The anisotropic Rabi model

Qiong-Tao Xie*, 1, 2 Shuai Cui*, 1 Jun-Peng Cao, 1, 3 Luigi Amico, 4, 5, 6, and Heng Fan 1, 3, 6

1 Beijing National Laboratory for Condensed Matter Physics, and Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China
2 School of Physics and Electronic Engineering, Hainan Normal University, Haikou 571158, China
3 Collaborative Innovation Center of Quantum Matter, Beijing, China
4 CNR-MATIS-IMM & Dipartimento di Fisica e Astronomia, Via S. Soa 64, 95127 Catania, Italy
5 Center for Quantum Technologies, National University of Singapore, 3 Science Drive 2, 117543 Singapore
6 National Institute of Education and Institute of Advanced Studies, Nanyang Technological University, 1 Nanyang Walk, 637616 Singapore

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We propose the anisotropic Rabi model as the generalization of the spin-boson Rabi model and present the exact energy spectrum and eigenstates of this model. The Hamiltonian system breaks the parity symmetry; the rotating and counter-rotating interactions are governed by two different coupling constants; a further parameter introduces a phase factor in the counter-rotating terms. The solution is obtained as an elaboration of the Braak method for the isotropic limit of the model. In this way, we provide a long sought solution of a cascade of models with immediate relevance in different physical fields, including i) quantum optics: two-level atom in single mode cross electric and magnetic fields; ii) solid state physics: electrons in semiconductors with Rashba and Dresselhaus spin-orbit coupling; iii) mesoscopic physics: Josephson junctions flux-quant quantum circuits.

There are very simple settings in physics whose understanding has very far reaching implications. This is the case of the Rabi type models, that are possibly the simplest ‘organisms’ describing the interaction between a spin-half degree of freedom with a single boson. Originally formulated in quantum optics to describe the atom-field interaction [1], such kind of models play a crucial role in many other fields, especially with the advent of the quantum technologies. The simplest version of that family, the isotropic Rabi model, was solved exactly recently by Braak [7]. In this paper, we introduce and discuss the exact energies and eigenstates of an anisotropic generalization of the Rabi model. In this way, we provide a long sought solution of a cascade of models with immediate relevance in different fields, including i) quantum optics: atom in cross electric and magnetic fields; ii) solid state physics: electrons in semiconductors with Rashba and Dresselhaus spin-orbit coupling; iii) mesoscopic physics: Josephson junctions flux-quant quantum circuits.

The Rabi type models provide the paradigm for key applications in a variety of different physical contexts, including quantum optics [2], solid state and mesoscopic physics. Despite its importance, such models remained intractable with exact means for many years. Nevertheless, the physical community could thoroughly analyze the Rabi model physics, essentially because the physical settings allowed to easily adjust the field frequency to be resonating with the atomic bandwidth. In this way, assuming as well that the field intensity is weak, the Rabi model could be drastically simplified to the Jaynes-Cummings (JC) model, through the celebrated ‘rotating wave’ approximation. The situation radically changed in the last decade, when Quantum Technology has been advancing towards more and more realistic applications. In most of the cases, if not all, the rotating wave approximation cannot be applied. In the solid state applications, for example, the electric field is an intrinsic quantity, that cannot be adjusted. On the other hand, in the applications in mesoscopic physics (like superconducting or QED circuits), the most interesting regimes correspond to very strong coupling between the spin variable and the bosonic degree of freedom.

The class of the anisotropic Rabi model we consider in the present paper are described by the following Hamiltonian

\[ H = \omega a^\dagger a + \sigma_x + \Delta \sigma_z + g(H_r + \lambda H_{cr}), \]

\[ H_r = (a^\dagger \sigma^- + a \sigma^+) , \]

\[ H_{cr} = e^{i\theta} a^\dagger \sigma^+ + e^{-i\theta} a \sigma^- . \]

Here \( a^\dagger \) and \( a \) are the annihilation and creation operators for a bosonic mode of frequency \( \omega \), \( \sigma^\pm = (\sigma_x \pm i \sigma_y)/2 \), \( \sigma_{x,y,z} \) are Pauli matrices for a two-level system, \( 2\Delta \) is the energy difference between the two levels, \( g \) denotes the coupling strength of the rotating wave interaction \( a^\dagger \sigma^- + a \sigma^+ \) between the two-level system and the bosonic mode. For simplicity, we already take the unit of \( \hbar = 1 \). In the Hamiltonian Eq.(1), the relative weight between rotating and counter-rotating terms, denoted respectively as \( H_r \) and \( H_{cr} \), can be adjusted by tuning the parameter \( \lambda \). When \( \epsilon = 0 \), the Hamiltonian enjoys a discrete \( Z_2 \) symmetry meaning that the parity of bosonic and spin excitations is conserved. Several attempts of solving these type of models were tried employing Bethe ansatz and Quantum Inverse Scattering techniques [11, 12]. The isotropic Rabi model corresponding to \( \theta = 0 \) and \( \lambda = 1 \) was solved exactly in a seminal paper by Braak [7]. Such an achievement has allowed to explore the physics of the Rabi model in full generality.

In this Letter, we present the exact solution of the anisotropic Rabi models Eq.(2). We discuss how the models can be applied to important physical settings in quantum optics, mesoscopic and solid state physics. We also observe...
that such model can be realized with cold atoms with arbitrary spin-orbit couplings.

