Finite Element Methods for Maxwell’s Equations

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Abstract. We survey finite element methods for approximating the time harmonic Maxwell equations. We concentrate on comparing error estimates for problems with spatially varying coefficients. For the conforming edge finite element methods, such estimates allow, at least, piecewise smooth coefficients. But for Discontinuous Galerkin (DG) methods, the state of the art of error analysis is less advanced (we consider three DG families of methods: Interior Penalty type, Hybridizable DG, and Trefftz type methods). Nevertheless, DG methods offer significant potential advantages compared to conforming methods.

1. Introduction

Maxwell’s equations govern the propagation of electromagnetic waves, and so applications are widespread in science and technology. When the material through which the waves propagate is inhomogeneous (i.e. has spatially varying electromagnetic parameters) or is anisotropic, it is common to discretize the Maxwell system directly (rather than via integral equations). In this paper we shall survey finite element methods for approximating the solution of frequency domain (or time-harmonic) electromagnetic wave propagation. The main emphasis of the paper is on discontinuous Galerkin (DG) methods, but we shall first recall a brief history of conforming methods and mention the important input that mathematics has provided in their development.

An obvious question is: Why use DG methods? The general answer is flexibility. It is easy to change the local discretization spaces (for example, polynomial degree) from element to element, and typically hanging nodes are permissible. More recently, the ability to use general polyhedral mesh elements has emerged as an attractive feature. Other attractive features of DG methods include their element-wise conservation properties. Especially, comparing with $H(\text{curl}; \Omega)$-conforming edge element methods, it is much easier to implement a DG method when using hp-adaptivity for Maxwell’s equations.

The usual objection to DG methods is that, for a given mesh and polynomial degree, they have more degrees of freedom than conforming methods. But this problem can be ameliorated by using a hybridizable DG method, or a Trefftz method.

2000 Mathematics Subject Classification. Primary
We shall only consider the simplest boundary value problem for Maxwell’s equations. To describe this problem, let \( \Omega \subset \mathbb{R}^3 \) denote a bounded connected Lipschitz polyhedral domain with connected boundary \( \Gamma := \partial \Omega \), and let \( \nu \) denote the unit outward normal. We suppose that the material in \( \Omega \) has relative magnetic permeability \( \mu_r = 1 \) (i.e. magnetic effects are absent), but that the relative permittivity \( \epsilon_r \) is a function of position and for simplicity is piecewise in \( W^{1, \infty} \) (see [4]). In general \( \Re(\epsilon_r) \) may be positive (for example in a dielectric) or negative (as for certain metals like gold at optical frequencies). However the conductivity \( \Im(\epsilon_r) \geq 0 \) and \( \Im(\epsilon_r) > 0 \) when \( \Re(\epsilon_r) < 0 \). For simplicity we shall assume \( \Re(\epsilon_r) \) is strictly positive in accordance with most of the papers we shall reference (where it is assumed that the wavelength is such that metals can be modelled as impenetrable regions).

In our upcoming discussion vector quantities or spaces of vector functions will be in bold typeface.

Suppose the electromagnetic field has angular frequency \( \omega \). We define the wave number \( \kappa = \omega / c \) where \( c \) is the speed of light in vacuum. Then the electric field \( E \) satisfies the following Maxwell system written in second order form:

\[
\begin{align*}
\nabla \times \nabla \times E - \kappa^2 \epsilon_r E &= F \text{ in } \Omega, \\
\nu \times E &= 0 \text{ on } \Gamma := \partial \Omega.
\end{align*}
\]

Here \( F \) describes imposed currents, and for simplicity we shall assume \( F \in L^2(\Omega) \) and \( \nabla \cdot F = 0 \). The boundary condition \( \text{eq. (1.1b)} \) is termed the perfect electrical conductor or PEC boundary condition and is appropriate for metal at microwave frequencies.

In order to write down variational formulations of \( \text{eq. (1.1)} \) we need some notation. Let \( \Lambda \subset \mathbb{R}^3 \) denote a bounded Lipschitz domain. Then the standard function space for analysing \( \text{eq. (1.1)} \) is

\[
\mathcal{H}(\text{curl}; \Lambda) := \{ u \in L^2(\Lambda) \mid \nabla \times u \in L^2(\Lambda) \}
\]

in the case when \( \Lambda = \Omega \). Taking into account the PEC boundary condition \( \text{eq. (1.1b)} \) the appropriate subspace for the solution is

\[
\mathcal{H}_0(\text{curl}; \Lambda) := \{ u \in \mathcal{H}(\text{curl}; \Lambda) \mid \nu \times u = 0 \text{ on } \Gamma \},
\]

with norm

\[
\| u \|_{\text{curl}, \Lambda}^2 = \| \nabla \times u \|_{\Lambda}^2 + \| u \|_{\Lambda}^2
\]

where \( \| \cdot \|_\Lambda \) denotes the \( L^2 \) norm on \( \Lambda \).

The main tool to derive variational formulations is the following integration by parts identity

\[
\int_\Lambda \nabla \times A \cdot B \ dV = \int_\Lambda A \cdot \nabla \times B \ dV + \int_{\partial \Lambda} \nu \times A \cdot B_T \ dA,
\]

where \( B_T = \nu \times (B \times \nu) \). This identity is valid for functions \( A, B \in \mathcal{H}(\text{curl}; \Lambda) \) and for Lipschitz domains [13, 39]. We also use the notation

\[
(u, v)_\Lambda := \int_\Lambda u \cdot \overline{v} \ dV,
\]

where the over-bar denotes complex conjugation.
Proceeding formally we can derive a weak form for eq. (1.1) by multiplying the first equation by the complex conjugate of a smooth test vector $\xi \in C_0^\infty(\Omega)$ and using the integration by parts identity eq. (1.2) with $\Lambda = \Omega$, to obtain

\begin{equation}
\int_{\Omega} \nabla \times \nabla \times E \cdot \overline{\xi} \, dV = \int_{\Omega} \nabla \times E \cdot \nabla \times \overline{\xi} \, dV + \int_{\Gamma} \nu \times E \cdot \overline{\xi}_T \, dA,
\end{equation}

where the boundary term vanishes because $\xi_T$ vanishes on $\partial\Omega$. Using this equality, we derive the variational problem of finding $E \in H_0^\text{curl}(\Omega)$ such that

\begin{equation}
(\nabla \times E, \nabla \times \xi)_\Omega - \kappa^2 (\epsilon_r E, \xi)_\Omega = (F, \xi)_\Omega \quad \text{for all } \xi \in H_0^\text{curl}(\Omega).
\end{equation}

The above problem (1.4) reveals two issues that complicate the analysis of Maxwell’s equations:

(1) Only the curl of $E$ appears in the derivative term in this equation. There is no explicit control over the divergence. But we note that by choosing $\xi \in H_0^1(\Omega) := \{ u \in L^2(\Omega) | \nabla u \in L^2(\Omega) \text{ and } u = 0 \text{ on } \Gamma \}$,

we have $\nabla \xi \in H_0^\text{curl}(\Omega)$. Using $\xi = \nabla \xi$ in eq. (1.4) we obtain, using our assumption that $\nabla \cdot F = 0$,

\begin{equation}
(\epsilon_r E, \nabla \xi)_\Omega = 0 \quad \text{for all } \xi \in H_0^1(\Omega).
\end{equation}

Thus $\nabla \cdot (\epsilon_r E) = 0$. This suggests that the correct space for the analysis of eq. (1.4) is then

$$V := H_0^\text{curl}(\Omega) \cap H(\text{div}^0_{\epsilon_r}; \Omega),$$

where

$$H(\text{div}^0_{\epsilon_r}; \Omega) := \{ u \in L^2(\Omega) | \nabla \cdot (\epsilon_r u) = 0 \text{ in } \Omega \}.$$

Note that this implies that $E$ generally has a discontinuous normal component across surfaces where $\epsilon_r$ is discontinuous.

