THE CHEEGER CONSTANT OF A QUANTUM GRAPH

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ABSTRACT. We review the theory of Cheeger constants for graphs and quantum graphs and their present and envisaged applications.

The Laplacian matrix $L$ of a graph $G$ (without loops) has a long history, appearing and being rediscovered several times; be it as related to electrical circuits [Kir45], to the discretisation of PDEs [Boo60], to the theory of time-continuous Markov chains [Kat54] or to the formalism of Dirichlet forms [BD59]. In the 1970s Fiedler made the case for the study of the lowest non-zero eigenvalue $\lambda_1(G)$ of $L$: since the multiplicity of 0 as an eigenvalue of $L$ is equal to the number of connected components of the underlying graph $G$, one may conjecture that if 0 is a simple eigenvalue, then the smaller $\lambda_1(G)$, the closer the graph is to being disconnected. And indeed, the following important relation between $\lambda_1(G)$ and the edge connectivity $e(G)$ of $G$ (i.e., the minimal number of edges that have to be removed to make $G$ disconnected) was proved in [Fie73, § 4].

Proposition 1 (Fiedler 1973). Let $G$ be a connected graph on $V$ vertices. Then

$$2e(G) \left(1 - \cos \frac{\pi}{V}\right) \leq \lambda_1(G) \leq e(G).$$

Thus, $e(G)$ and $\lambda_1(G)$ have the same asymptotic behaviour, although the scaling of $e(G)$ is sub-optimal as it penalises smaller graphs. Adapting an idea developed in [Che70] for manifolds, several authors have studied since the beginning of the 1980s a renormalised version of $e(G)$: the Cheeger constant $h(G)$ of $G$ is

$$h(G) := \inf \frac{\lvert \partial S \rvert}{\min\{\lvert S \rvert, \lvert S^c \rvert\}},$$

where $\inf$ is taken over all vertex sets $S$ and $\partial S$ is the set of all edges having exactly one endpoint in $S$ [Dod84, AM85, Chu97].

Proposition 2 (Dodziuk 1984, Alon–Milman 1985). Let $G$ be a connected graph of maximal degree $\deg_{\max}$. Then

$$\frac{h^2(G)}{2\deg_{\max}(G)} \leq \lambda_1(G) \leq 2h(G).$$

These estimates thus provide a variational relaxation of the NP-hard problem of determining $h(G)$ [BH09].
Cheeger-type inequalities similar to (1) hold for the Laplacian on quantum graphs as well: recall that a quantum graph $G$ is obtained from a graph $G$ by identifying each edge $e$ with an interval $(0, \ell_e)$. The standard Laplacian on $G$ is then a collection of second derivative operators on each edge, complemented with continuity and Kirchhoff (no flux loss) conditions in each edge [BK13, Mug14]. Nicaise introduced in [Nic87] a Cheeger-type constant for quantum graphs by

$$h(G) := \inf \frac{|\partial S|}{\min\{|S|, |S^c|\}}$$

where $\inf$ is taken over all Lebesgue measurable open subsets $S$ of the quantum graph: $|\partial S|$ is the number of edges that depart from such $S$ and $|S|$ is its measure; [DPR16] Thm. 6.2 characterises $h(G)$ as the lowest non-zero eigenvalue of the $1$-Laplacian on $G$. The straightforward estimate $h(G) \geq \frac{2}{L}$ holds for all quantum graphs $G$ of total length $L = \sum_{e \in E} \ell_e < \infty$; while the upper estimate $h(G) \leq \frac{2E}{L}$ – with equality (among others) for flower graphs with edges of equal length – follows from [DPR16] Thm. 6.2 and (a straightforward extension of) [KKMM16] Lemma 2.3. Here $E$ is the number of essential edges, i.e., the number of edges in $G$ once vertices of degree $2$ (irrelevant for the standard Laplacian) have been removed.

**Proposition 3.** Let $G$ be a connected quantum graph with $E$ essential edges. Then the lowest non-zero eigenvalue $\lambda_1(G)$ of the standard Laplacian on $G$ satisfies

$$\max\left\{ \frac{h^2(G)}{4}, \frac{\pi^2 h^2(G)}{E^2} \right\} \leq \lambda_1(G) \leq \frac{\pi^2 E^2 h^2(G)}{4}.$$

The lower estimates in (2) follow from [Nic87] Théo. 3.2 and [Nic87] Théo. 3.1 along with $h(G) \leq \frac{2E}{L}$; the upper estimate follows from [KKMM16] Thm. 4.2 and $h(G) \geq \frac{2}{L}$. We also mention the different but related upper estimate in [Kur13] Thm. 1. In analogy with a result obtained in [Par15] for convex subsets of $\mathbb{R}^2$, we conjecture that $\frac{\pi^2 h^2(G)}{4} \leq \lambda_1(G)$.

