Dynamic Pricing in Social Networks:
The Word of Mouth Effect*

Amir Ajorlou† Ali Jadbabaie† Ali Kakhbod‡

September 2014

Abstract

We study the problem of optimal dynamic pricing for a monopolist selling a product to consumers in a social network. The only means of spread of information about the product is via Word of Mouth communication; consumers’ knowledge of the product is only through friends who have already made a purchase. By analyzing the structure of the underlying endogenous process, we show that the optimal dynamic pricing policy for durable products drops the price to zero infinitely often, giving away the immediate profit in full to expand the informed network in order to exploit it in future. We provide evidence for this behavior from smartphone applications, where price histories indicate frequent free-offerings for many apps. Moreover, we show that despite infinitely often drops of the price to zero, the optimal price trajectory does not get trapped near zero. When externalities are present, we show that a strong enough network externality can push the price drops to a nonzero level, but similar price fluctuations to this new price floor still remain. When the product is nondurable, we show that the fluctuations disappear after a finite time.

Keywords: Information diffusion, word of mouth, price fluctuations, social networks, network externality, monopoly pricing.

JEL Classification: D42, O33, D85.

*We are grateful to Daron Acemoglu, Asu Ozdaglar, Xuanming Su, Alireza Tahbaz-Salehi, Rakesh Vohra, and the attendees of the 2014 W-PIN+NetEcon Workshop for helpful comments. This work was supported by ARO MURI W911NF-12-1-0509.

†Department of Electrical and Systems Engineering, University of Pennsylvania, Philadelphia, PA 19104, USA. E-mail: {ajorlou,jadbabaie}@seas.upenn.edu.

‡Laboratory for Information and Decision Systems (LIDS), Massachusetts Institute of Technology (MIT), Cambridge, MA 02139, USA. Email: akakhbod@mit.edu.
1 Introduction

How should a monopolist offering a product in a social network price its product over time? Does the profit-maximizing strategy always keep the prices monotone or steady? This paper introduces a new framework to investigate these questions by considering the mechanism by which information about a product diffuses in networks. In particular, our goal is to investigate the role of word of mouth (WOM) communications of consumers\(^1\), in the optimal pricing policy of the monopoly firm. The growing market for smartphone applications, where word of mouth is often the only means of spread of information about the product, is a great real world example of such a scenario.

During the last decade, there has been significant growth in the market for smartphone applications. Smartphone applications (apps) are typically cheap, and the only low budget\(^2\) means by which many of these apps spread is the word of mouth communication of their users.\(^3\) A good example is the smartphone application WhatsApp which was sold to Facebook in early 2014 for $19 billion. WOM was the key to WhatsApp popularity. As noted by Bloomberg (Satariano (2014)), “They [WhatsApp management] eschewed marketing and did not employ a public relations person, relying on the word of mouth recommendations of its users instead”. The price an app offers for its product is also a big driver for spreading the information. As such, posting time-varying prices is a common marketing tool for spreading information about the existence of a new app among the users. Figure 1 depicts the price history for tadaa 3D, an iPhone photo and video application since its release.\(^4\) An interesting observation from this chart is the frequent drops of the price to zero. The same pattern can be seen in the price trends of many other smartphone applications (e.g., XnShape, The Curse, Equalizer PRO\(^TM\), Color Vacuum, and ContactFlow, only to name a few\(^5\)).

Motivated by the above observations, we study the problem of optimal dynamic pricing of a profit-maximizing firm selling a product in a large social network where agents can only get informed about the product via word of mouth from a friend who has already bought

---

\(^1\) “Word of mouth communication involves the passing of information between a non-commercial communicator (i.e. someone who is not rewarded) and a receiver concerning a brand, a product, or a service”, Dichter (1966).

\(^2\) A recent survey by AppFlood (McCloughan (2013)) over 1000 independent small, medium and large app developers shows that the majority (78%) of developers surveyed had a per app marketing budget of $5000 or less.

\(^3\) Many apps ask users for permission to send notifications about the product to contacts in their address books or to post a message on their social pages, when they start using the app.

\(^4\) The price data is gathered using an application called appsfire.

\(^5\) These examples are chosen from various app categories of Photos & Videos, Games, Music, Education, and Utilities.
the product. A key feature of our work is modeling the effect of the price on the information diffusion via WOM. The (dynamic) price is a control variable by which the firm directly affects the information diffusion of its new product through the underlying social network. Firm’s problem is then to decide, at each time step, between optimally exploiting the existing informed network or charging a lower price in favor of a higher spread of information.

The main contribution of this paper is to study pricing in social networks through the channel of information diffusion. We show that when the spread of a durable product is only via word of mouth of its users, the optimal pricing policy is neither monotone nor steady. Rather, the optimal policy drops the price to zero infinitely often, essentially giving away the immediate profit in full to expand the informed network in order to exploit it in future. This is consistent with the real world evidence from smartphone applications described above. The key intuition behind this result is that frequent zero-price sales allow the firm to attract consumers who would not buy the product unless offered for free. Through the word of mouth of the zero-price buyers, the firm is able to reach sizeable parts of the network that would remain uninformed otherwise. By properly timing the drop, the firm can ensure that the marginal gain in future profit by selling the product in this previously unexplored part of the network prevails the loss in the immediate profit caused by offering the product for

---

6 We deal with smartphone applications as durable products. This is because when a user buys an app, she usually does not need to buy the same product any more.
free, making the drop of the price to zero a profitable course of action. We also show that, although the optimal policy drops the price to zero infinitely often, price will not get trapped near zero.

We further show that the durability of the product is a key driver for these frequent price drops. For a nondurable product, although the firm may initially make some free offers to expand its network, after a finite time it will fix the price at a level that extracts the maximum profit from the already informed population. In addition to the type of the product, the WOM nature of the information diffusion is another effective factor in play. Indeed, in the absence of WOM where the agents are assumed to be (ex ante) informed about the product, the optimal pricing policy will become monotonically decreasing over time. The infinitely often drops of the price to zero fully depend on the durability of the product as well as the nature of the strategic information diffusion via WOM in the social network. As for the final size of the informed population, it turns out that the pricing policy maximizing the profit for a durable product also maximizes the spread of the information in the network, while for a nondurable product the maximum profit is obtained by exploiting a smaller set of informed agents.

We also investigate the effect of network externalities on the price drops of a durable product. We show that the optimal policy still uses frequent price drops to sell the product to agents with low valuations of the product, in order to access new parts of the network that would remain untouched otherwise. In the presence of externalities, the valuations of the agents are augmented by the purchases made by their friends. This may enable the firm to sell the product to agents with low valuations at a nonzero price level, depending on the strength of the externality. As a result, we show that a strong enough network externality may shift the drops to a price level above zero.

1.1 Literature

This work is related to the literature on diffusion dynamics and pricing in social networks. One of the main challenges in information diffusion in social networks is developing tractable models. This is because combinatorial nature of networks with heterogeneity often makes analysis prohibitively difficult. Several modeling approaches have been developed to reduce the inherent complexity. An early model of diffusion is the Bass (1969) model. Although the proposed model does not capture any explicit social network structure, it still incorporates imitation from others. Some recent models use the concept of mean field theory to model
diffusion over the network. The main idea of this approach is to replace all interactions to an agent with an average or effective interaction. These tractable models have been used by Jackson and Rogers (2007b) to relate stochastic dominance properties of the degree distribution of the network to the depth of diffusion, by Young (2009) to provide methodologies for characterizing different models of social influence by the time path of adoption, by Jackson and Rogers (2007a) to infer how the formation process affects average utility in the network, and by Jackson and Yariv (2007) and López-Pintado (2008) to evaluate strategic adoption decisions of individuals.\(^7\)

Random graph theory has been used as a convenient modeling abstraction, which can facilitate modeling and analysis of the information diffusion in networks. Random networks find their origin in studies of random graph by Rapoport (1957) and Erdős and Rényi (1959, 1960, 1961). Random graph theory is also widely used by network scientists. For instance, it has been used by Watts and Strogatz (1998) to present their seminal small-world idea by creating highly clustered networks with small diameters, by Newman et al. (2001) to model the world-wide web and collaboration networks of company directors and scientists, and by Watts (2002) to model collective actions and the diffusion of norms and innovations. The main reason for the success of these models is that they relate the extent of the diffusion to an abstract model of the network. The current paper builds on this approach by using selling prices as a dynamic control for the information diffusion in a social network whose structure is captured by a Poisson random graph (Bollobás (2001)). The proposed model allows a firm to strategically affect the information diffusion about the existence of a new product within the social network by means of dynamic and different prices it charges over time.

Dynamic pricing has a rich history in economics and operation research.\(^8\) In general, varying prices over time may have different roots. It might be caused by the inability of the firms to commit to future actions (e.g. Conlisk et al. (1984), Sobel (1991)), or due to learning new experience goods (e.g. Bergemann and Välimäki (2006), Caminal and Vives (1999)), or the result of the inability of boundedly rational buyers to pay immediate attention to price changes (e.g. Radner et al. (2013)). Scarcity of the products with regard to the number of buyers (e.g. Gallego and van Ryzin (1994), Gershkov and Moldovanu (2009)),

\(^7\)Mean field theory is also used in revenue management, in particular, to study and model complex dynamic demand systems with the objective of maximizing performance, e.g. Gallego and van Ryzin (1994).
\(^8\)Talluri and van Ryzin (2004) and Phillips (2005) provide an extensive review of this topic.
\(^9\)These models are typically either two sided or one sided. Bergemann and Välimäki (1997, 2000), Ifrach et al. (2013), and Yu et al. (2013) consider two-sided learning models where buyers and sellers both learn the true value of a new product through consumer experiences. Papanastasiou et al. (2013) and Ifrach et al. (2011) analyze one-sided learning models when firm knows the product quality, buyers report their experiences and subsequent customers learn from these reports.
network externalities (e.g. Cabral et al. (1999)), stochastic incoming demand (e.g. Board (2008)), and time-varying values of buyers (e.g. Garrett (2013)) are among other causes suggested in the literature for varying prices over time. In particular, Garrett (2013) studies profit-maximizing prices in an environment where buyers arrive over time and have values for the good which evolve stochastically. The author shows that, for a range of parameter values, optimal prices fluctuate over time. Prices gradually fall up to sales dates and jump thereafter, mainly due to the inter-temporal price discrimination effect introduced by Stokey (1979). In contrast to the present paper, none of these works relate pricing to the extent of the information diffusion.

This paper is also related to the growing literature studying strategic interactions in social networks (e.g. Ballester et al. (2006), Bramoulle and Kranton (2007), Galeotti et al. (2010)). In contrast to these works, competition is absent in our setting. Moreover, for the most part, in these models prices are static. The closest result in literature to ours is the recent work in Campbell (2012), where the author studies pricing for a nondurable product under word of mouth communications. Campbell shows price fluctuations for a nondurable product during the introductory stages, when the size of the informed population is still very small. This is in line with our result for the nondurable product where we show that the firm may use price drops to zero during the early stages in order to expand its network. However, we show that fluctuations disappear as the size of the spread gets sufficiently large. Campbell models the word of mouth as a branching process, emphasizing that a branching model is only valid at the early stages of introducing a product to the network. Therefore, his model cannot explain the asymptotic properties of the process, which is the main focus of our analysis.

The rest of the paper is organized as follows. Section 2 presents a tractable model for strategic information diffusion via WOM in a large social network whose structure is represented by a Poisson random graph. Section 3 discusses the main challenge of the firm as deciding between spread and exploit, and elaborates on the main results of the paper. In Section 4, we extend the model to the case of networks with externalities, where we investigate the effect of network externalities on the price drops of a durable product. Finally, our conclusions are presented in Section 5.

10Other relevant studies include: strategic information exchange in social networks (e.g. Acemoglu et al. (2014)), optimal static pricing under presence of local network effect (Sundararajan (2008), Hartline et al. (2008), Candogan et al. (2012), Jadabaie et al. (2013)), and optimal advertising strategies in social networks (Galeotti and Goyal (2009), Galeotti and Mattozzi (2011)). More recently, Campbell (2013) studies static pricing strategies for selling a new product in a model of social interaction that builds on percolation theory.

11See also Su (2007), Nocke and Peitz, and Hörner and Samuelson (2011) for dynamic pricing with strategic customers. However, the price paths are found to be monotone in these works.


