The Hamilton-Jacobi characteristic equations for topological invariants:
Pontryagin and Euler classes

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By using the Hamilton-Jacobi [HJ] framework the topological theories associated with Euler and Pontryagin classes are analyzed. We report the construction of a fundamental $HJ$ differential where the characteristic equations and the symmetries of the theory are identified. Moreover, we work in both theories with the same phase space variables and we show that in spite of Pontryagin and Euler classes share the same equations of motion their symmetries are different. In addition, we report all HJ Hamiltonians and we compare our results with other formulations reported in the literature.

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I. INTRODUCTION

It is well-known that the study of topological theories is an interesting topic in mathematics or physics from either or both directions. In fact, for the former the study of topological structures of manifolds (with particular interest in four dimensional manifolds) based on the identification of topological invariants is one of the research subjects for the mathematical physics community. Examples of topological invariants in four dimensions can be cited, the so-called Euler and Pontryagin invariants. These invariants are fundamental objects in the characterization of the topological structure of a manifold, they label topologically distinct four-geometries; the Pontryagin invariant gives the relation between the number of selfdual and anti-selfdual harmonic connections on the manifold. Moreover, the Euler invariant gives a relation between the number of harmonic p-forms on the manifold \[1\]. From the physical point of view, there exist a close relation between these topological invariants and physical theories just like gravity and field theory. In this respect, the Euler and Pontryagin classes are fundamental blocks for constructing the noncommutative form of topological gravity \[2\]. In the $BF$-gravity context there exists also a relation between these invariants and $BF$-gravity formulation. In fact, in the MacDowell-Mansouri formulation of gravity based on a $SO(5)$ topological $BF$ theory, the symmetry group $SO(5)$ is broken down into $SO(4)$ in order to obtain the Palatini action plus the addition of the Pontryagin and Euler topological invariants \[3,5\], due to these topological classes have trivial local variations that do not contribute.
to the dynamics, hence one obtains essentially general relativity. In this respect, both the Euler and Pontryagin invariants can be written as a $BF$-like theory and this fact has allowed to study its canonical structure in more convenient way \cite{6}. In addition, the study of the canonical structure of Pontryagin invariant in the presence of boundaries has been analyzed in \cite{7}. In fact, topological gauge theories defined on spacetime regions with boundary are good toy models for studying the emerging of physical degrees of freedom and observables localized on the boundary; these theories are natural objects for researching the relation between the canonical structure of topological gauge theory and the existence of dualities a la $AdS/CFT$ \cite{8–10}.

On the other hand, from the Hamiltonian point of view, the Euler and Pontryagin invariants treated as field theories give rise the same equations of motion, are devoid of physical degrees of freedom, diffeomorphisms covariant and there exist reducibility conditions between the constraints \cite{11}. Furthermore, form the quantum point of view due to the $BF$ structure of these topological theories and the symmetries commented above, these invariants are good laboratories for studying the classical and quantum structure of a background independent theory and this fact could contribute to the spin foam formulation developed for $BF$ theories in the Loop Quantum Gravity [LQG] program \cite{12}.

With all these ideas in mind, the purpose of this paper is to develop the Hamilton-Jacobi [HJ] analysis of the Euler and Pontryagin invariants. As far as we know the Hamilton-Jacobi analysis of these invariants has not been carried out. In this respect, we wish to extend the results reported in \cite{11,13} where these invariants within the Dirac and Fadeev-Jackiw context have been analyzed. We shall use the Hamilton-Jacobi [HJ] scheme developed by G"uler \cite{14–18} because is a good alternative for analyzing gauge systems. In fact, the G"uler approach is based on the construction of a fundamental differential defined on the full phase space, and the elementary blocks for constructing the fundamental differential are the constraints of the theory called Hamiltonians. The HJ Hamiltonians can be involutives or noninvolutives and they are elementary for obtaining the characteristic equations from which one can identify the gauge symmetries, the equations of motion and the physical degrees of freedom. The construction of the fundamental differential is direct; in general the Hamiltonians of HJ approach do not coincide with the constraints obtained in the Dirac formalism, and the process for identifying the symmetries in HJ framework is more economical than other approaches; in this sense the HJ framework is a good alternative for analyzing gauge systems.

The paper is organized as follows. In Section II, the HJ analysis of the Pontryagin invariant is performed. We will write the invariant in a $BF$-like form, which will be useful in order to introduce a new set of variables that will allow us to compare the symmetries of both invariants in a better form. We identify all Hamiltonians of the theory and the fundamental differential is constructed, then all symmetries of the theory are identified. In Section III, we rewrite the Euler invariant also in a $BF$-like form and we use the same variables introduced in the Section I. Then the HJ analysis is performed; we construct the fundamental differential of the theory and the characteristic equations are obtained. We compare the symmetries of both theories and we will show that in spite of the invariants share the same equations of motion their symmetries are different to each other. Finally,
in Section IV we add some remarks and conclusions.

**II. HAMILTON-JACOBI ANALYSIS OF THE PONTRYAGIN INVARIANT**

It is well-known that the Pontryagin invariant is given by the following action

\[ S[A_{\mu}^{IJ}] = \Xi \int_{\mathcal{M}} F^{IJ} \wedge F_{IJ}, \]  

(1)

where \( \Xi \) is a constant, \( \mathcal{M} \) is a four-dimensional manifold without boundary, \( I, J, K = 0, 1, 2, 3 \) are \( SO(3,1) \) indices that can be raised and lowered with the metric \( \eta_{IJ} = (-1, 1, 1, 1) \), \( F_{\alpha\beta}^{IJ} \) is the strength curvature of the 1-form connexion \( A^{IJ} \) defined as

\[ F_{\alpha\beta}^{IJ} = \partial_{\alpha} A_{\beta}^{IJ} - \partial_{\beta} A_{\alpha}^{IJ} + A_{\alpha K}^{I} A_{\beta}^{KJ} - A_{\alpha J}^{I} A_{\beta K}^{I}. \]  

The action (1) was analyzed in [11, 13] by using the Dirac and Faddeev-Jackiw approaches respectively, in those papers were reported that the system presents first and second class constraints and a symplectic tensor was constructed, however, the symmetries of the constraints were not reported. Moreover, a direct comparison between Pontryagin and Euler invariants was not performed. We can observe that in the action (1) the dynamical variable is the connection \( A \) and \( F \) is a label; we can write the action (1) in a new fashion form, in a \( BF \)-like theory, this step is convenient for future computations, for which a new set of variables will be introduced allowing us to perform the comparison of both theories under study in more convenient form. The Pontryagin invariant in a \( BF \) fashion is expressed by

\[ S[A_{\mu}^{IJ}, B_{\mu\nu}^{IJ}] = \Xi \int_{\mathcal{M}} [F^{IJ} \wedge B_{1IJ} - \frac{1}{2} B^{IJ} \wedge B_{1IJ}], \]  

(2)

where \( B_{IJ} = \frac{1}{2} B_{\alpha\beta}^{IJ} dx^\alpha \wedge dx^\beta \) is a set of six two-forms. From (2) the following equations of motion arise

\[ DB = 0, \]
\[ F = B, \]  

