A New Formulation of General Relativity - Part III: GR as Scalar Field Theory

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Abstract

The aim of this paper (Part III) is formulating GR as a scalar field theory. The basic structural elements of it are a generating function, a generalized density and a generalized temperature. One of the axioms of this theory is a generalized Einstein equation which determines the generating function directly. It is shown that basic concepts like orientation, time orientation and isometry are expressible in terms of generating functions. At the end of the paper six problems are formulated which are still unsolved and can act as a stimulant for further research.

9 Heuristic Considerations

9.1: In Chapter 5 of Part I (cf. e.g. Remark (5.4), Proposition (5.15)) we have seen that the function Ψ (cf. Definition (3.7)) within the frame theory Φₘ (or Φₘ*) (cf. Notation (2.1) and Chapter 6.) generates the metric g and the velocity v. It turned out (cf. Proposition (7.1)) that Ψ is a generating function in the sense of Definition (7.2).

In this paper (the last part of a series of three papers) I want to establish a frame theory Φₙ which has only two base sets, the set of events M and the reals ℝ, and three structural terms Ψ, η and ϑ, where the two latter terms are generalizations of the density and the temperature. Hence, the class of systems to be considered is the same as in Part I.
The problem to be solved in this section is this: find axioms governing the base sets and the structural terms such that there are reasonable models of the theory $\Phi_{sc}$.

The method applied for this purpose is a heuristic argumentation. Clearly, such reasoning is not logically compelling, i.e. the axioms can not be deduced in the strict sense.

**9.2:** In a first step the geometrical and kinematical axioms of $\Phi_{sc}$ are considered, i.e. all those axioms which solely refer to $\Psi$. Since $\Psi$ is intended to be a generating function it should have the properties P1 to P5 of Section 7.1.2. But in our case these conditions cannot serve as axioms for $\Psi$ because they contain a metric $g$ and a velocity $v$ besides $\Psi$.

Nevertheless, let us consider that part of P1 to P5 which refers only to $\Psi$. The result are the following three conditions Q1 to Q3:

**Q 1:** $\Psi$ is a function, $\Psi : \cup_{q\in M} V_q \times \{q\} \to \mathbb{R}^4$, where $V_q$ is a subset of $M$ and $q \in V_q$, such that $\mathcal{A} = \{(V_q, \Psi(\cdot,q)) : q \in M\}$ is a $C^k$-atlas, $k \geq 3$, on $M$ and such that $(M, \mathcal{A})$ is a connected Hausdorff manifold; the function $\Psi$ is of class $C^k, k \geq 3$.

**Q 2:** For each $q \in M$ define $\gamma_q$ by

\begin{equation}
\gamma_q(t) = \Psi(\cdot,q)^{-1}(0,0,0,t)
\end{equation}

for all $(0,0,0,t) \in \text{ran } \Psi(\cdot,q)$. Then $\text{dom } \gamma_q =: J_q$, is an interval and there is a $t_q \in J_q$ such that $\gamma_q(t_q) = q$.

**Q 3:** For all $q' \in W_q := \text{ran } \gamma_q$ the equation $\Psi(\cdot,q') = \Psi(\cdot,q)$ holds.

Now these conditions determine $\Psi$ to the following extent:

**Proposition (9.1):** If $\Psi$ satisfies Condition Q1 then there is exactly one metric $g$ and one velocity field $v$ such that $\Psi$ is a generating function satisfying the conditions P1 to P3. If, in addition, $\Psi$ satisfies Q2 and Q3 then $\Psi$ fulfills also P4 and P5.

**Proof:** First of all, because of Q1 Condition P1 is satisfied for $\mathcal{A}^+ = \mathcal{A}$. Next define
\[ (9.2) \quad \Theta^\alpha(p) := d_p \Psi^\alpha(p,q)|_{q=p} \quad \text{and} \quad e^\beta(p) := \partial \Psi^\beta(p,q)|_{q=p}. \]

Then, in order that \( \Psi \) is a generating function, the metric \( g \) and the velocity \( v \) have to be given uniquely by

\[ (9.3) \quad g = \eta_{\alpha\beta} \Theta^\alpha \otimes \Theta^\beta \quad \text{and} \quad v = e_4. \]

Hence the Conditions P2 and P3 are fulfilled. From Q2 it follows that \( \Psi^\alpha(\cdot, q) \circ \gamma_q(t) = t \delta^\alpha_1 \). Hence \( \dot{\gamma}_q(t) = \partial_{\Psi^4(\cdot, q)} \gamma_q(t) \). Using Q3 we have \( \Psi(\cdot, q) = \Psi(\cdot, \gamma_q(t)) \) so that \( \dot{\gamma}_q(t) = e_4(\gamma_q(t)) \). Hence, for each \( q \) the path \( \gamma_q \) is a solution of the differential equation \( \dot{\gamma}_q = v(\gamma_q) \) which has unique solutions because \( v \) is of class \( C^r, r \geq 2 \). Since for each \( q \in M \) we have \( \gamma_q(t_q) = q \) all integral curves are of the form 9.1. This means that also P4 and P5 are satisfied.

This result suggests that the axioms governing \( \Psi \) we are looking for are the Conditions Q1 to Q3 or some equivalents of them.

9.3: In order to complete the axioms of the theory \( \Phi_{sc} \) one has to set up equations which determine the fields \( \Psi, \tilde{\eta} \) and \( \tilde{\vartheta} \) where up to now we only know that \( \tilde{\eta} \) and \( \tilde{\vartheta} \) must have something to do with density and temperature. Clearly, the starting point for our heuristic search are the axioms EM and EE of Chapter 4 and 6 in Part I. At the same time it is clear that the equations of motion and Einstein’s equation written in terms of \( g, v, \eta \) and \( \vartheta \) are not suitable to determine \( \Psi \) directly. But, since \( g \) and \( v \) are generated by \( \Psi \) or, more precisely, since they can be expressed in terms of the tetrads \( \Theta^\alpha \) and \( e^\beta, \alpha, \beta = 1, \cdots, 4 \), the equations of motion and the Einstein equation are also expressible in these terms. Thus, the problem of determining \( \Psi \) can be split up into two parts: first solve these equations for \( \Theta^\alpha, \eta \) and \( \vartheta \), and then determine \( \Psi \) from the equations \( d_p \Psi^\alpha(\cdot, q)|_{p=q} = \Theta^\alpha(p) \), e.g. via the methods developed in Chapter 8. Such procedure is possible. But the theory \( \Phi_{vs} \) thus obtained is not a scalar theory, rather it is a mixed one having vector fields and scalar fields as basic structural terms. Moreover the generating function \( \Psi \) is not a basic structural term, it is a derived quantity. Since we want to establish a theory which has no other structural terms than \( \Psi, \tilde{\eta} \) and \( \tilde{\vartheta} \) the following heuristic idea is helpful: write down the equations of motion and the Einstein equation in terms of the tetrad components \( \Lambda^\alpha_\beta \) for arbitrary coordinates, and in terms of density \( \eta \) and temperature \( \vartheta \).
Then remove in $\Lambda^\alpha_\beta(x) = \frac{\partial \phi^\alpha}{\partial x^\beta}(x, z)|_{z=x}$ the restriction $x = z$, i.e. substitute

