On the zeta functions on the projective complex spaces.

Mounir Hajli*

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Abstract

In this article, we study the zeta function $\zeta_q$ associated to the Laplace operator $\Delta_q$ acting on the space of the smooth $(0, q)$-forms with $q = 0, \ldots, n$ on the complex projective space $\mathbb{P}^n(\mathbb{C})$ endowed with its Fubini-Study metric. In particular, we show that the values of $\zeta_q$ at non-positive integers are rational. Moreover, we give a formula for $\sum_{q\geq 0}(-1)^{q+1}q\zeta_q'(0)$, the associated holomorphic analytic torsion.

MSC: 11M41.

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1 Introduction

Let $\mathbb{P}^n(\mathbb{C})$ be the complex projective space of dimension $n$ endowed with its Fubini-Study metric. For $q = 0, 1, \ldots, n$, we denote by $\zeta_q$ the zeta function of the Laplace operator $\Delta_q$ associated to the Fubini-Study metric and acting on $A^{0,q}(\mathbb{P}^n(\mathbb{C}))$, the space of smooth $(0, q)$-forms on $\mathbb{P}^n(\mathbb{C})$. The spectrum of $\Delta_q$ was computed by Ikeda and Taniguchi in [7].

It is known that $\zeta_q$ is a holomorphic function on $\mathbb{R}(s) > n$ and it admits a meromorphic extension to $\mathbb{C}$ which is holomorphic at $s = 0$ and has poles in $\{1, \ldots, n\}$. This follows from the general theory of Laplace operators on compact Kähler manifolds, see for instance [1].

The zeta function $\zeta_0$ associated to $\Delta_0$ associated to the Fubini-Study metric was studied in [10].

In this article, we study $\zeta_q$ for $1 \leq q \leq n$. Our first goal is to give explicit formulas for its values at non-positive integers. First, let us recall the expression of $\zeta_q$. By the computation of [7] or see [5, p. 33], we have for any $1 \leq q < n$,

$$\zeta_q(s) = \zeta_q(s) + \zeta_{q+1}(s),$$  

(1)

*Institute of Mathematics, Academia Sinica, Astronomy-Mathematics Building, 6F, No.1, Sec.4, Roosevelt Road, Taipei 10617, Taiwan. E-mail: hajli@math.sinica.edu.tw, hajlimounir@gmail.com

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and

\[ \zeta_n(s) = \overline{\zeta_n(s)}, \]

where

\[ \zeta_q(s) = \sum_{k \geq q} \frac{d_{n,q}(k)}{(k+q+1)!} \left( \frac{1}{k!} + \frac{1}{k+n+1-q} \right) \]

for \( q = 1, \ldots, n \),

with \( d_{n,q}(k) = \left( \frac{1}{k} + \frac{1}{k+n+1-q} \right) \frac{(k+n+1-q)!}{k!k!(n-q)!(q-1)!} \).

In Theorem (3.3) we give a formula for \( \zeta_q(-m) \) for any \( m \in \mathbb{N} \). The method of the proof is a generalization of the approach given in [11]. Roughly speaking, the idea of the proof consists of introducing an auxiliary function \( \xi_q \) and to write \( \zeta_q \) as an infinite sum involving the \( \xi_q \) (see Claim (3.4)). Then, we show that the computation of \( \overline{\zeta}_q(-m) \) reduces to the computation of the values of \( \xi_q \) at non-positive integers, which is done in Proposition (3.3).

When \( n = 1 \), we have a more simple expression. Namely, if we denote by \( \zeta^{(0)} \) the zeta function associated to the Fubini-Study metric on \( \mathbb{P}^1(\mathbb{C}) \), then we have

**Theorem 1.1 (Theorem (2.1)).** For any \( m \in \mathbb{N} \),

\[ \zeta^{(0)}(-m) = -\frac{1}{m+1} \sum_{k=m+1}^{2m+2} (-1)^k \left( \frac{m+1}{2m+2-k} \right) B_k, \]

where \( B_k \) is the \( k \)-th Bernoulli number.

The second goal of this article is the study of the holomorphic analytic torsion on \( \mathbb{P}^n(\mathbb{C}) \). Recall that the holomorphic analytic torsion associated to \( \mathbb{P}^n(\mathbb{C}) \) endowed with its Fubini-Study metric is by definition the following real number

\[ \sum_{q \geq 0} (-1)^{q+1} q \zeta_q'(0), \]

also called the regularized determinant. This notion was first introduced by Ray and Singer in [9]. In [8], Quillen uses the holomorphic analytic torsion in order to define a smooth metric on the determinant of cohomology of the direct image of a given holomorphic vector bundle; Roughly speaking, given \( f : (X, \omega_X) \to (Y, \omega_Y) \) a Kähler fibration between Kähler compact manifolds and \( E \) a hermitian vector bundle on \( X \) we can endow \( \chi(E) := \det(f_*E) \) (the determinant of cohomology of \( f_*E \)) with a smooth metric \( h_q \) given for any \( y \in Y \) as the product of the \( L^2 \)-metric on the fiber \( f^{-1}(y) \) and the holomorphic analytic torsion of the restriction \( E_y \) on \( f^{-1}(y) \) endowed with \( \omega_y := \omega_{X|f^{-1}(y)} \) (see also [2], [3] and [4]).

The holomorphic analytic torsion enters into the formulation of the Arithmetic Riemann Roch theorem of Gillet and Soulé (see [5] and [6]). In order to prove this theorem, one needs to compute the holomorphic analytic torsion of \( \mathbb{P}^n(\mathbb{C}) \) for the Fubini-Study metric. This is was done in [5] §2.3. They gave a formula for \( \sum_{q \geq 0} (-1)^{q+1} q \zeta_q'(0) \). One can see that \( \sum_{q \geq 0} (-1)^{q+1} q \zeta_q'(0) \) is a \( \mathbb{Q} \)-linear combination of \( \zeta_q(-m) \) and \( \zeta_q'(0) \) for even with \( 1 \leq m \leq n \) and the logarithm of a positive integer. A similar formula is given in [10] in low dimensional case.

We introduce some notations (see Paragraph (2.2) for more details). Let \( 1 \leq q \leq n \). For \( i = 0, \ldots, 2n \), we denote by \( P_i(q) \) (resp. \( Q_i(q) \)) the following integers given by

\[ \binom{l+q-1}{n} \binom{l-1}{l-n-1} = \sum_{i=0}^{2n} P_i(q) t^i \quad \forall t \in \mathbb{N}_{\geq 1}. \]
Theorem 2.1. We have, for any \( m \in \mathbb{N} \)

\[
\zeta_{p,s}(-m) = \zeta^{(0)}(-m) = -\frac{1}{m+1} \sum_{k=m+1}^{2m+2} (-1)^k \binom{m+1}{2m+2-k} B_k.
\]
where \( B_k \) is the \( k \)-th Bernoulli number.

