Hamiltonian theory for Quantum Hall systems in a tilted magnetic field: robustness of activation gaps

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We use the Hamiltonian theory developed by Shankar and Murthy to study a quantum Hall system in a tilted magnetic field. With a finite width of the system in the z direction, the parallel component of the magnetic field introduces anisotropy into the effective two-dimensional interactions. The effects of such an anisotropy can be effectively captured by the recently proposed generalized pseudo-potentials. We find that the off-diagonal components of the pseudo-potentials lead to mixing of composite fermions Landau levels, which is a perturbation to the picture of p filled Landau levels in composite-fermion theory. By changing the internal geometry of the composite fermions, such a perturbation can be minimized and one can find the corresponding activation gaps for different tilting angles, and we calculate the associated optimal metric. Our results show that the activation gap is remarkably robust against the in-plane magnetic field in the lowest Landau level.

I. INTRODUCTION

The fractional quantum Hall effect (FQHE) is a paradigmatic platform to realize phases beyond conventional classification, i.e. phases with topological order [1]. Among the properties of the topological phase, one of the most important ones is that the liquid phase of FQHE supports gapped quasi-particles satisfying anyonic statistics, which itself is the origin of the topological characteristics such as the ground-state degeneracy. The gap of the quasi-particle excitation is related to the stability of the fractional quantum Hall (FQH) liquid.

Besides the liquid phase, there are plenty of other compressible and incompressible candidates for quantum Hall systems. The stability of this liquid state under different perturbations can help us to better understand the competition between different phases. Perturbing quantum Hall systems by anisotropy is an attractive approach to look for instabilities of liquid states. Two of the easiest realizations are to add an anisotropic mass tensor or tilt the magnetic field. For a FQH liquid, how it reacts to anisotropic perturbation remains interesting both in theory [2, 3] and in experiments [4, 5]. The stripe phase [6, 7] and the nematic phase [8] with broken rotation symmetry are possible candidates in such cases. The former is common in higher ($\nu \geq 2$) Landau levels (LLs) while the latter is assumed to be the rotation-symmetry broken phase after the collective excitation mode is softened.

One method to describe the FQH liquid is to use the composite fermion (CF) language [9–11]. The FQH problem is transformed into an integer quantum Hall problem of CFs. Beyond Jain’s extremely successful wave-function approach [9, 10], a second-quantized Hamiltonian version has been developed by Shankar and Murthy [12]. The excitations of quasi-particles/holes in the FQH liquid phase translate to particle excitations in empty CF Landau levels or hole excitations in filled CF Landau levels. The activation gap for a free particle-hole pair can be computed within straightforward analytic calculation and a good quantitative agreement is obtained in systems with finite width [12], which is the case for our problem, in this scheme.

In this paper, we study the evolution of the CF activation gaps in the presence of a parallel inplane magnetic field. The finite width of the QH system allows the electronic orbits to be tilted away from the original plane of motion and thus to introduce an anisotropy in the effective two-dimensional (2D) motion. It turns out that in the lowest LL, the activation gap is always robust against the parallel magnetic field, in accordance with the conclusion of a generalized pseudo-potential study [13]. Meanwhile, the collective excitation gap, the magneto-roton minimum, can be subject to the anisotropy, as numerically computed in a recent preprint [14]. A variational metric is defined to look for the best geometry of the liquid phase perturbed by anisotropy. When the tilting angle of the magnetic field is increased, the optimal metric starts to deviate from the flat case, suggesting an anisotropic liquid phase is favored.

The structure of this paper is as follows. In Sec. II, we briefly review the Hamiltonian theory by Shankar and Murthy, in view of the activation gaps. In Sec. III, we combine Haldane’s geometric variational approach with the Hamiltonian theory, showing how to characterize an anisotropic liquid state in this sense and how this metric reacts to the external anisotropy. The single-particle excitation spectrum for a QH system in a tilted magnetic field is calculated in Sec. IV. There, the activation gaps and optimal metrics are obtained for different tilting angles and thicknesses of the sample.
II. REVIEW OF THE HAMILTONIAN THEORY

The Hamiltonian theory developed by Shankar and Murthy [12] has been proved successful to provide a microscopic approach for FQHE problems. Namely, this theory allows one to calculate, within standard second-quantization methods, the activation gap for a pair of well-separated quasiparticle and quasihole.

For a 2D electron moving in a perpendicular magnetic field, the kinetic energy gives rise to LL quantization, while the Coulomb interaction is usually treated as a smaller scale compared to the LL spacing. When a LL is partially filled, inter-LL transitions can then, in a first approximation, be neglected and the physical properties of the system arise from electrons projected into a single LL. In such a case, the kinetic energy is frozen and the remaining degrees of freedom are associated with the guiding-center coordinates $\mathbf{R}$ of electrons. Physically, this coordinate represents the classical center of cyclotron motion, which is a constant of motion for a homogeneous magnetic field, and its components obey the commutation relation $[R_{ex}, R_{ey}] = -i\ell^2$, where $\ell = \sqrt{\hbar/eB}$ is the magnetic length. The projected Hamiltonian merely contains the interaction part, with the density operator replaced by the projected one:

$$
H^p = \frac{1}{2} \sum_{\mathbf{q}} V_{\text{eff}}(\mathbf{q}) \rho_e(\mathbf{q}) \rho_e(-\mathbf{q}),
$$

where $\rho_e(\mathbf{q}) = \sum_i \exp(-i\mathbf{q} \cdot \mathbf{R}_i)$ is the projected density operator. It is constructed by keeping only the guiding-center coordinate in the density operator while averaging the cyclotron motion over the LL wave function. The latter averaging results in a form factor that is then absorbed into the effective interaction $V_{\text{eff}}(\mathbf{q})$. The effective interaction also encodes other effects such as the finite width of the sample and the anisotropy induced by the tilted magnetic field, as we will discuss in detail below.

