GLOBAL EXISTENCE AND UNIQUENESS
FOR A VOLUME-SURFACE
REACTION-NONLINEAR-DIFFUSION SYSTEM

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Dedicated to Alexander Mielke on the occasion of his 60th birthday

Abstract. We prove a global existence, uniqueness and regularity result for a two-species reaction-diffusion volume-surface system that includes nonlinear bulk diffusion and nonlinear (weak) cross diffusion on the active surface. A key feature is a proof of upper $L^\infty$-bounds that exploits the entropic gradient structure of the system.

1. Introduction. There has been a lot of recent interest in the mathematical analysis of reaction-diffusion processes that are coupled across volume and surface domains, due to their relevance in particular in the modelling of biological cells and technological processes, e.g. involving catalysis by surfactants [2, 7, 8, 10]. For systems coupled across volume and surface, a particular modelling issue and mathematical difficulty is the nonlinearity of diffusion that is typical of many biological processes (due to porous media, non-Newtonian flow, ...) and naturally associated to diffusion that is modelled on multiple (spatial) scales ([1, 11, 13, 14, 15]). The aim of this paper is to prove a global well-posedness result for reaction-diffusion volume-surface systems that include typical general classes of nonlinear diffusion (like slow and fast diffusion) and nonlinear (weak) cross diffusion on the active surface. This generalizes some of the recent results on volume-surface reaction-diffusion [2, 7, 8] to the nonlinear diffusion case and to the case that only part of the boundary of the bulk domain is active.

Besides the well-posedness result based on the analysis for quasilinear volume-surface systems in [5], the aim of the paper is to introduce a proof of $L^\infty$-estimates that is different from the classical proof of comparison principles for the linearized part [7, 8]. Instead, we explicitly exploit the entropic gradient structure of the system [11] to show that the (chemical) logarithmic potentials remain pointwise bounded. This translates to pointwise estimates on the concentrations. A feature of this estimate is that it does not use the positivity of the (quasi)linear diffusion. The method was inspired by the boundedness-by-entropy method developed by Jüngel [12], but the argument here is more direct and thus does not promise as much generality. So it is unclear whether a strategy of this type can also be used.

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to obtain results on more complex cases of multicomponent diffusion or reaction. This is the aim of future work.

1.1. Model equations and assumptions. We consider the reaction-cross-diffusion system

\[ \dot{u} - \text{div}(\mu(u)\nabla u) = 0, \quad \text{in } (0, \infty) \times \Omega, \]

\[ \mu(u) \frac{\partial u}{\partial \nu} = 0, \quad \text{on } (0, \infty) \times \partial \Omega \setminus \Gamma, \]

\[ \mu(u) \frac{\partial u}{\partial \nu} + k\alpha(u^\alpha - \kappa v^\beta) = 0, \quad \text{on } (0, \infty) \times \Gamma, \]

\[ \dot{v} - \text{div}_\Gamma(\mu_\Gamma(u, v)\nabla_\Gamma v) - k\beta(u^\alpha - \kappa v^\beta) = 0, \quad \text{on } (0, \infty) \times \Gamma, \]

with initial conditions

\[ u(0, x) = u_0(x) > 0, x \in \Omega \]

\[ v(0, y) = v_0(y) > 0, y \in \Gamma, \]

with the following assumptions on the geometry and coefficients:

**Assumption 1.** \( \Omega \subset \mathbb{R}^d, d = 2, 3 \) is a bounded strong Lipschitz domain, and \( \Gamma \subset \partial \Omega \) is an open connected \( C^1 \)-part of its boundary. If \( d = 3 \), \( \Gamma \) has Lipschitz boundary \( \partial \Gamma \).

The unit outer normal vector field at \( \partial \Omega \) is called \( \nu \) and if \( d = 3 \), \( \nu_\Gamma \) is the unit outer normal vector field at \( \partial \Gamma \) (for a (canonical) definition and discussion of the operators \( \text{div}_\Gamma \) and \( \nabla_\Gamma \) in this non-smooth case, see e.g. [6]).

**Assumption 2.** The diffusion coefficients \( \mu(\cdot, u), \mu_\Gamma(\cdot, u, v) \) satisfy:

1. Locally uniformly in \( (u, v) \in \mathbb{R}^+ \times \mathbb{R}^+ \), \( \mu(\cdot, u), \mu_\Gamma(\cdot, u, v) \) are measurable, bounded and elliptic,
2. uniformly in \( x \in \Omega, y \in \Gamma, u \mapsto \mu(x, u) \) and \( (u, v) \mapsto \mu_\Gamma(y, u, v) \) are locally Lipschitz,
3. if \( d = 3 \), additionally, \( \mu \) is of the form \( \mu(x, u) = \mu_0(u)m(x) \), where \( \mu_0 : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is a scalar function that is locally Lipschitz and \( m : \Omega \rightarrow \mathbb{R}^{3 \times 3}_{\text{sym}} \) is continuous.

