1-D shallow water equation, with no viscosity and no rotation, for a topography represented by a quadratic function

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Abstract

Based on the linearized shallow water equations with no rotation and no viscosity, in a rectangular channel with topography, the analytical solution for elevation and velocity was obtained. The bathymetry of the rectangular channel is described by a quadratic function. The analytical solution was determined from generalized power series, resulting in numerically convergent power series. This methodology is a useful and easy tool, which can be used by teachers in the teaching of differential equations and modeling.

1. Introduction

Modeling bathymetry in coastal areas allows us to understand and determine changes due to different physical and hydrological processes, which can be used for conservation studies of natural systems and coastal platforms [1].

The most important physical factor in wave propagation in shallow waters is the bathymetry. A change in the seabed represents a change in the frequency and wavelength of the natural modes of oscillation. Its effects can result in human, economic, and material losses [2, 3].

To date, several analytical models of linearized shallow water flow without rotation have been published, with specific characteristics of bathymetry, allowing to establish the resonant modes of the study area. Among the specified bathymetries are flat bottom [4], parabolic [5], exponential [6] and hyperbolic square cosine [7].

The analytical models mentioned use mathematical functions such as Legendre’s differential equation and Bessel’s equation [8]. For example, for a port of rectangular shape, Wang et al (2014) use Legendre’s differential equation and consider a hyperbolic square cosine seabed, and Wang et al (2015) use the Bessel equation considering an exponential seabed. The analytical model proposed by Hernandez-Walls et al (2017), for a flat bottomed channel, uses finite differences and generates a tridiagonal system of equations that is then solved using a recursive form.

Although several studies about numerical solutions have found a way out of hydrological and environmental problems, some of them still lack an analytical model to be validated [9]. In the present work, an analytical method to solve the system of linearized shallow water equations with no rotation or viscosity is presented. We considered a bathymetry that does not depend on the modeling by a specific mathematical function, but it is described by the equation of a curve defined by three points. The solution for elevation and velocity is given in power series, which makes the solution easier to obtain.

The method for the solution of elevation and velocity is based on a system of two partial differential equations for shallow water that depends on velocity flow, elevation around mean sea level, and seafloor topography. This system of equations is decoupled and simplified by considering elevation and velocity as plane waves. In addition, a set of boundary conditions is imposed on the equations system.

1.1. Shallow water

The coordinate system $x, z$ is defined as shown in figure 1. The theory of linearized shallow water, without rotation and viscosity, is reduced to solving the system of partial differential equations [10]:

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where \( \hat{u}(x, t) \) is the average velocity flow that varies with position and time; \( \hat{h}(x, t) \) is the elevation around mean sea level that changes with position and time; \( H(x) \) is the topography of the seafloor, and \( g \) is the acceleration of gravity (see figure 1). The topography is assumed to be invariant in the \( y \) direction, allowing the problem to be simplified to one dimension and visualized as a cross-section on the \( y \) axis.

In order to achieve the solution of the system of equations (1) and (2), independently for elevation and velocity, we decouple the system. This decoupling is achieved by deriving with respect to the time in both equations. The higher order cross derivatives of elevation and velocity are continuous at each point on the bottom of the channel, and this allows the partial derivatives to be permuted \([11]\). Then the equations (1) and (2) can be replaced. The new system of partial differential equations is expressed by the same function in each equation (\( \hat{u} \) and \( \hat{h} \)), which corresponds to,

\[
\frac{\partial^2 \hat{h}}{\partial t^2} = g \frac{\partial^2 \hat{h}}{\partial x^2} + \hat{u} \frac{\partial^2 H}{\partial x^2},
\]

\[
\frac{\partial^2 \hat{h}}{\partial t^2} = g H \frac{\partial^2 \hat{h}}{\partial x^2} + g \left( \frac{\partial \hat{H}}{\partial x} \right) \left( \frac{\partial H}{\partial x} \right).
\]

