New Effects in Gauge Theory 
from pp-wave Superstrings

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Abstract

It has recently been observed that IIB string theory in the pp-wave background can be used to calculate certain quantities, such as the dimensions of BMN operators, as exact functions of the effective coupling $\lambda' = \lambda/J^2$. These functions interpolate smoothly between the weak and strong effective coupling regimes of $\mathcal{N} = 4$ SYM theory at large $R$ charge $J$. In this paper we use the pp-wave superstring field theory of\textsuperscript{hep-th/0204146} to study more complicated observables. The expansion of the three-string interaction vertex suggests more complicated interpolating functions which in general give rise to fractional powers of $\lambda'$ in physical observables at weak effective coupling.
1. Introduction

Recently the exact solvability\footnote{This result was rederived in \cite{5} (following earlier suggestions in \cite{6}) via semiclassical analysis of the $AdS_5 \times S^5$ sigma model, valid for large $\lambda$ and large $J$.} of type IIB string theory in the pp-wave background\footnote{This result was rederived in \cite{5} (following earlier suggestions in \cite{6}) via semiclassical analysis of the $AdS_5 \times S^5$ sigma model, valid for large $\lambda$ and large $J$.} has been used to understand the AdS/CFT correspondence in the limit of large $R$ charge\footnote{This result was rederived in \cite{5} (following earlier suggestions in \cite{6}) via semiclassical analysis of the $AdS_5 \times S^5$ sigma model, valid for large $\lambda$ and large $J$.}. It was discovered that string theory makes new exact statements about the $\mathcal{N} = 4$ SYM theory that may be checked in perturbation theory. The simplest such prediction concerns the dimensions of the BMN operators of $R$ charge $J$:\footnote{This result was rederived in \cite{5} (following earlier suggestions in \cite{6}) via semiclassical analysis of the $AdS_5 \times S^5$ sigma model, valid for large $\lambda$ and large $J$.}

\[ \Delta - J = \sum_{n=-\infty}^{\infty} N_n \sqrt{1 + \lambda' n^2} \] (1.1)

This formula shows that, for large $J$, these dimensions are functions of $\lambda' = \lambda / J^2$ where $\lambda$ is the ‘t Hooft coupling.\footnote{This result was rederived in \cite{5} (following earlier suggestions in \cite{6}) via semiclassical analysis of the $AdS_5 \times S^5$ sigma model, valid for large $\lambda$ and large $J$.} Therefore, even though the stringy derivation of this formula assumes that $\lambda$ is large, the effective coupling $\lambda'$ is a parameter that may assume arbitrary values. The interpolating formula (1.1) is remarkable: not only does it have the correct strong and weak coupling limits, but it constitutes a string theoretic prediction for perturbative gauge theory, which has recently been checked successfully\footnote{This result was rederived in \cite{5} (following earlier suggestions in \cite{6}) via semiclassical analysis of the $AdS_5 \times S^5$ sigma model, valid for large $\lambda$ and large $J$.}. Further interesting gauge theory results for correlators of the BMN operators were obtained in\footnote{This result was rederived in \cite{5} (following earlier suggestions in \cite{6}) via semiclassical analysis of the $AdS_5 \times S^5$ sigma model, valid for large $\lambda$ and large $J$.}. In order to compare these results with string theory, it is important to develop a string theoretic approach to observables more complicated than the operator dimensions; for example, the 3-point functions of the BMN operators. Since the RR-charged pp-wave background is solvable in the light-cone gauge, it is appropriate to use the techniques of light-cone superstring field theory\footnote{This result was rederived in \cite{5} (following earlier suggestions in \cite{6}) via semiclassical analysis of the $AdS_5 \times S^5$ sigma model, valid for large $\lambda$ and large $J$.}. Extension of this formalism from flat space to the pp-wave background was presented in\footnote{This result was rederived in \cite{5} (following earlier suggestions in \cite{6}) via semiclassical analysis of the $AdS_5 \times S^5$ sigma model, valid for large $\lambda$ and large $J$.} and further explored in\footnote{This result was rederived in \cite{5} (following earlier suggestions in \cite{6}) via semiclassical analysis of the $AdS_5 \times S^5$ sigma model, valid for large $\lambda$ and large $J$.}. In this
paper we present some additional calculations which shed new light on the $\lambda'$ dependence of various observables.

