Deviation equations of Synge and Schild over 
\((\overline{L}_n, g)\)-spaces

S. Manoff
Bulgarian Academy of Sciences
Institute for Nuclear Research
and Nuclear Energy
Department of Theoretical Physics
Blvd. Tzarigradsko chaussee 72
1784 Sofia - Bulgaria

e-mail address: smanov@inrne.bas.bg

Abstract

Deviation equation of Synge and Schild has been investigated over differentiable manifolds with contravariant and covariant affine connections (whose components differ not only by sign) and metric \([((\overline{L}_n, g))\]-spaces\). It is shown that the condition \(\mathcal{L}_u = 0\) for obtaining this equation is only a sufficient (but not necessary) condition. By means of a non-isotropic (non-null) vector field \(u\) \([g(u, u) = \epsilon \neq 0]\) and the metric \(h_u\), orthogonal to it, a projected deviation equation of Synge and Schild has been obtained for the orthogonal to \(u\) vector field \(\xi_\perp\) and its square \(L^2 = g(\xi_\perp, \xi_\perp)\). For a given non-isotropic, auto-parallel and normalized vector field \(u\) this equation could have some simple solutions.

PACS numbers: 02.90; 04.50+h; 04.90.+e: 04.30.+x

1 Introduction

In the last decades the models of space-time have been generalized from (pseudo) Riemannian spaces without torsion (denoted \(V_n\)-spaces) or with torsion (denoted \(U_n\)-spaces) to spaces with contravariant and covariant affine connections (whose components differ only by sign) and metrics [denoted \((L_n, g)\)-spaces] as well as to spaces with contravariant and covariant affine connections (whose components differ not only by sign) [denoted \((\overline{L}_n, g)\)-spaces]. It has been proved that in these spaces the principle of equivalence holds \([3, 5, 6]\), and special type of transports (called Fermi-Walker transports) \([7, 8]\) exist which do not deform a Lorentz basis. Therefore, the law of causality is not abuse in \((L_n, g)\)- and \((\overline{L}_n, g)\)-spaces if one uses a Fermi-Walker transport instead of a parallel transport (used in a \(V_n\)-space). Moreover, there also exist other types of transports (called...
conformal transports) \cite{9, 10} under which a light cone does not deform. At the same time, the auto-parallel equation can play the same role in $(L_n, g)$- and $(\mathcal{L}_n, g)$-spaces as the geodesic equation does in the Einstein theory of gravitation (ETG) \cite{11, 12}. On this basis, many of the differential-geometric constructions used in the ETG in $V_4$-spaces could be generalized for the cases of $(L_n, g)$- or $(\mathcal{L}_n, g)$-spaces. Bearing in mind this background a question arises about applications of generalizations of well constructed mathematical models in the ETG to theories in $(L_n, g)$- and $(\mathcal{L}_n, g)$-spaces. Such models, for instance, are deviation equations used as theoretical basis for construction of gravitational wave detectors in ETG. They can be generalized for $(L_n, g)$- and $(\mathcal{L}_n, g)$-spaces and are worth being investigated.

The task of this paper is to show that deviation equations can be used in the same way as in the ETG in gravitational theories using $(\mathcal{L}_n, g)$-spaces as a model of space-time. *Deviation equations are independent of the gravitational equations* conditions for finding out the relative acceleration (as kinematic characteristic) between moving particles in space with affine connections and metrics. Gravitational equations of the type of those in the ETG (as dynamic characteristics) impose only additional conditions on the curvature tensor and at the same time they give the explicit form of these quantities for the corresponding theoretical model.

In the present paper the deviation equation of Synge and Schild is generalized for $(\mathcal{L}_n, g)$-spaces and specialized for description of the variation of the second covariant derivative of a vector field $\xi$, orthogonal to the non-isotropic (non-null) (time like for $n = 4$) vector field $u$. The vector field $\xi$ is interpreted as a deviation vector. In Sec. 3. the generalized deviation equation of Synge and Schild and its projective form (projective deviation equation of Synge and Schild) are considered in $(\mathcal{L}_n, g)$-spaces. An analogous deviation equation for the square of a non-isotropic vector (which is space like for $n = 4$) is found and investigated. In Sec. 4. the projective deviation equation of Synge and Schild for the square of an auto-parallel ($\nabla_u u = 0$) (non-isotropic) and normalized $[g(u, u) = e = const. \neq 0]$ vector field $u$ in $(\mathcal{L}_n, g)$-spaces as well as in $\mathcal{V}_n$- and $\mathcal{V}_n$-spaces [as special cases of $(\mathcal{L}_n, g)$-spaces] is considered. Some simple solutions are found and examples for the case of $\mathcal{V}_n$-spaces are given which can lead to an equation in the form of an oscillator equation. The results can be easily specialized for canonical contraction operator $C$ in (pseudo) Riemannian spaces with and without torsion ($U_n$- and $V_n$-spaces).

## 2 Deviation equations

In the general relativity, as a basis for the theoretical scheme for gravitational wave detectors proposed by (Weber 1958-1961) and discussed by many authors (Zacharov 1972), (Amaldi, Pizzella 1979), (Will 1979, 1981), (Bicak, Rudenko 1987), the geodesic deviation equation (proposed by Levi-Civita in 1925 in a
co–ordinate basis) in the form
\[ D^2 \xi^i ds^2 = R_{jkl} u^j u^k \xi^l, \quad u^i \cdot \xi = a^i = 0, \quad (1) \]
or in the index free form
\[ \nabla_u \nabla_u \xi = [R(u, \xi)] u, \quad a = \nabla_u u = 0, \quad (2) \]
has been used. Its generalization for non-geodesic trajectories \((a \neq 0)\) (proposed by Synge and Schild in 1956 in a co–ordinate basis) in the form
\[ D^2 \xi^i ds^2 = R_{jkl} u^j u^k \xi^l + a_i \xi^i, \quad a_i = u^i \cdot \xi, \quad (3) \]
or in index free form
\[ \nabla_u \nabla_u \xi = [R(u, \xi)] u + \nabla_u a, \quad (4) \]
has also been used by Weber in a special form for construction of gravitational waves detectors of the type of massive cylinders reacting to periodical gravitational processes. The application of these equations in experiments for detecting gravitational waves turned the attention of many authors to considerations and proposals for new deviation equations.