For a fractional filling $\nu = n_e/n_g$, that is the ratio between the 2D electronic $n_e$ and flux $n_g = eB/\hbar$ densities, there is a huge degeneracy inside each LL. This degeneracy of the non-interacting ground state prohibits diagrammatic approaches that treat the interaction as a perturbation to a non-degenerate reference state. In order to eliminate this degeneracy, the strategy is to bind each electron with vortices carrying 2s magnetic flux quanta opposite to the external one, forming the CF [9–11]. The CFs thus feel a weaker magnetic field and the filling changes from a fractional electronic value $\nu_e = p/(2ps + 1)$ to an integer CF filling $\nu_{CF} = p$. The latter CF state can then be viewed as a non-degenerate reference state for perturbative treatments. In the language of the Hamiltonian theory, the Hilbert space is enlarged to account for the guiding-center coordinates $\mathbf{R}_v$ of these vortices, which satisfy the commutation relation $[R_{ex}, R_{ey}] = i\ell^2/c^2$ with $c = 2ps/(2ps + 1)$ characterizing the vortex charge. A canonical transformation involving the guiding-center coordinates $\mathbf{R}_c$ and $\mathbf{R}_v$ allows us then to introduce the CF cyclotron $\eta$ and guiding-center coordinates $\mathbf{R}$,

$$
\eta = \frac{c}{1 - c^2} (R_c - R_v), \quad \mathbf{R} = \frac{R_c - c^2 R_v}{1 - c^2},
$$

which satisfy the commutation relations $[\eta_x, \eta_y] = il^2$ and $[R_{cx}, R_{cy}] = -i\ell^2$, while $[\eta_x/R_{ex}, R_{ey}/R_{ey}] = 0$, in terms of an effective CF magnetic length $l^* = \sqrt{2ps + 1}$.

As vortices are collective configuration of electrons, the newly introduced vortex operators in fact doubly count the physical degrees of freedom. To heal this double counting, we need to restrict the dynamical variables to the physical sub-Hilbert space, which is subject to a constraint $\chi(q)|\text{phys} = 0$, where $\chi(q) = \sum_i \exp(-i\mathbf{q} \cdot \mathbf{R}_i)$ is the vortex density operator [12]. Reversing (2), the electron density operator can be expressed as $\rho_e(\mathbf{q}) = \sum_i \exp[-i\mathbf{q} \cdot (\mathbf{R}_c + c\mathbf{R}_v)]$.

In the lowest LL, the Hamiltonian is simply given by the two-body interaction Eq. (1). According to Eq. (2), the magnetic field felt by the CFs is reduced to $B/(2ps + 1)$. Since the CF density is equal to the density of electrons, the CFs now fill completely $p$ CF LLs. Therefore the huge degeneracy for fractional fillings is lifted and our calculation can be based on a state with these $p$ filled CF LLs. It serves as the reference state for further diagrammatic approaches, such as the Hartree-Fock approximation, which we use to calculate the activation gap given by

$$
\Delta = \langle p + PH|H^p|p + PH \rangle - \langle p|H^p|p \rangle,
$$

where $|p\rangle$ stands for the ground state with $p$ filled LLs and $PH$ symbolizes a widely separated quasiparticle-quasihole pair. In practice, the quasi-particle and quasihole are so far away that they have no correlations and both of their gaps (called as charged gaps) are computed individually in the single-particle/hole excitation Hilbert space. The activation gap is the sum of two,

$$
\Delta = \Delta_P + \Delta_H,
$$

where their explicit expressions are given by creating a particle in the $p$-th CF LL by applying $d_{p-1}^\dagger|p\rangle$ or a hole in the $(p - 1)$-th CF LL via $d_{p-1}|p\rangle$,

$$
\Delta_P = \langle p + p'|H^p|p + P \rangle = \langle p|d_p H^p d_p^\dagger|p\rangle,
$$

$$
\Delta_H = \langle p + H|H^p|p + H \rangle = \langle p|d_{p-1}^\dagger H^p d_{p-1}|p\rangle.
$$

The above picture naturally follows from the CF transformation. However the Hartree-Fock ground state $|p\rangle$ does not obey the physical constraint $\chi(q)|\text{phys} = 0$. If we plug in the projected Hamiltonian Eq. (1) and the electron density $\rho_e$, this naive procedure therefore suffers from strong corrections. A practical solution is to employ the preferred density $\rho^p = \rho_e - c^2\chi$ instead of $\rho_e$ in the Hamiltonian. This density has the merits of complying with Kohn’s theorem and giving the correct charge in the small-$q$ limit [12]. The Hamiltonian is finally written as

$$
H^p = \frac{1}{2} \sum_{\mathbf{q}} \rho^p(\mathbf{q}) V_{\text{eff}}(\mathbf{q}) \rho^p(-\mathbf{q}).
$$
and will serve as the starting point in our calculations presented in the following sections.

III. A VARIATIONAL METRIC IN THE HAMILTONIAN THEORY

A. The deformed liquid state

Quantum Hall systems with anisotropy are of particular interest due to possible phase transitions and competitions between them. In particular, the deformation of the Laughlin state is a good starting point for such systems. In this section, we show how Haldane’s geometric point of view [15] can be implemented in the Hamiltonian theory and provide interpretation of generalized pseudo-potentials in this language.

