Some Orthogonal Polynomials in Four Variables*

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Abstract. The symmetric group on 4 letters has the reflection group $D_3$ as an isomorphic image. This fact follows from the coincidence of the root systems $A_3$ and $D_3$. The isomorphism is used to construct an orthogonal basis of polynomials of 4 variables with 2 parameters. There is an associated quantum Calogero–Sutherland model of 4 identical particles on the line.

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1 Introduction

The symmetric group on $N$ letters acts naturally on $\mathbb{R}^N$ (for $N = 2, 3, \ldots$) but not irreducibly, because the vector $(1, 1, \ldots, 1)$ is fixed. However the important basis consisting of nonsymmetric Jack polynomials is defined for $N$ variables and does not behave well under restriction to the orthogonal complement of $(1, 1, \ldots, 1)$, in general. In this paper we consider the one exception to this situation, occurring when $N = 4$. In this case there is a coordinate system, essentially the $4 \times 4$ Hadamard matrix, which allows a different basis of polynomials, derived from the type-B nonsymmetric Jack polynomials for the subgroup $D_3$ of the octahedral group $B_3$. We will construct an orthogonal basis for the $L^2$-space of the measure

$$
\prod_{1 \leq i < j \leq 4} \left| x_i - x_j \right|^{2\kappa} \left| x_1 + x_2 + x_3 + x_4 \right|^{2\kappa'} \exp \left( -\frac{1}{2} \sum_{i=1}^{4} x_i^2 \right) \, dx
$$

on $\mathbb{R}^4$, with $\kappa, \kappa' > 0$.

We will use the following notations: $\mathbb{N}_0$ denotes the set of nonnegative integers; $\mathbb{N}_0^N$ is the set of compositions (or multi-indices), if $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$ then $|\alpha| := \sum_{i=1}^{N} \alpha_i$ and the length of $\alpha$ is $\ell(\alpha) := \max \{ i : \alpha_i > 0 \}$. Let $\mathbb{N}_0^{N,+}$ denote the subset of partitions, that is, $\lambda \in \mathbb{N}_0^{N,+}$ and $\lambda_i \geq \lambda_{i+1}$ for $1 \leq i < N$. For $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ let $x^\alpha = \prod_{i=1}^{N} x_i^{\alpha_i}$, a monomial of degree $|\alpha|$; the space of polynomials is $\mathcal{P} = \text{span}_\mathbb{R} \{ x^\alpha : \alpha \in \mathbb{N}_0^N \}$. For $x, y \in \mathbb{R}^N$ the inner product is $\langle x, y \rangle := \sum_{i=1}^{N} x_i y_i$, and $|x| := (\langle x, x \rangle)^{1/2}$; also $x^\perp := \{ y : \langle x, y \rangle = 0 \}$. The cardinality of a set $E$ is denoted by $|E|$.

Consider the elements of $S_N$ as permutations on $\{1, 2, \ldots, N\}$. For $x \in \mathbb{R}^N$ and $w \in S_N$ let $(xw)_i := x_{w(i)}$ for $1 \leq i \leq N$ and extend this action to polynomials by $(wf)(x) = f(xw)$. Monomials transform to monomials: $w(x^\alpha) := x^{w\alpha}$ where $(w\alpha)_i := \alpha_{w^{-1}(i)}$ for $\alpha \in \mathbb{N}_0^N$. (Consider $x$ as a row vector, $\alpha$ as a column vector, and $w$ as a permutation matrix, with 1’s at the

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(w(\(j\), \(j\) entries.) For \(1 \leq i \leq N\) and \(f \in \mathcal{P}\) the Dunkl operators are

\[
\mathcal{D}_i f(x) := \frac{\partial}{\partial x_i} f(x) + \kappa \sum_{j \neq i} \frac{f(x) - f(x(i,j))}{x_i - x_j},
\]

and

\[
\mathcal{U}_i f(x) := \mathcal{D}_i (x_i f(x)) - \kappa \sum_{j=1}^{i-1} (j, i) f(x).
\]

Then \(\mathcal{U}_i \mathcal{U}_j = \mathcal{U}_j \mathcal{U}_i\) for \(1 \leq i, j \leq N\) and these operators are self-adjoint for the following pairing

\[
\langle f, g \rangle_{\kappa} := \langle f, \mathcal{D}_1, \ldots, \mathcal{D}_N, g(x) \rangle_{x=0}.
\]