Consider, for example, 3-point functions of the BMN operators with large R charge. While the position dependence is fixed in terms of the operator dimensions by the conformal invariance, the normalization $C_{ijk}$ is an interesting observable. If we restrict ourselves to the planar limit, $C_{ijk}$ may depend on the 't Hooft coupling $\lambda$ and the R charges $J_i$ through combinations $\lambda' = \lambda/J_i^2$ and $J_2/J_1$, and there is no a priori reason to believe that the dependence is particularly simple. We will argue that in this case the interpolating function may be far more complex than (1.1) and will present some evidence for this.

An analogy we have in mind is to another non-BPS observable: the free energy of the $\mathcal{N} = 4$ SYM theory at a finite temperature $T$. On general field theoretic grounds we expect that in the 't Hooft large $N$ limit the entropy is given by

$$F/V = -\frac{\pi^2}{6} N^2 f(\lambda) T^4.$$ (1.2)

The AdS/CFT correspondence predicts the following behavior of $f$ for large $\lambda$ [20,21]:

$$f(\lambda) = \frac{3}{4} + \frac{45}{32} \zeta(3) \lambda^{-3/2} + \ldots.$$ (1.3)

On the other hand, perturbative field theory gives the following small $\lambda$ behavior [22,23]:

$$f(\lambda) = 1 - \frac{3}{2\pi^2} \lambda + \frac{3 + \sqrt{2}}{\pi^3} \lambda^{3/2} + \ldots.$$ (1.4)

Calculation of the full interpolating function is an interesting challenge which seems to be beyond the scope of presently available methods: supergravity methods are not sufficient for studying small $\lambda$ while full string theoretic methods have not been developed far enough. The expansions (1.3) and (1.4) indicate, however, that the interpolating function is far more complicated than in (1.1). For instance, at small $\lambda$ we observe the appearance of a term of order $\lambda^{3/2}$ [23] which is due to a resummation of diagrams with insertions of the thermal mass induced at one loop, $m^2 \sim \lambda T^2$. This non-analytic term is an infrared effect: it follows from the fact that the free energy depends on the mass as $F/V \sim m^3 T$. In this paper we will see hints of similar effects in the pp-wave light-cone string field theory. Luckily, in this case methods are available for studying the string field theory at small $\lambda'$ (or, equivalently, at large $\mu$). We turn to this analysis in the next section.

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2 In this paper we use $\mu$ as shorthand for the dimensionless variable $1/\sqrt{\lambda'}$. This is a departure from the more conventional relation $\lambda' = \frac{1}{(\mu p^+\alpha')^2}$, where $p^+$ is the largest light-cone momentum involved in the process of interest.
2. The Light-Cone String Vertex at Large $\mu$

The three string splitting-joining interaction in the pp-wave background has been worked out in [14]. The interaction consists of a delta-functional overlap which expresses continuity of the string worldsheet, and an operator required by supersymmetry which is inserted at the point where the string splits [13]. In this paper we focus on the overlap, which we express as a state in the three-string Hilbert space of the form

$$|V\rangle = \exp \left[ \frac{1}{2} \sum_{r,s=1}^{3} \sum_{m,n=-\infty}^{\infty} a_{m(r)}^{I_{r}} \overline{N}_{mn}^{(rs)} a_{n(s)}^{J_{s}} \delta_{IJ} \right] |0\rangle.$$  \hspace{1cm} (2.1)

Like the dimensions (1.1), the Neumann coefficients $\overline{N}_{mn}^{(rs)}$ are smooth functions of $\lambda'$ which interpolate between the flat space expressions of [12] at $\lambda' = \infty$ and the very simple expressions of [15,16] at $\lambda' = 0$. They encode a wealth of information about the interacting gauge theory, but unlike (1.1) they are highly nontrivial functions of $\lambda'$ which have not been computed explicitly. In this section we report some progress in this direction. We highlight the difficulty of calculating even $O(\lambda')$ effects, and point out the existence of non-analytic terms involving half-integer powers of $\lambda'$ as well as $e^{-1/\sqrt{\lambda'}}$.

