Brane Worlds and the Cosmic Coincidence Problem

Massimo Pietroni

INFN – Sezione di Padova,
Via F. Marzolo 8, I-35131 Padova, Italy

Brane world models with ‘large’ extra dimensions with radii in the \( r_i \sim 10 - 100 \ \mu m \) range and smaller ones at \( r_s \lesssim (1 \text{ TeV})^{-1} \) have the potential to solve the cosmic coincidence problem, i.e. the apparently fortuitous equality between dark matter and dark energy components today. The main ingredient is the assumption of a stabilization mechanism fixing the total volume of the compact submanifold, but allowing for shape deformations. The latter are associated with phenomenologically safe ultra-light scalar fields. Bulk fields Casimir energy naturally plays the role of dark energy, which decreases in time because of expanding \( r_i \). Stable Kaluza Klein states may play the role of dark matter with increasing, \( O(1/r_s) \), mass. The cosmological equations exhibit attractor solutions in which the global equation of state is negative, the ratio between dark energy and dark matter is constant and the observed value of the ratio is obtained for two large extra dimensions.

Experimental searches of large extra dimensions should take into account that, due to the strong coupling between dark matter and radii dynamics, the size of the large extra dimensions inside the galactic halo may be smaller than the average value.

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I. INTRODUCTION

The energy content of the Universe is nowadays, more than ever, the subject of intense research. The picture emerging is the following [1]: the total energy density of the Universe equals the critical value \( \rho_c \simeq (2 \times 10^{-3} \text{ eV})^4 \simeq (100 \mu m)^{-4} \) (i.e. \( \Omega \simeq 1 \)), non-relativistic dark matter (DM) accounts for at most forty percent of it (\( \Omega_{DM} \simeq 0.2 - 0.4 \)), and the rest is in the form of a smooth - i.e. non-clusterized - component, named Dark Energy (DE) (plus a small amount of baryons, \( \Omega_b \simeq 0.05 \)). Luminosity-redshift measurements of supernovae Ia [2], as well as the resolution of the so called age problem [3], agree with this picture and point towards a DE with negative equation of state \( W = p/\rho \), where \( p \) and \( \rho \) are pressure and energy density, respectively. A viable DE candidate is then Einstein’s cosmological constant \( (W = -1) \), but more general fluids with \( W < 0 \) have been considered, emerging, for instance, from the dynamics of scalar fields with appropriate potentials. The old cosmological constant problem [4] now becomes double-faced. On one hand we still have the old puzzle of the cosmological constant – or the scalar field potential– being so tiny compared to any common particle physics scale. On the other hand – and this is the new face of the problem– the presence of DE today in roughly the same amount as DM energy density is embarrassing. Indeed, the ratio between the two energy densities scales as

\[
\frac{\rho_{DM}}{\rho_{DE}} \sim a^{3W},
\]

where \( a \) is the scale factor of the Universe. So, the approximate equality between the two components today looks quite an amazing coincidence in the cosmic history if DE has \( W \neq 0 \) as required by observations.

In principle, brane world scenarios offer a suggestive contact with the DE problem. Indeed, extra spatial dimensions compactified to a size as large as 100\( \mu m \) have been shown to be a viable possibility, provided that no Standard Model (SM) field propagates through them [5]. It might then be tempting to attribute the observed value of the DE energy density to the Casimir energy associated to some field propagating in these large extra dimensions of size \( r \), i.e. \( \rho_{DE} \simeq B/r^4 \), with a constant typically \( O(10^{-5} - 10^{-4}) \) (see sect. IV).

If the radius \( r \) is stabilized, then the energy density behaves exactly as a cosmological constant, and we have made no substantial progress with respect to any of the two problems mentioned above. Indeed, we have no clue about why the Casimir energy would be the only non-vanishing contribution to the cosmological constant and, moreover, the equality between the latter and the matter energy density would appear as fortuitous as before.

In order to gain some insight at least on the cosmic coincidence problem it seems then necessary to make the radius \( r \) dynamical, so that it may evolve on a cosmological time-scale. But this turns out to be quite dangerous. The ‘radion’ field, whose expectation value fixes \( r \), couples to the trace of the SM energy momentum tensor with gravitational strength. Moreover, in order to evolve with \( \dot r/r \sim H \) (\( H \) being the Hubble parameter), it must be extremely light, \( m \simeq H_0 \simeq 10^{-33} \) eV. It then behaves as a massless Jordan-Brans-Dicke scalar with \( O(1) \) couplings to matter, whereas present bounds are \( O(10^{-3}) \) [6]. Thus, the geometric explanation of a dynamical DE looks quite unlikely (see however [7]).

