1. Introduction

In the present paper, we review a supersymmetric extension of the Hadamard model, the classical and quantum motions of a superparticle on the super Riemann surface (SRS).

The conventional Hadamard model [1] represents the free motion of a nonrelativistic particle on the compact Riemann surface of a constant negative curvature [2]. The classical motion is known for its strongly chaotic property. The Hadamard model was originally proposed as a simple example of the dynamical system which possesses the ergodic property. This model has various advantages in the study of the chaotic properties mathematically. One of the essential features of this model is that there exists a well-defined quantum dynamics where the Laplace-Beltrami operator on the Riemann surface acts as the Hamiltonian. The model may be useful in physics to examine the properties of a quantum chaos. Specifically, the model shows us the relation between a classical and a quantum chaos. A quantized energy sum rule is actually the celebrated Selberg trace formula [3,4]. The quantum energy spectrum is complicated, however, it would be obtained through the Selberg zeta function.

Here, we intend to bring the supersymmetry into the Hadamard model. That is, we investigate the system of a superparticle moving freely on a compact super Riemann surface of genus $g \geq 2$. The supersymmetrical Hadamard model will offer

1. an application of the superanalog of the analytic theories on a Grassmann algebra [5].

2. an example of integrable classical and quantum dynamical systems with supersymmetry (the motion on SH, a universal covering space of the SRS, is expected to be integrable).

3. the notion of supersymmetrized chaos (however it seems to be rather puzzling).

4. superanalogs of the trace formula and the zeta function, which are important for the superstring theory (the notion of an SRS comes naturally in the superspace approach of superstrings [6]).
Motivated by them and armed with the mathematical tools for the supersymmetry, we develop the theory along the conventional study of the Hadamard model. This paper is organized as follows. In the next section the notations and the conventions of super Riemann surfaces are presented and the Lagrangian of a superparticle on a super Riemann surface is given. Section 3 is devoted to the classical mechanics for the system and quantization is carried out in Sec. 4. Superanalogs of the Selberg trace formula and the zeta function are given in Sec. 5. Section 6 is devoted to the discussions on the classical chaos. In the final section we comment on the moduli space of the super Riemann surface. This paper is based on our previous works [7 − 11].

2. Preliminaries

This section is devoted to compiling the notations and conventions of super Riemann surfaces (SRSs) of genus \( g \geq 2 \). As in the conventional Hadamard model, we will employ here the convenient way to represent the SRS. We will see that it is represented as a fundamental domain of a certain universal covering space. And this brings us an advantage that we can investigate the motions on an SRS by imposing a periodic boundary condition on the motions on the covering space. The free motion on an SRS is supposed to be generated by the distance-proportional Lagrangian. We will give a metric tensor on the covering space parametrized by a real parameter \( a \) and also give the Lagrangian for a superparticle based on the distance.

As is well known, the Poincaré upper half-plane \( H = \{ z \in \mathbb{C} \mid \text{Im} \, z > 0 \} \) together with the group of its conformal automorphisms \( \text{PSL}(2, \mathbb{R}) \) (Möbius group) is a model of the hyperbolic geometry. Turning to the supergeometry, we can extend all the standard constructions to the superplanes. The super Poincaré upper half-plane \( \text{SH} \) is expressed by one Grassmann even and one odd complex coordinate \( z \) and \( \theta \), respectively;

\[
\text{SH} = \{ Z = (z, \theta) \mid \text{Im} \, z > 0 \}. \tag{1}
\]

× Im \( z > 0 \) means that Im \( z_0 > 0 \) with \( z_0 \) being the body part of \( z \). We shall use such a convention for inequalities throughout this paper for simplicity.

The superconformal automorphisms \( \text{SPL}(2, \mathbb{R}) \) of \( \text{SH} \) consist of such transformation \( k : (z, \theta) \mapsto (\tilde{z}, \tilde{\theta}) \) as

\[
\begin{align*}
\tilde{z} &= \frac{az + b}{cz + d} + \theta \frac{\alpha z + \beta}{(cz + d)^2}, \\
\tilde{\theta} &= \frac{\alpha z + \beta}{cz + d} + \theta \frac{1 + \frac{1}{2} \beta \alpha}{(cz + d)^2},
\end{align*}
\]

(2)

where \( a, b, c \) and \( d \) are Grassmann even and \( \alpha \) and \( \beta \) are Grassmann odd parameters with

\[
ad - bc = 1, \quad a, b, c, d \in \mathbb{R}. \tag{3}
\]

The above transformation (2) is, of course, superanalytic and is also superconformal,

\[
D \tilde{z} - \tilde{\theta} D \tilde{\theta} = 0,
\]

(4)

\[
D \equiv \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}.
\]

(5)

If we introduce homogeneous coordinates \((z_1, z_2, \xi)\) of the complex projective space, we can rewrite (2) as a linear transformation \((z = z_1 z_2^{-1}, \theta = \xi z_2^{-1})\),

\[
\begin{pmatrix}
\tilde{z}_1 \\
\tilde{z}_2 \\
\tilde{\xi}
\end{pmatrix} = A_k
\begin{pmatrix}
z_1 \\
z_2 \\
\xi
\end{pmatrix},
\]

(6)

\[
A_k = \left( 1 + \frac{1}{2} \beta \alpha \right)^{-1} \times
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}^{-1} + \alpha \beta \left( 1 + \frac{3}{2} \beta \alpha \right), \quad \text{sdet} \, A_k = 1.
\]

Similarly to the ordinary Riemann surfaces, every compact super Riemann surface with genus \( g \geq 2 \) can be represented as a quotient space \( \text{SH}/\Sigma \) [12], where the

† As to complex conjugation, we adopt such a convention that \( \bar{\alpha} = i \alpha, \bar{\beta} = i \beta \).
universal covering space of a super Riemann surface (SRS) is the super Poincaré upper half-plane SH and the covering group \(S\Gamma\) (called the super Fuchsian group) is a discrete subgroup of superconformal automorphisms \(\text{SPL}(2,\mathbb{R})\) having no fixed points on \(SH\).

The super Fuchsian group \(S\Gamma\) is generated by \(2g\) elements \(\{A_i, B_i, i = 1, \cdots, g\}\) satisfying a condition,

\[
\prod_{i=1}^{g} (A_i B_i A_i^{-1} B_i^{-1}) = 1. \tag{7}
\]

Each element of the generators contains three Grassmann even and two odd parameters and the condition (7) is invariant under \(A_i \mapsto M A_i M^{-1}, \ B_i \mapsto M B_i M^{-1}\) where \(M \in \text{SPL}(2,\mathbb{R})\). Thus the set of the generators and hence the SRS with genus \(g \geq 2\) are specified by \(6g - 6\) Grassmann even and \(4g - 4\) odd parameters. \(S\Gamma\) acts discontinuously on \(SH\) and all its elements are hyperbolic, i.e., the reduced subgroup, where odd parameters are put to be zero, consists of the hyperbolic elements, \(|a + d| > 2\). Let \(\text{Conj}(S\Gamma)\) be the set of all conjugacy classes of \(S\Gamma\) and \(\text{Prim}(S\Gamma) = \{p \in \text{Conj}(S\Gamma); p \neq k^m\text{ for any } k \in \text{Conj}(S\Gamma)\text{ and } m \geq 2\}\) the set of all primitive conjugacy classes of \(S\Gamma\). Then we have

\[
Q(\text{Prim}(S\Gamma)) = \text{Conj}(S\Gamma), \text{ where } Q(P) = \{p^m, p \in P, m \geq 0\}. \tag{8}
\]

An element \(k \neq 1\) of \(S\Gamma\) causes such a transformation as (2). \(S\Gamma\) acts effectively on \(SH\), however, \(k \in S\Gamma\) has two fixed points, \((u, \mu)\) and \((v, \nu)\), on the “super” real axis \(\mathbb{R}_s \equiv \{Z = (z, \theta) \mid \text{Im } z = 0, \ \tilde{\theta} = i\theta\},\)

\[
u, v = \frac{a - d \pm \sqrt{(a + d)^2 - 4}}{2c}, \quad \mu = \frac{\alpha u + \beta}{cu + d - 1}, \quad \nu = \frac{\alpha v + \beta}{cv + d - 1}. \tag{9}
\]

These fixed points are repelling and attractive points, respectively,

\[
k^{-n} : (z, \theta) \rightarrow (u, \mu), \quad k^n : (z, \theta) \rightarrow (v, \nu), \quad \text{for } n \rightarrow \infty. \tag{10}
\]

Let us define the quantities \(N_k\) (norm function) and \(\chi(k)\) (sign factor) for \(k \in S\Gamma\)

\[
\chi(k)(N_k^{1/2} + N_k^{-1/2}) = a + d - \frac{a + d + 2}{2} \beta \alpha = \text{str } A_k + 1. \tag{11}
\]

