A recognition algorithm for adjusted interval digraphs

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Abstract
Min orderings give a vertex ordering characterization, common to some graphs and digraphs such as interval graphs, complements of threshold tolerance graphs (known as co-TT graphs), and two-directional orthogonal ray graphs. An adjusted interval digraph is a reflexive digraph that has a min ordering. Adjusted interval digraph can be recognized in $O(n^4)$ time, where $n$ is the number of vertices of the given graph. Finding a more efficient algorithm is posed as an open question. This note provides a new recognition algorithm with running time $O(n^3)$. The algorithm produces a min ordering if the given graph is an adjusted interval digraph.

Keywords: Adjusted interval digraphs, Min ordering, Recognition algorithm

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1. Introduction

All graphs and directed graphs (digraphs for short) considered in this paper are finite and have no multiple edges but may have loops. We write $uv$ for the undirected edge joining a vertex $u$ and a vertex $v$; we write $(u, v)$ for the directed edge from $u$ to $v$. We write $V(H)$ for the vertex set of a digraph $H$; we write $E(H)$ for the edge set of $H$. We say that $u$ dominates $v$ (and that $v$ is dominated by $u$) in a digraph $H$ if $(u, v) \in E(H)$, and denote it by $u \rightarrow v$ or $v \leftarrow u$.

A digraph $H$ is an interval digraph [5] if for each vertex $v$ of $H$, there is a pair of intervals $I_v$ and $J_v$ on the real line such that $u \rightarrow v$ in $H$ if and only if $I_u$ intersects $J_v$. An interval digraph is an adjusted interval digraph [2] if the two intervals $I_v$ and $J_v$ have the same left endpoint for each vertex $v$. A digraph is called reflexive if every vertex has a loop, and every adjusted interval digraph is reflexive by definition.

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Adjusted interval digraphs have been introduced by Feder et al. [2] in connection with the study of list homomorphisms. They have shown two characterizations and a recognition algorithm of this graph class.

A min ordering of a digraph $H$ is a linear ordering $\prec$ of the vertices of $H$ such that for any two edges $(u, v)$ and $(u', v')$ of $H$, we have $(u, v') \in E(H)$ if $u < u'$ and $v' < v$. We remark that $(u, v)$ can be a loop, and similarly for $(u', v')$. A reflexive digraph has a min ordering if and only if it is an adjusted interval digraph [2]. Min orderings give similar characterizations of some graph classes such as interval graphs, complements of threshold tolerance graphs (known as co-TT graphs) [4], two-directional orthogonal ray graphs [6], and signed-interval digraphs [3]. See [3] for details.

Suppose that a digraph $H$ has a min ordering $\prec$, and let $(u, v), (u', v')$ be two edges of $H$ with $(u, v') \notin E(H)$. We have $v \neq v'$ from $(u, v) \in E(H)$ and $(u, v') \notin E(H)$; similarly, we have $u \neq u'$ from $(u', v) \in E(H)$ and $(u, v') \notin E(H)$. If $u < u'$ and $v' < v$, then $(u, v') \in E(H)$ by the property of min orderings, a contradiction. Thus, $u < u'$ implies $v < v'$ and $v' < v$ implies $u' < u$. We can capture this forcing relation with an auxiliary digraph. The pair digraph $H^+$ associated with a digraph $H$ is a digraph such that the vertex set $V(H^+)$ is the set of ordered pair of two vertices of $H$, and $(u, u') \to (v, v')$ and $(v', v) \to (u', u)$ in $H^+$ if and only if $(u, v), (u', v') \in E(H)$ and $(u, v') \notin E(H)$.

An invertible pair of a digraph $H$ is a pair of two vertices $u, v$ of $H$ such that in $H^+$, the vertices $(u, v)$ and $(v, u)$ are in the same strong component. It is clear that if $H$ has an invertible pair, then $H$ does not have any min ordering. Feder et al. [2] have shown that the converse also holds; therefore, a reflexive digraph has no invertible pairs if and only if it has a min ordering.