From mathematical point of view many of the proposed by different authors deviation equations can be obtained from the s.c. generalized deviation identity (generalized deviation equation) in \((L_n, g)-spaces\) -
\[ \nabla_u \nabla_u \xi \equiv [R(u, \xi)] u + \nabla_u \xi + T(\xi, a) - \nabla_u [T(\xi, u)] + [\mathcal{L}_\xi \Gamma(\xi, u)] u, \quad (5) \]
or in a (co-ordinate or non-co-ordinate) basis
\[ (\xi^i \cdot \xi^j) u^k \equiv R_{kij} u^k u^j \xi^i + \xi^i \cdot \xi^j u^i + T_{k}^{i} u^k a^i - \left(T_{k}^{i} u^k u^j \right) \cdot \xi^i + \mathcal{L}_\xi \Gamma_{ijkl} u^k u^l, \quad (6) \]
where the components \(\mathcal{L}_\xi \Gamma_{ijkl}\) are the Lie derivatives of the components \(\Gamma_{ijkl}\) of the contravariant affine connection \(\Gamma\) and
\[ a = \nabla_u u = u^i \cdot \xi^j, \quad e_i = a^i, \quad u \in T(M), \quad (7) \]
\[ e_i = \partial_i = \partial/\partial x^i \quad \text{(in a co–ordinate basis)}, \quad u^i \cdot \xi^j = e_j u^i + \Gamma_{kji}^i u^k, \quad \Gamma_{kji}^i \neq \Gamma_{ijkl}, \quad (7) \]
The operator \(R(\xi, u)\) is the curvature operator
\[ R(\xi, u) = -R(u, \xi) = \nabla_\xi \nabla_u - \nabla_u \nabla_\xi - \nabla_\xi u = [\nabla_\xi, \nabla_u] - \nabla_{[\xi, u]} = \nabla_\xi, \quad \xi, u \in T(M), \quad (8) \]
The operator $\mathcal{L} \Gamma(\xi, u)$ is the deviation operator \[15\]

$$\mathcal{L} \Gamma(\xi, u) = \mathcal{L}_\xi \nabla_u - \nabla_\xi \mathcal{L}_u - \mathcal{L}_\xi \mathcal{L}_u =$$

$$= [\mathcal{L}_\xi, \nabla_u] - \nabla_{[\xi,u]} , \quad \xi, u \in T(M) ,$$

$\mathcal{L}_\xi u$ is the Lie derivative of the contravariant vector field $u$ along the contravariant vector field $\xi$,

$$\mathcal{L}_\xi u = [\xi, u] = \nabla_\xi u - \nabla_u \xi - T(\xi, u) ,$$

$\nabla_u \xi$ is the covariant derivative of the vector field $\xi$ along the vector field $u$, $T(\xi, u)$ is the torsion vector field

$$T(\xi, u) = T_{kl}^i \xi^k . u^l . e_i ,$$

$$T_{kl}^i = \Gamma_{lk}^i - \Gamma_{ki}^l \quad \text{(in a co-ordinate basis \{\partial_i\})},$$

$$T_{kl}^i = \Gamma_{lk}^i - C_{kl}^i \quad \text{(in a non-co-ordinate basis \{e_i\})},$$

$$\mathcal{L}_e e_i = [e_k, e_l] = C_{kl}^i . e_i .$$

Since the deviation equations are related to the second covariant derivative of a deviation vector $\xi$ they could be represented by means of the kinematic characteristics related to the notions of relative accelerations (shear, rotation and expansion accelerations) \[18\] and to their corresponding relative velocities (shear, rotation and expansion velocities). In Einstein’s theory of gravitation notions such as shear (shear velocity) $\sigma$, rotation (rotation velocity) $\omega$ and expansion (expansion velocity) $\theta$ are used for invariant classification of solutions of the Einstein’s field equations. These notions \[19\] can be defined for vector fields over $(L_n, g)$-spaces in analogous way as in $V_n$- and $U_n$-spaces by means of representation of the covariant derivative of a vector field $\xi$ along (another) non-isotropic (non-null) vector field $u \ [g(u, u) = e \neq 0]$ in the form \[8\]

$$\nabla_u \xi \equiv \frac{1}{e} g(u, \nabla_u \xi) . u + \tilde{g} [h_u (\frac{1}{e} a - \mathcal{L}_\xi u)] +$$

$$+ \tilde{g} [\sigma(\xi)] + \tilde{g} [\omega(\xi)] + \frac{1}{n-1} \theta h_u (\xi) ,$$

or in a given basis in the form

$$\xi^i \cdot u^j \equiv \frac{1}{e} g_{kl} u^k . \xi^l . m u^m . u^i + g^{ij} [h \Gamma_k^j (\xi^k a^l - \mathcal{L}_\xi u^l)] +$$

$$+ (\sigma \Gamma_k^j + \omega \Gamma_k^j + \frac{1}{n-1} \theta h \Gamma_j^k) \xi^k ,$$