By [11] C4, p.197, we have

\[
\zeta^{(0)}(s) = \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{(2s + k - 2)\Gamma(s + k - 1)}{\Gamma(k + 1)} \zeta_Q(2s + k - 1, 2), \tag{3}
\]

where \( \zeta_Q(\cdot, a) \) is the generalized Riemann zeta function defined for \( \Re(s) > 1 \) by

\[
\zeta_Q(s, a) := \sum_{l=0}^{\infty} \frac{1}{(l + a)^s} \quad \text{for } a \in \mathbb{N}_{\geq 1}.
\]

Recall that \( z \to \Gamma(z) \) is a meromorphic function on \( \mathbb{C} \) with simple pole at \( z = -m \) for any \( m \in \mathbb{N} \) with residue \((-1)^m/m!\). Let \( m \in \mathbb{N} \). We have in a neighbourhood of \(-m\) the following

\[
\zeta^{(0)}(s) = \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{(2s + k - 2)\Gamma(s + k - 1)}{\Gamma(k + 1)} (\zeta_Q(2s + k - 1) - 1)
\]

\[
= \frac{1}{\Gamma(s)} \sum_{k=0}^{m+1} \frac{(2s + k - 2)\Gamma(s + k - 1)}{\Gamma(k + 1)} (\zeta_Q(2s + k - 1) - 1)
\]

\[
+ \frac{1}{\Gamma(s)} \sum_{k=m+2}^{\infty} \frac{(2s + k - 2)\Gamma(s + k - 1)}{\Gamma(k + 1)} (\zeta_Q(2s + k - 1) - 1)
\]

\[
= \frac{1}{\Gamma(s)} \sum_{k=0}^{m+1} \frac{(2s + k - 2)\Gamma(s + k - 1)}{\Gamma(k + 1)} (\zeta_Q(2s + k - 1) - 1) + O(s + m)
\]

\[
= (-1)^m m! (s + m) \sum_{k=0}^{m+1} \frac{(-2m + k - 2)(-1)^{m-k}}{\Gamma(k + 1)(m + 1 - k)! (s + m)} (\zeta_Q(-2m + k - 1) - 1) + O(s + m)
\]

\[
= m! \sum_{k=0}^{m+1} \frac{(-1)^k (-2(m + 1) + k)}{k!(m + 1 - k)!} \left( - \frac{B_{2(m+1)-k}}{2(m + 1) - k} - 1 \right) + O(s + m)
\]

\[
= - \frac{1}{m + 1} \sum_{k=0}^{m+1} (-1)^k \binom{m + 1}{k} B_{2(m+1)-k} + O(s + m).
\]

Thus,

\[
\zeta^{(0)}(-m) = - \frac{1}{m + 1} \sum_{k=0}^{m+1} (-1)^k \binom{m + 1}{k} B_{2(m+1)-k} = - \frac{1}{m + 1} \sum_{k=m+1}^{2m+2} (-1)^k \binom{m + 1}{2m + 2 - k} B_k.
\tag{4}
\]

This ends the proof of Theorem (2.1).

### 3 The general case

We keep the same notations as in the introduction. For any \( n \) and \( q \) in \( \mathbb{N}_{\geq 1} \) with \( n \geq q \), we set

\[
S_{n,q}(z) = \sum_{k=q}^{\infty} \alpha_{n,q}(k) z^k \quad \forall |z| < 1, \tag{5}
\]
with \( \alpha_{n,q}(k) := \frac{1}{n!(q-1)!} \binom{k+q}{k} \frac{(k+q+n)!}{k!} \) for any \( k \geq q \).

**Claim 3.1.** Let \( 1 \leq q \leq n \). We have

\[
S_{n,q}(z) = \frac{R_{n,q}(z)}{(1-z)^{2n+1}} = \frac{(n+q-1)}{q(n+1)} z^q \quad \forall |z| < 1,
\]

where \( R_{n,q}(z) = \sum_{k=0}^{2n-1} c_{n,q,j} z^j \) is a polynomial of degree \( \leq 2n-1 \) and has a zero at \( z = 0 \) of order \( \geq q \).

**Proof.** Let \( T_{n,q}(z) = \sum_{k \geq 0} \binom{k+q+n}{k} (k+q)^k \) for any \( |z| < 1 \). It is clear that \( T_{n,q} \) converges on \( \{|z| < 1\} \). We can check easily the following equality

\[
\frac{d}{dz} \left( z^{n+2-q} \frac{d}{dz} S_{n,q}(z) \right) = z^n T_{n,q}(z) \quad \forall |z| < 1.
\]

Let \( A(z) = \sum_{k=0}^{\infty} \binom{k+q+n}{k} z^k \) and \( B(z) = \sum_{k=0}^{\infty} \binom{k+q}{k} z^k \). We have \( B(z) = \frac{1}{(1-z)^{q+1}} \) and

\[
A(z) = \frac{1}{(1-z)^n} z^q \sum_{k=0}^{\infty} \binom{k+q+n}{k} z^k \quad \text{for any } 0 < |z| < 1.
\]

Consider \( \alpha > 1 \). We have for any \( 0 < r < 1 \),

\[
\int_0^{2\pi} A(e^{i\theta}) B(re^{-i\theta}) d\theta = T_{n,q}(\frac{r}{\alpha}).
\]

Note that \( \frac{1}{(1-z)^{n+1}} = G_{n+1}(\frac{1}{z}) + G_{n+1}(\frac{1}{1-z}) \) where \( G_{n+1}(z) = \sum_{k=0}^{n} a_{n+1,k} z^k \) is a polynomial of degree \( n \) with rational coefficients. Since,

\[
A(e^{i\theta}) B(re^{-i\theta}) = \frac{\alpha^{n+1} e^{(n+1-q)\theta}}{(\alpha - e^{i\theta})^{n+1}(e^{i\theta} - r)^{n+1}} - \alpha^q \sum_{k=0}^{q-1} \binom{k+n}{k} e^{(n+1+k-q)\theta},
\]

Then,

\[
\int_0^{2\pi} A(e^{i\theta}) B(re^{-i\theta}) d\theta = \int_{|z|=1} \frac{\alpha^{n+1} z^{n+1-q}}{(\alpha - z)^{n+1}(z - r)^{n+1}} \frac{dz}{z} - \alpha^q \sum_{k=0}^{q-1} \binom{k+n}{k} \int_{|z|=1} \frac{z^{n+1+k-q}}{(z - r)^{n+1}} \frac{dz}{z} \quad (7)
\]