(3)

by taking into account (3) in (2) we obtain again (1), from this point of view either \( A \) or \( B \) are dynamical fields and this fact will be taken into account in the analysis. By performing the 3+1 decomposition and breaking down the Lorentz covariance we obtain the following Lagrangian

\[ \mathcal{L} = \Xi \eta^{abc} \int_{\mathcal{M}} \left[ B_{bc0i} \dot{A}_a^{0i} + \frac{1}{2} B_{bcij} \dot{A}_a^{ij} + \frac{1}{2} \left( \partial_a B_{bcij} + 2B_{bca0k} A_{aj}^{k} + 2B_{bca0} A_{a0j}^{ij} \right) A_0^{ij} \right] \]

\[ + \left( \partial_a B_{bc0i} + B_{bcij} A_{a0}^{ij} + B_{bca0j} A_{a}^{0j} \right) A_0^{0i} + \left( \partial_a A_{b0i}^{0j} - \partial_b A_{a0i}^{0} + A_{a}^{i} A_{b}^{0j} - A_{a}^{i} A_{b}^{0j} \right) B_{0c0i} \]

\[ + \frac{1}{2} \left( \partial_a A_{bcij} - \partial_b A_{acij} + A_{a}^{i} A_{bc0j}^{0j} - A_{b}^{0i} A_{a0}^{0j} + A_{a}^{i} A_{b}^{k} A_{c}^{kj} - A_{b}^{k} A_{a}^{ij} A_{c}^{kj} \right) B_{0c0ij} \]

\[ - \frac{1}{4} \left( B_{0a}^{ij} B_{bcij} + B_{abcij} B_{0c0ij} \right) - \frac{1}{2} \left( B_{a0}^{0i} B_{bc0i} + B_{ab0}^{0i} B_{0c0i} \right) \right] d^3x, \]  

(4)
now we introduce the following variables

\[\begin{align*}
A_{aij} &\equiv -\epsilon_{ijk}A^k_a, \\
A_{0ij} &\equiv -\epsilon_{ijk}A^k_0, \\
B_{abij} &\equiv -\epsilon_{ijk}B_{ab}^k, \\
B_{0aij} &\equiv -\epsilon_{ijk}B_{0a}^k, \\
A_{ai} &\equiv \Upsilon_{ai}, \\
A_0^i &\equiv -T^i, \\
A_{00i} &\equiv -\Lambda_i, \\
B_{0a}^0 &\equiv \frac{1}{2}s^a, \\
B_{0ai} &\equiv -\frac{1}{2}\chi_{ai},
\end{align*}\]

and the Lagrangian is rewritten in the following form

\[
\mathcal{L} = \Xi \eta^{abc} \int_M \left[ B_{abij} \dot{A}_i \dot{B}_i - (\partial_c B_{abk} - \epsilon_{ijk} B_{ab} \Upsilon_{cij} - \epsilon_{ijk} B_{a0} A_{c0j}) T^k \\
- (\partial_c B_{ab0i} - \epsilon_{ijk} B_{abk} A_{c0j} + \epsilon_{ijk} B_{ab0j} \Upsilon_{c0k}) \Lambda^i \\
- \frac{1}{2}(\partial_b A_{c0} \dot{c} - \partial_c A_{b0} \dot{i} + \epsilon_{ijk} A_{b0j} \Upsilon_{c0k} - \epsilon_{ijk} A_{c0j} \Upsilon_{b0k} + B_{bc} \dot{0i}) s_{ai} \\
- \frac{1}{2}(\partial_b \Upsilon_{ci} - \partial_c \Upsilon_{bi} - \epsilon_{ijk} A_{b0j} A_{c0k} + \epsilon_{ijk} \Upsilon_{b0j} \Upsilon_{c0k} - B_{abij}) \chi_{ai} \right] d^3x.
\tag{5}
\]

From the definition of the momenta \((\rho^{i}^{a0i}, \pi^{ai}, \dot{T}^{i}, \dot{\Lambda}^{i}, \dot{\chi}^{ai}, \dot{\chi}^{a0i}, p^{abi}, \dot{p}^{abi})\) canonically conjugate to \((A_{a0i}, \Upsilon_{ai}, T_i, \Lambda_i, \chi_{ai}, B_{a0i}, B_{abi})\) we identify the following \(H J\) Hamiltonians of the theory

\[
H' = \Pi + H_0 = 0, \\
\phi_1^{a0i} = p^{a0i} - \Xi \eta_{abc} B_{bc} \dot{0i} = 0, \\
\phi_2^{ai} = \pi^{ai} - \Xi \eta_{abc} B_{bc} \dot{i} = 0, \\
\phi_3^{ij} = \dot{T}^{i} = 0, \\
\phi_4^{i} = \dot{\Lambda}^{i} = 0, \\
\phi_5^{ai} = \dot{\chi}^{ai} = 0, \\
\phi_6^{a0} = \dot{\chi}^{a0} = 0, \\
\phi_7^{abi} = p^{abi} = 0, \\
\phi_8^{abi} = p^{abi} = 0,
\tag{6}
\]

where \(\Pi = \partial_0 S\), \(S\) is the action and \(H_0\) is identified as the canonical Hamiltonian expressed by

\[
H_0 = \left( \partial_0 \pi^{ai} - \epsilon_{ijk} \pi^{aj} \Upsilon^{ak}_{c0} - \epsilon_{ijk} \rho^{ai} A_{c0}^j \right) T_i + (\partial_0 p^{a0i} + \epsilon_{ijk} \pi^{aj} A_{a0}^k - \epsilon_{ijk} \rho^{a0j} \Upsilon_{a0}^k) \Lambda^i \\
+ \frac{1}{2} \eta_{abc} \left( \partial_b A_{c0} \dot{c} - \partial_c A_{b0} \dot{i} + \epsilon_{ijk} A_{b0j} \Upsilon_{c0k} - \epsilon_{ijk} A_{c0j} \Upsilon_{b0k} \right) s_{ai} \\
+ \frac{1}{2} \eta_{abc} \left( \partial_b \Upsilon_{ci} - \partial_c \Upsilon_{bi} - \epsilon_{ijk} A_{b0j} A_{c0k} + \epsilon_{ijk} \Upsilon_{b0j} \Upsilon_{c0k} \right) \chi_{ai} \\
+ \frac{1}{2} s_{ai} \rho^{a0i} - \frac{1}{2} \chi_{ai} \pi^{ai}.
\tag{7}
\]
now, from the definition of the momenta we also identify the fundamental Poisson brackets between dynamical variables

\[
\{A_{a0i}(x), p^{b0j}(y)\} = \delta^b_c \delta^j_d \delta^3(x - y),
\]

\[
\{\ddot{Y}_{ai}(x), \pi^{bj}(y)\} = \delta^b_c \delta^j_d \delta^3(x - y),
\]

\[
\{T_i(x), \dot{T}^j(y)\} = \delta^b_c \delta^j_d \delta^3(x - y),
\]

\[
\{A_i(x), \dot{A}^j(y)\} = \delta^b_c \delta^j_d \delta^3(x - y),
\]

\[
\{\zeta_{ai}(x), \chi^{bj}(y)\} = \delta^b_c \delta^j_d \delta^3(x - y),
\]

\[
\{\chi_{ai}(x), \chi^{bj}(y)\} = \delta^b_c \delta^j_d \delta^3(x - y),
\]

\[
\{B_{ab0i}(x), \rho^{cd0i}(y)\} = \frac{1}{2} (\delta^e_c \delta^d_b - \delta^d_c \delta^e_b) \delta^j_d \delta^3(x - y),
\]

\[
\{B_{abi}(x), p^{cdi}(y)\} = \frac{1}{2} (\delta^e_c \delta^d_b - \delta^d_c \delta^e_b) \delta^j_d \delta^3(x - y).
\]

(8)

