DENSE ENTIRE CURVES IN RATIONALLY CONNECTED MANIFOLDS

FRÉDÉRIC CAMPANA & JÖRG WINKELMANN

Abstract. We show the existence of metrically dense entire curves (of growth order 0) in rationally connected complex projective manifolds, confirming for this case a conjecture formulated in [10], according to which such entire curves on projective manifolds exist if and only if these are ‘special’ in the sense defined in loc.cit. For unirational manifolds, the statement above is an easy consequence of function theoretic methods. Our proof rests on the ‘comb smoothing’ technique of Kollár-Miyaoka-Mori, and may be seen as a substitute of the power series expansion of entire functions with values in $\mathbb{C}^n$, in the absence of global coordinates.

We next show the existence of dense entire curves, avoiding the singular locus, in certain log-terminal normal rational surfaces. This implies via results of Grassi and Oguiso the existence of dense entire curves into any Calabi-Yau threefold fibered in Abelian surfaces or elliptic curves.

We then show that a dense entire curve may be chosen on any rationally connected manifold in such a way that it does not lift to any of its ramified covers, answering in this case a question of [21] about the Nevanlinna analog of the ‘weak Hilbert property’ of arithmetic geometry. We consider briefly the other test case of the conjecture, namely manifolds with $c_1 = 0$, and most ‘special’ surfaces.

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1. Introduction

Our main result is the existence of dense entire curves (i.e. a holomorphic map from $\mathbb{C}$ with metrically dense image) in rationally connected manifolds.

More precisely, we prove that, given a rationally connected complex projective manifold $X$ with a countable subset $M$, a hypersurface $D$ and a non-trivially ramified cover $Y \to X$ we can find a dense entire curve $c : \mathbb{C} \to X$ such that

- The countable set $M$ is contained in the image $c(\mathbb{C})$,
- $c(\mathbb{C})$ meets $D$ tranversally somewhere,
- $c$ can not be lifted to an entire curve in $Y$,
- The order $\rho_c$ (in the sense of Nevanlinna theory) equals zero.

The non-liftability confirms (for rationally connected manifolds) a conjecture of Corvaja and Zannier.

We also prove some results in these directions for normal projective surfaces and for compact Kähler manifolds with $c_1 = 0$.

The existence of a dense entire curve for any rationally connected manifolds confirms, for this case, a conjecture, which motivated the present text, that for compact Kähler manifolds, the existence of a dense entire curve is equivalent to being ‘special’ (definition recalled
below). Special manifolds are in a precise sense ‘opposite’ to manifolds of general type. While, following a conjecture of S. Lang, the non-existence of Zariski dense entire curves on manifolds of general type has been investigated since decades, our result seems to be the first one in the opposite direction, beyond the classical case of unirational or abelian manifolds.

Although such dense curves exist on any unirational manifold for simple function-theoretic interpolation reasons, it is presently unknown whether rationally connected are all unirational. In fact the contrary is expected to be true quite generally. Moreover our proof, which is mainly algebraic, based on the ‘comb smoothing’ technique of [33], provides a (to our knowledge) new approach to these interpolation properties, and plays a rôle similar to the power series expansion of entire functions in this broader context. We don’t know however if any entire curve in a rationally connected manifold can be approximated by algebraic maps, as in our construction.

The conjectural links formulated by S. Lang between arithmetic and hyperbolicity properties lead to conjecture that the existence of a Zariski dense entire curve should occur on $X$ defined over a number field $k$ if and only if $X(k')$ is Zariski dense for some finite extension $k'$ of $k$. As this arithmetic property is unknown for $X$ rationally connected, our result give some support to this arithmetic conjecture.

We show finally that the entire curves we construct in any rationally connected manifolds may be chosen so as to have growth order $\rho_f = 0$.

In particular, any rationally connected complex projective manifold $X$ admits dense entire curves $h : \mathbb{C} \to X$ with $\rho_h = 0$. In the opposite direction we showed in [16] that any complex projective manifold $X$ of dimension $n$ admitting a non-degenerate holomorphic map $F : \mathbb{C}^n \to X$ of order $\rho_F < 2$ is rationally connected.

For normal projective surfaces $X$ with only quotient singularities we show: If there is an effective non-zero $\mathbb{Q}$-divisor $\Delta$ such that $K + \Delta$ is $\mathbb{Q}$-trivial and such that the pair $(X, \Delta)$ is log terminal, then there exists a dense entire curve avoiding the singular locus. Here the 2-dimensional MMP plays a crucial rôle in the proof.

2. Brief review of special manifolds

Let $X$ be an $n$-dimensional smooth connected compact complex manifold (either projective or compact Kähler). Its Kodaira dimension is denoted with $\kappa(X)$.

Recall (see [10]) that a compact Kähler manifold $X$ is said to be ‘special’ if no rank one subsheaf $L \subset \Omega^p_X$ has top Kodaira dimension $p$, ...
this for any $p > 0$. These manifolds are higher dimensional generalisations of rational and elliptic curves. In particular, manifolds $X$ which are either rationally connected (see next section for a brief reminder), or with $\kappa(X) = 0$ are special\footnote{However, among $n$-dimensional manifolds with $\kappa = k$, there are both special and non-special examples, for any $k \neq 0, n$.}. In fact, the decomposition $c = (j \circ r)$ of the ‘core map’ (See [10] for details) shows that the ‘building blocks’ of special manifolds should be (smooth) ‘orbifold pairs’ either ‘rationally connected’\footnote{in the sense of having $\kappa^+ = -\infty$. When there is no orbifold structure, a classical conjecture claims that this is equivalent to rational connectedness.}, or with $\kappa = 0$. Moreover ([10], Corollary 8.11) $X$ is ‘special’ if it is $\mathbb{C}^n$-dominable (i.e. if there exists a meromorphic map from $\mathbb{C}^n$ to $X$ regular and submersive at some point). The converse is not expected to hold in general (but partially weaker versions are conjectured below).

We conjecture that the special manifolds are also characterised by the two (conjecturally equivalent properties) that their Kobayashi pseudometric $d_X$ vanishes identically, and that they admit a holomorphic map $h : \mathbb{C} \to X$ with metrically dense image. This conjecture is motivated by the above-mentioned decomposition $c = (j \circ r)^n$ of the core map, which essentially reduces these conjectures to the two cases of orbifold pairs either rationally connected in the above sense, or with $\kappa = 0$.

For rationally connected manifolds, $d_X$ vanishes obviously. We prove the existence of a dense $h(\mathbb{C})$ in Theorem 5.2 using the comb-smoothing technique of [33]. We also deduce a Nevanlinna analogue of the Hilbert Irreducibility Property introduced in [21]. Similar results hold more classically when $X$ is a complex torus.

We give some brief remarks on the much more challenging class of manifolds with zero first Chern class in section 8 below.

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\end{itemize}

3. Review of Rationally connected manifolds

Recall that a projective manifold $X$ is said to be:
1. **rational** if it is bimeromorphic to \( \mathbb{P}^n \),
2. **unirational** if dominated by \( \mathbb{P}^n \) (which means that there exists a non-degenerate (i.e.: dominant) meromorphic map \( \Phi : \mathbb{P}^n \to X \)), and:
3. **rationally connected** if any two generic points of \( X \) are connected by a rational curve in \( X \).
4. **Fano** if its anticanonical bundle is ample.

These properties (except for 4) are birational. One has the implications:

\[
\text{Fano} \quad \Downarrow \\
\text{rational} \implies \text{unirational} \implies \text{rationally connected}
\]

For curves (resp. surfaces), these properties (resp. except being Fano) are equivalent. But, starting in dimension 3, many unirational manifolds are known to be non-rational. On the other hand, although it is expected that most rationally connected threefolds are not unirational, there is presently no known invariant to distinguish them.

Examples of rationally connected threefolds not known to be unirational, and possibly non-unirational, are ‘general’ quartics in \( \mathbb{P}^4 \), double covers of \( \mathbb{P}^3 \) branched over a smooth sextic, and standard conic bundles over \( \mathbb{P}^2 \) with smooth discriminant of large degree.

### 4. Conjectures on Special manifolds

**Definition 4.1.** ([10], Definition 2.24 and Theorem 2.27) A compact Kähler manifold \( X \) is said to be ‘special’ if \( \kappa(X, L) < p \) for any rank-one coherent sheaf \( L \subset \Omega^p_X \), and any \( p > 0 \).

We refer to [10] for details on this class of manifolds, and some of the reasons to introduce them, the main one being the existence of the ‘core map’ which splits any \( X \) into its opposite parts: special vs general type. We shall here simply mention the two basic examples of ‘special’ manifolds: Rationally connected manifolds and manifolds with \( \kappa = 0 \).

Below we summarize what we expect to be true for special manifolds. Many more variants can be formulated, we state only the simplest ones.

**Conjecture 4.2.** ([10], §9.2, 9.8, 9.5, 9.20.)

(i) A compact connected Kähler manifold \( X \) is ‘special’ if and only if any one of the following properties is satisfied:

1.1. The Kobayashi pseudo-metric of \( X \) vanishes identically.
1.2. There is a entire curve \( h : \mathbb{C} \to X \) with Zariski-dense image.
1.3. There is a entire curve \( h : \mathbb{C} \to X \) with metrically dense image.

1.4. Any two generic points of \( X \) are joined by some entire curve.

1.5. Any countable subset of \( X \) is contained in the image of some entire curve.

(ii) An arithmetic analogue is the following:

\( X \) is potentially dense (i.e.: \( X(k) \) is Zariski-dense for \( k \) a sufficiently large number field \( k \) of definition of \( X \)).

(iii) Moreover we conjecture that \( \pi_1(X) \) is almost Abelian if \( X \) is special ([10], Conjecture 7.1).

- The property 1.4 means that special manifolds are expected to be analogues of rationally connected manifolds, replacing algebraic maps from \( \mathbb{C} \to X \) by their transcendental version: entire curves \( h : \mathbb{C} \to X \), with \( h \) holomorphic, non-constant.
- In general, we have the obvious implications:
  
  \[
  1.5 \implies 1.3 \implies 1.1, 1.3 \implies 1.2, \text{ and } 1.5. \implies 1.4 \implies 1.1.
  \]
- If \( X \) is unirational, all the properties above (1.1–1.5, 2 and 3) are satisfied.
- If there exists a surjective holomorphic map \( p : \mathbb{C}^n \to X \), then all these properties 1.1-1.5 are satisfied.
- Properties 1.1 and 1.4 are obvious for rationally connected manifolds.

The main purpose of this article is to show that every rationally connected manifold admits a dense entire curve. In fact, we prove 1.2, 1.3 and 1.5 for rationally connected manifolds. (Theorem 5.2). Thus all the properties 1.k (1 ≤ k ≤ 5) hold for every rationally connected manifold, i.e., we confirm part 1.5 of the conjecture for the simplest class of special manifolds, namely rationally connected manifolds.

- It is proved in [7] that a projective surface is 'special' in the above sense if and only if it is \( \mathbb{C}^2 \)-dominable (With the possible exception of non-elliptic and non-Kummer \( K3 \) surfaces.). This might however be a low-dimensional phenomenon, and it is not expected to remain true in dimension 3.

**Remark.** Vojta [45] has introduced a dictionary between entire curves and infinite sequences of rational points. The conjectures above imply that if \( X \) is defined over a number field, the set of \( k' \)-rational points is infinite for some finite extension \( k'/k \) if and only if \( X \) contains an entire curve, Vojta’s analogy thus becomes an equivalence between entire curves and arithmetic geometry. The Nevanlinna version of the Hilbert property discussed in §9 is another illustration of this statement.
5. **Entire Curves in Rationally Connected Manifolds**

We recall here classical results used in the next subsections.

5.1. **Comb smoothing.** We recall the technique of "comb smoothing", introduced in [33], see also [22].

In the sequel, we always see $\mathbb{P}_1 = \mathbb{C} \cup \{\infty\}$, i.e., with a distinguished point at infinity.

Let $C$ be the reducible curve given by glueing two rational curves together transversally in one point.

Let $V$ be the variety obtained from blowing up the point $(\infty, 0)$ in $\mathbb{P}_1 \times \mathbb{C}$. Using $[y_0 : y_1], t$ as coordinates on $\mathbb{P}_1 \times \mathbb{C}$, we may embed $C$ into $V$ as the total transform of $t = 0$. With $L$ denoting an affine line, the function $t$ realizes $V$ as a flat projective $L$-scheme, with a special fibre isomorphic to $C$ and all other fibers isomorphic to smooth rational curves. By abuse of notation, we denote this special fiber again as $C$.