$$\Lambda^\alpha_\beta(x) \quad \text{by} \quad \Pi^\alpha_\beta(x, z) := \frac{\partial \Phi^\alpha}{\partial x^\beta}(x, z),$$

and generalize $\eta(x)$ by $\tilde{\eta}(x, z)$ and $\vartheta(x)$ by $\tilde{\vartheta}(x, z)$.

The equations thus gained are taken for the remaining axioms of the theory $\Phi_{sc}$. This program is carried through more detailed in the next chapter.

10 Generalized Field Equations

10.1 The tetrad form of the field equations

In this section the field equations, i.e. the equation of continuity, the balance of energy and momentum and Einstein’s equation are formulated in terms of the components $\Lambda^\alpha_\beta, \Lambda^{-1^\alpha}_\beta$ of $\Theta^\alpha$ and $e_\beta, \alpha, \beta = 1, \cdots, 4$ with respect to an arbitrary coordinate system $\chi$ (cf. Remark (5.14)). These equations are obtained from the usual formulation in terms of the $\chi$-components $g_{\alpha\beta}$ and $v_\alpha$ by inserting (cf. Formulae (5.1) and (5.2))

$$g_{\alpha\beta} = \Lambda^\kappa_\alpha \Lambda^\lambda_\beta \eta_{\kappa\lambda} \quad \text{and} \quad v_\alpha = -\Lambda^4_\alpha.$$  \hfill (10.1)

In what follows, for the sake of convenience the abbreviations

$$\Lambda := ((\Lambda^\alpha_\beta)), \quad \Lambda^\alpha_\beta := \Lambda^{-1^\alpha}_\beta \quad \text{and} \quad \bar{\Lambda} := ((\bar{\Lambda}^\kappa_\lambda)) = \Lambda^{-1}$$  \hfill (10.2)

are used. Then the following proposition holds.

**Proposition (10.1):** The equation of continuity, $\text{div}(\eta v) = 0$, reads:

$$\bar{\Lambda}^\beta_4 \frac{\partial \eta}{\partial x^\beta} + \eta(\bar{\Lambda}^\beta_4 \bar{\Lambda}^\alpha_\sigma - \bar{\Lambda}^\beta_\sigma \bar{\Lambda}^\alpha_4) \frac{\partial}{\partial x^\beta} \Lambda^\sigma_\alpha = 0$$  \hfill (10.3)

The proof is based on the formula:

$$\Gamma^\alpha_\beta_\gamma = \left[ \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right] \frac{\partial}{\partial x^\kappa} \Lambda^\sigma_\lambda$$  \hfill (10.4)
where

\[
2 \left[ \frac{\alpha}{\beta \gamma} \right] = \tilde{\Lambda}^\alpha_{\sigma} \left( \delta^\gamma_{\sigma} \delta^\lambda_{\beta} + \delta^\lambda_{\beta} \delta^\gamma_{\gamma} \right) - \tilde{\Lambda}^\alpha_{\nu} \tilde{\Lambda}^{\nu \epsilon_{\rho \sigma}} \eta_{\sigma \mu} \left( \Lambda^\mu_{\gamma} \delta^\lambda_{\beta} + \Lambda^\mu_{\beta} \delta^\lambda_{\gamma} \right) + \tilde{\Lambda}^\alpha_{\nu} \tilde{\Lambda}^{\nu \epsilon_{\rho \sigma}} \eta_{\sigma \mu} \left( \Lambda^\mu_{\gamma} \delta^\lambda_{\beta} + \Lambda^\mu_{\beta} \delta^\lambda_{\gamma} \right).
\]

It can be derived by a straightforward but lengthy calculation.

Since the balance of energy and momentum depends strongly on the constitutive equation

(10.5) \[ T = T(g, v, \eta, \vartheta) \]

it cannot be written down explicitly for all the different functionals \( T \). One case of major interest is that of a Eulerian or ideal fluid. In this case \( T \) is given by its components:

(10.6) \[ T_{\alpha \beta} = p g_{\alpha \beta} + h v_\alpha v_\beta \]

where \( p = \tilde{p}(\eta, \vartheta) \) and \( h = \tilde{h}(\eta, \vartheta) \).

**Proposition (10.2):** The balance of energy and momentum, \( \text{div}(T) = 0 \), reads in terms of \( \Lambda \) for a Eulerian fluid:

(10.7) \[
\tilde{\Lambda}^\beta_j \frac{\partial p}{\partial x^\beta} + h(\tilde{\Lambda}^\alpha_j \tilde{\Lambda}^\beta_j - \tilde{\Lambda}^\alpha_4 \tilde{\Lambda}^\beta_j) \frac{\partial}{\partial x^\sigma} \Lambda^\alpha_4 = 0, \quad j = 1, 2, 3
\]

\[
\tilde{\Lambda}^\beta_4 \frac{\partial p}{\partial x^\beta} - \tilde{\Lambda}^\beta_4 \frac{\partial h}{\partial x^\sigma} - h(\tilde{\Lambda}^\alpha_4 \tilde{\Lambda}^\beta_4 - \tilde{\Lambda}^\alpha_4 \tilde{\Lambda}^\beta_4) \frac{\partial}{\partial x^\sigma} \Lambda^\alpha_4 = 0.
\]

Again, the proof is based on (10.4) together with some lengthy calculations.

Since the right-hand side of Einstein’s equation (in its usual form!) depends on the constitutive equation (10.5) one can write down only the left-hand side in an explicit way.

