A CHARACTERIZATION OF GVZ GROUPS IN TERMS OF FULLY RAMIFIED CHARACTERS

SHAWN T. BURKETT AND MARK L. LEWIS

Abstract. In this paper, we obtain a characterization of GVZ-groups in terms of commutators and monolithic quotients. This characterization is based on counting formulas due to Gallagher.

Throughout this paper, all groups are finite. For a group $G$, we write $\text{Irr}(G)$ for the set of irreducible characters of $G$. In this paper, we present a new characterization of GVZ-groups. A group $G$ is a GVZ-group if every irreducible character $\chi \in \text{Irr}(G)$ satisfies that $\chi$ vanishes on $G \setminus Z(\chi)$.

The term GVZ-group was introduced by Nenciu in [12]. Nenciu continued the study of GVZ-groups in [13] and the second author further continued these studies in [10]. In our paper [2], we showed that GVZ-groups can be characterized in terms of another class of groups that have appeared in the literature.

An element $g \in G$ is called flat if the conjugacy class of $G$ is $g[|g,G|]$. In [14], they defined a group $G$ to be flat if every element in $G$ is flat. In fact, groups satisfying this condition had been studied even earlier. Predating each of these references, Murai [11] referred to such groups as groups of Ono type. In [14], they proved that if $G$ is nilpotent and flat, then $G$ is a GVZ-group. Improving this result, we prove in [2] that a group $G$ is a GVZ-group if and only if it is flat.

In this paper, we characterize GVZ-groups using fully ramified characters. For a normal subgroup $N$ of $G$, we say that the character $\chi \in \text{Irr}(G)$ is fully ramified over $N$ if $\chi_N$ is homogeneous and $\chi(g) = 0$ for every element $g \in G \setminus N$.

Following the literature, a group $G$ is called central type if there is an irreducible character of $G$ that is fully ramified over the center $Z(G)$. Results about central type groups are in [4], [5], [6], and [7].

With this as motivation, we define an irreducible character $\chi$ of $G$ to be central type if $\chi$, considered as a character of $G/\ker(\chi)$, is fully ramified over $Z(G/\ker(\chi))$. (I.e., $G/\ker(\chi)$ is a group of central type with faithful character $\chi$.) It is not difficult to see that $G$ is a GVZ-group if and only if every character $\chi \in \text{Irr}(G)$ is of central type.

Recall from the literature that a group is called monolithic if it has a unique minimal normal subgroup. It is easy to see that if $N$ is a normal subgroup of $G$ and $G/N$ is monolithic, then $N$ appears as the kernel of some irreducible character of $G$. Also an irreducible character $\chi$ is called monolithic if the quotient group $G/\ker(\chi)$ is monolithic. Thus, monolithic quotients correspond to monolithic characters.

The purpose of this paper is to give a new characterization of central type characters based on ideas of Gallagher that are encapsulated in [9, Theorem 1.19 and
Lemma 1.20], thereby obtaining a new characterizations of GVZ-groups. In particular, we prove the following theorem.

**Theorem 1.** Let $G$ be a nonabelian group. Then the following are equivalent:

1. $G$ is a GVZ-group.
2. For every monolithic character $\chi \in \text{Irr}(G)$ and for every element $g \in G \setminus Z(\chi)$, there exists an element $x \in G$ so that $[g, x] \in Z(\chi) \setminus \ker(\chi)$.
3. $G$ is nilpotent, and for every normal subgroup $N$ of $G$ for which $G/N$ is monolithic and for every element $g \in G$ satisfying $[g, G] \not\leq N$, there exists an element $x \in G$ such that $[g, x] \not\in N$ and $[[g, x], G] \leq N$.

Our proof relies on the following lemma, which we will see is an immediate consequence of some arguments of Gallagher that can be found in [9, Theorem 1.19 and Lemma 1.20]. For an element $g \in G$, we set $D_G(g) = \{x \in G' \mid [x, g] \in Z(G)\}$. Observe that $D_G(g)/Z(G) = C_{G/Z}(gZ(G))$, so $D_G(g)$ is always a subgroup of $G$.

**Lemma 2.** Let $G$ be a group. If the character $\vartheta^G \in \text{Irr}(Z(G))$ is faithful, then $\vartheta$ is fully ramified with respect to $G/Z(G)$ if and only if $[g, D_G(g)] \neq 1$ for every element $g \in G \setminus Z(G)$.