This satisfies \(\langle f, g \rangle_{\kappa} = \langle g, f \rangle_{\kappa}\) for \(f, g \in \mathcal{P}\) and \(w \in \mathcal{S}_N\); furthermore \(\langle f, f \rangle_{\kappa} > 0\) when \(f \neq 0\) and \(\kappa \geq 0\). The operators \(\mathcal{U}_i\) have the very useful property of acting as triangular matrices on the monomial basis furnished with a certain partial order. However the good properties depend completely on the use of \(\mathbb{R}^N\) even though the group \(\mathcal{S}_N\) acts irreducibly on \((1,1,\ldots,1)^\perp\). We suggest that an underlying necessity for the existence of an analog of \(\{\mathcal{U}_i\}\) for any reflection group \(W\) is the existence of a \(W\)-orbit in which any two points are orthogonal or antipodal (as in the analysis of the hyperoctahedral group \(B_N\)). This generally does not hold for the action of \(\mathcal{S}_N\) on \((1,1,\ldots,1)^\perp\). We consider the exceptional case \(N = 4\) and exploit the isomorphism between \(\mathcal{S}_4\) and the group of type \(D_3\), that is, the subgroup of \(B_3\) whose simple roots are \((1, -1, 0), (0, 1, -1), (0, 1, 1)\). We map these root vectors to the simple roots \((0, 1, -1, 0), (0, 0, 1, -1), (1, -1, 0, 0)\) of \(\mathcal{S}_4\), in the same order. This leads to the linear isometry

\[
y_1 = \frac{1}{2} (x_1 + x_2 - x_3 - x_4), \quad y_2 = \frac{1}{2} (x_1 - x_2 + x_3 - x_4), \quad y_3 = \frac{1}{2} (x_1 - x_2 - x_3 + x_4), \quad y_0 = \frac{1}{2} (x_1 + x_2 + x_3 + x_4).
\]

Consider the group \(D_3\) acting on \((y_1, y_2, y_3)\) and use the type-\(B_3\) Dunkl operators with the parameter \(\kappa' = 0\) (associated with the class of sign-changes \(y_i \mapsto -y_i\) which are not in \(D_3\)). Let \(\sigma_{ij}, \tau_{ij}\) denote the reflections in \(y_i - y_j = 0, y_i + y_j = 0\) respectively. Then for \(i = 1, 2, 3\) let

\[
\mathcal{D}_i^B f(y) = \frac{\partial}{\partial y_i} f(y) + \kappa \sum_{j=1, j \neq i}^{3} \left( \frac{f(y) - f(y\sigma_{ij})}{y_i - y_j} + \frac{f(y) - f(y\tau_{ij})}{y_i + y_j} \right),
\]

\[
\mathcal{U}_i^B f(y) = \mathcal{D}_i^B (y_i f(y)) - \kappa \sum_{1 \leq j < i} (\sigma_{ij} + \tau_{ij}) f(y).
\]

The operators \(\{\mathcal{U}_i^B\}\) commute pairwise and are self-adjoint for the usual inner product. The simultaneous eigenvectors are expressed in terms of nonsymmetric Jack polynomials with argument \((y_1^2, y_2^2, y_3^2)\). In the sequel we consider polynomials with arguments \(x\) or \(y\) with the convention that \(y\) is given in terms of \(x\) by equation (1).

## 2 Nonsymmetric Jack polynomials

Nonsymmetric Jack polynomials (NSJP) are the simultaneous eigenfunctions of \(\{\mathcal{U}_i\}_{i=1}^N\). We consider the formulae for arbitrary \(N\) since there is really no simplification for \(N = 3\).
Definition 1. For $\alpha \in \mathbb{N}_0^N$, let $\alpha^+$ denote the unique partition such that $\alpha^+ = w\alpha$ for some $w \in S_N$. For $\alpha, \beta \in \mathbb{N}_0^N$ the partial order $\alpha \succ \beta$ ($\alpha$ dominates $\beta$) means that $\alpha \neq \beta$ and $\sum_{j=1}^i \alpha_i \geq \sum_{j=1}^i \beta_i$ for $1 \leq j \leq N$; $\alpha \triangleright \beta$ means that $|\alpha| = |\beta|$ and either $\alpha^+ \succ \beta^+$ or $\alpha^+ = \beta^+$ and $\alpha \succ \beta$.

For example $(2, 6, 4) \triangleright (5, 4, 3) \triangleright (3, 4, 5)$. When acting on the monomial basis $\{ x^\alpha : \alpha \in \mathbb{N}_0^N, |\alpha| = n \}$ for $n \in \mathbb{N}_0$ the operators $U_i$ have on-diagonal coefficients given by the following functions on $\mathbb{N}_0^N$.

