THE SKEWNESS FOR UNCERTAIN RANDOM VARIABLE AND APPLICATION TO PORTFOLIO SELECTION PROBLEM

Bo Li
School of Applied Mathematics, Nanjing University of Finance and Economics
Nanjing 210023, China

Yadong Shu
School of Mathematics and Statistics
Nanjing University of Information Science and Technology
Nanjing 210044, China

(Communicated by Yuanguo Zhu)

Abstract. Uncertainty and randomness are two basic types of indeterminacy, where uncertain variable is used to represent quantities with human uncertainty and random variable is applied for modeling quantities with objective randomness. In many real systems, uncertainty and randomness often exist simultaneously. Then uncertain random variable and chance measure can be used to handle such cases. We know that the skewness is a measure of distributional asymmetry. However, the concept of skewness for uncertain random variable has not been clearly defined. In this paper, we first propose a concept of skewness for uncertain random variable and then present a formula for calculating the skewness via chance distribution. Applying the presented formula, the skewnesses of three special uncertain random variables are derived. Finally, a portfolio selection problem is carried out for showing the efficiency and applicability of skewness and presented formula.

1. Introduction. Indeterministic phenomena exist widely in our daily life. In order to quantify indeterministic phenomena, many mathematical theories have been developed, such as probability theory and fuzzy set theory. Probability theory is a classical mathematical tool for describing the random phenomena. A complete axiomatic foundation was given by Kolmogorov [11]. And, fuzzy set theory is commonly used to model the fuzzy phenomena, which was introduced by Zadeh [30]. What’s more, randomness and fuzziness usually exist in a complex system simultaneously. For describing the fuzzy random phenomenon, Kwakernaak [12, 13] introduced the notion of fuzzy random variable, which is a function from a probability space to a collection of fuzzy variables.

As we known, the skewness is an important statistical indicator in the fields of probability theory and fuzzy set theory, which represents the asymmetric characteristics of statistical data. It has been widely used in many practical problems, especially in finance field. Ibbotson [10] and Prakash et al. [23] showed that when skewness is included in the decision process, an investor can get a higher return.
Brie et al. [3] used skewness as the risk measure for the stochastic portfolio selection problem. Fernandez-Perez et al. [7] demonstrated that the skewness of commodity futures returns contains information about subsequent returns. Li et al. [15] defined a concept of skewness for fuzzy variable and formulated a mean-variance-skewness model for the fuzzy portfolio selection problem. Xu et al. [25] proposed a project portfolio model with skewness risk constraints. Chatterjee et al. [4] constructed a mean-semivariance-skewness portfolio selection model in fuzzy random environment.

A premise of applying probability theory is that the available distribution is close enough to the real frequency, which means that we must find enough samples to estimate the distribution via statistic method. However, we often encounter situations with no date or little data (see Liu [16] for details). For example, we have recently seen the COVID-19 spreading rapidly across the world. At the beginning, it was hard to predict the changes of the novel coronavirus because there are not enough reliable data of confirmed cases and suspected cases. When there is not enough data, we have to invite some domain experts to evaluate the belief degree that each indeterministic event happens. According to Kahneman and Tversky [14], human tends to overweight unlikely events, so the belief degree generally has a much larger range than the real frequency. Hence, human uncertainty cannot be treated via probability theory. In addition, although fuzzy set theory has been studied and applied widely, it is still challenged by many scholars because some surveys show that human uncertainty does not behave like fuzziness (see Huang and Ying [9] and Liu [17] for details).

In order to better deal with human uncertainty, an uncertainty theory was founded by Liu [16] in 2007 and refined by Liu [17] in 2010. Nowadays, uncertainty theory has become a branch of mathematics for modeling the indeterministic phenomena with human uncertainty ([5, 6, 8, 18, 26, 27, 28, 29]). The concept of skewness for uncertain variable was first presented by Bhattacharyya et al. [2]. Then, Chen et al. [5] and Zhai et al. [29] used skewness as the risk measure for the uncertain portfolio selection problem. However, uncertainty and randomness often exist simultaneously in many real systems. In order to describe such kinds of systems, Liu [19] pioneered the concepts of uncertain random variable and chance measure. Meanwhile, Liu [19] defined the concepts of chance distribution, expected value and variance for uncertain random variable. Ahmadzade et al. [1] provided some formulas for calculating the moments of uncertain random variable.

