PERIODIC MODULES OVER GORENSTEIN LOCAL RINGS

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Abstract. It is proved that the minimal free resolution of a module \( M \) over a Gorenstein local ring \( R \) is eventually periodic if, and only if, the class of \( M \) is torsion in a certain \( \mathbb{Z}[t^{\pm 1}] \)-module associated to \( R \). This module, denoted \( J(R) \), is the free \( \mathbb{Z}[t^{\pm 1}] \)-module on the isomorphism classes of finitely generated \( R \)-modules modulo relations reminiscent of those defining the Grothendieck group of \( R \). The main result is a structure theorem for \( J(R) \) when \( R \) is a complete Gorenstein local ring; the link between periodicity and torsion stated above is a corollary.

1. Introduction

This work makes a contribution to the study of eventually periodic modules over (commutative, Noetherian) local rings. In 1990, Avramov \cite{Avramov1990} posed the problem of characterizing rings that have a periodic module. Eisenbud \cite{Eisenbud1989} had previously shown that every complete intersection ring has a periodic module, but the question remained unanswered for other rings, even those which are Gorenstein. In Corollary \ref{cor:complete}, we prove that a complete Gorenstein local ring \( R \) has a periodic module if and only if there is torsion in a certain \( \mathbb{Z}[t^{\pm 1}] \)-module associated to \( R \), where \( \mathbb{Z}[t^{\pm 1}] \) denotes the ring of Laurent polynomials. This module, which we denote \( J(R) \), is the free \( \mathbb{Z}[t^{\pm 1}] \)-module on the isomorphism classes of finitely generated \( R \)-modules modulo relations reminiscent of those defining the Grothendieck group of \( R \); see Definition \ref{def:J} and Proposition \ref{prop:J}

The main result of this paper is a structure theorem for \( J(R) \) when \( R \) is a Gorenstein local ring with the Krull-Remak-Schmidt property; see Theorem \ref{thm:structure}. As a corollary, we deduce that an \( R \)-module is eventually periodic if and only if its class in \( J(R) \) is annihilated by some non-zero element of \( \mathbb{Z}[t^{\pm 1}] \). This leads to a characterization of hypersurface rings in terms of \( J(R) \); see Corollary \ref{cor: hypersurface}

This paper is motivated by work of D.R. Jordan \cite{Jordan2018}, who defined the module \( J(R) \) and proved that if the class of a module in \( J(R) \) is torsion then the module has a rational Poincaré series. The converse, however, does not hold. Indeed, Jordan proved that, for an Artinian complete intersection ring \( R \) with codimension at least two, the class of its residue field is not torsion in \( J(R) \). Corollary \ref{cor: Jordan} contains this result, since the residue field of a complete intersection ring is eventually periodic if and only if the codimension is at most one.

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2. The module $J(R)$

All rings considered in this paper are commutative and Noetherian. Let $R$ be a ring and $\mathcal{C}(R)$ the set of isomorphism classes of finitely generated $R$-modules; write $[M]$ for the class of an $R$-module $M$ in $\mathcal{C}(R)$. When the ring is clear from context, we write $\mathcal{C}$ instead of $\mathcal{C}(R)$.

Definition 2.1. Let $F$ be the free $\mathbb{Z}[t^{\pm 1}]$-module $\mathbb{Z}[t^{\pm 1}]^{(C)}$, that is,

$$F = \bigoplus_{[M] \in \mathcal{C}} \mathbb{Z}[t^{\pm 1}][M],$$

and let $I$ be the $\mathbb{Z}[t^{\pm 1}]$-submodule generated by the following elements:

1. $(R1)$ $[M] - [M']$ for every exact sequence of finitely generated $R$-modules $0 \to P \to M \to M' \to 0$ with $P$ projective;
2. $(R2)$ $[M] - t[M']$ for every exact sequence of finitely generated $R$-modules $0 \to M' \to P \to M \to 0$ with $P$ projective;
3. $(R3)$ $[M + M'] - [M] - [M']$ for all finitely generated $R$-modules $M$ and $M'$.

The main object of study in this article is the $\mathbb{Z}[t^{\pm 1}]$-module:

$$J(R) = F/I.$$

In the following remark, we make a few observations about the module $J(R)$.

Remark 2.2. Let $M, M'$, and $P$ be finitely generated $R$-modules with $P$ projective.

1. $[P] = 0$ in $J(R)$.
2. If $0 \to M \to M' \to P \to 0$ is exact, then $[M] - [M'] = 0$ in $J(R)$.

Indeed, for (1), note that there is an exact sequence $0 \to P = P = 0$, and so the desired result follows from $(R1)$.

To prove (2), notice that $M' \cong M \oplus P$ since $P$ is projective. Then in $J(R)$, $[M'] = [M] + [P]$ by $(R3)$. Since $[P] = 0$ in $J(R)$, it follows that $[M'] = [M]$.

The module $J(R)$ was defined by D.R. Jordan in [1] and called the Grothendieck module. In Jordan’s definition, the submodule $I$ is generated by four types of elements: the three given in Definition 2.1 as well as elements of the form $[M] - [M']$ where $M$ and $M'$ are modules as in Remark 2.2 (2).

Remark 2.3. The Grothendieck group $\mathcal{G}$ of a ring $R$ is the free $\mathbb{Z}$-module $\mathbb{Z}^{(C)}$ modulo the subgroup generated by the Euler relations, that is, elements of the form $[M'] - [M] + [M'']$ for each exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely generated $R$-modules. The reduced Grothendieck group $\overline{\mathcal{G}}$ of $R$ is the group $\mathcal{G}$ modulo the subgroup generated by classes of modules of finite projective dimension. We note that $\overline{\mathcal{G}} = J(R)/L$, where $L$ is the submodule generated by the Euler relations.

Syzygies. In order to discuss syzygies, we recall Schanuel’s Lemma; a proof can be found in [12] Thm 4.1.A.

Schanuel’s Lemma. Given exact sequences of $R$-modules

$$0 \to K \to P \to M \to 0 \quad \text{and} \quad 0 \to K' \to P' \to M \to 0$$

with $P$ and $P'$ projective, there is an isomorphism $K \oplus P' \cong K' \oplus P$ of $R$-modules. $\square$

Let $R$ be a ring and $M$ an $R$-module. Denote by $\Omega_R M$ any $R$-module that is the kernel of a homomorphism of $R$-modules $P \to M$ with $P$ a finitely generated projective. While $\Omega_R M$ depends on the choice of $P$, Schanuel’s Lemma shows that
$M$ determines $\Omega_RM$ up to a projective summand. Any module isomorphic to a module $\Omega_RM$ is called a syzygy of $M$. For any $d > 1$, a $d$th syzygy of $M$ is a module $\Omega^d_RM$ such that $\Omega^d_RM = \Omega_R(\Omega^{d-1}_R M)$ for some $(d-1)$st syzygy of $M$. By Schanuel’s Lemma, $\Omega^d_RM$ is also determined by $M$ up to a projective summand. For any $n \geq 0$, we write $\Omega^nM$ when the ring is clear from context.

The syzygy gives a well-defined functor on $J(R)$, as shown in Lemma 2.5. The following remark will aid in this discussion.

**Remark 2.4.** If $0 \to P \to M \to M' \to 0$ is an exact sequence of $R$-modules with $P$ projective, then there is a module that is a syzygy of both $M$ and $M'$.