It is clear that \( \frac{1}{(z-\frac{\alpha}{z-\frac{\alpha}{z}})^{n+1}} = \frac{1}{(z-\frac{\alpha}{z})^{n+1}} (G_{n+1}(\frac{\alpha-z}{z-r}) + G_{n+1}(\frac{\alpha-z}{z-r})) \). This gives

\[
\int_0^{2\pi} A(e^{i\theta}) B(re^{-i\theta}) d\theta = \frac{\alpha^{n+1}}{(\alpha - r)^{2n+2}} \int_{|z|=1} z^{n+1-q} \left(G_{n+1}(\frac{\alpha-r}{z-r}) + G_{n+1}(\frac{\alpha-r}{z-\alpha})\right) \frac{dz}{z} - \alpha^q \sum_{k=0}^{q-1} \binom{q-1+k}{n} \int_{|z|=1} z^{n+1+k-q} \frac{dz}{z} \quad (8)
\]

\[
= \frac{\alpha^{n+1}}{(\alpha - r)^{2n+2}} \int_{|z|=1} z^{n+1-q} G_{n+1}(\frac{\alpha-r}{z-r}) \frac{dz}{z} - \alpha^q \binom{n+q-1}{n} \quad (9)
\]

\[
= \frac{\alpha^{n+1}}{(\alpha - r)^{2n+2}} \sum_{k=0}^{n} a_{n+1,k}(\alpha-r)^{k+1} \int_{|z|=1} z^{n+1-q-k} \frac{dz}{z} - \alpha^q \binom{n+q-1}{n} \quad (10)
\]

\[
= \frac{\alpha^{n+1}}{(\alpha - r)^{2n+2}} \sum_{k=0}^{n+1-q} a_{n+1,k}(\alpha-r)^{k+1} \binom{n+1-q-k}{k} r^{n+1-q-k} - \alpha^q \binom{n+q-1}{n}.
\]
It follows that

\[ T_{n,q}(r) = \frac{\alpha^{n+1} \sum_{k=0}^{n+1-q} a_{n+1,k+1}(\alpha - r)^{k+1}(n+1-q)}{(\alpha - r)^{2n+2}} = \frac{P_{n,q}(r)}{(1-r)^{2n+1}} \alpha (n+q-1) \]

We let \( \alpha \to 1 \). Then we obtain the following

\[ T_{n,q}(r) = \frac{1}{(1-r)^{2n+1}} \sum_{k=0}^{n+1-q} a_{n+1,k+1}(1-r)^k(n+1-q)_k - \frac{n+q-1}{n} \]

That is

\[ T_{n,q}(z) = \frac{P_{n,q}(z)}{(1-z)^{2n+1}} - \frac{n+q-1}{n} \] \((\forall z < 1)\)

where \( P_{n,q}(z) = \sum_{k=0}^{n+1-q} a_{n+1,k+1}(n+1-q)_k(1-z)^k \). Now using \((7)\), we obtain

\[
\frac{d}{dz} \frac{z^{n+2-q} dz}{dz} S_{n,q}(z) = z^n P_{n,q}(z) = \frac{(n+q-1)}{n} z^n - \sum_{k=0}^{n+1} a_{n+1,k+1} \frac{(n+1-q)_k}{(1-z)^{2n+1}} - \frac{n+q-1}{n} z^n.
\]

This equality gives

\[
\frac{d}{dz} \frac{z^{n+2-q} dz}{dz} S_{n,q}(z) = \frac{z^m Q_{n,q}(z)}{(1-z)^{2n+1}} - \frac{(n+q-1)_n}{(n+1)} z^{n+q-1}.
\]

with \( m \) is a nonnegative integer and \( Q_{n,q} \) is a polynomial such that \( m + \text{deg}(Q_{n,q}) \leq 2n \) and \( Q_{n,q}(0) \neq 0 \). Since \( S_{n,q} \) has a zero of order \( q \) at \( z = 0 \), then \( n + 2 - q + 1 = n + 1 \leq m + \text{deg}(Q_{n,q}) \leq 2n \). So,

\[
\frac{d}{dz} S_{n,q}(z) = \frac{z^{m-n+q-2} Q_{n,q}(z)}{(1-z)^{2n+1}} - \frac{(n+q-1)_n}{(n+1)} z^{q-1},
\]

and \( \text{deg}(Q_{n,q}) \leq n - 1 \). We conclude that there exists a rational polynomial \( R_{n,q} \) of degree \( \leq 2n - 1 \) such that

\[
S_{n,q}(z) = \frac{R_{n,q}}{(1-z)^{2n+1}} - \frac{(n+q-1)_n}{q(n+1)} z^q.
\]

And, we can see that the order of \( R_{n,q} \) at \( z = 0 \) is \( \geq q \).

We set

\[
\xi_q(s) = \frac{1}{1(s)} \int_0^\infty t^{s-1} S_{n,q}(e^{-t}) e^{-(n+1-q)t} dt \quad \forall \Re(s) \gg 1.
\]

From Claim \((3.1)\), we get

\[
\xi_q(s) = \sum_{k=q}^{2n-1} E_{n,q,k} \frac{e^{-(n+1-q)t}}{q(n+1)^{s+1}}.
\]
where \( \zeta_{2n-1}(s, a) \) is the multiple Hurwitz zeta function.

We write

\[
\left( \frac{z + 2n - 2}{2n - 2} \right) = \frac{1}{(2n - 2)!} (z + 2n - 2)(z + 2n - 3) \cdots (z + 1) = \sum_{i=0}^{2n-2} b_{2n-1,i} z^i,
\]

where \( b_{k,i} \) are rational numbers related to Stirling numbers.

**Claim 3.2.** Let \( k \in \mathbb{N} \). We have for any \( l \in \mathbb{N} \)

\[
\zeta_{2n-1}(-l, k + n + 1 - q) = \sum_{i=0}^{2n-2} b_{2n-1,i} \left( \sum_{p=0}^{\lfloor i/2 \rfloor} (q - n - 1 - k)^{i-p} \binom{i}{p} B_{l+p+1}(k + n + 1 - q) \right),
\]

where \( B_t(\cdot) \) is the \( t \)-th Bernoulli polynomial. Moreover, the residue of \( \zeta_{2n-1}(s, k + n + 1 - q) \) at \( s = l \in \{1, \ldots, 2n - 1\} \) is equal to

\[
\sum_{i=l-1}^{2n-2} b_{2n-1,i} \left( \sum_{p=0}^{\lfloor i/2 \rfloor} (q - n - 1 - k)^{i-p} \binom{i}{p} \right) \zeta_Q(s - p, \alpha).
\]

**Proof.** Let \( \alpha \) be a positive integer. We have

\[
\zeta_{2n-1}(s, \alpha) = \sum_{k=0}^{\infty} \frac{(k+2n-2)}{(2n-2)!} \frac{k^s}{(k + \alpha)^s} = \sum_{i=0}^{2n-2} b_{2n-2,i} \sum_{k=0}^{\infty} \frac{k^i}{(k + \alpha)^s} = \sum_{i=0}^{2n-2} b_{2n-1,i} \sum_{k=0}^{\infty} \frac{(k + \alpha - \alpha)^i}{(k + \alpha)^s}
\]

\[
= \sum_{i=0}^{2n-2} b_{2n-1,i} \sum_{p=0}^{\lfloor i/2 \rfloor} (-\alpha)^{i-p} \binom{i}{p} \zeta_Q(s - p, \alpha).
\]

Then the claim follows from the following well-known formula

\[
\zeta_Q(-l, \alpha) = -\frac{B_{l+1}(\alpha)}{l+1},
\]

and the fact that \( \zeta_Q \) has residue 1 at \( s = 1 \).

