ON SOME VERTEX ALGEBRAS RELATED TO $V_{-1}(sl(n))$ AND THEIR CHARACTERS

DRAŽEN ADAMOVIĆ AND ANTUN MILAS

Dedicated to Mirko Primc on the occasion of his 70th birthday

ABSTRACT. We consider several vertex operator (super)algebras closely related to $V_{-1}(sl(n)), n \geq 3$ : (a) the parafermionic subalgebra $K(sl(n), -1)$ for which we completely describe its inner structure, (b) the vacuum algebra $\Omega(V_{-1}(sl(n)))$, and (c) an infinite extension $\mathcal{U}$ of $V_{-1}(sl(n))$ constructed by combining certain irreducible ordinary modules with integral weights. It turns out that $\mathcal{U}$ is isomorphic to the coset vertex algebra $psl(n|n)_{1}/sl(n)_{1}$, $n \geq 3$. We show that $V_{-1}(sl(n))$ admits precisely $n$ ordinary irreducible modules, up to isomorphism. This leads to the conjecture that $\mathcal{U}$ is quasi-lisse. We present strong evidence in support of this conjecture: we prove that the (super)character of $\mathcal{U}$ is quasi-modular of weight one by virtue of being the constant term of a meromorphic Jacobi form of index zero. Explicit formulas and MLDE for characters and super-characters are given for $g = sl(3)$ and outlined for general $n$. We finish with a conjecture pertaining to characters of $psl(n|n)$ and $\mathcal{U}$-modules.

1. INTRODUCTION

Orbifolding, coset constructions, and simple current extensions, are standard methods for producing new examples of vertex algebras. For irrational vertex algebras it is also important to consider infinite simple current extension. For instance, lattice vertex algebras are infinite simple current extensions of the Heisenberg vertex algebras. Infinite simple current extensions are also important in logarithmic conformal field theory. As demonstrated by the authors, the triplet vertex algebra $W(p)$ (which is $C_{2}$-cofinite) is indeed an infinite simple current extension of the non $C_{2}$-cofinite singlet vertex superalgebra [3].

In this paper the aim is to study the simple affine vertex operator algebra of level $-1$, denoted by $V_{-1}(sl(n))$ and some of its subalgebras and infinite extensions. This vertex algebra is known to be irrational and non $C_{2}$-cofinite and has been studied from several points of view. Early work [22] was focused primarily on the properties of characters. The first author and Peršč obtained a complete classification of ordinary (atypical) irreducible $V_{-1}(sl(n))$-modules and fusion rules of ordinary modules [11] (see also [10]). They also showed that $V_{-1}(sl(n))$ admits generic (or typical) series of irreducible representations. Kac and Wakimoto recently obtained a Weyl-Kac type character formula [23] involving higher rank partial theta series (cf. also [13]). Asymptotic and modular-type properties of characters of $V_{-1}(sl(n))$-modules were studied recently in the work of Bringmann, Mahlburg and the second author [15]. Although characters are mixed quantum modular forms [13, 15], presently it seems difficult to formulate and prove a continuous version of the Verlinde formula of characters even for ordinary modules.
Instead of studying the vertex algebra $V_{-1}(\mathfrak{sl}(n))$, here we focus on two somewhat better behaved objects: the parafermionic algebra(s) and a certain infinite (simple current) extension which we denote by $U$. Both vertex algebras have interesting properties from an algebraic and number theoretic standpoints.

Let us outline the content and the main results. Throughout we assume that $n \geq 3$. We first review construction of the simple vertex algebra $V_{-1}(\mathfrak{sl}(n))$. Here we utilize the rank $n$ symplectic fermion vertex algebra $\mathcal{A}(n)$, a certain lattice vertex algebra $V_L$, and the beta-gamma system (or the Weyl vertex algebra) $W(n)$. Then $V_{-1}(\mathfrak{gl}(n))$ (and then of course $V_{-1}(\mathfrak{sl}(n))$) embeds inside the zero "charge" subalgebra $W^{(0)}(n) \subset A(n) \otimes V_L$. Similarly, we obtain explicit realization of irreducible ordinary $V_{-1}(\mathfrak{sl}(n))$-modules denoted by $V_s$, $s \in \mathbb{Z}$ (see Proposition 2.3).

Then we move on study parafermionic and vacuum subalgebras. Recall that the parafermionic subalgebra $K(\mathfrak{sl}(n), -1)$ is defined as

$$K(\mathfrak{sl}(n), -1) := \{v \in V_{-1}(\mathfrak{sl}(n)) : a(m)v = 0, \ a \in M(1), m \geq 0\},$$

where $M(1)$ is the Heisenberg subalgebra, and the vacuum algebra is similarly defined as

$$\Omega_n := \{v \in V_{-1}(\mathfrak{sl}(n)) : a(m)v = 0, \ a \in M(1), m > 0\}.$$

Our next result pertains to the structure of these vertex algebras.

**Theorem 1.1.** We have

(1) $K(\mathfrak{sl}(n), -1) \cong \mathcal{M}(1)^{\otimes n}$, where $\mathcal{M}(1)$ is the singlet vertex algebra of central charge $-2$ (cf. [13, 28]),

(2) and

$$\Omega_n \cong \mathcal{A}(n)^{(0)}$$

the charge zero subalgebra of the symplectic fermion vertex algebra.

Then we consider an infinite extension of $V_{-1}(\mathfrak{sl}(n))$. We first prove that for every $n \geq 3$

$$U^{(n)} := \bigoplus_{s \in \mathbb{Z}} V_{ns}$$

has a simple vertex algebra structure for $n$ even, and $\mathbb{Z}$-graded vertex superalgebra if $n$ is odd.

Then we can prove

**Theorem 1.2.** (1) The vertex (super)algebra $U := U^{(n)}$ has precisely $n$ ordinary irreducible modules $U_i$, $0 \leq i \leq n - 1$, such that

$$U_i \cong \bigoplus_{s \equiv i \mod n} V_s.$$

(2) For $n \geq 3$, we have

$$U \cong \frac{\mathfrak{psl}(n,n)_{1}}{\mathfrak{sl}(n)_{1}}.$$
Since our newly introduced vertex algebra has finitely many ordinary modules, it is natural to ask whether it is quasi-lisse in the sense of [12]. As we are currently unable to prove this property, instead, we investigate the (super)characters of $\mathcal{U}$ and of its modules. If a vertex algebra is quasi-lisse, then necessarily characters and supercharacters must be solutions of modular linear differential equation (MLDE) [12]. In particular, solutions of such equations are known to be either modular (as in the case of ordinary admissible representations) or quasimodular (as in the case of Deligne’s series at non-admissible levels). We prove

**Theorem 1.3.** Characters and supercharacters of $\mathcal{U}$-modules are quasi-modular forms. Moreover, for $n$ even, the character $\text{ch}[\mathcal{U}](\tau)$ is a quasi-modular form of weight 1 and depth 1 on $\Gamma_0(2n)$ (with multiplier), while for $n$ odd, the supercharacter $\text{sch}[\mathcal{U}](\tau)$ is quasimodular of weight 1 and depth 1 on $\Gamma_0(n)$.

Motivated again by [12] we conjecture that the (super)character of $\mathcal{U}$ is a component of a vector-valued modular form coming from a modular linear differential equations (MLDEs). Compared to Deligne’s series where this differential equation is of order two, here the situation is much more complicated as the order of the equation grows with $n$. We hope to return to vector-valuedness and properties of MLDE in our future publications. Here we only analyze an MLDE corresponding to $g = \mathfrak{sl}(3)$ (see Proposition 7.3).

The vertex algebras associated to $\mathfrak{psl}(n|n)$ have attracted much attention in the literature (cf. [7], [8], [18], [17]). In the present paper we identify the coset $\mathfrak{psl}(n|n)/\mathfrak{sl}(n)$ as a vertex algebra $\mathcal{U}$ for $n \geq 3$. We conjecture that the super character for the simple vertex algebra $V_1(\mathfrak{psl}(n,n))$ is for every $n \geq 3$ equal to the supercharacter of symplectic fermion vertex algebra, and equals to $\eta(\tau)^2$. We prove this conjecture in the case $n = 3$, by using we use the (super)character of $\mathcal{U}$ from previous section, together with a branching rules for conformal embeddings in the case $n = 3$.

We believe that a similar phenomena exists for a broader family of affine vertex algebras. We have the following conjecture based on the analysis in the case $\mathfrak{psl}(n|n)$ and results from the paper [7] and [12].

**Conjecture 1.4.** For every $n \geq 0$ even we have

$$\text{sch}[V_{-2}(\mathfrak{osp}(n + 8|n))](\tau) = \text{ch}[V_{-2}(\mathfrak{so}(8))](\tau).$$

We should also mention that the vertex algebra $V_{-2}(\mathfrak{osp}(n + 8|n))$ has recently appeared in the work of K. Costello and D. Gaiotto [17] Section 5 in the context of $SU(2)$-gauge theory with $N \geq 4$ flavors.

**Acknowledgments:**

D.A. is partially supported by the Croatian Science Foundation under the project 2634 and by the QuantiXLie Centre of Excellence, a project cofinanced by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (KK.01.1.1.01). A.M. was partially supported by the NSF Grant DMS-1601070.

2. **The affine vertex algebra $V_{-1}(\mathfrak{sl}(n))$**

In this section we recall the basic properties of the affine vertex algebra $V_{-1}(\mathfrak{sl}(n))$. Here we use the standard notation: $V^k(g)$ denotes the universal vertex algebra of level $k$ and $V_k(g)$ is
the corresponding simple vertex algebra. All affine vertex algebras are equipped with the usual conformal structure (via Sugawara’s construction).

2.1. Symplectic fermions and the \( c = -2 \) singlet vertex algebra. The symplectic fermion vertex algebra \( \mathcal{A}(n) \) (see [1] for more details) is the universal vertex superalgebra generated by odd fields/vectors \( b_i \) and \( c_i \) (\( i = 1, \ldots, n \)) with the following non-trivial \( \lambda \)–bracket

\[
[(b_i)_\lambda c_j] = \delta_{i,j}\lambda.
\]

\( \mathcal{A}(n) \) can be realized on the irreducible level one module for the Lie superalgebra with generators

\[
\{ K, b_i(n), c_i(n), n \in \mathbb{Z} \}
\]

and relations

\[
\{ b_i(n), b_j(m) \} = \{ c_i(n), c_j(m) \} = 0, \quad \{ b_i(n), c_j(m) \} = n\delta_{i,j}\delta_{n+m,0}K.
\]

Here \( K \) is central and other super-commutators are trivial. As a vector space, \( \mathcal{A}(n) = \bigwedge \text{span} \{ b_i(-m), c_i(-m), m \in \mathbb{Z}_{>0}, i = 1, \ldots, n \} \).

