SHOCK REFLECTION-DIFFRACTION, VON NEUMANN’S CONJECTURES, AND NONLINEAR EQUATIONS OF MIXED TYPE

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ABSTRACT. Shock waves are fundamental in nature. One of the most fundamental problems in fluid mechanics is shock reflection-diffraction by wedges. The complexity of reflection-diffraction configurations was first reported by Ernst Mach in 1878. The problems remained dormant until the 1940s when John von Neumann, as well as other mathematical/experimental scientists, began extensive research into all aspects of shock reflection-diffraction phenomena. In this paper we start with shock reflection-diffraction phenomena and historic perspectives, their fundamental scientific issues and theoretical roles in the mathematical theory of hyperbolic systems of conservation laws. Then we present how the global shock reflection-diffraction problem can be formulated as a boundary value problem in an unbounded domain for nonlinear conservation laws of mixed hyperbolic-elliptic type, and describe the von Neumann conjectures: the sonic conjecture and the detachment conjecture. Finally we discuss some recent developments in solving the von Neumann conjectures and establishing a mathematical theory of shock reflection-diffraction, including the existence, regularity, and stability of global regular configurations of shock reflection-diffraction by wedges.

1. Introduction

Shock waves are steep fronts that propagate in the compressible fluids when the convective motion dominates the diffusion. They are fundamental in nature. Examples include shock waves formed by solar winds (bow shocks), supersonic or near sonic aircrafts (transonic shocks around the body or supersonic bubbles with transonic shocks), explosions (blast waves), and various other natural processes. When a shock hits an obstacle (steady or flying), shock reflection-diffraction phenomena occur. One of the most fundamental problems in fluid mechanics is the problem of shock reflection-diffraction by wedges; see Ben-Dor [4], Courant-Friedrichs [13], von Neumann [31, 32, 33], and the references cited therein. When a plane shock hits a wedge head on, it experiences a reflection-diffraction process, and then a fundamental question is what types of wave patterns of reflection-diffraction configurations may be formed around the wedge.

The complexity of reflection-diffraction configurations was first reported by Ernst Mach [27] in 1878, who first observed two patterns of reflection-diffraction configurations: regular reflection (two-shock configuration; see Fig. 1 (left)) and Mach reflection (three-shock/one-vortex-sheet configuration; see Fig. 1 (center)); also see [4, 7, 13, 30]. The issues remained dormant until the 1940s when John von Neumann, as well as other mathematical/experimental scientists, began extensive research into all aspects of shock reflection-diffraction phenomena, due to its importance in applications. See von Neumann [31, 32, 33], Courant-Friedrichs [13], Glimm-Majda [17], and Ben-Dor [4]; also see [7, 30] and the references cited therein. It has been found that the situations are much more complicated than what Mach originally observed: The Mach reflection can be further divided into more specific sub-patterns, and various other...
patterns of shock reflection-diffraction may occur such as the double Mach reflection, von Neumann reflection, and Guderley reflection; see [4, 7, 13, 17, 30] and the references cited therein. The fundamental scientific issues include the following:

(i) Structure of the shock reflection-diffraction configurations;
(ii) Transition criteria between the different patterns of shock reflection-diffraction configurations;
(iii) Dependence of the patterns upon the physical parameters such as the wedge angle $\theta_w$, the incident-shock-wave Mach number, and the adiabatic exponent $\gamma \geq 1$.

In particular, several transition criteria between the different patterns of shock reflection-diffraction configurations have been proposed, including the sonic conjecture and the detachment conjecture by von Neumann [31, 32, 33].

