TORUS BUNDLES NOT DISTINGUISHED BY TQFT INVARIANTS

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WITH AN APPENDIX BY LOUIS FUNAR AND ANDREI RAPINCHUK

Abstract. We show that there exist infinitely many pairs of non-homeomorphic closed oriented SOL torus bundles with the same quantum (TQFT) invariants. This follows from the arithmetic behind the conjugacy problem in SL(2,Z) and its congruence quotients, the classification of SOL (polycyclic) 3-manifold groups and an elementary study of a family of Pell equations. A key ingredient is the congruence subgroup property of modular representations, as it was established by Coste and Gannon, Bantay, Xu for various versions of TQFT, and lastly by Ng and Schauenburg for the Drinfeld doubles of spherical fusion categories. On the other side we prove that two torus bundles over the circle with the same quantum invariants are (strongly) commensurable. The examples above show that this is the best that it could be expected.

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1. Introduction and statements

Two fundamental constructions of TQFTs are due to Reshetikhin-Turaev (see [53]), using link invariants and quantum groups, and to Turaev-Viro ([60]), using quantum 6j-symbols. The Reshetikhin-Turaev method was further extended in [58] to a very general construction of TQFTs, whose input is a modular tensor category, namely an algebraic structure which seems to be the most general data needed for building invariants of arbitrary closed 3-manifolds.

If \(A\) is such a modular tensor category (see [58]) we denote by \(RT_A\) the Reshetikhin-Turaev TQFT invariant of 3-manifolds constructed out of the category \(A\). In the particular case when the modular tensor category \(A\) is the Drinfeld double \(D(C)\) of a spherical fusion category (also called the center of \(C\)) the associated invariant \(RT_{D(C)}\) will be denoted as \(TV_C\) and it will be called the Turaev-Viro TQFT invariant of 3-manifolds associated to \(C\). If \(C\) were itself a modular tensor category then \(RT_{D(C)}\) would indeed coincide with the usual Turaev-Viro invariants \(|M|_C\) constructed out of \(C\) by intrinsic methods (see [58], section V). We chose to single out this family of TQFT invariants because they are somewhat easier to handle than the more general Reshetikhin-Turaev invariants, as they lead to anomaly-free TQFTs. Observe also that spherical fusion categories are more general than modular tensor categories although their Drinfeld doubles account only for part of the anomaly-free modular tensor categories.

According to Muger’s results (see [40]) the Drinfeld double \(D(C)\) of a spherical fusion category \(C\) is a modular tensor category. As a matter of terminology, the Turaev-Viro invariants \(TV_C\) should not be confused with the Turaev-Viro-Barrett-Westbury invariant \(|M|_C\), which extends the intrinsic state-sum definition of a 3-manifold invariant associated to an arbitrary spherical fusion category \(C\) (see [7]). Nevertheless this source of confusion is not relevant, as Turaev and Virelizier proved recently (see [59]) that the Turaev-Viro-Barrett-Westbury invariant \(|M|_C\) actually coincides with \(RT_{D(C)}(M)\), for any spherical fusion category \(C\) of non-zero dimension. Notice that, according to ([19, Thm.2.3]) all spherical fusion categories over \(C\) have non-zero dimension. All fusion categories considered here will be \(C\)-linear categories, unless the opposite is explicitly stated.
A natural question in the area is to what extent the collection of all these 3-manifolds invariants determine the topology of the manifolds. The aim of this article is to solve this question for a particular class of 3-manifolds, namely the SOL manifolds.

Every closed SOL manifold has a finite cover of degree at most 8 which is a torus bundle over a circle. Given \( A \in \text{SL}(2, \mathbb{Z}) \) we denote by \( M_A \) the torus bundle over the circle whose monodromy is given by the matrix \( A \). It is well-known that the manifold has geometry SOL if and only if \( A \) is hyperbolic (or Anosov).

The first result of this paper is the following:

**Theorem 1.1.** There exist infinitely many pairs of Anosov matrices \( A, B \) such that \( M_A \) and \( M_B \) have non-isomorphic fundamental groups although for every spherical fusion category \( C \) their Turaev-Viro invariants agree:

\[
TV_C(M_A) = TV_C(M_B)
\]

The simplest series of examples is the following:

\[
A = \left( \begin{array}{cc} 1 & kq^2 \\ kv & 1 + k^2q^2v \end{array} \right), \quad B = \left( \begin{array}{cc} 1 & k \\ kvq^2 & 1 + k^2q^2v \end{array} \right)
\]

where \( k \in \mathbb{Z}, k \neq 0, q \) is an odd prime number \( q \equiv 1 \pmod{4} \), \( v \) is a positive integer such that \(-v\) is a non-zero quadratic residue mod \( q \) and \( v \) is divisible either by a prime \( p \) satisfying \( p \equiv 3 \pmod{4} \), or by 4.

**Remark 1.1.** Notice that the manifolds \( M_A \) and \( M_B \) are prime SOL manifolds.

As an immediate consequence we obtain a negative answer to a question due to Turaev (see [58], Problem 5, p.571).

**Corollary 1.1.** There exist infinitely many pairs of matrices \( A \) and \( B \) as in Theorem 1.1 such that \( M_A \) and \( M_B \) have non-isomorphic fundamental groups but for every modular tensor category \( C \) their Reshetikhin-Turaev TQFT invariants agree:

\[
RT_C(M_A \# M_A) = RT_C(M_B \# M_B)
\]

Here \( \overline{M} \) denotes the manifold \( M \) with the reversed orientation.

**Proof.** This follows from the fact that

\[
RT_C(M_A \# M_A) = TV_C(M_A)
\]

according to Proposition 3.4. Moreover prime decomposition of 3-manifolds, as well as splittings of groups as free amalgamated products are unique by classical results of Milnor and Stallings. Therefore the fundamental groups are non-isomorphic since their factors are not isomorphic.

We will show later (see Theorem 1.3) that there are also examples of pairs of prime manifolds, but we cannot provide yet infinite families.

Recall now that a quotient of \( SL(2, \mathbb{Z}) \) is a congruence quotient if it is of the form \( SL(2, \mathbb{Z}/m\mathbb{Z}) \) for some non-zero integer \( m \). The key steps in the proof of Theorem 1.1 are the following. We will prove first:

**Proposition 1.1.** If \( M_A \) and \( M_B \) are torus bundle as above then \( TV_C(M_A) = TV_C(M_B) \) for any spherical fusion category provided that the matrices \( A \) and \( B \) are conjugate in every congruence quotient of \( SL(2, \mathbb{Z}) \).

**Remark 1.2.** It seems plausible that \( RT_A(M_A) = RT_A(M_B) \) for every modular tensor category \( A \) whose TQFT is anomaly-free if and only if \( A \) and \( B \) are conjugate in every congruence quotient of \( SL(2, \mathbb{Z}) \).

Lackenby was the first to observe in [35] that quantum \( SU(2) \)-invariants behave well with respect to modular transformations from congruence subgroups. Specifically he defined the \( f \)-congruence of manifolds, for \( f \in \mathbb{Z}_+ \setminus \{0, 1\} \), as follows. Two closed 3-manifolds are \( f \)-congruent if they can be obtained by Dehn surgeries on framed links related by a sequence of moves which consists in Kirby
moves and changes of the framings by adding integral multiples of \( f \). This was further explored and refined (to weak and strong \( f \)-congruence) by Gilmer in \([27]\) where it was shown that quantum invariants are natural obstructions to the \( f \)-congruence of given 3-manifolds. 

We will say that two closed 3-manifolds are \textit{congruent} if they are \( f \)-congruent, for every integral \( f \). Therefore the meaning of our Proposition 1.1 is that the torus bundles \( M_A \) and \( M_B \) are congruent. 

Now there exists an explicit classification of the manifolds of the form \( M_A \). For the sake of simplicity we will restrict ourselves to Anosov matrices \( A, B \). In this case \( M_A \) is a SOL manifold and it is easy to see that it is Haken since the fiber is incompressible. Therefore it suffices to understand its fundamental group, which is the polycyclic group \( \Gamma_A \) with the presentation:

\[
\Gamma_A = \langle t, a, b | ab = ba, tat^{-1} = a^{\alpha_{11}}b^{\alpha_{12}}, tbt^{-1} = a^{\alpha_{21}}b^{\alpha_{22}} \rangle
\]

where \( A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \). We have then the following:

**Proposition 1.2.** Let \( A \) and \( B \) be matrices from \( SL(2, \mathbb{Z}) \) whose traces are different from 2. Then the groups \( \Gamma_A \) and \( \Gamma_B \) are isomorphic if and only if \( A \) is conjugate to either \( B \) or to \( B^{-1} \) within \( GL(2, \mathbb{Z}) \).

Although considered a folklore statement going back as far as Poincaré the result above seems to be first appeared with a sketch of proof in \([28]\), Appendix 1, Prop.2) and then with all details in the unpublished \([4]\). For the sake of completeness we give a detailed proof below. Notice that Proposition 1.2 actually gives the classification of torus bundles up to homeomorphism, since these are aspherical Haken manifolds and hence completely determined by their fundamental groups.

Eventually the problem of finding 3-manifolds \( M_A \) and \( M_B \) as in the statement of Theorem 1.1 is reduced to a purely arithmetic question on integral matrices. This amounts to find whether there exist Anosov integral matrices which are conjugate in every congruence subgroup but are not conjugate within \( GL(2, \mathbb{Z}) \). This question was already answered affirmatively by Stebe in \([56]\), who gave such an example. We are able to give infinitely many such pairs of examples having a slightly stronger property (as needed in Proposition 1.2), as follows:

**Proposition 1.3.** There exist infinitely many pairs of matrices \( A \) and \( B \) in \( SL(2, \mathbb{Z}) \) which are conjugate in every congruence quotient, such that \( A \) is conjugate neither to \( B \) nor to \( B^{-1} \) in \( GL(2, \mathbb{Z}) \). For instance we can take \( A = \begin{pmatrix} 1 & kq^2 \\ kv & 1 + k^2q^2v \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & k \\ kvq^2 & 1 + k^2q^2v \end{pmatrix} \), where \( k \in \mathbb{Z}, q \) is an odd prime number \( q \equiv 1(\text{mod} \ 4) \), \( v \) is a positive integer such that first \( -v \) is a non-zero quadratic residue \( \text{mod} \ q \), and second \( v \) is divisible either by a prime \( p \equiv 3(\text{mod} \ 4) \), or by 4.

**Remark 1.3.** Stebe’s example from \([56]\) is \( A = \begin{pmatrix} 188 & 275 \\ 121 & 177 \end{pmatrix} \) and \( B = \begin{pmatrix} 188 & 11 \\ 3025 & 177 \end{pmatrix} \).

This implies that any pair of integral Anosov matrices as in Proposition 1.3 gives raise to SOL 3-manifolds which are not distinguished by their Turaev-Viro TQFT invariants, thus proving Theorem 1.1.

In the examples above the manifolds \( M_A \) and \( M_B \) obtained throughout Proposition 1.3 are actually commensurable SOL manifolds. This is not a fortuitous coincidence since we have the following:

**Theorem 1.2.** If the torus bundles SOL manifolds \( M \) and \( N \) have the same Turaev-Viro invariants for the \( U(1) \) and \( SU(2) \) TQFTs then they are strongly commensurable.

Let us explain briefly the terminology used for the commensurability above. Two groups are said to be \textit{commensurable} if they have finite index subgroups which are isomorphic.

Barbot (\([4]\), see also \([5]\)) proved that the groups \( \Gamma_A \) and \( \Gamma_B \) are commensurable if and only if the quotient of their discriminants \( D_A/D_B \) is the square of a rational number. Here the discriminant of \( A \) is \( D_A = \text{Tr}(A)^2 - 4 \text{det}(A) \), when \( \text{Tr}(A) \) is odd. Moreover, this is equivalent to the fact that \( A^p \) and \( B^q \) are conjugate within \( GL(2, \mathbb{Q}) \), for some \( p, q \in \mathbb{Z} \setminus \{0\} \). We will call the matrices \( A \) and \( B \) in \( SL(2, \mathbb{Z}) \).
Corollary 1.4. stated by Long and Reid in [36] (see also Remark 3.7 in [8]), as follows:

of hyperbolic fibered 3-manifolds with fiber of given genus?

(2) $N$ with fiber of genus $g$ constructed out of torus bundles we ask:

of this geometry.

Conjecture 1.1. If $N$ has fiber of genus $g$ having the same Turaev-Viro invariants for all spherical fusion categories.

Proposition 1.4. If $M$ is a SOL torus bundle over the circle. The subset of the torus bundles homeomorphism classes in $\mathcal{X}^{U(1)}(M)$ injects into $I(M)/\iota$ and, in particular, it is bounded by the class number of the corresponding totally real quadratic field.

Remark 1.4. One might consider the set of torus bundles $N$ having the same Turaev-Viro invariants as $M$, up to an orientation preserving homeomorphism. Observe that $M_A$ and $M_B$ are orientation-preserving homeomorphic if and only if $A$ and $B$ are conjugate within $SL(2,\mathbb{Z})$ or else $A$ and $B^{-1}$ are conjugate in $GL(2,\mathbb{Z})$ by a matrix of determinant $-1$.

The few examples we know would suggest that the subset of torus bundles homeomorphism classes in $\mathcal{X}^{TV}(M)$ is a quite small proper subset of $I(M)/\iota$. As a consequence of our proof of Theorem 1.1 and results of Platonov and Rapinchuk (see [48, 51], [49, section 8.8.5]) on the genus problem in arithmetic groups we obtain a stronger but less precise statement as follows:

Corollary 1.3. The number of homeomorphisms classes of torus bundles in $\mathcal{X}^{TV}(M)$, for $M$ running over all torus bundles, is unbounded. Alternatively, for each $m \geq 2$ there exist examples of $m$ pairwise non-homeomorphic torus bundles having the same Turaev-Viro invariants for all spherical fusion categories.

We can slightly improve the finiteness result in Corollary 1.2 as follows:

Proposition 1.4. If $M$ is a closed irreducible orientable SOL manifold then $|X^{TV}(M)|$ is finite.

These results give some evidence for the following general conjecture:

Conjecture 1.1. If $M$ and $N$ are closed irreducible geometric 3-manifolds having the same abelian and $SU(2)$ Turaev-Viro invariants then $M$ and $N$ should be commensurable and, in particular, they share the same geometry.

On the other hand, we don’t know whether the unboundedness of the number of classes of torus bundles is a general phenomenon, valid in higher genus as well. In order to dismiss obvious examples constructed out of torus bundles we ask:

Conjecture 1.2. The number of homeomorphism classes in $\mathcal{X}^{TV}(M)$ of hyperbolic fibered 3-manifolds $N$ with fiber of genus $g \geq 1$ is finite for every $M$. Is this number unbounded, when $M$ runs over the set of hyperbolic fibered 3-manifolds with fiber of given genus?

The pairs of manifolds from Theorem 1.1 and Corollary 1.3 also give a negative answer to a question stated by Long and Reid in [30] (see also Remark 3.7 in [8]), as follows:

Corollary 1.4. For any $m \geq 2$ there exist torus bundles whose fundamental groups have isomorphic profinite completions although they are pairwise not isomorphic.
This consequence was independently noticed by G. Masbaum.

We don’t know if the SU(2) Turaev-Viro invariants alone determine already the profinite completion of the fundamental group.

**Remark 1.5.** We expect that the topological content of the Turaev-Viro invariants is precisely this kind of arithmetic information. An over-optimistic conjecture would be that two closed irreducible geometric 3-manifolds $M$ and $N$ with infinite fundamental groups define the same class in $\mathcal{X}^{TV}(M)$ if and only if the profinite completions of their fundamental groups are isomorphic. The ”only if” part is immediate (see the proof of Corollary [14]). In particular, if the closed hyperbolic 3-manifolds $M$ is determined up to homeomorphism by the profinite completion of its fundamental group, then $\mathcal{X}^{TV}(M)$ will be a singleton. This would connect the quantum invariants to some version of Grothendieck’s problem for 3-manifold groups which is stated in [36].

Eventually, the result of Theorem 1.1 can be improved (with a loss of effectivity), as follows:

**Theorem 1.3.** There exist infinitely many pairs of Anosov matrices $A$, $B$ such that $M_A$ and $M_B$ have non-isomorphic fundamental groups although for every modular tensor category $C$ their Reshetikhin-Turaev invariants agree:

$$RT_C(M_A) = RT_C(M_B)$$

The simplest four examples are the following:

(7) \[ A = \begin{pmatrix} 1 & 21 \\ 21 & 442 \end{pmatrix}, \quad B = \begin{pmatrix} 106 & 189 \\ 189 & 337 \end{pmatrix}, \]

(8) \[ A = \begin{pmatrix} 1 & 51 \\ 51 & 2602 \end{pmatrix}, \quad B = \begin{pmatrix} 562 & 1479 \\ 1479 & 2041 \end{pmatrix}, \]

(9) \[ A = \begin{pmatrix} 1 & 53 \\ 53 & 2810 \end{pmatrix}, \quad B = \begin{pmatrix} 425 & 1007 \\ 1007 & 2386 \end{pmatrix}, \]

(10) \[ A = \begin{pmatrix} 1 & 55 \\ 55 & 3026 \end{pmatrix}, \quad B = \begin{pmatrix} 881 & 1375 \\ 1375 & 2146 \end{pmatrix}. \]

The equivalence relation on torus bundles induced by the equality of all Turaev-Viro invariants is the local equivalence of matrices determining a fixed genus, in the sense studied by Platonov and Rapinchuk (see [48, 51, 49, section 8.8.5]). Specifically, $M_B$ and $M_A$ represent the same class in $\mathcal{X}^{TV}(M)$ if and only if $A$ and $B$ are locally conjugate, namely their images mod $m$ are conjugate in $GL(2, \mathbb{Z}/m\mathbb{Z})$, for any positive integer $m$. Notice that this implies automatically that $A$ and $B$ are conjugate in $GL(2, \mathbb{Q})$.