Examples of admissible diffusion coefficients are slow and fast diffusion like

\[ \mu(x, u) = u^\delta \text{Id}_d, \quad \mu(x, u) = e^{\delta u} \text{Id}_d, \quad \delta \in \mathbb{R}. \]

Correspondingly, these and expressions like

\[ \mu_\Gamma(u, v) = \frac{v}{u + \delta v} \text{Id}_{d-1}, \]

are admissible for the diffusion coefficient on the surface. With natural boundary conditions, starting from positively bounded initial values, no degeneracy will develop for this system. The reversible chemical reaction on the surface is described by mass-action kinetics with reaction rate constants \( k, \kappa > 0 \) and stoichiometric coefficients \( \alpha, \beta \geq 1 \).

1.2. Outline. Section 2 on preliminaries is mostly technical, in that it is used to introduce the notation and arguments needed to fit system (1) in the framework of the main result in [5] on existence and uniqueness for quasilinear dissipative bulk-interface dynamics. It comes before the main result as it is needed for the detailed/precise statement of the main result in Theorem 3.1 in Section 3.
\(L^\infty\)-estimate derived from the entropic gradient structure of system (1) is shown in Section 4.

2. Preliminary results and notation. Due to the quasilinear structure of system (1), global existence and uniqueness do not follow from standard linearization techniques, if only \(L^\infty\) bounds are available globally a priori. To see that system (1) fits with the theory in [5], for convenience and in order to give a precise statement of the main result, we introduce some of the functional analytic framework used for volume-surface systems in [5].

2.1. Notation and function spaces. For \(q, q_\Gamma \in [1, \infty] \) and \(0 \leq \delta, \delta_\Gamma < 1\), define

\[
L^{q, q_\Gamma} := L^q(\Omega) \times L^{q_\Gamma}(\Gamma),
\]

\[
W^{1, q, q_\Gamma} := W^{1, q}(\Omega) \times W^{1, q_\Gamma}(\Gamma),
\]

\[
C^{\alpha, q, q_\Gamma} := C^\alpha(\Omega) \times C^{\alpha q_\Gamma}(\Gamma),
\]

with components \(L^{q, q_\Gamma} \ni \varphi = (\varphi_u, \varphi_v)\), where \(L^q\), \(W^{1, q}\), \(C^\alpha\) denote the usual Lebesgue, Sobolev and Hölder spaces. In the following, \(q'\) is the dual exponent of \(q\) with \(1 = \frac{1}{q} + \frac{1}{q'}\) and we denote dual Sobolev spaces by

\[
W^{-1, q}(\Omega) = \left(W^{1, q'}(\Omega)\right)', \quad W^{-1, q, q_\Gamma}(\Gamma) = \left(W^{1, q_\Gamma}(\Gamma)\right)'
\]

and

\[
W^{-1, q, q_\Gamma} := \left(W^{1, q', q_\Gamma}\right)'.
\]

Let \((u, v) \in C^{0,0}_\Gamma\) with \(u, v > 0\). Then using the assumptions on \(\mu, \mu_\Gamma\), the bilinear form \(I_{u,v} : W^{1,2,2}_\Gamma \times W^{1,2,2}_\Gamma \to \mathbb{R}\) given by

\[
I_{u,v}(\psi, \varphi) := \int_\Omega \nabla \psi_u \cdot \mu(u) \nabla \varphi_u \, dx + \int_{\Gamma} \nabla \psi \cdot \mu(u, v) \nabla \varphi \, dy
\]

is well-defined, continuous and coercive (after shifting). It induces the two-component Neumann Laplacian \(\mathcal{L}_{u,v} : W^{1,2,2}_\Gamma \to W^{-1,2,2}_\Gamma\) with \(\mathcal{L}_{u,v}(\psi)(\varphi) := I_{u,v}(\psi, \varphi)\). For \(q, q_\Gamma \in [2, \infty]\), let \(L^{q, q_\Gamma}_{u,v}\) be the closed and densely defined restriction of \(\mathcal{L}_{u,v}\) to \(W^{-1, q, q_\Gamma}\).

