Sensitivity Analysis for Constructing Optimal Treatment Regimes in the Presence of Non-compliance and Two Active Treatment Options

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Abstract

Existing literature on constructing optimal regimes often focuses on intention-to-treat analyses that completely ignore the compliance behavior of individuals. Instrumental variable-based methods have been developed for learning optimal regimes under endogeneity. However, when there are two active treatment arms, the average causal effects of treatments cannot be identified using a binary instrument, and thus the existing methods will not be applicable. To fill this gap, we provide a procedure that identifies an optimal regime and the corresponding value function as a function of a vector of sensitivity parameters. We also derive the canonical gradient of the target parameter and propose a multiply robust classification-based estimator of the optimal regime. Our simulations highlight the need for and usefulness of the proposed method in practice. We implement our method on the Adaptive Treatment for Alcohol and Cocaine Dependence randomized trial.

Keywords: Canonical gradients, endogeneity, instrumental variables, value function, weighted classification
1 Introduction

An important goal in recent clinical research is to use the available data to understand “what works best for whom?” This can be achieved by constructing individualized treatment strategies that map a person’s characteristics to a treatment option. An optimal individualized treatment strategy is one that optimizes a specified health outcome. In certain research areas, such as mental health and substance use disorder, the high rate of non-compliance to treatments imposes a substantial challenge in constructing optimal strategies that are generalizable to the general population.

An optimal strategy can be estimated using policy (strategy) learning or conditional outcome model-based (e.g., Q-learning and A-learning) methods (Schulte et al., 2014; Nahum-Shani et al., 2017; Watkins & Dayan, 1992; Shi et al., 2018; Chakraborty & Moodie, 2013; Davidian et al., 2016; Zhao et al., 2015; Kosorok & Moodie, 2015; Ertefaie et al., 2021). The main drawback of the outcome model-based approaches is that the quality of the constructed optimal strategy relies on the specified regression models (Zhao et al., 2009, 2011). Policy learning approaches, however, circumvent the need for the conditional outcome models by framing the problem of estimating the optimal strategy as a weighted classification problem, thereby directly optimizing the expected outcome among a class of rules (Zhao et al., 2012, 2015; Kosorok & Moodie, 2015; Zhao et al., 2009; Zhang et al., 2013). The preceding methods rely on the no unmeasured confounder assumption, which is not verifiable using the observed data. The latter assumption imposes an even stronger limitation in randomized trials with non-compliance. This is because individuals may fail to comply with the prescribed treatment due to many reasons that are not recorded in the data. Hence, when randomized trial data are available, intention-to-treat analyses are often performed to avoid the no unmeasured confounder assumption. However, such anal-
yses have two important limitations. First, they estimate the effect of randomization to a treatment sequence and not the actual treatment effect that is of main interest. In fact, the treatment effect estimates are often biased toward the null effect (Marasinghe & Amarasinghe, 2007; Lin et al., 2008). Second, the concluding results may not be reproducible due to the potential differential compliance behavior in real-world settings (Frangakis & Rubin, 1999; Robins & Tsiatis, 1991; Hewitt et al., 2006; Lin et al., 2008).

Instrumental variable (IV) methods are well-studied for providing unbiased treatment effect estimates in the presence of unmeasured confounders. An instrument is a random or haphazard encouragement to adopt some change in behavior, where encouragement can affect outcomes only indirectly through its manipulation of the treatment and also is independent of unmeasured confounders. The simplest and clearest case is a randomized encouragement (Angrist et al., 1996; Angrist & Krueger, 2001). Recently, Cui & Tchetgen Tchetgen (2021) developed a semiparametric IV-based approach to construct an optimal strategy. Qiu et al. (2021) also proposed an IV analysis that can estimate an optimal strategy under resource constraints. However, these approaches are not applicable to settings with two (or more) active treatment options, which is the case in many comparative effectiveness studies and randomized trials (Swanson et al., 2015; Ertefaie et al., 2016b,a). This is because when there are multiple active treatments, the compliance values will have at least one level more than the instrument, which leads to an identifiability issue (Cheng & Small, 2006).

In this paper, we provide a method for constructing an optimal treatment strategy in the presence of non-compliance and two active treatments. We propose sensitivity analyses which generalize the weighted classification methods by allowing the compliance to be endogenous and have more levels than the instrument. To this end, we (1) show that
the optimal strategy is identifiable as a function of sensitivity parameters; (2) propose a multiply robust sensitivity analysis approach that identifies the optimal regime as a function of sensitivity parameters; and (3) derive multiply robust estimator for the mean outcome under a treatment strategy.

2 Problem Setting

2.1 Notation

Our data consists of $n$ independent, identically distributed trajectories of $O = (X, Z, A, Y) \sim P_0$. The vector $X \in \mathcal{X}$ includes all available baseline covariates measured before the instrumental variable $Z \in \mathcal{Z} = \{-1, 1\}$. Let $A$ be the observed compliance value where $A \in \{-1, 0, 1\}$. Let $A(z)$ be the potential compliance value given $Z = z$, where $A(z) \in \{-1, 0, 1\}$, e.g., $A(1)$ is the level of compliance if the instrument was 1. In this formulation, $A(z) = 0$ indicates that no treatment was received under $Z = z$. The potential outcomes are $Y(z, a)$ for $Z = z$ and $A = a$. Let $\mathcal{D}$ be a class of decision rules. We define a treatment regime as a function $\pi \in \mathcal{D}$ where $\pi : \mathcal{X} \mapsto \mathcal{Z}$. The potential outcome under a regime $\pi$ is defined as

$$Y(\pi) = \sum_z \sum_a Y(z, a) I\{\pi(X) = z\} I\{A = a\}.$$

For any $\pi$, we define a value function $\mathcal{V}(\pi) = E\{Y(\pi)\}$. An optimal regime $\pi^{opt}$ satisfies that $\mathcal{V}(\pi^{opt}) \geq \mathcal{V}(\pi)$ for all $\pi \in \mathcal{D}$. Our goal is to construct an optimal regime among subjects who would comply with their assigned treatment (i.e., compliers) which is defined as $\pi_c^{opt} = \arg \max_{\pi \in \mathcal{D}} E\{Y(\pi) \mid A(1) = 1, A(-1) = -1\}$. We also define an $L_2$-norm $\|f\|_{2,\mu}^2 = \int f^2(x) d\mu(x)$ where $\mu$ is an appropriate measure (i.e., the Lebesgue or the counting measure).
Table 1: The list of principal strata

| Principal strata | Compliance value $A(-1)$ | Compliance value $A(+1)$ |
|-----------------|-------------------------|-------------------------|
| S1 (always -1 taker) | -1                      | -1                      |
| S2 (always +1 taker) | +1                      | +1                      |
| S3 (never taker) | 0                        | 0                        |
| S4 (complier) | -1                      | +1                      |
| S5 | -1                        | 0                        |
| S6 | 0                        | +1                      |
| S7 (defier) | +1                      | -1                      |
| S8 | +1                        | 0                        |
| S9 | 0                        | -1                      |

2.2 Identification assumptions

Our identification results rely on the following assumptions.

**Assumption 1** *(Fundamental assumptions in IV analyses)*

*(1A) If $Z = z$ then $A = A(z)$ and if $Z = z$ and $A = a$ then $Y = Y(z,a)$. In other words, the treatment only affects the subject taking the treatment and there is only one version of IV and treatment.*

*(1B) $p(A = 1 \mid Z = 1, X = x) > p(A = 1 \mid Z = -1, X = x)$*

*(1C) The IV has no direct effect on the outcome: $Y(Z = -1, A = a) = Y(Z = 1, A = a)$.*

*(1D) The instrument $Z$ is independent of the potential outcomes of $Y$ and $A$ given $X$: $Z \perp \perp \{A(-1), A(1), Y(-1, -1), Y(-1, 0), Y(-1, 1), Y(1, -1), Y(1, 0), Y(1, 1)\} \mid X$.*

**Assumption 2** $Y(a) \perp \perp A \mid X, U$.

**Assumption 3** $Z \mid X$ has a positive density with respect to a dominating measure on $\mathcal{Z}$. In particular, we assume $\text{Var}(Z \mid X)$ exists and is between $1/C$ and $C$ for some constant $C > 1$ and all $X \in \mathcal{X}$. 
Assumptions 1A-1D are standard IV assumptions (Angrist et al., 1996; Baiocchi et al., 2014). Assumption 2 suggests that there exists unmeasured confounders that influence the compliance status of each subject. Assumption 3 is the positivity assumption that ensures the possibility of statistical inference for the treatment effect. Furthermore, we make the following two monotonicity assumptions.

**Assumption 4 (Monotonicity)** The potential outcomes of \( A \) satisfy the following properties:

\[(4A) \ p\{A(1) = -1, A(-1) = 1\} = 0 \ (i.e. \ there \ are \ no \ defiers).\]

\[(4B) \ p\{A(-1) = 1, A(1) = 0\} = 0\]

\[(4C) \ p\{A(-1) = 0, A(1) = -1\} = 0\]

In our setting, we can classify subjects into the nine principal strata listed in Table 1 (Frangakis & Rubin, 2002). The monotonicity assumptions eliminate the principal strata S7, S8, and S9. They reduce the number of free parameters, thereby enabling the estimation of an optimal treatment strategy among compliers. Importantly, these assumptions are not restrictive in our setting because it is irrational for the subjects to refuse the given treatment but seek it when they are given a different treatment.

### 2.3 Preliminaries

One of the key assumptions made in constructing optimal strategies is that of no unmeasured confounders. Cui & Tchetgen Tchetgen (2021) relaxed this assumption by generalizing the weighted classification approach of Zhao et al. (2012) which breaks the endogeneity of the compliance variable using an instrument. In the presence of a binary treatment...
and a binary compliance value, Cui & Tchetgen Tchetgen (2021) showed that under their assumptions, the compliers’ value function $V_c(\pi)$ can be identified by

$$E \left[ \frac{ZAYI\{A = \pi(X)\}}{f(Z|X)\{p(A = 1|Z = 1) - p(A = 1|Z = -1)\}} \right],$$

where $I(.)$ is an indicator function. The function $V_c(\pi)$ is defined as $E[h\{\pi, X, A(1), A(-1)\} | A(1) > A(-1)]$ where $h\{\pi, X, A(1), A(-1)\} = I\{\pi(X) = 1\}E\{Y_1 | A(1) > A(-1), X\} + I\{\pi(X) = -1\}E\{Y(-1) | A(1) > A(-1), X\}$. The key building block of the latter result is that under a monotonicity assumption with binary instruments and binary compliances, the group of subjects with $A(1) > A(-1)$ consists of the compliers. This implies that $p\{A(1) > A(-1)\}$ can be identified using the observed data by $p(A = 1 | Z = 1) - p(A = 1 | Z = -1)$. However, with a binary instrument and a three level compliance variable, the event $A(1) > A(-1)$ does not correspond to the complier group (e.g., the inequality is also satisfied among individuals in S6). Moreover, the probability of being a complier (i.e., $p\{A(1) = 1, A(-1) = -1\}$) is no longer identifiable using the observed data. To see where the issue arises, we let $\eta^S$ denotes the the proportion of strata $S$ and consider the following system of linear equations

$$p(A = 1|Z = 1) = \eta^{S2} + \eta^{S4} + \eta^{S6}, \quad p(A = -1|Z = -1) = \eta^{S1} + \eta^{S4} + \eta^{S5},$$

$$p(A = 1|Z = -1) = \eta^{S2}, \quad p(A = 0|Z = -1) = \eta^{S3} + \eta^{S6},$$

$$p(A = -1|Z = 1) = \eta^{S1}, \quad p(A = 0|Z = 1) = \eta^{S3} + \eta^{S5},$$

The system of equations do not have a unique solution for $\eta^{S4}$ (i.e., $p\{A(1) = 1, A(-1) = -1\}$).
3 Methodology

We will show that the compliers’ value function $E\{Y(\pi)|S4\}$ can be identified using a vector of sensitivity parameters. Let $f(y \mid S4, X, A(Z) = Z)$ denote the conditional density of the outcome given $S4$, baseline covariates and $A(Z) = Z$. Then by the Bayes rule,

$$f(y \mid S4, X, A(Z) = Z) = \frac{p(S4 \mid Y = y, X, A(Z) = Z)f(y \mid X, A(Z) = Z)}{p(S4 \mid X, A(Z) = Z)}.$$  \hspace{1cm} (1)

**Assumption 5** For any $z$, $p(S4 \mid Y = y, X, A(z) = z) = p(S4 \mid Y = y, A(z) = z)$.

Assumption 5 implies that the information in the outcome is sufficient to model the conditional probability of belonging to the compliers’ stratum. This is a plausible assumption as the outcome is a summary function of both the measured and unmeasured confounders.

Let $w_{\alpha}(A, Z, Y) = p(S4 \mid A, Z, Y)$. Assuming logit models for the probabilities $p(S4 \mid Y = y, A(1) = 1)$ and $p(S4 \mid Y = y, A(-1) = -1)$, we define

$$w_{\alpha}(A, Z, Y) = I(A = Z)\frac{\exp(\alpha Z Y)}{1 + \exp(\alpha Z Y)},$$

where $(\alpha_{+1}, \alpha_{-1})$ are the sensitivity parameters. The parameter $\exp(\alpha Z)$ represents the conditional (on $X$ and $A = Z$) odds ratio of being a complier for patients who differ by one unit of $Y$. Specifically, among individuals with $Z = 1$, $\alpha_{+1} > 0$ ($\alpha_{+1} < 0$) implies that the odds of being a complier among patients who have complied with $Z = 1$ (i.e., $p(S4 \mid A = 1, Z = 1, Y = y)$) is higher (lower) for a larger $Y(1) = y$. The parameter $\alpha_{-1}$ have a similar interpretation but among those with $Z = -1$.

Let $\gamma(A, Z, X) = \int w_{\alpha}(A, Z, y)f(y \mid X, A, Z)dy$. Then, $f(y \mid S4, X, A, Z) = w_{\alpha}(y, A, Z)f(y \mid X, A, Z)/\gamma(A, Z, X)$. The proposed sensitivity analyses can still be performed when Assumption 5 is violated at the cost of increasing the number of sensitivity parameters which
may negatively impact the interpretability of the results.

Remark 1 Often there is a scientifically informed range for the sensitivity parameters, which can improve the interpretability of the results. Specifically, as suggested by our simulation studies, when the signs are correctly specified, the concluding results of the proposed methods remain similar and outperform the existing methods.

Theorem 1 Under Assumptions 1 - 5, for any \( \pi \in \mathcal{D} \), the compliers’ value function \( V_c(\pi) = E\{Y(\pi)|S4\} \) is nonparametrically identified for a given set of sensitivity parameters \( (\alpha_{-1}, \alpha_{+1}) \),

\[
V_c(\pi) = E\left[ I\{\pi(X) = Z\}A(A + Z)Y \omega_{\alpha Z}(A, Z, Y) \right] / 2\gamma(A, Z, X)f(A, Z|X). \tag{2}
\]

Accordingly, the optimal regime among compliers is identified as

\[
\pi_c^{opt} = \arg\max_{\pi \in \mathcal{D}} E\left[ I\{\pi(X) = Z\}A(A + Z)Y \omega_{\alpha Z}(A, Z, Y) \right] / 2\gamma(A, Z, X)f(A, Z|X). \tag{3}
\]

3.1 Weighted Learning with sensitivity parameters

Our result in Theorem 1 indicates that under certain assumptions, the optimal regime can be identified as a function of sensitivity parameters. The optimization task of equation (3) is equivalent to

\[
\arg\min_{\pi \in \mathcal{D}} E\left[ |W|I\{\text{sign}(W)Z \neq \pi(X)\} \right], \tag{4}
\]

where \( W \equiv W(\gamma, f) = \frac{A(A+Z)Y \omega_{\alpha Z}(A, Z, Y)}{2\gamma(A, Z, X)f(A, Z|X)} \) (Chen et al., 2018). Also, for \(|W| > 0\), \( \text{sign}(W) = W/|W| \) and for \(|W| = 0\), \( \text{sign}(W) = 0 \). The minimization of (4) is difficult due to discontinuity and non-convexity of 0-1 loss. To overcome this, we follow Zhao et al. (2012) and proceed with convex relaxation via the use of the hinge loss function with the added
penalty term to avoid over-fitting. Thus, we estimate the optimal strategy by minimizing the following objective function

\[ \frac{1}{n} \sum_{i=1}^{n} |W_i| \phi\{\text{sign}(W_i)Z_i g(X_i)\} + \frac{\lambda}{2} \|g\|^2, \]  

where \( \phi \) is the hinge loss function, \( \lambda \) is penalty term, \( g \) is the decision function, and \( \|g\| \) is the Euclidean norm of \( g \). We refer to Zhao et al. (2012) for selecting the appropriate norm. Let \( \hat{W} \equiv W(\hat{\gamma}, \hat{f}) \) denote an estimator of \( W \). Accordingly, we define our inverse probability weighted optimal strategy estimator as \( \hat{\pi}_{ipw} = \text{sign}(\hat{g}) \) where

\[ \hat{g} = \arg\min_{g} \frac{1}{n} \sum_{i=1}^{n} |\hat{W}_i| \phi\{\text{sign}(\hat{W}_i)Z_i g(X_i)\} + \frac{\lambda}{2} \|g\|^2. \]

For any \( \pi \in D \), the asymptotic linearity of the corresponding inverse probability weighted value estimator \( \hat{V}_{IPW,c}(\pi) \) relies on

\[ \|\hat{\gamma} - \gamma\|_{2,\mu} + \|\hat{f} - f\|_{2,\mu} = o_p(n^{-1/2}) \]

where

\[ \hat{V}_{IPW,c}(\pi) = \frac{1}{n} \sum_{i=1}^{n} \hat{W}_i I\{\pi(X_i) = Z_i\} Y_i. \]

The asymptotic linearity of the latter estimator follows from the results in Robins et al. (1995).

3.2 Multiply robust sensitivity analyses

We propose a multiply-robust estimator of \( W \) and \( V_c(\pi) \), which leads to our proposed multiply-robust sensitivity analysis approach. The optimal strategy considered can be equivalently defined as the minimizer of

\[ E[Z\Delta(X)I\{\pi(X) \neq Z\}] \]
with respect to regime \( \pi \in \mathcal{D} \) where \( \Delta(X) = E[Y(1) - Y(-1) \mid X, S4] \) is a blip function.