A related equivalence relation is the one corresponding to the Pickel genus of groups (see [47]). Two finitely generated groups are in the same Pickel genus if the corresponding sets of finite quotients are the same. This is equivalent, following a deep result of Nikolov and Segal (see [43]) to the fact that their profinite completions are isomorphic. The groups of torus bundles $\pi_1(M_B)$ and $\pi_1(M_A)$ have isomorphic profinite completions if and only if the subgroups $\langle A \rangle$ and $\langle B \rangle$ are locally conjugate, namely their images mod $m$ are conjugate in $GL(2, \mathbb{Z}/m\mathbb{Z})$, for any positive integer $m$. This is coarser than the former equivalence relation.

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2. Preliminaries about modular tensor categories

2.1. Fusion categories. For simplicity we will consider only strict monoidal categories below, meaning that the associativity morphisms are identities. We follow the definitions from [2, 10].

A left/right rigid monoidal category is a a strict monoidal category \( \mathcal{C} \) with unit object \( 1 \) such that to each object \( X \in \text{Ob}(\mathcal{C}) \) there are associated a dual object \( X^* \in \text{Ob}(\mathcal{C}) \) and four morphisms

\[
\begin{align*}
ev_X & : X^* \otimes X \to 1, \\
\coev_X & : 1 \to X \otimes X^*, \\
\tilde{\ev}_X & : X \otimes X^* \to 1, \\
\tilde{\coev}_X & : 1 \to X^* \otimes X
\end{align*}
\]

such that, for every \( X \in \text{Ob}(\mathcal{C}) \), the pair \( (\ev_X, \coev_X) \) is a left duality for \( X \) and the pair \( (\tilde{\ev}_X, \tilde{\coev}_X) \) is a right duality for \( X \), namely:

\[
(1_X \otimes \ev_X)(\coev_X \otimes 1_X) = 1_X \quad \text{and} \quad (\ev_X \otimes 1_{X^*})(1_{X^*} \otimes \coev_X) = 1_X.
\]

The category is rigid if it is both left and right rigid.

A pivotal category is a left rigid monoidal category equipped with an isomorphism \( j \) of monoidal functors between identity and \((-)^{*}\), called pivotal structure. One should notice that the formulas

\[
\tilde{\ev}_X = \coev_X(1_{X^*} \otimes j_X^{-1}), \quad \tilde{\coev}_X = (j_X \otimes 1_{X^*})\ev_X
\]

define a right duality so that a pivotal category is rigid.

It is known that every pivotal category is equivalent to a strict pivotal category, namely one where the associativity isomorphisms, the pivotal structure and the canonical isomorphisms \( (V \otimes W)^* \to W^* \otimes V^* \) are identities.

The morphisms \( \ev_1 \) and \( \coev_1 \) (respectively, \( \tilde{\ev}_1 \) and \( \tilde{\coev}_1 \)) are mutually inverse isomorphisms and

\[\ev_1 = \tilde{\ev}_1 : 1^* \to 1.\]

Now, for an endomorphism \( f \) of an object \( X \) of a pivotal category \( \mathcal{C} \), one defines the left/right traces \( \text{tr}_l(f), \text{tr}_r(f) \in \text{End}_\mathcal{C}(1) \) by

\[
\text{tr}_l(f) = \ev_X(1_{X^*} \otimes f)\coev_X \quad \text{and} \quad \text{tr}_r(f) = \tilde{\ev}_X(f \otimes 1_{X^*})\tilde{\coev}_X
\]

Both traces are symmetric: \( \text{tr}_l(gh) = \text{tr}_l(hg) \) and \( \text{tr}_r(gh) = \text{tr}_r(hg) \) for any morphisms \( g : X \to Y \) and \( h : Y \to X \) in \( \mathcal{C} \). Also \( \text{tr}_l(f) = \text{tr}_r(f^{**}) = \text{tr}_r(f^{**}) \) for any endomorphism \( f \) of an object (and similarly for \( l \) exchanged with \( r \)).

The left and right dimensions of \( X \in \text{Ob}(\mathcal{C}) \) are defined by \( \text{dim}_l(X) = \text{tr}_l(1_X) \) and \( \text{dim}_r(X) = \text{tr}_r(1_X) \). Note that isomorphic objects have the same dimensions and \( \text{dim}_l(1) = \text{dim}_r(1) = 1_1 \).

A spherical category is a pivotal category whose left and right traces are equal, i.e., \( \text{tr}_l(f) = \text{tr}_r(f) \) for every endomorphism \( f \) of an object. Then they are denoted \( \text{tr}(f) \) and called the trace of \( f \). The left (and right) dimensions of an object \( X \) are denoted \( \text{dim}(X) \) and called the dimension of \( X \). In a (strict) spherical category we can make free use of the graphical calculus.

Let \( K \) be a field, which for the moment is not supposed to be of characteristic zero, although in the next section we will consider \( K = \mathbb{C} \).

A monoidal \( K \)-linear category is a monoidal category \( \mathcal{C} \) such that its Hom-sets are (left) \( K \)-modules and the composition and monoidal product of morphisms are \( K \)-bilinear. An object \( V \in \text{Ob}(\mathcal{C}) \) is called simple if the map \( K \to \text{End}_\mathcal{C}(1), a \mapsto a1_1 \) is a \( K \)-algebra isomorphism.

An additive category is said to be semisimple if every object is a direct sum of finitely many simple objects. In the case of Ab-categories from [55] we can weaken our requirements by asking that every object be dominated by finitely many simple objects. A monoidal \( K \)-linear category is called semisimple if the underlying \( K \)-linear category is semisimple with finite dimensional Hom spaces and \( 1 \) is a simple object.

Now a fusion category over \( K \) is a rigid semisimple \( K \)-linear category \( \mathcal{C} \) with finitely many simple objects. The fusion categories which are considered in the next sections will always be spherical.
A monoidal category $\mathcal{C}$ is braided if there exist natural isomorphisms $c_{U,V,W} : V \otimes W \rightarrow W \otimes V$ for every objects $V, W$, such that for any $U, V, W \in \text{Ob}(\mathcal{C})$ we have:

$$c_{U,V\otimes W} = (1_V \otimes c_{U,W})(c_{U,V} \otimes 1_W), \quad c_{U\otimes V,W} = (c_{U,W} \otimes 1_V)(1_U \otimes c_{V,W})$$

Let now $\mathcal{C}$ be a left rigid braided monoidal category. We do not require that $V^{**} = V$. A twist of $\mathcal{C}$ is an automorphism $\theta$ of the identity functor of $\mathcal{C}$ satisfying

$$\theta_{V\otimes W} = c_{W,V}c_{V,W}(\theta_V \otimes \theta_W), \quad \text{and} \quad \theta_1 = 1_1$$

The twist $\theta$ is a ribbon structure on $(\mathcal{C}, c)$ if it also satisfies $\theta^{-1}_V = \theta_V$, for every $V \in \text{Ob}(\mathcal{C})$, and the (left) duality is compatible with the ribbon and twist structures, namely:

$$(\theta_V \otimes 1_V)c_{eV} = (1_V \otimes \theta_V)c_{eV}$$

In this case $(\mathcal{C}, c, \theta)$ is called a ribbon category. In a ribbon category one associates naturally a pivotal structure by using the (canonical) isomorphism $u_X : X \rightarrow X^{**}$ given by:

$$u_X = (\text{ev}_X \otimes 1_X)(1_X \otimes c_{X,X***})(\text{coev}_X \otimes 1_X)$$

and setting $\theta = u^{-1}j$. Moreover this pivotal structure $j$ is spherical.

A modular tensor category over $K$ is a ribbon fusion category $(A, c, \theta)$ over $K$ such that the matrix $S$ having entries $S_{ij} = \text{tr}((c_{U_i,U_j}c_{U_j,U_i}^{-1}))$ is non-singular, where $i, j \in I$ and $I$ is the set indexing the simple objects $U_i$, $i \in I$ in $A$. This matrix is called the $S$-matrix of the category $A$. Notice that $I$ has induced a duality $*$ such that $U_i^* = U_i^+$, for any $i \in I$ and there exists a label (also called color) $0 \in I$ such that $U_0 = 1$. Since the object $U_i$ is simple the twist $\theta_{U_i}$ acts on $U_i$ as a scalar $\omega_i \in K$.

The left Drinfeld double (also called the center) of a (strict) monoidal category $\mathcal{C}$ is a category $D(\mathcal{C})$ whose objects are pairs $(V, \sigma_V)$, where $V \in \text{Ob}(\mathcal{C})$ and the half-braiding $\sigma_V(W) : V \otimes W \times W \otimes V$ are natural isomorphisms satisfying for every $U, V, W \in \text{Ob}(\mathcal{C})$ the identities:

$$(V \otimes \sigma_V(W))(\sigma_U(V) \otimes W) = \sigma_U(V \otimes W), \quad \sigma_V(1) = 1_X$$

There is a natural monoidal structure on $D(\mathcal{C})$ by defining the tensor product $(U, \sigma_U) \otimes (V, \sigma_V) = (U \otimes V, \sigma_U \otimes \sigma_V)$, where

$$\sigma_{U\otimes V}(W) = (\sigma_U(W) \otimes V)(U \otimes \sigma_V(W))$$

and the unit object $(1, \sigma_1)$, where $\sigma_1(V) = 1_V$, for any $U, V, W \in \text{Ob}(\mathcal{C})$. More interesting is the fact that $D(\mathcal{C})$ has a braiding given by $c_{V,\sigma_U(W),\sigma_V(W)} = \sigma_V(W)$ so that $\mathcal{C}$ is a braided monoidal category. If $\mathcal{C}$ is left rigid/pivotal/spherical then $D(\mathcal{C})$ is also left rigid/pivotal/spherical respectively.

2.2. $SL(2,\mathbb{Z})$ representations from modular tensor categories. Any modular tensor category $\mathcal{C}$ defined over the algebraically closed field $K$ has associated the modular data (see [28]), which contains a projective representation $\overline{\mathcal{P}}_\mathcal{C} : SL(2,\mathbb{Z}) \rightarrow PGL(K_0(\mathcal{C}))$, where $K_0(\mathcal{C})$ is the Grothendieck ring of $\mathcal{C}$ with $\mathbb{C}$-coefficients. However, we have slightly more than that, namely a lift of $\overline{\mathcal{P}}_\mathcal{C}$ to an almost linear representation, by means of the matrices $S$ and $T$. The almost linear representation comes with a 2-cocycle which was completely described by Turaev. An essential feature of the genus 1 situation is that projective representations could always be lifted (in more than one way) to genuine linear representations, which contrasts with the higher genus case.

The matrices entering in the definition of $\overline{\mathcal{P}}_\mathcal{C}$ are the $S$-matrix defined above and the $T$-matrix associated to the twist. Specifically, $T$ has the entries $T_{ij} = \omega_i \delta_{ij}$, $i, j \in I$. Moreover there is also the so-called charge conjugation matrix $C$ having entries $C_{ij} = \delta_{j,i}$, $i, j \in I$, which is actually $S^2$.

The Gauss sums of $\mathcal{C}$ are given by $p_{\mathcal{C}}^\pm = \sum_{i} \omega_i^{\pm 1} \dim(U_i)^2$ and these are non-zero scalars satisfying:

$$p_{\mathcal{C}}^+ p_{\mathcal{C}}^- = \sum_{i,j} \dim(U_i)^2 = \dim(\mathcal{C})$$

In [58] Turaev used the notation $\Delta_{\mathcal{C}} = p_{\mathcal{C}}^-$ so that $p_{\mathcal{C}}^+ = \dim(\mathcal{C})\Delta_{\mathcal{C}}^{-1}$. Further one chooses a rank (also called quantum order), which is an element $\lambda \in K$ such that $\lambda^2 = \dim\mathcal{C}$. This was denoted by $D$ in [58].
The group $SL(2, \mathbb{Z})$ is generated by the matrices $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The usual presentation of $SL(2, \mathbb{Z})$ in the generators $s, t$ has the relations $(st)^3 = s^2$ and $s^4 = 1$.

The projective representation $\overline{\rho}_C : SL(2, \mathbb{Z}) \rightarrow PGL(K_0(C))$ is defined by

$$\overline{\rho}_C(s) = S, \overline{\rho}_C(t) = T$$

However the choice of a rank $\lambda$ and a third root of unity $\zeta \in \mathbb{K}$ of the anomaly $\zeta^3 = D\Delta^{-1} = p_C^+\lambda^{-1}$ enables us to define a lift of $\overline{\rho}_C$ to an ordinary linear representation $\rho^\lambda_\zeta : SL(2, \mathbb{Z}) \rightarrow GL(K_0(C))$ by setting

$$\rho^\lambda_\zeta(s) = \lambda^{-1}S, \rho^\lambda_\zeta(t) = \zeta^{-1}T$$

These lifts are called the modular representations associated to $C$. It is known that, given a rank $\lambda$ then the modular tensor category defines a TQFT with anomaly in the group generated by $\zeta^3$, so that 3-manifold invariants associated to the data $(C, \lambda)$ do not depend on the particular choice of $\zeta$.

3. Proof of Proposition 1.1

3.1. TQFT coming from centers of spherical fusion categories. In the case when the modular tensor category is the Drinfeld double $D(C)$ of a spherical fusion category $C$ a number of simplifications occur.

For every $SL(2, \mathbb{Z})$ representation $\rho$ we define its dual representation $\overline{\rho}$ by means of $\overline{\rho}(x) = \rho(JxJ^{-1})$, where $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{Z})$ acts by conjugacy as an outer automorphism of $SL(2, \mathbb{Z})$.

Notice that in this case we have the following:

**Lemma 3.1.**

1. The anomaly of the TQFT coming from $D(C)$ is trivial, i.e. $\zeta^3 = 1$ and thus there exists a privileged modular representation $\rho^\lambda_\zeta^1$.

2. Further we have $\rho^\lambda_{D(C)} = \rho^\lambda_\zeta \otimes \overline{\rho}^\lambda_\zeta^1$. Here $\zeta$ is arbitrary and in fact the right hand side tensor product is well-defined even when we have only projective representations.

**Proof.** See (14), Lemma 6.2. \qed

The invariants of mapping tori have a very simple expression when the TQFT is anomaly-free. In fact we have the following well-known result:

**Lemma 3.2.** Assume that the TQFT associated to the modular tensor category $C$ is anomaly-free, namely that $\zeta^3 = 1$. Then the invariant of the mapping torus $M_A$ of $A \in SL(2, \mathbb{Z})$ is expressed as:

$$RT_C(M_A) = Tr(\overline{\rho}^\lambda_{\zeta^1}(A))$$

**Proof.** For the sake of completeness here is the proof. Turaev defined in ([58], section IV.5, (5.1.a)) an almost linear representation $\epsilon : SL(2, \mathbb{Z}) \rightarrow GL(K_0(C))$ satisfying the cocycle law:

$$\epsilon(A_1A_2) = \epsilon(A_2)^{A_1 (L_1, L_2)} \epsilon(A_1)$$

where $\mu(L_1, L_2, L_3)$ denotes the Maslov index (see [58], section IV.3, p.179) of the triple $(L_1, L_2, L_3)$ of Lagrangian subspaces of $H_1(\Sigma; \mathbb{R})$ and $L$ a fixed Lagrangian subspace. Further $\epsilon$ is determined by its values on the generators $\epsilon(s) = \lambda^{-1}S$ and $\epsilon(t) = T$. If $\zeta^3 = 1$ then $\epsilon$ is a linear representation which coincides with $\rho^\lambda_{\zeta^1}$. Moreover, one also knows from ([58], section III.2.8, Ex.1) that

$$RT_C(M_A) = Tr(\epsilon(A))$$

This proves the claim. \qed

We will prove now:
Proposition 3.1. Let \( \mathcal{C} \) be an anomaly-free modular tensor category such that the modular representation \( \rho_\mathcal{C}^{λ,1} \) factors through \( SL(2, \mathbb{Z}/N\mathbb{Z}) \). Let \( A \) and \( B \) be two integral matrices from \( SL(2, \mathbb{Z}) \) whose reductions mod \( N \) are conjugate. Then \( RT_\mathcal{C}(M_A) = RT_\mathcal{C}(M_B) \).