In explicit coordinates

$$V = \{(y_0 : y_1; [s_0 : s_1]) \in \mathbb{P}_1 \times \mathbb{C} \times \mathbb{P}_1 : s_1 y_0 = s_0 y_1 t\}$$

$$\mapsto t \in L$$

For a given morphism $f : C \to X$ with values in a projective manifold $X$, the morphism $f$ can be extended to a holomorphic map $F$ defined on an open neighborhood of $C$ in $V$ if the following condition $(\ast)$ is fulfilled:

$$f^*TX \text{ is spanned by global sections on } C \text{ and } H^1(C, f^*TX) = \{0\}$$

Let $r \geq 0$ be an integer, recall ([22], Definition 4.5) that a (parametrised) rational curve $f : \mathbb{P}_1 \to X$ is said to be $r$-free if $f^*(TX) \otimes \mathcal{O}_{\mathbb{P}_1}(-(r+1))$ is ample on $\mathbb{P}_1$, or equivalently, if $H^1(\mathbb{P}_1, f^*(TX) \otimes \mathcal{O}_{\mathbb{P}_1}(-(r+1))) = 0$. This is an open condition on the space of such maps.

The condition "1-free" is frequently called "very free".

If $f : C \to X$ is a morphism whose restriction to the two irreducible components of $C$ are $r_1$-free resp. $r_2$-free, then the condition $(\ast)$ is fulfilled. Hence $f$ extends to a holomorphic map $F$ defined on some neighborhood of $C$ in $V$. For each fiber of $V \to L$ contained in this neighborhood we obtain a rational curve $f_t$. Moreover in this case the curves $f_t$ are $(r_1 + r_2)$-free rational curves.

Summarizing, we have the following:

**Proposition 5.1.** Let $X$ be a projective manifold, equipped with a distance function $d$ induced by a hermitian metric on $X$, and let $g_i :
$\mathbb{P}_1 = \mathbb{C} \cup \{\infty\} \to X$ be $r_i$-free rational curves ($i = 1, 2$, $r_i \geq 0$) with $g_1(\infty) = g_2(0)$.

Then for every $R, R', \epsilon > 0$ there is a $(r_1 + r_2)$-free rational curve $c : \mathbb{P}_1 = \mathbb{C} \cup \{\infty\} \to X$ and a parameter $\lambda > 0$ such that $d(c(z), g_1(z)) < \epsilon$ for all $z$ with $|z| < R$ and $d(c(\lambda z), g_2(z)) < \epsilon$ for all $z$ with $|z| > R'$.

Moreover given $p_0, p_1, \ldots, p_{r_1} \in \mathbb{C}$ and $q_1, \ldots, q_{r_2} \in \mathbb{C}^*$ the curve $c$ may be chosen in such a way that there are points $\tilde{p}_i, \tilde{q}_i$ on $C$ with $c(\tilde{p}_i) = g_1(p_i)$, $|\tilde{p}_i - p_i| < \epsilon$ and $c(\tilde{q}_i) = g_2(q_i)$.

Proof. Let $V$ be as above. We embed the first rational curve into $V$ as $\mathbb{C} \cup \{\infty\} \ni x \mapsto (1 : x, 0, [1 : 0])$ and the second one as $\mathbb{C} \cup \{\infty\} \ni x \mapsto (0 : 1, 0, [1 : x])$.

Now the pair $(g_1, g_2)$ defines a morphism from $C$ to $X$ which extends to holomorphic map $F$ defined on an open neighborhood. Thus for small enough $t$ we obtain a rational curve $f_t : \mathbb{C} \cup \{\infty\} \to X$ as $\mathbb{C} \cup \{\infty\} \ni x \mapsto F([1 : x], t, [1 : xt])$.

We define $\lambda = \frac{1}{t}$ and obtain $\mathbb{C} \cup \{\infty\} \ni \lambda x \mapsto F([1 : \lambda x], t, [1 : \lambda xt]) = F([t : x], t, [1 : x])$.

It is now easy to check explicitly that for sufficiently small $t$ the rational curve $f_t$ has the desired properties. $\square$

### 5.2. Dense entire curves in Rationally connected manifolds.

We present our main result on the existence of dense entire curves in rationally connected manifolds. Later we deduce from it stronger forms by applying it to projective jet bundles of $X$ (see corollary 5.7 below).

**Theorem 5.2.** Let $X$ be a rationally connected complex projective manifold and let $A$ be a closed analytic subset of codimension at least two, and let $M$ be a countable subset of $X \setminus A$.

Then there exists an entire curve $h : \mathbb{C} \to X$ such that

(i) $M \subset h(\mathbb{C})$,

(ii) $h(\mathbb{C}) \subset X \setminus A$,

In particular, if $M$ is dense in $X$, so is $h(\mathbb{C})$.

**Corollary 5.3.** Rationally connected manifolds satisfy all the properties of part (i) of Conjecture 4.2.

**Assumptions 5.4.** In the sequel, $X$ always denotes a rationally connected complex projective manifold, $A$ a closed analytic subset of $X$ of
Lemma 5.5. Let $X, A, d$ be as defined above and $r \in \mathbb{N}$. Let $f : \mathbb{P}_1 \to X$ be a $(r - 1)$-free rational curve such that $f(\mathbb{P}_1) \cap A = \emptyset$. Let $a_i \in \mathbb{C}, i = 1, \ldots, r$ be given, let $R > 0, \varepsilon > 0$ be given, with $R \geq |a_i| + \varepsilon, \forall i$. Let $q \in X \setminus A$.

There then exists a rational curve $h : \mathbb{P}_1 \to X$ and $a_i' \in \mathbb{C}, i = 1, \ldots, r + 1$ such that:

1. $h$ is $r$-free.
2. $h(\mathbb{P}_1) \cap A = \emptyset$.
3. $d(h(z), f(z)) \leq \varepsilon$ if $|z| \leq R$.
4. $h(a_i') = f(a_i), i = 1, \ldots, r$.
5. $|a_i' - a_i| \leq \varepsilon, i = 1, \ldots, r$.
6. $h(a_{r+1}') = q$.

Proof. We first prove the existence of a 1-free curve $g : \mathbb{P}_1 \to X$ avoiding $A$ and containing $q$ and $f(\infty)$. Indeed, from [22], Corollary 4.28, we get the existence of an 1-free curve $g : \mathbb{P}_1 \to X$ such that $g(0) = f(\infty)$ and $g(1) = q$. Since the deformations of $g$ are unobstructed, there is a holomorphic map $G : T \times \mathbb{P}_1 \to X$, where $T$ is an open neighborhood of 0 in $H^0(\mathbb{P}_1, f^*(TX) \otimes \mathcal{O}_{\mathbb{P}_1}(-\{0, 1\}))$ inducing deformations $G(t, .) = g_t$ of $g$ fixing $g(0)$ and $g(1)$, with $dG(0,z)(t) = t(z)$, for $z \in \mathbb{P}_1$, and $t \in H^0(\mathbb{P}_1, f^*(TX))$. Now $g^*(TX)$ is generated outside of 0 and 1 by its sections, and so $G$ has maximal rank near $\{0\} \times (\mathbb{P}_1 \setminus \{0, 1\})$, so that $G^{-1}(A)$ has codimension at least 2 in $T \times \mathbb{P}_1$. Its projection in $T$ has thus codimension at least one, and hence the generic $g_t$ avoids $A$.

We now apply the “comb smoothing” technique (see Proposition 5.1) to the comb defined by the rational curves $f$ and $g$. We obtain a $r$-free rational curve $h$ fulfilling the properties 1, 3, 4, 5, 6. Since $f(\mathbb{P}_1) \cup g(\mathbb{P}_1)$ do not intersect $A$, we see that $C \cap F^{-1}(A)$ (in the notation of prop. 5.1) is empty. It follows that $F^{-1}(A)$ does not intersect an open neighborhood of $C$ in $V$. Therefore we may choose $h$ such that $h(\mathbb{P}_1) \subset X \setminus A$ (i.e., such that condition 2 is fulfilled).

Proof. We now prove Theorem 5.2. Let $M := \{x_1, x_2, \ldots, x_n, \ldots\}$ be the given sequence in $X \setminus A$. We construct inductively a sequence of rational curves $h_n : \mathbb{P}_1 = \mathbb{C} \cup \{\infty\} \to X, R_n > 0, \varepsilon_n > 0, a^k_n \in \mathbb{C}, \forall k \geq n \geq 1$ such that:

0. $\varepsilon_n \to 0$ when $n \to +\infty$, the sequence $\varepsilon_n$ being decreasing.
1. $h_n$ is $n$-free.
2. $d(h_n(\mathbb{P}_1), A) \geq \varepsilon_n > 0$, in particular: $h_n(\mathbb{P}_1) \cap A = \emptyset$.
3. $d(h_{n+1}(z), h_n(z)) < \frac{\varepsilon_n}{2^{n+1}}$ for $|z| \leq R_n$. 
Corollary 5.6. Let $X$ be a rationally connected projective manifold with a free rational curve $c : \mathbb{P}_1 \to X$ and a distance function $d(\cdot, \cdot)$ induced by some hermitian metric.

Then for every $R > 0, \epsilon > 0$ there exists an entire curve $h : \mathbb{C} \to X$ with dense image and such that $d(h(z), c(z)) < \epsilon$ for all $z \in \mathbb{C}$ with $|z| < R$.

Corollary 5.7. Let $X, A, M$ be as above in \[5.4\], let $D \subset X$ be a reduced divisor (not necessarily irreducible nor connected), and let $m > 0$ be an integer. For each $x_n \in M$, fix an (unparametrized) $m$-jet $j_n$ of map from the unit disc in $\mathbb{C}$ to $X$ at $x$. The map $h$ of Theorem \[5.2\] which sends $a_n \in \mathbb{C}$ to $x_n$ can be chosen to have its $m$-jet at $a_n$ coincide with $j_n$, this for each $n$, and moreover: to avoid $D^{\text{sing}}$, and to be transversal to $D^{\text{reg}}$ at each point where $h(\mathbb{C})$ meets $D^{\text{reg}}$. (Here $D^{\text{sing}}$ is the singular locus of $D$, and $D^{\text{reg}}$ is its regular part).
Proof. We prove this first for $m = 1$: let $\pi : X_1 := \mathbb{P}(TX)$ be the natural projection. Since $X$ is rationally connected, so is $X_1$.

Lift arbitrarily the sequence $M$ to the given sequence $M_1$ in $X_1$, and put $A_1 := [\pi^{-1}(A \cup D^{\text{reg}})] \cup \mathbb{P}(TD^{\text{reg}}) \subset X_1$. Apply Theorem 5.2 to $X_1, A_1, M_1$ to get the result, since $A_1$ is algebraic, closed, and of codimension at least 2 in $X_1$. Then proceed inductively on $m$, by defining $X_m := (X_{m-1})_1$ to get the general case if $m > 1$.

Remark. By composing $h$ with a suitable Weierstrass product $w : \mathbb{C} \to \mathbb{C}$, one can even realise the jets $j_m$ themselves, and not only at the level of their projectivisation. This gives an entire curve analogue of the Weak Approximation Property in arithmetic geometry on rationally connected manifolds. This connection with the Nevanlinna Hilbert Property treated below was pointed to us by P. Corvaja.

One can refine Corollary 5.7 by imposing to $h$ to meet $D$ transversally at every of its regular intersection point:

**Theorem 5.8.** In the set-up of theorem 5.2, assume that in addition a (reduced) hypersurface $D$ in $X$ is given. Then we may chose the entire curve $h : \mathbb{C} \to X$ provided by theorem 5.2 in such a way that every intersection point of $h(\mathbb{C})$ and $D$ is transversal.

Before proving the theorem, we need some auxiliary lemmata.

**Lemma 5.9.** Let $D$ be a reduced hypersurface in a projective manifold $X$ and let $u : \mathbb{P}_1 \to X$ be a 2-free rational curve.

Then there exist arbitrarily small deformations $\tilde{u}$ of $u$ such that every intersection point of $\tilde{u}(\mathbb{P}_1) \cap D$ is transversal.

**Proof.** A generic deformation of $u$ will avoid any given analytic subset of codimension at least two. Therefore a generic deformation of $u$ avoids $\text{Sing}(D)$. Now let $p \in \mathbb{P}_1$ with $u(p) \in D \setminus \text{Sing}(D)$. By assumption $u^*TX \otimes I(2\{p\})$ is an ample vector bundle on $\mathbb{P}_1$. By the theorem of Grothendieck $u^*TX$ is a direct sum of line bundles. We choose a direct summand $L$ of $u^*TX$ which is complimentary to $u_p^*TD$ at $p$. Now we take a section of $u^*TX$ which vanishes at $p$, but has non-zero derivative in the direction of $L$. Then the corresponding deformations yield rational curves which intersect $D$ with multiplicity 1 in $p$. Since $D \cap u(\mathbb{P}_1)$ is finite (in fact its cardinality is bounded by $\deg(u^*\mathcal{L}(D))$), it follows that a generic deformation will have transversality at every point of intersection.

**Lemma 5.10.** Let $X$ be a complex manifold, $D$ a (reduced) hypersurface on $X$, and $h : \mathbb{C} \to X$ an entire curve. Let $G$ be a relatively
compact open subset of $\mathbb{C}$ with $h(\partial G) \cap D = \emptyset$. Let $d$ be a distance function on $X$ induced by some hermitian metric.

