ON THE UNIVERSAL COEFFICIENTS FORMULA FOR SHAPE HOMOLOGY

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Abstract. In this paper it is investigated whether various shape homology theories satisfy the Universal Coefficients Formula (UCF). It is proved that pro-homology and strong homology satisfy $UCF$ in the class $\text{FAB}$ of finitely generated abelian groups, while they do not satisfy $UCF$ in the class $\text{AB}$ of all abelian groups. Two new shape homology theories (called $UCF$-balanced) are constructed. It is proved that balanced pro-homology satisfies $UCF$ in the class $\text{AB}$, while balanced strong homology satisfies $UCF$ only in the class $\text{FAB}$.

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0. Introduction

The purpose of this paper is to investigate whether various shape homology theories satisfy the Universal Coefficients Formula (UCF). It is proved (Theorem 1.1) that pro-homology and strong homology satisfy UCF in the class $F\text{AB}$ of finitely generated abelian groups, while they do not satisfy UCF in the class $AB$ of all abelian groups (Theorems 1.3 and 1.4). Two new shape homology theories (called UCF-balanced, or simply balanced) are constructed. It is proved that balanced pro-homology satisfies UCF in the class $AB$ (Theorem 1.2). It happens that our balanced strong homology is not “balanced enough”. It satisfies UCF in the class $F\text{AB}$ (Theorem 1.1), but not in the class $AB$ (Theorem 1.5). The latter theorem is the most difficult part of the paper. Two counter-examples are constructed. The first one is simpler, but depends on the Continuum Hypothesis (in fact, on a weaker assumption “$d = \aleph_1$”). Hence, that example cannot be considered as a “final” counter-example. Another counter-example is much more complicated, but does not depend on any extra assumption, therefore can be considered as “final”.

To deal with UCF, one needs to develop the torsion functor $\text{Tor}_k^A$ on the category of pro-modules, which is done in Section 2. The problem is that the category $\text{Pro}(k)$ of pro-modules does not have enough projectives (Remark 2.23). It does have, however, enough quasi-projectives (Proposition 2.24). It follows also that quasi-projectives are flat (Proposition 2.30) provided $k$ is quasi-noetherian. The above two facts allow defining the torsion functors using quasi-projective resolutions (Proposition 2.35).

To calculate the strong homology groups (balanced or non-balanced), one needs spectral sequences from Section 3. Those spectral sequences are of inverse limit type (in contrast to the direct limit type sequences), and are concentrated in the I and IV quadrants. It is not very easy to treat the convergence of such sequences. There are several papers that deal with the convergence of inverse limit type spectral sequences. See, e.g., the most general treatment in [Boa99], Part II, spectral sequences for towers of fibrations and homotopy limits in [BK72], §IX.4 and §XI.7, spectral sequences for homotopy limits of spectra in [Tho85], 1.16, 5.44-5.48, spectral sequences for $\Gamma$-spaces in [Pra01], Theorem 2, and a spectral sequence for strong homology in [Pra88]. None of the approaches above suits 100% our purposes. That is why in Section 3 spectral sequences for homotopy limits of diagrams of chain complexes are developed (Theorem 3.11). The most important result in that Section is Theorem 3.14 where homotopy limits of pro-complexes are treated.

Let $G \in \text{Mod}(\mathbb{Z})$ be an abelian group (see the notations for $\text{Mod}(k)$ and $\text{Pro}(k)$ from Example 2.7 (1)). Throughout this paper, $h_n(\_ , G)$ will be one of the following homology theories:

1. Pro-homology $H_n^p(X, G) \in \text{Pro}(\mathbb{Z})$ (see Definition 4.1).
2. Strong homology $\overline{H}_n(X, G) \in \text{Mod}(\mathbb{Z})$ (see Definition 4.2).
3. Balanced pro-homology $H_n^b(X, G) \in \text{Pro}(\mathbb{Z})$ (see Definition 4.6).
4. Balanced strong homology $\overline{H}_n^b(X, G) \in \text{Mod}(\mathbb{Z})$ (see Definition 4.8).

Let $\otimes_{\mathbb{Z}}$ denote either the usual tensor product

$$\otimes_{\mathbb{Z}} : \text{Mod}(\mathbb{Z}) \times \text{Mod}(\mathbb{Z}) \to \text{Mod}(\mathbb{Z})$$

or the tensor product

$$\otimes_{\mathbb{Z}} : \text{Pro}(\mathbb{Z}) \times \text{Mod}(\mathbb{Z}) \to \text{Pro}(\mathbb{Z})$$
from Theorem 2.17.

Proposition 0.1. For each of the four theories there exists a natural (on $X$ and $G$) pairing

$$h_n (X, \mathbb{Z}) \otimes \mathbb{Z} G \rightarrow h_n (X, G).$$

Remark 0.2. Since all the four theories are defined on the strong shape category $SSh$, “natural” here means that for each $n \in \mathbb{Z}$ the pairing above is a morphism of functors

$$h_n (?, \mathbb{Z}) \otimes \mathbb{Z} ? \rightarrow h_n (?, ?) : SSh \times \text{Mod} (\mathbb{Z}) \rightarrow C$$

where $C$ is either $\text{Mod} (\mathbb{Z})$ or $\text{Pro} (\mathbb{Z})$.

Proof. See Section 5.1.

Let $\mathcal{C}$ be either the class $AB$ or the subclass $FAB \subseteq AB$.

Definition 0.3. We say that the homology theory $h_\ast$ satisfies UCF in the class $\mathcal{C}$ iff for any $n \in \mathbb{Z}$, $X \in \text{TOP}$ and $G \in \mathcal{C}$:

- The pairing $h_n (X, \mathbb{Z}) \otimes \mathbb{Z} G \rightarrow h_n (X, G)$ is a monomorphism.
- The cokernel

$$\text{coker} (h_n (X, \mathbb{Z}) \otimes \mathbb{Z} G \rightarrow h_n (X, G))$$

is naturally (on $X$ and $G$) isomorphic to $\mathbb{F} \text{or}^Z (h_{n-1} (X, \mathbb{Z}), G)$ where $\mathbb{F} \text{or}$ is either the usual torsion functor

$$\mathbb{F} \text{or}^Z : \text{Mod} (\mathbb{Z}) \times \text{Mod} (\mathbb{Z}) \rightarrow \text{Mod} (\mathbb{Z})$$

or the functor

$$\mathbb{F} \text{or}^Z : \text{Pro} (\mathbb{Z}) \times \text{Mod} (\mathbb{Z}) \rightarrow \text{Pro} (\mathbb{Z})$$

from Definition 2.31.

Remark 0.4. Roughly speaking, the theory $h_\ast$ satisfies UCF in the class $\mathcal{C}$ iff there are natural (on $X \in SSh$ and $G \in \mathcal{C}$) exact sequences ($n \in \mathbb{Z}$)

$$0 \rightarrow h_n (X, \mathbb{Z}) \otimes \mathbb{Z} G \rightarrow h_n (X, G) \rightarrow \mathbb{F} \text{or}^Z (h_{n-1} (X, \mathbb{Z}), G) \rightarrow 0.$$

1. Main results

Theorem 1.1. All the four theories $h_n$ satisfy UCF in the class $FAB$.

Proof. See Section 5.2.

Theorem 1.2. The balanced pro-homology $H^b_\ast$ satisfies UCF in the class $AB$.

Proof. See Section 5.3.

Theorem 1.3. The pro-homology $H_\ast$ does not satisfy UCF in the class $AB$.

Proof. See Section 5.4.

Theorem 1.4. The strong homology $H^s_\ast$ does not satisfy UCF in the class $AB$.

Proof. See Section 5.5.

Theorem 1.5. The balanced strong homology $H^s_b_\ast$ does not satisfy UCF in the class $AB$.

Proof. See Section 5.6.
2. Pro-modules

2.1. Pro-category. We will often refer to Chapter 6 of [KS06] where ind-objects are considered. Since the category of pro-objects $\text{Pro}(k)$ is isomorphic to the category $\text{Ind}(\text{Mod}(k)^{op})^{op}$ where $\text{Ind}(C)$ is the category of ind-objects, all properties of $\text{Pro}(k)$ are dual to the properties of $\text{Ind}(\text{Mod}(k)^{op})$. We can therefore use the statements that are dual to the corresponding statements in [KS06].

Pro-objects are represented by inverse systems. In [MS82] and [Mar00], inverse systems are indexed by filtrant (called directed in [MS82], §I.1.1) ordered sets $I$.
We use here more general inverse systems indexed by cofiltrant index categories, i.e. an inverse system in $C$ (Definition 2.1.1 below) will be a functor $X : I \to C$ where $I$ is a small cofiltrant category (see [KS06], Definition 3.1.1, for the definition of (co)filtrant categories). Without loss of generality, let us denote such a functor by $(X_i)_{i \in I}$. The so-called Mardešić trick ([MS82], Theorem I.1.4) shows that our construction (indexing by a category) is equivalent to the Mardešić-Segal construction (indexing by an ordered set). The difference in terminology (cofiltrant categories instead of filtrant sets) is due to the fact that our construction uses covariant functors, while the Mardešić-Segal construction uses contravariant ones.

Definition 2.1. (compare with [Pra01], Definition 2.1.3) For a category $C$, let $\text{Inv}(C)$ be the following category (of inverse systems). The objects are functors $X : I \to C$ where $I$ is a small cofiltrant category. A morphism $X \to Y$ where $X : I \to C$ and $Y : J \to C$, is a pair $(\varphi, \psi)$ where $\varphi : J \to I$ is a functor and $\psi$ is a morphism of functors $\psi : X \circ \varphi \to Y$. The morphisms are composed as follows:

$$(\varphi_2, \psi_2) \circ (\varphi_1, \psi_1) = (\varphi_1 \circ \varphi_2, \psi_2 \circ \psi_1 (\varphi_1))$$

where

$$\varphi_1 : J \to I, \varphi_2 : K \to J,$$
$$\psi_1 : X \circ \varphi_1 \to Y, \psi_2 : Y \circ \varphi_2 \to Z,$$

and $\psi_1 (\varphi_2) : X \circ \varphi_1 \circ \varphi_2 \to Y \circ \varphi_2$ is the morphism given by

$$\psi_1 (\varphi_2)_k = (\psi_1)_{\varphi_2(k)} : X_{\varphi_1(\varphi_2(k))} \to Y_{\varphi_2(k)}.$$

The identity morphisms $1_X$ are given by families $1_X = \{1 : I \to I, (1_i : X_i \to X_i)_{i \in I}\}$.

It is easily checked that $\text{Inv}(C)$ is indeed a category.

Definition 2.2. If $I = J$ and $\varphi$ is the identity functor, the morphism $(\varphi, \psi)$ is called a level morphism.

Definition 2.3. Let $C$ be a category. The pro-category $\text{Pro}(C)$ (see [KS06], Definition 6.1.1, [MS82], Remark I.1.4, or [AM86], Appendix) is the category with the same class of objects as $\text{Inv}(C)$, and the following sets of morphisms between two objects $X = (X_i)_{i \in I}$ and $Y = (Y_j)_{j \in J}$:

$$\text{Hom}_{\text{Pro}(C)}(X, Y) = \varprojlim_{j \in J} \varprojlim_{i \in I} \text{Hom}_C(X_i, Y_j).$$

Remark 2.4. It follows from the definition that a morphism in $\text{Pro}(C)$ between $X = (X_i)_{i \in I}$ and $Y = (Y_j)_{j \in J}$ can be represented by a pair $(\varphi, \psi)$ where $\varphi : \text{Ob}(J) \to \text{Ob}(I)$ is a mapping between the sets of objects, and $\psi = (\psi_j) : X_{\varphi(j)} \to Y_j$ is a family of morphisms satisfying the standard inverse limit relations. A morphism $(\varphi, \psi)$ in $\text{Inv}(C)$ can be therefore interpreted as a morphism in $\text{Pro}(C)$, and one has an evident functor $\text{Inv}(C) \to \text{Pro}(C)$. 


Definition 2.5. A morphism \((\varphi, \psi) \in \text{Hom}_{\text{Inv}(C)}(X, Y)\) is called cofinal, if \(\varphi : J \to I\) is a cofinal functor (left cofinal in \([BK72]\), XI.9.1, see also \([AM86]\), Appendix (1.5)), \(X \circ \varphi = Y\), and
\[
\psi = 1_Y : X \circ \varphi \longrightarrow X \circ \varphi = Y.
\]

Theorem 2.6. The category \(\text{Pro}(C)\) is equivalent to the category of fractions \(\text{Inv}(C)[\Sigma^{-1}]\) where \(\Sigma\) is the class of cofinal morphisms.

**Proof.** See \([Pra01]\), Theorem 2.1.10.

Example 2.7.

1. Let \(k\) be an associative commutative ring with unit. The category of \(k\)-modules will be denoted \(\text{Mod}(k)\). The pro-category \(\text{Pro}(\text{Mod}(k))\) of pro-modules will be shortly denoted \(\text{Pro}(k)\). Since the category \(AB\) of abelian groups can be naturally identified with \(\text{Mod}(\mathbb{Z})\), the category \(\text{Pro}(AB)\) of abelian pro-groups can be naturally identified with \(\text{Pro}(\mathbb{Z})\).

2. We will consider both the categories \(\text{Pro}(\text{TOP})\) and \(\text{Pro}(\text{HTOP})\) where \(\text{TOP}\) is the category of topological spaces and \(\text{HTOP}\) is the category of homotopy types having topological spaces as objects and homotopy classes of mappings as morphisms. There are full subcategories \(\text{Pro}(\text{POL}) \subseteq \text{Pro}(\text{TOP})\) and \(\text{Pro}(\text{HPOL}) \subseteq \text{Pro}(\text{HTOP})\) where \(\text{POL}\) is the category of topological spaces having the homotopy type of a polyhedron.

3. Let \(\text{CHAIN}(k)\) be the category of chain complexes of \(k\)-modules. We will consider the category \(\text{Pro}(\text{CHAIN}(k))\) of chain pro-complexes.

It is well-known that the category \(\text{Pro}(C)\) admits coproducts if \(C\) does (see, e.g., \([Pra01]\), Proposition 2.4.1). Below is the explicit description of a coproduct.

Example 2.8. Let \((X^\alpha = (X^\alpha_i)_{i \in I^\alpha})_{\alpha \in A}\) be a family of pro-objects. Define a pro-object \(Y = (Y_j)_{j \in J}\) as follows:
\[
J = \prod_{\alpha \in A} I^\alpha, \quad Y_j = \prod_{\alpha \in A} X^\alpha_{j(\alpha)},
\]
where \(j = (j(\alpha) \in I^\alpha)_{\alpha \in A}\). One has for another pro-object \(Z = (Z_k)_{k \in K}\)
\[
\text{Hom}_{\text{Pro}(C)}(Y, Z) \simeq \lim_k \lim_j \text{Hom}_{\text{Pro}(C)}(Y_j, Z_k) \simeq \lim_k \lim_j \text{Hom}_{\text{C}} \left( \prod_{\alpha \in A} X^\alpha_{j(\alpha)}, Z_k \right) \simeq \lim_k \left( \prod_{\alpha \in A} \text{Hom}_{\text{C}}(X^\alpha_{j(\alpha)}, Z_k) \right) \simeq \lim_k \left( \prod_{\alpha \in A} \text{Hom}_{\text{Pro}(C)}(X^\alpha, Z_k) \right) \simeq \prod_{\alpha \in A} \text{Hom}_{\text{Pro}(C)}(X^\alpha, Z),
\]
and \(Y\) is indeed a coproduct of \(\{X^\alpha\}_{\alpha \in A}\).

Definition 2.9. Given a pro-complex
\[
C_* = (C_{*i})_{i \in I} \in \text{Pro}(\text{CHAIN}(k))
\]
let \(C^\#_n = (C^\#_{n,i})_{i \in I}, d_n = (d_{n,i})_{i \in I}\). There are two ways of defining the pro-homologies of
Proposition 2.10. \( H_n(C_\ast) \simeq H^\#_n(C_\ast) \).

Proof. Straightforward. Compare with [MSS2, §II.2]. We need to describe kernels, cokernels, images and coinages.