Proof. According to the Lemma 3.2, the invariant \( RT_\mathcal{C}(M_A) \) is the trace of the endomorphism \( \rho_\mathcal{C}^{λ,1}(A) \). By hypothesis \( \rho_\mathcal{C}^{λ,1} \) factorizes as \( \rho_\mathcal{C}^{λ,1} = \rho_\mathcal{C}^{λ,1,N} \circ \nu_N \), where \( \nu_N : SL(2, \mathbb{Z}) \to SL(2, \mathbb{Z}/N\mathbb{Z}) \) is the homomorphism of reduction mod \( N \).

Since \( \nu_N \) is surjective there exists \( T \in SL(2, \mathbb{Z}) \) such that \( \nu_N(A) = \nu_N(T^{-1}BT) \). Therefore

\[
\text{Tr}(\rho_\mathcal{C}^{λ,1}(A)) = \text{Tr} \left( \rho_\mathcal{C}^{λ,1,N}(\nu_N(T)) \cdot \rho_\mathcal{C}^{λ,1,N}(\nu_N(B)) \cdot \left( \rho_\mathcal{C}^{λ,1,N}(\nu_N(T)) \right)^{-1} \right) = \text{Tr}(\rho_\mathcal{C}^{λ,1}(B))
\]

Then Lemma 3.2 yields the equality of quantum invariants of \( M_A \) and \( M_B \). \( \square \)

The final ingredient in the proof of Proposition 3.1 is the following result due to Ng and Schauenburg for RCFT (see [55]). Let \( \rho \) be the Drinfeld double of a spherical fusion category \( \mathcal{C} \) and to Coste, Gannon and Bantay for RCFT (see [9, 3]). Recall that a congruence subgroup of \( SL(2, \mathbb{Z}) \) is the kernel of one of the reducing mod \( m \) homomorphisms \( SL(2, \mathbb{Z}) \to SL(2, \mathbb{Z}/m\mathbb{Z}) \), for some non-zero integer \( m \).

Theorem 3.1. Let \( D(\mathcal{C}) \) be the Drinfeld double of a spherical fusion category \( \mathcal{C} \). Then the modular representations \( \rho_\mathcal{C}^{λ,1} \) have the congruence property, namely the kernels contain congruence subgroups.

This proves Proposition 3.1, namely if \( A \) and \( B \) are conjugate in every congruence quotient of \( SL(2, \mathbb{Z}) \) then \( TV_\mathcal{C}(M_A) = TV_\mathcal{C}(M_B) \) for any spherical tensor category \( \mathcal{C} \).

Corollary 3.1. Assume that the modular representation of every anomaly-free modular tensor category has the congruence property. Then there exist pairs of matrices \( A, B \) such that \( M_A \) and \( M_B \) are not homeomorphic but \( RT_A(M_A) = RT_A(M_B) \) for any anomaly-free modular tensor category \( \mathcal{A} \).

3.2. General modular tensor categories. Turaev constructed in ([55], p.198-199) some almost linear representations of the mapping class group \( \mathcal{M}_g \) of genus \( g \) surfaces, for every \( g \). We have to choose first some Lagrangian subspace \( L \subset H_1(\Sigma_g; \mathbb{R}) \) with respect to the usual symplectic form \( \omega \) in homology coming from the intersection form. We denote by \( Z_\mathcal{C}(\Sigma_g) \) the space of conformal blocks in genus \( g \) associated to the modular tensor category \( \mathcal{C} \).

It is known that there exist maps (which will be called almost linear representations) \( f^{λ,L}_\mathcal{C} : \mathcal{M}_g \to GL(Z_\mathcal{C}(\Sigma_g)) \) into the automorphisms of the space of conformal blocks \( Z_\mathcal{C}(\Sigma_g) \) satisfying the following 2-cocycle condition:

\[
f^{λ,L}_\mathcal{C}(\varphi_1 \varphi_2) = \zeta^{3\mu(\varphi_2, (L), L, \varphi_1, \zeta^{-1}(L))} f^{λ,L}_\mathcal{C}(\varphi_1) f^{λ,L}_\mathcal{C}(\varphi_2)
\]

where \( \mu(L_1, L_2, L_3) \) denotes the Maslov index (see [55], section IV.3, p.179) of the triple \( (L_1, L_2, L_3) \) of Lagrangian subspaces of \( H_1(\Sigma_g; \mathbb{R}) \). This can be found for instance either in ([55], section IV.5, (5.1.a)) and also in an rather equivalent context in ([55], section IV.6, Lemma 6.3.2, (6.3.c)).

We introduce now the Rademacher Phi function (see [50]) \( \Phi_R : SL(2, \mathbb{Z}) \to \mathbb{Z} \) defined as follows.

\[
\Phi_R \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) = \begin{cases} \frac{\alpha + \delta}{\gamma} - 12 \text{sgn}(\gamma) s(\alpha, |\gamma|), & \text{if } \gamma \neq 0 \\ \frac{\alpha}{\gamma}, & \text{otherwise} \end{cases}
\]

Here \( s(m, n) \), for \( n > 0 \), denotes the Dedekind sum

\[
s(m, n) = \sum_{j=1}^{n-1} \left( \left( \frac{j}{n} \right) \left( \frac{jm}{n} \right) \right), \quad s(0, 1) = 0
\]

where

\[
((x)) = \begin{cases} 0, & \text{if } x \in \mathbb{Z} \\ x - [x] - \frac{1}{2}, & \text{otherwise} \end{cases}
\]
Alternatively we have

\[(21) \quad s(m, n) = \frac{1}{4n} \sum_{j=1}^{n-1} \cot \left( \frac{\pi j}{n} \right) \cot \left( \frac{\pi jm}{n} \right), \]

We have then the following result, which seems to be well-known to the specialists:

**Lemma 3.3.** Let \( L_0 \) be the integral Lagrangian subspace of the homology \( H_1(\Sigma_1; \mathbb{R}) = \mathbb{R}^2 \) generated by the vector \((1, 0)\). Then Turaev’s almost linear representation \( f^\lambda_{\mathcal{C}L_0} \) in genus \( g = 1 \) is related to the modular representation \( \rho^\lambda_{\mathcal{C}L} \) of \( SL(2, \mathbb{Z}) \), by means of the formula

\[(22) \quad \rho^\lambda_{\mathcal{C}L}(A) = \zeta^{-\Phi_R(A)} f^\lambda_{\mathcal{C}L_0}(A) \]

for every \( A \in SL(2, \mathbb{Z}) \).

**Proof.** Consider \( A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) and \( B = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \) and let \( BA = \begin{pmatrix} \alpha'' & \beta'' \\ \gamma'' & \delta'' \end{pmatrix} \). By direct computation we obtain

\[(23) \quad \mu(BA(L_0), B(L_0), L_0) = -\text{sgn}(\gamma' \gamma'') \]

On the other hand Rademacher proved that \( \Phi_R \) is a 1-cocycle whose boundary is 3 times the signature 2-cocycle, in other words we have the identities:

\[(24) \quad \Phi_R(BA) = \Phi_R(A) + \Phi_R(B) - 3 \text{sgn}(\gamma' \gamma'') \]

for \( A, B \) as above. Therefore the equation above, the cocycle identity \[17\] for \( f^\lambda_{\mathcal{C}L_0} \) and \[22\] yield:

\[(25) \quad \zeta^{-\Phi_R(BA)} f^\lambda_{\mathcal{C}L_0}(BA) = \zeta^{-\Phi_R(B)} f^\lambda_{\mathcal{C}L_0}(B) \cdot \zeta^{-\Phi_R(A)} f^\lambda_{\mathcal{C}L_0}(A) \]

This means that \( \zeta^{-\Phi_R} f^\lambda_{\mathcal{C}L_0} \) is a linear representation of \( SL(2, \mathbb{Z}) \). Since \( \Phi_R(s) = 0 \) and \( \Phi_R(t) = 1 \) the two linear representations \( \zeta^{-\Phi_R} f^\lambda_{\mathcal{C}L_0} \) and \( \rho^\lambda_{\mathcal{C}L} \) agree. \( \square \)

**Proposition 3.2.** The quantum invariant of a mapping torus \( M_A \) of \( A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{Z}) \) is given by the formula

\[(26) \quad RT_{\mathcal{C}}(M_A) = \zeta^{-3 \varphi(A)} \text{Tr}(\rho^\lambda_{\mathcal{C}L}(A)) \]

where \( \varphi : SL(2, \mathbb{Z}) \to \mathbb{Z} \) is the modified Meyer function

\[(27) \quad 3 \varphi(A) = \Phi_R(A) - 3 \text{sgn}(\gamma(\alpha + \delta - 2)) \]

**Proof.** The main reason to introduce the almost linear representations \( f^\lambda_{\mathcal{C}L} \) is the following result of Turaev (see \[58\], section IV.7, Thm.7.2.1, p.209) which expresses the quantum invariant of a mapping torus as follows:

**Proposition 3.3.** Let \( M_h \) be the mapping torus of some homeomorphism whose mapping class is \( h \in \mathfrak{M}_g \). Then

\[(28) \quad RT_{\mathcal{C}}(M_h) = \zeta^{3 \mu(\Lambda(h), L \oplus h_+(L), \text{Diag})} \text{Tr}(f^\lambda_{\mathcal{C}L}(h)) \]

where \( \mu \) is the Maslov index of the Lagrangian subspace of \(-H_1(\Sigma_g; \mathbb{R}) \oplus H_1(\Sigma_g; \mathbb{R})\) endowed with the symplectic form \(-\omega \oplus \omega\), \( \Lambda(h_+) \) denotes the graph of \( h_+\), i.e. the subspace of vectors \( x \oplus h_+(x) \), where \( x \in H_1(\Sigma_g; \mathbb{R}) \) and Diag is the diagonal subspace \( \Lambda(1_{H_1(\Sigma_g; \mathbb{R})}) \).

Observe that the manifold \( M_h \) and its invariant \( RT_{\mathcal{C}}(M_h) \) do not depend on the choice of the Lagrangian \( L \), although \( f^\lambda_{\mathcal{C}L}(h) \) does.

Now it suffices to check that

\[(29) \quad \mu(\Lambda(A), L_0 \oplus A(L_0), \text{Diag}) = -\text{sgn}((\alpha + \delta - 2) \gamma) \]
when \( A \in SL(2, \mathbb{Z}) \). If \( \gamma = 0 \) then one verifies that the Maslov index is 0. Suppose now that \( \gamma \neq 0 \). A direct inspection shows that \((\Lambda(A) + L_0 \oplus A(L_0)) \cap \text{Diag}\) is the one-dimensional subspace generated by the vector \( e = (\alpha - 1, \gamma) \oplus (\alpha - 1, \gamma) \). The quadratic form associated to \( e \) has value \( \omega(e_2, e) \), where \( e = e_1 + e_2 \) is any decomposition with \( e_1 \in \Lambda(A) \) and \( e_2 \in L_0 \oplus A(L_0) \). We can take \( e_1 = (0, \gamma) \oplus (\beta \gamma, \delta \gamma) \) and \( e_2 = (\alpha - 1, 0) \oplus (\alpha (1 - \delta), \gamma (1 - \delta)) \). This implies that

\[
\omega(e_2, e) = (2 - \alpha - \delta) \gamma
\]

Now the signature of this quadratic form is the value of the Maslov index and the formula above follows. \( \square \)

Since the orientation preserving homeomorphism type of the manifolds \( M_A \) depends only on the conjugacy class of \( A \) we obtain immediately the following property of Meyer’s function:

**Corollary 3.2.** Meyer’s function \( \varphi \) is conjugacy invariant.

**Remark 3.1.** There exist a slight difference between the usual Meyer’s function \( \varphi_M \) from \([33]\) and the modified Meyer function \( \varphi(M) \) considered by here, following Turaev. This does not makes a big difference since it only affects the invariants for \( M_A \) where \( A \) is parabolic. Specifically we have:

\[
\varphi(A) - \varphi_M(A) = \begin{cases} \frac{1}{2} (1 + \text{sgn}(\delta)) \text{sgn}(\beta), & \text{if } \gamma = 0 \\ 0, & \text{otherwise} \end{cases}
\]

However the function \( \phi - \phi_M \) is an integral 1-cocycle so that the boundary \( \delta(\phi) \) and \( \delta(\phi_M) \) are cohomologous. Notice that \( \delta(\phi_M) \) is Meyer’s signature 2-cocycle (see \([11, 39]\)).

### 3.3. Reshetikhin-Turaev invariants vs Turaev-Viro invariants.

**Proposition 3.4.** For any modular tensor category \( \mathcal{C} \) we have the identity:

\[
RT_{\mathcal{C}}(M_A \# M_A) = TV_{\mathcal{C}}(M_A)
\]

**Proof.** Since \( RT_{\mathcal{C}} \) behaves multiplicatively with respect to connected sums we have

\[
RT_{\mathcal{C}}(M_A \# M_A) = RT_{\mathcal{C}}(M_A) \cdot RT_{\mathcal{C}}(M_A) = RT_{\mathcal{C}}(M_A) \cdot RT_{\mathcal{C}}(M_A)
\]

from \([58], I.2, (2.5.a)\). Here \( \mathcal{C} \) denotes the mirror category of the modular tensor category \( \mathcal{C} \) according to \([58], I.1.4\). It is known that the rank \( \lambda = \lambda_C \) for \( \mathcal{C} \) is equivalently a rank \( \lambda = \lambda_{\mathcal{C}} \) for \( \mathcal{C} \), although the roles of \( p_C^+ \) are inverted, namely we have \( p_C^+ = p_{\mathcal{C}}^- \) and \( p_C^- = p_{\mathcal{C}}^+ \). It follows also that \( \dim_{\mathcal{C}} C_i = \dim_{\mathcal{C}} i \), for every \( i \in I \), but \( \omega_{\mathcal{C}} C_i = \omega_{\mathcal{C}} i \). Furthermore the anomalies \( \zeta_i^3 = \rho_{\mathcal{C}} C_i / \lambda_{\mathcal{C}} \) are inverse to each other, namely \( \zeta_i^3 = \zeta_{\mathcal{C}}^{-3} \). Therefore the \( S \) matrix associated to the mirror category has its entries \( S(\mathcal{C})_{ij} \) equal to \( S(\mathcal{C})_{ij} \) (from \([58], I.1.9, Ex. 1.9.(2)\)). At the last the matrix \( T(\mathcal{C})_{ij} \) is the inverse of \( T(\mathcal{C})_{ij} \) since \( T(\mathcal{C})_{ij} = \omega_{\mathcal{C}}^{\lambda} C_{ij} \).

On the other hand we have the representation \( \rho_{\mathcal{C}}^{\lambda C}(x) = \rho_{\mathcal{C}}^{\lambda C}(J x J^{-1}) \) defined above. We have the following identities, where \( \zeta \) stands for \( \zeta_C \):

\[
\rho_{\mathcal{C}}^{\lambda C}(s) = \lambda^{-1}(S(\mathcal{C})_{ij}^{-1}) = \lambda^{-1} S(\mathcal{C})^{-1} = \rho_{\mathcal{C}}^{\lambda C}(s)
\]

\[
\rho_{\mathcal{C}}^{\lambda C}(t) = \zeta T(\mathcal{C})^{-1} = \rho_{\mathcal{C}}^{\lambda C}(t)
\]

Therefore the two representations agree on every element

\[
\rho_{\mathcal{C}}^{\lambda C}(x) = \rho_{\mathcal{C}}^{\lambda C}(x), \text{ for any } x \in SL(2, \mathbb{Z})
\]

Now using Proposition 3.2 and Lemmas 3.1 and 3.2 we obtain the identities:

\[
RT_{\mathcal{C}}(M_A) \cdot RT_{\mathcal{C}}(M_A) = \text{Tr}(\rho_{\mathcal{C}}^{\lambda C}(A) \text{Tr}(\rho_{\mathcal{C}}^{\lambda C}(B)) = \text{Tr}(\rho_{\mathcal{C}}^{\lambda C} \otimes \rho_{\mathcal{C}}^{\lambda C}(A)) = \text{Tr}(\rho_{D(C)}^{\lambda C}(A) = TV_{\mathcal{C}}(M_A)
\]

A more direct proof of Proposition 3.4 comes from the recent proof by Turaev and Virelizier (see \([59]\)) of the formula \( |M| = RT_{D(C)}(M) \) for any oriented 3-manifold, and spherical fusion category of non-zero
there are modular categories whose anomaly $M$ Thus we have equality of quantum invariants of $C$.

Remark 3.2. Eholzer conjectured in ([14]) that modular representations $\rho_{C,\lambda}^\lambda$ have the congruence property for every RCFT, or in somewhat equivalent terms, for every modular tensor category. If Eholzer’s conjecture were true, then Proposition 3.5 would imply that $RT_C(M_A) = RT_C(M_B)$ for every modular tensor category $C$ if $A$ and $B$ are conjugate in every congruence quotient of $SL(2,\mathbb{Z})$ and $\varphi(A) = \varphi(B)$.