**Proposition (10.3):** The Einstein equation \( G + \Lambda_0 g = \kappa_0 T \) reads:
\[ E^\kappa_{jk\mu} \left( \frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x^\kappa} \Lambda^\mu_\sigma \right) + D^\kappa_{jk\mu\nu} \left( \frac{\partial}{\partial x^\kappa} \Lambda^\mu_\sigma \right) \left( \frac{\partial}{\partial x^\lambda} \Lambda^\nu_\sigma \right) + \Lambda_0 \eta_{\kappa\lambda} \Lambda^\kappa_j \Lambda^\lambda_k = \kappa_0 T_{jk} \]

where all indices run from 1 to 4. Here \( \Lambda_0 \) is an (unspecified) cosmological constant and \( \kappa_0 \) is Einstein’s gravitational constant as usual. The coefficients \( E:: \) and \( D:: \) are explicitly given. They are polynomials in \( \Lambda: \) and its inverse \( \Lambda: \).

The proof is extremely lengthy, but also straightforward.

For later purposes it be noticed that the components of the Ricci tensor have a similar form as (10.8). They read

\[ R_{jk} = S^\kappa_{jk\mu} \left( \frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x^\kappa} \Lambda^\mu_\sigma \right) + T^\kappa_{jk\mu\nu} \left( \frac{\partial}{\partial x^\kappa} \Lambda^\mu_\sigma \right) \left( \frac{\partial}{\partial x^\lambda} \Lambda^\nu_\sigma \right). \]

Hence

\[ E^\kappa_{jk\mu} = S^\kappa_{jk\mu} - \frac{1}{2} \eta_{\alpha\beta} \Lambda^\alpha_j \Lambda^\beta_k \eta^\gamma_{\alpha\gamma} \Lambda^\mu_\gamma \Lambda^\nu_{\mu\nu} \]

and similarly for \( D:: \) and \( T:: \).

### 10.2 The generalizing procedure

In this section the heuristic ideas presented at the end of Section 9.3 are to be worked out in detail. For this purpose let us again consider a chart \((V, \chi)\), and let the terms \( \Lambda^\alpha_j, \eta, \vartheta \) be functions of \( x = \chi(p), p \in V \). Moreover, let \( \Phi(x, z) := \Psi(\chi^{-1}(x), \chi^{-1}(z)) \) for all \( x, z \in \chi[V] \) for which the right-hand side is defined. Finally it is assumed that the constitutive equation (10.5) is given in the form

\[ T_{\alpha\beta} = T'_{\alpha\beta}(\Lambda, \eta, \vartheta). \]

Especially for a Eulerian fluid it follows from (10.6) that
Then the generalized field equations are obtained from (10.3), (10.8) and from \( \text{div}(T) = 0 \), e.g. from (10.7), by omitting the restriction \( z = x \) in \( \Lambda(x) = \frac{\partial}{\partial z} \Phi(x, z)|_{z=x} \). More precisely, this means we have to carry out in (10.3), (10.8) and (10.7) (or more general in \( \text{div}(T) = 0 \)) the following

**Substitution (10.4):**

1.  
   \[ \Lambda(x) = \frac{\partial}{\partial x} \Phi(x, z) \bigg|_{z=x} \longrightarrow \Pi(x, z) := \frac{\partial}{\partial x} \Phi(x, z) \]

2.  
   \[ \Lambda(x) := \Lambda^{-1}(x) \longrightarrow \Pi(x, z) := \Pi^{-1}(x, z). \]

3.  
   \[ \eta(x) \longrightarrow \tilde{\eta}(x, z) \quad \text{and} \quad \vartheta(x) \longrightarrow \tilde{\vartheta}(x, z) \]
   
   where

4.  
   \[ \tilde{\eta}(x, x) = \eta(x) \quad \text{and} \quad \tilde{\vartheta}(x, x) = \vartheta(x). \]

3. For the derivatives of \( \Lambda \) it is natural to set

   \[ \frac{\partial}{\partial x^\alpha} \Lambda^\kappa_\alpha \longrightarrow \frac{\partial}{\partial x^\alpha} \Pi^\kappa_\alpha + \frac{\partial}{\partial z^\alpha} \Pi^\kappa_\alpha = \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\alpha} \Phi^\kappa + \frac{\partial}{\partial z^\alpha} \frac{\partial}{\partial x^\alpha} \Phi^\kappa, \]

4. Likewise the derivatives of \( \eta \) and \( \vartheta \) are replaced by

   \[ \frac{\partial}{\partial x^\alpha} \eta \longrightarrow \frac{\partial}{\partial x^\alpha} \tilde{\eta} + \frac{\partial}{\partial z^\alpha} \tilde{\eta}, \quad \frac{\partial}{\partial x^\alpha} \vartheta \longrightarrow \frac{\partial}{\partial x^\alpha} \tilde{\vartheta} + \frac{\partial}{\partial z^\alpha} \tilde{\vartheta}. \]

(10.12)  
\[ T_{\alpha\beta} = p\Lambda^\kappa_\alpha \Lambda^\lambda_\beta \eta_{\kappa\lambda} + h\Lambda^4_\alpha \Lambda^4_\beta. \]
**Remark (10.5):** By definition we have $\Pi(x, x) = \Lambda(x)$, $\tilde{\eta}(x, x) = \eta(x)$ and $\tilde{\vartheta}(x, x) = \vartheta(x)$. The same holds for (10.17), (10.18) and (10.19). This means, if $z = x$ on the right-hand side the arrow $\to$ can be replaced by $=$.

### 10.3 Results

#### 10.3.1: Applying the rules of Substitution (10.4) to Equation (10.3) we arrive at the following

**Proposition (10.6):** The generalization of the equation of continuity yields

$$C(\Phi, \tilde{\eta}) = 0$$

where

$$\begin{equation}
(10.20) \quad C(\Phi, \tilde{\eta}) = \Pi_4^\beta \left( \frac{\partial \tilde{\eta}}{\partial x^\beta} + \frac{\partial \tilde{\eta}}{\partial z^\beta} \right) + \tilde{\eta} \left( \Pi_4^\beta \Pi_4^\kappa - \Pi_n^\beta \Pi_n^\kappa \right) \frac{\partial}{\partial z^\beta} \frac{\partial}{\partial x^n} \Phi^\lambda.
\end{equation}$$

The proof is trivial.

Since the generalizing procedure cannot be carried through for arbitrary constitutive equations (10.11) we confine again to the case of a Eulerian fluid.

