ENERGY CONDITIONS AND THEIR COSMOLOGICAL IMPLICATIONS

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The energy conditions of general relativity permit one to deduce very powerful and general theorems about the behaviour of strong gravitational fields and cosmological geometries. However, the energy conditions these theorems are based on are beginning to look a lot less secure than they once seemed: (1) there are subtle quantum effects that violate all of the energy conditions, and more tellingly (2), there are also relatively benign looking classical systems that violate all the energy conditions. This opens up a Pandora’s box of rather disquieting possibilities — everything from negative asymptotic mass, to traversable wormholes, to warp drives, up to and including time machines.

1 Introduction

Einstein gravity (general relativity) is a tremendously complex theory even if you restrict attention to the purely classical regime. The field equations are

\[ G^{\mu\nu} = \frac{8\pi G_{\text{Newton}}}{c^4} T^{\mu\nu}. \] (1)

The left-hand-side, the Einstein tensor \( G^{\mu\nu} \), is complicated enough by itself, but is at least a universal function of the spacetime geometry. In contrast the right-hand-side, the stress-energy tensor \( T^{\mu\nu} \), is not universal but instead depends on the particular type of matter and interactions you choose to insert in your model. Faced with this situation, you must either resign oneself to performing an immense catalog of special-case calculations, one special case for each conceivable matter Lagrangian you can write down, or try to decide on some generic features that “all reasonable” stress-energy tensors should satisfy, and then try to use these generic features to develop general theorems concerning the strong-field behaviour of gravitational fields.

One key generic feature that most matter we run across experimentally seems to share is that energy densities (almost) always seem to be positive. The so-called “energy conditions” of general relativity are a variety of different ways of making this notion of locally positive energy density more precise. The (pointwise) energy conditions take the form of assertions that various linear combinations of the components of the stress-energy tensor (at any specified point in spacetime) should be positive, or at least non-negative. The so-called “averaged energy conditions” are somewhat weaker, they permit localized violations of the energy conditions, as long as “on average” the energy conditions hold when integrated along null or timelike geodesics.

The refinement of the energy conditions paralleled the development of powerful mathematical theorems, such as the singularity theorems (guaranteeing, under certain circumstances, gravitational collapse and/or the existence of a big bang singularity), the positive energy theorem, the non-existence of traversable wormholes (topological censorship), and limits on the extent to which light cones can “tip over” in strong gravitational fields (superluminal censorship). All these theorems require some form of energy condition, some notion of positivity of the stress-energy tensor as an input hypothesis, and the variety of energy conditions in use in the relativity community is driven largely by the technical requirements of how much you have to assume to easily prove certain theorems.

Over the years, opinions have changed as to how fundamental some of the specific energy conditions are. One particular energy condition has now been completely abandoned, and there is general agreement that another is on the verge of being relegated to the dustbin. There are however, some more general issues that make one worry about the whole programme. Specifically:

(1) Over the last decade or so it has become increasingly obvious that there are subtle quantum effects that are capable of violating all the energy conditions, even the weakest of the standard energy conditions. Now because these are quantum effects, they are by definition small (proportional to \( \hbar \)) so the general consensus for many years was to not worry too much.

(2) More recently, it has become clear that there are quite reasonable looking classical systems, field theories that are compatible with all known experimental data, and that are in some sense very natural from a quantum field theory point of view, which violate all the energy conditions. Because these are now classical violations of the energy conditions they can be made arbitrarily large, and seem to lead to rather weird physics. (For instance, it is possible to demonstrate that Lorentzian-signature traversable wormholes arise as classical solutions of the field equations.)
Faced with this situation, you will either have to learn to live with some rather peculiar physics, or you will need to make a radical reassessment of the place of the energy conditions in general relativity and cosmology.

2 Energy conditions

To set some basic nomenclature, the pointwise energy conditions of general relativity are:

*Trace energy condition* (TEC), now abandoned.
*Strong energy condition* (SEC), almost abandoned.
*Null energy condition* (NEC).
*Weak energy condition* (WEC).
*Dominant energy condition* (DEC).

