On perfect 2-colorings of infinite circulant graphs with a continuous set of odd distances*

Parshina O. G.,\textsuperscript{1,2,†} Lisitsyna M. A.\textsuperscript{3}

\textsuperscript{1}Université de Lyon, Université Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 boulevard du 11 novembre 1918, F69622 Villeurbanne Cedex, France
\textsuperscript{2}Sobolev Institute of Mathematics of the Siberian Branch of the Russian Academy of Sciences, 4 Acad. Koptyug avenue, 630090 Novosibirsk, Russia
\textsuperscript{3}Marshal Budyonny Military Academy of Telecommunications, Tikhoretskii pr. 3, St. Petersburg, 194064 Russia

Abstract

A vertex coloring of a given simple graph $G = (V, E)$ with $k$ colors ($k$-coloring) is a map from its vertex set to the set of integers $\{1, 2, 3, \ldots, k\}$. A coloring is called perfect if the multiset of colors appearing on the neighbors of any vertex depends only on the color of the vertex. We consider perfect colorings of Cayley graphs of the additive group of integers with generating set $\{1, -1, 3, -3, 5, -5, \ldots, 2n - 1, 1 - 2n\}$ for a positive integer $n$. We enumerate perfect 2-colorings of the graphs under consideration and state the conjecture generalizing the main result to an arbitrary number of colors.

Keywords: perfect coloring, circulant graph, Cayley graph, equitable partition

1 Introduction

Let $G$ be a simple graph, $k$ be a positive integer and $M = (m_{ij})_{i,j=1}^k$ be a non-negative integer matrix of order $k$. A coloring of vertices of the graph $G$ with $k$ colors is a map $\varphi : V \to \{1, 2, 3, \ldots, k\}$. The value $\varphi(v) = s$ is said to be the color of $v$. Hereinafter by coloring we mean a coloring of its vertex set. A coloring of the graph $G$ is called perfect with the parameter matrix $M$, if for any $i,j$ from 0 to $k - 1$ any vertex with color $i$ has exactly $m_{ij}$ neighbors with color $j$. In this case the matrix $M$ is called admissible for the graph $G$. The corresponding partition of the vertex set of $G$ is known as an equitable partition.

In this text we use the notion of a perfect coloring, since results and proofs presented here are combinatorial, and it is more convenient to describe them in

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\†Corresponding author, parolja@gmail.com
these terms. The concept of a perfect coloring plays an important role in graph theory, algebraic combinatorics, and coding theory. The notion of a perfect coloring is closely related to the notion of a perfect code. Namely, a perfect code in distance regular graph induces a perfect 2-coloring of the vertex set of the graph; a distance partition of a graph in accordance to a perfect code is a perfect coloring.

Hereinafter \( n \) and \( k \) are positive integers. In this paper we aim to classify perfect colorings of graphs from the family of infinite circulants. The graphs under consideration are Cayley graphs of the additive group of integers with generating set \( \{1, -1, 3, -3, 5, -5, \ldots, 2n - 1, 1 - 2n\} \). We call such graphs infinite circulant graphs with the set of distances \( \{1, 3, 5, \ldots, 2n - 1\} \). Perfect 2-colorings of infinite circulant graphs with the set of distances \( \{1, 2, 3, \ldots, n\} \) are enumerated in [1]. The conjecture generalizing described result to the case of arbitrary number of colors is posed in [2]. Partial results on the conjecture one can see in [3].

There is a natural homomorphism from the \( n \)-dimensional rectangular grid graph \( G(\mathbb{Z}^n) \) to an infinite circulant graph with \( n \) distances [4]. That means every perfect coloring of an infinite circulant induces a perfect coloring of the grid. Perfect colorings of the infinite rectangular grid graph have been widely studied. Admissible for the graph \( G(\mathbb{Z}^2) \) parameter matrices of order 3 are enumerated by S. A. Puzynina [5]. Perfect \( s \)-colorings of the graph \( G(\mathbb{Z}^2) \) for \( s \geq 3 \) are described by D. S. Krotov [6].