By definition an optimal regime is the one that maps \( X \) to sign\( \{\Delta(X)\} \). That is when the assigned treatment is suboptimal \( Z\Delta(X) < 0 \). Hence, the objective function (7) identifies a regime that if not followed would result in the largest loss (i.e., the most negative value of the objective function). Using (1), we can show that the blip function can be written as a function of our sensitivity parameters as

\[
\Delta(X) = \frac{Q(1,1,X)}{\gamma(1,1,X)} - \frac{Q(-1,-1,X)}{\gamma(-1,-1,X)},
\]

where \( Q(A,Z,X) = E\{Yw_\alpha(A,Z,Y)\mid A,Z,X\} \). The parameter \( \Delta(X) \) is not pathwise differentiable, which makes the construction of a multiply robust estimator challenging. However, the expected value of \( \Delta(X) \) with respect to the distribution of \( X \) (i.e., \( \psi = E\{\Delta(X) \mid S4\} \)) is pathwise differentiable. Hence the canonical gradient of \( \psi \) exists in nonparametric models, which is also the corresponding efficient influence function (Bickel et al., 1998). This is helpful because an intuitive candidate for the unknown function \( \Delta(X) \) would be a component of the efficient influence function for \( \psi \).

**Theorem 2** Under a non-parametric model, the efficient influence function for \( \psi = E\{\Delta(X) \mid S4\} \) is given by

\[
\phi_\psi(O,Q,\gamma,\delta,f) = \frac{ZA(A+Z)}{2\gamma(A,Z,X)f(A,Z|X)} \left\{ Yw_\alpha(A,Z,Y) - Q(A,Z,X) \right. \\
- \left. \delta(A,Z,X)[w_\alpha(A,Z,Y) - \gamma(A,Z,X)] \right\} + \Delta(X) - \psi,
\]

where \( \delta(A,Z,X) = \frac{Q(A,Z,X)}{\gamma(A,Z,X)} \).

In the supplementary material we show that \( E\{\phi_\psi(O,Q,\gamma,\delta,f)\} = 0 \) in the union of
the nuisance models $\mathcal{M}_1$ and $\mathcal{M}_2$ where

$$
\mathcal{M}_1 : \text{Models for } Q(A, Z, X), \gamma(A, Z, X) \text{ are correctly specified}
$$

$$
\mathcal{M}_2 : \text{Models for } f(A, Z|X), \gamma(A, Z, X) \text{ are correctly specified}
$$

Let $\phi(O, Q, \gamma, \delta, f) = \phi_\psi(O, Q, \gamma, \delta, f) + \psi$. Corollary 1 shows that the estimator $\hat{\psi} = P_n \phi(O, \hat{Q}, \hat{\gamma}, \hat{\delta}, \hat{f})$ is asymptotically linear with influence function $\phi_\psi(O, Q, \gamma, \delta, f)$ when the nuisance parameter estimators satisfy the following rate assumption.

**Assumption 6** *(Accuracy of the nuisance models)* For any $z$, $(\|f - \hat{f}\|_{2,\mu} + \|\gamma - \hat{\gamma}\|_{2,\mu})(\|\Delta - \hat{\Delta}\|_{2,\mu} + \|\gamma - \hat{\gamma}\|_{2,\mu} + \|Q - \hat{Q}\|_{2,\mu}) = o_p(n^{-1/2}).$

**Corollary 1** Let $\hat{\psi} = P_n \phi(O, \hat{Q}, \hat{\gamma}, \hat{\delta}, \hat{f})$ and $\psi_0 = P_0 \phi(O, Q, \gamma, \delta, f)$. Under Assumptions 1 - 5, we have

$$
\hat{\psi} - \psi_0 = P_n \phi_\psi(O, Q, \gamma, \delta, f) + R(\mathcal{M}_1, \mathcal{M}_2),
$$

where the remainder term $R(\mathcal{M}_1, \mathcal{M}_2)$ is given in Section D of the Supplementary Material. Moreover, under Assumption 6, $R(\mathcal{M}_1, \mathcal{M}_2) = o_p(n^{-1/2})$ which implies the asymptotic linearity of the estimator $\hat{\psi}$.

The result of Theorem 2 motivates the use of a multiply robust weight function

$$
W_{mr} \equiv W_{mr}(Q, \gamma, \delta, f) = \frac{A(A + Z)}{2\gamma(A, Z, X)f(A, Z|X)} \left\{ Yw_{\alpha z}(A, Z, Y) - Q(A, Z, X) ight. \\
- \delta(A, Z, X)[w_{\alpha z}(A, Z, Y) - \gamma(A, Z, X)] \left\} + Z\Delta(X).
$$

Let $W_{mr}^* \equiv W_{mr}^*(Q^*, \gamma^*, \delta^*, f^*)$. Then, the multiply robust objective function $E[W_{mr}^* I(D(L) \neq Z)]$ maximizes the value function when at least one of nuisance modeling conditions $\mathcal{M}_1$ and $\mathcal{M}_2$ holds.
Let \( \hat{W}_{mr} \equiv W_{mr}(\hat{Q}, \hat{\gamma}, \hat{\delta}, \hat{f}) \). We define our multiply robust optimal strategy estimator as 
\[
\hat{\pi}_{mr} = \text{sign}(\hat{g}_{mr}) \quad \text{where} \quad \hat{g}_{mr} = \arg\min_{g} \frac{1}{n} \sum_{i=1}^{n} |\hat{W}_{mr,i}| \phi\{\text{sign}(\hat{W}_{mr,i})Z_i g(X_i)\} + \frac{1}{2} \|g\|^2.
\]

Treatment strategies are often quantified by the value function. Theorem 3 proposes a multiply robust estimator of a compliers' value function with an influence function \( \xi_V(O,Q,\gamma,\kappa,f_A,f_Z) \). The results shows that the corresponding estimator is asymptotically linear and 
\[
E\{\xi_V(O,Q,\gamma,\kappa,f_A,f_Z)\} = 0 \text{ in the union of the nuisance models } \mathcal{M}_1^\dagger, \mathcal{M}_2^\dagger, \text{ and } \mathcal{M}_3^\dagger. \]

The nuisance models are defined as
\[
\begin{align*}
\mathcal{M}_1^\dagger & : f(A|Z,X), \gamma(A,Z,X), f(Z|X) \text{ are correctly specified.} \\
\mathcal{M}_2^\dagger & : f(A|Z,X), \gamma(A,Z,X), \kappa(Z,X) \text{ are correctly specified.} \\
\mathcal{M}_3^\dagger & : \gamma(A,Z,X), f(Z|X), Q(A,Z,X) \text{ are correctly specified.}
\end{align*}
\]

where \( \kappa(Z,X) = E\{\frac{A(Z+A)Yw_{oZ}(A,Z,Y)}{2\gamma(A,Z,X)f_A(A|Z,X)}|Z,X\} \).

**Assumption 7** *(Accuracy of the nuisance models for value function)* Let \( f_A \equiv f(A|Z,X) \) and \( f_Z \equiv f(Z|X) \). For any \( z \),
\[
(\|f_Z - \hat{f}_Z\|_2 + \|f_A - \hat{f}_A\|_2 + \|\gamma - \hat{\gamma}\|_2 + \|\gamma - \hat{\gamma}\|_2 + (\|f_A - \hat{f}_A\|_2 + \|\gamma - \hat{\gamma}\|_2)\|Q - \hat{Q}\|_2 \\
+ \|\gamma - \hat{\gamma}\|_2\|\delta - \hat{\delta}\|_2 + \|\kappa - \hat{\kappa}\|_2\|f_Z - \hat{f}_Z\|_2 = o_p(n^{-1/2}).
\]

**Theorem 3** Under Assumptions 1 - 5 and 7, the estimator \( \hat{V}_{mr}^c(\pi) = P_n\xi(O,Q,\gamma,\kappa,f_A,f_Z) \) is an asymptotic linear estimator of \( V_{mr}^c(\pi) \) such that
\[
\hat{V}_{mr}^c(\pi) - V_{mr}^c(\pi) = P_n\xi_V(O,Q,\gamma,\kappa,f_A,f_Z) + o_p(n^{-1/2}),
\]
where \( \xi_V(O, Q, \gamma, \kappa, f_A, f_Z) = \xi(O, Q, \gamma, \kappa, f_A, f_Z) - V^c(\pi) \) with

\[
\xi(O, Q, \gamma, \kappa, f_A, f_Z) = \frac{A(Z + A)I\{\pi(X) = Z\}Yw_{az}(A, Z, Y)}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \\
- \left\{ \frac{I\{\pi(X) = Z\}}{f(Z|X)}E\left[ \frac{A(Z + A)Yw_{az}(A, Z, Y)}{2\gamma(A, Z, X)f(A|Z, X)} | Z, X \right] \right\} \\
- \sum_z I\{\pi(X) = z\}E\left[ \frac{A(A + z)Yw_{az}(A, z, Y)}{2\gamma(A, z, X)f(A|z, X)} | Z = z, X \right] \\
- \left\{ \frac{A(Z + A)I\{\pi(X) = Z\}\delta(A, Z, X)}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} [w_{az}(A, Z, Y) - \gamma(A, Z, X)] \right\} \\
- \left\{ \frac{A(Z + A)I\{\pi(X) = Z\}E[Yw_{az}(A, Z, Y)|A, Z, X]}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \\
- \sum_a \frac{a(A + Z)I\{\pi(X) = Z\}E[Yw_{az}(a, Z, Y)|A = a, Z, X]}{2\gamma(a, Z, X)f(Z|X)} \right\}.
\]

The estimator \( \hat{V}_{mr}^c(\pi) \) is consistent under \( \mathcal{M}_1^\dagger \cup \mathcal{M}_2^\dagger \cup \mathcal{M}_3^\dagger \). Furthermore, \( \hat{V}^c(\pi) \) is semi-parametric locally efficient in \( \mathcal{M}_{\text{union}} \) at the intersection sub-model \( \mathcal{M}_{\text{int}} = \mathcal{M}_1^\dagger \cap \mathcal{M}_2^\dagger \cap \mathcal{M}_3^\dagger \).

4 Simulation Studies

4.1 Scenarios and competing methods

We conduct simulation studies to demonstrate the performance of our methods. First, we compare our proposed inverse probability weighted (IPW) and multiply robust estimators (MR) to the outcome weighted learning estimator (OWL) of Zhao et al. (2012) and the IV-based estimator (IVT) of Cui & Tchetgen Tchetgen (2021). In our simulations, we violate the no unmeasured confounders assumption and allow the levels of the endogenous variable (i.e., compliance) to be greater than the IV, thereby the latter two methods are expected to fail. Second, we will assess the robustness of our proposed estimators presented in Theorems 1 and 3 to the misspecification of nuisance parameters. In this section, we
refer to the value function estimators in equation (6) and Theorem 3 as the IPW the MR
estimators, respectively.

Our simulation studies consist of two main sections. In the first section, we assume that
the sensitivity parameters $\alpha$ are known to assess the robustness of our estimators to certain
misspecifications. In the second section, the true values of those sensitivity parameters are
unknown. Throughout, we consider the following scenarios to demonstrate the robustness
properties of the classification weight function $W_{mr}$:

1. All nuisance parameters are correctly specified ($\mathcal{M}_1 \cup \mathcal{M}_2$).

2. $f(A, Z|X)$ is incorrectly specified ($\mathcal{M}_1$).

3. $Q(A, Z, X)$ is incorrectly specified ($\mathcal{M}_2$).

Moreover, to examine the robustness of the multiply robust value function estimator,
we consider the following scenarios:

1. All nuisance parameters are correctly specified ($\mathcal{M}_1^\dagger \cup \mathcal{M}_2^\dagger \cup \mathcal{M}_3^\dagger$).

2. $f(Z|X)$ is incorrectly specified ($\mathcal{M}_2^\dagger$).

3. $f(A|Z, X)$ is incorrectly specified ($\mathcal{M}_3^\dagger$).

4. $Q(A, Z, X)$ is incorrectly specified ($\mathcal{M}_1^\dagger$).

4.2 Generative models

We set the true values of the sensitivity parameters ($\alpha_{-1}, \alpha_{+1}$) to be (-0.5,0.5) and (0.5,-0.5)
for the first and second section of the simulation, respectively. We generate the covariates
$X_1$ and $X_2$ from the uniform distributions $(-1,1)^2$ and $Z$ is generated from a Bernoulli
distribution with \( P(Z = 1 \mid X_1, X_2) = \frac{\exp(0.3 - 2X_1 + 2X_2)}{1 + \exp(0.3 - 2X_1 + 2X_2)} \). The unmeasured confounder, \( U \) is drawn from the bridge distribution with density \( g_b(u) = \frac{1}{2\pi \cosh(0.5u) + \cos(0.5\pi)}; \(-\infty < u < \infty\) \) (Wang & Louis, 2003). We generate the proportion strata as following

\[
\begin{align*}
\log \left\{ \frac{P(PS = S1 \mid X, Z, U)}{P(PS = S3 \mid X, Z, U)} \right\} &= 0.5X_1 + 0.5Z + U \\
\log \left\{ \frac{P(PS = S2 \mid X, Z, U)}{P(PS = S3 \mid X, Z, U)} \right\} &= -0.5X_1 + 0.5Z + U \\
\log \left\{ \frac{P(PS = S4 \mid X, Z, U)}{P(PS = S3 \mid X, Z, U)} \right\} &= -0.5X_1 + 0.5Z - U \\
\log \left\{ \frac{P(PS = S5 \mid X, Z, U)}{P(PS = S3 \mid X, Z, U)} \right\} &= 0.5X_1 + 0.5Z - U \\
\log \left\{ \frac{P(PS = S6 \mid X, Z, U)}{P(PS = S3 \mid X, Z, U)} \right\} &= 0.5X_1 + 0.5Z - U.
\end{align*}
\]

The compliance level \( A \) is then determined based on the subject’s treatment and type of principal strata. The outcome of individuals with \( A = Z = -1 \) and \( A = Z = 1 \) are generated from the normal distribution \( f_{-1}(y \mid X_1, X_2) = N(1 + 2X_1 + 2X_2, 0.5^2) \) and \( f_{+1}(y \mid X_1, X_2) = N(1, 0.5^2) \). From the distribution \( f_{-1}(y) \) and \( f_{+1}(y) \) and the pre-specified value of \( \alpha \)'s, we generate the compliers outcome using rejection sampling (see Section G of the supplementary material). The outcome of individuals with \( (Z = -1, A = 1), \ (Z = 1, A = -1), \ (Z = 1, A = 0), \ (Z = -1, A = 0) \) are generated from \( N(3 + X_1 + X_2, 0.5^2) \), \( N(-1 + X_1 + X_2, 0.5^2) \), \( N(5, 0.1^2) \), \( N(-5, 0.1^2) \), respectively. We generate 500 datasets with sample size 500 using our generative model. Our generative model results in roughly 240 compliers in each data set.

### 4.3 Nuisance parameters approximation and estimation

Our sensitivity analyses rely on correctly specified \( \gamma(a, z, x) \). However, it cannot be directly inferred from the generative model. To overcome this issue, we approximate \( \gamma(a, z, x) \approx \)
Table 2: Simulation Result: Mean (sd) of correct classification rate. The sensitivity parameter are \((\alpha_{-1}, \alpha_{+1}) = (0.5,0.5)\)

| Case                              | OWL       | IVT       | IPW       | MR        |
|-----------------------------------|-----------|-----------|-----------|-----------|
| 1 All correctly specified         | 0.51 (0.03) | 0.73 (0.07) | 0.88 (0.04) | 0.95 (0.02) |
| 2 \(f(A,Z|X)\) misspecified      | 0.51 (0.03) | 0.73 (0.06) | 0.78 (0.04) | 0.97 (0.02) |
| 3 \(Q(A,Z,X)\) misspecified      | 0.51 (0.03) | 0.72 (0.07) | 0.88 (0.05) | 0.88 (0.05) |

\[
\frac{1}{5000} \sum_{j=1}^{5000} y_{jz}w_{\alpha_z}(a, z, y_{jz}) \text{ where } y_{jz} \text{ is a random sample from } f_{-1}(y|x_1, x_2) \text{ when } z = -1, \\
\text{and } f_{+1}(y|x_1, x_2) \text{ otherwise.}
\]

In our simulations, to estimate \(f_z(y|x_1, x_2)\), we assume a normal distribution for the outcome \(Y\) and estimate the corresponding means and variances using linear regressions. Alternatively, to gain robustness, one can estimate the conditional cumulative function \(Y\) given \((A = z, Z = z, X = x)\) (i.e., \(F_z(y|x_1, x_2)\)) using a single-index model (Chiang & Huang, 2012). We estimate the propensity score \(f(Z|X)\) using logistic regression models and \(f(A|Z,X)\) using multinomial logistic regression. To estimate the value function, we will use the cross-fitted Highly Adaptive Lasso (HAL) (Benkeser & Van Der Laan, 2016) to estimate \(\kappa(Z, X)\). The misspecified nuisance models are fitted similarly where the design matrix \(X\) is replaced by an intercept term only (i.e., ignoring \(X_1\) and \(X_2\)).

4.4 Known \(\alpha\’s\)

Tables 2 and 3 report the correct classification rate of the estimated optimal rules and their corresponding Monte Carlo approximated value function, respectively. The latter is obtained by generating 500 datasets of sizes 500 according to the estimated rule and averaging the resulting outcome. As expected, because of the presence of unmeasured confounders, OWL method fails. The correct classification rate for OWL is 51% resulting in the lowest value function among the methods considered in all three scenarios. The IVT method results in the correct classification rates of roughly 73% and the value function of roughly 1.4. In contrast, the proposed IPW estimator produces correct classification
Table 3: Simulation Result: Mean (sd) of value functions. The true optimal value function is 1.68. The sensitivity parameter are \((\alpha_{-1}, \alpha_{+1}) = (0.5,0.5)\)

| Case          | OWL  | IVT  | IPW  | MR   |
|---------------|------|------|------|------|
| 1 All correctly specified | 1.05 (0.00) | 1.49 (0.12) | 1.64 (0.06) | 1.67 (0.06) |
| 2 \(f(A, Z|X)\) misspecified | 1.05 (0.01) | 1.49 (0.10) | 1.55 (0.07) | 1.68 (0.06) |
| 3 \(Q(A, Z, X)\) misspecified | 1.05 (0.00) | 1.49 (0.12) | 1.65 (0.07) | 1.64 (0.07) |

rates of 88% and the value function estimators are around 1.64 when all the nuisance parameters are correctly specified. Notably, even when the nuisance parameter, \(f(A, Z|X)\), is misspecified, the IPW estimator still outperforms the existing methods. Importantly, the proposed multiply robust estimator shows the best performance among the methods considered. Specifically, the corresponding rule matches the true rule 95% of the times when all the nuisance parameters are correctly specified. The accuracy of the rules estimated using multiply robust method drops slightly to 88% when \(Q(A, Z, X)\) is misspecified.

