Extension of tetration to real and complex heights

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Abstract
The continuous tetrational function \( x^r = \tau(r, x) \), the unique solution of equation \( \tau(r, x) = r^{\tau(r, x-1)} \) and its differential equation \( \tau'(r, x) = q \tau(r, x) \tau'(r, x-1) \), is given explicitly as \( x^r = \exp^{[\{x\}]}(x), \) where \( x \) is a real variable called height, \( r \) is a real constant called base, \( \{x\} = x - \lfloor x \rfloor \) is the sawtooth function, \( \lfloor x \rfloor \) is the floor function of \( x \), and \( \{x\}_q = (q^{\{x\}} - 1)/(q - 1) \) is a q-analog of \( \{x\} \) with \( q = \ln r \), respectively. Though \( x^r \) is continuous at every point in the real \( r - x \) plane, extensions to complex heights and bases have limited domains. The base \( r \) can be extended to the complex plane if and only if \( x \in \mathbb{Z} \). On the other hand, the height \( x \) can be extended to the complex plane at \( \Re(x) \notin \mathbb{Z} \). Therefore \( r \) and \( x \) in \( x^r \) cannot be complex values simultaneously. Tetrational laws are derived based on the explicit formula of \( x^r \).

1 Introduction
Tetration, i.e., iterated exponentiation, is the fourth hyperoperation after addition, multiplication, and exponentiation \([1]\). Tetration is defined as

\[ n^r := r^{r^{\cdots^{r_n}}}, \]

meaning that copies of \( r \) are exponentiated \( n \) times in a right-to-left direction, or recursively defined as

\[ n^r := \begin{cases} 1 & (n = 0) \\ r^{(n-1)r} & (n \geq 1). \end{cases} \]

Obviously \( n^{-1}r = \log_r(n^r) \). The parameter \( r \) and \( n \) are referred to as the base and height, respectively. In this paper, we use the notations to express tetration as

\[ n^r = \tau(r, n) = \exp^n_{1}, \]

and we write \( \tau(n) \) for simplicity if \( r \) is a constant.
Tetration has been applied in fundamental physics [2], degree of connection in air transportation system [3], signal interpolation [4], data compression [5], etc.

Tetration is known as a rapid growing function that may be useful for expressing superexponential growth or huge numbers, but it also shows saturation and oscillation. As in Fig. 1 the protean dynamics can be illustrated by cobweb plots.

Euler [6] proved that the limit of tetration $^n r$, as $n$ goes to infinity, converges if $e^{(1/e)} \leq r \leq e^{1/e}$. Eisenstein [7] gave the closed form of $^\infty r$ for complex $r$, and Corless et al. [8] expressed it as $^\infty r = W(-\ln r)/(-\ln r)$ by using Lambert’s W function.

Different from established extension of base $r$ to real or complex values under integers or infinite height, extension of height $n$ to real or complex values is still under debate. The object of this paper is to derive rigorously the explicit formula of continuous tetration.

**Definition 1.1.** Let $r > 0$ be real constant and $x > -2$ be real variable. Tetrational function $\tau(x)$ is a continuous function defined as:

$$\tau(0) := 1, \quad \tau(x) := r^{\tau(x-1)}.$$  

**Corollary 1.2.** Independent of $r$, every tetrational function $\tau(x)$ goes through $\tau(-1) = 0$ and $\tau(0) = 1$. 

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Figure 1: Behaviors of discrete tetration for different $r$ (left), and cobweb plots (right).
**Corollary 1.3.** If a curve segment connecting between $\tau(-1) = 0$ and $\tau(0) = 1$ is defined, and is extended to other region by exponentiation or logarithm, then the whole curve satisfies Definition 1.1.

Because of Corollary 1.3, it is taken for granted that $\tau(x)$ cannot be uniquely determined without an extra requirement. It follows that there exist many unique solutions for different extra conditions. Hooshmand [9] applied a linear segment between $\tau(-1) = 0$ and $\tau(0) = 1$, and proved uniqueness of the function $\tau(x) = \exp^{|x|+1}\{x\}$, where $[x]$ is the floor function of $x$ and $\{x\} = x - [x]$ is the sawtooth function. He found that the first derivative exists at the connecting points of $x = n \in \mathbb{N}$ only for the case of $r = e$, and the second derivative does not exist at the connecting points. Kouznetsov [10] gave a numerical solution for $r > e^{1/e}$ in the complex plane under the requirement that $\tau(x+iy)$ is holomorphic, and Paulsen and Cowgill [11] proved that such a holomorphic solution is unique.

In this paper, we analytically derive an unique solution only from the minimum requirement of differentiability: the delay-differential equation in Lemma 1.4.

**Lemma 1.4.** Let $q = \ln r$ be a real constant. If a tetrational function is differentiable, the following delay-differential equation holds at every point in $x > -2$:

$$\frac{d\tau(x)}{dx} = q\tau(x) \frac{d\tau(x - 1)}{dx}$$

**Proof.** The derivative of $\tau(x)$ in Definition 1.1

This paper consists of eight sections. In Section 2 a new function, named multi-tetrational function, is defined to express the restriction placed in the delay-differential equation. In Section 3 the tetrational function and multi-tetrational function are then expressed in Taylor series with coefficients characteristic to these functions. In Section 4 based on the general forms of coefficients, the Taylor series are transformed into new expressions. In Section 5 explicit formulae of tetrational functions and multi-tetrational functions, the main results of this paper, are derived. In Section 6 Analytical properties as well as the extension to the complex heights and bases are discussed. In Section 7 calculation rules of tetration are presented. Concluding remarks are given in section 8.

## 2 Multi-tetrational function

In this section we define a key function, and by using this, we transform the delay-differential function into an ordinary differential equation.
Let us consider a continuous function $\mu(x)$ which goes through the following discrete values:

$$
\begin{align*}
\mu(0) &= \tau(0) = 1 \\
\mu(1) &= \tau(1)\tau(0) = r \\
\mu(2) &= \tau(2)\tau(1)\tau(0) = r^2 r \\
&\vdots \\
\mu(n) &= \tau(n)\mu(n-1)
\end{align*}
$$

It is reasonable to define $\mu(x)$ by replacing $n$ with $x$.

**Definition 2.1.** A continuous function $\mu(x)$ is called a multi-tetrational function if it satisfies the following relations:

$$
\mu(0) := \tau(0), \quad \mu(x) := \tau(x)\mu(x-1)
$$

**Lemma 2.2.** The equation $\mu(-1) = \mu(0) = 1$ holds independent of $r$.

**Proof.** From Definition 2.1 we get $\mu(0) = \tau(0) = 1$. Assigning $x = 0$ to second equation in Definition 2.1 gives $\mu(-1) = \mu(0)/\tau(0) = 1$. □

The delay-differential equation in Lemma 1.4 is transformed to an ordinary differential equation including $\mu(x)$ as in Theorem 2.3, which places strong restriction on both $\tau(x)$ and $\mu(x)$.

**Theorem 2.3.** Let $q = \ln r$ be a real constant.

Let $\omega \in \mathbb{C}$ be $q$-dependent constant.

The delay differential equation 1.4 is equivalent to the differential equation:

$$
\frac{d\tau(x)}{dx} = \omega q \mu(x)
$$

**Proof.** By assigning integer heights $x = n \in \mathbb{N}$ to the equation 1.4 we get:

$$
\begin{align*}
\tau'(0) &= q\tau(0)\tau'(-1) \\
\tau'(1) &= q\tau(1)\tau'(0) = q^2 \tau(1)\tau(0)\tau'(-1) \\
\tau'(2) &= q\tau(2)\tau'(1) = q^3 \tau(2)\tau(1)\tau(0)\tau'(-1) \\
&\vdots \\
\tau'(n) &= q^{n+1} \mu(n)\tau'(-1).
\end{align*}
$$

By defining $\omega := q\tau'(-1)$, we get

$$
\tau'(n) = \omega q^n \mu(n).
$$
By definitions 1.1 and 2.1, the continuous functions \( \tau(x) \), \( \mu(x) \) and \( q^x \) go through all of the discrete points \( \tau(n) \), \( \mu(n) \) and \( q^n \), respectively. Therefore equation above is extended to real values:

\[
\tau'(x) = \omega q^x \mu(x).
\]

Conversely, if \( \frac{\tau'(x)}{\tau'(x-1)} = \frac{\omega q^x \mu(x)}{\omega q^{x-1} \mu(x-1)} = \frac{q \mu(x)}{\mu(x-1)} = q \tau(x) \).

Therefore the delay-differential equation \( \tau'(x) = q \tau(x) \tau'(x-1) \) is reproduced.

\[\square\]

### 3 Taylor series

In this section, we will find general forms of the coefficients in Taylor series of the tetrational function and multi-tetrational function.