In this manner, with the Hamiltonians at hand, we construct the fundamental differential which describes the evolution of any function, say \(f\), on the phase space [14-17]

\[
df(x) = \int d^3y \left[ \{f(x), H'(y)\} dt + \{f(x), \phi^{00i}_a\} d\rho_{a0i} + \{f(x), \phi^{0ti}_a\} d\tau_{ai} + \{f(x), \phi^{1i}_a\} d\lambda_i
\]

\[
+ \{f(x), \phi^{00i}_b\} d\sigma_{a0i} + \{f(x), \phi^{0ti}_b\} d\zeta_{ai} + \{f(x), \phi^{01i}_a\} d\theta_{ab0i} + \{f(x), \phi^{10i}_b\} d\zeta_{abi}\right],
\]

(9)

where \((\rho_{a0i}, \phi^{00i}_a, \tau_{ai}, \lambda_i, \sigma_{a0i}, \zeta_{ai}, \theta_{ab0i}, \zeta_{abi})\) are parameters associated with the Hamiltonians. On the other hand, we observe that the Hamiltonians \(\phi^{00i}_1, \phi^{00i}_2, \phi^{00i}_3\) and \(\phi^{01i}_1\) are involutives and \(\phi^{00i}_4, \phi^{00i}_2\) and \(\phi^{01i}_2, \phi^{01i}_3\) are non-involutives. Involutive Hamiltonians, are those whose Poisson brackets with all Hamiltonians, including themselves, vanish; otherwise, they are called non-involutives. The presence of non-involutive Hamiltonians introduce the generalized HJ brackets defined by [14-17]

\[
\{A, B\}^* = \{A, B\} - \{A, H'_2\} (C^{-1}_{\hat{a} \hat{b}}) \{H'_b, B\},
\]

(10)

where \((C_{\hat{a} \hat{b}})\) is the matrix whose entries are given by the Poisson brackets between non-involutives Hamiltonians and \((C^{-1}_{\hat{a} \hat{b}})\) its inverse matrix; explicitly

\[
C_{\hat{a} \hat{b}} = \begin{pmatrix}
0 & 0 & \Xi_{\gamma^{abc}} \delta^{il} & 0 \\
0 & 0 & 0 & -\Xi_{\gamma^{abc}} \delta^{il} \\
-\Xi_{\gamma^{abc}} \delta^{il} & 0 & 0 & 0 \\
0 & \Xi_{\gamma^{abc}} \delta^{il} & 0 & 0
\end{pmatrix} \delta^3(x - y),
\]

(11)

and

\[
C^{-1}_{\hat{a} \hat{b}} = \begin{pmatrix}
0 & 0 & -\frac{1}{\Xi} \gamma^{def} \delta_{ij} & 0 \\
0 & 0 & 0 & \frac{1}{\Xi} \gamma^{def} \delta_{ij} \\
\frac{1}{\Xi} \gamma^{def} \delta_{ij} & 0 & 0 & 0 \\
0 & -\frac{1}{\Xi} \gamma^{def} \delta_{ij} & 0 & 0
\end{pmatrix} \delta^3(x - y),
\]

(12)
hence, the generalized brackets between the dynamical variables read
\[
\{A_{a0i}(x), p^{b0j}(y)\}^* = \delta^j_a \delta^i_b \delta^3(x - y),
\]
\[
\{\chi_{ai}(x), \phi_i^j(y)\}^* = \delta^j_a \delta^3(x - y),
\]
\[
\{\chi_{ai}(x), \phi_i^j(y)\}^* = \delta^j_a \delta^i_y \delta^3(x - y),
\]
\[
\{\chi_{ai}(x), \phi_i^j(y)\}^* = \delta^j_a \delta^i_y \delta^3(x - y),
\]
\[
\{\chi_{ai}(x), \phi_i^j(y)\}^* = \delta^j_a \delta^i_y \delta^3(x - y),
\]
\[
\{B_{abi}(x), p^{c0j}(y)\}^* = 0,
\]
\[
\{B_{abi}(x), p^{c0j}(y)\}^* = 0,
\]
\[
\{B_{abi}(x), A_{c0j}(y)\}^* = \frac{1}{2\Xi} \eta_{abc} \delta_{ij} \delta^3(x - y),
\]
\[
\{B_{abi}(x), \chi_{c0j}(y)\}^* = -\frac{1}{2\Xi} \eta_{abc} \delta_{ij} \delta^3(x - y),
\]
where we observe that there is a contribution in these brackets due to the parameter \(\Xi\). The introduction of the generalized brackets allows us to rewrite the fundamental differential in terms of involutives Hamiltonians \[14\] \[18\]
\[
df(x) = \int d^3y \left[ \{f(x), H'(y)\}^* dt + \{f(x), \phi_i^j(y)\}^* d\tau_i + \{f(x), \phi_i^j(y)\}^* d\lambda_i + \{f(x), \phi_i^j(y)\}^* d\sigma_{ai} + \{f(x), \phi_i^j(y)\}^* d\zeta_{ai} \right],
\]
where we can observe that the noninvolutive Hamiltonians have been removed. In fact, this is an advantage of the HJ formalism with respect to the Dirac formulation. From one hand, by introducing the generalized brackets in HJ framework we remove constraints from the beginning. On the other hand, in Dirac’s formulation we must to identify future constraints by means of consistency, at the end we need to perform the classification of the constraints in first class and second class, then Dirac’s brackets are introduced and second class constraints can be eliminated; in HJ scheme we will have at the end less number of constraints than Dirac’s scheme.

Furthermore, the Frobenius integrability conditions on the involutive Hamiltonians \(\phi_i^j, \phi_i^j, \phi_i^j\) and \(\phi_i^j\) could introduce new HJ Hamiltonians. In fact, integrability conditions are relevant because ensure that the system is integrable. From the integrability conditions the following Hamiltonians arise
\[
d\phi_i^j(x) = \int d^3y \left[ \{\phi_i^j(x), H'(y)\}^* dt + \{\phi_i^j(x), \phi_i^j(y)\}^* d\tau_j + \{\phi_i^j(x), \phi_i^j(y)\}^* d\lambda_j + \{\phi_i^j(x), \phi_i^j(y)\}^* d\sigma_{aj} + \{\phi_i^j(x), \phi_i^j(y)\}^* d\zeta_{aj} \right] = 0,
\]
\[
\Rightarrow - (\partial_a \pi_i^a - \epsilon_{ijk} \pi^{a0j} T_a^k - \epsilon_{ijk} \pi^{a0j} A_{a0k}) = 0,
\]
\[
d\phi_i^j(x) = \int d^3y \left[ \{\phi_i^j(x), H'(y)\}^* dt + \{\phi_i^j(x), \phi_i^j(y)\}^* d\tau_j + \{\phi_i^j(x), \phi_i^j(y)\}^* d\lambda_j + \{\phi_i^j(x), \phi_i^j(y)\}^* d\sigma_{aj} + \{\phi_i^j(x), \phi_i^j(y)\}^* d\zeta_{aj} \right] = 0,
\]
\[
\Rightarrow - (\partial_a \pi_i^a - \epsilon_{ijk} \pi^{a0j} T_a^k - \epsilon_{ijk} \pi^{a0j} A_{a0k}) = 0,
\]
\[ d\phi^{ai}_5(x) = \int d^3y \left[ \{\phi^{ai}_5(x), H'(y)\}^* dt + \{\phi^{ai}_5(x), \phi^b_5(y)\}^* d\tau_j + \{\phi^{ai}_5(x), \phi^b_5(y)\}^* d\lambda_j + \{\phi^{ai}_5(x), \phi^b_5(y)\}^* d\sigma_{ij} \ight. \]
\[ \left. + \{\phi^{ai}_5(x), \phi^b_5(y)\}^* d\zeta_{ij} \right] = 0, \tag{19} \]
\[ d\phi^{ai}_6(x) = \int d^3y \left[ \{\phi^{ai}_6(x), H'(y)\}^* dt + \{\phi^{ai}_6(x), \phi^b_6(y)\}^* d\tau_j + \{\phi^{ai}_6(x), \phi^b_6(y)\}^* d\lambda_j + \{\phi^{ai}_6(x), \phi^b_6(y)\}^* d\sigma_{ij} \ight. \]
\[ \left. + \{\phi^{ai}_6(x), \phi^b_6(y)\}^* d\zeta_{ij} \right] = 0, \tag{20} \]
\[ \Rightarrow \frac{\Xi}{2} \eta^{abc} \left( \partial_b \mathcal{Y}^c_i - \partial_c \mathcal{Y}^b_i - \epsilon^i_{jk} A_{b0}^j \mathcal{Y}^c_k - \epsilon^i_{jk} A_{c0}^j \mathcal{Y}^b_k \right) + \frac{1}{2} \rho^{a0i} = 0, \tag{22} \]