Indeed, pick a surjective map $G' \to M'$, with $G'$ a projective $R$-module. Consider the following diagram. Let $X$ be the pullback of $M \to M'$ and $G' \to M'$. Since $G' \to M'$ is surjective, $X \to M$ is also surjective. Since $G'$ and $P$ are projective, $X$ is projective. Hence the kernel of $X \to M$ is a syzygy of $M$; let $N$ be this kernel. Let $N'$ denote the kernel of $G \to M'$. Then there is a commutative diagram with exact rows as follows.

$$
\begin{array}{cccccc}
0 & \to & P & \to & M & \to & M' & \to & 0 \\
0 & \to & P & \to & X & \to & G' & \to & 0 \\
\end{array}
$$

This justifies the claim.

**Lemma 2.5.** Assigning $[M]$ to $[\Omega M]$ induces a $\mathbb{Z}[t^{\pm 1}]$-linear map

$$
\Omega : J(R) \to J(R).
$$

**Proof.** By Schanuel’s Lemma, the assignment $[M] \mapsto [\Omega M]$ gives a homomorphism

$$
\hat{\Omega} : \bigoplus_{[M] \in C} \mathbb{Z}[t^{\pm 1}][M] \to J(R)
$$

of $\mathbb{Z}[t^{\pm 1}]$-modules. It is enough to check that (R1), (R2), and (R3) from Definition 2.1 are in $\text{Ker}(\hat{\Omega})$, so that $\hat{\Omega}$ factors through $J(R)$; the induced map is $\Omega$.

For (R3), note that for any syzygies $\Omega M$ of $M$ and $\Omega M'$ of $M'$, the $R$-module $\Omega M \oplus \Omega M'$ is a syzygy of $M \oplus M'$. Since $[\Omega M \oplus \Omega M'] = [\Omega M] + [\Omega M']$ in $J(R)$, one finds that $\hat{\Omega}([M \oplus M']) = \hat{\Omega}([M]) \oplus \hat{\Omega}([M'])$.

Next, we consider (R2). Given an exact sequence of finitely generated $R$-modules $0 \to M' \to P \to M \to 0$ with $P$ projective, we show that $[M'] - t^{-1}[M]$ is in $\text{Ker}(\hat{\Omega})$. In $J(R)$, one has $\hat{\Omega}([M']) = t^{-1}[M']$. Since $M'$ is a syzygy of $M$, we have $[M'] = [\Omega M]$ in $J(R)$. As $[\Omega M] = \hat{\Omega}([M])$, the $\mathbb{Z}[t^{\pm 1}]$-linearity of $\hat{\Omega}$ implies that

$$
\hat{\Omega}([M']) = t^{-1}\hat{\Omega}([M]) = \hat{\Omega}(t^{-1}[M]).
$$

Therefore (R2) is in $\text{Ker}(\hat{\Omega})$.

Finally, we verify that (R1) is in $\text{Ker}(\hat{\Omega})$. Given an exact sequence of finitely generated $R$-modules $0 \to P \to M \to M' \to 0$ with $P$ projective, Remark 2.4 implies there is a module $L$ that is a syzygy of both $M$ and $M'$. Therefore $\hat{\Omega}([M]) = [L] = \hat{\Omega}([M'])$. □
For $i > 1$, define $\Omega^i : J(R) \to J(R)$ by $\Omega^i = \Omega \circ \Omega^{i-1}$. The next remark demonstrates a relationship in $J(R)$ between the class of a module and the classes of its syzygies.

**Remark 2.6.** Let $M$ be a finitely generated $R$-module. Then $[M] = t^n[\Omega^n M]$ in $J(R)$ for any $n \in \mathbb{N}$.

Indeed, (R2) implies that $t[\Omega M] = [M]$ in $J(R)$. Iterating this, one finds that $[M] = t^n[\Omega^n M]$ for all $n \in \mathbb{N}$.

In the next proposition, we give an alternate description of $J(R)$ which makes the relations in this module more transparent.

**Proposition 2.7.** Let $F$ be the free $\mathbb{Z}[t^{\pm 1}]$-module $\mathbb{Z}[t^{\pm 1}] \langle C \rangle$, and let $L$ be the $\mathbb{Z}[t^{\pm 1}]$-submodule generated by the following elements:

1. $[P]$ for every finitely generated projective $R$-module $P$;
2. $[M] - t[\Omega M]$ for every finitely generated $R$-module $M$;
3. $[M + M'] - [M] - [M']$ for all finitely generated $R$-modules $M$ and $M'$.

There is an isomorphism of $\mathbb{Z}[t^{\pm 1}]$-modules

$$J(R) \cong F/L.$$

**Proof.** Via a proof similar to the proof of Lemma 2.6, it can be shown that assigning $[M]$ to $[\Omega M]$ induces a $\mathbb{Z}[t^{\pm 1}]$-linear map $\Omega : F/L \to F/L$.

Let $\tilde{q} : F \to J(R)$ be the quotient map. We show that (R1'), (R2'), and (R3) are in Ker$(\tilde{q})$, and hence $\tilde{q}$ factors through the quotient $F/L$ via a map $\hat{q} : F/L \to J(R)$.

The elements given by (R1') are in Ker($\tilde{q}$) by Remark 2.2(1), and those from (R2') are in Ker($\tilde{q}$) by Remark 2.5. The elements given by (R3) are in Ker($\tilde{q}$) by the definition of $J(R)$.

Let $\bar{q} : F \to F/L$ be the quotient map. We show that (R1), (R2), and (R3) are in Ker$(\bar{q})$, and hence $\bar{q}$ factors through the quotient $J(R)$ by a map $\bar{p} : J(R) \to F/L$. Note that the elements given by (R3) are in Ker($\bar{q}$) by the definition of $L$. It remains to verify that (R1) and (R2) are in Ker$(\bar{p})$.

First, consider (R1). Let $0 \to P \to M \to M' \to 0$ be an exact sequence of finitely generated $R$-modules with $P$ projective. By (R2'), one has

$$[M] - [M'] = [M] - t[\Omega M']$$

in $F/L$ for any syzygy $\Omega M'$ of $M'$. Given a syzygy $\Omega M$ of $M$, Remark 2.4 shows that there is a projective $R$-module $G$ such that $\Omega M \oplus G$ is a syzygy of $M'$. Hence $[\Omega M'] = [\Omega M \oplus G] = [\Omega M]$ in $F/L$, and thus

$$[M] - [M'] = [M] - t[\Omega M] = 0$$

in $F/L$. Therefore (R1) is in Ker($\bar{p}$).

Finally, we show that (R2) is in Ker($\bar{p}$). Let $0 \to M' \to P \to M \to 0$ be an exact sequence of finitely generated $R$-modules. Then $M'$ is a syzygy of $M$, so $t[M'] - [M] = t[\Omega M'] - [M] = 0$ by (R2'). Hence (R2) is in Ker($\bar{p}$).

Note that $p \circ q$ is the identity map on $F/L$; thus $p$ is injective. Since $p$ is a quotient map and hence also surjective, $p$ is an isomorphism. □

Recall that a homomorphism of rings $\varphi : R \to S$ is flat if $S$ is flat as an $R$-module via $\varphi$. A straightforward argument yields the following result.
Lemma 2.8. Let \( \varphi : R \to S \) be a homomorphism of rings. When \( \varphi \) is flat, the assignment \([M] \mapsto [S \otimes_R M]\) induces a homomorphism of \( \mathbb{Z}[t^{\pm 1}]\)-modules
\[
J(\varphi) : J(R) \to J(S).
\]
\(\square\)

Finite projective dimension. In [11] Prop 3] Jordan proves the following: if \( R \) is a commutative local Noetherian ring and \( M \) a finitely generated \( R \)-module, then the projective dimension of \( M \) is finite if and only if \([M] = 0\) in \( J(R) \). In Proposition 2.12 we extend this result to all commutative Noetherian rings.