(2) Even using $V$ in place of $H_0^\text{curl}(\Omega)$ in eq. (1.4) the sesquilinear form on the left hand side of eq. (1.4) is not coercive. However, it does satisfy a Gårding inequality, and then the compact embedding of $V$ into $L^2(\Omega)$ shows that eq. (1.4) has a unique solution provided $\kappa$ is not an eigenvalue for the curl-curl operator. We assume this restriction on $\kappa$ from now on.

For more information about the mathematical theory of Maxwell’s equations see [44], [39], [4].

The outline of the paper is as follows. In the next section we shall briefly discuss conforming discretizations of Maxwell’s equations before turning to the main subject of this paper: DG methods. There are many versions of DG methods, and we have chosen just three examples to discuss here. In Section 3 we discuss one of the first DG methods for Maxwell’s equations, the Interior Penalty Discontinuous Galerkin Method (IPDG). We then move, in Section 4 to an alternative approach that allows for new solution strategies, the Hybridizable Discontinuous Galerkin (HDG) method, before continuing with a Trefftz type method, the Ultra Weak Variational Formulation (UWVF) in Section 5. We end with some conclusions.
2. Finite Elements and Maxwell’s Equations

In this section we shall discuss the standard conforming approximation to Maxwell’s equations. The most direct approach would be to construct a conforming approximation in $V$, but this brings certain dangers that we shall mention in Section 2.1. Instead we prefer to work in $H_0(\text{curl}; \Omega)$ (see Section 2.2).

Although discontinuous Galerkin methods can be used on rather general meshes, we restrict ourselves in this paper to the simplest case, a tetrahedral mesh. Let $\{T_h\}_{h>0}$ denote a family of meshes of shape regular tetrahedra that cover $\Omega$ and satisfy the usual finite element meshing constraints [22]. The parameter $h$ denotes the maximum diameter of the elements in $T_h$.

2.1. Stabilized $H^1$ elements. Let us suppose in this subsection that $\epsilon_r$ is constant. We could then define a finite element subspace of $V$ of polynomials of degree $k$ by

$$V_h^{(1)} := \{ u \in V \mid u|_K \in \mathbb{P}_k^3 \text{ for all } K \in T_h \},$$

where $\mathbb{P}_k$ denotes the space of polynomials of total degree at most $k$ in three variables. If $u_h \in V_h^{(1)}$, the requirement that $u_h \in V$ implies it must have a well defined divergence and curl. This necessarily requires that $u_h$ is continuous and hence $V_h^{(1)} \subset H^1(\Omega)$ (both normal and tangential components of the field must be continuous across faces in the mesh). Thus we can construct $V_h^{(1)}$ from three copies of the standard scalar continuous piecewise $k$-degree finite element spaces. This is a major practical attraction.

Imposition of the divergence free constraint suggests to modify eq. (1.4) to include a penalty term for the divergence [30], so choosing $\gamma > 0$ we seek $E_h \in V_h^{(1)}$ such that

$$\text{(2.1)} \quad (\nabla \times E_h, \nabla \times \xi)_\Omega + \gamma (\nabla \cdot E_h, \nabla \cdot \xi)_\Omega - \kappa^2 (\epsilon_r E_h, \xi)_\Omega = (F, \xi)_\Omega$$

for all $\xi \in V_h^{(1)}$.

Unfortunately this formulation has subtle problems that were finally understood by Costabel and Dauge [25]: in domains with reentrant corners or edges this method may produce discrete solutions that converge to a function that is not the solution of Maxwell’s equations. In more detail, the above problem requires us to consider convergence in the norm (recall that $\epsilon_r$ is constant)

$$\| u \|_{V_h^{(1)}}^2 := \| \nabla \times u \|^2_{\Omega} + \| \nabla \cdot u \|^2_{\Omega} + \| u \|^2_{\Omega}.$$ 

Necessarily the solution $E_h$ of eq. (2.1) will have $\| E_h \|_{V_h^{(1)}}^2 < \infty$ and hence lie in the space $H := \{ u \in H_0(\text{curl}; \Omega) \mid \| u \|_{V_h^{(1)}} < \infty \}$. In a convex domain, $H = V$ and convergence to the true solution $E$ will occur. But if $\Omega$ has reentrant corners or edges then $H$ is a closed subspace of $V$ with infinite codimension [25]. Thus there are solutions of eq. (1.1) that cannot be approximated by finite elements in $V_h^{(1)} \subset H$. Worse, in this bad case, the finite element solutions will converge as $h$ decreases, but to a function that does not solve Maxwell’s equations. Understanding this subtle problem was a major contribution by mathematics to computational electromagnetism.

Of course it is possible to “rehabilitate” continuous elements. Indeed Costabel and Dauge [25] achieve this by weakening the divergence penalty term replacing $\gamma (\nabla \cdot E_h, \nabla \cdot \xi)$ by $(\tilde{\gamma} \nabla \cdot E_h, \nabla \cdot \xi)$ where now $\tilde{\gamma} \geq 0$ is a position dependent weight.
function that tends to zero approaching a reentrant corner or edge in the right way \([25]\).

An alternative to using weighted spaces is instead to use negative norms of the divergence to stabilize the system \([9]\). Another alternative approach is to supplement the \(H^1\) approximation space using singular functions \([5]\).

As we have seen, across surfaces where \(\epsilon_r\) is discontinuous, the normal component of the electric field will generally also jump. So to handle discontinuous \(\epsilon_r\) with \(H^1\) elements it is necessary to break the continuity of the finite element functions across material interfaces. Then suitable transmission conditions can be imposed by Nitsche’s method \([6]\) which is essentially the same as the IPDG method discussed in Section 3.

2.2. Edge Elements. In the previous section we described the use of standard continuous piecewise polynomial finite elements, and outlined some of the issues associated with them. While it has proved possible to obtain an appropriate variational formulation, and make technical modifications to enable the use of these elements even when \(\epsilon_r\) is discontinuous, some of the simplicity is lost. In this subsection we describe the standard conforming finite element approach to approximating the time harmonic Maxwell system but which drops the goal of working in \(V\) and instead seeks conforming elements in the larger space \(H_0(\text{curl};\Omega)\).

Finite elements in \(H(\text{curl};\Omega)\) date back to the work of Nédélec \([46,47]\). In \([46]\) he proposed a family of “first kind” elements that are obtained by augmenting function in \(P_3^{k-1}\) by some extra polynomials in \(P_3^k\) to give a conforming family of elements in \(H(\text{curl};\Omega)\). The lowest order finite element space (when \(k = 1\)) is given by

\[
V_h^{N_1} = \{ u \in H_0(\text{curl};\Omega) \mid u|_K = a_K + b_K \times x \text{ for } a_K, b_K \in \mathbb{C}^3 \text{ and all } K \in \mathcal{T}_h \}
\]

Functions of this type also appear in the work of Whitney \([53]\), and so these elements are sometimes called Whitney elements, Nédélec elements or edge elements. This family is likely the most commonly used choice of elements, although usually higher order spaces are used in practice \([44]\). An important work on the mathematical foundations of edge elements and the use of these elements for Maxwell’s equations is due to Hiptmair \([31]\), and for a different approach see \([44]\).

