Inhomogeneous Fokker-Planck equation as an homogeneous one from deformed derivative framework

Bruno G. da Costa,1,* Ignacio S. Gomez,2,3,† and Ernesto P. Borges2,3,‡

1Instituto Federal de Educação, Ciência e Tecnologia do Sertão Pernambucano, BR 407, km 08, 56314-520 Petrolina, Pernambuco, Brazil
2Instituto de Física, Universidade Federal da Bahia, R. Barão de Jeremoabo s/n, 40170-115 Salvador, Bahia, Brazil
3National Institute of Science and Technology for Complex Systems, Brazil
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Abstract

We present the Fokker-Planck equation (FPE) for an inhomogeneous medium with a position-dependent mass particle by making use of the Langevin equation in the context of an algebra inspired in nonextensive statistics. We show that the FPE for an inhomogeneous medium with non-constant diffusion coefficient is equivalent to a deformed FPE where the position space is deformed, the usual derivative is generalized, and the mass and diffusion coefficient remain constant. We obtain a deformed version of Theorem $H$, from which emerges a functional entropy that can be identified as the Kullback-Leibler one between the probability distributions of the particles and that of the medium. The formalism is illustrated when the confining potential is given by a square infinite well potential and a linear one. As a consequence, an asymmetry of the diffusion is reported in both cases. In absence of a potential confining, the mean squared displacement is associated with an exponential law spreading corresponding to an hyper-ballistic diffusion.

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* brunocosta@ifsertao-pe.edu.br
† nachosky@fisica.unlp.edu.ar
‡ ernesto@ufba.br
Diffusion is understood as the disorder molecular motion from the macroscopically viewpoint. A standard way to quantify this important phenomena is from individual particles that are subjected to drag forces (properly of the fluid) and random ones (Brownian motion), which gives place to the Langevin equation \([1]\).

In turn, for a useful probabilistic characterization, the next usual step is to rewrite the Langevin equation in terms of the probability density function (PDF), thus obtaining the Fokker-Planck equation (FPE) \([2]\). The FPE has been widely investigated in the literature, mainly applied to the study of different types of diffusion (including the normal and anomalous ones) \([3-5]\). Subsequent applications in multiple kind of phenomena have displayed the relevance of the FPE in the field of statistical physics \([6-12]\). In particular, the FPE in specific media have presented an increasing interest since it allows to characterize electron diffusion \([13]\), photoinduction in nonequilibrium processes \([14]\), rarefied gases and heterogeneous media \([15, 16]\), interfaces-membranes \([17]\), among others.

In addition, theoretical investigations have shown an intimate connection between generalized FPE, theorem \(H\), master equations and entropic forms, highlighting the role played by the nonextensive statistics \([18-21]\). Along to this progress, the mathematical structure inherited by nonextensive statistics turned out to be a useful tool to generalize concepts of statistical mechanics. Some mathematical structures have been presented \([22-25]\), referred as nonextensive algebras.

The goal of this paper is to present the FPE for an inhomogeneous medium with a non-constant diffusion coefficient within a position-dependent mass scenario \([26-31]\) using a nonextensive \(q\)-algebra \([22, 23]\), in which \(q\) stands for a real and continuous deformation parameter of the statistics. As a consequence, we find an equivalence between the FPE in an inhomogeneous medium and a \(q\)-deformed FPE with a constant mass and diffusion coefficient. The solutions exhibit an asymmetric spatial distribution that corresponds physically to the inhomogeneity of the medium. We present a generalized version of the theorem \(H\) where the deformed entropy is the Boltzmann one plus an additional term associated to the inhomogeneity of the medium.

The work is organized as follows. In Section II we review the FPE construction from Langevin equation along with the nonextensive algebra. Section III is devoted to generalize
the FPE for an inhomogeneous medium (that we called deformed FPE) from its corresponding Langevin equation, by employing a generalized derivative operator determined by the mass functional and the properties of the medium. Here we also present a generalized version of the theorem $H$ for the FPE in a deformed position space in general. In Section IV we introduce a $q$-deformed FPE associated with the $q$-algebra as well as an analytical expression for the general solution in a deformed space. Next, in Section V we illustrate the formalism presented for two types of confining potential: the infinite square well and the linear potential. Finally, in Section VI some conclusions and perspectives are outlined.

II. PRELIMINARIES

We present a brief review of the Langevin, Klein-Krammers and Fokker-Planck equations, along with the $q$-algebra.

A. Langevin, Klein-Krammers and Fokker–Planck equations

A single particle of mass $m_0$ in a fluid of viscosity coefficient $\lambda_0$ subjected to an external potential $V(x)$ (i.e. an external force $F(x) = -dV(x)/dx$) and a random force $R(t)$ has a motion equation that can be obtained from the Lagrangian [32]

$$L(x, \dot{x}, t) = \frac{1}{2}m_0\dot{x}^2 - U(x, t)$$

and using the Euler-Lagrange equation

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} + \frac{\partial Q}{\partial \dot{x}} = 0,$$

where $Q = \frac{1}{2}m_0\lambda_0\dot{x}^2$ is a Rayleigh dissipation function, and $U(x, t) = V(x) - xR(t)$ is the potential due to conservative and random forces. Thus, the corresponding Langevin equation is

$$\ddot{x} = -\lambda_0 \dot{x} + f(x) + \xi(t),$$

with $f(x) = F(x)/m_0$ and $\xi(t) = R(t)/m_0$.