**Proposition (10.7):** The generalization of the balance of energy and momentum yields for a Eulerian fluid

$$E_n(\Phi, \tilde{\eta}, \tilde{\vartheta}) = 0, \quad \varrho = 1, \cdots, 4$$

where

$$\begin{equation}
(10.21) \quad E_n(\Phi, \tilde{\eta}, \tilde{\vartheta}) = \Pi_n^\beta \left( \frac{\partial \tilde{\vartheta}}{\partial x^\beta} + \frac{\partial \tilde{\vartheta}}{\partial z^\beta} \right) - \tilde{h} \left( \Pi_4^\beta \Pi_n^\kappa - \Pi_n^\beta \Pi_4^\kappa \right) \frac{\partial}{\partial z^\beta} \frac{\partial}{\partial x^n} \Phi^4
\end{equation}$$

for $n = 1, 2, 3$ and

$$\begin{equation}
(10.22) \quad E_4(\Phi, \tilde{\eta}, \tilde{\vartheta}) = \Pi_4^\beta \left( \frac{\partial \tilde{\vartheta}}{\partial x^\beta} - \frac{\partial \tilde{\vartheta}}{\partial x^\beta} + \frac{\partial \tilde{\vartheta}}{\partial z^\beta} - \frac{\partial \tilde{\vartheta}}{\partial z^\beta} \right) + \tilde{h} \left( \Pi_4^\beta \Pi_4^\kappa - \Pi_4^\beta \Pi_4^\kappa \right) \frac{\partial}{\partial z^\beta} \frac{\partial}{\partial x^n} \Phi^\lambda;
\end{equation}$$
moreover, using Equation (10.6) we have

\begin{equation}
\tilde{p} = \hat{p}(\tilde{\eta}, \tilde{\vartheta}) \text{ and } \tilde{h} = \hat{h}(\tilde{\eta}, \tilde{\vartheta}).
\end{equation}

The proof is again simple.

Finally the replacement rules are applied to Equation (10.8).

**Proposition (10.8):** If the replacement rules applied to the constitutive equation (10.11) make sense, the generalization of the Einstein equation yields

\begin{equation}
G_{jk} + \Lambda_0 \Pi^\lambda_j \Pi^\kappa_k \eta_{\lambda \kappa} = \kappa_0 \tilde{T}_{jk}
\end{equation}

where

\begin{equation}
\tilde{T}_{jk} = T'_{jk}(\Pi, \tilde{\eta}, \tilde{\vartheta})
\end{equation}

and

\begin{equation}
G_{jk}(\Phi) = M^{\lambda \kappa}_j \frac{\partial^3 \Phi^\mu}{\partial z^\lambda \partial z^\kappa \partial x^\sigma} + M^{\lambda \kappa}_{jk} \frac{\partial^3 \Phi^\mu}{\partial z^\lambda \partial x^\kappa \partial x^\sigma}
\end{equation}

\begin{equation}
+ K^{\lambda \kappa \varrho}_j \frac{\partial^2 \Phi^\mu}{\partial z^\lambda \partial x^\kappa} \left( \frac{\partial^2 \Phi^\nu}{\partial x^\varrho \partial x^\sigma} \right)
\end{equation}

\begin{equation}
+ K^{\lambda \kappa \varrho}_{jk} \frac{\partial^2 \Phi^\mu}{\partial z^\lambda \partial x^\kappa} \left( \frac{\partial^2 \Phi^\nu}{\partial x^\varrho \partial x^\sigma} \right)
\end{equation}

(All indices run from 1 to 4.) Again the proof is simple but very lengthy.

The properties of the substituted terms mentioned in Remark (10.5) have some consequences which are important later on.
Remark (10.9): It follows from the construction of the terms $C, E_\varrho$ and $G_{jk}, \varrho, j, k = 1, \ldots, 4$ that

\[(10.27) \quad C(\Phi, \bar{\eta})(x, x) = \text{div}(\eta v)(x)\]

where $\text{div}(\eta v)(x)$ is the left-hand side of (10.3), and that

\[(10.28) \quad G_{jk}(\Phi)(x, x) = G_{jk}(x) = R_{jk}(x) - \frac{1}{2}g_{jk}(x)\bar{R}(x)\]

where $R_{jk}$ is given by (10.9) and $G_{jk}$ by the first and the second term of the left-hand side of (10.8). Moreover, $E_\varrho(\Phi, \bar{\eta}, \bar{\vartheta})(x, x)$ is equal to the left-hand side of (10.7).

10.3.2: In this section the results of the Propositions (10.6) to (10.8) are generalized further. In order to do this let us introduce the following.

Notation (10.10): For a given chart $(V, \chi)$ let $V \subset \chi[V] \times \chi[V]$ be the domain of $\Phi = \Psi(\chi^{-1}, \chi^{-1})$. Then a real function $O$ defined on $V$ such that $O(x, x) = 0$, is called quasi-zero.

Now the following proposition holds:

Proposition (10.11): Let the functions $\Phi, \bar{\eta}, \bar{\vartheta}$ be defined on $V$, and let $C, E_\varrho, G_{jk}$ be given by the Formulae (10.20), (10.21), (10.22) and (10.26). If $\Phi, \bar{\eta}, \bar{\vartheta}$ satisfy any subset of the equations

\[(10.29) \quad C(\Phi, \bar{\eta}) = O,\]

\[(10.30) \quad E_\varrho(\Phi, \bar{\eta}, \bar{\vartheta}) = O_\varrho\]

\[(10.31) \quad G_{jk}(\Phi) + \Lambda_0 \Pi^j_k \Pi^\lambda_k \eta_{\lambda k} = \kappa_0 T^l_{jk}(\Pi, \bar{\eta}, \bar{\vartheta}) + O_{jk},\]

where $O, O_\varrho$ and $O_{jk}, \varrho, j, k = 1, \ldots, 4$ are quasi-zeros, then the terms $\Lambda, \eta, \vartheta$ defined by $\Lambda(x) = \Pi(x, x)$, $\eta(x) = \bar{\eta}(x, x)$, $\vartheta(x) = \bar{\vartheta}(x, x)$ satisfy the corresponding subset of the Equations (10.3), (10.7) and (10.8).
The proof results immediately from Remark (10.9).

**Notation (10.12):** The equations (10.29) to (10.31) are called the generalized equation of continuity, the generalized balance of energy and momentum and the generalized Einstein equation.

**Remark (10.13):** If one has a solution $\Phi, \tilde{\eta}, \tilde{\vartheta}$ of the generalized Einstein equation such that $\tilde{\eta}$ is a quasi-zero, then $\Lambda$ is a vacuum solution of Einstein’s equation.