First we express \( \tau(x) \) and \( \mu(x) \) as Taylor series.

**Definition 3.1.** Let \( a_n, b_n, c_n \) and \( d_n \) be real Taylor coefficients. Taylor series of tetrational and multi-tetrational functions near the points \( x = 0 \) or \( x = -1 \) are expressed as:

\[
\begin{align*}
\tau(x) &:= \sum_{n=0}^{\infty} \frac{1}{n!} a_n x^n, \\
\tau(x-1) &:= \sum_{n=0}^{\infty} \frac{1}{n!} b_n x^n, \\
\mu(x) &:= \sum_{n=0}^{\infty} \frac{1}{n!} c_n x^n, \\
\mu(x-1) &:= \sum_{n=0}^{\infty} \frac{1}{n!} d_n x^n,
\end{align*}
\]

**Corollary 3.2.** The following coefficients are given as follows independent of \( r \):

\[
\begin{align*}
a_0 &= 1, & b_0 &= 0, & c_0 &= 1, & d_0 &= 1
\end{align*}
\]

**Proof.** Corollary 1.3 gives \( a_0 = \tau(0) = 1 \) and \( b_0 = \tau(-1) = 0 \). Lemma 2.2 gives \( c_0 = \mu(0) = 1 \), and \( d_0 = \mu(-1) = 1 \).

Next we find the correlations of coefficients \( a_n, b_n, c_n \) and \( d_n \).

**Lemma 3.3.** Let \( s = \ln q \) be the real or complex constant dependent on \( q \), where \( s \) is the principal value for \( q < 0 \).

Let \( \binom{n}{k} \) be the binomial coefficient.

Taylor coefficients \( a_n, b_n, c_n \), and \( d_n \) for \( n \geq 1 \) are related to each other in the...
following ways:

\[
\begin{align*}
  a_n &= \omega \sum_{k=1}^{n} \binom{n-1}{k-1} c_{k-1} s^{n-k}, \\
  q b_n &= \omega \sum_{k=1}^{n} \binom{n-1}{k-1} d_{k-1} s^{n-k}, \\
  c_n &= \sum_{k=0}^{n} \binom{n}{k} a_{n-k} d_{k}.
\end{align*}
\]

(1)  \qquad (2)  \qquad (3)

**Proof.** Repeated differentiation of the equation in Theorem 2.3 and assigning \(x = 0\) and \(x = -1\) to it give relations (1) and (2). The relation (3) is given by assigning \(x = 0\) to the equation in Definition 2.1 after repeated differentiation of it.

From Lemma 1.4, we can derive another relationship between \(a_n\) and \(b_n\) similar to Lemma 3.3. The relation, however, can be derived from equations in Lemma 3.3.

Finally we show that general formulae of \(a_n, b_n, c_n\) and \(d_n\) have the common structure with the Stirling numbers of the second kind \(\{n\}^m\) and constants \(A_n\) or \(B_n\) e.g.,

\[
\begin{align*}
  a_0 &= A_0, \\
  a_1 &= A_0 \omega, \\
  a_2 &= A_0 s \omega + A_1 \omega^2, \\
  a_3 &= A_0 s^2 \omega + 3A_1 s \omega^2 + A_2 \omega^3, \\
  a_4 &= A_0 s^3 \omega + 7A_1 s^2 \omega^2 + 6A_2 s \omega^3 + A_3 \omega^4.
\end{align*}
\]

Generally,

\[
a_0 = A_0, \quad a_n = \sum_{k=1}^{n} \binom{n}{k} A_k s^{n-k} \omega^k \quad (n \geq 1).
\]

The proof is given in Theorem 3.4.

**Theorem 3.4.** Let \(A_n\) and \(B_n\) be a real constants having the relation:

\[
A_0 := 1, \quad A_1 := 1, \quad B_0 := 0, \quad B_1 := 1,
\]

\[
A_n := \sum_{k=1}^{n} \binom{n}{k-1} A_{n-k} B_k. \quad (n \geq 2)
\]

Let \(\{n\}^m\) be the Stirling numbers of the second kind.

Then the coefficients \(a_n, b_n, c_n\) and \(d_n\) for \(n \geq 1\) with the following formulae satisfy the relations given in Lemma 3.3:

\[
\begin{align*}
  a_n &= \sum_{k=1}^{n} \binom{n}{k} A_k s^{n-k} \omega^k, \\
  q b_n &= \sum_{k=1}^{n} \binom{n}{k} B_k s^{n-k} \omega^k, \\
  c_n &= \sum_{k=1}^{n} \binom{n}{k} A_{k+1} s^{n-k} \omega^k, \\
  d_n &= \sum_{k=1}^{n} \binom{n}{k} B_{k+1} s^{n-k} \omega^k.
\end{align*}
\]
Proof. We shall prove this by induction. For \( n = 1 \), relations in Lemma 3.3 (1), (2) and (3) are \( a_1 = \omega, qb_1 = \omega \) and 
\( c_1 = a_1 d_0 + a_0 d_1 \), respectively. Clearly, the formulae of \( a_1 = A_1 \omega = \omega \) and 
\( qb_1 = B_1 \omega = \omega \) satisfy the first and second relations above, respectively. The relation 3.3 (3) also holds since the left hand side is \( c_1 = A_2 \omega \), and the right hand side is \( a_1 d_0 + a_0 d_1 = (A_1 B_1 + A_0 B_2) \omega = A_2 \omega \).

Then suppose the given formulae hold for \( n = k \).

By using \( c_n (n \leq k) \) and the following relationship [12][13]:
\[
\binom{n+1}{k+1} = \sum_{j=k}^{n} \binom{n}{j} \binom{j}{k},
\]
the relation of 3.3 (1) for \( n = k + 1 \) is expressed as
\[
a_{k+1} = \omega \sum_{j=1}^{k} \binom{k}{j-1} c_{j-1} s^{k+1-j} = \omega \left[ \binom{k}{0} c_0 s^k + \binom{k}{1} c_1 s^{k-1} + \cdots + \binom{k}{k} c_k \right] = \binom{k}{0} A_1 s^k \omega + \binom{k}{1} A_2 s^{k-1} \omega^2 + \cdots + \binom{k}{k} A_{k+1} s^{k-k} \omega^{k+1} = \binom{k+1}{1} A_1 s^k \omega + \binom{k+1}{2} A_2 s^{k-1} \omega^2 + \cdots + \binom{k+1}{k+1} A_{k+1} \omega^{k+1}
\]
Therefore, if the relation holds for \( n = k \), it holds for \( n = k + 1 \). It follows that the first relation is generally satisfied by the given formulae.

Similarly, proof for the relation of 3.3 (2) is given by replacing \( a_n \) and \( c_n \) with \( q b_n \) and \( d_n \), respectively. Again, if the relation holds for \( n = k \), it holds for \( n = k + 1 \).

Therefore relation 3.3 (1) and (2) are generally satisfied by given expressions. Next we check the given formulae of \( a_n, c_n \) and \( d_n \) generally satisfy the relation of 3.3 (3) at any \( k \).
\[
c_k = \sum_{j=0}^{k} \binom{k}{j} a_{k-j} d_j
\]
\[
= \binom{k}{0} a_k d_0 + \binom{k}{1} a_{k-1} d_1 + \cdots + \binom{k}{k} a_0 d_k
\]
By replacing each \( a_n \) and \( d_n \) by the given formulae, we have
\[
c_k = \binom{k}{0} \left[ \binom{k}{1} A_1 s^{k-1} \omega + \cdots + \binom{k}{k} A_k s^0 \omega^k \right] B_1
\]
\[
+ \binom{k}{1} \left[ \binom{k-1}{1} A_1 s^{k-2} \omega + \cdots + \binom{k-1}{k-1} A_{k-1} s^0 \omega^{k-1} \right] B_2 \omega + \cdots
\]
Now we have the general form of Taylor series of the tetrational and multi-tetra-
tional functions near the points $x = 0$ or $x = -1$:

$$
\tau(x) = A_0 + A_1 \omega x + \frac{1}{2}(A_1 s \omega + A_2 \omega^2) x^2 + \frac{1}{3!}(A_1 s^2 \omega + 3 A_2 s \omega^2 + A_3 \omega^3) x^3 + \cdots
$$

$$
\mu(x) = A_1 + A_2 \omega x + \frac{1}{2}(A_2 s \omega + A_3 \omega^2) x^2 + \frac{1}{3!}(A_2 s^2 \omega + 3 A_3 s \omega^2 + A_4 \omega^3) x^3 + \cdots
$$

$$
\tau(x-1) = \frac{B_0}{q} + \frac{B_1 \omega}{q} x + \frac{1}{2} \frac{B_1 s \omega + B_2 \omega^2}{q^2} x^2 + \frac{1}{3!} \frac{B_1 s^2 \omega + 3 B_2 s \omega^2 + B_3 \omega^3}{q^3} x^3 + \cdots
$$

$$
\mu(x-1) = B_1 + B_2 \omega x + \frac{1}{2} (B_2 s \omega + B_3 \omega^2) x^2 + \frac{1}{3!} (B_2 s^2 \omega + 3 B_3 s \omega^2 + B_4 \omega^3) x^3 + \cdots
$$
4 Tetrational series

In this section, the Taylor series of the tetrational functions and multi-tetrational functions are transformed into new expressions based on the property of Stirling numbers of the second kind.