Remark 3.3. Notice that Ng and Schauenburg proved in ([44]) that the projective representation $\mathcal{P}_C$ has the congruence property for every modular tensor category $C$. However this does not imply that some linear lift $\rho_{C,\lambda}^\lambda$ of it has also the congruence property (see [44], section 7). Moreover, it is not clear whether the fact that $\rho_{C,\lambda}^\lambda$ has the congruence property for one particular value of $(\lambda, \zeta)$ would imply that all modular representations $\rho_{C,\lambda}^\lambda$ do have it.

Example 3.1. Consider now the torus bundles $M_A$ and $M_B$, where $A = \begin{pmatrix} 188 & 275 \\ 121 & 177 \end{pmatrix}$ and $B = \begin{pmatrix} 188 & 11 \\ 3025 & 177 \end{pmatrix}$. We proved above that the absolute values of all TQFT invariants agree on $M_A$ and
\( M_B \). In order to find the phase factors of these invariants we have to compute \( \varphi(A) \) and \( \varphi(B) \). We use the reciprocity law for Dedekind sums which reads:

\[
12s(\alpha, \gamma) + 12s(\gamma, \alpha) = \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma} + \frac{1}{\alpha \gamma} - 3\text{sgn}(\alpha \gamma)
\]

and the obvious \( s(\alpha, \gamma) = s(\alpha', \gamma) \) if \( \alpha \equiv \alpha' \pmod{\gamma} \). We find therefore

\[
\varphi(A) = -1, \varphi(B) = -21
\]

### 4. Proof of Proposition 1.2

Recall from the Introduction that

\[
\Gamma_A = \langle t, a, b | ab = ba, tat^{-1} = a^{\alpha_1}b^{\alpha_2}, tbt^{-1} = a^{\alpha_2}b^{\alpha_2} \rangle
\]

where \( A \in SL(2, \mathbb{Z}) \) is a matrix with entries \( \alpha_{ij} \), \( 1 \leq i, j \leq 2 \), such that \( \text{Tr}(A) \neq 2 \). We have the following exact sequence:

\[
1 \to \mathbb{Z}^2 \xrightarrow{i_A} \Gamma_A \xrightarrow{p_A} \mathbb{Z} \to 1
\]

defined by

\[
i_A(1, 0) = a, i_A(0, 1) = b, p_A(t) = 1, p_A(a) = p_A(b) = 0
\]

**Proposition 4.1.** The abelian subgroup \( i_A(\mathbb{Z}^2) \subset \Gamma_A \) is the radical set \( R_A \) of \( [\Gamma_A, \Gamma_A] \) in \( \Gamma_A \), namely the set of those \( x \in \Gamma_A \) for which there exists some \( k \neq 0 \) such that \( x^k \in [\Gamma_A, \Gamma_A] \).

**Proof.** Consider \( x \in R_A \). By the definition of the radical set there exists \( k \neq 0 \) such that \( x^k \in [\Gamma_A, \Gamma_A] \) and thus the image of \( x^k \) vanishes in every abelian quotient of \( \Gamma_A \). This implies that \( p_A(x^k) = 0 \). Since \( k \neq 0 \) we have \( p_A(x) = \frac{1}{k} p_A(x^k) = 0 \), which means that \( x \in \ker p_A = i_A(\mathbb{Z}^2) \).

**Lemma 4.1.** Every element of \( \Gamma_A \) can be uniquely written in the form \( t^s a^a b^b \).

**Proof.** For every \( x \in \mathbb{Z}^2 \) the conjugacy by the stable letter \( t \) can be expressed as follows:

\[
ti_A(x)t^{-1} = i_A(A(x)), t^{-1}i_A(x)t = i_A(A^{-1}(x))
\]

where \( A(x) \) denotes the left multiplication by the matrix \( A \) of the vector \( x \in \mathbb{Z}^2 \).

Consider now a word in the generators containing at least one letter \( t \). We use the conjugacy relations above to move to the left every occurrence of the letter \( t \) (or \( t^{-1} \)). If a leftmost subword of the new word is of the form \( \iota_A(x)t^\epsilon \), with non-zero \( x \in \mathbb{Z}^2 \) then rewrite it as \( t^\epsilon(t^{-\epsilon}\iota_A(x)t^\epsilon) = t^\epsilon \cdot i_A(A^{-\epsilon}(x)) \) and continue. This process will stop eventually because there are only finitely many occurrences of \( t \) and the resulting word will have the desired form.

For the uniqueness it suffices to see that \( \Gamma_A \) is a HNN extension with one stable letter and conclude by the classical Britton’s Lemma.

**Lemma 4.2.** Let \( U \) denote the integral matrix \( A - 1 \). Then \( [\Gamma_A, \Gamma_A] \) is the subgroup \( i_A(U(\mathbb{Z}^2)) \) of \( R_A \).

**Proof.** If \( x \in \mathbb{Z}^2 \) then we have the identities:

\[
[t, i_A(x)] = ti_A(x)t^{-1}i_A(x^{-1}) = i_A(A(x) - x)
\]

This shows that \( i_A(U(\mathbb{Z}^2)) \subset [\Gamma_A, \Gamma_A] \).

Conversely, let us consider \( u \in i_A(U(\mathbb{Z}^2)) \). Then \( tut^{-1} \in i_A(U(\mathbb{Z}^2)) \) and \( t^{-1}ut \in i_A(U(\mathbb{Z}^2)) \) because

\[
ti_A(A(x) - x)t^{-1} = i_A(A^2(x) - A(x)) = i_A(A(y) - y), \text{ where } y = Ax
\]

and

\[
t^{-1}i_A(A(x) - x)t = i_A(x - A^{-1}(x)) = i_A(A(y) - y), \text{ where } y = A^{-1}x
\]

Further \( p_A([\Gamma_A, \Gamma_A]) = 0 \) so that \( [\Gamma_A, \Gamma_A] \) is a subgroup of \( R_A \), in particular it is abelian. Thus the action of \( a \) and \( b \) (or any \( x \in i_A(U(\mathbb{Z}^2)) \)) by conjugacy on \( i_A(U(\mathbb{Z}^2)) \) is trivial.
Now $[\Gamma_A, \Gamma_A]$ is generated by the commutators of elements of $\Gamma_A$. The Hall identities
\begin{equation}
[xy, z] = y^{-1}[x, z]y \cdot [y, z], \quad [x, yz] = [x, z] \cdot z^{-1}[x, y]z
\end{equation}
Then a double recurrence on the number of letters in the words representing $x, y \in \Gamma_A$ and the previous observations about the conjugacy by generators prove that $[x, y] \in i_A(U(Z^2))$. \hfill \Box

**Lemma 4.3.** If $A$ has no eigenvalue equal to 1 then $[\Gamma_A, \Gamma_A]$ is a finite index subgroup of $R_A$.

**Proof.** Since $\text{Tr}(A) \neq 2$ we have $|\text{det}(U)| = |\text{Tr}(A) - 2| \neq 0$ so that $UZ^2 \subset Z^2$ is a subgroup of index $|\text{det}(U)|$. Since $i_A$ is an isomorphism the Lemma follows. \hfill \Box

In particular, for every $x \in i_A(Z^2)$ there exists some $k$ (which divides $|\text{det}(U)|$) for which $x^k \in \left[\Gamma_A, \Gamma_A\right]$, as claimed. \hfill \Box

Now any isomorphism $\phi: \Gamma_A \to \Gamma_B$ should restrict to an isomorphism $\phi: i_A(Z^2) \to i_B(Z^2)$. In fact, if $x \in R_A$ there exists $k \neq 0$ such that $x^k \in \left[\Gamma_A, \Gamma_A\right]$ so that $\phi(x^k) \in \left[\Gamma_B, \Gamma_B\right]$, meaning that $\phi(x) \in R_B$. Now isomorphisms $\phi$ between free abelian groups are determined by some invertible matrix, namely
\begin{equation}
\phi(i_A(x)) = i_B(V(x)), \quad x \in Z^2
\end{equation}
where $V \in GL(2, Z)$.

**Lemma 4.4.** There exists $E \in Z^2$ such that either $\phi(t) = ti_B(E)$ or $\phi(t) = t^{-1}i_B(E)$.

**Proof.** In fact $\phi$ induces an isomorphism $\phi_* = \Gamma_A/R_A \to \Gamma_B/R_B$. Now both groups $\Gamma_A/R_A$ and $\Gamma_B/R_B$ are isomorphic to $Z$ and so $\phi_* \in \varepsilon \Gamma_2$ where $\varepsilon \in \{-1, 1\}$. This is precisely the claim of the Lemma. \hfill \Box

In order to get rid of the translation factor in $\phi$ we need the following extension result:

**Lemma 4.5.** For every $E \in Z^2$ there exists an automorphism $L_E: \Gamma_B \to \Gamma_B$ such that
\begin{equation}
L_E(ti_B(x)) = ti_B(x + E), \quad L_E(i_B(x)) = i_B(x), \quad \text{for every} \quad x \in Z^2
\end{equation}

**Proof.** We have to show that the homomorphism defined on the generators by:
\begin{equation}
L_E(t) = ti_B(E), \quad L_E(a) = a, \quad L_E(b) = b,
\end{equation}
is well-defined. First we compute:
\begin{equation}
L_E(t^{-1}) = (ti_B(E))^{-1} = i_B(-E)t^{-1} = t^{-1} \cdot ti_B(-E)t^{-1} = t^{-1}i_B(-B(E))
\end{equation}
It suffices now to verify that the relations in $\Gamma_B$ are preserved, namely at first:
\begin{equation}
L_E(tx^{-1}) = ti_B(x + E)t^{-1}i_B(-B(E)) = i_B(B(x + E))i_B(-B(E)) = i_B(B(x))
\end{equation}
for $x \in Z^2$, and second $L_E(ab) = L_E(ba)$, which is obvious. Hence $L_E$ defines a homomorphism, whose inverse is $L_{-E}$ which implies that $L_E$ is an automorphism of $\Gamma_B$. An immediate computation shows that
\begin{equation}
L_C(t^s i_B(x)) = \begin{cases} 
t^s i_B(x + E + B(E) + B^2(E) + \cdots + B^{s-1}(E)), & \text{if} \quad s \geq 1 \\
t^s i_B(x - B^{-1}(E) - B^{-2}(E) - \cdots - B^{-s}(E)), & \text{if} \quad s \leq -1
\end{cases}
\end{equation}
This proves the Lemma. \hfill \Box

We replace now the isomorphism $\phi$ by the composition $L_{-E} \circ \phi: \Gamma_A \to \Gamma_B$, which has trivial translation part and keep the same notation $\phi$ for the new isomorphism which has the property that
\begin{equation}
\phi(t) = t^\varepsilon, \quad \text{where} \quad \phi_* = \varepsilon \Gamma_2
\end{equation}
Recall now that $ti_A(x)t^{-1} = i_A(A(x))$, for any $x \in Z^2$. If $\varepsilon = 1$ then on one hand we have
\begin{equation}
\phi(t_iA(x)t^{-1}) = t\phi((i_A(x))t^{-1} = ti_B(V(x)t^{-1} = i_B(BV(x))
\end{equation}
and on the other hand:
\begin{equation}
\phi(t_iA(x)t^{-1}) = \phi(i_A(A(x)) = i_B(VA(x))
\end{equation}
The two right hand side terms from above must coincide, so \( i_B(VA(x)) = i_B(BV(x)) \) for every \( x \in i(\mathbb{Z}^2) \), which implies \( VA = BV \) so that \( A \) and \( B \) are conjugate within \( GL(2, \mathbb{Z}) \).

**Lemma 4.6.** There is an automorphism \( J : \Gamma_B \rightarrow \Gamma_{B^{-1}} \) given by \( J(t) = t^{-1}, J(a) = a, J(b) = b \).

**Proof.** Clear, by direct computation. \( \square \)

Assume now that \( \varepsilon = -1 \). We consider then the isomorphism \( J \circ \phi : \Gamma_A \rightarrow \Gamma_{B^{-1}} \), which satisfies:

\[
J \circ \phi(t) = t, J \circ \phi(i_A(x)) = i_B(V(x)), \quad x \in \mathbb{Z}^2
\]

The argument from above shows now that \( A \) and \( B^{-1} \) are conjugate by the matrix \( V \in GL(2, \mathbb{Z}) \). This proves Proposition 1.2.

**Remark 4.1.** The result holds more generally when \( A \) and \( B \in GL(2, \mathbb{Z}) \) and \( |\text{Tr}(A)| \neq 2 \neq |\text{Tr}(B)| \), with the same proof.

5. Proof of Proposition 1.3

**Proposition 5.1.** Let \( A = \begin{pmatrix} 1 & kq^2 \\ kv & 1 + k^2q^2v \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & k \\ kvq^2 & 1 + k^2q^2v \end{pmatrix} \) denote matrices from \( SL(2, \mathbb{Z}) \), where \( k \in \mathbb{Z}, q \) is an odd prime number \( q \equiv 1(\text{mod } 4) \), \( v \) is a positive integer such that \( -v \) is a non-zero quadratic residue mod \( q \) which is divisible either by a prime \( p \equiv 3(\text{mod } 4) \), or by \( 4 \). Then the following hold:

1. For every natural \( N \) there exists \( T_N \in SL(2, \mathbb{Z}) \) such that \( \nu_N(A) \) and \( \nu_N(B) \) are conjugate by the matrix \( \nu_N(T) \), where \( \nu_N : \mathbb{Z} \rightarrow \mathbb{Z}/N\mathbb{Z} \) is the reduction mod \( N \).

2. The matrix \( A \) is conjugate neither to \( B \) nor to \( B^{-1} \) within \( GL(2, \mathbb{Z}) \).

**Proof.** The conjugacy condition \( TA = BT \) is equivalent to a linear system of equations having the 2-parameter family of solutions \( T(x, y) = \begin{pmatrix} x \\ vy \end{pmatrix} = \begin{pmatrix} x/y \\ q^2x + kq^2vy \end{pmatrix} \). The matrix \( T \) belongs to \( SL(2, \mathbb{Z}) \) if and only if \( x, y \in \mathbb{Z} \) and the determinant of \( T \) is 1, namely if and only if the quadratic Diophantine equation

\[
q^2x^2 + kq^2xy - vy^2 = 1
\]

has integral solutions.

In a similar way \( \nu_N(A) \) and \( \nu_N(B) \) are conjugate by a matrix \( T \in SL(2, \mathbb{Z}/N\mathbb{Z}) \) if and only if the equation \( (59) \) has solutions \( x, y \in \mathbb{Z}/N\mathbb{Z} \). We can improve this last statement as follows:

**Lemma 5.1.** The homomorphism \( \nu_N : SL(m, \mathbb{Z}) \rightarrow SL(m, \mathbb{Z}/N\mathbb{Z}) \) is surjective.

**Proof.** It is known that the group \( SL(m, \mathbb{Z}/N\mathbb{Z}) \) is generated by the matrices of the form \( I + E_{ij} \), where \( E_{ij} \) has only one non-zero entry, which is 1, sitting in position \( ij \). See (31, Thm. 4.3.9) for a proof.

Here is an explicit construction when \( m = 2 \). Let \( U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \) be an integral matrix whose reduction mod \( N \) is a given matrix of \( SL(2, \mathbb{Z}/N\mathbb{Z}) \). There exist integers \( \alpha, \beta \) such that \( \alpha u_{12} - \beta u_{11} = \gcd(u_{11}, u_{12}) \). Set then:

\[
T = \begin{pmatrix}
u_{11} + N(\alpha + u_{11}\gamma) + 1 - \det U & \nu_{12} + Nb + N(\beta + u_{12}\gamma) + 1 - \det U \\ u_{21} - \frac{1}{\gcd(u_{11}, u_{12})u_{11}\gamma(\det U - 1)} & u_{22} - \frac{1}{\gcd(u_{11}, u_{12})u_{12}\gamma(\det U - 1)}
\end{pmatrix}
\]

where \( \gamma = \beta u_{21} - \alpha u_{22} \). Now \( \nu_N(T) = \nu_N(U) \) because \( \nu_N(\det U) \equiv 1 \in \mathbb{Z}/N\mathbb{Z} \), and \( \det T = 1 \) so that \( T \in SL(2, \mathbb{Z}) \). \( \square \)

Therefore, if the Diophantine equation \( (59) \) has solutions in \( \mathbb{Z}/q^2\mathbb{Z} \) for every prime \( q \) then, for every natural \( N \) there exists \( T_N \in SL(2, \mathbb{Z}) \) such that \( \nu_N(A) \) and \( \nu_N(B) \) are conjugate by the matrix \( \nu_N(T) \).

**Lemma 5.2.** If \( -v \) is a non-zero quadratic residue mod \( q \) then the equation \( (59) \) has solutions in \( \mathbb{Z}/N\mathbb{Z} \) for every \( N \).
Proof. Let us show that this equation has solutions mod $p^l$ for every prime $p$ and positive integer $l$, which will imply that there exist solutions mod $N$ for every $N$.

If $p \neq q$ then take $x = q$ and $y = 0$, where $\pi$ denotes the inverse of $a \mod p^l$. If $p = q$ then $-v$ is also a quadratic residue mod $q^l$ for every positive $l$, by the quadratic reciprocity law and the fact that $q \equiv 1 \pmod{4}$. Thus there exists an invertible $z$ such that $-v \equiv z^2 \pmod{q^l}$. Therefore $x = 0$ and $y = \pi$ is a solution mod $h^l$.