**Exact Solution.** Our solution elaborates on the approach originally developed by Braak[7]. In order to find a concise solution, we perform a unitary transformation on the spin degree of freedom of $\phi_2$. The eigenvalues can be found as (see the Methods section for the derivation)

$$E_n = x_n - \frac{\lambda g^2}{\omega}$$

where $x_n$ are the zeros or poles of the transcendental function

$$G_\epsilon(x) = \phi_1 \phi_2 - \phi_2 \phi_1$$

where $\phi_1(z) = \exp(-\frac{\sqrt{\lambda g^2}}{\omega} z) \sum_{n=0}^{\infty} L_n^+(z + \frac{\sqrt{\lambda g^2}}{\omega})^n$, $\phi_2(z) = \exp(-\frac{\sqrt{\lambda g^2}}{\omega} z) \sum_{n=0}^{\infty} K_n^+(z + \frac{\sqrt{\lambda g^2}}{\omega})^n$, and $\phi_1(-z) = \phi_1(z)$, $\phi_2(-z) = \phi_2(z)$. Fig. 1 and Fig. 2 displays the actual behavior of $G_\epsilon(x)$ in different parameter regimes. For $\epsilon = 0$, the $Z_2$ symmetry is recovered; in this case the transcendental function can be discussed through the functions $G_+ = -e^{i\theta} \phi_1 + \sqrt{\lambda} \phi_2$, $G_- = e^{i\theta} \phi_2 + \sqrt{\lambda} \phi_1$, living in the two parity sectors, separately (see Fig 2). The explicit form of eigenfunctions $\phi_{1,2}(z)$ can also be obtained, see Supplemental Material.

FIG. 1: (Color online) Trascendental function $G_\epsilon(x)$ for $\epsilon \neq 0$. The parameters are $\omega = 1$, $g = 0.1$, $\lambda = 0.5$, $\Delta = 0.4$, $\epsilon = 0.2$, and $\theta = -\pi/2$, the zero points whose real (blue-solid line) and imaginary part (red-dashed line) of $G_\epsilon$ both equal 0 correspond the eigenvalues of Hamiltonian.

For vanishing $\epsilon$ or multiple of $\omega/2$, the system enjoys a $Z_2$ (parity) symmetry. In this case, the energy spectrum can be labelled by the two eigenvalues of the parity operator (corresponding to green-with-circle and purple-with-square lines in Fig. 3(b)). At the points of level crossings the energy is doubly degenerate. For the isotropic case, those solutions were found previously by Judd[7,8]. For our anisotropic Rabi model, the crossing points are found as $E_n = n\omega - (1 + \lambda^2)/2\omega$, which corresponding to poles of the transcendental function in Eq. (3).

FIG. 2: (color online) Trascendental functions $G_\epsilon^+(x)$ (green-solid line) and real part of $G_\epsilon^-(x)$ (purple-dashed line) for $\epsilon = 0$. The parameters are $\omega = 1$, $g = 0.7$, $\Delta = 0.4$, $\lambda = 0.5$ and $\theta = -\pi/2$. The regular parts of the energy spectrum are determined by the zeros of the transcendental function $G_\epsilon^+(x)$, and the dotted vertical lines denote the poles $x = n - 0.0612$, $n = 0, 1, 2, ...$. Notice that the imaginary part of $G_\epsilon^-(x)$ gives the same zero and poles as the real part, which is not shown.

FIG. 3: (Color online) Comparison between exact solution and the numerical results. (a) Energy spectrum for $\omega = 1$, $\Delta = 0.4$, $\lambda = 0.5$, $\epsilon = 0.2$ and $\theta = -\pi/2$. Solid lines are exact results, and the energy levels are differentiated by colors. Numerical results are represented by stars. (b) Energy spectrum for $\omega = 1$, $\Delta = 0.4$, $\lambda = 0.5$, $\epsilon = 0$, and $\theta = 0$ in the spaces with positive (blue lines) and negative (red lines) parities. Small squares and circles represent numerical results with different parities, where circles are with positive parity and squares are for negative parity. The first crossing point is at $g = 4/\sqrt{15} \approx 1.0328$ and $E = -2/3$ which has no definite parity.

For non vanishing generic values of $\epsilon$, the $Z_2$ symmetry is lost. This is manifested in the spectrum; in particular, there are no degeneracies (see Fig. 3(a)). As we shall see, the parameter $\theta$ is important to capture general spin-orbit couplings. We remark that with $Z_2$ symmetry preserved, $\theta$ can be deleted by a unitary transformation, and thus it does not
change the energy spectrum. When $Z_2$ symmetry is broken with non-vanishing $\epsilon$, the parameter $\theta$ enters the energy spectrum through $\sigma_x$ and this unitary transformation will induce term $\sigma_x \rightarrow \cos(\theta/2)\sigma_x + \sin(\theta/2)\sigma_y$.