Generally, when having $N$ stochastic variables $\vec{y} = \{y_1, ..., y_N\}$ with $M$ white Gaussian noises $\vec{\xi} = \{\xi_1, ..., \xi_M\}$ and a diffusion coefficient $D_0$ (i.e. $\langle \xi_j(t) \rangle = 0$ and $\langle \xi_j(t)\xi_l(t') \rangle = \begin{cases} D_0 & \text{for } j = l, \\ 0 & \text{for } j \neq l \end{cases}$)
\[ 2D_0 \delta_{jl} \delta(t' - t) \] for all \( j, l = 1, \ldots, M \) and for all \( t \) the Langevin equation is
\[
\frac{dy_i}{dt} = A_i(\vec{y}, t) + \sum_j B_{ij}(\vec{y}, t) \xi_j(t), \quad (i = 1, \ldots, N),
\] (4)
from which the generalized diffusion equation results [2]
\[
\frac{\partial P}{\partial t} = -\sum_i \frac{\partial}{\partial y_i} \left\{ \left[ A_i(\vec{y}, t) + \frac{\Gamma}{2} \sum_{jl} B_{jl}(\vec{y}, t) \frac{\partial B_{il}}{\partial y_j} \right] P \right\} + D_0 \sum_{ij} \frac{\partial^2}{\partial y_i \partial y_j} \left\{ \sum_l B_{il}(\vec{y}, t) B_{jl}(\vec{y}, t) \right\} P.
\] (5)
Using the relations
\[
\begin{cases}
\frac{dx}{dt} = v, \\
\frac{dv}{dt} = -\lambda_0 v + f(x) + \xi(t)
\end{cases}
\] (6)
and comparing Eq. (6) with the Eqs. (4) and (5) it follows that \( y_1 = x, \ y_2 = v, \ A_x = v, \ A_v = -\lambda_0 v + f(x), \ B_{xx} = B_{xx} = B_{v} = 0, \) and \( B_{vv} = 1. \) This leads to the Klein-Krammer equation for the PDF \( P = P(x, v, t) \)
\[
\frac{\partial P}{\partial t} = -v \frac{\partial P}{\partial x} - f(x) \frac{\partial P}{\partial v} + \lambda_0 \left( \frac{\partial}{\partial v} v + \frac{D_0}{\lambda_0} \frac{\partial^2}{\partial v^2} \right) P.
\] (7)
In the overdamped limit of the Langevin equation, (i.e. \( \lambda_0 \gg \tau^{-1} \) with \( \tau \) a coarse-grained time scale) the inertia term \( dv/dt \) is negligible compared with \( \lambda_0 v, \) so \( dx/dt = [f(x) + \xi(t)]/\lambda_0. \) Substituting \( y = x, \ A = f(x)/\lambda_0 \) and \( B = 1/\lambda_0 \) in Eq. (5) we obtain the unidimensional Fokker-Planck equation
\[
\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} [A(x) P(x, t)] + \frac{\Gamma}{2} \frac{\partial^2 P}{\partial x^2},
\] (8)
with \( A(x) \) the confining potential and \( \Gamma/2 = D_0/\lambda_0^2 \) the diffusion coefficient. The general solution of Eq. (8) depends on the confining potential and the initial conditions. For long times \( t \to \infty, \) the solution of the FPE tends to the stationary distribution
\[
P^{(st)}(x) = C \exp \left[ \frac{2}{\Gamma} \int_0^x A(x') dx' \right],
\] (9)
where \( C \) is the normalization constant. In some cases analytical solutions can be obtained. We briefly review two typical cases [2]. In absent of confining potential \( A(x) = 0 \) (free particle case) with a Dirac delta centered in \( x = 0 \) as the initial condition \( P(x, t = 0) = \delta(x), \) the probability distribution is simply the Gaussian one
\[
P(x, t) = \frac{1}{\sqrt{2\pi \Gamma t}} e^{-x^2/(2\Gamma t)}. \] (10)
corresponding to normal diffusion. For a linear potential \( A(x) = -\alpha x \) with the same initial condition \( P(x, t) = 0 \) and \( \partial P(x, t)/\partial x \big|_{x \to \pm \infty} = [A(x)P(x, t)] \big|_{x \to \pm \infty} = 0 \ \forall t \), the solution is

\[
P(x, t) = \sqrt{\frac{\alpha}{\pi \Gamma(1 - e^{-2\alpha t})}} \exp \left[ -\frac{\alpha x^2}{\Gamma(1 - e^{-2\alpha t})} \right], \tag{11}
\]

which tends asymptotically for \( t \to \infty \) to the Gaussian stationary solution

\[
P^{(st)}(x) = \sqrt{\frac{\alpha}{\pi \Gamma}} e^{-\alpha x^2/\Gamma}. \tag{12}
\]

**B. Nonextensive algebra and deformed derivatives**

Generalizations of differential and integral operators can be formulated using deformed algebraic structures. One of these structures is motivated by nonextensive statistical mechanics \([10, 11]\) and it is based on the generalized functions: the \( q \)-exponential \( \exp_q(u) \equiv \left[ 1 + (1 - q)u \right]^{1/(1-q)} \) with \( [A]_+ = \max\{A, 0\} \) (where the parameter \( q \) has several physical interpretations \([11]\)) and its inverse, the \( q \)-logarithm \( \ln_q(u) \equiv u^{1 - q} - 1 \left/ (1 - q) \right. \) \((u > 0)\). The properties of these functions allow to define a generalization of the operations of the real numbers \([22, 23]\): the \( q \)-sum \( a \oplus_q b = a + b + (1 - q)ab \), the \( q \)-subtraction \( a \ominus_q b = \frac{a - b}{1 + (1 - q)b} \) \((b \neq \frac{1}{q-1})\), the \( q \)-product \( a \otimes_q b = [a^{1-q} + b^{1-q} - 1]_+^{1/q} \) \((a, b > 0)\), and the \( q \)-division \( a \oslash_q b = [a^{1-q} - b^{1-q} + 1]_+^{1/q} \) \((a, b > 0)\).

A deformed calculus has been developed from the \( q \)-algebra \([23]\) where the derivative and integral operators allow to formulate a \( q \)-deformed linear Schrödinger equation associated with a position-dependent effective mass system \([27–31]\). Broadly, a generalized derivative operator can be defined by

\[
D_{[h]} f(u) = \frac{1}{h(u)} \frac{df}{du} = \frac{df(u)}{d_h u}, \tag{13}
\]

along with its dual derivative

\[
\tilde{D}_{[h]} f(u) = h(f) \frac{df}{du} = \frac{d_{[h]} f(u)}{du}, \tag{14}
\]

where the generalized infinitesimal element \( dh = d_{[h]} u = h(u) du \) leads to a deformed spatial variable \( \eta(u) = \int^u h(u') du' \). The deformed derivative operator \( D_{[h]} f(u) \) may be understood as the rate of variation of the function \( f(u) \) with respect to a generalized variation of the
independent variable $u$, denoted by $d_{[h]}u$. Analogously, the deformed derivative operator $\tilde{D}_{[h]}f(u)$ may be viewed as the rate of a generalized variation of the function $f(u)$, i.e. $d_{[h]}f$, with respect to the ordinary variation of the independent variable $u$. It is straightforward verified that $D_{[h]}f = 1/\tilde{D}_{[h]}f^{-1}$ which expresses the duality between these derivatives.

The $q$-deformed calculus can be obtained through the transformation $u_q = \ln[\exp_q(u)] = \ln[1 + (1 - q)u]/(1 - q)$ [23], and using the $q$-deformed infinitesimal element

$$d_qu = \lim_{u' \to u} u' \otimes_q u = \frac{du}{1 + (1 - q)u}, \quad (15)$$

we obtain $h(u) = 1/[1 + (1 - q)u]$, which in turn leads to the $q$-deformed derivative operator

$$D_qf(u) \equiv \lim_{u' \to u} \frac{f(u') - f(u)}{u' \otimes_q u} = [1 + (1 - q)u] \frac{df}{du}, \quad (16)$$

together with its associated $q$-deformed integral

$$\int f(u)d_qu = \int \frac{f(u)}{1 + (1 - q)u} du. \quad (17)$$

These operators satisfy the properties $D_q\exp_q(u) = \exp_q(u)$ and $\int \exp_q(u)d_qu = \exp_q(u)$. In this case the dual $q$-derivative is

$$\tilde{D}_qf(u) \equiv \lim_{u' \to u} \frac{f(u') \otimes_q f(u)}{u' - u} = \frac{1}{1 + (1 - q)f(u)} \frac{df(u)}{du}, \quad (18)$$

satisfying $\tilde{D}_q\ln_q(u) = 1/u$. In the following sections, we will discuss how generalized derivatives operators of the form (13), (14), (16), and (18) allow to describe the phenomena diffusion in inhomogeneous media.