Definition 2. For $\alpha \in \mathbb{N}_0^N$ and $1 \leq i \leq N$ let

$$r (\alpha, i) := \# \{ j : \alpha_j > \alpha_i \} + \# \{ j : 1 \leq j \leq i, \alpha_j = \alpha_i \},$$

$$\xi_i (\alpha) := (N - r (\alpha, i)) \kappa + \alpha_i + 1.$$

Clearly for a fixed $\alpha \in \mathbb{N}_0^N$ the values $\{ r (\alpha, i) : 1 \leq i \leq N \}$ consist of all of $\{1, \ldots, N\}$; let $w$ be the inverse function of $i \mapsto r (\alpha, i)$ so that $w \in S_N$, $r (\alpha, w (i)) = i$ and $\alpha^+ = w\alpha$ (note that $\alpha \in \mathbb{N}_0^N$ if and only if $r (\alpha, i) = i$ for all $i$). Then

$$U_i x^\alpha = \xi_i (\alpha) x^\alpha + q_{\alpha, i} (x)$$

where $q_{\alpha, i} (x)$ is a sum of terms $\pm \kappa x^\beta$ with $\alpha \succ \beta$.

Definition 3. For $\alpha \in \mathbb{N}_0^N$, let $\zeta_\alpha$ denote the $x$-monic simultaneous eigenfunction (NSJP), that is, $U_i \zeta_\alpha = \xi_i (\alpha) \zeta_\alpha$ for $1 \leq i \leq N$ and

$$\zeta_\alpha = x^\alpha + \sum_{\alpha \succ \beta} A_{\beta\alpha} x^\beta,$$

with coefficients $A_{\beta\alpha} \in \mathbb{Q} (\kappa)$, rational functions of $\kappa$.

There are norm formulae for the pairing $\langle \cdot, \cdot \rangle_\kappa$. Suppose $\alpha \in \mathbb{N}_0^N$ and $\ell (\alpha) = m$; the Ferrers diagram of $\alpha$ is the set $\{ (i, j) : 1 \leq i \leq m, 0 \leq j \leq \alpha_i \}$. For each node $(i, j)$ with $1 \leq j \leq \alpha_i$ there are two special subsets of the Ferrers diagram, the arm $\{ (i, l) : j < l \leq \alpha_i \}$ and the leg $\{ (l, j) : l > i, j \leq \alpha_l \leq \alpha_i \} \cup \{ (l, j - 1) : l < i, j - 1 \leq \alpha_l < \alpha_i \}$. The node itself, the arm and the leg make up the hook. (For the case of partitions the nodes $(i, 0)$ are customarily omitted from the Ferrers diagram.) The cardinality of the leg is called the leg-length, formalized by the following:

Definition 4. For $\alpha \in \mathbb{N}_0^N$, $1 \leq i \leq \ell (\alpha)$ and $1 \leq j \leq \alpha_i$ the leg-length is

$$L (\alpha ; i, j) := \# \{ l : l > i, j \leq \alpha_l \leq \alpha_i \} + \# \{ l : l < i, j \leq \alpha_l + 1 \leq \alpha_i \}.$$

For $t \in \mathbb{Q} (\kappa)$ the hook-length and the hook-length product for $\alpha$ are given by

$$h (\alpha, t) := \prod_{i=1}^{\ell (\alpha)} \prod_{j=1}^{\alpha_i} h (\alpha; t, i, j),$$

$$h (\alpha, t) := \prod_{i=1}^{\ell (\alpha)} \prod_{j=1}^{\alpha_i} h (\alpha; t, i, j),$$

and for $\lambda \in \mathbb{N}_0^{N,+}$ and $t \in \mathbb{Q} (\kappa)$ the generalized Pochhammer symbol is

$$(t)_\lambda := \prod_{i=1}^{N} \prod_{j=0}^{\lambda_i-1} (t - (i - 1) \kappa + j).$$
(The product over $j$ is an ordinary Pochhammer symbol.)

**Proposition 1.** For $\alpha, \beta \in \mathbb{N}_0^N$, the following orthogonality and norm formula holds:

$$
\langle \zeta_\alpha, \zeta_\beta \rangle_\kappa = \delta_{\alpha\beta} (N\kappa + 1)_{\alpha^+} \frac{h(\alpha, 1)}{h(\alpha, \kappa + 1)}.
$$

