Cavity locking with spatial modulation of optical phase front for laser stabilization

Sheng Feng1 · Songqing You1,2 · Peng Yang1 · Fenglei Zhang1 · Yunlong Sun1 · Boya Xie2

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Abstract
We study optical cavity locking for laser stabilization through spatial modulation of the phase front of a light beam. A theoretical description of the underlying principle is developed, showing that an error signal similar to the Pound-Drever-Hall one may be created for cavity locking with this method. Residual amplitude modulation (RAM) can be caused by experimental imperfections, especially the manufacture errors of the phase modulators, and we discuss how RAM noises may be suppressed. Although there are technical issues waiting for practical solutions before a mature technology is developed based on the proposed method, it holds great potential for implementation of compact laser stabilization systems locked to Fabry-Pérot cavities without use of expensive bulky devices including signal generators and electro-optic modulators, which will be useful in space-based research missions such as gravitational-wave signals searching and inter-spacecraft laser ranging.

1 Introduction
Ultra-stable lasers locked to high-finesse Fabry-Pérot (FP) cavities delivering single-mode electromagnetic waves play an essential role in a variety of advanced research fields, such as optical atomic clocks [1–3], tests of relativity [4, 5], gravitational wave detectors [6], and photonic microwave synthesizers [7]. Despite several cavity locking schemes of potential interest [8–12], most of the performance demonstrations have been accomplished by use of the standard Pound-Drever-Hall (PDH) one [13] in which the phase front of the optical carrier undergoes temporal modulation. A well-known issue with the PDH modulation technique is residual amplitude modulation (RAM) that couples frequency offset noise to the servo error signal and thereby aggravates the laser frequency instability. A few RAM reduction schemes [14–17] have been implemented with the most remarkable one demonstrated recently [18] that utilized an active servo loop involving both DC electric field and temperature corrections applied to the phase modulator. RAM control on the $10^{-6}$ level was reported with a RAM-induced frequency instability comparable to or lower than thermal noise limit [18]. Using PDH cavity locking scheme, state-of-the-art stable lasers with high-spectral purity have been realized with integral linewidths below 1 Hz [19, 20]. As a promising alternative to the PDH scheme, self-injection locking technique can also be explored to achieve a short-term linewidth of about 1 Hz [21].

Another research direction of relevance is to construct portable stable laser systems [21, 22] towards applications wherein robustness and integration are primary concerns, with a 20 ms stability of $10^{-13}$ already reported [23]. In space missions, e.g., gravitational-wave signals searching [24] and inter-spacecraft laser ranging applications [25], the bulkiness, power hunger, environment sensitivity of the standard laser stabilization scheme are becoming limiting factors of the relevant research activities. It has been demonstrated that the tilt locking scheme might be another promising alternative to the standard PDH one to construct compact laser stabilization systems for space-borne applications [22, 26].

In analog to the PDH method [13] that makes use of temporal modulation, we explore in this work spatial modulation of the optical phase front of the carrier for FP cavity locking. Aside from its own merit of fundamental interest, a notable advantage of our proposed method is that it can be implemented with passive modulators whereas the
PDH method necessitates active modulators, in the sense that an active modulator demands an electronic driver but a passive modulator needs no driver at all. Examples of active modulators are electro-optic modulators (EOM’s) which are indispensable core components in the PDH method. Active modulators are usually subject to thermal noise and electronic noise and often make the cavity locking systems bulky. In contrast, as will be explained soon in the following, for spatial phase modulation a passive modulator can be made out of a piece of machined glass which will be beneficiary to the construction of low-cost compact laser systems like the tilt-locking method [26]. Moreover, the passive modulator in principle is free of electronic noise and has low thermal noise if it is composed of material with low thermal expansion.

