Learning with Changing Features

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Abstract

In this paper we study the setting where features are added or change interpretation over time, which has applications in multiple domains such as retail, manufacturing, finance. In particular, we propose an approach to provably determine the time instant from which the new/changed features start becoming relevant with respect to an output variable in an agnostic (supervised) learning setting. We also suggest an efficient version of our approach which has the same asymptotic performance. Moreover, our theory also applies when we have more than one such change point. Independent post analysis of a change point identified by our method for a large retailer revealed that it corresponded in time with certain unflattering news stories about a brand that resulted in the change in customer behavior. We also applied our method to data from an advanced manufacturing plant identifying the time instant from which downstream features became relevant.

To the best of our knowledge this is the first work that formally studies change point detection in a distribution independent agnostic setting, where the change point is based on the changing relationship between input and output.

1 Introduction

In domains such as advanced manufacturing which involve thousands to tens of thousands of processing stages spread over months, measurements that are taken to monitor the quality of the products are usually very expensive. This is because not only do the measurement tools, which have to be extremely high precision, cost millions of dollars; but each measurement slows down the line resulting in significant loss of throughput/productivity. Hence, unless these measurements provide an accurate indication of product health they are simply an incurred cost. Thus as new tools or old tools after maintenance are added to the production line, which results in additional/altered measurements, it is critically important to know the time instant from when these tools significantly

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impact quality. Usually manufacturers/engineers will randomly infrequently measure the products, but a more intelligent strategy can potentially save them billions of dollars as premeptive/corrective actions can be taken on likely to be faulty/subpar products, which can be predicted with much improved accuracy. Moreover, it is quite possible that there was a process change at the detected time instant and hence many of the recent products followed a different route (viz. new tools). Based on this, the intelligent strategy could be used to recommend that for products that follow the old route we do not need the measurements corresponding to the new tools, while those in the new route we should heavily sample these measurements. Such a dynamic policy, where we decide how to distribute measurements for better prediction of product quality, can significantly improve overall profitability. For instance in chip manufacturing, typically 1% increase in yield – i.e. percentage of within spec chips – is worth over a billion dollars in revenue. Similar gains can be seen in other types of manufacturing such as pharmaceutical or processed food industries.

Our solution is also applicable to other domains such as retail, finance, document classification, sensor networks, where features are added or change interpretation over time. For instance in finance, different financial indicators may be added over time to better determine the health of a deal. Moreover, change in the competitive landscape can lead to different outcomes even with the same values for these indicators after a certain point in time. Identifying this time instant can be extremely important in adapting to the changing environment. A similar issue can be witnessed in document classification, where for instance recent documents with the word ”tweet” may be incorrectly classified as those about birds rather than technology if the change point was not promptly identified.

In particular, we make the following contributions: In Section 3, we provide distribution independent excess risk guarantees for the problem of statistical regression with no assumptions on the nature of the change. Furthermore, our guarantees are not significantly worse than settings with no feature changes making our approach effective in adapting to the change. Our results in the main article are for spaces of finite pseudo-dimension, so that our analysis is applicable to commonly used empirical risk minimization methods (viz. generalized least squares, logistic regression, etc.). The supplemental material (section 7) extends the theory to cover regularized learning rules (viz. lasso, ridge, regularized logistic regression, etc.) and function classes of unbounded pseudo-dimension (viz. kernel regression). This extension, which is conceptually based on similar ideas that we have in the main article, nicely generalizes our theory with the added complexity of reasoning about tail conditions on the noise. In Section 3, we also show how our analysis can be applied to provably identify not just a single but multiple change points. In Section 4 we provide a more efficient version of our algorithm based on our analysis in section 3 and related work, with essentially the same asymptotic performance as our original algorithm. In Section 5 we perform experiments on synthetic data as well as two real industrial datasets. The first real dataset is from a large retailer. Post analysis by domain experts of a change point identified by our method for a certain brand of interest
for this retailer revealed that certain unflattering news stories around that time tarnished the brand image resulting in this change. This was an independent justification for our identified change point and a potential confirmation that it was possibly not just noise. The second real dataset is from an advanced manufacturing plant. In this case we effectively identified the time instant from which measurements from a downstream process started becoming relevant. We report results with three base learning methods namely; SVM with RBF kernel, Logistic regression and L1-regularized logistic regression to showcase the fact that the performance of our detection methods is not limited to any specific learning technique.

2 Related Work

There has been recent work [5] studying a dynamically changing environment in which multiple new features are added at each stage of a multistage process. However, the goal there was to suggest an efficient and accurate meta-algorithm to update an already existing regression model with the added features. Their strategy is one component in making our algorithm in section 4 more efficient when learning and detecting the change. However, the aim of the present work overall is different in that we focus on adapting to the time index when the new/changed features actually become relevant to the learning problem.

Another piece of closely related work is that of change point detection which has been heavily studied in statistics [2]. Typically, the goal is to find an instant in a time series from where the values/distribution are significantly different than the prior period. There are many statistics (viz. CUSUM based, stability based) that have been developed to detect this change. Some of these ideas have also been extended to the multivariate setting [10]. All of this work however is different from ours, as it is primarily unsupervised.

This topic has intimate connections to the general subject of concept drift in statistical learning [4,7], where the function to be learned varies over time. It is also related to the topic of distribution drift [13], in which the marginal distribution over \( x \) changes over time. This captures phenomena such as the fact that certain terminology has a life cycle, so that terms that were previously common (such as “milliner”) are later much less common, and vice versa (such as “click”). However, the present work differs significantly from the prior literature on both concept drift and distribution drift. Indeed, our setting itself differs, in that these prior works focus on prediction problems, where the objective is to generalize to new unseen test points; in contrast, we are interested in a data set where features may be added/change over time, and the task is to estimate the regression function at these given points. Thus, we have access to all of the response variables when estimating the time index at which the change occurs. Furthermore, we are mainly interested in more of a sudden change described by varying of the features to the estimated function, rather than some notion of gradual drift of a target concept, or general changes to a data distribution.

Our setting is also different from the problem of domain adaptation [3].
main distinction between our task and traditional domain adaptation is that in
our case the learner isn’t aware of the time instant at which the new/changed
features gain significance; indeed, this is our main challenge.

## 3 Framework and Theoretical Analysis

Suppose that \(x_1, \ldots, x_m\) are data points in a space \(X\), and \(Y_1, \ldots, Y_m\) are independent \(\mathbb{R}\)-valued random variables, with range contained in \([-B, B]\) for some \(B \in [1, \infty)\) denote \(\eta_i = E[Y_i]\). The restriction \(B \geq 1\) is merely for convenience.

The results clearly also have implications for values bounded in a range \([-b, b]\) with \(b \in (0, 1)\), obtained simply by multiplying every function \(h \in \mathcal{H}_1 \cup \mathcal{H}_2\) and response \(Y_i\) by \(1/b\), calculating the bound below for the case \(B = 1\), and multiplying the resulting bound by \(b^2\). First, in this abstract version of our setting, we consider general function classes \(\mathcal{H}_i\) for each \(i \in \{1, 2\}\): that is, \(p_i\) is the largest integer \(p \in \mathbb{N}\) such that \(\exists (z_1, y_1), \ldots, (z_p, y_p) \in X \times \mathbb{R}\) for which the collection of \(p\)-dimensional binary vectors \(\{(\mathbb{I}[h(z_1) \leq y_1], \ldots, \mathbb{I}[h(z_p) \leq y_p]) : h \in \mathcal{H}_i\}\) has cardinality \(2^p\). For instance, the set of linear functions mapping an \(r\)-dimensional representation of points \(x \in X\) to \(\mathbb{R}\) has pseudo-dimension \(r + 1\) (see e.g., \([1]\)), and this remains an upper bound on the pseudo-dimension for any fixed monotone transformation of such linear functions (such as is used in logistic regression) \([1]\). Throughout this section, we suppose \(m \geq \max\{p_1, p_2\}\).