However, the concept of skewness for uncertain random variable has not been clearly defined. Considering the importance of skewness in statistical analysis, in this paper, we will propose a concept of skewness for uncertain random variable and present a formula for calculating the skewness. The rest of this paper is organized as follows. In Section 2, we review some concepts and theorems of uncertain random variable and chance measure. A concept of skewness for uncertain random variable is defined and a formula is presented in Section 3. In Section 4, we derive the skewnesses of three special uncertain random variables. A portfolio selection problem is presented in Section 5. The last section gives a conclusion.

2. Preliminary.

2.1. Uncertain variable. In 2007, Liu [16] proposed the concept of uncertain measure to indicate the belief degree that a possible event happens. Let \( \Gamma \) be a nonempty set, and \( \mathcal{L} \) a \( \sigma \)-algebra over \( \Gamma \). Each element \( \Lambda \in \mathcal{L} \) is called an event. A
set function $\mathcal{M}$ defined on the $\sigma$-algebra over $\mathcal{L}$ is called an uncertain measure if it satisfies the following axioms: (normality axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set $\Gamma$; (duality axiom) $\mathcal{M}\{\Lambda\} + M\{\Lambda^c\} = 1$ for any event $\Lambda$; (subadditivity axiom) $\mathcal{M}\{\bigcup_{i=1}^\infty \Lambda_i\} \leq \sum_{i=1}^\infty \mathcal{M}\{\Lambda_i\}$ for every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$.

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. A product uncertain measure $\mathcal{M}$ is defined by Liu [20] to produce an uncertain measure of compound event: (product axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^\infty \Lambda_k\right\} = \bigwedge_{k=1}^\infty \mathcal{M}_k\{\Lambda_k\},$$

where $\Lambda_k$ are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

An uncertain variable is a function $\xi$ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that for any Borel set of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$ is an event. The uncertainty distribution $\Phi : \mathbb{R} \to [0, 1]$ of an uncertain variable $\xi$ is defined by

$$\Phi(x) = \mathcal{M}\{\gamma \in \Gamma \mid \xi(\gamma) \leq x\}.$$

The expected value of uncertain variable $\xi$ is defined in Liu [16] as

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\} dx - \int_{-\infty}^0 \mathcal{M}\{\xi \leq x\} dx$$

provided that at least one of the two integrals is finite.

**Definition 1.** (Bhattacharyya et al. [2]) Let $\xi$ be an uncertain variable with finite expected value $E[\xi]$. Then the skewness of $\xi$ is defined by

$$S[\xi] = E[(\xi - E[\xi])^3].$$

2.2. Uncertain random variable. An uncertain variable is essentially a measurable function from an uncertainty space to the set of real numbers. A random variable is a measurable function from a probability space to the set of real numbers. In 2013, Liu [19] first proposed the chance measure, which is a mathematical methodology for modeling complex systems with both uncertainty and randomness.

The chance space refers to the product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$, where $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space and $(\Omega, \mathcal{A}, \Pr)$ is a probability space. An uncertain random variable is a function $\xi$ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ to the set of real numbers such that $\{\xi \in B\}$ is an event in $\mathcal{L} \times \mathcal{A}$ for any Borel set $B$ of real numbers.

**Definition 2.** (Liu [19]) Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ be a chance space, and let $\Theta \in \mathcal{L} \times \mathcal{A}$ be an event. Then the chance measure of $\Theta$ is defined as

$$\text{Ch}\{\Theta\} = \int_0^1 \Pr\left\{\omega \in \Omega, \mathcal{M}\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta\} \geq x\right\} dx.$$

**Remark 1.** An uncertain variable or a random variables can be regarded as a special uncertain random variable. The chance distribution of an uncertain variable is just its uncertainty distribution and the chance distribution of a random variable is just its probability distribution. Let $\eta$ be a random variable, $\tau$ be an uncertain variable. Then $\eta + \tau$ and $\eta \times \tau$ are both uncertain random variables.

