Finite Metabelian Group Algebras
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Abstract. Given a finite metabelian group G, whose central quotient is abelian (not cyclic) group of order \( p^2 \), \( p \) odd prime, the objective of this paper is to obtain a complete algebraic structure of semisimple group algebra \( \mathbb{F}_q[G] \) in terms of primitive central idempotents, Wedderburn decomposition and the automorphism group.

1. Introduction

Let \( F \) be a field and \( G \) be a finite group such that the group algebra \( F[G] \) is semisimple. A fundamental problem in the theory of group algebras is to understand the complete algebraic structure of semisimple group algebra \( F[G] \). In the recent years, a lot of work has been done to solve this problem [1,2,5,7,8,9]. Bakshi et.al [3] have solved this problem for semisimple finite group algebra of certain groups whose central quotient is Klein’s four-group. In the present paper, a complete algebraic structure of semisimple group algebra \( \mathbb{F}_q[G] \) for some finite groups \( G \), whose central quotient \( G/Z(G) \), is the direct product of two cyclic groups of order \( p \), \( p \) odd prime, is obtained. It is known [6] that finitely generated groups \( G \), whose central quotient is isomorphic to \( \mathbb{Z}_p \times \mathbb{Z}_p \) break into nine classes. The complete algebraic structure of \( \mathbb{F}_q[G] \), for group \( G \) in the two of the nine classes, is obtained in the present paper.

2. Notation

Let \( G \) be a finite group of order coprime to \( q \) and \( \text{Irr}(G) \) denotes the set of all irreducible characters of \( G \) over \( \mathbb{F}_q \), the algebraic closure of \( \mathbb{F}_q \). Let \( H \lhd K \leq G \) such that \( K/H \) is cyclic of order \( n \) and \( T = N_G(H) \cap N_G(K) \), where \( N_G(H) \) denotes the normalizer of \( H \) in \( G \). Let \( C(K/H) \) denotes the set of \( q \)-cyclotomic sets of \( \text{Irr}(K/H) \) containing the generators of \( \text{Irr}(K/H) \). Suppose that \( T \) acts on \( C(K/H) \) by conjugation, then it is easy to see that stabilizer of any \( C \in C(K/H) \) remains the same. Let \( E_C(K/H) \) denotes the stabilizer of any \( C \in C(K/H) \) and let \( \mathcal{R}(K/H) \) denotes the set of distinct orbits of \( C(K/H) \) under the action of \( T \) on \( C(K/H) \). Observe that

\[
|\mathcal{R}(K/H)| = \frac{\phi(n)|E_C(K/H)|}{|T|\text{ord}_n(q)},
\]

where \( \text{ord}_n(q) \) denotes the order of \( q \) modulo \( n \).
For $C \in C(K/H), \chi \in C$ and $\zeta_n$ a primitive $n$th root of unity in $\overline{F}_q$, set

$$e_C(K/H) = |K|^{-1} \sum_{g \in C} (tr_{F_q(\zeta_n)/F_q}(\chi(g)))g^{-1},$$

and $e_C(G, K, H)$ as the sum of distinct $G$-conjugates of $e_C(K/H)$.

3. Metabelian group algebras

The notation used in [4] will be followed: For a normal subgroup $N$ of $G$, let $A_N/N$ be an abelian normal subgroup of $G/N$ of maximal order. Let $D_N$ be the set of subgroups $D/N$ of $A_N/N$ such that $A_N/D$ is cyclic and $T_{G/N}$ be the set of representatives of $D_N$ under the equivalence relation of conjugacy of subgroups of $G/N$. Define

$$S_{G/N} := \{(D/N, A_N/N) \mid D/N \in T_{G/N}, D/N \text{ core-free in } G/N\}.$$

Let

$$S := \{(N, D/N, A_N/N) \mid N \triangleleft G, S_{G/N} \neq \emptyset, (D/N, A_N/N) \in S_{G/N}\}.$$

We are now ready to recall the theorem describing the complete algebraic structure of semisimple finite metabelian group algebras:

**Theorem 1 [3]:** Let $G$ be a finite metabelian group of order coprime to $q$. Then,

(i) A complete set of primitive central idempotents of semisimple group algebra $\mathbb{F}_q[G]$ is given by the set $\{e_C(G, A_N, D) \mid (N, D/N, A_N/N) \in S, C \in \mathfrak{R}(A_N/D)\}$;

(ii) the simple component corresponding to primitive central idempotent $e_C(G, A_N, D)$ is

$$F_q[G]e_C(G, A_N, D) \cong M_{\mathfrak{R}(A_N/D)}(F_q),$$

where $M_n(R)$ denotes the ring of $n \times n$ matrices over the ring $R$ and

$$o(A_N, D) = \frac{\text{ord}_{(A_N, D)}(q)}{[E_G(A_N, D) : A_N]}.$$  

Moreover the number of such simple components is $|\mathfrak{R}(A_N, D)|$.

4. Groups whose central quotient is abelian (not cyclic) group of order $p^2$

Conelissen and Milies [6] have classified indecomposable finitely generated groups $G$, such that $G/Z(G) \cong C_p \times C_p$, into nine classes. In all of these classes, $G = \langle a, b, Z(G) \rangle$ with some more relations as described in following table:
| Group $G$ | $Z(G)$ | Relations |
|---|---|---|
| $\mathfrak{G}_1$ | $\langle c \rangle$ | $a^p, b^p, c^{p^n}b^{-1}a^{-1}bac^{-1(p^{m-1})}$, $m \geq 1$ |
| $\mathfrak{G}_2$ | $\langle c \rangle$ | $a^p c^{-1}, b^p c^{-1}, c^{p^n}, b^{-1}a^{-1}bac^{-1(p^{m-1})}$, $m \geq 1$ |
| $\mathfrak{G}_3$ | $\langle c_1 \rangle \times \langle c_2 \rangle$ | $a^p, b^p c_2^{-1}, c_1^{p^n}, c_2^{p^n}, b^{-1}a^{-1}bac_1^{-1(p^{m-1})}$, $m_1, m_2 \geq 1$ |
| $\mathfrak{G}_4$ | $\langle c_1 \rangle \times \langle c_2 \rangle$ | $a^p c_1^{-1}, b^p c_2^{-1}, c_1^{p^n}, c_2^{p^n}, b^{-1}a^{-1}bac_1^{-1(p^{m-1})}$, $m_1, m_2 \geq 1$ |
| $\mathfrak{G}_5$ | $\langle c_1 \rangle \times \langle u_i \rangle$ | $a^p, b^p u_i^{-1}, c_1^{p^n}, b^{-1}a^{-1}bac_1^{-1(p^{m-1})}$, $m_i \geq 1$ |
| $\mathfrak{G}_6$ | $\langle c_1 \rangle \times \langle u_i \rangle$ | $a^p c_1^{-1}, b^p u_i^{-1}, c_1^{p^n}, b^{-1}a^{-1}bac_1^{-1(p^{m-1})}$, $m_i \geq 1$ |
| $\mathfrak{G}_7$ | $\langle c_1 \rangle \times \langle c_2 \rangle \times \langle c_3 \rangle$ | $a^p c_2^{-1}, b^p c_3^{-1}, c_1^{p^n}, c_2^{p^n}, c_3^{p^n}, b^{-1}a^{-1}bac_1^{-1(p^{m-1})}$, $m_1, m_2, m_3 \geq 1$ |
| $\mathfrak{G}_8$ | $\langle c_1 \rangle \times \langle c_2 \rangle \times \langle u_i \rangle$ | $a^p c_2^{-1}, b^p u_i^{-1}, c_1^{p^n}, c_2^{p^n}, b^{-1}a^{-1}bac_1^{-1(p^{m-1})}$, $m_1, m_2 \geq 1$ |
| $\mathfrak{G}_9$ | $\langle c_1 \rangle \times \langle u_i \rangle \times \langle u_2 \rangle$ | $a^p u_i^{-1}, b^p u_2^{-1}, c_1^{p^n}, b^{-1}a^{-1}bac_1^{-1(p^{m-1})}$, $m_i \geq 1$ |

$o(u_i), i = 1, 2$ is infinite.