We will present a theoretical model and show that the studied cavity locking method with spatial phase modulation can produce considerably large error signals for the servo loop. Our theoretical analysis also reveals how RAM noises may be introduced into the error signals due to imperfections of the spatial modulators and then we will discuss possible ways for RAM reduction. Because of the way of error signal production in our method, it may be realized with standing-wave cavities like the PDH scheme; in contrast, the tilt-locking method utilizes travelling-wave cavities to achieve the ultimate limit of laser stabilization [26]. Therefore, our proposed method may share the same optical cavity platform with the PDH one and might pave the way to develop low-cost portable compact laser stabilization systems locked to FP cavities [22, 23] by use of passive phase modulators.

In the next section, a theoretical model will be established for the proposed cavity locking method. In this model, a transparent optical element, namely a spatio-optic modulator (SOM), is exploited as a passive modulator for the spatial modulation of light. In principle, the SOM could be a piece of carefully machined glass or a liquid crystal Spatial Light Modulator (SLM) that may be programmed following specific experimental requirements. However, a commercial SLM is not a passive modulator and does not meet the requirement of low noise and compactness necessary for constructing low-cost portable compact laser systems. Therefore, SLM’s are not in the scope of consideration in the current work. In section three, with theoretical calculations we will show how an error signal may be created in the studied scheme and how RAM noises will be introduced into the error signal, from which one may find handy ways to suppress RAM. Thereby, we intend to provide a theoretical framework for cavity locking with spatial phase modulation. The fourth section is devoted to the discussions about the error signal and RAM reduction that may be instructive for follow-on works towards a full-blown cavity locking technology based on the proposed method.

2 Theoretical model

In what follows, let describe the principle underlying the studied cavity locking method through spatial modulation of optical phase front. For simplicity without loss of generality, let consider a collimated laser beam in a Gaussian mode, $E_1(\vec{r}, t)$, that is coupled into an FP cavity through a focal lens (Fig. 1) and the SOM is inserted into the incident light beam before the lens such that the phase front of the beam undergoes spatial modulation. Then the spatially modulated light field $E_m(\vec{r}, t)$ may be described as,

$$E_m(\vec{r}, t) = E_1(\vec{r}, t)U(\vec{r}) = E_1(\vec{r}, t)e^{i\varphi(\vec{r})}, \quad (1)$$

wherein $U(\vec{r})$ is the SOM modulation function, and the vector $\vec{r}$ represents the two-dimensional (2D) coordinates in the plane perpendicular to the propagation axis of the light beam. Unity transmittance is assumed for the SOM and $\varphi(\vec{r})$ is the spatial distribution of the change to the light field phase front caused by the SOM.

After passing through the SOM, a minor part of the incident light beam is reflected by a polarizing beamsplitter (BPS) and directed to a photodetector (PD$_1$, Fig. 1) to sense the beam steering/jitter for RAM cancellation, and major part of the beam flies in succession through the PBS, a 1/4 waveplate whose optic axis is aligned 45° relative to the PBS transmission axis, and the focal lens before it is coupled into the FP cavity. As will be shown soon in

![Fig. 1 (color online) Schematics for cavity locking by spatial modulation of optical phase front. SOM: Spatio-optic modulator. PBS: Polarizing beam-splitter. $\lambda/4$: Quarter wave plate. L$_1$: Focal lens coupling the incident light into the cavity. M$_1$,$M_2$: Cavity mirrors. PZT: Piezoelectrical transducer. L$_{2,3}$: Imaging lenses used to map the SOM images to the corresponding photo-detectors. PD$_2$: Dual-quadrant photo-detectors. F: An optional optical element used to tailor the spatial mode amplitudes of the light field received by PD$_1$ for RAM cancellation.](image)
the following, the incident beam is split into two parts of spatial mode by spatial phase modulation, with one part in fundamental Gaussian mode and the other in a superposition of high-order transverse modes. While the fundamental mode after the focal lens will enter into the cavity and experience some phase delay due to optical resonance, most of the rest of the beam will be directly reflected back serving as a phase reference for cavity locking. Eventually the fundamental mode will exit from the cavity and merge with the directly reflected part, both flying towards to the photodetector (PD) where an error signal will be created. The output from the detector will be sent to control the laser for frequency stabilization.