Table 4 demonstrates the performance of our proposed value function estimators in equation (6) and Theorem 3. In case 1, when all nuisance parameters are correctly specified, the estimated value functions using IPW and MR estimators are relatively close to the true value. When a nuisance parameter is misspecified (i.e., cases 2-4), the estimated value of the MR estimator closely matches the corresponding true value. However, the IPW estimator overestimates the true value function when either \(f(Z|X)\) or \(f(A|Z, X)\) are misspecified. The misspecification of \(Q(A, Z, X)\) does not impact the IPW estimator as the estimator does not depend on it. As expected, the standard error of the estimated value function using MR is uniformly smaller than those obtained by IPW.
Table 4: Simulation Result: Comparing the performance of the proposed value function estimators. The sensitivity parameter are $(\alpha_{-1}, \alpha_{+1}) = (0.5,0.5)$

| Case | True | IPW | MR |
|------|------|-----|----|
| 1 All correctly specified | 1.65 | 1.65 (0.14) | 1.62 (0.09) |
| 2 $f(Z|X)$ misspecified | 1.52 | 1.92 (0.12) | 1.48 (0.07) |
| 3 $f(A|Z,X)$ misspecified | 1.64 | 2.55 (0.31) | 1.60 (0.14) |
| 4 $Q(A,Z,X)$ misspecified | 1.64 | 1.66 (0.13) | 1.62 (0.10) |

Figure 1: Sensitivity analysis of the proposed method (IPW): Sensitivity parameters are $(\alpha_{-1}, \alpha_{+1})$. The darker area indicates a higher correct classification rate/value function. The true vector of sensitivity parameters are $(\alpha_{-1}, \alpha_{+1}) = (0.5,-0.5)$.

4.5 Unknown $\alpha$’s

We now consider the case that the true parameters $(\alpha_{-1}, \alpha_{+1})$ are unknown. Hence, we conduct a sensitivity analysis to determine our methods’ performance under the misspecification of $\alpha_{+1}$’s. In this simulation, the true set of parameters $(\alpha_{-1}, \alpha_{+1})$ is (0.5,-0.5).

With this set of parameters, the correct classification rate obtained using the IVT method is 68.96% with the corresponding value function of 1.37. Figure 1 displays the performance of the IPW estimators. The heat map in the left column presents correct classification rates and value functions when the $\alpha$’s are varied from their true value. For most of the specified grids, the IPW method outperforms the IVT. However, when $\alpha_{-1} < 0$, and $\alpha_{+1} > -0.5$,
Figure 2: Sensitivity analysis of the multiply robust method when all nuisance parameters are correctly specified: Sensitivity parameters are \((\alpha_{-1}, \alpha_{+1})\). The darker area indicates a higher correct classification rate/value function. The true vector of sensitivity parameters are \((\alpha_{-1}, \alpha_{+1}) = (0.5, -0.5)\).

The classification rate drops to nearly 0.5 with the corresponding value of nearly 1.0. In contrast, the MR estimator with correctly specified nuisance parameters is less sensitive to the deviation from the true \(\alpha\)'s and remains superior to IVT for substantially larger area of the grid points (Figure 2). When some of the nuisance parameters are misspecified, the correct classification rate drops slightly but still outperforms both the IVT and the IPW approach (see Figures 5 - 7 in the Supplementary Material).

Our simulation results highlight several important points. First, the MR method is more robust to the misspecification of \(\alpha\)'s compared with the IPW. Second, any knowledge about the sign of \(\alpha\)'s can improve the interpretability of the results, particularly for the IPW-based approach. Third, the existing IV-based methods (e.g., IVT) may result in severely suboptimal rules.
5 Application

We implement our approach to the Adaptive Treatment for Alcohol and Cocaine Dependence (ENGAGE) study (McKay et al., 2015). This study recruited 500 individuals to enter the intensive outpatient program (IOP) consisting of attending three sessions per week for two weeks. Those who failed to attend at least two sessions in Week 2 were eligible to get randomized to one of two telephone motivational interviewing (MI) based interventions \((n = 189)\). One intervention was to encourage patients to engage in IOP (MI-IOP), and the other included a choice of IOP or three other treatment options (MI-PC). The binary outcome of interest is whether an individual has any cocaine use at the end of the eight-week program (i.e., \(Y = 0\) indicates cocaine use and \(Y = 1\) indicates no cocaine use).

Our analyses include the following baseline variables: race, sex, education, smoking status, treatment readiness, and general health. We include all of these covariates in the weighted SVM models for treatment classification and models for estimating nuisance parameters. The compliance level is decided based on the number of MI sessions attended by an individual. If a person attends more than the median of the attended sessions of other patients, we classify that person as a complier. Otherwise, we will classify that person as a non-complier. The instrument variable \(Z\) is the assigned intervention, either MI-IOP \((Z = -1)\) or MI-PC \((Z = 1)\).

In order to compare our method to existing methods, we randomly split the data set into a train and a test sets in a 60:40 ratio. We repeat this process one hundred times, then fit the four models IPW, MR, IVT, and OWL on the resulting data sets. Let \(\hat{\pi}_{ipw}, \hat{\pi}_{mr}, \hat{\pi}_{ivt}\) and \(\hat{\pi}_{owl}\) denote the estimated optimal policy of the corresponding methods.

We note that since treatments in the ENGAGE study is randomized, we know the true
This allows us to simplify our multiply robust estimator as follow

\[
\hat{\tilde{V}}_{mr}^c(\pi) = P_n\frac{A(Z + A)I\{\pi(X) = Z\}Yw_{\alpha z}(A, Z, Y)}{2\hat{\gamma}(A, Z, X)f(Z|X)f(A|Z, X)}
- \left[ \frac{A(A + Z)I\{\pi(X) = Z\}\hat{\delta}(A, Z, X)}{2\hat{\gamma}(A, Z, X)f(Z|X)f(A|Z, X)}\left\{w_{\alpha z}(A, Z, Y) - \hat{\gamma}(A, Z, X)\right\} \right]
- \left[ \frac{A(Z + A)I\{\pi(X) = Z\}\hat{Q}(A, Z, X)}{2\hat{\gamma}(A, Z, X)f(Z|X)f(A|Z, X)} - \sum_a^{\alpha} \frac{a(a + Z)I\{\pi(X) = Z\}\hat{Q}(a, Z, X)}{2\hat{\gamma}(a, Z, X)f(Z|X)} \right].
\]

We will use this modified multiply robust estimator to estimate the value function for each test set and obtain the estimated value function by taking the average of the value function over 100 test sets.

The key difference between \(\hat{\tilde{V}}_{mr}^c(\pi)\) and \(\hat{V}_{mr}^c(\pi)\) is that, in the former, the corresponding influence function is no longer orthogonal to the nuisance tangent space of \(f(Z|X)\). This choice is made for two reasons: (1) the treatment is randomized so from the consistency point of view, there is no need to be robust to possible misspecification of \(f(Z|X)\); (2) due to a relatively small sample size, the estimation of the complex functional \(\kappa(Z, X)\) may lead to a biased estimate of the value function. We acknowledge that estimating known nuisance parameters leads to efficiency gain (Hirano et al., 2003; Van der Laan & Robins, 2003) but in our case, we decided to sacrifice the slight efficiency gain in the favor of reducing bias (Liang & Yu, 2020).

The nuisance parameters \(f(Z|X)\) and \(f(A|Z, X)\) are fitted using the logistic regression and multinomial logistic regression, respectively, and \(Q(A, Z, X)\) is obtained using a random forest model. We approximate \(\gamma(a, z, x) = \frac{1}{5000} \sum_{i=1}^{5000} y^i_z w_{\alpha z}(a, z, y)\) where \(y^i_z\) is a sample from the Bernoulli distribution with the success probability of \(p_z(Y = 1|X)\) is estimated using a random forest model. Note that estimating \(f(Z|X)\) may lead to a slightly conservative estimate of the variance of \(\hat{\tilde{V}}_{mr}^c(\pi)\).
The parameter $\alpha_{-1}$ is the log odds coefficient of cocaine use in predicting the probability that a patient would take MI-PC (i.e., $A = 1$) if she were assigned to MI-PC (i.e., $Z = 1$) given that the patient would take MI-IOP (i.e., $A = -1$) if she were assigned to MI-IOP (i.e., $Z = -1$). The parameter $\alpha_{+1}$ is defined similarly. There is a clinical sense that individuals with more severe substance use (e.g., $Y = 0$) are less likely to comply with the intervention. We may focus on the plausible range of $\alpha_{-1} > 0$ and $\alpha_{+1} > 0$. Moreover, because, for any given patient with $Y = y$, the intensive outpatient program (IOP) intervention is more burdensome than the patient choice (PC) intervention option, we may consider $\alpha_{-1} > \alpha_{+1}$.

Let $d(\pi, \pi') = \hat{\hat{V}}_{mr}^c(\pi) - \hat{\hat{V}}_{mr}^c(\pi')$ denote the difference between the estimated multiply robust value function estimators under policies $\pi$ and $\pi'$. Figure 3 and 4 compare the performance of the proposed methods to IVT and OWL methods, respectively. In Figure 3, the left heat maps show the difference between the estimated value of the estimated optimal policy using the proposed (i.e., IPW and MR) and IVT estimators (i.e., $d(\hat{\pi}_{mr}, \hat{\pi}_{ivt})$ and $d(\hat{\pi}_{ipw}, \hat{\pi}_{ivt})$). The darker blue indicates the area of $\alpha_{+1}$'s where the magnitude of the difference is larger (i.e., the darker the better performance of our proposed methods). The right heat maps show whether the value functions of the constructed optimal regimes using our methods outperform the IVT method where the red color indicates $d(\hat{\pi}_{mr}, \hat{\pi}_{ivt}) < 0$ or $d(\hat{\pi}_{ipw}, \hat{\pi}_{ivt}) < 0$. Figure 4 shows the same information but for the OWL method. Table 5 shows the value functions and the standard errors of the proposed methods, IVT, and OWL for some selected $\alpha$’s. The MR method yields a higher value compared with both OWL and IVT methods in the quarter, where $\alpha_{+1} > 0$ and $\alpha_{-1} > 0$. 

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Figure 3: Sensitivity analysis of the difference the proposed methods (MR and IPW) and the IVT method. The sensitivity parameters are \((\alpha_{+1}, \alpha_{-1})\). The left panel shows the magnitude of the difference between the proposed and IVT methods (i.e., \(d(\hat{\pi}_{mr}, \hat{\pi}_{ivt})\) and \(d(\hat{\pi}_{ipw}, \hat{\pi}_{ivt})\)) and the darker color the better performance of our methods. The right panel shows the area on the grid in which our methods outperform the IVT method where the red color indicates \(d(\hat{\pi}_{mr}, \hat{\pi}_{ivt}) < 0\) or \(d(\hat{\pi}_{ipw}, \hat{\pi}_{ivt}) < 0\).
Figure 4: Sensitivity analysis of the difference the proposed methods (MR and IPW) and the OWL method. The sensitivity parameters are \((\alpha_+, \alpha_-)\). The left panel shows the magnitude of the difference between the proposed and OWL methods \((d(\hat{\pi}_{mr}, \hat{\pi}_{owl})\) and \(d(\hat{\pi}_{ipw}, \hat{\pi}_{owl})\)) and the darker color the better performance of our methods). The right panel shows the area on the grid in which our methods outperform the IVT method where the red color indicates \(d(\hat{\pi}_{mr}, \hat{\pi}_{owl}) < 0\) or \(d(\hat{\pi}_{ipw}, \hat{\pi}_{owl}) < 0\).
Table 5: Real Data Application: Mean (se) of estimated value functions.

| $\alpha = (\alpha_{-1}, \alpha_{+1})$ | IPW | MR  | IVT | OWL |
|--------------------------------------|-----|-----|-----|-----|
| 1 (-0.58, -0.58)                    | 0.21 (0.15) | 0.25 (0.15) | 0.26 (0.15) | 0.25 (0.15) |
| 2 (-0.58, 0.05)                     | 0.27 (0.15) | 0.30 (0.15) | 0.30 (0.15) | 0.29 (0.15) |
| 3 (-0.58, 0.58)                     | 0.34 (0.17) | 0.35 (0.17) | 0.33 (0.16) | 0.32 (0.16) |
| 4 (0.05, -0.58)                     | 0.44 (0.19) | 0.49 (0.19) | 0.43 (0.18) | 0.39 (0.18) |
| 5 (0.05, 0.05)                      | 0.44 (0.17) | 0.49 (0.17) | 0.47 (0.17) | 0.43 (0.16) |
| 6 (0.05, 0.58)                      | 0.46 (0.17) | 0.51 (0.17) | 0.49 (0.17) | 0.46 (0.16) |
| 7 (0.58, -0.58)                     | 0.60 (0.20) | 0.64 (0.20) | 0.53 (0.18) | 0.48 (0.18) |
| 8 (0.58, 0.05)                      | 0.60 (0.18) | 0.68 (0.18) | 0.57 (0.16) | 0.52 (0.16) |
| 9 (0.58, 0.58)                      | 0.58 (0.20) | 0.68 (0.20) | 0.60 (0.18) | 0.55 (0.17) |

SUPPLEMENTARY MATERIAL

A Proof of theorem 1

By definition, we have the value function

$$V_c(\pi) = E[I\{\pi(X) = 1\}Y(1) + I\{\pi(X = -1)\}Y(-1)|PS = S4]$$

$$= E\left[E[I\{\pi(X) = 1\}Y(1)|S4, X] + E[I\{\pi(X) = -1\}Y(-1)|S4, X]\right]$$

$$= E\left[I\{\pi(X) = 1\}E[Y|Z = 1, A = 1, S4, X] + I\{\pi(X) = -1\}E[Y|Z = -1, A = -1, S4, X]\right]$$

We can verify that

$$f(y|Z = 1, A = 1, X) = E\left[f(y|Z, A, X)\frac{A(A + Z)(A + 1)}{4f(Z, A|X)}\right]|X]$$

and

$$f(y|Z = -1, A = -1, X) = E\left[f(y|Z, A, X)\frac{A(A + Z)(1 - A)}{4f(Z, A|X)}\right]|X]$$

We have
\[ E[Y|Z = 1, A = 1, S4, X] = \int \frac{yw_{\alpha+1}(1,1,y)}{\gamma(1,1,x)} f(y|Z = 1, A = 1, X) dy \]

\[ = \frac{1}{\gamma(1,1,X)} \int yw_{\alpha+1}(1,1,y) E[f(y|Z, A, X)\frac{A(A + Z)(A + 1)}{4f(Z, A|X)} | X] dy \]

\[ = \frac{1}{\gamma(1,1,X)} \int \int yw_{\alpha+1}(1,1,y) \frac{a(a + z)(a + 1)}{4f(z, a|X)} f(y|z, a, X) f(z, a|X)d(z)a)dy \]

\[ = \frac{1}{\gamma(1,1,X)} E[\frac{A(A + Z)(A + 1)Yw_{\alpha+1}(1,1,Y)}{4f(A, Z|X)} | X] \]

\[ = E[\frac{A(A + Z)(A + 1)Yw_{\alpha+1}(1,1,Y)}{4f(A, Z|X)\gamma(1,1,X)} | X] \]

Similarly, we have

\[ E[Y|Z = -1, A = -1, S4, X] = E[\frac{A(A + Z)(1 - A)Yw_{\alpha-1}(-1,-1,Y)}{4f(A, Z|X)\gamma(-1,-1,X)} | X] \]

Hence,

\[ \nu_c(\pi) = E[E\left[I\{\pi(X) = 1\}A(A + Z)(1 + A)Yw_{\alpha+1}(1,1,Y)\right.\left.\frac{4f(A, Z|X)\gamma(1,1,X)}{4f(A, Z|X)\gamma(1,1,X)} | X\right]] \]

\[ + E[E\left[I\{\pi(X) = -1\}A(A + Z)(1 - A)Yw_{\alpha-1}(-1,-1,Y)\right.\left.\frac{4f(A, Z|X)\gamma(-1,-1,X)}{4f(A, Z|X)\gamma(-1,-1,X)} | X\right]] \]

\[ = E[E[I\{\pi(X) = Z\}A(A + Z)Yw_{\alpha_2}(A, Z, Y)\right.\left.\frac{\gamma(A, Z, X)f(A, Z|X)}{\gamma(A, Z, X)f(A, Z|X)} | X\right]] \]

\[ = E[E[I\{\pi(X) = Z\}A(A + Z)Yw_{\alpha_2}(A, Z, Y)\right.\left.\frac{\gamma(A, Z, X)f(A, Z|X)}{\gamma(A, Z, X)f(A, Z|X)} | X\right]] \]
B Robust Estimator of $\Delta(X)$

\[
\frac{\partial}{\partial t} \Delta|_{t=0} = \frac{\partial}{\partial t} \big|_{t=0} E_t \big[ E_t[Y|Z = 1, PS = 4, X] - E_t[Y|Z = -1, PS = 4, X] \big] = \frac{\partial}{\partial t} \big|_{t=0} \int \left[ \int \frac{yw_{a+1}(1, 1, y)}{\gamma_t(1, 1, X)} f_t(y|Z = 1, A = 1, X)dy \right. \\
\left. - \int \frac{yw_{a-1}(-1, -1, y)}{\gamma_t(-1, -1, X)} f_t(y|Z = -1, A = -1, X)dy \right] f_t(X) \\
= \frac{\partial}{\partial t} \big|_{t=0} E_t \left[ \int \frac{yw_{a+1}(1, 1, y)f_t(y|A = 1, Z = 1, X)dy}{\gamma_t(1, 1, X)} \\
- \frac{\int yw_{a-1}(-1, -1, y)f_t(y|A = -1, Z = -1, X)dy}{\gamma_t(-1, -1, X)} \right] \right] \\
\]

We let

\[
Q_t(1, 1, X) = \int yw_{a+1}(1, 1, y)f_t(y|A = 1, Z = 1, X)dy \\
Q_t(-1, -1, X) = \int yw_{a-1}(-1, -1, y)f_t(y|A = -1, Z = -1, X)dy \\
\]

We continue

\[
= \frac{\partial}{\partial t} \big|_{t=0} E_t \left[ \frac{Q_t(1, 1, X)}{\gamma_t(1, 1, X)} - \frac{Q_t(-1, -1, X)}{\gamma_t(-1, -1, X)} \right] \\
= \int \left( \frac{Q(1, 1, X)}{\gamma(1, 1, X)} - \frac{Q(-1, -1, X)}{\gamma(-1, -1, X)} \right) \frac{\partial}{\partial t} f_t(x)dx + \int f(x) \frac{\partial}{\partial t} \left( \frac{Q(1, 1, X)}{\gamma(1, 1, X)} - \frac{Q(-1, -1, X)}{\gamma(-1, -1, X)} \right) dx \\
= \int \left( \frac{Q(1, 1, X)}{\gamma(1, 1, X)} - \frac{Q(-1, -1, X)}{\gamma(-1, -1, X)} \right) \frac{\partial}{\partial t} \log f_t(x)f_t(x)dx + E \left[ \frac{\partial}{\partial t} \left( \frac{Q(1, 1, X)}{\gamma(1, 1, X)} - \frac{Q(-1, -1, X)}{\gamma(-1, -1, X)} \right) \right] \\
= E \left[ \frac{Q(1, 1, X)}{\gamma(1, 1, X)} - \frac{Q(-1, -1, X)}{\gamma(-1, -1, X)} \right] S(X) + E \left[ \frac{\partial}{\partial t} \left( \frac{Q(1, 1, X)}{\gamma(1, 1, X)} - \frac{Q(-1, -1, X)}{\gamma(-1, -1, X)} \right) \right] \\
= E \left[ \frac{Q(1, 1, X)}{\gamma(1, 1, X)} - \frac{Q(-1, -1, X)}{\gamma(-1, -1, X)} \right] S(\mathcal{O}) + E \left[ \frac{\partial}{\partial t} \left( \frac{Q(1, 1, X)}{\gamma(1, 1, X)} \right) \right] - E \left[ \frac{\partial}{\partial t} \left( \frac{Q(-1, -1, X)}{\gamma(-1, -1, X)} \right) \right] \\
= (I) + (II) + (III) \\
\]
B.1 Term (II)

\[(II) = E \left[ \frac{\gamma(1,1,X) \frac{\partial}{\partial t} Q_t(1,1,X) - Q(1,1,X) \frac{\partial}{\partial t} \gamma_t(1,1,X)}{[\gamma(1,1,X)]^2} \right] \]

