Duality in Regret Measures and Risk Measures

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Abstract

Optimization models based on coherent regret measures and coherent risk measures are of essential importance in financial management and reliability engineering. This paper studies the dual representations of these two measures. The relationship between risk envelopes and regret envelopes are established by using the Lagrangian duality theory. The notion of effective scaling domain is introduced and its properties are discussed.

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1 Introduction

It is often desirable to quantify “regret” in models of financial management and reliability engineering. In general, the word “regret” refers to how individuals feel after having made a decision and whether they experiencing lingering doubt about their choice. In mathematical science, however, the notion of regret is associated with a random variable $X$ in a probability

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space \((\Omega, \mathcal{F}, \mathbb{P})\) and is usually regarded as a functional \(\mathcal{V} : \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) \to (-\infty, +\infty]\), where \(\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) (\mathcal{L}^2\ for\ short)\) is the Lebesgue square-integrable space. Equivalently, in utility models, the regret (Or regret measure, as is more popularly called in the literature) is simply defined as the negative utility. That is

\[\mathcal{V}(X) = -\mathcal{U}(-X)\]

where \(\mathcal{U}\) is the utility functional of \(-X\), noting that \(-X\) is the “gain” if \(X\) stands for the “loss” (which is adopted throughout this paper). Therefore, all our subsequent results can have a corresponding interpretation in utility models.

Paired with the regret measure, there is the notion of risk measure, denoted by \(\mathcal{R}\). The risk measure could be understood as the “certainty-uncertainty trade-off” of the regret measure, namely, we define the risk measure \(\mathcal{R} : \mathcal{L}^2 \to (-\infty, +\infty]\) as

\[\mathcal{R}(X) = \inf_C \{C + \mathcal{V}(X - C)\}. \quad (1.1)\]

This formula generalizes the well-known formula for conditional value at risk (CVaR for short) of Rockafellar and Uryasev [6].

For both theoretical and practical purposes, we concentrate on the “coherent” cases. Let us begin with the definitions of coherent risk measures and coherent regret measures.

Let \(X \in \mathcal{L}^2\) be a random variable. A risk measure \(\mathcal{R}\) is coherent if it satisfies the following axioms (Rockafellar [5]).

(A1) \(\mathcal{R}(C) = C\) for all constant \(C\),

(A2) \(\mathcal{R}((1 - \lambda)X + \lambda X') \leq (1 - \lambda)\mathcal{R}(X) + \lambda \mathcal{R}(X')\) for \(\lambda \in [0, 1]\) (“convexity”),

(A3) \(\mathcal{R}(X) \leq \mathcal{R}(X')\) if \(X \leq X'\) almost everywhere (“monotonicity”),

(A4) \(\mathcal{R}(X) \leq 0\) when \(\|X_k - X\|_2 \to 0\) with \(\mathcal{R}(X_k) \leq 0\) (“closedness”),

(A5) \(\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)\) for \(\lambda > 0\) (“positive homogeneity”).

Furthermore, we say that the coherent risk measure \(\mathcal{R}\) is finite, if \(\mathcal{R}(X) \in (-\infty, +\infty)\) for any \(X \in \mathcal{L}^2\).

The coherent regret \(\mathcal{V}\) (Rockafellar and Uryasev [7]) is defined similarly. A functional \(\mathcal{V} : \mathcal{L}^2 \to (-\infty, +\infty]\) is called a coherent regret measure if it satisfies the following.
(B1) \( \mathcal{V}(0) = 0, \)

(B2) \( \mathcal{V}((1 - \lambda)X + \lambda X') \leq (1 - \lambda)\mathcal{V}(X) + \lambda \mathcal{V}(X') \) for \( \lambda \in [0, 1] \) ("convexity"),

(B3) \( \mathcal{V}(X) \leq \mathcal{V}(X') \) if \( X \leq X' \) almost everywhere ("monotonicity"),

(B4) \( \mathcal{V}(X) \leq 0 \) when \( \|X_k - X\|_2 \to 0 \) with \( \mathcal{V}(X_k) \leq 0 \) ("closedness"),

(B5) \( \mathcal{V}(\lambda X) = \lambda \mathcal{V}(X) \) for \( \lambda > 0 \) ("positive homogeneity").

Furthermore, we say that the coherent regret measure \( \mathcal{V} \) is finite, if \( \mathcal{V}(X) \in (-\infty, +\infty) \) for any \( X \in \mathcal{L}^2 \).