The fields \( b_i, c_i \) can be identified as formal Laurent series acting on \( \mathcal{A}(n) \).

\[
b_i(x) = \sum_{n \in \mathbb{Z}} b_i(n)x^{-n-1}, \quad c_i(x) = \sum_{n \in \mathbb{Z}} c_i(n)x^{-n-1}
\]

The vertex algebra \( \mathcal{A}(n) \) has the following Virasoro element of central charge \( c = -2n \):

\[
\omega_{\mathcal{A}(n)} = \sum_{i=1}^{n} b_i c_i.
\]

There is a charge operator \( J \in \text{End}(\mathcal{A}(n)) \) such that

\[
[J, b_i(n)] = b_i(n), \quad [J, c_i(n)] = -c_i(n)
\]

which defines on \( \mathcal{A}(n) \) the \( \mathbb{Z} \)–gradation:

\[
\mathcal{A}(n) = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{A}(n)^{(\ell)}, \quad \mathcal{A}(n)^{(\ell)} = \{ v \in \mathcal{A}(n) \mid Jv = \ell v \}.
\]

The vertex algebra \( \mathcal{A}(1)^{(0)} \) is isomorphic to the singlet vertex algebra \( \mathcal{M}(1) \) of central charge \( c = -2 \) (cf. [28], [2]). For every \( i \in \{0, \ldots, n\} \) we set \( \mathcal{G}_i^0 = 1 \), and for \( m \in \mathbb{Z}_{\geq 1} \) we define

\[
\mathcal{G}_i^m = b_i(-m) \cdots b_i(-1), \quad \mathcal{G}_i^{-m} = c_i(-m) \cdots c_i(-1).
\]

Each \( u_r = \mathcal{G}_i^r.1, r \in \mathbb{Z} \), is a singular vector for the singlet vertex algebra, which generates an irreducible module \( \pi_r \) (note that we drop the index \( i \)).

It was proven in [4] that modules are simple current \( \mathcal{A}(1)^{(0)} \)–modules with the following fusion rules:

\[
\pi_r \times \pi_s = \pi_{r+s}.
\]
2.2. The Weyl vertex algebra and its bosonization. The Weyl vertex algebra \( W(n) \) is the universal vertex algebra generated by the even fields \( a^\pm_i \) and the following non-trivial \( \lambda \)-bracket:

\[
[(a^+_i)_i a^-_j] = \delta_{i,j}, \quad (i, j = 1 \ldots , n).
\]

The vertex algebra \( W(n) \) has the structure of the irreducible level one module for the Lie algebra with generators \( \{ K, a^\pm (n + 1/2) \mid n \in \mathbb{Z} \} \) and commutation relations:

\[
[a^+_i (r), a^-_j (s)] = \delta_{i,j} \delta_{r+s,0} K, \quad [a^+_i (r), a^+_j (s)] = 0 \quad (r, s \in \frac{1}{2} + \mathbb{Z}, \ i, j = 1, \ldots , n),
\]

where \( K \) is central element. The fields \( a^\pm_i \) acts on \( W(n) \) as the following Laurent series

\[
a^\pm_i (z) = \sum_{n \in \mathbb{Z}} a^\pm (n + \frac{1}{2}) z^{-n-1}.
\]

For \( i = \{1, \ldots, n\} \) and \( r \in \mathbb{Z} \), we define

\[
X^r_i := a^+_i (-1/2)^r, \quad r \geq 0 \quad \text{and} \quad X^r_i := a^-_i (-1/2)^r \mathbf{1}, \quad r < 0.
\]

Let \( V_L = M_n(1) \otimes \mathbb{C}[L] \) be the lattice vertex superalgebra associated to the lattice

\[
L = \mathbb{Z} \varphi_1 \oplus \cdots \oplus \mathbb{Z} \varphi_n
\]

with products:

\[
\langle \varphi_i, \varphi_j \rangle = -\delta_{i,j} \quad (i, j = 1, \ldots , n).
\]

Here \( M_n(1) \) denotes the level one module for the Heisenberg vertex algebra associated to the Heisenberg Lie algebra \( \hat{h}_n = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h}_n \oplus \mathbb{C} K \), where \( \mathfrak{h}_n = \mathbb{C} \otimes_{\mathbb{Z}} L \).

2.3. Realization of \( V^{-1}(sl(n)) \) and its ordinary modules. We have the embedding

\[
W(n) \to A(n) \otimes V_L
\]

such that

\[
a^+_i =: b_i e^{\varphi_i} : \quad a^-_i = -: c_i e^{-\varphi_i} :
\]

Define \( c := -(\varphi_1 + \cdots \varphi_n) \). Then \( c(0) \) defines on \( W(n) \) the natural \( \mathbb{Z} \)-gradation:

\[
W(n) = \bigoplus_{s \in \mathbb{Z}} W^{(s)} (n).
\]

Let \( M_c(1) \) be the Heisenberg vertex algebra of level 1 generated by \( c \). Let \( M_c(1, r) \) be the irreducible \( M_c(1) \)-module on which \( c(0) \) acts as \( r \text{Id} \).

The vertex subalgebra of \( W^{(0)} (n) \) generated by the vectors

\[
\{ e_{i,j} = -: a^+_i a^-_j : \mid i, j = 1, \ldots , n \}
\]

is isomorphic to the simple affine vertex algebra \( V^{-1}(gl(n)) \) at level \(-1\) (cf. [11]).

We also have for \( i \neq j \):

\[
e_{i,j} :=: b_i c_j e^{\varphi_i - \varphi_j} :.
\]

Then we have
Theorem 2.1. $W^{(0)}_{(n)}$ is a simple vertex algebra and the following holds:

- [24] For $n = 1$, $W^{(0)}_{(n)}$ is a $W_{1+\infty}$-algebra at central charge $c = -1$.
- [19] Theorem 5.2] For $n = 2$, $W^{(0)}_{(n)} \cong \mathcal{W} \otimes M_c(1)$, where $\mathcal{W}$ is a certain $W$-algebra of type $W(1,1,1,2,2,2)$ at central charge $c = -3$ (conjecturally isomorphic to $W_{-5/2}(\mathfrak{sl}(4), f_{sh})$ (cf. [19], [5]) where $f_{sh}$ is a short nilpotent element of $\mathfrak{sl}(4)$
- [11] For $n \geq 3$: $W^{(0)}_{(n)} \cong V_{-1}(\mathfrak{sl}(n)) \otimes M_c(1)$.

We need the following result on fusion rules.

Proposition 2.2. [11] Assume that $n \geq 3$. For $s \in \mathbb{Z}_{\geq 0}$, let

$$ V_s := L_{\mathfrak{sl}(n)}(-(1+s)\Lambda_0 + s\Lambda_1), \quad V_{-s} := L_{\mathfrak{sl}(n)}(-(1+s)\Lambda_0 + s\Lambda_{n-1}). $$

- The set $\{V_s \mid s \in \mathbb{Z}\}$ provides a complete list of irreducible $V_{-1}(\mathfrak{sl}(n))$ modules in the category $KL_{-1}$ ($= \text{the category of ordinary modules}$).
- The following fusion rules hold in the category $KL_{-1}$.

$$(2) \quad V_{s_1} \times V_{s_2} = V_{s_1+s_2} \quad (s_1, s_2 \in \mathbb{Z}).$$

Let us now present a realization of irreducible $V_{-1}(\mathfrak{sl}(n))$-modules. Let

$$ Q_n = \{z_1\varphi_1 + \cdots + z_n\varphi_n \mid z_1 + \cdots + z_n = 0\}. $$

Since $V_{-1}(\mathfrak{sl}(n)) \subset \mathcal{A}(n) \otimes V_{Q_n}$, we have that for every $\lambda \in Q_n^0$ (= the dual lattice of $Q_n$), $\mathcal{A}(n) \otimes V_{\lambda+Q_n}$ is a $V_{-1}(\mathfrak{sl}(n))$-module. Let

$$ \omega_1 = \frac{1}{n}((n-1)\varphi_1 - \varphi_2 - \cdots - \varphi_n), \quad \omega_{n-1} = \frac{1}{n}(\varphi_1 + \cdots + \varphi_{n-1} - (n-1)\varphi_n). $$

We set $v^{(0)} = 1$. For $i \in \mathbb{Z}_{\geq 0}$ we define

$$ v^{(j)} = b_1(-j) \cdots b_1(-1) \otimes e^{j\omega_1} \quad v^{(-j)} = c_n(-j) \cdots c_n(-1) \otimes e^{j\omega_{n-1}} $$

Proposition 2.3. For $s \in \mathbb{Z}$ we have:

$$ V_s \cong V_{-1}(\mathfrak{sl}(n)), v^{(s)}. $$

Proof. First we notice that $v^{(s)}$ is a singular vector for $\mathfrak{sl}(n)$. Then $\tilde{U}_s = V_{-1}(\mathfrak{sl}(n)), v^{(s)}$ is a highest weight $V_{-1}(\mathfrak{sl}(n))$-module, having the same highest weight as $U_s$. By using the bosonization of the Weyl vertex algebra, we show that as a $V_{-1}(\mathfrak{gl}(n)) = V_{-1}(\mathfrak{sl}(n)) \otimes M_c(1)$-module $W^{(s)}_{(n)} \cong \tilde{U}_s \otimes M_c(1, s)$. Since $W^{(s)}_{(n)}$ is irreducible $V_{-1}(\mathfrak{gl}(n))$-module, we conclude that $\tilde{U}_s$ is irreducible $V_{-1}(\mathfrak{sl}(n))$-module, and thus $\tilde{U}_s \cong U_s$. \qed
3. Parafermionic algebra and the vacuum of $V_{-1}(\mathfrak{sl}(n))$

Recall the definition of the parafermion vertex algebra of level $k$:

$$K(g, k) := \{ v \in V_k(g) \mid (\mathfrak{h} \otimes t^m).v = 0, \; m \in \mathbb{Z}_{\geq 0} \}.$$

**Theorem 3.1.** Assume that $n \geq 3$. Then

$$K(\mathfrak{sl}(n), -1) \cong (\mathcal{A}(1)^0)^{\otimes n}.$$

**Proof.** Let $M_{n-1}(1)$ (resp. $M_n(1)$) be the Heisenberg vertex algebra generated by the Cartan Lie subalgebra of $\mathfrak{sl}(n)$ (resp. $\mathfrak{gl}(n)$). Let $\mathfrak{gl}(n) = \mathfrak{sl}(n) \oplus \mathfrak{c}$. As usual we identify $x = x_{(-1)} 1$ for $x \in \mathfrak{sl}(n)$. Then $M_n(1) = M_{n-1}(1) \otimes M_c(1)$, where $M_c(1)$ is the Heisenberg vertex algebra generated by $c$.

