Relative quantum coherence, incompatibility, and quantum correlations of states

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Quantum coherence, incompatibility, and quantum correlations are fundamental features of physics. A unified view of those features is crucial for revealing quantitatively their intrinsic connections. We define the relative quantum coherence of two states as the coherence of one state in the reference basis spanned by the eigenvectors of another one and establish its quantitative connections with the extent of mutual incompatibility of two states. We also show that the proposed relative quantum coherence, which can take any form of measures such as $l_1$ norm and relative entropy, can be interpreted as or connected to various quantum correlations such as quantum discord, symmetric discord, entanglement of formation, and quantum deficits. Our results reveal conceptual implications and basic connections of quantum coherence, mutual incompatibility, and quantum correlations.

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I. INTRODUCTION

Quantum coherence is rooted in the superposition of quantum states, and also remains as a research focus since the early days of quantum mechanics [1]. It plays a key role for nearly all the novel quantum phenomena in the fields of quantum optics [1], quantum thermodynamics [2–6], and quantum biology [7]. However, its characterization and quantification from a mathematically rigorous and physically meaningful perspective has been achieved only very recently [8], when Baumgratz et al. introduced the defining conditions for a bona fide measure of coherence, and proved that the $l_1$ norm and relative entropy satisfy the required conditions.

In the past few years, many other coherence measures that satisfy the defining conditions have been proposed from different aspects [9–15]. Moreover, the freezing phenomenon of coherence in open systems [16–21], the coherence-preserving channels [22], and the creation of coherence by local or nonlocal operations [23–26] have been extensively studied. Other topics such as the coherence distillation [27–29], the complementarity relations of coherence [30, 31], the connections of coherence with path distinguishability [32, 33] and asymmetry [34, 35], the coherence averaged over all basis sets [30] or the Haar distributed pure states [36], and the role of coherence in state merging [37] were also studied.

For bipartite and multipartite systems, quantum coherence also underpins different forms of quantum correlations [9, 24, 38–44]. The mutual incompatibility of states, which represents another fundamental feature of the nonclassical systems, is also intimately related to quantum correlations, e.g., for any nondiscordant state there must exist local measurements which commute with it [45]. Intuitively, coherence, incompatibility, and quantum correlations are all closely related concepts. A direct and quantitative connection between them can provide a whole view and a measure in characterizing the quantumness of the nonclassical systems. In this paper, by defining the relative quantum coherence (RQC) of two states as the coherence of one state in the basis spanned by the eigenvectors of another one, we find connections between coherence, incompatibility, and quantum correlations (Fig. 1). Our observations include: (i) the quantitative connection between RQC and mutual incompatibility of states, and (ii) the interpretation of quantum discord (QD), symmetric discord, measurement-induced disturbance, quantum deficits, entanglement of formation, and distillable entanglement via the proposed RQC. We expect the connections established in this paper may contribute to a unified view of the resource theory of coherence, incompatibility and quantum correlations.

II. QUANTIFYING THE RQC

Consider two states $\rho$ and $\sigma$ in the same Hilbert space $\mathcal{H}$. When $\sigma$ is nondegenerate with the eigenvectors $\Xi = \{|\psi_i\rangle\}$, i.e., $\sigma = \sum \epsilon_i |\psi_i\rangle\langle \psi_i|$, (i.e., the eigenvalues of $\sigma$), we define the RQC of $\rho$ with respect to $\sigma$ as

$$C(\rho, \sigma) = C^\Xi(\rho),$$

where $C^\Xi(\rho)$ denotes any bona fide measure of quantum coherence defined in the reference basis $\Xi$.

The rationality for this choice of reference basis lies in that the RQC of a state with respect to itself equals zero. Although this definition is essentially the same as that of the coherence measure introduced by Baumgratz et al. [8], the choice of the eigenvectors of another state as the reference basis allows one to establish quantitatively the connections between quantum coherence, incompatibility, and quantum correlations of states (e.g., the excess of RQC of the total state with respect to the postmeasurement state and the sum of RQC of the reduced states with respect to the local postmeasurement states gives an interpretation of the QD), hence can deepen our understanding about distribution of quantumness in a composite.
system. Moreover, by choosing $\sigma = \rho_0$ as the initial state and $\rho = \rho_1$ as the evolved state, $C(\rho_1, \rho_0)$ also allows one to characterize the decoherence or coherence process of a system relative to its initial state.