To a certain extent also the converse of Proposition (10.11) holds:

**Proposition (10.14):** Let be given a solution $\Lambda, \eta, \vartheta$ of the Equations (10.3) and (10.8), i.e. of the equation of continuity and of Einstein’s equation in tetrad form. Moreover, let $\Phi$ be given by Definition (8.12) of Part II and define $\tilde{\eta}, \tilde{\vartheta}$ by

$$
\tilde{\eta}(x, z) = \eta(z^1, z^2, z^3, x^4), \quad \tilde{\vartheta}(x, z) = \vartheta(z^1, z^2, z^3, x^4)
$$

Finally, let the energy-momentum tensor $T$ be defined by a functional $T'$ such that for the generalization $\tilde{T}$ of $T$ the following relations hold for $j, k = 1, \ldots, 4$:

$$
\tilde{T}'_{jk}(x, z) = T'_{jk}(\Pi, \tilde{\eta}, \tilde{\vartheta})(x, z) = T'_{jk}(\Lambda, \eta, \vartheta)(\bar{z}, x^4) + O_{jk}
$$

with $O_{jk}$ being quasi-zeros and $\bar{z} = (z^1, z^2, z^3)$. Then the triple $\Phi, \tilde{\eta}, \tilde{\vartheta}$ is a (local) solution of the generalized equations (10.29) and (10.31).

The proof is again very lengthy but straightforward.

11 The Theory $\Phi_{sc}$

11.1 Geometry and Kinematics

In this chapter the theory $\Phi_{sc}$ is formulated in the sense of Section 1.2 (i) of Part I, i.e. the terminology of Section 2.2 is used. Since the inductive procedure for setting up $\Phi_{sc}$ is described extensively in the Chapters 9 and 10 it suffices to write down the elements of $\Phi_{sc}$ without any further comment.
The base sets of $\Phi_{sc}$ are $M$ and $\mathbb{R}$, and $\Psi, \tilde{\eta}, \tilde{\vartheta}$ are its structural terms.

The physical interpretation of these terms is this:
$M$ is the set of signs for events;
$\Psi$ determines an atlas of pre-radar charts;
$\tilde{\eta}$ is a generalized density which determines the density $\eta$ by $\tilde{\eta}(p, p) = \eta(p)$;
$\tilde{\vartheta}$ is a generalized temperature which determines the (empirical) temperature $\vartheta$ by $\tilde{\vartheta}(p, p) = \vartheta(p)$.

At first the axioms for geometry and kinematics are formulated:

**GK$_{sc}$1:** $\Psi$ is a function, $\Psi : M \rightarrow \mathbb{R}^d$ where $M := \cup_{q \in M} V_q \times \{q\}$ with $V_q \subset M$, $V_q \neq \emptyset$ and $q \in V_q$.

**GK$_{sc}$2:** The term $A := \{(V_q, \Psi(\cdot, q)) : q \in M\}$ is a $C^k$-atlas, $k \geq 3$ on $M$ so that $(M, A)$ is a connected Hausdorff manifold.

**GK$_{sc}$3:** $\Psi$ is of class $C^k$, $k \geq 3$

**GK$_{sc}$4:** For each $q \in M$ there is a maximal open interval $J_q$ such that $(0, 0, 0, \tau) \in \text{ran} \Psi(\cdot, q)$ for each $\tau \in J_q$.

It is useful to introduce the following

**Notation (11.1):** 1. For each $q \in M$ the function $\gamma_q : J_q \rightarrow M$ is defined by $\gamma_q(t) = \Psi(\cdot, q)^{-1}(0, 0, 0, t)$. Moreover, we write $W_q := \text{ran} \gamma_q$.
2. $\mathcal{D}$ is the differential structure which contains all charts $(V, \chi)$ which are $C^k$-compatible, $k \geq 3$, with $A$.

**GK$_{sc}$5:** For each $q \in M$ there is a $t_q \in J_q$ such that $\gamma_q(t_q) = q$.

**GK$_{sc}$6:** For all $q' \in W_q$ it holds that $\Psi(\cdot, q') = \Psi(\cdot, q)$.

### 11.2 Field equations

In the next step the axioms of the motion of matter are formulated. For this purpose a notation is used which depends on coordinates. Moreover,
for any function \( F \) depending on \( p, q \in M \) and its coordinate form the same symbol is used, i.e. we write \( F(p, q) = F(x, z) \) for \( x = \chi(p) \) and \( z = \chi(q) \).

**EM\(_{sc}\)1:** The terms \( \tilde{\eta} \) and \( \tilde{\vartheta} \) are functions, \( \tilde{\eta} : M \to \mathbb{R} \), \( \tilde{\vartheta} : M \to \mathbb{R} \) which are of class \( C^r, r \geq 2 \) and where \( M \) is the set introduced in \( GK_{sc}1 \).

**EM\(_{sc}\)2:** Let \((V, \chi) \in D\) be any chart and let \((x, y) \in V := (\chi \times \chi)[M \cap (V \times V)]\). Then the \( \chi \)-components of the generalized energy-momentum tensor \( \tilde{T}_{jk} \) are given by

\[
\tilde{T}_{jk}(x, z) = T'_{jk}(\Pi, \tilde{\eta}, \tilde{\vartheta})(x, z)
\]

where \( T'_{jk} \) is the same functional as in Formula (10.11) for which Substitution (10.4) makes sense.

**EM\(_{sc}\)3:** For each chart \((V, \chi) \in D\) there is a quasi-zero \( O \) such that the equation \( \mathcal{C}(\Phi, \tilde{\eta}) = O \) holds.

**EM\(_{sc}\)4:** For each chart \((V, \chi) \in D\) there are quasi-zeros \( O^\alpha, \alpha = 1, \ldots, 4 \) so that the generalization of the balance of energy and momentum holds for \( \tilde{T}^{\alpha\beta} \) and with \( O^\alpha \) at the right-hand side. Especially, for Euler fluids the resulting equation reads \( \mathcal{E}_\varrho(\Phi, \tilde{\eta}, \tilde{\vartheta}) = O_\varrho, \varrho = 1, \ldots, 4 \), where \( O_\varrho \) are quasi-zeros and where \( \mathcal{E}_\varrho \) is defined in Proposition (10.7).