2 Model

2.1 General Description

The economy consists of a unit measure continuum of agents indexed by $i \in I = [0, 1]$. Agents reside in a social network, the structure of which is captured by an undirected random graph $G$ with Poisson degree distribution with mean $\lambda$. More precisely, each agent $i \in I$ has a total of $d_i \sim \text{Poiss}(\lambda)$ friends uniformly distributed in $I$.\footnote{This can be thought of as a limit case of the well-known Erdős-Rényi graph (Erdős and Rényi (1959)), keeping the mean degree equal to $\lambda$ and with $I = [0, 1]$ as the limit vertex set. This network model inherits independence of the edges from the Erdős-Rényi graph which proves very convenient in analyzing network behavior. A similar network model is used by Oberfield (2012), Larson (2013), and Galeotti and Goyal (2009).} We denote the set of the friends of $i$ in $G$ with $N_i$. For every $i \in I$, the set of her friends $N_i$ forms a Poisson process in $I$.

At each time step $t = 0, 1, 2, \cdots$, a firm offers a product to the continuum of agents in the network at price $u(t) \in U$, where $U$ is a finite set of admissible prices. The set of admissible prices $U$ can represent any set of quantized price levels in $[0, 1]$. In particular, we assume $0 \in U$ to allow for the free offer of the product. We can write the set of admissible prices as $U = \{p_0 = 0 < p_1 < \ldots < p_m \leq 1\}$, where $m \geq 1$ is the number of nonzero price levels.

Each agent has a private valuation $\theta \sim \text{Unif}[0, 1]$ for the product.\footnote{The results can be easily extended to a general distribution under mild assumptions on the corresponding CDF. However, for the sake of readability, we consider a uniform distribution for $\theta$ in developing the results, and discuss the extension to a general CDF in Appendix B.} The valuations of the agents are time-invariant and are independent of their degrees and the valuations of their friends. Moreover, agents’ valuations and their positions in the network are their private information, and hence, not known to the firm.

In order for an agent to buy the product, she should first be informed about its existence. At $t = 0$ to initiate the spread of information, a uniformly randomly chosen subset of the population becomes informed about the product directly by the firm. Later on, at any time $t \geq 1$, other agents can only get informed via word of mouth from a friend who has already bought the product. Note that at any time $t$, an informed agent buys the product if the offered price does not exceed her valuation, i.e. $u(t) \leq \theta$. Upon buying the product, she will inform her friends about the product.

In this framework, firm’s objective is to devise an optimal dynamic pricing policy...
maximizing its accumulated discounted profit over an infinite time horizon. We first study this problem for the case of a durable product, such as many smartphone applications, in order to justify the behavior pointed out in the previous section. We then investigate the role of the type of the product in the price drops to zero by studying the problem for a nondurable product. It is to be noted that an informed agent may buy a nondurable product at each time step, given that its price is lower than her valuation. However, if she buys a durable product at some time, she will not buy it thereafter. Finally, we extend our model to the case of networks with externalities, where we study the effect of network externalities on the price drops of a durable product.

2.2 WOM Diffusion Dynamics

In this subsection, we first present a few notations, definitions, and observations that will be used later to derive the dynamics of the information diffusion in the network. We denote the set of informed agents at time \( t \) by \( X(t) \) and its size by \( x(t) \). \( X(0) \) is therefore the set of those agents directly informed by the firm, with \( x(0) = x_0 \) denoting the size of this set. Considering that we are dealing with a unit measure continuum of agents, an informal use of the strong law of large numbers implies that \( x(t) = \text{Prob}(i \in X(t)) \). As we will see in the sequel, this will prove very convenient in deriving the dynamics of the information diffusion.\(^{14}\) The set of informed agents \( X(t) \) is increasing, that is \( X(t - 1) \subseteq X(t) \), and \( Y(t) = X(t) - X(t - 1) \) represents the set of freshly informed agents at time \( t \) whose size is denoted by \( y(t) \).

The set of agents that contribute to diffusion is comprised of two subsets. One set is the freshly informed agents in \( Y(t) \) that buy the product, hereafter denoted by \( B_Y(t) \). Noting that \( \theta \) has the same uniform distribution in \( Y(t) \) as it has in \( I \), the fraction of agents from \( Y(t) \) that buy the product when offered the price \( u(t) \) is \((1 - u(t))y(t)\). These new buyers spread the information about the product and their friends will constitute a fraction of \( Y(t + 1) \).

Another contribution to diffusion comes from the set of agents previously informed about the product who have not yet purchased. For such agents, the price has not fallen below their valuation since the time they were informed about the product. We denote the set of such agents at time \( t \) by \( Z(t) \). An agent in this set may buy the product at time \( t \) and thus inform her friends, if the offered price at time \( t \) is below her valuation. Unlike \( Y(t) \),

\(^{14}\)Following the same logic, we may exchangeably use the words size, fraction, and probability in the paper.
the distribution of \( \theta \) is not uniform for the agents in \( Z(t) \) and depends on the price history. However, we can use a stack of \( m \) variables (recall that \( m \) is the number of nonzero price levels) to fully describe the distribution of \( \theta \) in \( Z(t) \). We can partition \( Z(t) \) as \( \bigcup_{j=1}^{m} Z_j(t) \), where \( Z_j(t) \) is the set of those agents in \( Z(t) \) whose valuations lie between price levels \( p_{j-1} \) and \( p_j \), that is, \( Z_j(t) = \{ i \in Z(t) | p_{j-1} \leq \theta_i < p_j \} \). Then, the distribution of \( \theta \) in \( Z(t) \) is fully determined by the sizes of these sets, denoted by \( z(t) = [z_1(t) \ldots z_m(t)]^T \). If the firm chooses \( u(t) = p_r \) as the price at time \( t \), then all the agents in \( B_Z(t) = \bigcup_{j=r+1}^{m} Z_j(t) \) will buy the product and will subsequently inform their friends, while the rest of the agents in \( Z(t) \) will be carried over to \( Z(t+1) \). However, \( Z(t+1) \) also has another component coming from those freshly informed agents in \( Y(t) \) whose valuations are below \( p_{j-1} \) and \( p_j \) and have not yet bought the product:

\[
 z_j(t + 1) = \begin{cases} 
 z_j(t) + (p_j - p_{j-1})y(t), & 1 \leq j \leq r \\
 0, & \text{otherwise}
\end{cases} \tag{1}
\]

assuming that the offered price at time \( t \) is \( p_r \). As can be seen from the above discussion, in a WOM model of diffusion, the fresh buyers \( B(t) = B_Y(t) \cup B_Z(t) \) are responsible for expanding the network of informed agents. The size of this set is given by

\[
b(t) = (1 - p_r)y(t) + \sum_{j=r+1}^{m} z_j(t), \tag{2}
\]

where \( p_r \) is the price level chosen by firm at time \( t \) (\( u(t) = p_r \)). An agent \( i \in I \) will be uninformed at time \( t + 1 \) if and only if she neither is informed nor has a friend among fresh buyers at time \( t \). This can be written as

\[
\text{Prob}(i \notin X(t + 1)) = \text{Prob}(i \notin X(t) \land N_i \cap B(t) = \emptyset) \\
= \text{Prob}(i \notin X(t)) \times \text{Prob}(N_i \cap B(t) = \emptyset | i \notin X(t)). \tag{3}
\]

Using the stationarity increments property\(^{15}\) of Poisson processes, the number of friends agent \( i \notin X(t) \) has in \( B(t) \) is a Poisson random variable with mean \( \lambda b(t) \). Therefore, the probability of having no friend in \( B(t) \) for an agent \( i \notin X(t) \) is given by \( e^{-\lambda b(t)} \). It is worth mentioning that \( i \notin X(t) \) implies that \( i \) does not have any friend among the previous buyers \( \cup_{\tau=0}^{t-1} B(\tau) \).

\(^{15}\)According to the stationary increments property of Poisson processes, the probability distribution of the number of occurrences (herein friends) in any subset only depends on the size of the subset (Billingsley (1995)).
Figure 2: An overall view of the information diffusion via word of mouth. Agents in $B_Y(t) \cup B_Z(t)$ buy the product and inform their friends about it, forming $Y(t+1)$, the set of the freshly informed agents at time $t+1$.

However, this does not affect the distribution of her friends in $B(t)$ due to the independent increments property\textsuperscript{16} of Poisson processes. Using (3), we can write the dynamics of the informed population $x(t)$ as

$$1 - x(t + 1) = (1 - x(t))e^{-\lambda b(t)},$$

(4)

where $b(t)$ is given by (2), $z(t)$ is updated using (1), and $y(t + 1) = x(t + 1) - x(t)$. Moreover, $y(0) = x(0) = x_0$ and $z_j(0) = 0$ for $1 \leq j \leq m$. The overall structure of the information diffusion via WOM is depicted in Figure 2.

### 2.3 Durable and Nondurable Products

The profit of the firm for a durable product is given by

$$\Pi^D(u(\cdot)) = \sum_{t=0}^{\infty} \beta^t u(t)b(t),$$

(5)

where $0 < \beta < 1$ is the discount factor, and the marginal cost of the product is assumed to be zero. Firm’s objective is to find a pricing policy that maximizes the above profit, which

\textsuperscript{16}The independent increments property of Poisson processes states that the number of occurrences (herein friends) in disjoint subsets are independent of each other (Billingsley (1995)).
we denote as $u^D(\cdot)$.

For a nondurable product, every agent $i \in X(t)$ can buy the product if the offered price is below her valuation. Recalling that $\theta$ has a uniform distribution in $X(t)$, the size of the buyers at time $t$ is $(1 - u(t))x(t)$, and therefore the accumulated discounted profit of the firm over an infinite time horizon is given by

$$\Pi^{ND}(u(\cdot)) = \sum_{t=0}^{\infty} \beta^t u(t)(1 - u(t))x(t).$$  \hspace{1cm} (6)

Firm’s objective is to find the optimal pricing policy $u^{ND}(\cdot)$ that maximizes the above profit.

It is to be noted that the dynamics of the diffusion is the same for both cases, as in both an agent informs her friends about the product as soon as she buys it.

## 3 Firm’s Problem: To Spread or Exploit?

Given the dynamics of the information diffusion for the WOM model developed in previous section and the profit of the firm given by (5) and (6), the firm’s problem is to decide at each time step, between optimally exploiting the network it already has by offering a price that results in the maximum immediate profit, or offering a lower price in favor of a higher spread.

A related problem is to find the maximum achievable size of the informed network via WOM. For any price function $u(\cdot)$, $x(t)$ is bounded and increasing thus having a limit as $t \to \infty$. Define $q(x_0; u(\cdot)) = \lim_{t \to \infty} x(t)$. $q$ is the asymptotic size of the population that can be informed about the product via WOM, starting from a uniformly randomly chosen informed population of size $x_0$ and following a given pricing policy $u(\cdot)$. It is easy to see that for $x_0 < 1$ this asymptotic size is always less than 1, implying that the product cannot take over the entire population $I$ via only WOM. This is simply due to the fact that there are $e^{-\lambda}$ isolated agents (with no friend) in $I$, out of which $(1 - x_0)e^{-\lambda}$ of them are not in $X(0)$ and therefore will never hear about the product via WOM.

The case where the product is offered for free, i.e. $u \equiv 0$, gives an upperbound on the asymptotic size of the informed population, that is $q(x_0; u(\cdot)) \leq q(x_0; 0)$. In this case, every agent that gets informed about the product will in turn inform her friends. The information will then spread throughout the network and all the agents that are reachable from an agent.
\(i \in X(0)\) will eventually know about the product. In this case, \(Z(t) = \emptyset\) and \(B(t) = Y(t)\), thus the dynamics of diffusion governed by (1), (2), and (4) simplifies to

\[
1 - x(t + 1) = (1 - x(t))e^{-\lambda y(t)},
\]

\[
y(t + 1) = x(t + 1) - x(t),
\]

where \(y(0) = x(0) = x_0\). Using this recursively for \(t, t - 1, \ldots, 0\), we obtain

\[
1 - x(t + 1) = (1 - x_0)e^{-\lambda x(t)}.
\]

The asymptotic size of the informed network for a free product can be obtained noting that \(q(x_0; 0)\) should satisfy the above relation:

\[
1 - q(x_0; 0) = (1 - x_0)e^{-\lambda q(x_0; 0)}.
\]

Based on this equation, we present several properties for \(q(x_0; 0)\) in the following proposition.

