Delta expansion at low temperatures

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In the low temperature phase of the square Ising model, we describe the inverse temperature \( \beta \) as the function of a squared mass \( M \) and study the critical behavior of \( \beta(M) \) via the large \( M \) expansion. Using the \( \delta \)-expansion by which the large mass expansion is transformed into a series exhibiting expected scaling behavior, we perform the estimation of the critical inverse temperature \( \beta_c \) with the help of linear differential equation to be satisfied by ansatz of \( \beta(M) \) near the critical point \( M = 0 \). To improve the estimation, the leading correction exponent \( \nu \) is independently estimated from \( \beta(3)/\beta(1) \) and is used in the estimation of \( \beta_c \), giving rise to remarkable accuracy improvement.

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I. INTRODUCTION

The \( \delta \) expansion introduced in ref. [1] has been applied to various models and developed to the level of being effective in the estimation of critical exponents in the cubic Ising model [2]. Due to the space-time discretization, the field theoretic and condensed matter models share a similar form of the action and allow similar calculation methods. The Boltzmann weight usually takes the form of \( \exp(-\beta H) \) where \( H \) means the energy function in Euclidean lattice and \( \beta \) stands for the inverse bare coupling for field theory models or inverse temperature in condensed matter models.

In models where there are separated phases, the play ground of the \( \delta \) expansion has limited so far to the region of small \( \beta < \beta_c \), where \( \beta_c \) denotes the critical inverse temperature (or critical inverse coupling). In the present paper, we consider for the first time the application of the \( \delta \) expansion method to the system in the low temperature phase. As an explicit example, we take up two dimensional square Ising model. The square Ising model has a long history due to Ising, Onsager, Yang and Lee, and many successors [3]. Making use of exact related results, we explore the \( \delta \)-expansion applied to a new arena of low temperature phase. In this article, we confine ourselves with the very basic points of \( \delta \)-expansion such as the estimation of \( \nu \) and \( \beta_c \), from the low temperature expansion. In particular, we place emphasis on the investigation how to use \( \nu \) to be independently estimated for the improvement of \( \beta_c \) estimate.

This article is organized as follows: In section II, we prepare the estimation task by presenting low temperature expansion and basic strategy of our approach. In section III, we turn to the explicit estimation of \( \beta_c \) and the exponent \( \nu \). We make an effort to improve the accuracy of \( \beta_c \) estimation. In section IV, we summarize this work.

II. LOW TEMPERATURE EXPANSION AND BASIC STRATEGY FOR THE ESTIMATION

The square Ising model is controlled by the Boltzman weight

\[
\exp(-\beta \sum_{<i,j>} s_i s_j),
\]

(2.1)

where the spin variable \( s_n \in \{+1,-1\} \) is on the site \( n \) connected to compose square lattice system. Here, \( <i,j> \) means that the site \( i \) and \( j \) are nearest neighbor pair and the summation should be taken over all nearest neighbor pairs.

As in the high temperature phase, the squared mass is extracted from the two-point function at large enough separation. At low temperature, there is non-vanishing magnetization per site \( M = \langle s_0 \rangle \) and the two point function fluctuates around \( M^2 \). At present, the large separation limit of the fluctuation is explicitly known for the cases where the two sites are on diagonal or parallel lines to axes \([3,4]\). Here, we employ the later case as was considered in ref. [5]. The correlation length \( \xi \) of the fluctuation is then known as \([3,4]\)

\[
\xi^{-1} = \log(\tanh \beta) + 2\beta.
\]

(2.2)

At low enough temperatures, \( \xi^{-1} = 2\beta - 2e^{-2\beta} - 2e^{-6\beta}/3 + \cdots \). Since for the application of \( \delta \)-expansion the corresponding mass squared \( M \) proves to be convenient than \( \xi \), we use \( M \) defined by \([3,4]\)

\[
M = 2(\cosh \xi^{-1} - 1).
\]

(2.3)

From \([2.2]\) and \([2.3]\), the squared mass \( M \) can be expanded as \( M = e^{\beta} - 4 + 3e^{-\beta} + 4e^{-6\beta} + 4e^{-10\beta} + \cdots \). Then, inversion gives

\[
\beta = \frac{1}{2} \log M + \frac{2}{M} - \frac{11}{2M^2} + \frac{68}{3M^3} - \frac{451}{4M^4} + \cdots.
\]