Likewise, the generalized Einstein equation is introduced by an axiom:

**EE\(_{sc}\):** For each chart \((V, \chi) \in D\) there are quasi-zeros \( O_{jk}, j, k = 1, \ldots, 4 \) so that the equations

\[
\mathcal{G}_{jk}(\Phi) + \Lambda_0 \Pi^j_\lambda \Pi^k_\kappa \eta_{\lambda\kappa} = \kappa_0 T'_{jk}(\Pi, \tilde{\eta}, \tilde{\vartheta}) + O_{jk}
\]

hold where \( \Lambda_0 \) is an unspecified cosmological constant and where \( \kappa_0 \) is Einstein’s gravitational constant.
11.3 Additional Conditions

Finally, in order to complete the axioms of $\Phi_{sc}$ the additional conditions (AC) have to be formulated. As in Section 4.3 we only illustrate the subject by three examples:

(i) Initial conditions;
(ii) Boundary conditions;
(iii) Symmetry conditions.

In Section 12.2 we come back to the formulation of symmetry conditions for the function $\Psi$.

11.4 Some consequences of $\Phi_{sc}$

11.4.1: If one defines the tetrads $\Theta^\alpha$ and $e_\beta, \alpha, \beta = 1, \cdots, 4$ by

$$\Theta^\alpha(p) = d\Psi^\alpha(\cdot, p)|_p \quad \text{and} \quad e_\beta(p) = \partial_{\Psi^\beta(\cdot, p)}|_p$$

and the metric $g$ and the velocity $v$ by

$$g = \eta_{\alpha\beta} \Theta^\alpha \otimes \Theta^\beta \quad \text{and} \quad v = e_4,$$

then $\Psi$ is a full generating function in the sense of Definition (7.2), i.e. $\Psi$ satisfies the Conditions P1 to P5 of Section 7.1.2. This follows directly from the Axioms GK$_{sc}$1 to 6 and from Proposition (9.1). If one is interested only in a partial generating function $\Psi$ satisfying the Conditions P1 to P3 then $\Phi_{sc}$ can be weakened by omitting the Axioms GK$_{sc}$4 to 6.

11.4.2: Let $\gamma_q$ be defined by Notation (11.1). Then $\gamma_q$ is bijective and of class $C^k, k \geq 3$. This follows directly from $\Psi(\cdot, q) \circ \gamma_q(t) = (0,0,0,t)$.

11.4.3: From Axiom GK$_{sc}$5 and from Notation (11.1) it follows that $q \in W_q$ for each $q \in M$, and therefore $\cup_{q \in M} W_q = M$. Moreover, from Axiom GK$_{sc}$6 we conclude that $\gamma_{q'} = \gamma_q$ for $q' = W_q$. Hence $J_{q'} = J_q$ and $W_{q'} = W_q$ for $q' \in W_q$. 
11.4.4: If \( q' \notin W_q \) then \( W_q \cap W_{q'} = \emptyset \). For assume that \( \bar{q} \in W_q \cap W_{q'} \), then \( \bar{q} \in W_q \) and \( \bar{q} \in W_{q'} \), so that \( W_q = W_{\bar{q}} = W_{q'} \).

11.4.5: For each \( q \in M \) let us select exactly one element \( \hat{q} \) from \( W_q \) so that \( \hat{q} \) is also selected from \( W_{q'} \) for \( q' \in W_q \), and let \( N \subset M \) be the set of all these selected \( \hat{q} \). Now, let \( P \) be any set of the same cardinality as \( N \). Then, identifying a particle in \( \Phi_{sc} \) with a worldline \( W_q \) the set \( P \) is a set of indices for particles as is used in the theories \( \Phi_R \) and \( \Phi^*_R \) (cf. Notation (2.1) and (6.1)). Let \( A \in P \) denote a particle and let \( \hat{q} \leftrightarrow A \). Then by \( \gamma_A := \gamma_{\hat{q}} \) and \( W_A := W_{\hat{q}} \) the notation used in \( \Phi_R \) and \( \Phi^*_R \) is regained. Also the function \( F \) (cf. Remark (3.6)) is defined in \( \Phi_{sc} \) by \( F(q) = A \) for all \( q \in W_A \).

11.5 Remarks concerning models of \( \Phi_{sc} \)

By Notation (4.1) of Part I the concept of a model was explicitly introduced for the theory \( \Phi_R \). It can be transferred quite easily to each theory \( \tilde{\Phi} \) which is formulated according to the scheme (i) in the Sections 1.2 and 2.2 as follows:

**Notation (11.2):** If one replaces the base sets and the structural terms of \( \tilde{\Phi} \) by explicit terms of mathematical analysis (or of the theory of sets) such that these terms satisfy the axioms of \( \tilde{\Phi} \) within mathematical analysis (or the theory of sets), then we say that these terms define an analytical (or a set theoretical) model of \( \tilde{\Phi} \).

In the usual formulations of GR the additional conditions (AC) are chosen such that the models are unique (or unique up to some diffeomorphisms). In case of the theory \( \Phi_{sc} \) the situation is different. Uniqueness is not needed for the models. Rather the generalized density \( \tilde{\eta} \) and the generalised temperature \( \tilde{\vartheta} \) only have to be unique up to quasi-zeros. Then different \( \tilde{\eta} \) and \( \tilde{\vartheta} \) define the same physically interpretable fields \( \eta \) and \( \vartheta \) by \( \tilde{\eta}(p,p) = \eta(p) \) and \( \tilde{\vartheta}(p,p) = \vartheta(p) \). Likewise, the function \( \Psi \) need not be unique. Any two model functions \( \Psi \) and \( \Psi' \) describe the same physical situation if they generate the same differential structure \( \mathcal{D} \), the same metric \( g \) and the same velocity \( v \).

This suggests the following

**Definition (11.3):** Any two arrays of terms \( M, \Psi, \tilde{\eta}, \tilde{\vartheta} \) and \( M, \Psi', \tilde{\eta}', \tilde{\vartheta}' \) forming models of the theory \( \Phi_{sc} \) are called **physically equivalent** if \( \Psi \) and \( \Psi' \)
generate the same $\mathcal{D}, g$ and $v$ and if $\bar{\eta}, \bar{\vartheta}$ and $\bar{\eta}', \bar{\vartheta}'$ differ only by a quasi-zero.

Clearly, physical equivalence is an equivalence relation within the models of $\Phi_{sc}$. Consequently, the axioms AC should be such that they determine uniquely a class of physically equivalent models of $\Phi_{sc}$. But up to now, the mathematical question is still open how to formulate a well-posed initial value problem for the generalized Einstein equation and the generalized equation of continuity so that an equivalence class is uniquely determined.

