Time-Reversal Symmetry in Non-Hermitian Systems

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For ordinary Hermitian Hamiltonians, the states support the mathematical structure of quaternion and show the Kramers degeneracy when the system has a half-odd-integer spin and the time-reversal operator obeys $\Theta^2 = -1$, while no such structure exists when $\Theta^2 = +1$. Here, we point out that for non-Hermitian systems, there exists a similar mathematical structure called split-quaternion even when $\Theta^2 = +1$. It is also found that a degeneracy similar to the Kramers degeneracy follows from the split-quaternion structure if the Hamiltonian is pseudo-Hermitian with the metric operator $\eta$ satisfying $\{\eta, \Theta\} = 0$. Furthermore, we show that particle/hole symmetry gives rise to a pair of states with opposite energies on the basis of the split-quaternion in a class of non-Hermitian Hamiltonians. As concrete examples, we examine in detail $N \times N$ Hamiltonians with $N = 2$ and 4, which are non-Hermitian generalizations of spin $1/2$ Hamiltonian and quadrupole Hamiltonian of spin $3/2$, respectively.

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§1. Introduction

The original observation between time-reversal (TR) invariance and statistical mechanics is traced back to Dyson who pointed out that TR operator $\Theta$ is naturally incorporated in the algebra of quaternions if the system has a half-odd-integer spin and $\Theta^2 = -1$. He showed that the Kramers degeneracy comes from mathematical structures of quaternion, and its statistical properties are described using the symplectic group. On the basis of these, Avron et al. explored topological properties of fermionic systems with TR symmetry, and the second Chern number was introduced as an extension of the TKNN topological number of quantum Hall effect. The TR symmetry and the resultant Kramers degeneracy also play a central role in recent developments of the quantum spin Hall effects and topological insulators. Indeed, the Kramers degeneracy enables us to introduce a new class of topological numbers characterizing these phases. The mathematical structures of the topological insulators have been studied in Refs. 10, 11, and 12.

Meanwhile, if the system has an integer spin and $\Theta^2 = +1$, such as boson systems and systems with even number of electrons, we have no such Kramers degeneracy. Correspondingly, its topological structure is rather simple and the Hamiltonian has a real structure in general. However, such a consequence changes if we allow the non-Hermiticity of Hamiltonians. Indeed, as shown below, there is generally a degeneracy similar to the Kramers degeneracy even when $\Theta^2 = +1$ in a class of non-Hermitian Hamiltonians.

Although we usually assume Hermiticity of Hamiltonian, non-Hermitian Haml-
tonians also have applications to interesting problems such as open chaotic scattering,\textsuperscript{13} dissipative quantum maps,\textsuperscript{14} and delocalization of pinned vortices in superconductors.\textsuperscript{15} We also have non-Hermitian Hamiltonians as effective theories of Hermitian systems. Moreover, non-Hermitian Hamiltonians might be meaningful themselves if a kind of TR symmetry such as $PT$ symmetry\textsuperscript{16} or pseudo-Hermiticity\textsuperscript{17} is imposed. They are a part of the reasons why we pursue the present work.

We investigate TR symmetry with $\Theta^2 = +1$ in non-Hermitian Hamiltonians. From a general argument, it is shown that such symmetry is naturally incorporated in the algebra of split-quaternion, instead of quaternion. Then, a new kind of degeneracy is obtained from structures of split-quaternion. As concrete examples, we examine $N \times N$ non-Hermitian Hamiltonians up to $N = 4$. The structure of split-quaternion is identified in these Hamiltonians, and we find that it has a close similarity to the quaternion structure of the spin $1/2$ Hamiltonian for $N = 2$ and quadrupole Hamiltonian of spin $3/2$ for $N = 4$. Furthermore, it is shown that the particle/hole symmetry also gives rise to a pair of states with opposite energies $(E, -E)$ in a class of non-Hermitian Hamiltonians. A random matrix classification of the non-Hermitian models is also provided.

The organization of the paper is as follows. In §2, a generalized Kramers degeneracy in pseudo-Hermitian quantum mechanics is discussed. We point out relations between split-quaternion and TR operation for $\Theta^2 = +1$, and show the existence of generalized Kramers degeneracy in pseudo-Hermitian systems. We also discuss the relations between the particle/hole symmetry and split-quaternions. In §3, we show how the generalized Kramers theorem in pseudo-Hermitian systems is incorporated in the non-Hermitian random matrix classification. As a concrete example of the pseudo-Hermitian model with particle/hole symmetry, the $SU(1,1)$ model is introduced in §4, and basic properties of the model are investigated. In §5, we discuss the properties of the $SO(3,2)$ model with time-reversal symmetry $\Theta^2 = +1$ as a simple exemplification of the generalized Kramers degeneracy. In §6, we show that the $SO(3,2)$ model is realized as an $SU(1,1)$ quadrupole model with $SU(1,1)$ spin $3/2$. The split-quaternion structure of $SU(2)$ integer spin systems is also clarified. Section 7 is devoted to summary and discussion.

§2. Generalized Kramers degeneracy and split-quaternion

2.1. Split-quaternion and time-reversal symmetry

Let us start with a brief review of the split-quaternion. The split-quaternion\textsuperscript{18} is a variant of the quaternion,\textsuperscript{19} which is written as

$$q = w + xi + yj + zk,$$

(2.1)

with real numbers $(w, x, y, z)$ in the basis $(1, i, j, k)$. The algebra of the basis for the split-quaternion is different from that for the quaternion, and it is given by

$$ij = k = -ji, \quad jk = -i = -kj, \quad ki = j = -ik, \quad i^2 = -1, \quad j^2 = 1, \quad k^2 = 1.$$

(2.2)
Note that $j^2 = 1$ and $k^2 = 1$, not $j^2 = -1$ and $k^2 = -1$ as in the quaternion case.

The structure of the quaternion naturally arises in time-reversal (TR) invariant systems. The TR operator is antiunitary and anticommutes with $i$:

$$\Theta = UK,$$

(2.3)

where $U$ is a unitary operator and $K$ complex conjugates everything to its right. For systems with an integer spin, we have $\Theta^2 = +1$, while for systems with a half-integer spin, $\Theta^2 = -1$.\(^*$) In the latter case, the TR invariance results in the structure of the quaternion for the Hamiltonian. Then, what is a natural mathematical structure in the former?

The answer is the split-quaternion. The TR operator is antiunitary,

$$\Theta i = -i \Theta.$$

(2.4)

By identifying $j$ and $k$ with $\Theta$ and $i\Theta$, respectively, one finds a correspondence between the triplet of the TR algebra and the split-quaternion,

$$(i, \Theta, i\Theta) \leftrightarrow (i, j, k).$$

(2.5)

Thus, the split-quaternion also fits into the TR symmetry with $\Theta^2 = +1$.

In spite of the argument above, such a split-quaternion structure has been missed in former studies of TR invariant Hamiltonians with $\Theta^2 = +1$. The Hamiltonian supports only a real structure instead.\(^1\) This is because, usually, the Hamiltonians are supposed to be Hermitian. This implicit assumption makes the split-quaternion into a real number. Nevertheless, physical phenomena are not always described using Hermitian Hamiltonians. Non-Hermitian Hamiltonians also have interesting physical applications. Thus, if we consider a class of non-Hermitian Hamiltonians, the hidden split-quaternion structure becomes evident, as will be shown in the following sections.

2.2. Pseudo-Hermiticity

A non-Hermitian Hamiltonian $H$ is called pseudo-Hermitian\(^17\) when it satisfies pseudo-Hermiticity,

$$H^\dagger = \eta H \eta^{-1},$$

(2.6)

where $\eta$ is a Hermitian operator referred to as the metric operator. For example, consider a non-unitary transformation $G$ on a Hermitian Hamiltonian $H_0$; then, $H = GH_0G^{-1}$ is not Hermitian, but pseudo-Hermitian,

$$H^\dagger = G^\dagger -1 H_0 G^\dagger = G^\dagger -1 G^{-1} H G G^\dagger,$$

(2.7)

with a metric operator $\eta = (GG^\dagger)^{-1}$.

The reason why $\eta$ is called the metric operator is that the time-independent inner product of a state is given by a metric $\eta$. For a non-Hermitian Hamiltonian,\(^*$) The action of the TR operator is two, i.e. the system comes back to the original if we apply $\Theta$ twice. Thus, $\Theta^2$ should be a phase $e^{i\alpha}$, which implies $U = e^{i\alpha} U^T$ and $U^T = U e^{i\alpha}$. This yields $U = U e^{2i\alpha}$, so the phase is $e^{i\alpha} = \pm 1$.\(^*)
there generally exists a set of states, $|\phi\rangle$ and $|\varphi\rangle$ that satisfy

$$i \frac{\partial}{\partial t} |\phi\rangle = H |\phi\rangle, \quad i \frac{\partial}{\partial t} |\varphi\rangle = H^\dagger |\varphi\rangle.$$ \hspace{1cm} (2.8)

The time-independent inner product is constructed as $\langle \phi | \varphi \rangle$, as shown by

$$i \frac{\partial}{\partial t} \langle \phi | \varphi \rangle = i \frac{\partial}{\partial t} \langle \phi | \varphi \rangle + i \langle \phi | \frac{\partial}{\partial t} | \varphi \rangle = -\langle \phi | H^\dagger | \varphi \rangle + \langle \phi | H^\dagger | \varphi \rangle = 0.$$ \hspace{1cm} (2.9)

For a pseudo-Hermitian Hamiltonian $H$, (2.6) leads to

$$i \frac{\partial}{\partial t} \eta |\phi\rangle = H^\dagger \eta |\phi\rangle, \quad i \frac{\partial}{\partial t} \eta^{-1} |\varphi\rangle = H \eta^{-1} |\varphi\rangle.$$ \hspace{1cm} (2.10)

Thus, we also have additional time-independent inner products, $\langle \phi | \eta | \varphi \rangle$, $\langle \varphi | \eta^{-1} | \varphi \rangle$.

In the following, we mainly use $\langle \phi | \varphi \rangle$ as the inner product unless explicitly written.

For the pseudo-Hermiticity to be consistent with the TR symmetry, the condition (2.6) should be commutative with the TR operation. This leads to $\eta^* = \pm U^\dagger \eta U$, that is, $\Theta \eta = \pm \eta \Theta$. Therefore, possible metric operators are classified into two. The first one satisfies $[\eta, \Theta] = 0$, and the second, $\{\eta, \Theta\} = 0$.

2.3. Generalized Kramers degeneracy

Let us assume two conditions: one is the TR symmetry with $\Theta^2 = 1$

$$[H, \Theta] = 0, \hspace{1cm} (2.11)$$

and the other is the anticommutation relation

$$\{\eta, \Theta\} = 0. \hspace{1cm} (2.12)$$

Let $|\phi_n\rangle$ be (right) eigenstates of $H$,

$$H |\phi_n\rangle = E_n |\phi_n\rangle.$$ \hspace{1cm} (2.13)

Then, the corresponding eigenstates $|\phi_n\rangle$ of $H^\dagger$ (i.e., ket states of left eigenstates of $H$),

$$H^\dagger |\phi_n\rangle = E_n^* |\phi_n\rangle,$$ \hspace{1cm} (2.14)

which satisfy

$$\langle \phi_n | \phi_m \rangle = \langle \phi_m | \phi_n \rangle = \delta_{nm}. \hspace{1cm} (2.15)$$

The eigenstates $|\phi_n\rangle$ and $|\phi_n\rangle$ satisfying (2.15) are known as the bi-orthonormal basis.\textsuperscript{20,21} By using the pseudo-Hermiticity (2.6), (2.14) is rewritten as

$$H \eta^{-1} |\phi_n\rangle = E_n^* \eta^{-1} |\phi_n\rangle.$$ \hspace{1cm} (2.16)

\textsuperscript{*} Here, the bra state is a Hermitian conjugate of the ket state, i.e., $\langle \phi | = |\phi \rangle^\dagger$, $\langle \varphi | = |\varphi \rangle^\dagger$. 

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We apply $\Theta$ from the left to both sides of (2.16) to have

$$H\Theta\eta^{-1}|\phi_n\rangle = E_n\Theta\eta^{-1}|\phi_n\rangle,$$

(2.17)

where on the left-hand side, we utilized the time-reversal invariance of the Hamiltonian (2.11). Thus, $|\phi_n\rangle$ and $\Theta\eta^{-1}|\phi_n\rangle$ have the same eigenvalue $E_n$. Therefore, if they are linearly independent, we have degeneracy in the eigenvalues of $H$. Note

$$\langle\langle \phi_n|\Theta\eta^{-1}|\phi_n\rangle \rangle = \langle\langle \Theta^2\eta^{-1}\phi_n|\Theta\phi_n\rangle \rangle = \langle\langle \phi_n|\eta^{-1}\Theta\phi_n\rangle \rangle = -\langle\langle \phi_n|\Theta\eta^{-1}|\phi_n\rangle \rangle.$$  

(2.18)

In the first equation, we have used the antiunitary property of $\Theta$, and the second equation follows from $\Theta^2 = +1$ and the Hermiticity of $\eta^{-1}$. In the third equation, the anticommutation relation between $\Theta$ and $\eta^{-1}$ was utilized.

Thus we have $\langle\langle \phi_n|\Theta\eta^{-1}|\phi_n\rangle \rangle = 0$. On the other hand, $\langle\langle \phi_n|\phi_n\rangle \rangle = 1$ from (2.15). Therefore, $|\phi_n\rangle$ and $\Theta\eta^{-1}|\phi_n\rangle$ are linearly independent.

As a result, we have two fold degeneracy in eigenstates of $H$, which is the generalized Kramers degeneracy.

