Lie bialgebra structures on the Schrödinger-Virasoro Lie algebra

Jianzhi Han†, Junbo Li†‡, Yucai Su†
†Department of Mathematics, University of Science and Technology of China, Hefei 230026, China
‡Department of Mathematics, Changshu Institute of Technology, Changshu 215500, China

E-mail: jzzhan@mail.ustc.edu.cn, sd_junbo@163.com, ycsu@ustc.edu.cn

Abstract. In this paper we investigate Lie bialgebra structures on the Schrödinger-Virasoro algebra \( L \). Surprisingly, we find out an interesting fact that not all Lie bialgebra structures on the Schrödinger-Virasoro algebra are triangular coboundary, which is different from the related known results of some Lie algebras related to the Virasoro algebra.

Key words: Lie bialgebras, Yang-Baxter equation, Schrödinger-Virasoro algebras.

Mathematics Subject Classification (2000): 17B05, 17B37, 17B62, 17B68.

§1. Introduction

To search for the solutions of the Yang-Baxter quantum equation, Drinfeld [1] introduced the notion of Lie bialgebras in 1983. Since then, many papers on Lie bialgebras appeared, e.g., [3, 11, 13, 16, 18, 19, 22, 23]. Witt type Lie bialgebras introduced in [19] were classified in [16], whose generalized cases were considered in [18, 22]. Lie bialgebra structures on generalized Virasoro-like and Block Lie algebras were investigated in [23, 11]. The Schrödinger-Virasoro Lie algebra [6] was introduced in the context of non-equilibrium statistical physics during the process of investigating the free Schrödinger equations. There are two sectors of this type Lie algebras, i.e., the original one and the twisted one, both of which are closely related to the Schrödinger algebra and the Virasoro algebra, which play important roles in many areas of mathematics and physics (e.g., statistical physics) and have been investigated in a series of papers [5, 6, 8–10, 12, 17, 20]. However, Lie bialgebra structures on the Schrödinger-Virasoro Lie algebra have not yet been considered. Drinfel’d [2] posed the problem whether or not there exists a general way to quantilize all Lie bialgebras. Although Etingof and Kazhdan [3] gave a positive answer to the question, they did not provide a uniform method to realize quantilizations of all Lie bialgebras. As a matter of fact, investigating Lie bialgebras and quantilizations is a complicated problem. In this paper we shall determine Lie bialgebra structures on the Schrödinger-Virasoro algebra \( L \). It is known that every Lie bialgebra structure on the Lie algebras considered in [11, 16, 18, 23] is triangle coboundary. Surprisingly, we find out an interesting fact that not all Lie bialgebra structures on the Schrödinger-Virasoro algebra are triangular coboundary.

The Schrödinger-Virasoro algebra \( L \) [6] is an infinite-dimensional Lie algebra over a field \( \mathbb{F} \) of characteristic 0 with basis \( \{ L_n, Y_p, M_n \mid n \in \mathbb{Z}, p \in \frac{1}{2} + \mathbb{Z} \} \) and the following non-vanishing
It has an infinite-dimensional ideal $S$ with basis $\{Y_{n+\frac{1}{2}}, M_n \mid n \in \mathbb{Z}\}$ and a Witt subalgebra (the centerless Virasoro algebra) $W$ with basis $\{L_n \mid n \in \mathbb{Z}\}$. And $\mathbb{F}M_0$ is the center of $L$.

Let us recall the definitions related to Lie bialgebras. Let $L$ be any vector space. Denote $\xi$ the cyclic map of $L \otimes L \otimes L$, namely, $\xi(x_1 \otimes x_2 \otimes x_3) = x_2 \otimes x_3 \otimes x_1$ for $x_1, x_2, x_3 \in L$, and $\tau$ the twist map of $L \otimes L$, i.e., $\tau(x \otimes y) = y \otimes x$ for $x, y \in L$. The definitions of a Lie algebra and Lie coalgebra can be reformulated as follows. A Lie algebra is a pair $(L, \delta)$ of a vector space $L$ and a bilinear map $\delta : L \otimes L \to L$ with the conditions:

$$\text{Ker}(1 - \tau) \subset \text{Ker}\delta, \quad \delta \cdot (1 \otimes \delta) \cdot (1 + \xi + \xi^2) = 0 : L \otimes L \otimes L \to L.$$  

Dually, a Lie coalgebra is a pair $(L, \Delta)$ of a vector space $L$ and a linear map $\Delta : L \to L \otimes L$ satisfying:

$$\text{Im}\Delta \subset \text{Im}(1 - \tau), \quad (1 + \xi + \xi^2) \cdot (1 \otimes \Delta) \cdot \Delta = 0 : L \otimes L \otimes L \to L.$$  

We shall use the symbol "·" to stand for the diagonal adjoint action:

$$x \cdot (\sum_i a_i \otimes b_i) = \sum_i ([x, a_i] \otimes b_i + a_i \otimes [x, b_i]).$$

A Lie bialgebra is a triple $(L, \delta, \Delta)$ such that $(L, \delta)$ is a Lie algebra, $(L, \Delta)$ is a Lie coalgebra, and the following compatible condition holds:

$$\Delta \delta(x \otimes y) = x \cdot \Delta y - y \cdot \Delta x, \quad \forall \ x, y \in L.$$  

(1.3)

Denote $\mathcal{U}$ the universal enveloping algebra of $L$, and $1$ the identity element of $\mathcal{U}$. For any $r = \sum_i a_i \otimes b_i \in L \otimes L$, define $c(r)$ to be elements of $\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U}$ by

$$c(r) = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}],$$

where $r^{12} = \sum_i a_i \otimes b_i \otimes 1$, $r^{13} = \sum_i a_i \otimes 1 \otimes b_i$, $r^{23} = \sum_i 1 \otimes a_i \otimes b_i$. Obviously

$$c(r) = \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j + \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j + \sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j].$$

**Definition 1.1** (1) A coboundary Lie bialgebra is a 4-tuple $(L, \delta, \Delta, r)$, where $(L, \delta, \Delta)$ is a Lie bialgebra and $r \in \text{Im}(1 - \tau) \subset L \otimes L$ such that $\Delta = \Delta_r$ is a coboundary of $r$, where $\Delta_r$ is defined by

$$\Delta_r(x) = x \cdot r \text{ for } x \in L.$$  

(1.4)
(2) A coboundary Lie bialgebra \((L, \delta, \Delta, r)\) is called triangular if it satisfies the following classical Yang-Baxter Equation (CYBE):
\[
c(r) = 0. \tag{1.5}
\]

(3) An element \(r \in \text{Im}(1-\tau) \subset L \otimes L\) is said to satisfy the modified Yang-Baxter equation (MYBE) if
\[
x \cdot c(r) = 0, \quad \forall \, x \in L. \tag{1.6}
\]

Denote \(V = L \otimes L\). Then \(L\) and \(V\) are both \(\frac{1}{2}\mathbb{Z}\)-graded. Denote (see \S3) \(\text{Der}(L, V)\) (resp. \(\text{Inn}(L, V)\)) the space of derivations (resp. inner derivations) from \(L\) to \(V\), and \(H^1(L, V)\) the first cohomology group of \(L\) with coefficients in \(V\). For any 6 elements \(\alpha, a^\dagger, \beta, \beta^\dagger, \gamma, \gamma^\dagger \in \mathbb{F}\), one can easily verify that the linear map \(D : L \to V\) defined below is a derivation:
\[
\begin{align*}
D(L_n) &= (n\alpha + \gamma) M_0 \otimes M_n + (n\alpha^\dagger + \gamma^\dagger) M_n \otimes M_0, \\
D(Y_{n-\frac{1}{2}}) &= \beta M_0 \otimes Y_{n-\frac{1}{2}} + \beta^\dagger Y_{n-\frac{1}{2}} \otimes M_0, \\
D(M_n) &= 2(\beta M_0 \otimes M_n + \beta^\dagger M_n \otimes M_0), \quad n \in \mathbb{Z}. \tag{1.7}
\end{align*}
\]

Denote \(D\) the 6-dimensional space spanned by the such elements \(D\). Let \(D_1\) be the subspace of \(D\) consisting of elements \(D\) such that \(D(L) \subseteq \text{Im}(1-\tau)\). Namely, \(D_1\) is the 3-dimensional subspace of \(D\) consisting of elements \(D\) with \(\alpha = -\alpha^\dagger, \beta = -\beta^\dagger, \gamma = -\gamma^\dagger\).

The main results of this paper can be formulated as follows.

**Theorem 1.2**

(i) \(\text{Der}(L, V) = \text{Inn}(L, V) \oplus D\) and \(H^1(L, V) = \text{Der}(L, V)/\text{Inn}(L, V) \cong D\).

(ii) Let \((L, [\cdot, \cdot], \Delta)\) be a Lie bialgebra such that \(\Delta\) has the decomposition \(\Delta_r + D\) with respect to \(\text{Der}(L, V) = \text{Inn}(L, V) \oplus D\), where \(r \in V \mod M_0 \otimes M_0\) and \(D \in D\). Then, \(r \in \text{Im}(1-\tau)\) and \(D \in D_1\). Furthermore, \((L, [\cdot, \cdot], D)\) is a Lie bialgebra provided \(D \in D_1\).

