Analogue of Gauss-Lucas theorem for non convex set on the complex plane

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Abstract

Let $S(\phi) = \{z : |\arg(z)| \geq \phi\}$ be a sector on the complex plane $\mathbb{C}$. If $\phi \geq \pi/2$, then $S(\phi)$ is a convex set and, according to the Gauss-Lucas theorem, if a polynomial $p(z)$ has all its zeros on $S(\phi)$, then the same is true for the zeros of all its derivatives. In this paper is proved that if the polynomial $p(z)$ is with real and non negative coefficients, then the same is true also for $\phi < \pi/2$, when the sector is not a convex set on the complex plane.

Keywords: Gauss-Lucas theorem, polynomials with non negative coefficients.

1 Introduction

We start with defining several notations.

Let $R_n$ be the set of all monic algebraic polynomials with real coefficients and $R_n^+$ be the subset of $R_n$ containing all polynomials with non negative coefficients. Obviously $R_n$ and $R_n^+$ are convex sets. Let

$$p(z) = (z - z_1(p))(z - z_2(p)) \cdots (z - z_n(p))$$

$$= z^n + \sum_{k=0}^{n-1} a_k(p) z^k; \quad z_k(p) = r_k(p) e^{i\varphi_k(p)}, \quad k = 1, 2, \ldots, n.$$ 

For every $t \geq 0$ and $p(z) \in R_n^+$ define the polynomial $p_t(z) \in R_n^+$ by

$$a_0(p_t) = t, \quad a_k(p_t) = a_k(p); \quad k = 1, 2, \ldots, n - 1.$$ 

Let $z_k(t) = r_k(t) e^{i\varphi_k(t)}; \quad k = 1, 2, \ldots, n$ be the zeros of $p_t(z)$. The arguments of the zeros of $p_t(z)$ do not change by the transformation $z \Rightarrow t^{1/n} z$ and $p_t^n(z) = t^{-1} p_t(t^{1/n} z) \in R_n^+$. On the other hand, for $q(z) = z^n + 1$, we have

$$\lim_{t \to \infty} |p_t^n(z) - q(z)| = 0,$$

hence, according to the Hurwitz theorem,

$$\lim_{t \to \infty} \varphi_k(p_t^n) = \frac{(2k - 1)\pi}{n}; \quad k = 1, 2, \ldots, [n/2]. \quad (1)$$

We index the zeros of $p(t)$ on the upper half plane by order of increasing argument of the limit of the zeros of $p_t(z)$ for $t \to \infty$, or

$$\lim_{t \to \infty} \varphi_1(p_t) \leq \lim_{t \to \infty} \varphi_2(p_t) \leq \cdots \leq \lim_{t \to \infty} \varphi_{[n/2]}(p_t) \leq \pi. \quad (2)$$

Denote by $S(\phi)$ the sector on the complex plane $\mathbb{C}$,

$$S(\phi) = \{z : |\arg(z)| \geq \phi\} \quad (3)$$
and by \( R_n(\phi) \left( \mathcal{R}_n(\phi) \right) \), the set of all polynomials \( p(z) \in R_n \left( p(z) \in \mathcal{R}_n \right) \), with zeros on the sector \([0, \pi/2]\).

Define the function

\[
\varphi_p(t) = \min \left\{ \varphi_1(p_1), \varphi_2(p_2), \ldots, \varphi_{n/2}(p_t) \right\}; \quad t \geq 0 \quad \text{and} \quad \varphi(p) = \varphi_p(a_0(p)).
\] (4)

If \( \varphi(p) \geq \pi/2 \), then the sector \( S(\varphi(p)) \) is a convex set and according to the Gauss-Lucas theorem, we have:

**Statement 1** If \( p(z) \in R_n; n \geq 2 \) and \( \varphi(p) \geq \pi/2 \), then all the zeros of the derivative \( p'(z) \) are on \( S(\varphi(p)) \).

Our goal is to show that in **Statement 1** the condition \( \varphi(p) \geq \pi/2 \) is redundant, if \( p(z) \in R_n^+ \), and to prove:

**Theorem 1** If \( p(z) \in R_n^+; n \geq 2 \) and all the zeros of \( p(z) \) are on the sector \([0, \pi] \), then all the zeros of its derivative \( p'(z) \) are also on the sector \([0, \pi]\).