First we define a key function to express new series.

**Definition 4.1.** Let \( \varphi(x) \) be a function defined as

\[
\varphi(x) := \frac{\omega(q^x - 1)}{s}.
\]

**Corollary 4.2.** The following equation holds:

\[
\frac{d\varphi(x)}{dx} = \omega q^x.
\]

By using \( \varphi(x) \), the Taylor series are then simplified as in Theorem 4.3.

**Theorem 4.3.** Let Taylor series of tetrational functions and multiple tetrational functions near the points \( x = 0 \) or \( x = -1 \) be absolutely convergent, then they are expressed in power series of \( \varphi(x) \), named tetrational series:

\[
\tau(x) = \sum_{n=0}^{\infty} \frac{1}{n!} A_n \varphi^n(x), \quad \tau(x - 1) = \frac{1}{q} \sum_{n=0}^{\infty} \frac{1}{n!} B_n \varphi^n(x),
\]

\[
\mu(x) = \sum_{n=0}^{\infty} \frac{1}{n!} A_{n+1} \varphi^n(x), \quad \mu(x - 1) = \frac{1}{q} \sum_{n=0}^{\infty} \frac{1}{n!} B_{n+1} \varphi^n(x).
\]

**Proof.** Since the Taylor series of tetrational function is absolutely convergent, it can be rearranged as follows:

\[
\tau(x) = A_0 + A_1 \omega x + \frac{1}{2} (A_1 s \omega + A_2 \omega^2) x^2 + \frac{1}{3!} (A_1 s^2 \omega + 3 A_2 s \omega^2 + A_3 \omega^3) x^3 + \ldots
\]

\[
= A_0 + A_1 \omega \left( x + \frac{1}{2} s^2 x^2 + \frac{1}{3!} s^3 x^3 + \ldots \right) + A_2 \omega^2 \left( \frac{1}{2} x^2 + \frac{1}{3!} 3 s x^3 + \ldots \right) + \ldots
\]

\[
= A_0 + A_1 \omega \left( s x + \frac{1}{2} s^2 x^2 + \frac{1}{3!} s^3 x^3 + \ldots \right) + A_2 \omega^2 \left( \frac{1}{2} s^2 x^2 + \frac{1}{3!} 3 s^3 x^3 + \ldots \right) + \ldots
\]

By using the exponential generating function of the Stirling numbers of the second kind [12][13]:

\[
\frac{1}{k!} (e^x - 1)^k = \sum_{n=k}^{\infty} \binom{n}{k} x^n
\]

we have

\[
\tau(x) = A_0 + A_1 \left[ \frac{\omega(e^{sx} - 1)}{s} \right] + \frac{1}{2} A_2 \left[ \frac{\omega(e^{sx} - 1)}{s} \right]^2 + \frac{1}{3!} A_3 \left[ \frac{\omega(e^{sx} - 1)}{s} \right]^3 + \ldots.
\]
Since \( s = \ln q \), we can replace \( e^{sx} \) with \( q^x \). The Taylor series are expressed as the power series of \( \varphi(x) \). Similarly, we can transform \( \tau(x-1), \mu(x) \) and \( \mu(x-1) \) into the tetrational series:

\[
\tau(x) = A_0 + A_1 \varphi(x) + \frac{1}{2} A_2 \varphi^2(x) + \frac{1}{3!} A_3 \varphi^3(x) + \ldots,
\]

\[
\mu(x) = A_1 + A_2 \varphi(x) + \frac{1}{2} A_3 \varphi^2(x) + \frac{1}{3!} A_4 \varphi^3(x) + \ldots,
\]

\[
\tau(x-1) = B_0 + \frac{B_1}{q} \varphi(x) + \frac{1}{2} \frac{B_2}{q} \varphi^2(x) + \frac{1}{3!} \frac{B_3}{q} \varphi^3(x) + \ldots,
\]

\[
\mu(x-1) = B_1 + \frac{B_2}{q} \varphi(x) + \frac{1}{2} \frac{B_3}{q} \varphi^2(x) + \frac{1}{3!} \frac{B_4}{q} \varphi^3(x) + \ldots.
\]

\[\square\]

**Corollary 4.4.** The following equations hold near \( x = 0 \):

\[
\frac{\partial \tau(x)}{\partial \varphi} = \mu(x), \quad \frac{\partial \tau(x-1)}{\partial \varphi} = q^{-1} \mu(x-1).
\]

We check the consistency of the formulae as shown in Lemma 4.5.

**Lemma 4.5.** Formulae 4.3 satisfy the differential equation 2.3:

\[
\frac{d \tau(x)}{dx} = \omega q^x \mu(x).
\]

**Proof.** Since 4.2 and 4.4 with chain rule,

\[
\frac{d \tau(x)}{dx} = \frac{\partial \tau(x)}{\partial \varphi} \frac{\partial \varphi(x)}{dx} = \omega q^x \mu(x),
\]

\[
\frac{d \tau(x-1)}{dx} = \frac{\partial \tau(x-1)}{\partial \varphi} \frac{\partial \varphi(x)}{dx} = \omega q^{x-1} \mu(x-1).
\]

\[\square\]

We further describe the general case of tetrational series with an integer shift \( n \) as in Lemma 4.6. The formulae above are special case of \( n = 0 \).

**Lemma 4.6.** Let \( 0 \leq x < 1 \) be the real heights.

Let \( n \) and \( m \) be integers and natural numbers, respectively. Let \( A_n^{[n]} \) and \( B_m^{[n]} \) be \( n \)-dependent real constants.
The tetrational and multi-tetrational functions, having horizontal translation by integers \( n \) or \( n - 1 \), are expressed as:

\[
\tau(x + n) = q^n \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=1}^{k} \binom{k}{j} A_j^{[n]} s^{k-j} \omega^j = q^n \sum_{k=0}^{\infty} \frac{1}{k!} A_k^{[n]} \varphi^k(x),
\]

\[
\mu(x + n) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=1}^{k} \binom{k}{j} A_j^{[n]} s^{k-j} \omega^j = \sum_{k=0}^{\infty} \frac{1}{k!} A_k^{[n]} \varphi^k(x),
\]

\[
\tau(x + n - 1) = q^{n-1} \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=1}^{k} \binom{k}{j} B_j^{[n]} s^{k-j} \omega^j = q^{n-1} \sum_{k=0}^{\infty} \frac{1}{k!} B_k^{[n]} \varphi^k(x),
\]

\[
\mu(x + n - 1) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=1}^{k} \binom{k}{j} B_j^{[n]} s^{k-j} \omega^j = \sum_{k=0}^{\infty} \frac{1}{k!} B_k^{[n]} \varphi^k(x).
\]

Coefficients have the relationships of:

\[
A_0^{[n]} = \tau(n), \quad B_0^{[n]} = \tau(n - 1),
\]

\[
A_m^{[n]} = q^n \sum_{k=1}^{m} \frac{(m-1)!}{(k-1)!} A_{m-k}^{[n]} B_k^{[n]} \quad (m \geq 1).
\]

Generally, \( \tau(x) \) and \( \mu(x) \) having horizontal translation by \( n \) have the relationship of:

\[
\frac{\partial \tau(x + n)}{\partial \varphi} = q^n \mu(x + n).
\]

Proof. Obviously the tetrational series satisfy \( \partial \tau(x + n)/\partial \varphi = q^n \mu(x + n) \). It follows that these series satisfy the differential equation \( \tau'(x + n) = \omega q^{x+n} \mu(x+n) \), as similarly in 4.3. Tetrational series and the Taylor series are converted into each other with the exponential generating function of the Stirling numbers of the second kind as in Theorem 4.3. The relationships of coefficients are given by assigning \( x = 0 \) to the repeated partial derivatives by \( \varphi \) of \( \mu(x + n) = \tau(x + n) \mu(x + n - 1) \).

\[\square\]

5 Explicit formulae

In this section, we derive the explicit formulae of tetrational and multi-tetrational function by determining the coefficients \( A_n \) and \( B_n \).