Details can be found in the book by Xu and the author [2, Chapter 8], the concept of leg-length and its use in the norm formula is due to Knop and Sahi [3]. The (symmetric) Jack polynomial with leading term $x^\lambda$ for $\lambda \in \mathbb{N}_0^{N^+}$ is obtained by symmetrizing $\zeta_\lambda$. The coefficients involve, for $\alpha \in \mathbb{N}_0^N$, $\varepsilon = \pm 1$:

$$
E_\varepsilon(\alpha) := \prod_{i<j, \alpha_i<\alpha_j} \left( 1 + \frac{\varepsilon \kappa}{(r(\alpha, i) - r(\alpha, j)) \kappa + \alpha_j - \alpha_i} \right),
$$

in fact, [1, Lemma 3.10],

$$
E_1(\alpha) x^\lambda = E_{-1}(\alpha) x^{\lambda^R},
$$

for $\alpha \in \mathbb{N}_0^N$. Then

$$
\langle j_\lambda, j_\lambda \rangle_\kappa = \# \{ \alpha : \alpha^+ = \lambda \} \frac{(N\kappa + 1)_\lambda h(\lambda, 1)}{E_1(\lambda^R) h(\lambda, \kappa + 1)},
$$

where $\lambda_i^R = \lambda_{N+1-i}$ for $1 \leq i \leq N$ (the reverse of $\lambda$). Note $\{ \alpha : \alpha^+ = \lambda \} = \{ w\lambda : w \in S_N \}$.

### 3 The groups $S_4$ and $D_3$

By using the $x \leftrightarrow y$ correspondence (equation (1)) we obtain operators which behave well on $(1, \ldots, 1)^\perp$. Here are the lists of reflections in corresponding order:

$$
[\sigma_{12}, \tau_{12}, \sigma_{13}, \tau_{13}, \sigma_{23}, \tau_{23}],
$$

$$
[(23), (14), (24), (13), (34), (12)].
$$

The following orthonormal basis is used in the directional derivatives:

$$
v_0 = \frac{1}{2} (1, 1, 1, 1),
$$

$$
v_1 = \frac{1}{2} (1, 1, -1, -1),
$$

$$
v_2 = \frac{1}{2} (1, -1, 1, -1),
$$

$$
v_3 = \frac{1}{2} (1, -1, -1, 1).
$$

That is, $y_i = \langle x, v_i \rangle$ and $\frac{\partial}{\partial y_i} = \sum_{j=1}^4 (v_i)_j \frac{\partial}{\partial x_j}$ for $0 \leq i \leq 3$. Note that $\{ \pm v_1, \pm v_2, \pm v_3 \}$ is an octahedron and an $S_4$-orbit. Then

$$
D^B_1 f(x) = \sum_{j=1}^4 (v_1)_j \frac{\partial f(x)}{\partial x_j} + \kappa \left( \frac{1 - (23)}{x_2 - x_3} + \frac{1 - (14)}{x_1 - x_4} + \frac{1 - (24)}{x_2 - x_4} + \frac{1 - (13)}{x_1 - x_3} \right) f(x),
$$
Suppose \( y \) so that \( w \) permutations \( \{ \) preserves order on \( E \) result from \( [2, \text{Corollary 9.3.3, p. 342}] \).

**Proposition 2.** Suppose \( \alpha \in \mathbb{N}_0^3 \) and \( k = 0, 1, 2, 3 \), then for \( 1 \leq i \leq k \)

\[
U_i y_{E_k} \zeta_\alpha (y^2) = 2 \xi_i (\alpha) y_{E_k} \zeta_\alpha (y^2),
\]
and for \( k < i \leq 3 \)

\[
U_i y_{E_k} \zeta_\alpha (y^2) = (2 \xi_i (\alpha) - 1) y_{E_k} \zeta_\alpha (y^2).
\]

The polynomial \( y_{E_k} \zeta_\alpha (y^2) \) is labeled by \( \beta \in \mathbb{N}_0^3 \) where \( \beta_i = 2 \alpha_i + 1 \) for \( 1 \leq i \leq k \) and \( \beta_i = 2 \alpha_i \) for \( k < i \leq 3 \). The difference \( \beta - \alpha \in \mathbb{N}_0^3 \) and appears in the norm formula (the result for the pairing \((f,g) \mapsto f(D_1^B, D_2^B, D_3^B) g(y)|_{y=0} \) applies because of the isomorphism).

**Proposition 3.** Suppose \( \beta \in \mathbb{N}_0^3 \) and \( \beta_i \) is odd for \( 1 \leq i \leq k \) and is even otherwise, then for \( \alpha_i = \left\lfloor \frac{\beta_i}{2} \right\rfloor \), \( 1 \leq i \leq 3 \)

\[
\langle y_{E_k} \zeta_\alpha (y^2), y_{E_k} \zeta_\alpha (y^2) \rangle = 2^{[\beta]} (3 \kappa + 1)_{\alpha^+} \left( \frac{2\kappa + 1}{2} \right)^{2 - \alpha^+} \frac{h(\alpha,1)}{h(\alpha,\kappa + 1)}.
\]