It can be see easily that $G$ is finite metabelian group only in five classes. Out of these five classes, we will give a complete algebraic structure of $\mathbb{F}_q[G]$, for $G = \mathfrak{G}_3$ and $\mathfrak{G}_2$ only. The rest of the cases can be dealt similarly. Throughout this section $\mathbb{F}_q$ is a finite field with $q$ elements and $\gcd(p, q) = 1$. Let $ord_p(q)$, the order of $q$ modulo $p$, be $f$ and $e = \frac{p-1}{f}$. Write $q^l = 1 + p^d c$, where $p$ does not divide $c$. Then for $l \geq 1$,

$$ord_{p^l}(q) = \begin{cases} f, & l \leq d, \\ fp^{l-d}, & l \geq d+1. \end{cases}$$

4.1. Structure of $\mathbb{F}_q[\mathfrak{G}_1]$

Let $G$ be a group of type $\mathfrak{G}_1$. Thus $G$ has following representation:

$$G = \langle a, b, c \mid a^p = b^p = c^{p^m} = 1, b^{-1}a^{-1}ba = c^{p^{m-1}} , c \text{ central in } G \rangle (1)$$
where \( p \) is prime and \( m \geq 1 \). For \( p = 2 \), the complete algebraic structure of \( \mathbb{F}_q[G] \) can be read from [3]. Suppose \( p \) is an odd prime. For \( m \geq 2 \), define

\[
K_0 := \langle 1 \rangle, \quad K_1 := \langle c, a \rangle, \quad K_2^{(i)} := \langle c, a^i b \rangle, \quad K_3^{(i)} := \langle a, b, c^p \rangle, \quad 0 \leq i \leq p - 1,
\]

\[
K_4^{(i,j)} := \langle a, c^p b^j \rangle, \quad K_5^{(i,j)} := \langle b, c^p a^j \rangle, \quad K_6^{(i,j,k)} := \langle c^p a^j, c^p b^k \rangle,
\]

\[
0 \leq i \leq m - 2, \quad 1 \leq j, k \leq p - 1.
\]

**Theorem 2.** A complete set of primitive central idempotents of semisimple group algebra \( \mathbb{F}_q[G] \), \( G \) of type \( \mathfrak{G}_1 \), is given as follows:

**Primitive central idempotents of \( \mathbb{F}_q[G] \) for \( m = 1 \):**

\[
e_c(G, G, G), \quad C \in \mathfrak{R}(G/G);
\]

\[
e_c(G, G, \langle c, a^i b \rangle), \quad C \in \mathfrak{R}(G/\langle c, a^i b \rangle), \quad 0 \leq i \leq p - 1;
\]

\[
e_c(G, G, \langle a, c \rangle), \quad C \in \mathfrak{R}(G/\langle a, c \rangle);
\]

\[
e_c(G, \langle a, c \rangle, \langle a \rangle), \quad C \in \mathfrak{R}(\langle a, c \rangle/\langle a \rangle).
\]

**Primitive central idempotents of \( \mathbb{F}_q[G] \) for \( m \geq 2 \):**

\[
e_c(G, K_1, \langle a \rangle), \quad C \in \mathfrak{R}(K_1/\langle a \rangle);
\]

\[
e_c(G, G, K_1), \quad C \in \mathfrak{R}(G/K_1);
\]

\[
e_c(G, G, K_2^{(i)}), \quad C \in \mathfrak{R}(G/K_2^{(i)}) \quad 0 \leq i \leq p - 1;
\]

\[
e_c(G, G, K_3^{(i)}), \quad C \in \mathfrak{R}(G/K_3^{(i)}) \quad 0 \leq i \leq p - 1;
\]

\[
e_c(G, G, K_4^{(i,j)}), \quad C \in \mathfrak{R}(G/K_4^{(i,j)}) \quad 0 \leq i \leq m - 2, \quad 1 \leq j \leq p - 1;
\]

\[
e_c(G, G, K_5^{(i,j)}), \quad C \in \mathfrak{R}(G/K_5^{(i,j)}) \quad 0 \leq i \leq m - 2, \quad 1 \leq j \leq p - 1;
\]

\[
e_c(G, G, K_6^{(i,j,k)}), \quad C \in \mathfrak{R}(G/K_6^{(i,j,k)}) \quad 0 \leq i \leq m - 2, \quad 1 \leq j, k \leq p - 1.
\]

To prove this Theorem, we first need to find the normal subgroups of \( G \).

