6]. The string-theoretic Hodge numbers $h^p_q \text{st}(V)$ coincide with the usual Hodge numbers $h^p_q(V)$ if $V$ is smooth, and with the usual Hodge numbers of a crepant desingularization $\hat{V}$ of $V$ if such a desingularization exists. Therefore the property [4] must be reformulated as follows:

Let $(V, W)$ be a mirror pair of singular $n$-dimensional Calabi-Yau varieties. Then the string-theoretic Hodge numbers satisfy the duality:

$$h^p_q \text{st}(V) = h^{n-p}_q \text{st}(W), \quad 0 \leq p, q \leq n.$$  \hspace{1cm} (2)

The string-theoretic Hodge numbers for Gorenstein algebraic varieties with toroidal or quotient singularities were introduced and studied in [6]. It was also conjectured in [4] that the combinatorial construction of mirror pairs of Calabi-Yau complete intersections in Gorenstein toric Fano varieties satisfies the duality (2). This conjecture has been proved in [6] for mirror pairs of Calabi-Yau hypersurfaces of arbitrary dimension that can be obtained by the Greene-Plesser construction [19]. Some other results supporting this conjecture have been obtained in [4, 5, 27]. Additional evidence in favor of the conjecture has been received by explicit computations of instanton sums using generalized hypergeometric functions [3, 20, 22, 24].

The paper is organized as follows:

In Section 2, we introduce some polynomials $B(P; u, v)$ of an Eulerian partially ordered set $P$ using results of Stanley [29]. It seems that the polynomial $B(P; u, v)$ have independent interest in combinatorics. For our purposes, their most important property is the relation between $B(P; u, v)$ and $B(P^*; u, v)$, where $P^*$ is the dual to $P$ Eulerian poset (Theorem 2.13).

In Section 3, we give an explicit formula for the polynomial $E(Z; u, v)$ which describes the mixed Hodge structure of an affine hypersurface $Z$ in an algebraic torus $T$ (Theorem 3.18). We remark the following: the explicit formula for $E(Z; 1, 1)$ is due to Bernstein, Khovanski and Kushnirensko [21, 23]; the computation of the polynomial $E(Z; t, 1)$ which describes the Hodge filtration on $H^*_c(Z)$ is due to Danilov and Khovanskiii [11] (see also [1]); the polynomial $E(Z; t, t)$ which describes the weight filtration on $H^*_c(Z)$ has been computed by Denef and Loeser [13].

In Section 4, we derive an explicit formula for the polynomial $E_{st}(V; u, v)$ where $V$ is a Calabi-Yau complete intersection in a Gorenstein toric Fano variety (Theorem
4. The coefficients of $E_{st}(V; u, v)$ are equal up to a sign to string-theoretic Hodge numbers of $V$. Since our formula is written in terms of $B$-polynomials as a sum over pairs of lattice points contained in the corresponding pair of dual to each other reflexive Gorenstein cones $C$ and $C^*$, the mirror duality for string-theoretic Hodge numbers becomes immediate consequence of the duality for $B$-polynomials after the transposition $C \leftrightarrow C^*$ (Theorem 4.15). Following some recent development of ideas of Witten [30] by Morisson and Plesser [25], we conjecture that the formula obtained in this paper gives the spectrum of the abelian gauge theory in two dimensions which could be constructed from any pair $(C, C^*)$ of two dual to each other reflexive Gorenstein cones.

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2 Combinatorial polynomials of Eulerian posets

Let $P$ be a finite poset (i.e., finite partially ordered set). Recall that the Möbius function $\mu_P(x, y)$ of a poset $P$ is a unique integer valued function on $P \times P$ such that for every function $f : P \to A$ with values in an abelian group $A$ the following Möbius inversion formula holds:

$$f(y) = \sum_{x \leq y} \mu_P(x, y)g(x), \quad \text{where} \quad g(y) = \sum_{x \leq y} f(x).$$

From now on we always assume that the poset $P$ has a unique minimal element $\hat{0}$, a unique maximal element $\hat{1}$, and that every maximal chain of $P$ has the same length $d$ which will be called the rank of $P$. For any $x \leq y$ in $P$, define the interval $[x, y] = \{ z \in P : x \leq z \leq y \}$.

In particular, we have $P = [\hat{0}, \hat{1}]$. Define the rank function $\rho : P \to \{0, 1, \ldots, d\}$ of $P$ by setting $\rho(x)$ equal to the length of any saturated chain in the interval $[\hat{0}, x]$.

**Definition 2.1** [29] A poset $P$ as above is said to be Eulerian if for any $x \leq y$ $(x, y \in P)$ we have

$$\mu_P(x, y) = (-1)^{\rho(y)-\rho(x)}.$$

**Remark 2.2** It is easy to see that any interval $[x, y] \subset P$ in an Eulerian poset $P$ is again an Eulerian poset with the rank function $\rho(z) - \rho(x)$ for any $z \in [x, y]$. If an Eulerian poset $P$ has rank $d$, then the dual poset $P^*$ is again an Eulerian poset with the rank function $\rho^*(x) = d - \rho(x)$.
Example 2.3 Let $C$ be an $d$-dimensional finite convex polyhedral cone in $\mathbb{R}^d$ such that $-C \cap C = \{0\} \in \mathbb{R}^d$. Then the poset $P$ of faces of $C$ satisfies all above assumptions with the maximal element $C$, the minimal element $\{0\}$, and the rank function $\rho$ which is equal to the dimension of the corresponding face. It is easy to show that $P$ is an Eulerian poset of rank $d$.

Definition 2.4 Let $P = [\hat{0}, \hat{1}]$ be an Eulerian poset of rank $d$. Define two polynomials $G(P, t), H(P, t) \in \mathbb{Z}[t]$ by the following recursive rules:

\[
G(P, t) = H(P, t) = 1 \text{ if } d = 0;
\]

\[
H(P, t) = \sum_{\hat{0} < x < \hat{1}} (t - 1)^{\rho(x) - 1} G([x, \hat{1}], t) \quad (d > 0),
\]

\[
G(P, t) = \tau_{<d/2} \left( (1 - t)H(P, t) \right) \quad (d > 0),
\]

where $\tau_{<r}$ denotes the truncation operator $\mathbb{Z}[t] \to \mathbb{Z}[t]$ which is defined by

\[
\tau_{<r} \left( \sum_i a_i t^i \right) = \sum_{i < r} a_i t^i.
\]

Theorem 2.5 Let $P$ be an Eulerian poset of rank $d \geq 1$. Then

\[
H(P, t) = t^{d-1} H(P, t^{-1}).
\]

Proposition 2.6 Let $P$ be an Eulerian poset of rank $d \geq 0$. Then

\[
t^d G(P, t^{-1}) = \sum_{\hat{0} \leq x \leq \hat{1}} (t - 1)^{\rho(x)} G([x, \hat{1}], t).
\]

Proof. The case $d = 0$ is obvious. Using 2.5, we obtain

\[
(t - 1)H(P, t) = t^d G(P, t^{-1}) - G(P, t) \quad (d > 0).
\]

Now the statement follows from the formula for $H(P, t)$ in 2.4.