A perfect \( k \)-coloring is called distance regular if its parameter matrix is tridiagonalizable. In this case colors of the perfect coloring can be arranged in a way that every vertex of color \( i \in \{2, 3, \ldots, k - 1\} \) can only be adjacent to vertices of color \( i - 1 \) and \( i + 1 \). Moreover, the set of vertices of color 1 and the set of vertices of color \( k \) are completely regular codes. Parameters of distance regular colorings of the infinite rectangular grid graph are enumerated by S. V. Avgustinovich, A. Yu. Vasil’eva and I. V. Sergeeva [7].

Along with perfect colorings of infinite rectangular grid graph, perfect colorings of triangle and hexagonal infinite grid graphs have been studied. S. A. Puzynina proved that for every perfect coloring of infinite triangle or hexagonal grids there exists a periodic coloring of the grid with the same parameter matrix [8]. Distance regular colorings of the infinite triangle grid graph are enumerated by A. Yu. Vasil’eva [9], of hexagonal grid graph – by S. V. Avgustinovich, D. S. Krotov and A. Yu. Vasil’eva [10].

Let \( G = (V, E) \) be a simple graph, \( M = (m_{ij})_{i,j=1}^{k} \) be a square matrix of order \( k \), and \( r \geq 1 \). A coloring of the vertex set of the graph \( G \) is called perfect of radius \( r \) with parameter matrix \( M \) if each element \( m_{ij} \) of the matrix is the number of vertices of color \( j \) at the distance at most \( r \) from the vertex of color \( i \) for \( i, j \in \{1, 2, 3, \ldots, k\} \).

Admissible parameter matrices of perfect 2-colorings of radius 1 of the graph \( G(\mathbb{Z}^2) \) are enumerated by M. Axenovich [4]. In the same paper the author states several necessary conditions on a parameter matrix to be admissible for \( G(\mathbb{Z}^2) \) in the case \( r \geq 2 \). Parameters and properties of perfect colorings of \( G(\mathbb{Z}^2) \) have been studied by S. A. Puzynina in her PhD thesis [5]. In particular, she showed that all perfect colorings of radius \( r > 1 \) of this graph are periodic. Several results on perfect 2-colorings of circulant graphs were obtained by D. B. Khoroshilova [11, 12].

Let us mention several results on perfect colorings of graphs similar to circulants
and to infinite grid graphs.

Perfect 2-colorings of the hypercube graph have been studied by D. G. Fon-Der-Flaass. He obtained necessary conditions on parameters of perfect 2-colorings of this graph and presented an infinite series of such colorings [13]. Later he obtained a bound on correlation immunity of non-constant unbalanced Boolean functions that allows to obtain a new necessary condition for a perfect coloring with given parameters to exist in the hypercube graph [14]. Fon-Der-Flaass constructed perfect colorings of the 12-dimensional hypercube graph that attain this bound [15]. Another method to construct perfect 2-colorings via parameter matrices was provided by D. G. Fon-Der-Flaass and K. V. Vorobev [16]. Let us note that the set of parameter matrices admissible for the hypercube graph has not been described yet even in the case of two colors.

A Johnson graph $J(n, \omega)$ is the graph with the set of boolean vectors of weight $\omega$ as the set of vertices; two vertices are adjacent in $J(n, \omega)$, if they differ in exactly two coordinates. W. J. Martin showed that a coloring of $J(n, \omega)$ obtained by coloring vertices of blocks of $(\omega - 1) - (n, \omega, \lambda)$-scheme with color 1 and all the other vertices of $J(n, \omega)$ with the color 2 is a perfect 2-coloring of this graph [17].

A systematic study of perfect 2-colorings in Johnson graphs is performed in the thesis of I. Yu. Mogilnykh [18]. He constructed several series of perfect 2-colorings of Johnson graphs and provided several necessary conditions for such colorings to exist. These results were used for enumeration of parameters of perfect 2-colorings of Johnson graphs $J(n, \omega)$, where $n \leq 8$. In [19] one can find the complete description of admissible for the graph $J(n, 3), n$ is odd, parameter matrices of order 2. The problem of Johnson graphs perfect colorings classification is not solved even in the case of 2-colors.

Perfect 2-colorings of transitive cubic graphs with the set of vertices of cardinality up to 18 are enumerated by S. V. Avgustinovich and M. A. Lisitsyna in [20]. In the later work the authors list perfect colorings of the infinite prism graph with arbitrary number of colors [21].