Because of their relevance to discontinuous Galerkin methods, in this paper we shall consider the second family of edge elements of degree \(k\) proposed by Nédélec in 1986 \([47]\):

\[
V_h^{N_2} = \{ u \in H_0(\text{curl};\Omega) \mid u|_K \in P_3^k \text{ and all } K \in \mathcal{T}_h \}.
\]

This family differs from the first family by the addition of the gradients of certain polynomials. The degrees of freedom (DOF) for this element are associated with edges, faces and tetrahedra in the mesh. In particular, specification of the following quantities uniquely determines a polynomial in \(P_3^k\) and globally guarantees a conforming space \([44]\):

\[
(2.2a) \quad \left\{ \int_e (u \cdot \boldsymbol{\tau}) q ds \text{ for all } q \in P(k)(e) \text{ and all edges } e \text{ of } K \right\},
\]

\[
(2.2b) \quad \left\{ \int_F u_T \cdot q dA \text{ for all } q \in D_{k-1}(F) \text{ and all faces } F \text{ of } K \right\},
\]

\[
(2.2c) \quad \left\{ \int_K u \cdot q dV \text{ for all } q \in D_{k-2}(K) \right\},
\]
where $\tau$ is a unit tangent vector to the edge $e$. The spaces $D_{k-1}(F)$ and $D_{k-2}(K)$ are defined using the space of homogeneous polynomials of degree $k$ (denoted $\mathbb{P}_k$) as follows: $D_{k-1}(F) = (\mathbb{P}_{k-2}(F))^2 \oplus \mathbb{P}_{k-1}(F) \mathbf{x}$ of vector functions tangential to $F$, and $D_{k-2}(K) = (\mathbb{P}_{k-2}(K))^3 \oplus \mathbb{P}_{k-1}(K) \mathbf{x}$ (see [44]).

We remark that the DOFs in (2.2a) are associated with the edges of the mesh and this accounts for the name “edge elements”. Although the DOFs defined in eq. (2.2) implicitly define a basis for $V_{h}^{N_k^2}$ they are not convenient (at least for $k > 2$) and other basis functions more suited to implementation have been defined (for example those in [54]).

Taken together, for any sufficiently smooth vector function $u$ on $\Omega$, the degrees of freedom (2.2) define an interpolation operator $r_h$ such that $r_h u \in V_{h}^{N_k^2}$. Unfortunately functions in $\tilde{V}_h^{N_k^2}$ are not divergence free. If

$$\epsilon \mathbf{E}_h = \nabla \times \nabla \times \xi \in \mathbb{H}(\text{curl}; \Omega)$$

for some $\xi \in V_{h}^{N_k^2}$ then

$$\nabla \cdot r_h u = 0$$

in $\Omega$ for all $\xi \in V_{h}^{N_k^2}$. Choosing $\xi = \nabla p_h$ for any $p_h \in S_h^{k+1}$ in eq. (2.3) shows that

$$-\kappa^2 (\epsilon, \mathbf{E}_h, \nabla p_h) = 0.$$

Comparing to eq. (1.5) we say that $\epsilon, \mathbf{E}_h$ is discrete divergence free. We can also write a discrete Helmholtz decomposition

$$V_{h}^{N_k^2} = \tilde{V}_h^{N_k^2} \oplus \nabla S_h^{k+1}$$

where

$$\tilde{V}_h^{N_k^2} = \{ v_h \in V_{h}^{N_k^2} \mid (\epsilon, v_h, \nabla p_h) = 0 \text{ for all } p_h \in S_h^{k+1} \}.$$

Unfortunately functions in this space are not divergence free.

A key property of this space is discrete compactness. In particular, if $\{ \mathbf{u}_n \}$, $n = 1, \ldots$, with $h_n \to 0$ as $n \to \infty$ and if $\mathbf{u}_n \in \tilde{V}_h^{N_k^2}$ is bounded in $\mathbf{H}(\text{curl}; \Omega)$, then $\{ \mathbf{u}_n \}$ contains a subsequence converging to $\mathbf{u} \in L^2(\Omega)$ and $\nabla (\epsilon, \mathbf{u}) = 0$. This property was introduced by Kikuchi [38] to study eigenvalue problems, and is central also to the study of the source problem. Boffi has shown that it is equivalent to the Fortin condition from the theory of mixed methods [8].

Even though functions in $\tilde{V}_h^{N_k^2}$ are not divergence free, if $\epsilon = 1$, given $\mathbf{w}_h \in \tilde{V}_h^{N_k^2}$, and if we define $\mathbf{H} \mathbf{w}_h \in V$ by requiring that $\nabla \times \mathbf{H} \mathbf{w}_h = \nabla \times \mathbf{w}_h$ in $\Omega$ then

$$\| \mathbf{w}_h - \mathbf{H} \mathbf{w}_h \|_{\Omega} \leq C h^\sigma \| \nabla \times \mathbf{w}_h \|_{\Omega}$$

for some $\sigma > 0$ depending on the domain $\Omega$. Thus the discrete divergence free function $\mathbf{w}_h$ is close to a divergence free function. This verifies that the crucial
GAP property of [12] is satisfied. Then applying the general theory developed by Buffa in [12] gives the following theorem:

**Theorem 2.1 (Theorem 3.7 [12]).** Suppose that $\epsilon_r$ is piecewise constant with positive real part, and $\kappa$ is not a Maxwell eigenvalue. Then for all $h$ sufficiently small there exists a unique solution $E_h$ to eq. (2.3). In addition

$$\| E - E_h \|_{\text{curl}, \Omega} \leq C \| E - v \|_{\text{curl}, \Omega}$$

for any $v \in V_h^{N2}$.

Buffa’s theory has been applied to more general scattering type problems, for example, in [28].

We have already noted that $\nabla S_h^{k+1} \subset V_h^{N2}$, but there is a deeper connection. Using the following DOFs $\mathbb{P}_{k+1}$ on an element $K$:

1. vertex degrees: $p(a_i), \ 1 \leq i \leq 4$, for the four vertices $a_i$ of $K$,
2. edge degrees: $\left\{ \frac{1}{\text{length}(e)} \int e p q \, ds \right\}$ for all $q \in P_{k-1}(e)$, for all edges of $K$,
3. face degrees: $\left\{ \frac{1}{\text{area}(f)} \int f p q \, dA \right\}$ for all $q \in P_{k-2}(f)$, for all faces of $K$,
4. volume degrees: $\left\{ \frac{1}{\text{volume}(K)} \int_K p q \, dV \right\}$ for all $q \in P_{k-3}$,

we can define an interpolation operator $\pi_h$. Then for sufficiently smooth scalar functions $p$

$$\nabla (\pi_h p) = r_h(\nabla p).$$

The recognition by Bossavit [11] (see also [31]) that this, and a similar equality for $H(\text{curl}; \Omega)$ and $H(\text{div}; \Omega)$ interpolation, constitute a discrete de Rham complex motivated in part the Finite Element Exterior Calculus of Arnold, Falk and Winther [31-2]. This has resulted, for example, in a much better understanding of edge elements, new elements on pyramids [49], and a general theory for Laplace type equations (i.e. involving the Hodge Laplacian).

### 3. Interior Penalty DG Methods

One of the first discontinuous Galerkin methods to be developed for Maxwell’s equations followed the Interior Penalty Discontinuous Galerkin (IPDG) approach [1]. An early paper explicitly including control of the divergence [50] was soon superseded by improvements that do not require explicit divergence control. The version we shall describe here is from [36].