In an isotropic translationally invariant quantum Hall system, any two-body interaction $V$ can be expanded in terms of the relative-angular-momentum basis, $\hat{V} = \sum_m V_m |m\rangle\langle m|$, where $|m\rangle$ is the two-body state with relative angular momentum $m$ [16]. For fermions only odd values of $m$ are relevant to insure the antisymmetry of the wave function. The Laughlin state for filling factor $\nu = 1/q$ is characterized as the unique zero-energy eigenstate of a certain class of Haldane’s pseudo-potential [16], $H_M = \sum_{m,n=1}^{q-2} V_m P_m$ to the space with relative angular momentum $m$ [see Eq. (27) for explicit forms] and pseudo-potentials $V_m$ are arbitrary positive energy coefficients. In an anisotropic but translationally invariant system, the interaction is no longer diagonal in the angular-momentum basis, and a generalization of Haldane’s pseudo-potentials has recently been proposed to incorporate those off-diagonal parts $|m+n\rangle\langle m|$ ($n \neq 0$) [17]:

$$H_M = \sum_{m,n,\sigma} V_{m,n,\sigma}^\sigma P_{m,n}^\sigma. \qquad (8)$$

$\sigma = \pm$ is for symmetric and anti-symmetric combinations of $|m+n\rangle\langle m|$ and $|m\rangle\langle m+n|$ in order to make the interaction Hermitian. $P_{m,n,\sigma}^\sigma$ can be viewed as a complete basis for any two-body translationally invariant interaction $V_{\text{eff}}(\mathbf{q})$ and $V_{m,n,\sigma}^\sigma$ are the expanding coefficients. The diagonal components $P_{m,0,\sigma} = P_m$ are Haldane’s original pseudo-potentials and the other components are referred to as off-diagonal pseudo-potentials. In their presence, no model wave function is known as the zero-energy ground state. The off-diagonal parts serve as perturbations to the liquid state. Their relative magnitude with respect to the diagonal pseudo-potentials, which define the Laughlin state, provides good criterion for stability issues [13, 18].

It is pointed out by Haldane [15] that the quantum Hall system may have a hidden variational geometric parameter, parameterized by a metric. This can be manifested from the definition of Laughlin states according to pseudo-potentials. A metric $g$ can be introduced in its definition and therefore, a family of generalized Laughlin wave functions is obtained,

$$P_m(g)|\Psi_g(g)\rangle = 0, \quad m < q. \quad (9)$$

The exact form of this generalized Laughlin state has been constructed [19] by combining the metric with guiding-center coordinates to form ladder operators. For an isotropic system, the kinetic part and the interaction part of the Hamiltonian have the same flat metric. The effective interaction is therefore isotropic. The Laughlin state $|\Psi(g)\rangle$ with a flat metric is then the most favored state. When the kinetic part is perturbed by an anisotropic mass tensor or the interaction deviates from the isotropic case, the flat metric in the wave function will change to an anisotropic one in order to minimize the energy. Two approaches have been established to find the optimal metric in this situation. For example, in the presence of a parallel magnetic field, the optimal metric is obtained by finding the anisotropic Laughlin state $|\Psi(g)\rangle$ with highest overlap with the numerical exact diagonalization result [2]. Or it can be determined by finding the set of generalized pseudo-potentials with minimal coefficients for the off-diagonal components of $P_{m,n,\sigma}^\sigma(g)$ [13]. As we have mentioned below Eq. (8), any effective potential can be expanded in terms of $P_{m,n,\sigma}^\sigma(g)$, by varying $g$, the off-diagonal components may have minimum expansion coefficients.

Now we introduce the metric variable into the Hamiltonian theory formalism. As introduced in the previous section, the Hamiltonian theory is based on the fact that CFs fill $p$ LLs. For a charged particle under magnetic field, the cyclotron and guiding-center coordinates satisfy the following commutation relation:

$$[a^\dagger, a] = i\hbar^2 c_{ab}, \quad [R^a, R^b] = -i\hbar^2 c_{ab}. \qquad (10)$$

They can be combined with a complex vector $v_a$ to form ladder operators which define the CF LLs and their angular momenta,

$$\hat{a} = \frac{v_a R^a}{l_s}, \quad \hat{a}^\dagger = \frac{v_a^* R^a}{l_s}, \quad \hat{b} = \frac{v_b R^a}{l_s}, \quad \hat{b}^\dagger = \frac{v_b R^a}{l_s}. \qquad (11)$$

where the complex vector $v$ satisfies $v_a v_b^* - v_b v_a^* = -i\epsilon_{ab}$. Here and in the remainder of this paper, we use Einstein’s convention according to which we sum over repeated indices of co- and contra-variant vectors. A complete basis of the one-body Hilbert space is defined as $|m, n\rangle \propto (\hat{a}^\dagger)^m (\hat{b}^\dagger)^n |0\rangle$. By changing the form of this vector, the eigenstates of ladder operators exhibit different orbital shapes of $\eta$ and $R$. In general, the complex vector multiplied with the cyclotron coordinate and that with the guiding center coordinate are different, as is shown in the original idea by Haladane of anisotropic Laughlin states. However, in the Hamiltonian theory case, both of the two CF coordinates come from the guiding-center coordinates of electrons, because the pseudo-vortex is only an auxiliary variable composed of electronic degrees of freedom, so that $\eta$ and $R$ are related to the same vector
A metric \( g \) conserving the area \( \det(g) = 1 \) is associated with such a complex vector,
\[
g_{ab} = v_a v_b^* + v_a^* v_b. \tag{12}
\]
When \( v = (1, i)/2 \), the metric is flat and the corresponding ladder operators are those defined by Shankar and Murthy [12]. As one varies the vector \( v \), the metric associated with it is a direct reflection of the internal geometry.