It is well known \[5\] that the coherent risk measure has a dual representation; that is, there is a convex and closed set \( \mathcal{Q} \subset \mathcal{L}^2 \), which can be shown to be unique, called "the risk envelope" of \( \mathcal{R} \), such that for any \( X \in \mathcal{L}^2 \),

\[
\mathcal{R}(X) = \sup_{Q \in \mathcal{Q}} \mathbb{E}(XQ). \tag{1.2}
\]

Moreover, \( \mathcal{Q} \) is a subset of

\[
\mathcal{P} := \{0 \leq Q \in \mathcal{L}^2 : \mathbb{E}(Q) = 1\}.
\]

More detailed analysis can be seen in \[1\]. It is obvious that \( \mathcal{R}(\cdot) \) is finite if \( \mathcal{Q} \) is compact.

The dual representation for \( \mathcal{V}(\cdot) \) can be similarly established. By convex analysis, any functional that satisfies (B1)-(B5) can be represented as a specific support function. That is, there is a unique, convex and closed \( \tilde{\mathcal{Q}} \), such that for any \( X \in \mathcal{L}^2 \),

\[
\mathcal{V}(X) = \sup_{Q \in \tilde{\mathcal{Q}}} \mathbb{E}(XQ), \tag{1.3}
\]

where \( \tilde{\mathcal{Q}} \) is a subset of

\[
\tilde{\mathcal{P}} := \{Q \geq 0 : Q \in \mathcal{L}^2\}.
\]

Furthermore, \( \mathcal{V}(\cdot) \) is finite if \( \tilde{\mathcal{Q}} \) is compact in \( \mathcal{L}^2 \).

This paper is concerned with the relationship of the dual representations of coherent regret measures and coherent risk measures. Starting with the basic equation (1.1), we investigate the relationship between \( \mathcal{Q} \) and \( \tilde{\mathcal{Q}} \). We then present several explicit descriptions of \( \mathcal{Q} \) and \( \tilde{\mathcal{Q}} \) for popular risk and regret measures. We shall demonstrate by examples that
the relations of maximum and positive combination do not pass from regret measures to risk measures. We then introduce a new concept called effective scaling domain and discuss its properties.

2 Risk envelope and Lagrangian dual

The next proposition sets up a basic relationship between coherent risk measures and coherent regret measures.

Proposition 2.1 For any coherent risk measure $\mathcal{R}$, there exists at least one coherent regret measure $\mathcal{V}$, such that

$$
\mathcal{R}(X) = \inf_{C \in \mathbb{R}} \{ C + \mathcal{V}(X - C) \}.
$$

(2.1)

Proof. Just note that $\mathcal{R}(-)$ itself can be a candidate for $\mathcal{V}(-)$ to satisfy (2.1)

Remarks. 

(1) It should be noted that even if $\mathcal{V}(-)$ is a finite coherent regret measure, the functional $\mathcal{R}(-)$ defined via (2.1) may not be a risk measure. For example, let $\mathcal{V}(X) = 2\mathbb{E}(X)$ for $X \in \mathcal{L}^2$, then $\mathcal{V}(-)$ is a finite coherent regret measure on $\mathcal{L}^2$ by the definition of coherent regret measure, but if we define $\mathcal{R}(-)$ via (2.1), then $\mathcal{R}(X) \equiv -\infty$ for any $X \in \mathcal{L}^2$ and therefore $\mathcal{R}(-)$ is not a risk measure. Hence it is important to find the conditions for $\mathcal{V}(-)$ to guarantee $\mathcal{R}(-)$ defined via (2.1) being a risk measure.

(2) For fixed coherent risk measure $\mathcal{R}(-)$, there may be more than one candidate $\mathcal{V}(-)$ satisfying relationship (2.1).

The next theorem illustrates the relationship between $\mathcal{R}(-)$, $\mathcal{V}(-)$, $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$.

Theorem 2.1 Suppose $\mathcal{R}(-)$ is a coherent risk measure with the dual representation (1.2) and $\mathcal{V}(-)$ is a coherent regret measure with the dual representation (1.3), where $\tilde{\mathcal{Q}}$ is compact. Then $\mathcal{R}(-)$ and $\mathcal{V}(-)$ have relationship (2.1) if and only if $\mathcal{Q} = \tilde{\mathcal{Q}} \cap \mathcal{P}$.