To get an insight into the cavity locking method, let assume $|\psi(\vec{r})|<<1$ for the moment. Under this assumption, the modulated light field $E_m(\vec{r}, t)$ reads

$$E_m(\vec{r}, t) = E_i(\vec{r}, t) + i\psi(\vec{r})E_i(\vec{r}, t),$$  

(2)

from which it follows that the modulated light field becomes a superposition of the incident light field, $E_i(\vec{r}, t)$, and another effective field, $E_e(\vec{r}, t) \equiv i\psi(\vec{r})E_i(\vec{r}, t)$, as a consequence of spatial modulation. For the purpose of this work, the SOM may be designed for $\psi(\vec{r})$ to satisfy

$$\iint d\vec{r} E_i^*(\vec{r}, t) E_e(\vec{r}, t) = \iint d\vec{r} \psi(\vec{r}) |E_i(\vec{r}, t)|^2 = 0,$$

(3)

in which the integral area covers the whole cross section of the incident beam. In other words, the spatial mode of the effective field, $E_e(\vec{r}, t)$, is orthogonal to that of the incident field, $E_i(\vec{r}, t)$, and hence the two fields may not simultaneously resonate inside a narrow-linewidth optical cavity due to their spatial mode orthogonality [27].

To that end, however, an FP cavity of perfect confocal configuration must be avoided to break the transverse mode degeneracy. When the number of transverse modes is large, it may be difficult in experiment to ensure frequency non-degeneracy for all high-order modes. To highlight the key point, nevertheless, we still assume frequency non-degeneracy for all transverse modes for the moment and will turn to the problem of mode degeneracy in the discussion section.

Now let suppose that the incident field $E_i(\vec{r}, t)$ is near resonance in the FP cavity of Fig. 1, and that the cavity linewidth is narrow enough so that the effective field $E_e(\vec{r}, t)$ is far-off resonance. Then after reflection from the cavity, the modulated light field becomes [8]

$$E_e(\vec{r}, t) = \left\{ \sqrt{R_1} [1 + i\psi(\vec{r})] - \frac{T_i}{\sqrt{R_1}} e^{i\delta} \sqrt{R} \right\} E_i(\vec{r}, t)$$

$$= \chi e^{i\theta} + i\psi(\vec{r}) \sqrt{R_1} E_i(\vec{r}, t),$$

(4)

in which

$$\chi e^{i\theta} \equiv 1 - \frac{T_i}{R_1} \frac{e^{i\delta} \sqrt{R}}{1 - e^{i\delta} \sqrt{R}}.$$

(5)

and $R_1, T_i$ are respectively the reflectivity and transmissivity of the cavity entrance mirror $M_1$, $R \equiv R_1 R_2$ ($R_2$ the reflectivity of the other cavity mirror $M_2$), and $\delta$ stands for the round trip phase delay of the light field inside the cavity.

In what follows, we will show how to exploit an optical element of spatial modulation, i.e., an SOM, to implement cavity locking based on Eq. (4). Given that the incident light field $E_i(\vec{r}, t)$ is in a Gaussian mode which is symmetric with respect to the light propagation axis, i.e.,

$$E_i(\vec{r}, t) = E_i(-\vec{r}, t),$$

(6)

one may utilize an SOM with a spatial modulation function that introduces an anti-symmetric phase change to the incident beam around its axis,

$$\psi(\vec{r}) = -\psi(-\vec{r}),$$

(7)

wherein $\psi(\vec{r})$ is usually a space-varying function. Here one must stress that $\vec{r}$ is not a one-dimensional (1D) radial coordinate, instead it as a vector represents 2D coordinates. From Eqs. (6) and (7) it immediately follows that Eq. (3) holds true and, hence, so does Eq. (4). An example of a designed SOM satisfying Eq. (7) is illustrated in Fig. 2 and the phase front change $\psi(\vec{r})$ of the light field due to the spatial modulation reads