(2.4)

This is the large mass expansion at low temperature. A difference compared to the high temperature case: As the leading contribution, there exists a logarithmic term of the mass squared.
The behavior of $\beta(M)$ near the critical point $\beta_c = \log(1 + \sqrt{2})/2$, given from (2.2), by solving $\xi^{-1} = 0$, is easily derived from (2.2) and (2.3). The result of expansion provides

$$\beta = \beta_c + \frac{M^{1/2}}{4}(1 - \frac{M}{24} + \frac{3M^2}{640} - \cdots) + R, \quad (2.5)$$

where $R$ denotes the analytic background given by

$$R = \frac{M}{16\sqrt{2}}(1 - \frac{3M}{32} + \frac{19M^2}{1536} - \cdots). \quad (2.6)$$

From the definition of the critical exponent $\nu$, $\xi \sim (\beta - \beta_c)^\nu$ and $\beta - \beta_c \sim M^{1/\nu}$. Then, we find from (2.5) that $\nu = 1$.

Our motivation of recent series of works is to open a new way of quantitative computational method of critical behaviors from simply accessible series expansions, such as high and low temperature expansions. The $\delta$-expansion [1, 5] plays here a key role of handling the series and enables estimation of critical quantities. Suppose a given truncated expansion of $f(M)$ to the order $N$, $f(M) = \sum_{n=0}^{N} a_n (1/M)^n$. Minimal result of $\delta$-expansion needed in this study is that it induces transformation summarized as

$$D_N[M^{-\lambda}] = C_{N, \lambda} t^\lambda, \quad (2.7)$$

where

$$C_{N, \lambda} = \frac{\Gamma(N + 1)}{\Gamma(\lambda + 1) \Gamma(N - \lambda + 1)}. \quad (2.8)$$

Here the symbol $D_N$ means the transformation which is $N$ dependent. Note that $D_N[1] = 1$ and $D_N[M^\ell] = 0$ for positive integer $\ell$. Then, we have

$$D_N[f] =: \tilde{f}(t) = \sum_{n=0}^{N} C_{N, n} a_n t^n. \quad (2.9)$$

The coefficients are dependent on the truncation order $N$. For many examples investigated so far, it is numerically verified that $\tilde{f}(t)$ recovers its effective region the small $M$ behavior of $f(M)$ [7].

In the low temperature phase, $\beta(M)$ at large $M$ involves $\log M$ as shown in (2.4). The transformation rule of the logarithmic function can also be drawn from (2.7). For instance, putting $\lambda = \epsilon$, expanding in $\epsilon$, and the comparison of coefficients of $\epsilon$ in the result, we find

$$D_N[\log M] = -\log t - \sum_{n=1}^{N} \frac{1}{n} a_n. \quad (2.10)$$

In what follows we use notation $f_\epsilon$ for the series expansion of $f$ at large $M$. Expansion at small $M$ is denoted as $f_\epsilon$. Transformed series obeys the same notations. For instance, we obtain

$$D_N[\beta >] =: \tilde{\beta}_\epsilon(t) = \frac{1}{2}(\log t - \sum_{k=1}^{N} \frac{1}{k} a_n)$$

$$+ C_{N, 1} 2t - C_{N, 2} \frac{11}{2} t^2 + C_{N, 3} \frac{68}{3} t^3 - \cdots. \quad (2.11)$$

where the last term is $const \times t^N$. We now see the behaviors of $\tilde{\beta}_\epsilon$ and its derivatives with respect to $\log t$ in FIG. 1. From the plots, we find that $\tilde{\beta}_\epsilon$ gradually approaches $\beta_c$ but the speed of convergence is too slow. The reason is the existence of $M^{1/\nu}$ in the second place of the expansion (2.5). If one can effectively subtract the correction terms, then the accurate estimation of $\beta_c$ would become possible. This strategy is conveniently carried out by setting up the differential equation approximately satisfied by $\tilde{\beta}_\epsilon$ [2, 3].