By [11], we have for $n \geq 3$:

$$V_{-1}(\mathfrak{gl}(n)) = V_{-1}(\mathfrak{sl}(n)) \otimes M_c(1) \cong (W(n))_0 = \text{Ker} W(n)c(0).$$

A. Linshaw in [25, Theorem 7.2] proved that

$$\text{Com}(M_n(1), W(n)) \cong (\mathcal{A}(1)^0)^{\otimes n}.$$

(This corresponds to the case $m = n$ in [25]). Therefore for $n \geq 3$:

$$\text{Com}(M_n(1), W(n)) \cong \text{Com}(M_n(1), V_{-1}(\mathfrak{gl}(n))) \cong K(\mathfrak{sl}(n), -1).$$

The proof follows. \qed

3.1. The vacuum space. The vacuum space is defined as

$$\Omega(V_k(g)) = \{ v \in V_k(g) \mid h(j)v = 0 \; \; j \geq 1, \; h \in \mathfrak{h} \},$$

and it has the structure of a generalized vertex algebra [26], [21].

**Theorem 3.2.**

1. Assume that $n \geq 2$. The vacuum algebra

$$\Omega_n := \Omega(V_{-1}(\mathfrak{sl}(n))) = \{ v \in V_{-1}(\mathfrak{sl}(n)) \mid h(j)v = 0 \; \; j \geq 1, \; h \in \mathfrak{h} \}$$

is isomorphic to a vertex subalgebra of $\mathcal{A}(n)$ generated by

$$\{ Z_{i,j} := b_i c_j : \; \; 1 \leq i \neq j \leq n \}.$$

2. Assume that $n \geq 3$. Then $\Omega_n \cong \mathcal{A}(n)^{(0)}$

3. The $q$–character of $\Omega_n$ is given by

$$\text{ch}[\mathcal{A}(n)^{(0)}](\tau) = q^{\frac{1}{12}} \text{CT} \prod_{i=1}^{\infty} (1 + q^i \zeta)^n (1 + q^i \zeta^{-1})^n,$$

where CT is the constant term.
Proof. The proof uses the explicit realization, the bosonization and the formula for $Z$–operators.

By using [26, Theorem 6.4], we see that

$$\Omega(V_{-1}(\mathfrak{sl}(n)))$$

is generated by the following (generalized) vertex operators

$$Z_{i,j}(z) = Y_{\Omega}(e_{i,j}, z) = E^-(h_{i,j}, z)E^+(h_{i,j}, z)z^{h_{i,j}(0)}$$

where $h_{i,j} = \varphi_i - \varphi_j$ and

$$E^\pm(\alpha, z) = \exp\left(\sum_{n=1}^{\infty} \frac{\alpha(\pm n)}{\pm n} z^{\pm n}\right).$$

Using (1) we see that on $\Omega_n = \Omega(V_{-1}(\mathfrak{sl}(n)))$ we have that

$$Z_{i,j} =: b_i c_j :$$

Therefore $\Omega_n$ is generated by $\{Z_{i,j} =: b_i c_j : 1 \leq i \neq j \leq n\}$. This proves (1).

(2) Since $\Omega_n$ is generated by $Z_{i,j}$, $i \neq j$ we have:

$$u_{i,j} = (Z_{i,j})_1 Z_{j,i} = (b_i(-1)c_j(-1)1)b_j(-1)c_i(-1)1 =: b_i c_i : + : b_j c_j : \in \Omega_n.$$

This implies

$$(*) : b_i c_i : \in \Omega_n$$

Since $\Omega_n \subset \mathcal{A}(n)^0$, then (2) will follow from the following claim

$$(2')$$

For $n \geq 3$, $\mathcal{A}(n)^0$ is generated by the set $\{b_i c_j : i, j = 1, \ldots, n\}$.

The claim (2') can be proved using completely analogue methods to those from [19, Section 5] (we omit details).

The proof of assertion (3) is clear.

Remark 1. In [19], the authors denoted the maximal vertex operator subalgebra of the generalized vertex operator algebra $\Omega(V_k(g))$ by $E_{k,g}$ (see [19, Example 3]). In our case, $\Omega(V_{-1}(\mathfrak{sl}(n)))$ is a vertex algebra, so we have

$$E_{-1,\mathfrak{sl}(n)} \cong \mathcal{A}(n)^{0}.$$
(2) In the category of ordinary modules, \(\mathcal{U}\) has \(n\)-non-equivalent irreducible ordinary modules:

\[
\mathcal{U}_i := \bigoplus_{s \in \mathbb{Z}} V_{ns+i}, \quad (i = 0, \ldots, n-1)
\]

with the following fusion rules

\[
\mathcal{U}_i \times \mathcal{U}_j = \mathcal{U}_{i+j \mod n}.
\]

**Proof.** The proof of assertion (1) is based on the explicit realization discussed in Section 2. Note that vector \(v_n\) is even (resp. odd). Therefore, \(e^{mn\omega_1} \mathcal{U}_i \mathcal{U}_{i+j} \mathcal{U}_0\) are even vectors in \(V_L\).

Consider the vertex subalgebra \(\tilde{\mathcal{U}}\) of \(\mathcal{A}(n) \otimes V_L\) generated by \(U_0\) and highest weight vectors

\[
v^{(n)} = b_1(-n) \cdots b_1(-2)b_1(-1) \otimes e^{n\omega_1} \quad v^{(-n)} = c_1(-n) \cdots c_1(-2)c_1(-1) \otimes e^{n\omega_{n-1}}
\]

Note that vector \(v^{(\pm n)}\) have conformal weight \(n\). Moreover, vectors \(v^{(\pm n)}\) are even (resp. odd) if \(n\) is even (resp. odd). Therefore, \(\tilde{\mathcal{U}}\) is a vertex operator algebra if \(n\) is even, and a \(\mathbb{Z}\)-graded vertex operator superalgebra if \(n\) is odd.

By Proposition 2.3 we have that \(m \in \mathbb{Z}\geq 0\) modules \(V_{\pm mn}\) are realized as \(V_{\pm mn} = V_0 \otimes v^{(\pm mn)}\). Since \(v^{(\pm mn)} \in \tilde{\mathcal{U}}\) we get that \(\tilde{\mathcal{U}} \supset \mathcal{U}(n)\). By using fusion rules (2) we see that \(\tilde{\mathcal{U}}(n)\) is a vertex subalgebra of \(\tilde{\mathcal{U}}\). Since both vertex algebras are generated by \(U_0\) and \(v^{(\pm n)}\) we conclude that \(\mathcal{U}(n) = \tilde{\mathcal{U}}\). This proves the assertion (1).

Let us now discuss the construction and classification of irreducible \(\mathcal{U}(n)\)-modules. Clearly \(L_i = \mathcal{U}(n) \mathcal{U}(i) = \bigoplus_{s \in \mathbb{Z}} V_{ns+i}\) is an irreducible \(\mathcal{U}(n)\)-module for \(i = 0, \ldots, n-1\).

Assume that \(M\) is an irreducible ordinary module for \(\mathcal{U}(n)\). Then \(M\) is in the category \(KL_{n-1}\) as a \(V_{-1}(sl(n))\)-module. Since the top component \(\Omega(M)\) is a finite-dimensional module for \(U(sl(n))\), we conclude that \(\Omega(M)\) contains a singular vector for \(sl(n)\). Thus, \(M\) contains a \(V_{-1}(sl(n))\)-submodule isomorphic to \(V_i\) for certain \(i \in \mathbb{Z}\). By using the fusion rules (2) again, we conclude that \(M \cong \bigoplus_{s \in \mathbb{Z}} V_{ns+i} = \mathcal{U}_i\). The proof follows.

**Conjecture 3.4.** The vertex algebra \(\mathcal{U}(n)\) is quasi-lisse in the sense of [12].

**Remark 2.** In our paper we present some evidence for Conjecture 3.4:

- There are finitely many (ordinary) irreducible \(\mathcal{U}(n)\)-modules.
- Characters and super-characters of (ordinary) \(\mathcal{U}(n)\)-modules are quasi-modular forms. For \(g = sl_3\), the supercharacters are solutions of an MLDE (see Proposition 7.3).
- The vacuum spaces is a \(C_2\)-cofinite vertex operator algebra.
- For simplicity, let us discuss the case \(n = 3\). Then we will see that the vacuum space \(\Omega(\mathcal{U}(3))\) is a \(\mathbb{Z}_3\)-orbifold of the symplectic fermion vertex algebra \(A(3)\). Since every cyclic orbifold of a \(C_2\)-cofinite vertex algebra is \(C_2\)-cofinite (cf. [27]), then the vacuum \(\Omega(\mathcal{U}(3))\) is \(C_2\)-cofinite.

**Remark 3.** Assume that \(V\) is a vertex operator (super) algebra containing a Heisenberg vertex subalgebra \(M(1)\). We believe that if the vacuum space \(\Omega(V)\) is \(C_2\)-cofinite, then \(V\) is quasi-lisse.
4. A DECOMPOSITION OF $V_{-1}(\mathfrak{sl}(n))$ AS $K(\mathfrak{sl}(n), -1) \otimes M_{n-1}(1)$–MODULE

4.1. A decomposition of $V_{-1}(\mathfrak{sl}(3))$ as $K(\mathfrak{sl}(3), -1) \otimes M_2(1)$–modules: from the realization. Let $Q$ be the root lattice of $\mathfrak{sl}(3)$. For $(r, s) \in \mathbb{Z}^2$, we set $\gamma_{r,s} = r \varphi_1 + s \varphi_2 - (r + s) \varphi_3$. We have:

$$V_{-1}(\mathfrak{sl}(3)) = \bigoplus_{(r, s) \in \mathbb{Z}^2} (K(\mathfrak{sl}(3), -1) \otimes M_2(1)) \cdot P_{r,s}$$

$$= \bigoplus_{(r, s) \in \mathbb{Z}^2} (K(\mathfrak{sl}(3), -1) \otimes M_2(1)) \cdot (v_{r,s} \otimes e^{\gamma_{r,s}})$$

$$= \bigoplus_{(r, s) \in \mathbb{Z}^2} K_{r,s} \otimes M_2(1) \cdot e^{\gamma_{r,s}}$$

where

$$P_{r,s} = X_1^r X_2^s X_3^{-r-s} \mathbf{1} = v_{r,s} \otimes e^{\gamma_{r,s}},$$

and

$$K_{r,s} = \{ v \in V_{-1}(\mathfrak{sl}(3)) | h(n)v = \delta_{n,0} \langle h, \gamma_{r,s} \rangle v \ \forall h \in \mathfrak{h}, \ n \in \mathbb{Z}_{\geq 0} \}$$

is irreducible $K(\mathfrak{sl}(3), -1)$–module generated by lowest weight vector

$$v_{r,s} = G_1^r G_2^s G_{-r-s} \mathbf{1}.$$ We conclude that $K_{r,s} = \pi_r \otimes \pi_s \otimes \pi_{-r-s}$. In this way we have proved the following theorem:

**Theorem 4.1.** The vertex algebra $V_{-1}(\mathfrak{sl}(3))$ is a simple current extension of $(A(1)^0)^{\otimes 3} \otimes M_2(1)$ and

$$V_{-1}(\mathfrak{sl}(3)) = \bigoplus_{(r, s) \in \mathbb{Z}^2} \pi_r \otimes \pi_s \otimes \pi_{-r-s} \otimes M_2(1) e^{\gamma_{r,s}}.$$

4.2. A decomposition of $V_{-1}(\mathfrak{sl}(3))$ as $K(\mathfrak{sl}(3), -1) \otimes M_2(1)$–modules II: from character formulas. It is possible to prove Theorem 4.1 directly from the character [14]. Observe a well-known identity

$$\frac{1}{\prod_{n \geq 1} (1 - z q^n - 1)(1 - z^{-1} q^n)} = \sum_{m \in \mathbb{Z}} F_m(q) z^m \prod_{n \geq 1} (1 - q^n)^2,$$

where

$$F_m(q) = \sum_{r \geq 0} q^{(2r+1)r + 2mr} - \sum_{r \geq 0} q^{(2r-1)r + m(2r-1)}, \quad m \geq 0$$

and

$$F_m(q) = q^m F_{-m}(q), \quad m < 0.$$

We use this formula to expand the character (three times). Then we extract the coefficients of $z_1$ and $z_2$ which corresponds the modules for the vacuum algebra. Finally we use a well-known formula for $\text{ch}[K_{r,s}]$ irreducible module for the tensor product of three copies of the singlet algebra. This gives

$$\text{ch}[V_{-1}(\mathfrak{sl}(3))] (\tau) = \sum_{(r, s) \in \mathbb{Z}^2} \text{ch}[K_{r,s}] (\tau) \text{ch}[M_2(1) \cdot e^{\gamma_{r,s}}] (\tau).$$
4.3. $q$-hypergeometric formula for $\text{ch}[V_{-1}(\mathfrak{sl}(3))](\tau)$. Now we use discussion from the last section to prove

**Proposition 4.2.**

$$
(q; q)_{\infty}^{2} q^{-\frac{1}{6}} \text{ch}[V_{-1}(\mathfrak{sl}(3))](\tau) = \sum_{k_{1}, k_{2} \in \mathbb{Z}^{2}} \sum_{n_{1}, n_{2}, n_{3} \geq 0} q^{n_{1}^{2} + n_{2}^{2} + n_{3}^{2} + \frac{(k_{1} + 1)n_{1} + (k_{2} + 1)n_{2} + (k_{1} - k_{2})+1)n_{3} + |k_{1}| + |k_{2}| + |k_{1} - k_{2}|}{2}
$$

**Proof.** Follows directly from the $q$-hypergeometric representations of the $p = 2$ false theta functions which are known to be characters of modules for the $p = 2$ singlet algebra (here $k \in \mathbb{Z}$):

$$
\frac{q^{\frac{k}{2}} \sum_{n \geq 0} \left( q^{2n^{2} + n(2k+1)} - q^{2n^{2} + n(2k+3)+k+1} \right)}{(q; q)_{\infty}^{2}} = \sum_{n \geq 0} q^{n^{2} + n(|k|+1)+\frac{|k|}{2}}.
$$

These relations are proven along the lines of [16] generalizing a well-known Ramanujan’s identity corresponding to $k = 0$. 

4.4. A realization of the vertex superalgebra $U^{(n)}$. Let

$$\omega_{1} = \frac{1}{n}((n-1)\varphi_{1} - \varphi_{2} - \cdots - \varphi_{n}), \quad \omega_{n-1} = \frac{1}{n}(\varphi_{1} + \cdots + \varphi_{n-1} - n\varphi_{n}).$$

Note that $m\omega_{1}, m\omega_{n-1} \in L$ and that $e^{m\omega_{1}}$ and $e^{m\omega_{n-1}}$ are even vectors in $V_{L}$.

The vertex superalgebra $U^{(n)}$ can be realized as a subalgebra of $A(n) \otimes V_{L}$ generated by $U_{0}$ and highest weight vectors

$$
v^{(n)} = b_{1}(-n) \cdots b_{1}(-2)b_{1}(-1) \otimes e^{n\omega_{1}}
$$

$$
v^{(-n)} = c_{n}(-n) \cdots c_{n}(-2)c_{n}(-1) \otimes e^{n\omega_{n-1}}
$$

Note that vector $v^{(\pm n)}$ have conformal weight $n$. Therefore, $U$ is $\mathbb{Z}$-graded vertex operator superalgebra.

4.5. The vacuum space $\Omega(U^{(3)})$. Let us again consider the case $n = 3$, so that $U = U^{(3)}$.

Moreover, for $m \in \mathbb{Z}_{>0}$ modules $U_{\pm 3m}$ are realized as $U_{\pm 3m} = U_{0}v^{(\pm 3m)}$ where

$$
v^{(3m)} = b_{3}(-3m) \cdots b_{1}(-1) \otimes e^{3m\omega_{1}}
$$

$$
v^{(-3m)} = c_{3}(-3m) \cdots c_{3}(-1) \otimes e^{3m\omega_{2}}
$$

**Theorem 4.3.** We have:

$$\Omega(U) \cong A(3)^{\mathbb{Z}_{3}}.$$

**Proof.** First we notice that

$$
A(3)^{\mathbb{Z}_{3}} \cong \bigoplus_{m_{1}+m_{2}+m_{3} \in 3\mathbb{Z}} \pi_{m_{1}} \otimes \pi_{m_{2}} \otimes \pi_{m_{3}} = \bigoplus_{m_{1}+m_{2}+m_{3} \in 3\mathbb{Z}} (A(1)^{0})^{\otimes 3}.G_{1}^{m_{1}}G_{2}^{m_{2}}G_{3}^{m_{3}}1
$$

Using realization we see that
12 DRAŽEN ADAMOVIĆ AND ANTUN MILAS

• $\Omega(\mathcal{U}) \subset A(3)^{\mathbb{Z}_3}$;
• $G_1^m G_2^{m_2} G_3^{m_3} 1 \in \Omega(\mathcal{U})$ for all $(m_1, m_2, m_3) \in \mathbb{Z}^3$, $m_1 + m_2 + m_3 \in 3\mathbb{Z}$.

Now claim follows by (3). □

4.6. Decomposition for the general case $n \geq 3$. For $s \in \mathbb{Z}$ we define

$$Q_n^{(s)} = \{ z = (z_1, \ldots, z_n) \in \mathbb{Z}^n \mid z_1 + \cdots + z_n = s \}.$$ 

For $z \in Q_n^{(s)}$ we define

$$\gamma_z = z_1 \varphi_1 + \cdots + z_n \varphi_n$$

$$P_z = \psi_1^{z_1} \cdots \psi_n^{z_n} 1 \in W_{(n)}^{(0)} \cong V_{-1}(\mathfrak{sl}(n))$$

$$v_z = G_1^{z_1} \cdots G_3^{z_3} 1 \in A(n)^{(s)}$$

We have

$$P_z = \nu v_z \otimes e^{\gamma_z} \quad (\nu = \pm 1).$$

Set $Q_n := Q_n^{(0)}$.

**Theorem 4.4.** The vertex algebra $V_{-1}(\mathfrak{sl}(n))$ is a simple current extension of $(A(1)^0)^{\otimes n} \otimes M_{n-1}(1)$ and

$$V_{-1}(\mathfrak{sl}(n)) = \bigoplus_{z \in Q_n} \pi_{z_1} \otimes \pi_{z_2} \cdots \otimes \pi_{z_n} \otimes M_{n-1}(1) e^{\gamma_z}.$$ 

For $s \in \mathbb{Z}$ we have:

$$V_s = \bigoplus_{z \in Q_n^{(s)}} \pi_{z_1} \otimes \pi_{z_2} \cdots \otimes \pi_{z_n} \otimes M_{n-1}(1) e^{\gamma_z}.$$ 

4.7. Decomposition in the case $n = 2$. Let us consider also the case $n = 2$.

**Theorem 4.5.** The vertex algebra $W = \text{Com}(M_{s_1}(1), W_{(2)})$ (which is isomorphic to affine $W$–algebra $W_{-5/2}(\mathfrak{sl}(4), f_{sh})$) is a simple current extension of $(A(1)^0)^{\otimes 2} \otimes M_1(1)$ and

$$W = \bigoplus_{m \in \mathbb{Z}} \pi_m \otimes \pi_{-m} \otimes M_1(1) e^{m(\varphi_1 - \varphi_2)}.$$

**Remark 4.** A detailed study of the representation theory of the vertex algebra $W$ will be discussed in our forthcoming paper [6].

5. The Character $\text{ch}[\mathcal{U}]$ for $\mathfrak{g} = \mathfrak{sl}(3)$

We now discuss graded dimensions (or characters) of ordinary $V_{-1}(\mathfrak{sl}(3))$-modules. This was thoroughly analyzed in [15].