To outline the derivation of the method, we apply eq. (1.2) to an element $K$ in the mesh and using eq. (1.1a) we obtain for any smooth vector function $\xi$ on $K$

$$\nabla \times (E, \nabla \times \xi)_K - k^2(\epsilon_r, E, \xi)_K - (F, \xi)_K + \int_{\partial K} \nu_K \times \nabla \times E \cdot \xi_T \, dA = 0,$$

where $\nu_K$ is the unit outward normal to $K$. To proceed, we will add the above equality over all elements, and to do this we need more notation. If $K^+$ and $K^-$ meet at a face $F$ we define the standard jump and average value of a piecewise smooth vector function $v$ taking values $v^+$ on $K^+$ and $v^-$ on $K^-$ by

$$[v] := \nu_{K^+} \times v^+ + \nu_{K^-} \times v^-,$$

$$\{v\} = (v^+ + v^-)/2$$

For a face $F \subset \Gamma$, $[v] = \nu \times v$ and $\{v\} = v$. 


Adding eq. (3.1) over all elements, and using the following “DG magic formula” (see e.g. [26, 50])
\[
\sum_K \int_{\partial K} \nu_K \times \nabla \times E : \tilde{\xi} T dA = \int_{F_I} \left[ \nabla \times E \cdot [\xi] - [\nabla \times E] \cdot \tilde{\xi} \right] dA + \int_{F_B} \left[ \nabla \times E \cdot [\xi] \right] dA
\]
where \( F_I \) is the union of all interior faces of the mesh and \( F_B \) is the union of all boundary faces, using the PEC boundary condition and the continuity of the tangential component of \( \nabla \times E \) across internal faces gives
\[
(3.2) \quad (\nabla \times E, \nabla \times \xi)_{T_h} - \kappa^2 (\varepsilon, E, \xi)_{T_h} - (F, \xi)_{T_h} - \int_{F_I} \left[ \nabla \times E \cdot [\xi] \right] dA = 0.
\]
This identity is the basic result for writing down the upcoming IPDG method after discretization.

To discretize the problem we introduce the space of discontinuous vector polynomials of degree \( k \):
\[
V_h^{IP} = \{ v \in L^2(\Omega) \mid v|_K \in \mathbb{P}_k^2 \text{ for all } K \in T_h \}.
\]
Then using (3.2) we are led to the IPDG method proposed in [36]: find \( E_h \in V_h^{IP} \) such that
\[
(3.3) \quad a^{IP}(E_h, \xi) = \int_K F \cdot \tilde{\xi} dV \text{ for all } \xi \in V_h^{IP}
\]
where
\[
a^{IP}(E_h, \xi) = (\nabla \times E_h, \nabla \times \xi)_{T_h} - \kappa^2 (\varepsilon, E_h, \xi)_{T_h} - \int_{F_I} \left[ \nabla \times E_h \cdot [\xi] + [\nabla \times \xi] \cdot [E_h] \right] dA + \int_{F_B} \alpha_l |\xi|_{[E_h]} |\tilde{\xi}| dA.
\]
Here, following the usual IPDG philosophy [1], we have added a symmetrizing (but consistent) term and a term that penalizes the jumps across element faces. In addition, \( \alpha > 0 \) is a penalty parameter that must be chosen sufficiently large and
\[
h(x) = h_F \text{ for } x \in F
\]
where \( h_F \) denotes the diameter of the face \( F \).

From the definition of \( a^{IP}(., .) \), the appropriate norm to measure the error in the solution is the DG norm
\[
\|v\|_{DG}^2 = \|h^{-1/2}v\|^2_{F_I} + \|v\|^2_{T_h} + \sum_{K \in T_h} \|\nabla \times v\|^2_K.
\]
Then under the condition that \( \varepsilon \) is constant in \( \Omega \), the following theorem can be shown:

**Theorem 3.1 (Th. 3.2 of [36]).** Assume that the true solution satisfies \( E \in H^s(\Omega) \) and \( \nabla \times E \in H^s(\Omega) \) for some \( s > 1/2 \) and assume that the penalty parameter is chosen sufficiently large. Then for all \( h \) small enough, there is a unique solution to eq. (3.3). In addition the following error estimate holds:
\[
\|E - E_h\|_{DG} \leq C h^{\min(s,k)} \left( \|E\|_{H^s(\Omega)} + \|\nabla \times E\|_{H^s(\Omega)} \right)
\]

While we won’t give full details (see [36]), the proof of this result proceeds as follows. First, by using the definition of the sesquilinear form, the following estimate holds:
\[
\|E - E_h\|_{DG} \leq C \left( \inf_{v \in V_h^{IP}} \|E - v\|_{DG} + R_h(E) + \mathcal{E}_h(E - E_h) \right)
\]
where, after some calculation, and defining $\Pi_h$ to be the $L^2$ projection onto $V^{IP}_h$, we have (see Lemma 4.9 of [36])

$$ R_h(E) = \sup_{v \in V^{IP}_h} \frac{\int_{\Omega} \langle [v] \cdot \{\nabla \times E - \Pi_h \nabla \times E \} \rangle \, dA}{\|v\|_{DG}}. $$

This can then be estimated in the usual way using error estimates for $\Pi_h$.

The term $E_h(E - E_h)$ is given by

$$ E_h(E - E_h) = \sup_{v \in V^{IP}_h} \langle (E - E_h, v)_{\Omega} \rangle, $$

and arises from the lower order term in Maxwell’s equations (as is usual for time harmonic wave equations). In [36] this is analyzed using an adjoint problem (see [44] for a similar analysis in the case of edge elements). In particular, for a given $v \in V^{IP}_h$, let $v^c \in V^{N2}_h$ be a conforming edge element approximation of $v$ such that

$$ |v - v^c|_{\Omega} \leq Ch \|v\|_{DG}, \quad (3.4) $$

Then let $v^c = v_0^c + \nabla p_h$ according to the discrete Helmholtz decomposition eq. (2.4) (i.e. $v_0^c$ is discrete divergence free). Using the operator $H$ from section 2 we have (cancelling the gradient term because $E - E_h$ is discrete divergence free)

$$ (E - E_h, v) = (E - E_h, v - v^c) + (E - E_h, v_0^c - Hv_0^c) + (E - E_h, Hv_0^c). $$

The first term on the left hand side is approximated by eq. (3.4), the second from the eq. (2.5) and the third is estimated using duality via the solution $z \in H_0(\text{curl}; \Omega)$ of the adjoint Maxwell problem

$$ \nabla \times \nabla \times z - \kappa^2 \epsilon_r z = Hv_0^c \text{ in } \Omega. $$

The key here is that the right hand side is exactly divergence free, and so $z$ is smoother than $E - E_h$. Using the adjoint problem, and approximation properties of the Nédélec interpolant $r_h$ of $z$, it is then possible to show that

$$ |(E - E_h, Hv_0^c)| \leq Ch^\sigma \|E - E_h\|_{DG} \|v\|_{DG} $$

for some $\sigma > 0$ and the usual kickback argument completes the proof.

The direct use of the adjoint problem here limits the proof to constant $\epsilon_r$. For the case of piecewise smooth $\epsilon_r$, a more sophisticated analysis in [15] Section 6 proves an inf-sup condition in this case, and results in a quasi-optimal error estimate (see in particular [15] Remark 7.11). Our discussion was intended illustrates the interplay of conforming and DG approximations and the use of duality.