All of these energy conditions can be modified by averaging along null or timelike geodesics.

The trace energy condition is the assertion that the trace of the stress-energy tensor should always be negative (or positive depending on metric conventions), and was popular for a while during the 1960’s. However, once it was realized that stiff equations of state, such as those appropriate for neutron stars, violate the TEC this energy condition fell into disfavor. It has now been completely abandoned and is no longer cited in the literature — we mention it here as a concrete example of an energy condition being outright abandoned.

The strong energy condition is currently the subject of much discussion, sometimes heated. (1) The most naive scalar field theory you can write down, the minimally coupled scalar field, violates the SEC and indeed curvature-coupled scalar field theories also violate the SEC; there are fermionic quantum field theories where interactions engender SEC violations of specific models of point-like particles with two-body interactions that violate the SEC. (2) If you believe in cosmological inflation, the SEC must be violated during the inflationary epoch, and the need for this SEC violation is why inflationary models are typically driven by scalar inflaton fields. (3) If you believe the recent observational data regarding the accelerating universe, then the SEC is violated on cosmological scales right now! (4) Even if you are somewhat more conservative, and regard the alleged present-day acceleration of the cosmological expansion as “unproven”, the tension between the age of the oldest stars and the measured present-day Hubble parameter makes it very difficult to avoid the conclusion that the SEC must have been violated in the cosmologically recent past, sometime between redshift 10 and the present. Under the circumstances it would be rather quixotic to take the SEC too seriously as a fundamental guide.

In contrast, the null, weak, and dominant energy conditions are still extensively used in the general relativity community. The weakest of these is the NEC, and it is in many cases also the easiest to work with and analyze. The standard wisdom for many years was that all reasonable forms of matter should at least satisfy the NEC. After it became clear that the NEC (and even the ANEC) was violated by quantum effects two main lines of retrenchment developed:

(1) Many researchers simply decided to ignore quantum mechanics, relying on the classical NEC to prevent grossly weird physics in the classical regime, and hoping that the long sought for quantum theory of gravity would eventually deal with the quantum problems. This is not a fully satisfactory response in that NEC violations already show up in semiclassical quantum gravity (where you quantize the matter fields and keep gravity classical), and show up at first order in \( \hbar \). Since semiclassical quantum gravity is certainly a good approximation in our immediate neighborhood, it is somewhat disturbing to see widespread (albeit small) violations of the energy conditions in the here and now. Many experimental physicists and observational astrophysicists react quite heatedly when the theoreticians tell them that according to our best calculations there should be plenty of “negative energy” (energy densities less than that of the flat-space Minkowski vacuum) out there in the real universe. However, to avoid the conclusion that quantum effects can and do lead to locally negative energy densities, and even violations of the ANEC, requires truly radical surgery to modern physics, and in particular we would have to throw away almost all of quantum field theory.

(2) A more nuanced response is based on the Ford–Roman Quantum Inequalities. These inequalities, which are currently still being developed and extended, and whose implications are still a topic of considerable activity, are based on the facts that while quantum-induced violations of the energy conditions are widespread they are also small, and on the observation that a negative energy in one place and time always seems to be compensated for (indeed, over-compensated for) by positive energy elsewhere in spacetime. This is the so-called Quantum Interest Conjecture. While the positive pay-back is not enough to prevent violation of the ANEC (based on averaging the NEC along a null geodesic) the hope is that it will be possible to prove some type of space-time averaged energy condition from first principles, and that such a space-time averaged energy condition might be sufficient to enable us to recover the singularity/positive-mass/censorship theorems under weaker hypotheses than currently employed.

A fundamental problem for this type of approach is the more recent realization that there are also serious classical violations of the energy conditions. These classical violations can easily be made arbitrarily large,
3 Scalar Fields: Background

Scalar fields play a somewhat ambiguous role in modern theoretical physics: on the one hand they provide great toy models, and are from a theoretician's perspective almost inevitable components of any reasonable model of empirical reality; on the other hand the direct experimental/observational evidence is spotty.