If one takes the $l_1$ norm or the relative entropy as the coherence measure \[ C_{l_1}(\rho, \sigma) = \sum_{i \neq j} |\langle \psi_i | \rho | \psi_j \rangle|, \]
then
\[
C_{l_1}(\rho, \sigma) = \frac{1}{2} \sum_{i \neq j} |\langle \psi_i | \rho | \psi_j \rangle|,
\]
and
\[
C_{l_1}(\rho, \sigma) = \frac{1}{2} \sum_{i \neq j} |\langle \psi_i | \rho | \psi_j \rangle|.
\]

III. LINKING RQC TO INCOMPATIBILITY OF STATES

We establish connections between the RQC and mutual incompatibility of two states in this section. For this purpose, we rewrite $\rho$ in the basis $\Xi$ as $\rho = \sum_{kl} \lambda_{kl} |\psi_k\rangle \langle \psi_k|$, where $\lambda_{kl} = \langle \psi_k | \rho | \psi_l \rangle$. Then by Eq. (2) we obtain
\[
C_{l_1}(\rho, \sigma) = \sum_{i \neq j} |\lambda_{ij}|.
\]
On the other hand, the commutator $[\rho, \sigma]$ of states $\rho$ and $\sigma$ can be calculated as
\[
[\rho, \sigma] = \sum_{i \neq j} \lambda_{ij} (\epsilon_j - \epsilon_i) |\psi_i\rangle \langle \psi_j|,
\]
then if we quantify the extent of the mutual incompatibility of the two states by $l_1$ norm of the commutator, we have
\[
Q_{l_1}(\rho, \sigma) = \sum_{i \neq j} |\lambda_{ij}|,
\]
where $Q_{l_1}(\rho, \sigma)$ is by definition nonnegative, vanishes if and only if the commutator $[\rho, \sigma]$ vanishes, and is unitary invariant. Different from the mutual incompatibility $Q_F(\rho, \sigma) = 2 ||[\rho, \sigma]||^2_2$ measured by square of the Frobenius norm, which takes the value between 0 and 1 \[46\], the maximum possible value of $Q_{l_1}(\rho, \sigma)$ is $\sqrt{d-1}$. But apart from the single-qubit case, this maximum is reached on the pure but not maximally coherent states (see Appendix B).

Here, by comparing Eqs. (5) and (7), one can obtain a quantitative connection between $C_{l_1}(\rho, \sigma)$ and $Q_{l_1}(\rho, \sigma)$,
\[
C_{l_1}(\rho, \sigma) \geq Q_{l_1}(\rho, \sigma),
\]
due to $|\epsilon_j - \epsilon_i| \leq 1, \forall i, j$. For the special case that $\sigma$ is pure, it has only one nonvanishing eigenvalue; this gives $Q_{l_1}(\rho, \sigma) = 2 \sum_{i \neq 1} |\lambda_{i1}|$. Thus when the elements $\lambda_{ij}$ of $\rho$ not in the first row, the first column, and the main diagonal equal to zero, we have $C_{l_1}(\rho, \sigma) = Q_{l_1}(\rho, \sigma)$. That is to say, the bound in Eq. (8) is tight. In particular, for pure state $\sigma$ and arbitrary state $\rho$ of one qubit, the bound is always saturated.

From Eq. (8), one can see that the mutual incompatibility of two states provides a lower bound for their RQC. Alternatively, the RQC provides an upper bound for the extent of their mutual incompatibility. It also implies that when there is no RQC between two states, then they must commute with each other. On the other hand, when two states commute, they either share a common eigenbasis or are orthogonal, for which we always have the vanishing RQC. What is more, by using Eqs. (6), (7), and the definition of $Q_F(\rho, \sigma)$ \[46\], one can show that $Q_{l_1}(\rho, \sigma) \geq Q_F(\rho, \sigma)/2$. As $Q_F(\rho, \sigma)$ can be measured via an interferometric setup and without performing the full state tomography \[46\], it provides an experimentally accessible lower bound for the mutual incompatibility and RQC of two quantum states.

