Quantum criticality and state engineering in the simulated anisotropic quantum Rabi model

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Abstract

Promising applications of the anisotropic quantum Rabi model (AQRM) in broad parameter ranges are explored, which is realized with superconducting flux qubits simultaneously driven by two-tone time-dependent magnetic fields. Regarding the quantum phase transitions (QPTs), with assistance of fidelity susceptibility, we extract the scaling functions and the critical exponents, with which the universal scaling of the cumulant ratio is captured by rescaling the parameters related to the anisotropy. Moreover, a fixed point of the cumulant ratio is predicted at the critical point of the AQRM with finite anisotropy. In respect of quantum information tasks, the generation of the macroscopic Schrödinger cat states and quantum controlled phase gates are investigated in the degenerate case of the AQRM, whose performance is also investigated by numerical calculation with practical parameters. Therefore, our results pave the way to explore distinct features of the AQRM in circuit QED systems for QPTs, quantum simulations and quantum information processing.

1. Introduction

Recent experimental progresses in solid-state-based quantum systems have allowed the advent of the so-called ultrastrong coupling (USC) regime [1–3] and the deep strong coupling (DSC) regime [4, 5] of light–matter interactions, where the coupling strength is comparable to (USC) or larger than (DSC) appreciable fractions of the mode frequency. In these regimes, the celebrated rotating-wave approximation (RWA) breaks down and the quantum Rabi model (QRM) is invoked [6, 7]. In addition to the relatively complex quantum dynamics provided by the QRM, it brings about novel quantum phenomena [8–10] and challenges in implementing quantum information tasks [11–14]. Although exciting, natural implementations of the QRM in the USC/DSC regime in other platforms remain very challenging since they are confined by fundamental limitations. However, different schemes have been used to simulate the QRM in the USC/DSC regime using superconducting circuits [15–18], quantum optical systems [19], trapped ions [20, 21], cold atoms [22], and so on.

In the other aspect, the fascinating promises of the QRM has triggered many studies of the anisotropic quantum Rabi model (AQRM), see e.g., [23–25]

\[
H_{\text{AQRM}} = \hbar \omega a^{\dagger}a + \frac{\hbar}{2} \tilde{\omega}_q \sigma_z + \hbar \tilde{g}[\sigma_z a + a^\dagger \sigma_z] + \hbar \tilde{\lambda} (\sigma_z a^\dagger + a \sigma_z),
\]

(1)

where \(a\) and \(a^\dagger\) are the annihilation and creation operators of the bosonic mode with frequency \(\tilde{\omega}_q\), \(\sigma_z\) and \(\sigma_z = \sigma_x + \sigma_y\) are Pauli operators associated with a qubit with ground state \(|\tilde{g}\rangle\), excited state \(|\tilde{e}\rangle\), and transition frequency \(\tilde{\omega}_q\). It is a generalization of the QRM affiliated with \(\tilde{\lambda}\) denoting the asymmetry between rotating and
counter-rotating terms. The AQRM returns to the Jaynes–Cummings model (JCM) with $\lambda = 0$ [26], or the original QRM with $\lambda = 1$.

Since individual addressing of the two coupling constants are allowed, the AQRM presents a favorable test bed for many valuable theoretical issues, such as the role of the counter-rotating terms [27–29], the Fisher information [30], the universality scaling of quantum phase transition (QPT) [10, 25, 31, 32], the enhanced squeezing [24], and thus it may bridge the gap between the JCM and the QRM in the dynamics [33]. The discussions of the anisotropy in the standard AQRM were further extended to the semi-classical case [34], the multi-qubit case [25, 35] (namely the anisotropic Dicke model). As a consequence, theoretical advancements bring up experimental requests for individual adjustability of the rotating and counter-rotating terms in wide parameter ranges to demonstrate the innovative features of the AQRM. Although there have already been some experimental proposals for the realization of the AQRM in some systems, i.e., quantum well with spin–orbit coupling [30, 36], and circuit QED systems [35, 37], they are quite limited on the tunability and the achievable parameter ranges. Therefore, the demand to explore new platforms to study the dynamics of the AQRM is put forward.

In this work, we propose an experimentally feasible scheme to simulate the controllable AQRM demonstrating the USC and DSC dynamics with superconducting flux qubits. We show through analytical and numerical calculations that our schematic setup has the distinct advantage that the parameters in the effective AQRM can be individually controlled by the frequencies and the amplitudes of the bichromatic magnetic driving fields [17, 18, 38]. The all-round tunability of parameters in this model provides a powerful tool for exploring a few appealing issues. It should be aware that the present AQRM cannot only reduces to the QRM but also produces the primitive JCM with a sufficiently strong coupling strength, providing the opportunity to experimentally observe the gapless Nambu–Goldstone mode [39]. First, we focus on the long-sought QPT in a few-body system, which is initially thought as a privilege of quantum many-body systems, so the critical phenomena and the universal properties of the effective AQRM can be addressed. The critical exponents can be extracted from the scaling behavior of the fidelity susceptibility. Independent of the diversities in the anisotropy and the frequency ratio, a fixed point of a cumulant ratio is predicted and a universal scaling of the cumulant ratio is obtained. In addition, two-qubit quantum gates and Schrödinger cat states can be produced in the special degenerate case of the AQRM. Therefore, our proposal paves not only a way to implement quantum simulators [40] and quantum information tasks, but also the way to explore the QPTs for rich coupling regimes of light–matter interaction in systems where they are experimentally inaccessible.

The paper is organized as follows. We first describe in section 2 the Hamiltonian of our qubit–resonator setup, where the flux qubit is controlled by the bichromatic time-dependent magnetic fluxes. The effective AQRM is obtained when the frequency conditions are well-respected. In section 3, we study the QPTs of the simulated AQRM with the method of fidelity susceptibility and scaling theory. In section 4, we discuss quantum information applications with the degenerate AQRM, such as the generation of the macroscopic Schrödinger cat states and the quantum controlled phase gates. The conclusions are presented in section 5.

2. The qubit-resonator circuit

For simplicity, but here without loss of generality, we use three-junction flux qubits in our scheme [41, 42]. As shown in figure 1(a), a flux qubit is coupled to a LC circuit with an inductance $L$ and a capacitance $C$. The mutual inductance between the flux qubit and the LC circuit is denoted by $M$. The applied magnetic flux $\Phi$ through the flux qubit loop in figure 1(a), which controls the qubit–resonator coupling, is assumed to include a static magnetic flux $\Phi_0$, and also two time-dependent magnetic fields (TDMFs), $\Phi_1(t) = A_j \cos(\omega_j t + \phi_j)$. Here $j = r, b$ label the two TDMFs, individually. Considering one three-junction flux qubit, the qubit’s Hamiltonian $H_q$ reads $H_q = \sum_{j=1}^{3} \left[ C_j \Phi_0^2 \Phi_j^2 / (8\pi^2) - E_j \cos \phi_j \right]$, where we have assumed each junction in the flux qubit has a capacitance $C_j$, phase drop $\phi_j = 2\pi \Phi_j / \Phi_0$, Josephson energy $E_j$, and critical current $I_{ij} = 2\pi E_j / \Phi_0$. Here, $\Phi_0 = h / 2e$ is the magnetic flux quantum. With current-phase relation, the super–current for each junction reads $I_j = I_{0j} \sin \phi_j$, and the persistent current in the qubit loop is given by $I_p = C_q \sum_{j=1}^{3} I_{0j} \sin \phi_j / G_j$ [42–44], where $C_q$ is the total capacitance of the flux qubit, $C_q^{-1} = \sum_i C_i^{-1}$, with the convention $G_j = \eta G_0 = \eta G_2$, and $\eta$ being the relative size of the Josephson junction. Taking into account the TDMFs, the flux quantization around the qubit’s loop imposes a constraint on the phase drop across the three junctions [41–43], $\sum_{j=1}^{3} \phi_j + 2\pi (\Phi_0^2 + \Phi_1^2(t) + \Phi_2^2(t)) / \Phi_0 = 0$. In order to define an effective qubit within the junction architecture, we diagonalize the Hamiltonian $H_q$ containing only the junctions in absence of the TDMFs, i.e., $A_j = 0$. The two lowest eigenstates are labeled as the eigenstates of $\sigma_z$, i.e., $| g \rangle$ and $| e \rangle$, and the spanned two-dimensional subspace describes the effective qubit.