The only scalar fields for which we have really direct "hands-on" experimental evidence are the scalar mesons (pions \( \pi \); kaons \( K \); and their "charmed", "truth" and "beauty" relatives, plus a whole slew of resonances such as the \( \eta, f_0, \eta', a_0, \ldots \)). Not one of these particles are fundamental, they are all quark-antiquark bound states, and while the description in terms of scalar fields is useful when these systems are probed at low momenta (as measured in their rest frame) we should certainly not continue to use the scalar field description once the system is probed with momenta greater than \( h/(\text{bound state radius}) \). In terms of the scalar field itself, this means you should not trust the scalar field description if gradients become large, if

\[
||\nabla \phi|| > \frac{||\phi||}{\text{bound state radius}}.
\]  

Similarly you should not trust the scalar field description if the energy density in the scalar field exceeds the critical density for the quark-hadron phase transition. Thus scalar mesons are a mixed bag: they definitely exist, and we know quite a bit about their properties, but there are stringent limitations on how far we should trust the scalar field description.

The next candidate scalar field that is closest to experimental verification is the Higgs particle responsible for electroweak symmetry breaking. While in the standard model the Higgs is fundamental, and while almost everyone is firmly convinced that some Higgs-like scalar field exits, there is a possibility that the physical Higgs (like the scalar mesons) might itself be a bound state of some deeper level of elementary particles (e.g., technicolor and its variants). Despite the tremendous successes of the standard model of particle physics we do not (currently) have direct proof of the existence of a fundamental Higgs scalar field.

A third candidate scalar field of great phenomenological interest is the axion: it is extremely difficult to see how one could make strong interaction physics compatible with the observed lack of strong CP violation, without something like an axion to solve the so-called "strong CP problem". Still, the axion has not yet been directly observed experimentally.

A fourth candidate scalar field of phenomenological interest specifically within the astrophysics/cosmology community is the so-called "inflaton". This scalar field is used as a mechanism for driving the anomalously fast expansion of the universe during the inflationary era. While observationally it is a relatively secure bet that something like cosmological inflation (in the sense of anomalously fast cosmological expansion) actually took place, and while scalar fields of some type are the only known reasonable way of driving inflation, we must again admit that direct observational verification of the existence of the inflaton field (and its variants, such as quintessence) is far from being accomplished.

A fifth candidate scalar field of phenomenological interest specifically within the general relativity community is the so-called "Brans–Dicke scalar". This is perhaps the simplest extension to Einstein gravity that is not ruled out by experiment. (It is certainly greatly constrained by observation and experiment, and there is no positive experimental data guaranteeing its existence, but it is not ruled out.) The relativity community views the Brans–Dicke scalar mainly as an excellent testing ground for alternative ideas and as a useful way of parameterizing possible deviations from Einstein gravity. (And experimentally and observationally, Einstein gravity still wins.)

Finally, the membrane-inspired field theories (low-energy limits of what used to be called string theory) are literally infested with scalar fields. In membrane theories it is impossible to avoid scalar fields, with the most ubiquitous being the so-called "dilaton". However, the dilaton field is far from unique, in general there is a large class of so-called "moduli" fields, which are scalar fields corresponding to the directions in which the background spacetime geometry is particularly "soft" and easily deformed. So if membrane theory really is the fundamental theory of quantum gravity, then the existence of fundamental scalar fields is automatic, with the field theory description of these fundamental scalars being valid at least up to the Planck scale, and possibly higher.

(For good measure, by making a conformal transformation of the spacetime geometry it is typically possible to put membrane-inspired scalar fields into a framework which closely parallels that of the generalized Brans–Dicke fields. Thus there is a potential for much cross-pollination between Brans–Dicke inspired variants of general relativity and membrane-inspired field theories.)

So overall, we have excellent theoretical reasons to expect that scalar field theories are an integral part of reality, but the direct experimental/observational verification of the existence of fundamental fields is still an
open question. Nevertheless, we think it fair to say that there are excellent reasons for taking scalar fields seriously, and excellent reasons for thinking that the gravitational properties of scalar fields are of interest cosmologically, astrophysically, and for providing fundamental probes of general relativity.