For any functions \(h_1 \in \mathcal{H}_1\) and \(h_2 \in \mathcal{H}_2\), and any \(t_0 \in \{1, \ldots, m+1\}\), denote

\[
R^*(h_1, h_2, t_0) = \frac{1}{m} \left( \sum_{t=1}^{t_0-1} (h_1(x_t) - \eta_t)^2 + \sum_{t=t_0}^{m} (h_2(x_t) - \eta_t)^2 \right)
\]

and

\[
\hat{R}(h_1, h_2, t_0) = \frac{1}{m} \left( \sum_{t=1}^{t_0-1} (h_1(x_t) - Y_t)^2 + \sum_{t=t_0}^{m} (h_2(x_t) - Y_t)^2 \right).
\]

To be clear, in the cases of \(t_0 \in \{1, m+1\}\), we are defining \(\sum_{t=1}^{0} \cdot = \sum_{t=m+1}^{m} \cdot = 0\) in these definitions (and below). We then denote by

\[
(h_1^*, h_2^*, t^*) = \arg\min_{(h_1, h_2, t_0) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \{1, \ldots, m+1\}} R^*(h_1, h_2, t_0).
\]

We are interested in obtaining \(\hat{h}_1 \in \mathcal{H}_1, \hat{h}_2 \in \mathcal{H}_2\), and \(\hat{t} \in \{1, \ldots, m+1\}\) based only on \(Y_1, \ldots, Y_m\), such that \(R^*(\hat{h}_1, \hat{h}_2, \hat{t})\) is not too much larger than \(R^*(h_1^*, h_2^*, t^*)\).
In particular, let us choose

\((\hat{h}_1, \hat{h}_2, \hat{t}) = \arg\min_{(h_1, h_2, t_0) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \{1, \ldots, m+1\}} \hat{R}(h_1, h_2, t_0)\).

We refer to this general strategy as the **Search-and-Split** algorithm (abbreviated as **SaS** below).

This method is essentially a variant of *empirical risk minimization* for this setting in which there is an unknown change time. Specifically, we have the following theorem for the above Search-and-Split method.

**Theorem 1.** With probability at least \(1 - \delta\),

\[
R^* (\hat{h}_1, \hat{h}_2, \hat{t}) \leq R^* (h_1^*, h_2^*, t^*) + 22B \sqrt{\frac{2 \ln(2(m + 1)/\delta)}{m} + \frac{\sum_{j=1}^{2} 3p_j \ln(eb/p_j)}{m}}.
\]

**Proof.** Let \(\tilde{Y}_1, \ldots, \tilde{Y}_m\) be equal in distribution to \(Y_1, \ldots, Y_m\) but independent of \(Y_1, \ldots, Y_m\), and define \(\hat{R}(h_1, h_2, t_0) = \frac{1}{m} \left( \sum_{i=1}^{m-1} (h_1(x_i) - \tilde{Y}_i)^2 + \sum_{i=t_0}^{m} (h_2(x_i) - \tilde{Y}_i)^2 \right). \)

For any fixed choices of \(h_1, h_2, t_0 \in \{1, \ldots, m + 1\}\), Hoeffding’s inequality implies that with probability at least \(1 - \delta'\),

\[
\left| \hat{R}(h_1, h_2, t) - \mathbb{E}[\hat{R}(h_1, h_2, t)] \right| \leq \sqrt{\frac{2B^2 \ln(2/\delta')}{m}}.
\]

Fix \(\epsilon = \sqrt{\max\{p_1, p_2\}/m}\). Now, for each \(j \in \{1, 2\}\), let \(\mathcal{H}_{j, \epsilon}\) denote a minimal \(\epsilon\)-cover of \(\mathcal{H}_j\) with respect to the pseudo-metric \((h, g) \mapsto \rho_m(h, g) = \max_{1 \leq i \leq m} |h(x_i) - g(x_i)|\). It is known (see e.g., [1]) that

\[
|\mathcal{H}_{j, \epsilon}| \leq (eb/\epsilon\rho_j)^{p_j}.
\]

By a union bound, [1] holds simultaneously for every choice of \(h_1 \in \mathcal{H}_{1, \epsilon}\), \(h_2 \in \mathcal{H}_{2, \epsilon}\), and \(t \in \{1, \ldots, m + 1\}\), with probability at least \(1 - \delta'(|\mathcal{H}_{1, \epsilon}| |\mathcal{H}_{2, \epsilon}|/(m + 1))\), this holds with probability at least \(1 - \delta\).

Also define \(\hat{h}_{j, \epsilon} = \arg\min_{h \in \mathcal{H}_{j, \epsilon}} \rho_m(h, h_j^*)\) and \(\hat{h}_{j, \epsilon} = \arg\min_{h \in \mathcal{H}_{j, \epsilon}} \rho_m(h, \hat{h}_j)\), for each \(j \in \{1, 2\}\).

To simplify notation, denote \(\hat{y}_t = \hat{h}_1(x_t)\) and \(\hat{y}_{t, \epsilon} = \hat{h}_{1, \epsilon}(x_t)\) for \(t \leq \hat{t} - 1\), and denote \(\hat{y}_t = \hat{h}_2(x_t)\) and \(\hat{y}_{t, \epsilon} = \hat{h}_{2, \epsilon}(x_t)\) for \(t \geq \hat{t}\). Similarly, denote \(\hat{y}_t^* = h_1^*(x_t)\) and \(\hat{y}_{t, \epsilon}^* = h_{1, \epsilon}^*(x_t)\) for \(t \leq t^* - 1\), and denote \(\hat{y}_t^* = h_2^*(x_t)\) and \(\hat{y}_{t, \epsilon}^* = h_{2, \epsilon}^*(x_t)\) for
\[ t \geq t^*. \] Then note that, by straightforward calculations,

\[
R^*(\hat{h}_1, \hat{h}_2, \hat{t}) - R^*(h_{1}^{*}, h_{2}^{*}, t^*)
\]

\[
= \mathbb{E} \left[ \hat{R}(\hat{h}_1, \hat{h}_2, \hat{t}) | \hat{h}_1, \hat{h}_2, \hat{t} \right] - \mathbb{E} \left[ \hat{R}(h_{1}^{*}, h_{2}^{*}, t^*) \right]
\]

\[
\leq \frac{1}{m} \sum_{i=1}^{m} \mathbb{E} \left[ (\hat{y}_{i,e} - \hat{y}_i)^2 + (\hat{y}_{i,e} - \hat{y}_i)^2 + 2\epsilon |\hat{y}_{i,e} - \hat{y}_i| \right] \]

\[
\leq 2\epsilon^2 + 8\epsilon B + \mathbb{E} \left[ \hat{R}(\hat{h}_1, \hat{h}_2, \hat{t}) | \hat{h}_1, \hat{h}_2, \hat{t} \right] - \mathbb{E} \left[ \hat{R}(h_{1}^{*}, h_{2}^{*}, t^*) \right].
\]

