Piecewise Linear Structures on Topological Manifolds
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Preface

In his paper Novikov \[N3\] p.409 wrote:
Sullivan’s Hauptvermutung theorem was announced first in early 1967. After the careful analysis made by Bill Browder and myself in Princeton, the first version in May 1967 (before publication), his theorem was corrected: a necessary restriction on the 2-torsion of the group \(H_3(M)\) was missing. This gap was found and restriction was added. Full proof of this theory has never been written and published. Indeed, nobody knows whether it has been finished or not. Who knows whether it is complete or not? This question is not clarified properly in the literature. Many pieces of this theory were developed by other topologists later. In particular, the final Kirby–Siebenmann classification of topological multi-dimensional manifolds therefore is not proved yet in the literature.

I do not want to discuss here whether the situation is so dramatic as Novikov wrote. However, it is definitely true that, up to now, there is no detailed enough and well-ordered exposition of Kirby–Siebenmann classification, such that it can be recommended to advanced students which are willing to learn the subject. The fundamental book of Kirby–Siebenmann [KS2] was written by pioneers and, in a sense, in hot pursuit. It contains all the necessary ingredient for the proof, but it is really “Essays”, and one have to do a certain work in order to make it easy readable for general audience.

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Notation and Conventions

We work mainly with $CW$-spaces and topological manifolds. However, when we quit these classes by taking products or functional spaces, we equip the last ones with the compactly generated topology, (following Steenrod [St] and McCord [McC], see e.g. [Rud] for the exposition). All maps are supposed to be continuous. All neighborhoods are supposed to be open.

We denote the one-point space by $pt$.

A pointed space is pair $(X, \{x_0\})$ where $x_0$ is a point of $X$. We also use that notation $(X,x_0)$ and call $x_0$ the base point of $X$. If we do not need to indicate the base point, we can write $(X,*)$ (or even $X$ if it is clear that $X$ is a pointed space). Given two pointed spaces $(X,x_0)$ and $(Y,y_0)$, a pointed map is a map $f : X \to Y$ such that $f(x_0) = y_0$.

Given two topological spaces $X,Y$, we denote by $[X,Y]$ the set of homotopy classes of maps $X \to Y$. We also use the notation $[X,Y]^*$ for the set of pointed homotopy classes of pointed maps $X \to Y$ of pointed spaces.

It is quite standard to denote by $[f]$ the homotopy class of a map $f$. However, frequently we do not distinguish a map and its homotopy class and use the same symbol, say $f$ for a map as well as for the homotopy class. In this paper this does not lead to any confusion.

We use the term inessential map for null-homotopic maps; otherwise a map is called essential.

We use the sign $\simeq$ for homotopy of maps or homotopy equivalence of spaces. We use the sign $\cong$ for bijection of sets or isomorphism of groups. We use the notation := for “is defined to be”.

We reserve the term bundle for locally trivial bundles and the term fibration for Hurewicz fibrations.

Given a space $F$, an $F$-bundle is a bundle whose fibers are homeomorphic to $F$, and an $F$-fibration is a fibration whose fibers are homotopy equivalent to $F$. 
We denote the \textit{trivial} $F$-bundle $X \times F \to X$ over $X$ by $\theta_X^F$, or merely $\theta^F$. Also, we denote the trivial $\mathbb{R}^n$-bundle over $X$ by $\theta^n$ or $\theta^n$.

We do not mention \textit{microbundles} at all, because in the topological and in the PL category every $n$-dimensional microbundle over a space $X$ contains an $\mathbb{R}^n$-bundle over $X$, and these bundles are unique up to equivalence, see Kister \cite{Kis} for the topological category and Kuiper-Lashof \cite{KL} for the PL category. For this reason, any claim on microbundles can be restated in terms of bundles. The reader should keep it in the mind when we cite (quote about) something concerning microbundles.

Given a bundle or fibration $\xi = \{p : E \to B\}$, the space $B$ is called the \textit{base} of $\xi$ and denote also by $\text{bs}(\xi)$, i.e. $\text{bs}(\xi_1) = B$. The space $E$ is called the \textit{total space} of $\xi$. Furthermore, given a space $X$, we set $\xi \times X = \{p \times 1 : E \times X \to B \times X\}$.

Given two bundles $\xi = \{p : E \to B\}$ and $\eta = \{q : Y \to X\}$, a \textit{bundle morphism} $\varphi : \xi \to \eta$ is a commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{g} & Y \\
\downarrow p & & \downarrow q \\
B & \xrightarrow{f} & X.
\end{array}
$$

We say that $f$ is the \textit{base} of the morphism $\varphi$ or that $\varphi$ is a \textit{morphism over} $f$. We also say that $g$ is a \textit{map over} $f$. If $X = B$ and $f = 1_B$ we say that $g$ is a \textit{map over} $B$ (and $\varphi$ is a morphism over $B$).

Given a map $f : Z \to B$ and a bundle (or fibration) $\xi = \{p : E \to B\}$, we use the notation $f^*\xi$ for the induced bundle over $Z$. Recall that $f^*(\xi) = \{r : D \to Z\}$ where

$$
D = \{(z,e) \in Z \times E \mid f(z) = p(e)\}
$$

. There is a canonical bundle morphism

$$
I = \mathcal{I}_f = \mathcal{I}_{f,\xi} : f^*\xi \to \xi
$$

given by the map $D \to E, (z,e) \mapsto e$ over $f$, see \cite{Rud} (or \cite{FR} where it is denoted by $\text{ad}(f)$). Following \cite{FR}, we call $\mathcal{I}_{f,\xi}$ the \textit{adjoint morphism of} $f$, or just the \textit{$f$-adjoint} morphism. Furthermore, given a bundle morphism $\varphi : \xi \to \eta$ with the base $f$, there exists a unique bundle morphism $c(\varphi) : \xi \to f^*\eta$ over the base of $\xi$ such that the composition

$$
\xi \xrightarrow{c(\varphi)} f^*\eta \xrightarrow{\mathcal{I}_{f,\eta}} \eta
$$

coincides with $\varphi$. Following \cite{FR}, we call $c(\varphi)$ the \textit{correcting morphism}. 

Given a subspace $A$ of a space $X$ and a bundle $\xi$ over $X$, we denote by $\xi|_A$ the bundle $i^*\xi$ where $i : A \subset X$ is the inclusion.

Given a map $p : E \to B$ and a map $f : X \to B$, a $p$-lifting of $f$ is any map $g : X \to E$ with $pg = f$. Two $p$-lifting $g_0, g_1$ of $f$ are \textit{vertically homotopic} if there exists a homotopy $G : X \times I \to E$ between $g_0$ and $g_1$ such that $pg_t = f$ for all $t \in I$. The set of vertically homotopic $p$-liftings of $f$ is denoted by $[\text{Lift}_p f]$.

We denote by $p_k, w_k$ and $L_k$ the Pontryagin, Stiefel–Whitney, and Hirzebruch characteristic classes, respectively. We denote by $\sigma(M)$ the signature of a manifold $M$. See [MS] for the definitions.
Introduction

Throughout the paper we use abbreviation PL for “piecewise linear”.

Hauptvermutung (main conjecture) is an abbreviation for die Hauptvermutung der kombinatorischen Topologie (the main conjecture of combinatorial topology). It seems that the conjecture was first formulated in the papers of Steinitz [Ste] and Tietze [Ti] in 1908.

The conjecture claims that the topology of a simplicial complex determines completely its combinatorial structure. In other words, two simplicial complexes are simplicially isomorphic whenever they are homeomorphic. This conjecture was disproved by Milnor [Mi2] in 1961.

However, for manifolds one can state a refined version of the Hauptvermutung by considering simplicial complexes with additional restrictions. A PL manifold is defined to be a simplicial complex such that the star of every point (the union of all closed simplexes containing the point) is simplicially isomorphic to the n-dimensional ball. Such simplicial complexes are also called combinatorial triangulations. Equivalently, a PL manifold can also be defined a manifold equipped with a maximal PL atlas.

There exist topological manifolds that are homeomorphic to a simplicial complex but do not admit a PL structure (non-combinatorial triangulations), see Example 21.4. Furthermore, there exist topological manifolds that are not homeomorphic to any simplicial complex, see Example 21.6.

Now, the Hauptvermutung for manifolds asks if any two homeomorphic PL manifolds are PL homeomorphic. Furthermore, the related question asks whether every topological manifold is homeomorphic to a PL manifold. Both these questions were solved (negatively) by Kirby and Siebenmann [KS1, KS2]. In fact, Kirby and Siebenmann classified PL structures on high-dimensional topological manifolds. It turned out that a topological manifold can have different PL structures, as well as not to have any. Now we give a brief description of these results.
Let $BTOP$ and $BPL$ be the classifying spaces for stable topological and PL bundles, respectively. We regard the forgetful map $\alpha : BPL \to BTOP$ as a fibration and denote its homotopy fiber by $TOP/PL$.

Let $f : M \to BTOP$ classify the stable tangent bundle of a topological manifold $M$. By main properties of classifying spaces, every PL structure on $M$ gives us a $\alpha$-lifting of $f$ and that every two such liftings for the same PL structure are fiberwise homotopic.

It is remarkable that the inverse is also true provided that $\dim M \geq 5$. In greater detail, $M$ admits a PL structure if $f$ admits a $\alpha$-lifting (the Existence Theorem 6.3), and PL structures on $M$ are in a bijective correspondence with fiberwise homotopy classes of $\alpha$-liftings of $f$ (the Classification Theorem 6.2). Kirby and Siebenmann proved these theorems and, moreover, they proved the following Main Theorem:

$TOP/PL$ is the Eilenberg–MacLane space $K(\mathbb{Z}/2, 3)$.

Thus, there is only one obstruction

$\kappa(M) \in H^4(M; \mathbb{Z}/2)$

to an $\alpha$-lifting of $f$, and the set of fiberwise homotopic $\alpha$-liftings of $f$ (if they exist) is in bijective correspondence with $H^3(M; \mathbb{Z}/2)$. In other words, a topological manifold $M$, $\dim M \geq 5$ admits a PL structure if and only if $\kappa(M) = 0$. Furthermore, every homeomorphism $h : V \to M$ of two PL manifolds assigns a class

$\kappa(h) \in H^3(M; \mathbb{Z}/2)$,

and $\kappa(h) = 0$ if and only if $h$ is concordant to a PL homeomorphism (or, equivalently, to the identity map $1_M$, see Remark 3.2(2)). Finally, every class $a \in H^3(M; \mathbb{Z}/2)$ has the form $a = \kappa(h)$ for some homeomorphism $h : V \to M$ of two PL manifolds.

These results give us the complete classification of PL structures on a topological manifold of dimension $\geq 5$. In particular, the situation with Hauptvermutung turns out to be understandable. See Section 20 for more detailed exposition.

We must explain the following. It can happen that two different PL structures on $M$ yield PL homomorphic PL manifolds (like that two $p$-liftings $f_1, f_2 : M \to BPL$ of $f$ can be non-fiberwise homotopic). Indeed, roughly speaking, a PL structure on a topological manifold $M$ is a concordance class of PL atlases on $M$ (see Section 3 for precise definitions). However, a PL automorphism of a PL manifold can turn the atlas into a non-concordant to the original one, see Example 21.2.

So, in fact, the set of pairwise non-isomorphic PL manifolds which are
homeomorphic to a given PL manifold is in a bijective correspondence with the set $H^3(M; \mathbb{Z}/2)/R$ where $R$ is the following equivalence relation: two PL structure are equivalent if the corresponding PL manifolds are PL homeomorphic. The *Hauptvermutung* for manifolds claims that the set $H^3(M; \mathbb{Z}/2)/R$ is a singleton for all $M$. But this is wrong in general.

Namely, there exists a PL manifold $M$ which is homeomorphic but not PL isomorphic to $\mathbb{R}P^n$, $n \geq 5$, see Example 21.1. So, here we have a counterexample to the *Hauptvermutung*.

To complete the picture, we mention again that there are topological manifolds that do not admit any PL structure, see Example 21.3. Moreover, there are manifold that cannot be triangulated as simplicial complexes, see Example 21.6.

Comparing the classes of smooth, PL and topological manifolds, we see that there is a big difference between first and second classes, and not so big difference between second and third ones. From the homotopy-theoretical point of view, one can say that the space $PL/O$ (which classifies smooth structures on PL manifold, see Remark 6.7) has many non-trivial homotopy groups, while the space $TOP/PL$ is an Eilenberg–MacLane space. Geometrically, one can mention that there are many smooth manifolds which are PL homeomorphic to $S^n$ but pairwise non-diffeomorphic, while any PL manifold $M^n, n \geq 5$ is PL homeomorphic to $S^n$ provided that it is homeomorphic to $S^n$.

It is worthwhile to go one step deeper and explain the following. Let $M^{4k}$ be a closed connected almost parallelizable manifold (i.e. $M$ becomes parallelizable after deletion of a point). Let $\sigma_k$ denote the minimal natural number which can be realized as the signature of the manifold $M^{4k}$. In fact, for every $k$ we have three numbers $\sigma^{S}_k, \sigma^{PL}_k$ and $\sigma^{TOP}_k$ while $M^{4k}$ is a smooth, PL or topological manifold, respectively. Milnor and Kervaire [MK] proved that

$$\sigma^{S}_k = c_k(2k - 1)!$$

where $c_k \in \mathbb{N}$. On the other hand,

$$\sigma^{PL}_1 = 16 \text{ and } \sigma^{PL}_k = 8 \text{ for } k > 1.$$  

Finally,

$$\sigma^{TOP}_k = 8 \text{ for all } k.$$  

So, here we can see again the big difference between smooth and PL cases. On the other hand, $\sigma^{PL}_k = \sigma^{TOP}_k$ for $k > 1$. Moreover, we
will see below that the number
\[ 2 = 16/8 = \sigma_1^{PL}/\sigma_1^{TOP} \]
is another guise of the number
\[ 2 = \text{the order of the group } \pi_3(TOP/PL). \]

In this context, it makes sense to notice about low dimensional manifolds, because of the following remarkable contrast. There is no difference between PL and smooth manifolds in dimension \(< 7\): every PL manifold \(V^n, n < 7\) admits a unique smooth structure. However, there are infinitely many smooth manifolds which are homeomorphic to \(\mathbb{R}^4\) but pairwise non-diffeomorphic, see Section 21, Summary.

Concerning the description of the homotopy type of \(TOP/PL\), we have the following. Because of the Classification Theorem, if \(k+n \geq 5\) then the group \(\pi_n(TOP/PL)\) is in a bijective correspondence with the set of PL structures on \(\mathbb{R}^k \times S^n\). However, this set of PL structures looks wild and uncontrollable. In order to make the situation more manageable, we consider PL structures on the compact manifold \(T^k \times S^n\) and then extract the necessary information on the universal covering \(\mathbb{R}^k \times S^n\) from here. We can’t do it directly, but there is a trick (the Reduction Theorem 8.7) which allows us to estimate PL structures on \(\mathbb{R}^k \times S^n\) in terms of so-called homotopy PL structures on \(T^k \times D^n\) (more precisely, we should consider the homotopy PL structures on \(T^k \times D^n\) modulo the boundary), see Section for the definitions. Now, using results of Hsiang and Shaneson [HS] or Wall [W2, W3] about homotopy PL structures on \(T^k \times D^n\), one can prove that \(\pi_i(TOP/PL) = 0\) for \(i \neq 3\) and that \(\pi_3(TOP/PL)\) has at most 2 elements. Finally, there exists a high-dimensional topological manifold which does not admit any PL structure. Hence, by the Existence Theorem, the space \(TOP/PL\) is not contractible. Thus, \(TOP/PL = K(\mathbb{Z}/2, 3)\).

For better arrangement of the previous paragraph, look at the graph located at the end of the Introduction. Here we formulate without proofs the boxed claims (and provide the necessary preliminaries and references), while in Chapter II we explain how a claim (box) can be deduced from other ones, accordingly with the arrows in the graph.

Let me tell you something more about the graph. As we have already seen, the classification theory of PL structures on topological manifolds splits into two parts. The first one reduces the original geometric problem to a homotopy one (a classification of \(p\)-liftings of a map \(M \to BTOP\) to \(BPL\)), the second part solves this homotopy problem by proving that \(TOP/PL = K(\mathbb{Z}/2, 3)\).
The Product Structure Theorem [6.1] is a very important ingredient for the proof. Roughly speaking, this theorem establishes a bijection between PL structures on $M$ and $M \times \mathbb{R}$. The Classification Theorem [6.2] and the Existence Theorem [6.3] are the consequences of the Product Structure Theorem.

Now I say some words about the top box of the above graph. Let $F_n$ be the monoid of pointed homotopy equivalences $S^n \to S^n$, let $BF_n$ be the classifying space for $F_n$, and let $BF = \lim_{n \to \infty} BF_n$. There is an obvious forgetful map $BPL \to BF$, and we denote by $F/PL$ the homotopy fiber of this map. For every homotopy equivalence of closed PL manifolds $h : V \to M$ Sullivan [Sul1, Sul2] defined the normal invariant of $h$ to be a certain homotopy class $j_F(h) \in [M, F/PL]$, see Section 4.

Let $M$, dim $M \geq 5$ be a closed PL manifold. Sullivan proved that, for every homeomorphism $h : V \to M$, we have $j_F(h) = 0$ whenever $H_3(M)$ is 2-torsion free. Moreover, this theorem implies that if, in addition, $M$ is simply-connected then $h$ is homotopic to a PL homeomorphism. Thus the Hauptvermutung holds for simply-connected closed manifolds $M$, dim $M \geq 5$ with $H_3(M)$ 2-torsion free, see Section [19].

Definitely, the above formulated Sullivan Theorem on the Normal Invariant of a Homeomorphism is interesting by itself. However, in the paper on hand this theorem plays also an additional important role. Namely, the Sullivan Theorem for $T^k \times S^n$ is a lemma in classifying of homotopy structures on $T^k \times D^n$. For this reason, we first prove the Sullivan Theorem for $T^k \times S^n$, then use it in the proof of the Main Theorem, and then (in Chapter III) use the Main Theorem in order to prove the Sullivan Theorem in full generality.

You can also see that the proof of the Main Theorem uses the difficult Freedman’s example of a 4-dimensional almost parallelizable topological manifold of signature 8. This example provides the equality $\sigma_1^{TOP} = 8$. Actually, the original proof of the Main Theorem appeared before Freedman’s Theorem and therefore did not use the last one. However, as we already mentioned, the Freedman’s results clarify the relations between PL and topological manifolds, and thus they should be incorporated in the exposition of the global picture.

The paper is organized as follows. The first chapter contains the architecture of the proof of the Main Theorem: $TOP/PL \simeq K(\mathbb{Z}/2, 3)$. In fact, here we comment the above mentioned graph.

The second chapter contains a proof of the Sullivan Theorem on the normal invariant of a homeomorphism for $T^k \times S^n$, i.e. we attend
the top box of the graph. We also discuss the Browder–Novikov Theorem [4.5] about homotopy properties of normal bundles: we need this discussion in order to clarify the concept of normal invariant.

The third chapter contains some applications if the Main Theorem. We complete the proof of the Sullivan Theorem on the normal invariant of a homeomorphism and tell more on classification of PL manifolds an, in particular, on Hauptvermutung. Several interesting examples are considered. Finally, we discuss the homotopy and topological invariants of certain characteristic classes.
The Graph

Theorem on the normal invariant of a homeomorphism for $T^k \times S^n$

Classification of homotopy PL structures on $T^k \times D^n$

Classification Theorem

Product Structure Theorem

Reduction Theorem

Local contractibility of the homeomorphism group

$TOP/PL = K(\pi, 3), \pi \subset \mathbb{Z}/2$

Existence Theorem

Main Theorem: $TOP/PL = K(\mathbb{Z}/2, 3)$

Existence of high-dimensional topological manifolds that admit no PL structures

Rokhlin Signature Theorem

Freedman’s Example
CHAPTER I

Architecture of the Proof

1. Principal Fibrations

Recall that an $H$-space is a space $F$ with a base point $f_0$ and a multiplication map $\mu : F \times F \to F$ such that $f_0$ is a homotopy unit, i.e. the maps $f \mapsto \mu(f, f_0)$ and $f \mapsto \mu(f_0, f)$ are homotopic to the identity rel $\{f_0\}$. For details, see [BV].

1.1. Definition. (a) Let $(F, f_0)$ be an $H$-space with the multiplication $\mu : F \times F \to F$. A principal $F$-fibration is an $F$-fibration $p : E \to B$ equipped with a map $m : E \times F \to E$ such that the following holds:

(i) the diagrams

$$
\begin{array}{ccc}
E \times F \times F & \xrightarrow{m \times 1} & E \times F \\
1 \times \mu & \downarrow & m \\
E \times F & \xrightarrow{m} & E
\end{array}
\quad
\begin{array}{ccc}
E \times F & \xrightarrow{m} & E \\
p_1 & \downarrow & p \\
E & \xrightarrow{p} & B
\end{array}
$$

commute;

(ii) the map

$$
E \to E, \quad e \mapsto m(e, f_0)
$$

is a homotopy equivalence;

(iii) for every $e_0 \in E$, the map

$$
F \to p^{-1}(p(e_0)), \quad f \mapsto m(e_0, f)
$$

is a homotopy equivalence.

(b) A trivial principal $F$-fibration is the fibration $p_2 : X \times F \to F$ with the action $m : E \times F \to E$ of the form

$$
m : X \times F \times F \to X \times F, \quad m(x, f_1, f_2) = (x, \mu(f_1, f_2)).
$$

It is easy to see that if the fibration $\eta$ is induced from a principal fibration $\xi$ then $\eta$ turns into a principal fibration in a canonical way.

1.2. Definition. Let $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ be two principal $F$-fibrations over the same base $B$. We say that a map $h :
$E_1 \to E_2$ is an $F$-equivariant map over $B$ if $h$ is a map over $B$ and the diagram

$$
\begin{array}{ccc}
E_1 \times F & \xrightarrow{h \times 1} & E_2 \times F \\
m_1 \downarrow & & \downarrow m_2 \\
E_1 & \xrightarrow{h} & E_2
\end{array}
$$

commutes up to homotopy over $B$.

Note that, for every $b \in B$, the map

$$h_b : \pi_1^{-1}(b) \to \pi_2^{-1}(b), \quad h_b(x) = h(x)$$

is a homotopy equivalence.

Now, let $p : E \to B$ be a principal $F$-fibration, and let $f : X \to B$ be an arbitrary map. Given a $p$-lifting $g : X \to E$ of $f$ and a map $u : X \to F$, consider the map

$$g_u : X \xrightarrow{\Delta} X \times X \xrightarrow{g \times u} E \times F \xrightarrow{m} E.$$ 

It is easy to see that the correspondence $(g, u) \mapsto g_u$ yields a well-defined map (right action)

$$(1.1) \quad [\text{Lift}_p f] \times [X, F] \to [\text{Lift}_p f].$$

In particular, for every $p$-lifting $g$ of $f$ the correspondence $u \mapsto g_u$ induces a map

$$T_g : [X, F] \to [\text{Lift}_p f].$$

1.3. Theorem. Let $\xi = \{p : E \to B\}$ be a principal $F$-fibration, and let $f : X \to B$ be a map where $X$ is assumed to be paracompact and locally contractible. If $F$ is a homotopy associative $H$-space with a homotopy inversion, then $[X, F]$ is a group the above action (1.1) is free and transitive provided $[\text{Lift}_p f] \neq \emptyset$. In particular, for every $p$-lifting $g : X \to E$ of $f$ the map $T_g$ is a bijection.

Proof. We start with the following lemma.

1.4. Lemma. The theorem holds if $X = B$, $f = 1_X$ and $\xi$ is the trivial principal $F$-fibration.

Proof. In this case every $p$-lifting $g : X \to X \times F$ of $f = 1_X$ determines and is completely determined by the map

$$\overline{g} : X : \xrightarrow{g} X \times F \xrightarrow{p_2} F.$$
In other words, we have the bijection $[\text{Lift}_p f] \cong [X, F]$, and under this bijection the action \eqref{1.1} turns into the multiplication $[X, F] \times [X, F] \to [X, F]$.

Now the result follows since $[X, F]$ is a group. ☐

We finish the proof of the theorem. Consider the induced fibration $f^* \xi = \{q : Y \to X\}$ and note that there is an $[X, F]$-equivariant bijection

\begin{equation}
[\text{Lift}_p f] \cong [\text{Lift}_q 1_X].
\end{equation}

Now, suppose that $[\text{Lift}_p f] \neq \emptyset$ and take a $p$-lifting $g$ of $f$. Regarding $Y$ as the subset of $X \times E$, define the $F$-equivariant map

$$h : X \times F \to Y, \quad h(x, a) = (x, g(x)a), \quad x \in X, a \in F.$$

It is easy to see that the diagram

$$\begin{diagram}
X \times F \arrow{s}{p_1} \arrow{r}{h} & Y \\
X \arrow{r} & X \\
\end{diagram}$$

commutes, i.e. $h$ is a map over $X$. Since $X$ is a locally contractible paracompact space, and by a theorem of Dold \[Dold\], there exists a map $k : Y \to X \times F$ over $X$ which is homotopy inverse over $X$ to $h$. It is easy to see that $k$ is an equivariant map over $X$. Indeed, if $m_1 : X \times F \times F \to X \times F$ and $m_2 : Y \times F \to Y$ are the corresponding actions then

$$m_1(k \times 1) \simeq k m_1(k \times 1) \simeq k m_2(h \times 1)(k \times 1) \simeq k m_2(h k \times 1) \simeq k m_2,$$

where $\simeq$ denotes the homotopy over $X$.

In particular, there is an $[X, F]$-equivariant bijection

$$[\text{Lift}_q 1_X] \cong [\text{Lift}_{p_1} 1_X]$$

where $p_1 : X \times F \to X$ is the projection. Now we compose this bijection with \eqref{1.2} and get $[X, F]$-equivariant bijections

$$[\text{Lift}_p f] \cong [\text{Lift}_q 1_X] \cong [\text{Lift}_{p_1} 1_X],$$

and the result follows from Lemma \[1.4\] ☐

1.5. Example. If $p : E \to B$ is an $F$-fibration then $\Omega p : \Omega E \to \Omega B$ is a principal $\Omega F$-fibration. Here $\Omega$ denotes the loop functor.
2. Preliminaries on Classifying Spaces

Here we give a brief recollection on $\mathbb{R}^n$ bundles, spherical fibrations, and their classifying spaces. For details, see [Rud, Chapter IV].

2.1. Definition. We define a topological $\mathbb{R}^n$-bundle over a space $B$ to be an $\mathbb{R}^n$-bundle $p : E \to B$ equipped with a fixed section $s : B \to E$ (the zero section). Given two topological $\mathbb{R}^n$-bundles $\xi = \{ p : E \to B \}$ and $\eta = \{ q : Y \to X \}$, we define a topological $\mathbb{R}^n$-morphism $\varphi : \xi \to \eta$ to be a commutative diagram

\[
\begin{array}{ccc}
E & \xrightarrow{g} & Y \\
\downarrow p & & \downarrow q \\
B & \xrightarrow{f} & X
\end{array}
\]

where $g$ preserves the sections and induces a homeomorphism on each of fibers. The last one means that, for every $b \in B$, the map

\[ g_b : \mathbb{R}^n = p^{-1}(b) \to q^{-1}(f(b)) = \mathbb{R}^n, \quad g_b(a) = g(a) \text{ for all } a \in p^{-1}(b) \]

is a homeomorphism. As usual, we call $f$ the base of the morphism $\varphi$. and denote it also $\text{bs}(\varphi)$, i.e. $\text{bs}(\varphi) = f$.