**Proposition 1.** For every \(0 < x_0 \leq 1\), the asymptotic size of the informed population for a free product is given by the unique solution of \(1 - q(x_0; 0) = (1 - x_0)e^{-\lambda q(x_0; 0)}\) in \([0, 1]\). The solution is concave and monotonically increasing in \(x_0\). Moreover, \(q(x_0; 0) > 1 - \frac{1}{\lambda} \) \(^{17}\)

*Proof.* See the appendix. \(\blacksquare\)

One interesting consequence of Proposition 1 is the discontinuity in \(q(x_0; 0)\) at \(x_0 = 0\) for \(\lambda > 1\). Although \(q(0; 0) = 0\), for any nonzero \(x_0\) and \(\lambda > 1\), \(q(x_0; 0)\) is lowerbounded by a positive constant independent of \(x_0\). This implies that no matter how small the size of the initially informed population is, a free product can take over a large portion of the network via WOM given the typically large average number of friends in the networks.

Another interesting problem is to identify the pricing policies under which the informed network achieves the maximum asymptotic size \(q(x_0; 0)\). We claim that this happens if and only if the price drops to zero infinitely often. The rough idea for the sufficiency of this condition is that, an agent who is informed by time \(t\) under the zero price policy, is also surely informed under the pricing policy \(u(\cdot)\) after the \(t\)-th drop of the price to 0. The

\(^{17}\)\(q(x_0; 0)\) can also be represented in terms of the Lambert W function, which is defined as the solution to the equation \(W(z)e^{W(z)} = z\) (Corless et al. (1996)). Using this notation, we can easily show that \(\lambda(1 - q(x_0; 0)) = -W(-\lambda e^{-\lambda}(1 - x_0))\). W is known to have two branches. It follows from Proposition 1 that \(\lambda(1 - q(x_0; 0)) < 1\), requiring \(W > -1\). This identifies the principal branch of the Lambert W function, denoted by \(W_0\). Therefore, we can write \(\lambda(1 - q(x_0; 0)) = -W_0(-\lambda e^{-\lambda}(1 - x_0))\). This representation enables us to use the properties of the Lambert W function, if ever needed.
necessity part is more involved and is proved by finding a nonzero measure subset of agents that will be part of the informed population for a zero price function, but cannot be informed about the product if the price drops to zero only a finite number of times. This result is presented in the next proposition.

**Proposition 2.** For a pricing policy \( u(\cdot) \), the asymptotic size of the informed population \( q(x_0; u(\cdot)) \) achieves its maximum \( q(x_0; 0) \) if and only if the price drops to zero infinitely often, that is, \( q(x_0; u(\cdot)) = q(x_0; 0) \) if and only if there exists an infinite sequence of time instants \( 0 \leq t_0 < t_1 < \ldots \) such that \( u(t_j) = 0 \) for \( j \in \mathbb{N}_0 \).

*Proof.* Consider a pricing policy \( u(\cdot) \) with infinitely many drops to zero. Denote by \( X^0(t) \) the set of informed agents for the zero pricing policy. Then, it is straightforward to see that \( X^0(\tau) \subseteq X(t_{\tau-1} + 1) \), for any \( \tau \in \mathbb{N} \), yielding \( q(x_0; 0) \leq q(x_0; u(\cdot)) \). This, along with \( q(x_0; u(\cdot)) \leq q(x_0; 0) \) implies that \( q(x_0; u(\cdot)) = q(x_0; 0) \).

To prove the necessity, assume that \( u(\cdot) \) has a finite number of drops to zero. Therefore, there exists \( T \geq 0 \) such that \( u(t) \geq p_1 \) for \( t \geq T \), where \( p_1 \) is the smallest nonzero price level in \( U \). Let \( Y_1(T) \subset Y(T) \) be those freshly informed agents at time \( T \) whose valuations are below \( p_1 \), i.e. \( Y_1(T) = \{ i \in Y(T) | 0 \leq \theta_i < p_1 \} \). Recall, also, that \( Z_1(T) \) is the set of those previously informed agents whose valuations are below \( p_1 \) and have not yet bought the product. Note that the agents in \( Y_1(T) \cup Z_1(T) \) will never buy the product under the price policy \( u(\cdot) \). Therefore, if we define \( \Delta \) as those agents \( i \notin X(T) \) that have at least one friend and that their friends are only among agents in \( Y_1(T) \cup Z_1(T) \), then they will never get informed about the product under the price policy \( u(\cdot) \), while they would have been informed under the zero price policy. All we need to do now is to calculate the size of this set in order to show that it has a nonzero measure. The size of this set is given by

\[
\delta = \text{Prob}(i \notin X(T) \land N_i \subset Y_1(T) \cup Z_1(T) \land d_i \neq 0)
= \text{Prob}(i \notin X(T)) \times \text{Prob}(N_i \subset Y_1(T) \cup Z_1(T) \land d_i \neq 0 | i \notin X(T)). \tag{11}
\]

For \( i \notin X(T) \), the condition \( N_i \subset Y_1 \cup Z_1 \) is equivalent to \( N_i \cap (X^c \cup (Z - Z_1) \cup (Y - Y_1)) = \emptyset \), where \( X^c = I - X(T) \) and we have dropped the argument \( (T) \) for the sake of readability. The number of friends of \( i \notin X(T) \) in \( X^c \cup (Z - Z_1) \cup (Y - Y_1) \) is a Poisson random variable with a mean of \( \lambda b_1(T) \), where

\[
b_1(T) = 1 - x(T) + (1 - p_1)y(T) + \sum_{j=2}^{m} z_j(T). \tag{12}
\]
So, the probability of not having a friend in $X^c \cup (Z - Z_1) \cup (Y - Y_1)$ for $i \notin X(T)$ will be $e^{-\lambda b_1(T)}$. Similarly, the probability of having at least one friend in $Y_1 \cup Z_1$ can be obtained to be $1 - e^{-\lambda(p_1 y(T) + z_1(T))}$. Putting all these together, we can use (11) to get

$$\delta = (1 - x(T))e^{-\lambda b_1(T)}(1 - e^{-\lambda(p_1 y(T) + z_1(T))}),$$

from which we can easily see that $\delta \neq 0$. The proof is now complete by noting that $q(x_0; u_{\cdot}) \leq q(x_0; 0) - \delta < q(x_0; 0)$.

As the main objective of this paper, we next consider the case of a durable product and we show that under the optimal policy price should drop to zero infinitely often. This is in line with the real world evidence from smartphone applications discussed in Section 1, where price histories witness frequent drops of the price to zero for many apps. Moreover, using these frequent price drops to zero we can show that the optimal pricing policy maximizing the profit for a durable product also maximizes the spread of the information in the network. These results are summarized in the next theorem.

**Theorem 1.** Under the optimal pricing policy $u^D(\cdot)$ for a durable product, the price drops to zero infinitely often. Moreover, $q(x_0; u^D(\cdot)) = q(x_0; 0)$, that is, the network of informed agents achieves its maximum asymptotic size under this optimal policy.

**Proof.** First we note that for any pricing policy, there exists at least one price level that holds infinitely often. This follows from the finiteness of the set of admissible prices $U$. Let $p_r \in U$ be the smallest price level which holds infinitely often for $u^D(\cdot)$. Then, any price level below $p_r$ is used finitely in $u^D(\cdot)$. Therefore, there exists $T \geq 0$ such that $u^D(t) \geq p_r$ for all $t \geq T$.

Having infinitely many drops to zero under the optimal policy $u^D(\cdot)$ is clearly equivalent to $p_r = 0$ (that is $r = 0$). Therefore, to prove the theorem, we assume $r \geq 1$ and try to reach contradiction by constructing a new policy with a profit higher than that of $u^D(\cdot)$. For this purpose, we show that by zeroing the price to sell the product to a subset of informed agents that would not buy it otherwise, we can reach out a part of the network that would remain unexplored under $u^D(\cdot)$. Dropping the price to zero to access this part of the network at a proper time, we then introduce a new policy yielding a profit higher than that of $u^D(\cdot)$.

Let $Y^D_r(T) \subset Y^D(T)$ denote those freshly informed agents at time $T$ whose valuations are below $p_r$, i.e. $Y^D_r(T) = \{i \in Y^D(T) | 0 \leq \theta_i < p_r\}$, with a size of $y^D_r(T) = p_r y^D(T)$.

---

\[\text{We use superscript } D \text{ to indicate that the variables correspond to the pricing policy } u^D(\cdot).\]
None of the agents in $\bigcup_{j=1}^{r} Z_j^D(T) \cup Y_r^D(T)$ will ever buy the product under the pricing policy $u^D(\cdot)$, where $\bigcup_{j=1}^{r} Z_j^D(T)$ is the set of those previously informed agents at time $T$ whose valuations are below $p_r$. Now, consider the set of agents that will remain uniformed under $u^D(\cdot)$. The size of this set is clearly $1 - q^D$, where $q^D$ is the asymptotic size of the informed population under $u^D(\cdot)$, i.e., $q^D = q(x_0; u^D(\cdot))$. Define $\Delta_r$ as the subset of these agents who have at least a friend in $\bigcup_{j=1}^{r} Z_j^D(T) \cup Y_r^D(T)$, that is

$$\Delta_r = \{ i \in I | (\exists t \in \mathbb{N}_0 : i \in x^D(t)) \land d_i(\bigcup_{j=1}^{r} Z_j^D(T) \cup Y_r^D(T)) \neq 0 \}.$$  

(14)

(19)

Noting that the number of friends of an uninformed agent among $\bigcup_{j=1}^{r} Z_j^D(T) \cup Y_r^D(T)$ has a Poisson distribution with mean $\lambda(p_r y^D(T) + \sum_{j=1}^{r} z_j^D(T))$, we can easily show that the size of this set is given by

$$\delta_r = (1 - q^D)(1 - e^{-\lambda(p_r y^D(T) + \sum_{j=1}^{r} z_j^D(T))}),$$  

(15)

from which it follows that $\delta_r \neq 0$. The idea is to show that after a while there is so little profit left to be made in future under $u^D(\cdot)$ that it is profitable to zero the price to reach out agents in $\Delta_r$, as will be elaborated below.

Let $t_k$, $k = 1, 2, \ldots$, denote the $k$-th price drop to $p_r$ after time $T$ under the optimal policy $u^D(\cdot)$. If an agent $i \in X^D(t_k)$ does not buy the product at this time, neither will she buy it in future. This means that agents in $X^D(t_k)$ do not contribute to the set of buyers $B^D(t)$ for $t > t_k$. Therefore, the size of the buyers from time $t_k + 1$ to $t_k + \tau$ for any $\tau \geq 1$ can be upperbounded by $x^D(t_k + \tau) - x^D(t_k)$, that is

$$\sum_{t=t_k+1}^{t_k+\tau} b^D(t) \leq x^D(t_k + \tau) - x^D(t_k).$$  

(16)

Shifting $\tau \to \infty$, yields

$$\sum_{t=t_k+1}^{\infty} b^D(t) \leq q^D - x^D(t_k).$$  

(17)

---

19For any $S \subseteq I$ and $i \in I$, we denote the number of friends of agent $i$ in $S$ with $d_i(S)$. 