In fact, for every basis state $|\psi_i\rangle$ of $\sigma$, we can associate with it a pure state $\sigma_i = |\psi_i\rangle \langle \psi_i|$. Then by summing $Q_{l_1}(\rho, \sigma_i)$ over the set $\{\sigma_i\}$, one can obtain
\[
\sum_i Q_{l_1}(\rho, \sigma_i) = 2C_{l_1}(\rho, \sigma),
\]
that is to say, the sum of $Q_i(\rho, \sigma_i)$ over \{\sigma_i\} equals twice the RQC between $\rho$ and $\sigma$. This establishes another connection between RQC and the mutual incompatibility of two states.

IV. LINKING RQC TO QUANTUM CORRELATIONS

The RQC and incompatibility of states are also intimately related to quantum correlations such as QD \cite{47}. For bipartite state $\rho_{AB}$ with the reduced state $\rho_A$, if the QD $D_A(\rho_{AB})$ defined with respect to party $A$ equals zero, then $\rho_{AB}$ and $\rho_A \otimes \mathbb{I}_B$ commute. Equivalently, if $\rho_{AB}$ does not commute with $\rho_A \otimes \mathbb{I}_B$, then it is quantum discordant \cite{48}. By Eq. (8), we know that the nonvanishing mutual incompatibility implies the nonvanishing RQC of $\rho_{AB}$ with respect to $\rho_A \otimes \mathbb{I}_B$. In fact, a necessary and sufficient condition for $\rho_{AB}$ to have zero discord has also been proved, which says that $D_A(\rho_{AB}) = 0$ if and only if all the operators $\rho_{AB}[l,b] = \langle b_l | \rho_{AB} | b_2 \rangle$ commute with each other for any orthonormal basis \{|$b_l$\} in $\mathcal{H}_B$ \cite{49}. From the analysis below Eq. (8), we know that the commutativity of two operators corresponds to the vanishing RQC of them. Hence, the RQC can be linked to a QD state.

By denoting $\mathcal{P} = \{\Pi^A_k\}$ ($\Pi^B_k = |k\rangle\langle k|$) the local projective measurements on party $A$, and likewise for $\mathcal{Q} = \{\Pi^B_l\}$, the postmeasurement states after the measurements $\mathcal{P} \otimes \mathbb{I}_B$ and $\mathcal{P} \otimes \mathcal{Q}$ are given, respectively, by

$$\rho_{PB} = \sum_k p_k \Pi^A_k \otimes \rho_{B|k},$$

$$\rho_{PQ} = \sum_{kl} p_{kl} \Pi^A_k \otimes \Pi^B_l,$$

where $\rho_{B|k} = \text{tr}_A(\Pi^A_k \rho_{AB} \Pi^A_k)/p_k$, $p_k = \langle k | \rho_{AB} | k \rangle$, $p_{kl} = \langle kl | \rho_{AB} | kl \rangle$. Then, by the definition of QD \cite{47}, we obtain

$$D_A(\rho_{AB}) = S(\rho_A) - S(\rho_{AB}) - S(\tilde{\rho}_P) + S(\tilde{\rho}_{PB}) \leq S(\rho_A) - S(\rho_{AB}) - S(\tilde{\rho}_P) + S(\tilde{\rho}_{PQ}) = C_{\rho_{AB}, \tilde{\rho}_{PQ}} - C_{\rho_{AB}, \tilde{\rho}_P} = \delta(\rho_{AB}),$$

where $S(\tilde{\rho}_{PB}) \leq S(\tilde{\rho}_{PQ})$ as project measurements do not decrease entropy. Moreover, $\tilde{\rho}_{PB}$ and $\tilde{\rho}_{PQ}$ denote, respectively, the postmeasurement states of $\{\Pi^A_k\}$ and $\{\Pi^A_k \otimes \Pi^B_l\}$, where $\{\Pi^A_k\}$ are the optimal measurements for obtaining $D_A(\rho_{AB})$, while $\{\Pi^B_l\}$ can in fact be arbitrary projective measurements, and here we fix it to be the measurement that gives the minimal entropy increase of $\tilde{\rho}_{PB}$ for tightening the above bound. Moreover, the eigenbasis of $\tilde{\rho}_{PQ}$ ($\tilde{\rho}_P$) for obtaining the RQC in Eq. (11) are chosen to be \{|$k\rangle \otimes |l\rangle\}$ (\{|$k\rangle\}$), which corresponds to the optimal measurement $\{\Pi^A_k \otimes \Pi^B_l\}$, i.e., here we do not perform the optimization of Eq. (3) even if the postmeasurement states are degenerate. The same holds for other discussions of this section.