Hence, we identify the following set of new HJ Hamiltonians
\[ \phi^{i}_5 = \partial_a \pi^i_a - \epsilon^i_{jk} \pi^a j \mathcal{Y}_a^k - \epsilon^i_{jk} \rho^{a0j} A_{a0}^k = 0, \]
\[ \phi^{i}_{10} = \partial_a \pi^{a0i} + \epsilon^i_{jk} \pi^a j \mathcal{Y}_a^k - \epsilon^i_{jk} \rho^{a0j} A_{a0}^k = 0, \]
\[ \phi^{ai}_{11} = \frac{\Xi}{2} \eta^{abc} \left( \partial_b \mathcal{Y}^c_i - \partial_c \mathcal{Y}^b_i - \epsilon^i_{jk} A_{b0}^j \mathcal{Y}^c_k - \epsilon^i_{jk} A_{c0}^j \mathcal{Y}^b_k \right) + \frac{1}{2} \rho^{a0i} = 0, \]
\[ \phi^{ai}_{12} = \frac{\Xi}{2} \eta^{abc} \left( \partial_b \mathcal{Y}^c_i - \partial_c \mathcal{Y}^b_i - \epsilon^i_{jk} A_{b0}^j \mathcal{Y}^c_k + \epsilon^i_{jk} \mathcal{Y}^b j \mathcal{Y}_c^k \right) - \frac{1}{2} \pi^a = 0. \tag{23} \]

The generalized brackets between these new Hamiltonians are given by
\[ \{\phi^{i}_5(x), \phi^{j}_5(y)\}^* = \epsilon^{ij}_k \phi^{k}_5 \delta^3(x - y), \]
\[ \{\phi^{i}_5(x), \phi^{j}_{10}(y)\}^* = \epsilon^{ij}_k \phi^{k}_{10} \delta^3(x - y), \]
\[ \{\phi^{i}_5(x), \phi^{a}_{11}(y)\}^* = \epsilon^{ij}_k \phi^{k}_{11} \delta^3(x - y), \]
\[ \{\phi^{i}_5(x), \phi^{a}_{12}(y)\}^* = \epsilon^{ij}_k \phi^{k}_{12} \delta^3(x - y), \]
\[ \{\phi^{j}_{10}(x), \phi^{j}_{10}(y)\}^* = -\epsilon^{ij}_k \phi^{k}_{10} \delta^3(x - y), \]
\[ \{\phi^{j}_{10}(x), \phi^{a}_{11}(y)\}^* = \epsilon^{ij}_k \phi^{k}_{11} \delta^3(x - y), \]
\[ \{\phi^{j}_{10}(x), \phi^{a}_{12}(y)\}^* = -\epsilon^{ij}_k \phi^{k}_{12} \delta^3(x - y), \]
\[ \{\phi^{a}_{11}(x), \phi^{j}_{11}(y)\}^* = 0, \]
\[ \{\phi^{a}_{11}(x), \phi^{a}_{12}(y)\}^* = 0, \]
\[ \{\phi^{a}_{12}(x), \phi^{j}_{12}(y)\}^* = 0, \tag{24} \]
since the algebra is closed we conclude that these Hamiltonians are involutive, therefore we do not expect new Hamiltonians. Moreover, we observe that the Hamiltonians \( \phi^{i}_5(x) \) and \( \phi^{j}_{10}(x) \) are identified as generators of rotations and boost respectively. The rest of the Hamiltonians \( \phi^{ai}_{11}(x) \) and \( \phi^{ai}_{12}(x) \) are reducible Hamiltonians, namely, they are not independent.; but a linear combination of
involution Hamiltonians, as it will be showed. With all involutive Hamiltonians and by using the generalized brackets the following generalized differential is constructed

\[
df(x) = \int d^4y \left[ \{f(x), H'(y)\}^* dt + \{f(x), \phi^i_1(y)\}^* d\tau_i + \{f(x), \phi^i_2(y)\}^* d\lambda_i + \{f(x), \phi^i_{11}(y)\}^* d\sigma_{ai} + \{f(x), \phi^i_{12}(y)\}^* d\zeta_{ai} \right]
\]

where \(\bar{\tau}_i, \bar{\lambda}_i, \bar{\zeta}_i, \bar{\lambda}_{ai}\) are parameters related with the Hamiltonians \(\phi^i_1, \phi^i_{10}, \phi^i_{11}, \phi^i_{12}\) respectively. Therefore, from the fundamental differential we can obtain the relevant symmetries of the theory. In fact, the symmetries are exposed by the characteristic equations 14–17, and they are given by

\[
dA_{0i} = \left[ \epsilon_{ij}^k A_{0i} B^k T_j - \partial_a \Lambda^i + \epsilon_{ij}^k \Upsilon^k_{ai} \Lambda^j - B_{0a} \right] dt
\]  

\[
+ \left[ \epsilon_{ij}^k A_{0i} \right] d\tau_j - \left[ \delta^{ij} \partial_a - \epsilon_{ij}^k \Upsilon^k_{ai} \right] d\lambda_j + \frac{1}{2} d\sigma_{ai},
\]

\[
d\Upsilon^i = \left[ - \partial_a T^i - \epsilon_{ij}^k \Upsilon^k_{ai} T_j - \epsilon_{ij}^k A_{ab}^k \Lambda^j + B_{0a} \right] dt
\]  

\[
- \left[ \delta^{ij} \partial_a + \epsilon_{ij}^k \Upsilon^k_{ai} \right] d\tau_j - \left[ \epsilon_{ij}^k A_{ab}^k \right] d\lambda_j - \frac{1}{2} d\zeta_{ai},
\]

\[
dT_i = d\tau_i,
\]

\[
d\Lambda_i = d\lambda_i,
\]

\[
d\zeta_{ai} = d\sigma_{ai},
\]

\[
d\chi_{ai} = d\zeta_{ai},
\]

\[
\]

\[
dB_{ab0} = \left[ \epsilon_{ij}^k (B_{0a} B^k - B_{0a}^0 T_j) + \epsilon_{ij}^k (\Upsilon_{ai} B_{0b}^0 - \Upsilon_{ai} B_{0b}^0) - \epsilon_{ij}^k (A_{ab}^k B_{0b}) - A_{ab}^k B_{0b} - (\partial_a B_{ab}^0 i - \partial_b B_{ab}^0 i) \right] dt
\]  