For any \( l \in \mathbb{N} \) and \( 0 \leq p \leq i \leq 2n - 2 \) we set

\[
\beta_{n,q}(l, i, p) := \sum_{k=q}^{2n-1} c_{n,q,k} (q - n - 1 - k)^{i-p} B_{l+p+1}(k + n + 1 - q),
\]

and for any \( l \in \{1, \ldots, 2n - 1\} \)

\[
\gamma_{n,q}(l) := \sum_{k=q}^{2n-1} c_{n,q,k} \sum_{i=l-1}^{2n-2} b_{2n-1,i} \left( \sum_{p=0}^{\lfloor i/2 \rfloor} (q - n - 1 - k)^{i-p} \binom{i}{p} \right) \zeta_Q(s - p, \alpha).
\]
Proposition 3.3. The function $\xi_q$ is holomorphic on $\Re(s) > 2n - 1$ and admits a meromorphic extension to $\mathbb{C}$ such that its poles are included into $\{1, 2, \ldots, 2n - 1\}$. Moreover, the residue of $\xi_q$ at $l = 1, \ldots, 2n - 1$ is a rational number equal to $\gamma_{n,q}(l)$, and we have

$$\xi_q(-l) = \sum_{i=0}^{2n-2} b_{2n-1,i} \left( \frac{i}{p} \right) \frac{\beta_{n,q}(l, i, p)}{p + l + 1} - \frac{(n+q-1)}{n}(n+1)^{l-1} \quad \forall l \in \mathbb{N}.$$

Proof. The proof follows from Equality (12), the fact that $\xi_q$ has residue 1 at $s = 1$ and Claim 3.2.

Claim 3.4. We have for any $1 \leq q \leq n$, and $\Re(s) > n$

$$\zeta_q(s) = \frac{1}{\Gamma(s)} \sum_{j=0}^{\infty} (n+1-q)^j \frac{\Gamma(s+j-1)(2s-2+j)}{\Gamma(j+1)} \zeta_q(2s+j-1).$$

Proof. Obviously, we have

$$\zeta_q(s) = \sum_{k \geq q} \frac{1}{\Gamma(k+q)} \left( \frac{1}{k^{s+1}(k+n+1-q)^{s+1}} + \frac{1}{k^{s+1}(k+n+1-q)^{s+1}} \right). \quad (16)$$

Recall that $\Gamma(s)l^{-s} = \int_0^\infty t^{s-1}e^{-lt}dt$. Then,

$$\zeta_q(s) = \frac{1}{\Gamma(s-1)\Gamma(s)} \int_0^\infty \int_0^1 (uv)^{s-2}(u+v) \sum_{k \geq q+1} \frac{\alpha_{n,q}(k)}{\Gamma(k+q)} \exp(-k(u+v) - (n+1-q)v) du dv$$

$$= \frac{1}{\Gamma(s-1)\Gamma(s)} \int_0^\infty \int_0^1 (\theta(1-\theta))^{s-2} \sum_{k \geq q+1} \frac{\alpha_{n,q}(k)}{\Gamma(k+q)} \exp(-kt)(1-\theta)(1-\theta) t d\theta dt$$

$$= \frac{1}{\Gamma(s-1)\Gamma(s)} \int_0^\infty \int_0^1 (\theta(1-\theta))^{s-2} \sum_{k \geq q+1} \frac{\alpha_{n,q}(k)}{\Gamma(k+q)} \exp(-(n+1-q)\theta) t d\theta dt$$

We set $u := \theta t$ and $v := (1-\theta) t$. We have

$$\int_0^1 (\theta(1-\theta))^{s-2} \exp((n+1-q)\theta) d\theta = \sum_{j=0}^{\infty} (n+1-q)^j \int_0^1 (\theta^{s+j-2}(1-\theta)^{s-2} d\theta) t^j \quad (17)$$

$$= \sum_{j=0}^{\infty} (n+1-q)^j \Gamma(s+j-1) \Gamma(j+1) \Gamma(2s-2+j) \Gamma(j+1) \Gamma(2s-2+j) t^j. \quad (18)$$

Thus,

$$\zeta_q(s) = \frac{1}{\Gamma(s)} \sum_{j=0}^{\infty} (n+1-q)^j \frac{\Gamma(s+j-1)(2s-2+j)}{\Gamma(j+1)\Gamma(2s-2+j)} \int_0^\infty t^{2s+j-2} S_{n,q}(e^{-t}) \exp(-(n+1-q)t) dt.$$

Therefore,

$$\zeta_q(s) = \frac{1}{\Gamma(s)} \sum_{j=0}^{\infty} (n+1-q)^j \frac{\Gamma(s+j-1)(2s-2+j)}{\Gamma(j+1)} \zeta_q(2s+j-1).$$
3.1 The values of $\zeta_q$ at non-positive integers

In this paragraph, we study the zeta function $\zeta_q$ at non-positive integers, also the value of its derivative at $s = 0$. The following theorem gives formulas for the values of $\zeta_q$ and thus of $\zeta$ at non-positive integers.

**Theorem 3.5.** For any $1 \leq q \leq n$ and any $m \in \mathbb{N}$,

$$
\zeta_q(-m) = \sum_{j=0}^{2m+1} (-1)^{j+1} \frac{(n+1-q)^j}{m^j} \left( \frac{m+1}{j} \right) \xi_q(-2m + j - 1)
$$

$$
+ (-1)^m m! \sum_{j=2m+2}^{2n+2m} (n+1-q)^j \frac{\Gamma(-m+j-1)(-2m - 2 + j)}{\Gamma(j+1)} \gamma_{n,q}(-2m + j - 1),
$$

Recall that the value of $\xi_q(-2m + j - 1)$ is given in Proposition (3.3). In particular,

$$
\zeta_q(-m) \in \mathbb{Q}.
$$

**Proof.** Let $m \in \mathbb{N}$. By Claim (3.4) we have for any $s$ in a small open neighborhood of $-m$ the following

$$
\zeta_q(s) = \frac{1}{\Gamma(s)} \sum_{j=0}^{2m+1} (n+1-q)^j \frac{\Gamma(s+j-1)(2s - 2 + j)}{\Gamma(j+1)} \xi_q(2s + j - 1)
$$

$$
+ \frac{1}{\Gamma(s)} \sum_{j=2m+2}^{2n+2m} (n+1-q)^j \frac{\Gamma(s+j-1)(2s - 2 + j)}{\Gamma(j+1)} \xi_q(2s + j - 1)
$$

(20)

$$
+ \frac{1}{\Gamma(s)} \sum_{j=2n+2m+1}^{\infty} (n+1-q)^j \frac{\Gamma(s+j-1)(2s - 2 + j)}{\Gamma(j+1)} \xi_q(2s + j - 1).
$$

First, we can show by a classical argument that

$$
\sum_{j=2n+2m+1}^{\infty} (n+1-q)^j \frac{\Gamma(s+j-1)(2s - 2 + j)}{\Gamma(j+1)} \xi_q(2s + j - 1)
$$

converges normally on $\Re(s) > -m$. Thus, the third sum in (20) vanishes at $s = -m$.