Definition 2.7 Let $P$ be an Eulerian poset of rank $d$. Define the polynomial $B(P; u, v) \in \mathbb{Z}[u, v]$ by the following recursive rules:

\[
B(P; u, v) = 1 \text{ if } d = 0,
\]

\[
\sum_{\hat{0} \leq x \leq \hat{1}} B([\hat{0}, x]; u, v) u^{d-\rho(x)} G([x, \hat{1}], u^{-1}v) = G(P, uv).
\]
Example 2.8 Let $P$ be the boolean algebra of rank $d \geq 1$. Then $G(P, t) = 1$, $H(P, t) = 1 + t + \cdots + t^{d-1}$, and $B(P; u, v) = (1 - u)^d$.

Example 2.9 Let $C \subset \mathbb{R}^3$ be a 3-dimensional finite convex polyhedral cone with $k$ 1-dimensional faces $(-C \cap C = \{0\}) \in \mathbb{R}^3$, $P$ the Eulerian poset of faces of $C$. Then $G(P, t) = 1 + (k-3)t$, $H(P, t) = 1 + (k-2)t + (k-2)t^2 + t^3$, and

$$B(P; u, v) = 1 - (k - (k - 3)v)u + (k - (k - 3)v)u^2 - u^3.$$ We notice that $B(P; u, v)$ satisfies the relation

$$B(P; u, v) = (-u)^3 B(P; u^{-1}, v)$$

which is a consequence of the selfduality $P \cong P^*$ and a more general property 2.13.

Proposition 2.10 Let $P$ be an Eulerian poset of rank $d > 0$. Then $B(P; u, v)$ has the following properties:

(i) $B(P; u, 1) = (1 - u)^d$ and $B(P; 1, v) = 0$;

(ii) the degree of $B(P; u, v)$ with respect to $v$ is less than $d/2$.

Proof. The statement (i) follows immediately from 2.6 and the recursive definition of $B(P; u, v)$. In order to prove (ii) we use induction on $d$. By assumption, the degree of $B([0, x]; u, v)$ with respect to $v$ is less than $\rho(x)/2$. On the other hand, the $v$-degree of $G([x, 1]; u^{-1}v)$ is less than $(d - \rho(x))/2$ (see 2.4). It remains to apply the recursive formula of 2.7.

Proposition 2.11 Let $P$ be an Eulerian poset of rank $d$. Then $B$-polynomials of intervals $[0, x]$ and $[x, \hat{1}]$ satisfy the following relation:

$$\sum_{\hat{0} \leq x \leq \hat{1}} B([\hat{0}, x]; u^{-1}, v^{-1})(uv)^{\rho(x)}(v - u)^{d - \rho(x)} = \sum_{\hat{0} \leq x \leq \hat{1}} B([x, \hat{1}]; u, v)(uv - 1)^{\rho(x)}.$$ 

Proof. Let us substitute $u^{-1}, v^{-1}$ instead of $u, v$ in the recursive relation 2.7. So we obtain

$$\sum_{\hat{0} \leq x \leq \hat{1}} B([\hat{0}, x]; u^{-1}, v^{-1})u^{-d+\rho(x)}G([x, \hat{1}], uv^{-1}) = G(P, u^{-1}v^{-1}). \quad (3)$$

By 2.6, we have

$$G(P, u^{-1}v^{-1}) = (uv)^{-d} \sum_{\hat{0} \leq x \leq \hat{1}} (uv - 1)^{\rho(x)}G([x, \hat{1}], uv) \quad (4)$$

and

$$G([x, \hat{1}], uv^{-1}) = \sum_{x \leq y \leq \hat{1}} (u^{-1}v - 1)^{\rho(y)-\rho(x)} u^{d-\rho(x)} v^{\rho(x)-d} G([y, \hat{1}], u^{-1}v)$$
Proof. We set
\[
Q([y, \hat{1}], u, v) = \sum_{y \leq x \leq \hat{1}} u^{d-\rho(y)} v^{\rho(x)-d} (v-u)^{\rho(y)-\rho(x)} G([y, \hat{1}], u^{-1} v).
\] (5)

By (2.7), we also have
\[
G([x, \hat{1}], uv) = \sum_{x \leq y \leq \hat{1}} u^{d-\rho(y)} B([x, y]; u, v) G([y, \hat{1}], u^{-1} v).
\] (6)

By substitution (1) in (1), and two equations (1), (2) in (1) we obtain:
\[
\sum_{\delta \leq x \leq y \leq \hat{1}} B([\hat{0}, x]; u^{-1}, v^{-1}) u^{\rho(x)-\rho(y)} v^{\rho(x)-d} (v-u)^{\rho(y)-\rho(x)} G([y, \hat{1}], u^{-1} v)
\]
\[
= \sum_{\delta \leq x \leq y \leq \hat{1}} B([x, y]; u, v) u^{-\rho(y)} v^{-d} (uv-1)^{\rho(x)} G([y, \hat{1}], u^{-1} v).
\] (7)

Now we use induction on d. It is easy to see that the equation (4) and the induction hypothesis for \(y < \hat{1}\) immediately imply the statement of the proposition. \(\square\)

Proposition 2.12 The B-polynomials are uniquely determined by the relation 2.11, by the property of v-degree from 2.10(ii), and by the initial condition \(B(P; u, v) = 1\) if \(d = 0\).