4 Scalar Fields and Gravity

In setting up the formalism for a scalar field coupled to gravity we first need to specify a few conventions: the metric signature will be taken to be \((-__,++,+)\) and we adopt Landau–Lifshitz spacelike conventions (LLSC). This is equivalent to MTW conventions with latin indices for the tensors, and is equivalent to Flanagan–Wald in the MTW classification this corresponds to \((+,+,{\text{Riemann}},+{\text{Einstein}})\). We take the total action to be:

\[ S = S_{\phi} + S_{\text{bulk}}. \]  

(3)

The gravity action is standard

\[ S_{\phi} = \int d^4 x \sqrt{-g} \frac{1}{2} \kappa R, \]

(4)

with the ordinary Newton constant being defined by

\[ \kappa = \frac{c^4}{8 \pi G_{\text{Newton}}}. \]

(5)

We shall permit the scalar field to exhibit an arbitrary curvature coupling \(\xi\)

\[ S_{\phi} = \int d^4 x \sqrt{-g} \left( -\frac{1}{2} \left[ (\nabla \phi)^2 + \xi R \phi^2 \right] - V(\phi) \right). \]

(6)

Finally the action for ordinary bulk matter is taken as

\[ S_{\text{bulk}} = \int d^4 x \sqrt{-g} f(\phi) L_{\text{matter}}. \]

(7)

Here we assume \(f(\phi)\) is algebraic, and that \(L_{\text{matter}}\) does not involve \(\phi\). We also assume for technical reasons that \(L_{\text{matter}}\) does not contain any terms involving second derivatives of the metric. (For example, let \(L_{\text{matter}}\) be the Lagrangian of the ordinary standard model of particle physics.) Under these circumstances the equations of motion (EOM) for gravity can be written

\[ \kappa G_{ab} = T_{ab}^{\phi} + f(\phi) T_{\text{matter}}^{\phi}. \]

(8)

The EOM of the \(\phi\) field are

\[ (\nabla^2 - \kappa R)\phi - V'(\phi) + f'(\phi) L_{\text{matter}} = 0. \]

(9)

The EOM for the bulk matter fields can be phrased as

\[ \nabla_b \left[ f(\phi) T_{ab}^{\text{matter}} \right] = 0. \]

(10)

There are a few hidden subtleties here: First because of the curvature coupling term \(\xi R \phi^2\) the stress-energy tensor for the scalar field contains a term proportional to the Einstein tensor

\[ T_{ab}^{\phi} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} (\nabla \phi)^2 - g_{ab} V(\phi) \]

\[ + \xi \left[ g_{ab} \phi^2 - 2 \nabla_a (\phi \nabla_b \phi) + 2 g_{ab} \nabla^c (\phi \nabla_c \phi) \right]. \]

(11)

Second, the way we have defined \(T_{\text{matter}}^{\phi}\) it is to be calculated from \(S_{\text{bulk}}\) by simply ignoring the factor \(f(\phi)\).

Because the RHS contains a term proportional to the Einstein tensor it is best to rearrange the gravity EOM by isolating all such terms on the LHS, in which case the gravity EOM is equivalent to

\[ \kappa G_{ab} = \left[ T_{\text{eff}}^{\phi} \right]_{ab} + \frac{\kappa}{\kappa - \xi \phi^2} T_{ab}^{\text{matter}}. \]

(12)

Here the “effective” stress-energy for the scalar field is

\[ \left[ T_{\text{eff}}^{\phi} \right]_{ab} = \frac{\kappa}{\kappa - \xi \phi^2} \left\{ \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} (\nabla \phi)^2 - g_{ab} V(\phi) \right. \]

\[ - \left. \xi \left[ 2 \nabla_a (\phi \nabla_b \phi) - 2 g_{ab} \nabla^c (\phi \nabla_c \phi) \right] \right\}. \]

(13)

Note that this effective stress-energy can change sign for certain values of the scalar field, which is our first hint of peculiar physics. More importantly, observe that it is the energy conditions defined in terms of this effective stress-energy tensor that are the physically interesting ones: energy conditions imposed upon this effective stress-energy imply constraints on the Einstein tensor, placing constraints on the curvature, which is what is really needed for deriving singularity/positive-mass/censorship theorems.