(The formulae in \([2, \text{Chapter 9}] \) are given for the p-monic polynomials, here we use the x-monic type, see \([2, \text{pp. 323–324}] \). There is an evaluation formula for \( \zeta_\alpha (1,1,1) \) which provides the value at \( x = (2,0,0,0) \), corresponding to \( y = (1,1,1,1) \). Indeed for \( \alpha \in \mathbb{N}_0^3 \) (see \([2, \text{p. 324}] \))

\[
\zeta_\alpha (1,1,1) = \left( \frac{3 \kappa + 1}{2} \right)_{\alpha^+} h(\alpha,\kappa + 1).
\]

For any point \( (\pm 2,0,0,0) \) \( w \) with \( w \in S_4 \) the corresponding \( y \) satisfies \( y_i = \pm 1 \) for \( 1 \leq i \leq 3 \), so that \( y^2 = (1,1,1) \). For any other subset \( E \subset \{1,2,3\} \) with \#\(E = k \) let \( w \in S_3 \) be such that \( w(i) \in E \) for \( 1 \leq i \leq k \), \( 1 \leq i < j \leq k \) or \( k < i < j \leq 3 \) implies \( w(i) < w(j) \) (that is, \( w \) preserves order on \( \{1,\ldots,k\} \) and on \( \{k+1,\ldots,3\} \)). Here is the list of sets with corresponding permutations \((w(i))^3_{i=1}\):

\[
E = \{2\}, \quad w = (2,1,3),
E = \{3\}, \quad w = (3,1,2),
E = \{1,3\}, \quad w = (1,3,2),
E = \{2,3\}, \quad w = (2,3,1).
\]

Then (letting \( w \) act on \( y \)) \( w y_{E_k} = y_E \) and for \( \alpha \in \mathbb{N}_0^3 \) the polynomial \( w(y_{E_k} \zeta_\alpha (y^2)) \) is a simultaneous eigenfunction and

\[
U_i^B w y_{E_k} \zeta_\alpha (y^2) = 2 \xi_i (\alpha) w y_{E_k} \zeta_\alpha (y^2), \quad 1 \leq i \leq k,
\]

\[
U_i^B w y_{E_k} \zeta_\alpha (y^2) = (2 \xi_i (\alpha) - 1) w y_{E_k} \zeta_\alpha (y^2), \quad k < i \leq 3.
\]
Define $\beta$ as before ($\beta_i = 2\alpha_i + 1$ for $1 \leq i \leq k$ and $\beta_i = 2\alpha_i$ for $k < i \leq 3$) then the label for the polynomial $w_{yE_k} \zeta_\alpha (y^2)$ is $w \beta$ (recall $(w \beta)_i = \beta_{w^{-1}(i)}$). Denote

$$p_{w \beta} (y) := w_{yE_k} \zeta_\alpha (y^2).$$

This defines a polynomial $p_\gamma$ for any $\gamma \in \mathbb{N}_0^3$. The norm of $w_{yE_k} \zeta_\alpha (y^2)$ is the same as that of $y_{E_k} \zeta_\alpha (y^2)$ since any $w \in S_3$ acts as an isometry for $\langle \cdot , \cdot \rangle_\kappa$. Suppose $E, E' \subset \{1, 2, 3\}$ and $E \neq E'$ and $f, g \in \mathcal{P}^{(3)}$ then $\langle y_{E} \zeta_\alpha (y^2), y_{E'} \zeta_\alpha (y^2) \rangle_\kappa = 0$. The root system $D_3$ is an orbit of the subgroup of diagonal elements of $B_3$ (isomorphic to $\mathbb{Z}_2^3$). Denote the sign change $y_i \rightarrow -y_i$ by $\sigma_i$ for $1 \leq i \leq 3$. From the $B_3$ results we have $\sigma_i D^B_j = D^B_j \sigma_i$ for $1 \leq i, j \leq 3$ and this implies $\langle y_{E} \zeta_\alpha (y^2), y_{E'} \zeta_\alpha (y^2) \rangle_\kappa = \langle \sigma_i y_{E} \zeta_\alpha (y^2), \sigma_i y_{E'} \zeta_\alpha (y^2) \rangle_\kappa = -\langle y_{E} \zeta_\alpha (y^2), y_{E'} \zeta_\alpha (y^2) \rangle_\kappa$ for any $i \in (E, E') \cup (E' \setminus E)$ (the symmetric difference). Thus $\{ p_\gamma : \gamma \in \mathbb{N}_0^3 \}$ is an orthogonal basis for $\langle \cdot , \cdot \rangle_\kappa$.