We can verify that

\[ f(Y|Z=1, A = 1, X) = E \left[ f(Y|Z, A, X) \frac{A(A+Z)(A+1)}{4f(Z, A|X)} \right] \]

and

\[ f(Y|Z=-1, A = -1, X) = E \left[ f(Y|Z, A, X) \frac{A(A+Z)(1-A)}{4f(Z, A|X)} \right] \]

Now we have

\[
\frac{\partial}{\partial t} Q_t(1,1,X) = \int yw_{a+1}(1,1,y) \frac{\partial}{\partial t} f_t(y|A = 1, Z = 1, X) dy
\]

\[ = \int yw_{a+1}(1,1,y) E \left[ \frac{\partial}{\partial t} f_t(y|A, Z, X) \frac{A(A+Z)(A+1)}{4f(Z, A|X)} \right] dy \]

\[ = \int yw_{a+1}(1,1,y) E \left[ S(y|A, Z, X) f(y|Z, A, X) \frac{A(A+Z)(A+1)}{4f(Z, A|X)} \right] dy \]

\[ = \int \int yw_{a+1}(1,1,y) S(y|a, z, X) \frac{a(a+z)(a+1)}{4f(z|x)} f(y|z, a, X) f(z, a|X) dy d(az) \]

\[ = \int \int yw_{a+1}(1,1,y) (S(\mathcal{O}) - E[S(\mathcal{O})|a, z, X]) \frac{a(a+z)(a+1)}{4f(z, a|X)} f(y|z, a, X) f(z, a|X) dy d(az) \]

\[ = E \left[ \frac{Yw_{a+1}(1,1,Y) A(A+Z)(A+1)}{4f(Z|X)} S(\mathcal{O}) \right] X \]

\[ - \int Q^+(A, Z, X) \int S(\mathcal{O}) \frac{a(a+z)(a+1)}{4f(a, z|X)} f(y|z, a, X) f(z, a|X) dy d(az) \]

\[ = E \left[ \frac{Yw_{a+1}(1,1,Y) A(A+Z)(A+1)}{4f(A, Z|X)} S(\mathcal{O}) \right] X - E \left[ \frac{Q^+(A, Z, X) A(A+Z)(A+1)}{4f(A, Z|X)} S(\mathcal{O}) \right] X \]

\[ = E \left[ \frac{A(A+Z)(A+1)}{4f(Z, A|X)} [Yw_{a+1}(1,1,Y) - Q^+(A, Z, X)] S(\mathcal{O}) \right] X \]
where, for \( z = 1 \) or \(-1\),

\[
Q_z(A, Z, X) = \int y w_{az}(z, z, y) f(y|A, Z, X) dy = E[Y w_{az}(a, z, y)|A = Z = z, X] = Q(z, z, X)
\]

This is because \( w_{az}A, Z, Y = 0 \) for \( A \neq Z \).

Following the same step, we can calculate

\[
\frac{\partial}{\partial t} \gamma_t(1, 1, X) = \int w_{a+1}(1, 1, Y) \frac{\partial}{\partial t} f(y|A = 1, Z = 1, X) dy
\]

\[
= E\left[ \frac{A(A + Z)(A + 1)}{4f(Z, A|X)} [w_{a+1}(1, 1, Y) - \gamma(1, 1, X)] S(O|X) \right]
\]

Therefore, we have

\[
(II) = E\left[ \frac{\gamma(1, 1, X) \frac{\partial}{\partial t} Q_t(1, 1, X) - Q(1, 1, X) \frac{\partial}{\partial t} \gamma_t(1, 1, X)}{\gamma(1, 1, X)} \right]
\]

\[
= E\left[ \frac{\partial}{\partial t} Q_t(1, 1, X) \right] - E\left[ \frac{Q(1, 1, X) \frac{\partial}{\partial t} \gamma_t(1, 1, X)}{\gamma(1, 1, X)} \right]
\]

\[
= E\left[ \frac{A(A + Z)(A + 1)}{4\gamma(1, 1, X)f(A, Z|X)} [Y w_{a+1}(1, 1, Y) - Q(1, 1, X)] S(O) \right] - E\left[ \frac{Q(1, 1, X) A(A + Z)(A + 1)}{4\gamma(1, 1, X)f(A, Z|X)} [w_{a+1}(1, 1, Y) - \gamma(1, 1, X)] S(O) \right]
\]

\[
= E\left[ \frac{A(A + Z)(A + 1)}{\gamma(1, 1, X)4f(A, Z|X)} [Y w_{a+1}(1, 1, Y) - Q(1, 1, X)] S(O) \right] - E\left[ \frac{Q(1, 1, X) A(A + Z)(A + 1)}{4\gamma(1, 1, X)^2 f(A, Z|X)} [w_{a+1}(1, 1, Y) - \gamma(1, 1, X)] S(O) \right]
\]
B .2 Term (III)

\[(III) = E\left[ \frac{\gamma(-1, -1, X) \frac{\partial}{\partial t} Q_t(-1, -1, X) - Q(-1, -1, X) \frac{\partial}{\partial t} \gamma_t(-1, -1, X)}{[\gamma(-1, -1, X)]^2} \right] \]

Following the same argument in the previous part, we have

\[\frac{\partial}{\partial t} Q_t(-1, -1, X) = E\left[ \frac{A(A + Z)(1 - A)}{4f(A, Z|X)} [Yw_{\alpha-1}(-1, -1, Y) - Q(-1, -1, X)]S(O)|X \right] \]

\[\frac{\partial}{\partial t} \gamma_t(-1, -1, X) = E\left[ \frac{A(A + Z)(1 - A)}{4f(A, Z|X)} [w_{\alpha-1}(-1, -1, Y) - \gamma(-1, -1, X)]S(O)|X \right] \]

Hence, we have

\[(III) = E\left[ \frac{\gamma(-1, -1, X) \frac{\partial}{\partial t} Q_t(-1, -1, X) - Q(-1, -1, X) \frac{\partial}{\partial t} \gamma_t(-1, -1, X)}{[\gamma(-1, -1, X)]^2} \right] \]

\[= E\left[ \frac{A(A + Z)(1 - A)}{4\gamma(-1, -1, X)f(Z, A|X)} [Yw_{\alpha-1}(-1, -1, Y) - Q(-1, -1, X)]S(O) \right] \]

\[ - E\left[ \frac{Q(-1, -1, X)A(A + Z)(1 - A)}{4\gamma(-1, -1, X)^2 f(A, Z|X)} [w_{\alpha-1}(-1, -1, Y) - \gamma(-1, -1, X)]S(O) \right] \]

\[= E\left[ \frac{A(A + Z)(1 - A)}{4\gamma(-1, -1, X)f(A, Z|X)} [Yw_{\alpha-1}(-1, -1, Y) - Q(-1, -1, X)]S(O) \right] \]

\[ - E\left[ \frac{Q(-1, -1, X)A(A + Z)(1 - A)}{4\gamma(-1, -1, X)^2 f(A, Z|X)} [w_{\alpha-1}(-1, -1, Y) - \gamma(-1, -1, X)]S(O) \right] \]

The efficient curve for \( \Delta(X) \) is
\[ EIF_{\Delta} = \left( \frac{Q(1, 1, X)}{\gamma(1, 1, X)} - \frac{Q(-1, -1, X)}{\gamma(-1, -1, X)} \right) \]
\[ + \frac{A(A + Z)(A + 1)}{4f(Z, A|X)} \left( \frac{1}{\gamma(1, 1, X)} \left[ Yw_{\alpha+1}(1, 1, Y) - Q(1, 1, X) \right] \right. \]
\[ - \frac{Q(1, 1, X)}{\gamma(1, 1, X)^2} \left[ w_{\alpha+1}(1, 1, Y) - \gamma(1, 1, X) \right) \]
\[ - \frac{A(A + Z)(1 - A)}{4f(A, Z|X)} \left( \frac{1}{\gamma(-1, -1, X)} \left[ Yw_{\alpha-1}(-1, -1, Y) - Q(-1, -1, X) \right] \right. \]
\[ - \frac{Q(-1, -1, X)}{\gamma(-1, -1, X)^2} \left[ w_{\alpha-1}(-1, -1, Y) - \gamma(-1, -1, X) \right] \]
\[ (9) \]

We let

\[ \delta(1, 1, X) = \frac{Q(1, 1, X)}{\gamma(1, 1, X)} \]
\[ \delta(-1, -1, X) = \frac{Q(-1, -1, X)}{\gamma(-1, -1, X)} \]

Then (9) becomes
\[ EIF_\Delta = \frac{A(A + Z)(A + 1)}{4f(A, Z|X)\gamma(1, 1, X)} \left( Yw_{\alpha+1}(1, 1, Y) - Q(1, 1, X) - \delta(1, 1, X)\{w_{\alpha+1}(1, 1, Y) - \gamma(1, 1, X)\} \right) \]
\[ - \frac{A(A + Z)(1 - A)}{4f(A, Z|X)\gamma(-1, -1, X)} \left( Yw_{\alpha-1}(-1, -1, Y) - Q(-1, -1, X) \right. \]
\[ - \left. \delta(-1, -1, X)\{w_{\alpha-1}(-1, -1, Y) - \gamma(-1, -1, X)\} \right) \]
\[ + \delta(1, 1, X) - \delta(-1, -1, X) \]
\[ = \frac{ZA(A + Z)}{2f(A, Z|X)\gamma(A, Z, X)} \left( Yw_{\alpha z}A, Z, Y) - Q(A, Z, X) - \delta(A, Z, X)\{w_{\alpha z}A, Z, Y) - \gamma(A, Z, X)\} \right) \]
\[ + \delta(1, 1, X) - \delta(-1, -1, X) \]

The multiply robust estimators for \( \Delta(X) \) is

\[ \Delta_{mr} = \delta(1, 1, X; \hat{\theta}) - \delta(-1, -1, X; \hat{\theta}) \]
\[ + \frac{A(A + Z)(A + 1)}{4f(A, Z|X, \hat{\eta})\gamma(1, 1, X; \hat{\alpha}_1)} \left( Yw_{\alpha+1}(1, 1, Y) - Q(1, 1, X; \hat{\beta}) \right. \]
\[ - \left. \delta(1, 1, X; \hat{\theta})\{w_{\alpha+1}(1, 1, Y) - \gamma(1, 1, X; \hat{\alpha})\} \right) \]
\[ - \frac{A(A + Z)(1 - A)}{4f(A, Z|X; \hat{\eta})\gamma(-1, -1, X; \hat{\alpha}_2)} \left( Yw_{\alpha-1}(-1, -1, Y) - Q(-1, -1, X; \hat{\beta}_2) \right. \]
\[ - \left. \delta(-1, -1, X, \hat{\theta})\{w_{\alpha-1}(-1, -1, Y) - \gamma(-1, -1, x; \hat{\alpha}_2)\} \right) \]

We propose two models

\[ M_1 : Q(A, Z, X), \gamma(A, Z, X) \text{ for any } z, \text{ are correctly specified} \]
\[ M_2 : f(A, Z|X), \gamma(A, Z, X) \text{ for any } z, \text{ are correctly specified} \]

The asterisk (*) symbol denotes a model being incorrectly specified. Suppose that \( M_1 \)
holds, we have

\[ E[\Delta_{mr}] = E[\delta(1, 1, X)] - E[\delta(-1, -1, X)] \\
+ E \frac{A(A + Z)(A + 1)}{4f^*(A, Z|X)\gamma(1, 1, X)} \left( E[Y_{w_{\alpha+1}}(1, 1, Y)|A, Z, X] - Q(1, 1, X) \\
- \delta(1, 1, X)\{E[w_{\alpha+1}(1, 1, Y)|A, Z, X] - \gamma(1, 1, X)\} \right) \\
- E \frac{A(A + Z)(1 - A)}{4f^*(A, Z|X)\gamma(-1, -1, X)} \left( E[Y_{w_{\alpha-1}}(-1, -1, Y)|A, Z, X] - Q(-1, -1, X) \\
- \delta(-1, -1, X)\{E[w_{\alpha-1}(-1, -1, Y)|A, Z, X] - \gamma(-1, -1, X)\} \right) \\
= E[\delta(1, 1, X) - \delta(-1, -1, X)] = \Delta(X) \]

Suppose that only \( M_2 \) holds, then we have.

\[ E[\Delta_{mr}] = E[\delta^*(1, 1, X)] - E[\delta^*(-1, -1, X)] \\
+ E \frac{A(A + Z)(A + 1)}{4f^*(A, Z|X)\gamma(1, 1, X)} \left( E[Y_{w_{\alpha+1}}(1, 1, Y)|A, Z, X] - Q^*(1, 1, X) \\
- \delta^*(1, 1, X)\{E[w_{\alpha+1}(1, 1, Y)|A, Z, X] - \gamma(1, 1, X)\} \right) \\
- E \frac{A(A + Z)(1 - A)}{4f^*(A, Z|X)\gamma(-1, -1, X)} \left( E[Y_{w_{\alpha-1}}(-1, -1, Y)|A, Z, X] - Q^{*-}(X) \\
- \delta^{*-}(X)\{E[w_{\alpha-1}(-1, -1, Y)|A, Z, X] - \gamma(-1, -1, X)\} \right) \\
= E[\delta^*(1, 1, X)] - E \left[ \frac{A(A + Z)(A + 1)Q^*(1, 1, X)}{4\gamma(-1, -1, X)f(A, Z|X)} \right] \\
+ E \left[ \frac{A(A + Z)(A + 1)Y_{w_{\alpha+1}}(1, 1, Y)}{4\gamma(-1, -1, X)f(A, Z|X)} \right] \\
- E[\delta^*(-1, -1, X)] + E \left[ \frac{A(A + Z)(1 - A)Q^{*-}(-1, -1, X)}{4\gamma(-1, -1, X)f(A, Z|X)} \right] \\
- E \left[ \frac{A(A + Z)(1 - A)Y_{w_{\alpha-1}}(-1, -1, Y)}{4\gamma(-1, -1, X)f(A, Z|X)} \right] \\
= E \left[ \frac{A(A + Z)(A + 1)Y_{w_{\alpha+1}}(1, 1, Y)}{4\gamma(-1, -1, X)f(A, Z|X)} - \frac{A(A + Z)(1 - A)Y_{w_{\alpha-1}}(-1, -1, Y)}{4\gamma(-1, -1, X)f(A, Z|X)} \right] \\
= E[\delta(1, 1, X) - \delta(-1, -1, X)] = \Delta(X) \]
C Influence Function of Value function

We define $\frac{\partial}{\partial t} = 0$. The value function is

$$\mathcal{V}(\mathcal{D}) = E\left[ \frac{A(Z + A)Yw_{\alpha_2}(A, Z, Y)I\{\pi(X) = Z\}}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \right]$$

Hence, we have

$$\frac{\partial}{\partial t} \mathcal{V}_t(\mathcal{D}) = \frac{\partial}{\partial t} E_t\left[ \frac{A(Z + A)Yw_{\alpha_2}(A, Z, Y)I\{\pi(X) = Z\}}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \right]$$

$$= \int \frac{A(Z + A)Yw_{\alpha_2}(Z, A, Y)I\{\pi(X) = Z\}}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \frac{\partial}{\partial t} f_t(x) dx$$

$$+ \int \frac{\partial}{\partial t} \left( \frac{A(Z + A)Yw_{\alpha_2}(A, Z, Y)I\{\pi(X) = Z\}}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \right) f(x) dx$$

$$= E\left[ \frac{A(Z + A)Yw_{\alpha_2}(A, Z, Y)I\{\pi(X) = Z\}}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} S(O) \right]$$

$$+ E\left[ \frac{\partial}{\partial t} \left( \frac{A(Z + A)Yw_{\alpha_2}(A, Z, Y)I\{\pi(X) = Z\}}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \right) f_t(Z|X) \right]$$

$$= E\left[ \frac{A(Z + A)Yw_{\alpha_2}(A, Z, Y)I\{\pi(X) = Z\}}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} S(O) \right]$$

$$- E\left[ \frac{A(Z + A)Yw_{\alpha_2}(A, Z, Y)I\{\pi(X) = Z\}}{2\gamma(A, Z, X)^2f(Z|X)^2f(A|Z, X)^2} \left\{ \gamma(A, Z, X)f(A | Z, X) \frac{\partial}{\partial t} f_t(Z|X) \right. \right.$$

$$+ f_t(Z|X) \frac{\partial}{\partial t} \gamma_t(A, Z, X) + \gamma(A, Z, X) f(Z|X) \frac{\partial}{\partial t} f_t(A | Z, X) \right\}$$

$$= (I) - (II) - (III) - (IV)$$

We have
\[(II) = E \left\{ \frac{A(Z + A) Y w_{\alpha z} (A, Z, Y) I \{\pi(X) = Z\} \partial}{2\gamma(A, Z, X) f(Z|X)^2 f(A|Z, X)} \right\} \]
\[= E \left\{ \frac{A(Z + A) Y w_{\alpha z} (A, Z, Y) I \{\pi(X) = Z\} \partial}{2\gamma(A, Z, X) f(Z|X) f(A|Z, X)} \right\} \]
\[= E \left\{ \frac{A(Z + A) Y w_{\alpha z} (A, Z, Y) I \{\pi(X) = Z\}}{2\gamma(A, Z, X) f(Z|X) f(A|Z, X)} S(Z|X) \right\} \]
\[= E \left\{ \left( E \left[ \frac{A(Z + A) Y w_{\alpha z} (A, z, Y) I \{\pi(X) = Z\}}{2\gamma(A, z, X) f(Z|X) f(A|z, X)} \right] \right) S(Z, X) \right\} \]
\[= E \left[ \left( E \left[ \frac{A(Z + A) Y w_{\alpha z} (A, Z, Y) I \{\pi(X) = Z\}}{2\gamma(A, Z, X) f(Z|X) f(A|Z, X)} \right] \right) S(|Z, X|) \right] \]
\[= E \left[ \frac{I \{\pi(X) = Z\} f(Z|X)}{f(Z|X)} E \left[ \frac{A(Z + A) Y w_{\alpha z} (A, Z, Y) I \{\pi(X) = Z\}}{2\gamma(A, Z, X) f(A|Z, X)} \right] S(Z, X) \right] \]
\[= E \left[ \frac{\sum_{z} I \{\pi(X) = z\} E \left[ \frac{A(A + z) Y w_{\alpha z} (z, A, Y)}{2\gamma(A, z, X) f(A|z, X)} \right] S(Z, X) \right] \right] \]