Next formula is a consequence of the explicit construction of modules (here $s \geq 0$):

$$\text{ch}[V_s](\tau) = q^{h_{s+1/6}} \text{Coeff}_s \prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 - \zeta q^{n-1})^3(1 - \zeta^{-1} q^n)^3},$$

(4)
where \( h_s \) is the lowest conformal weight of \( V_s \) and we also used that \( c = -4 \). We also have the full character formula of Kac and Wakimoto \(^{22}\)

\[
\text{ch}[V_s](\tau; z_1, z_2) = q^{h_s + 1/6} \cdot \text{Coeff}_{\mathcal{C}} \prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 - \zeta q^{n-1})(1 - \zeta z_2 q^{n-1})(1 - \zeta^{-1} z_1^{-1} z_2^{-1} q^n)(1 - \zeta^{-1} z_2^{-1} q^n)(1 - \zeta^{-1} q^n)}.
\]

Very recently, Kac and Wakimoto \(^{23}\) gave another Weyl-Kac type character formula for \( \text{ch}[V_s] \) expressed as a rank two Jacobi false theta function (see also \(^{15}\) for a different formula). After a specialization \((z_1, z_2) \to (1, 1)\) we easily get

\[
F_0(q) := (q; q)^{8} q^{-1/6} \text{ch}[V_0](\tau)
= 4 \sum_{n_1 \in \mathbb{N}_0 \atop n_2 \in \mathbb{Z}} \left( 2n_1 - n_2 + \frac{1}{2} \right) \left( 2n_2 - n_1 + \frac{1}{2} \right) (n_1 + n_2 + 1) q^{2n_1^2 + 2n_2^2 - 2n_1 n_2 + n_1 + n_2}.
\]

We also have

\[
F_s(q) := (q; q)^{8} q^{-1/6 - h_s} \text{ch}[V_s](\tau)
= 4 \sum_{n_1 \in \mathbb{N}_0 \atop n_2 \in \mathbb{Z}} \left( 2n_1 - n_2 + \frac{s}{2} + \frac{1}{2} \right) \left( 2n_2 - n_1 + \frac{1}{2} \right) (n_1 + n_2 + \frac{s}{2} + 1) q^{2n_1^2 + 2n_2^2 - 2n_1 n_2 + (s+1)n_1 + n_2}.
\]

Observe that the summation over \( n_1 \) is only over the set of non-negative integers. On the other hand, it is easy to see that the sum over the integers vanishes (by changing \( n_j \mapsto -n_j - 1 \))

\[
\sum_{n_1 \in \mathbb{Z} \atop n_2 \in \mathbb{Z}} \left( 2n_1 - n_2 + \frac{1}{2} \right) \left( 2n_2 - n_1 + \frac{1}{2} \right) (n_1 + n_2 + 1) q^{2n_1^2 + 2n_2^2 - 2n_1 n_2 + n_1 + n_2} = 0.
\]

Let

\[
G(\tau) := 4 \sum_{n_1 \in \mathbb{Z} \atop n_2 \in \mathbb{Z}} \left( 2n_1 - n_2 + \frac{1}{2} \right) \left( 2n_2 - n_1 + \frac{1}{2} \right) (n_1 + n_2 + 1) q^{2n_1^2 + 2n_2^2 - 2n_1 n_2 + n_1 + n_2}
\]

**Proposition 5.1.** We have

\[
G(\tau) = q^3 F_3(\tau).
\]

**Proof.** Straightforward computation with \( q \)-series. \( \square \)

**Corollary 5.2.**

\[
F_{n_2=0}(q) := F_0(q) + q^3 F_3(q),
\]

where

\[
F_{n_2=0}(q) := 4 \sum_{n_1 \in \mathbb{Z}} \left( 2n_1 + \frac{1}{2} \right) \left( -n_1 + \frac{1}{2} \right) (n_1 + 1) q^{2n_1^2 + n_1}.
\]

Moreover, this series is quasi-modular.
Proof. Follows directly from the previous proposition and vanishing of the sum over the full lattice. Quasi-modularity is clear as this series is obtained by differentiating unary theta functions.

Now we combine characters of modules appearing in the decomposition of $\mathcal{U}$ in pairs:

$$q^{-1/6}(q; q)^8_{\infty}\text{ch}[\mathcal{U}](\tau) = \sum_{m \geq 0} \left(q^{3m^2 + 4m} F_{3m}(q) + q^{3(m+1)^2} + 3(m+1)} F_{3m+3}(q)\right),$$

where $(q; q)_{\infty} = \prod_{i=1}^{\infty} (1 - q^i)$. For each pair in the summation we have a similar $q$-series identity (the proof is almost identical)

**Lemma 5.3.** For every $m \geq 0$,

$$q^{3m^2 + 4m} F_{3m}(q) + q^{3(m+1)^2} + 3(m+1)} F_{3m+3}(q)$$

$$= 4 \sum_{n, m \in \mathbb{Z}} (2n + 1/2)^2 (n + 1/2)(n + 3/2)q^{2n^2 + 3n + 1/2}. $$

**Theorem 5.4.** We have (i)

$$\text{ch}[\mathcal{U}](\tau) = \text{tr}_{\mathcal{U}} q^{L(0) - c/24}$$

$$= \frac{4}{2\eta(\tau)^8} \sum_{n, m \in \mathbb{Z}} (2n + 1/2)(-n + 3/2 + 1/2)(n + 3/2)q^{2(n+1/4)^2 + 3/2}.$$

(ii) Denote by $\mathcal{U}_{\pm 1} = \oplus_{n \in \mathbb{Z}} U_{3n \pm 1}$. Then $\text{ch}[\mathcal{U}_{\pm 1}](\tau) = \text{ch}[\mathcal{U}_{\pm 1}](\tau)$ and

$$\text{ch}[\mathcal{U}_{\pm 1}](\tau) = \text{tr}_{\mathcal{U}_{\pm 1}} q^{L(0) - c/24}$$

$$= \frac{4}{2\eta(\tau)^8} \sum_{n, m \in \mathbb{Z}} (2n + 1/2)(-n + 3/2 + 1/2)(n + 3/2)q^{2(n+1/4)^2 + 3/2}.$$

(iii) For the supercharacter we have

$$\text{sch}[\mathcal{U}](\tau) = \text{tr}_{\mathcal{U}} q^{L(0) - c/24}$$

$$= \frac{4}{\eta(\tau)^8} \sum_{n, m \in \mathbb{Z}} (2n + 1/2)(-n + 3/2 + 1/2)(n + 3/2)q^{2(n+1/4)^2 + 3/2}.$$

Proof. We only prove (i) here - formula (ii) can be proven using similar ideas. Proof of (iii) is slightly different and is postponed for Section 8 (see Remark 7).

We first apply Lemma 5.3 to write

$$\text{ch}[\mathcal{U}](\tau) = \text{tr}_{\mathcal{U}} q^{L(0) + 1/6} = \sum_{m \geq 0} \frac{q^{1/3}}{(q; q)_{\infty}^8} F(m)$$

where $F(m) := q^{3m^2 + 4m} F_{3m}(q) + q^{3(m+1)^2} + 3(m+1)} F_{3m+3}(q)$, in the form

$$\text{ch}[\mathcal{U}](\tau) = \frac{4}{\eta(\tau)^8} \sum_{m \geq 0} \sum_{n \in \mathbb{Z}, m \geq 0} (2n + 1/2)(-n + 3/2 + 1/2)(n + 3/2)q^{2(n+1/4)^2 + 3/2}.$$
Now observe that
\[ \text{ch}[\mathcal{U}](\tau) = \frac{4}{\eta(\tau)} \sum_{m \geq 0} \sum_{n \in \mathbb{Z}, m < 0} (2n + \frac{1}{2})(-n + \frac{3m}{2} + \frac{1}{2})(n + \frac{3m}{2} + 1)q^{2(n+1/4)^2 + \frac{3}{2}(m+1/2)^2}, \]
so if we take summation over \( m \in \mathbb{Z} \) we have to divide by 2. This implies formula (i).

\[ \square \]

Remark 5. Observe that irreducible \( \mathcal{U} \)-modules are also \( \mathbb{Z}_2 \)-graded. Their supercharacters are given by
\[
\text{sch}[\mathcal{U}_1] = - \text{ch}[V_1] + \sum_{i \geq 1}(-1)^{i-1}(\text{ch}[V_{3i+1}] + \text{ch}[V_{-3i+1}]),
\]
\[
\text{sch}[\mathcal{U}_2] = \text{ch}[V_2] + \sum_{i \geq 1}(-1)^{i-1}(\text{ch}[V_{3i+2}] + \text{ch}[V_{-3i+2}]).
\]
Because of \( \text{ch}[V_i] = \text{ch}[V_{-i}] \) this easily implies \( \text{sch}[\mathcal{U}_1] = \text{sch}[\mathcal{U}_2] \). One can also show
\[
\text{sch}[\mathcal{U}_1](\tau) = \frac{4}{\eta(\tau)^8} \sum_{n,m \in \mathbb{Z}} (2n + \frac{1}{2})(-n + \frac{3m}{2} + \frac{1}{2})(n + \frac{3m}{2} + 1)q^{2(n+1/4)^2 + \frac{3}{2}(m+1/2)^2}.
\]

6. Modular properties of \( \text{ch}[\mathcal{U}] \) for \( \mathfrak{g} = \mathfrak{sl}(3) \)

In this section we discuss modular properties of \( \text{ch}[\mathcal{U}] \) for \( \mathfrak{g} = \mathfrak{sl}(3) \). Since its denominator is a power of \( \eta(\tau) \) we only have to analyze its numerator. We first rewrite the numerator in a more convenient form and then we discuss modular transformation properties under \( \tau \mapsto -\frac{1}{2} \). Next result is easy to check using standard methods (e.g. Jacobi triple product identity).

Lemma 6.1.
\[
\sum_{m \in \mathbb{Z}} q^{\frac{3}{2}(m+1/2)^2} = \frac{1}{2} \eta(3\tau)f_2(3\tau)^2,
\]
\[
\sum_{m \in \mathbb{Z}} (n + 1/4)q^{2(n+1/4)^2} = \frac{1}{4} \eta(\tau)^3,
\]
where \( f_2(\tau) = q^{\frac{1}{12}} \prod_{n \geq 0} (1 + q^n) \) is Weber’s modular function.

Using
\[
(2n + \frac{1}{2})(-n + \frac{3m}{2} + \frac{1}{2})(n + \frac{3m}{2} + 1) = -2(n + 1/4)^3 + 9/2(m + 1/2)^2(n + 1/4).
\]
we get
\[
\sum_{n,m \in \mathbb{Z}} (-2(n + 1/4)^3 + 9/2(m + 1/2)^2(n + 1/4))q^{2(n+1/4)^2 + \frac{3}{2}(m+1/2)^2}
\]
\[
= - \sum_{m \in \mathbb{Z}} q^{\frac{3}{2}(m+1/2)^2} \Theta_q \left( \sum_{n \in \mathbb{Z}} (n + 1/4)q^{2(n+1/4)^2} \right) + 3\Theta_q \left( \sum_{m \in \mathbb{Z}} q^{\frac{3}{2}(m+1/2)^2} \right) \sum_{n \in \mathbb{Z}} (n + 1/4)q^{2(n+1/4)^2},
\]
where $\Theta_q := q^{d/6}$. 