\[ \begin{align*}
\Delta \equiv (\varphi_0/2\pi) \cdot \lambda/(n-1) \\
&\text{with } \lambda \text{ being the wavelength of the light beam and } n \text{ the refractive index of the SOM material. (Top): Front view of the SOM. (Bottom): The SOM surface profile along the dash line as indicated on the top.}
\end{align*} \]
\[ \varphi(\vec{r}) = \begin{cases} \varphi_0 & |\vec{r} - r_0\hat{x}| < r_0 \\ 0 & \text{otherwise} \\ -\varphi_0 & |\vec{r} + r_0\hat{x}| < r_0 \end{cases}, \quad (8) \]

in which \( \pm \varphi_0 \) are constant modulation-induced phase changes in the corresponding areas, \( S_{1,2} \), which are circular here centered at the points of \((\pm r_0, 0)\) respectively with the same radius of \( r_0 \), and \( \delta \) denotes the unit vector along the horizontal direction. Eq. (8) can also be expressed in polar coordinates where one can see that the spatial modulation is dependent on both the radial and azimuthal coordinates as long as \( r_0 \neq 0 \).

### 3 Error signal and RAM issue

Next we will show how an error signal may be obtained in the cavity locking method. To this end, one may use a dual-quadrant photo-detector to receive the light beam reflected from the cavity. An optical imaging system is utilized to project the image of the SOM onto the photo-detector, with each quadrant aligned with respect to each of the circles, \( S_{1,2} \), on the SOM (Fig. 1). The output photo-electric currents from the two quadrants are subtracted to produce an error signal \( \epsilon(\delta) \) as described by,

\[ \epsilon(\delta) = \int \int_{S_1} d\vec{r} \ | E_1(\vec{r}, t) |^2 - \int \int_{S_2} d\vec{r} \ | E_2(\vec{r}, t) |^2 . \quad (9) \]

To make it clearer, one may plug Eqs. (4) and (8) into Eq. (9), leading to

\[ \epsilon(\delta) = \int \int_{S_1} d\vec{r} R_1 | (\chi e^{i\theta} + i\varphi_0)E_1(\vec{r}, t) |^2 \\
- \int \int_{S_2} d\vec{r} R_1 | (\chi e^{i\theta} - i\varphi_0)E_1(\vec{r}, t) |^2 \\
= 4R_1 I_0 \varphi_0 \chi \sin \theta \\
= 4R_1 I_0 \varphi_0 \ \text{Im} \left( 1 - \frac{T_1}{R_1} \frac{e^{i\delta} \sqrt{R}}{1 - e^{i\delta} \sqrt{R}} \right), \quad (10) \]

in which \( \text{Im}(\cdot) \) stands for the imaginary part of a complex number and one has invoked Eq. (6) from which it follows that

\[ \int \int_{S_1} d\vec{r} \ | E_1(\vec{r}, t) |^2 = \int \int_{S_2} d\vec{r} \ | E_1(\vec{r}, t) |^2 \equiv I_0 . \quad (11) \]

The error signal for cavity locking varies as a function of cavity detuning \( \delta \) from its peak resonance according to Eq. (10). Here one should note that the coefficient \( I_0 \) is dependent on the value of the SOM parameter \( r_0 \) through which the amplitude of the error signal may be optimized.

In what follows, let turn to the RAM problem associated with the studied cavity locking method due to SOM manufacture errors. Specifically, the anti-symmetry given by Eq. (7) may be broken to some extent in practice resulting from the difference between the areas, optical transmissivities, and modulation depths of the two circles, \( S_1 \) and \( S_2 \); the SOM symmetry breaking, together with unwanted input beam drifting/jitter, thermal sensitivities of the SOM, and unbalanced quantum efficiencies and unequal gains of the two quadrants of the photo-detector, will invalidate the assumption of Eqs. (8) and (11), causing RAM noises in the error signal.