(iii) A Lie bialgebra \((L, [\cdot, \cdot], \Delta)\) is triangular coboundary if and only if \(\Delta\) is an inner derivation (thus \(\Delta = \Delta_r, \) where \(r \in \text{Im}(1-\tau)\) is some solution of CYBE).

Note that Theorem 1.2(ii) shows that there exist Lie bialgebras which are not triangular coboundary. Moreover, Theorem 1.2(iii) (resp. (ii)) gives a description of Lie bigalebra structures determined by \(\text{Inn}(L, V)\) (resp. \(D\)). But one cannot expect that \((L, [\cdot, \cdot], \Delta_r + D)\) would automatically become a Lie bialgebra even if both \((L, [\cdot, \cdot], \Delta_r)\) and \((L, [\cdot, \cdot], D)\) are Lie bialgebras, since the equation in (1.2) does not satisfies the linear relation for derivations. This is one of the reasons why it is difficult to classify all Lie bialgebra structures (in case when all Lie bialgebra structures are triangular coboundary, the classification of Lie bialgebra structures is equivalent to solving all solutions of CYBE (cf. Lemma 2.2), which is not done even for the Virasoro algebra \([16]\)).
§2. Some preliminary results

Throughout the paper, we denote by $\mathbb{Z}_+$ the set of all nonnegative integers and $\mathbb{Z}^*$ (resp. $\mathbb{F}^*$) the set of all nonzero elements of $\mathbb{Z}$ (resp. $\mathbb{F}$).

Lemma 2.1 Regard $L^\otimes n$ (the tensor product of $n$ copies of $L$) as an $L$-module under the adjoint diagonal action of $L$. Suppose $r \in L^\otimes n$ satisfying $x \cdot r = 0$, $\forall x \in L$. Then $r \in FM^3_0$.

Proof It can be proved directly by using the similar arguments as those presented in the proof of Lemma 2.2 of [23].

Lemma 2.2

(i) $r$ satisfies CYBE in (1.5) if and only if it satisfies MYBE in (1.6).

(ii) Let $L$ be a Lie algebra and $r \in \text{Im}(1 - \tau) \subset L \otimes L$, then

$$(1 + \xi + \xi^2) \cdot (1 \otimes \Delta_r) \cdot \Delta_r(x) = x \cdot c(r), \forall x \in L,$$

and the triple $(L, [\cdot, \cdot], \Delta_r)$ is a Lie bialgebra if and only if $r$ satisfies CYBE (1.5).

Proof (i) If $r$ satisfies MYBE, by Lemma 2.1, $c(r) \in FM^3_0$. As in [16], one has $c(r) = 0$. The reverse statement is obvious.

(ii) The result can be found in [1, 2, 16].

The following technical result gives some descriptions of solutions of CYBE.

Proposition 2.3 Let $r = \sum_{q \in \frac{1}{2} \mathbb{Z}} r_q \in \text{Im}(1 - \tau)$ be a nonzero solution of CYBE, and $p$ be the maximal index with $r_p \neq 0$. If $p \in \mathbb{Z}$, then $r_p \in \cup_{i=0}^5 V_i$, where $V_1, ..., V_5$ are subspaces spanned respectively by

$$
\begin{align*}
L_0 & \otimes L_p - L_p \otimes L_0, & M_0 & \otimes L_p - L_p \otimes M_0; \\
L_0 & \otimes L_p - L_p \otimes L_0, & L_0 & \otimes M_p - M_p \otimes L_0; \\
M_0 & \otimes L_p - L_p \otimes M_0, & M_0 & \otimes M_p - M_p \otimes M_0; \\
L_0 & \otimes M_p - M_p \otimes L_0, & M_0 & \otimes M_p - M_p \otimes M_0; \\
M_j & \otimes M_{p-j} - M_{p-j} \otimes M_j & \text{for all } j \in \mathbb{Z}.
\end{align*}
$$

If $p \in \frac{1}{2} + \mathbb{Z}$, then $r_p \in V_6 \cup V_7 \cup (\cup_{i \in \mathbb{Z}} V^{(i)}_8)$, where $V_6, V_7, V_8^{(i)}$ are subspaces spanned respectively by

$$
\begin{align*}
L_0 & \otimes Y_p - Y_p \otimes L_0, & M_0 & \otimes Y_p - Y_p \otimes M_0; \\
L_0 & \otimes Y_p - Y_p \otimes L_0, & M_0 & \otimes Y_p - Y_p \otimes M_0; \\
M_i & \otimes Y_{p-i} - Y_{p-i} \otimes M_i.
\end{align*}
$$

Here and below, we treat $L_a, Y_b$ as zero if $a \notin \mathbb{Z}$, $b \notin \frac{1}{2} + \mathbb{Z}$.

Proof First assume $p \neq 0$. We can suppose $p > 0$ otherwise the arguments are similar. Let $a_i \otimes b_{p-i} - b_{p-i} \otimes a_i$ be a term in $r_p$ with nonzero coefficient, where $a_i, b_i \in \{X_i, M_i, Y_i, -\frac{i}{2}\}$. Now we prove it case by case.
Case 1 \( a_i \otimes b_{p-i} = L_i \otimes L_{p-i} \) or \( Y_{i-\frac{1}{2}} \otimes Y_{p+\frac{1}{2}-i} \).

Changing the sign of the coefficient of \( a_i \otimes b_{p-i} - b_{p-i} \otimes a_i \) if necessary, we may assume that \( i > 0 \), since \( p > 0 \). Moreover, we can assume that \( i \) is maximal. Assume that \( a_i \otimes b_{p-i} = L_i \otimes L_{p-i} \). Then \( r_p \) cannot contain terms of the form \( d(X_{j} \otimes W_{p-j} - X_{p-j} \otimes W_{j}) \) with \( d \in \mathbb{F}^* \) and \( j \neq i \), where \( X, W \in \{L, M\} \). Suppose the contrary. We could take a nonzero term 

\[
d_0(X_{j_0} \otimes W_{p-j_0} - W_{p-j_0} \otimes X_{j_0})\]

of \( r_p \) with \( j_0 \neq i \) being maximal, and one could easily see that 

\[
[L_i, X_{j_0}] \otimes L_{p-i} \otimes W_{p-j_0} \]

would be a term in \( c(r)_{2p} \) with nonzero coefficient, contradicting \( c(r)_{2p} = 0 \). In particular, we have shown that the \( L_i \otimes L_{p-i} \) is the unique term of the form \( L_k \otimes L_{p-k} - L_{p-k} \otimes L_k \) in \( r_p \). Thus, \( 0 = c(L_i \otimes L_{p-i} - L_{p-i} \otimes L_i) \in \mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L} \), since the terms of the type \( L_{k_1} \otimes L_{k_2} \otimes L_{k_3} \) of \( c(r)_{2p} \) can only be obtained from \( L_i \otimes L_{p-i} \) in \( r_p \). It follows that \( i = p \), i.e., \( d(L_p \otimes L_0 - L_0 \otimes L_p) \) is the only term in \( r_p \) of the form \( L_k \otimes L_{p-k} - L_{p-k} \otimes L_k \) in \( r_p \). Applying the similar arguments as above to the case \( a_i \otimes b_{p-i} = Y_{i-\frac{1}{2}} \otimes Y_{p+\frac{1}{2}-i} \) one can see that 

\[
Y_{i-\frac{1}{2}} \otimes Y_{p+\frac{1}{2}-i} - Y_{p+\frac{1}{2}-i} \otimes Y_{i-\frac{1}{2}}
\]

is the unique term of the form \( Y_{k-\frac{1}{2}} \otimes Y_{p+\frac{1}{2}-k} - Y_{p+\frac{1}{2}-k} \otimes Y_{k-\frac{1}{2}} \) in \( r_p \). Clearly, one also should have \( c(Y_{i-\frac{1}{2}} \otimes Y_{p+\frac{1}{2}-i} - Y_{p+\frac{1}{2}-i} \otimes Y_{i-\frac{1}{2}}) = 0 \), which is impossible. Thus, the situation \( a_i \otimes b_{p-i} = Y_{i-\frac{1}{2}} \otimes Y_{p+\frac{1}{2}-i} \) cannot occur. Whence we conclude that \( r_p \) must lie in the subspace spanned by \( X_0 \otimes W_p - W_p \otimes X_0 \), where \( X, W \in \{L, M\} \). Furthermore, observe that the coefficient of \( M_0 \otimes M_p - M_p \otimes M_0 \) must be zero. Thus,

\[
r_p \in \text{Span}\{L_0 \otimes L_p - L_p \otimes L_0, L_0 \otimes M_p - M_p \otimes L_0, M_0 \otimes L_p - L_p \otimes M_0\}.
\]

Now one can check that either \( r_p \in V_1 \) or \( r_p \in V_2 \). Namely, \( r_p \in V_1 \cup V_2 \).

Case 2 \( a_i \otimes b_{p-i} = L_i \otimes M_{p-i} \) and Case 3 does not occur.