2 Preliminaries

Let us consider a simple graph $G = (V, E)$ with vertex set $V$ and edge set $E$. Given a vertex $v \in V$, by $N(v)$ we denote the set of vertices adjacent to $v$, i.e. its neighborhood.

We are interested in graphs defined as follows. Given a positive integer $n$, let us consider a set $D = \{d_1, d_2, \ldots, d_n\}$ of positive integers enumerated in ascending order. We say that a graph $Ci_\infty(D) = (\mathbb{Z}, E)$, where $E = \{(i, i+d) | i \in \mathbb{Z}, d \in D\}$, is an infinite circulant graph with the set of distances $D$. This graph can be regarded as the Cayley graph of the additive group of $\mathbb{Z}$ with the generating set $\{\pm d_j\}_{j=1}^n$. Along with infinite circulant graphs we consider finite ones. Let $t$ be a positive integer. Let us consider a graph $Ci_t(D)$ with the set $\mathbb{Z}_t$ as the vertex set and the set $\{(i, i+d \mod t) | i \in \mathbb{Z}_t, d \in D\}$ as the edge set. We call $Ci_t(D)$ a finite circulant graph with the set of distances $D$. Such graphs can have multiedges and loops, namely they are pseudographs. A coloring of a pseudograph is called perfect
if for two vertices of the same color the multisets of colors of their neighborhoods coincide. By multiset of colors of a vertex neighborhood \( N(v) \) we mean the multiset where the number of occurrences of a color \( i \) is equal to the number of edges between the vertex \( v \) and vertices of color \( i \).

To study and enumerate perfect colorings of graphs we use the following notion of homomorphism. A \textit{homomorphism} from a graph \( G_1 = (V_1, E_1) \) to a graph \( G_2 = (V_2, E_2) \) is a surjective map \( h : V_1 \rightarrow V_2 \) preserving the relation of adjacency of vertices.

**Proposition 1.** Given graphs \( G_1 \) and \( G_2 \). If there exists a homomorphism from \( G_1 \) to \( G_2 \), then every perfect coloring of the graph \( G_2 \) induces a perfect coloring of the graph \( G_1 \) with the same parameter matrix.

The proof of this statement follows immediately from the definition of homomorphism. Proposition 1 provides us a method of constructing perfect colorings of a given graph using perfect colorings of other graphs. There is a natural homomorphism from \( Ci_{\infty}(D) \) to a pseudograph \( Ci_{i}(D) \) corresponding to the homomorphism from \( \mathbb{Z} \) to \( \mathbb{Z}_i \). This homomorphism allows us to pass from infinite to finite graphs, which are easier to work with.

Let \( t \) be a positive integer. A coloring \( \varphi \) of the circulant graph \( Ci_{\infty}(D) \) is \textit{periodic} with the length of period \( t \), if \( \varphi(i) = \varphi(i + t) \) for every \( i \in \mathbb{Z} \). We will write \([\varphi(i + 1)\varphi(i + 2)\cdots\varphi(i + t)]\) for an integer \( t \) to depict the period of \( \varphi \).

Hereinafter \( D_n \) stands for the set of distances \( \{1, 3, 5, \ldots, 2n - 1\} \) for a positive integer \( n \). In the paper we consider finite and infinite circulants with the set of distances \( D_n \). In finite case we are interested in circulants with even number of vertices.

Let us call graphs \( Ci_{\infty}(D_n) \) and \( Ci_i(D_n), t \in 2\mathbb{N}, \) \textit{infinite and finite circulant graphs with a continuous set of odd distances} respectively. These graphs are regular of degree \( 2n \) and are bipartite. One part of the bipartite graph \( Ci_i(D_n) = (V, E) \), where \( l \in 2\mathbb{N} \cup \{\infty\} \), is composed of vertices with even indices, the other one — of vertices with odd indices. Thus we can represent this graph in the following way: \( Ci_i(D_n) = (V_e \cup V_o, E) \), where \( V_e \) is the set of vertices with even indices, and \( V_o \) is the set of vertices with odd indices (\textit{even} and \textit{odd} parts of the graph respectively). We will write \( v_e \) or \( v_o \) when it is necessary to indicate that a vertex belongs to the even or to the odd part of the graph respectively.

