ASYMPTOTIC BEHAVIORS OF CONVOLUTION POWERS OF THE RIEmann Zeta Distribution

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Abstract. In probability theory, there exist discrete and continuous distributions. Generally speaking, we do not have sufficient kinds and properties of discrete ones compared to the continuous ones. In this paper, we treat the Riemann zeta distribution as a representative of few known discrete distributions with infinite supports. Some asymptotic behaviors of convolution powers of the Riemann zeta distribution are discussed.

1. Introduction and main results

Let $\varphi$ be a complex-valued function whose support $\mathcal{S} = \mathcal{S}_1 \subset \mathbb{R}$ is at most countable. For $n \in \mathbb{N}$, the $n$-th convolution power of the function $\varphi$ is defined by $\varphi^1 := \varphi$ and

$$\varphi^n(x) := \sum_{y \in \mathcal{S}} \varphi^{(n-1)}(x - y) \varphi(y), \quad n = 2, 3, \ldots, x \in \mathcal{S}_n,$$

where $\mathcal{S}_n$ denotes the countable support of $\varphi^n$ which may be different from $\mathcal{S}$ in general.

It has been interesting problems to investigate some asymptotic behaviors of the convolution powers $\varphi^n$ as $n \to \infty$ from not only analytic but probabilistic perspectives. Let us first look at the cases where $\mathcal{S}$ is finite. One of motivations behind such kinds of problems was found in the context of numerical difference schemes for partial differential equations. We refer to e.g. Thomeé [13, 14] for details. Moreover, motivated by the de Forest’s local limit theorems and statistical data smoothing procedures, Greville [5] and Schoenberg [12] treated finitely supported functions $\varphi$ with values in $\mathbb{R}$ and established their local limit theorems. Namely, the uniform convergence of the convolution power $\varphi^n(x)$ as $n \to \infty$ was obtained and its leading term was shown to be an analytic function like heat kernels. We note that some well-known properties of Fourier transforms play a crucial role in obtaining such asymptotic behaviors. Afterwards, Randles and Saloff-Coste [10] extended the local limit theorems to the cases where the finitely supported function $\varphi$ is complex-valued. Furthermore, the corresponding leading term is shown to be governed by a complex-valued analytic function which is regarded as an evaluation of a function similar to the heat kernel at some imaginary time. We also refer to Diaconis–Saloff-Coste [9] for related results on such local limit theorems and Randles–Saloff-Coste [11] for extensions of the results in [10] to multidimensional cases with concrete examples.

In view of probability theory, these results exactly help us to reveal the detailed asymptotic behaviors of random walks whose one-step distribution consists of finitely many
mass points. In spite of such developments, asymptotic behaviors of convolution powers of countably supported functions have not been investigated so much, though there exist a number of discrete probability distributions whose support is countable. We emphasize that main difficulties in considering these problems can be often replaced by the complexities of convolution powers of functions with countable supports.

Therefore, our aim is to provide some asymptotic behaviors of convolution powers of treatable functions whose support is countable. Particularly, in the present paper, we try to focus on the so-called Riemann zeta distribution on $\mathbb{R}$, which is known as an important example of countably supported probability distribution on $\mathbb{R}$.

In order to introduce this distribution, we start with the definition of the Riemann zeta function $\zeta(s)$.

**Definition 1.1 (Riemann zeta function).** The Riemann zeta function $\zeta(s)$ is a function of a complex variable $s = \sigma + it$ for $\sigma > 1$ and $t \in \mathbb{R}$ defined by

$$\zeta(s) := \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad (1.1)$$

where we denote by $\mathbb{P}$ the set of all prime numbers. The product representation of (1.1) is called the Euler product.

It is known that the function $\zeta(s)$ converges absolutely in the half-plane $\{\sigma + it \mid \sigma > 1\}$ and uniformly in every compact subset of the half-plane. We also note that the Riemann zeta function can be extended to a meromorphic function on the complex plane $\mathbb{C}$ having a single pole at $s = 1$ by analytic continuation. See e.g., Apostol [3] for more details on the Riemann zeta function.

The Riemann zeta distribution is defined as a probability distribution on $\mathbb{R}$ generated by $\zeta(s)$.

**Definition 1.2 (Riemann zeta distribution).** Fix $\sigma > 1$. A probability distribution $\mu_\sigma$ on $\mathbb{R}$ is called a Riemann zeta distribution if

$$\mu_\sigma(\{-\log m\}) = \frac{m^{-\sigma}}{\zeta(\sigma)}, \quad m \in \mathbb{N}. \quad (1.2)$$