In general, the generalized Kramers partner $\Theta\eta^{-1}|\phi_n\rangle$ is not coincident with $\Theta|\phi_n\rangle$. Actually, unlike the TR symmetry with $\Theta^2 = -1$, TR symmetry with $\Theta^2 = +1$ does not imply that $|\phi_n\rangle$ and $\Theta|\phi_n\rangle$ are linearly independent. Nevertheless, we can say that if eigenvalues of $H$ are real, the generalized Kramers partner is essentially the same as $\Theta|\phi_n\rangle$.

To see this, let us consider the eigenstate $|\phi_n\rangle$ satisfying (2.13). Then, the pseudo-Hermiticity leads to

$$H^\dagger\eta|\phi_n\rangle = E_n\eta|\phi_n\rangle.$$  

(2.19)

Therefore, $\eta|\phi_n\rangle$ can be expanded as

$$\eta|\phi_n\rangle = \sum_m |\phi_m\rangle c_{mn},$$

(2.20)

where the sum is taken for $|\phi_m\rangle$'s satisfying (2.14) and (2.15) with $E_m^* = E_n$. (Note that if there is a degeneracy in the spectrum, we may have multiple such $|\phi_m\rangle$'s.)

Applying $\langle\phi_m|$ from the left, we obtain

$$c_{mn} = \langle\phi_m|\eta|\phi_n\rangle.$$  

(2.21)

Because of the Hermiticity of $\eta$, $c_{mn}$ is Hermitian for the indices $m$ and $n$. Thus, it can be diagonalized using a unitary matrix $G$

$$\sum_{lk} G^\dagger_{ml} c_{lk} G_{kn} = \lambda_m \delta_{mn},$$  

(2.22)

\*\* For usual Hermitian Hamiltonians, we have $\eta = 1$ in general. Thus, the anticommutativity (2.12) does not hold. This is the reason why even if a Hermitian Hamiltonian has TR symmetry with $\Theta^2 = +1$, there is no generalized Kramers pair.

\*\* Though $|\phi_n\rangle$ and $\Theta\eta^{-1}|\phi_n\rangle$ are linearly independent, they are not orthogonal, i.e. $\langle\phi_n|\Theta\eta^{-1}|\phi_n\rangle \neq 0$, in general.
with real $\lambda_m$. The eigenvalue $\lambda_n$ is not zero because $c_{mn}$ is invertible. Thus, taking the following new bi-orthonormal basis:

$$
|\phi'_n\rangle = \sum_m |\phi_m\rangle G_{mn} / \sqrt{|\lambda_n|}, \quad |\phi'_n\rangle = \sum_m |\phi_m\rangle G_{mn} \sqrt{|\lambda_n|}, \quad \langle \phi'_n | \phi'_m \rangle = \delta_{mn}, \quad (2.23)
$$

we have

$$
\eta |\phi'_n\rangle = \text{sgn}(\lambda_n) |\phi'_n\rangle. \quad (2.24)
$$

Now, suppose that $E_n$ is real; then, $E_m$ for $|\phi_m\rangle$ in (2.20) is also real and coincides with $E_n$. This yields that all $|\phi_m\rangle$’s on the right-hand side of the first equation in (2.23) have the same energy $E_n$. In other words, $|\phi'_n\rangle$ remains to be an eigenstate of $H$ with the eigenvalue of $E_n$. Applying $\Theta \eta^{-1}$ from the left to both sides of (2.24), we find that the (generalized) Kramers partner $\Theta \eta^{-1} |\phi'_n\rangle$ is the same as $\Theta |\phi'_n\rangle$ up to an irrelevant overall sign,

$$
\Theta |\phi'_n\rangle = \text{sgn}(\lambda_n) \Theta \eta^{-1} |\phi'_n\rangle. \quad (2.25)
$$

### 2.4. Particle/hole symmetry and split-quaternion

In addition to the TR invariance, we can have the particle/hole symmetry $C$, which is antiunitary. Here, we briefly see the split-quaternion structure of the particle/hole symmetric system.

We say that a Hamiltonian $H$ has the particle/hole symmetry $C$ if it satisfies

$$
C H C^{-1} = -H. \quad (2.26)
$$

If we write $C$ as $C = \Gamma K$ with a unitary operator $\Gamma$, (2.26) is recast into

$$
\Gamma H \Gamma^\dagger = -H^*. \quad (2.27)
$$

One can show that $C^2 = \pm 1$.

In a manner similar to the TR symmetry, we have the split-quaternion structure if $C^2 = +1$. The correspondence between the particle/hole symmetry and the split-quaternion is

$$(i, C, iC) \leftrightarrow (i, j, k). \quad (2.28)$$

When $H$ is pseudo-Hermitian, $H^\dagger = \eta H \eta^{-1}$ with $\{\eta, C\} = 0$, we find that eigenstates of $H$ are paired with eigenvalues $(E_n, -E_n)$. Consider

$$
H |\phi_n\rangle = E_n |\phi_n\rangle, \quad H^\dagger |\phi_n\rangle = E_n^* |\phi_n\rangle, \quad (2.29)
$$

with $\langle \phi_n | \phi_m \rangle = \langle \phi_m | \phi_n \rangle = \delta_{mn}$. It is found that $|\phi_n\rangle$ and $C \eta^{-1} |\phi_n\rangle$ have the eigenenergies $E_n$ and $-E_n$, respectively. Then, if $\eta$ satisfies $\{\eta, C\} = 0$, we can show that $|\phi_n\rangle$ and $C \eta^{-1} |\phi_n\rangle$ are linearly independent for any $E_n$, in a manner similar to that described in §2.3. Thus, the eigenstates of $H$ are paired. In particular, if we have a zero energy state with $E = 0$, then it should be degenerated.
Note that the particle/hole symmetry itself implies that if $|\phi_n\rangle$ is an eigenstate of $H$ with eigenenergy $E_n$, then $C|\phi_n\rangle$ is the one with $-E_n^*$. For a non-Hermitian Hamiltonian, however, this does not always mean an additional pair of states. Indeed, if $E_n$ is pure imaginary, then $E_n$ is the same as $-E_n^*$, and $|\phi_n\rangle$ and $C|\phi_n\rangle$ can be the same. We can also say that if $E_n$ is real, $C|\phi_n\rangle$ is essentially the same as $C\eta^{-1}|\phi_n\rangle$:

\[ C|\phi_n\rangle = \text{sgn}(\lambda_n)\eta^{-1}|\phi_n\rangle. \]  

(2.30)

Thus, $C|\phi_n\rangle$ and $C\eta^{-1}|\phi_n\rangle$ coincide with each other up to a sign factor.

Formally, we can treat the particle/hole symmetry as the TR symmetry by redefining $H \rightarrow iH$. In this case, however, the pseudo-Hermiticity is replaced by “pseudo-anti-Hermiticity”,

\[ H^\dagger = -\eta H \eta^{-1}. \]  

(2.31)

### 2.5. Example: $2 \times 2$ matrix

In this subsection, we will see the split-quaternion structure in a concrete example. Consider a $2 \times 2$ matrix as the simplest nontrivial Hamiltonian. In general, by using a $2 \times 2$ unit matrix $1_2$ and the Pauli matrices $\sigma_i$ ($i = x, y, z$), any $2 \times 2$ matrix can be written as

\[ H = h_1 1_2 + \sum_{i=x,y,z} h^i \sigma_i, \]  

(2.32)

with complex numbers $h$ and $h^i$ ($i = x, y, z$). Then, suppose that $H$ is invariant under the TR symmetry $\Theta = UK$ with $\Theta^2 = +1$,

\[ [H, \Theta] = 0. \]  

(2.33)

$\Theta^2 = +1$ implies that $U$ is a symmetric (unitary) matrix, $U = U^T$. Following Ref. 1), $U$ can be $U = 1_2$ in a proper basis of the Hamiltonian. The TR invariance yields

\[ h^* = h, \quad h^x^* = h^x, \quad h^y^* = -h^y, \quad h^z^* = h^z, \]  

(2.34)

and thus, we obtain

\[ H = w 1_2 + x\sigma_x + yi\sigma_y + z\sigma_z, \]  

(2.35)

with real numbers, $w, x, y, z$. The split-quaternion structure of $H$ is evident if we notice the following identification between the Pauli matrices and the basis for the split-quaternion,

\[ (i\sigma_y, \sigma_x, \sigma_z) \leftrightarrow (i, j, k), \]  

(2.36)

which reproduces the algebra (2.2). Thus, the TR invariant Hamiltonian (2.35) is a split-quaternion.
Let us further impose the pseudo-Hermiticity. To satisfy \( \{ \Theta, \eta \} = 0 \), the Hermitian matrix \( \eta \) should be pure imaginary, \( \eta^* = -\eta \). Thus, it can be written as \( \eta = c \sigma_y \) with a real number \( c \). If \( H \) is pseudo-Hermitian (2.6), we obtain \( x = y = z = 0 \). Therefore, the Hamiltonian becomes \( H = \sigma_1 \). While this Hamiltonian is rather trivial, we have a two fold degeneracy. This can be considered as the generalized Kramers degeneracy explained in §2.3. In this case, any column vectors \( |\phi\rangle = (a, b)^T \) are eigenstates of \( H \). The corresponding \( |\phi\rangle \) satisfying \( \langle \phi | \phi \rangle = 1 \) is

\[
|\phi\rangle = \frac{1}{|a|^2 + |b|^2} \left( \begin{array}{c} a \\ b \end{array} \right).
\]

Thus, the generalized Kramers partner, \( \Theta \eta^{-1} |\phi\rangle \), is given by

\[
\Theta \eta^{-1} |\phi\rangle = -\frac{c^{-1}}{|a|^2 + |b|^2} \sigma_y \left( \begin{array}{c} a^* \\ b^* \end{array} \right) = \frac{c^{-1}}{|a|^2 + |b|^2} \left( \begin{array}{c} ib^* \\ -ia^* \end{array} \right).
\]

One can easily check that \( |\phi\rangle \) and \( \Theta \eta^{-1} |\phi\rangle \) are linearly independent if \( |a|^2 + |b|^2 \neq 0 \).

Next, impose the pseudo-anti-Hermiticity, which implies \( w = 0 \) in (2.35), so

\[
H = x \sigma_x + y i \sigma_y + z \sigma_z.
\]

The eigenvalues are \( E_{\pm} = \pm \sqrt{x^2 + z^2 - y^2} \). The corresponding eigenstates \( |\phi_{\pm}\rangle \) are given by

\[
|\phi_{\pm}\rangle = c_{\pm} \left( \begin{array}{c} x + y \\ \pm E - z \end{array} \right), \quad E = \sqrt{x^2 + z^2 - y^2},
\]

where \( c_{\pm} \) are constants. In accordance with the general argument in §2.4, the eigenstate with the eigenvalue \( E \) is paired with the one with \( -E \).

We can also check that if we assume the Hermiticity of \( H \), instead of the pseudo-Hermiticity or pseudo-anti-Hermiticity, the split-quaternion structure is replaced by the real structure, \( y \) in (2.35) being zero; thus, \( H \) reduces to a \( 2 \times 2 \) real symmetric matrix.

§3. Random matrix classification

The idea of random matrix ensembles, pioneered by Wigner and Dyson, is based on classifying matrices by discrete symmetries (e.g. see 22)). Altland and Zirnbauer established the Hermitian random matrix theory in the context of superconductivity, \(^{23,24} \) which contains 10 classes.

Bernard and LeClair extended the random matrix classification for non-Hermitian matrices.\(^ {25} \) In this section, we apply the arguments in §2 to the scheme of the random matrix classification.

A possible application of this classification is an extension of topological phases in non-Hermitian systems. It has been revealed that the random matrix classification is a useful framework to investigate topological phases in non-interacting Hermitian systems.\(^ {11} \) The consideration in this section gives an extension of the framework to non-Hermitian topological phases.\(^ {26} \)
3.1. Non-Hermitian random matrix classification

Following Ref. 25), let us consider discrete symmetries on non-Hermitian random matrices. Suppose that the discrete symmetries are implemented by unitary transformations, and the system comes back to the original up to a phase factor if they are applied twice. Then, there are four possible transformations on a non-Hermitian random matrix $H$:

$$K \ sym. : \ H = k H^* k^{-1}, \quad k k^* = \pm 1,$$
$$Q \ sym. : \ H = \epsilon_q q H^\dagger q^{-1}, \quad q^\dagger q^{-1} = 1,$$
$$C \ sym. : \ H = \epsilon_c c H^T c^{-1}, \quad c^T c^{-1} = \pm 1,$$
$$P \ sym. : \ H = -p H p^{-1}, \quad p^2 = 1,$$

(3.1a)

(3.1b)

(3.1c)

(3.1d)

where $\epsilon_c$ and $\epsilon_q$ are signs, i.e., $\epsilon_c = \pm 1$ and $\epsilon_q = \pm 1$, and $k$, $q$, $c$, and $p$ are unitary matrices,

$$k k^\dagger = 1, \quad q q^\dagger = 1, \quad c c^\dagger = 1, \quad p p^\dagger = 1.$$

(3.2)

Demanding that the transformations (3.1a)–(3.1d) commute, we have

$$q^* = \pm k^{-1} q k^\dagger^{-1}, \quad k^T c^{-1} k c^* = \pm 1, \quad p^* = \pm k^{-1} p k,$$

$$q^T = \pm c^\dagger q^{-1} c, \quad q = \pm p q p^\dagger, \quad c = \pm p c p^t.$$

(3.3)

We refer to K symmetry as the TR symmetry. We can add the minus sign to the right-hand side of the first equation in (3.1a) by redefining $H \rightarrow iH$. Thus, we can also consider K symmetry as the particle/hole symmetry.

Q symmetry corresponds to the pseudo-Hermiticity ($\epsilon_q = 1$) or the pseudo-anti-Hermiticity ($\epsilon_q = -1$), defined in the previous section, by identifying $q$ with $\eta^{-1}$. We note that the correspondence is not one-to-one. While $q$ is a unitary operator, $\eta$ is not always. (Both $\eta$ and $q$ are Hermitian.) Thus, Q symmetry is a part of the pseudo-(anti-)Hermiticity.

