The spin-parities of the 13.35 MeV state and high-lying excited states around 20 MeV in $^{12}$C nucleus

Alla Demyanova$^{1,a}$, Viktor Staratsin$^{1}$, Alexey Ogloblin$^{1}$, Andrey Danilov$^{1}$, Sergey Dmitriev$^{1}$, Władysław Trzaska$^{2}$, Pauli Heikkinen$^{3}$, Tatjana Belyaeva$^{3}$, Sergey Goncharov$^{4}$, Vladimir Maslov$^{5}$, Yuri Sobolev$^{5}$, Yury Gurov$^{6}$, Boris Chernyshev$^{6}$, Nassurlla Burtebaev$^{7,8}$, Daniyar Janseitov$^{5,7,8}$, Sergey Khlebnikov$^{9}$

$^{1}$ National Research Centre Kurchatov Institute, Akademika Kurchatova pl. 1, 123182 Moscow, Russia
$^{2}$ Department of Physics, University of Jyväskylä, Jyväskylä, Finland
$^{3}$ Universidad Autonoma del Estado de Mexico, Toluca, Mexico
$^{4}$ M.V. Lomonosov Moscow State University, Moscow, Russia
$^{5}$ Flerov Laboratory of Nuclear Reactions, JINR, Dubna, Russia
$^{6}$ National Research Nuclear University MEPhI, Moscow, Russia
$^{7}$ Institute of Nuclear Physics, Almaty, Republic of Kazakhstan
$^{8}$ Al-Farabi Kazakh National University, Almaty, Republic of Kazakhstan
$^{9}$ V. G. Khlopin Radium Institute, St. Petersburg, Russia

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Abstract
A study of the $^{11}$B($^3$He,d)$^{12}$C reaction at incident $^3$He energy $E_{lab} = 25$ MeV has been performed at the K-130 cyclotron at the University of Jyväskylä, Finland. Differential cross sections have been measured for the 13.35 MeV state and for the states with excitation energy around 20 MeV in $^{12}$C. The data were analyzed with the DWBA method. A tentative assignment, $4^-$, is given for the state at 13.35 MeV in a joint study of the reaction and inelastic scattering of $\alpha$-particles with the energy of 110 MeV. For the state at 20.98 MeV, the possible spin-parity $3^-$ and the isospin $T = 0$ are assigned for the first time. Our model description of the broad state at 21.6 MeV is consistent with the previous assignments of isospin $T = 0$ and spin-parity of $2^+$ or $3^-$. The excited state at 22.4 MeV may have possible spin-parities of either $6^+$ or $5^-$. The collected statistics was insufficient to solve this question. Rotational bands which can exist in $^{12}$C were presented.

1 Introduction

Recently, $^{12}$C became a key nucleus in the study and description of nucleon clustering in light nuclei. For the instance, identification of the states with abnormally large radii, observation of alpha-cluster rotational bands, and the prevailing absence of adequate understanding of the structure of the famous Hoyle state ($0^+_{2, g.s., 7.65}$ MeV), evoked numerous theoretical and experimental studies (as example Refs. [1–19]).

The structure of the $0^+_{2, 7.65}$ MeV Hoyle state of $^{12}$C permanently attracts attention due to its importance in understanding clustering in nuclei and the key role in nucleosynthesis in the Universe. During the last decade, there appeared several new theoretical approaches that predicted some unusual features of this state. The most ambitious among them is the model of $\alpha$-particle condensation (APC) [20–24] according to which the Hoyle state resembles as a gas of almost noninteracting alpha particles. In this state $^{12}$C is expected to have an anomalously enhanced radius, 60–80% larger than that in the ground state (g.s.). Other theoretical calculations also predict an enlarged radius of $^{12}$C in the Hoyle state, but not as large as the APC radius (see, for example, Refs. [19,25] and references therein).

The analysis of inelastic scattering of various nuclei on $^{12}$C by Modified Diffraction Model (MDM) [26] gave an increase of 25% of the $rms$ radius (2.89 ± 0.04 fm) [26] of $^{12}$C in the Hoyle state comparing with that in the ground

This article is dedicated to memory of our leader and colleague, Professor Alexey Ogloblin. He passed on February 23, 2021. Alexey Ogloblin will be greatly missed as a Head of department of Nuclear Physics in Kurchatov Institute. His work in light exotic nuclei was widely cited and earned him great respect. He had the highest standards in research, publication, mentoring, and international collaboration. We will miss the contributions he could made else to the nuclear physics and to our lives as a leader, colleague and friend.

*e-mail: a.s.demyanova@bk.ru (corresponding author)
state (2.34 fm) [26]. A similar result was obtained by the Antisymmetrized Molecular Dynamics (AMD) calculations [15].

A special interest is generated by the proposed existence of the excited states “genetically related” to the Hoyle state. The idea that the Hoyle state might be the head of a rotational band became quite natural after appearance of the Morinaga’s model [27] presenting this level as a chain-like configuration of three α-particles. Recently, a suitable candidate for the second member of the rotational band based on the Hoyle state was identified either at 9.75 or at 10.13 MeV [16,17,28]. The radius of this state was determined to be equal ≈ 3.1 fm [29], i.e. practically the same as that for the Hoyle state. Thus this state can be regarded as a second member of the rotational band based on the Hoyle state.