In Eq. (11), $C_{\rho_{AB}, \tilde{\rho}_{PQ}}$ is the RQC (defined by the relative entropy) between $\rho_{AB}$ and $\tilde{\rho}_{PQ}$, while $C_{\rho_{AB}, \tilde{\rho}_P}$ is that between $\rho_A$ and $\tilde{\rho}_P$, which can be recognized as the quantum coherence localized in subsystem $A$. From this point of view, the QD of a state is always smaller than or equal to $\delta(\rho_{AB})$, which characterizes the discrepancy between the RQCs for the total system and that for the subsystem to be measured in the definition of QD, see Fig. (b). When the RQC discrepancy $\delta(\rho_{AB})$ vanishes, there will be no QD in the state.

For the case of quantum-classical state $\chi_{AB} = \sum_l p_l |\varphi_l\rangle \langle \varphi_l|$, with $\rho_{AB}$ being the density operator in $\mathcal{H}_A$, $|\varphi_l\rangle$ the orthonormal basis (also the eigenvectors of $\chi_{AB} = \text{tr}_X \chi_{AB}$) in $\mathcal{H}_B$, and $\{p_l\}$ the probability distribution, we have $\rho_{PB} = \tilde{\rho}_{PQ}$ if we choose $Q = \{|\varphi_l\rangle \langle \varphi_l|\}$. Then the RQC discrepancy $\delta(\chi_{AB})$ equals exactly the QD of $\chi_{AB}$, i.e.,

$$D_A(\chi_{AB}) = C_{\rho_{AB}, \tilde{\rho}_{PQ}} - C_{\rho_{AB}, \tilde{\rho}_P} = \delta(\chi_{AB}),$$

and hence the bound given in Eq. (11) is tight.

Besides the QD defined with one-sided measurements \cite{47}, one can also demonstrate the role of RQC discrepancy in interpreting the symmetric discord $D_s(\rho_{AB}) = D(\rho_{AB}) - D(\rho)$ defined via two-sided optimal measurements $\{\Pi^A_k \otimes \Pi^B_l\}$ \cite{50}. For this case, one can prove that

$$D_s(\rho_{AB}) = C_{\rho_{AB}, \tilde{\rho}_{PQ}} - C_{\rho_{AB}, \tilde{\rho}_P} - C_{\rho_{PB}, \tilde{\rho}_Q} = \delta(\rho_{AB}),$$

with $\delta(\rho_{AB})$ being the RQC discrepancy. It implies that the symmetric discord $D_s(\rho_{AB})$ is nonzero if and only if there exists RQC not localized in the subsystems, see Fig. (b). This establishes a direct connection between the RQC discrepancy and the symmetric discord.

The RQC can also be linked to other discord-like correlation measures such as measurement-induced disturbance \cite{51}, measurement-induced nonlocality \cite{52}, and quantum deficits \cite{53, 54}. (i) For the measurement-induced disturbance, it is just the RQC of $\rho_{AB}$ with respect to $\rho_{AB} = \sum_{ij} \xi_{ij} \rho_{ij}$, i.e.,

$$M(\rho_{AB}) = C_{\rho_{AB}, \tilde{\rho}_{PQ}} \leq C_{\rho_{AB}, \tilde{\rho}_P} \leq \delta(\rho_{AB}).$$