As it will be later argued to be important for many applications, we quantify on the energy correction due to the counter-rotating term (Bloch-Siegert shift [36]). Based on the exact solution, we can give closed expressions in several interesting limits. For $2\Delta \approx \omega$, $g \ll \omega$ the shift is $g^2/\omega$. For $\epsilon = 0$, at the degenerate points, and setting $|\Delta| = (1 - \lambda^2)g^2/2\omega$, the ground state energy gap between the JC model and the anisotropic Rabi model can be found as,

$$\Delta E_0 = \frac{\lambda^2 g^2}{\omega}. \quad (4)$$

For $\lambda = 1$, it is just the standard Bloch-Siegert shift[36]. Such gap can be obtained also for the excitations. For the first and the second excited states at degenerate points, it reads $\sim \lambda^2 g^4/\omega^3$ (see Supplemental Material for details).

We now discuss how our solution can be applied to solve long standing problems in different physical contexts.

**Application to quantum optics:** two level atom in cross electric and magnetic field. When an atom is subjected of a crossed electric and magnetic field, the selection rules are not dictated by the possible values of the atomic angular momentum. Therefore, both the electric dipole and magnetic dipole transition are allowed. The Hamiltonian describing the system is

$$H = H_0 - d \cdot E - \mu \cdot B \quad (5)$$

where we have assumed that the quadrupole transitions can be neglected. Inserting the standard expressions of the quantized electric and magnetic fields are respectively $E \sim (a + a^\dagger)$ and $B \sim i(a - a^\dagger)$, Eq. (5) can be recast into our anisotropic Rabi model Eq. (2) with

$$g = \frac{\langle |d\rangle \sigma_y - \langle |\mu\rangle \sigma_x \rangle}{2} \quad (6)$$

$$\lambda = \frac{\langle |d\rangle \sigma_y - \langle |\mu\rangle \sigma_x \rangle + \langle |\mu\rangle \sigma_y \rangle}{2} \quad (7)$$

being $H_0|\pm\rangle = E_{\pm}|\pm\rangle$.

**Application to superconducting circuits.** Superconducting circuits exploits the inherent coherence of superconductors for a variety of technological applications, including quantum computation[13]. In this case, the bosonic fields typically represent the electromagnetic fields generated by the superconducting currents. The spin degree of freedom describes the two states of the qubit.

As immediate application, we consider two inductively coupled dc-Superconducting Quantum Interference Devices (SQUIDs)[14,16]: a primary SQUID $p$ (assumed large enough to produce an electromagnetic field characterized by a bosonic mode) control the qubit realized by the secondary SQUID. In the limit of negligible capacitive coupling between the two SQUID’s, the circuit Hamiltonian is

$$H_{\text{circuit}} = \omega_p a^\dagger a - 2E_j^s \sigma^x - 2L_p(a + a^\dagger)\sigma^z - iM(a - a^\dagger)\sigma^y \quad (8)$$

where $\omega_p$ is the “frequency” of the primary and $E_j^s = E_j^s(\text{ext})$ is the Josephson energy of secondary SQUID, controlled by the external magnetic field; $M$ is the mutual inductance and a gate voltage $V_g$ is tuned to the charge degeneracy point. The Eq. (8) can be recast into the anisotropic Rabi model[9]: $\{\omega_p, E_j^s, 2L_p, M\} \rightarrow \{\omega, \epsilon, g(1 + \lambda), g(1 - \lambda)\}$.

We comment that the implications of the simultaneous presence of the rotating and counter-rotating terms have been evidenced experimentally [17,18]. The experimental system is an LC resonator magnetically coupled to a superconducting flux qubit in the ultrastrong coupling regime. Indeed the experimental data were interpreted as Bloch-Siegert energy correction of the Jaynes-Cummings dynamics. Here we point out that the experimental results can be also fitted very well in terms of our anisotropic Rabi model, see supplementary material.

**Applications to electrons in semiconductors with spin-orbit coupling.** Spin-orbit coupling effects have been opening up new perspectives in solid state physics, both for fundamental research (including topological insulators and spin-Hall effects[19,20]) and applications (notably spintronics[21]). Electronic spin orbit coupling can be induced by the electric field acting at the two dimensional interfaces of semiconducting heterostructure devices[24-29]. The effective Hamiltonian reads

$$H = \frac{1}{2m} \pi^2 + \frac{1}{2} g\mu B \sigma_z + H_{\text{so}}$$

$$H_{\text{so}} = H_R + H_D$$

$$H_R = \alpha(\pi_\sigma \sigma_y - \pi_y \sigma_x), \quad H_D = \beta(\pi_x \sigma_x - \pi_y \sigma_y) \quad (9)$$

where $\pi = \{\pi_x, \pi_y, \pi_z\}$ is the electrons canonical momentum $\pi = \{p - \frac{2}{3}A\}$. $H_R$ and $H_D$ are the Rashba and Dresselhaus spin-orbit interactions. The coupling constant $\alpha$ depends on the electric field across the well, while the Dresselhaus coupling $\beta$ is determined by the geometry of the heterostructure. The perpendicular magnetic field couples both to the electronic spin and orbital angular momentum. Applying the standard procedure leading to the Landau levels, the Hamiltonian (9) can be recast into our anisotropic Rabi model:

$$\alpha = \sqrt{g^2 + (1 + \lambda)^2 \sin \theta}, \quad \beta = \sqrt{g^2 + (1 + \lambda)^2 \cos \theta}.$$ 

Incidentally, we observe that the simultaneous presence of Dresselhaus and Rashba contributions couples all the Landau levels, making our exact solution immediately relevant for the physics of the system.

