Transformation design approach to a fermionic spacetime cage

De-Hone Lin

Department of Physics, National Sun Yat-sen University, Kaohsiung, Taiwan

(Dated: January 1, 2016)

Abstract

This paper is concerned with the application of a spacetime structure which emerges from the transformation design of spin-1/2 fermions to a three-dimensional quantum system. There are three components. First, the main part of this paper presents the constraint conditions which build the relation of a spacetime structure and a form invariance solution to the covariant Dirac equation. The second is to devise a spacetime cage for fermions with chosen constraints. The third part discusses the feasibility of the cage with an experiment.

PACS numbers: 04.40.-b, 03.65.Pm, 11.80.Et, 03.75.-b

Keywords: Dirac Equation; Constraint Conditions; Spacetime Structure; Form Invariance Solution

Email: dhlin@mail.nsYSu.edu.tw
I. INTRODUCTION

Searching for a form invariance solution to the Maxwell equations is the central issue in making an invisibility cloak for electromagnetic waves with the transformation design method [1, 2]. Recently, it was shown that the form invariance solution for the Dirac equation establishes the relation between the quantum rule of two-dimensional (2D) fermions in a central force field and a spacetime structure through constraint conditions [3].

In this paper, we present the conditions to obtain a 3D quantum system with a form invariance spinor. As an application, a spacetime cage for fermions is coined. The paper is arranged as follows: In Section II, the constraints which determine the spacetime and the 3D form invariance solution for fermions in the central force field are presented. We devote Section III to constructing the spacetime cage which traps fermion states accompanying an invisible region by the chosen constraints and thereby the spacetime structure emerged. The physical approach to the trapping effect by way of an experiment is outlined. Our conclusion and several notes are summarized in Section IV. Finally, the appendices add some detailed mathematical support to the discussions involving the length computations in Section II.

II. CONSTRAINTS AND THE FORM INVARIANCE DIRAC SPINOR

We discuss the conditions to achieve a form invariance wave function for fermions in this section. The evolution of spin-1/2 fermions with mass \( M \) in curved spacetime is depicted by the spinor \( \Phi \) which satisfies the covariant Dirac equation (e.g. [4])

\[
\{ \tilde{\gamma}^\mu (\partial_\mu - \Gamma_\mu) + M \} \Phi = 0,
\]

where \( \partial_\mu = \partial / \partial x^\mu \) with \( x^\mu = (x^1, x^2, x^3, x^4) = (x, y, z, ict) \) in our convention, the symbol \( \tilde{\gamma}^\mu \) labels the spin matrices in curved space, and \( \Gamma_\mu \) is the spin connection. The fundamental connection between spin and spacetime is through the anticommutation relation

\[
\tilde{\gamma}^\mu \tilde{\gamma}^\nu + \tilde{\gamma}^\nu \tilde{\gamma}^\mu = 2g^{\mu\nu},
\]

in which \( g^{\mu\nu} \) is the inverse of the metric of the line element

\[
ds^2 = \sum_{\mu,\nu=1}^4 g_{\mu\nu} dx^\mu dx^\nu,
\]
satisfying $\sum_\lambda g_{\mu\lambda}g^{\lambda\nu} = \delta^\nu_\mu$. Throughout this paper, the natural units $\hbar = c = 1$ are adopted if not explicitly stated otherwise. In the following discussions, we shall restrict ourselves to spacetime described by the line element

$$ds^2 = f_1^2(dx^1)^2 + f_2^2(dx^2)^2 + f_3^2(dx^3)^2 + f_4^2(dx^4)^2.$$  \hfill{(4)}

So we have the metric of spacetime

$$g_{\mu\nu} = \text{diag}(g_{11}, g_{22}, g_{33}, g_{44}) = \text{diag}(f_1^2, f_2^2, f_3^2, f_4^2),$$  \hfill{(5)}

where $f_\mu$ are the arbitrary functions of the variables $x^1, x^2, x^3,$ and $x^4$. Based on the orthogonal assumption, the spin connection has the representation \[3, 5\]

$$\Gamma_\mu = \frac{1}{4} \left[ (\partial_\mu g_{\mu\mu}) g^{\mu\mu} - (\partial_\lambda g_{\mu\mu}) \gamma^\mu_\lambda \gamma^\lambda_\mu \right].$$  \hfill{(6)}

There is no summation over $\mu$ here. In the light of (6), the covariant Dirac equation can be expressed in terms of

$$\sum_{i=1}^4 \frac{\gamma^i}{f_i} (\partial_i - A_i) + M \} \tilde{\Psi} = 0,$$  \hfill{(7)}

where the spinor $\tilde{\Psi}$ is connected to the old one by $\tilde{\Psi} = (f_1f_2f_3f_4)\Phi = \sqrt{H}\Phi$ in which $H$ denotes the determinant of the metric $g_{\mu\nu}$, the effective four vector

$$A_i = \frac{1}{2} \partial_i [\ln (f_1f_2f_3f_4) + \ln f_i],$$  \hfill{(8)}

and $\gamma^i$ are the spin matrices in the flat spacetime of special relativity which satisfy the anticommutation relation

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 2\delta^{ij}.$$  \hfill{(9)}

For a static spacetime, the metric is only the spacial function, i.e., $g_{\mu\mu} = f_\mu^2(x)$. There exists a solution to the steady state for $\tilde{\Psi}$ which is that the Dirac spinor can be decomposed as $\tilde{\Psi}(x,t) = \Psi(x) \exp\{-iEt\}$. Eq. (7) then reduces to

$$\left\{ \left[ \sum_{i=1}^3 \frac{\gamma^i}{f_i} (\partial_i - A_i) \right] - \frac{\gamma^4}{f_4} E + M \right\} \Psi(x) = 0.$$  \hfill{(10)}

We are now in a position to seek a form invariance solution in the line below with the formulation. For this, we assume the metric $g_{\mu\mu} = f_\mu^2(R(r))$ with $R(r)$ being arbitrary
radial function, the four component spinor is taken as $\Psi(x) = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)^T$, and the spin matrices conforming to (9) are chosen to be

$$
\gamma^k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3, \text{ and } \gamma^4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.
$$

(11)

Substituting the representation into (10), the Dirac equation has the decomposition

$$
-i \left\{ \frac{1}{f_1} \partial_x \Phi_4 - \frac{xR'(r)}{2rf_1} \left[ \frac{d}{dR} (\ln f_1 f_2 f_3 f_4 + \ln \Phi_1) \right] \Phi_1 \right\} - (E/f_4 - \mathcal{M}) \Phi_1 = 0,
$$

(12)

$$
-i \left\{ \frac{1}{f_2} \partial_y \Phi_4 - \frac{yR'(r)}{2rf_2} \left[ \frac{d}{dR} (\ln f_1 f_2 f_3 f_4 + \ln f_2) \right] \Phi_2 \right\} + \left\{ \frac{1}{f_2} \partial_y \Phi_3 - \frac{yR'(r)}{2rf_2} \left[ \frac{d}{dR} (\ln f_1 f_2 f_3 f_4 + \ln f_2) \right] \Phi_3 \right\} - (E/f_4 - \mathcal{M}) \Phi_2 = 0,
$$

(13)

$$
+i \left\{ \frac{1}{f_3} \partial_z \Phi_4 - \frac{zR'(r)}{2rf_3} \left[ \frac{d}{dR} (\ln f_1 f_2 f_3 f_4 + \ln f_3) \right] \Phi_3 \right\} + (E/f_4 + \mathcal{M}) \Phi_3 = 0,
$$

(14)

and

$$
-i \left\{ \frac{1}{f_1} \partial_x \Phi_1 - \frac{xR'(r)}{2rf_1} \left[ \frac{d}{dR} (\ln f_1 f_2 f_3 f_4 + \ln f_1) \right] \Phi_1 \right\} - \left\{ \frac{1}{f_2} \partial_y \Phi_1 - \frac{yR'(r)}{2rf_2} \left[ \frac{d}{dR} (\ln f_1 f_2 f_3 f_4 + \ln f_2) \right] \Phi_1 \right\} - i \left\{ \frac{1}{f_3} \partial_z \Phi_2 - \frac{zR'(r)}{2rf_3} \left[ \frac{d}{dR} (\ln f_1 f_2 f_3 f_4 + \ln f_3) \right] \Phi_2 \right\} + (E/f_4 + \mathcal{M}) \Phi_4 = 0,
$$

(15)
where $R'(r) = dR(r)/dr$. A further reduction is made by assuming that the angular part of the spinor $\Psi$ can be separated by the spin spherical harmonics which are classified by the total angular momentum $j$, and then $\Psi$ can be represented by

$$
\begin{align*}
\Phi_1 &= \sqrt{\frac{l+m+1}{2l+1}} F(R(r)) Y_{l,m-1/2} \\
\Phi_2 &= -\sqrt{\frac{l-m+1}{2l+1}} F(R(r)) Y_{l,m+1/2} \\
\Phi_3 &= -i \sqrt{\frac{l-m+3}{2l+3}} G(R(r)) Y_{l+1,m-1/2} \\
\Phi_4 &= -i \sqrt{\frac{l+m+3}{2l+3}} G(R(r)) Y_{l+1,m+1/2}
\end{align*}, \quad \text{for } j = \frac{l+1}{2}, l = 0, 1, \ldots, (16)
$$

and

$$
\begin{align*}
\Phi_1 &= \sqrt{\frac{l-m+1}{2l+1}} F(R(r)) Y_{l,m-1/2} \\
\Phi_2 &= \sqrt{\frac{l+m+1}{2l+1}} F(R(r)) Y_{l,m+1/2} \\
\Phi_3 &= -i \sqrt{\frac{l-m+1}{2l+1}} G(R(r)) Y_{l-1,m-1/2} \\
\Phi_4 &= i \sqrt{\frac{l+m+1}{2l+1}} G(R(r)) Y_{l-1,m+1/2}
\end{align*}, \quad \text{for } j = l - 1/2, l = 1, 2, \ldots, (17)
$$

where $m = m_l + 1/2$ with $-l \leq m_l \leq l$. It is easy to show that the chosen angular parts of the spinors are normalized to unity. After a tedious calculation for the solution (16), one verifies that equation (12) is equivalent to the following four independent equalities (see Appendix A):

$$
\left(-\frac{1}{f_1} + \frac{1}{f_2}\right) \left[ \frac{dG}{dr} - \left(\frac{l+1}{r}\right) G \right] + \left[ -\frac{(\Delta_1)}{f_1} + \frac{(\Delta_2)}{f_2} \right] \frac{R'}{2} G = 0, (18)
$$

$$
\left(\frac{1}{f_1} - \frac{1}{f_2}\right) \left[ \frac{dG}{dr} + \left(\frac{l+2}{r}\right) G \right] + \left[ \frac{(\Delta_1)}{f_1} - \frac{(\Delta_2)}{f_2} \right] \frac{R'}{2} G = 0, (19)
$$

$$
\left(\frac{1}{2f_1} + \frac{1}{2f_2} - \frac{1}{f_3}\right) \left[ \frac{dG}{dr} - \left(\frac{l+1}{r}\right) G \right] + \left[ \frac{(\Delta_1)}{2f_1} + \frac{(\Delta_2)}{2f_2} + \frac{(\Delta_3)}{f_3} \right] \frac{R'}{2} G = 0, (20)
$$

and

$$
-\left(\frac{l+m+3/2}{2(2l+3)}\right) \left\{ \left(\frac{1}{f_1} + \frac{1}{f_2}\right) \left[ \frac{dG}{dr} + \left(\frac{l+2}{r}\right) G \right] + \left[ \frac{(\Delta_1)}{f_1} + \frac{(\Delta_2)}{f_2} \right] \frac{R'}{2} G \right\} + \left(\frac{l-m+3/2}{2l+3}\right) \left\{ -\frac{1}{f_3} \left[ \frac{dG}{dr} + \left(\frac{l+2}{r}\right) G \right] + \frac{(\Delta_3)}{f_3} \frac{R'}{2} G \right\} + \left( -\frac{E}{f_4} + M \right) F = 0. (21)
$$