2.1. The Matrix $\Gamma_+$

The difficulty in obtaining explicit formulas for the Neumann coefficients starts with the problem of inverting a certain infinite dimensional matrix $\Gamma_+$. In appendix A we define this matrix and evaluate its components explicitly. It can be expressed as

$$\Gamma_+ = \Gamma_0 - H$$  \hspace{1cm} (2.2)

where $\Gamma_0$ is diagonal,

$$[\Gamma_0]_{mn} = 2 \frac{\sqrt{m^2 + \mu^2}}{m} \delta_{mn},$$  \hspace{1cm} (2.3)

(for positive integers $m, n$) and the matrix elements of $H$ are

$$H_{mn} = \frac{8}{\mu^2 \pi^2} (-1)^{m+n} \sqrt{mn} \sin(\pi m y) \sin(\pi n y) \int_{1}^{\infty} \frac{dz}{(z^2 + m^2/\mu^2)(z^2 + n^2/\mu^2)} F(z) \sqrt{z^2 - 1},$$  \hspace{1cm} (2.4)

where $y = p_{(1)}^+/p_{(3)}^+$ lies in the range $0 < y < 1$ and

$$F(z) = \frac{1}{2} \left[ \coth(\pi \mu y z) + \coth(\pi \mu (1 - y) z) \right].$$  \hspace{1cm} (2.5)
Note that $H$ has a finite limit as $\mu \to 0$, which must be the case since in this limit $\Gamma_+$ goes over smoothly to the flat space matrix $\Gamma$ of [12], which is not diagonal.

In the opposite limit $\mu \to \infty$ or $\lambda' \to 0$, we note that $\Gamma_0$ is of order $\mu$ while $H$ is of order $\mu^{-2}$. Furthermore $\Gamma_0$ has a power series expansion around $\mu = \infty$ in which only odd powers of $1/\mu$ appear, while $H$ has an expansion with two kinds of terms: even powers of $1/\mu$ and non-perturbative terms of order $e^{-2\pi \mu y}$ and $e^{-2\pi \mu (1-y)}$ which come from the function $F(z)$. The last kind of terms correspond to field theory effects of order $e^{-1/\sqrt{\lambda'}}$, which are reminiscent of D-branes rather than instantons.

We refer to $H$ as the ‘non-analytic’ part of $\Gamma_+$ for two reasons. First, it is directly responsible for the half-integer powers of $\lambda'$ and non-perturbative $e^{-1/\sqrt{\lambda'}}$ effects. Secondly, it is shown in appendix B how these terms arise from a certain branch cut in the complex plane which was missed in the analytic continuation argument of [L7]. We will see however that $H$ also contributes to integer powers of $\lambda'$ in observables.

Having now an explicit expression for the elements of the matrix $\Gamma_+$, the next step is to find $\Gamma_+^{-1}$. Since $\Gamma_0$ is easy to invert and is larger than $H$ by a factor of $\mu^3$ for large $\mu$, it seems sensible to employ the expansion

$$
\Gamma_+^{-1} = (\Gamma_0 - H)^{-1} = \Gamma_0^{-1} - \Gamma_0^{-1} H \Gamma_0^{-1} + \Gamma_0^{-1} H \Gamma_0^{-1} H \Gamma_0^{-1} + \cdots. \tag{2.6}
$$

In order to establish the validity of this expansion, two issues must be addressed: the first is whether each term on the right-hand side is finite, and the second is whether the sum of all of the terms converges.