**Proposition 2.** Every perfect coloring of the graph \( Ci_{\infty}(D_n), n \in \mathbb{N}, \) is periodic.

**Proof.** Let \( \varphi \) be a perfect coloring of \( Ci_{\infty}(D_n), n \in \mathbb{N} \) with parameter matrix \( M \). Let us take an arbitrary integer \( i \) and consider a vertex \( v_i \) with its neighborhood \( N(v_i) = v_{i-2n+1}v_{i-2n-1}v_{i-2n-2}v_{i-3}v_{i-3}v_{i-1}v_{i+1}v_{i+3}v_{i+3}v_{i+2n-1} \) perfectly colored with \( \varphi \). Let us consider the vertex \( v_{i+2n+1} \). Since it is the only vertex from the set \( N(v_{i+2}) \setminus N(v_i) \), its color is uniquely determined by the color of the vertex \( v_{i+2} \) and the parameter matrix \( M \). The same holds for the vertex \( v_{i-2n+3} \) by symmetry. This property induces the periodicity of the coloring \( \varphi \).

The following proposition concerns perfect colorings of the infinite path graph, which is, in our terms, the infinite circulant graph \( Ci_{\infty}(\{1\}) \).
Proposition 3. Let $k$ be a positive integer. The list of perfect $k$-colorings of the graph $Ci_{\infty}(\{1\})$ is exhausted by colorings with the following four periods:

1. $[123 \cdots (k-1)k]$;
2. $[k(k-1)(k-2) \cdots 212 \cdots (k-2)(k-1)]$;
3. $[k(k-1)(k-2) \cdots 212 \cdots (k-2)(k-1)k]$;
4. $[k(k-1)(k-2) \cdots 2112 \cdots (k-2)(k-1)k]$.

The proof of this statement can be found, for example, in [21] (lemma 2). Let us note that these colorings are perfect for every infinite circulant graph under consideration.

We state the following conjecture.

Conjecture 1. Let $k$ and $n$ be positive integers. The set of perfect $k$-colorings of the graph $Ci_{\infty}(D_n)$ consists of perfect colorings induced from perfect colorings of the infinite path graph and of graphs $Ci_t(D_n)$ for $t = 4n - 2, 4n, 4n + 2$.

In this paper we prove the conjecture 1 in the case when $k = 2$.

2.1 Perfect colorings of finite bipartite circulants

In this section we consider perfect colorings of graphs $Ci_t(D_n)$ for $t \in \{4n, 4n - 2, 4n + 2\}$ and $n \in \mathbb{N}$.

We say that a coloring $\varphi$ of a bipartite graph is bipartite if sets of colors of even and odd parts of the graph are disjoint.

2.1.1 The case $t = 4n$

The graph $Ci_{4n}(D_n)$ is the complete bipartite graph $K_{2n, 2n}$. A coloring of this graph is perfect if it is bipartite or if odd and even parts of the graph contain the same number of vertices of each color. Figure 1 shows the graph $Ci_8(\{1, 3\})$ perfectly colored with three colors.

![Figure 1: Perfect 3-coloring of the graph $Ci_8(\{1, 3\})$.](image-url)
2.1.2 The case $t = 4n + 2$

Let us remind that a perfect matching of a graph is an independent edge set in which every vertex of the graph is incident to exactly one edge of the matching.

We may say that the graph $Ci_{4n+2}(D_n) = (V_e \cup V_o, E)$ is the complete bipartite graph $K_{2n+1,2n+1}$ without a perfect matching $P_{2n+1} = \{(i, i + 2n + 1) | i = 0, 1, 2, \ldots, 2n\}$. In other words, every vertex $i$ of one part of the graph is adjacent to all vertices of another part except for the vertex $j$ such that $(i, j) \in P_{2n+1}$.

Let $\varphi$ be a coloring of $Ci_{4n+2}(D_n)$. The necessary condition on $\varphi$ to be perfect follows. Let $(i, j)$ be an edge from $P_{2n+1}$, and let $\varphi(i), \varphi(j)$ be (not necessary distinct) colors of vertices $i$ and $j$. Then any other edge from $P_{2n+1}$ having one endpoint colored with $\varphi(i)$ must have another endpoint colored with $\varphi(j)$, and vice versa. In particular, that means that the set of colors of a bipartite perfect coloring must be of even cardinality.