In the other aspect, the Hamiltonian of the total system is written as $H = H_q + H_c + \text{IML}_q$, where $H_c = Q^2 / 2C + \Phi^2 / 2L$ is the Hamiltonian of the LC circuit, with $Q$ being the capacitor’s charge, $\Phi = IL$ being
the magnetic flux through the LC circuit loop and $I$ is the inductor’s current. The Hamiltonian of the LC circuit $H_c$ can be simply quantized by introducing the annihilation and creation operators $\Phi = \sqrt{\hbar/(2C\omega)}(a + a^\dagger)$ and $Q = -i\sqrt{\hbar/(2C\omega)}(a - a^\dagger)$, with the frequency $\omega = 1/\sqrt{LC}$. After projecting the total Hamiltonian into the qubit’s bases $\{ |g\rangle, |e\rangle \}$, we obtain

$$
\hat{H} = \frac{\hbar}{2}\omega q\sigma_z + \hbar\omega a^\dagger a + \hbar g\sigma_z(a^\dagger + a) + \sum_{j=r,b} \cos(\omega_j t + \varphi_j)(\Omega_j\sigma_j - \Lambda_j\sigma_j(a^\dagger + a)).
$$

The first two terms in equation (2) denote the free Hamiltonians of both the qubit and the LC circuit, where $\omega q$ is the transition frequency of the effective qubit. The third term in equation (2) represents the qubit-resonator interaction with the coupling strength being $g = M\sqrt{\omega/2}\mathcal{L} \langle \langle e|I|g\rangle \rangle$. Here $I_q = C_q \sum_{j=r,b} I_{q,j} \sin\phi_{q,j}/C_{q,j} + C_p I_{p,0}\sin\phi_{p,0}/C_{p,0}$ is the super-current through the qubit loop when $\Phi(t) = 0$, where $\phi_{q,j} = -\phi_{q} + \phi_{b} + 2\pi f_{c,j}$ and $f_{c,j} \equiv \Phi_{f,j}/\Phi_{b}$ is the reduced dc bias magnetic flux. The fourth term in equation (2) plays the role of a driving Hamiltonian representing the interaction between the qubit and TDMFs with the respective driving strength being $\Omega_j = A_j \langle |e| \langle 0|I|g\rangle \rangle$. The fifth term of equation (2) is the controllable nonlinear interaction among the qubit, the resonator, and the TDMFs, with the respective coupling strength being $\Lambda_j = 4\pi^2 A_j M_{J0} C_{q,j} \Phi_{b,0}^2 \sqrt{\omega/(2\mathcal{L})} \langle \langle e| E_b \cos\phi_{b,0} \rangle \rangle$. As noticed above, the TDMFs $\Phi_j(t)$ equal to zero when calculating the coupling strengths $g$, $\Omega_j$, and $\Lambda_j$. It is worth noting that in the above derivations, we keep the time-dependent amplitudes small such that the reduced time-dependent magnetic fluxes satisfy $|f_{c,j}(t)| < 10^{-3}$. This leads to: (1) the approximation of $\sin[2\pi f_{c,j}(t)] \approx 2 f_{c,j}(t)$ and $\cos[2\pi \Phi_j(t)/\Phi_{b,0}] \approx 1$; (2) safely neglecting the interaction terms controlled by two simultaneously applied TDMFs in the form of $\sim f_{c,j}(t)f_{c,\bar{j}}(t)$. As a result, when expanding the potential energy in qubit’s Hamiltonian $H_q$ and the qubit’s loop current $I_{p,0}$ we only need to keep the first order of the small reduced flux $\Phi_j(t)/\Phi_{b,0}$.

With realistic parameters discussed in [42], the frequency of the LC oscillator can be designed to be $\omega \approx 2\pi \times 3$ GHz, the qubit’s frequency is approximately $\omega q \approx 2\pi \times 18$ GHz, with $g \approx 2\pi \times 37$ MHz when $f_{c} = 0.49$. Therefore, the conditions $g \ll |\omega q \pm \omega|$, is well satisfied and the effect of the always-on qubit-resonator interaction term is negligibly small. Equation (2) can be written as

$$
\hat{H}_\text{int}' = \sum_{j=r,b} \Omega_j \cos(\omega_j t + \varphi_j)(\sigma_j e^{i\omega_j t} + \text{h.c.}) - \sum_{j=r,b} \Lambda_j \cos(\omega_j t + \varphi_j)(a^\dagger e^{i\omega_j t} + ae^{-i\omega_j t})(\sigma_j e^{i\varphi_j} + \text{h.c.}),
$$

where we have performed the RWAs and neglected all terms that are fast oscillating in the interaction picture with respect to the system’s free Hamiltonian $\hbar\omega_q\sigma_z/2 + \hbar\omega a^\dagger a$. We now consider the case where the TDMFs are inducing the respective first order red ($\delta_r$) and blue ($\delta_b$) sideband transitions with small detunings $\delta_r$ and $\delta_b$ onto the qubit-resonator system, e.g., $\omega_r = \omega_q - \omega - \delta_r$ and $\omega_b = \omega_q + \omega - \delta_b$ [38]. In such a scenario, the terms in the first line of equation (3) representing the direct driving on the qubit can be ignored for weak drivings such that $\Omega_j/\hbar \ll \min\{\omega_{b,0},|\omega_q \pm \omega|\}$ is fulfilled, since $\Omega_j/\hbar$ is a few megahertz and $|\omega_q \pm \omega|$ is on the order of gigahertz. Similarly, when the rest of the frequency detunings are large compared to the coupling parameters, i.e., $|\omega - \omega_r \pm \omega_b| > \Lambda_j/\hbar$, $|\omega - \omega_b \pm \omega_q| > \Lambda_q/\hbar$, one may neglect the rest of the fast oscillating terms. These approximations lead to a simplified time-dependent Hamiltonian

$$
\hat{H}_\text{eff} = \frac{\Lambda_r}{2}(\sigma_r e^{i\delta_r t + i\varphi_r} + \text{h.c.}) + \frac{\Lambda_b}{2}(\sigma_b e^{i\delta_b t + i\varphi_b} + \text{h.c.}).
$$