We comment that the Hamiltonian (9), has been realized with cold fermionic atoms systems, opening the avenue to study the spin-orbit effects with controllable parameters and in extremely clean environments[30,32].

**Summary and Conclusion.** In conclusion, we discussed a carefully chosen generalization of the Rabi model: The Hamiltonian system breaks the parity symmetry; the rotating and counter-rotating interactions are governed by two diff-
ferent coupling constants; a further parameter introduces a phase factor in the counter-rotating terms. Such system captures the physics of notoriously important problems in different physical contexts, including two dimensional electron gas with general spin-orbit interaction, two level atom in electromagnetic field, and superconducting circuits in ultra-strong regimes. We obtained exact energies and eigenstates of the system through the analytical properties of a transcendental function. In some cases, we explained how our results are immediately relevant for the comprehension of the experiments.

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1 Electronic address: lamico@dmfc.unict.it
2 Electronic address: hfan@iphy.ac.cn

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Supplemental Material

EXACT SOLUTION OF THE ANISOTROPIC RABI MODEL

The Hamiltonian of the anisotropic Rabi model is

\[ H = \omega a^\dagger a + \Delta \sigma_z + g[\sigma^+ a + \sigma^- a^\dagger + \lambda(e^{i\theta} \sigma^+ a^\dagger + e^{-i\theta} \sigma^- a)] + \epsilon \sigma_x. \]  \hspace{1cm} (10)

The parameter \( \lambda \) controls the anisotropy between the rotating and the counter rotating terms and \( \theta \) introduce a phase into the counter rotating terms only; the term \( \epsilon \sigma_x \) breaks the \( \mathbb{Z}_2 \) symmetry, and therefore the eigenspace of the model \( (10) \) cannot be split in invariant subspaces. Nevertheless, the model \( (10) \) can still be solved exactly with the approach originally developed by Braak[7] for the isotropic model \( \lambda = 1, \theta = \epsilon = 0 \).

Firstly, for technical convenience (we comment further below), we perform a unitary transformation \( U(\lambda, \theta) \),

\[ U(\lambda, \theta) = \begin{pmatrix} \cos \eta e^{i\theta/2} & -\sin \eta e^{-i\theta/2} \\ \sin \eta e^{i\theta/2} & \cos \eta e^{-i\theta/2} \end{pmatrix} = \frac{1}{\sqrt{1 + \lambda}} \begin{pmatrix} \xi & -\sqrt{\lambda} \\ \sqrt{\lambda} & \xi^* \end{pmatrix}, \]  \hspace{1cm} (11)

where \( \tan \eta = \sqrt{\lambda} \), \( \xi = e^{i\theta/2} \), when \( \theta = 0, \xi = 1 \), it is a orthogonal transformation. The Hamiltonian \( (10) \) becomes

\[ U^\dagger H U \left( \begin{array}{cc} \omega^a_1 a + \sqrt{\lambda} g(\xi^a_1 a + \xi_1 a^\dagger) + c & \xi^{a^2}(1 - \lambda) g a - d^* \\ \xi^2/(1 - \lambda) g a^\dagger - d & \omega^a_1 a - \sqrt{\lambda} g(\xi^a_1 a + \xi_1 a^\dagger) - c \end{array} \right). \]  \hspace{1cm} (12)

where \( c = \frac{-\lambda}{1 + \lambda} \Delta + \frac{(\xi + \xi^*)\sqrt{\lambda}}{1 + \lambda} \). We exploit the Bargmann representation of bosonic operators in terms of analytic functions: \( a^\dagger \rightarrow z, a \rightarrow \frac{d}{dz} \), and consider the eigenfunction of the Hamiltonian as \( (\phi_1, \phi_2)^T \), we have

\[ \left( \begin{array}{cc} \omega z \frac{d}{dz} + \sqrt{\lambda} g(\xi^a z + \xi z) + c + \xi^2(1 - \lambda) g \frac{d}{dz} - d^* \end{array} \right) \phi_1 = E \phi_1, \]

\[ \left( \begin{array}{cc} \omega z \frac{d}{dz} - \sqrt{\lambda} g(\xi^a z + \xi z) - c \end{array} \right) \phi_2 = E \phi_2. \]  \hspace{1cm} (13)

For convenience, we introduce the notations \( \phi_{1,2}(z) = \exp(-\sqrt{\lambda} \xi z) \psi_{1,2}(y), y = z + \sqrt{\lambda} \xi^*, x = E + \frac{\lambda z^2}{\omega}, f = d + (1 - \lambda) \sqrt{\lambda} \xi \).