Actually, \(N_k(> 1)\) is the square of the maximum eigenvalue of the matrix \(A_k\) and \(\chi(k)\) has to be chosen as

\[
\chi(k) = \begin{cases} 1, & \text{if } \text{str } A_k + 1 > 2; \\ -1, & \text{if } \text{str } A_k + 1 < -2, \end{cases} \tag{12}
\]

Using the transformation of \(\text{SPL}(2,\mathbb{R})\), we see that any element \(k \neq 1\) of \(S\Gamma\) is always conjugate in \(\text{SPL}(2,\mathbb{R})\) to the magnification

\[
\bar{w} = N_k w, \quad \bar{\eta} = \chi(k) N_k^{1/2} \eta, \quad N_k > 1, \quad \text{or in the matrix representation:}
\]

\[
A_f A_k A_f^{-1} = A_{f k f^{-1}} = \text{diag}\left(\chi(k) N_k^{1/2}, \chi(k) N_k^{-1/2}, 1\right) = A_{\text{mag}}. \tag{13}
\]

Apparently, the magnification depends on the element of \(\text{Conj}(S\Gamma)\)

\[
N_{g k g^{-1}} = N_k, \quad \chi(g k g^{-1}) = \chi(k). \tag{14}
\]

Now we will introduce a \(\text{SPL}(2,\mathbb{R})\)-invariant metric tensor on \(SH\) which is a superanalog of the Poincaré metric on \(H\). The latter, \(dS_0^2 = |dz|^2/(\text{Im } z)^2\), is invariant under \(\text{PSL}(2,\mathbb{R})\) and gives a constant negative curvature. The corresponding volume element is

\[
\frac{dz \, dy}{y^2}, \quad (\text{Re } z = x, \ \text{Im } z = y). \tag{15}
\]

To construct the \(\text{SPL}(2,\mathbb{R})\)-invariant metric tensor on \(SH\), we use the technique developed in the supergravity theory on 2 + 2 dimensional curved superspace.
The basic quantities are the super vielbein $E^A_M$ which, however, are not completely independent superfields. It was shown that 2 + 2 dimensional superspace is superconformally flat [13] where the basis one-forms $E^A$ are

$$\dot{E}^{+} = dz + \theta d\theta, \quad \dot{E}^{-} = d\bar{z} - \bar{\theta} d\bar{\theta}, \quad \dot{E}_z = d\theta, \quad \dot{E}^z = d\bar{\theta}. \quad (17)$$

By the super Weyl transformation [13] the vielbein $\hat{E}^A_M$ changes as

$$\hat{E}^A_M \rightarrow E^A_M = \begin{cases} E^A_M = \Lambda(Z)\hat{E}^A_M, \\
E^A_M = \Lambda^{1/2}(Z)\hat{E}^A_M - i\hat{E}^A_M(\gamma_\alpha)\alpha\beta\hat{E}^N_N \partial_N \Lambda^{1/2}(Z), \quad (a = +, -, \alpha = +, -) \end{cases} \quad (18)$$

where $\hat{E}^A_M = (\hat{E}^A_M)^{-1}, \Lambda(Z)$ is the scaling function and $(\gamma_\alpha)$ is the gamma matrix:

$$(\gamma_+)^{\alpha\beta} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad (\gamma_-)^{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}. \quad (19)$$

We take a SPL$(2,\mathbb{R})$-covariant function for $\Lambda(Z)$,

$$\Lambda(Z) = Y^{-1}, \quad (20)$$

where $Y$ is given by

$$Y = \text{Im} \ z + \frac{1}{2} \theta \bar{\theta}, \quad (21)$$

which is the superanalog of $y = \text{Im} \ z$ on $H$. The $E^A$ are now given as

$$E^{++} = Y^{-1} (dz + \theta d\theta), \quad E^{-} = E^{++},$$

$$E^+ = Y^{-3/2} \left[ (Y + \frac{1}{2} \theta \bar{\theta}) d\theta + \frac{1}{2} (i\theta - \bar{\theta}) dz \right], \quad E^- = E^+. \quad (22)$$

The SPL$(2,\mathbb{R})$-invariant line element can be constructed by

$$ds^2 = E^{++} E^- - 2a E^+ E^- , \quad (23)$$

where $a(\neq 0)$ is an arbitrary real Grassmann even number. Rewriting $ds^2 = dq^A g_{AB} dq^B$, $(q^x, q^\bar{z}, q^\theta, q^\bar{\theta}) = (z, \bar{z}, \theta, \bar{\theta})$, we obtain the metric tensor on SH,

$$(g_{AB}) = \begin{pmatrix} 0 & \frac{1}{2}Y^2 & 0 & -\frac{\bar{\theta} + a(Y - \theta)}{2Y^2} \\
\frac{1}{2}Y^2 & 0 & \frac{\theta - a(Y + i\bar{\theta})}{2Y^2} & 0 \\
0 & \frac{\theta - a(Y + i\bar{\theta})}{2Y^2} & 0 & \frac{\theta - 2a(Y + \bar{\theta})}{2Y^2} \\
-\frac{\bar{\theta} + a(Y - \theta)}{2Y^2} & 0 & \frac{\theta - 2a(Y + \bar{\theta})}{2Y^2} & 0 \end{pmatrix}, \quad (24)$$

and the corresponding volume element is given by

$$dV = \frac{1}{2aY} dxdyd\theta d\bar{\theta}. \quad (25)$$

Since we have now a SPL$(2,\mathbb{R})$-invariant line element (23), we give our Lagrangian of a superparticle with mass $m$ on SH [9],

$$L = \frac{m}{2} \left( \frac{ds}{dt} \right)^2 = \frac{m}{2} q^A g_{AB} q^B. \quad (26)$$

This is SPL$(2,\mathbb{R})$-invariant and hence, of course, $\Sigma$-invariant, thus it is also the Lagrangian for a superparticle on the SRS.

**3. Classical Mechanics**

In this section we examine the classical dynamics of a superparticle on the SRS. Firstly, we solve the motion on SH. The motion on SH is expected to be integrable. According to the canonical theory of supermechanics, if we find out the adequate number of integrals of motion which commute each other with respect to the Poisson bracket, we can construct the general solution out of the integrals. However, it is rather difficult to do it explicitly. There is no systematical way to get such integrals, and hence we take another path. We solve the Hamilton-Jacobi equation. The calculation is considerably cumbersome but the general solution for the metric tensor with an arbitrary parameter $a$ is given. Since the SRS is represented by the fundamental domain of SH, $SH/\Sigma$, the motion on the SRS is given by imposing the periodic boundary condition on the motion on SH. As we have expected, the motion on the SRS shows chaotic properties.
The Euler-Lagrange equations from \( L \) in (26) are geodesic equations,

\[
\dot{q}^A + \Gamma^A_{BC} \dot{q}^B q^C = 0,
\]

where \( \Gamma^A_{BC} \) is the Cristoffel’s symbol [11]. Eq.(27) is given explicitly,

\[
\begin{align*}
\ddot{z} + \frac{1}{Y}(iz^2 - \dot{z} \dot{\theta}) + \frac{1}{2a} \left( \frac{i}{Y} \dot{\theta} \dot{z} \ddot{z} - \frac{2}{Y} \dot{z} \dot{\theta} \right) &= 0, \\
\ddot{\theta} + \frac{1}{Y} \dot{z} \dot{\theta} + \frac{1}{2a} \left( \frac{2Y + \dot{\theta}}{Y^2} \dot{z} \dot{\theta} - \frac{\theta + i \dot{\theta}}{Y^2} \ddot{z} - \frac{2}{Y} \ddot{\theta} \dot{\theta} \right) &= 0,
\end{align*}
\]

and their complex conjugated ones. The body part of the Eqs.(28) is

\[
\ddot{z}_0 + \frac{i}{y_0} \dot{z}_0^2 = 0,
\]

which is the geodesic equation on \( H \) with the Poincaré metric [1]. The solutions to (29) are give by,

\[
z_0(t) = c_1 \sinh X_0 + i + c_2, \quad \text{and} \quad ie^{X_0} + c_2,
\]

where

\[
X_0 \equiv \omega(t + t_0), \quad c_1, c_2, \omega, t_0 \in \mathbb{R}.
\]

A classical motion is determined uniquely with the boundary conditions which are the position and the velocity at the initial point. Thus the constants of the integration for the Euler-Lagrange equation (27) or (28) are four real Grassmann even and also four odd constants. So expanding \( z \) and \( \theta \) in the Grassmann odd constants, say, \( c_1, \bar{c}_1, c_2, \bar{c}_2 \), we have a set of differential equations for the coefficients of the Grassmann even functions. However, it is actually not easy to solve those equations. So instead of solving (28) directly, we will take a roundabout