We say that topological $\mathbb{R}^n$-morphism $\varphi$ is a morphism over $B$ if the map $f$ in (2.1) is equal to $1_B$.

A topological $\mathbb{R}^n$-morphism is a topological $\mathbb{R}^n$-isomorphism if the above mention $g$ is a homeomorphism.

We define two topological $\mathbb{R}^n$-morphisms $\varphi_0, \varphi_1 : \xi \to \eta$ to be bundle homotopic if there exists a topological $\mathbb{R}^n$-morphism $\Phi : \xi \times I \to \eta$ such that $\Phi|_{\xi \times \{i\}} = \varphi_i, i = 0, 1$.

A topological $\mathbb{R}^n$-morphism $\varphi : \xi \to \eta$ is a bundle homotopy equivalence if there exists a topological $\mathbb{R}^n$-morphism $\psi : \eta \to \xi$ such that $\varphi \psi$ and $\psi \varphi$ are bundle homotopic to the corresponding identity maps.

Frequently, we will say merely “homotopy” instead of “bundle homotopy”, etc. if this does not lead to confusions.

2.2. Theorem–Definition. There exists a topological $\mathbb{R}^n$-bundle $\gamma^n_{\text{TOP}}$ with the following universal property: For every topological $\mathbb{R}^n$-bundle $\xi$ over a CW-space $B$, every CW-subspace $A$ of $B$ and every morphism

\[ \psi : \xi_A \to \gamma^n_{\text{TOP}} \]

of topological $\mathbb{R}^n$-bundles, there exists a morphism $\varphi : \xi \to \gamma^n_{\text{TOP}}$ which is an extension of $\psi$. The base of $\gamma^n_{\text{TOP}}$ is called the classifying space for topological $\mathbb{R}^n$-bundles ans denoted by $B\text{TOP}_n$. 
We can regard topological $\mathbb{R}^n$-bundles as $(\text{TOP}_n, \mathbb{R}^n)$-bundles, i.e. $\mathbb{R}^n$-bundles with the structure group $\text{TOP}_n$. Here $\text{TOP}_n$ is the topological group of self-homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n, f(0) = 0$. The classifying space $B\text{TOP}_n$ of the group $\text{TOP}_n$ turns out to be a classifying space for topological $\mathbb{R}^n$-bundles.

Consider a topological $\mathbb{R}^n$-bundle $\xi$ over a CW space $B$. By the definition of universal bundle, there exists a topological $\mathbb{R}^n$-morphism $\varphi : \xi \to \gamma^n_{\text{TOP}}$. We call such $\varphi$ a classifying morphism for $\xi$. The base $f : B \to B\text{TOP}_n$ of $\varphi$ is called a classifying map for $\xi$. It is clear that $\xi$ is isomorphic over $B$ to $f^*\gamma^n_{\text{TOP}}$.

2.3. Proposition. If $\varphi_0, \varphi_1 : \xi \to \gamma^n_{\text{TOP}}$ be two classifying morphisms for $\xi$, then there are homotopic. In particular, a classifying map $f$ for $\xi$ is determined by $\xi$ uniquely up to homotopy.

Proof. This follows from the universal property 2.2 applied to $\xi \times I$, if we put $A = X \times \{0, 1\}$ where $X$ denotes the base of $\xi$. □

2.4. Remark. This is important to understand the difference between classifying maps and classifying morphisms. Non-homotopic classifying morphisms can induce homotopic classifying maps. On the other hand, not every map $X \to B\text{TOP}_n$ is a classifying map, while every morphism $\xi \to \gamma^N$ is a classifying morphism. To feel the difference, consider a trivial $\mathbb{R}^n$-bundle $\theta^N$ over a space $X$. Then there is a classifying map $X \to \ast \in B\text{TOP}_n$, while a corresponding morphism is a trivialization of $\theta^N$, i.e. a morphism $X \times \mathbb{R}^n \to \mathbb{R}^n$.

A piecewise linear (in future PL) $\mathbb{R}^n$-bundle is a topological $\mathbb{R}^n$-bundle $\xi = \{p : E \to B\}$ such that $E$ and $B$ are polyhedra and $p : E \to B$ and $s : B \to E$ are PL maps. Furthermore, we require that, for every simplex $\Delta \subset B$, there is a PL homeomorphism $h : p^{-1}(\Delta) \cong \Delta \times \mathbb{R}^n$ with $h(s(\Delta)) = \Delta \times \{0\}$. (For definitions of PL maps, see [Hud, RS].)

A PL morphism of PL $\mathbb{R}^n$-bundles is a topological $\mathbb{R}^n$-morphism where the maps $g$ and $f$ in (2.1) are PL maps.

There exists a universal PL $\mathbb{R}^n$-bundle $\gamma^n_{\text{PL}}$ over a certain space $B\text{PL}_n$. This means that the universal property 2.2 remains valid if we replace $\gamma^n_{\text{TOP}}$ by $\gamma^n_{\text{PL}}$ and “topological $\mathbb{R}^n$ bundle” by “PL $\mathbb{R}^n$-bundle” there. So, $B\text{PL}_n$ is a classifying space for PL $\mathbb{R}^n$-bundles.

This is worthy to mention that $B\text{PL}_n$ can be chosen to be a locally finite simplicial complex, [KS2, Essay IV §8].
Note that $BPL_n$ can also be regarded as the classifying space of a certain group $PL_n$ (which is constructed as the geometric realization of a certain simplicial group), $[KL, LR]$.

A sectioned $S^n$-fibration is defined to be an $S^n$-fibration $p : E \to B$ equipped with a section $s : B \to E$. Morphisms of sectioned $S^n$-fibrations are defined similarly to Definition 2.1 where each map $g_b$ and the total map $g$ is assumed to be a pointed proper homotopy equivalence. We shall use the brief term “$(S^n, \ast)$-fibration” for sectioned $S^n$-fibrations and “$(S^n, \ast)$-morphism” for morphisms of sectioned $S^n$-fibrations.

There exists a universal sectioned $S^n$-fibration $\gamma_n$. To define it, replace $\gamma_n^{TOP}$ by $\gamma_n^F$ and “topological $R^n$ bundle” by “sectioned $S^n$-fibration” in 2.2. The base $BF_n$ of $\gamma_n^F$ is called the classifying space for sectioned $S^n$-fibrations. The space $BF_n$ can also be regarded as the classifying space for the monoid $F_n$ of pointed homotopy equivalences $(S^n, \ast) \to (S^n, \ast)$.

We need also to recall the space $BO_n$ which classifies $n$-dimensional vector bundles. This well-known space is described in many sources, e.g. $[MS]$. The universal vector bundle over $BO_n$ is denoted by $\gamma_n^O$.

It is worthy to notice that the spaces $BO_n$, $BPL_n$, $BTOP_n$ and $BF_n$ are defined uniquely up to weak homotopy equivalence.

We regard $\gamma_n^{PL}$ as the (underlying) topological $R^n$-bundle and get the classifying morphism

$$\omega = \omega^{PL}_{TOP}(n) : \gamma_n^{PL} \to \gamma_n^{TOP}.$$  

We denote by $\alpha = \alpha^{PL}_{TOP}(n) : BPL_n \to BTOP_n$ the base of this morphism.

Given a topological $R^n$-bundle $\xi = \{p : E \to B\}$, let $\xi^\ast$ denote the $S^n$-bundle

$$\xi^\ast = \{p^\ast : E^\ast \to B\}$$

where $E^\ast$ is the fiberwise one-point compactification of $E$. Note that the added points (“infinities”) give us a certain section of $\xi^\ast$.

In other words, the $TOP_n$-action on $R^n$ extends uniquely to a $TOP_n$-action on the one-point compactification $S^n$ of $R^n$, and $\xi^\ast$ is the $(TOP_n, S^n)$-bundle associated with $\xi$. Furthermore, the fixed point $\infty$ of the $TOP_n$-action on $S^n$ yields a section of $\xi^\ast$.

So, $\xi^\ast$ can be regarded as an $(S^n, \ast)$-fibration over $B$. In particular, $(\gamma_n^{TOP})^\ast$ can be regarded as an $(S^n, \ast)$-fibration over $BTOP_n$. So, there
is a classifying morphism
\[ \omega_{F}^{TOP}(n) : \gamma_{TOP}^{n} \rightarrow \gamma_{F}^{n}. \]
We denote by \( \alpha_{F}^{TOP}(n) : BTOP_n \rightarrow BF_n \) the base of \( \omega_{F}^{TOP}(n) \).

Finally, we note that an \( n \)-dimensional vector bundle over a polyhedron \( X \) has a canonical structure of PL \( \mathbb{R}^n \)-bundle over \( X \). Similarly to above, this gives us a (forgetful) map
\[ \alpha_{PL}^{O}(n) : BO_n \rightarrow BPL_n. \]

So, we have a sequence of forgetful maps
\[ BO_n \xrightarrow{\alpha'} BPL_n \xrightarrow{\alpha''} BTOP_n \xrightarrow{\alpha'''} BF_n \]
where \( \alpha' = \alpha_{PL}^{O}(n) \), etc.

2.5. CONSTRUCTIONS. 1. Given an \( F \)-bundle \( \xi = \{ p : E \to B \} \) and an \( F' \)-bundle \( \xi' = \{ p' : E' \to B' \} \), we define the smash product \( \xi \wedge \xi' \) to be the \( F \times F' \)-bundle
\[ p \times p' : E \times E' \to B \times B'. \]

2. Given an \( F \)-bundle \( \xi = \{ p : E \to B \} \) with a section \( s : B \to E \) and an \( F' \)-bundle \( \xi' = \{ p' : E' \to B' \} \) with a section \( s' : B' \to E' \), we define the smash product \( \xi \wedge \xi' \) to be the \( F \wedge F' \)-bundle as follows. The map \( p \times p' : E \times E' \to B \times B' \) passes through the quotient map \( q : E \times E' \to E \times E'/\{(E \times s(B')) \cup E' \times s(B) \} \), and we set
\[ \xi \wedge \eta = \{ \pi : E \times E'/\{(E \times s(B')) \cup E' \times s(B) \} \to B \times B' \}, \]
where \( \pi \) is the unique map with \( p \times p' = \pi q \). Finally, the section \( s \) and \( s' \) yield an obvious section of \( \pi \).

3. Given an \( \mathbb{R}^m \)-bundle \( \xi \) and an \( \mathbb{R}^n \)-bundle \( \eta \) over the same space \( X \), the *Whitney sum* of \( \xi \) and \( \eta \) is the \( \mathbb{R}^{m+n} \)-bundle \( \xi \oplus \eta = d^*(\xi \times \eta) \) where \( d : X \to X \times X \) is the diagonal.

Note that if \( \xi \) and \( \eta \) are a PL \( \mathbb{R}^m \) and PL \( \mathbb{R}^n \)-bundle, respectively, then \( \xi \times \eta \) is a PL \( \mathbb{R}^{m+n} \)-bundle.

4. Given a sectioned \( S^m \)-bundle \( \xi \) and sectioned \( S^n \)-bundle \( \eta \) over the same space \( X \), we set \( \xi \uplus \eta = d^*(\xi \wedge \eta). \)

We denote by \( r_n = r_{TOP}^{n} : BTOP_n \to BTOP_{n+1} \) the map which classifies \( \gamma_{TOP}^{n} \oplus \theta_{BTOP_n}^{n} \). The maps \( r_{PL}^{n} : BPL_n \to BPL_{n+1} \) and \( r_{O}^{n} : BO_n \to BO_{n+1} \) are defined in a similar way.

We can also regard the above map \( r_n : BTOP_n \to BTOP_{n+1} \) as a map induced by the standard inclusion \( TOP_n \subset TOP_{n+1} \). Using this
approach, we define $r_n^F : BF_n \to BF_{n+1}$ as the map induced by the standard inclusion $F_n \subset F_{n+1}$, see [MM, p. 45].

2.6. REMARKS. 1. Regarding $R^n_m$ as the bundle over the point, we see that $(R^n_m)^* = (S^n_m)$ and, moreover,

$$(R^m \times R^n)^* = S^m \wedge S^n, \text{ i.e. } (R^m \oplus R^n)^* = S^m \pitchfork S^n.$$ 
Therefore $(\xi \oplus \eta)^* = \xi^* \pitchfork \eta^*$ for every $R^m$-bundle $\xi$ and $R^n$-bundle $\eta$.

2. Generally, the smash product of (sectioned) fibrations is not a fibrations. But we apply it to bundles only and so do not have any troubles. On the other hand, there is an operation $\wedge^h$, the homotopy smash product, such that $\xi \wedge^h \eta$ is the $(F \wedge G)$-fibration over $X \times Y$ if $\xi$ is an $F$-fibration over $X$ and $\eta$ is an $G$-fibration over $Y$, see [Rud]. In particular, one can use it in order to define an analog of Whitney sum for spherical fibrations and then use this one in order to construct the map $BF_n \to BF_{n+1}$.

Now we consider the classifying spaces $BO_n, BPL_n, BTOP_n$ and $BF_n$ as $n \to \infty$. In greater detail, we do the following.

Choose classifying spaces $B'F_n$ for $(S^n, *)$-fibrations (i.e., in the weak homotopy type $BF_n$) and consider the maps $r_n^F : B'F_n \to B'F_{n+1}$ as above. We can assume that every $B'F_n$ is a $CW$-complex and every $r_n$ is a cellular map. We define $BF$ to be the telescope (homotopy direct limit) of the sequence

$$\cdots \longrightarrow B'F_n \xrightarrow{r_n} B'F_{n+1} \longrightarrow \cdots,$$

see e.g. [Rud, Definition I.3.19]. Furthermore, we define $BF_n$ to be the telescope of the finite sequence

$$\cdots \longrightarrow B'F_{n-1} \xrightarrow{r_{n-1}} B'F_n.$$ 
(Note that $BF_n \simeq B'F_n$.) So, we have the sequence (filtration)

$$\cdots \subset BF_n \subset BF_{n+1} \subset \cdots.$$ 

So, $BF = \bigcup BF_n$ and $BF_n$ is closed in $BF$. Moreover, $BF$ has the direct limit topology with respect to the filtration $\{BF_n\}$. Furthermore, if $f : K \to BF$ is a map of a compact space $K$ then there exists $n$ such that $f(K) \subset BF_n$.

Now, for every $n$ consider a $CW$-space $B'TOP_n$ in the weak homotopy type $BTOP_n$ and define $B''TOP$ to be the telescope of the sequence

$$\cdots \longrightarrow B'TOP_n \xrightarrow{r_n} B'TOP_{n+1} \longrightarrow \cdots.$$
Furthermore, we define $B''TOP_n$ to be the telescope of the finite sequence

$$\cdots \longrightarrow B'TOP_{n-1} \overset{r_{n-1}}{\longrightarrow} B'TOP_n.$$ 

So, we have the diagram

$$\cdots \subset B''TOP_n \subset B''TOP_{n+1} \subset \cdots \subset B''TOP$$

(2.5)

$$\cdots \subset BF_n \subset BF_{n+1} \subset \cdots \subset BF$$

where the map $p$ is induced by maps $\alpha^{TOP}_F(n)$. Now we apply the Serre construction and replace every vertical map in the diagram (2.5) by its fibrational substitute. Namely, we set

$$BTOP = \{(x, \omega) \mid x \in B''TOP, \omega \in (BF)^I, \ p(x) = \omega(0)\}$$

and define $\alpha^{TOP}_F : BTOP \to BF$ by setting $\alpha^{TOP}_F(x, \omega) = \omega(1)$. Finally, we set

$$BTOP_n = \{(x, \omega) \in BTOP \mid x \in B''TOP_n, \omega \in (BF_n)^I \subset (B''TOP)^I\}$$

and get the commutative diagram

$$\cdots \subset BTOP_n \subset BTOP_{n+1} \subset \cdots \subset BTOP$$

$$\cdots \subset BF_n \subset BF_{n+1} \subset \cdots \subset BF$$

where all the vertical maps are fibrations.

Now it is clear how to proceed and get the diagram

$$\cdots \subset BO_n \subset BO_{n+1} \subset \cdots \subset BO$$

$$\cdots \subset BPL_n \subset BPL_{n+1} \subset \cdots \subset BPL$$

(2.6)

$$\cdots \subset BTOP_n \subset BTOP_{n+1} \subset \cdots \subset BTOP$$

$$\cdots \subset BF_n \subset BF_{n+1} \subset \cdots \subset BF$$

where all the vertical maps are fibrations. Moreover, each of limit spaces has the direct limit topology with respect to the corresponding filtration, and every compact subspace of, say, $BO$ is contained in some $BO_n$. 
2.7. Convention. Let \( \xi \) classify a map \( f_n : X \to BF_n \) (say). It is convenient for us to speak about \( n = \infty \) and write that a map \( f : X \to BF \) classify \( \xi \) if \( f \) can be expressed as
\[
f : X \xrightarrow{\sim} BF_n \xrightarrow{\subset} BF.
\]

Take a point \( b \in BTOP \), put \( (TOP/PL)_b := (\alpha^\text{PL}_{TOP})^{-1}(b) \) to be the fiber of \( \alpha = \alpha^\text{PL}_{TOP} \), and put
\[
\beta = \beta_b : (TOP/PL)_b \to BPL
\]
to be the inclusion of the fiber. In further we allow us to omit the subscript \( b \) and write the fibration \( \alpha \) as
\[
\text{(2.7)} \quad TOP/PL \xrightarrow{\beta^\text{PL}_{TOP}} BPL \xrightarrow{\alpha^\text{PL}_{TOP}} BTOP.
\]
This will not lead to confusions because, if we choose another point \( b' \in BTOP \), then the maps \( \beta_b \) and \( \beta_{b'} \) occur to be homotopy equivalent. We also use the notation \( TOP/PL \) for the homotopy fiber of the map \( \alpha : BPL \to BTOP \).

The homotopy fiber of \( \alpha^O_{PL} : BO \to BPL \) is denoted by \( PL/O \), the fiber of \( \alpha^F_{TOP} \) is denoted by \( F/TOP \), etc. Similarly, the homotopy fiber of the composition, say,
\[
\alpha^F_{PL} := \alpha^F_{TOP} \circ \alpha^\text{PL}_{TOP} : BPL \to BF
\]
is denoted by \( F/PL \). In particular, we have a fibration
\[
\text{(2.8)} \quad TOP/PL \xrightarrow{a} F/PL \xrightarrow{b} F/TOP.
\]
Finally, note that \( F/TOP = \bigcup F_n/TOP \) where \( F_n/TOP \) denotes the fiber of the fibration \( BTOP_n \to BF_n \), and \( F/TOP \) has the direct limit topology with respect to the filtration \( \{F_n/TOP\} \). The same holds for other “homogeneous spaces” \( F/PL, TOP/PL \), etc.

Because of well-known results of Milnor [Mil], all these “homogeneous spaces” have the homotopy type of \( CW \)-spaces. Furthermore, all the spaces \( BO, BPL, BTOP, BF, F/PL, TOP/PL \), etc. are infinite loop spaces, and the maps like in \((2.7)\) \((2.8)\) are infinite loop maps, see [BV]. In particular, the classifying spaces \( BO \), etc. are homotopy associative and invertible \( H \)-spaces, and the fibrations \((2.7), (2.8)\), etc. are principal fibrations.

We mention also the following useful fact.

2.8. Lemma. Let \( Z \) denote one of the symbols \( O, PL, F \). The above described map \( BZ_n \to BZ_{n+1} \) induces an isomorphism of homotopy groups in dimensions \( \leq n - 1 \) and an epimorphism in dimension \( n \).
3. Structures on Manifolds and Bundles

2.9. Remark. An analog of Lemma 2.8 holds for TOP as well, see Remark 3.10.

2.10. Remark. Let $G_n$ denote the topological monoid of homotopy self-equivalences $S^{n-1} \to S^{n-1}$. Then the classifying space $BG_n$ of $G_n$ classifies $S^{n-1}$-fibrations (non-sectioned). Every $h \in TOP_n$ induces a map $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ which, in turn, yields a self-map

$$\pi_h : S^{n-1} \to S^{n-1}, \quad \pi_h(x) = h(x)/\|h(x)\|.$$ 

So, we have a map $TOP_n \to G_n$ which, in turn, induces a map $BTOP_n \to BG_n$ of classifying spaces. In the language of bundles, this map converts a topological $\mathbb{R}^n$-bundle into a (non-sectioned) spherical fibration via deletion of the section.

We can also consider the space $BG$ by tending $n$ to $\infty$. In particular, we have the spaces $G/PL$ and $G/TOP$.

There is an obvious forgetful map $F_n \to G_{n+1}$ (ignore sections), and it turns out that the induced map $BF \to BG$ (as $n \to \infty$) is a homotopy equivalence, see e.g. [MM, Chapter 3]. In particular, $F/PL \simeq G/PL$ and $F/TOP \simeq G/TOP$.

3. Structures on Manifolds and Bundles

A PL atlas on a topological manifold is an atlas such that all the transition maps are PL ones. We define a PL manifold as a topological manifold with a maximal PL atlas. Furthermore, given two PL manifolds $M$ and $N$, we say that a homeomorphism $h : M \to N$ a PL homeomorphism if $h$ is a PL map. (One can prove that in this case $h^{-1}$ is a PL map as well, [Hud].)

3.1. Definition. (a) We define a $\partial_{PL}$-manifold to be a topological manifold whose boundary $\partial M$ is a PL manifold. In particular, every closed topological manifold is a $\partial_{PL}$-manifold. Furthermore, every PL manifold can be canonically regarded as a $\partial_{PL}$-manifold.

(b) Let $M$ be a $\partial_{PL}$-manifold. A PL structuralization of $M$ is a homeomorphism $h : V \to M$ such that $V$ is a PL manifold and $h$ induces a PL homeomorphism $\partial V \to \partial M$ of boundaries (or, equivalently, PL homeomorphism of corresponding collars). Two PL structuralizations $h_i : V_i \to M, i = 0, 1$ are concordant if there exist a PL homeomorphism $\varphi : V_0 \to V_1$ and a homeomorphism $H : V_0 \times I \to M \times I$ such that
\( H|_{V \times \{0\}} = h_0 \) and \( H|_{V_0 \times \{1\}} = h_1 \varphi \) and, moreover, \( H : \partial V_0 \times I \to \partial M \times I \) coincides with \( h_0 \times 1_I \). Any concordance class of PL structuralizations is called a PL structure on \( M \). We denote by \( \mathcal{T}_{PL}(M) \) the set of all PL structures on \( M \).

(c) If \( M \) on its own is a PL manifold then \( \mathcal{T}_{PL}(M) \) contains the distinguished element: the concordance class of \( 1_M \). We call it the trivial element of \( \mathcal{T}_{PL}(M) \).

3.2. REMARKS. 1. Clearly, every PL structuralization of \( M \) equips \( M \) with a certain PL atlas. Conversely, if we equip \( M \) with a certain PL atlas then the identity map can be regarded as a PL structuralization of \( M \).

2. If \( M \) by itself is a PL manifold then the concordance class of any PL homeomorphism \( h : V \to M \) is the trivial element of \( \mathcal{T}_{PL}(M) \). Indeed, to prove this, we must find a homeomorphism \( H : V \times I \to M \times I \) and a PL homeomorphism \( \varphi : V \to M \) such that \( H|_{V \times \{0\}} = h \) and \( H|_{V \times \{1\}} = 1_M \varphi = \varphi \). But this is easy: put \( \varphi = h \) and \( H(v,t) = (h(v),t) \).

3. Recall that two homeomorphism \( h_0, h_1 : X \to Y \) are isotopic if there exists a homeomorphism \( H : X \times I \to Y \times I \) (isotopy) such that \( p_2H : X \times I \to Y \times I \to I \) coincides with \( p_2 : X \times I \to I \). Given \( A \subseteq X \), we say that \( h_0 \) and \( h_1 \) are isotopic rel \( A \) if there exists an isotopy \( H \) such that \( H(a,t) = h_0(a) \) for every \( a \in A \) and every \( t \in I \). In particular, if two PL structuralization \( h_0, h_1 : V \to M \) are isotopic rel \( \partial V \) then they are concordant.

4. Given two PL structuralizations \( h_i : V_i \to M, i = 0, 1 \), they are not necessarily concordant if \( V_0 \) and \( V_1 \) are PL homeomorphic. We are not able to give such examples here, but we do it later, see Remark 3.11(2) and Example 21.2.

3.3. DEFINITION (cf. [Br, Rud]). Given a topological \( \mathbb{R}^n \)-bundle \( \xi \), define a PL structuralization of \( \xi \) to be a topological \( \mathbb{R}^n \)-morphism \( \varphi : \xi \to \gamma^n_{PL} \). We define a PL structure on \( \xi \) to be a homotopy class of PL structuralizations of \( \xi \).

Let \( f_n : X \to B\text{TOP}_n \) classify a topological \( \mathbb{R}^n \)-bundle \( \xi \), and assume that there is an \( \alpha_{\text{TOP}}^n \)-lifting
\[
g_n : X \to B\text{PL}_n
\]
of \( f_n \). Take the \( g_n \)-adjoint classifying morphism
\[
\mathcal{J} : \mathcal{J}_{g_n} : g_n^* \gamma^n \to \gamma^n.
\]
and consider the morphism
\[
\xi \cong f_n^* \gamma^n_{\text{TOP}} = g_n^* \alpha(n)^* \gamma^n_{\text{TOP}} = g_n^* \gamma^n_{\text{PL}} \xrightarrow{\mathcal{J}} \gamma^n_{\text{PL}}, \quad \alpha(n) := \alpha_{\text{TOP}}^n(n).
\]
This morphism \( \xi \to \gamma_{PL} \) is a PL structuralization of \( \xi \). It is easy to see that in this way we have a correspondence

\[
\text{(3.1)} \quad \text{[Lift}_{\alpha(n)} f_n] \to \text{\{PL structures on } \xi \}.\]

3.4. Theorem. The correspondence (3.1) is a bijection.

Proof. See [Rud, Theorem IV.2.3], cf. also [Br, Chapter II]. \( \square \)

Consider now the map

\[
f : X \xrightarrow{f_n} BTOP_n \subset BTOP
\]

and the map \( \alpha = \alpha_{PL}^{TOP} : BPL \to BTOP \) as in (2.6). Every \( \alpha(n) \)-lifting \( g_n : X \to BPL_n \) of \( f_n \) gives us the \( \alpha \)-lifting

\[
X \xrightarrow{g_n} BPL_n \rightarrow BPL
\]

of \( f \). So, we have a correspondence

\[
\text{(3.2)} \quad u_\xi : \text{\{PL structures on } \xi \} \to \text{[Lift}_{\alpha(n)} f_n] \to \text{[Lift}_{\alpha} f]
\]

where the first map is the inverse to (3.1). Furthermore, there is a canonical map

\[
v_\xi : \text{\{PL structures on } \xi \} \to \text{\{PL structures on } \xi \oplus \theta^1\},
\]

and these maps respect the maps \( u_\xi \), i.e. \( u_{\xi \oplus \theta^1} = v_\xi u_\xi \). So, we have the map

\[
\text{(3.3)} \quad \lim_{n \to \infty} \text{\{PL structures on } \xi \oplus \theta^n\} \to \text{[Lift}_{\alpha} f]
\]

where \( \lim \) means the direct limit of the sequence of sets.