III. INHOMOGENEOUS FOKKER-PLANCK EQUATION AS AN HOMOGENEOUS ONE USING DEFORMED DERIVATIVE

We revisit the path outlined in Section II provided with a position-dependent effective mass $m(x)$ in an inhomogeneous medium, and we express the FPE as an homogeneous one by means of the deformed derivative associated to the $q$-algebra. Then, we provide a version of the Theorem $H$ along with a discussion about the nonlinear FPE within the context of the deformed derivative.

The Lagrangian for a position-dependent mass system is

$$L(x, \dot{x}, t) = \frac{1}{2}m(x)\dot{x}^2 - U(x, t), \quad (19)$$
where \( U(x, t) \) and \( V(x) \) are defined as previously in Subsection II.A. From Eq. (2) and \( Q = \frac{1}{2}m(x)\lambda(x)\dot{x}^2 \) (now the damping coefficient depends on the position \( x \)) we obtain the following Langevin equation

\[
m(x)\ddot{x} + \frac{1}{2}m'(x)\dot{x}^2 = -m(x)\lambda(x)\dot{x} + F(x) + R(t). \tag{20}
\]

Let us define the deformed derivative operator \( \tilde{D}_{[\xi]}x(t) = \zeta(x)\frac{dx}{dt} \) with \( \zeta(x) = \sqrt{m(x)/m_0} \) (the usual one is recovered for \( m(x) \to m_0 \)). Then, \( \tilde{D}_{[\xi]}^2x(t) = \zeta(x)\frac{d^2x}{dt^2} \) and Eq. (20) can be written as (compare it with Eq. (3))

\[
\tilde{D}_{[\xi]}^2x(t) = -\tilde{\lambda}(x)\tilde{D}_{[\xi]}x(t) + f(x) + \xi(t). \tag{21}
\]

with \( \tilde{\lambda}(x) = \lambda(x)\sqrt{m(x)/m_0}, f(x) = F(x)/m_0, \) and \( \xi(t) = R(t)/m_0 \). It follows that

\[
\begin{aligned}
\frac{dx}{dt} &= v, \\
\frac{dv}{dt} &= -\frac{1}{2}m'(x)v^2 - \lambda(x)v + \frac{m_0f(x)}{m(x)} + \frac{m_0\xi(t)}{m(x)}.
\end{aligned} \tag{22}
\]

We identify \( y_1 = x, \ y_2 = v, \ A_x = v, \ A_v = -\left[\frac{1}{2}m'(x)v^2 + m(x)\lambda(x)v - m_0f(x)/m(x)\right], \)

\( B_{xx} = B_{xv} = B_{vx} = 0, \) and \( B_{vv} = m_0/m(x) \) (compare (4) with (22)). From Eq. (5) the diffusion equation for the probability distribution function \( P(x, v, t) \) associated with (22) is

\[
\frac{\partial P}{\partial t} = -v\frac{\partial P}{\partial x} + \frac{1}{2}\frac{m'(x)}{m(x)}\frac{\partial(v^2P)}{\partial v} - f(x)\frac{m_0}{m(x)}\frac{\partial P}{\partial v} \\
+ \lambda(x)\frac{m_0}{m(x)}\frac{\partial(vP)}{\partial v} + D\left[\frac{m_0}{m(x)}\right]^2\frac{\partial^2 P}{\partial v^2}, \tag{23}
\]

corresponding to a Klein-Kramers equation of particles having a position-dependent mass \( m(x) \) and with non-constant damping coefficient \( \lambda(x) \).

In this case, the overdamped limit of the Langevin equation (21) is \( \tilde{D}_{[\xi]}x(t) = [f(x) + \xi(t)]/\tilde{\lambda}(x), \) i.e.

\[
\frac{dx}{dt} = \frac{m_0}{m(x)\lambda(x)}[f(x) + \xi(t)]. \tag{24}
\]

As before, from Eq. (4) we have \( \tilde{A}(x) = A(x)/\kappa(x) \) and \( \tilde{B}(x) = 1/|\lambda_0\kappa(x)| \) with \( \kappa(x) = \lambda(x)m(x)/(|\lambda_0|m_0) \), this later associated with the non-isotropy of the position space. Using Eq. (5) we obtain the Fokker-Planck equation for an inhomogeneous medium of mass \( m(x) \).
and dumping coefficient $\lambda(x)$

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} \left[ \frac{A(x)}{\kappa(x)} P(x, t) \right] + \frac{D_0}{\lambda^2} \frac{\partial}{\partial x} \left\{ \left[ -\frac{\kappa'(x)}{\kappa^3(x)} + \frac{1}{\kappa^2(x)} \frac{\partial}{\partial x} \right] P(x, t) \right\}$$

$$= -\frac{\partial}{\partial x} \left[ \frac{A(x)}{\kappa(x)} P(x, t) \right] + \frac{\Gamma}{2} \frac{\partial}{\partial x} \left\{ \frac{1}{\kappa(x)} \frac{\partial}{\partial x} \left[ \frac{1}{\kappa(x)} P(x, t) \right] \right\}. \quad (25)$$

If we define $D(x) \equiv D_0/\kappa^2(x) \geq 0$ as the diffusion coefficient, then we can recast the Fokker-Planck equation for inhomogeneous medium (25)

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} \left[ \sqrt{\frac{D(x)}{D_0}} A(x) P(x, t) \right] + \frac{D_0}{\lambda^2} \frac{\partial}{\partial x} \left\{ \sqrt{\frac{D(x)}{D_0}} \frac{\partial}{\partial x} \left[ \sqrt{\frac{D(x)}{D_0}} P(x, t) \right] \right\}. \quad (26)$$

It is worthy to mention that other formulations for the FPE in inhomogeneous media have been investigated [21]. The given by Eq. (26) has contributions of the viscosity and mass of the particles that may be position-dependent due to the the non-isotropy of the position space. The stationary solution, for reflecting boundary with a diffusion coefficient $D(x)$, has the integral form

$$P^{(st)}(x) = \frac{C}{\sqrt{D(x)/D_0}} \exp \left[ \frac{2}{\Gamma} \int_x^x \frac{A(x')}{\sqrt{D(x')/D_0}} dx' \right]. \quad (27)$$