\[
- \left[ \epsilon_{ij}^k B_{ab} \right] d\tau_j
\]  

\[
+ \left[ \epsilon_{ij}^k B_{ab} \right] d\lambda_j
\]  

\[
+ \frac{1}{2} \left[ \delta^{ij} \partial_a - \epsilon_{ij}^k \Upsilon^k_{ai} \right] d\lambda_j - \frac{1}{2} \left[ \delta^{ij} \partial_b - \epsilon_{ij}^k \Upsilon^k_{ai} \right] d\sigma_{aj}
\]  

\[
+ \frac{1}{2} \left[ \epsilon_{ij}^k A_{ab}^k \right] d\chi_{bj} - \frac{1}{2} \left[ \epsilon_{ij}^k A_{ab}^k \right] d\zeta_{aj},
\]

\[
dB_{ab} = \left[ - \epsilon_{ij}^k (B_{ab} T_j - B_{ab}^0 \Lambda_j) - \epsilon_{ij}^k (\Upsilon_{ab} B_{0b}^0 - \Upsilon_{ab} B_{0b}^0) - \epsilon_{ij}^k (A_{ab} \Upsilon_{ab} B_{0b}) - \Upsilon_{ab} B_{0b} - (\partial_a B_{ab} i - \partial_b B_{ab} i) \right] dt
\]  

\[
- \left[ \epsilon_{ij}^k B_{ab} \right] d\tau_j
\]  

\[
+ \left[ \epsilon_{ij}^k B_{ab} \right] d\lambda_j
\]  

\[
+ \frac{1}{2} \left[ \delta^{ij} \partial_a - \epsilon_{ij}^k \Upsilon^k_{ai} \right] d\lambda_j - \frac{1}{2} \left[ \delta^{ij} \partial_b - \epsilon_{ij}^k \Upsilon^k_{ai} \right] d\sigma_{aj}
\]  

\[
- \frac{1}{2} \left[ \delta^{ij} \partial_a - \epsilon_{ij}^k \Upsilon^k_{ai} \right] d\lambda_j + \frac{1}{2} \left[ \epsilon_{ij}^k B_{ab} \right] d\chi_{aj} + \frac{1}{2} \left[ \epsilon_{ij}^k B_{ab} \right] d\zeta_{aj},
\]

\[
\]
\[ d\hat{T}^i = -\phi^i_0 dt = 0, \]
\[ d\hat{\Lambda}^i = -\phi^i_{10} dt = 0, \]
\[ d\hat{\varsigma}^i = -\phi^i_{11} dt = 0, \]
\[ d\hat{\chi}^i = -\phi^i_{12} dt = 0, \]
\[ dp^{a0i} = 0, \]
\[ dp^{abi} = 0. \]  

(31)

From the characteristic equations the following equations of motion arise

\[ \dot{A}^{a0i} = \varepsilon^{ijk} A^{a0k} T_j - \partial_a A^i + \varepsilon^{ijk} \Upsilon_a^k A_j - B_{0a}^{0i}, \]
\[ \dot{\Upsilon}_a^i = -\partial_a T^i - \varepsilon^{ijk} \Upsilon_a^k T_j - \varepsilon^{ijk} A^{a0k} A_j + B_{0a}^{0i}, \]  

(32)

and by taking \( dt = 0 \), the following gauge transformations are identified

\[ \delta A^{a0i} = [\varepsilon^{ijk} A^{a0k}] \delta \tilde{r}_j - \left[ \delta^{ij} \partial_a - \varepsilon^{ijk} \Upsilon_a^k \right] \delta \tilde{\lambda}_j + \frac{1}{2} \delta \tilde{\sigma}_a^i, \]
\[ \delta \Upsilon_a^i = - \left[ \delta^{ij} \partial_a + \varepsilon^{ijk} \Upsilon_a^k \right] \delta \tilde{r}_j - \left[ \varepsilon^{ijk} A^{a0k} \right] \delta \tilde{\lambda}_j - \frac{1}{2} \delta \tilde{\varsigma}_a^i. \]  

(33)

Furthermore, we can identify from (28) and (31) the non dynamical variables. In fact, the former implies that the variables \( T_i, \Lambda_i, \varsigma_i, \chi_i \) are identified as Lagrange multipliers and the later says that the variables \( B_{a0i}^{0i} \) and \( B_{abi}^{abi} \) are non dynamical because the characteristic equations of their momenta do not present any evolution. Therefore, the dynamical variables are given finally by \( A^{a0i} \) and \( \Upsilon_a^i \) and they will be taken into account in the counting of physical degrees of freedom; all these results were not reported in [11, 13].

On the other hand, we commented above that the involutive constraints \( \phi_{11i}^i, \phi_{12i}^i \) are reducible; in fact, we can observe that there are 6 reducibility conditions [13]

\[ \partial_a \phi_{11i}^i = -\varepsilon^{ijk} \Upsilon_a^j \phi_{11}^{ak} - \varepsilon^{ijk} A^{a0j} \phi_{12}^{ak} + \frac{1}{2} \phi_{10}^i, \]
\[ \partial_a \phi_{12i}^i = \varepsilon^{ijk} A^{a0j} \phi_{11}^{ak} - \varepsilon^{ijk} \Upsilon_a^j \phi_{12}^{ak} + \frac{1}{2} \phi_{10}^i. \]  

(34)

they are linear combination of involutive Hamiltonians, in this manner, eq. [34] implies that there are a total of 18 independent involutive Hamiltonians \( \phi_9^i, \phi_{10i}^i, \phi_{11i}^i, \phi_{12i}^i \), and we identified above that there are 18 dynamical variables; \( A^{a0i} \) and \( \Upsilon_a^i \). Hence, the counting of physical degrees of freedom is carried out as follows; \( \text{DOF}= \text{Dynamical variables} - \text{involutive constraints}=18-18=0 \), therefore the theory is devoid of physical degrees of freedom as expected. We can observe in [11] that in Dirac’s approach used more constraints and dynamical variables in order to perform the counting of physical degrees of freedom than in HJ scheme. On the other hand, the HJ generalized brackets and the Fadeev-Jackiw brackets reported in [11] coincide to each other, however, in Fadeev-Jackiw scheme we had to fix the gauge in order to obtain the brackets; in HJ formalism this was not a necessary step, in this sense we have completed the results reported in [11, 13] and we can say that the HJ scheme is more direct and economical.
III. HAMILTON-JACOBI ANALYSIS FOR THE EULER CLASS