Definition 2.9. Let \((R, m, k)\) be a local ring with maximal ideal \( m \) and residue field \( k \), and let \( M \) be a finitely generated \( R \)-module. Set
\[
\beta_i(M) = \text{rank}_k \text{Tor}_i^R(M, k);
\]
this is the \( i \)th Betti number of \( M \). The Poincaré series of \( M \) is given by
\[
P_M^R(t) = \sum_{i=0}^{\infty} \beta_i(M) t^i
\]
viewed as an element in the formal power series ring \( \mathbb{Z}[t^{\pm 1}] \).

Let \( \mathbb{Z}(t) \) denote the ring of formal Laurent series, \( \mathbb{Z}[\lbrack t\rbrack][\frac{1}{t}] \); we view it as a module over \( \mathbb{Z}[t^{\pm 1}] \). Notice that \( \mathbb{Z}[t^{\pm 1}] \) is a \( \mathbb{Z}[t^{\pm 1}] \)-submodule of \( \mathbb{Z}(t) \).

The following proposition is [11] Lem 1. We include the statement here for ease of reference.

Proposition 2.10. Let \( R \) be a local ring. The assignment \([M] \mapsto P_M^R(t)\) induces a homomorphism of \( \mathbb{Z}[t^{\pm 1}]\)-modules
\[
\pi : J(R) \to \mathbb{Z}(t) / \mathbb{Z}[t^{\pm 1}].
\]
\(\square\)

Definition 2.11. An \( R \)-module \( M \) has finite projective dimension if an \( i \)th syzygy module \( \Omega^i M \) is projective for some \( i \geq 0 \); in this case, we write \( \text{pd}_R M < \infty \).

By Schanuel’s Lemma, an \( i \)th syzygy module is projective if and only if every \( i \)th syzygy module is projective. Observe that if \( \Omega^i M \) is projective, then \( \Omega^j M \) is projective for all \( j \geq i \). When \( R \) is local, an \( R \)-module \( M \) has finite projective dimension if and only if \( \beta_i(M) = 0 \) for \( i \gg 0 \); see [3] Cor 1.3.2.

The following proposition was proved in [11] Prop 3] for local rings.

Proposition 2.12. Let \( R \) be a commutative Noetherian ring and \( M \) a finitely generated \( R \)-module. Then \([M] = 0\) in \( J(R) \) if and only if the projective dimension of \( M \) is finite.

Proof. If the projective dimension of \( M \) is finite, then \([\Omega^n M] = 0\) for some \( n \in \mathbb{N} \). Hence \([M] = 0\) by Remark 2.6.

Suppose \([M] = 0\) in \( J(R) \). First, we consider the case when \( R \) is local. Using the homomorphism \( \pi \) from Proposition 2.10], one finds that \( P_R(M) \in \mathbb{Z}[t^{\pm 1}] \). Hence \( P_R(M) \) is a polynomial, and it follows that \( \beta_i(M) = 0 \) for \( i \gg 0 \). Thus the projective dimension of \( M \) is finite.

For a general ring \( R \), the map \( R \to R_m \) is flat for each maximal ideal \( m \). Lemma 2.8 gives a homomorphism \( J(R) \to J(R_m) \) with \([M] \mapsto [M_m] \). Thus \([M_m] = 0\) in \( J(R_m) \), and hence \( \text{pd}_R M_m < \infty \). Hence the projective dimension of \( M \) over \( R \) is finite by [2] Thm 4.5]. \( \square \)
3. **MCM modules over Gorenstein local rings**

In this section, we collect known, but hard to document, properties of MCM modules over Gorenstein local rings.

For the remainder of this article, let \( R \) be a local ring with residue field \( k \) and \( M \) a finitely generated \( R \)-module. Set \((-)^* = \text{Hom}_R(-, R)\).

A **free cover** of \( M \) \cite[Def 5.1.1]{5} is a homomorphism \( \varphi : G \to M \) with \( G \) a free \( R \)-module such that

1. for any homomorphism \( g : G' \to M \) with \( G' \) free there exists a homomorphism \( f : G' \to G \) such that \( g = \varphi f \), and
2. any endomorphism \( f \) of \( G \) with \( \varphi = \varphi f \) is an automorphism.

A free cover is unique up to isomorphism.

Let \( \nu_R(M) \) denote the minimal number of generators of an \( R \)-module \( M \), i.e., \( \nu_R(M) = \text{rank}_k(k \otimes_R M) \).

**Remark 3.1.** Every \( R \)-module admits a free cover. A homomorphism \( \varphi : R^n \to M \) is a free cover of \( M \) if and only if \( \varphi \) is surjective and \( n = \nu_R(M) \).

A **free envelope** of \( M \) \cite[Def 6.1.1]{5} is a homomorphism \( \varphi : M \to G \) with \( G \) a free \( R \)-module such that

1. for any homomorphism \( g : M \to G' \) with \( G' \) free there exists a homomorphism \( f : G \to G' \) such that \( g = \varphi f \), and
2. any endomorphism \( f \) of \( G \) with \( \varphi = \varphi f \) is an automorphism of \( G \).

A free envelope is unique up to isomorphism.

**Remark 3.2.** Every finitely generated \( R \)-module \( M \) admits a free envelope. Indeed, the homomorphism \( f = (f_1, \ldots, f_n) : M \to R^n \), where \( f_1, \ldots, f_n \) is a minimal system of generators of \( M^* \), is a free envelope of \( M \).

The free envelope of \( M \) can also be constructed as follows. Let \( R^n \to M^* \) be the free cover of \( M^* \). Applying \((-)^* \) to this map, one has an injection \( M^{**} \to R^n \). The composite map \( M \to M^{**} \to R^n \) is the free envelope of \( M \), where \( M \to M^{**} \) is the natural biduality map.

**Remark 3.3.** For a local ring \( R \), one can choose \( \Omega M \) so that it is unique up to isomorphism by selecting \( \Omega M = \text{Ker}(\varphi) \) for a free cover \( \varphi \) of \( M \). Hence from this section on, \( \Omega(-) \) is well-defined, up to isomorphism, on the category of \( R \)-modules.

**Definition 3.4.** The **cosyzygy module** of \( M \) is \( \Omega_R^{-1}M = \text{Coker}(\varphi) \), where \( \varphi \) is the free envelope of \( M \). For \( n > 1 \), the **\( n \)th cosyzygy module** of \( M \) is

\[
\Omega_R^{-n}M = \Omega_R^{-1}(\Omega_R^{-n-1}M).
\]

In \cite[Sect 8.1]{5}, the authors refer to the cosyzygy module as the **free cosyzygy module**; since this is the only cosyzygy module studied in this article, we simply call it the cosyzygy module. We note that, when the module \( M \) is torsion-free, the cosyzygy module is also called the pushforward; see \cite{3}.

**Maximal Cohen-Macaulay modules.** A non-zero \( R \)-module \( M \) is said to be **maximal Cohen-Macaulay** (abbreviated to MCM) if \( \text{depth}_R M = \dim R \).