For simplicity of presentation, here we have used $\Delta_i$ to denote

$$
\begin{align*}
\Delta_1 &= \frac{d}{dt} \left( \ln f_1 f_2 f_3 f_4 + \ln f_1 \right), \\
\Delta_2 &= \frac{d}{dt} \left( \ln f_1 f_2 f_3 f_4 + \ln f_2 \right), \\
\Delta_3 &= \frac{d}{dt} \left( \ln f_1 f_2 f_3 f_4 + \ln f_3 \right).
\end{align*} (22)
$$
Substituting (16) into (13), one obtains the four equations equivalent to (13) (see Appendix A):

\[
\left( 1 - \frac{1}{f_2} \right) \left[ \frac{dG}{dr} - \left( \frac{l + 1}{r} \right) G \right] + \left[ \frac{\left( \Delta_1 \right)}{f_1} - \frac{\left( \Delta_2 \right)}{f_2} \right] \frac{R'}{2} G = 0, \tag{23}
\]

\[
\left( - \frac{1}{f_1} + \frac{1}{f_2} \right) \left[ \frac{dG}{dr} + \left( \frac{l + 2}{r} \right) G \right] + \left[ \frac{\left( \Delta_1 \right)}{f_1} + \frac{\left( \Delta_2 \right)}{f_2} \right] \frac{R'}{2} G = 0, \tag{24}
\]

\[
\left( 2f_1 - 1 \right) \left[ \frac{dG}{dr} - \left( \frac{l + 1}{r} \right) G \right] + \left[ \frac{\left( \Delta_1 \right)}{2f_1} + \frac{\left( \Delta_2 \right)}{2f_2} + \frac{\left( \Delta_3 \right)}{f_3} \right] \frac{R'}{2} G = 0, \tag{25}
\]

and

\[
\left( \frac{l - m + 3/2}{2(2l + 3)} \right) \left\{ \left( \frac{1}{f_1} + \frac{1}{f_2} \right) \left[ \frac{dG}{dr} + \left( \frac{l + 2}{r} \right) G \right] + \left[ \frac{\left( \Delta_1 \right)}{f_1} + \frac{\left( \Delta_2 \right)}{f_2} \right] \frac{R'}{2} G \right\} + \left( \frac{l + m + 3/2}{2l + 3} \right) \left( \frac{1}{f_3} \left[ \frac{dG}{dr} + \left( \frac{l + 2}{r} \right) G \right] - \frac{\left( \Delta_3 \right)}{f_3} \frac{R'}{2} G \right) + \left( \frac{E}{f_4} - M \right) F = 0. \tag{26}
\]

Note that Eqs. (18)-(20) are the same as (23)-(25). If we subtract (21) from (26), it follows that

\[
\left\{ \left( \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} \right) \left[ \frac{dG}{dr} + \left( \frac{l + 2}{r} \right) G \right] + \left[ \frac{\left( \Delta_1 \right)}{2f_1} + \frac{\left( \Delta_2 \right)}{2f_2} - \frac{\left( \Delta_3 \right)}{f_3} \right] \frac{R'}{2} G \right\}
+ 2 \left( \frac{E}{f_4} - M \right) F = 0. \tag{27}
\]

Eliminating the final term \((\Delta_3)R'G/2f_3^2\) in the second middle bracket from Eq. (25), we get

\[
\left( \frac{1}{2f_1} + \frac{1}{2f_2} \right) \frac{dG}{dr} + \left[ \left( \frac{1}{2f_1} + \frac{1}{2f_2} \right) \frac{1}{2} + \left( \frac{\left( \Delta_1 \right)}{2f_1} + \frac{\left( \Delta_2 \right)}{2f_2} \right) \frac{R'}{2} + \frac{1}{f_3} \left( \frac{l + 3/2}{r} \right) \right] G
+ \left( \frac{E}{f_4} - M \right) F = 0. \tag{28}
\]

Changing the variable of differentiation by letting \(d/dr = R'd/dR\), it turns into

\[
\frac{dG}{dR} + \frac{1}{R} \left[ \frac{R}{2rR'} + \frac{R f_1(\Delta_2) + f_2(\Delta_1)}{f_1 + f_2} \right] G + \left[ \frac{R}{r f_3 R'} \left( \frac{2 f_1 f_2}{f_1 + f_2} \right) \left( \frac{l + 3/2}{R} \right) \right] G
+ \frac{1}{R'} \left( \frac{2 f_1 f_2}{f_1 + f_2} \right) \left( \frac{E}{f_4} - M \right) F = 0. \tag{29}
\]

Substituting (16) into (14) and (15), and using the same argument as above gives

\[
\frac{dF}{dR} + \frac{1}{R} \left[ \frac{R}{2rR'} + \frac{R f_1(\Delta_2) + f_2(\Delta_1)}{f_1 + f_2} \right] F - \left[ \frac{R}{r f_3 R'} \left( \frac{2 f_1 f_2}{f_1 + f_2} \right) \left( \frac{l + 1/2}{R} \right) \right] F
- \frac{1}{R'} \left( \frac{2 f_1 f_2}{f_1 + f_2} \right) \left( \frac{E}{f_4} + M \right) G = 0. \tag{30}
\]

Combining with the chosen constraint conditions, Eqs. (29) and (30) will lead to form invariance solutions for fermions. Before doing this, let’s turn our attention to the equations
that satisfy the trial solution (17). Substituting (17) for (12) gives the four equivalent equations (see Appendix B)

\[
\left( \frac{1}{f_1} - \frac{1}{f_2} \right) \left[ \frac{dG}{dr} - \left( \frac{l - 1}{r} \right) G \right] + \left[ \frac{(\Delta_1)}{f_1} - \frac{(\Delta_2)}{f_2} \right] \frac{R'}{2} G = 0, \tag{31}
\]

\[
\left( -\frac{1}{f_1} + \frac{1}{f_2} \right) \left[ \frac{dG}{dr} + \frac{l}{r} G \right] + \left[ -\frac{(\Delta_1)}{f_1} + \frac{(\Delta_2)}{f_2} \right] \frac{R'}{2} G = 0, \tag{32}
\]

\[
\left( \frac{1}{2f_1} + \frac{1}{2f_2} - \frac{1}{f_3} \right) \left[ \frac{dG}{dr} + \frac{l}{r} G \right] + \left[ \frac{(\Delta_1)}{2f_1} + \frac{(\Delta_2)}{2f_2} + \frac{(\Delta_3)}{f_3} \right] \frac{R'}{2} G = 0, \tag{33}
\]

and

\[
- \left( \frac{l - m - 1/2}{2(2l - 1)} \right) \left\{ \left( \frac{1}{f_1} + \frac{1}{f_2} \right) \left[ \frac{dG}{dr} - \left( \frac{l - 1}{r} \right) G \right] + \left[ \frac{(\Delta_1)}{f_1} + \frac{(\Delta_2)}{f_2} \right] \frac{R'}{2} G \right\}
- \left( \frac{l + m - 1/2}{2l - 1} \right) \left\{ \frac{1}{f_3} \left[ \frac{dG}{dr} - \left( \frac{l - 1}{r} \right) G \right] - \frac{(\Delta_3)}{f_3} \frac{R'}{2} G \right\} - \left( \frac{E}{f_4} + M \right) F = 0. \tag{34}
\]

Substituting (17) for (13) results in the equalities (see Appendix B)

\[
\left( \frac{1}{f_1} - \frac{1}{f_2} \right) \left[ \frac{dG}{dr} - \left( \frac{l - 1}{r} \right) G \right] + \left[ \frac{(\Delta_1)}{f_1} - \frac{(\Delta_2)}{f_2} \right] \frac{R'}{2} G = 0, \tag{35}
\]

\[
\left( -\frac{1}{f_1} + \frac{1}{f_2} \right) \left[ \frac{dG}{dr} + \frac{l}{r} G \right] + \left[ -\frac{(\Delta_1)}{f_1} + \frac{(\Delta_2)}{f_2} \right] \frac{R'}{2} G = 0, \tag{36}
\]

\[
\left( \frac{1}{2f_1} + \frac{1}{2f_2} - \frac{1}{f_3} \right) \left[ \frac{dG}{dr} + \frac{l}{r} G \right] + \left[ \frac{(\Delta_1)}{2f_1} + \frac{(\Delta_2)}{2f_2} + \frac{(\Delta_3)}{f_3} \right] \frac{R'}{2} G = 0, \tag{37}
\]

and

\[
- \left( \frac{l + m - 1/2}{2(2l - 1)} \right) \left\{ \left( \frac{1}{f_1} + \frac{1}{f_2} \right) \left[ \frac{dG}{dr} - \left( \frac{l - 1}{r} \right) G \right] + \left[ \frac{(\Delta_1)}{f_1} + \frac{(\Delta_2)}{f_2} \right] \frac{R'}{2} G \right\}
- \left( \frac{l - m - 1/2}{2l - 1} \right) \left\{ \frac{1}{f_3} \left[ \frac{dG}{dr} - \left( \frac{l - 1}{r} \right) G \right] - \frac{(\Delta_3)}{f_3} \frac{R'}{2} G \right\} - \left( \frac{E}{f_4} - M \right) F = 0. \tag{38}
\]

Again, we see the same structure among (31)-(33) and (35)-(37). Adding (34) and (38), one gets the equation

\[
\left\{ \left( \frac{1}{2f_1} + \frac{1}{2f_2} + \frac{1}{f_3} \right) \left[ \frac{dG}{dr} - \left( \frac{l - 1}{r} \right) G \right] + \left[ \frac{(\Delta_1)}{2f_1} + \frac{(\Delta_2)}{2f_2} - \frac{(\Delta_3)}{f_3} \right] \frac{R'}{2} G \right\}
+ 2 \left( \frac{E}{f_4} - M \right) F = 0. \tag{39}
\]
Solving \((\Delta_3) R' G/2 f_3\) from (37), and substituting its representation into the corresponding term in (39), it yields
\[
\left( \frac{1}{2f_1} + \frac{1}{2f_2} \right) \frac{dG}{dr} + \left[ \left( \frac{1}{2f_1} + \frac{1}{2f_2} \right) \frac{1}{2r} + \left( \frac{(\Delta_1)}{2f_1} + \frac{(\Delta_2)}{2f_2} \right) \frac{R'}{r f_3} \right] \frac{(l - 1/2)}{r} G + \left( \frac{E}{f_4} - M \right) F = 0. \tag{40}
\]

It follows from the change of variable \(dG/dr = \left( \frac{dG}{dR} \right) \left( \frac{dR}{dr} \right)\) that
\[
\frac{dG}{dR} + \frac{1}{R} \left[ R \left( \frac{2f_1(\Delta_2) + f_2(\Delta_1)}{2f_1 + f_2} \right) \right] G - \frac{R}{r f_3 R'} \left( \frac{2f_1 f_2}{f_1 + f_2} \right) \left( \frac{l - 1/2}{R} \right) G
+ \frac{1}{R'} \left( \frac{2f_1 f_2}{f_1 + f_2} \right) \left( \frac{E}{f_4} - M \right) F = 0. \tag{41}
\]
Substituting (17) into (14) and (15), an argument similar to the one used above leads to
\[
\frac{dF}{dR} + \frac{1}{R} \left[ R \left( \frac{2f_1(\Delta_2) + f_2(\Delta_1)}{2f_1 + f_2} \right) \right] F + \frac{R}{r f_3 R'} \left( \frac{2f_1 f_2}{f_1 + f_2} \right) \left( \frac{l + 1/2}{R} \right) F
- \frac{1}{R'} \left( \frac{2f_1 f_2}{f_1 + f_2} \right) \left( \frac{E}{f_4} + M \right) G = 0. \tag{42}
\]
Eqs. (41) and (42) are an alternative set of equations to obtain the form invariance solution.