4. HDG Methods

In the Introduction we have argued that DG methods offer several advantages compared to conforming methods. However, the total number of global degrees of freedom (DOF) of a DG method is much more than that of an $H(\text{curl}; \Omega)$-conforming edge element method on the same mesh and of the same order.

Hybridizable discontinuous Galerkin (HDG) methods [24] were recently introduced with the aim of reducing the dimension of the discrete global linear system that needs to be solved and deliver superconvergent recovery of variables of interest. By design, degrees of freedom associated with the volume of each element can be
eliminated from the global discrete problem in a process akin to static condensation. The resulting global system of an HDG method only involves DOFs on the skeleton of the mesh (i.e. all faces of the mesh) and so the global system after condensation is much smaller.

We now follow closely [48] in order to write down a standard HDG method for Maxwell’s equations. We first introduce a vector variable \( \mathbf{t} = \nabla \times \mathbf{E} \) (this is just a scaled magnetic field variable which is a quantity of interest in most simulations). We can then rewrite (1.1) into the following first order system: find \((\mathbf{t}, \mathbf{E})\) such that

\[
\begin{align*}
(4.1a) & \quad \mathbf{t} - \nabla \times \mathbf{E} = 0 \quad \text{in } \Omega, \\
(4.1b) & \quad \nabla \times \mathbf{t} - \kappa^2 \epsilon_r \mathbf{E} = \mathbf{F} \quad \text{in } \Omega, \\
(4.1c) & \quad \mathbf{\nu} \times \mathbf{E} = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Given a choice of three finite dimensional polynomial spaces \( \mathbf{V}(K) \subset H^1(K), \mathbf{W}(K) \subset H(\text{curl}; K) \) and \( \mathbf{M}(F) \subset L^2(F) \), where \( K \) is an arbitrary element in the mesh and \( F \) is an arbitrary face (or edge in 2D), we define the global spaces by

\[
\mathbf{V}_h := \{ \mathbf{v} \in L^2(\mathcal{T}_h) : \mathbf{v}|_K \in \mathbf{V}(K), K \in \mathcal{T}_h \}, \\
\mathbf{W}_h := \{ \mathbf{w} \in L^2(\mathcal{T}_h) : \mathbf{w}|_K \in \mathbf{W}(K), K \in \mathcal{T}_h \}, \\
\mathbf{M}_h := \{ \mathbf{\mu} \in L^2(\mathcal{E}_h) : \mathbf{\mu}|_F \in \mathbf{M}(F), F \in \mathcal{F}_h \}.
\]

Note that if \( \mathbf{W}(K) = [P_k(K)]^d \) where \( d = 2, 3 \) is the spatial dimension, then \( \mathbf{W}_h = \mathbf{V}_h^{IP} \) introduced earlier.

We can now derive the HDG method for (4.1) by multiplying each equation by the complex conjugate of an appropriate discrete test function, integrating element by element and using the integration by parts identity eq. (1.2) element by element. Summing the results over all elements, the HDG methods seeks by element and using the integration by parts identity eq. (1.2) element by element

\[
\begin{align*}
(4.2a) & \quad \langle \mathbf{t}_h, \mathbf{v}_h \rangle_{\mathcal{T}_h} - \langle \nabla \times \mathbf{v}_h, \mathbf{t}_h \rangle_{\partial \mathcal{T}_h} = 0, \\
(4.2b) & \quad \langle \mathbf{t}_h, \nabla \times \mathbf{w}_h \rangle_{\mathcal{T}_h} + \langle \nabla \times \mathbf{t}_h, \mathbf{w}_h \rangle_{\partial \mathcal{T}_h} - \langle \kappa^2 \epsilon_r \mathbf{E}_h, \mathbf{w}_h \rangle_{\mathcal{T}_h} = \langle \mathbf{F}, \mathbf{w}_h \rangle_{\mathcal{T}_h}, \\
(4.2c) & \quad \langle \mathbf{\nu} \times \mathbf{t}_h, \mathbf{\bar{v}}_h \rangle_{\partial \mathcal{T}_h/\partial \Omega} = 0, \\
(4.2d) & \quad \langle \mathbf{\nu} \times \mathbf{\bar{E}}_h, \mathbf{\nu} \times \mathbf{\bar{E}}_h \rangle_{\partial \Omega} = 0
\end{align*}
\]

for all \((\mathbf{r}_h, \mathbf{v}_h, \mathbf{\bar{v}}_h) \in \mathbf{V}_h \times \mathbf{W}_h \times \mathbf{M}_h\).

Different choices of the space \( \mathbf{V}(K) \times \mathbf{W}(K) \times \mathbf{M}(F) \) and numerical flux \( \mathbf{\nu} \times \mathbf{\bar{t}}_h \) give different HDG methods.

Two HDG methods were presented in [48] to approximate Maxwell’s equations. Both these HDG methods use

\[
\mathbf{V}(K) \times \mathbf{W}(K) \times \mathbf{M}(F) = [P_k(K)]^d \times [P_k(K)]^d \times [P_k(F)]^{d-1},
\]

where \( d = 2, 3 \) is the space dimension. The numerical flux is written including a stabilization term as

\[
\mathbf{\nu} \times \mathbf{\bar{t}}_h = \mathbf{\nu} \times \mathbf{t}_h + \tau (\mathbf{E}_h - \mathbf{\bar{E}}_h)_T,
\]

where \( \tau \) is a positive constant.

The first HDG method of [48] extends (1.2) by enforcing the divergence-free condition on the electric induction \( \epsilon_r \mathbf{E} \) and introduces a Lagrange multiplier to accomplish this. After hybridization (static condensation), it produces a linear
system for the DOF of the approximate traces of both the tangential components of the vector field $\hat{E}_h$ and the global Lagrange multiplier. The second HDG method does not enforce the divergence-free condition and results in a linear system only for the DOF of the approximate trace of the tangential components of the vector field $\hat{E}$. The well-posedness, conservativity and consistency of the two HDG methods, together with a numerical demonstration, was shown in [48]. However, this paper does not include an error analysis and the numerical experiments are only for $d = 2$ (two dimensional domains).

The case of HDG on a three dimensional domain was discussed in [41]. The resulting linear system was solved using a domain decomposition technique but without a convergence analysis.

There are some papers with a complete convergence analysis. The first was proposed in [27]: the authors use $k = 1$, $d = 3$ and $\tau = 1/h$ in (4.3) and (4.4), respectively. A detailed hp and a posteriori convergence analysis is found in [42, 19] for arbitrary polynomial degree and $d = 3$. However, these works only obtained a suboptimal convergence rate for $t$.

In a very recent paper [18], we used the concept of an $M$-decomposition, which was proposed by Cockburn et al in [23] for elliptic PDEs to analyze HDG schemes for Maxwell’s equations in two dimensions. This analysis provides conditions on the HDG spaces need to obtain optimal convergence, and superconvergence of some variables. The extension of this approach to 3D is challenging, and remains to be done.

Another way to design an optimally convergent HDG method is to define a special numerical flux and the new spaces $V(K)$, $W(K)$ and $M(\partial K)$, even though the spaces do not admit $M$-decompositions. In [20], Chen et al proposed a new HDG method for Maxwell’s equations by augmenting the boundary space and choosing $V(K) \times W(K) \times M(\partial K) = [P_k(K)]^3 \times [P_{k+1}(K)]^3 \times [\tilde{P}_k(F) \oplus \nabla \tilde{P}_{k+2}(F)]^2$, where $\tilde{P}_{k+2}(F)$ denotes the set of homogeneous polynomials of degree $k + 2$ on $F$. The stabilization has the form

$$n \times t_h = n \times t_h + \frac{1}{h}(P_M \hat{E}_h - \tilde{E}_h)_T,$$

where $P_M$ is an elementwise $L^2$ orthogonal projection into $M(F)$. They obtain an optimal convergent rate for the solution by some regularity assumptions on the dual problem at the expense of a larger trace space.