If \( R \) is Cohen-Macaulay, the set of isomorphism classes of MCM, non-free, indecomposable modules generates \( J(R) \) as a module over \( \mathbb{Z}[t^{\pm 1}] \) since \cite[Prop 1.2.9]{5} implies that each \( R \)-module has a syzygy that is either MCM or zero. If \( R \) is
Gorenstein and has the Krull-Remak-Schmidt property, one can do better: \( J(R) \) is generated over \( \mathbb{Z} \) by the isomorphism classes of MCM, non-free, indecomposable modules; see Theorem 3.2.

The ring \( R \) is Gorenstein if it has finite injective dimension as a module over itself. Equivalently, \( R \) is Gorenstein provided it is Cohen-Macaulay and \( \text{Ext}_R^i(M, R) = 0 \) for all MCM modules \( M \) and all \( i \geq 1 \); this equivalence can be seen from [10, Satz 2.6] and [3, Prop 3.1.10].

For the remainder of this article, we focus on Gorenstein rings. The following are well-known results on MCM modules that will be used throughout the paper; for lack of adequate references, some of the proofs are given here.

**Remark 3.5.** Let \( R \) be a Gorenstein local ring and \( M \) an MCM \( R \)-module.

1. Let \( N \) be an \( R \)-module. If \( d \geq \dim R \), then \( \Omega^d N \) is MCM or zero.
2. The natural homomorphism \( M \to M^{**} \) is an isomorphism.
3. The free envelope of \( M \) is an injective homomorphism.
4. The modules \( \Omega M \) and \( \Omega^{-1} M \) are MCM.
5. \( (\Omega^{-1} M)^* \cong \Omega(M^*) \).
6. If \( M \) is indecomposable, then \( \Omega M \) and \( \Omega^{-1} M \) are also indecomposable.
7. If \( M \) has no free summands, then the modules \( \Omega M \) and \( \Omega^{-1} M \) also have no free summands.
8. If \( M \) has no free summands, then \( \Omega^{-n} \Omega^n M \cong M \) for all \( n \in \mathbb{Z} \).

Property (1) follows from the Depth Lemma [3, Prop 1.2.9]. Property (2) is proved in [15, Cor 2.3]. For (4), a proof that \( \Omega M \) is an MCM module is given in [8, Lem 1.3] and [9, Prop 1.6.(2)] shows that \( \Omega^{-1} M \) is MCM.

**Proof of (3).** The free envelope of \( M \) is the composition

\[ M \to M^{**} \to F^* \]

where \( F \to M^* \) is the free cover of \( M^* \). So (3) follows from (2).

**Proof of (4).** Let \( \pi : F \to M^* \) be the free cover of \( M^* \). Since the natural map \( M \to M^{**} \) is an isomorphism, \( \pi^* : M \to F^* \) is the free envelope of \( M \) by Remark 3.2. Thus \( \Omega^{-1} M \) is defined by an exact sequence

\[ 0 \to M \xrightarrow{\pi^*} F^* \to \Omega^{-1} M \to 0. \]

Applying \((-)^*\) to this sequence yields the exact sequence

\[ 0 \to (\Omega^{-1} M)^* \to (\Omega^{-1} M)^* \to M^* \to 0. \]

As \( \pi \) is the free cover of \( M^* \), one gets \( \Omega(M^*) \cong (\Omega^{-1} M)^* \).

**Proof of (5).** A proof that \( \Omega M \) is indecomposable is given in [8, Lem 1.3]. We prove that \( \Omega^{-1} M \) is indecomposable. Let \( G \) be the free envelope of \( M \). By (4), the following sequence is exact:

\[ 0 \to M \to G \to \Omega^{-1} M \to 0. \]

Since \( \Omega^{-1} M \) is MCM, applying \((-)^*\) to this sequence yields the exact sequence

\[ 0 \to (\Omega^{-1} M)^* \to G^* \to M^* \to 0. \]
Proposition 3.6. Let $M$ be an $R$-module. Since $M$ is indecomposable and isomorphic to $M^*$, it follows that $M^*$ is indecomposable. As $M^*$ is an indecomposable MCM module, $(\Omega^{-1}M)^*$ is indecomposable by the result for syzygies. Thus $\Omega^{-1}M$ is also indecomposable.

Proof of (1). Suppose $\Omega M \cong N \oplus R$. Let $G$ be the free cover of $M$, and let $X$ be the pushout of $N \oplus R \to R$ and $N \oplus R \to G$. Then we have the following commutative diagram with exact rows.

\[
\begin{array}{c}
0 \to N \oplus R \to G \to M \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \to R \to X \to M \to 0
\end{array}
\]

Since $M$ is MCM one has $\operatorname{Ext}^1_R(M, R) = 0$, so $X \cong M \oplus R$. Then $\nu_R(G) \geq \nu_R(M) + 1$ as $G$ maps onto $M \oplus R$. However, this is a contradiction since $G$ is the free cover of $M$. Hence $\Omega M$ has no free summand.

Since $M$ has no free summand, $M^*$ has no free summand. By property (1) and the result for syzygies, $(\Omega^{-1}M)^*$ has no free summand. Hence $\Omega^{-1}M$ also has no free summand.

Proof of (2). First, note that $M \cong \Omega(\Omega^{-1}M) \oplus F'$ for some free module $F'$ by Schanuel’s Lemma. Since $M$ has no free summands, $M \cong \Omega(\Omega^{-1}M)$. Next, we show that $\Omega^{-1}(\Omega M) \cong M$; the result then follows by induction on $n$.

Let $\Omega M \to G$ be the free envelope of $\Omega M$, and let $G' \to M$ be the free cover of $M$. We have the following commutative diagram.

\[
\begin{array}{c}
0 \to \Omega M \overset{i}{\to} G \overset{f}{\to} \Omega^{-1}(\Omega M) \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \to \Omega M \overset{j}{\to} G' \to M \to 0
\end{array}
\]

Indeed, since $M$ is MCM there are maps $f : G' \to G$ and $g : G \to G'$ such that $f \circ j = i$ and $g \circ i = j$. Then $i = is(f \circ g)$, and $f \circ g$ is an isomorphism since $i$ is the free envelope of $\Omega M$. Hence $f : G' \to G$ is surjective. Thus the map $M \to \Omega^{-1}(\Omega M)$ is also surjective. Note that the kernels of $G' \to G$ and $M \to \Omega^{-1}(\Omega M)$ are isomorphic by the Snake Lemma. Since $\Omega^{-1}(\Omega M)$ is MCM and $K$ is free, $M \cong \Omega^{-1}(\Omega M) \oplus K$. Since $M$ has no free summands, $M \cong \Omega^{-1}(\Omega M)$.

Proposition 3.6. Let $R$ be a Gorenstein local ring and $M$ a finitely generated $R$-module.

1. If $M$ is an MCM module, then $t^{-n}[\Omega^{-n}M] = [M]$ in $J(R)$ for each $n \in \mathbb{Z}$.
2. There is an MCM $R$-module $N$ with $[M] = [N]$ in $J(R)$.

Proof. Remark 2.6 showed (1) for $n \leq 0$. Using Remark 3.3 (3), a proof similar to that of Remark 2.6 yields the desired result.