Careful asymptotic analysis has been made for various reflection-diffraction configurations in Lighthill [23, 24], Keller-Blank [22], Hunter-Keller [21], Harabetian [19], Morawetz [28], and the references cited therein; also see Glimm-Majda [17]. Large or small scale numerical simulations have been also made; cf. [4, 17, 34] and the references cited therein. However, most of the fundamental issues for shock reflection-diffraction phenomena have not been understood, especially the global structure and transition between different patterns of shock reflection-diffraction configurations. This is partially because physical and numerical experiments are hampered by various difficulties and have not yielded clear transition criteria between different patterns. In particular, numerical dissipation or physical viscosity smear the shocks and cause boundary layers that interact with the reflection-diffraction patterns and can cause spurious Mach steams; cf. Woodward-Colella [34]. Furthermore, some different patterns occur when the wedge angles are only fractions of a degree apart, a resolution even by sophisticated modern experiments (cf. [26]) has been unable to reach. For this reason, it is almost impossible to distinguish experimentally between the sonic and detachment criteria, as pointed out in [4]. In this regard, the necessary approach to understand fully the shock reflection-diffraction phenomena, especially the transition criteria, is still via rigorous mathematical analysis. To achieve this, it is essential to establish first the global existence, regularity, and structural stability of solutions of the shock reflection-diffraction problem.

Furthermore, shock reflection-diffraction configurations are the core configurations in the structure of global entropy solutions of the two-dimensional Riemann problem for hyperbolic conservation laws, while the Riemann solutions are building blocks and local structure of general solutions and determine global attractors and asymptotic states of entropy solutions, as time tends to infinity, for multidimensional hyperbolic systems of conservation laws. See [7, 17, 29, 36] and the references cited therein. In this sense, we have to understand the
shock reflection-diffraction phenomena, in order to understand fully global entropy solutions to multidimensional hyperbolic systems of conservation laws.

In Section 2, we first formulate the shock reflection-diffraction problem into an initial-boundary value problem. Based on the invariance of the problem under the self-similar scaling and some basic properties of the governing equations, we reformulate the problem as a boundary value problem in an unbounded domain for nonlinear conservation laws of mixed hyperbolic-elliptic type. In Section 3, we describe the von Neumann’s conjectures: the sonic conjecture and the detachment conjecture. In Section 4, we discuss some recent developments in solving the von Neumann’s conjectures and establishing a mathematical theory of shock reflection-diffraction, including the existence, regularity, and stability of global regular configurations of shock reflection-diffraction by wedges. The shock reflection-diffraction problems involve several core mathematical difficulties we have to face in solving nonlinear partial differential equations in mechanics and geometry; also see [10, 11, 18, 25, 35] and the references cited therein. These include nonlinear equations of mixed hyperbolic-elliptic type, nonlinear degenerate elliptic equations, nonlinear degenerate hyperbolic equations, free boundary problems for nonlinear degenerate equations, corner singularity/regularity especially when free boundaries meet degenerate curves, and a priori estimate techniques.

2. Mathematical Formulation and Nonlinear Equations of Mixed Type

In this section we first formulate the shock reflection-diffraction problem into an initial-boundary value problem and then reformulate the problem as a boundary value problem in an unbounded domain for a nonlinear conservation laws of mixed elliptic-hyperbolic type.

2.1. Mathematical Problems. The potential flow is governed by the conservation law of mass and the Bernoulli law:

\[ \partial_t \rho + \nabla_x \cdot (\rho \nabla_x \Phi) = 0, \]
\[ \partial_t \Phi + \frac{1}{2} |\nabla_x \Phi|^2 + h(\rho) = B \]  
(2.1) (2.2)

for the density \( \rho \) and the velocity potential \( \Phi \), where \( B \) is the Bernoulli constant determined by the incoming flow and/or boundary conditions, and

\[ h'(\rho) = \frac{p'(\rho)}{\rho} = \frac{c^2(\rho)}{\rho} \]

with \( c(\rho) \) being the sound speed and \( p \) the pressure. For an ideal polytropic gas, the pressure \( p \) and the sound speed \( c \) are given by

\[ p(\rho) = \kappa \rho^\gamma, \quad c^2(\rho) = \kappa \gamma \rho^{\gamma-1} \]

for constants \( \gamma > 1 \) and \( \kappa > 0 \). By scaling, we may choose \( \kappa = 1/\gamma \) to have

\[ h(\rho) = \rho^{\gamma-1} - \frac{1}{\gamma - 1}, \quad c^2(\rho) = \rho^{\gamma-1}. \]  
(2.3)