The lemma and logarithmic differentiation now implies the following identities:

$$h_1(q) := \Theta_q \left( \sum_{m \in \mathbb{Z}} (n + 1/4)q^{2(n+1/4)^2} \right) = \frac{1}{4} \eta(\tau)^3 \left( \frac{1}{8} - 3 \sum_{n \geq 1} \frac{nq^n}{1 - q^n} \right)$$

and

$$h_2(q) := \Theta_q \left( \sum_{m \in \mathbb{Z}} q^{3(m+1/2)^2} \right) = \frac{1}{2} \eta(3\tau)^3 f_2(3\tau)^2 \left( \frac{1}{8} - 3 \sum_{n \geq 1} \frac{nq^{3n}}{1 - q^{3n}} + \frac{1}{4} + 6 \sum_{n \geq 1} \frac{nq^{3n}}{1 + q^{3n}} \right)$$

Three infinite series appearing on the right-hand are Eisenstein series $E_2(\tau)$, $E_2(3\tau)$ and $E_{2,1}(3\tau)$, respectively. As $E_2(\tau)$ is quasi-modular of weight 2 and depth 1 and $E_{2,1}(\tau)$ is an ordinary modular form of weight 2 on $\Gamma_0(2)$, the expression inside the parentheses is a quasi-modular form of weight 2 and depth 1 on $\Gamma_0(6)$. The $q$-derivative of two unary series transform invariantly under the same group (again with a character).

Then we utilize well-known modular transformation properties for the Dedekind $\eta$-function:

$$g_1(q) := \eta \left( -\frac{1}{\tau} \right)^3 = (-i\tau)^{\frac{3}{2}} \eta(\tau)^3.$$

$$g_2(q) := \eta \left( \frac{3}{\tau} \right)^{\frac{1}{2}} f_2 \left( \frac{3}{\tau} \right)^2 = \left( -\frac{i\tau}{3} \right)^{\frac{3}{2}} \eta \left( \frac{\tau}{3} \right) f_1 \left( \frac{\tau}{3} \right)^2$$

where $f_1(\tau) = \sqrt{2}q^{-1/48} \prod_{n \geq 1} (1 - q^{n-1/2})$ is another Weber’s modular function. Combined we can write

$$\Theta_q \left( \sum_{m \in \mathbb{Z}} (n + 1/4)q^{2(n+1/4)^2} \right)_{\tau \to -\frac{1}{\tau}} = \mu_1 \tau^{\frac{3}{2}} \eta(\tau)^3 + \mu_2 \tau^{\frac{7}{2}} \Theta_q(\eta(\tau)^3)$$

and

$$\Theta_q \left( \sum_{m \in \mathbb{Z}} q^{3(m+1/2)^2} \right)_{\tau \to -\frac{1}{3}} = \lambda_1 \tau^{\frac{3}{2}} \eta \left( \frac{\tau}{3} \right) f_1 \left( \frac{\tau}{3} \right)^2 + \lambda_2 \tau^{\frac{7}{2}} \Theta_q \left( \eta \left( \frac{\tau}{3} \right) f_1 \left( \frac{\tau}{3} \right)^2 \right)$$

for some scalars $\lambda_i$ and $\mu_j$. Overall the numerator will now transform as a quasi modular form of weight 4 and depth 1 and thus the character is of weight 0 and depth 1. We will show that the supercharacter satisfies even better modular invariance properties, which is expected for $\mathbb{Z}_{\geq 0}$-graded vertex superalgebras.

**Remark 6.** The above approach to modularity is difficult to generalize to $\mathfrak{sl}(n)$ because it requires explicit formulae as in Theorem 5.4. But these formulas are non-trivial to extract from [23]. In the remaining of the paper we show how to solve the (quasi)-modularity problem via meromorphic Jacobi forms.
7. QUASI-MODULARITY OF \( \text{ch}[\U](\tau) \)

In this part we prove the quasi-modularity of \( \text{ch}[\U](\tau) \), generalizing our previous computation for \( g = \mathfrak{sl}(3) \). Let

\[
(a)_{\infty} := \prod_{i \geq 1} (1 - aq^{i-1}).
\]

We will make use of a Jacobi theta function

\[
\vartheta(z; \tau) := (-i)^{q_1/8} \zeta^{-1/2} (q)_{\infty} \zeta(q^{-1})_{\infty},
\]

where \( \zeta = e^{2\pi iz} \). Recall the elliptic and modular transformation formulae (here \( \lambda, \mu \in \mathbb{Z} \), \( (a b c d) \in \text{SL}_2(\mathbb{Z}) \))

\[
\vartheta(z + \lambda \tau + \mu) = (-1)^{\lambda + \mu} q^{-\lambda^2/2} \zeta^{-\lambda} \vartheta(z),
\]

\[
\vartheta \left( \frac{z}{c \tau + d} \right) = \chi \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) (c \tau + d)^{\frac{1}{2}} e^{\frac{\pi i c z^2}{c \tau + d}} \vartheta(z; \tau),
\]

where \( \chi \) is a certain multiplier. In particular

\[
\vartheta \left( \frac{z}{\tau}; -\frac{1}{\tau} \right) = -i \sqrt{-i \tau e^{\frac{\pi i z^2}{\tau}}} \vartheta(z; \tau).
\]

As in the \( \mathfrak{sl}(3) \) case, from the explicit construction [22],

\[
(5) \quad \hat{\text{ch}}(V_s) = \text{Coeff}_{\zeta^n} \left( \frac{(q)_{\infty}}{(\zeta^n)_{\infty} (q\zeta^{-1})_{\infty}} \right)
\]

where \( \hat{\text{ch}}(V_s) \) is the character of \( V_s \) up to a multiplicative \( q \)-shift. More precisely, for \( s \geq 0 \)

\[
\hat{\text{ch}}(V_s) = \dim(L(s\omega_1)) + O(q)
\]

and for \( s < 0 \)

\[
\hat{\text{ch}}(V_s) = q^{-s}(\dim(L(s\omega_{n-1})) + O(q)),
\]

where \( L(m\omega_1) \) denotes an irreducible \( \mathfrak{sl}(3) \)-module of highest weight \( m\omega_1 \). Thus in order to compute the genuine character we must multiply with

\[
q^{h_{V_s} - c_n/24},
\]

for \( s \geq 0 \), and in addition shift with \( q^s \) for \( s < 0 \). It is easy to see that for \( s \geq 0 \)

\[
h_{V_s} = h_{V_{-s}} = \frac{s^2}{2n} + \frac{s}{2}
\]

and the central charge is

\[
c_n = -(n + 1).
\]

Combined

\[
h_{V_{n,s}} - \frac{c_n}{24} = \frac{s^2 n}{2} + \frac{sn}{2} + \frac{n + 1}{24}.
\]

Putting this together with (5), and taking into account \( q \)-multiplicative shift for \( s < 0 \), we get
Next we multiply the numerator and the denominator with $\frac{q^n}{24}$ so that in the denominator we have a power of $\theta(z, \tau)$, a weight $\frac{1}{2}$ Jacobi form of index $\frac{1}{2}$. So we obtain

$$\text{ch}[\mathcal{U}](\tau) = C T \zeta \sum_{s \in \mathbb{Z}} q^{\frac{s^2}{2} + \frac{sn}{2} + \frac{n+1}{24}} \zeta^{-sn} \prod_{i=1}^{\infty} (1 - q^i)$$

Next we multiply the numerator and the denominator with $\frac{q^n}{24}$ so that in the denominator we have a power of $\theta(z, \tau)$, a weight $\frac{1}{2}$ Jacobi form of index $\frac{1}{2}$. So we obtain

$$\text{ch}[\mathcal{U}](\tau) = C T \zeta \sum_{s \in \mathbb{Z}} q^{\frac{s^2}{2} + \frac{sn}{2} + \frac{n+1}{24}} \zeta^{-sn} \prod_{i=1}^{\infty} (1 - q^i)$$

Where we used the Jacob triple product identity

$$\vartheta(z, \tau) = (-i) q^{1/8} \zeta^{-1/2} \prod_{n \in \mathbb{Z} + 1/2} e^{2\pi i (n+1/2) z} = i \sum_{n \in \mathbb{Z}} (-1)^n q^{(n+1/2)^2 / 8} e^{2\pi i (n+1/2) z}.$$
7.4. Characters of modules. Straightforward computation gives for $0 \leq k \leq n - 1$,
\[ \text{ch}[U_k](\tau) = \eta(\tau)^{n+1} C_T \zeta \sum_{s \in \mathbb{Z}} q^{\frac{(s+k+n)^2}{2n}} \frac{\vartheta(z,\tau)^n}{\vartheta(z,\tau)^n}. \]
For $n$ even we can write this as
\[ \text{ch}[U_k](\tau) = \eta(\tau)^{n+1} C_T \zeta \left( \frac{\vartheta((z + \frac{k}{2n})n, n\tau)}{\vartheta(z,\tau)^n} \right). \]
Similarly we can compute \( \text{sch}[U_k](\tau) \) - we omit details.

7.5. Quasimodularity. Here we prove a general theorem on quasimodularity which extends our previous observation for $\mathfrak{sl}(3)$.

**Theorem 7.1.** The supercharacter of $U$ (for $n$ odd) and the character of $U$ (for $n$ even) are quasi-modular forms of "weight" one for a suitable congruence group.

**Proof.** We let first $\zeta = e(z)$ where $z \in \mathbb{C}$. First we observe that
\[ G(\tau, z) := \eta(\tau)^{n+1} \left( \frac{\vartheta(zn,n\tau)}{\vartheta(z,\tau)^n} \right) \]
is a meromorphic Jacobi form of weight $\frac{n+1}{2} + \frac{1}{2} - \frac{n}{2} = 1$. We multiply it with $\frac{1}{\eta(\tau)^2}, H(\tau; z) := G(\tau, z) \eta(\tau)^2$, so we have a meromorphic Jacobi function of weight zero.

**Claim:** $G(\tau, z)$ is a Jacobi form of index zero for the congruence group $\Gamma_0(n)$.

We consider $\left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \in \Gamma_0(n)$ (thus $ad - cb = 1$ and $n|c$). Then
\[ \vartheta(nz;n\tau)|_{(\tau \rightarrow \frac{a\tau+b}{cn+d}; z \rightarrow \frac{az+b}{cn+d})} = \vartheta \left( \frac{n z}{nc + d}; \frac{n(a\tau + b)}{cn + d} \right) = \vartheta \left( \frac{n z}{n(\tau + d) + \frac{a(n\tau) + nb}{n(\tau + d)}}; \frac{a(n\tau) + nb}{n(\tau + d)} \right) \]
\[ = \chi'(c\tau + d)^{1/2} e^{\frac{\pi i n^2}{c\tau + d}} \vartheta(nz;n\tau) \]
where we used that $\left[ \begin{array}{cc} a & bn \\ c & d \end{array} \right] \in \Gamma(1)$. On the other hand,
\[ \vartheta(z;\tau)^n|_{(\tau \rightarrow \frac{a\tau+b}{cn+d}; z \rightarrow \frac{az+b}{cn+d})} = \chi^n (c\tau + d)^{n/2} e^{\frac{\pi i n^2}{c\tau + d}} \vartheta(z;\tau)^n. \]
After taking quotient this implies the claim.