The FPE (26) can be generalized in terms of the deformed operator derivative $\mathcal{D}_{[\kappa]} = \frac{1}{\kappa(x)} \frac{\partial}{\partial x}$ and PDF $\mathcal{P}_{[\kappa]}(x, t) = P(x, t)/\kappa(x)$, thus obtaining

$$\frac{\partial \mathcal{P}_{[\kappa]}}{\partial t} = -\mathcal{D}_{[\kappa]} \left[ A(x) \mathcal{P}_{[\kappa]}(x, t) \right] + \frac{\Gamma}{2} \mathcal{D}^2_{[\kappa]} \mathcal{P}_{[\kappa]}(x, t). \quad (28)$$

Therefore, we have expressed the usual Fokker-Planck equation for the PDF $P(x, t)$ in an inhomogeneous medium (26) as an homogeneous one for the deformed PDF $\mathcal{P}_{[\kappa]}(x, t)$ (28), the later having a constant mass and diffusion coefficient, and deformed derivatives $\mathcal{D}_{[\kappa]}$, $\mathcal{D}^2_{[\kappa]}$. The deformed PDF obeys a generalized normalization condition $\int \mathcal{P}_{[\kappa]}(x, t) d_{[\kappa]} x = 1$, and its stationary solution has the form

$$\mathcal{P}_{[\kappa]}^{(st)}(x) = C \exp \left[ \frac{2}{\Gamma} \int_x^x A(x') d_{[\kappa]} x' \right]. \quad (29)$$
A. Theorem $H$ for inhomogeneous FPE: entropy deformed by the medium

A generalized version of the theorem $H$ for the inhomogeneous FPE (28) can be also established. It is sufficient to repeat the steps given in Ref. [19] but using the generalized free energy functional

$$F[\mathcal{P}_{[\kappa]}] = \int \Phi_{[\kappa]}[x, \mathcal{P}_{[\kappa]}(x,t)]d_{[\kappa]}x = U - \theta S,$$  \hspace{1cm} (30)

with $\theta$ the temperature parameter. The first term is

$$U[\mathcal{P}_{[\kappa]}] = \int \vartheta_{[\kappa]}(x)\mathcal{P}_{[\kappa]}(x,t)d_{[\kappa]}x,$$  \hspace{1cm} (31)

where $\vartheta_{[\kappa]}(x)$ corresponds to an auxiliar potential while the second term is a deformed entropy functional along with the usual convex conditions

$$S[\mathcal{P}_{[\kappa]}] = \int s_{[\kappa]}[\mathcal{P}_{[\kappa]}(x,t)]d_{[\kappa]}x \quad , \quad s_{[\kappa]}[0] = s_{[\kappa]}[1] = 0 \quad , \quad \frac{d^2 s_{[\kappa]}}{d\mathcal{P}^2_{[\kappa]}} \leq 0.$$  \hspace{1cm} (32)

The time derivative of the functional (30) is

$$\frac{dF}{dt} = \int \left[ \vartheta_{[\kappa]}(x) - \theta \frac{ds_{[\kappa]}}{d\mathcal{P}_{[\kappa]}} \right] \frac{\partial \mathcal{P}_{[\kappa]}}{\partial t} d_{[\kappa]}x$$

$$= \int \left[ \vartheta_{[\kappa]}(x) - \theta \frac{ds_{[\kappa]}}{d\mathcal{P}_{[\kappa]}} \right] \times \mathcal{D}_{[\kappa]} \left[ -A(x)\mathcal{P}_{[\kappa]} + \frac{\Gamma}{2} \mathcal{P}_{[\kappa]} \mathcal{P}_{[\kappa]} \right] d_{[\kappa]}x$$

$$= - \int \mathcal{P}_{[\kappa]} \left[ \mathcal{D}_{[\kappa]} \vartheta_{[\kappa]}(x) - \theta \frac{d^2 s_{[\kappa]}}{d\mathcal{P}^2_{[\kappa]}} \mathcal{D}_{[\kappa]} \mathcal{P}_{[\kappa]} \right]$$

$$\times \left[ -A(x) + \frac{\Gamma}{2} \mathcal{D}_{[\kappa]} \mathcal{P}_{[\kappa]} \right] d_{[\kappa]}x.$$  \hspace{1cm} (33)

In order that the time derivative of the free energy is well defined and satisfies inequality

$$\frac{dF}{dt} \leq 0, \quad \forall t \geq 0,$$  \hspace{1cm} (34)

we impose $\theta = \Gamma/2$, $\mathcal{D}_{[\kappa]} \vartheta_{[\kappa]}(x) = -A(x)$, and $d^2 s_{[\kappa]}/d\mathcal{P}^2_{[\kappa]} = -1/\mathcal{P}_{[\kappa]}$. Thus, we obtain

$$\vartheta_{[\kappa]}(x) = - \int A(x)d_{[\kappa]}x,$$

$$s_{[\kappa]}[\mathcal{P}_{[\kappa]}] = -\mathcal{P}_{[\kappa]} \ln \mathcal{P}_{[\kappa]}$$

that correspond to $q$-deformed versions of the Boltzmann case. The deformed entropy of the system is given by

$$S = - \int \mathcal{P}_{[\kappa]}(x,t) \ln \mathcal{P}_{[\kappa]}(x,t)d_{[\kappa]}x$$

$$= - \int \mathcal{P}_{[\kappa]}(\eta,t) \ln \mathcal{P}_{[\kappa]}(\eta,t)d\eta.$$  \hspace{1cm} (35)
Substituting $P_{\kappa}(x,t) = P(x,t)/\kappa(x)$ we have
\[ S = -\int P(x,t) \ln[P(x,t)/\kappa(x)]dx = S_{\text{BG}} + \langle \ln[\kappa(x)] \rangle, \quad (36) \]
which physically means that the deformed entropy $S$ for an inhomogeneous medium is the sum of two terms: the Boltzmann-Gibbs entropy $S_{\text{BG}} = -\int P(x,t) \ln P(x,t) dx$ associated with the distribution of particles plus a residual contribution resulting from the inhomogeneity of the medium $S_{\text{medium}} = \int P(x,t) \ln[\kappa(x)]dx$. It is worthwhile to notice that $S$ corresponds to the Kullback-Leibler relative entropy $S_{\text{KL}}(P,P_0) = -\int P(x,t) \ln[P(x,t)/P_0(x,t)]dx$ between $P$ and $P_0$ with $P_0(x,t) = \kappa(x)$.