Lemma 5.3. If $q$ is an odd prime and $v$ is a positive integer then the equation (59) has not integral solutions.

Proof. The discriminant is a perfect square $w^2$ such that

$$w^2 - (k^2q^4v^2 + 4q^2v)y^2 = 4q^2$$

If $k = 2n$ is even then $w$ is even and divisible by $q$ so that we can put $w = 2qu$, for some integer $u$ satisfying:

$$u^2 - (n^2q^2v^2 + v)y^2 = 1$$

If $k$ is odd then $w = qu$ and the equation reads:

$$u^2 - (k^2q^2v^2 + 4v)y^2 = 4$$

Lemma 5.4. If $(u, y)$ is an integer solution for either one of the equations (61) or (62) then $y$ is divisible by $q$.

Proof. Consider first $k$ even when the equation (61) is a Pell equation. Let us remind briefly the theory of the Pell equation:

$$u^2 - Dy^2 = 1$$

where $D$ is a positive integer, which is not a square. There exists only one minimal solution which can be constructed following classical results (see [46, 45]) as follows. We set

$$P_0 = 0, Q_0 = 1, a_0 = \lfloor \sqrt{D} \rfloor$$

(64)

$$H_{-2} = 1, H_{-1} = 0, G_{-2} = -P_0, G_{-1} = Q_0$$

(65)

We define inductively:

$$H_i = a_iH_{i-1} + H_{i-2}, G_i = a_iG_{i-1} + G_{i-2}$$

(66)

$$P_i = a_{i-1}Q_{i-1} - P_{i-1}, Q_i = (D - P_i^2)/Q_{i-1}$$

(67)

$$a_i = \left\lfloor \frac{P_i + \sqrt{D}}{Q_i} \right\rfloor$$

(68)

We have therefore

$$G_{i-1}^2 - DH_{i-1}^2 = (-1)^iQ_i$$

(69)

The algorithm for solving the Pell equation is as follows. Find the smallest even integer $l \geq 1$ such that $Q_l = 1$. Then $(G_{l-1}, H_{l-1})$ is the minimal non-trivial solution $(u_0, y_0)$ to the Pell equation (63).

Moreover, any other (positive) integral solution can be obtained from the minimal one by means of the following recurrence:

$$u_s + 1 = u_0u_s + Dy_0y_s, y_{s+1} = y_0u_s + x_0y_s, \text{ for } s \geq 0$$

(70)

The previous algorithm (notice that $D$ is not a square) gives us the minimal solution for (61)

$$u_0 = 2n^2q^2v + 1, y_0 = 2nq$$

(71)

A recurrence on $s$ shows that $y_s$ is a multiple of $q$ for every $s \geq 0$.

Assume now that $k$ is odd where the equation (62) is a Pell-type equation.
When \( v \) is odd we can use the same algorithm as used for the Pell equation above to solve \( (62) \) but starting from the initial data:

\[
P_0 = 1, \quad Q_0 = 2
\]

because we are in the situation when \( D \equiv 1 \pmod{4} \). Then the minimal solution is

\[
u_0 = k^2 q^2 v + 2, \quad y_0 = kq
\]

and the same arguments show that all solutions \( y \) are multiple of \( q \).

Eventually assume that \( v \) is even, \( v = 2v' \), such that \( u = 2u' \) for some integer \( u' \) and the equation \( (62) \) becomes:

\[
u'^2 - (k^2 q^2 v'^2 + 2v')y^2 = 1
\]

One finds the minimal solutions

\[
u_0' = k^2 q^2 v' + 1, \quad y_0 = kq
\]

Thus all solutions \( y \) are multiple of \( v \). This proves Lemma 5.3. \( \Box \)

**Remark 5.1.** If \( v \) is negative the minimal solutions are different, for instance when \( v = -1 \) and \( k \) is even we have \( u_0 = \frac{1}{2}q, \ y_0 = 1 \), so that the previous Lemma cannot be extended to negative \( v \).

Going back to the original equation \( (59) \) if \( y \) were a multiple of \( q \) it would imply that \( q \) divides 1, which is a contradiction. Thus \( (59) \) has not integral solutions and hence Lemma 5.3 is proved. \( \Box \)

Another pair of matrices satisfying the claims from Proposition 5.1 was obtained by Stebe in [56], as follows:

\[
A = \begin{pmatrix} 188 & 275 \\ 121 & 177 \end{pmatrix}, \quad B = \begin{pmatrix} 188 & 11 \\ 3025 & 177 \end{pmatrix}
\]

**Remark 5.3.** We have \( \varphi(A) = \varphi(B) = 2k(q^2 - 1)(v + 1) \). Since \( v \) is positive all pairs \( (A, B) \) furnished by Proposition 5.1 have \( \varphi(A) \neq \varphi(B) \) and hence the manifolds \( M_A \) and \( M_B \) can be distinguished by their Reshetikhin-Turaev invariants.

**Remark 5.4.** There exist always rational solutions to the Diophantine equation above and thus the matrices \( A \) and \( B \) are always conjugate within \( SL(2, \mathbb{Q}) \). This implies that the associated 3-manifolds \( M_A \) and \( M_B \) are commensurable.
6. Proof of Theorem 1.3

6.1. Outline of the proof. According to the previous sections it suffices to show that there exist pairs of Anosov matrices $A$ and $B$ such that their images $A$ and $B$ are conjugate within $SL(2, \mathbb{Z}/m\mathbb{Z})$ for every $m$, neither $A$ and $B$, nor $A$ and $B^{-1}$ are conjugate in $GL(2, \mathbb{Z})$ (thus satisfying the claims of Proposition 6.1) and moreover $\varphi(A) = \varphi(B)$. The key ingredient is to reformulate these requirements as follows:

**Proposition 6.1.** There exist infinitely many pairs of Anosov matrices $A$ and $B$ such that:

1. $A$ and $B$ are conjugate in $SL(2, \mathbb{Z}/m\mathbb{Z})$ for every $m$;
2. $A$ and $B$ are reciprocal, namely they are conjugate in $SL(2, \mathbb{Z})$ to $A^{-1}$ and $B^{-1}$ respectively;
3. $A$ and $B$ are inert, namely they are conjugate in $SL(2, \mathbb{Z})$ to $wA^{-1}$ and $wBw^{-1}$ respectively, where $w = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;
4. $A$ and $B$ are not conjugate in $SL(2, \mathbb{Z})$.

**Proof of Theorem 1.3 assuming Proposition 6.1** If $A$ were conjugate to $B$ in $GL(2, \mathbb{Z})$, namely $A = sBs^{-1}$, with $s \in GL(2, \mathbb{Z})$ then $\det(s) = -1$ and $wAw^{-1} = wsB(ws)^{-1}$, with $\det(ws) = 1$. Since $A$ is inert, this would imply that $A$ is conjugate in $SL(2, \mathbb{Z})$ to $B$, which contradicts our assumption. Since $A$ and $B$ are reciprocal $A$ cannot be conjugate to $B^{-1}$ in $GL(2, \mathbb{Z})$ either.

Eventually recall that $\varphi$ is constructed from $\Phi_R$ in such a way that it becomes a quasi-homomorphism $\varphi : SL(2, \mathbb{Z}) \to \mathbb{Z}$. Namely, the following hold (see [59]) :

\begin{equation}
\varphi(CAC^{-1}) = \varphi(A), \quad \text{for } C \in SL(2, \mathbb{Z}),
\end{equation}

\begin{equation}
\phi(A^{-1}) = -\varphi(A)
\end{equation}

In particular, if $A$ and $B$ are reciprocal, then $\varphi(A) = \varphi(B) = 0$ and this actually holds for any quasi-homomorphism $\varphi$. This will settle Theorem 1.3.

6.2. Proof of Proposition 6.1. Reciprocal (conjugacy) classes in $SL(2, \mathbb{Z})$ were recently discussed by Sarnak in [55]. Let $A^\perp$ denote the transpose of $A$. Since the transpose $(A^{-1})^\perp = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, it follows that $A$ is reciprocal if and only if it is conjugate to its transpose $A^\perp$ (see also [55], p.218). Recall that $A$ is ambiguous if $A$ is conjugate within $SL(2, \mathbb{Z})$ to $w^{-1}A^{-1}w$.

We say that $A$ and $B$ are in the same genus if their images are conjugate within $SL(2, \mathbb{Z}/m\mathbb{Z})$, for every $m$. Our aim is to find reciprocal and inert conjugacy classes in the same genus.

Let $D$ be an odd (square-free) fundamental discriminant. Following Gauss (see [6]) there are $2^{\sigma(D)-1}$ genera of primitive integral binary forms, where $\sigma(D)$ is the number of distinct prime divisors of $D$.

Denote by $D^-$ the set of those $D$ for which the negative Pell equation

\begin{equation}
X^2 - DY^2 = -4
\end{equation}

has integral solutions. It is known that $D \in D^-$ if and only if the narrow class group $C_D$ coincides with the class group $Cl_D$ of $\mathbb{Q}(\sqrt{D})$ (see [57], Lemma 2.1).

Recall that the 4-rank of an abelian group $C$ is the rank of $C^2/C^4$ which counts the number of distinct cyclic factors of order 4.

**Lemma 6.1.** Every $D \in D^-$ such that the 4-rank of $C_D$ is non-trivial gives raise to a pair of non-conjugate reciprocal matrices in the same principal genus.
Proof. According to Gauss (see [4]) the group of genera is $C_D/C_D^2$. The classes in the kernel of the projection $C_d \rightarrow C_d/C_D^2$ form the principal genus. The set of ambiguous classes is identified with the kernel of the square homomorphism $\delta : C_D \rightarrow C_D$ given by $\delta(x) = x^2$. Therefore the elements of order $2^n$ in $C_D$, with $n \geq 2$ are ambiguous classes in the principal genus.

When $D \in D^-$ it is known that every class is inert and every ambiguous class is reciprocal and viceversa (see [53], p.214). In particular, if the 4-rank of $C_D$ is positive then there are at least two inert and reciprocal classes in the principal genus. They are non-conjugate as they are distinct classes in $C_D$. □

There exists a simple method developed by Rédei and Reichardt (see [52]) to find the 4-rank of the narrow class group $C_D$. Let $D = p_1p_2 \cdots p_n$ be the decomposition in odd prime numbers of $D$. The Rédei matrix $M_D$ is the $n$-by-$n$ matrix over $\mathbb{Z}/2\mathbb{Z}$ whose entries $a_{ij}$ are:

$$a_{ij} = \begin{cases} 1, & \text{if } i \neq j, \text{ and } \left( \frac{p_i}{p_j} \right) = -1 \\ 0, & \text{if } i \neq j, \text{ and } \left( \frac{p_i}{p_j} \right) = 1 \end{cases}$$

(82)

$$a_{jj} = \sum_{i\neq j, 1 \leq i \leq n} a_{ij}$$

(83)

Here $\left( \frac{p}{q} \right) \in \{-1,1\}$ is the Legendre symbol, equal to 1 if and only if $p$ is a quadratic residue mod $q$.

Eventually, following [52] the 4-rank of $C_D$ is given by $\sigma(D) - 1 - \text{rank}_{\mathbb{Z}/2\mathbb{Z}} M_D$.

We will consider from now on $D$ of the form $D = u^2 + 4$, so that the negative Pell equation has obvious solutions $X = u, Y = 1$. We seek for those $D$ which are odd square-free and such that the associated Rédei matrix is identically zero. If $D$ has at least two prime factors then the 4-rank of $C_D$ is non-trivial. In this case it is easy to find explicit matrices $A$ and $B$ corresponding to ambiguous, inert and reciprocal pairs of classes in the principal genus. The trace $t$ of $A$ will be

$$t = D - 2$$

(84)

so that it verifies

$$t^2 - Du^2 = 4$$

(85)

For each positive integral solutions $(a, b)$ of the equation $4a^2 + b^2 = D$ we have associated the classes of binary forms $(a, b, -a)$, which correspond to the symmetric matrices:

$$A_{a,b} = \begin{pmatrix} \frac{1}{2}(t - ub) & au \\ au & \frac{1}{2}(t + ub) \end{pmatrix}$$

(86)

These are obviously reciprocal classes in the principal genus $C_D^2$ of $C_D$. The examples in Theorem 1.3 arise when choosing $u \in \{21, 51, 53, 55\}$ for which $\sigma(D) = 2$ and $M_D = 0$.

Eventually, there are infinitely many $D \in D^-$ for which the 4-rank of $C_D$ is positive. Let $\mathcal{D}$ denote the set of special discriminants, namely the set of those $D$ whose prime factorization has only distinct odd primes of the form $p \equiv 1(\text{mod } 4)$ and possibly 8. Then in [19], Theorem 2 the authors state that the subset of those $D \in \mathcal{D}$ for which the 4-rank of $C_D$ equals 1 and the 8-rank vanishes (and hence $D \in D^-$) has positive density within the set $\mathcal{D}$. In particular this set is infinite.

Remark 6.1. We conjecture that the number of distinct cyclic factors of order $2^m \geq 4$ of the class group $C_D$, where $D$ runs over the odd square-free $D$ of the form $n^2 + 4$ is unbounded.

7. Proof of Theorem 1.2

7.1. Abelian invariants. We will consider the $U(1)$ gauge theory as defined in [22, 23, 29, 41] and then generalized in [10, 11]. One chooses a root of unity $q$ of order $k$ for odd $k$ and of order $2k$ for even $k$. Then in [11] there is defined the invariant $Z_q(M, q)$ for 3-manifolds $M$ as follows. Set $L$ be a framed link with $n$ components in $S^3$ such that the 3-manifold $M$ is obtained by Dehn surgery on $L$. Let $A_L$
denote the linking matrix of $L$. We define then after ([41], (1.1)) the MOO invariant of the 3-manifold $M$ as being:

$$Z_k(M, q) = \left( \frac{G_k(q)}{|G_k(q)|} \right)^{-\sigma(A_L)} \sum_{x \in (\mathbb{Z}/k\mathbb{Z})^n} q^{x A_L x}$$  \hspace{1cm} (87)$$

where $\sigma$ denotes the signature of the matrix and the Gaussian sums are given by:

$$G_k(q) = \sum_{h \in \mathbb{Z}/k\mathbb{Z}} q^{h^2}$$  \hspace{1cm} (88)$$

Notice that for even $k$ the value of $q^{x A_L x}$ is defined by taking arbitrary lifts $\tilde{x} \in (\mathbb{Z}/2k\mathbb{Z})^n$ and setting $q^{x A_L x} = q^{\tilde{x} A_L \tilde{x}}$, which is independent on the choice of the lifts, since $A$ is symmetric.

These invariants where further extended by Deloup in [10] by making use of general quadratic forms and finally extended to TQFTs in [11]. These TQFT correspond to suitable modular tensor categories, which are related to the Drinfeld double $D(\mathbb{Z}/k\mathbb{Z})$ of the finite group $\mathbb{Z}/k\mathbb{Z}$ and to the geometric $U(1)$ Chern-Simons gauge theories. A more precise statement is given in ([10], Appendix A) where the invariants $Z_k$ and their generalizations are identified with the Reshetikhin-Turaev invariants associated to a modular category $A$ coming from an abelian group, which is described by Turaev in ([58], p.29).

The Turaev-Viro invariants $TV_A$ are therefore the absolute values of $|Z_k(M, q)|$. The main result of this section is the following:

**Proposition 7.1.** Let $M_A$ and $M_B$ be SOL torus bundles with the same absolute value MOO invariants $|Z_k(M, q)|$, for all $k$. Then, either:

$$\text{Tr}(A) = \text{Tr}(B)$$  \hspace{1cm} (89)$$
or else:

$$\text{Tr}(A) + \text{Tr}(B) = 4$$  \hspace{1cm} (90)$$

Consequently those torus bundles having the same abelian Turaev-Viro invariants as $M_A$ fall into two commensurability classes.

**Proof.** We have first the following explicit computation of the MOO invariants from [41]:

**Lemma 7.1.** If $k$ is odd then we have:

$$|Z_k(M, q)| = |H^1(M, \mathbb{Z}/k\mathbb{Z})|^{1/2}$$  \hspace{1cm} (91)$$

If $k$ is even then:

$$|Z_k(M, q)| = \begin{cases} |H^1(M, \mathbb{Z}/k\mathbb{Z})|^{1/2}, & \text{if } \alpha \cup \alpha \cup \alpha = 0, \text{for every } \alpha \in H^1(M, \mathbb{Z}/k\mathbb{Z}) \\ 0, & \text{otherwise} \end{cases}$$  \hspace{1cm} (92)$$

**Proof.** See ([41], Thm.3.2). \hfill \Box

Further the cohomology of SOL torus bundles is given by:

**Lemma 7.2.** If $M = M_A$ with $A \in SL(2, \mathbb{Z})$ hyperbolic then

$$H^1(M_A, \mathbb{Z}/k\mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z} \oplus \ker \nu_k(A^T - 1)$$  \hspace{1cm} (93)$$

where $A^T$ denotes the transposed of the matrix $A$.