Then there exists a number $\delta > 0$ such that the following two properties hold for every entire curve $\tilde{h} : \mathbb{C} \to X$ with $d(\tilde{h}(z), h(z)) < \delta$, $\forall z \in \bar{G}$:

(i) $\tilde{h}(\partial G) \cap D = \emptyset$.
(ii) $\deg_B(h^*D) = \deg_B(\tilde{h}^*D)$.

Proof. Follows easily using the theorem of Rouché. \qed

Lemma 5.11. Let $X$ be a complex manifold, $D$ a (reduced) hypersurface on $X$, and $h : \mathbb{C} \to X$ an entire curve. Let $G$ be a relatively compact open subset of $\mathbb{C}$ with $h(\partial G) \cap D = \emptyset$. Let $d$ be a distance function on $X$ induced by some hermitian metric. Assume that $h^*D$ is reduced (i.e. all the multiplicities are one).

Then there exists a number $\delta > 0$ such that the following two properties hold for every entire curve $\tilde{h} : \mathbb{C} \to X$ with $d(\tilde{h}(z), h(z)) < \delta$, $\forall z \in \bar{G}$:

(i) $\tilde{h}(\partial G) \cap D = \emptyset$.
(ii) $\tilde{h}^*D$ is reduced on $G$.

Proof. Let $p_1, \ldots, p_n$ be the points of $h^*D$. We choose small balls $B_i$ such that $p_i \in B_i \subset G$ and such that $p_j \notin \overline{B_i}$ for $i \neq j$. Then we choose $\epsilon_i > 1$ such that $\deg_{B_i}(\tilde{h}^*D) = 1$ for every entire curve $\tilde{h} : \mathbb{C} \to X$ with $d(\tilde{h}(z), h(z)) < \epsilon_i$, $\forall z \in B_i$. (This we can do due to the preceding lemma). Finally we choose $\epsilon = \min_i \epsilon_i$. \qed

Proof of theorem 5.8. We proceed as in the proof of theorem 5.2 with the following modifications. We choose the rational curve $h_n$ such that $h_n^*D$ is reduced on $\{ z \in \mathbb{C} : |z| \leq R_n \}$ (This is possible due to Lemma 5.9). Thanks to Lemma 5.11 there exists a number $\delta_n > 0$ such that any entire curve $\tilde{h} : \mathbb{C} \to X$ with $d(h_n(z), \tilde{h}(z)) < \delta_n$, $\forall |z| \leq R_n$ preserve this property (i.e., $\tilde{h}^*D$ is reduced on the disc with radius $R_n$). We choose $\epsilon_n$ such that $\epsilon_n < \delta_n$. In this way we finally arrive at an entire curve $h$ with $h^*D$ being reduced everywhere. \qed

Remark. The arithmetic version (i.e.: the potential density of Rationally connected manifolds $X$ defined over a number field) of the Theorem 5.2 is open when $X$ is not known to be unirational, even for conic bundles over $\mathbb{P}_n, n \geq 2$. There are (at least) two cases known: smooth quartics in $\mathbb{P}_4$ ([28]), and some conic bundles over $\mathbb{P}_2$ ([6], where $X(\mathbb{Q})$ is shown to be already Zariski-dense).
5.3. **An orbifold situation.** In some situations, we can impose tangency conditions as well. These are simple instances of an extension of the preceding results to the ‘orbifold pair’ situation, for which the appropriate techniques have not been presently developed.

**Corollary 5.12.** Let $S \subset \mathbb{P}^3$ be a smooth sextic surface. There exists entire curves $g : \mathbb{C} \to \mathbb{P}^3$ with metrically dense image $C$, and such that $C$ is tangent to $S$ at each point where it meets $S$.

**Proof.** Let $\pi : X \to \mathbb{P}^3$ be the double cover ramified along $S$: $X$ is rationally connected since it is Fano. Let $h : \mathbb{C} \to X$ be a dense entire curve. Then $g := \pi \circ h : \mathbb{C} \to \mathbb{P}^3$ is a dense entire curve with the claimed property. \qed

We don’t know how to prove Corollary 5.12 directly on $(\mathbb{P}^3, S)$, without going to the double cover.

**Remark.** Observe that in the situation of the above corollary $K_{\mathbb{P}^3} + (1 - \frac{1}{2})S$ has negative degree (i.e.: the smooth ‘orbifold pair’ $(\mathbb{P}^3, (1 - \frac{1}{2})S)$ is Fano). The statement of the preceding corollary thus means that this orbifold pair admits metrically dense ‘orbifold entire curves’ (in the ‘divisible’ sense). This is conjectured in [9] to be the case for arbitrary smooth orbifolds which are Fano (or, more generally, Rationally connected in the orbifold sense).

5.4. **Brody curves.** An entire curve $f : \mathbb{C} \to X$ is a “Brody curve” if its derivative is uniformly bounded with respect to the euclidean metric on $\mathbb{C}$ and an arbitrary hermitian metric on $X$. The theorem of Brody states that on a compact complex manifold $X$ there is a non-constant Brody curve if and only if there is a non-constant entire curve.

However, there is no such statement concerning dense curves. Indeed, an Abelian threefold blown-up along a curve is constructed in [46], which admits (lots of) dense entire curves, although every Brody curve is contained in the exceptional divisor. The existence of dense Brody curves is thus not a bimeromorphic property.

Our results give no information about the existence of (Zariski-)dense Brody curves on arbitrary rationally connected manifolds (although they obviously exist on some of them).

6. **Normal rational surfaces.**

If $X$ is a singular complex algebraic variety with desingularization $\pi : \tilde{X} \to X$, then there exists a (Zariski-)dense entire curve $h : \mathbb{C} \to X$ if and only if there exists a (Zariski-)dense entire curve $\tilde{h} : \mathbb{C} \to \tilde{X}$. 
(This is clear, because we can lift every holomorphic map \( h : \mathbb{C} \to X \) whose image \( h(\mathbb{C}) \) is not contained in the singular locus \( X^{\text{sing}} \).

However, it is a much more delicate question whether there exists a (Zariski-)dense entire curve \( h : \mathbb{C} \to X \) which avoids the singular locus \( X^{\text{sing}} \) of \( X \).

In this section we prove the existence of such curves for certain normal rational surfaces (Theorem 6.1). Note that most normal rational surfaces have no Zariski dense entire curves in their smooth locus. See examples 6.9 and 6.10 at the end of this section.

We consider the following situation: \( S \) is a normal projective surface with only quotient singularities, such that \( -K_S = \Delta \), an effective nonzero \( \mathbb{Q} \)-divisor on \( S \) with \( \text{Coeff}(\Delta) \leq 1 \), where \( \text{Coeff}(\Delta) \) is the largest of the coefficients of the irreducible components of the support of \( \Delta \).

This permits to apply to \( S \) the MMP, which preserves the condition on \( \text{Coeff}(\Delta) \), and abuts, after finitely many \( K \)-negative contractions, to a pair \((S', \Delta')\) such that either \( K_{S'} \) is nef, or anti-ample with Picard number 1, or admitting a Fano-contraction \( f : S' \to B \) on a smooth projective curve \( B \) with relative Picard number 1. This allows us to characterise the existence of dense entire curves in the smooth locus of \( S \).

**Theorem 6.1.** Let \( S \) be a normal rational surface with only quotient singularities, and \( F \subset S \) be a finite subset containing the singular points of \( S \). Let \( \Delta \) be an effective nonzero \( \mathbb{Q} \)-divisor such that \( (K_S + \Delta) \) is \( \mathbb{Q} \)-trivial.

(i) If \( \text{Coeff}(\Delta) < 1 \), then for any two generic points of \( S \setminus F \), there is a 1-free rational curve through these two points avoiding \( F \).

(ii) If \( \text{Coeff}(\Delta) \leq 1 \), then there is an entire curve \( h : \mathbb{C} \to S \setminus F \) with dense image.

**Proof.** Applying the Minimal Model Program (MMP) for surfaces (see [34], Theorem 3.47, 4.11 and 4.18), we obtain a birational map \( \mu : S \to S' \) to a normal surface \( S' \) with only quotient singularities and a \( \mathbb{Q} \)-divisor \( \Delta' := \mu_*(\Delta) \) on \( S' \) with \( \text{Coeff}(\Delta') \leq 1 \), and \( K_{S'} + \Delta' \) \( \mathbb{Q} \)-trivial such that one of the following possibilities hold:

(i) The canonical divisor \( K_{S'} \) is nef.

(ii) The anticanonical divisor \( -K_{S'} \) is ample.

(iii) There is a Fano fibration \( f : S' \to B = \mathbb{P}_1 \) of relative Picard number 1.

In our situation, the first case does not occur. Indeed, in this case we had: \( 0 \leq \kappa(S') = \kappa(S) = \kappa(S, -\Delta) = -\infty \), since: \( \Delta \) is \( \mathbb{Q} \)-effective and
nonzero, the MMP preserves the Kodaira dimension, and \( \kappa(S') \geq 0 \) if \( K_{S'} \) is nef.

In the second case, our first claim has been proved in [47]. Claim 2 now follows from claim 1 just as in the smooth case, proved in Theorem 5.2.

In the third case the existence of a dense entire curve and of rational curves with the said properties follow from Proposition 6.8. \( \Box \)

**Remark.** When \( \Delta = 0 \) above, one also expects the existence of dense entire curves, but this is an open question. See Example 6.9 for an example in which this conclusion does not follow from the arguments given here, but from an explicit description.

It were interesting also to get a criterion to recognise from the initial data \((S, \Delta)\) itself, whether one abuts, for some suitable MMP, to the \( \mathbb{Q} \)-Fano case with \( \rho = 1 \), or to the fibered case for any MMP-sequence of contractions.

We need some preparatory lemmas.

**Assumptions 1.**

\( S \) denotes a projective normal surface with only quotient singularities, and there exists a surjective Fano-fibration \( f : S \to B \) of relative Picard number 1. This implies that all fibers \( S_b = f^{-1}\{b\} \) are irreducible. The \( \mathbb{Q} \)-divisor \( \Delta_f := \sum b (1 - \frac{1}{m_b})\{b\} \) (with \( m_b \) being the multiplicity of the fiber \( f^{-1}(b) \) for each \( b \in B \)) denotes the 'orbifold base' of \( f \).

**Lemma 6.2.** Let \( S \) be a normal surface with a holomorphic map \( f : S \to B \) with irreducible fibers. If there exists an entire curve \( h : \mathbb{C} \to S_{\text{reg}} \) such that \( f \circ h : \mathbb{C} \to B \) is nonconstant, then \( \deg(K_B + \Delta_f) \leq 0 \). Thus: either \( B \) is elliptic and \( \Delta_f = 0 \), or \( (B, \Delta_f) = (\mathbb{P}_1, \Delta_f) \) is in the short classical ‘platonic’ list of orbifolds on \( \mathbb{P}_1 \) with (semi-)negative canonical bundle.

**Proof.** This follows from [15], since \( f \circ h \) is then an orbifold morphism to \((B, \Delta_f)\). \( \Box \)

We shall next use the following properties:

1. If \( m_b = 1 \), \( S_b \cong \mathbb{P}_1 \) is smooth ([31], 11.5.1), and \( f \) is thus a \( \mathbb{P}_1 \)-bundle over \( B \) if and only if \( \Delta_f = 0 \).

2. Assume\(^3\) there exists a finite ramified cover \( \beta : B' \to B \) which ramifies at order exactly \( m_b \) over \( b \), this for each \( b \in B \). Let \( \sigma : S' \to S \)

\(^3\)This is true unless \( B = \mathbb{P}_1 \), and the support of \( \Delta_f \) consists of one or two points with different multiplicities. However, in these two cases there is still such a dominant finite algebraic map \( \beta : \mathbb{C} \to B \) which suffices for our purposes.
and $f' : S' \to B'$ be deduced by base-changing $f$ by $\beta$, and normalising the base-change $S \times_B B'$. Then $f'$ has generically reduced fibres, $S'$ has still quotient singularities, with $K_{S'} + \Delta'$ $\mathbb{Q}$-trivial ([34], Proposition 5.20 and Proposition 4.18), and $\text{Coeff}(\Delta') \leq 1$. We can still apply to $S'$ the MMP and abut now to a fibration $f'' : S'' \to B'$ which has only reduced fibres, and is thus a $\mathbb{P}_1$-bundle.

**Lemma 6.3.** Let $f : S \to B$ be as in the assumptions above. Let $f' : S' \to B'$ be deduced from $f$ by a finite base-change $B' \to B$. Assume that all fibres of $f' : S' \to B'$ have a reduced component. Let $F \subset S^{\text{reg}}$ be a finite set. Then $S^{\text{reg}} \setminus F$ contains a dense entire curve (resp. a 1-free rational curve through any two of its generic points) if $B'$ is elliptic (resp. rational).