**B. Nonlinear FPE associated to Tsallis entropy in terms of deformed derivatives**

Besides the deformed derivative inherited by the $q$-algebra other generalizations have been considered. Nobre et al. [33] have formulated a deformed derivative operator associated with a nonlinear formulation of the quantum mechanics within the nonextensive framework. Generalized derivative operators can be introduced through a functional given by a power law $h(u) = u^{1-q}$, expressed in the form
\[ D_q f(u) = \frac{1}{u^{1-q}} \frac{df(u)}{du}, \quad (37) \]
with its associated dual derivative
\[ \tilde{D}_q f(u) = [f(u)]^{1-q} \frac{df(u)}{du}. \quad (38) \]
From (37) and (38) we obtain $D_q \ln_q(u) = 1/u$ and $\tilde{D}_q \exp_q(u) = \exp_q(u)$. As an application, the dual deformed derivative (38) can be used to formulate the nonlinear Fokker-Planck equation proposed in [4]. In this case, we have that
\[ \tilde{D}_{q,t} P_q(x,t) = \tilde{D}_{q,x}[A(x)P_q(x,t)] + \frac{\Gamma}{2} \tilde{D}_{q,x}^2 P_q(x,t) \quad (39) \]
is equivalent to
\[ \frac{\partial P_q(x,t)}{\partial t} = \frac{\partial}{\partial x}[A(x)P_q(x,t)] + \frac{\Gamma_q}{2} \frac{\partial^2}{\partial x^2} [P_q(x,t)]^{2-q}, \quad (40) \]
where $\Gamma_q = \Gamma/(2-q)$. By means of the Theorem $H$ it can be shown that the nonlinear FPE (40) is associated with the Tsallis entropy $S_q = -\int P_q(x,t) \ln_q P_q(x,t)dx$. Recently, Sicuro et al. [21] have proved that the FPE (40) is related with an inhomogeneous media in which the pressure of the fluid obeys a power law of the density.
IV.  

$q$-DEFORMED FOKKER-PLANCK EQUATION

In this Section, we formulate a $q$-deformed FPE based on the deformed derivative associated to the $q$-algebra [23]. For this purpose, we consider a medium with a diffusion coefficient depending on the position $x$ of the form

$$D(x) = D_0 (1 + \gamma_q x)^2,$$

(41)

where $\gamma_q \equiv (1 - q) / l_0$ is an adimensional parameter measuring the deformation and $l_0$ is a characteristic length. This diffusion coefficient can be obtained, for instance, from a position-dependent mass system having a mass given by

$$m(x) = \frac{m_0}{(1 + \gamma_q x)^2},$$

(42a)

and a damping coefficient

$$\lambda(x) = \lambda_0 (1 + \gamma_q x).$$

(42b)

Using the deformed PDF $P_q(x, t) = (1 + \gamma_q x) P(x, t)$ and the $q$-derivative $D_q = (1 + \gamma_q x) \partial_x$, the $q$-deformed Fokker-Planck equation can be recasted as

$$\frac{\partial P_q(x, t)}{\partial t} = -D_q [A(x) P_q(x, t)] + \frac{\Gamma}{2} D_q^2 P_q(x, t)$$

(43)

provided the deformed normalization condition

$$\int P_q(x, t) dq x = 1.$$  

(44)

Interestingly, from Eqs. (8) and (43) we can see that the $q$-deformed Fokker-Planck equation is simply the standard one but replacing the usual derivative $d/dx$ and the PDF $P(x, t)$ by their $q$-deformed versions $D_q$ and $P_q(x, t)$. This remark indicates that, when the diffusion coefficient depends on the position then it is possible to express the inhomogeneous FPE into an homogeneous one having a constant diffusion coefficient, where the inhomogeneity is contained in the deformed versions of the derivatives and of the PDF. In next sections we analyze the effect of the deformation on the solutions and what are the physical consequences about the diffusion.

A. Stationary solution

In order to obtain the stationary solution of the $q$-deformed FPE (43) we rewrite it as

$$\frac{\partial P_q}{\partial t} = -D_q J_q(x, t),$$

(45)
where \( J_q(x,t) \) is a \( q \)-deformed probability current density

\[
J_q(x,t) = A(x)P_q(x,t) - \frac{\Gamma}{2}D_qP_q(x,t).
\]

From the \( q \)-deformed integral we have

\[
\frac{\partial}{\partial t} \int_{x_i}^{x_f} P_q(x,t) dq x = J_q(x_f,t) - J_q(x_i,t),
\]

and using the normalization condition (44) the conservation of the total probability is guaranteed only if \( J_q(x_f,t) = J_q(x_i,t) \). For the case of reflecting boundary conditions \((J_q(x,t) = 0 \ \forall x)\) the stationary solution \((\partial P_q^{(st)}/\partial t = 0)\) satisfies

\[
A(x)P_q^{(st)}(x) = \frac{\Gamma}{2}D_qP_q^{(st)}(x),
\]

so we obtain

\[
P_q^{(st)}(x) = C_q \exp \left[ \frac{2}{\Gamma} \int_0^x A(x') dq x' \right],
\]

where \( C_q \) is a normalization constant.

**B. General solution**

The general solution for the \( q \)-deformed FPE can be obtained by the method of separation of variables. In this direction, let us consider the \( q \)-deformed FPE (43) expressed as

\[
\frac{\partial P_q(x,t)}{\partial t} = \hat{L}_q P_q(x,t)
\]

where \( \hat{L}_q \) is a deformed Fokker-Planck operator whose action over a function \( f(x) \) is \( \hat{L}_q f(x) = -D_q[A(x)f(x)] \) \(+ \frac{1}{2}\Gamma D_q^2 f(x) \). Thus, the general solution of the \( q \)-deformed FPE can be expanded in a power series of the eigenfunctions \( \phi_{q,n}(x) \) with the coefficients the eigenvalues \( \Lambda_n \) of \( \hat{L}_q \), i.e.

\[
P_q(x,t) = \sum_n c_n \phi_{q,n}(x)e^{\Lambda_n t}.
\]

By the boundary conditions in \( x = x_i \) and \( x = x_f \) we have

\[
\int_{x_i}^{x_f} \hat{L}_q \phi_{q,n}(x) dq x = \Lambda_n \int_{x_i}^{x_f} \phi_{q,n}(x) dq x = 0.
\]

Next step is to employ an associated Schrödinger equation for obtaining the explicit formula of the general solution of the \( q \)-deformed FPE. For accomplish this, we use the operator \( \hat{K}_q \) defined by

\[
\hat{K}_q \psi_{q,n}(x) = \frac{\hat{L}_q[\psi_{q,0}(x)\phi_{q,n}(x)]}{\psi_{q,0}(x)},
\]
where \( \psi_{q,n}(x) = \phi_{q,n}(x)/\psi_{q,0}(x) \) with \( \psi_{q,0}(x) = \sqrt{\mathcal{P}_q^{(st)}(x)} \). It is straightforward to show that \( \hat{\mathcal{K}}_q \psi_{q,n}(x) = \Lambda_n \psi_{q,n}(x) \), and then from the relation \( \mathcal{D}_q[\ln \psi_{q,0}(x)] = A(x)/\Gamma \) and using the operator \( \hat{\mathcal{L}}_q \) we obtain

\[
\hat{\mathcal{K}}_q \psi_{q,n}(x) = \frac{\Gamma}{2} \mathcal{D}_q^2 \psi_{q,n}(x) - \frac{1}{2} \left\{ \frac{1}{\Gamma} [A(x)]^2 + \mathcal{D}_q A(x) \right\} \psi_q(x).
\]  