It is time to choose the constraints for the form invariance. By comparing Eqs. (29), (30), (41), and (42), we choose
\[
\frac{R}{r R'} + \frac{f_1(\Delta_2) + f_2(\Delta_1)}{f_1 + f_2} R = 1, \tag{43}
\]
\[
\frac{R}{r f_3 R'} \left( \frac{2f_1 f_2}{f_1 + f_2} \right) = 1, \tag{44}
\]
\[
\frac{1}{R'} \left( \frac{2f_1 f_2}{f_1 + f_2} \right) = 1, \tag{45}
\]
\[
\frac{1}{f_4} = \text{any function of } R(r), \tag{46}
\]
which result in the four equations
\[
\begin{cases}
\frac{dG}{dR} + \left( \frac{l + 2}{R} \right) G + \left( \frac{E}{f_4} - M \right) F = 0, \\
\frac{dF}{dR} - \left( \frac{l}{R} \right) F - \left( \frac{E}{f_4} + M \right) G = 0,
\end{cases}
\text{ for } j = l + 1/2, \ l = 0, 1, 2, \ldots, \tag{47}
\]
and
\[
\begin{cases}
\frac{dG}{dR} - \left( \frac{l - 1}{R} \right) G + \left( \frac{E}{f_4} - M \right) F = 0, \\
\frac{dF}{dR} + \left( \frac{l + 1}{R} \right) F - \left( \frac{E}{f_4} + M \right) G = 0,
\end{cases}
\text{ for } j = l - 1/2, \ l = 1, 2, 3, \ldots. \tag{48}
\]
We leave \(1/f_4\) in the equations intact since it can be any function of \(R\) and depends on what kind of interaction we consider. Note that the constraints, Eqs. (43)-(46), build the connection between a spacetime structure and a quantum system. By defining the quantum numbers

\[
\kappa = \begin{cases} 
-(j + 1/2) = -(l + 1) & \text{for } j = l + 1/2 \\
+(j + 1/2) = l & \text{for } j = l - 1/2 
\end{cases} = \mp 1, \mp 2, \cdots, 
\]

(49)
two systems (47) and (48) can be unified and reduced to a single one for \(F\) and \(G\)

\[
\frac{dF}{dR} + \left(\frac{\kappa + 1}{R}\right) F - \left(\frac{E}{f_4} + M\right) G = 0, 
\]

(50)
and

\[
\frac{dG}{dR} - \left(\frac{\kappa - 1}{R}\right) G + \left(\frac{E}{f_4} - M\right) F = 0. 
\]

(51)
Together with constraints (43)-(46), the set serves as the basic equations to obtain a form invariance solution in a central force field from a spacetime structure.

III. A SPACETIME CAGE FOR FERMIONS

As an application of the formulation in section II, we coin a cage of spacetime for fermions in this section. For this, we choose a trapping zone with axial symmetry along the \(z\) axis. This can be achieved by letting \(g_{11} = g_{22}\), and thus \(f_1 = f_2\). The choice also makes equations (18), (19) and the similar equations in the remaining sets from (23) to (42) are automatically satisfied. As a result, the form of the spatial components of metric is immediately determined by conditions (44) and (45). They are

\[
f_1 = R', \; f_2 = R', \; \text{and} \; f_3 = \frac{R}{r}. 
\]

(52)
The temporal component of the metric is set to

\[
\frac{1}{f_4} = 1 - \frac{V}{E}, 
\]

(53)
with \(V\) serving as the potential function, and chosen here to be

\[
V = \begin{cases} 
-V_0, & r \leq r_b, \\
0, & r > r_b, 
\end{cases} 
\]

(54)
where \(V_0\) is a positive constant. It is obvious that the number \(r_b\) labels the boundary of the cage. The explicit representation of an invariance solution needs to know the function
$R(r)$. It can be determined by Eq. (43). Since $f_4$ is a constant under consideration, it is not difficult to verify that it is an arbitrary linear function of $r$ under the conditions (52) and (54), i.e.,

$$R(r) = \alpha r + \beta,$$

where $\alpha$ and $\beta$ are constants. It is time to calculate the form invariance spinor. By substituting (53) into (50) and (51), the form invariant radial equations for region $r \leq r_b$ are given by

$$\frac{dF}{dR} + \left(\frac{\kappa + 1}{R}\right) F - (E + V_0 + M) G = 0,$$

and

$$\frac{dG}{dR} - \left(\frac{\kappa - 1}{R}\right) G + (E + V_0 - M) F = 0.$$

The equations which $F$ and $R$ satisfy can be found by taking the derivative of (56) and (57) with respect to $R$ again, which gives

$$\frac{d^2 F}{dR^2} + \frac{2}{R} \frac{dF}{dR} + \left(k^2 - \frac{\kappa(\kappa + 1)}{R^2}\right) F = 0,$$

and

$$\frac{d^2 G}{dR^2} + \frac{2}{R} \frac{dG}{dR} + \left(k^2 - \frac{\kappa(\kappa - 1)}{R^2}\right) G = 0,$$

where $k^2 = (E + V_0)^2 - M^2$. This is the well-known spherical Bessel equation. The physical solutions can be divided into two classes. First, when $k^2 = (E + V_0)^2 - M^2 > 0$, the solution to (58) is

$$F(R) = \begin{cases} a_1 j_\kappa(kR) + a_2 n_\kappa(kR), & \text{for } \kappa > 0, \\ a_1 j_{|\kappa| - 1}(kR) + a_2 n_{|\kappa| - 1}(kR), & \text{for } \kappa < 0. \end{cases}$$

Here $a_1$ and $a_2$ are constants, and $j_\kappa$ and $n_\kappa$ are the spherical Bessel functions. The solution to (59) can be found through Eq. (56) which shows

$$G = \frac{1}{E + V_0 + M} \left[ \frac{dF}{dR} + \left(\frac{\kappa + 1}{R}\right) F \right].$$

Substituting (60) into (61) gives rise to the answer

$$G(R) = \begin{cases} \frac{k}{E + V_0 + M} \left[ a_1 j_{|\kappa| - 1}(kR) + a_2 n_{|\kappa| - 1}(kR) \right], & \text{for } \kappa > 0, \\ \frac{-k}{E + V_0 + M} \left[ a_1 j_\kappa(kR) + a_2 n_\kappa(kR) \right], & \text{for } \kappa < 0. \end{cases}$$

To have the representation, we have invoked the recurrence relations

$$\frac{n+1}{z} W_n(z) + \frac{d}{dz} W_n(z) = W_{n-1}(z),$$

and

$$\frac{n}{z} W_n(z) - \frac{d}{dz} W_n(z) = W_{n+1}(z).$$
where the function $W_n(z)$ labels $j_n(z)$ and $n_n(z)$ (p. 439, [6]).

For $k^2 = (E + V_0)^2 - M^2 < 0$, the solution to (58) is

$$F(R) = \begin{cases} 
\sqrt{\frac{2k}{\pi R}} [b_1 I_{\kappa+1/2}(\bar{k}R) + b_2 K_{\kappa+1/2}(\bar{k}R)] , & \text{for } \kappa > 0, \\
\sqrt{\frac{2k}{\pi R}} [b_1 I_{|\kappa|-1/2}(\bar{k}R) + b_2 K_{|\kappa|-1/2}(\bar{k}R)] , & \text{for } \kappa < 0,
\end{cases}$$

(64)

where we define the symbol $\bar{k}^2 = M^2 - (E + V_0)^2 > 0$, and $I_\nu$ and $K_\nu$ are the modified spherical Bessel functions. The corresponding solution to $G$ is

$$G(R) = \begin{cases} 
\frac{k}{E+V_0+M} \sqrt{\frac{2k}{\pi R}} [b_1 I_{\kappa-1/2}(\bar{k}R) - b_2 K_{\kappa-1/2}(\bar{k}R)] , & \text{for } \kappa > 0, \\
\frac{k}{E+V_0+M} \sqrt{\frac{2k}{\pi R}} [b_1 I_{|\kappa|+1/2}(\bar{k}R) - b_2 K_{|\kappa|+1/2}(\bar{k}R)] , & \text{for } \kappa < 0.
\end{cases}$$

(65)

The recurrence relations

$$\frac{n+1}{z} W_n(z) + \frac{d}{dz} W_n(z) = W_{n-1}(z), \text{ and}$$

$$\frac{n}{z} W_n(z) - \frac{d}{dz} W_n(z) = W_{n+1}(z),$$

(66)

are used to yield the representation in which $W_n(z)$ labels $\sqrt{\pi/2z} I_{n+1/2}(z)$ and $(-1)^{n+1} \sqrt{\pi/2z} K_{n+1/2}(z)$ (p. 444, [6]). Using the fact that the wave function has to be continuous, the radial functions $F(R)$ and $G(R)$ are connected by the functions satisfying the equations in the region $r > r_b$,

$$\frac{dF}{dr} + \left(\frac{\kappa + 1}{r}\right) F - (E + M) G = 0,$$

(67)

and

$$\frac{dG}{dr} - \left(\frac{\kappa - 1}{r}\right) G + (E - M) F = 0.$$

(68)

Obviously, the corresponding solutions to the system are given by

$$F(r) = \begin{cases} 
a_1 j_\kappa(k_1r) + a_2 n_\kappa(k_1r) , & \text{for } \kappa > 0, \\
a_1 j_{|\kappa|-1}(k_1r) + a_2 n_{|\kappa|-1}(k_1r) , & \text{for } \kappa < 0,
\end{cases}$$

(69)

and

$$G(r) = \begin{cases} 
\frac{k_1}{E+M} [a_1 j_{\kappa-1}(k_1r) + a_2 n_{\kappa-1}(k_1r)] , & \text{for } \kappa > 0, \\
-\frac{k_1}{E+M} [a_1 j_{|\kappa|}(k_1r) - a_2 n_{|\kappa|}(k_1r)] , & \text{for } \kappa < 0,
\end{cases}$$

(70)

for $k_1^2 = E^2 - M^2 > 0$, and

$$F(r) = \begin{cases} 
\sqrt{\frac{2k}{\pi r}} [b_1 I_{\kappa+1/2}(\bar{k}_1r) + b_2 K_{\kappa+1/2}(\bar{k}_1r)] , & \text{for } \kappa > 0, \\
\sqrt{\frac{2k}{\pi r}} [b_1 I_{|\kappa|-1/2}(\bar{k}_1r) + b_2 K_{|\kappa|-1/2}(\bar{k}_1r)] , & \text{for } \kappa < 0,
\end{cases}$$

(71)

and

$$G(r) = \begin{cases} 
\frac{k_1}{E+M} \sqrt{\frac{2k}{\pi r}} [b_1 I_{\kappa-1/2}(\bar{k}_1r) - b_2 K_{\kappa-1/2}(\bar{k}_1r)] , & \text{for } \kappa > 0, \\
\frac{k_1}{E+M} \sqrt{\frac{2k}{\pi r}} [b_1 I_{|\kappa|+1/2}(\bar{k}_1r) - b_2 K_{|\kappa|+1/2}(\bar{k}_1r)] , & \text{for } \kappa < 0,
\end{cases}$$

(72)

for $\bar{k}_1 = M^2 - E^2 > 0$. 