Thus, on the above event of probability \( 1 - \delta \),

\[
R^*(\hat{h}_1, \hat{h}_2, \hat{t}) - R^*(h_{1}^{*}, h_{2}^{*}, t^*)
\]

\[
\leq 2\epsilon^2 + 8\epsilon B + 2\sqrt{\frac{2B^2 \ln(2/\delta)}{m}} + R(\hat{h}_1, \hat{h}_2, \hat{t}) - \hat{R}(h_{1}^{*}, h_{2}^{*}, t^*)
\]

Then note that

\[
\hat{R}(\hat{h}_1, \hat{h}_2, \hat{t}) - \hat{R}(h_{1}^{*}, h_{2}^{*}, t^*)
\]

\[
\leq \frac{1}{m} \sum_{i=1}^{m} ((\hat{y}_{i,e} - Y_i)^2 + (\hat{y}_{i,e} - Y_i)^2 + 2\epsilon |\hat{y}_{i,e} - Y_i|) \]

\[
\leq 2\epsilon^2 + 8\epsilon B + \hat{R}(\hat{h}_1, \hat{h}_2, \hat{t}) - \hat{R}(h_{1}^{*}, h_{2}^{*}, t^*)
\]

\[
\leq 2\epsilon^2 + 8\epsilon B.
\]

Altogether, and combined with \((2)\), we have that, with probability at least \( 1 - \delta \),

\[
R^*(\hat{h}_1, \hat{h}_2, \hat{t}) - R^*(h_{1}^{*}, h_{2}^{*}, t^*)
\]

\[
\leq 4\epsilon^2 + 16\epsilon B + 2\sqrt{\frac{2B^2 \ln(2/\delta)}{m}}
\]

\[
\leq 20B \sqrt{\max\{p_1, p_2\}}/m
\]

\[
+ 2B \sqrt{\frac{2 \ln(2(m+1)/\delta) + \sum_{j=1}^{2} 3p_j \ln(emB/p_j)}{m}}
\]

\[
\leq 22B \sqrt{\frac{2 \ln(2(m+1)/\delta) + \sum_{j=1}^{2} 3p_j \ln(emB/p_j)}{m}}.
\]
Application to Addition of New Features: Next, consider the special case in which there exist feature functions $\phi_1, \ldots, \phi_{d+k} : \mathcal{X} \rightarrow \mathbb{R}$, for $d, k \in \mathbb{N}$, and there exist function classes $\mathcal{F}_1, \mathcal{F}_2$ such that every $f \in \mathcal{F}_1$ maps $\mathbb{R}^d \rightarrow [-B, B]$, while every $f \in \mathcal{F}_2$ maps $\mathbb{R}^{d+k} \rightarrow [-B, B]$. Then we can use the above framework to discuss the scenario in which the learner is tasked with identifying a time $t_0$ before which the first $d$ features suffice for good performance, and after which the full $d+k$ features are needed to obtain good performance. Specifically, in this case, the class $\mathcal{H}_1$ is the set of functions $x \mapsto f(\phi_1(x), \ldots, \phi_d(x))$ s.t. $f \in \mathcal{F}_1$, and the class $\mathcal{H}_2$ is the set of functions $x \mapsto f(\phi_1(x), \ldots, \phi_{d+k}(x))$ s.t. $f \in \mathcal{F}_2$. Then overloading the above notation, so that

$$\hat{R}(f_1, f_2, t_0) = \sum_{t=1}^{t_0-1} (f_1(\phi_1(x_t), \ldots, \phi_d(x_t)) - Y_t)^2$$

$$+ \sum_{t=t_0}^m (f_2(\phi_1(x_t), \ldots, \phi_{d+k}(x_t)) - Y_t)^2$$

and

$$R^*(f_1, f_2, t_0) = \sum_{t=1}^{t_0-1} (f_1(\phi_1(x_t), \ldots, \phi_d(x_t)) - \eta_t)^2$$

$$+ \sum_{t=t_0}^m (f_2(\phi_1(x_t), \ldots, \phi_{d+k}(x_t)) - \eta_t)^2,$$

consider the algorithm that chooses

$$(\hat{f}_1, \hat{f}_2, \hat{t}) = \arg\min_{(f_1, f_2, t_0) \in \mathcal{F}_1 \times \mathcal{F}_2 \times \{1, \ldots, m+1\}} \hat{R}(f_1, f_2, t_0).$$

Then for

$$(f^*_1, f^*_2, t^*) = \arg\min_{(f_1, f_2, t_0) \in \mathcal{F}_1 \times \mathcal{F}_2 \times \{1, \ldots, m+1\}} R^*(f_1, f_2, t_0),$$

Theorem 1 implies the following corollary.

**Corollary 2.** With probability at least $1 - \delta$, 

$$R^*(\hat{f}_1, \hat{f}_2, \hat{t}) \leq R^*(f^*_1, f^*_2, t^*)$$

$$+ 22B \sqrt{\frac{2 \ln(2(m+1)/\delta) + \sum_{j=1}^2 3p_j \ln(emB/p_j)}{m}}.$$ 

Remarks on Computational Speedups: For the sake of reducing the computational burden of searching over values of $t_0$ to identify $\hat{t}$, we can alternatively search over a grid of values $i \lfloor \sqrt{m}/B \rfloor$, $i \in \mathbb{N}$ with $i \lfloor \sqrt{m}/B \rfloor \leq m+1$ (supposing $B \leq \sqrt{m}$ for simplicity). Denoting

$$(\hat{g}_1, \hat{g}_2, \hat{t}_\sqrt{m}) = \arg\min_{(f_1, f_2, t_0) \in \mathcal{F}_1 \times \mathcal{F}_2 \times \{i \lfloor \sqrt{m}/B \rfloor : i \lfloor \sqrt{m}/B \rfloor \leq m+1\}} \hat{R}(f_1, f_2, t_0),$$

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the above analysis provides a similar guarantee as Corollary 2. Specifically, the only step requiring modification to accommodate this change is the inequality in (3), to account for the fact that \((\hat{g}_1, \hat{g}_2, \hat{t}_\sqrt{m})\) is not quite the minimizer. However, for \((\hat{f}_1, \hat{f}_2, \hat{t})\) as above, denoting by \(t'_\sqrt{m}\) the value in \(\{i\lfloor\sqrt{m}/B\rfloor : i\lfloor\sqrt{m}/B\rfloor \leq m + 1\}\) closest to \(\hat{t}\), we have

\[
\hat{R}(\hat{g}_1, \hat{g}_2, \hat{t}_\sqrt{m}) \leq \hat{R}(\hat{f}_1, \hat{f}_2, t'_\sqrt{m}) \leq \frac{4B}{\sqrt{m}} + \hat{R}(\hat{f}_1, \hat{f}_2, \hat{t}).
\]

Plugging this into the above analysis yields that, with probability at least \(1 - \delta\),

\[
R^*(\hat{g}_1, \cdots, \hat{g}_{K+1}, \hat{t}_1, \cdots, \hat{t}_K) \leq R^*(\hat{f}_1, \cdots, \hat{f}_{K+1}, \hat{t}_1, \cdots, \hat{t}_K) + \frac{26B}{m} \left( 2\ln\left(\frac{2(m+1)}{\delta}\right) + \sum_{j=1}^{K+1} 3p_j \ln\left(\frac{e m B}{p_j}\right) \right)
\]

