THE CONTACT STRUCTURE INDUCED BY A LINE FIBRATION OF $\mathbb{R}^3$ IS STANDARD

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Abstract. Building on the work of and answering a question by Michael Harrison, we show that any contact structure on $\mathbb{R}^3$ induced by a line fibration of $\mathbb{R}^3$ is diffeomorphic to the standard contact structure.

1. Introduction

A fibration of $\mathbb{R}^3$ by oriented lines (or line fibration, for short) is described by a smooth vector field $V$ on $\mathbb{R}^3$ whose integral curves are straight lines. We may write

$$V = V_1 \partial_x + V_2 \partial_y + V_3 \partial_z$$

and, assuming $V$ to be of unit length, regard

$$V = (V_1, V_2, V_3): \mathbb{R}^3 \to S^2 \subset \mathbb{R}^3$$

as a map into the unit 2-sphere. We write

$$\ell_p = \{p + tV(p): t \in \mathbb{R}\}$$

for the line through $p$ defined by $V$, and $\ell_p^+ \subset \mathbb{R}^3$ for the affine plane through $p$, orthogonal to $\ell_p$. We shall write $\{V\}$ for the line fibration defined by the vector field $V$. Any line fibration $\{V\}$ defines a tangent 2-plane distribution $\xi$ on $\mathbb{R}^3$ by

$$\xi_p = \langle V(p) \rangle^+ \subset T_p \mathbb{R}^3.$$

Line fibrations of $\mathbb{R}^3$ were studied by M. Salvai [4]. Building on Salvai’s work, M. Harrison [3] showed that the 2-plane distribution $\xi$ defined by any non-degenerate line fibration $\{V\}$ is a tight contact structure, and hence (by Eliashberg’s classification) diffeomorphic (perhaps orientation-reversingly) to the standard contact structure

$$\xi_{st} = \ker(dz + x\,dy)$$

on $\mathbb{R}^3$; see Section 2 for the definition of non-degeneracy. Harrison also gave a sufficient criterion for tightness under the a priori assumption that the induced 2-plane distribution $\xi$ be a contact structure. He then posed the question whether there exists a line fibration of $\mathbb{R}^3$ that induces an overtwisted contact structure, and conjectured the answer to be no.

In the present note we confirm this conjecture. In fact, we show directly (in the case not covered by Harrison’s work) that any contact structure arising from a line fibration is diffeomorphic to the standard one.

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2. NON-DEGENERATE IMPLIES SKEW

A line fibration \( \{V\} \) is called skew if it does not contain any distinct parallel lines. It is called non-degenerate if the differential \( dV \) vanishes only in the direction of \( V \), in other words, if \( d_p V|_{\xi_p} \) is of rank 2 everywhere.

The plane field \( \xi \) defined by the line fibration \( \{V\} \) is given as the kernel of the 1-form

\[ \alpha = V_1 \, dx_1 + V_2 \, dx_2 + V_3 \, dx_3. \]

A straightforward computation shows that the contact condition \( \alpha \wedge d\alpha \neq 0 \) is equivalent to

\[ \langle V, \text{curl} \, V \rangle \neq 0. \]

In [3, Theorem 1] it is shown that this condition is satisfied whenever the line fibration is non-degenerate.

The following lemma, which says that non-degenerate line fibrations are skew, is essentially [4, Lemma 6], but we give a more specific statement and a more direct proof.

**Lemma 2.** If \( d_{p_0} V \) is of rank 2 at some point \( p_0 \in \mathbb{R}^3 \), then the line fibration \( \{V\} \) does not contain any line \( \ell \neq \ell_{p_0} \) parallel to \( \ell_{p_0} \).

