ON THE COHOMOLOGY OF REAL GRASSMANN MANIFOLDS

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Abstract. We give an explicit and simple construction of the incidence graph for the integral cohomology of real Grassmann manifold \( \text{Gr}(k, n) \) in terms of the Young diagrams filled with the letter \( q \) in checkered pattern. It turns out that there are two types of graphs, one for the trivial coefficients and other for the twisted coefficients, and they compute the homology groups of the orientable and non-orientable cases of \( \text{Gr}(k, n) \) via the Poincaré-Verdier duality. We also give an explicit formula of the Poincaré polynomial for \( \text{Gr}(k, n) \) and show that the Poincaré polynomial is also related to the number of points on \( \text{Gr}(k, n) \) over a finite field \( \mathbb{F}_q \) with \( q \) being a power of prime which is also used in the Young diagrams.

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1. Introduction

The Grassmann manifolds (sometimes, referred to as simply the Grassmannian) which parameterize vector subspaces of fixed dimensions of a given vector space are fundamental objects, and appear in many areas of mathematics. Their study dates back to Plücker who, in the 19th century, considered the case of vector subspaces of dimension \( k = 2 \) of a space of dimension \( n = 4 \). In this note we study the integral cohomology of real Grassmann manifolds. The cohomology of complex Grassmann manifolds is well-known but the real case poses additional challenges. We give an explicit description of the Betti numbers and the incidence graph which leads to the integral cohomology.

There is a well-known decomposition of a real Grassmann manifold into Schubert cells. The Schubert cells can be parametrized with Young diagrams and these Young diagrams will then also appear in a co-chain complex to calculate the cohomology. In this note the Young diagrams are filled in with a letter \( q \) in a checkered pattern and this gives rise to a power of \( q \) for each Young diagram and therefore to a polynomial by considering certain alternating sum. This polynomial is

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computed explicitly and, it is shown to contain the information of all the Betti numbers of the real Grassman manifolds. That is, it gives the Poincaré polynomial of the manifold. There is also an alternative construction of these polynomials in terms of varieties over a finite field $\mathbb{F}_q$ where this time $q$ is a power of a prime number. One can consider varieties over $\mathbb{F}_q$ that naturally correspond to the real Grassmann manifolds and then count the number of $\mathbb{F}_q$ points. This again gives rise to a polynomial in $q$ which agrees with the polynomial that was obtained directly from the Young diagrams.

The results in this paper were initially motivated by considering real Grassmann manifolds in terms of the KP hierarchy [5].

2. The real Grassmann manifold $\text{Gr}(k, n)$

The real Grassmannian $\text{Gr}(k, n)$ is defined by

$$\text{Gr}(k, n) := \{ \text{the set of all } k \text{-dimensional subspaces in } \mathbb{R}^n \}.$$ 

Let $\{e_j : j = 1, \ldots , n\}$ be a basis of $\mathbb{R}^n$, and $\{f_i : i = 1, \ldots , k\}$ a basis of $k$-dimensional subspace $V_k(e_1, \ldots , e_n)$. Then there exists a full rank $k \times n$ matrix $A$ such that we have

$$(f_1, \ldots , f_k) = (e_1, \ldots , e_n)A^T,$$

where $A^T$ is the transpose of the matrix $A$. Since the left action of $\text{GL}_k(\mathbb{R})$ on $A$ does not change the subspace $V_k$, $\text{Gr}(k, n)$ can be expressed as

$$\text{Gr}(k, n) \cong \text{GL}_k(\mathbb{R}) \setminus M_{k \times n}(\mathbb{R}),$$

where $M_{k \times n}(\mathbb{R})$ is the set of all $k \times n$ matrices of rank $k$. The $\text{GL}_k(\mathbb{R})$-action puts $A$ in the canonical form called the reduced row echelon form (RREF). For example, the RREF for the generic matrix has the form,

$$A = \begin{pmatrix} * & \cdots & * & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ * & \cdots & 0 & \cdots & 1 \end{pmatrix} \in \text{Gr}(k, n).$$

The 1’s in the matrix are called the pivots of the RREF. In general, each $k$-element subset $I = \{i_1, \ldots , i_k\}$ of $[n] = \{1, 2, \ldots , n\}$ is called the Schubert symbol and represents the index set of the pivots for the matrix $A$. Then the set of all $k$-element subsets $\binom{[n]}{k}$ gives a parametrization for the Schubert decomposition of $\text{Gr}(k, n)$, i.e.

$$\text{Gr}(k, n) = \bigsqcup_{I \in \binom{[n]}{k}} \Omega^I_f,$$

where $\Omega^I_f$ is the Schubert cell, and $A \in \Omega^I_f$ means that the RREF of $A$ has the pivot index $I$.

2.1. The Schubert decomposition of $\text{Gr}(k, n)$ and the Bruhat poset $S_n^{(k)}$. Let $S_n^{(k)}$ denote the set of minimal coset representatives defined by (see [1])

$$S_n^{(k)} := \{ t \in S_n : \ell(ts) \leq \ell(t), \forall s \in P_k \},$$

where $\ell(s)$ is the length of $s$, and $P_k$ is the maximal parabolic subgroup generated by

$$P_k := \langle s_1, \ldots , s_{k-1} \rangle \cong S_k \times S_{n-k}.$$

Since there is a bijection between $\binom{[n]}{k}$ and $S_n^{(k)}$, the Schubert decomposition can be also expressed in terms of $S_n^{(k)}$, i.e.

$$\text{Gr}(k, n) = \bigsqcup_{w \in S_n^{(k)}} \Omega^w_w,$$
where we identify $\Omega^2_w = \Omega^0_w$. More precisely, we have $I = \{i_1, i_2, \ldots, i_k \} = w \cdot \{1, 2, \ldots, k\}$, i.e. $i_j = w(j)$. Note that the dimension of $\Omega^0_w$ is given by $\dim(\Omega^0_w) = \ell(w)$.