The first sum is equal at $s = -m$ to $\sum_{j=0}^{2m+1} (-1)^{j+1} \frac{(n+1-q)^j}{m^j} \left( \frac{m+1}{j} \right) \xi_q(-2m + j - 1)$. For $2m+2 \leq j \leq 2n+2m$, the function $\xi_q(2s + j - 1)$ may have a pole at $s = -m$. In this case, we know (see Proposition (3.3)) that

$$
\xi_q(2s + j - 1) = \frac{\gamma_{n,q}(-2m + j - 1)}{s+1} + o(1), \quad 2m+2 \leq j \leq 2n+2m.
$$

as $s \to -m$.

Gathering all these computations, we obtain that

$$
\zeta_q(-m) = \sum_{j=0}^{2m+1} (-1)^{j+1} \frac{(n+1-q)^j}{m^j} \left( \frac{m+1}{j} \right) \xi_q(-2m + j - 1)
$$

$$
+ (-1)^m m! \sum_{j=2m+2}^{2n+2m} (n+1-q)^j \frac{\Gamma(-m+j-1)(-2m - 2 + j)}{\Gamma(j+1)} \gamma_{n,q}(-2m + j - 1)
$$

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Recall that, by Proposition (3.3), we have

\[ \zeta_q(-2m + j - 1) = \sum_{i=0}^{2n-2} b_{2n-1,i} \sum_{p=0}^{i} \begin{pmatrix} i \\ p \end{pmatrix} \beta_{n,q}(2m - j + 1, i, p) \frac{(n+q-1)_j}{q} (n+1)^{2m-j}. \]

We conclude that \( \zeta_q(-m) \in \mathbb{Q} \).

Let \( \Re(s) > 2n - 1 \) and \( 1 \leq q \leq n \), we set

\[ \eta_q(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} S_{n,q}(e^{-t}) dt. \]

Then,

\[ \eta_q(s) = \sum_{k=q}^{2n-1} c_{n,q,k} \zeta_{2n-1}(s,k) = \frac{(n+q-1)}{(n+1)^q} \cdot (s+1). \] (21)

By a similar argument as in Proposition (3.3) we can show that \( \eta_q \) is holomorphic on \( \Re(s) > 2n - 1 \) and admits a meromorphic extension to \( \mathbb{C} \) with poles in \( \{1, \ldots, 2n - 1\} \).

The following theorem provides the first formula for \( \zeta_q(0) \). This formula will be simplified in the sequel.

**Theorem 3.6.** For any \( 1 \leq q \leq n \), we have

i. \[ \zeta_q'(-1) = \sum_{k=q}^{2n-1} \sum_{i=0}^{2n-2} b_{2n-1,i} \sum_{p=0}^{i} \begin{pmatrix} i \\ p \end{pmatrix} \zeta_q(-p-1) + \frac{(n+q-1)}{q} \log(n+1), \] (22)

ii. \[ \zeta_q'(-1) = \sum_{k=q}^{2n-1} \sum_{i=0}^{2n-2} b_{2n-1,i} \sum_{p=0}^{i} \begin{pmatrix} i \\ p \end{pmatrix} \zeta_q(-p-1) + \frac{(n+q-1)}{q} \log(n+1), \] (23)

\[ \eta_q'(-1) = \sum_{k=q}^{2n-1} \sum_{i=0}^{2n-2} b_{2n-1,i} \sum_{p=0}^{i} \begin{pmatrix} i \\ p \end{pmatrix} \zeta_q(-p-1) + \frac{(n+q-1)}{q} \log(n+1), \] (24)

\[ \eta_q'(-1) = \sum_{k=q}^{2n-1} \sum_{i=0}^{2n-2} b_{2n-1,i} \sum_{p=0}^{i} \begin{pmatrix} i \\ p \end{pmatrix} \zeta_q(-p-1) + \frac{(n+q-1)}{q} \log(n+1). \] (25)

ii. We have the following expression for \( \zeta_q(0) \)

\[ \zeta_q(0) = 6\zeta_q'(-1) - 3(n+1-q)\zeta_q'(0) - 2\eta_q'(-1) + (n+1-q)\eta_q'(0) + 2(n+1-q)\zeta_q(0) + \sum_{j=2}^{2n} (n+1-q)^j w_j \gamma_{n,q}(j-1). \]

where \( w_j \) is a rational number equal to \( \frac{\Gamma(s+j-1)/(2s+2+j)}{\Gamma(s+1)(s+1)} \) for \( j = 2, \ldots, 2n \).
Proof. i. These expressions follow directly from (12), (13) and (21).

ii. From Claim (3.4) and using similar arguments as before, we have for any \( s \) in a small open neighborhood of \( 0 \), the following equality

\[
\mathcal{R}_q(s) = 2\xi_q(2s - 1) + (n + 1 - q)(2s - 1)\xi_q(2s) + \sum_{j=2}^{2n} (n + 1 - q)^j \frac{\Gamma(s + j - 1)(2s - 2 + j)}{\Gamma(j + 1)} (\gamma_{n,q}(j - 1) \\
+ \text{so}(1)) + \frac{1}{\Gamma(s)} \sum_{j=2n+1}^{\infty} (n + 1 - q)^j \frac{\Gamma(s + j - 1)(2s - 2 + j)}{\Gamma(j + 1)} \xi_q(2s + j - 1).
\]

Observe that \( \sum_{j=2n+1}^{\infty} (n + 1 - q)^j \frac{\Gamma(s + j - 1)(2s - 2 + j)}{\Gamma(j + 1)} \xi_q(2s + j - 1) \) converges normally in an open neighbourhood of \( s = 0 \). Then, we obtain easily the following

\[
\mathcal{R}_q(0) = 4\xi_q(0) - 2(n + 1 - q)(\xi_q(0) - \xi_q'(0)) + \sum_{j=2}^{2n} (n + 1 - q)^j w_j \gamma_{n,q}(j - 1)
\]

\[
+ \sum_{j=2n+1}^{\infty} (n + 1 - q)^j \frac{j - 2}{j(j - 1)} \xi_q(j - 1),
\]

where \( w_j = \frac{d}{dt} \left( \frac{\Gamma(s + j - 1)(2s - 2 + j)}{\Gamma(j + 1)(s + 1)} \right) \bigg|_{s=0} \) which is clearly a rational number.