**Proof.** By the Universal Coefficient Theorem $H^1(M_A, \mathbb{Z}/k\mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z}$ = Hom($H_1(M_A), \mathbb{Z}/k\mathbb{Z}$). Since $A$ is hyperbolic $H_1(M_A) = \mathbb{Z} \oplus \text{Tors}(H_1(M_A))$. The torsion part can be computed by abelianizing $\Gamma_A$ and we find $\text{Tors}(H_1(M_A)) = \mathbb{Z}^2/(A - 1)(\mathbb{Z}^2)$, which is a finite abelian group of order $|\det(A - 1)| = |\text{Tr}(A) - 2|$.

Then Hom($\text{Tors}(H_1(M_A)), \mathbb{Z}/k\mathbb{Z}$) is naturally identified with $\ker(A - 1)^{\ast}_k$, where $(A - 1)^{\ast}_k : \text{Hom}(\mathbb{Z}^2, \mathbb{Z}/k\mathbb{Z}) \to \text{Hom}(\mathbb{Z}^2, \mathbb{Z}/k\mathbb{Z})$ is the linear map given by $(A - 1)^{\ast}_k(f) = f \circ (A - 1)$, for $f \in \text{Hom}(\mathbb{Z}^2, \mathbb{Z}/k\mathbb{Z})$. We have a (non-canonical) isomorphism $(\mathbb{Z}/k\mathbb{Z})^2 \to \text{Hom}(\mathbb{Z}^2, \mathbb{Z}/k\mathbb{Z})$ which sends
Consider now two SOL manifolds $M_A$ and $M_B$ having the same absolute value MOO invariants. If the MOO invariants as well as their generalizations from [10] were the same for the two manifolds then the result would be a simple consequence of the main theorem from [12]. In fact these invariants determine the linking pairing of the 3-manifold and in particular the torsion group $\text{Tors}(H_1(M))$.

The case where we know that the absolute value of the MOO invariants agree is only slightly more complicated. First, when $k$ is odd Lemma 7.4 and Lemma 7.2 imply that

$$(94)\quad |\ker \nu_k(A^T - 1)| = |\ker \nu_k(B^T - 1)|$$

In order to compute the orders of the kernels above we have to recall some standard facts concerning the normal forms of integral matrices. Let $C : \mathbb{Z}^n \to \mathbb{Z}^n$ be a non-singular linear map $C : \mathbb{Z}^n \to \mathbb{Z}^n$. Then there exists a (unique) collection of positive integers $r_1, r_2, \ldots, r_n$, called the invariant factors of $C$ with $r_1$ dividing $r_{j+1}$ (when $1 \leq j < n$) such that $C = VD_W$, where $V, W \in GL(n, \mathbb{Z})$ are invertible integral matrices and $D$ is diagonal with entries $r_1, r_2, \ldots, r_n$. Moreover $|\det(C)| = r_1r_2 \cdots r_n$. This is the so-called Smith normal form (see [12], II.15).

This normal form is particularly useful if one seeks for counting the solutions of the congruences system $C(x) \equiv 0(\text{mod } k)$. By above this is equivalent to the system of congruences $r_jx_j \equiv 0(\text{mod } k)$, for $1 \leq j \leq n$. Each congruence above gives g.c.d. $(r_j, k)$ distinct solutions $x_j \text{ mod } k$, so that the total number of solutions of the system is $\prod_{j=1}^n \text{g.c.d.}(r_j, k)$.

Notice that the invariant factors for a 2-by-2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are simply $r_1(A) = \text{g.c.d.}(a, b, c, d)$ and $r_2(A) = \det(A)/r_1(A)$.

Now, for fixed $C$ and $k$ of the form $k = p^r$, with prime $p$, if we choose $r$ large enough such that $r \geq m_{j, p}(C)$, where $r_j = p^{m_{j, p}(C)}s_j$, with g.c.d. $(p, s_j) = 1$ then the previous discussion shows that

$$(95)\quad |\ker \nu_k(C)| = \text{g.c.d.}(|\det(C)|, k)$$

We will apply this formula to $C = A - 1$ and respectively $C = B - 1$, where $k = p^r$ for odd prime $p$ and $r$ is chosen large enough such that

$$(96)\quad r \geq \max(m_{j, p}(A - 1), m_{j, p}(B - 1))$$

Then the relations above imply that

$$(97)\quad \text{g.c.d.}(\text{Tr}(A) - 2, p^r) = \text{g.c.d.}(\text{Tr}(B) - 2, p^r)$$

for every odd prime $p$ and $r$ large enough. Therefore the numbers $|\text{Tr}(A) - 2|$ and $|\text{Tr}(B) - 2|$ have the same odd divisors.

Let now call the even number $k$ to be good for $M$ if we have $\alpha \cup \alpha \cup \alpha = 0$, for every $\alpha \in H^1(M, \mathbb{Z}/k\mathbb{Z})$. Lemma 7.4 shows that $k$ is good if and only if $|Z_k(M, q)| \neq 0$. On the other hand in [11], Cor. 5.3 one founds the following explicit criterion. The number $k$ is not good for $M$, i.e. $Z_k(M, q) = 0$, if and only if there exists $x \in \text{Tors}(H_1(M))$ of order $2^m$ such that $L_M(x, x) = \frac{2^m}{k}$, where $k = 2mb$, with odd $b$, $L_M$ denotes the linking pairing $L_M : \text{Tors}(H_1(M)) \times \text{Tors}(H_1(M)) \to \mathbb{Q}/\mathbb{Z}$, and $c$ is odd.

Now, if $M_A$ and $M_B$ have the same absolute value of MOO invariants then $k$ is good for $M_A$ if and only if $k$ is good for $M_B$. On the other hand, we know that $|\text{Tr}(A) - 2| = 2^{m_A}s$ and $|\text{Tr}(A) - 2| = 2^{m_B}s$, with odd $s$. Observe that any $k$ of the form $k = 2^r$, with $r \geq m_A + 1$ is good for $M_A$ since the torsion $\text{Tors}(H_1(M_A))$ has no elements of order $2^r$. In particular if $r \geq \max(m_A, m_B) + 1$ then $2^r$ is good for both $M_A$ and $M_B$.

Choose now $p = 2$, $k = 2^r$ with $r$ large enough as in (96) and such that $2^r$ is good for both $M_A$ and $M_B$. Then the equality of MOO invariants of $M_A$ and $M_B$ implies

$$(98)\quad |\ker \nu_k(A^T - 1)| = |\ker \nu_k(B^T - 1)|$$
which, by above, is equivalent to the following:

\[ \text{g.c.d.}(\text{Tr}(A) - 2, 2^{n}) = \text{g.c.d.}(\text{Tr}(B) - 2, 2^{n}) \]

Thus \( m_{A} = m_{B} \) and this completes the proof of the fact that:

\[ |\text{Tr}(A) - 2| = |\text{Tr}(B) - 2| \]

If \( \text{Tr}(A) = \text{Tr}(B) \) then \( A \) and \( B \) have the same trace and the same determinant and thus the equation \( XA = BX \) has solutions in \( GL(2, \mathbb{Q}) \), so that \( M_{A} \) and \( M_{B} \) are (strongly) commensurable.

In fact, recall that Barbot and later Bridson and Gersten \((\text{[4]} \text{[5]})\) proved the following:

**Lemma 7.3.** The groups \( \Gamma_{A} \) and \( \Gamma_{B} \) are commensurable if and only if the quotient of their discriminants \( D_{A}/D_{B} \) is the square of an rational. Here the discriminant of \( A \) is \( D_{A} = \text{Tr}(A)^{2} - 4 \det(A) \). Moreover, this is equivalent to the fact that \( A^{p} \) and \( B^{q} \) are conjugate within \( GL(2, \mathbb{Q}) \), for some \( p, q \in \mathbb{Z} \).

Further if \( \text{Tr}(A) + \text{Tr}(B) = 4 \) we have again only one (strong) commensurability class allowed for \( B \). Thus the torus bundles as in the statement of the Proposition fall into two commensurability classes.

\[ \square \]

We will give now several examples to show that all abelian invariants (of Reshetikhin-Turaev type, not only their absolute values) fail to distinguish the two distinct commensurability classes above.

**Proposition 7.2.** Set

\[ A = \begin{pmatrix} 1 & n \\ 1 & n+1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 - 2n & n \\ -1 - 2n & n+1 \end{pmatrix} \quad n \in \mathbb{Z}_{+} \]

The manifolds \( M_{A} \) and \( M_{B} \) have the same quantum abelian invariants although \( \text{Tr}(A) + \text{Tr}(B) = 4 \) and \( \text{Tr}(A) \neq \text{Tr}(B) \), if \( n \geq 1 \) and \( n \neq 4 \). In particular the trace is not detected by the quantum abelian invariants of torus bundles. Moreover, if \( \frac{4n+1}{4n-1} \notin \mathbb{Q}^{2} \) then \( M_{A} \) and \( M_{B} \) (equivalently \( \Gamma_{A} \) and \( \Gamma_{B} \)) are not commensurable.

**Proof.** The quantum abelian invariants from \([10]\) are identical for two manifolds if and only if their first Betti numbers agree and their linking pairings are isomorphic (see \([12]\)).

Let \( T \) be the torus fiber of \( M_{A} \). The we have the exact sequence:

\[ H_{2}(M_{A}) \to H_{1}(T) \xrightarrow{\Lambda} H_{1}(M_{A}) \to H_{1}(T) \to H_{1}(M_{A}) \to \mathbb{Z} \]

Therefore \( H_{1}(M_{A}) = \mathbb{Z} \oplus \text{Tors}(H_{1}(M_{A})) \), where the torsion \( \text{Tors}(H_{1}(M_{A})) \) is the image of \( H_{1}(T) \) into \( H_{1}(M_{A}) \).

The linking pairing \( L_{A} : \text{Tors}(H_{1}(M_{A})) \times \text{Tors}(H_{1}(M_{A})) \to \mathbb{Q}/\mathbb{Z} \) is defined as follows. For every \( \xi \in \text{Tors}(H_{1}(M_{A})) \) we choose a lift of it as an element in \( H_{2}(M_{A}; \mathbb{Q}/\mathbb{Z}) \), namely an element \( \hat{\xi} \) whose image by the boundary connecting homomorphism \( \delta : H_{2}(M_{A}; \mathbb{Q}/\mathbb{Z}) \to H_{1}(M, \mathbb{Z}) \) is exactly \( \xi \). Here the connecting homomorphism comes from the long exact sequence associated to the coefficients exact sequence:

\[ H_{2}(M_{A}; \mathbb{Q}) \to H_{2}(M_{A}; \mathbb{Q}/\mathbb{Z}) \to H_{1}(M, \mathbb{Z}) \to H_{1}(M, \mathbb{Q}) \to \]

We take then \( L_{A}(\eta, \xi) = \eta \cdot \hat{\xi} \in \mathbb{Q}/\mathbb{Z} \) where the intersection product is the one \( H_{1}(M, \mathbb{Z}) \times H_{2}(M_{A}; \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z} \).

If we have a 1-cycle \( \xi \) representing the class \( [\xi] \in H_{1}(T^{2}) \) then its product with \([0, 1]\) yields a 2-chain whose boundary is \( (A - 1)\xi \). This implies that the linking pairing of \( M_{A} \) is given by:

\[ L_{A}(\eta, \xi) = \omega((A - 1)^{-1}(\eta), \xi) \in \mathbb{Q}/\mathbb{Z} \]

where \( \eta, \xi \in H_{1}(T) \cong \mathbb{Z}^{2} \) are representing (torsion) classes in \( \mathbb{Z}^{2}/(A - 1)(\mathbb{Z}^{2}) \subset H_{1}(M_{A}) \) and \( \omega \) is the usual (symplectic) intersection form on \( H_{1}(T) \), namely

\[ \omega((v_{1}, v_{2}), (w_{1}, w_{2})) = v_{1}w_{2} - v_{2}w_{1} \]

The torsion group of 1-homologies of \( M_{A} \) and \( M_{B} \) are both cyclic groups of order \( |\text{Tr}(A) - 2| \) since the first invariant factors for the integral matrices \( A - 1 \) and \( B - 1 \) are both equal to 1. Thus the torsion
homology groups are isomorphic. Now we can verify that \((A - 1)^{-1} - (B - 1)^{-1}\) is the integral matrix
\[
\begin{pmatrix}
2 & -2 \\
2 & -2
\end{pmatrix}
\]
such that the linking pairings of \(M_A\) and \(M_B\) are isomorphic. If \(n \geq 1\) and \(n \neq 4\) then
these torus bundles are SOL manifolds. Their Betti numbers coincide as all SOL manifolds have their
first Betti number equal to 1.

The statement concerning the commensurability is a consequence of the commensurability criterion
for the polycyclic groups from Lemma 7.3 saying that \(\Gamma_A\) and \(\Gamma_B\) are commensurable if and only if
\(D_A/D_B \in \mathbb{Q}^2\). \(\Box\)

Remark 7.1. If \(n = 4\) then \(M_A\) is a SOL torus bundle but \(M_B\) is a NIL manifold. Although their linking
pairings are isomorphic their first Betti numbers are different, as the Nil manifold has Betti number 2.
Another pairs with the same property are
\[
(106) \quad A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 \\ -8 & -5 \end{pmatrix}
\]

7.2. \(SU(2)\)-invariants and the metaplectic representations. Denote by \(\rho_{SU(2),k}\) (and respectively
\(\rho_{U(1),k}\)) the \(SU(2, \mathbb{Z})\) representation associated to the modular tensor category constructed out of
\(SU(2)\) (and respectively \(U(1)\) or \(\mathbb{Z}/k\mathbb{Z}\)) in level \(k\) (see \([35]\)). It should be noticed that the parameters
\(\lambda_{SU(2),k}, \zeta_{SU(2),k}\) do not agree with \(\lambda_{U(1),k}, \zeta_{U(1),k}\). For instance \(\zeta_{U(1),k} = \exp\left(\frac{\pi i}{k}\right)\) is independent
on \(k\). The choice of the rank and anomaly will be irrelevant in the arguments below.

Recall first that both representations \(\rho_{SU(2),k}\) and \(\rho_{U(1),k}\) factor through the finite congruence group
\(SL(2, \mathbb{Z}/k\mathbb{Z})\).

Now explicit formulas for the values of \(SU(2)\) quantum invariants of torus bundles were obtained in
\([22]\) by Jeffrey. Nevertheless it seems difficult to extract explicit topological information out of them.

The key point in our computation is the existence of simple formulas for the characters of the \(SU(2)\)
quantum representations:

Proposition 7.3. We have
\[
(107) \quad 2 \text{Tr}(\rho_{SU(2),k}(A)) = \text{Tr}(\rho_{U(1),k}(A)) - \text{Tr}(\rho_{U(1),k}(-A))
\]

Proof. The finite symplectic groups \(Sp(2g, \mathbb{Z}/k\mathbb{Z})\) are endowed with (projective) representations into
some complex vector space \(V_k\), which are known under the name of Segal-Shale-Weil metaplectic
representations. Although these were classically constructed only for prime \(k\) there exist now several
constructions valid for every \(k\). In \([22, 23, 29]\) one constructed such representations for every even
\(k\) (and for a congruence quotient of the Theta group \(\Gamma[2]\) when \(k\) is odd) in any dimension \(g\) using level
\(k\) theta functions. The monodromy representations from \([11]\) agree with the previous constructions
and work for every odd \(k\) as well. Later in \([18]\) one described a direct construction of the metaplectic
\(SL(2, \mathbb{Z}/k\mathbb{Z})\) representations which were further generalized in \([31]\) to higher dimensions.

The following seems to be widely known among experts:

Lemma 7.4. The \(SL(2, \mathbb{Z})\) quantum representations \(\rho_{U(1),k}\) are lifts of the projective metaplectic
representations.

The theta functions construction was generalized in \([24]\) to quantizations of multidimensional tori
endowed with Coxeter group actions. This leads to finite symplectic group representations depending
on a semisimple Lie group \(G\) or, equivalently on a Coxeter group \(W\) (corresponding to the Weyl group
of \(G\)). It was already noticed in \([24]\) that the \(SL(2, \mathbb{Z}/k\mathbb{Z})\) representations associated to \(W = \mathbb{Z}/2\mathbb{Z}\)
coincide (projectively) with \(\rho_{SU(2),k}\).

Lemma 7.5. Let \(\tau = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\) \(\in SL(2, \mathbb{Z})\). The space \(V_k\) splits into eigenspaces for the metaplectic
action of \(\tau\) as \(V_k = V_k^+ \oplus V_k^-\), where
\[
(108) \quad V_k^\pm = \{ x \in V_k; \rho_{U(1),k}(\tau)(x) = \pm x \}.
\]
Then the representation $\rho_{SU(2),k}$ of $SL(2, \mathbb{Z})$ is isomorphic to the restriction $\rho_{U(1),k}|_{V_k^{-}}$ of the metaplectic representation to the invariant sub-module $V_k^{-}$.

Proof. This was made so by the explicit construction in [24]. The result was also formulated explicitly in (21), section 5) for prime $k$, but the same argument is valid for all $k$ when comparing with the formulas in [18]. A more precise result was given by Larsen and Wang in [37] and independently by Gilmer in ([27], Thm.5.2).