**Definition 3.** (Liu [19]) Let $\xi$ be an uncertain random variable. Then its chance distribution is defined by

$$\Phi(x) = \text{Ch}\{\xi \leq x\}.$$
for any $x \in \mathbb{R}$.

**Theorem 1.** (Liu [21]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Phi_1, \Phi_2, \cdots, \Phi_m$, respectively, and let $\tau_1, \tau_2, \cdots, \tau_n$ be uncertain variables. Then the uncertain random variable $\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ has a chance distribution

$$
\Psi(x) = \int_{\mathbb{R}^n} F(x, y_1, \cdots, y_m) d\Phi_1(y_1) \cdots d\Phi_m(y_m),
$$

where $F(x, y_1, \cdots, y_m)$ is the uncertain distribution of uncertain variable

$$
f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n).
$$

**Definition 4.** (Liu [19]) Let $\xi$ be an uncertain random variable. Then its expected value is defined by

$$
E[\xi] = \int_{0}^{+\infty} \text{Ch}\{\xi \geq r\} dr - \int_{-\infty}^{0} \text{Ch}\{\xi \leq r\} dr
$$

provided that at least one of the two integrals is finite.

**Definition 5.** (Liu [19]) Let $\xi$ be an uncertain random variable with finite expected value $E[\xi]$. Then the variance of $\xi$ is defined by

$$
V[\xi] = E[(\xi - E[\xi])^2].
$$

### 3. Skewness for uncertain random variable.

In this section, we define a concept of skewness for uncertain random variable and present a formula for calculating the skewness.

**Definition 6.** Let $\xi$ be an uncertain random variable with a finite expected value $E[\xi]$. Then the skewness of $\xi$ is

$$
S[\xi] = E[(\xi - E[\xi])^3].
$$

**Theorem 2.** Let $\xi$ be an uncertain random variable with a chance distribution $\Psi$ and a finite expected value $E[\xi]$. Then the skewness of $\xi$ is

$$
S[\xi] = \int_{-\infty}^{+\infty} (x - E[\xi])^3 d\Psi(x). 
$$

**Proof.** According to Definitions 4 and 6, it holds that

$$
S[\xi] = \int_{0}^{+\infty} \text{Ch}\{(\xi - E[\xi])^3 \geq r\} dr - \int_{-\infty}^{0} \text{Ch}\{(\xi - E[\xi])^3 \leq r\} dr
$$

$$
= \int_{0}^{+\infty} \text{Ch}\{(\xi - E[\xi]) \geq \sqrt[3]{r}\} dr - \int_{-\infty}^{0} \text{Ch}\{(\xi - E[\xi]) \leq \sqrt[3]{r}\} dr
$$

$$
= \int_{0}^{+\infty} \text{Ch}\{\xi \geq \sqrt[3]{r} + E[\xi]\} dr - \int_{-\infty}^{0} \text{Ch}\{\xi \leq \sqrt[3]{r} + E[\xi]\} dr
$$

$$
= \int_{0}^{+\infty} (1 - \Psi(\sqrt[3]{r} + E[\xi])) dr - \int_{-\infty}^{0} (\Psi(\sqrt[3]{r} + E[\xi])) dr.
$$
Substituting $\sqrt{\tau} + E[\xi]$ with $x$ and using integration by parts, we have
\[
\int_{0}^{\infty} (1 - \Psi(\sqrt{\tau} + E[\xi]))dr = \int_{0}^{\infty} (1 - \Psi(x))d(x - E[\xi])^3 = \int_{E[\xi]}^{\infty} (x - E[\xi])^3d\Psi(x),
\]
\[
\int_{-\infty}^{0} (\Psi(\sqrt{\tau} + E[\xi]))dr = \int_{-\infty}^{E[\xi]} \Psi(x)d(x - E[\xi])^3 = \int_{-\infty}^{E[\xi]} (x - E[\xi])^3d\Psi(x).
\]
Obviously,
\[
S[\xi] = \int_{-\infty}^{\infty} (x - E[\xi])^3d\Psi(x).
\]
The proof is completed.