We claim that \( L_i \otimes M_{p-i} - M_{p-i} \otimes L_i \) is the only term in \( r_p \) of the form \( L_j \otimes M_{p-j} - M_{p-j} \otimes L_j \). Indeed, let \( i \) be maximal. Then it is not difficult to see that the result holds for \( i > 0 \). If \( i \leq 0 \), and suppose that there exists \( j_0 \neq i \) such that \( L_{j_0} \otimes M_{p-j_0} - M_{p-j_0} \otimes L_{j_0} \) is a term in \( r_p \) with nonzero coefficient. Take \( j_0 \) to be minimal. Then \( L_i \otimes [M_{p-i}, L_{j_0}] \otimes M_{p-j_0} \) is a term in \( c(r)_{2p} \) with nonzero coefficient, a contradiction. This proves the claim. Now the condition \( c(r)_{2p} = 0 \) yields that \( c(L_i \otimes M_{p-i} - M_{p-i} \otimes L_i) = 0 \). It follows that \( i = 0 \) or \( p - i = 0 \), i.e., \( L_i \otimes M_{p-i} - M_{p-i} \otimes L_i \) is equal to \( L_p \otimes M_0 - M_0 \otimes L_p \) or \( L_0 \otimes M_p - M_p \otimes L_0 \). Now by our assumption that Case 1 does not occur, we need only to consider terms of the form \( M_k \otimes M_{p-k} - M_{p-k} \otimes M_k \) for all \( k \in \mathbb{Z} \). It is not difficult to check that the only possible term of the form \( M_k \otimes M_{p-k} - M_{p-k} \otimes M_k \) is \( M_p \otimes M_0 - M_0 \otimes M_p \). Thus \( r_p \in V_3 \cup V_4 \).

Case 3 \( a_i \otimes b_{p-i} = L_i \otimes Y_{p-i} \) or \( M_i \otimes Y_{p-i} \).

In this case, \( p \in \frac{1}{2} + \mathbb{Z} \) and \( r_p \in \text{Span}\{L_j \otimes Y_{p-j} - Y_{p-j} \otimes L_j, M_j \otimes Y_{p-j} - Y_{p-j} \otimes M_j \mid j \in \mathbb{Z}\} \). Assume that \( a_i \otimes b_{p-i} = L_i \otimes Y_{p-i} \) and \( i \) is the maximal integer such that \( L_i \otimes Y_{p-i} - Y_{p-i} \otimes L_i \).
is a nonzero term in $r_p$. If $i > 0$, then we conclude that only $j \leq 0$ and $2(p-j) = i$ can $L_j \otimes Y_{p-j} - Y_{p-j} \otimes L_j$ be a nonzero term in $r_p$. Meanwhile, the condition $c(r)_{2p} = 0$ yields the coefficient of $L_i \otimes [Y_{p-i}, Y_{p-j}] \otimes L_j$ in $c(r)_{2p}$ to be zero and so is the coefficient of $L_j \otimes Y_{p-j} - Y_{p-j} \otimes L_j$. If $i \leq 0$, then it is easy to see that the coefficient of $L_j \otimes Y_{p-j} - Y_{p-j} \otimes L_j$ with $j \neq i$ must be zero. Thus, we conclude that $L_i \otimes Y_{p-i} - Y_{p-i} \otimes L_i$ is the only term of the form $L_k \otimes Y_{p-k} - Y_{p-k} \otimes L_k$ in $r_p$. While for the case $a_i \otimes b_{p-i} = M_i \otimes Y_{p-i}$, the same result can be obtained. Thus, $c(L_i \otimes Y_{p-i} - Y_{p-i} \otimes L_i) = 0$. It follows from that $i = 0$ or $2(p-i) = i$, i.e., $L_i \otimes Y_{p-i} - Y_{p-i} \otimes L_i$ is equal to $L_0 \otimes Y_p - Y_p \otimes L_0$ or $L_i \otimes Y_{2p} - Y_{2p} \otimes L_i$ with $i = \frac{2}{3}p \in \mathbb{Z}$. In the former case, $M_0 \otimes Y_p - Y_p \otimes M_0$ is the only possible nonzero term of the form $M_k \otimes Y_{p-k} - Y_{p-k} \otimes M_k (k \in \mathbb{Z})$ in $r_p$, while in the latter case, the only possibility is $M_i \otimes Y_{2p} - Y_{2p} \otimes M_i$. Thus we conclude that $r_p \in V_0 \cup V_1$ if $a_i \otimes b_{p-i} = L_i \otimes Y_{p-i}$, otherwise $r_p \in V_8^{(i)}$.

**Case 4** $a_i \otimes b_{p-i} = M_i \otimes M_{p-i}$ and Cases [1] and [2] do not occur.

Then one must have $r_p \in \sum_{j \in \mathbb{Z}} F(M_j \otimes M_{p-j} - M_{p-j} \otimes M_j)$, i.e., $r_p \in V_5$.

Now consider the case $p = 0$. By the similar argument as in Case [1] one knows that the terms of the form $Y_{i-\frac{1}{2}} \otimes Y_{i-\frac{1}{2}} - Y_{i-\frac{1}{2}} \otimes Y_{i-\frac{1}{2}} (i \in \mathbb{Z})$ cannot occur in $r_0$. So $r_0$ is in the subspace spanned by

$$\{L_i \otimes L_{-i} - L_{-i} \otimes L_i, L_i \otimes M_{-i} - M_{-i} \otimes L_i, M_i \otimes M_{-i} - M_{-i} \otimes M_i \mid i \in \mathbb{Z}\}.$$ 

Let $i$ be the maximal index such that $a_i \otimes b_{-i} - b_{-i} \otimes a_i$ is a term with nonzero coefficient. We may assume that $i > 0$, since the case $i = 0$ is trivial. If $L_j \otimes L_{-j} - L_{-j} \otimes L_j$ is a term of $r_p$ with nonzero coefficient and $j \in \mathbb{Z}_{>0}$, then by the similar argument as in Case 1 one has that $L_j \otimes L_{-j} - L_{-j} \otimes L_j$ is the unique term of the form $L_k \otimes L_{-k} - L_{-k} \otimes L_k$ with $k \in \mathbb{Z}_{>0}$ and $c(L_j \otimes L_{-j} - L_{-j} \otimes L_j) = 0$. But this implies $j = 0$, contradicting the choice of $j$. Similarly, for each $j \in \mathbb{Z}^*$ we deduce that $L_j \otimes M_{-j} - M_{-j} \otimes L_j$ cannot be a term of $r_0$. Thus we conclude that $r_0 \in \text{Span}\{L_0 \otimes M_0 - M_0 \otimes L_0, M_j \otimes M_{-j} - M_{-j} \otimes M_j \mid j \in \mathbb{Z}\}$. Now one can easily see that either $r_0 \in V_1$ or $r_0 \in V_5$. □

Although not all solutions to CYBE in $\mathcal{L}$ can be solved (even in the Witt algebra spanned by the set $\{L_i \mid i \in \mathbb{Z}\}$, cf. [16]), Proposition [2,3] nevertheless provides us some rule to decide when $r \in \text{Im}(1 - \tau)$ is not a solution to CYBE. Indeed, Proposition [2,3] classifies all possible highest components $r_p$ of $r$ for which $r \in \text{Im}(1 - \tau)$ and $c(r) = 0$. Similarly the form of the lowest components $r_q$ can also be determined.

### §3. Proof of Theorem [1.2]

Regard $\mathcal{V} = \mathcal{L} \otimes \mathcal{L}$ as a $\mathcal{L}$-module under the adjoint diagonal action. Denote by $\text{Der}(\mathcal{L}, \mathcal{V})$
the set of derivations $D : \mathcal{L} \to \mathcal{V}$, namely, $D$ is a linear map satisfying

$$D([x, y]) = x \cdot D(y) - y \cdot D(x),$$  

(3.1)

and $\text{Inn}(\mathcal{L}, \mathcal{V})$ the set consisting of the derivations $v_{\text{inn}}, v \in \mathcal{V}$, where $v_{\text{inn}}$ is the inner derivation defined by $v_{\text{inn}} : x \mapsto x \cdot v$. Then it is well known that $H^1(\mathcal{L}, \mathcal{V}) \cong \text{Der}(\mathcal{L}, \mathcal{V})/\text{Inn}(\mathcal{L}, \mathcal{V})$, where $H^1(\mathcal{L}, \mathcal{V})$ is the first cohomology group of the Lie algebra $\mathcal{L}$ with coefficients in the $\mathcal{L}$-module $\mathcal{V}$.

A derivation $D \in \text{Der}(\mathcal{L}, \mathcal{V})$ is homogeneous of degree $\alpha \in \frac{1}{2} \mathbb{Z}$ if $D(\mathcal{L}_p) \subseteq \mathcal{V}_{\alpha + p}$ for all $p \in \frac{1}{2} \mathbb{Z}$. Denote $\text{Der}(\mathcal{L}, \mathcal{V})_\alpha = \{ D \in \text{Der}(\mathcal{L}, \mathcal{V}) \mid \deg D = \alpha \}$ for $\alpha \in \frac{1}{2} \mathbb{Z}$. Let $D$ be an element of $\text{Der}(\mathcal{L}, \mathcal{V})$. For any $\alpha \in \frac{1}{2} \mathbb{Z}$, define the linear map $D_\alpha : \mathcal{L} \to \mathcal{V}$ as follows: For any $\mu \in \mathcal{L}_q$ with $q \in \frac{1}{2} \mathbb{Z}$, write $D(\mu) = \sum_{p \in \frac{1}{2} \mathbb{Z}} \mu_p$ with $\mu_p \in \mathcal{V}_p$, then we set $D_\alpha(\mu) = \mu_{q + \alpha}$. Obviously, $D_\alpha \in \text{Der}(\mathcal{L}, \mathcal{V})_\alpha$ and we have

$$D = \sum_{\alpha \in \frac{1}{2} \mathbb{Z}} D_\alpha,$$

(3.2)

which holds in the sense that for every $u \in \mathcal{L}$, only finitely many $D_\alpha(u) \neq 0$, and $D(u) = \sum_{\alpha \in \frac{1}{2} \mathbb{Z}} D_\alpha(u)$ (we call such a sum in (3.2) summable).