**Lemma 1.** Let \( G \) be a group as defined in (1) and \( \mathcal{N} \) be the set of distinct normal subgroups of \( G \). Then

(i) For \( m = 1 \), \( \mathcal{N} = \{(1), (c), (c, a), (c, a, b)(c, a^i b), 0 \leq i \leq p - 1 \}

and \( S = \{(1), \langle a \rangle, \langle c, a \rangle\} \cup \{(1, \langle a, c \rangle), \langle c, a, b \rangle, \langle c, a^i b \rangle, \langle a, b, c \rangle, \langle a, c, b \rangle | 0 \leq i \leq p - 1\};

(ii) For \( m \geq 2 \), \( \mathcal{N} = \{(c^p)^i, (c^p, a), (c^p, b), (c^p, a, b) | 0 \leq i \leq m - 1 \} \cup \{(c^p, a^j b^k), (c^p, a^j, c^p b^k) | 0 \leq i \leq m - 2, 1 \leq j, k \leq p - 1\}

and \( S = \{(K_0, \langle a \rangle), (K_1)\} \cup \{(K_0, G/K_1), (K_2^{(i)}), (K_3^{(i)}, G/K_3^{(i)}), (K_4^{(i,j)}, G/K_4^{(i,j)}), (K_5^{(i,j)}, G/K_5^{(i,j)}), (K_6^{(i,j,k)}, G/K_6^{(i,j,k)}) | 0 \leq i \leq m - 1, 1 \leq j \leq p - 1, 0 \leq i \leq m - 2, 1 \leq j \leq p - 1\}.\)
Proof. It can be seen easily that in (i) and (ii), the subgroups listed are distinct and normal in \( G \). Also if \( N \triangleleft G \), then it can be shown easily, as in \([3], \text{Lemma 4}\), that \( N \) is one of the subgroups listed in the statement of Lemma.

Observe that in both (i) and (ii), for \( N = \langle l \rangle, A_N / N = \langle c, a \rangle \). Hence \( S_{G/N} = \{\langle (l), G/N \rangle, \phi, G \} \), if \( G/N \) is cyclic, otherwise.

Thus to complete the proof, we need to find only those \( N \in \mathcal{N} \) for which \( G/N \) is cyclic. In (i), the subgroups \( \langle c, a \rangle, \langle c, a, b \rangle, \langle c, a'b \rangle, 0 \leq i \leq p - 1 \) have cyclic quotient with \( G \), whereas in (ii), the following normal subgroups have cyclic quotient with \( G \):

\[
K_i, K_2^{(i)}, K_3^{(i)}, 0 \leq i \leq p - 1, \\
K_4^{(i,j)}, K_5^{(i,j)}, K_6^{(i,j,k)}, 0 \leq i \leq m - 2, 1 \leq j, k \leq p - 1.
\]

Thus the proof of the lemma is complete.

Proof of Theorem 2. The list of primitive central idempotents of group algebra \( F_q[G] \) can now be easily obtained with the help of Theorem 1 and Lemma 1.

Theorem 3. The Wedderburn decomposition and the automorphism group of semisimple group algebra \( F_q[G] \), \( G \) of type \( \mathfrak{S}_1 \), are given as follows:

**Wedderburn decomposition**

\[
F_q[G] \cong \begin{cases} 
F_q \oplus F_q^{((p+1)e)} \oplus M_p \left(F_q^{(e)}\right), & m = 1, \\
F_q \oplus F_q^{(m+1)} \oplus M_p \left(F_q^{(m-1)e}\right), & 2 \leq m \leq d, \\
F_q \oplus F_q^{(d+2-1)} \oplus \sum_{i=d+1}^{m-1} F_{q^{p^{d+2-1}}} \oplus M_p \left(F_{q^{p^{d+1}}}\right), & m \geq d + 1.
\end{cases}
\]