**Proof.** By assumption, the restriction of the differential \( d_{p_0} V \) to the plane \( \xi_{p_0} \subset T_{p_0} \mathbb{R}^3 \) defines an isomorphism \( \xi_{p_0} \to T_{V(p_0)} S^2 \). It follows that we can find a small disc \( D_\varepsilon^2 \subset \ell_{p_0}^\perp \) about \( p_0 \) that is mapped diffeomorphically by \( V \) onto a small neighbourhood of \( V(p_0) \) in \( S^2 \). In particular, the map \( V \) never attains the value \( \pm V(p_0) \) on \( D_\varepsilon^2 \setminus \{p_0\} \).

This means that the vector \( W(p) \) obtained by projecting \( V(p) \) orthogonally to \( \ell_{p_0}^\perp \) is non-zero for \( p \in D_\varepsilon^2 \setminus \{p_0\} \). We also know that \( W(p) \) cannot point along the line through \( p_0 \) and \( p \), or else \( \ell_p \) would intersect \( \ell_{p_0} \). This implies that \( W(p) \) completes exactly one positive turn as \( p \) ranges along \( \partial D_\varepsilon^2 \), see Figure 1 which shows the situation in the affine plane \( \ell_{p_0}^\perp \).

Now suppose there was a line \( \ell_{p_1} \neq \ell_{p_0} \) parallel to \( \ell_{p_0} \) in the line fibration \( \{V\} \). We may take \( p_1 \) to be the intersection point of this line with \( \ell_{p_0}^\perp \). Notice that \( p_1 \) has to lie outside the 2-disc \( D_\varepsilon^2 \). As \( p \) ranges along \( \partial D_\varepsilon^2 \), the vector \( p - p_1 \) never completes a full turn.

It would follow that there have to be at least two points \( p \in \partial D_\varepsilon^2 \) where \( W(p) \) is a scalar multiple of \( p - p_1 \), which would imply that \( \ell_p \) intersects \( \ell_{p_1} \), contradicting the assumption that \( V \) defines a line fibration.

\[ \square \]

3. PROOF OF THEOREM 1

We begin with a simple lemma.

**Lemma 3.** If the line fibration \( \{V\} \) defines a contact structure, and for every line \( \ell \in \{V\} \) there exists a parallel line \( \ell' \neq \ell \) in \( \{V\} \), then \( dV \) has rank 1 at every point of \( \mathbb{R}^3 \).

**Proof.** If \( \alpha = V_1 \, dx_1 + V_2 \, dx_2 + V_3 \, dx_3 \) is a contact form, then \( dV \) is nowhere trivial, since \( d\alpha = \sum_i dV_i \wedge dx_i \). On the other hand, the assumption on parallel lines implies with Lemma 2 that the rank of \( dV \) can nowhere be equal to 2. \[ \square \]
Figure 1. Non-degenerate implies skew.

Proof of Theorem 1. As shown by Harrison [3, Theorem 2], if there is a line in \( \{ V \} \) not parallel to any other line in \( \{ V \} \), then \( \xi \) is tight, and hence diffeomorphic to \( \xi_{st} \) by Eliashberg’s classification [1], cf. [2, Theorem 4.10.1].

It remains to consider the case when for every line \( \ell \in \{ V \} \) there exists a parallel line \( \ell' \neq \ell \) in \( \{ V \} \). We are then in the situation of Lemma 3, so \( dV \) has rank 1 at every point of \( \mathbb{R}^3 \).

Thus, \( \ker dV \cap \xi \) defines a line field on \( \mathbb{R}^3 \). This line field is necessarily trivial, and we choose a vector field \( X \) of constant length 1 spanning it. Any flow line of \( X \) stays within a compact region in finite time. It follows that \( X \) has a global flow, defined for all times.