In the case of Hermitian matrices, K symmetry is nothing but C symmetry. Thus, one often refers to C symmetry as the TR symmetry if $\epsilon_c = 1$ or particle/hole symmetry if $\epsilon_c = -1$. For non-Hermitian matrices, however, they are different. Thus, in this paper, we do not refer to C symmetry as the TR symmetry or particle/hole symmetry. C symmetry is obtained by combining K and Q symmetries. Specifically, from K (3.1a) and Q (3.1b) symmetries, C symmetry (3.1c) is obtained with $c = k q^*$ (up to a phase factor) and $\epsilon_c = \epsilon_q$.

Finally, $P$ symmetry is called the chiral symmetry in the literature.

3.2. Random matrix classification and split-quaternion

Write

$$k = U,$$

(3.4)

* The terminology “symmetry” is usually used for the operations that commute with Hamiltonian, but in this section, “symmetry” refers to the operations (3.1a)–(3.1d).
then (3.1a) gives the TR symmetry with \( \Theta = UK \) as
\[
[\Theta, H] = 0, \quad \Theta^2 = \pm 1.
\] (3.5)

Next, write
\[
q = \eta^{-1}.
\] (3.6)

Then (3.1b) reads
\[
H^\dagger = \epsilon_q \eta H \eta^{-1}, \quad \eta^\dagger = \eta.
\] (3.7)

Thus, Q symmetry with \( \epsilon_q = 1 \) is considered as the pseudo-Hermiticity, and with \( \epsilon_q = -1 \) as the pseudo-anti-Hermiticity.

If the first equation of (3.3) holds, a system has both K and Q symmetries. In terms of \( \Theta \) and \( \eta \), the commutativity between K and Q, i.e., \( q^* = \pm k^{-1} q k^\dagger \), is written as
\[
\Theta \eta = \eta \Theta = 0.
\] (3.8)

Thus, K and Q symmetries are equipped with all the properties used in the arguments in the previous section.

The arguments in the previous section lead to the following.

1. When a non-Hermitian matrix has K symmetry with \( kk^* = 1 \), the matrix supports the split-quaternion structure.

2. If the non-Hermitian matrix also has Q symmetry with \( \epsilon_q = 1 \) and \( q^* = -k^{-1} q k^\dagger \), at the same time, each eigenvalue of the non-Hermitian matrix has two fold degeneracy.

3. If the sign of Q symmetry is minus, i.e., \( \epsilon_q = -1 \) (and if \( q \) and \( k \) satisfy \( q^* = -k^{-1} q k^\dagger \)), each eigenstate with the eigenvalue \( E \) of the non-Hermitian matrix has a partner state with \( -E \).

The second result is nothing but the generalized Kramers degeneracy in §2.3, and the last one comes from particle/hole symmetry arguments in §2.4.

§4. Random matrix class of the \( SU(1,1) \) model

As a concrete realization of the statements in §3.2, we introduce the \( SU(1,1) \) models. First consider K symmetry realized by \( k = \sigma_x \) with \( kk^* = 1 \). Although \( k \) can be \( k = 1_z \) by taking a proper basis\(^1\) as mentioned in §2.5 and explicitly shown below, the present form of \( k = \sigma_x \) is convenient to see the \( SU(1,1) \) structure.

Any arbitrary 2×2 matrix can be expanded by a 2×2 unit matrix and the Pauli matrices
\[
H = h 1_2 + \sum_{i=x,y,z} h^i \sigma_i,
\] (4.1)

where \( h \) and \( h^i \) (\( i = x, y, z \)) stand for complex parameters. The imposition of K symmetry specifies \( h \) and \( h^i \) as
\[
h^* = h, \quad h^{x*} = h^x, \quad h^{y*} = h^y, \quad h^{z*} = -h^z.
\] (4.2)
Thus, $h, h^x, h^y$ are real parameters, while $h^z$ is a pure imaginary parameter. Thus, the Hamiltonian is rewritten as

$$H = w_1^2 + x\sigma_x + y\sigma_y + iz\sigma_z,$$

(4.3)

with real parameters $w, x, y, z$. We further impose Q symmetry with $q = \sigma_z$. There are two types of Q symmetry, corresponding to $\epsilon_q = \pm 1$.

4.1. $\epsilon_q = +1$: pseudo-Hermiticity

In this case, the Hamiltonian takes the form of

$$H = w_1^2.$$

(4.4)

The Kramers degeneracy of this Hamiltonian was discussed in §2.5. Indeed, the $K$ symmetry here corresponds to $\Theta = \sigma_x K$. Thus, by applying the following unitary transformation,

$$\Theta \to V^\dagger \Theta V,$$

$$V = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & -i \\ i & 1 \end{array} \right),$$

(4.5)

$\Theta$ reduces to the one in §2.5, $\Theta = K$. At the same time, in this unitary transformation, $q$ and $H$ become

$$q \to V^\dagger \sigma_z V = \sigma_y, \quad H \to V^\dagger w_1^2 V = w_1^2,$$

(4.6)

which are also the same as those in §2.5. Thus, the degeneracy here can be understood as a consequence of the generalized Kramers.

4.2. $\epsilon_q = -1$: pseudo-anti-Hermiticity

When $\epsilon_q = -1$, the Hamiltonian becomes

$$H = x\sigma_x + y\sigma_y + iz\sigma_z.$$

(4.7)

From the general theorem 3 in §3.2, $H$ is expected to have a partner state with $E$ and $-E$. Actually, the Hamiltonian has eigenvalues $\pm \sqrt{x^2 + y^2 - z^2}$; thus, the energy eigenvalues are paired. The eigenvalues are real when $x^2 + y^2 - z^2 \geq 0$, which corresponds to the model in quantum optics. In this case, the constant energy surface in the parameter space is a one-leaf hyperboloid, $H^{1,1}$. On the other hand, when $x^2 + y^2 - z^2 \leq 0$, the constant energy surface in the parameter space is described using the two-leaf hyperboloid, $H^{2,0}$. Here, we will consider the properties of the latter.

The corresponding Hamiltonian takes the pure imaginary eigenvalues with opposite sign. Therefore, we transform $H$ into $iH$, and then deal with the following Hamiltonian,

$$H = -x\tau_x - y\tau_y + z\tau_z = \left( \begin{array}{cc} z & -ix - y \\ -ix + y & -z \end{array} \right).$$

(4.8)

Here, $\tau_i$ ($i = x, y, z$) are “Pauli matrices” of $SU(1, 1)$:

$$\tau_x = i\sigma_x, \quad \tau_y = i\sigma_y, \quad \tau_z = \sigma_z.$$

(4.9)
As a consequence of the transformation $H \to iH$, the TR symmetry in the original Hamiltonian is converted into the particle/hole symmetry

$$\mathcal{C}H\mathcal{C}^{-1} = -H, \quad \mathcal{C} = \sigma_x \cdot K,$$

and the pseudo-anti-Hermiticity becomes the pseudo-Hermiticity,

$$H^\dagger = \eta H \eta^{-1}, \quad \eta = \sigma_z. \quad (4.11)$$

Let us consider situations where all the eigenvalues of the Hamiltonian (4.8) are real. Then, the parameters $x, y,$ and $z$ give coordinates on a two-leaf hyperboloid $H^2$:

$$x^2 + y^2 - z^2 = -r^2 \leq 0, \quad (4.12)$$

where $r$ is a real positive constant. Since $x$ and $y$ are real, $z$ is taken as either $z \geq r$ (upper leaf) or $z \leq -r$ (lower leaf). The eigenvalues of the $SU(1,1)$ Hamiltonian (4.8) are given by

$$E_{\pm} = \pm r. \quad (4.13)$$

On the upper leaf ($z \geq r$), the eigenvectors are given by

$$|\phi_\pm\rangle = \frac{1}{\sqrt{2r(r+z)}} \begin{pmatrix} r + z \\ y - ix \end{pmatrix}, \quad |\phi_-\rangle = \frac{1}{\sqrt{2r(r+z)}} \begin{pmatrix} y + ix \\ r + z \end{pmatrix}, \quad (4.14)$$

which satisfy

$$\langle \phi_\pm | \eta | \phi_\pm \rangle = \langle \phi_\pm | \sigma_z | \phi_\pm \rangle = \pm 1. \quad (4.15)$$

Meanwhile, the eigenvectors of the hermite-conjugate Hamiltonian

$$H^\dagger = x\tau_x + y\tau_y + z\tau_z = \begin{pmatrix} z & ix + y \\ ix - y & -z \end{pmatrix}, \quad (4.16)$$

are

$$|\varphi_\pm\rangle = \frac{1}{\sqrt{2r(r+z)}} \begin{pmatrix} r + z \\ -y + ix \end{pmatrix}, \quad |\varphi_-\rangle = \frac{1}{\sqrt{2r(r+z)}} \begin{pmatrix} -y - ix \\ r + z \end{pmatrix}, \quad (4.17)$$

which satisfy

$$\langle \langle \varphi_\pm | \eta^{-1} | \varphi_\pm \rangle \rangle = \langle \langle \varphi_\pm | \sigma_z | \varphi_\pm \rangle \rangle = \pm 1. \quad (4.18)$$

The bi-orthonormal bases, $|\phi_\pm\rangle$ and $|\varphi_\pm\rangle$, satisfy

$$\langle \phi_m | \varphi_n \rangle = \delta_{mn}, \quad \sum_{m=\pm} |\phi_m\rangle \langle \varphi_m | = |\phi_+\rangle \langle \varphi_+ | + |\phi_-\rangle \langle \varphi_- | = 1_2, \quad (4.19)$$
and are related as\(^*\)

\[ |\phi_\pm\rangle \underset{\eta}{\longrightarrow} \eta|\phi_\pm\rangle = \sigma_z|\phi_\pm\rangle = \pm|\varphi_\pm\rangle. \tag{4.23} \]

From the general arguments in §2.4, the particle/hole pair of \(|\phi_m\rangle\) is given by \(C\eta^{-1}|\varphi_m\rangle\). With \(\eta = \sigma_z\) and \(C = \sigma_xK\), the particle/hole pair of \(|\phi_\pm\rangle\) is given by

\[ C\eta^{-1}|\varphi_\pm\rangle = -i\sigma_yK|\varphi_\pm\rangle = \pm|\phi_\mp\rangle. \tag{4.24} \]

Thus, the particle/hole pair of \(|\phi_\mp\rangle\) is \(|\phi_\pm\rangle\). Indeed, \(|\phi_\pm\rangle\) and \(|\phi_\mp\rangle\) are eigenstates of \(H\) with opposite energies, \(E_+ = r\) and \(E_- = -r\). Although \(|\phi_\pm\rangle\) and \(|\phi_\mp\rangle\) are not orthogonal in the usual sense, i.e., \(\langle\phi_\pm|\phi_\mp\rangle \neq 0\), they are linearly independent. They are orthogonal in the sense of the pseudo-inner product:

\[ \langle\phi_-|\sigma_z|\phi_+\rangle = 0. \tag{4.25} \]

Applying \(C\eta^{-1}\) from the left to both sides of (4.23), \(\eta|\phi_\pm\rangle = \pm|\varphi_\pm\rangle\), we have

\[ C|\phi_\pm\rangle = \pm C\eta^{-1}|\varphi_\pm\rangle. \tag{4.26} \]

Therefore, \(C\eta^{-1}|\varphi_\pm\rangle\) is equal to \(C|\phi_\pm\rangle\) up to the \(\pm\) sign. (See also (2.30) in §2.4.) Indeed,

\[ |\phi_\pm\rangle \underset{C}{\longrightarrow} |\phi_\mp\rangle = C|\phi_\pm\rangle. \tag{4.27} \]

Thus, we find that the particle/hole pair of \(|\phi_\mp\rangle\) is simply given by \(C|\phi_\pm\rangle\) in the present case.

The relations of bi-orthonormal bases, \(|\phi_\pm\rangle\) and \(|\varphi_\mp\rangle\), are summarized in Fig. 1.

---

\(^*\) Similarly, for the lower-leaf \((z \leq -r)\), the eigenvectors are

\[ |\phi'_+\rangle = \frac{1}{\sqrt{2r(r-z)}} \begin{pmatrix} -y - ix \\ r - z \end{pmatrix}, \quad |\phi'_-\rangle = \frac{1}{\sqrt{2r(r-z)}} \begin{pmatrix} r - z \\ -y + ix \end{pmatrix}, \tag{4.20} \]

and

\[ |\varphi'_+\rangle = \frac{1}{\sqrt{2r(r-z)}} \begin{pmatrix} y + ix \\ r - z \end{pmatrix}, \quad |\varphi'_-\rangle = \frac{1}{\sqrt{2r(r-z)}} \begin{pmatrix} r - z \\ y - ix \end{pmatrix}. \tag{4.21} \]

They satisfy

\[ \langle\varphi'_\pm|\sigma_z|\varphi'_\mp\rangle = \langle\varphi'_m|\sigma_z|\varphi'_n\rangle = \mp 1, \quad \langle\phi'_m|\varphi'_n\rangle = \delta_{mn}. \tag{4.22} \]
§5. Random matrix class of $SO(3, 2)$ model

As a nontrivial realization of the generalized Kramers degeneracy, we introduce the $SO(3, 2)$ model. First consider K symmetry realized by

$$k = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix},$$

(5.1)

which satisfies $kk^* = 1$. Any arbitrary $4 \times 4$ matrix can be expanded as

$$H = h \, 1_4 + \sum_{a=1}^{5} h^a \gamma_a + \sum_{a<b=1}^{5} h^{ab} \gamma_{ab},$$

(5.2)

where $1_4$ is a $4 \times 4$ unit matrix, $h, h^a$ and $h^{ab}$ stand for complex parameters, $\gamma_a$ denotes $SO(3, 2)$ gamma matrices

$$\gamma_1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & -i\sigma_z \\ i\sigma_z & 0 \end{pmatrix},$$

$$\gamma_4 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix},$$

(5.3)

and $\gamma_{ab}$ are $SO(3, 2)$ generators constructed by

$$\gamma_{ab} = \frac{1}{4i} [\gamma_a, \gamma_b].$$

(5.4)

The sixteen matrices, $1_4, \gamma_a, \gamma_{ab}$, amount to complete matrix bases that span an arbitrary $4 \times 4$ matrix. The imposition of K symmetry specifies $h$ and $h^a$ as real parameters and $h^{ab}$ as pure imaginary parameters,

$$h^* = h, \quad h^{a*} = h^a, \quad h^{ab*} = -h^{ab}.$$  

(5.5)

Thus, the Hamiltonian becomes

$$H = w \, 1_4 + \sum_{a=1}^{5} x^a \gamma_a + i \sum_{a<b=1}^{5} x^{ab} \gamma_{ab},$$

(5.6)

with real parameters, $w, x^a$ and $x^{ab}$. We further impose Q symmetry

$$q = \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix},$$

(5.7)

which satisfies $q^* = -k^{-1} q k^{\dagger -1}$. According to two types of Q symmetry, $\epsilon_q = \pm 1$, the Hamiltonian takes two different forms as one of which is shown in §5.1 and the other is in §5.2.
5.1. $\epsilon_q = +1$: pseudo-Hermiticity

In this case, the Hamiltonian becomes

$$H = w 1_4 + \sum_{a=1}^{5} x^a \gamma_a.$$  \hspace{1cm} (5.8)

From the general theorem 2 in §3.2, this Hamiltonian is expected to exhibit Kramers degeneracy. The Hamiltonian is invariant under the time-reversal symmetry

$$\Theta H \Theta^{-1} = H,$$ \hspace{1cm} (5.9)

where

$$\Theta = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} \cdot K,$$ \hspace{1cm} (5.10)

with $\Theta^2 = +1$.