To account for these effects in theory, one may define different phase modulations \( \varphi_{1,2} \) and center positions \( r_{0_{1,2}} \) for each of the two circular areas \( S_{1,2} \) respectively with radii \( r_1 \leq r_{0_{1}} \) and \( r_2 \leq r_{0_{2}} \).

\[ \varphi(\vec{r}) = \begin{cases} \varphi_1 & |\vec{r} - r_{0_{1}}\hat{x}| < r_1 \\ 0 & \text{otherwise} \\ -\varphi_2 & |\vec{r} + r_{0_{2}}\hat{x}| < r_2 \end{cases}, \quad (12) \]

and photo-currents \( I_{1,2} \).

\[ I_1 \equiv G_1 \int \int_{S_1} d\vec{r} \ | E_1(\vec{r}, t) |^2, \quad I_2 \equiv G_2 \int \int_{S_2} d\vec{r} \ | E_1(\vec{r}, t) |^2 . \quad (13) \]

Here the normalized coefficients \( G_{1,2} \) account for all the effects resulting from the SOM symmetry breaking and thermal sensitivities, beam drifting/jitter, and the differences in the optical transmissivities of the areas \( S_{1,2} \), the gains and quantum efficiencies of the two detector quadrants.

With Eqs. (4), (12), and (13), the error signal becomes

\[ \epsilon(\delta) = G_1 \int \int_{S_1} d\vec{r} R_1 | (\chi e^{i\theta} + i\varphi_1)E_1(\vec{r}, t) |^2 \\
- G_2 \int \int_{S_2} d\vec{r} R_1 | (\chi e^{i\theta} - i\varphi_2)E_1(\vec{r}, t) |^2 \\
= 4R_1 (I_1 \varphi + \Delta I \Delta \varphi) \ \text{Im} \left( 1 - \frac{T_1}{R_1} \frac{e^{i\delta} \sqrt{R}}{1 - e^{i\delta} \sqrt{R}} \right) \\
+ 2R_1 [2I_1 \varphi \Delta \varphi + \Delta I (\chi^2 + \varphi^2 + (\Delta \varphi)^2)] . \quad (14) \]

in which \( I = (I_1 + I_2)/2, \Delta I = (I_1 - I_2)/2, \varphi = (\varphi_1 + \varphi_2)/2, \) and \( \Delta \varphi = (\varphi_1 - \varphi_2)/2 \). The first term on the right hand side of Eq. (14) is the error signal required for cavity locking, whereas the second term describes RAM in the studied scheme that introduces frequency offset noise into the servo error signal leading to degradation of the laser frequency stability.

To get rid of RAM noises in the scheme, one may adjust the coefficients \( G_{1,2} \) to change the value of \( \Delta I = (I_1 - I_2)/2 \) according to Eq. (13) such that
\[ \Delta I = -\frac{2\bar{\rho}\Delta \varphi}{\chi^2 + \bar{\rho}^2 + (\Delta \varphi)^2}, \]  

(15)

which guarantees zero RAM, i.e., the second term in Eq. (14) becomes null. Equality (15) can be realized without much difficulty in practice by varying the optical transmissivities of the areas \( S_{1,2} \) on the SOM and/or the gains of the two quadrants of the detector. Moreover, unwanted drifting/jitter of the phase-modulated beam can be continuously monitored with another dual-quadrant photo-detector (PD1 in Fig. 1) and corrections may be applied to the input beam or the error signal accordingly for further RAM reduction.