Fix a point \( p_0 \in \mathbb{R}^3 \) and consider the maximal flow line \( \gamma_0 : \mathbb{R} \to \mathbb{R}^3 \) of \( X \) through \( p_0 = \gamma_0(0) \). Then \( V(\gamma_0(t)) = V(p_0) =: V_0 \) for all \( t \in \mathbb{R} \), since

\[
d_{\gamma_0(t)}V((\gamma_0'(t))) = d_{\gamma_0(t)}V(X(\gamma_0(t))) = 0.
\]

Write \( \pi : \mathbb{R}^3 \to \ell_{p_0}^\perp \) for the orthogonal projection of \( \mathbb{R}^3 \) along \( V_0 \) onto the affine plane \( \ell_{p_0}^\perp \). The projected curve \( \overline{\gamma}_0 = \pi \circ \gamma_0 \) is regular, since \( X \) is orthogonal to \( V_0 \) along \( \gamma_0 \). Notice that \( V \) is also constant equal to \( V_0 \) along \( \overline{\gamma}_0 \).

We claim that \( \overline{\gamma}_0 \) has to be a straight line. Otherwise, \( \overline{\gamma}_0 \) would have non-vanishing curvature at some point \( \overline{\gamma}_0(t_0) \). Then, for \( \delta > 0 \) sufficiently small, the arc \( A = \overline{\gamma}_0(I_0) \), where \( I_0 = [t_0 - \delta, t_0 + \delta] \), would lie to one side of the secant \( S \) joining \( \overline{\gamma}_0(t_0 - \delta) \) with \( \overline{\gamma}_0(t_0 + \delta) \). It follows that \( V(q) = V_0 \) also for all \( q \in S \), or else \( \ell_q \) would intersect one of the \( \ell_p \) with \( p \in A \). So the lines in \( \{ V \} \) through \( A \) and \( S \) form a straight cylinder. But then in fact \( V \equiv V_0 \) on the whole disc-like region in \( \ell_{p_0}^\perp \), bounded by \( A \) and \( S \), which would mean that \( dV \equiv 0 \) there, contradicting the fact that \( dV \) has rank 1 everywhere.

Hence, the \( V \)-lines through \( \overline{\gamma}_0 \) form an affine 2-plane. Choose coordinates on \( \mathbb{R}^3 \) such that this plane coincides with the \( xy \)-plane, \( \overline{\gamma}_0 \) with the \( y \)-axis, and \( V_0 = \partial_x \).
(hence $V = \partial_z$ along the $xy$-plane). Then all lines in $\{V\}$ must lie in affine planes parallel to the $xy$-plane, or else there would be an intersection of lines, and the lines in each of these horizontal planes must be parallel. It follows that

$$V(x, y, z) = (\cos \theta(z), -\sin \theta(z), 0)$$

for some smooth function $\theta : \mathbb{R} \to \mathbb{R}$, with $\theta(0) = 0$ and, since $dV$ has rank 1, satisfying $\theta'(z) \neq 0$ for all $z \in \mathbb{R}$.

A diffeomorphism of $\mathbb{R}^3$ pulling back

$$\alpha = \cos \theta(z) \, dx - \sin \theta(z) \, dy$$

to the standard contact form $\alpha_{st} = dz + x \, dy$ is given by

$$(x, y, z) \mapsto \left( z \cos \theta(y) + \frac{x}{\theta'(y)} \sin \theta(y), -z \sin \theta(y) + \frac{x}{\theta'(y)} \cos \theta(y), y \right).$$

This completes the proof of Theorem 1. \qed

Remark 4. Observe that this argument yields another sufficient condition for a line fibration of $\mathbb{R}^3$ to induce a contact structure: if $dV$ has constant rank 1, then $\{V\}$ defines the standard contact structure $\xi_{st}$ on $\mathbb{R}^3$.

References

[1] Ya. Eliashberg, Contact 3-manifolds twenty years since J. Martinet’s work, Ann. Inst. Fourier (Grenoble) 42 (1992), 165–192.

[2] H. Geiges, An Introduction to Contact Topology, Cambridge Stud. Adv. Math. 109, Cambridge University Press, Cambridge (2008).

[3] M. Harrison, Contact structures induced by skew fibrations of $\mathbb{R}^3$, arXiv:1904.00405.

[4] M. Salvai, Global smooth fibrations of $\mathbb{R}^3$ by oriented lines, Bull. London Math. Soc. 41 (2009), 155–163.