The characterizations of adjusted interval digraphs yield a recognition algorithm with running time $O(m^2 + n^2)$, where $n$ and $m$ are the number of vertices and edges of the given graph, respectively [2]. Finding a linear-time recognition algorithm is posed as an open question [2, 3]. In this paper, we show an $O(n^3)$-time recognition algorithm for adjusted interval digraphs. The algorithm produces a min ordering or finds an invertible pair of the given graph if it exists. As a byproduct, we also give an alternative proof to show that a reflexive digraph has a min ordering if and only if it has no invertible pairs.

2. Algorithm

In the case of reflexive digraphs, there is an equivalent simpler definition of min orderings.
Theorem 1 (Feder et al. [2]). Let $H$ be a reflexive digraph. A linear ordering $<$ of the vertices of $H$ is a min ordering if and only if for any three vertices $u, v, w$ with $u < v < w$,

- $(u, w) \in E(H)$ implies $(u, v) \in E(H)$, and
- $(w, u) \in E(H)$ implies $(v, u) \in E(H)$.

In other words, a linear ordering $<$ of the vertices of a reflexive digraph is a min ordering if it contains no triples of vertices $u, v, w$ with $u < v < w$ such that $(u, w) \in E(H)$ and $(u, v) \notin E(H)$, or $(w, u) \in E(H)$ and $(v, u) \notin E(H)$. We call such triples of vertices the forbidden patterns.

Let $H$ be an adjusted interval digraph with a min ordering $\prec$. Let $u, v, w$ be distinct vertices of $H$ with $(u, w) \in E(H)$ and $(u, v) \notin E(H)$, or $(w, u) \in E(H)$ and $(v, u) \notin E(H)$. In both cases, if $u < v < w$ then we have a forbidden pattern. Thus, $u < v$ implies $w < v$ and $v < w$ implies $v < u$. To capture this forcing relation, we define an auxiliary digraph associated with $H$.

Definition 2. Let $H$ be a reflexive digraph. The implication graph $H^*$ of $H$ is a digraph such that the vertex set $V(H^*)$ is the set $\{(u, v) : u \neq v\}$ of ordered pair of two vertices of $H$, and for any three vertices $u, v, w$ of $H$, we have $(u, v) \to (w, v)$ and $(v, w) \to (v, u)$ in $H^*$ if and only if

- $(u, w) \in E(H)$ and $(u, v) \notin E(H)$, or
- $(w, u) \in E(H)$ and $(v, u) \notin E(H)$.

We can use the implication graphs for recognizing adjusted interval digraphs.

Lemma 3. Let $H$ and $H^*$ be a reflexive digraph and its implication graph, respectively. A pair of two vertices $u, v \in V(H)$ is an invertible pair if and only if in $H^*$, the vertices $(u, v)$ and $(v, u)$ are in the same strong component.

Proof. Let $H^+$ be the pair digraph of $H$. Let $u, v, w$ be three vertices of $H$ such that $(u, v) \to (w, v)$ in $H^+$ (or equivalently, $(v, w) \to (v, u)$ in $H^+$). By definition, $(u, w) \in E(H)$ and $(u, v) \notin E(H)$, or $(w, u) \in E(H)$ and $(v, u) \notin E(H)$. Since the vertex $v$ has a loop, in both cases $(u, v) \to (w, v)$ and $(v, w) \to (v, u)$ in $H^+$. Therefore, $H^+$ is a subgraph of $H^*$.

Assume that $(u, v) \to (u', v')$ in $H^+$. By definition, $(u, u'), (v, v') \in E(H)$ and $(u, v') \notin E(H)$, or $(u', u), (v', v) \in E(H)$ and $(v', u) \notin E(H)$. In both cases, if $(u, u')$ or $(v, v')$ is a loop, then $(u, v) \to (u', v')$ in $H^+$. Thus we may assume $u \neq u'$ and $v \neq v'$. We have $u \neq v'$ since $H$ is reflexive. Recall that $u \neq v$ and $u' \neq v'$. Thus,
the vertices $u, v, v'$ are distinct, and $(u, v) \rightarrow (u, v')$ in $H^*$. Similarly, the vertices $u, u', v'$ are distinct, and $(u, v') \rightarrow (u', v')$ in $H^*$. Therefore, if $(u, v) \rightarrow (u', v')$ in $H^+$, then $H^*$ has a directed path from $(u, v)$ to $(u', v')$.