*Proof.* It is sufficient to show this for $(S')^{\text{reg}} \setminus F'$, where $F'$ contains the inverse image in $S'$ of $S^{\text{sing}}$ and of $F$. But $S'$ dominates a $\mathbb{P}_1$-bundle over $B'$, after the preceding observations, which implies the claims. □

The following corollary then follows from the existence of a suitable ‘orbifold-étale’ base change $B' \to B$, with $B'$ either elliptic or rational, when $(B, \Delta_f)$ is in this list of ‘platonic’ orbifolds.

**Corollary 6.4.** Let $f : S \to B$ be a Fano fibration of relative Picard rank one, $S$ a normal projective surface with only quotient singularities. The following properties are then equivalent:

1. $\deg(K_B + \Delta_f) < 0$ (resp. $\deg(K_B + \Delta_f) \leq 0$), where $(B, \Delta_f)$ is the orbifold basis of $f$.

2. There exists a rational curve (resp. an entire curve) on $S^{\text{reg}}$ which is not $f$-vertical\footnote{i.e.: not contained in some fibre of $f$.}

3. There exists a 1-free rational curve through any two generic points of $S$ (resp. a dense entire curve) which is contained in $S^{\text{reg}}$.

• We shall now relate $K_S$ to $K_B + \Delta_f$ by means of a ‘canonical bundle’ formula for $K_S$.

Let $r : \tilde{S} \to S$ be a minimal resolution, and $t : \tilde{S} \to S_0$ be a relative minimal model of $\tilde{S}$ over $B$, so that $f_0 : S_0 \to B$ is a $\mathbb{P}_1$-bundle, with $f_0 \circ t = f \circ r$. Then $S_0 = \mathbb{F}_m$, the Hirzebruch surface of index $m$ for some $m \in \mathbb{N}_0$. In this case, we denote by $G$ (resp. $G'$) the section of $f_0$ with $G.G = -m$ (resp. a section of $f_0$ with $G.G' = 0$, or equivalently, disjoint from $G$). We have: $K_{S_0} = -(G + G') + (f_0)^*(K_B)$. We shall then denote with $C$ (resp. $C'$) the strict transforms in $S$ of $G, G'$. 

Lemma 6.5. Under the above assumptions, we have:
\[ K_S \sim f^*(K_B + \Delta_f) - (C + C'). \] Moreover, \( C \) and \( C' \) are still disjoint.

Proof. Each singular fibre of the fibration \( f \circ r : \tilde{S} \to B \) has two reduced components, each of which is met once by either \( \tilde{G} \) or \( \tilde{G}' \), the strict transforms of \( G, G' \) in \( \tilde{S} \). The contraction \( r : \tilde{S} \to S \) then sends each of these components to 2 distinct singular points of the corresponding fibre of \( f \), by [31], 11.5. 5, which implies that \( C, C' \) are disjoint, too. The first claim then follows from the fact that \( F_b := (S_b)_{\text{red}} \) is irreducible for each \( b \in B \), so that \( K_S = aF_b - C - C' \) locally near \( F_b \), with \( a = (m_b - 1) \) by adjunction, since \( F_b \) has multiplicity \( m_b \) in \( f \).

Lemma 6.6. Under the above assumptions \( C, C' = 0 \) and \( C' \sim C + k\Phi \) for some \( k \geq 0 \) (possibly exchanging \( C, C' \)). Moreover, for each \( f \)-horizontal irreducible curve \( H \subset S \) different from \( C, C' \), we have: \( H \sim d(C' + \ell \Phi) \) with \( d := H.\Phi > 0, \ell = \frac{H.C}{d} \geq 0 \).

Proof. \( S \) is \( \mathbb{Q} \)-factorial (because it has only quotient singularities) and its Picard number is 2, since \( f : S \to B \) has relative Picard number one. Hence every divisor is numerically equivalent to a linear combination of \( C' \) and \( \Phi \). The assertions of the lemma are now obvious.

Assumptions 2.

Let \( f : S \to B \) be a Fano fibration with \( S \) normal and having only quotient singularities. Assume now that \( K_S + \Delta \) is \( \mathbb{Q} \)-trivial, where \( \Delta \) is a nonzero effective \( \mathbb{Q} \)-divisor on \( S \). We write \( \Delta = \Delta^h + \Delta^v \), where \( \Delta^h \) (resp. \( \Delta^v \)) is the \( f \)-horizontal (resp. \( f \)-vertical) part of \( \Delta \).

We write \( \text{Coeff} f^h(\Delta) \) for the largest coefficient in \( \Delta \) of the irreducible components of \( \Delta^h \).

Lemma 6.7. Under the above assumptions, \( \deg(K_B + \Delta_f) \leq 0 \). Moreover, \( \deg(K_B + \Delta_f) < 0 \) if \( \text{Coeff} f^h(\Delta) < 1 \).

Proof. Since: \( \Delta = \Delta^h + \Delta^v \sim K_S \sim (C + C') - f^*(K_B + \Delta_f) \), \( f^*(K_B + \Delta_f) \sim \Delta^v + \Delta^h - (C + C') \), and it is sufficient to show that: \( \Delta^h - (C + C') \) is \( \mathbb{Q} \)-effective. We may write: \( \Delta^h = aC + a'C' + R \), where \( R = \sum_j a_j C_j \) with \( C_j \) being irreducible multisections of \( f \) distinct from \( C, C' \), each of degree \( d_j > 0 \) over \( B \), and \( a_j > 0 \) are rational numbers.

We have, for each \( j \): \( C_j = d_j(C' + k_j\Phi) \) for some \( k_j \in \mathbb{Q} \). Thus \( R = \sum_j a_j(C' + k_j\Phi) \). Calculating intersection numbers with \( \Phi \) gives:
\[ 2 = (C + C').\Phi = \Delta^h.\Phi = a + a' + \sum_j a_j. \] Therefore:
\[ \Delta^h - (C + C') \sim aC + a'C' + (2 - a - a') \sum_j(C' + k_j\Phi) - (C + C') \]
\[(1 - a)(C' - C) + (2 - a - a')\left(\sum_j k_j\right)\Phi.\]

Recall that \(k_j \geq 0\). Furthermore \((C' - C)\) is effective (lemma [6.6]). Hence the above equation implies that \((\Delta^h - C - C')\) is \(\mathbb{Q}\)-effective, and so is \((\Delta - C - C')\) which implies: \(\deg(\Delta_f + K_B) \leq 0\).

If \(\text{Coeff}_h(\Delta) < 1\), we have: \(a, a' < 1\). Consequently \((1 - a) > 0\) which implies that \((\Delta^h - C - C')\) is non-zero, and so: \(\deg(\Delta_f + K_B) < 0\). \(\square\)

We have the following criterion for the existence of 1-free rational curves (resp. dense entire curves) in the regular part of such Fano fibrations. It immediately implies Theorem [6.1].

**Proposition 6.8.** Let \(S\) be a normal projective surface with only quotient singularities, with a non-zero effective \(\mathbb{Q}\)-divisor \(\Delta\) on \(S\) such that \(K_S + \Delta\) \(\mathbb{Q}\)-trivial.

Let \(f : S \to B\) be a Fano fibration to a smooth curve \(B\) with relative Picard number 1. Assume that with \(\text{Coeff}_f^h(\Delta) < 1\) (resp. \(\text{Coeff}_f^h(\Delta) \leq 1\)).

The following three properties then hold and are equivalent:

1. \(\deg(K_B + \Delta_f) < 0\) (resp. \(\deg(K_B + \Delta_f) \leq 0\)), where \((B, \Delta_f)\) is the orbifold basis of \(f\).
2. There exists a rational curve (resp. an entire curve) on \(S_{\text{reg}}\) which is not \(f\)-vertical.
3. There exists a 1-free rational curve through any two generic points of \(S\) (resp. a dense entire curve) which is contained in \(S_{\text{reg}}\).

**Proof.** The equivalence of (1), (2) and (3) is due to Corollary [6.4]. Property (1) holds because of Lemma [6.7]. \(\square\)

6.1. Examples.

**Example 6.9.** Let \(S := \mathbb{A}/\mathbb{Z}_4\), where \(A = \mathbb{E} \times \mathbb{E}\) and \(E := \mathbb{C}/\mathbb{Z}[\sqrt{-1}]\), and \(\mathbb{Z}_4\) is generated by the multiplication by \(\sqrt{-1}\) on the two factors simultaneously. Then \(S\) (known as “Ueno surface”) has quotient non-canonical singularities and a \(\mathbb{Q}\)-trivial canonical bundle. If \(\pi : S' \to S\) is its minimal resolution, then \(-2K_{S'} = \sum E_j\), where \(E_j\) are the \(-4\)-curves lying over the non-canonical singularities. The preceding Proposition applies to \(S'\), but not to \(S\) (since all rational curves on \(S\) meet some of the non-canonical singularities of \(S\)). See [9] for a description of the MMP on \(S'\). Remark also that \(S_{\text{reg}}\) contains many dense entire curves (coming from \(A\)) although it contains no rational curves.

On most normal rational surfaces, every entire curve meets the singular locus, as shown (for \(m \geq 5\)) in the simplest example below:
Example 6.10. Let $f : S \to \mathbb{P}_1$ be a Fano fibration from a normal surface $S$ with Picard number 2 and with only $2m$ ordinary double points as singularities, these lying in pairs on $m \geq 0$ double fibres of $f$. Such surfaces are obtained by applying the following steps on $m$ distinct fibres, starting from any Hirzebruch surface $S_0$. Blow-up one point, then the intersection point of the two $-1$-curves on the first blow-up. One thus gets a fibre with 3 components: one double $-1$-curve meeting two reduced $-1$-curves. Then blow-down these last two curves to ordinary double points.

From Theorem 6.1 and [15] (applied as in Lemma 6.2), we get that $S^{\text{reg}}$ contains 1-free rational curves (resp. dense entire curves) if and only if $m \leq 3$ (resp. $m \leq 4$).

7. Dense entire curves transversal to all divisors.

One can strengthen the results of §5 to deal with all divisors $D$ simultaneously:\footnote{This is motivated by Theorem 9.4 below.}

Theorem 7.1. Let $X$ be a rationally connected complex projective manifold, $X$ an analytic subset of codimension at least two and $M$ a countable subset of $X \setminus A$.

Then there exists an entire curve $h : \mathbb{C} \to X \setminus A$ with $M \subset h(\mathbb{C})$ such that its image meets any irreducible divisor $D \subset X$ transversally at some of its smooth points.

Proof. Due to theorem 5.2 there exists an entire curve $h : \mathbb{C} \to X \setminus A$ with $M \subset h(\mathbb{C})$. We have to show that this map $h$ may be chosen in such a way that its image meets any irreducible divisor $D \subset X$ transversally at some of its smooth points.

Let $S \subset \text{Chow}(X)$ be the subset parametrizing irreducible divisors: it is a countable union of irreducible quasi-projective varieties\footnote{Since $h^1(X, \mathcal{O}_X) = 0$ here, these quasi-projective varieties are Zariski open subsets of projective spaces, and in particular, smooth.}. And $S$ is thus the union of an increasing sequence of compact subsets $S_k, k \geq 1$, closures of their interiors, with $S_k \subset S_{k+1}$ for each $k$. For each $s \in S$ and $g : \mathbb{P}_1 = \mathbb{C} \cup \{\infty\} \to X$, we consider the following property $T$

- There is a complex number $z \in D(1, \frac{1}{2})$ such that $c(\mathbb{P}_1)$ intersects $D_s$ transversally in $g(z)$ (i.e., $g'(z) \not\in T_{g(z)}D_s$, in particular $g(z) \in D^{\text{reg}}_s$).
- There is no other point $w$ in $D(1, \frac{1}{2})$ with $g(w) \in D_s$. 


This property \( T \) is open in \( c \) and \( s \). We may choose a 1-free rational curve \( g_s : \mathbb{P}_1 \to X \) such that \( g_s(\mathbb{P}_1) \) is transverse to \( D_s \) at \( q_s := g_s(1) \in D_s \text{reg} \), and has no other zero on the closed disk \( \overline{D(1, \frac{1}{2})} \). In particular, \( g_s \) fulfills property \( T \) with respect to \( s \). By the openness of property \( T \) there exists an open neighborhood \( s \in B_s \subset S \) and a positive number \( \alpha_s > 0 \) such that property \( T \) still holds for all \((s', \tilde{g})\) where \( s' \in B_s \) and where \( \tilde{g} : \overline{D(1, \frac{1}{2})} \to X \) is holomorphic such that \( d(\tilde{g}(z), g_{s_m}(z)) \leq \alpha_m, \forall z \in D(1, \frac{1}{2}) \). In other words, \( \tilde{g}(D(1, \frac{1}{2})) \) meets each \( D_{s'} \) \((s' \in B_s)\) transversally at a single point which is the interior of the disc.

For each \( k > 0 \), there thus exist finitely many \( s \in S_{k+1} \setminus S_k \) such that the open sets \( B_s \) cover the (compact) closure \( S_{k+1} \setminus S_k \) of \( S_{k+1} \setminus S_k \).