Apparently, the members of the rotational band based on the Hoyle state are not the only states of $^{12}$C with an enlarged radius. A considerable size increase was found also for other states located above the $^{12}$C $\rightarrow$ 3α threshold. The radius of the 3− state at 9.64 MeV was determined to be $R = 2.88 \pm 0.11$ fm [26]. The exotic 3α and α + 3Be structures of the states in $^{12}$C near and above the α - emission threshold remain objects of intense theoretical studies [25]. They include calculations of the 3Be direct transfer [30] contributing to the elastic and inelastic α + $^{12}$C scattering to the $2^+_1$, $0^+_2$ and $3^-_1$ states. The measurements were done at 110 MeV [31,32] over the full angular range.

The relative contributions of different angular momenta of α and 3Be in the four lowest states of $^{12}$C were calculated as a ratio of the extracted spectroscopic factors corresponding to the angular momenta $L = 0, 2$, and $4$ for the $0^+_1$, $2^+_1$, and $2^+_2$ states, and $L = 1, 3$ and $5$ for the $3^-_1$ state. A comparison of these ratios revealed interesting regularities. Namely, the occupation probability in the g.s. and the 2+ 4.44-MeV state (members of the g.s. rotational band) was found almost evenly distributed between all orbital momenta. The occupation probabilities in the $0^+_2$ Hoyle state and the 3− 9.64-MeV state (the first members of the positive and negative rotational bands) were found predominantly concentrated in the lowest orbit with $L = 0$ and $1$, and probabilities 62 and 69%, respectively. This fact indicates that the structure of the $0^+_2$ and the 3− 9.64-MeV states is completely different from the structure of the lowest $^{12}$C states.

There are few open questions regarding excited states of $^{12}$C and the rotational bands that can exist in this nucleus. The g.s. rotational band ($0^+_1$, g.s.; $2^+_1$, 4.44 MeV; $4^+_1$, 14.1 MeV) in $^{12}$C is well known. Recently, the authors of Ref. [33], based on the $D_{3h}$ symmetry in the Algebraic Cluster Model (ACM) of $^{12}$C, which predicts two vibrational bands with ($v_1$, 0$^0$) with allowed values of angular momenta and parity $0^+$, $2^+$, $4^+$, ..., ($K = 0$) and $3^−$, $4^−$, $5^−$, ..., ($K = 3$), have suggested to see as the ground state rotational band the states: 3−, 9.64 MeV; 4−, 13.35 MeV and the new 5− state at 22.4(2) MeV. The branch (1, 0$^0$) can be considered as the rotational bands based on the Hoyle state. The situation with these bands is also very interesting. Negative parity states related to this branch have not yet been identified. Unfortunately, the spin-parity of the 13.35 MeV state remains ambiguous with contradicting assignments of either 2− [34,35] or 4− [36,37]. Resolving this ambiguity is important for the understanding the structure of $^{12}$C as a whole.

While the $2^+_2$ state at 9.75 or 10.13 MeV [16,17,28] is quite certain today, the $4^+_2$ state has raised many questions, in particular, related to the part of the spectrum near 13-14 MeV. Recently, we have identified a new level in $^{12}$C at 13.75 ± 0.12 MeV ($\Gamma = 1.4 \pm 0.15$ MeV) [38] with a spin-parity assignment of 4+. Apparently, this state coincides with the state at 13.3 ± 0.2 MeV ($\Gamma = 1.7 \pm 0.2$ MeV) in $^{12}$C previously determined in Ref. [39]. If so, the state at 13.75 MeV can be included into the Hoyle-state based rotational band as a third member [19,25]. The next challenge is identification of spin-parities of all the states with excitation energy between 18 and 23 MeV. In this region, one would expect to find the higher members of the $^{12}$C rotational bands.

To address these questions, we have studied the $^{11}$Bi($^3$He,d)$^{12}$C reaction at the incident $^3$He energy of the 25 MeV. Accordingly to our estimates, this energy is optimal for a spin-parity determination of the higher-excited levels in the 18–23 MeV region. For the 13.35-MeV state, the shape of the theoretical angular distributions in the c.m. angular range 4°–60° is quite different for the transferred angular momenta $L = 0$ and $L = 2$. This should allow for an unambiguous determination of spin-parity for this state. The available published data on the transfer reaction $^{11}$Bi($^3$He,d)$^{12}$C at $E(\text{He}) = 44$ MeV [35] are too poor (the angular distribution contains only 3 points) to allow for a reliable determination of the spin-parity of the 13.35-MeV state.

2 Experimental procedure

The measurements were carried out in the 150 cm diameter Large Scattering Chamber (LSC) [40] at the Accelerator Laboratory of the University of Jyväskylä (Finland). The $^3$He beam at $E(\text{He}) = 25$ MeV was extracted from the K-130 cyclotron.

The LSC was equipped with three sets of $\Delta E - E$ detector telescopes, each containing two independent $\Delta E$ detectors and one common $E$ detector. So each device allowed carrying out measurements at two angles. The measurements in c.m. angular range 10° were conducted in one exposure.

