Lattice calculations of meson correlators and spectral functions at finite temperature

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Abstract. I review recent progress in relating meson spectral function to imaginary time correlation function at finite temperature calculated on isotropic as well as on anisotropic lattices. Special attention is paid for the lattice artifacts present in calculation of meson spectral functions. Results in the case of light quarks as well as heavy quarks are reviewed which indicate in particular that even in the chiral limit meson spectral functions have non-trivial structure and the ground state quarkonia survive up to temperature $1.5 T_c$.

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1. Introduction

Spectral functions of mesonic operators play an important role in finite temperature QCD. Many experimental results in high energy heavy ion collisions (e.g. low invariant mass dilepton enhancement, anomalous $J/\psi$ suppression etc.) can be understood in terms of medium modifications of meson spectral functions [1].

It is commonly believed that lattice QCD is capable only for calculation of static quantities at finite temperatures, such as the transition temperature, equation of state, screening lengths etc. However, it was shown by Asakawa, Hatsuda and Nakahara that using the Maximum Entropy Method one can in principle reconstruct also meson spectral functions. The method was successfully applied at zero temperature [3, 4] and later also at finite temperature [5, 6, 7, 8, 9, 10, 11]. Though systematic uncertainties in the spectral function calculated on lattice are not yet completely understood, it was shown in Ref. [5] that precise determination of the imaginary time correlator can alone provide stringent constraints on the spectral function at finite temperature.

In this contribution I am going to review recent results on meson correlators and spectral functions in finite temperature QCD. The rest of the paper is organized as follows. In section 2 I will discuss the relation between the imaginary time correlators and meson spectral functions. In section 3 numerical results for the light quarks will be discussed. Section 4 contains results on charmonia at finite temperature. Finally conclusions are given in section 5.

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2. Meson correlators and spectral function in continuum and lattice QCD

In finite temperature field theory one usually considers two types of real time correlation function of some operator $\hat{O}$
\[
D^>(t,t') = \langle \hat{O}(t)\hat{O}(t') \rangle_T, \quad D^<(t,t') = \langle \hat{O}(t')\hat{O}(t) \rangle_T,
\] (1)
where $\langle ... \rangle_T = \langle ... e^{-\hat{H}/T} \rangle_T$ denotes the thermal average [12]. The imaginary time correlation function which can be calculated also by using lattice simulations is defined as simple analytic continuation
\[
G(\tau) = \langle T\hat{O}(-i\tau)\hat{O}(0) \rangle.
\] (2)
\[ T \text{ stands for time ordered product}. \]

The spectral function can be defined through the Fourier transform of $D^>(\omega)$ as
\[
\sigma(\omega) = D^>(\omega) - D^<(\omega) = \frac{1}{\pi} Im D_R(\omega),
\] (3)
where $D_R(\omega)$ is the retarded correlator. Using Eq. (2) and KMS condition on $D^>(\omega)$ [12] one can easily derive the following integral relation between the imaginary time correlator and the spectral function
\[
G(\tau) = \int_0^\infty d\omega \sigma(\omega) \frac{\cosh(\omega(\tau - 1/(2T)))}{\sinh \frac{\omega}{2T}} = \int_0^\infty d\omega \sigma(\omega) K(\omega,\tau)
\] (4)
Using this relation one can in principle reconstruct the spectral function by calculating $G(\tau)$ on lattice. At zero temperature the kernel, $K(\omega,\tau)$ reduces to simple exponential and at large Euclidean times the correlation function picks up the contribution from the lowest lying meson state in $\sigma(\omega)$, i.e $G(\tau) = \exp(-m\tau)$. At finite temperature the analysis of the asymptotic behavior of the correlators is no longer possible as $\tau$ is limited to the interval $[0, 1/T]$ where excited states are equally important as the ground state. Additional complications arise in lattice calculations where correlators are calculated only at finite set of Euclidean times $\tau T = k/N_T$, $k = 0,...,N_T - 1$ with $N_T$ being the temporal extent of the lattice. In order to reconstruct the spectral functions from this limited information it is necessary to include in the statistical analysis of the numerical results also prior information on the structure of $\sigma(\omega)$ (e.g. such as $\sigma(\omega) > 0$ for $\omega > 0$). This can be done through the application of the Maximum Entropy Method (MEM) ‡.