Proof. Fix $X \in \mathcal{L}^2$, let

$$
L(Q,C) := \mathbb{E}(XQ) + C[1 - \mathbb{E}(Q)].
$$
for $C \in \mathbb{R}$ and $Q \in \tilde{P}$. From compactness of $\tilde{Q}$ and Sion’s theorem [8], we then have

$$\sup_{Q \in \tilde{Q}} \inf_{C \in \mathbb{R}} L(Q, C) = \inf_{C \in \mathbb{R}} \sup_{Q \in \tilde{Q}} L(Q, C).$$

(2.2)

Since

$$\inf_{C \in \mathbb{R}} L(Q, C) = \begin{cases} \mathbb{E}(XQ) & \text{if } Q \in \mathcal{P}, \\ -\infty & \text{otherwise}, \end{cases}$$

we have

$$\sup_{Q \in \tilde{Q}} \inf_{C \in \mathbb{R}} L(Q, C) = \sup_{Q \in \tilde{Q} \cap \mathcal{P}} \mathbb{E}(XQ).$$

Then, by (1.3) we have

$$\inf_{C \in \mathbb{R}} \sup_{Q \in \tilde{Q}} L(Q, C) = \inf_{C \in \mathbb{R}} \{ C + \mathcal{V}(X - C) \}.$$ 

Thus, by (2.2), we get

$$\sup_{Q \in \tilde{Q} \cap \mathcal{P}} \mathbb{E}(XQ) = \inf_{C \in \mathbb{R}} \{ C + \mathcal{V}(X - C) \}. \quad (2.3)$$

On one hand, if $Q = \tilde{Q} \cap \mathcal{P}$, then by (2.3), we have

$$\sup_{Q \in \tilde{Q} \cap \mathcal{P}} \mathbb{E}(XQ) = \inf_{C \in \mathbb{R}} \{ C + \mathcal{V}(X - C) \}. \quad (2.4)$$

Together with (1.2), it follows that the relationship (2.1) holds for $\mathcal{R}(\cdot)$ and $\mathcal{V}(\cdot)$.

On the other hand, if $\mathcal{R}(\cdot)$ and $\mathcal{V}(\cdot)$ have relationship (2.1), then by (2.1) and (2.3), we get

$$\mathcal{R}(X) = \sup_{Q \in \tilde{Q} \cap \mathcal{P}} \mathbb{E}(XQ).$$

It is easy to see that $\tilde{Q} \cap \mathcal{P}$ is a nonempty, convex and closed subset of $L^2$, and therefore, it is the risk envelope of $\mathcal{R}(\cdot)$. By the uniqueness of risk envelope, we get $Q = \tilde{Q} \cap \mathcal{P}$. \qed

Theorem 2.1 provides us a way to determine a coherent regret measure $\mathcal{V}(\cdot)$ corresponding to a given coherent risk measure $\mathcal{R}(\cdot)$ as follows. Given a coherent risk measure $\mathcal{R}(\cdot)$, find its risk envelope $\tilde{Q}$, relax the condition “$\mathbb{E}(Q) = 1$” to get $\tilde{Q}$, then (1.3) determines the corresponding $\mathcal{V}(\cdot)$. Note that since there may be more than one way to relax the condition “$\mathbb{E}(Q) = 1$”, there may be more than one $\mathcal{V}(\cdot)$ corresponding to one $\mathcal{R}(\cdot)$ as well. Furthermore, if we want to find a finite $\mathcal{V}(\cdot)$ given finite $\mathcal{R}(\cdot)$, then we should let $\tilde{Q}$ be bounded in $L^2$. 

5
3  Examples

3.1 Optimized certainty equivalence (OCE)

Given $0 \leq \gamma_2 < 1 \leq \gamma_1$, let $S(\cdot)$ be the OCE-measure introduced by Ben-Tal and Teboulle [2]. It is shown in [1] that the OCE is a coherent risk measure with risk envelope

$$Q_{\gamma_1, \gamma_2} = \{Q: \gamma_2 \leq Q \leq \gamma_1, E(Q) = 1\}.$$

Removing the condition “$E(Q) = 1$”, we get

$$\tilde{Q}_{\gamma_1, \gamma_2} = \{Q: \gamma_2 \leq Q \leq \gamma_1\}.$$

Therefore, the corresponding regret measure is

$$V_{\gamma_1, \gamma_2}(X) = \sup_{Q \in \tilde{Q}_{\gamma_1, \gamma_2}} E(XQ) = \gamma_1E(X_+) - \gamma_2E(X_-).$$

In particular, if we take $\gamma_1 = \frac{1}{1 - \alpha}$ and $\gamma_2 = 0$, where $0 \leq \alpha < 1$, then the OCE-measure becomes the CVaR measure $\text{CVaR}_\alpha(\cdot)$.