**Remarks on Adapting to Multiple Change Times:** Rather than allowing only a single change time, it is a simple matter to generalize the above procedure to allow any number \(K\) of change times. Specifically, with \(K+1\) spaces \(H_1, K, \ldots, H_{K+1}, K\), where \(p_{j,K}\) denotes the pseudo-dimension of \(H_j, K\), defining \(t_0 = 0, t_{K+1} = m, \)

\[
R^*(h_1, \ldots, h_{K+1}, t_1, \ldots, t_K) = \frac{1}{m} \sum_{j=0}^{K} \sum_{t=t_{j+1}}^{t_{j+1}} (h_{j+1}(x_t) - \eta_t)^2,
\]

\[
\hat{R}(h_1, \ldots, h_{K+1}, t_1, \ldots, t_K) = \frac{1}{m} \sum_{j=0}^{K} \sum_{t=t_{j+1}}^{t_{j+1}} (h_{j+1}(x_t) - Y_t)^2,
\]

\[
(h^*_{1,K}, \ldots, h^*_{K+1,K}, t^*_{1,K}, \ldots, t^*_{K,K}) = \arg\min_{(h_1, \ldots, h_{K+1}) \in \times_{j=1}^{K+1} H_{j,K}, 0 \leq t_1 \leq \cdots \leq t_K \leq m} R^*(h_1, \ldots, h_{K+1}, t_1, \ldots, t_K),
\]

and

\[
(\hat{h}_1, \ldots, \hat{h}_{K+1}, \hat{t}_1, \ldots, \hat{t}_K) = \arg\min_{(h_1, \ldots, h_{K+1}) \in \times_{j=1}^{K+1} H_{j,K}, 0 \leq t_1 \leq \cdots \leq t_K \leq m} \hat{R}(h_1, \ldots, h_{K+1}, t_1, \ldots, t_K),
\]

we have with probability at least \(1 - \delta,\)

\[
R^*(\hat{h}_1, \ldots, \hat{h}_{K+1}, \hat{t}_1, \ldots, \hat{t}_K) \leq R^*(h^*_{1,K}, \ldots, h^*_{K+1,K}, t^*_{1,K}, \ldots, t^*_{K,K}) + 22B \sqrt{2\ln\left(\frac{2(m+1)^K}{\delta}\right) + \sum_{j=1}^{K+1} 3p_j \ln\left(\frac{e m B}{p_{j,K}}\right)} \cdot \frac{1}{m}.
\]
The proof follows analogously to that of Theorem 1.

If $K$ is unknown, via the method of structural risk minimization \cite{11,12}, it is still possible to effectively learn. Specifically, for any $K \in \{0, \ldots, m\}$, based on analogous arguments to Theorem 1 we have that with probability $1 - \delta/(K+2)^2$,

\[
R^*(\hat{h}_1, K, \ldots, h_{K+1}, K, \hat{t}_1, K, \ldots, \hat{t}_{K}, K) + \frac{1}{m} \sum_{t=1}^{m} \text{Var}(Y_t) \\
\leq \hat{R}(\hat{h}_1, K, \ldots, h_{K+1}, K, \hat{t}_1, K, \ldots, \hat{t}_{K}, K) \\
+ \frac{2 \ln \left( \frac{(m+1)^K (K+2)^2}{\delta} \right) + \sum_{j=1}^{K+1} 3p_jK \ln \left( \frac{e m B}{p_j K} \right)}{m},
\]

while

\[
\hat{R}(\hat{h}_1, K, \ldots, h_{K+1}, K, \hat{t}_1, K, \ldots, \hat{t}_{K}, K) \\
\leq \hat{R}(h_1^*, K, \ldots, h_{K+1}^*, K, t_1^*, K, \ldots, t_{K}^*, K),
\]

and

\[
\hat{R}(h_1^*, K, \ldots, h_{K+1}^*, K, t_1^*, K, \ldots, t_{K}^*, K) \\
\leq R^*(h_1^*, K, \ldots, h_{K+1}^*, K, t_1^*, K, \ldots, t_{K}^*, K) + \frac{1}{m} \sum_{t=1}^{m} \text{Var}(Y_t) \\
+ \frac{2 \ln \left( \frac{(m+1)^K (K+2)^2}{\delta} \right) + \sum_{j=1}^{K+1} 3p_jK \ln \left( \frac{e m B}{p_j K} \right)}{m},
\]

Therefore, choosing

\[
\hat{K} = \arg\min_{K \in \{0, \ldots, m\}} \hat{R}(\hat{h}_1, K, \ldots, h_{K+1}, K, \hat{t}_1, K, \ldots, \hat{t}_{K}, K)
\]

by a union bound, we have that with probability $1 - \delta$,

\[
R^*(\hat{h}_1, K, \ldots, h_{K+1}, K, \hat{t}_1, K, \ldots, \hat{t}_{K}, K) \\
\leq \min_{K \in \{0, \ldots, m\}} R^*(h_1^*, K, \ldots, h_{K+1}^*, K, t_1^*, K, \ldots, t_{K}^*, K) \\
+ 22B \frac{2 \ln \left( \frac{(m+1)^K (K+2)^2}{\delta} \right) + \sum_{j=1}^{K+1} 3p_jK \ln \left( \frac{e m B}{p_j K} \right)}{m}.
\]

We are thus able to achieve roughly the same guarantee as available above for any fixed $K$. 

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Figure 1: Above we see the performance of the different split point detection approaches with a) SVM-RBF, b) logistic regression and c) L1 reg. logistic regression, as the base learning methods on the synthetic data. The red bold line at 5000 is where the true change has occurred. In d), we see the running time of the different methods.

4 Efficient Version (SaSF)

We have seen that in our algorithm we learn a model at every time instant leading to \( m \) iterations of model building with the new (or newly interpreted) features. Moreover, we learn from scratch with the old and new features at these instances. This can be computationally intensive as we might have thousands of old features while only a few new and we would be learning \( m \) times over the entire set.

We thus suggest a couple of efficient approximations that could be done to scale our algorithm. Firstly, based on our remarks in section 3 on computational speedups, we could consider only time instances that are \( \Theta(\sqrt{m}) \) apart and hence learn only \( \Theta(\sqrt{m}) \) models as opposed to \( m + 1 \) models. This would potentially degrade the performance only by a constant factor as confirmed by the bound derived in that subsection. Secondly, we could use the meta-
Figure 2: Above we see the performance on the real datasets. a) and c) are the accuracy results on the retail dataset and the manufacturing dataset respectively, with SVM-RBF as the base learning method. Due to space constraints results with logistic and L1-logistic regression are in the supplementary material. b) and d) are the timing results on the retail dataset and manufacturing dataset respectively.

algorithm suggested in [5] to efficiently update our models with the newly added features. This would lead to a (learning) time complexity proportional only to the newly added features. Moreover, their algorithm is optimal for generalized linear models and more accurate than optimizing the residual for other models.

The above two strategies should help in making our SaS algorithm significantly faster, while maintaining accuracy; we thus refer to the resulting algorithm based on these two speedups[^1] as SaSF, an acronym for SaS Fast.