The above construction can be viewed as building an anisotropic liquid phase from the Hamiltonian theory, parallel to the construction of anisotropic Laughlin wave functions [19]. There the metric is also parameterized by a complex vector. The polynomial factors are replaced by the creation operators \( b_i^\dagger \) of electronic guiding centers
\[
\Psi_{LLL}(g) = \prod_{i<j} [(\hat{b}_i^\dagger - \hat{b}_j^\dagger)]^g \Psi_{LLL}, \tag{13}
\]
where \( \Psi_{LLL} \) is the lowest Landau level wave function annihilated by \( \hat{a}_i \) and \( \hat{b}_i \) for every electron \( i \), \( \hat{a}_i \Psi_{LLL} = 0 \) and \( \hat{b}_i \Psi_{LLL} = 0 \). This definition is easily shown to be equivalent to Eq. (9). Since the cyclotron and guiding-center coordinates are true degrees of freedom, they are free to choose different metrics. The metric of the guiding center determines the geometry of the liquid state while the cyclotron metric determines the LL shape. In contrast, in the Hamiltonian theory, the electronic guiding-center metric induces that of the vortices. There is therefore a common metric for the CF cyclotron and guiding-center coordinates. The reference state of \( p \) filled CF LLs, built using anisotropic CF cyclotron coordinate, can thus be regarded as the operator description of the anisotropic Laughlin wave functions.

\section*{B. The response of the metric to anisotropy}

Now we study how this metric responds to anisotropy in the effective interaction. First, in the Hamiltonian theory the ladder operators \( \hat{a} \) and \( \hat{b} \) determine the matrix entries for the interaction \( V_{\text{eff}} \). Therefore, the first point we need to verify is the Hartree-Fock nature of the Hamiltonian theory, namely whether the anisotropy changes the structure of the Hartree-Fock ansatz. According to the CF construction, the ground state consists of \( p \) filled LLs. We need to know if the interaction \( V_{\text{eff}} \) causes inter-Landau level transitions and thus mixes different Landau levels. In the isotropic case, the answer is negative because, as we show below,
\[
\langle p | d_j V_{\text{eff}} d_i^\dagger | p \rangle \sim \delta_{ji}, \quad \text{for isotropic } V_{\text{eff}}(q) = V(q), \tag{14}
\]
i.e. on the Hartree-Fock level the states \( d_i^\dagger | p \rangle \) obtained by adding CF to state in an empty level, are eigenstates of the Hamiltonian. The state \( d_i^\dagger | p \rangle \) is obtained by adding a CF particle at an empty level \( i \). It is also possible to show that the above equation holds for creating a hole in filled levels. However, this relation strongly relies on the rotation invariance of the interaction \( V \). When rotation symmetry is broken, the interaction leads to mixing of LLs. And in the following we show that this mixing effect is proportional to the off-diagonal parts of the generalized pseudo-potentials.

As long as the anisotropy is small, one can still assume that the ground state is composed of \( p \) filled CF LLs because it is protected by the quasi-particle and quasi-hole gaps. Hamiltonian (7) is written in second quantization as
\[
H_p = \frac{1}{2} \sum_{1,2,3,4} V_{\text{eff}}(q) \rho_{12}^p(q) \rho_{34}^p (-q) d_1^\dagger d_2^\dagger d_3 d_4. \tag{15}
\]
The subscripts 1, 2, 3, 4 are abbreviations of \((m,n)\) corresponding to the CF states defined before. \( d_i \) and \( d_i^\dagger \) are annihilation and creation operators of CFs in this basis, and \( \rho_{ij}^p(q) \) is the matrix element of the preferred density operator:
\[
\rho_{ij}^p(q)_{mn;n',n''} = \langle m,n | \rho_c(q) - c^2 \chi(q) | m',n' \rangle. \tag{16}
\]
The above matrix has a product structure for the indices of angular momenta and Landau levels, \( \rho^p(q)_{m,n;m',n'} = \rho^p(q)_{m,m'} \otimes \rho^p(q)_{n,n''} \). With the Hartree-Fock assumption, the activation gap can be calculated using Wick’s theorem. The matrix element of the Hamiltonian between two one-particle excitation states is:
\[
\langle p | d_j V_{\text{eff}} d_i^\dagger p \rangle = \frac{1}{2} \int \frac{d^2 q}{4\pi^2} V_{\text{eff}}(q) \rho_{12}^p(q) \rho_{34}^p (-q) \times \sum_{1,2,3,4} [\delta_{j1}(1-n_1)\delta_{23}(1-n_2)\delta_{41}(1-n_4) \\
- \delta_{j3}(1-n_3)\delta_{14} n_4 \delta_{21}(1-n_2)], \tag{17}
\]
and a similar expression for hole excitations. Because of the product structure of \( \rho \), it can be shown that the interaction preserves the CF angular momentum. Using the identity \( \exp(-iq \cdot R) \times \exp(iq \cdot R) = 1 \), the above expression is thus proportional to \( \delta_{m_1,m_j} \). It only mixes different Landau levels of the composite fermion. By summing up all guiding center parts, one obtains:
\[
\langle p | d_j H_p d_i^\dagger p \rangle = \frac{1}{2} \int \frac{d^2 q}{4\pi^2} V_{\text{eff}}(q) \Theta(n_j-p) \Theta(n_i-p) \times \\
\sum_{n_2=p}^\infty \rho_{n_1 n_2}^p(q) \rho_{n_2 n_1}^p (-q) \\
- \sum_{n_1=0}^{p-1} \rho_{n_1 n_2}^p(q) \rho_{n_2 n_1}^p (-q) \delta_{m_j,m_i}, \tag{18}
\]
and a similar formula for hole excitations. \( \Theta \) is the step function with \( \Theta(x < 0) = 0 \) and \( \Theta(x \geq 0) = 1 \). In order to study the role of anisotropy in \( V_{\text{eff}} \), we need
In the second equality we use the property $g$ with the help of the inverse of the metric, where the contra-variant vector is obtained from expanding the density operator in terms of coordinates naturally appear in the exponent. Thus the expectation value can be written as:

$$\langle \hat{z}|e^{-i\eta \cdot \eta}|\hat{z}\rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{\hat{z}_{n_1}}{\sqrt{n_1!}} \frac{\hat{z}_{n_2}}{\sqrt{n_2!}} \langle n_2|e^{-i\eta \cdot \eta}|n_1\rangle. \tag{21}$$