For (2), let $d = \dim R$. By Remark 2.6, $t^d[\Omega^dM] = [M]$. By (1), $t^{-d}[\Omega^{-d}\Omega^dM] = [\Omega^dM]$.
Hence $[M] = [\Omega^{-d} \Omega^d M]$. □

4. Gorenstein local rings: structure of $J(R)$

The main result of this section is a structure theorem for $J(R)$ when $R$ is a Gorenstein local ring; see Theorem 4.2.

Throughout this section $R$ will be a Gorenstein local ring. Recall that $\Omega^n(\_\_)$ denotes the $n$th (co)syzygy module, which is well-defined up to isomorphism.

Recall that a local ring $R$ is said to have the Krull-Remak-Schmidt property if the following condition holds: given an isomorphism of finitely generated $R$-modules

$$
\bigoplus_{i=1}^m M_i \cong \bigoplus_{j=1}^n N_j
$$

where $M_i$ and $N_j$ are indecomposable and non-zero, $m = n$ and, after renumbering if necessary, $M_i \cong N_i$ for each $i$.

Henselian local rings, and in particular complete local rings, have the Krull-Remak-Schmidt property; see [14, Thm 1.8] and [14, Cor 1.9].

Remark 4.1. In order to set up notation for the next theorem, we first discuss a special type of $\mathbb{Z}[t^{\pm 1}]$-module. For this, we view $\mathbb{Z}[t^{\pm 1}]$ as the group algebra over $\mathbb{Z}$ of the free group $G = \langle t \rangle$ on a single generator $t$; that is, $G \cong (\mathbb{Z}, +)$. Let $X$ be a set with a $G$-action. Let $\mathbb{Z}X = \mathbb{Z}(X)$, the free $\mathbb{Z}$-module with basis given by the elements of $X$, and let $\mathbb{Z}G$ be the group algebra over $\mathbb{Z}$ of $G$. Then $\mathbb{Z}X$ is naturally a $\mathbb{Z}G$-module [13, Ch.III, §1].

In what follows, we let

$$
\mathcal{M}(R) = \left\{ [M] \in \mathcal{C}(R) \mid M \text{ is MCM, non-free, and indecomposable} \right\}.
$$

When $R$ is clear from context, we write $\mathcal{M}$ for $\mathcal{M}(R)$.

Remark 3.5 properties (4), (6), and (7) imply that $[\Omega M]$ and $[\Omega^{-1} M]$ are in $\mathcal{M}$ if $[M] \in \mathcal{M}$. Thus there is an action of $G$ on $\mathcal{M}$ with $t[M] = [\Omega^{-1} M]$ and $t^{-1}[M] = [\Omega M]$. Let $\mathcal{A} = \mathcal{Z}(\mathcal{M})$ be the corresponding $\mathbb{Z}[t^{\pm 1}]$-module. The canonical map $\mathcal{M} \to J(R)$ induces a $\mathbb{Z}[t^{\pm 1}]$-linear homomorphism:

$$
\Phi : \mathcal{A} \to J(R).
$$

Assume $R$ has the Krull-Remak-Schmidt property. We define a $\mathbb{Z}[t^{\pm 1}]$-linear homomorphism

$$
\psi : \bigoplus_{[M] \in \mathcal{C}} \mathbb{Z}[t^{\pm 1}][M] \longrightarrow \mathcal{A} \tag{4.1}
$$

by setting $\psi([M]) = \sum_{i=1}^n [M_i]$, where each $M_i$ is indecomposable and

$$
\Omega^{-(d+1)} \Omega^{d+1} M \cong \bigoplus_{i=1}^n M_i
$$

with $d = \dim R$. Since $\Omega^{-(d+1)} \Omega^{d+1} M$ is either zero or MCM with no free summands, $\sum_{i=1}^n [M_i]$ is indeed in $\mathcal{A}$. Since $R$ has the Krull-Remak-Schmidt property, $\psi$ is well-defined.
Theorem 4.2. Let $R$ be a Gorenstein local ring that has the Krull-Remak-Schmidt property. Then the $\mathbb{Z}[t^{\pm 1}]$-linear map
\[ \Phi : \mathcal{A} \to J(R) \]
is an isomorphism with inverse $\Psi$ induced by $\psi$ described in (4.1).

Proof. To show that $\psi$ induces a $\mathbb{Z}[t^{\pm 1}]$-linear map $\Psi : J(R) \to \mathcal{A}$, it suffices to show that the elements described in (R1), (R2), and (R3) from Definition 3.1 are in the kernel of $\psi$.

For elements given by (R3), note that
\[ \Omega^{-d+1}(\Omega^d(M \oplus N)) \cong \Omega^{-d+1}(\Omega^dM \oplus \Omega^{-d+1}\Omega M^j N). \]
Then $\psi([M \oplus N]) = \psi([M]) + \psi([N])$, and hence (R3) is in Ker($\psi$).

Next, we consider (R2): given an exact sequence $0 \to M' \to P \to M \to 0$ of finitely generated $R$-modules with $P$ projective, we show that $\psi([M']) = \psi(t^{-1}[M])$. By Schanuel’s Lemma there exists a free $R$-module $G$ such that $M' \cong \Omega M \oplus G$. Since $\psi([G]) = 0$, we have $\psi([M']) = \psi([\Omega M])$ in $J(R)$. Next, we show that $\psi([\Omega M]) = \psi(t^{-1}[M])$. By Remark 3.5.(8).
\[ \Omega^{-d+1}(\Omega^dM) \cong \Omega(\Omega^{-d+1}\Omega^dM). \]
Note that $\Omega^{-d+1}\Omega^dM$ determines $\psi([\Omega M])$ and $\Omega(\Omega^{-d+1}\Omega^dM)$ determines $\psi(t^{-1}[M])$. Hence $\psi([\Omega M]) = \psi(t^{-1}[M])$, and therefore (R2) is in Ker($\psi$).

It remains to verify that (R1) is in Ker($\psi$). Let $0 \to P \to M \to M' \to 0$ be an exact sequence of $R$-modules with $P$ projective. By Remark 2.4, there are free $R$-modules $G$ and $G'$ such that $\Omega M \oplus G \cong \Omega M' \oplus G'$. Since $\psi([G]) = \psi([G']) = 0$, $\psi([\Omega M] = \psi([\Omega M']).$ As (R2) is in Ker($\psi$), one finds that $\psi(t^{-1}[M]) = \psi(t^{-1}[M'])$ and thus $\psi([M]) = \psi([M'])$. Hence (R1) is in Ker($\psi$).

Thus $\psi$ factors through the quotient $J(R)$ via a homomorphism $\Psi : J(R) \to \mathcal{A}$. Notice that $\Psi \circ \Phi$ is the identity. Indeed, if $M$ is an MCM module with no free summands, then $\Omega^{-d+1}\Omega^dM \cong M$ by Remark 3.5.(8). Hence $\Phi$ is injective. For each $R$-module $N$, Proposition 3.6(2) shows that there is an MCM $R$-module $M$ such that $[M] = [N]$. Thus $\Phi$ is also surjective and hence an isomorphism. \(\square\)

Remark 4.3. If $R$ is Gorenstein, Theorem 4.2 implies that $J(R)$ is torsion-free as an abelian group. We do not know whether this holds for a general local ring $R$.

In [11] Lem 8], the following result is proved for Artinian Gorenstein rings.

Corollary 4.4. Let $R$ be a Gorenstein local ring, and let $M$ and $N$ be finitely generated MCM $R$-modules. Then $[M] = [N]$ in $J(R)$ if and only if
\[ M \oplus R^m \cong N \oplus R^n \]
for some $m,n \in \mathbb{Z}_{>0}$. Thus if neither $M$ nor $N$ has a free summand, $[M] = [N]$ in $J(R)$ if and only if $M \cong N$.