Let $\sigma > 1$. As is seen later in Section 2, it is verified that the normalized function

$$f_\sigma(t) := \frac{\zeta(\sigma + it)}{\zeta(\sigma)}, \quad t \in \mathbb{R},$$

is a characteristic function of the Riemann zeta distribution $\mu_\sigma$. Moreover, we can also see that the $\mu_\sigma$ is infinitely divisible (see Propositions 2.1 and 2.2). Note that we often identify $\mu_\sigma$ with a countably supported function $\mu_\sigma = \mu_\sigma(x)$ given by

$$\mu_\sigma(x) = \sum_{m=1}^{\infty} \frac{m^{-\sigma}}{\zeta(\sigma)} 1_{\{-\log m\}}(x), \quad x \in \Lambda_1,$$
where \( \Lambda_1 = \{- \log m \mid m \in \mathbb{N} \} \). Then, for \( n \in \mathbb{N} \), the \( n \)-th convolution power of the function \( \mu_\sigma \) is recursively given by \( \mu_\sigma = \mu_\sigma \) and

\[
\mu_\sigma^n(x) = \sum_{y \in \Lambda_{n-1}} \mu_\sigma^{(n-1)}(x-y) \mu_\sigma(y)
= \sum_{m_1, m_2, \ldots, m_n = 1}^{\infty} \frac{(m_1 m_2 \cdots m_n)^{-\sigma}}{\zeta(\sigma)^n} \mathbf{1}_{\{- \log (m_1 m_2 \cdots m_n)\}}(x), \quad x \in \Lambda_n, \ n = 2, 3, \ldots,
\]

where the countable support \( \Lambda_n \) of \( \mu_\sigma^n \) is given by

\[
\Lambda_n = \{ x \in \mathbb{R} \mid y \in \Lambda_{n-1}, x-y \in \Lambda_1 \}
= \{ - \log (m_1 m_2 \cdots m_n) \mid m_1, m_2, \ldots, m_n \in \mathbb{N} \}, \quad n = 2, 3, \ldots.
\]

By applying the Taylor formula to the function \( \Gamma_\sigma(t) := \log f_\sigma(t) \) on a neighborhood of 0, we know that there exist \( \alpha_\sigma \in \mathbb{R} \) and \( \beta_\sigma > 0 \) such that

\[
\Gamma_\sigma(t) := i \alpha_\sigma t - \beta_\sigma t^2 + o(t^2)
\]
as \( t \to 0 \), where the constants \( \alpha_\sigma \) and \( \beta_\sigma \) can be regarded as the expectation and the variance of the Riemann zeta distribution \( \mu_\sigma \), respectively. The explicit representations of these constants will be given in Lemma 3.3.

The main results of the present paper consist of two claims. The first main result is the very local limit theorem for the \( n \)-th convolution power \( \mu_\sigma^n \) of the Riemann zeta distribution \( \mu_\sigma \), which reveals the leading term of the asymptotic behavior of \( \mu_\sigma^n \) as \( n \to \infty \).

**Theorem 1.3.** Let \( \sigma > 1 \). Then, we have

\[
\mu_\sigma^n(x) = \frac{1}{\sqrt{n}} p_\sigma \left( \frac{x - \alpha_\sigma n}{\sqrt{n}} \right) + o \left( \frac{1}{\sqrt{n}} \right), \quad x \in \Lambda_n,
\]
as \( n \to \infty \), where the function

\[
p_\sigma(x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixu} e^{-\beta_\sigma u^2} du = \frac{1}{\sqrt{4\pi \beta_\sigma}} \exp \left( - \frac{x^2}{4\beta_\sigma} \right), \quad x \in \mathbb{R},
\]
is the heat kernel evaluated at time \( \beta_\sigma \).

The second main result of the present paper is the upper bound of the supremum norm of \( \mu_\sigma^n \), which is stated as follows:

**Theorem 1.4.** Let \( \sigma > 1 \). Then, there exists a positive constant \( C_\sigma > 0 \) such that

\[
\| \mu_\sigma^n \|_\infty := \sup_{x \in \Lambda_n} \mu_\sigma^n(x) \leq \frac{C_\sigma}{\sqrt{n}}, \quad n \in \mathbb{N}.
\]

Let \( \{ X_\sigma^{(n)} \}_{n=1}^\infty \) be a sequence of i.i.d. random variables whose common law is \( \mu_\sigma \) with \( \sigma > 1 \). Then, the usual central limit theorem implies that the random variable defined by

\[
\frac{X_\sigma^{(1)} + X_\sigma^{(2)} + \cdots + X_\sigma^{(n)} - n \alpha_\sigma}{\sqrt{n}}, \quad n \in \mathbb{N},
\]
converges in law to the normal distribution \( N(0, 2\beta_\sigma) \) as \( n \to \infty \). Namely, we obtain
\[
\lim_{n \to \infty} P\left( a \leq \frac{X^{(1)}_\sigma + X^{(2)}_\sigma + \cdots + X^{(n)}_\sigma - n\alpha_\sigma}{\sqrt{n}} \leq b \right) = \int_a^b \frac{1}{\sqrt{4\pi\beta_\sigma}} \exp\left( -\frac{x^2}{4\beta_\sigma} \right) dx
\]
for all \(-\infty \leq a < b \leq \infty\). Our local limit theorem (Theorem 1.3) can be regarded as a refinement of the central limit theorem for the sequence of Riemann zeta random variables \( \{X^{(n)}_\sigma\}_{n=1}^\infty \). Moreover, it turns out from Theorem 1.4 that the supremum norm of \( \mu^{*n}_\sigma(x) \) decays on at most the order of \( \sqrt{n} \). We claim that our results are to be a specific case of known rather unknown examples of local limit theorems usually discussed in probability theory.