In addition, the “effective” gravitational coupling of “normal matter” to the gravitational field is

\[ \kappa_{\text{eff}} = \frac{\kappa}{\kappa - \xi \phi^2}. \]

(14)

In terms of Newton’s constant

\[ G_{\text{Newton}}^{\phi}(\phi) = G_{\text{Newton}} f(\phi) \frac{\kappa}{\kappa - \xi \phi^2}. \]

(15)

It is particularly convenient to pick \(f(\phi) = (\kappa - \xi \phi^2)/\kappa\). With this simplifying choice bulk matter couples to gravity in the ordinary way, while the only gravitational peculiarities are now concentrated in the gravity-\(\phi\) sector.

In the remainder of the technical discussion we shall, for simplicity, treat the normal matter in the test-field limit. That is, whatever normal matter is present is assumed to be sufficiently diffuse to not appreciably affect the spacetime geometry, while test particles of normal
matter can still be used as convenient probes of the space-
time geometry. Making this test-matter approximation
thus simplifies life to the extent that we are looking at a
gravity EOM
\[ \kappa G_{ab} = [T_{\text{eff}}]_{ab}, \] (16)
coupled to the scalar EOM
\[ (\nabla^2 - \xi R)\phi - V'(\phi) = 0. \] (17)

This system is now sufficiently simple that a number of
exact analytic solutions are known. We shall not
re-derive any of these exact analytic solutions but instead
will quote them as examples when we want to illustrate
some aspect of the generic situation.

5 Scalar Fields and Energy Conditions

5.1 SEC

Technically the SEC is defined (in 4 dimensions) by using
the trace-reversed stress energy tensor
\[ T_{ab} \equiv T_{ab} - \frac{1}{2} g_{ab} \text{tr}(T). \] (18)
The SEC is then the assertion that for any timelike vector
\[ T_{ab} v^a v^b \geq 0. \] (19)
The reason for wanting this condition is that, via the
Einstein equations, the SEC would imply the Ricci conver-
gence condition
\[ R_{ab} v^a \geq 0. \] (20)

This convergence condition on the Ricci curvature tensor
is then used to prove that nearby timelike geodesics are
always focussed towards each other, and this focussing
lemma is then a first critical step in proving singularity
theorems and the like. Unfortunately if you actually cal-
culate this quantity for the effective stress-energy of the
scalar field you find
\[ [T_{\text{eff}}^\phi]_{ab} v^a v^b = \frac{\kappa}{\kappa - \xi \phi^2} \left\{ (v^a \nabla_a \phi)^2 - V(\phi) \right. \]
\[ \left. - \xi \left[ 2v^a v^b \nabla_a (\phi \nabla_b \phi) - \nabla_c (\phi \nabla_c \phi) \right] \right\}. \] (21)

It’s very easy to make this quantity negative. There is
a specific example in Hawking–Ellis\cite{Hawking:1974sw} page 95: flat space
with \( \xi = 0 \) and \( V(\phi) = \frac{1}{2} m^2 \phi^2 \), but the phenomenon is
much more general. For minimal coupling (\( \xi = 0 \)) we see
\[ [T_{\text{eff}}^\phi]_{ab} v^a v^b = (v^a \nabla_a \phi)^2 - V(\phi). \] (22)

Thus any static field with a positive potential violates the
SEC. (For example, a slowly changing scalar with a mass
or quartic self-interactions, this is the simplest toy model
for cosmological inflation and/or quintessence; also any
positive cosmological constant violates the SEC.)

Adding a non-minimal coupling (\( \xi \neq 0 \)) is not help-
ful, though it does make the algebra messier. (In particu-
lar, adding non-minimal coupling will not by itself switch
off cosmological inflation.) The key point regarding the
SEC is that a positive potential energy \( V(\phi) > 0 \) tends
to violate the SEC, and you can always think of getting
SEC violations by going to a region of field space with
high potential energy.