It is worth noting that equation (4) corresponds to the interaction picture of the generalized AQRM with respect to the uncoupled Hamiltonian $H_0 = \hbar(\delta_r + \delta_b)\sigma_z/4 + \hbar(\delta_b - \delta_r)\sigma_x^2/2$ [20], such that
\[ H'_{\text{AQRM}} = \frac{\hbar}{2} \tilde{\omega}_d \sigma_z + \hbar \tilde{\alpha} a^\dagger a + \hbar \tilde{g} (\sigma_x a e^{i\tilde{\varphi}} + \text{h.c.}) + \hbar \tilde{g}_w (\sigma_y a^\dagger e^{i\tilde{\varphi}} + \text{h.c.}), \]  

with the effective parameters being \( \tilde{\omega}_d = (\delta_1 + \delta_2)/2, \tilde{\omega} = (\delta_0 - \delta_1)/2, \tilde{g}_c = \Lambda_c/(2\hbar), \tilde{g}_w = \Lambda_w/(2\hbar) \). Here the qubit’s and the resonator’s frequencies are represented by \( \tilde{\omega}_d \) and \( \tilde{\omega} \), respectively. The tunability of these parameters permits study of all coupling regimes of the AQRM via suitable choices of the amplitudes and the detunings of the TDMFs. It is noteworthy that the complex coupling strengths \( \tilde{g}_c \) and \( \tilde{g}_w \) can be realized by choosing the phases \( \varphi \) and \( \varphi_0 \) of the TDMFs. For example, equation (5) leads to the standard AQRM in equation (1) when \( \varphi = \varphi_0 = 0 \) and \( \tilde{\lambda} = \tilde{g}_c/\tilde{g}_w \). To go beyond the USC, we only need the condition that \( \lambda_j \gtrsim h/|\delta | \). In particular, with only one single frequency TDMF on the flux qubit, i.e., \( \Lambda_w = 0 \), equation (5) reduces to the standard JCM, which possesses the continuous \( U(1) \) symmetry. One interesting point is that it provides different phase diagrams with sufficiently strong coupling strength, such as the emergence of the gapless excitation spectrum, namely the so-called Nambu–Goldstone mode [31, 39]. It should be emphasized that a pure JCM with a very large coupling strength does not naturally exist due to the breakdown of the RWA, which in a way prevents the experimental observation of the Nambu–Goldstone modes. Therefore, our scheme sheds bright light on the possibility of experimentally demonstrating the gapless excitation spectrum in circuit QED systems.

3. QPT and finite frequency scaling in AQRM

QPT in systems with few degrees of freedom has recently been a topic of major interest recently [10, 25, 31, 32, 45]. Although the quantum criticality is commonly believed to take place in a many-body system in the thermodynamic limit, it is newly realized that a few-body system may undergo QPTs provided the energy barrier between two local-minima states is infinite. In particular, for large frequency ratio of \( \tilde{\eta} = \tilde{\omega}_d/\tilde{\omega} \rightarrow \infty \), the AQRM with finite counter-rotating terms displays a second-order QPT from a normal phase of \( \tilde{g} < \tilde{g}_c \) (with zero mean photon population) to a superradiant phase of \( \tilde{g} > \tilde{g}_c \) (with nonzero photon population) when passing through the normalized critical coupling strength \( \tilde{g}_c = (\sqrt{\tilde{\omega}_d/\tilde{\omega}} + \lambda)/\bar{\lambda} \) [10, 25]. In the theory of critical phenomena, second-order QPTs can be classified into universality classes determined only by a few properties characterizing the system and may not depend on microscopic details. The critical exponents and scaling functions are the same for all systems belonging to a given universality class. The studies in [25] show that the AQRM with any finite values of \( \bar{\lambda} = 0 \) belongs to the same universality class as the QRM, but the special case of \( \bar{\lambda} = 0 \) for the JCM belongs to a different universality class [31]. It is also worth mentioning that, due to the negligible quantum fluctuations, it is still a debate whether the zero-temperature superradiant phase transition in the AQRM can be understood as a QPT or not [32]. With the simulated AQRM, i.e., equation (5) in hand, the free adjustment of parameters allows us to investigate many interesting issues like the critical phenomena and the universal properties. In this section, we study the QPTs in the AQRM from the perspective of fidelity susceptibility and scaling theory.

3.1. Fidelity susceptibility with AQRM

The fidelity susceptibility [46] is originally proposed to elucidate the changing rate of fidelity [47, 48] under an infinitesimal variation of the driving parameter:

\[ \left| \langle \Psi_\epsilon(\tilde{g}) | \Psi_\epsilon(\tilde{g} + \delta \tilde{g}) \rangle \right|^2 = 1 - \chi_F \delta \tilde{g}^2 / 2 + O(\delta \tilde{g}^3), \]  

where \( |\Psi_\epsilon(\tilde{g})\rangle \) is the ground state wave function of a Hamiltonian \( \hat{H}(\tilde{g}) = H_0 + \tilde{g} H_1 \), \( \tilde{g} \) is the external driving parameter, and \( \delta \tilde{g} \) is a tiny variation of the external parameter. Though borrowed from the quantum information theory, the fidelity susceptibility has been proved to be an effective sensor to detect and characterize QPTs in condensed matter physics [49]. As an informational metric, the quantum fidelity susceptibility can be also devised to seize the criticality in the perspective of the Riemannian metric tensor form

\[ \chi_F = \langle \partial_\tilde{g} \Psi_\epsilon | \partial_\tilde{g} \Psi_\epsilon \rangle - |\langle \partial_\tilde{g} \Psi_\epsilon | \partial_\tilde{g} \Psi_\epsilon \rangle|^2. \]  

The fidelity susceptibility is of fundamental interest in QPTs because it provides direct manifestation of their universal properties without prior knowledge of order parameters. More importantly, the fidelity susceptibility can be in general reconstructed from the power spectral density of driving field [45, 50], which is accessible experimentally for example via Rabi oscillation measurements, spin-echo measurements, and energy relaxation measurements [51]. Last but not the least, there is a refreshing proposed duality, which connects the fidelity susceptibility and the max volume of a codimension-one time slice in antide Sitter (AdS) space [32, 53]. Such duality bridges quantum information theory and holography, and may deepen our understanding of quantum gravity [54, 55].

For a second-order QPT, the gap in the excitation spectrum vanishes as \( (\tilde{g} - \tilde{g}_c)^{\nu z} \), where \( \tilde{g}_c \) is the critical point, \( \nu \) and \( z \) are the correlation-length and dynamic exponents, respectively. We can account for the divergence around quantum critical points (QCPs) in the AQRM by formulating a so-called finite-\( \tilde{\eta} \) scaling
Figure 2. (a) Fidelity susceptibility in the AQRM as a function of the normalized coupling strength $\tilde{g}/\tilde{g}_c$ for different values of $\tilde{\eta}$ with $\tilde{\lambda} = 0.1$. (b) The finite frequency $\eta$-dependence of the maximized fidelity susceptibility, $\log(\chi_F^{\text{max}})$ as a function of $\log(\tilde{\eta})$. With the linear fits of the numerical data, we find $\chi_F^{\text{max}} \approx \tilde{\eta}^{d_f}$ with $d_f^+ \approx 0.333$ for $\tilde{\lambda} = 0.1$ (blue dashed line), $d_f^+ \approx 0.351$ for $\tilde{\lambda} = 1$ (red dashed–dotted line), $d_f^+ \approx 0.337$ for $\tilde{\lambda} = 10$ (green dotted line). (c) The position of the maximized fidelity susceptibility, $\log(1 - \tilde{g}/\tilde{g}_c^{\text{max}})$ as a function of $\log(\tilde{\eta})$. The linear fittings to the numerical data leads to $|1 - \tilde{g}/\tilde{g}_c^{\text{max}}| \sim \tilde{\eta}^{-1/\nu}$ with $1/\nu \approx 0.652$ for $\tilde{\lambda} = 0.1$ (blue dashed line), $1/\nu \approx 0.668$ for $\tilde{\lambda} = 1$ (red dashed–dotted line), $1/\nu \approx 0.697$ for $\tilde{\lambda} = 10$ (green dotted line).