The rest of the present paper is organized as follows: We review some basics of the Riemann zeta distribution \( \mu_\sigma, \sigma > 1 \), in Section 2. In particular, the expectation and the variance of the Riemann zeta random variable are computed. We establish local behaviors of the characteristic function \( f_\sigma(t) \) of \( \mu_\sigma \) on a neighborhood of 0 in Section 3. More precisely, we establish both the lower and the upper estimates of \( |f_\sigma(t)| \) on the neighborhood by making use of the Lévy–Khintchine representation of \( f_\sigma(t) \) (see Lemmas 3.2 and 3.4). Section 4 is devoted to the proof of the local limit theorem for \( \mu^{*n}_\sigma \) as \( n \to \infty \) (see Theorem 1.3). We employ the standard Fourier analysis technique and the upper estimate of \( f_\sigma(t) \) in order to find out the leading term of \( \mu^{*n}_\sigma \) as \( n \) tends to infinity. We also give the proof of Theorem 1.4 which provides the upper bound of the supremum norm of \( \mu^{*n}_\sigma \), in Section 5. We give some further comments towards possible extensions of our study in Section 6 as well.

2. The Riemann zeta distribution

Let \( \sigma > 1 \) and consider the Riemann zeta distribution \( \mu_\sigma, \sigma > 1 \), defined by (1.2). In this section, several properties of the Riemann zeta distribution are exhibited. We refer to Lin–Hu [8] for related topics. The following fact is well-known.

**Proposition 2.1** (see e.g., Gnedenko–Kolmogorov [1]). The characteristic function \( f_\sigma(t) \) of (1.2) is given by
\[
f_\sigma(t) := \frac{\zeta(\sigma + it)}{\zeta(\sigma)}, \quad t \in \mathbb{R}.
\]
We note that the function \( f_\sigma(t) \) is also obtained by the Fourier transform of the function \( \mu_\sigma(x) \). The Riemann zeta distribution is known to be infinitely divisible.

**Proposition 2.2** (see e.g., Gnedenko–Kolmogorov [1]). Let \( \mu_\sigma \) be a Riemann zeta distribution on \( \mathbb{R} \) with the characteristic function \( f_\sigma(t) \). Then, \( \mu_\sigma \) is compound Poisson on \( \mathbb{R} \) and it holds that
\[
\log f_\sigma(t) = \sum_{p \in \mathbb{P}} \sum_{r=1}^{\infty} p^{-r\sigma} r \left( e^{-itr \log p} - 1 \right) = \int_0^\infty \left( e^{-itx} - 1 \right) N_\sigma(dx), \quad t \in \mathbb{R}, \tag{2.1}
\]
where \( N_\sigma(dx) \) is a finite Lévy measure on \( \mathbb{R} \) given by
\[
N_\sigma(dx) = \sum_{p \in \mathbb{P}} \sum_{r=1}^{\infty} p^{-r\sigma} \delta_r \log p(dx). \tag{2.2}
\]
Here, $\delta_a(dx)$ denotes the delta measure at $a \in \mathbb{R}$.

By direct calculations, the first and the second derivatives of $\zeta(s)$ are given by

$$
\zeta'(s) = \sum_{m=1}^{\infty} \frac{-\log m}{m^s}, \quad \zeta''(s) = \sum_{m=1}^{\infty} \frac{(\log m)^2}{m^s},
$$

respectively. Let $X_\sigma$ be a random variable whose distribution is $\mu_\sigma$. Then, the expectations $E[X_\sigma]$, $E[X_\sigma^2]$ and the variance $\text{Var}(X_\sigma)$ of $X_\sigma$ are given in the following.

**Proposition 2.3.** It holds that

$$
E[X_\sigma] = \frac{\zeta'(\sigma)}{\zeta(\sigma)}, \quad E[X_\sigma^2] = \frac{\zeta''(\sigma)}{\zeta(\sigma)}, \quad \text{Var}(X_\sigma) = \frac{1}{\zeta(\sigma)^2} \{\zeta(\sigma)\zeta''(\sigma) - \zeta'(\sigma)^2\}.
$$

**Proof.** It follows from (2.3) that

$$
E[X_\sigma] = \sum_{m=1}^{\infty} \frac{-\log m}{\zeta(\sigma)} = \sum_{m=1}^{\infty} \frac{-\log m}{m^\sigma} \frac{1}{\zeta(\sigma)} = \frac{\zeta'(\sigma)}{\zeta(\sigma)},
$$

$$
E[X_\sigma^2] = \sum_{m=1}^{\infty} \frac{(\log m)^2}{m^\sigma} \frac{1}{\zeta(\sigma)} = \sum_{m=1}^{\infty} \frac{(\log m)^2}{m^\sigma} \frac{1}{\zeta(\sigma)} \frac{1}{\zeta(\sigma)} = \frac{\zeta''(\sigma)}{\zeta(\sigma)},
$$