Now, obtain,

\[ (\omega y \frac{d}{dy} - x + c) \psi_1 = \left[ f^* - \xi^2(1 - \lambda) g \frac{d}{dy} \right] \psi_2, \]  \hspace{1cm} (15)

\[ \left( \omega y - 2\sqrt{\lambda} g \xi^* \right) \frac{d}{dy} - 2\sqrt{\lambda} g \xi y + \frac{4\lambda g^2}{\omega} - x - c \right) \psi_2 = \left[ f - (1 - \lambda) g y \right] \psi_1. \]  \hspace{1cm} (16)

Assuming that the functions \( \psi_{1,2} \) can be expanded as \( \psi_2 = \sum_{n=0}^{\infty} K_n^+(x)y^n, \psi_1 = \sum_{n=0}^{\infty} L_n^+(x)y^n \), from Eq. \( (15) \), the relation between \( K_n^+ \) and \( L_n^+ \) is found as

\[ L_n^+ = \frac{f^* K_n^+ - \xi^2(1 - \lambda) g K_{n+1}^+(n + 1)}{n\omega - x + c} \]  \hspace{1cm} (17)

Then from Eq. \( (16) \), the recursive relation of \( K_n^+ \) is obtained,

\[ a_n(x)K_{n+1}^+ = b_n(x)K_n^+ + c_n(x)K_{n-1}^+, \]  \hspace{1cm} (18)

\[ a_n(x) = \left[ 2\sqrt{\lambda} - \frac{(1 - \lambda) f}{n\omega - x + c} \right] (n + 1)g\xi^*, \]  \hspace{1cm} (19)

\[ b_n(x) = \frac{4\lambda g^2}{\omega} + n\omega - x - c - \frac{f^* f}{n\omega - x + c} - \frac{(1 - \lambda)^2 g^2 n}{(n - 1)\omega - x + c}, \]  \hspace{1cm} (20)

\[ c_n(x) = -2\sqrt{\lambda} g \xi + \frac{(1 - \lambda) g f^* \xi^2}{(n - 1)\omega - x + c}. \]  \hspace{1cm} (21)
Then from Eq. (26) we obtain the recursive relation
\begin{align}
\phi_1(z) &= \exp\left(-\frac{\sqrt{\lambda g} \xi}{\omega} z\right) \sum_{n=0}^{\infty} L_n^\pm (z + \frac{\sqrt{\lambda g} \xi^*}{\omega})^n, \quad (22) \\
\phi_2(z) &= \exp\left(-\frac{\sqrt{\lambda g} \xi}{\omega} z\right) \sum_{n=0}^{\infty} K_n^\pm (z + \frac{\sqrt{\lambda g} \xi^*}{\omega})^n, \quad (23)
\end{align}

Substituting \( z \to -z \) in Eq. (13), \( \phi_1(-z) = \overline{\phi}_1(z), \phi_2(-z) = \overline{\phi}_2(z) \) are eigenfunctions of the spectral problem (14) as well. Such functions can be obtained applying the procedure led to (22) and (23). The differential equations for \( \phi_1(z) \) and \( \phi_2(z) \) are
\begin{align}
\left[\omega z \frac{d}{dz} - \sqrt{\lambda g} (\xi^* \frac{d}{dz} + \xi) + c\right] \phi_1 + \left[-\xi^2 (1-\lambda) g \frac{d}{dz} - d^*\right] \phi_2 &= E \phi_1, \quad (24) \\
\left[-\xi^2 (1-\lambda) g z - d\right] \phi_1 + \left[\omega z \frac{d}{dz} + \sqrt{\lambda g} (\xi^* \frac{d}{dz} + \xi) - c\right] \phi_2 &= E \phi_2. \quad (25)
\end{align}

Using the following notations, \( \overline{\phi}_{1,2}(z) = \exp\left(-\frac{\sqrt{\lambda g} \xi}{\omega} z\right) \overline{\psi}_{1,2}(y), \ y = z + \frac{\sqrt{\lambda g} \xi^*}{\omega}, \ x = E + \frac{\lambda g^2}{\omega}, \ \mathcal{J} = d - \frac{(1-\lambda) \sqrt{\lambda g} \xi^*}{\omega} \), the above equations can be rewritten as,
\begin{align}
\left[\omega y - 2\sqrt{\lambda g} \xi^* \frac{d}{dy} - 2\sqrt{\lambda g} \xi y + 4\lambda g^2 \omega^2 - x + c\right] \overline{\psi}_1 &= \left[\mathcal{J}^* + \xi^2 (1-\lambda) g \frac{d}{dy}\right] \overline{\psi}_2, \quad (26) \\
\left[\omega y \frac{d}{dy} - x - c\right] \overline{\psi}_2 &= \left[\mathcal{J} + \xi^2 (1-\lambda) g y\right] \overline{\psi}_1, \quad (27)
\end{align}

Expand function \( \overline{\psi}_{1,2} \) as \( \overline{\psi}_1 = \sum_{n=0}^{\infty} K_n^-(x) y^n, \overline{\psi}_2 = \sum_{n=0}^{\infty} L_n^-(x) y^n \), from Eq. (27) we find the relation of \( K_n^- \) and \( L_n^- \),
\begin{equation}
L_n^- = \frac{\mathcal{J} K_n^- + \xi^2 (1-\lambda) g K_{n-1}^-}{n \omega - x - c} \quad (28)
\end{equation}

Then from Eq. (26) we obtain the recursive relation
\begin{align}
\overline{\tau}_n(x) K_{n+1}^- &= \overline{\tau}_n(x) K_n^- + \overline{\tau}_n(x) K_{n-1}^- , \quad (29) \\
\overline{\tau}_n(x) &= \left[2\sqrt{\lambda} + \frac{(1-\lambda) \mathcal{J}^*}{(n+1) \omega - x - c}\right] (n+1) g \xi^*, \quad (30) \\
\overline{b}_n(x) &= \frac{4\lambda g^2}{\omega} + n \omega - x + c - \frac{\mathcal{J} \mathcal{J}^*}{n \omega - x - c} - \frac{(1-\lambda)^2 g^2 (n+1)}{(n+1) \omega - x - c}, \quad (31) \\
\overline{\tau}_n(x) &= -2\sqrt{\lambda} g \xi - \frac{(1-\lambda) g \mathcal{J} \xi^2}{n \omega - x - c}. \quad (32)
\end{align}

where \( K_{-1}^- = 0, K_0^- = 1, n = 0, 1, 2, ... \)