To construct a perfect bipartite coloring $\varphi$ of $Ci_{4n+2}(D_n)$, we split the set of colors $C$ into two disjoint subsets of the same cardinality $C = C_e \cup C_o$. We arrange colors in pairs $(c_i, c_j)$, where $i, j \in \{1, 2, \ldots, |C_e|\}$, and $c_i \in C_e$, $c_j \in C_o$. We color every edge $(v_e, v_o)$ of $P_{2n+1}$ with one of the assigned pairs of colors such that $\varphi(v_e) \in C_e$, $\varphi(v_o) \in C_o$.

Let us provide a method of a non-bipartite perfect coloring construction. In this case each part of the graph must have the same number of vertices of each color. Let $\varphi$ denote the coloring we are going to construct.

Let us split the set of colors into two disjoint subsets $C = C_1 \cup C_2$, where $|C_2|$ is even. Along with that we split the edges of the perfect matching $P_{2n+1}$ into two disjoint subsets $P_1$ and $P_2$, where $|P_2|$ is even. We color the endpoints of edges from the set $P_1$ with colors from the set $C_1$ in a way that endpoints of every edge get the same color.

We arrange colors of $C_2$ and edges from $P_2$ in pairs. Let us consider a pair $(c_1, c_2)$ of colors from $C_2$ and suppose that we would like to color several pairs of edges from $P_2$ with $c_1$ and $c_2$. We color each pair of edges $(v_e, v_o), (u_e, u_o)$ of the set $P_2$ in a way that endpoints of every edge get different colors, but $\varphi(v_e) = \varphi(u_o)$, and $\varphi(u_e) = \varphi(v_o)$.

The fact that the obtained coloring $\varphi$ is perfect follows directly from the definition. The perfect 4-coloring of the graph $Ci_{4n+2}(D_n)$, $n = 2$, is shown in Figure 2. The absent perfect matching is $P_{10} = \{(0, 5), (1, 6), (2, 7), (3, 8), (4, 9)\} = P_1 \cup P_2$, where $P_1 = \{(0, 5), (1, 6), (3, 8)\}$ and $P_2 = \{(2, 7), (4, 9)\}$.

![Figure 2: Perfect 4-coloring of the graph Ci_{10}(\{1, 3\}).](image-url)
Let us note that in the case of two colors $C = \{0, 1\}$ there are two possibilities:

1. $C_1 = C = \{0, 1\}$, $C_2 = \emptyset$. Endpoints of every edge $(i, i + 2n + 1)$, $i = 0, 1, \ldots, 2n$, of the perfect matching $P_{2n+1}$ are either both colored with 0, or both colored with 1.

2. $C_1 = \emptyset$, $C_2 = \{0, 1\}$. The only possible perfect 2-coloring in this case is the bipartite one.

### 2.1.3 The case $t = 4n - 2$

Let us consider the perfect matching on $4n - 2$ vertices $P_{2n-1} = \{(i, i + 2n - 1)\mid i = 0, 1, 2, \ldots, 2n - 2\}$. Every vertex $i \in V_e$ of the bipartite pseudograph $C_{i4n-2}(D_n) = (V_e \cup V_o, E)$ is adjacent to all vertices of $V_o$ and has an extra edge to the vertex $j$ such that $(i, j) \in P_{2n-1}$. The same holds for every vertex of $V_o$. Informally speaking, $C_{i4n-2}(D_n)$ is the complete bipartite graph $K_{2n-1, 2n-1}$ with extra perfect matching $P_{2n-1}$.

The coloring procedure for this graph coincides with the coloring procedure for the graph $C_{i4n-2}(D_n)$. Figure 3 shows two examples of perfect 2-colorings of $C_{i6}(\{1, 3\})$. In the first case the set of colors $C = \{\text{white, gray}\}$ coincides with the set $C_1$, while sets $C_2$ and $P_2$ are empty. In the second picture the bipartite coloring is shown.