1. **Kernels.** Let 
   \( (f : A \rightarrow B) = (f_i : A_i \rightarrow B_i)_{i \in I} \)
   be a level morphism, and let
   \( C = (\ker (f_i))_{i \in I} \).
   We have to prove that \( C \) is the kernel of \( f \). Let \( D = (D_j)_{j \in J} \in \text{Pro} (k) \).
   Then
   \[
   \text{Hom}_{\text{Pro} (k)} (D, C) = \lim_{\leftarrow \leftarrow \leftarrow} \lim_{\leftarrow \leftarrow \leftarrow} \text{Hom} (D_j, C_i) = \\
   \lim_{\leftarrow \leftarrow \leftarrow} \lim_{\leftarrow \leftarrow \leftarrow} \ker \left( \text{Hom} (D_j, A_i) \rightarrow \text{Hom} (D_j, B_i) \right) = \\
   \lim_{\leftarrow \leftarrow \leftarrow} \ker \left( \lim_{\leftarrow \leftarrow \leftarrow} \text{Hom} (D_j, A_i) \rightarrow \lim_{\leftarrow \leftarrow \leftarrow} \text{Hom} (D_j, B_i) \right) = \\
   \ker \left( \lim_{\leftarrow \leftarrow \leftarrow} \lim_{\leftarrow \leftarrow \leftarrow} \text{Hom} (D_j, A_i) \rightarrow \lim_{\leftarrow \leftarrow \leftarrow} \lim_{\leftarrow \leftarrow \leftarrow} \text{Hom} (D_j, B_i) \right) = \\
   \ker \left( \text{Hom}_{\text{Pro} (k)} (D, A) \rightarrow \text{Hom}_{\text{Pro} (k)} (D, B) \right).
   \]

2. **Cokernels.** Let now
   \( E = (\coker (f_i))_{i \in I} \).
   We have to prove that \( E \) is the cokernel of \( f \). Let \( D = (D_j)_{j \in J} \in \text{Pro} (k) \).
   Then
   \[
   \text{Hom}_{\text{Pro} (k)} (E, D) = \lim_{\rightarrow \rightarrow \rightarrow} \lim_{\rightarrow \rightarrow \rightarrow} \text{Hom} (E_i, D_j) = \\
   \lim_{\rightarrow \rightarrow \rightarrow} \lim_{\rightarrow \rightarrow \rightarrow} \ker \left( \text{Hom} (B_i, D_j) \rightarrow \text{Hom} (A_i, D_j) \right) = \\
   \lim_{\rightarrow \rightarrow \rightarrow} \ker \left( \lim_{\rightarrow \rightarrow \rightarrow} \text{Hom} (B_i, D_j) \rightarrow \lim_{\rightarrow \rightarrow \rightarrow} \text{Hom} (A_i, D_j) \right) = \\
   \ker \left( \lim_{\rightarrow \rightarrow \rightarrow} \lim_{\rightarrow \rightarrow \rightarrow} \text{Hom} (B_i, D_j) \rightarrow \lim_{\rightarrow \rightarrow \rightarrow} \lim_{\rightarrow \rightarrow \rightarrow} \text{Hom} (A_i, D_j) \right) = \\
   \ker \left( \text{Hom}_{\text{Pro} (k)} (B, D) \rightarrow \text{Hom}_{\text{Pro} (k)} (A, D) \right).
   \]

3. **Images.**
   \( \text{Im} (f) = \ker \text{coker} (f) = \ker (B_i \rightarrow \text{coker} (f_i))_{i \in I} = (\text{Im} (f_i))_{i \in I} \).

4. **Coinages.**
   \( \text{Coim} (f) = \text{coker} \ker (f) = \text{coker} (\ker (f_i) \rightarrow A_i)_{i \in I} = (A_i / \ker (f_i))_{i \in I} \simeq (\text{Im} (f_i))_{i \in I} \).
Actually, $\text{Im}(f) \simeq \text{Coim}(f)$ because $\text{Pro}(k)$ is an abelian category (see [KS06], dual to Theorem 8.6.5). Finally,

$$H_n^\#(C) = \text{coker}(C_{n+1} \rightarrow \ker(C_n \rightarrow C_{n-1})) \simeq \left( \frac{\ker d_{n,i}}{\text{Im}(d_{n+1,i})} \right)_{i \in I} = H_n(C).$$

Proposition 2.11. Let $(C_\alpha \in \text{Pro}(\text{CHAIN}(k)))_{\alpha \in A}$ be a family of chain pro-complexes of $k$-modules. Then, for any $n \in \mathbb{Z},$

$$H_n \left( \prod_{\alpha \in A} C_\alpha \right) \simeq \prod_{\alpha \in A} H_n(C_\alpha).$$

Proof. Straightforward, due to the explicit description in Example 2.8. □

Remark 2.12. Since both pro-complexes and pro-modules form additive (even abelian) categories, one can use the symbol $\oplus$ (direct sum) instead of $\sqcup$:

$$H_n \left( \bigoplus_{\alpha \in A} C_\alpha \right) \simeq \bigoplus_{\alpha \in A} H_n(C_\alpha).$$

2.2. Strong shape. Theorem 2.16 below gives a definition of the strong shape category which is equivalent to the Lisitsa-Mardešić one ([Mar00], §8.2).

Definition 2.13. A level morphism $f: X \rightarrow Y$ in $\text{Pro}(\text{TOP})$ is called a level equivalence iff $f_i: X_i \rightarrow Y_i$ are homotopy equivalences for all $i \in I.$

Definition 2.14. ([Pra01], Definition 2.1.9). A morphism

$$(\varphi, \psi): X \rightarrow Y$$

in $\text{Inv}(\text{TOP})$ is called special iff $\varphi$ is a cofinal functor and $\psi_j: X_{\varphi(j)} \rightarrow Y_j$ is a homotopy equivalence for all $j \in J.$

Proposition 2.15. Any special morphism $f: X \rightarrow Y$ is a composition $f = h \circ g$ of a cofinal morphism $g$ and a level equivalence $h.$

Proof. Given a special morphism $(\varphi, \psi): X \rightarrow Y,$ let

$$Z = (Z_j = X_{\varphi(j)})_{j \in J} \in \text{Inv}(\text{TOP})$$

and

$$g = (\varphi, 1_{X_{\varphi(j)}}): X \rightarrow Z,$$

$$h = (1_J, \psi): Z \rightarrow Y.$$  

Clearly $g$ is cofinal, and $h$ is a level equivalence, while $f = h \circ g.$ □

Theorem 2.16. The strong shape category $\text{SSh}$ is a full subcategory of the category of fractions $\text{Pro}(\text{POL})[\Sigma^{-1}]$ where $\Sigma$ is the class of special morphisms.

Proof. Due to [Pra01], Theorem 1, the category $\text{SSh},$ being a full subcategory of $\text{SSh}(\text{Pro}(\text{TOP})),$ is therefore a full subcategory of $\text{Pro}(\text{ANR})[\Sigma^{-1}].$ It is easy (and standard in strong shape theory) to substitute the category $\text{Pro}(\text{ANR})[\Sigma^{-1}]$ by the alternative category $\text{Pro}(\text{POL})[\Sigma^{-1}]$ because any $\text{ANR}$ is homotopy equivalent to a polyhedron. □
2.3. **Tensor product.** Let \( P \in \text{Pro}(k) \), \( M \in \text{Mod}(k) \). Consider the functor

\[
F_{P,M} : \text{Pro}(k) \longrightarrow \text{Mod}(k),
\]

\[
F_{P,M}(N) = \text{Hom}_{\text{Mod}(k)}(M, \text{Hom}_{\text{Pro}(k)}(P,N)).
\]

**Theorem 2.17.**

1. The functor \( F_{P,M} \) is representable. It means that there exists a pro-module (unique up to an isomorphism) \( P \otimes_k M \) such that

\[
\text{Hom}_{\text{Pro}(k)}(P \otimes_k M, N) \simeq \text{Hom}_{\text{Mod}(k)}(M, \text{Hom}_{\text{Pro}(k)}(P,N))
\]

naturally on \( N \in \text{Pro}(k) \).

2. Moreover, since the mapping \((P, M) \mapsto F_{P,M}\) is functorial, then the mapping \((P, M) \mapsto P \otimes_k M\) is in fact an bi-additive (even \(k\)-bilinear) functor

\[
\otimes_k : \text{Pro}(k) \times \text{Mod}(k) \longrightarrow \text{Pro}(k).
\]

3. The functor \( \otimes_k \) is right exact with respect to both variables.

4. If \( M \) is projective, then the functor \( ? \otimes_k M : \text{Pro}(k) \to \text{Pro}(k) \) is exact.

**Proof.** **Representability.** Compare Theorem 2.1.8 in [Sug01] or Proposition 2.1 in [Sch87].

Assume \( M = k^X \) is a free \( k \)-module generated by the set \( X \), and let

\[
P \otimes_k M := \bigoplus_X P
\]

be the direct sum in \( \text{Pro}(k) \). Then

\[
\text{Hom}_{\text{Pro}(k)}(P \otimes_k M, N) \simeq \text{Hom}_{\text{Pro}(k)} \left( \bigoplus_X P, N \right) \simeq
\]

\[
\prod_X \text{Hom}_{\text{Pro}(k)}(P, N) \simeq \text{Hom}_{\text{Mod}(k)}(M, \text{Hom}_{\text{Pro}(k)}(P,N))
\]

as desired. Consider now a general \( k \)-module \( M \). It can be represented as

\[
M = \text{coker} \left( k^Y \overset{f}{\longrightarrow} k^X \right)
\]

where \( f \) is given by a (infinite in general) \( X \times Y \) matrix with coefficients in \( k \). Define

\[
P \otimes_k M = \text{coker} \left( \bigoplus_Y P \overset{F}{\longrightarrow} \bigoplus_X P \right)
\]

where \( F \) is given by the same \( X \times Y \) matrix. Then

\[
\text{Hom}_{\text{Pro}(k)}(P \otimes_k M, N) \simeq
\]

\[
\ker \left( \text{Hom}_{\text{Pro}(k)} \left( \bigoplus_X P, N \right) \longrightarrow \text{Hom}_{\text{Pro}(k)} \left( \bigoplus_Y P, N \right) \right) \simeq
\]

\[
\ker \left( \text{Hom}_{\text{Mod}(k)}(k^X, \text{Hom}_{\text{Pro}(k)}(P,N)) \longrightarrow \text{Hom}_{\text{Mod}(k)}(k^Y, \text{Hom}_{\text{Pro}(k)}(P,N)) \right) \simeq
\]

\[
\text{Hom}_{\text{Mod}(k)} \left( \text{coker}(k^Y \to k^X), \text{Hom}_{\text{Pro}(k)}(P,N) \right) \simeq
\]

\[
\simeq \text{Hom}_{\text{Mod}(k)}(M, \text{Hom}_{\text{Pro}(k)}(P,N))
\]

as desired.
Exactness. We will prove even more: \( \otimes_k \) commutes with arbitrary direct limits (with respect to both variables). Indeed, let \( P \in \text{Pro}(k) \). The functor

\[
P \otimes_k ? : \text{Mod}(k) \to \text{Pro}(k)
\]
is left adjoint to the functor

\[
\text{Hom}_{\text{Pro}(k)}(P, ?) : \text{Pro}(k) \to \text{Mod}(k).
\]

Hence, it commutes with direct limits, and therefore is right exact.

Let now \( M \in \text{Mod}(k) \), and let \( P = \varprojlim_i P_i \) be the direct limit of a diagram (not necessarily filtrant!) of pro-modules. For any pro-module \( N \),

\[
\text{Hom}_{\text{Pro}(k)}\left( \left( \varprojlim_i P_i \right) \otimes_k M, N \right) \simeq \text{Hom}_{\text{Mod}(k)}\left( M, \text{Hom}_{\text{Pro}(k)}\left( \left( \varprojlim_i P_i \right), N \right) \right) \simeq \\
\text{Hom}_{\text{Mod}(k)}\left( M, \text{Hom}_{\text{Pro}(k)}(P, N) \right) \simeq \\
\varprojlim_i \text{Hom}_{\text{Pro}(k)}(P_i \otimes_k M, N).
\]

Therefore, \( \varprojlim_i P_i \otimes_k M \simeq \varprojlim_i (P_i \otimes_k M) \), and \( ? \otimes_k M : \text{Pro}(k) \to \text{Pro}(k) \) is right exact as well.

Finally, let \( M \) be a free \( k \)-module, \( M \simeq k^X \) for some set \( X \). Since \( \otimes_k \) commutes with direct limits, the functor \( ? \otimes_k M \) is the direct sum of the identity functors \( \otimes_k k \). Hence, \( ? \otimes_k M \) is exact as a direct sum of exact functors. If \( M \) is projective, then it is a retract of a free module \( F \). Therefore, \( ? \otimes_k M \), being a retract of the exact functor \( ? \otimes_k F \), is exact as well. \( \square \)

**Remark 2.18.** Let \( P_* \in \text{Pro}(\text{CHAIN}(k)) \) be a pro-complex of \( k \)-modules, and let \( M \) be a \( k \)-module. Then the tensor product \( P_* \otimes_k M \) can be represented by a complex of inverse systems and level differentials indexed by the same index category \( J \):

\[
P_n \otimes_k M = \left( (Q_n)_j \right)_{j \in J},
\]

\[
(d_n : P_n \to P_{n-1}) = \left( d_n : (Q_n)_j \to (Q_{n-1})_j \right)_{j \in J}.
\]

Indeed, the complex of pro-modules

\[
Q_n = P_n \otimes_k M = \text{coker} \left( \bigoplus_{Y} P_n \xrightarrow{F_j} \bigoplus_{X} P_n \right),
\]

as in the proof above, can be represented by the following inverse system. Let \( J \) be the product of the \( X \times Y \) copies of \( I \); \( J = I^X \times Y \). Given \( j = (i, x, y) \in X \times Y \), let

\[
(Q_n)_j = \text{coker} \left( \bigoplus_{y \in Y} (P_n)_{j(x, y)} \xrightarrow{F_j} \bigoplus_{x \in X} (P_n)_{j(x, y)} \right),
\]

\[
(d_n : Q_n \to Q_{n-1})_j = \text{coker} \left( \bigoplus_{y \in Y} (d_n)_{j(x, y)} \xrightarrow{F_j} \bigoplus_{x \in X} (d_n)_{j(x, y)} \right),
\]

where \( F_j \) is the following constant (not depending on \( j \)) \( X \times Y \) matrix: \( (F_j)_{x,y} = f_{x,y} \).
Definition 2.19. There is another (weak) tensor product
\[ \tilde{\otimes}_k : (\text{Pro}(k)) \times \text{Mod}(k) \to \text{Pro}(k) \]
which is defined as follows:
\[ P \tilde{\otimes}_k M := (P_i \otimes_k M)_{i \in I} \]
where the pro-module \( P \) is defined by an inverse system \( (P_i)_{i \in I} \).

Theorem 2.20. There is a natural homomorphism \( \otimes \to \tilde{\otimes} \). If \( M \) is finitely generated, the homomorphism above becomes an epimorphism \( P \otimes_k M \to P \tilde{\otimes}_k M \). If \( M \) is finitely presented, the homomorphism becomes an isomorphism.

Proof. It is easy to see that, if \( M = k \), then the homomorphism becomes an isomorphism:
\[ P \otimes_k M = P \otimes_k k = P = P \tilde{\otimes}_k k = P \tilde{\otimes}_k M. \]
Since both functors are additive (even \( k \)-linear), the same is true when \( M = k^n \) is a free finitely generated module:
\[ P \otimes_k M = P \otimes_k k^n = \bigoplus P = P \tilde{\otimes}_k k^n = P \tilde{\otimes}_k M. \]
If \( M \) is finitely generated, then
\[ M = \text{coker} (k^X \to k^n), \]
\[ P \otimes_k M = \text{coker} \left( \bigoplus X P \to P^n \right), \]
\[ P \tilde{\otimes}_k M = \text{coker} \left( P \otimes_k k^X \to P^n \right), \]
and both pro-modules are factormodules of the same pro-module \( P^n \), hence \( P \otimes_k M \to P \tilde{\otimes}_k M \) is an epimorphism. Finally, if \( M \) is finitely presented, then
\[ M = \text{coker} (k^m \to k^n), \]
\[ P \otimes_k M = \text{coker} (P^m \to P^n) = \text{coker} \left( P \otimes_k k^m \to P \otimes_k k^n \right) = P \tilde{\otimes}_k M. \]

2.4. Quasi-projects.

Definition 2.21. (dual to [KS06], Definition 15.2.1) A pro-module \( P \) is called quasi-projective if the functor
\[ \text{Hom}_{\text{Pro}(k)} (P, ?) : \text{Mod}(k) \to \text{Mod}(k) \]
is exact.