The Hamilton-Jacobi equation is given by

\[
\frac{\partial S}{\partial t} + \frac{1}{2m} g^{AB} \frac{\partial S}{\partial q^A} \frac{\partial S}{\partial q^B} = 0.
\]

Since the action which the classical solutions are plugged into satisfies the above equation (32), we express \( S \) as

\[
S(q_1, q_2; t_2 - t_1) = \frac{m}{2} \int_{t_1}^{t_2} dt \dot{q}^A(t) g_{AB}(q(t)) \dot{q}^B(t),
\]

where \( \dot{q}^A(t) \) is a solution of the geodesic equation (27) connecting the initial point \( q_1 = q(t_1) \) and the final point \( q_2 = q(t_2) \). It can be easily shown that the integrand is time-independent and its body part is non-negative. Taking them into account, we set

\[
\dot{q}^A(t) g_{AB}(t) \dot{q}^B(t) = (\text{const.}) \equiv \omega^2,
\]

and define a superanalog of the hyperbolic distance,

\[
d(q_1, q_2) = \int_{t_1}^{t_2} dt \sqrt{\dot{q}^A(t) g_{AB}(t) \dot{q}^B(t)} = |\omega|(t_2 - t_1).
\]

From (33), (34) and (35), we get

\[
S(q_1, q_2; t) = \frac{m}{2} \frac{|d(q_1, q_2)|^2}{t}.
\]

Note that the hyperbolic distance \( d_0(q_1, q_2) \) between \( (q_1)_0 = (z_0, \bar{z}_0) \) and \( (q_2)_0 = (w_0, \bar{w}_0) \), which should be the body part of \( d(q_1, q_2) \), is given by

\[
\cosh d_0 = 1 + \frac{|z_0 - w_0|^2}{2 \text{Im } z_0 \text{Im } w_0} = 1 + \frac{1}{2} R_0,
\]

which is \( \text{PSL}(2,\mathbb{R}) \)-invariant. And hence \( d(q_1, q_2) \) should be symmetric under the exchange of \( q_1 \) and \( q_2 \) and \( \text{SPL}(2,\mathbb{R}) \)-invariant. There exist two basic function

* The second solution is in fact obtained by taking a proper limit of the first solution. The first solution is always transformed into the second one by an appropriate Möbius transformation.
on $\text{SH} \times \text{SH}$ with such properties [14],

\[
R(q_1, q_2) = \frac{|z_1 - z_2 - \theta_1 \theta_2|^2}{Y_1(1) Y_2(2)},
\]

\[
r(q_1, q_2) = \left\{ \frac{2\theta_1 \theta_1 + i(\theta_2 - i\bar{\theta}_2)(\theta_1 + i\bar{\theta}_1) + (1 \leftrightarrow 2)}{4Y_1(1)} \right\} \left( \frac{\theta_2 + i\bar{\theta}_2(\theta_1 + i\bar{\theta}_1)\text{Re}(z_1 - z_2 - \theta_1 \theta_2)}{4Y_1(1) Y_2(2)} \right),
\]

where $Y_i = \text{Im } z_i + \frac{1}{2} \theta_i \bar{\theta}_i$ for $i = 1, 2$. Here $R$ is the superanalog of $R_0$ in (37) and $r$ is nilpotent, and hence we can expect that $d(q_1, q_2)$ takes in general the following form,

\[
\cosh d = f(R) + k(R) r.
\]

The Hamilton-Jacobi equation leads to the differential equations for the unknown functions $f$ and $k$, and we find that the “super” hyperbolic distance $d(q_1, q_2)$ is given by [10],

\[
\cosh[d(q_1, q_2)] = 1 + \frac{1}{2} R(q_1, q_2) + k(R) r(q_1, q_2),
\]

where

\[
k(R) = \cosh l - 1 - \sinh l \ \coth \frac{l}{2a},
\]

\[
l = l(q_1, q_2) = \cosh^{-1} \left( 1 + \frac{1}{2} R \right).
\]

The next step is to solve $q_1 \equiv q$ in terms of $q_2$ and its canonical conjugated quantity, say, $p^{(2)}$. This can be done by solving the following algebraic equations with respect to $q$,

\[
\frac{\partial S}{\partial q_2} = -p_2^{(2)},
\]

where $q_2$’s and $p^{(2)}$’s actually correspond to the constants of integration for the differential equations (28). The calculation is cumbersome but rather straightforward and we obtain the solution of the Euler-Lagrange equations (28), $(z^{(1)}(t), \theta^{(1)}(t))$, $(z^{(0)}(t), \theta^{(0)}(t))$ or $(z^{(3)}(t), \theta^{(3)}(t))$, [10],

\[
z^{(1)}(t) = \left[ c_1 - \frac{2}{\cosh X} \left( i\xi_1 \xi_2 e^{-\frac{\bar{X}}{2}} - i\xi_3 \xi_4 e^{\frac{\bar{X}}{2}} \right) - \xi_1 \xi_4 e^{(1 - \frac{\omega}{2})X} + \xi_3 \xi_4 e^{(\frac{\omega}{2} - 1)X} \right] \sinh X + i \bar{c}_2,
\]

\[
\theta^{(1)}(t) = \frac{\sinh X + i}{\cosh X} + 1 \right\} \left( \xi_1 e^{-\frac{X}{2}} - i\xi_2 e^{\frac{X}{2}} + i\xi_3 e^{(1 - \frac{\omega}{2})X} + \xi_4 \right),
\]

\[
z^{(2)}(t) = i\bar{e} X + c_2
\]

\[
+ i\xi_1 \xi_2 e^{(1 - \frac{\omega}{2})X} - i\xi_3 \xi_4 e^{(1 + \frac{\omega}{2})X} - \xi_1 \xi_4 e^{(2 - \frac{\omega}{2})X} + \xi_3 \xi_4 e^{\frac{\bar{X}}{2}},
\]

\[
\theta^{(2)}(t) = i\xi_1 e^{(1 - \frac{\omega}{2})X} + \xi_2 - \xi_3 e^{\frac{\bar{X}}{2}} + i\xi_4 e^X,
\]

\[
z^{(3)}(t) = i\bar{c}_1 + c_2 - 2ac_1 \omega_s t
\]

\[
- \omega_s \left\{ 2ia^2 c_1 \omega_s - a \bar{e}_2 c_1 + (1 - a) \bar{e}_2 \right\} t^2 - \frac{1}{3} e_1 \bar{e}_1 \omega_s t^3,
\]

\[
\theta^{(3)}(t) = c_2 + \epsilon_1 t + \left\{ i\bar{a} e_1 + (1 - a) \bar{e}_1 \right\} \omega_s - \frac{1 - a}{2a c_1} \epsilon_1 \bar{e}_1 \epsilon_2 t^2,
\]

where

\[
X \equiv \omega(t + t_0),
\]

$\omega, t_0, c_1, c_2$: real Grassmann even constants, $c_1 > 0$,

\[
\xi_k (k = 1, 2, 3, 4): \text{Grassmann odd constants with } \xi_k = i\bar{\xi}_k,
\]

$\epsilon_1, \epsilon_2$: complex Grassmann odd constants,

$\omega_s$: Grassmann even constant with no body part.

The first $(z^{(1)}, \theta^{(1)})$ and the second $(z^{(2)}, \theta^{(2)})$ solutions correspond to the first and the second solutions in (30), respectively and the third one $(z^{(3)}, \theta^{(3)})$ corresponds to the solution with $\omega = 0$ in (30). And actually $(z^{(2)}, \theta^{(2)})$ is obtained by taking a proper limit of $(z^{(1)}, \theta^{(1)})$.