We have

\[(III) = E \left\{ \frac{A(Z + A) Y w_{\alpha z} (A, Z, Y) I \{\pi(X) = Z\} \partial}{2\gamma(A, Z, X)^2 f(Z|X) f(A|Z, X)} \right\} \]
\[= E \left\{ \frac{A(Z + A) (A + 1) Y w_{\alpha_{z+1}} (1, 1, Y) \{\pi(X) = Z\} \partial}{4\gamma(1, 1, X)^2 f(Z|X) f(A|Z, X)} \right\} \]
\[+ E \left\{ \frac{A(Z + A) (1 - A) Y w_{\alpha_{z-1}} (-1, -1, Y) \{\pi(X) = Z\} \partial}{4\gamma(-1, -1, X)^2 f(Z|X) f(A|Z, X)} \right\} \]

We have
\[
E \left[ \frac{A(Z + A)(A + 1)Yw_{\alpha+1}(1, 1, Y)I(\pi(X) = Z)}{4\gamma(1, 1, X)^2f(Z|X)f(A|Z, X)} \frac{\partial}{\partial t} \gamma(1, 1, X) \right]
\]
\[
= E \left[ \sum_a \sum_z a(z + a)(a + 1)Yw_{\alpha+1}(1, 1, Y)I\{\pi(X) = z\} \right] \left[ A = a, Z = z, X \right] \frac{\partial}{\partial t} \gamma(1, 1, X) \right]
\]
\[
= E \left[ \frac{I(\pi(X) = 1)\delta(1, 1, X)}{\gamma(1, 1, X)} \frac{\partial}{\partial t} \gamma(1, 1, X) \right]
\]
\[
= E \left[ \frac{I(\pi(X) = 1)\delta(1, 1, X)}{\gamma(1, 1, X)} \frac{A(A + Z)(A + 1)}{4f(Z, A|X)} \left[ w_{\alpha+1}(1, 1, Y) - \gamma(1, 1, X) \right] S(\mathcal{O}) \right] \]
\]

Similarly, we have
\[
E \left[ \frac{A(Z + A)(1 - A)Yw_{\alpha-1}(-1, -1, Y)I(\pi(X) = Z)}{4\gamma(-1, -1, X)^2f(Z|X)f(A|Z, X)} \frac{\partial}{\partial t} \gamma(-1, -1, X) \right]
\]
\[
= E \left[ \frac{I(\pi(X) = -1)\delta(-1, -1, X)}{\gamma(-1, -1, X)} \frac{A(A + Z)(1 - A)}{4f(Z, A|X)} \left[ w_{\alpha-1}(-1, -1, Y) - \gamma(-1, -1, X) \right] S(\mathcal{O}) \right]
\]

Putting the two terms together, we have
\[
(III) = E \left[ \frac{A(A + Z)I\{\pi(X) = Z\} \delta(A, Z, X)}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \left[ w_{\alpha_2}(Z, A, Y) - \gamma(A, Z, Y) \right] S(\mathcal{O}) \right]
\]

We have
\[(IV) = E \left[ \frac{A(Z + A)Y w_{az}(A, Z, Y)I(\pi(X) = Z)}{\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \frac{\partial}{\partial t} f_i(A|Z, X) \right] \]

\[= E \left[ \frac{A(Z + A)Y w_{az}(A, Z, Y)I\{\pi(X) = Z\}}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \right] \frac{\partial}{\partial t} f_i(A|Z, X) \]

\[= E \left[ \frac{A(Z + A)Y w_{az}(A, Z, Y)I\{\pi(X) = Z\}}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \right] S(A|Z, X) \]

\[= E \left[ \left( E \left[ \frac{A(Z + A)Y w_{az}(A, Z, Y)I\{\pi(X) = Z\}}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \right] \right) |Z, A, X] \right] S(A|Z, X) \]

\[= E \left[ \left( E \left[ \frac{A(Z + A)Y w_{az}(A, Z, Y)I\{\pi(X) = Z\}}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \right] \right) |A, Z, X] \right] \]

\[= E \left[ \left( E \left[ \frac{A(Z + A)Y w_{az}(a, Z, X)I(\pi(X) = Z)}{2\gamma(a, Z, X)f(Z|X)f(A|Z, X)} \right] \right) S(O) \right] \]

\[= E \left[ \frac{A(Z + A)I(\pi(X) = Z)}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \right] \]

Thus, the \( \frac{\partial}{\partial t} \mathcal{V}(O) |_{t=0} \) is
The canonical gradient is

\[
E\left[ \frac{A(Z + A)Yw_{\alpha Z}(A, Z, Y)I\{\pi(X) = Z\}}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \right]S(O)
\]

\[
- E\left[ \frac{A(Z + A)Yw_{\alpha Z}(A, Z, Y)}{2\gamma(A, Z, X)f(A|Z, X)} | Z, X \right]
\]

\[
- \sum_z I\{\pi(X) = z\} E\left[ \frac{A(A + z)Yw_{\alpha Z}(A, z, Y)}{2\gamma(z, A, X)f(A|z, X)} | Z = z, X \right] S(O)
\]

\[
- E\left[ \frac{A(A + Z)I\{\pi(X) = Z\}\delta(A, Z, X)}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \{w_{\alpha Z}(A, Z, Y) - \gamma(A, Z, Y)\} S(O) \right]
\]

\[
- E\left[ \frac{A(A + Z)I\{\pi(X) = Z\} E[Yw_{\alpha Z}(a, Z, Y)|A, Z, X]}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \right] S(O)
\]

\[
- \sum_a a(a + Z)I\{\pi(X) = Z\} E[Yw_{\alpha Z}(a, Z, Y)|A = a, Z, X] S(O)
\]

We let

\[
\kappa(Z, X) = E\left[ \frac{A(Z + A)Yw_{\alpha Z}(A, Z, Y)}{2\gamma(A, Z, X)f(A|Z, X)} | Z, X \right]
\]

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We propose the following three models for our multiply robust estimator. Under each or union of those models, our robust estimator will work

\( \mathcal{M}_1^\dagger : f(A|Z, X), \gamma(A, Z, X), f(Z|X) \) are correctly specified.

\( \mathcal{M}_2^\dagger : f(A|Z, X), \gamma(A, Z, X), \kappa(Z, X) \) are correctly specified.

\( \mathcal{M}_3^\dagger : \gamma(A, Z, X), f(Z|X), Q(A, Z, X) \) are correctly specified.

We let the If \( \mathcal{M}_1 \) is correct, we have

\[
E \left[ \frac{A(Z + A)Y w_{oz}(A, Z, Y) I\{\pi(X) = Z\}}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} - \left\{ \frac{I(\pi(X) = z)}{\gamma(A, Z, X)f(Z|X)} \kappa^*(Z, X) - \sum_z \frac{I(\pi(X) = z)}{\gamma(A, z, X)} \kappa^*(z, X) \right\} 
- \frac{A(Z + A)I(\pi(X) = Z)\delta^*(A, Z, X)}{2\gamma(A, Z, X)f(A, Z|X)} \left[ w_{oz}(A, Z, Y) - \gamma(A, Z, X) \right] 
- \left( \frac{A(Z + A)I(\pi(X) = Z)Q^*(A, Z, X)}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} - \sum_a \frac{Z(a + Z)I\{\pi(X) = Z\}Q^*(a, Z, X)}{2\gamma(A, Z, X)f(Z|X)} \right) \right] 
= E \left[ \frac{A(Z + A)Y w_{oz}(A, Z, Y) I\{\pi(X) = Z\}}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \right] = E[Y_{\pi(X)}] 
\]
If $\mathcal{M}_2$ is correct, we have

$$
E \left[ \frac{A(Z + A)Y w_{\alpha z}(A, Z, Y) I(\pi(X) = Z)}{2\gamma(A, Z, X) f^*(Z|X) f(A|Z, X)} - \left\{ \frac{I(\pi(X) = Z)}{\gamma(A, Z, X) f^*(Z|X)} \kappa(Z, X) - \sum_z \frac{I(\pi(X) = z)}{\gamma(A, z, X)} \kappa(z, X) \right\} \right]
$$

$$
- \frac{A(Z + A) I(\pi(X) = Z) \delta^*(A, Z, X)}{2\gamma(A, Z, X) f^*(Z|X) f(A|Z, X)} \left[w_{\alpha z}(A, Z, Y) - \gamma(A, Z, X)\right]
$$

$$
- \left( \frac{A(Z + A) I(\pi(X) = Z) Q^*(A, Z, X)}{2\gamma(A, Z, X) f^*(Z|X) f(A|Z, X)} - \sum_a Z(a + Z) I\{\pi(X) = Z\} Q^*(a, Z, X) \right)
$$

$$
= E \left[ \frac{A(Z + A) Y w_{\alpha z}(A, Z, Y) I(\pi(X) = Z)}{2\gamma(A, Z, X) f^*(Z|X) f(A|Z, X)} - \frac{I(\pi(X) = Z)}{\gamma(A, Z, X) f^*(Z|X)} \kappa(Z, X) \right]
$$

$$
+ E \left[ \sum_z \frac{I(\pi(X) = z)}{\gamma(A, z, X)} \kappa(z, X) \right]
$$

$$
= E[Y_{\pi(X)}]
$$

If $\mathcal{M}_3$ is correct, we have

$$
E \left[ \frac{A(Z + A) Y w_{\alpha z}(A, Z, Y)}{2\gamma(A, Z, X) f(Z|X) f^*(A|Z, X)} - \frac{A(Z + A) I(\pi(X) = Z) Q(A, Z, X)}{2\gamma(A, Z, X) f(Z|X) f^*(A|Z, X)} \right]
$$

$$
- E \left[ \frac{I(\pi(X) = Z)}{\gamma(A, Z, X) f(Z|X)} \kappa^*(Z, X) - \sum_z \frac{I(\pi(X) = z)}{\gamma(A, z, X)} \kappa^*(z, X) \right]
$$

$$
- E \left[ \frac{A(Z + A) I(\pi(X) = Z) \delta(A, Z, X)}{2\gamma(A, Z, X) f(Z|X) f^*(A|Z, X)} \left[w_{\alpha z}(A, Z, Y) - \gamma(A, Z, X)\right] \right]
$$

$$
+ E \left[ \sum_a Z(a + Z) I\{\pi(X) = Z\} Q(a, Z, X) \right]
$$

$$
= E \left[ \sum_a Z(a + Z) I\{\pi(X) = Z\} Q(a, Z, X) \right] = E[Y_{\pi(X)}]
$$
D Proof of Corollary 1

By definition, \( \Psi = \phi_\psi(O,..) = (P_0 - P_n)\phi_\psi(O,..) + P_n\phi_\psi(O,..) \) and \( \hat{\Psi} = P_n\hat{\phi}_\psi(O,..) \)

\[
\hat{\Psi} - \Psi = \hat{\Psi} - (P_0 - P_n)\phi_\psi(O,..) - P_n\phi_\psi(O,..) \\
= (P_n - P_0)\phi_\psi(O,..) + \hat{\Psi} - P_n\phi_\psi(O,..)
\]

Hence, the remainder term is defined
\[ R = \hat{\Psi} - P_n \phi_{\psi}(O, ...) \]

\[
= P_n \frac{ZA(Z + A) \left\{ [Y - \hat{Q}(A, Z, X)] - \delta(A, Z, X) [w_{\alpha_z}(A, Z, Y) - \hat{\gamma}(A, Z, X)] \right\}}{2\hat{\gamma}(A, Z, X) \hat{f}(A, Z|X)} + P_n \hat{\Delta}(X) \\
- P_n \frac{ZA(Z + A) \left\{ Y - Q(A, Z, X) - \delta(A, Z, X) [w_{\alpha_z}(A, Z, Y) - \gamma(A, Z, X)] \right\}}{2\gamma(A, Z, X) f(Z|X)} - P_n \Delta(X) \\
\pm P_n \frac{ZA(Z + A) \left\{ Y - Q(A, Z, X) - \delta(A, Z, X) [w_{\alpha_z}(A, Z, Y) - \gamma(A, Z, X)] \right\}}{2\gamma(A, Z, X) f(A, Z|X)} \\
= P_n \frac{ZA(Z + A)}{2\hat{\gamma}(A, Z, X) \hat{f}(A, Z|X)} \left\{ [Q(A, Z, X) - \hat{Q}(A, Z, X)] + \delta(A, Z, X) [w_{\alpha_z}(A, Z, Y) - \gamma(A, Z, X)] \\
- \delta(A, Z, X) [w_{\alpha_z}(A, Z, Y) - \hat{\gamma}(A, Z, X)] \right\} \\
+ P_n \left[ \Delta(X) - \hat{\Delta}(X) \right] \\
+ P_n \frac{ZA(Z + A)}{2} \left\{ Y - Q(A, Z, X) - \delta(A, Z, X) [w_{\alpha_z}(A, Z, Y) - \gamma(A, Z, X)] \right\} \\
\times \left[ \frac{1}{\hat{\gamma}(A, Z, X) \hat{f}(A, Z|X)} - \frac{1}{\gamma(A, Z, X) f(A, Z|X)} \right] \\
= P_n \frac{ZA(Z + A)}{2\hat{\gamma}(A, Z, X) \hat{f}(A, Z|X)} \left\{ [Q(A, Z, X) - \hat{Q}(A, Z, X)] \right\} \\
+ P_n \frac{ZA(Z + A)}{2\gamma(A, Z, X) f(A, Z|X)} \left\{ \delta(A, Z, X) [w_{\alpha_z}(A, Z, Y) - \gamma(A, Z, X)] \\
- \delta(A, Z, X) [w_{\alpha_z}(A, Z, Y) - \hat{\gamma}(A, Z, X)] \right\} \\
+ P_n \left[ \hat{\Delta}(X) - \Delta(X) \right] + o_p(n^{-1/2}) \\
= (I) + (II) + (III) + o_p(n^{-1/2})
\]

Consider the term (I)
\[ (I) = P_n \frac{ZA(Z + A)}{2\gamma(A, Z, X)f(A, Z|X)} \left[ Q(A, Z, X) - \hat{Q}(A, Z, X) \right] \]

\[
\pm P_n \frac{ZA(Z + A)}{2\gamma(A, Z, X)f(A, Z|X)} \left[ Q(A, Z, X) - \hat{Q}(A, Z, X) \right] \frac{\gamma(A, Z, X) - \hat{\gamma}(A, Z, X)}{\gamma(A, Z, X)\hat{\gamma}(A, Z, X)}
\]

\[
= P_n \frac{ZA(Z + A)}{2f(A, Z|X)} \left[ Q(A, Z, X) - \hat{Q}(A, Z, X) \right] \left[ \gamma(A, Z, X) - \hat{\gamma}(A, Z, X) \right] \frac{\gamma(A, Z, X)\hat{\gamma}(A, Z, X)}{\gamma(A, Z, X)\hat{\gamma}(A, Z, X)}
\]

\[
+ P_n \frac{ZA(Z + A)}{2\gamma(A, Z, X)f(A, Z|X)} \left[ Q(A, Z, X) - \hat{Q}(A, Z, X) \right] \frac{\gamma(A, Z, X) - \hat{\gamma}(A, Z, X)}{\gamma(A, Z, X)\hat{\gamma}(A, Z, X)}
\]

Consider term (II)
\[(II) = \frac{ZA(Z + A)}{2\hat{\gamma}(A, Z, X)\hat{f}(A, Z|X)}\left\{ \delta(A, Z, X)\left[w_{\alpha_2}(A, Z, Y) - \gamma(A, Z, X)\right] \right. \\
- \delta(A, Z, X)\left[w_{\alpha_2}(A, Z, Y) - \hat{\gamma}(A, Z, X)\right] \right\} \\
\pm P_n\frac{ZA(Z + A)}{2\hat{\gamma}(A, Z, X)\hat{f}(A, Z|X)}\delta(A, Z, X)\left[w_{\alpha_2}(A, Z, Y) - \hat{\gamma}(A, Z, X)\right] \\
= P_n\frac{ZA(Z + A)}{2\hat{\gamma}(A, Z, X)\hat{f}(A, Z|X)}\left\{ \delta(A, Z, X)\left[\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)\right] \\
- \left[\delta(A, Z, X) - \delta(A, Z, X)\right]\left[w_{\alpha_2}(A, Z, Y) - \hat{\gamma}(A, Z, X)\right] \right\} \\
= P_n\frac{ZA(Z + A)}{2\hat{\gamma}(A, Z, X)\hat{f}(A, Z|X)}\left\{ \delta(A, Z, X)\left[\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)\right] + o_p(n^{-1/2}) \right\} \\
= P_n\frac{ZA(Z + A)}{2\hat{\gamma}(A, Z, X)\hat{f}(A, Z|X)}\delta(A, Z, X)\left[\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)\right] \\
+ P_n\frac{ZA(Z + A)}{2\hat{\gamma}(A, Z, X)\hat{f}(A, Z|X)}\left[\frac{f(A, Z|X) - \hat{f}(A, Z|X)}{f(A, Z|X)\hat{f}(A, Z|X)}\right] \\
+ P_n\frac{ZA(Z + A)}{2\hat{\gamma}(A, Z, X)\hat{f}(A, Z|X)}\delta(A, Z, X)\left[\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)\right] + o_p(n^{-1/2}) \\
= P_n\frac{ZA(Z + A)}{2\hat{\gamma}(A, Z, X)}\left[\frac{f(A, Z|X) - \hat{f}(A, Z|X)}{f(A, Z|X)\hat{f}(A, Z|X)}\right] \\
+ P_n\frac{ZA(Z + A)}{2\hat{\gamma}(A, Z, X)}\delta(A, Z, X)\left[\frac{\gamma(A, Z, X) - \hat{\gamma}(A, Z, X)}{\gamma(A, Z, X)\hat{\gamma}(A, Z, X)}\right] \\
+ P_n\frac{ZA(Z + A)}{2\hat{\gamma}(A, Z, X)}\delta(A, Z, X)\left[\frac{\gamma(A, Z, X) - \hat{\gamma}(A, Z, X)}{\gamma(A, Z, X)\hat{\gamma}(A, Z, X)}\right] + o_p(n^{-1/2}) \\
= P_0\frac{Q(1, 1, X)}{\gamma(1, 1, X)}\left[\frac{\hat{\gamma}(1, 1, X) - \gamma(1, 1, X)}{\gamma(1, 1, X)}\right] - P_0\frac{Q(-1, -1, X)}{\gamma(-1, -1, X)}\left[\frac{\hat{\gamma}(-1, -1, X) - \gamma(-1, -1, X)}{\gamma(-1, -1, X)}\right] \\
+ P_n\frac{ZA(Z + A)}{2\hat{\gamma}(A, Z, X)}\delta(A, Z, X)\left[\frac{\gamma(A, Z, X) - \hat{\gamma}(A, Z, X)}{\gamma(A, Z, X)\hat{\gamma}(A, Z, X)}\right] + o_p(n^{-1/2}) \]
Putting everything together