We label the Schubert cell $\Omega^0_w$ for $w \in S_n^{(k)}$ by the Young diagram $Y_w$. Given a Schubert symbol $\{w(1) < \cdots < w(k)\}$, first define $\lambda := (\lambda_1, \ldots, \lambda_r)$ with $\lambda_j = w(j) - j$ (note $\lambda_1 \leq \cdots \leq \lambda_r$) and $|\lambda| := \sum_{j=1}^r \lambda_j$. This $\lambda$ gives a partition of number $|\lambda|$. Then the Young diagram with shape $\lambda$ is a collection of $|\lambda|$ top-left-justified boxes with $\lambda_j$ boxes in the $j$-th row from the bottom as shown below:

$$
\begin{array}{c|c|c|c|c|c|c}
  k+1 & k+2 & \cdots & k+\lambda_k & \cdots & n \\
  \hline
  k & s_k & s_{k+1} & \cdots & s_{i_k-1} & i_k = w(k) \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  j & s_j & s_{j+1} & \cdots & s_{i_j-1} & i_j = w(j) \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  1 & s_1 & s_2 & \cdots & s_{i_1-1} & i_1 = w(1) \\
  \hline
  1 & 2 & \cdots & i_1-1 \\
\end{array}
$$

In the diagram, one can consider a lattice path starting from the bottom left corner and ending to the top right corner with the label $\{1, 2, \ldots, n\}$ in the counterclockwise direction, then the indices in the Schubert symbol $I = \{w(1) < \cdots < w(k)\}$ appear at the vertical paths. In the case of $I = \{n-k+1, \ldots, n\}$, i.e. the top cell, corresponding to the longest element $w_0 \in S_n^{(k)}$, we write $Y_{w_0} = (n-k)^k$ representing the $k \times (n-k)$ rectangular diagram.

Here a reduced expression of $w$ can be written by

$$
w = w_1 w_2 \cdots w_k \quad \text{with} \quad w_j := s_{i_j-1} \cdots s_j, \quad j = 1, \ldots, k,
$$

where each $w_j$ is the product of $s_j$ in the $j$-th row from the bottom. Note that each $Y_w$ is a sub-diagram of the $k \times (n-k)$ of the top cell, and we have the inclusion relation,

$$
Y_w \subseteq Y_{w'} \quad \iff \quad w \leq w'.
$$

(Notice the direction of the Bruhat order in this paper.) We then define:

**Definition 2.1.** Based on a weak Bruhat order, we define the weak Bruhat graph,

$$
B(k, n) := \{ (Y_w, \to) : w \in S_n^{(k)} \}.
$$

That is, the vertices are given by the Young diagrams $Y_w$, and the edges are defined by

$$
Y_w \rightarrow Y_{w'} \quad \text{if} \quad w' = s_j w \quad \text{with} \quad \ell(w') = \ell(w) + 1,
$$

for some $s_j \in S_n$. This graph is also referred to as Young’s lattice of $\text{Gr}(k, n)$.

Below we define additional graphs $G(k, n)$, $G_r(k, n)$ whose vertices have parameters involving the Young diagrams $Y_w$. The blanket notational convention is that if there is an edge in any of these graphs involving a pair $w, w'$ then there is a simple reflection $s_j$ such that $w' = s_j w$ with $\ell(w') = \ell(w) + 1$. These graphs can therefore be considered as subgraphs of the weak Bruhat graph even if this is not explicitly restated.
Example 2.1. The graph $B(2, 5)$ is given by

Each arrow $\rightarrow$ represents the action $s_j$ to the pivot set, and new box added by this action is labeled by $s_j$. For example, $s_2 : (1, 2) \rightarrow (1, 3)$ in the top left one. Notice that a reduced expression of $w \in S_{n}^{(k)}$ can be obtained by following the paths $\rightarrow$ starting from $\emptyset$ to the diagram $Y_w$. For example, the diagram at the right side in the second row can be written by $w = s_1 s_3 s_2 = s_4 s_3 s_1 s_2 = s_4 s_1 s_3 s_2$, depending on the three different paths.

3. The integral cohomology of $Gr(k, n)$ and the incidence graph

Let us first denote the Schubert variety marked by $Y_w$ as

Let $\sigma_w$ denote the Schubert class associated to $\Omega_w$, i.e. $\sigma_w := [\Omega_w]$. Then the co-chain complex of $Gr(k, n)$ is given in the form,

The co-boundary operator, $\delta_j : C^j \rightarrow C^{j+1}$, gives the form,

where the coefficient $[w; w']$ is called the incidence number and it takes either 0 or $\pm 2$ (see e.g. [9, 6]). We denote the incidence graph of $Gr(k, n)$ by

Then identifying $\sigma_w$ with the Young diagram $Y_w$ in the Bruhat graph $B(k, n)$, the arrow $\rightarrow$ in the Bruhat graph $B(k, n)$ can be considered as the action of the co-boundary operator $\delta_j$. The incidence graph $G(k, n)$ is then obtained by giving specific information on the incidence numbers for those arrows. The main result of this paper is to provide a "simple" representation of the incidence graph in terms of the Bruhat graph with those specific incidence numbers: For this purpose, let us first define:

Definition 3.1. We define a $q$-weighted Schubert variety $\langle Y_w; \eta(w) \rangle$ where $\eta(w)$ is defined as follows:

1. Insert the letter $q$ and 1 alternatively in the boxes of the Young diagram $Y_w$, such that $q$ locates at the northwest corner box of the diagram. That is, the first box with $s_k$ has $q$, then the boxes with $s_{k \pm 2j}$ have $q$, and the boxes with $s_{k \pm (2j+1)}$ have 1.
(2) Then $\eta(w)$ for each diagram $Y_w$ filled with 1 and $q$ in the previous step is defined by

$$\eta(w) = \text{the total number of } q\text{'s in the diagram } Y_w.$$ 

We will also need to refer to the $q$ weighted Schubert variety $\langle Y_w; \eta^*(w) \rangle$ defined in exactly the same way but now the boxes with $s_{k \pm 2j}$ have 1, and the boxes with $s_{k \pm (2j+1)}$ have $q$. To make a distinction, the first arrangement of $q$'s in the Young diagram will be referred to as standard and the second as $q$-shifted. We will show in subsection 3.1 below that the monomial $q^{\eta(w)}$ can be expressed in terms of the Hecke algebra action, a $q$-deformation of the Weyl group $W$.

For each $w \in S_n^{(k)}$, one can easily find the formula of $\eta(w)$.

**Lemma 3.1.** The function $\eta(w)$ is given by

$$\eta(w) = \sum_{j=1}^{k} \left\lfloor \frac{w(j) - j + \sigma(j)}{2} \right\rfloor$$

where $\sigma(j) = 1$ if $k - j = \text{even}$, and $\sigma(j) = 0$ if $k - j = \text{odd}$.

**Definition 3.2.** We define a graph $G(k, n)$ for the $q$-weighted Schubert varieties by modifying the Bruhat graph $B(k, n)$,

$$G(k, n) = \{ (\langle Y_w; \eta^*(w) \rangle, \Rightarrow) : Y_w \Rightarrow Y_w' \text{ if } \eta(w) = \eta^*(w') \}.$$ 

That is, identifying each Young diagram with $q$'s as the $q$-weighted Schubert variety, and replacing the edge $\rightarrow$ between $Y_w$ and $Y_w'$ in the Bruhat graph $B(k, n)$ with $\Rightarrow$ if $\eta(w) = \eta^*(w')$. 

There is an alternative graph $G^*(k, n)$ in which an edge $\Rightarrow$ corresponds instead to $\eta^*(w) = \eta^*(w')$, i.e.

$$G^*(k, n) = \{ (\langle Y_w; \eta^*(w) \rangle, \Rightarrow) : Y_w \Rightarrow Y_w' \text{ if } \eta^*(w) = \eta^*(w') \}.$$ 

**Example 3.1.** The graph $G(2, 5)$ is given by

\[
\begin{array}{ccc}
\emptyset & \rightarrow & q \\
\downarrow & & \downarrow \\
q & \Rightarrow & q \, 1 \\
\downarrow & & \downarrow \\
q \, 1 & \Rightarrow & q \, 1 \, q \\
\downarrow & & \downarrow \\
q \, 1 \, 1 & \Rightarrow & q \, 1 \, q \\
\downarrow & & \downarrow \\
q \, 1 \, q & \Rightarrow & q \, 1 \, q \\
\downarrow & & \downarrow \\
q \, 1 \, q \, 1 & \Rightarrow & q \, 1 \, q \, 1 \\
\end{array}
\]

Then we have the main theorem of this paper:

**Theorem 3.2.** The graph $G(k, n)$ with the edges $\Rightarrow$ having the incidence number $\pm 2$ (others are zero) gives the incidence graph $G(k, n)$ of the real Grassmann manifold $Gr(k, n)$.

Based on this theorem, we can compute the $\mathbb{Z}$-cohomology of $Gr(k, n)$. For example, from the above example, one can easily compute the $\mathbb{Z}$-cohomology of $Gr(2, 5)$, and we have

\[
H^0 = \mathbb{Z}, \quad H^1 = 0, \quad H^2 = \mathbb{Z}_2, \quad H^3 = \mathbb{Z}_2 \\
H^4 = \mathbb{Z} \oplus \mathbb{Z}_2, \quad H^5 = 0, \quad H^6 = \mathbb{Z}_2
\]

Note $Gr(2, 5)$ is not orientable. This is always true if $M$ is odd, that is, we have:
Corollary 3.3. $\text{Gr}(k, n)$ is orientable iff $n$ is even.

Proof. Since the box at the southeast corner of the top Young diagram, $(n-k)^k$, has the letter $q$ iff $n$ is even. This implies the incidence number $[w; w_0] = 0$ for the longest element $w_0$. □

3.1. Outline of the proof of the main theorem. The proof of Theorem 3.2 relies on the description given in [4] on the incidence graph for real flag manifolds. In [4], each local system on the real flag manifold is parametrized in terms of a vector of signs $\epsilon = (\epsilon_1, \cdots, \epsilon_{n-1}) \in \{\pm\}^{n-1}$, which encodes the structure of the local system (see Definition 4.3 of [4]). The sign $\epsilon_i = -$ indicates that the local system is constant along $s_i$ (restricted to a fiber which corresponds to the flag manifold of an $\text{SL}_2(\mathbb{R})$). Similarly, the sign $\epsilon_i = +$ corresponds to the local system which is twisted along $s_i$.