Proof. Indeed, if we know \(B([x, y]; u, v)\) for all \(\rho(y) - \rho(x) < d\), then we know all terms in 2.11 except for \(B(P; u, v)\) on the right hand side and \(B(P; u^{-1}, v^{-1})(uv)^d\) on the left hand side. Because the v-degree of \(B(P; u, v)\) is less than \(d/2\), the possible degrees of monomials with respect to variable v in \(B(P; u, v)\) and \(B(P; u^{-1}, v^{-1})(uv)^d\) do not coincide. This allows us to determine \(B(P; u, v)\) uniquely. \(\square\)

Theorem 2.13 Let \(P\) be an Eulerian poset of rank \(d\), \(P^*\) be the dual Eulerian poset. Then
\[
B(P; u, v) = (-u)^d B(P^*; u^{-1}, v).
\]

Proof. We set
\[
Q(P; u, v) = (-u)^d B(P^*; u^{-1}, v).
\]

It is clear that \(Q(P; u, v) = 1\) and v-degree of \(Q(P; u, v)\) is the same as v-degree of \(B(P; u, v)\). By 2.12, it remains to establish the same recursive relations for \(Q(P; u, v)\) as for \(B(P; u, v)\) in 2.11. The last property follows from straightforward computations. Indeed, the equality
\[
\sum_{\delta \leq x \leq \hat{1}} Q([\hat{0}, x]; u^{-1}, v^{-1})(uv)^{\rho(x)} (v-u)^{d-\rho(x)} = \sum_{\delta \leq x \leq \hat{1}} Q([x, \hat{1}]; u, v)(uv-1)^{\rho(x)}
\] (8)

is equivalent to the relation 2.11 for \(B(P^*; u, v^{-1})\):
\[
\sum_{\delta \leq x \leq \hat{1}} B([x, \hat{1}]; u^{-1}, v)(uv^{-1})^{d-\rho(x)} (v^{-1}-u)^{\rho(x)}
\]
\[
= \sum_{\hat{0} \leq x \leq \hat{1}} B((\hat{0}, x)^*; u, v^{-1})(uv^{-1} - 1)^{d-\rho(x)},
\]

because

\[Q([x, \hat{1}]; u, v) = (-u)^{d-\rho(x)}B([x, \hat{1}]^*; u^{-1}, v)\]

and

\[Q((\hat{0}, x]; u^{-1}, v^{-1}) = (-u)^{-\rho(x)}B((\hat{0}, x)^*; u, v^{-1}).\]

\[\square\]

3 E-polynomials of toric hypersurfaces

Let \(M\) and \(N\) be two free abelian groups of rank \(d\) which are dual to each other; i.e., \(N = \text{Hom}(M, \mathbb{Z})\). We denote by \(\langle *, * \rangle : M \times N \to \mathbb{Z}\) the canonical bilinear pairing, and by \(M_\mathbb{R}\) (resp. by \(N_\mathbb{R}\)) the real scalar extensions of \(M\) (resp. of \(N\)).

**Definition 3.1** A subset \(C \subset M\) is called a \(d\)-dimensional rational convex polyhedral cone with vertex \(\{0\} \in M\) if there exists a finite set \(\{e_1, \ldots, e_k\} \subset M\) such that

\[C = \{\lambda_1 e_1 + \cdots + \lambda_k e_k \in M_\mathbb{R} : \text{where} \quad \lambda_i \in \mathbb{R}_{\geq 0} \quad (i = 1, \ldots, k)\}\]

and \(-C + C = M_\mathbb{R}, -C \cap C = \{0\} \in M\).

**Remark 3.2** If \(C \subset M\) is a \(d\)-dimensional rational convex polyhedral cone with vertex \(\{0\} \in M\), then the dual cone

\[C^* = \{z \in N_\mathbb{R} : \langle e_i, z \rangle \geq 0 \text{ for all } i \in \{1, \ldots, k\} \}\]

is also a \(d\)-dimensional rational convex polyhedral cone with vertex \(\{0\} \in N_\mathbb{R}\). Moreover, there exists a canonical bijective correspondence \(F \leftrightarrow F^*\) between faces \(F \subset C\) and faces \(F^* \subset C^*\) (dim \(F + \text{dim } F^* = d\)):

\[F \leftrightarrow F^* = \{z \in C^* : \langle z', z \rangle = 0 \text{ for all } z' \in F\}\]

which reverses inclusion relation between faces.

Let \(P\) be the Eulerian poset of faces of a \(d\)-dimensional rational convex polyhedral cone \(C \subset M_\mathbb{R}\) with vertex in \(\{0\}\). For convenience of notations, we use elements \(x \in P\) as indices and denote by \(C_x\) the face of \(C\) corresponding to \(x \in P\), in particular, we have \(C_0 = \{0\}, C_1 = C, \) and \(\rho(x) = \text{dim } C_x\). The dual Eulerian poset \(P^*\) can be identified with the poset of faces \(C_x^*\) of the dual cone \(C^* \subset N_\mathbb{R}\).
Definition 3.3 A $d$-dimensional cone $C$ ($d \geq 1$) as in [3,4] is called Gorenstein if there exists an element $n_C \in \mathbb{N}$ such that $\langle z, n_C \rangle > 0$ for any nonzero $z \in C$ and all vertices of the $(d-1)$-dimensional convex polyhedron

$$\Delta(C) = \{ z \in C : \langle z, n_C \rangle = 1 \}$$

belong to $M$. This polyhedron will be called the supporting polyhedron of $C$. For convenience, we consider $\{0\}$ as a 0-dimensional Gorenstein cone with the supporting polyhedron $\Delta(\{0\}) := \emptyset$. For any $m \in C \cap M$, we define the degree of $m$ as

$$\text{deg } m = \langle m, n_C \rangle.$$ 

Remark 3.4 It is clear that any face $C_x$ of a Gorenstein cone is again a Gorenstein cone with the supporting polyhedron

$$\Delta(C_x) = \{ z \in C_x : \langle z, n_C \rangle = 1 \}.$$ 

Now we recall standard facts from theory of toric varieties [9, 11, 26] and fix our notations:

Let $P(C)$ be the $(d-1)$-dimensional projective toric variety associated with a Gorenstein cone $C$. By definition,

$$P(C) = \text{Proj} \mathbb{C}[C \cap M]$$

where $\mathbb{C}[C \cap M]$ is a graded semigroup algebra over $\mathbb{C}$ of lattice points $m \in C \cap M$. Each face $C_x \subset C$ of positive dimension defines an irreducible projective toric subvariety

$$P(C_x) = \text{Proj} \mathbb{C}[C_x \cap M] \subset P(C)$$

which is a compactification of a $(\rho(x) - 1)$-dimensional algebraic torus

$$T_x := \text{Spec} \mathbb{C}[M_x],$$

where $M_x \subset M$ is the subgroup of all lattice points $m \in (-C_x + C_x) \cap M$ such that $\langle m, q \rangle = 0$. Moreover, the multiplicative group low on $T_x$ extends to a regular action of $T_x$ on $P(C_x)$ so that one has the natural stratification

$$P(C_x) = \bigcup_{0 < y \leq x} T_y$$

by $T_x$-orbits $T_y$. We denote by $\mathcal{O}_P(C)(1)$ the ample tautological sheaf on $P(C)$. In particular, lattice points in $\Delta(C)$ can be identified with a torus invariant basis of the space of global sections of $\mathcal{O}_P(C)(1)$. We denote by $\mathcal{Z}$ the set of zeros of a generic global section of $\mathcal{O}_P(C)(1)$ and set

$$Z_x := \mathcal{Z} \cap T_x \ (\hat{0} < x \leq \hat{1}).$$

Thus we have the natural stratification:

$$\mathcal{Z} = \bigcup_{\hat{0} < x \leq \hat{1}} Z_x,$$

where each $Z_x$ is a smooth affine hypersurface in $T_x$ defined by a generic Laurent polynomial with the Newton polyhedron $\Delta(C_x)$. 

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Definition 3.5 Define two functions
\[ S(C_x, t) := (1 - t)^{\rho(x)} \sum_{m \in C_x \cap M} t^{\deg m} \]
and
\[ T(C_x, t) := (1 - t)^{\rho(x)} \sum_{m \in \text{Int}(C_x) \cap M} t^{\deg m}, \]
where \( \text{Int}(C_x) \) denotes the relative interior of \( C_x \subset C \).

The following statement is a consequence of the Serre duality (see [10, 1]):

Proposition 3.6 \( S(C_x, t) \) and \( T(C_x, t) \) are polynomials satisfying the relation
\[ S(C_x, t) = t^d T(C_x, t^{-1}). \]

Definition 3.7 [10] Let \( X \) be a quasi-projective algebraic variety over \( C \). For each pair of integers \( (p, q) \), one defines the following generalization of Euler characteristic:
\[ e^{p, q}(X) = \sum_k (-1)^k h^{p, q}(H^k_c(X)), \]
where \( h^{p, q}(H^k_c(X)) \) is the dimension of the \( (p, q) \)-component of the mixed Hodge structure of \( H^k_c(X) \) [12]. The sum
\[ E(X; u, v) := \sum_{p, q} e^{p, q}(X) u^p v^q \]
is called \( E \)-polynomial of \( X \).

Next statement is also due to Danilov and Khovanskiǐ (see [10] §4, or another approach in [1]):

Proposition 3.8 We set \( E(Z_x; t, 1) := (t - 1)^{-1} \). Then
\[ E(Z_x; t, 1) = \frac{(t - 1)^{\rho(x)-1} + (-1)^{\rho(x)} S(C_x, t)}{t} \]
for \( \rho(x) \geq 0 \).

The purpose of this section is to give an explicit formula for \( E \)-polynomials of affine hypersurfaces \( Z_x \subset T_x \). Following the method of Denef and Loeser [13] combined with ideas of Danilov and Khovanskiǐ [10], we compute \( E(Z_x; u, v) \) using intersection cohomology introduced by Goresky and MacPherson [17]. Recall that intersection cohomology \( IH^*(X) \) of a quasiprojective algebraic variety \( X \) of pure dimension \( n \) over \( C \) can be defined as hypercohomology of the so called intersection complex \( IC^*_X \). Moreover, the intersection complex carries a natural mixed Hodge structure. The weight filtration on the \( l \)-adic version of intersection cohomology has been introduced and studied by Beilinson, Bernstein, Deligne and Gabber using theory of perverse sheaves [8]. The Hodge filtration on intersection cohomology of algebraic varieties over \( C \) has been introduced by M. Saito using his theory of mixed Hodge modules [28] (see also [14]).
**Definition 3.9** Let $X$ be a quasi-projective algebraic variety over $\mathbb{C}$. For each pair of integers $(p, q)$, one defines the following generalization of Euler characteristic for intersection cohomology:

$$e_{\text{int}}^{p,q}(X) = \sum_k (-1)^k h^{p,q}(IH^k_c(X)),$$

where $h^{p,q}(H^k_c(X))$ is the dimension of the $(p, q)$-component in the mixed Hodge structure of $IH^k_c(X)$. The sum

$$E_{\text{int}}(X; u, v) := \sum_{p,q} e_{\text{int}}^{p,q}(X) u^pv^q$$

is called intersection cohomology $E$-polynomial of $X$.

The following statement has been discovered by Bernstein, Khovanskiǐ and MacPherson (two independent proofs are contained in [13] and [15]):

**Theorem 3.10**

$$E_{\text{int}}(\mathbb{P}(C); u, v) = H(P, uv) = \sum_{0<x<1} (uv - 1)^{\rho(x) - 1} G([x, \hat{1}], uv).$$

Moreover, the cohomology sheaves $H^i(I\mathbb{C}^*_x\mathbb{P}(C))$ are constant on torus orbits $T_x$ and $G([x, \hat{1}], uv)$ is the Poincaré polynomial describing their dimensions.

**Corollary 3.11** Let $W \subset \mathbb{P}(C)$ be a hypersurface that meets transversally all toric strata $T_x \subset \mathbb{P}(C)$ that it intersects ($W$ is not necessary ample). Then

$$E_{\text{int}}(W; u, v) = \sum_{0<x<1} E(W_x; u, v) G([x, \hat{1}], uv),$$

where $W_x = W \cap T_x$ ($0 < x \leq \hat{1}$).

**Proof.** The statement is essentially contained in [13] (Lemma 7.7). The key fact is that singularities of $W$ along $W_x$ are toroidal (see [6]), i.e., locally isomorphic to toric singularities which appear on $\mathbb{P}(C)$. \hfill $\square$

Applying 3.8, we obtain:

**Corollary 3.12**

$$E_{\text{int}}(\mathbb{P}(C); t, 1) = \sum_{0<x<1} \left( \frac{(t - 1)^{\rho(x) - 1} + (-1)^{\rho(x)} S(C_x, t)}{t} \right) G([x, \hat{1}], t).$$
Definition 3.13 Define $H_{\text{Lef}}(P, t)$ to be the polynomial of degree $(d - 2)$ with the following properties:

(i) $H_{\text{Lef}}(P, t) = t^{d-2}H_{\text{Lef}}(P, t^{-1})$;
(ii) $\tau_{\leq (d-2)/2}H_{\text{Lef}}(P, t) = \tau_{\leq (d-2)/2}H(P, t)$.