In the preceding analysis it was assumed that \( |\varphi(\vec{r})| \ll 1 \), which is nonetheless usually invalid in practice when large error signal sizes are desired. In the following, let relax this assumption and discuss how to implement cavity locking through spatial modulation of optical phase front. To that end, Eq. (2) may be replaced by

\[ E_m(\vec{r}, t) = A E_1(\vec{r}, t) + E_e(\vec{r}, t), \]

(16)

wherein the effective field now becomes

\[ E_e(\vec{r}, t) = [\bar{e}^{i\varphi(\vec{r})} - A] E_1(\vec{r}, t). \]

(17)

Here the constant coefficient \( A \) is defined as

\[ A \equiv \frac{\int \int d\vec{r} \cos \varphi(\vec{r}) |E_1(\vec{r}, t)|^2}{\int \int d\vec{r} |E_1(\vec{r}, t)|^2}, \]

(18)

which ensures the orthogonality between the effective field \( E_e(\vec{r}, t) \) and the incident field \( E_1(\vec{r}, t) \), i.e.,

\[ \int \int d\vec{r} E_1^*(\vec{r}, t) E_e(\vec{r}, t) = 0, \]

(19)

as long as the anti-symmetry (7) of the phase modulation function holds valid. Then after reflection from the cavity, the modulated light field becomes [8]

\[ E_r(\vec{r}, t) = \left\{ \sqrt{R_1} e^{i\varphi(\vec{r})} - A \frac{T_1}{\sqrt{R_1}} e^{i\delta \sqrt{R_1}} \right\} E_1(\vec{r}, t) \]

\[ \equiv [\chi e^{i\theta} + i \sin \varphi(\vec{r})] \sqrt{R_1} E_1(\vec{r}, t), \]

(20)

in which

\[ \chi e^{i\theta} \equiv \cos \varphi(\vec{r}) - A \frac{T_1}{\sqrt{R_1}} e^{i\delta \sqrt{R_1}}. \]

(21)

After substituting Eqs. (20) and (8) into Eq. (9), one arrives at

\[ \varepsilon(\delta) = \int \int_{S_1} d\vec{r} R_1 |(\chi e^{i\theta} + i \sin \varphi_0) E_1(\vec{r}, t)|^2 \]

\[ - \int \int_{S_1} d\vec{r} R_1 |(\chi e^{i\theta} - i \sin \varphi_0) E_1(\vec{r}, t)|^2 \]

\[ = 4R_1 I_0 \sin \varphi_0 \text{ Im} \left( \cos \varphi_0 - A \frac{T_1}{R_1} e^{i\delta \sqrt{R_1}} \right), \]

(22)

from which it follows that the error signal for cavity locking varies as the cavity detunes from its peak resonance (Fig. 3) as long as \( \sin \varphi_0 \neq 0 \).

### 4 Discussions

To gain a full evaluation of the proposed cavity locking method with respect to the standard PDH and tilt-locking ones, comprehensive studies must be carried out on the error signal optimization and the suppression of the noise levels inherent to this method based on Eqs. (22), (14), and (15) respectively. This is surely a task beyond the scope of the current work and in the following we would intend to give only brief discussions that may be instructive for follow-on works on the topic of relevance.

Regarding the error signal it follows from Fig. 3 that, when the spatial modulation depth \( \varphi_0 = \pi/3 \), its size can be almost twice that of the PDH one with temporal modulation depth of \( \beta = 1.08 \), for which the PDH scheme needs to pay the price of high-order harmonic r.f. modulation side-bands in the optical spectrum of the incident beam. Although our result is based only on numerical simulation,
it is encouraging and deserves further experimental pursuit. Moreover, we stress that the error signal in this scheme is generated as the differential photo-current from the two quadrants of the detector and, thereby, low-frequency optical and detection noises can be substantially suppressed.