We consider the $S_4$-invariant polynomials: they are generated by $y_0, \sum_{i=1}^3 y_i^2, y_1 y_2 y_3, \sum_{i=1}^3 y_i^4$. Any invariant is a sum of terms of the form $y_0^n (y_1 y_2 y_3)^s f(y^2)$ where $n \in \mathbb{N}_0, s = 0$ or $1$, and $f$ is a symmetric polynomial in three variables. For now consider only polynomials in $\{ y_1, y_2, y_3 \}$. Let $\lambda \in \mathbb{N}_0^{3,+}$, then there are two corresponding simultaneous eigenfunctions of $\sum_{i=1}^3 (U_i^B)^n$ (it suffices to take $n = 1, 2, 3$ to generate the commutative algebra of $S_4$-invariant operators). From [2, Theorem 8.5.10] let

$$A_\lambda = \# \{ \alpha : \alpha^+ = \lambda \} \frac{(3\kappa + 1)\lambda}{\mathcal{S}_1(\lambda^R)} \frac{\kappa (\lambda, 1)}{\kappa (\lambda, \kappa + 1)},$$

$$F^0_\lambda (x) = j \lambda (y^2),$$

$$\langle F^0_\lambda, F^0_\lambda \rangle_\kappa = 2^{2\lambda |\lambda|} \left( \frac{2\kappa + 1}{2} \right) A_\lambda,$$

$$F^1_\lambda (x) = y_1 y_2 y_3 j \lambda (y^2),$$

$$\langle F^1_\lambda, F^1_\lambda \rangle_\kappa = 2^{2\lambda |\lambda|} \left( \frac{2\kappa + 1}{2} \right) \frac{1}{(\lambda_1 + 1, \lambda_2 + 1, \lambda_3 + 1)} A_\lambda.$$

The polynomials $\{ F^0_\lambda, F^1_\lambda : \lambda \in \mathbb{N}^{3,+}_0 \}$ are pairwise orthogonal.

Up to now we have mostly ignored the fourth dimension, namely, the coordinate $y_0$. The reflection $\sigma_0$ along $v_0$ (given by $x \sigma_0 = x - (\sum_{i=1}^4 x_i) v_0$) commutes with the $S_4$-action. We introduce another parameter $\kappa'$ and let

$$D_0 f (x) = \frac{1}{2} \sum_{i=1}^4 \frac{\partial}{\partial x_i} f (x) + \frac{\kappa'}{\langle x, v_0 \rangle} (f (x) - f (x \sigma_0)), $$

$$D'_i f (x) = D_i f (x) + \frac{\kappa'}{2 \langle x, v_0 \rangle} (f (x) - f (x \sigma_0)). $$

The operators $\{ D'_i : 1 \leq i \leq 4 \}$ are the Dunkl operators for the group $W = S_4 \times \mathbb{Z}_2$ (the reflection group generated by $\{(1, 2), (2, 3), (3, 4), \sigma_0\}$. Then $D_0 y_0^{2n} = 2n y_0^{2n-1}$ and $D_0 y_0^{2n+1} = (2n + 1 + 2\kappa') y_0^{2n}$. We define the extended pairing for polynomials

$$\langle f (x), g (x) \rangle_{\kappa, \kappa'} = f (D'_1, \ldots, D'_4) g (x) |_{x=0};$$

in terms of $y$

$$\langle f_0 (y_0), f_1 (y_1, y_2, y_3), g_0 (y_0), g_1 (y_1, y_2, y_3) \rangle_{\kappa, \kappa'}$$

$$= f_0 (D_0) g_0 (y_0) |_{y_0=0} \times f_1 (D^B_1, \ldots) g_1 (y_1, y_2, y_3) |_{y=0}$$

$$= f_0 (D_0) g_0 (y_0) |_{y_0=0} \times \langle f_1, g_1 \rangle_\kappa.$$
It is easily shown by induction that for $n \in \mathbb{N}_0$

$$\langle y_0^{2n}, y_0^{2n} \rangle_{\kappa, \kappa'} = 2^{2n} n! \left( \frac{\kappa' + 1}{2} \right)_n,$$

$$\langle y_0^{2n+1}, y_0^{2n+1} \rangle_{\kappa, \kappa'} = 2^{2n+1} n! \left( \frac{\kappa' + 1}{2} \right)_{n+1}.$$

The direct product structure implies that $\{ p_{(\gamma_1, \gamma_2, \gamma_3)} (y) y_0^n : \gamma \in \mathbb{N}_0^3 \}$ is an orthogonal basis for $\langle \cdot, \cdot \rangle_{\kappa, \kappa'}$.