Lemma 3 gives an algorithm to find an invertible pair if it exists. Given a reflexive digraph $H$, the algorithm first construct the implication graph $H^*$ of $H$, then compute the strong components of $H^*$, and finally check for the existence of a pair $(u, v)$ and $(v, u)$ within one strong component. The implication graph $H^*$ has $n(n - 1)$ vertices, and at most $2nm$ edges since $H^*$ has at most two edges for each pair of a vertex and an edge of $H$. Therefore, we can construct $H^*$ in time $O(nm)$, and check for the existence of invertible pairs in the same time bound.

We next describe the algorithm for producing a min ordering of an adjusted interval digraph. Let $H$ and $H^*$ be a reflexive digraph and its implication graph, respectively. As an auxiliary graph, we use a complete graph $K$ with the vertex set $V(H)$. An orientation of $K$ is a digraph obtained from $K$ by orienting each edge of $K$, that is, replacing each edge $uv \in E(K)$ with either $(u, v)$ or $(v, u)$. An orientation of $K$ is acyclic if it contains no directed cycles; an acyclic orientation of $K$ is equivalent to a linear ordering of the vertices of $H$.

We say that a vertex $(u, v)$ of $H^*$ is an implicant of a vertex $(u', v')$ if $H^*$ has a directed walk from $(u', v')$ to $(u, v)$. We say that an orientation $T$ of $K$ is consistent with $H$ if for each vertex $(u, v)$ of $H^*$, we have $u \rightarrow v$ in $T$ implies $u' \rightarrow v'$ for every implicant $(u', v')$ of $(u, v)$. It is clear that an acyclic orientation of $K$ is consistent with $H$ if and only if it contains no forbidden patterns of min orderings. Therefore, a min ordering of $H$ is equivalent to an orientation of $K$ that is acyclic and consistent with $H$.

It is sufficient for the existence of a min ordering of $H$ that there is an orientation of $K$ consistent with $H$.

**Lemma 4.** There is a min ordering of $H$ if and only if there is an orientation of $K$ consistent with $H$.

Let $T$ be an orientation of $K$ consistent with $H$ that is not acyclic. In order to prove Lemma 4, we provide an algorithm for producing another orientation of $K$ that is acyclic and consistent with $H$.

A directed triangle is a directed cycle of length 3. It is well known that an orientation of a complete graph is acyclic if and only if it contains no directed triangles. Let $u$ be a vertex of $K$, and let $E_u$ be the set of all the edges $(v, w) \in E(T)$ such that $u \rightarrow v$, $v \rightarrow w$, and $w \rightarrow u$ in $T$. The reversal $E_u^-$ of $E_u$ is the set of directed edges obtained from $E_u$ by reversing the direction of all the edges
in $E_u$, that is, $E_u^- = \{(x, y) : (y, x) \in E_u\}$. We define that $T'$ is the orientation of $K$ obtained from $T$ by reversing the direction of all the edges in $E_u$, that is, $E(T') = (E(T) - E_u) \cup E_u^-.$

We will show that the orientation $T'$ has the following properties: $T'$ is still consistent with $H$; $T'$ contains no directed triangles having the vertex $u$; the reversing the direction of edges in $E_u$ generates no directed triangles. Therefore, by repeated application of this procedure for each vertex of $K$, we can obtain an orientation of $K$ that is acyclic and consistent with $H$; the complexity of the algorithm is $O(n^2)$.

To show that $T'$ is still consistent with $H$, we prove a lemma for directed triangles in the orientation of $K$ consistent with $H$.

**Lemma 5.** Let $T$ be an orientation of $K$ consistent with $H$. Suppose that $T$ has three vertices $u, v, w$ such that $u \to v, v \to w, and w \to u$ in $T$. If $v' \to w'$ in $T$ and $(v', w') \to (v, w)$ in $H^+$, then $u \to v', v' \to w'$, and $w' \to u$ in $T$.

**Proof.** We say that a set of vertices $S \subseteq V(H)$ is complete if $(x, y), (y, x) \in E(H)$ for any two vertices $x, y \in S$, and is independent if $(x, y), (y, x) \notin E(H)$. We claim that the set of vertices $\{u, v, w\}$ is complete or independent. Suppose $(u, v) \in E(H)$. If $(w, v) \notin E(H)$, then $(v, w) \to (u, w)$ in $H^+$, a contradiction. Thus $(w, v) \in E(H)$.