Proof. If $M \oplus R^m \cong N \oplus R^n$ for some $m,n \in \mathbb{Z}_{>0}$, then in $J(R)$ we have
\[ [M] = [M \oplus R^m] = [N \oplus R^n] = [N]. \]

Suppose that $[M] = [N]$ in $J(R)$. We may assume $M$ and $N$ have no free summands. We first prove the result under the assumption that $R$ is complete with respect to the maximal ideal. Complete rings have the Krull-Remak-Schmidt property for finitely generated modules; see for example [14 Cor 1.10]. Hence
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Theorem 4.2 applies. Let \( d = \dim R \), and let \( \Psi : J(R) \to \mathcal{A} \) be the isomorphism given in Theorem 4.2. Suppose

\[
M = \bigoplus_{[M_{\lambda}] \in \mathcal{M}} M_{\lambda}^{e_{\lambda}} \quad \text{and} \quad N = \bigoplus_{[M_{\lambda}] \in \mathcal{M}} M_{\lambda}^{f_{\lambda}}
\]

where \( e_{\lambda}, f_{\lambda} \geq 0 \). From Remark 5.5 (7) and the definition of \( \psi \) given in (4.1), one gets an equality

\[
\sum_{[M_{\lambda}] \in \mathcal{M}} e_{\lambda}[M_{\lambda}] = \Psi([M]) = \Psi([N]) = \sum_{[M_{\lambda}] \in \mathcal{M}} f_{\lambda}[M_{\lambda}].
\]

Since \( \mathcal{A} \) is free on \( \mathcal{M} \), we have \( e_{\lambda} = f_{\lambda} \) for all \( \lambda \). Therefore \( M \cong N \) as \( R \)-modules.

Now suppose that \( R \) is any local ring with maximal ideal \( m \). Write \( \hat{R} \) for the \( m \)-adic completion of \( R \). If \( [M] = [N] \) in \( J(R) \), then \( [M \otimes R \hat{R}] = [N \otimes R \hat{R}] \) in \( J(\hat{R}) \) by Lemma 2.8.

Note that an \( R \)-module \( M \) has a free summand if and only if the evaluation map \( ev : M^* \otimes R \to R \), where \( \varphi \otimes m \mapsto \varphi(m) \), is surjective. But if this map is surjective for \( M \), then the map \( ev \otimes R \hat{R} \) is also surjective. So since \( M \) and \( N \) have no free summands, \( M \otimes R \hat{R} \) and \( N \otimes R \hat{R} \) also have no free summands.

The result for complete rings then shows that \( M \otimes R \hat{R} \cong N \otimes R \hat{R} \) as \( \hat{R} \)-modules, and [13] Cor. 1.15 implies that \( M \cong N \).

Note that cancellation of direct summands is valid over local rings [14] Cor. 1.16]. Then \( M \oplus R^m \cong N \oplus R^n \) implies that \( M \oplus R^m' \cong N \) or \( M \cong N \oplus R^n' \). Thus if neither \( M \) nor \( N \) has a free summand, \( M \cong N \).

5. Gorenstein local rings: torsion in \( J(R) \)

Let \( R \) be a Gorenstein local ring. The main result of this section, Theorem 5.6, is that the class of a module is torsion in \( J(R) \) if and only if the module is eventually periodic. This result does not extend verbatim to Cohen-Macaulay local rings; see Example 5.11.

In the next lemma, we give a decomposition for the special type of \( \mathbb{Z}[t^{\pm 1}] \)-modules discussed in Remark 4.1.

**Lemma 5.1.** Let \( G = \langle t \rangle \), and let \( X \) be a set with a \( G \)-action. Then there is an isomorphism of \( \mathbb{Z}G \)-modules

\[
\mathbb{Z}X \cong \bigoplus_{n=1}^{\infty} \left( \frac{\mathbb{Z}[t]}{(t^n - 1)} \right)^{b_n} \oplus (\mathbb{Z}G)^{b_\infty}
\]

where \( b_{\infty}, b_n \in \mathbb{Z}_{\geq 0} \cup \{ \infty \} \) for all \( n \).

**Proof.** For any \( x \in X \), either \( t^n x \neq x \) for all \( n \neq 0 \) and the orbit of \( x \) is

\[
Gx = \{ t^i x : i \in \mathbb{Z} \},
\]

or there is an \( n > 0 \) with \( t^n x = x \) and

\[
Gx = \{ x, tx, t^2 x, \ldots, t^{n-1} x \}.
\]

Then for each \( x \in X \) either \( \mathbb{Z}Gx \cong \mathbb{Z}G \) or \( \mathbb{Z}Gx \cong \mathbb{Z}[t]/(t^n - 1) \) as \( \mathbb{Z}G \)-modules; in either case, the map assigning \( x \) to \( 1 \) induces an isomorphism. Thus the decomposition of \( X \) into orbits gives the desired isomorphism. \( \square \)
Recall that the torsion submodule of a \(ZG\)-module \(L\) is
\[
T_{ZG}(L) = \{ u \in L : ru = 0 \text{ for some } r \in ZG \setminus \{0\} \}.
\]
An element \(u \in T_{ZG}(L)\) is said to be a torsion element of \(L\).

**Proposition 5.2.** An element \(u \in ZX\) is torsion if and only if there exists an \(n \in \mathbb{N}\) such that \((t^n - 1)u = 0\).

**Proof.** Suppose \(u\) is torsion in \(ZX\). Identifying \(ZX\) with the right hand side of the isomorphism in Lemma 5.1, one finds that \(u\) belongs to the submodule \(\oplus_{n=1}^{\infty} (L_n)^{b_n}\) of \(ZX\) where
\[
L_n = \frac{\mathbb{Z}[t]}{(t^n - 1)}.
\]
Consider the case when \(u = v + w\) where \(v \in L_\ell\) and \(w \in L_m\) for some \(\ell, m \in \mathbb{N}\). Then \((t^\ell - 1)v = 0\) and \((t^m - 1)w = w\), and hence \((t^{\ell m} - 1)(v + w) = 0\), since \((t^\ell - 1)\) and \((t^m - 1)\) both divide \(t^{\ell m} - 1\). By induction on the number of terms in \(u\), there exists an \(n \in \mathbb{N}\) such that \((t^n - 1)u = 0\).

The reverse implication is immediate. \(\Box\)

In light of the preceding results, Theorem 4.2 has the following corollaries.

**Corollary 5.3.** Let \(R\) be a Gorenstein local ring that has the Krull-Remak-Schmidt property. The following statements hold.

1. An element \(u \in J(R)\) is torsion if and only if there exists an \(n \in \mathbb{N}\) such that \((t^n - 1)u = 0\).
2. The \(\mathbb{Z}[t^{\pm 1}]\)-module \(J(R)\) has nonzero torsion if and only if there is a finitely generated \(R\)-module \(M\) such that \([M]\) is torsion.

**Proof.** For (1), note that \(G = \langle t \rangle\) acts on \(M\). By Theorem 4.2, \(J(R) \cong A = ZM\) as \(\mathbb{Z}[t^{\pm 1}]\)-modules. The result then follows from Proposition 5.2.