**Automorphism group**

\[
Aut(F_q[G]) \cong \begin{cases} 
\left(\mathbb{Z}_f^{(p+1)e} \rtimes S_{(p+1)e}\right) \oplus \left(SL_p(F_q) \rtimes \mathbb{Z}_f^{(e)}\right) \rtimes S_e), m = 1, \\
\left(\mathbb{Z}_f^{(m+1)} \rtimes S_{(m+1)}\right) \oplus \\
\left(SL_p(F_q) \rtimes \mathbb{Z}_f^{(m-1)e}\right) \rtimes S_{p^{m-1}e} \right), 2 \leq m \leq d, \\
\left(\mathbb{Z}_f^{(d+2-1)} \rtimes S_{p^{d+2-1}}\right) \oplus \sum_{i=d+1}^{m-1} \left(S_{F_{p^{d+1}}^{p^{d+1}e}} \rtimes S_{p^{d+1}e}\right) \right), m \geq d + 1.
\end{cases}
\]

where \( \mathbb{Z}_m \) denotes the cyclic group of order \( n \), \( S_n \) denotes the symmetric group of degree \( n \) and for a group \( H \), \( H^{(n)} \) a direct sum of \( n \) copies of \( H \).
**Proof of Theorem 3.** In order to find the Wedderburn decomposition of $\mathbb{F}_q[G]$, we need to find the simple component corresponding to each primitive central idempotent. More precisely, for each $(N, D/N, A_N/N) \in S, C \in \mathcal{R}(A_N/D)$, we need to calculate $o(A_N, D)$ and $|\mathcal{R}(A_N/D)|$, as given by the following tables:

**Case I : $m = 1$**

| $(N, D/N, A_N/N)$ | $E_G(A_N/D)$ | $o(A_N, D)$ | $|\mathcal{R}(A_N/D)|$ |
|--------------------|--------------|--------------|------------------------|
| $((1), (a), \langle a, c \rangle)$ | $\langle a, c \rangle$ | $f$ | $e$ |
| $((c, a), (1), G/\langle c, a \rangle)$ | $G$ | $f$ | $e$ |
| $((G), (1), (1))$ | $G$ | $1$ | $1$ |
| $((c, a'b), (1), G/\langle c, a'b \rangle)$ $0 \leq i \leq p - 1$ | $G$ | $f$ | $e$ |

**Case II : $m \geq 2$**

| $(N, D/N, A_N/N)$ | $E_G(A_N/D)$ | $o(A_N, D)$ | $|\mathcal{R}(A_N/D)|$ |
|--------------------|--------------|--------------|------------------------|
| $(K_0, (a), K_1)$ | $K_1$ | $\begin{cases} f, & m \leq d, \\ fp^{m-d}, & m \geq d + 1. \end{cases}$ | $\begin{cases} p^{m-1}e, & m \leq d, \\ p^{d-1}e, & m \geq d + 1. \end{cases}$ |
| $(K_1, K_0, G/K_1)$ | $G$ | $f$ | $e$ |
| $(K_2^{(i)}, K_0, G/K_2^{(i)})$ $0 \leq i \leq p - 1$ | $G$ | $\begin{cases} 1, & i = 0 \\ f, & 1 \leq i \leq d \\ fp^{i-d}, & i \geq d + 1 \end{cases}$ | $\begin{cases} 1, & i = 0, \\ p^{i-1}e, & 1 \leq i \leq d, \\ p^{d-1}e, & i \geq d + 1. \end{cases}$ |
| $(K_3^{(i)}, K_0, G/K_3^{(i)})$ $0 \leq i \leq p - 1$ | $G$ | $\begin{cases} 1, & i = 0 \\ f, & 1 \leq i \leq d \\ fp^{i-d}, & i \geq d \end{cases}$ | $\begin{cases} 1, & i = 0, \\ p^{i-1}e, & 1 \leq i \leq d, \\ p^{d-1}e, & i \geq d. \end{cases}$ |
| $(K_4^{(i,j)}, K_0, G/K_4^{(i,j)})$ $0 \leq i \leq m - 2, 1 \leq j \leq p - 1$ | $G$ | $\begin{cases} f, & i \leq d - 1 \\ fp^{j-d+1}, & i \geq d \end{cases}$ | $\begin{cases} p^{i-1}e, & i \leq d - 1, \\ p^{d-1}e, & i \geq d. \end{cases}$ |
| $(K_5^{(i,j)}, K_0, G/K_5^{(i,j)})$ $0 \leq i \leq m - 2, 1 \leq j \leq p - 1$ | $G$ | $\begin{cases} f, & i \leq d - 1 \\ fp^{j-d+1}, & i \geq d \end{cases}$ | $\begin{cases} p^{i-1}e, & i \leq d - 1, \\ p^{d-1}e, & i \geq d. \end{cases}$ |
| $(K_6^{(i,j,k)}, K_0, G/K_6^{(i,j,k)})$ $0 \leq i \leq m - 2, 1 \leq j, k \leq p - 1$ | $G$ | $\begin{cases} f, & i \leq d - 1 \\ fp^{j-d+1}, & i \geq d \end{cases}$ | $\begin{cases} p^{i-1}e, & i \leq d - 1, \\ p^{d-1}e, & i \geq d. \end{cases}$ |