In order to study meson properties at finite temperature appropriate choice of the operator $\hat{O}$ should be made. One possible choice is local meson operator bilinear in quark-antiquark fields (current) [3, 4, 5, 6]
\[
O_H(\tau,\vec{x}) = Z_H q(\tau,\vec{x}) \Gamma H q(\tau,\vec{x}), \quad \Gamma H = 1, \gamma_5, \gamma_\mu, \gamma_5 \gamma_\mu
\] (5)
for scalar, pseudoscalar, vector and axial vector channels correspondingly. The normalization constant $Z_H$ relates the current calculated in lattice regularization scheme to current in $\overline{MS}$ scheme with $\mu^2_{\overline{MS}} = a^{-1}$ (with $a$ being the lattice spacing). We can define then the temporal correlators at finite spatial momentum $\vec{p}$
\[
G_H(\tau,\vec{p}) = \langle O_H(\tau,\vec{p})\hat{O}_H^\dagger(\tau,-\vec{p}) \rangle, O(\tau,\vec{p}) = \sum_{\vec{x}} e^{i\vec{p}\vec{x}} O_H(\tau,\vec{x})
\] (6)
‡ Other methods of introducing prior information into the statistical analysis have been also discussed in Ref. [13] for the zero temperature case and in Ref. [14] for finite temperature QCD.
One can also consider correlators of extended operators defined as
\[ \tilde{O}_H(\tau, \vec{x}) = \sum_{\vec{y}} \phi(\vec{y}) \bar{q}(\tau, \vec{x}) \Gamma q(\tau, \vec{x} + \vec{y}). \] (7)

Here
\[ \phi(\vec{y}) = \exp(-b|\vec{y}|^p) \] (8)
is the trial wave function which controls the size of the meson source and can be regarded as a physical input to the problem. The use of extended operators makes the reconstruction of the spectral function easier as the correlator is dominated by a single peak in the spectral function \[10\]. However, the corresponding spectral function is not related to a physically observable quantity, it can provide information only about the mass and the width of the resonance but not about the corresponding decay constant. This also only works for sharp resonances. At high temperature “mesons” will appear as broad structures in the spectral function and the above method is no longer applicable.

Though the in-medium properties of mesons are encoded primarily in the temporal correlator and the corresponding spectral function, in finite temperature QCD it is customary to study spatial correlators defined as
\[ G_H = \langle O(z)O^\dagger(0)\rangle, \quad O(z) = \sum_{\tau, x, y} O(\tau, x, y, z). \] (9)
Since the lattice extent is not limited in the spatial directions one can study their large distance behavior. At large distances the spatial correlators decay exponentially and the exponential decay governed by the so-called screening mass. The spatial correlators can be related to the spectral function via the following relation
\[ G_H(\omega) = \int_{-\infty}^{\infty} dp_z \int_0^{1/T} G(\tau, 0, 0, p_z) = \int_{-\infty}^{\infty} dp_z e^{i p_z z} \int_0^{\infty} d\omega \frac{\sigma(\omega, 0, 0, p_z)}{\omega} \] (10)

In general the screening masses are different from the pole masses, however, from the above equation one can easily see that in the special case when the spectral function is dominated by a single \( \delta \)-function for small \( \omega \) the screening and the pole masses are equal.

The relation between the imaginary time correlator and the spectral function \[14\] is valid only in the continuum theory. As the correlators are calculated on lattice the question arises whether the spectral representation of the form \[14\] can be derived also for the lattice correlator. In ref. \[16\] it was shown that this is indeed the case in the free theory and that the cutoff effects present in the correlator at small separations are contained in the spectral function. Because of the asymptotic freedom calculation in the free theory are relevant at high temperature and/or large energies \( \omega \). In Fig. 1 I show the ratio of of the lattice spectral function to the corresponding continuum spectral function versus \( a\omega \) as well as the lattice pseudoscalar spectral function versus \( \omega/T \) in the free theory. As one can see from the figures the lattice spectral functions start to deviate from the continuum one for \( \omega a > 0.5 \) and in the pseudoscalar and vector channels show a peak like structure around \( \omega a \simeq 2 \). The lattice spectral functions vanishes for \( \omega > \omega_{max} a \sim 4N_\tau \). As the temporal extent of the lattice \( N_\tau \) is increased for fixed temperature, i.e. the lattice spacing decreases \((a = 1/(N_\tau T))\) the peak structure moves to larger value of \( \omega/T \) (see Fig. 1). Peak structure around \( \omega a \simeq 2 \) was also observed in the interacting theory \[14\]. The main reason for the large cutoff effects in the lattice spectral function is that on lattice the quark dispersion relation...
gets modified at large momenta, $p_a > 1$, relative to its continuum counterpart. Cutoff effects in the spectral function can be substantially reduced by using an improved fermion action, the so-called truncated perfect action on hypercube [17] for which the lattice quark dispersion relation is much closer to the continuum even for $p_a > 1$ [16].