Suppose we have no information on the exponents and let $\tilde{\beta}_\epsilon = \beta_c + \sum_{n=1}^{K} C_{N, n} a_n t^{-p_n}$. Then, the $\delta$-expansion to order $N$ is given by

$$\tilde{\beta}_\epsilon = \beta_c + \sum_{n=1}^{K} C_{N, n} a_n t^{-p_n}. \quad (2.12)$$

Here, the set of exponents $\{p_n\}$ is the subset of $\{\lambda_n\}$ obtained from removing the integer ones (Note that $D_N[M^0] = 0$, $(\ell = 1, 2, 3, \cdots)$. Analytic part $R$ thus becomes negligible.).

Truncating the summation at order $K$ in (2.12), resulting series $\tilde{\beta}_\epsilon = \beta_c + \sum_{n=1}^{K} C_{N, n} a_n t^{-p_n}$ is called ansatz to the $K$th order. It satisfies

$$\prod_{k=1}^{K} [1 + p_k^{-1} \frac{d}{d\log t}] \tilde{\beta}_\epsilon = \beta_c. \quad (2.13)$$

In this linear differential equation (LDE), the highest derivative order is $K$. If the transformed function $\tilde{\beta}_\epsilon^{(K)}$ shows the expected scaling, it is allowed to substitute $\tilde{\beta}_\epsilon^{(n)}$ ($n = 0, 1, 2, \cdots, K$) into $\tilde{\beta}_\epsilon^{(n)}$ included in the above LDE.

Now, in reality, LDE is only valid locally around a certain $t$ due to the truncation of the expansion. Then, at $t = \epsilon^\ell$ in the neighborhood of $t$, we have an expansion,

$$\prod_{k=1}^{K} [1 + p_k^{-1} \frac{d}{d\log t}] \tilde{\beta}(\epsilon^\ell) = \prod_{k=1}^{K} [1 + p_k^{-1} \frac{d}{d\log t} \tilde{\beta}(t)]$$

$$+ \sum_{k=1}^{K} [1 + p_k^{-1} \frac{d}{d\log t}] \tilde{\beta}^{(1)}(t) \epsilon + O(\epsilon^2). \quad (2.14)$$
Good adjustment of values of unknown exponents \( p_1, p_2, \cdots \) should make LDE be approximately valid in a wide region of a plateau and the plateau indicates \( \beta_c \). Thus, we employ an extended version of the principle of minimum sensitivity [8] (PMS) to fix exponents and \( t \) at which \( \beta_c \) is estimated. The extended PMS reads

\[
\prod_{k=1}^{K} [1 + p_k^{-1} \frac{d}{d \log t} \bar{\beta}_c^{(n)}] = 0, \quad (2.15)
\]

for some set of \( n \). In the above LDE, unknown variables are \( (t; p_1, p_2, \cdots, p_K) \). Hence, we need \( K + 1 \) equations to estimate all values. Then, the last LDE includes \( 2K + 1 \) th order derivative. At low orders, less number of derivatives exhibit scalings and at large orders derivatives to several orders are available. When the solution of the set \( (t^*; p_1^*, \cdots, p_K^*) \) are obtained, we estimate \( \beta_c \) by

\[
\prod_{k=1}^{K} [1 + p_k^{-1} \frac{d}{d \log t} \bar{\beta}_c^{(n)}]^{k-1} \beta_c = \beta_c. \quad (2.16)
\]

This is a basic strategy to be used in our approach. Similar approach will be taken for the estimation of \( v \) and the improved estimation of \( \beta_c \), which shall be explored in later.

### III. ESTIMATION

#### A. Naive estimation of \( \beta_c \)

In this subsection, we present estimation study in the manner detailed in the previous section. In the protocol, we introduced exponents as adjustable ones to satisfy the extended PMS condition, which was adopted in [9]. We here call this protocol as the naive protocol.