Proposition 2.22. A pro-module \( P \) is quasi-projective iff it is isomorphic to a pro-module \( (Q_i)_{i \in I} \) where all modules \( Q_i \in \text{Mod}(k) \) are projective.

Proof. The statement is dual to [KS06], Proposition 15.2.3.

Remark 2.23. The category \( \text{Pro}(k) \) does not have enough projectives (compare with [KS06], Corollary 15.1.3). However, it has enough quasi-projectives (see Proposition 2.24 below).

Proposition 2.24. For any pro-module \( M \) there exists an epimorphism \( P \to M \) where \( P \) is quasi-projective.
Proof. The statement is dual to the rather complicated Theorem 15.2.5 from [KS06]. However, the proof is much simpler in our case. Given \( M = (M_i)_{i \in I} \), let \( P = (P_i = F(M_i))_{i \in I} \), where \( F(M_i) \) is the free \( k \)-module generated by the set of symbols \( (\{ m \})_{m \in M} \). A family of epimorphisms

\[
f_i : P_i \rightarrow M_i \quad \left( f_i \left( \sum_j k_j [m_j] \right) = \sum_j k_j m_j, k_j \in k, m_j \in M_i \right),
\]
defines the desired level epimorphism \( f : P \rightarrow M \). □

2.5. Quasi-noetherian rings.

**Definition 2.25.** A commutative ring \( k \) is called **quasi-noetherian** if for any quasi-projective \( P \in \text{Pro}(k) \), and any injective \( J \in \text{Mod}(k) \), the \( k \)-module \( \text{Hom}_{\text{Pro}(k)}(P, J) \) is injective.

**Remark 2.26.** In [Sug01], Definition 2.1.10, such rings are called “satisfying condition A”. However, Proposition 2.28 below justifies our name.

**Lemma 2.27.** Let \( k \) be a noetherian ring. Then any filtrant direct limit of injective \( k \)-modules is injective.

**Proof.** See [Lam99], Theorem 3.46. □

**Proposition 2.28.** A noetherian ring is quasi-noetherian.

**Remark 2.29.** This proposition answers positively Conjecture 2.1.11 from [Sug01].

**Proof.** A quasi-projective \( P \) can be represented by an inverse system \( (P_i)_{i \in I} \) where all \( P_i \) are projective \( k \)-modules. Therefore, the \( k \)-module

\[
\text{Hom}_{\text{Pro}(k)}(P, J) = \lim_{\leftarrow i \in I} \text{Hom}_{\text{Mod}(k)}(P_i, J),
\]

being a filtrant direct limit of injectives, is injective as well, by Lemma 2.27. □

The proposition below shows that quasi-projective pro-modules over a quasi-noetherian ring are flat.

**Proposition 2.30.** If \( k \) is quasi-noetherian and \( P \in \text{Pro}(k) \) is quasi-projective, then the functor \( P \otimes_k ? : \text{Mod}(k) \rightarrow \text{Mod}(k) \) is exact.

**Proof.** See [Sug01], Theorem 2.1.12. □

2.6. Tor for pro-modules. We define the torsion functors \( \text{Tor}^k_* : \text{Pro}(k) \rightarrow \text{Pro}(k) \) as the left derived functors of the functor

\[
\otimes_k : \text{Pro}(k) \times \text{Mod}(k) \rightarrow \text{Pro}(k)
\]

with respect to the second variable. Later, in Proposition 2.35 we will show that these functors can be equally defined by using the first variable (provided \( k \) is quasi-noetherian).

**Definition 2.31.** Let \( M \in \text{Pro}(k) \), \( G \in \text{Mod}(k) \), and let

\[
0 \leftarrow G \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow ...
\]
be a projective resolution of \( G \). Define

\[
\text{Tor}^n_k (M, G) := H_n (M \otimes_k P_*), \ n \geq 0,
\]

to be the pro-homology of \( (M \otimes_k P_*) \in \text{CHAIN}(\text{Pro}(k)) \).
Remark 2.32. Using the standard homological techniques, one can easily show that $\Tor_n^k$ are well defined bi-additive (even $k$-bilinear) functors from $\text{Pro} (k) \times \text{Mod} (k)$ to $\text{Pro} (k)$. Moreover, since $\otimes_k$ is right exact, $\Tor_0^k$ is naturally isomorphic to $\otimes_k$.

Proposition 2.33. (The first long exact sequence) Let $G \in \text{Mod} (k)$, and let

$$0 \rightarrow M \rightarrow N \rightarrow K \rightarrow 0$$

be a short exact sequence of pro-modules. Then there exists a natural long exact sequence of pro-modules

$$0 \leftarrow K \otimes_k G \leftarrow N \otimes_k G \leftarrow M \otimes_k G \leftarrow \Tor_1^k (K, G) \leftarrow \ldots$$

$$\ldots \leftarrow \Tor_n^k (K, G) \leftarrow \Tor_n^k (N, G) \leftarrow \Tor_n^k (M, G) \leftarrow \Tor_{n+1}^k (K, G) \leftarrow \ldots$$

Proof. Let

$$0 \leftarrow G \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \ldots$$

be a projective resolution of $G$. Due to Theorem 2.17, there is a short exact sequence of complexes of pro-modules

$$0 \rightarrow M \otimes_k P_* \rightarrow N \otimes_k P_* \rightarrow K \otimes_k P_* \rightarrow 0.$$  

The corresponding long exact sequence of pro-homologies is as desired. □

Proposition 2.34. Let $C_* \in \text{Pro} (\text{CHAIN} (k))$ be a pro-complex of $k$-modules, and let $G \in \text{Mod} (k)$. There is a natural on $C_*$ and $G$ pairing

$$H_* (C_*) \otimes_k G \rightarrow H_* (C_* \otimes_k G).$$

Proof. For each $a \in G$ we will define morphisms

$$\overline{\pi}_*: H_* (C_*) \rightarrow H_* (C_* \otimes_k G)$$

which satisfy the condition $(sa + tb)_* = \overline{\pi}_* + \overline{\pi} t$, $a, b \in G$, $s, t \in k$. Since, due to Theorem 2.17

$$\text{Hom}_{\text{Mod} (k)} (G, \text{Hom}_{\text{Pro} (k)} (C_*, C_* \otimes_k G)) \simeq \text{Hom}_{\text{Pro} (k)} (C_* \otimes_k G, C_* \otimes_k G),$$

let

$$\varphi_n : G \rightarrow \text{Hom}_{\text{Pro} (k)} (C_n, C_n \otimes_k G)$$

be a family of morphisms corresponding to $1_n \in \text{Hom}_{\text{Pro} (k)} (C_n \otimes_k G, C_n \otimes_k G)$. The morphisms

$$\varphi_n (a) : C_n \rightarrow C_n \otimes_k G$$

define a chain mapping inducing the desired morphisms $\overline{\pi}_n : H_n (C_*) \rightarrow H_n (C_* \otimes_k G)$.

From now on, we assume that $k$ is quasi-noetherian. Proposition 2.35 below shows that quasi-projective resolutions can be used to define the torsion functors.

Proposition 2.35. Let $M \in \text{Pro} (k)$, $G \in \text{Mod} (k)$, and let

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \ldots$$

be an exact sequence of pro-modules where all $P_i$ are quasi-projective. Then the homology groups of the complex $P_* \otimes_k G$ are naturally isomorphic to the torsion pro-modules:

$$H_n (P_* \otimes_k G) \simeq \Tor_n^k (M, G), n \geq 0.$$
Proof. Induction on \( n \).

**Step** \( n = 0 \): since \( \otimes_k \) commutes with direct limits,

\[
H_0 (P_\ast \otimes_k G) \simeq \text{coker} (P_1 \otimes_k G \to P_0 \otimes_k G) \simeq \text{coker} (P_1 \to P_0) \otimes_k G \simeq M \otimes_k G \simeq \text{Tor}_0^k (M, G).
\]

**Step** \( n \Rightarrow n + 1 \): let \( N = \ker (M \leftarrow P_0) \). The short exact sequence

\[
0 \to N \to P_0 \to M \to 0
\]

induces, due to Proposition 2.33, a long exact sequence

\[
\ldots \leftarrow 0 = \text{Tor}_n^k (P_0, G) \leftarrow \text{Tor}_n^k (N, G) \leftarrow \text{Tor}_{n+1}^k (M, G) \leftarrow 0 = \text{Tor}_n^k (P_0, G) \leftarrow \ldots,
\]

which, in turn, implies an isomorphism \( \text{Tor}_n^k (N, G) \simeq \text{Tor}_{n+1}^k (M, G) \). There is an evident resolution for \( N \):

\[
0 \leftarrow N \leftarrow P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow \ldots
\]

By the induction hypothesis,

\[
\text{Tor}_n^k (N, G) \simeq H_n (P_{n+1} \otimes_k G) \simeq H_{n+1} (P_\ast \otimes_k G).
\]

Finally, \( \text{Tor}_{n+1}^k (M, G) \simeq \text{Tor}_n^k (N, G) \simeq H_{n+1} (P_\ast \otimes_k G) \), as desired.

□

**Proposition 2.36.** *(The second long exact sequence)* Let \( M \in \text{Pro} (k) \), and let

\[
0 \to A \to B \to C \to 0
\]

be a short exact sequence of \( k \)-modules. Then there exists a natural long exact sequence of pro-modules

\[
\ldots \leftarrow M \otimes_k C \leftarrow M \otimes_k B \leftarrow M \otimes_k A \leftarrow \text{Tor}_n^k (M, C) \leftarrow \ldots
\]

\[
\ldots \leftarrow \text{Tor}_n^k (M, C) \leftarrow \text{Tor}_n^k (B, B) \leftarrow \text{Tor}_n^k (M, A) \leftarrow \text{Tor}_{n+1}^k (M, C) \leftarrow \ldots
\]

**Proof.** Let

\[
0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \ldots
\]

be a quasi-projective resolution of \( M \). Due to Proposition 2.30, there is a short exact sequence of complexes of pro-modules

\[
0 \to P_\ast \otimes_k A \to P_\ast \otimes_k B \to P_\ast \otimes_k C \to 0.
\]

Because of Proposition 2.35, the corresponding long exact sequence of pro-homologies is isomorphic to the desired sequence. □

**Theorem 2.37.** Let \( P_\ast \in \text{Pro} (\text{CHAIN} (\mathbb{Z})) \) be a pro-complex of abelian groups, such that \( P_n \) is quasi-projective for all \( n \in \mathbb{Z} \), and let \( G \in \text{Mod} (\mathbb{Z}) \). Then for any \( n \in \mathbb{Z} \) the pairing

\[
H_n (P_\ast) \otimes_\mathbb{Z} G \to H_n (P_\ast \otimes_\mathbb{Z} G)
\]

is a monomorphism, with the cokernel naturally (on \( P_\ast \) and \( G \)) isomorphic to \( \text{Tor}_1^\mathbb{Z} (H_{n-1} (P_\ast), G) \). In other words, there exist natural on \( P_\ast \) and \( G \) short exact sequences \((n \in \mathbb{Z})\)

\[
0 \to H_n (P_\ast) \otimes_\mathbb{Z} G \to H_n (P_\ast \otimes_\mathbb{Z} G) \to \text{Tor}_1^\mathbb{Z} (H_{n-1} (P_\ast), G) \to 0.
\]
Proof. Let
\[ 0 \leftarrow G \leftarrow Q_0 \leftarrow Q_1 \leftarrow 0 \]
be a free resolution for \( G \). Due to Proposition 2.30, there is a short exact sequence in \( \text{CHAIN}(\text{Pro}(\mathbb{Z})) \):
\[ 0 \rightarrow P_* \otimes \mathbb{Z} Q_1 \rightarrow P_* \otimes \mathbb{Z} Q_0 \rightarrow P_* \otimes \mathbb{Z} G \rightarrow 0. \]
Since the functors \( ? \otimes \mathbb{Z} Q_0 \) and \( ? \otimes \mathbb{Z} Q_1 \) are exact, the induced long exact sequence of pro-homologies will have the following form:
\[ \cdots \rightarrow H_n(P_*) \otimes \mathbb{Z} Q_1 \rightarrow H_n(P_*) \otimes \mathbb{Z} Q_0 \rightarrow H_n(P_* \otimes \mathbb{Z} G) \rightarrow \]
\[ \cdots\]
The latter sequence splits into the desired pieces:
\[ 0 \rightarrow \text{coker}(H_n(P_*) \otimes \mathbb{Z} Q_1) \rightarrow H_n(P_*) \otimes \mathbb{Z} Q_0 \rightarrow H_n(P_* \otimes \mathbb{Z} G) \rightarrow \]
\[ \text{Tor}^1_{\mathbb{Z}}(H_{n-1}(P_*), G) \rightarrow 0. \]
\( \square \)

3. Spectral sequences

3.1. Towers. Let
\[ A = \left( \ldots \rightarrow A_n \xrightarrow{i_n} A_{n-1} \rightarrow \ldots \rightarrow A_1 \rightarrow A_0 \right) \]
be a tower of abelian groups, and let
\[ A^{(r)} = \left( \ldots \rightarrow A^{(r)}_n \xrightarrow{p} A^{(r)}_{n-1} \rightarrow \ldots \rightarrow A^{(r)}_1 \rightarrow A^{(r)}_0 \right) \]
where
\[ A^{(r)}_n = i^r A_n. \]

Remark 3.1. We assume that \( i_n : A_n \rightarrow A_{n-1} \) is zero when \( n \geq 0 \), hence \( A^{(r)}_n = 0 \) when \( r > n \).

Definition 3.2. Let \( A = (A_n, i) \) be a tower. The first derived inverse limit is defined as follows:
\[ \lim_\leftarrow^1 A_n = \text{coker} \left( \prod_{n \geq 0} A_n \xrightarrow{1-i} \prod_{n \geq 0} A_n \right). \]

Remark 3.3. It is clear that
\[ \lim_\leftarrow A_n = \ker \left( \prod_{n \geq 0} A_n \xrightarrow{1-i} \prod_{n \geq 0} A_n \right). \]

Theorem 3.4. (Mittag-Leffler)
(1) For a short exact sequence of towers
\[ 0 \rightarrow (A_n) \rightarrow (B_n) \rightarrow (C_n) \rightarrow 0 \]
there exist an 8 term exact sequence
\[ 0 \rightarrow \lim_\leftarrow A_n \rightarrow \lim_\leftarrow B_n \rightarrow \lim_\leftarrow C_n \rightarrow \lim_\leftarrow^1 A_n \rightarrow \lim_\leftarrow^1 B_n \rightarrow \lim_\leftarrow^1 C_n \rightarrow 0. \]
(2) If \( i_n \) are epimorphisms for all \( n \), then \( \lim_{\leftarrow}^1 A_n = 0 \). The condition can be substituted by a weaker “Mittag-Leffler condition”, but we do not need this generalization.

(3) For any \( r \),
\[
\lim_n A_{n+r} \simeq \lim_n A_n, \quad \lim_n^1 A_{n+r} \simeq \lim_n^1 A_n.
\]

(4) For any \( r \),
\[
\lim_n^1 A_n^{(r)} \simeq \lim_n^1 A_n, \quad \lim_n^1 A_n^{(r)} \simeq \lim_n^1 A_n.
\]

(5)
\[
\lim_n \lim_n^r A_n \simeq \lim_n A_n.
\]

(6) There is a short exact sequence
\[
0 \rightarrow \lim_n^1 \lim_n^r A_n^{(r)} \rightarrow \lim_n^1 A_n \rightarrow \lim_n \lim_n^1 A_n^{(r)} \rightarrow 0.
\]

(7)
\[
\lim_n^1 \lim_n^1 A_n^{(r)} = 0
\]

(8)
\[
\lim_n^1 A_n^{(r)} \rightarrow \lim_n^1 A_{n-1}^{(r)}
\]
are epimorphisms for all \( n \in \mathbb{Z} \).

