Self-intersection on pair of pants

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Abstract

In this paper, we use the coding developed by R. Bowen and C. Series to compute the number of self-intersections of a closed geodesic on a pair of pants. We give lower and upper bounds on the number of self-intersections of a closed geodesic on a pair of pants. We prove a conjecture of Moira Chas and Anthony Phillips in [4]. We get also bounds for the number of closed geodesics whose self-intersection number is very close to the maximal self-intersection number on a pair of pants.

1 Introduction

Let $S = \mathbb{D}/\Gamma$ be a hyperbolic surface where $\Gamma$ is a Schottky group purely hyperbolic generated by the isometries $a_1, b_1, \ldots, a_p, b_p$. We note by $\Sigma$ the set of all bi-infinite reduced words of $\Gamma$. The coding introduce by R. Bowen and C. Series (see [3], [18], [19]) gives a bijective correspondence between the set of closed geodesics $G^c$ of $S$ and the set $\Sigma$ under cyclic permutation; in particular a closed geodesic $\gamma \in G^c$ is associated to a unique finite word $w(\gamma) = s_1^{i_1} r_1^{j_1} \cdots s_n^{i_n} r_n^{j_n}$ under cyclic permutation where $s_i \in \{a_1, \ldots, a_p, \bar{a}_1, \ldots, \bar{a}_p\}$ and $r_i \in \{b_1, \ldots, b_p, \bar{b}_1, \ldots, \bar{b}_p\}$, $\bar{a}_i = a_i^{-1}$ and $\bar{b}_i = b_i^{-1}$ for all $1 \leq i \leq n$. The integer $L(\gamma) = \sum_{k=1}^{n} (i_k + j_k)$ is the combinatorial length of $\gamma$.

In this paper, we are interested in the relationship between the self-intersection and the combinatorial length of a closed geodesic on a pair of pants. The methods on this paper are combinatorial and we use the ideas developed by Cohen and Lustig in [5] to compute the self-intersection number of a closed geodesic. In [4], A. Basmajian study...
the relationship between self-intersection and the geometric length of a closed geodesic on a hyperbolic surface by using hyperbolic geometric.

Let $\mathbb{D}$ denotes the Poincaré disk endowed with the hyperbolic metric and let $a$ and $b$ two hyperbolic isometries of $\mathbb{D}$ verifying the following conditions:

1. there are four disjoint euclidean closed half-disks $D(a)$, $D(\bar{a})$, $D(b)$ and $D(\bar{b})$ of $\mathbb{D}$ which are orthogonal to $S^1$ and such that $a(D(a)) = \mathbb{D} - D(\bar{a})$ et $b(D(b)) = \mathbb{D} - D(\bar{b})$.

2. the half-disks $D(a)$ and $D(\bar{a})$ side by side (see figure 1), where $\bar{a}$ and $\bar{b}$ are respectively the inverse of $a$ and $b$.

Throughout this paper we will consider the configuration in figure 1. The group $\Gamma := \langle a, b \rangle$ generated by $a$ and $b$ is a Schottky group and the surface $P = \mathbb{D}/\Gamma$ is a pair of pants (see [2], [6]). Let us denote $\Gamma = \{a, \bar{a}, b, \bar{b}\}$ the set of generators of $\Gamma$ and their inverse. The set $D = \bigcap_{e \in \{a, \bar{a}, b, \bar{b}\}} \mathbb{D} - \hat{D}(e)$ is a fundamental domain of $\mathbb{D}$ under the action of $\Gamma$.

Using this symbolic point of view, we give a method to calculate the self-intersection number of any geodesic in $\mathfrak{F}$. Let $\gamma$ be a closed geodesic of $P$ and $w(\gamma) = s_1^{i_1} r_1^{j_1} \cdots s_n^{i_n} r_n^{j_n}$ the word associated to $\gamma$ where $n, i_1, j_1, \cdots, i_n, j_n$ are non-zero positive integers and $s_k \in \{a, \bar{a}\}$ and $r_k \in \{b, \bar{b}\}$ for all $1 \leq k \leq n$. Each cyclic permutation of $w(\gamma)$ corresponds to a unique lift $\gamma_i$ of $\gamma$ in $\mathbb{D}$ which crosses the fundamental domain $\Delta_{\Gamma}$ for $1 \leq i \leq L(\gamma)$ . Then:

\[ i(\gamma; \gamma) = \# \{ \gamma_i \cap \gamma_j \cap D \neq \emptyset; \ 1 \leq i < j \leq L(\gamma) \}. \]
The self-intersection number $i(\gamma; \gamma)$ of $\gamma$ satisfies the following inequalities:

**Theorem 1.1.** Let $\gamma$ be a non-simple closed geodesic of $P$ and $w(\gamma) = s_1^{i_1} r_1^{j_1} \cdots s_n^{i_n} r_n^{j_n}$ the word associated to $\gamma$, then:

$$L(\gamma) - n - 1 \leq i(\gamma; \gamma) \leq nL(\gamma) - n^2.$$

In [4], M. Chas and A. Phillips use the same method to tabulate self-intersection numbers for closed curves on pair of pants and they proved that the self-intersection numbers are bounded below and these bounds are sharp. They also proved that the self-intersection of a closed curve of combinatorial length $L(\gamma)$ is bounded above and this bound is sharp when $L(\gamma)$ is even and when $L$ is odd, they conjectured that the maximal self-intersection number is $\frac{L^2(\gamma) - 1}{4}$. The theorem [1.1] enables us to prove this conjecture for closed geodesics on the pair of pants.

**Theorem 1.2.** Let $\gamma$ be a non-simple closed geodesic of $P$ and $L(\gamma)$ combinatorial length then we have:

$$
\begin{cases}
\frac{L(\gamma) - 1}{2} & \text{if } L(\gamma) \text{ is even} \\
\frac{L(\gamma)}{2} & \text{if } L(\gamma) \text{ is odd}
\end{cases}
\leq i(\gamma; \gamma) \leq
\begin{cases}
\frac{L^2(\gamma)}{4} & \text{if } L(\gamma) \text{ is even} \\
\frac{L^2(\gamma) - 1}{4} & \text{if } L(\gamma) \text{ is odd}
\end{cases}
$$

These bounds are sharp.

We want to determine the proportion of closed geodesics of $P$ whose self-intersection number is very close to the maximal self-intersection number. Set

$$A_\epsilon(L) = \{ \gamma \in \mathcal{C} | L(\gamma) = L, \ i(\gamma; \gamma) \geq \left( \frac{1}{4} - \epsilon \right)L^2 \}$$

where $L$ is an integer and $\epsilon$ a positive real. In this paper, we get the following result for a pair of pants.

**Theorem 1.3.** For $L$ large enough, we have:

$$\frac{4 \times 3^{6\epsilon L}}{L} \leq \#A_\epsilon(L) \leq \frac{2^L}{L} \left[ \frac{3}{2} \right]^6 \sqrt{\epsilon}L - 3\sqrt{\epsilon}L$$

and

$$\lim_{L \to +\infty} \frac{\#A_\epsilon(L)}{\#\mathcal{C}(L)} = 0 \text{ for } \epsilon < \frac{1}{36}.$$
The problem of counting closed geodesics has been studied in many contexts (see [9], [10], [12], [8], [7]). For more details, there is a survey of the history of this problem by Richard Sharp that was published in conjunction with Margulis’s thesis in [20].

Recently, there has been work on the dependence of the number of closed geodesics and their self-intersection number as well as length (see [11] [14], [13] [16] and [17]).

In Section 2, we recall the construction of a combinatorial model for closed geodesics on a pair of pants. We show that each geodesic can be represent as a cyclically reduced non-periodic word. In section 3, we use this combinatorial model to develop a method which enables us to calculate the number of self-intersections of a given closed geodesic on a pair of pants. In section 4, we use the formula obtained in section 3 to prove the theorems previously enunciated.

2 Coding closed geodesics

Let $a$ and $b$ be two hyperbolic isometries such that we have the configuration described in the introduction (see figure 1).

Denote by $\Gamma = \langle a, b \rangle$ the group generated by $a$ and $b$ and by $\overline{\Gamma} = \{a, \bar{b}, b, \bar{a}\}$ the set of generators of $\Gamma$ and their inverse. Our goal in this section is to give a way to code the closed geodesics of the hyperbolic surface $P = \mathbb{D}/\Gamma$. We begin to code the points of the limit set $L_\Gamma$ of $\Gamma$. Any geodesic of the hyperbolic surface $P$ admits a lift in the Poincaré disc which both endpoints lie in the limit set $L_\Gamma$. Together with the coding of the limit set, this implies the coding of the geodesics of $P$.

2.1 Coding of the limit set

In this section, we will give a coding of the points of $L_\Gamma$. For any $e \in \overline{\Gamma}$, define $A(e)$ to be the arc of $\partial \mathbb{D}$ inside $D(e)$ (see figure 1). We will call these four arcs the first order intervals on $\partial \mathbb{D}$.

**Definition 2.1.**

- A word in $\Gamma$ is a sequence $e_1 e_2 \cdots$ where $e_i \in \Gamma$.
- A finite word $e_1 e_2 \cdots e_n$ is reduced if for all $1 \leq i \leq n$, $e_i \neq \bar{e}_{i+1}$. It is said cyclically reduced if $e_1 \neq \bar{e}_n$.
- A infinite or bi-infinite word is reduced if each of his finite sub-word is reduced.

In order to have a better localisation of the points of the limit set and to construct the coding of the limit set we need to use the sub-arcs of the first order intervals on $\partial \mathbb{D}$.
Definition 2.2. For any reduced word $e_1 e_2 \cdots$ in $\Gamma$, the subset of $\partial \mathbb{D}$ defined by:

$$A(e_1 e_2 \cdots e_m) = e_1 \cdots e_{m-1}A(e_m)$$

is called $m$-th order interval.

Proposition 2.3. 1. For any reduced word $e_1 e_2 \cdots$, we have

$$A(e_1 e_2 \cdots e_m) \subset A(e_1 e_2 \cdots e_{m-1}) \subset \cdots \subset A(e_1).$$

2. If $e_1 e_2 \cdots e_m$ and $f_1 f_2 \cdots f_m$ are two reduced words such that $e_1 e_2 \cdots e_m \neq f_1 f_2 \cdots f_m$ then:

$$A(e_1 e_2 \cdots e_m) \cap A(f_1 f_2 \cdots f_m) = \emptyset.$$ 

Proof. 1. We will show this point by recurrence. Because $e_1 e_2 \cdots$ is reduced, $e_1 \neq \bar{e}_2$ and then $A(e_2) \subset \partial \mathbb{D} \setminus A(\bar{e}_1)$. The condition on the isometries $a$ and $b$ implies that $A(e_1 e_2) = e_1 A(e_2) \subset A(e_1)$. We suppose that $A(e_2 e_3 \cdots e_{m}) \subset A(e_2 e_3 \cdots e_{m-1})$. Because $e_1$ is an isometry, calculate the image of these sets by $e_1$ gives us the result.