2.2. Interpretation of (1). In order to re-write system (1) as a quasilinear abstract Cauchy problem in \(W^{-1, q, q_\Gamma}\), consider the reaction terms as components of a functional \(\mathcal{F}(u, v) \in W^{-1, q, q_\Gamma}\) through

\[
\mathcal{F}(u, v)(\varphi) = \begin{pmatrix}
\mathcal{F}_u(u, v)(\varphi) \\
\mathcal{F}_v(u, v)(\varphi)
\end{pmatrix} = \begin{pmatrix}
-\alpha \int_{\Gamma} k(u^\alpha - \kappa v^\beta) \varphi_u \, dy \\
+\beta \int_{\Gamma} k(u^\alpha - \kappa v^\beta) \varphi_v \, dy
\end{pmatrix},
\]

for all \(\varphi \in W^{1, q', q_\Gamma}\). With the existence of the traces \(u|_\Gamma \in C^{0}(\Gamma)\) and \(\varphi_u|_\Gamma \in L^{q_\Gamma}(\Gamma)\), \(\mathcal{F}(u, v)\) is well-defined for all \(t > 0\). Note that this is just the usual way of casting inhomogeneous Neumann boundary conditions in a weak form. Hence, system (1) can be written as the abstract Cauchy problem

\[
\begin{pmatrix}
u \\
v
\end{pmatrix}(t) + L^{q, q_\Gamma}_{u(t), v(t)}(u(t), v(t)) = \mathcal{F}(u(t), v(t)) \quad (2)
\]

in \(W^{-1, q, q_\Gamma}\) with initial data \((u, v)(0) = (u_0, v_0)\). Due to the uniqueness and regularity of solutions (Theorem 3.1), solutions of (2) solve system (1) in an adequate sense.
2.3. Regularity results. To solve (2) on any time interval \( J_T = (0, T), T > 0 \), we use that for \( r \in (1, \infty) \), \( \mathcal{L}_{u,v}^{q,q_r} \) has maximal parabolic regularity in \( L^r(J_T; W^{-1,q,q_r}) \) and that there are \( q > d, q_r > 2 \) such that \( \text{dom}(\mathcal{L}_{u,v}^{q,q_r} + \lambda) = W^{1,q,q_r} \) for \( \lambda > 0 \) (for details and proofs, see [5][Sect. 2.3]).

We define
\[

\text{MR}_{q,q_r}^r := L^r(J_T; W^{1,q,q_r}) \cap W^{1,r}(J_T; W^{-1,q,q_r})
\]
as the space of solutions corresponding to the setting of system (2) and
\[

X_{q,q_r}^r := (W^{1,q,q_r}, W^{-1,q,q_r})_{1-\frac{1}{r}},
\]
as the corresponding time trace space, satisfying
\[

\text{MR}_{q,q_r}^r \hookrightarrow C^0(J_T; X_{q,q_r}^r).
\]
The following embedding result is essential [5][Lemma 2.4(2)]: If \( q > d, q_r > d - 1 \) and \( r > \max(\frac{2q}{q-d}, \frac{2q}{q_r-d+1}) \), then
\[

X_{q,q_r}^r \hookrightarrow C^{\gamma,q_r} \hookrightarrow C^{0,0},
\]
for \( 0 < \gamma \leq 1 - \frac{d}{q} = \frac{2}{q} \) and \( 0 < \gamma_r \leq 1 - \frac{d-1}{q} \). In particular, this implies that for any \((u,v) \in \text{MR}_{q,q_r}^r\), at any \( t > 0 \), the linearized operator \( \mathcal{L}_{u(t),v(t)}^{q,q_r} \) is well-defined.

2.4. Equilibria. Note that the system (1) preserves the total mass
\[

m = \beta \int \Omega u(t,x) \, dx + \alpha \int \Gamma v(t,y) \, dy.
\]
Given initial values \((u_0,v_0) \geq 0\) with mass
\[

m = \beta \int \Omega u_0(x) \, dx + \alpha \int \Gamma v_0(y) \, dy,
\]
the conservation of mass,
\[

m = \beta |\Omega| u_* + \alpha |\Gamma| v_*,
\]
and the zero reaction rate
\[

0 = u_*^\alpha - \kappa v_*^\beta,
\]
determine uniquely the equilibrium \((u_*,v_*) \in \mathbb{R}^2_+\) corresponding to \((u_0,v_0)\) (to see this here, note that the function \( v \mapsto \beta |\Omega| u_*^{1/\alpha} v^{\beta/\alpha} + \alpha |\Gamma| v_* \) is strictly monotone and thus invertible from \( \mathbb{R}_+ \) to \( \mathbb{R}_+ \)).

3. Main result.

**Theorem 3.1.** Under Assumptions 1 and 2, there exist \( q > d, q_r > 2 \) such that for all \( r > \max(\frac{2q}{q-d}, \frac{2q_r}{q_r-d+1}) \), and for all positive \((u_0,v_0) \in X_{q,q_r}^r\), there exists a unique global positive solution \((u,v) \in \text{MR}_{q,q_r}^r\) of system (2).