Proof. To prove (1), consider the following short exact sequence of cochain complexes:
\[
0 \rightarrow \prod_{n \geq 0} A_n \rightarrow \prod_{n \geq 0} B_n \rightarrow \prod_{n \geq 0} C_n \rightarrow 0
\]

\[
\begin{array}{ccc}
0 & \rightarrow & \prod_{n \geq 0} A_n \\
\downarrow & & \downarrow \\
\prod_{n \geq 0} B_n & \rightarrow & \prod_{n \geq 0} C_n \\
\downarrow & & \downarrow \\
\prod_{n \geq 0} A_{n-1} & \rightarrow & 0
\end{array}
\]

The long exact sequence of cohomologies reduces to the desired 8 term exact sequence, because
\[
H^n \left( \prod_{n \geq 0} A_n \rightarrow \prod_{n \geq 0} A_n \right) = \left\{ \begin{array}{ll}
\lim_n^1 A_n & \text{if } n = 0 \\
\lim_n^1 A_n & \text{if } n = 1 \\
0 & \text{if } n < 0 \text{ or } n > 1
\end{array} \right.,
\]

and similarly for \((B_n)\) and \((C_n)\).

To prove (2), notice that
\[
\prod_{n \geq 0} A_n \rightarrow \prod_{n \geq 0} A_n
\]
is an epimorphism whenever all \( i_n \) are epimorphisms.
To prove (3), let
\begin{align*}
0 & \longrightarrow C^0 = \prod_{n \geq 0} A_{n+1} \\
& \xrightarrow{d = 1 - i} C^1 = \prod_{n \geq 0} A_{n+1} \\
& \xrightarrow{\sim} 0 \\
& \xrightarrow{i} j \\
0 & \longrightarrow D^0 = \prod_{n \geq 0} A_n \\
& \xrightarrow{d = 1 - i} D^1 = \prod_{n \geq 0} A_n \\
& \xrightarrow{\sim} 0
\end{align*}
be a pair of morphisms
\[ C^* \xrightarrow{i} D^* \xrightarrow{j} C^* \]
of cochain complexes where \( j \) is the natural projection.

Define cochain homotopies
\begin{align*}
0 & \longrightarrow C^0 = \prod_{n \geq 0} A_{n+1} \\
& \xrightarrow{1 - i} C^1 = \prod_{n \geq 0} A_{n+1} \\
& \xrightarrow{\sim} 0 \\
& \xrightarrow{ji} 1 \\
0 & \longrightarrow C^0 = \prod_{n \geq 0} A_{n+1} \\
& \xrightarrow{1 - i} C^1 = \prod_{n \geq 0} A_{n+1} \\
& \xrightarrow{\sim} 0 \\
& \xrightarrow{ij} 1
\end{align*}
by
\begin{align*}
S(a_1, a_2, ..., a_n, ...) &= (-a_1, -a_2, ..., -a_n, ...) \\
(ji - 1)(a_1, a_2, ..., a_n, ...) &= (ia_2 - a_1, ia_3 - a_2, ..., ia_{n+1} - a_n, ...) , \\
ji - 1 &= Sd + dS,
\end{align*}
and
\begin{align*}
0 & \longrightarrow D^0 = \prod_{n \geq 0} A_n \\
& \xrightarrow{1 - i} D^1 = \prod_{n \geq 0} A_n \\
& \xrightarrow{\sim} 0 \\
& \xrightarrow{ij} 1 \\
0 & \longrightarrow D^0 = \prod_{n \geq 0} A_n \\
& \xrightarrow{1 - i} D^1 = \prod_{n \geq 0} A_n \\
& \xrightarrow{\sim} 0 \\
& \xrightarrow{ji} 1
\end{align*}
by
\begin{align*}
T(a_0, a_1, ..., a_n, ...) &= (-a_0, -a_1, ..., -a_n, ...) \\
(ij - 1)(a_0, a_1, ..., a_n, ...) &= (ia_1 - a_0, ia_2 - a_0, ..., ia_{n+1} - a_n, ...) , \\
ij - 1 &= Td + dT.
\end{align*}

Therefore, \( i \) and \( j \) are cochain homotopy equivalences which induce the isomorphisms
\[ \lim_{\leftarrow} A_{n+1} \simeq \lim_{\leftarrow} A_n, \lim_{\leftarrow} A_{n+1} \simeq \lim_{\leftarrow} A_n. \]

By induction on \( r \), one can easily deduce the desired isomorphisms
\[ \lim_{\leftarrow} A_{n+r} \simeq \lim_{\leftarrow} A_n, \lim_{\leftarrow} A_{n+r} \simeq \lim_{\leftarrow} A_n. \]
To prove (H), let

\[ 0 \rightarrow C^0 = \prod_{n \geq 0} A_n \xrightarrow{d = 1 - i} C^1 = \prod_{n \geq 0} A_n \rightarrow 0 \]

\[ 0 \rightarrow D^0 = \prod_{n \geq 0} A_n^{(1)} \xrightarrow{d = 1 - i} D^1 = \prod_{n \geq 0} A_n^{(1)} \rightarrow 0 \]

be a pair of morphisms

\[ C^* \xrightarrow{i} D^* \xrightarrow{j} C^* \]

of cochain complexes where \( j \) is the natural inclusion.

Define cochain homotopies

\[ 0 \rightarrow C^0 = \prod_{n \geq 0} A_n \xrightarrow{1 - i} C^1 = \prod_{n \geq 0} A_n \rightarrow 0 \]

\[ 0 \rightarrow D^0 = \prod_{n \geq 0} A_n^{(1)} \xrightarrow{1 - i} D^1 = \prod_{n \geq 0} A_n^{(1)} \rightarrow 0 \]

by

\[ S(a_0, a_1, ..., a_n, ...) = (-a_0, -a_1, ..., -a_n, ...) \]

\[ (ji - 1)(a_0, a_1, ..., a_n, ...) = (ia_1 - a_0, ia_1 - a_0, ..., ia_{n+1} - a_n, ...) \]

\[ ji - 1 = Sd + dS. \]

and

\[ 0 \rightarrow D^0 = \prod_{n \geq 0} A_n^{(1)} \xrightarrow{1 - i} D^1 = \prod_{n \geq 0} A_n^{(1)} \rightarrow 0 \]

\[ 0 \rightarrow D^0 = \prod_{n \geq 0} A_n^{(1)} \xrightarrow{1 - i} D^1 = \prod_{n \geq 0} A_n^{(1)} \rightarrow 0 \]

by

\[ T(a_0, a_1, ..., a_n, ...) = (-a_0, -a_1, ..., -a_n, ...) \]

\[ (ij - 1)(a_0, a_1, ..., a_n, ...) = (ia_1 - a_0, ia_1 - a_0, ..., ia_{n+1} - a_n, ...) \]

\[ ij - 1 =Td +dT. \]

Hence, \( i \) and \( j \) are cochain homotopy equivalences which induce the isomorphisms

\[ \lim_{\leftarrow} A_n^{(1)} \simeq \lim_{\leftarrow} A_n, \lim_{\leftarrow} A_n^{(1)} \simeq \lim_{\leftarrow} A_n. \]

By induction on \( r \), one can easily deduce the desired isomorphisms

\[ \lim_{\leftarrow} A_n^{(r)} \simeq \lim_{\leftarrow} A_n, \lim_{\leftarrow} A_n^{(r)} \simeq \lim_{\leftarrow} A_n. \]
To prove (4)-(7), consider the cochain bicomplex $F^{**}$:

$$
0 \longrightarrow F^{01} = \prod_{n \geq 0} \prod_{r \geq 0} A_n^{(r)} \xrightarrow{(1 - i, 1)} F^{11} = \prod_{n \geq 0} \prod_{r \geq 0} A_n^{(r)} \longrightarrow 0
$$

$$
(1, 1 - i)
$$

$$
0 \longrightarrow F^{00} = \prod_{n \geq 0} \prod_{r \geq 0} A_n^{(r)} \xrightarrow{(1 - i, 1)} F^{10} = \prod_{n \geq 0} \prod_{r \geq 0} A_n^{(r)} \longrightarrow 0
$$

($F^{st} = 0$ if $s \neq 0, 1$, or if $t \neq 0, 1$). It produces two “classical” spectral sequences converging to the cohomology of the total complex $T^*$:

$$
E^{st}_r \Longrightarrow H^{s+t}(T^*),
$$

$$
\mathcal{E}^{st}_r \Longrightarrow H^{s+t}(T^*),
$$

where

$$
E^{00}_2 = \lim_{n} \lim_{r} A_n^{(r)}, \quad E^{10}_2 = \lim_{n} \lim_{r} A_n^{(r)},
$$

$$
E^{01}_2 = \lim_{n} \lim_{r} A_n^{(r)}, \quad E^{11}_2 = \lim_{n} \lim_{r} A_n^{(r)},
$$

and, due to (4),

$$
\mathcal{E}^{00}_2 = \lim_{n} \lim_{r} A_n^{(r)} = \lim_{n} \lim_{r} A_n = \lim_{n} A_n,
$$

$$
\mathcal{E}^{10}_2 = \lim_{n} \lim_{r} A_n^{(r)} = \lim_{n} \lim_{r} A_n = \lim_{n} A_n,
$$

$$
\mathcal{E}^{01}_2 = \lim_{n} \lim_{r} A_n^{(r)} = \lim_{n} \lim_{r} A_n = 0,
$$

$$
\mathcal{E}^{11}_2 = \lim_{n} \lim_{r} A_n^{(r)} = \lim_{n} \lim_{r} A_n = 0.
$$

It follows that

$$
\lim_{n} \lim_{r} A_n^{(r)} = E^{00}_2 = H^0(T^*) = \mathcal{E}^{00}_2 = \lim_{n} A_n,
$$

$$
\lim_{n} \lim_{r} A_n^{(r)} = E^{11}_2 = H^2(T^*) = \mathcal{E}^{11}_2 = 0.
$$

Moreover, from the second spectral sequence,

$$
H^1(T^*) = \mathcal{E}^{10}_2 = \lim_{n} A_n,
$$

while from the first one, $H^1(T^*)$ has a two-fold filtration with subquotients $E^{01}_2$ and $E^{10}_2$, giving the desired exact sequence

$$
0 \longrightarrow E^{10}_2 = \lim_{n} \lim_{r} A_n^{(r)} \longrightarrow \lim_{n} A_n \longrightarrow \lim_{n} \lim_{r} A_n^{(r)} = E^{01}_2 \longrightarrow 0.
$$

It remains only to prove (7). Fix $n$, and consider an exact sequence of towers (with respect to $r$):

$$
0 \longrightarrow \left( B^{(r)} \right) \longrightarrow \left( A_n^{(r)} \right) \xrightarrow{i} \left( A_{n-1}^{(r+1)} \right) \longrightarrow 0
$$

where

$$
B^{(r)} = \ker \left( A_n^{(r)} \xrightarrow{i} A_{n-1}^{(r+1)} \right).
$$

This exact sequence induces the 8 term exact sequence from (1)

$$
0 \longrightarrow \lim_{r} B^{(r)} \longrightarrow \lim_{r} A_n^{(r)} \longrightarrow \lim_{r} A_{n-1}^{(r+1)} \longrightarrow \lim_{r} B^{(r)} \longrightarrow \lim_{r} A_n^{(r)} \longrightarrow \lim_{r} A_{n-1}^{(r+1)} \longrightarrow 0.
$$
It follows that, due to (4),
\[ \lim_{r \to 1} A^{(r+1)}_{n-1} = \lim_{r \to 1} A^{(r)}_{n-1}, \]
and
\[ \lim_{r \to 1} A^{(r)}_n \to \lim_{r \to 1} A^{(r+1)}_{n-1} = \lim_{r \to 1} A^{(r)}_{n-1} \]
is an epimorphism. □

3.2. Bicomplexes.

**Definition 3.5.** Let \((C^*, d, \delta)\) be a cochain bicomplex, with the horizontal differential \(d\) and the vertical differential \(\delta\), concentrated in the I and IV quadrant, i.e. \(C^{st} = 0\) if \(s < 0\). Let \(\text{Tot}(C)\) be the following cochain complex:
\[ \text{Tot}(C)^n = \prod_{s+t=n} C^{st} \]
with the differential \((\partial c)^{st} = dc^{s-1,t} + (-1)^s \delta c^{s,t-1}\) where
\[ (c^{st}) \in \prod_{s+t=n-1} C^{st} = \text{Tot}(C)^{n-1}, \]
\[ ((\partial c)^{st}) \in \prod_{s+t=n-1} C^{st} = \text{Tot}(C)^n. \]

Clearly, \(\partial \partial = 0\):
\[ (\partial \partial c)^{st} = d(\partial c)^{s-1,t} + (-1)^s \delta(\partial c)^{s,t-1} = d (dc^{s-2,t} + (-1)^t \delta c^{s-1,t-1}) + (-1)^s \delta (dc^{s-1,t-1} + (-1)^s \delta c^{s,t-2}) = (-1)^s d c^{s-1,t-1} + (-1)^s \delta d c^{s-1,t-1} = (-1)^s (-d \delta + \delta d) = 0, \]
and \((\text{Tot}(C), \partial)\) is a cochain complex.

**Remark 3.6.** Notice that we use **products** instead of **direct sums**.

Let now
\[ \text{Tot}^{(p)}(C)^n = \prod_{s=0}^p C^{s,n-s}. \]

Denote by letter \(\partial^{(p)}\) the differential
\[ (\partial^{(p)})^s = dc^{s-1,n-s} + (-1)^s \delta c^{s,n-s}, \]
\[ (\text{Tot}^{(p)}(C), \partial^{(p)}) \]
become cochain complexes, and there are natural projections
\[ \text{Tot}(C) \to \text{Tot}^{(p)}(C) \]
that are epimorphisms of cochain complexes. Clearly,
\[ \text{Tot}(C) = \varprojlim_p \text{Tot}^{(p)}(C). \]

Denote by
\[ D^{st} = D_1^{st} = H^{s+t} \left( \text{Tot}^{(s)}(C) \right), \]
\[ E^{st} = E_1^{st} = H^{s+t} \Gamma^{(s)}(C), \]
where
\[ \Gamma^{(s)}(C) = \ker \left( \text{Tot}^{(s)}(C) \to \text{Tot}^{(s-1)}(C) \right) \]

The long exact sequence corresponding to the short exact sequence of complexes
\[ 0 \to \Gamma^{(s)}(C) \to \text{Tot}^{(s)}(C) \to \text{Tot}^{(s-1)}(C) \to 0 \]
gives an exact couple
\[ D = D_1 \xrightarrow{i} (0, 0) \xleftarrow{k} E = E_1 \]
of bi-graded abelian groups and homogeneous morphisms with the bi-degrees written at the corresponding arrows.

One can construct the derived exact couples
\[ D_r = D_{r-1} \xrightarrow{i_r} (0, 0) \xleftarrow{k_r} E_r \xrightarrow{j_r} (1, 0) \]
where \( D_r = i_r^{-1}D, E_r = H(E_{r-1}, d = j_{r-1} k_{r-1}) \) is the cohomology of the complex build on the previous couple, \( i_r \) is induced by \( i_{r-1} \), \( k_r \) is induced by \( k_{r-1} \), while \( j_r = j_{r-1} (i_{r-1})^{-1} \).