![Figure 3: Perfect 2-colorings of the graph $C_{i6}(\{1, 3\})$.](image)

### 3 Main result

In this section we consider perfect 2-colorings of the graph $C_{i\infty}(D_n)$. As a matter of convenience hereinafter we name colors of any 2-coloring of a graph black ($\bullet$) and white ($\circ$). The parameter matrix of a perfect 2-coloring has the following form: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since the graph under consideration is regular of degree $2n$, the parameters $a$ and $d$ can be represented as $2n - b$ and $2n - c$ respectively. Sometimes instead of considering the parameter matrix of a coloring we will take into account parameters $b$ and $c$, which are called outer degrees of black and white color...
respectively. A pair \((b, c)\) is called *admissible* for the graph \(C_{i\infty}(D_n)\) if there exists a perfect 2-coloring of \(C_{i\infty}(D_n)\) with the parameter matrix \(\begin{pmatrix} 2n - b & b \\ c & 2n - c \end{pmatrix}\).

**Theorem 1.** Let \(n\) be a positive integer, and \(C_{i\infty}(D_n)\) be the infinite circulant graph with a continuous set of odd distances. The set of perfect 2-colorings of \(C_{i\infty}(D_n)\) consists of perfect colorings induced by perfect colorings of the infinite path graph and of graphs \(C_{i\infty}(D_n)\) for \(t = 4n - 2, 4n, 4n + 2\).

Let us state and prove several preliminary lemmas.

**Lemma 1.** Let \(n\) be a positive integer. A pair of positive integers \((b, c)\) is *admissible* for the graph \(C_{i\infty}(D_n)\) if and only if \(b + c \in \{4n, 2n, 2n + 1, 2n - 1\}\).

**Proof.** The parameter matrix of the bipartite coloring of \(C_{i\infty}(D_n)\) is \(\begin{pmatrix} 0 & 2n \\ 2n & 0 \end{pmatrix}\), and \(b + c = 4n\). The period of this coloring is \([••]\).

Let \(\varphi\) be a perfect coloring of \(C_{i\infty}(D_n)\) with the period length longer than 2. That means there exists a positive integer \(i\) such that \(\varphi(v_i) \neq \varphi(v_{i+2})\). Without loss of generality let \(\varphi(v_i) = •\). The neighborhoods \(N(v_i)\) and \(N(v_{i+2})\) share \(2n - 2\) vertices, and the following holds: \(N(v_i) \setminus N(v_{i+2}) = \{v_{i-2n+1}\}, N(v_{i+2}) \setminus N(v_i) = \{v_{i+2n+1}\}\). Let us consider the pair of vertices \((v_{i-2n+1}, v_{i+2n+1})\) and their possible colors.

1. If \(\varphi(v_{i-2n+1}) = \varphi(v_{i+2n+1})\), neighborhoods \(N(v_i)\) and \(N(v_{i+2})\) have the same number of black and white vertices. That means every vertex is adjacent to the same number of black and white vertices regardless of its own color. The parameter matrix of the coloring in this case is \(\begin{pmatrix} c & b \\ c & b \end{pmatrix}\), and \(b + c = 2n\). Moreover, vertices \(v_{i-2n+1}\) and \(v_{i+2n+1}\) are of the same color for every \(i \in \mathbb{Z}\), which means the coloring \(\varphi\) is periodic with the period length \(4n\).

2. Let \((\varphi(v_{i-2n+1}), \varphi(v_{i+2n+1})) = (\circ, •)\). In this case every black vertex has one more white vertex in its neighborhood than the white one. The parameter matrix of the coloring is \(\begin{pmatrix} c - 1 & b \\ c & b - 1 \end{pmatrix}\), and \(b + c = 2n + 1\).

3. Let \((\varphi(v_{i-2n+1}), \varphi(v_{i+2n+1})) = (•, \circ)\). In this case every black vertex has one more black vertex in its neighborhood than the white one. The parameter matrix of the coloring is \(\begin{pmatrix} c + 1 & b \\ c & b + 1 \end{pmatrix}\), which means \(b + c = 2n - 1\).

Since all possibilities are listed, then \(b + c \in \{4n, 2n, 2n + 1, 2n - 1\}\). □

**Lemma 2.** Let \(n, b, c\) be positive integers, and the pair \((b, c)\) be admissible for the graph \(C_{i\infty}(D_n)\) with corresponding to it perfect coloring \(\varphi\). Let us consider vertices \(i\) and \(i + 2\) of the graph \(C_{i\infty}(D_n)\). The following holds:

1. If \(\varphi(i) = \varphi(i + 2)\), then \(\varphi(i - 2n + 1) = \varphi(i + 2n + 1)\);
2. If \( \varphi(i) \neq \varphi(i + 2) \) and \( b + c = 2n + 1 \), then \( \varphi(i - 2n + 1) = \varphi(i + 2) \), and 
\( \varphi(i + 2n + 1) = \varphi(i) \);

3. If \( \varphi(i) \neq \varphi(i + 2) \) and \( b + c = 2n - 1 \), then \( \varphi(i - 2n + 1) = \varphi(i) \) and 
\( \varphi(i + 2n + 1) = \varphi(i + 2) \).