Number of exponents in ansatz (2.12) agrees with the order \( K \) of LDE (kLDE) to be imposed. At 1LDE, we use ansatz \( \beta_c = \beta_c + A_1 M^n \) and impose \( [1 + p_1^{-1}(d/d \log t)] \bar{\beta}_c = \beta_c \). Then, we first consider

\[
[1 + p_1^{-1}(d/d \log t)] \bar{\beta}_c^{(k)} = 0, \quad k = 1, 2. \quad (3.1)
\]

This set of LDEs includes derivatives to the 3rd order. By the substitution of \( \beta_c \) into \( \bar{\beta}_c \) (2.15), we obtain the solution of the set \( (t^*; p_1^*) \). For some details on the estimation process, see [5]. Then, by the substitution of the set \( (t^*; p_1^*) \) into the left-hand-side of \( [1 + p_1^{-1}(d/d \log t)] \bar{\beta}_c = \beta_c \), we obtain the estimation of \( \beta_c \) as suggested in (2.16). We also performed for 2- and 3-parameter ansatz the same estimation protocol to the 50th order. The result is depicted in Fig. 2 and summarized in Table I.

The behavior of the left-hand-side of LDE showing scaling sets in at some order depending the number of exponent parameters of ansatz (in this context, see [5, 9]). At \( K = 1 \), it is no sharp transition for odd orders. For even orders, rough scaling behavior begins to appear from 6th order. At \( K = 2 \), the onset orders are 11th for odd order and 14th for even order. At \( K = 3 \), the orders are 19th for odd and 22th for even orders.

As expected, increasing number of terms in the ansatz improves the accuracy. However, reliability of estimates for each \( K \) actually depends on the order \( N \) of large mass expansion, as suggested just before by the concerned function’s behaviors. Though the convergence to the exact value of \( \beta_c \) is strongly indicated at least with 2- and 3-parameters, we like to improve the accuracy. As in the high temperature case demonstrated in ref. [8], we turn to the estimation of \( \nu \) and then revisit estimation of \( \beta_c \) under the bias of the estimated \( \nu \).

#### B. Estimation of \( \nu \)

As long as the order \( N \) of large mass expansion is large enough, the additional estimation of \( \nu = (2p_1)^{-1} \) is effective in the naive estimation presented just before. However, accurate value is not obtained at moderate orders. We expect that if one is able to obtain more accurate value of \( p_1 \) in the independent way via an appropriate other function, it would help to estimate \( \beta_c \) since the leading correction \( t^{-p_1} \) is effectively subtracted in LDE. To resolve the problem, it proves convenient to consider the function \( \beta^{(2)} / \beta^{(1)} = f_\beta \) where \( \beta^{(k)} (M) = (d/d \log M)^k \beta (M) \). From (2.25), we find the critical behavior

\[
f_\beta (M) = \frac{1}{2v} + \frac{M^{1/2}}{4 \sqrt{2}} - \frac{3M}{16} + \cdots. \quad (3.2)
\]

The exponent \( p_1 \) appears as the leading term of constant and its estimation may become accurate. Writing ansatz as \( \beta_c (M) = p_1 + \sum_{k=1} B_k M^{\ell k} \), we obtain its \( \delta \)-expansion as

\[
f_\beta (t) = p_1 + \sum_{k=1} C_N q_k B_k t^{-q_k}. \quad (3.3)
\]

The truncated series at \( t^{-q_k} \) satisfies the following LDE,

\[
\prod_{k=1}^{K} [1 + q_k^{-1} \frac{d}{d \log t}] \bar{\beta}_c = p_1. \quad (3.4)
\]
As in the previous section, the above LDE should be valid at certain $r$ and the estimation of $p_1$ needs the optimal set $(t^*; q_1, \cdots, q_K)$. The set is determined by the extended PMS.

Now we attempt to estimate $p_1$ from the large mass expansion obtained from (2.4)

$$f_{\beta^+}(x) = \frac{4}{M} - \frac{28}{M^2} + \frac{208}{M^3} + \frac{1616}{M^4} + \cdots. \quad (3.5)$$

We substitute $f_{\beta^+}$ into $f_{\beta^+}$ in the derivatives of (4.4) and $K_{\ell=1} = 1 + q_k^{-1}(d/d \log t)|f_{\beta^+}^{(k)} = 0 \ (i = 1 \sim K + 1)$. The solution determines the optimal set $(t^*; q_1^*, \cdots, q_K^*)$ and the substitution into (3.4) provides $p_1$ estimate. The result is depicted in FIG. 3 and summarized in Table II.