5. The Ultra Weak Variational Formulation

Discontinuous Galerkin methods are not limited to piecewise polynomial elements. In this section we present a DG method that uses solutions of Maxwell’s equations element by element. Methods that use solutions of the underlying equation in the approximation scheme are termed Trefftz methods [51], and the method we shall describe falls into the general class of Trefftz DG schemes. We shall also consider a more general version of the boundary value problem that brings us closer to practical applications.

The particular method we will describe is the Ultra Weak Variational Formulation (UWVF) of Maxwell’s equations due to Cessenat [17] and developed further in [37]. This method applies to a more general problem that includes the previously considered PEC boundary condition as a special case, but is restricted to equations
in which \( F = 0 \). In addition it is necessary to assume that \( \epsilon_r \) is piecewise constant on the mesh. Therefore we now assume that \( E \) satisfies

\begin{align}
\nabla \times \nabla \times E - \kappa^2 \epsilon_r E &= 0 \text{ in } \Omega, \\
\nu \times \nabla \times E - i \kappa \lambda E_T &= Q(\nu \times \nabla \times E + i \kappa \lambda E_T) + g \text{ on } \Gamma,
\end{align}

where \( \lambda > 0 \) is a real parameter, \( Q \) is another real parameter with \( |Q| < 1 \) and \( g \in L^2_\Gamma(\Gamma) := \{ g \in L^2(\Gamma) \mid \nu \cdot g = 0 \} \) is data. Note that \( Q = 1 \) gives the PEC boundary condition considered before.

To derive the UWVF we apply eq. (1.2) again to the curl term in eq. (3.1) (with \( F = 0 \)) to obtain

\begin{equation}
(E, \nabla \times \nabla \times \xi - \kappa^2 \epsilon_r \xi)_K + \int_{\partial K} \nu_K \times \nabla \times E \cdot \xi_T + \nu_K \times E \cdot \nabla \times \xi_T \, dA = 0.
\end{equation}

Clearly, if \( \xi \) solves the adjoint Maxwell system

\begin{equation}
\nabla \times \nabla \times \xi - \kappa^2 \epsilon_r \xi = 0 \text{ in } K,
\end{equation}

then we obtain the identity

\begin{equation}
\int_{\partial K} \nu_K \times \nabla \times E \cdot \xi_T + \nu_K \times E \cdot \nabla \times \xi_T \, dA = 0.
\end{equation}

Using this identity we can prove the following “isometry” result (see [17]):

\begin{equation}
\int_{\partial K} \frac{1}{\lambda} (\nu_K \times \nabla \times E + i \kappa \lambda E_T) \cdot (\nu_K \times \nabla \times \xi + i \kappa \lambda \xi_T) \, dA = \int_{\partial K} \frac{1}{\lambda} (\nu_K \times \nabla \times E - i \kappa \lambda E_T) \cdot (\nu_K \times \nabla \times \xi - i \kappa \lambda \xi_T) \, dA
\end{equation}

Now consider an arbitrary element \( K \) in the mesh. The four faces are either faces of another element in the mesh (interior faces) or boundary faces. Using the above equality and taking into account the difference in sign of the normal vector on adjacent elements we obtain

\begin{equation}
\int_{\partial K} \frac{1}{\lambda} (\nu_K \times \nabla \times E + i \kappa \lambda E_T) \cdot (\nu_K \times \nabla \times \xi + i \kappa \lambda \xi_T) \, dA
= \int_{\partial K \cap \partial K'} \frac{1}{\lambda} (-\nu_{K'} \times \nabla \times E - i \kappa \lambda E_T) \cdot (\nu_K \times \nabla \times \xi - i \kappa \lambda \xi_T) \, dA
+ \int_{\partial K \cap \Gamma} \frac{1}{\lambda} [Q (\nu_K \times \nabla \times E + i \kappa \lambda E_T) + g] \cdot (\nu_K \times \nabla \times \xi - i \kappa \lambda \xi_T) \, dA.
\end{equation}

Assuming that \( E \) is sufficiently regular such that \( \nu \times \nabla \times E + i \kappa \lambda E_T \in L^2_T(\partial K) \), and defining the space

\[ W = \Pi_{K \in \mathcal{T}_h} L^2_T(\partial K), \]

we define the unknown boundary impedance flux \( \chi_K = \nu \times \nabla \times E|_K + i \kappa \lambda (E|_K)_T \) so that \( \xi := \Pi_{K \in \mathcal{T}_h} \chi_K \in W \) satisfies

\begin{equation}
\int_{\partial K} \frac{1}{\lambda} \chi_K \cdot (\nu_K \times \nabla \times \xi + i \kappa \lambda \xi_T) \, dA
= - \int_{\partial K \cap \partial K'} \frac{1}{\lambda} \chi_{K'} \cdot (\nu_K \times \nabla \times \xi - i \kappa \lambda \xi_T) \, dA
+ \int_{\partial K \cap \Gamma} \frac{1}{\lambda} [Q \chi_K + g] \cdot (\nu_K \times \nabla \times \xi - i \kappa \lambda \xi_T) \, dA
\end{equation}

for all test functions \( \xi \) such that \( \nu_K \times \nabla \times \xi + i \kappa \lambda \xi_T \in L^2_T(\partial K) \) and \( \xi \) satisfies eq. (5.3) on \( K \), for all \( K \in \mathcal{T}_h \).
The key idea from [17] is to use a space of plane wave solutions to the adjoint problem to discretize (5.1). For each element $K \in T_h$ we take $p_K$ independent real direction vectors $d^K_j$ with $|d^K_j| = 1$ and for each $j$ we choose two mutually orthogonal real unit polarization vectors $p^K_{j,\ell}$ with $p^K_{j,\ell} \cdot d^K_j = 0, \ell = 1, 2$. Then the field $p^K_{j,\ell} \exp(i \kappa \sqrt{\varepsilon_r} d^K_j \cdot x)$ satisfies the adjoint problem in $K$. Then we define the local plane wave space on an element $K$ by

$$PW^K_h = \text{span} \{ p^K_{j,\ell} \exp(i \kappa \sqrt{\varepsilon_r} d^K_j \cdot x) | 1 \leq j \leq p_K, \ell = 1, 2 \}.$$ 

Using this space, the global test and trial space is

$$W_h = \Pi_{K \in T_h} \{ \nu \times \nabla \times \xi_h + i \kappa \lambda(\xi_h)_T | \xi_h \in PW^K_h \}.$$ 

With this in hand, we seek $\chi_h \in W_h$ such that (5.5) holds for all $\xi_h \in PW^K_h$ and all elements $K \in T_h$.

The choice of direction vectors $d^K_j$ can be made in a variety of ways. An example with good theoretical properties is given in [32], but we use the convenient Hammersley points on the unit sphere.

Cessenat [17] shows that, if $|Q| < 1$, this discrete UWVF has a unique solution and derives error estimates on $\Gamma$. A computational study relating the UWVF to upwind DG methods, and deriving the Perfectly Matched Layer (a mesh truncation technique) in this case was performed in [37].