**Proof.** By Theorem 1.19 and Lemma 1.20 of [9], the number of irreducible constituents of $\vartheta^G$ equals the number of conjugacy classes of cosets $gZ(G) \in G/Z(G)$ that satisfy $[g, D_G(g)] = 1$. Observe that if $g \in Z(G)$, then $[g, D_G(g)] = 1$. Hence, the only way that there can be only one conjugacy class of elements of in $G/Z(G)$ satisfying this condition is if $[g, D_G(g)] \neq 1$ for all elements $g \in G \setminus Z(G)$. Since $\vartheta$ is fully ramified with respect to $G/Z(G)$ if and only if $\vartheta^G$ has a unique irreducible constituent, it follows that $\vartheta$ is fully ramified with respect to $G/Z(G)$ if and only if there is only one conjugacy class satisfying the condition. This gives the desired result. \hfill $\square$

We get a slightly stronger statement without much difficulty.

**Lemma 3.** Let $G$ be a group. If $\lambda \in \text{Irr}(Z(G))$ is a character, then $\lambda$ is fully ramified with respect to $G/Z(G)$ if and only if $[g, D_G(g)] \not\leq \ker(\lambda)$ for every element $g \in G \setminus Z(G)$.

**Proof.** Let $Z = Z(G)$ and let $K = \ker(\lambda)$. Suppose first that $\lambda$ is fully ramified with respect to $G/Z$. Since $\lambda$ is fully ramified with respect to $G/Z$, it follows that $Z/K = Z(G/K)$. Applying Lemma 2, we have that $[gK, D_{G/K}(gK)] \neq 1$ for all cosets $gK \in G/K \setminus Z/K$. It is not difficult to see that this implies that $[g, D_G(g)] \not\leq K$ for all elements $g \in G \setminus Z$. Conversely, suppose that $[g, D_G(g)] \not\leq K$ for all $g \in G \setminus Z$. Hence, we have $[gK, D_{G/K}(gK)] \neq 1$ for all $gK \in G/K \setminus Z/K$. This implies that $[gK, G/K] \neq 1$ for all cosets $gK \in G/K \setminus Z/K$, and so $Z(G/K) \leq Z/K$. Since $Z/K \leq Z(G/K)$ obviously holds, we have $Z(G/K) = Z/K$. Notice that $\lambda$ is a faithful character of $Z/K$, so we may apply Lemma 2 to see that $\lambda$ is fully ramified with respect to $G/K$. \hfill $\square$

Let $G$ be a group, fix a character $\chi \in \text{Irr}(G)$, and write $\chi_{Z(G)} = \chi(1)\lambda$ for some character $\lambda \in \text{Irr}(Z(G))$. Note that $\ker(\lambda) = \ker(\chi) \cap Z(G)$. Consider an element $g \in G$. Since $[g, D_G(g)] \leq Z(G)$, we have $[g, D_G(g)] \not\leq \ker(\lambda)$ if and only if $[g, D_G(g)] \not\leq \ker(\chi)$. Furthermore, $[g, D_G(g)] \not\leq \ker(\chi)$ if and only if there exists an
element \(x \in G\) so that \([g, x] \in Z(G) \setminus \ker(\chi)\). Hence, Lemma 3 can be equivalently stated as follows.

**Lemma 4.** Let \(G\) be a group. A character \(\chi \in \text{Irr}(G)\) is fully ramified over \(Z(G)\) if and only if for every element \(g \in G \setminus Z(G)\), there exists an element \(x \in G\) for which \([g, x] \in Z(G) \setminus \ker(\chi)\).

This yields the desired characterization of central type characters.

**Theorem 5.** The character \(\chi \in \text{Irr}(G)\) has central type if and only if for every element \(g \in G \setminus Z(\chi)\), there exists an element \(x \in G\) for which \([g, x] \in Z(\chi) \setminus \ker(\chi)\).

**Proof.** Note that \(\chi\) is a faithful irreducible character of \(G/\ker(\chi)\) and \(Z(G/\ker(\chi)) = Z(\chi)/\ker(\chi)\). Thus we see from Lemma 4 that \(\chi\), regarded as a character of \(G/\ker(\chi)\), has central type if and only if for every element \(g \in G \setminus Z(\chi)\), there exists an element \(x \in G\) for which \(1 \neq [g, x] \ker(\chi) \in Z(G/\ker(\chi))\). It is easy to see that this is equivalent to the statement that was to be proved. \(\square\)

**Remark 6.** Observe that Theorem 5 implies the well-known result that \(\chi\) has central type if \(G/Z(\chi)\) is abelian (see [8, Theorem 2.31], for example).

Before proceeding, we discuss monolithic groups and characters. We need one more result to prove Theorem 1. This result is proved in our paper [1].

**Theorem 7.** The group \(G\) is nilpotent if and only if \(Z(\chi) > \ker(\chi)\) for each nonprincipal, monolithic character \(\chi \in \text{Irr}(G)\).