Let us evaluate the sum \( \sum_{j=2n+1}^{\infty} (n + 1 - q)^j \frac{j - 2}{j(j - 1)} \xi_q(j - 1) \). Notice that \( \sum_{j=2n+1}^{\infty} \frac{j - 2}{j(j - 1)} t^{j-2} = (t^{-1} - 2t^{-2}) (e^t - \sum_{j=0}^{2n-1} \frac{t^j}{j!}) + 2 \frac{t^{2n-2}}{(2n)!} \) for any \( t > 0 \). This gives

\[
\int_0^{\infty} \left[ ((n + 1 - q)t^{n-1} - 2t^{-2}) (e^{(n+1-q)t} - \sum_{j=0}^{2n-1} (n + 1 - q)^j \frac{t^j}{j!}) + 2(n + 1 - q)^{2n} S_{n,q}(e^{-t}) e^{-(n+1-q)t} \right] dt
\]

We introduce the following function \( \Omega \) given on \( \Re(s) > 1 \) by

\[
\Omega(s) := (n + 1 - q)\Gamma(s)\eta_q(s) - 2\Gamma(s - 1)\eta_q(s - 1) - \sum_{j=0}^{2n-1} (n + 1 - q)^j \left[ (n + 1 - q)\Gamma(s + j)\xi_q(s + j) \right. \\
- 2\Gamma(s + j - 1)\xi_q(s + j - 1) + 2(n + 1 - q)(s + 2n - 2)\xi_q(s + 2n - 2) \\
\left. - 4(n + 1 - q)^{2n} \Gamma(s + 2n - 3)\xi_q(s + 2n - 3) \right].
\]

We can see easily that \( \Omega \) is holomorphic on \( \Re(s) > 2n \) and

\[
\Omega(s) = \int_0^{\infty} \left[ ((n+1-q)t^{n-1}-2t^{-2}) (e^{(n+1-q)t} - \sum_{j=0}^{2n-1} (n + 1 - q)^j \frac{t^j}{j!}) + 2(n + 1 - q)^{2n} t^{2n-2} S_{n,q}(e^{-t}) e^{-(n+1-q)t} \right] dt
\]
In particular $\Omega$ admits a meromorphic extension to $\mathbb{C}$ which is moreover holomorphic at $s = 0$ (this follows from (26)).

From (27) we have
\[
\Omega(s) = (n + 1 - q)\Gamma(s + 1)\eta_q'(0) - 2\Gamma(s + 1)\eta_q'(-1) + (n + 1 - q)\Gamma(s)\eta_q(0) - 2\Gamma(s - 1)\eta_q(-1) + h_0(s)
\]
\[- (n + 1 - q)\Gamma(s)\xi_q(0) - (n + 1 - q)\Gamma(s + 1)\xi_q'(0) + 2\Gamma(s - 1)\xi_q(-1) + 2\Gamma(s + 1)\xi_q'(0) + h_0(s)
\]
\[- \sum_{j=1}^{2n-1} \frac{(n + 1 - q)j}{j!} \left[ (n + 1 - q)\Gamma(s + j)\frac{\gamma_{n,q}(j)}{s} - 2\Gamma(s + j - 1)\frac{\gamma_{n,q}(j - 1)}{s} \right] + h_2(s)
\]
\[+ 2(n + 1 - q)^{2n+1} \Gamma(s + 2n - 2)\frac{\gamma_{n,q}(2n - 2)}{s} - 4(n + 1 - q)^{2n} \Gamma(s + 2n - 3)\frac{\gamma_{n,q}(2n - 3)}{s} + h_3(s).
\]

where $h_0$, $h_1$, $h_2$ and $h_3$ are smooth functions in a open neighborhood of $s = 0$, and $h(0) = h_1(0) = h_2(0) = h_3(0) = 0$.

Thus,
\[
\Omega(s) = (n + 1 - q)\eta_q'(0) - 2\eta_q'(-1) + (n + 1 - q)\xi_q'(0) + 2\xi_q'(-1) + \frac{w}{s} + h(s),
\]

where $w$ is a rational number and $h$ is a smooth function in a open neighborhood of $s = 0$ such that $h(0) = 0$. We claim that $w = 0$. Indeed, the formula (28) and Equation (26) show that the meromorphic extension of $\Omega$ is in fact holomorphic at $s = 0$, that is $w = 0$.

Then,
\[
\sum_{j=2}^{\infty} (n + 1 - q)^j \frac{j - 2}{j(j - 1)} \xi_q(j - 1) = \Omega(0) = (n + 1 - q)\eta_q'(0) - 2\eta_q'(-1) + (n + 1 - q)\xi_q'(0) + 2\xi_q'(-1).
\]

We conclude that,
\[
\tilde{\zeta}_q(0) = 4\xi_q'(-1) + 2(n + 1 - q)(\xi_q(0) - \xi_q'(0)) + \sum_{j=2}^{2n} (n + 1 - q)^j w_j \gamma_{n,q}(j - 1)
\]
\[- (n + 1 - q)\eta_q'(0) - 2\eta_q'(-1) - (n + 1 - q)\xi_q'(0) + 2\xi_q'(-1)
\]
\[= 6\xi_q'(-1) - 3(n + 1 - q)\xi_q'(0) - 2\eta_q'(-1) + (n + 1 - q)\eta_q'(0) + 2(n + 1 - q)\xi_q(0)
\]
\[+ \sum_{j=2}^{2n} (n + 1 - q)^j w_j \gamma_{n,q}(j - 1).
\]

This ends the proof of the theorem.

\[\square\]

### 3.2 Toward a formula for the regularized determinant

In this paragraph, we propose an alternative approach for the computation of $\sum_{q=0}^{\infty} (-1)^q q^{\zeta_q'(0)}$. We will prove that $\tilde{\zeta}_q(0)$, and hence $\xi_q'(0)$, is given in terms of the derivatives of $\zeta_q$ at non-positive integers.
Let $1 \leq q \leq n$. There exist integers denoted $P_i(q) \in \mathbb{Z}$ for $i = 0, \ldots, 2n$ such that

$$
(\frac{l + q - 1}{n})(\frac{l - 1}{l - n - 1}) = \sum_{i=0}^{2n} P_i(q) l^i \quad \forall l \in \mathbb{N}_{\geq 1}.
$$

(29)

$P_i$ can be seen as a polynomial in the variable $q$. We can prove that its degree is $\leq 2n - i$. In fact, it suffices to note that $P_i(q)$ is expressed in terms of the zeros of the polynomial $(\frac{l + q - 1}{n})(\frac{l - 1}{l - n - 1})$ with $l$ as a variable.