**Proof.** The proof of the lemma follows directly from the definition of perfect coloring and the proof of lemma 1.

Let \( G \) be an infinite circulant graph, \( \varphi \) be its 2-coloring, and \( s \) be a positive integer. The sequence of vertices \( \{v_{i+j}\}_{j \in \mathbb{Z}} \) of \( G \) for an integer \( i \) is called \( s \)-chain. If the inequality \( \varphi(v_{i+j}) \neq \varphi(v_{i+(j+1)s}) \) holds for every \( j \), then the sequence \( \{v_{i+j}\}_{j \in \mathbb{Z}} \) is called an alternating \( s \)-chain.

**Lemma 3.** Let \( n, b \) and \( c \) be positive integers. Let the pair \((b, c)\) be admissible for the graph \( Ci_{\infty}(D_n) \). If \( b + c = 2n + 1 \), then every perfect 2-coloring \( \varphi \) corresponding to the pair \((b, c)\) either has the period length \( 2n + 1 \), or consists of alternating \((2n-1)\)-chains.

**Proof.** Let us suppose that \( b+c = 2n+1 \) and that there is a vertex \( i \) of \( Ci_{\infty}(D_n) \) such that \( \varphi(i) \neq \varphi(i + 2n + 1) \). Let \( \varphi(i) = x \in \{0, 1\} \), then \( \varphi(i + 2n + 1) = 1 - x = \overline{x} \). Let us consider the vertex \( i + 2 \). It cannot be colored with \( \overline{x} \), since that would contradict item 2 of lemma 2. Thus \( \varphi(i + 2) = x \). According to item 1 of lemma 2 the vertex \((i - 2n + 1)\) is colored with \( \overline{x} \). Following the same logic we obtain that the vertex \((i + 2n - 1)\) is colored with \( \overline{x} \), and the vertex \((i + 4n)\) is colored with \( x \). Applying item 2 and item 1 of lemma 2 to vertices \((i + 2n - 1), (i + 4n)\), we obtain equalities \( \varphi(i + 4n - 2) = x \) and \( \varphi(i + 6n - 1) = \overline{x} \).

Alternatively applying item 2 and item 1 of lemma 2 to pairs of vertices \((i + (2n - 1)j, i + 2n + 1 + (2n - 1)j)\) \( j \in \mathbb{N} \) and \((i + 2 - (2n + 1)j, i - 2n - 1 - (2n + 1)j)\) \( j \in \mathbb{N} \), we obtain two alternating \((2n - 1)\)-chains \( \{i + (2n - 1)j\}_{j \in \mathbb{Z}} \) and \( \{i + 2 + (2n - 1)j\}_{j \in \mathbb{Z}} \). Two alternating \((2n - 1)\)-chains in \( Ci_{\infty}(D_n) \) built in described way are shown in Figure 4. The edges colored with gray do not exist in the graph, they are shown by illustrative reasons and connect pair of vertices \((l, l + 2n + 1)\) \( l \in \mathbb{Z} \).

\[ \text{Figure 4: Alternating } (2n-1)\text{-chains in } Ci_{\infty}(D_n). \]

In view of two alternating chains shown in Figure 4 let us consider vertices \((i+4)\) and \((i + 2n + 3)\). Using the inequality \( \varphi(i + 2) \neq \varphi(i - 2n + 3) \) and applying item 1 and item 2 of lemma 2, we obtain the inequality \( \varphi(i + 4) \neq \varphi(i + 2n + 3) \). Acting the
same way we obtain the alternating \((2n - 1)\)-chain \(\{i + 4 + (2n - 1)j\}_{j \in \mathbb{Z}}\). Finally, the whole graph is colored with alternating \((2n - 1)\)-chains; the period length of the obtained coloring is \(4n - 2\), and the number of black and white vertices in the period is the same. This coloring can be considered as the one induced from a coloring of \(Ci_{4n-2}(D_n)\).