2. Because $e_1 e_2 \cdots e_m \neq f_1 f_2 \cdots f_m$, there exists an integer $p$ between 1 and $m - 1$ such that $e_1 e_2 \cdots e_p = f_1 f_2 \cdots f_p$ and $e_{p+1} \neq f_{p+1}$. Using the point 1 of this proposition and the fact that $A(e_{p+1}) \cap A(f_{p+1}) = \emptyset$ gives us the result.

\[ \square \]

Set $A = \bigcup_{e \in \Gamma} A(e)$ and define the following map $f : A \to \partial \mathbb{D}$ by $f_{|A(e)}(x) = \bar{e}(x)$. Let $\eta \in L_{\Gamma} \subset A$ be a point of the limit set then there exists $e_1 \in \Gamma$ such that $\eta \in A(e_1)$ and $f(\eta) = \bar{e}_1(\eta) \in \partial \mathbb{D} \setminus A(\bar{e}_1)$. If $f(\eta) \in A$ then there exists $e_2 \in \Gamma$ such that $f(\eta) \in A(e_2)$ and in this case we have $f^2(\eta) = \bar{e}_2(\bar{e}_1(\eta))$. By this way, we show that any point $\eta$ of $A$ admits a finite or an infinite expansion $e_1 e_2 \cdots, e_1 \in \Gamma$ defined by $f^n(\eta) \in A(e_n), n \geq 1$ where the sequence terminates at $e_n$ if and only if $f^n(\eta) \in A$ and $f^{n+1}(\eta) \notin A$.

The sequence is infinite if and only if $\eta \in L_{\Gamma}$. Conversely, for any infinite reduced word $e_1 e_2 \cdots$, the set $\bigcap_{n=1}^{\infty} A(e_1 e_2 \cdots e_n) = \bigcap_{n=1}^{\infty} f^{-n-1} A(e_n)$ is non-empty and it is reduced to a point belonging on the limit set because the diameter of the sequence of sets $A(e_1 e_2 \cdots e_n)$ goes to 0 whenever $n$ goes to $\infty$.

Denote by $\Sigma^+$ the set of all infinite reduced words in $\Gamma$. We have showed the following result:
Proposition 2.4. The map
\[ p^+ : \Sigma^+ \to L_{\Gamma} \]
\[ e_1 e_2 \ldots \mapsto \lim_{n \to +\infty} e_1 e_2 \ldots e_n(o). \]
is a bijection.

This proposition enables us to represent a point of the limit set \( L_{\Gamma} \) as an infinite reduced word. We will write \( \eta = e_1 e_2 \ldots \) when the infinite reduced word \( e_1 e_2 \ldots \) corresponds to the limit point \( \eta \).

Remark 2.5. Let \( \eta = e_1 e_2 \ldots \) be a point of the limit set then \( \bar{e}_1(\eta) = e_2 e_3 \ldots \) and for any \( f \in \Gamma \setminus \{\bar{e}_1\} \) we have \( f(\eta) = f e_1 e_2 \ldots \).

2.2 Representation of geodesics

In this section, we use the coding of the limit set in order to give a representation of the oriented closed geodesics of the pair of pants.

We begin with the geodesics of the Poincaré disc \( \mathbb{D} \). Remember that a geodesic in \( \mathbb{D} \) can be specified by its two endpoints. Recall that the non-wandering set of a flow is the set of points which return infinitely often within bounded distance of some given fixed point.

Denote by \( \mathcal{A} \) be the set of oriented geodesics on \( \mathbb{D} \) which intersect the fundamental domain \( \mathcal{D} \) and both his endpoints belong on \( L_{\Gamma} \). Let \( \gamma \) be a geodesic of \( \mathcal{A} \) and denote by \( \gamma^+ \) his positive endpoint and by \( \gamma^- \) his negative endpoint. By the proposition 2.4, there exists two infinite reduced words \( e_1 e_2 \ldots e_n \ldots \) and \( f_1 f_2 \ldots f_n \ldots \) such that \( \gamma^+ = e_1 e_2 \ldots e_n \ldots \) and \( \gamma^- = f_1 f_2 \ldots f_n \ldots \). Because of \( \gamma \cap \mathcal{D} \neq \emptyset \), we have \( e_1 \neq f_1 \) and the bi-infinite word \( \gamma^+ \ast \gamma^- = \cdots f_n \cdots f_2 f_1 e_1 e_2 \cdots e_n \cdots \) is reduced. We have the following result of C. Series [19]

Proposition 2.6. The map \( p : \mathcal{A} \to \Sigma \) defined by \( p(\gamma) = \gamma^+ \ast \gamma^- \) for all \( \gamma \in \mathcal{A} \) is a bijection.

The following result tells us that the action of the shift \( \sigma \) on \( \Sigma \) and the action of \( \Gamma \) on \( \mathcal{A} \) permutes.

Proposition 2.7. Let \( \gamma \) be a geodesic of \( \mathcal{A} \) and \( g \) an isometry in \( \Gamma \). There exists an integer \( N \in \mathbb{Z} \) such that \( p(g(\gamma)) = \sigma^N p(\gamma) \). where \( \sigma^N = \sigma \sigma \cdots \sigma \)

Remark 2.8. Let \( \gamma \) be an oriented geodesic in \( \mathbb{D} \) such that both of his endpoints lie in \( L_{\Gamma} \) but \( \gamma \) doesn’t intersect the fundamental domain \( \mathcal{D} \). If \( \gamma^+ = e_1 e_2 \ldots \) and \( \gamma^- = f_1 f_2 \ldots \) then there exists an integer \( m \) such that \( e_1 e_2 \cdots e_m = f_1 f_2 \cdots f_m \) and \( e_{m+1} \neq f_{m+1} \). Thus the geodesic \( \gamma \) is map by the isometry \( g = \bar{e}_m e_{m-1} \cdots \bar{e}_1 \) to an oriented geodesic which intersects the fundamental domain \( \mathcal{D} \) and both of his endpoints lie in \( L_{\Gamma} \)
The two last results tell us that there is a bijection between the set of all orbits of the action of \( \sigma \) on the set of bi-infinite reduced words, \( \Sigma \) and the set of orbits of the action of \( \Gamma \) on the set of all oriented geodesics in \( D \) which intersect the fundamental domain and both of whose endpoints lie in \( L_\Gamma \) up to cyclic permutation. In other words, \( \Sigma \) is in bijection with the non-wandering set of the geodesic flow on \( P \) up to cyclic permutation.

Now we will focus on the closed geodesics and on the bi-infinite reduced periodic words of \( \Sigma \). A word \( W \) is said to be periodic if there exists an integer \( n \) such that \( \sigma^n(W) = W \). Thus if \( W \) is periodic, there exists a cyclically reduced word \( w = e_1e_2\cdots e_n \) such that \( W = \cdots www \cdots \). In this case we will write \( W = \langle w \rangle = \langle e_1e_2\cdots e_n \rangle \).

Let \( \gamma \) be an oriented geodesic of \( P \) in the non-wandering set of the geodesic flow. Assume that the word \( W = \langle e_1e_2\cdots e_n \rangle \) associated to \( \gamma \) is periodic. For any integer \( 1 \leq i \leq n \), denote by \( \gamma_i^+ = e_ie_{i+1}\cdots e_{i-1} \) and by \( \gamma_i^- = \bar{e}_{i-1}\bar{e}_{i-2}\cdots \bar{e}_{i} \) the points of \( L_\Gamma \) associated to the infinite reduced periodic words \( e_ie_{i+1}\cdots \) and \( \bar{e}_{i-1}\bar{e}_{i-2}\cdots \). Because \( W \) is reduced, the geodesic of \( D \) denoted by \( \gamma_i \) which endpoints are \( \gamma_i^+ \) and \( \gamma_i^- \) intersects the fundamental domain \( D \).

Lemma 2.9. The unique lifts of \( \gamma \) in the Poincaré disc \( D \) which intersect the fundamental domain \( D \) are the geodesics \( \gamma_i \) for \( 1 \leq i \leq n \).

Proof. Let \( \alpha = (\alpha^+;\alpha^-) \) be a lift of \( \gamma \) in \( D \) which intersect the fundamental domain \( D \), then there exists an isometry \( g = f_1f_2\cdots f_p \in \Gamma \) where \( f_i \in \Gamma \) such that \( g(\gamma_i) = \alpha \). Thus, by the remark \( 2.5 \), we have \( \alpha^+ = f_1f_2\cdots f_pe_1e_2\cdots e_n e_1e_2\cdots e_n \cdots \) and \( \alpha^- = f_1f_2\cdots f_pe_n\cdots \bar{e}_1 \cdots \). Without loss of generality, we can suppose that \( p < n \). Then the geodesic \( \alpha \) intersects the fundamental domain \( D \) if and only if \( f_1f_2\cdots f_p = (e_1e_2\cdots e_p)^{-1} \) or \( f_1f_2\cdots f_p = (\bar{e}_ne_{n-1}\cdots \bar{e}_{n-p})^{-1} \). In the first case, \( \alpha = \gamma_p \) and in the second \( \alpha = \gamma_{n-p} \).

Together with the fact that any non-wandering geodesic of \( P \) is closed if and only if it admits a finite number of lifts in \( D \) which intersect the fundamental domain \( D \), this lemma implies the following result:

Proposition 2.10. Any closed geodesic of \( P \) is associated to a unique finite cyclically reduced word up to cyclic permutation.

We will use this symbolic point of view in section 3 to calculate the number of self-intersections of the closed geodesics of \( P \).
3 Self-intersection of closed geodesic

In this section, we will give a method which enables us to determine the number of self-intersections of a closed geodesic of $P$ by using the coding of closed geodesics of $P$ introduced in section 2.

3.1 Cyclically lexicographical ordering

Let $\gamma_1 = (\gamma_1^+; \gamma_1^-)$ and $\gamma_2 = (\gamma_2^+; \gamma_2^-)$ be two oriented geodesics in $\mathbb{D}$. The geodesic $\gamma_1$ intersects the geodesic $\gamma_2$ if and only if the endpoints of $\gamma_1$, $\gamma_1^+$ and $\gamma_1^-$ separates the endpoints of $\gamma_2$, $\gamma_2^+$ and $\gamma_2^-$. A cyclic alphabet is a cyclically ordered set of distinct symbols and an alphabet is a finite ordered set of distinct symbols.