Proposition 3.14

$H_{\text{Lef}}(P, t) = (1 - t)^{-1}(G(P, t) - t^{d-1}G(P, t^{-1})).$

Proof. Let us set $Q(P, t) := (1 - t)^{-1}(G(P, t) - t^{d-1}G(P, t^{-1})).$

We check that the properties 3.13(i)-(ii) are satisfied for $Q(P, t)$. Indeed 3.13(i) follows immediately from the definition of $Q(P, t)$. If

$H(P, t) = \sum_{0 \leq i \leq d-1} h_i t^i$

and

$G(P, t) = h_0 + \sum_{1 \leq i < d/2} (h_i - h_{i-1}) t^i,$

then

$Q(P, t) = h_0 \frac{1 - t^{d-1}}{1 - t} + \sum_{1 \leq i < d/2} (h_i - h_{i-1}) \frac{t^i - t^{d-1-i}}{1 - t}.$

This shows (ii) and the fact that $Q(P, t)$ is a polynomial.

Proposition 3.15 Define $E_{\text{int}}^{\text{prim}}(\overline{Z}; u, v)$ to be the polynomial

$E_{\text{int}}^{\text{prim}}(\overline{Z}; u, v) := E_{\text{int}}(\overline{Z}; u, v) - H_{\text{Lef}}(P, uv).$

Then $E_{\text{int}}^{\text{prim}}(\overline{Z}; u, v)$ is a homogeneous polynomial of degree $(d - 2)$.

Proof. By the Lefschetz theorem for intersection cohomology [18], we have isomorphisms

$IH^i(P(C)) \cong IH^i(\overline{Z}), \quad (0 \leq i < d - 2)$

and the short exact sequence

$0 \to IH^{d-2}(P(C)) \to IH^{d-2}(\overline{Z}) \to IH_{\text{prim}}^{d-2}(\overline{Z}) \to 0,$

where $IH_{\text{prim}}^{d-2}(\overline{Z})$ denotes the primitive part of intersection cohomology of $\overline{Z}$ in degree $(d - 2)$. By purity theorem for intersection cohomology [16] (see also [14]), the Hodge structure of $IH_{\text{prim}}^{d-2}(\overline{Z})$ is pure. On the other hand, it follows from the Poincaré duality for intersection cohomology that $E_{\text{int}}^{\text{prim}}(\overline{Z}; u, v)$ is the $E$-polynomial of this Hodge structure.
Theorem 3.16  We set $E(Z_0; u, v) := (uv - 1)^{-1}$. Then $E$-polynomials $E(Z_x; u, v)$ of affine toric hypersurfaces satisfy the following recursive relation

$$
\sum_{0 \leq x \leq 1} (E(Z_x; u, v) - (uv)^{-1}(uv - 1)^{\rho(x)-1})G([x, \hat{1}], uv)
$$

$$
= v^{d-2} \sum_{0 \leq x \leq 1} (u^{-1}v)^{-1}(uv - 1)^{\rho(x)}S(C_x, uv^{-1})G([x, \hat{1}], uv^{-1}).
$$

Proof. By 3.12 and 3.14, we have

$$
E_{\text{prim}}^{\text{int}}(Z; t, 1) = E_{\text{int}}(Z; t, 1) - H_{\text{Lef}}(P, t) =
$$

$$
= \sum_{0 \leq x \leq 1} t^{-1}((t - 1)^{\rho(x)-1} + (1 - t)^{\rho(x)}S(C_x, t))G([x, \hat{1}], t)
$$

$$
- (1 - t)^{-1}(G(P, t) - t^{d-1}G(P, t^{-1})).
$$

Using 2.6, we obtain

$$
\sum_{0 \leq x \leq 1} t^{-1}(t - 1)^{\rho(x)-1}G([x, \hat{1}], t) = t^{-1}(t - 1)^{-1}t^{d-1}G(P, t^{-1}) - G(P, t)).
$$

This yields

$$
E_{\text{prim}}^{\text{int}}(Z; t, 1) = \sum_{0 \leq x \leq 1} t^{-1}(1 - t)^{\rho(x)}S(C_x, t))G([x, \hat{1}], t). \quad (9)
$$

On the other hand, by 3.11 and 3.14, we have

$$
E_{\text{prim}}^{\text{int}}(Z; u, v) = E_{\text{int}}(Z; u, v) - H_{\text{Lef}}(P, uv)
$$

$$
= \sum_{0 \leq x \leq 1} E(Z_x; u, v)G([x, \hat{1}], uv) - (1 - uv)^{-1}(G(P, uv) - (uv)^{d-1}G(P, (uv)^{-1})).
$$

Using 2.6, we obtain

$$
\sum_{0 \leq x \leq 1} (uv)^{-1}(uv - 1)^{\rho(x)-1}G([x, \hat{1}], uv) = (uv)^{d-1}(uv - 1)^{-1}(G(P, (uv)^{-1}).
$$

This yields

$$
E_{\text{prim}}^{\text{int}}(Z; u, v) = \sum_{0 \leq x \leq 1} (E(Z_x; u, v) - (uv)^{-1}(uv - 1)^{\rho(x)-1})G([x, \hat{1}], uv). \quad (10)
$$

By 3.13, we have

$$
E_{\text{prim}}^{\text{int}}(Z; u, v) = v^{d-2}E_{\text{prim}}^{\text{int}}(Z; uv^{-1}, 1).
$$

It remains to combine (9) and (10).
**Definition 3.17** Let \( m \) be a lattice point in \( C \cap M \). We denote by \( x(m) \) the minimal element among \( x \in P \) such that the face \( C_x \subset C \) contains \( m \). The interval \([x(m), \hat{1}] \subset P\) parametrizes the set of all faces of \( C \) containing \( m \). We identify the dual interval \([x(m), \hat{1}]^* \) with the Eulerian poset of all faces \( C_x^* \subset C \) such that \( \langle m, z \rangle = 0 \) for all \( z \in C_x^* \).