**Theorem 3.** Let $\xi$ be an uncertain random variable with a finite expected value $E[\xi]$. For any constants $a$ and $b$, we have
\[
S[a\xi + b] = a^3S[\xi].
\]

**Proof.** It follows from Definition 6 that
\[
S[a\xi + b] = E \left[ (a\xi + b - (aE[\xi] + b))^3 \right] = a^3E \left[ (\xi - E[\xi])^3 \right] = a^3S[\xi].
\]
The proof is completed.

**Example 1.** Let $\eta$ be a random variable with probability distribution $\Phi$, and let $\tau$ be an uncertain variable with uncertainty distribution $\Upsilon$. It follows from Theorem 2 that the skewness of the sum $\xi = \eta + \tau$ can be written as
\[
S[\xi] = \int_{-\infty}^{\infty} (x - E[\xi])^3d\Psi(x),
\]
where chance distribution $\Psi(x) = \int_{-\infty}^{\infty} \Upsilon(x - y)d\Phi(y)$ and $E[\xi] = E[\eta] + E[\tau]$.

**Example 2.** Let $\eta$ be a random variable with probability distribution $\Phi$, and let $\tau$ be an uncertain variable with uncertainty distribution $\Upsilon$. It follows from Theorem 2 that the skewness of the product $\xi = \eta\tau$ is
\[
S[\xi] = \int_{-\infty}^{\infty} (x - E[\xi])^3d\Psi(x),
\]
where chance distribution $\Psi(x) = \int_{-\infty}^{\infty} \Upsilon(x/y)d\Phi(y)$ and $E[\xi] = E[\eta]E[\tau]$.

4. **Three special uncertain random variables.** In this section, we calculate the skewnesses of three special uncertain random variables. Assume that $\eta$ is a standard normal distribution with the probability density function
\[
f(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right).
\]

**Theorem 4.** If $\tau$ is a linear uncertain variable with uncertain distribution $\Upsilon(x)$ defined as
\[
\Upsilon(x) = \begin{cases} 
0, & \text{if } x \leq a \\
(x - a)/(b - a), & \text{if } a \leq x \leq b \\
1, & \text{if } x \geq b.
\end{cases}
\]
Then the skewness of uncertain random variable $\xi = \eta + \tau$ is
\[
S[\xi] = 0.
\]
Proof. It follows from Theorem 1 that the chance distribution of uncertain random variable $\xi$ is

$$\Psi(x) = \int_{-\infty}^{+\infty} \Upsilon(x-y)f(y)dy = \int_{-\infty}^{x-a} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)dy + \int_{x-a}^{x-b} x-y-a \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)dy.$$

Then

$$\frac{d\Psi(x)}{dx} = \frac{1}{\sqrt{2\pi}(b-a)} \int_{x-b}^{x-a} \exp\left(-\frac{y^2}{2}\right)dy.$$

Applying equation (3.1), the skewness of $\xi$ is

$$S[\xi] = \int_{-\infty}^{+\infty} \left( x - \frac{a+b}{2} \right)^3 d\Psi(x)$$

$$= \frac{1}{\sqrt{2\pi}(b-a)} \int_{-\infty}^{x-b} \left( x - \frac{a+b}{2} \right)^3 \exp\left(-\frac{y^2}{2}\right)dydx$$

$$= \frac{1}{\sqrt{2\pi}(b-a)} \int_{-\infty}^{x-a} \exp\left(-\frac{y^2}{2}\right) \int_{y+a}^{y+b} \left( x - \frac{a+b}{2} \right)^3 dx dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right)[4y^3 + y(a-b)^2]dy.$$

Because $\exp\left(-\frac{y^2}{2}\right)[4y^3 + y(a-b)^2]$ is an odd function in $(-\infty, +\infty)$, we have $S[\xi] = 0$. The proof is completed.