It is well-known that the Euler class can be expressed as a BF-like theory \[ S[A_{\mu \nu}^{IJ}, B_{\mu \nu}^{IJ}] = \Omega \int_M [F^{IJ} \wedge B_{IJ} - \frac{1}{2} * B^{IJ} \wedge B_{IJ}], \] (35)
where * = \epsilon^{IJKL} is the dual of SO(3,1) and \( \Omega \) is a constant. The equations of motion arising from the variation of (35) are

\[ D* B = 0, \]
\[ *F = *B, \] (36)

by applying the star product to the equations (36) we obtain again (3); the Euler and Pontryagin classes share the same equations of motion. What about their symmetries? By performing the 3+1 decomposition, break down the Lorentz covariance and introducing the variables defined in previous sections we obtain that the Euler Lagrangian takes the form

\[ L = \Omega \eta^{abc} \int_M \left[ B_{ab}^{0i} \dot{T}_{ci} + B_{abi} \dot{\Lambda}_c^{0i} \right. \]
\[ - \left( \partial_a B_{bc}^{0i} - \epsilon^i_{jk} B_{bc}^{0j} \gamma_a^k + \epsilon^i_{jk} A_a^{0j} B_{bc}^{k} \right) T_i \]
\[ + \left( \partial_a B_{bc}^{i} + \epsilon^i_{jk} \gamma_a^{i} B_{bc}^{k} + \epsilon^i_{jk} A_{a0}^{i} B_{bc}^{0k} \right) \Lambda_i \]
\[ - \frac{1}{2} \left( \partial_a \gamma_b^i - \partial_b \gamma_a^i - \epsilon^i_{jk} A_{a0}^j \gamma_b^k + \epsilon^i_{jk} \gamma_a^i \gamma_b^k \right) \gamma^j_{ci} \]
\[ + \frac{1}{2} \left( \partial_a A_{b0}^i - \partial_b A_{a0}^i + \epsilon^i_{jk} A_{a0}^{i} \gamma_b^k - \epsilon^i_{jk} A_{b0}^{i} \gamma_a^k + B_{ab}^{0i} \right) \chi_{ci} \] (37)

now, in order to compare the symmetries of both theories, we will use the same phase space variables that we used in the Pontryagin invariant. Hence, by using the following Hamiltonians for Euler class

\[ H' = H + H_0 = 0 \]
\[ \phi_1^{0i} = p^{0i} - \Xi \eta^{abc} B_{bc}^{0i} = 0, \]
\[ \phi_2^{ai} = \pi_i^{ai} - \Xi \eta^{abc} B_{bc}^{i} = 0, \]
\[ \phi_3^{ai} = \dot{T}^{ai} = 0, \]
\[ \phi_4^{ai} = \dot{\Lambda}^{ai} = 0, \]
\[ \phi_5^{ai} = \dot{\gamma}^{ai} = 0, \]
\[ \phi_6^{ai} = \dot{\chi}^{ai} = 0, \]
\[ \phi_7^{abi} = p^{abi} = 0, \]
\[ \phi_8^{abi} = p^{abi} = 0, \] (38)
where the canonical Hamiltonian for the Euler invariant given in terms of the Hamiltonians (38) is expressed as

\[
H_0 = \frac{\Omega}{\Xi} \left[ \partial_a p^{a0} + \epsilon_{ijk} \pi^{a0} J_a + \epsilon_{ijk} \pi^{a0} J_a - \epsilon_{ijk} \pi^{a0} J_a - \epsilon_{ijk} \pi^{a0} J_a \right] T_i - \frac{\Omega}{\Xi} \left[ \partial_a \pi^{ai} - \epsilon_{ijk} \pi^{a0} J_a - \epsilon_{ijk} \pi^{a0} J_a \right] A_i \\
+ \frac{\Omega}{\Xi} \left[ \partial_a \pi^{ai} - \epsilon_{ijk} \pi^{a0} J_a - \epsilon_{ijk} \pi^{a0} J_a \right] A_i \\
- \frac{1}{2} \frac{\partial_a \pi^{ai} - \epsilon_{ijk} \pi^{a0} J_a - \epsilon_{ijk} \pi^{a0} J_a}{\Xi} \chi_{ai} \\
- \frac{1}{2} \frac{\partial_a \pi^{ai} - \epsilon_{ijk} \pi^{a0} J_a - \epsilon_{ijk} \pi^{a0} J_a}{\Xi} \chi_{ai},
\]

(39)

then, from the Lagrangian we identify the following fundamental Poisson brackets

\[
\{ A_{a0}(x), \pi^{bj}(y) \} = -\frac{\Xi}{\Omega} \delta^a_b \delta^i_j \delta^3(x - y), \\
\{ T_{ai}(x), p^{b0j}(y) \} = \frac{\Xi}{\Omega} \delta^a_b \delta^i_j \delta^3(x - y), \\
\{ \Lambda_i(x), \Lambda^j(y) \} = \delta^a_b \delta^i_j \delta^3(x - y), \\
\{ \zeta_{ai}(x), \zeta^{bj}(y) \} = \delta^a_b \delta^i_j \delta^3(x - y), \\
\{ \chi_{ai}(x), \chi^{bj}(y) \} = \delta^a_b \delta^i_j \delta^3(x - y), \\
\{ B_{abi}(x), p^{cdj}(y) \} = \frac{1}{2} (\delta^c_d \delta^a_b - \delta^a_b \delta^c_d) \delta^i_j \delta^3(x - y),
\]

(40)

it is appreciable in these brackets the contribution of the constants \( \Omega \) and \( \Xi \). Moreover, the role of the canonically conjugate variables has changed; now \( \pi^{bj} \) is canonical conjugated to \( A_{a0} \) and \( p^{b0j} \) is canonical conjugated to \( T_{ai} \) whereas in Pontryagin they were interchanged. Furthermore, by using the HJ Hamiltonians (38) the fundamental differential for the Euler theory is given by

\[
\frac{df(x)}{dt} = \int d^3 y \left[ \{ f(x), H'(y) \} dt + \{ f(x), \phi^{00i} \} d\omega_{a0} + \{ f(x), \phi^{00i} \} d\varphi_{ai} + \{ f(x), \phi^{01i} \} d\tau_i + \{ f(x), \phi^{01i} \} d\lambda_i \\
+ \{ f(x), \phi^{01i} \} d\sigma_{ai} + \{ f(x), \phi^{02i} \} d\zeta_{ai} + \{ f(x), \phi^{02i} \} d\theta_{a0i} + \{ f(x), \phi^{02i} \} d\chi_{abi} \right],
\]

(41)

where \( \omega_{a0}, \varphi_{ai}, \tau_i, \sigma_{ai}, \zeta_{ai}, \theta_{a0i}, \chi_{abi} \) are parameters related with the Hamiltonians. Moreover, we can notice that \( \phi^{01}_i, \phi^{02}_i \) and \( \phi^{01}_a \) are involutives and \( \phi^{00}_a, \phi^{02}_a \) \( \phi^{01}_a \) and \( \phi^{02}_a \) are noninvolutives. In this manner, noninvolutives Hamiltonians allow us to introduce the generalized brackets (10), where

\[
C_{ab} = \begin{pmatrix}
0 & 0 & \Xi h^{abc} \delta^i_j & 0 \\
0 & 0 & 0 & -\Xi h^{abc} \delta^i_j \\
-\Xi h^{abc} \delta^i_j & 0 & 0 & 0 \\
0 & \Xi h^{abc} \delta^i_j & 0 & 0
\end{pmatrix} \delta^3(x - y),
\]