**Theorem 3.18** Let us set \( Z := Z_1 \). Then there exists the following explicit formula for \( E(Z; u, v) \) in terms of \( B \)-polynomials:

\[
E(Z; u, v) = \frac{(uv - 1)^{d-1}}{uv} + \frac{(-1)^d}{u^v} \sum_{m \in C \cap M} (v - u)^{\rho(x(m))} B([x(m), \hat{1}]^*; u, v) \left( \frac{u}{v} \right)^{\deg m} .
\]

**Proof.** By induction, \( E \)-polynomials are uniquely determined from the recursive formula \( 3.10 \). Therefore, it suffices to show that the functions

\[
\frac{(uv - 1)^{\rho(x)-1}}{uv} + \frac{(-1)^{\rho(x)}}{uv} \sum_{m \in C \cap M} (v - u)^{\rho(x(m))} B([x(m), \hat{1}]^*; u, v) \left( \frac{u}{v} \right)^{\deg m}
\]

satisfy the same recursive formula as polynomials \( E(Z_x; u, v) \). Indeed, let us substitute these functions instead of \( E \)-polynomials in the left hand side of \( 3.10 \) and expand

\[
(-1)^{\rho(x)} S(C_x, uv^{-1}) = \left( \frac{u}{v} - 1 \right)^{\rho(x)} \sum_{m \in C \cap M} \left( \frac{u}{v} \right)^{\deg m}
\]

on the right hand side of \( 3.10 \). Now we choose a lattice point \( m \in C \cap M \), collect terms containing \( (u/v)^{\deg m} \) in right and left hand sides, and use the equality \( 2.10 \)

\[
\sum_{x(m) \leq x \leq 1} \left( \frac{u}{v} - 1 \right)^{\rho(x)} G([x, \hat{1}], uv^{-1}) = \left( \frac{u}{v} - 1 \right)^{\rho(x(m))} \left( \frac{u}{v} \right)^{d - \rho(x(m))} G([x(m), \hat{1}], u^{-1}v)
\]

on the right hand side. By the duality \( 2.13 \)

\[
B([x(m), \hat{1}]^*; u, v) = (-u)^{\rho(x) - \rho(x(m))} B([x(m), x]^*; u^{-1}, v),
\]

it remains to establish the recursive relation:

\[
\frac{(v - u)^{\rho(x(m))}}{uv} \sum_{x(m) \leq x \leq 1} (-1)^{\rho(x)} (-u)^{\rho(x) - \rho(x(m))} B([x(m), x]^*; u^{-1}, v) G([x, \hat{1}], uv) =
\]

\[
= \left( \frac{u}{v} - 1 \right)^{\rho(x(m))} \frac{v^{d-1}}{u} \left( \frac{u}{v} \right)^{d - \rho(x(m))} G([x(m), \hat{1}], u^{-1}v)
\]

which is equivalent to the recursive relation in \( 2.7 \) after the substitution \( u^{-1} \) instead of \( u \).  

\[\blacksquare\]
4 Mirror duality

Let $\overline{M}$ and $\overline{N} = \text{Hom}(\overline{N}, \mathbb{Z})$ be dual to each other free abelian groups of rank $d$, $\overline{M}_\mathbb{R}$ and $\overline{N}_\mathbb{R}$ the real scalar extensions of $M$ and $N$, $\langle *, * \rangle : \overline{M} \times \overline{N} \to \mathbb{Z}$ the natural pairing.

Definition 4.1 \[4\] Let $C \subset \overline{M}_\mathbb{R}$ be a $d$-dimensional Gorenstein cone. The cone $C$ is called reflexive if the dual cone $C^* \subset \overline{N}_\mathbb{R}$ is also Gorenstein; i.e., there exists a lattice element $m_{C^*} \in M$ such that all vertices of the supporting polyhedron $\Delta(C^*) = \{z \in C^* : \langle m_{C^*}, z \rangle = 1 \}$ are contained in $M$. In this case, we call $r = \langle m_{C^*}, n_C \rangle$ the index of $C$.

Definition 4.2 \[2\] Let $M$ be a free abelian group of rank $d$. A $d$-dimensional polyhedron in $M_\mathbb{R}$ with vertices in $M$ is called reflexive if it can be identified with a supporting polyhedron of some $(d + 1)$-dimensional reflexive Gorenstein cone of index 1.

Recall the definition of string-theoretic Hodge numbers of an algebraic variety $X$ with at most Gorenstein toroidal singularities \[6\]:

Definition 4.3 \[6\] Let $X = \bigcup_{i \in I} X_i$ be a $k$-dimensional stratified algebraic variety over $\mathbb{C}$ with at most Gorenstein toroidal singularities such that for any $i \in I$ the singularities of $X$ along the stratum $X_i$ of codimension $k_i$ are defined by a $k_i$-dimensional finite rational polyhedral cone $\sigma_i$; i.e., $X$ is locally isomorphic to

$\mathbb{C}^{k-k_i} \times U_{\sigma_i}$

at each point $x \in X_i$ where $U_{\sigma_i}$ is a $k_i$-dimensional affine toric variety which is associated with the cone $\sigma_i$ (see \[3\]). Then the polynomial

$E_{\text{st}}(X; u, v) := \sum_{i \in I} E(X_i; u, v) \cdot S(\sigma_i, uv)$

is called the string-theoretic $E$-polynomial of $X$. If we write $E_{\text{st}}(X; u, v)$ in form

$E_{\text{st}}(X; u, v) = \sum_{p,q} a_{p,q} u^p v^q$,

then the numbers $h_{\text{st}}^{p,q}(X) := (-1)^{a+q}a_{p,q}$ are called the string-theoretic Hodge numbers of $X$.

Remark 4.4 Comparing with \[3\].10 and \[3\].11, the definition of the string-theoretic Hodge numbers looks as if there were a complex $ST_X^*$ whose hypercohomology groups have natural Hodge structure which assumed to be pure if $X$ is compact. We remark that the construction of such a complex $ST_X^*$ (an analog of the intersection complex) is still an open problem.
Let \( V = D_1 \cap \cdots \cap D_r \) be a generic Calabi-Yau complete intersection of \( r \) semi-ample divisors \( D_1, \ldots, D_r \) in a \( d \)-dimensional Gorenstein toric Fano variety \( X \) \((k \geq r)\). According to [4], there exists a \( d \)-dimensional reflexive polyhedron \( \Delta \) and its decomposition into a Minkowski sum

\[
\Delta = \Delta_1 + \cdots + \Delta_r,
\]

where each lattice polyhedron \( \Delta_i \) is the supporting polyhedron for global sections of a semi-ample invertible sheaf \( L_i \cong \mathcal{O}_X(D_i) \) \((i = 1, \ldots, r)\).