Naive counting of $\mu$’s suggests that each term in (2.6) is suppressed relative to the previous term by a factor of $\mu^{-3}$. However, the matrix product in $H \Gamma_0^{-1} H$ involves a sum of the form

$$
\sum_{p=1}^{\infty} \frac{\sin^2(\pi py)}{\sqrt{p^2 + \mu^2} \left( (p^2 + \mu^2 x^2)(p^2 + \mu^2 z^2) \right)} \tag{2.7}
$$

We evaluate this sum in appendix B and find that it behaves for large $\mu$ like $\mu^{-2}$ rather than the naive $\mu^{-5}$. This ‘renormalization’ by $\mu^3$ is a direct consequence of the large $p$ behavior of (2.7), which would equal $\mu^{-5}$ times a cubically divergent sum if one tried to take $\mu \to \infty$ before evaluating the sum.

So the good news is that in the expansion (2.6), each term on the right-hand side exists (indeed we present an explicit formula for the $k$-th term in appendix B), but the bad news is that all of the terms (except the first) are of order $\mu^{-4}$! Therefore it is not
clear that the sum of these terms converges, although this can still be the case if the $k$-th term is suppressed by a coefficient which decreases sufficiently rapidly with $k$. While we have not been able to prove convergence, numerical evidence suggests that the expansion (2.6) is indeed sensible and converges rapidly to $\Gamma^{-1}$.

To summarize, we have shown that for large $\mu$,

$$\mu \left[ \Gamma_{+}^{-1} \right]_{mn} = \left[ \frac{m}{2} - \frac{m^3}{4} \lambda' + \mathcal{O}(\lambda'^2) \right] \delta_{mn} + \lambda'^{3/2} R_{mn} + \mathcal{O}(\lambda'^{5/2}), \quad (2.8)$$

where the term in brackets is the expansion of $\Gamma_0^{-1}$ and $R_{mn}$ is nonzero and nondiagonal but has eluded explicit evaluation since it requires summing an infinite number of terms in (2.6). This result highlights the fact that (2.6), while a true formula, is not very useful for studying the small-$\lambda'$ expansion. Hopefully a more clever method of inverting $\Gamma_{+}$ can be found.

2.2. Some Neumann Matrix Elements

In some Neumann matrix elements $[\Gamma_{+}^{-1}]_{mn}$ appears on its own, but in others it must be multiplied on the left and/or right by certain $\mu$-independent matrices or vectors (see appendix A). In this subsection we show that these summations renormalize the contribution of the non-analytic terms $H$ by additional powers of $\mu$, allowing them to contribute at order $\lambda'$ or even $\sqrt{\lambda'}$ to the Neumann matrix elements.

The simplest Neumann matrix is

$$\mathbf{N}^{(33)}_{mn} = \delta_{mn} - 2 \left( \frac{m^2 + \mu^2}{\sqrt{mn}} \right)^{1/4} \left( \frac{n^2 + \mu^2}{\sqrt{mn}} \right)^{1/4} \left[ \Gamma_{+}^{-1} \right]_{mn}. \quad (2.9)$$

Using (2.8) we see immediately that for large $\mu$,

$$\mathbf{N}^{(33)}_{mn} = - \frac{2}{\sqrt{mn}} \lambda'^{3/2} R_{mn} + \frac{3}{8} n^4 \delta_{mn} \lambda'^2 + \cdots, \quad (2.10)$$

which demonstrates the existence of half-integer powers of $\lambda'$ in string theory observables.

Next consider the Neumann coefficient $\mathbf{N}^{(13)}_{0m}$, which at large $\mu$ involves $\mu[\Gamma_{+}^{-1}B]_m$, where the vector $B$ is defined in appendix A. Using the expansion (2.6), we expect

$$\Gamma_{+}^{-1}B = \Gamma_0^{-1}B + \Gamma_0^{-1}HT_0^{-1}B + \cdots. \quad (2.11)$$