Now we examine the classical motion on the SRS. Since we have obtained the classical paths (44), (45) and (46) on the covering space SH of the SRS, we can deduce the classical motion on the SRS through projecting the paths on SH. 


onto the fundamental domain \(SH/\Sigma\). We study closed orbits on the SRS at first. A path \(Z(t) = (z(t), \theta(t))\) on \(SH\) gives a closed loop on the SRS if it satisfies the condition that there exist such an element \(k \neq 1\) in \(\Sigma\) and a time interval \(T\) that

\[
Z(t + T) = k(Z(t)).
\] (48)

Since \(k\) is characterized by the two fixed points, \((u, \mu)\) and \((v, \nu)\), the sign factor \(\chi(k)\) and the norm function \(N_k\) (see Sect.2), the above condition gives a necessary condition,

\[
\frac{z(t + T) - u - \theta(t + T)\mu}{z(t + T) - v - \theta(t + T)\nu} = N_k \frac{z(t) - u - \theta(t)\mu}{z(t) - v - \theta(t)\nu}.
\] (49)

We find that the classical motions \(Z^{(i)}(t)\) (45) and \(Z^{(i)}(t)\) (46) do not satisfy the above condition (49) and only the motions \(Z^{(s)}(t)\) (44) with the parameters having values, the two fixed points, \((u, \mu)\) and \((v, \nu)\), the sign factor \(\chi(k)\) and the norm function \(N_k\) (see Sect.2), the above condition gives a necessary condition,

\[
\frac{\theta(t + T) + \frac{\mu - \nu}{v - u} z(t + T) + \frac{\nu - u}{v - u} \theta(t + T)\mu}{z(t + T) - v - \theta(t + T)\nu} = \chi(k) N_k^{1/2} \frac{\theta(t) + \frac{\mu - \nu}{v - u} z(t) + \frac{\nu - u}{v - u} \theta(t)\mu}{z(t) - v - \theta(t)\nu}.
\]

\[
\theta(t + T) + \frac{\mu - \nu}{v - u} z(t + T) + \frac{\nu - u}{v - u} \theta(t + T)\mu = \chi(k) N_k^{1/2} \frac{\theta(t) + \frac{\mu - \nu}{v - u} z(t) + \frac{\nu - u}{v - u} \theta(t)\mu}{z(t) - v - \theta(t)\nu}.
\] (49)

The path \(Z^{(i)}(t)\) with (50), which we denote \(Z_k(t)\) associated with the element \(k\), is the geodesic curve connecting the two fixed points of the element \(k \neq 1\) in \(\Sigma\)

\[
Z_k(t \to +\infty) \to (v, \nu),
\]

\[
Z_k(t \to -\infty) \to (u, \mu), \quad \omega > 0.
\] (50)

A segment \([Z_k(t), Z_k(t + T)]\) of the geodesic curve becomes a closed loop on the SRS and the length of the loop \(l(k)\) is given by

\[
l(k) \equiv d(Z_k(t), Z_k(t + T)) = d(Z_k(t), k(Z_k(t))) = \log N_k,
\] (51)

which in fact depends only on the element \(k \in \Sigma\). Equation (53) yields

\[
l(k^n) = |n| l(k).
\] (55)

The geodesic segment \([Z_k(t), Z_k(t + nT)]\) becomes a closed loop lying \(|n|\)-fold exactly on the closed loop coming from the segment \([Z_k(t), Z_k(t + T)]\). So \([Z_k(t), Z_k(t + nT)]\) and \([Z_k(t), Z_k(t + T)]\) determine the same primitive periodic orbit and we conclude that two elements \(k^m\) and \(k^n \neq 1(m, n : \text{integers})\) in \(\Sigma\) are associated with the same primitive periodic orbit on the SRS. Furthermore, due to \(\text{SPL}(2, \mathbb{R})\)-invariance of \(d(q_1, q_2)\), we get

\[
l(k) = d(gZ_k(t), gZ_k(t + T)) = d(gZ_k(t), g k g^{-1}(gZ_k(t))), \quad g \in \Sigma.
\] (55)

This implies that \(gZ_k(t)\) is the geodesic curve connecting the two fixed points of the element \(g k g^{-1} \in \Sigma\). Since \(gZ_k(t)\) and \(Z_k(t)\) become the same trajectory on the SRS, we conclude that every geodesic curve connecting the fixed points of each element of \(\text{Conj}(\Sigma)\) becomes the same orbit on the SRS. Thus we find that each pair \((p, p^{-1}) \in \text{Prim}(\Sigma)\) is associated with a primitive periodic orbit on the SRS and its length is given by

\[
l(p) = \log N_p = \log N_{p^{-1}},
\] (56)

where \(N_p\) is the norm function associated with \(p\). Conversely any periodic orbit can be lifted to a geodesic segment \([Z(t), k(Z(t))]\) on \(SH\) with some element \(k\).
1 in $\Gamma$. Since there exists a unique geodesic curve connecting the two points $Z(t)$ and $k(Z(t)) = Z(t + T)$, the geodesic curve is in fact a solution $Z^{(1)}(t)$ connecting the two fixed points of $k$. Then we conclude that there exists a one-to-one correspondence between primitive periodic orbits on the SRS and pairs of inconjugate primitive elements $(p, p^{-1})$. Any geodesic curve $Z(I)$ not connecting two fixed points of any element in $\Gamma$ becomes a nonperiodic orbit on the SRS and such geodesic curves are dense on SH. Hence the classical motion on the SRS is chaotic.

Signals of Chaos:

1. the lagrangian (26) is SPL(2, $\mathbb{R}$)-invariant, however, after projecting out onto the SRS, we find that the symmetry generators on SH become no longer those on the SRS and only two Grassmann even quantities are conserved ones, which are the Hamiltonian $H$ and a nilpotent quantities $H^{(2)}$ essentially corresponding to $E^\theta E^\bar{\theta}$. The fact that there are two kind of conserved quantities has been already presented in constructing the Lagrangian which consists of two SPL(2, $\mathbb{R}$)-invariant pieces. However, the dimension of the hyper surface determined by $H = E$ and $H^{(2)} = E^{(2)}$ constants in the total super space becomes less by one bosonic degree than that of total space according to the (super) implicit function theorem [5].

2. we will study the Anosov property [15] which describes the behavior of the initially neighboring trajectories at large times and is suitable to study the strongly chaotic systems [16]. Let us take two geodesic curves $Z^{(1)}(t)$ ($\omega > 0, t_0 = 0$) with the conditions (50) and another one with

\[
c_1 = \frac{u + \delta v - u}{2}, \quad c_2 = \frac{u + v + \delta v}{2}, \quad \xi_2 = \frac{v + \delta v}{2}, \quad \xi_4 = \frac{u}{2}, \quad \xi_1 = \xi_3 = 0.
\]

These two trajectories start from the same point $(u, \mu)$ at $t \to -\infty$, however, arrive at slightly different points $(v, \nu)$ and $(v + \delta v, \nu + \delta \nu)$ when $t \to \infty$. At $t = 0$ the value of separation is

\[
d_{t=0} \sim \delta v + \frac{\mu + \nu}{2} \delta \mu,
\]

However as $t \to \infty$ the trajectories separate exponentially,

\[
d_{t \to \infty} \sim (\delta v + \nu \delta \nu) e^{\omega t}.
\]

The velocity $\omega$ is the Liapunov exponent. This implies that trajectories are unstable, which is characteristic of classical chaos.

3. we comment on the Kolmogorov-Sinai entropy $h$ [17] which, roughly speaking, measures unpredictability of the motions. This number comes out from the asymptotic formula for the counting function of primitive orbits of period $T(p) \leq T$,

\[
\# \{p, T(p) \leq T \} \sim \frac{e^{hT}}{hT}, \quad T \to +\infty,
\]

which indicates the exponential proliferation of the periodic orbits. From (51) and (56), this formula yields

\[
\# \{p, l(p) \leq x \} \sim \frac{e^{\alpha x}}{\alpha x}, \quad x \to \infty \text{ with } \alpha = \frac{h}{\omega}.
\]

The asymptotic formula (61) will be discussed in the quantum mechanical framework in Sect. 6.
4. Quantum Mechanics

In this section we develop the quantum theory for a particle moving on the SRS. As was the conventional Hadamard model, we cannot quantize the classical motion on the SRS because of its ergodicity. However there exists a well-defined quantum mechanics on the SRS where the Laplace-Beltrami operator acts as the quantum Hamiltonian. To construct the quantum mechanics on the SRS, we firstly develop the quantum mechanics on SH where the quantum Hamiltonian is also the Laplace-Beltrami operator. The quantum motion on the SRS is obtained by imposing the periodic boundary conditions upon the motion on SH just as the case of the classical motion. The quantum motion on SH is also expected to be integrable. In fact, as we will see, we can solve the Schrödinger equation explicitly and obtain the exact energy spectrum for the quantum motion on SH. A wave function on the SRS will be obtained by folding that on SH, however the function is quite complicated.

First we give the quantum Hamiltonian. Our Lagrangian (26) is nonlinear in a sense that \( g_{AB} \) are functions of supercoordinates. Omote and Sato [18] developed a procedure to construct Hamiltonian for a system with a (purely bosonic) nonlinear Lagrangian of a form \( L_B = \frac{1}{2} g_{ij}(q) \dot{q}^i \dot{q}^j \) with having considered a symmetry as a guiding principle (see also [19]). We can follow their arguments paying attention to sign factors. We find the quantum Hamiltonian on SH,

\[
H_Q = \frac{(-)^A}{2m} g^{-1/4} p_A g^{1/2} g^{AB} p_B g^{-1/4},
\]

(62)

where \( g \equiv |\text{sdet} g_{AB}| = (4a^2 Y^2)^{-1} \),

(63)

with the canonical commutation relations,

\[
[p_A, q^B]_\pm = -i \hbar \delta_A^B.
\]

(64)

The Hamiltonian \( H_Q \) is also SPL(2,IR)-invariant and hence is the Hamiltonian on SRS.