All detectors have similar diaphragms with holes 3 mm which define angular resolution, we estimates it $\sim 0.3$ deg. Every detector has solid angle $\sim 2.1 \times 10^{-5}$ sr.
Two silicon pin diodes of 380 and 100 μm were operating as ΔE detectors and 3.6 mm lithium-drifted silicon detectors as E detectors. The differential cross sections of the $^{11}$B($^{3}$He,d)$^{12}$C reaction were measured over the 4°–60° range in c.m. The beam intensity was about 20 particle nA. A self-supported, 0.275 mg/cm² thick, enrichment (95%) boron foil was used as a $^{11}$B target.

Beam intensity was measured using Faraday Cup and Ortec Digital Current Integrator 439. According to device manual, measurement may be in error by as much as 1% for currents ranging from 100 nA to 1 μA while in this experiment current range was from 5–30 nA.

The $^{10}$B nuclei is the main impurity in the target. Also signs of $^{12}$C and $^{16}$O as impurity are seen. We didn’t notice any increase of impurities quantity. Measurements at similar angles at the beginning and the end of the experiment confirm it. Presence of the lighter isotope added excited states of $^{11}$C to the experimental spectra. Fortunately, this did not compromise our measurements as all the levels below the excitation energy of 19 MeV were well separated.

This was possible thanks to the total energy resolution of about 100–150 keV. This very good energy resolution was needed to resolve the 4° or 2° and 1° states in $^{12}$C. It was achieved with a monochromatization method described in Ref. [40]. The procedure reduces the energy spread of the native cyclotron beam by a factor of 2–3 making this measurement possible. Figure 1 shows a sample of the registered deuteron spectrum from the reaction $^{11}$B($^{3}$He,d)$^{12}$C.

The background was of a physical nature and was described by the following phase volumes [41]: $^{8}$Be + $^{1}$α or $^{11}$B + p + d and $^{11}$C + n + d. Figure 1 (left and right panels) shows the spectrum with the background subtracted. The similar background subtraction procedure was applied, for example, in Ref. [42].

Peaks corresponding to the relevant transitions in the collected deuteron spectra were identified and parametrized using a standard Gaussian decomposition method.

We think that it is reasonable. To confirm this, we compared two variants of decomposition using Gaussian and Voigt line shapes. The systematic uncertainty caused by this choice is small and observed integrals differ only by 10-15% which is the same order as our statistical errors.

With the known energy calibration, the peak positions and widths were fixed in accordance with the generally accepted values while the areas under the peaks were treated as free parameters. At excitation energies above 19 MeV (Fig. 1c) the procedure became more complex as some of the states could not be fully resolved.

We present here the $\chi^2$/DoF values in the energy excitation regions of the peaks we are focused on. In Fig. 1a—$\chi^2$/DoF is 1.93 for the region 12.71–14.1 MeV; Fig. 1b—$\chi^2$/DoF is 1.97 for the region 15.11–17.76 MeV; Fig. 1c—$\chi^2$/DoF is 0.23 for the region 19.55–23.52 MeV.

In the ($^{3}$He,d) reaction, both the isospin $T = 0$ and $T = 1$ states are excited while in the inelastic scattering of $^{1}$α-particle, only the states with $T = 0$ are excited. Therefore, if one observes the same state in the ($^{3}$He,d) reaction and in the inelastic scattering spectrum, one can confidently assign $T = 0$ to this state. This comparative procedure was applied to the experimental data on the inelastic scattering of $^{1}$α-particles on $^{12}$C at $E_{lab} = 110$ MeV [31, 32].

In Fig. 2, we show a sample spectrum from the $^{4}$He + $^{12}$C scattering at $\theta_{lab} = 23^\circ$ showing the excitation of $^{12}$C states around 21 MeV. The spectra in Fig. 2 (left and right panels) are shown with a subtracted background. The background has a physical nature and the main contribution to it is determined by the phase volume of $^{8}$Be + $^{1}$α + $^{1}$α.

We also present here the $\chi^2$/DoF values: Fig. 2a—$\chi^2$/DoF is 0.6 for the region 12.71–14.1 MeV; Fig. 2b—$\chi^2$/DoF is 3.26 for the region 18.69–22.4 MeV.

So, in both cases obtained values of the $\chi^2$/DoF confirms reasonable fits of experimental spectra.

In total, deuteron angular distribution for the g.s. and eight excited states of $^{12}$C were extracted using the $^{11}$B($^{3}$He,d)$^{12}$C. The resulting differential cross sections for the g.s. and the excited states at $E_x = 4.44, 7.65, 9.64, 13.35, 16.57, 20.98, 21.6$ and 22.4 MeV are presented in Figs. 3, 4, 5, 6, 7 and 10.

3 Theoretical analysis

Theoretical analysis of the experimental differential cross sections are carried out in the framework of the finite range distorted-wave Born approximation (DWBA) [43] using the FRESCO code [44]. The contributions of all allowed combinations of transmitted angular momenta $L$ and spins $J$ are coherently accounted.

It is well known that the reactions of direct nucleon transfer are surface phenomena, and the angular distribution of the emitted particles has a predominant orientation, forming in the forward angular region the so-called main maximum, the position and shape of which are determined by the transferred angular momenta. One of the main goals of our work was just the determination of the transferred momenta, and from them the spin-parities of the states under study.