Similarly, we define $Q_i(q) \in \mathbb{Z}$ for $i = 0, \ldots, 2n$ verifying

$$
(\frac{l + n}{n})(\frac{l - q + n}{l - q}) = \sum_{i=0}^{2n} Q_i(q) l^i \quad l \in \mathbb{N}_{\geq q}.
$$

(30)

Also, $Q_i$ is a polynomial of degree $\leq 2n - i$.

We set $\tilde{\alpha}_{n,q}(l) := \alpha_{n,q}(l - n - 1 + q)$ for any $l \geq n + 1 - q$ (see (5) for the definition of $\alpha_{n,q}$).

From Claim (3.1), we obtain the following

$$
\alpha_{n,q}(l) = \sum_{k=q}^{l} c_{n,q,k} \binom{l - k + 2n - 2}{2n - 2}
$$

$$
= \sum_{k=q}^{l} c_{n,q,k} \sum_{i=0}^{2n-2} b_{2n-1,i} (l - k)^i \quad \forall l \geq 2n - 1.
$$

Which is equivalent to

$$
\alpha_{n,q}(l) = \sum_{k=q}^{2n-2} c_{n,q,k} \sum_{i=0}^{2n-2} b_{2n-1,i} \sum_{p=0}^{i} (-k)^{i-p} \binom{i}{i-p} l^p \quad l \geq 2n - 1.
$$

In a similar way, we obtain

$$
\tilde{\alpha}_{n,q}(l) = \sum_{k=q}^{2n-2} c_{n,q,k} \sum_{i=0}^{2n-2} b_{2n-1,i} \sum_{p=0}^{i} (q - n - 1 - k)^{i-p} \binom{i}{i-p} l^p \quad \forall l \geq 2n - 1.
$$

We denote by $c_{q,i}^{(n)}$ (resp. $\tilde{c}_{q,i}^{(n)}$) the $i$-th coefficient of $\alpha_{n,q}$ (resp. $\tilde{\alpha}_{n,q}$). That is,

$$
\alpha_{n,q}(l) = \sum_{i=0}^{2n-2} c_{q,i}^{(n)} l^i \quad \text{and} \quad \tilde{\alpha}_{n,q}(l) = \sum_{i=0}^{2n-2} \tilde{c}_{q,i}^{(n)} l^i \quad \forall l \geq 2n - 1.
$$

(31)

Claim 3.7. Let $1 \leq q \leq n$. We have,
1. 

\[
\xi_q(-1) = \sum_{p=0}^{2n-2} \tilde{c}_{q,p}^{(n)} \zeta_q(-p - 1) + \frac{(n+q-1)}{n} \log(n + 1),
\]

(32) 

\[
\xi'_q(0) = \sum_{p=0}^{2n-2} \tilde{c}_{q,p}^{(n)} \zeta'_q(-p) + \frac{(n+q-1)}{q(n+1)} \log(n + 1),
\]

(33) 

\[
\eta'_q(-1) = \sum_{p=0}^{2n-2} \tilde{c}_{q,p}^{(n)} \zeta'_q(-p - 1) + \frac{(n+q-1)}{n} \log(q),
\]

(34) 

\[
\eta'_q(0) = \sum_{p=0}^{2n-2} \tilde{c}_{q,p}^{(n)} \zeta'_q(-p) + \frac{(n+q-1)}{(n+1)q} \log(q).
\]

(35) 

2. 

\[
\zeta_q(0) = (6\tilde{c}_{q,2n-2}^{(n)} - 2c_{q,2n-2}^{(n)}) \zeta_q(-2n + 1) + (n + 1 - q)(-3\tilde{c}_{q,0}^{(n)} + c_{q,0}^{(n)}) \zeta'_q(0)
\]

\[
+ \sum_{p=1}^{2n-2} (6\tilde{c}_{q,p-1}^{(n)} - 3(n + 1 - q)c_{q,p-1}^{(n)} - 2c_{q,p-1}^{(n)} + (n + 1 - q)c_{q,p}^{(n)}) \zeta_q(-p)
\]

\[
+ 3(n + 1 + q) \frac{(n+q-1)}{q(n+1)} \log(n + 1) + (n + 1 - 3q) \frac{(n+q-1)}{q(n+1)} \log(q)
\]

\[
+ 2(n + 1 - q)\xi_q(0) + \sum_{j=2}^{2n} (n + 1 - q)^j w_j \gamma_{n,q}(j - 1).
\]

Recall that \(\xi_q(0)\) and \(\gamma_{n,q}(j - 1)\) are given explicitly in Proposition \([3.3]\).

Proof. 1. By definition,

\[
c_{q,p}^{(n)} = \sum_{k=q}^{2n-1} c_{n,q,k} \sum_{i \geq p} b_{2n-1,i} (q - n - 1 - k)^{i-p} \binom{i}{p},
\]

and

\[
c_{q,p}^{(n)} = \sum_{k=q}^{2n-1} c_{n,q,k} \sum_{i \geq p} b_{2n-1,i} (-k)^{i-p} \binom{i}{p}.
\]

Then 1. follows from Theorem \([3.6]\).

2. The expression of \(\tilde{\zeta}_q(0)\) follows from Theorem \([3.6]\) and 1.

The following proposition gives an explicit formula for the rational numbers \(c_{q,p}^{(n)}\) and \(\tilde{c}_{q,p}^{(n)}\) for \(p = 0, \ldots, 2n - 2\).

**Proposition 3.8.** For any \(n \geq 1\) and \(1 \leq q \leq n\), we have

1. 

\[
c_{q,j}^{(n)} = \frac{(-1)^{q-j}}{(n)!^2} \sum_{i=j+2}^{2n} (-1)^i Q_i(q)(n + 1 - q)^{i-2-j},
\]

(36)
\[ \sum_{q=1}^{n} (-1)^q c_{q,j}^{(n)} = 0 \quad \text{and} \quad \sum_{q=1}^{n} (-1)^q \tilde{c}_{q,j}^{(n)} = 0 \quad \forall j \geq n. \]