To prove (2), suppose \(u\) is a nonzero torsion element of \(J(R)\). In the notation of Lemma 5.1, there is an \(n \in \mathbb{N}\) such that \(b_n \neq 0\). By Theorem 4.2, there are some \([M_n] \in \mathcal{M}\) that generate \(L_n\), and thus \([M_n]\) is torsion for each \(a\).

The reverse implication is immediate. \(\Box\)

**Torsion in \(J(R)\).** Let \((R, m)\) be a local ring. An \(R\)-module \(M\) is said to be periodic if there exists an \(n \in \mathbb{N}\) such that \(M \cong \Omega^n M\). The module \(M\) is said to be eventually periodic if there exists an \(n \in \mathbb{N}\) and \(\ell \in \mathbb{Z}_{\geq 0}\) such that \(\Omega^\ell M \cong \Omega^{n + \ell} M\).

In either case, the minimal such integer \(n\) is called the period of \(M\). We make some observations about torsion in \(J(R)\) and eventually periodic modules.

**Remark 5.4.** Let \(M\) be a finitely generated \(R\)-module. It is easy to see that the following statements hold.

1. \([M]\) is torsion in \(J(R)\) if and only if \([\Omega^n M]\) is torsion for some (equivalently, all) \(n \in \mathbb{N}\).
2. \(M\) is eventually periodic if and only if \(\Omega^n M\) is eventually periodic for some (equivalently, all) \(n \in \mathbb{N}\).

In what follows, we write \(\widehat{M}\) for the \(m\)-adic completion of the \(R\)-module \(M\).

**Lemma 5.5.** Let \(M\) be a finitely generated \(R\)-module, and let \(i, j \in \mathbb{Z}_{\geq 0}\). Then \(\Omega^n R \cong \Omega^n R\) if and only if \(\Omega^n \widehat{M} \cong \Omega^n \widehat{M}\). In particular, \(M\) is eventually periodic if and only if the \(\widehat{R}\)-module \(\widehat{M}\) is eventually periodic as an \(\widehat{R}\)-module.
Proof. Given that $\Omega^n_R M \cong \Omega^n_R M$, one has
$$\Omega^j_R(M) \cong \Omega^j_R(M) \cong \Omega^j_R(M) \cong \Omega^j_R(M).$$
Suppose that $\Omega^j_R M \cong \Omega^j_R M$. Then we have the following isomorphisms:
$$\Omega^j_R M \cong \Omega^j_R(M) \cong \Omega^j_R(M) \cong \Omega^j_R(M),$$
and so $\Omega^j_R M \cong \Omega^j_R M$ by [14 Cor 1.15].

**Theorem 5.6.** Let $R$ be a Gorenstein local ring, and let $M$ be a finitely generated $R$-module. Then $[M]$ is torsion in $J(R)$ with respect to the $\mathbb{Z}[t^{\pm 1}]$-action if and only if $M$ is eventually periodic. Moreover, for any $n \in \mathbb{N}$, the following conditions are equivalent:

1. $(t^n - 1)[M] = 0$ in $J(R)$.
2. $\Omega^f M \cong \Omega^{n+\ell} M$ for $\ell \gg 0$.

Proof. Suppose $M$ is eventually periodic. Then there are $i, j \in \mathbb{Z}_{\geq 0}$ with $i \neq j$ such that $\Omega^i M \cong \Omega^j M$. In $J(R)$, $t^{-i}[M] = [\Omega^i M] = [\Omega^j M] = t^{-j}[M]$, and hence $(t^{-i} - t^{-j})[M] = 0$.

Assume $[M]$ is torsion in $J(R)$. We first show that we can reduce to the case when $R$ is complete with respect to the maximal ideal $\mathfrak{m}$. Let $\hat{M}$ denote the $\mathfrak{m}$-adic completion of $M$. Since the canonical homomorphism $\varphi : R \to \hat{R}$ is flat, Lemma 2.8 implies that there is a homomorphism of $\mathbb{Z}[t^{\pm 1}]$-modules $J(\varphi) : J(R) \to J(\hat{R})$ with $J(\varphi)([M]) = [\hat{M}]$. Hence $[M]$ torsion implies that $[\hat{M}]$ is torsion. If the result holds for complete rings, then $\hat{M}$ is eventually periodic as an $\hat{R}$-module. Hence Lemma 5.5 implies that $M$ is eventually periodic as an $R$-module.

Assume $R$ is complete. To show that $M$ is eventually periodic, it is enough to show that some syzygy of $M$ is eventually periodic. We may assume $M$ is MCM with no free summands.

Remark 3.3 [1] implies that $\Omega^d M = 0$ or is MCM for $d \gg 0$. If $\Omega^d M = 0$, the proof is complete. If not, then replacing $M$ by $\Omega^d M$ we may assume that $M$ is MCM. If $M = N \oplus R$, then $[M] = [N]$ and hence $[M]$ is torsion in $J(R)$ if and only if $[N]$ is torsion. Note that $M$ is eventually periodic if and only if $N$ is eventually periodic, since $\Omega M \cong \Omega N$. Thus we may assume $M$ has no free summands.

As $[M]$ is torsion, Corollary 5.3 [1] implies that there is an $n \in \mathbb{N}$ such that $(t^n - 1)[M] = 0$ in $J(R)$. Proposition 5.3 [1] shows that $[\Omega^{-n} M] = [\Omega^n M] = [M]$. By Remark 3.3 [4], the $R$-module $\Omega^{-n} M$ is MCM, and thus Corollary 4.4 implies that $\Omega^{-n} M \cong \Omega^n M$ for some free $R$-modules $F$ and $G$. Then, as $R$-modules, $\Omega^n (\Omega^{-n} M \oplus F) \cong \Omega^n (M \oplus G)$, and thus $\Omega^n \Omega^{-n} M \cong \Omega^n M$. Since $M$ is MCM with no free summands, $M \cong \Omega^n \Omega^{-n} M$ by Remark 3.3 [6]. Hence $M \cong \Omega^n M$, and therefore $M$ is eventually periodic.

It is clear that (2) implies (1). The argument in the previous paragraph along with Lemma 3.3 [1] shows that (1) implies (2).

**Corollary 5.7.** Let $R$ be a Gorenstein local ring and $M$ a finitely generated $R$-module. Then $[M]$ is torsion in $J(R)$ with respect to the $\mathbb{Z}[t^{\pm 1}]$-action if and only if $[\hat{M}]$ is torsion in $J(\hat{R})$ with respect to the $\mathbb{Z}[t^{\pm 1}]$-action.

Proof. It follows from Lemma 2.8 and was already used in the proof of Theorem 5.6 that if $[M]$ is torsion in $J(R)$, then $[\hat{M}]$ is torsion in $J(\hat{R})$. 

□
The reverse implication is immediate from Theorem 5.6 and Lemma 5.0.

The following corollary gives the result announced in the abstract.

**Corollary 5.8.** Let $R$ be a Gorenstein local ring that has the Krull-Remak-Schmidt property. The ring $R$ has a periodic module if and only if $J(R)$ has nonzero torsion.

**Proof.** Corollary 5.8 and Theorem 5.6 give the desired result.

**Corollary 5.9.** Suppose $M = \oplus_{i=1}^{m} M_i$ for some $R$-modules $M_i$. Then $[M]$ is torsion in $J(R)$ if and only if $[M_i]$ is torsion in $J(R)$ for all $i$.