If $(w, u) \notin E(H)$, then $(w, u) \to (v, u)$ in $H^+$, a contradiction. Thus $(w, u) \in E(H)$. If $(v, u) \notin E(H)$, then $(u, v) \to (w, v)$ in $H^+$, a contradiction. Thus $(v, u) \in E(H)$.

Continuing in this way, we have that $\{u, v, w\}$ is complete.

We have either $v' = v$ or $w' = w$. Suppose $v' = v$. Since $(v', w') \to (v, w)$ in $H^+$, we have $(v, w) \notin E(H)$ and $(w', w) \in E(H)$, or $(w, v) \notin E(H)$ and $(w, w') \in E(H)$. In both cases, the set of vertices $\{u, v, w\}$ is independent. Since $(u, w) \notin E(H)$ and $(w', w) \in E(H)$, or $(w, u) \notin E(H)$ and $(w, w') \in E(H)$, we have $(w, u) \to (w', u)$ in $H^+$; therefore, $w' \to u$ in $T$.

We next suppose $w' = w$. Since $(v', w') \to (v, w)$ in $H^+$, we have $(v', w) \notin E(H)$ and $(v', v) \in E(H)$, or $(w', v') \notin E(H)$ and $(v, v') \in E(H)$.

Due to symmetry, we may assume $(v', w) \notin E(H)$ and $(v', u) \in E(H)$. If $(v', u) \in E(H)$, we have $(v', w') \to (u, w)$ in $H^+$, a contradiction. Thus $(v', u) \notin E(H)$. We now have $(u, v) \to (u, v')$ in $H^+$, and therefore, $u \to v'$ in $T$.

Suppose that $T'$ is not consistent with $H$. Then, there exist three vertices $x, y, z$ such that $x \to y$ and $y \to z$ in $T'$ but $(x, y) \to (z, y)$ in $H'$ (or equivalently, $(y, z) \to (y, x)$ in $H^+$). Since $T$ is consistent with $H$, we have $(x, y) \in E_u^- or (y, z) \in E_u^-$. Suppose $(x, y), (y, z) \in E_u^-$. We have that $(x, y) \in E_u^-$ implies $u \to y$ in $T$ and $(y, z) \in E_u^-$ implies $y \to u$ in $T$, a contradiction. If $(x, y) \in E_u^-$ and
(y, z) \notin E^+_u$, then $(y, x) \in E_u$ and $y \to z$ in $T$. Since $(y, z) \to (y, x)$ in $H^*$, we have from Lemma 5 that $(y, z) \in E_u$, a contradiction. Similarly, if $(x, y) \notin E^+_u$ and $(y, z) \in E^+_u$, then $x \to y$ in $T$ and $(z, y) \in E_u$. Since $(x, y) \to (z, y)$ in $H^*$, we have from Lemma 5 that $(x, y) \in E_u$, a contradiction. Therefore, $T'$ is still consistent with $H$.

Trivially, $T'$ contains no directed triangles having the vertex $u$.

Let $x, y, z$ be three vertices such that $x \to y, y \to z$, and $z \to x$ in $T'$. Suppose $(x, y), (y, z) \in E^+_u$. We have that $(x, y) \in E^-_u$ implies $u \to y$ in $T$ and $(y, z) \in E^-_u$ implies $y \to u$ in $T$, a contradiction. Thus at most one edge on the directed triangle is in $E^-_u$. Suppose $(x, y) \in E^-_u$ and $(y, z), (z, x) \notin E^-_u$. We have $u \to y, y \to z, z \to x$, and $x \to u$ in $T$. If $u \to z$ in $T$ then $(z, x) \in E_u$; if $z \to u$ in $T$ then $(y, z) \in E_u$, a contradiction. Therefore, the reversing the direction of edges in $E_u$ generates no directed triangles, and we have Lemma 4.