The $SO(3, 2)$ Hamiltonian (5.8) is rewritten as

$$H = \begin{pmatrix} x^5_{12} & x^4_{12} - i x^i \tau_i \\ x^4_{12} + i x^i \tau_i & -x^5_{12} \end{pmatrix};$$ \hspace{1cm} (5.11)

where $x^i \tau_i \equiv x_\tau + y_\tau y + z_\tau z$. ($\sum_{i=1}^{3} x^i \tau_i$ will be abbreviated as $x^i \tau_i$ hereafter.) The eigenvalues of $H$ are derived as $E_\pm = \pm \sqrt{- (x^1)^2 - (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2}$. Note that we have twofold degeneracy in the spectrum, which comes from the generalized Kramers theorem mentioned in §2.3.

Let us consider situations where all the eigenvalues of (5.11) are real:

$$(x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2 - (x^5)^2 = -r^2 \leq 0,$$ \hspace{1cm} (5.12)

where $r$ is a real positive constant. (When $x^4 = x^5 = 0$, the $SO(3, 2)$ model is reduced to two independent $SU(1, 1)$ models.) The eigenvalues of the $SO(3, 2)$ model (5.11) are

$$E_\pm = \pm r.$$ \hspace{1cm} (5.13)

Here, $E_+$ and $E_-$ are doubly degenerate. For $x^5 > -r$, the eigenvectors are given by

$$|\psi_+\rangle = \frac{1}{\sqrt{2r(r + x^5)}} \begin{pmatrix} (r + x^5) \phi_+ \rangle \\ (x^4 + ix^i \tau_i) \phi_+ \rangle \end{pmatrix},$$

$$|\psi_-\rangle = \frac{1}{\sqrt{2r(r + x^5)}} \begin{pmatrix} (-x^4 + ix^i \tau_i) \phi_- \rangle \\ (r + x^5) \phi_- \rangle \end{pmatrix},$$ \hspace{1cm} (5.14)

where $\phi_\alpha (\alpha = \pm)$ represent two-component spinors that account for double degeneracy. Take $\phi_\pm$ as

$$\phi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \phi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ \hspace{1cm} (5.15)

\footnote{This is the first nontrivial form of self-dual real split-quaternion matrix. See Appendix C for details.}
The Hermitian conjugate of the Hamiltonian (5.11) is

\[
H^\dagger = \begin{pmatrix}
x^5 1_2 & x^4 1_2 - ix^i \tau_i^\dagger \\
x^4 1_2 + ix^i \tau_i & -x^5 1_2
\end{pmatrix}.
\]

(5.16)

Since

\[
H^\dagger H^\dagger = H^2 = r^2 1_4,
\]

(5.17)

the eigenvalues of \(H^\dagger\) are also given by \(\pm r\), and the corresponding eigenvectors are\(^*\)

\[
|\chi_{+\alpha}\rangle = \frac{1}{\sqrt{2r(r + x^5)}} \begin{pmatrix} (r + x^5) \phi_\alpha \\ x^4 + ix^i \tau_i \phi_\alpha \end{pmatrix},
\]

\[
|\chi_{-\alpha}\rangle = \frac{1}{\sqrt{2r(r + x^5)}} \begin{pmatrix} (r - x^5) \phi_\alpha \\ -x^4 + ix^i \tau_i \phi_\alpha \end{pmatrix}.
\]

(5.22)

\(|\psi_{m\alpha}\rangle\) and \(|\chi_{m\alpha}\rangle\) are related as

\[
|\psi_{m\alpha}\rangle \xrightarrow{\eta} \eta|\psi_{m\alpha}\rangle = \alpha|\chi_{m\alpha}\rangle.
\]

(5.23)

With (5.14) and (5.22), it is straightforward to confirm that \(|\psi_{m\alpha}\rangle\) and \(|\chi_{m\alpha}\rangle\) indeed constitute the bi-orthonormal basis:

\[
\sum_{m,\alpha=+,-} |\psi_{m\alpha}\rangle \langle \chi_{m\beta}| = |\psi_{+++}\rangle \langle \chi_{++}| + |\psi_{++-}\rangle \langle \chi_{+-}| + |\psi_{-+-}\rangle \langle \chi_{-+}| + |\psi_{---}\rangle \langle \chi_{--}| = 1_4.
\]

(5.24)

\(*\) For \(x^5 < r\), the eigenvectors are

\[
|\psi'_{+\alpha}\rangle = \frac{1}{\sqrt{2r(r - x^5)}} \begin{pmatrix} -x^4 + ix^i \tau_i \phi_\alpha \\ -(r - x^5) \phi_\alpha \end{pmatrix},
\]

\[
|\psi'_{-\alpha}\rangle = \frac{1}{\sqrt{2r(r - x^5)}} \begin{pmatrix} (r - x^5) \phi_\alpha \\ -x^4 + ix^i \tau_i \phi_\alpha \end{pmatrix}.
\]

(5.18)

\(|\psi_{\pm\alpha}\rangle\) and \(|\psi'_{\pm\alpha}\rangle\) are related by the \(SU(1, 1)\) transformation

\[
g_{\pm} = \frac{1}{\sqrt{r^2 - (x^5)^2}} (-x^4 \pm ix^i \tau_i),
\]

(5.19)

where \(g_- = g_+^{-1}\). The eigenvectors of \(H^\dagger\) are

\[
|\chi'_{+\alpha}\rangle = \frac{1}{\sqrt{2r(r - x^5)}} \begin{pmatrix} -x^4 + ix^i \tau_i \phi_\alpha \\ -(r - x^5) \phi_\alpha \end{pmatrix},
\]

\[
|\chi'_{-\alpha}\rangle = \frac{1}{\sqrt{2r(r - x^5)}} \begin{pmatrix} (r - x^5) \phi_\alpha \\ -x^4 + ix^i \tau_i \phi_\alpha \end{pmatrix}.
\]

(5.20)

\(|\chi_{\pm\alpha}\rangle\) and \(|\chi'_{\pm\alpha}\rangle\) are related by the \(SU(1, 1)\) transformation

\[
g'_{\pm} = \frac{1}{\sqrt{r^2 - (x^5)^2}} (-x^4 \pm ix^i \tau_i),
\]

(5.21)

where \(g'_- = g'_+^{-1}\).
With (5.23), the time-independent inner products induced by $\eta$ are given as

$$\langle \psi_m | \eta | \psi_n \rangle = \langle \chi_m | \eta | \chi_n \rangle = \alpha \delta_{mn} \delta_{\alpha \beta}.$$  (5.25)

Note that the signs of these products depend on their “spin” directions.

As mentioned above, twofold degeneracy is a consequence of the generalized Kramers theorem. The generalized Kramers pair of $|\psi_{m\alpha}\rangle$ is given by $\Theta \eta^{-1} |\chi_{ma}\rangle$.

Here, $\eta$ and $\Theta$ are given by $\eta = q$ (5.7) and $\Theta$ (5.10). Therefore, the generalized Kramers pair is derived as

$$\Theta \eta^{-1} |\chi_{ma}\rangle = - \left( \begin{array}{cc} i\sigma_y & 0 \\ 0 & i\sigma_y \end{array} \right) K |\chi_{ma}\rangle = \alpha |\psi_{m\bar{\alpha}}\rangle,$$  (5.26)

where $\bar{\alpha}$ is the opposite spin of $\alpha$; $\bar{\alpha} = -\alpha$, i.e. $(+) = -$, $(-) = +$. Thus, $|\psi_{ma}\rangle$ and $|\psi_{ma}\rangle$ are indeed the generalized Kramers pair. Notice that the “spins” $\alpha$ of the generalized Kramers pair are opposite to each other. Although they are not orthogonal in the ordinary sense, they are linearly independent. In the present case, they are orthogonal in the following inner product:

$$\langle \psi_{m\bar{\alpha}} | \eta | \psi_{m\alpha} \rangle = 0.$$  (5.27)

Since we have the relation (5.23) that corresponds to (2.24) with real eigenenergies, the generalized Kramers pair $\Theta \eta^{-1} |\chi_{ma}\rangle$ is equivalent to $\Theta |\psi_{ma}\rangle$ (see §2.3). Indeed,

$$|\psi_{ma}\rangle \xrightarrow{\Theta} \Theta |\psi_{ma}\rangle = |\psi_{m\bar{\alpha}}\rangle.$$  (5.28)

Thus, we find that the generalized Kramers pair is simply given by $\Theta |\psi_{ma}\rangle$.

The relations between $|\psi_{ma}\rangle$ and $|\chi_{ma}\rangle$ are summarized in Fig. 2.

5.2. $\epsilon_q = -1$: pseudo-anti-Hermiticity

When $\epsilon_q = -1$, the Hamiltonian takes the form of

$$H = i \sum_{a<b=1}^{5} x^{ab} \gamma_{ab}.$$  (5.29)

The Hamiltonian is invariant under the particle/hole symmetry

$$CHC^{-1} = -H,$$  (5.30)
where
\[ C = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} \cdot K, \quad (5.31) \]

with \( C^2 = +1 \). From theorem 3 in §3.2, we expect \( H \) has partner states whose energies are \( E \) and \( -E \). Indeed, an explicit calculation shows that the Hamiltonian has two paired states with energies \((E_1, -E_1)\) and \((E_2, -E_2)\) (in general \( E_1 \neq E_2 \)).

§6. \( SU(2), SU(1, 1) \) spins and time-reversal operation with \( \Theta^2 = +1 \)

In this section, we consider \( SU(2) \) integer spins and \( SU(1, 1) \) spins in the context of time-reversal operation with \( \Theta^2 = +1 \) and split-quaternions. It will be shown that \( SU(2) \) integer spin Hamiltonians invariant under time-reversal operations with \( \Theta^2 = +1 \) are expressed using split-quaternions. We then consider \( SU(1, 1) \) spins as a non-Hermitian generalization of \( SU(2) \) spins. We will show that the \( SO(3, 2) \) Hamiltonian discussed in §5 is realized as an \( SU(1, 1) \) spin 3/2 quadrupole model.

6.1. \( SU(2) \) integer spin systems

In this subsection, we will see the split-quaternion structure of integer spin systems in which \( \Theta^2 = +1 \). The spin, \( S_i \) \((i = x, y, z)\), changes its sign under the TR transformation
\[ \Theta S_i \Theta^{-1} = -S_i. \quad (6.1) \]
Write \( \Theta = UK \), and the condition (6.1) is written as
\[ US_x U^{-1} = -S_x, \quad US_y U^{-1} = S_y, \quad US_z U^{-1} = -S_z, \quad (6.2) \]
where we have assumed the standard matrix realization of \( S_i \) in which only \( S_y \) is complex and pure imaginary. Then, \( U \) is given by
\[ U = e^{i\pi S_y}, \quad (6.3) \]
and
\[ \Theta^2 = e^{2i\pi S_y} = (-1)^{2S}, \quad (6.4) \]
where \( S \) represents the magnitude of the spin. Thus, for integer \( S \), \( \Theta^2 = +1 \), while for half-integer \( S \), \( \Theta^2 = -1 \). For low spins,
\[ S = 1/2 : \ \Theta = i\sigma_y K, \quad (6.5a) \]
\[ S = 1 : \ \Theta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} K, \quad (6.5b) \]
\[ S = 3/2 : \ \Theta = \begin{pmatrix} 0 & i\sigma_y & 0 \\ i\sigma_y & 0 & 0 \end{pmatrix} K. \quad (6.5c) \]

*) The expressions of \( E_1 \) and \( E_2 \) are rather lengthy, so we omit their explicit formulae.

*) The form of the TR operator depends on the physical meaning of the operator. For instance, when \( S_i \) denotes “isospin” labeling two different energy levels, the TR operator \( \Theta \) is simply given by \( \Theta = K \).
Let us first consider the $S = 1$ system. The TR operator is given by (6.5b). Since $U$ is a real symmetric matrix, it can be diagonalized by an orthogonal matrix $O$,

$$
\Theta = O \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} O^T K. \tag{6.6}
$$

Then, $\Theta$ is written as

$$
\Theta = OV V^T O^T K = (OV) K (OV)^\dagger \tag{6.7}
$$

with $V = \text{diag}(1, i, i)$. Therefore, performing the unitary transformation $OV$ on the basis, $\Theta$ is recast into $\Theta = K$.