From (2.2) and (2.3), we have

\[ \rho(\partial_t \Phi, \nabla_x \Phi) = h^{-1}(B - \partial_t \Phi - \frac{1}{2} |\nabla_x \Phi|^2). \]  
(2.4)

Then system (2.1), (2.2) can be rewritten as

\[ \partial_t \rho(\partial_t \Phi, \nabla_x \Phi) + \nabla_x \cdot (\rho(\partial_t \Phi, \nabla_x \Phi) \nabla_x \Phi) = 0 \]
(2.5)

with \( \rho(\partial_t \Phi, \nabla_x \Phi) \) determined by (2.4).
When a vertical planar shock perpendicular to the flow direction $x_1$ and separating two uniform states $(0)$ and $(1)$, with constant velocities $(u_0, v_0) = (0, 0)$ and $(u_1, v_1) = (u_1, 0)$, and constant densities $\rho_1 > \rho_0$ (state $(0)$ is ahead or to the right of shock in the figure and state $(1)$ is behind the shock), hits a symmetric wedge:

$$W := \{ |x_2| < x_1 \tan(\theta_w), x_1 > 0 \}$$

head on at time $t = 0$, then a reflection-diffraction process takes place when $t > 0$. Since state $(1)$ does not satisfy the slip boundary condition $\partial_\nu \phi = 0$ prescribed on the wedge boundary, the solution must differ from state $(1)$ on and near the wedge boundary. Mathematically, the shock reflection-diffraction problem is a multidimensional lateral Riemann problem in the domain $\mathbb{R}^2 \setminus W$.

**Problem 1** (Lateral Riemann Problem). *Piecewise constant initial data, consisting of state $(0)$ on $\{ x_1 > 0 \} \setminus W$ and state $(1)$ on $\{ x_1 < 0 \}$, are prescribed at $t = 0$. Seek a solution of (1) for $t \geq 0$ subject to these initial data and the boundary condition $\nabla \Phi \cdot \nu = 0$ on $\partial W$.*

Notice that the initial-boundary value problem for (2.1)–(2.2) is invariant under the scaling:

$$(t, \xi) \rightarrow (\alpha t, \alpha \xi), \quad (\rho, \Phi) \rightarrow (\rho \frac{\Phi}{\alpha}) \quad \text{for} \quad \alpha \neq 0.$$  

Thus, we seek self-similar solutions in the form of

$$\rho(t, \xi) = \rho(\xi, \eta), \quad \Phi(t, \xi) = t \phi(\xi, \eta) \quad \text{for} \quad (\xi, \eta) = \frac{\xi}{t}.$$  

Then the pseudo-potential function $\phi = \phi - \frac{1}{2}(\xi^2 + \eta^2)$ satisfies the following nonlinear conservation laws of mixed type:

$$\text{div}(\rho(|D\phi|^2, \phi)D\phi) + 2\rho(|D\phi|^2, \phi) = 0,$$  

for

$$\rho(|D\phi|^2, \phi) = (\rho_0^{-1} - (\gamma - 1)(\frac{1}{2}|D\phi|^2 + \phi)))^{\frac{1}{\gamma - 1}},$$  

where the divergence div and gradient $D$ are with respect to $(\xi, \eta)$, and we have used the initial condition to fix the Bernoulli constant $B$ in (2.2). Then equation (2.6) is a second-order nonlinear conservation law of mixed elliptic-hyperbolic type. It is elliptic if and only if

$$|D\phi|^2 < c^2(|D\phi|^2, \phi) := \rho_0^{-1} - (\gamma - 1)(\frac{1}{2}|D\phi|^2 + \phi),$$  

which is equivalent to

$$|D\phi| < c_*(\phi, \rho_\infty) := \sqrt{\frac{2}{\gamma + 1}(\rho_0^{-1} - (\gamma - 1)\phi)).}$$  

The type of equation, due to its nonlinearity, depends on the solution of the equation, which makes the problem truly fundamental and challenging. Elliptic regions of the solution correspond to the subsonic flow, and hyperbolic regions to the supersonic flow.