$\Gamma = \{a, \bar{b}, b, \bar{a}\}$ is the cyclic alphabet whose letters are the generating set arranging in the order in which the first orders intervals (see figure 1) occur around $\partial \mathbb{D}$ anticlockwise. Thus to the cyclic alphabet $\Gamma$ we can associate four distinct alphabets $\Gamma_a = \{a, \bar{a}, b\}$, $\Gamma_b = \{\bar{b}, b, a\}$, $\Gamma_{\bar{a}} = \{\bar{a}, a, \bar{b}\}$ and $\Gamma_{\bar{b}} = \{\bar{b}, a, \bar{a}\}$.

In [3], J. Birman and C.Series establish the next theorem:

**Theorem 3.1 (Thm A, [3])**. Let $\eta = e_1 e_2 \cdots$ and $\zeta = f_1 f_2 \cdots$ be two distinct points on $\partial \mathbb{D}$. Then $\eta$ precedes $\zeta$ in anticlockwise order around $\partial \mathbb{D}$ starting from the point $I$ (see figure 1) if and only if:

1. $e_1$ precedes $f_1$ in the alphabet $\Gamma_a$, or
2. $e_i = f_i$ for $i = 1, \cdots, m$ and $e_{m+1}$ precedes $f_{m+1}$ in the alphabet $\Gamma_{\bar{a}}$.

This theorem enables us to define an order on the set of infinite reduced words which is compatible with the representation of points of $L_\Gamma$. From now to see if an oriented geodesic in $\mathbb{D}$ intersect another oriented geodesic, it suffices to look the infinite reduced words associated to the endpoints of these geodesics.

3.2 Self-intersection and coding

Let $\gamma$ be a closed geodesic on $P$, it admits a self-intersection on $P$ if there exists two lifts of $\gamma$ on $\mathbb{D}$ which intersect each other in $\mathbb{D}$. If one of those lifts belongs entirely on $D(e)$ for $e \in \Gamma$, there exists an isometry $g \in \Gamma$ which maps these two geodesics on two lifts of $\gamma$ on $\mathbb{D}$ which intersect the fundamental domain $D$ and intersect each other. Thus in order to calculate the number of self-intersections of a closed geodesic, it suffices to consider the lifts which intersect the fundamental domain. The lemma 2.9 enables us to describe all of these lifts if we know the word associated to the closed geodesic $\gamma$. 

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Let \( w = e_1e_2 \cdots e_n \) be a cyclically reduced word. Let \( \gamma = \gamma(w) \) be the closed geodesic of \( P \) associated to this word. The lifts of \( \gamma \) in \( \mathbb{D} \) which intersect the fundamental domain \( \mathcal{D} \) are the oriented geodesics \( \gamma_i \) of positive endpoint \( \gamma_i^+ = e_ie_{i+1} \cdots e_{i-1} \) and negative endpoint \( \gamma_i^- = \bar{e}_{i-1} \cdots \bar{e}_1 \bar{e}_n \cdots \bar{e}_i \) for all \( 1 \leq i \leq n \). If two geodesics of \( \mathbb{D} \), \( \gamma_i \) and \( \gamma_j \) intersect each other we will note by \( (\gamma_i; \gamma_j) \) the self-intersection point.

Denote by \( A_\gamma = \{ (\gamma_i; \gamma_j); \gamma_i \cap \gamma_j \neq \emptyset; 1 \leq i \neq j \leq n \} \) the set of intersection points between two lifts of \( \gamma \) which intersect the fundamental domain \( \mathcal{D} \). To calculate the number of points of self-intersections of the closed geodesic \( \gamma \), it suffices to determine the number of points of \( A_\gamma \) lying in the fundamental domain. But it is not always easy to know if two lifts intersect each other in the fundamental or no.

For example the closed geodesic \( \gamma \) associated to the cyclically reduced non-periodic word \( w = a^b b \) has two self-intersections (see figure 2). It admits three lifts whose intersect the fundamental domain (see figure 2) \( \alpha_0(\alpha_0^+ = a^2 b; \alpha_0^- = b a^2) \), \( \alpha_1(\alpha_1^+ = a b a; \alpha_1^- = a b a) \) and \( \beta_0(\beta_0^+ = b a^2; \beta_0^- = a b a) \). The geodesic \( \alpha_0 \) intersects the geodesic \( \beta_0 \) and the intersection point lies in the fundamental domain \( \mathcal{D} \). Using the cyclically lexicographical ordering, we see that the geodesic \( \alpha_1 \) intersects the geodesics \( \alpha_0 \) and \( \beta_0 \). But we can not say if the points \( (\alpha_0; \alpha_1) \) and \( (\beta_0; \alpha_1) \) belongs in the fundamental domain. It is easy to see that the isometry \( a \) maps the geodesics \( \beta_0 \) and \( \alpha_1 \) on the geodesics \( \alpha_1 \) and \( \alpha_0 \) and so the the points \( (\alpha_0; \alpha_1) \) and \( (\beta_0; \alpha_1) \) are projected on the same point on \( P \) and this shows that the number of self-intersections of \( \gamma \) is actually two.

In order to solve this problem, we define the following equivalence
relation on $A$; we will say that two points $(\gamma_i; \gamma_j)$ and $(\gamma_k; \gamma_l)$ are equivalent and we write $(\gamma_i; \gamma_j) \sim (\gamma_k; \gamma_l)$ if there exists an isometry in $\Gamma$ which maps the two geodesics $\gamma_i$ and $\gamma_j$ on the geodesics $\gamma_k$ and $\gamma_l$.

The number of self-intersections of closed geodesic $\gamma$ is equal to the cardinal of the quotient of $A_{\gamma}$ by this equivalence relation:

$$i(\gamma; \gamma) = \#A_{\gamma}/\sim.$$ 

Remark 3.2. Let $\gamma$ and $\gamma_0$ be two closed geodesics and $m$ an integer such that $\gamma = (\gamma_0)^m$, then:

$$i(\gamma; \gamma) = m^2 i(\gamma_0; \gamma_0).$$

### 3.3 Calculation of the number of self-intersections of a closed geodesic

A cyclically reduced word in $\Gamma$ has the form $w = s_1^{i_1}r_1^{j_1} \cdots s_n^{i_n}r_n^{j_n}$ where $n, i_1, \cdots, i_n, j_1, \cdots, j_n$ are non-zero positive integers and $s_k \in \{a, \overline{a}\}$ and $r_k \in \{b, \overline{b}\}$ for any $1 \leq k \leq n$. From now, every cyclically reduced word will be write in this form. We consider the following cyclic permutations of the word $w$ for all $1 \leq k \leq n$, $0 \leq m_k \leq i_k - 1$ and $0 \leq p_k \leq j_k - 1$:

$$w_{k, m_k} = s_k^{i_k-m_k} r_k^{j_k} \cdots s_n^{i_n} r_n^{j_n} \cdots s_{k-1}^{i_{k-1}} r_{k-1}^{j_{k-1}} s_k^{m_k}$$

and

$$v_{k, p_k} = r_k^{j_k-p_k} s_{k+1}^{j_{k+1}} \cdots s_n^{i_n} r_n^{i_n} \cdots s_k^{i_k} r_k^{p_k}.$$

For $1 \leq k \leq n$, we denote respectively by $\alpha_{k, m_k}^+$ and $\alpha_{k, m_k}^-$ the points of $L_{\Gamma}$ associated to the infinite reduced periodic words of period $w_{k, m_k}$ and $w_{k, m_k}^{-1}$ and by $\beta_{k, p_k}^+$ and $\beta_{k, p_k}^-$ the points of $L_{\Gamma}$ associated respectively to the infinite reduced periodic words of period $v_{k, p_k}$ et $v_{k, p_k}^{-1}$. The lifts of the closed geodesic $\gamma$ which intersect the fundamental domain are the geodesics: $\alpha_{k, m_k} = (\alpha_{k, m_k}^+; \alpha_{k, m_k}^-)$ and $\beta_{k, p_k} = (\beta_{k, p_k}^+; \beta_{k, p_k}^-)$ for all $1 \leq k \leq n$, $0 \leq m_k \leq i_k - 1$ and $0 \leq p_k \leq j_k - 1$.

We will establish some results which enable us to understand better the intersections between these lifts. In the next lemma, we show that for every integer $1 \leq k \leq n$, the geodesics $\alpha_{k, m_k}$ and $\alpha_{k, m'_k}$ for $0 \leq m_k < m'_k < i_k$ intersect each other and the intersections points of type $(\alpha_{k, m_k}; \alpha_{k, m'_k})$ are equivalent to intersection points of type $(\alpha_{k, 0}; \alpha_{k, m'_k})$ with $0 \leq m'_k < i_k$. We show also the same thing for the intersection points of type $(\beta_{k, p_k}; \beta_{k, p'_k})$ for $0 \leq p_k < p'_k \leq j_k - 1$. 

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Lemma 3.3. Let \( k \) be an integer between 1 and \( n \).