Here $\xi_{ij} = |\langle \varphi^A_i | \varphi^B_j \rangle|^2$ and $\{|\varphi^A_i\rangle, |\varphi^B_j\rangle\}$ are local eigenvectors of the reduced states. (ii) For the measurement-induced nonlocality defined as $N_v(\rho_{AB}) = \max_{\Pi_A} S(\Pi^A_A|\rho|) - S(\rho)$ ($\Pi^A_A$ are restricted to the locally invariant measurements), from Eq. (9) of Ref. \cite{52} one can obtain that $N_v(\rho_{AB}) \leq C_{\rho_{AB}, \tilde{\rho}_{PQ}}$. Here, $\tilde{\rho}_{PQ}$ is similar to that in Eq. (11), and the difference
that \( \{ \Pi^A_k \} \) should be locally invariant. (iii) For the zero-way deficit \( \Delta^\theta = S(\tilde{\rho}_{PQ}) - S(\rho_{AB}) \) and one-way deficit (equal to the thermal discord [43]) \( \Delta^\gamma = S(\tilde{\rho}_{P|B}) - S(\rho_{AB}) \) (where \( \tilde{\rho}_{PQ} \) and \( \tilde{\rho}_{P|B} \) denote, respectively, the corresponding optimal postmeasurement states) [54], it is direct to see that

\[
\begin{align*}
\Delta^\theta(\rho_{AB}) &= C_{\text{re}}(\rho_{AB}, \tilde{\rho}_{PQ}), \\
\Delta^\gamma(\rho_{AB}) &\leq C_{\text{re}}(\rho_{AB}, \tilde{\rho}_{PQ}),
\end{align*}
\]

where for the quantum-classical states, the inequality becomes equality when \( Q = \{ \{ \hat{\rho}_i \} \{ \hat{\rho}_i \} \} \). These relations give interpretations of the corresponding correlation measures in terms of RQC, and hence bridge the gap between quantum coherence for a single quantum system and quantum correlations for a system with two parties, which are two fundamentals of quantum physics.

For certain cases, the RQC can also be linked to quantum entanglement. For example, by resorting to the Koashi-Winter equality [56], the chain inequality [57], and equality condition of the Araki-Lieb inequality [58], one can obtain

\[
E_I(\rho_{AB}) \leq \delta_l(\rho_{AB}),
\]

when the conditional entropy \( S(B|A) \) is nonnegative or when the equality \( S(\rho_{AB}) = |S(\rho_A) - S(\rho_B)| \) is satisfied (see Appendix C). Here, \( E_I(\rho_{AB}) \) denotes the entanglement of formation for \( \rho_{AB} \) [59]. The relation shows that for these cases, the entanglement of formation for a state is always bounded from above by its RQC discrepancy \( \delta_l(\rho_{AB}) \).

When one performs local measurements \( \{ \Pi^A_k \} \) on a system \( \rho_{AB} \), there will be entanglement created between the measurement apparatus \( M \) and the system \( AB \). Streltsov et al. showed that the minimal distillable entanglement \( E^\text{min}_{D}(\tilde{\rho}_{M:AB}) = \min_U E_D(\tilde{\rho}_{M:AB}) \) created between \( M \) and \( AB \) equals \( \Delta^\gamma(\rho_{AB}) \) [60]. Here, \( \tilde{\rho}_{M:AB} = U(|0_M\rangle\langle 0_M| \otimes \rho_{AB})U^\dagger \), and \( U \) are unitaries acting on \( MAB \) which realize a von Neumann measurement on \( A \), i.e., \( \text{tr}_M(\tilde{\rho}_{M:AB}) = \sum_k \Pi^A_k \tilde{\rho}_{M:AB} \Pi^A_k \). Then by Eq. (13) we obtain \( E^\text{min}_{D}(\tilde{\rho}_{M:AB}) \leq C_{\text{re}}(\tilde{\rho}_{M:AB}, \tilde{\rho}_{PQ}) \).

Moreover, the minimal partial entanglement \( E^\text{min}_{E_D}(\tilde{\rho}_{M:AB}) = \min_U [E_D(\tilde{\rho}_{M:AB}) - E_D(\tilde{\rho}_{M:AB})] \) created in a von Neumann measurement on \( A \) equals \( D_A(\rho_{AB}) \) [60], and this gives \( E^\text{min}_{E_D}(\tilde{\rho}_{M:AB}) \leq \delta_l(\rho_{AB}) \). All these relations show the role of RQC in interpreting quantum entanglement.