The quality of the series of $f_{\beta^+}$ is lower than that of $\bar{\beta}$. In the 1-parameter ansatz, the orders at which the solution $(t^*; q_1)$ is found at the rough scaling region (we call such solution as proper) are 17th for odd order and 20th for even order. In the 2-parameter ansatz, the behavior characteristic to 2-parameter ansatz appears at 19th (for odd) and 20th (for even) orders and proper solution sets in from 27th (for odd) and 30th (for even) in the 3-parameter ansatz.

This quality difference affects the accuracy of estimate at low orders. For given same orders, the estimate from $f_{\beta^+}$ is slightly more accurate than the estimate from $\bar{\beta}$. As the order grows, the estimate from $f_{\beta^+}$ is more and more improved. As for the relation of accuracy with number of parameters, the positive correlation is clearly confirmed. In conclusion, this experiment proved that estimation of critical exponent $v$ is better in $f_{\beta^+}$.

C. Improved estimation of $\bar{\beta}$

In this subsection we like to investigate experimentally the best way to estimate $\bar{\beta}$. In addition to the naive way presented in the former, we here examine two ways: (i) One is to use $p_1$ estimated from $f_{\beta^+}$ and the rest exponents are adjusted under the extended PMS as usual. (ii) The other is to use exactly known values of exponents $p_i$ to all exponents included in the ansatz. For instance, at $K = 2$, we simply substitute $p_1 = 1/2$, $p_2 = 3/2$ and at $K = 3$, substitute in addition $p_3 = 5/2$. This prescription supplies us the standard reference of the estimation accuracy.

As shown in FIG. 4, in 1-parameter cases, both of the prescriptions (i) and (ii) provide better estimates compared to the naive one and (i) yields the best one among all three prescriptions. In 2 parameter cases, we first note that used $p_1$ value is the one from 2-parameter ansatz of $f_{\beta^+}$ (see FIG. 5). The result is that the whole trend is same as that of 1-parameter ansatz. Still we note that the prescription (i) becomes the best one from 22nd order. In 3-parameter cases, we have examined two versions of the prescription (i). One is to use $p_1$ from 2-parameter ansatz of $f_{\beta^+}$ and the other from 3-parameter ansatz of $f_{\beta^+}$. In the later case, stable estimation sets in from 30th order (see FIG. 6). In 2- and 3- parameter ansatz, the prescrip-
We have carried through a basic task of estimating critical quantities \( \nu \) and \( \beta_\nu \). First we have demonstrated that the large mass expansion with \( \delta \)-expansion works also at the low temperature phase. In addition, we have examined that the prescription (i), the substitution of the leading correction exponent \( p_1 = 1/(2\nu) \) estimated with \( f_\beta \) under extended PMS, provided better value of \( \beta_\nu \) than naive estimation of \( \beta_\nu \) with extended PMS. Moreover, (i)-prescription results in better estimate than the prescription where the exact exponents in the ansatz is used. This would shed light on an accurate estimation of \( \beta_\nu \) in the cubic model in the approach of ref. [2].

Encouraged with the results so far, it is of interest to attempt computation of the exponent of spontaneous magnetization \( \mathcal{M} \) in the present approach and additional estimation of other critical quantities from low temperature phase. By these further studies on the method of large mass expansion with \( \delta \)-expansion, its full aspects and power will be grasped. Such a thorough examination may stimulate the improvement of the method and help us to apply the method to more complicated and interesting physics models.

IV. SUMMARY

Fig. 6. Plots of \( \beta_\nu \) estimates. 3p (exact) means the result under the substitution \( p_1 = 1/2 \) (prescription (ii)). 3p (naive) means the result under the naive prescription. 3p (kp-imp, \( k = 1, 2 \)) means the result under the substitution \( p_1 = p'_1 \) where \( p'_1 \) is estimated in \( k \)-parameter ansatz of \( f_\beta \) (prescription (i)).

From Table III, we find in 2-parameter case that from moderate orders or as the order growing to higher, the best estimation is realized in the prescription (i) where \( p'_1 \) at \( K_\nu = 2 \) is substituted and \( p_2 \) is adjusted to satisfy extended PMS. Also in 3-parameter case, similar trend is observed, though 50th order is exceptional. Though rigid reasoning is not known to us, we can conclude that the best estimation does not come from the substitution of the exact exponents in the ansatz, and the use of the leading exponent value obtained from \( f_\beta \) with extended PMS to adjust the rest and higher order exponents provided best results.