(12)
where
\[ h_u = g - \frac{1}{2}g(u) \otimes g(u) = h_{ij}.e^i.e^j, \quad g = g_{kl}.e^k.e^l, \]
\[ e^i.e^j = \frac{1}{2}(e^i \otimes e^j + e^j \otimes e^i), \]
\[ e^i = dx^i \text{ (in a co-ordinate basis)}, \]
\[ h_u(u) = h_{ij}.u^i.e^j = h_{ij}.u^i.e^j = u(h_u) = 0, \]
\[ h_{ij} = g_{ij} - \frac{1}{c}.u_i.u_j, \]
\[ h_{ij} = f^k.j.h_{ik}, \quad u^j = f^k.k.\dot{u}^k, \quad u_i = g_{ik}.u^k = g_{ik}.u^k, \]
\[ f^i.j = S(e^i.e^j) = S(e_j.e^i) = e^i(e_j) \in C(M), \]
\[ \text{Remark. In } (L_n, g)-\text{spaces (dim } M = n) S \text{ is the contraction operator. In } \]
\[ (L_n, g)-\text{spaces [as special case of } (L_n, g)-\text{spaces] } S = C : C(e^i.e^j) = g^i_j = e^i(e_j), \]
\[ g^j_i = 1 \text{ for } i = j \text{ and } g^j_i = 0 \text{ for } i \neq j. \]

The tensor \( \sigma \) is the shear velocity tensor (shear) which can be written in the form
\[ \sigma = \frac{1}{2}\{h_u(\nabla_u \bar{g} - L_u \bar{g})h_u - \frac{1}{n-1}(h_u[\nabla_u \bar{g} - L_u \bar{g}]).h_u\} = \sigma_{ij}.e^i.e^j, \]
\[ \bar{g} = g^{ij}.e_i.e_j, \quad e_i.e_j = \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i), \]
\[ \sigma_{ij} = \frac{1}{2}\{h_u(g^{kl}.m^i.u^m - L_u g^{kl})h_{ij} - \frac{1}{n-1}.h_u(g^{kl}.m^i.u^m - L_u g^{kl}).h_{ij}\}, \]

where
\[ g(u) = g_{ik}.u^k.e^i = g_{ik}.\dot{u}^k.e^i, \quad g_{ik} = f^i.k.\dot{g}_k, \quad \bar{g}[g(u)] = u, \]
\[ g[\bar{g}(p)] = p, \quad p \in T^*(M), \quad g^{ij}.g_{jk} = g^k_i, \quad g_{jk} = g_{jk}, \]
\[ g^{jk} = f^j.k.f^m.m.\dot{g}_{lm}, \quad g^{ij} = f^i.k.f^j.l.g^{kl}. \]

The tensor \( \omega \) is the rotation velocity tensor (rotation),
\[ \omega = h_u(k_u)h_u = h_u(s)h_u - h_u(q)h_u = S - Q = \omega_{ij}.e^i.e^j, \]
\[ s = \frac{1}{2}(u^k.m^i.g^{ml} - u^l.m^i.g^{mk})e_k \wedge e_l, \]
\[ q = \frac{1}{2}(T_{mn}^lg^{ml} - T_{mn}^lg^{mk})u^n.e_k \wedge e_l, \]
\[ S = h_u(s)h_u = h_{ik}.s_{kl}.h_{ij}.e^i.e^j + Q = h_u(q)h_u, \]
\[ e^i.e^j = \frac{1}{2}(e^i \otimes e^j - e^j \otimes e^i), \quad e_k \wedge e_l = \frac{1}{2}(e_k \otimes e_l - e_l \otimes e_k), \]
The invariant $\theta$ is the *expansion velocity* invariant (expansion),

$$\theta = \frac{1}{2} h_u[\nabla_u \mathcal{F} - \mathcal{F}_u] = \frac{1}{2} h_u(g^{ij} : u^k - \mathcal{E}_u g^{ij}) \quad . \quad (18)$$

In this way the notions of shear, rotation and expansion are generalized for $(\mathcal{T}_n, g)$-spaces. In analogous way (after some more complicated computations) for the second covariant derivative $\nabla_u \nabla_u \xi$ notions such as shear acceleration, rotational acceleration and expansion acceleration can be introduced in $\nabla_n$-spaces [(pseudo) Riemannian spaces without torsion with contraction operator $S \neq C$], in $\mathcal{T}_n$-spaces [(pseudo) Riemannian spaces with torsion and with contraction operator $S \neq C$] and in $(\mathcal{T}_n, g)$-spaces. These notion can also be connected with the generalized deviation identity which can be written in the form

$$\nabla_u \nabla_u \xi \equiv \frac{1}{e} g(u, \nabla_u \nabla_u \xi) \cdot u + \mathcal{F}[\nabla_u (\nabla_u \nabla_u \xi)] \quad .$$

After straightforward calculations the orthogonal to $u$ part $\mathcal{F}[\nabla_u (\nabla_u \nabla_u \xi)]$ of $\nabla_u \nabla_u \xi$ can be represented in the form

$$\mathcal{F}[\nabla_u (\nabla_u \nabla_u \xi)] = \mathcal{F}(h_u)[\frac{1}{e} \nabla_u a - \nabla_u \xi u - \mathcal{E}_u (\xi u) + \mathcal{F}(\xi u)] +$$

$$+ \mathcal{F} [sD(\xi) + W(\xi) + \frac{1}{n \cdot 1} U(h_u(\xi))] \quad , \quad (19)$$

or in a given basis (in index form)

$$g^{ij} h_{\xi k} (\xi^k \cdot u^l) \cdot m u^m = g^{ij} h_{\xi k} (\xi^k \cdot u^l - u^k \cdot \xi^l u^l - \mathcal{E}_u (\xi^k u^l) + T_{mn} \xi^k u^m u^n) +$$

$$+ g^{ij} (sD_{jk} + W_{jk} + \frac{1}{n \cdot 1} U(h_{jk}) \xi^k) \quad . \quad (20)$$