The way the error signal is produced in the studied method is a spatial analog to that of the PDH one. Although a quantitative investigation of the RAM effects due to beam steering and jitter is desired, we have shown that these effects can be monitored and corrections may be applied to the input beam for RAM suppression. Another way to reduce RAM due to beam steering and jitter is to utilize an SOM with higher degree of azimuthal symmetry in comparison with the type of SOM described in Fig. 2 where the SOM produces a phase modulation like a TEM$_{10}$ mode. For example, a higher-degree SOM may be designed to generate a phase modulation similar to a TEM$_{41}$ mode, a supposition of a TEM$_{10}$ mode and a TEM$_{01}$ one, such that the system is more robust against transverse beam jitter. In contrast, to suppress RAM is not a trivial work in the standard PDH scheme [18]; therefore, our proposed scheme sounds a promising alternative to the PDH one in applications where RAM noise suppression to lower levels may be demanded for portable compact laser stabilization systems [22, 23].

From the above discussions, it follows that our proposed scheme may allow one to implement laser stabilization with FP cavities as the PDH scheme does. Both the techniques of spatial modulation (our method) and temporal modulation (the PDH one) of optical phase front can share the same optical platform for laser stabilization, which will save much time and man power for cavity design. On the contrary the error signal is created in a different way with the tilt-locking method, which is modulation free [9], as follows. By slightly tilting or laterally displacing the input beam of a TEM$_{10}$ mode, a supposition of a TEM$_{10}$ mode and a TEM$_{01}$ one, such that the system is more robust against transverse beam jitter. In contrast, to suppress RAM is not a trivial work in the standard PDH scheme [18]; therefore, our proposed scheme sounds a promising alternative to the PDH one in applications where RAM noise suppression to lower levels may be demanded for portable compact laser stabilization systems [22, 23].

Accordingly, the error signal will become

$$E_t (\vec{r}, t) = \left[ \chi e^{i \theta} + i \phi (\vec{r}) \right] \sqrt{R_1} E_1 (\vec{r}, t)$$

$$\quad + \left[ \chi e^{i \theta} - i \phi (\vec{r}) \right] \sqrt{R_1} C_{st} u_{st} (\vec{r}, t) . (24)$$

provided that the $st$-order mode function $u_{st} (\vec{r}, t)$ is symmetric relative to the $y$-axis. On the other hand, if $u_{st} (\vec{r}, t)$ is anti-symmetric with respect to the $y$-axis, the error signal will then become

$$E_t (\vec{r}, t) = \sum_{n+m \geq 0} C_{nm} u_{nm} (\vec{r}, t) , (23)$$

wherein $u_{nm} (\vec{r}, t)$ is the $nm$-order Hermite-Gaussian mode function with $C_{nm}$ being the corresponding expansion coefficient. To consider the mode degeneracy effect, let suppose that one of the high-order transverse modes, say the $st$-order mode, is degenerate with the fundamental mode, then Eq. (4) needs to be rewritten as,

$$E_t (\vec{r}, t) = \left[ \chi e^{i \theta} + i \phi (\vec{r}) \right] \sqrt{R_1} E_1 (\vec{r}, t)$$

$$\quad + \left[ \chi e^{i \theta} - i \phi (\vec{r}) \right] \sqrt{R_1} C_{st} u_{st} (\vec{r}, t) . (24)$$

Accordingly, the error signal will become

$$\epsilon (\delta) = 4R_1 I_0 \varphi_0 \chi \sin \theta + 4R_1 I_{st} \varphi_0 \chi \cos \theta$$

$$= 4R_1 I_0 \varphi_0 \Im \left( 1 - \frac{T_1}{R_1} \frac{e^{i \delta} \sqrt{R}}{1 - e^{i \delta} \sqrt{R}} \right)$$

$$+ 4R_1 I_{st} \varphi_0 \Re \left( 1 - \frac{T_1}{R_1} \frac{e^{i \delta} \sqrt{R}}{1 - e^{i \delta} \sqrt{R}} \right) , (25)$$

in which

$$I_{st} = \int \int_{S_1} d \vec{r} \left[ E_1 (\vec{r}, t) C_{st}^* u_{st}^* (\vec{r}, t) - E_{st}^* (\vec{r}, t) C_{st} u_{st} (\vec{r}, t) \right] . (26)$$

Surely an SOM composed of a piece of machined glass will be of very low cost and free of SLM noises, e.g., the instability of the phase modulation function. Moreover, if the SOM is made with ultra-low expansion glass (ULE), its thermal sensitivity may be significantly suppressed for the sake of RAM reduction. It is reasonable to expect that ULE SOM’s could be powerful tools for construction of low-cost portable compact laser stabilization systems.