[^1]: Of course, the second speedup is only relevant when new features are added.
To support the theory, in this section, we evaluate our approach on synthetic as well as on two real industrial datasets. Since our setting is novel and we do not have direct competitors, we compare against methods from related frameworks described before. In particular, we compare against i) two concept drift methods one based on gradual forgetting (CGF) and the second based on abrupt forgetting (CAF) \[6\], ii) a multivariate unsupervised change point detection method namely, multivariate cusum based on sequential probability ratio test (MCSPRT) and iii) a randomized strawman which randomly determines the split point (RS). Moreover, to showcase the fact that our approach’s performance is not tied to any specific learning algorithm we experiment with three algorithms namely, logistic regression, L1-regularized logistic regression and SVM with an RBF kernel.

For our approaches, RS and MCSPRT each learning algorithm is trained separately on the data before and after the identified change/split by these methods. If no split point is detected by a particular method, we train the algorithm over the entire dataset. We report the mean squared error (MSE) for each datapoint on the time ordered datasets we consider, based on the prediction of the learned hypothesis that the point corresponds to i.e. either before or after the split. In other words, points before the identified split point are predicted with the hypothesis trained on the data before the split and analogously points following the split point are predicted with hypothesis trained on the data after the split. For RS we average the results over 100 different randomly determined splits.

For the concept drift approaches the algorithms are trained based on a discounted weighting scheme for CGF and windowing for CAF as described in \[6\]. We experimented with 100 different $\lambda$ parameters for CGF and 100 different window sizes for CAF and chose the best.

### 5.1 Synthetic Setup

We generate data using two 1001 dimensional gaussians with mean zero – corresponding to data before and after the split – where the last feature is the output variable. Hence, we have 1000 input variables. We set the correlation between all the input variables to be a small value such as 0.2, as it is more realistic than having extremely high correlation or at the other end of the spectrum all of them being independent. Usually in practice many statisticians working on applied problems consider a correlation less than 0.2 to be a proxy for independence. So a value of 0.2 is indicative of weak dependence. In the first gaussian, we set the first 500 features to have very low correlation ($\rho = 0.1$) with the output, while the next 500 features have high correlation ($\rho = 0.7$) with the output. In the second gaussian the situation is reversed and the first 500 features have high correlation ($\rho = 0.7$) with the output while the next 500 features have very low correlation ($\rho = 0.1$) with the output.

We generate 100 datasets of size 10000 (i.e. $m = 10000$) where the first
5000 points in each dataset are generated by the first gaussian, while the second 5000 are generated by the second gaussian. We generate the points sequentially so that there is an explicit ordering. We also normalize the outputs between zero and one so as to effectively apply logistic regression and its regularized counterpart.

5.2 Real Data Setup

The first dataset we consider is from a large retailer. We have 2 years of online customer data from the beginning of 2015 to the end of 2016. This is around 4TB of data containing information of roughly 80 million customers. The goal was to predict daily sales of a line/brand based on derived (aggregated) attributes from customer online visits that day such as the most common geo or zip, number of loyalty customers, average time spent on a visit, average number of pages visited per visit, average number of visits, average brand affinity across the customers that visited, average color affinity and average finish affinity for such customers. Although actual daily sales would be recorded at the end of day, having a good model can inform the business of which factors are important in identifying sales and provide confidence that they are in fact monitoring the right factors. Moreover, if sales drastically change or if factors change it is important to detect and consequently investigate the reasons. We report the results on one of the brands that was high priority for the business.

The second real dataset we consider is a real semi-conductor process of microprocessor or chip production. In our data, a single datapoint is a wafer, which is a group of chips, and measurements, which correspond to input features (temperatures, pressures, etc.), are made on this wafer throughout its production. The target that is used to evaluate the quality of the wafer, in this case, is the (normalized) speed of the wafer, which is the median of the speeds of its chips. We consider two critical stages of the manufacturing process where we have 2287 measurements at the first stage and another 1030 added at the second stage. The total number of wafers is 8926. The goal is to figure out if and when the second stage features start affecting wafer speed. Hence, any virtual metrology models (i.e. regression models) that have been built with the historical data have to be updated from this point onwards.

5.3 Observations

Synthetic Experiments: In figures 1a, 1b and 1c we see the performance of the different approaches on the synthetic dataset. We observe that our SaS method is by far the best at adapting to the split point as it seamlessly chooses the appropriate hypothesis for points before and after the change. SaSF is not as adept, however it too quickly switches to the right hypothesis after the change. CAF seems to be the best amongst the competitors, however it takes significantly longer to adapt with its performance being consistently slightly worse after the change. CGF slowly improves after the change point but is significantly worse than both SaS and SaSF. Random and MCSPRT do not improve
with time. However random is better than MCSPRT as with random at least we have two hypothesis that are used to predict, while with MCSPRT since there is no change in the input distribution, no change point is identified and we hence learn a single regressor over the whole dataset. In figure 1d, we see that SaSF is amongst the fastest methods and significantly faster than SaS. Hence, although SaS might be the best in performance, SaSF is definitely preferable if efficiency is a concern.

Real Data Experiments: In figures 2a and 2b we see the performance and timing results on the retail dataset. Figure 2a shows the results with SVM-RBF, however results with the other two methods that are in the supplementary material are qualitatively similar. We observe that around 16th of October our method SaS identifies a change point. At this juncture the other methods fail to detect the change point and keep using the same hypothesis resulting in bad predictions. SaSF although not as good as SaS adapts faster than the other competitors. Moreover, it is significantly faster than all the other methods as is seen in figure 2b. Post analysis of the change point by the domain experts resulted in the finding that around that time certain unflattering news stories were released which tarnished the brand image resulting in this change. This was an independent justification for the identified change point and a potential confirmation that it was not just noise.

In figures 2c and 2d we see the performance and timing results on the manufacturing dataset. In figure 2c, we observe that after our identified change point the performance suddenly improves. If we investigate the regressor after the change point we see that the new features start having high correlation with the target which results in this dramatic improvement. As before we see that SaS is the fastest in detecting, while SaSF although slower is still much better than the other methods. In terms of time, we observe in figure 2d, that SaSF has the biggest gain when compared with the previous experiments. The reason being that we can use both the speedups i.e. only $\sqrt{m}$ models to learn and the efficient feature updating strategy.

6 Discussion

Based on the theory and experiments we find that our method of determining the empirically optimal choice of split point leads to the best models in these applications.

In the future it would be interesting to experiment with multiple change points, where the number is not a priori known. The method based on structural risk minimization, discussed in Section 3 should be effective for this. This method appears to present computational challenges when searching over many change points. However, it may be possible to reduce the time complexity through dynamic programming.
7 Supplement: Results for Regularized Learners with Large Function Classes

In this supplement, we prove an extension of Theorem 1 from the main article, to allow for regularization terms in the optimization defining the learning algorithm. This extends the theory to cover methods such as $L_1$ regularized logistic regression and Support Vector Regression. The derived results are analogous to those presented for empirical risk minimization over classes with finite pseudo-dimension in the main article, but are included here for completeness.