Shocks are discontinuities in the pseudo-velocity $D\phi$. If curve $S$ subdivides the domain into subdomains $\Omega^\pm$, and $\phi \in C^1(\Omega^\pm \cup S) \cap C^2(\Omega)$, then $\phi$ is a weak solution of (2.6) in $\Omega$ across $S$ if and only if $\phi$ satisfies (2.6) in $\Omega^\pm$ and the Rankine-Hugoniot conditions on $S$:

$$[\phi]_S = 0, \quad [\rho(|D\phi|^2, \phi)D\phi \cdot \nu]_S = 0,$$  

where $[\cdot]_S$ is the jump across $S$, and $\nu$ is the unit normal to $S$. 

Then Problem 1 is reformulated as the following boundary value problem in unbounded domain

\[ \Lambda := \mathbb{R}^2 \setminus \{|\eta| \leq \xi \tan(\theta_w), \xi > 0\} \]

in the self-similar coordinates \((\xi, \eta)\).

**Problem 2 (Boundary Value Problem).** *Seek a solution \( \varphi \) of equation (2.6) in the self-similar domain \( \Lambda \) with the slip boundary condition \( D\varphi \cdot \nu|_{\partial\Lambda} = 0 \) on the wedge boundary \( \partial\Lambda \) and the asymptotic boundary condition at infinity:

\[ \varphi \to \bar{\varphi} = \begin{cases} 
\varphi_0 & \text{for } \xi > \xi_0, |\eta| > \xi \tan(\theta_w), \\
\varphi_1 & \text{for } \xi < \xi_0.
\end{cases} \]

when \( \xi^2 + \eta^2 \to \infty. \)

By symmetry we can restrict to the upper half-plane \( \{\eta > 0\} \cap \Lambda \), with condition \( \partial_\nu \varphi = 0 \) on \( \{\eta = 0\} \cap \Lambda \).

3. **Von Neumann’s Conjectures on Shock Reflection-Diffraction Configurations**

In this section, we discuss von Neumann’s conjectures, including the sonic conjecture and the detachment conjecture, on shock reflection-diffraction configurations.

If a solution has the regular reflection-diffraction configurations as shown in Fig. 2 and Fig. 3 and if \( \varphi \) is smooth in the subregion between the reflected shock and the wedge, then it should satisfy the boundary condition \( D\varphi \cdot \nu = 0 \) and the Rankine-Hugoniot conditions (2.10) at \( P_0 \) across the reflected shock separating it from state (1). We define the uniform state (2) with pseudo-potential \( \varphi_2(\xi, \eta) \) such that

\[ D\varphi_2(P_0) = D\varphi(P_0), \]

and the constant density \( \rho_2 \) of state (2) is equal to \( \rho(|D\varphi|^2, \varphi)(P_0) \) defined by (2.6):

\[ \rho_2 = \rho(|D\varphi|^2, \varphi)(P_0). \]

Then \( D\varphi_2 \cdot \nu = 0 \) on the wedge boundary, and the Rankine-Hugoniot conditions (2.10) hold on the flat shock \( S_1 = \{\varphi_1 = \varphi_2\} \) between states (1) and (2), which passes through \( P_0 \).