Lemma 5.1. Let \( 0 \leq x < 1 \). The following equations hold:

\[
\mu(x - 1) = 1,
\]

\[
\mu(x) = \tau(x),
\]

\[
\mu(x + n) = \prod_{k=0}^{n} \tau(x + k).
\]
Proof. We defined $\mu(x)$ in Definition 2.1 as a continuous function satisfying

$$\mu(x) = \tau(x)\mu(x - 1),$$

which goes through the following discrete points:

- $\mu(0) = \tau(0)$
- $\mu(1) = \tau(1)\tau(0)$
- $\mu(2) = \tau(2)\tau(1)\tau(0)$

$$\vdots$$

- $\mu(n) = \prod_{k=0}^{n} \tau(k) = \tau(n)\mu(n - 1)$.

Similarly, $\mu(x)$ also goes through the discrete points with changes of $0 \leq x < 1$ as follows, since continuous functions $\mu(x)$ and $\tau(x)$ go through $\mu(n)$ and $\tau(n)$ respectively:

- $\mu(x) = \tau(x)$
- $\mu(x + 1) = \tau(x + 1)\tau(x)$
- $\mu(x + 2) = \tau(x + 2)\tau(x + 1)\tau(x)$

$$\vdots$$

- $\mu(x + n) = \prod_{k=0}^{n} \tau(x + k) = \tau(x + n)\mu(x + n - 1)$

The relationship $\mu(x + n) = \tau(x + n)\mu(x + n - 1)$ is consistent with the definition 2.1.

Since from the definition $\mu(x - 1) = \mu(x)/\tau(x)$ and $\mu(x) = \tau(x)$ as given above, we have

$$\mu(x - 1) = \frac{\tau(x)}{\tau(x)} = 1.$$

\[ \square \]

Corollary 5.2. Tetrational function $\tau(x)$ and $\mu(x)$ are piecewise connected functions.

Proof. Obviously from Lemma 5.1, multi-tetrational function $\mu(x)$ is a piecewise connected function. Since tetralional function have the relationship of $\tau'(x) = \omega q^x \mu(x)$, tetrational function $\tau(x)$ is also a piecewise connected function. \[ \square \]

Lemma 5.3. The coefficients $A_n$ and $B_n$ are determined as

- $A_n = 1$,
- $B_{n+1} = 0$, $B_1 = 1$. 

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Proof. From Theorem 4.3

\[ \tau(x) = \sum_{n=0}^{\infty} \frac{1}{n!} A_n \phi^n(x), \quad \mu(x) = \sum_{n=0}^{\infty} \frac{1}{n!} A_{n+1} \phi^n(x). \]

Since \( \frac{\partial \tau(x)}{\partial \phi} = \mu(x) \) from 4.4 and \( \tau(x) = \mu(x) \) from 5.1 we have

\[ 1 = A_0 = A_1 = A_2 = \ldots. \]

From the relationship of coefficients \( A_n = \sum_{k=1}^{n} \binom{n}{k-1} A_{n-k} B_k \), we get \( B_n \neq 1 \) and \( B_1 = 1 \).

Similarly, we can get the values of \( A_m^{[n]} \) and \( B_m^{[n]} \) given in Lemma 4.6.

Lemma 5.4. The constant \( \omega \) is determined as

\[ \omega = \frac{sq}{q-1}. \]

Proof. Since we determined the values of \( B_n \) as in Lemma 5.3 and with the expression of \( \phi(x) \) in Definition 4.1, we can express \( \tau(x-1) \) for \( 0 \leq x < 1 \) as

\[ \tau(x-1) = \frac{1}{q} \phi(x) = \frac{\omega(q^x - 1)}{sq} \]

Then by assigning \( x = 1 \), we get the equation \( \omega(q - 1)/sq = 1 \) to derive \( \omega \).

Now in Theorem 5.5 we can derive the explicit solution for the delay-differential equation 1.4. The domains of the tetrational function as well as the multi-tetrational function are naturally extended by applying the sawtooth function.

Theorem 5.5. Let \( r \) be real constant.

Let \( x \) be real variable heights in \( x \in \mathbb{R} \setminus \{ n : n \in \mathbb{Z}, n \leq -2 \} \).

Let \( \lfloor x \rfloor \) and \( \{ x \} := x - \lfloor x \rfloor \) be the floor function and the sawtooth function of \( x \), respectively.

Let \( [x]_q \) be the \( q \)-analog of \( x \) defined as

\[ [x]_q := \frac{q^x - 1}{q - 1}, \]

where \( q = \ln r \) is the principal branch if \( r < 0 \).

Let \( \exp_r^n = \log_{r^{-n}} \) be the extended iterated exponential operator defined as

\[ \exp_r^n f := \begin{cases} \exp_r \exp_r \cdots \exp_r f & (n \geq 0) \\ \log_r \log_r \cdots \log_r f & (n < 0), \end{cases} \]
where \( f \) is the operand.

Let \( \prod_{k=m}^n f_k \) be the extended product operator defined as

\[
\prod_{k=m}^n f_k := \begin{cases} 
\frac{f_m f_{m+1} \cdots f_{n-1} f_n}{1} & (0 \leq m \leq n) \\
\frac{1}{f_{m+1} f_{m+2} \cdots f_{n+1}} & (n \leq m < 0), 
\end{cases}
\]

where \( f_k \) are the operands.

Then the tetrational function and the multi-tetrational functions are expressed as

\[
x_r = \tau(x) = \exp_r[x] \quad \text{and} \quad \mu(x) = \prod_{k=0}^n \exp_{r+1}[k] q = q^{-1} \partial \tau(x) \quad \text{in Theorem 4.3 and Lemma 5.4,}
\]

Proof. Let us consider the case of \( 0 \leq x < 1 \).

From Definition 4.1 and Lemma 5.4, we have

\[
\varphi(x) = \frac{q(q^x - 1)}{q - 1}.
\]

By using the notation of q-analog of the variable, we can express it as

\[
\varphi(x) = q[x].
\]

Then the tetrational and multi-tetrational functions for \( n \geq 0 \) are given as follows since \( \tau(x) = \sum_{n=0}^\infty (q[x]) n^x/n! = \exp_r[x] \) in Theorem 4.3, \( \tau(x + n + 1) = \exp_r \tau(x + n) \) as well as \( \mu(x + n) = \tau(x + n) \mu(x + n - 1) \):

\[
\begin{align*}
\tau(x) &= \exp_r[x] q, \quad \mu(x) = \exp_r[x] q \\
\tau(x + 1) &= \exp_r^2[x] q, \quad \mu(x + 1) = \exp_r[x] q \exp_r[x] q \\
\tau(x + 2) &= \exp_r^3[x] q, \quad \mu(x + 2) = \exp_r[x] q \exp_r^2[x] q \exp_r[x] q \\
& \quad \vdots \\
\tau(x + n) &= \exp_r^{n+1}[x] q, \quad \mu(x + n) = \prod_{k=0}^n \exp_{r+1}[x] q
\end{align*}
\]

In the case for \( n < 0 \). The relation \( \tau(x - 1) = [x] q \) and \( \tau(x - n - 1) = \log_r \tau(x - n) \), as well as \( \mu(x - 1) = 1 \) and \( \mu(x - n) = \mu(x - n + 1)/\tau(x - n + 1) \), give the following series:

\[
\begin{align*}
\tau(x - 1) &= [x] q, \quad \mu(x - 1) = \frac{\tau(x)}{\tau(x)} = 1, \\
\tau(x - 2) &= \log_r[x] q, \quad \mu(x - 2) = \frac{\tau(x)}{\tau(x) \tau(x - 1)} = \frac{1}{[x] q}, \\
\tau(x - 3) &= \log_r^2[x] q, \quad \mu(x - 3) = \frac{\tau(x)}{\tau(x) \tau(x - 1) \tau(x - 2)} = \frac{1}{[x] q \log_r[x] q}, \\
& \quad \vdots \\
\tau(x - n) &= \log_r^{n+1}[x] q, \quad \mu(x - n) = \frac{\prod_{k=0}^{n-1} \tau(x - k)}{\prod_{k=0}^{n-1} \tau(x - k)} = \frac{\prod_{k=0}^{n} \tau(x + k)}{\prod_{k=0}^{n} \tau(x + k)}.
\end{align*}
\]
In the last line, we applied the extended operator. Therefore we can also give the same expression for \( n < 0 \) as

\[
\tau(x + n) = \exp_r^{n+1}[x]_q, \quad \mu(x + n) = \prod_{k=0}^{n} \exp_r^{k+1}[x]_q.
\]

Obviously, the following equation holds for both cases of \( n \geq 0 \) and \( n < 0 \):

\[
\mu(x + n) = q^{-n-1} \frac{\partial \tau(x + n)}{\partial [x]_q}.
\]

By replacing \( x \) and \( n \) above with \( \{x\} \) and \( [x] \), respectively, we have the expressions of tetrational and multi-tetrational functions for \( x \in (-\infty, \infty) \):

\[
\tau(x) = \exp_r^{[x]+1}[\{x\}]_q, \quad \mu(x) = \prod_{k=0}^{[x]} \exp_r^{k+1}[\{x\}]_q = q^{-[x]-1} \frac{\partial \tau(x)}{\partial [\{x\}]_q}.
\]

**Corollary 5.6.** The following equation holds:

\[
\{x\}_r = \{\{x\}\}_q.
\]

**Proof.** Since \( 0 \leq \{x\} < 1 \) and \( [\{x\}] = 0 \), the equation is given from Theorem 5.2.

Let us consider the special case of \( r = e \). If \( r \to e \), hence \( q = \ln r \to 1 \), then \( q \)-analog becomes \( [\{x\}]_q \to \{x\} \). So we have \( \tau(x) = \exp_r^{[x]+1}\{x\} \). This formula is the same as Hooshmand’s ultra exponential function for \( r = e \).