Theorem 5. If $\tau$ is a zigzag uncertain variable with uncertain distribution

$$\Upsilon(x) = \begin{cases} 
0, & \text{if } x \leq a \\
(x-a)/(b-a), & \text{if } a \leq x \leq b \\
(x+c-2b)/(2c-b), & \text{if } b \leq x \leq c \\
1, & \text{if } x \geq c.
\end{cases}$$

Then the skewness of uncertain random variable $\xi = \eta + \tau$ is

$$S[\xi] = \frac{1}{32} (c-a)^2 (a+c-2b).$$

Proof. It follows from Theorem 1 that

$$\Psi(x) = \int_{-\infty}^{+\infty} \Upsilon(x-y)f(y)dy$$

$$= \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{-c} \exp\left(-\frac{y^2}{2}\right)dy + \int_{-c}^{x-c} x-y-c \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)dy + \int_{x-c}^{x-a} x-y-a \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)dy \right) + \frac{1}{\sqrt{2\pi}/2} \exp\left(-\frac{y^2}{2}\right)dy.$$

Then

$$\frac{d\Psi(x)}{dx} = \frac{1}{2\sqrt{2\pi}(b-a)} \int_{x-b}^{x-a} \exp\left(-\frac{y^2}{2}\right)dy + \frac{1}{2\sqrt{2\pi}(c-b)} \int_{x-c}^{x-b} \exp\left(-\frac{y^2}{2}\right)dy.$$
Applying equation (3.1), the skewness of $\xi$ is

$$S[\xi] = \int_{-\infty}^{+\infty} \left( x - \frac{a + 2b + c}{4} \right)^3 d\Psi(x)$$

$$= \frac{1}{2\sqrt{2\pi}(b-a)} \int_{-\infty}^{+\infty} \int_{x-a}^{x-b} \left( x - \frac{a + 2b + c}{4} \right)^3 \exp\left(-\frac{y^2}{2}\right) dy dx$$

$$+ \frac{1}{2\sqrt{2\pi}(c-b)} \int_{-\infty}^{+\infty} \int_{x-c}^{x-b} \left( x - \frac{a + 2b + c}{4} \right)^3 \exp\left(-\frac{y^2}{2}\right) dy dx$$

$$= \frac{1}{2\sqrt{2\pi}(b-a)} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) \left( \int_{y+a}^{y+b} \left( x - \frac{a + 2b + c}{4} \right)^3 dx \right) dy$$

$$+ \frac{1}{2\sqrt{2\pi}(c-b)} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) \left( \int_{y+b}^{y+c} \left( x - \frac{a + 2b + c}{4} \right)^3 dx \right) dy$$

$$= \frac{1}{8\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left\{ 8y^3 + 2[(a - A)^2 + 2(b - A)^2 + (c - A)^2]y \right. $$

$$+ \left. \frac{(c - a)[(c - A)^2 - (a - A)^2]}{2} \right\} \exp\left(-\frac{y^2}{2}\right) dy$$

$$= \frac{1}{8\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{(c - a)[(c - A)^2 - (a - A)^2]}{2} \exp\left(-\frac{y^2}{2}\right) dy$$

$$= \frac{1}{32\sqrt{2\pi}} (c - a)^2 (a + c - 2b) \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) dy,$$

where $A = \frac{a + 2b + c}{4}$. Since $\int_{-\infty}^{+\infty} \exp(-y^2) dy = \sqrt{\pi}$, we have $S[\xi] = \frac{1}{\sqrt{2\pi}} (c - a)^2 (a + c - 2b)$. The proof is completed.

**Theorem 6.** If $\tau$ is a normal uncertain variable with uncertain distribution

$$\Upsilon(x) = \left( 1 + \exp\left( \frac{\pi(e - x)}{\sqrt{3\delta}} \right) \right)^{-1}.$$

Then the skewness of uncertain random variable $\xi = \eta + \tau$ is

$$S[\xi] = 0.$$

**Proof.** It follows from Theorem 1 that

$$\Psi(x) = \int_{-\infty}^{+\infty} \Upsilon(x - y)f(y)dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left( 1 + \exp\left( \frac{\pi(e - x + y)}{\sqrt{3\delta}} \right) \right)^{-1} \exp\left(-\frac{y^2}{2}\right) dy.$$