Let us now impose the TR invariance $\Theta$ on the Hamiltonian. In general, the Hamiltonian $H$ is written as

$$
H = \begin{pmatrix} h + h^x \sigma_x + i h^y \sigma_y + h^z \sigma_z & a^T \\ b & c \end{pmatrix}, \tag{6.8}
$$

where $h$ and $h^i$ ($i = x, y, z$) are complex numbers, $a$ and $b$ are two component complex vectors, and $c$ is a complex number. Take $[H, \Theta] = 0$ with $\Theta = K$; then, all the elements of $H$ are real. In particular, from the correspondence (2.36), this implies that the left upper part of $H$ is given by a split-quaternion, $h^r = w + x \sigma_x + i y \sigma_y + z \sigma_z$ with real coefficients $w, x, y, z$.

Next, consider the $S = 2$ case, in which $S_y$ is given by

$$
S_y = \frac{i}{2} \begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}, \tag{6.9}
$$

and the corresponding TR operator $\Theta$ becomes

$$
\Theta = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} K. \tag{6.10}
$$

In a manner similar to the above, we have $\Theta = K$ by choosing the basis properly. The TR invariant Hamiltonian is given by

$$
H = \begin{pmatrix} h_{11}^R & h_{12}^R & a_1^T \\ h_{21}^R & h_{22}^R & a_2^T \\ b_1 & b_2 & c \end{pmatrix}, \tag{6.11}
$$

where $h_{IJ}^R$ ($I, J = 1, 2$) are split-quaternions of the form

$$
h_{IJ}^R = w_{IJ} + x_{IJ} \sigma_x + iy_{IJ} \sigma_y + z_{IJ} \sigma_z, \tag{6.12}
$$
with real coefficients \( w_{IJ}, x_{IJ}, y_{IJ}, z_{IJ} \). Here, \( a_I \) and \( b_I \) \((I = 1, 2)\) are real two-component row vectors, and \( c \) is a real constant.

In general, for \( S = M \) with integer \( M \), we can always choose the basis in which \( \Theta \) is given by \( \Theta = K \). Then, a TR invariant Hamiltonian \( H \) is given by

\[
H = \begin{pmatrix}
  h^R_{IJ} & a^T_I \\
b_I & c
\end{pmatrix},
\]

where \( h^R_{IJ} \) \((I, J = 1, \cdots, M)\) are split-quaternions, and \( a_I \) and \( b_I \) \((I = 1, \cdots, M)\) are real two-component row vectors, and \( c \) is a real constant.

Note that if we impose the Hermiticity on the Hamiltonian (6.13), \( H \) reduces to a real symmetric Hamiltonian. Thus, our result here is consistent with the known result that the Hermitian TR invariant systems with integer spins belong to the orthogonal ensembles.\(^1\),\(^2\)

6.2. \( SO(3, 2) \) Hamiltonian as \( J = 3/2 \) \( SU(1, 1) \) quadrupole model

In Ref. 2), Avron et al. demonstrated that the \( S = 3/2 \) quadrupole Hamiltonian can be expressed by an \( SO(5) \) Hamiltonian. Here, we demonstrate how such arguments are generalized to the present non-Hermitian case. The correspondences between \( SU(2) \) and \( SO(5) \), and \( SU(1, 1) \) and \( SO(3, 2) \) suggest that the \( SU(1, 1) \) spin \( 3/2 \) quadrupole Hamiltonian may be expressed by an \( SO(3, 2) \) Hamiltonian. The \( SU(1, 1) \) spin \( J_i \) \((i = x, y, z)\) is defined so as to satisfy

\[
[J_i, J_j] = i\epsilon_{ijk}J^k,
\]

where \( \epsilon_{ijk} \) denotes the totally antisymmetric 3-rank tensor with \( \epsilon_{xyz} = 1 \), and \( J^i = (J_x, J_y, -J_z) \). With real \( 3 \times 3 \) quadrupole coefficients \( Q^{ij} \), we introduce the \( SU(1, 1) \) quadrupole Hamiltonian as

\[
H(Q) = \sum_{i,j=x,y,z} Q^{ij} J_i J_j.
\]

The \( SU(1, 1) \) quadrupole Hamiltonian is invariant under the \( SU(1, 1) \) spin flipping transformation

\[
J_i \rightarrow -J_i.
\]

The five basis elements of \( Q^{ij} \) are taken as

\[
Q_1 = \frac{1}{\sqrt{3}} \begin{pmatrix}
  0 & 0 & 1 \\
  0 & 0 & 0 \\
  1 & 0 & 0
\end{pmatrix}, \quad Q_2 = \frac{1}{\sqrt{3}} \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 1 \\
  0 & 1 & 0
\end{pmatrix}, \quad Q_3 = \frac{1}{\sqrt{3}} \begin{pmatrix}
  1 & 0 & 0 \\
  0 & -1 & 0 \\
  0 & 0 & 0
\end{pmatrix},
\]

\[
Q_4 = \frac{1}{\sqrt{3}} \begin{pmatrix}
  0 & 1 & 0 \\
  1 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix}, \quad Q_5 = \frac{1}{3} \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 2
\end{pmatrix},
\]

\[6.17\]
which are orthonormal

\[(Q_a, Q_b) = \frac{3}{2} \text{Tr}(Q_a Q_b) = \delta_{ab}. \quad (6.18)\]

With the use of \(Q_a\), an arbitrary quadrupole matrix is expanded as

\[Q = \sum_{a=1}^{5} x^a Q_a, \quad (6.19)\]

where \(x^a\) are real, and the \(SU(1,1)\) quadrupole Hamiltonian (6.15) is expressed as

\[H(Q) = \sum_{a=1}^{5} x^a H(Q_a), \quad (6.20)\]

where

\[H(Q_a) = \sum_{i,j=x,y,z} (Q_a)^{ij} J_i J_j. \quad (6.21)\]

More explicitly, they are

\[H(Q_1) = \frac{1}{\sqrt{3}} \{J_x, J_z\}, \quad H(Q_2) = \frac{1}{\sqrt{3}} \{J_y, J_z\}, \quad H(Q_3) = \frac{1}{\sqrt{3}} (J_x^2 - J_y^2), \quad (6.22)\]

In particular, for \(J = 3/2\), the \(SU(1,1)\) spin is given by \(4 \times 4\) matrices

\[J_x = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3}i & 0 & 0 \\ \sqrt{3}i & 0 & 2i & 0 \\ 0 & 2i & 0 & \sqrt{3}i \\ 0 & 0 & \sqrt{3}i & 0 \end{pmatrix}, \quad J_y = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}, \quad (6.23)\]

The spin flipping operator \(\Theta'\) satisfying

\[\Theta' J_i \Theta'^{-1} = -J_i, \quad (6.24)\]

is given by

\[\Theta' = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} \cdot K. \quad (6.25)\]

\(^{a)}\) Note that \(Q_5\) is different from the traceless quadrupole matrix, \(\frac{1}{3} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\), used in Ref. 2.
Since $\Theta'$ satisfies
\[ \Theta'^2 = +1, \] (6.26)
$\Theta'$ can be regarded as the TR operator for $\Theta'^2 = +1$. (See Appendix D for more details about the $SU(1, 1)$ spin flipping operator.) Since $H(Q_a)$ denotes the quadratic forms of the $SU(1, 1)$ spins (6.22), they are invariant under the $SU(1, 1)$ spin flipping operation. By substituting (6.23) into (6.22), $H(Q_a)$ is explicitly derived as
\[ H(Q_a) = \gamma'_a, \] (6.27)
with $\gamma'_a$ being $SO(3, 2)$ gamma matrices
\[
\begin{align*}
\gamma'_1 &= \begin{pmatrix} i\sigma_x & 0 \\ 0 & -i\sigma_x \end{pmatrix}, & \gamma'_2 &= \begin{pmatrix} i\sigma_y & 0 \\ 0 & -i\sigma_y \end{pmatrix}, & \gamma'_3 &= \begin{pmatrix} 0 & -1_2 \\ -1_2 & 0 \end{pmatrix}, \\
\gamma'_4 &= \begin{pmatrix} 0 & i1_2 \\ -i1_2 & 0 \end{pmatrix}, & \gamma'_5 &= \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix}. 
\end{align*}
\] (6.28)

Therefore, the $J = 3/2$ $SU(1, 1)$ quadrupole model is expressed by
\[ H = \sum_{a=1}^{5} x^a \gamma'_a, \] (6.29)
and the Hamiltonian is invariant under the TR transformation of $\Theta'$, $[H, \Theta'] = 0$. By rearranging the basis, $\gamma'_a$ (6.28) is transformed to the previous $SO(3, 2)$ gamma matrices $\gamma_a$ (5.3), and $\Theta'$ (6.25) is also transformed to $\Theta$ (5.10). Thus, we have shown that the $J = 3/2$ $SU(1, 1)$ quadrupole Hamiltonian is equivalent to the $SO(3, 2)$ Hamiltonian.

§7. Summary and discussion

We explored the generalized Kramers degeneracy for $\Theta^2 = +1$ in pseudo-Hermitian quantum mechanics. As the quaternions realize the TR operation for $\Theta^2 = -1$, the split-quaternions realize the TR operation for $\Theta^2 = +1$. We showed, by passing from the quaternions to split-quaternions, the following generalized theorems in pseudo-Hermitian quantum mechanics:

- If the system is invariant under the TR transformation $\Theta^2 = +1$ and also TR operator $\Theta$ is anticommutative with the metric operator, the system has at least doubly degenerate states: the generalized Kramers pair.
- When the system is invariant under the particle/hole transformation $C^2 = +1$ and also charge-conjugation operator $C$ is anticommutative with the metric operator, the system has paired states with $E$ and $-E$: the particle/hole pair.

In both cases, the Hamiltonians necessarily possess the split-quaternion structure. We also identified TR, particle/hole, and pseudo-(anti-)Hermitian symmetries in the non-Hermitian category proposed by Bernard and LeClair, and reconsidered the

\[ \gamma'_a \] and $\gamma_a$ are related by unitary transformation, $U = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_z & -i\sigma_z \\ -i\sigma_z & \sigma_z \end{pmatrix}$.\]
above theorems in view of a non-Hermitian random matrix. As a concrete example of the second theorem stated above, we investigated the $SU(1,1)$ model, and confirmed that the theorem indeed holds. Similarly, as an example of the first theorem, we introduced the $SO(3,2)$ model. We confirmed that the $SO(3,2)$ Hamiltonian is invariant under TR transformation, and the TR symmetry brings double degeneracy to the $SO(3,2)$ model, exactly analogous to the Kramers degeneracy of the $SO(5)$ model. The correspondence between the $SO(3,2)$ Hamiltonian and the $SU(1,1)$ spin $3/2$ model was also clarified.

As pointed out in Ref. 2), the structure of the original $SO(5)$ model is related to instantons and the twistor theory. Similarly, the present $SO(3,2)$ model is related to split-instantons and the twistor theory. The present work was inspired by recent developments of the pseudo-Hermitian quantum mechanics and the topological insulator in condensed matter physics. This work may hopefully be regarded as the first step of the interplay between these two developments. It is intriguing to speculate realizations of the pseudo-Hermitian quantum mechanics in condensed matter physics. We would like to pursue the issue in a future research.

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Appendix A

SU(1,1) and SO(3,2)

Here, we briefly review the noncompact groups, $SU(1,1)$ and $SO(3,2)$.

The $SU(1,1)$ group consists of $2 \times 2$ matrices $g$ satisfying the following relations,

$$g^\dagger \sigma_z g = \sigma_z, \quad \det g = 1. \quad (A.1)$$

Expanding $g$ by its generators, $\tau_i$ ($i = x, y, z$), $g = 1 + i \sum_i \theta_i \tau_i + \cdots$ with real parameters $\theta_i$, we obtain

$$\tau_i^\dagger \sigma_z - \sigma_z \tau_i = 0, \quad \text{tr} \tau_i = 0. \quad (A.2)$$

Thus, the generators of $SU(1,1)$ are given by

$$\tau_x = i\sigma_x, \quad \tau_y = i\sigma_y, \quad \tau_z = \sigma_z, \quad (A.3)$$

where $\sigma_i$ ($i = x, y, z$) denotes the standard Pauli matrices for $SU(2)$ group. The $SU(1,1)$ Pauli matrices, $\tau_i = \langle \tau_x, \tau_y, \tau_z \rangle = \langle i\sigma_x, i\sigma_y, \sigma_z \rangle$, satisfy the following relations:

$$\sigma_x \tau_i \sigma_x^{-1} = -\tau_i^*, \quad (A.4a)$$
\[
\sigma_y \tau_i \sigma_y^{-1} = -\tau_i^t, \quad (A.4b)
\]
\[
\sigma_z \tau_i \sigma_z^{-1} = \tau_i^t. \quad (A.4c)
\]

The \(\text{SO}(3, 2)\) group denotes linear transformations with unit determinant acting on a five-dimensional vector \((x_1, x_2, x_3, x_4, x_5)\) and preserving the following norm,
\[
-x_1^2 - x_2^2 + x_3^2 + x_4^2 + x_5^2. \quad (A.5)
\]
Its element is given by a \(5 \times 5\) matrix \(G\), which satisfies
\[
\sum_{cd} G_{ac} G_{bd} \eta_{cd} = \eta_{ab}, \quad \det G = 1, \quad (A.6)
\]
with \(\eta_{ab} = \text{diag}(-1, -1, 1, 1, 1)\). Expanding \(G\) by its generators \(M^{ab}\) \((a, b = 1, 2, \ldots, 5)\), we find
\[
(M^{ab})_{cc} \eta_{cd} + (M^{ab})_{de} \eta_{cc} = 0, \quad (M^{ab})_{cc} = 0, \quad (A.7)
\]
where we use the convention in which the repeating indices are summed. These relations are met by the following \(M^{ab}\),
\[
(M^{ab})_{cd} = i(\delta_{ca} \eta_{bd} - \delta_{cb} \eta_{ad}), \quad (A.8)
\]
which satisfies
\[
[M^{ab}, M^{cd}] = -i(\eta_{ac} M^{bd} - \eta_{bc} M^{ad} - \eta_{ad} M^{bc} - \eta_{bd} M^{ac}). \quad (A.9)
\]