Definition 4.5 [7] Denote by \( E_1, \ldots, E_k \) the closures of \((d-1)\)-dimensional torus orbits in \( X \) and set \( I := \{1, \ldots, k\} \). A decomposition into a Minkowski sum \( \Delta = \Delta_1 + \cdots + \Delta_r \) as above is called a nef-partition if there exists a decomposition of \( I \) into a disjoint union of \( r \) subsets \( I_j \subset I \) \((j = 1, \ldots, r)\) such that

\[
\mathcal{O}(D_j) \cong \mathcal{O}(\sum_{l \in I_j} E_l), \quad (j = 1, \ldots, r).
\]

Now we put \( \overline{M} = \mathbb{Z}^r \oplus M, \overline{d} = d + r \), and define the \( \overline{d} \)-dimensional cone \( C \subset \overline{M}_R \) as

\[
C := \{(\lambda_1, \ldots, \lambda_r, \lambda_1 z_1 + \cdots + \lambda_r z_r) \in \overline{M}_R : \lambda_i \in \mathbb{R}_{\geq 0}, z_i \in \Delta_i, i = 1, \ldots, r\}.
\]

We extend the pairing \( \langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z} \) to the pairing between \( \overline{M} \) and \( \overline{N} := \mathbb{Z}^r \oplus N \) by the formula

\[
\langle (a_1, \ldots, a_r, m), (b_1, \ldots, b_r, n) \rangle = \sum_{i=1}^r a_i b_i + \langle m, n \rangle.
\]

Theorem 4.6 [7] Let \( \Delta = \Delta_1 + \cdots + \Delta_r \) be a nef-partition. Then it defines canonically a \( d \)-dimensional reflexive polyhedron \( \nabla \subset N_R \) and a nef-partition \( \nabla = \nabla_1 + \cdots + \nabla_r \), which are uniquely determined by the property that

\[
C^* := \{(\lambda_1, \ldots, \lambda_r, \lambda_1 z_1 + \cdots + \lambda_r z_r) \in \overline{N}_R : \lambda_i \in \mathbb{R}_{\geq 0}, z_i \in \nabla_i, i = 1, \ldots, r\}
\]

is the dual reflexive Gorenstein cone \( C^* \subset \overline{N}_R \).

Definition 4.7 [7] The nef-partition \( \nabla = \nabla_1 + \cdots + \nabla_r \) as in 4.6 is called the dual nef-partition.

We set

\[
Y := \mathbb{P}(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r).
\]

Recall the standard construction of the reduction of complete intersection \( V \subset X \) to a hypersurface \( \tilde{V} \subset Y \) [3]. Let \( \pi \) be the canonical projection \( Y \to X \) and \( \mathcal{O}_Y(-1) \) the tautological Grothendieck sheaf on \( Y \). Since

\[
\pi_* \mathcal{O}_Y(1) = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r,
\]

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we obtain the isomorphism
\[ H^0(Y, \mathcal{O}_Y(1)) \cong H^0(X, \mathcal{L}_1) \oplus \cdots \oplus H^0(X, \mathcal{L}_r). \]
Assume that \( D_i \) is the set of zeros of a global section \( s_i \in H^0(X, \mathcal{L}_i) \) (\( 1 \leq i \leq r \)). We define \( V \) as the zero set of the global section \( s \in H^0(Y, \mathcal{O}_Y) \) which corresponds to the \( r \)-tuple \( (s_1, \ldots, s_r) \) under above isomorphism. Our main interest is the following standard property ([4]):

**Proposition 4.8** The restriction of \( \pi \) on \( Y \setminus \bar{V} \) is a locally trivial \( \mathbb{C}^{r-1} \)-bundle in Zariski topology over \( X \setminus V \).

Let us set
\[ P = \text{Proj} \bigoplus_{i \geq 0} H^0(Y, \mathcal{O}_Y(i)). \]
The following statement is contained in [4]:

**Proposition 4.9** The tautological sheaf \( \mathcal{O}_Y(1) \) is semi-ample and the natural toric morphism
\[ \alpha : Y \to P \]
is crepant. Moreover, \( \mathcal{O}_Y(r) \) is the anticanonical sheaf of \( Y \), \( P \) is a Gorenstein toric Fano variety, and \( Z := \alpha(\bar{V}) \) is an ample hypersurface in \( P \).

There is the following explicit formula for \( E_{st}(V; u, v) \) in terms of \( E_{st}(P; u, v) \) and \( E_{st}(Z; u, v) \):

**Theorem 4.10**
\[ E_{st}(V; u, v) = ((uv - 1)((uv)^r - 1)^{-1})E_{st}(P; u, v) - (uv)^{1-r}E_{st}(P \setminus Z; u, v). \]

**Proof.** Since \( V \) is transversal to all toric strata in \( X \) we have:
\[ E_{st}(V; u, v) = E_{st}(X; u, v) - E_{st}(X \setminus V; u, v). \]
Using the \( \mathbb{CP}^{r-1} \)-bundle structure of \( Y \) over \( X \), we obtain:
\[ E_{st}(X; u, v) = ((uv)^r - 1)^{-1}(uv - 1)E_{st}(Y; u, v). \]
By 4.8, we also have
\[ E_{st}(X \setminus V; u, v) = (uv)^{1-r}E_{st}(Y \setminus \bar{V}; u, v). \]
Since birational crepant toric morphisms do not change string-theoretic Hodge numbers (see [3]), by 4.9, we conclude
\[ E_{st}(Y; u, v) = E_{st}(P; u, v), \quad E_{st}(Y \setminus \bar{V}; u, v) = E_{st}(P \setminus Z; u, v). \]
\( \square \)
Definition 4.11 Let $C \subset \overline{M}_\mathbb{R}$ be a reflexive Gorenstein cone, $C^* \subset \overline{N}_\mathbb{R}$ the dual reflexive Gorenstein cone. We define 

$$\Lambda(C, C^*) := \{(m, n) \in \overline{M} \oplus \overline{N} : m \in C, n \in C^*, \text{ and } \langle m, n \rangle = 0\}.$$ 

Definition 4.12 Let $(m, n)$ be an element of $\Lambda(C, C^*)$. We define the Eulerian poset $P_{(m,n)}$ as the subset of all faces $C_x \subset C$ such that $C_x$ contains $m$ and $\langle z, n \rangle = 0$ for all $z \in C_x$. We denote by $\rho(x^*(n))$ the dimension of the intersection of $C$ with the hyperplane $\langle z, n \rangle = 0$.