Suppose first $n$ is even. Notice that $H(\tau; z)$ is even with respect to $z$ has a pole of order $n$ at $z = 0$, so we can write Laurent expansion [14] (see also [20])

\[ H(\tau; z) = \frac{H_n(\tau)}{(2\pi iz)^n} + \frac{H_{n-2}(\tau)}{(2\pi iz)^{n-2}} + \cdots + \frac{H_2(\tau)}{(2\pi iz)^2} + H_0(\tau), \]
where $H_{2j}(\tau)$ is a modular form of weight $-2j$ with respect to the congruence subgroup.
Then by using \[20\] we can write the "finite" part as

$$H^F(\tau) := H_0(\tau) + \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} H_{2j}(\tau) E_{2j}(\tau)$$

which is quasi-modular of weight zero. Here $E_{2j}(\tau)$ denotes Eisenstein series and $B_{2j}$ are Bernoulli numbers. Finally, the constant term is

$$\text{ch}[U](\tau) = \eta(\tau)^2 H^F(\tau)$$

is of weight one. Similar argument can be used for $n$ odd.

\[7.6. \text{ Odd case: explicit example } n=3.\] Here we compute the constant term of

$$G(\tau, z) := \eta(\tau)^4 \left( \frac{\vartheta(3z; 3\tau)}{\vartheta(z; \tau)^3} \right).$$

The same method can be used to compute $G(\tau; z)$ for every $n$. We have to compute modular forms appearing inside the series \([6]\). Here we use a standard method of Laurent expansion following \([14]\). We write

$$\vartheta^*(z; \tau) := \frac{1}{z} \vartheta(z; \tau),$$

where we suppress $\tau$ from the formula for brevity. Then we have

$$\vartheta^*(z; \tau) = \vartheta^*(0; \tau) + \vartheta^{*\prime\prime}(0; \tau) \frac{z^2}{2!} + O(z^4)$$

and

$$\vartheta^*(3z; 3\tau) = \vartheta^*(0; 3\tau) + \vartheta^{*\prime\prime}(0) + \frac{9 z^2}{2!} + O(z^4)$$

$$G(z) = \frac{3z \left( \vartheta^*(0; 3\tau) + \vartheta^{*\prime\prime}(0; 3\tau) \frac{9 z^2}{2!} + O(z^4) \right)}{z^3 \left( \vartheta^*(0) + \vartheta^{*\prime\prime}(0) + \frac{9 z^2}{2!} + O(z^4) \right)}$$

(8)

It is clear that

$$\vartheta^*(0; \tau) = -2\pi \eta^3(\tau) = -2\pi \sum_{n \in \mathbb{Z}} (-1)^n (n + 1/2) q^{\frac{1}{2}(n+1/2)^2}$$

from the infinite expansion of $\vartheta(z; \tau)$ and Euler’s theorem and

$$\vartheta^{*\prime\prime}(0; \tau) = \frac{1}{12} (2\pi i)^2 E_2(\tau) \eta^3(\tau).$$

Expanding \((8)\) gives only even powers of $z$ and in particular

$$H_0(\tau) + \frac{H_{-2}(\tau)}{(2\pi i z)^2} + O(1).$$
Finally
\[
\begin{align*}
CT_2 \left\{ \eta(\tau)^4 \left( \frac{\partial(3z; 3\tau)}{\partial(\zeta, \tau)^3} \right) \right\} &= H_0(\tau) + \frac{B_2}{2} H_2(\tau) E_2(\tau) \\
&= -\frac{9}{8} E_2(\tau) \eta(\tau)^3 \eta(3\tau)^3 \frac{\eta(\tau)^8}{\eta(3\tau)^8} + \frac{9}{8} \eta(\tau)^3 \eta(3\tau)^3 \frac{\sum_{n \geq 0} (-1)^n (2n + 1)^3 q^{3n(n+1)/2}}{\eta(\tau)^8} + \frac{\eta(\tau)^3 \sum_{n \geq 0} (-1)^n (2n + 1)^3 q^{3n(n+1)/2}}{\eta(\tau)} \\
&= -\frac{1}{8} \left( E_2(\tau) - 9 E_2(3\tau) \right) \eta(3\tau)^3 \frac{\eta(\tau)^8}{\eta(3\tau)^8}.
\end{align*}
\]

**Remark 7.** Observe that the above formulas, with \( \eta(\tau)^3 \) and \( \eta(3\tau)^3 \) expanded as sums, immediately imply relation Theorem 5.4 (iii).

It is clear that \( E_2(\tau) \) is a quasi-modular form of weight 2 and depth 1 on \( \Gamma(1) \). It is easy to show that \( E_{2,3}(\tau) := E_2(3\tau) \) is a quasi-modular form of weight 2 and depth 1 on \( \Gamma_0(3) \), i.e.
\[
E_{2,3} \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 E_{2,3}(\tau) + \frac{6c(c\tau + d)}{3i\pi}.
\]
As index of \( \Gamma_0(3) \) in \( \Gamma(1) \) is 4, \( E_{2,3}(\tau) \) combines into a vector-valued quasi-modular form under the full modular group.

**Lemma 7.2.** \( \eta(\tau/3)^3 \) and \( \eta(3\tau)^3 \) form a 2-dimensional vector-valued modular form of weight \( \frac{3}{2} \) under the full modular group.

**Proof.** Straightforward computation with Shimura’s theta series of weight \( \frac{3}{2} \) together with Jacobi’s identity for \( \eta(\tau)^3 \).

Using this lemma and previous discussion one can explicitly write down a vector space of quasi-modular forms closed under the modular group, which also contains the supercharacter. However, this space is difficult to analyze and does not give much evidence for the quasi-lisseness of \( \mathcal{U} \) conjectured earlier. As demonstrated in [12], characters of quasi-lisse vertex algebras must satisfy a particular type of linear modular differential equation with modular Eisenstein series being coefficients (usually abbreviated as MLDE). For quasi-lisse \( \mathbb{Z}_{\geq 0} \)-graded vertex superalgebras we expect the same property to hold for supercharacters. By analyzing the leading behavior of the above function (which is quasi-modular) combined with computer computations we can conclude

**Proposition 7.3.** \( \text{sch}[\mathcal{U}](\tau) \) satisfies a 5-th order MLDE
\[
\theta^5(y(q)) - \frac{7}{36} E_4(\tau) \theta^3(y(q)) + \frac{19}{216} E_6(\tau) \theta^2(y(q)) - \frac{5}{324} E_4(\tau) \theta(y(q)) + \frac{5}{1944} E_4(\tau) E_6(\tau) y(q) = 0,
\]
where Ramanujan-Serre’s \( n \)-th derivative is defined by
\[
\theta^n := \theta_{2n} \circ \cdots \circ \theta_0 ; \quad \theta_k := \left( q \frac{d}{dq} - \frac{k E_2(\tau)}{12} \right).
\]
As usual, the Eisenstein series in the equation are given by

\[
E_2(\tau) = 1 - 24 \sum_{n \geq 1} \frac{nq^n}{1 - q^n}
\]
\[
E_4(\tau) = 1 + 240 \sum_{n \geq 1} \frac{n^3q^n}{1 - q^n}
\]
\[
E_6(\tau) = 1 - 504 \sum_{n \geq 1} \frac{n^5q^n}{1 - q^n}.
\]

Two supercharacters (that are equal) of ordinary \(U\)-modules also satisfy this modular equation. Two additional solutions are expected to come from \(\sigma\)-twisted \(U\)-modules, not analyzed in this paper. These four solutions together with a logarithmic solution form a fundamental system of this modular linear equation.

7.7. Even case: explicit example \(n = 2\). Here we essentially repeat the same procedure with a notable difference that

\[
\vartheta \left(2z + \frac{1}{2}; 2\tau\right)
\]
admits Taylor expansion in even powers of the \(z\) variable:

\[
\vartheta \left(2z + \frac{1}{2}; 2\tau\right) = \vartheta \left(\frac{1}{2}; 2\tau\right) + O(z^2)
\]

so that for \(n\) even

\[
\frac{\vartheta(2z + \frac{1}{2}; 2\tau)}{\vartheta(z; \tau)^2} = \frac{1}{z^2} a_0(\tau) + a_1(\tau) + O(z^2)
\]

Repeating the same procedure using easily computable expressions gives

\[
\text{ch}[U](\tau) = \frac{1}{3} \frac{\eta(4\tau)^2}{\eta(\tau)^3 \eta(2\tau)} \left(4E_2(2\tau) - E_2(4\tau)\right).
\]

8. Vertex superalgebra \(V_1(\text{psl}(n, n))\) and \(V_{-1}(\text{sl}_n)\)

Let \(g = \text{psl}(n, n)\). We consider the simple vertex algebra \(V_1(g)\). We have the following result which identifies our vertex algebra \(U^{(n)} = U_0\) as a coset subalgebra in \(V_1(g)\).

**Proposition 8.1.** Assume than \(n \geq 3\). Then we have:

1. The vertex algebras \(U^{(n)}\) and \(L_{\text{sl}_n}(\Lambda_0)\) form a Howe dual pair inside \(V_1(g)\). In particular,

\[
\frac{\text{psl}(n, n)}{\text{sl}(n)_1} := \text{Com}_{V_1(g)}(L_{\text{sl}_n}(\Lambda_0)) \cong U^{(n)}.
\]

2. \(K(g, 1) \cong K(\text{sl}(n), -1) \cong (A(1)^0)^{\otimes n}\).
Proof. By using the decomposition of conformal embedding \( \mathfrak{sl}(n) \times \mathfrak{sl}(n) \hookrightarrow g \) (cf. [9]) we get

\[
V_1(g) = \bigoplus_{i=0}^{n-1} \mathcal{U}_i \otimes L(A_i),
\]

where for brevity we omit the superscript \( \mathfrak{sl}_n \). Alternatively, relation (9) can be directly proved by using the fusion rules result from Theorem 3.3 and the well-known fact that all \( V_1(\mathfrak{sl}(n)) \)-modules are simple currents.