theory [10, 25], in parallel with the scaling theory for finite-size effects in a many-body system, and here $\tilde{\eta}$ plays a similar role as system size in the latter case. Universal information could be decoded from the scaling behavior of the fidelity susceptibility [56–58]. The fidelity susceptibility exhibits stronger dependence on $\tilde{\eta}$ across the critical point than in the non-critical region. Referring to standard arguments in finite-size scaling analysis [59], one obtains that the fidelity susceptibility can exhibit a finite-$\eta$ scaling. For a finite-$\eta$ system, the position of a divergence peak defines a pseudocritical point $\tilde{g}_c$ as $\tilde{\eta} \to \infty$, implying an abrupt change in the ground state of the system at the QCP in the classical oscillator limit. The maximum point of the fidelity susceptibility at $\tilde{g}_c$ scales as

$$\chi_F(\tilde{g}_c^{\text{max}}) \sim \tilde{\eta}^{d_f},$$  

where $d_f$ denotes the critical adiabatic dimension. Meanwhile, the dependence of the fidelity susceptibility on $\tilde{\eta}$ defines the critical exponent as $d_f^\pm$ above(below) the critical point in the following way:

$$\chi_F(\tilde{g}) \sim \tilde{\eta}^{d_f^\pm}.$$  

On the other hand, the position of the pseudocritical point obeys such a scaling behavior as

$$|\tilde{g}_c - \tilde{g}_c^{\text{max}}| \sim \tilde{\eta}^{-1/\nu}.$$  

Thus, the behavior of $\chi_F$ on finite systems in the vicinity of a second-order QCP can be estimated as

$$\chi_F \approx C\tilde{\eta}^{d_f} f(|\tilde{g} - \tilde{g}_c|^{1/\nu}),$$

where $f$ is an unknown regular scaling function, and $C$ is a constant independent of $\tilde{g}$ and $\tilde{\eta}$.

In figure 2(a), we show the fidelity susceptibility $\chi_F$ in the AQRM as a function of the normalized coupling strength $\tilde{g}/\tilde{g}_c$ for different values of $\tilde{\eta}$ with $\tilde{\lambda} = 0.1$. Relatively away from critical points, the fidelity susceptibility $\chi_F(\tilde{g})$ is independent of $\tilde{\eta}$, suggesting $d_f^\pm = 0$. The fidelity susceptibilities are always peaked around the critical point of AQRM, i.e., $\tilde{g}_c$ in the classical oscillator limit. As $\tilde{\eta}$ increases, the value of the fidelity susceptibility becomes larger and its distribution becomes narrower, and the coupling strength $\tilde{g}_c$ corresponding to the maximized fidelity susceptibility approaches to the quantum critical point $\tilde{g}_c$, which separates a normal phase at $\tilde{g} \leq \tilde{g}_c$ from a superradiant phase at $\tilde{g} \geq \tilde{g}_c$. The critical exponents appearing in equations (8) and (10), we next study the finite frequency scalings of the fidelity susceptibility for different values of $\tilde{\lambda} = 0.1, 1, 10$, as shown in figures 2(b) and (c). Here in both figures, the colored-markers represent the calculated numerical results, and the color-coded lines are the numerical fittings to the corresponding data sets. The maximal fidelity susceptibility $\chi_F^{\text{max}}$ follows a
universal power law with respect to $\tilde{\eta}$, as indicated from Figure 2(b). The slopes of the fitting indicate that the adiabatic dimension $d_{\eta} = 1/3$ [10, 25]. Figure 2(c) exhibits the position of the coupling strength corresponding to $\log(\chi_{\eta}^{\text{max}})$, i.e., $\tilde{g}_{\text{max}}$, as a function of $\log(\tilde{\eta})$. Similarly, the slopes of the fitting lines in Figure 2(c) suggest the value of the other critical exponent $\nu \approx 3/2$. The gap scaling shows that $\nu z = 1/2$ [10, 25], and this accounts for that the dynamic exponent $z = 1/3$, which agrees well with the excitation energy according to $\epsilon \sim \tilde{\eta}^{-z}$. A simple relation $d_{\eta} = 2/\nu - 1 = 1$ is also verified [60].

### 3.2. The cumulant ratio and the fixed point with AQRM

With the extracted values of the critical exponents, i.e., $d_{\eta}' = 1/3$ and $\nu = 3/2$, we are now ready to discuss the finite frequency scaling properties of the field’s position quadrature operator, $X_n = (a + a^\dagger)/\sqrt{2}$. Taking average over the ground state of the AQRM, the scaling behavior of the field’s position quadrature can be written in the form of [25]

$$\langle X_n^{2\eta'} \rangle = \tilde{\eta}'^d_n \chi_n([\tilde{g} - \tilde{g}_c] \tilde{\eta}'^{1/\nu}),$$

(12)

Here, $\tilde{\eta}' = (1 + \tilde{\lambda})/(2\sqrt{\tilde{\lambda}|l|})$ is the relative scaling variable modified by the anisotropy parameter $\tilde{\lambda}$, and $\chi_n$ is the universal scaling function. The scaling form of equation (12) includes the anisotropic effect, which is beyond the traditional scaling frame. Actually, the experimental verification of this anisotropic involved universality is urgently needed. Being different from typical physical quantities whose features have already been investigated in the AQRM [10, 25, 31, 32], here for finite frequency ratio, we design a cumulant ratio as

$$U_X = \frac{\langle X_n^2 \rangle}{\langle X_n^2 \rangle^2} = \frac{\chi_n([\tilde{g} - \tilde{g}_c] \tilde{\eta}'^{1/\nu})}{\chi_n^2([\tilde{g} - \tilde{g}_c] \tilde{\eta}'^{1/\nu}),}$$

(13)

which can be considered as an analogy to the famous Binder cumulant ratio of the dimensional criticality in a statistical system [61, 62]. The cumulant ratio $U_X$, which needs to measure the quadrature of the displacement, may possibly be performed with squeezed quadrature quantum tomography of the electro-magnetic field using homodyne or heterodyne detection [63, 64] and together with photon number measurement [16]. Obviously, at the critical point $\tilde{g} = \tilde{g}_c$, $U_X$ is independent of $\tilde{\eta}'$, and locates at a universal value of $\chi_n(0)/\chi_n^2(0)$. It means that there is a fixed point at $[\tilde{g}_c, \chi_n(0)/\chi_n^2(0)]$, which is universal for different values of anisotropic strength $\tilde{\lambda}$ and frequency ratio $\tilde{\eta}'$. We confirm this fixed point involved universality via numerical calculations. In fact, we show in Figure 3(a) that, the numerical values of $U_X$ for different values of $\tilde{\lambda}$ and $\tilde{\eta}'$ crossover each other at the fixed point of $\tilde{g}^2/\tilde{g}_{\text{c}} = 1$, which is exactly the critical point in the AQRM. Moreover, Figure 3(b) implies that, despite the differences in $\tilde{\lambda}$ and $\tilde{\eta}'$, the cumulant ratio $U_X$ collapse into a single curve when appropriately scaled. This unambiguously reveals the observable-dependent scaling function $\chi_n$ in equation (13). Therefore, we can draw the same conclusion as that in [25], which is, in a word, the AQRM with finite anisotropy of $\tilde{\lambda} \neq 0$ is of the same universality class as the QRM. Moreover, we hope that the fixed point discussion presents an alternative possibility to explore the universality of the AQRM. It is important to address also the special case of the JCM with $\tilde{\lambda} = 0$, where the QPT is given by a succession of level crossings and thus gives rise to discrete step-like

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**Figure 3.** (a) The cumulant ratio $U_X$ as a function of the normalized coupling strength $\tilde{g}/\tilde{g}_c$ for different values of $\tilde{\eta}$ and $\tilde{\lambda}$. All the lines coincide with each other at the fixed point of $\tilde{g}/\tilde{g}_c$. (b) The universal scaling of the cumulant ratio $U_X$ as a function of $\tilde{g}/\tilde{g}_c$ for different values of $\tilde{\eta}$ and $\tilde{\lambda}$. All numerical data collapse into the single function.
behaviors in physical quantities, including the cumulant ratio [25, 31]. In fact, due to the discontinuous features of the JCM when $\tilde{\eta}$ is finite, a fixed point may not be well defined.