Though the SEC violations we are talking about here
are most commonly used in building models of cosmolog-
ical inflation, perhaps the most serious long term issue is
that the singularity theorem we use to prove the existence
of a big bang singularity in FLRW cosmologies depends
crucially on the SEC. Not only do SEC violations per-
mit cosmological inflation but they also open the door
to replacing the big bang singularity with a “bounce”,
or “Tolman wormhole”\cite{Hawking:1974sw}. In fact it is now known that
genERICALLY the SEC is the only energy condition you need
to violate to get a bounce, and that it is still possible to
satisfy all the other energy conditions at and near the
bounce.\cite{Hawking:1974sw} (In counterpoint, if one replaces the SEC by
strong enough inhomogeneity assumptions then it is pos-
sible to prove that certain classes of chaotic inflationary
models must nevertheless possess a big-bang singularity.)

Note that scalar fields do not guarantee the preven-
tion of a big bang singularity, they merely raise this pos-
sibility — and this issue is interesting enough to warrant
further investigation. Another interesting point is that
the reason that most inflationary models are based on
minimally coupled scalars (\( \xi = 0 \)) is purely a histori-
ical one, there simply was no need to go to non-minimal
coupling to get the SEC violations that are needed for
cosmological inflation, and it is only once you get down
to rather specific model building that non-minimal cou-
bining becomes interesting.

5.2 NEC

The NEC has two great advantages over the SEC: it is
the simplest energy condition to deal with algebraically,
and because it is the weakest pointwise energy condition
it leads to the strongest theorems. The NEC is the as-
sertion that for any null vector \( k^a \) we should have
\[ T_{ab} k^a k^b \geq 0. \] (23)
Unfortunately when we actually calculate this quantity for the scalar field we find
\[ [T^\phi_{\text{eff}}]_{ab} k^a k^b = \frac{\kappa}{\kappa - \xi \phi^2} \left\{ (k^a \nabla_a \phi)^2 - \xi \left[ 2k^a \nabla_a (\phi \nabla_b \phi) \right] \right\}. \]

(24)

For minimal coupling
\[ [T^\phi_{\text{eff}}]_{ab} k^a k^b = (k^a \nabla_a \phi)^2 \geq 0. \]

(25)

The accident that the NEC is satisfied for minimal coupling led to a situation where researchers just did not look under enough rocks to see where the problems lay.

For non-minimal coupling we start, as a convenience, by extending \( k \) to be a geodesic vector field around the point of interest, so that \( k^a \nabla_a k^b = 0 \). Then using the affine parameter \( \lambda \) we have \( k^a \nabla_a = d/d\lambda \) so that
\[ [T^\phi_{\text{eff}}]_{ab} k^a k^b = \frac{\kappa}{\kappa - \xi \phi^2} \left\{ \left( \frac{d\phi}{d\lambda} \right)^2 - \xi \left( \frac{d^2[\phi^2]}{d\lambda^2} \right) \right\}. \]

(26)

Pick any local extremum of \( \phi \) along the null geodesic, then
\[ [T^\phi_{\text{eff}}]_{ab} k^a k^b = -\frac{\kappa}{\kappa - \xi \phi^2} \left\{ \xi \left( \frac{d^2[\phi^2]}{d\lambda^2} \right) \right\}. \]

(27)

It is easy to make this negative (for any \( \xi \neq 0 \)). If \( \xi < 0 \) consider any local maximum of the field \( \phi \), the NEC is violated. If \( \xi > 0 \) and \( \phi < \sqrt{\kappa/\xi} \) then any local minimum does the job, while if \( \xi > 0 \) and \( \phi > \sqrt{\kappa/\xi} \) one again needs a local maximum of \( \phi \) to violate the NEC.

Now classical violations of the NEC are much more disturbing than classical violations if the SEC. In particular, traversable Lorentzian-signature wormholes are known to be associated with violations of the NEC (and ANEC) \( \square \) as are warp-drives \( \square \), time machines \( \square \), and similar exotica — this should start to make you feel just a little nervous.