In the case of cohomology with constant coefficients, we define an action of the Weyl group on the signs starting with $(-, \cdots, -)$ representing a constant local system:

Definition 3.3. [4] Definition 2.10] Each simple reflection $s_i \in W$ acts on the sign $\epsilon_j$ in the vector $\epsilon = (\epsilon_1, \cdots, \epsilon_{n-1}) \in \{\pm\}^{n-1}$ as

$$s_i : \epsilon_j \to \epsilon_j \epsilon_i C_{ij},$$

where $C_{ij}$ is the Cartan matrix of $\mathfrak{sl}_n(\mathbb{R})$. That is, $s_i(\epsilon_i) = \epsilon_i$, $s_i(\epsilon_{i \pm 1}) = \epsilon_{i \pm 1} \epsilon_i$, and $s_i(\epsilon_j) = \epsilon_j$ for $|i - j| \geq 2$.

With the $W$-action on $\{\pm\}^{n-1}$, we also define a number $\hat{\eta}(w, \epsilon)$ with $\hat{\eta}(e, \epsilon) = 0$ by

$$\hat{\eta}(s_i w, \epsilon) = \begin{cases} \hat{\eta}(w, \epsilon) & \text{if } s_i(w(\epsilon)) = w(\epsilon), \\ 1 + \hat{\eta}(w, \epsilon) & \text{if } s_i(w(\epsilon)) \neq w(\epsilon). \end{cases}$$

The $\hat{\eta}(w, \epsilon)$ is then simply the number of times that the simple reflections in a fixed reduced expression of $w$ have acted non-trivially (changed a sign) starting with $\epsilon = (\epsilon_1, \cdots, \epsilon_{n-1})$ (see Definition 4.6 and Proposition 4.2 of [4]). The numbers $\hat{\eta}(w, \epsilon)$ have the algebraic description in terms of the Hecke algebra $\mathcal{H} = \mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[W]$ (see Section 4 of [4] for the details).

We now recall the connection with the incidence graph $\mathcal{G}(k, n)$. There is an edge $\Rightarrow$ between two vertices, $\sigma_w \Rightarrow \sigma_{w'}$, related by a simple reflection $s_i$ exactly when the $s_i$-action does not change the signs, i.e., $w(\epsilon) = s_i w(\epsilon)$ starting with $\epsilon = (-, \cdots, -)$. Then defining a weighted Schubert class by $\langle \sigma_w; \hat{\eta}(w, \epsilon) \rangle$, we have the following theorem which is proven in [6], and also in Theorem 4.9 in [4]:

Theorem 3.4. With $\epsilon = (-, \cdots, -)$, the incidence graph $\mathcal{G}(k, n)$ is expressed by

$$\mathcal{G}(k, n) = \{(\langle \sigma_w; \hat{\eta}(w, \epsilon) \rangle, \Rightarrow) : \sigma_w \Rightarrow \sigma_{w'} \text{ if } \hat{\eta}(w, \epsilon) = \hat{\eta}(w', \epsilon)\},$$

that is, one can replace $[w; w'] \neq 0$ with the equation $\hat{\eta}(w, \epsilon) = \hat{\eta}(w', \epsilon)$.

For $\text{Gr}(k, n)$ there are exactly two initial signs $\epsilon$ corresponding to local systems on $\text{Gr}(n, k)$. Let $\epsilon_\pm$ denote those initial signs which are defined by

$$\epsilon_\pm := (-, \cdots, -, \pm, -, \cdots, -).$$

As explained above, $\epsilon_-$ represents a constant local system, while $\epsilon_+$ represents a non-constant local system. Then we have:

Lemma 3.5. For any $w \in S_n^{(k)}$, one can show that

(a) $w(\epsilon_-) = s_i w(\epsilon_-)$ if and only if $\eta(w) = \eta(s_i w)$, and
(b) $w(\epsilon_+) = s_i w(\epsilon_+)$ if and only if $\eta^*(w) = \eta^*(s_i w)$. 

Proof. We give an outline of the proof by induction on $n$ using the weak Bruhat graph $B(k, n)$ (see Example 2.1). We first consider the partial graph consisting of the Young diagrams $Y_w$ with one row, i.e. $w = s_{k+m-1} \cdots s_{k+1}s_k$, or in terms of the Schubert symbol $w \cdot \{1, \ldots, k\}$ which is

$$\{1, 2, \ldots, k-1, k+m\} \quad \text{for} \quad m = 0, \ldots, n-k.$$ 

In terms of the signs representing local systems, we compute $w(\epsilon_-)$ with $\epsilon_- = (-, \ldots, -)$. It is easy to check with a direct calculation that Lemma 3.5 holds for this partial graph. One recovers the incidence graph of $Gr(1, n-k+1) \cong \mathbb{R}P^{n-k}$ in this first step. This graph has an alternating pattern in $\Rightarrow$. This edges $\Rightarrow$ can be simultaneously described in terms of agreement of $\eta(w)$ and in terms of the $W$-action on signs. Lemma 3.5 is then true for the edges in this partial graph.

If one starts with $\epsilon_+$ then the graph starts with $\Rightarrow$ and this corresponds to a $q$-shifted arrangement which is the the incidence graph of $Gr(1, n-k+1)$ with local coefficients.