3.5. Proposition. If \( X \) is a finite CW-space then the map (3.3) is a bijection.

Proof. The surjectivity follows since every compact subset of \( BTOP \) is contained in some \( BTOP_n \). Similarly, every map \( X \times I \to BPL \) passes through some \( BPL_n \), and therefore the injectivity holds. \( \square \)

The space \( TOP/PL \) is a homotopy associative and homotopy invertible \( H \)-space, and hence the set \( [X, TOP/PL] \) has a natural group structure. Here the neutral element is the homotopy class of inessential map \( X \to TOP/PL \). Now, consider a principal \( F \)-fibration \( F \to E \to B \) as in Definition 1.1 and apply it to the case

\[
TOP/PL \xrightarrow{\beta} BPL \xrightarrow{\alpha} BTOP.
\]

Then for every map \( f : X \to BTOP \) we have a right action

\[
r : \text{[Lift}_{\alpha} f] \times [X, TOP/PL] \to \text{[Lift}_{\alpha} f]
\]
3.6. Proposition. Suppose that the map \( f : X \to B\text{TOP} \) lifts to \( B\text{PL} \). Then the action \( r \) is transitive. Furthermore, for every \( \alpha \)-lifting \( g \) of \( f \) the map
\[
[X, \text{TOP}/\text{PL}] \longrightarrow [\text{Lift}_\alpha f], \quad \varphi \mapsto r(g, \varphi)
\]
is a bijection.

Proof. See Theorem 1.3. \( \square \)

Note that, in view of Propositions 3.3 and 3.6, if a topological bundle \( \xi \) admits a PL structure then the bijection (3.3) turns into the bijection (3.4)
\[
\lim_{n \to \infty} \{\text{PL structures on } \xi \oplus \theta^n\} \longrightarrow [X, \text{TOP}/\text{PL}]
\]
provided \( X \) is a finite CW space.

3.7. Definition. Let \( M \) be a \( \partial\text{PL} \)-manifold. A homotopy PL structuralization of \( M \) is a homotopy equivalence \( h : (V, \partial V) \to (M, \partial M) \) such that \( V \) is a PL manifold and \( h|_{\partial V} : \partial V \to \partial M \) is a PL homeomorphism. Two homotopy PL structuralizations \( h_i : V_i \to M, i = 0, 1 \) are equivalent if there exists a PL homeomorphism \( \varphi : V_0 \to V_1 \) such that \( h_1 \varphi \) is homotopic to \( h_0 \) rel \( \partial V \). In detail, there is a homotopy \( H : V_0 \times I \to M \) such that \( H|_{V_0 \times \{0\}} = h_0 \) and \( H|_{V_0 \times \{1\}} = h_1 \varphi \) and, moreover, \( H|_{V_0 \times \{t\}} : \partial V_0 \to \partial M \) coincides with \( h_0 \). Any equivalence class of homotopy PL structuralizations is called a homotopy PL structure on \( X \). We denote by \( S_{\text{PL}}(X) \) the set of all homotopy PL structures on \( X \).

If \( M \) itself is a PL manifold, we define the trivial element of \( S_{\text{PL}}(M) \) as the equivalence class of \( 1_M : M \to M \).

Pay attention to the map
\[
T_{\text{PL}}(M) \xrightarrow{\phi} S_{\text{PL}}(M)
\]
that regards a PL structuralization as the homotopy PL structuralization.

3.8. Definition. Given an \((S^n, \ast)\)-fibration \( \xi \) over \( X \), a homotopy PL structuralization of \( \xi \) is an \((S^n, \ast)\)-morphism \( \varphi : \xi \to (\gamma^n_{\text{PL}})\ast \). We say that two PL structuralizations \( \varphi_0, \varphi_1 : \xi \to (\gamma^n_{\text{PL}})\ast \) are equivalent if there exists a morphism \( \Phi : \xi \times 1_I \to (\gamma^n_{\text{PL}})\ast \) of \((S^n, \ast)\)-fibrations such that \( \Phi|_{\xi \times \{i\}} = \varphi_i, i = 0, 1 \). Every such an equivalence class is called a homotopy PL structure on \( \xi \).
Now, similarly to (3.4), for a finite CW-space $X$ we have a bijection
\[
\lim_{n \to \infty} \{\text{homotopy PL structures on } \xi \oplus \theta^n\} \to [X, F/PL].
\]
However, here we can say more.

3.9. **Proposition.** The sequence
\[
\{\{\text{homotopy PL structures on } \xi \oplus \theta^n\}\}_{n=1}^{\infty}
\]
stabilizes. In particular, the map
\[
\{\text{homotopy PL structures on } \xi \oplus \theta^n\} \to [F/PL]
\]
is a bijection if $\dim \xi > > \dim X$.

*Proof.* This follows from 2.8. \qed

Thus, for every $\mathbb{R}^N$-bundle $\xi$ that admits a PL structure we have a commutative diagram
\[
\begin{array}{ccc}
\{\text{PL structures on } \xi\} & \longrightarrow & [X, TOP/PL] \\
\downarrow & & \downarrow a_* \\
\{\text{homotopy PL structures on } \xi^*\} & \longrightarrow & [X, F/PL]
\end{array}
\]

Here the right vertical map $a$ in (2.8) induces the map $a_* : [X, TOP/PL] \to [X, F/PL]$. The left vertical arrow converts a morphism of $\mathbb{R}^N$-bundles into a morphism of $(S^N, *)$-bundles and regards the last one as a morphism of $(S^N, *)$-fibrations.

For a finite CW-space $X$, the horizontal arrows turn into bijections if we stabilize the picture, i.e. pass to the limit as in (3.4). Furthermore, the bottom arrow is an isomorphism if $N > > \dim X$.

3.10. **Remark.** Actually, following the proof of the Main Theorem, one can prove that $TOP_m/PL_m = K(\mathbb{Z}/2, 3)$ for $m \geq 5$, see [KS2, Essay V, §5]. So, an obvious analog of 2.8 holds for $TOP$ also, and therefore the top map of the above diagram is a bijection for $N$ large enough. But, of course, we are not allowed to use these a posteriori arguments here, until we accomplish the proof of the Main Theorem.

3.11. **Remark.** We can also consider smooth (= differentiable $C^\infty$) structures on topological manifolds. To do this, we must replace the words “PL” in Definition 3.3 by the word “smooth”. The related set of smooth concordance classes is denoted by $\mathcal{T}_D(M)$.

The set $\mathcal{S}_D(M)$ of homotopy smooth structures is defined in a similar way: replace the words “PL” in Definition 3.7 by the word “smooth”.

\[3. \text{ STRUCTURES ON MANIFOLDS AND BUNDLES}]\]
Moreover, every smooth manifold can be canonically converted into a PL manifold (S. Cairns and J. Whitehead Theorem \cite{Cai, W}, see e.g. \cite{HM}). So, we can define the set $\mathcal{P}_D(M)$ of smooth structures on a PL manifold $M$. To do this, we must modify definition 3.1 as follows: $M$ is a PL manifold with a compatible smooth boundary, $V_i$ are smooth manifolds, $h_i$ and $H$ are PL isomorphisms.

For convenience of references, we fix here the following theorem of Smale \cite{Sma}. Actually, Smale proved it for smooth manifolds, a good proof can also be found in Milnor \cite{Mi4}. However, the proof can be transmitted to the PL case, see Stallings \cite{Sta, 8.3, Theorem A}.

3.12. Theorem. Let $M$ be a closed PL manifold that is homotopy equivalent to the sphere $S^n$, $n \geq 5$. Then $M$ is PL homeomorphic to $S^n$. \hfill $\blacksquare$

3.13. Example. Now we construct an example of two smooth structuralizations $h_i : V \to S^n$, $i = 1, 2$ that are not concordant. First, note that there is a bijective correspondence between $\mathcal{S}_D(S^n)$ and the Kervaire–Milnor group $\Theta_n$ of homotopy spheres, \cite{KM}. Indeed, $\Theta_n$ consists of equivalence classes of oriented homotopy spheres: two oriented homotopy spheres are equivalent if they are orientably diffeomorphic (= h-cobordant). Now, given a homotopy smooth structuralization $h : \Sigma^n \to S^n$, we orient $\Sigma^n$ so that $h$ has degree 1. In this way we get a well-defined map $u : \mathcal{S}_D(S^n) \to \Theta_n$. Conversely, given a homotopy sphere $\Sigma^n$, consider a homotopy equivalence $h : \Sigma^n \to S^n$ of degree 1. In this way we get a well-defined map $\Theta_n \to \mathcal{S}_D(S^n)$ which is inverse to $u$.

Note that, because of the Smale Theorem, every smooth homotopy sphere $\Sigma^n$, $n \geq 5$, possesses a smooth function with exactly two critical points. Thus, $\mathcal{S}_D(S^n) = \mathcal{T}_D(S^n) = \mathcal{P}_D(S^n)$ for $n \geq 5$. Kervaire and Milnor \cite{KM} proved that $\Theta_7 = \mathbb{Z}/28$, i.e., because of what we said above, $\mathcal{S}_D(S^7) = \mathcal{T}_D(S^7) = \mathcal{P}_D(S^7)$ consists of 28 elements.

On the other hand, there are only 15 classes of diffeomorphism of smooth manifolds which are homeomorphic (and PL homeomorphic, and homotopy equivalent) to $S^7$ but mutually non-diffeomorphic. Indeed, if an oriented smooth 7-dimensional manifold $\Sigma$ is homeomorphic to $S^7$ then $\Sigma$ bounds a parallelizable manifold $W_\Sigma$, \cite{KM}. We equip $W$ an orientation which is compatible with $\Sigma$ and set

$$a(\Sigma) = \frac{\sigma(W_\Sigma)}{8} \mod 28$$
where $\sigma(W)$ is the signature of $W$. Kervaire and Milnor [KM] proved that the correspondence

$$\Theta_7 \to \mathbb{Z}/28, \quad \Sigma \mapsto a(W_\Sigma)$$

is a well-defined bijection.

However, if $a(\Sigma_1) = -a(\Sigma_2)$ then $\Sigma_1$ and $\Sigma_2$ are diffeomorphic: namely, $\Sigma_2$ is merely the $\Sigma_1$ with the opposite orientation. So, there are only 15 smooth manifolds which are homeomorphic (and homotopy equivalent, and PL homeomorphic) to $S^7$ but mutually non-diffeomorphic.

In terms of structures, it can be expressed as follows. Let $\rho : S^n \to S^n$ be a diffeomorphism of degree -1. Then the smooth structuralizations $h : \Sigma^7 \to S^7$ and $\rho h : \Sigma^7 \to S^7$ are not concordant, if $a(\Sigma^7) \neq 0, 14$.

4. From Manifolds to Bundles

Recall that, for every topological manifold $M^n$, its tangent bundle $\tau_M$ and normal (with respect to an embedding $M \subset \mathbb{R}^{N+n}$, $N \gg n$) PL $\mathbb{R}^N$-bundle $\nu_M$ are defined. Here $\tau_M$ is a topological $\mathbb{R}^n$-bundle, and we can regard $\nu_N$ as a topological $\mathbb{R}^N$-bundle. Furthermore, if $M$ is a PL manifold then $\tau_M$ and $\nu_M$ turns into PL bundles in a canonical way, see [KS2, Rud].

Concerning tangent and normal (micro)bundles and their properties, see Milnor [Mi3] for the topological category and Haefliger–Wall [HW] for the PL category.

4.1. Construction. Consider a manifold $M$ (possibly with boundary) and a PL structuralization $h : V \to M$. Let $g = h^{-1} : M \to V$. Since $g$ is a homeomorphism, it yields a topological morphism $\lambda_g : \tau_M \to \tau_V$ where $\tau_V, \tau_M$ denote the tangent bundles to $V, M$ respectively, and so we have the correcting topological morphism $c(\lambda_g) : \tau_M \to \lambda^* \tau_V$. Now, let $\nu = \nu_M$ be a normal bundle of $M$ in $\mathbb{R}^{N+n}$ with $N$ large enough. Consider the topological morphism

$$\theta_{M+\nu}^N = \tau_M \oplus \nu_M \longrightarrow g^* \tau_V \oplus \nu_M \xrightarrow{\text{classify}} \gamma_{PL}$$

and regard it as a PL structuralization of $\theta_{M+\nu}^N$. It is easy to see that in this way we have the correspondence

$$(4.1) \quad \jmath_{TOP} : \mathcal{T}_{PL}(M) \to \lim_{n \to \infty} \{\text{PL structures on } \theta_M^N\} \to [M, TOP/PL]$$

where the last map comes from (3.4).
Moreover, since \( g : \partial M \to \partial V \) is a PL homeomorphism, we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{T}_{PL}(M) & \xrightarrow{j_{TOP}} & [M, TOP/PL] \\
\downarrow & & \downarrow \\
\mathcal{T}_{PL}(\partial M) & \longrightarrow & pt \\
\end{array}
\]

cf. Remark 3.2(2). So, we can (and sometimes shall) regard the map

\[
j_{TOP} \text{ from (4.1) as the map} \quad j_{TOP} : \mathcal{T}_{PL}(M) \longrightarrow [(M, \partial M), (TOP/PL,*)].
\]

Now we construct a map \( j_F : S_{PL}(M) \to [M, F/PL] \), a “homotopy analogue” of \( j_{TOP} \). This construction is more delicate, and we treat here the case of closed manifolds only. So, let \( M \) be a connected closed PL manifold.

4.2. Definition. Given an \((S^n,\ast)-fibration\) \( \xi = \{E \to B\} \) with a section \( s : B \to E \), we define its Thom space \( T\xi \) as the quotient space \( E/s \). Given a topological \( \mathbb{R}^N \)-bundle \( \eta \), we define the Thom space \( T\eta \) as \( T\eta := T(\eta^*) \).

Given a morphism \( \varphi : \xi \to \eta \) of \((S^n,\ast)-fibrations\), we define \( T\varphi : T\xi \to T\eta \) to be the unique map such that the diagram

\[
\begin{array}{ccc}
E & \longrightarrow & E' \\
\downarrow & & \downarrow \\
T\xi & \xrightarrow{T\varphi} & T\eta
\end{array}
\]

commutes. Here \( E' \) is the total space of \( \eta \).

4.3. Definition. A pointed space \( X \) is called reducible if there is a pointed map \( f : S^m \to X \) such that \( f_* : \tilde{H}_i(S^m) \to \tilde{H}_i(X) \) is an isomorphism for \( i \geq m \). Every such map \( f \) (as well as its homotopy class or its stable homotopy class) is called a reducibility for \( X \).

We embed \( M^n \) in \( \mathbb{R}^{N+n} \), \( N \gg n \) and let \( \nu = \nu_M, \dim \nu = N \) be a normal bundle of this embedding. Recall that \( \nu \) is a PL bundle \( E \to M \) whose total space \( E \) is PL homeomorphic to a (tubular) neighborhood \( U \) of \( M \) in \( \mathbb{R}^{N+n} \). We choose such isomorphism and denote it by \( \varphi : U \to E \).

4.4. Construction–Definition. Let \( T\nu \) be the Thom space of \( \nu \). Then there is a unique map

\[
\psi : \mathbb{R}^{N+n}/(\mathbb{R}^{N+n} \setminus U) \to T\nu
\]
such that $\psi|_U = \varphi$. We define the collapse map $\iota : S^{n+N} \to T\nu_M$ (the Browder–Novikov map, cf. [Br, NT]) to be the composition

$$\iota : S^{n+N} \xrightarrow{\text{quotient}} S^{n+N}/(S^{n+N} \setminus U) \xrightarrow{\psi} T\nu.$$

It is well known and easy to see that $\iota$ is a reducibility for $T\nu$, see Corollary 10.7 below.

It turns out that, for $N$ large enough, the normal bundle of a given embedding $M \to \mathbb{R}^{N+n}$ exists and is unique up to isomorphism. For detailed definitions and proofs, see [HW, KL, LR]. The uniqueness gives us the following important fact. Let $\nu' =\begin{cases} E' \to M \end{cases}$ be another PL homeomorphism. Let $\iota : S^{n+N} \to T\nu$ and $\iota' : S^{n+N} \to T\nu'$ be the corresponding Browder–Novikov maps. Then there is a morphism $\nu \to \nu'$ of PL bundles which carries $\iota$ to a map homotopic to $\iota'$.

4.5. Theorem. Consider a PL $\mathbb{R}^N$-bundle $\eta$ over $M$ such that $T\eta$ is reducible. Let $\alpha \in \pi_{N+n}(T\eta)$ be an arbitrary reducibility for $T\eta$. Then there exist an $(S^N,*)$-equivalence $\mu : \nu^*_M \to \eta^*$ such that $(T\mu)_*(\iota) = \alpha$, and such a $\mu$ is unique up to fiberwise homotopy over $M$.

The Theorem is a version of the Spivak Theorem [Spi, Theorem A], cf. also [Br I.4.19]. Note that our version does not require the simply-connectedness of $M$. We postpone the proof to the next Chapter, see Section 10.7.

Given a homotopy equivalence $h : V \to M$ of closed connected PL manifolds, let $\nu_V$ be a normal bundle of a certain embedding $V \subset \mathbb{R}^{N+n}$, and let $u \in \pi_{N+n}(T\nu_V)$ be the homotopy class of the collapsing map $S^{n+N} \to T\nu_V$. Let $g : M \to V$ be homotopy inverse to $h$ and set $\eta = g^*\nu_V$. The $g$-adjoint morphism

$$\mathcal{J} = \mathcal{J}_g : \eta = g^*\nu_V \to \nu_V$$

yields the map $T\mathcal{J} : T\eta \to T\nu_V$. It is easy to see that $T \mathfrak{frak}{\mathcal{J}}$ is a homotopy equivalence, and so there exists a unique $\alpha \in \pi_{N+n}(T\eta)$ with $(T\mathcal{J})_*(\alpha) = u$. Since $u$ is a reducibility for $T\nu_V$, we conclude that $\alpha$ is a reducibility for $T\eta$. By Theorem 4.5, we get an $(S^N,*)$-equivalence $\mu : \nu^*_M \to \eta^*$ with $(T\mu)_*(\iota) = \alpha$. Now, the morphism

$$(\nu_M)^* \xrightarrow{\mu} \eta^* \xrightarrow{\text{classif}} \gamma^N_F$$

is a homotopy PL structuralization of $\nu_M$. Because of the uniqueness of the normal bundle, the homotopy class of this structuralization is
well defined. So, in this way we have the function

\[ j_F : S_{PL}(M) \to \{\text{homotopy PL structures on } \nu_M\} \cong [M, F/PL] \]

where the last bijection comes from 3.9.

4.6. DEFINITION. The function \( j_F \) is called the normal invariant, and its value on a homotopy PL structure (as well as on its PL structuralization) is called the normal invariant of this structure (structuralization).

5. Homotopy PL Structures on \( T^k \times D^n \)

Below \( T^k \) denotes the \( k \)-dimensional torus.

5.1. Theorem. Assume that \( k + n \geq 5 \). If \( x \in S_{PL}(T^k \times S^n) \) can be represented by a homeomorphism \( M \to T^k \times S^n \) then \( j_F(x) = 0 \).

This is a special case of the Sullivan Normal Invariant Homeomorphism Theorem. We prove 5.1 (in fact, a little bit general result) in the next chapter.

We also prove the Sullivan Theorem in full generality in Chapter III, Section 19. We must do this loop (repetition) since the proof in Chapter III uses Main Theorem and hence Theorem 5.1.

5.2. Construction–Definition. Let \( x \in S_{PL}(M) \) be represented by a map \( h : V \to M \), and let \( p : \tilde{M} \to M \) be a covering. Then we have a commutative diagram

\[
\begin{array}{ccc}
\tilde{V} & \xrightarrow{\tilde{h}} & \tilde{M} \\
q \downarrow & & \downarrow p \\
V & \xrightarrow{h} & M
\end{array}
\]

where \( q \) is the induced covering. Since \( \tilde{h} \) is defined uniquely up to deck transformations, the concordance class of \( \tilde{h} \) is well defined. So, we have a well-defined map

\[ p^* : S_{PL}(M) \to S_{PL}(\tilde{M}) \]

where \( p^*(x) \) is the concordance class of \( \tilde{h} \).

If \( p \) is a finite covering, we say that a class \( p^*(x) \in S_{PL}(\tilde{M}) \) finitely covers the class \( x \).

5.3. Theorem. Let \( k + n \geq 5 \) Then the following holds:

(i) if \( n > 3 \) then the set \( S_{PL}(T^k \times D^n) \) consists of precisely one (trivial) element;
(ii) if $n < 3$ then every element of $S_{PL}(T^k \times D^n)$ can be finitely covered by the trivial element;
(iii) the set $S_{PL}(T^k \times D^3)$ contains at most one element which cannot be finitely covered by the trivial element.

Some words about the proof. First, we mention the proof given by Wall, [W2] and [W3, Section 15 A]. Wall proved the bijection $w : S_{PL}(T^k \times D^n) \to H^{3-n}(T^k)$. Moreover, he also proved that finite coverings respect this bijection, i.e. if $p : T^k \times D^n \to T^k \times D^n$ is a finite covering then there is the commutative diagram

\[
\begin{array}{ccc}
S_{PL}(T^k \times D^n) & \xrightarrow{w} & H^{3-n}(T^k; \mathbb{Z}/2) \\
p^* & \uparrow & p^* \\
S_{PL}(T^k \times D^n) & \xrightarrow{w} & H^{3-n}(T^k; \mathbb{Z}/2).
\end{array}
\]

Certainly, this result implies all the claims (i)–(iii). Wall’s proof uses difficult algebraic calculations.

Another proof of the theorem can be found in [HS, Theorem C]. Minding the complaint of Novikov concerning Sullivan’s results (see Preface), we mention that the nice paper of Hsiang and Shaneson [HS] use a Sullivan’s result. Namely, Hsiang and Shaneson consider the so-called surgery exact sequence

\[
\partial \to S_{PL}(S^k \times T^n) \xrightarrow{j_F} [S^k \times T^n, F/PL] \to \cdots
\]

and write (page 42, Section 10):

By [44], every homomorphism $h : M \to S^k \times T^n, k = n \geq 5$, represents an element in the image of $\partial$.

Here the item [44] of the citation is our bibliographical item [Su1]. So, in fact, Hsiang and Shaneson use Theorem 5.1. As I already said, we prove 5.1 in next Chapter. □

6. The Product Structure Theorem, or from Bundles to Manifolds

Let $M$ be an $n$-dimensional $\partial_{PL}$-manifold. Then every PL structuralization $h : V \to M$ yields a PL structuralization $h \times 1 : V \times \mathbb{R}^k \to M \times \mathbb{R}^k$.

Thus, we have a well-defined map $e : T_{PL}(M) \to T_{PL}(M \times \mathbb{R}^k)$.

6.1. Theorem (The Product Structure Theorem). For every $n \geq 5$ and every $k \geq 0$, the map $e : T_{PL}(M) \to T_{PL}(M \times \mathbb{R}^k)$ is a bijection.
In particular, if $\mathcal{T}_{PL}(M \times \mathbb{R}^k) \neq \emptyset$ then $\mathcal{T}_{PL}(M) \neq \emptyset$.

Kirby and Siebenmann made the breakthrough for 6.1 [K1, KS1, KS2]. Quinn [Q2] gave a nice short proof of 6.1 by developing his theory of ends of maps, [Q1].

6.2. COROLLARY (The Classification Theorem). If $\dim M \geq 5$ and $M$ admits a PL structure, then the map

$$j_{TOP} : \mathcal{T}_{PL}(M) \rightarrow [(M, \partial M), (TOP/PL, \ast)]$$

is a bijection.

Proof. We construct a map

$$(6.1) \quad \sigma : [(M, \partial M), (TOP/PL, \ast)] \rightarrow \mathcal{T}_{PL}(M)$$

which is inverse to $j_{TOP}$. Take an element $a \in [(M, \partial M), (TOP/PL, \ast)]$

and, using (3.4), interpret it as a homotopy class of a topological $\mathbb{R}^N$-morphism $\phi : \theta^N_M \rightarrow \gamma^N_{PL}$ such that $\phi|_{\partial M}$ is a PL $\mathbb{R}^N$-morphism. The morphism $\phi$ yields a correcting isomorphism $\theta^N_M \rightarrow b^*\gamma^N_{PL}$ of topological $\mathbb{R}^N$-bundles over $M$, where $b : M \rightarrow BPL$ is the base of the morphism $\phi$. So, we have the commutative diagram

$$\begin{array}{ccc}
M \times \mathbb{R}^N & \xrightarrow{h} & W \\
\downarrow & & \downarrow \\
M & = & M
\end{array}$$

where $h$ is a fiberwise homeomorphism and $W \rightarrow M$ is a PL $\mathbb{R}^N$-bundle $b^*\gamma^N_{PL}$. In particular, $W$ is a PL manifold. Regarding $h^{-1} : W \rightarrow M \times \mathbb{R}^N$ as a PL structuralization of $M \times \mathbb{R}^N$, we conclude that, by the Product Structure Theorem 6.1, $h^{-1}$ is concordant to a map $g \times 1$ for some PL structuralization $g : V \rightarrow M$. We define $\sigma(a) \in \mathcal{T}_{PL}(M)$ to be the concordance class of $g$. One can check that $\sigma$ is a well-defined map which is inverse to $j_{TOP}$. Cf. [KS2] Essay IV Theorem 4.1. □

6.3. COROLLARY (The Existence Theorem). A topological manifold $M$ with $\dim M \geq 5$ admits a PL structure if and only if the tangent bundle of $M$ admits a PL structure.

Proof. Only claim “if” needs a proof. Let $\tau = \{\pi : D \rightarrow M\}$ be the tangent bundle of $M$, and let $\nu = \{\nu : E \rightarrow M\}$ be a stable normal bundle of $M$, $\dim \nu = N$. Then $E$ is homeomorphic to an open subset of $\mathbb{R}^{N+n}$, and therefore we can (and shall) regard $E$ as a PL manifold. Since $\tau$ is a PL bundle, we conclude that $r^*\tau$ is a PL bundle over $E$. 
In particular, the total space $M \times \mathbb{R}^{N+n}$ of $r^*\tau$ turns out to be a PL manifold, cf. [Mi3]. Now, because of the Product Structure Theorem 6.1, $M$ admits a PL structure. Cf. [KS2], Essay IV Theorem 3.1 \[ \Box \]

Let $f : M \rightarrow BTOP$ classify the stable tangent bundle of a closed topological manifold $M$, $\dim M \geq 5$.

6.4. COROLLARY. The following conditions are equivalent:
(i) $M$ admits a PL structure;
(ii) $\tau$ admits a PL structure;
(iii) there exists $k$ such that $\tau \oplus \theta^k$ admits a PL structure;
(iv) the map $f$ admits an $\alpha_{TOP}^{PL}$-lifting to $BPL$.

Proof. It suffices to prove that (iv) $\implies$ (iii) $\implies$ (i). The implication (iii) $\implies$ (i) can be proved similarly to 6.3. Furthermore, since $M$ is compact, we conclude that $f(M) \subset BTOP_m$ for some $m$. So, if (iv) holds then $f$ lifts to $BPL_m$, i.e. $\tau \oplus \theta^{m-k}$ admits a PL structure. \[ \Box \]

6.5. REMARK. It follows from [1.3], 6.3 and 6.2 that the set $\mathcal{T}_{PL}(M)$ of PL structures on $M$ is in a bijective correspondence with the set of fiber homotopy classes of $\alpha_{TOP}^{PL}$-liftings of $f$.

6.6. REMARK. It is well known that $j_F$ is not a bijection in general. The “kernel” and “cokernel” of $j_F$ can be described in terms of so-called Wall groups, [W3]. (For $M$ simply-connected, see also Theorem [12.2]) On the other hand, the bijectivity of $j_{TOP}$ (the Classification Theorem) follows from the Product Structure Theorem. So, informally speaking, kernel and cokernel of $j_F$ play the role of obstructions to splitting of structures. It seems interesting to develop and clarify these naive arguments.

6.7. REMARK. Since tangent and normal bundles of smooth manifolds turn out to be vector bundles, one can construct a map

$$k : \mathcal{P}_D(M) \rightarrow [M, PL/O]$$

which is an obvious analog of $j_{TOP}$. Moreover, the obvious analog of the Product Structure Theorem (as well as of the Classification and Existence Theorems) holds without any dimensional restriction. In particular, $k$ is a bijection for every smooth manifold, [HM].