4 Hermite polynomials

The pairing $\langle \cdot, \cdot \rangle_{\kappa, \kappa'}$ is related to a measure on $\mathbb{R}^4$: let $\kappa, \kappa' \geq 0$ and

$$dm (x) := (2\pi)^{-2} \exp \left( -\frac{1}{2} |x|^2 \right) dx, \quad x \in \mathbb{R}^4,$$

$$h (x) := \prod_{1 \leq i < j \leq 4} |x_i - x_j|^{\kappa} |y_0|^{\kappa'},$$

$$c_{\kappa, \kappa'}^{-1} := \int_{\mathbb{R}^4} h (x)^2 dm (x),$$

$$d\mu_{\kappa, \kappa'} (x) := c_{\kappa, \kappa'} h (x)^2 dm (x).$$

In fact

$$c_{\kappa, \kappa'}^{-1} = 2^{\kappa'} \Gamma \left( \frac{\kappa' + 1}{2} \right) \Gamma (2\kappa + 1) \Gamma (3\kappa + 1) \Gamma (4\kappa + 1) \Gamma (\frac{1}{2}) \Gamma (\kappa + 1)^3.$$

The integral is a special case of the general formula (any suitably integrable function $f$ on $\mathbb{R}$):

$$(2\pi)^{-N/2} \int_{\mathbb{R}^N} \prod_{1 \leq i < j \leq N} |x_i - x_j|^{2\kappa} f \left( \sum_{i=1}^N x_i \right) \exp \left( -\frac{1}{2} |x|^2 \right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f \left( t\sqrt{N} \right) e^{-t^2/2} dt \cdot \prod_{j=2}^N \frac{\Gamma (j\kappa + 1)}{\Gamma (j\kappa + 1)};$$

this follows from the Macdonald–Mehta–Selberg integral for $S_N$ and the use of an orthogonal coordinate system for $\mathbb{R}^N$ in which $\sum_{i=1}^N x_i/\sqrt{N}$ is one of the coordinates. The Laplacian is $\Delta_h := \sum_{i=1}^4 (D_i^h)^2 = \sum_{i=1}^3 (D_i^B)^2 + D_0^2$. Also set $\Delta_B := \sum_{i=1}^3 (D_i^B)^2$. Then for $f, g \in \mathcal{P}$ [2, Theorem 5.2.7]

$$\langle f, g \rangle_{\kappa, \kappa'} = \int_{\mathbb{R}^4} \left( e^{-\Delta_h/2} f (x) \right) \left( e^{-\Delta_h/2} g (x) \right) d\mu_{\kappa, \kappa'} (x).$$

The orthogonal basis elements $p_{\gamma} (y) y_0^n (\gamma \in \mathbb{N}_0^3, n \in \mathbb{N}_0)$ are transformed to orthogonal polynomials in $L^2 (\mathbb{R}^4, \mu_{\kappa, \kappa'})$ under the action of $e^{-\Delta_h/2} = \sum_{n=0}^\infty \frac{1}{n!} \left( -\frac{1}{2} \right)^n \Delta_h^n$ (only finitely many terms are nonzero when acting on a polynomial). We have

$$e^{-\Delta_h/2} p_{\gamma} (y) y_0^n = \left( e^{-\Delta_B/2} p_{\gamma} (y) \right) \left( e^{-\Delta_0^2/2} y_0^n \right).$$
Then for \( n \in \mathbb{N}_0 \)
\[
e^{-D_0^2/2} y_0^{2n} = (-2)^n n! L_n^{\kappa - \frac{1}{2}} \left( \frac{y_0^2}{2} \right),
\]
\[
e^{-D_0^2/2} y_0^{2n+1} = (-2)^n n! y_0 L_n^{\kappa + \frac{1}{2}} \left( \frac{y_0^2}{2} \right).
\]

Recall the Laguerre polynomials \( L_n^\alpha \) (see \([2, \text{Proposition 9.4.5}]\)). For the symmetric cases \( j\lambda (y^2) \) and \( y_1 y_2 y_3 j\lambda (y^2) \), \( \lambda \in \mathbb{N}_0^3 \) these coefficients were investigated by Lassalle \([4] \) and Okounkov and Olshanski \([5, \text{equation (3.2)}] \); in the latter paper there is an explicit formula.

Finally we can use our orthogonal basis to analyze a modification of the type-\( A \) quantum Calogero–Sutherland model with four particles on a line and harmonic confinement. By rescaling, the Hamiltonian (with exchange terms) can be written as:
\[
\mathcal{H} = -\Delta + \frac{|x|^2}{4} + 2\kappa \sum_{1 \leq i < j \leq 4} \frac{\kappa - (i,j)}{(x_i - x_j)^2} + \frac{4\kappa'(\kappa' - \sigma_0)}{(x_1 + x_2 + x_3 + x_4)^2}.
\]