$$
\text{Var}(X_\sigma) = E[X_\sigma^2] - (E[X_\sigma])^2 = \frac{\zeta''(\sigma)}{\zeta(\sigma)} - \left(\frac{\zeta'(\sigma)}{\zeta(\sigma)}\right)^2 = \frac{1}{\zeta(\sigma)^2} \{\zeta(\sigma)\zeta''(\sigma) - \zeta'(\sigma)^2\},
$$

which are the desired equalities. \qed

3. Local behaviors of the Riemann zeta distribution

Let us fix $\sigma > 1$ and consider the characteristic function $f_\sigma : \mathbb{R} \rightarrow \mathbb{C}$ of the Riemann zeta distribution $\mu_\sigma$ given in Proposition 2.1. By definition, we have $f_\sigma(0) = 1$ and $|f_\sigma(t)| \leq 1$ for $t \in \mathbb{R}$. In particular, $|f_\sigma(t)|$ attains its maximum 1 at $t = 0$. Therefore, we are interested in some local behaviors of $f_\sigma(t)$ at $t = 0$. In this section, we give both the lower and the upper bounds of $|f_\sigma(t)|$ on a neighborhood of $t = 0$, which play a key role in the proof of main theorems.

We put $\Gamma_\sigma(t) \equiv \log f_\sigma(t)$, $t \in \mathbb{R}$. Since the function $\Gamma_\sigma(t)$ is analytic on a neighborhood of 0, we can consider its convergent Taylor series

$$
\Gamma_\sigma(t) = \sum_{l=1}^{\infty} a_l(\sigma) t^l
$$
on a neighborhood of 0. We give the explicit representation of the coefficient $a_l(\sigma)$, $l = 1, 2, \ldots$, by using Proposition 2.2.

**Lemma 3.1.** The coefficient $a_l(\sigma)$, $l = 1, 2, \ldots$, is given by

$$
a_l(\sigma) = \sum_{p \in \mathbb{P}} \sum_{r=1}^{\infty} \frac{p^{-r \sigma}}{r} \frac{(-ir \log p)^l}{l!}, \quad l = 1, 2, \ldots.
$$
Proof. By applying Proposition 2.2, we have
\[
\Gamma_\sigma(t) = \sum_{p \in \mathbb{P}} \sum_{r=1}^{\infty} \frac{p^{-r\sigma}}{r} \left( e^{-itr \log p} - 1 \right)
\]
\[
= \sum_{p \in \mathbb{P}} \sum_{r=1}^{\infty} \frac{p^{-r\sigma}}{r} \left( \sum_{l=1}^{\infty} \frac{(-itr \log p)^l}{l!} \right) = \sum_{l=1}^{\infty} \left( \sum_{p \in \mathbb{P}} \sum_{r=1}^{\infty} \frac{p^{-r\sigma}}{r} \frac{(-itr \log p)^l}{l!} \right) t^l
\]
for every \( t \) in a neighborhood of 0. \( \square \)

The following proposition gives the lower estimate of the function \( |f_\sigma(t)| \) on a neighborhood of 0.

Lemma 3.2. There exists a positive constant \( C_\sigma > 0 \) such that \( \exp(-C_\sigma t^2) \leq |f_\sigma(t)| \) on a neighborhood of 0.

Proof. It follows from Lemma 3.1 that
\[
|f_\sigma(t)| = \left| \exp \left( \sum_{l=1}^{\infty} \sum_{p \in \mathbb{P}} \sum_{r=1}^{\infty} \frac{p^{-r\sigma}}{r} \frac{(-itr \log p)^l}{l!} t^l \right) \right|
\]
\[
= \exp \left( \sum_{j=1}^{\infty} \sum_{p \in \mathbb{P}} \sum_{r=1}^{\infty} \frac{p^{-r\sigma}}{r} \frac{(r \log p)^{4j}}{(4j)!} t^{4j} - \sum_{j=1}^{\infty} \sum_{p \in \mathbb{P}} \sum_{r=1}^{\infty} \frac{p^{-r\sigma}}{r} \frac{(r \log p)^{4j-2}}{(4j-2)!} t^{4j-2} \right)
\]
\[
\geq \exp \left( - \sum_{j=1}^{\infty} \sum_{p \in \mathbb{P}} \sum_{r=1}^{\infty} \frac{p^{-r\sigma}}{r} \frac{(r \log p)^{4j-2}}{(4j-2)!} t^{4j-2} \right) \geq \exp \left( - \sum_{j=1}^{\infty} \sum_{p \in \mathbb{P}} \sum_{r=1}^{\infty} \frac{p^{-r\sigma}}{r} \frac{(r \log p)^{4j-2}}{(4j-2)!} t^{2} \right) = \exp(-C_\sigma t^2)
\]
for every \( t \) in a neighborhood of 0. \( \square \)