1. For any integers \( 0 \leq m_k < m'_k \leq i_k - 1 \), \( \alpha_{k,m_k} \cap \alpha_{k,m'_k} \neq \emptyset \) and we have:

\[
(\alpha_{k,m_k}; \alpha_{k,m'_k}) \sim (\alpha_{k,0}; \alpha_{k,i_k+m_k-m'_k}).
\]

2. For any integers \( 0 \leq p_k < p'_k \leq j_k - 1 \), \( \beta_{k,p_k} \cap \beta_{k,p'_k} \neq \emptyset \) and we have:

\[
(\beta_{k,p_k}; \beta_{k,p'_k}) \sim (\beta_{k,0}; \beta_{k,i_k+p_k-p'_k}).
\]

Proof. 1. For any integer \( 0 \leq m_k < m'_k \leq i_k - 1 \), with the alphabet \( A_a \), we have \( \alpha_{k,m_k} < \alpha_{k,m'_k} < \alpha_{k,m_k} \) if \( s_k = a \) and \( \alpha_{k,m'_k} < \alpha_{k,m_k} < \alpha_{k,m'_k} \) if \( s_k = \bar{a} \). So in every case the pair of points \( (\alpha_{k,m_k}, \alpha_{k,m'_k}) \) separates the pair \( (\alpha_{k,m'_k}, \alpha_{k,m'_k}) \) and therefore the geodesics \( \alpha_{k,m_k} \) and \( \alpha_{k,m'_k} \) intersect each other. Moreover we have:

\[
s_k^{m_k} (\alpha_{k,m_k}) = \alpha_{k,0}, \quad s_k^{m'_k} (\alpha_{k,m'_k}) = \alpha_{k,m'_k-m_k},
\]

\[
s_k^{m_k-i_k} (\alpha_{k,m_k}) = \beta_{k,0} \quad \text{and} \quad s_k^{m'_k-i_k} (\alpha_{k,m'_k}) = \alpha_{k,i_k+m_k-m'_k}.
\]

It follows that:

\[
(\alpha_{k,m_k}; \alpha_{k,m'_k}) \sim (\alpha_{k,0}; \alpha_{k,i_k+m_k-m'_k}).
\]

2. Likewise for any integers \( 0 \leq p_k < p'_k \leq j_k - 1 \), with the alphabet \( A_a \), we obtain \( \beta_{k,p_k} < \beta_{k,p'_k} < \beta_{k,p_k} \) if \( r_k = b \) and \( \beta_{k,p'_k} < \beta_{k,p_k} < \beta_{k,p'_k} \) if \( r_k = \bar{b} \). Thus the pair of points \( (\beta_{k,p_k}, \beta_{k,p'_k}) \) separates the pair \( (\beta_{k,p'_k}, \beta_{k,p'_k}) \) and therefore the geodesics \( \beta_{k,p_k} \) and \( \beta_{k,p'_k} \) intersect each other. Moreover we have:

\[
r_k^{p_k} (\beta_{k,p_k}) = \beta_{k,0} \quad \text{and} \quad r_k^{p'_k} (\beta_{k,p'_k}) = \beta_{k,j_k+p_k-p'_k},
\]

and also \( r_k^{j_k-p_k} (\beta_{k,p'_k}) = \alpha_{k+1,0} \) and \( r_k^{j_k-p_k} (\beta_{k,p_k}) = \beta_{k,j_k+p_k-p'_k} \). It follows that:

\[
(\beta_{k,p_k}; \beta_{k,p'_k}) \sim (\beta_{k,0}; \beta_{k,j_k+p_k-p'_k}) \sim (\alpha_{k+1,0}; \beta_{k,j_k+p_k-p'_k}).
\]

In the lemma 3.3 we fix the integer \( k \). Now we want to know what happens between intersection points of type \( (\alpha_{k,m_k}; \alpha_{l,m_l}) \) or \( (\beta_{k,p_k}; \beta_{l,p_l}) \) whenever \( k \leq l \). The lemma 3.4 give us similar results to those of lemma 3.3.

Notation: Set \( \alpha_{k,i_k} = \beta_k \) for all \( 1 \leq k \leq n \) and \( \beta_{k,j_k} = \alpha_{k+1,0} \) for all \( 1 \leq k \leq n \) et \( \beta_n,i_n = \alpha_{1,i_1} \).

Lemma 3.4. Let \( k \) and \( l \) be two integers between 1 and \( n \).
1. For any integers $0 \leq m_k \leq i_k - 1$, $0 \leq m_l \leq i_l - 1$, if $\alpha_{k,m_k} \cap \alpha_{l,m_l} \neq \emptyset$, then we have:

- if $s_k = s_l$,
  \[
  \begin{cases}
    (\alpha_{k,m_k}; \alpha_{l,m_l}) \sim (\alpha_{k,0}; \alpha_{l,m_l-m_k}) \sim (\beta_{l,0}; \alpha_{k,i_l+m_k-m_l}) \\
    \text{or} \\
    (\alpha_{k,m_k}; \alpha_{l,m_l}) \sim (\alpha_{l,0}; \alpha_{k,m_k-m_l}) \sim (\beta_{l,0}; \alpha_{k,i_k+m_l-m_k})
  \end{cases}
  \]

- if $s_k = \overline{s}_l$,
  \[
  \begin{cases}
    (\alpha_{k,m_k}; \alpha_{l,m_l}) \sim (\alpha_{l,0}; \alpha_{k,m_k+m_l}) \sim (\beta_{l,0}; \alpha_{k,m_l+m_k}) \\
    \text{or} \\
    (\alpha_{k,m_k}; \alpha_{l,m_l}) \sim (\beta_{l,0}; \alpha_{l,m_l+m_k-m_l}) \sim (\beta_{l,0}; \alpha_{k,m_l+m_k-1})
  \end{cases}
  \]

2. For any integers $0 \leq p_k \leq j_k - 1$, $0 \leq p_l \leq j_l - 1$, if $\beta_{k,p_k} \cap \beta_{l,p_l} \neq \emptyset$, then we have:

- if $r_k = r_l$,
  \[
  \begin{cases}
    (\beta_{k,p_k}; \beta_{l,p_l}) \sim (\beta_{l,0}; \beta_{k,p_k-p_l}) \sim (\alpha_{l,0}; \beta_{k,j_l-p_k+p_l}) \\
    \text{or} \\
    (\beta_{k,p_k}; \beta_{l,p_l}) \sim (\beta_{l,0}; \beta_{k,p_l-p_k}) \sim (\alpha_{k,0}; \beta_{l,j_k+p_l-p_k})
  \end{cases}
  \]

- if $r_k = \overline{r}_l$,
  \[
  \begin{cases}
    (\beta_{k,p_k}; \beta_{l,p_l}) \sim (\beta_{k,0}; \beta_{l,p_k+p_l}) \sim (\beta_{l,0}; \beta_{k,p_l+p_k}) \\
    \text{or} \\
    (\beta_{k,p_k}; \beta_{l,p_l}) \sim (\alpha_{l,0}; \beta_{k,p_k+p_l-j_l}) \sim (\alpha_{k,0}; \beta_{l,p_l+p_k-j_l})
  \end{cases}
  \]

Proof. 1. We assume that $s_k = s_l$. If the geodesics $\alpha_{k,m_k}$ and $\alpha_{l,m_l}$ intersect each other, then we have:

- either $i_k - m_k \leq i_l - m_l$ and $m_l \leq m_k$
- $i_k - m_k \geq i_l - m_l$ and $m_l \geq m_k$

In the first case the $s_{k}^{m_k}$ maps the geodesics $\alpha_{k,m_k}$ and $\alpha_{l,m_l}$ respectively on the geodesics $\alpha_{k,0}$ and $\alpha_{l,m_l-m_k}$ and the isometry $s_{l}^{m_l-i_l}$ maps the geodesics $\alpha_{k,m_k}$ and $\alpha_{l,m_l}$ respectively on the geodesics $\alpha_{k,i_l+m_k-m_l}$ and $\beta_{l,0}$. Therefore the intersection point $(\alpha_{k,m_k}; \alpha_{l,m_l})$ is equivalent to the intersection points $(\alpha_{k,0}; \alpha_{l,m_l-m_k})$ and $(\beta_{l,0}; \alpha_{k,i_l+m_k-m_l})$. In the second case the isometries $s_{k}^{m_k}$ and $s_{k}^{i_k-m_k}$ enable us to have:

$$
(\alpha_{k,m_k}; \alpha_{l,m_l}) \sim (\alpha_{l,0}; \alpha_{k,m_k-m_l}) \sim (\beta_{l,0}; \alpha_{l,i_l+m_l-m_k}).
$$

Likewise, if $s_k = \overline{s}_l$ and if $\alpha_{k,m_k} \cap \alpha_{l,m_l} \neq \emptyset$, we have two cases:

- $i_k - m_k \geq m_l$ and $i_l - m_l \geq m_k$
- $i_k - m_k \leq m_l$ and $i_l - m_l \leq m_k$.

In the first case the isometries $s_{k}^{m_k}$ and $s_{l}^{m_l}$ give us the following relations: $(\alpha_{k,m_k}; \alpha_{l,m_l}) \sim (\alpha_{k,0}; \alpha_{l,m_l+m_k}) \sim (\alpha_{l,0}; \alpha_{k,m_k+m_l})$. In the second case we have:

$$
(\alpha_{k,m_k}; \alpha_{l,m_l}) \sim (\beta_{l,0}; \alpha_{k,m_l+m_k-i_l}) \sim (\beta_{k,0}; \alpha_{l,m_l+m_k-i_l}).
$$
because of the isometries $s_k^{m_k-i_k}$ and $s_l^{m_l-i_l}$.

2. To prove this part, it suffices to replace $s_k, s_l, \alpha_{k,m_k}$ and $\alpha_{l,m_l}$ respectively by $r_k, r_l, \beta_{k,p_k}$ and $\beta_{l,p_l}$ and follow the same way like the point 1.

\[ \square \]

**Remark 3.5.** Let $k$ and $l$ be two integers in $[1; n]$:

1. Let $m_k$ and $m_l$ such that $\alpha_{k,m_k} \cap \beta_{l,m_l} \neq \emptyset$.
   - If $s_k = s_l$ and $m_l - m_k = i_l - i_k$ then
     \[(\alpha_{k,0}; \alpha_{l,i_l-i_k}) \sim (\beta_{k,0}; \beta_{l,0}).\]
     Furthermore if $i_k = i_l$ then $(\alpha_{k,0}; \alpha_{l,0}) \sim (\beta_{k,0}; \beta_{l,0})$.
   - If $s_k = s_l$ and $m_l + m_k = k$ then
     \[(\alpha_{k,0}; \alpha_{l,i_l}) \sim (\beta_{k,0}; \alpha_{l,0}).\]
     Furthermore if $i_k = i_l$ then $(\beta_{k,0}; \alpha_{l,0}) \sim (\alpha_{k,0}; \beta_{l,0})$.

2. Let $p_k$ and $p_l$ such that $\beta_{k,p_k} \cap \beta_{l,p_l} \neq \emptyset$.
   - If $r_k = r_l$ and $p_l - p_k = j_l - j_k$ then
     \[(\beta_{k,0}; \beta_{l,j_l-j_k}) \sim (\alpha_{k+1,0}; \alpha_{l+1,0}).\]
     Furthermore if $j_k = j_l$ then $(\beta_{k,0}; \beta_{l,0}) \sim (\alpha_{k+1,0}; \alpha_{l+1,0})$.
   - If $r_k = r_l$ and $p_l + p_k = k$ then
     \[(\beta_{k,0}; \beta_{l,j_k}) \sim (\alpha_{k+1,0}; \beta_{l,0}).\]
     Furthermore if $j_k = j_l$ then $(\alpha_{k+1,0}; \beta_{l,0}) \sim (\beta_{k,0}; \alpha_{l+1,0})$.