(42)

with inverse given by
hence, the generalized brackets are given by

\[ \{ A_{a0i}(x), \pi^{bj}(y) \}^* = -\frac{\Xi}{\Omega} \delta^b_a \delta^j_i \delta^3(x - y), \]
\[ \{ \Upsilon_{ai}(x), p^{b0j}(y) \}^* = \frac{\Xi}{\Omega} \delta^b_a \delta^j_i \delta^3(x - y), \]
\[ \{ T_i(x), \hat{T}^j(y) \}^* = \delta^b_a \delta^j_i \delta^3(x - y), \]
\[ \{ \Lambda_i(x), \Lambda^j(y) \}^* = \delta^b_a \delta^j_i \delta^3(x - y), \]
\[ \{ \varsigma_{ai}(x), \varsigma^{bj}(y) \}^* = \delta^b_a \delta^j_i \delta^3(x - y), \]
\[ \{ \chi_{ai}(x), \chi^{bj}(y) \}^* = \delta^b_a \delta^j_i \delta^3(x - y), \]
\[ \{ B_{ab0i}(x), p^{c0j}(y) \}^* = 0, \]
\[ \{ B_{abi}(x), p^{c0j}(y) \}^* = 0, \]
\[ \{ B_{ab0i}(x), \Upsilon_{cj}(y) \}^* = \frac{1}{2\Omega} \eta_{abc} \delta_{ij} \delta^3(x - y), \]
\[ \{ B_{abi}(x), A_{c0j}(y) \}^* = \frac{1}{2\Omega} \eta_{abc} \delta_{ij} \delta^3(x - y), \] (44)

we can observe that the Euler and Pontryagin theories have different generalized brackets; the contribution of the constants \( \Omega \) and \( \Xi \) is manifested. Furthermore, the generalized brackets \([44]\) coincide with those reported in \([13]\) where the Fadeev-Jackiw approach was developed, in addition, for obtaining those brackets a temporal gauge fixing was necessary. The introduction of the generalized brackets allow us to introduce a new fundamental differential

\[ df(x) = \int d^3y \left[ \{ f(x), H'(y) \}^* dt + \{ f(x), \phi^i_1(y) \}^* d\tau_i + \{ f(x), \phi^i_2(y) \}^* d\lambda_i + \{ f(x), \phi^i_3(y) \}^* d\sigma_{ai} \right. \]
\[ + \left. \left\{ f(x), \phi^i_6(y) \right\}^* d\zeta_{ai} \right]. \] (45)

On the other hand, integrability conditions on the involutive Hamiltonians \( \phi^3_3, \phi^4_1, \phi^5_i \) and \( \phi^6_6 \) introduce new \( HJ \) Hamiltonians. In fact, from integrability conditions the following Hamiltonians arise

\[ d\phi^i_3(x) = 0 \implies \phi^i_3 : \frac{\Omega}{2} (\partial_a p^{0ai} + \epsilon^i_{jk} \pi^{aj} A_{a0k} - \epsilon^i_{jk} p^{0aj} Y^a_k) = 0, \]
\[ d\phi^i_4(x) = 0 \implies \phi^i_4 : \frac{\Omega}{2} (\partial_a \pi^{ai} - \epsilon^i_{jk} \pi^{aj} Y^a_k - \epsilon^i_{jk} p^{0aj} A_{a0k}) = 0, \]
\[ d\phi^i_5(x) = 0 \implies \phi^i_5 : -\frac{\Omega}{2} \eta^{abc} (\partial_b Y^c_k - \partial_c Y^b_k - \epsilon^i_{jk} A_{b0j} A_{c0k} + \epsilon^i_{jk} Y^b_k Y^c_k) + \frac{\Omega}{2} \pi^{ai} = 0, \]
\[ d\phi^i_6(x) = 0 \implies \phi^i_6 : \frac{\Omega}{2} \eta^{abc} (\partial_b A_{c0i} - \partial_c A_{b0i} + \epsilon^i_{jk} A_{b0j} Y^c_k - \epsilon^i_{jk} A_{c0j} Y^b_k) + \frac{\Omega}{2} p^{0ai} = 0, \] (46)
the generalized algebra between these new Hamiltonians read

\[
\begin{align*}
\{\phi^i_0(x), \phi^j_0(y)\}^* &= \epsilon^{ij}_{k} \phi^k_0 \delta^3(x - y), \\
\{\phi^i_0(x), \phi^j_{10}(y)\}^* &= \epsilon^{ij}_{k} \phi^k_{10} \delta^3(x - y), \\
\{\phi^i_0(x), \phi^j_{11}(y)\}^* &= \epsilon^{ij}_{k} \phi^k_{11} \delta^3(x - y), \\
\{\phi^i_0(x), \phi^j_{12}(y)\}^* &= \epsilon^{ij}_{k} \phi^k_{12} \delta^3(x - y), \\
\{\phi^i_{10}(x), \phi^j_{10}(y)\}^* &= -\epsilon^{ij}_{k} \phi^k_0 \delta^3(x - y), \\
\{\phi^i_{10}(x), \phi^j_{11}(y)\}^* &= \epsilon^{ij}_{k} \phi^k_{12} \delta^3(x - y), \\
\{\phi^i_{10}(x), \phi^j_{12}(y)\}^* &= -\epsilon^{ij}_{k} \phi^k_{11} \delta^3(x - y), \\
\{\phi^i_{11}(x), \phi^j_{12}(y)\}^* &= 0, \\
\{\phi^i_{11}(x), \phi^j_{12}(y)\}^* &= 0, \\
\{\phi^i_{12}(x), \phi^j_{12}(y)\}^* &= 0,
\end{align*}
\]

(47)

which is closed, therefore we do not expect new Hamiltonians. Furthermore, now we observe that the Hamiltonians \(\phi^i_0(x)\) and \(\phi^i_{10}(x)\) are identified as generators of rotations and boost respectively. This is a relevant result because the generators for Euler theory are interchanged with respect Pontryagin’s invariant; this result implies that the theories are classically different. The Hamiltonians (46) are involutives and this fact allows us to introduce a new generalized differential

\[
df(x) = \int d^3y \left[ \left\{ f(x), H'(y) \right\}^* dt + \left\{ f(x), \phi^i_0(y) \right\}^* d\tau_i + \left\{ f(x), \phi^i_{10}(y) \right\}^* d\lambda_i + \left\{ f(x), \phi^i_{11}(y) \right\}^* d\sigma_{ai} + \left\{ f(x), \phi^i_{12}(y) \right\}^* d\zeta_{ai} \right],
\]

(48)

where \(\tau_i, \lambda_i, \sigma_{ai}, \zeta_{ai}\) are parameters associated to the Hamiltonians (46). It is worth to comment that the generalized brackets (44) make the fundamental differentials (25) and (48) to be completely different. In fact, due to the generalized brackets of the theories are different to each other, the fundamental differentials will describe different scenarios on the phase space. From the generalized differential (48) we can obtain the characteristic equations of the theory, then we can identify the symmetries. The characteristic equations are given by

\[
dA^i_a = \left[ \epsilon^{ij}_{k} A_{a0}^k T^j - \partial_a \Lambda^i + \epsilon^{ij}_{k} \Upsilon^k_a \Lambda^j + \frac{1}{2} \zeta_{ai} \right] dt \\
+ \left[ \epsilon^{ij}_{k} A_{a0}^k \right] d\tau_j + \left[ \delta^{ij} \partial_a - \epsilon^{ij}_{k} \Upsilon^k_a \right] d\lambda_j + \frac{1}{2} d\sigma_{ai},
\]

(49)

\[
d\Upsilon^i_a = \left[ - \partial_a T^i + \epsilon^{ij}_{k} \Upsilon^k_a T^j - \epsilon^{ij}_{k} A_{a0}^k \Lambda^j - \frac{1}{2} \lambda_{ai} \right] dt \\
- \left[ \delta^{ij} \partial_a - \epsilon^{ij}_{k} \Upsilon^k_a \right] d\tau_j - \left[ \epsilon^{ij}_{k} A_{a0}^k \right] d\lambda_j + \frac{1}{2} d\zeta_{ai},
\]