Now, consider the spinor representation of \(\text{SO}(3, 2)\). The spinor representation is given by the \(4 \times 4\) matrices \(\gamma_a\) with anticommutation relations
\[
\{\gamma_a, \gamma_b\} = 2\eta_{ab}. \quad (A.10)
\]
The anticommutation relation is obtained by
\[
\gamma_1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & -i\sigma_z \\ i\sigma_z & 0 \end{pmatrix},
\]
\[
\gamma_4 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \quad (A.11)
\]
and the \(\text{SO}(3, 2)\) algebra \((A.9)\) is realized by
\[
M^{ab} = \gamma_{ab} \equiv \frac{1}{4i}[\gamma_a, \gamma_b]. \quad (A.12)
\]
The gamma matrices \((A.11)\) and generators \((A.12)\) of \(\text{SO}(3, 2)\) satisfy the following relations,
\[
k\gamma_a k^{-1} = \gamma_a^*, \quad k\gamma_{ab} k^{-1} = -\gamma_{ab}^*, \quad (A.13a)
\]
\[
c\gamma_a c^{-1} = \gamma_a^t, \quad c\gamma_{ab} c^{-1} = -\gamma_{ab}^t, \quad (A.13b)
\]
\[
q\gamma_a q^{-1} = \gamma_a^t, \quad q\gamma_{ab} q^{-1} = \gamma_{ab}^t. \quad (A.13c)
\]
where
\[
k = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}, \quad c = \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}, \quad q = \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}, \quad (A.14)
\]
Appendix B

Quaternion and Split-Quaternion

The quaternion \( (1, e_1, e_2, e_3) \) is defined so as to satisfy

\[
(e_1)^2 = -1, \quad (e_2)^2 = -1, \quad (e_3)^2 = -1, \\
e_i e_j = -e_j e_i \quad (i \neq j), \quad e_1 e_2 e_3 = -1. \tag{B.1}
\]

The “imaginary” quaternions are realized as Pauli matrices as

\[
(e_1, e_2, e_3) = (-i \sigma_x, -i \sigma_y, -i \sigma_z). \tag{B.2}
\]

The split-quaternion algebra \( (1, q_1, q_2, q_3) \) is simply obtained by flipping two signs of squares of quaternions:

\[
(q_1)^2 = +1, \quad (q_2)^2 = +1, \quad (q_3)^2 = -1, \\
q_i q_j = -q_j q_i \quad (i \neq j), \quad q_1 q_2 q_3 = -1. \tag{B.3}
\]

The split-quaternions are realized by non-Hermitian matrices as

\[
(q_1, q_2, q_3) = (i \tau_x, i \tau_y, i \tau_z) = (-\sigma_x, -\sigma_y, i \sigma_z), \tag{B.4}
\]

where \( \tau_i \) (\( i = x, y, z \)) denotes the \( SU(1, 1) \) “Pauli matrices” (A.3). They satisfy

\[
[\tau_i, \tau_j] = 2i \varepsilon_{ijk} \tau^k, \quad \{\tau_i, \tau_j\} = -2 \eta_{ij}, \tag{B.5}
\]

where \( \varepsilon_{ijk} \) is the three rank antisymmetric tensor with \( \varepsilon_{xyz} = 1 \), while \( \eta_{ij} = \text{diag}(+1, +1, -1) \) and \( \tau^i = (\tau_x, \tau_y, -\tau_z) \). By replacing the imaginary unit \( i \) in \( \sigma_y \) (B.2) with three imaginary quaternions, the Pauli matrices are “enhanced” to yield \( SO(5) \) gamma matrices:

\[
\begin{pmatrix}
0 & i \sigma_x \\
-i \sigma_x & 0
\end{pmatrix}, \quad 
\begin{pmatrix}
0 & i \sigma_y \\
-i \sigma_y & 0
\end{pmatrix}, \quad 
\begin{pmatrix}
0 & i \sigma_z \\
-i \sigma_z & 0
\end{pmatrix}, \\
\begin{pmatrix}
0 & 1_2 \\
1_2 & 0
\end{pmatrix}, \quad 
\begin{pmatrix}
1_2 & 0 \\
0 & -1_2
\end{pmatrix}. \tag{B.6}
\]

It is straightforward to see that (B.6) satisfies \( \{\gamma_a, \gamma_b\} = 2 \delta_{ab} \). By applying such substitution in the case of split-quaternions, we obtain \( 4 \times 4 \) non-Hermitian gamma matrices of \( SO(3, 2) \) (A.11). The correspondence can also be naturally understood by noticing the isomorphism of groups: \( SU(2) \cong USp(2) \) and \( SO(5) \cong USp(4); \quad SU(1, 1) \cong Sp(2, R) \) and \( SO(3, 2) \cong Sp(4, R) \).

Appendix C

Definitions and Relations for Split-Quaternions

We introduce the terminology for split-quaternions in the same spirit for quaternions (see Refs. 1 and 22) for instance). The split-quaternion generally takes the form of

\[
q = c q_0 + c_i q_i, \tag{C.1}
\]
where $c$ and $c_i$ ($i = 1, 2, 3$) are complex numbers. There are three types of conjugation for split-quaternion: the complex conjugate, split-quaternionic conjugate, and split-quaternionic Hermitian conjugate, which are respectively defined by

$$q^* = c^*q_0 + c_i^*q_i,$$  
$$\overline{q} = cq_0 - c_iq_i,$$  
$$q^\dagger \equiv (q^*)^\dagger = c^*q_0 - c_i^*q_i. \quad (C.2)$$

Such conjugations have the following properties:

$$(q_1 \cdot q_2)^* = q_1^* \cdot q_2^*, \quad (q_1 \cdot q_2)^\dagger = q_2^\dagger \cdot q_1^\dagger. \quad (C.3)$$

With the matrix realization (B.4), the split-quaternion (C.1) is expressed as

$$q = c + c_i i\tau_i = c - c_1 \sigma_x - c_2 \sigma_y + c_3 i\sigma_z = \begin{pmatrix} c + ic_3 & -c_1 + ic_2 \\ -c_1 - ic_2 & c - ic_3 \end{pmatrix}. \quad (C.4)$$

Correspondingly, the three types of conjugate (C.2) are

$$q^* = c^* + c_i^* i\tau_i = c^* - c_1^* \sigma_x - c_2^* \sigma_y + c_3^* i\sigma_z = \begin{pmatrix} c^* + ic_3^* & -c_1^* + ic_2^* \\ -c_1^* - ic_2^* & c^* - ic_3^* \end{pmatrix}, \quad (C.5a)$$

$$\overline{q} = c - c_i i\tau_i = c + c_1 \sigma_x + c_2 \sigma_y - c_3 i\sigma_z = \begin{pmatrix} c - ic_3 & c_1 - ic_2 \\ c_1 + ic_2 & c + ic_3 \end{pmatrix}, \quad (C.5b)$$

$$q^\dagger = c^* - c_i^* i\tau_i = c^* + c_1^* \sigma_x + c_2^* \sigma_y - c_3^* i\sigma_z = \begin{pmatrix} c^* - ic_3^* & c_1^* - ic_2^* \\ c_1^* + ic_2^* & c^* + ic_3^* \end{pmatrix}. \quad (C.5c)$$

Owing to the non-Hermitian property of the split-quaternions, $\tau_i^\dagger = (-\tau_x, -\tau_y, \tau_z)$, the split-quaternionic Hermitian conjugate (C.5c) does not coincide with the ordinary definition of the Hermitian conjugate

$$q^\dagger = c^* - c_i^* i\tau_i^\dagger = c^* - c_1^* \sigma_x - c_2^* \sigma_y - c_3^* i\sigma_z = \begin{pmatrix} c^* - ic_3^* & -c_1^* + ic_2^* \\ -c_1^* - ic_2^* & c^* + ic_3^* \end{pmatrix}. \quad (C.6)$$

(In the quaternion case, the quaternionic Hermitian conjugate coincides with the ordinary Hermitian conjugate.)

The real split-quaternion is defined as

$$q^r = wq_0 + x^i q_i = \begin{pmatrix} w + ix^3 \\ -x^1 - ix^2 \\ -x^1 + ix^2 \\ w - ix^3 \end{pmatrix}, \quad (C.7)$$

where $w$ and $x^i$ ($i = 1, 2, 3$) are real numbers. The necessary and sufficient condition for the real split-quaternion is given by

$$q^\dagger = \overline{q}. \quad (C.8)$$

An $M \times M$ split-quaternion matrix ($2M \times 2M$ matrix in the usual sense) $Q$ is defined as a matrix whose matrix elements are split-quaternions:

$$(Q)_{IJ} = q_{IJ}. \quad (C.9)$$
where \( I, J = 1, 2, \cdots, M \). The complex conjugation, split-quaternionic conjugation, and split-quaternionic Hermitian conjugation of \( Q \) are respectively defined as

\[
(Q^*_{IJ}) = q_{IJ}^*, \quad (\overline{Q})_{IJ} = \overline{q_{JI}}, \quad (Q^\dagger_{IJ}) = q_{IJ}^\dagger.
\]

(C.10a) \hspace{2cm} (C.10b) \hspace{2cm} (C.10c)

We call \( \overline{Q} \) the “dual” of \( Q \). The split-quaternionic Hermitian matrix is a split-quaternion matrix that satisfies

\[
Q^\dagger = Q.
\]

(C.11)

Unlike the quaternion matrix, the split-quaternionic Hermitian matrix is not Hermitian in the usual sense. For instance, \( q = w + ix^i q_i = w - x^i \tau_i = \begin{pmatrix} w - x^3 & -ix^1 - x^2 \\ -ix^1 + x^2 & w + x^3 \end{pmatrix} \) (with real numbers \( w, x^1, x^2, x^3 \)) is a split-quaternionic Hermitian, but not Hermitian in the usual sense. A real split-quaternionic matrix refers to the matrix whose components are real split-quaternions \( q_{rIJ}^r \),

\[
(Q^r_{IJ}) = q_{rIJ}^r,
\]

and then \( Q^r \) satisfies the relation

\[
Q^\dagger = \overline{Q}.
\]

(C.12) \hspace{2cm} (C.13)

The self-dual real split-quaternion matrix is defined as a split-quaternion matrix that satisfies both (C.11) and (C.13):

\[
Q = Q^\dagger = \overline{Q}.
\]

(C.14)

Thus, the split-quaternionic Hermitian real split-quaternion matrix is equivalent to the self-dual real split-quaternion matrix. (The condition \( \overline{Q} = Q \) is the self-dual condition.) The terminology “split-quaternionic Hermitian real split-quaternion” is rather clumsy, so we use “self-dual real split-quaternion” instead. Such self-dual real split-quaternion matrix generally accommodates the generalized Kramers degeneracy for \( \Theta^2 = +1 \).\(^*)\) In low dimensions, the self-dual real split-quaternion matrices are given by

\[
M = 1: \quad Q_1 = \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix} = w_{12},
\]

\[
M = 2: \quad Q_2 = \begin{pmatrix} w + x^5 & 0 & x^4 - ix^3 & x^1 - ix^2 \\ 0 & w + x^5 & x^1 + ix^2 & x^4 + ix^3 \\ -x^1 - ix^2 & -x^1 + ix^2 & w - x^5 & 0 \\ -x^1 + ix^2 & x^4 - ix^3 & 0 & w - x^5 \end{pmatrix},
\]

(C.15)

\(^*)\) Real split-quaternion matrix does not accommodate the generalized Kramers degeneracy in general. The split-quaternionic Hermitian condition has to be imposed as well.
where \( w, x^1, \cdots, x^5 \) are real parameters. With the \( SO(3,2) \) gamma matrices (5.3), \( Q_2 \) is concisely represented as

\[
Q_2 = w \gamma_4 + \sum_{a=1}^{5} x^a \gamma_a. \tag{C.16}
\]

\( Q_1 \) and \( Q_2 \) are exactly equal to the matrices (4.4) and (5.8), respectively. They have both K and Q symmetries \( (\epsilon_q = +1) \), and their eigenvalues are

\[
M = 1 : \quad w,
M = 2 : \quad w \pm \sqrt{-(x^1)^2 - (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2}, \tag{C.17}
\]

with double degeneracy (of the generalized Kramers).