State (2) can be either subsonic or supersonic at \( P_0 \). This determines the subsonic or supersonic type of regular reflection-diffraction configurations. The supersonic regular reflection-diffraction configuration as shown in Fig. 2 consists of three uniform states (0), (1), (2), and a non-uniform state in the domain \( \Omega \), where the equation is elliptic. The reflected shock \( P_0P_1P_2 \)
has a straight part $P_0P_1$. The elliptic domain $\Omega$ is separated from the hyperbolic region $P_0P_1P_4$ of state (2) by a sonic arc $P_1P_4$. The subsonic regular reflection-diffraction configuration as shown in Fig. 3 consists of two uniform states (0) and (1), and a non-uniform state in the domain $\Omega$, where the equation is elliptic and $D(\varphi_2')(P_0) = D\varphi_2(P_0)$.

Thus, a necessary condition for the existence of regular reflection solution is the existence of the uniform state (2) determined by the conditions described above. These conditions lead to an algebraic system for the constant velocity $(u_2, v_2)$ and density $\rho_2$ of state (2), which has solutions for some but not all of the wedge angles. Specifically, for fixed densities $\rho_0 < \rho_1$ of states (0) and (1), there exist a sonic angle $\theta_w^s$ and a detachment angle $\theta_w^d$ satisfying

$$0 < \theta_w^d < \theta_w^s < \frac{\pi}{2}$$

such that state (2) exists for all $\theta_w \in (\theta_w^d, \frac{\pi}{2})$ and does not exist for $\theta_w \in (0, \theta_w^d)$, and the weak state (2) is

(i) supersonic at the reflection point $P_0(\theta_w)$ for $\theta_w \in (\theta_w^s, \frac{\pi}{2})$;

(ii) sonic for $\theta_w = \theta_w^s$;

(iii) subsonic for $\theta_w \in (\theta_w^d, \theta_w^s)$

for some $\theta_w^s \in (\theta_w^d, \theta_w^s)$. In fact, for each $\theta_w \in (\theta_w^d, \frac{\pi}{2})$, there exists also a strong state (2) with $\rho_2^{\text{strong}} > \rho_2^{\text{weak}}$. There had been a long debate to determine which one is physical for the local theory; see Courant-Friedrichs [13], Ben-Dor [4], and the references cited therein. It is expected that strong reflection-diffraction configurations are non-physical; indeed, it is shown that the weak reflection-diffraction configuration tends to the unique normal reflection, but the strong reflection-diffraction configuration does not, when the wedge angle $\theta_w$ tends to $\frac{\pi}{2}$, as proved in Chen-Feldman [6].

If the weak state (2) is supersonic, the propagation speeds of the solution are finite and state (2) is completely determined by the local information: state (1), state (0), and the location of point $P_0$. That is, any information from the region of reflection-diffraction, especially the disturbance at corner $P_3$, cannot travel towards the reflection point $P_0$. However, if it is subsonic, the information can reach $P_0$ and interact with it, potentially altering a different reflection-diffraction configuration. This argument motivated the following conjecture von Neumann in [31, 32]:

**The Sonic Conjecture:** There exists a supersonic reflection-diffraction configuration when $\theta_w \in (\theta_w^a, \frac{\pi}{2})$ for $\theta_w^a > \theta_w^d$, i.e., the supersonicity of the weak state (2) implies the existence of a supersonic regular reflection-diffraction solution, as shown in Fig. 3.

Another conjecture is that global regular reflection-diffraction configuration is possible whenever the local regular reflection at the reflection point is possible:

**The Detachment Conjecture:** There exists a regular reflection-diffraction configuration for all wedge angles $\theta_w \in (\theta_w^d, \frac{\pi}{2})$, i.e., that the existence of state (2) implies the existence of a regular reflection-diffraction solution, as shown in Fig. 3.