**Remark (11.4):** 1. If $\Psi$ and $\Psi'$ belong to two physically equivalent models they both satisfy the Conditions P1 to P5 of Section 7.1.2 in Part II and are related by Formula (7.8) where the Lorentz matrix $L$ and the function $R$ obey the relations (7.12), (7.14), (7.15) and $d_p R(p, q)|_{q=p} = 0$. This follows directly from Proposition (9.1) and the Propositions (7.5), (7.11) and (7.12) together with Corollary (7.10). Especially from Formula (7.14) one concludes that $R$ is a quasi-zero because there is a $t_q \in J_q$ such that $\gamma_q(t_q) = q$ for each $q \in M$.

2. Since one is only interested in a class of physically equivalent models the theory $\Phi_{sc}$ is a gauge theory.

12 Further Properties of Generating Functions

12.1 Orientation and time orientation

In this chapter a connected Hausdorff manifold $(M, \mathcal{A}^+)$ is considered where $\mathcal{A}^+$ is of class $C^k, k \geq 3$. Moreover, it is assumed that a partial generating function $\Psi$ in the sense of Definition (7.2) is defined on $M$. This implies that $\Psi$ satisfies Condition P1 of Section 7.1.2, i.e. that the atlas $\mathcal{A}$ generated by $\Psi$ is $C^k$-compatible with $\mathcal{A}^+$. In other words, $\Psi$ generates a differential structure $\mathcal{D}$ which contains $\mathcal{A}^+$. These assumptions are satisfied by the theories $\Phi_R$ and $\Phi^*_R$ treated in Part I and the theory $\Phi_{sc}$ introduced in Chapter 11. Later on some additional assumptions are needed. The above assumptions allow to introduce the tetrads $\Theta^\alpha, e_\beta, \alpha, \beta = 1, \cdots, 4$ in the usual way by

\begin{equation}
(12.1) \quad \Theta^\alpha(p) = d\Psi^\alpha(\cdot, p)|_p \quad \text{and} \quad e_\beta(p) = \partial\Psi_\beta(\cdot, p)|_p
\end{equation}
as well as the fields $g$ and $v$ by (9.3). Then the following simple result holds:

**Proposition (12.1):** The 4-form $\omega$ defined by

\[
\omega = \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge \Theta^4
\]

determines an orientation on $M$.

For the **proof** one has to show that $\omega$ nowhere vanishes. This follows directly from $\omega(e_1, \cdots, e_4) = 1$ throughout $M$.

For the next step we need the assumption that a Lorentz metric $g$ and a velocity $v$ is defined on $M$ and that $g(v,v) = -1$.

Then the manifold $(M, \mathcal{A}^+, g)$ is time orientable (cf. e.g. [11] p. 26).

In a further step it is assumed that $\Psi$ in addition satisfies the conditions P2 and P3. This means that the differential structure $\mathcal{D}$ defined by $\mathcal{A}$ is generated by $\Psi$ and that $\Psi$ also generates $g$ and $v$. In this case a time orientation is given by

**Definition (12.2):** Let $u \in T_pM$ be timelike or lightlike. Then $u$ is called future-pointing if $\Theta^4(u) = -g(v,u) > 0$ and past-pointing if $\Theta^4(u) = -g(v,u) < 0$.

**Remark (12.3):** Independently of the fact that $(M, \mathcal{A}^+, g)$ is time orientable if a velocity field $v$ exists on $M$ it can be seen that Definition (12.2) makes sense. For, it can be shown that a future-pointing vector can not be transferred into a past-pointing only by parallel transport, and vice versa.

### 12.2 Isometries

The aim of this section is defining the concept of isometry solely in terms of a generating function $\Psi$, i.e. without using explicitly a metric $g$. For this purpose we use again the notation and the suppositions introduced at the beginning of Section 12.1.
Moreover, throughout this section it is assumed that a bijective function \( f : M \to M \) is given. Then let \( \Psi' := \Psi(f,f) \). Finally, let the term \( \mathcal{A}' \) be defined by

\[
\mathcal{A}' = \left\{ (V'_q, \Psi'(\cdot, q')) : q' \in M, V'_q = f^{-1}[V_q], V_q = \text{dom}\Psi(\cdot, q), q = f(q') \right\}.
\]

Then we obtain the following result:

**Remark (12.4):** If \( f \) is a homeomorphism then \( \mathcal{A}' \) is a \( C^k \)-atlas, \( k \geq 3 \) on \( M \). For, \( V'_q \) is open because \( V_q \) is open and \( f \) is continuous. Moreover, \( \Psi'(\cdot, q') = \Psi(\cdot, f(q')) \circ f \) is a homeomorphism. Hence \( (V'_q, \Psi'(\cdot, q')) \) is a chart. Each two charts are \( C^k \)-compatible because

\[
(12.4) \quad \Psi'(\cdot, q'_1) \circ \Psi'^{-1}(\cdot, q'_2) = \Psi(\cdot, f(q'_1)) \circ \Psi^{-1}(\cdot, f(q'_2)).
\]

In the next step it is assumed that \( f \) is differentiable.

**Proposition (12.5):** The function \( f \) is a \( C^k \)-diffeomorphism, \( k \geq 3 \) exactly if \( \mathcal{A} \) and \( \mathcal{A}' \) are \( C^k \)-compatible atlases, \( k \geq 3 \).

**Proof:**
1. If \( f \) is diffeomorphic then \( \mathcal{A}' \) is an atlas. Let \( \Psi'(\cdot, q'), \Psi(\cdot, q) \) be arbitrary coordinate functions from \( \mathcal{A}' \) resp. from \( \mathcal{A} \). Then the function

\[
(12.5) \quad \Psi'(\cdot, q') \circ \Psi^{-1}(\cdot, q) = \Psi(\cdot, f(q')) \circ f \circ \Psi^{-1}(\cdot, q)
\]

is of class \( C^k, k \geq 3 \), and likewise its inverse, so that \( \mathcal{A} \) and \( \mathcal{A}' \) are \( C^k \)-compatible.
2. If \( \mathcal{A} \) and \( \mathcal{A}' \) are \( C^k \)-compatible atlases the left-hand side of (12.5) and its inverse are of class \( C^k \), so that \( f \) is a \( C^k \)-diffeomorphism.