The first assertion follows directly from (9). The second assertion follows again from (9) and from

\[
K(\mathfrak{sl}(n), 1) \cong \mathbb{C}, \quad \text{Com}_{U(n)}(M_{n-1}(1)) = K(\mathfrak{sl}(n), -1).
\]

□

The case \( n = 1 \) corresponds exactly to the symplectic fermion vertex algebra \( \mathcal{A}(1) \) of central charge \( c = -2 \). We conjecture that for all \( n \geq 2 \), the supertrace \( \text{sch}[V_1(g)](\tau) \) are the same, and therefore they satisfy the same MLDE

\[
\theta^2(y(\tau)) + \frac{1}{144} E_4(\tau) y(\tau) = 0.
\]

Conjecture 8.2. We have:

\[
\text{sch}[V_1(g)](\tau) = \eta(\tau)^2.
\]

8.1. Proof of Conjecture 8.2 for \( n = 3 \). We have

\[
V_1(g) = \mathcal{U}_0 \otimes L(A_0) \bigoplus \mathcal{U}_1 \otimes L(A_1) \bigoplus \mathcal{U}_2 \otimes L(A_2).
\]

This gives

\[
\text{sch}[V_1(g)](\tau) = \sum_{i=0}^{n-1} \text{sch}[\mathcal{U}_i](\tau) \text{ch}[L(A_i)](\tau).
\]

Since both left and right hand side are (quasi)modular in theory it would be sufficient to compute a few first coefficients in the \( q \)-expansion. Here we present a more conceptual proof. We need an auxiliary result

Lemma 8.3.

\[
\text{ch}[\Lambda_0](\tau) = \frac{\sum_{m,n \in \mathbb{Z}} q^{m^2+n^2-mn}}{\eta(\tau)^2} = \frac{1}{\eta(\tau)^3} \left( 3\eta(3\tau)^3 + \eta(\tau/3)^3 \right)
\]

\[
\text{ch}[\Lambda_1](\tau) = \frac{\sum_{m,n \in \mathbb{Z}} q^{m^2+n^2+m+1+mn}}{\eta(\tau)^2} = \frac{3\eta(3\tau)^3}{\eta(\tau)^3},
\]

Proof. The second identity is equivalent to Macdonald’s denominator identity for \( A_2 \). By Lemma 7.2 we have

\[
\mathcal{V} := \text{Span} \left\{ \frac{3\eta(3\tau)^3}{\eta(\tau)^3}, \frac{3\eta(\tau/3)^3}{\eta(\tau)^3} \right\}
\]
is modular invariant. On the other hand
\[
W := \text{Span}\{\text{ch}[\Lambda_0](\tau), \text{ch}[\Lambda_1](\tau)\}
\]
is also two-dimensional modular invariant subspace. Since \(\text{ch}[\Lambda_1](\tau) \in \mathcal{V}\) we must have \(\text{ch}[\Lambda_0](\tau) \in \mathcal{V}\). This quickly gives the formula by comparing the leading coefficients in the \(q\)-expansion.

**Proposition 8.4.** The Conjecture 8.2 holds for \(n = 3\).

**Proof.** The above lemma gives
\[
\text{ch}[\Lambda_0](\tau) = \frac{1}{\eta(\tau)^3} (3\eta(3\tau)^3 + \eta(\tau/3)^3)
\]
\[
\text{ch}[\Lambda_i](\tau) = \frac{3\eta(3\tau)^3}{\eta(\tau)^3}.
\]

As in the previous section, for \(1 \leq i \leq 2\) we get
\[
\text{sch}[\mathcal{U}_i](\tau) = - \sum_{n \geq 0; n \equiv \pm 1 \text{ mod } 6} \text{ch}[V_n] + \sum_{n \geq 0; n \equiv \pm 2 \text{ mod } 6} \text{ch}[V_n]
\]
\[
= \frac{1}{6} \frac{\eta(\tau)^5}{\eta(3\tau)^3} + \frac{(E_2(\tau) - 9E_2(3\tau))(\eta(\tau/3)^3 + 3\eta(3\tau)^3)}{48\eta(\tau)^5}.
\]

We previously derived the formula
\[
\text{sch}[\mathcal{U}_0](\tau) = - \frac{1}{8} (E_2(\tau) - 9E_2(3\tau)) \frac{\eta(3\tau)^3}{\eta(\tau)^3}.
\]

Plugging-in these \(q\)-series gives
\[
\sum_{i=0}^{2} \text{sch}[\mathcal{U}_i](\tau) \text{ch}[\Lambda_i](\tau) = \eta(\tau)^2
\]
as desired.

**Remark 8.** It seems difficult to prove Conjecture 8.2 for general \(n\) using the above method because the weight of the numerator of \(\text{ch}[\mathcal{U}_i]\) grows quadratically with respect to \(n\).

9. **On super-characters of \(V_{-2}(osp(n+8|n))\)**

In [12], T. Arakawa and K. Kawasetsu proved the character formula for the vertex operator algebras associated with the Deligne exceptional series at level \(k = -h^\vee_6 - 1\). In [7] and [8], the authors discovered a family of Lie superalgebras such that the associated vertex algebras also have level \(k = -h^\vee_6 - 1\) and similar properties as in the case of the Deligne exceptional series. Vertex algebras \(V_1(psl(n,n)) = V_{-1}(psl(n,n))\) belong to the series. Since we have demonstrated in previous section that the supercharacters of \(V_1(psl(n,n))\) should not depend on the parameter \(n\), one can ask if the similar situation can be happened in other cases. A natural example is \(V_{-2}(osp(n+8|n))\) which is a super-generalization of the affine vertex algebra \(V_{-2}(so(8))\). We have the following conjecture.
Conjecture 9.1. For every even $n \geq 0$, we have

$$\text{sch} [V_{-2}(\mathfrak{osp}(n + 8|n))] (\tau) = \text{ch} [V_{-2}(\mathfrak{so}(8))] (\tau) = \frac{(qd/dq)^4 E_4(\tau)}{240\eta(\tau)^{10}}.$$ 

We plan to discuss a proof in our forthcoming papers.

REFERENCES

[1] T. Abe, A $\mathbb{Z}_2$–orbifold model of the symplectic fermionic vertex operator superalgebra, Math. Z., 255 (2007), pp. 755–792.
[2] D. Adamović, Classification of irreducible modules of certain subalgebras of free boson vertex algebra, J. Algebra 270 (2003), pp. 115–132.
[3] D. Adamović, A. Milas, On the triplet vertex algebra $W(p)$, Advances in Mathematics 217 (2008) 2664-2699.
[4] D. Adamović, A. Milas, Some applications and constructions of intertwining operators in LCFT, Contemporary Mathematics (2017).
[5] D. Adamović, A. Milas, M. Penn, in preparation.
[6] D. Adamović, A. Milas, V. Pedić, in preparation.
[7] D. Adamović, V. G. Kac, P. Möseneder Frajria, P. Papi, O. Perše, Conformal embeddings of affine vertex algebras in minimal W-algebras I: structural results, Journal of Algebra 500 (2018) pp. 117–152 (Special Issue dedicated to Efim Zelmanov).
[8] D. Adamović, V. G. Kac, P. Möseneder Frajria, P. Papi, O. Perše, An application of collapsing levels to the representation theory of affine vertex algebras, [arXiv:1801.09880]
[9] D. Adamović, V. Kac, P. Moseneder Frajria, P. Papi and O. Perše, Conformal embeddings of some affine vertex algebras in affine vertex superalgebras, to appear.
[10] D. Adamović and O. Perše, Representations of certain non-rational vertex operator algebras of affine type, Journal of Algebra 319 (2008), 2434-2450.
[11] D. Adamović, O. Perše, Fusion rules and complete reducibility of certain modules for affine Lie algebras, J. Algebra Appl. 13 (2014), 1350062, 18 pp.
[12] T. Arakawa, K. Kawasetsu, Quasi-lisse vertex algebras and modular linear differential equations, [arXiv:1610.05865]
[13] K. Bringmann, T. Creutzig, and L. Rolen, Negative index Jacobi forms and quantum modular forms, Res. Math. Sci. 1 (2014), 1-32.
[14] K. Bringmann, A. Folsom and K. Mahlburg, Quasimodular forms and $s\ell(m|m)^c$ characters, Ramanujan Journal 36, (2015), 103-116.
[15] K. Bringmann, K. Mahlburg and A. Milas, On characters of $L_{\mathfrak{sl}_n}(-\Lambda_0)$-modules, [arXiv:1803.08029]
[16] K. Bringmann and A. Milas, W-algebras, false theta functions and quantum modular forms, IMRN, 21 (2015), 11351-11387.
[17] K. Costello and D. Gaiotto, Vertex Operator Algebras and 3d $N = 4$ gauge theories, [arXiv:1804.06460]
[18] T. Creutzig and Gaiotto, Vertex Algebras for $S$-duality, [arXiv:1708.00875]
[19] T. Creutzig, S. Kanade, A. Linshaw, D. Ridout, Schur-Weyl duality for Heisenberg cosets, to appear in Transformation Groups.[arXiv:1611.00305]
[20] A. Dabholkar, S. Murthy, and D. Zagier, Quantum black holes, wall crossing, and mock modular forms, [arXiv:1206.4074]
[21] C. Dong, J. Lepowsky, Generalized Vertex Algebras and Relative Vertex Operators, Progress in Math. Vol. 112, Birkhäuser, Boston (1993).
[22] V. Kac and M. Wakimoto, Integrable highest weight modules over affine superalgebras and Appell’s function, Communications in Mathematical Physics 215 (2001): 631-682.
[23] V. Kac, M. Wakimoto, On characters of irreducible highest weight modules of negative integer level over affine Lie algebras, [arXiv:1706.08387]
[24] V. Kac, A. Radul, Representation theory of the vertex algebra $W_{1+\infty}$, Transform. Groups, 1 (1996) 41–70
[25] A. Linshaw, Invariant chiral differential operators and the $W_3$ algebra, Journal of Pure and Applied Algebra 213 (2009) 632–648
[26] H. Li, On abelian coset generalized vertex algebras, Commun. Contemp. Math. 3 (2001), no. 2, 287–340.
[27] M. Miyamoto, $C_2$-cofiniteness of cyclic-orbifold models, Comm. Math. Phys. 335 (2015), no. 3, 1279?-1286
[28] W. Wang, $W_{1+\infty}$-algebra, $W_3$-algebra, and Friedan–Martinec–Shenker bosonization Comm. Math. Phys., 195 (1998), pp. 95-111

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ZAGREB, CROATIA
E-mail address: adamovic@math.hr

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY AT ALBANY (SUNY), ALBANY, NY 12222
E-mail address: amilas@math.albany.edu