It is well known (although difficult to prove) that $\pi_i(PL/O) = 0$ for $i \leq 6$. (See [Rud, IV.4.27(iv)] for the references.) Thus, every PL manifold $M$ of dimension $\leq 7$ admits a smooth structure, and this structure is unique if $\dim M \leq 6$. 
7. Non-contractibility of $\text{TOP}/\text{PL}$

7.1. THEOREM (Rokhlin Signature Theorem). Let $M$ be a closed 4-dimensional PL manifold with $w_1(M) = 0 = w_2(M)$. Then the signature of $M$ is divisible by 16.

Proof. See [MK], [K2, XI], or the original work [Ro]. In fact, Rokhlin proved the result for smooth manifolds, but the proof works for PL manifolds as well. On the other hand, in view of 6.7, there is no difference between smooth and PL manifolds in dimension 4. □

7.2. THEOREM (Freedman’s Example). There exists a closed simply-connected topological 4-dimensional manifold $V$ with $w_2(V) = 0$ and such that $E_8$ is the matrix of the intersection form $H^2(V) \otimes H^2(V) \to \mathbb{Z}$. In particular, the signature of $V$ is equal to 8. Furthermore, such a manifold $V$ is unique up to homeomorphism.

Proof. See [FU], [FQ], or the original work [F]. □

7.3. COMMENT. Some words on constructing of $V$. Take the manifold $W$ (plumbing) from Browder [Br, Complement V.2.6]. This is a smooth 4-dimensional simply-connected parallelizable manifold whose boundary $\partial W$ is a homology sphere. Furthermore, $E_8$ is the matrix of the intersection form $H_2(W) \otimes H_2(W) \to \mathbb{Z}$. A key (and very difficult) result of Freedman [F] claims that $\partial W$ bounds a compact contractible topological manifold $P$. Now, put $V = W \cup_{\partial W} P$.

7.4. COROLLARY. The topological manifolds $V$ and $V \times T^k$, $k \geq 1$ do not admit any PL structure.

Proof. The claim about $V$ follows from 7.1. (Note that $w_1(V) = 0$ because $V$ is simply-connected.) Suppose that $V \times T^k$ has a PL structure. Then $V \times \mathbb{R}^k$ has a PL structure. So, because of the Product Structure Theorem [6.1], $V \times \mathbb{R}$ has a PL structure. Hence, by [6.7] it possesses a smooth structure. Then the projection $p_2 : V \times \mathbb{R} \to \mathbb{R}$ can be $C^0$-approximated by a map $f : V \times \mathbb{R} \to \mathbb{R}$ which coincides with $p_2$ on $V \times (-\infty, 0)$ and is smooth on $V \times (1, \infty)$. Take a regular value $a \in (0, \infty)$ of $f$ (which exists because of the Sard Theorem) and set $U = f^{-1}(a)$. Then $U$ is a smooth manifold (by the Implicit Function Theorem), and it is easy to see that $w_1(U) = 0 = w_2(U)$ (because it holds for both manifolds $\mathbb{R}$ and $V \times \mathbb{R}$). On the other hand, both manifolds $V$ and $U$ cut the “tube” $V \times \mathbb{R}$. So, they are (topologically) bordant, and therefore $U$ has signature 8. But this contradicts the Rokhlin Theorem 7.1. □
7.5. Corollary. The space $\text{TOP}/\text{PL}$ is not contractible.

Proof. Indeed, suppose that $\text{TOP}/\text{PL}$ is contractible. Then every map $X \to B\text{TOP}$ lifts to $B\text{PL}$, and so, by 6.3, every closed topological manifold of dimension greater than 4 admits a PL structure. But this contradicts 7.4. □

7.6. Remark. Kirby and Siebenmann [KS2, Annex C] constructed the original example of a topological manifold which does not admit a PL structure. Namely, they considered the space $X^4 = T^4 \# \text{cone of } W$ with $W$ as in 7.3 and proved that $X \times S^1$ is a topological manifold. If we assume that $X \times S^1$ and argue as in the end of the proof of 7.4, we construct a manifold $Y$ (an analog of $U$ in 7.4) with $w_1(Y) = 0 = w_2(Y)$ and $\sigma(Y) = 8$. Thus, the 5-manifold $X \times S^1$ does not admit PL structure.

8. Homotopy groups of $\text{TOP}/\text{PL}$

Let $M$ be a compact topological manifold equipped with a metric $\rho$. Then the space $\mathcal{H}$ of self-homeomorphisms $M \to M$ gets a metric $d$ with $d(f, g) = \sup\{x \in M \mid \rho(f(x), g(x))\}$.

8.1. Theorem. The space $\mathcal{H}$ is locally contractible.

Proof. See [Ch, EK]. □

8.2. Corollary. There exists $\varepsilon > 0$ such that every homeomorphism $h \in \mathcal{H}$ with $d(H, 1_M) < \varepsilon$ is isotopic to $1_M$. □

8.3. Construction. We regard the torus $T^k$ as a commutative Lie group (multiplicative) and equip it with the invariant metric $\rho$. Consider the map $p_\lambda : T^k \to T^k$, $p_\lambda(a) = a^\lambda$, $\lambda \in \mathbb{N}$. Then $p_\lambda$ is a $\lambda^k$-sheeted covering. It is also clear that all the deck transformations of the covering torus are isometries. Hence the diameter of each of (isometric) fundamental domain for $p_\lambda$ tends to zero as $\lambda \to \infty$.

8.4. Lemma. Let $h : T^k \times D^n \to T^k \times D^n$ is a self-homeomorphism which is homotopic rel $\partial(T^k \times D^n)$ to the identity. Then there exist $\lambda \in \mathbb{N}$ and a commutative diagram

\[
\begin{array}{ccc}
T^k \times D^n & \xrightarrow{h} & T^k \times D^n \\
\downarrow p_\lambda & & \downarrow p_\lambda \\
T^k \times D^n & \xrightarrow{h} & T^k \times D^n
\end{array}
\]
where the lifting \( \tilde{h} \) of \( h \) is isotopic rel \( \partial(T^k \times D^n) \) to the identity.

**Proof.** (Cf. [KS2, Essay V].) First, consider the case \( n = 0 \). Without loss of generality we can assume that \( h(e) = e \) where \( e \) is the neutral element of \( T^k \). Consider a covering \( p_\lambda : T^k \to T^k \) as in [8,3] and take a covering \( \tilde{h} : T^k \to T^k, p_\lambda \tilde{h} = \tilde{h}p_\lambda \) of \( h \) such that \( \tilde{h}(e) = e \). In order to distinguish the domain and the range of \( p_\lambda \), we denote the domain of \( p_\lambda \) by \( \tilde{T} \) and the range of \( p_\lambda \) by \( T \). Since \( \tilde{h} \) is homotopic to \( 1_\tilde{T} \), we conclude that every point of the lattice \( L := p_\lambda^{-1}(e) \) is fixed under \( \tilde{h} \).

Given \( \varepsilon > 0 \), choose \( \delta \) such that \( \rho(\tilde{h}(x), \tilde{h}(y)) < \varepsilon/2 \) whenever \( \rho(x, y) < \delta \). Furthermore, choose \( \lambda \) so large that the diameter of any closed fundamental domain is less than \( \min\{\varepsilon/2, \delta\} \). Now, given \( x \in \tilde{T} \), choose \( a \in L \) such that \( a \) and \( x \) belong to the same closed fundamental domain. Now, \[
\rho(x, \tilde{h}(x)) \leq \rho(x, a) + \rho(a, \tilde{h}(x)) = \rho(x, a) + \rho(\tilde{h}(a), \tilde{h}(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Thus, by [8,2] \( \tilde{h} \) is isotopic to \( 1_\tilde{T} \) for \( \lambda \) large enough.

The proof for \( n > 0 \) is similar but a bit more technical. Let \( D_\eta \subset D^n \) be the disk centered at 0 and having the radius \( \eta \). We can always assume that \( h \) coincides with identity outside of \( T^k \times D_\eta \). Now, asserting as for \( n = 0 \), take a covering \( p_\lambda \) as above and choose \( \lambda \) and \( \eta \) so small that the diameter of every fundamental domain in \( \tilde{T} \times D_\eta \) is small enough. Then

\[
\tilde{h} : \tilde{T} \times D_\eta \to \tilde{T} \times D_\eta
\]
is isotopic to the identity and coincides with identity outside \( \tilde{T} \times D_\eta \). This isotopy is not an isotopy rel \( \tilde{T} \times \partial D_\eta \). Nevertheless, we can easily extend it to the whole \( \tilde{T} \times D^n \) so that this extended isotopy is an isotopy rel \( \partial(\tilde{T} \times D^n) \).

If you want an explicit formula, do the following. Given \( a = (b, c) \in \tilde{T} \times D_\eta \), set \( |a| = |c| \). Consider an isotopy

\[
\varphi : \tilde{T} \times D_\eta \times I \to \tilde{T} \times D_\eta \times I, \quad \varphi(a, 0) = a, \ \varphi(a, 1) = \tilde{h}(a), \ a \in \tilde{T} \times D_\eta.
\]

Define \( \overline{\varphi} : \tilde{T} \times D_\eta \times I \to \tilde{T} \times D_\eta \times I \) by setting

\[
\overline{\varphi}(a, t) = \begin{cases} 
\varphi(a, t) & \text{if } |a| \leq \eta, \\
\varphi(a, \frac{|a| - 1}{\eta - 1} t) & \text{if } |a| \geq \eta.
\end{cases}
\]

Then \( \overline{\varphi} \) is the desired isotopy rel \( \partial(\tilde{T} \times D^n) \). \( \square \)
8.5. COROLLARY. Let \( \phi : T_{PL}(T^k \times D^n) \rightarrow S_{PL}(T^k \times D^n) \) be the forgetful map as in \([3.5]\). If \( \phi(x) = \phi(y) \) then there exists a finite covering \( p : T^k \times D^n \rightarrow T^k \times D^n \) such that \( p^*(x) = p^*(y) \). □

Consider the map

\[
\psi : \pi_n(TOP/PL) = [(D^n, \partial D^n), (TOP/PL, \ast)] \xrightarrow{p_2^*} [(T^k \times D^n, \partial(T^k \times D^n)), (TOP/PL, \ast)] \xrightarrow{\sigma} T_{PL}(T^k \times D^n)
\]

where \( \sigma \) is the map from \([6.1]\) (the inverse map to \( j_{TOP} \)).

8.6. LEMMA. The map \( \psi \) is injective. Moreover, if \( p^*\psi(x) = p^*\psi(y) \) for some finite covering \( p : T^k \times D^n \rightarrow T^k \times D^n \) then \( x = y \).

In particular, if \( p^*\psi(x) \) is the trivial element of \( T_{PL}(T^k \times D^n) \) then \( x = 0 \).

Proof. The injectivity of \( \psi \) follows from the injectivity of \( p_2^* \) and \( \sigma \). Furthermore, for every finite covering \( p : T^k \times D^n \rightarrow T^k \times D^n \) we have the commutative diagram

\[
\pi_n(TOP/PL) \xrightarrow{\psi} T_{PL}(T^k \times D^n) \xrightarrow{p^*} \pi_n(TOP/PL) \xrightarrow{\psi} T_{PL}(T^k \times D^n)
\]

Therefore \( x = y \) whenever \( p^*\psi(x) = p^*\psi(y) \). Finally, if \( p^*\psi(x) \) is trivial element then \( p^*\psi(x) = p^*\psi(0) \), and thus \( x = 0 \). □

Consider the map

\[
\varphi : \pi_n(TOP/PL) \xrightarrow{\psi} T_{PL}((T^k \times D^n) \xrightarrow{\phi} S_{PL}((T^k \times D^n)
\]

where \( \phi \) is the forgetful map described in \([3.5]\).

8.7. THEOREM (The Reduction Theorem, cf. \([K1]\)). The map \( \varphi \) is injective.

Moreover, if \( p^*\varphi(x) = p^*\varphi(y) \) for some finite covering

\[
p : T^k \times D^n \rightarrow T^k \times D^n
\]

then \( x = y \).

In particular, if \( p^*\varphi(x) \) is the trivial element of \( T_{PL}(T^k \times D^n) \) then \( x = 0 \).

We call it the Reduction Theorem because it reduces the calculation of the group \( \pi_i(TOP/PL) \) to the calculation of the sets \( S_{PL}(T^k \times D^n) \).
Proof. If $\varphi(x) = \varphi(y)$ then $\phi(\psi(x)) = \phi(\psi(y))$. Hence, by Corollary 8.5, there exists a finite covering $\pi: T^k \times D^n \to T^k \times D^n$ such that $\pi^*\psi(x) = \pi^*\psi(y)$. So, by Lemma 8.6, $x = y$, i.e. $\varphi$ is injective.

Now, suppose that $p^*\varphi(x) = p^*\varphi(y)$ for some finite covering $p: T^k \times D^n \to T^k \times D^n$. Then $\phi(p^*\psi(x)) = \phi(p^*\psi(y))$. Now, by Corollary 8.5, there exists a finite covering

$$q: T^k \times D^n \to T^k \times D^n$$

such that $q^*p^*\psi(x) = q^*p^*\psi(y)$, i.e. $(pq)^*\psi(x) = (pq)^*\psi(y)$. Thus, by Lemma 8.6, $x = y$. \qed

8.8. Corollary (The Main Theorem). $\pi_i(TOP/PL) = 0$ for $i \neq 3$. Furthermore, $\pi_3(TOP/PL) = \mathbb{Z}/2$. Thus, $TOP/PL = K(\mathbb{Z}/2, 3)$.

Proof. The equality $\pi_i(TOP/PL) = 0$ for $i \neq 3$ follows from Theorem ??(i,ii) and Theorem 8.7. Furthermore, again because of ?? and 8.7, we conclude that $\pi_3(TOP/PL)$ has at most two elements. In other words, $TOP/PL = K(\pi, 3)$ where $\pi = \mathbb{Z}/2$ or $\pi = 0$. Finally, by Corollary 7.5, the space $TOP/PL$ is not contractible. Thus, $TOP/PL = K(\mathbb{Z}/2, 3)$. \qed
CHAPTER II

Normal Invariant

The goal of this chapter is to prove Theorem 5.1. The proof uses the Sullivan’s result on the homotopy type of \( F/PL \). \[ \text{[Sul1], [Sul2]} \]. Note that Madsen and Milgram \[ \text{[MM]} \] gave a detailed proof of those Sullivan result.

We also need Theorem 4.5. We prove it in Section 10 in the form that is suitable for our aim.

9. Stable equivalences of spherical bundles

Given a sectioned spherical bundle \( \xi \) over a finite CW-space \( X \), let \( \text{aut} \xi \) denote the group of fiberwise homotopy classes of self-equivalences \( \xi \to \xi \) over \( X \), where we assume that all homotopies preserve the section.

We denote by \( \sigma^k = \sigma^k_X \) the trivial \( S^k \)-bundle over \( X \) with a fixed section. In another words, \( \sigma^k = (\theta^k)^* \).

9.1. Proposition. There is a natural bijection

\[ \text{aut} \sigma^k = [X, F_k]. \]

Proof. Because of the exponential law, every map \( X \to F_k \) yields a section-preserving map \( X \times S^k \to X \times S^k \) over \( X \), and vice versa. Cf. \[ \text{[Br], Prop. I.4.7]} \].

Consider the map

\[ \mu : F_k \times F_k \to F_{2k}, \quad \mu(a, b) = a \wedge b : S^{2k} = S^k \wedge S^k \to S^k \wedge S^k = S^{2k} \]

where we regard \( a, b \in F_k \) as pointed maps \( S^k \to S^k \). Let \( T : F_k \times F_k \to F_k \times F_k \) be the transpose map, \( T(a, b) = (b, a) \).

9.2. Lemma. The maps \( \mu : F_k \times F_k \to F_{2k} \) and \( \mu T : F_k \times F_k \to F_{2k}, k > 0 \) are homotopic.

Proof. Consider the map

\[ \tau : S^{2k} = S^k \wedge S^k \to S^k \wedge S^k = S^{2k}, \quad \tau(u, v) = (v, u) \]
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and note that, for every \(a, b \in F_k\), we have
\[
(\mu \circ T)(a, b) = \tau \circ \mu(a, b) \circ \tau.
\]

First, consider the case of \(k\) odd. Then there is a pointed homotopy \(h_t\) between \(\tau\) and \(1_{S^{2k}}\). Now, the pointed homotopy \(h_t \circ \mu(a, b) \circ h_t\) is a pointed homotopy between \((\mu \circ T)(a, b)\) and \(\mu(a, b)\) which yields a homotopy between \(\mu T\) and \(\mu\).

Now consider the case of \(k\) even. We regard \(S^{2k}\) as \(\mathbb{R}^{2k} \cup \infty\) with \(\mathbb{R}^{2k} = \{(x_1, \ldots, x_{2k})\}\) and define \(\tau', \tau'': S^{2k} \to S^{2k}\) by setting
\[
\tau'(x_1, x_2, x_3, \ldots, x_{2k}) = (x_2, x_1, x_3, \ldots, x_{2k}),
\tau''(x_1, \ldots, x_{2k-2}, x_{2k-1}, x_{2k}) = (x_1, \ldots, x_{2k-2}, x_{2k-1}, x_{2k}),
\]
(i.e. \(\tau'\) permutes the first two coordinates and \(\tau''\) permutes the last two coordinates). Since \(k\) is even, we conclude that \(\tau' \simeq \tau \simeq \tau''\). Furthermore, \(\tau'' \tau' \simeq 1_{S^{2k}}\). If we fix such pointed homotopies then we get the pointed homotopies
\[
(\mu \circ T)(a, b) = \tau \circ \mu(a, b) \circ \tau \simeq \tau'' \circ \mu(a, b) \circ \tau' = (a \wedge 1) \circ (\tau'' \tau')(a \wedge b) = a \wedge b = \mu(a, b)
\]
which yield the homotopy \(\mu \circ T \simeq \mu\). \(\square\)

9.3. Corollary. Let \(\varphi, \psi : \sigma^k \to \sigma^k\) be two automorphisms of \(\sigma^k\). Then the automorphisms \(\varphi \uparrow \psi\) and \(\psi \uparrow \varphi\) of \(\sigma^{2k}\) are fiberwise homotopic. \(\square\)

Given two spherical bundles \(\xi\) and \(\eta\) over \(X\), consider the bundle \(\xi \wedge \eta\) over \(X \times X\). We denote by \(\Delta : X \to X \times X\) the diagonal and consider the \(\Delta\)-adjoint bundle morphism
\[
J := \mathfrak{J}_{\Delta, \xi \wedge \eta} : \xi \uparrow \eta \to \xi \wedge \eta.
\]

9.4. Proposition. For every automorphism \(\varphi : \eta \to \eta\) the diagram
\[
\begin{array}{cccc}
\xi \uparrow \eta & \xrightarrow{J} & \xi \wedge \eta \\
1 \uparrow \varphi \downarrow & & \downarrow 1 \wedge \varphi \\
\xi \uparrow \eta & \xrightarrow{J} & \xi \wedge \eta
\end{array}
\]
commutes. \(\square\)
9.5. Corollary. The diagram

\[
\begin{array}{c}
\xi \uparrow \eta \uparrow \eta \xrightarrow{J} (\xi \uparrow \eta) \wedge \eta \xrightarrow{(1+1) \wedge \varphi} (\xi \uparrow \eta) \wedge \eta \\
\xi \uparrow \eta \uparrow \eta \xrightarrow{J} (\xi \uparrow \eta) \wedge \eta \xrightarrow{(1+\varphi) \wedge 1} (\xi \uparrow \eta) \wedge \eta
\end{array}
\]

commutes up to homotopy. \(\Box\)

10. Proof of Theorem 4.5

We need certain preliminaries on stable duality [Spa1]. Given a pointed map \(f : X \to Y\), let \(Sf : SX \to SY\) denote the reduced suspension over \(f\). So, we have a well-defined map

\[
S : [X, Y]^\bullet \to [SX, SY]^\bullet, \quad [f] \to [Sf]
\]

10.1. Proposition. Suppose that \(\pi_i(Y) = 0\) for \(i < n\) and that \(X\) is a CW-space with \(\dim X < 2n - 1\). Then the map \(S : [X, Y]^\bullet \to [SX, SY]^\bullet\) is a bijection.

Proof. This is the famous Freudenthal Suspension theorem, see e.g. [FFG, H, Spa2, Sw].

Given two pointed spaces \(X, Y\), we define \(\{X, Y\}\) to be the direct limit of the sequence

\[
[X, Y]^\bullet \xrightarrow{S} [SX, SY]^\bullet \xrightarrow{S} \cdots \xrightarrow{S} [S^n X, S^n Y]^\bullet \xrightarrow{S} \cdots
\]

In particular, we have the obvious maps

\[
\Sigma : (Y, *)^{(X, *)} \to [X, Y]^\bullet \to \{X, Y\}.
\]

Given a pointed map \(f : X \to Y\), the element \(\Sigma(f) \in \{X, Y\}\) is called the stable homotopy class of \(f\). The standard notation for this one is \(\{f\}\), but, as usual, in several cases we use the same notation \(f\) for \([f]\) and \(\{f\}\).

It is well known that, for \(n \geq 2\), the set \([S^n X, S^n Y]^\bullet\) has a natural structure of the abelian group, and the corresponding maps \(S\) are homomorphisms, [Sw]. So, \(\{X, Y\}\) turns out to be a group. Furthermore, by Theorem [10.1] if \(X\) is a finite CW-space then the map

\[
[S^N X, S^N Y]^\bullet \to \{S^N X, S^N Y\}
\]

is a bijection for \(N\) large enough.
10.2. Definition. A map \( f : S^d \to A \wedge A^\perp \) is called a (stable) \( d \)-duality if, for every space \( E \), the maps
\[
u_E : \{A, E\} \to \{S, E \wedge A^\perp\}, \quad \nu_E(\varphi) = (\varphi \wedge 1_{A^\perp})u
\]
and
\[
u^E : \{A^\perp, E\} \to \{S, A \wedge E\}, \quad \nu^E(\varphi) = (1_A \wedge \varphi)u
\]
are isomorphisms.

10.3. Proposition-Construction. Let \( u : S^d \to A \wedge A^\perp \) be a \( d \)-duality between two finite CW-spaces. Then, for all \( i \) and \( \pi \), the map \( u \) yields an isomorphism
\[
\tilde{H}_i(u; \pi) : \tilde{H}^i(A^\perp; \pi) \to \tilde{H}_{d-i}(A, \pi).
\]

Proof. Recall that
\[
H^n(A^\perp; \pi) = [A, K(\pi, n)]^* = [S^N A, K(\pi, N + n)]^*
\]
where \( K(\pi, i) \) is the Eilenberg–MacLane space. Because of Theorem [10.1], the last group coincides with \( \{S^N A, K(\pi, N + n)\} \) for \( N \) large enough, and therefore
\[
H^n(A^\perp; \pi) = \{S^N A, K(\pi, N + n)\} \text{ for } N \text{ large enough}.
\]

Furthermore, let \( \varepsilon_n : SK(\pi, n) \to K(\pi, n + 1) \) be the adjoint map to the standard homotopy equivalence \( K(\pi, n) \to \Omega K(\pi, n + 1) \), see e.g. [Sw]. G. Whitehead [Wh] noticed that
\[
\tilde{H}_n(A; \pi) = \lim_{\longrightarrow} [S^{N+n}, K(\pi, N) \wedge A]^*.
\]
Here \( \lim_{\longrightarrow} \) is the direct limit of the sequence
\[
[S^{N+n}, K(\pi, N) \wedge A]^* \longrightarrow [S^{N+n+1}, SK(\pi, N) \wedge A]^* \xrightarrow{\varepsilon_n*} [S^{N+n+1}, K(\pi, N) \wedge A]^*
\]
(see [Gray] Ch 18 or [Rud], II.3.24 for greater details). Since \( \varepsilon_n \) is an \( n \)-equivalence, and because of Theorem [10.1], we conclude that
\[
\tilde{H}_n(A; \pi) = [S^{N+n}, K(\pi, N) \wedge A] \text{ for } N \text{ large enough}.
\]

So, again because of Theorem [10.1]
\[
\tilde{H}_n(A; \pi) = \{S^{N+n}, K(\pi, N) \wedge A\}
\]
for \( N \) large enough.

Now, consider a \( d \)-duality \( u : S^d \to A \wedge A^\perp \). Fix \( i \) and choose \( N \) large enough such that
\[
\tilde{H}^i(A^\perp; \pi) = \{S^N A^\perp, K(\pi, N + i)\},
\]
\[
\tilde{H}_{d-i}(A; \pi) = \{S^{N+d}, K(\pi, N + i) \wedge A\}.
\]
10. PROOF OF THEOREM 4.5

Put \( K = K(\pi, N + i) \). By suspending the domain and the range, we get a duality (denoted also by \( u \))

\[ u : S^{N+d} \to A \land S^N A^\perp. \]

This duality yields the isomorphism

\[ u^K : \tilde{H}^i(A^\perp; \pi) = \{S^N A^\perp, K\} \to \{S^{N+d}, K \land A\} = \tilde{H}_{d-i}(A; \pi), \]

and we set \( H_i(u; \pi) := u^K \).

10.4. Definition. We dualize 4.3 and say that a pointed map \( a : A \to S^k \) (or its stable homotopy class \( a \in \{A, S^k\} \)) is a coreducibility if the induced map

\[ a^* : \tilde{H}^i(S^k) \to \tilde{H}^i(A) \]

is an isomorphism for \( i \leq k \).

10.5. Proposition. Let \( u : S^d \to A \land A^\perp \) be a \( d \)-duality between two finite CW-spaces, and let \( k \leq d \). A class \( \alpha \in \{A^\perp, S^k\} \) is a coreducibility if and only if the class \( \beta := u^{s_k} \alpha \in \{S^{d-k}, A\} \) is a reducibility.

Proof. Let \( H_i(u) : \tilde{H}^i(A^\perp) \to \tilde{H}_{d-i}(A) \) be the isomorphism as in 10.3. Note that the standard homeomorphism \( v : S^d \to S^k \land S^{d-k} \) is a \( d \)-duality. It is easy to see that the diagram

\[
\begin{array}{ccc}
\tilde{H}^i(A^\perp) & \xrightarrow{H_i(u)} & \tilde{H}_{d-i}(A) \\
\uparrow a^* & & \uparrow \beta^* \\
\tilde{H}^i(S^k) & \xrightarrow{H_i(v)} & \tilde{H}_{d-i}(S^{d-k})
\end{array}
\]

commutes. In particular, the left vertical arrow is an isomorphism if and only if the right one is.

Consider a closed connected \( n \)-dimensional PL manifold \( M \) and embed it in \( \mathbb{R}^{N+n+k} \) with \( N \) large enough. Let \( \iota : S^{N+n+k} \to T\nu^{N+k} \) be a collapse map as in 4.4 and let

\[ J : (\nu^{N+k})^* = (\nu^N)^* \uparrow \sigma^k \to (\nu^N)^* \land \sigma^k \]

be the morphism as in 9.4 where \( \sigma^k = \sigma^k_M \).

10.6. Theorem. The map

\[ S^{N+n+k} \xrightarrow{\iota} T\nu^{N+k} \xrightarrow{TJ} T\nu^N \land \sigma^k \]

is an \((N + n + k)\)-duality map.

Proof. This is actually proved in [DP]. For greater detail, see [Rud, V.2.3(i)].
10.7. **Corollary.** If the manifold \( M \) is orientable then the collapse map \( \iota : S^{N+n} \to T\nu^N \) is a reducibility.