Elevation and velocity are considered a flat wave with an angular frequency of \( \omega \), so:

\[
\hat{h}(x, t) = \eta(x) e^{i \omega t}
\]

\[
\hat{u}(x, t) = u(x) e^{i \omega t}.
\]

The system of equations (3) and (4), when replaced by the equations (5) and (6) becomes the new system of differential equations with total derivatives,

\[
-\frac{\omega^2}{g} \eta = H \frac{d^2 \eta}{dx^2} + \frac{dH}{dx} \frac{d\eta}{dx}
\]

\[
-\frac{\omega^2}{g} u = 2 \frac{dH}{dx} \frac{du}{dx} + H \frac{d^2 u}{dx^2} + \frac{d^2 H}{dx^2} u.
\]

For the elevation \( \eta(x) \) and the velocity \( u(x) \) the boundary conditions Robin type and Dirichlet type are imposed, respectively, both in the domain of each function \( \Omega \) \([12]\),

\[
x \in \Omega: \quad \gamma \frac{df}{dx} + \beta f = \alpha.
\]

The constants for Robin type and Dirichlet type boundary conditions, which correspond to our case study, are expressed in the following table.

According to the boundary conditions described for a shallow-water channel (table 1 and figure 1), the elevation is interpreted as an incident wave with a given height (\( \eta(0) = \eta_0 \)) that is then fully reflected on the shoreline (\( x = 1, \frac{dx}{d\xi}(1) = 0 \)); the velocity of the water column coming from the open sea is determined by the elevation, the depth of the channel at this point, and the acceleration of gravity \( u(0) = \eta_0 \sqrt{\frac{g}{H_0}} \) and once it reaches the shoreline, the velocity is equal to 0.
2. Methods

The methodology for finding the solution of the differential equation system for shallow water without rotation and viscosity will be shown below. The seabed is determined by bathymetry. Bathymetry is modeled using a quadratic function determined by three points.

2.1. Bathymetry modeled by a quadratic function

It is considered a unit channel \((0 \leq x \leq 1)\) with a bathymetry given by the function,

\[ H(x) = A + Bx + Cx^2, \]

its particular shape is defined by three points on the bathymetric curve (figure 2), determined by the constants \(r\) and \(s\) taken freely. The three points are defined below,

\[
\begin{align*}
P_0&=(0, H_0) \\
P_1&=(r, sH_0), \quad \forall r, s \in [0, 1] \\
P_2&=(1, 0).
\end{align*}
\]

To determine the values of the constants \(A, B\) and \(C\) in the equation (9), the equations system is solved,

\[
\begin{align*}
H_0 &= A \\
sh_0 &= A + Br + Cr^2 \\
0 &= A + B + C.
\end{align*}
\]

The particular function \(H(x) = H_0 + H_0r\alpha + H_0r^2\tau^2\) is obtained, where:

\[
\begin{align*}
\nu &= \frac{r^2 + s - 1}{r(1 - r)} \\
\tau &= \frac{1 - r - s}{r(1 - r)}.
\end{align*}
\]

This particular function is replaced (7) and (8), such that

\[
(1 + \nu x + \tau x^2) \frac{d^2 \eta}{dx^2} + (\nu + 2\tau x) \frac{d\eta}{dx} + k^2 \eta = 0
\]
To determine the elevation and velocity, each is posed as a general power series,

\[ \eta(x) = \sum_{n=0}^{\infty} a_n x^n \]  

\[ u(x) = \sum_{n=0}^{\infty} b_n x^n, \]  

which are subsequently replaced in (11) and (12).

3. Results

For the results, it is necessary to consider the differential equation system (11) and (12), which is replaced by the power series (13) and (14). Boundary conditions for elevation and velocity are considered.