For \( \phi < \pi/2 \), the sector \([0, \pi]\) is not a convex set on the complex plane, so **Theorem 1** is an analogue of the Gauss-Lucas theorem for non convex set. Observe that the differentiation do not change sine of the coefficients, hence:

**Corollary 1** If \( p(z) \in R_n^+; n \geq 2 \) and all the zeros of \( p(z) \) are on the sector \([0, \pi]\), then all the zeros of all its derivatives \( p^{(k)}(z); k = 1, 2, \ldots, n - 1 \) are also on the sector \([0, \pi]\).

## 2 Preliminaries

It is obvious that a polynomial \( p(z) \in R_n^+ \) has no real positive zeros, hence \( \varphi(p) > 0 \) and, more precisely:

**Statement 2** If \( p(z) \in R_k^+ \), then \( \varphi(p) \geq \pi/k; k = 1, 2, 3, \ldots \)

Really, \( \text{Im}(p(z_1(p))) = \sum_{s=1}^{k} a_s r(p)^s \sin(s \varphi(p)) > 0 \), if \( \varphi(p) < \pi/k \).

**Statement 3** The set \( R_2(\phi) = \left\{ z^2 - 2\nu rz + r^2 : v \in [-1, \cos \phi], r \geq 0 \right\}; \phi \in (0, \pi) \) is convex.

**Proof.** As \( R_2 \) is a half strip, it is sufficient to prove the convexity separately in respect to \( v \) and in respect to \( r \).

Let \( v < w; v, w \in [-1, \cos \phi] \) and consider the polynomial

\[
h(z) = \lambda(z^2 - 2\nu rz + r^2) + (1 - \lambda)(z^2 - 2\nu rz + r^2); \quad \lambda \in [0, 1].\]

We have

\[
h(z) = z^2 - 2[\lambda v + (1 - \lambda)w]r + r^2 = (z - z_1)(z - \overline{z}_1) .
\]

Denote \( u = \cos \phi = \lambda v + (1 - \lambda)w \in [v, w] \). Then \( z_1 = r(\cos \phi + i \sin \phi) \) and convexity in respect to \( v \) is proved.

Let \( r_1 < r_2 \) and consider the function

\[
h(z) = \lambda(z^2 - 2\nu r_1 z + r_1^2) + (1 - \lambda)(z^2 - 2\nu r_2 z + r_2^2)
\]

\[
= z^2 - 2\nu[\lambda r_1 + (1 - \lambda)r_2]z + \lambda r_1^2 + (1 - \lambda)r_2^2 = (z - z_1)(z - \overline{z}_1),
\]
where
\[ z_1 = v[\lambda r_1 + (1 - \lambda)r_2] + i\sqrt{\lambda r_1^2 + (1 - \lambda)r_2^2 - v^2[\lambda r_1 + (1 - \lambda)r_2]^2}. \]

As \( v \in (0, 1) \), we have
\[ \lambda r_1^2 + (1 - \lambda)r_2^2 - v[\lambda r_1 + (1 - \lambda)r_2]^2 \geq \lambda r_1^2 + (1 - \lambda)r_2^2 - [\lambda r_1 + (1 - \lambda)r_2]^2 = \lambda(1 - \lambda)(r_1 - r_2)^2 > 0. \]

Hence
\[ \tau = \frac{\lambda r_1^2 + (1 - \lambda)r_2^2}{v^2[\lambda r_1 + (1 - \lambda)r_2]^2} > 1 \]
and
\[ |z_1|^2 = \left| v[\lambda r_1 + (1 - \lambda)r_2] + i\sqrt{\lambda r_1^2 + (1 - \lambda)r_2^2 - v^2[\lambda r_1 + (1 - \lambda)r_2]^2} \right|^2 = \lambda r_1^2 + (1 - \lambda)r_2^2. \]

We prove that \( r_1 < |z_1| < r_2 \). On the other hand, we have
\[ z_1 = [\lambda r_1 + (1 - \lambda)r_2] \cos \phi + i\sqrt{\tau - \cos^2 \phi} = [\lambda r_1 + (1 - \lambda)r_2] \left( \cos \phi + i\sqrt{\sin^2 \phi + \tau - 1} \right). \]

As \( \tau > 1 \), from the last equality follows that \( \arg(z_1) > \phi \), which completes the proof.

**Corollary 2** The set
\[ A_n(g; \phi) = \left\{ p(z) : p(z) = u(z)g(z), u(z) \in R_2(\phi), g(z) \in R_{n-2}(\phi) \right\} \]
is a convex subset of \( R_n(\phi) \).