We may therefore choose a sequence \( s_m \) (together with a sequence \( g_m = g_{s_m} \)) such that \( S = \bigcup_m B_{s_m} \).

Let \( M \) be the set consisting of the given sequence \((x_n)_{n>0}\) of points. We now consider the following sequence of points \( y_n, n \geq 1 : y_{3m} := x_m, y_{3m+1} := q_m', \) where \( q_m := g_m(0) \in g_m(\mathbb{C}) \subset g_m(\mathbb{P}_1) \). Finally, let \( y_{3m+2} := q_m = g_m(1), \) this for every \( m \geq 0 \). We now construct inductively on \( m \) the sequence of rational curves \( h_n \) according to this sequence of points \( y_n \) as in the proof of Theorem 5.2 with the only restriction that for each \( m > 0 \), the curve \( h_{3m+2} \) is obtained from \( h_{3m+1} \) using the comb defined by \( h_{3m+1} \) and \( g_m \), that is smoothing this comb. Observe indeed that \( h_{m+1}(\mathbb{P}_1) \) contains the point \( q_m' \in g_m(\mathbb{P}_1) \), in addition to the points \( x_1, \ldots, x_m, q_1', \ldots, q_m', q_1, \ldots, q_{m-1} \).

Choose \( h_{3m+2} \) such that \( d(g'(z), g_m(z)) \leq \frac{\alpha_m}{2^{3m+3}}, \forall z \in D(1, \frac{1}{2}) \), where \( g'(z) = h_{3m+2}(y) \), where \( \sigma y = z \) and \( h_{3m+2} := H(\sigma, \_0) \) in the notations of the proof of Lemma 5.5. The transversality of \( h_{3m+2} \) to \( D_s \) then holds for \( s \in B(s, \frac{\alpha_m}{2}) \), and also for the subsequent \( h_n, n > 3m + 2 \), provided the sequence \( \epsilon_n \) introduced in the proof of Theorem 5.2 is sufficiently small (it can be inductively choosen). The fact that \( h_n \) converges uniformly on compacts of \( \mathbb{C} \) to an entire map \( h \) satisfying the claims of Theorem 5.1 is checked as in the proof of Theorem 5.2 \( \square \)

7.1. **An analogue for** \( \mathbb{C}^n \). Here we discuss dense entire curves in \( \mathbb{C}^n \), proving a strong existence result which will be useful later on.

**Theorem 7.2.** Let \( E, D \) be closed analytic subsets in \( \mathbb{C}^n \). Assume \( \text{codim}(E) \geq 2, \text{codim}(D) \geq 1 \). Let \( S \) be a countable subset of \( \mathbb{C}^n \setminus E \).

Then there exists a holomorphic map \( F : \mathbb{C} \to \mathbb{C}^n \) such that

(i) \( F(\mathbb{C}) \) is dense in \( \mathbb{C}^n \),

(ii) \( S \subset F(\mathbb{C}) \),

(iii) \( F(\mathbb{C}) \) does not intersect \( E \), and
(iv) $F(C)$ and $D$ are transversal in every point of intersection.

Proof. By enlarging $S$ we may and do assume that $S$ is a dense subset of $\mathbb{C}^n \setminus E$. We assume that $E$ contains the singular locus of $D$. Let $g : \mathbb{C} \to \mathbb{C}^n$ be a holomorphic map with $g(Z) = S$. Let $H$ denote the Fréchet vector space of holomorphic maps from $\mathbb{C}$ to $\mathbb{C}^n$. For every $v \in H$ we define

$$F_v(z) = g(z) + (e^{2\pi i z} - 1)v(z)$$

and observe that $F_v(Z) = S$.

For each compact subset $K \subset \mathbb{C}$ we define a subset $W_K$ of $H$ by choosing all those $v \in H$ for which the following properties hold:

(i) $F_v(\partial K) \cap D$ is empty,
(ii) $F_v(K) \cap E$ is empty,
(iii) $F_v(K)$ intersects $D$ transversally, i.e., $F_v'(z) \notin T_{F_v(z)}D$ for $z \in K$ with $F_v(z) \in D$.

We claim: If $\partial K$ is a smooth real curve, then $W_K$ is open and dense in $H$.

First we explain, how the claim implies the theorem: As a Fréchet space, $H$ has the Baire property. We cover $\mathbb{C}$ by countably many balls $B_i$ and set $W_i = W_{B_i}$. Then each $W_i$ is open and dense in $H$ and due to the Baire property $\bigcap_i W_i$ is not empty. Hence we may choose an element $v \in \bigcap_i W_i$ and define $F = F_v$.

Second, we prove the claim. The topology on $H$ is defined by the sup-Norm. Hence properties (i) and (ii) are obviously open. Since we discuss holomorphic functions, locally uniform convergence of functions implies locally uniform convergence of their derivatives. Therefore it is clear that (iii) is likewise an open property.

We still have to show density.

We fix a finite-dimensional vector subspace $V$ of $H$ such that the induced map $V \times \mathbb{C} \to \mathbb{C}^n$ is everywhere submersive. Then the map

$$\Phi : V \times (\mathbb{C} \setminus Z) \to \mathbb{C}^n$$

given by $(v, z) \mapsto F_v(Z)$ is also everywhere submersive. It follows that $\Phi^{-1}(D)$ and $\Phi^{-1}(E)$ have at least codimension 1 resp. 2 in $V \times (\mathbb{C} \setminus Z)$. Since $\partial K$ has real dimension one and $K$ has real dimension 2, it follows that the projection map onto $V$ maps both $\Phi^{-1}(D) \cap (V \times \partial K)$ and $\Phi^{-1}(E) \cap (V \times K)$ to zero measure subsets of $V$. This implies density for (i) and (ii).

Finally we have to show the density claim for (iii). We have already seen that the set of all $v \in H$ for which $F_v(\partial K) \cap D = \emptyset$ holds, is dense. Fix an element $w \in H$ with $F_w(\partial K) \cap D = \emptyset$. Using the
theorem of Rouché, we know that the degree \( \deg(F^*_vD)|_K \) is constant in a neighbourhood of \( w \) in \( H \). If \( D \) is locally defined by a holomorphic function \( h \), then
\[
v \mapsto \Pi_{x \in K \cap F^*_v(D)} dh(F'(x))
\]
is holomorphic on \( H \), which implies that the complement of its zero set is dense. This completes the proof of the claim and as we have seen before, the claim implies the theorem. \( \square \)

8. Manifolds with \( c_1 = 0 \).

We now consider the second main class for testing Conjecture 4.2: compact Kähler manifolds \( X \) with \( c_1(X) = 0 \). For them, S. Kobayashi already conjectured that their Kobayashi pseudo-distance was identically zero.

Bogomolov decomposition. A compact Kähler manifold with vanishing \( c_1 \) is (up to finite etale cover) a direct product of:

- compact complex tori,
- Hyperkähler manifolds, (equivalently: compact Kähler manifolds which are holomorphically symplectic).
- Calabi-Yau manifolds which are not hyperkähler. They admit a nowhere vanishing holomorphic \( n \)-form (\( n \) being the dimension of the manifold), but no other non-zero holomorphic differential form.

See [2] and [3].

8.1. Abelian varieties. Our result on entire curves in rationally connected manifolds easily extends to abelian varieties and other compact complex tori.

**Theorem 8.1.** Let \( A \) be a compact complex torus, let \( Z \) be a closed analytic subset of codimension at least two and let \( M \) be a countable subset of \( A \setminus Z \).

Then there exists an entire curve \( h : \mathbb{C} \to A \setminus Z \) with dense image and \( M \subset h(\mathbb{C}) \).

Moreover, every hypersurface \( D \) intersects \( h(\mathbb{C}) \) transversally in some point, i.e., there is a point \( p = h(t) \in h(\mathbb{C}) \cap D \) with \( h'(t) \notin T_pD \).

**Proof.** We use the universal covering \( u : \mathbb{C}^n \to A \). Let \( E = u^{-1}(Z) \). Let \( S \) be a dense countable subset of \( \mathbb{C}^n \setminus E \) with \( M \subset u(S) \).

Due to theorem 7.2 we obtain a holomorphic map \( F : \mathbb{C} \to \mathbb{C}^n \) with \( S \subset F(\mathbb{C}) \) and \( F(\mathbb{C}) \cap E = \emptyset \). Define \( h \overset{def}{=} u \circ F \). Now \( h : \mathbb{C} \to A \) is a dense entire curve which avoids \( Z \) and contains \( M \) in its image.
Finally observe that due to a result from Value Distribution Theory (see [39], Theorem 6.6.1.) for every hypersurface $D$ in $A$ and every Zariski dense entire curve $h$ there is a point in which $D$ and $h(\mathbb{C})$ intersect transversally. \hfill \square

**Remark.** Recall that, by [48], a compact Kähler manifold admits a finite étale cover which is a complex torus if and only if it $c_1(X) = c_2(X) = 0$.

### 8.2. Manifolds without nontrivial analytic subvarieties

The following trivial remark nevertheless leads to interesting examples.

**Proposition 8.2.** Let $X$ be a connected normal compact complex space. Assume that $X$ does not contain any non-trivial irreducible complex subvariety (except for $X$ itself and points). If $X$ is not Kobayashi-hyperbolic, then it contains a Zariski-dense entire curve.

Indeed, By Brody’s Lemma ([5], or [32], Theorem 3.6.3), there is a non-constant entire curve on $X$, which is Zariski-dense, because the whole space $X$ is its only positive-dimensional analytic subset.

These manifolds are connected to those with $c_1 = 0$ by means of the following:

**Remark.** A compact Kähler manifold $X$ without non-trivial subvariety is conjectured to be either a simple compact complex torus or to be hyperkähler, in particular, $c_1(X) = 0$. See [11], Question 1.4, and [12], Conjecture 1.1). This conjecture is proved in dimension 2 (by Kodaira’s classification), and in dimension 3 in [12], which proves that a smooth compact Kähler threefold without subvarieties is a simple torus.

**Example 8.3.** 1. The general $7$ deformation of the Hilbert scheme $\text{Hilb}^m(K3)$ of $m$ points on a $K3$-surface has no non-trivial subvariety, by [42] and is not Kobayashi-hyperbolic, by [43]. Hence Proposition 8.2 applies and such a manifold does admit a Zariski dense entire curve.

2. If $X$ is a compact Kähler threefold without subvarieties, it is biholomorphic to a compact torus (see [12]) and therefore contains a dense entire curve.

**Remark.** Unfortunately our argument gives no information on the ‘size’ (measured say by its Hausdorff dimension) of the closure of the entire curve obtained from Brody’s Theorem. Hence we can not deduce property 1.3 or 1.5. or even 1.4. by this method.

\[\text{i.e.: in the countable intersection of Zariski open subsets of the relevant moduli space.}\]
Remark. The result of [43] is based on the following construction ([8]):

of entire curves in some Hyperkähler manifolds: if \( X \) is a compact Kähler Hyperkähler manifold with twistor space \( Z \) associated to the Ricci-flat Kähler metric on \( X \) of a given class \( \omega \), then some member \( X_s \) of this twistor family contains an entire curve. This entire curve is obtained by deforming the twistor fibres (which are \( \mathbb{P}^1 \)'s with normal bundles direct sums of \( \mathcal{O}(1) \)) and taking suitable limits. The proof of Verbitsky then shows that this holds in fact for all members of such a family, using the ergodicity of the action of the mapping class group on the Teichmüller space and period domain.

It is quite interesting (but much more difficult) to extend the preceding remarks to the case of ‘simple’ compact Kähler manifolds, which are those which are not covered by subvarieties of intermediate dimension, or equivalently, such that their general point is not contained in a strict irreducible compact subvariety. In [11], it is conjectured that a ‘simple’ compact Kähler manifold is either bimeromorphic to a quotient of torus by a finite group of automorphisms, or is of even complex dimension and carries a holomorphic 2-form which is symplectic on a nonempty Zariski open subset. In particular: ‘simple’ compact Kähler manifolds should have \( \kappa = 0 \), and are of algebraic dimension zero, and are thus ‘special’. We thus expect them to have dense entire curves. For surfaces, the existence of Zariski dense entire curves is known by classification: every surface with algebraic dimension zero is simple, and bimeromorphic to either a torus or a \( K3 \) surface, so far confirming the conjecture, as we see by the Proposition below.

**Proposition 8.4.** Let \( S \) be a compact Kähler surface of algebraic dimension zero. Then \( S \) contains a Zariski dense entire curve.

**Proof.** If \( S \) is bimeromorphic to a torus, this is clear. If \( S \) is bimeromorphic to a \( K3 \) surface, we can assume that it is a \( K3 \) surface. In this case, by [11], VIII, 3.6, the only connected curves on \( S \) are chains of \(-2\)-curves, and there exists a holomorphic bimeromorphic map \( f : S \to S' \) which contracts all curves of \( S \) to (singular, normal, Du

---

8These are the main ‘building blocks’ in the construction of arbitrary (non-projective) compact Kähler manifolds.

9outside a countable union of strict subvarieties.