The curves of continuous tetrational functions \( ^r x = \tau(x) \) for different bases \( r \) are shown in Fig. 2. It is obvious that the tetrational functions for \( r < 1 \) have complex values.

Figure 3 shows the behaviours of the multiple tetrational functions \( \mu(x) \) for different bases \( r \). In general the functions are not differentiable at \( x = n \in \mathbb{Z} \).

### 6 Analytical properties

First, let us study the properties of \( ^y x = \tau(x, y) \) as the multivariable function of base \( x \) and height \( y \). From this point, we shall freely choose the letters to express the bases and integers.

**Theorem 6.1.** Let \( x > 0 \) be real bases and let \( q = \ln x \).

Let \( y \) be real heights in \( y \in \mathbb{R} \setminus \{ n: n \in \mathbb{Z}, n \leq -2 \} \).

Then the tetrational function \( ^y x = \tau(x, y) = \exp_r^{[y]+1}[\{y\}]_q \) is totally differentiable:

\[
d\tau(x, y) = \frac{\partial \tau(x, y)}{\partial x} dx + \frac{\partial \tau(x, y)}{\partial y} dy.
\]
Figure 2: Behaviours of the continuous tetrational functions $x^r$ for different bases $r$, where only the real component are expressed for $r = 1/e$ and $r = e^{-e}$ (left). The tetrational functions of $r = 1/e$ and $r = e^{-e}$ are expressed in the complex plane (right).

Figure 3: Behaviors of the multi-tetrational functions $\mu(x)$ for different bases $r$, where only the real component are expressed for $r = 1/e$ and $r = e^{-e}$ (left). The multi-tetrational functions of $r = 1/e$ and $r = e^{-e}$ are expressed in the complex plane (right).
Proof. We shall prove this by induction. Let \(-1 \leq y < 0\), then the tetranational function is

\[ \tau(x, y) = q^{y+1} - 1 = e^{(\ln x)(y+1)} - 1. \]

Then the partial derivatives for \(-1 \leq y < 0\) are:

\[ \frac{\partial \tau(x, y)}{\partial x} = ye^{(\ln x)(y+1)} \frac{e^{(\ln x)(y+1)} - 1}{x \ln x - 1} \]
\[ \frac{\partial \tau(x, y)}{\partial y} = \frac{(\ln x)e^{(\ln x)(y+1)}}{\ln x - 1} \]

The following equation holds for \(-1 \leq y < 0\).

\[ \frac{\partial^2 \tau(x, y)}{\partial x \partial y} = \frac{\partial^2 \tau(x, y)}{\partial y \partial x} = \frac{(1 + (y + 1) \ln x)e^{(\ln x)(y+1)}}{x \ln x - 1} - \frac{(\ln x)e^{(\ln x)(y+1)}}{x (\ln x - 1)^2} \]

Therefore

\[ d\tau(x, y) = \frac{\partial \tau(x, y)}{\partial x} \, dx + \frac{\partial \tau(x, y)}{\partial y} \, dy. \]

Next, let us consider the general case.

From the Definition 1.1

\[ y+1x = \tau(x, y + 1) = e^{(\ln x}\tau(x, y)), \]

the partial derivatives of this general equation are

\[ \frac{\partial \tau(x, y + 1)}{\partial x} = \tau(x, y + 1) \left[ \frac{\tau(x, y)}{x} + (\ln x) \frac{\partial \tau(x, y)}{\partial x} \right], \]
\[ \frac{\partial \tau(x, y + 1)}{\partial y} = (\ln x) \tau(x, y + 1) \frac{\partial \tau(x, y)}{\partial y}. \]

Let \(\tau(x, y)\) be defined in \(n \leq y < n + 1\), and \(\tau(x, y + 1)\) be defined in \(n + 1 \leq y + 1 < n + 2\).

Suppose the following equation holds for \(n \leq y < n + 1\):

\[ \frac{\partial^2 \tau(x, y)}{\partial x \partial y} = \frac{\partial^2 \tau(x, y)}{\partial y \partial x} \]

Since \(\tau(x, y)\) in Theorem 5.5 is determined so as to be continuous at integer heights \(y\), and differentiation along base \(x\) is not affected by the existence of boundary, obviously the connection at \(y = n\) is continuous as follows:

\[ \lim_{y \to n+1} \frac{\partial \tau(x, y)}{\partial x} = \lim_{y \to n} \frac{\partial \tau(x, y + 1)}{\partial x} = \frac{\partial \tau(x, n + 1)}{\partial x}, \]
\[ \lim_{y \to n+1} \frac{\partial \tau(x, y)}{\partial y} = \lim_{y \to n} \frac{\partial \tau(x, y + 1)}{\partial y} = \frac{\partial \tau(x, n + 1)}{\partial y}. \]

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The following equation holds:

\[
\frac{\partial^2 \tau(x, y + 1)}{\partial x \partial y} = \frac{\partial^2 \tau(x, y + 1)}{\partial y \partial x}
\]

\[
= (\ln x) \tau(x, y + 1) \left[ \frac{\tau(x, y)}{x} + (\ln x) \frac{\partial \tau(x, y)}{\partial x} \right]
\]

\[
+ \tau(x, y + 1) \left[ \frac{1}{x} \frac{\partial \tau(x, y)}{\partial y} + (\ln x) \frac{\partial^2 \tau(x, y)}{\partial y \partial x} \right]
\]

Hence:

\[
d\tau(x, y + 1) = \frac{\partial \tau(x, y + 1)}{\partial x} dx + \frac{\partial \tau(x, y + 1)}{\partial y} dy.
\]

In general, therefore, tetralational function is totally differentiable.

\[
\square
\]

The property above allow us to define the inverse operations of tetration, super-logarithm or super-root, in monotonic changing range: base \( r \geq 1 \) and height \( h > -2 \).

Next we study the behaviours of the tetralational function \( ^h x = \tau(x, h) \) with a variable base \( x \) and a constant height \( h \).

**Lemma 6.2.** let \( n \) be the integer constant.

Let \( x > 0 \) be the real variable.

\[
\lim_{x \to 0} (^n x) = \begin{cases} 
0 & (n: \text{even}) \\
1 & (n: \text{odd}) 
\end{cases}
\]

**Proof.** We shall prove this by induction. From Definition 1.1 the relations \( ^{-1} x = 0 \) and \( ^0 x = 1 \) hold for \( x \to 0 \).

Obviously we have \( \lim_{x \to 0} (\ln x) = -\infty \).

If \( \lim_{x \to 0} (^{2n} x) = 1 \), then

\[
\lim_{x \to 0} (^{2n} x \ln x) = -\infty.
\]

It follows

\[
\lim_{x \to 0} (^{2n+1} x) = \lim_{x \to 0} [e^{(2n) \ln x}] = 0,
\]

If \( \lim_{x \to 0} (^{2n+1} x) = 0 \), then

\[
\lim_{x \to 0} (^{2n+1} x \ln x) = \lim_{x \to 0} [\ln(x^{(2n+1) x})] = \lim_{x \to 0} [\ln (^{2n} x)] = 0.
\]

It follows

\[
\lim_{x \to 0} (^{2n+2} x) = \lim_{x \to 0} [e^{(2n+1) \ln x}] = 1.
\]
Therefore the relation generally holds.

The relation in Lemma 6.2 can be confirmed by the curves shown in Figure 4 (left). The tetrational functions $n^x$ for integers $n$ have real values and go to 0 or 1 for even $n$ or odd $n$ respectively under $x \to 0$. As shown in the right graph of Figure 4, in the general case including non-integers $h$, the functions $h^x$ have complex values and non-monotonic behaviours for $x < 1$ while the functions have real values and monotonic changes for $x \geq 1$.