We put
\[
\alpha_\sigma := \sum_{p \in \mathbb{P}} \frac{- \log p}{p^\sigma - 1}, \quad \beta_\sigma := \sum_{p \in \mathbb{P}} \frac{(\log p)^2}{2 \left( p^\sigma - 1 \right)^2}.
\]
Since it holds that \( a_1(\sigma) = i\alpha_\sigma \) and \( a_2(\sigma) = -\beta_\sigma \), we obtain
\[
\Gamma_\sigma(t) = i\alpha_\sigma t - \beta_\sigma t^2 + \sum_{l=3}^{\infty} a_l(\sigma) t^l \tag{3.1}
\]
on a neighborhood of 0. Moreover, it follows from Proposition 2.2 that
\[
\Gamma_\sigma(t) = \int_{0}^{\infty} \sum_{l=1}^{\infty} \frac{(-itx)^l}{l!} N_\sigma(dx) = \sum_{l=1}^{\infty} \left( \int_{0}^{\infty} \frac{(-ix)^l}{l!} N_\sigma(dx) \right) t^l \tag{3.2}
\]
on a neighborhood of 0. Therefore, each coefficient \( a_l(\sigma) \) is also represented as
\[
a_l(\sigma) = \int_{0}^{\infty} \frac{(-ix)^l}{l!} N_\sigma(dx), \quad l = 1, 2, \ldots.
\]
Particularly, look at the cases where $l = 1, 2$. It follows from (2.2) that

\[ a_1(\sigma) = -i \int_0^\infty x \sum_{p \in \mathbb{P}} \sum_{r=1}^\infty \frac{p^{-r\sigma}}{r} \delta_{r \log p}(dx) = -i \sum_{p \in \mathbb{P}} \sum_{r=1}^\infty p^{-r\sigma} \log p, \]

\[ a_2(\sigma) = -\frac{1}{2} \int_0^\infty x^2 \sum_{p \in \mathbb{P}} \sum_{r=1}^\infty \frac{p^{-r\sigma}}{r} \delta_{r \log p}(dx) = -\frac{1}{2} \sum_{p \in \mathbb{P}} \sum_{r=1}^\infty rp^{-r\sigma}(\log p)^2. \]

By putting it all together, we have the following.

**Lemma 3.3.** It holds that

\[ \alpha_\sigma = -\int_0^\infty x N_\sigma(dx) = \sum_{p \in \mathbb{P}} \frac{-\log p}{p^\sigma - 1} = -\sum_{p \in \mathbb{P}} \sum_{r=1}^\infty p^{-r\sigma} \log p, \]

\[ \beta_\sigma = \frac{1}{2} \int_0^\infty x^2 N_\sigma(dx) = \sum_{p \in \mathbb{P}} \frac{(\log p)^2}{2} \cdot \frac{1}{p^\sigma - 1} = \frac{1}{2} \sum_{p \in \mathbb{P}} \sum_{r=1}^\infty rp^{-r\sigma}(\log p)^2. \]

Here, we also give another expressions of the constants $\alpha_\sigma$ and $\beta_\sigma$ in terms of the Riemann zeta random variable. Let $X_\sigma$ be a random variable whose distribution is $\mu_\sigma$. Then, we can show that

\[ \alpha_\sigma = \mathbb{E}[X_\sigma] = \frac{\zeta'(\sigma)}{\zeta(\sigma)}, \quad \beta_\sigma = \frac{1}{2} \text{Var}(X_\sigma) = \frac{1}{2\zeta(\sigma)^2} \{\zeta(\sigma)\zeta''(\sigma) - \zeta'(\sigma)^2\}. \]  \hfill (3.3)

Indeed, we have

\[ \frac{d}{dt} f_\sigma(t) = \frac{i \zeta'(\sigma + it)}{\zeta(\sigma)}, \quad \frac{d^2}{dt^2} f_\sigma(t) = -\frac{\zeta''(\sigma + it)}{\zeta(\sigma)} \]

for $t \in \mathbb{R}$. By letting $t = 0$ and by using Proposition 2.3 we obtain

\[ \frac{d}{dt} f_\sigma(0) = i \frac{\zeta'(\sigma)}{\zeta(\sigma)} = i \mathbb{E}[X_\sigma], \quad \frac{d^2}{dt^2} f_\sigma(0) = -\frac{\zeta''(\sigma)}{\zeta(\sigma)} = -\mathbb{E}[X^2]. \]  \hfill (3.4)

On the other hand, the Lebesgue convergence theorem and Proposition 2.2 imply

\[ \frac{d}{dt} f_\sigma(t) = -if_\sigma(t) \left( \int_0^\infty xe^{-itx} N_\sigma(dx) \right), \quad t \in \mathbb{R}, \]

\[ \frac{d^2}{dt^2} f_\sigma(t) = f_\sigma(t) \left\{ - \left( \int_0^\infty xe^{-itx} N_\sigma(dx) \right)^2 - \int_0^\infty x^2 e^{-itx} N_\sigma(dx) \right\}. \quad t \in \mathbb{R}. \]

By letting $t = 0$, we also have

\[ \frac{d}{dt} f_\sigma(0) = -i \int_0^\infty x N_\sigma(dx), \]  \hfill (3.5)

\[ \frac{d^2}{dt^2} f_\sigma(0) = -\left( \int_0^\infty x N_\sigma(dx) \right)^2 - \int_0^\infty x^2 N_\sigma(dx). \]  \hfill (3.6)

Therefore, by combining Lemma 3.3 with (3.4), (3.5) and (3.6), we obtain

\[ \mathbb{E}[X_\sigma] = \alpha_\sigma, \quad \mathbb{E}[X^2_\sigma] = \alpha^2_\sigma + 2\beta_\sigma, \quad \text{Var}(X_\sigma) = \mathbb{E}[X^2_\sigma] - (\mathbb{E}[X_\sigma])^2 = 2\beta_\sigma, \]

which are the desired equalities (3.3).
The following lemma gives the upper estimate of $|f_\sigma(t)|$ on a neighborhood of 0.