The tensor $sD = sF D_0 - sT D_0 + s M$ is the *shear acceleration* tensor (shear acceleration) constructed by three terms: the tensor $sF D_0$ is the curvature- and torsion-free shear acceleration, the tensor $sT D_0$ is the shear acceleration, induced by torsion, the tensor $s M$ is the shear acceleration, induced by curvature; the tensor $W = F W_0 - T W_0 + N$ is the *rotation acceleration* tensor (rotation acceleration) which has also three terms: the tensor $F W_0$ is the curvature- and torsion-free rotation acceleration, the tensor $T W_0$ is the rotation acceleration, induced by torsion, the tensor $N$ is the rotation acceleration, induced by curvature; the invariant $U = F U_0 - T U_0 + I$ is the *expansion acceleration* invariant (expansion acceleration) with the three terms: the invariant $F U_0$ is the curvature- and torsion-free expansion acceleration, the invariant $T U_0$ is the expansion acceleration, induced by torsion, the invariant $I$ is the expansion acceleration, induced by curvature [this term appears as a generalization of the Raychaudhuri identity \[16\] for $(\mathcal{T}_n, g)$-spaces].

By means of different representations of the generalized deviation identity possibilities can be considered for writing down theoretical schemes in gravitational theories (and particular in the ETG) for construction of gravitational wave detectors.
3 Deviation equation of Synge and Schild

The deviation equation of Synge and Schild in \((L_n, g)\)-spaces can be obtained from the generalized deviation identity by means of the additional condition \(\mathcal{L}_\xi u = 0\) or \(\mathcal{L}_\xi u^i = 0\) in the form

\[
\nabla_u \nabla \xi = [R(u, \xi)]u + \nabla \xi a - \nabla [T(\xi, u)] , \quad a = \nabla_u u ,
\]

or in an arbitrary basis (or in index form)

\[
(\xi^i \cdot \jmath^j)\cdot k u^k = R_{klj}^i u^k u^l \xi^l + a^i \cdot j \xi^j - (T_{kl}^i \xi^k u^l)\cdot j u^j .
\]

At the same time the conditions

\[
\nabla_u \xi = \nabla \xi u - T(\xi, u) \quad \text{or} \quad \xi^i \cdot j u^j = u^i \cdot j \xi^j - T_{kl}^i \xi^k u^l ,
\]

\[
\mathcal{L}_\xi a = [\mathcal{L} \Gamma(\xi, u)]u \quad \text{or} \quad \mathcal{L}_\xi \Gamma_j^i k u^k u^l = \mathcal{L}_\xi a^i ,
\]

are fulfilled.

The way of getting the deviation equation of Synge and Schild gives the possibility for proving the following proposition:

**Proposition 1.** Every vector field \(\xi\), which satisfies the equation \(\mathcal{L}_\xi u = 0\) \((\mathcal{L}_\xi u^i = 0)\) for an arbitrary vector field \(u\) is a solution of the deviation equation of Synge and Schild.

Proof: There are at least two ways for proving this proposition:

1. The proof follows immediately from the generalized identity and the condition \(\mathcal{L}_\xi u = 0\).
2. From the condition \(\mathcal{L}_\xi u = 0\) and after covariant differentiation along \(u\) of the expression for \(\nabla_u \xi\) (s. above) the deviation equation follows.

**Corollary.** The condition \(\mathcal{L}_\xi u = 0\) is a "first integral" for the deviation equation of Synge and Schild (for arbitrary vector field \(u\)).

**Remark.** Under "first integral" here one can define a quantity whose covariant derivative along an arbitrary vector field \(u\) leads to the deviation equation of a concrete type (here of Synge and Schild).

**Proposition 2.** The necessary and sufficient condition for the existence of the deviation equation of Synge and Schild is the condition \(\mathcal{L}_\xi a = [\mathcal{L} \Gamma(\xi, u)]u\) or \(\mathcal{L}_\xi a^i = \mathcal{L}_\xi \Gamma^i_{jk} u^k u^l\).

Proof: (a) Necessity: From the generalized deviation identity and the deviation equation of Synge and Schild the condition follows.
(b) Sufficiency: From the condition and the generalized deviation identity the deviation equation of Synge and Schild follows.

**Remark.** In finding out deviation equations different authors used only sufficient (or "first integrals") conditions for these equations (like those in proposition 2.). They don’t take into account that the obtained equations can fulfill also other sufficient conditions than the considered one (s. for example [14].

In a \((L_n, g)\)-space, the second covariant derivative of a vector field \(\xi\) along a non-isotropic (non-null) vector field \(u\) can be written in two parts: the one
is collinear to \( u \), the other is orthogonal to the vector field \( u \). The second term can be interpreted as a relative acceleration between two points, lying on a hyper-surface orthogonal to the vector field \( u \). Since the (infinitesimal) deviation vector has also to lie on this hyper-surface, then in this case \( \xi \) has to obey the condition

\[
g(\xi, u) = 0 ,
\]

or \( \xi \) has to be in the form

\[
\xi = g[\xi_u(\nabla_u \nabla \xi)] = g[\xi_u(\xi)] = g[\xi_u(\xi)] ,
\]

\[
g(\xi, u) = 0 .
\]

**Definition 1.** The deviation equation which is obtained for \( g[\xi_u(\nabla_u \nabla \xi)] \) or for \( h_u(\nabla_u \nabla \xi) \)

under the conditions

\[
\xi \perp u = 0 , \quad g(u, \xi \perp) = 0 , \quad \xi = g[\xi_u(\xi)] ,
\]

is called \textit{projective deviation equation of Synge and Schild}.