Then

$$\frac{d\Psi(x)}{dx} = \frac{\sqrt{\pi}}{\sqrt{6\delta}} \int_{-\infty}^{+\infty} \left( 1 + \exp\left( \frac{\pi(e - x + y)}{\sqrt{3\delta}} \right) \right)^{-2} \exp\left( \frac{\pi(e - x + y)}{\sqrt{3\delta}} \right) \exp\left(-\frac{y^2}{2}\right) dy.$$
Applying equation (3.1), the skewness of $\xi$ is

$$S[\xi] = \int_{-\infty}^{+\infty} (x-e)^3 \, d\Phi(x)$$

$$= \frac{\sqrt{\pi}}{\sqrt{6\delta}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x-e)^3 \left(1 + \exp \left(\frac{\pi(x-x+y)}{\sqrt{3\delta}}\right)\right)^{-2} \exp \left(\frac{\pi(x-x+y)}{\sqrt{3\delta}}\right) \exp\left(-\frac{y^2}{2}\right) \, dy \, dx$$

$$= \frac{\sqrt{\pi}}{\sqrt{6\delta}} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) \int_{-\infty}^{+\infty} (x-e)^3 \left(1 + \exp \left(\frac{\pi(x-x+y)}{\sqrt{3\delta}}\right)\right)^{-2} \exp \left(\frac{\pi(x-x+y)}{\sqrt{3\delta}}\right) \, dx \, dy.$$

By making the change of variables $z = \frac{\pi(x-x+y)}{\sqrt{3\delta}}$, we have

$$S[\xi] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) \int_{-\infty}^{+\infty} \left(\frac{\sqrt{3\delta}}{\pi} y\right)^3 \left(\exp[z]\right)^{\frac{3}{2}} \frac{\exp(z)}{[1+\exp(z)]^2} \, dz \, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) \int_{-\infty}^{+\infty} \left(\frac{3\sqrt{3\delta}}{\pi} y^3 - \frac{9\delta y}{\pi} z^2 - \frac{3\sqrt{3\delta} y}{\pi} z^2 - \frac{y^3}{3}\right) \frac{\exp(z)}{[1+\exp(z)]^2} \, dz \, dy$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) \left(3\sqrt{3\delta} y + y^3\right) \, dy$$

$$= 0.$$

The proof is completed. \qed

5. **Portfolio selection problem.** Portfolio selection aims at selecting an optimal investment strategy to achieve different investment goals. Markowitz [22] first proposed the well-known mean-variance model. Although, variance has been widely used, it is not a good risk measure when the distribution of security return is asymmetric. It is well known that the skewness is an important statistical indicator, which represents the asymmetric characteristics of statistical data. Then more and more scholars introduce skewness into portfolio optimization models ([4, 15, 25]). In this section, we present a mean-variance-skewness model under uncertain random environment.

Suppose there are two risky securities in the market: one is an existing risky security which has enough historical data and the other is a newly listed risky security which is lack of historical data. Here, the return rates of the existing risky security and the newly listed risky security are described as normal random variable $N(0, 1)$ and zigzag uncertain variable $Z(-0.1, 0.03, 0.2)$, respectively. When minimal expected return and maximal risk are given, the investors prefer a portfolio with large skewness (see Li et al. [15] and Bhattacharyya et al. [2]). Therefore, we establish the following mean-variance-skewness model:

$$\max \ S[x_1 \eta + x_2 \tau]$$

s.t. \quad \begin{align*}
E[x_1 \eta + x_2 \tau] &\geq p, \\
V[x_1 \eta + x_2 \tau] &\leq q,
\end{align*}

$$x_1 + x_2 = 1,$$

$$x_1, x_2 \geq 0,$$

where $p$ and $q$ represent the lower bound of expected return and the upper bound of risk, respectively.
Table 1. The computational results for different $p$ and $q$