**Proof.** Recall that \( T\sigma^k = (M \times S^k)/M = S^k(M^+) \). Consider a surjective map \( e : M^+ \to S^0 \) and define \( \varepsilon = S^k e : T\sigma^k \to S^k \). The map

\[
S^k \iota : S^k S^{N+n} = S^{N+n+k} \to T\nu^{N+k} = S^k T\nu
\]

can be written as

\[
S^{N+n+k} \xrightarrow{S^k} T\nu^{N+k} = T\nu^N \wedge T\sigma^k \xrightarrow{1 \wedge \varepsilon} T\nu^N \wedge S^k = S^k T\nu,
\]

where the composition of first two maps is the duality from \([10.6]\). Hence \( \iota \) is dual to \( \varepsilon \) with respect to duality \((10.2)\). Clearly, \( \varepsilon \) is a coreducibility because \( M \) is orientable. Thus, the result follows from \([10.5]\). \( \square \)

For technical reasons, it will be convenient for us to consider the duality

\[
(10.1) \quad S^{N+n+2k} \to T\nu^{N+2k} \xrightarrow{TJ} T\nu^{N+k} \wedge T\sigma^k.
\]

This duality yields an isomorphism

\[
(10.2) \quad D := u^{S^k} : \{T\sigma^k, S^k\} \to \{S^{N+n+2k}, T\nu^{N+k}\} \to \{S^{N+n+k}, T\nu^{N+k}\}.
\]

10.8. **Proposition.** For every automorphism \( \varphi : \sigma^k \to \sigma^k \) the following diagram commutes up to homotopy:

\[
\begin{array}{ccc}
S^{N+n+2k} & \xrightarrow{\iota} & T\nu^{N+2k} \\
\| & & \| \\
S^{N+n+2k} & \xrightarrow{TJ} & T\nu^{N+k} \wedge T\sigma^k \\
\| & & \| \\
S^{N+n+2k} & \xrightarrow{TJ} & T\nu^{N+k} \wedge T\sigma^k \\
\end{array}
\]

**Proof.** This follows from \([9.5]\). \( \square \)

Every automorphism \( \varphi : \sigma^k \to \sigma^k \) yields a homotopy equivalence

\[
T(1 \dagger \varphi) : T\nu^{N+k} = T(\nu^N \dagger \sigma^k) \longrightarrow T(\nu^N \dagger \sigma^k) = T\nu^{N+k}
\]

and hence an isomorphism

\[
T(1 \dagger \varphi)_* : \{S^{N+n+k}, T\nu^{N+k}\} \to \{S^{N+n+k}, T\nu^{N+k}\}.
\]

So, we have the aut \( \sigma^k \)-action

\[
a_\nu : \text{aut} \sigma^k \times \{S^{N+n+k}, T\nu^{N+k}\} \to \{S^{N+n+k}, T\nu^{N+k}\},
\]

\[
a_\nu(\varphi, \alpha) = T(1 \dagger \varphi)_*(\alpha).
\]
Similarly, every automorphism $\varphi$ of $\sigma^k$ induces a homotopy equivalence $T\sigma^k \to T\sigma^k$, and therefore we have the action

$$a_\sigma : \text{aut} \sigma^k \times \{T\sigma^k, S^k\} \to \{T\sigma^k, S^k\}.$$  

10.9. **Theorem.** The diagram

\[
\begin{array}{ccc}
\text{aut} \sigma^k \times \{T\sigma^k, S^k\} & \xrightarrow{a_\sigma} & \{T\sigma^k, S^k\} \\
1 \times D & \downarrow & D \\
\text{aut} \sigma^k \times \{S^{N+n+k}, T\nu^{N+k}\} & \xrightarrow{a_\nu} & \{S^{N+n+k}, T\nu^{N+k}\}
\end{array}
\]

commutes.

**Proof.** This follows from 10.8 and the definition of $D, a_\nu$ and $a_\sigma$. □

Because of Theorem 10.1, for $k$ large enough we have

$$\{T\sigma^k, S^k\} = \pi^k(T\sigma^k) \text{ and } \{S^{N+n+k}, T\nu^{N+k}\} = \pi_{N+n+k}(T\nu^{N+k}).$$

Then we can rewrite the diagram from Theorem 10.9 as

\[
\begin{array}{ccc}
\text{aut} \sigma^k \times \pi^k(T\sigma^k) & \xrightarrow{a_\sigma} & \pi^k(T\sigma^k) \\
1 \times D & \downarrow & D \\
\text{aut} \sigma^k \times \pi_{N+n+k}(T\nu^{N+k}) & \xrightarrow{a_\nu} & \pi_{N+n+k}(T\nu^{N+k})
\end{array}
\]

(10.3)

Let $R \in \pi_{N+n+k}(T\nu^{N+k})$ be the set of reducibilities, and let $C \in \pi^k(T\sigma^k)$ be the set of coreducibilities. Then, clearly, $a_\nu(R) \subset R$ and $a_\sigma(C) \subset C$. Therefore, in view of Proposition 10.5, the diagram (10.3) yields the commutative diagram

\[
\begin{array}{ccc}
\text{aut} \sigma^k \times C & \xrightarrow{a_\sigma} & C \\
1 \times D & \downarrow & D \\
\text{aut} \sigma^k \times R & \xrightarrow{a_\nu} & R
\end{array}
\]

(10.4)

10.10. **Theorem.** For every $\alpha, \beta \in C$ there exists an automorphism $\varphi$ of $\sigma^k$ such that $a_\sigma(\varphi, \alpha) = \beta$. Moreover, this $\varphi$ is unique up to fiberwise homotopy. In other words, the action $a_\sigma : \text{aut} \sigma^k \times C \to C$ is free and transitive.

**Proof.** Recall that $T\sigma^k = (M \times S^k)/M$. So, for every $m \in M$, a pointed map $f : T\sigma^k \to S^k$ yields a pointed map $f_m : S^k_m \to S^k$ where $S^k_m$ is the fiber over $m$. Furthermore, $f$ represents a coreducibility if and only if all maps $f_m$ belong to $F_k$. In other words, every coreducibility $T\sigma^k \to S^k$ yields a homotopy class $M \to F_k$, and in fact we have a
bijectioj C → [M, F_k] there. Moreover, it is easy to see that, in view of Proposition 9.1, the action a_σ coincides with the map

\[ [M, F_k] \times [M, F_k] \to [M, F_k] \]

induced by the product in F_k, and the result follows. □

Since D is an isomorphism, Theorem 10.10 yields the following corollary.

10.11. Corollary. The action \( a_\nu : \text{aut}^k \times \mathcal{R} \to \mathcal{R} \) is free and transitive. □

Now we can finish the proof of Theorem 4.5. Assuming \( \dim \eta = N + k \) to be large enough, we conclude that \( \nu^* \) and \( \eta^* \) are homotopy equivalent over \( \bar{M} \), see Atiyah [At, Prop. 3.5]. (Note that Atiyah works with non-sectioned bundles, but there is no problem to adapt the proof for sectioned ones.) Choose any such \( F_{N+k} \)-equivalence \( \varphi : \eta^* \to \nu^* \) and consider the induced homotopy equivalence \( T\varphi : T\eta \to T\nu \). Clearly, the composition

\[ \beta : S^{N+n+k} \to T\eta \xrightarrow{T\varphi} T\nu \]

is a reducibility. So, by 10.11 there exists an \( F_{N+k} \)-equivalence \( \lambda : \nu^* \to \eta^* \) over \( M \) with \( (T\lambda)_\ast(\beta) = \iota \). Now, we define \( \mu : \nu^* \to \eta^* \) to be the fiber homotopy inverse to \( \lambda \varphi \). (The existence of an inverse equivalence can be proved following Dold [Dold], cf. also [May]). Clearly, \( \mu_\ast \iota = \alpha \). This proves the existence of the required equivalence \( \mu \).

Furthermore, if there exists another equivalence \( \mu' : \eta^* \to \nu^* \), then \( \mu' \circ \mu^{-1}(\iota) = \iota \), and so \( \mu \) and \( \mu' \) are homotopic over \( M \). This proves the uniqueness of \( \mu \). Thus, Theorem 4.5 is proved. □

11. Normal Morphisms, Normal bordisms, and F/PL

Throughout the section we fix a closed orientable n-dimensional PL manifold \( M \).

11.1. Definition ([Br]). A normal morphism at \( M \) is a PL \( \mathbb{R}^N \)-morphism \( \varphi : \nu_V \to \xi \) where \( \xi \) is a PL \( \mathbb{R}^N \)-bundle over \( M \), \( V \) is a closed PL manifold, and \( \nu_V \) is PL \( \mathbb{R}^N \)-bundle of \( V \) in \( \mathbb{R}^{N+n} \).

11.2. Example. Let \( h : V \to M \) be a homotopy equivalence and \( g : M \to V \) a homotopy inverse map to \( h \). Consider the normal bundle \( \nu \) of \( V \) and put \( \xi := g^*(\nu) \). Then \( h^*(\xi) = h^*g^*\nu = \nu \). The correcting morphism \( \nu = h^*(\xi) \to \xi \) is a normal morphism.
A normal morphism is called reducible if the map
\[ S^{N+n} \xrightarrow{\text{collapse}} T\nu \xrightarrow{T\varphi} T\xi \]
is a reducibility.

Because of the Thom Isomorphism Theorem, a normal morphism is reducible whenever its base \( V \to M \) is a map of degree 1. (One can prove that \( \xi \) is orientable if \( V \) and \( M \) are.)

We denote the set of all reducible normal morphisms at \( M \) by \( \text{Nor}(M) \). (For persons who asks whether \( \text{Nor}(M) \) is a set, we take the space \( \mathbb{R}^\infty \) and assume that all the spaces in 11.1 are contained in \( \mathbb{R}^\infty \).

11.3. Construction–Definition. Represent a map (homotopy class) \( f : M \to F/PL \) by an \((S^N,\ast)\)-morphism \( \varphi : \nu_M^* \to (\gamma^N_{PL})^* \) with \( N \) large enough, see [3.9]. Set \( \xi = (b\varphi)^*\gamma^N_{PL} \). Then the correcting \((S^N,\ast)\)-morphism \( \nu_M^* \to \xi^* \) is a commutative diagram
\[
\begin{array}{ccc}
U^* & \xrightarrow{g} & U'^* \\
q \downarrow & & \downarrow p \\
M & = & M
\end{array}
\]
where \( \nu_M = \{q : U \to M\} \), \( \xi = \{p : U' \to M\} \) are PL \( \mathbb{R}^N \)-bundles, and \( U^*, U'^* \) are fiberwise one-point compactifications of \( U \) and \( U' \), respectively.

We regard \( M \) as the zero section of \( \xi \), \( M \subset U' \) and deform \( g \) to a map \( t : U^* \to U'^* \) which is transverse to \( M \). Set \( V = t^{-1}(M) \) and \( b = t|_V : V \to M \). We can assume that \( V \subset U \). So, we get the \( b \)-adjoint PL \( \mathbb{R}^N \)-morphism
\[
\mathcal{J}_b : b^*\xi \to \xi, \quad \text{bs}(\mathcal{J}_b) = b : V \to M.
\]
Note that \( b^*\xi \) is the normal bundle of \( V \) in \( U \), and therefore it is the normal bundle \( \nu_V \) of \( V \) in \( \mathbb{R}^{N+n} \) because \( U \) is the open subset of \( \mathbb{R}^{N+n} \). In other words, the morphism \( (11.2) \) is a normal morphism at \( M \). We say that the normal morphism \( (11.2) \) is associated with a map (homotopy class) \( f : M \to F/PL \).

Clearly, there are many normal morphisms that are associated with a given map \( f : M \to F/PL \).

11.4. Construction–Definition. Let
\[
\varphi : \nu_V \to \xi
\]
be a reducible normal morphism at \( M \) and assume that \( \dim \nu_V \) is large. Consider a collapse map (homotopy class) \( t : S^{N+n} \to T\nu_M \) as in [4.4].
Since the map 
\[ \alpha : S^{N+n} \text{ collapse} \xrightarrow{\nu} T\nu \xrightarrow{T\varphi} T\xi \]
is a reducibility, there exists, by Theorem 4.5, a unique \((S^N, *)\)-morphism \(\mu : \nu_M \to \xi^*\) with \(\mu_*(i) = \alpha\). Now, the morphism
\[ \nu_M^* \xrightarrow{\mu_*} \xi^* \xrightarrow{\text{classif}} (\gamma_{PL})^* \]
is a homotopy PL structuralization of \(\nu_M\). Thus, in view of (3.9), we get a homotopy class in \([M, F/PL]\). We denote by \(f_\varphi : M \to F/PL\) any representative of this class.

11.5. **Proposition.** *The normal morphism* (11.3) *is associated with the map* \(f_\varphi : M \to F/PL\). \(\square\)

11.6. **Definition.** *The function*
\[(11.4) \quad \Gamma = \Gamma_M : \text{Nor}(M) \longrightarrow [M, F/PL], \quad \varphi \mapsto [f_\varphi].\]

is called the normal invariant for \(M\).

 Probably, a reader noticed that we already defined normal invariant \(j_F\) in Definition 11.6. Now we show that these two definitions (for homotopy structures and for normal morphisms) are quite close to each other. The relation between \(\Gamma\) and \(j_F\) appears in the commutative diagram
\[
\begin{array}{ccc}
\text{he}(M) & \longrightarrow & \text{Nor } M \\
\downarrow & & \downarrow \Gamma \\
\mathcal{S}_{PL}(M) & \xrightarrow{j_F} & [M, F/PL]
\end{array}
\]
where \(\text{he}(M)\) is the set of homotopy equivalences whose targets is \(M\). The horizontal top map is explained in Example 11.2, the left vertical map send a homotopy equivalence to its equivalence class as Definition 3.7.

11.7. **Definition (11.7).** A *normal bordism* between two normal morphisms \(\varphi_i : \nu_{V_i} \to M, i = 0, 1\) at \(M\) is a PL \(\mathbb{R}^N\)-morphism \(\Phi : \nu_W \to \xi\) where \(W\) is a compact PL manifold with \(\partial W = V_0 \sqcup V_1\) and \(\Phi|_{V_i} = \xi, i = 0, 1\). Furthermore, \(\nu_M\) is the PL normal \(\mathbb{R}^N\)-bundle of \(W\).
where $D \to W$ is a normal bundle of $W$ and $E \to M$ is the same bundle as in Definition [11.1]. Furthermore, $W$ is a compact manifold with $\partial W = V_0 \sqcup V_1$ and $(\bar{c}, c)|_{V_i} = (\bar{b}_i, b_i), i = 0, 1$.

We say that two normal morphisms are \textit{normally bordant} if there exists a normal bordism between these two normal morphisms. Clearly, “to be normally bordant” is an equivalence relation. The equivalence classes are called the \textit{normal bordism classes}. We denote by $[\text{Nor } M]$ the set of normal bordism classes at $M$.

11.8. \textbf{Theorem.} If $\varphi_0, \varphi_1$ are two normally bordant normal morphisms at $M$ then $f_{\varphi_0} \cong f_{\varphi_1} : \text{Nor } M \to F/\text{PL}$. So, the map $\Gamma$ yields a map

$$\tilde{\Gamma} : [\text{Nor } M] \to F/\text{PL}, \quad \tilde{\Gamma}[\varphi] = [\Gamma(\varphi)].$$

Moreover, the map $\tilde{\Gamma}$ is a bijection;

\textit{Proof (sketch).} This is a version of the Pontryagin–Thom theorem. We give a sketch and leave the detail to the reader. Let $\Phi$ be a normal bordism as in Definition [11.7]. Follow [11.4] and construct a map $F_\phi : M \to F/\text{PL}$. Then $F_\phi|_{M \times \{i\}} = f_{\varphi_i}$ for $i = 0, 1$. So, the above mentioned map $\tilde{\Gamma}$ is well-defined.

To construct an inverse map $\Delta$ to $\tilde{\Gamma}$, take a map $f : M \to F/\text{PL}$ and put $\Delta[f]$ to be the normal bordism class that is associated to $f$. Check that this normal bordism class is well-defined and that $\tilde{\Gamma}$ and $\Delta$ are inverse to each other. \hfill $\square$

Recall that a closed manifold is called \textit{almost parallelizable} if it becomes parallelizable after deleting of a point. Note that every almost parallelizable manifold is orientable (e.g., because its first Stiefel–Whitney class is equal to zero).

11.9. \textbf{Proposition.} For every $V^m$ be an almost parallelizable PL manifold $V^m$ there exists a reducible normal morphism with a base $V \to M$.

\textit{Proof.} We regard $S^m = \{(x_1, \ldots, x_{m+1}) \mid \sum x_i^2 = 1\}$ as the union of two discs, $S^m = D_+ \cup D_-$, where

$$D_+ = \{x \in S^m | x_{m+1} \geq 0\}, \quad D_- = \{x \in S^m | x_{m+1} \leq 0\}.$$

Take a map $b : V \to M$ of degree 1. We can assume that there is a small closed disk $D_0$ in $V$ such that $b_+ := b|_{D_0} : D_0 \to D_+$ is a PL homeomorphism. We set $W = V \setminus (\text{Int } D_0)$. Since $W$ is parallelizable, there exists a PL morphism $\varphi_- : \nu_V|_W \to \theta_{D_-}$ such that $b|_W : W \to D_-$ is the base of $\varphi$. Furthermore, since $b_+$ is a PL homeomorphism, there
exists a morphism $\varphi_+ : \nu_V|_{D_0} \to \theta|_{D_0}$ over $b_+$ such that $\varphi_+$ and $\varphi_-$ coincide over $b|_{\partial W} : \partial W \to S^{m-1}$. Together $\varphi_+$ and $\varphi_-$ give us a PL morphism $\varphi : \nu_V \to \xi$ where $\xi$ is a PL bundle over $S^m$. Clearly, $\varphi$ is a normal morphism with the base $b$, and it is reducible because $\deg b = 1$. \hfill \Box

12. The Sullivan Map $s : [M, F/PL] \to P_{\dim M}$

We define the groups $P_i$ by setting

$$P_i = \begin{cases} 
\mathbb{Z} & \text{if } i = 4k, \\
\mathbb{Z}/2 & \text{if } i = 4k + 2, \\
0 & \text{if } i = 2k + 1
\end{cases}$$

where $k \in \mathbb{N}$.

Given a closed connected $n$-dimensional PL manifold $M$ (which is assumed to be orientable for $n = 4k$), we define a map

(12.1) \hspace{1cm} s : [M, F/PL] \to P_n

as follows. Given a homotopy class $f : M \to F/PL$, consider a normal morphism $\mathcal{J}_b : b^* \xi \to \xi$, $\text{bs}(\mathcal{J}) = b : V \to M$ associated with $f$, see (11.2).

For $n = 4k$, let $\psi$ be the symmetric bilinear intersection form on $\text{Ker}\{b_* : H_{2k}(V; \mathbb{Q}) \to H_{2k}(M; \mathbb{Q})\}$. We define $s(u) = \sigma(\psi)/8$ where $\sigma(\psi)$ is the signature of $\psi$. It is well known that $\sigma(\psi)$ is divisible by 8, (see e.g. [Br]), and so $s(u) \in \mathbb{Z}$.

Also, it is easy to see that $\sigma(\psi) = \sigma(V) - \sigma(M)$, and so

$s(u) = \frac{\sigma(V) - \sigma(M)}{8}$

where $\sigma(V), \sigma(M)$ is the signature of the manifold $V, M$, respectively.

For $n = 4k + 2$, we define $s(u)$ to be the Kervaire invariant of the normal morphism $\mathcal{J}_b$, see e.g. [Br].

The routine arguments show that $s$ is well-defined, i.e. it does not depend on the choice of the associated normal morphism. See [Br] Ch. III, §4 or [N1] for details.

In particular, if $b$ is a homotopy equivalence then $s(u) = 0$.

One can prove that, for all $M$, the map $s$ is a homomorphism of abelian groups, where the abelian group structure on $[M, F/PL]$ is given by the $H$-space structure on $F/PL$. 
Given a map \( f : M \to F/PL \), it is useful to introduce the notation 
\( s(M, f) := s([f]) \) where \([f]\) is the homotopy class of \( f \).

12.1. Theorem. (i) The map \( s : [S^{4i}, F/PL] \to \mathbb{Z} \) is surjective for all \( i > 1 \),
(ii) The map \( s : [S^{4i-2}, F/PL] \to \mathbb{Z}/2 \) is surjective for all \( i > 0 \).
(iii) The image of the map \( s : [S^4, F/PL] \to \mathbb{Z} \) is the subgroup of index 2.

Proof. (i) For every \( k > 1 \) Milnor constructed a parallelizable \( 4k \)-dimensional smooth manifold \( W^{4k} \) of signature 8 and such that \( \partial W \) is a homotopy sphere, see [Br, V.2.9]. Since, by Theorem 3.12, every homotopy sphere of dimension \( \geq 5 \) is PL homeomorphic to the standard one, we can form a closed PL manifold 
\( V := W \cup S^{4k-1}D^{4k} \)
of the signature 8. Because of Proposition 11.9 there exists a reducible normal morphism with the base \( V^{4k} \to S^{4k} \). Because of Proposition 11.5 this normal morphism is associated with a certain map (homotopy class) \( f : S^{4k} \to F/PL \). Thus,
\( s(S^{4k}, f) = \frac{\sigma(V^{4k}) - gs(S^{4k})}{8} = 1 \).

(ii) The proof is similar to that of (i), but we must use \((4k+2)\)-dimensional parallelizable Kervaire manifolds \( W, \partial W = S^{4k+1} \) of the Kervaire invariant one, see [Br] V.2.11.

(iii) The Kummer algebraic surface \([K2]\) gives us an example of 4-dimensional almost parallelizable smooth manifold of the signature 16. So, \( \text{Im } s \supseteq 2\mathbb{Z} \).

Now suppose that there exists \( f : S^4 \to F/PL \) with \( s(S^4, f) = 1 \). Then there exists a normal morphism with the base \( V^4 \to S^4 \) and such that \( V \) has signature 8. Since normal bundle of \( V \) is induced from a bundle over \( S^4 \), we conclude that \( w_1(V) = 0 = w_2(V) \). But this contradicts the Rokhlin Theorem 7.1.

□

12.2. Theorem (Sullivan [Sul2]). For any closed simply-connected PL manifold \( M \) of dimension \( \geq 5 \), the sequence
\[ 0 \to S_{PL}(M) \xrightarrow{j_F} [M, F/PL] \xrightarrow{s} P_{\dim M} \]
is exact, i.e. \( j_F \) is injective and \( \text{Im } j_F = s^{-1}(0) \).

Proof. See [Br] II.4.10 and II.4.11. Note that the map \( \omega \) in loc. cit. is the zero map because, by Theorem 3.12, every homotopy sphere of dimension \( \geq 5 \) is PL homeomorphic to the standard sphere. □
12.3. Corollary. We have $\pi_{4i}(F/PL) = \mathbb{Z}$, $\pi_{4i-2}(F/PL) = \mathbb{Z}/2$, and $\pi_{2i-1}(F/PL) = 0$ for every $i > 0$. Moreover, the homomorphism

$$s : [S^k, F/PL] \to P_k$$

is an isomorphism for $k \neq 4$, while for $k = 4$ it has the form

$$\mathbb{Z} = \pi_4(F/PL) \xrightarrow{s} P_4 = \mathbb{Z}, \ a \mapsto 2a.$$

Proof. First, if $k > 4$ then, because of the Smale Theorem 3.12, $S_{PL}(S^k)$ is the one-point set. Now the result follows from Theorem 12.2 and 12.1.

If $k \leq 4$ then $\pi_k(PL/O) = 0$, cf. Remark 6.7. So, $\pi_k(F/PL) = \pi_k(F/O)$. Moreover, the forgetful map $\pi_k(BO) \to \pi_k(BF)$ coincides with the Whitehead $J$-homomorphism. So, we have the long exact sequence

$$\cdots \to \pi_k(F/O) \to \pi_k(BO) \xrightarrow{J} \pi_k(BF) \to \pi_{k-1}(F/O) \to \cdots.$$ 

For $k \leq 5$ all the groups $\pi_k(BO)$ and $\pi_k(BF)$ are known (note that $\pi_k(BF)$ is the stable homotopy group $\pi_{k+N-1}(S^N)$), and it is also known that $J$ is an epimorphism for $k = 1, 2, 4, 5$, see e.g. [Ad]. Thus, $\pi_k(F/O) \cong P_i$ for $k \leq 4$.

The last claim follows from Theorem 12.1. \hfill \qed

13. The Homotopy Type of $F/PL[2]$

Recall that, given a space $X$ and an abelian group $\pi$, we allow us to ignore the distinction between elements of $H^*(X; \pi)$ and maps (homotopy classes) $X \to K(\pi, n)$.

13.1. Notation. Given a prime $p$, let $\mathbb{Z}[p]$ be the subring of $\mathbb{Q}$ consisting of all irreducible fractions with denominators relatively prime to $p$, and let $\mathbb{Z}[1/p]$ be the subgroup of $\mathbb{Q}$ consisting of the fractions $m/p^k, m \in \mathbb{Z}$. Given a simply-connected space $X$, we denote by $X[p]$ and $X[1/p]$ the $\mathbb{Z}[p]$- and $\mathbb{Z}[1/p]$-localization of $X$, respectively. Furthermore, we denote by $X[0]$ the $\mathbb{Q}$-localization of $X$. For the definitions, see [HMR].

13.2. Proposition (Sullivan [Sul1, Sul2]). For every $i > 0$ there are cohomology classes

$$K_{4i} \in H^{4i}(F/PL; \mathbb{Z}[2]), \ K_{4i-2} \in H^{4i-2}(F/PL; \mathbb{Z}/2)$$

such that

$$s(M^{4i}, f) = \langle f^*K_{4i}, [M] \rangle$$

for every closed connected oriented PL manifold $M$, and

$$s(N^{4i-2}, f) = \langle f^*K_{4i-2}, [N]_2 \rangle.$$
for every closed connected manifold $N$. Here $[M] \in H^{4i}(M)$ is the fundamental class of $M$, $[N]_2 \in H^{4i-2}(N; \mathbb{Z}/2)$ is the modulo 2 fundamental class of $N$, and $\langle - , - \rangle$ is the Kronecker pairing.

**Proof.** Let $MSO_*(-)$ denote the oriented bordism theory, see e.g. [Rud]. Recall that if two maps $f : M^{4i} \to F/PL$ and $g : N^{4i} \to F/PL$ are bordant (as oriented singular manifolds) then $s(M, f) = s(N, g)$. Thus, $s$ defines a homomorphism

$$\tilde{s} : MSO_{4i}(F/PL) \to \mathbb{Z}.$$ 

It is well known that the Steenrod–Thom map $t : MSO_*(-) \otimes \mathbb{Z}[2] \to H_*(-; \mathbb{Z}[2])$ splits, i.e. there is a natural map $v : H_*(-; \mathbb{Z}[2]) \to MSO_*(-) \otimes \mathbb{Z}[2]$ such that $tv = 1$ (a theorem of Wall [W1], see also [St, Rud, As]. In particular, we have a natural homomorphism

$$\hat{s} : H_{4i}(F/PL; \mathbb{Z}[2]) \xrightarrow{u} MSO_{4i}(F/PL) \otimes \mathbb{Z}[2] \xrightarrow{\tilde{s}} \mathbb{Z}.$$ 

Since the evaluation map $ev : H^*(X; \mathbb{Z}[2]) \to \text{Hom}(H_*(X; \mathbb{Z}[2]), \mathbb{Z}[2])$, $$(ev(u)(v) = \langle u, v \rangle, \ u \in H^*(X; \mathbb{Z}[2]), \ v \in H_*(X; \mathbb{Z}[2])$$ is surjective for all $X$, there exists a class $K_{4i} \in H^{4i}(F/PL; \mathbb{Z}[2])$ such that $ev(K_{4i}) = \hat{s}$. Now

$$s(M, f) = \hat{s}(f_*[M]) = \langle K_{4i}, f_*[M] \rangle = \langle f^*K_{4i}, [M] \rangle.$$ 

So, we constructed the desired classes $K_{4i}$.