After some calculations, it follows from the form of \( g[\xi_u(\nabla_u \nabla \xi)] \) that this equation can be written in the form [18]

\[
\bar{g}[\xi_u(\nabla_u \nabla \xi)] = \bar{g}[A(\xi_{\perp})] = \bar{g}[s D(\xi_{\perp})] + \bar{g}[W(\xi_{\perp})] + \frac{1}{n-1} U \cdot \xi_{\perp} ,
\]

or in index form

\[
g^{ij} h_{jk}(\xi_{\perp}^k u^l m^m u^m = g^{ij} A_{jk} \xi_{\perp}^k =
\]

\[
g^{ij} (s D_{jk} + W_{jk}) \xi_{\perp}^k + \frac{1}{n-1} U \cdot \xi_{\perp} ,
\]

where

\[
\xi_{\perp}^k = g^{kl} h_{mn} \xi^m , \quad h_u(\xi_{\perp}) = h_u(\bar{g}) h_u(\xi) = h_u(\xi) ,
\]

\[
\bar{g}[h_u(\xi_{\perp})] = \bar{g}[h_u(\xi)] = \xi_{\perp} .
\]

The projective deviation equation can also be written in an equivalent form

\[
h_u(\nabla_u \nabla \xi_{\perp}) = \bar{s} D(\xi_{\perp}) + \bar{W}(\xi_{\perp}) + \frac{1}{n-1} U \cdot g(\xi_{\perp})
\]

Every vector field \( \xi_{\perp} \) [for an arbitrary non-isotropic (non-null) vector field \( u \)] which fulfills the conditions \( \xi_{\perp} u = 0, \xi_{\perp} = g[\xi_u(\xi)] \), is a solution of the projective deviation equation of Synge and Schild. Therefore, the solution of equation \( \xi_{\perp} u = 0 \) (or \( \xi_{\perp} u \xi_{\perp} = 0 \)) for a vector field \( \xi_{\perp}(x^k) \) and a given vector field \( u(x^k) \) is also a solution of the projective deviation equation. It follows in this case that, if the components of the vector field \( \xi = \xi^i e_i = \xi^k \delta_k \) should be
solutions of a homogeneous (or non-homogeneous) oscillator equation, then an additional equation for the vector field $u$ has to be proposed, which could lead to such properties of $\xi$.

A deviation equation under the same conditions $\mathcal{L}_{\xi_\perp}u = 0$, $\xi_\perp = \mathcal{F}[h_u(\xi)]$, can also be written for the square of $\xi_\perp$, i.e., for $g(\xi_\perp, \xi_\perp) = L^2 \neq 0$. If the vector field $u$ is considered as a time like vector field which is orthogonal to $\xi_\perp$, then $\xi_\perp$ could be interpreted as a space like vector field which length is considered as the length of a material object or the length of the distance between two particles, lying on an orthogonal to $u$ hypersurface.

By means of the relations

\[
\nabla_u \xi_\perp =_{\text{rel}} v + \mathcal{F}[\nabla_u \xi](\xi) + (\nabla_u \mathcal{G})(h_u(\xi)),
\]

\[
_{\text{rel}}v = \mathcal{F}[h_u(\nabla_u \xi)] = g^{ij} h \xi_{jk} \xi_j^k \xi^i, e_i,
\]

\[
\nabla_u \nabla_u \xi_\perp =_{\text{rel}} a + 2\mathcal{G}(\nabla_u h_u)(\nabla_u \xi) + \mathcal{G}(\nabla_u \nabla_u h_u)(\xi) + 2(\nabla_u \mathcal{G})g(r_{\text{rel}}v) + 2(\nabla_u \mathcal{G})g(\mathcal{F}[h_u(\mathcal{G})]g(r_{\text{rel}}v) + \mathcal{G}(\nabla_u h_u)(\xi) + (\nabla_u \nabla \mathcal{G})g(\xi_\perp),
\]

\[
r_{\text{rel}}a = \mathcal{F}[h_u(\nabla_u \nabla_u \xi)] = g^{ij} h \xi_{jk} \xi_j^k \xi^i, m u^m, e_i,
\]

\[
(\nabla_u \mathcal{G})g(r_{\text{rel}}v) = (\nabla_u \mathcal{G})(g(r_{\text{rel}}v)) = g^{ij} k g \xi_{jk} \xi_j^k \xi_i, e_i,
\]

the deviation equation for $L^2$ can be obtained in the form

\[
u(uL^2) = 2g(\xi_\perp, r_{\text{rel}} a) + 2g(\xi_\perp, \mathcal{F}(\nabla_u h_u)(\nabla_u \xi)) + g(\xi_\perp, \mathcal{F}(\nabla_u \nabla_u h_u)(\xi)) + 2g(\xi_\perp, (\nabla_u \mathcal{G})g(r_{\text{rel}}v)) + 2g(\xi_\perp, (\nabla_u \mathcal{G})(\nabla_u h_u)(\xi)) + 2g(\xi_\perp, (\nabla_u \mathcal{G})(\nabla_u \nabla_u h_u)(\xi)) + 2g(\mathcal{G}(\nabla_u h_u)(\xi), (\nabla_u \mathcal{G})g(\xi_\perp)) + g((\nabla_u \mathcal{G})g(\xi_\perp), (\nabla_u \mathcal{G})g(\xi_\perp)) + 4(\nabla_u g)(\xi_\perp, \nabla_u \xi_\perp) + (\nabla_u \nabla_u g)(\xi_\perp, \xi_\perp).
\]