It is clear that the supersonic/subsonic regular reflection-diffraction configurations are not possible without a local two-shock configuration at the reflection point on the wedge, so this is the weakest possible criterion for the existence of (supersonic/subsonic) regular shock reflection-diffraction configurations.
4. Solution to the von Neumann Equations of Mixed Type

We observe that the key obstacle to the existence of regular shock reflection-diffraction configurations as conjectured by von Neumann is an additional possibility that, for some wedge angle \( \theta_w \in (\theta_w^d, \frac{\pi}{2}) \), shock \( P_0P_2 \) may attach to the wedge tip \( P_3 \), as observed by experimental results (cf. [30, Fig. 238]). To describe the conditions of such attachments, we note that

\[
\rho_1 > \rho_0, \quad u_1 = (\rho_1 - \rho_0) \sqrt{\frac{2(\rho_1^{-1} - \rho_0^{-1})}{\rho_1^2 - \rho_0^2}}.
\]

Then, for each \( \rho_0 \), there exists \( \rho_1^{cr} > \rho_0 \) such that

\[
u_1 \leq c_1 \text{ if } \rho_1 \in (\rho_0, \rho_1^{cr}], \quad \nu_1 > c_1 \text{ if } \rho_1 \in (\rho_1^{cr}, \infty).
\]

If \( u_1 \leq c_1 \), we can rule out the solutions with shocks attached to the wedge tip.

If \( u_1 > c_1 \), this is unclear and could not be even true as experiments show (e.g. [30, Fig. 238]).

Thus, in [6, 7], we have obtained the following results:

(i) If \( \rho_0 \) and \( \rho_1 \) are such that \( u_1 \leq c_1 \), then the supersonic/subsonic regular reflection-diffraction solution exists for each wedge angle \( \theta_w \in (\theta_w^d, \frac{\pi}{2}) \);

(ii) If \( \rho_0, \rho_1 \) are such that \( u_1 > c_1 \), then there exists \( \theta_w^a \in [\theta_w^d, \frac{\pi}{2}) \) such that the regular reflection solution exists for each wedge angle \( \theta_w \in (\theta_w^a, \frac{\pi}{2}) \). Moreover, if \( \theta_w^a > \theta_w^d \), then, for the wedge angle \( \theta_w = \theta_w^a \), there exists an attached solution, i.e., a solution of Problem 2 with \( P_2 = P_3 \).

The type of regular reflection (supersonic, Fig. 2 or subsonic, Fig. 3) is determined by the type of state (2) at \( P_0 \). The reflected shock \( P_4P_2 \) is \( C^{2,\alpha} \)-smooth. The solution \( \varphi \) is \( C^{1,1} \) across the sonic arc for the supersonic reflection-diffraction solution, which is optimal. For the subsonic and sonic reflection-diffraction case (Fig. 3), the solution is \( C^{2,\alpha} \) in \( \Omega \) near \( P_0 \). Furthermore, the regular reflection-diffraction solution tends to the unique normal reflection, when the wedge angle \( \theta_w \) tends to \( \frac{\pi}{2} \).

These results are obtained by solving a free boundary problem for \( \varphi \) in \( \Omega \). The free boundary is the curved part of the reflected shock: \( P_1P_2 \) on Fig. 2 and \( P_3P_2 \) on Fig. 3. We seek a solution \( \varphi \) of equation (2.6) in \( \Omega \), satisfying the Rankine-Hugoniot conditions (2.10) on the free boundary, \( \varphi = 0 \) on the boundary of the wedge and on \( P_2P_3 \), and

\[
\varphi = \varphi_2, \quad D\varphi = D\varphi_2
\]

on \( P_1P_4 \) in the supersonic case as shown in Fig. 2 and at \( P_0 \) in the subsonic case as shown in Fig. 3. The equation is expected to be elliptic in \( \Omega \) for \( \varphi \) (unknown a priori before obtaining the solution), which is a part of the results.