V. CONCLUSION

In this paper, we explored the connections among quantum coherence, incompatibility, and quantum correlations. We defined the RQC and proved several of its connections to the extent of mutual incompatibility of states. We also gave interpretations of various quantum correlations (QD, entanglement of formation, symmetric discord, measurement-induced disturbance, measurement-induced nonlocality, and quantum deficits, etc.) via the RQC discrepancy between the total system and those localized in the respective subsystems.

We mainly considered the \( l_1 \) norm and relative entropy of coherence, but we remark that the coherence measure in other forms can also be extended to the case of RQC, and more connections among coherence, mutual incompatibility, and quantum correlations can be expected in future research.

Noted added. Recently Yao et al. presented a similar study on the problem of maximum coherence under generic basis [61].

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Appendix A: Proof of Eq. (4)

For any \( d \)-dimensional state \( \rho \), it can always be decomposed as

\[
\rho = \frac{1}{d} \mathbb{I}_d + \frac{1}{2} \sum_{i=1}^{d^2-1} x_i X_i, \tag{A1}
\]

where \( \{ X_i \} \) constitutes the orthonormal operator bases, and the \( l_1 \) norm of coherence can be obtained as [31]

\[
C_{l_1}(\rho) = \sum_{r=1}^{d_0} (x_{2r-1}^2 + x_{2r}^2)^{1/2}, \tag{A2}
\]

where \( d_0 = d(d - 1)/2 \).

The maximization of \( C_{l_1}(\rho) \) over all possible reference basis is equivalent to its maximization over all the unitary transformations of \( \rho \) which keep \( \bar{x} \) unchanged. Then, by denoting \( \bar{x}_i = \text{tr}(\rho U X_i) \) with \( \rho U = U \rho U^\dagger \), and by using the mean inequality, we obtain

\[
C_{l_1}(\rho_U) \leq \sqrt{d_0} \sum_{k=1}^{2d_0} \bar{x}_k^2 \tag{A3}
\]

and \( C_{l_1}(\rho_U) = \sqrt{d_0} |\bar{x}| \) when \( U \) gives rise to \( x_{2r-1}^2 + x_{2r}^2 = x_{2r'-1}^2 + x_{2r'}^2 \) \( \forall r, r' \in [1, d_0] \), and \( \bar{x}_i = 0 \) \( \forall i \geq 2d_0 + 1 \). Thus we arrive at the first equality of Eq. (4).

Second, the maximization of \( C_{\text{re}}(\rho, \sigma_m) \) over all the reference basis is equivalent to the maximization of \( S(\rho_{\text{diag}}^{(U)}) \) over all \( U \). Still, by using the mean inequality, we obtain

\[
S(\rho_{\text{diag}}^{(U)}) = -\sum_{i=1}^{d} \lambda_i^{(U)} \log_2 \lambda_i^{(U)} \leq \frac{1}{d} \sum_{i=1}^{d} (\lambda_i^{(U)} \log_2 \lambda_i^{(U)})^2 \tag{A4}
\]

\[
\leq \log_2 d,
\]
where $S(\rho_{\text{diag}}^{ij}) = \log_2 d$ when $\rho_{ij}^{ij} \log_2 \rho_{ij}^{ij} = \rho_{ij}^{ij} \log_2 \rho_{ij}^{ij}$, $\forall i, j$, and $\bar{x}_l = 0, \forall l \geq 2d_0 + 1$. This completes the proof of the second equality of Eq. (4).

Appendix B: Maximum of the mutual incompatibility

To obtain the maximum of $Q_l(\rho, \sigma)$, we consider

$$\rho^\Psi = |\Psi\rangle\langle\Psi|, \quad \text{with} \quad |\Psi\rangle = \sum_{i=1}^d |a_i^d\rangle|\psi_i\rangle \quad \text{and} \quad \sum_{i=1}^d |a_i^d\rangle|\psi_i\rangle = 1, \quad \text{and} \quad \sigma_1 = |\psi_1\rangle\langle\psi_1|,$$

for which we have

$$Q_l(\rho^\Psi, \sigma_1) = 2 \sum_{i \neq 1} |a_i^d| \langle a_i^d|,$$

(B1)