For $\mathcal{G}_{n-}$ and $\mathcal{G}_{n}$-spaces ($\nabla_u g = 0$ for $\forall u \in T(M)$) this equation will have the form

\[
u(uL^2) = 2g(\xi_\perp, r_{\text{rel}} a) + 2g(\xi_\perp, \mathcal{F}(\nabla_u h_u)(\nabla_u \xi)) + g(\xi_\perp, \mathcal{F}(\nabla_u \nabla_u h_u)(\xi)) + 2g(\mathcal{G}(\nabla_u h_u)(\xi), (\nabla_u \mathcal{G})g(\xi_\perp)) + 2g((\nabla_u \mathcal{G})g(\xi_\perp), (\nabla_u \mathcal{G})g(\xi_\perp)) + 4(\nabla_u g)(\xi_\perp, \nabla_u \xi_\perp) + (\nabla_u \nabla_u g)(\xi_\perp, \xi_\perp).
\]
If the additional condition (parallel transport of $h_u$ along $u$)
\[ \nabla_u h_u = 0 \] (33)
is required, then the equation for $L^2$ will have the form
\[ u(uL^2) = 2[g(\xi_{\perp, \text{rel}} a) + g(\text{rel} v, \text{rel} v)] , \] (34)
or in index form
\[ ((L^2)_i u^i)_j u^j = 2(g_{\perp b \text{rel} a}^k u^k + g_{\perp \text{rel} v}^k - \text{rel} v^l) . \] (35)

**Remark.** If $u = d/ds$ is a tangent vector at a curve $x(s)$ then $u(uL^2) = d^2 L^2 / ds^2$.

The next task is to consider the deviation equation for $L^2$ for auto-parallel ($\nabla_u h_u = a = 0$), non-isotropic (non-null) ($g(u, u) = e \neq 0$) and normalized ($e = \text{const.} \neq 0$) vector field $u$.

### 4 Projective deviation equation of Synge and Schild for $L^2$ in the case of auto-parallel vector field $u$ in $\bar{U}_n$- and $\bar{V}_n$-spaces

If the condition for auto parallelism is given for the vector field $u$, i.e. if
\[ \nabla_u u = a = u^i \cdot j u^j \cdot e_i = a^i e_i = 0 \] (36),
then by means of the expression for $\nabla_u h_u$ in $(\bar{L}_n, g)$-spaces
\[ \nabla_u h_u = \nabla_u g + \frac{1}{4}(\xi(ue) g(u) \otimes g(u) - [g(a) \otimes g(u) + g(u) \otimes g(a)] - \nabla u g)(u) - \nabla g)(u)] \] (37)
the following proposition can be proved for the case of $\bar{U}_n$-spaces [$\nabla_v g = 0$ for $\forall v \in T(M)$];

**Proposition 3.** For a non-isotropic, normalized and auto parallel vector field $u$ in $\bar{U}_n$-space the condition for $L^2 = g(\xi_{\perp}, \xi_{\perp})$
\[ u(uL^2) = 2[g(\xi_{\perp, \text{rel}} a) + g(\text{rel} v, \text{rel} v)] \] (38)
is fulfilled.

**Proof:** The last condition follows immediately from the expression for $u(uL^2)$ and the condition $\nabla_u h_u = 0$ (which is fulfilled in this case).

Let us now use the representation for $\nabla_u \xi$ by means of the kinematic characteristics $d, \sigma, \omega, \theta$ and for $\nabla_u \nabla u \xi_{\perp}$ by means of the kinematic characteristics $A, sD, W, U$ and their structure under the conditions of proposition 3. Then, the last expression for $u(uL^2)$ can be written in the form
\[ u(uL^2) - \frac{2}{n-1} U L^2 = 2[sD(\xi_{\perp}, \xi_{\perp}) + \overline{\gamma}(d(\xi_{\perp}), d(\xi_{\perp}))] , \] (39)
where
\[ sD(\xi_\perp, \xi_\perp) = (\xi_\perp)(sD(\xi_\perp)) = sD_k\xi_\perp \xi_\perp, \]
\[ d(\xi_\perp) = d_s\xi_\perp e^k, \]
and the following relations are fulfilled
\[ g(\text{rel} v, \text{rel} v) = (\text{rel} v)^2 = g(d(\xi_\perp), d(\xi_\perp)), \]
\[ g(\xi_\perp, \text{rel} a) = g(\xi_\perp, [sD(\xi_\perp)]) + \frac{1}{n-1} U L^2, \]
\[ g(\xi_\perp, [W(\xi_\perp)]) = 0. \]

(40)

If we use the explicit form of \[ d = \sigma + \omega + \frac{1}{n-1} \theta. h_u \] and introduce the following abbreviations
\[ \lambda = -2 \left\{ \overline{g}[\sigma(g)] + \overline{g}[\omega(g)] + \theta^2 + 2 \frac{n-1}{n} \theta^2 \right\}, \]
\[ sD(\xi_\perp, \xi_\perp) = D^2, \theta = u \theta = u^i \partial_i \theta = u^i e^i \theta, \]
\[ \sigma(\xi_\perp) = \delta, \omega(\xi_\perp) = \eta, \]
\[ \overline{g}[D(\xi_\perp, \xi_\perp)] = \overline{g}[\sigma(\xi_\perp)] = \sigma^2, \]
\[ (\delta + \eta)^2 = \delta^2 + 2 \delta \eta + \eta^2 = \overline{g}(\delta, \delta) + 2 \overline{g}(\delta, \eta) + \overline{g}(\eta, \eta), \]
\[ L^2 = y, \quad u = \frac{d}{ds} = (dx^i/ds) \partial_i = u^i \partial_i, \]
then after some computations we can obtain the equation for \[ u(u L^2) = d^2 L^2 / ds^2 \]
(for \( u = d / ds \)) in the form
\[ \frac{d^2 y}{ds^2} + \lambda(s) y = f(s), \]