First we claim that if $\alpha \in \frac{1}{2} \mathbb{Z}$ then $D_\alpha \in \text{Inn}(\mathcal{L}, \mathcal{V})$. To see this, denote $\gamma = \alpha^{-1} D_\alpha(L_0) \in \mathcal{V}_\alpha$. Then for any $x_n \in \mathcal{L}_n$, applying $D_\alpha$ to $[L_0, x_n] = nx_n$ and using $D_\alpha(x_n) \in \mathcal{V}_{n + \alpha}$, we obtain $(\alpha + n)D_\alpha(x_n) - x_n \cdot D_\alpha(L_0) = L_0 \cdot D_\alpha(x_n) - x_n \cdot D_\alpha(L_0) = nD_\alpha(x_n)$, i.e., $D_\alpha(x_n) = \gamma_{\text{inn}}(x_n)$. Thus $D_\alpha = \gamma_{\text{inn}}$ is inner.

In the following we always use the symbol “$\equiv$” to denote modulo $\mathbb{F}(M_0 \otimes M_0)$. Then we can claim that $D_0(L_0) \equiv 0$. Indeed, for any $p \in \frac{1}{2} \mathbb{Z}$ and $x_p \in \mathcal{L}_p$, applying $D_0$ to $[L_0, x_p] = px_p$, one has $x_p \cdot D_0(L_0) = 0$. Thus by Lemma 2.1 $D_0(L_0) \equiv 0$.

Now we claim that for any $D \in \text{Der}(\mathcal{L}, \mathcal{V})$, (3.2) is a finite sum. To see this, one can suppose $D_n = (v_n)_{\text{inn}}$ for some $v_n \in \mathcal{V}_n$ and $n \in \frac{1}{2} \mathbb{Z}$. If $\mathbb{Z}' = \{ n \in \frac{1}{2} \mathbb{Z} \mid v_n \neq 0 \}$ is an infinite set, then $D(L_0) = D_0(L_0) + \sum_{n \in \mathbb{Z}'} L_0 \cdot v_n = D_0(L_0) + \sum_{n \in \mathbb{Z}'} n v_n$ is an infinite sum, which is not an element in $\mathcal{V}$, contradicting the fact that $D$ is a derivation from $\mathcal{L}$ to $\mathcal{V}$. This together with the proposition below proves Theorem 1.2(i). (To complete proof of Theorem 1.2(i), one still needs to show $\text{Inn}(\mathcal{L}, \mathcal{V}) \cap \mathcal{D} = \{ 0 \}$. For this, suppose $D = u_{\text{inn}} \in \mathcal{D}$, where $u$ is a linear combination of $a_i \otimes b_j$ for some $a_i, b_j \in \{ L_m, Y_{m - \frac{1}{2}}, M_m \mid m \in \mathbb{Z} \}$. Applying $D$ to generators of $\mathcal{L}$ and using (1.7), one immediately obtains $D = 0$.)

**Proposition 3.1** Replacing $D_0$ by $D_0 - u_{\text{inn}}$ for some $u \in \mathcal{V}_0$, one can suppose $D_0 \in \mathcal{D}$ (where $\mathcal{D}$ is defined in (1.7)).

**Proof** The proof seems to be technical. We shall prove that after a number of steps in each of which $D_0$ is replaced by $D_0 - u_{\text{inn}}$ for some $u \in \mathcal{V}_0$, we obtain $D_0 \in \mathcal{D}$. This will be done by some lengthy calculations.
For any \( n \in \mathbb{Z} \), one can write \( D_0(Y_{n-\frac{1}{2}}) \), \( D_0(L_n) \) and \( D_0(M_n) \) as follows

\[
D_0(Y_{n-\frac{1}{2}}) = \sum_{i \in \mathbb{Z}} \left( \alpha_{n,i} L_i \otimes Y_{n-\frac{1}{2}} - i + \alpha_{n,i}^\dagger Y_{i-\frac{1}{2}} \otimes L_{n-i} + \beta_{n,i} M_i \otimes Y_{n-\frac{1}{2}} - i + \beta_{n,i}^\dagger Y_{i-\frac{1}{2}} \otimes M_{n-i} \right),
\]

\[
D_0(L_n) = \sum_{i \in \mathbb{Z}} \left( a_{n,i} L_i \otimes L_{n-i} + b_{n,i} L_i \otimes M_{n-i} + b_{n,i}^\dagger M_i \otimes L_{n-i} + c_{n,i} M_i \otimes M_{n-i} + d_{n,i} Y_{i-\frac{1}{2}} \otimes Y_{n-i+\frac{1}{2}} \right),
\]

\[
D_0(M_n) = \sum_{i \in \mathbb{Z}} \left( e_{n,i} L_i \otimes M_{n-i} + f_{n,i} L_i \otimes M_{n-i} + f_{n,i}^\dagger M_i \otimes L_{n-i} + g_{n,i} M_i \otimes M_{n-i} + h_{n,i} Y_{i-\frac{1}{2}} \otimes Y_{n-i+\frac{1}{2}} \right),
\]

where all coefficients of the tensor products are in \( \mathbb{F} \), and the sums are all finite. For any \( n \in \mathbb{Z} \), the following identities hold,

\[
L_1 \cdot (M_n \otimes M_{-n}) = nM_{n+1} \otimes M_{-n} - nM_n \otimes M_{1-n}, \\
L_1 \cdot (L_n \otimes M_{-n}) = (n-1)L_{n+1} \otimes M_{-n} - nL_n \otimes M_{1-n}, \\
L_1 \cdot (M_n \otimes L_{-n}) = nM_{n+1} \otimes L_{-n} - (1+n)M_n \otimes L_{1-n}, \\
L_1 \cdot (L_n \otimes L_{-n}) = (n-1)L_{n+1} \otimes L_{-n} - (1+n)L_n \otimes L_{1-n}, \\
L_1 \cdot (Y_{n-\frac{1}{2}} \otimes Y_{-\frac{1}{2}}) = (n-1)Y_{n+\frac{1}{2}} \otimes Y_{-\frac{1}{2}} - nY_{n-\frac{1}{2}} \otimes Y_{3/2-n}.
\]

Let \( \triangle \) denote the set consisting of 5 symbols \( a, b, b^\dagger, c, d \). For each \( x \in \triangle \) we define \( M_x = \max \{ |p| \mid x_{1,p} \neq 0 \} \). For \( n = 1 \), using the induction on \( \sum_{x \in \triangle} M_x \) in the above identities, and replacing \( D_0 \) by \( D_0 - u_{1,1,n} \), where \( u \) is a proper linear combination of \( L_p \otimes L_{-p}, L_p \otimes M_{-p}, M_p \otimes L_{-p}, M_p \otimes M_{-p} \) and \( Y_{p-\frac{1}{2}} \otimes Y_{\frac{1}{2}+p} \) with \( p \in \mathbb{Z} \), one can safely suppose

\[
a_{1,i} = b_{1,j} = b_{1,k}^\dagger = c_{1,m} = 0 \quad \text{for} \quad i \neq -1, 2, j \neq 0, 2, k \neq \pm 1, m \neq 0, 1, n \neq 0, 2. \quad (3.3)
\]

Applying \( D_0 \) to \([-L_1, L_{-1}] = -2L_0 \) and using the fact that \( D_0 (L_0) = dM_0 \otimes M_0 \) for some \( d \in \mathbb{F} \), we obtain

\[
\sum_{p \in \mathbb{Z}} \left( (p-2)a_{-1,p-1} - (p+2)a_{-1,p} + (p-2)a_{1,p} - (p+2)a_{1,1+p} \right) L_p \otimes L_{-p} \\
+ \left( (p-2)b_{-1,p-1} - (1+p)b_{-1,p} + (p-1)b_{1,p} - (p+2)b_{1,1+p} \right) L_p \otimes M_{-p} \\
+ \left( (p-1)b_{-1,p-1}^\dagger - (p+2)b_{-1,p}^\dagger + (p-2)b_{1,p}^\dagger - (p+1)b_{1,1+p}^\dagger \right) M_p \otimes L_{-p} \\
+ \left( (p-1)c_{-1,p-1} - (p+1)c_{-1,p} + (p-1)c_{1,p} - (p+1)c_{1,1+p+1} + 2\delta_{0,p}d \right) M_p \otimes M_{-p} \\
+ \left( (p-2)d_{-1,p-1} - (1+p)d_{-1,p} + (p-2)d_{1,p} - (p+1)d_{1,1+p} \right) Y_{p-\frac{1}{2}} \otimes Y_{\frac{1}{2}-p} \right) = 0.
\]