The setting is similar to that in Theorem 1, except that now $p_i$ may be infinite, and we will specify $\hat{h}_i$ using a penalized empirical risk minimization setup. Specifically, we now denote by $p_i(q, t)$ a value in $[0, \infty)$, for any given $q \in [0, \infty)$. Then, for each $i \in \{1, 2\}$ and $t \in \{1, \ldots, m + 1\}$, we let $\mathcal{H}_{i,t}^{(q)}$ be a family of sets (indexed by $q \in [0, \infty)$) that is nondecreasing in $q$: that is, for $q < q'$, $\mathcal{H}_{i,t}^{(q)} \subseteq \mathcal{H}_{i,t}^{(q')}$. Then we let

$$(\hat{h}_1, \hat{h}_2, \hat{t}, \hat{q}_1, \hat{q}_2) = \arg\min_{(h_1, h_2, t_0, q_1, q_2): h_1 \in \mathcal{H}_{i,t}^{(q_1)}, \quad h_2 \in \mathcal{H}_{i,t}^{(q_2)}, \quad t_0 \in \{1, \ldots, m + 1\}, \quad q_1, q_2 \in [0, \infty)} \left( m\tilde{R}(h_1, h_2, t_0) + p_1(q_1, t_0) + p_2(q_2, t_0) \right).$$

Also denote by

$$(h_1^*, h_2^*, t^*, q_1^*, q_2^*) = \arg\min_{(h_1, h_2, t_0, q_1, q_2): h_1 \in \mathcal{H}_{i,t}^{(q_1)}, \quad h_2 \in \mathcal{H}_{i,t}^{(q_2)}, \quad t_0 \in \{1, \ldots, m + 1\}, \quad q_1, q_2 \in [0, \infty)} \left( mR^*(h_1, h_2, t_0) + p_1(q_1, t_0) + p_2(q_2, t_0) \right).$$

As in the proof of Theorem 1, let $\mathcal{H}_{i,t}^{(q)}$ denote a minimal $\epsilon$-cover of $\mathcal{H}_{i,t}^{(q)}$ with respect to the pseudo-metric $\rho_{m}$. Then let $N_i(\epsilon, q)$ be any value satisfying $N_i(\epsilon, q) \geq \max_{t \in \{1, \ldots, m + 1\}} |\mathcal{H}_{i,t}^{(q)}|$. Also let $\epsilon_i(q)$ denote any value in $(0, B]$ such that

$$\epsilon_i(q) \leq B \sqrt{\frac{\ln(N_i(\epsilon_i(q), q))}{m}}.$$

For brevity, also denote $N_i(q) = N_i(\epsilon_i(q), q)$.

We have the following result.

**Theorem A1.** With probability at least $1 - \delta$,

$$R^*(\hat{h}_1, \hat{h}_2, \hat{t}) \leq R^*(h_1^*, h_2^*, t^*) + \frac{p_1(q_1^*, t^*) + p_2(q_2^*, t^*)}{m} + 22B^2 \sqrt{\frac{2\ln(\frac{2(m+1)}{\delta}) + \sum_{i=1}^{2} \sum_{q \in (\hat{q}_i, q_i^*)} \ln((q+3)^2 N_i([q]))}{m}}.$$

**Proof.** We begin similarly to the proof of Theorem 1. Let $\hat{Y}_1, \ldots, \hat{Y}_m$ be equal in distribution to $Y_1, \ldots, Y_m$ but independent of $Y_1, \ldots, Y_m$, and define $\tilde{R}(h_1, h_2, t_0)$
as in the proof of Theorem 1. Again, for any \( h_1, h_2, \) and any \( t \in \{1, \ldots, m+1\} \), for any \( \delta'_{q_1, q_2} \in (0, 1) \), Hoeffding’s inequality implies that with probability at least \( 1 - \delta'_{q_1, q_2} \),

\[
\left| \hat{R}(h_1, h_2, t) - \mathbb{E}[\hat{R}(h_1, h_2, t)] \right| \leq \sqrt{\frac{2B^2 \ln \left( \frac{2}{\delta'_{q_1, q_2}} \right)}{m}}. \tag{4}
\]

For any given \( q_1, q_2 \in [0, \infty) \), by the union bound, \( \text{(4)} \) holds simultaneously for every choice of \( t \in \{1, \ldots, m+1\} \), \( h_1 \in \mathcal{H}^{(q_1)}_{1,t,\epsilon_1(q_1)} \), and \( h_2 \in \mathcal{H}^{(q_2)}_{2,t,\epsilon_2(q_2)} \), with probability at least \( 1 - \delta'_{q_1, q_2} \). Thus, for any \( \delta_{q_1, q_2} \in (0, 1) \), taking \( \delta'_{q_1, q_2} = \frac{\delta}{N_1(q_1)N_2(q_2)(m+1)} \), we have that with probability at least \( 1 - \delta_{q_1, q_2} \),

\[
\left| \hat{R}(h_1, h_2, t) - \mathbb{E}[\hat{R}(h_1, h_2, t)] \right| \leq \sqrt{\frac{2B^2}{m} \ln \left( \frac{2(m+1)N_1(q_1)N_2(q_2)}{\delta_{q_1, q_2}} \right)}.
\]

Furthermore, by the union bound, this fact holds simultaneously for all \( q_1, q_2 \in \mathbb{N} \cup \{0\} \) with probability at least \( 1 - \sum_{q_1, q_2} \mathbb{P}(\delta_{q_1, q_2} < \delta) \). In particular, taking \( \delta_{q_1, q_2} = \frac{\delta}{(q_1^2+q_2^2)^2} \), we have \( \sum_{q_1, q_2} \mathbb{P}(\delta_{q_1, q_2} < \delta) \). Thus, with probability at least \( 1 - \delta \), \( \forall q_1, q_2 \in \mathbb{N} \cup \{0\} \), \( \forall t \in \{1, \ldots, m+1\} \), \( \forall h_1 \in \mathcal{H}^{(q_1)}_{1,t,\epsilon_1(q_1)} \), \( \forall h_2 \in \mathcal{H}^{(q_2)}_{2,t,\epsilon_2(q_2)} \),

\[
\left| \hat{R}(h_1, h_2, t) - \mathbb{E}[\hat{R}(h_1, h_2, t)] \right| \leq \sqrt{\frac{2B^2}{m} \ln \left( \frac{2(m+1)(q_1+2)(q_2+2)^2N_1(q_1)N_2(q_2)}{\delta} \right)}.
\]

To extend this to allow general values of \( q_1, q_2 \in [0, \infty) \), we simply round up to the next integer: that is, on the above event of probability at least \( 1 - \delta \), for any \( q_1, q_2 \in [0, \infty) \), \( \forall t \in \{1, \ldots, m+1\} \), \( \forall h_1 \in \mathcal{H}^{(q_1)}_{1,t,\epsilon_1([q_1])} \), \( h_2 \in \mathcal{H}^{(q_2)}_{2,t,\epsilon_2([q_2])} \),

\[
\left| \hat{R}(h_1, h_2, t) - \mathbb{E}[\hat{R}(h_1, h_2, t)] \right| \leq \sqrt{\frac{2B^2}{m} \ln \left( \frac{2(m+1)(q_1+3)(q_2+3)^2N_1([q_1])N_2([q_2])}{\delta} \right)}.
\]