Thus, the operator \(-\hat{\mathcal{K}}_q\) is the \( q \)-deformed Hamiltonian operator [27–31]

\[
\hat{H}_q = -\frac{\hbar^2}{2m_0} \mathcal{D}_q^2 + V_{ef}(\hat{x})
\]

which is associated to a quantum system having a position-dependent mass given by Eq. (42a) and subjected to an effective potential of the form

\[
V_{ef}(x) = \frac{1}{2} \left\{ \frac{1}{\Gamma} [A(x)]^2 + \mathcal{D}_q A(x) \right\}.
\]

Hence, by comparison with the solutions of (55) the general solution of the \( q \)-deformed FPE results

\[
\mathcal{P}_q(x,t) = \psi_{q,0}(x) \sum_n c_n \psi_{q,n}(x) e^{t\Lambda_n}.
\]  

V. APPLICATIONS OF THE \( q \)-DEFORMED FOKKER-PLANCK EQUATION

We illustrate the \( q \)-deformed FPE with two examples of confining potential: the infinite square well and the linear potential.

A. Infinite square well potential

Consider the \( q \)-deformed FPE for an infinite square well potential, where \( A(x) = 0 \) for \( |x| \leq L/2 \) and \( A(x) = \infty \) otherwise. Using (49) we obtain \( \mathcal{P}_q^{(st)}(x) = C_q \) for the stationary solution and from the normalization \( 1/C_q = \int_{-L/2}^{L/2} d_q x \) we have

\[
\mathcal{P}_q^{(st)}(x) = \frac{\gamma_q}{\ln \left( \frac{1+\gamma_q L/2}{1-\gamma_q L/2} \right)} = \frac{1}{L_q},
\]

where \( L_q \) is a deformed characteristic length. The eigenfunctions of the associated FPE operator satisfy

\[
\mathcal{D}_q^2 \phi(x) = -k^2 \phi(x),
\]

13
with $k^2 = -2\Lambda/\Gamma$. The solution of (59) for the boundary conditions $\mathcal{D}_q \phi(x) = 0$ in $x = \pm L/2$ is

$$
\phi_n(x) = \frac{1}{L_q} \cos \left( \frac{k_{q,n}}{\gamma_q} \ln \left( \frac{1 + \gamma_q x}{1 + \frac{1}{2} \gamma_q L} \right) \right)
$$

(60)

where $k_{q,n} = n\pi/L_q$, $n$ is a positive integer, and the constant $1/L_q$ has been chosen such that $\phi_0(x) = \mathcal{P}_q^{(st)}(x)$. Then, the general solution for $t = 0$ is

$$
\mathcal{P}_q(x,0) = \sum_n c_n \cos \left( \frac{k_{q,n}}{\gamma_q} \ln \left( \frac{1 + \gamma_q x}{1 + \frac{1}{2} \gamma_q L} \right) \right).
$$

(61)

The coefficients of the expansion above are obtained from the following $q$-integrals

$$
c_0 = \int_{-L/2}^{L/2} \mathcal{P}_q(x,0) d_q x,
$$

(62)

$$
c_n = 2 \int_{-L/2}^{L/2} \mathcal{P}_q(x,0) \phi_n(x) d_q x, \quad (n \neq 0).
$$

(63)

As usual, assuming a delta function for the initial condition $P(x,0) = \mathcal{P}_q(x,0)/(1 + \gamma_q x) = \delta(x)$, we obtain $c_0 = 1/L_q$ and $c_n = 2/L_q \cos[k_{q,n}\gamma_q^{-1} \ln(1 + \gamma_q L/2)]$ for $n \neq 0$. Thus, we have

$$
\mathcal{P}_q(x,t) = \frac{1}{L_q} \left\{ 1 + 2 \sum_{n=1}^{\infty} \left( \cos \left( \frac{k_{q,n}}{\gamma_q} \ln \left( \frac{1 + \gamma_q L}{2} \right) \right) \times \cos \left( \frac{k_{q,n}}{\gamma_q} \ln \left( \frac{1 + \gamma_q x}{1 + \frac{1}{2} \gamma_q L} \right) \right) e^{-t \Gamma k_{q,n}^2/2} \right) \right\}.
$$

(64)

Consistently, when $\gamma_q \rightarrow 0$ the standard case is recovered:

$$
P(x,t) = \frac{1}{L} \left[ 1 + 2 \sum_{n=1}^{\infty} e^{-t \Gamma k_{n}^2/2} \cos(k_n x) \right],
$$

(65)

with $k_n = k_{1,n} = n\pi/L$. In the limit of a large well $L \rightarrow \infty$ the general solution (64) takes the form

$$
\mathcal{P}_q(x,t) = \frac{2}{\pi} \lim_{L \rightarrow \infty} \int_0^{\infty} \left\{ \cos \left( \frac{k}{\gamma_q} \ln \left( \frac{1 + \gamma_q L}{2} \right) \right) \times \cos \left( \frac{k}{\gamma_q} \ln \left( \frac{1 + \gamma_q x}{1 + \frac{1}{2} \gamma_q L} \right) \right) e^{-t \Gamma k^2/2} \right\} dk
$$

$$
= \frac{1}{\sqrt{2\pi \Gamma t}} \exp \left[ -\frac{\ln^2(1 + \gamma_q x)}{(2\Gamma \gamma_q^2)} \right]
$$

$$
+ \lim_{L \rightarrow \infty} \exp \left[ -\frac{1}{(2\Gamma \gamma_q^2)} \ln^2 \left( \frac{1 + \gamma_q x}{1 + \frac{1}{2} \gamma_q L} \right) \right],
$$

(66)
and since the second term goes to zero, we obtain

\[ P_q(x,t) = \frac{1}{\sqrt{2\pi\Gamma t}} \exp \left[ -\ln^2(1 + \gamma_q x) \right]. \tag{67} \]

Now recalling the deformed space \( x_q(x) = \gamma_q^{-1} \ln(1 + \gamma_q x) \) and making \( \sigma^2(t) = \Gamma t \) then the Eq. (67) can be recasted as

\[ P_q(x,t) = \frac{1}{\sqrt{2\pi\sigma^2(t)}} \exp \left[ -\frac{x_q^2(x)}{2\sigma^2(t)} \right] \]

that corresponds to the \( q \)-deformed solution of the free particle case. As expected, in the limit \( q \rightarrow 1 \) since \( x_q \rightarrow x \), the standard stationary solution (12) is recovered. Figs. 1 and 2 illustrate the solution (67) for some representative values of the adimensional parameter \( \gamma_q l_0 \).