We can now decompose the graph $B(k, n)$ into a disjoint union of sub-graphs. Each sub-graph consists of the Young diagrams with the same first row having $m$ boxes for $m = 1, \ldots, n-k$. This sub-graph can be identified with $B(k-1, k+m-1)$ after removing the first common top row (in the Example 2.1 those subgraphs correspond to the columns of $B(2, 5)$). We note, however, that if start with a Young diagram with a standard $q$ arrangement, the Young diagrams in the decomposition, excluding that the common top row, correspond to $Gr(k-1, k+m-1)$ but are $q$-shifted. This is a trivial observation related to the fact that $q$’s are placed in an alternating pattern and we are starting from the second row which now has $q$ and not 1 at the beginning. We thus have a decomposition into a disjoint union of subgraphs associated to $Gr(k-1, k+m-1)$. Then by induction, Lemma 3.5 holds for the subgraphs and this allows us to complete the proof. \hfill \Box

This Lemma with the definition of $\eta(w, \epsilon)$ then leads to the following Proposition:

Proposition 3.6. We have $\eta(w, \epsilon_-) = \eta(w)$ and $\eta(w, \epsilon_+) = \eta^*(w)$.

This Proposition gives the proof of Theorem 3.2. That is, we have $\mathcal{G}(k, n) = G(k, n)$ by identifying $\langle \sigma_w; \eta(w, \epsilon_-) \rangle$ with $\langle Y_w; \eta(w) \rangle$. Also, for $G^*(k, n)$, we have the incidence graph $\mathcal{G}^*(k, n)$ corresponding to a non-constant local system, i.e.

$$\mathcal{G}^*(k, n) := \{\langle \sigma_w; \eta(w, \epsilon_+) \rangle; \Rightarrow \} : \sigma_w \Rightarrow \sigma_w' \text{ if } \eta(w, \epsilon_+) = \eta(w')\epsilon_+\}.$$ 

As the simplest but important example (used in the proof as the first step), we give the incidence graph of $Gr(1, n) \cong \mathbb{R}P^{n-1}$ whose the longest element is $w_o = s_{n-1} \cdots s_2s_1$. The graph $G(1, n)$ is given by

$$\emptyset \xrightarrow{s_1} q \xrightarrow{s_2} q \mathbf{1} \xrightarrow{s_3} q \mathbf{1} \mathbf{1} \xrightarrow{s_4} q \mathbf{1} \mathbf{1} \mathbf{1} \quad \Rightarrow \cdots$$

Recall that the boxes marked by $s_m$ with odd $m$ has $q$ and others have 1. Then we recover the well-known results of the integral cohomology of $\mathbb{R}P^{n-1}$,

$$H^k(\mathbb{R}P^{n-1}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } n-1 = \text{ odd} \\ \mathbb{Z}_2 & \text{if } k = \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Also the graph $G^*(1, n)$ is given by

$$\emptyset \xrightarrow{s_1} 1 \xrightarrow{s_2} 1 \mathbf{q} \xrightarrow{s_3} 1 \mathbf{q} \mathbf{1} \xrightarrow{s_4} 1 \mathbf{q} \mathbf{1} \mathbf{1} \quad \Rightarrow \cdots$$

Then we have the cohomology with twisted coefficient $\mathcal{L}$,

$$H^k(\mathbb{R}P^{n-1}; \mathcal{L}) = \begin{cases} \mathbb{Z} & \text{if } k = n-1 = \text{ even} \\ \mathbb{Z}_2 & \text{if } k = \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$
As we will show in Section 5, this cohomology is related to the homology of the non-orientable case $\mathbb{R}P^{n-1}$ with $n =$odd via the Poincaré-Verdier duality, i.e.

$$H_k(\mathbb{R}P^{n-1}; \mathbb{Z}) = H^{n-1-k}(\mathbb{R}P^{n-1}; \mathbb{L}).$$

4. The $F_q$-points on $\text{Gr}(k, n)$

Let $F_q$ be a finite field with $q$ elements. Here $q$ is a power of a prime. Then based on the $q$-weighted Schubert variety defined in the previous section, one can find the number of $F_q$ points on $\text{Gr}(k, n)$. We first recall that in the complex case, the $F_q$-points of $\text{Gr}(k, n)$ over different fields such as the algebraically closed fields fields such as $\mathbb{C}$, $F_q$, or the finite field $F_q$. This allows us to consider $\text{Gr}(k, n)$ over different fields such as the algebraically closed fields fields such as $\mathbb{C}$, $F_q$, or the finite field $F_q$. We refer to as the $F_q$-points on $\text{Gr}(k, n)$. Here we give an explicit formula of the $F_q$-points in terms of $\eta(w)$ describing the number of $q$’s in the Young diagram $Y_w$. Let us first give:

**Definition 4.1.** We define a polynomial $p(q)$ by

$$p(q) = (-1)^{k(n-k)} \sum_{w \in S_n^{(k)}} (-1)^{f(w)} q^{\eta(w)}.$$  

Then from the incidence graph $G(k, n)$, one can find the explicit formula of the polynomial $p(q)$.