3. Numerical results in the light quark sector

Meson correlators and spectral functions for small quark masses were intensively studied during recent years [5, 6, 7, 18]. In Refs. [5, 6, 7] spectral functions were studied using non-perturbatively $O(a)$ improved Wilson action [19] and isotropic $32^3 \times 16, 48^3 \times 12, 64^3 \times 16$ and $64^3 \times 24$ lattices. In these studies quark masses corresponding to the pion masses in the range 400MeV to 1Gev were considered in the confined phase, while in the deconfined phase calculation were performed in the chiral limit and the temperature interval was $0.4T_c - 3T_c$ with $T_c$ being the deconfinement temperature. The renormalization factors $Z_H$ appearing in Eq. 5 were determined non-perturbatively in Ref. [19] for the vector and axial vector channels, while for the scalar and pseudoscalar channels they were calculated using tadpole improved perturbation theory. Asakawa, Hatsuda and Nakahara have studied meson spectral functions using $32^3 \times N_\tau$ anisotropic lattices with $a_\tau/a_\sigma = 4$ and $N_\tau = 96 - 32$ corresponding to temperatures $0.8T_c - 3T_c$ [9]. The quark masses used in this study correspond to $m_\pi/m_\rho = 0.7$. Below the deconfinement temperature neither the correlators nor the spectral functions indicate in-medium change of meson properties contrary to existing theoretical predictions [4]. This is likely due to the quenched approximation and large quark masses used in these studies.

Before discussing the results on the spectral function above deconfinement let us first look at the correlators. The correlation functions alone can provide some constraints on the spectral functions, for example the analysis of the vector correlation function provides stringent constraint on the thermal dilepton rate [5]. At high temperature one would expect that the quark propagators should be close to the free ones. Therefore in what follows we will always normalize the meson correlators by the corresponding lattice correlators in the free theory. We will restrict the
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Figure 2. The ratio of meson correlators at $1.5T_c$ in different channels to the corresponding correlators in the free theory calculated on $64^3 \times 16$ lattice.

Figure 3. The ratio of meson correlators at $1.5T_c$ and $3T_c$ in the pseudoscalar (left) and vector (right) channels to the corresponding correlators in the free theory calculated on lattice.

discussion to the case of zero spatial momentum $\vec{p} = 0$. In Fig. 2 I show the meson correlators at $1.5T_c$ in different channels calculated on $64^3 \times 16$ lattice. As one can see from the figure the scalar-pseudoscalar and vector-axial-vector correlators become degenerate except at very small $\tau T$ where lattice artifacts present in the Wilson formulation explicitly break chiral symmetry. This can be viewed as an indication of the $U_A(1)$ symmetry restoration at high temperature. In Fig. 3 I show the vector and pseudoscalar correlators for $1.5T_c$ and $3T_c$ for different values of $N_\tau$. From the figure it is evident that the vector correlator stays close to its free value even at $1.5T_c$ while the pseudoscalar correlator is considerably enhanced compared to the corresponding free correlators. Note also that in the vector channel the correlators is $N_\tau$ (i.e. lattice spacing) independent, while in the pseudoscalar case there is a small $N_\tau$ dependence which is probably due to the perturbative error in calculation of $Z_{PS}$. In Fig. 4 I show the spectral functions for the pseudoscalar and vector channels. The spectral functions in both channels show a peak like structure at $\omega \simeq (5 - 6)T$ which is more pronounced in the pseudoscalar case and probably leads to the large enhancement of
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**Figure 4.** The spectral functions in the deconfined phase for the pseudoscalar channel (left) and the vector channel (right) reconstructed using MEM on $64^3 \times 16$ lattice.

**Figure 5.** The spectral functions $\rho(\omega) = \sigma(\omega)/\omega^2$ in different channels calculated at $1.4T_c$ (left) and at $1.9T_c$ (right) using anisotropic lattices [3]. Note that normalization of the spectral functions is different from the case on isotropic lattices.

the pseudoscalar correlator over the free case discussed above. Note that the position of the peak appears to be proportional to the temperature. Spectral function calculated on anisotropic lattice are shown in Fig. 4 and exhibit a similar peak like structure, roughly for the same value of $\omega/T$. However, at lower temperature, $T = 1.4T_c$ the corresponding structure in the spectral functions appears to be more sharp (see Fig. 5).

Meson correlators in the spatial directions and the corresponding screening masses have been studied since long time both in full and quenched QCD [20, 21, 22, 23, 25, 26, 27, 28]. However, reliable determination of the screening masses is available only in quenched approximation [24, 27, 28] as lattices with temporal extent $N_t \geq 8$ are needed in such analysis [27]. The most recent results for the screening masses are summarized in Fig. 6. The screening masses are close to their asymptotic ($T \rightarrow \infty$) value $2\pi T$ already at temperature for $T = 1.5T_c$. This could lead to the conclusion that there is an almost free quark propagation in spatial directions in the deconfined phase at temperatures as low as $1.5T_c$. However, a closer look on the spatial correlators in
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**Figure 6.** The screening masses (left) and spatial correlators (right) for vector and pseudoscalar channels at $T > T_c$. The screening masses have been calculated in quenched QCD using improved Wilson fermions [24, 28] and Kogut-Susskind fermions [27].