The purpose of this paper is to show that the above conclusion is indeed premature, and that standard brane world scenarios with large extra dimensions may quite naturally solve the cosmic coincidence problem. Unfortunately, we will add nothing new with respect to the first problem, that of the vanishing of all present contributions...
to the cosmological constant larger than $O(100\,\mu m)^{-4}$.

The key point in our discussion is the realization that besides large extra-dimensions in the $10 - 100\,\mu m$ range there could also be smaller ones. Considering for simplicity only two compact subspaces, each characterized by a single radius, we note that the trace of the four-dimensional energy-momentum tensor couples with the total volume of the compact space, that is with the combination $r_n^s r_l^n$, where $r_n$ and $r_l$ are the radii and dimensionalities of the two subvolumes, respectively. If we assume a stabilization mechanism that fixes the total volume $V$ of the compactified manifold but not its shape, the potentially dangerous combination of radion fields associated with volume fluctuations is made harmless, whereas the orthogonal one, associated with shape deformation, is decoupled from normal matter and may then be ultra-light. As a consequence, $r_l$ can grow on a cosmological time-scale (such that the associated Casimir energy $B/r_l^4$ decreases) and at the same time $r_s$ shrinks so that $V$ keeps a fixed value.

The coincidence problem is solved if we associate the dark matter with some stable state living in the extra dimensions, because the resulting cosmological equations exhibit an attractor solution in which the Casimir energy and $\rho_{DM}$ are redshifted at the same rate with their ratios fixed at a $O(1)$ value.

If the DM candidate has a mass independent on $r_s$ then the attractor corresponds to the equation of state of non-relativistic matter, i.e. $W = 0$, which is disfavoured by cosmological observations.

On the other hand, if the mass of the DM particle is inversely proportional to some power of $r_s$, as for a Kaluza-Klein (KK) state ($m_{DM} = m_{KK} \sim r_s^{-1}$), then a more interesting behavior is obtained. Indeed, on the attractor now we have

$$\rho_{DM} \sim 1/(r_s a^3) \sim \rho_{DE};$$

which, due to the shrinking of $r_s$, decreases slower than $a^{-3}$ during the cosmic expansion. Remembering that a fluid with equation of state $W$ scales as $\rho \sim a^{-3(W+1)}$ we see that the energy densities scale now with a negative effective equation of state. Moreover, the values of $W$, $\Omega_{DM}$, and $\Omega_{DE}$ depend only on the two parameters, $n_s$ and $n_l$.

The above results rely on naive dimensional reduction, without assuming drastic modifications to the radion kinetic terms as was done for instance in [7].

For sake of clarity, we summarize our assumptions here. They are:

1) the vanishing of any contributions to the effective 4-dimensional cosmological constant larger than $O(100\,\mu m)^{-4}$ today;

2) a stabilization mechanism for the total volume (rather than for each single radius separately) of the compactified space effective at low energies.

The paper is organized as follows. In sect. II we introduce our setting and derive the effective action after dimensional reduction. In sect. III we discuss a mechanism to stabilize the volume of the compact manifold, allowing at the same time its shape to vary. In sect. IV we will study the dynamics of the modulus associated to shape deformations of the compact manifold, showing that an (attractor) solution exists on which dark energy and dark matter scale at the same ratio and the universe has a negative effective equation of state. Finally, in sect. V we discuss our results.

II. THE SETTING

The starting point is the action

$$S = S_{bulk} + S_{4+n_s} + S_4. \quad (1)$$

The bulk action is given by

$$S_{bulk} = \int d^{4+\mathcal{N}}X \sqrt{-g} \left[ \frac{\mathcal{R}(G_{AB})}{16\pi G_{4+\mathcal{N}}} + \Lambda_B + \ldots \right], \quad (2)$$

where $\mathcal{N} = n_s + n_l$ and the $4 + \mathcal{N}$-dimensional metric is given by

$$ds^2 = G_{AB} dx^A dx^B = g_{\mu\nu} dx^\mu dx^\nu + \sum_{i=s,l} b_i(x)^2 \gamma_{i\alpha\beta} dy^\alpha dy^\beta, \quad (3)$$

where $\mu, \nu = 0, \ldots, 3$, $\alpha, \beta = 1, \ldots, n_i$, and $b_i(x) = r_i(x)/r_0^i$, with $r_0^i$'s the average values of the radii today. $\Lambda_B$ is the bulk cosmological constant, whereas the dots in eq. (2) represent extra fields living in the bulk which will contribute to the stabilization of the compact volume, as we will discuss in the next section.