The corresponding regret measure is

$$V_\alpha(X) = \frac{1}{1 - \alpha}E(X_+).$$

Then formula (2.1) is in fact the “minimization formula” of CVaR, i.e.,

$$\text{CVaR}_\alpha(X) = \min_{C \in \mathbb{R}} \left\{ C + \frac{1}{1 - \alpha}E(X - C)_+ \right\}.$$

3.2 Expectation as risk measure

This is a special case of CVaR when $\alpha = 0$ and $Q = \{1\}$. That is,

$$\mathcal{R}(X) = E(X).$$

By the result of subsection 3.1, a candidate for the corresponding regret measure is

$$V(X) = E(X_+).$$

\footnote{It is interesting to observe that OCE can be representable by CVaR, namely $S(X) = \gamma_2E(X) + \text{CVaR}_\alpha(X)$, where $\alpha = 1 - (\gamma_1 - \gamma_2)^{-1}$. Thus OCE and CVaR are in a sense equivalent.}
On the other hand, since $\overline{Q} = \{ Q : 1 \leq Q \leq 2 \}$ also satisfies $\overline{Q} \cap \mathcal{P} = \{1\}$, and it is bounded in $L^2$, we find that

$$\mathcal{V}(X) = \sup_{Q \in \overline{Q}} \mathbb{E}(XQ) = \mathbb{E}(X) + \mathbb{E}(X_+)$$

is another candidate for the corresponding regret measure. Therefore, the corresponding regret measure of $\mathbb{E}(X)$ is not unique.

### 3.3 Worst case as risk measure

This risk measure is defined as

$$\mathcal{R}(X) = \text{ess.sup}(X),$$

where ess.sup is the essential sup function. Note that the worst case risk measure is not finite and the corresponding risk envelope $Q = \mathcal{P}$ is not bounded. However we can directly verify that Theorem 2.1 is still true with $\tilde{Q} = \tilde{\mathcal{P}}$. Namely,

$$\mathcal{V}(X) = \begin{cases} 0 & \text{if } X \leq 0 \text{ almost surely}, \\ +\infty & \text{otherwise}. \end{cases}$$

### 3.4 Mean-deviation risk measure

Fix $0 \leq \lambda \leq 1$. Define

$$\mathcal{R}(X) = \mathbb{E}X + \lambda \cdot \| (X - \mathbb{E}X)_+ \|_2$$

for all $X \in L^2$, where $\| \cdot \|_2$ denotes the $L^2$-norm, that is,

$$\| X \|_2 := \left[ \mathbb{E}(X^2) \right]^{\frac{1}{2}}$$

for all $X \in L^2$. From Ang et al [1], we know that $\mathcal{R}(\cdot)$ is a coherent risk measure with risk envelope

$$Q = \left\{ 0 \leq Q \in L^2 : \mathbb{E}(Q) = 1, \| Q - \inf Q \|_2 \leq \lambda \right\} .$$

(3.1)

We next try to find the corresponding coherent regret measure $\mathcal{V}(\cdot)$ for it. Note that by simply getting rid of the restriction “$\mathbb{E}(Q) = 1$”, we will get an unbounded subset of $L^2$ and therefore may get a non-finite $\mathcal{V}(\cdot)$. To avoid it, note that $Q \geq 0$ and $\mathbb{E}(Q) = 1$ together imply $0 \leq \inf Q \leq 1$. Therefore,

$$\tilde{Q} = \left\{ Q \in L^2 : 0 \leq \inf Q \leq 1, \| Q - \inf Q \|_2 \leq \lambda \right\}$$
is bounded and satisfies $\tilde{Q} \cap P = Q$. Thus, we prefer to use $\tilde{Q}$ for calculating $\mathcal{V}(\cdot)$.