3.3. The simulated AQRM with finite large frequency ratio

In the following, with realistic parameters in circuit QED systems [42, 43], we demonstrate that the present proposal allows reaching a large value of the frequency ratio $\tilde{\eta}$ and extreme coupling regimes, so in principle it can approach the limit where the phase transition takes place. As an illustration, we make comparisons between the original Hamiltonian in equation (2) and the effective Hamiltonian in equation (5) for the ground state probability $P_c(t) = \langle \Psi_c | g | \Psi_c \rangle^2$ and the ground state entanglement entropy $S_G = -\text{Tr} [\rho_G^a \log_2 (\rho_G^a)]$, where $| \Psi_G \rangle$ is the ground state of the total system, $\rho_G^a = \text{Tr}_b [\rho_G]$ is the reduced density matrix of the qubit’s subsystem by tracing out the field’s degree of freedom. The ground state probability $P_c(t)$ indicates the atomic-excitation probability in the ground state $| \Psi_G \rangle$ of the total system, and the entanglement entropy $S_G$ measures the entanglement between the qubit and the resonator.

Clearly shown in figure 4, the results from the original Hamiltonian (red solid lines) are in consistent with the ones from the effective Hamiltonian (blue dashed lines with circles) for the two sets of parameters: (a) and (b) $\omega_q = 2\pi \times 15.448$ GHz and $\omega_b = 2\pi \times 20.548$ GHz, $\Omega_1 / \hbar = \Lambda_0 / \hbar = \Omega_b / \hbar = 2\pi \times 10.5$ MHz; (c) and (d) $\omega_q = 2\pi \times 15.089$ 7 GHz and $\omega_b = 2\pi \times 20.909$ GHz, $\Omega_1 / \hbar = \Lambda_0 / \hbar = \Omega_b / \hbar = 2\pi \times 15$ MHz. A large value of frequency ratio, i.e. $\tilde{\eta} = 300$, $\tilde{g}_c / \tilde{g}_b = \tilde{g}_e / \tilde{g}_b = 0.80$ for (a), (b) and $\tilde{g}_c / \tilde{g}_b = \tilde{g}_e / \tilde{g}_b = 2.88$ for (c), (d). For the simulation, the system is initially prepared in the ground state of the whole system $|g, 0\rangle$, and the rest of the parameters are chosen as $\omega = 2\pi \times 3$ GHz, $\omega_d = 2\pi \times 18$ GHz, and $g = 2\pi \times 37$ MHz [42].

**Figure 4.** The evolution of the atomic ground state probability $P_c(t)$ (a), (c) and the entanglement entropy $S_G(t)$ (b), (d) as a function of time, obtained by numerically integrating the original Hamiltonian in equation (2) (red solid line), and the effective Hamiltonian in equation (5) (blue dashed lines with circles), respectively. Two sets of parameters are considered: (a), (b) $\omega_q = 2\pi \times 15.448$ GHz and $\omega_b = 2\pi \times 20.548$ GHz, $\Omega_1 / \hbar = \Lambda_0 / \hbar = \Omega_b / \hbar = 2\pi \times 10.5$ MHz; (c), (d) $\omega_q = 2\pi \times 15.089$ GHz and $\omega_b = 2\pi \times 20.909$ GHz, $\Omega_1 / \hbar = \Lambda_0 / \hbar = \Omega_b / \hbar = 2\pi \times 15$ MHz. This leads to simulated effective parameters of $\eta = 300$, $\tilde{g}_c / \tilde{g}_b = \tilde{g}_e / \tilde{g}_b = 0.80$ for (a), (b) and $\eta = 300$, $\tilde{g}_c / \tilde{g}_b = \tilde{g}_e / \tilde{g}_b = 2.88$ for (c), (d). For the simulation, the system is initially prepared in the ground state of the whole system $|g, 0\rangle$, and the rest of the parameters are chosen as $\omega = 2\pi \times 3$ GHz, $\omega_d = 2\pi \times 18$ GHz, and $g = 2\pi \times 37$ MHz [42].
state in the USC/DSC regime with adiabatic evolution. More precisely, we can start with the trivial ground state of a weakly coupled circuit QED system, i.e., ground state \(|g, 0\rangle\) of equation (2) in the absence of the TDMFs with \(A_j = 0\), and then adiabatically prepare the ground state of the ultrastrong or deep strong coupling AQRM by slowly ramping up the drive amplitudes \(A_j = 0\). Once the desired state is achieved, the microwave drives can be turned off, returning the system to the weak-coupling dynamics for furthermeasurements [18, 21, 65]. For this adiabatic preparation of the ground state, the considered evolution time is determined by \(T = 20 \times 2 \pi / \omega\). A possible set of achievable parameters for the simulated scheme in the circuit setup could be \(\omega = 2\pi \times 18\) GHz, \(\omega_b = 2\pi \times 15.158\) GHz and \(\omega_c = 2\pi \times 20.836\) GHz, so that the frequency ratio \(\eta \approx 54\) with an effective resonator frequency \(\tilde{\omega} = 2\pi \times 3\) MHz can be explored. This implies \(T \approx 7\) \(\mu\)s, which approximately satisfies the adiabatic condition. In combination with the dedicated microwave manipulation techniques, the simulation protocol by principle is suited for exploring novel QPTs in the AQRM [10, 25, 31, 45]. For practice experimental implementations, it is worthy of mentioning that since the state preparation time \(T\) is inverse proportional to the effective resonator frequency, longer \(T\) is required for larger \(\eta\), which certainly leads to more errors accumulated from the imperfect operations and the decay processes.

4. Quantum information processing with degenerate AQRM

Without lose of generality, our scheme of simulating the controllable AQRM can be generalized to the multi-qubit case, where multiple flux quibbs are coupled to the LC circuit as shown in figure 1(b). When the corresponding conditions for each qubit to realize the effective anisotropic Rabi model as in equation (5) are well satisfied, and the parameters for the \(l\)th qubit \((l = 1, 2, \ldots N)\) are chosen such that, \(\delta^{(l)}_b = -\delta^{(l)}_s = \delta\), \(\varphi^{(l)}_b = -\varphi^{(l)}_s = \varphi\), and \(\Lambda^{(l)}_s = \Lambda^{(l)}_b = \Lambda\), the effective multi-qubit Hamiltonian can be written as

\[ \mathcal{H}^N_{\Lambda,\varphi} = g J (a e^{-i\varphi} + a^* e^{i\varphi}), \]  

where \(J = \sum_{l=1}^{N} \sigma^{(l)}_x\) are the collective qubit operators and we have defined \(g = \Lambda / 2\). Hereafter we name the case of degenerate qubits with \(\tilde{\omega}^{(l)}_q = 0\) as the degenerate AQRM. The evolution operator \(U_{\Lambda,\varphi}\) can then be found as

\[ U_{\Lambda,\varphi} = \exp[i\Phi(t)J_{\Lambda}^2] D(\alpha(t)J_{\Lambda}), \]  

where \(\Phi(t) = (g/\delta^2)(bt - \sin bt)\) and \(D(\chi) = \exp[\chi(t)a^t - \chi^t(t)a]\) is the displacement operator with \(\chi(t)\) being the collective displacement amplitude of the oscillator.