\[ R = (I) + (II) + (III) \]
\[
= P_n \frac{ZA(Z + A)}{2 \hat{f}(A, Z|X)} \left[ Q(A, Z, X) - \hat{Q}(A, Z, X) \right] \left[ \gamma(A, Z, X) - \gamma(A, Z, X) \right] \\
+ P_n \frac{ZA(Z + A)}{2 \hat{f}(A, Z|X)} \left[ Q(A, Z, X) - \hat{Q}(A, Z, X) \right] \left[ f(A, Z|X) - \hat{f}(A, Z|X) \right] \\
+ P_n \frac{ZA(Z + A)}{2 \hat{f}(A, Z|X)} \left[ Q(A, Z, X) - \hat{Q}(A, Z, X) \right] \left[ \frac{f(A, Z|X) - \hat{f}(A, Z|X)}{f(A, Z|X)\hat{f}(A, Z|X)} \right] \\
+ P_n \frac{ZA(Z + A)}{2 \hat{f}(A, Z|X)} \left[ Q(A, Z, X) - \hat{Q}(A, Z, X) \right] \left[ \frac{f(A, Z|X) - \hat{f}(A, Z|X)}{f(A, Z|X)\hat{f}(A, Z|X)} \right] \\
+ P_n \left[ \hat{\gamma}(A, Z, X) - \gamma(A, Z, X) \right] + o_p(n^{-1/2}) \\
= \frac{ZA(Z + A)}{2 \hat{f}(A, Z|X)} \left[ Q(A, Z, X) - \hat{Q}(A, Z, X) \right] \left[ \gamma(A, Z, X) - \gamma(A, Z, X) \right] \\
+ \frac{ZA(Z + A)}{2 \hat{f}(A, Z|X)} \left[ Q(A, Z, X) - \hat{Q}(A, Z, X) \right] \left[ f(A, Z|X) - \hat{f}(A, Z|X) \right] \\
+ \frac{ZA(Z + A)}{2 \hat{f}(A, Z|X)} \left[ Q(A, Z, X) - \hat{Q}(A, Z, X) \right] \left[ \frac{f(A, Z|X) - \hat{f}(A, Z|X)}{f(A, Z|X)\hat{f}(A, Z|X)} \right] \\
+ \frac{ZA(Z + A)}{2 \hat{f}(A, Z|X)} \left[ Q(A, Z, X) - \hat{Q}(A, Z, X) \right] \left[ \frac{f(A, Z|X) - \hat{f}(A, Z|X)}{f(A, Z|X)\hat{f}(A, Z|X)} \right] \\
+ \left\{ P_n \left[ \hat{\gamma}(A, Z, X) - \gamma(A, Z, X) \right] + P_n \frac{ZA(Z + A)}{2 \hat{f}(A, Z|X)} \left[ Q(A, Z, X) - \hat{Q}(A, Z, X) \right] \right\} \\
+ \frac{P_n Q(1, 1, X)}{\gamma(1, 1, X)} \left[ \frac{\hat{\gamma}(1, 1, X) - \gamma(1, 1, X)}{\gamma(1, 1, X)} \right] \\
- \frac{P_n Q(-1, -1, X)}{\gamma(-1, -1, X)} \left[ \frac{\hat{\gamma}(-1, -1, X) - \gamma(-1, -1, X)}{\gamma(-1, -1, X)} \right] + o_p(n^{-1/2})
\]
Consider the term

\[
P_n \frac{ZA(Z + A)}{2\gamma(A, Z, X) f(A, Z|X)} \left[ Q(A, Z, X) - \hat{Q}(A, Z, X) \right] + P_n \left[ \hat{\Delta}(X) - \Delta(X) \right]
\]

\[
= -P_0 \left[ \frac{\hat{Q}(1, 1, X)}{\gamma(1, 1, X)} - \frac{\hat{Q}(-1, -1, X)}{\gamma(-1, -1, X)} \right] + P_0 \left[ \frac{\hat{\gamma}(1, 1, X)}{\hat{\gamma}(1, 1, X)} - \frac{\hat{\gamma}(-1, -1, X)}{\hat{\gamma}(-1, -1, X)} \right] + o_p(n^{-1/2})
\]

\[
= -P_0 \hat{Q}(1, 1, X) \left[ \frac{\hat{\gamma}(1, 1, X) - \gamma(1, 1, X)}{\gamma(1, 1, X) \hat{\gamma}(1, 1, X)} \right] + P_0 \hat{Q}(-1, -1, X) \left[ \frac{\hat{\gamma}(-1, -1, X) - \gamma(-1, -1, X)}{\gamma(-1, -1, X) \hat{\gamma}(-1, -1, X)} \right] + o_p(n^{-1/2})
\]

The remainder term becomes
Thus, the remainder $R$ will go to $o_p(n^{-1/2})$ under the assumption

$$\|f - \hat{f}\|_{R_0} + \|\gamma - \hat{\gamma}\|_{R_0} (\|\Delta - \hat{\Delta}\|_{R_0} + \|\gamma - \hat{\gamma}\|_{R_0} + \|Q - \hat{Q}\|_{R_0}) = o_p(n^{-1/2}).$$
E Proof of Corollary 2

We let

\[
\kappa(Z, X) = E\left[\frac{A(Z + A)Yw_{\alpha Z}(A, Z, Y)}{2\gamma(A, Z, X)f(A|Z, X)} | Z, X\right]
\]

\[
\kappa'(X) = \sum_z I\{\pi(X) = z\}E\left[\frac{A(z + A)Yw_{\alpha z}(A, z, Y)}{2\gamma(A, z, X)f(A|z, X)} | Z = z, X\right]
\]

\[
= \sum_z I\{\pi(X) = z\}\kappa(Z = z, X)
\]

\[
\theta(Z, X) = \sum_a \frac{a(a + Z)}{2\gamma(a, Z, X)} E[Yw_{\alpha Z}(a, Z, X)|A = a, Z, X]
\]

\[
= \sum_a \frac{a(a + Z)}{2\gamma(a, Z, X)} Q(a, Z, X)
\]

We have

\[
\hat{\mathcal{V}}^c(\pi) - \mathcal{V}^c(\pi) = \hat{\mathcal{V}}^c(\pi) + (P_n - P_\pi)\xi_V - P_\pi\xi_V
\]

We have the remainder term
\[
R = \hat{V}^e(\pi) - P_n \xi_V \\
= P_n \frac{A(Z + A) Y w_{az}(A, Z, Y) I\{\pi(X) = Z\}}{2 \hat{\gamma}(A, Z, X) f(Z|X)f(A|Z, X)} - \left\{ P_n \frac{I\{\pi(X) = Z\}\hat{\kappa}(Z, X)}{f(Z|X)} - P_n \hat{\kappa}'(X) \right\} \\
- P_n \frac{A(Z + A) \delta(A, Z, X)[\omega_{az}(A, Z, Y) - \hat{\gamma}(A, Z, X)]}{2 \hat{\gamma}(A, Z, X) f(Z|X)f(A|Z, X)} \\
- \left\{ P_n \frac{A(Z + A) Y w_{az}(A, Z, Y) I\{\pi(X) = Z\}}{2 \hat{\gamma}(A, Z, X) f(Z|X)f(A|Z, X)} - P_n \frac{\hat{\theta}(Z, X)}{f(Z|X)} \right\} \\
- P_n \frac{A(Z + A) I\{\pi(X) = Z\}\delta(A, Z, X)[\omega_{az}(A, Z, Y) - \gamma(A, Z, X)]}{2 \gamma(A, Z, X) f(Z|X)f(A|Z, X)} + P_n \left\{ \frac{I\{\pi(X) = Z\}\kappa(Z, X)}{f(Z|X)} - \kappa'(X) \right\} \\
+ P_n \left\{ \frac{A(Z + A) Y w_{az}(A, Z, Y) I\{\pi(X) = Z\}}{2 \gamma(A, Z, X) f(Z|X)f(A|Z, X)} \right\} \\
+ \left\{ P_n \frac{A(Z + A) Y w_{az}(A, Z, Y) I\{\pi(X) = Z\}}{2 \gamma(A, Z, X) f(Z|X)f(A|Z, X)} - P_n \frac{\theta(Z, X)}{f(Z|X)} \right\} \\
= P_n \left\{ \frac{A(Z + A) Y w_{az}(A, Z, Y) I\{\pi(X) = Z\}}{2} \left( \frac{1}{\gamma(A, Z, X) f(Z|X)f(A|Z, X)} \right) \right\} \\
+ P_n \left\{ \frac{I\{\pi(X) = Z\}\kappa(Z, X)}{f(Z|X)} - \frac{I\{\pi(X) = Z\}\hat{\kappa}(Z, X)}{f(Z|X)} \right\} \\
+ P_n \left\{ \frac{\hat{\theta}(Z, X)}{f(Z|X)} - \frac{\theta(Z, X)}{f(Z|X)} \right\} \\
+ P_n \left\{ \frac{A(Z + A) I\{\pi(X) = Z\}\delta(A, Z, X)[\omega_{az}(A, Z, Y) - \gamma(A, Z, X)]}{2 \gamma(A, Z, X) f(Z|X)f(A|Z, X)} \\
- \frac{A(Z + A) I\{\pi(X) = Z\}\delta(A, Z, X)[\omega_{az}(A, Z, Y) - \hat{\gamma}(A, Z, X)]}{2 \hat{\gamma}(A, Z, X) f(Z|X)f(A|Z, X)} \right\} \\
+ P_n \left\{ \frac{A(Z + A) Y w_{az}(A, Z, Y) Q(A, Z, X)}{2 \gamma(A, Z, X) f(Z|X)f(A|Z, X)} - \frac{A(Z + A) \hat{Q}(A, Z, X) I\{\pi(X) = Z\}}{2 \gamma(A, Z, X) f(Z|X)f(A|Z, X)} \right\} \\
= (1) + (2.1) + (2.2) + (4.2) + (3) + (4.1)
\]

We will keep term (1), (2.2) and simplify (2.1), (3), (4.1), (4.2)
\((2.1) = P_n \left( \frac{I\{\pi(X) = Z\}}{f(Z|X)} \kappa(Z, X) - \frac{I\{\pi(X) = Z\}}{f(Z|X)} \hat{\kappa}(Z, X) \right) \\
= P_n \left[ \frac{I\{\pi(X) = Z\}}{f(Z|X)} \kappa(Z, X) - \frac{I\{\pi(X) = Z\}}{f(Z|X)} \hat{\kappa}(Z, X) \right] \pm P_n \left[ \frac{I\{\pi(X) = Z\}}{f(Z|X)} \kappa(Z, X) \right] \\
= P_n \kappa(Z, X) \left[ \frac{I\{\pi(X) = Z\}}{f(Z|X)} - \frac{I\{\pi(X) = Z\}}{f(Z|X)} \right] - P_n \hat{\kappa}(Z, X) \frac{I\{\pi(X) = Z\}}{f(Z|X)} \\
+ P_n \frac{I\{\pi(X) = Z\}}{f(Z|X)} \kappa(Z, X) \\
= P_n \kappa(Z, X) \left[ \frac{I\{\pi(X) = Z\}}{f(Z|X)} - \frac{I\{\pi(X) = Z\}}{f(Z|X)} \right] + P_n \frac{I\{\pi(X) = Z\}}{f(Z|X)} \left[ \kappa(Z, X) - \hat{\kappa}(Z, X) \right] \\
\pm P_n \frac{I\{\pi(X) = Z\}}{f(Z|X)} \left[ \kappa(Z, X) - \hat{\kappa}(Z, X) \right] \\
= P_n \kappa(Z, X) \left[ \frac{I\{\pi(X) = Z\}}{f(Z|X)} - \frac{I\{\pi(X) = Z\}}{f(Z|X)} \right] + P_n \frac{I\{\pi(X) = Z\}}{f(Z|X)} \left[ \kappa(Z, X) - \hat{\kappa}(Z, X) \right] \\
+ P_n \left[ \kappa(Z, X) - \hat{\kappa}(Z, X) \right] \left[ \frac{I\{\pi(X) = Z\}}{f(Z|X)} - \frac{I\{\pi(X) = Z\}}{f(Z|X)} \right] \\
\pm P_0 \left[ \kappa(Z, X) - \hat{\kappa}(Z, X) \right] \left[ \frac{I\{\pi(X) = Z\}}{f(Z|X)} - \frac{I\{\pi(X) = Z\}}{f(Z|X)} \right] \\
= P_n \kappa(Z, X) I\{\pi(X) = Z\} \left[ \frac{1}{f(Z|X)} - \frac{1}{f(Z|X)} \right] + P_n \frac{I\{\pi(X) = Z\}}{f(Z|X)} \left[ \kappa(Z, X) - \hat{\kappa}(Z, X) \right] \\
+ (P_n - P_0) \left[ \kappa(Z, X) - \hat{\kappa}(Z, X) \right] \left[ \frac{I\{\pi(X) = Z\}}{f(Z|X)} - \frac{I\{\pi(X) = Z\}}{f(Z|X)} \right] \\
+ P_0 \left[ \kappa(Z, X) - \hat{\kappa}(Z, X) \right] \left[ \frac{I\{\pi(X) = Z\}}{f(Z|X)} - \frac{I\{\pi(X) = Z\}}{f(Z|X)} \right] \\
= P_n \kappa(Z, X) I\{\pi(X) = Z\} \left[ \frac{1}{f(Z|X)} - \frac{1}{f(Z|X)} \right] + P_n \frac{I\{\pi(X) = Z\}}{f(Z|X)} \left[ \kappa(Z, X) - \hat{\kappa}(Z, X) \right] \\
+ P_0 \left[ \kappa(Z, X) - \hat{\kappa}(Z, X) \right] \left[ \frac{I\{\pi(X) = Z\}}{f(Z|X)} - \frac{I\{\pi(X) = Z\}}{f(Z|X)} \right] + o_p(n^{-1/2})
\begin{align*}
(4.2) &= P_n \left[ \frac{\hat{\theta}(Z,X)}{f(Z|X)} - \theta(Z,X) \right] - \frac{\hat{\theta}(Z,X)}{f(Z|X)} + P_n \left[ \hat{\theta}(Z,X) - \frac{1}{f(Z|X)} \right] \\
&= P_n \hat{\theta}(Z,X) \left[ \frac{1}{f(Z|X)} - \frac{1}{f(Z|X)} \right] + P_n \hat{\theta}(Z,X) \left[ \frac{1}{f(Z|X)} - \frac{1}{f(Z|X)} \right] + P_n \hat{\theta}(Z,X) - \theta(Z,X) \\
&= P_n \hat{\theta}(Z,X) \left[ \frac{1}{f(Z|X)} - \frac{1}{f(Z|X)} \right] + P_n \hat{\theta}(Z,X) \left[ \frac{1}{f(Z|X)} - \frac{1}{f(Z|X)} \right] \\
&+ P_n \hat{\theta}(Z,X) - \theta(Z,X) \\
&= P_0 \left[ \hat{\theta}(Z,X) - \theta(Z,X) \right] \left[ \frac{1}{f(Z|X)} - \frac{1}{f(Z|X)} \right] + P_n \hat{\theta}(Z,X) \left[ \frac{1}{f(Z|X)} - \frac{1}{f(Z|X)} \right] \\
&+ P_n \hat{\theta}(Z,X) - \theta(Z,X) \\
&= P_0 \left[ \hat{\theta}(Z,X) - \theta(Z,X) \right] \left[ \frac{1}{f(Z|X)} - \frac{1}{f(Z|X)} \right] + P_n \hat{\theta}(Z,X) \left[ \frac{1}{f(Z|X)} - \frac{1}{f(Z|X)} \right] \\
&+ P_n \hat{\theta}(Z,X) - \theta(Z,X)
\end{align*}
\( P_n \left\{ \frac{A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [w_{az}(A, Z, Y) - \gamma(A, Z, X)]}{2\gamma(A, Z, X) f(Z|X) f(A|Z, X)} \right\} \\
- \frac{A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [w_{az}(A, Z, Y) - \gamma(A, Z, X)]}{2\gamma(A, Z, X) f(Z|X) f(A|Z, X)} \\
\pm \frac{A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [w_{az}(A, Z, Y) - \gamma(A, Z, X)]}{2\gamma(A, Z, X) f(Z|X) f(A|Z, X)} \\
= P_n A(Z + A) \frac{2}{2} I\{\pi(X) = Z\} \delta(A, Z, X) \left( \frac{w_{az}(A, Z, Y) - \gamma(A, Z, X)}{\gamma(A, Z, X) f(Z|X) f(A|Z, X)} - \frac{w_{az}(A, Z, Y) - \gamma(A, Z, X)}{\gamma(A, Z, X) f(Z|X) f(A|Z, X)} \right) \\
+ P_n A(Z + A) \frac{2}{2} I\{\pi(X) = Z\} \delta(A, Z, X) \left( \frac{w_{az}(A, Z, Y) - \gamma(A, Z, X)}{\gamma(A, Z, X) f(Z|X) f(A|Z, X)} - \frac{w_{az}(A, Z, Y) - \gamma(A, Z, X)}{\gamma(A, Z, X) f(Z|X) f(A|Z, X)} \right) \\
- P_n A(Z + A) \frac{2}{2} I\{\pi(X) = Z\} \delta(A, Z, X) \left( \frac{w_{az}(A, Z, Y) - \gamma(A, Z, X)}{\gamma(A, Z, X) f(Z|X) f(A|Z, X)} - \frac{w_{az}(A, Z, Y) - \gamma(A, Z, X)}{\gamma(A, Z, X) f(Z|X) f(A|Z, X)} \right) \\
= (P_n - P_0) A(Z + A) \frac{2}{2} I\{\pi(X) = Z\} \delta(A, Z, X) \left( \frac{w_{az}(A, Z, Y) - \gamma(A, Z, X)}{\gamma(A, Z, X) f(Z|X) f(A|Z, X)} - \frac{w_{az}(A, Z, Y) - \gamma(A, Z, X)}{\gamma(A, Z, X) f(Z|X) f(A|Z, X)} \right) \\
+ P_n A(Z + A) \frac{2}{2} I\{\pi(X) = Z\} \delta(A, Z, X) \left( \frac{w_{az}(A, Z, Y) - \gamma(A, Z, X)}{\gamma(A, Z, X) f(Z|X) f(A|Z, X)} - \frac{w_{az}(A, Z, Y) - \gamma(A, Z, X)}{\gamma(A, Z, X) f(Z|X) f(A|Z, X)} \right) \\
- P_n A(Z + A) \frac{2}{2} I\{\pi(X) = Z\} \delta(A, Z, X) \left( \frac{w_{az}(A, Z, Y) - \gamma(A, Z, X)}{\gamma(A, Z, X) f(Z|X) f(A|Z, X)} - \frac{w_{az}(A, Z, Y) - \gamma(A, Z, X)}{\gamma(A, Z, X) f(Z|X) f(A|Z, X)} \right) \\
= P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) \left[ w_{az}(A, Z, Y) - \gamma(A, Z, X) \right] \\
- P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) \left[ w_{az}(A, Z, Y) - \gamma(A, Z, X) \right] \\
+ o_p(n^{-1/2}) \ \{ \text{since} \ P_0 \left\{ w_{az}(A, Z, Y) - \gamma(A, Z, X) \right\} = o_p(n^{-1/2}) \} \\
= P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) \left[ w_{az}(A, Z, Y) - \gamma(A, Z, X) \right] \\
- P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) \left[ w_{az}(A, Z, Y) - \gamma(A, Z, X) \right] \\
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\[\pm P_n \frac{A(Z + A)I\{\pi(X) = Z\} \delta(A, Z, X)[w_{\alpha z}(A, Z, Y) - \hat{\gamma}(A, Z, X)]}{2\hat{\gamma}(A, Z, X)f(Z|X)f(A|Z, X)} + o_p(n^{-1/2})\]