For any $X \in \mathcal{L}^2$ and $Q \in \tilde{Q}$, we have

$$\mathbb{E}(XQ) \leq \mathbb{E}[X_+(Q - \inf Q)] + \inf Q \cdot \mathbb{E}X \leq \lambda \cdot \|X_+\|_2 + (\mathbb{E}X)_+.$$  

Furthermore, the equation holds when $Q = 1_{\{\mathbb{E}X \geq 0\}} + \frac{\lambda X_+}{\|X_+\|_2}$. (0/0 is defined as 0.) Therefore,

$$\mathcal{V}(X) = \lambda \cdot \|X_+\|_2 + (\mathbb{E}X)_+$$

is a candidate for the regret measure corresponding to the mean-deviation risk measure.

We may check Theorem 2.1 for this case directly. For $C \in \mathbb{R}$, we have

$$C + \mathcal{V}(X - C) = C + (\mathbb{E}X - C)_+ + \lambda \cdot \|(X - C)_+\|_2 = \begin{cases} C + \lambda \cdot \|(X - C)_+\|_2 & \text{if } C \geq \mathbb{E}X, \\ \mathbb{E}X + \lambda \cdot \|(X - C)_+\|_2 & \text{if } C < \mathbb{E}X. \end{cases}$$

Therefore, $C + \mathcal{V}(X - C)$ is decreasing in $C$ when $C < \mathbb{E}X$ and increasing in $C$ when $C \geq \mathbb{E}X$, and so it reaches its minimum when $C = \mathbb{E}X$. Then (2.1) holds.

**Remark.** We can generalize the definition of mean-deviation risk measure in the following way. Fix $p \geq 1$ and $0 \leq \lambda \leq 1$. Define

$$\mathcal{R}(X) = \mathbb{E}X + \lambda \cdot \|(X - \mathbb{E}X)_+\|_p$$

for all $X \in \mathcal{L}^p$, where $\cdot \|_p$ denotes the $\mathcal{L}^p$-norm, that is,

$$\|X\|_p := \left[\mathbb{E}(|X|^p)\right]^{\frac{1}{p}}$$

for all $X \in \mathcal{L}^p$. We say that $\mathcal{R}(\cdot)$ is the mean-deviation risk measure with parameter $p$, which is a coherent risk measure in $\mathcal{L}^p$. It’s risk envelope is a subset of $\mathcal{L}^q$, where $q$ is the “conjugate number” of $p$, that is, $\frac{1}{p} + \frac{1}{q} = 1$. (Let the conjugate of 1 be $+\infty$.) By Theorem 2.1 we can write the corresponding regret measure.

### 3.5 More properties on the relationship between risk and regret

In Subsection 3.2 an example was given to demonstrate that there may be more than one regret measures corresponding to the same risk measure. In this subsection, we use it to demonstrate the relationship between expectation risk measure and mean-deviation risk measure.
Let \( \tilde{Q}_1 := \{Q : 0 \leq Q \leq 1\} \), \( \tilde{Q}_2 := \{Q : 1 \leq Q \leq 2\} \). Then

\[
\sup_{Q \in \tilde{Q}_1} \mathbb{E}(XQ) = \mathbb{E}(X^+), \quad \sup_{Q \in \tilde{Q}_2} \mathbb{E}(XQ) = \mathbb{E}X + \mathbb{E}(X^+)
\]

and

\[
\tilde{Q}_1 \cap \mathcal{P} = \tilde{Q}_2 \cap \mathcal{P} = \{1\}.
\]

Therefore, regret measures \( \mathbb{E}(X^+) \) and \( \mathbb{E}X + \mathbb{E}(X^+) \) correspond to the same risk measure \( \mathbb{E}(X) \). Meanwhile, the maximum of the two regret measures is

\[
\max\{\mathbb{E}(X^+), \mathbb{E}X + \mathbb{E}(X^+)\} = (\mathbb{E}X)_+ + \mathbb{E}(X^+),
\]

which correspond to a generalized mean-deviation risk measure of \( p = 1 \) and \( \lambda = 1 \) mentioned in the remark of Subsection 3.4.

This example demonstrates a relationship between expectation risk measure and mean-deviation risk measure. In addition, it demonstrates that the maximum of several regret measures does not correspond to the maximum of risk measures corresponding to the above regret measures. Simply speaking, the maximum relation does not preserve between regret measures and risk measures. We further note that the convex combination relation does not preserve between regret measures and risk measures, either. The following example demonstrates this.