We now show an algorithm for finding an orientation of $K$ consistent with $H$. We use an algorithm for the 2-satisfiability problem. An instance of the 2-satisfiability problem is a 2CNF formula, a Boolean formula in conjunctive normal form with at most two literals per clause. We construct the 2CNF formula $\phi_H$ associated with $H$. Assume that the vertices of $H$ are linearly ordered, and let $x_{(u,v)}$ be a Boolean variable if a vertex $u$ of $H$ precedes a vertex $v$ in this ordering. We denote the negation of $x_{(u,v)}$ by $x_{(v,u)}$. We define that $\phi_H$ is a 2CNF formula consisting of all the clauses $(x_{(u,v)} \lor x_{(v,u)})$ such that $(u, w) \in E(H)$ and $(u, v) \notin E(H)$, or $(w, u) \in E(H)$ and $(v, u) \notin E(H)$.

Let $\tau$ be a truth assignment of $\phi_H$. We define that an orientation of $K$ associated with $\tau$ is an orientation $T_\tau$ such that $u \to v$ in $T_\tau$ if and only if $x_{(u,v)} = 0$ in $\tau$ for any two vertices $u, v$ of $K$. It is clear from the construction of $\phi_H$ that $T_\tau$ is consistent with $H$ if and only if $\tau$ satisfies $\phi_H$.

**Lemma 6.** There is an orientation of $K$ consistent with $H$ if and only if $\phi_H$ is satisfiable.

The 2CNF formula $\phi_H$ has at most $n(n - 1)/2$ Boolean variables, and at most $nm$ clauses since $\phi_H$ has at most one clause for each pair of a vertex and an edge of $H$. Thus $\phi_H$ can be constructed in $O(nm)$ time. Since a satisfiability truth assignment of $\phi_H$ can be computed in time linear to the size of $\phi_H$ (see [1] for example), we can find an orientation of $K$ consistent with $H$ in $O(nm)$ time.

Let $\phi$ be a 2CNF formula. For a Boolean variable $x_i$ in $\phi$, the negation of $x_i$ is denoted by $\overline{x_i}$. The implication graph $G(\phi)$ of $\phi$ is the digraph constructed as follows: for each variable $x_i$, we add two vertices named $x_i$ and $\overline{x_i}$ to $G(\phi)$; for each clause $(x_i, x_j)$, we add two edges to $G(\phi)$ so that $\overline{x_i} \rightarrow x_j$ and $\overline{x_j} \rightarrow x_i$. A
2CNF formula $\phi$ is satisfiable if and only if in $G(\phi)$, any pair of vertices $x_i$ and $\overline{x_i}$ are not in the same strong component [1].

For a reflexive digraph $H$, it is clear from the construction of $\phi_H$ and $H^*$ that $G(\phi_H)$ is isomorphic to the subgraph of $H^*$ obtained by removing all the isolated vertices of $H^*$. Therefore, we have the following.

**Lemma 7.** The 2CNF formula $\phi_H$ is satisfiable if and only if $H$ has no invertible pairs.

From Lemmas 4, 6, and 7, we now have an alternative proof for the theorem of Feder et al. [2].

**Theorem 8.** A reflexive digraph has a min ordering if and only if it contains no invertible pairs.

We finally summarize our algorithm for recognizing adjusted interval graphs. This algorithm produces a min ordering of the given graph if it is an adjusted interval digraph, and finds an invertible pair if otherwise.

**Algorithm 9.** Let $H$ be a reflexive digraph.

*Step 1.* Compute a 2CNF formula $\phi_H$ from $H$.

*Step 2.* Find a satisfying truth assignment of $\phi_H$.

  If $\phi_H$ is satisfiable, go to Step 3. Otherwise, go to Step 4.

*Step 3.* Let $\tau$ be a satisfying truth assignment of $\phi_H$.

  Compute an orientation $T_\tau$ of $K$ associated with $\tau$.

  Compute a min ordering of $H$ from $T_\tau$ if $T_\tau$ is not acyclic.

  Output the min ordering of $H$, and halt.

*Step 4.* Construct the implication graph $H^*$ of $H$. Then, find an invertible pair.

  Output the invertible pair of $H$, and halt.

The correctness of the algorithm follows from Lemmas 4, 6, and 7. Steps 1, 2, and 4 can be performed in $O(nm)$ time; Step 3 can be performed in $O(n^3)$ time.

**Theorem 10.** Adjusted interval digraphs can be recognized in $O(n^3)$ time.

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