We now prove Theorem 1.

**Proof of Theorem 1.** First note the the statement (1) implies (2) follows immediately from Theorem 5.

Next we show that (2) implies (3). Let \(\chi \in \text{Irr}(G)\) be monolithic. By Theorem 5, \(\chi\) has central type. In particular \(\chi(1)^2 = |G : Z(\chi)|\), from which we deduce that \(Z(\chi) > \ker(\chi)\) if \(\chi\) is nonprincipal. Thus \(G\) is nilpotent by Theorem 7. Now, let \(N\) be a normal subgroup of \(G\) for which \(G/N\) is monolithic. Then \(G/N\) has a faithful irreducible character, and thus \(N = \ker(\chi)\) for some character \(\chi \in \text{Irr}(G)\). Let \(g \in G\) such that \([g, G] \not\leq N\). Then \(gN \not\in Z(G/N) = Z(\chi)/N\) and so \(g \not\in Z(\chi)\). By (1), there exists \(x \in G\) such that \([g, x] \in Z(\chi) \setminus N\). Since \(Z(\chi)/N = Z(G/N)\), we see that \([g, x], G] \leq N\).

To complete the proof, we show that (3) implies (1). Fix a prime \(p\) that divides \(|G|\), a Sylow subgroup \(P \in \text{Syl}_p(G)\), and a character \(\psi \in \text{Irr}(P)\). Consider the character \(\xi = \psi \times 1_H \in \text{Irr}(G)\), where \(H\) is a normal \(p\)-complement of \(G\). Then \(G/\ker(\xi) \cong P/\ker(\psi)\) is monolithic, by [8, Theorem 2.32]. So \(\xi\) is fully ramified over \(Z(\xi) = Z(\psi) \times H\) by Theorem 5, and this implies that \(\psi\) is fully ramified over \(Z(\psi)\). Now, consider a character \(\chi \in \text{Irr}(G)\). To show that \(G\) is a GVZ-group, it suffices to show that \(\chi\) is fully ramified over \(Z(\chi)\). Suppose that \(G = P_1 \times \cdots \times P_r\) is a factorization of \(G\) into a direct product of its Sylow subgroups. Then there exist characters \(\nu_i \in \text{Irr}(P_i)\) so that \(\chi = \nu_1 \times \cdots \times \nu_r\). Observe that \(Z(\chi) = Z(\nu_1) \times \cdots \times Z(\nu_r)\). We have already shown that each \(\nu_i\) is fully ramified over \(Z(\nu_i)\) and so it follows that \(\chi\) is fully ramified over \(Z(\chi)\), as desired. This proves (1). \(\square\)
References

[1] S. T. Burkett and M. L. Lewis, Characters with nontrivial center modulo their kernel, preprint.
[2] S. T. Burkett and M. L. Lewis, GVZ-groups, flat groups, and CM-groups, preprint.
[3] S. T. Burkett and M. L. Lewis, Partial GVZ-groups, preprint.
[4] F. R. DeMeyer and G. J. Janusz, Finite groups with an irreducible character of large degree, Math. Z. 108 (1969), 145-153.
[5] A. Espuelas, On certain groups of central type, Proc. Amer. Math. Soc. 97 (1986), 16-18.
[6] S. M. Gagola, Jr., Characters fully ramified over a normal subgroup, Pacific J. Math. 55 (1974), 107-126.
[7] R. B. Howlett and I. M. Isaacs, On groups of central type, Math. Z. 179 (1982), 552-569.
[8] I. M. Isaacs, Character theory of finite groups, Dover Publications, Inc., New York, 1994.
[9] I. M. Isaacs, Characters of solvable groups, American Mathematical Society, Providence, RI, 2018.
[10] M. L. Lewis, Groups where the centers of the irreducible characters form a chain, Monatsh. Math. 192 (2020), 371–399.
[11] M. Murai, Characterizations of $p$-nilpotent groups, Osaka J. Math. 31 (1994), 1–8.
[12] A. Nenciu, Isomorphic character tables of nested GVZ-groups, J. Algebra Appl. 11 (2012), 1250033, 12 pp.
[13] A. Nenciu, Nested GVZ-groups, J. Group Theory 19 (2016), 693-704.
[14] H. Tandra, and W. Moran, Flatness conditions on finite $p$-groups, Comm. Algebra, 32 (2004), 2215–2224.

Department of Mathematical Sciences, Kent State University, Kent, Ohio 44242, U.S.A.

Email address: sburkett@math.kent.edu

Department of Mathematical Sciences, Kent State University, Kent, Ohio 44242, U.S.A.

Email address: lewis@math.kent.edu