Apart from the overall $b_i$'s factors, the metrics $\gamma^{(i)}$'s are assumed to be non-dynamical.

$S_{4+n_s}$ is the action for the fields living in all the $4 + n_s$ dimensions. In the following, we will only assume that there is at least one stable KK state, which we will associate to DM.

The SM fields are confined to our four-dimensional brane, with action given by

$$S_4 = \int d^4x \sqrt{-g} [\mathcal{L}_{SM}(g_{\mu\nu}, \psi) + \Lambda_4], \quad (4)$$

where $\psi$ represent SM fields and we have singled out the contributions to the cosmological constant from the brane, $\Lambda_4$. The dimensional reduction of $S_{bulk}$ is a straightforward procedure (see for instance [8]). We define new `radion' variables, $\phi_a(x)$, as $b_i(x) = A_{ia} \phi_a$ with $i = s, l$, $a = 1, 2$, and

$$A = \frac{1}{\sqrt{\mathcal{N}}} \left[ \sqrt{\frac{\mathcal{N}^2}{2}} \sqrt{\frac{n_s}{n_l}} \sqrt{\frac{\mathcal{N}^2}{2}} - \sqrt{\frac{n_s}{n_l}} \right]. \quad (5)$$

By also rescaling the 4-dimensional metric as
\[ g_{\mu\nu} = e^{-C_N\phi_1} \delta_{\mu\nu}, \]  

where \( C_N = \sqrt{2N/(2 + N)} \), we obtain the ‘Einstein frame’ action,

\[ S_{\text{bulk}} = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{1}{16\pi G_4} \tilde{R}(\tilde{g}_{\mu\nu}) \right. \]
\[ -\frac{M_4^2}{2} \sum_{a=1}^2 \tilde{g}^{\mu\nu} \partial_\mu \phi_a \partial_\nu \phi_a + e^{-C_N\phi_1} \mathcal{V}(\phi) + \left. \frac{1}{4g_4^2} F_{AB} F^{AB} \right], \]

where we assumed that both \( R(\gamma^{(i)}) = 0 \), as in toroidal compactifications, and \( \mathcal{V} = G_{4+N}/G_4 \) is the volume of the compact manifold today.

In the same frame, \( S_4 \) takes the form

\[ S_4 = \int d^4x \sqrt{-g} e^{-2C_N\phi_1} \left[ \mathcal{L}_{\text{SM}}(e^{-C_N\phi_1} \tilde{g}_{\mu\nu}, \psi) \right. \]
\[ \left. + \mathcal{L}_4 \right], \]

from which we see that only \( \phi_1 \), which we might call ‘the volumon’, couples to matter. Indeed, from eq. (5) one realizes that \( \phi_1 \) is the scalar field controlling volume deformations, since

\[ e^{C_N\phi_1} = b_a^s b_1^t. \]

The orthogonal combination, \( \phi_2 \), controls shape deformations of the compact manifold and does not couple to SM fields directly. On the other hand, it couples to fields living in the bulk, or on the \( 4 + n_s \)-dimensional brane. In the 4-dimensional language, \( \phi_2 \) couples – besides the graviton– to non-zero KK states, since their mass terms are proportional to \( 1/r_i \sim \exp(-A_{<2}\phi_2) \) \((i = s, l)\).

### III. VOLUME STABILIZATION

The field \( \phi_1 \) couples to the SM as a Jordan-Brans-Dicke scalar, i.e. in a purely metric way. As is well known, this type of couplings enforces the (weak) equivalence principle, but is constrained from gravity tests in the solar system and in the laboratory [6]. A massless \( \phi_1 \) would modify gravity at an unacceptable level unless \( C_N < O(10^{-3}) \), which is clearly ruled out in the present model, as \( C_N \) ranges from 1 \((n_s = n_l = 1) \) to \( \sqrt{2} \) \((n_s + n_l \to \infty) \). It is then necessary to give a mass to \( \phi_1 \), that is, to stabilize the volume of the compact manifold. In order to fulfill the experimental bounds, the \( \phi_1 \) range should be smaller than about a hundred microns, that is, \( m_{\phi_1} \geq 10^{-3}\text{eV} \). At the same time, we would like to keep \( \phi_2 \) massless such that it may play the role of dynamical dark energy today, that is, \( m_{\phi_2} \lesssim H_0 \sim 10^{-33}\text{eV} \). Thus, we are looking for a stabilization mechanism which fixes only the total volume of the compact manifold, allowing the different dimensions to vary accordingly.