When this is applied to a \( W \)-invariant the reflections \((i,j)\) and \( \sigma_0 \) are replaced by the scalar 1. We combine the type-\( B \) results from \([2, \text{Section 9.6.5}] \) (setting \( \kappa' = 0 \) in the formulae) with simple \( \mathbb{Z}_2 \) calculations. The nonnormalized base state is
\[
\psi_0 (x) := \prod_{1 \leq i < j \leq 4} |x_i - x_j|^\kappa |y_0|^\kappa' \exp \left( -\frac{1}{4} |x|^2 \right).
\]

Then
\[
\psi_0^{-1} \mathcal{H} \psi_0 = -\Delta_B - D_0^2 + \sum_{i=0}^3 y_i \frac{\partial}{\partial y_i} + 6\kappa + \kappa' + 2.
\]

This operator has polynomial eigenfunctions and the eigenvalues are the energy levels of the associated states. From \([2, \text{Section 9.6.5}] \) we have
\[
e^{-\Delta_B/2} \sum_{i=1}^3 U_i^B e^{\Delta_B/2} = -\Delta_B + \sum_{i=1}^3 y_i \frac{\partial}{\partial y_i} + 6\kappa + 3,
\]
and by direct calculations
\[
D_0^2 = \frac{\partial^2}{\partial y_0^2} + \frac{2\kappa'}{y_0} \frac{\partial}{\partial y_0} - \kappa' \frac{1 - \sigma_0}{y_0^2},
\]
\[
e^{-D_0^2/2} (D_0 y_0 - \kappa' \sigma_0) e^{D_0^2/2} = -D_0^2 + y_0 \frac{\partial}{\partial y_0} + \kappa' + 1.
\]

Combine these results:
\[
\psi_0^{-1} \mathcal{H} \psi_0 = e^{-\Delta_h/2} \left( \sum_{i=1}^3 U_i^B + D_0 y_0 - \kappa' \sigma_0 - 2 \right) e^{\Delta_h/2}.
\]
Thus \( \left( e^{-\Delta h/2} (p_{\gamma}(y) y^0) \right) \psi_0 \) is an eigenfunction of \( \mathcal{H} \) for each \( \gamma \in \mathbb{N}_0^3, n \in \mathbb{N}_0 \). It suffices to consider \( y_{E_k} \zeta_\alpha \left( y^2 \right) y^0_n \). We have
\[
(D_0 y_0 - \kappa' \sigma_0) y^0_n = \left( (2n + 1 + 2\kappa') - \kappa' \right) y^0_n,
\]
\[
(D_0 y_0 - \kappa' \sigma_0) y^0_{n+1} = \left( (2n + 2 + \kappa') \right) y^0_n,
\]
\[
(D_0 y_0 - \kappa' \sigma_0) y^0_n = (n + 1 + \kappa') y^0_n.
\]
Furthermore \( \sum_{i=1}^3 U^B_i \left( y_{E_k} \zeta_\alpha \left( y^2 \right) \right) = \left( 2 \sum_{i=1}^3 \xi_i (\alpha) - (3 - k) \right) y_{E_k} \zeta_\alpha \left( y^2 \right) \); the eigenvalue is \( (2 |\alpha| + k) + 6 \kappa + 3 = |\beta| + 6 \kappa + 3 \) (where \( \beta_i = 2\alpha_i + 1 \) for \( 1 \leq i \leq k \) and \( \beta_i = 2\alpha_i \) for \( k < i \leq 3 \)). The energy level for \( \left( e^{-\Delta h/2} (p_{\beta}(y) y^0_0) \right) \psi_0 \) is \( |\beta| + n + 6 \kappa + \kappa' + 2 \). Observe the degeneracy of the energy levels; only the total degree \( |\beta| + n \) appears. The (nonnormalized) \( W \)-invariant eigenfunctions are \( (\lambda \in \mathbb{N}_0^3) \)
\[
\left( e^{-\Delta B/2} (j_\lambda \left( y^2 \right) \right) L^{\kappa' - 1/2} \left( \frac{y^0_0}{2} \right) \right) \psi_0 \left( x \right),
\]
\[
\left( e^{-\Delta B/2} (y_1 y_2 y_3 j_\lambda \left( y^2 \right) \right) L^{\kappa' - 1/2} \left( \frac{y^0_0}{2} \right) \right) \psi_0 \left( x \right).
\]
The \( L^2 \)-norms can be found by using equation (2).

In conclusion, we have found an unusual basis for polynomials which allowed an extra parameter in the action of \( S_4 \) on \( \mathbb{R}^4 \). This exploited the fact that \( v^+_0 \) has an orthogonal basis which together with its antipodes forms an \( S_4 \)-orbit. The pairing \( \langle \cdot, \cdot \rangle_\kappa \) has an analog for each reflection group and weight function. We are left with the interesting problem of how to construct orthogonal bases for groups not of type \( A \) or \( B \).

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