Then by taking $a_1^2 = \cos \theta_2$ and $a_2^2 = \sin \theta_2$ for $d = 2$, and $a_1^d = \cos \theta_d$, $a_2^d = \sin \theta_d a_{d-1}^d$ for $d \geq 2$ and $d \geq 3$ (the phases of $a_i^d$ do not affect the incompatibility), one can show

$$Q_l^{\text{max}}(\rho^\Psi, \sigma_1) = \sqrt{d-1},$$

(B2)

which is obtained with $\theta_2 = \theta = (2n + 1)\pi/4$ (n in Z), and $\theta_k = \pi/2 - \arctan(1/\sqrt{k-1})$, $\forall k \geq 3, d - 1$.

Now, we further prove that $\sqrt{d-1}$ is also the maximum of $Q_l(\rho, \sigma)$ for general cases. First, for the pure states $\rho^\Psi$ and general $\sigma$, the convexity of the l norm gives

$$Q_l(\rho^\Psi, \sigma) \leq \sum_i e_i Q_l(\rho^\Psi, \sigma_1) \leq \sqrt{d-1},$$

(B3)

where $\rho^\Psi$ is the optimal state for obtaining $Q_l^{\text{max}}(\rho^\Psi, \sigma_1)$, and the second inequality comes from the fact that $\rho^\Psi$ is not necessarily the optimal state for $Q_l^{\text{max}}(\rho^\Psi, \sigma_1)$ when $i \geq 2$.

Second, by denoting $\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|$, any pure state decomposition of the general state $\rho$, we have

$$Q_l(\rho, \sigma) \leq \sum_i p_i Q_l(|\phi_i\rangle\langle\phi_i|, \sigma) \leq \sqrt{d-1},$$

(B4)

and the second inequality is due to $Q_{l_i}(|\phi_i\rangle\langle\phi_i|, \sigma) \leq \sqrt{d-1}$ for any $|\phi_i\rangle$. Therefore, we proved $Q_{l_i}^{\text{max}}(\rho, \sigma) = \sqrt{d-1}$.

Appendix C: Proof of Eq. (15)

First, $S(B|A) \geq 0$ implies $S(\rho_C) \geq S(\rho_A)$ for any pure state $|\Psi\rangle_{ABC}$. Then, by using the chain inequality [57], one can obtain

$$S(\rho_C) + E_f(\rho_{AB}) \leq S(\rho_A) + E_f(\rho_{BC}),$$

(C1)

which is equivalent to $E_f(\rho_{AB}) \leq E_f(\rho_{BC}) - S(B|A)$. This, together with Eq. (11) and the Koashi-Winter equality [56], gives

$$D_A(\rho_{AB}) + S(B|A) = E_f(\rho_{BC}),$$

(C2)

Second, the equality $S(\rho_{AB}) = S(\rho_B) - S(\rho_A)$ if and only if the Hilbert space $\mathcal{H}_B$ can be decomposed as $\mathcal{H}_B = \mathcal{H}_{BL} \otimes \mathcal{H}_{BR}$ such that $\rho_{AB} = |\psi\rangle_{AB} \langle \psi| \otimes \rho_{BR}$ [58]. For this case, it is direct to show that

$$E_f(\rho_{AB}) = D_A(\rho_{AB}) = D_B(\rho_{AB}) = S(\rho_A).$$

(C3)

This, together with Eq. (11), gives $E_f(\rho_{AB}) \leq \delta_1(\rho_{AB})$.

Similarly, $S(\rho_{AB}) = S(\rho_A) - S(\rho_B)$ if and only if there exists a decomposition $\mathcal{H}_A = \mathcal{H}_A^L \otimes \mathcal{H}_A^R$ such that $\rho_{AB} = \rho_{AL} \otimes |\psi\rangle_{A^RB} \langle \psi|$, and for this kind of states we have

$$E_f(\rho_{AB}) = D_A(\rho_{AB}) = D_B(\rho_{AB}) = S(\rho_B).$$

(C4)

Hence, we still have $E_f(\rho_{AB}) \leq \delta_1(\rho_{AB})$.

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