In the \( q \)-representation the coordinates \( q^A \) and the momenta \( p_A \) are given by

\[
q^A = q^A, \quad p_A = -i\hbar g^{-1/4} \frac{\partial}{\partial q^A} g^{1/4} = -i\hbar g^{-1/4} \partial_A g^{1/4},
\]

(65)

so that \( H_Q \) is a super Laplace-Beltrami operator,

\[
H_Q = -\frac{\hbar^2}{2m} (-)^A g^{-1/2} \partial_A(g^{1/2} g^{AB} \partial_B)
\]

\[
= -\frac{\hbar^2}{2m} \left[ (2Y D \overline{D})^2 + \frac{1 - a}{a} (2Y D \overline{D}) \right] \equiv \frac{\hbar^2}{2m} \Box_{\text{SLB}},
\]

(66)

where \( D \) is given in (5) and \( \overline{D} \) is its complex conjugate. The Hilbert space \( \mathcal{H} \) of our model is the space of superfunctions on SH with the inner product

\[
\langle \Psi_1 | \Psi_2 \rangle \equiv \int d^4 q \, g^{1/2}(q) \langle \Psi_1 | q \rangle \langle q | \Psi_2 \rangle,
\]

(67)

and \( H_Q \) is hermitian with respect to this product.

We will study the spectral properties of \( H_Q \) on SH. In order to do that we examine the eigenvalue problem of the operator \( \Box_0 \),

\[
\Box_0 \equiv 2Y D \overline{D}.
\]

(68)

The Grassmann even (odd) eigenfunction \( e_A(\psi_A) \), with eigenvalue \( \lambda^B(\lambda^F) \),

\[
\Box_0 e_A = \lambda^B e_A, \quad \Box_0 \psi_A = \lambda^F \psi_A,
\]

(69)

may be expanded as

\[
e_A = A_A + \bar{\theta} \bar{B}_A, \quad \psi_A = \frac{1}{\sqrt{g}} (\theta \rho_A + \bar{\theta} \varphi_A),
\]

(70)

where \( \{A_A, B_A\} \) and \( \{\rho_A, \varphi_A\} \) are functions of Grassmann even coordinates \( z \).
Eqs. (69) and (70) yield

\[ B_\lambda = \frac{\lambda B}{2y} A_\lambda , \]

\[ \left\{ y^2 (\partial_x^2 + \partial_y^2) - \lambda B (\lambda B - 1) \right\} A_\lambda = 0 . \]

\[ \varphi_\lambda = -2y\sqrt{y} \partial_y \left( \frac{\rho_\lambda}{\sqrt{y}} \right) , \]

\[ \left\{ y^2 (\partial_x^2 + \partial_y^2) - iy \partial_y - ((\lambda F)^2 - \frac{1}{4}) \right\} \rho_\lambda = 0 , \]

and the above differential equations can be solved as *

\[ A_\lambda = C_{\lambda B,k} e^{ikx} \sqrt{y} K_{\lambda B - \frac{1}{4}} (|k|y) , \]

\[ \rho_\lambda = C_{\lambda F,k} e^{ikx} W_{\frac{3}{4},\lambda F} (2|k|y) , \]

where \( \sigma_k \equiv \text{sign}(k) \), \( k \neq 0 \) and \( K_{\mu} \) and \( W_{\kappa,\mu} \) are a modified Bessel function and a Whittaker function, respectively. The normalization constants \( C_{\lambda B,k} \) and \( C_{\lambda F,k} \) are determined as follows. Since SH is noncompact and the spectrum is continuous, the normalization condition should be

\[ \langle e_A | e_{A'} \rangle \propto \delta (\lambda B - (\lambda B')') , \]

\[ \langle \psi_A | \psi_{A'} \rangle \propto \delta (\lambda F - (\lambda F')') , \]

The above condition determines the regions of the eigenvalues [11],

\[ \lambda B = \frac{1}{2} + ip , \]

\[ \lambda F = c + ip , \]

where \( p \in (-\infty, +\infty) \) and \( \{ c : \text{real constant, } |c| \leq \frac{1}{2} \} \). And the normalized eigenfunctions are given by,

\[ e_{p,k}(Z) = \left( \frac{2a \sinh \pi p}{\pi^3} \right)^{1/2} \left( 1 + \frac{1 + 2ip \rho \theta}{4y} \right) e^{ikx} \sqrt{y} K_{ip} (|k|y) , \]

\[ \psi^c_{p,k}(Z) = \frac{a \cos [\pi (c + ip)]}{2\pi^2 k (c + ip) \sigma_k - 1} \frac{1}{\sqrt{y}} e^{ikx} \]

\[ \times \left\{ \theta W_{\frac{3}{4}, c+ip} (2|k|y) + i(c + ip) \sigma_k \tilde{\theta} W_{\frac{3}{4}, c+ip} (2|k|y) \right\} , \]

with

\[ \langle e_{q,l} | e_{p,k} \rangle = \delta (p + q) \delta (k - l) , \]

\[ \langle \psi_{q,l}^c | \psi_{p,k}^c \rangle = \delta (k - l) \delta (p + q) . \]

The eigenvalues of \( H_Q \) are

\[ E^B_{p,k} = \frac{\hbar^2}{2m} \left\{ \left( \frac{1}{2a} \right)^2 + \left( \frac{p - i}{2a} \right)^2 \right\} \equiv \frac{\hbar^2}{2m} \gamma_B (p) , \]

\[ E^F_{p,k,c} = \frac{\hbar^2}{2m} \left\{ \left( \frac{1}{2a} \right)^2 + \left( \frac{p - i - c - \frac{1}{2a}}{2} \right)^2 \right\} \equiv \frac{\hbar^2}{2m} s^c (p) . \]

Although \( H_Q \) is a hermite operator, the eigenvalue is complex. This is because the space of eigenstates contains isovectors [20] as is seen in (76). Notice that except when \( c = \frac{1}{2} \) or \( \frac{a - 2}{2a} \) with \( a \geq 1 \), the energy spectra of the Grassmann even states and the odd ones do not coincide with each other,

\[ \{ E^B_{p,k} \} \neq \{ E^F_{q,l,c} \} , \quad c \neq \frac{1}{2} \text{ and } \frac{a - 2}{2a} \quad (a \geq 1) . \]

A set of eigenfunctions for each \( c \), \( \{ e_{p,k}, \psi^c_{p,k} \} \), do satisfy the completeness relation,

\[ \int_{-\infty}^{\infty} dpdk \left[ e_{p,k}(q_2) \bar{e}_{-p,k}(q_1) + \psi^c_{p,k}(q_2) \bar{\psi}^c_{-p,k}(q_1) \right] \]

\[ = \left[ g(q_1)g(q_2) \right]^{-1/4} \delta (q_1 - q_2) . \]

For each \( c \), we have a Hilbert space for the Grassmann odd states \( \mathcal{H}_c^F \), and hence ...
the total Hilbert space is 

\[ \mathcal{H}_c = \mathcal{H}^B \oplus \mathcal{H}^F, \]

(80)

where \( \mathcal{H}^B \) is the Hilbert space for the Grassmann even states.

We now turn our attention to the eigenfunctions on the SRS or \( F \), a fundamental domain of \( ST \). The \( ST \)-invariance of the eigenfunction \( \Psi \) imposes the periodic boundary condition. The action of \( g \in ST \) on \( \Psi \) reads

\[ \Psi'(q) = [g\Psi](q) = \Psi(g^{-1}q), \quad q \in SH. \]

(81)

Hence, the periodicity is expressed as

\[ \Psi(g^{-1}q) = \Psi(q) \quad \text{for all} \quad g \in ST. \]

(82)

The spectrum of the operator \( \Delta_{SLB} \) on \( SH \), \( \{ \gamma^B(p) \} \) and \( \{ \gamma^F(p) \} \) in (77), will become discrete on the SRS. We then write the discrete spectrum of \( \Delta_{SLB} \) as

\[ \{ \gamma^B_n \} \quad (n = 0, 1, 2, \cdots) \quad \text{for Grassmann even state}, \]

\[ \{ \gamma^F_n \} \quad (n = 0, 1, 2, \cdots) \quad \text{for Grassmann odd state}, \]

(83)

and that of the operator \( \Box^0 \) in (68) as

\[ \{ \lambda^B_n \} \quad (n = 0, 1, 2, \cdots) \quad \text{for Grassmann even state}, \]

\[ \{ \lambda^F_n \} \quad (n = 0, 1, 2, \cdots) \quad \text{for Grassmann odd state}, \]

(84)

where

\[ \gamma^B(F)_n = - (\lambda^B(F)_n)^2 - \left( \frac{1-a}{a} \right) \lambda^B(F)_n. \]

(85)

However, because the periodic condition is complicated it is very difficult to see the spectrum on the SRS explicitly. We comment on the ground state. The Grassmann even ground state is given by a constant function. It is a trivial periodic function with the normalization

\[ \int \frac{dV}{\lambda} \text{(const.)} < \infty, \]

(86)

and has the energy \( \lambda^B_0 = 0 \) (\( \gamma^B_0 = 0 \)).