**Lemma 3.4.** There exists a positive constant $B_\sigma > 0$ such that

$$|\Gamma_\sigma(t) - i\alpha_\sigma t + \beta_\sigma t^2| \leq B_\sigma |t|^3,$$

and

$$|f_\sigma(t)| \leq \exp\left(-\frac{1}{2}\beta_\sigma t^2\right)$$

(3.7)

hold on a neighborhood of 0.

*Proof.* On a neighborhood of 0, we have

$$|\Gamma_\sigma(t) - i\alpha_\sigma t + \beta_\sigma t^2| = \left| \sum_{l=3}^{\infty} a_l(\sigma)t^l \right| = \left| \sum_{l=3}^{\infty} \sum_{p \in \mathbb{P}} \sum_{r=1}^{\infty} \frac{p^{-r\sigma}}{r} \left(-ir\log p\right)^l \frac{t^l}{l!} \right| \leq \sum_{l=2}^{\infty} \sum_{p \in \mathbb{P}} \sum_{r=1}^{\infty} \frac{p^{-r\sigma}}{r} \left(r\log p\right)^l \frac{|t|^3}{l!} = B_\sigma |t|^3.$$

Moreover, for every $t$ in a neighborhood of 0, we have

$$|f_\sigma(t)| = \left| \exp\left(i\alpha_\sigma t - \beta_\sigma t^2 + \sum_{l=3}^{\infty} a_l(\sigma)t^l \right) \right| \leq \exp\left(-\beta_\sigma t^2 + \left| \sum_{l=3}^{\infty} a_l(\sigma)t^l \right| \right) \leq \exp\left((-\beta_\sigma + B_\sigma |t|)t^2\right).$$

By taking the neighborhood of 0 being sufficiently small, we have

$$-\beta_\sigma + B_\sigma |t| \leq -\frac{1}{2}\beta_\sigma,$$

which concludes (3.7) on the neighborhood of 0. □

4. **Proof of Theorem 1.3**

The aim of this section is to give a proof of the local limit theorem for the convolution power of the Riemann zeta distribution $\mu_\sigma$ with fixed $\sigma > 1$ (Theorem 1.3). The following function will play a crucial role in the proof. We define a function $p_\sigma : \mathbb{R} \to \mathbb{R}$ by

$$p_\sigma(x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixu} e^{-\beta_\sigma u^2} du = \frac{1}{\sqrt{4\pi\beta_\sigma}} \exp\left(-\frac{x^2}{4\beta_\sigma}\right), \quad x \in \mathbb{R},$$

which is referred to the heat kernel on $\mathbb{R}$ evaluated at time $\beta_\sigma > 0$. In probability theory, it is well-known that the convolution powers of probability distributions are often approximated by the heat kernel. In this section, we also see that such local limit theorems are valid for the case of Riemann zeta distributions. For this sake, we first show the following lemma.

**Lemma 4.1.** For any $\varepsilon > 0$, there exist $\delta > 0$ and $N \in \mathbb{N}$ such that

$$\left| \frac{\sqrt{n}}{2\pi} \int_{|t|<\delta} f_\sigma(t)^n e^{-ixt} dt - p_\sigma\left(\frac{x - \alpha_\sigma n}{\sqrt{n}}\right) \right| < \varepsilon$$

(4.1)