\footnote{All expressions in this subsection are valid for positive indices $m, n$. These are sufficient to determine the elements with negative indices via fairly simple relations \footnote{F}.}
Our counting from the previous subsection suggests that the first term is $O(\mu^{-1})$, while the second and all higher terms are $O(\mu^{-4})$. However, the vector product in $H\Gamma_0^{-1}B$ involves a sum of the form
\[
\frac{1}{\mu} \sum_{p=1}^{\infty} \frac{\sin^2(\pi py)}{\sqrt{p^2 + \mu^2 p^2 + \mu^2 x^2}},
\] (2.12)
which is $O(\mu^{-3})$ rather than the naive $O(\mu^{-4})$ for large $\mu$. This renormalization by one power of $\mu$ is again the direct result of the large $p$ behavior of $(2.12)$, which would be linearly divergent if one tried to first set $\mu = \infty$ and then perform the sum. In appendix B we show that
\[
[\Gamma_0^{-1} H \Gamma_0^{-1} B]_m = \frac{1}{2\pi^2 \mu^3} m^3 B_m + O(\mu^{-5}).
\] (2.13)
The complete answer therefore has the form
\[
\mu [\Gamma^{-1} B]_m = \frac{m}{2} B_m + m^3 \lambda' \left[ -\frac{1}{4} + \frac{1}{2\pi^2} + \cdots \right] B_m + O(\lambda'^2).
\] (2.14)
Calculating the exact coefficient in brackets would require summing up the infinite number of terms on the right-hand side of $(2.6)$, which we have not been able to do, but numerical evidence suggests that the quantity converges rapidly (to $-\frac{1}{4} + x$, where $x \approx \frac{1}{16}$). Although $(2.14)$ shows that no half-integer powers of $\lambda'$ enter in the Neumann coefficients $N_{0m}^{(13)}$, it is remarkable that the coefficient of the $O(\lambda')$ term receives a finite renormalization due to the non-analytic contribution from $H$.

A similar analysis holds for the Neumann coefficients $N_{0m}^{(23)}$, as well as $N_{mn}^{(r3)}$ for $r \in \{1, 2\}$, although the latter involve a sum of the form
\[
\frac{1}{\mu} \sum_{p=1}^{\infty} \frac{\sin^2(\pi py)}{\sqrt{p^2 - m^2/y^2}} \frac{1}{\sqrt{p^2 + \mu^2 p^2 + \mu^2 x^2}},
\] (2.15)
rather than $(2.12)$. Like $(2.12)$, this sum behaves as $O(\mu^{-3})$ for large $\mu$. Therefore these Neumann matrix elements have no half-integer powers, but it seems difficult to calculate explicitly even the $O(\lambda')$ term since all of the terms in $(2.9)$ contribute, just as in $(2.14)$.

Finally, we remark that the remaining Neumann coefficients involve $\Gamma_+^{-1}$ multiplied both on the left and on the right. For example, for $r, s \in \{1, 2\}$, $N_{mn}^{(rs)}$ involves $\mu[A^{(r)}^T \Gamma_+^{-1} A^{(s)}]_{mn}$, while $N_{0m}^{(rs)}$ involves $\mu[A^{(r)}^T \Gamma_+^{-1} B]_m$. In these cases there are two summations which each provide an extra factor of $\mu$, so that these Neumann coefficients have contributions starting at $O(\sqrt{\lambda'})$.

\footnote{This introduces an apparent disagreement with the field theory calculation of \cite{17}, since we find that the correction factor in (56) and (58) should be $1 - (\frac{1}{2} - 4x)\lambda'n^2$ instead of $1 - \frac{1}{2}\lambda'n^2$. However, in the string field theory calculation one also needs to include the prefactor which may further modify the $O(\lambda')$ correction.}
3. Conclusion