We now construct the kernel function on \( SH \). The kernel function is given by

\[ K(q_1, q_2; t) \equiv \langle q_2 | e^{-\frac{i}{\hbar} \hat{H} t} | q_1 \rangle, \]

\[ = \int_{-\infty}^{\infty} dpdk \left\{ e^{-\frac{i}{\hbar} E^B_{p,k} q_2} \langle q_2 | e_{p,k} \rangle \langle e_{-p,k} | q_1 \rangle + e^{-\frac{i}{\hbar} E^F_{p,k,c} q_2} \langle q_2 | \psi^F_{p,k,c} \rangle \langle \psi^F_{-p,k,c} | q_1 \rangle \right\} \equiv K(q_1, q_2 | \tau), \]

(87)

where

\[ \tau \equiv \frac{i \hbar}{2m} t. \]

(88)

Plugging (75) into (87), we get [10, 11],

\[ K(q_1, q_2 | \tau) = K^{(0)}(l; \tau) + r(q_1, q_2) K^{(1)}(l; \tau), \]

(89)

where \( l = l(q_1, q_2) \) is given in (42) and,

\[ K^{(0)}(l; \tau) = \frac{-2a}{\pi \sqrt{2 \pi \tau}} e^{-\left( \frac{i \pi \alpha}{4 \tau} \right)^2 \tau} \int_{l}^{\infty} db \; e^{-\frac{\beta}{2b} \sinh \frac{b}{2a}} \frac{1}{(\cosh b - \cosh l)^{1/2}}, \]

(90)

\[ K^{(1)}(l; \tau) = \frac{-2a}{\pi \sqrt{2 \pi \tau}} e^{-\left( \frac{i \pi \alpha}{4 \tau} \right)^2 \tau} \int_{l}^{\infty} db \; \frac{1}{(\cosh b - \cosh l)^{1/2}} \times \frac{d}{db} \left( e^{-\frac{i}{\hbar} \frac{\beta}{2b} \sinh \frac{b}{2a}} \sinh b \right), \]
\[ e^{-\frac{b^2}{2a} \left( \frac{b}{2a} \cosh \frac{b}{2a} + \frac{a-1}{2a} \sinh \frac{b}{2a} \right)} \] .

(91)

So the time development for a wave function on SH is given by

\[ \Psi(q,t) = \int_{\text{SH}} dV(q') K(q,q';t-t') \Psi(q',t') . \]

(92)

As for the wave function on the SRS, we should have

\[ \Psi_{\text{SRS}}(q,t) = \int_{\text{SRS}} dV(q') K_{\text{SRS}}(q,q';t-t') \Psi_{\text{SRS}}(q',t') \]

(93)

The periodicity of \( \Psi_{\text{SRS}} \) implies that a kernel \( K_{\text{SRS}} \) on the SRS is written as

\[ K_{\text{SRS}}(q_1,q_2|\tau) = \sum_{g \in \Sigma} K(q_1,g(q_2)|\tau). \]

(94)

5. Trace Formula and Zeta Function

In the preceding sections, we have eventually solved the quantum mechanics on SH and obtained the heat kernel on SH, which yields that on the SRS. Here in this section, we will concentrate on the quantum energy spectrum on the SRS.

As we have seen that the spectrum on the SRS was too complicated, it is quite difficult to estimate it explicitly. However, in the conventional case, that is, in the case of a particle on the Riemann surface of genus \( g \geq 2 \), the energy spectrum is related to the length spectrum through the Selberg trace formula [2]. We may expect that a similar relation will exist in our model.

First we present a formula of supertrace of a function \( G_{\text{SRS}}(q_1,q_2) \) on \( \text{SH}/\Sigma \),

\[ G_{\text{SRS}}(q_1,q_2) = \sum_{g \in \Sigma} G(q_1,g(q_2)) , \]

(95)

with

\[ G(q_1,q_2) = \Phi(l(q_1,q_2)) + r(q_1,q_2) \Psi(l(q_1,q_2)) , \]

(96)

where \( \Phi \) and \( \Psi \) are some functions and \( l \) and \( r \) are given in (42) and (39), respectively. We find that the supertrace of \( G_{\text{SRS}} \) defined by,

\[ \text{str} \ G_{\text{SRS}} = \int_{F} dV G_{\text{SRS}}(q,q) = \sum_{g \in \Sigma} \int_{F} dV G(q,g(q)) , \]

(97)

where \( F \) is a fundamental domain of \( \Sigma \), is calculated as

\[ \text{str} \ G_{\text{SRS}} = \text{Area}(\text{SRS}) \Phi(0) + \sum_{p \in \text{Prim}(\Sigma)} \sum_{n=1}^{\infty} \int_{F_p} dV G(q,p^n(q)) , \]

(98)

where use has been made of a formula,

\[ \sum_{g \in \Sigma} f(g) = f(1) + \sum_{p \in \text{Prim}(\Sigma)} \sum_{n=1}^{\infty} \sum_{g \in \Sigma/Z(p)} f(gp^n g^{-1}) . \]

(99)

and \( F_p \) is a fundamental domain for the centralizer of \( p \), \( Z(p) \),

\[ F_p = \bigcup_{g \in \Sigma/Z(p)} g^{-1} F . \]

(100)

We can assume that \( p \) is a magnification with a matrix \( A_p = \text{diag}(\chi(p)N_p^{1/2}) \).
\( \chi(p)N_p^{-1/2}, 1 \) (see (14)) and we can choose a convenient domain for \( F_p \) [14],

\[
\int_{F_p} dV \mapsto \int_{1}^{N_p} dy \int_{-\infty}^{\infty} dx \int d\theta d\bar{\theta} \frac{1}{2ay + a\theta} .
\]  

(101)

Then we finally get

\[
\text{str} G_{\text{SRS}} = \frac{\pi(g-1)}{a} \Phi(0) - \sum_{p \in \text{Prim}(S^1)} \sum_{n=1}^{\infty} \frac{l(p)}{2a\sqrt{\cosh l(p^n) - 1}} \left\{ \left(1 - \chi(p^n) \cosh \frac{l(p^n)}{2} \right) \int_{l(p^n)}^{\infty} ds \frac{\sinh s}{(\cosh b - \cosh l(p^n))^{1/2}} \Psi(s) \right. \\
\left. + (1 - \cosh l(p^n)) \int_{l(p^n)}^{\infty} ds \frac{1}{(\cosh b - \cosh l(p^n))^{1/2}} \frac{d\Phi(s)}{ds} \right\},
\]

(102)

where \( g \) is the genus of the SRS and

\[
l(p^n) = |n|l(p) = |n| \log N_p .
\]

(103)

Now we apply the above formula to the heat kernel on the SRS (94) which can be written as

\[
K_{\text{SRS}}(q_1, q_2|\tau) = \sum_{g \in S^1} \langle q_1 | e^{-\tau \Delta_{\text{SLB}}} | g(q_2) \rangle .
\]

(104)

Then we get

\[
\text{str} K_{\text{SRS}} = \text{str} \left(e^{-\tau \Delta_{\text{SLB}}} \right) = \sum_{n=0}^{\infty} \left( e^{-\tau \gamma_n^B} - e^{-\tau \gamma_n^F} \right) .
\]

(105)

Plugging \( K^{(0)} \) (90) and \( K^{(1)} \) (91) respectively into \( \Phi \) and \( \Psi \) in (102) and integrating with respect to \( s \), we finally obtain a superanalog of Selberg trace formula [11],

\[
\sum_{n=0}^{\infty} \left( e^{-\tau \gamma_n^B} - e^{-\tau \gamma_n^F} \right) = A(\tau) + \Theta(\tau) ,
\]