**Theorem 3.7.** (spectral sequence of a bicomplex)

1. \( E^{st}_1 = (H_{ver}(C))^{st} \) where \( H_{ver} \) is the cohomology in the vertical direction.
2. \( E^{st}_2 = (H_{hor} H_{ver}(C))^{st} \) where \( H_{hor} \) is the cohomology in the horizontal direction.
3. \( D^{0, n}_2 \simeq E^{0, n}_2 \).
4. The groups \( D^{st}_2 \) are included in long exact sequences
   \[
   0 \to E^{1, n-1}_2 \to D^{1, n-1}_2 \to D^{0, n}_2 \to E^{2, n-1}_2 \to D^{2, n-1}_2 \to D^{1, n}_2 \to E^{3, n-1}_2 \to \cdots \\
   \cdots \to E^{s, n-1}_2 \to D^{s, n-1}_2 \to D^{s-1, n}_2 \to E^{s+1, n-1}_2 \to \cdots 
   \]
5. There are short exact sequences \((n \in \mathbb{Z})\)
   \[
   0 \to \lim_{\rightarrow} H^{n-1} \text{Tot}^{(s)}(C) \to H^n \text{Tot}(C) \to \lim_{\leftarrow} H^n \text{Tot}^{(s)}(C) \to 0. 
   \]
6. There are short exact sequences \((r \geq 1, n \in \mathbb{Z})\)
   \[
   0 \to \lim_{\rightarrow} D^{s, n-s-1}_r \to H^n \text{Tot}(C) \to \lim_{\leftarrow} D^{s, n-s}_r \to 0. 
   \]
7. If \( C \to C' \) is a morphism of bicomplexes inducing isomorphisms \( E^{st}_2(C) \simeq E^{st}_2(C') \) for all \( s, t \), then
   \[ H^n \text{Tot}(C) \simeq H^n \text{Tot}(C'). \]
If
\[ \lim_{r \to -1} E_r^{st} = 0 \]
then \( E_r^{st} \) completely (in the sense of \([\text{BK72}], \text{IX.5.3}\)) or strongly (in the sense of \([\text{Boa99}], \text{Definition 5.2}\)) converges to \( H^{s+t}(\text{Tot}(C)) \). The latter means that \( H^n(\text{Tot}(C)) \) is an inverse limit: \( H^n(\text{Tot}(C)) \simeq \lim_{s \to -1} (\ldots \to Q^{s,n-s} \to Q^{s-1,n-s+1} \to \ldots \to Q^{0,n} \to Q^{-1,n+1} = 0) \)
of epimorphisms with the kernels
\[ \ker (Q^{s,n-s} \to Q^{s-1,n-s+1}) \simeq E_{\infty} := \lim_{r \to -1} E_r^{st}. \]

**Proof.** We could not find in the literature the proof of the statements above directly in the form we need. The ideas of the desired proof can be found, however, in \([\text{Boa99}], \text{BK72}, \text{Tho85}, \text{Pra01}, \text{Pra89} \) and \([\text{Mar00}]\).

(1) The \( n \)-th component of the kernel \( \Gamma^{(s)}(C) \) consists of the elements
\[ c = (0, 0, \ldots, a) \in \prod_{i=0}^{s} C^{n-i}. \]
Let us compute \( \partial c \):
\[ (\partial c)^{j,n+1-j} = dc^{j-1,n+1-j} + (-1)^s \delta c^{j,n-j} = 0 \]
when \( j < s \), and \( = (-1)^s \delta a \) when \( j = s \), therefore
\[ \partial c = (0, 0, \ldots, (-1)^s \delta a). \]
It means that (up to sign of the differentials) the complex \( \Gamma^{(s)}(C) \) is isomorphic to the vertical \( s \)-th line of the bicomplex \( C^{**} \), hence its cohomology is the vertical cohomology of \( C^{**} \).

(2) It is enough to prove that the differential
\[ jk : E_1^{st} = H_{ver}(C)^{st} \to H_{ver}(C)^{s+1,t} \]
is induced by the horizontal differential \( d \). To do this, we need to calculate \( j \) and \( k \). Clearly, a class
\[ e' = [e] \in E_1^{st} = \frac{\ker \delta}{\text{Im}(\delta)} \]
is mapped (under the mapping \( k \)) to the class \( ke' = [\varepsilon] \) of
\[ \varepsilon = (0, 0, \ldots, e) \in \ker \partial \subseteq \text{Tot}^{(s)}(C)^{s+t}. \]
Since \( \text{Tot}^{(s+1)}(C)^{s+t} \to \text{Tot}^{(s)}(C)^{s+t} \) is an epimorphism, it is possible to find an element \( \theta \in \text{Tot}^{(s+1)}(C)^{s+t} \) "lying over" \( \varepsilon \). We have the right to choose
\[ \theta = (0, 0, \ldots, e, 0) \in \text{Tot}^{(s+1)}(C)^{s+t}. \]
Apply \( \partial \):
\[ \partial \theta = (0, 0, \ldots, (-1)^s \delta e, de) = (0, 0, \ldots, 0, de) \in \Gamma^{(s+1)}(C) \subseteq \text{Tot}^{(s+1)}(C)^{s+t+1}. \]
Finally, \( jke' = de \), and the differential \( d' \) in the cochain complex \( E_i^{s,t} \) is induced by the horizontal differential \( d \). Therefore,

\[
E_2^{s,t} = \frac{\ker \left( d' : E_i^{s,t} \to E_i^{s+1,t} \right)}{\text{Im} \left( d' : E_i^{s,t} \to E_i^{s+1,t} \right)} = H_{\text{hor}} H_{\text{ver}} \left( C \right)^{s,t}.
\]

(3) Consider the following parts of the long exact sequences of the derived couple

\[
\begin{array}{c}
D_2 \\
\downarrow i \\
E_2 \\
\downarrow \phi
\end{array}
\]

where \( \phi \) is the natural projection. The long exact sequence of cohomologies

\[
0 = D_2^{-2,n+1} \to E_2^{0,n} \to D_2^{0,n} \to D_2^{-1,n+1} = 0.
\]

It follows that \( E_2^{0,n} \simeq D_2^{0,n} \).

(4) Consider again long exact sequences of the derived couple

\[
0 = D_2^{-1,n} \to E_2^{1,n-1} \to D_2^{1,n-1} \to D_2^{0,n} \to \\
\to E_2^{2,n-1} \to D_2^{2,n-1} \to D_2^{1,n} \to E_2^{3,n-1} \to \ldots \\
\to E_2^{s,n-1} \to D_2^{s,n-1} \to D_2^{s-1,n} \to E_2^{s+1,n-1} \to \ldots
\]

(5) Consider a short exact sequence of cochain complexes:

\[
0 \to \text{Tot}(C) \to \prod_{s \geq 0} \text{Tot}^{(s)}(C) \xrightarrow{1-p} \prod_{s \geq 0} \text{Tot}^{(s)}(C) \to 0
\]

where

\[
p : \text{Tot}^{(s)}(C) \to \text{Tot}^{(s-1)}(C)
\]

is the natural projection. The long exact sequence of cohomologies

\[
\ldots \to \prod_{s \geq 0} H^{n-1} \text{Tot}^{(s)}(C) \xrightarrow{1-p} \prod_{s \geq 0} H^{n-1} \text{Tot}^{(s)}(C) \to H^n \text{Tot}(C) \to \\
\to \prod_{s \geq 0} H^n \text{Tot}^{(s)}(C) \xrightarrow{1-p} \prod_{s \geq 0} H^n \text{Tot}^{(s)}(C) \to \ldots
\]

splits into the desired pieces

\[
0 \to \text{coker} \left( \prod_{s \geq 0} H^{n-1} \text{Tot}^{(s)}(C) \xrightarrow{1-p} \prod_{s \geq 0} H^{n-1} \text{Tot}^{(s)}(C) \right) = \lim_{s \to 1} H^{n-1} \text{Tot}^{(s)}(C) \to \\
\to H^n \text{Tot}(C) \to \ker \left( \prod_{s \geq 0} H^n \text{Tot}^{(s)}(C) \xrightarrow{1-p} \prod_{s \geq 0} H^n \text{Tot}^{(s)}(C) \right) = \lim_{s \to 1} H^n \text{Tot}^{(s)}(C) \to 0
\]

(6) It follows from Theorem 3.4 [11] that

\[
\lim_{s \to 1} D_{r^{s,n-s}} \simeq \lim_{s \to 1} D_{1^{s,n-s}} = \lim_{s \to 1} H^n \text{Tot}^{(s)}(C),
\]

\[
\lim_{s \to 1} D_{r^{1,n-s-1}} \simeq \lim_{s \to 1} D_{1^{1,n-s-1}} = \lim_{s \to 1} H^{n-1} \text{Tot}^{(s)}(C).
\]
(7) It follows from (3) that
\[
D_0^{0,n}(C) \simeq E_2^{0,n}(C) \simeq E_2^{0,n}(C') \simeq D_0^{0,n}(C').
\]
The morphism \( C \to C' \) induces morphisms of exact sequences from (4) for \( C \) and \( C' \). Applying the 5-lemma several times, one gets
\[
D_2^{s,t}(C) \simeq D_2^{s,t}(C').
\]
There is a morphism (which is an isomorphism in the second and the fourth term) of short exact sequences from (6) for \( r = 2 \). Using again the 5-lemma, one gets the desired isomorphism
\[
H^n \text{Tot} (C) \simeq H^n \text{Tot} (C') .
\]

(8) Consider the \( r \)-th derived couple

\[
\begin{array}{ccc}
D_r & \xrightarrow{i_r} & D_r \\
\downarrow \phi_r & & \downarrow \psi_r \\
E_r & \xrightarrow{j_r} & E_r
\end{array}
\]

Fix \( s \) and \( t \). In the long exact sequence
\[
\ldots \to D_{r,s+r,t+r-1}^s \to E_{r}^{st} \to D_{r}^{s,t} \to D_{r,s+1,t+1}^s \to \ldots
\]
the term \( D_{r,s+r,t+r-1}^s \) equals zero for \( r \) large enough \( (r > s) \), and one gets a short exact sequence
\[
0 \to E_{r}^{st} \to D_{r}^{s,t} \to D_{r+1}^{s-1,t+1} \to 0.
\]
Since \( \varprojlim_s E_r^{st} = 0 \), one gets, due to Theorem 3.4 [1 4], a short exact sequence
\[
0 \to \varprojlim_s E_r^{st} \to \varprojlim_s D_r^{s,t} \to \varprojlim_s D_{r,s+1,t+1}^s \to 0,
\]
and an isomorphism
\[
\varprojlim_s D_r^{s,t} \simeq \varprojlim_s D_{r,s+1,t+1}^s.
\]
Varying \( s \), one gets
\[
\varprojlim_s D_r^{s,t} \simeq \varprojlim_s D_{r,s+1,t+1}^s \simeq \ldots \simeq \varprojlim_s D_r^{0,s+t} \simeq \varprojlim_s D_{r-1,s+t+1}^s = 0.
\]
Let us denote
\[
Q_r^{s,n-s} := \varprojlim_s D_{r}^{s,n-s} = D_{r}^{s,n-s}.
\]
It follows from Theorem 3.4 [5] that
\[
\varprojlim_s H^n \left( \text{Tot}^{(s)} (C) \right) \simeq \varprojlim_s \varprojlim_r D_r^{s,n-s} \simeq \varprojlim_s Q_r^{s,n-s}.
\]
Since
\[
\ldots \to Q_r^{s,n-s} \to Q^{s-1,n-s+1}_r \to \ldots
\]
is a tower of epimorphisms, \( \varprojlim_s Q_r^{s,n-s} = 0 \), due to Theorem 3.4 [2]. It follows from Theorem 3.4 [6], since \( \varprojlim_s D_{r,n-s-1}^r = 0 \), that
\[
\varprojlim_s H^{n-1} \left( \text{Tot}^{(s)} (C) \right) = 0.
\]
The short exact sequence (5) gives the desired isomorphism
\[ H^n (\text{Tot} (C)) \simeq \lim_{\leftarrow s} H^n \left( \text{Tot}^{(s)} (C) \right) \simeq \lim_{\leftarrow s} Q^{s,n-s}. \]

We have proved that \( H^n (\text{Tot} (C)) \) is isomorphic to the inverse limit of the tower \((Q^{s,n-s})_{s=0}^{\infty}\) of epimorphisms with kernels \(E^{s}_{n+s}\), hence \(E^{s}_{n}\) converges completely (in the sense of [BK72], IX.5.3) or strongly (in the sense of [Boa99], Definition 5.2) to \(H^{s+t} (\text{Tot} (C))\).

\[ \square \]

3.3. **Homotopy inverse limits.** On homotopy inverse limits of topological spaces, see [BK72], Chapter XI. Here we define and investigate the homotopy inverse limits for the diagrams of chain complexes.

**Definition 3.8.** Let \( I \) be a small category, and let \( C : I \to \text{CHAIN} (\mathbb{Z}) \) be a functor to the category of chain complexes of abelian groups. Negating the indices, one gets a functor to the category of cochain complexes. Take the **cosimplicial replacement** ([BK72], XI.5.1), which is a cosimplicial cochain complex, and, finally, a cochain bicomplex
\[ (RC)^{st} = \prod_{(i_0 \to i_1 \to \ldots \to i_s) \in I} C_{-t} (i_s) \]
with the horizontal differentials
\[ d : (RC)^{s-1,t} \to (RC)^{s,t}, \]
\[ dc (i_0 \to i_1 \to \ldots \to i_s) = \sum_{k=0}^{s-1} (-1)^k c (i_0 \to \ldots \to \widehat{i_k} \to \ldots \to i_s) + (-1)^s C (i_{s-1} \to i_s) c (i_0 \to \ldots \to i_{s-1}), \]
and the vertical differentials
\[ \delta : (RC)^{s,t-1} \to (RC)^{s,t}, \]
\[ \delta c (i_0 \to i_1 \to \ldots \to i_s) = d_{-1} (i_s) c (i_0 \to i_1 \to \ldots \to i_s) \]
where
\[ d_{-1} (i_s) : C_{-t+1} (i_s) \to C_{-t} (i_s) \]
is the differential in the chain complex \( C_s (i_s) \). Take the total complex \( \text{Tot} (RC) \) (using products, as in Definition 3.3), and negate the indices again. The resulting **chain complex** is called the homotopy inverse limit \( \underline{\text{holim}}_I C \) of the functor \( C \):
\[ \left( \underline{\text{holim}}_I C \right)_n = \prod_{\left( i_0 \to i_1 \to \ldots \to i_s \right) \in I} C_{n+s} (i_s) \]
with the differential
\[ \partial : \left( \underline{\text{holim}}_I C \right)_{n+1} \to \left( \underline{\text{holim}}_I C \right)_n \]
given by
\[ \partial c (i_0 \to i_1 \to \ldots \to i_s) = \]
\[ \sum_{k=0}^{s-1} (-1)^k c (i_0 \to \ldots \to \widehat{i_k} \to \ldots \to i_s) + (-1)^s C (i_{s-1} \to i_s) c (i_0 \to \ldots \to i_{s-1}) + (-1)^s d_{n+s} (i_s) c (i_0 \to i_1 \to \ldots \to i_s). \]
Definition 3.9. Let \( \varphi : J \to I \) be a cofinal functor (left cofinal in [BK72], XI.9.1), and let \( C : I \to CHAIN(\mathbb{Z}) \) be a functor. The natural homomorphisms

\[
\prod_{(i_0 \to i_1 \to \ldots \to i_s) \in I} C(i_s) \to \prod_{(j_0 \to j_1 \to \ldots \to j_s) \in J} C(\varphi(j_s)),
\]

\( c \mapsto c' \),

\[
c'(j_0 \to j_1 \to \ldots \to j_s) = c(\varphi(j_0) \to \varphi(j_1) \to \ldots \to \varphi(j_s))
\]

induce the morphism of chain complexes

\[
\varphi^* : \text{holim}_{i \in I} C \to \text{holim}_{j \in J} (C \circ \varphi).
\]

Remark 3.10. Notice that we do not require that the index categories \( I \) and \( J \) are cofiltrant.

Theorem 3.11. (spectral sequence of a homotopy inverse limit). Let \( E_r^{st} \) be the spectral sequence from Theorem 3.7 for the bicomplex \( RC \). Then:

1. \[
E_1^{st} = \prod_{(i_0 \to i_1 \to \ldots \to i_s) \in I} H_{-t}C(i_s).
\]

2. \[
E_2^{st} = \lim_{r \in I} H_{-t}(C(i)).
\]

3. \( D_r^{0,n} \approx E_r^{0,n} \approx \lim_{r \in I} H_{-n}(C(i)) \).

4. The groups \( D_r^{st} \) are included in long exact sequences

\[
0 \to E_r^{1,n-1} \to D_r^{1,n-1} \to D_r^{0,n} \to \ldots \\
\to E_r^{2,n-1} \to D_r^{2,n-1} \to D_r^{1,n} \to E_r^{3,n-1} \to \ldots \\
\ldots \to E_r^{s,n-1} \to D_r^{s,n-1} \to D_r^{s-1,n} \to E_r^{s+1,n-1} \to \ldots
\]

5. There are short exact sequences \( (n \in \mathbb{Z}) \)

\[
0 \to \lim_{r \in J} H_{n+1} \text{Tot}^{(r)}(C) \to H_n \left( \text{holim}_{i \in I} C \right) \to \lim_{r \in J} H_n \text{Tot}^{(r)}(C) \to 0.
\]

6. There are short exact sequences \( (r \geq 1, n \in \mathbb{Z}) \)

\[
0 \to \lim_{r \in J} D_r^{s,n-s-1} \to H_n \left( \text{holim}_{i \in I} C \right) \to \lim_{r \in J} D_r^{s,n-s} \to 0.
\]

7. If \( C \to C' \) is a morphism of functors inducing isomorphisms \( E_r^{st}(C) \approx E_r^{st}(C') \) for all \( s, t \), then

\[
H_n \left( \text{holim}_{i \in I} C \right) \approx H_n \left( \text{holim}_{i \in I} C' \right).
\]

8. Let \( \varphi : J \to I \) be a cofinal functor, and let

\[
\varphi^* : \text{holim}_{i \in I} C \to \text{holim}_{j \in J} (C \circ \varphi)
\]

be the corresponding morphism of complexes. Then \( \varphi^* \) is a weak equivalence (induces an isomorphism of homologies).