Remark 4.13 The dual Eulerian poset $P^*_x(m,n)$ can be identified with the subset of all faces $C^*_x \subset C^*$ such that $C^*_x$ contains $n$ and $\langle m, z \rangle = 0$ for all $z \in C^*_x$.

Theorem 4.14 Let us set $d = d + r$ and 

$$A_{(m,n)}(u,v) = \frac{(-1)^{\rho(x^*(n))}}{(uv)}(v-u)^{\rho(x(m))}B(P^*_x(m,n); u,v)(uv-1)^{d-\rho(x^*(n))}.$$ 

Then 

$$E_{st}(V; u, v) = \sum_{(m,n) \in \Lambda(C, C^*)} \left(\frac{1}{uv}\right)^{\deg m} A_{(m,n)}(u,v) \left(\frac{1}{uv}\right)^{\deg n}$$

Proof. By Definition 4.13, 

$$E_{st}(P; u, v) = \sum_{0 < x \leq 1} (uv-1)^{\rho(x)-1}S(C^*_x, uv)$$

$$= \sum_{0 < x \leq 1} (uv-1)^{\rho(x)-1}(uv-1)^{d-\rho(x)}T(C^*_x, (uv)^{-1})$$

$$= (uv-1)^{d-1} \sum_{0 < x \leq 1} \left(\sum_{n \in \text{Int}(C^*_x) \cap \overline{N}} (uv)^{-\deg n}\right) = (uv-1)^{d-1} \sum_{n \in \partial C^* \cap \overline{N}} (uv)^{-\deg n},$$

where $\partial C^* = C^* \setminus \text{Int}(C^*)$ is the boundary of $C^*$. Since $\overline{N} \cap \text{Int}(C^*) = p + \overline{N} \cap C^*$ and $\deg p = r$, we conclude:

$$E_{st}(P; u, v) = (1 - (uv)^{-r})(uv-1)^{d-1} \sum_{n \in C^* \cap \overline{N}} (uv)^{-\deg n}$$

$$= ((uv)^{r} - 1)(uv-1)^{d-1} \sum_{n \in \text{Int}(C^*) \cap \overline{N}} (uv)^{-\deg n}.$$

On the other hand, 

$$E_{st}(P \setminus \overline{Z}; u, v) = E_{st}(P; u, v) - E_{st}(\overline{Z}; u, v).$$
By Definition 4.3 and Theorem 3.18,
\[ E_{st}(\overline{Z}; u, v) = \sum_{0 < x \leq 1} \left( \frac{(uv - 1)^{\rho(x) - 1}}{uv} \right) S(C_x^*, uv) \]
\[ + \sum_{0 < x \leq 1} \frac{(-1)^{\rho(x)}}{uv} \sum_{m \in C_x \cap M} (v - u)^{\rho(x(m))} B([x(m), x]^*; u, v) \left( \frac{u}{v} \right)^{\deg m} S(C_x^*, uv) \]
\[ = (uv)^{-1} E_{st}(P; u, v) + \]
\[ + \sum_{0 < x \leq 1} \frac{(-1)^{\rho(x)}}{(uv)^r} \sum_{m \in C_x \cap M} (v - u)^{\rho(x(m))} B([x(m), x]^*; u, v) \left( \frac{u}{v} \right)^{\deg m} S(C_x^*, uv). \]

By 4.10,
\[ E_{st}(V; u, v) = ((uv - 1)((uv)^r - 1)^{-1} - (uv)^1 - (uv)^r) E_{st}(P; u, v) \]
\[ + \sum_{0 < x \leq 1} \frac{(-1)^{\rho(x)}}{(uv)^r} \sum_{m \in C_x \cap M} (v - u)^{\rho(x(m))} B([x(m), x]^*; u, v) \left( \frac{u}{v} \right)^{\deg m} S(C_x^*, uv) \]
\[ = (uv)^{-r} (uv - 1)^{\overline{d}} \sum_{n \in \text{Int}(C^*) \cap \overline{N}} (uv)^{-\deg n} \]
\[ + \sum_{0 < x \leq 1} \frac{(-1)^{\rho(x)}}{(uv)^r} \sum_{m \in C_x \cap M} (v - u)^{\rho(x(m))} B([x(m), x]^*; u, v) \left( \frac{u}{v} \right)^{\deg m} S(C_x^*, uv) \]
\[ = \sum_{0 < x \leq 1} \frac{(-1)^{\rho(x)}}{(uv)^r} \sum_{m \in C_x \cap M} (v - u)^{\rho(x(m))} B([x(m), x]^*; u, v) \left( \frac{u}{v} \right)^{\deg m} S(C_x^*, uv). \]

It remains to use the formula
\[ S(C_x^*, uv) = (uv - 1)^{\overline{d} - \rho(x)} \sum_{n \in \text{Int}(C_x^*) \cap \overline{N}} (uv)^{-\deg n} \]
\[ (0 \leq x \leq \overline{1}) \]
and notice that \( \rho(x) = \rho(x^*(n)) \) if \( n \) is an interior lattice point of \( C_x^* \) (see 4.12).

**Theorem 4.15** Let \( V \) be a \((d - r)\)-dimensional Calabi-Yau complete intersection defined by a nef-partition \( \Delta = \Delta_1 + \cdots + \Delta_r \), \( W \) a \((d - r)\)-dimensional Calabi-Yau complete intersection defined by the dual nef-partition \( \nabla = \nabla_1 + \cdots + \nabla_r \). Then
\[ E_{st}(V; u, v) = (-u)^{d-r} E_{st}(W; u^{-1}, v), \]
i.e.,
\[ h_{st}^{p,q}(V) = h_{st}^{d-r-p,q}(W) \]
\[ 0 \leq p, q \leq d - r. \]

**Proof.** If we use the duality between two \( \overline{d} \)-dimensional reflexive Gorenstein cones \( C \subset \overline{M}_R \) and \( C^* \subset \overline{N}_R \), then the statement of Theorem follows immediately from the explicit formula in 4.14 and from the duality for \( B \)-polynomials 2.13.
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