(106)

where

\[
A(\tau) = \frac{1}{\sqrt{\pi} \tau} (1 - g) e^{-\left(\frac{1}{4\tau}\right)^2} \int_{0}^{\infty} db e^{-\frac{b^2}{2}} \frac{\sinh b}{\sinh \frac{b}{2}} ;
\]

\[
\Theta(\tau) = \frac{1}{4\sqrt{\pi} \tau} \sum_{p \in \text{Prim}(S^1)} \sum_{n=1}^{\infty} \text{str} \left( K(p^n|a) \right) \frac{l(p^n)}{\sinh \frac{l(p^n)}{2}} e^{-\frac{2(l(p^n) - \left(\frac{1}{4\tau}\right)^2)}{4}},
\]

(107)

\[
K(p|a) = \text{diag} \left( e^{\frac{l(p)}{a}}, e^{-\frac{l(p)}{a}}, \chi(p) e^{\frac{l(p)}{a}}, \chi(p) e^{-\frac{l(p)}{a}} \right) .
\]

(108)

\( A(\tau) \) is the contribution of the “trivial motion” on the SRS (zero length term) and for \( \tau \to 0 \) it can be expanded into a positive power series,

\[
A(\tau) \sim -\frac{\text{Area}(\text{SRS})}{\pi} (b_0 + b_1 \tau + b_2 \tau^2 + \cdots) ,
\]

(109)

with

\[
b_0 = 1 ,
\]

\[
b_1 = \frac{(a - 1)(1 - 2a)}{6a^2} ,
\]

\[
b_2 = \frac{(a - 1)(2a - 1)(2a^2 - 2a + 1)}{60a^4} ,
\]

(110)

This series approximates “\( \text{str} \left(e^{-\tau \Delta_{\text{SLB}}} \right) \)” up to an exponentially small error. On the other hand, \( \Theta(\tau) \) is the contribution from the periodic motions on the SRS and consistent with the semiclassical approximation [9].
Let us introduce a zeta function $Z(s|a)$ [10] with one parameter $a$ associated with our model. The function is defined by,

$$Z(s|a) \equiv \prod_{p \in \text{Prim}(S\Gamma)} \prod_{n=0}^{\infty} \text{sdet} \left( 1 - K(p|a)e^{-(s+n)i(p)} \right) ,$$

(112)

and we see that the zeta function is related to the trace formula [11],

$$\frac{d}{ds} \log Z(s|a) = (2s - 1) \int_0^{\infty} dt e^{-(s-\frac{1}{2})^2 t + \left( \frac{1-2a}{2a} \right)^2 t} \Theta(t) .$$

(113)

We point out that the zero-points and the poles of $Z(s|a)$ give directly the eigenvalues of $\triangle_{\text{SLB}}$ (energy spectrum) on the SRS. More precisely, the zero-points give the eigenvalues of the Grassmann even functions and the poles give those of the odd functions. Using the trace formula, we find

$$\frac{d}{ds} \log Z(s|a)$$

$$= (2s - 1) \int_0^{\infty} dt e^{-(s-\frac{1}{2})^2 t + \left( \frac{1-2a}{2a} \right)^2 t} \left\{ \sum_{n=0}^{\infty} (e^{-t\gamma^B_n} - e^{-t\gamma^F_n}) - A(t) \right\}$$

$$= (2s - 1) \sum_{n=0}^{\infty} \left[ \frac{1}{(s-\frac{1}{2})^2 - \left( \frac{1-2a}{2a} \right)^2 + \gamma^B_n} - \frac{1}{(s-\frac{1}{2})^2 - \left( \frac{1-2a}{2a} \right)^2 + \gamma^F_n} \right]$$

$$+ 2(g-1) \sum_{n=0}^{\infty} \left( \frac{1}{s+n-\frac{1}{2a}} - \frac{1}{s+n+\frac{1}{2a}} \right) .$$

(114)

Note that the last term in (114) becomes a finite sum when $|a|^{-1} = m$ ($m$: positive integers),

$$2(g-1) \text{sign}(a) \sum_{n=0}^{m-1} \frac{1}{s+n-m} .$$

(115)

This formula implies that $Z(s|a)$ has a meromorphic continuation onto the whole complex plane $\mathbb{C}$. The zero-points (poles) of order 1 exist at

$$s = \frac{1}{2} \pm \sqrt{\left( \frac{1-a}{2a} \right)^2 - \gamma^B_F} .$$

(116)

Other trivial zero-points (ZP) and poles (P) exist respectively at,

I. $a^{-1} \neq m$ ($m$: positive integers);

ZP : $s = -n + \frac{1}{2a}$,  

P : $s = -n - \frac{1}{2a}$,  ($n = 0, 1, 2, \cdots$) ,

II. $a^{-1} = m$;

ZP : $s = -n + \frac{m}{2}$,  

P : none  ,  ($n = 0, 1, \cdots, m-1$) ,

III. $a^{-1} = -m$;

ZP : none ,  

P : $s = -n - \frac{m}{2}$,  ($n = 0, 1, \cdots, m-1$) ,

(117)

(118)

(119)

where the order of each ZP and P is $2(g-1)$ .

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6. Classical Chaos from the Trace Formula

Our aim in this section is to discuss the exponential growth of the counting function \( \Pi(x) \) for the lengths of primitive periodic orbits in (61),

\[
\Pi(x) = \# \{ p, l(p) \leq x \} \sim \frac{e^{\alpha x}}{\alpha x}, \quad \text{for } x \to \infty. \tag{120}
\]

For this purpose, we prefer the general Selberg supertrace formula, established in Refs. [14, 21]. Specifically, if \( h \) is a test function with the properties

(i) \( h(1/2 + ip) \in C^\infty(\mathbb{R}) \)

(ii) \( h(1/2 + ip) \sim O\left(\frac{1}{p^2}\right) \) for \( p \to \pm \infty \)

(iii) \( h(1/2 + ip) \) is holomorphic in the strip \( |\text{Im}p| \leq 1 + \epsilon, \epsilon > 0 \),

then the supertrace formula on the SRS with genus \( g \geq 2 \) is given by \( ^* \) [14],

\[
\sum_{n=0}^{\infty} \left[ h(\lambda_n^B) - h(\lambda_n^F) \right] = (1-g) \int_0^\infty \frac{g(u) - g(-u)}{\sinh \left( \frac{u}{2} \right)} \, du \tag{121}
\]

\[
+ \sum_{p \in \text{Prim}(\mathbb{F})} \sum_{n=1}^{\infty} \frac{l(p)}{2 \sinh \left( \frac{n \pi}{2p} \right)} \left[ g(nl(p)) + g(-nl(p)) \right. \\
\left. - \chi^p_n \left( g(nl(p)) e^{\frac{-u(p)}{2}} + g(-nl(p)) e^{\frac{u(p)}{2}} \right) \right],
\]

where \( \{ \lambda_n^{B(F)} \} \) is the spectrum of the operator \( \Box_0 \) (see (84)) and

\[
g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \, e^{-iup} h(1/2 + ip). \tag{122}
\]

To get the information for \( \Pi(x) \), it is convenient to choose a test function \( h \) so that the term proportional to \( \chi^p_n(p) \) in (121) cancel, i.e.,

\[
g(nl(p)) e^{\frac{-u(p)}{2}} + g(-nl(p)) e^{\frac{u(p)}{2}} = 0. \tag{123}
\]

We take a function \( (\mathrm{Re} \, s > 1, \mathrm{Re} \, \sigma > 1) \)

\[
h(\lambda) = 2\lambda \left( \frac{1}{s^2 - \lambda^2} - \frac{1}{\sigma^2 - \lambda^2} \right), \tag{124}
\]

with the Fourier transform \( g(u) \) given by [21],

\[
g(u) = \sign(u) e^{\frac{s}{2}} (e^{-s|u|} - e^{-\sigma|u|}) . \tag{125}
\]