---
Thus, the contribution of the buyers to the firm’s profit after $t_k$ can be upperbounded by

$$
\Pi_{\geq t_k}^D(u^D(\cdot)) = \sum_{t=t_k+1}^{\infty} \beta^t u^D(t)b^D(t)
\leq \beta^{t_k+1} \sum_{t=t_k+1}^{\infty} u^D(t)b^D(t)
\leq \beta^{t_k+1} p_m(q^D - x^D(t_k)).
$$

(18)

Now, consider a new policy $\tilde{u}(\cdot)$ having the same value as $u^D(\cdot)$ at all times except $t_k + 1$ and $t_k + 2$. Let $\tilde{u}(t_k + 1) = 0$ and $\tilde{u}(t_k + 2) = u^*$, where $u^*(1 - u^*) = \max_{u \in \mathcal{U}} u(1 - u)$. Note that agents in $\Delta_r$ are among the freshly informed agents $\tilde{Y}(t_k + 2)$ since the agents in $\bigcup_{j=1}^r Z_j^D(T) \cup Y_r^D(T)$ buy the product at time $t_k + 1$. $\theta$ has a uniform distribution on $\Delta_r$, hence the discounted profit made from $\Delta_r$ at time $t_k + 2$ is $\beta^{t_k+2} u^*(1 - u^*)\delta_r$. Considering that $x^D(t) \to q^D$ as $t \to \infty$, we can choose $k$ large enough such that

$$
q^D - x^D(t_k) < \beta \delta_r u^*(1 - u^*) \frac{1}{p_m},
$$

(19)
in which case the profit resulted from $\tilde{u}(\cdot)$ will be clearly higher than that from $u^D(\cdot)$. This contradicts the optimality of $u^D(\cdot)$, hence rejecting the initial assumption of $p_r \neq 0$, which completes the proof. The second part of the theorem directly follows from Proposition 2. ■

The above theorem shows infinitely many price drops to zero for a durable product under the optimal policy. A question that arises here is that is it possible for the optimal price trajectory to get trapped in a vicinity of zero? Noting that the price cannot stay at zero forever (due to the zero profit from such stay), this question translates to the possibility of getting stuck between price levels 0 and $p_1$ for small values of $p_1$. If such thing happens, one may even falsely attribute the price drops to a continuous-valued optimal price trajectory asymptotically converging to zero which is being simulated by a quantized price path bouncing between 0 and $p_1$. The following proposition rejects the possibility of such a price lockdown.

**Proposition 3.** Under the optimal pricing policy $u^D(\cdot)$, the price jumps to a level above $p_1$ infinitely often, given that $p_1 < \frac{1 - \sqrt{1 - 4c}}{2}$, where

$$
c = \max_{u \in \mathcal{U} \setminus \{p_1\}} \frac{u(1 - u)(1 - \beta \lambda (1 - q^0))}{1 - \beta \lambda (1 - q^0)(1 - u)},
$$

(20)
and \( q^0 = q(x_0; 0) \) is the asymptotic size of the informed population under the zero price policy.

**Proof.** See the appendix. ■

Based on this result, how small should \( p_1 \) be to guarantee frequent price jumps to levels above \( p_1 \)? In fact, as we will see in the sequel, \( p_1 \) does not need to be very small to satisfy the above condition for a wide range of parameters. It is easy to verify that \( c \) is decreasing with both \( \beta \) and \( \lambda(1 - q^0) \). So, the less the value of each, the looser the bound on \( p_1 \). Although it may look strange at first, the term \( \lambda(1 - q^0) \) is very helpful in loosening the bound on \( p_1 \). This term is less than 1 (from Proposition 1), and indeed is decreasing with \( \lambda \) for \( \lambda \geq 1 \).

We now use this background to study a few cases in order to get an insight on the values of \( p_1 \) satisfying the above condition. We assume \( 0.5 \in \mathcal{U} \) to use it as a (sub)maximizer in (20). If \( \lambda \geq 2 \), then \( \lambda(1 - q^0) < 0.41 \),\(^{21}\) which along with \( \beta < 1 \) yields \( c > 0.18 \), for which the condition in the above proposition reduces to \( p_1 < 0.24 \). Some information on \( \beta \) can loosen up this bound even further. For example, if we also know that \( \beta \leq 0.5 \), then the condition on \( p_1 \) becomes \( p_1 < 0.33 \). This bound gets closer to 0.5 for larger \( \lambda \) (or smaller \( \beta \)), assuring infinitely many price jumps to levels above \( p_1 \) even for values of \( p_1 \) that are not very small.

The WOM nature of the information diffusion is apparently a key driver for dropping the price to zero. If agents (users) are not involved in spreading the information about the product, the firm will not have any incentive to drop the price to zero. In fact, it is easy to show that for the case of full information, in which everybody is directly informed by the firm, the optimal pricing policy is monotone (decreasing) exploiting those who are willing to pay more first and then gradually lowering the price. Beside the WOM nature of the information diffusion, these drops are also rooted in the type of the product being offered. It can be shown that, for a nondurable product, when the size of the informed population gets large enough, the optimal policy is to set the price at a level that extracts the maximum profit out of the already informed population. The price level maximizing the immediate profit is identified in the lemma below.

**Lemma 1.** Let \( u^* \) denote the closest price level in \( \mathcal{U} \) to \( \frac{1}{2} \), that is

\[
  u^* = \arg\min_{u \in \mathcal{U}} |\frac{1}{2} - u|.
\]

\(^{20}\)Recall that \( \lambda(1 - q^0) = -W_0(-\lambda e^{-\lambda}(1 - x_0)) \), where \( W_0 \) is the principal branch of the Lambert W function (see Footnote 17). \( W_0 \) is an increasing function, and \( -\lambda e^{-\lambda} \) is also increasing with \( \lambda \) for \( \lambda \geq 1 \). This implies that \( \lambda(1 - q^0) \) is decreasing with \( \lambda \) for \( \lambda \geq 1 \).

\(^{21}\)\( \lambda(1 - q^0) \leq -W_0(-2e^{-2}) = 0.4064 \), for \( \lambda \geq 2 \).
If there are two such price levels in $\mathcal{U}$, denote the smaller one with $u^*$. Then, $u^*$ maximizes the immediate profit $u(t)(1 - u(t))x(t)$ for a nondurable product. More importantly, $u_{\text{ND}}(t) \leq u^*$ for all $t \geq 0$.

Proof. First part is obvious. Second part follows from the fact that lowering the price to $u^*$ increases both the immediate profit and the size of the informed population at future times. ■

The steady state fixed-price property for a nondurable product described above is presented in the next theorem.

**Theorem 2.** Given the optimal pricing policy $u_{\text{ND}}(\cdot)$ for a nondurable product, there exists a finite time $T$ after which the price is set to the fixed level $u^*$ maximizing the immediate profit, that is $u_{\text{ND}}(t) = u^*$ for $t \geq T$.

Proof. Denote with $q_{\text{ND}}$ the asymptotic size of the informed population under the optimal policy $u_{\text{ND}}(\cdot)$, i.e., $q_{\text{ND}} = q(x_0; u_{\text{ND}}(\cdot))$. We claim that when the size of the informed population gets large enough, then no price other than $u^*$ can be used by the optimal policy. In particular, we show that if $x_{\text{ND}}(t) > \gamma q_{\text{ND}}$, where

$$\gamma = \max \left\{ u \in \mathcal{U} | u < u^* \right\} \frac{\beta u^* (1 - u^*)}{u^* (1 - u^*) - (1 - \beta) u (1 - u)},$$

then $u_{\text{ND}}(t) = u^*$. Clearly $\gamma < 1$ on noting that $u(1 - u) < u^* (1 - u^*)$ for every $u < u^*$ in $\mathcal{U}$, according to Lemma 1.

In order to prove the above claim, we assume there is some time $t_0$ at which $x_{\text{ND}}(t_0) > \gamma q_{\text{ND}}$, but $u_{\text{ND}}(t_0) \neq u^*$, and we try to reach contradiction by constructing a new policy with a higher profit. We construct the new policy $\tilde{u}(\cdot)$, by shifting $u_{\text{ND}}(\cdot)$ one step to right for $t > t_0$, changing the price to $u^*$ at $t_0$, and keeping the policy unchanged for $t < t_0$. Writing this in a more formal language, we have

$$\tilde{u}(t) = \begin{cases} u_{\text{ND}}(t), & t < t_0 \\ u^*, & t = t_0 \\ u_{\text{ND}}(t - 1), & t > t_0 \end{cases}$$

The key observation here is to note that defining $\tilde{u}$ in this way, any agent who gets informed about the product under the optimal policy $u_{\text{ND}}(\cdot)$ will also get informed under the new policy $\tilde{u}(\cdot)$ with at most one step delay. This assures $X_{\text{ND}}(t - 1) \subseteq \tilde{X}(t)$ for $t > t_0$, implying
that \( x^{ND}(t - 1) \leq \tilde{x}(t) \) for \( t > t_0 \). Using this, we can lowerbound the accumulated discounted profit under the new policy \( \tilde{u}(\cdot) \) from time \( t_0 \) on by the immediate profit under this policy, plus the accumulated discounted profit under the optimal policy \( u^{ND}(\cdot) \) from time \( t_0 \) on discounted by \( \beta \) to account for the one step delay. This can be written as

\[
\Pi_{\geq t_0}^{ND}(\tilde{u}(\cdot)) = \sum_{t=t_0}^{\infty} \beta^t \tilde{u}(t)(1 - \tilde{u}(t))\tilde{x}(t) \\
\geq \beta^{t_0} u^*(1 - u^*)x^{ND}(t_0) + \beta \Pi_{\geq t_0}^{ND}(u^{ND}(\cdot)),
\]

where \( \Pi_{\geq t_0}^{ND}(u^{ND}(\cdot)) \) is the accumulated discounted profit under the optimal policy \( u^{ND}(\cdot) \) from time \( t_0 \) on.\(^{22}\)

The profit of the firm for \( t < t_0 \) is the same under both policies. Therefore, in order to prove that the new policy \( \tilde{u}(\cdot) \) results in a higher profit than \( u^{ND}(\cdot) \), we need to show that \( \Pi_{\geq t_0}^{ND}(\tilde{u}(\cdot)) > \Pi_{\geq t_0}^{ND}(u^{ND}(\cdot)) \). Applying (24), it thus suffices to show that

\[
(1 - \beta)\Pi_{\geq t_0}^{ND}(u^{ND}(\cdot)) < \beta^{t_0} u^*(1 - u^*)x^{ND}(t_0).
\]

We can use Lemma 1 to find an upperbound for \( \Pi_{\geq t_0}^{ND}(u^{ND}(\cdot)) \) as\(^{23}\)

\[
\Pi_{\geq t_0}^{ND}(u^{ND}(\cdot)) \leq \beta^{t_0} u^{ND}(t_0)(1 - u^{ND}(t_0))x^{ND}(t_0) + \frac{\beta^{t_0+1}}{1 - \beta} u^*(1 - u^*)q^{ND}.
\]

Using the above upperbound along with the fact that \( x^{ND}(t_0) > \gamma q^{ND} \), where \( \gamma \) is defined in (22), and after some simplifications, we can easily show that (25) holds. This completes the proof.\( \blacksquare \)

Although the above theorem assures that there will be no free offer of the product after some finite time in the nondurable case, it is possible that the firm drops the price to zero during the early stages in order to expand its network, as shown in the next theorem.

**Theorem 3.** Consider the optimal pricing policy \( u^{ND}(\cdot) \) for a nondurable product and assume that \( \beta > \frac{1 - u^*}{\varphi(\lambda^* - 1)} \), where \( \lambda^* = (1 - u^*)\lambda \). Then, there exists \( x^c > 0 \) such that for \( x^{ND}(t) < x^c \):

i) \( u^{ND}(t)u^{ND}(t + 1) = 0 \),

ii) \( x^{ND}(t + 1) > \lambda^* x^{ND}(t) \).

\(^{22}\)See Appendix A for more details on how to obtain (24).

\(^{23}\)See Appendix A for more details on how to obtain (26) and to use it in proving (25).
Proof. The proof is based on induction. It is a bit long and is included in the appendix. Part i) is proved by showing that if \( u^{ND}(t)u^{ND}(t+1) \neq 0 \) for some time \( t \) at which \( x^{ND}(t) < x^c \), then a new policy that is obtained from \( u^{ND}(\cdot) \) by only changing \( u^{ND}(t) \) to 0 will result in a higher profit. Part ii) provides a lower bound on the growth rate of the informed population for small \( x^{ND}(t) \). It can also be viewed as a lower bound on the relative size of the freshly informed population to the whole informed set. To see this, we can use \( y^{ND}(t+1) = x^{ND}(t+1) - x^{ND}(t) \) to write ii) as \( y^{ND}(t+1) > \frac{x^c}{\lambda^*}x^{ND}(t+1). \) This view will play a significant role in proving part i).

Part i) of the above theorem implies that, as long as the size of the informed population is below a certain threshold, there are no successive nonzero price levels in the optimal policy. This means that the firm should initially offer the product for free at least half of the time in order to expand its network. Part ii) presents a lower bound on the effectiveness of these free offers. Using this, it is easy to see that for \( \lambda^* > 1 \), these drops can result in an exponential growth of the informed population.