The proof is based on the main result in [5] that provides global existence and uniqueness in the functional analytic framework of system (2) if the right-hand-side \( \mathcal{F} \) is Lipschitz and preserves \( L^\infty\)-bounds (a posteriori). Regarding regularity of solutions, note that
\[

\text{MR}_{q,q_r}^r \hookrightarrow C^\nu(J_T; C^{\gamma,q_r})
\]
for suitable \( \nu, \gamma, \gamma_r > 0 \) [5][Lemma 2.4(3)] and note that \( W^{1,q,q_r} \hookrightarrow X_{q,q_r}^r \) by definition (as an upper limit on the regularity required of the initial data – with parabolic smoothing, less than \( X_{q,q_r}^r \) can be shown to be sufficient).
Proof. To be precise, we first note how the assumptions of Theorem 3.1 in [5] can be satisfied (there may be several choices): in [5], set \( m_r = m_+ = m_- = 0 \), \( \Omega_+ = \Omega \), \( \Omega_- = \emptyset \), \( u_+ = u \), \( u_0 = v \), \( f_+ = h_+ = h \), \( f_- = g_+ = 0 \), \( g_+ = \rho_0 = 0 \), \( h = k_1^\beta(u^\alpha - kv^\beta) \), \( f_0(u, v) = k_2^\beta(u^\alpha - kv^\beta) \), and \( f_1(u, v) = \mu(u) \), and \( h_1(u, v) = \mu_1(u, v) \). Then the assumptions in Section 2.1 in [5] are satisfied and it remains to check Assumption 2.5 there (local Lipschitzianity of \( F \)): for \((u_1, v_1), (u_2, v_2) \in X^\gamma,q_t^r \) with \(||(u_1, v_1)||_{X^\gamma,q_t^r}, ||(u_2, v_2)||_{X^\gamma,q_t^r} \leq \tilde{L} \), we have

\[ ||(u_1, v_1)||_{L^\infty}, ||(u_2, v_2)||_{L^\infty} \leq L, \]

by embedding (3), so

\[ ||F(u_1, v_1) - F(u_2, v_2)||_{W^{-1,q,x}} \leq C||u_1^\alpha - u_2^\alpha + \kappa(v_2^\beta - v_1^\beta)||_{L^\infty(\Gamma)} \leq C(L_{\infty})||(u_1 - u_2), (v_1 - v_2)||_{L^\infty} \leq C(L)||u_1 - u_2), (v_1 - v_2)||_{X^\gamma,q_t^r}, \]

The global result Theorem 3.1 follows from Theorem 3.1 in [5] as in Corollary 3.6 there, using Lemma 4.1 below on \( L^\infty \)-bounds. In fact, a Schaefer fixed point argument is used for the proof of existence of solutions. It requires uniform \( L^\infty \)-bounds for solutions of system (2) with right-hand-side \( \lambda F(u, v) \), \( 0 < \lambda \leq 1 \) and initial value \( \lambda(u_0, v_0) \). Clearly, this also follows from the proof of Lemma 4.1. \( \square \)

Remark 1. Theorem 3.1 also holds for spatially dependent reaction rates \( 0 < k \in L^\infty(\Gamma) \). The functional \( F \) still satisfies the assumptions of Theorem 3.1 in [5] in this case and the corresponding equilibria \((u_*, v_*)\) are unchanged. The method used in the proof of Lemma 4.1 still applies.

4. \( L^\infty \)-estimates based on the entropic gradient structure. The main result of this section is the following Lemma. It is sufficient for proving Theorem 3.1, but also the aim is to introduce a method that derives pointwise upper bounds from the entropy-producing structure of the reaction, without using the diffusivity.

Lemma 4.1. Under Assumptions 1 and 2, let \( r, q, q_t^r \) be as in Theorem 3.1 and let \( 0 < (u_0, v_0) \in X^\gamma,q_t^r \). Then the corresponding equilibrium \((u_*, v_*)\) is positive and there are constants \( c_\gamma, c_\delta, C_\gamma, C_\delta > 0 \) such that

\[ 0 < c_\gamma \leq \frac{u_0}{u_*} \leq C_\gamma < +\infty, \quad 0 < c_\delta \leq \frac{v_0}{v_*} \leq C_\delta < +\infty \]

for all \( x \in \Omega, y \in \Gamma \). Assume that \((u, v) \in MR^\gamma_{q_t^r} \) is a solution of (2) with initial data \((u_0, v_0) \). Define \( l := \min(c_\gamma^\alpha, \kappa c_\delta^\beta) \) and \( L := \max(C_\gamma^\alpha, C_\delta^\beta) \), then

\[ \left( \frac{u(t,x)}{u_*} \right)^\alpha, \left( \frac{v(t,y)}{v_*} \right)^\beta \leq L, \quad \text{and} \quad l \leq u^\alpha(t,x), kv^\beta(t,x), \]

for all \( t \geq 0, x \in \Omega \) and \( y \in \Gamma \).