In particular, for any \( p \in \mathbb{Z} \) one has

\[
(p-2)a_{-1,p-1} - (p+2)a_{-1,p} + (p-2)a_{1,p} - (p+2)a_{1,1+p} = 0,
\]

which together with the fact that \( \{ p \in \mathbb{Z} \mid a_{-1,p} \neq 0 \} \) is finite, forces

\[
a_{-1,p} = a_{-1,0} + 3a_{-1,1} + 3a_{1,2} = 3a_{-1,-2} + a_{-1,-1} + 3a_{1,-1} = a_{-1,-1} + a_{-1,0} = 0 \quad (3.4)
\]
for $p \in \mathbb{Z}\{-2, 0, \pm 1\}$. Similarly, comparing the coefficients of $L_p \otimes M_{-p}$, $M_p \otimes L_{-p}$, $M_p \otimes M_{-p}$ and $Y_{p - \frac{1}{2}} \otimes Y_{\frac{1}{2} - p}$, one has

\begin{align*}
 b_{1,q} = b_{1,p_1}^\dagger = b_{-1,p_2}^\dagger = c_{-1,p_3} = d_{-1,p_4} = 0, \\
 b_{-1,0} + 2b_{-1,-1} = b_{-1,-1} - b_{-1,-1} = b_{-1,1}^\dagger + 2b_{-1,-2} = c_{-1,-1} + c_{-1,0} + c_{1,1} + c_{1,0} - 2d \\
 = 2d_{-1,1} - d_{-1,0} + 2d_{1,0} = d_{-1,0} + 2d_{-1,1} + 2d_{1,2} = 0, 
\end{align*}

(3.5)

for any $q \in \mathbb{Z}$, $p_1 \in \mathbb{Z} \setminus \{0, \pm 1\}$, $p_2 \in \mathbb{Z} \setminus \{0, -1, -2\}$, $p_3 \in \mathbb{Z} \setminus \{-1, 0\}$ and $p_4 \in \mathbb{Z} \setminus \{0, \pm 1\}$.

Note that $L_1 \cdot (L_1 \otimes M_{-1} - L_0 \otimes M_0) = 0$. Replacing $D_0$ by $D_0 + b_{-1,1}(L_1 \otimes M_{-1} - L_0 \otimes M_0)$, one can assume $b_{-1,-1} = b_{-1,0} = b_{-1,1} = 0$ by (3.5). Similarly, replacing $D_0$ by $D_0 + b_{-1,-2}(M_{-1} \otimes L_1 - M_0 \otimes L_0)$, one can suppose $b_{-1,-2} = b_{1,-1} = b_{1,0} = 0$. Hence $b_{-1,p} = b_{1,p} = b_{1,p}^\dagger = b_{1,0} = 0$ for all $p \in \mathbb{Z}$. Applying $D_0$ to $[L_2, -L_1] = -3L_1$, one has

$$
\sum_{p \in \mathbb{Z}} (p - 4)a_{-1,p-2} - (3 + p)a_{-1,p} - (p + 2)a_{2,p+1} - (3 - p)a_{2,p} + 3a_{1,p})L_p \otimes L_{-1,p} \\
- ((p + 2)b_{2,p+1} + (2 - p)b_{2,p})L_p \otimes M_{-1,p} - ((p + 1)b_{2,p+1}^\dagger + (3 - p)b_{2,p}^\dagger)M_p \otimes L_{-1,p} \\
+ ((p - 2)c_{-1,p-2} - (1 + p)c_{-1,p} - (p + 1)c_{2,p+1} - (2 - p)c_{2,p} + 3c_{1,p})M_p \otimes M_{-1,p} \\
+ ((p - 2)d_{-1,p-2} - (3/2 + p)d_{-1,p} - (p + 1)d_{2,p+1} - (3 - p)d_{2,p} + 3d_{1,p})Y_{p - \frac{1}{2}} \otimes Y_{3/2 - p}) = 0.
$$

By computing the coefficient of $L_p \otimes L_{-1,p}$ and using (3.3), (3.4), and that $\{p \mid a_{2,p} \neq 0\}$ is finite, one has

\begin{align*}
 0 = a_{2,p} = a_{-1,-2} = a_{-1,1} & \text{ for } p \in \mathbb{Z} \setminus \{0, \pm 1, 2, 3\} \\
& = 4a_{-1,-1} - (3a_{-1,0} - a_{2,0}) = 2a_{2,1} + 3(a_{-1,0} + a_{2,0}) \\
& = a_{2,2} - (2a_{-1,0} + a_{2,0}) = 4a_{2,3} + (5a_{-1,0} + a_{2,0}). 
\end{align*}

(3.6)

Similarly, one can obtain that

\begin{align*}
 d_{-1,-1} = d_{-1,1} = 0, \\
 b_{2,p_1} = b_{2,p_2}^\dagger = c_{2,p_3} = d_{2,p_4} = 0, 
\end{align*}

(3.7)

\begin{align*}
 b_{2,0} + 3b_{2,-1} = b_{2,1} - 3b_{2,-1} = b_{2,2} + b_{2,-1} = b_{2,0}^\dagger + b_{2,3}^\dagger = b_{2,1}^\dagger - 3b_{2,3}^\dagger = b_{2,2}^\dagger + 3b_{2,3}^\dagger \\
= c_{2,1} - (3c_{1,0} - c_{-1,0} - 2c_{2,0}) \\
= 2c_{2,2} - (3c_{1,1} - c_{-1,-1} - 3c_{1,0} + c_{-1,0} + 2c_{2,0}) \\
= d_{2,1} - 3(2d_{1,0} - d_{2,0}) = d_{2,2} + 3(2d_{1,0} - d_{2,0}) = d_{2,3} - (4d_{1,0} - d_{2,0}) = 0, 
\end{align*}

(3.9)
for any $p_1 \in \mathbb{Z}\{\pm 1, 0, 2\}, p_2 \in \mathbb{Z}\{0, 1, 2, 3\}, p_3 \in \mathbb{Z}\{0, 1, 2\}$ and $p_4 \in \mathbb{Z}\{0, 1, 2, 3\}$. From the equation $[L_1, L_{-2}] = -3L_{-1}$, we obtain

$$
\sum_{p \in \mathbb{Z}} \left( (p - 2)a_{-2,p-1} - (3 + p)a_{-2,p} - (p + 4)a_{1,p+2} + (p - 3)a_{1,p} + 3a_{-1,p} \right) L_p \otimes L_{-1-p} \\
+ ((p - 2)b_{-2,p-1} - (2 + p)b_{-2,p}) L_p \otimes M_{-1-p} + ((p - 1)b_{1,-2,p-1} - (3 + p)b_{1,-2,p}) M_p \otimes L_{-1-p} \\
+ ((p - 1)c_{-2,p-1} - (2 + p)c_{-2,p} - (p + 2)c_{1,p+2} + (p - 1)c_{1,p} + 3c_{-1,p}) M_p \otimes M_{-1-p} \\
+ ((p - 2)d_{-2,p-1} - (2 + p)d_{-2,p} - (p + 5/2)d_{1,p+2} + (p - 5/2)d_{1,p} + 3d_{-1,p}) Y_{p-2} \otimes Y_{-p-2} = 0.
$$

It follows from the above formula and (3.3)–(3.7) that

$$
a_{\pm 1,p} = d_{\pm 1,p} = 0, \quad (3.10)
$$

$$
a_{-2,p_1} = b_{-2,p_2} = b_{-2,p_3} = c_{-2,p_4} = d_{-2,p_5} = 0, \quad (3.11)
$$

$$
a_{-2,-3} = a_{-2,1} = a_{-2,1} + 4a_{-2,1} = a_{-2,1} - 6a_{-2,1} = a_{-2,0} + 4a_{-2,1} = b_{-2,-1} + 3b_{-2,-2} = b_{-2,0} - 3b_{-2,-2} = b_{-2,0} + 3b_{-2,-3} = b_{-2,0} + b_{-2,3} = c_{-2,1} - (3c_{-1,1} - 2c_{-2,-2} - c_{1,1}) = 2c_{-2,0} - (3c_{-1,0} - c_{1,0} - 3c_{-1,1} + 2c_{-2,-2} + c_{1,1}) = d_{-2,-1} + 3d_{-2,-2} = d_{-2,0} - 3d_{-2,-2} = d_{-2,1} + d_{-2,2} = 0, \quad (3.12)
$$

for all $p \in \mathbb{Z}, p_1 \in \mathbb{Z}\{-3,-2,0,\pm 1\}, p_2 \in \mathbb{Z}\{-2,0,\pm 1\}, p_3 \in \mathbb{Z}\{-3,-2,-1,0\}, p_4 \in \mathbb{Z}\{-2,-1,0\}, p_5 \in \mathbb{Z}\{-2,0,\pm 1\}$. Applying $D_0$ to $[L_2, L_{-2}] = -4L_0$, one has