4 Price Drops in the Presence of Network Externalities

A firm offering a product in a social network can enjoy a positive effect on the spread of its product from network externalities. An informed agent who does not buy the product at a given price may do so later on if many of her friends buy the product, even if the firm does not lower the price. This raises another interesting question as to whether the price drops would be still profitable in the presence of network externalities? The aim of this section is to investigate the effect of network externalities on the price drops of a durable product.

In the presence of network externalities, an informed agent buys the product if the offered price does not exceed the sum of her valuation and the total externalities from her friends who are already using the product. Denote as \( B(t) = \bigcup_{\tau=0}^{t-1} B(\tau) \) the set of all previous buyers at time \( t \) and let \( 0 < \alpha \leq 1 \) represent the network externality effect. Then, an informed agent \( i \in I \) buys the product at time \( t \) if the offered price \( u(t) \) does not exceed her augmented valuation defined as \( \theta^a_i(t) = \theta_i + \alpha d_i(B(t)) \), where \( d_i(B(t)) \) denotes the number of friends at time \( t \) who have already bought the product.

The first step is to identify \( B(t) \), the set of the fresh buyers at time \( t \). This is required for both deriving the dynamics of information diffusion and determining the profit of the firm in the presence of network externalities. As before, fresh buyers are either among the freshly
informed agents \( Y(t) \) or among those previously informed agents who have not yet bought the product, denoted as \( Z(t) \). Unlike \( \theta \), the augmented valuation does not have a uniform distribution in \( Y(t) \). Define the set of those agents in \( Y(t) \) whose augmented valuations are below \( \theta^a \) as \( Y(\theta^a, t) \) and its size by \( y(\theta^a, t) \). Note that \( y(\theta^a, t) \) fully characterizes the distribution of the augmented valuation in \( Y(t) \). Similarly, we use \( Z(\theta^a, t) \) and \( z(\theta^a, t) \) to represent the set of those agents in \( Z(t) \) whose augmented valuations are below \( \theta^a \) and its size, respectively. Having been offered a price \( u(t) \in U \), those agents in \( Y(t) \cup Z(t) \) whose augmented valuations are higher than (or equal to) \( u(t) \) will buy the product, resulting in a set of buyers of size \( b(t) = b_Y(t) + b_Z(t) \), where \( b_Y(t) = y(t) - y(u(t), t) \) and \( b_Z(t) = z(t) - z(u(t), t) \). Agents in \( Y(u(t), t) \cup Z(u(t), t) \) do not have a high enough augmented valuation to buy the product at this price and will form \( Z(t+1) \).

Upon buying the product, the new buyers \( B(t) \) will inform their friends about the product. A freshly informed agent \( i \in Y(t+1) \), has at least one friend in \( B(t) \) but no friend in the set of buyers before \( t \) (otherwise, she would have been already informed by time \( t \)). For such an agent, the augmented valuation is \( \theta^a_i(t+1) = \theta_i + \alpha d_i(B(t)) \). The number of friends of an uninformed agent at time \( t \) in \( B(t) \) has a Poisson distribution with mean \( \lambda b(t) \). Using this along with the uniform distribution of \( \theta \), we can find the update rule for \( y(\theta^a, t+1) \), the size of the freshly informed agents at time \( t+1 \) whose augmented valuations are below \( \theta^a \), as

\[
y(\theta^a, t+1) = (1 - x(t))e^{-\lambda b(t)} \sum_{d=1}^{\infty} \frac{\lambda^d b(t)^d}{d!} U(\theta^a - \alpha d),
\]

where \( U \) is the CDF of Unif[0,1]. To find the update rule for \( z(\theta^a, t+1) \), consider an agent \( i \in Z(t+1) = Y(u(t), t) \cup Z(u(t), t) \), that is, an agent which was informed by time \( t \) but has not yet bought the product. The augmented valuation of such an agent may increase by time \( t+1 \) as some of her friends may buy the product at time \( t \). More precisely, \( \theta^a_i(t+1) = \theta^a_i(t) + \alpha d_i(B(t)) \). Noting again that \( d_i(B(t)) \) has a Poisson distribution with mean \( \lambda b(t) \), and that \( Y(u(t), t) \) and \( Z(u(t), t) \) are obtained by truncating the augmented valuations in \( Y(t) \) and \( Z(t) \) with \( u(t) \), we can write

\[
z(\theta^a, t+1) = e^{-\lambda b(t)} \sum_{d=0}^{\infty} \frac{\lambda^d b(t)^d}{d!} (y(\min(\theta^a - \alpha d, u(t)), t) + z(\min(\theta^a - \alpha d, u(t)), t)).
\]

---

24 In fact, \( y(\theta^a, t) \) is the CDF of \( \theta^a \) in \( Y(t) \) multiplied by its size \( y(t) \).

25 In general, for any \( S(t) \subseteq I \), we use the notation \( S(\theta^a, t) \) to denote the set of those agents in \( S(t) \) whose augmented valuations are below \( \theta^a \) and \( s(\theta^a, t) \) to denote its size.

26 Note that in our model, we only use the value of \( z(\theta^a, t) \) for \( \theta^a \in U \). However, updating \( z \) at these points
From the above model, it is clear that the firm may not need to drop the price to zero in order to sell the product to agents with low valuations and reach out new customers via their word of mouth. Any informed agent who is not directly informed by the firm, has at least a friend among previous buyers, shifting her augmented valuation above (or equal to) \( \alpha \). Therefore, the lowest price that firm has to offer to ensure that such an agent buys the product is \( \alpha \). If \( \alpha \) is not an admissible price level, i.e. \( \alpha \notin \mathcal{U} \), firm may instead drop the price to the closest price level to \( \alpha \) from below, which we denote with \( u^\alpha \). We show in the next theorem that, under the optimal policy, the firm should drop the price to \( u^\alpha \) infinitely often. Firm may also make a few drops to price levels below \( u^\alpha \) to expand its network via those low-valued agents in \( X_0 \) (directly informed by the firm) who do not have any friend among buyers.

**Theorem 4.** Let \( u^\alpha \) denote the closest admissible price level to \( \alpha \) from below, i.e. \( u^\alpha = \max\{u \in \mathcal{U} | u \leq \alpha \} \), where \( 0 < \alpha \leq 1 \) represents the network externality effect. Then, the optimal pricing policy for a durable product should drop the price to \( u^\alpha \) infinitely often. Moreover, the number of drops to price levels below \( u^\alpha \) is finite.

**Proof.** Following the same line as of the proof of Theorem 1, let \( p_r \in \mathcal{U} \) denote the smallest price level which holds infinitely often for the optimal pricing policy \( u^D(\cdot) \). Since any price level below \( p_r \) is used only finitely by \( u^D(\cdot) \), there exists \( T \geq 0 \) such that \( u^D(t) \geq p_r \) for all \( t \geq T \). The above theorem is then equivalent to having \( p_r = u^\alpha \).

We start by showing that \( p_r \leq \alpha \). To prove this, we assume \( p_r > \alpha \) and try to reach contradiction by constructing a new policy with a profit higher than that of \( u^D(\cdot) \). We do this by first identifying a nonzero measure set of informed agents who will not buy the product under \( u^D(\cdot) \). Then, we show that by dropping the price to \( u^\alpha \) at a proper time and selling the product to these agents we can reach a part of the network which would remain unexplored under \( u^D(\cdot) \), using which we show that the modified policy yields a higher profit than \( u^D(\cdot) \).

Consider \( Y^D(p_r, T) \), the set of freshly informed agents at time \( T \) whose augmented valuations are below \( p_r \). We can lowerbound the size of this set using only the term corresponding to \( d = 1 \) in (27), as

\[
y^D(p_r, T) \geq (1 - x^D(T - 1)) e^{-\lambda b^D(T - 1)} \lambda b^D(T - 1)(p_r - \alpha),
\]

also requires its value at \( \theta^a - \alpha d \), where \( 0 \leq d \leq \lfloor \frac{\theta^a}{\alpha} \rfloor \). This means that, unlike the case with no externalities, characterizing \( z \) requires more than \( m \) states but still can be done using a finite number of states.
which implies \( y^D(p_r, T) > 0 \) using the assumption \( p_r > \alpha \). Noting that \( u^D(t) \geq p_r \) for \( t \geq T \), agents in \( Y^D(p_r, T) \) will not buy the product at time \( T \). If there were no externalities, we could conclude that these agents would never buy the product under \( u^D(\cdot) \). However, due to network externalities, agents in this set may eventually buy the product if many friends do so, elevating their augmented valuations above \( p_r \). We could conclude that these agents would never buy the product under the optimal policy. Hence, we have shown that the probability of not having any friend among the buyers at time \( T \) is

\[
\Pr(d_i(\cup_{\tau=T}^{\infty} B^D(\tau)) = 0 | i \in Y^D(p_r, T)) = e^{-\lambda \sum_{\tau=T}^{\infty} b^D(\tau)}
\]

where we have also used the obvious fact that the total size of the buyers at all times cannot exceed the asymptotic size of the informed population \( q^D \). (30) implies that at least \( e^{-\lambda q^D} \) fraction of the agents in \( Y^D(p_r, T) \) will have no friend among the set of future buyers, and hence will never buy the product under the optimal policy \( u^D(\cdot) \). Denote the set of such agents with \( Y_0^D(p_r, T) \), i.e. \( Y_0^D(p_r, T) = \{i \in Y^D(p_r, T) | d_i(\cup_{\tau=T}^{\infty} B^D(\tau)) = 0\} \). Then, \( y_0^D(p_r, T) \geq e^{-\lambda q^D} y(p_r, T) \). Now, define \( \Delta_r \) as the set of those agents who will never get informed under the optimal policy \( u^D(\cdot) \) and who have at least a friend in \( Y_0^D(p_r, T) \), that is

\[
\Delta_r = \{i \in I | (\exists \tau \in \mathbb{N}_0 : i \in x^D(\tau)) \land d_i(Y_0^D(p_r, T)) \neq 0\}.
\]

Using the Poisson distribution of the number of friends of an uninformed agent in \( Y_0^D(p_r, T) \) and the fact that the size of the agents who will never get informed under \( u^D(\cdot) \) is \( 1 - q^D \), we can easily show that

\[
\delta_r = (1 - q^D)(1 - e^{-\lambda y^D(p_r, T)}) \\
\geq (1 - q^D)(1 - e^{-\lambda y(p_r, T)e^{-\lambda q^D}}),
\]

where \( \delta_r \) denotes the size of \( \Delta_r \). Note that \( \delta_r > 0 \) as \( q^D < 1 \) and \( y^D(p_r, T) > 0 \). By dropping the price to \( u^* \) at any time \( t \geq T \), agents in \( Y_0^D(p_r, T) \) will buy the product and will subsequently inform their friends including the agents in \( \Delta_r \) about the product.

Now, consider a new policy \( \tilde{u}(\cdot) \) having the same value as \( u^D(\cdot) \) at all times except \( t \) and \( t + 1 \), where we will specify \( t \) later. Let \( \tilde{u}(t) = u^* \) and \( \tilde{u}(t + 1) = u^* \), where \( u^* \) is the price level maximizing \( u(1 - u) \) given by (21). Using an approach similar to that used in the proof of Theorem 1, it is quite straightforward to show that by a proper choice of \( t \), this new
policy will yield a profit higher than that of \( u^D(\cdot) \). Firm’s profit from time \( t \) onward under the optimal policy can be upperbounded by

\[
\Pi^D_{\geq t}(u^D(\cdot)) = \sum_{\tau = t}^{\infty} \beta^\tau u^D(\tau)b^D(\tau) \\
\leq \beta^t p_m \sum_{\tau = t}^{\infty} b^D(\tau). 
\]

(33)

Under the new policy \( \tilde{u}(\cdot) \), firm makes a discounted profit of \( \beta^t u^\alpha y_0^D(p_r, T) \) from the agents in \( Y_0^D(p_r, T) \) at time \( t \). The discounted profit made from \( \Delta_r \) at time \( t+1 \) can be lowerbounded by \( \beta^{t+1} u^*(1 - u^*) \delta_r \). Noting that \( \sum_{\tau = 0}^{\infty} b^D(\tau) \leq q^D \), we can choose \( t \) large enough to ensure that

\[
\sum_{\tau = t}^{\infty} b^D(\tau) < \frac{u^\alpha y_0^D(p_r, T) + \beta u^*(1 - u^*) \delta_r}{p_m},
\]

(34)
in which the profit from \( \tilde{u}(\cdot) \) will be clearly higher than that of \( u^D(\cdot) \). This contradicts the optimality of \( u^D(\cdot) \), hence rejecting the initial assumption of \( p_r > \alpha \). This proves that indeed \( p_r \leq \alpha \).