3. For any integers $1 \leq m_k \leq i_k - 1$ and $1 \leq p_k \leq j_l - 1$ we have:
   
   \[ \alpha_{k,m_k} \cap \beta_{l,p_k} = \emptyset. \]

For any integers $k$ and $l$ between 1 and $n$:

- If $s_k = s_l$, then $\alpha_{k,0} \cap \alpha_{l,m_l} \neq \emptyset$ if $i_k > i_l - m_l$.
- If $s_k = s_l$, then $\alpha_{k,0} \cap \alpha_{l,m_l} \neq \emptyset$ if $i_k > m_l$ and $\beta_{k,0} \cap \alpha_{l,m_l} \neq \emptyset$ if $i_k > i_l - m_l$.

This remark, the lemmas 3.3 and 3.4 and the remark 3.5 motivates the definition of the following sets for any integer $1 \leq k \leq n$:

\[ A^l_k = \{ (\alpha_{k,0}; \alpha_{l,m_l}) ; \ 1 \leq l \leq n ; \ s_k = s_l ; \ \max(0; i_l - i_k) < m_l < i_l \}; \]
\[ A_k^2 = \{(\alpha_{k,0}; \alpha_{k,m_l}); \ k < l \leq n; \ s_k = \bar{s}_l; \ 0 < m_l < \max(i_k; i_l)\}; \]
\[ A_k^3 = \{(\beta_{k,0}; \alpha_{l,m_l}); \ k < l \leq n; \ s_k = \bar{s}_l; \ \max(0; i_l - i_k) < m_l < i_l\}. \]

Likewise, considering geodesics \( \beta_{k,0}; \alpha_{k+1,0} \) and \( \beta_{l,m_l} \), we have the same situation and we define the following sets:

\[ B_k^1 = \{(\beta_{k,0}; \beta_{l,m_l}); \ 1 \leq l \leq n; \ r_k = r_l; \ \max(0; j_l - j_k) < p_l < j_l\}; \]
\[ B_k^2 = \{(\beta_{k,0}; \beta_{l,m_l}); \ k < l \leq n; \ r_k = \bar{r}_l; \ 0 < p_l < \max(j_k; j_l)\}; \]
\[ B_k^3 = \{(\alpha_{k+1,0}; \beta_{l,m_l}); \ k < l \leq n; \ r_k = \bar{r}_l; \ \max(0; j_l - j_k) < p_l < j_l\}. \]

**Proposition 3.6.** Any two points of the set \( \bigcup_{i=1}^{3} \bigcup_{k=1}^{n} A_k^i \) are not equivalent.

**Proof.** We shall first prove that two points \( (\alpha_{k,0}; \alpha_{l,m_l}) \) and \( (\alpha_{l,0}; \alpha_{q,m_q}) \) of the set \( \bigcup_{k=1}^{n} A_k^1 \cup A_k^2 \) are not equivalent. We assume that there exists an isometry \( g \in \Gamma \) which maps the geodesics \( \alpha_{k,0} \) and \( \alpha_{l,m_l} \) on the geodesics \( \alpha_{l,0} \) and \( \alpha_{q,m_q} \). Without loss of generality, we can suppose that \( g(\alpha_{k,0}) = \alpha_{q,m_q} \) and \( g(\alpha_{l,m_l}) = \alpha_{l,0} \), because the other can be solved with the same arguments. Letting

\[ f = \begin{cases} \bar{s}_q^{m_q} \bar{s}_r^{m_q} \cdots \bar{s}_q^{j_q-1} s_q^{m_q} & \text{if } k = q \vspace{1em} \\ s_k^{i_k} s_r^{j_k} \cdots s_q^{j_q-1} s_q^{m_q} & \text{if } k \neq q \end{cases} \]

and \( h^{-1}(\alpha_{l,m_l}) = \alpha_{l,0} \) for \( h = \begin{cases} s_l^{i_l} s_r^{j_l} \cdots s_l^{j_l} s_l^{m_l} & \text{if } l = t \vspace{1em} \\ s_l^{i_l} s_r^{j_l} \cdots s_l^{j_l} s_l^{m_l} & \text{if } l \neq t \end{cases} \).

This implies that \( fg(\alpha_{k,0}) = \alpha_{k,0} \) and \( hg(\alpha_{l,m_l}) = \alpha_{l,m_l} \). As the isometries \( w_k \) and \( w_{l,m_l} \) fix respectively the geodesics \( \alpha_{k,0} \) and \( \alpha_{l,m_l} \), then there exists two integers \( u \) and \( v \) such that:

\[ g = f^{-1}(w_k)^u = h^{-1}(w_{l,m_l})^v. \]

Thus \( g \) is represented by two different reduced words. It is absurd because \( \Gamma \) is a free group.

It remains us to show that any point of the set \( \bigcup_{k=1}^{n} A_k^1 \cup A_k^2 \) is not equivalent to a point of \( \bigcup_{k=1}^{n} A_k^3 \). Let \( (\alpha_{k,0}; \alpha_{l,m_l}) \) be a point of \( \bigcup_{k=1}^{n} A_k^1 \cup A_k^2 \) and \( (\beta_{l,0}; \alpha_{q,m_q}) \) be two points and assume that there exists an isometry \( g \in \Gamma \) such that \( g(\alpha_{k,0}) = \beta_{l,0} \) and \( g(\alpha_{l,m_l}) = \alpha_{q,m_q} \).
We have \( f(\alpha_{q,m}) = \alpha_{k,0} \) and \( h(\alpha_{l,m}) = \beta_{t,0} \) with

\[
f = \begin{cases} \alpha_{k,0} & \text{if } k = q \\ s_k^{i_k} s_{q-1}^{i_{q-1}} \cdots s_1^{i_1} & \text{if } k \neq q \end{cases}
\]

and

\[
h = \begin{cases} \beta_{t,0} & \text{if } l = t \\ s_l^{i_l} s_{t-1}^{i_{t-1}} \cdots s_1^{i_1} & \text{if } l \neq t \end{cases}
\]

Then the isometries \( fg \) and \( hg \) fix respectively the geodesics \( \alpha_{k,0} \) and \( \alpha_{l,m} \). So there exists two integers \( u \) and \( v \) such that

\[
g = f^{-1}(w_k)^u = h^{-1}(w_l,m_l)^v.
\]

As the precedent point, \( g \) is represented by two different reduced words in a free group.

By replacing \( \alpha_{k,0} \) by \( \beta_{k,0} \) and \( \alpha_{l,m} \) by \( \beta_{l,p_l} \) we have:

**Proposition 3.7.** Any two points of the set \( \bigcup_{i=1}^{3} \bigcup_{k=1}^{n} B_i^k \) are not equivalent.

With the same arguments we prove the following result.

**Proposition 3.8.** The points of \( \bigcup_{i=1}^{3} \bigcup_{k=1}^{n} A_i^k \) are not equivalent to the points of \( \bigcup_{i=1}^{3} \bigcup_{k=1}^{n} B_i^k \).

Now, to compute the number of self-intersections, it remains us to examine the intersection points of type \( (\alpha_{k,0}; \alpha_{l,0}), (\alpha_{k,0}; \beta_{l,0}) \) and \( (\beta_{k,0}; \beta_{l,0}) \).

We will show first that these points are not equivalent to the points of \( A_i \) and \( B_i \) for \( i = 1, 2, 3 \).

**Proposition 3.9.** The intersection points of type \( (\alpha_{k,0}; \alpha_{l,0}), (\alpha_{k,0}; \beta_{l,0}) \) and \( (\beta_{k,0}; \beta_{l,0}) \) are not equivalent to the point of the set \( \bigcup_{i=1}^{3} \bigcup_{k=1}^{n} A_i^k \cup B_i^k \).

**Proof.** It suffices to take \( m_l = 0 \) or \( p_l = 0 \) in the proofs of propositions 3.6, 3.7 and 3.8.
The remark 3.5 motivates the definition of the following sets for any integer $1 \leq k \leq n$:

\[
C_k^1(w) = \{(\alpha_{k,0}; \alpha_{l,0}) ; \alpha_{k,0} \cap \alpha_{l,0} \neq \emptyset ; s_{k}^l \neq s_{l}^k ; k < l \leq n\};
\]

\[
C_k^2(w) = \{(\beta_{k,0}; \alpha_{l,0}) ; \beta_{k,0} \cap \alpha_{l,0} \neq \emptyset ; s_{k}^l \neq s_{l}^k ; k < l \leq n\};
\]

\[
D_k^1(w) = \{(\beta_{k,0}; \beta_{l,0}) ; \beta_{k,0} \cap \beta_{l,0} \neq \emptyset ; r_{k}^l \neq r_{l}^k ; k < l \leq n\};
\]

\[
D_k^2(w) = \{(\alpha_{l,0}; \beta_{l,0}) ; \alpha_{l,0} \cap \beta_{l,0} \neq \emptyset ; 1 \leq l \leq n\};
\]

and for all $2 \leq k \leq n$,

\[
D_k^2(w) = \{(\alpha_{k,0}; \beta_{l,0}) ; \alpha_{k,0} \cap \beta_{l,0} \neq \emptyset ; r_{k}^{j_{k-1}} \neq r_{l}^{j_{l}}; k \leq l \leq n\}.
\]

**Proposition 3.10.** Let $k$ and $l$ be two integers between 1 and $n$ such that $k \leq l$ and let $w = s_1^1 r_1^1 \cdots s_n^r r_n^j$ a finite non-periodic word, then we have:

1. Every intersection point of type $(\alpha_{k,0}; \alpha_{l,0})$ or $(\beta_{k,0}; \beta_{l,0})$ is equivalent to a point of $\bigcup_{k=1}^n C_k^1(w) \cup D_k^1(w)$.

2. Every intersection point of type $(\alpha_{k,0}; \beta_{l,0})$ is equivalent to a point of $\bigcup_{k=1}^n C_k^2(w) \cup D_k^2(w)$.

3. Two different points of $\bigcup_{i=1}^n \bigcup_{k=1}^n C_k^i(w) \cup D_k^i(w)$ are not equivalent.

**Proof.** We assume that $\alpha_{k,0} \cap \alpha_{l,0} \neq \emptyset$. If $s_{k}^l \neq s_{l}^k$, then $(\alpha_{k,0}; \alpha_{l,0}) \in \bigcup_{k=1}^n C_k^1(w)$ . In the case where $s_{k}^l = s_{l}^k$, then $s_k(\alpha_{k,0}) = \beta_{k,0}$ and $\bar{s}_l(\alpha_{l,0}) = \beta_{l,0}$. Thus $\beta_{k,0} \cap \beta_{l,0} \neq \emptyset$ and the points $(\alpha_{k,0}; \alpha_{l,0})$ and $(\beta_{k,0}; \beta_{l,0})$ are equivalent. If $r_{k}^l \neq r_{l}^k$, then $(\beta_{k,0}; \beta_{l,0}) \in \bigcup_{k=1}^n D_k^1(w)$.