The two lemmas above prove the claim, since the characters of the factors $V^\pm$ are precisely the $\pm$-invariant part of the character of $V_k$. \hfill \blacksquare

Recall now from Proposition 5.2 and equation (26) that the Reshetikhin-Turaev quantum invariants of the torus bundle $M_A$ are suitable multiples of the corresponding characters, as follows:

$$RT_{SU(2),k}(M_A) = \zeta^{-3\varphi(A)} Tr(\rho_{SU(2),k}(A)), \quad RT_{U(1),k}(M_A) = \zeta^{-3\varphi(A)} Tr(\rho_{U(1),k}(A))$$

The Turaev-Viro abelian invariant is known to be the same as the absolute value of the MOO invariant (up to a scalar) and this can be extended as follows:

Lemma 7.6. For any oriented 3-manifolds we have:

$$RT_{U(1),k}(M) = k^{-1/2} Z_k(M, q)$$

Proof. We know that $TV_{U(1),k}(M_A) = k^{-1/2} Z_k(M, q)$ and the associated projective representations are isomorphic (see [11], [22, 23, 29]). The anomalies are the same and thus the associated Reshetikhin-Turaev invariants agree.

Another proof is given in ([11], Appendix A) where one uses the modular tensor category from ([28], p.29).

Assume now that $RT_{SU(2),k}(M_A) = RT_{SU(2),k}(M_B)$ and $RT_{U(1),k}(M_A) = RT_{U(1),k}(M_B)$. Then Proposition 7.3 and relations (109) imply that $RT_{U(1),k}(M_{\tau A}) = RT_{U(1),k}(M_{\tau B})$. In particular, applying the result of Proposition 7.1 we obtain that either $\text{Tr}(A) = \text{Tr}(B)$ or else $\text{Tr}(-A) + \text{Tr}(-B) = 4$. The only possibility is that $\text{Tr}(A) = \text{Tr}(B)$.

The case when the Turaev-Viro invariants of the two manifolds agree is only slightly more complicated. The key point is that Proposition 7.3 leads to a closed formula for the $SU(2)$ quantum invariants of torus bundles. We restrict, for the sake simplicity, to the case of Turaev-Viro invariants, which are central in our argument.

Proposition 7.4. Let $A \in SL(2, \mathbb{Z})$ and $k$ be large enough such that whenever $p^m$, with prime $p$ and $m \geq 1$, divides some invariant factors of $A - 1$ or $A + 1$ then it also divides $k$. Then the $SU(2)$-Turaev-Viro invariant of $M_A$ is given by

$$|\text{Tr}(\rho_{SU(2),k}(A))|^2 = TV_{SU(2),k}(M_A) =$$

$$\left(\sqrt{\text{gcd}(\text{Tr}(A) - 2, k)} - \exp \left(\frac{\pi i}{4} (f_k(M_A))\right) \sqrt{\text{gcd}(\text{Tr}(A) + 2, k)}\right)^2$$

where $f_k(M_A) = \phi_k(M_{\tau A}) - \phi_k(M_A)$, $\tau A = -A$ and $\phi_k(M_A) \in \mathbb{Z}/8\mathbb{Z}$ is the function introduced in ([11], section 4).

Proof. We need first the following:

Lemma 7.7. If $A$ is hyperbolic then $\varphi(A) = \varphi(\tau A)$.

Proof. By definition $\Phi_R(A) = \Phi_R(-A)$ since the Rademacher function is defined on $PSL(2, \mathbb{Z})$. Further, by (3.22) the function $\varphi(A) - \Phi_R(A)$ is equal to $\text{sgn}(\gamma(\alpha + \delta - 2))$ when $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and so it also satisfies $\varphi(A) - \Phi_R(A) = \varphi(-A) - \Phi_R(-A)$ when $A$ is hyperbolic, by direct inspection. \hfill \blacksquare
The last lemma implies that
\begin{equation}
Z_k(M_A, q) - Z_k(M_{rA}, q) = \zeta^{-3\varphi(A)} \left(\Tr(\rho_{U(1), k}(A)) - \Tr(\rho_{U(1), k}(rA))\right)
\end{equation}
We have the following:

**Lemma 7.8.** If $A$ is hyperbolic and $k$ is good for $M_A$ and sufficiently large then
\begin{equation}
\Tr(\rho_{U(1), k}(A)) = \exp \left(\frac{\pi i}{4} (\varphi(A) + \phi_k(M_A))\right) |\ker \nu_k(A - 1)|
\end{equation}
where $\phi_k(M_A) \in \mathbb{Z}/8\mathbb{Z}$ is the function introduced in (41), section 4.

**Proof.** The MOO invariant was computed in (41), Thm.4.5 for those $k$ for which the invariant is non-zero, as being
\begin{equation}
Z_k(M_A, q) = \exp \left(\frac{\pi i}{4} (\varphi(A) + \phi_k(M_A))\right) |H^1(M_A, \mathbb{Z}/k\mathbb{Z})|
\end{equation}
Since $\zeta^3 \left(\frac{\pi}{4} \right)$ we obtain:
\begin{equation}
\Tr(\rho_{U(1), k}(A)) = \zeta^{3\varphi(A)} RT_{U(1), k}(M_{rA}) = \exp \left(\frac{\pi i}{4} (\varphi(A) + \phi_k(M_A))\right) k^{-1/2}|H^1(M_A, \mathbb{Z}/k\mathbb{Z})|^{1/2}
\end{equation}
which implies the claim. □

Now, if $A$ is hyperbolic and $k$ is large enough then use Lemma 7.8 to derive:
\begin{equation}
|\Tr(\rho_{SU(2), k}(A))|^2 = TV_{SU(2), k}(M_A) = |RT_{SU(2), k}(M_A)|^2 = k^{-1}|Z_k(M_A, q) - Z_k(M_{rA}, q)|^2 =
\end{equation}
\begin{equation}
|\ker \nu_k(A - 1)|^{1/2} - \exp \left(\frac{\pi i}{4} (\varphi(M_{rA}) - \phi_k(M_A))\right) |\ker \nu_k(A + 1)|^{1/2}|^2
\end{equation}
Then the closed formula (111) follows. □

### 7.3. End of the proof of Theorem 1.2

It remains to prove the following:

**Proposition 7.5.** If the SOL torus bundles $M_A$ and $M_B$ have the same abelian and $SU(2)$ Turaev-Viro invariants then $\Tr(A) = \Tr(B)$.

**Proof.** We have to recall (see e.g. 58, section VI) that the modular tensor category $\mathcal{C}_{SU(2)}$ which is leading to the $SU(2)$ invariants is defined only when the level is of the form $4n$, with $n \geq 3$.

We assume that $a = \Tr(A) \neq \Tr(B)$. According to Proposition 7.4 we must have $\Tr(B) = 4 - a$. Let $k$ be large enough in order to be good for $M_A$ and $M_B$ and also to verify (100). We put $\gcd(a - 2, k) = v$ and $\gcd(a - 6, k) = w$. Then (110) implies that
\begin{equation}
-2 \cos \left(\frac{\pi i}{4} f_A\right) \sqrt{uw} + v = -2 \cos \left(\frac{\pi i}{4} f_A\right) \sqrt{vw} + w
\end{equation}
where $f_A = \phi_k(M_{rA}) - \phi_k(M_A)$.

This is equivalent to the equation:
\begin{equation}
(\sqrt{v} - \sqrt{w}) \left(\sqrt{v} + \sqrt{w} - 2 \cos \left(\frac{\pi i}{4} f_A\right) \sqrt{w}\right) = 0
\end{equation}

**Lemma 7.9.** The prime divisors of $a - 6$ are the same as the prime divisors of $a + 2$.

**Proof.** Suppose that there exists some odd $p$ which divides $a - 6$ but not $a + 2$. We write $a - 6 = 2^sp^r c$, with $c$ odd and coprime with $p$, $r \geq 1$.

Assume first $s \geq 3$. We chose $k$ of the form $k = 2^mp^r$ (with $m$ large with respect to $r$ and $s$). Then $w = 2^sp^r, u = \gcd(4(2^{s-2}p^r c + 1), 2^mp^r) = 4, v = \gcd(8(2^{s-3}p^r c + 1), 2^mp^r) = 2^r$, where $t \geq 3$. Actually we have $t = 3$ if $s \geq 4$. Then equation (118) implies that:
\begin{equation}
\sqrt{2^sp^r} + \sqrt{2^r} = 4 \cos \left(\frac{\pi i}{4} f_A\right)
\end{equation}
Since $4 \cos \left( \frac{\pi}{4} f_A \right) \in \{0, \pm 2\sqrt{2}, \pm 4\}$ this equation is impossible for any odd prime $p$.

Consider now $s = 1$. We choose again $k$ of the form $k = 2^m p^n$, with $m$ large with respect to $r$. Then $w = 2p^r$, $u = \gcd(2(p^r c + 2), 2^m p^n) = 2$, $v = \gcd(2(p^r c + 4), 2^m p^n) = 2$, so that equation (118) implies that:

\begin{equation}
\sqrt{2^s p^r} + \sqrt{2^s} = 2 \cos \left( \frac{\pi}{4} f_A \right) \sqrt{2^s}
\end{equation}

Its only integral solution is $p = 1$ which is not convenient.

Let now $s = 0$. Then chose again $k$ of the form $k = 2^m p^n$ (with $m$ large with respect to $r$). We find that $w = p^r$, $u = \gcd(p^r c + 4, 2^m p^n) = 1$, $v = \gcd(p^r c + 8, 2^m p^n) = 1$, so that equation (118) above yields:

\begin{equation}
\sqrt{p^r} + 1 = 2 \cos \left( \frac{\pi}{4} f_A \right)
\end{equation}

The only integral solution is again $p = 1$.

Eventually, let us consider the case when $s = 2$. We write $p^r c + 1 = 2^u d$, with odd $d$. Chose now $k = 2^m p^m c^m$, for some large enough $m$ so that $k$ is good for $M_A$ and $M_A$ and verifies (99). Then $w = 4p^r c$, $u = \gcd(4(p^r c + 1), 2^m p^m c^m) = 2^{u+2}$, $v = \gcd(4(p^r c + 2), 2^m p^m c^m) = 4$. In this case equation (118) gives us:

\begin{equation}
\sqrt{p^r c + 1} = 2 \cos \left( \frac{\pi}{4} f_A \right) \sqrt{2^u}
\end{equation}

Suppose that $\cos \left( \frac{\pi}{4} f_A \right) = 1$ so that we have to find integral solutions of:

\begin{equation}
1 + \sqrt{2^u d - 1} = \sqrt{2^{u+2}}
\end{equation}

If $d \leq 5$ then for every $u \geq 1$ we have:

\begin{equation}
1 + \sqrt{2^u d - 1} \geq 1 + \sqrt{5 \cdot 2^u - 1} > 2\sqrt{2^u}
\end{equation}

If $d = 3$ then the previous equation is equivalent to

\begin{equation}
1 + \sqrt{3 \cdot 2^u - 1} = 2\sqrt{2^u}
\end{equation}

By taking the square and collecting together the terms we derive that $2^{2u-2} = 3 \cdot 2^u - 1$. This is impossible when $u \geq 1$ because of modulo 2 considerations. If $d = 1$ then

\begin{equation}
1 + \sqrt{2^u - 1} < 2\sqrt{2^u}
\end{equation}

The only possibility left is that $2 \cos \left( \frac{\pi}{4} f_A \right) = \sqrt{2}$ so that the equation reads:

\begin{equation}
1 + \sqrt{2^u d - 1} = \sqrt{2^{u+2}}
\end{equation}

If $d \geq 3$, as $u \geq 1$, we have:

\begin{equation}
1 + \sqrt{2^u d - 1} \geq 1 + \sqrt{3 \cdot 2^u - 1} > 2\sqrt{2^u}
\end{equation}

If $d = 1$ then the equation reads:

\begin{equation}
1 + \sqrt{2^u - 1} = 2\sqrt{2^u}
\end{equation}

Squaring both sides and collecting the terms we obtain $2^{2u-2} = 2^u - 1$, which is impossible by mod 2 considerations. This proves that any odd prime dividing $a - 6$ also divides $a + 2$. A similar proof shows that conversely, if an odd prime $p$ divides $a + 2$ then $p$ divides $a - 6$. This proves the Lemma.

Thus the prime divisors of $a - 6$ and $a + 2$ are the same and this implies that they divide their difference, so actually the only prime divisor of these two numbers is 2. Thus $a - 6 = \pm 2^m$ and $a + 2 = \pm 2^n$, for some integers $m, n$. This is impossible when $m \geq 5$ since it implies that $8(\pm 2^{m-3} + 1) = \pm 2^n$, but $\pm 2^{m-3} + 1$ is a non-trivial odd number. Inspecting the remaining cases when $0 \leq m \leq 4$ leads us to the following solutions $a = 2, a = 14$ and $a = -10$. The first is not convenient since $A$ was supposed
hyperbolic. The other ones do not satisfy the constraint (118). This contradiction shows that the only possibility is that $\text{Tr}(A) = \text{Tr}(B)$, as claimed. □

7.4. Ideal class groups and proofs of Corollaries 1.2 and 1.3. We want to prove that the set of those $M_B$ having the same abelian and $SU(2)$ Turaev-Viro invariants as $M_A$ is finite, and it can be identified with a subset of a quotient of $\mathcal{I}(M_A)$ by the involution $i$ which acts as $X \to X^{-1}$ on matrices with given trace.

Let $\alpha$ be a root of $x^2 - \text{Tr}(A)x + 1 = 0$, where $|\text{Tr}(A)| \neq 2$. A construction due to Latimer, MacDuffee and Taussky-Todd (see [61] and [122], III.16 for details) establishes a one to one correspondence between the ideal class group $\mathcal{I}(M_A)$ of the order $\mathbb{Z}[\alpha]$ and the classes of matrices $C \in SL(2, \mathbb{Z})$ with trace $\text{Tr}(C) = \text{Tr}(A)$, considered up to conjugacy in $GL(2, \mathbb{Z})$.

The order $\mathbb{Z}[\alpha]$ is sometimes (though not always) the ring of integers of a real quadratic field. Specifically, set $D_A = \text{Tr}(A)^2 - 4$, for odd $\text{Tr}(A)$ and $D_A = \frac{1}{4}\text{Tr}(A)^2 - 1$, for even $\text{Tr}(A)$, respectively. If $D_A$ is squarefree, then $\mathbb{Z}[\alpha]$ is the ring of integers $O_{\sqrt{D_A}}$ of the real quadratic field $\mathbb{Q}(\sqrt{D_A})$.

For any $SL(2, \mathbb{Z})$ matrix $C$ having trace $\text{Tr}(A)$ one defines an ideal of $\mathbb{Z}[\alpha]$ as follows. Consider an eigenvector $(u_1, u_2)$ of $C$ associated to the eigenvalue $\alpha$, which could be chosen to lie within $\mathbb{Z}[\alpha] \times \mathbb{Z}[\alpha]$. Therefore $\{u_1, u_2\}$ form the basis of an ideal $I(C) \subset \mathbb{Z}[\alpha]$. Conversely, the choice of a basis of an ideal $I \subset \mathbb{Z}[\alpha]$ determines a matrix $C(I) \in SL(2, \mathbb{Z})$ corresponding to the multiplication by $\alpha$. This matrix is uniquely determined by $I$, up to conjugacy in $GL(2, \mathbb{Z})$.

In the ideal class group $\mathcal{I}(M_A)$ of $\mathbb{Z}[\alpha]$ two ideals $I$ and $J$ are identified if there exist nonzero elements $v, w \in \mathbb{Z}[\alpha]$ such that $vI = wJ$. Further, if $B = UCU^{-1}$, with $U \in GL(2, \mathbb{Z})$, then the ideals $I(B)$ and $I(C)$ are equivalent. Therefore the class of $I(C)$ is well-defined in $\mathcal{I}(M_A)$, independently on the representative $C$ in its conjugacy class.

Now recall that two torus bundles manifolds $M_A$ and $M_B$ are homeomorphic if and only if their fundamental groups are isomorphic, since they are aspherical. According to Proposition 1.2 this corresponds to the fact that $A$ is conjugate to $B$ or to $B^{-1}$ within $GL(2, \mathbb{Z})$. If we take into account the involution $B \to B^{-1}$ we obtain the first claim of the Corollary 1.2. Eventually Dedekind’s Theorem states the finiteness of the ideal class group and it permits to conclude.

Although the statement of Proposition 1.2 was only stated for hyperbolic matrices $A$ and $B$ this extends naturally to all matrices from $SL(2, \mathbb{Z})$.

Eventually, stronger results due to Platonov and Rapinchuk (see [19], section 8.8.5) show that the number of classes in an arithmetic group belonging to the same $G$-genus (where $G$ is a connected linear algebraic group defined over $\mathbb{Q}$) is finite and unbounded. In particular, the number of classes in $\mathcal{X}_{TV}(M)$ is unbounded. This settles Corollary 1.3.