### 5.3 ANEC

The ANEC is technically much more interesting. Pick some complete null geodesic \( \gamma \) and consider the integral
\[ \mathcal{I} = \oint [T^\phi_{\text{eff}}]_{ab} k^a k^b \, d\lambda. \]

(28)

Then
\[ \mathcal{I} = \oint \frac{\kappa}{\kappa - \xi \phi^2} \left\{ \left( \frac{d\phi}{d\lambda} \right)^2 - 2\xi \frac{d}{d\lambda} \left( \phi \frac{d\phi}{d\lambda} \right) \right\} \, d\lambda. \]

(29)

Integrate by parts, discarding the boundary terms (that is, assume sufficiently smooth asymptotic behaviour)
\[ \mathcal{I} = \oint \frac{\kappa}{\kappa - \xi \phi^2} \left\{ \left( \frac{d\phi}{d\lambda} \right)^2 + \frac{4\xi^2 \phi^2 (d\phi/d\lambda)^2}{\kappa - \xi \phi^2} \right\} \, d\lambda. \]

(30)

Now assemble the pieces:
\[ \mathcal{I} = \oint \frac{\kappa \xi (1 - 4\xi \phi^2)}{(\kappa - \xi \phi^2)^2} \left( \frac{d\phi}{d\lambda} \right)^2 \, d\lambda. \]

(31)

The integrand is not positive definite, and ANEC can be violated, provided there is a region along the geodesic where
\[ \xi (1 - 4\xi \phi^2) > \kappa. \]

(32)

This can only happen for \( \xi \in (0, 1/4) \), so there is something very special about this range of curvature couplings. In particular, \( 1/6 \in (0, 1/4) \), so conformal coupling in \( d = 4 \) lies in this range. This is important because there are technical issues in quantum field theory that seem to almost automatically imply that real physical scalars should be conformally coupled. (Conformal coupling \( \xi = 1/6 \) is typically an infrared fixed point of the renormalization group flow, furthermore setting \( \xi = 1/6 \) and then going to flat spacetime automatically reproduces the so-called “new-improved stress-energy tensor”, a stress-energy tensor that is much better behaved in quantum field theory than the unimproved minimally-coupled stress-energy tensor.)

Also note that ANEC violations require
\[ \phi^2 > \frac{\kappa}{\xi (1 - 4\xi)} > \frac{\kappa}{\xi}. \]

(33)

This implies that the prefactor in the effective stress-energy tensor for scalars must go through a zero and become negative in order to get ANEC violations. Thus ANEC violations are considerably more constrained than NEC violations, and the ANEC violating regions are always associated with regions where the effective stress-energy has “reversed sign”.

Furthermore in the ANEC violating region
\[ \phi^2 > \frac{\kappa}{\xi (1 - 4\xi)} > 16\kappa, \]

(34)

implying that the scalar field must take on enormous (super-Planckian) values in order to provide ANEC violations. Now super-Planckian values for scalar fields are not that unusual, they are part and parcel of many (though not all) inflationary models, and the standard lore in cosmology is to not worry about a super-Planckian value for the scalar field unless the energy density is also super-Planckian.
6 Conclusions

Even with the caveats provided above, the fact that the ANEC can be violated by classical scalar fields is significant and important — in particular the ANEC is the weakest of the energy conditions in current use, and violating the ANEC short circuits all the standard singularity/positive-mass/censorship theorems. This observation piqued our interest and we decided to see just how weird the physics could get once you admit scalar fields into your models.

In particular, it is by now well-known that traversable wormholes are associated with violations of the NEC and ANEC, so we became suspicious that there might be an explicit class of exact traversable wormhole solutions to the coupled gravity-scalar field system. In a recent paper, we presented such a class of solutions — for algebraic simplicity we restricted attention to conformal coupling ($\xi \rightarrow 1/6$) and set the potential to zero $V(\phi) \rightarrow 0$. We found a three-parameter class of exact solutions to the coupled Einstein–scalar field equations with the three parameters being the mass, the scalar charge, and the value of the scalar field at infinity. Within this three-dimensional parameter space we found a two-dimensional subspace that corresponds to exact traversable wormhole solutions.