(42)

where \[ y = y(x^k(s)) = y(s), \quad x^k = x^k(s), \] and
\[ f(s) = 2[D^2 + (\delta + \eta)^2 + \sigma^2], \]
\[ D^2 = D^2(x^k(s)) = D^2(s), \quad \delta = \delta(x^k(s)), \quad \eta = \eta(x^k(s)), \quad \sigma = \sigma(x^k(s)). \]

The explicit forms of \( \lambda(s) \) and \( f(s) \) determine the explicit form of the equation for \( y \) and therefore its solutions as well.

It is worth to mention that the explicit form of \( \lambda \) and \( f \) can be found after solving the equations for the vector fields \( u \) and \( \xi \): \( \nabla_u u = 0 \), \( \mathcal{L}_{\xi} u = 0 \) under the additional conditions \( g(u, u) = e = \text{const.} \neq 0 \), \( g(u, \xi) = 0 \).
From the form of the equation for \( y \) one can draw a conclusion that the equation for \( y \) could have a form of oscillator equation (homogeneous or non-homogeneous) under the condition

\[
\lambda(s) = \lambda_0 = \text{const.} \neq 0 ,
\]

which is a very special case, requiring additional discussions.

In the case of \( U_n \)-space admitting non-isotropic (non-null), auto-parallel and normalized vector field \( u \) with shear \( \sigma = 0 \) and rotation \( \omega = 0 \)

\[
\lambda = -\frac{2}{n-1} (\theta + \frac{2}{n-1} \theta^2) , \quad D^2 = 0 , \quad \delta = 0 , \quad \eta = 0 , \quad \sigma^2 = 0 , \quad (43)
\]

the equation for \( y \) will have the form

\[
y'' = \left\{ [g(s)]^2 + g'(s) \right\} y , \quad y' = \frac{ds}{d\sigma} , \quad y'' = \frac{d^2 y}{d\sigma^2} , \quad (44)
\]

One solution of the last equation has been found by Ielchin [2] in the form

\[
y = \exp \int g(s) ds . \quad (45)
\]

In the case of \( V_n \)-space (\( n = 4 \)) under the conditions \( \mathcal{L}_{\xi} u = 0 , g(\xi, u) = 0 \), and the conditions for \( u \) to be non-isotropic, normalized and auto parallel vector field for \( L^2 = g(\xi, \xi) \) the following deviation equation can be obtained

\[
u(u L^2) - \frac{2}{n-1} I. L^2 = 2[s M(\xi, \xi) + \mathcal{G}(\xi, \xi) - \mathcal{M}(\xi, \xi)] , \quad (46)
\]

where \( U = I \) (\( V U = 0 \)) , \( s D = s M \).

The last equation for \( L^2 \) (if \( u = \frac{ds}{d\sigma} , \nabla u = \frac{D}{d\sigma} \)) can be written therefore in the form

\[
\frac{d^2 y}{ds^2} + \lambda(s) y = f(s) ,
\]

where

\[
\lambda(s) = \frac{2}{n-1} (I + \frac{1}{n-1} \theta^2) ,
\]

\[
I = \mathcal{G}[\sigma \mathcal{G}] \sigma + \mathcal{G}[\omega \mathcal{G}] \omega + \theta^2 + \frac{1}{n-1} \theta^2 , \quad \text{(Raychaudhuri identity)} ,
\]

\[
f(s) = 2[M^2 + (\delta + \eta)^2 + \sigma^2] ,
\]

\[
M^2 = s M(\xi, \xi) = s M_{\xi \xi} \xi^k \xi^l .
\]
For \( \overline{V}_n \)-spaces with \( \text{Ricci} = 0 \) \( (R_{ij} = 0) \) the equation for \( y \) takes the form

\[
y'' - \frac{2}{(n-1)^2} \theta^2 y = f(s) , \quad y = \frac{dy}{ds} .
\]

(48)

If for such type of spaces the conditions \( \sigma = 0, \omega = 0 \) are fulfilled, then

\[
x = M = 0, \quad \delta = 0, \quad \eta = 0, \quad \sigma^2 = 0, \quad f(s) = 0 .
\]

(49)

The equation for \( y \) will have the form

\[
y'' = \frac{2}{(n-1)^2} \theta^2 y ,
\]

(50)

which by means of the substitutions \[21\] \( y' = y.v(s) \) can be transformed in a Riccati equation

\[
v' + v^2 = \frac{2}{(n-1)^2} \theta^2 .
\]

(51)

If \( v(s) \) is one solution of this equation, then the solutions for \( y \) are solutions of a 1st order linear differential equation

\[
y' - v(s).y = C. \exp(- \int v ds) , \quad C = \text{const} .
\]

(52)

For the special case, when \( \theta = \theta_0 = \text{const} . \neq 0 \), the equation \( y'' = \frac{2}{(n-1)^2} \theta^2 y \) has the form

\[
y'' + \lambda_0.y = 0 , \quad \lambda_0 = -\frac{2}{(n-1)^2} \theta_0^2 < 0 ,
\]

(53)

and solutions of a type

\[
y = L^2 = a. \cosh\left(\frac{\sqrt{2}}{n-1} \theta_0.s\right) + b. \sinh\left(\frac{\sqrt{2}}{n-1} \theta_0.s\right) , \quad a, b = \text{const} .
\]

(54)

Therefore, a deviation equation with non-isotropic, normalized and auto-parallel (time like) vector field \( u \) for the square of a non-isotropic orthogonal (space like) to \( u \) vector field \( \xi \) can be considered as an eventual candidate for the theoretical scheme of gravitational waves detectors because such equations of this type could have, under certain conditions in \( \overline{U}_n \)- and \( \overline{V}_n \)-spaces, the form of an oscillator equation.