9. If

\[
\lim_{r \in I} E_r^{st} = 0
\]
then $E^s_2$ completely (in the sense of [BK72], IX.5.3) or strongly (in the sense of [Boa99], Definition 5.2) converges to $H_{-s-t} \left( \text{holim}_{i \in I} C \right)$. The latter means that $H_n \left( \text{holim}_{i \in I} C \right)$ is isomorphic to an inverse limit of epimorphisms with the kernels

$$\ker \left( Q^{s-n-s} \to Q^{s-1,n-s+1} \to \cdots \to Q^{0,n} \to Q^{-1,n+1} = 0 \right)$$

of homomorphisms, where

$$\lim_s \left( \cdots \to Q^{s-n-s} \to Q^{s-1,n-s+1} \to \cdots \to Q^{0,n} \to Q^{-1,n+1} = 0 \right)$$

is isomorphic to $E^s_{\infty} := \lim_r E^s_{r-n-s}$.

Proof.

(1) Follows from Theorem 3.7 (1).

(2) Follows from Theorem 3.7 (2) and Proposition XI.6.2 in [BK72].

(3) Follows from Theorem 3.7 (3).

(4) Follows from Theorem 3.7 (4).

(5) Negate indices in Theorem 3.7 (5).

(6) Negate indices in Theorem 3.7 (6).

(7) Negate indices in Theorem 3.7 (7).

(8) A cofinal functor induces an isomorphism of higher limits. This is a rather well-known fact. It can be proven similarly to the Cofinality Theorem ([BK72], XI.9.2). Moreover, the statement follows from ([BK72], XI.9.2 and XI.7.2). Therefore

$$E^s_2 \left( \text{holim}_{i \in I} C \right) \to E^s_2 \left( \text{holim}_{j \in J} (C \circ \varphi) \right)$$

is an isomorphism for all $s$, $t$. The desired isomorphism of homologies follows from (8).

(9) Negate indices in Theorem 3.7 (9).

$\square$

Definition 3.12. Let $C_* = (C_i)_{i \in I} \in \text{Pro} \left( \text{CHAIN} \left( \mathbb{Z} \right) \right)$ be a pro-complex. Define its homotopy inverse limit $\text{holim}$ as follows:

$$\text{holim} C_* := \text{holim}_{i \in I} C_i.$$

Remark 3.13. It follows from Proposition 3.14 below that the homotopy inverse limit of a pro-complex is well defined up to weak equivalence.

Proposition 3.14. The homotopy inverse limit from Definition 3.12 is a well-defined functor

$$\text{holim} : \text{Pro} \left( \text{CHAIN} \left( \mathbb{Z} \right) \right) \to \text{Ho} \left( \text{CHAIN} \left( \mathbb{Z} \right) \right)$$

where $\text{Ho} \left( \text{CHAIN} \left( \mathbb{Z} \right) \right)$ is the category of fractions of the category $\text{CHAIN} \left( \mathbb{Z} \right)$ with respect to weak equivalences of complexes.

Proof. The functor is well-defined on the category $\text{Inv} \left( \text{CHAIN} \left( \mathbb{Z} \right) \right)$. In order to use Theorem 2.6 one needs only to check that cofinal morphisms are mapped into weak equivalences. The latter fact, however, follows from Theorem 3.11 (8). $\square$

Theorem 3.15. (spectral sequence of a pro-complex). Let $C \in \text{Pro} \left( \text{CHAIN} \left( \mathbb{Z} \right) \right)$ be a chain pro-complex. Then there exists a spectral sequence $E^s_2 (C)$, natural on $C$ from $E_2$ on, such that:
Proof.

(1) \[ E_2^{st} = \lim^s H_{-t}(C). \]

(2) \[ D_2^{0,n} \simeq E_2^{0,n} \simeq \lim H_{-n}(C). \]

(3) The groups \( D_2^{s,t} \) are included in long exact sequences

\[
0 \rightarrow E_2^{1,n-1} \rightarrow D_2^{1,n-1} \rightarrow D_2^{0,n} \rightarrow \\
E_2^{2,n-1} \rightarrow D_2^{2,n-1} \rightarrow D_2^{1,n} \rightarrow E_2^{3,n-1} \rightarrow \ldots \\
\ldots \rightarrow E_2^{s,n-1} \rightarrow D_2^{s,n-1} \rightarrow D_2^{s-1,n} \rightarrow E_2^{s+1,n-1} \rightarrow \ldots 
\]

(4) There are short exact sequences \((r \geq 2, n \in \mathbb{Z})\)

\[
0 \rightarrow \lim_1 D_2^{r,-n-s-1} \rightarrow H_n \left( \lim \text{holim} \ C \right) \rightarrow \lim_1 D_2^{r,-n-s} \rightarrow 0. 
\]

(5) If \( C \rightarrow C' \) is a morphism of pro-complexes inducing isomorphisms

\[ E_2^{st}(C) \simeq E_2^{st}(C') \]

for all \( s, t \), then

\[ H_n \left( \lim \text{holim} \ C \right) \simeq H_n \left( \lim \text{holim} \ C' \right). \]

(6) If

\[ \lim_1 E_r^{st} = 0 \]

then \( E_r^{st} \) completely \((\text{in the sense of } [\text{BK}72, \text{IX.5.3}] \text{ or strongly (in the sense of } [\text{Boa99}, \text{Definition 5.2}] \text{ converges to } H_{-s-1}(\lim \text{holim} C). \). The latter means that \( H_n \left( \lim \text{holim} C \right) \simeq \)

\[ \lim_{n,s} \left( \ldots \rightarrow Q^{s,-n-s} \rightarrow Q^{s-1,-n-s+1} \rightarrow \ldots \rightarrow Q^{0,-n} \rightarrow Q^{-1,-n+1} = 0 \right) \]

of epimorphisms with kernels

\[ \ker \left( Q^{s,-n-s} \rightarrow Q^{s-1,-n-s+1} \right) \simeq E_{\infty,-n-s} := \lim_{r} E_{r}^{s,-n-s}. \]

Proof.

(1) The formula for \( E_2^{st} \) follows from Theorem 3.11 [3]. To check naturality, use Theorem 2.6 and the facts that both \( H_{-t} \) and \( \lim^s \) map cofinal morphisms into isomorphisms.

(2) Follows from Theorem 3.11 [3].

(3) Follows from Theorem 3.11 [4].

(4) Follows from Theorem 3.11 [6].

(5) It follows from [3] that

\[ D_2^{s,n}(C) \simeq E_2^{0,n}(C) \simeq E_2^{0,n}(C') \simeq D_2^{0,n}(C'). \]

The morphism \( C \rightarrow C' \) induces morphisms of exact sequences from [3] for \( C \) and \( C' \). Applying the 5-lemma several times, one gets

\[ D_2^{s,t}(C) \simeq D_2^{s,t}(C'). \]

There is a morphism \((\text{which is an isomorphism in the second and the forth term})\) of short exact sequences from [3] for \( r = 2 \). Using again the 5-lemma, one gets the desired isomorphism

\[ H^n \left( \lim \text{holim} C \right) \simeq H^n \left( \lim \text{holim} C' \right). \]

(6) Follows from Theorem 3.11 [3].
4. **Shape homology**

4.1. **Pro-homology.** Let $\text{Pro}(\text{TOP}), \text{Pro}(\text{HTOP}), \text{Pro}(\text{POL})$, and $\text{Pro}(\text{HPOL})$ be the pro-categories from Example 2.7 (2).

**Definition 4.1.** Let $G$ be an abelian group. Given a pro-space $X = (X_i)_{i \in I} \in \text{Pro}(\text{TOP})$, or a pro-homotopy type $X = (X_i)_{i \in I} \in \text{Pro}(\text{HTOP})$, define

$$C_*^{\text{pro}}(X, G) = (C_*(X_i, G))_{i \in I}$$

to be the corresponding chain pro-complex (see Example 2.7 (3)) if $X \in \text{Pro}(\text{TOP})$ or a family of chain complexes (if $X \in \text{Pro}(\text{HTOP})$) where $C_*(X_i, G)$ is the singular chain complex for $X_i$ with coefficients in $G$. Let $H_*(X, G)$ be the corresponding pro-homology group (or a family of abelian groups):

$$H_n(X, G) = (H_n(C_*(X_i, G)))_{i \in I}.$$

For a topological space $X$, let $X \to X = (X_i)_{i \in I}$ be an $\text{HPol}$-expansion ([MS82], §I.2.1), or a strong polyhedral expansion ([Mar00], §7.1.), i.e. $X \in \text{Pro}(\text{HPOL})$ or $X \in \text{Pro}(\text{POL})$. Let finally

$$H_*(X, G) := H_n(X, G).$$

It follows from [MS82], §II.3.1, that $H_*(X, G)$ are well defined abelian pro-groups that do not depend on the choice of an expansion $X$.

4.2. **Strong homology.** We define strong homology as in [Pra01], Definition 3.1.3. The definition is equivalent to that in [Mar00], Chapter 17 and 18, see [Pra01], Theorem 3(a).

**Definition 4.2.** Given a pro-space $X = (X_i)_{i \in I} \in \text{Pro}(\text{TOP})$, let

$$C_*^{\text{pro}}(X, G) = (C_*(X_i, G))_{i \in I} \in \text{Pro}(\text{CHAIN}(\mathbb{Z}))$$

be the corresponding singular chain pro-complex from Definition 4.1. Define

$$\overline{H}_n(X, G) := H_n \left( \text{holim}_{\leftarrow} \left( C_*(X, G) \right) \right)$$

where $\text{holim}$ is the homotopy inverse limit from Definition 3.12. Given a topological space $X$, let $X \to X = (X_i)_{i \in I}$ be a strong polyhedral expansion. The homology of $\text{holim}(C_*(X, G))$ is called the strong homology of $X$ with coefficients in $G$:

$$\overline{H}_n(X, G) := H_n \left( \text{holim}(C_*(X, G)) \right).$$

**Remark 4.3.** Strong homology is strong shape invariant (see [Mar00], Theorem 18.12). Compare with Proposition 4.10 (3).

**Remark 4.4.** $\overline{H}_n$ is defined for all $n \in \mathbb{Z}$ (the negative values of $n$ included).

**Theorem 4.5.** (Spectral sequence for strong homology). Let $X$ be a topological space as an object of the strong shape category $\text{SSh}$. Then there exists a spectral sequence $E^\text{st}_r(X)$, natural on $X \in \text{SSh}$ from $E^\text{st}_2$ on, such that:

1. $E^\text{st}_2 = \lim^s H_{-t}(X, G)$.
2. $D^0_{2,n} \simeq E^\text{st}_{2,n} \simeq \overline{H}_{-n}(X, G)$ where $\overline{H}^*$ is Čech homology.
(3) The groups $D^k_t$ are included in long exact sequences

\[
\begin{align*}
0 & \rightarrow E^{1}_{2} & D^{1}_{2} & \rightarrow D^{0}_{2} \\
& \rightarrow E^{2}_{2} & D^{2}_{2} & \rightarrow D^{1}_{2} \\
& \rightarrow \vdots & \vdots & \vdots \\
& \rightarrow E^{k}_{2} & D^{k}_{2} & \rightarrow D^{k-1}_{2} \\
& \rightarrow \ldots & \rightarrow E^{k+1}_{2} & D^{k+1}_{2} \\
\end{align*}
\]

(4) There are short exact sequences ($r \geq 2, n \in \mathbb{Z}$)

\[
0 \rightarrow \lim_{\leftarrow} D^{r-s}_{\alpha} \rightarrow H_{n}(X, G) \rightarrow \lim_{\leftarrow} D^{r-s}_{\alpha} \rightarrow 0.
\]

(5) If

\[
\lim_{\leftarrow} E^{st}_{r} = 0
\]

then $E^{r+s}_{s}$ completely (in the sense of [BK72, IX.5.3]) or strongly (in the sense of [Boa99], Definition 5.2) converges to $H_{-n}(X, G)$. The latter means that $H_{-n}(X, G)$ converges to $E^{r+s}_{s}$.

Proof. Most statements of this Theorem were proved in [Pra89]. Let $X \rightarrow X' = (X_i)_{i \in I}$ be a strong polyhedral expansion. Apply Theorem 2.16 to the pro-complex $(C_\ast(X_i, G))_{i \in I}$ where $C_\ast(X_i, G)$ is the singular complex for $X_i$ with coefficients in $G$.

To check that the spectral sequence is natural on $E_2$ on, apply Theorem 2.16. It is enough to check that both cofinal morphisms and level equivalences induce an isomorphism on $E_2$.

If $X \rightarrow X'$ is cofinal, then the corresponding morphism $C_\ast(X, G) \rightarrow C_\ast(X', G)$ is isomorphisms in $\text{Pro}(\text{CHAIN}(\mathbb{Z}))$, and induces therefore an isomorphism

\[
E^{r+s}_{s}(X) = \lim_{\leftarrow} H_{-t}(X, G) \rightarrow E^{r+s}_{s}(X') = \lim_{\leftarrow} H_{-t}(X', G).
\]

Finally, if $X = (X_i)_{i \in I} \rightarrow X' = (X'_i)_{i \in I}$ is a level equivalence, the homomorphisms $C_\ast(X_i, G) \rightarrow C_\ast(X'_i, G)$ are weak equivalences. Hence, $H_{-t}(X, G) \rightarrow H_{-t}(X', G)$ and $E^{r+s}_{s}(X) \rightarrow E^{r+s}_{s}(X')$ are isomorphisms, as desired. \hfill \square

4.3. Balanced homology.

**Definition 4.6.** Given a pro-space $X = (X_i)_{i \in I} \in \text{Pro}(\text{TOP})$, let $C^b_\ast(X, G)$ be the tensor product in the sense of Theorem 2.14

\[
C^b_\ast(X, G) := C_\ast(X, \mathbb{Z}) \otimes_{\mathbb{Z}} G \in \text{Pro}(\text{CHAIN}(\mathbb{Z})).
\]

where $C_\ast(X, \mathbb{Z})$ is the pro-complex from Definition 4.1. $C^b_\ast(X, G)$ can be represented by an inverse system

\[
C^b_\ast(X, G) = \left( (Y_j)_{j \in J} \right)_{j \in J}.
\]

Let $H^b_n(X, G)$ be the corresponding pro-homology group:

\[
H^b_n(X, G) := (H_n(Y_j))_{j \in J}.
\]

Given a topological space $X$, let $X \rightarrow X = (X_i)_{i \in I}$ be a strong polyhedral expansion. Define

\[
H^b_n(X, G) := H^b_n(X, G).
\]
Remark 4.7. Strictly speaking, the items $Y_n$ above are abelian pro-groups that are represented by different index categories $J_n$: 

$$C^b_n(X, G) = \left( (Y_n)_j \right)_{j \in J_n}.$$ 

However, Remark 2.18 guarantees that the index categories $J_n$ can be chosen to be equal, and the differentials to be level morphisms (Definition 2.2).

Definition 4.8. Given a pro-space $X = (X_i)_{i \in I} \in \text{Pro} (\text{TOP})$, define 

$$H^b_n (X, G) := H_n \left( \text{holim} \left( C^b_n (X, G) \right) \right)$$

where the complex in the brackets is the homotopy inverse limit from Definition 3.12. Given a topological space $X$, let $X \rightarrow X = (X_i)_{i \in I}$ be a strong polyhedral expansion. Define 

$$\overline{H}^b_n (X, G) := \overline{H}_n (X, G).$$

Remark 4.9. Due to Theorem 2.20, there are natural (on $X$ and $G$) morphisms 

$$C^b_n (X, G) = C_\ast (X, Z) \otimes Z G \rightarrow C_\ast (X, Z) \tilde\otimes Z G = C_\ast (X, G)$$

of pro-complexes inducing the morphisms $H^b_n (X, G) \rightarrow H_n (X, G)$ and $\overline{H}^b_n (X, G) \rightarrow \overline{H}_n (X, G)$.