Going back to the original notations, we have,
\begin{align}
\overline{\phi}_1(z) &= \exp\left(-\frac{\sqrt{\lambda g} \xi}{\omega} z\right) \sum_{n=0}^{\infty} K_n^-(z + \frac{\sqrt{\lambda g} \xi^*}{\omega})^n, \quad (33) \\
\overline{\phi}_2(z) &= \exp\left(-\frac{\sqrt{\lambda g} \xi}{\omega} z\right) \sum_{n=0}^{\infty} L_n^-(z + \frac{\sqrt{\lambda g} \xi^*}{\omega})^n. \quad (34)
\end{align}

Considering the relation of these two sets of eigenstates which mentioned above, \( \phi_1(-z) = C \overline{\phi}_1(z), \phi_2(-z) = C \overline{\phi}_2(z) \), then canceling the arbitrary constant \( C \), a transcendental function can be constructed as,
\begin{equation}
G_c(x; z) = \phi_1 \overline{\phi}_2 - \phi_2 \overline{\phi}_1, \quad (35)
\end{equation}
Because $G_{s}(x; z)$ is well defined at $z = \pm \sqrt{\lambda g} \xi^*/\omega$ within the convergent radius $R = 2\sqrt{\lambda g} \xi^*/\omega$, we can set $z = 0$ \cite{7}. The function $G_{s}(x; 0)$ is analytic in the complex plane except in the simple poles

\[ x_{n}^{pole} = n\omega - \frac{(1 - \lambda)^2 g^2}{2\omega} + \frac{(\xi + \lambda \xi^* e^{-i\theta})}{2\sqrt{\lambda}}, \quad (n = 0) \]

\[ \pi_{n}^{pole} = (n + 1)\omega - \frac{(1 - \lambda)^2 g^2}{2\omega} - \frac{(\xi + \lambda \xi^*)e^{-i\theta}}{2\sqrt{\lambda}}, \quad (n = 0) \]

which follows from the zeros of the denominator of $K^\pm_n$: $a_n(x) = 0$ and $\pi_n(x) = 0$ in Eq. \cite{18} and Eq. \cite{29}, respectively. Then, the eigenvalues and eigenstates can be obtained by solving $G_{s}(x) = 0$,

\[ E_n = x_n - \frac{\lambda g^2}{\omega}, \]

\[ \Psi_n = U(\lambda, \theta) \left( \phi_1(x_n), \phi_2(x_n) \right) = U(\lambda, \theta) \left( \sum_{n=0}^{\infty} L_n^+ |n\rangle \right), \]

Using the second solution, the eigenstates of Hamiltonian with $a \rightarrow -a, a^\dagger \rightarrow -a^\dagger$ can be obtained:

\[ \Psi_n = U(\lambda, \theta) \left( \tilde{\phi}_1(x_n), \tilde{\phi}_2(x_n) \right) = U(\lambda, \theta) \left( \sum_{n=0}^{\infty} K_n^- |n\rangle \right), \]

where

\[ |n\rangle \equiv (a^\dagger + \frac{\sqrt{\lambda g} \xi^*}{\omega})^n|0\rangle - \frac{\sqrt{\lambda g} \xi}{\omega} e^{-\frac{\lambda g^2}{2\omega^2} - \frac{\lambda g^2 a^\dagger}{2\omega}} |0\rangle. \]

In case $\lambda = 1$ and $\theta = 0$, we can recover the results given by Braak \cite{7} and in Ref. \cite{37}.

\[ \text{The case } \epsilon = 0 \]

For $\epsilon = 0$, the model \cite{10} enjoys a $Z_2$ symmetry reflecting the conservation of the parity of the operator

\[ \hat{N} = a^\dagger a + \frac{1}{2}(\sigma_z + 1), \]

In this case, the phase factors $e^{\pm i\theta}$ in the Hamiltonian can be canceled by a unitary transformation $R(\theta)$,

\[ R(\theta) = e^{i\frac{\theta}{2}(\hat{N} - \hat{\pi})} = e^{i\frac{\theta}{2}(\hat{N} + a^\dagger a)}, \]

\[ R^\dagger(\theta)H R(\theta) = \omega a^\dagger a + \Delta \sigma_z + g[\sigma^+ a + \sigma^- a^\dagger + \lambda(\sigma^+ a^\dagger + \sigma^- a)]. \]

Therefore the parameter $\theta$ gives no contribution to the energy spectra, but enters the wave functions only.

We shall see that the $Z_2$ symmetry effectively simplifies the procedure of finding the exact spectrum since the transcendental function $G(x)$, can be discussed into the different parity sectors separately.