**Appendix D**

*Spin Flipping Operators and Quadrupole Hamiltonians for Low SU(1,1) Spins*

The \( SU(1,1) \) algebra is given by

\[
[J_i, J_j] = i \epsilon_{ijk} J^k, \tag{D.1}
\]

where \( \epsilon_{ijk} \) denotes a totally antisymmetric tensor with \( \epsilon_{xyz} = 1 \), and \( J^i = (J_x, J_y, -J_z) \). Explicitly,

\[
[J_x, J_y] = -i J_z, \quad [J_y, J_z] = i J_x, \quad [J_z, J_x] = i J_y. \tag{D.2}
\]

From \( SU(2) \) spins \( S_x, S_y, S_z \), the \( SU(1,1) \) spins are constructed with the identification

\[
J_x = i S_x, \quad J_y = i S_y, \quad J_z = S_z. \tag{D.3}
\]

Note that \( J_x \) is pure imaginary; \( J_y \) and \( J_z \) are real. The magnitude of \( SU(1,1) \) spin \( J \) is defined as

\[
-J_x^2 - J_y^2 + J_z^2 = J(J+1). \tag{D.4}
\]

For instance,

\[
J = 1/2 : \quad J_x = \frac{1}{2} \tau_x, \quad J_y = \frac{1}{2} \tau_y, \quad J_z = \frac{1}{2} \tau_z, \tag{D.5a}
\]

\[
J = 1 : \quad J_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad J_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{D.5b}
\]

The \( SU(1,1) \) spin flipping operator \( \Xi \) is defined so as to satisfy

\[
J_i \rightarrow \Xi J_i \Xi^{-1} = -J_i. \tag{D.6}
\]
Express $\Xi = U \cdot K$, and $U$ satisfies
\[ U J_x U^{-1} = J_x, \quad U J_y U^{-1} = -J_y, \quad U J_z U^{-1} = -J_z. \] (D.7)

Here, $U$ is a unitary matrix given by
\[ U = (-i)^{2J} e^{\pi J_x}, \] (D.8)

where the factor $(-i)^{2J}$ is added for convenience. Consequently, $\Xi$ is given by
\[ \Xi = (-i)^{2J} e^{\pi J_x} \cdot K, \] (D.9)

which satisfies
\[ \Xi^2 = +1, \] (D.10)

independent of the magnitude of the $SU(1,1)$ spin.\(^\ast\) For low $SU(1,1)$ spins,
\[ J = 1/2 : \quad \Xi_{1/2} = \sigma_x \cdot K, \] (D.11a)
\[ J = 1 : \quad \Xi_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot K, \] (D.11b)
\[ J = 3/2 : \quad \Xi_{3/2} = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} \cdot K. \] (D.11c)

Note that $\Xi_{1/2}$ is equal to the charge conjugation operator $\mathcal{C}$ (4.10) of the $SU(1,1)$ model, and $\Xi_{3/2}$ is the time-reversal operator $\Theta'$ (6.25) of the $SO(3,2)$ model. Similarly, for low $SU(1,1)$ spins, the quadrupole Hamiltonians (6.21) introduced in §6.2 are given by
\[ J = 1/2 : \quad H(Q_0) = 0, \] (D.12a)
\[ J = 1 : \quad H(Q_1) = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & i & 0 \\ i & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, \quad H(Q_2) = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \] (D.12b)
\[ H(Q_3) = -\frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad H(Q_4) = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \] (D.12c)
\[ H(Q_5) = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \] (D.12d)

\section*{Appendix E}
\textit{P, T, and C Operators for SU(1,1) and SO(3,2) Models}

For non-Hermitian Hamiltonians, there is a systematic procedure for constructing a metric operator $\eta_0$, and symmetry operators that commute with the Hamiltonian.\(^\ast\) In this appendix, we briefly review the procedure and apply it to the $SU(1,1)$ and $SO(3,2)$ models.

\(^\ast\) As discussed in §6.1, the square of the $SU(2)$ spin flipping operator takes $-1$ for half-integer spins, while the $SU(1,1)$ spin flipping operator yields $+1$ even for half-integer $SU(1,1)$ spins.
For a non-Hermitian Hamiltonian $H$, the Schrödinger equation is

$$H|\phi_n\rangle = E_n|\phi_n\rangle, \quad H^\dagger|\varphi_n\rangle = E_n^*|\varphi_n\rangle,$$

(E.1)

where the eigenvalue $E_n$ is complex in general. As discussed in §2.3, the eigenvectors, $|\phi_n\rangle$ and $|\varphi_n\rangle$, give a bi-orthonormal basis satisfying the orthonormal and complete relations

$$\langle \varphi_m|\varphi_n\rangle = \delta_{mn}, \quad \sum_n |\phi_n\rangle \langle\varphi_n| = \sum_n |\varphi_n\rangle \langle\varphi_n| = 1.$$  

(E.2)

The Hamiltonian and its Hermitian conjugate are expanded as

$$H = \sum_n E_n|\phi_n\rangle \langle\varphi_n|, \quad H^\dagger = \sum_n E_n^*|\varphi_n\rangle \langle\varphi_n|.$$  

(E.3)

A non-Hermitian Hamiltonian is called pseudo-hermite when it satisfies

$$H^\dagger = \eta H \eta^{-1},$$  

(E.4)

where $\eta$ is Hermitian and called a metric operator. A state given by

$$|\phi_n\rangle' \equiv \eta^{-1} |\varphi_n\rangle,$$

(E.5)

is an eigenvector of $H$ with an eigenvalue $E_n^*$ as seen from (E.1) and (E.4). Thus, the eigenvalues of the pseudo-Hermitian Hamiltonian are classified into two types. One is a set of real eigenvalues and the other is a set of complex conjugate pairs. (For more details, see 17.)

Suppose all the eigenvalues of a non-Hermitian Hamiltonian are real, i.e., $E_n^* = E_n$. Following Ref. 33), a metric operator $\eta_0$ is constructed as

$$\eta_0^{-1} = \sum_{l=1}^N |\phi_l\rangle \langle\phi_l|, \quad \eta_0 = \sum_{l=1}^N |\varphi_l\rangle \langle\varphi_l|.$$  

(E.6)

Here, note that the metric operator satisfying (E.4) is not unique. $\eta_0$ in the above generally depends on parameters of the Hamiltonian, while we may have a constant metric operator for some models. For example, see §§E.1 and E.2.

Let us now define the following operators $P$, $T$, and $C$,

$$P|\varphi_n\rangle = (-1)^{n+1} |\phi_n\rangle,$$  

(E.7a)

$$T|\phi_n\rangle = |\varphi_n\rangle,$$  

(E.7b)

$$C|\phi_n\rangle = (-1)^{n+1} |\phi_n\rangle.$$  

(E.7c)

These operators were originally introduced in analogy with parity, TR, and charge conjugation, respectively; however they are not, in fact, directly related. With the orthonormal and complete conditions (E.2), they are explicitly written as

$$P = \sum_{l=1}^N (-1)^{l+1} |\phi_l\rangle \langle\phi_l|.$$  

(E.8a)
\[ T = \sum_{l=1}^{N} |\varphi_l\rangle K \langle \varphi_l|, \]  
\[ C = \sum_{l=1}^{N} (-1)^{l+1} |\varphi_l\rangle \langle \varphi_l|, \]  
(E.8b)

which are respectively Hermitian, anti-Hermitian, and pseudo-Hermitian

\[ P^\dagger = P, \]  
\[ T^\dagger = T, \]  
\[ C^\dagger = \eta_0 C \eta_0^{-1}. \]  
(E.9c)

From (E.8), \( PT \) and \( CPT \) operators are given by

\[ PT = \sum_{l=1}^{N} (-1)^{l+1} |\varphi_l\rangle K \langle \varphi_l|, \]  
\[ CPT = \sum_{l=1}^{N} |\varphi_l\rangle K \langle \varphi_l|, \]  
(E.10b)

and

\[ PT |\varphi_n\rangle = (-1)^{n+1} |\varphi_n\rangle, \]  
\[ CPT |\varphi_n\rangle = |\varphi_n\rangle. \]  
(E.11b)

They are pseudo-antiunitary

\[ (PT)^\dagger = \eta_0 (PT)^{-1} \eta_0^{-1}, \]  
\[ (CPT)^\dagger = \eta_0 (CPT)^{-1} \eta_0^{-1}. \]  
(E.12b)

It is readily seen that these operators satisfy

\[ [H, PT] = 0, \]  
\[ [H, C] = [H, CPT] = 0, \]  
\[ [PT, C] = 0, \]  
\[ [H, P] \neq 0, \]  
(E.13)

and

\[ (PT)^2 = C^2 = (CPT)^2 = 1, \]  
\[ P^2 \neq 1, \quad T^2 \neq 1. \]  
(E.14)

As realized in the first and second lines of (E.13), the Hamiltonian always displays “\( PT \) symmetry” and “\( C \) symmetry” (or “\( CPT \) symmetry”) with respect to the \( PT \) and \( C \) operators constructed above.

\(^*\) The terminology “anti-Hermitian” usually refers to the operator whose Hermitian conjugate is equal to the minus of the original, but here, the terminology “anti-Hermitian” refers to Hermitian operator that anticommutes with an imaginary unit.
E.1. \textit{SU}(1, 1) model

With the bi-orthonormal bases (4.14) and (4.17), we construct the metric operator $\eta_0$, and $P$, $T$, and $C$ operators for the \textit{SU}(1, 1) model. From (E.6),

$$\eta_0 = |\varphi_+\rangle\langle\varphi_+| + |\varphi_-\rangle\langle\varphi_-| = \frac{1}{r} \left( \begin{array}{cc} z & -ix-y \\ ix-y & z \end{array} \right), \quad (E.15)$$

from (E.8),

$$P = |\phi_+\rangle\langle\phi_+| - |\phi_-\rangle\langle\phi_-| = \sigma_z,$$

$$T = (|\varphi_+\rangle\langle\varphi_+^*| + |\varphi_-\rangle\langle\varphi_-^*|) \cdot K$$

$$= \frac{1}{2r(r+z)} \left( \begin{array}{cc} (r+z)^2 + (ix+y)^2 & -2(r+z)y \\ -2(r+z)y & (r+z)^2 + (ix+y)^2 \end{array} \right) \cdot K,$$

$$C = |\phi_+\rangle\langle\varphi_-| - |\phi_-\rangle\langle\varphi_+| = \frac{1}{r} \left( \begin{array}{cc} z & -ix-y \\ -ix+y & z \end{array} \right) = \frac{1}{r} H, \quad (E.16)$$

with $H$ (4.8), and from (E.10),

$$PT = (|\varphi_+\rangle\langle\varphi_+^*| - |\varphi_-\rangle\langle\varphi_-^*|) \cdot K$$

$$= \frac{1}{2r(r+z)} \left( \begin{array}{cc} (r+z)^2 + (ix+y)^2 & -2(r+z)y \\ -2(r+z)y & (r+z)^2 - (ix+y)^2 \end{array} \right) \cdot K,$$

$$CPT = (|\varphi_+\rangle\langle\varphi_+^*| + |\varphi_-\rangle\langle\varphi_-^*|) \cdot K$$

$$= \frac{1}{2r(r+z)} \left( \begin{array}{cc} (r+z)^2 - (ix+y)^2 & 2i(r+z)x \\ -2i(r+z)x & (r+z)^2 - (ix-y)^2 \end{array} \right) \cdot K. \quad (E.17)$$

The metric operator (E.15) is different from $\eta = \sigma_z$ used in §4: This “discrepancy” stems from the nonuniqueness of the metric operator.

E.2. \textit{SO}(3, 2) model

For the \textit{SO}(3, 2) model, with use of the bi-orthonormal basis $|\psi_{ma}\rangle$ (5.14) and $|\chi_{ma}\rangle$ (5.22), the metric operator is constructed as

$$\eta_0 = \sum_{\alpha=+,-} (|\chi_+\rangle\langle\chi_+| + |\chi_-\rangle\langle\chi_-|)$$

$$= \frac{1}{2r(r+x^5)} \left( (r+x^5)^2 - (x^4-ix^4\tau_i^\dagger)(x^4+ix^4\tau_j) + i(r+x^5)x^i(\tau_i - \tau_j^\dagger) \\ -i(r+x^5)x^i(\tau_i - \tau_j^\dagger) \right), \quad (E.18)$$

Similar to the case of the \textit{SU}(1, 1) model, the metric operator (E.18) is different from the one given by (5.7). (5.7) is anticommutative with $\Theta$ (5.10), while (E.18) is commutative with $\Theta$. From (E.8), $P$, $T$ and $C$ are derived as

$$P = \sum_{\alpha=+,-} (|\psi_{+}\rangle\langle\psi_+| - |\psi_-\rangle\langle\psi_-|)$$
ogy in phase space.

(5.10), and 

\begin{align}
\sum_{\alpha=+,-} \langle |\psi_{+\alpha}\rangle \langle \chi_{+\alpha}^* | + |\psi_{-\alpha}\rangle \langle \chi_{-\alpha}^* | \rangle \cdot K
\end{align}

with \( H \) (5.11), and \( PT \) and \( CPT \) are

\begin{align}
PT &= \sum_{\alpha=+,-} \langle |\psi_{+\alpha}\rangle \langle \chi_{+\alpha}^* | - |\psi_{-\alpha}\rangle \langle \chi_{-\alpha}^* | \rangle \cdot K \\
&= \frac{1}{2r(r+x^5)} \times \begin{pmatrix}
(x^4 - ix^3 \tau_i)(x^4 - ix^3 \tau_j^*) & (r + x^5)(2x^4 - ix^4(\tau_i - \tau_j^*)) \\
(r + x^5)(2x^4 + ix^3(\tau_i + \tau_j^*)) & -(r + x^5)^2 + (x^4 + ix^4 \tau_i)(x^4 + ix^4 \tau_j^*)
\end{pmatrix} \\
&\cdot K,
\end{align}

\begin{align}
CPT &= \sum_{\alpha=+,-} \langle |\psi_{+\alpha}\rangle \langle \chi_{+\alpha}^* | + |\psi_{-\alpha}\rangle \langle \chi_{-\alpha}^* | \rangle \cdot K \\
&= \frac{1}{2r(r+x^5)} \times \begin{pmatrix}
(x^4 - ix^3 \tau_i)(x^4 - ix^3 \tau_j^*) & i(r + x^5)x^4(\tau_i + \tau_j^*) \\
(r + x^5)(2x^4 + ix^3(\tau_i + \tau_j^*)) & -(r + x^5)^2 + (x^4 + ix^4 \tau_i)(x^4 + ix^4 \tau_j^*)
\end{pmatrix} \\
&\cdot K.
\end{align}