10that is: have non nonconstant meromorphic function, or equivalently: only finitely many irreducible divisors.

11Indeed, all line bundles on \( S \) have negative self-intersections, so all nodal classes are classes of \(-2\)-curves, and by easy computation, these have to meet transversally if not disjoint, never with triple intersections. Although certainly well-known, we do not know a reference.
Val) points of $S'$. Since $d_S \equiv 0$, so is $d_{S'} \equiv 0$, too. Now, $S'$ does not contain nontrivial subvarieties (i.e. curves), and so the conclusion follows from Proposition 8.2.

We conclude with the observation that the same holds true in dimension 3 by the classification of non-projective compact Kähler threefolds given in [14].

**Proposition 8.5.** Let $X$ be a connected compact Kähler threefold of algebraic dimension zero. Then $X$ contains a Zariski dense entire curve.

**Proof.** The claim is obvious, taking Proposition 8.4 into account, since by [14], §9, we get that $X$ is either:

1. ‘Simple’ (hence meromorphically covered by a torus), or bimeromorphically:
2. a $\mathbb{P}_1$-fibration with empty discriminant over a surface of algebraic dimension zero (the surface may be singular, without curves, if $K3$).

The obstruction to extending the above result to all ‘special’ compact Kähler threefolds lies in the cases of ‘general’ projective $K3$ surfaces, normal rational surfaces with quotient singularities and torsion canonical bundle (as the one in Example 6.9), and Calabi-Yau threefolds.

### 8.3. Some Calabi-Yau Manifolds.

These are even more difficult to handle than Hyperkähler manifolds. In general, we do not know whether the Kobayashi pseudodistance vanishes. Thus for an arbitrary Calabi-Yau manifold none of the properties of conjecture 4.2 is known to be true (except (3) which states that $\pi_1$ should be almost abelian).

We just mention 2 peculiar families for which we at least know that the Kobayashi pseudo-distance degenerates. However, also in these cases the other properties (1.2-1.5) are not known.

1. ‘General’ quintics in $\mathbb{P}_4$: they contain $(1,1)$ rational curves of arbitrarily large degree by [19]. This still works in higher dimensions for general smooth hypersurfaces of degree $(n+2)$ in $\mathbb{P}_{n+1}, n \geq 3$.
2. Double covers of $\mathbb{P}_3$ ramified over a smooth octic: they are covered by elliptic curves by [44], Example 2.17. This still works for double covers of $\mathbb{P}_n$ ramified over a smooth hypersurface of degree $2(n+1)$.

### 8.4. Elliptic Fibrations and Elliptic Calabi-Yau Threefolds.

**Proposition 8.6.** Let $f : X \to B$ be an elliptic fibration from a compact Kähler manifold over a rationally connected manifold $B$ (i.e. $f$ is proper, flat and the generic fibers are elliptic curves). Assume that $f$ has no multiple fibres in codimension one over $B$. Then $X$ contains dense entire curves.
Remark. With a “multiple fiber” we mean a fiber such that every irreducible component of it has multiplicity at least two.

Proof. Let $h : C \to B$ be any dense entire curve avoiding the subset over which $f$ has multiple fibres. By Theorem 5.2, many such entire curves exist. Let
$$Y = E \times_B \mathbb{C} = \{(p, t) \in E \times C : h(t) = f(p)\}.$$ Now we may regard the natural projection $Y \to C$ which is again an elliptic fibration. Moreover, there are no multiple fibers by construction of $h$. Following the arguments of [7] we obtain the existence of a holomorphic map $G : \mathbb{C}^2 \to Y$ with dense image. Because there exists a dense entire curve $\zeta : \mathbb{C} \to \mathbb{C}^2$ (see e.g. theorem 7.2), we are now in a position to define a dense entire curve $g : \mathbb{C} \to E$ as
$$g(z) \overset{\text{def}}{=} \pi(G(\zeta(z)))$$ where $\pi : Y \to E$ is the natural projection map. □

Theorem 8.7. Let $X$ be a simply-connected Calabi-Yau threefold with terminal singularities and $c_2(X) \neq 0$. Assume that $X$ admits an elliptic fibration $f : X \to B$ over a normal surface $B$.

Then $X$ contains a dense entire curve.

Proof. It follows from [26], [41] that $B$ is a normal rational surface with only quotient singularities and with $-(K_B + D)$ effective for some effective divisor $D$ on $B$ such that $(B, D)$ is Log-terminal, the multiple fibres of $f$ lying over a finite set $F$. From Theorem 6.1, we obtain the existence of a dense entire curve inside $(B^{\text{reg}} \setminus F)$. Now the existence of a dense entire curve in $X$ follows in the same way as for the preceding Proposition 8.6. □

Remark. When $c_2(X) = 0$, and $X$ is smooth of any dimension, the conclusion still holds since, by [18], $X$ is then covered by a complex torus. We thank S. Diverio for this observation.

The conclusion and method of proof of Theorem 8.7 should still apply to Calabi-Yau Manifolds fibered over $\mathbb{P}_1$ (possibly even an elliptic curve), using the results of [23].

9. The Nevanlinna version of the Hilbert property

9.1. Its statement.

Definition 9.1. ([21], §2.2) Let $X$ be a (smooth) projective variety defined over a number field $k$. Then $X$ is said to have the ‘Weak
Hilbert Property’ over $k$ (WHP for short) if $(X(k) - \cup_j \pi_j(Y_j(k)))$ is Zariski-dense in $X$, for any finite set of covers $\pi_j : Y_j \to X$ defined over $k$, each ramified over a non-empty divisor $D_j$ of $X$.

Note that $X(k)$ being Zariski-dense, Conjecture 4.2 implies that $X$ should be special, and its fundamental group should be almost abelian.

In [21], Corvaja-Zannier propose a Nevanlinna version of the WHP in the following form ([21], §2.4):

**Question-Conjecture:** Let $X$ be a simply-connected smooth projective manifold. Assume that there exists a Zariski dense entire curve $g : \mathbb{C} \to X$. For any finite cover $\pi : Y \to X$ ramified over a non-empty divisor, with $Y$ irreducible, there exists an entire curve $h : \mathbb{C} \to X$ which does not lift to an entire curve $h' : \mathbb{C} \to Y$ (i.e.: such that $\pi \circ h' = h$).

Let us extend their question to the case of ‘special’ manifolds:

**Question-Conjecture:** Let $X$ be a ‘special’ compact Kähler manifold. For any finite cover $\pi : Y \to X$ ramified over a non-empty divisor, with $Y$ irreducible, there exists a Zariski dense entire curve $h : \mathbb{C} \to X$ which does not lift to an entire curve $h' : \mathbb{C} \to Y$ (i.e.: such that $\pi \circ h' = h$).

We shall abbreviate with $NHP(X)$ if $X$ possesses this property, and say that $X$ has $NHP$ (for Nevanlinna-Hilbert Property).

A stronger version is the question whether there is a single entire curves $h$ on $X$ such that for any finite cover $\pi : Y \to X$ ramified over a non-empty divisor of $X$, $h$ does not lift to $Y$. We shall denote with $NHP^+(X)$ this property.

These $NHP$ properties are preserved by finite étale covers and smooth blow-ups.

**Lemma 9.2.** The $NHP$ and $NHP$ for Galois covers properties are preserved by finite étale covers, and bimeromorphic equivalence.

**Proof.** Let $f : X' \to X$ be a surjective holomorphic map between connected compact complex manifolds.

Assume first that $f$ is finite étale. Let $\pi : Y \to X$ be actually ramified, and $\pi' : Y' \to X'$ be deduced from $\pi$ by the base-change $f$. If $h : \mathbb{C} \to X$ is Zariski dense and lifts to $Y$, it lifts to $Y'$ which is étale over $Y$. This lifts then lifts some lifting of $h$ to $X'$. A contradiction if $NHP(X')$ holds, and so $NHP(X')$ implies $NHP(X)$. Conversely, 

\[12\] The classical Hilbert property does not require the covers $Y_j \to X$ to be ramified, it thus implies, by the Chevalley-Weil Theorem, that $X$ to be algebraically simply-connected.
let $\pi' : Y' \to X'$ be given, actually ramified. If $h' : C \to X'$ is Zariski dense and liftable to $Y'$, then $h := f \circ h' : C \to X$ is Zariski dense and liftable to $Y'$ (which is actually ramified over $X$), thus $NHP(X)$ is violated as well.

Let us now assume that $f$ is bimeromorphic. Since the fundamental groups of $X$ and $X'$ coincide, so do (up to bimeromorphic equivalence) the covers of $X$ and $X'$ which actually ramify. This (easily) implies the equivalence of $NHP(X)$ and $NHP(X')$, since the existence of lifts of Zariski dense entire curves is also a bimeromorphic property.

The proof for Galois covers is the similar.

A simple tool in finding non-liftable curves is the following:

**Proposition 9.3.** Let $h : C \to X$ be an entire curve and $H$ an hypersurface of $X$ such that there exists a regular point $a \in H$ in which $h(C)$ and $H$ intersect with order of contact $t$.

Let $\pi : X_1 \to X$ be a finite Galois covering with branch locus containing $H$, such that $\pi$ ramifies at order $s \geq 2$ over $H$ at $a$. Then $h$ cannot be lifted to an entire curve $\tilde{h} : C \to X_1$ if $t$ does not divide $s$.

In particular, if $h(C)$ and $H$ meet transversally at $a$, $h$ does not lift to $Y$.

**Proof.** Since $\pi$ is Galois, it ramifies at order $s$ at any point of $Y$ over $a \in H$. Since $h(C)$ intersect at order $s$ at $a$, if it lifted to $Y$, its order of contact with $H$ were a multiple of $s$.

9.2. Rationally connected and Abelian manifolds. We have the following stronger form for rationally connected manifolds, in which a fixed entire curve $h$ does not lift to any Galois ramified cover $\pi : Y \to \mathbb{P}^n$.

**Theorem 9.4.** Let $X$ be a rationally connected complex projective manifold or a complex compact torus.

Then there exists an entire curve $f : C \to X$ such that

(i) The image $f(C)$ is dense.

(ii) $f$ can not be lifted to any ramified Galois covering $\tau : X' \to X$.

**Proof.** Combine Theorem 5.2 resp. Theorem 8.1 with Proposition 9.3.

9.3. Special surfaces.

**Proposition 9.5.** Let $S$ be a compact Kähler surface. We denote by $S'$ an arbitrary finite étale cover of $S$. The following are equivalent:

1. $S$ is special
2. No $S'$ maps meromorphically onto a variety of general type.
3. $\kappa(S) \leq 1$ and $q(S') \leq 2, \forall S'$.
4. $\kappa(S) \leq 1$ and $\pi_1(S)$ is virtually abelian.
5. $S$ is one of the following:
   5.1. Rational
   5.2. Birational to $\mathbb{P}_1 \times E$, with $E$ elliptic.
   5.3. Some $S'$ is birational to Enriques, Bielliptic, $K3$, or torus.
   5.4. Some $S'$ admits an elliptic fibration $f : S' \to B$ with $B$ either
elliptic and no multiple fibre, or $B = \mathbb{P}_1$ and at most 2 multiple fibres.

**Theorem 9.6.** Let $S$ be a special compact Kähler surface. Then $S$ satisfies the Nevanlinna-Hilbert property for Galois covers, except (maybe) if $S$ is a $K3$ surface which is neither elliptic nor Kummer.

If we assume the Green-Griffiths Lang conjecture, a non-special surface does not fulfill this property.

**Proof.** We shall first check the property in the cases 5.1-5.3 of Proposition 9.5, case 5.4 being treated in Lemma 9.7 below. If $S$ is rational, it is rationally connected and so satisfies the NHP, by Theorem 9.4.

If $S$ is in the class 5.2, some birational model has an elliptic fibration, and so the conclusion follows from Lemma 9.7. Now if $S$ is in the class 5.3, it has a finite étale cover bimeromorphic to either an elliptic $K3$ surface, or to a compact torus. In the first case, the conclusion follows from Lemma 9.7, in the second from Theorem 9.4.

□

**Lemma 9.7.** A special surface $S$ admitting an elliptic fibration $f : S \to B$ has the NHP for Galois covers.

**Proof.** We shall check that $S$ satisfies the condition given by Proposition 9.3. Let $\pi : S' \to S$ be a Galois cover, actually ramified over an irreducible divisor $R \subset S$. Let $(B, D_f)$, with $D_f := \sum_j (1 - \frac{1}{m_j}) \{b_j\}$ be the orbifold base of the fibration $f$, where $b_j \in B, \forall j$, and $m_j$ is the multiplicity of the fibre $S_{b_j}$ of $f$ over $b_j$.

Since $S$ is special, $\deg(K_B + D_f) \leq 0$. For any $b \in B$, there thus exists an entire map $h : \mathbb{C} \to (B, D_f)$, which goes through $b$, and which is an orbifold entire curve (ie: such that its order of contact with any $b' \in B$ is equal to the multiplicity of $D_f$ at $b'$, which is equal to 1 if $b'$ is not any one of the $b_j$).