$$
\sum_{p \in \mathbb{Z}} \left( (p - 4)a_{-2,p-2} - (p + 4)a_{-2,p} - (p + 4)a_{2,p+2} + (p - 4)a_{2,p} \right) L_p \otimes L_{-p} \\
+ ((p - 4)b_{-2,p-2} - (p + 4)b_{-2,p} - (p + 4)b_{2,p+2} + (p - 4)b_{2,p}) L_p \otimes M_{-p} \\
+ ((p - 4)b_{1,-2,p-2} - (p + 4)b_{1,-2,p} - (p + 4)b_{1,-2,p+2} + (p - 4)b_{1,-2,p}) M_p \otimes L_{-p} \\
+ ((p - 2)c_{-2,p-2} - (p + 2)c_{-2,p} - (p + 2)c_{2,p+2} + (p - 2)c_{2,p} + 4d_{0,p}) M_p \otimes M_{-p} \\
+ ((p - 7/2)d_{-2,p-2} - (p + 5/2)d_{-2,p} - (p + 5/2)d_{2,p+2} + (p - 7/2)d_{2,p}) d_{p-2} \otimes Y_{2-p} = 0,
$$

which combined with (3.6) and (3.8)–(3.12) yields the follows:

$$
b_{\pm 2,p} = b_{\pm 2,p} = 0, \quad \forall \ p \in \mathbb{Z},
$$

$$
a_{-2,1} + a_{-2,-1} + c_{-2,1} + d_{-2,-2} + d_{2,0} = 0, \quad (3.13)
$$

$$
c_{2,0} + c_{2,2} + c_{-2,0} + c_{-2,-2} = 2d.
$$
Set \( u = L_{-1} \otimes L_1 - 2L_0 \otimes L_0 + L_1 \otimes L_{-1} \). Observe that \( L_{\pm 1} \cdot u = 0 \), so, by equations (3.6), (3.10), (3.12) and (3.13), one can assume
\[
a_{2,-1} = a_{2,0} = a_{2,1} = a_{2,2} = a_{2,3} = a_{-2,-3} = a_{-2,-2} = a_{-2,-1} = a_{-2,0} = a_{-2,1} = 0,
\]
when \( D_0 \) is replaced by \( D_0 + a_{2,-1}(L_{-1} \otimes L_1 - 2L_0 \otimes L_0 + L_1 \otimes L_{-1}) \). Similarly, set \( u = Y_{\frac{1}{2}} \otimes Y_{-\frac{1}{2}} - Y_{-\frac{1}{2}} \otimes Y_{\frac{1}{2}} \), one can assume \( d_{2,0} = d_{2,1} = d_{2,2} = d_{-2,-2} = d_{-2,-1} = d_{-2,0} = d_{-2,1} = 0 \).

Hence so far we have obtained that \( a_{\pm 1, p} = a_{\pm 2, p} = b_{\pm 1, p} = b_{\pm 2, p} = b_{\pm 1, p}^{\dagger} = b_{\pm 2, p}^{\dagger} = d_{\pm 1, p} = d_{\pm 2, p} = 0 \) for any \( p \in \mathbb{Z} \) and
\[
\begin{align*}
&c_{-1,-1} + c_{-1,0} + c_{1,1} + c_{1,0} - 2d = c_{1,p} = c_{-1,-p} = 0 \quad \forall \ p \in \mathbb{Z} \setminus \{0, 1\} \\
c_{2,p} = c_{-2,-p} = 0 \quad \forall \ p \in \mathbb{Z} \setminus \{0, 1, 2\} \\
&= c_{2,1} - (3c_{1,0} - c_{1,-1} - 2c_{2,0}) = 2c_{2,2} - (3c_{1,1} - c_{1,-1} - 3c_{1,0} + c_{1,0} + 2c_{2,0}) \\
&= c_{-2,-1} - (3c_{-1,-1} - 2c_{-2,-2} - c_{1,1}) = 2c_{-2,0} - (3c_{-1,0} - c_{-1,1} - 3c_{-2,-2} + c_{1,1}) \\
&= c_{-2,-1} + c_{2,1} = c_{2,0} + c_{2,2} + c_{-2,0} + c_{-2,-2} - 2d. \quad (3.14)
\end{align*}
\]

It follows from by repeatedly applying \( adL_1 \) to \( L_2 \) and using the fact that for all \( n \in \mathbb{Z} \), \( D_0(L_n) \) is a finite sum that \( c_{2,1} = 0 \). Hence by (3.14), \( c_{-2,-1} = 0 \). Now replacing \( c_{\pm 1,0}, c_{\pm 1,1}, c_{\pm 2,0} \) and \( c_{\pm 2,\pm 2} \) by \( \gamma \pm \alpha, \gamma^\dagger \pm \alpha^\dagger, \gamma \pm 2\alpha \) and \( \gamma^\dagger \pm 2\alpha^\dagger \) in (3.14) respectively, \( D_0(L_{\pm 1}) \) and \( D_0(L_{\pm 2}) \) have the following more concise expressions:
\[
\begin{align*}
D_0(L_{\pm}) &= (\gamma \pm \alpha)M_0 \otimes M_{\pm 1} + (\gamma^\dagger \pm \alpha^\dagger)M_{\pm 1} \otimes M_0, \quad (3.15) \\
D_0(L_{\pm 2}) &= (\gamma \pm 2\alpha)M_0 \otimes M_{\pm 2} + (\gamma^\dagger \pm 2\alpha^\dagger)M_{\pm 2} \otimes M_0. \quad (3.16)
\end{align*}
\]

Thus for any \( n \in \mathbb{Z} \) one can deduce \( D_0(L_n) = (n\alpha + \gamma)M_0 \otimes M_n + (n\alpha^\dagger + \gamma^\dagger)M_n \otimes M_0 \), since \( \mathcal{W} \) can be generated by \( L_{\pm 1} \) and \( L_{\pm 2} \).

To prove the proposition we still need to show \( D_0(Y_{\frac{1}{2}}) = \beta M_0 \otimes Y_{\frac{1}{2}} + \beta^\dagger Y_{-\frac{1}{2}} \otimes M_0 \). Applying \( D_0 \) to \( [L_m, Y_{n-m-\frac{1}{2}}] = (n - (m + 1)/2)Y_{n+m-m} - \frac{1}{2} \) and noticing that \( Y_{n-m-\frac{1}{2}} \cdot D_0L_m = 0 \), we obtain
\[
\begin{align*}
& (i - 2m)\alpha_{n,i,m} + (n - i - (m + 1)/2)\alpha_{n,i} - (n - (m + 1)/2)\alpha_{n+m,i} = 0, \\
& (i - (3m + 1)/2)\alpha^\dagger_{n,i,m} + (n - m - i)\alpha^\dagger_{n,i} - (n - (m + 1)/2)\alpha^\dagger_{n+m,i} = 0, \\
& (i - m)\beta_{n,i,m} + (n - i - (m + 1)/2)\beta_{n,i} - (n - (m + 1)/2)\beta_{n+m,i} = 0, \\
& (i - (3m + 1)/2)\beta^\dagger_{n,i,m} + (n - i)\beta^\dagger_{n,i} - (n - (m + 1)/2)\beta^\dagger_{n+m,i} = 0. \quad (3.17)
\end{align*}
\]

In the above equations, putting \( n = m = 1 \) and using the fact that the rank of \( \{ x_{1,p} \mid x = \alpha, \alpha^\dagger, \beta \text{ or } \beta^\dagger \} \) is finite, one has
\[
\begin{align*}
\alpha_{1,1} + \alpha_{1,0} &= \alpha^\dagger_{1,1} + \alpha^\dagger_{1,0} = 0, \\
\alpha_{1,p} &= \alpha^\dagger_{1,p} = \beta_{1,p} = \beta^\dagger_{1,p} = 0, \quad \forall \ p \in \mathbb{Z} \setminus \{0, 1\}, \quad p_2 \in \mathbb{Z} \setminus \{0\}, \quad p_3 \in \mathbb{Z} \setminus \{1\}. \quad (3.18)
\end{align*}
\]
Similarly, letting $m = -1$ and $n = 0$, then one has

$$\begin{align*}
\alpha_{0,-1} + \alpha_{0,0} &= \alpha_{0,1}^\dagger + \alpha_{0,0}^\dagger = 0, \\
\alpha_{0,p_1} &= \alpha_{0,p_2} = \beta_{0,p_3} = \beta_{0,0}^\dagger = 0, \quad \forall \ p_1 \in \mathbb{Z}\backslash\{0,-1\}, \ p_2 \in \mathbb{Z}\backslash\{0,1\}, \ p_3 \in \mathbb{Z}\backslash\{0\}.
\end{align*}$$

(3.19)