11
A. Bound states

Let us now discuss the bound states caught by the spacetime cage. For fermions with energy satisfying the condition $k^2 = (E + V_0)^2 - M^2 > 0$, their radial amplitudes in the cage are described by (60) and (62) with the coefficient $a_2 = 0$ since $n_n(z) \longrightarrow 1/z^{n+1}$ as $z \rightarrow 0$. Outside the cage, the representations (71) and (72) with $b_1 = 0$ are the radial solutions of fermions since the wave functions are bounded, and $I_{n+1/2}(z) \rightarrow e^{z}/\sqrt{z}$ as $z \rightarrow \infty$. The energy levels of the states are determined by the continuous condition of wave functions at the boundary $r = r_b$. It is straightforward to show that the spectrum is determined by the equation

$$\frac{j_{l_1}(kr_b)}{j_{l_2}(kr_b)} = -\epsilon_\kappa \frac{k}{k_1} \left( \frac{E + M}{E + V_0 + M} \right) \frac{K_{l_1+1/2}(\kappa_1 r_b)}{K_{l_2+1/2}(\kappa_1 r_b)},$$

(73)

where $\epsilon_\kappa = 1 (-1)$ for $\kappa > 0 (\kappa < 0)$, and the symbols $l_1$ and $l_2$ denote

$$l_1 = \begin{cases} \kappa, & \text{for } \kappa > 0, \\ |\kappa| - 1, & \text{for } \kappa < 0, \end{cases}$$

and $l_2 = \begin{cases} \kappa - 1, & \text{for } \kappa > 0, \\ |\kappa|, & \text{for } \kappa < 0. \end{cases}$

(74)

For $s$ states, the quantum number $\kappa = -1$. The energy levels satisfy the relation

$$\frac{j_0(kr_b)}{j_1(kr_b)} = \frac{k}{k_1} \left( \frac{E + M}{E + V_0 + M} \right) \frac{K_{1/2}(\kappa_1 r_b)}{K_{3/2}(\kappa_1 r_b)}.$$

(75)

Fig. 1 shows the energy spectrum allowed by the $s$ states. The boundaries of the cage are taken as $r_b = 10\lambda_F$ and $100\lambda_F$ with $\lambda_F = \hbar/Mc$, the Compton wavelength of fermions. The spectrum equation (73) is actually the same as that in the usual spherical box for fermions [7]. However, the presented spacetime approach to the quantization rule has more implications hidden in the freedom of solutions with the form invariance. In equation (55), we have shown that the radial amplitude of fermions in the cage is with the function $R(r) = \alpha r + \beta$ as a variable. One can choose

$$\alpha = \frac{r_b}{r_b - r_a}, \text{ and } \beta = \frac{-r_ar_b}{r_b - r_a}.$$

(76)

This choice implies that the spacetime cage can capture fermion states accompanying an invisible region. Fig. 2 shows the component $F(R(r))$ for the $1s_{1/2}$ state in the cage. It demonstrates the zone sustaining a fermion state accompanying a zero amplitude invisible region.
B. Scattering states

The presented cage can be regarded as a quantum cloak for bound states since it establishes an invisible region for them and does not alter the behavior of scattering states. For completeness, the scattering behavior of fermions from the cage is discussed in this subsection. The allowed physical solution in the cage is given by

\[ F_i(R) = a_1 j_{l_1}(kR), \]
\[ G_i(R) = \epsilon \frac{k}{E + V_0 + M} a_1 j_{l_2}(kR), \]

for every \( \kappa \). Outside the cage, the scattering radial functions are

\[ F_o(r) = c_1 j_{l_1}(k_1 r) + c_2 n_{l_1}(k_1 r), \]
\[ G_o(r) = \epsilon \frac{k_1}{E + M} [c_1 j_{l_2}(k_1 r) + c_2 n_{l_2}(k_1 r)], \]

where \( k_1^2 = E^2 - M^2 \). An asymptotic analysis of the outgoing wave can be made with the formulas \( j_{\nu}(z) \rightarrow \sin(z - \nu \pi/2)/z \) and \( n_{\nu}(z) \rightarrow -\cos(z - \nu \pi/2)/z \) as \( z \rightarrow \infty \). Consequently, the radial parts of the outgoing wave in the far zone are given by

\[ F_o(r) \rightarrow A \frac{k_1}{k_1 r} \sin(k_1 r - l_1 \pi/2 + \delta_\kappa), \]
\[ G_o(r) \rightarrow \epsilon \frac{k_1}{E + M} A \frac{k_1}{k_1 r} \sin(k_1 r - l_2 \pi/2 + \delta_\kappa), \]

where the phase shift is defined by \( c_2/c_1 = -\tan \delta_\kappa \), and \( A = c_1/\cos \delta_\kappa \). Using the fact that the spinor is continuous at \( r = r_b \),

\[ c_1 j_{l_1}(k_1 r_b) + c_2 n_{l_1}(k_1 r_b) = a_1 j_{l_1}(k r_b), \]

and

\[ \epsilon \frac{k_1}{E + M} [c_1 j_{l_2}(k_1 r_b) + c_2 n_{l_2}(k_1 r_b)] = \epsilon \frac{k}{E + V_0 + M} a_1 j_{l_2}(k r_b). \]

It is easy to show that the phase shift has the analytical representation

\[ \tan \delta_\kappa = \frac{j_{l_1}(k r_b) j_{l_2}(k_1 r_b) - B j_{l_1}(k r_b) j_{l_2}(k r_b)}{j_{l_1}(k r_b) n_{l_2}(k_1 r_b) - B j_{l_2}(k r_b) n_{l_1}(k_1 r_b)}, \]

where the constant

\[ B = \sqrt{\left(\frac{E + M}{E - M}\right) \left(\frac{E + V_0 - M}{E + V_0 + M}\right)}. \]
A phase shift often reflects the subtle structure of the bound states \([9, 10]\). Fig. 3 shows the phase shift of the \(s\) states for \(V_0 = 2Mc^2\) and radius \(r_b = 10\lambda_F\), where the phase shift has absorbed \(k_1r_b\), i.e. \(\delta = \delta_{x=-1} + k_1r_b\), since it can show that all \(s\) states have the factor.

C. Experiment

We here explain how it is possible to coin the trapping cage with a physical method. Eqs. (52) and (53) show that the cage is created by a curved space depicted by the line element

\[
ds^2 = \alpha^2(dx^1)^2 + \alpha^2(dx^2)^2 + \left(\alpha + \frac{\beta}{r}\right)^2(dx^3)^2 - \left(\frac{1}{1 + V_0/E}\right)^2(dt)^2. \tag{84}\]

Constant \(\alpha\) and \(\beta\) are as in (76). Note that there is the same proper time on every point in the cage. This is easy to be seen by

\[
d\tau = ids = \sqrt{g_{44}}dt = \left(\frac{1}{1 + V_0/E}\right) dt. \tag{85}\]

The element of spatial distance for all space is given by (p. 235, \([8]\))

\[
dL^2 = \left(g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}\right)dx^i dx^j, \ i, j = 1, 2, 3. \tag{86}\]

Since there are no off diagonal components here, the element of spatial distance has the simple representation

\[
dL^2 = \alpha^2(dx^1)^2 + \alpha^2(dx^2)^2 + \left(\alpha + \frac{\beta}{r}\right)^2(dx^3)^2. \tag{87}\]

The first step to the cage is to coin a spatial zone with the line element \(dL^2\). In a laboratory, the interval of both space and time for the zone is given by

\[
ds^2 = \alpha^2(dx^1)^2 + \alpha^2(dx^2)^2 + \left(\alpha + \frac{\beta}{r}\right)^2(dx^3)^2 - (dt)^2. \tag{88}\]

Here \(t\) is defined by a running cloak in the laboratory. It shows that the elapsed time on each point of the zone is the same as the laboratory. To have the effect of the proper time shown as (85), one pushes the 3D trapping zone along the \(x^1\) or \(x^2\) axis, say \(x^1\), to have a constant velocity \(v\). Eq. (88) shows that

\[
ds^2 = \left[\alpha^2\left(\frac{dx^1}{dt}\right)^2 - 1\right] (dt)^2 = \left(\alpha^2v^2 - 1\right) (dt)^2, \tag{89}\]
that is,

\[ d\tau = ids = \sqrt{1 - \alpha^2 v^2} dt. \]  

(90)

A comparison of this equation with (85) gives

\[ \sqrt{1 - \alpha^2 v^2} = \left( \frac{E}{E + V_0} \right). \]  

(91)

Therefore, the velocity to create the constant global proper time in the cage is

\[ v = \frac{c}{\alpha} \sqrt{1 - \left( \frac{E}{E + V_0} \right)^2}, \]  

(92)

where we put the light speed \( c \) back clearly. According to the numerical result of Fig. 1, for the cage with radius \( r_b = 100\lambda_F \), the first bound state \( 1s_{1/2} \) appears at \( V_0/E_0 = 0.008059 \) with energy \( E/E_0 = 0.992 \). The corresponding velocity in (92) is about \( v = 0.1267c/\alpha \).

A large number \( \alpha \) will greatly reduce the velocity. The constant velocity of the zone can actually also be offered by a circular motion. The Lorentz contraction due to the velocity can be ignored here since it is less than 1\% for the maximum \( v = 0.1267c \). Before closing the section, let’s make two notes on the asymptotic behavior of the cage. First, the parameter \( \alpha = 1 \) when \( r_a = 0 \). The spacetime structure becomes

\[ ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - \left( \frac{1}{1 + V_0/E} \right)^2 (dt)^2. \]  

(93)

Therefore, a pure curved structure of time with \( g_{44} = [E/(E + V_0)]^2 \) suffices to obtain the trapping effect of an ideal potential well for fermions. However, as stated above, the lowest velocity to have the effect is about 13\% of the light speed. Second, the parameter \( \alpha \to \infty \) when \( r_a \to r_b \). The velocity \( v \) for the cage is easier to reach. However, the spatial geometry becomes complicated. A balance between the velocity which we can push and the difficulty of making a space with the metric (87) should be estimated.