4.1. The generation of macroscopic Schrödinger cat states

It has been proved that, the Schrödinger cat states have promising applications in hardware-efficient quantum memory and quantum error corrections [66, 67]. Below we show the performance of our scheme in generating this class of non-classical states with both theoretical and numerical approaches. In the single-qubit case, \(J_{\Lambda} = 1\), the four types of the quasi-orthogonal states \(\{|\alpha\rangle, |\hat{\alpha}\rangle, \pm |\alpha\rangle, \pm |\hat{\alpha}\rangle\}\) are well defined. For practice experimental implementations, it is worthy of mentioning that since the state preparation time \(T\) is inverse proportional to the effective resonator frequency, longer \(T\) is required for larger \(\eta\), which certainly leads to more errors accumulated from the imperfect operations and the decay processes.

\[ |\Psi(t)\rangle = e^{i\Phi(t)} \frac{1}{\sqrt{2}} (|+\rangle |\alpha\rangle - |-\rangle -|\alpha\rangle), \]  

where \(|\pm\rangle = (|\alpha\rangle \pm |\hat{\alpha}\rangle) / \sqrt{2}\), and \(|\pm|\alpha\rangle\) being the coherent states of the harmonic oscillator, which are of the same amplitude but opposite phase in the phase space. \(\alpha(t) = \hat{g}(1 - e^{\tilde{\omega}t}) e^{i\varphi}/\delta\) is the coherent state amplitude for the single-qubit case. It is worth of mention from equation (16) that, since \(\sigma^2_\alpha = 1\), the first term in equation (15) behaves only as a global phase factor in equation (16). Obviously, depending on the states of the flux qubits \(|\pm\rangle\), the coherent states undergo different displacements \(|\pm|\alpha\rangle\), respectively. With the basis states \(|\{\alpha\rangle, |\hat{\alpha}\rangle\}\), the state in equation (16) can be rewritten as

\[ |\Psi(t)\rangle = \frac{1}{2\sqrt{N}} (|\alpha\rangle |C^\alpha_{\alpha}\rangle + |\hat{\alpha}\rangle |C^\hat{\alpha}_{\alpha}\rangle), \]  

where \(\{C^\alpha_{\alpha}\} \equiv \mathcal{N}(|\alpha\rangle \pm |\hat{\alpha}\rangle)\) with \(\mathcal{N} = 1/\sqrt{2(1 + e^{-2|\alpha|^2})}\) are the so-called even \((|C^\alpha_{\alpha}\rangle\) and odd \((|C^\hat{\alpha}_{\alpha}\rangle\) Schrödinger cat states. Note that the normalizing factor is approximately \(\mathcal{N} \approx 1/\sqrt{2}\) for coherent state with amplitude of \(|\alpha| > 1\). By choosing the phase \(\varphi\), the four types of the quasi-orthogonal states \(|\pm|\alpha\rangle\) and \(|\pm|\hat{\alpha}\rangle\) [66], i.e., \(|\alpha|\rangle \langle \alpha| \approx 1\) (note that for \(\alpha = 2\), \(|\alpha|\rangle \langle \alpha| \approx 10^{-1}\)), can be generated by measuring the qubit in the basis states \(|\pm\rangle\). By performing projective measurements in the qubit’s basis states \(|\alpha\rangle, |\hat{\alpha}\rangle\), the oscillator will collapse into the Schrödinger cat states with probability of \((1 + e^{-2|\alpha|^2})/2\), respectively. As shown in equation (17) and displayed in figure 5, the even cat state \(|C^\alpha_{\alpha}\rangle\) is generated with a projective measurement onto the qubit’s ground state \(|\hat{\alpha}\rangle\). See from equation (17) that, the maximum displacement amplitude is \(|\alpha|_{\max} = 2g/\delta\), and it can be obtained at the times \(t = (2m + 1)\pi/\delta\) for natural number \(m\). By choosing a
small value for \( \delta \) and a large effective coupling strength \( g \), we can create macroscopically distinct Schrödinger cat states of considerable size of \( |\alpha| > 1 \).

In another aspect, the displacement amplitude of the Schrödinger cat states can be further enhanced with even number of flux qubits by exploring the multi-qubit case and preparing the system in the state of \( \{|+, +, \ldots, +\} - |-, -, \ldots, -\}\rangle/\sqrt{2} \). In this case, the state after evolution is given by

\[
|\Psi_N(t)\rangle = \frac{e^{i\delta N(t)}}{\sqrt{2}}\{|+, +, \ldots, +\}N\alpha\rangle - |-, -, \ldots, -\rangleN\alpha\rangle,
\]

(18)

where the coherent state amplitude is enhanced by a factor \( N \), and the first term in equation (15) remains as a global phase factor. However, collective measurements on the flux qubit in the basis states \( \{|+, +, \ldots, +\} \pm |-, -, \ldots, -\rangle\rangle/\sqrt{2} \) are required to obtain the Schrödinger cat states with an enhanced amplitude.

4.2. The two-qubit controlled quantum phase gate generation

As seen from the evolution operator in equation (15), the Hamiltonian in equation (14) introduces qubit–qubit interaction between any pair of qubits. Thus, our circuit can be used to generate quantum gates and produce highly-entangled states between qubits. Let the system evolve for a time period of \( T = 2\pi/\dot{\theta} \), we obtain \( \chi(T) = 0 \) and up to an overall phase factor, the evolution operator can be recast as

\[
U(T) = \exp\left[ i \frac{\dot{\theta}}{N} \sum_{k=1}^{N} \sigma_3^k \sigma_3^k \right],
\]

(19)

with \( \dot{\theta} = 2\dot{\Phi}(T) = 4\pi \dot{g}^2/\delta^2 \). In the following, we show that the generation of a two-qubit quantum controlled-NOT gate is straightforward from equation (19). In the two-qubit bases of \( \{|e\rangle, |g\rangle\}, |\pm\rangle \), the evolution operator can be expressed as

\[
U(T) = \begin{pmatrix}
\cos \theta & 0 & 0 & i \sin \theta \\
0 & \cos \theta & i \sin \theta & 0 \\
i \sin \theta & i \sin \theta & \cos \theta & 0 \\
i \sin \theta & 0 & 0 & \cos \theta
\end{pmatrix},
\]

(20)

which represents a non-trivial two-qubit gate when \( \dot{\theta} = m\pi \) (\( m = 0, 1, 2, \cdots \)). Specifically, when \( \dot{\theta} = \pi/4 \) (i.e., \( g = \delta/4 \)), \( U(T) \) is locally equivalent to the controlled-NOT (CNOT) gate.

4.3. The simulated doubly degenerate AQRM

By performing numerical calculations with practical parameters [42], we show that our proposal performs well in simulating the AQRM. Without loss of generality, we display in figure 6 the simulation of the doubly degenerate AQRM, where both \( \tilde{\omega} \) and \( \dot{\omega} \) are zero. The atomic ground state probability \( P_g(t) \) and the ground state entanglement entropy \( S_0 \) are plotted for three sets of parameters, \( \Omega/\hbar = \Lambda/\hbar = 2\pi \times 15 \text{ MHz} \), \( \Omega_\alpha/\hbar = \Lambda_\alpha/\hbar = 2\pi \times 3 \text{ MHz} \) for figures 6(a) and (b); \( \Omega/\hbar = \Lambda/\hbar = 2\pi \times 15 \text{ MHz}, \Omega_\alpha/\hbar = \Lambda_\alpha/\hbar = 2\pi \times 15 \text{ MHz} \) for figures 6(c) and (d); and \( \Omega/\hbar = \Lambda/\hbar = 2\pi \times 3 \text{ MHz} \), \( \Omega_\alpha/\hbar = \Lambda_\alpha/\hbar = 2\pi \times 15 \text{ MHz} \) for

![Figure 5](image-url)
The frequency of the red and blue drivings are chosen to be $\omega_r = 2\pi \times 15$ GHz and $\omega_b = 2\pi \times 21$ GHz. The curved lines for the original Hamiltonian in equation (2) (red solid line) reproduce the ones calculated for the effective Hamiltonian in equation (5) (blue dashed lines with circles) with high accuracy. The numerical agreements shown in figure 6 confirm that our scheme has excellent performance in simulating the properties of the doubly degenerate AQRM. It is also interesting to illustrate the simulation protocol reversely, that is, the properties of the artificial AQRM help to predict the observed dynamics of the original Hamiltonian when appropriate parameters are chosen. For example, starting from the state $|g, 0\rangle$, no population transfer takes place if $\tilde{g}_r = 0$ and the state stays as it is; partial population transfer between states $|g, 0\rangle$ and $|e, 1\rangle$ happens when $\tilde{g}_r > \tilde{g}_r = 0$, see figures 6(a) and (b); and complete Rabi oscillation between states $|g, 0\rangle$ and $|e, 1\rangle$ appears when $\tilde{g}_r > \tilde{g}_r = 0$, see figures 6(e) and (f). A special case is when $\tilde{g}_r = \tilde{g}_r$, the state $|g, 0\rangle$ evolves to an eigenstate of the QRM, which is in the form of $\propto (|+\rangle + |\beta\rangle - |-\beta\rangle)$ and thus gives balanced atomic populations. This phenomenon is characteristic for the doubly degenerate QRM, and ideally the atomic
population and the amount of entanglement are no longer changing after reaching its eigenstate, as displaced in figures 6(c) and 6(d).