Now, for any \( q_1, q_2 \in [0, \infty) \), define

\[ h^*_i = \arg\min_{h \in \mathcal{H}^{(q_i)}_{1,t,\epsilon_i([q_i])}} \rho_m(h, h^*_i) \]

and

\[ \hat{h}_{i,q} = \arg\min_{h \in \mathcal{H}^{(q_i)}_{1,t,\epsilon_i([q_i])}} \rho_m(h, \hat{h}_i) \]

for \( i \in \{1, 2\} \). Denote \( \hat{y}_t = \hat{h}_1(x_t) \) and \( \hat{y}_{t,q_1,q_2} = \hat{h}_{1,q_1}(x_t) \) for each \( t \leq t - 1 \), and \( \hat{y}_t = \hat{h}_2(x_t) \) and \( \hat{y}_{t,q_1,q_2} = \hat{h}_{2,q_2}(x_t) \) for each \( t \geq t \). Also let \( \hat{\epsilon} = \max_{t \in \{1, 2\}} \epsilon_t([\hat{y}_t]) \). Similarly, define \( y^*_i = h^*_i(x_t) \) and \( y^*_{i,q_1,q_2} = h^*_{i,q_1}(x_t) \) for each \( t \leq t^* - 1 \), and \( y_t = h_2^*(x_t) \) and \( y_{t,q_1,q_2} = h_2^*_{q_2}(x_t) \) for each \( t \geq t^* \).
for each $t \geq t^*$, and let $\epsilon^* = \max_{i \in \{1, 2\}} \epsilon_i([q_i^*])$. Then by straightforward calculations, we have

\[
R^*(\hat{h}_1, \hat{h}_2, \hat{t}) - R^*(h_1^*, h_2^*, t^*) = \mathbb{E}\left[\tilde{R}(\hat{h}_1, \hat{h}_2, \hat{t})|\hat{h}_1, \hat{h}_2, \hat{t}, \hat{q}_1, \hat{q}_2\right] - \mathbb{E}\left[\tilde{R}(h_1^*, h_2^*, t^*)\right]
\]

\[
\leq \frac{1}{m} \sum_{t=1}^{m} \mathbb{E}\left[(\tilde{y}_t - \tilde{Y}_t)^2 + \epsilon^2 + 2\epsilon|\tilde{y}_t - \tilde{Y}_t|\big|\hat{h}_1, \hat{h}_2, \hat{t}, \hat{q}_1, \hat{q}_2\right] - \mathbb{E}\left[\tilde{R}(h_1^*, h_2^*, t^*)\right]
\]

\[
\leq \epsilon^2 + (\epsilon^*)^2 + 4(\epsilon + \epsilon^*)B
\]

Thus, on the above event of probability $1 - \delta$,

\[
R^*(\hat{h}_1, \hat{h}_2, \hat{t}) - R^*(h_1^*, h_2^*, t^*) \leq \epsilon^2 + (\epsilon^*)^2 + 4(\epsilon + \epsilon^*)B
\]

\[
+ \sqrt{\frac{2B^2}{m} \ln \left(\frac{2(m + 1)(q_1^* + 3)^2(q_2^* + 3)^2N_1([\hat{q}_1])N_2([\hat{q}_2])}{\delta}\right)}
\]

\[
+ \sqrt{\frac{2B^2}{m} \ln \left(\frac{2(m + 1)(q_1^* + 3)^2(q_2^* + 3)^2N_1([q_1^*])N_2([q_2^*])}{\delta}\right)}
\]

\[
+ \tilde{R}(\hat{h}_1, \hat{q}_1, \hat{h}_2, \hat{q}_2, \hat{t}) - \tilde{R}(h_1^*, h_2^*, t^*, h_1^*, h_2^*, t^*)
\]

Then note that

\[
\tilde{R}(\hat{h}_1, \hat{q}_1, \hat{h}_2, \hat{q}_2, \hat{t}) - \tilde{R}(h_1^*, h_2^*, t^*, h_1^*, h_2^*, t^*)
\]

\[
\leq \frac{1}{m} \sum_{t=1}^{m} ((\tilde{y}_t - Y_t)^2 + \epsilon^2 + 2\epsilon|\tilde{y}_t - Y_t|)
\]

\[
- \frac{1}{m} \sum_{t=1}^{m} ((y_t^* - Y_t)^2 - (\epsilon^*)^2 - 2\epsilon|y_t^* - Y_t|)
\]

\[
\leq \epsilon^2 + (\epsilon^*)^2 + 4B(\epsilon + \epsilon^*) + \tilde{R}(\hat{h}_1, \hat{h}_2, \hat{t}) - \tilde{R}(h_1^*, h_2^*, t^*)
\]

\[
\leq \epsilon^2 + (\epsilon^*)^2 + 4B\epsilon^* + \tilde{R}(\hat{h}_1, \hat{h}_2, \hat{t})
\]

\[
+ \frac{p_1(\hat{q}_1, \hat{t}) + p_2(\hat{q}_2, \hat{t})}{m} - \tilde{R}(h_1^*, h_2^*, t^*)
\]

\[
\leq \epsilon^2 + (\epsilon^*)^2 + 4B(\epsilon + \epsilon^*) + \tilde{R}(\hat{h}_1, \hat{h}_2, \hat{t})
\]

\[
+ \frac{p_1(q_1^*, t^*) + p_2(q_2^*, t^*)}{m} - \tilde{R}(h_1^*, h_2^*, t^*)
\]

\[
= \epsilon^2 + (\epsilon^*)^2 + 4B(\epsilon + \epsilon^*) + \frac{p_1(q_1^*, t^*) + p_2(q_2^*, t^*)}{m}
\]
 Altogether, we have that with probability at least $1 - \delta$,

$$R^*(\hat{h}_1, \hat{h}_2, \hat{t}) - R^*(h_1^*, h_2^*, t^*) \leq 2\epsilon^2 + 2(\epsilon^*)^2 + 8(\epsilon + \epsilon^*)B + \frac{p_1(q_1^*, t^*) + p_2(q_2^*, t^*)}{m}$$

$$+ \sqrt{\frac{2B^2}{m} \ln \left( \frac{2(m + 1)(\hat{q}_1 + 3)(\hat{q}_2 + 3)2N_1([\hat{q}_1])N_2([\hat{q}_2])}{\delta} \right)}$$

$$+ \sqrt{\frac{2B^2}{m} \ln \left( \frac{2(m + 1)(q_1^* + 3)(q_2^* + 3)2N_1([q_1^*])N_2([q_2^*])}{\delta} \right)}$$

$$\leq 11B^2 \sqrt{\frac{2}{m} \ln \left( \frac{2(m + 1)(q_1^* + 3)(q_2^* + 3)2N_1([q_1^*])N_2([q_2^*])}{\delta} \right)}$$

$$+ 22B^2 \sqrt{\frac{2}{m} \ln \left( \frac{2(m + 1)}{\delta} \prod_{i \in \{1, 2\}} \prod_{q \in \{q_1^*, q_2^*\}} (q + 3)2N_i([q]) \right)}$$

$$+ \frac{p_1(q_1^*, t^*) + p_2(q_2^*, t^*)}{m}.$$ 

As an example application of Theorem A1, consider the case of regularized kernel regression, with kernel functions $K_1, K_2$ (e.g., radial basis functions), along with feature functions $\phi_1, \ldots, \phi_{d+k}$ as in Section 3 of the main article. For brevity, denote by $\phi_{1:d}(x) = (\phi_1(x), \ldots, \phi_d(x))$ and $\phi_{1:(d+k)}(x) = (\phi_1(x), \ldots, \phi_{d+k}(x))$. In this case, fix any $\lambda_1, \lambda_2 > 0$, and consider choosing