Otherwise, because $w$ is non-periodic, there exists a positive non-zero integer $p$ such that for any integer $0 \leq m < p$ we have:

\[
\begin{cases}
    s_{k+m}^l = s_{l+m}^k \\
    r_{k+m}^l = r_{l+m}^k \\
    r_{k+p}^l \neq r_{l+p}^k
\end{cases}
\]

or

\[
\begin{cases}
    s_{k+m}^l = s_{l+m}^k \\
    r_{k+m}^l = r_{l+m}^k \\
    s_{k+p}^l = s_{l+p}^k \\
    r_{k+p}^l \neq r_{l+p}^k
\end{cases}
\]
In the first case we have \( g(\alpha_{k,0}) = \alpha_{k+p,0} \) and \( g(\alpha_{l,0}) = \alpha_{l+p,0} \) with 
\[
g = \tilde{r}_{k+p-1} \cdots \tilde{r}_{k+m-1} \tilde{s}_{k+m-1} \cdots \tilde{r}_{k} \tilde{s}_{k}.
\]
Thus the points \((\alpha_{k,0}; \alpha_{l,0})\) and \((\alpha_{k+p,0}; \alpha_{l+p,0})\) are equivalent and \((\alpha_{k+p,0}; \alpha_{l+p,0}) \in \bigcup_{k=1}^{n} C_{k}^{i}(w)\).

In the second case the point \((\alpha_{k,0}; \alpha_{l,0})\) is equivalent to the point \((\beta_{k+p,0}; \beta_{l+p,0})\) which lies in \(\bigcup_{k=1}^{n} D_{k}^{i}(w)\).

We prove the point 2 with the same arguments.
We prove the point 3 with the same proof of the proposition 3.6.

Now, we can determine the number of self-intersections \(i(\gamma; \gamma)\) of a closed geodesic \(\gamma\) of \(P\) associated to the cyclically reduced word \(w = s_{1}^{i_{1}} r_{1}^{j_{1}} \cdots\); set:
\[
H(w) = 2 \sum_{i=1}^{2} \sum_{k=1}^{n} \#C_{i}^{k}(w) + \#D_{i}^{k}(w).
\]
Then the the number of self-intersections of \(\gamma\):
\[
i(\gamma; \gamma) = H(w) + \sum_{i=1}^{3} \sum_{k=1}^{n} \#A_{i}^{k} + \#B_{i}^{k}.
\]

**Remark 3.11.** Let \(w\) and \(v\) two finite cyclically reduced words and \(m\) a non-zero integer such that \(w = v^{m}\). Then we have:
\[
H(w) = m^{2}H(v).
\]

Let us compute \(\sum_{i=1}^{3} \sum_{k=1}^{n} \#A_{i}^{k} + \#B_{i}^{k}\) as a function of the integers \(i_{k}\) and \(j_{k}\) for \(1 \leq k \leq n\).

\[
\sum_{k=1}^{n} \#A_{k}^{i} = \sum_{k=1}^{n} \sum_{l=1}^{n} \#\{(\alpha_{k,0}; \alpha_{l,m_l}); s_{k} = s_{l}; \text{max}(0; i_l - i_k) < m_l < i_l\}
= \sum_{k=1}^{n} \sum_{l=1}^{n} (\min(i_k; i_l) - 1) \delta_{s_k; s_l}
= \sum_{k=1}^{n} i_k - 1 + 2 \sum_{k=1}^{n} \sum_{l=k+1}^{n} (\min(i_k; i_l) - 1) \delta_{s_k; s_l}
\]
\[
\sum_{k=1}^{n} \#A_k^2 = \sum_{k=1}^{n} \sum_{l=1}^{n} \left\{ (\alpha_k; \alpha_l; m_l); s_k = s_l; 0 < m_l < \min(i_k; i_l) \right\}
\]
\[
= \sum_{k=1}^{n} \sum_{l=1}^{n} \left( \min(i_k; i_l) - 1 \right) \delta_{s_k; s_l}.
\]
\[
\sum_{k=1}^{n} \#A_k^3 = \sum_{k=1}^{n} \sum_{l=1}^{n} \left\{ (\beta_k; \alpha_l; m_l); s_k = s_l; \max(0; i_l - i_k) < m_l < i_l \right\}
\]
\[
= \sum_{k=1}^{n} \sum_{l=k+1}^{n} \left( \min(i_k; i_l) - 1 \right) \delta_{s_k; s_l}.
\]

Because \(\delta_{s_k; s_l} + \delta_{s_k; s_t} = 1\), then we have:
\[
3 \sum_{i=1}^{n} \sum_{k=1}^{n} \#A_k^3 = n \sum_{k=1}^{n} i_k - 1 + 2 \sum_{k=1}^{n} \sum_{l=k+1}^{n} \min(i_k; i_l) - 1.
\]

Similarly, we show that:
\[
3 \sum_{k=1}^{n} \#B_k = \sum_{k=1}^{n} j_k - 1 + 2 \sum_{k=1}^{n} \sum_{l=k+1}^{n} \min(j_k; j_l) - 1.
\]

Recall that \(L(\gamma) = \sum_{k=1}^{n} i_k + j_k\) is the combinatorial length of \(\gamma\). We have just proved the following result:

**Theorem 3.12.** Let \(\gamma\) be a closed geodesic of \(P\) associated to the cyclically reduced word \(w = s_1^{i_1} r_1^{j_1} \cdots s_n^{i_n} r_n^{j_n}\). Then we have:
\[
i(\gamma; \gamma) - H(w) = nL(\gamma) - 2n^2 - \sum_{k=1}^{n} \sum_{l=k+1}^{n} |i_k - i_l| + |j_k - j_l|.
\]

### 4  Proofs of theorems \(1.1\) and \(1.2\)

To understand better the self-intersections \(i(\gamma; \gamma)\) of a closed geodesic \(\gamma\) of \(P\), it is necessary to understand \(H(w)\). In this section, we use the theorem \(3.12\) in order to prove the theorems \(1.1\) and \(1.2\). In this section set: \(\alpha_k = \alpha_{k, 0}\) and \(\beta_k = \beta_{k, 0}\). We begin with the following result on \(H(w)\) where \(w\) is a cyclically reduced non-periodic word.

**Theorem 4.1.** Let \(\gamma\) be a closed geodesic of \(P\) associated to the cyclically reduced word \(w(\gamma) = s_1^{i_1} r_1^{j_1} \cdots s_n^{i_n} r_n^{j_n} = (w_0)^q\) where \(q \in \mathbb{N}\) and \(w_0\) is a cyclically reduced non-periodic word. Then we have:
\[
q(n - q) \leq H(w) \leq n^2 + (n - q)^2.
\]
In particular if $w$ is cyclically reduced non-periodic word,

$$(n - 1) \leq H(w) \leq n^2 + (n - 1)^2.$$  

We prove first this theorem for the cyclically reduced non-periodic word containing two distinct letters of $\Gamma$.

**Proposition 4.2.** Let $\gamma$ be a closed geodesic of $P$ associated to the cyclically reduced non-periodic word $w(\gamma) = s_{i_1}^{r_{i_1}} \cdots s_{i_n}^{r_{i_n}}$, then we have:

1. If $s_{i_k} = a$ and $r_{i_k} = \bar{b}$ for all $1 \leq i \leq n$, then:

   $$n^2 + n - 1 \leq H(w) \leq n^2 + (n - 1)^2.$$  

2. If $s_{i_k} = a$ and $r_{i_k} = b$ for all $1 \leq i \leq n$, then:

   $$n - 1 \leq H(w) \leq (n - 1)^2.$$  

**Proof.** We consider the cyclically reduced non-periodic word $w = a_{i_1} b_{i_1} \cdots a_{i_n} b_{i_n}$. For all $1 \leq k; l \leq n$, the geodesics $\alpha_k$ and $\beta_l$ intersect each other, $s_k = s_l$ and $r_{k-1} = r_l$, then we have:

$$\sum_{k=1}^{n} \# C^1_k(w) + \# D^1_k(w) = n^2.$$  

It remains us now to prove that:

$$n - 1 \leq \sum_{k=1}^{n} \# C^1_k(w) + \# D^1_k(w) \leq (n - 1)^2.$$  

Because $s_k = s_l$ and $r_k = r_l$ for $1 \leq k \leq n$,

$$C^1_k(w) = \# \{k < l \leq n : \alpha_k \cap \alpha_l \neq \emptyset ; \ i_k \neq i_l\}$$  

and

$$D^1_k(w) = \# \{k < l \leq n : \beta_k \cap \beta_l \neq \emptyset ; \ j_k \neq j_l\}.$$  

It follows that:

$$\sum_{k=1}^{n} \# C^1_k(w) \leq \sum_{k=1}^{n} \# \left\{ l : \begin{array}{c} i_k < i_l \ \ j_{k-1} \leq j_{l-1} \end{array} \right\}$$  

and

$$\sum_{k=1}^{n} \# D^1_k(w) \leq \sum_{k=1}^{n} \# \left\{ l : \begin{array}{c} i_k \leq i_l \ \ j_k < j_l \end{array} \right\}.$$  

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There exists a permutation $\sigma$ of $\{i_1, i_2, \cdots, i_n\}$ and a permutation $\tau$ of $\{j_1, j_2, \cdots, j_n\}$ such that $\sigma(i_k) = x_k$ and $\tau(j_k) = y_k$ $\forall$ $1 \leq k \leq n$ and for all $1 \leq i \leq n - 1$, $x_i \leq x_{i+1}$ and $y_i \leq y_{i+1}$. Thus for any integer $1 \leq k \leq n$, there exists three integers $1 \leq p \leq n$, $1 \leq q \neq m \leq n$ such that $i_k = x_p$, $j_k = y_q$ and $j_{k-1} = y_m$. So

$$\sum_{k=1}^{n} \#C^1_k(w) \leq \sum_{k=1}^{n} \# \left\{ l : \begin{array}{c} x_p < i_l \\ y_m \leq j_{l-1} \end{array} \right\}$$

and

$$\sum_{k=1}^{n} \#D^1_k(w) \leq \sum_{k=1}^{n} \# \left\{ l : \begin{array}{c} x_p \leq i_l \\ y_q < j_{l} \end{array} \right\}.$$

We can assume that $m \leq q - 1$ and thus:

$$\sum_{k=1}^{n} \# \left\{ l : \begin{array}{c} x_p < i_l \\ y_m \leq j_{l-1} \end{array} \right\} + \# \left\{ l : \begin{array}{c} x_p \leq i_l \\ y_q < j_{l} \end{array} \right\} \leq 2n - 2m - 1.$$

This implies that:

$$\sum_{k=1}^{n} \#C^1_k(w) + \#D^1_k(w) \leq \sum_{m=1}^{n} 2n - 2m - 1 = (n - 1)^2.$$

We use the method to show that the cardinal of the complementary of the set $\bigcup_{k=1}^{n} C^1_k(w) \cup D^1_k(w)$ is less than or equal $(n - 1)^2$ and this proves that $\sum_{k=1}^{n} \#C^1_k(w) + \#D^1_k(w) \geq n - 1$.