IV. CONCLUSION

Constructing the connection between a spacetime structure and a quantum rule has long attracted considerable attention since the general invariance principle was established [11]. In this paper, we give the constraint conditions for the covariant Dirac equation which determines the spacetime structure of constructing a quantum system with the form
invariance solution in the central force field. As an application, constructing a spacetime cage that confines fermions forming bound states and builds the quantum rules of a spherical well is presented. Let us remark on several consequences of our discussion: (i) The quantum rule of fermions created by a spacetime structure accompanies novel fermion states. As shown in Sec. III, the quantum rule of fermions in a spherical well can be built from a spacetime geometry, while the corresponding states accompany an invisible region. Obviously, any quantum rules using the spacetime approach correspond to the new type of fermion states due to the form invariance variable. (ii) There may be a chance to design a new quantum device with a macroscopic scale using the presented spacetime approach. In general, the curved structure of a space is often involved in the discussion of the motion of macroscopic objects. Establishing the connection between a quantum system and a metric structure makes the construction of a macroscopic quantum system possible. For instance, a neutrino mass is less than 2eV [12]. The corresponding Compton wavelength $\lambda_{\text{neutrino}} > 10^{-5} cm$. Making a spacetime cage for neutrinos with radius $r_b = 500\lambda_{\text{neutrino}}$ could have the width $w = 10^3\lambda_{\text{neutrino}} > 10^{-2} cm$. (iii) The trapping cage coined from the spacetime structure may serve as a bag to get the bizarre spin-1/2 particles. The trapping force emerging from our discussion is constructed from the shape of spacetime. It is applicable to any spin-1/2 particles. The approach here may be a ladder to get to the trapping job for spin-1/2 dark particles. Specifically, relic neutrinos have very low energy, which carry information of our universe about one second old. The presented construct may be applicable to catch these marvelous particles.
Appendix A:

Proof of Eqs. (18)-(21) and (23)-(26)

In deriving the form invariance Dirac spinors, we need the formulas for the action of the coordinates on the spherical harmonics. We list them as follows:

\[
\frac{2x}{r} Y_{lm} = \sqrt{\frac{(l-m-1)(l-m)}{(2l+1)(2l-1)}} Y_{l-1,m+1} - \sqrt{\frac{(l+m+2)(l+m+1)}{(2l+3)(2l+1)}} Y_{l+1,m+1} \\
- \sqrt{\frac{(l+m)(l+m-1)}{(2l+1)(2l-1)}} Y_{l-1,m-1} + \sqrt{\frac{(l+m+2)(l+m+1)}{(2l+3)(2l+1)}} Y_{l+1,m-1},
\]

\[
\frac{2iy}{r} Y_{lm} = \sqrt{\frac{(l-m-1)(l-m)}{(2l+1)(2l-1)}} Y_{l-1,m+1} - \sqrt{\frac{(l+m+2)(l+m+1)}{(2l+3)(2l+1)}} Y_{l+1,m+1} \\
+ \sqrt{\frac{(l+m)(l+m-1)}{(2l+1)(2l-1)}} Y_{l-1,m-1} - \sqrt{\frac{(l+m+2)(l+m+1)}{(2l+3)(2l+1)}} Y_{l+1,m-1},
\]

and

\[
\frac{z}{r} Y_{lm} = \sqrt{\frac{(l+m+1)(l-m-1)}{(2l+3)(2l+1)}} Y_{l+1,m+1} + \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}} Y_{l-1,m}.
\]

Also, derivatives of the wave function with respect to the coordinates occur. They have relations (see, e.g., pp. 346-349 [13])

\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (GY_{lm}) = \sqrt{\frac{(l+m+2)(l+m+1)}{(2l+3)(2l+1)}} Y_{l+1,m+1} \left( \frac{dG}{dr} - \frac{G}{r} \right) \\
- \sqrt{\frac{(l-m)(l-m-1)}{(2l+1)(2l-1)}} Y_{l-1,m+1} \left( \frac{dG}{dr} + (l+1) \frac{G}{r} \right),
\]

\[
\left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (GY_{lm}) = -\sqrt{\frac{(l+m+2)(l-m+1)}{(2l+3)(2l+1)}} Y_{l+1,m-1} \left( \frac{dG}{dr} - l \frac{G}{r} \right) \\
+ \sqrt{\frac{(l+m)(l+m-1)}{(2l+1)(2l-1)}} Y_{l-1,m-1} \left( \frac{dG}{dr} + (l+1) \frac{G}{r} \right),
\]

and

\[
\frac{\partial}{\partial z} (GY_{lm}) = \sqrt{\frac{(l+m+1)(l-m+1)}{(2l+3)(2l+1)}} Y_{l+1,m} \left( \frac{dG}{dr} - \frac{G}{r} \right) \\
+ \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}} Y_{l-1,m} \left( \frac{dG}{dr} + (l+1) \frac{G}{r} \right).
\]
With the relations (A1)–(A3), we get
\[
\frac{2x}{r} Y_{l+1,m+1/2} = \sqrt{\frac{(l - m - 1/2)(l - m + 1/2)}{(2l + 3)(2l + 1)}} Y_{l,m+3/2} - \sqrt{\frac{(l + m + 7/2)(l + m + 5/2)}{(2l + 5)(2l + 3)}} Y_{l+2,m+3/2} - \sqrt{\frac{(l + m + 3/2)(l + m + 1/2)}{(2l + 3)(2l + 1)}} Y_{l,m-1/2} + \sqrt{\frac{(l - m + 5/2)(l - m + 3/2)}{(2l + 5)(2l + 3)}} Y_{l+2,m-1/2}, \quad (A7)
\]
\[
2iy \frac{r}{r} Y_{l+1,m+1/2} = \sqrt{\frac{(l - m - 1/2)(l - m + 1/2)}{(2l + 3)(2l + 1)}} Y_{l,m+3/2} - \sqrt{\frac{(l + m + 7/2)(l + m + 5/2)}{(2l + 5)(2l + 3)}} Y_{l+2,m+3/2} + \sqrt{\frac{(l + m + 3/2)(l + m + 1/2)}{(2l + 3)(2l + 1)}} Y_{l,m-1/2} - \sqrt{\frac{(l - m + 5/2)(l - m + 3/2)}{(2l + 5)(2l + 3)}} Y_{l+2,m-1/2}, \quad (A8)
\]
and
\[
\bar{z} \frac{r}{r} Y_{l+1,m-1/2} = \sqrt{\frac{(l + m + 3/2)(l - m + 5/2)}{(2l + 5)(2l + 3)}} Y_{l+2,m-1/2} - \sqrt{\frac{(l + m + 1/2)(l - m + 3/2)}{(2l + 3)(2l + 1)}} Y_{l,m+1/2}, \quad (A9)
\]
With the relations (A4)–(A6), we have
\[
2 \frac{\partial}{\partial x} \left( G Y_{l+1,m+1/2} \right) = \sqrt{\frac{(l + m + 7/2)(l + m + 5/2)}{(2l + 5)(2l + 3)}} Y_{l+2,m+3/2} \left( \frac{dG}{dr} - \left( \frac{l + 1}{r} \right) G \right) - \sqrt{\frac{(l - m + 1/2)(l - m - 1/2)}{(2l + 3)(2l + 1)}} Y_{l,m+3/2} \left( \frac{dG}{dr} + \left( \frac{l + 2}{r} \right) G \right) - \sqrt{\frac{(l - m + 5/2)(l - m + 3/2)}{(2l + 5)(2l + 3)}} Y_{l+2,m-1/2} \left( \frac{dG}{dr} - \left( \frac{l + 1}{r} \right) G \right) + \sqrt{\frac{(l + m + 3/2)(l + m + 1/2)}{(2l + 3)(2l + 1)}} Y_{l,m-1/2} \left( \frac{dG}{dr} + \left( \frac{l + 2}{r} \right) G \right), \quad (A10)
\]
\[
2i \frac{\partial}{\partial y} \left( G Y_{l+1,m+1/2} \right) = \sqrt{\frac{(l + m + 7/2)(l + m + 5/2)}{(2l + 5)(2l + 3)}} Y_{l+2,m+3/2} \left( \frac{dG}{dr} - \left( \frac{l + 1}{r} \right) G \right) - \sqrt{\frac{(l - m + 1/2)(l - m - 1/2)}{(2l + 3)(2l + 1)}} Y_{l,m+3/2} \left( \frac{dG}{dr} + \left( \frac{l + 2}{r} \right) G \right) + \sqrt{\frac{(l - m + 5/2)(l - m + 3/2)}{(2l + 5)(2l + 3)}} Y_{l+2,m-1/2} \left( \frac{dG}{dr} - \left( \frac{l + 1}{r} \right) G \right) - \sqrt{\frac{(l + m + 3/2)(l + m + 1/2)}{(2l + 3)(2l + 1)}} Y_{l,m-1/2} \left( \frac{dG}{dr} + \left( \frac{l + 2}{r} \right) G \right), \quad (A11)
\]
and
\[
\frac{\partial}{\partial z} (GY_{l+1,m-1/2}) = \sqrt{\frac{(l + m + 3/2)(l - m + 5/2)}{(2l + 5)(2l + 3)}} Y_{l+2,m-1/2} \left( \frac{dG}{dr} - \left( \frac{l + 1}{r} \right) G \right) + \sqrt{\frac{(l + m + 1/2)(l - m + 3/2)}{(2l + 3)(2l + 1)}} Y_{l,m-1/2} \left( \frac{dG}{dr} + \left( \frac{l + 2}{r} \right) G \right).
\] (A12)

Apply (A7)-(A12) to (12) for the trial solution (16). Eq. (12) has the representation

\[
\frac{1}{2(l + 3)} \sqrt{\frac{(l + m + 3/2)(l + m + 5/2)(l + m + 3/2)}{(2l + 5)}} \left( \frac{1}{f_1} \right) Y_{l+2,m+3/2} \left( \frac{dG}{dr} - \left( \frac{l + 1}{r} \right) G \right) + \frac{1}{2(l + 3)} \sqrt{\frac{(l + m + 3/2)(l - m + 1/2)(l - m - 1/2)}{(2l + 1)}} \left( \frac{1}{f_1} \right) Y_{l,m+3/2} \left( \frac{dG}{dr} + \left( \frac{l + 2}{r} \right) G \right) + \frac{1}{2(l + 3)} \sqrt{\frac{(l + m + 3/2)(l - m + 5/2)(l - m + 3/2)}{(2l + 5)}} \left( \frac{1}{f_1} \right) Y_{l+2,m-1/2} \left( \frac{dG}{dr} - \left( \frac{l + 1}{r} \right) G \right)
\]

\[
- \frac{1}{2(l + 3)} \sqrt{\frac{(l + m + 3/2)(l + m + 1/2)}{(2l + 1)}} \left( \frac{1}{f_1} \right) Y_{l,m-1/2} \left( \frac{dG}{dr} + \left( \frac{l + 2}{r} \right) G \right) + \frac{1}{2(l + 3)} \sqrt{\frac{(l + m + 3/2)(l - m + 5/2)(l - m - 1/2)}{(2l + 1)}} \left( \frac{R'}{2f_1} \right) G Y_{l,m+3/2} \Delta_1
\]

\[
- \frac{1}{2(l + 3)} \sqrt{\frac{(l + m + 7/2)(l + m + 5/2)(l + m + 3/2)}{(2l + 5)}} \left( \frac{R'}{2f_1} \right) G Y_{l+2,m+3/2} \Delta_1
\]

\[
- \frac{1}{2(l + 3)} \sqrt{\frac{(l + m + 3/2)(l + m + 1/2)}{(2l + 1)}} \left( \frac{R'}{2f_1} \right) G Y_{l,m-1/2} \Delta_1 + \frac{1}{2(l + 3)} \sqrt{\frac{(l + m + 3/2)(l - m + 5/2)(l - m + 3/2)}{(2l + 5)}} \left( \frac{R'}{2f_1} \right) G Y_{l+2,m-1/2} \Delta_1
\]

\[
+ \frac{1}{2(l + 3)} \sqrt{\frac{(l + m + 7/2)(l + m + 5/2)(l + m + 3/2)}{(2l + 5)}} \left( \frac{1}{f_2} \right) Y_{l+2,m+3/2} \left( \frac{dG}{dr} - \left( \frac{l + 1}{r} \right) G \right) - \frac{1}{2(l + 3)} \sqrt{\frac{(l + m + 3/2)(l - m + 1/2)(l - m - 1/2)}{(2l + 1)}} \left( \frac{1}{f_2} \right) Y_{l,m+3/2} \left( \frac{dG}{dr} + \left( \frac{l + 2}{r} \right) G \right)
\]