On the other hand, the above expectation can be obtained from expanding the density operator in terms of ladder operators:

$$q_a \eta^a = q_a \delta^a_b \eta^b = q_a (v^a v^*_b + v^{a*} v_b) \eta^b = q_a v^a \hat{a}^\dagger + q_a v^{a*} \hat{a},$$

where the contra-variant vector $v^a = g^{ab} v_b$ is defined with the help of the inverse of the metric, $g^{ab} = (g^{-1})_{bc}$. In the second equality we use the property $g^{ac} g_{cd} = \delta^a_c \rightarrow v^a v^*_b + v^{a*} v_b = \delta^a_b$. So the expectation value can be written as:

$$\langle \hat{z}|e^{-i\eta \cdot \eta}|\hat{z}\rangle = \langle \hat{z}| \exp [-iu^a (q_+ a^\dagger + q_- a)] |\hat{z}\rangle = \langle \hat{z}| -iu^a q_+ + -iu^b q_- \rangle e^{q^2/2}/4 = \langle \hat{z}| -il^a (q_+ z + -il^a q_+ z - il^b q_-) \rangle e^{-q^2/2}/4, \tag{23}$$

where $q_+ = q_a v^a, q_- = q_a v^{a*}$. Comparing the above equation with Eq. (21), one obtains:

$$\langle n_2|e^{-i\eta \cdot \eta}|n_1\rangle = \sqrt{\frac{n_2!}{n_1!}} e^{-x/2} \left( -i q + t^* \right)^{n_1-n_2} L_{n_2}^{n_1-n_2}(x), \tag{24}$$

where $x = q_a q_b g^{ab} t^* /2$ and $L_n^m$ are associated Laguerre polynomials. The result is valid for $n_1 \geq n_2$. For $n_1 \leq n_2$, the matrix elements are obtained from the complex conjugation $\langle n_2|e^{-i\eta \cdot \eta}|n_1\rangle = \langle n_1|e^{i\eta \cdot \eta}|n_2\rangle$.

With these expressions for $\rho'(\mathbf{q})$ at hand, we are able to evaluate the Hamiltonian in one-particle/hole excitation space. The interaction preserves particle numbers. Therefore for the excitation of a CF, both the initial state and the final state should have the quasiparticle on an empty CF LL, such that there is no mixing between hole excitations and particle excitations. Now if we choose the metric $g_{ab}$ to be flat, i.e. $v = (1,i)/\sqrt{2}$, the reduced matrix behaves like $\rho'(\mathbf{q})_{n_1,n_2} \sim \exp [-i(\phi n_1 - \phi n_2)]$, where $\phi$ is the angle between $\mathbf{q}$ and the $x$-axis. Hence according to Eq. (18) we observe that

$$\langle p|d_H^\dagger d^\dagger H|p\rangle \propto \int d\phi V_{\text{eff}}(\mathbf{q}) e^{-i(n_f - n_i)\phi}. \tag{25}$$

This actually gives an angular decomposition of the effective potential, which means, if $V_{\text{eff}}$ is isotropic, there is no mixing between different single-particle excitation states, as mentioned above. The Hartree-Fock assumption is stable and self-consistent. However, if $V_{\text{eff}}$ is anisotropic, it will induce transition from one CF LL to another. When the transition is small, the anisotropy is simply a perturbation to the picture of $p$ filled CF LLs. When the transition is sufficiently large, the spacing between Landau levels may even close, in which case the liquid state is no longer stable.

In particular, we can also interpret this in the language of generalized pseudo-potentials. The explicit expressions of them are as follows [17]:

$$P_{m,n}^+(g) = \lambda_n N_{m,n} \left( L_m^m(|q|^2)e^{-\frac{1}{2}(|q|^2)q^2_{m} + c.c.} \right), \tag{26}$$

$$P_{m,n}^-(g) = -i N_{m,n} \left( L_m^m(|q|^2)e^{-\frac{1}{2}(|q|^2)q^2_{m} - c.c.} \right), \tag{27}$$

where $\lambda_n = 1/\sqrt{2}$ when $n = 0$ and $\lambda_n = 1$ for $n \neq 0$. $N_{m,n}$ is normalization for this expansion of two-body interaction. They are chosen to be real and the components $P_{m,n}^+$ with negative superscripts are only nonvanishing for $n \neq 0$ and therefore do not exist in an isotropic case. Further we have $|q|^2 = q_a q_b g^{ab}$ and $q_+$ is defined as before. When choosing the flat metric $g = 1$, $q_+ = |q|^2 e^{i\phi}/\sqrt{2}$, one can immediately observe that a generalized pseudo-potential $P_{m,n}^+$ introduces transition from a CF LL $n$ to $n' \pm n$. Notice, however, that since the CF does not have the same magnetic length as an electron, the off-diagonal pseudo-potential $P_{m,n}^+$ are not one-to-one correspondence to different LL transitions, but they rather mix all the CF LLs separated by a multiple of $n$. In Ref. [13], the optimal $g$ is chosen such as to minimize the off-diagonal components $n \neq 0$, because they are perturbations to the Haldane’s pseudo-potentials which define the liquid state. The Hamiltonian theory provides us a complementary picture: the off-diagonal pseudo-potentials cause transitions between CF LLs and thus destroy the Hartree-Fock nature of the ground state. However, the variation of the metric $g$ allows us to minimize CF LL mixing and thus to save the picture of $p$ filled CF LLs. Since this transition amplitude is proportional to the off-diagonal pseudo-potentials, the optimal metric obtained in this way agrees with that of Ref. [13], as we show in the following section.