Proof. The aim is to show that solutions of system (2) satisfy the uniform positive bounds in (6). We first modify (2) to a system (2) in both reaction and diffusion in a way that changes the equation only if \((u, v) \) would violate (6). Then we show the bounds in (6) for solutions of (2) and derive global existence and uniqueness as in Theorem 3.1. But then the solution \((u, v) \) of (2) is positive and we derive that it is also the unique global solution of (2).
Modification of (2): First note that since \( m > 0 \), \((u_* , v_*)\) is positive, and the embedding (3) implies that \((u_0 , v_0)\) are positively bounded from above and below. Hence, in (2), the diffusion coefficients \( \mu, \mu_l \) can be replaced by \( \mu^L_l(u) := \mu(u^L_l), \quad \mu^L_{l,l}(u,v) := \mu^L_l(u^L_l , v^L_l) \), where

\[
\hat{u}^L_l := \begin{cases} 
  u, & \frac{l}{2} \leq \left( \frac{u}{u_*} \right)^\alpha < 2L,
  {u_* (}\frac{l}{2})^{1/\alpha}, & \left( \frac{u}{u_*} \right)^\alpha \leq \frac{l}{2},
  {u_* (}2L)^{1/\alpha}, & 2L \leq \left( \frac{u}{u_*} \right)^\alpha,
\end{cases}
\]

and

\[
\hat{v}^L_l := \begin{cases} 
  v, & \frac{l}{2} \leq \left( \frac{v}{v_*} \right)^\alpha < 2L,
  {v_* (}\frac{l}{2})^{1/\alpha}, & \left( \frac{v}{v_*} \right)^\alpha \leq \frac{l}{2},
  {v_* (}2L)^{1/\alpha}, & 2L \leq \left( \frac{v}{v_*} \right)^\alpha.
\end{cases}
\]

By the assumptions in Subsection 1.1, \( \mu^L_l \) and \( \mu^L_{l,l} \) are uniformly bounded and elliptic, so there are constants

\[
\bar{\mu}, \overline{\mu_l}, \overline{\mu_{l,l}}, \overline{\mu_1} > 0,
\]

such that

\[
X \cdot \mu_l^L(x,u)X \geq ||X||^2, \quad X \cdot \mu_{l,l}^L(x,u)X' \leq \overline{\mu_1} ||X|| ||X'||,
\]

\[
Y \cdot \mu_l^L(y,u,v)Y \geq ||Y||^2, \quad Y \cdot \mu_{l,l}^L(y,u,v)Y' \leq \overline{\mu_1} ||Y|| ||Y'||,
\]

for a.a. \( x \in \Omega \) and \( y \in \Gamma \), for all \( X, X' \in \mathbb{R}^d \) and \( Y, Y' \in \mathbb{R}^{d-1} \), and for all \((u,v) \in \mathbb{R}^2\).

To show that the reaction in (1) is entropy-producing, the reaction rate can be reformulated as

\[
f(u,v) := k(u^\alpha - \kappa v^\beta)
= k\Lambda(u^\alpha , \kappa v^\beta)(\alpha \ln u - \beta \ln v - \ln \kappa)
= k\Lambda(u^\alpha , \kappa v^\beta)(\alpha \ln \frac{u}{u_*} - \beta \ln \frac{v}{v_*}),
\]

where the last equality follows from (4). Here,

\[
\Lambda(a , b) = \begin{cases} 
  \frac{a-b}{\ln a - \ln b}, & \text{if } a,b > 0, \; a \neq b, \\
  0, & \text{if } a = 0 \text{ or } b = 0, \\
  a, & \text{if } a = b > 0,
\end{cases}
\]

denotes the logarithmic mean. The equality in (8) is justified if \( u,v \geq 0 \). Since \( u,v \geq 0 \) will hold a posteriori (even a uniform positive lower bound), we replace the rate function \( f(u,v) = k(u^\alpha - \kappa v^\beta) \) with

\[
\hat{f}(u,v) := \begin{cases} 
  f(u,v) = k\Lambda(u^\alpha , \kappa v^\beta)(\alpha \ln \frac{u}{u_*} - \beta \ln \frac{v}{v_*}), & \text{if } u,v \geq 0,
  0, & \text{if } u < 0 \text{ or } v < 0.
\end{cases}
\]

In the next two steps, we show that a solution \((u,v) \in \text{MR}_{q,r}^\alpha \) of the adapted system (2) with \( \mu^L_l , \mu^L_{l,l} \) replacing \( \mu, \mu_l \) and \( \hat{f} \) replacing \( f \) in (2) satisfies (6).