Taking $n = 1$ and $m = -1$, one has $\alpha_{1,0} = -\alpha_{0,0}, \ \alpha_{1,0}^\dagger = \alpha_{0,0}^\dagger, \ \beta_{1,0} = \beta_{0,0}, \ \beta_{1,1}^\dagger = \beta_{0,0}^\dagger$. Thus $D_0(Y_{\frac{1}{2}})$ and $D_0(Y_{-\frac{1}{2}})$ can be written as

$$\begin{align*}
D_0(Y_{\frac{1}{2}}) &= -\alpha_{0,0} L_0 \otimes Y_{\frac{1}{2}} + \alpha_{0,0} L_1 \otimes Y_{-\frac{1}{2}} + \alpha_{0,0}^\dagger Y_{-\frac{1}{2}} \otimes L_1 - \alpha_{0,0}^\dagger Y_{\frac{1}{2}} \otimes L_0 \\
&\quad + \beta_{0,0} M_0 \otimes Y_{\frac{1}{2}} + \beta_{0,0}^\dagger Y_{\frac{1}{2}} \otimes M_0, \\
D_0(Y_{-\frac{1}{2}}) &= -\alpha_{0,0} L_{-1} \otimes Y_{\frac{1}{2}} + \alpha_{0,0} L_0 \otimes Y_{-\frac{1}{2}} - \alpha_{0,0}^\dagger Y_{\frac{1}{2}} \otimes L_{-1} + \alpha_{0,0}^\dagger Y_{-\frac{1}{2}} \otimes L_0 \\
&\quad + \beta_{0,0} M_0 \otimes Y_{-\frac{1}{2}} + \beta_{0,0}^\dagger Y_{-\frac{1}{2}} \otimes M_0.
\end{align*}$$

While for $M_n$ we have

$$\begin{align*}
(i-2m)e_{n,i-m} + (n-m-i)e_{n,i} - ne_{n+m,i} &= (i-2m)f_{n,i-m} + (n-i)f_{n,i} - nf_{n+m,i} = 0, \\
(i-(3m+1)/2)h_{n,i-m} + (n-i-(m-1)/2)h_{n,i} - nh_{n+m,i} &= 0, \quad \forall \ n, \ i, \ m, \\
(i-m)f_{n,i-m}^\dagger + (n-m-i)f_{n,i}^\dagger - nf_{n+m,i}^\dagger &= (i-m)g_{n,i-m} + (n-i)g_{n,i} - ng_{n+m,i} = 0.
\end{align*}$$

(3.20)

For fixed $n = 0$, putting $m = 1$ and $m = -1$, respectively, one can deduce

$$\begin{align*}
e_{0,-1} - e_{0,1} &= e_{0,0} + 2e_{0,1} = \alpha_{0,1} + \alpha_{0,0} = 0, \\
e_{0,p_1} &= f_{0,p_2} = f_{0,p_2}^\dagger = g_{0,p_3} = h_{0,p_4} = 0, \quad \forall \ p_1 \in \mathbb{Z}\backslash\{0,\pm 1\}, \ p_2 \in \mathbb{Z}, \ p_3 \in \mathbb{Z}^*, \ p_4 \in \mathbb{Z}\backslash\{0,1\}.
\end{align*}$$

Thus $D_0(M_0)$ can be written as

$$\begin{align*}
D_0(M_0) &= e_{0,1} L_{-1} \otimes L_1 - 2e_{0,1} L_0 \otimes L_0 + e_{0,1} L_1 \otimes L_{-1} + g_{0,0} M_0 \otimes M_0 \\
&\quad - h_{0,1} Y_{-\frac{1}{2}} \otimes Y_{\frac{1}{2}} + h_{0,1} Y_{\frac{1}{2}} \otimes Y_{-\frac{1}{2}}.
\end{align*}$$

Applying $D_0$ to the equation $[Y_{-\frac{1}{2}}, Y_{\frac{1}{2}}] = M_0$, we get

$$e_{0,1} = 0, \ g_{0,0} = 2(\beta_{0,0} + \beta_{0,0}^\dagger), \ h_{0,1} = \frac{3}{2}(\alpha_{0,0} - \alpha_{0,0}^\dagger).$$

Thus, one can rewrite $D_0(M_0)$ as

$$D_0(M_0) = 2(\beta_{0,0} + \beta_{0,0}^\dagger) M_0 \otimes M_0 - h_{0,1} Y_{-\frac{1}{2}} \otimes Y_{\frac{1}{2}} + h_{0,1} Y_{\frac{1}{2}} \otimes Y_{-\frac{1}{2}}.$$
Notice that \([M_0, Y_{\frac{1}{2}}] = 0\), applying \(D_0\) to which one would have \(h_{0,1} = 0\), that is to say \(\alpha_{0,0} = \alpha_{0,0}^\dagger\). So we can further simplify \(D_0(Y_{\frac{1}{2}}), D_0(Y_{-\frac{1}{2}})\) and \(D_0(M_0)\) as follows:

\[
D_0(M_0) = 2(\beta_{0,0} + \beta_{0,0}^\dagger)M_0 \otimes M_0,
\]

\[
D_0(Y_{\frac{1}{2}}) = -\alpha_{0,0}L_0 \otimes Y_{\frac{1}{2}} + \alpha_{0,0}L_1 \otimes Y_{-\frac{1}{2}} + \alpha_{0,0}Y_{\frac{1}{2}} \otimes L_1
\]

\[-\alpha_{0,0}Y_{\frac{1}{2}} \otimes L_0 + \beta_{0,0}M_0 \otimes Y_{\frac{1}{2}} + \beta_{0,0}^\dagger Y_{\frac{1}{2}} \otimes M_0,
\]

\[
D_0(Y_{-\frac{1}{2}}) = -\alpha_{0,0}L_1 \otimes Y_{\frac{1}{2}} + \alpha_{0,0}L_0 \otimes Y_{\frac{1}{2}} - \alpha_{0,0}Y_{\frac{1}{2}} \otimes L_{-1}
\]

\[+ \alpha_{0,0}Y_{\frac{1}{2}} \otimes L_0 + \beta_{0,0}M_0 \otimes Y_{\frac{1}{2}} + \beta_{0,0}^\dagger Y_{\frac{1}{2}} \otimes M_0.
\]

Using the equation \([L_2, Y_{-\frac{1}{2}}] = -\frac{3}{2}Y_{3/2}\), we can deduce

\[
D_0(Y_{3/2}) = -2\alpha_{0,0}L_1 \otimes Y_{\frac{1}{2}} - 2\alpha_{0,0}Y_{\frac{1}{2}} \otimes L_1 - \frac{\alpha_{0,0}}{3}L_{-1} \otimes Y_{5/2} - \frac{\alpha_{0,0}}{3}Y_{5/2} \otimes L_{-1}
\]

\[+ \frac{4\alpha_{0,0}}{3}L_2 \otimes Y_{-\frac{1}{2}} + \frac{4\alpha_{0,0}}{3}Y_{-\frac{1}{2}} \otimes L_2 + \alpha_{0,0}L_0 \otimes Y_{3/2} + \alpha_{0,0}Y_{3/2} \otimes L_0
\]

\[+ \beta_{0,0}M_0 \otimes Y_{3/2} + \beta_{0,0}^\dagger Y_{3/2} \otimes M_0.
\]

Applying \(D_0\) to \([L_{-2}, Y_{3/2}] = \frac{5}{2}Y_{-\frac{1}{2}}\) and noticing \(Y_{3/2} \cdot D_0(L_{-2}) = 0\), one has \(\alpha_{0,0} = 0\), which yields

\[
D_0(Y_{\frac{1}{2}}) = \beta_{0,0}M_0 \otimes Y_{\frac{1}{2}} + \beta_{0,0}^\dagger Y_{\frac{1}{2}} \otimes M_0.
\]

Now the statement in Proposition 3.1 can be obtained immediately, since \(L_{\pm 1}, L_{\pm 2}\) and \(Y_{\frac{1}{2}}\) is a system of generators of \(L\). \(\square\)

To prove the second part of the main theorem, we need the following lemma.

**Lemma 3.2** Suppose \(v \in \mathcal{V}\) such that \(x \cdot v \in \text{Im}(1 - \tau)\) for all \(x \in \mathcal{L}\). Then \(v - d_0M_0 \otimes M_0 \in \text{Im}(1 - \tau)\) for some \(d_0 \in \mathbb{F}\).

**Proof** First note that \(\mathcal{L} \cdot \text{Im}(1 - \tau) \subseteq \text{Im}(1 - \tau)\). We prove that after several steps, by replacing \(v\) with \(v - u\) for some \(u \in \text{Im}(1 - \tau)\), we shall have \(v - d_0M_0 \otimes M_0 = 0\) for some \(d_0 \in \mathbb{F}\) and thus \(v - d_0M_0 \otimes M_0 \in \text{Im}(1 - \tau)\). Write \(v = \sum_{n \in \mathbb{Z}} v_n\). Obviously,

\[
v \in \text{Im}(1 - \tau) \iff v_n \in \text{Im}(1 - \tau), \quad \forall n \in \frac{1}{2}\mathbb{Z}.
\]  

(3.21)

Then \(\sum_{n \in \frac{1}{2}\mathbb{Z}} nv_n = L_{0} \cdot v \in \text{Im}(1 - \tau)\). By (3.21), \(nv_n \in \text{Im}(1 - \tau)\), in particular, \(v_n \in \text{Im}(1 - \tau)\) if \(n \neq 0\). Thus by replacing \(v\) by \(v - \sum_{n \in \frac{1}{2}\mathbb{Z}} v_n\), one can suppose \(v = v_0 \in \mathcal{V}_0\). Write

\[
v = \sum_{p \in \mathbb{Z}} (a_pL_p \otimes L_{-p} + b_pL_p \otimes M_{-p} + c_pM_p \otimes L_{-p} + d_pM_p \otimes M_{-p} + e_pY_{\frac{1}{2}} \otimes Y_{\frac{1}{2} - p}),
\]