Let \( \mathcal{V}_1(X) := \mathbb{E}(X^+) \) and \( \mathcal{V}_2(X) = \mathbb{E}X + \mathbb{E}(X^+) \) be two regret measures. Fix \( 0 < \lambda < 1 \) and let

\[
\mathcal{V}(X) := (1 - \lambda)\mathcal{V}_1(X) + \lambda \mathcal{V}_2(X) = \lambda \mathbb{E}X + \mathbb{E}(X^+).
\]

It is known that \( \mathcal{V}_1(X) \) and \( \mathcal{V}_2(X) \) correspond to the same risk measure \( \mathbb{E}X \). Next, we calculate the risk measure corresponding to \( \mathcal{V}(X) \). It is easy to see that

\[
C + \mathcal{V}(X - C) = \lambda \mathbb{E}X + (1 - \lambda)C + \mathbb{E}[(X - C)^+].
\]

Hence we have

\[
\min_{C \in \mathbb{R}}\{C + \mathcal{V}(X - C)\} = \lambda \mathbb{E}X + (1 - \lambda) \cdot \min_{C \in \mathbb{R}} \left\{ C + \frac{1}{1 - \lambda} \cdot \mathbb{E}[(X - C)^+] \right\}
\]

\[
= \lambda \mathbb{E}X + (1 - \lambda) \text{CVaR}_\lambda(X),
\]

where the minimum is reached when \( C = \text{VaR}_\lambda(X) \). Therefore, the risk measure corresponding to \( \mathcal{V}(X) \) is the convex combination of the expectation measure \( \mathbb{E}(\cdot) \) and the CVaR measure \( \text{CVaR}_\lambda(\cdot) \). Thus, the risk measure generated by \( \lambda \mathcal{V}_1 + (1 - \lambda) \mathcal{V}_2 \) is not \( \lambda \mathcal{R}_1 + (1 - \lambda) \mathcal{R}_2 \).
This example shows that the convex combination relationship does not preserve between regret measures and risk measures.

4 Proper regret measures and effective scaling domain

From the remarks after Proposition 2.1 we know that a coherent regret measure \(\mathcal{V}(\cdot)\) may not generate a coherent risk measure \(\mathcal{R}(\cdot)\) via (2.1) due to \(\mathbb{R}(X) = -\infty\) for some \(X\). We first give the following definition.

**Definition 4.1** Suppose \(\mathcal{V}(\cdot)\) is a coherent regret measure satisfying axioms (B1) – (B5). If \(\mathcal{R}(\cdot)\), as defined in (2.1), is a coherent risk measure satisfying \(\mathcal{R}(X) > -\infty\) for any \(X \in \mathcal{L}^{2}\), then we say that \(\mathcal{V}(\cdot)\) is proper.

The next proposition gives a necessary and sufficient condition for a coherent regret measure being proper.

**Proposition 4.1** Suppose \(\mathcal{V}(\cdot)\) is a coherent regret measure satisfying (B1) – (B5) with envelope \(\widetilde{Q}\). Then \(\mathcal{V}(\cdot)\) is proper if and only if \(\widetilde{Q} \cap \mathcal{P} \neq \emptyset\).

**Proof.** By Theorem 2.1 if \(\widetilde{Q}\) is the envelope of \(\mathcal{V}(\cdot)\), and \(\mathcal{R}(\cdot)\) is defined via (2.1), then we have \(\mathcal{R}(X) = \sup_{Q \in \widetilde{Q} \cap \mathcal{P}} \mathbb{E}(XQ)\) for any \(X \in \mathcal{L}^{2}\). Therefore, \(\mathcal{R}(X) > -\infty\) for any \(X \in \mathcal{L}^{2}\) if and only if \(\widetilde{Q} \cap \mathcal{P} \neq \emptyset\). \(\square\)

Note that if \(\mathcal{V}(\cdot)\) is a coherent regret measure satisfying (B1) – (B5), then so is \(a\mathcal{V}(\cdot)\) for any \(a \geq 0\). We say that \(a\mathcal{V}(\cdot)\) is a scaling of \(\mathcal{V}(\cdot)\). Note that the scaling of a proper regret measure may not be proper. For example, for \(\mathcal{V}(\cdot) = \mathbb{E}(\cdot)\), the scaling of \(\mathcal{V}(\cdot)\) is proper if and only if \(a = 1\), while for \(\mathcal{V}(X) = \mathbb{E}(X_{+})\), the scaling of \(\mathcal{V}(\cdot)\) is proper for \(a \geq 1\) and is not proper for \(0 \leq a < 1\). In fact, for \(a \geq 1\), the coherent risk measure corresponding to \(a\mathcal{V}(\cdot)\) via (2.1) is CVaR\(\frac{a}{a+1}\)(\(\cdot\)).