**Figure 7.** The ratio of spatial and temporal correlators in the deconfined phase to the corresponding free correlators for the pseudoscalar (left) and vector (right) channels.

Fig. 6 reveals that this is not the case even at $3 T_c$. The correlators calculated by using lattice simulations are several times larger than the corresponding free ones. This is not unexpected since it is well known that the physics at distances $z > 1/(g^2 T)$ is non-perturbative even at very high temperature [29]. In Fig. 7 I show the vector and pseudoscalar spatial correlators at short distances together with the temporal one. As one can see from the figures at small separations temporal and spatial correlators show very similar behavior.

4. Heavy quarkonia correlators and spectral functions

Heavy quarkonia correlators and spectral functions were studied in Refs. [10, 15] using anisotropic lattices and extended operators as well as on isotropic lattices and point correlators [8]. In Ref. [8] pseudoscalar, vector, scalar and axial vector channels were considered which correspond to $^1S_0 \ (\eta_c)$, $^3S_1 \ (J/\psi)$, $^3P_0 \ (\chi_{c0})$ and $^3P_1 \ (\chi_{c1})$ charmonia states respectively. As in the light quark sector some statements about
in-medium properties of charmonia can be made just by analyzing the behavior of the correlators. In Fig. 8 I show the ratio of different quarkonia correlators to the corresponding free correlators. As one can see from the figures the correlators in the pseudoscalar and vector channels show very little change across $T_c$ while in the case of the scalar and axial vector channels larger changes in the correlators are visible. This implies that the ground state charmonia ($^1S_0$ and $^3S_1$) are very likely to survive in the deconfined phase. The spectral function reconstructed using MEM are shown in Fig. 9 which shows indeed the ground state peak survives in the deconfined phase while the excited $^3P_0$ state is dissociated already at $T = 1.25T_c$. Spectral functions were also reconstructed for $1.5T_c$ and show again that the ground state charmonia survive while the excited P-states are dissociated. Ground state charmonia ($^1S_0$, $^3S_1$ states) spectral functions were also studied in Ref. [10] using extended operators. The results for the vector channel are shown in Fig. 10. The $J/\psi$ mass (the peak position) changes very little across $T_c$ while the corresponding peak in the spectral function gets broader above $T_c$. This broadening of the ground state peak is also visible in calculations with local operators. However, such broadening is not necessarily a physical effect and may come from the fact that less data points $N_\tau$ are available at higher temperature.
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Figure 10. The $J/\psi$ spectral function reconstructed from the correlation function of extended operators using MEM [10] for $0.9T_c$ and $1.1T_c$ and different input wave functions (different values of $b$).

and also the temporal extent of the lattice in physical units become smaller [10]. The authors of Ref. [10] have found that the width of the peak in the spectral function also depends on the trial wave function $\phi(y)$ used in constructing the extended operator (see Fig. [10]), the peak is broader for larger value of $b$ in the trial wave function $\phi(y)$. Therefore MEM cannot yet provide a reliable estimate of the width of the $J/\psi$ in the plasma. To overcome this difficulty $\chi^2$-fits to the Breit-Wigner form of the spectral function was used [10]. This led to a first hints for a non-zero widths of charmonium states above $T_c$.

The existence of ground state charmonia at $1.5T_c$ is in sharp disagreement with predictions of potential models based on color screening [30, 31]. A possible reason for this discrepancy is an oversimplified picture of color screening used in these studies. In fact recent analysis of the free energy of quark-antiquark pairs suggest quite complicated medium dependence of inter-quark forces [32, 33, 34].

5. Conclusions

In this paper recent results on meson spectral functions and correlators calculated on lattice were discussed. Though systematic uncertainties are present in the spectral functions the results appear to be very interesting and even intriguing to some extent. The presence of the peak like structures in the meson spectral functions well above deconfinement temperature is, too large extent, unexpected. These peak structures seems to be present in the spectral functions calculated on isotropic lattices [5, 6, 8] as well as on anisotropic lattices [9, 10] and in some case their presence is supported by direct inspection of the corresponding correlation functions (c.f. Fig. 8 (left) and Fig. 5). In the case of heavy quarkonia the interpretation of the peaks in the spectral function is clear, they correspond to the quarkonia states which survive in the deconfined phase up to temperatures $1.5T_c$. We have seen that the position of these
peaks does not change considerably compared to the zero temperature case. The interpretation of peak-like structures in the light quark sector is less evident as the position of these peaks seems to be proportional to the temperature. Certainly much more work is needed in order to be able to interpret the present findings in physical terms as well as for the detailed understanding of the systematic uncertainties involved in extraction of the spectral functions from lattice correlators.

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