(50)
\[ dT_i = d\tau_i, \]
\[ d\Lambda_i = d\lambda_i, \]
\[ ds_{ai} = d\sigma_{ai}, \]
\[ d\chi_{ai} = d\zeta_{ai}, \]
(51)

\[ dB_{ab0i} = \left[ \epsilon_{ij}k(B_{ab}^k \Lambda_j - B_{ab}^{0k}T_j) + \epsilon_{ij}k(\Upsilon_a^k B_{0b}^0 - \Upsilon_b^k B_{0a}^0) - \epsilon_{ij}k(A_{a0}^k B_{0bj} - \Upsilon_{a0}^k B_{0aj}) \right. \]
\[ - A_{b0}^k B_{0aj}) - (\partial_a B_{0b0}^0 - \partial_b B_{0a0}^0) \right] dt \]
\[ - \left[ \epsilon_{ij}k B_{ab}^{0k} \right] d\tilde{\tau}_j \]
\[ + \left[ \epsilon_{ij}k B_{ab}^k \right] d\tilde{\lambda}_j \]
\[ + \frac{1}{2} \left[ \delta_{ij} \partial_a - \epsilon_{ij}k \Upsilon_a^k \right] d\tilde{\sigma}_{b}j - \frac{1}{2} \left[ \delta_{ij} \partial_b - \epsilon_{ij}k \Upsilon_b^k \right] d\tilde{\sigma}_{a}j \]
\[ + \frac{1}{2} \left[ \epsilon_{ij}A_{a0}^k \right] d\tilde{\zeta}_{b}j - \frac{1}{2} \left[ \epsilon_{ij}k A_{a0}^k \right] d\tilde{\zeta}_{a}j \]
(52)

\[ dB_{ai} = \left[ \epsilon_{ij}k(B_{ab}^k T_j + B_{ab}^{0k} \Lambda_j) - \epsilon_{ij}k(A_{a0}^k B_{0a}^0 - A_{0b}^k B_{0b}^0) - \epsilon_{ij}k(\Upsilon_a^k B_{0bj} - \Upsilon_{b}^k B_{0aj}) \right. \]
\[ - \Upsilon_{b}^k B_{0aj} \right] + (\partial_a B_{0b0}^0 - \partial_b B_{0a0}^0) \right] dt \]
\[ + \left[ \epsilon_{ij}k B_{ab}^k \right] d\tilde{\tau}_j \]
\[ + \left[ \epsilon_{ij}k B_{ab}^{0k} \right] d\tilde{\lambda}_j \]
\[ + \frac{1}{2} \left[ \epsilon_{ij}k A_{a0}^k \right] d\tilde{\sigma}_{b}j - \frac{1}{2} \left[ \epsilon_{ij}k A_{b0}^k \right] d\tilde{\sigma}_{a}j \]
\[ + \frac{1}{2} \left[ \delta_{ij} \partial_a - \epsilon_{ij}k \Upsilon_a^k \right] d\tilde{\zeta}_{b}j - \frac{1}{2} \left[ \delta_{ij} \partial_b - \epsilon_{ij}k \Upsilon_b^k \right] d\tilde{\zeta}_{a}j, \]
(53)

\[ d\hat{T}_i = -\phi_9 dt = 0, \]
\[ d\hat{\Lambda}_i = -\phi_{10} dt = 0, \]
\[ d\hat{\xi}_i = -\phi_{11} dt = 0, \]
\[ d\hat{\chi}_i = +\phi_{12} dt = 0, \]
\[ dp_{ab0i} = 0, \]
\[ dp_{abbi} = 0, \]
(54)

thus, from the characteristics equations we identify the Euler’s equations of motions given by

\[ \dot{A}_{a0} = \epsilon_{ij}k A_{a0}^k T_j - \partial_a \Lambda_i + \epsilon_{ij}k \Upsilon_a^k \Lambda_j + \frac{1}{2} \zeta_a^i, \]
\[ \dot{\Upsilon}_a = -\partial_a T_i + \epsilon_{ij}k \Upsilon_a^k T_j - \epsilon_{ij}k A_{a0}^k \Lambda_j - \frac{1}{2} \chi_a^i, \]
(55)

and by taking \( dt = 0 \) the following gauge transformations arise

\[ \delta A_{a0}^i = \left[ \epsilon_{ij}k A_{a0}^k \right] \delta \tilde{\tau}_j + \left[ \delta_{ij} \partial_a - \epsilon_{ij}k \Upsilon_a^k \right] \delta \tilde{\lambda}_j + \frac{1}{2} \delta \tilde{\sigma}_a^i, \]
\[ \delta \Upsilon_a = - \left[ \delta_{ij} \partial_a - \epsilon_{ij}k \Upsilon_a^k \right] \delta \tilde{\tau}_j - \left[ \epsilon_{ij}k A_{a0}^k \right] \delta \tilde{\lambda}_j + \frac{1}{2} \delta \tilde{\zeta}_a^i. \]
(56)
Moreover, from eq. (51) we can observe that $T_i, \Lambda_i, \varsigma_i, \chi_i$ are identified as Lagrange multipliers and (54) says that the variables $B_{a0}^i$ and $B_{ab}^i$ are not dynamical. Therefore, the dynamical variables are given by $A_{a0}^i$ and $\Upsilon_a^i$ just like in Pontryagin invariant.

On the other hand, we observe that the constrains $\phi_{11}^a, \phi_{12}^a$ are not independent and present the following 6 reducibility conditions

$$\partial_a \phi_{11}^a = -\epsilon_{ijk} \Upsilon_{a}^j \phi_{11}^{ak} + \epsilon_{ijk} A_{a0}^j \phi_{12}^{ak} - \frac{1}{2} \phi_{10}^i,$$

$$\partial_a \phi_{12}^a = -\epsilon_{ijk} A_{a0}^j \phi_{11}^{ak} - \epsilon_{ijk} \Upsilon_{a}^j \phi_{12}^{ak} + \frac{1}{2} \phi_{10}^i.$$  \hspace{1cm} (57)

hence, eq. (49) and eq. (50) implies that there are a total of 18 dynamical variables, and there are 18 independent involutive constraints $\phi_{10}^a, \phi_{11}^a, \phi_{12}^a$. It is worth to mention that the reducibility conditions (54) and (57) do not affect integrability. In fact, these conditions are linear combination of involutive Hamiltonians, then the generalized brackets between reducibility conditions with other Hamiltonians vanish. Thus, the counting of physical degrees of freedom yield to conclude that the Euler theory lacks of physical degrees of freedom.

**IV. CONCLUSIONS**

In this paper a complete HJ analysis for the Pontryagin and Euler classes has been performed. We have developed our study by using in both theories the same phase space variables and this fact has allowed us to compare the emergent symmetries. The full set of Hamiltonians of the theories were identified and different generalized HJ differentials have been constructed. From the generalized differential all symmetries of the theories have been found, we observed that in spite of the invariants share the same equations of motion and the same dynamical variables, the symmetries are different. In fact, the generators of rotations and boost are interchanged; the generators of boost and rotations for Pontryagin are generators of rotations and boost for Euler respectively. Moreover, we found that the generalized brackets are also different because there is a direct contribution of the parameters $\Omega$ and $\Xi$, which could be relevant in the quantization process or the identification of the observables of the theories. In this respect, the generalized brackets between the dynamical variables for both theories are not the same to each other, and this implies that the corresponding uncertainty principles will be different. It is worth to comment, that all these results will be important when a boundary is added. In fact, we can see in [2] that the knowledge of the canonical structure of topological theories with a boundary is a mandatory step to perform in order to know the symmetries and physical degrees of freedom at the boundary, thus, due to the close relation between Pontryagin and Euler invariants a work is in progress [19].

In this manner, we can observe that our results are generic and we have extended those reported in [11, 13]; we have also showed that the HJ formulation is an elegant and complete framework for studying topological gauge theories.

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