Figure 4: Behaviors of $n^x$ for integer heights $-1 \leq n \leq 4$ (left). The tetrational function $h^x$ for various heights $-1 \leq h \leq 3$ with an interval of 0.2 (right). In general, the functions have complex values for non-integer heights if $x < 1$, and only the real components are shown in the right figures.

In the right graph of Figure 4 only the real components are shown, while trajectories for $x < 1$ in complex plane are given in Figures 5. It is obvious that $h^x$ do not always go to 0 or 1 for $x \to 0$, e.g., the trajectory for $h = 0.5$ circulates around 0 and hence never reach 0 or 1.

It is also shown in Figure 5 (a) that the trajectories for $h = -0.5$ is a semicircle and the distorted semicircles for $h$ and $-(1+h)$ are axially symmetric each other, e.g., -0.2 and -0.8. The relation between $h^x$ and $-(1+h)^y$ is given as Lemma 6.3 which explains not only the symmetry but also existence of a pair function convertible each other if $-1 \leq h \leq 0$.

**Lemma 6.3.** Let $-1 \leq h \leq 0$. Let $x > 0$ and $y > 0$. Let $(h^x)^*$ be the complex conjecture of $h^x$. 

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Then the following equation holds:

\((h \, x)^* = 1 - (-1 + h) y)\),

or with the q-analog notation:

\[[h + 1]_{\ln x} = 1 - [-h]_{\ln y},\]

where

\[\ln y = \frac{1}{\ln x}.\]

**Proof.** Let \(q = \ln x\) and \(k > 0\).

For \(0 < x < 1\), obviously \(q < 0\), and \(q\) can be written as \(q = -k = ke^{i\pi}\). Hence we have

\[q^{h+1} = k^{h+1}[\cos \pi (h + 1) + i \sin \pi (h + 1)]\]

The tetrational function \(h \, x\) for \(-1 \leq h \leq 0\) is then expressed as

\[h \, x = [h + 1]_q = \frac{q^{h+1} - 1}{q - 1} = \frac{k^{h+1} \cos \pi (h + 1)}{-k - 1} + i \frac{k^{h+1} \sin \pi (h + 1)}{-k - 1}.\]
Therefore
\[(^h x)^* = -\frac{h^{k+1} \cos \pi(h + 1)}{k + 1} + i \frac{k^{h+1} \sin \pi(h + 1)}{k + 1}.\]

On the other hand,
\[\ln y = \frac{1}{q} = k^{-1} e^{i\pi}.\]

Hence we get
\[\left(\frac{1}{q}\right)^{-(h+1)} = k^{h+1} \left[\cos \pi(h + 1) - i \sin \pi(h + 1)\right].\]

Then
\[1 - (-1+y) = 1 - [h]_{1/q} = 1 - \left(\frac{1}{q}\right)^{-(h+1)} - \frac{1 - q}{1 - q}\]
\[= 1 - \frac{h^{k+1} \cos \pi(h + 1) + k}{1 + k} + i \frac{k^{h+1} \sin \pi(h + 1)}{1 + k}\]
\[= \frac{-h^{k+1} \cos \pi(h + 1) + k}{k + 1} + i \frac{k^{h+1} \sin \pi(h + 1)}{k + 1}.\]

Therefore \((^h x)^* = 1 - (-1+y)\) holds.

For \(x \geq 1\), obviously \(q \geq 0\), and we have
\[(^h x)^* = h_x = \frac{q^{h+1} - 1}{q - 1}.\]

On the other hand, we have
\[1 - (-1+y) = 1 - \left(\frac{1}{q}\right)^{-(h+1)} - \frac{1 - q}{q - 1} = \frac{q^{h+1} - 1}{q - 1}.\]

Therefore generally \((^h x)^* = 1 - (-1+y)\) holds.

Now let us consider the extension of the tetration to complex bases and heights.

**Theorem 6.4.** Let \(r\) be real constant.
Let \(z\) be a complex variable.
Then the following statements hold:

- \(z^r\) is holomorphic if and only if \(\Re(z) \notin \mathbb{Z}\).
- \(r^z\) is holomorphic if and only if \(r \in \mathbb{Z}\).
Proof. Let \( z = x + iy \).
(1) Proof of the first statement.
We shall prove this by induction.
Let \( s = \ln q = \ln r \).
Then \( q^{x+1} = e^{s(x+1+iy)} = e^{s(x+1)}[\cos(sy) + i\sin(sy)] \).
For \(-1 \leq x < 0\) we have
\[
x_r = \frac{q^{x+1+iy} - 1}{q - 1} = \frac{e^{s(x+1)} \cos(sy) - 1}{q - 1} + i \frac{e^{s(x+1)} \sin(sy)}{q - 1} = u_0 + iv_0,
\]
where \( u_0 \) and \( v_0 \) are real and imaginary component of \( z_r \), respectively.
Obviously the Cauchy-Riemann equations holds:
\[
\begin{align*}
\frac{\partial u_0}{\partial x} &= \frac{\partial v_0}{\partial y} = \frac{se^{s(x+1)} \cos(sy)}{q - 1}, \\
\frac{\partial u_0}{\partial y} &= -\frac{\partial v_0}{\partial x} = -\frac{se^{s(x+1)} \sin(sy)}{q - 1}.
\end{align*}
\]
Therefore \( z_r \) is holomorphic in \(-1 \leq x < 0\).
Let \( n \in \mathbb{Z} \). Suppose \((x+iy)_r = u_n + iv_n\) for \((n-1) \leq x < n\) is holomorphic.
Then
\[(x+1+iy)_r = e^{qu_n} \cos(qv_n) + ie^{qu_n} \sin(qv_n) = u_{n+1} + iv_{n+1}.
\]
By using the chain rule, e.g.,
\[
\frac{\partial u_{n+1}}{\partial x} = \frac{\partial u_{n+1}}{\partial u_n} \frac{\partial u_n}{\partial x} + \frac{\partial u_{n+1}}{\partial v_n} \frac{\partial v_n}{\partial x},
\]
we can prove that Cauchy-Riemann equations holds:
\[
\begin{align*}
\frac{\partial u_{n+1}}{\partial x} &= \frac{\partial v_{n+1}}{\partial y} = qe^{qu_n} \cos(qv_n) \frac{\partial u_n}{\partial x} - qe^{qu_n} \sin(qv_n) \frac{\partial v_n}{\partial x}, \\
\frac{\partial v_{n+1}}{\partial x} &= -\frac{\partial u_{n+1}}{\partial y} = qe^{qu_n} \sin(qv_n) \frac{\partial u_n}{\partial x} - qe^{qu_n} \cos(qv_n) \frac{\partial v_n}{\partial x}.
\end{align*}
\]
Therefore \( z_r \) is generally holomorphic in inside of each segment \((n-1) < x < n\).
However the Cauchy-Riemann equations do not hold at boundaries of segments \( x = n \) as follows:
The real and imaginary components of \( z_r \) in segment \(-1 \leq x < 0\) approach to the following values as \( x \to 0 \):
\[
\lim_{x \to 0} u_0 = \frac{q \cos(sy) - 1}{q - 1}, \quad \lim_{x \to 0} v_0 = \frac{q \sin(sy)}{q - 1}.
\]
On the other hand, \( z_r \) approaches the following values as \( x \to 0 \) since \( u_1 + iv_1 = e^{qu_0} \cos(qv_0) + ie^{qu_0} \sin(qv_0)\):
\[
\lim_{x \to 0} u_1 = \exp \left[ \frac{q \cos(sy) - 1}{q - 1} \right] \cos \left( \frac{q \sin(sy)}{q - 1} \right)
\]
Similarly, the partial derivatives of \( u \) where
\[
\frac{\partial u}{\partial x} = \frac{q \cos(sy) - 1}{q - 1} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{q \sin(sy)}{q - 1}.
\]
Obviously, in general \( r_z \) is not continuous at the segment boundary. It is continuous if and only if the heights are real, \( y = 0 \) hence \( v_1 = v_0 = 0 \). Similarly,
\[
\lim_{x \to 0} u_{n+1} = e^{qu_n} \cos(qv_n), \quad \lim_{x \to 0} v_{n+1} = e^{qu_n} \sin(qv_n).
\]
Then \( \lim_{x \to n} (u_{n+1} + iv_{n+1}) = \lim_{x \to n} (u_n + iv_n) \) holds if and only if \( v_{n+1} = v_n = 0 \). Therefore, the first statement holds.