A mechanism with this property has been discussed for instance in refs. [9], and is based on the introduction of a \( U(1) \) gauge monopole configuration in the extra dimensions, besides the bulk cosmological constant \( \Lambda_B \) and the ‘brane tension’, \( \Lambda_4 \), that we have already introduced.

For definitness, we will first consider a two dimensional torus of radii \( b_l \) and \( b_s \), and then we will discuss the extension of the mechanism to higher dimensional manifolds. The key point is that \( U(1) \) gauge fields can have non-zero magnetic flux through the two-dimensional surface, and the flux is quantized and topological invariant,

\[ \Phi = \int d^2y \epsilon^{mn} F_{mn} = 2\pi k, \]

where \( m, n = 4, 5 \), \( \epsilon^{45} = -\epsilon^{54} = 1 \), and \( k \) is an integer. Adding the gauge kinetic term,

\[ \int d^{4+2}x \sqrt{-g} \frac{M_4^2}{4g_4^2} F_{AB} F^{AB} \]

\((A, B = 0, \ldots, 5)\), to \( S_{\text{bulk}} \) and solving the Euler-Lagrange equations for \( F_{AB} \), a solution can be found of the form

\[ F_{\mu\nu} = F_{\mu\mu} = 0, \quad F_{mn} = \frac{\epsilon_{mn}B}{b_rb_s}, \]

with \( \partial_\mu B = 0 \). \( M_* \) in (11) is a mass scale typically \( O(G_5^{-1/4}) \). Inserting this solution in eq. (10) we find that flux quantization forces the electromagnetic field to be inversely proportional to the 2-dimensional volume,

\[ B = \frac{k\pi}{\sqrt{b_rb_s}}, \]

so that eq. (11) becomes

\[ \frac{k^2\pi^2 M_4^2}{2g_4^2} \int d^4x \sqrt{-g} e^{-3C_2\phi_1}. \]

Like the terms proportional to \( \Lambda_B \) in eq. (7), and to \( \Lambda_4 \) in eq. (8), eq. (12) is only sensitive to the volume, that is to \( \phi_1 \), and not to the shape modulus, \( \phi_2 \). These contributions give the effective potential for \( \phi_1 \) in the Einstein frame, which is the appropriate quantity to minimize in order to study stabilization*, i.e.

\[ V(\phi_1) = D e^{-C_2\phi_1} + A e^{-3C_2\phi_1} + \Lambda_4 e^{-2C_2\phi_1}, \]

with

\[ A = \frac{k^2\pi^2 M_4^2}{2g_4^2 M_*^2}, \quad D = \mathcal{V}\Lambda_B. \]