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TABLE I. Naive $K$-parameter estimation results of $\beta_c = 0.4406867935 \cdots$ via $\hat{\beta}$ under the naive protocol.

|   | 20   | 25   | 30   | 35   | 40   | 45   | 50   |
|---|------|------|------|------|------|------|------|
| $K = 1$ | 0.44243483 | 0.44178017 | 0.44129454 | 0.44119466 | 0.44108312 | 0.44103340 |
| $K = 2$ | 0.44093637 | 0.44079480 | 0.44072337 | 0.44071252 | 0.44069980 |
| $K = 3$ | 0.44083163 | 0.44071513 | 0.44069812 | 0.44069176 | 0.44068834 | 0.44068783 |

TABLE II. $K_\nu$- and $K$-parameter estimation results of $p_1 = 0.5$ via $\hat{f}_\beta$ and $\hat{\beta}$, respectively. At $K_\nu = 1$, proper estimate appears from 17th for odd orders and 20th for even orders. At $K_\nu = 2$, proper estimate appears from 27th for odd orders and 30th for even orders. At $K_\nu = 3$, proper estimate appears from 37th for odd orders and 42th for odd orders.

|   | 20   | 25   | 30   | 35   | 40   | 45   | 50   |
|---|------|------|------|------|------|------|------|
| $K_\nu = 1$ | 0.50208109 | 0.50129729 | 0.50098433 | 0.50071069 | 0.50058813 | 0.50045795 | 0.50039751 |
| $K = 1$ | 0.51821317 | 0.51317267 | 0.51100350 | 0.50882559 | 0.50781555 | 0.50660710 | 0.50603627 |
| $K_\nu = 2$ | 0.49843886 | 0.5008775 | 0.50006051 | 0.50002454 | 0.50001244 |
| $K = 2$ | 0.50380104 | 0.50199096 | 0.50132581 | 0.50085852 | 0.50065241 | 0.50046897 | 0.50038216 |
| $K_\nu = 3$ | 0.50058851 | 0.5000213 | 0.5000098 | 0.5000066 |
| $K = 3$ | 0.50243597 | 0.50064853 | 0.50030467 | 0.50015361 | 0.50005811 | 0.50004131 |

TABLE III. Relative error ($\beta_{estimated}/\beta_c - 1$) in 2-parameter estimation results of $\beta_c = 0.4406867935 \cdots$ via three prescriptions.

|   | 20   | 25   | 30   | 35   | 40   | 45   | 50   |
|---|------|------|------|------|------|------|------|
| naive | 0.00056634 | 0.00024509 | 0.00014498 | 0.00008301 | 0.00005839 | 0.00003832 | 0.00002952 |
| exact | 0.00013657 | 0.00006330 | 0.00003965 | 0.00002339 | 0.00001698 | 0.00001135 | 0.00000893 |
| $p_1(K_\nu = 2)$ | -0.00035963 | -0.00004278 | -0.00002146 | -0.00001539 | -0.00000959 | -0.00000751 | -0.00000517 |

TABLE IV. Relative error ($\beta_{estimated}/\beta_c - 1$) in 3-parameter estimation results of $\beta_c = 0.4406867935 \cdots$ via three prescriptions.

|   | 20   | 25   | 30   | 35   | 40   | 45   | 50   |
|---|------|------|------|------|------|------|------|
| naive | 0.000328649 | 0.000064305 | 0.000025697 | 0.000011261 | 0.000006458 | 0.000003519 | 0.000002342 |
| exact | -0.000025312 | -0.000009856 | -0.000003990 | -0.000002217 | -0.000001156 | -0.000000759 | -0.000000455 |
| $p_1(K_\nu = 2)$ | -0.000216960 | 0.00000710 | 0.000002244 | 0.000001161 | 0.000000944 | 0.000000567 | 0.000000464 |
| $p_1(K_\nu = 3)$ | -0.000014723 | -0.000001518 | -0.000000711 | -0.000000483 | -0.000000282 |