Proof of lower bounds for solutions of (2): The uniform lower bound can be proved by a standard comparison principle (dissipation for the \( L^2 \)-norm): Let
\( \sigma_u := l^{1/\alpha} \) and \( \sigma_v := (\frac{1}{\alpha})^{1/\beta} \), then \( \sigma_u \leq u_0 \) and \( \sigma_v \leq v_0 \) and \( (\sigma_u, \sigma_v) \) is a (stationary) solution of system (1). Define

\[
\begin{align*}
 u_-(t,x) = & \begin{cases}
 0, & u(t,x) \geq \sigma_u, \\
 u(t,x) - \sigma_u, & u(t,x) < \sigma_u,
\end{cases} \\
 v_-(t,x) = & \begin{cases}
 0, & v(t,y) \geq \sigma_v, \\
 v(t,y) - \sigma_v, & v(t,y) < \sigma_v,
\end{cases}
\end{align*}
\]

then \( (u_-, v_-)(0) = 0 \) and \( (u_-, v_-) \in L^1 (\Omega; \mathbb{W}^{1,q}; \mathbb{W}_0^{1,q}) \) is a suitable test function for (2), so that

\[
\frac{d}{dt} \left( \int_\Omega u_-^2(t,x) \, dx + \int_\Gamma v_-^2(t,y) \, dy \right) + \int_\Omega \mu |\nabla u_-|^2 \, dx + \int_\Gamma \mu_t |\nabla v_-|^2 \, dy \leq -\int_\Gamma k(u^\alpha - \kappa v^\beta)(\alpha u_- - \beta v_-) \, dy,
\]

where \( \Gamma_0 \subset \Gamma \) is the subset of \( \Gamma \), where \( (u, v) \geq 0 \). We estimate the reaction term in detail: for each \( t > 0 \), let

\[
\begin{align*}
 \Gamma_u := & \{(t,y) \in (0, \infty) \times \Gamma_0 : u(t,y) < \sigma_u, v_-(t,y) = 0\}, \\
 \Gamma_v := & \{(t,y) \in (0, \infty) \times \Gamma_0 : u_-(t,y) = 0, v(t,y) < \sigma_v\}, \\
 \Gamma_{u,v} := & \{(t,y) \in (0, \infty) \times \Gamma_0 : u(t,y) < \sigma_u, v(t,y) < \sigma_v\},
\end{align*}
\]

then for some (generic) constant \( C > 0 \),

\[
-\int_{\Gamma_u} k(u^\alpha - \kappa v^\beta)(\alpha u_- - \beta v_-) \, dy = -\int_{\Gamma_u} \alpha k(u^\alpha - \sigma_u^\alpha - \kappa(v^\beta - \sigma_v^\beta))u_- \, dy \leq 0,
\]

\[
-\int_{\Gamma_v} k(u^\alpha - \kappa v^\beta)(\alpha u_- - \beta v_-) \, dy \leq 0,
\]

and using the mean value theorem on the functions \( u \mapsto u^\alpha, v \mapsto v^\beta \),

\[
-\int_{\Gamma_{u,v}} k(u^\alpha - \kappa v^\beta)(\alpha u_- - \beta v_-) \, dy = -\int_{\Gamma_{u,v}} k(u^\alpha - \sigma_u^\alpha - \kappa(v^\beta - \sigma_v^\beta))(\alpha u_- - \beta v_-) \, dy
\]

\[
\leq \alpha \beta k(\sigma_u^{\alpha-1} + \kappa \sigma_v^{\beta-1}) \int_{\Gamma_0} u_- \, dy
\]

\[
\leq C \int_{\Gamma_{u,v}} |v_-|^2 \, dy + \frac{\mu}{2} \int_\Omega |\nabla u_-|^2 \, dx + |u_-|^2 \, dx,
\]

where the last inequality follows from Young’s inequality and the trace estimate \( \int_\Gamma |u_-|^2 \, dy \leq C \|u_-\|^2_{H^1(\Omega)} \). Hence,

\[
\frac{d}{dt} \left( \int_\Omega u_-^2(t,x) \, dx + \int_\Gamma v_-^2(t,y) \, dy \right) \leq C \left( \int_\Omega u_-^2(t,x) \, dx + \int_\Gamma v_-^2(t,y) \, dy \right)
\]

and thus, by Gronwall’s inequality, \( (\sigma_u, \sigma_v) \leq (u,v)(t) \) for all \( t \geq 0 \).

**Proof of upper bounds for solutions of (2):** The upper bounds in Lemma 4.1 are shown using the entropy-producing structure of the system. In the usual proof of a comparison principle, the dissipation of the \( L^2 \)-energy of truncated solutions is used to show pointwise bounds. Here, the idea is that, similarly, the dissipation of the free energy of truncated potentials provides pointwise bounds.

For this method to work, the solution must be constructed to have an entropy estimate and it must be sufficiently regular to allow for the truncation. Here, we are dealing with (fairly) regular solutions and it is straightforward to do everything rigorously (see (10) below).
Remark 2. Regarding weaker notions of solutions, in [9], the corresponding relative entropy estimate is shown for a general class of reaction-diffusion systems, for renormalized (and thus for weak) solutions. A similar result should hold for this simple volume-surface case, even with nonlinear diffusion (the truncated version may not follow automatically for weak solutions though). In [7], the entropy estimate is assumed and used to show the exponential equilibration of the linear volume-surface system.