In this paper we have studied the Neumann coefficients of the three-string vertex in the pp-wave background. These matrices are highly nontrivial functions of $\lambda'$ which smoothly interpolate between the weak and strong effective coupling regimes of the SYM gauge theory and potentially encode a wealth of information about non-BPS observables in the field theory. We have shown that these coefficients contain half-integer powers of $\lambda'$ in the weak effective coupling expansion. Recall, however, that the plane wave limit is carried out at large 't Hooft coupling $\lambda$. Therefore, there are two possibilities. The first one is that $(\lambda')^{n/2}$ may be replaced literally by $\lambda^{n/2}/J^n$, so that we find fractional powers of $\lambda$ at weak coupling, as in the free energy (1.4). The second possibility is that $(\lambda')^{n/2}$ should be interpreted as $g(\lambda)/J^n$ where $g(\lambda)$ has a weak coupling expansion in integer powers of $\lambda$ but approaches $\lambda^{n/2}$ for large $\lambda$. It would be very desirable to decide which of the two possibilities is correct.

We also remark that the precise relation between the Neumann coefficients and gauge theory three-point functions is not well-understood at finite coupling. This is both because the dictionary between pp-wave string theory and SYM theory is not precisely known away from $\lambda' = 0$ (see [14,24]), and because we have not included the prefactor of the cubic string interaction [13] in our analysis, although we do not expect the latter to change our conclusions qualitatively. Finally, the dictionary is also complicated by mixing between single- and multi-trace operators.

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Appendix A. Some Matrices

We start by defining for $m, n > 0$ the matrices

\begin{align*}
A^{(1)}_{mn} &= (-1)^{m+n+1} \frac{2 \sqrt{mn} y \sin(\pi my)}{\pi n^2 - m^2 y^2}, \\
A^{(2)}_{mn} &= (-1)^m \frac{2 \sqrt{mn} (1 - y) \sin(\pi my)}{\pi n^2 - m^2 (1 - y)^2}, \\
A^{(3)}_{mn} &= \delta_{mn}, \\
C_{mn} &= m \delta_{mn}, \\
C^{(1)}_{mn} &= \delta_{mn} \sqrt{m^2 + \mu^2 y^2}, \\
C^{(2)}_{mn} &= \delta_{mn} \sqrt{m^2 + \mu^2 (1 - y)^2}, \\
C^{(3)}_{mn} &= \delta_{mn} \sqrt{m^2 + \mu^2}
\end{align*}

and the vector

\begin{equation}
B_m = \frac{2}{\pi y(1 - y)\alpha' p^+} \frac{(-1)^{m+1} \sin(\pi my)}{m^{3/2}}.
\end{equation}

Note that $\mu = 1/\sqrt{\lambda}$ stands for what was called $\mu |\alpha^{(3)}|$ in [14], $p^+ = p^{(3)}_+$ is the momentum of the big string, and $y = p^{(3)}_+ / p^+$ is the fraction of $p^+$ carried by little string number 1.

The matrix $\Gamma_+$ whose inverse appears in the Neumann coefficients for positive $m, n$ is given by [13]

\begin{equation}
\Gamma_+ = \sum_{r=1}^{3} A^{(r)} C_{r} C^{-1} A^{(r)T} + \frac{1}{2} \mu y (1 - y) (\alpha' p^+)^2 B B^T.
\end{equation}

It is manifest that $\Gamma_+$ goes over smoothly to the matrix $\Gamma$ of [12] as $\mu \to 0$. The Neumann matrices are then given for $m, n > 0$ by

\begin{equation}
\Gamma_{mn}^{(rs)} = \delta^{rs} \delta_{mn} - 2 \left[ C^{1/2}_{(r)} C^{-1/2} A^{(r)T} \Gamma^{-1}_+ A^{(s)} C^{-1/2} C^{1/2}_{(s)} \right]_{mn}.
\end{equation}

Appendix B. Some Sums and Integrals

Let us first calculate $\Gamma_+$. From (A.3) and the definitions (A.1) it is easy to see that we need to evaluate sums of the form

\begin{equation}
\sum_{p=1}^{\infty} f(p), \quad f(z) = \frac{\sqrt{z^2 + \mu^2 y^2}}{(z^2 - m^2 y^2)(z^2 - n^2 y^2)},
\end{equation}

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Fig. 1: The analytic structure of the function $f(z) \pi \cot(\pi z)$ for $f(z)$ given in (B.1). The poles lie at all integer $z$, with four additional poles on the real axis at $z = \pm my, \pm ny$. The branch cuts on the imaginary axis start at $z = \pm i\mu$. The top, bottom and central contours correspond to $I_t$, $I_b$ and $I_c$ respectively.