*In the second of refs. [9] they minimized the effective potential in the frame with metric \( \tilde{g}_{\mu\nu} \), where the field \( \phi_1 \) appears also in front of the Ricci scalar, which should be taken in into account in the minimization. As a consequence, their results on the mass of the scalar field differ significantly from ours.
Notice that $A > 0$, which ensures that the potential is bounded from below. Requiring that the contribution of the potential to the cosmological constant vanishes in the minimum, we get the fine tuning condition
\[ \Lambda_4^2 = 4AD, \]
which forces $D > 0$, and then $\Lambda_4 > 0$. The potential has a minimum only if $\Lambda_4 < 0$, that is, for $\Lambda_4 = -2\sqrt{AD}$, in
\[ \phi_1^{\text{min}} = \frac{1}{C_2} \log \sqrt{\frac{A}{D}}. \]
On this minimum, the scalar field $\phi_1$ has a mass
\[ m_{\phi_1}^2 = V''(\phi_1^{\text{min}})/M_p^2 \]
\[ = \frac{2^{3/2}C^2}{k\pi}(r_0^0 M_s)^2 (r_1^0 M_s)^2 \left( \frac{\Lambda_B}{M_s^4} \right)^{3/2} \frac{M_s^4}{M_p^2}. \]
The above mechanism can be generalized to compact spaces of higher dimensions. In order to stabilize the volume of a $N$-dimensional manifold, one needs a $N-1$-form $U(1)$ gauge field, with $N$-form field strength $F^{(N)}$. Then, the flux (10) generalizes to the topological invariant
\[ M_s^{-2} \int_{M_N} F^{(N)} = 2\pi k, \]
and the kinetic energy is again inversely proportional to the compact volume [9]. Considering again toroidal compactifications, the $N$-dimensional case gives a scalar mass
\[ m_{\phi_1}^2 \lesssim \frac{2^{3/2}C^2}{k\pi} (r_s^0 M_s)^{3n_s/n_l} (r_l^0 M_s)^{2n_l} \left( \frac{\Lambda_B}{M_s^4} \right)^{3/2} \frac{M_s^4}{M_p^2}, \]
where we have taken $k = 1$. Assuming $r_s M_s \gtrsim 1$ and $M_s \gtrsim 10 \text{TeV}$ we see that the requirement $m_{\phi_1} \gtrsim 10^{-3} \text{eV}$ is in no conflict with the bound above for any $n_l \geq 2$.

Since $m_{\phi_1} > H$ ($H$ being the Hubble parameter) for temperatures $T \lesssim M_s$, in the following we will consider the field $\phi_1$ fixed during the late time evolution of the Universe, and will study the dynamics of the light $\phi_2$ field. When $\phi_1$ is constant and massive, the two frames related by eq. (6) are equivalent.

### IV. VAMPS FROM EXTRA-DIMENSIONS

Integrating the non-zero KK modes out leaves ‘Casimir’ contributions to the free energy proportional to $1/r_s^4$. For instance, a scalar field compactified on a circle of radius $r$ with periodic (antiperiodic) boundary conditions gives a contribution $-5.06 \cdot 10^{-5}/r^4$ ($4.74 \cdot 10^{-5}/r^4$) [11], whereas a fermion gives $2.02 \cdot 10^{-4}/r^4$ ($-1.90 \cdot 10^{-4}/r^4$). For higher dimensional tori, the numerical coefficients change, but the order of magnitude is the same (on a two-torus with $r_1 = r_2 = r$, we find $-4.87 \cdot 10^{-5}/r^4$ ($1.95 \cdot 10^{-4}/r^4$) for a scalar (fermion) with periodic boundary conditions.

Consistently with our assumption 1) in the Introduction, we will neglect the $O(r_s^{-4})$ term, since it must be cancelled by the (unknown) mechanism solving the cosmological constant problem. A possible mechanism could be supersymmetry in the bulk or, even without supersymmetry, a particle content in the $4 + N$-dimensional bulk, which we require to be positive.

As we have anticipated, the other ingredient of our model is dark matter. We associate it to a stable KK brane, cancelling exactly the $1/r_s^4$ contribution, in the same spirit of ref. [12].

Thus, we are left with the Casimir contribution from the ‘large’ dimensions, given by
\[ V(\phi_2) = \frac{B}{r_1^4} = \frac{B}{(r_1^0)^4} \exp \left( 4\sqrt{\frac{n_s}{n_l} \phi_2} \right), \]
where we have fixed $\phi_1 = 0$ and $B$ is a $O(10^{-4} - 10^{-5})$ coefficient depending on the particle content and the dimensionality of the $4 + N$-dimensional bulk, which we require to be positive.

As we have anticipated, the other ingredient of our model is dark matter. We associate it to a stable KK state, whose mass scales as $O(1/r_s)$. It will be an example of varying mass dark matter particles, or Vamps, which were discussed in a different context in refs. [10].

The important point is that the cosmological abundance of this non-relativistic relic will scale in this case as
\[ \rho_{DM} \sim r_s^{-1} a^{-3} \sim \exp \left( -\sqrt{\frac{n_l}{n_s N}} \phi_2 \right) a^{-3}. \]

Since the runaway potential, eq. (18), pushes the field $\phi_2$ to $-\infty$, the mass of the dark-matter particle increases during the cosmological expansion, and its energy density redshifts less than for common dark matter.