Now, the required Wedderburn decomposition and automorphism group can be easily read from these two tables and [3, Theorem 3].
4.2. Structure of $\mathbb{F}_q[\mathfrak{G}_2]$ 

Observe that if group $G$ is of type $\mathfrak{G}_2$, then it has the following presentation:

$$G = \langle a, b \mid a^{p^{m+1}} = 1, b^p = a^{p^2}, b^{-1}a^{-1}ba = a^{p^{m+1}}, a^p \text{ central in } G \rangle,$$

where $p$ is a prime and $m \geq 1$. For $p = 2$, the complete algebraic structure of $\mathbb{F}_q[G]$ can be read from [3]. Suppose $p$ is an odd prime. For $m \geq 1$, set:

$$L_0 := \langle 1 \rangle, L_1 := \langle a \rangle, L_2 := \langle a^p \rangle, 1 \leq i \leq m, L_3 := \langle a, b \rangle,$$

$$L_4 := \langle a^p, ab \rangle, 0 \leq i \leq p-1,$$

$$L_5 := \langle a^p, a^{b^{i-1}}b \rangle, 2 \leq i \leq m, 1 \leq j \leq p.$$

The following Theorems give a complete algebraic structure of semisimple group algebra $\mathbb{F}_q[G]$:

**Theorem 4.** A complete set of primitive central idempotents of semisimple group algebra $\mathbb{F}_q[G]$, $G$ of type $\mathfrak{G}_2$, is given as follows:

**Primitive central idempotents of $\mathbb{F}_q[G]$**

$$e_C(G, G, G), C \in \mathcal{R}(G/G);$$

$$e_C(G, G, L_1), C \in \mathcal{R}(G/L_1);$$

$$e_C(G, G, L_2), C \in \mathcal{R}(G/L_2);$$

$$e_C(G, L_1, L_0), C \in \mathcal{R}(L_1/L_0);$$

$$e_C(G, G, L_4^{(i)}), C \in \mathcal{R}(G/L_4^{(i)}), 0 \leq i \leq p-1;$$

$$e_C(G, G, L_5^{(i,j)}), C \in \mathcal{R}(G/L_5^{(i,j)}), 2 \leq i \leq m, 1 \leq j \leq p.$$

**Proof of Theorem 4.** In view of Theorem 1, to find a complete list of primitive central idempotents of $\mathbb{F}_q[G]$ we first need to list all normal subgroups of $G$. It can be see easily that the set $\mathcal{N}$ of distinct normal subgroups of $G$ is equal to

$$\{L_0, L_1, L_2^{(i)}, 1 \leq i \leq m, L_3, L_4^{(i)}, 0 \leq i \leq p-1, L_5^{(i,j)}, 2 \leq i \leq m, 1 \leq j \leq p\}.$$

For $N = L_0$, $A_N / N = L_1$. Hence $S_{G/N} = \{\langle L_0, L_1 \rangle\}$. Moreover, for non-identity $N \in \mathcal{N}, S_{G/N}$ is non-empty if and only if $G/N$ is cyclic. The following $N \in \mathcal{N}$ have cyclic quotient with $G$:

$$L_1, L_3, L_4^{(i)}, 0 \leq i \leq p-1, L_5^{(i,j)}, 2 \leq i \leq m, 1 \leq j \leq p.$$

Thus (i) follows from Theorem 1.