Lemma 4. Let \(n, b\) and \(c\) be positive integers. Let the pair \((b, c)\) be admissible for the graph \(Ci_{\infty}(D_n)\). If \(b + c = 2n - 1\), then every perfect 2-coloring \(\varphi\) corresponding to the pair \((b, c)\) either has the period length \(2n - 1\), or consists of alternating \((2n + 1)\)-chains.

\[\begin{proof}\]
The proof of this lemma is similar to the previous one. One should suppose that there is a vertex \(i\) such that \(\varphi(i) \neq \varphi(i + 2n - 1)\) and then apply item 3 and item 1 of lemma 2 one after other to build an alternating \((2n + 1)\)-chain. The corresponding to this algorithm picture is shown in Figure 5. Edges colored with gray represent parts of chains.

\[\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Alternating \((2n + 1)\)-chains in \(Ci_{\infty}(D_n)\).}
\end{figure}\]

Applying item 3 and item 1 of lemma 2 to \((2n + 1)\)-chains \(\{i + 2s + (2n + 1)j\}_{j \in \mathbb{Z}}\) for \(s = 1, 2, \ldots, 2n - 1\), we obtain the coloring of \(Ci_{\infty}(D_n)\) with alternating \((2n + 1)\)-chains. It has the period length \(4n + 2\) and can be considered as the coloring induced from one of the colorings of \(Ci_{4n+2}(D_n)\).

\[\begin{proofof}[\text{of theorem 1}]\]
In the case of two colors the set of perfect colorings of the graph \(Ci_{\infty}(\{1\})\) consists of three elements with periods \([\bullet \circ \circ \circ], [\bullet \circ \circ]\) and \([\bullet \circ]\). Colorings of the graph \(Ci_{\infty}(D_n)\) with these periods are perfect for every positive integer \(n\).

According to lemma 1, the sum \(b + c\) can be equal to \(4n, 2n, 2n + 1\) or \(2n - 1\).

The only possible perfect coloring corresponding to the admissible pair \((b, c)\) with \(b + c = 4n\) is bipartite and is already listed as a coloring of \(Ci_{\infty}(\{1\})\). Let us consider other possible values of the sum \(b + c\).

1. Let \(b + c = 2n\). By lemma 1, every perfect coloring corresponding to the pair \((b, c)\) is periodic with the period length \(4n\).

Let us consider the graph \(Ci_{4n}(D_n) = (V, E)\) with \(V = \{0, 1, 2, \ldots, 4n - 1\}\).

According to proposition 1 every perfect coloring \(\phi\) of this graph induces the perfect coloring of \(Ci_{\infty}(D_n)\) with period \([\phi(0)\phi(1)\phi(2)\cdots\phi(4n - 1)]\).
2. Let \( b + c = 2n + 1 \). According to lemma 3, every perfect coloring corresponding to the pair \((b, c)\) either consists of alternating \((2n - 1)\)-chains, or has the period length \(2n + 1\).

Let us consider the graph \( Ci_{4n+2}(D_n) \). The set of its perfect colorings is described in Subsection 2.1.2. Every coloring \( \phi \) of the graph \( Ci_{4n+2}(D_n) \) induces the perfect coloring of \( Ci_{\infty}(D_n) \) with period \( [\phi(0)\phi(1)\phi(2)\cdots\phi(4n+1)] \).

3. Let \( b + c = 2n - 1 \). By lemma 4 every perfect coloring corresponding to the pair \((b, c)\) either consists of alternating \((2n - 1)\)-chains, or has the period length \(2n - 1\).

Let us consider the graph \( Ci_{4n-2}(D_n) \). The set of its perfect colorings is described in Subsection 2.1.3. Every coloring \( \phi \) of the graph \( Ci_{4n-2}(D_n) \) induces the perfect coloring of \( Ci_{\infty}(D_n) \) with period \( [\phi(0)\phi(1)\phi(2)\cdots\phi(4n-3)] \).

The main result of the paper confirms conjecture 1 in the case of two colors. The case when the number of colors is greater than two remains open. The main obstacle on the way of further classification of perfect colorings of infinite circulants is a large number of cases to study. The techniques of case reduction and examination of perfect colorings of such graphs are yet to be described.

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