Making use of the operator expression of $\rho(\mathbf{q})/\rho(-\mathbf{q})$,
IV. ACTIVATION GAP IN TILTED MAGNETIC FIELD

For an ideally 2D system, adding an in-plane magnetic field has no effect, apart from an enhancement of the global Zeeman effect, since it only causes movements in the restricted perpendicular direction. However, in realistic systems, usually the sample has a finite thickness. While the motion of the electrons remains restricted in the z-direction, they can respond to the parallel component of the magnetic field. Indeed, the electrons tends to perform its cyclotron motion in a plane perpendicular to the total magnetic field such that the plane of motion is tilted, as sketched in Fig. 2(a). The projection of the associated wave function is therefore an ellipse, as shown in Fig. 2(b). One thus notices already one origin of an anisotropy in the effective interaction potential. For a more quantitative analysis, we model the restriction in the z-direction by a parabolic confining potential $m\Omega z^2/2$, which, together with the kinetic energy in the perpendicular direction, needs to be added to the one-particle Hamiltonian,

$$H_z = \frac{\Omega_z^2}{2m} + \frac{m\Omega_z^2 z^2}{2}.$$  \hspace{1cm} (30)

Such a Hamiltonian is a quadratic form and can be solved exactly as a harmonic oscillator. To obtain the effective 2D motion, we project the system to the ground state of this harmonic oscillator. The projection changes the interaction between electrons in a similar way of projecting the physics to the lowest Landau level. In the absence of the in-plane magnetic field, the effective two-dimensional interaction has a cut-off at short distance:

$$V_\parallel(q) = \frac{2\pi e^2}{q} e^{(qbl)^2} \text{Erfc}(qbl),$$ \hspace{1cm} (31)

where $b = \omega_{\perp}/2\Omega$, $\omega_{\perp} = eB/m$ is the cyclotron frequency, and $bl$ is the typical length of this potential.

When the magnetic field is tilted towards the plane by an angle $\theta$, the cyclotron motion of the electron is also tilted towards the direction of the magnetic field, as already mentioned above. In addition to the elliptic shape, also the relevant overlap between the wave functions is drastically altered – while one maintains a substantial overlap between the wave functions of adjacent electrons in the y direction (perpendicular to the inplane component of the magnetic field), the tilted wave functions can be “stacked” in the z-direction and their overlap substantially reduced [see Fig. 2(c) and (d)]. As a consequence, the effective interaction is anisotropic with a space inversion symmetry. The effective interaction for the n-th Landau level of two-dimensional electrons under a tilted magnetic field can be written as (see Ref. [2, 21] and the
obtained by diagonalizing this matrix. First, we calculate $\nu$ for filling interaction only mixes CF LLs different by an even index. According to the discussion in last section, this anisotropic since the latter does not have an inversion symmetry. As the in-plane magnetic field picks a preferred direction and the interaction is inversion invariant, the system has two principal axes. Due to this symmetry, the ansatz of optimal metrics only mixes CF LLs different by an even index.

The above interaction is invariant under $q_x \rightarrow -q_x, q_y \rightarrow -q_y$. From the decomposition of generalized pseudopotentials, it has thus no component of $P_{r,s}^{\alpha}$ with $s$ odd since the latter does not have an inversion symmetry. According to the discussion in last section, this anisotropic interaction only mixes CF LLs different by an even index.

The spectrum of exciting a single particle/hole is obtained by diagonalizing this matrix. First, we calculate matrix elements in the flat metric case. For $\nu = 1/3$, the ground state has the $n = 0$ Landau level filled. Holes are created in the $n = 0$ Landau levels and particles are created in $n > 0$ Landau levels. The hole excitation has no mixing with the particle excitation because of particle number conservation. So the perturbation to the activation gap comes from the transition from $n = 1$ to higher CF Landau levels. According to the previous arguments, the first three states are deemed to be physical. As the system has an inversion symmetry, the lowest Landau level that perturbs the activation gap is the $n = 3$ level, at the brim of physical states. Therefore, to obtain the leading perturbation, it is reasonable to take only the transition from $n = 1$ to $n = 3$ into account. The relevant Landau level mixing amplitude is $M_{1,3} = M_{3,1}^*$. It is the direct reflection of the anisotropic effects.

Let us now add the variational metric to the Hamiltonian theory. The optimal metric is the one which minimizes the inter-Landau level transition amplitude $M_{1,3}$. As the in-plane magnetic field picks a preferred direction and the interaction is inversion invariant, the system has two principal axes. Due to this symmetry, the ansatz of the metric takes the form:

$$g_{ab} = \begin{pmatrix} \alpha^2 & 0 \\ 0 & 1/\alpha^2 \end{pmatrix}.\] (35)

The corresponding complex vector of this metric can be parameterized by $v_n = (\alpha, i/\alpha)/\sqrt{2}$. The results of $\alpha$ are shown in Fig. 3a. The optimal metric starts to deviate from one at around $\tan \theta = 0.5 (\theta = 27^\circ)$ for all thicknesses. As the anisotropic effects come from the
connect our calculation to tilted magnetic field experi-
however, that the deformation is always a small one. To
\[ \tan \theta \]
rapidly the optimal metric, which is approximately linear
ior in our results. The thicker the sample is, the more
prominent for large thickness. This is indeed the behav-
\[ z \]
ments, for a wide quantum well of width 50 nm [5], the
ratio between the confining frequency and the cyclotron
frequency is estimated to be \( \Omega/\omega_c = 2/|B| \). At the fill-
ing \( \nu = 1/3 \) and 2/5, the confining frequency can be
much smaller than the cyclotron frequency for such wide
quantum wells, \( \Omega/\omega_c \approx 0.1 \sim 0.2 \). So we set \( \Omega/\omega_c = 0.1 \)
as our lowest value in calculation. We see that even at
the extreme case, with \( \Omega/\omega_c = 0.1 \) and \( \tan \theta = 5 \), the
optimal metric deviates from the flat one by only 20%.
Therefore, the deformation remains small. This agrees
with the results of generalized pseudo-potential analysis
[13].