The elastic scattering wave functions in the entrance ($^{3}$He + $^{11}$B) and the exit (d + $^{12}$C) reaction channels are calculated in the framework of the fenomenological optical model. A standard Woods–Saxon form of the real part of optical potentials and a combination of the volume $W_s$ and surface $W_d$ potentials for the imaginary part are used. For the entrance channel, we have applied parameters of the global potential [45]. For the exit channel, also global potential parameters were used [46]. These parameterizations give a good description of the elastic scattering data in the forward angular range.
Fig. 1 Left panel: sample of a deuteron spectrum from the \(^{11}\text{B}(^{3}\text{He},d)^{12}\text{C}\) reaction at \(E_{\text{lab}} = 25\) MeV registered at \(\theta_{\text{lab}} = 11^\circ\). The black rectangular areas labeled a–c are shown expanded in the right panels. Right panel a: region near 13.35 MeV in excitation energy of \(^{12}\text{C}\) after background subtraction. Right panel b: region near 16.57 MeV in excitation energy of \(^{12}\text{C}\) after background subtraction. Right panel c: region near 21 MeV in excitation energy of \(^{12}\text{C}\) after background subtraction. Red color (online) indicates states that are of particular interest to us (13.35, 16.57, 20.98, 21.6 and 22.4 MeV). Blue color (online) indicates other states in \(^{12}\text{C}\).

Fig. 2 Left panel: a sample spectrum from the \(^{4}\text{He} + ^{12}\text{C}\) scattering at \(E_{\text{lab}} = 110\) MeV for \(\theta_{\text{lab}} = 23^\circ\). Right panel a: an expanded part of the spectrum region near 13.35 MeV excitation energy of \(^{12}\text{C}\). Right panel b: an expanded part of the spectrum region near 21 MeV excitation energy of \(^{12}\text{C}\). Red color (online) indicates the states that are of particular interest to us (13.35, 20.98, 21.6 and 22.4 MeV). Blue color (online) indicates other states in \(^{12}\text{C}\).
Fig. 3 The experimental differential cross sections of the $^{11}$B($^3$He,d)$^{12}$C reaction at $E_{\text{lab}} = 25$ MeV for (a) ground state, (b) 4.44-MeV state, (c) 7.65-MeV state, and (d) 9.64-MeV state. The (1) (red online) solid curves show DWBA calculations, the (2) (blue online) solid curves show CRC calculations for these states.

Fig. 4 Left panel: the experimental differential cross section of the $^{11}$B($^3$He,d)$^{12}$C reaction at $E_{\text{lab}} = 25$ MeV with the excitation of the 16.57-MeV state (black dots). The solid curve (red online) corresponds to the DWBA calculation for this state with spin-parity $2^-$. Right panel: the experimental differential cross section of the $^{11}$B($^3$He,d)$^{12}$C reaction at $E_{\text{lab}} = 25$ MeV with the excitation of the 16.57-MeV state (black dots). The solid (red online) and dashed (blue online) lines correspond to the CRC calculations with the bound and resonant wave functions (from Ref. [48]).
The advantages of global potentials are the dependence of force parameters on energy, which is very important due to the lack of experimental data on elastic scattering at specific energies in the input and output channels in the cases we are considering, as well as the obvious belonging of these potentials to one family. In addition, the number of free parameters is orders of magnitude less than the number of experimental data. In our case significantly all this reduces the ambiguity introduced by the optical model. The disadvantage of the global potentials used is a linear dependence of the depth of the real part of the potential on the energy which, at low energies, does not take into account the deviation from the linear dependence due to the threshold anomaly. The quality of the description of experimental differential cross sections with these potentials at close energies is shown in Figures 1 and 2 in Ref. [45] and in Figure 2 in Ref. [46], which show that in the regions of energies of interest to us these potentials give a good description in the angular range up to 60–70°.
Note that some correction of the real part of the potential at low energies, taking into account the threshold anomaly within the same family, can improve the description of elastic scattering in the angular range beyond 60°. However, we verified that such a correction does not lead to a noticeable change in the description of the reaction in the region of the main peak. As for the reaction form factors, precisely in order to minimize uncertainty through the use of a smaller number of parameters, we considered the transfer of protons to single-particle states of various configurations \((LSJ)\), which are formed in the same single-particle potential.

The transfer form factors to the bound states are modeled by the normalized single-particle wave functions in the Woods–Saxon potentials. The depth of the potential is automatically varied to give a binding energy of the transmitted particle (Well-Depth-Prescription procedure, WDP). The relative contributions \(A_{LJ}\) of components corresponding to different moments \((LJ)\) and the geometric parameters \(r_0\) and \(\alpha\) of the Woods–Saxon potential are adjusted to fit the experimental angular distributions especially in the main front peak.

The geometrical parameters and the relative contribution of components \(A_{LJ}\) for each bound state of the final nucleus are presented in Table 1.