As a consequence of the particular form of the functional mass (42a), we can see that the diffusion is asymmetrical and the PDF is accumulated in a zone near to the mass asymptote \( x_d = -1/\gamma_q \), where physically the particle tends to have an infinite mass. By contrast, in the region \( x \geq -x_d \) the PDF tends rapidly to zero as the solution evolves. Moreover, as \( \gamma_q l_0 \) increases the particle becomes more localized at \( x = 0 \) because the region where the PDF can be diffused becomes small, as is shown in Fig. 2 for a value of the mass asymptote \( x_d = -10^{-2} \sim 0 \).

In order to analyze the type of diffusion involved, we calculate the first two moments of \( x \), whose general formula is given by

\[ \langle x^n(t) \rangle = \int_{-\infty}^{\infty} x^n P(x,t)dx = \int_{-\infty}^{\infty} x^n P_q(x,t)d_qx \tag{69} \]

for the \( n \)-th moment of \( x \). Using Eq. (67) and the transformation \( x \rightarrow x_q \) we have

\[ \langle x^n(t) \rangle = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\Gamma t}} \frac{x^n}{1 + \gamma_q x} \exp \left[ -\ln^2(1 + \gamma_q x) \right] dx \]

\[ = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\Gamma t}} \left( \frac{e^{\gamma_q x} - 1}{\gamma_q} \right)^n e^{-x^2_q/(2\Gamma t)} dx_q. \]

The first and second moment can be written so that the deformation is shown explicitly

\[ \langle x(t) \rangle = \frac{e^{(\Gamma t)\gamma_q^2/2} - 1}{\gamma_q}, \tag{70a} \]

\[ \langle x^2(t) \rangle = \frac{e^{2(\Gamma t)\gamma_q^2} - 2e^{(\Gamma t)\gamma_q^2/2} + 1}{\gamma_q^2}, \tag{70b} \]

that tend to the standard moments \( \langle x(t) \rangle = 0 \) and \( \langle x^2(t) \rangle = \Gamma t \) in the limit \( \gamma_q \rightarrow 0 \), as expected. Fig. 3 shows \( \langle (\Delta x)^2(t) \rangle \) as a function of time for \( \gamma_q l_0 = 0.01, 0.1, 0.2, 0.4 \). The
is reflected into a diffusion only over the semi-straight $x < x_d$ of the mass $m$ in the region more asymmetrical as the parameter $t/\tau$ increases exponentially for spreading is hyperdiffusive, faster than the power-law of the superballistic diffusion, and it increases exponentially for $t/\tau \gg 1$ with a characteristic time $\tau = l_0^2/\Gamma$.

FIG. 1. FPE for an inhomogeneous media for the case of an infinite square well potential as the confining one. Upper line: evolution of $P(x,t)$ for the values of $\gamma l_0 = 0.1, 0.2$ and $0.4$ and for the times $t/\tau = 0.1, 0.5, 1.0, 2.0, 5.0$ with $\tau = l_0^2/\Gamma$ the characteristic time. The diffusion becomes more asymmetrical as the parameter $\gamma l_0$ increases. Bottom line: some surfaces of $P(x,t)$ within the region $|x| < 10l_0$, $0 < t/\tau \leq 20$ for the same parameters $\gamma l_0$. The asymmetrical dependence of the mass $m(x) = m_0/(1 + \gamma q x)^2$ implies a divergence of the PDF when $x \rightarrow x_d = -1/\gamma q$, which is reflected into a diffusion only over the semi-straight $x < x_d$.

FIG. 2. FPE for an inhomogeneous media for the case of an infinite square well potential as the confining one. Evolution of $P(x,t)$ for $\gamma l_0 = 10^2$. The particle remains quasi-localized at $x = 0$. The spreading is hyperdiffusive, faster than the power-law of the superballistic diffusion, and it increases exponentially for $t/\tau \gg 1$ with a characteristic time $\tau = 1/(\gamma q^2 \Gamma)$. The normal
diffusion is recovered for $\gamma_q \to 0$, corresponding to an infinite characteristic time $\tau$.

![Plot of $\langle (\Delta x)^2(t) \rangle = \langle x^2(t) \rangle - \langle x(t) \rangle^2$ in function of time for the free particle case at inhomogeneous medium for the values $\gamma_q l_0 = 0.01, 0.1, 0.2$ and 0.4. The axes are displayed in logarithmic scale with the aim to show a wide range of values. For $t$ very small $\langle (\Delta x)^2 \rangle \approx \Gamma t$ for all $\gamma_q l_0$, while for $t/\tau \gg 1$ we have $\langle (\Delta x)^2 \rangle \propto e^{2t/\tau}$ corresponding to an exponential hyperdiffusion.]

**B. Linear potential**

The $q$-deformed FPE for $A(x) = -\alpha x$ is

$$\frac{\partial P_q(x,t)}{\partial t} = \alpha D_q[xP_q(x,t)] + \frac{\Gamma}{2} D_q^2 P_q(x,t). \quad (71)$$

In this case the associated effective potential (56) is given by

$$V_{ef}(x) = \frac{\alpha^2}{2\Gamma} x^2 - \frac{\alpha \gamma_q}{2} x - \frac{\alpha}{2}. \quad (72)$$

The eigenfunctions $\psi_q(x)$ for the operator $\hat{K}_q$ (see Eq. (54)) can be obtained from a comparison with the solutions of the $q$-deformed time-independent Schrödinger equation for an harmonic oscillator frequency with $\omega_0$ (for usual case $q = 1$) and electric charge $e$ in an uniform electric field $\vec{E} = E\hat{x}$ [31]:

$$-\frac{\hbar^2}{2m_0} D_q^2 \psi_q + \left(\frac{1}{2} m_0 \omega_0^2 x^2 - eE x + V_0\right) \psi_q(x) = E \psi_q(x), \quad (73)$$

where $V_0$ is a constant and $E$ is the eigenvalue of $\psi_q(x)$. In absence of an electric field, the solutions of Eq. (73) has been studied [28, 30], the eigenfunctions and energies are obtained by means of a canonical point transformation that maps the system into a Morse
The number of bound states of the deformed oscillator is eigenfunctions of the Eq. (73) are

\[ \psi_{q,n}(x) = \chi_n(s(x)) = A_n e^{-\frac{s}{2}(1+\frac{q}{2})} L_n^{(\nu)}(s(x)), \] (75)

where \( z(x) = 2d(1+\gamma q x), d = m_0 \omega_0/(\hbar \gamma^2), \nu = 2d(1+\gamma q x_0) - 1 - 2n > 0, A_n^2 = \nu \gamma q n!/(\nu + n)!, \) and \( L_n^{(\nu)}(z) \) are the Laguerre polynomials. The energy eigenvalues of the Eq. (73) are

\[ E_n = V_0 - \frac{e^2 \mathcal{E}^2}{2m_0 \omega_0^2} + \hbar \omega_0 \left( 1 + \frac{\gamma q e \mathcal{E}}{m_0 \omega_0^2} \right) \left( n + \frac{1}{2} \right) \]

\[ - \frac{\hbar^2 \gamma^2}{2m_0} \left( n + \frac{1}{2} \right)^2. \] (76)