Thus only \( \chi(p) \)-independent term remains in the supertrace formula. Plugging (124) and (125) into (121), we have

\[
2 \sum_{n=0}^{\infty} \left\{ \frac{\lambda_n^B}{s^2 - (\lambda_n^B)^2} - \frac{\lambda_n^F}{s^2 - (\lambda_n^F)^2} \right\} + \sum_{p \in \text{Prim}(\mathbb{F})} \int_{-\infty}^{\infty} dp \, \frac{l(p)}{2 \sinh \left( \frac{n \pi}{2p} \right)} \\
= 4(g-1) \{ \Psi(1+s) + \Psi(s) - \Psi(1+s) - \Psi(s) \}
\]

\[
+ \sum_{p \in \text{Prim}(\mathbb{F})} l(p) \frac{e^{-s|p|}}{1 - e^{-s|p|}} - \frac{e^{-\sigma|p|}}{1 - e^{-\sigma|p|}}
\]

where \( \Psi(z) = \Gamma'(z)/\Gamma(z) \). Defining

\[
F(s) = \sum_{p \in \text{Prim}(\mathbb{F})} l(p) \frac{e^{-s|p|}}{1 - e^{-s|p|}}, \tag{126}
\]

and using a formula,

\[
\Psi(z) = -\gamma - \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+1} \right) , \tag{127}
\]
we obtain

\[ F(s) - F(\sigma) = 4(g - 1)\left\{ \sum_{n=0}^{\infty} \frac{1}{1 + s + n} + \sum_{n=0}^{\infty} \frac{1}{s + n} - \sum_{n=0}^{\infty} \frac{1}{1 + \sigma + n} - \sum_{n=0}^{\infty} \frac{1}{\sigma + n} \right\} \]

\[ + \sum_{n=0}^{\infty} \left\{ \frac{1}{s - \lambda^B_n} - \frac{1}{s + \lambda^B_n} - \frac{1}{s - \lambda^F_n} + \frac{1}{s + \lambda^F_n} \right\} \]

\[ - \sum_{n=0}^{\infty} \left\{ \frac{1}{\sigma - \lambda^B_n} - \frac{1}{\sigma + \lambda^B_n} - \frac{1}{\sigma - \lambda^F_n} + \frac{1}{\sigma + \lambda^F_n} \right\}. \]  

(129)

Now we can read the analytic properties of \( F(s) \) from the above trace formula \( \star \):

(i) \( F(s) \) has a meromorphic continuation onto the whole plane \( \mathbb{C} \),

(ii) \( F(s) \) has poles in the following points

\[
\begin{align*}
    s &= 0, \quad \text{residue } 4(g - 1), \\
    s &= -n \ (n = 1, 2, \cdots), \quad \text{residue } 8(g - 1), \\
    s &= \pm \lambda^B_n \text{ and } \lambda^B_n \neq \lambda^F_n, \quad \text{residue } \pm 1, \\
    s &= \pm \lambda^F_n \text{ and } \lambda^B_n \neq \lambda^F_n, \quad \text{residue } \mp 1.
\end{align*}
\]

(130)

Thus the largest positive eigenvalue \( \lambda_{\text{max}} \) should exist and it takes the value \( \frac{1}{2} \) or \( c(>0) \) if we assume its existence and ignore so called small eigenvalues, which we have no knowledge except for the ground energy \( \lambda^B_0 = 0 \). In the second step, we relate the first singularity of \( F(k) \) to the proliferation of periodic orbits. In fact, we have

\[ F(k) \sim \sum l(p) e^{-kl(p)} \quad l \to \infty, \]

(131)

\[ \sim \int xe^{-kx} d\Pi(x) \quad x \to \infty. \]

This implies by the above arguments

\[ \int xe^{-kx} d\Pi(x) \sim \frac{1}{k - \lambda_{\text{max}}}. \]

(132)

If we now perform the inverse Laplace transformation, we get

\[ \Pi(x) \sim e^{\lambda_{\text{max}}x} / x, \]

(133)

and hence

\[ \alpha = \lambda_{\text{max}}. \]

(134)

Here we stress the following points:

\[ * \]  

We have assumed the convergence of the series (129).
(i) the asymptotic for of $\Pi(x)$ has been determined from the real eigenvalue $\lambda_{\text{max}}$ which is quite different in comparison to the dynamical system on the ordinary Riemann surface. In the latter model, the ground energy ($\lambda_0 = 0$) controls the exponential proliferation of the periodic orbits through the trace formula and leads to the asymptotic formula

$$\Pi(x) \sim e^{\alpha x}/x.$$  \hfill (137)

(ii) Our derivation includes the delicate arguments so the rigorous proof should be developed. One way to the direction is to investigate the analytic properties of $L$-function defined by

$$L(s) = Z(s + 1)/Z(s),$$

$$Z(s) = \prod_{p \in \text{Prim}(S^\Gamma)} \prod_{n=0}^{\infty} \left(1 - e^{-(s+n)(l(p))}\right),$$  \hfill (138)

Then the following theorem is known [22],

**Theorem**

If $L(s)$ satisfies the conditions:

(I) $L(s)$ converges absolutely for $\Re(s) > \alpha$,

(II) $L(s)$ has a meromorphic continuation onto some region including $\Re(s) > \alpha$,

(III) $L(s)$ has no zero point on $\Re(s) > \alpha$,

(IV) $L(s)$ is holomorphic on $\Re(s) > \alpha$ and has a simple pole at $s = \alpha$, then the asymptotic formula exists

$$\pi(x) \sim e^{\alpha x}/\alpha x.$$  \hfill (139)

Our previous arguments support the conditions (I)-(IV), however, the constant $\alpha$ has not been determined strictly due to small eigenvalue problems.

7. Moduli

The energy spectrum is controlled by the length spectrum, or equivalently, the periodic orbits on SRS through the trace formula and the length of a periodic orbit is determined by the norm function of the corresponding $S\Gamma$-element. This permits us to think of the *moduli* for SRS which is the free parameters in $S\Gamma$. In the theory of Riemann surface, it is known that some lengths corresponding to $6g - 6$ primitive periodic orbits can be chosen as the moduli parameters. The similar situation may be expected in the case of SRS. In Sec. 2, we saw that the moduli of SRS is the $6g - 6$ Grassmann even and $4g - 4$ odd parameters and hence, lengths of $6g - 6$ should be used as the even moduli parameter. Unfortunately, we have no quantity for the odd-moduli, i.e., the mechanical observable is Grassmann even.

We first note that the length $l(k), k \in \Gamma$ is written as

$$l(k) = \log(k(S), S, U_k, V_k)$$  \hfill (140)

Here $U_k(V_k)$ is the repelling (attractive) fixed point of $k$ (see (9) and (10)) and $S$ is an arbitrary point in $R_s - \{U_k, V_k\}$. The Grassmann even bracket of 4-points $(Z_1, Z_2, Z_3, Z_4)$ on $SH \cup R_s$ is defined by

$$\langle Z_1, Z_2, Z_3, Z_4 \rangle = \frac{z_{13} z_{24}}{z_{14} z_{23}},$$  \hfill (141)

where $Z_i = (z_i, \theta_i)$ and $z_{ij} = z_i - z_j - \theta_i \theta_j$. This is invariant under $\text{SPL}(2,\mathbb{R})$ and its body is actually the ordinary cross ratio invariant under the Möbius transformations. Next we introduce the Grassmann odd $\text{SPL}(2,\mathbb{R})$ invariant quantity. This is defined by the bracket of 3-points on $SH \cup R_s$,

$$\langle Z_1, Z_2, Z_3 \rangle = \frac{\theta_{123}}{(z_{12} z_{23} z_{31})^{1/2}},$$  \hfill (142)

where $\theta_{123} = \theta_1 z_{23} + \theta_2 z_{31} + \theta_3 z_{12} + \theta_1 \theta_2 \theta_3$. The ordering of 3-points in the above formula is fixed (up to cyclic permutations) by demanding that $(z_1 - z_2)(z_2 - z_3)$...
$z_3 (z_3 - z_1) > 0$. Using this invariant, we can provide 4g − 4 odd moduli parameters. Let \( \{ A_i, B_i \} \) \( (i = 1 \sim g) \) be the generators of \( S^1 \) in Eq. (7), and \( \{ U_i^A, V_i^A \} \) and \( \{ U_i^B, V_i^B \} \) be the fixed points of the generators \( A_i \) and \( B_i \), respectively. Then odd moduli parameters \( \{ \lambda(k) \} \) \( (k = 1 \sim 4g - 4) \) are given by [23],

\[
\{ \lambda(k) \} = \{ (U_1^A, V_1^A, U_1^A, V_1^A), (U_1^A, V_1^A, U_1^A, V_1^A), \ldots \}.
\] (143)

Since the condition (7) on the generators is invariant under conjugation, one may regard \( A_1 \) as diagonal, i.e., the fixed points of \( A_1 \) can be put to \( \{ (0,0), (\infty,0) \} \).

Then the condition reveals that the parameters in \( B_1 \) is written by the parameters in \( \{ A_j, B_j \} \) \( (j = 2, \cdots, g) \). The \( \{ \lambda(k) \} \) represent essentially all the odd parameters in the remaining generators, \( \{ A_j, B_j \} \) \( (j = 2, \cdots, g) \). Thus \( \{ l(i), \lambda(j) \} \) \( (i = 1, \cdots, 6g - 6, j = 1, \cdots, 4g - 4) \) provide the moduli of SRS. These moduli parameters have the manifest SPL(2, IR) invariance by the construction and offer the good coordinates on the moduli space of SRS.

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