**Proposition 4.1.** For $\text{Gr}(k, n)$, the polynomial $p(q)$ takes the following form,

(i) if $(k, n)$ equals $(2j, 2m), (2j, 2m + 1)$ or $(2j + 1, 2m + 1)$, then

$$p(q) = \left[ \frac{m}{j} \right]_{q^2},$$

(ii) if $(k, n) = (2j + 1, 2m)$, then

$$p(q) = (q^m - 1) \left[ \frac{m - 1}{j} \right]_{q^2}.$$  

**Proof.** Let $Y_w$ be a Young diagram satisfying the following condition:

(i) If $(k, n)$ equals $(2j, 2m), (2j, 2m + 1)$ or $(2j + 1, 2m + 1)$, then $Y_w$ consists of only the sub-diagram given by

$$\begin{array}{c}
q \\
1 \\
1 \\
q
\end{array}$$
(ii) If \((k, n) = (2j + 1, 2m)\), \(Y_w\) consists of the above sub-diagram and the hook diagram, \(Y_w = (n - k) \times 1^{k - 1}\) with \(w = s_1 s_2 \cdots s_{k-1} s_n s_{n-1} \cdots s_k\), e.g. for \(\text{Gr}(3, 8)\), \(w = s_1 s_2 s_7 s_6 \cdots s_3\) and the Young diagram is given by \(Y = 5 \times 1^2\),

\[
\begin{array}{cccc}
q & 1 & q & 1 \\
1 & q
\end{array}
\]

The number of \(q\)’s in this hook diagram is \(m\), i.e. it has the weight \(q^m\), and the length (total number of boxes) of the diagram is \(n - 1 = 2m - 1\).

Then for those Young diagrams in the graph \(G(k, n)\), the arrows coming in or going out are just \(\rightarrow\), not \(\Rightarrow\). That is, \(\eta(w)\) in the \(q\)-weighted Schubert variety \((Y_w : \eta(w))\) changes under those nontrivial \(W\)-actions, and those \(w\) contribute the sum. One can also show that those are only ones which contribute the sum. Then for example, in the case \((k, n) = (2j, 2m)\) in the case (i), one can see that the polynomial \(p(q)\) agrees with \(|\text{Gr}(j, m, \mathbb{F}_q)|\) in \(\{1\}\), which is the formula in (i). Other cases in (i) can be obtained in the similar manner.

For the case (ii), i.e. \((k, n) = (2j+1, 2m)\), there are two types of Young diagrams which contribute the sum \(p(q)\). The Young diagrams having just type (i) give the second term \(-\frac{m-1}{j} q^2\) in \(p(q)\) \((-\) sign is due to \((-1)^{k(n-k)}\)), and those with the hook diagram give \(q^m\frac{m-1}{j} q^2\), the first term in \(p(q)\). \(\square\)

One can also define the polynomial \(p^*(q)\) using \(\eta^*(w)\) by

\[
p^*(q) = (-1)^{k(n-k)} \sum_{w \in S_n} (-1)^{\ell(w)} q^{\eta^*(w)}.
\]

This polynomial will be important for the non-orientable case of \(\text{Gr}(k, n)\) with \(n =\text{odd}\), and we have:

**Proposition 4.2.** For \(\text{Gr}(k, 2m + 1)\), the polynomial \(p^*(q)\) takes the form,

\[
p^*(q) = q^s \frac{m}{j} q^2 = q^s p(q),
\]

where \(s = j\) if \(k = 2j\), and \(s = m - j\) if \(k = 2j + 1\).

We then have the following theorem showing that \(p(q)\) is related to the \(\mathbb{F}_q\) points on \(\text{Gr}(k, n)\):

**Theorem 4.3.** We assume \(\sqrt{-1} \in \mathbb{F}_q\). Then we have

\[
|\text{Gr}(k, n)_{\mathbb{F}_q}| = q^r p(q) \text{ with } r = k(n-k) - \text{deg}(p(q)).
\]

In the following subsection, we give an outline of the proof based on the action of the Hecke algebra and the Frobenius eigenvalues in étale cohomology (the Weil étale cohomology theory).

**Example 4.1.** Let us some explicit examples:

(a) \(|\text{Gr}(1, 2m)_{\mathbb{F}_q}| = |(\mathbb{R}P^{2m-1})_{\mathbb{F}_q}| = q^{m-1}(q^m - 1),\)

(b) \(|\text{Gr}(1, 2m + 1)_{\mathbb{F}_q}| = |(\mathbb{R}P^{2m})_{\mathbb{F}_q}| = q^{2m},\)

(c) \(|\text{Gr}(2, 4)_{\mathbb{F}_q}| = q^2(1 + q^2),\)

(d) \(|\text{Gr}(3, 6)_{\mathbb{F}_q}| = q^4(1 + q^2)(q^3 - 1).\)
4.2. Outline of the proof of Theorem 4.3: We recall that the number of points $|\text{Gr}(k, n)_{\bar{\mathbb{F}}_q}|$ is given by a Lefschetz fixed-point Theorem applied to the Frobenius map, $Fr : x \to x^q$ on $\text{Gr}(k, n)_{\bar{\mathbb{F}}_q}$ with the algebraically closed field $\bar{\mathbb{F}}_q$. That is, we have the formula relating $|\text{Gr}(k, n)_{\bar{\mathbb{F}}_q}|$ to the Frobenius eigenvalues,

$$
\sum_{s=0}^{k(n-k)} (-1)^s \text{Tr} \left((Fr)|_{H^s_c(\text{Gr}(k, n)_{\bar{\mathbb{F}}_q}; \mathbb{Q}_l)}\right) = |\text{Gr}(k, n)_{\bar{\mathbb{F}}_q}|,
$$

where $H^s_c(X, \mathbb{Q}_l)$ is the étale cohomology with compact support of $X$ with the values in the $l$-adic number field $\mathbb{Q}_l$.

We also have the following Proposition which corresponds to Proposition 6.1 in [4]:

**Proposition 4.4.** The induced action of the Frobenius map on $H^s(\text{Gr}(k, n)_{\bar{\mathbb{F}}_q}; \mathbb{Q}_l)$ with the $l$-adic number field $\mathbb{Q}_l$ has the eigenvalues of the form $q^i$ where $i$ is an integer.