for all $n \geq N$ and $x \in \mathbb{R}$.
Proof. Since the function \( u \mapsto \exp(-\beta u^2/2) \) is Lebesgue integrable on \( \mathbb{R} \), for any \( \varepsilon > 0 \), there exists \( M > 0 \) such that
\[
\int_{|u| \geq M} \exp\left(-\frac{1}{2}\beta u^2\right) \, du < \frac{2\pi\varepsilon}{3}. \tag{4.2}
\]
For such \( M > 0 \), we take a sufficiently large \( n \in \mathbb{N} \) with \( M < \delta \sqrt{n} \). By using Lemma 3.4, there exists \( \delta > 0 \) such that \( |u| < \delta \sqrt{n} \) implies
\[
\left| \exp\left(-\frac{i\alpha u}{\sqrt{n}}\right) f_{\sigma}\left(\frac{u}{\sqrt{n}}\right) \right|^n \leq \exp\left(-\frac{1}{2}\beta_{\sigma}\left(\frac{u}{\sqrt{n}}\right)^2\right)^n = \exp\left(-\frac{1}{2}\beta_{\sigma} u^2\right). \tag{4.3}
\]
Since it holds that
\[
\frac{\sqrt{n}}{2\pi} \int_{|t| < \delta} f_{\sigma}(t)^n e^{-iut} \, dt
\]
\[
= \frac{1}{2\pi} \int_{|u| < \delta \sqrt{n}} \left\{ \exp\left(-\frac{i\alpha u}{\sqrt{n}}\right) f_{\sigma}\left(\frac{u}{\sqrt{n}}\right) \right\}^n \exp\left(-iu\frac{x-\alpha_n}{\sqrt{n}}\right) \, du,
\]
we have
\[
\left| \frac{\sqrt{n}}{2\pi} \int_{|t| < \delta} f_{\sigma}(t)^n e^{-iut} \, dt - p_{\sigma}\left(\frac{x-\alpha_n}{\sqrt{n}}\right) \right|
\]
\[
= \left| \frac{1}{2\pi} \int_{|u| < \delta \sqrt{n}} \left\{ \exp\left(-\frac{i\alpha u}{\sqrt{n}}\right) f_{\sigma}\left(\frac{u}{\sqrt{n}}\right) \right\}^n \exp\left(-iu\frac{x-\alpha_n}{\sqrt{n}}\right) \, du
\]
\[
- \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left(-iu\frac{x-\alpha_n}{\sqrt{n}}\right) e^{-\beta_{\sigma} u^2} \, du \right|\]
\[
\leq \frac{1}{2\pi} \int_{|u| < M} \left| \left\{ \exp\left(-\frac{i\alpha u}{\sqrt{n}}\right) f_{\sigma}\left(\frac{u}{\sqrt{n}}\right) \right\}^n - e^{-\beta_{\sigma} u^2} \right| \, du
\]
\[
+ \frac{1}{2\pi} \int_{M \leq |u| < \delta \sqrt{n}} \left| \left\{ \exp\left(-\frac{i\alpha u}{\sqrt{n}}\right) f_{\sigma}\left(\frac{u}{\sqrt{n}}\right) \right\}^n \, du + \frac{1}{2\pi} \int_{|u| \geq M} e^{-\beta_{\sigma} u^2} \, du
\]
\[
=: I_1 + I_2 + I_3,
\]
We first consider the term \( I_1 \). Since
\[
\lim_{n \to \infty} \left| \left\{ \exp\left(-\frac{i\alpha u}{\sqrt{n}}\right) f_{\sigma}\left(\frac{u}{\sqrt{n}}\right) \right\}^n - e^{-\beta_{\sigma} u^2} \right|
\]
\[
= \lim_{n \to \infty} e^{-\beta_{\sigma} u^2} \left| \exp\left(\frac{1}{\sqrt{n}} \sum_{l=3}^{\infty} a_l(\sigma) \frac{u^l}{n(l-3)/2} \right) - 1 \right| = 0, \quad |u| \leq M,
\]
there exists \( N \in \mathbb{N} \) such that \( N > (M/\delta)^2 \) implies \( I_1 < \varepsilon/3 \) by applying the Lebesgue convergence theorem. As for the terms \( I_2 \) and \( I_3 \), (4.2) and (4.3) yield
\[
I_2 \leq \frac{1}{2\pi} \int_{M \leq |u| < \delta \sqrt{n}} \exp\left(-\frac{1}{2}\beta_{\sigma} u^2\right) \, du \leq \frac{1}{2\pi} \int_{|u| \geq M} \exp\left(-\frac{1}{2}\beta_{\sigma} u^2\right) \, du < \frac{\varepsilon}{3},
\]
\[
I_3 \leq \frac{1}{2\pi} \int_{|u| \geq M} e^{-\beta_{\sigma} u^2} \, du \leq \frac{1}{2\pi} \int_{|u| \geq M} \exp\left(-\frac{1}{2}\beta_{\sigma} u^2\right) \, du < \frac{\varepsilon}{3}.
\]
By putting it all together, we have established (4.1). \( \square \)
We are now ready for the proof of Theorem 1.3.

**Proof of Theorem 1.3.** Since the characteristic function $f_{\sigma}(t)$ is obtained as the Fourier transform of the function $\mu_{\sigma}(x)$, we also obtain

$$\mu_{\sigma}^n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_{\sigma}(t)^n e^{-ixt} \, dt, \quad n \in \mathbb{N}, \, x \in \Lambda_n.$$ (4.4)

Let $\varepsilon > 0$. By taking a sufficiently small $\delta > 0$, we have

$$\sqrt{n} \mu_{\sigma}^n(x) = \frac{\sqrt{n}}{2\pi} \int_{|t| < \delta} f_{\sigma}(t)^n e^{-ixt} \, dt$$

$$= \frac{\sqrt{n}}{2\pi} \int_{|t| < \delta} f_{\sigma}(t)^n e^{-ixt} \, dt + \frac{\sqrt{n}}{2\pi} \int_{\delta \leq |t| < \pi} f_{\sigma}(t)^n e^{-ixt} \, dt$$

$$=: J_1 + J_2$$

for all $x \in \Lambda_n$. Let us consider the term $J_1$. By virtue of Lemma 4.1, there exist $\delta > 0$ and $N \in \mathbb{N}$ such that

$$\left| \frac{\sqrt{n}}{2\pi} \int_{|t| < \delta} f_{\sigma}(t)^n e^{-ixt} \, dt - p_{\sigma} \left( \frac{x - \alpha_{\sigma}n}{\sqrt{n}} \right) \right| < \frac{\varepsilon}{2}$$

for $n \geq N$ and $x \in \Lambda_n$. Hence, we have

$$|J_1| \leq p_{\sigma} \left( \frac{x - \alpha_{\sigma}n}{\sqrt{n}} \right) + \frac{\varepsilon}{2}, \quad x \in \Lambda_n.$$