The construction of classes $K_{4i-2}$ is similar. Let $MO_*(-)$ denoted the non-oriented bordism theory. Then the map $s$ yields a homomorphism

$$\tilde{s} : MO_{4i-2}(F/PL) \to \mathbb{Z}/2.$$ 

Furthermore, there exists a natural map $H_*(-; \mathbb{Z}/2) \to MO_*(-)$ which splits the Steenrod–Thom homomorphism, and so we have a homomorphism

$$\hat{s} : H_{4i-2}(F/PL; \mathbb{Z}/2) \longrightarrow MO_{4i-2}(F/PL) \otimes \mathbb{Z}[2] \xrightarrow{\hat{s}} \mathbb{Z}/2$$ 

with $\hat{s}(f_*([M]_2)) = s(M, f)$. Now we can complete the proof similarly to the case of classes $K_{4i}$. $\square$

We set

$$(13.1) \quad \Pi := \prod_{i>1} (K(\mathbb{Z}[2], 4i) \times K(\mathbb{Z}/2, 4i - 2)).$$
Together the classes $K_{4i} : F/PL \to K(\mathbb{Z}[2], 4i), i > 1$ and $K_{4i-2} : F/PL \to K(\mathbb{Z}[2, 4i-2], i > 1$ yield a map
\begin{equation}
(13.2) \quad K : F/PL \to \Pi
\end{equation}
such that for each $i > 1$ the map
\[
F/PL \xrightarrow{K} \Pi \xrightarrow{\text{projection}} K(\mathbb{Z}[2], 4i)
\]
coincides with $K_{4i}$ and the map
\[
F/PL \xrightarrow{K} \Pi \xrightarrow{\text{projection}} K(\mathbb{Z}/2, 4i-2)
\]
coincides with $K_{4i-2}$.

13.3. **Lemma.** The map
\begin{equation}
(13.3) \quad K[2] : F/PL[2] \to \Pi
\end{equation}
induced an isomorphism of homotopy groups in dimensions $\geq 5$.

**Proof.** This follows from Theorem 12.1 and Corollary 12.3.

Let $Y$ be the Postnikov 4-stage of $F/PL$. So, we have a map
\begin{equation}
(13.4) \quad \psi : F/PL \to Y
\end{equation}
that induces an isomorphism of homotopy groups in dimension $\leq 4$. Consider the map
\[
\phi : F/PL[2] \to Y[2] \times \Pi, \quad \phi(x) = (\psi[2](x), K[2](x)).
\]

13.4. **Theorem.** The map
\[
\phi : F/PL[2] \to Y[2] \times \Pi
\]
is a homotopy equivalence.

**Proof.** The maps
\[
\phi_* : \pi_i(F/PL[2] \to \pi_i(Y[2] \times \Pi)
\]
are isomorphism for all $i$. Indeed, for $i \leq 4$ the holds since $\psi$ is the Postnikov 4-approximation of $F/PL$, for $\geq 5$ it follows from 13.3. Thus, $\phi$ is a homotopy equivalence by the Whitehead Theorem.

Now we discuss the space $Y$ in greater detail. We have $\pi_2(Y) = \mathbb{Z}/2$, $\pi_4(Y) = \mathbb{Z}$, and $\pi_i(Y) = 0$ otherwise. So, we have a $K(\mathbb{Z}, 4)$-fibration
\begin{equation}
(13.5) \quad K(\mathbb{Z}, 4) \xrightarrow{i} Y \xrightarrow{p} K(\mathbb{Z}/2, 2)
\end{equation}
whose characteristic class is the Postnikov invariant
\[
\kappa \in H^5(K(\mathbb{Z}/2, 2)) := \kappa \in H^5(K(\mathbb{Z}/2, 2); \mathbb{Z})
\]
of $Y$. We shall see in Theorem 13.8 below that $\kappa = \delta Sq^2 \iota_2 \neq 0$. Hence, $\kappa$ is also the first non-trivial Postnikov invariant of $F/PL$. 
13.5. **Lemma.** There exists a map $g : \mathbb{CP}^2 \to F/PL$ such that $s(\mathbb{CP}^2, g) = 1$.

**Proof.** Let $\eta$ denote the canonical complex line bundle over $\mathbb{CP}^2$. First, we prove that $24\eta$ is fiberwise homotopy trivial. Let $H : S^3 \to S^2$ be the Hopf map. Consider the Puppe sequence

$$S^3 \xrightarrow{H} S^2 \longrightarrow \mathbb{CP}^2 \longrightarrow S^4$$

and the induced exact sequence

$$[S^3, BF] \xleftarrow{H^*} [S^2, BF] \xleftarrow{} [\mathbb{CP}^2, BF] \xleftarrow{} [S^4, BF]$$

Let $\Pi^S_n$ denotes the $n$-th stable homotopy group $\pi_{n+N}(S^N)$, $N$ large. Note that $[S^n, BF] = \Pi^S_{n-1}$. We have $[S^4, BF] = \Pi^S_3 = \mathbb{Z}/24$, $[H]$. Furthermore, the homomorphism

$$\mathbb{Z}/2 = [S^3, BF] \xleftarrow{H^*} [S^2, BF] = \mathbb{Z}/2$$

is an isomorphism, because the suspension $S^N H : S^{N+3} \to S^{N+2}$ is the generator of $\pi_{n+3}S^{N+2} = \Pi_n = \mathbb{Z}/2$ for $N$ large. Hence, $[\mathbb{CP}^2, BF]$ is a quotient group of $\mathbb{Z}/24$. In particular, $24\eta$ is fiberwise homotopy trivial.

So, the classifying map $\mathbb{CP}^2 \to BO \to BPL$ for $24\eta$ lifts to $F/PL$, In other words, there exist a map $g : \mathbb{CP}^2 \to F/PL$ such that the map

$$\mathbb{CP}^2 \xrightarrow{g} F/PL \longrightarrow BPL$$

classifies $24\eta$. Since $\langle p_1(\eta), [\mathbb{CP}^2] \rangle = 1$, we have $\langle p_1(24\eta), [\mathbb{CP}^2] \rangle = 24$, and therefore $\langle L_1(24\eta), [\mathbb{CP}^2] \rangle = 8$ (here $p_1$ and $L_1$ denote the first Pontryagin class and first Hirzebruch class, respectively), see [MS]. Thus, $s(\mathbb{CP}^2, g) = 8/8 = 1$, and therefore $\langle K_4, g_*[\mathbb{CP}^2] \rangle = 1$. □

Let $h : \pi_4(F/PL) \to H_4(F/PL)$ be the Hurewicz homomorphism. Let tors denotes the torsion subgroup of $H_4(F/PL)$.

13.6. **Lemma.** The map

$$a : \mathbb{Z} = \pi_4(F/PL) \xrightarrow{h} H_4(F/PL) \xrightarrow{\text{quotient}} H_4(F/PL)/\text{tors} = \mathbb{Z}$$

is not surjective.

**Proof.** Consider the Leray–Serre spectral sequence of the fibration (13.5) and note that $H_4(Y)/\text{tors} = \mathbb{Z}$, because $H_4(K(\mathbb{Z}, 4)) = \mathbb{Z}$ and all the groups $\tilde{H}_i(K(\mathbb{Z}/2, 2))$ are finite. Furthermore, $H_4(F/PL) \cong H_4(Y)$ since $Y$ is a Postnikov 4-stage of $F/PL$. Thus, $H_4(F/PL)/\text{tors} = \mathbb{Z}$.

Because of 12.3 and 13.2 the subgroup $\langle K^4, \text{Im } a \rangle$ of $\mathbb{Z}$ consist of even numbers. On the other hand, $\langle K_4, g_*[\mathbb{CP}^2] \rangle = 1$ by Lemma 13.5.
Thus, the image of $g_*[\mathbb{CP}^2]$ in $H_4(F/PL)/\text{tors}$ does not belong to $\text{Im}\ a$.

Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z}[2] \xrightarrow{\rho} \mathbb{Z}/2 \longrightarrow 0$$

where $2$ over the arrow means multiplication by $2$ and $\rho$ is the modulo $2$ reduction. This exact sequence yields the Bockstein exact sequence

$$\cdots \longrightarrow H^n(X; \mathbb{Z}) \xrightarrow{2} H^n(X; \mathbb{Z}) \xrightarrow{\rho_*} H^n(X; \mathbb{Z}/2) \xrightarrow{\delta} H^{n+1}(X; \mathbb{Z}) \longrightarrow \cdots .$$

Put $X = K(\mathbb{Z}/2, n)$ and consider the fundamental class $\iota_n \in H^n(K(\mathbb{Z}/2, n); \mathbb{Z}/2)$. Then we have the class $\delta := \delta(\iota_n) \in H^{n+1}(K(\mathbb{Z}/2, n), \mathbb{Z})$. According to what we said above, we regard $\delta$ as a map $\delta : K(\mathbb{Z}/2, n) \to K(\mathbb{Z}, n+1)$ and/or the cohomology operation

$$\delta : H^n(\cdot; \mathbb{Z}/2) \to H^{n+1}(\cdot, \mathbb{Z}).$$

13.7. Lemma. We have: $H^{n+3}(K(\mathbb{Z}/2, n)) = \mathbb{Z}/2 = \delta Sq^2 \iota_n$ for all $n \geq 4$.

Proof. Put $\iota = \iota_n$. We have

$$H^{n+3}(K(\mathbb{Z}/2, n), \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 = \{Sq^3 \iota, Sq^2 Sq^1 \iota\},$$

see [MT, Ch. 9]. Let

$$\beta := \rho_* \delta : H^i(\cdot; \mathbb{Z}/2) \to H^{i+1}(\cdot; \mathbb{Z}/2)$$

be the Bockstein homomorphism. Since $\beta(Sq^3 \iota) \neq 0$, we conclude that $Sq^3 \iota \notin \text{Im}\{\rho_* : H^{n+3}(K(\mathbb{Z}/2, n)) \to H^{n+3}(K(\mathbb{Z}/2, n), \mathbb{Z}/2)\}$, see [MT, Ch. 11]. Since the homomorphism

$$\rho_* \otimes 1 : H_*(\cdot) \otimes \mathbb{Z}/2 \to H_*(\cdot; \mathbb{Z}/2)$$

is injective, we conclude that the group $H^{n+3}(K(\mathbb{Z}/2, n))$ is cyclic, and this group is $2$-primary in view of Serre Class Theory, [MT, Ch. 10]. Hence, the group $H^{n+3}(K(\mathbb{Z}/2, n))$ generates $\delta Sq^2 \iota$ since $\rho_* \delta Sq^2 \iota = Sq^3 \iota$ generates $H^{n+3}(K(\mathbb{Z}/2, n); \mathbb{Z}/2)$. Finally, $\delta Sq^2 \iota$ has the order $2$ since $2\delta = 0$.

Recall that we denote by $\kappa \in H^5(K(\mathbb{Z}/2, 2))$ the characteristic class of the fibration (13.5).
13.8. Theorem. We have $\kappa = \delta Sq^2 \iota_2$, and it is an element of order 2 in the group $H^5(K(\mathbb{Z}/2, 2)) = \mathbb{Z}/4$. So, $\kappa$ is the first non-trivial Postnikov invariant of $F/PL$.

Proof. Note that $\kappa \neq 0$ because of 13.6. Indeed, otherwise $Y \cong K(\mathbb{Z}, 4) \times K(\mathbb{Z}/2, 2)$. But this contradicts Lemma 13.6.

Let $\Omega$ be the loop functor on the category of topological spaces and maps. Since $F/PL$ is an infinite loop space, the Postnikov invariant $\kappa$ of $F/PL$ can be written as $\Omega^N a_N$ for all $N$ and suitable $a_N \in [K(\mathbb{Z}/2, N + 2), K(\mathbb{Z}, N + 5)] = H^{N+5}(K(\mathbb{Z}/2, N + 2); \mathbb{Z})$.

By Lemma 13.7, for $N > 5$ the last group is equal to $\mathbb{Z}/2$. Thus, $\kappa$ has the order 2. It is easy to see that $H^5(K(\mathbb{Z}/2, 2)) = \mathbb{Z}/4 = \{x\}$ with $2x = \delta Sq^2 \iota_2$, see e.g. [Rud, Lemma VI.2.7]. Thus, $\kappa = \delta Sq^2 \iota_2$. \qed

13.9. Lemma. Let $X$ be a finite $CW$-space such that the group $H_*(X)$ is torsion free. Let $Z$ be an infinite loop space such the groups $\pi_i(Z)$ have no odd torsion for all $i$. Then the group $[X, Z]$ is torsion free. In particular, the group $[X, F/PL[1/2]]$ is torsion free.

Proof. It suffices to prove that $[X, Z[p]]$ is torsion free for every odd prime $p$. Note that $Z[p]$ is an infinite loop space since $Z$ is. So, there exists a connected $p$-local spectrum $E$ such that $\tilde{E}^0(\cdot) = [-, Z[p]] = [-, Z \otimes Z[p]]$.

Moreover, $E^{-i}(pt) = \pi_i(E) = \pi_i(Z) \otimes Z[p]$, So, because of the isomorphism $\tilde{E}^0(X) \cong [X, Z[p]]$, it suffices to prove that $E^*(X)$ is torsion free. Consider the Atiyah–Hirzebruch spectral sequence for $E^*(X)$. Its initial term is torsion free because $E^*(pt)$ and $H^*(X)$ are torsion free. Hence, the spectral sequence degenerates, and thus the group $E^*(X)$ is torsion free. \qed

13.10. Proposition. Let $X$ be a finite $CW$-space such that the group $H_*(X)$ is torsion free. Let $f : X \to F/PL$ be a map such that $f^*K_{4i} = 0$ and $f^*K_{4i+2} = 0$ for all $i \geq 1$. Then $f$ is null-homotopic.

Proof. Consider the commutative square

$$
\begin{array}{ccc}
F/PL & \xrightarrow{l_1} & F/PL[2] \\
\downarrow{l_2} & & \downarrow{l_3} \\
F/PL[1/2] & \xrightarrow{l_4} & F/PL[0]
\end{array}
$$
where the horizontal maps are the \( \mathbb{Z}[2]\)-localizations and the vertical maps are the \( \mathbb{Z}[1/2]\)-localizations. Because of \([13.4]\) \([X,F/PL]\) is a finitely generated abelian group, and so it suffices to prove that both \( l_1 \circ f \) and \( l_2 \circ f \) are null-homotopic. First, we remark that \( l_2 \circ f \) is null-homotopic whenever \( l_1 \circ f \) is. Indeed, since \( H_*(X) \) is torsion free, the group \([X,F/PL[1/2]]\) is torsion free by \([13.9]\). Now, if \( l_1 \circ f \) is null-homotopic then \( l_3 \circ l_1 \circ f \) is null-homotopic, and hence \( l_4 \circ l_2 \circ f \) is null-homotopic. Thus, \( l_2 \circ f \) is null-homotopic since \([X,F/PL[1/2]]\) is torsion free.

So, it remains to prove that \( l_1 \circ f \) is null-homotopic.

Clearly, the equalities \( f^* K_{4i} = 0 \) and \( f^* K_{4i-2} = 0, \ i > 1 \), imply that the map

\[
X \longrightarrow F/PL \xrightarrow{l_1} F/PL[2] \cong Y[2] \times \Pi \xrightarrow{p_2} \Pi
\]

is null-homotopic. So, it remains to prove that the map

\[
g : X \xrightarrow{f} F/PL \xrightarrow{l_1} F/PL[2] \cong Y[2] \times \Pi \xrightarrow{p_1} Y[2]
\]

is null-homotopic.

It is easy to see that \( H^4(Y[2];\mathbb{Z}[2]) = \mathbb{Z}[2] \), see e.g. \([Rud], \text{VI.2.9(i)}\]. Let \( u \in H^4(Y;\mathbb{Z}[2]) \) be a free generator of the free \( \mathbb{Z}[2] \)-module \( H^4(Y;\mathbb{Z}[2]) \).

The fibration \([13.5]\) gives us the following diagram with the exact row:

\[
\begin{array}{ccc}
H^4(X;\mathbb{Z}[2]) & \xrightarrow{i_*} & [X,Y[2]] \\
& \downarrow^{u_*} & \\
H^4(X;\mathbb{Z}[2])
\end{array}
\]

Note that

\[
u_* i_* : \mathbb{Z}[2] \to \mathbb{Z}[2]
\]

is the multiplication by \( 2\varepsilon \) where \( \varepsilon \) is an invertible element of the ring \( \mathbb{Z}[2] \), see e.g. \([Rud], \text{VI.2.9(ii)}\]. Since \( f^* K_2 = 0 \), we conclude that \( p_*(g) = 0 \), and so \( g = i_*(a) \) for some \( a \in H^4(X;\mathbb{Z}[2]) \). Now,

\[
0 = u_*(g) = u_*(i_*(a)) = 2a\varepsilon.
\]

But \( H^*(X;\mathbb{Z}[2]) \) is torsion free, and thus \( a = 0 \). \( \square \)

For completeness, we mention also that \( F/PL[1/2] \cong BO[1/2] \). So, there is a Cartesian square (see \( \text{MM, [Sn2]} \))

\[
\begin{array}{ccc}
F/PL & \longrightarrow & \Pi \times Y \\
\downarrow & & \downarrow \\
BO[1/2] & \xrightarrow{\text{ph}} & \prod K(Q,4i)
\end{array}
\]

where \( \text{ph} \) is the Pontryagin character.
14. Splitting Theorems

14.1. Definition. Let $A^{n+k}$ and $W^{n+k}$ be two connected PL manifolds (without boundaries), and let $M^n$ be a closed PL submanifold of $A$. We say that a map $g : W^{n+k} \to A^{n+k}$ splits along $M^n$ if there exists a homotopy $g_t : W^{n+k} \to A^{n+k}$, $t \in I$

such that:

(i) $g_0 = g$;
(ii) there is a compact subset $K$ of $W$ such that $g_t|_{W \setminus K} = g|_{W \setminus K}$ for all $t \in I$;
(iii) the map $g_1$ is transverse to $M$ (and hence $g_1^{-1}(M)$ is a closed PL submanifold of $M_1$), and the map $b := g_1|_{g_1^{-1}(M)} : g_1^{-1}(M) \to M$ is a homotopy equivalence.

We also say that the homotopy $G : W \times I \to A, G(w, t) = g_t(w)$

realizes the splitting of $g$.

An important special case is when $A^{n+k} = M^n \times B^k$ for some connected manifold $B^k$. In this case we can regard $M$ as submanifold $M \times \{b_0\}, b_0 \in B$ of $A$ and say that $g : W \to A$ splits along $M$ if it splits along $M \times \{b_0\}$ if and only if it splits along $M \times \{b\}$ with any other $b \in B$.

Recall that a map $f$ is called proper if $f^{-1}(C)$ is compact whenever $C$ is compact. A map $f : X \to Y$ is called a proper homotopy equivalence if there exist a map $g : Y \to X$ and homotopies $F : gf \simeq 1_X$, $G : fg \simeq 1_Y$ such that all the four maps $f, g, F : X \times I \to X$, and $G : Y \times I \to Y$ are proper.

14.2. Theorem. Let $M^n, n \geq 5$ be a closed connected $n$-dimensional PL manifold such that $\pi_1(M)$ is a free abelian group. Then every proper homotopy equivalence $h : W^{n+1} \to M^n \times \mathbb{R}$ splits along $M^n$.

Proof. Because of the Thom transversality theorem, there is a homotopy $h_t : W \to M \times \mathbb{R}$ satisfies condition (ii) of [14.1] and such that $h_1$ is transversal to $M$ . We let $f = h_1$. Because of a crucial theorem of Novikov [N2, Theorem 3], there is a sequence of surgeries of the inclusion $f^{-1}(M) \subset W$ in $W$ such that the final result of these surgeries is a homotopy equivalence $V \subset W$. Using the Pontryagin–Thom construction, we can realize this sequence of surgeries via a homotopy $f_t$ such that $f_t$ satisfies conditions (i)–(iii) of [14.1] with $f_1^{-1}(M) = V$. 

14.3. Theorem. Let $M^n$ be a manifold as in [14.2]. Then every homotopy equivalence $W^{n+1} \to M^n \times S^1$ splits along $M^n$. 

Proof. See [Fa], cf. also [FH]. □

14.4. COROLLARY. Let \( M^n \) be a manifold as in 14.2. Let \( T^k \) denote the \( k \)-dimensional torus. Then every homotopy equivalence \( W^{n+k} \to M^n \times T^k \) splits along \( M^n \).

Proof. This follows from 14.3 by induction. □

14.5. THEOREM. Let \( M^n \) be a manifold as in 14.2. Then every homeomorphism \( h : W^{n+k} \to M^n \times \mathbb{R}^k \) splits along \( M^n \).

Proof. We use the Novikov’s torus trick. The inclusion \( T^{k-1} \times \mathbb{R} \subset \mathbb{R}^k \) yields the inclusion
\[
M \times T^{k-1} \times \mathbb{R} \subset M \times \mathbb{R}^k.
\]
We set \( W_1 := h^{-1}(M \times T^{k-1} \times R) \). Note that \( W_1 \) is a smooth subset of \( W \). Now, set

\[
u = h|_{W_1} : W_1 \to M \times T^{k-1} \times \mathbb{R}.
\]

Then, by 14.2 \( u \) splits along \( M \times T^{k-1} \), i.e. there is a homotopy \( u_t \) as in 14.1. We set \( f := u_1, V := f^{-1}(M \times T^{k-1}) \), and \( g := f|_V \). Because of 14.4 \( g : V \to M \times T^{k-1} \) splits along \( M \). Hence, \( f \) splits along \( M \), and therefore \( u \) in (14.1) splits along \( M \). This splitting yields a homotopy \( \overline{u}_t \) with \( \overline{u}_0 = u \) as in Definition 14.1. Now, we define the homotopy
\[
k_t : W \to M \times \mathbb{R}^k
\]
by setting \( k_t|_{W_1} := \overline{u}_t|_{W_1} \) and \( k_t|_{W \setminus W_1} := h|_{W \setminus W_1} \). Note that \( \{k_t\} \) is a well-defined and continuous family since the family \( \{\overline{u}_t\} \) satisfies 14.1 ii). It is clear that \( k_t \) satisfies the conditions (i)—(iii) of 14.1 and that \( k_1 \) extends \( f \) on the whole \( W \), i.e. \( h \) splits along \( M \). Thus, \( h \) splits along \( M \). □

14.6. REMARKS. 1. The above used Theorems 14.2, 14.3, and 14.5 of Novikov and Farrell–Hsiang were originally proved for smooth manifolds, but they are valid for PL manifolds as well, because there is an analog of the Thom Transversality Theorem for PL manifolds, [Wil].

2. In the above mentioned theorems we require the spaces to have free abelian fundamental groups. For arbitrary fundamental groups, there are obstructions to the splittings that involves algebraic \( K \)-theory of the fundamental group \( \pi \). In fact, in Theorem 14.3 there is an obstruction that is in an element of the Whitehead group \( \text{Wh}(\pi) \) of \( \pi \). For Theorem 14.2 there are two obstructions: in \( K^0(\pi) \) and in \( \text{Wh}(\pi) \).
14.7. Lemma. Suppose that a map $g : W \to A$ splits along a submanifold $M$ of $A$. Let $\xi = \{ E \to A \}$ be a PL bundle over $A$, let $g^*\xi = \{ D \to W \}$, and let $\mathcal{I}_g : g^*\xi \to \xi$ be the $g$-adjoint bundle morphism. Finally, let $l : D \to E$ be the map of the total spaces induced by $k$. Then $l$ splits over $M$. (Here we regard $A$ as the zero section of $\xi$, and so $M$ turns out to be a submanifold of $E$).

Proof. Let $G : W \times I \to A$ be a homotopy which realizes the splitting of $g$. Recall that $g^*\xi \times I$ is equivalent to $G^*\xi$. Now, the morphism

$$g^*\xi \times I \cong G^*\xi \xrightarrow{3g} \xi$$

gives us the homotopy $D \times I \to E$ which realizes the splitting of $l$. □

14.8. Lemma. Let $M$ be a manifold as in 14.2. Consider two PL $\mathbb{R}^N$-bundles $\xi = \{ U \to M \}$ and $\eta = \{ E \to M \}$ over $M$ and a topological morphism $\varphi : \xi \to \eta$ over $M$ of the form

$$
\begin{array}{ccc}
U & \to^g & E \\
\downarrow & & \downarrow \\
M & \cong & M.
\end{array}
$$

Then there exists $k$ such that the map

$$g \times 1 : U \times \mathbb{R}^k \to E \times \mathbb{R}^k$$

splits along $M$, where $M$ is regarded as the zero section of $\eta$.

Proof. Take a PL $\mathbb{R}^m$-bundle $\zeta$ such that $\eta \oplus \zeta = \mathbb{R}^{60N+m}$ and let $W$ be the total space of $\xi \oplus \zeta$. Then the morphism

$$\varphi \oplus 1 : \xi \oplus \zeta \to \eta \oplus \zeta = \theta^{N+m}$$

yields a map of the total spaces

$$
\Phi : W \to M \times \mathbb{R}^{N+m}.
$$

Because of Theorem 14.5, the map $\Phi$ splits along $M$. Furthermore, the morphism

$$\varphi \oplus 1 \oplus 1 : \xi \oplus \zeta \oplus \eta \to \eta \oplus \zeta \oplus \eta$$

yields a map of the total spaces

$$g \times 1 : U \times \mathbb{R}^{2N+m} \to E \times \mathbb{R}^{2N+m}.$$ 

In view of Lemma 14.7, this map splits over $M$ because $\Phi$ does. So, we can put $k = 2N + m$. □

Now, let $a : \text{TOP/PL} \to F/\text{PL}$ be a map as in (2.8).
14.9. Theorem. Let $M$ be as in 14.2. Then the composition
\[ [M, TOP/PL] \xrightarrow{a_*} [M, F/PL] \xrightarrow{s} P_{\dim M} \]
is trivial, i.e., $sa_*(v) = 0$ for every $v \in [M, TOP/PL]$. In other words, $s(M, af) = 0$ for every $f : M \to TOP/PL$.

Proof. In view of (3.4), every element $v \in [M, TOP/PL]$ gives us a (class of a) topological morphism
\[ \varphi : \nu_M^N \longrightarrow \gamma_{PL}^N \]
of PL $\mathbb{R}^N$-bundles. To map the class $v \in [M, TOP/PL]$ to the class $a_*v \in [M, F/PL]$, we must convert $\varphi$ to the (equivalence class of the) $(S^N, \ast)$-morphism $\varphi^* : (\nu_M)^* \to (\gamma_{PL}^N)^*$. Now, we follow and construct a commutative diagram
\[ \begin{array}{ccc}
U^{\ast} & \xrightarrow{g} & U'^{\ast} \\
q \downarrow & & \downarrow p \\
M & \xrightarrow{=} & M
\end{array} \tag{14.3} \]
lke (11.1). However, here $g$ is a homeomorphism. Thus, $g(U) = U'$, and so we get the diagram
\[ \begin{array}{ccc}
U & \xrightarrow{g} & U' \\
q \downarrow & & \downarrow p \\
M & \xrightarrow{=} & M
\end{array} \tag{14.4} \]
which is a topological morphism of PL bundles over $M$.