**Theorem 5.** The Wedderburn decomposition and the automorphism group of semisimple group algebra $\mathbb{F}_q[G]$, $G$ of type $\mathfrak{G}_2$, are as follows:

**Wedderburn decomposition**

$$\mathbb{F}_q[G] \cong \begin{cases} 
\bigoplus \mathbb{F}_q \bigoplus \mathbb{F}_q \left( \frac{p^{m+1}}{f} \right) \bigoplus M_p \left( \mathbb{F}_q \right)^{\left( \frac{p^{m+1}-1}{f} \right)}, & m \leq d - 1, \\
\bigoplus \mathbb{F}_q \bigoplus \mathbb{F}_q \left( \frac{p^{d+1}}{f} \right) \bigoplus \sum_{i=d+1}^{m} \mathbb{F}_q \left( \frac{p^{d-1}}{f} \right) \bigoplus M_p \left( \mathbb{F}_q \right)^{\left( \frac{p^{d-1}-1}{f} \right)}, & m \geq d.
\end{cases}$$
Proof of Theorem 5. We will first find \( E_G(L_1/L_0) \). Observe that \( |L_1/L_0| = p^{m+1} \) and \( L_1 \subseteq E_G(L_1/L_0) \subseteq G \). Let \( m \leq d - 1 \). In this case, \( b \in E_G(L_1/L_0) \), if and only if \( \zeta^{p^{m+1}} = \zeta^{p^d} \) for some \( i, 1 \leq i \leq f \), where \( \zeta \) is a primitive \( p^{m+1} \) th root of unity. This implies that \( p^{m+1} = q^i (\text{mod } p^{m+1}) \), i.e., \( p = \frac{f}{\gcd(i, f)} \), which gives that \( p \) divides \( p - 1 \), a contradiction. Hence in this case \( E_G(L_1/L_0) = L_1 \). For \( m \geq d \), \( E_G(L_1/L_0) = G \). Thus we have the following:

| \((N, D)/N, A_N/N\) | \(E_G(A_N/D)\) | \(o(A_N, D)\) | \(|\mathfrak{M}(A_N/D)|\) |
|---------------------|-----------------|-----------------|-----------------|
| \((G, L_0, L_0)\)   | \(G\)           | 1               | 1               |
| \((L_1, L_0, G/L_1)\)| \(G\)             | \(f\)           | \(e\)           |
| \((L_4^{(i)}, L_0, G/L_4^{(i)})\), \(0 \leq i \leq p - 1\) | \(G\)           | \(f\)           | \(e\)           |
| \((L_5^{(i,j)}, L_0, G/L_5^{(i,j)})\), \(2 \leq i \leq m, 1 \leq j \leq p\) | \(G\)           | \(\begin{cases} f, & i \leq d, \\ fp^{i-d}, & i > d + 1. \end{cases}\) | \(\begin{cases} p^{i-1}e, & i \leq d, \\ p^{d-1}e, & i > d + 1. \end{cases}\) |
| \((L_0, L_0, L_1)\)  | \(\begin{cases} L_1, & m \leq d - 1, \\ G, & m \geq d. \end{cases}\) | \(\begin{cases} f, & m \leq d - 1, \\ fp^{m-d}, & m \geq d. \end{cases}\) | \(\begin{cases} p^{m-1}e, & m \leq d - 1, \\ p^{d-1}e, & m \geq d. \end{cases}\) |

The Wedderburn decomposition and automorphism group of \( \mathbb{F}_q[G] \) can now be easily read with the help of this table and [3, Theorem 3].

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