Due to \((u, v) \in \text{MR}_{q,q}^r\), is a solution of the adapted system (2) that satisfies the lower bound \((\sigma_u, \sigma_v) \leq (u, v)(t)\) for all \(t \geq 0\). Let \(\mathcal{E}\) be the (negative) relative entropy for the system given by

\[
\mathcal{E}((u, v); (u_*, v_*))(t) = \int_{\Omega} u_* e\left(\frac{u(t, x)}{u_*}\right) \, dx + \int_{\Gamma} v_* e\left(\frac{v(t, y)}{v_*}\right) \, dy,
\]

where

\[
e(z) = \begin{cases} z \ln z - z + 1, & z > 0, \\ 1, & z = 0. \end{cases}
\]

Note that due to the lower bounds shown before, in this a posteriori definition it is sufficient to consider positive \((u, v)\). Let

\[
\mathcal{E}_L((u, v); (u_*, v_*))(t) = L^{1/\alpha} \int_{\Omega} u_* e\left(\frac{u(t, x)}{u_*}\right) \, dx + L^{1/\beta} \int_{\Gamma} v_* e\left(\frac{v(t, y)}{v_*}\right) \, dy,
\]

be a relative entropy adapted to the bound \(L\). Define

\[
u^L(t, x) = \begin{cases} u_*, & \frac{u(t, x)}{u_*} \leq L, \\ \frac{u(t, x)}{u_*}, & \frac{u(t, x)}{u_*} > L, \end{cases} \quad u^L(t, x) = \begin{cases} v_*, & \frac{v(t, x)}{v_*} \leq L^{1/\beta}, \\ \frac{v(t, x)}{v_*}, & \frac{v(t, x)}{v_*} > L^{1/\beta}, \end{cases}
\]

so that

\[
\mathcal{E}_L((u^L, v^L); (u_*, v_*))(0) = \mathcal{E}_L((u_*, v_*); (u_*, v_*)) = 0.
\]

Then by Brézis’ Lemma [3][Ch. III, Lemma 3.3] and the convexity of \(e\) on \((0, \infty)\), we obtain

\[
\frac{d}{dt} \mathcal{E}_L((u^L, v^L); (u_*, v_*))(t) = L^{1/\alpha} \int_{\Omega} \nu^L(t, x) e'\left(\frac{\nu^L(t, x)}{u_*}\right) \, dx + L^{1/\beta} \int_{\Gamma} \nu^L(t, y) e'\left(\frac{\nu^L(t, y)}{v_*}\right) \, dy.
\]

With the definition

\[
\xi^L(t, x) := e'\left(\frac{\nu^L(t, x)}{u_*}\right) = \begin{cases} 0, & \frac{u(t, x)}{u_*} \leq L, \\ \ln\left(\frac{u(t, x)}{u_*}\right) - \frac{1}{\alpha} \ln L, & \frac{u(t, x)}{u_*} > L, \end{cases}
\]

and

\[
\chi^L(t, y) := e'\left(\frac{\nu^L(t, y)}{v_*}\right) = \begin{cases} 0, & \frac{v(t, y)}{v_*} \leq L^{1/\beta}, \\ \ln\left(\frac{v(t, y)}{v_*}\right) - \frac{1}{\beta} \ln L, & \frac{v(t, y)}{v_*} > L^{1/\beta}, \end{cases}
\]

this can be written as

\[
\frac{d}{dt} \mathcal{E}_L((u^L, v^L); (u_*, v_*))(t) = \int_{\Omega} \hat{u}(t, x) \xi^L(t, x) \, dx + \int_{\Gamma} \hat{v}(t, y) \chi^L(t, y) \, dy.
\]

Due to \((u, v) \in \text{MR}_{q,q}^r\) and since \(e'\) is uniformly Lipschitz on \((1, +\infty)\),

\[
((\xi^L, \chi^L) \in L^r(J_T; W^{1,q'}(\Omega)),
\]

where
so apply equation (2) to \((\xi^L, \chi^L)\) and add both components to get that, for a.e. \(t > 0\),
\[
\frac{d}{dt} \mathcal{E}_L((u^L, v^L); (u_*, v_*)) = -\int_\Omega \nabla u \cdot \mu^L_t(u) \nabla \xi^L \, dx - \int_\Gamma \nabla v \cdot \mu^L_t(u, v) \nabla \chi^L \, dy \\
- \int_{\Gamma_0} k\Lambda(u^\alpha, \kappa \nu^\beta)(\alpha \ln \frac{u}{u_*} - \beta \ln \frac{v}{v_*})(\alpha \xi^L - \beta \chi^L) \, dy.
\]
(10)