Let us remind that $\text{FAB}$ is the class of finitely generated abelian groups.

Proposition 4.10.

1. Balanced pro-homology is strong shape invariant.
2. The mappings $H^b_n (X, G) \rightarrow H_n (X, G)$ are isomorphisms if $G \in \text{FAB}$.
3. Balanced strong homology is strong shape invariant.
4. The mappings $\overline{H}^b_n (X, G) \rightarrow \overline{H}_n (X, G)$ are isomorphisms if $G \in \text{FAB}$.

Proof.

1. The functor 

$$X \mapsto C_\ast (X, Z) :? \rightarrow \text{Pro} (\text{CHAIN} (Z))$$

is evidently defined on the category $\text{Inv} (\text{TOP})$ (see Definition 2.1), therefore also on the category $\text{Inv} (\text{POL})$. Let $G \in \text{Mod} (Z)$. Applying the tensor product from Theorem 2.17 one gets a functor 

$$X \mapsto C^b_n (X, G) := C_\ast (X, Z) \otimes Z G : \text{Inv} (\text{POL}) \rightarrow \text{Pro} (\text{CHAIN} (Z))$$

and functors 

$$X \mapsto H^b_n (X, G) := H_n (C^b_n (X, G)) : \text{Inv} (\text{POL}) \rightarrow \text{Pro} (Z).$$

In order to use Theorem 2.16 let us check that special morphisms are mapped into isomorphisms. Due to Proposition 2.15 it is enough to check this for cofinal morphisms and for level equivalences. If $f : X \rightarrow Y$ is a cofinal morphism in $\text{Inv} (\text{TOP})$ then 

$$C_\ast (f, Z) : C_\ast (X, Z) \rightarrow C_\ast (Y, Z)$$

is a cofinal morphism in $\text{Inv} (\text{CHAIN} (Z))$, inducing therefore an isomorphism in $\text{Pro} (\text{CHAIN} (Z))$. It follows that 

$$C^b_n (f, G) : C^b_n (X, G) \rightarrow C^b_n (Y, G)$$
and

\[ H^*_n (f, G) : H^*_n (X, G) \rightarrow H^*_n (Y, G) \]

are isomorphisms as well (in the categories Pro(CHAIN(Z)) and Pro(Z) respectively). Let now

\[ f = (f_i)_{i \in I} : X = (X_i)_{i \in I} \rightarrow Y = (Y_i)_{i \in I} \]

be a level equivalence. Let \( n \in \mathbb{Z} \). It follows that both \( H_n (X, Z) \rightarrow H_n (Y, Z) \) and \( H_{n-1} (X, Z) \rightarrow H_{n-1} (Y, Z) \) are isomorphisms in Pro(Z).

Hence, both

\[ H_n (X, Z) \otimes_Z G \rightarrow H_n (Y, Z) \otimes_Z G \]

and

\[ Tor^2_1 (H_{n-1} (X, Z), G) \rightarrow Tor^2_1 (H_{n-1} (Y, Z), G) \]

are isomorphisms. The singular chain complexes \( C_\ast (X, Z) \) and \( C_\ast (Y, Z) \) consist of free abelian groups, therefore the pro-complexes \( C_\ast (X, Z) \) and \( C_\ast (Y, Z) \) consist of quasi-projective \( Z \)-modules (Definition 2.21 and Proposition 2.22), and one can apply Theorem 2.37: there exists a morphism of short exact sequences

\[
\begin{array}{c}
0 \rightarrow H_n (X, Z) \otimes_Z G \rightarrow H_n^b (X, G) \rightarrow Tor^2_1 (H_{n-1} (X, Z), G) \rightarrow 0 \\
0 \rightarrow H_n (Y, Z) \otimes_Z G \rightarrow H_n^b (Y, G) \rightarrow Tor^2_1 (H_{n-1} (Y, Z), G) \rightarrow 0
\end{array}
\]

The 5-lemma gives the desired isomorphism \( H^*_n (f, G) : H^*_n (X, G) \rightarrow H^*_n (Y, G) \).

Hence, the functors \( H^*_n (?, G) \) maps special morphisms into isomorphisms. Theorem 2.16 implies that one has well-defined functors \( H^*_n (?, G) : SSh \rightarrow Mod(Z) \).

(2) Follows from Theorem 2.20 because \( Z \) is noetherian, therefore any finitely generated \( Z \)-module is finitely presented.

(3) The functors

\[
\begin{aligned}
X &\mapsto C^b_\ast (X, G) := \holim C^b_\ast (X, G) : \text{Inv(POL)} \rightarrow \text{CHAIN(Z)}, \\
X &\mapsto \overline{H}^b_n (X, G) := H_n \left( \overline{C}^b_\ast (X, G) \right) : \text{Inv(POL)} \rightarrow \text{Mod(Z)},
\end{aligned}
\]

are well-defined. In order to extend the definitions to the strong shape category \( SSh \), it is enough, due to Theorem 2.16 and Proposition 2.15, to check whether cofinal morphisms and level equivalences are mapped into isomorphisms. If \( f : X \rightarrow Y \) is cofinal, it follows from Theorem 2.11 (3), that

\[
\overline{C}^b_\ast (X, G) = \holim C^b_\ast (X, G) \rightarrow \holim C^b_\ast (Y, G) = \overline{C}^b_\ast (Y, G)
\]

is a weak equivalence of complexes, hence

\[
\overline{H}^b_n (X, G) = H_n \left( \holim C^b_\ast (X, G) \right) \rightarrow H_n \left( \holim C^b_\ast (Y, G) \right) = \overline{H}^b_n (Y, G)
\]

is an isomorphism. Let now

\[ f = (f_i)_{i \in I} : X = (X_i)_{i \in I} \rightarrow Y = (Y_i)_{i \in I} \]
be a level equivalence. It follows from (11) that
\[ E_{2}^{st} (C^{b}_{s} (X, G)) = \lim_{t} H_{t} (C^{b}_{s} (X, G)) \] is an isomorphism for all \( s, t \in \mathbb{Z} \). Using Theorem 3.11 (11), one concludes that
\[ \Pi^{b}_{s} (X, G) = H_{n} \left( \holim_{t} C^{b}_{s} (X, G) \right) \] is an isomorphism. Finally, Theorem 2.16 guarantees that \( \Pi^{b}_{n} (? , G) \) are well-defined functors from \( \text{SSh} \) to \( \text{Mod} (\mathbb{Z}) \).

(4) Follows from Theorem 2.20.

\[ \square \]

5. Proof of the main results

5.1. Proof of Proposition 0.1

Proof. For each \( a \in G \) and for each of the four theories \( h_{s} \) we will define morphisms
\[ \pi_{s} : h_{s} (X, \mathbb{Z}) \longrightarrow h_{s} (X, G) \]
which satisfy the condition \( \left( a + b \right)_{s} = \bar{a}_{s} + \bar{b}_{s} \), \( a, b \in G \). Let \( X \rightarrow X = (X_{i})_{i \in I} \) be a polyhedral expansion, and let \( (C_{s} (X_{i}, \mathbb{Z}))_{i \in I} \) and \( (C_{s} (X_{i}, G))_{i \in I} \) be the corresponding pro-complexes

(1) Pro-homology. The mappings \( c \mapsto c \otimes a \) define morphisms of pro-complexes
\[ \pi_{s} : (C_{s} (X_{i}, \mathbb{Z}))_{i \in I} \longrightarrow (C_{s} (X_{i}, G))_{i \in I} \]
and morphisms of their pro-homology groups
\[ \pi_{s} : (H_{s} (X_{i}, \mathbb{Z}))_{i \in I} \longrightarrow (H_{s} (X_{i}, G))_{i \in I} . \]
Clearly \( \left( \bar{a} + \bar{b} \right)_{s} = \overline{a}_{s} + \overline{b}_{s} \), and we get a homomorphism of abelian groups
\[ a \mapsto \pi_{s} : G \longrightarrow \text{Hom}_{\text{Pro} (\mathbb{Z})} (H_{s} (X, \mathbb{Z}), H_{s} (X, G)) . \]
Theorem 2.17 gives the desired homomorphism
\[ H_{s} (X, \mathbb{Z}) \otimes_{\mathbb{Z}} G \longrightarrow H_{s} (X, G) . \]

(2) Strong homology. The mappings
\[ \pi_{s} : (C_{s} (X_{i}, \mathbb{Z}))_{i \in I} \longrightarrow (C_{s} (X_{i}, G))_{i \in I} \]
from (1) define morphisms of the homotopy inverse limits
\[ \pi_{s} : \holim (C_{s} (X_{i}, \mathbb{Z}))_{i \in I} \longrightarrow \holim (C_{s} (X_{i}, G))_{i \in I} \]
and their homologies
\[ \pi_{s} : \Pi^{s}_{s} (X, \mathbb{Z}) \longrightarrow \Pi^{s}_{s} (X, G) . \]
Clearly \( \left( \bar{a} + \bar{b} \right)_{s} = \overline{a}_{s} + \overline{b}_{s} \), and we get a homomorphism of abelian groups
\[ a \mapsto \pi_{s} : G \longrightarrow \text{Hom}_{\text{Mod} (\mathbb{Z})} (\Pi^{s}_{s} (X, \mathbb{Z}), \Pi^{s}_{s} (X, G)) . \]
The usual properties of the tensor product give the desired homomorphism
\[ \Pi^{s}_{s} (X, \mathbb{Z}) \otimes_{\mathbb{Z}} G \longrightarrow \Pi^{s}_{s} (X, G) . \]
(3) **Balanced pro-homology.** Due to Theorem 2.17 there is a natural isomorphism of abelian groups  

\[
\text{Hom}_{\text{Mod}(\mathbb{Z})} \left( G, \text{Hom}_{\text{Pro}(\mathbb{Z})} \left( C_*(X, \mathbb{Z}), C^b_*(X, G) \right) \right) \cong \\
\text{Hom}_{\text{Pro}(\mathbb{Z})} \left( C_*(X, \mathbb{Z}) \otimes \mathbb{Z} G, C^b_*(X, G) \right).
\]

Since \( C_*(X, \mathbb{Z}) \otimes \mathbb{Z} G = C^b_*(X, G) \) by definition, let  

\[ \varphi \in \left( G, \text{Hom}_{\text{Pro}(\mathbb{Z})} \left( C_*(X, \mathbb{Z}), C^b_*(X, G) \right) \right) \]

be the morphism corresponding to the identity morphism \( C_*(X, \mathbb{Z}) \otimes \mathbb{Z} G \to C^b_*(X, G) \) under the isomorphism above. For \( a \in G \), let  

\[ \overline{\varphi}(a) : H_*(X, \mathbb{Z}) \to H^b_*(X, G). \]

be the induced mapping of pro-homologies. The correspondence \( a \mapsto \overline{\varphi}(a) \) defines a homomorphism  

\[ G \to \text{Hom}_{\text{Pro}(\mathbb{Z})} \left( H_*(X, \mathbb{Z}), H^b_*(X, G) \right). \]

Applying again Theorem 2.17 one gets the desired morphism of abelian pro-groups  

\[ H_*(X, \mathbb{Z}) \otimes \mathbb{Z} G \to H^b_*(X, G). \]

(4) **Balanced strong homology.** The morphisms  

\[ \varphi(a) \in \text{Hom}_{\text{Pro}(\mathbb{Z})} \left( C_*(X, \mathbb{Z}), C^b_*(X, G) \right) \]

from (3) define morphisms  

\[ \text{holim} \varphi(a) : \text{holim} C_*(X, \mathbb{Z}) \to \text{holim} C^b_*(X, G) \]

and the corresponding morphisms of homologies  

\[ \overline{\pi}_* : \overline{H}_*(X, \mathbb{Z}) = \text{holim} H_*(X, \mathbb{Z}) \to \text{holim} H^b_*(X, G). \]

Clearly \( \overline{a+b} = \overline{a} + \overline{b} \), and one gets a homomorphism of abelian groups  

\[ a \mapsto \overline{\pi}_* : G \to \text{Hom}_{\text{Mod}(\mathbb{Z})} \left( \overline{H}_*(X, \mathbb{Z}), \overline{H}_*(X, G) \right). \]

The usual properties of the tensor product give the desired homomorphism  

\[ \overline{H}^b_*(X, \mathbb{Z}) \otimes \mathbb{Z} G \to \overline{H}^b_*(X, G). \]

\[ \square \]

5.2. **Proof of Theorem 1.1**

**Proof.** Since for \( G \in \text{FAB} \), the natural morphisms \( H^b_n(X, G) \to H_n(X, G) \) and \( \overline{H}^b_n(X, G) \to \overline{H}_n(X, G) \) are isomorphisms (Theorem 1.10 (1)), it is enough to prove our statement for the two non-balanced homology theories. Let \( X \to \mathbf{X} \) be a strong polyhedral expansion. Consider first the case \( G = \mathbb{Z}^n \) is a free finitely generated abelian group. Clearly, \( \text{Tor}^2(X, G) = 0 \) and \( \text{Tor}^2(\overline{H}_n(X), G) = 0 \), and it is enough to notice that both  

\[ H_n(X, \mathbb{Z}) \otimes \mathbb{Z} G = \bigoplus H_n(X, \mathbb{Z}) \to H_n(X, G) \]

and  

\[ \overline{H}_n(X, \mathbb{Z}) \otimes \mathbb{Z} G = \bigoplus \overline{H}_n(X, \mathbb{Z}) \to \overline{H}_n(X, G) \]

are isomorphisms.
Let now 
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be a resolution of $G$ where $F_i$ are free finitely generated abelian groups. There is a short exact sequence of pro-complexes and level morphisms

$$0 \to C_\ast (X, F_1) \to C_\ast (X, F_0) \to C_\ast (X, G) \to 0$$

inducing a short exact sequence of homotopy inverse limits

$$0 \to \underline{\text{holim}} C_\ast (X, F_1) \to \underline{\text{holim}} C_\ast (X, F_0) \to \underline{\text{holim}} C_\ast (X, G) \to 0.$$  

The first sequence gives rise to long exact sequence of pro-homologies

$$\ldots \to H_n (X, F_1) \to H_n (X, F_0) \to H_n (X, G) \to H_{n-1} (X, F_1) \to H_{n-1} (X, F_0) \to \ldots$$

while the second one induces a long exact sequence of strong homologies

$$\ldots \to \overline{H}_n (X, F_1) \to \overline{H}_n (X, F_0) \to \overline{H}_n (X, G) \to \overline{H}_{n-1} (X, F_1) \to \overline{H}_{n-1} (X, F_0) \to \ldots$$

Consider a morphism of exact sequences

$$
\begin{array}{ccc}
H_n (X, F_1) & \to & H_n (X, F_0) \\
\alpha & & \beta \\
H_n (X, Z) \otimes \mathbb{Z} F_1 & \to & H_n (X, Z) \otimes \mathbb{Z} F_0 \\
\gamma & & \gamma \\
H_n (X, Z) \otimes \mathbb{Z} G & \to & 0
\end{array}
$$

Since $\alpha$ and $\beta$ are isomorphisms, $\gamma$ is a monomorphism with the cokernel equal to

$$\ker (H_{n-1} (X, F_1) \to H_{n-1} (X, F_0)) \simeq \ker (H_{n-1} (X, Z) \otimes \mathbb{Z} F_1 \to H_{n-1} (X, Z) \otimes \mathbb{Z} F_0) \simeq \text{Tor}_1 (H_{n-1} (X, Z), G).$$

Similarly, $\overline{H}_n (X, Z) \otimes \mathbb{Z} G \to \overline{H}_n (X, G)$ is a monomorphism with the cokernel equal to

$$\ker (\overline{H}_{n-1} (X, F_1) \to \overline{H}_{n-1} (X, F_0)) \simeq \ker (\overline{H}_{n-1} (X, Z) \otimes \mathbb{Z} F_1 \to \overline{H}_{n-1} (X, Z) \otimes \mathbb{Z} F_0) \simeq \text{Tor}_1 (\overline{H}_{n-1} (X, Z), G).$$

\[\square\]