Historically, regarding error analysis, in [14, 29] it was noted that the UWVF for the Helmholtz equation is a special case of the IPDG method (i.e. a special choice of parameters in IPDG) when a Trefftz plane wave basis is used and when the coefficient functions in the Helmholtz equation are real. This leads to an error analysis for the Helmholtz equation (see for example [34] and the survey article [35]).

The UWVF for Maxwell’s equations is also equivalent to an IPDG scheme when $\varepsilon_r$ is real (and of course, piecewise constant). This theory depends on having the same plane wave space for trial and test space, and the equivalence of UWVF and IPDG does not hold for absorbing media (complex parameters). The plane wave IPDG scheme was analyzed in [32] using the interesting stability result from [33] and convergence was verified. Hence convergence of the UWVF is also verified. The convergence of UWVF for complex $\varepsilon_r$ is not proved.

Because of the isometry result (5.4), solving the discrete UWVF system can be accomplished by simple iterative techniques [17, 37]. Although not justified theoretically, we typically use the BiCGstab algorithm [52]. This takes many hundreds of iterations but is easy to parallelize.

In comparison to HDG, we can see that HDG is a single trace formulation ($\tilde{E}_h$) whereas UWVF involves two traces $\chi_h$ on a face between an element $K$ and its neighbor $K'$.

The main drawback of the UWVF and the Trefftz IPDG scheme is that the discrete problem becomes rapidly ill-conditioned as $p := \max_{K \in T_h} p_K$ increases. In [37] we used several ad-hoc techniques to control conditioning by varying $P_K$ from element to element, but this in turn limits accuracy. It is likely that a combined Trefftz and finite element basis (Trefftz on some larger elements and finite elements on small elements or near singularities) could be attractive [45].

Modern implementations of UWVF use general element types (besides simplices, also prisms and hexahedra) to help with meshing layers such as the PML.
In principle much more complicated elements are allowed by the convergence theory [32].

As an example of the use of the UWVF, we consider the problem of computing exterior scattering from a unit ball with $\epsilon_r = 1$ and $\kappa = 44$. The parameter $\lambda = 1$ and $Q$ is set to $Q = 1$ on the surface of the sphere (PEC boundary condition). The computational domain is the annular region inside the cube $[-1.8568, 1.8568]^3$ outside of the ball. We choose $Q = 0$ on the outer surface of the domain (this is a simple absorbing boundary condition that approximates scattering on an infinite domain). In addition, a PML with parameter $\sigma = 2$ is used to cut down reflection outside the cube $[-1.5712, 1.5712]^3$ (see [37] for how to implement a simple PML in the UWVF). The mesh consists of 127,113 tetrahedra. The number of directions per element is between 29 and 67 and is chosen to equilibrate the local condition number of the inner product matrix on each element [37]. The directions themselves are Hammersley points. This results in 10,743,064 degrees of freedom. The BICG solver used 154 iterations to reduce the residual by a factor of $10^{-5}$. This calculation was performed in 432 seconds using 20 MPI parallel processes on a 20-core Linux computer using an Intel(R) Xeon(R) Gold 6138 CPU at 2.00GHz. This time includes assembling the matrices, solving via BiCGstab and computing the far field pattern at 720 points.

In Fig. 1, the left panel shows a slice through the mesh. The right hand panel shows the real part of the $y$ component of the scattered electric field $E_y$ in the $x-z$ plane that was created by an incident plane wave along the $x$-axis. This is a typical model problem for scattering calculations since the exact solution is known [44]. Generally, it is more efficient to use the UWVF with elements that have a geometric diameter that is at least one wavelength. This may then introduce an unacceptable error due to approximating any curved boundaries of scatterers by facets. Therefore it is essential to include curved elements in the mesh. Although
The example shows some of the issues for a useful code which must be able to approximate scattering problems and also handle much more complex domains and higher frequencies than that shown here. Of course, realistic calculations generally demand many more tetrahedra and degrees of freedom.

6. Conclusions and the Future

From the point of view of error analysis, the conforming edge finite element method is the best understood. Furthermore, such elements provide critical conforming approximation results for proofs of convergence for other methods such as IPDG. However, the three DG schemes we have discussed are less well justified theoretically, especially for inhomogeneous media.

All the analysis mentioned here does not explicitly track the $\kappa$ dependence of the constants appearing in the error bounds. Obtaining $\kappa$ dependent bounds for Maxwell solvers along the lines of those for the Helmholtz equation in \cite{43} would also be interesting from the point of view of high wave number applications.

Looking ahead, the main problem with all these methods is to obtain a fast solver for the discrete matrix problem after discretization. Numerical results for the UWVF (not shown) show that the number of iterations to solve the discrete problem are not negatively affected by changes in wave number $\kappa$ for fixed $h$ and $p$. This UWVF results suggests some hope that other hybridized solvers, working only on the skeleton of the mesh, may help with the $\kappa$ dependence of the iteration number.

Acknowledgements

The research of P.B. Monk and Y. Zhang is partially supported by the US National Science Foundation (NSF) under grant number DMS-1619904. Research of P.B. Monk is also supported in part by AFOSR under grant FA9550-17-1-0147.

References

1. D.N. Arnold, An interior penalty finite element method with discontinuous elements, SIAM J. Numer. Anal. 19 (1982), 742–760.
2. D.N. Arnold, R.S. Falk, and R. Winther, Finite element exterior calculus: From Hodge theory to numerical stability, Bulletin of the American Mathematical Society 47 (2010), 281–354.
3. D.N. Arnold, R.S. Falk, and R. Winther, Multigrid in $H(\text{div})$ and $H(\text{curl})$, Numer. Math. 85 (2000), 197–217.
4. F. Assous, P. Ciarlet Jr., and S. Labrunie, Mathematical foundations of computational electromagnetism, Springer, Cham, Switzerland, 2018.
5. F. Assous, P. Ciarlet Jr., and E. Sonnendrücker, Resolution of the Maxwell equations in a domain with reentrant corners, ESAIM: Mathematical Modeling and Numerical Analysis 32 (1998), 359–389.
6. F. Assous and M. Michaeli, A numerical method for handling boundary and transmission conditions in some linear partial differential equations, Procedia Computer Science 9 (2012), 422–431.
7. D Boffi, Finite element approximation of eigenvalue problems, Acta Numerica 19 (2010), 1–120.
8. Daniele Boffi, Fortin operator and discrete compactness for edge elements, Numerische Mathematik 87 (2000), 229–246.
9. A. Bonito and J.-L. Guermond, *Approximation of the eigenvalue problem for the time harmonic Maxwell system by continuous Lagrange elements*, Math. Comput. **80** (2011), 1887–1910.

10. A.-S. Bonnet-Ben Dhia, L. Chesnel, and P. Ciarlet Jr., *T-coercivity for the Maxwell problem with sign-changing coefficients*, Communications in Partial Differential Equations **39** (2014), 1007–1031.

11. A. Bossavit, *Computational electromagnetism*, Academic Press, San Diego, 1998.

12. A. Buffa, *Remarks on the discretization of some noncoercive operator with applications to heterogeneous Maxwell equations*, SIAM J. Numer. Anal. **43** (2005), 1–18.

13. A. Buffa and P. Ciarlet Jr., *On traces for functional spaces related to Maxwell’s equations Part I: An integration by parts formula in Lipschitz polyhedra*, Math. Meth. Appl. Sci. **24** (2001), 9–30.

14. A. Buffa and P. Monk, *Error estimates for the Ultra Weak Variational Formulation of the Helmholtz equation*, ESAIM: Mathematical Modeling and Numerical Analysis **42** (2008), 925–40.