The simplifications of conformal coupling and zero potential were made only for the sake of algebraic simplicity and we expect that there are more general classes of wormhole solutions waiting to be discovered. In addition, deviations from spherical symmetry are also of interest. It should be borne in mind that although the present analysis indicates that there must be super-Planckian scalar fields somewhere in the wormhole spacetime, this does not necessarily mean that a traveller needs to traverse one of these super-Planckian regions to cross to the other side: If the wormhole is not spherically symmetric it is typically possible to minimize the spatial extent of the regions of peculiar physics. Work on these topics is continuing.

Now traversable wormholes, while certainly exotic, are by themselves not enough to get the physics community really upset: The big problem with traversable wormholes is that if you manage to acquire even one inter-universe traversable wormhole then it seems almost absurdly easy to build a time machine — and this does get the physics community upset. There is a conjecture (Hawking’s Chronology Protection Conjecture) that quantum physics will save the universe by destabilizing the wormhole just as the time machine is about to form, but it must certainly be emphasized that there is considerable uncertainty as to how serious these causality problems are.

There are two responses to the current state of affairs: either we can learn to live with wormholes, and other strange physics engendered by energy condition violations, or we need to patch up the theory. We cannot just say that some improved version of the energy conditions will do the job for us, since we already have an explicit solution of the Einstein equations that contains traversable wormholes — we would have to do something more drastic, like attack the notion of a scalar field, or forbid conformal couplings [we would need to forbid the entire range $\xi \in (0,1/4)$], or forbid super-Planckian field values — each one of these particular possibilities however is in conflict with cherished notions of some segments of the particle physics/ membrane theory/ relativity/ astrophysics communities. Most physicists would be loathe to give up the notion of a scalar field, and conformal coupling is so natural that it is difficult to believe that banning it would be a viable option. Banishing super-Planckian field values is more plausible, but this runs afoot of at least some segments of the inflationary community.

References

1. S.W. Hawking and G.F.R. Ellis, The large scale structure of spacetime, (Cambridge University Press, England, 1973).
2. R.M. Wald, General Relativity, (University of Chicago Press, Chicago, 1984).
3. M. Visser, Lorentzian wormholes, (AIP Press, New York, 1995).
4. E.E. Flanagan and R.M. Wald, Phys. Rev. D 54, 6233 (1996).
5. C. Barceló and M. Visser, Phys. Lett. B466, 127 (1999).
6. Ya.B. Zeldovich, and I.D. Novikov, Stars and Relativity (Relativistic Astrophysics, Vol. 1), (University of Chicago Press, Chicago, 1971), see esp. p. 197.
7. B. Rose, Class. Quantum Grav. 3, 975 (1986); 4, 1019 (1987).
8. L. Parker and Y. Wang, Phys. Rev. D 42, 1877 (1990).
9. M. Visser, Science 276, 88 (1997).
10. L.H. Ford and T.A. Roman, Phys. Rev. D 51, 4277 (1995); D 60, 104018 (1999);
L.H. Ford, M.J. Pfenning, and T.A. Roman, Phys. Rev. D 57, 4839 (1998).
11. Particle Data Group, http://pdg.lbl.gov. Euro. Phys. J. C3, 1 (1998).
12. C.W. Misner, K.S. Thorne, and J.A. Wheeler, Gravitation, (Freeman, San Francisco, 1973).
13. L.Z. Fisher, Zh. Exp. Th. Fiz. 18, 636 (1948); gr-qc/9911108.
14. O. Bergmann and R. Leipnik, Phys. Rev. 107,
15. H. Ellis, J. Math. Phys. 14, 104 (1973); Errata 15, 520 (1974).

16. D. Hochberg, C. Molina–París, and M. Visser, Phys. Rev. D 59, 044011 (1999); C. Molina–París and and M. Visser, Phys. Lett. B455, 90 (1999).

17. M.S. Morris and K.S. Thorne, Am. J. Phys. 56, 395 (1988).

18. H. Bondi, Rev. Mod. Phys. 29, 423 (1957).

19. M. Alcubierre, Class. Quantum Grav. 11, L73 (1994).

20. M. Visser, Phys. Rev. D 39, 3182 (1989).