As we explained before, firm may only need to use price levels below \( u^\alpha \) to sell the product to low-valued agents who were directly informed by the firm. By the first time that firm offers the product at a price level \( p_r < u^\alpha \), all agents in \( X_0 \) whose valuations are above \( p_r \) and have not yet bought the product will buy the product, eliminating the need for a second offer at this price. This means that any price level below \( u^\alpha \) is used at most once by the optimal policy. This, along with the above result that \( p_r \leq \alpha \) yields \( p_r = u^\alpha \), completing the proof.

The above analysis reveals that the same intuition of dropping the price to reach out new parts of the network that would remain uninformed without the drops still holds in the presence of network externalities. As shown in the analysis above, such price drops can be profitable if used at proper times. The only difference with the case of no externalities is that firm may be able to sell the product to low-valued agents at a nonzero price, since the externality effect elevates the augmented valuations of the informed agents above (or equal to) \( \alpha \). This may lead to a shift of the drops to a nonzero price level. In fact, according to the above theorem, for externalities weaker than \( p_1 \) (i.e. \( \alpha < p_1 \)), the optimal pricing policy for a durable product still exhibits frequent price drops to zero. However, an externality effect stronger than \( p_1 \) will push these price drops away from zero. As for apps, considering the price levels commonly used by most apps (e.g., \( \{0, 0.99, 1.99, 2.99, 3.99\} \) for tadaa 3D in
Figure 1), and the typically small value of $\alpha$ as the effect of purchasing decision of only one friend, we can expect the condition $\alpha < p_1$ to hold in many cases.

5 Conclusions

In this paper, we analyzed optimal dynamic pricing in social networks from the information diffusion point of view. We developed a tractable, yet rich, model for information diffusion via word of mouth, where an agent can only get informed about a product through a friend who has already bought the product. Word of mouth is the only means by which many apps spread among smartphone users. Using this model, we showed that the optimal pricing policy for a durable product, such as many smartphone applications, should drop the price to zero infinitely often. The rationale for this behavior is that by dropping the price to zero and selling the product to agents with low valuations of the product, firm can reach out a new part of the network that would remain untouched otherwise. By timing the drop properly, firm can make sure that the marginal growth in future profit by exploiting this new part of the network prevails the loss in the immediate profit caused by dropping the price to zero. We also showed that although the optimal policy drops the price to zero infinitely often, the price trajectory cannot get trapped in a vicinity of zero meaning that it jumps away from this vicinity infinitely often.

We showed that beside the word of mouth nature of the information diffusion, this behavior is also rooted in the type of product being offered. For a nondurable product, although the firm may initially make some free offers to expand its network, after a while it will set the price at a fixed level which extracts the maximum profit from the already informed population. When the network gets large, the loss in the immediate profit by dropping the price in favor of a higher spread would become too large to compare with the marginal gain in future resulted from the excess expansion of the network. As for the final size of the spread, we showed that the pricing policy maximizing the profit for a durable product also maximizes the spread of the information in the network, while for a nondurable product the maximum profit is obtained by exploiting a smaller set of informed agents.

We also investigated the effect of network externalities on the price drops of a durable product. We showed that the same intuition of the profitability of timely drops of the price to reach out new parts of the network still holds in the presence of network externalities. However, a strong enough externality effect may enable the firm to reach out these regions using a nonzero price level, shifting the price drops to a level above zero.
References

Daron Acemoglu, Kostas Bimpikis, and Asuman Ozdaglar. Dynamics of information exchange in endogenous social networks. *Theoretical Economics*, 9(1):41–97, 2014.

Coralio Ballester, Antoni Calvó-Armengol, and Yves Zenou. Who’s who in networks. wanted: The key player. *Econometrica*, 74(5):1403–1417, 2006.

Frank Bass. A new product growth for model consumer durables. *Management Science*, 15 (5):215–227, 1969.

Dirk Bergemann and Juuso Välimäki. Market diffusion with two-sided learning. *RAND Journal of Economics*, 28(4):773–795, 1997.

Dirk Bergemann and Juuso Välimäki. Experimentation in markets. *Review of Economic Studies*, 67(2):213–234, 2000.

Dirk Bergemann and Juuso Välimäki. Dynamic pricing of new experience goods. *Journal of Political Economy*, 114(4):713–743, 2006.

Patrick Billingsley. *Probability and Measure*. Wiley Series in Probability and Mathematical Statistics, 1995.

Simon Board. Durable-goods monopoly with varying demand. *Review of Economic Studies*, 75(2):391–413, 2008.

Béla Bollobás. *Random Graphs*. Academic Press, 2001.

Yann Bramoulle and Rachel Kranton. Public goods in networks. *Journal of Economic Theory*, 135(1):478–494, 2007.

Luis M.B. Cabral, David J. Salant, and Glenn A. Woroch. Monopoly pricing with network externalities. *International Journal of Industrial Organization*, 17(2):199–214, 1999.

Roman Caminal and Xavier Vives. Price dynamics and consumer learning. *Journal of Economics & Management Strategy*, 8(1):95–131, 1999.

Arthur Campbell. Word of mouth model of sales. *Working paper*, 2012.

Arthur Campbell. Word-of-mouth communication and percolation in social networks. *American Economic Review*, 67(6):2466–2498, 2013.
Ozan Candogan, Kostas Bimpikis, and Asuman Ozdaglar. Optimal pricing in networks with externalities. *Operations Research*, 60(4):883–905, 2012.

John Conlisk, Eitan Gerstner, and Joel Sobel. Cyclic pricing by a durable goods monopolist. *Quarterly Journal of Economics*, 99(3):489–505, 1984.

Robert M. Corless, Gaston H. Gonnet, David E. G. Hare, David J. Jeffrey, and Donald E. Knuth. On the Lambert W function. *Advances in Computational Mathematics*, 5(1):329–359, 1996.

Ernest Dichter. How word-of-mouth advertising works. *Harvard Business Review*, 44(6):147–166, 1966.

Paul Erdős and Alfred Rényi. On random graphs. *Publicationes Mathematicae*, 6:290–297, 1959.

Paul Erdős and Alfred Rényi. On the evolution of random graphs. *Publications of the Mathematical Institute of the Hungarian Academy of Sciences*, 5:17–61, 1960.

Paul Erdős and Alfred Rényi. On the strength of connectedness of a random graph. *Acta Mathematica Scienta Hungary*, 12(1-2):261–267, 1961.

Andrea Galeotti and Sanjeev Goyal. Influencing the influencers: a theory of strategic diffusion. *RAND Journal of Economics*, 40(3):509–532, 2009.

Andrea Galeotti and Andrea Mattozzi. Personal influence: Social context and political competition. *American Economic Journal: Microeconomics*, 3(1):307–327, 2011.

Andrea Galeotti, Sanjeev Goyal, Matthew O. Jackson, Fernando Vega-Redondo, and Leeat Yariv. Network games. *Review of Economic Studies*, 77(1):218–244, 2010.

Guillermo Gallego and Garrett van Ryzin. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Science*, 40(8):999–1020, 1994.

Daniel F. Garrett. Incoming demand with private uncertainty. *Working paper*, 2013.

Alex Gershkov and Benny Moldovanu. Dynamic revenue maximization with heterogeneous objects: A mechanism design approach. *American Economic Journal: Microeconomics*, 1(2):168–198, 2009.

Jason Hartline, Vahab Mirrokni, and Mukund Sundararajan. Optimal marketing strategies over social networks. *Proc. 17th Internat. Conf. World Wide Web (WWW ’08) (ACM, New York)*, pages 189–198, 2008.
Johannes Hörner and Larry Samuelson. Managing strategic buyers. *Journal of Political Economy*, 119(3):379–425, 2011.

Bar Ifrach, Costis Maglaras, and Marco Scarsini. Monopoly pricing in the presence of social learning. *Working paper*, 2011.

Bar Ifrach, Costis Maglaras, and Marco Scarsini. Bayesian social learning from consumer reviews. *Working paper*, 2013.

Matthew O. Jackson and Brian W. Rogers. Meeting strangers and friends of friends: How random are social networks? *American Economic Review*, 97(3):890 – 915, 2007a.

Matthew O. Jackson and Brian W. Rogers. Relating network structure to diffusion properties through stochastic dominance. *B. E. Journal of Theoretical Economics: Advances in Theoretical Economics*, 7(1):1 – 13, 2007b.

Matthew O. Jackson and Leeat Yariv. Diffusion of behavior and equilibrium properties in network games. *American Economic Review*, 97(2):92 – 98, 2007.

Ali Jadbabaie, Ali Kakhbod, and Alireza Tahbaz Salehi. Optimal contracting in networks. *Working paper*, 2013.

Nathan Larson. Dynamic viral marketing on a social network. *Working paper*, 2013.

Dunia López-Pintado. Diffusion in complex social networks. *Games and Economic Behavior*, 62(2):573 – 590, 2008.

Kaitlin McCloughan. New report on app developers’ attitudes towards mobile advertising. *Available at: http://appflood.com/blog/app-marketing-report-2013.html*, 2013.

Mark E. Newman, Steven H. Strogatz, and Duncan J. Watts. Random graphs with arbitrary degree distributions and their applications. *Physical Review E*, 64(2):026118, 2001.

Volker Nocke and Martin Peitz. A theory of clearance sales. *Economic Journal*, 117(522):964–990.

Ezra Oberfield. Business networks, production chains, and productivity: A theory of input-output architecture. *Working paper*, 2012.

Yiangos Papanastasiou, Nitin Bakhshi, and Nicos Savva. Social learning from early buyer reviews: Implications for new product launch. *Working paper*, 2013.

Robert Phillips. *Pricing and Revenue Optimization*. Stanford Business Books, 2005.
Roy Radner, Ami Radunskaya, and Arun Sundararajan. Dynamic pricing of network goods with boundedly rational consumers. Proceedings of the National Academy of Sciences of the United States of America (PNAS), 111(1):99–104, 2013.

Anatol Rapoport. A contribution to the theory of random and biased nets. Bulletin of Mathematical Biophysics, 19(4):257–277, 1957.

Adam Satariano. WhatsApp’s founder goes from food stamps to billionaire. Bloomberg, available at: http://www.bloomberg.com/news/2014-02-20/whatsapp-s-founder-goes-from-food-stamps-to-billionaire.html, 2014.

Joel Sobel. Durable goods monopoly with entry of new consumers. Econometrica, 59(5):1455–1485, 1991.

Nancy L. Stokey. Intertemporal price discrimination. Quarterly Journal of Economics, 93(3):355–371, 1979.

Xuanming Su. Intertemporal pricing with strategic customer behavior. Management Science, 53(5):726–741, 2007.

Arun Sundararajan. Local network effects and complex network structure. The B.E. Journal of Theoretical Economics, 7(1):1935–1704, 2008.

Kalyan T. Talluri and Garrett van Ryzin. The Theory and Practice of Revenue Management (International Series in Operations Research & Management Science). Springer, 2004.

Duncan J. Watts. A simple model of cascades on random networks. Proceedings of the National Academy of Sciences of the United States of America (PNAS), 99(9):5766–5771, 2002.

Duncan J. Watts and Steven H. Strogatz. Collective dynamics of small world networks. Nature, 393:440–442, 1998.

Peyton H. Young. Innovation diffusion in heterogeneous populations: Contagion, social influence, and social learning. American Economic Review, 99(5):1899 – 924, 2009.