What coming along with the atomic population transfers are the collapses and revivals of the photon wave packets and the variation of the photon statistics. In the following, by employing the Wigner quasi-probability distribution function (WF), we show some interesting features of the field statistical properties of the doubly degenerate AQRM with $\omega = \tilde{\omega}_q = 0$. In figure 7, we plot the WF of the AQRM at different time intervals for

Figure 7. The Wigner function $W(X, P)$ of the field state at different interaction times after tracing out the qubit’s degree of freedom, calculated ab initio from equation (2) with $|g, 0\rangle$ being the initial state of the system. We have considered four cases: (a)–(d) $\Omega_r = \Lambda_r = 0$ and $\Omega_b/\hbar = \Lambda_b/\hbar = 2\pi \times 15$ MHz, which corresponds to simulated effective parameters of $\omega = \tilde{\omega} = \tilde{\omega}_r = 0$, $\tilde{g}_{cr} = 1$. (e)–(h) $\Omega_r = \Lambda_r = 2\pi \times 7.5$ MHz and $\Omega_b/\hbar = \Lambda_b/\hbar = 2\pi \times 15$ MHz, which corresponds to simulated effective parameters of $\omega = \tilde{\omega} = \tilde{\omega}_r = 0$, $\tilde{g}_{cr} = 1$, $\tilde{g}_{br} = 0.5$. (i)–(l) $\Omega_r = \Lambda_r = 2\pi \times 15$ MHz and $\Omega_b/\hbar = \Lambda_b/\hbar = 2\pi \times 15$ MHz, which corresponds to simulated effective parameters of $\omega = \tilde{\omega} = 0$, $\tilde{g}_{cr} = \tilde{g}_{br} = 1$, $\tilde{g}_{cr} = 0.5$. For this simulation, the system is also initially prepared in the ground state of the whole system $|g, 0\rangle$, and the rest of the parameters are chosen to the same as for figure 5.
four sets of parameters with the initial state $|g, 0\rangle$ and $\omega_g = 2\pi \times 15$ GHz and $\omega_b = 2\pi \times 21$ GHz. The top row of figures 7(a)–(d) depicts the evolution of the WF of the field generated when $\Omega_r = \Lambda_r = 0$, $\Omega_b/h = \Lambda_b/h = 2\pi \times 15$ MHz, which corresponds to the population transfer between the states of $|g, 0\rangle$ and $|e, 1\rangle$, and the WF of the single photon Fock state is shown in figure 7(c) at time $\Delta \tau t/2\pi = 0.25$. The third row of figures 7(i)–(l) shows the evolution of the WF of the field generated when $\Omega_r/h = \Lambda_r = 0$, $\Omega_b/h = \Lambda_b/h = 2\pi \times 15$ MHz, which describes a mixture of two coherent states $|\pm \alpha\rangle$ with time-dependent displacement amplitude $\alpha(t) = -i \mathcal{G} r(t) [68]$. The amplitudes of the coherent states ideally increase linearly and practically, they will be prevented from diverging into instability by the damping of the oscillator and the finite duration of the evolution. It is noted that the small distortion of the WF from the ones of the ideal coherent state is due to a small deviation of our scheme from the effective ones for longer evolution time. The second row and the bottom row of figure 7 display the field properties with unbalanced and nonzero rotating and counter-rotating coupling terms in the degenerate AQRM, i.e., $\Omega_r/h = \Lambda_r = 2\pi \times 7.5$ MHz and $\Omega_b/h = \Lambda_b/h = 2\pi \times 15$ MHz for figures 7(e)–(h), and $\Omega_r/h = \Lambda_r/h = 2\pi \times 15$ MHz, $\Omega_b/h = \Lambda_b/h = 2\pi \times 7.5$ MHz for figures 7(m)–(p). In these two cases, both the rotating and counter-rotating terms contribute to the dynamics of the system, but unbalanced. An intuitive picture to understand these figures could be the following. Started from $|g, 0\rangle$, the photons spread independently along the even parity chain, and thus produce a qubit-resonator entangled state. Such an entangled state has the properties of the displaced squeezed states, whose squeezing parameters are functions of the relative ratio $\mathcal{G}_r/\mathcal{G}_g$.

5. Discussions and conclusions

So far, we have shown that the simulation scheme exhibits many distinguishing characteristics in good agreement with the ideal AQRM in equilibrium. However, in practice, the performance of engineering the artificial Hamiltonian is conditioned on other device parameters, including coherence time and microwave manipulation precision. In systems with good coherence, it is expected that although the decays certainly influence the fidelity of the target-state preparation and other properties of the system, the major qualitative features are still accessible [21]. The non-equilibrium dynamics and the non-equilibrium phase diagram of such a simulation scheme in the dissipation-dominated regime are also interesting and deserve investigation in the future. The possible experimental limitations imposed by the low coherence may be mitigated with fast growing experimental techniques in the future by employing a high-quality resonator and a dedicated qubit circuit, e.g., the C-shunt flux qubit [69], of which the decoherence time can be much improved by shunting a large capacitor to the smaller Josephson junction to reduce the effect of the charge noise [70]. The limitations on microwave manipulation precision may be surmounted by an offset frequency and subsequent drive pulses for phase corrections [16]. Our simulation scheme may serve as a starting point to experimentally investigate the properties of the AQRM in a well-controlled circuit QED system with superconducting flux qubits.

In summary, by manipulating flux qubits with bichromatic time-dependent magnetic fields, we propose an experimentally-accessible method to approach the physics of the AQRM in broad parameter ranges. With all-round tunability of the AQRM, we investigate its rich applications for QPTs from the perspective of information metric. Universal information like critical exponents can be well extracted from the scaling behavior of the fidelity susceptibility. Despite the differences in the finite anisotropy ($\lambda \neq 0$) and the frequency ratio, a fixed point of a cumulant ratio is predicted at the critical point of the QPTs and a universal scaling of the cumulant ratio is obtained with appropriate rescaling of the parameters. With numerical calculations we show that our proposal permits pursuing the extreme coupling regimes of the AQRM with a large value of the frequency ratio, which enables the possibility to investigate the quantum criticality in the AQRM. Hence, our scheme may in general serve as a favorable platform to explore the Rabi physics and testify the universal scaling of quantum critical phenomena in few-body systems. Especially, our scheme may also open the appealing possibility of experimentally exhibiting the gapless Nambu–Goldstone mode, which appears in the pure Jaynes–Cummings model with sufficiently strong coupling strength. This is forbidden in natural systems with very large coupling due to the failure of the rotating-wave approximation. Moreover, we find that our scheme acts well for the generation of the macroscopic Schrödinger cat states and the quantum controlled phase gates, and therefore, it may find promising applications in quantum simulation and quantum information processing.