(and the same with $y \to 1 - y$) for positive integers $m, n$ and $0 < y < 1$. To this end consider the integral

$$I_C = \oint_C \frac{dz}{2\pi i} f(z) \pi \cot(\pi z)$$

for the various contours shown in Fig. 1.

It is easy to evaluate

$$I_c = \frac{\mu}{y^3 m^2 n^2} + \frac{\pi}{y^2} \left[ \frac{\cot(\pi my)}{m(m^2 - n^2)} \sqrt{m^2 + \mu^2} + (m \leftrightarrow n) \right] + 2 \sum_{p=1}^{\infty} f(p),$$

$$I_t = I_b = \frac{1}{\mu^2 y^2} \int_1^{\infty} dx \frac{\sqrt{x^2 - 1}}{(x^2 + m^2/\mu^2)(x^2 + n^2/\mu^2)} \coth(\pi \mu y x).$$

Now, since

$$I_t + I_b + I_c = 0,$$

we conclude that

$$\sum_{p=1}^{\infty} f(p) = -\frac{\mu}{2 y^3 m^2 n^2} - \frac{\pi}{2 y^2} \left[ \frac{\cot(\pi my)}{m(m^2 - n^2)} \sqrt{m^2 + \mu^2} + (m \leftrightarrow n) \right] - I_t.$$

Note that for very large $\mu$ we can set $F(z) = 1$ and evaluate the integral $I_t$, obtaining

$$I_t = \frac{1}{y^2} \frac{1}{(m^2 - n^2)} \left[ \frac{\sqrt{m^2 + \mu^2}}{m} \arcsinh(m/\mu) - \frac{\sqrt{n^2 + \mu^2}}{n} \arcsinh(n/\mu) \right].$$
This is valid up to corrections of order $e^{-2\pi \mu y}$ and $e^{-2\pi \mu (1-y)}$.

Using the sum \((B.5)\) and the definitions in appendix A, it takes only a little algebra to show that the contribution from the first two terms in \((B.5)\) is such that the $r = 1, 2$ terms in \((A.3)\) cancel the $BB^T$ term, leaving only a diagonal piece $\frac{1}{2} \Gamma_0$. The other $\frac{1}{2} \Gamma_0$ comes from $r = 3$ in \((A.3)\). The net result is that omitting $I_t$ in \((B.5)\) would lead one to the conclusion that $\Gamma_+ = \Gamma_0$, as in the analytic continuation argument of [17]. Instead, we find that the branch cut terms $I_t$ precisely account for the matrix $H$ as written in \((2.4)\) after summing over $r = 1, 2$ in \((A.3)\).

Let us list some other useful sums which can be derived using similar techniques. For $v > 1$ we have

$$\sum_{n=1}^{\infty} \frac{\sin^2(\pi ny)}{n^2 + \mu^2 v^2} \frac{1}{\sqrt{n^2 + \mu^2}} = -\frac{1}{2\mu^2} \int_{1}^{\infty} \frac{dz}{\sqrt{z^2 - 1}} \frac{1}{z^2 - v^2} F(z),$$

(B.7)

where the symbol $P$ stands for the principal value of the integral. For large $\mu$ we can set $F(z) = 1$ and evaluate the integral, giving

$$\sum_{n=1}^{\infty} \frac{\sin^2(\pi n y)}{n^2 + \mu^2 v^2} \frac{1}{\sqrt{n^2 + \mu^2}} = \frac{1}{2\mu^2} \frac{\arccosh(v)}{v\sqrt{v^2 - 1}},$$

(B.8)

up to exponential corrections. A variant of \((B.7)\) which we will need is

$$\sum_{n=1}^{\infty} \frac{\sin^2(\pi ny)}{n^2 - m^2/ y^2} \frac{1}{\sqrt{n^2 + \mu^2}} = -\frac{1}{2\mu^2} \int_{1}^{\infty} \frac{dz}{\sqrt{z^2 - 1}} \frac{1}{z^2 + m^2/(\mu^2 y^2)} F(z),$$

(B.9)

where $m$ is an integer.