(2) Proof of the second statement
Let \( h, k \in \mathbb{R} \) and let \( \ln z = e^{i \theta} = e^h \cos k + ie^h \sin k \).
Since \( z = x + iy = e^h \cos k \cos(e^h \cos k) + ie^h \cos k \sin(e^h \cos k) \), we can confirm that the Cauchy-Riemann equations hold and expressed as follows:
\[
\frac{\partial h}{\partial x} = \frac{\partial k}{\partial y} = \frac{1}{e^{(e^h \cos k)} \cos(e^h \sin k + k)}, \quad \frac{\partial h}{\partial y} = -\frac{\partial k}{\partial x} = \frac{1}{e^{(e^h \cos k)} \sin(e^h \sin k + k)}.
\]
Let \(-1 \leq r < 0\). Then we have
\[
r_z = \frac{(\ln z)^{1+r} - 1}{\ln z - 1} = \frac{e^{(r+1)^h} \cos k(r + 1) + ie^{(r+1)^h} \sin k(r + 1) - 1}{e^h \cos k + ie^h \sin k - 1}
\]
\[
= \frac{(e^{h \cos k - 1})^2 + e^{2h \sin^2 k}}{e^{(r+1)^h} \cos k(r + 1) - e^h \cos k + 1 + i e^{(r+1)^h} \sin k(r + 1) + e^h \sin k}
\]
\[
= \frac{e^{h \cos k - 1} + 1}{e^{2h} - 2e^h \cos k + 1} = u + iv,
\]
where \( u \) and \( v \) are real and imaginary component of \( r_z \), respectively.
The partial derivatives of \( u \) and \( v \) by \( h \) or \( k \) are:
\[
\frac{\partial u}{\partial h} = \frac{(r + 1)e^{(r+2)^h} \cos kr - (r + 1)e^{(r+1)^h} \cos k(r + 1) - e^h \cos k}{e^{2h} - 2e^h \cos k + 1} - \frac{e^{(r+2)^h} \cos kr - e^{(r+1)^h} \cos k(r + 1) - e^h \cos k + 1}{(2e^{2h} - 2e^h \cos k)^2},
\]
\[
\frac{\partial u}{\partial k} = \frac{-re^{(r+2)^h} \sin kr + (r + 1)^2e^{(r+1)^h} \sin k(r + 1) + e^h \sin k}{e^{2h} - 2e^h \cos k + 1} - \frac{e^{(r+2)^h} \cos kr - e^{(r+1)^h} \cos k(r + 1) - e^h \cos k + 1}{(2e^{2h} - 2e^h \cos k + 1)^2},
\]
\[
\frac{\partial v}{\partial h} = \frac{(r + 1)e^{(r+2)^h} \sin kr - (r + 1)e^{(r+1)^h} \sin k(r + 1) - e^h \sin k}{e^{2h} - 2e^h \cos k + 1} - \frac{e^{(r+2)^h} \sin kr - e^{(r+1)^h} \sin k(r + 1) - e^h \sin k + 1}{(2e^{2h} - 2e^h \cos k + 1)^2},
\]
\[
\frac{\partial v}{\partial k} = \frac{-re^{(r+2)^h} \cos kr + (r + 1)^2e^{(r+1)^h} \cos k(r + 1) + e^h \cos k}{e^{2h} - 2e^h \cos k + 1} - \frac{e^{(r+2)^h} \cos kr - e^{(r+1)^h} \cos k(r + 1) - e^h \cos k + 1}{(2e^{2h} - 2e^h \cos k + 1)^2},
\]
Hence it is obvious that the Cauchy-Riemann equations do not hold.

\[
\frac{\partial v}{\partial h} = \frac{(r+2)e^{(r+2)h} \sin kr - (r+1)e^{(r+1)h} \sin k(r+1) + e^h \sin k}{e^{2h} - 2e^h \cos k + 1} - \frac{e^{(r+2)h} \sin kr - e^{(r+1)h} \sin k(r+1) + e^h \sin k}{2e^{2h} - 2e^h \cos k} \frac{(e^{2h} - 2e^h \cos k + 1)^2}{(e^{2h} - 2e^h \cos k + 1)^2},
\]

\[
\frac{\partial v}{\partial k} = \frac{re^{(r+2)h} \cos kr - (r+1)2e^{(r+1)h} \cos k(r+1) + e^h \cos k}{e^{2h} - 2e^h \cos k + 1} - \frac{e^{(r+2)h} \sin kr - e^{(r+1)h} \sin k(r+1) + e^h \sin k}{2e^{2h} \sin k} \frac{(e^{2h} - 2e^h \cos k + 1)^2}{(e^{2h} - 2e^h \cos k + 1)^2}.
\]

It is obvious that the Cauchy-Riemann equations do not hold.

\[
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial h} \frac{\partial h}{\partial x} + \frac{\partial u}{\partial k} \frac{\partial k}{\partial x} = \frac{\partial v}{\partial h} \frac{\partial h}{\partial y} + \frac{\partial v}{\partial k} \frac{\partial k}{\partial y} = \frac{\partial v}{\partial y}.
\]

Similarly the tetrational function in other segments \((r+n)z = \exp_n^r(z)\) are generally not holomorphic inside the segment \((n-1) < x < n\).

However, at the boundaries of the segment \(r = n\), the function is expressed as \(^n_z = \exp_n^r[1]\) and is holomorphic proved by induction as follows.

Let \(z = e^{x+iy}\) and \((n+1)z = x_n + iy_n\).

For \(n = 1\),

\[^1_z = e^{x+iy} = e^x \cos y + ie^x \sin y = u_1 + iv_1\]

The Cauchy-Riemann equations hold:

\[
\frac{\partial u}{\partial y} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -e^x \sin y.
\]

Hence \(^1_z\) is holomorphic.

Suppose \(^n_z = u_{n-1} + iv_{n-1}\) is holomorphic:

\[
\frac{\partial u_{n-1}}{\partial x} = \frac{\partial v_{n-1}}{\partial y}, \quad \frac{\partial u_{n-1}}{\partial y} = -\frac{\partial v_{n-1}}{\partial x}.
\]

Then \((n+1)z\) is expressed as:

\[
(n+1)z = z^{(n)} = e^{(x+iy)(u_{n-1}+iv_{n-1})} = e^{(xu_{n-1} - yv_{n-1} - ivu_{n-1} + ivv_{n-1})}
\]

\[
= e^{(xu_{n-1} - yv_{n-1})} \cos(xv_{n-1} + yu_{n-1}) + ie^{(xu_{n-1} - yv_{n-1})} \sin(xv_{n-1} + yu_{n-1})
\]

\[= u_n + iv_n.\]

By using the relations above, it is shown that the Cauchy-Riemann equations hold:

\[
\frac{\partial u_n}{\partial x} = \frac{\partial v_n}{\partial y}
\]

\[
= \left( u_n + x \frac{\partial u_{n-1}}{\partial x} - y \frac{\partial v_{n-1}}{\partial x} \right) e^{(xu_{n-1} - yv_{n-1})} \cos(xv_{n-1} + yu_{n-1})
\]

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\[-(v_{n-1} + x \frac{\partial v_{n-1}}{\partial x} + y \frac{\partial u_{n-1}}{\partial x}) e^{(xu_{n-1} - yv_{n-1})} \sin(xv_{n-1} + yu_{n-1}),\]

\[-(v_{n-1} + x \frac{\partial v_{n-1}}{\partial y} + y \frac{\partial u_{n-1}}{\partial x}) e^{(xu_{n-1} - yv_{n-1})} \cos(xv_{n-1} + yu_{n-1}),\]

\[-(u_{n-1} + y \frac{\partial u_{n-1}}{\partial x} + y \frac{\partial v_{n-1}}{\partial x}) e^{(xu_{n-1} - yv_{n-1})} \sin(xv_{n-1} + yu_{n-1}).\]

Therefore the second statement holds.

**Corollary 6.5.** If both the base and height are complex values without integer components, \(x, y \in \mathbb{C}\) and \(x, y \notin \mathbb{Z}\), then the tetragonal function \(y^x\) is not holomorphic.

### 7 Calculation rules of tetration

In this section, we study calculation rules of tetration based on the explicit formulae and properties of the functions.

First let us extend the exponentiation operator.

**Definition 7.1.** Let \(r, x, y \in \mathbb{R}\).

Extended exponentiation operator \(\exp^x_r\) is defined as:

\[\exp^x_r 1 := \exp^{|x|}_r (r^{[x]} x) = |x| r,

where the notations in Theorem 5.5 are used.