Let $f_h : S_h := S \times_B \mathbb{C} \to \mathbb{C}$ be deduced form $f$ by the base change $h : \mathbb{C} \to B$. Then $f_h$ has no multiple fibre, and (by [7]) admits a holomorphic section $s : \mathbb{C} \to S_h$ going through any given reduced component $F$ of any of the fibres of $f_h$.

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13The ‘classical’ and ‘non-classical’ versions coincide for elliptic fibrations.
The main result of [7] then implies the following:

**Lemma 9.8.** Assume that \( R \) is either a reduced component of some fibre of \( f_h \), or that \( f(R) = B \). There exists then a holomorphic map \( H : \mathbb{C}^2 \to X \) which is unramified over its image, which is a Zariski open subset of \( S \) meeting \( R \).

Composing \( H \) with an suitable injection \( j : \mathbb{C} \to \mathbb{C}^2 \) with dense image and meeting transversally \( H^{-1}(R) \) at some point, Proposition 9.3 and Lemma 9.2 imply \( NHP(S) \) for Galois covers.

We still need to deal with the case when \( R \) is a non-reduced component of some fibre \( F := f_h^{-1}(0) \) of \( f_h \). Let then \( m > 1 \) be the multiplicity of \( R \) in \( F \). Let \( \mu : \mathbb{C} \to \mathbb{C} \) be defined by \( \mu(z) = z^m \), let \( k := h \circ \mu : \mathbb{C} \to B \), and let \( f_k : S_k' \to \mathbb{C} \) be deduced from \( f_h \) by the base-change \( \mu : \mathbb{C} \to C \) after taking a smooth bimeromorphic model \( S_k' \) of \( S_k := S \times_B \mathbb{C} \).

The natural map generically finite map \( \sigma_k : S_k' \to S_h \) is not étale, but it is étale over the generic point of \( R \), and also outside the fibre \( F \). Applying again [7] to the fibration \( f_k : S_k' \to \mathbb{C} \), we obtain a holomorphic map \( H_k : \mathbb{C}^2 \to S_k' \) which is unramified over the generic point of the inverse image \( R_k \) of \( R \), and thus a Zariski dense holomorphic map \( h_k : \mathbb{C} \to S_k' \) which is transversal to \( R_k \). The map \( \sigma_k \circ h_k : \mathbb{C} \to S_k \) is thus transversal to \( R \) at some of its generic points, establishing \( NHP(S) \) for Galois covers. \( \square \)

### 9.4. Removing the Galois condition.

**Theorem 9.9.** Let \( X \) be a complex projective manifold. Assume that \( X \) is rationally connected or that there exists a surjective and submersive holomorphic map \( \rho : \mathbb{C}^N \to X \).

Let \( \pi : Y \to X \) be a finite map with non-empty ramification.

Then there exists a dense entire curve \( h : \mathbb{C} \to X \) which can not be lifted to \( Y \) (i.e., there is no entire curve \( \tilde{h} : \mathbb{C} \to Y \) with \( h = \pi \circ \tilde{h} \)).

This result differs from theorem 9.4 in two points: First, the ramified covering \( \pi \) is no longer required to be Galois. Second (this is the price we pay for dropping the Galois condition), here the non-liftable dense entire curve \( h \) may depend on \( \pi \).

**Proof.** First we discuss the case \( \dim_{\mathbb{C}} X = 1 \). Then \( X \) is a rational or elliptic curve. If \( X \) is an elliptic curve, \( Y \) is of genus at least 2 and therefore hyperbolic, i.e., there is no non-constant holomorphic map from \( \mathbb{C} \) to \( Y \). If \( X \) is a rational curve, we chose the natural injection \( \mathbb{C} \subset \mathbb{P}_1 \) as \( h : \mathbb{C} \to \mathbb{P}_1 \). Using the theorem of Liouville one may show easily: For every non-constant holomorphic map \( \tilde{h} \) from \( \mathbb{C} \) to a compact
Riemann surface \( Y \) the image must be dense. This implies that such a map \( \tilde{h} \) cannot be a lift of \( h \).

Now we start the proof of the general case, i.e., \( \dim \mathbb{C} X \geq 2 \). Let \( R \subset Y \) denote the ramification divisor and let \( B = \pi(R) \subset X \) be the branching locus. Fix \( p \in X \setminus B \). For every \( q \in \pi^{-1}(p) \) we choose a real continuous curve \( c_q : [0,1] \to Y \) with \( c_q(0) = q \), \( c_q(1) \in R \) and \( \pi(c_q(t)) \notin B \) for \( 0 \leq t < 1 \). Since \( \dim \mathbb{C} X \geq 2 \), we may, by slightly perturbing the curves \( c_q \) if necessary, assume that for \( q \neq q' \) we have \( \pi(c_q([0,1])) \cap \pi(c_{q'}([0,1])) = \{ p \} \). Then \( T = \bigcup_q \pi(c_q([0,1])) \) is a tree, i.e., a simply-connected real one-dimensional simplicial complex which may be visualized as \( d = \# \pi^{-1}(p) \) line segments glued together in one point.

Due to Proposition 9.11 resp. Proposition 9.12 we obtain an entire curve \( h_n : \mathbb{C} \to X \) with embeddings \( \zeta_n : T \to \mathbb{C} \) such that \( \lim_{n \to \infty} h_n \circ \zeta_n = i \) where \( i : T \to X \) is the inclusion map. For sufficiently large \( n \) we now may replace \( T \) by \( (h_n \circ \zeta_n)(T) \) and obtain an obstruction to the lifting of the entire curve \( h_n \) by using proposition 9.10.

Finally, it is a consequence of Corollary 5.6 that the entire curve may be chosen in such a way that it has dense image. \( \square \)

**Proposition 9.10.** Let \( \pi : Y \to X \) be a finite ramified covering of complex manifolds with ramification divisor \( R \subset Y \) and branching locus \( B = \pi(R) \).

Let \( p \in X \setminus B \). For every \( q \in \pi^{-1}(p) \) let \( c_q : [0,1] \to Y \) be a continuous real curve with \( c_q(0) = q \) and \( c_q(1) \in R \) and \( \pi(c_q(t)) \notin B \) for all \( 0 \leq t < 1 \).

Let \( h : \mathbb{C} \to X \) be an entire curve such that for every \( q \in \pi^{-1}(p) \) there is a smooth real curve \( \gamma_q : [0,1] \to \mathbb{C} \) with \( \gamma_q(0) = 0 \), \( h \circ \gamma_q = \pi \circ c_q \) and \( h'(\gamma_q(1)) \notin TB \).

Then there does not exist a lift \( \tilde{h} : \mathbb{C} \to Y \), i.e., there is no holomorphic map \( \tilde{h} : \mathbb{C} \to Y \) with \( \pi \circ \tilde{h} = h \).

**Proof.** Assume the converse. Since \( \gamma_q(0) = 0 \) for all \( q \), we have

\[
h(0) = h(\gamma_q(0)) = \pi(c_q(0)) = \pi(q) = p.
\]

Fix \( q \in \pi^{-1}(p) \) such that \( \tilde{h}(0) = q \). Observe that

- \( \pi \circ \tilde{h} \circ \gamma_q = h \circ \gamma_q = \pi \circ c_q \),
- \( \tilde{h} \circ \gamma_q(0) = q = c_q(0) \),
- \( \pi \circ c_q(t) \notin B \) for all \( t < 1 \).

It follows that \( \tilde{h} \circ \gamma_q = c_q \) and therefore \( (\tilde{h} \circ \gamma_q)(1) \in R \). In combination with \( h'(\gamma_q(1)) \notin TB \) this yields a contradiction. \( \square \)

**Proposition 9.11.** Let \( T \) be a (real) tree, i.e., a simply-connected finite one-dimensional simplicial complex.
Let $X$ be a complex projective manifold with a surjective and submersive holomorphic map $\rho : \mathbb{C}^N \to X$.

Let $c : T \to X$ be a continuous map. Let $\epsilon > 0$ and let $d(\cdot, \cdot)$ be a distance function on $X$ induced by a hermitian metric. Then there exists an entire curve $h : \mathbb{C} \to X$ and a continuous map $\tilde{c} : T \to \mathbb{C}$ such that $h(\mathbb{C})$ is dense in $X$ and $d(c(x), h(\tilde{c}(x))) < \epsilon$ for all $x \in T$.

**Proof.** We chose a closed embedding $\tilde{c} : T \to \mathbb{C}$ and an infinite discrete subset $S \subset \mathbb{C}$ with $S \cap \tilde{c}(T) = \emptyset$. Upon replacing $c$ by a small deformation, we may assume that $c(T)$ contains no critical values of $\rho$. Since $\rho$ is surjective, we may lift $c : T \to X$ to a continuous map $\hat{c} : T \to \mathbb{C}^N$. The theorem of Arakelyan/Nersesyan \cite{38} implies that continuous maps from $\tilde{c}(T) \cup S$ to $\mathbb{C}^N$ may uniformly be approximated by holomorphic maps from $\mathbb{C}$ to $\mathbb{C}^N$. Hence the continuous map $\hat{c} \circ (\tilde{c})^{-1} : \tilde{c}(T) \to \mathbb{C}^N$ may be approximated uniformly by holomorphic maps $\tilde{h}$ from $\mathbb{C}$ to $\mathbb{C}^N$ in such a way that $\tilde{h}(\mathbb{C})$ is dense in $\mathbb{C}^N$. This implies the assertion, by taking $h \overset{\text{def}}{=} \rho \circ \tilde{h}$. \hfill $\square$

**Proposition 9.12.** Let $T$ be a (real) tree, i.e., a simply-connected finite one-dimensional simplicial complex.

Let $X$ be a rationally connected complex projective manifold.

Let $c : T \to X$ be a continuous map. Let $\epsilon > 0$ and let $d(\cdot, \cdot)$ be a distance function on $X$ induced by a hermitian metric. Then there exists a free rational curve $h : \mathbb{P}_1 \to X$ and a continuous map $\tilde{c} : T \to \mathbb{P}_1$ such that $d(c(x), h(\tilde{c}(x))) < \epsilon$ for all $x \in T$.

**Proof.** We decompose the tree into arcs. Locally, each arc admits an approximation by free rational curves, see lemma 9.13. Due to compactness, we obtain a decomposition into finitely any arcs such that each arc admits an approximation by a free rational curve. Using comb smoothing (in the form given by corollary 9.15) we obtain an approximation by one free rational curve. \hfill $\square$

**Lemma 9.13.** Let $X$ be a rationally connected projective manifold with a distance function $d(\cdot, \cdot)$ (induced by some hermitian metric). Let $\epsilon > 0, r \in \mathbb{N}$ and let $\gamma : [-1,1]$ be a regular real curve (i.e. $\gamma$ is $C^1$ with $\gamma'(t) \neq 0 \forall t$).

Then there exists a positive number $0 < \delta < 1$ such that for every $-\delta < a < b < \delta$ there is an $r$-free rational curve $c : \mathbb{P}_1 \to X$ satisfying

(i) $d(c(t), \gamma(t)) < \epsilon$ for every $t \in [a, b]$.
(ii) $c(a) = \gamma(a), c(b) = \gamma(b)$,
(iii) $c'(a) = \gamma'(a), c'(b) = \gamma'(b)$. 
Proof. Without loss of generality we assume $r \geq 2$. We start by choosing an $r$-free rational curve $h : \mathbb{P}_1 \to X$ with $h(0) = \gamma(0)$ and $h'(0) = \gamma'(0)$. We consider the deformations of $h$. Since $h$ is free, these deformations are unobstructed. Thus we may fix a ball $B \subset V = H^0(\mathbb{P}_1, h^*TX)$ parametrizing deformations of $h$ by a map $\Phi : B \times \mathbb{P}_1 \to X$.

Choosing the ball small enough, we may assume $d(\Phi(p, t), h(t)) < \epsilon/2$ for all $p \in B$, $t \in [-1, +1]$. We may furthermore assume that $\Phi$ extends to $\bar{B} \times \mathbb{P}_1$ where $\bar{B}$ is the closure in $V$ and compact.

Because $h$ is $1$-free, for every pair $(a, b)$, $-1 \leq a < b \leq +1$ the evaluation map $B \ni p \mapsto \Phi(p, a) \times \Phi(p, b)$ contains an open neighborhood of $(a, b)$ in $X \times X$ in its image. Using compactness, we choose $\eta > 0$ such that for all $a, b \in [-1, 1]$ and all $a', b' \in X$ with $d(\gamma(a), a') < \eta$, $d(\gamma(b), b') < \eta$ there exists a parameter $p \in B$ with $\Phi(p, a) = a'$ and $\Phi(p, b) = b'$.