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**References**

[1] Niemczyk T et al 2010 Nat. Phys. 6 772–6
[2] Form-Diaz P, Lisenthal J, Marcos D, Garcia-Ripoll J, Solano E, Harman C J P M and Moolij J E 2010 Phys. Rev. Lett. 105 237001
[3] Chen Z et al 2017 Phys. Rev. A 96 012325
[4] Yoshihara F, Fusse T, Ashhab S, Kakuyanagi K, Saito S and Semba K 2017 Nat. Phys. 13 44–7
[5] Form-Diaz P, Garcia-Ripoll J J, Peropadre B, Orgiazzi J L, Yurtalan M A, Belyansky R, Wilson C M and Lupascu A 2017 Nat. Phys. 13 39–43
[6] Rabi I I 1936 Phys. Rev. 49 324–8
[7] Braak D 2011 Phys. Rev. Lett. 107 100401
[8] Ridelof A, Leib M, Savasta S and Hartmann M J 2012 Phys. Rev. Lett. 109 193602
[9] Sanchez-Burillo E, Zueco D, Garcia-Ripoll J J and Martin-Moreno I 2014 Phys. Rev. Lett. 113 263604
[10] Hwang M J, Puebla R and Plenio M B 2015 Phys. Rev. Lett. 115 180404
[11] Romero G, Ballester D, Wang Y M, Scarani V and Solano E 2012 Phys. Rev. Lett. 108 120501
[12] Kawy T H, Felicetti S, Romero G, Solano E and Kwek L C 2015 Sci. Rep. 5 8621
[13] Wang Y, Zhang J, Wu C, You J and Romero G 2016 Phys. Rev. A 94 012328
[14] Wang Y, Guo C, Zhang G Q, Wang G and Wu C 2017 Sci. Rep. 7 44251
[15] Ballester D, Romero G, Garcia-Ripoll J J, Deppe F and Solano E 2012 Phys. Rev. X 2 021007
[16] Langford N K, Susskind L, Gnacinski M, Dickel C, Bruno A, Luthe F, Thoern D J, Eisa M and DiCarlo L 2017 Nat. Commun. 8 1715
[17] Braumüller J, Marthaler M, Schneider A, Stellli R, Rottinger H, Weides M and Ustinov A V 2017 Nat. Commun. 8 779
[18] Leroux C, Govia I C G and Clerk A A 2018 Phys. Rev. Lett. 120 093602
[19] Crespi A, Longhi S and Osellame R 2012 Phys. Rev. Lett. 108 163601
[20] Pedernales J S, Lizaun I, Felicetti S, Romero G, Lamata L and Solano E 2015 Sci. Rep. 5 15472
[21] Lv D, An S, Liu Z, Zhang J N, Pedernales J S, Garcia-Ripoll J J, Solano E, Harmans C J P M and Mooij J E 2010 Phys. Rev. A 96 033839
[22] Xie Q T, Cui S, Cao J P, Hwang M J and Plenio M B 2016 Phys. Rev. A 95 042382
[23] Liu M, Chee S, Ying Z J, Chen X, Luo H G and Lin H Q 2017 Phys. Rev. Lett. 119 220601
[24] Jaynes E T and Cummings F W 1963 Proc. IEEE 51 89–109
[25] Huang J F and Law C K 2015 Phys. Rev. A 91 023806
[26] Zhang G F and Zhu H J 2015 Sci. Rep. 5 8736
[27] Yu W and Liu M 2017 Phys. Rev. A 96 032123
[28] Wang Z H, Zheng Q, Wang X and Li Y 2016 Sci. Rep. 6 22347
[29] Hwang M J and Plenio M B 2016 Phys. Rev. Lett. 117 130602
[30] Larson J and Irish E K 2017 J. Phys. A: Math. Theor. 50 174002
[31] Wang Y and Haw J Y 2015 Phys. Lett. A 379 779–86
[32] Dai K, Wu H, Zhao P, Li M, Liu Q, Xue G, Tan X, Yu H and Yu Y 2017 Appl. Phys. Lett. 111 242601
[33] Baksi A and Ciuti C 2014 Phys. Rev. Lett. 112 173601
[34] Schliemann J, Eiges J C and Loss D 2003 Phys. Rev. B 67 085302
[35] Yang W J and Wang X B 2017 Phys. Rev. A 95 043823
[36] Fedorchenko S, Felicetti S, Markovic D, Jezouin S, Keller A, Coudreau T, Huard B and Milman P 2017 Phys. Rev. A 95 042313
[37] Fan J, Yang Z, Zhang Y, Ma J, Chen G and Jia S 2014 Phys. Rev. A 89 023812
[38] Georgescu I M, Ashhab S and Nori F 2014 Rev. Mod. Phys. 86 153–85
[39] Orlando T P, Moosj E, Tian L, van der Wal C H, Levitov L S, Lloyd S and Mazo J J 1999 Phys. Rev. B 60 15398–413
[40] Liu X Y, Wei L F, Johansson J R, Tsai J S and Nori F 2007 Phys. Rev. B 76 144518
[41] Huang S Y and Goan H S 2014 Phys. Rev. A 90 012318
[42] Qiu Y, Xiong W, He X L, Li T F and You J Q 2016 Sci. Rep. 6 28622
[43] Wei B B and Lv X C 2018 Phys. Rev. A 97 012345
[44] You W L, Li Y W and Gu S J 2007 Phys. Rev. E 76 022101
[45] Quan H T, Song Z, Liu X F, Zanardi P and Sun C P 2006 Phys. Rev. Lett. 96 140604
[46] Zanardi P and Paunkovic N 2006 Phys. Rev. E 74 031123
[47] Gu S J 2010 Int. J. Mod. Phys. B 24 4171–418
[48] You W L and He L 2015 J. Phys.: Condens. Matter 27 205601
[49] Yoshihara F, Nakamura Y, Yan F, Gustavsson S, Bylander J, Oliver W D and Tsai J S 2014 Phys. Rev. B 89 020503
[50] Miyaji M, Numasawa T, Shiba N, Takayanagi T and Watanabe K 2015 Phys. Rev. Lett. 115 261602
[51] Gan W C and Shu F W 2017 Phys. Rev. D 96 026008
[52] Susskind L 2016 Fortschr. Phys. 64 24–43
[53] Susskind L 2016 Fortschr. Phys. 64 49–71
[54] Kwok H M, Ning W Q, Gu S J and Lin H Q 2008 Phys. Rev. E 78 032103
[55] Gu S J, Kwok H M, Ning W Q and Lin H Q 2008 Phys. Rev. B 77 245109
[56] Yu C W, Kwok H M, Cao J and Gu S J 2009 Phys. Rev. E 80 021108
[57] Continetal M A 2001 Quantum Scaling in Many-Body Systems 1st edn (Singapore: World Scientific)
[58] Schwandt D, Alet F and Capponi S 2009 Phys. Rev. Lett. 103 170501
[61] Binder K 1981 Phys. Rev. Lett. 47 693–6
[62] Binder K 1981 Z. Phys. B 43 119–40
[63] Wiseman H M and Milburn G J 1993 Phys. Rev. A 47 642–62
[64] Mallet F, Castellanos-Beltran M A, Ku H S, Glancy S, Knill E, Irwin K D, Hilton G C, Vale L R and Lehnert K W 2011 Phys. Rev. Lett. 106 220502
[65] Puebla R, Hwang M J, Casanova J and Plenio M B 2017 Phys. Rev. Lett. 118 073001
[66] Leghtas Z, Kirchmair G, Vlastakis B, Schoelkopf R J, Devoret M H and Mirrahimi M 2013 Phys. Rev. Lett. 111 120501
[67] Olek N et al 2016 Nature 536 441–5
[68] Ashhab S and Nori F 2010 Phys. Rev. A 81 042311
[69] Yan F et al 2016 Nat. Commun. 7 12964
[70] You J Q, Hu X, Ashhab S and Nori F 2007 Phys. Rev. B 75 140515