Man Yu, Laurens Debo, and Roman Kapuscinski. Strategic waiting for consumer-generated quality information: Dynamic pricing of new experience goods. Working paper, 2013.
Appendix: Proofs

Proof of Proposition 1. We can use (10) to write $x_0$ as a function of $q$

$$x_0 = 1 - (1 - q)e^{\lambda q}.$$  \hfill (35)

The existence of a solution $q \in [0, 1]$ for any $0 < x_0 \leq 1$ now follows from the continuity of $x_0$ in $q$ and that $x_0(q = 0) = 0$ and $x_0(q = 1) = 1$. Taking derivatives from the above equation, we obtain

$$\frac{dx_0}{dq} = (1 - \lambda(1 - q))e^{\lambda q},$$  \hfill (36)

$$\frac{d^2x_0}{dq^2} = \lambda(2 - \lambda(1 - q))e^{\lambda q}. \hfill (37)$$

It follows from (36) that $x_0$ attains its minimum at $q^* = 1 - \frac{1}{\lambda}$ and is strictly increasing (decreasing) for $q \geq q^*$ ($q \leq q^*$). It also follows from (37) that $x_0$ is convex for $q \geq q^*$. Next, we show that for $q \in [0, 1]$ the constraint $0 < x_0 \leq 1$ implies $q > q^*$. This is automatically satisfied for the case where $\lambda < 1$ since $q^* < 0$. For $\lambda \geq 1$, we have $q^* \geq 0$. However, $x_0(q)$ is decreasing for $0 \leq q \leq q^*$ resulting in $x_0(q) \leq x_0(q = 0) = 0$. This shows that also in this case we should have $q > q^*$.

Now, the uniqueness of the solution in $[0, 1]$ for $0 < x_0 \leq 1$ follows from the fact that $x_0$ is strictly increasing for $q \geq q^*$. Also, since $x_0$ is strictly increasing and convex for $q \geq q^*$, thus $q$ is strictly increasing, but is concave in $x_0$. \hfill ■

Proof of Proposition 3. In order to prove the proposition, we show that if the optimal policy $u^D(\cdot)$ gets stuck between 0 and $p_1$ after a finite time, then we should have $p_1 \geq 1 - \sqrt{1 - 4\epsilon}$. Suppose that there is $T \geq 0$ such that $u^D(t) \in \{0, p_1\}$ for all $t \geq T$. Denote with $t_k$ the $k$-th drop of the price to zero after $T$ under $u^D(\cdot)$. Note that there are infinitely many such drops according to Theorem 1. We first try to find an upperbound for the accumulated profit of the firm under $u^D(\cdot)$ after $t_k$. Consider a new policy $\tilde{u}(\cdot)$ which has the same values as $u^D(\cdot)$ before $t_k$ but is zero afterwards, that is,

$$\tilde{u}(t) = \begin{cases} 
 u^D(t), & t \leq t_k \\
 0, & t > t_k 
 \end{cases} \hfill (38)$$
The profit made by the firm under \( u^D(\cdot) \) after \( t_k \) is upperbounded by

\[
\Pi^D_{>t_k}(u^D(\cdot)) \leq p_1(1 - p_1) \sum_{t = t_k + 1}^{\infty} \beta^t \hat{y}(t). \tag{39}
\]

To see this, note that \( \hat{u}(\cdot) \) offers the product for free after \( t_k \). So, the summation \( \sum_{t = t_k + 1}^{\infty} \beta^t \hat{y}(t) \) in fact discounts the informed population after \( t_k \) by the earliest time they may get informed about the product. No agent can know about the product under \( u^D(\cdot) \) earlier than when it would have known about the product under \( \hat{u}(\cdot) \). The upperbound in (39) now follows on noting that firm sells the product to \( (1 - p_1) \) fraction of any set of newly informed agents when it offers the product at price \( p_1 \). The dynamics of \( \hat{y}(t) \) for \( t > t_k \) has the simple form of

\[
\hat{y}(t + 1) = (1 - \hat{x}(t))(1 - e^{-\lambda \hat{y}(t)}), \tag{40}
\]

where \( \hat{y}(t_k + 1) = y^D(t_k + 1) \) and \( \hat{x}(t_k + 1) = x^D(t_k + 1) \). Using this, we can easily obtain

\[
\hat{y}(t + 1) \leq \lambda(1 - x^D(t_k + 1)) \hat{y}(t), \tag{41}
\]

for all \( t > t_k \). This means that \( \hat{y}(t) \) for \( t > t_k \) is upperbounded by a geometric sequence with common ratio \( \lambda(1 - x^D(t_k + 1)) \). This common ratio can be made less than 1 by choosing \( k \) sufficiently large. To see this, first note that the asymptotic size of the informed population under \( u^D(\cdot) \) is the same as the zero policy according to Theorem 1, that is \( q^D = q^0 \). Also, from Proposition 1 we know that \( \lambda(1 - q^0) < 1 \). Therefore, on noting that \( \lim_{t \to \infty} x^D(t) = q^D \), one can choose \( k \) sufficiently large to ensure \( \lambda(1 - x^D(t_k + 1)) < 1 \). Using this along with (39), the profit of the firm under \( u^D(\cdot) \) after \( t_k \) can be upperbounded by

\[
\Pi^D_{>t_k}(u^D(\cdot)) \leq \frac{\beta^{t_k + 1} p_1 (1 - p_1) y^D(t_k + 1)}{1 - \beta \lambda (1 - x^D(t_k + 1))}, \tag{42}
\]

for sufficiently large \( k \). Next, we compare this profit with that of a modified policy \( \tilde{u}(\cdot) \) having the same value as \( u^D(\cdot) \) for \( t \leq t_k \) and a fixed value \( u^c \) for \( t > t_k \), where

\[
u^c = \arg\max_{u \in \mathcal{U} \setminus \{p_1\}} \frac{u(1 - u)(1 - \beta \lambda (1 - q^0))}{1 - \beta \lambda (1 - q^0)(1 - u)}, \tag{43}\]

that is the maximizer in (20). The profit of the firm for policy \( \tilde{u}(\cdot) \) after \( t_k \) is given by

\[
\Pi^D_{>t_k}(\tilde{u}(\cdot)) = u^c (1 - u^c) \sum_{t = t_k + 1}^{\infty} \beta^t \hat{y}(t). \tag{44}\]
The dynamics of \( \tilde{y}(t) \) for \( t > t_k \) is given by

\[
\tilde{y}(t + 1) = (1 - \tilde{x}(t))(1 - e^{-\lambda(1 - u^c)}\tilde{y}(t)),
\]

where \( \tilde{y}(t + 1) = y^D(t + 1) \) and \( \tilde{x}(t + 1) = x^D(t + 1) \). We next aim to lowerbound \( \tilde{y}(t) \) for \( t > t_k \) with a geometric sequence, in order to find a closed form lowerbound for the profit of the firm for \( \tilde{u}(\cdot) \) after \( t_k \) given by (44). For any \( \omega < 1 \), there exists \( y^\omega > 0 \) such that for every \( 0 < \tilde{y} < y^\omega \) we have \( 1 - e^{-\lambda(1 - u^c)}\tilde{y} > \omega\lambda(1 - u^c)\tilde{y} \). Choosing \( k \) sufficiently large such that \( q^D - x^D(t_k) < y^\omega \), we can ensure \( \tilde{y}(t) < y^\omega \) for \( t > t_k \) on noting that \( \tilde{y}(t) = \tilde{x}(t) - \tilde{x}(t - 1) < q^D - x^D(t_k) \) for \( t > t_k \).\(^{27}\) Using this along with (45), we can obtain

\[
\tilde{y}(t + 1) > (1 - q^0)\omega\lambda(1 - u^c)\tilde{y}(t),
\]

for sufficiently large \( k \) and all \( t > t_k \). This implies that \( \tilde{y}(t) \) for \( t > t_k \) is lowerbounded by a geometric sequence with common ratio \( (1 - q^0)\omega\lambda(1 - u^c) \), and hence the profit of the firm after \( t_k \) for \( \tilde{u}(\cdot) \) can be lowerbounded by

\[
\Pi^D_{> t_k}(\tilde{u}(\cdot)) > \frac{\beta^{k+1}u^c(1 - u^c)y^D(t_k + 1)}{1 - \beta(1 - q^0)\omega\lambda(1 - u^c)}.
\]

Noting that \( u^D(\cdot) \) is the optimal policy, we should have \( \Pi^D_{> t_k}(u^D(\cdot)) \geq \Pi^D_{> t_k}(\tilde{u}(\cdot)) \). This, along with (42) and (47) yields

\[
\frac{p_1(1 - p_1)}{1 - \beta\lambda(1 - x^D(t_k + 1))} > \frac{u^c(1 - u^c)}{1 - \beta(1 - q^0)\omega\lambda(1 - u^c)},
\]

for any \( \omega < 1 \) and sufficiently large \( k \). Shifting \( \omega \to 1 \) and \( k \to \infty \), we can obtain

\[
\frac{p_1(1 - p_1)}{1 - \beta\lambda(1 - q^0)} \geq \frac{u^c(1 - u^c)}{1 - \beta(1 - q^0)\lambda(1 - u^c)},
\]

yielding \( p_1(1 - p_1) \geq c \) from the definition of \( c \) given by (20). This implies that \( p_1 \geq \frac{1 - \sqrt{1 - 4c}}{2} \), which completes the proof.

**Proof of Theorem 2.** Below, we provide more details on parts of the proof of Theorem 2.

i) **Proof of the lowerbound on \( \Pi^N_{D \geq t_0}(\tilde{u}(\cdot)) \) given by (24):** For \( t > t_0 \), we have \( \tilde{x}(t) \geq x^{N^D}(t - 1) \)

\(^{27}\)This follows from the fact that \( \tilde{x}(t - 1) \geq \tilde{x}(t_k) = x^D(t_k) \) for \( t > t_k \) and that \( \tilde{x}(t) < q^0 = q^D \) as \( q^0 \) is the maximum asymptotic size of the informed population.
and \( \bar{u}(t) = u^{ND}(t - 1) \). Also, \( \bar{x}(t_0) = x^{ND}(t_0) \) and \( \bar{u}(t_0) = u^* \). Therefore, we can write

\[
\Pi_{\geq t_0}^{ND}(\bar{u}(\cdot)) = \sum_{t=t_0}^{\infty} \beta^t \bar{u}(t)(1 - \bar{u}(t)) \bar{x}(t)
\]

\[
\geq \beta^{t_0} \bar{u}(t_0)(1 - \bar{u}(t_0)) \bar{x}(t_0) + \sum_{t=t_0+1}^{\infty} \beta^t u^{ND}(t)(1 - u^{ND}(t - 1)) x^{ND}(t - 1)
\]

\[
= \beta^{t_0} u^*(1 - u^*) x^{ND}(t_0) + \beta \Pi_{\geq t_0}^{ND}(u^{ND}(\cdot)). \tag{50}
\]

ii) **Proof of the upperbound on** \( \Pi_{\geq t_0}^{ND}(u^{ND}(\cdot)) \) **given by** \( (26) \): Using Lemma 1 and on noting that \( x^{ND}(t) \leq q^{ND} \), we have

\[
\Pi_{\geq t_0}^{ND}(u^{ND}(\cdot)) = \sum_{t=t_0}^{\infty} \beta^t u^{ND}(t)(1 - u^{ND}(t)) x^{ND}(t)
\]

\[
\leq \beta^{t_0} u^{ND}(t_0)(1 - u^{ND}(t_0)) x^{ND}(t_0) + \sum_{t=t_0+1}^{\infty} \beta^t u^*(1 - u^*) q^{ND}
\]

\[
= \beta^{t_0} u^{ND}(t_0)(1 - u^{ND}(t_0)) x^{ND}(t_0) + \frac{\beta^{t_0+1}}{1 - \beta} u^*(1 - u^*) q^{ND}. \tag{51}
\]

iii) **Proof of the inequality given by** \( (25) \): Applying the upperbound given by \( (26) \) and simple algebra, we can easily see that this inequality is satisfied if

\[
(1 - \beta) u^{ND}(t_0)(1 - u^{ND}(t_0)) x^{ND}(t_0) + \beta u^*(1 - u^*) q^{ND} < u^*(1 - u^*) x^{ND}(t_0), \tag{52}
\]

or equivalently

\[
x^{ND}(t_0) > \frac{\beta u^*(1 - u^*)}{u^*(1 - u^*) - (1 - \beta) u^{ND}(t_0)(1 - u^{ND}(t_0))} q^{ND}, \tag{53}
\]

where \( u^{ND}(t_0) < u^* \) (this follows from the assumption \( u^{ND}(t_0) \neq u^* \) and Lemma 1). The above follows then from the fact that \( x^{ND}(t_0) > \gamma q^{ND} \) and the definition of \( \gamma \) in \( (22) \).

**Proof of Theorem 3.** The proof is by induction. However, in order to use induction, we will need a more accurate but dirtier version of the theorem as follows.