15. A. Buffa and I. Perugia, *Discontinuous Galerkin approximation of the Maxwell eigenproblem*, SIAM J. Numer. Anal. **44** (2006), 2198–2226.

16. A. Cangiani, Z. Dong, E.H. Georgoulis, and P. Houston, *hp-version discontinuous Galerkin methods on polygonal and polyhedral meshes*, Springer International Publishing, 2017.

17. O. Cessenat, *Application d’une nouvelle formulation variationnelle aux équations d’ondes harmoniques. Problèmes de Helmholtz 2D et de Maxwell 3D.*, Ph.D. thesis, Université Paris IX Dauphine, 1996.

18. G. Chen, P. Monk, and Y. Zhang, *Superconvergent HDG methods for Maxwell’s equations via the M-decomposition*, https://arxiv.org/abs/1905.07383, 2019.

19. H. Chen, W. Qiu, and K. Shi, *A priori and computable a posteriori error estimates for an HDG method for the coercive Maxwell equations*, Comput. Methods Appl. Mech. Engrg. **333** (2018), 287–310.

20. H. Chen, W. Qiu, K. Shi, and M. Solano, *A superconvergent HDG method for the Maxwell equations*, J. Sci. Comput. **70** (2017), 1010–1029.

21. S.H. Christiansen and R. Winther, *Smoothed projections in finite element exterior calculus*, Math. Comput. **77** (2008), 813–829.

22. P.G. Ciarlet, *The finite element method for elliptic problems*, Studies In Mathematics and Its Applications, vol. 4, North-Holland, New York, 1978.

23. B. Cockburn, G. Fu, and F.J. Sayas, *Superconvergence by M-decompositions. Part I: General theory for HDG methods for diffusion*, Math. Comp. **86** (2017), 1609–1641.

24. B. Cockburn, J. Gopalakrishnan, and R. Lazarov, *Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems*, SIAM J. Numer. Anal. **47** (2009), 1319–1365.

25. M. Costabel and M. Douge, *Weighted regularization of Maxwell equations in polyhedral domains*, Numer. Math. **93** (2002), 239–277.

26. S. Esterhazy and J.M. Melenk, *On stability of discretizations of the Helmholtz equation*, Numerical Analysis of Multiscale Problems (I.G. Graham, T.Y. Hou, O. Lakkis, and R. Scheichl, eds.), Lecture Notes on Computational Science and Engineering, vol. 83, Springer, 2012, pp. 285-324.

27. X. Feng, P. Lu, and X. Xu, *A hybridizable discontinuous Galerkin method for the time-harmonic Maxwell equations with high wave number*, Comput. Methods Appl. Math. **16** (2016), 429–445.

28. G. Gatica and S. Meddahi, *Finite element analysis of a time harmonic Maxwell problem with an impedance boundary condition*, IMA J. Numer. Anal. **32** (2011), 534–552.

29. C. Gittelson, R. Hiptmair, and I. Perugia, *Plane wave discontinuous Galerkin methods*, ESAIM: Mathematical Modeling and Numerical Analysis **43** (2009), 297–331.

30. C. Hazard and M. Lenoir, *On the solution of time-harmonic scattering problems for Maxwell’s equations*, SIAM J. Math. Anal. **27** (1996), 1597–630.

31. R. Hiptmair, *Finite elements in computational electromagnetism*, Acta Numerica **11** (2002), 237–339.

32. R. Hiptmair, A. Moiola, and I. Perugia, *Error analysis of Trefftz-discontinuous Galerkin methods for the time-harmonic Maxwell equations*, Math. Comput. **82** (2011), 247–268.
33. , Stability results for the time-harmonic Maxwell equations with impedance boundary conditions, Math. Meth. Appl. Sci. 21 (2011), 2263–87.
34. , Plane wave discontinuous Galerkin methods: Exponential convergence of the hp-version, Foundations of Computational Mathematics 16 (2016), 637–675.
35. , A survey of Trefftz methods for the Helmholtz equation, Building Bridges: Connections and Challenges in Modern Approaches to Numerical Partial Differential Equations (Cham) (G.R. Barrenechea, F. Brezzi, A. Cangiani, and E.H. Georgoulis, eds.), Springer International Publishing, 2016, pp. 237–279.
36. P. Houston, I. Perugia, A. Schneebeli, and D. Schötzau, Interior penalty method for the indefinite time-harmonic Maxwell equations, Numer. Math. 100 (2005), 485–518.
37. T. Huttunen, M. Malinen, and P.B. Monk, Solving Maxwell’s equations using the Ultra Weak Variational Formulation, J. Comput. Phys. 223 (2007), 731–58.
38. F. Kikuchi, On a discrete compactness property for the Nédélec finite elements, J. Fac. Sci. Univ. Tokyo, Sect. IA Math. 36 (1989), 479–90.
39. A. Kirsch and F. Hettlich, The mathematical theory of time-harmonic Maxwell’s equations, Applied Mathematical Sciences, vol. 190, Springer, Switzerland, 2015.
40. R. Leis, Initial boundary value problems in mathematical physics, Wiley, New York, 1988.
41. L. Li, S. Lanteri, and R. Perrusel, A hybridizable discontinuous Galerkin method combined to a Schwarz algorithm for the solution of 3d time-harmonic Maxwell’s equation, J. Comput. Phys. 256 (2014), 563–581.
42. P. Lu, H. Chen, and W. Qiu, An absolutely stable hp-HDG method for the time-harmonic Maxwell equations with high wave number, Math. Comp. 86 (2017), 1553–1577.
43. J.M. Melenk and S. Sauter, Wavenumber explicit convergence analysis for Galerkin discretizations of the Helmholtz equation, SIAM J. Numer. Anal. 49 (2011), 1210–1243.
44. P. Monk, Finite element methods for Maxwell’s equations, Oxford University Press, Oxford, 2003.
45. P. Monk, J. Schöberl, and A. Sinwel, Hybridizing Raviart-Thomas elements for the Helmholtz equation, Electromagnetics 30 (2010), 149–76.
46. J.C. Nédélec, Mixed finite elements in $\mathbb{R}^3$, Numer. Math. 35 (1980), 315–41.
47. , A new family of mixed finite elements in $\mathbb{R}^3$, Numer. Math. 50 (1986), 57–81.
48. N. C. Nguyen, J. Peraire, and B. Cockburn, Hybridizable discontinuous Galerkin methods for the time-harmonic Maxwell’s equations, J. Comput. Phys. 230 (2011), 7151–7175.
49. N. Nigam and J. Phillips, High-order conforming finite elements on pyramids, IMA J. Numer. Anal. 32 (2012), 448–483.
50. I. Perugia, D. Schötzau, and P. Monk, Stabilized interior penalty methods for the time-harmonic Maxwell equations, Comput. Methods Appl. Mech. Eng. 191 (2002), 4675–4697.
51. E. Trefftz, Ein gegenstück zum Ritz’schen verfahren, Proc. 2nd Int. Congr. Appl. Mech. (Zurich), 1926, pp. 131–137.
52. H.A. Van der Vorst, Bi-CGSTAB: A fast and smoothly converging variant of Bi-CG for the solution of nonsymmetric linear systems, SIAM J. Sci. Stat. Comput. 13 (1992), 631–644.
53. H. Whitney, Geometric integration theory, Princeton University Press, Princeton, 1957.
54. S. Zaglmayr, High order finite element methods for electromagnetic field computation, Ph.D. thesis, Johannes Kepler Universität Linz, 2006.

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