**Proof.** Let \( l \gg 1 \), we have

\[
\alpha_{n,q}(l) = \frac{{l+n \choose l} {l-q+n \choose l-q}}{n!(q-1)!(n-q)!(l+n+1-q)} = \frac{l^{2n-2}}{n!(n-1)!(n-q)!} \sum_{i=0}^{2n} Q_{2n-i}(q) \frac{1}{i!} \sum_{l=0}^{\infty} \frac{(-1)^i (n+1-q)^i}{l^i} = \frac{l^{2n-2}}{n!(n-1)!(n-q)!} \sum_{i=0}^{\infty} \sum_{j=0}^{i} Q_{2n-i}(q) (-1)^i (n+1-q)^{i-j} \frac{1}{l^i},
\]

and

\[
\tilde{\alpha}_{n,q}(l) = \frac{{l-1+q \choose l-1-n+1} {l-1 \choose l-n+1}}{n!(l-1)!(l-n+1+q)} = \frac{l^{2n-2}}{n!(n-1)!(q-n)!} \sum_{i=0}^{2n} P_{2n-i}(q) \frac{1}{i!} \sum_{l=0}^{\infty} \frac{(n+1-q)^i}{l^i} = \frac{l^{2n-2}}{n!(n-1)!(n-q)!} \sum_{i=0}^{\infty} \sum_{j=0}^{i} P_{2n-i}(q) (n+1-q)^{i-j} \frac{1}{l^i}.
\]

For any \( j = 0, 1, \ldots, 2n-2 \), we deduce that

\[
\tilde{c}_{q,j}^{(n)} = \frac{1}{n!(q-1)!(n-q)!} \sum_{i=0}^{2n-2-j} Q_{2n-i}(q) (-1)^i (n+1-q)^{2n-2-j-i} = \left( \frac{n}{q-1} \right) \frac{1}{(n!)^2} \sum_{i=j+2}^{2n} (-1)^i Q_i(q) (n+1-q)^{i-2-j}.
\]

and,

\[
\tilde{c}_{q,j}^{(n)} = \frac{1}{n!(q-1)!(n-q)!} \sum_{i=0}^{2n-2-j} P_{2n-i}(q) (n+1-q)^{2n-2-j-i} = \left( \frac{n}{q-1} \right) \frac{1}{(n!)^2} \sum_{i=j+2}^{2n} P_i(q) (n+1-q)^{i-2-j}.
\]
In particular, \( c_{q,j}^{(n)} = 0 \) and \( \tilde{c}_{q,j}^{(n)} = 0 \) for \( j \geq 2n - 1 \). If \( j \geq n \), we have \( \sum_{i=j+2}^{2n}(-1)^i Q_i(q)(n+1-q)^{i-2-j} \) (resp. \( \sum_{i=j+2}^{2n} P_i(q)(n+1-q)^{i-2-j} \)), as a polynomial with \( q \) as an indeterminate has a degree \( \leq 2n - 2 - j \leq n - 2 \). Then by a classical argument of analysis we get
\[
\sum_{q=1}^{n} (-1)^q c_{q,j}^{(n)} = 0 \quad \text{and} \quad \sum_{q=1}^{n} (-1)^q \tilde{c}_{q,j}^{(n)} = 0 \quad \forall j \geq n.
\]

\( \square \)

**Corollary 3.9.** Let \( 1 \leq q \leq n \), we have

1. 
\[
\zeta_q(0) = 1 - \frac{n}{(n+1)q^n} \zeta_q^n(-2n+1) + \frac{n}{(n+1)q^n} (-3P_1(q) - Q_1(q)) \zeta_q^n(0) + \sum_{p=1}^{2n-2} \left( 5\tilde{c}_{q,p-1}^{(n)} - c_{q,p-1}^{(n)} + 3 \frac{n}{(n+1)q^n} P_{p+1}(q) + (-1)^p \frac{n}{(n+1)q^n} Q_{p+1}(q) \right) \zeta_q^n(-p)
\]
\[
+ 3(n+1+q) \frac{n+q-1}{q(n+1)} \log(n+1) + (n+1 - 3q) \frac{n+q-1}{q(n+1)} \log(q)
\]
\[
+ 2(n+1-q) \zeta_q^n(0) + \sum_{j=2}^{2n} (n+1-q)^j w_j \gamma_{j,q}(j-1).
\]

2. 
\[
\sum_{q=1}^{n} (-1)^q \zeta_q^n(0) = \sum_{p=1}^{n} (5\tilde{c}_{q,p-1}^{(n)} - c_{q,p-1}^{(n)} + d_p) \zeta_q^n(-p) - d_0 \zeta_q^n(0) + e_0.
\]

With \( c_p^{(n)} = \sum_{q=1}^{n} (-1)^q \zeta_q^n(p) \), \( \tilde{c}_p^{(n)} = \sum_{q=1}^{n} (-1)^q \zeta_q^n(p) \), \( d_p = \sum_{q=1}^{n} (-1)^q \zeta_q^n(p) \), \( Q_{p+1}(q) \) for \( p = 0, \ldots, n \) and \( e_0 = \sum_{q=1}^{n} (n+1-q) \zeta_q^n(0) \).

**Proof.** 1. By using Proposition 3.8, we have for any \( p = 1, \ldots, 2n - 2 \)
\[
6\tilde{c}_{q,p-1}^{(n)} - 3(n+1-q) c_{q,p}^{(n)} - 2 c_{q,p-1}^{(n)} + (n+1-q) c_{q,p}^{(n)} = 5 \tilde{c}_{q,p-1}^{(n)} + 3 \frac{n}{(n+1)q^n} P_{p+1}(q)
\]
\[
- c_{q,p-1}^{(n)} + (-1)^p \frac{n}{(n+1)q^n} Q_{p+1}(q).
\]

We can deduce also that
\[
c_{q,2n-2}^{(n)} = \frac{(-1)^{n+1}}{(n+1)q^n} P_{2n}(q) \quad \text{and} \quad \tilde{c}_{q,2n-2}^{(n)} = \frac{(-1)^n}{(n+1)q^n} Q_{2n}(q).
\]

But, \( P_{2n}(q) = Q_{2n}(q) = \frac{1}{(n+1)q^n} \) which follows from 29 and 30. Also, note that
\[
\sum_{i=0}^{2n} (-1)^i Q_i(q)(n+1-q)^i = 0, \quad \sum_{i=0}^{2n} P_i(q)(n+1-q)^i = 0, \quad Q_0(q) = 0 \quad \text{and} \quad P_0(q) = 0,
\]

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because \( n + 1 - q \) and 0 are zeros for the polynomials in (39) and (40).

Then 1. follows from 2. of Claim (3.7).

2. By combining 2. of Proposition (3.8) and 1. we conclude the proof of 2.

**Theorem 3.10.** We keep the same notations as in Corollary (3.9). We have,

\[
\sum_{q=1}^{n} (-1)^{q+1} q \zeta_q'(0) = \sum_{p=1}^{n} (5 \tilde{c}_{p-1}^{(n)} - c_{p-1}^{(n)} + d_p) \zeta'_Q(-p) - d_0 \zeta'_Q(0) + e_0.
\]

**Proof.** By (I), we have

\[
\sum_{q=0}^{n} (-1)^{q+1} q \zeta_q(s) = \sum_{q=1}^{n} (-1)^{q+1} \zeta_q(s).
\]

Then the theorem follows from Corollary (3.9).

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