Figure 4a and 4b show our results for the activation
gap, for two different values of the confinement poten-
tial, as a function of the tilt angle. The yellow squares
indicate those calculated for the optimal metric evaluated
for each value of the tilt angle, in comparison with the
activation gaps obtained from a purely flat metric (blue
circles). We note that for moderate parameters, the inter-
Landau level transition \( M_{1,3} \) is always much smaller than
the Landau level spacing \( M_{1,1} + M_{0,0} \). For \( \Omega/\omega_c = 2 \) and
\( \theta = 35^\circ \), \( M_{1,3} = 0.002 \) while \( M_{1,1} + M_{0,0} = 0.147 \), the
former is only one percent of the latter. Notice that even
in the extreme case of a wide quantum well \( \Omega/\omega_c = 0.1 \)
with \( \tan \theta = 5 \), i.e. an inplane component of the mag-
netic field that is five times larger than its perpendicular
component, the ratio between \( M_{1,1} + M_{0,0} = 0.050 \) and
\( M_{1,3} = 0.017 \) is roughly 1/3 only, such that our vari-
ational metric can still cope with this situation. Hence
the activation is robust against the parallel magnetic field
— when the CFs pick the optimal metric, the activation
gap decreases much more slowly than that obtained from
a flat metric even for weak confining potential and large
tilting angles.

We also perform the same calculation for the \( \nu = 2/5 \)
filling [see Figs. 4c and 4d]. Among the physically rele-
vant CF LLs, the lowest one perturbing the activation
gap is the \( n = 4 \) level. The optimal metric, which we have
calculated for \( \Omega/\omega_c \) between 0.1 and 5, is almost the same
as \( \nu = 1/3 \) [see Fig. 3b]. This is in accordance with our
discussion in Sec. III.B. Indeed, all we are minimizing is
the \( \delta n = 2 \) transition, which is proportional to the \( F_{m,2}^\pm \)
pseudo-potential components. When the metric is chosen
to minimize these components, the transition amplitude
is minimal, so that the optimal metric for \( \nu = 1/3 \) and
\( \nu = 2/5 \) should be close. On the other hand, as the activ-
ation gap itself for \( \nu = 2/5 \) is smaller, the perturbation
due to the parallel magnetic field is stronger. One can
see that for large tilting, where the LL mixing amplitude
\( M_{2,4} \) is of the same magnitude as the Landau level spac-
ing \( M_{2,2} + M_{1,1} \), meaning that the Hartree-Fock ansatz
receives strong correction.

As a supplement to the truncation that we use, we also
verify the perturbations from higher LLs. Take \( \nu = 2/5 \)
for example, the mixing between the second and the sixth
LLs \( M_{2,6} \) is always one magnitude smaller than the mix-
ing between the second and the fourth LLs \( M_{2,4} \). In
table I, the extreme case of large tilting and thickness

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{(a) The activation gap for \( \nu = 1/3 \) of a weak
confining potential \( \Omega/\omega_c = 0.1 \). (b) The activation
gap for \( \nu = 1/3 \) of a strong confining potential \( \Omega/\omega_c = 5 \).
(c) The activation gap for \( \nu = 2/5 \) of a weak confining
potential \( \Omega/\omega_c = 0.1 \). (d) The activation gap for
\( \nu = 2/5 \) of a strong confining potential \( \Omega/\omega_c = 5 \).}
\end{figure}
is listed. The matrix elements $M_{2,4}$ and $M_{4,6}$ are comparable to the activation gap, but the higher transition $M_{2,6}$ is merely one tenth of them. So it is valid to neglect the mixing from higher LLs and restrict the perturbative approach to the first five LLs in order to illustrate the result qualitatively.

We finally remark that the above computation is only valid in the perturbation sense. The Hartree-Fock ground state of $p$ filled LLs should be calculated in a self-consistent way in order to give the accurate activation gap. However there is no such a self-consistent approach in FQH problems. Nonetheless, our calculation is valid when the inter-CF LL transitions are small, as reflected in our work. It provides a qualitative criterion for the robustness of activation gaps. In particular, in the computation of the optimal metric, we observe that when tuning the metric, the transition amplitude goes from positive values to negative values. So at some point we can turn off the transition from, for example, $n = 1$ to $n = 3$ Landau levels. This means for the optimal metric, the Hartree-Fock ansatz is self-consistent in the first few LLs.

### V. CONCLUSION

In this paper, we build an anisotropic liquid state from the Hamiltonian point of view. The CF LLs and angular momenta can be combined with a geometrical deformation of the Landau sites. A system under tilted magnetic field is studied to illustrate how the metric of this anisotropic liquid is deformed in response to external anisotropy. We find that our results can be directly connected to the generalized pseudo-potential studies, as the inter-CF LL transitions are proportional to the off-diagonal components of generalized pseudo-potentials. For $\nu = 1/3$ and $\nu = 2/5$ states, they almost have the same optimal metric, defined as the minimal inter-LL transitions, implying the underlying off-diagonal pseudo-potentials. In the variation of the metric, we also find the off-diagonal transitions turn from positive to negative, indicating that they actually vanish at some specific metric. So when we take this metric, the CF description is almost self-consistent.