5.3. **Proof of Theorem 1.2**

*Proof.* Let $X \to X$ be a strong polyhedral expansion. Apply Theorem 2.37 to the pro-complex $C_\ast (X, Z)$. \[\square\]

5.4. **Proof of Theorem 1.3**

**Definition 5.1.** For an arbitrary abelian group $H$, let $P (H), P^\omega (H) \in \text{Pro} (\mathbb{Z})$ be the following pro-groups:

$$P (H) = (H \leftarrow H \oplus H \leftarrow H \oplus H \leftarrow \ldots),$$

$$P^\omega (H) = \bigoplus_{0}^{\infty} P (H).$$
Proof. Let $G$ be a free countably generated abelian group. Since $\mathbf{Tor}_1^\mathbb{Z}(H_{n-1}(X,\mathbb{Z}),G) = 0$, it is enough to construct an example $X$ such that $H_n(X,\mathbb{Z}) \otimes \mathbb{Z} G \to H_n(X,G)$ is not an isomorphism. Let

$$X_k = \bigvee_{0}^{\infty} S^k$$

be the $k$-dimensional Hawaiian ear-ring, i.e. a compact wedge (or the cluster) of countably many $k$-spheres. Assume for simplicity that $k \neq 0$. $X_k$ can be described as a subspace of $\mathbb{R}^{k+1}$:

$$X_k = \bigcup_{i=0}^{\infty} S_k^{(i)}$$

where $S_k^{(i)}$ is the sphere of radius $\frac{1}{i+1}$ with the center in $\left(\frac{1}{i+1}, 0, \ldots, 0\right) \in \mathbb{R}^{k+1}$. It follows from [Pra05], Section 4.2, p. 505) that (see Definition 5.1)

$\mathbf{H}_n(X_k,\mathbb{Z}) = \left\{ \begin{array}{ll} \mathbb{P}(\mathbb{Z}) & \text{if } n = k, \\ 0 & \text{if } n \neq k, 0, \\ \mathbb{Z} & \text{if } n = 0. \end{array} \right.$

$\mathbf{H}_k(X_k,\mathbb{Z}) \otimes \mathbb{Z} G = \left\{ \begin{array}{ll} \mathbb{P}^{\infty}(\mathbb{Z}) & \text{if } n = k, \\ 0 & \text{if } n \neq k, 0, \\ G & \text{if } n = 0. \end{array} \right.$

$\mathbf{H}_k(X_k,G) = \left\{ \begin{array}{ll} \mathbb{P}(G) & \text{if } n = k, \\ 0 & \text{if } n \neq k, 0, \\ G & \text{if } n = 0. \end{array} \right.$

Assume on the contrary that $\mathbf{H}_k(X_k,\mathbb{Z}) \otimes \mathbb{Z} G \to \mathbf{H}_k(X_k,G)$ is an isomorphism. It would follow that

$\text{Hom}_{\mathbf{Pro}(\mathbb{Z})} (\mathbf{H}_k(X_k,G),B) \to \text{Hom}_{\mathbf{Pro}(\mathbb{Z})} (\mathbf{H}_k(X_k,\mathbb{Z}) \otimes \mathbb{Z} G, B)$

is an isomorphism for any abelian group $B$. Direct computation shows that

$\text{Hom}_{\mathbf{Pro}(\mathbb{Z})} (\mathbf{H}_k(X_k,G),B) = \lim \left( \prod_{0}^{\infty} B \to \prod_{0}^{\infty} B \times B \to \prod_{0}^{\infty} B \times B \times B \to \ldots \right)$,

$\text{Hom}_{\mathbf{Pro}(\mathbb{Z})} (\mathbf{H}_k(X_k,\mathbb{Z}) \otimes \mathbb{Z} G, B) = \lim \prod_{0}^{\infty} (B \to B \times B \to B \times B \times B \to \ldots)$.

Notice direct (instead of inverse) limits. Elements of $\prod_{0}^{\infty} B^{s+1}$ can be represented by infinite $s \times \infty$ matrices $(b_{ij})_{0 \leq i \leq s,j \geq 0}$, or, equivalently, by $\infty \times \infty$ matrices $(b_{ij})_{i \geq 0,j \geq 0}$ with $b_{ij} = 0$ whenever $i > s$. It follows that

$\lim \left( \prod_{0}^{\infty} B \to \prod_{0}^{\infty} B \times B \to \prod_{0}^{\infty} B \times B \times B \to \ldots \right)$

can be represented by $\infty \times \infty$ matrices $(b_{ij})_{i \geq 0,j \geq 0}$ matrices with $b_{ij} = 0$ whenever $i > s$ for some $s$. Similarly, elements of

$\lim (B \to B \times B \to B \times B \times B \to \ldots)$
can be represented by $1 \times \infty$ matrices $(b_j)_{j \geq 0}$ where $b_j = 0$ if $j > s$ for some $s$. Finally, elements of

$$\prod_0^\infty \lim \rightarrow (B \rightarrow B \times B \rightarrow B \times B \times B \rightarrow ...)$$

can be represented by $\infty \times \infty$ matrices $(b_{ij})$ with $b_{ij} = 0$ whenever $i > s_j$ for some $s_j$ (depending on $j$). Let $B$ be any non-trivial abelian group, let $c \in B$, $c \neq 0$. Define

$$b_{ij} = \begin{cases} 0 & \text{if } i > j, \\ c & \text{if } i \leq j. \end{cases}$$

Clearly

$$(b_{ij}) \in \prod_0^\infty \lim \rightarrow (B \rightarrow B \times B \rightarrow B \times B \times B \rightarrow ...),$$

but $(b_{ij})$ does not lie in the image of

$$\lim \rightarrow \left( \prod_0^\infty B \rightarrow \prod_0^\infty B \times B \rightarrow \prod_0^\infty B \times B \times B \rightarrow ... \right).$$

The pairing $H_k(X_k, Z) \otimes \mathbb{Z} G \rightarrow H_k(X_k, G)$ is not an isomorphism. Contradiction. $\square$

5.5. **Proof of Theorem 1.4**

**Proof.** Let $X_k$, $k \geq 1$, and $G$, be the same as in Section 5.4. It follows from ([Pra05], Section 4.2, p. 505) that

$$\prod_0^\infty F \simeq \prod_0^\infty Z,$$

$$\prod_0^\infty \prod_0^\infty F \simeq \prod_0^\infty \prod_0^\infty Z,$$

$$\prod_0^\infty \prod_0^\infty F \simeq \prod_0^\infty \prod_0^\infty Z.$$}

Let us repeat the proof here. Let $F$ be any abelian group. We will calculate $\prod_0^\infty F$ using the spectral sequence from Theorem 4.5 (5), and the calculations from Section 5.4. Clearly, since $\mathbf{P}(F)$ (see Definition 5.1) is a tower of epimorphisms,

$$E_2^{st}(\mathbf{C}_s(X, F)) = \begin{cases} \lim \rightarrow \mathbf{P}(F) \simeq \prod_0^\infty F & \text{if } s = 0, t = -k, \\ \lim \rightarrow \mathbf{P}(F) = 0 & \text{if } s = 1, t = -k, \\ 0 & \text{otherwise}. \end{cases}$$

The spectral sequence degenerates, implying $\lim \rightarrow E_r^{st} = 0$. It follows that

$$\prod_0^\infty F \simeq E_\infty^{0,-k} \simeq E_2^{0,-k} \simeq \prod_0^\infty F.$$
Moreover, $\text{Tor}_k^G(\mathcal{H}_{k-1}(X_k, \mathbb{Z}), G) = 0$ because $G$ is a free abelian group. The elements of $\mathcal{H}_k(X_k, \mathbb{Z}) \otimes G$ are represented by $\infty \times \infty$ matrices $(b_{ij})$ where

$$b_{ij} = \begin{cases} 0 & \text{if } i > j, \\ 1 & \text{if } i \leq j. \end{cases}$$

Clearly $(b_{ij}) \in \mathcal{H}_k(X_k, \mathbb{Z})$, but $(b_{ij})$ does not lie in the image of $\mathcal{H}_k(X_k, \mathbb{Z}) \otimes \mathbb{Z}G \to \mathcal{H}_k(X_k, G)$ is not an isomorphism. Contradiction.

5.6. Proof of Theorem 1.5. This is the most complicated argument.

5.6.1. The counter-example depending on the Continuum Hypothesis.

Proof. Take again $G$ a countably generated abelian group. The first counter-example will be the same $X_k$ (the cluster of $k$-spheres) as above. However, the statement that $\mathcal{H}_{k-1}(X_k, \mathbb{Z}) \otimes \mathbb{Z}G = \mathcal{H}_{k-1}(X_k, G)$ is not an isomorphism, would depend on the Continuum Hypothesis (in fact, on a weaker assumption $d = \aleph_1$). A counter-example which is independent on the Continuum Hypothesis, will be much more complicated (see Section 5.6.2). Assume for simplicity that $k \geq 2$. Let $X_k \to X$ be a strong expansion. Consider the spectral sequence from Theorem 3.15 (6) for the pro-complex

$$C_*^b(X, G) = \bigoplus C_*^b(X, \mathbb{Z}) \otimes \mathbb{Z}G = \bigoplus_{0}^{\infty} C_*^b(X, \mathbb{Z}).$$

Reasoning exactly as in ([Pra05], Theorem 4.5 and Proposition 4.10), one calculates the $E_2$ terms (see Definition 5.1):

$$E_2^{st}(C_*^b(X, G)) = \begin{cases} \lim_{\leftarrow}^s \mathbb{P}^w(\mathbb{Z}) = \bigoplus_{0}^{\infty} \mathbb{Z} & \text{if } s = 0, t = -k, \\ \lim_{\leftarrow}^s \mathbb{P}^w(\mathbb{Z}) = \bigoplus_{0}^{\infty} \mathbb{Z} & \text{if } s > 0, t = -k, \\ \bigoplus_{0}^{\infty} \mathbb{Z} & \text{if } s = t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The spectral sequence degenerates, implying $\lim_{\leftarrow}^1 E_2^{st} = 0$, and it follows that

$$\mathcal{H}_{k-1}^b(X, G) \simeq \lim_{\leftarrow}^1 \mathbb{P}^w(\mathbb{Z}).$$

In [vDS84], §3, the two cardinals were defined: $b$ (bounding number) and $d$ (dominating number). It is known that $\aleph_1 \leq b \leq d \leq c$ where $c$ is the cardinality of continuum. Moreover (see [vDS84], §5), for any integers $1 \leq p \leq l \leq m$ the statement

$$(b = \aleph_p) \& (d = \aleph_l) \& (c = \aleph_m)$$

is consistent with ZFC (Zermelo–Fraenkel axioms plus the Axiom of Choice). Assume now $d = \aleph_1$ (this assumption is weaker than the Continuum Hypothesis.
c = \aleph_1). It is proved in (Pra05), Theorem 4, that under this assumption the cardinality of \( \lim^1 \mathbf{P}^\omega (\mathbb{Z}) \) is very large (equal to \((\aleph_0)^{\aleph_1}\)). Since

\[
\mathcal{H}_{k-1}^b (X,Z) = \mathcal{H}_{k-1} (X,Z) = 0,
\]

it follows that

\[
0 = \mathcal{H}_{k-1}^b (X_k,Z) \otimes \mathbb{Z} G \rightarrow \mathcal{H}_{k-1}^b (X_k,G) = \lim^1 \mathbf{P}^\omega (\mathbb{Z}) \neq 0
\]
is not an isomorphism. Contradiction. □

Remark 5.2. In [DSV89] it is proved that under PFA (the Proper Forcing Axiom) \( \lim^1 \mathbf{P}^\omega (\mathbb{Z}) = 0 \). Therefore, the statement “\( \mathcal{H}_{k-1}^b (X_k,Z) \otimes \mathbb{Z} G \rightarrow \mathcal{H}_{k-1}^b (X_k,G) \)” does not depend on ZFC.

5.6.2. The “absolute” counter-example. Let \( \omega_1 \) be the first uncountable ordinal.

Definition 5.3. (compare with Pra05, Section 6) Let \( C \) be an abelian group. Define \( I (\omega_1) \) to be the category with the elements \( \alpha < \omega_1 \) as objects, and inequalities \( \beta < \alpha \) as morphisms \( \alpha \rightarrow \beta \). Notice the inverse order in which \( \alpha \) and \( \beta \) appear in the morphisms. Consider the following two abelian pro-groups:

\[
\mathbf{A}_k (C) = \left( A_{\alpha} (C) \right)_{\alpha \in I (\omega_1)},
\]

\[
\mathbf{A}^\omega (C) = \bigoplus_{0} A (C),
\]

where

\[
A_{\alpha} (C) = \bigoplus_{\gamma \in [\alpha, \omega_1)} C,
\]

and for \( \beta < \alpha \) the morphisms

\[
(\alpha \rightarrow \beta)_*: A_{\alpha} (C) = \bigoplus_{\gamma \in [\alpha, \omega_1)} C \rightarrow A_{\beta} (C) = \bigoplus_{\gamma \in [\beta, \omega_1)} C
\]

are induced by the inclusions \([\alpha, \omega_1) \subseteq [\beta, \omega_1)\).

Proof. Let \( m \geq 1 \) be an integer. Denote by \( X_m \) the space \( X (m,0,\omega_1) \) from [Mar96]. As a set, \( X_m \) is a wedge

\[
\bigvee_{\alpha < \omega_1} B^m
\]

of \( m \)-dimensional balls equipped with a special paracompact topology. In [Mar96, Theorem 3 and 6], a polyhedral resolution \( X_m \rightarrow X \) was constructed, and the pro-homology of \( X_m \) was calculated. Namely,

\[
\mathbf{H}_n (X_m, \mathbb{Z}) = \mathbf{H}_n (C_*(X_m, \mathbb{Z})) = \left\{\begin{array}{cc}
\mathbb{Z} & \text{if } n = 0, \\
A (\mathbb{Z}) & \text{if } n = m \\
0 & \text{otherwise.}
\end{array}\right.
\]

Let us now proceed similarly to [Pra05], Proposition 7.3. Let \( G \) be, as in Section 5.6.1, a countably generated free abelian group. Assume for simplicity that \( m > 2 \). In the two spectral sequences from Theorem 3.15 (6) for the pro-complexes
$C_\ast(X_m, \mathbb{Z})$ and $C_b^\ast(X_m, \mathbb{Z}) = C_\ast(X_m, \mathbb{Z}) \otimes \mathbb{Z}G$ one obtains, using [Pra05], Proposition 6.1:

$$E_2^{st}(C_\ast(X_m, \mathbb{Z})) = \begin{cases} \mathbb{Z} & \text{if } s = t = 0, \\ \lim_{\leftarrow}^2 A(\mathbb{Z}) & \text{if } s = 2, t = -m, \\ 0 & \text{otherwise.} \end{cases}$$

$$E_2^{st}(C_b^\ast(X_m, G)) = \begin{cases} \bigoplus \mathbb{Z} & \text{if } s = t = 0, \\ \lim_{\leftarrow}^2 A^\omega(\mathbb{Z}) & \text{if } s = 2, t = -m, \\ 0 & \text{otherwise.} \end{cases}$$

Both spectral sequences degenerate, implying $\lim_{\leftarrow}^1 E_2^{st} = 0$. It follows from Theorem 3.15 (6) that

$$\mathcal{T}_n^s(X_m, \mathbb{Z}) = \mathcal{T}_n^s(X_m, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \lim_{\leftarrow}^2 A(\mathbb{Z}) & \text{if } n = m - 2, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{T}_n^b(X_m, G) = \begin{cases} \bigoplus \mathbb{Z} & \text{if } n = 0, \\ \lim_{\leftarrow}^2 A^\omega(\mathbb{Z}) & \text{if } n = m - 2, \\ 0 & \text{otherwise.} \end{cases}$$

The pairing

$$\mathcal{T}_{m-2}^t(X_m, \mathbb{Z}) \otimes \mathbb{Z}G = \bigoplus_{\mathbb{Z}} \lim_{\leftarrow}^2 A(\mathbb{Z}) \to \mathcal{T}_{m-2}^b(X_m, G) = \lim_{\leftarrow}^2 A^\omega(\mathbb{Z})$$

is not an isomorphism, due to [Pra05], Proposition 6.1 and Corollary 6.5. Contradiction. □

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