With the same method, we prove that $n - 1 \leq H(w) \leq (n - 1)^2$ for any cyclically reduced non-periodic word of type $w = a^{i_1}b^{j_1} \cdots a^{i_n}b^{j_n}$.

\begin{itemize}
  \item Remark 4.3. If $w = a^{i_1}b^{j_1} \cdots a^{i_n}b^{j_n}$ is periodic then there exists two integers $p, q \in \mathbb{N}^*$ such that $w' = a^{i_1}b^{j_1} \cdots a^p b^q$ is a non-periodic sub-word of $w$ and $w = (w')^q$. By using the proposition 4.2 and the fact that $H(w) = q^2 H(w')$ we have the following inequalities for these words:

$$n^2 + q(n - q) \leq H(w) \leq n^2 + (n - q)^2.$$

  \item If we replace $b$ by $b$ in the word $w$, we have:

$$q(n - q) \leq H(w) \leq (n - q)^2.$$
\end{itemize}
Now we will prove the theorem 4.1 for the words with 3 or 4 distinct letters of $\Gamma$. We can suppose that every cyclically reduced word can be written in the following form: $w = a^{i_1}b^{j_1}a^{i_2}b^{j_2} \cdots a^{i_r}b^{j_r}a^{j_{r+1}}$ or $w = a^{i_1}b^{j_1}a^{i_2}b^{j_2} \cdots a^{i_r}b^{j_r}\bar{a}^{r+1}x$ where $x$ is a sub-word of $w$. We will just deal with the case where $w = a^{i_1}b^{j_1}a^{i_2}b^{j_2} \cdots a^{i_r}b^{j_r}\bar{a}^{r+1}x$ because the other case can be deduced from the first. Set $u = b^{j_1}a^{i_2}b^{j_2} \cdots a^{i_r}b^{j_r}$ and consider the cyclically reduced word $w' = a^{i_1}u a^{j_{r+1}}x$. We have:

**Proposition 4.4.** Let $w = a^{i_1}u a^{j_{r+1}}x$ and $w' = a^{i_1}\bar{u} a^{j_{r+1}}x$ two cyclically reduced words, then we have $H(w) \leq H(w')$.

**Proof.** Consider the geodesics $\beta_1(\beta_1^+ = b^{j_1} \cdots x a^{i_1}; \beta_1^- = \bar{a}^{i_1} \bar{x} \cdots \bar{b}^{j_1})$ and $\alpha_{p+1}(\alpha_{p+1}^+ = a^{j_{r+1}}x \cdots b^{j_r}; \alpha_{p+1}^- = \bar{b}^{j_r} \cdots \bar{x} \bar{a}^{r+1})$; the geodesic $\beta_1$ doesn’t intersect $\alpha_{p+1}$. The transformation introduced changes these two geodesics into two geodesics $\beta_1'(\beta_1'^+ = \bar{b}^{j_r}a^{j_{r+1}}x \cdots a^{i_1}; \beta_1'^- = \bar{a}^{i_1} \bar{x} \cdots \bar{b}^{j_r}b^{j_r})$ and $\alpha_{p+1}'(\alpha_{p+1}'^+ = a^{j_{r+1}}x a^{i_1} \cdots \bar{b}^{j_r}; \alpha_{p+1}'^- = b^{j_r} \cdots \bar{x} \bar{a}^{r+1})$ and these two geodesics intersect each other. By using the definition of $H(w)$, we show that for any integer $1 \leq k \leq p$,

- if $(\alpha_{p-k+2}; \alpha_{p+1}') \notin C_{p-k+2}^1(w')$ then:
  $$(\beta_k; \alpha_{p+1}) \notin C_{k}^2(w) \text{ or } (\beta_k; \beta_1) \notin D_1^1(w)$$

and

- if $(\beta_{p-k+2}; \beta_1') \notin D_1^1(w')$ then:
  $$(\beta_p; \beta_{k-1}) \notin D_{k-1}^1(w) \text{ or } (\alpha_k; \alpha_{p+1}) \notin C_k^1(w) \text{ or } (\beta_1; \alpha_k) \notin C_1^2(w).$$

By proceeding in the same way, we also show without difficulty that for $p+1 \leq l \leq n$, if $s_{l+1} = a$ and $r_l = b$, then if $(\alpha_{l+1}; \alpha_{p+1}') \notin C_{p+1}^1(w')$ then $(\beta_1; \alpha_{l+1}) \notin C_{l+1}^2(w)$ or $(\alpha_{l+1}; \alpha_{p+1}) \notin C_{p+1}^1(w)$.

The remaining cases are treated in the same way. $\square$

The idea is to "straighten" the geodesics of type $(W^+ = axb; W^- = \bar{b}\vec{a})$ and $(V^+ = bya; V^- = \bar{a}\vec{b})$ and to turn them into geodesics of type $(W'^+ = axb; W'^- = \bar{x}\vec{a})$ or $(V'^+ = bya; V'^- = \bar{a}\vec{b})$. By repeating this procedure, we get at the end a word containing only the letters $a$ and $b$, we obtain:

**Proposition 4.5.** Let $w = s_{i_1}^1r_{j_1}^1 \cdots s_{i_n}^n r_{j_n}^n$ a cyclically reduced word then there exists a permutation $\sigma$ of $\{i_1, i_2, \cdots, i_n\}$ and a permutation $\tau$ of $\{j_1, j_2, \cdots, j_n\}$ and a cyclically reduced word $w' = a^{\sigma(i_1)}b^{\tau(j_1)} \cdots a^{\sigma(i_n)}b^{\tau(j_n)}$ such that $H(w) \leq H(w')$.

To prove the other inequality of Theorem 4.1 for words containing at least 3 letters of $\Gamma$, we introduce a transformation that consists of
doing the "opposite" of what we did to prove the last proposition \[\text{4.5}\]
Without loss of generality, we can assume that every word is written as follows: \( w = a^{i_1}b^{j_1}\bar{a}^{i_2}\cdots\bar{a}^{i_p}b^{j_p}a^{i_{p+1}}x \) or \( w = a^{i_1}b^{j_1}\bar{a}^{i_2}\cdots\bar{a}^{i_p}b^{j_p}a^{i_{p+1}}x \) where \( x \) is a sub-word of \( w \).

Let \( v = \bar{b}^{j_1}\bar{a}^{i_2}\cdots\bar{a}^{i_p}b^{j_p} \) and \( w'' = a^{i_1}\bar{v}a^{i_{p+1}}x \), the method used in the proof of the proposition 4.5 can be used to prove that \( H(w'') \leq H(w) \). By repeating this procedure, we get at the end a word containing only the letters \( a \) and \( b \), we obtain:

**Proposition 4.6.** Let \( w = s_1^{i_1}r_1^{j_1}\cdots s_n^{i_n}r_n^{j_n} \) a cyclically reduced word then there exists a permutation \( \sigma \) of \( \{i_1, i_2, \ldots, i_n\} \) and a permutation \( \tau \) of \( \{j_1, j_2, \ldots, j_n\} \) and a cyclically reduced word \( w'' = a^{\sigma(i_1)}b^{\tau(j_1)}\cdots a^{\sigma(i_n)}b^{\tau(j_n)} \) such that \( H(w'') \leq H(w) \).

**Proof of theorem 1.1**
From theorem 4.1 and theorem 3.12 we will deduce the theorem 1.1.

We begin with the first inequality.

**Proposition 4.7.** Let \( \gamma \) be a non-simple closed geodesic of \( P \) associated to the cyclically reduced word \( w(\gamma) = s_1^{i_1}r_1^{j_1}\cdots s_n^{i_n}r_n^{j_n} \), then we have:

\[ i(\gamma; \gamma) \geq L(\gamma) - n - 1. \]

**Proof.** Thanks to theorem 3.12 we have:

\[ i(\gamma; \gamma) = L - 2n + H(w) + 2 \sum_{1 \leq k < l \leq n} \min(i_k; i_l) + \min(j_k; j_l) - 2. \]

If \( \sum_{1 \leq k < l \leq n} \min(i_k; i_l) + \min(j_k; j_l) - 2 \neq 0 \) then:

\[ \sum_{1 \leq k < l \leq n} \min(i_k; i_l) + \min(j_k; j_l) - 2 \geq n - 1 \]

and we have the result.

If \( \sum_{1 \leq k < l \leq n} \min(i_k; i_l) + \min(j_k; j_l) - 2 = 0 \), then:

\[ \#\{1 \leq k \leq n : i_k = j_k = n - 1\} \geq n - 1 \]

and we can have two different cases:

1. there exists \( 1 \leq k_0 \leq n \) such that \( i_{k_0} \neq 1 \) or \( j_{k_0} \neq 1 \). In this case, proposition 4.1 gives us \( H(w) \geq n - 1 \).

2. for any \( 1 \leq k \leq n \), \( i_k = j_k = 1 \), because \( \gamma \) is non-simple \( w \neq (ab)^n \). Thanks to proposition 4.6 \( H(w) \geq \min(H(w_1); H(w_1)) \) where \( w_1 = a(b\bar{a})^p(b(ab))^{n-p-1} \) and \( w_2 = a(b\bar{a})^p(b(ab))^{n-p-1} \). It is not difficult to show that \( H(w_1) = n - 1 \) and \( H(w_2) = n \). Thus we have the result.