\[
+ \frac{1}{2(l + 3)} \sqrt{\frac{(l + m + 3/2)(l - m + 5/2)(l - m + 3/2)}{(2l + 5)}} \left( \frac{1}{f_2} \right) Y_{l+2,m-1/2} \left( \frac{dG}{dr} - \left( \frac{l + 1}{r} \right) G \right) - \frac{1}{2(l + 3)} \sqrt{\frac{(l + m + 3/2)^2(l + m + 1/2)}{(2l + 1)}} \left( \frac{1}{f_2} \right) Y_{l,m-1/2} \left( \frac{dG}{dr} + \left( \frac{l + 2}{r} \right) G \right)
\]
\[
\frac{1}{2(2l + 3)} \sqrt{\frac{(l + m + 3/2)(l - m + 1/2)(l - m - 1/2)}{(2l + 1)}} \left( \frac{R'}{2f_2} \right) GY_{l,m+3/2}\Delta_2 \\
+ \frac{1}{2(2l + 3)} \sqrt{\frac{(l + m + 7/2)(l + m + 5/2)(l + m + 3/2)}{(2l + 5)}} \left( \frac{R'}{2f_2} \right) GY_{l+2,m+3/2}\Delta_2 \\
- \frac{1}{2(2l + 3)} \sqrt{\frac{(l + m + 3/2)^2(l + m + 1/2)}{(2l + 1)}} \left( \frac{R'}{2f_2} \right) GY_{l,m-1/2}\Delta_2 \\
+ \frac{1}{2(2l + 3)} \sqrt{\frac{(l + m + 3/2)(l - m + 5/2)(l - m + 3/2)}{(2l + 5)}} \left( \frac{R'}{2f_2} \right) GY_{l+2,m-1/2}\Delta_2 \\
- \frac{1}{(2l + 3)} \sqrt{\frac{(l + m + 3/2)(l - m + 5/2)(l - m + 3/2)}{(2l + 5)}} \left( \frac{1}{f_3} \right) Y_{l+2,m-1/2} \left( \frac{dG}{dr} - \left( \frac{l + 1}{r} \right) G \right) \\
- \frac{1}{(2l + 3)} \sqrt{\frac{(l + m + 1/2)(l - m + 3/2)^2}{(2l + 1)}} \left( \frac{1}{f_3} \right) Y_{l,m-3/2} \left( \frac{dG}{dr} + \left( \frac{l + 2}{r} \right) G \right) \\
+ \frac{1}{(2l + 3)} \sqrt{\frac{(l + m + 3/2)(l - m + 5/2)(l - m + 3/2)}{(2l + 5)}} \left( \frac{R'}{2f_3} \right) GY_{l+2,m-3/2}\Delta_3 \\
+ \frac{1}{(2l + 3)} \sqrt{\frac{(l + m + 1/2)(l - m + 3/2)^2}{(2l + 1)}} \left( \frac{R'}{2f_3} \right) GY_{l,m-1/2}\Delta_3 \\
- \left( \frac{E}{f_4} - M \right) \sqrt{\frac{(l + m + 1/2)}{(2l + 1)}} F Y_{l,m-1/2} = 0, \tag{A13}
\]

where $\Delta_i$ are as defined in (22). Multiplying this equation by $Y_{l+2,m+3/2}$, and integrating both sides of it with respect to the solid angle $d\Omega = \sin \theta d\theta d\varphi$ gives the equality

\[
\left( \frac{1}{f_1} + \frac{1}{f_2} \right) \left[ \frac{dG}{dr} - \left( \frac{l + 1}{r} \right) G \right] + \left[ - \frac{(\Delta_1)}{f_1} + \frac{(\Delta_2)}{f_2} \right] \frac{R'}{2} G = 0. \tag{A14}
\]

Multiplying both sides by $Y_{l,m+3/2}$, and performing integration with respect to the solid angle gets

\[
\left( \frac{1}{f_1} - \frac{1}{f_2} \right) \left[ \frac{dG}{dr} + \left( \frac{l + 2}{r} \right) G \right] + \left[ \frac{(\Delta_1)}{f_1} - \frac{(\Delta_2)}{f_2} \right] \frac{R'}{2} G = 0. \tag{A15}
\]

Multiplying both sides by $Y_{l+2,m-1/2}$, and performing the integration as above, it follows that

\[
\left( \frac{1}{2f_1} + \frac{1}{2f_2} - \frac{1}{f_3} \right) \left[ \frac{dG}{dr} - \left( \frac{l + 1}{r} \right) G \right] + \left[ \frac{(\Delta_1)}{2f_1} + \frac{(\Delta_2)}{2f_2} + \frac{(\Delta_3)}{f_3} \right] \frac{R'}{2} G = 0. \tag{A16}
\]
Multiplying both sides by $Y_{l,m-1/2}$, and performing the integration of the solid angle results in
\[
-\frac{l+m+3/2}{2(2l+3)} \left\{ \left( \frac{1}{f_1} + \frac{1}{f_2} \right) \left[ \frac{dG}{dr} + \left( \frac{l+2}{r} \right) G \right] + \left[ \frac{(\Delta_1)}{f_1} + \frac{(\Delta_2)}{f_2} \right] \frac{R'}{2} G \right\} \\
+ \frac{(l-m+3/2)}{2l+3} \left\{ -\frac{1}{f_3} \left[ \frac{dG}{dr} + \left( \frac{l+2}{r} \right) G \right] + \frac{(\Delta_3)}{f_3} \frac{R'}{2} G \right\} + \left( -\frac{E}{f_4} + M \right) F = 0. \tag{A17}
\]
Eqs. (A14)-(A17) are Eqs. (18)-(21). The proof is completed.

We turn to the proof of the equivalence between (13) and (23)-(26). It follows from the recurrence relations in (A1)-(A3) that
\[
\frac{2x}{r} Y_{l+1,m-1/2} = \sqrt{\frac{(l-m+3/2)(l-m+1/2)}{(2l+3)(2l+1)}} Y_{l,m-1/2} - \sqrt{\frac{(l+m+5/2)(l+m+3/2)}{(2l+5)(2l+3)}} Y_{l+2,m-1/2}
\]
\[
- \sqrt{\frac{(l+m+1/2)(l-m+1/2)}{(2l+3)(2l+1)}} Y_{l,m+1/2} + \sqrt{\frac{(l-m+7/2)(l-m+5/2)}{(2l+5)(2l+3)}} Y_{l+2,m+1/2}
\]
\[
\frac{2iy}{r} Y_{l+1,m-1/2} = \sqrt{\frac{(l-m+3/2)(l-m+1/2)}{(2l+3)(2l+1)}} Y_{l,m+1/2} - \sqrt{\frac{(l+m+5/2)(l+m+3/2)}{(2l+5)(2l+3)}} Y_{l+2,m+1/2}
\]
\[
+ \sqrt{\frac{(l+m+1/2)(l-m+1/2)}{(2l+3)(2l+1)}} Y_{l,m-1/2} - \sqrt{\frac{(l-m+7/2)(l-m+5/2)}{(2l+5)(2l+3)}} Y_{l+2,m-1/2}, \tag{A18}
\]
and
\[
\frac{2}{r} Y_{l+1,m+1/2} = \sqrt{\frac{(l+m+5/2)(l-m+3/2)}{(2l+5)(2l+3)}} Y_{l+2,m+1/2} + \sqrt{\frac{(l+m+3/2)(l-m+1/2)}{(2l+3)(2l+1)}} Y_{l,m+1/2}. \tag{A19}
\]

With the relations (A3)-(A6), we have
\[
2 \frac{\partial}{\partial x} \left( G Y_{l+1,m-1/2} \right) = \sqrt{\frac{(l+m+5/2)(l+m+3/2)}{(2l+5)(2l+3)}} Y_{l+2,m+1/2} \left( \frac{dG}{dr} - \left( \frac{l+1}{r} \right) G \right)
\]
\[
- \sqrt{\frac{(l-m+3/2)(l-m+1/2)}{(2l+3)(2l+1)}} Y_{l,m+1/2} \left( \frac{dG}{dr} + \left( \frac{l+2}{r} \right) G \right)
\]
\[
- \sqrt{\frac{(l-m+7/2)(l-m+5/2)}{(2l+5)(2l+3)}} Y_{l+2,m-1/2} \left( \frac{dG}{dr} + \left( \frac{l+1}{r} \right) G \right)
\]
\[
+ \sqrt{\frac{(l+m+1/2)(l-m+1/2)}{(2l+3)(2l+1)}} Y_{l,m-1/2} \left( \frac{dG}{dr} + \left( \frac{l+2}{r} \right) G \right), \tag{A20}
\]
\[
2i \frac{\partial}{\partial y} \left( G Y_{l+1,m-1/2} \right) = \sqrt{\frac{(l+m+5/2)(l+m+3/2)}{(2l+5)(2l+3)}} Y_{l+2,m+1/2} \left( \frac{dG}{dr} - \left( \frac{l+1}{r} \right) G \right)
\]
Substitution of (A18)-(A23) for (13) yields

\[-\sqrt{(l+m+3/2)(l-m+1/2)} Y_{l,m+1/2} \left( \frac{dG}{dr} + \left( \frac{l+2}{r} \right) G \right) + \sqrt{(l+m+7/2)(l-m+5/2)} Y_{l+2,m-3/2} \left( \frac{dG}{dr} - \left( \frac{l+1}{r} \right) G \right) - \sqrt{(l+m+1/2)(l-m-1/2)} Y_{l,m-3/2} \left( \frac{dG}{dr} + \left( \frac{l+2}{r} \right) G \right), \quad (A22)\]

and

\[
\frac{\partial}{\partial z} (GY_{l+1,m+1/2}) = \sqrt{(l+m+5/2)(l-m+3/2)} Y_{l+2,m+1/2} \left( \frac{dG}{dr} - \left( \frac{l+1}{r} \right) G \right) + \sqrt{(l+m+3/2)(l-m+1/2)} Y_{l,m+1/2} \left( \frac{dG}{dr} + \left( \frac{l+2}{r} \right) G \right), \quad (A23)\]

Substitution of [A18]-[A23] for [13] yields

\[-\frac{1}{2(2l+3)} \sqrt{(l+m+5/2)(l+m+3/2)(l-m+3/2)} (\frac{1}{f_1}) Y_{l+2,m+1/2} \left( \frac{dG}{dr} - \left( \frac{l+1}{r} \right) G \right) + \frac{1}{2(2l+3)} \sqrt{(l-m+3/2)^2(l-m+1/2)} (\frac{1}{f_1}) Y_{l,m+1/2} \left( \frac{dG}{dr} + \left( \frac{l+2}{r} \right) G \right) + \frac{1}{2(2l+3)} \sqrt{(l-m+7/2)(l-m+5/2)(l-m+3/2)} (\frac{1}{f_1}) Y_{l+2,m-3/2} \left( \frac{dG}{dr} - \left( \frac{l+1}{r} \right) G \right) - \frac{1}{2(2l+3)} \sqrt{(l+m+1/2)(l-m-1/2)(l-m+3/2)} (\frac{1}{f_1}) Y_{l,m-3/2} \left( \frac{dG}{dr} + \left( \frac{l+2}{r} \right) G \right) + \frac{1}{2(2l+3)} \sqrt{(l-m+1/2)(l-m+3/2)^2} \left( \frac{R'}{2f_1} \right) GY_{l,m+1/2} \Delta_1 - \frac{1}{2(2l+3)} \sqrt{(l+m+5/2)(l+m+3/2)(l-m+3/2)} \left( \frac{R'}{2f_1} \right) GY_{l+2,m+1/2} \Delta_1 - \frac{1}{2(2l+3)} \sqrt{(l+m+1/2)(l+m-1/2)(l-m+3/2)} \left( \frac{R'}{2f_1} \right) GY_{l,m-3/2} \Delta_1 + \frac{1}{2(2l+3)} \sqrt{(l-m+7/2)(l-m+5/2)(l-m+3/2)} \left( \frac{R'}{2f_1} \right) GY_{l+2,m-3/2} \Delta_1 - \frac{1}{2(2l+3)} \sqrt{(l+m+5/2)(l+m+3/2)(l-m+3/2)} \left( \frac{1}{f_2} \right) Y_{l+2,m+1/2} \left( \frac{dG}{dr} - \left( \frac{l+1}{r} \right) G \right) \]