**Remark 4.** 1) If $G$ is an interval, then $\lambda_1(G) = \frac{\pi^2}{L^2} = \pi^2 h^2(G)$. If $G$ is a flower, then $\lambda_1(G) = \frac{\pi^2}{L^2} = \pi^2 h^2(G)$. Unfortunately, the dependence on $E$ cannot in general be dropped in the upper estimate in (2). symmetric flower dumbbells (see Figure 1) obviously have Cheeger constant $\frac{2}{L}$, as the optimal Cheeger set $S$ is obtained by just cutting $G$ in the middle. At the same time, by adding more and more petals and simultaneously shortening all of them while making the handle shorter and shorter, one can produce symmetric flower dumbbells with same total length but arbitrarily high $\lambda_1(G)$.

2) While we do not know whether the upper estimate in (2) is sharp, symmetric flower dumbbells with $E = 2m + 1$ edges satisfy

$$\lambda_1(G) \approx \frac{\pi^2 m^2}{4} = \frac{\pi^2 (E - 1)^2 h^2(G)}{4},$$

which is the corresponding value of $\lambda_1$ for a flower with $E - 1$ edges, provided the symmetric flower dumbbell’s handle is arbitrarily short.
The main fascinating feature of the Cheeger constant of quantum graphs is its hybrid nature, partly combinatorial and partly metric (its numerator and denominator, respectively), in sharp contrast to its counterparts for manifolds and graphs. But is it meaningful at all to consider the Cheeger constant of a quantum graph? From the point of view of theoretical computer science the lowest non-zero eigenvalue is an elementary object that can be easily determined by variational methods and can in turn help to estimate the Cheeger constant – the really interesting quantity, for the purpose of machine learning.

We maintain that quantum graphs are not unnecessarily complicated gadgets, but rather useful tools delivering additional information. As an example, let us consider the first two quantum graphs in Figure 1, each of whose intervals is assumed to have unit length. One sees that the Cheeger constant of the cycle is $\frac{4}{5}$, while the butterfly has Cheeger constant $\frac{2}{3}$. On the other hand, both underlying discrete graphs have Cheeger constant 1. We argue that the information yielded by $h(G)$ may in critical cases be complemented by $h(\mathcal{G})$, upon turning a graph $G$ into a quantum graph $\mathcal{G}$ with edges of unit length, whenever the interaction-based description offered by a quantum graph is as relevant as the agent-based description offered by a graph.

REFERENCES

[AM85] N. Alon and V. D. Milman. $\lambda_1$, isoperimetric inequalities for graphs, and superconcentrators. J. Combin. Theory Ser. B, 38:73–88, 1985.

[BD59] A. Beurling and J. Deny. Dirichlet spaces. Proc. Natl. Acad. Sci. USA, 45:208–215, 1959.

[BH09] T. Bühler and M. Hein. Spectral clustering based on the graph $p$-Laplacian. In Proc. 26th Annual Int. Conf. Mach. Learning, pages 81–88, New York, 2009. ACM.

[BK13] G. Berkolaiko and P. Kuchment. Introduction to Quantum Graphs, volume 186 of Math. Surveys and Monographs. Amer. Math. Soc., Providence, RI, 2013.

[Boo60] G. Boole. A Treatise on the Calculus of Finite Differences. Macmillan, Cambridge, 1860.

[Che70] J. Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. In R.C. Gunning, editor, Problems in Analysis, pages 195–199, Princeton, NJ, 1970. Princeton Univ. Press.
[Chu97] F.R.K. Chung. *Spectral Graph Theory*, volume 92 of *Reg. Conf. Series Math. Amer. Math. Soc.*, Providence, RI, 1997.

[Dod84] J. Dodziuk. Difference equations, isoperimetric inequality and transience of certain random walks. *Trans. Amer. Math. Soc.*, 284:787–794, 1984.

[DPR16] L. M. Del Pezzo and J. D. Rossi. The first eigenvalue of the $p$-Laplacian on quantum graphs. *Analysis and Math. Phys.*, DOI:10.1007/s13324-016-0123-y, 2016.

[Fie73] M. Fiedler. Algebraic connectivity of graphs. *Czech. Math. J.*, 23:298–305, 1973.

[Kat54] T. Kato. On the semi-groups generated by Kolmogoroff’s differential equations. *J. Math. Soc. Jap.*, 6:1–15, 1954.

[Kir45] G. Kirchhoff. Ueber den Durchgang eines elektrischen Stromes durch eine Ebene, insbesondere durch eine kreisförmige. *Ann. Physik*, 140:497–514, 1845.

[KKMM16] J. B. Kennedy, P. Kurasov, G. Malenová, and D. Mugnolo. On the spectral gap of a quantum graph. *Ann. Henri Poincaré A*, DOI:10.1007/s00023-016-0460-2, 2016.

[Kur13] P. Kurasov. On the spectral gap for Laplacians on metric graphs. *Acta Phys. Pol. A*, 124:1060–1062, 2013.

[Mug14] D. Mugnolo. *Semigroup Methods for Evolution Equations on Networks*. Springer-Verlag, Berlin, 2014.

[Nic87] S. Nicaise. Spectre des réseaux topologiques finis. *Bull. Sci. Math., II. Sér.*, 111:401–413, 1987.

[Par15] E. Parini. Reverse cheeger inequality for planar convex sets. *arXiv:1501.04520*, 2015.

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