**Claim:** Choose some \( \lambda_1 \) satisfying \( \lambda^* < \lambda_1 < \frac{\lambda(\lambda^*-1)}{\lambda^*} + 1 \). Note that RHS is greater than LHS since \( \lambda^* < \lambda \). For any such \( \lambda_1 \), there exists \( x^c > 0 \) such that for \( x^{ND}(t) < x^c \):

i) \( u^{ND}(t) u^{ND}(t + 1) = 0 \),

ii) if \( u^{ND}(t) = 0 \) then \( x^{ND}(t + 1) > \lambda_1 x^{ND}(t) \). And, if \( u^{ND}(t) \neq 0 \) then \( x^{ND}(t + 1) > \lambda^* x^{ND}(t) \). Note that in either case in ii) we have \( x^{ND}(t + 1) > \lambda^* x^{ND}(t) \) since \( \lambda_1 > \lambda^* \).
Define
\begin{align}
g_1(x) &= 1 - (1 - x)e^{-\frac{\lambda^*(\lambda_1 - 1)}{\lambda_1}x} - \lambda^*x, \quad (54) \\
g_2(x) &= 1 - (1 - x)e^{-\frac{\lambda^*(\lambda_1 - 1)}{\lambda^*}x} - \lambda_1x, \quad (55) \\
g_3(x) &= (1 - x)(e^{-\frac{\lambda^*(\lambda_1 - 1)}{\lambda_1}x} - e^{-\frac{\lambda(\lambda_1 - 1)}{\lambda^*}x}) - \frac{\lambda\lambda^*(1 - u)}{\lambda^*}x, \quad (56)
\end{align}

where in \( g_3(x) \), \( u \) is a nonzero price level in \( U \). We can easily find the derivatives of these functions at \( x = 0 \). For \( g_1(x) \), we can write
\[
\frac{dg_1}{dx}(0) = \frac{\lambda^*(\lambda_1 - 1)}{\lambda_1} + 1 - \lambda^* = \frac{\lambda_1 - \lambda^*}{\lambda_1} > 0.
\] (57)

Similarly,
\[
\frac{dg_2}{dx}(0) = \frac{\lambda(\lambda^* - 1)}{\lambda^*} + 1 - \lambda_1 > 0.
\] (58)

Finally,
\[
\frac{dg_3}{dx}(0) = \frac{\lambda(\lambda_1 - 1)u}{\lambda_1} - \frac{\lambda\lambda^*(1 - u)}{\lambda^*} > 0.
\] (59)

This implies that all the three functions are strictly increasing in a vicinity of \( x = 0 \). Using this along with the fact that \( g_1(0) = g_2(0) = g_3(0) = 0 \), we can conclude that there exists \( \bar{x}^c > 0 \) such that \( g_1(x) > 0, g_2(x) > 0 \), and \( g_3(x) > 0 \) for all \( 0 < x < \bar{x}^c \). Also, define
\[
h(y) = e^{-\lambda(1-u)y} - e^{-\lambda y}.
\] (60)

It is quite straightforward to show that \( h \) is strictly increasing for \( 0 < y < \frac{-\ln(1-u)}{\lambda u} \), and that \( \frac{1}{\lambda} < \frac{-\ln(1-u)}{\lambda u} \) for \( 0 < u < 1 \). Thus, \( h \) is strictly increasing for \( 0 < y < \frac{1}{\lambda} \) for any nonzero price level \( u \in U \). We set \( x^c = \min\{\bar{x}^c, \frac{1}{\lambda}\} \) and prove the claim above for this choice of \( x^c \) with induction.

We start with the transition part of the induction. Assuming that the claim holds for
Using the assumption of induction, and that

Now, we try to lowerbound \( y^{ND}(t) \) in terms of \( x^{ND}(t) \). From the assumption of induction and that \( u^{ND}(t-1) = 0 \), we get \( x^{ND}(t) > \lambda_1 x^{ND}(t-1) \), which along with \( y^{ND}(t) = x^{ND}(t) - x^{ND}(t-1) \) yields

\[
y^{ND}(t) > \frac{\lambda_1 - 1}{\lambda_1} x^{ND}(t).
\]

where we have also used the fact that for the optimal policy \( u^{ND}(t) \leq u^* \) from Lemma 1. This proves that if \( u^{ND}(t) \neq 0 \) then \( x^{ND}(t+1) > \lambda^* x^{ND}(t) \). If, on the other hand, \( u^{ND}(t) = 0 \), then \( b^{ND}(t) \geq y^{ND}(t) \) from (2), and we can use (4) to obtain

\[
x^{ND}(t + 1) \geq 1 - (1 - x^{ND}(t)) e^{-\lambda y^{ND}(t)}.
\]

Using the assumption of induction, \( x^{ND}(t) > \lambda^* x^{ND}(t-1) \), which along with \( y^{ND}(t) = x^{ND}(t) - x^{ND}(t-1) \) yields

\[
y^{ND}(t) > \frac{\lambda^* - 1}{\lambda^*} x^{ND}(t).
\]

Using (64) and (65), and that \( g_2(x^{ND}(t)) > 0 \), we get

\[
x^{ND}(t + 1) > 1 - (1 - x^{ND}(t)) e^{-\lambda y^{ND}(t)}
\]

which completes the proof of part ii) for \( t \). Now, we get to the proof of part i). Assume that \( u^{ND}(t) u^{ND}(t+1) \neq 0 \), and construct a new policy \( \tilde{u}(\cdot) \) that is obtained from \( u^{ND}(\cdot) \) by only changing \( u^{ND}(t) \) to 0. We claim that the new policy will result in a profit higher than that of \( u^{ND}(\cdot) \). First of all, note that for all times \( \tau \geq 0 \), \( X^{ND}(\tau) \subseteq \bar{X}(\tau) \), thus \( x^{ND}(\tau) \leq \bar{x}(\tau) \).
In particular, we are interested in calculating \( \tilde{x}(t+1) - x^{ND}(t+1) \). From the assumption of induction, we should have \( u^{ND}(t-1)u^{ND}(t) = 0 \). Therefore, \( u^{ND}(t-1) = 0 \) since \( u^{ND}(t) \neq 0 \). This implied that \( Z^{ND}(t) = \emptyset \). Hence, using (2) and (4) we get

\[
\tilde{x}(t+1) - x^{ND}(t+1) = (1 - x^{ND}(t))(e^{-\lambda(1-u^{ND}(t))}y^{ND}(t) - e^{-\lambda y^{ND}(t)}).
\] (67)

It follows from \( u^{ND}(t-1) = 0 \) and the assumption of induction that \( x^{ND}(t) > \lambda_1 x^{ND}(t-1) \), which in turn implies

\[
y^{ND}(t) > \frac{\lambda_1 - 1}{\lambda_1} x^{ND}(t).
\] (68)

Now, considering that \( h(y) \) defined in (60) is strictly increasing for \( 0 < y < x^c \), (67) yields

\[
\tilde{x}(t+1) - x^{ND}(t+1) > (1 - x^{ND}(t))(e^{-\frac{\lambda(1-u^{ND}(t))x^{ND}(t)}{\lambda_1}} - e^{-\frac{\lambda(1-u^{ND}(t))x^{ND}(t)}{\lambda_1}})
\]
\[
> \frac{\lambda(\lambda^* - 1)u^{ND}(t)}{\lambda^*} x^{ND}(t),
\] (69)

where the last inequality comes from \( g_3(x^{ND}(t)) > 0 \).

In order to show that the new policy \( \tilde{u}(\cdot) \) results in a higher profit, it suffices to show that

\[
u^{ND}(t)(1 - u^{ND}(t))x^{ND}(t) + \beta u^{ND}(t+1)(1 - u^{ND}(t+1))x^{ND}(t+1) < \beta u^{ND}(t+1)(1 - u^{ND}(t+1))\tilde{x}(t+1),
\] (70)

or equivalently,

\[
u^{ND}(t)(1 - u^{ND}(t))x^{ND}(t) < \beta u^{ND}(t+1)(1 - u^{ND}(t+1))\tilde{x}(t+1) - x^{ND}(t+1)).
\] (71)

Applying (69) and some simplifications, we can see that the above is satisfied if

\[
1 - u^{ND}(t) < \beta u^{ND}(t+1)(1 - u^{ND}(t+1))\frac{\lambda(\lambda^* - 1)}{\lambda^*}.
\] (72)

Noting that \( u^{ND}(t) \neq 0 \), the LHS is maximized when \( u^{ND}(t) = p_1 \). Also, \( p_1(1 - p_1) \leq u^{ND}(t+1)(1 - u^{ND}(t+1)) \) for \( p_1 \leq u^{ND}(t+1) \leq u^* \). Therefore, it suffices to have

\[
1 - p_1 < \beta p_1(1 - p_1)\frac{\lambda(\lambda^* - 1)}{\lambda^*},
\] (73)

which holds if \( \beta > \frac{1-u^*}{p_1(\lambda^* - 1)} \). This shows that \( \tilde{u}(\cdot) \) has a higher profit that \( u^{ND}(\cdot) \) which
contradicts its optimality. This completes the proof of part i).

The only thing which is left is to verify the base of the induction, that is to prove the claim for \( t = 0 \). For part ii), it is easy to see that all the relations (61)-(66) also hold for \( t = 0 \), noting that \( y^{ND}(0) = x^{ND}(0) \) and \( Z(0) = \emptyset \). Similar story holds for part i).  ■
Appendix: Extension to Non-Uniform Valuations

In this appendix, we discuss the extension of the results to the case where the valuations of the agents are distributed according to a cumulative distribution function $F(\theta)$, with support $\theta \in [0, 1]$. Most of the results of the paper can be rewritten for the general case by simply replacing $u$ and $p_j$’s with $F(u)$ and $F(p_j)$’s when using the price to determine the size of a subset of the population based on their valuations. For example, the dynamics of the diffusion under $F(\theta)$ can be written as

$$1 - x(t + 1) = (1 - x(t))e^{-\lambda b(t)},$$
$$y(t + 1) = x(t + 1) - x(t),$$
$$z_j(t + 1) = \begin{cases} z_j(t) + (F(p_j) - F(p_j-1))y(t), & 1 \leq j \leq r \\ 0, & \text{otherwise} \end{cases}$$

where $b(t)$, the size of the set of fresh buyers at time $t$, is given by

$$b(t) = (1 - F(p_r))y(t) + \sum_{j=r+1}^{m} z_j(t).$$

The profit of the firm for a durable product will stay the same as in (5). For a nondurable product, the profit will become

$$\Pi_{ND}(u(\cdot)) = \sum_{t=0}^{\infty} \beta^t u(t)(1 - F(u(t)))x(t).$$

We assume that $F$ corresponds to a non-atomic pdf and is strictly increasing on $\mathcal{U}$, that is $F(p_{j-1}) < F(p_j)$ for $j = 1, 2, \ldots, M$. Under this assumption, it is quite straightforward to verify that Propositions 1-2 and Theorem 1 still hold exactly as they are. For Proposition 3, we have to replace $(1 - u)$ with $(1 - F(u))$ in the definition of $c$ in (20). The condition on $p_1$ then becomes $p_1(1 - F(p_1)) < c$ instead of $p_1(1 - p_1) < c$. The expression given by (21) for $u^*$ is not valid any more. However, the second part stating that $u_{ND}(t) \leq u^*$ still holds, where $u^*$ is the smallest price level $u \in \mathcal{U}$ which maximizes $u(1 - F(u))$. As a result, Theorem 2 is still valid. Theorem 3 requires minor changes. $\lambda^* = (1 - u^*)\lambda$ should be replaced by $\lambda^* = (1 - F(u^*))\lambda$. Also, the condition $\beta > \frac{1-u^*}{p_1(\lambda^*-1)}$ on the discount factor
should be replaced by \( \beta > \frac{1-u^*}{u^*(\lambda^*-1)} \), where

\[
u^F = \frac{\min_{p_1 \leq u \leq u^*} u(1 - F(u))}{\max_{p_1 \leq u \leq u^*} \frac{u(1-F(u))}{F(u)}}. \tag{77}
\]

Under these changes, the proof of Theorem 3 can be easily modified for the general case. Finally, Theorem 4 is still valid for the general case if we assume that \( F \) is strictly increasing over \( U \cup \{\alpha\} \). This is to ensure that \( F(p_r - \alpha) > 0 \) for \( p_r > \alpha \), which will be used in showing that \( y^D(p_r, T) > 0 \).