We embed $M$ in $U'$ as the zero section of $p$. By the definition of the map $s$, we conclude that $s(M, a_*v) = 0$ if the map $g : U \to U'$ splits along $M$ (because in this case the associated normal morphism is a map over a homotopy equivalence). Furthermore for any $k$, the topological morphisms $\varphi$ and
\[ \nu_M \oplus \theta^k \xrightarrow{(\varphi \oplus 1)} \gamma_{PL}^N \oplus \theta^k \longrightarrow \gamma_{PL}^{N+k} \]
represent the same element of $[M, TOP/PL]$. Hence, $s(M, a_*v) = 0$ provided there exists at least one $k$ such that the map
\[ g \times 1 : U \times \mathbb{R}^k \to U' \times \mathbb{R}^k \]
splits along $M$. But this follows from Lemma 14.8 since (14.4) is a topological morphism $\square$

Now we show that the condition $\dim M \geq 5$ in 14.9 is not necessary.
14.10. Corollary. Let $M$ be a closed connected PL manifold such that $\pi_1(M)$ is a free abelian group. Then $s(M, af) = 0$ for every map $f : M \to \text{TOP}/\text{PL}$.

Proof. Let $\mathbb{C}P^2$ denote the complex projective plane, and let

$$p_1 : M \times \mathbb{C}P^2 \to M$$

be the projection on the first factor. Then $s(M \times \mathbb{C}P^2, gp_1) = s(M, g)$ for every $g : M \to F/\text{PL}$, see [Br] Ch. III, §5. In particular, for every map $f ; M \to \text{TOP}/\text{PL}$ we have

$$s(M, af) = s(M \times \mathbb{C}P^2, (af)p_1) = s(M \times \mathbb{C}P^2, a(fp_1)) = 0$$

where the last equality follows from Theorem 14.9. □

15. Detecting Families

Recall the terminology: a singular smooth manifold in a space $X$ is a map $M \to X$ of a smooth manifold $M$.

Given a CW-space $X$, consider a connected closed smooth singular manifold $\gamma : M \to X$ in $X$. Then, for every map $f : X \to F/\text{PL}$, the invariant $s(M, f\gamma) \in P_{\dim M}$ is defined. Clearly, if $f$ is null-homotopic then $s(M, f\gamma) = 0$.

15.1. Definition. Let $\{\gamma_j : M_j \to X\}_{j \in J}$ be a family of closed connected smooth singular manifolds in $X$; here $J$ is an index set. We say that the family $\{\gamma_j : M_j \to X\}$ is a detecting family for $X$ if, for every map $f : X \to F/\text{PL}$, the validity of all the equalities $s(M_j, f\gamma_j) = 0, j \in J$ implies that $f$ is null-homotopic.

Note that $F/\text{PL}$ is an H-space, and hence, for every detecting family \(\{\gamma_j : M_j \to X\}\), the collection \(\{s(M_j, f\gamma_j)\}\) determine a map $f : X \to F/\text{PL}$ uniquely up to homotopy.

The concept of detecting family is related to Sullivan’s “characteristic variety”, but it is not precisely the same. If a family $\mathcal{F}$ of singular manifolds in $X$ contains a detecting family, then $\mathcal{F}$ on its own is a detecting family. On the contrary, the characteristic variety is in a sense “minimal” detecting family.

15.2. Lemma. Let $X$ be a finite CW-space such that the group $H_*(X)$ is torsion free. Let $\{\gamma_j : M_j \to X\}$ be a family of smooth oriented closed connected singular manifolds in $X$ such that, for each $m$, the elements $(\gamma_j)_*[M_j^{2m}]$ generate the group $H_{2m}(X)$. Then $\{\gamma_j\}$ is a detecting family for $X$.
Proof. Consider a map $f : X \to F/PL$ such that $s_j(M_j, f\gamma_j) = 0$ for all $j \in J$. We must prove that $f$ is null-homotopic.

Because of 13.10, it suffices to prove that $f^*K_i = 0$ and $f^*K_{4i-2} = 0$ for all $i \geq 1$. Furthermore, $H^*(X) = \text{Hom}(H_*(X), \mathbb{Z})$ because $H_*(X)$ is torsion free. So, it suffices to prove that

$$\langle f^*K_{4i}, z \rangle = 0 \text{ for all } z \in H_{4i}(X)$$

and

$$\langle f^*K_{4i-2}, z \rangle = 0 \text{ for all } z \in H_{4i-2}(X; \mathbb{Z}/2).$$

First, we prove (15.1). Since the classes $(\gamma_j)_*[M_j], \dim M_j = 4i$ generates the group $H_{4i}(X)$, it suffices to prove that

$$\langle f^*K_{4i}, (\gamma_j)_*[M_j] \rangle = 0 \text{ whenever } \dim M_j = 4i.$$

But, because of 13.2, for every $4i$-dimensional $M$ we have

$$0 = s(M_j, f\gamma_j) = \langle (f\gamma_j)^*K_{4i}, [M_j] \rangle = \langle f^*K_{4i}, (\gamma_j)_*[M_j] \rangle.$$

This completes the proof of the equality (15.1).

For the case $i = 4k - 2$, note that the group $H_{4i-2}(X; \mathbb{Z}/2)$ is generated by the elements $(\gamma_j)_*[M_j]_2, \dim M_j = 4k - 2$, since $H_*(X)$ is torsion free. Now the proof can be completed similarly to the case $i = 4k$.

15.3. Theorem. Let $X$ be a connected finite CW-space such that the group $H_*(X)$ is torsion free. Then $X$ admits a detecting family $\{\gamma_j : M_j \to X\}$ such that each $M_j$ is orientable.

Proof. Since $H_*(X)$ is torsion free, every homology class in $H_*(X)$ can be realized by a closed connected smooth oriented singular manifold, see e.g. [Co 15.2] or [Rud 6.6 and 7.32]. Now apply Lemma 15.2.

15.4. Example. Let $X$ be the space $T^k \times S^n$. Clearly, $H_{2m}(X)$ is generated by fundamental classes of submanifolds $T^{2m}$ and $T^{2m-n} \times S^n$ of $T^k \times S^n$. Hence, $X$ has a detected family $\{\gamma_j : M_j \to X\}$ such that each $M_j$ is either $T^r$ or $T^r \times S^n$.

16. Normal Invariant of a Homeomorphism: a Special Case

16.1. Theorem. If the element $x \in S_{PL}(T^k \times S^n)$ can be represented by a homeomorphism $h : V \to M$, then $j_F(x) = 0$.
Proof. Put $M = T^k \times S^n$. The maps $j_{\text{TOP}}$ and $j_F$ from section 3 can be included in the commutative diagram

\[
\begin{array}{ccc}
T_{PL}(M) & \xrightarrow{j_{\text{TOP}}} & [M,\text{TOP}/PL] \\
\downarrow & & \downarrow a_* \\
S_{PL}(M) & \xrightarrow{j_F} & [M,\text{F}/PL]
\end{array}
\]

(16.1)

where the left arrow is the obvious forgetful map and $a_*$ is induced by $a$ as in (2.8).

Suppose that $x$ can be represented by a homeomorphism $h : V \to M$. Consider a map $f : M \to \text{TOP}/PL$ such that $j_{\text{TOP}}(h)$ is homotopy class of $f$. Then, clearly, the class $j_F(x) \in [M,\text{F}/PL]$ is represented by the map

\[M \xrightarrow{f} \text{TOP}/PL \xrightarrow{a} \text{F}/PL.\]

As we explained in Example 15.4, $M$ possesses a detecting family $\{\gamma_j : M_j \to M\}$ where each $M_j$ is either $T^r$ or $T^r \times S^n$. Hence, by 14.9 and 14.10, $s(M_j,af\gamma_j) = 0$ for all $j$. So, $af$ is null-homotopic since $\{\gamma_j\}$ is a detecting family. Thus, $j_F(x) = 0$. \qed
CHAPTER III

Applications and Consequences

17. The Space $F/TOP$

Because of the Main Theorem and results of Freedman $F$ and Scharlemann $Sch$, the Transversality Theorem holds for topological manifolds and bundles. For the references, see Rudyak $Rud$ IV.7.18.

Since we have the topological transversality, we can define the maps

$$s' : [M, F/TOP] \to P_{\dim M}$$

where $M$ turns out to be a topological manifold. These map $s'$ are obvious analog of maps $s$ defined in (12.1): you need merely replace PL by TOP in Equation (12.1) and Theorem 13.2. We leave it to the reader.

The following proposition states the main difference between $F/PL$ and $F/TOP$.

17.1. Proposition. The map $s' : \pi_4(F/TOP) \to \mathbb{Z}$ is a surjection.

Proof. Note that the Freedman manifold $V$ from Theorem 7.2 is almost parallelizable and has signature 8. Now the proof can be completed just as 12.1(i). \hfill \square

17.2. Remark. Kirby and Siebenmann $KS_2$ used a homology 4-manifold of signature 8 in order to prove Proposition 17.1. The paper of Freedman appeared later.

17.3. Theorem. (i) For $i \neq 4$ the map $b : F/PL \to F/TOP$ induces an isomorphism

$$b_* : \pi_i(F/PL) \to \pi_i(F/TOP).$$

(ii) The homomorphism

$$b_* : \mathbb{Z} = \pi_4(F/PL) \to \pi_4(F/TOP) = \mathbb{Z}$$

is the multiplication by 2.
III. APPLICATIONS AND CONSEQUENCES

Proof. (i) Recall that $\text{TOP/PL} = K(\mathbb{Z}/2, 3)$ and $\pi_4(F/PL) = \mathbb{Z}$. So, the exactness of the homotopy sequence of the fibration

$$\text{TOP/PL} \xrightarrow{a} F/PL \xrightarrow{b} F/\text{TOP}$$

in (2.8) yields an isomorphism $b_* : \pi_i(F/PL) \cong \pi_i(F/\text{TOP})$ for $i \neq 4$.

(ii) We have the commutative diagram

$$\begin{array}{ccc}
0 &=& \pi_4(\text{TOP/PL}) \\
\downarrow a_* & & \downarrow s \\
\mathbb{Z} &=& \pi_4(F/\text{PL}) \\
\downarrow b_* & & \downarrow s' \\
\pi_4(F/\text{TOP}) &=& \mathbb{Z} \\
\downarrow & & \downarrow \\
\mathbb{Z}/2 &=& \pi_3(\text{TOP/PL}) \\
\downarrow & & \downarrow \\
0 &=& \pi_3(F/\text{PL})
\end{array}$$

where the middle vertical line is a short exact sequence. Therefore $\pi_4(F/\text{TOP}) = \mathbb{Z}$ or $\pi_4(F/\text{TOP}) = \mathbb{Z} \oplus \mathbb{Z}/2$. By Theorem 12.1(iii), $\text{Im}s$ is the subgroup $2\mathbb{Z}$ of $\mathbb{Z}$, while $s'$ is a surjection by 17.3. Thus, $\pi_4(F/\text{TOP}) = \mathbb{Z}$ and $b_*$ is the multiplication by 2. □

Now, following 13.2, we can introduce the classes

$$K'_{4i} \in H^{4i}(F/\text{TOP}, \mathbb{Z}[2]) \quad \text{and} \quad K'_{4i-2} \in H^{4i-2}(F/\text{TOP}, \mathbb{Z}/2)$$

such that

$$s'(M^{4i}, f) = \langle f^*K'_{4i}, [M] \rangle \quad \text{and} \quad s'(N^{4i-2}, f) = \langle f^*K'_{4i-2}, [N]_2 \rangle.$$ 

However, here $M$ and $N$ are allowed to be topological (i.e. not necessarily PL) manifolds.

Similarly to (13.2), together these classes yield the map

$$K' : F/\text{TOP} \longrightarrow \prod_{i>0} (K(\mathbb{Z}[2], 4i) \times K(\mathbb{Z}/2, 4i - 2)).$$

such that for each $i > 0$ the map

$$F/\text{TOP} \xrightarrow{K'} \prod \xrightarrow{\text{projection}} K(\mathbb{Z}/2, 4i - 2) \quad \text{(resp.} \quad K(\mathbb{Z}[2], 4i)$$

coincides with $K'_{4i-2}$ (resp. $K'_{4i}$).
17.4. **Theorem.** The map

\[ K'[2] : K : F/TOP[2] \longrightarrow \prod_{i>0} (K(\mathbb{Z}[2], 4i) \times K(\mathbb{Z}/2, 4i - 2)) \]

is a homotopy equivalence.

**Proof.** Together [12.1] and [17.1] imply that the homomorphisms

\[ s' : \pi_{2i}(F/TOP) \longrightarrow P_{2i} \]

are surjective. Now, in view of [17.3] all the homomorphisms \( s' \)'s are isomorphisms, and the result follows. \( \square \)

So, the only difference between the spaces \( F/PL \) and \( F/TOP \) is that \( F/TOP[2] \) has trivial Postnikov invariants, while \( F/PL[2] \) has exactly one non-trivial Postnikov invariant \( \delta Sq^2 \iota_2 \in H^5(K(\mathbb{Z}/2, 2); \mathbb{Z}[2]) \).

Now we discuss the groups \( \pi_i(BTOP) \). Consider the map

\[ \alpha = \alpha_{TOP}^{PL} : BPL \rightarrow BTOP \]

and the fibration

\[ TOP/PL \longrightarrow BPL \longrightarrow BTOP \]

as in (2.7). Since \( \pi_3(BPL) = 0 \) and \( \pi_i(TOP/PL) = 0 \) for \( i \neq 3 \), we conclude that

\[ \alpha_* : \pi_i(BPL) \longrightarrow \pi_i(BTOP) \]

is an isomorphism for \( i \neq 4 \). Furthermore, we have the exact sequence

\[ 0 \longrightarrow \pi_4(BPL) \xrightarrow{\alpha_*} \pi_4(BTOP) \longrightarrow \pi_3(TOP/PL) \longrightarrow \]

where \( \pi_4(BPL) = \mathbb{Z} \) and \( \pi_3(TOP/PL) = \mathbb{Z}/2 \). Hence we have that either \( \pi_4(BTOP) = \mathbb{Z} \) or \( \pi_4(BTOP) = \mathbb{Z} \oplus \mathbb{Z}/2 \).

Now, consider the diagram of fibrations

\[ \begin{array}{ccc}
F/PL & \longrightarrow & BPL \\
\downarrow & & \downarrow \\
F/TOP & \longrightarrow & BTOP \\
\end{array} \]

\[ \overset{\|}{\longrightarrow} \]

It is known that \( J \)-homomorphism

\[ J : \mathbb{Z} = \pi_4(BPL) \longrightarrow \pi_4(BSF) = \mathbb{Z}/24 \]

is surjective [Ad, MK] (recall that \( \pi_i(PL/O) = 0 \) for \( i < 7 \), and so there is no difference between \( \pi_i(BPL) \) and \( \pi_i(BO) \) up to dimension
Furthermore, $\pi_5(BF)$ is finite and $\pi_3(F/PL) = \pi_3(F/TOP) = 0$. Now, we apply $\pi_4$ and get the commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{Z} & \rightarrow^{24} & \mathbb{Z} & \rightarrow \mathbb{Z}/24 & \rightarrow 0 \\
2 & \downarrow & \downarrow & & \downarrow & = & \\
0 & \rightarrow & \mathbb{Z} & \rightarrow \pi_4(BTOP) & \rightarrow \mathbb{Z}/24 & \rightarrow 0.
\end{array}
\]

The assertion $\pi_4(BTOP)$ contradicts the commutativity of the diagram. Thus, $\pi_4(BTOP) = \mathbb{Z} \oplus \mathbb{Z}/2$. Cf. Milgram [Mil].

18. The Map $a : TOP/PL \rightarrow F/PL$

Recall that in (2.8) we described the fibration

\[TOP/PL \xrightarrow{a} F/PL \xrightarrow{b} F/TOP.\]

18.1. Proposition. The map $a : TOP/PL \rightarrow F/PL$ is essential.

Proof. For general reasons, the fibration

\[TOP/PL \xrightarrow{a} F/PL \rightarrow F/TOP\]

yields a fibration

\[\Omega(F/TOP) \xrightarrow{u} TOP/PL \xrightarrow{a} F/PL.\]

If $a$ is inessential then there exists a map

\[v : TOP/PL \rightarrow \Omega(F/TOP)\]

with $uv \simeq 1$. But this is impossible because $\pi_3(TOP/PL) = \mathbb{Z}/2$ while \(\pi_3(\Omega(F/TOP)) = \pi_4(F/TOP) = \mathbb{Z}\). \hfill \Box

Let $\ell : F/PL \rightarrow F/PL[2]$ denote the localization map. Let $\psi : F/PL \rightarrow Y$ be the Postnikov 4-approximation of $F/PL$ as in (13.4).

Take an arbitrary map $f : X \rightarrow TOP/PL$.

18.2. Proposition. The following three conditions are equivalent:

(i) the map

\[X \xrightarrow{f} TOP/PL \xrightarrow{a} F/PL\]

is essential;

(ii) the map

\[X \xrightarrow{f} TOP/PL \xrightarrow{a} F/PL \xrightarrow{\ell} F/PL[2]\]

is essential;

(iii) the map

\[X \xrightarrow{f} TOP/PL \xrightarrow{a} F/PL \xrightarrow{\ell} F/PL[2] \xrightarrow{\psi[2]} Y[2]\]
is essential.

Proof. It suffices to prove that (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii). To prove the first implication, recall that a map \(u : X \rightarrow F/PL\) is inessential if both localized maps

\[
X \xrightarrow{\ell u} F/PL \xrightarrow{\ell} F/PL[2], \quad X \xrightarrow{\ell u} F/PL \xrightarrow{\ell} F/PL[1/2]
\]

are inessential. Now, (i) \(\Rightarrow\) (ii) holds since \(TOP/PL[1/2]\) is contractible.

To prove the second implication, note that a map \(v : X \rightarrow F/PL[2]\) is inessential if both maps (we use notation as in 13.4)

\[
X \xrightarrow{v} F/PL[2] \xrightarrow{K} \Pi, \quad X \xrightarrow{v} F/PL[2] \xrightarrow{\ell v} Y
\]

are inessential. So, it suffices to prove that the map

\[
X \xrightarrow{\ell af} F/PL[2] \xrightarrow{\ell} \Pi
\]

is inessential. This holds, in turn, because the map \(TOP/PL \rightarrow F/PL \rightarrow F/TOP\) is inessential and the diagram

\[
\begin{array}{ccc}
F/PL[2] & \xrightarrow{K[2]} & \Pi \\
\downarrow[i[2]] & & \Pi \\
F/TOP[2] & \xrightarrow{K'[2]} & \prod_{i>0}(K(Z/2, 4i - 2) \times K(Z[2], 4i)) \xrightarrow{proj} \Pi
\end{array}
\]

commutes. \(\square\)

Consider the fibration

\[
K(Z[2], 4) \xrightarrow{i} Y[2] \xrightarrow{\psi[2]} K(Z/2, 2)
\]

that is the \(Z[2]\)-localization of the fibration (13.3).

18.3. Lemma. For every space \(X\), the homomorphism

\[
H^4(X; Z[2]) = [X, K(Z[2], 4)] \xrightarrow{i_*} [X, Y[2]]
\]

is injective. Moreover, \(i_*\) is an isomorphism if \(H^2(X; Z/2) = 0\).

Proof. The fibration (13.5) yields the exact sequence (see e.g. [MT])

\[
H^1(X; Z/2) \xrightarrow{\delta Sq^2} H^4(X; Z[2]) \xrightarrow{i_*} [X, Y[2]] \rightarrow H^2(X; Z/2)
\]

where \(\delta Sq^2(x) \equiv 0\) (because \(\delta Sq^2(x) = 0\) whenever \(\deg x = 1\)). \(\square\)

Let \(g : TOP/PL \rightarrow Y\) be the composition

\[
TOP/PL \xrightarrow{a} F/PL \xrightarrow{\ell} F/PL[2] \xrightarrow{\psi[2]} Y[2].
\]

Note that \(g\) is essential because of 18.1 and 18.2.
18.4. COROLLARY. The map
\[ \text{TOP/PL} = K(Z/2, 3) \xrightarrow{\delta} K(Z[2], 4) \xrightarrow{i} Y[2] \]
is homotopic to \( g \), i.e. \( g \simeq i\delta \).

Proof. Because of Lemma 18.3 applied to \( X = K(Z/2, 3) \), the set \([K(Z/2, 3), Y[2]]\) has exactly two elements. Since both maps \( g \) and \( i\delta \) are essential (the last one because of Lemma 18.3), we conclude that \( g \simeq i\delta \). \( \square \)

18.5. THEOREM. Given a map \( f : X \to \text{TOP/PL} \), the map
\[ X \xrightarrow{f} \text{TOP/PL} \xrightarrow{a} F/PL \]
is essential if and only if the map
\[ X \xrightarrow{f} \text{TOP/PL} = K(Z/2, 3) \xrightarrow{\delta} K(Z[2], 4) \]
is essential.

Proof. We have the chain of equivalences
\[ af \text{ is essential } \iff \text{18.2} \iff \text{18.4} \iff i\delta \text{ is essential} \\
\iff \text{18.3} \iff \delta f \text{ is essential.} \]
\( \square \)

19. Normal Invariant of a Homeomorphism

19.1. LEMMA. Let \( X \) be a finite CW-space such that \( H_n(X) \) is 2-torsion free. Then the homomorphism
\[ \delta : H^n(X; Z/2) \to H^{n+1}(X; Z[2]) \]
is zero.

Proof. Because of the exactness of the sequence (13.6)
\[ H^n(X; Z/2) \xrightarrow{\delta} H^{n+1}(X; Z[2]) \xrightarrow{2} H^{n+1}(X; Z[2]), \]
it suffices to prove that \( H^{n+1}(X; Z[2]) \) is 2-torsion free. Since \( H_n(X) \) is 2-torsion free, we conclude that \( \text{Ext}(H_n(X), Z[2]) = 0 \). (Indeed, \( \text{Ext}(Z/m, A) = A/mA \) for all \( A \).) Thus, because of the Universal Coefficient Theorem,
\[ H^{n+1}(X; Z[2]) = \text{Hom}(H_{n+1}(X; Z[2]) \oplus \text{Ext}(H_n(X); Z[2]) \]
\[ = \text{Hom}(H_{n+1}(X; Z[2]), \]
and the result follows. \( \square \)
19.2. **Theorem.** Let $M$ be a closed PL manifold such that $H_3(M)$ is 2-torsion free. Then the normal invariant of any homeomorphism $h : V \to M$ is trivial.

**Proof.** Since $h$ is a homeomorphism, the normal invariant $j_F(h)$ turns out to be the homotopy class of a map

$$M \xrightarrow{f} TOP/PL \xrightarrow{a} F/PL$$

where the homotopy class of $f$ is $j_{TOP}(h)$. Because of 18.2 and 18.3, it suffices to prove that the map

$$M \xrightarrow{f} TOP/PL = K(\mathbb{Z}/2, 3) \xrightarrow{\delta} K(\mathbb{Z}[2], 4)$$

is inessential. But this follows from Lemma 19.2. □

Now we have the following version of the Hauptvermutung, cf. [Cas, Corollary on p.68] and [Sul2, Theorem H on p. 93].

19.3. **Corollary.** Let $M, \dim M \geq 5$ be a closed simply-connected PL manifold such that $H_3(M)$ is 2-torsion free. Then every homeomorphism $h : V \to M$ is homotopic to a PL homeomorphism. In particular, $V$ and $M$ are PL homeomorphic.

**Proof.** This follows from 12.2 and 19.2. □

19.4. **Remark.** Rourke [Rou] suggested another proof of 19.2, using the technique of simplicial sets.

### 20. Kirby-Siebenmann and Casson-Sullivan Invariants

Recall some facts on obstruction theory [DK, FFG, H, MT, Spa2]. Let $F \to E \to B$ be a principal $F$-fibration such that $F$ is an Eilenberg-MacLane space $K(\pi, n)$, and assume that the $\pi_1(B)$-action on $\pi = \pi_n(F)$ is trivial. Let $\iota = \iota_n \in H^n(K(\pi, n); \pi)$ be the fundamental class of $F$, and let $\kappa = \tau \iota \in H^{n+1}(B; \pi)$ be the characteristic class of the fibration $F \to E \to B$, where $\tau : H^n(F; \pi) \to H^{n+1}(B; \pi)$ is the transgression. This is well-known that the fibration $F \to E \to B$ admits a section if and only if $\kappa = 0$ and, if a section exists then the vertically homotopy class of sections of the fibration are in a bijective correspondence with elements of $H^n(B; \pi)$. Hence, given a map $f : X \to B$, the map $f$ can be lifted to $E$ iff $f^*(\kappa) = 0$, and the vertical homotopy classes of liftings of $f$ to $E$ are in a bijective correspondence with elements $H^n(X; \pi)$ provided such a lifting exists.
Since $TOP/PL$ is the Eilenberg-MacLane space $K(Z/2, 3)$, we can apply previous arguments to the principal $TOP/PL$-fibration (2.7)

$$TOP/PL \longrightarrow BPL \xrightarrow{\alpha_{TOP}^{PL}} BTOP.$$ 

Then we get the characteristic class

(20.1) $\kappa = \tau_\iota \in H^4(BTOP; \pi_3(K(Z/2, 3))) = H^4(BTOP; \mathbb{Z}/2)$

where $\iota \in H^3(K(Z/2, 3); \mathbb{Z}/2)$ is the fundamental class. We call $\kappa$ the universal Kirby-Siebenmann class.

Let $M$ be a topological manifold, and let $f : M \to BTOP$ classify the stable tangent bundle of $M$. Since $f$ is unique up to homotopy, the class $f^*(\kappa) \in H^*(M; \mathbb{Z}/2)$ is a well-defined invariant of $M$. We put

(20.2) $\nu(M) := f^*(\kappa) \in H^*(M; \mathbb{Z}/2)$

and call it the Kirby-Siebenmann class of $M$.

20.1. Theorem. Let $M$ be a topological manifold. If $M$ admits a PL structure then $\nu(M) = 0$. If $\dim M \geq 5$ and $\nu(M) = 0$ then $M$ admits a PL structure. In particular, if $\dim M \geq 5$ and $H^4(M; \mathbb{Z}/2) = 0$ then $M$ admits a PL structure.

Proof. If $M$ admits a PL structure then the classifying map $f : M \to BTOP$ can be lifted to $BPL$, and hence $f^*(\kappa) = 0$, i.e. $\nu(M) = 0$. Conversely, if $\nu(M) = 0$ then $f$ can be lifted to $BPL$. Thus, in case $\dim M \geq 5$ the manifold $M$ admits a PL structure by Corollary 6.3.

20.2. Theorem. If a topological manifold $M, \dim M \geq 5$ admits a PL structure then set of concordance classes of PL structure on $M$ is in bijective correspondence with $H^3(M; \mathbb{Z}/2)$, i.e.

$$\mathcal{T}_{PL}(M) \cong H^3(M; \mathbb{Z}/2).$$

In particular, if $H^3(M; \mathbb{Z}/2) = 0$ then the Hauptvermutung holds for $M$.

Proof. Because of Corollary 6.2, we have a bijection

$$\mathcal{T}_{PL}(M) \cong [M, TOP/PL].$$

Thus, because of the Main Theorem $TOP/PL \simeq K(Z/2, 3)$, we get

$$\mathcal{T}_{PL}(M) \cong [M, TOP/PL] \cong [M, K(Z/2, 3)] \cong H^3(M; \mathbb{Z}/2).$$
20.3. Definition. Let $M$ be a PL manifold and $h : V \to M$ be a PL structuralization. In view of bijection from Theorem 20.2, the PL structure $h$ gives us a cohomology class $\kappa(h) \in H^3(M; \mathbb{Z}/2)$. This class is called the Casson-Sullivan invariant of $h$, and it measures the difference between $h : V \to M$ and $1_M$.