Appendix F

Exceptional Point and Monopole Structure in Non-Hermitian Hamiltonians

F.1. \( SU(1,1) \) model and \( U(1) \) monopole

The singularity of phase of the eigenstate generally reflects the nontrivial topology in phase space.\(^3\),\(^4\) For instance, the crossing point of two energy levels of the \( SU(2) \) model is called a diabolic point (an isolated point), which brings \( U(1) \)
holonomy in phase space.\(^{(34)}\) In the \(SU(1,1)\) model, the eigenenergies are given by 
\[ E_{\pm} = \pm r \] 
with \( r = \sqrt{z^2 - x^2 - y^2} \), and the exceptional point \( E_+ = E_- \) is achieved when \( r = 0 \). In the \( SU(2) \) model, because of the Euclidean signature, the condition, 
\[ r = \sqrt{x^2 + y^2 + z^2} = 0, \] 
is met only at the point \( x = y = z = 0 \). Meanwhile, in the \( SU(1,1) \) model, the signature is hyperbolic, and the condition is satisfied on the surface 
\[ x^2 + y^2 - z^2 = 0. \] 
\[(F.1)\]
In such a case, \( r = 0 \) point is called the exceptional point. Around the exceptional point \( (r \sim 0) \), the upper and lower energy eigenvectors \( (z \geq r) \) (4.14) behave as 
\[ |\phi_+\rangle \sim \frac{1}{\sqrt{2rz}} \left( \begin{array}{c} z \\ y - ix \end{array} \right), \quad |\phi_-\rangle \sim \frac{1}{\sqrt{2rz}} \left( \begin{array}{c} y + ix \\ z \end{array} \right), \] 
\[(F.2)\]
and
\[ |\varphi_+\rangle \sim \frac{1}{\sqrt{2rz}} \left( \begin{array}{c} z \\ -y + ix \end{array} \right), \quad |\varphi_-\rangle \sim \frac{1}{\sqrt{2rz}} \left( \begin{array}{c} -y - ix \\ z \end{array} \right). \] 
\[(F.3)\]
They are degenerate, as found,
\[ |\phi_-\rangle \sim e^{i\chi}|\phi_+\rangle, \]
\[ |\varphi_-\rangle \sim -e^{i\chi}|\varphi_+\rangle, \] 
\[(F.4)\]
where \( \tan \chi = \frac{x}{y} \). Then, the normalization condition is not satisfied but
\[ \langle \phi_+|\varphi_-\rangle = \langle \phi_-|\varphi_+\rangle \sim \frac{r}{2z} \sim 0, \quad \langle \phi_+|\varphi_-\rangle = 0, \quad \langle \phi_-|\varphi_+\rangle = 0. \] 
\[(F.5)\]
The holonomy of the exceptional points is
\[ A^+ = -i\langle \varphi_+|d\phi_+\rangle = -i\langle \phi_+|d\varphi_+\rangle = -i\langle \phi_+|\sigma_z d\phi_+\rangle = dx^i A^+_i, \] 
\[(F.6)\]
where
\[ A^+_i = \frac{1}{2r(r + z)} \epsilon_{ij3} x^j, \] 
\[(F.7)\]
which is the \( U(1) \) monopole gauge field in a hyperbolic space.\(^{(35)}\),\(^{(36)}\) The corresponding field strength is calculated as
\[ F^+_{ij} = \partial_i A^+_j - \partial_j A^+_i = \frac{1}{2r^3} \epsilon_{ijk} x^k. \] 
\[(F.8)\]
Similarly, the holonomy for the negative energy state is evaluated as
\[ A^- = -i\langle \varphi_-|d\phi_-\rangle = -i\langle \phi_-|d\varphi_-\rangle = -i\langle \phi_-|(-\sigma_z) d\phi_-\rangle = dx^i A^-_i, \] 
\[(F.9)\]
where the insertion of \( -\sigma_z \) is in accordance with the normalization (4.15), and
\[ A^-_i = -\frac{1}{2r(r + z)} \epsilon_{ij3} x^j = -A^+_i. \] 
\[(F.10)\]
The corresponding gauge field strength is

$$F_{ij}^- = \partial_i A_j^- - \partial_j A_i^- = -\frac{1}{2r^3}\epsilon_{ijk}x^k = -F_{ij}^+. \quad (F.11)$$

The field strength diverges at the exceptional point.

On the lower leaf \((z \leq -r)\), the holonomies are derived as

$$A_i^+ = -i\langle\langle \phi'_+ | d\phi'_+ \rangle \rangle = -i\langle\langle \phi'_+ | (-\sigma_z) | \phi'_+ \rangle \rangle = dx^i A_i^+, \quad (F.12a)$$
$$A_i^- = -i\langle\langle \phi'_- | d\phi'_- \rangle \rangle = -i\langle\langle \phi'_- | \sigma_z | \phi'_- \rangle \rangle = dx^i A_i^-, \quad (F.12b)$$

where

$$A_i^+ = -A_i^- = -\frac{1}{2r(r - z)}\epsilon_{ij3}x^j. \quad (F.13)$$

As found above, the holonomy of the exceptional point in the \(SU(1,1)\) model is regarded as the gauge field of hyperbolic \(U(1)\) monopole, and the monopole charges for the upper and lower energy states are opposite. Such effect is similar to the \(U(1)\) monopole holonomy of the diabolic point in the \(SU(2)\) model.\(^{34}\)

### F.2. \(SO(3,2)\) model and \(SU(1,1)\) monopole

Degeneracies in energy levels generally bring non-Abelian holonomy in the parameter space.\(^{37}\) For instance, the \(SO(5)\) Hamiltonian (Luttinger Hamiltonian\(^{38}\)) has \(SU(2)\) holonomy, which is crucial for the spin-Hall effect.\(^{39}\) Here, we consider what kind of holonomy could emerge in the \(SO(3,2)\) model. The energy levels of the \(SO(3,2)\) model are \(\pm r\), and the exceptional point is at \(r = 0\), namely,

$$(x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2 - (x^5)^2 = 0. \quad (F.14)$$

Near the exceptional point \((r \sim 0)\), the upper energy and lower energy eigenvectors behave as

$$|\psi_{+\alpha}\rangle \sim \frac{1}{\sqrt{2r}x^5} \left( \begin{array}{c} x^5\phi_{\alpha} \\ (x^4 + ix^4 \tau_1)\phi_{\alpha} \end{array} \right), \quad |\psi_{-\alpha}\rangle \sim \frac{1}{\sqrt{2r}x^5} \left( \begin{array}{c} -(x^4 - ix^4 \tau_1)\phi_{\alpha} \\ x^5\phi_{\alpha} \end{array} \right),$$

and

$$|\chi_{+\alpha}\rangle \sim \frac{1}{\sqrt{2r}x^5} \left( \begin{array}{c} x^5\phi_{\alpha} \\ (x^4 + ix^4 \tau_1^\dagger)\phi_{\alpha} \end{array} \right), \quad |\chi_{-\alpha}\rangle \sim \frac{1}{\sqrt{2r}x^5} \left( \begin{array}{c} -(x^4 - ix^4 \tau_1^\dagger)\phi_{\alpha} \\ x^5\phi_{\alpha} \end{array} \right). \quad (F.15)$$

Then, at the exceptional point, \(|\psi_{+\alpha}\rangle\) and \(|\psi_{-\alpha}\rangle\) are related by the \(SU(1,1)\) gauge transformation \(-\frac{x^4 - ix^4 \tau_1}{x^5}\), and \(|\chi_{+\alpha}\rangle\) and \(|\chi_{-\alpha}\rangle\) are by \(-\frac{x^4 - ix^4 \tau_1^\dagger}{x^5}\), and the normalization conditions are no longer satisfied:

$$\langle \psi_{\pm\alpha} | \chi_{\pm\alpha'} \rangle \overset{r\to 0}{\sim} \frac{r}{2x^5} \sim 0, \quad \langle \psi_{\pm\alpha} | \chi_{\mp\alpha'} \rangle = 0. \quad (F.17)$$

For \(x^5 > -r\), the upper energy degenerate eigenvectors bring the following holonomy:

$$A^+ = -i\langle\langle \chi_+ | d\psi_+ \rangle \rangle = -i\langle\langle \psi_+ | d\chi_+ \rangle \rangle = dx^a\phi_+^\dagger \sigma_z A_+^a \phi, \quad (F.18)$$
\[ A^+_{\mu} = -\frac{1}{2r(r + x^5)}\eta_{\mu\nu}\xi^\nu\tau^i, \quad A^+_5 = 0. \]  

(F.19)

Here, \( \mu, \nu = 1, 2, 3, 4 \), and \( \eta_{\mu\nu} \) are the “split”-'t Hooft symbol defined by \( \eta_{\mu\nu} = \epsilon_{\mu\nu} + \eta_{\mu} \eta_{\nu} - \eta_{\nu} \eta_{\mu} \) with \( \eta_{\mu\nu} = \text{diag}(+,-,-,-) \), and \( \epsilon_{\mu\nu} \) is a totally antisymmetric tensor, \( \epsilon_{123} = 1 \) and \( \epsilon_{\mu\nu} = 0 \) for \( \mu = 4 \) or \( \nu = 4 \). The corresponding field strength

\[ F^+_{ab} = \partial_a A^+_b - \partial_b A^+_a + i[A^+_a, A^+_b], \]  

is derived as

\[ F^+_{\mu\nu} = -\frac{1}{2r^2}(r + x^5)A^+_\mu A^+_\nu - \frac{1}{2r^2}\eta_{\mu\nu}\tau^i, \quad F^+_{\mu5} = \frac{1}{r^2}(r + x^5)A^+_\mu. \]  

(F.21)

The gauge field strength has the singularity at the exceptional point. Similarly, \( A^- \) is given by

\[ A^- = -i\langle \chi_+|d\psi_- \rangle = -i\langle \psi_-|d\chi_+ \rangle = dx^a\phi^\dagger \tau^i \sigma_z A^-_{a\phi}, \]  

where

\[ A^-_{\mu} = \frac{1}{2r(r + x^5)}\eta'_{\mu\nu}\xi^\nu\tau^i, \quad A^-_5 = 0, \]  

(F.23)

with \( \eta'_{\mu\nu} = \epsilon_{\mu\nu} - \eta_{\mu\nu} \eta_{4\nu} + \eta_{\nu\nu} \eta_{4\mu} \). The corresponding field strength is

\[ F^-_{\mu\nu} = \frac{1}{2r^2}(r + x^5)A^-_{\mu} A^-_{\nu} - \frac{1}{2r^2}\eta'_{\mu\nu}\tau^i, \quad F^-_{\mu5} = -\frac{1}{r^2}(r + x^5)A^-_{\mu}. \]  

(F.24)

With a different gauge choice \( (x^5 < r) \), the holonomy is calculated as

\[ A'^+ = -i\langle \chi'_+|d\psi'_+ \rangle = -i\langle \psi'_+|d\chi'_+ \rangle = dx^a\phi^\dagger \tau^i \sigma_z A'^{+}_{a\phi}, \]  

where

\[ A'^{+}_{\mu} = -\frac{1}{2r(r - x^5)}\eta_{\mu\nu}\xi^\nu\tau^i, \quad A'^+_5 = 0. \]  

(F.26)

The corresponding field strength is

\[ F'^+_{\mu\nu} = \frac{1}{2r^2}(r - x^5)A'^+_{\mu} A'^+_{\nu} - \frac{1}{2r^2}\eta_{\mu\nu}\tau^i, \quad F'^+_{\mu5} = -\frac{1}{r^2}(r - x^5)A'^+_{\mu}. \]  

(F.27)

Similarly, the lower energy degenerate states bring the holonomy

\[ A'^- = -i\langle \chi'_-|d\psi'_- \rangle = -i\langle \psi'_-|d\chi'_- \rangle = dx^a\phi^\dagger \tau^i \sigma_z A'^{+}_{a\phi}, \]  

where

\[ A'^{-}_{\mu} = -\frac{1}{2r(r - x^5)}\eta_{\mu\nu}\xi^\nu\tau^i, \quad A'^-_5 = 0. \]  

(F.29)
The corresponding field strength is

\[ F'_{\mu\nu} = \frac{1}{r^2} x_\mu A'^{-}_\nu - \frac{1}{r^2} x_\nu A'^{-}_\mu + \frac{1}{2r^2} \eta_{\mu\nu} r^i, \]

\[ F'_{5\mu} = -F'_{5\mu} = \frac{1}{r^2} (r - x^5) A'^{+}_\mu. \]  

(F.30)

The \( SU(1,1) \) gauge transformation relates \( A^+ \) and \( A'^+ \) as

\[ \sigma_z A'^{+}_a dx^a = g_+^\dagger (\sigma_z A^+_a dx^a) g_+ - ig_+^\dagger \sigma_z dg_+, \]  

(F.31)

and their field strengths as

\[ \sigma_z F'_{ab} = g_+^\dagger (\sigma_z F_{ab}) g_+, \]  

(F.32)

where \( g_\pm \) is given by (5.19). Similarly, \( A^- \) and \( A'^- \) are related by the \( SU(1,1) \) transformation,

\[ \sigma_z A'^{-}_a dx^a = g_-^\dagger (\sigma_z A^-_a dx^a) g_- - ig_-^\dagger \sigma_z dg_-, \]  

(F.33)

and

\[ \sigma_z F'_{-ab} = g_-^\dagger (\sigma_z F_{-ab}) g_- . \]  

(F.34)

Thus, the exceptional points of the \( SO(3,2) \) model act as the \( SU(1,1) \) monopole with opposite charges for the upper and lower energy states. Such noncompact gauge group monopoles have been introduced in the context of the noncompact Hopf maps.\(^{36}\)

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**Note added:** After completion of the present work, we became aware of Refs. 30) and 31). In these papers, a split-quaternion extension (co-quaternion as the terminology of the reference) was discussed in a context of $PT$ symmetric non-Hermitian quantum mechanics. The obtained split-quaternion structure in the two-level system$^{31)}$ is consistent with ours. We arrive at the split-quaternion in a completely different manner by investigating general structures of non-Hermitian quantum mechanics with time-reversal symmetry $\Theta^2 = +1$. We also discussed the split-quaternion structure in light of the non-Hermitian random matrix theory. We are grateful to Dorje C. Brody for notifying us of their work.