**Proof.** Assume $[M_i]$ is torsion in $J(R)$ for all $i$. For each $i \in \{1, \ldots, m\}$, there is an $f_i(t) \in \mathbb{Z}[t^{\pm 1}]$ such that $f_i(t) [M_i] = 0$ in $J(R)$. Then $f_1(t) \cdots f_m(t) [M] = 0$.

Suppose $[M]$ is torsion in $J(R)$. We first prove the result under the assumption that $R$ is complete. It suffices to consider the case when each $M_i$ is indecomposable. Since $[M]$ is torsion in $J(R)$, Theorem 5.6 implies that $M$ is eventually periodic. So there is an $n \in \mathbb{N}$ and an $\ell \in \mathbb{Z}_{\geq 0}$ such that $\Omega^{n+\ell} M \cong \Omega^{\ell} M$, and therefore

$$\bigoplus_{i=1}^{m} \Omega^{n+\ell}(M_i) \cong \bigoplus_{i=1}^{m} \Omega^{\ell}(M_i).$$

We prove that each $[M_i]$ is torsion by using induction on $m$, the number of indecomposable summands of $M$. Suppose $M = M_1 \oplus M_2$. Then by the Krull-Remak-Schmidt property, either $\Omega^{n+\ell}(M_i) \cong \Omega^{\ell}(M_i)$ for $i = 1, 2$ or $\Omega^{n+\ell}(M_1) \cong \Omega^{\ell}(M_2)$ and $\Omega^{n+\ell}(M_2) \cong \Omega^{\ell}(M_1)$. In the first case, it is clear that $M_1$ and $M_2$ are eventually periodic. In the second case, note that

$$\Omega^{2n+\ell}(M_1) \cong \Omega^{n+\ell}(M_2) \cong \Omega^{\ell}(M_1),$$

and hence $M_1$ is eventually periodic. Similarly, $M_2$ is eventually periodic. Then by Theorem 5.6 $[M_i]$ is torsion in $J(R)$ for $i = 1, 2$.

Suppose $M = \oplus_{i=1}^{m} M_i$ and that the conclusion holds for $s < m$. By the Krull-Remak-Schmidt property, for each $i$ there exists $j$ such that $\Omega^{n+\ell}(M_i) \cong \Omega^{\ell}(M_j)$. If there is an $i$ such that $\Omega^{n+\ell}(M_i) \cong \Omega^{\ell}(M_i)$, then the result follows from the inductive hypothesis. Without loss of generality, suppose $\Omega^{n+\ell}(M_i) \cong \Omega^{\ell}(M_{i+1})$ for $1 \leq i \leq m - 1$ and $\Omega^{n+\ell}(M_m) \cong \Omega^{\ell}(M_1)$. The following isomorphisms of $R$-modules show that $M_1$ is eventually periodic:

$$\Omega^{mn+\ell}(M_1) \cong \Omega^{(m-1)n+\ell}(M_2) \cong \cdots \cong \Omega^{n+\ell}(M_m) \cong \Omega^{\ell}(M_1).$$

Similarly $M_i$ is eventually periodic for $2 \leq i \leq m$, and consequently Theorem 5.6 implies that $[M_i]$ is torsion in $J(R)$ for all $i$.

Now suppose that $R$ is any local ring and $[M]$ is torsion in $J(R)$. By Corollary 5.7 $[M]$ is torsion in $J(R)$. By Corollary 5.7 $\widehat{M} \cong \bigoplus_{i=1}^{m} \left( \bigoplus_{j=1}^{a_i} M_{ij} \right)$ with $\widehat{M}_i = \bigoplus_{j=1}^{a_i} M_{ij}$ and each $M_{ij}$ an indecomposable $\widehat{R}$-module. The result for complete rings implies that $[M_{ij}]$ is torsion in $J(\widehat{R})$ for each $i$ and $j$. Then $[\widehat{M}_i]$ is torsion in $J(\widehat{R})$ for all $i$, and so Corollary 5.7 implies that $[M_i]$ is torsion in $J(R)$ for all $i$. 

\[\square\]
Theorem 5.6 also gives a characterization of hypersurface rings in terms of $J(R)$. The main result of [11, Thm 7] is that (3) implies (1) holds when $R$ is an Artinian complete intersection.

**Corollary 5.10.** Let $(R, m, k)$ be a Gorenstein local ring. Then the following conditions are equivalent:

1. $R$ is a hypersurface;
2. $(1 - t^2) \cdot J(R) = 0$;
3. $J(R)$ is a torsion module;
4. $[k]$ is torsion in $J(R)$ with respect to the $\mathbb{Z}[t^{\pm 1}]$-action.

**Proof.** (1) $\Rightarrow$ (2). For any module $M$ over a hypersurface one has $\Omega^{2+\ell} M \cong \Omega^{\ell} M$ for $\ell \gg 0$, by [4, Thm 6.1], and hence $(1 - t^2)[M] = 0$.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4). These implications are immediate.

(4) $\Rightarrow$ (1). By Theorem 5.6, the module $k$ is eventually periodic and hence the Betti numbers of $k$ are bounded. By [7, Cor 1], $R$ is a hypersurface. □

The following class of examples shows that the statement of Theorem 5.6 can fail for non-Gorenstein rings.

**Example 5.11.** Let $(R, m, k)$ be a local ring with $m^2 = 0$ and embedding dimension $e \geq 2$. Note that $R$ is Cohen-Macaulay but not Gorenstein because the rank of its socle (as a $k$-vector space) is $e$. Then $(1 - et)J(R) = 0$, but $R$ has no nonzero nonfree eventually periodic module.

First, we note that $k$ is not eventually periodic but $[k]$ is torsion in $J(R)$. Indeed, the sequence

$$0 \to m \to R \to k \to 0$$

is exact, and $\Omega k \cong m$ as $R$-modules. Therefore $\Omega k \cong k^e$, which implies that $k$ is not eventually periodic. On the other hand, $t^{-1}[k] = e[k]$ in $J(R)$, and therefore $(1 - et)[k] = 0$.

Let $M$ be a nonzero, nonfree $R$-module. Since $m^2 = 0$, we have $\Omega M \cong k^{\beta_1(M)}$. As $M$ is nonzero, $\beta_1(M) \geq 1$. Since $k$ is not eventually periodic, the module $M$ is not eventually periodic. However,

$$t^{-1}[M] = [\Omega M] = \beta_1(M)[k]$$

in $J(R)$, and therefore

$$(1 - et)[M] = t(1 - et)\beta_1(M)[k] = 0.$$ 

**Remark 5.12.** Using Corollary 5.3 [14], one can determine the torsion submodule of $J(R)$ for a Gorenstein local ring that has the Krull-Remak-Schmidt property:

$$T_{\mathbb{Z}[t^{\pm 1}]}(J(R)) = \bigcup_{n=1}^{\infty} \text{Ann}_{J(R)}(1 - t^n).$$

If $R$ is a complete intersection, then $T_{\mathbb{Z}[t^{\pm 1}]}(J(R)) = \text{Ann}_{J(R)}(1 - t^2)$ by Theorem 5.6 since [4, Thm 5.2] shows that a periodic module $M$ over a complete intersection has period at most two and hence $(1 - t^2)[M] = 0$ in $J(R)$. For a Gorenstein ring $R$, however, $[M]$ torsion in $J(R)$ for an $R$-module $M$ need not imply that $(1 - t^2)[M] = 0$. Indeed, for each $n \in \mathbb{N}$, there exists an Artinian Gorenstein local ring with a periodic module of period $n$; see [6, Ex 3.6].
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