13
where all the coefficients are in $\mathbb{F}$ and the sums are all finite. Since the elements of the form $u_{1,p} := L_p \otimes L_{-p} - L_{-p} \otimes L_p$, $u_{2,p} := L_p \otimes M_{-p} - M_{-p} \otimes L_p$, $u_{3,p} := M_p \otimes M_{-p} - M_{-p} \otimes M_p$ and $u_{4,p} := Y_{p-1} \otimes Y_{-p} - Y_{-p} \otimes Y_{p-1}$ are all in $\text{Im}(1 - \tau)$, replacing $v$ by $v - u$, where $u$ is a combination of some $u_{1,p}, u_{2,p}, u_{3,p}$ and $u_{4,p}$, one can suppose

$$c_p = 0, \forall \ p \in \mathbb{Z}; \quad a_p, \quad d_p \neq 0 \implies p > 0 \quad \text{or} \quad p = 0; \quad e_p \neq 0 \implies p > 0. \quad (3.22)$$

Then $v$ can be rewritten as

$$v = \sum_{p \in \mathbb{Z}^+} a_p L_p \otimes L_{-p} + \sum_{p \in \mathbb{Z}} b_p L_p \otimes M_{-p} + \sum_{p \in \mathbb{Z}^+} d_p M_p \otimes M_{-p} + \sum_{p \in \mathbb{Z}^+} e_p Y_{p-1} \otimes Y_{-p}. \quad (3.23)$$

Assume $a_p \neq 0$ for some $p > 0$. Choose $q > 0$ such that $q \neq p$. Then $L_{p+q} \otimes L_{-p}$ appears in $L_q \cdot v$, but (3.22) implies that the term $L_{-p} \otimes L_{p+q}$ does not appear in $L_q \cdot v$, a contradiction with the fact that $L_q \cdot v \in \text{Im}(1 - \tau)$. Then one can suppose $a_p = 0, \forall \ p \in \mathbb{Z}^*$. Similarly, one can also suppose $d_p = 0, \forall \ p \in \mathbb{Z}^*$ and $e_p = 0, \forall \ p \in \mathbb{Z}$. Then (3.23) becomes

$$v = \sum_{p \in \mathbb{Z}} b_p L_p \otimes M_{-p} + a_0 L_0 \otimes L_0 + d_0 M_0 \otimes M_0. \quad (3.24)$$

Recall the fact $\text{Im}(1 - \tau) \subset \text{Ker}(1 + \tau)$ and our hypothesis $\mathcal{L} \cdot v \subset \text{Im}(1 - \tau)$, one has

$$0 = (1 + \tau)L_1 \cdot v = -2a_0 (L_1 \otimes L_0 + L_0 \otimes L_1) + \sum_{p \in \mathbb{Z}} ((p - 2)b_{p-1} - pb_p) L_p \otimes M_{1-p} + \sum_{p \in \mathbb{Z}} ((p - 2)b_{p-1} - pb_p) M_{1-p} \otimes L_p.$$

Comparing the coefficients, and noting that the set $\{p \mid b_p \neq 0\}$ is finite, one gets

$$a_0 = b_0 + b_1 = b_p = 0, \quad \forall \ p \in \mathbb{Z} \setminus \{0, 1\}.$$

Then (3.24) can be rewritten as

$$v = b_1 (L_1 \otimes M_{-1} - L_0 \otimes M_0) + d_0 M_0 \otimes M_0. \quad (3.25)$$

Observing $(1 + \tau)L_2 \cdot v = 0$, one has $b_1 = 0$. Thus the lemma follows. □

Proof of Theorem 1.2 (ii) and (iii) Let $(\mathcal{L}, [\cdot, \cdot], \Delta)$ be a Lie bialgebra structure on $\mathcal{L}$. By (3.1), (1.3) and Theorem 1.2 (i), $\Delta = \Delta_r + D$, where $r \in \mathcal{V} (\text{mod} M_0 \otimes M_0)$ and $D \in \mathcal{D}$. By (1.2), $\text{Im} \Delta \subset \text{Im}(1 - \tau)$, so $\Delta_r (L_n) + D(L_n) \in \text{Im}(1 - \tau)$ for $n \in \mathbb{Z}$, which implies that $\alpha + \alpha^\dagger = \gamma + \gamma^\dagger = 0$. Similarly, $\beta + \beta^\dagger = 0$ by the fact that $\Delta_r (M_n) + D(M_n) \in \text{Im}(1 - \tau)$ for $n \in \mathbb{Z}$. Thus, $D(\mathcal{L}) \in \text{Im}(1 - \tau)$. So $\text{Im} \Delta_r \in \text{Im}(1 - \tau)$. It follows immediately from Lemma 3.2 that $r \in \text{Im}(1 - \tau) (\text{mod} M_0 \otimes M_0)$, proving the first statement of Theorem 1.2 (ii). If $D \in \mathcal{D}_1$, one can easily verify that $(1 + \xi + \xi^2) \cdot (1 \otimes D) \cdot D = 0$ by acting it on generators of $\mathcal{L}$, which shows $(\mathcal{L}, [\cdot, \cdot], D)$ is a Lie bialgebra, and the proof of Theorem 1.2 (ii) is completed. Theorem 1.2 (iii) follows immediately from (1.2), Definition 1.1 and Lemma 2.2. □
References

[1] V.G. Drinfeld, Constant quasiclassical solutions of the Yang-Baxter quantum equation, *Soviet Math. Dokl.* 28(3) (1983), 667–671.

[2] V.G. Drinfeld, Quantum groups, in: *Proceeding of the International Congress of Mathematicians*, Vol. 1, 2, Berkeley, Calif. 1986, Amer. Math. Soc., Providence, RI, 1987, pp. 798–820.

[3] P. Etingof, D. Kazhdan, Quantization of Lie bialgebras, I, *Selecta Math. (New Series)* 2 (1996) p. 1-41 MR 1403351 — Zbl 0863.17008

[4] C. Grunspan, Quantizations of the Witt algebra and of simple Lie algebras in characteristic $p$, *J. Alg.* 280 (2004), 145–161.

[5] S. Gao, C. Jiang, Y. Pei, Structure of the extended Schrödinger-Virasoro Lie algebra, *Alg. Colloq.* in press (2008).

[6] Henkel M. Schrödinger invariance and strongly anisotropic critical systems. *J. Stat. Phys.*, 75 (1994), 1023–1029.

[7] Henkel M. Phenomenology of local scale invariance: from conformal invariance to dynamical scaling. *Nucl. Phys. B*, 641 (2002), 405–410.

[8] Henkel M, Unterberger J. Schrödinger invariance and space-time symmetries. *Nucl. Phys. B*, 660 (2003), 407–412.

[9] J. Li, Y. Su, Representations of the Schrödinger-Virasoro algebras, *J. Math. Phys.*, 49 (2008), 053512.

[10] J. Li, Y. Su, The derivation algebra and automorphism group of the twisted Schrödinger-Virasoro algebra, arXiv:0801.2207v1, (2008).

[11] J. Li, Y. Su, B, Xin, Lie bialgebras of a family of Block type, *Chinese Annals of Math. (Series B)* 29 (2008), 487–500.

[12] J. Li, Y. Su, L. Zhu, 2-cocycles of original deformative Schrödinger-Virasoro algebras, *Science in China: Series A* 51 (2008), 1989–1999.

[13] W. Michaelis, A class of infinite-dimensional Lie bialgebras containing the Virasoro algebras, *Adv. Math.* 107 (1994), 365–392.

[14] W. Michaelis, Lie coalgebras, *Adv. Math.* 38 (1980), 1–54.

[15] W. Michaelis, The dual Poincare-Birkhoff-Witt theorem, *Adv.Math.* 57 (1985), 93–162.

[16] S.H. Ng, E.J. Taft, Classification of the Lie bialgebra structures on the Witt and Virasoro algebras, *J. Pure Appl. Alg.* 151 (2000), 67–88.

[17] C. Roger, J. Unterberger, The Schrödinger-Virasoro Lie group and algebra: representation theory and cohomological study, *Ann. Henri Poincaré*, 7 (2006), 1477–1529.

[18] G. Song, Y. Su, Lie bialgebras of generalized Witt type, *Science in China: Series A* 49 (2006), 533–544.
[19] E.J. Taft, Witt and Virasoro algebras as Lie bialgebras, *J. Pure Appl. Al.* 87 (1993), 301–312.

[20] S. Tan, X. Zhang, Automorphisms and Verma modules for Generalized Schrödinger-Virasoro algebras. arXiv:0804.1610v2.

[21] J. Unterberger, On vertex algebra representations of the Schrödinger-Virasoro algebra. arXiv:cond-mat/0703214v2, (2007).

[22] Y. Wu, G. Song, Y. Su, Lie bialgebras of generalized Witt type. II. *Comm. Algebra*, 35(6) (2007), 1992-2007.

[23] Y. Wu, G. Song, Y. Su, Lie bialgebras of generalized Virasoro-like type, *Acta Mathematica Sinica, English Series*, 22 (2006), 1915–1922.

[24] X. Zhang, S. Tan, Whittaker modules and a class of new modules similar as Whittaker modules for the Schrödinger-Virasoro algebra, arXiv:0812.3245v1.