On the other hand, as for the term $J_2$, we also have

$$|J_2| \leq \frac{\sqrt{n}}{2\pi} \int_{\delta \leq |t| < \pi} |f_{\sigma}(t)|^n \, dt \leq \sqrt{n} \left( \sup_{\delta \leq |t| < \pi} |f_{\sigma}(t)| \right)^n.$$

By noting the fact that $\sup_{\delta \leq |t| < \pi} |f_{\sigma}(t)| < 1$, we take a sufficiently large $N' \in \mathbb{N}$ such that $n \geq N'$ implies that $|J_2| < \varepsilon/2$. Therefore, we obtain

$$\left| \sqrt{n} \mu_{\sigma}^n(x) - p_{\sigma} \left( \frac{x - \alpha_{\sigma}n}{\sqrt{n}} \right) \right| \leq \varepsilon$$

for all $n \geq \max\{N, N'\}$ and $x \in \Lambda_n$, which leads to

$$\mu_{\sigma}^n(x) = \frac{1}{\sqrt{n}} p_{\sigma} \left( \frac{x - \alpha_{\sigma}n}{\sqrt{n}} \right) + o\left( \frac{1}{\sqrt{n}} \right)$$

as $n \to \infty$ for all $x \in \Lambda_n$. $\square$

5. **Proof of Theorem 1.4**

In order to show Theorem 1.4, we need to give the proof of the following lemma.

**Lemma 5.1.** There exists a sufficiently small $\delta > 0$ such that

$$\left| \frac{1}{2\pi} \int_{|t| < \delta} f_{\sigma}(t)^n e^{-ixt} \, dt \right| \leq \frac{1}{\sqrt{n} \beta_{\sigma}}, \quad n \in \mathbb{N}, \, x \in \Lambda_n.$$
Proof. By applying Lemma 3.4, there exists a sufficiently small δ > 0 such that
\[ |f_\sigma(t)| \leq \exp \left( -\frac{n}{2} \beta_\sigma t^2 \right), \quad |t| < \delta. \]
Therefore, we obtain
\[ \left| \frac{1}{2\pi} \int_{|t|<\delta} f_\sigma(t)^n e^{-ixt} dt \right| \leq \frac{1}{2\pi} \int_{|t|<\delta} |f_\sigma(t)|^n dt \leq \frac{1}{2\pi} \int_{\mathbb{R}} \exp \left( -\frac{n}{2} \beta_\sigma t^2 \right) dt = \frac{1}{\sqrt{n\beta_\sigma}} \]
for \( n \in \mathbb{N} \) and \( x \in \Lambda_n \).

At last, we give the proof of Theorem 1.4.

Proof of Theorem 1.4. By virtue of (4.4) and Lemma 3.4, there is a sufficiently small \( \delta > 0 \) such that
\[ \mu_\sigma^{*n}(x) \leq \frac{1}{2\pi} \int_{|t|<\delta} f_\sigma(t)^n e^{-ixt} dt + \frac{1}{2\pi} \int_{\delta \leq |t| < \pi} f_\sigma(t)^n e^{-ixt} dt \]
\[ \leq \frac{1}{\sqrt{n\beta_\sigma}} + \left( \sup_{\delta \leq |t| < \pi} |f_\sigma(t)| \right)^n, \quad n \in \mathbb{N}, \ x \in \Lambda_n. \]
By noting \( \sup_{\delta \leq |t| < \pi} |f_\sigma(t)| < 1 \) and by taking the supremum over \( x \in \Lambda_n \), there exists a sufficiently large \( C_\sigma > 0 \) such that
\[ \|\mu_\sigma^{*n}\|_\infty \leq \frac{C_\sigma}{\sqrt{n}}, \quad n \in \mathbb{N}, \]
which is the desired upper bound.

6. Conclusion
Throughout the present paper, we have discussed asymptotic behaviors including local limit theorems for \( n \)-th convolution powers of Riemann zeta distributions as \( n \to \infty \). We believe that our result exactly contributes to the study of convolution powers of complex functions in that the function treated in the present paper has a not finite but countable support. On the other hand, we can find several studies in which relations between generalizations of the Riemann zeta function and probability distributions on \( \mathbb{R}^d \) are discussed. We refer to Hu–Iksanov–Lin–Zakusylo [6] for probability distributions generated by the Hurwitz zeta function and Aoyama–Nakamura [1, 2] for those generated by certain classes of multidimensional zeta functions. Such zeta functions are also countably supported and have large amount of applications in both probability theory and number theory. Hence, further studies for investigating precise asymptotics of convolution powers of zeta distributions are expected, which should be interesting problems to reveal.

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