So, $\kappa(h) = 0$ if and only if $h : V \to M$ is concordant to the identity map of $M$. It is also worthy to mention that, for every $a \in H^3(M; \mathbb{Z}/2)$ there exists a homeomorphism $h : V \to M$ with $a = \kappa(h)$.

20.4. Remark. We know that Hauptvermutung holds for $T^k \times S^n$ with $k + n \geq 5$ and $n \geq 3$, [HS]. In other words, if two PL manifolds $M_1, M_2$ are homomorphic to $T^k \times S^n$ then there are PL homeomorphic. On the other hand, the group $H^3(T^k \times S^n; \mathbb{Z}/2)$ is quite large for $k$ large enough, i.e. $T^k \times S^n$ has many different PL structure. Is it a contradiction? No, it is not. The explanation comes because, given a homeomorphism $h : T^k \times S^n \to T^k \times S^n$, there are many PL concordance classes $T^k \times S^n \to T^k \times S^n$ that are homotopic to $h$.

21. Several Examples

21.1. Example. There are two closed PL manifolds that are homeomorphic but not PL homeomorphic.

Let $\mathbb{RP}^n$ denote the real projective space of dimension $n$ and assume that $n > 4$.

Recall that $j_{TOP} : T_{PL}(\mathbb{RP}^n) \to [\mathbb{RP}^n, TOP/PL]$ is a bijection. Consider a homeomorphism $k : M \to \mathbb{RP}^5$ such that

$$j_{TOP}(k) \neq 0 \in [\mathbb{RP}^n, TOP/PL] = H^3(\mathbb{RP}^n; \mathbb{Z}/2) = \mathbb{Z}/2.$$ 

Note that the Bockstein homomorphism

$$\beta : \mathbb{Z}/2 = H^3(\mathbb{RP}^n; \mathbb{Z}/2) \to H^4(\mathbb{RP}^n; \mathbb{Z}/2) = \mathbb{Z}/2$$

is an isomorphism, and hence $\delta(j_{TOP}(k)) \neq 0$ for $\delta : H^3(\mathbb{RP}^n; \mathbb{Z}/2) \to H^4(\mathbb{RP}^n)$. So, by Theorem [18.5], $a_*j_{TOP}(k) \neq 0$. In view of commutativity of the diagram (16.1), $j_F(k) = a_*j_{TOP}(k)$, i.e. $j_F(k) \neq 0$.

On the other hand, it follows from the obstruction theory that every homotopy equivalence $h : \mathbb{RP}^n \to \mathbb{RP}^n$ is homotopic to the identity map. In particular, $j_F(h) = 0$. Thus, $M$ is not PL homeomorphic to $\mathbb{RP}^n$.

21.2. Example. For every $n > 3$ there is a homeomorphism

$$h = h_n : S^3 \times S^n \to S^3 \times S^n, n > 3$$
which is homotopic to a PL homeomorphism but is not concordant to any PL homeomorphism.

Take an arbitrary homeomorphism \( f : M \to S^3 \times S^n, n > 3. \) Then \( j_F(f) \) is trivial by Theorem 19.2. Thus, by Theorem 12.2, \( f \) is homotopic to a PL homeomorphism. In particular, \( M \) is PL homeomorphic to \( S^3 \times S^n. \)

Now, we refine the situation and take a homeomorphism

\[ h : S^3 \times S^n \to S^3 \times S^n \]

such that

\[ j_{TOP}(h) \neq 0 \in \mathcal{T}_{PL}(S^3 \times S^n) = H^3(S^3 \times S^n; \mathbb{Z}/2) = \mathbb{Z}/2. \]

Such \( h \) exists because \( j_{TOP} \) is a bijection. So, \( h \) is not concordant to the identity map, and therefore \( h \) is not concordant to a PL homeomorphism, see Remark 3.2(2). But, as we have already seen, \( h \) is homotopic to a PL homeomorphism.

Note that the maps \( h \) and the identity map have the same domain while they are not concordant. So, this example serves also the Remark 3.2(3).

21.3. Examples. There are topological manifolds that do not admit any PL structure.

See manifold \( V \times T^n \) that are described in Corollary 7.4.

In 1970 Siebenmann [Sieb] published a paper with the intriguing title: Are nontriangulable manifolds triangulable? The paper cerebrated about the following problem: Are there manifolds that can be triangulated as simplicial complexes but do not admit any PL structure? Later, people made a big progress related to this issue, [AM, GaSt1, GaSt2, Mat, Man, Rand, Sav]. Here we give only a brief survey notice on the issue because non-combinatorial triangulations are far from the main line of our concern.

Recall that a homology \( k \)-sphere is defined to be a \( k \)-dimensional closed PL manifold \( \Sigma \) such that \( H_*(\Sigma) \cong H_*(S^k). \)

21.4. Examples. There are topological manifolds that can be triangulated as simplicial complexes but do not admit any PL structure.

21.5. Theorem. Every orientable topological 5-dimensional closed manifold can be triangulated as a simplicial complex.
21. SEVERAL EXAMPLES

Proof. Let us say that a homology 3-sphere $\Sigma$ is good if the double suspension $S^2\Sigma$ over $\Sigma$ is homeomorphic to $S^5$ and $\Sigma$ is bounded by a compact parallelizable manifold of signature 8. Siebenmann \cite{Sieb}, Assertion on p. 81 proved that every orientable topological 5-dimensional closed manifold can be triangulated as a simplicial complex provided that there exists a good homology 3-sphere. Cannon \cite{Ca} proved that, for any homology 3-sphere $\Sigma$, the double suspension $S^2\Sigma$ is homeomorphic to $S^5$. Now, note that the homology 3-sphere $\partial W$ from Comment \ref{C7.3} is good, and the theorem follows. \hfill \Box

Take $M = V \times S^1$. Then $M$ does not admit any PL structure by Corollary \ref{C7.4}. On the other hand, $M$ can be triangulated as a simplicial complex by Theorem \ref{Th21.5}. Because of this, for each $k \geq 1$ the manifold $V \times T^k$ also have these properties. This is remarkable that $V$ cannot be triangulated as a simplicial complex, see below.

21.6. EXAMPLES. There are topological manifolds that cannot be triangulated as simplicial complexes.

First, note that if a 4-dimensional topological manifold $M$ can be triangulated as a simplicial complex then $M$ admits a PL structure. In particular, $V$ cannot be triangulated as a simplicial complex, (Casson), see \cite{AM, Sav}.

Now we pass to higher dimensions.

Define two oriented homology 3-spheres $\Sigma_1, \Sigma_2$ to be equivalent if there exists an oriented PL bordism $W, \partial W = \Sigma_1 \sqcup \Sigma_2$ such that $H_1(W') = 0 = H_2(W)$. Let $\Theta^H_3$ denote the abelian group obtained from the set of equivalence classes using the operation of connected sum. We define a homomorphism $\mu : \Theta^H_3 \to \mathbb{Z}/2$ as follows.

It is well known that every homology 3-sphere (in fact, every orientable 3-manifold) $\Sigma$ bounds a 4-dimensional parallelizable manifold $P$. By the Rokhlin Theorem \ref{Rok} the signature $\sigma(P) \mod 16$ is a is well-defined invariant of $\Sigma$, and 8 divides $\sigma(\Sigma)$. Take $a \in \Theta^H_3$, let $\Sigma_a$ be a homology 3-sphere that represents $a$, and let $P_a$ be a 4-dimensional parallelizable manifold with $\partial P_a = \Sigma_a$. Now, put $\mu(a) = (\sigma(P_a) \mod 16)/8$ and get a well-defined homomorphism $\mu : \Theta^H_3 \to \mathbb{Z}/2$.

Consider the short exact sequence

$$0 \longrightarrow \ker \mu \longrightarrow \Theta^H_3 \xrightarrow{\mu} \mathbb{Z}/2 \longrightarrow 0$$

and let $\delta : H^4(\cdot; \mathbb{Z}/2) \to H^5(\cdot; \ker \mu)$ be the Bockstein homomorphims associated with this sequence.
21.7. **Theorem** (Galewski–Stern [GaSt2], Matumoto [Mat]). A topological manifold \( M \) of dimensional \( \geq 5 \) can be triangulated as a simplicial complex if and only if \( \delta \kappa(M) = 0 \). Here \( \kappa(M) \) denotes the Kirby-Siebenmann invariant of \( M \).

Manolescu [Man] proved that the above mentioned short exact sequence does not split. This allowed him to prove that, for any \( n \geq 5 \), there is a manifold \( M^n \) with \( \delta \kappa(M^n) \neq 0 \). Thus, for all \( n \geq 5 \) there exists an \( n \)-dimensional manifold that cannot be triangulated as a simplicial complex.

Concerning explicit constructions of such manifolds. Galewski and Stern [GaSt1] constructed a certain manifold \( N^5 \) with the following property: if \( N \) can be triangulated as a simplicial complex then every closed manifold of dimension \( \geq 5 \) can. So, \( N \) cannot be triangulated. In particular, \( \delta \kappa(N) \neq 0 \). Finally, \( N \times T^k \) cannot be triangulated as a simplicial complex because \( \delta \kappa(N \times T^k) \neq 0 \).

**Summary**

Here all manifolds are assumed to be connected and having the homotopy type of a finite CW complex.

1. Every manifold \( M^n \) with \( n \leq 3 \) admits a unique PL structure (trivial assertion for \( n = 1 \), Rado [Rad] for \( n = 2 \), Moise [Mo] for \( n = 3 \)).

2. There are uncountable set of mutually different PL manifolds that are homeomorphic to \( \mathbb{R}^4 \) (Taubes [Ta], cf also [GS, K2]). There are countably infinite set of mutually different closed 4-dimensional PL manifolds that are homeomorphic to the blow-up of \( \mathbb{CP}^2 \) at the nine points of intersection of two general cubics (Okonek–Van de Ven [OV]).

3. For every \( n \geq 5 \) there exist closed \( n \)-dimensional PL manifolds that are homeomorphic but not PL homeomorphic. So, the Hauptvermutung is wrong in general. However, any topological manifold \( M^n, n \geq 5 \) (not necessarily closed) possesses only finite number of PL structures (Kirby–Siebenmann [KS2]).

4. For every \( n \geq 4 \) there exist closed topological \( n \)-dimensional manifolds that do not admit any PL structure (Freedman [F] for \( n = 4 \), Kirby–Siebenmann [KS2] for \( n > 4 \)).

The item 4 can be bifurcated as follows:

4a. For every \( n \geq 5 \) there exists an \( n \)-manifold that does not possess any PL structure but can be triangulated as a simplicial complex
22. INVARIANCE OF CHARACTERISTIC CLASSES

Such examples do not exist for \( n \leq 4 \).

4b. For every \( n \geq 4 \) there exists an \( n \)-manifold that cannot be triangulated as a simplicial complex (Casson [AM, Sav] for \( n = 4 \), Manolescu [Man] for \( n \geq 5 \).)

22. Topological and Homotopy Invariance of Characteristic Classes

Given a real vector bundle \( \xi \) over a space \( X \), the \( k \)th Pontryagin class of \( \xi \) is a cohomology class \( p_k(\xi) \in H^{4k}(X) \). In particular, for every smooth manifold \( M \) we have the Pontryagin classes 

\[ p_k(M) := p_k(\tau M) \]

where \( \tau M \) is the tangent bundle of \( M \). Given a commutative ring \( \Lambda \) with unit, we can consider 

\[ p_k(\xi) \in H^{4k}(X; \Lambda) \]

under the coefficient homomorphism \( \mathbb{Z} \to \Lambda \). In particular, we have rational Pontryagin classes \( p_k(\xi) \in H^{4k}(X; \mathbb{Q}) \) and modulo \( p \) Pontryagin classes \( p_k(\xi) \in H^{4k}(X; \mathbb{Z}/p) \).

In this section we discuss homotopy and topological invariance of some characteristic classes. In particular, we prove that the Novikov’s Theorem [N2] on topological invariance of rational Pontryagin classes is a direct corollary of the Main Theorem. (It is worthy to note, however, that the proof of the Main Theorem uses ideas from [N2].) Concerning other proofs of the Novikov’s theorem see [G, ST, RW].

22.1. Definition. Given a class \( x \in H^*(BO; \Lambda) \), we say that \( x \) is topologically invariant if, for any two maps \( f_1, f_2 : B \to BO \) such that

\[ \alpha^O_{TOP} f_1 \cong \alpha^O_{TOP} f_2 : B \to BTOP, \]

we have

\[ f_1^*(x) = f_2^*(x) \text{ in } H^*(B; \Lambda). \]

Now we give some conditions for topological invariance. Similarly to the fibration (2.7), consider the fibration

\[ TOP/O \xrightarrow{\beta} BO \xrightarrow{\alpha} BTOP. \]

22.2. Proposition. (i) If

\( x \in \text{Im}\{\alpha^* : H^*(BTOP; \Lambda) \to H^*(BO; \Lambda)\} \)

then \( x \) is topologically invariant. In particular, if \( \Lambda \) is such that \( \alpha^* \) is epimorphic than every class \( x \in H^*(BO; \Lambda) \) is topologically invariant.

(ii) If \( x \in H^*(BO; \Lambda) \) is topologically invariant then \( \beta^*(x) = 0 \) for \( \beta^* : H^*(BO) \to H^*(TOP/O) \).
III. APPLICATIONS AND CONSEQUENCES

Proof. (i) is obvious. To prove (ii), note that $\alpha \beta$ is inessential. Hence $\alpha \beta \cong \alpha \varepsilon$ where $\varepsilon : \text{TOP/O} \to \text{BO}$ is a constant map. Since $x$ is topologically invariant, we conclude that $\beta^*(x) = \varepsilon^*(x) = 0$. □

Proposition 22.2(i) tells us a sufficient condition for topological invariance, while 22.2(ii) tells us a necessary condition. We will see below that 22.2(i) is not necessary and 22.2(ii) is not sufficient for topological invariance. Now we give a necessary and sufficient condition for invariance. Consider the map

$$\mu : \text{BO} \times \text{TOP/O} \xrightarrow{1 \times \beta} \text{BO} \times \text{BO} \xrightarrow{m} \text{BO}$$

where $m$ is the multiplication in the $H$-space $\text{BO}$.

22.3. THEOREM. The class $x \in \text{H}^* (\text{BO}; \Lambda)$ is topologically invariant if and only if $\mu^* (x) = x \otimes 1 \in \text{H}^* (\text{BO}; \Lambda) \otimes \text{H}^* (\text{TOP/O}; \Lambda)$

Proof. The map $\alpha \beta$ is topologically trivial, and hence $\alpha \mu$ is homotopic to the map

$$\alpha \nu : \text{BO} \times \text{TOP/O} \to \text{BO} \to \text{BTOP}$$

where $\nu : \text{BO} \times \text{TOP/O} \to \text{BO}$ is the projection on the first factor. Since $x$ is topologically invariant, we conclude that $\mu^* x = \nu^* (x) = x \otimes 1$.

Conversely, suppose that $\mu^* (x) = x \otimes 1$. Recall that, for all $X$, the infinite space structure in $\text{BO}$ turns $[X, \text{BO}]$ into an abelian group. Let $f_1, f_2 : B \to \text{BO}$ be two maps such that $\alpha f_1 \cong \alpha f_2$. Recall that $[X, \text{BO}]$ is an abelian group with respect to the infinite space structure in $\text{BO}$. Then $f_2 - f_1 : B \to \text{BO}$ lifts to a map $B \to \text{TOP/O}$. In other words, $f_2 = f_1 + g$ for some $g : B \to \text{TOP/O}$. Hence we have a homotopy commutative diagram

$$\begin{array}{ccc}
B & \xrightarrow{f_2} & \text{BO} \\
\downarrow \Delta & & \uparrow \mu \\
B \times B & \xrightarrow{f_1 \times g} & \text{BO} \times \text{TOP/O}
\end{array}$$

Now

$$f_2^* (x) = \Delta^* (f_1 \times g)^* \mu^* (x) = \Delta^* (f_1 \times g)^* (x \otimes 1) = \Delta^* (f_1^* (x) \otimes 1) = f_1^* (x).$$

22.4. REMARK. The items 22.1–22.3 are taken from the paper of Sharma [S].

The following lemma plays a crucial role for topological invariance of rational Pontryagin classes.
22.5. **Lemma.** The forgetful map $\alpha^O_{\text{TOP}} : BO[0] \to B\text{TOP}[0]$ is a homotopy equivalence. Thus, the forgetful map $\alpha^O_{\text{TOP}} : BO \to B\text{TOP}$ induces an isomorphism

$$(\alpha^O_{\text{TOP}})^* : H^*(B\text{TOP}; \mathbb{Q}) \to H^*(BO; \mathbb{Q}).$$

**Proof.** First, note that the homotopy groups $\pi_i(PL/O)$ are finite, see [Rud, IV.4.27(iv)] for the references. Hence, the space $PL/O[0]$ is contractible. Thus, $\alpha^O_{\text{PL}} : BO[0] \to B\text{PL}[0]$ is a homotopy equivalence.

Second, the homotopy groups $\pi_i(TOP/PL)$ are finite by the Main Theorem. Hence, the space $TOP/PL[0]$ is contractible. Thus, $\alpha^PL_{\text{TOP}} : B\text{PL}[0] \to \text{BTOP}[0]$ is a homotopy equivalence.

Now, since $\alpha^O_{\text{TOP}} = \alpha^PL_{\text{TOP}} \alpha^O_{\text{PL}}$, we conclude that $\alpha^O_{\text{TOP}}[0]$ is a homotopy equivalence. \qed

Recall that $H^*(BO; \mathbb{Q}) = \mathbb{Q}[p_1, \ldots, p_i, \ldots]$ where $p_k, \dim p_k = 4k$ is the universal Pontryagin class, [MS]. It follows from Lemma $22.5$ that $H^*(B\text{TOP}; \mathbb{Q}) = \mathbb{Q}[p'_1, \ldots, p'_k, \ldots]$ where $p'_k$ are the cohomology classes determined by the condition $\alpha^*(p'_k) = p_k \in H^*(BO; \mathbb{Q})$.

Now, given an arbitrary topological $\mathbb{R}^n$ bundle $\lambda$ over $B$, we define its rational Pontryagin classes $p'_k(\lambda) \in H^{4i}(B; \mathbb{Q})$ by setting

$$p'_k(\lambda) = t^*p'_k$$

where $t : B \to B\text{TOP}$ classifies $\lambda$.

22.6. **Theorem.** Every class in $H^*(BO; \mathbb{Q})$ is topologically invariant. In other words, if $\xi_i = \{\pi_i : E_i \to B\}, i = 1, 2$ be two topologically isomorphic vector bundles over a space $B$ then $p_k(\xi_1; \mathbb{Q}) = p_k(\xi_2; \mathbb{Q})$.

This is the famous Novikov theorem on topological invariance of rational Pontryagin classes.

**Proof.** This follows from Lemma $22.5$ and Proposition $22.2$ immediately. \qed

For completeness, we state the original Novikov version of topological invariance, see [N2].

22.7. **Theorem.** Let $f : M_1 \to M_2$ be a homeomorphism of closed smooth manifolds. and let $f^* : H^*(M_2; \mathbb{Q}) \to H^*(M_1; \mathbb{Q})$ be the induced isomorphism. Then $f^*p_k(M_2; \mathbb{Q}) = p_k(M_1; \mathbb{Q})$ for all $k$.

**Proof.** Let $t_s : M_s \to BO \to B\text{TOP}, s = 1, 2$ classify the stable tangent bundle of $M_s$. Then $t_1 \simeq t_2 f$. Now

$$f^*p_k(M_2; \mathbb{Q}) = f^*t_2^*p'_k = (t_2 f)^*p'_k = t_1^*p'_k = p_k(M_1; \mathbb{Q}),$$
and we are done.

22.8. Remark. We can pass the previous issues to PL category. To define PL invariance, we should replace topological isomorphism by PL isomorphism of PL bundles and require $B$ to be a polyhedron in Definition 22.1. Rokhlin and Švarc [RS] and Thom [T] proved PL invariance of rational Pontryagin classes in 1957-58th. Of course, this result follows from the Novikov Theorem 22.7 on topological invariance of rational Pontryagin classes, but the Novikov Theorem appeared almost 10 years later.

So, rational Pontryagin classes are topological invariants. What about integral Pontryagin classes? It turns out to be that they are not even PL invariant. Milnor [Mi3, §9] constructed two smooth manifolds $M_1, M_2$ that are PL homeomorphic while $p_2(M_1) = 0, p_2(M_2) \neq 0$ (and $7p_2(M_2) = 0$). Nevertheless, there are certain topological invariance results for integral Pontryagin classes.

22.9. Notation. Because of Lemma 22.5, the index of the image subgroup

$$\text{Im}\{(\alpha_{TOP}^O)^* : H^m(BTOP) \to H^m(BO)\}$$

in $H^m(BO)$ is finite for each $m$. Let $\varepsilon_k$ denote this index for $m = 4k$. Clearly, the class $\varepsilon_kp_k \in H^{4k}(BO)$ (the multiple of the integral Pontryagin class) is topologically invariant.

Define $e_k \in \mathbb{N}$ to be the smallest number such that $e_kp_k$ is topologically invariant.

22.10. Comment. To evaluate $e_k$, Sharma [S] proved the following. Let $d_k$ be the smallest positive integer such that $d_kp_k \in \text{Ker}\{\beta* : H^*(BO) \to H^*(TOP/O)\}$

Then $e_k = \text{LCM}(d_1, \ldots, d_k)$. In particular, $e_k | e_{k+1}$.

To compute $d_k$, let $\gamma_k = (2^{2k-1} - 1)\text{Num}(B_{2k}/4k)$. Here $B_m$’s are the Bernoulli numbers in notation where $B_{2n+1} = 0$ and Num denotes the numerator. Now, if $p$ is an odd prime which divides $\gamma_k$ but does not divide $\gamma_i$ with $i < k$, then $\nu_p(d_k) = \nu_p(\gamma_k)$. Here, as usual, $m = p^{\nu_p(m)}a$ with $(a, p) = 1$.

Sharma [S, Theorem 1.6] used these results in order to evaluate $e_k$ for $k \leq 8$. In particular, $e_1 = 1, e_2 = 7, e_3 = 7 \cdot 31, e_4 = 7 \cdot 31 \cdot 127$. It is remarkable to note that $e_4 < \varepsilon_4$ (strict inequality!), [S, Prop. 1.7 ff]. So, there are topologically invariant classes that do not come from $BTOP$, i.e. the sufficient condition 22.2(i) for topological invariance is not necessary. To see that the necessary condition 22.2(ii) is not
sufficient, note that $31p_3$ is not topological invariant because $e_3$ does not divide $31$, while $31p_3 \in \text{Ker} \beta$, see [S] Section 4, proof of Theorem 1.3).

Another kind of topological invariance appears when we consider $p_k \mod m$, the modulo $m$ Pontryagin classes. Here we will not give detailed proofs but give a sketch/survey only. As a first example, note that $p_k \mod 2 = w_2^k$, and hence $p_k \mod 2$ is topologically (and even homotopy) invariant in view of homotopy invariance of Stiefel–Whitney classes, [MS]. So, the question about topological invariance of modulo $p$ Pontryagin classes is not vacuous. In fact, we have the following result:

**22.11. Theorem (SS).** Given an odd prime $p$, let $n(p)$ be the smallest value of $k$ such that $p$ divides $e_k$. Then $p_k \mod p$ is a topological invariant for $k < n(p)$ and is not a topological invariant for $k \geq n(p)$. In particular, if $p$ does not divide $e_k$, for every $k \geq 1$, then $p_k \mod p$ is a topological invariant.

Because of Theorem 22.11 and Comment 22.10 one can prove that the classes $p_k \mod p$ are topologically invariant for all $k$ and $p = 3, 5, 11, 13, 17$, while $p_k \mod 7$ is not a topological invariant. (For $p = 3$ it is an old theorem of Wu, see Theorem 22.14)

Now some words about homotopy invariance.

**22.12. Definition.** Given a class $x \in H^*(BO; \Lambda)$, we say that $x$ is homotopy invariant if, for any two maps $f_1, f_2 : B \rightarrow BO$ such that

$$\alpha^O_f f_1 \cong \alpha^O_f f_2 : B \rightarrow BF,$$

we have

$$f_1^*(x) - r_2^*(x) \in H^*(B; \Lambda).$$

The obvious analogs of Proposition 22.2 and Theorem 22.3 remains valid if we speak about homotopy invariant instead of topological invariance and replace $TOP$ by $F$.

**22.13. Proposition.** Rational Pontryagin classes are not homotopy invariant.

*Proof.* Note that $\pi_i(BF)$ is isomorphic to the stable homotopy group $\pi^S_{i-1}(S)$ and therefore is finite because of a well-known theorem of Serre, [Se]. Hence, $\pi_i(BF) \otimes \mathbb{Q} = 0$, and so $BF[0]$ is contractible. Now consider the fibration $F/O \rightarrow BO \rightarrow BF$ and conclude that $\beta[0] : F/O[0] \rightarrow BO[0]$ is a homotopy equivalence, and hence $\beta^* :$
$H^*(BO; \mathbb{Q}) \rightarrow H^*(F/O; \mathbb{Q})$ is an isomorphism. Thus, because of the homotopy analog of (22.2(ii)), we see that $x \in \tilde{H}^*(BO; \mathbb{Q})$ is homotopy invariant iff $x = 0$. \hfill \Box

On the other hand, we have $p_i \mod 2 = w_{2i}^2$, [MS] i.e. $p_i \mod 2$ is a homotopy invariant. So, it seems reasonable to ask about homotopy invariance of $p_i \mod p$ for odd prime $p$.

Recall that the homotopy invariance of Stiefel-Whitney follows from the Thom-Wu formula $w_i(\xi) = \varphi^{-1} Sq^i u$ where $u$ is the Thom class of $\xi$ and $\varphi : H^*(B; \mathbb{Z}/2) \rightarrow \tilde{H}^{*+n}(T\xi; \mathbb{Z}/2)$ is the Thom isomorphism, [MS]. (Here $T\xi$ is the Thom space of the $\mathbb{R}^n$-bundle $\xi$ over $B$.)

We apply this idea modulo $p$. So, let $p$ be an odd prime and $P^k : H^*(; \mathbb{Z}/p) \rightarrow H^{*+2k(p-1)}(; \mathbb{Z}/p)$ be the Steenrod power. Given an oriented $\mathbb{R}^n$-bundle (or an $(S^n, *)$-fibration) $\xi$ over $B$, let $T\xi$ be the Thom space of $\xi$, let $u \in H^n(T\xi; \mathbb{Z}/p)$ be the Thom class, and let $\varphi : H^*(B; \mathbb{Z}/p) \rightarrow \tilde{H}^{*+n}(T\xi; \mathbb{Z}/p$ be the Thom isomorphism. Then

$$q_k(\xi) := \varphi^{-1} P^k(u) \in H^{2k(p-1)}(X)$$

is a characteristic class, and it is homotopy invariant by construction. For $X = BO$ we get a universal characteristic class, and it is a polynomial of universal Pontryagin classes mod $p$. Wu [Wu] proved that $q_k = p_k$ if $p = 3$. So, we get the following theorem.

22.14. THEOREM (Wu). The Pontryagin classes $p_k \mod 3, k \geq 1$ are homotopy invariant.

Madsen [M] proved that the classes $p_k \mod 8, k \geq 1$ are homotopy invariant. So, we have the following result:

22.15. COROLLARY. The classes $p_k \mod 24, k \geq 1$ are homotopy invariant.
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