$$\hat{h}_1(x) = \sum_{t=1}^{\hat{t}-1} \hat{\alpha}_t K_1(\phi_{1:d}(x_t), \phi_{1:d}(x))$$

and

$$\hat{h}_2(x) = \sum_{t=\hat{t}}^{m} \hat{\alpha}_t K_2(\phi_{1:(d+k)}(x_t), \phi_{1:(d+k)}(x)),$$

with parameters $\hat{\alpha}_1, \ldots, \hat{\alpha}_m$ chosen so that $\hat{\alpha}_1, \ldots, \hat{\alpha}_{\hat{t}-1}$ minimizes the expression

$$\sum_{t=1}^{\hat{t}-1} \left( Y_t - \sum_{i=1}^{\hat{t}-1} \hat{\alpha}_i K_1(\phi_{1:d}(x_i), \phi_{1:d}(x_t)) \right)^2 + \lambda_1 \sum_{i=1}^{\hat{t}-1} \sum_{j=1}^{\hat{t}-1} \hat{\alpha}_i \hat{\alpha}_j K_1(\phi_{1:d}(x_i), \phi_{1:d}(x_j)),$$
and \( \hat{\alpha}_t, \ldots, \hat{\alpha}_m \) minimizes the expression

\[
\sum_{t=\bar{t}}^{m} \left( Y_t - \sum_{i=\bar{t}}^{m} \hat{\alpha}_i K_2(\phi_{1:(d+k)}(x_i), \phi_{1:(d+k)}(x_t)) \right)^2 + \lambda_2 \sum_{i=\bar{t}}^{m} \sum_{j=\bar{t}}^{m} \hat{\alpha}_i \hat{\alpha}_j K_2(\phi_{1:(d+k)}(x_i), \phi_{1:(d+k)}(x_j)),
\]

and where \( \bar{t} \) is chosen to minimize the sum of these two expressions obtained at these minimizing \( \hat{\alpha}_1, \ldots, \hat{\alpha}_m \) values. This corresponds to choosing \((\hat{h}_1, \hat{h}_2, \bar{t})\) as above, with \( \mathcal{H}^{(q)}_{1,t} \) defined as the set of functions \( x \mapsto \sum_{i=1}^{t-1} \alpha_i K_1(\phi_{1:d}(x_i), \phi_{1:d}(x)) \) with \( \sum_{i=1}^{t-1} \sum_{j=1}^{t-1} \alpha_i \alpha_j K_1(\phi_{1:d}(x_i), \phi_{1:d}(x_j)) \leq q \), and similarly with \( \mathcal{H}^{(q)}_{2,t} \) defined as the set of functions \( x \mapsto \sum_{i=1}^{m} \alpha_i K_2(\phi_{1:(d+k)}(x_i), \phi_{1:(d+k)}(x)) \) with \( \sum_{i=1}^{t} \sum_{j=t}^{m} \alpha_i \alpha_j K_2(\phi_{1:(d+k)}(x_i), \phi_{1:(d+k)}(x_j)) \leq q \), and with \( p_1(q, t) = \lambda_1 q \) and \( p_2(q, t) = \lambda_2 q \).

In this example, the covering numbers \(|\mathcal{H}^{(q)}_{i,t,\epsilon}|\) will generally depend on the specific kernel function \( K_i \), and have been the subject of much study. As one concrete example, consider unit-bandwidth Gaussian kernels: \( K_1(u, v) = \exp\{-\|u - v\|^2\} \), for \( u, v \in \mathbb{R}^d \), and \( K_2(u, v) = \exp\{-\|u - v\|^2\} \), for \( u, v \in \mathbb{R}^{d+k} \).

In this case, denoting \( n_i = d + k \mathbb{1}[i = 2] \), \( \left[14\right] \) argues that for \( 0 < \epsilon \leq q \exp\{-90n_i^2 - 11n_i - 3\} \), we have

\[
\ln(|\mathcal{H}^{(q)}_{i,t,\epsilon}|) \leq 4n_i(6n_i + 2)(\ln(q/\epsilon))^{n_i + 1},
\]

for \( i \in \{1, 2\} \). Since the right hand side does not depend on \( t \), let us take \( N_i(\epsilon, q) \) equal to this value (for \( \epsilon \) in the range specified above). Thus, for instance, we may take

\[
\epsilon_i(q) = \min \left\{ q \exp\{-90n_i^2 - 11n_i - 3\}, \sqrt{1 \over m} \right\}
\]

to satisfy the constraint

\[
\epsilon_i(q) \leq \min \left\{ q \exp\{-90n_i^2 - 11n_i - 3\}, B2^{n_i} (\ln(q/\epsilon_i(q)))^{2n_i + \frac{1}{2}} \sqrt{6n_i + 2 \over m} \right\}.
\]

Now, denoting by \( \alpha^*_1, \ldots, \alpha^*_m \) the values such that \( \hat{h}_1^*(x) = \sum_{i=1}^{t-1} \alpha^*_i K_1(x_i, x) \)
and \( h_2^*(x) = \sum_{i=t^*}^m \alpha_i^* K_2(x_i, x) \), and denoting

\[
A_1^* = \sum_{i=1}^{t^*-1} \sum_{j=1}^{t^*-1} \alpha_i^* \alpha_j^* K_1(\phi_{1:d}(x_i), \phi_{1:d}(x_j)),
\]

\[
A_2^* = \sum_{i=t^*}^{m} \sum_{j=t^*}^{m} \alpha_i^* \alpha_j^* K_2(\phi_{1:(d+k)}(x_i), \phi_{1:(d+k)}(x_j)),
\]

\[
\hat{A}_1 = \sum_{i=1}^{i-1} \sum_{j=1}^{i-1} \hat{\alpha}_i \hat{\alpha}_j K_1(\phi_{1:d}(x_i), \phi_{1:d}(x_j)),
\]

\[
\hat{A}_2 = \sum_{i=t}^{m} \sum_{j=t}^{m} \hat{\alpha}_i \hat{\alpha}_j K_2(\phi_{1:(d+k)}(x_i), \phi_{1:(d+k)}(x_j)),
\]

the above theorem guarantees that, for some finite numerical constants \( c_1, c_2 \), with probability at least \( 1 - \delta \),

\[
R^*(\hat{h}_1, \hat{h}_2, \hat{t}) \leq R^*(h_1^*, h_2^*, t^*) + \frac{1}{m} (\lambda_1 A_1^* + \lambda_2 A_2^*) + \frac{c_1 B^2}{\sqrt{m}} \left( \sqrt{\ln(1/\delta)} + (d + k)^c_2 d + \ln(\max\{A_1^*, \hat{A}_1\} \sqrt{m})^c_2 d \right)
\]

\[
+ \ln(\max\{A_2^*, \hat{A}_2\} \sqrt{m})^c_2 (d + k)
\]

supposing \( d \geq 2 \) to simplify the expression. This expression is exponential in the dimension (reflecting the well-known “curse of dimensionality” commonly observed in nonparametric regression), but may be decreasing in \( m \), at a rate determined by the \( A_1^* \) and \( \hat{A}_1 \) values (which themselves can be analyzed in terms of characterizations of the smoothness of the regression function).

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