Remark 7.2. One should notice that there exist classes of matrices $B$ in $I(M_A)/i$ such that $B$ and $A$ are not conjugate in every congruence quotient. If $\mathcal{I}^{A,loc}(M_A) \subset I(M_A)$ is the set of conjugacy classes of matrices $B$ which are conjugate in every congruence quotient to a given $A$ it would be interesting to know the behavior of the $|\mathcal{I}^{A,loc}(M_A)|$ when $D_A$ goes to infinity and also of the relative density of this subset in $\mathcal{I}(M_A)$.

7.5. Proof of Corollary 1.4. For every finite group $F$ there is associated a modular category whose associated invariants are the so-called Dijkgraaf-Witten invariants (see e.g. [58]). The simplest of them is the untwisted Dijkgraaf-Witten invariant $RT_F$ given by the following explicit counting formula in terms of the fundamental group of the closed 3-manifold $M$ (according to [58] or [20], (5.14)):

$$RT_F(M) = \frac{1}{|F|} |\text{Hom}(\pi_1(M), F)|,$$

(130)

We have now the following easy lemma:

Lemma 7.10. Let $\Gamma_1$ and $\Gamma_2$ be finitely generated groups such that:

$$|\text{Hom}(\Gamma_1, F)| = |\text{Hom}(\Gamma_2, F)|,$$

(131)

holds for any finite group $F$. Then the sets of finite quotients of $\Gamma_1$ and $\Gamma_2$ respectively, coincide.
Eventually, recall that the profinite completions of two finitely generated groups are isomorphic as topological groups if and only if the sets of their finite quotients are the same (see [13]). However, two profinite completions are isomorphic as topological groups if and only if they are isomorphic as discrete groups, because finite index subgroups in profinite groups are open, according to a fundamental result of Nikolov and Segal ([43] and the discussion in [13]). This settles Corollary 1.4.

Proof of Lemma 7.10. Let $\text{Hom}^{\text{surj}}(\Gamma, F)$ denotes the set of surjective homomorphisms between the groups $\Gamma$ and $F$. We claim first that, under the assumptions of the lemma, we have for any finite group $F$ the equality:

$$|\text{Hom}^{\text{surj}}(\Gamma_1, F)| = |\text{Hom}^{\text{surj}}(\Gamma_2, F)|$$

Otherwise, pick up some $F$ for which the claim above is false and such that $F$ is a minimal group, with respect to the inclusion, with this property. Then $F$ is nontrivial and

$$|\text{Hom}^{\text{surj}}(\Gamma_1, F)| \neq |\text{Hom}^{\text{surj}}(\Gamma_2, G)|$$

By the induction hypothesis we have:

$$|\text{Hom}^{\text{surj}}(\Gamma_1, G)| = |\text{Hom}^{\text{surj}}(\Gamma_2, G)|$$

for any subgroup $G \subset F$ such that $G \neq F$. However, we also have:

$$|\text{Hom}(\Gamma_1, F)| = \sum_{G \subset F} |\text{Hom}^{\text{surj}}(\Gamma_1, G)|$$

The inequality above implies then

$$|\text{Hom}(\Gamma_1, F)| \neq |\text{Hom}(\Gamma_2, G)|$$

contradicting our assumptions. This proves the claim.

Eventually, observe that $F$ is a finite quotient of the group $\Gamma_i$ if and only if $|\text{Hom}^{\text{surj}}(\Gamma_i, F)| \neq 0$. Then the claim above implies that the set of finite quotients of the groups $\Gamma_i$ should coincide. □

7.6. Proof of Proposition 1.4. Let $G$ be the fundamental group of a closed orientable irreducible SOL manifold $M$. Then $G$ is solvable and according to a result of Evans and Moser (see [17], Theorem 5.2) $G$ is polycyclic.

Consider the fundamental group $H$ of a closed 3-manifold whose class is in $\mathcal{A}^{TV}(M)$. According to Lemma 7.10 the finite quotients of $H$ coincide with the finite quotients of $G$. Moreover, by classical results of Hempel and Perelman’s solution to the geometrization conjecture the 3-manifold groups are residually finite. Sabagh and Wilson have proved in [53] that any residually finite group $H$ having the same quotients as a polycyclic group is also polycyclic. In particular $H$ is polycyclic. Now the finiteness statement is a consequence of a deep theorem of Grunewald, Pickel and Segal (see [31]) which states that the number of polycyclic groups with the same profinite completion is finite.

8. Comments

8.1. Higher genus. A direct extension of these results to higher genus surface bundles does not seem to work. In the case of the closed torus the kernel of all modular representations of level $k$ is a congruence subgroup of level $k$ and hence strictly larger than the normal subgroup generated by the $k$-th powers of Dehn twists. In higher genus one expects the kernel of $SU(2)$ quantum representation to be precisely the normal subgroup generated by the $k$-th powers of Dehn twists.

The first case to analyze is the the mapping class group of the 1-punctured torus $\mathcal{M}_1^1$ (isomorphic to $SL(2, \mathbb{Z})$). Its quantum representations are known not being always congruence anymore. Moreover, the kernel of the quantum $SU(2)$-representations (where the puncture is colored with every possible color) is now the subgroup $\mathcal{M}_1^1[k]$ generated by the $k$-th powers of Dehn twists (see [25, 38]). The following shows that the analog of Proposition 1.3 does not hold:

Proposition 8.1. If two matrices $A, B \in SL(2, \mathbb{Z}) = \mathcal{M}_1^1$ are conjugate in each quotient $\mathcal{M}_1^1/\mathcal{M}_1^1[k]$ then $A$ and $B$ are conjugate in $SL(2, \mathbb{Z})$. 

Proof. Let $F$ be a finite quotient of $SL(2, \mathbb{Z})$. Then the image of the Dehn twist corresponding to a parabolic in $SL(2, \mathbb{Z})$ is of finite order, say $k$. The Dehn twists on $\Sigma_1$ are conjugate so that $F$ is a quotient of $\mathcal{M}_1^1/\mathcal{M}_1^k$. This implies that the images of $A$ and $B$ are conjugate in any finite quotient $F$. According to Stebe (see [56]) the group $SL(2, \mathbb{Z})$ is conjugacy separable and this means that $A$ and $B$ are conjugate. \hfill \Box

8.2. Equivalence relations on 3-manifolds. There are some natural equivalence relations on the set of closed 3-manifolds which are inspired by the present constructions.

At first there is Lackenby’s congruence relation from the Introduction. Further two manifolds are said Turaev-Viro equivalent if their Turaev-Viro invariants agree, for every spherical fusion category.

The third equivalence relation is to consider a slight generalization of the transformations arising in Proposition [11] which we will call local equivalence. Specifically we have an elementary locally equivalence between $M_1$ and $M_2$ if there exists a third closed 3-manifold $N$ with a non-separating embedded 2-torus $T \subset N$ and a pair of matrices $A_1$ and $A_2$ with the properties:

1. The matrices $A_1$ and $A_2$ are locally equivalent, meaning that they are conjugate in every congruence quotient.
2. We obtain $M_i$ by cutting open $N$ along $T$ and gluing back the two torus components obtained after twisting by $A_i$.

Eventually $M$ and $N$ are called locally equivalent if there is a sequence of elementary local equivalences connecting $M$ and $N$.

An easy extension of Proposition [11] shows that closed 3-manifolds which are locally equivalent are Turaev-Viro equivalent. On the other hand from [27, 35] one derive that congruent manifolds are also Turaev-Viro equivalent.

It is not clear whether the three above relations are actually the same. It would be interesting to have examples of equivalent hyperbolic 3-manifolds, if they ever exist.

The set of homeomorphisms types of torus bundles $M_B$ which are locally equivalent to $M_A$ is then a subset $\mathcal{X}^{loc}(M) = \mathcal{X}^{TV}(M) \cap \mathcal{I}(M)/\mathcal{L}$. It is not clear a priori that all elements in $\mathcal{X}^{loc}(M)$ are of the form $M_B$ with $A$ locally equivalent to $B$. If true, this will permit to compute effectively the subset $\mathcal{X}^{loc}(M)$.

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Appendix A. Counting matrices in a given genus

by Louis Funar and Andrei Rapinchuk

For $t \in \mathbb{Z}$, we let

\[ \mathcal{M}_t = \{ A \in SL(2, \mathbb{Z}) \mid \text{tr}(A) = t \}, \]

and let $\mathcal{X}_t$ denote the set of $GL(2, \mathbb{Z})$-conjugacy classes of matrices in $\mathcal{M}_t$. We define the discriminant of $A \in \mathcal{M}_t$ to be

\[ D = D(t) = \begin{cases} 
    t^2 - 4 & \text{for } t \text{ even,} \\
    t^2/4 - 1 & \text{for } t \text{ odd.}
\end{cases} \]

Furthermore, the genus $\mathfrak{G}(A)$ of $A \in \mathcal{M}_t$ is the set of $B \in SL(2, \mathbb{Z})$ that are conjugate to $A$ in $SL(2, \hat{\mathbb{Z}})$ where $\hat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$ (we note that obviously $\mathfrak{G}(A) \subset \mathcal{M}_t$). Equivalently, $B \in \mathfrak{G}(A)$ if the images of $A$ and $B$ are conjugate in $SL(2, \mathbb{Z}/m\mathbb{Z})$ for all $m > 1$. It may appear that to comply with the general definition of genus adopted in \cite{B}[§8.5] we would also need to require that $B$ must also be conjugate to $A$ in $SL(2, \mathbb{Q})$, but here this condition follows automatically from local conjugacy in view of the Hasse norm theorem for quadratic extensions. On the other hand, one can consider a variation of this definition of genus by requiring that the the images of $A$ and $B$ in $SL(2, \mathbb{Z}/m\mathbb{Z})$ be conjugate in $SL^\pm(2, \mathbb{Z}/m\mathbb{Z})$, the group of matrices over $\mathbb{Z}/m\mathbb{Z}$ with determinant $\pm 1$, for all $m > 1$; the genus of $A$ thus defined will be denoted by $\mathfrak{G}^\pm(A)$. Finally, we let $\mathfrak{A}(A)$ denote the set of $SL(2, \mathbb{Z})$-conjugacy classes in $\mathfrak{G}(A)$.

Our main result is the following.

**Theorem A.1.** There exists an increasing sequence of integers $\{t_n\}$ such that:

1. $D_n := D(t_n)$ is square-free for all $n$;
2. We have:

\[
\max_{A \in \mathcal{M}_{t_n}} |\mathfrak{A}(A)| \geq 0.1023 \cdot 10^{-4} \cdot D_n^{0.49} \frac{1}{2\log 2 + \log(D_n + 2)}
\]
and therefore \( \max_{A \in M_n} |\mathfrak{A}(A)| \to \infty \) as \( n \to \infty \);

(3) Eventually we have

\[
\max_{A \in M_n} |\mathfrak{A}(A)| \geq \frac{1}{64}
\]

In particular,

\[
\limsup_{t \to \infty} \frac{1}{|X_t|} \max_{A \in M_t} |\mathfrak{A}(A)| \geq \frac{1}{64}
\]

According to Propositions 1.1 and 1.2 above, matrices \( A_1, A_2 \in \Phi(A) \) such that neither of \( A_1 \pm 1 \) and \( A_2 \pm 1 \) are conjugate in \( GL(2, \mathbb{Z}) \) (we will call such matrices strongly nonconjugate) give rise to nonhomeomorphic torus bundles having the same quantum invariants. This, in particular, yields nonisomorphic 3-manifold groups having the same profinite completion, answering the Grothendieck-type question raised in [7]. Theorem A.1 above implies an asymptotic lower bound on the size of a set of pairwise strongly nonconjugate matrices in a genus inquired about in Remark 6.2 above, which gives an effective version of Corollary 1.3. This effective version is closely related to the more general results of the second author from [10, 11]. The proof below is based on the (well-known) connection between the conjugacy of 2-by-2 matrices and the equivalence of binary quadratic forms (see [3]), although one can also give a direct argument.

**Proof.** Assume henceforth that \( |t| \geq 3 \) and set \( D = D(t) \). First, we prove the following result about the number of genera, which is based on the analysis of local conjugacy (it should be noted that there are easy algorithms to determine if two matrices in \( SL(n, \mathbb{Z}) \) are conjugate, for any \( n \) (see [1]), but all we need for \( n = 2 \) is the classical result about binary quadratic forms).

**Proposition A.1.** Let \( D = 2^n p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n} \) be the prime factorization of \( D \).

1. The number of distinct genera \( \Phi(A) \) contained in \( M_t \) is

\[
s(t) = 2^{n+\nu(D)} \cdot \tau(D)
\]

where \( \tau(D) \) is the number of divisors of \( D \) and \( \nu(D) \) is the number of primitive binary quadratic forms of discriminant \( D \).

2. The number of distinct genera \( \Phi^\pm(A) \) in \( M_t \) is

\[
s_{\pm}(t) = 2^{n_1+\nu(D)} \cdot \tau(D)
\]

where \( \tau(D) \) and \( \nu(D) \) are the same as above and \( n_1 \) is the number of odd prime factors \( p_i \equiv 1 \) (mod 4).

**Proof.** For a matrix \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) which is not scalar one defines the Jorgensen invariant to be

\[
J(A) = \gcd(a - d, b, c)
\]

As pointed out in [14], \( J(A) \) is an invariant of the conjugacy class of \( A \).

Furthermore, if one associates to the matrix \( A \) the primitive bilinear form

\[
\frac{\text{sgn}(\text{tr}(A))}{J(A)} (bx^2 - (a - d)xy - cy^2)
\]

then conjugacy classes in \( SL(2, \mathbb{Z}) \) will correspond to equivalence classes of bilinear forms.
Then $J(A)$ can take $\tau(D)$ distinct values. Moreover the number of genera of primitive bilinear forms over $\mathbb{Z}$ was basically computed by Gauss ([3]), see ([3], chap. 14, section 3, p.339–340, Lemmas 3.1-3.3) for a modern treatment. The case of improper equivalence classes is similar.

Denote by $\omega$ the element

\begin{equation}
\omega = \begin{cases}
\frac{1+\sqrt{D}}{2}, & \text{for odd } t,
\frac{\sqrt{D}}{2}, & \text{for even } t,
\end{cases}
\end{equation}

According to the Latimer-MacDuffee-Taussky correspondence (see [8]) there is a bijection between the elements of $X_t$ and the ideal class group $I(\mathbb{Z}[\omega])$ of the order $\mathbb{Z}[\omega]$. Denote then by $h(D) = |I(\mathbb{Z}[\omega])|$ the class number of $\mathbb{Z}[\omega]$. Notice that $\mathbb{Z}[\omega]$ might not be the maximal order in $\mathbb{Q}(\sqrt{D})$ unless $D$ is square-free.

Therefore there exists some $A \in M_t$ such that the number of conjugacy classes in $\mathfrak{A}(A)$ is at least:

\begin{equation}
N(t) = 2^{-a(t)}h(D)
\end{equation}

According to a celebrated theorem of Jing Run Chen (see [4]) revisited by Halberstam (see [9]) and Richert (see [12], Thm. 13.2) there exist infinitely many primes $p_n$ such that $p_n + 4$ has at most two factor primes. Assuming that $p_n > 5$ the two factor primes have to be distinct and different from $p_n$. If we set $t_n = p_n + 2$ then $D_n = t_n^2 - 4$ are odd square-free and have at most 3 prime divisors (counted with their multiplicities). In particular $h(D_n) = h_{D_n}$ where this time $h_D$ denotes the class number of the quadratic field $\mathbb{Q}(\sqrt{D})$ (namely of its ring of integers).

It remains to prove that for this subsequence we also have $\lim \sup h(D_n) = \infty$. This is already classical. Indeed the Dirichlet class number formula for real quadratic fields reads:

\begin{equation}
h_d = \frac{1}{2 \log \epsilon_D} \sqrt{D} \cdot L(1, \chi_d)
\end{equation}

where $d = 4^s D$ is the discriminant of $\mathbb{Q}(\sqrt{D})$, $\epsilon_D$ is the fundamental unit, $\chi_d$ is the mod $d$ Dirichlet primitive character and $L(\cdot, \chi_d)$ the associated $L$-series. In our case $D_n \equiv 1 (\mod 4)$ so that $\delta_{D_n} = 0$ and the fundamental unit is $\epsilon_D = \frac{1+\sqrt{D}}{2}$, if $D = t^2 - 4$. Thus $\epsilon_D < 2\sqrt{D+2}$.

The Tatuzawa effective version of Siegel’s theorem (see [13], Thm. 2) states the following lower bound for the $L$-series:

\begin{equation}
L(1, \chi_d) > 0.655 \frac{s^{-1}}{d^{1/s}}
\end{equation}

for all $d \geq \max(\exp(s), \exp(11.2))$ with one possible exception and all $s > 2$. Eventually consider $s = 100$ and $t_n$ large enough for which the inequality above holds. This gives our estimate.

\[\square\]

Remark A.1. The congruence subgroup property implies that the estimates of Theorem [A,1] also hold in $SL(n,\mathbb{Z})$, with $n \geq 3$, by considering matrices of the form $A \oplus 1_{n-2}$, with $A \in SL(2,\mathbb{Z})$.

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