**Lemma 7.2.** Let \(r, x, y \in \mathbb{R}\). The following equation holds:

\[x^r = \exp_{r^y} (y^r)\]

**Proof.** Since \(x = [x] + \{x\}\), by using the relation 5.6, we have the following relations:

\[x^r = \exp^x_r 1 = \exp^{|x|}_{r^y} (x^{[x]} r) = \exp_x^{|x|^r} (x^{[x]} r).\]

Then for any \(w \in \mathbb{R}\),

\[\exp^x_r 1 = \exp^{|x|-w + |x|+w} r 1 = \exp_x^{|x|-w} (\{x\}+w).\]

By using \(y = \{x\} + w\) and hence \([x] - w = [x] + |x| - y = x - y\), we get the relation.

**Lemma 7.3.** Let \(r, x, y \in \mathbb{R}\). The following equation holds:

\[\exp^x_r \exp^y_r 1 = \exp^{x+y}_r 1 = x^y r.\]
Proof. From Lemma 7.2

\[ x^r = \exp_{x^r} \left( a^r \right) = \exp_{x^r}^a \exp_{x^r}^1 \]

By rewriting \( y = x - a \), we get the relation. \( \square \)

Next, we distinguish the exponentiation (right associative) from the power (left associative) for clarity.

**Definition 7.4.** Let \( \lor \) and \( \land \) be the symbols of the exponentiation (right associative) and the power (left associative), respectively.

\[ x^r \lor f \coloneqq \exp_{x^r}^f, \]
\[ f \land x^r \coloneqq f^{(x^r)}, \]

where \( f \) is the operand.

The left-hand side of \( \lor \) is not a number but the part of the exponentiation operator. The iterated tetration is given as:

\[ x^{(y^r)} \lor f \coloneqq (\exp_y^x)^f = \exp_{x^y}^f. \]

The notation \( x^r \lor y^r \) refers to \( \exp_{x^r}^y \), where the operand \( y^r \) is a number while \( x^r \lor \) is not a number but the extended exponent operator, not referring to \( \exp_{(x^r)} \). On the other hand, the notation \( x^r \land y^r \) refers to \( (x^r)^{y^r} = \exp_{(x^r)}^{(y^r)} \), where both \( x^r \) and \( y^r \) are numbers. Similarly, \( x^{(y^r)} \lor 1 \) is not \( x^t \) with \( t = y^r \lor 1 \).

Some calculation rules are given as in Theorem 7.5

**Theorem 7.5.** Let \( r, t, x, y, z \in \mathbb{R} \). The following relations hold:

\[ x^r \lor 1 = x^r. \] \( (1) \)
\[ -x^r = -1 \lor \left( \frac{1}{x+1} \right). \] \( (2) \)
\[ \frac{1}{x^r} = r \lor (-x^{-1}r). \] \( (3) \)
\[ x^r + y^r = -1 \lor (x+1r \cdot y+1r). \] \( (4) \)
\[ x^r - y^r = -1 \lor \left( \frac{x+1r}{y+1r} \right). \] \( (5) \)
\[ x^r \land y^r = y+1r \land x^{-1}r. \] \( (6) \)
\[ x^r \cdot y^r = -1 \lor (x^r \land y^r) = r \lor (x^{-1}r + y^{-1}r). \] \( (7) \)
\[ \frac{x^r}{y^r} = -1 \lor (x+1r \land \frac{1}{y}) = r \lor (x^{-1}r - y^{-1}r). \] \( (8) \)
\[
x_r \lor y_r = y_r \lor x_r = x \lor y,
\]
(9)

\[
y_r(x_r) = x_r(y_r) = x \cdot y.
\]
(10)

\[
z(x_r \lor y_r) = z(x \lor y).
\]
(11)

\[
(x_r)^r = r \lor \frac{\ln r(x_r) - 1}{\ln r - 1}
\]
(12)

\[
x_r = |x_r| \lor \{x_r\}
\]
(13)

\[
\{x\}^r = \{x\} \lor + (\ln r(x)) \cdot (\{y\}^r), \quad \{x\} + \{y\} < 1.
\]
(14)

\[
x_r = (\{x\}^r) \lor (1 - (\{x\}^r))^*, \quad \ln r \cdot \ln t = 1.
\]
(15)

\[
\frac{\partial (x_r)}{\partial (\{x\}^r)} = (\ln r(x) + \frac{\partial |x|}{\partial (\{x\}^r)} \sum_{k=0}^{\{x\}^r - 1} (r^k \lor \{x\}^r).
\]
(16)

**Proof.** The proofs are given as fellows respectively:

1. Directly given from Definition 7.1. This is the relation between operator \(x_r \lor\) and number \(x_r\):
   \[
x_r = \exp_r^* 1 = x_r \lor 1.
\]

2. By Definition 7.4, we have \(\log_r f = -1 r \lor f\).
   \[
   -x_r = -\log_r(x_r^{x+1}) = \log_r \left( \frac{1}{x+1} \right) = \frac{1}{x+1}.
   \]

3. By the exponent rule, we have
   \[
   \frac{1}{x_r} = \left( x_r \right)^{-1} = (r^{-1}) = r^{-0} = r \lor (-x^{-1}_r).
   \]

4. By the logarithm rule,
   \[
x_r + y_r = \log_r(x^{x+1}) + \log_r(y^{x+1}) = \log_r(x^{x+1} \cdot y^{x+1}) = -1 r \lor (x^{x+1} \cdot y^{x+1}).
   \]

5. By the logarithm rule,
   \[
x_r - y_r = \log_r(x^{x+1}) - \log_r(y^{x+1}) = \log_r \left( \frac{x^{x+1}}{y^{x+1}} \right) = -1 r \lor \left( \frac{x^{x+1}}{y^{x+1}} \right).
   \]

6. By the exponent rule,
   \[
x_r \lor y_r = \left( x_r \right)^{(y_r)} = r^{(x_r)}(y_r) = r^{(x_r + y_r)} = y_r \lor (x_r \lor y_r).
   \]

7. By the logarithm rule,
   \[
x_r \cdot y_r = x_r \cdot \log_r(y^{x+1}) = \log_r \left( y^{x+1} \right)^{(x_r)} = -1 r \lor (y_{x+1} \lor x_r) = -1 r \lor (x_r \lor y_r),
   \]
where the (6) is used at the last step. By the exponent rule,
   \[
x_r \cdot y_r = r^{(x+1)} \cdot r^{(y+1)} = r^{(x+1) + (y+1)} = r \lor (x+1 + y+1).
   \]
By the logarithm rule,

\[
\frac{\log_r (x^{x+1})}{y} = \log_r \left( \frac{x+1}{y} \right) = -1 \log_r \left( \frac{1}{y} \right).
\]

By the exponent rule,

\[
\frac{\log_r (x^{x+1})}{y} = \log_r (x^{x+1}) \cdot y = \log_r (x^{x+1}) \cdot \frac{1}{y} = \log_r (x^{x+1} \cdot \frac{1}{y}).
\]

(9) Directly give from Lemma 7.2 and 7.3.

(10) From definition 7.4, we have the relation by assigning \( f = 1 \).

(11) From definition 7.4, we have the relation by assigning \( f = 1 \).

(12) Corollary 5.6 directly give the relation.

(13) We have the relation from Definition 7.1 and (12).

(14) This relation is the property of q-numbers:

\[
\log_r (x^{x+1}) \cdot y = \log_r (x^{x+1}) \cdot \frac{1}{y} = \log_r (x^{x+1} \cdot \frac{1}{y}).
\]

(15) From Lemma 6.3, we have the following relation.

\[
\left( \log_r (x^{x+1}) \cdot y \right)^* = 1 - \left( \log_r (x^{x+1}) \cdot y \right).
\]

Then \( (x)^{-1}_r = \left( 1 - \left( x \right)_r \right)^* \).

By applying the operation of \( (\log_r (x^{x+1}) \cdot y) \), we have

\[
(\log_r (x^{x+1}) \cdot y)^* = \left( (\log_r (x^{x+1}) \cdot y) \right)^*.
\]

Then we use (13) and get the relation.

(16) We get the relation from Theorem 5.5 and Corollary 5.6.

8 Concluding remarks

We derived the simple explicit formulae of the continuous tetrational function as well as multi-tetrational function based only on the delay-differential equation. The definition of the multi-tetrational function and finding the Stirling numbers of the second kind in Taylor coefficients are the key steps of our approach. The solutions is piecewise connected function and is class \( C^1 \) at connecting point for real heights, and the height is extended to complex heights except the segment boundaries. Our solution has advantages in simple expression as well as in continuity in real \( r-x \) plane. The series for each segment converges absolutely and has infinite radius of convergence. The solution should be compared with those of numerical approaches of class \( C^\infty \) with limited domain of base \( r > e^{1/e} \) and finite radius of convergence for Taylor expansion. The calculation rules of tetrations are given by distinguishing exponentiation from power operation.
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