| $(p, q)$ | $(x_1^*, x_2^*)$ | Expected value | Variance | Skewness |
|----------|------------------|----------------|----------|----------|
| $(0.04, 0.2)$ | $(0, 1)$ | 0.04 | 0.0075 | $1.125 \times 10^{-4}$ |
| $(0.04, 0.5)$ | $(0, 1)$ | 0.04 | 0.0075 | $1.125 \times 10^{-4}$ |
| $(0.04, 0.8)$ | $(0, 1)$ | 0.04 | 0.0075 | $1.125 \times 10^{-4}$ |
| $(0.02, 0.2)$ | $(0, 1)$ | 0.04 | 0.0075 | $1.125 \times 10^{-4}$ |
| $(0.02, 0.5)$ | $(0, 1)$ | 0.04 | 0.0075 | $1.125 \times 10^{-4}$ |
| $(0.02, 0.8)$ | $(0, 1)$ | 0.04 | 0.0075 | $1.125 \times 10^{-4}$ |
| $(0.01, 0.2)$ | $(0, 1)$ | 0.04 | 0.0075 | $1.125 \times 10^{-4}$ |
| $(0.01, 0.5)$ | $(0, 1)$ | 0.04 | 0.0075 | $1.125 \times 10^{-4}$ |
| $(0.01, 0.8)$ | $(0, 1)$ | 0.04 | 0.0075 | $1.125 \times 10^{-4}$ |

According to Theorem 5 and Theorem 2 in Qin [24], the formulated model can be written as

$$\max_{x} x^3 S[\tau]$$

s.t. $x_1 E[\eta] + x_2 E[\tau] \geq p,$

$$x_1^2 V[\eta] + x_2^2 V[\tau] \leq q,$$

$$x_1 + x_2 = 1,$$

$$x_1, x_2 \geq 0,$$

i.e.,

$$\max 1.125 \times 10^{-4} \times x_2^3$$

s.t. $0.04x_2 \geq p,$

$$x_1^2 + 0.0075x_2^2 \leq q,$$

$$x_1 + x_2 = 1,$$

$$x_1, x_2 \geq 0.$$

Here, we consider 9 different combinations of $(p, q)$ and the computational results are shown in Table 1. For the given $p$ and $q$, the optimal investment strategies can be obtained easily. The results show that the optimal investment proportions are $x_1^* = 0$ and $x_2^* = 1$, i.e., the investor needs to invest 100% of total capital to the newly listed risky security. Obviously, the same investment proportion will lead to same expected return, risk and skewness. The maximum of skewness is $1.125 \times 10^{-4}$.

6. Conclusion. The uncertain random variable has been used to describe the mixture of uncertainty and randomness. This paper studied the skewness for uncertain random variable. A concept of skewness for uncertain random variable was defined and a formula was presented for calculating the skewness via chance distribution. At the same time, the skewnesses of three special uncertain random variables were derived for showing the effectiveness of presented formula. Finally, a portfolio selection problem was carried out for showing the efficiency and applicability of skewness and presented formula.

Acknowledgments. This work is supported by the Natural Science Foundation of Jiangsu Province (No.BK20190787) and the Natural Science Research of the Jiangsu Higher Education Institutions of China (No.18KJB110012).
REFERENCES

[1] H. Ahmadzade, Y. Sheng and F. Hassantabar Darzi, Some results of moments of uncertain random variables, *Iran. J. Fuzzy Syst.*, 14 (2017), 1–21.

[2] R. Bhattacharyya, A. Chatterjee and S. Kar, Mean-variance-skewness portfolio selection model in general uncertain environment, *Indian J. Ind. Appl. Math.*, 3 (2012), 45–61.

[3] W. Briec, K. Kerstens and I. Van de Woestyne, Portfolio selection with skewness: A comparison of methods and a generalized one fund result, *Eur. J. Oper. Res.*, 230 (2013), 412–421.

[4] A. Chatterjee, R. Bhattacharyya, S. Mukherjee and S. Kar, Optimization of mean-semivariance-skewness portfolio selection model in fuzzy random environment, in *ICOMOS 2010, American Institute of Physics conference proceedings*, 1298 (2010), 516–521.

[5] W. Chen, Y. Wang, P. Gupta and M. K. Mehlawat, A novel hybrid heuristic algorithm for a new uncertain mean-variance-skewness portfolio selection model with real constraints, *Appl. Intell.*, 48 (2018), 2996–3018.