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\[ + \frac{1}{2(2l + 3)} \sqrt{\frac{(l + m + 3/2)^2(l - m + 1/2)}{(2l + 1)}} (\frac{1}{f_2}) Y_{l,m+1/2} \left( \frac{dG}{dr} + \left( \frac{l + 1}{r} \right) G \right) \]

\[- \frac{1}{2(2l + 3)} \sqrt{\frac{(l - m + 5/2)(l - m + 3/2)}{(2l + 1)}} (\frac{1}{f_2}) Y_{l+2,m-3/2} \left( \frac{dG}{dr} - \left( \frac{l + 1}{r} \right) G \right) \]

\[+ \frac{1}{2(2l + 3)} \sqrt{\frac{(l + m + 5/2)(l + m - 1/2)(l - m + 3/2)}{(2l + 1)}} (\frac{1}{f_2}) Y_{l,m-3/2} \left( \frac{dG}{dr} + \left( \frac{l + 2}{r} \right) G \right) \]

\[+ \frac{1}{2(2l + 3)} \sqrt{\frac{(l - m + 7/2)(l - m + 5/2)(l - m + 3/2)}{(2l + 1)}} \left( \frac{R'}{2f_2} \right) G Y_{l,m+1/2} \Delta_2 \]

\[- \frac{1}{2(2l + 3)} \sqrt{\frac{(l + m + 5/2)(l + m + 3/2)(l - m + 3/2)}{(2l + 1)}} \left( \frac{R'}{2f_2} \right) G Y_{l+2,m+1/2} \Delta_2 \]

\[+ \frac{1}{2(2l + 3)} \sqrt{\frac{(l + m + 3/2)(l + m + 3/2)(l - m + 3/2)}{(2l + 1)}} (\frac{1}{f_3}) Y_{l+2,m+1/2} \left( \frac{dG}{dr} - \left( \frac{l + 1}{r} \right) G \right) \]

\[+ \frac{1}{2(2l + 3)} \sqrt{\frac{(l + m + 3/2)(l - m + 1/2)}{(2l + 1)}} (\frac{1}{f_3}) Y_{l,m+1/2} \left( \frac{dG}{dr} + \left( \frac{l + 2}{r} \right) G \right) \]

\[- \frac{1}{2(2l + 3)} \sqrt{\frac{(l + m + 5/2)(l + m + 3/2)(l - m + 3/2)}{(2l + 1)}} \left( \frac{R'}{2f_3} \right) G Y_{l+2,m+1/2} \Delta_3 \]

\[- \frac{1}{2(2l + 3)} \sqrt{\frac{(l + m + 3/2)^2(l - m + 1/2)}{(2l + 1)}} \left( \frac{R'}{2f_3} \right) G Y_{l+2,m+1/2} \Delta_3 \]

\[+ \left( \frac{E}{f_4} - M \right) \sqrt{\frac{(l - m + 1/2)}{(2l + 1)}} F Y_{l,m+1/2} = 0. \] (A24)

Multiplying both sides by \( Y_{l+2,m-3/2}, Y_{l,m-3/2}, Y_{l+2,m+1/2}, \) and \( Y_{l,m+1/2}, \) and performing the integration with respect to the solid angle gives the four independent equalities

\[ \left( \frac{1}{f_1} - \frac{1}{f_2} \right) \left[ \frac{dG}{dr} - \left( \frac{l + 1}{r} \right) G \right] + \left[ \frac{(\Delta_1)}{f_1} - \frac{(\Delta_2)}{f_2} \right] \frac{R'}{2} G = 0, \] (A25)

\[ \left( - \frac{1}{f_1} + \frac{1}{f_2} \right) \left[ \frac{dG}{dr} + \left( \frac{l + 2}{r} \right) G \right] + \left[ - \frac{(\Delta_1)}{f_1} + \frac{(\Delta_2)}{f_2} \right] \frac{R'}{2} G = 0. \] (A26)
\[
\left(\frac{1}{2f_1} + \frac{1}{2f_2} - \frac{1}{f_3}\right) \left[ \frac{dG}{dr} - \left(\frac{l+1}{r}\right)G \right] + \left[\frac{(\Delta_1)}{2f_1} + \frac{(\Delta_2)}{2f_2} + \frac{(\Delta_3)}{f_3}\right] \frac{R'}{2} G = 0, \quad (A27)
\]

and
\[
\left(\frac{l-m+3/2}{2(2l+3)}\right) \left\{ \left(\frac{1}{f_1} + \frac{1}{f_2}\right) \left[ \frac{dG}{dr} + \left(\frac{l+2}{r}\right)G \right] + \left[\frac{(\Delta_1)}{f_1} + \frac{(\Delta_2)}{f_2}\right] \frac{R'}{2} G \right\} + \left(\frac{l+m+3/2}{2l+3}\right) \left\{ \frac{1}{f_3} \left[ \frac{dG}{dr} + \left(\frac{l+2}{r}\right)G \right] - \frac{(\Delta_3)}{f_3} \frac{R'}{2} G \right\} + \left(\frac{E}{f_4} - M\right) F = 0. \quad (A28)
\]

These are the equations given in (23)-(26). This completes the proof of the equivalence.

**Appendix B: Proof of Eqs. (31) - (34) and (35) - (38)**

We proceed to prove the equivalence between (12) and (31)-(34) for the second invariance solution (17). Using the relations in (A1)-(A6), (12) can be expressed by

\[
\frac{1}{2(2l-1)} \sqrt{\frac{(l+m+3/2)(l+m+1/2)(l-m-1/2)}{(2l+1)}} \left(\frac{1}{f_1}\right) Y_{l,m+3/2} \left(\frac{dG}{dr} - \left(\frac{l-1}{r}\right)G\right)
\]

\[
- \frac{1}{2(2l-1)} \sqrt{\frac{(l-m-1/2)(l-m-3/2)(l-m-5/2)}{(2l-3)}} \left(\frac{1}{f_1}\right) Y_{l-2,m+3/2} \left(\frac{dG}{dr} + \left(\frac{l}{r}\right)G\right)
\]

\[
- \frac{1}{2(2l-1)} \sqrt{\frac{(l-m+1/2)(l-m-1/2)^2}{(2l+1)}} \left(\frac{1}{f_1}\right) Y_{l,m-1/2} \left(\frac{dG}{dr} - \left(\frac{l-1}{r}\right)G\right)
\]

\[
+ \frac{1}{2(2l-1)} \sqrt{\frac{(l+m-3/2)(l+m-1/2)(l-m-1/2)}{(2l-3)}} \left(\frac{1}{f_1}\right) Y_{l-2,m-1/2} \left(\frac{dG}{dr} + \left(\frac{l}{r}\right)G\right)
\]

\[
- \frac{1}{2(2l-1)} \sqrt{\frac{(l-m-1/2)(l-m-3/2)(l-m-5/2)}{(2l-3)}} \left(\frac{R'}{2f_1}\right) GY_{l-2,m+3/2} \Delta_1
\]

\[
+ \frac{1}{2(2l-1)} \sqrt{\frac{(l+m+3/2)(l+m+1/2)(l-m-1/2)}{(2l+1)}} \left(\frac{R'}{2f_1}\right) GY_{l,m+3/2} \Delta_1
\]

\[
+ \frac{1}{2(2l-1)} \sqrt{\frac{(l+m-1/2)(l+m-3/2)(l-m-1/2)}{(2l-3)}} \left(\frac{R'}{2f_1}\right) GY_{l-2,m-1/2} \Delta_1
\]

\[
- \frac{1}{2(2l-1)} \sqrt{\frac{(l-m+1/2)(l-m-1/2)^2}{(2l+1)}} \left(\frac{R'}{2f_1}\right) GY_{l,m-1/2} \Delta_1
\]

\[
- \frac{1}{2(2l-1)} \sqrt{\frac{(l+m+3/2)(l+m+1/2)(l-m-1/2)}{(2l+1)}} \left(\frac{1}{f_2}\right) Y_{l,m+3/2} \left(\frac{dG}{dr} - \left(\frac{l-1}{r}\right)G\right)
\]

\[
+ \frac{1}{2(2l-1)} \sqrt{\frac{(l-m-1/2)(l-m-3/2)(l-m-5/2)}{(2l-3)}} \left(\frac{1}{f_2}\right) Y_{l-2,m+3/2} \left(\frac{dG}{dr} + \left(\frac{l}{r}\right)G\right)
\]
\[-\frac{1}{2(2l-1)} \sqrt{\frac{(l-m+1/2)(l-m-1/2)^2}{(2l+1)}} \left(\frac{1}{f_2}\right) Y_{l,m-1/2} \left(\frac{dG}{dr} - \left(\frac{l-1}{r}\right) G\right)\]
\[+ \frac{1}{2(2l-1)} \sqrt{\frac{(l-m+3/2)(l+m-1/2)(l-m-1/2)}{(2l-3)}} \left(\frac{1}{f_2}\right) Y_{l-2,m-1/2} \left(\frac{dG}{dr} + \left(\frac{l}{r}\right) G\right)\]
\[+ \frac{1}{2(2l-1)} \sqrt{\frac{(l-m-1/2)(l-m-3/2)(l-m-5/2)}{(2l-3)}} \left(\frac{R'}{2f_2}\right) G Y_{l-2,m+3/2}\Delta_2\]
\[+ \frac{1}{2(2l-1)} \sqrt{\frac{(l+m+3/2)(l+m+1/2)(l-m-1/2)}{(2l+1)}} \left(\frac{R'}{2f_2}\right) G Y_{l,m+3/2}\Delta_2\]
\[+ \frac{1}{2(2l-1)} \sqrt{\frac{(l+m-1/2)(l+m-3/2)(l-m-1/2)}{(2l-3)}} \left(\frac{R'}{2f_2}\right) G Y_{l-2,m-1/2}\Delta_2\]
\[-\frac{1}{2(2l-1)} \sqrt{\frac{(l-m+1/2)(l-m-1/2)^2}{(2l+1)}} \left(\frac{1}{f_3}\right) Y_{l,m-1/2} \left(\frac{dG}{dr} - \left(\frac{l-1}{r}\right) G\right)\]
\[+ \frac{1}{2(2l-1)} \sqrt{\frac{(l+m-3/2)(l+m-1/2)(l-m-1/2)}{(2l-3)}} \left(\frac{1}{f_3}\right) Y_{l-2,m-1/2} \left(\frac{dG}{dr} + \left(\frac{l}{r}\right) G\right)\]
\[+ \frac{1}{2(2l-1)} \sqrt{\frac{(l+m-1/2)^2(l-m+1/2)}{(2l+1)}} \left(\frac{R'}{2f_3}\right) G Y_{l,m-1/2}\Delta_3\]
\[+ \frac{1}{2(2l-1)} \sqrt{\frac{(l+m-1/2)(l+m-3/2)(l-m-1/2)}{(2l-3)}} \left(\frac{R'}{2f_3}\right) G Y_{l-2,m-1/2}\Delta_3\]
\[-\left(\frac{E}{f_4} - M\right) \sqrt{\frac{(l-m+1/2)}{(2l+1)}} F Y_{l,m-1/2} = 0.\] (B1)