3.1. Elevation

Equation (13) is replaced in (11),

\[ k^2 \sum_{n=0}^{\infty} a_n x^n + (1 + \nu x + \tau x^2) \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + (\nu + 2\tau x) \sum_{n=0}^{\infty} na_n x^{n-1} = 0. \]  

(15)

Similar terms are grouped together with respect to the variable \( x \). A change of the variable is made for the summation indexes, such as the powers of the variable \( x \) coincide,

\[ \sum_{n=0}^{\infty} [(n(n+1)\nu + k^2)a_n + (n+1)^2 \nu a_{n+1} + (n+2)(n+1)a_{n+2}] x^n = 0. \]  

(16)

From the above, the succession is obtained,

\[ a_{n+2} = \frac{-1}{(n+2)(n+1)} [(n(n+1)\nu + k^2)a_n + (n+1)^2 \nu a_{n+1}]. \]  

(17)

To determine the solution of the elevation, two cases are contemplated where \( a_0 \) and \( a_1 \) will have null or non-null values. Each case determines a term of the general solution,

\[ \eta(x) = \eta_1(x) + \eta_2(x). \]  

(18)

For the first case \( a_0 \neq 0 \) and \( a_1 = 0 \) you get the following solution,

\[ \eta_1 = a_0 \left[ 1 + \sum_{n=2}^{\infty} (-1)^{n+1} W_n x^n \right], \]  

(19)

here, succession \( W_n \) \( \forall n \geq 2 \) is given by,

\[ W_0 = -1 \]
\[ W_1 = 0 \]
\[ W_2 = \frac{k^2}{2} \]
\[ W_3 = \frac{\nu k^2}{3} \]
\[ \vdots \]
\[ W_n = \frac{(n-1)\nu}{n} W_{n-1} - \frac{(n-1)(n-2)\nu + k^2}{n(n-1)} W_{n-2}. \]
For the second case \( a_0 = 0 \) and \( a_1 = 0 \) you get the following solution,
\[
\eta_2(x) = a_1 \left[ x + \sum_{n=2}^{\infty} (-1)^{n+1} V_n x^n \right].
\]
(20)

here, succession \( V_n \ \forall \ n \geq 2 \) is given by,
\[
\begin{align*}
V_0 &= 0 \\
V_1 &= 1 \\
V_2 &= \frac{1}{2} \nu \\
V_3 &= -\frac{2 \tau + k^2}{6} + \frac{1}{3} \nu^2 \\
& \vdots \\
V_n &= \frac{n-1}{n} \nu V_{n-1} - \frac{(n-1)(n-2) \tau + k^2}{n(n-1)} V_{n-2}.
\end{align*}
\]

Using equation (18) we have,
\[
\eta(x) = a_0 \left[ 1 + \sum_{n=2}^{\infty} (-1)^{n+1} W_n x^n \right] + a_1 \left[ x + \sum_{n=2}^{\infty} (-1)^{n+1} V_n x^n \right].
\]
(21)

here, we start from equation (21) and consider the Robin-type boundary conditions for the elevation, from which we obtain the coefficients \( a_0 \) and \( a_1 \),
\[
a_0 = \eta_0 \\
a_1 = -\frac{\eta_0 \sum_{n=2}^{\infty} (-1)^{n+1} n W_n}{1 + \sum_{n=2}^{\infty} (-1)^{n+1} V_n}
\]

By finding \( a_0 \) and \( a_1 \), the analytical solution of the elevation for the shallow water equations, without rotation and viscosity, is obtained in a channel with bathymetry modeled by a quadratic equation determined by three points.

3.2. Velocity

Equation (14) is replaced in (12),
\[
(1 + \nu x + \tau x^2) \sum_{n=0}^{\infty} n(n-1)b_n x^{n-2} + 2(\nu + 2 \tau x) \sum_{n=0}^{\infty} n b_n x^{n-1} + (2\tau + k^2) \sum_{n=0}^{\infty} b_n x^n = 0.
\]
(22)

Similar terms are grouped together with respect to the variable \( x \). A change of the variable is made for the summation indexes, such as the powers of the variable \( x \) coincide,
\[
\sum_{n=0}^{\infty} \left[ ((n+2)(n+1) \tau + k^2) b_n + (n+2)(n+1) \nu b_{n+1} + (n+2)(n+1) b_{n+2} \right] x^n = 0.
\]
(23)