Next we chose $\delta > 0$ in such a way that

$$d(c(t), \gamma(t)) < \min\{\eta, \epsilon/2\} \forall t, |t| < \delta.$$ 

Given any two numbers $a, b$ satisfying $-\delta < a < b < \delta$, we have $d(c(a), \gamma(a)) < \eta$ and $d(c(b), \gamma(b)) < \eta$ and therefore may choose $p \in B$ with $\Phi(p, a) = \gamma(a)$ and $\Phi(p, b) = \gamma(b)$. Then we define $c(t) := \Phi(p, t)$. We obtain a $r$-free rational curve $c$. Evidently $c$ satisfies property $(ii)$ of the assertion. Furthermore, $c$ satisfies $(i)$ because

$$d(c(t), \gamma(t)) \leq d(c(t), h(t)) + d(h(t), \gamma(t)) \leq \epsilon/2 + \epsilon/2 = \epsilon.$$ 

Finally we note that $(iii)$ may be achieved in the same way as in Corollary 5.7. \)

Remark. As pointed out to us by F. Forstneric, there is a related result by A. Gournay, who proved- under a certain additional ‘regularity’ hypothesis- a Runge type approximation theorem (see [24]) on domains in Riemann surfaces mapped to compact almost complex manifolds containing ‘free’ rational curves. Where applicable, the statement of the above lemma can be deduced from Gournay result as follows: First approximate a given map $\gamma : [0, 1] \to X$ by a real-analytic map, then extend it to a holomorphic map $g$ on some simply-connected open neighborhood of $[0, 1]$. If a certain ‘regularity’ condition is fulfilled, Gournay’s theorem then implies that $g$ may be approximated by a rational curve, implying the statement of the above lemma. The ‘regularity’ assumption, although satisfied for ‘generic’ almost complex structures, seems however difficult to check on a given complex structure.
Lemma 9.14. Let $T$ be a tree, and let $c : T \to C$ be a continuous closed embedding with $C = \{(z,w) \in \mathbb{C}^2 : zw = 0\}$.

Then there exists a sequence of non-zero complex numbers $\epsilon_n$ with $\lim_{n \to \infty} \epsilon_n = 0$ and a sequence of continuous maps $c_n : T \to C_{\epsilon_n}$ such that $\lim_{n \to \infty} c_n = c$ uniformly on $T$.

Proof. The assertion is easily verified if $c(T)$ is contained in one of the two irreducible components of $C$. Hence we assume that $c(T)$ is not contained in one of the irreducible components, implying that $c(T)$ contains $(0,0)$, since this is the only point in which the two components intersect.

Let $0 \in T$ denote the point with $c(0) = (0,0)$. Let $T_i^\circ$ denote the connected components of $T \setminus \{(0)\}$ and let $T_i = T_i^\circ \cup \{(0)\}$ be its closure.

We choose an injective continuous curve $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) = 0$ such that $\gamma(t) + x^2 \neq 0$ for all $t \in [0,1]$ and $(0,x), (x,0) \in T$. Since $T$ is a tree, we know that $T_i \times [0,1]$ is simply-connected. The choice of $\gamma$ allows us thus to choose a branch of $\sqrt{\gamma(t) + x^2}$ (with $(x,0)$ resp. $(0,x)$ in $T_i$ and $t \in [0,1]$). We choose the branch $h_i(x,t)$ with $\sqrt{0 + x^2} = x$ for some point $(x,0)$ resp. $(0,x)$ in $T_i$. By continuity it follows that $\sqrt{0 + x^2} = x$ for all $x$. Consequently $\lim_{t \to 0} h_i(t,x) = x$ for all $(x,0)$ resp. $(0,x)$ in $T_i$. We define $c_{t,i} : T_i \to \mathbb{C}^2$ as

$$c_{t,i}(x) = \frac{1}{2} (h_i(t,x) + x, h_i(t,x) - x)$$

if $T_i \subset \mathbb{C} \times \{(0)\}$ and

$$c_{t,i}(x) = \frac{1}{2} (h_i(t,x) - x, h_i(t,x) + x)$$

if $T_i \subset \{(0)\} \times \mathbb{C}$. We observe that

$$\lim_{t \to 0} c_{t,i} = c.$$

Now $h_i(t,0)^2 = \gamma(t)$. We fix a branch of the square root along $\gamma$:

$$\xi(t) \overset{def}{=} \sqrt{\gamma(t)}, \ t \in [0,1]$$

Then for each $i$ there is an element $\sigma_i \in \{+1,1\}$ such that

$$h_{t,i}(0) = \sqrt{\gamma(t)} + 0 = \sigma_i \xi(t)$$

and consequently

$$c_{t,i}(0) = (\sigma_i \xi(t), \sigma_i \xi(t))$$

The problem is that for distinct $i, j$ we may have $\sigma_i \neq \sigma_j$.

We enumerate the $T_i$ as $T_1, \ldots, T_r$. Since $T$ is a tree, and hence contains no closed loops, each subtree $T_i$ contains a unique edge ending at $(0,0)$. 
Now we define $c_t : T \to \mathbb{C}^2$ recursively as follows: If $c_t$ is defined on $\cup_{i \leq k} T_i$, then we choose a small curve inside $C_{\gamma^2(t)}$ starting at $c_t(0)$ and ending at $c_{t,k+1}(0)$. Because

$$||(\xi(t), \xi(t) - (-\xi(t), -\xi(t)))|| = 2\sqrt{2}\gamma(t)$$

and $\lim_{t \to 0} \gamma(t) = 0$, this small curve can be chosen as small as desired if $t$ is small enough. Thus we may attach this curve to $c_{t,k+1}(T_{k+1})$ without changing this subtree too much.

In this way we can construct a family of continuous maps $c_t : T \to C_{\gamma^2(t)}$ with $\lim_{t \to 0} c_t = c$ (uniformly on $T$).

**Corollary 9.15.** Let $C$ be a comb in a rationally connected projective manifold, let $T$ be a tree and let $c : T \to C$ be a closed continuous embedding.

Then $c$ may be approximated by $h_n \circ c_n$ where $h_n$ are free rational curves and $c_n : T \to \mathbb{P}_1$ are closed continuous embeddings.

**10. Dense entire curves of order zero**

For an entire curve $f : \mathbb{C} \to X$ with values in a Kähler manifold $X$ the characteristic function (in its Ahlfors-Shimizu form) is defined as

$$T_f(r) = \int_0^r \left( \int_{f^{-1}(\Delta_t)} f^* \omega \right) \frac{dt}{t}$$

where $\omega$ is a Kähler form on $X$ and $\Delta_t = \{ z \in \mathbb{C} : |z| < t \}$.

The order $\rho_f$ of $f$ is defined as

$$\rho_f = \limsup_{r \to \infty} \frac{\log T_f(r)}{\log r}$$

We will show that the entire curves in rationally connected manifolds constructed above in this article may be constructed in such a way that the order $\rho_f$ equals zero.

This is interesting in view of the following facts:

- For every rational map $f : \mathbb{C} \to X$ we have $T_f(r) = O(\log r)$ and consequently $\rho_f = 0$.
- For an abelian variety $A$ and a non-constant entire curve $f : \mathbb{C} \to A$ we have $\rho_f \geq 2$.
- The notion of the order $\rho_f$ may generalized to non-degenerate holomorphic maps from $\mathbb{C}^n$ to $X$. If (for $n = \dim(X)$) there exists a non-degenerate holomorphic map $f : \mathbb{C}^n \to X$ with $\rho_f < 2$, then $X$ is rationally connected (see [16],[40]).

We observe that from the definition of the characteristic function given above the following may be easily deduced:
If \( f_n \) is a sequence of holomorphic maps from \( \mathbb{C} \) to some Kähler manifold \((X, \omega)\) which converges locally uniformly on \( \mathbb{C} \) to a holomorphic map \( f : \mathbb{C} \to X \), then the sequence of characteristic functions \( T_{f_n} : [1, \infty[ \to \mathbb{R}^+ \) converges locally uniformly on \( [1, \infty[ \) to \( T_f : [1, \infty[ \to \mathbb{R}^+ \).

Let \( X \) be a projective rationally connected manifold. We fix a Kähler class \( \omega \). For any rational curve \( f : \mathbb{P}_1 \to X \) the degree \( \deg(f) \) is defined as the volume of its image with respect to the Kähler form, i.e.,

\[
\deg(f) = \int_{\mathbb{P}_1} f^* \omega
\]

For any \( c > 0 \) let \( F_c \) denote the set of all free rational curves of degree at most \( c \). Due to the Kähler assumption the degree is invariant under deformations. Hence there is a constant \( U > 0 \) such that for any two given points \( p, q \) there is a free rational curve \( f \) in the family \( F_U \) connecting \( p \) with \( q \).

Let \( g : [1, \infty[ \to [1, \infty[ \) be a continuous function with the following properties

(i) \( r \mapsto \frac{g(r)}{\log r} \) is monotone increasing on \( [2, \infty[ \) and unbounded.

(ii) \( \lim_{r \to \infty} \frac{\log g(r)}{\log r} = 0 \).

(iii) \( g(2) \geq 2U \log 2 \)

For example, we may take \( g(r) = \min\{ (\log r)^2, 2U \log r \} \), or: \( g(r) = \min\{ (\log(\log r)) \cdot \log r, 2U \log r \} \).

We choose recursively a sequence \( f_n \) of rational curves in \( X \), with \( \deg(f_n) = nU \).

We start with a rational curve \( f_1 \) of degree \( \leq U \). For \( r \geq 1 \) we have:

\[
T_{f_1}(r) \leq \int_1^r \left( \int_{D_t} f_1^* \omega \right) \frac{dt}{t} \leq U \log r
\]

Since \( g(2) \geq 2U \log 2 \), we have \( T_{f_1}(2) \leq \frac{1}{2} g(2) \). By the monotonicity of \( g(r)/\log r \) it follows that

\[
T_{f_1}(r) \leq \frac{1}{2} g(r) \quad \forall r \geq 2
\]

We claim that the recursive construction of the rational curves \( f_n \) as made above may be carried out in such a way that

\[
T_{f_n}(r) \leq \left( 1 - \frac{1}{2^{n+1}} \right) g(r) \quad \forall r \geq 2, \forall n,
\]
Given $f_n$ with

$$T_{f_n}(r) \leq \left(1 - \frac{1}{2^{n+1}}\right) g(r) \quad \forall r \geq 2,$$

first we choose $R_n > 2$ such that

$$\frac{g(R_n)}{\log R_n} > 2(n + 1)U$$

Now we choose $f_{n+1}$ such that on $\{z : |z| \leq R_n\}$ it is close enough to $f_n$ in order to ensure

$$T_{f_{n+1}}(r) \leq \left(1 - \frac{1}{2^{n+2}}\right) g(r) \quad \forall r \leq R_n$$

Let us now consider $T_f(r)$ for $r > R_n$:

$$T_{f_{n+1}}(r) = T_{f_{n+1}}(R_n) + \int_{R_n}^{r} \left(\int_{D_t} f_{n+1}^* w\right) \frac{dt}{t}$$

$$\leq T_{f_{n+1}}(R_n) + \int_{R_n}^{r} (n + 1)U \frac{dt}{t}$$

$$= T_{f_{n+1}}(R_n) + (n + 1)U \log r - \log R_n$$

By the choice of $R_n$, the condition $r \geq R_n$ and the monotonicity of $g(r)/\log r$ we have $2U(n + 1) < g(r)/\log r$. Therefore

$$T_{f_{n+1}}(r) \leq T_{f_{n+1}}(R_n) + \frac{1}{2}g(r) - (n + 1)U \log R_n$$

Because of $\deg(f_{n+1}) = (n + 1)U$ we have

$$T_{f_{n+1}}(R_n) \leq \int_{1}^{R_n} \left(\int_{D_t} f_{n+1}^* w\right) \frac{dt}{t}$$

$$\leq (n + 1)U \log R_n$$

Hence

$$T_{f_{n+1}}(R_n) - (n + 1)U \log R_n < 0$$

and consequently

$$T_{f_{n+1}}(r) \leq \frac{1}{2}g(r) \quad \forall r \geq R_n$$

It follows that

$$T_{f_{n+1}}(r) \leq \left(1 - \frac{1}{2^{n+2}}\right) g(r) \quad \forall r \geq 2$$
Finally, we obtain an entire curve $f : \mathbb{C} \to X$ as $f = \lim f_n$ and our construction implies
\[ T_f(r) \leq g(r) \]
which implies that $f$ is of order zero:
\[ \rho_f = \limsup_{r \to \infty} \frac{\log T_f(r)}{\log r} \leq \limsup_{r \to \infty} \frac{\log g(r)}{\log r} = 0 \]
The entire curves so constructed have thus growth order zero.

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Frédéric Campana, Département de Mathématiques, Université Nancy 1, Vandoeuvre-lès-Nancy, 54500, France
E-mail address: frederic.campana@univ-lorraine.fr

Jörg Winkelmann, IB 3-111, Lehrstuhl Analysis II, Fakultät für Mathematik, Ruhr-Universität Bochum, 44780 Bochum, Germany
E-mail address: joerg.winkelmann@rub.de
ORCID: 0000-0002-1781-5842