\[= P_n \frac{A(Z + A)I\{\pi(X) = Z\} \delta(A, Z, X)[\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2\hat{\gamma}(A, Z, X)f(Z|X)f(A|Z, X)}\]

\[+ P_n \frac{A(Z + A)I\{\pi(X) = Z\}[w_{\alpha z}(A, Z, Y) - \hat{\gamma}(A, Z, X)]\delta(A, Z, X) - \hat{\delta}(A, Z, X]}{2\hat{\gamma}(A, Z, X)f(Z|X)f(A|Z, X)} + o_p(n^{-1/2})\]

\[= P_n \frac{A(Z + A)I\{\pi(X) = Z\} \delta(A, Z, X)[\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2\hat{\gamma}(A, Z, X)f(Z|X)f(A|Z, X)}\]

\[+ \frac{1}{\hat{\gamma}(A, Z, X)} - \frac{1}{\gamma(A, Z, X)}\]

\[= P_n \frac{A(Z + A)I\{\pi(X) = Z\} \delta(A, Z, X)[\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2\hat{\gamma}(A, Z, X)f(Z|X)f(A|Z, X)}\]

\[+ \frac{1}{\hat{\gamma}(A, Z, X)} - \frac{1}{\gamma(A, Z, X)}\]

\[= P_n \frac{A(Z + A)I\{\pi(X) = Z\} \delta(A, Z, X)[\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2\hat{\gamma}(A, Z, X)f(Z|X)f(A|Z, X)}\]

\[+ \frac{1}{\hat{\gamma}(A, Z, X)} - \frac{1}{\gamma(A, Z, X)}\]
+ \frac{P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2g(\gamma(A, Z, X)) f(Z|X) / f(A|Z, X)} 

+ \frac{P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2g(\gamma(A, Z, X)) f(Z|X) / f(A|Z, X)} 

+ \frac{P_n A(Z + A) I\{\pi(X) = Z\} [w_{\alpha Z}(A, Z, Y) - \hat{\gamma}(A, Z, X)] [\delta(A, Z, X) - \hat{\delta}(A, Z, X)]}{2g(\gamma(A, Z, X)) f(Z|X) / f(A|Z, X)} + o_p(n^{-1/2}) 

= P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)] 

+ \frac{P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2g(\gamma(A, Z, X)) f(Z|X) / f(A|Z, X)} 

+ \frac{P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2g(\gamma(A, Z, X)) f(Z|X) / f(A|Z, X)} 

+ \frac{P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2g(\gamma(A, Z, X)) f(Z|X) / f(A|Z, X)} 

+ \frac{P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2g(\gamma(A, Z, X)) f(Z|X) / f(A|Z, X)} 

+ \frac{P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2g(\gamma(A, Z, X)) f(Z|X) / f(A|Z, X)} 

+ \frac{P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2g(\gamma(A, Z, X)) f(Z|X) / f(A|Z, X)} 

+ \frac{P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2g(\gamma(A, Z, X)) f(Z|X) / f(A|Z, X)} 

+ \frac{P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2g(\gamma(A, Z, X)) f(Z|X) / f(A|Z, X)} 

+ \frac{P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2g(\gamma(A, Z, X)) f(Z|X) / f(A|Z, X)} 

+ \frac{P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2g(\gamma(A, Z, X)) f(Z|X) / f(A|Z, X)} 

+ \frac{P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2g(\gamma(A, Z, X)) f(Z|X) / f(A|Z, X)} 

+ \frac{P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2g(\gamma(A, Z, X)) f(Z|X) / f(A|Z, X)} 

+ \frac{P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2g(\gamma(A, Z, X)) f(Z|X) / f(A|Z, X)}
\begin{equation}
(4.1) = P_n A(Z + A)I\{\pi(X) = Z\}Q(A, Z, X) \frac{1}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} - P_n A(Z + A)I\{\pi(X) = Z\} \hat{Q}(A, Z, X) \frac{1}{2\hat{\gamma}(A, Z, X)f(Z|X)f(A|Z, X)} \\
\pm P_n A(Z + A)I\{\pi(X) = Z\}Q(A, Z, X) \frac{2}{2\hat{\gamma}(A, Z, X)f(Z|X)f(A|Z, X)} \\
= P_n A(Z + A)I\{\pi(X) = Z\}Q(A, Z, X) \left[ \frac{1}{\gamma(A, Z, X)f(Z|X)f(A|Z, X)} - \frac{1}{\hat{\gamma}(A, Z, X)f(Z|X)f(A|Z, X)} \right] \\
\pm P_n A(Z + A)I\{\pi(X) = Z\}Q(A, Z, X) \frac{1}{2\hat{\gamma}(A, Z, X)f(Z|X)f(A|Z, X)} \\
= P_n A(Z + A)I\{\pi(X) = Z\}Q(A, Z, X) \left[ \frac{1}{\gamma(A, Z, X)f(Z|X)f(A|Z, X)} - \frac{1}{\hat{\gamma}(A, Z, X)f(Z|X)f(A|Z, X)} \right] \\
\pm P_n A(Z + A)I\{\pi(X) = Z\}Q(A, Z, X) \frac{1}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \\
\end{equation}
\[
= P_n A(Z + A)I\{\pi(X) = Z\}[Q(A, Z, X) - \hat{Q}(A, Z, X)]\left[\frac{1}{\hat{f}(A|Z, X)} - \frac{1}{\hat{f}(A|Z, X)}\right]
\]
\[
+ P_n A(Z + A)I\{\pi(X) = Z\}[Q(A, Z, X) - \hat{Q}(A, Z, X)]\left[\frac{1}{\hat{f}(Z|X)f(A|Z, X)} - \frac{1}{\hat{f}(Z|X)f(A|Z, X)}\right]
\]
\[
+ P_n A(Z + A)I\{\pi(X) = Z\}[Q(A, Z, X) - \hat{Q}(A, Z, X)]\left[\frac{1}{\hat{f}(A|Z, X)} - \frac{1}{\hat{f}(A|Z, X)}\right]
\]
\[
+ P_n A(Z + A)I\{\pi(X) = Z\}[Q(A, Z, X) - \hat{Q}(A, Z, X)]\left[\frac{1}{\hat{f}(Z|X)f(A|Z, X)} - \frac{1}{\hat{f}(Z|X)f(A|Z, X)}\right]
\]
\[
= P_n A(Z + A)I\{\pi(X) = Z\}[Q(A, Z, X) - \hat{Q}(A, Z, X)]\left[\frac{1}{\hat{f}(Z|X)f(A|Z, X)} - \frac{1}{\hat{f}(Z|X)f(A|Z, X)}\right]
\]
\[
= P_n \frac{A(Z + A)I\{\pi(X) = Z\}[Q(A, Z, X) - \hat{Q}(A, Z, X)]}{2\gamma(A, Z, X)f(Z|X)f(A|Z, X)} \\
+ P_0 \frac{A(Z + A)I\{\pi(X) = Z\}[Q(A, Z, X) - \hat{Q}(A, Z, X)]}{2\gamma(A, Z, X)f(A|Z, X)} \left[ \frac{1}{\hat{f}(Z|X)} - \frac{1}{f(Z|X)} \right] \\
+ P_0 \frac{A(Z + A)I\{\pi(X) = Z\}[Q(A, Z, X) - \hat{Q}(A, Z, X)]}{2\gamma(A, Z, X)f(Z|X)} \left[ \frac{1}{\hat{f}(A|Z, X)} - \frac{1}{f(A|Z, X)} \right] \\
+ P_0 \frac{A(Z + A)I\{\pi(X) = Z\}Q(A, Z, X)}{2\hat{f}(Z|X)f(A|Z, X)} \left[ \frac{1}{\gamma(A, Z, X)f(Z|X)f(A|Z, X)} - \frac{1}{\hat{\gamma}(A, Z, X)\hat{f}(Z|X)\hat{f}(A|Z, X)} \right]
\]
In summary, we have

\[(2.1) = P_n \kappa(Z, X) I\{\pi(X) = Z\} \left[ \frac{1}{f(Z|X)} - \frac{1}{f(Z|X)} \right] + P_n I\{\pi(X) = Z\} [\kappa(Z, X) - \hat{\kappa}(Z, X)] \]

\[+ P_0 [\kappa(Z, X) - \hat{\kappa}(Z, X)] \left[ \frac{I\{\pi(X) = Z\}}{f(Z|X)} - \frac{I\{\pi(X) = Z\}}{f(Z|X)} \right] + o_p(n^{-1/2}) \]

\[(3) = P_0 \frac{A(Z + A)I\{\pi(X) = Z\} \delta(A, Z, X)[\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2\gamma(A, Z, X) f(Z|X)} \left[ \frac{1}{\hat{f}(A|Z, X)} - \frac{1}{f(A|Z, X)} \right] \]

\[+ P_n \frac{A(Z + A)I\{\pi(X) = Z\} \delta(A, Z, X)[\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2\gamma(A, Z, X) \hat{f}(A|Z, X)} \left[ \frac{1}{\hat{f}(A|Z, X)} - \frac{1}{f(A|Z, X)} \right] \]

\[+ P_0 \frac{A(Z + A)I\{\pi(X) = Z\} \delta(A, Z, X)[\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2\hat{f}(Z|X) \hat{f}(A|Z, X)} \left[ \frac{1}{\hat{\gamma}(A, Z, X)} - \frac{1}{\gamma(A, Z, X)} \right] \]

\[+ P_0 \frac{A(Z + A)I\{\pi(X) = Z\}[w_{a\gamma}(A, Z, Y) - \hat{\gamma}(A, Z, X)]\delta(A, Z, X) - \hat{\delta}(A, Z, X)]}{2\hat{\gamma}(A, Z, X) f(Z|X) \hat{f}(A|Z, X)} + o_p(n^{-1/2}) \]

\[(4.1) = P_n \frac{A(Z + A)I\{\pi(X) = Z\} [Q(A, Z, X) - \hat{Q}(A, Z, X)]}{2\gamma(A, Z, X) f(Z|X) \hat{f}(A|Z, X)} \]

\[+ P_0 \frac{A(Z + A)I\{\pi(X) = Z\} [Q(A, Z, X) - \hat{Q}(A, Z, X)]}{2\gamma(A, Z, X) \hat{f}(A|Z, X)} \left[ \frac{1}{\hat{f}(A|Z, X)} - \frac{1}{f(A|Z, X)} \right] \]

\[+ P_0 \frac{A(Z + A)I\{\pi(X) = Z\} [Q(A, Z, X) - \hat{Q}(A, Z, X)]}{2\hat{f}(Z|X) \hat{f}(A|Z, X)} \left[ \frac{1}{\hat{\gamma}(A, Z, X)} - \frac{1}{\gamma(A, Z, X)} \right] \]

\[+ P_0 \frac{A(Z + A)I\{\pi(X) = Z\} [Q(A, Z, X) - \hat{Q}(A, Z, X)]}{2\gamma(A, Z, X) f(Z|X) \hat{f}(A|Z, X)} \left[ \frac{1}{\gamma(A, Z, X) f(Z|X) \hat{f}(A|Z, X)} - \frac{1}{\hat{\gamma}(A, Z, X) f(Z|X) \hat{f}(A|Z, X)} \right] \]

\[(4.2) = P_0 [\hat{\theta}(Z, X) - \theta(Z, X)] \left[ \frac{1}{\hat{f}(Z|X)} - \frac{1}{f(Z|X)} \right] \]

\[+ P_n [\hat{\theta}(Z, X)] \left[ \frac{1}{f(Z|X)} - \frac{1}{f(Z|X)} \right] + P_n \frac{\hat{\theta}(Z, X) - \theta(Z, X)}{f(Z|X)} \]

Putting everything together, we have
\[ R = P_n \left\{ \frac{A(Z + A)}{2} Y w_{\alpha z}(A, Z, Y) I\{\pi(X) = Z\} \left( \frac{1}{\hat{\gamma}(A, Z, X) f(Z|X) \hat{f}(A|Z, X)} \right) \right. \]

\[
- \frac{1}{\gamma(A, Z, X) f(Z|X) f(A|Z, X)} \right) \}
\]

\[ + P_n \kappa(Z, X) I\{\pi(X) = Z\} \left[ \frac{1}{f(Z|X)} - \frac{1}{\hat{f}(Z|X)} \right] + P_n I\{\pi(X) = Z\} [\kappa(Z, X) - \hat{\kappa}(Z, X)] \]

\[ + P_0 [\kappa(Z, X) - \hat{\kappa}(Z, X)] \left[ \frac{I\{\pi(X) = Z\}}{\hat{f}(Z|X)} - \frac{I\{\pi(X) = Z\}}{f(Z|X)} \right] + P_n \left\{ \hat{\kappa}'(X) - \kappa'(X) \right\} \]

\[ + P_0 \left\{ \frac{A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2\gamma(A, Z, X) f(Z|X) f(A|Z, X)} \left[ \frac{1}{\hat{f}(A|Z, X)} - \frac{1}{f(A|Z, X)} \right] \right. \]

\[ + P_0 \left\{ \frac{A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2\gamma(A, Z, X) f(Z|X) f(A|Z, X)} \left[ \frac{1}{\hat{f}(Z|X)} - \frac{1}{f(Z|X)} \right] \right. \]

\[ + P_0 \left\{ \frac{A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]}{2\hat{f}(Z|X) \hat{f}(A|Z, X)} \left[ \frac{1}{\hat{\gamma}(A, Z, X)} - \frac{1}{\gamma(A, Z, X)} \right] \right. \]

\[ + P_0 \left\{ \frac{A(Z + A) I\{\pi(X) = Z\} [w_{\alpha z}(A, Z, Y) - \hat{\gamma}(A, Z, X)] [\delta(A, Z, X) - \hat{\delta}(A, Z, X)]}{2\hat{\gamma}(A, Z, X) \hat{f}(Z|X) \hat{f}(A|Z, X)} \right. \]

\[ + P_n \left\{ \frac{A(Z + A) I\{\pi(X) = Z\} Q(A, Z, X) - \hat{Q}(A, Z, X)}{2\gamma(A, Z, X) f(Z|X) f(A|Z, X)} \left[ \frac{1}{\hat{f}(Z|X)} - \frac{1}{f(Z|X)} \right] \right. \]

\[ + P_0 \left\{ \frac{A(Z + A) I\{\pi(X) = Z\} Q(A, Z, X) - \hat{Q}(A, Z, X)}{2\gamma(A, Z, X) f(Z|X) f(A|Z, X)} \left[ \frac{1}{\hat{f}(A|Z, X)} - \frac{1}{f(A|Z, X)} \right] \right. \]

\[ + P_0 \left\{ \frac{A(Z + A) I\{\pi(X) = Z\} Q(A, Z, X) - \hat{Q}(A, Z, X)}{2\gamma(A, Z, X) f(Z|X) f(A|Z, X)} \left[ \frac{1}{\hat{\gamma}(A, Z, X)} - \frac{1}{\gamma(A, Z, X)} \right] \right. \]

\[ + P_n \left\{ \frac{A(Z + A) I\{\pi(X) = Z\} Q(A, Z, X)}{2} \left[ \frac{1}{\gamma(A, Z, X) f(Z|X) f(A|Z, X)} \right. \right. \]

\[ \left. - \frac{1}{\hat{\gamma}(A, Z, X) \hat{f}(Z|X) \hat{f}(A|Z, X)} \right) \}

\[ + P_0 \left[ \hat{\theta}(Z, X) - \theta(Z, X) \right] \left[ \frac{1}{\hat{f}(Z|X)} - \frac{1}{f(Z|X)} \right] + P_n \theta(Z, X) \left[ \frac{1}{\hat{f}(Z|X)} - \frac{1}{f(Z|X)} \right] \]

\[ + P_n \frac{\hat{\theta}(Z, X) - \theta(Z, X)}{f(Z|X)} + o_p(n^{-1/2}) \]
We let $R_1$ be the term with $P_n$ and $R_2$ be the term with $P_0$ in $R$

\[
R_1 = P_n A(Z + A) I\{\pi(X) = Z\} [Y w_{az} Z, A, X) - Q(A, Z, X)] \frac{1}{2} \left[ \frac{1}{\hat{\gamma}(A, Z, X) \hat{f}(Z|X) \hat{f}(A|Z, X)} - \frac{1}{\gamma(A, Z, X) f(Z|X) f(A|Z, X)} \right]
\]

\[
+ P_n \left[ \kappa(Z, X) I\{\pi(X) = Z\} - \theta(Z, X) \right] \left[ \frac{1}{f(Z|X)} - \frac{1}{\hat{f}(Z|X)} \right]
\]

\[
+ P_n \left\{ \hat{\kappa}'(X) - \kappa'(X) + \frac{I\{\pi(X) = Z\}}{f(Z|X)} [\kappa(Z, X) - \hat{\kappa}(Z, X)] \right\}
\]

\[
+ P_n A(Z + A) I\{\pi(X) = Z\} \delta(A, Z, X) [\hat{\gamma}(A, Z, X) - \gamma(A, Z, X)]
\]

\[
+ P_n A(Z + A) I\{\pi(X) = Z\} \left[ Q(A, Z, X) - \hat{Q}(A, Z, X) \right] \frac{1}{2\gamma(A, Z, X) f(Z|X) f(A|Z, X)} + P_n \frac{\hat{\theta}(Z, X) - \theta(Z, X)}{f(Z|X)}
\]