In a last step the main result of this section is formulated. For this purpose it is assumed that a metric \( g \) is defined on the manifold \((M, \mathcal{A}^+)) \). Then a diffeomorphism \( f \) is called an isometry if \( g(p') = f^*_p g(p) \) where \( p = f(p') \) and \( f^*_p \) is the pull back of \( f \) at \( p' \).
Proposition (12.6): Let $\Psi$ satisfy the Conditions P1 and P2 of Section 7.1.2, i.e. $\Psi$ generates the differential structure $D$ of class $C^k$, $k \geq 3$ containing $A^+$, and the metric $g$. Then the (bijective) function $f$ is a $C^k$-isometry, $k \geq 3$ exactly if $\Psi'$ satisfies also P1 and P2, i.e. if $\Psi'$ generates $D$ and $g$.

Proof: 1. First of all some auxiliary formulae are proved. Let $f$ be a diffeomorphism, and let $p = f(p')$ and $q = f(q')$. Then

\[ d\Psi'(\cdot, q')|_{p'} = d\Psi(f, q)|_p = f_\star^p d\Psi(\cdot, q)|_p. \]

This equation is seen to be true by the following short calculation. Let $w' \in T_{p'}M$, then

\[ d_{p'}\Psi'^\alpha(\cdot, q')(w') = w'(\Psi'^\alpha(\cdot, q')) = w'(\Psi^\alpha(\cdot, q) \circ f) = d_{p'}\Psi^\alpha(\cdot, q)(f_\star^p w'). \]

(12.7)

Now let $g'$ be the metric which is generated by $\Psi'$:

\[ g'(p') := \eta_{\alpha\beta} d\Psi'^\alpha(\cdot, p') \otimes d\Psi'^\beta(\cdot, p')|_{p'}. \]

Then from (12.6) it follows that

\[ g'(p') = f_\star^p g(p). \]

(12.9)

2. If $f$ is a $C^k$-isometry then by Proposition (12.5) the atlases $A$ and $A'$ are $C^k$-compatible. Hence $\Psi$ and $\Psi'$ satisfy P1. Moreover, the isometry $f$ satisfies the relation

\[ g(p') = f_\star^p g(p). \]

(12.10)

Hence from (12.9) it follows that $g' = g$. This means that Condition P2 is satisfied by $\Psi$ and $\Psi'$ for the same metric $g$.

3. If $\Psi$ and $\Psi'$ satisfy P1 generating the same $D$ then $f$ is a diffeomorphism
by Proposition (12.5) so that (12.9) is true. If Ψ and Ψ' satisfy P2 with 
\( g' = g \) we find that (12.10) holds. Hence f is an isometry.

**Corollary (12.7):** Let Ψ be a partial generating function which satisfies 
condition P1 and P2 of Section 7.1.2, i.e. Ψ generates the metric g and the 
differential structure D, and let f be a \( C^k \)-diffeomorphism, \( k \geq 3 \). Then f 
is an isometry exactly if

\[
\Psi(f(p), f(q)) = L(q) \cdot \Psi(p, q) + R(p, q)
\]

where \( L \) is a field of Lorentz matrices and where \( dR(\cdot, q)|_{p=q} = 0 \). 
This follows directly from the Propositions (12.6) and (7.5).

Equation (12.11) is the symmetry condition of Ψ for a given isometry f 
which we are looking for in this section. It can be helpful to derive a special 
form for Ψ which reduces the very complicated generalized Einstein equa-
tion. Similar results hold for conformal mappings, too.

13 Final Remarks

13.1 Results

The main point of all three parts of this treatise is the existence of a function 
Ψ which generates an atlas \( \mathcal{A} \) of pre-radar charts, a metric g, a velocity field 
v and the integral curves of v. It is shown that also an orientation and a 
time orientation is defined by Ψ. Finally, the concept of isometry can be 
formulated directly with the help of Ψ, i.e. without using the metric g. 
This illustrates the significance Ψ has: it determines like a "potential" al-
most all of the fundamental concepts considered in GR and it is itself phys-
ically interpretable as a set of coordinate functions.

Since the existence of Ψ guarantees the existence of a smooth field of tetrads, 
it imposes restrictions on space-times. On the other hand, by space-time 
theory the existence of pre-radar or even radar charts is indispensable for 
space-times so that the restrictions imposed on them by Ψ are physically 
motivated and natural.

Finally, in Chapter 11 it is shown that GR can be formulated as a scalar field 
theory. But the price is doubling the independent variables and a generalized 
Einstein equation which is of third order.
13.2 Open problems

13.2.1: The main problem of any axiomatic formulation of GR, e.g. of the theories \( \Phi_R, \Phi^*_R \) and \( \Phi_{sc} \), is how to get models. For this purpose the additional conditions have to be concreted. This can be done for \( \Phi_R \) and \( \Phi^*_R \) in the usual way, but for \( \Phi_{sc} \) it is not known up to now how to formulate a well-posed Cauchy problem for the generalized Einstein equation together with the generalized equation of continuity. In both cases solving Einstein’s equation is the most difficult step in obtaining models, but it is not all one has to do. The other axioms must be satisfied, too.

13.2.2: An open practical problem is the exploitation of Equation (12.11), the definition of isometry in terms of \( \Psi \). The aim is obtaining a restricted form of \( \Psi \) for a given symmetry \( f \). For this purpose one needs a sufficiently large set of representation theorems. But, little is known in this field.

13.2.3: A solution of the inverse problem described in Section 8.1 is of physical significance because the existence of a generating function imposes restrictions on a space-time. Therefore it would be of great interest to find necessary and sufficient conditions for the solution of the (non-local) inverse problem.

13.2.4: A more principal problem is the formulation of the equations \( EM_{sc} \) and \( EE_{sc} \) in geometrical terms without use of coordinates. It seems to me that for this goal the product manifold \( M \times M \) must be considered. Up to now the problem is unsolved.

13.2.5: The generalized equation of continuity and the generalized Einstein equation form a system of 11 equations for the 6 unknown functions \( \Psi, \tilde{\eta}, \tilde{\vartheta} \). This fact seems to be a hint that there are internal dependencies between these equations which are not known up to now. It could also be the case that the generalized equations of continuity and of motion together with a reduced version of the generalized Einstein equation, e.g. its trace, suffice to determine \( \Psi, \tilde{\eta}, \tilde{\vartheta} \).

13.2.6: It is a hard task to calculate the coefficients \( M \) and \( K \) which occur in Equation (10.26) for a given special ansatz of \( \Psi \) based on symmetries. A well adapted computer algebra could be helpful in this field. I think that this problem is solvable, though it is not yet solved.

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