The fact that $\phi_2$ is the canonically normalized version of a radion field is the reason for the exponential dependences in (18) and (19). This is of crucial importance for what follows, because exponentials, once inserted in the cosmological equations, allow scaling solutions, in which $\rho_{DE}$ and $\rho_{DM}$ redshift at the same rate.

Defining
\[ \beta = 4 \sqrt{\frac{n_s}{n_l N}}, \quad \lambda = \sqrt{\frac{n_l}{n_s N}}, \]
the cosmological equations are,
\[ \phi_2 + 3H\phi_2 = -\beta V + \lambda \rho_{DM} \]
\[ H^2 = \frac{1}{3M_p^2} \left( \rho_{DM} + \frac{M_p^2}{2} \phi_2^2 + V \right) \]
\[ \frac{\ddot{a}}{a} = -\frac{1}{6M_p^2} \left( \rho_{DM} + 2M_p^2 \phi_2^2 - 2V \right). \] (21)

They admit a solution of the form
\[ \phi_2 = -\frac{3}{\lambda + \beta} \log a, \]
\((a_0 = 1)\) which is such that \(\rho_{\phi_2} \sim \rho_{DM} \sim a^{-3(W+1)}\), with the equation of state
\[ W = -\frac{\lambda}{\lambda + \beta} = -\left(1 + \frac{4}{n_s} \frac{n_l}{n_l} \right)^{-1}, \] (22)
and fixed ratio,
\[ \Omega_{DE} = \frac{\rho_{\phi_2}}{\rho_{DM} + \rho_{\phi_2}} = \frac{3 + \lambda(\beta + \lambda)}{(\beta + \lambda)^2}, \]
\[ = \frac{n_l(3n_lN + 4n_s + n_l)}{(4n_s + n_l)^2}, \] (23)
independent on the scale factor \(a\). The solution above is an attractor over solution space if
\[ n_s > n_l \left(3n_l - 4\right) \frac{16 - 3n_l}{16 - 3n_l}. \] (24)

Differently from the case of a cosmological constant, or even from quintessence models with inverse power law potentials, once the attractor is reached \(\Omega_{DE}\) and \(\Omega_{DM}\) become independent on the cosmological era, thus solving the cosmic coincidence problem. Moreover, the equation of state (22) is negative, as required by observations.

The values of \(W\) and \(\Omega_{DE}\) on the attractor are functions of \(n_s\) and \(n_l\) only. In Tab. 1 we list the possible values of \(\Omega_{DE}, W\) and of \(H_0t_0\), \(t_0\) being the present age of the Universe. We limited the dimensionality of the compact space according to the theoretical prejudice coming from string theory, i.e. \(N \leq 6\).

| \(n_s\) | \(n_l\) | \(\Omega_{DE}\) | \(W\) | \(H_0t_0\) |
|---|---|---|---|---|
| 1-5 | 1 | \(\leq 0.44\) | - | - |
| 1 | 2 | 0.83 | -1/3 | 1 |
| 2 | 2 | 0.68 | -0.20 | 0.83 |
| 3 | 2 | 0.60 | -0.14 | 0.78 |
| 4 | 2 | 0.56 | -0.11 | 0.75 |
| 1-3 | 3 | \(\geq 0.92\) | - | - |
| 1-2 | 4 | no attractor | - | - |
| 1 | 5 | no attractor | - | - |

Tab.1 The values of \(\Omega_{DE}, W\), and \(H_0t_0\) for different values of \(n_s\) and \(n_l\) such that \(N = n_s + n_l \leq 6\).

V. DISCUSSION

The first noticeable fact about Tab. 1 is that the observed range for the dark energy density, \(0.6 \leq \Omega_{DE} \leq 0.8\) uniquely selects the number of ‘large’ extra dimensions to be \(n_l = 2\), the same value that is required by the totally unrelated issue of solving the hierarchy problem with ‘millimeter’ size extra dimensions [5].

Indeed, at the level of Tab. 1, we have not yet inserted any information about absolute scales, such as 100 μm, TeV, \(H_0\) and so on. We find it a remarkable and inspiring ‘coincidence’ that the observed balance between \(\Omega_{DM}\) and \(\Omega_{DE}\) points to the same value, \(n_l = 2\), obtained from scale dependent considerations, such as reproducing \(H_0\) or solving the hierarchy problem. For \(n_l \neq 2\) we either find that the attractor corresponds to energy densities outside the observed range or that the couplings \(\lambda\) and \(\beta\) lie outside the limit of eq. (24) and correspond to a different –unphysical – attractor.