The proof follows from the convexity of \( R_2(\phi) \).

**Statement 4** The set of polynomials
\[ B_n(g, g^*; \phi) = \left\{ h(z) : h(z) = \lambda p(z) + (1 - \lambda)f(z), p(z) \in A_n(g; \phi), f(z) \in A_n(g^*; \phi) \right\}, \quad \lambda \in [0, 1] \]
is a convex subset of \( R_n(\phi) \).

**Proof.** For \( n = 2 \), we have \( B_n(g, g^*; \phi) = R_2 \), hence the Statement is true, according to Statement 3. For \( n = 3 \), \( g(z) = z + c, \ g^*(z) = z + c^* \), \( c, c^* \geq 0 \) and \( \lambda g(z) + (1 - \lambda)g^*(z) \in R_1(\pi) \subset R_1(\phi) \) is convex. Then \( B_3(g, g^*; \phi) \) is a Cartesian product of a half strip and a convex set, hence it is convex.

Suppose that \( B_{n-2}(g, g^*; \phi) \) is convex and let \( p(z) \in R_n(\phi) \). It is clear that we may factored a quadratic polynomial \( u(z) = z^2 - 2rz \cos \varphi + z^2 \) with \( \varphi \in [\phi, \pi] \), such that \( p(z) = u(z)g(z) \), where \( g(z) \in R_{n-2}(\phi) \). By induction, \( B_n(g, g^*; \phi) \) is a Cartesian product of a half strip and a convex set, hence it is convex. This completes the proof.

**Lemma 1** Let \( p(z) \in R_n^+ \) and \( z_k(t) = r_k(t)e^{i\varphi_k(t)}; \ k = 1, 2, \ldots, n \) be the zeros of \( p(z) \). Then, the function \( \varphi_k(t) \) is monotone decreasing and the function \( r_k(t) \) is monotone increasing; \( k = 1, 2, \ldots, [n/2] \).

**Proof.** The functions \( \varphi_k(t) = \alpha(t) \) and \( r_k(t) = \rho(t) \) are smooth in \((0, \infty)\). If the Lemma is not true for \( \alpha(t) \), then for a given \( t^* \in (0, \infty) \), we shall have \( \alpha'(t^*) = 0 \). As the zeros of a polynomial are continuous functions of its coefficients, by a small local perturbation, we may have
\[ \Im \left( p'_* (z_1(p_1)) \right) \neq 0, \quad (5) \]
preserving the condition \(\alpha'(t^*) = 0\). Observe that we may suppose that all the coefficients of \(p(z)\) are strictly positive, as such polynomials are everywhere dense in \(R_+^n\). This allows local perturbations of \(p(z)\) without going out of \(R_+^n\).

Differentiating the equation \(p_t(z_1(p_t)) = 0\) in respect to \(t\) we get

\[
p_t'(z_1(p_t))e^{i\alpha(t^*)}(\rho'(t^*) + i\rho(t^*)\alpha'(t^*)) + 1 = 0. \tag{6}
\]

Denote \(\mathcal{R}e\left(p_t'(z_1(p_t))\right) = A\) and \(\mathcal{I}m\left(p_t'(z_1(p_t))\right) = B\). Then, from the equation \(5\) we get the system

\[
A\rho'(t^*) - B\rho(t^*)\alpha'(t^*) = -1, \quad B\rho'(t^*) + A\rho(t^*)\alpha'(t^*) = 0 \tag{7}
\]

with determinant \(\rho(t^*)(A^2 + B^2) > 0\). Solving the system \(7\), we get

\[
\alpha'(t^*) = \frac{B}{\rho(t^*)(A^2 + B^2)} \quad \text{and} \quad \rho'(t^*) = \frac{A}{(A^2 + B^2)}.
\]

If \(\alpha'(t^*)\), then \(B = 0\), which contradicts \(5\). This completes the proof that \(\alpha(t)\) is monotone.

In the same way, we prove that \(\rho(t)\) is monotone. From \(\lim_{t \to \infty} \rho(t) = \infty\), follows that \(\rho(t)\) is monotone increasing.