If
\[
\frac{d}{dt} \mathcal{E}_L((u^L, v^L); (u_*, v_*))(t) \leq 0,
\]
for all \(t > 0\), then
\[
\mathcal{E}_L((u^L, v^L); (u_*, v_*))(t) = \mathcal{E}_L((u^L, v^L); (u_*, v_*))(0) = 0,
\]
hence,
\[
(u^L, v^L)(t) = (u_*, v_*)
\]
for all \(t > 0\) (\(\mathcal{E}_L\) is a strict Lyapunov functional), and then the upper bound in Lemma 4.1 is proved. To show this, the terms on the right-hand-side of (10) can be considered in a pointwise manner. First the reaction term: for each \(t > 0\), let
\[
\Gamma_\xi = \{(t, y) \in (0, \infty) \times \Gamma_0 : \xi^L(t, y) > 0, \chi^L(t, y) = 0\},
\]
\[
\Gamma_\chi = \{(t, y) \in (0, \infty) \times \Gamma_0 : \xi^L(t, y) = 0, \chi^L(t, y) > 0\},
\]
\[
\Gamma_{\xi, \chi} = \{(t, y) \in (0, \infty) \times \Gamma_0 : \xi^L(t, y), \chi^L(t, y) > 0\},
\]
then \(\ln \left(\frac{u}{u_*}\right)^\alpha - \ln \left(\frac{v}{v_*}\right)^\beta > 0\) in \(\Gamma_\xi\) and \(\ln \left(\frac{u}{u_*}\right)^\alpha - \ln \left(\frac{v}{v_*}\right)^\beta < 0\) in \(\Gamma_\chi\), so
\[
(\alpha \ln \frac{u}{u_*} - \beta \ln \frac{v}{v_*})(\alpha \xi^L - \beta \chi^L) = \begin{cases} 
\alpha(\ln \left(\frac{u}{u_*}\right)^\alpha - \ln \left(\frac{v}{v_*}\right)^\beta)\xi^L > 0, & \text{in } \Gamma_\xi, \\
\beta(\ln \left(\frac{u}{u_*}\right)^\beta - \ln \left(\frac{v}{v_*}\right)^\alpha)\chi^L > 0, & \text{in } \Gamma_\chi, \\
(\ln \left(\frac{u}{u_*}\right)^\alpha - \ln \left(\frac{v}{v_*}\right)^\beta)^2 > 0, & \text{in } \Gamma_{\xi, \chi},
\end{cases}
\]
where in the last line, the adaptation of \(\mathcal{E}\) with the weights \(L^{1/\alpha}, L^{1/\beta}\) was used. This would not be necessary if the case that both \(u\) and \(v\) are pointwise large can be excluded a priori, an estimate that is often known and employed in the linear diffusion case, see [16]. In summary,
\[
-\int_{\Gamma_0} k\Lambda(u^\alpha, \kappa \nu^\beta)(\alpha \ln \frac{u}{u_*} - \beta \ln \frac{v}{v_*})(\alpha \xi^L - \beta \chi^L) \, dy \leq 0
\]
for the reaction term in (10).

For the diffusion, the estimate is
\[
-\int_\Omega \nabla u \cdot \mu^L_t(u) \nabla \xi^L \, dx = -4 \int_{\Omega_{\xi} = \{x \in \Omega : (\frac{u(x)}{u_*})^\alpha \geq L\}} \nabla \sqrt{u} \cdot \mu^L_t(u) \nabla \sqrt{u} \, dx \\
\leq -4 \int_{\Omega_{\xi}} \mu|\nabla \sqrt{u}|^2 \, dx \leq 0,
\]
with the bounds from (7), and, in the same way,
\[
-\int_\Gamma \nabla v \cdot \mu^L_t(u, v) \nabla \chi^L \, dy \leq -4 \int_{\Gamma_{\xi}} \mu_{\Gamma} |\nabla \sqrt{v}|^2 \, dy \leq 0.
\]
This proves the upper bound.

**Conclusion** As in the proof of Theorem 3.1, the preceding arguments imply that there exists a unique global regular solution \((u, v) \in \text{MR}^r_q, q, \Gamma \) of (2) that satisfies (6). But then,

\[
\hat{f}(u(t), v(t)) = f(u(t), v(t)), \mu^L_t(u(t)) = \mu(u(t)),
\]

and

\[
\mu^L_t(u(t), v(t)) = \mu^L_t(u(t), v(t)),
\]

for all \(t \geq 0\). Due to the Hölder regularity of \((u, v)\) in space and time (5) and due to the local Lipschitzianity of \(f, \mu, \) and \(\mu^L, (u, v)\) is also the unique solution of the original system (2). This concludes the proof of Lemma 4.1.

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