Furthermore, we find the activation gap is robust to the parallel magnetic field in the lowest Landau level, both for the flat metric and optimal metric. Only in very extreme case $\tan \theta > 3$ the activation gap is substantially modified in wide quantum wells. When the optimal metric is taken into account instead of the flat one, the activation gap is almost unaffected by the tilt-induced anisotropy in the effective interaction potential, exhibiting a great stability of the liquid phase, which is a reminiscent of the observation of enhanced FQHE in small tilting angle [5]. Even in wide quantum wells with an strong inplane component of the magnetic field, the activation gap decreases at $\nu = 1/3$ and $2/5$ by only $10 \ldots 20\%$.

In our study, the metric is uniform in both space and time, treated as a parameter rather than a dynamical variable. From the point view of a recently proposed bimetric theory [22, 23], the metric computed here can be understood as the ambient metric. How to build such a dynamical metric in the Hamiltonian theory will be an attractive extension of the construction here.

We also notice that in a recent work [14], the authors numerically calculated the magneto-roton gap for QH systems under tilted magnetic field. There they show that the magneto-roton gap for filling $\nu = 1/3$ closes for $\tan \theta > 6$ and the structure factor exhibits charge density waves. We emphasize that this is not in contradiction with respect to our results on the activation gap since the latter reflects the energy costed to activate a widely separated quasi-particle/hole pair. The anisotropy of the effective interaction potential is to great extent averaged in its computation. However, for wavevector dependent quantities, such as the magneto-roton excitation, the anisotropy manifests itself more directly. The magneto-roton gap is then likely to close before the charge gap, corresponding to a charge-density-wave instability with a well-defined wave vector. Combined with our result, this may be a signature of nematic phase transition [24], where the magneto-roton gap closes while the charged gap remains. The QH system under tilted magnetic field is thus a possible candidate to observe the nematic order.

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### Appendix: The effective interaction in the presence of a parallel magnetic field

As mentioned in the main text, the Hamiltonian of a quasi-2D system with a parabolic confinement potential in the $z$-direction is given by

$$H_{\text{conf}} = \sum_{i=x,y,z} \frac{(p_i + eA_i)^2}{2m} + \frac{m\Omega^2 z^2}{2}. \quad (A.1)$$

The gauge potential is chosen as $(0, -zB \tan \theta - xB, 0)$ in the Landau gauge. Neglecting the Coulomb interaction, the Hamiltonian is both quadratic in momenta and
positions. It can be transformed into a canonical form by a coordinate transformation:

\[
H_{\text{conf}} = \frac{p_x^2}{2m} + \frac{m\omega_x^2 \xi^2}{2} + \frac{p_y^2}{2m} + \frac{m\omega_y^2 \zeta^2}{2}.
\]  
(A.2)

The coordinates \(\xi\) and \(\zeta\) are related to the original coordinates by:

\[
\begin{pmatrix}
\xi \\
\zeta
\end{pmatrix} = \begin{pmatrix}
\cos \tilde{\theta} & -\sin \tilde{\theta} \\
\sin \tilde{\theta} & \cos \tilde{\theta}
\end{pmatrix} \begin{pmatrix}
x \\
z
\end{pmatrix},
\]  
(A.3)

where \(\tan 2\tilde{\theta} = \tan 2\theta/(1 - \tan^2 \theta - \Omega^2/\omega^2)\). Now the Hamiltonian is in its canonical form and describes two decoupled harmonic oscillators, one for the confined motion in the \(z\)-direction and another one for the cyclotron motion in the external magnetic field. Their corresponding frequencies are given by

\[
\frac{\omega_\pm^2}{\omega_c^2} = \frac{\lambda^2 + 1}{2} \pm \frac{\lambda^2 - 1}{2} \cos 2\tilde{\theta} \mp \tan \theta \sin 2\tilde{\theta},
\]  
(A.4)

where \(\lambda^2 = (\tan^2 \theta + \Omega^2/\omega_c^2)\). The two harmonic oscillators naturally define two length scales \(L^2_\pm = (\omega_c/\omega_\pm) L^2\).

With such a decoupled Hamiltonian, the one-body states are denoted as \(|N, n, m\rangle\), where the first index corresponding to the higher frequency represents the level in the confining potential and the \(n\) represents the cyclotron level, while the quantum number \(m\) denotes the LL degeneracy (associated with the guiding-center coordinate). The lowest LL is naturally the \(|0, 0, m\rangle\). To focus on the physics inside the lowest LL, the density operator is projected into this state:

\[
\rho(q_x, q_y, q_z) = \sum_{m, m'} \langle 0, 0, m|e^{-iq \cdot r}|0, 0, m'\rangle c_{m'}^\dagger c_m.
\]  
(A.5)

The indices of higher LLs depend on the exact strength of the confining potential: when \(\Omega \rightarrow \infty\), the level \(|0, n, m\rangle\) becomes the usual \(n\)-th LL for 2D electrons.

Inserting them into the interaction part of the Hamiltonian, after integrating out the \(z\) coordinate, we obtain the effective two-dimensional interaction:

\[
V_{\text{eff}}(q) = \int dq_x \frac{4\pi e^2}{2\pi} \sum_{n} \frac{1}{q^2} \left[ \frac{\Delta q_x^2 \sin^2 \tilde{\theta}}{\ell_n^2} + \frac{\Delta q_x^2 \cos^2 \tilde{\theta}}{\ell_n^2} \right] \times e^{-\frac{1}{2} \left( \frac{\sqrt{\Delta q_x^2 \sin^2 \tilde{\theta}}}{\ell_n^2} + \frac{\sqrt{\Delta q_x^2 \cos^2 \tilde{\theta}}}{\ell_n^2} \right)}.
\]  
(A.6)

In Ref. [17], the authors integrate out the above expression in an arbitrary gauge and the effective potential is expressed in terms of special functions. Here we stick to the integration expression in which a very practical Gaussian integral is available in numerical computations.

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