The proton transfer form factor to the continuum states of \(^{12}\text{C}\) is calculated using the wave functions \(\Phi(r)\) defined in Ref. [44]

\[
\Phi(r) = \sqrt{\frac{2}{\pi N}} \int_{k_1}^{k_2} w(k)\varphi_{k,LJ}(r)dk,
\]

where \(N\) is the normalization of the weight function

\[
w(k) = exp(-i\delta_k)\sin\delta_k.
\]

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### Table 1 Geometrical parameters of the single-particle potential of p+\(^{11}\text{B}\) in \(^{12}\text{C}\) and the relative contribution \(A_{LJ}\) of components for allowed combinations of angular momentum transfers \(LJ\)

| \(^{12}\text{C}\) states | \(r_0\) (fm) | \(\alpha\) (fm) | \(L\) | \(J\) | \(A_{LJ}\) |
|------------------------|-------------|--------------|-----|-----|------|
| g.s., 0\(^+\)           | 1.1         | 0.87         | 1   | 3/2 | 1.00 |
| 4.44, 2\(^+\)           | 1.0         | 1.0          | 1   | 1/2 | 1.00 |
| 7.65, 0\(^+\)           | 0.6         | 0.1          | 1   | 3/2 | 1.00 |
| 9.64, 3\(^-\)           | 1.2         | 1.5          | 2   | 3/2 | 1.00 |
| 13.35, 2\(^-\)          | 1.1         | 1.3          | 0   | 1/2 | 0.197|
| 13.35, 4\(^-\)          | 1.2         | 1.4          | 2   | 5/2 | 0.661|
|                        | 4           | 7/2          | 0.296|

The weight function \(w(k)\) is best chosen (ref. [44, p. 148]) to include some of the effects known to be caused by the variation of \(\varphi_k(r)\) within the bin range \(k_1 < k < k_2\). If \(w(k) = exp(-i\delta_k)\), where \(\delta_k\) is the scattering phase shift for \(\varphi_k(r)\), then it includes the effects of the overall phase variations of \(\varphi_k\) at least in the DWBA limit. If, however, \(w(k) = exp(-i\delta_k)\sin\delta_k = T_k^*\), where \(T_k\) is the T-matrix element for \(\varphi_k(r)\), then it includes in addition a scale factor which is useful if the \(T_k\) varies significantly, as it does, for example, over resonances. Both choices result in a real-valued wave function \(\Phi(r)\) (for real potentials), which is computationally advantageous. If the maximum radius \(R\) (say) is sufficiently large, then the wave functions \(\Phi(r)\) will be normalised to unity in the interval \([0, R]\). In our case, the maximum radius \(R = 150\) fm was sufficient, so a single-particle wave function \(\varphi_k(r)\) is averaged over energy “bin” in continuum [44]. The experimental level widths [34] were used as the resonance widths. The relative contribution of components \(A_{LJ}\), potential depths \(V\) and geometric parameters \(r_0\) and \(\alpha\) of Woods–Saxon potential selected for the best fit of the experimental angular distributions at forward angles are presented in Table 2 for each state of the continuum proton spectrum of \(^{12}\text{C}\).

The purpose of our work was not a detailed description of specific experimental curves in the entire range of angles, which requires a more complex formalism and an actual increase in the number of free parameters. To determine the transferred moments and, accordingly, the spin-parity, we consider it sufficient to use DWBA and a description in the region of the main maximum.

Nonetheless, we have checked the correctness of our results by using an independent theoretical method with a
Table 2  The potential parameters \((V, r_0 \text{ and } a)\) of the single-particle potential of \(p+^{11}\text{B}\) for the continuum states of \(^{12}\text{C}\), relative contribution \(A_{LJ}\) of components for allowed combinations of angular momentum transfers \(L\)\(J\)

| \(^{12}\text{C}\) states | \(V\) (MeV) | \(r_0\) (fm) | \(a\) (fm) | \(L\) | \(J\) | \(A_{LJ}\) |
|-------------------------|------------|-------------|-------------|-----|-----|--------|
| 16.57, 2\(^{-}\)       | 160        | 1.2         | 0.7         | 0   | 1/2 | 0.474  |
|                         |            |             |             | 2   | 3/2 | 0.623  |
|                         |            |             |             | 2   | 5/2 | 0.623  |
| 20.98, 3\(^{-}\)       | 175        | 1.7         | 0.8         | 2   | 3/2 | 0.639  |
|                         |            |             |             | 2   | 5/2 | 0.639  |
|                         |            |             |             | 4   | 7/2 | 0.303  |
|                         |            |             |             | 4   | 9/2 | 0.303  |
| 20.98, 5\(^{-}\)       | 161        | 1.6         | 0.9         | 4   | 7/2 | 0.5    |
|                         |            |             |             | 4   | 7/2 | 0.5    |
|                         |            |             |             | 6   | 7/2 | 0.5    |
|                         |            |             |             | 6   | 7/2 | 0.5    |
| 21.6, 2\(^{+}\)       | 180        | 1.2         | 1.2         | 1   | 1/2 | 0.5    |
|                         |            |             |             | 1   | 3/2 | 0.5    |
|                         |            |             |             | 1   | 5/2 | 0.5    |
|                         |            |             |             | 1   | 7/2 | 0.5    |
| 21.6, 3\(^{-}\)       | 185        | 1.2         | 1.2         | 2   | 3/2 | 0.697  |
|                         |            |             |             | 2   | 5/2 | 0.697  |
|                         |            |             |             | 4   | 7/2 | 0.116  |
|                         |            |             |             | 4   | 9/2 | 0.116  |
| 22.4, 5\(^{-}\)       | 155        | 1.8         | 0.7         | 4   | 7/2 | 0.588  |
|                         |            |             |             | 4   | 9/2 | 0.588  |
|                         |            |             |             | 6   | 11/2| 0.425  |
|                         |            |             |             | 6   | 13/2| 0.425  |
| 22.4, 6\(^{+}\)       | 150        | 1.8         | 0.8         | 5   | 9/2 | 0.5    |
|                         |            |             |             | 5   | 11/2| 0.5    |
|                         |            |             |             | 7   | 13/2| 0.5    |
|                         |            |             |             | 7   | 15/2| 0.5    |