The number of bound states of the deformed oscillator is \( N_b = \lfloor d(1+\gamma q x_0) - 1/2 \rfloor, \lfloor u \rfloor \) denoting the floor function, which tends to increase (decrease) for \( \gamma q x_0 > 0 (\gamma q x_0 < 0) \) in presence of an external electric field. By means of the relations \( \hbar^2/m_0 = \Gamma, m_0 \omega_0^2 = a^2/\Gamma, \) \( e \mathcal{E} = \alpha \gamma q/2 \) and \( V_0 = -\alpha/2, \) we have

\[ \psi_{q,n}(x) = A_n e^{-\mu(1+\gamma q x)} [2\mu(1+\gamma q x)]^{\mu-n} L_n^{(2\mu-2n)}(2\mu(1+\gamma q x)), \] (77)

where \( \mu = \alpha/(\Gamma \gamma^2) \) and \( A_n^2 = 2(\mu - n) \gamma_q n!/(2\mu - n)!. \) From Eq. (76), the eigenvalues for \( \hat{K}_q \) are

\[ \Lambda_n = -E_n = -\alpha n \left( 1 - \frac{\Gamma \gamma^2}{2 \alpha} n \right), \] (78)

with \( \Lambda_n < 0 \) for all \( n \in \mathbb{N} \) and \( \Lambda_0 = 0. \) The eigenfunctions of (77) are orthogonalized by means a deformed inner product, i.e., \( \int_{-\infty}^{+\infty} \psi_{q,n}(x) \psi_{q,n}(x) dx = \delta_{n,m} \). For the initial condition \( P(x,0) = P_q(x,0)/(1+\gamma q x) = \delta(x), \) the coefficients \( c_n \) of (57) are \( c_n = \psi_{q,n}(0)/\psi_{q,0}(0) \), so the general solution of Eq. (71) results

\[ P_q(x,t) = \frac{\psi_{q,0}(x)}{\psi_{q,0}(0)} \sum_n \psi_{q,n}(x) \psi_{q,0}(0) e^{-t \Lambda_n}. \] (79)
The summation in Eq. (79) has the form of a quantum propagator for the Morse oscillator, [35] from which we obtain its stationary solution

\[ P_{q}^{(st)}(x) = \frac{\gamma_q \left( \frac{1}{\sigma_0 \gamma_q} \exp \left\{ \left[ -(1 + \gamma_q x) + \ln(1 + \gamma_q x) \right] \right\} \right)^{\frac{1}{\sigma_0 \gamma_q}}}{\left( \frac{1}{\sigma_0 \gamma_q} \right)!}, \] 

where \( \Gamma/2\alpha = \sigma_0^2 \). Alternatively, the stationary solution may be obtained from (49) using \( A(x) = -\alpha x \). It is worthwhile to notice that the transformation \( \gamma_q \rightarrow -\gamma_q \) implies the following ones \( P_{q}^{(st)}(x) \rightarrow P_{q}^{(st)}(-x) \) for the PDF (80) and \( x_d \rightarrow -x_d \) for the mass asymptote. Thus, the inversion transformation \( x \rightarrow -x \) is equivalent to \( \gamma_q \rightarrow -\gamma_q \) which means physically to change the direction of the confining potential force \(-\partial A/\partial x\).

Fig. 4 (a) shows some plots of the stationary solution \( P^{(st)}(x) = P_{q}^{(st)}(x)/(1 + \gamma_q x) \) for some values of \( \sigma_0 \gamma_q \). For \( |\gamma_q| \rightarrow 1 \) the PDF (80) diverges in \( x_d = -1/\gamma_q \). Fig. 4 (b) shows the deformed entropy (36) in function of \( \gamma_q \) for the stationary PDF along with the entropic contributions of the particles and the medium, obtained by numerical integration. To guarantee a dimensionless argument in the logarithm of the Boltzmann-Gibbs entropy for the particles, we introduce an additive constant \( S_0 = -\ln(\bar{\sigma}) \) (where \( \bar{\sigma} \) has units of length), so we have \( S_{BG} = -\int P \ln(\bar{\sigma} P) dx \). To achieve that the limit \( S \rightarrow 0 \) as \( \gamma_q \sigma_0 \rightarrow 1 \) is satisfied, as a consequence of the localization of the particles at \( x_d \) for \( \gamma_q \sigma_0 \rightarrow 1 \), then we choose \( S_0 = -1 \). The greater the value of the parameter \( \gamma_q \) the greater (smaller) is the entropic contribution of the medium (particles) on the total entropy.

VI. CONCLUSIONS

We have obtained an alternative way to present the Fokker-Planck equation in an inhomogeneous medium for a position-dependent mass system, by making use of a deformed algebra-calculus inspired in nonextensive statistics. We have illustrated the results when the confining potential is the given by an infinite square well and by a linear one.

Our contribution is in multiple aspects. The equivalence between the FPE in an inhomogeneous medium with a position-dependent mass and the deformed FPE in an homogeneous medium with a constant mass, accomplished by means of deformed derivatives, allows to connect the characteristics of the medium with the deformed algebraic structure used in an univocal way. From this equivalence the free particle solution of the FPE in an inhomoge-
FIG. 4. FPE for an inhomogeneous media for the case of a linear potential as the confining one. Left panel (a): stationary solution for the values $\sigma_0 \gamma_q = 0, 0.4, 0.8, 0.99$. As in the free particle case of Fig. 1, asymmetry in the PDF is observed. Right panel (b): deformed entropy $S$ of the system (black line), as the sum of the Boltzmann-Gibbs one $S_{BG}$ of particles (blue line) plus a contribution due to the medium $-\ln(1 + \gamma_q x)$ (red line), in function of $\gamma_q \sigma_0$. The contribution of the medium becomes relevant as $\gamma_q \sigma_0$ increases.

For the type of mass functional studied, the asymmetrical diffusion of the PDF is connected with the asymptote $x_d$ of the mass functional in a explicit way, where PDF tends to accumulate in a region close to $x_d$, as was observed in the examples studied. Also, as a result of the mass functional chosen, an exponential hyperdiffusion is obtained for times larger than the characteristic time, which in turn depends on the deformation parameter. Hence, in the deformed FPE framework the interplay between the deformation and the behavior of the diffusion along with its characteristic time can be easily visible.

By last, the deformed calculus also allows to generalize the Theorem $H$ for the deformed FPE so that the entropy of the system is the sum of the Boltzmann-Gibbs one plus a term corresponding only to the medium. The growing of the entropy of the medium as the deformation increases was consistently reported for the case of the linear potential (Fig. 4).

We hope that the deformed FPE framework can motivate its exploration in other contexts. For example, through the deformed algebra introduced by Kaniadakis in an deformed calculus applied to relativistic statistical mechanics [36, 37].
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