It remained to prove that \(\alpha(t)\) is decreasing. We know that

\[
\lim_{t \to \infty} \alpha(t) = \frac{(2k - 1)\pi}{n}. \tag{8}
\]

There exists \(R_0 > 0\), such that for \(\rho(t) > R_0\), the inequality

\[
\frac{(2k - 1)\pi}{n} < \alpha(t) < \frac{2k\pi}{n} \tag{9}
\]

holds. Really, \(\rho(t)\) may go to \(\infty\) if \(\sin n\alpha(t) < 0\). This follows from the equation

\[
\mathcal{I}m\left(p_t(z_k(t))\right) = \rho(t)^n \sin n\alpha(t) + \cdots + a_1\rho(t)\sin \alpha(t) = 0.
\]

On the other hand, from

\[
\mathcal{R}e\left(p_t(z_k(t))\right) = \rho(t)^n \cos n\alpha(t) + \cdots + a_1\rho(t)\sin \alpha(t) + t = 0
\]

follows that \(\rho(t)\) may go to \(\infty\) if \(\cos n\alpha(t) < 0\). From \(\sin n\alpha(t) < 0\) and \(\cos n\alpha(t) < 0\) follows that

\[
\frac{(2k - 1)\pi}{n} < \alpha(t) < \frac{(4k - 1)\pi}{2n}.
\]

As \(\rho(t)\) is monotone increasing, from the last and \(8\), follows that when \(t\) is increasing, \(\alpha(t)\) is decreasing. This completes the proof.

**Corollary 3** For every polynomial \(p(z) \in R_+^n\) the function \(\varphi(p)\), see \([4]\), is monotone decreasing.

### 3 Proof of Theorem \([1]\)

The proof of Theorem \([1]\) will be by induction, based on the main theorem in \([1]\) p. 78, Theorem 1.1], which may be stated, using our notations, as follows:
Theorem 2 If \( p(z) \in R_n^+(\phi) \) and
\[
p(z) = \left(z^2 - 2zr(\cos \varphi(p) + r(p)^2)\right) \hat{p}(z),
\]
then \( \hat{p}(z) \in R_{n-2}^+(\phi) \).

This very important for us theorem says that a conjugate pair of zeros can be factored from every polynomial \( p(z) \in R_n^+(\phi); \ n \geq 3 \) so that the resulting polynomial is in \( R_{n-2}^+(\phi) \). That gives the opportunity to prove Theorem 1 by induction.

From Corollary 3 follows that it is sufficient to prove Theorem 1 only for polynomials \( p(z) = zq(z), \ q(z) \in R_{n-1}^+(\phi) \).

For \( n = 2 \), we have to consider the polynomials \( p(z) = zq(z) = z(z + c); \ c \geq 0 \). As \( p'(z) = 2z + c \), the Theorem 1 is true for \( n = 2 \).

For \( n = 3 \), we have to consider the polynomials \( p(z) = z(z^2 - 2rz \cos \varphi + r^2) \in R_3^+(\phi); \ \phi \in [\pi/3, \pi/2] \).

Hence \( \varphi = \pi/2 \), \( p(z) = z(z^2 + r^2) \). This proves Theorem 1 for \( n = 3 \).

Suppose that Theorem 1 is true for all natural numbers which are less or equal to \( n - 1 \). According to Theorem 2 for \( p(z) = zq(z) \in R_n^+(\phi) \), we have
\[
p(z) = zq(z) = (z^2 - 2vrz + r^2)zg(z), \quad g(z) \in R_{n-3}^+(\phi), \ v \in [0, \cos \phi].
\]

On the other hand
\[
n^{-1}p'(z) = n^{-1}[(3z^2 - 2vrz + r^2)g(z) + (z^2 - vrz + z^2)zg'(z)] \in R_{n-1}^+. \tag{10}
\]

We have to prove that \( n^{-1}p'(z) \in R_{n-1}^+(\phi) \). From Corollary 3 follows that \( (3z^2 - 2vrz + r^2)g(z) \in A(g, \phi) \), as the zeros of \( 3z^2 - 2vrz + r^2 = 0 \) are
\[
z_{1,2} = \frac{r}{3} \left(\cos \phi \pm i \sqrt{\sin^2 \phi + 2}\right)
\]
and the argument of \( z_1 \) is bigger than \( \phi \).

We have also, by induction, that \( g'(z) = zg'(z) \in A(\phi) \). From Statement 3 follows that \( n^{-1}p'(z) \in R_{n-1}^+(\phi) \) and according to 4, \( p'(z) \in R_{n-1}^+(\phi) \). This completes the proof of Theorem 1.

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References

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