larger number of parameters. The theoretical calculation of single-proton pickup in the \(^{11}\text{B}(3\text{He},d)^{12}\text{C}\) reaction was carried out for g.s, 4.44, 7.65, 9.64, 16.57 and 21.6 MeV states (Figs. 3, 5 left panel, correspondingly) using the coupled reaction channels method (CRC) [44].

Indeed, within this approximation, better agreement is achieved at large angles, while in the main peak range the description is practically the same as in our DWBA analysis.

4 Results and discussion

Figure 3 shows the experimental deuteron angular distributions in comparison with the DWBA calculations for the g.s (Fig 3a) and the states at 4.44 MeV (Fig. 3b), 7.65 MeV (Fig. 3c), and 9.64 MeV (Fig. 3d). For these levels, the values of spin-parity and isospin are well known [34].

For the lowest \(^{12}\text{C}\) states (g.s., 4.44 and 7.65), we considered only one configuration as the main one. The depth parameter is fixed by the proton separation energy. In this case, we have only two free parameters of the one-particle potential, which are selected for the best description in the region of the main peak. For the ground state and the 4.44 MeV state, a good description is obtained in the region of the main peak for known values of spin-parity and transferred angular momentum (Fig. 3a, b).

The fact that for the 7.65 MeV state the use of such a simple model of form factors did not lead to a qualitative description of the main maximum position and its shape at small angles (Fig. 3c) is quite expected, due to the more complex structure of the nucleus in this state (large cluster component). However, the study of this condition was not the goal of our work, therefore, we did not focus on it. It should be noted that in this reaction we observe a weak excitation of the 7.65 MeV state. As a result, the obtained differential cross sections have large statistical errors.

For states where more than one configuration is included, free parameters are added— relative normalization proportional to spectroscopic amplitudes. So for the 9.64 MeV state, two configurations are considered, while the position and shape of the main maximum are determined only by \(L = 2\) and are practically independent of \(J\). A good description of the position and shape of the main maximum is obtained. The analysis for known states shows that we can apply such a model approach with a sufficient degree of unambiguity and understanding of its limitations in cases of states with a more complex cluster structure.

The calculations satisfactorily describe the data, especially at forward angles, and allow us to correctly determine the spin-parity values and the transfer moments (see Table 1).

In the following chapters we summarize the outcome of the analysis of studying the differential cross sections for the \(^{12}\text{C}\) states at 16.57, 21.6, 20.98, 13.35, and 22.4 MeV excitation energy.

For these states above the proton separation threshold, the difference is that the proton goes into a state that is a resonance in the single-particle potential. The resonance width is of the order of experimental one, and the parameter of the potential depth is also free. We do not believe that adding one or two parameters will significantly increase the ambiguity of the analysis.

4.1 The state at 16.57 and 21.6 MeV

In the beginning, as an example, we considered such a state, namely 16.57, with the spin-parity value \(2^{-}\) already set [34]. As can be seen in Fig. 4 (left panel), the description in the region of the main maximum and even up to 50° is very good.

This gives us reason to use such a model of the form factor for states above the proton separation threshold and consider
reliable description of data in the angular range up to 40-50° and the determination of the transferred angular momenta and, accordingly, the permissible spin-parities of these states from this region.

The differential cross section of the $^{11}$B$(^3$He,d)$^{12}$C reaction with excitation of the 16.57-MeV state is presented in Fig. 4 in comparison with the DWBA calculation assigning to this state spin-parity of $2^-$ (solid red curve). As it was mentioned earlier, this excited state was not observed in the inelastic scattering of $\alpha$-particles on $^{12}$C at 110 MeV [31, 32] because of the accepted isospin of $T = 1$ [34].

On the other hand, the 16.57 MeV state, $2^-$ has a special status. In the isobar-analog triad $^{12}$B, 1.67 MeV—$^{12}$C, 16.57 MeV—$^{12}$N, 1.19 MeV with states $2^-$, the 16.57 state was considered as a candidate for a state with one-proton halo. On the Figure 4 (right panel) presented the experimental differential cross section with the CRC calculations taken from Ref. [48]. We calculate the reaction cross sections by using two different proton wave functions in the excited state of $^{12}$C: (a) the resonant wave functions $\Phi(r)$ and (b) the bound wave functions calculated with the small (fictitious) negative ($\sim -0.01$ MeV) binding energy of the valence proton. Calculations with both wave functions reproduce the experimental deuteron angular distributions in a similar way; therefore the calculations with bound wave functions could be taken as a good approximation. It is obvious that the absolute values of the experimental spectroscopic amplitudes $\langle n_1l_1j_1|Y_l|n_2l_2j_2\rangle$ determined from the calculations with continuum and bound wave functions are different, but the relative weights of the sp configurations are found to be the same (within the error bars) regardless of the choice of the wave functions.