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Finish the proof of theorem 1.1

**Proposition 4.8.** Let $\gamma$ be a closed geodesic of $P$ associated to the cyclically reduced word $w(\gamma) = s_1^{j_1} r_1^{j_1} \cdots s_n^{j_n} r_n^{j_n}$, then we have:

\[ i(\gamma; \gamma) \leq nL(\gamma) - n^2. \]

**Proof.** As
\[ i(\gamma; \gamma) = nL(\gamma) - 2n^2 + H(w) - \sum_{1 \leq k < l \leq n} |i_k - i_l| + |j_k - j_l|, \]

it suffices to prove the following inequality:
\[ H(w) - \sum_{1 \leq k < l \leq n} |i_k - i_l| + |j_k - j_l| \leq n^2. \]

By the definition of $H(w)$, we have:
\[ H(w) \leq 2n^2 - n - \#\{1 \leq k < l \leq n; i_k \neq i_l\} - \#\{1 \leq k < l \leq n; j_k \neq j_l\} \]

and thus
\[ H(w) \leq 2n^2 - n - \sum_{1 \leq k < l \leq n} \delta_{i_k,i_l} + \delta_{j_k,j_l}. \]

As $\delta_{i,j} = 1$ ou 0 for any integers $i$ and $j$, we have also:
\[ \sum_{1 \leq k < l \leq n} |i_k - i_l| + \delta_{i_k,i_l} + |j_k - j_l| + \delta_{j_k,j_l} \geq n^2 - n. \]

This implies the result.

This completes the proof of the theorem 1.1

**Proof of theorem 1.2**

The theorem 1.2 is deduced from the inequalities of theorem 1.1

**Corollary 4.9.** Let $\gamma$ be a closed geodesic of $P$ and $L(\gamma)$ his combinatorial length then we have:

\[ i(\gamma; \gamma) \geq \begin{cases} \frac{L(\gamma)}{2} - 1 & \text{if } L(\gamma) \text{ is even} \\ \frac{L(\gamma) - 1}{2} & \text{if } L(\gamma) \text{ is odd} \end{cases} \]
and these bounds are sharp. The geodesics associated to the following words \( w = a(ba)^p\bar{b}(ab)^{n-p-1} \) when \( L(\gamma) \) is even and \( w = a^2b(ab)^{n-1} \) when \( L(\gamma) \) is odd realize the minimal self-intersections number.

**Proof.** Consider the closed geodesics \( \gamma_1 \) and \( \gamma_3 \) of \( P \) associated respectively to the following cyclically reduced words \( w_1 = a(ba)^p\bar{b}(ab)^{n-p-1} \) and \( w_3 = a^2b(ab)^{n-1} \). By definition of \( H(w) \), we have

\[
H(w_1) = n - 1.
\]

Thus, by the theorem 3.12

\[
i(\gamma_1; \gamma_1) = i(\gamma_3; \gamma_3) = L(\gamma) - n - 1.
\]

This gives the proof.

\[\square\]

**Corollary 4.10.** Let \( \gamma \) be a closed geodesic of \( P \) and \( L(\gamma) \) his combinatorial length then we have:

\[
i(\gamma; \gamma) \leq \begin{cases} 
\frac{L^2(\gamma)}{4} & \text{if } L(\gamma) \text{ is even} \\
\frac{L^2(\gamma) - 1}{4} & \text{if } L(\gamma) \text{ is odd}
\end{cases}
\]

and these bounds are sharp. The geodesics associated to the following words \( w = (ab)^n \) when \( L(\gamma) \) is even and \( w = a^2b(ab)^{n-1} \) when \( L(\gamma) \) is odd realize the maximal self-intersections number.

**Proof.** Consider the closed geodesics \( \gamma'_1 \) and \( \gamma'_3 \) of \( P \) associated respectively to the following cyclically reduced words \( w'_1 = (ab)^n \) and \( w'_3 = a^2b(ab)^{n-1} \). By definition of \( H(w) \), we have

\[
H(w'_1) = n^2 \text{ and } H(w'_3) = n^2 + n - 1.
\]

Thus, by the theorem 3.12

\[
i(\gamma'_1; \gamma'_1) = i(\gamma'_3; \gamma'_3) = nL - n^2.
\]

This gives the proof.

\[\square\]

From these corollaries we deduce:

**Corollary 4.11.** Let \( \gamma \) be a closed geodesic of \( P \), \( L(\gamma) \) his combinatorial length and \( k \) an integer.
If \( i(\gamma; \gamma) = k \), then \( 2\sqrt{k} \leq L(\gamma) \leq 2k + 2 \).

**Remark 4.12.** This corollary tells us an important thing on the pair of pants; if we fix the number of self-intersections \( k \), there exists an integer \( L_0 \) such that for every closed geodesic \( \gamma \) of length \( L > L_0 \), his number of self-intersections is bigger than \( k \). The result of M. Mirzakhani ([11]) shows that for any hyperbolic compact surface different from the pair of pants, this result is false.

## 5 Proof of theorem 1.3

Now we are interested on the following set:

\[
A_\epsilon(L) = \{ \gamma \in \mathcal{G} | L(\gamma) = L, \ i(\gamma; \gamma) \geq (1/4 - \epsilon) L^2 \}
\]

for \( \epsilon > 0 \).

Let \( \gamma \) be a closed geodesic of \( P \) and let \( w = w(\gamma) = s_1^{i_1} r_1^{j_1} \cdots s_n^{i_n} r_n^{j_n} \) be the word associated to \( \gamma \). By using the theorem 1.1, we can deduce the following result:

If the closed geodesic \( \gamma \in A_\epsilon(L) \) then \( n \geq \frac{L}{2} - \sqrt{\epsilon L} \).

Set \( B_\epsilon(L) = \{ \gamma \in \mathcal{G}; \ L(\gamma) = L; L-2n \leq 2\sqrt{\epsilon L} \} \), we have the following inclusion:

\[ A_\epsilon(L) \subset B_\epsilon(L) \]

Let \( \gamma \) be a closed geodesic of \( P \) and \( w(\gamma) = s_1^{i_1} r_1^{j_1} \cdots s_n^{i_n} r_n^{j_n} \) the word associated to \( \gamma \). There exists a permutation of the powers of the letters of the word \( w \) and a positive integer \( N \) such that \( w(\gamma) \) can be write on the following form: \( w = s_1^{i_1} r_1^{j_1} \cdots s_N^{i_N} r_N^N s_{N+1} r_{N+1} \cdots s_n r_n \)

with \( i_k > 1 \) or \( j_k > 1 \) for any \( 1 \leq k \leq N \). Set \( L_N = \sum_{k=1}^{N} i_k + j_k \), then \( L = L_N + 2(n - N) \) and \( L_N - 2N \leq 2\sqrt{\epsilon L} \). We have:

\[ \#B_\epsilon(L) = \{ w = s_1^{i_1} r_1^{j_1} \cdots s_N^{i_N} r_N^N s_{N+1} r_{N+1} \cdots s_n r_n; L_N - 2N \leq 2\sqrt{\epsilon L} \} \]

For any \( 1 \leq k \leq N \), \( i_k > 1 \) or \( j_k > 1 \), then \( L_N \geq 3N \) and so \( L_N \leq 6\sqrt{\epsilon L} \). Thus

\#B_\epsilon(L) \leq \{ w = s_1^{i_1} r_1^{j_1} \cdots s_N^{i_N} r_N^N s_{N+1} r_{N+1} \cdots s_n r_n; L_N \leq 6\sqrt{\epsilon L} \} \]

A straightforward calculation gives us:

\[ \#B_\epsilon(L) \leq \frac{1}{L} \sum_{L_N=0}^{E(6\sqrt{\epsilon L})+1} 8(3^{L_N-2} - 2^{L_N-2})(2^{L-L_N-2}) \]

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and we deduce the following inequality:

\[ \#A_\epsilon(L) \leq \frac{9L}{L} [2^2 (\frac{3}{2}) 6\sqrt{\epsilon L} - 3\sqrt{\epsilon L}] . \]

Recall that \( \#G_c(L) = \frac{8 \times 3L-2}{L} \). When \( \epsilon < \frac{1}{36} \), this inequality implies that:

\[ \lim_{L \to +\infty} \frac{\#A_\epsilon(L)}{\#G_c(L)} = 0. \]

**Proposition 5.1.** Let \( N \) be the integer defined above and \( p \) a positive integer. Let \( \gamma \) be the closed geodesic of \( P \) associated to the word \( w = s_1^{i_1} r_1^{j_1} \cdots s_{N+1}^{i_{N+1}} r_{N+1} \cdots s_N^{i_N} r_N^{j_N} (ab)^{n-N-p} \) with \( i_k > 1 \) or \( j_k > 1 \) for any \( 1 \leq k \leq N \).

If \( L - 2n + 2p \leq 2\epsilon L \) then the closed geodesic \( \gamma \) lies in \( A_\epsilon(L) \).

**Proof.** To prove this proposition we will use the following result:

\[ i(\gamma; \gamma) = L - 2n + H(w) + 2 \sum_{k=1}^{n} \sum_{l=k+1}^{n} \min(i_k; i_l) + \min(j_k; j_l) - 2. \]

We have:

\[ \sum_{k=1}^{n} \sum_{l=k+1}^{n} \min(i_k; i_l) + \min(j_k; j_l) - 2 = \sum_{k=1}^{N} \sum_{l=k+1}^{N} \min(i_k; i_l) + \min(j_k; j_l) - 2. \]

The definition of \( N \) implies that \( \min(i_k; i_l) + \min(j_k; j_l) \geq 3 \) for any integers \( k \) and \( l \) between \( 1 \) and \( N \). This implies that:

\[ i(\gamma; \gamma) \geq H(w) + N^2 + L_N - 3N, \]

where \( L_N = \sum_{k=1}^{N} i_k + j_k \). The conditions on \( p \) and \( N \) give:

\[ H(w) \geq (n - p - N)^2 + 2nN. \]

As \( n - p \geq \frac{L}{2} - \epsilon L \), we have \( i(\gamma; \gamma) \geq \left( \frac{1}{4} - \epsilon \right)L^2 \). This achieve the proof. \( \blacksquare \)

This proposition shows that for every word \( w = w'ab \cdots ab \) such that \( w' \) is a finite word of length \( L(\gamma(w')) \leq 6\epsilon L \), the closed geodesic \( \gamma \) associated to this word, lies in \( A_\epsilon(L) \). This implies that:

\[ \# \{ w = w'(ab)^{n-p-N}; L(\gamma(w')) \leq 6\epsilon L \} \leq \#A_\epsilon(L) \]

and so

\[ \#A_\epsilon(L) \geq \frac{2}{3L} [3^{6\epsilon L} - 1]. \]

This gives the theorem [1.3]

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