It would be interesting to determine what scaling coefficients can make the coherent regret measure proper. We start with the following definition.

**Definition 4.2** Let \(\mathcal{V}(\cdot)\) be a coherent regret measure satisfying (B1) – (B5). We call

\[
\mathcal{D}_{\mathcal{V}} := \{a \geq 0 \mid a\mathcal{V}(\cdot) \text{ is proper}\} \tag{4.1}
\]

the effective scaling domain of \(\mathcal{V}(\cdot)\).
We next study properties for the effective scaling domain of a coherent regret measure.

**Theorem 4.1** For every coherent regret measure \( V(\cdot) \neq 0 \), the effective scaling domain \( D_V \) is a nonempty interval on \([0, +\infty)\). A singleton is also considered as a nonempty interval (with length 0).

**Proof.** First, we prove that \( D_V \) is nonempty. Suppose \( \tilde{Q} \) is the envelope of the regret measure \( V(\cdot) \), then we have \( \tilde{Q} \neq \{0\} \) since \( V(\cdot) \neq 0 \). Therefore, we can take \( Q_0 \in \tilde{Q} \) such that \( E(Q_0) = a > 0 \). It is not difficult to verify that the envelope of the regret measure \( \frac{1}{a} V(\cdot) \) is \( \frac{1}{a} \tilde{Q} := \left\{ \frac{Q}{a} : Q \in \tilde{Q} \right\} \). Note that \( \left( \frac{1}{a} \tilde{Q} \right) \cap \mathcal{P} \supseteq \left\{ \frac{Q_0}{a} \right\} \neq \emptyset \). Thus, by Proposition 4.1, \( \frac{1}{a} V(\cdot) \) is a proper coherent regret measure. Therefore, \( \frac{1}{a} \in D_V \), and \( D_V \) is nonempty.

Next, we prove that if \( a, b \in D_V \) and \( a < b \), then \( c \in D_V \) for any \( c \in (a, b) \). In fact, since \( a, b \in D_V \), we have \( \left( a \tilde{Q} \right) \cap \mathcal{P} \neq \emptyset \), \( \left( b \tilde{Q} \right) \cap \mathcal{P} \neq \emptyset \). Therefore, there exists \( Q_1, Q_2 \in \tilde{Q} \), such that \( E(Q_1) = \frac{1}{a} \), \( E(Q_2) = \frac{1}{b} \). Since \( c \in (a, b) \), there is unique \( \lambda \in (0, 1) \) such that \( \lambda \cdot \frac{1}{a} + (1 - \lambda) \cdot \frac{1}{b} = \frac{1}{c} \). Take \( Q_0 = \lambda Q_1 + (1 - \lambda)Q_2 \), then by the convexity of \( \tilde{Q} \), we have \( Q_0 \in \tilde{Q} \), and \( E(Q_0) = \frac{1}{c} \). Hence \( \left( c \tilde{Q} \right) \cap \mathcal{P} \neq \emptyset \), and then \( c \in D_V \). Therefore, \( D_V \) is a nonempty interval on \([0, +\infty)\). \( \Box \)

In fact, we can calculate the effective scaling domain \( D_V \) explicitly. We shall need the following result, which is a special case of Theorem 1 in Krokhmal [3].

**Proposition 4.2** The coherent regret measure \( V(\cdot) \) is proper if and only if \( V(C) \geq C \) for all constant \( C \).

**Remark.** By positive homogeneity of \( V(\cdot) \), \( V(C) \geq C \) is equivalent to \( V(1) \geq 1 \) and \( V(-1) \geq -1 \).

**Theorem 4.2** The effective scaling domain for the coherent regret measure \( V(\cdot) \) is the nonempty interval \( \left[ \frac{1}{V(1)}, \frac{-1}{V(-1)} \right] \), where \( a/0 \) is defined as \( +\infty \) for all constant \( a > 0 \).