To solve this free boundary problem, we define a class of admissible solutions of Problem 2, which are solutions \( \varphi \) with weak regular reflection-diffraction configuration, such that, in the supersonic reflection case, equation (2.6) is strictly elliptic for \( \varphi \) in \( \overline{\Omega} \setminus P_1P_4 \), inequalities \( \varphi_2 \leq \varphi \leq \varphi_1 \) hold in \( \Omega \), and the following monotonicity properties hold:

\[
\partial_\eta(\varphi_1 - \varphi) \leq 0, \quad D(\varphi_1 - \varphi) \cdot e \leq 0 \quad \text{in } \Omega,
\]

where \( e = \frac{P_0P_1}{P_0P_2} \). In the subsonic reflection case, admissible solutions are defined similarly, with changes corresponding to the structure of subsonic reflection solution. We derive uniform
a priori estimates for admissible solutions with wedge angles $\theta_w \in [\theta_w^d + \varepsilon, \frac{\pi}{2}]$ for each $\varepsilon > 0$, and then apply the degree theory to obtain the existence for each $\theta_w \in [\theta_w^d + \varepsilon, \frac{\pi}{2}]$ in the class of admissible solutions, starting from the unique solution for $\theta_w = \frac{\pi}{2}$, called the normal reflection. To derive the a priori bounds, we first obtain the estimates related to the geometry of the shock: Show that the free boundary has a uniform positive distance from the sonic circle of state (1) and from the wedge boundary away from $P_2$ and $P_0$. This allows to estimate the ellipticity of (2.6) for $\varphi$ in $\Omega$ (depending on the distance to the sonic arc $P_1P_4$ for the supersonic reflection and to $P_0$ for the subsonic reflection). Then we obtain the estimates near $P_1P_4$ (or $P_0$ for the subsonic reflection) in scaled and weighted $C^{2,\alpha}$ for $\varphi$ and the free boundary, considering separately four cases depending on $\frac{|D\varphi|}{c_2}$ at $P_0$:

(i) $\frac{|D\varphi|^2(P_0)}{c_2} \geq 1 + \delta$;

(ii) Supersonic (almost sonic): $1 + \delta > \frac{|D\varphi|^2(P_0)}{c_2} > 1$;

(iii) Subsonic (almost sonic): $1 \geq \frac{|D\varphi|^2(P_0)}{c_2} \geq 1 - \delta$;

(iv) Subsonic: $\frac{|D\varphi|^2(P_0)}{c_2} \leq 1 - \delta$.

In cases (i)–(ii), equation (2.6) is degenerate elliptic in $\Omega$ near $P_1P_4$ on Fig. 2. In case (iii), the equation is uniformly elliptic in $\Omega$, but the ellipticity constant is small near $P_0$ on Fig. 3. Thus, in cases (i)–(iii), we use the local elliptic degeneracy, which allows to find a comparison function in each case, to show the appropriately fast decay of $\varphi - \varphi_2$ near $P_1P_4$ in cases (i)–(ii) and near $P_0$ in case (iii); furthermore, combining with appropriate local non-isotropic rescaling to obtain the uniform ellipticity, we obtain the a priori estimates in the weighted and scaled $C^{2,\alpha}$-norms, which are different in each of cases (i)–(iii), but imply the standard $C^{1,1}$-estimates in cases (i)–(ii), and the standard $C^{2,\alpha}$-estimates in case (iii). This is an extension of methods of our earlier work [6]. In the uniformly elliptic case (iv), the solution is of subsonic reflection-diffraction configuration as shown in Fig. 3 and the estimates are more technically challenging than in cases (i)–(iii), due to the lower a priori regularity of the free boundary and since the uniform ellipticity does not allow a comparison function that shows the decay of $\varphi - \varphi_2$ near $P_0$. Thus, we prove $C^{1,\alpha}$-estimates of $D(\varphi - \varphi_2)$ near $P_0$, which imply the $C^{2,\alpha}$-estimates near $P_0$ for $\varphi$. With all of these, we provide a solution to the von Neumann’s conjectures.

In Chen-Feldman-Xiang [9], we have established the strict convexity of the curved (transonic) part of shock in the shock reflection-diffraction problem, as well as two other problems: the Prandtl-Meyer reflection (cf. Bae-Chen-Feldman [2]) and shock diffraction (cf. Chen-Xiang [12]). In order to prove the convexity, we employ global properties of admissible solutions, including the existence of the cone of monotonicity discussed above.

More details can be found in Chen-Feldman [7].

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