By Definition 4.1 of the polynomial $p(q)$, Proposition 4.4 then implies

$$
\sum_{s=0}^{k(n-k)} (-1)^s \text{Tr} \left((Fr)|_{H^s_c(\text{Gr}(k, n)_{\bar{\mathbb{F}}_q}; \mathbb{Q}_l)}\right) = (-1)^{k(n-k)} p(q)
$$

Then to compute $|\text{Gr}(k, n)_{\bar{\mathbb{F}}_q}|$, we simply have to make a replacement in the polynomial $p(q)$ corresponding to the dual of $H^s_c(\text{Gr}(k, n)_{\bar{\mathbb{F}}_q}; \mathbb{Q}_l)$. Duality has the effect of replacing $q^{-1}$ instead of $q$ to take into account the action of Frobenius on the dual space. Then there is the additional standard shift by $q^{k(n-k)}$ because of the Frobenius eigenvalues on stalks of the coefficient $\mathbb{V}$, the dual of a constant sheaf in the Poincaré duality formula. Thus we have $q^{k(n-k)} p(q^{-1})$. These observations leads to the proof of Theorem 4.3.

We need to consider the cases with $n=$even (orientable case) and $n=$odd (non-orientable case):

1. Assume that $n$ is even. By the Poincaré duality, $H^{2k(n-k)-s}_c(\text{Gr}(k, n)_{\bar{\mathbb{F}}_q}; \mathbb{Q}_l)$ is the dual of $H^s_c(\text{Gr}(k, n)_{\bar{\mathbb{F}}_q}; \mathbb{Q}_l)$. As has been already discussed above, this simply corresponds to considering the polynomial $q^{k(n-k)} p(q^{-1})$.

   We now use standard properties of the $q$ deformation of binomial coefficients. We note that if $D = \text{deg}(p(q))$ then $q^D p(q^{-1}) = (-1)^{k(n-k)} p(q)$. We obtain

$$
\sum_{s=0}^{k(n-k)} (-1)^s \text{Tr} \left((Fr)|_{H^s_c(\text{Gr}(k, n)_{\bar{\mathbb{F}}_q}; \mathbb{Q}_l)}\right) = |\text{Gr}(k, n)_{\bar{\mathbb{F}}_q}| = q^{k(n-k)-D} p(q).
$$

2. Assume that $n$ is odd, say $n = 2m + 1$. There are two cases, i.e. $k = 2j$ and $k = 2j + 1$. By the Poincaré-Verdier duality (see IX.4 and VI.3 in [1]), $H^{2k(n-k)-s}_c(\text{Gr}(k, n)_{\bar{\mathbb{F}}_q}; \mathbb{Q}_l)$ is the dual of $H^s_c(\text{Gr}(k, n)_{\bar{\mathbb{F}}_q}; \mathbb{Q}_l)$ where $\mathbb{L}$ is a twisted local system. We then have $|\text{Gr}(k, n)_{\bar{\mathbb{F}}_q}| = q^{k(n-k)} p^*(q^{-1}) = q^{k(n-k)-D} p^*(q)$ with $D^* = \text{deg}(p^*(q))$. Here the $q$-shifted polynomial $p^*(q)$ are given in Proposition 4.2 i.e. $p^*(q) = q^s p(q)$ with $s = j$ if $k = 2j$, and $s = m - j$ if $k = 2j + 1$. Then we have $D^* = s + D$, and obtain $|\text{Gr}(k, n)_{\bar{\mathbb{F}}_q}| = q^{k(n-k)-D} p^*(q)$. We may use the standard properties of the $q$ deformation of binomial coefficients to obtain $p^*(q)$.

5. THE POINCARE POLYNOMIALS

For $\text{Gr}(k, n)$, the Poincaré polynomial, denoted by $P_{(k, n)}$, is given by

$$
P_{(k, n)}(t) = \sum_{i=1}^{k(n-k)} \dim(H^i(\text{Gr}(k, n), \mathbb{C})) t^i.
$$

Here we show that $P_{(k, n)}(t)$ can be obtained from the polynomial $p(q)$ defined in the previous section.
Theorem 5.1. The Poincaré polynomial for \( \text{Gr}(k, n) \) is given as follows:

(i) If \((k, n)\) equals \((2j, 2m), (2j, 2m + 1)\) or \((2j + 1, 2m + 1)\), we have

\[
P_{(k,n)}(t) = p(t^2) = \left[ \begin{array}{c} m \\ j \end{array} \right]_{t^4}.
\]

(ii) If \((k, n) = (2j + 1, 2m)\), we have

\[
P_{(k,n)}(t) = (t^{2m-1} + 1) \left[ \begin{array}{c} m-1 \\ j \end{array} \right]_{t^4}.
\]

where \(p(q) = (q^m - 1) \left[ \begin{array}{c} m-1 \\ j \end{array} \right]_{q^2}\), that is, replace \((q^m - 1)\) by \((t^{2m-1} + 1)\) and set \(q = t^2\).

Note in particular that for the case (i), we have

\[
P_{(k,n)}(t) = P_{(\lfloor k/2 \rfloor, \lfloor n/2 \rfloor)}(t^2).
\]

These polynomials then lead to the well-known formulas of the Euler characteristic for \( \text{Gr}(k, n) \), that is, we have:

Corollary 5.2. The Euler characteristic \( \chi_E(\text{Gr}(k, n)) \) has the following form.