At this point, one must be reminded of that the SOM in principle may be either a commercially available SLM system that is based on a translucent or reflective liquid crystal micro-display, or simply a piece of transparent glass that is carefully manufactured following the requirement of Eq. (8). Surely an SOM composed of a piece of machined glass will be of very low cost and free of SLM noises, e.g., the instability...
\[ \epsilon(\delta) = 4R_1I_0 \, \varphi_0 \, \chi \sin \theta + 4R_1I'_{st} \left( \phi_0^2 + \chi^2 \right) \\
= 4R_1I_0 \, \varphi_0 \, \text{Im} \left( 1 - \frac{T_1}{R_1} \, e^{i\phi} \sqrt{R} \right) + 4R_1I'_{st} \left( \phi_0^2 + \chi^2 - 1 - \frac{T_1}{R_1} \, e^{i\phi} \sqrt{R} \right), \]
\tag{27}

in which

\[ I'_{st} = \int \int \bar{S} \, d\vec{r} \left[ E_1(\vec{r}, t) \, C_{m}^{*} \, u_{st}^*(\vec{r}, t) + E_1^*(\vec{r}, t) \, C_{m} \, u_{st}(\vec{r}, t) \right]. \]
\tag{28}

From Eqs. (25) and (27) it follows that both the correction terms to these two equations cause distortion (RAM) to the first term (the error signal) due to transverse mode degeneracy.

To suppress the error signal distortion due to the st-order mode degeneracy, one may design and fabricate an SOM so that the phase modulation function is approximately proportional to a Hermite-Gaussian function of the desired order, say a TEM$_{10}$ mode function $u_{10}(\vec{r}, t)$. Then the expansion coefficient $C_{10}$ will overwhelmingly dominate over all others in Eq. (23) and the specific expression for the first term in Eqs. (25) and (27) will be slightly changed though. As long as the value of the coefficient $C_{m}$ is small enough, i.e., $C_{m} \rightarrow 0$, then it follows that $I_{st}, I'_{st} \rightarrow 0$ from Eqs. (26) and (28) and hence the effect of the frequency degeneracy will be negligible for the st-order mode according to Eqs. (25), (26), (27). From the perspective of beam diffraction, the production of a TEM$_{10}$ transverse mode is another benefit from the suggested SOM design, i.e., a TEM$_{10}$ mode can co-propagate with the fundamental mode on the way to the detector for the production of the wanted error signal without being affected by diffraction, as evidenced by the investigation on the tilt-locking scheme [26].

5 Conclusion

To conclude, we have studied optical cavity locking for laser stabilization through spatial modulation of the phase front of an optical carrier. A theoretical description of the underlying principle has been developed showing that an error signal similar to that of the PDH scheme may be created for cavity locking with the studied method. RAM can be caused by experimental imperfections, especially the manufacture errors of the phase modulators, and we have discussed how RAM noises may be suppressed. In situations where cost and portability are a practical issue, the proposed cavity locking method may pave an alternative way to implement compact laser stabilization systems locked to FP cavities without use of expensive bulky devices including signal generators and EOM’s. It should deserve further investigations for a mature technology that is useful in space-based research missions such as gravitational-wave signals searching and inter-spacecraft laser ranging.

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Author contributions SF wrote the main manuscript text and did most of the calculations. SY did the simulations. PY and FZ did the double check on the calculations. YS prepared the figures. All authors reviewed the manuscript.

Declarations

Conflict of interest The authors declare no conflict of interest.

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