\[
= R_1(1) + R_1(2) + R_1(3) + R_1(4) + R_1(5)
\]
We simplify the term in $R_1$
\[ R_1(1) = o_p(n^{-1/2}) \text{ because } P_0 \left[ Y w_{az}(A, Z, Y) - Q(A, Z, X) \right] = o_p(n^{-1/2}) \]

\[ R_1(2) = o_p(n^{-1/2}) \text{ because } P_0 \left[ \kappa(Z, X) I \{ \pi(X) = Z \} - \theta(Z, X) \right] = o_p(n^{-1/2}) \]

\[ R_1(3) = P_n \left\{ \hat{\kappa}'(X) - \kappa'(X) + \frac{I(\pi(X) = Z)}{f(Z|X)} [\kappa(Z, X) - \hat{\kappa}(Z, X)] \right\} \]

\[ = P_0 \hat{\kappa}'(X) - P_0 \kappa'(X) + \left[ P_n - P_0 \right] \left[ \hat{\kappa}'(X) - \kappa'(X) \right] \]

\[ + P_0 \frac{I(\pi(X) = Z) \kappa(Z, X)}{f(Z|X)} - P_0 \frac{I(\pi(X) = Z) \hat{\kappa}(Z, X)}{f(Z|X)} + \left( P_n - P_0 \right) \frac{\kappa(X) - \hat{\kappa}(Z, X)}{f(Z|X)} \]

\[ = o_p(n^{-1/2}) \]

\[ \text{because } P_0 \hat{\kappa}'(X) = P_0 \frac{I(\pi(X) = Z) \hat{\kappa}(Z, X)}{f(Z|X)}, \quad P_0 \kappa'(X) = P_0 \frac{I(\pi(X) = Z) \kappa(Z, X)}{f(Z|X)} \]

\[ R_1(5) = P_n \frac{A(Z + A) I \{ \pi(X) = Z \} \left[ Q(A, Z, X) - \hat{Q}(A, Z, X) \right]}{2 \gamma(A, Z, X) f(Z|X) f(A|Z, X)} + P_n \frac{\hat{\theta}(Z, X) - \theta(Z, X)}{f(Z|X)} \]

\[ = -P_0 \left[ \frac{I(\pi(X) = 1) \hat{Q}(1, 1, X)}{\gamma(1, 1, X)} + \frac{I(\pi(X) = -1) \hat{Q}(-1, 1, X)}{\gamma(-1, 1, X)} \right] \]

\[ + P_0 \left[ \frac{I(\pi(X) = 1) \hat{Q}(1, 1, X)}{\gamma(1, 1, X)} + \frac{I(\pi(X) = -1) \hat{Q}(-1, 1, X)}{\gamma(-1, 1, X)} \right] + o_p(n^{-1/2}) \]

\[ = P_0 \left[ I(\pi(X) = 1) \hat{Q}(1, 1, X) \left( \frac{\gamma(1, 1, X) - \gamma(1, 1, X)}{\gamma(1, 1, X)} \right) \right] \]

\[ + P_0 \left[ I(\pi(X) = -1) \hat{Q}(-1, 1, X) \left( \frac{\gamma(-1, 1, X) - \gamma(-1, 1, X)}{\gamma(-1, 1, X)} \right) \right] + o_p(n^{-1/2}) \]

\[ R_1(4) = P_n \frac{A(Z + A) I \{ \pi(X) = Z \} \delta(A, Z, X) \left[ \hat{\gamma}(A, Z, X) - \gamma(A, Z, X) \right]}{2 \gamma(A, Z, X) f(Z|X) f(A|Z, X)} \]

\[ = P_0 \frac{I(\pi(X) = 1) Q(1, 1, X) \left[ \hat{\gamma}(1, 1, X) - \gamma(1, 1, X) \right]}{[\gamma(1, 1, X)]^2} \]

\[ + P_0 \frac{I(\pi(X) = -1) Q(-1, 1, X) \left[ \hat{\gamma}(-1, 1, X) - \gamma(-1, 1, X) \right]}{[\gamma(-1, 1, X)]^2} \]

Lastly, we simplify two terms in \( R_2 \)
\[
\begin{align*}
P_0 \frac{A(Z + A)I\{\pi(X) = Z\}}{2\gamma(A, Z, X)f(A|Z, X)} & \left[ Q(A, Z, X) - \hat{Q}(A, Z, X) \right] \left[ \frac{1}{f(Z|X)} - \frac{1}{f(Z|X)} \right] \\
& + P_0 \left[ \hat{\theta}(Z, X) - \theta(Z, X) \right] \left[ \frac{1}{f(Z|X)} - \frac{1}{f(Z|X)} \right] \\
& = P_0 \left[ \frac{1}{f(Z|X)} - \frac{1}{f(Z|X)} \right] \left[ \frac{A(Z + A)I\{\pi(X) = Z\}Q(A, Z, X)}{2\gamma(A, Z, X)f(A|Z, X)} - \theta(Z, X) \right] \\
& - P_0 \left[ \frac{1}{f(Z|X)} - \frac{1}{f(Z|X)} \right] \left[ \frac{A(Z + A)I\{\pi(X) = Z\}\hat{Q}(A, Z, X)}{2\gamma(A, Z, X)f(A|Z, X)} - \hat{\theta}(Z, X) \right] \\
& = o_p(\sqrt{n})
\end{align*}
\]

The second equation is due to

\[
P_0 \frac{A(Z + A)I\{\pi(X) = Z\}Q(A, Z, X)}{2\gamma(A, Z, X)f(A|Z, X)} = P_0\theta(Z, X)
\]

and

\[
P_0 \frac{A(Z + A)I\{\pi(X) = Z\}\hat{Q}(A, Z, X)}{2\gamma(A, Z, X)f(A|Z, X)} = P_0\hat{\theta}(Z, X)
\]

Putting all the component together
\[
R = P_0I\{\pi(X) = 1\} \frac{\hat{\gamma}(1, 1, X) - \gamma(1, 1, X)}{\gamma(1, 1, X)} \left[ \frac{\hat{Q}(1, 1, X) - Q(1, 1, X)}{\gamma(1, 1, X)} \right] \\
+ P_0I\{\pi(X) = -1\} \frac{\hat{\gamma}(-1, -1, X) - \gamma(-1, -1, X)}{\gamma(-1, -1, X)} \left[ \frac{\hat{Q}(-1, -1, X) - Q(-1, -1, X)}{\gamma(-1, -1, X)} \right] \\
+ P_0\left[ \kappa(Z, X) - \hat{\kappa}(Z, X) \right] \left( \frac{I\{\pi(X) = Z\}}{f(Z|X)} - \frac{I\{\pi(X) = Z\}}{f(Z|X)} \right) \\
+ P_0 \frac{A(Z + A)I\{\pi(X) = Z\} \delta(A, Z, X)}{2f(A|Z, X)f(Z|X)} \left[ \frac{1}{\gamma(A, Z, X)} - \frac{1}{\gamma(A, Z, X)} \right] \\
+ P_0 \frac{A(Z + A)I\{\pi(X) = Z\} \delta(A, Z, X)}{2\gamma(A, Z, X)f(Z|X)} \left[ \frac{1}{f(Z|X)} - \frac{1}{f(Z|X)} \right] \\
+ P_0 \frac{A(Z + A)I\{\pi(X) = Z\} \delta(A, Z, X)}{2\gamma(A, Z, X)f(Z|X)} \left[ \frac{1}{f(A|Z, X)} - \frac{1}{f(A|Z, X)} \right] \\
+ P_0 \frac{A(Z + A)I\{\pi(X) = Z\} \delta(A, Z, X)}{2f(Z|X)f(A|Z, X)} \left[ Q(A, Z, X) - \hat{Q}(A, Z, X) \right] \left[ \frac{1}{\gamma(A, Z, X)} - \frac{1}{\gamma(A, Z, X)} \right] \\
+ P_0 \frac{A(Z + A)I\{\pi(X) = Z\} \delta(A, Z, X)}{2\gamma(A, Z, X)f(Z|X)} \left[ Q(A, Z, X) - \hat{Q}(A, Z, X) \right] \left[ \frac{1}{f(A|Z, X)} - \frac{1}{f(A|Z, X)} \right] + o_p(n^{-1/2}) \\
\]

Hence, the remainder will go to \(o_p(n^{-1/2})\) under the assumption:
For any \( z \),

\[
(\|f - \hat{f}\|_{P_0} + \|p - \hat{p}\|_{P_0} + \|\gamma - \hat{\gamma}\|_{P_0})(\|\gamma - \hat{\gamma}\|_{P_0}) + (\|p - \hat{p}\|_{P_0} + \|\gamma - \hat{\gamma}\|_{P_0})(\|Q - \hat{Q}\|_{P_0}) \\
+ (\|\gamma - \hat{\gamma}\|)(\|\delta - \hat{\delta}\|) + (\|\kappa - \hat{\kappa}\|)(\|f - \hat{f}\|_{P_0}) = o_p(n^{-1/2})
\]

where \( f = f(Z|X) \) and \( p = f(A|Z, X) \)

\[\text{F} \quad \text{Addition Simulations with different value of } (\alpha_1, \alpha_{+1})\]

\[\text{F}.1 \quad \text{Sample Size 250}\]

Table 6: Simulation Result: Mean (sd) of value functions. The empirical optimal value function is 1.68. The sensitivity parameter are \((\alpha_1, \alpha_{+1}) = (0.5,0.5)\)

| Case                      | OWL    | IVT    | IPW    | MR     |
|---------------------------|--------|--------|--------|--------|
| 1 All correctly specified | 1.05 (0.00) | 1.46 (0.18) | 1.63 (0.09) | 1.67 (0.09) |
| 2 \( f(A,Z|X) \) misspecified | 1.06 (0.06) | 1.47 (0.16) | 1.54 (0.10) | 1.67 (0.08) |
| 3 \( Q(A,Z,X) \) misspecified | 1.04 (0.02) | 1.48 (0.16) | 1.58 (0.06) | 1.55 (0.09) |

Table 7: Simulation Result: Mean (sd) of correct classification rate. The sensitivity parameter are \((\alpha_1, \alpha_{+1}) = (0.5,0.5)\)

| Case                      | OWL    | IVT    | IPW    | MR     |
|---------------------------|--------|--------|--------|--------|
| 1 All correctly specified | 0.51 (0.04) | 0.72 (0.10) | 0.87 (0.06) | 0.94 (0.04) |
| 2 \( f(A,Z|X) \) misspecified | 0.52 (0.06) | 0.73 (0.09) | 0.78 (0.05) | 0.95 (0.03) |
| 3 \( Q(A,Z,X) \) misspecified | 0.52 (0.05) | 0.72 (0.09) | 0.87 (0.06) | 0.86 (0.07) |

Table 8: Simulation Result: Demonstration of the robustness of the multiply robust estimator of the value function of the IPW method. The sensitivity parameter are \((\alpha_1, \alpha_{+1}) = (0.5,0.5)\)

| Case                      | Empirical       | Non-robust Estimator | Multiply robust Estimator |
|---------------------------|-----------------|----------------------|--------------------------|
| 1 All correctly specified | 1.63 (0.09)     | 1.63 (0.21)          | 1.59 (0.13)              |
| 2 \( f(Z|X) \) misspecified | 1.52 (0.09)     | 1.93 (0.17)          | 1.49 (0.10)              |
| 3 \( f(A|Z,X) \) misspecified | 1.62 (0.09)     | 2.48 (0.40)          | 1.52 (0.21)              |
| 4 \( Q(A,Z,X) \) misspecified | 1.64 (0.06)     | 1.66 (0.13)          | 1.62 (0.10)              |
Table 9: Simulation Result: Mean (sd) of value functions. The empirical optimal value function is 1.62. The sensitivity parameter are \((\alpha_{-1}, \alpha_{+1}) = (0.5, -0.5)\)

| Case                      | OWL   | IVT   | IPW   | MR    |
|---------------------------|-------|-------|-------|-------|
| 1 All correctly specified | 0.93 (0.00) | 1.39 (0.14) | 1.58 (0.07) | 1.61 (0.07) |
| 2 \(f(A, Z|X)\) misspecified | 0.93 (0.02) | 1.40 (0.10) | 1.49 (0.07) | 1.62 (0.06) |
| 4 \(Q(A, Z, X)\) misspecified | 0.93 (0.00) | 1.38 (0.14) | 1.58 (0.06) | 1.59 (0.15) |

Table 10: Simulation Result: Mean (sd) of correct classification rate. The sensitivity parameter are \((\alpha_{-1}, \alpha_{+1}) = (0.5, -0.5)\)

| Case                      | OWL   | IVT   | IPW   | MR    |
|---------------------------|-------|-------|-------|-------|
| 1 All correctly specified | 0.48 (0.03) | 0.70 (0.07) | 0.89 (0.05) | 0.95 (0.03) |
| 2 \(f(A, Z|X)\) misspecified | 0.48 (0.03) | 0.71 (0.06) | 0.79 (0.04) | 0.97 (0.02) |
| 3 \(Q(A, Z, X)\) misspecified | 0.48 (0.03) | 0.69 (0.07) | 0.89 (0.03) | 0.87 (0.06) |

Table 11: Simulation Result: Demonstration of the robustness of the multiply robust estimator of the value function of the IPW method. The sensitivity parameter are \((\alpha_{-1}, \alpha_{+1}) = (0.5, -0.5)\)

| Case                      | Empirical | Non-robust Estimator | Multiply robust Estimator |
|---------------------------|-----------|----------------------|--------------------------|
| 1 All correctly specified | 1.58 (0.07) | 1.59 (0.14)         | 1.56 (0.09)              |
| 2 \(f(Z|X)\) misspecified | 1.47 (0.07) | 1.84 (0.11)         | 1.43 (0.08)              |
| 3 \(f(A|Z,X)\) misspecified | 1.57 (0.07) | 2.44 (0.26)         | 1.53 (0.13)              |
| 4 \(Q(A, Z, X)\) misspecified | 1.58 (0.06) | 1.60 (0.14)         | 1.56 (0.14)              |
Table 12: Simulation Result: Mean (sd) of value functions. The empirical optimal value function is 1.65. The sensitivity parameter are \((\alpha_{-1}, \alpha_{+1}) = (0,0)\)

| Case                  | OWL     | IVT     | IPW     | MR      |
|-----------------------|---------|---------|---------|---------|
| 1 All correctly specified | 1.00 (0.00) | 1.44 (0.13) | 1.61 (0.06) | 1.64 (0.06) |
| 2 f(A, Z|X) misspecified    | 1.00 (0.02) | 1.52 (0.10) | 1.65 (0.06) | 1.65 (0.06) |
| 3 Q(A, Z, X) misspecified | 1.00 (0.00) | 1.46 (0.19) | 1.61 (0.07) | 1.61 (0.07) |

Table 13: Simulation Result: Mean (sd) of correct classification rate. The sensitivity parameter are \((\alpha_{-1}, \alpha_{+1}) = (0,0)\)

| Case                  | OWL     | IVT     | IPW     | MR      |
|-----------------------|---------|---------|---------|---------|
| 1 All correctly specified | 0.51 (0.03) | 0.72 (0.07) | 0.89 (0.05) | 0.95 (0.03) |
| 2 f(A|Z|X) misspecified    | 0.51 (0.04) | 0.73 (0.06) | 0.79 (0.04) | 0.97 (0.02) |
| 3 Q(A, Z, X) misspecified | 0.51 (0.03) | 0.73 (0.07) | 0.89 (0.05) | 0.89 (0.05) |

Table 14: Simulation Result: Demonstration of the robustness of the multiply robust estimator of the value function of the IPW method. The sensitivity parameter are \((\alpha_{-1}, \alpha_{+1}) = (0,0)\)

| Case                  | Empirical | Non-robust Estimator | Multiply robust Estimator |
|-----------------------|-----------|----------------------|--------------------------|
| 1 All correctly specified | 1.61 (0.06) | 1.63 (0.15) | 1.59 (0.09) |
| 2 f(Z|X) misspecified    | 1.49 (0.07) | 1.88 (0.12) | 1.45 (0.07) |
| 3 f(A|Z, X) misspecified | 1.59 (0.07) | 2.48 (0.27) | 1.55 (0.13) |
| 4 Q(A, Z, X) misspecified | 1.61 (0.07) | 1.62 (0.13) | 1.60 (0.12) |
G  Generating the outcome of the compliers

The outcome of the compliers, \( y^{S4|x} \), follows the distribution \( Y|S4, X, A, Z \) with the density

\[
f(y|S4, X, A, Z) = \frac{w_{aZ}(y, A, Z)f_Z(y)}{\gamma(A, Z, X)}
\]

We will use the rejection sampling method to draw samples of \( Y|S4, X, A, Z \) from a normal distribution \( Y_N \sim N(1.5, 2^2) \) with the density

\[
f_{Y_N}(y) = \frac{1}{2\sqrt{2\pi}} \exp\{-\frac{1}{2} \left( \frac{y - 1.5}{2} \right)^2 \}
\]

using the following algorithm.

**Input:** \( x, a, z, y \)  
\( M \leftarrow 3.5; \)

**for** \( i = 1 \) **to** 8000 **do**

- Draw \( u_i \) from Uniform(0,1);
- Draw \( y_{N,i} \) from Normal(1.5, 4);
- \( C \leftarrow \frac{f(y_{N,i}|S4,x,a,z)}{Mf_{Y_N}(y_{N,i})} \)
- **if** \( u_i < C \) **then**
  - Accept \( y_{N,i} \)
- **else**
  - Reject \( y_{N,i} \)

**end**

**Output** \( y^{S4|x} = \) the mean of the accepted \( y_{N,i} \)

**Algorithm 1:** Rejection Sampling algorithm to draw \( y^{S4|x} \)

We apply the Algorithm 1 to every subject that has the compliance level \( A \) equals to the treatment \( Z \).

H  Additional Heat Maps

Heat maps of the multiply robust method when nuisance parameters are misspecified
Figure 5: Sensitivity analysis of the multiply robust method when the nuisance parameters $Q(1,1,X)$ and $Q(-1,-1,X)$ are incorrectly specified: Sensitivity parameters are $(\alpha_{-1}, \alpha_{+1})$. The darker area indicates a higher correct classification rate/value function. The true vector of sensitivity parameters are $(\alpha_{-1}, \alpha_{+1}) = (0.5,-0.5)$.

Figure 6: Sensitivity analysis of the multiply robust method when the nuisance parameter $f(A|Z,X)$ is incorrectly specified: Sensitivity parameters are $(\alpha_{-1}, \alpha_{+1})$. The darker area indicates a higher correct classification rate/value function. The true vector of sensitivity parameters are $(\alpha_{-1}, \alpha_{+1}) = (0.5,-0.5)$. 
Figure 7: Sensitivity analysis of the multiply robust method when the nuisance parameter $f(Z|X)$ is incorrectly specified: Sensitivity parameters are $(\alpha_{-1}, \alpha_{+1})$. The darker area indicates a higher correct classification rate/value function. The true vector of sensitivity parameters are $(\alpha_{-1}, \alpha_{+1}) = (0.5, -0.5)$.

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