(i) If \((k, n)\) equals \((2j, 2m), (2j, 2m + 1)\) or \((2j + 1, 2m + 1)\), then

\[
\chi_E(\text{Gr}(k, n)) = P_{(k,n)}(-1) = \left( \begin{array}{c} m \\ j \end{array} \right).
\]

(ii) If \((k, n) = (2j + 1, 2m)\), then

\[
\chi_E(\text{Gr}(k, n)) = P_{(k,n)}(-1) = 0.
\]

5.1. The Poincaré-Verdier duality. When the Grassmannian \( \text{Gr}(k, n) \) is orientable, \( n \) is even, we have the Poincaré duality

\[
H_j(\text{Gr}(k, n), \mathbb{Z}) = H^{k(n-k)-j}(\text{Gr}(k, n), \mathbb{Z}).
\]

In the non-orientable case, i.e. \( n \) is odd, we define the incidence graph with twisted coefficients \( \mathcal{L} \), denoted by \( G(k, n)^* \), that is, this graph is obtained by the Young diagrams with \( q \)-shifted arrangement. The graph then gives the cohomology of twisted coefficient,

\[
H^*(\text{Gr}(k, n), \mathcal{L}).
\]

which then gives the homology of the non-orientable Grassmannian \( \text{Gr}(k, n) \), via the Poincaré-Verdier duality (see e.g. Section IX.4 and VI.3 in [7]),

\[
H_j(\text{Gr}(k, n), \mathbb{Z}) = H^{k(n-k)-j}(\text{Gr}(k, n), \mathcal{L}).
\]
Example 5.1. The graph $G(2, 5)^*$ is given by

$$
\begin{array}{c}
\emptyset \quad \Rightarrow \quad 1 \\
\downarrow \\
1 \qquad \Rightarrow \quad 1 \quad q \\
\downarrow \\
1 \quad q \\
\downarrow \\
1 \quad q \\
\downarrow \\
1 \quad q \\
\downarrow \\
1 \quad q \\
\downarrow
\end{array}
$$

which is obtained by exchanging $q \leftrightarrow 1$ in the $q$-weighted Schubert varieties in the incidence graph $G(k, n)$. The cohomology obtained from this graph is

$$
H^0(Gr(2, 5), L) = 0, \quad H^4(Gr(2, 5), L) = \mathbb{Z}_2,
$$

$$
H^1(Gr(2, 5), L) = \mathbb{Z}_2, \quad H^5(Gr(2, 5), L) = \mathbb{Z}_2,
$$

$$
H^2(Gr(2, 5), L) = \mathbb{Z}, \quad H^6(Gr(2, 5), L) = \mathbb{Z}.
$$

Then the homology group $H_*(Gr(2, 5), \mathbb{Z})$ via the Poincaré-Verdier duality gives

$$
H_j(Gr(2, 5), \mathbb{Z}) = H^{6-j}(Gr(2, 5), L) \quad \text{for} \quad j = 0, 1, \ldots, 6.
$$

Notice that the homology generators are the Pontryagin classes in $H_0$ and $H_4$.

6. Conjecture on the ring structure of the cohomology

Recall (see e.g. [11]) that the cohomology ring of the complex Grassmannian $Gr(k, n, \mathbb{C})$ is given by

$$
H^*(Gr(k, n, \mathbb{C}), \mathbb{R}) \cong \mathbb{R}[c_1, \ldots, c_{n-k}, \bar{c}_1, \ldots, \bar{c}_k]_{\{c \cdot \bar{c} = 1\}}
$$

where $c = 1 + c_1 + \cdots + c_{n-k}$ and $\bar{c} = 1 + \bar{c}_1 + \cdots + \bar{c}_k$ with the Chern classes $c_j \in H^{2j}(Gr(k, n, \mathbb{C}, \mathbb{R})$.

The cohomology ring of the classifying space $BO(k) = Gr(k, \infty)$ is also known, and it is given by

$$
H^*(BO(k), \mathbb{R}) \cong \mathbb{R}[p_1, \ldots, p_{\left\lfloor \frac{k}{2} \right\rfloor}],
$$

where the generators $p_j$ of the ring are given by the Pontryagin classes $p_j \in H^{4j}(BO(k), \mathbb{R})$.

Then it is natural to make the following conjectures:

**Conjecture 6.1.** The homology ring $H^*(Gr(k, n), \mathbb{R})$ is given by the following:

(i) If $(k, n)$ equals $(2j, 2m)$, $(2j, 2m+1)$ or $(2j+1, 2m+1)$, then

$$
H^*(Gr(k, n), \mathbb{R}) \cong \mathbb{R}[p_1, \ldots, p_{m-j}, \bar{p}_1, \ldots, \bar{p}_j]_{\{p \cdot \bar{p} = 1\}},
$$

where $p = 1 + p_1 + \cdots + p_{m-j}$ and $\bar{p} = 1 + \bar{p}_1 + \cdots + \bar{p}_j$ with the Pontryagin classes $p_j \in H^{4j}(Gr(k, n), \mathbb{R})$. 


(ii) If $(k, n) = (2j + 1, 2m + 2)$, then

$$H^*(\text{Gr}(k, n), \mathbb{R}) \cong \mathbb{R}[p_1, \ldots, p_{m-j}, \bar{p}_1, \ldots, \bar{p}_j, r] \{p \cdot \bar{p} = 1, r^2\},$$

where $p_j \in H^{4j}(\text{Gr}(k, n), \mathbb{R})$ and the element $r$ corresponds to the Schubert class $\sigma_w$ with the hook diagram $Y_w = (n - k) \times 1^{k-1}$.

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