Next we study the $k$-th term in the expansion \((2.6)\). Using the integral representation \((2.4)\) for $H$, we find that the matrix multiplication $H \Gamma_0^{-1} H$ involves a sum of the form

$$P(x_1, x_2) \equiv \frac{1}{2} \sum_{p=1}^{\infty} \frac{\sin^2(\pi py)}{\sqrt{p^2 + \mu^2}} \frac{p^2}{(x_1^2 + p^2/\mu^2)(x_2^2 + p^2/\mu^2)}$$

$$= -\frac{\mu^2}{4} \int_{1}^{\infty} \frac{dz}{\sqrt{z^2 - 1}} \frac{1}{F(z)} \frac{z^2}{(z^2 - x_1^2)(z^2 - x_2^2)}.$$

(B.10)

Using this definition it is straightforward to derive the explicit though complicated formula

$$[H \Gamma_0^{-1} H \Gamma_0^{-1} \cdots H]_{mn} = \left( \frac{8}{\mu^2 \pi^2} \right)^k (-1)^{m+n} \sqrt{mn} \sin(\pi my) \sin(\pi ny)$$

$$\times \prod_{i=1}^{k} \int_{1}^{\infty} dx_i \sqrt{x_i^2 - 1} F(x_i) \frac{P(x_1, x_2) \times \cdots \times P(x_{k-1}, x_k)}{(x_1^2 + m^2/\mu^2)(x_2^2 + n^2/\mu^2)},$$

(B.11)
where $k$ is the number of times $H$ appears on the left. Since each of the $k-1$ ‘propagators’ $P$ has an explicit factor of $\mu^2$ from the result (B.10), we see that (B.11) is $O(\mu^{-2})$ for any $k$. This establishes the claim that all of the terms in (2.6) except for the first are $O(\mu^{-4})$ for large $\mu$. Note that for large $\mu$ we can set $F(z) = 1$ in (B.10) to obtain

$$P(x_1, x_2) = \frac{\mu^2}{4} \left[ \frac{x_1}{x_1^2 - x_2^2} \frac{\arccosh(x_1)}{\sqrt{x_1^2 - 1}} + (x_1 \leftrightarrow x_2) \right], \quad (B.12)$$

up to exponential corrections. Nevertheless we have not been able to evaluate the iterated integrals in (B.11) in a closed form.

Let us conclude by calculating the second term in brackets in (2.14). We have

$$[H\Gamma_{0}^{-1}B]_m = \frac{1}{2} \left[ HCC_{(3)}^{-1}B \right]_m$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{8}{\mu^2 \pi^2} (-1)^{m+n} \sqrt{mn} \sin(\pi my) \sin(\pi ny)$$

$$\times \int_{1}^{\infty} dz F(z) \frac{\sqrt{z^2 - 1}}{(z^2 + m^2/\mu^2)(z^2 + n^2/\mu^2)}$$

$$\times \frac{n}{\sqrt{n^2 + \mu^2}} \times \frac{2}{\pi y(1 - y)p^+} (-1)^{n+1} n^{-3/2} \sin(\pi ny). \quad (B.13)$$

The sum over $n$ can be evaluated for large $\mu$ using (B.8). The remaining integral over $z$ is then of the form

$$\int_{1}^{\infty} dz \frac{\arccosh(z)}{z^3} = \frac{1}{2}. \quad (B.14)$$

Putting everything together, we find for large $\mu$

$$[\Gamma_{0}^{-1}H\Gamma_{0}^{-1}B]_m = \frac{1}{2\pi^2 \mu^3} m^3 B_m. \quad (B.15)$$
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