[6] Y. Chen and Y. Zhu, Indefinite LQ optimal control with process state inequality constraints for discrete-time uncertain systems, *J. Ind. Manag. Optim.*, 14 (2018), 913–930.

[7] A. Fernandez-Perez, B. Frijns, A. M. Fuertes and J. Miffre, The skewness of commodity futures returns, *J. Bank. Financ.*, 86 (2018), 143–158.

[8] R. Gao and D. A. Ralescu, Elliptic entropy of uncertain set and its applications, *Int. J. Intell. Syst.*, 33 (2018), 836–857.

[9] X. Huang and H. Ying, Risk index based models for portfolio adjusting problem with returns subject to experts’ evaluations, *Econ. Model.*, 30 (2013), 61–66.

[10] R. G. Ibbotson, Price performance of common stock new issues, *J. Financ. Econ.*, 2 (1975), 235–272.

[11] A. Kolmogorov, *Grundbegriffe Der Wahrscheinlichkeitsrechnung*, Julius Springer, Berlin, 1933.

[12] H. Kwakernaak, Fuzzy random variables-I: Definitions and theorems, *Inform. Sciences*, 15 (1978), 1–29.

[13] H. Kwakernaak, Fuzzy random variables-II: Algorithms and examples for the discrete case, *Inform. Sciences*, 17 (1979), 253–278.

[14] D. Kahneman and A. Tversky, Prospect theory: An analysis of decision under risk, *Econometrica*, 47 (1979), 263–292.

[15] X. Li, Z. Qin and K. Kar, Mean-variance-skewness model for portfolio selection with fuzzy returns, *Eur. J. Oper. Res.*, 202 (2010), 239–247.

[16] B. Liu, *Uncertainty Theory*, Second ed., Springer-Verlag, Berlin, 2007.

[17] B. Liu, *Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty*, Springer-Verlag, Berlin, 2004.

[18] B. Li, Y. Sun, G. Aw and K. L. Teo, Uncertain portfolio optimization problem under a minimax risk measure, *Appl. Math. Model.*, 76 (2019), 274–281.

[19] Y. Liu, Uncertain random variables: A mixture of uncertainty and randomness, *Soft Comput.*, 17 (2013), 625–634.

[20] B. Liu, Some research problems in uncertainty theory, *J. Uncertain Syst.*, 3 (2009), 3–10.

[21] Y. Liu, Uncertain random programming with applications, *Fuzzy Optim. Decis. Ma.*, 12 (2013), 153–169.

[22] H. M. Markowitz, Portfolio selection, *J. Financ.*, 7 (1952), 77–91.

[23] A. J. Prakash, C. H. Chang and T. E. Pactwa, Selecting a portfolio with skewness: Recent evidence from US, European and Latin American equity markets, *J. Bank. Financ.*, 27 (2003), 1375–1390.

[24] Z. Qin, Mean-variance model for portfolio optimization problem in the simultaneous presence of random and uncertain returns, *Eur. J. Oper. Res.*, 245 (2015), 480–488.

[25] W. Xu, G. Liu, H. Li and W. Luo, A study on project portfolio models with skewness risk and staffing, *Int. J. Fuzzy Syst.*, 19 (2017), 2033–2047.

[26] H. Yan, Y. Sun and Y. Zhu, A linear-quadratic control problem of uncertain discrete-time switched systems, *J. Ind. Manag. Optim.*, 13 (2017), 267–282.

[27] X. Yang and J. Gao, Linear-quadratic uncertain differential games with application to resource extraction problem, *IEEE Trans. Fuzzy Syst.*, 24 (2016), 819–826.

[28] T. Ye and Y. Zhu, A metric on uncertain variables, *Int. J. Uncertain. Quan.*, 8 (2018), 251–266.
[29] J. Zhai, M. Bai and H. Wu, Mean-risk-skewness models for portfolio optimization based on uncertain measure, Optimization, 67 (2018), 701–714.

[30] L. A. Zadeh, Fuzzy sets, Inform. Control, 8 (1965), 338–353.

Received June 2020; revised August 2020.

E-mail address: libnust@163.com
E-mail address: 972718523@qq.com