The values for \(H_0t_0\) listed in the table are obtained from
\[ H_0(t_0 - t_{\text{att}}) = \frac{2}{3(W+1)} \left(1 - a_{\text{att}}^{\frac{2}{W+1}}\right), \]
where \(t_{\text{att}}\) and \(a_{\text{att}}\) are the time and scale factor when the attractor is reached, assuming \(a_{\text{att}} < 10^{-2}\). Taking the 95% confidence level lower limit on \(t_0\) from globular cluster age estimates, \(t_0 > 11\text{Gyr}\) [3] we obtain the lower bound
\[ H_0t_0 > (0.71 - 0.79) \]
where we have varied \(H_0\) in the range \((63 - 70)\text{ km s}^{-1}\text{Mpc}^{-1}\). The above bound is inconsistent with a flat, matter-dominated universe (for which \(W = 0\) and \(H_0t_0 = 2/3\)) while it is satisfied by all the \(n_l = 2\) models in Tab. 1.

Supernovae Ia data, taken at face value, point towards a more negative equation of state than those listed in Tab. 1, typically in the \(W \leq -0.6\) range [2]. However, the analyses have been done assuming two fluids with different equations of state, i.e. matter \((W = 0)\) and ‘quintessence’ \((W = W_x)\), whereas in the present case the two fluids scale with the same, negative, equation of state. Since quintessence begins to dominate the energy density quite recently, the negative equation of state takes over later than in our model, where it has been negative since much previous epochs. As a consequence, the supernovae bounds on the equation of state of the universe should be somehow relaxed in our model compared to quintessence.

Structure formation in a generic model of the type of eq. (21) was studied in [13] (where also the baryonic component was considered). There, it was shown that the non-zero – and constant – \(\Omega_{DM}\) allows the growth of perturbations even if the expansion is accelerated. This behavior is to be contrasted with the usual quintessence or cosmological constant case, where the perturbations
freeze out soon after the takeover because in that case \( \Omega_{DM} \) drops quickly to zero.

The model we have presented in this paper is just the simplest version of a large family. In order to discuss its phenomenological implications thoroughly, one should specify it in detail. Different dependences of the dark matter mass on the small radius could also be envisaged, possibly leading to different predictions for the equation of state. Also, the couplings of the KK sector to the SM are crucial. If the dark matter candidate is not a SM singlet, then its mass variation would induce variations in the gauge coupling constants. Imposing the existing strong constraints on the latter would imply that the field was frozen up to a not too distant past and that the attractor became effective quite recently.

It is, anyway, not trivial at all that the simplest choices, that is, tree-level dimensional reduction and the assumption of a \( O(10^{-4}/r_l^2) \) Casimir energy, lead to a model which solves the cosmic coincidence problem and is –if not fully compatible with– at least very close to the observations. A non-trivial point to consider is the persistence of the behavior of the tree-level solution once radiative corrections are included. The issue can be treated consistently only once a detailed model is given, however we refer the reader to the discussion in ref. [7], where supersymmetry in the bulk is argued to cancel all dangerous corrections larger than \( O(1/(r_l^2)^2M_p) \) to the radion mass.

A model-independent prediction of the framework outlined in this paper is the presence of extra dimensions in the \( 10-100 \mu m \) range. Whereas for the solution of the hierarchy problem discussed in [5] this value was allowed, the link with the cosmological expansion discussed in this paper makes it a true prediction. The expected value for \( r_l \) may be quite close to the present bound of \( 200 \mu m \), obtained from measurements of the Newton’s law at small distances [14]. However, one cautionary remark is due on this point. The predicted value is determined by global observables, i.e. \( H_0 \) and \( \delta_0/\delta_0 \). Since the radius and DM are strongly coupled in this model, it is plausible that the local value of \( r_l \), inside the galactic halo, would be somehow different. It is difficult to estimate the size of these effects, which would require a dedicated numerical study of the halo formation in this model. However, the sign would plausibly go in the direction of shrinking \( r_l \) inside the halo. This is because the system would use the additional degree of freedom represented by the varying mass, to decrease the gravitational potential in overdense regions, thus expanding \( r_\Delta \) – and then shrinking \( r_l \) – with respect to the average universe.

The effect of varying mass dark particles on cosmic perturbations and on the structure of halos will be the subject of a forthcoming publication [15].

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