The experimental differential cross section of the $^{11}$B$(^3$He,d)$^{12}$C reaction with an excitation of the 21.6-MeV state is plotted in Fig. 5 in comparison with the DWBA calculations assuming generally accepted $2^+$ or $3^-$ values of spin-parity [34] for this state.

In the right panels of Fig. 5, the DWBA calculations are shown in a linear scale for spin-parity of $3^-$ (green curve in the top panel) and $2^+$ (blue curve in the bottom panel). The behavior of both curves is very similar over the entire angular range.

However, in our case, it is not always possible to unambiguously select spin-parity from two variants that are possible by visual comparison in this region. In particular, for the 21.6 MeV state, the uncertainty of the choice between the previously assumed [34] values $2^+$ or $3^-$ also remained.

As the 21.6-MeV state was also observed in the inelastic scattering of $\alpha$-particles at 110 MeV [31, 32], we confirm the generally accepted values of spin-parity of $2^+$ or $3^-$ and the isospin of $T = 0$ for the 21.6-MeV state of $^{12}$C.

4.2 The state at 20.98 MeV

The spin-parity and the isospin have not been so far assigned for the state at 20.98 MeV. The differential cross sections of the $^{11}$B$(^3$He,d)$^{12}$C reaction with excitation of the 20.98-MeV state are presented in Fig. 6 with theoretical curves corresponding to the values of spin-parities $3^-$ and $5^-$. However, description with spin-parity $3^-$ is more preferable due to better description of behavior of differential cross section in the region of main peak and less $\chi^2$ ($\sim 30\%$).

Since the state at 20.98 MeV is observed in the inelastic scattering, we can assign to it the isospin of $T = 0$.

Thus, for the first time, we determine the possible values of spin-parity $J^T = 3^-$ and isospin $T = 0$ for the state at 20.98 MeV.

4.3 The state at 13.35 MeV

The differential cross section of the $^{11}$B$(^3$He,d)$^{12}$C reaction with excitation of the 13.35 state is presented in Fig. 7. The DWBA analysis reveals that the dominant transferred angular momentum is $L = 2$. For this moment, there are two possible values of spin-parity: $4^-$ and $2^-$. The DWBA calculations depicted in Fig. 7 with spin-parities of $4^-$ and $2^-$ (red and blue curves, respectively) differ little in the description of the data. In this case, a choice of the spin-parity value of the 13.35-MeV state is hindered and the question remains open.

To answer the question we tried to use our data on the inelastic scattering angular distribution of $\alpha$-particles at 110 MeV [31,32,38].

Since the assumed values of spin-parity are unnatural, one should consider two-step mechanism to excite this state in inelastic scattering. However, by using this approach we could not reproduce the experimental angular distributions, and moreover the absolute values of the calculated cross sections have been very small. We use an alternative way to estimate the relative contribution of various transferred angular momenta $L$ in the framework of one-step DWBA applying a simple cluster model of $^{12}$C. In this way, the inelastic form factor is considered in the frame of microscopic (cluster) interaction model

$$ f_L(r) = 4\pi V_0 \sqrt{(2J_A + 1)} \langle j^L J_B \parallel Y_L \parallel j^L J_A \rangle \times \int dr R_{Lj} \langle j^L j \parallel R_{Lj} (r) \rangle, $$

where $R_{Lj} (r)$ are the single-particle wave functions describing the relative motion (with orbital momentum $l$ and total momentum $j$) of $^8$Be (with spin $I$) in the potential of alpha-particle core (with spin $I_c$). These wave functions are calcu-
lated with a standard WDP procedure using a Woods–Saxon potential. The geometric parameters of the Woods–Saxon potential are selected based on the best fit of the experimental angular distribution. The Yukawa form for a radial part of the central interaction $\nu L$ is used with inverse range parameter $\mu = 0.7$. The strength $V_0$ is included in the resulting normalization.

We account coherent contributions from different combinations ($l'j',IIIj$) that allow transferred momenta $L = 1$, $3$, $5$. The calculations are carried out with code DWUCK4 [47].

Figure 8 shows that the best description of the angular distribution is given by a component with $L = 1$ including small corrections from other components. In addition to this calculation, the analysis in the framework of the MDM is performed (see Fig. 9). The shape of the angular distributions is approximated by combination of the Bessel functions $J_L(x)$, where $L$ is the transferred angular momentum. The MDM analysis showed that for the transferred momentum $L = 1$ diffraction radius is increased while for the $L = 3$ diffraction radius is normal. For transferred momentum $L = 5$ MDM analysis can’t be applied as we don’t have corresponding formulas.