**Proof.** From the above remark, we have

\[
 aV(\cdot) \text{ is proper } \iff aV(1) \geq 1 \text{ and } aV(-1) \geq -1 \iff a \in \left[ \frac{1}{V(1)}, \frac{-1}{V(-1)} \right].
\] \( \Box \)
Corollary 4.1 \( \mathcal{D}_V \) is a singleton if and only if \( V(\cdot) = a \mathcal{R}(\cdot) \) for certain coherent risk measure \( \mathcal{R}(\cdot) \) and certain constant \( a > 0 \).

**Proof.** By Theorem 4.2, \( \mathcal{D}_V \) is a singleton if and only if \( V(-1) = -V(1) \). On one hand, if \( V(\cdot) = a \mathcal{R}(\cdot) \) for some coherent risk measure \( \mathcal{R}(\cdot) \) and some constant \( a > 0 \), then we can directly check \( V(-1) = -V(1) = -a \). On the other hand, if \( V(-1) = -V(1) = -a \) where \( a > 0 \), then by the positive homogeneity of \( V(\cdot) \), we have \( V(C) = aC \) for all constant \( C \in \mathbb{R} \). Let \( \mathcal{R}(X) = \frac{1}{a} V(X) \) for \( X \in \mathcal{L}^2 \). Then \( \mathcal{R}(\cdot) \) is a coherent regret measure which satisfies \( \mathcal{R}(C) = C \) for all constant \( C \in \mathbb{R} \). Therefore, \( \mathcal{R}(\cdot) \) is a coherent risk measure, and \( V(\cdot) = a \mathcal{R}(\cdot) \). \( \Box \)

**Remark.** We say that \( V(\cdot) \) is a “trivial” regret measure if \( V(\cdot) = a \mathcal{R}(\cdot) \) for some coherent risk measure \( \mathcal{R}(\cdot) \) and some constant \( a > 0 \). Corollary 4.1 tells us that \( V(\cdot) \) is trivial if and only if \( \mathcal{D}_V \) is a singleton.

Corollary 4.2 Suppose \( V_1(\cdot), \cdots, V_n(\cdot) \) are coherent regret measures, then we have

\[
\mathcal{D}_{\max \atop 1 \leq i \leq n} V_i = \text{co} \left( \bigcup_{i=1}^{n} \mathcal{D}_{V_i} \right),
\]

where “co” stands for the convex hull in the sense of convex analysis.

**Proof.** By Theorem 4.2 we have

\[
\text{co} \left( \bigcup_{i=1}^{n} \mathcal{D}_{V_i} \right) = \text{co} \left( \bigcup_{i=1}^{n} \left[ \frac{1}{V_i(1)}, \frac{-1}{V_i(-1)} \right] \right) = \left[ \frac{1}{\max_{1 \leq i \leq n} V_i(1)}, \frac{-1}{\max_{1 \leq i \leq n} V_i(-1)} \right] = \mathcal{D}_{\max \atop 1 \leq i \leq n} V_i. \quad \Box
\]

Go back to the examples in Section 3. By Theorem 4.2, the effective scaling domain for \( V_{\gamma_1, \gamma_2} \), the regret measure corresponding to the OCE-measure \( S \) \((0 \leq \gamma_2 < 1 \leq \gamma_1)\), is \( \left[ \frac{1}{\gamma_1 \gamma_2}, \frac{1}{\gamma_1 \gamma_2} \right] \). The case of the conditional value at risk, expectation risk measure and the worst case risk measure are all special cases of it. On the other hand, the effective scaling domain for \( V(X) = \lambda \|X_+\|_2 + (\mathbb{E}(X))_+ \), the regret measure corresponding to the mean-deviation risk measure, is \( \left[ \frac{1}{1 + \lambda}, +\infty \right) \).
5 Concluding remarks

This paper is concerned with the relationship of the dual representations of coherent regret measures and coherent risk measures. We investigated the relationship between $Q$ and $\tilde{Q}$ and presented several explicit descriptions of $Q$ and $\tilde{Q}$ for popular regret and risk measures. We also presented some examples to demonstrate that the relations of maximum and positive combination do not pass from regret measures to risk measures. It would be interesting to find the natural and nontrivial regret measures for the maximum of several known coherent risk measures; for example, the worst case CVaR measure which was studied in Natarajan, Pachamanova & Sim [4], as well as for the weighted sum (discrete or continuous) of several known coherent risk measures; for example, the MAXVAR measure, which was recently studied in Sun and Yao [9].

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