To simplify the solution of the spectral problem, we resort to a similar trick we employed above. Namely, we apply the rotation

\[ V = U^\dagger(\lambda, \theta)W = \frac{1}{\sqrt{2(1 + \lambda)}} \begin{pmatrix} \xi^* + \sqrt{\lambda} & -\xi^* + \sqrt{\lambda} \\ \xi - \sqrt{\lambda} & \xi + \sqrt{\lambda} \end{pmatrix}. \]

with

\[ W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \]

to the Hamiltonian.
The eigenfunctions $\phi_1, \phi_2$ (and similarly $\overline{\phi}_1, \overline{\phi}_2$) in Eqs. (13,14) transform according to $(\varphi_1, \varphi_2)^T = V^\dagger(\phi_1, \phi_2)^T$:

$$
\varphi_1 = \frac{(\xi + \sqrt{\lambda})\phi_1 + (\xi^* - \sqrt{\lambda})\phi_2}{\sqrt{2(1+\lambda)}},
$$

$$
\varphi_2 = \frac{(-\xi + \sqrt{\lambda})\phi_1 + (\xi^* + \sqrt{\lambda})\phi_2}{\sqrt{2(1+\lambda)}}.
$$

Finally, $\phi_1, \phi_2$ read as

$$
\phi_1(z) = \exp(-\frac{\sqrt{\lambda}g\xi}{\omega}z) \sum_{n=0}^{\infty} L_n(x)(z + \frac{\sqrt{\lambda}g\xi^*}{\omega})^n,
$$

$$
\phi_2(z) = \exp(-\frac{\sqrt{\lambda}g\xi}{\omega}z) \sum_{n=0}^{\infty} K_n(x)(z + \frac{\sqrt{\lambda}g\xi^*}{\omega})^n.
$$

where

$$
L_n = \frac{f^*K_n - \xi^2(1-\lambda)gK_{n+1}(n+1)}{n\omega - x + c}.
$$

Here, the superindices + are omitted.

The $Z_2$ symmetry reflects into a symmetry in the eigenfunction: $\varphi_2(-z) = C\varphi_1(z)$, where $C$ is an arbitrary constant. Without loss of generality we take $\varphi_{1,2}$ normalized and real. In this case, $C = \pm 1$, and the transcendental function $G$ is

$$
G_{\pm}^\lambda(x; z) = \varphi_2(-z) \mp \varphi_1(z) = 0 \quad \forall z \in C
$$

Setting $z = 0$ as the above section, and substituting $\varphi_{1,2}$ by $\phi_{1,2},$

$$
G_{\pm}^\lambda(x) = -\xi\phi_1 + \sqrt{\lambda}\phi_2,
$$

$$
G_{\pm}^\lambda(x) = \sqrt{\lambda}\phi_1 + \xi^*\phi_2.
$$

The energy spectrum can be divided into two cases. One case is the regular solution which is determined by zeros of the transcendental function. Another case corresponds to the irregular solutions. They are both the poles and the zeros of the transcendental function determined by setting $a_n(x) = 0,$

$$
x_n^{pole} = n\omega - \frac{(1-\lambda)^2g^2}{2\omega},
$$

see Fig. 4. For special values of the parameter $g$ and $\Delta$, $K_{n+1}(x_n^{pole}) = 0$ the poles can be lifted, because the numerator of $G_{\pm}^\lambda$ is also vanishing. This special solution are Judd type solution for the anisotropic Rabi model, corresponding to the so called isolated integrability [8]. Owing to $G_{\pm}^\lambda \neq 0$, these eigenvalues have no definite parity, and a double degeneracy of the eigenvalues occurs (see FIG. 3(b) in main text). In this case, the infinite series solutions $K_n$ and $L_n$ can be terminated as finite series solutions. We note, in particular, that there is no crossing with the same parity. Incidentally, we observe a crossing between the ground state and the first excited state occurs for the anisotropic case which corresponds to the exact solutions obtained by Judd [8], which does not occur in the isotropic Rabi model. The position of this point can be analytically determined by the relation $K_1(x_0^{pole}) = 0$ as mentioned above, i.e., $a_0 = 0, b_0 = 0,$

$$
g = \sqrt{\frac{2|\Delta|\omega}{1-\lambda^2}} = \frac{4}{\sqrt{15}},
$$

$$
E_0^c = -\frac{(1+\lambda^2)g^2}{2\omega} = -\frac{2}{3}.
$$

Here let us see an example, for the crossing of the ground state and the first excited state, we find that the series terminates at the first term, as $K_0 = 1$ and $L_0 = 2\sqrt{\lambda}\xi^*/(1-\lambda),$ thereby resulting in

$$
\phi_1 = \frac{2\sqrt{\lambda}\xi^*}{1-\lambda} \exp(-\frac{\sqrt{\lambda}g\xi}{\omega}z),
$$

$$
\phi_2 = \exp(-\frac{\sqrt{\lambda}g\xi}{\omega}z).
$$
Consider that \( z \to a^\dagger \), and \( \Psi = U(\lambda, \theta)(\phi_1, \phi_2)^T \), the first degenerate eigenstate is essentially a coherent state.

Notice that the Bargmann representation of Bosonic operator used in our model can also be used in the JC model, which has \( U(1) \) symmetry. The eigenvalues and eigenstates can be obtained in the familiar steps, but simpler because the total number \( \hat{N} = a^\dagger a + 1 \) is conserved.