The most probable transferred angular momentum is $L = 1$. The MDM analysis allowed us to estimate the $rms$ radii of $^{12}\text{C}$ in the excited state at 13.35 MeV, given a value of $R_{rms} \approx 3$ fm. This increased value is close to the radius of the $3^−$, 9.64-MeV state of $^{12}\text{C}$ ($R_{rms} = 2.88 \pm 0.11$) [26]. Thus, if we assume that the 9.64-MeV and the 13.35-MeV states belong to a rotational band based on the 9.64-MeV state ($3^−$), then it is quite natural to assume that the spin-parity of the 13.35 MeV state should be $4^−$.

The differential cross section of the $^{11}\text{B}(^{3}\text{He},d)^{12}\text{C}$ reaction with excitation of the 22.4-MeV state in $^{12}\text{C}$ is presented in Fig. 10. The DWBA analysis of the data was carried out taking into account all possible spin-parities for this state. The best fit of the data was obtained for the spin-parities of $6^+ \text{ and } 5^−$. The calculated cross sections are also shown in Fig. 10 as green and blue curves. From our point of view, the fit of the data in the region of the main peak is better for the calculations corresponding to the spin-parity of $6^+$. This is especially evident if we compare the experimental and theoretical differential cross sections in a linear scale (see the right panels in Fig. 10).

The representation of the values of $\chi^2$ is additional justification. We have compared $\chi^2$ for local region up to 20 deg in c.m. for DWBA calculations with $5^− \text{ and } 6^+$. $\chi^2$ for $5^−$ is equal 1.24 and for $6^+$ is equal 0.26.

As the state at 22.4 MeV was also observed in the inelastic scattering of $\alpha$-particles at $E_{lab} = 110$ MeV [31,32], we can assign the isospin of $T = 0$ to the 22.4-MeV state of $^{12}\text{C}$. Besides, the presence of the $6^+$ state does not at all contradict the ACM ($D_3h$ symmetry) predictions [49]. Moreover, if we extrapolate the energies of the Hoyle band in Fig. 4 from [33], we obtain the energy of the $6^+$ state close to the $5^−$ state of the g.s band. So, our assumption about the possible coexistence of both states in this narrow energy range has a right to be.

If we accept that the spin-parity of the 22.4-MeV state is of $6^+$ and the isospin is $T = 0$ then it corresponds to the rotational trajectory $J(J + 1)$ of the Hoyle band, which also includes the $2^+$ state near 9.9 MeV [16,17,28] and the $4^+$ state at 13.75 MeV [32,38].
Fig. 10 Left panel: the experimental differential cross section of the $^{11}$B($^3$He,d)$^{12}$C reaction for the 22.4-MeV state (black dots) in comparison with the DWBA calculations with spin-parities of $5^-$ (1) curve (blue online) and $6^+$ (2) curve (green online). Right panel: a comparison of the data with the DWBA calculations in a linear scale.

4.5 New members of the rotational band in $^{12}$C

To summarize the analysis of the received data let us present an updated systematics of the states in $^{12}$C grouping them to the rotational bands shown in Fig. 11.

The well-established $K^\pi=0^+$ ground-state rotational band includes $2^+$ and $4^+$ states at 4.44, and 14.1 MeV (black line in Fig. 11). The $K^\pi=3^-$ rotational band proposed in Ref. [33] contains the negative-parity states $3^-$ and $4^-$ states at 9.64 and 13.35 MeV (blue line in Fig. 11). The rotational band based on the Hoyle state can include the $0^+$, $2^+$, $4^+$, and $6^+$ states at 7.65, 9.9 (9.75 [17] or 10.13 [28]), 13.75 and 22.4 MeV (shown in Fig. 11 by red line).

Recently, the observation of the state at 22.5 MeV with a tentative assignment, $5^-$, was announced [33]. However, the new data table assigns $T=1$ in the energy region 22.4 MeV [50]. The authors assumed that it is a good candidate to complete the negative-parity rotational band.

The matter is further complicated by the fact that in this high-excitation region of the spectrum, the majority of states overlap and therefore their separation and characterization is not straightforward. It is conceivable that in the excitation region of 22.4–22.5 MeV, there are two overlapping states with spin-parities of $5^-$ and $6^+$.

5 Conclusion

The $^{11}$B($^3$He,d)$^{12}$C reaction was chosen to study proton transfer at $E_{lab}=25$ MeV. Deuteron angular distributions for the g.s. and eight excited states at $E_x=4.44, 7.65, 9.64, 13.35, 16.57, 20.98, 21.6$ and $22.4$ MeV of $^{12}$C were measured in the c.m. angular range $4^\circ$–$60^\circ$. In addition, the data on inelastic scattering of $\alpha$-particles on $^{12}$C at 110 MeV with excitation of the 13.35 MeV state are presented for the first time.

The generally accepted spin-parity of $2^-$ and the isospin of $T=1$ for the state at 16.57 MeV, and the spin-parity of
$2^+ \text{ or } 3^-$, and the isospin of $T = 0$ for the state at 21.6 MeV were confirmed. For the first time, the possible spin-parity $3^-$ and the isospin $T = 0$ for the state at 20.98 MeV were determined.

The DWBA analysis of the differential cross section of the $^{11}$B($^3$He,$d$)$^{12}$C reaction with excitation of the 13.35 state shows the presence of several states. The parity and spin-parity of the states were determined to be $5^-$ and $6^-$, respectively. However, further analysis is required to confirm these assignments.

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