Multiplying both sides by $Y_{l,m+3/2}$, $Y_{l-2,m+3/2}$, $Y_{l-2,m-1/2}$, and $Y_{l,m-1/2}$, and integrating the equation with respect to the solid angle gives

\[\left(\frac{1}{f_1} - \frac{1}{f_2}\right) \left[\frac{dG}{dr} - \left(\frac{l-1}{r}\right) G\right] + \left[\frac{(\Delta_1)}{f_1} - \frac{(\Delta_2)}{f_2}\right] \frac{R'}{2} G = 0,\] (B2)

\[\left(-\frac{1}{f_1} + \frac{1}{f_2}\right) \left[\frac{dG}{dr} + \left(\frac{l}{r}\right) G\right] + \left[\frac{(\Delta_1)}{f_1} + \frac{(\Delta_2)}{f_2}\right] \frac{R'}{2} G = 0,\] (B3)

\[\left(\frac{1}{2f_1} + \frac{1}{2f_2} - \frac{1}{f_3}\right) \left[\frac{dG}{dr} + \left(\frac{l}{r}\right) G\right] + \left[\frac{(\Delta_1)}{2f_1} + \frac{(\Delta_2)}{2f_2} + \frac{(\Delta_3)}{f_3}\right] \frac{R'}{2} G = 0,\] (B4)

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and

\[- \left( \frac{l - m - 1/2}{2(2l - 1)} \right) \left\{ \left( \frac{1}{f_1} + \frac{1}{f_2} \right) \left[ \frac{dG}{dr} - \left( \frac{l - 1}{r} \right) G \right] + \left[ \frac{(\Delta_1)}{f_1} + \frac{(\Delta_2)}{f_2} \right] \frac{R'}{2} G \right\} \]

\[- \left( \frac{l + m - 1/2}{2l - 1} \right) \left\{ \frac{1}{f_3} \left[ \frac{dG}{dr} - \left( \frac{l - 1}{r} \right) G \right] - \left( \frac{(\Delta_3)}{f_3} \right) \frac{R'}{2} G \right\} \]

- \left( \frac{E}{f_4} + M \right) F = 0. \quad (B5)

These are equations (31)-(34). The proof is completed.

The proof of the equivalence between (13) and (35)-(38) is analogous. Using (A1)-(A6), Eq. (13) has the representation

\[- \frac{1}{2(2l - 1)} \sqrt{(l + m + 1/2)(l + m - 1/2)} \left( \frac{1}{f_1} \right) Y_{l,m+1/2} \left( \frac{dG}{dr} - \left( \frac{l - 1}{r} \right) G \right) \]

\[+ \frac{1}{2(2l - 1)} \sqrt{(l + m - 1/2)(l - m - 1/2)(l - m - 3/2)} \left( \frac{1}{f_1} \right) Y_{l-2,m+1/2} \left( \frac{dG}{dr} + \left( \frac{l}{r} \right) G \right) \]

\[+ \frac{1}{2(2l - 1)} \sqrt{(l + m - 1/2)(l - m + 1/2)} \left( \frac{1}{f_1} \right) Y_{l,m-3/2} \left( \frac{dG}{dr} - \left( \frac{l - 1}{r} \right) G \right) \]

\[- \frac{1}{2(2l - 1)} \sqrt{(l + m - 1/2)(l + m - 3/2)} \left( \frac{1}{f_1} \right) Y_{l-2,m-3/2} \left( \frac{dG}{dr} + \left( \frac{l}{r} \right) G \right) \]

\[+ \frac{1}{2(2l - 1)} \sqrt{(l + m - 1/2)(l - m - 1/2)(l - m - 3/2)} \left( \frac{R'}{2 f_1} \right) G Y_{l-2,m+1/2} \Delta_1 \]

\[- \frac{1}{2(2l - 1)} \sqrt{(l + m + 1/2)(l + m - 1/2)} \left( \frac{R'}{2 f_1} \right) G Y_{l,m+1/2} \Delta_1 \]

\[- \frac{1}{2(2l - 1)} \sqrt{(l + m - 1/2)(l + m - 3/2)(l + m - 5/2)} \left( \frac{R'}{2 f_1} \right) G Y_{l-2,m-3/2} \Delta_1 \]

\[+ \frac{1}{2(2l - 1)} \sqrt{(l + m - 1/2)(l - m + 3/2)(l - m + 1/2)} \left( \frac{R'}{2 f_1} \right) G Y_{l,m-3/2} \Delta_1 \]

\[+ \frac{1}{2(2l - 1)} \sqrt{(l + m + 1/2)(l + m - 1/2)} \left( \frac{1}{f_2} \right) Y_{l,m+1/2} \left( \frac{dG}{dr} - \left( \frac{l - 1}{r} \right) G \right) \]

\[+ \frac{1}{2(2l - 1)} \sqrt{(l + m - 1/2)(l - m - 1/2)(l - m - 3/2)} \left( \frac{1}{f_2} \right) Y_{l-2,m+1/2} \left( \frac{dG}{dr} + \left( \frac{l}{r} \right) G \right) \]

\[- \frac{1}{2(2l - 1)} \sqrt{(l + m - 1/2)(l - m + 3/2)(l - m + 1/2)} \left( \frac{1}{f_2} \right) Y_{l,m-3/2} \left( \frac{dG}{dr} - \left( \frac{l - 1}{r} \right) G \right) \]
Multiplying both sides by \( Y \) sides of the equation with respect to the solid angle gives

\[
\frac{1}{2(2l-1)} \sqrt{(l+m-1/2)(l+m-3/2)(l+m-5/2)} \left( \frac{1}{f_2} \right) Y_{1,m-3/2} \left( \frac{dG}{dr} + \left( \frac{1}{r} \right) G \right)
+ \frac{1}{2(2l-1)} \sqrt{(l+m-1/2)(l-m-1/2)(l-m-3/2)} \left( \frac{R'}{2f_2} \right) G Y_{1,m-1/2} \Delta_2
\]

\[
- \frac{1}{2(2l-1)} \sqrt{(l+m+1/2)(l+m-1/2)^2} \left( \frac{1}{f_2} \right) Y_{1,m+1/2} \left( \frac{dG}{dr} - \left( \frac{l-1}{r} \right) G \right)
\]

\[
- \frac{1}{2(2l-1)} \sqrt{(l+m-1/2)(l-m-1/2)(l-m-3/2)} \left( \frac{1}{f_2} \right) Y_{1,m+1/2} \left( \frac{dG}{dr} + \left( \frac{l}{r} \right) G \right)
\]

\[
+ \frac{1}{(2l-1)} \sqrt{(l+m+1/2)(l-m-1/2)^2} \left( \frac{R'}{2f_2} \right) G Y_{1,m+1/2} \Delta_2
\]

\[
+ \frac{1}{(2l-1)} \sqrt{(l+m-1/2)(l-m-1/2)(l-m-3/2)} \left( \frac{R'}{2f_2} \right) G Y_{1,m+1/2} \Delta_2
\]

\[
- \left( \frac{E}{f_4} - M \right) \sqrt{(l+m+1/2)} \left( \frac{dG}{dr} + \left( \frac{l}{r} \right) G \right) = 0. \quad (B6)
\]

Multiplying both sides by \( Y_{1,m-3/2}, Y_{1,m-1/2}, Y_{1,m+1/2}, \) and \( Y_{1,m+1/2}, \) and integrating both sides of the equation with respect to the solid angle gives

\[
\left( \frac{1}{f_1} - \frac{1}{f_2} \right) \left[ \frac{dG}{dr} - \left( \frac{l-1}{r} \right) G \right] + \left[ \frac{(\Delta_1)}{f_1} - \frac{(\Delta_2)}{f_2} \right] R' G = 0, \quad (B7)
\]

\[
\left( \frac{1}{f_1} + \frac{1}{f_2} \right) \left[ \frac{dG}{dr} + \left( \frac{l}{r} \right) G \right] + \left[ \frac{(\Delta_1)}{f_1} + \frac{(\Delta_2)}{f_2} \right] R' G = 0, \quad (B8)
\]

\[
\left( \frac{1}{2f_1} - \frac{1}{f_3} \right) \left[ \frac{dG}{dr} + \left( \frac{l}{r} \right) G \right] + \left[ \frac{(\Delta_1)}{2f_1} - \frac{(\Delta_2)}{2f_2} + \frac{(\Delta_3)}{f_3} \right] R' G = 0, \quad (B9)
\]
and
\[- \left( \begin{array}{c}
\frac{l + m - 1/2}{2(2l - 1)} \\
\frac{(l - m - 1/2)}{2l - 1}
\end{array} \right) \right\} \left\{ \left( \begin{array}{c}
\frac{1}{f_1} + \frac{1}{f_2} \\
\frac{1}{f_3}
\end{array} \right) \left[ \frac{dG}{dr} - \left( \frac{l - 1}{r} \right) G \right] + \left[ \frac{(\Delta_1)}{f_1} + \frac{(\Delta_2)}{f_2} \right] \frac{R'}{2} G \right\} \]
\[- \left( \begin{array}{c}
\frac{1}{f_3} \left[ \frac{dG}{dr} - \left( \frac{l - 1}{r} \right) G \right] - \frac{(\Delta_3)}{f_3} \frac{R'}{2} G \right] - \left( \frac{E}{f_4} - M \right) F = 0. \quad (B10)
\]
These are Eqs. (35)-(38). Hence the equivalence is proved.

ACKNOWLEDGMENTS

The author would like to thank C. R. Harrington for her careful reading of the manuscript. My work has been supported by the National Science Council of Taiwan under contract No. NSC 102-2112-M-110-012-MY3.

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FIG. 1: (Color online) Energy levels of trapped fermion states in the spacetime cage. Left: The figure shows the energy levels of the $s$ states as a function of the potential strength which is determined by the proper time, see (85), where $E_0 = M c^2$, the rest energy of a trapped spin-1/2 fermion. The radius of the cage is $r_b = 10\lambda_F$, where $\lambda_F = \hbar/Mc$ is the Compton wavelength of the fermions. The deeper the cage, the more bound levels it will allow. Right: The pattern shows the energy spectrum of the cage with radius $r_b = 100\lambda_F$. 
FIG. 2: (Color online) A bound state accompanying an invisible region. Since the radial amplitude $F$ is spherically symmetric, it suffices to show its distribution at the $x$-$y$ plane. Top left: The component $F(r)$ for $1s_{1/2}$ state in a regular spherical well, where $x_0 = y_0 = \lambda_F$. Top right: The presented cage creates a zero distribution region of the component around the center of the bound state, where the radius of the cage is $10\lambda_F$, and the radius of the zero amplitude region is chosen as $2\lambda_F$. To plot the patterns, the depth of the cage and the bound energy are chosen as $V_0/E_0 = 0.1043$, and $E/E_0 = 0.992$ which are extracted from the numerical data in Fig. 1. The amplitude is not normalized to clearly exhibit the solid profiles. The bottom pattern shows the projection of the top right on the $x$-$y$ plane.
FIG. 3: (Color online) The phase shift of $s$ states for the fermionic cage. The zeros of $\delta$ correspond to the resonances, where one has a larger probability of finding the fermions in the cage.