### Generalized Bloch-Siegert effect

In this section, we consider the energy gaps between the anisotropic Rabi model and the JC model. Such energy differences, generalizing the Bloch-Siegert effect of the isotropic Rabi model, play important roles in many physical applications where the strong coupling regime is the actual one (like in superconducting circuits).

Ordinarily, there is no general form for the gap, but we can analyse it at the degenerate points. The ground state energy of the JC model is \( E^{JC}_0 = -\Delta \), so the ground state gap at \( |\Delta| = (1 - \lambda^2)g^2/2\omega \) is,

\[
\Delta E_0 = -\Delta - E^c_0 = \frac{\lambda^2 g^2}{\omega}.
\] (58)

when \( \lambda = 0 \) (the JC limit), the gap vanishes; for \( \lambda = 1 \), the gap is just the standard Bloch-Siegert shift in the Rabi model, \( \Delta E_0 = g^2/\omega \). For \( \lambda \neq 1 \), the first excited state energy gap with the JC is

\[
\Delta E_1 = \frac{\omega}{2} - \sqrt{(\Delta - \frac{\omega}{2})^2 + g^2} \approx \frac{1 + \lambda^2}{2}\omega \] (59)

For small \( g/\omega \), \( \Delta E_1 \approx \lambda^2 g^4/\omega^3 \), remarkably different from the standard Bloch-Siegert shift \( g^2/\omega \). This is for case \( |\Delta| = (1 - \lambda^2)g^2/2\omega \), as we just mentioned.

In the Rabi model, there is a crossing between the second and the third energy levels at \( |\Delta| = \sqrt{\omega^2 - 4g^2} \), \( E = \omega - g^2/\omega \)

[7], the second energy level of the JC model in small \( g/\omega \) can be written as

\[
E^{JC}_2 = \frac{\omega}{2} + \sqrt{(\Delta - \frac{\omega}{2})^2 + g^2} \approx \omega - \frac{g^2}{\omega} + \frac{g^4}{\omega^3}.
\] (60)

Obviously, the second excited energy difference between the JC model and Rabi model is in \( g^4/\omega^3 \) scale, too. However, we compare the third excited energy difference at this point as

\[
E^{JC}_3 - E^{Rabi}_3 = \frac{3\omega}{2} - \sqrt{(\Delta - \frac{\omega}{2})^2 + 2g^2} \approx \frac{g^2}{\omega} - \frac{2g^4}{\omega^3}.
\] (61)

where the condition \( g/\omega \ll 1 \) is used, the difference is still in \( g^2/\omega \) scale. Maybe the energy differences of the second and third excited state are strange at the degenerate point. Those differences may be verified by the recent experiments with ultrastrong coupling.
Application of the anisotropic Rabi model to superconducting circuits

In ultrastrong coupling regime, the deviation from the JC model known as Bloch-Siegert shift was experimentally observed in Ref. [17], which is an LC resonator magnetically coupled to a superconducting flux qubit in the ultrastrong coupling regime, and the system can be modeled by the Hamiltonian,

$$H = \frac{\omega_q}{2} (\sigma_z + 1) + \omega_r a^\dagger a + g (\cos \vartheta \sigma_z - \sin \vartheta \sigma_x) (a + a^\dagger),$$  \hspace{1cm} (62)

with $$\omega_q \equiv \sqrt{\epsilon^2 + \Delta^2}$$, $$\epsilon = 2 \pi I_p (\Phi - \Phi_0) / 2$$ and $$\tan \vartheta = \Delta / \epsilon$$, where $$\hbar$$ is conventionally set to 1.

If we neglect the term $$g \cos \vartheta \sigma_z (a + a^\dagger)$$, which only contributes a constant $$-g^2 \cos^2 \vartheta / \omega_r$$, under the transformation $$U = \exp (g \cos \vartheta / \omega_r)$$ to the second order. And omit the counter-rotating term, the corresponding JC model is given by

$$H_{JC} = \frac{\omega_q}{2} (\sigma_z + 1) + \omega_r a^\dagger a - g \sin \vartheta (\sigma^- a^\dagger + \sigma^+ a).$$  \hspace{1cm} (63)

So in the ultrastrong coupling regime, the experimental results of this system do not agree with the JC model, the Bloch-Siegert shift caused by the counter-rotating term is evidently observed.

Then, we directly use our proposed anisotropic model to fit this system

$$H_{a-Rabi} = \frac{\omega_q}{2} (\sigma_z + 1) + \omega_r (a^\dagger a + \frac{1}{2}) - g \sin \vartheta (\sigma^- a^\dagger + \sigma^+ a + \lambda (\sigma^- a + \sigma^+ a^\dagger)).$$  \hspace{1cm} (64)

where the anisotropic parameter $$\lambda$$ is decided by fitting, and $$g = 0.74 \text{GHz}$$, $$\Delta = 4.21 \text{GHz}$$, $$I_p = 500 \text{nA}$$, $$\omega_r = 8.13 \text{GHz}$$, are the same as in Ref. [17]. As shown in Fig. 5 we can find that the experimental data agree perfectly with the case $$\lambda = 0.5$$ (red dashed dot line) which is neither the Rabi model nor the JC model, and remark that the result of $$\lambda = 0.7$$ case seems similar as the Hamiltonian (62) (solid black line), and $$\lambda = 0$$ case is the same as the JC Hamiltonian (63) (dashed black line).