Remark. The deviation equations of Synge and Schild have equivalent forms in \( (L_n, g) \) - and in \( (\overline{U}_n, g) \)-spaces. These equations are different in their forms if written in a given basis \[22\]. This allows a general consideration for the both types of spaces.
5 Conclusions

The deviation equation of Synge and Schild and its corresponding projective deviation equation in \((L_n, g)\)-spaces can be considered as a corollary of the equation \(\mathcal{L}_u \xi = 0\) (\(\mathcal{L}_u \xi = 0\)) for a vector field \(\xi\) and an arbitrary vector field \(u\). The last equation appears only as a sufficient, but not necessary condition for the existence of the deviation equation of Synge and Schild, which, therefore, allows other "first integrals" as well.

A deviation equation can also be considered for the square \(L^2\) of a non-isotropic (space like) vector field \(\xi\), which equation appears in fact as equation for an invariant, carrying information about the length of this vector field. In the case of non-isotropic, normalized and auto-parallel vector fields \(u\) in \(U_n\)-spaces and \(V_n\)-spaces this equation could have the form of an oscillator equation under certain conditions. This fact could be explored when theoretical schemes for gravitational waves detectors are considered in a fixed gravitational theory in \((L_n, g)\)-spaces, \(U_n\)-spaces and \(V_n\)-spaces.

Acknowledgments

This work is supported in part by the National Science Foundation in Bulgaria.

References

[1] S. Manoff, Spaces with contravariant and covariant affine connections and metrics. Physics of elementary particles and atomic nucleus (Particles and Nuclei) [Russian Edition: 30 5, 1211-1269 (1999)], [English Edition: 30 5, 527-549 (1999)].

[2] B. Z. Iliev, Normal frames and the validity of the equivalence principle: I. Cases in a neighbourhood and at a point. J. Phys. A: Math. Gen. 29, 6895-6901 (1996).

[3] B. Z. Iliev, Normal frames and the validity of the equivalence principle: II. The case along paths. J. Phys. A: Math. Gen. 30, 4327-4336 (1997).

[4] B. Z. Iliev, Normal frames and the validity of the equivalence principle: III. The case along smooth maps with separable points of self-interaction. J. Phys. A: Math. Gen. 31, 1287-1296 (1998).

[5] B. Z. Iliev, Is the principle of equivalence a principle? Journal of Geometry and Physics 24, 209-222 (1998).

[6] D. Hartley, Normal frames for non-Riemannian connections. Class. and Quantum Grav. 12, L103-L105 (1995).

[7] S. Manoff, Fermi derivative and Fermi-Walker transports over \((L_n, g)\)-spaces. Class. Quantum Grav. 15 2, 465-477 (1998).
[8] S. Manoff, Fermi derivative and Fermi-Walker transports over \((L_n,g)\)-spaces. Intern. J. Mod. Phys. A 13 25, 4289-4308 (1998).

[9] S. Manoff, Conformal derivative and conformal transports over \((L_n,g)\)-spaces. E-print (1999) gr-qc/99 07 095.

[10] S. Manoff, Conformal derivative and conformal transports over \((T_n,g)\)-spaces. Intern. J. Mod. Phys. A 15 5, 679-695 (2000).

[11] S. Manoff, Geodesic and autoparallel equations over differentiable manifolds. Intern. J. Mod. Phys. A 11 21, 3849-3874 (1996).

[12] S. Manoff, Auto-parallel equation as Euler-Lagrange’s equation over spaces with affine connections and metrics Gen. Rel. and Grav. 32 8 (2000) (to appear).

[13] I. Ciufolini, Generalized geodesic deviation equation, Phys. Rev. D34 4, 1014-1017 (1986).

[14] S. Manoff, Lie derivatives and deviation equations in Riemannian spaces. Gen. Rel. and Grav. 11 189-204 (1979).

[15] S. Manoff, Deviation operator in spaces with affine connection. 6th Sov. Grav. Conf. Contr. Papers. (UDN, Moscow, 1984), pp. 231-232.

[16] B. Iliev and S. Manoff, Deviation equations in \(U_n\)-spaces. 5th Sov. Grav. Conf. Contr. Papers. (MGU, Moscow, 1981), p. 122 (in Russian).

[17] B. Iliev and S. Manoff, Deviation equations in spaces with affine connection. Comm. JINR Dubna P2-83-897 1-16 (1983).

[18] S. Manoff, Kinematics of vector fields, in Complex Structures and Vector Fields, eds. S. Dimiev and K. Sekigawa (World Sci. Publ., Singapore, 1995) pp. 61-113.

[19] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, Exact Solutions of Einstein’s Field Equations (VEB Deutscher Verlag der Wissenschaften, Berlin, 1980).

[20] N. S. Swaminarayan, J. L. Safko, A coordinate-free derivation of a generalized geodesic deviation equations, J. Math. Phys. 24 4, 883-885 (1983).

[21] E. Kamke, Differentialgleichungen. Lösungsmethoden und Lösungen. Bd.1. Gewöhnliche Differentialgleichungen. (Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1961).

[22] S. Manoff, Deviation equations of Synge and Schild in spaces with affine connection and metric, and equations for gravitational waves detectors, Comm. JINR Dubna E5-92-19 1-12 (1992).