From the above, the succession is obtained,
\[
b_{n+2} = -\frac{1}{(n+2)(n+1)} \left[ ((n+2)(n+1) \tau + k^2) b_n + (n+2)(n+1) \nu b_{n+1} \right].
\]
(24)

To determine the solution of the velocity, two cases are contemplated where \( b_0 \) and \( b_1 \) will have null or non-null values. Each case determines a term of the general solution,
\[
u(x) = u_1(x) + u_2(x).
\]
(25)

For the first case \( b_0 \neq 0 \) and \( b_1 = 0 \) you get the following solution,
\[
u_1(x) = \frac{1}{b_0} \left[ x + \sum_{n=2}^{\infty} (-1)^{n+1} F_n x^n \right],
\]
(26)
here, succession $F_n \forall n \geq 2$ is given by,

\[
F_0 = -1 \\
F_1 = 0 \\
F_2 = \frac{2\tau + k^2}{2} \\
F_3 = \nu \frac{2\tau + k^2}{2} \\
\vdots \\
F_n = \nu F_{n-1} - \frac{n(n-1)\tau + k^2}{n(n-1)} F_{n-2}.
\]

For the second case $b_0 = 0$ and $b_1 = 0$ you get the following solution,

\[
u_2(x) = b_1 \left[ x + \sum_{n=2}^{\infty} (-1)^{n+1} G_n x^n \right],
\]

here, succession $G_n \forall n \geq 2$ is given by,

\[
G_0 = 0 \\
G_1 = 1 \\
G_2 = \nu \\
G_3 = \nu^2 - \frac{6\tau + k^2}{6} \\
\vdots \\
G_n = \nu G_{n-1} - \frac{n(n-1)\tau + k^2}{n(n-1)} G_{n-2}.
\]

Using equation (25) we have,

\[
u(x) = b_0 \left[ 1 + \sum_{n=2}^{\infty} (-1)^{n+1} F_n x^n \right] + b_1 \left[ x + \sum_{n=2}^{\infty} (-1)^{n+1} G_n x^n \right],
\]

here, we start from equation (28) and consider the Dirichlet-type boundary conditions for the velocity, from which we obtain the coefficients $b_0$ and $b_1$,

\[
b_0 = \eta_0 \sqrt{\frac{g}{H_0}} \\
b_1 = -\frac{\eta_0}{H_0} \left[ 1 + \sum_{n=2}^{\infty} (-1)^{n+1} F_n \right].
\]

By finding $b_0$ and $b_1$, the analytical solution of the velocity for the equations of shallow water, without rotation and viscosity, is obtained in a channel with bathymetry modeled by a quadratic equation determined by three points.

### 3.3. Graphic analysis of convergence

To determine the regions in the unitary channel where the methodological proposal is convergent, equations (21) and (28) have been evaluated. For the constants $r$ and $s$, which are the points that define the form of bathymetry (figure 2) and its quadratic equation (10), they are assigned values from zero to one. To determine the points in the channel $(r, s)$ where the solutions for elevation and velocity are converging, the first 1000 terms of the series were taken. The last two terms of these finite series are compared, in a difference under absolute value.

In figures 3 and 4, the area where the elevation and velocity are converging, respectively, is shown in the unit channel. For the system of differential equations (11) and (12) to have a solution it is necessary to determine the zone where both, elevation and velocity, are converging (figure 5).

### 4. Conclusions

In this work, the analytical solution for the system of differential equations of linearized shallow water, without rotation and viscosity, has been deduced for a channel with bathymetry modeled by a quadratic function defined by three points. Due to the simplicity of the model, and the consideration of the elevation and velocity as plane...
waves, this allowed the solution for the elevation and velocity to be given in power series, and this, in the future, will allow the generalization of the model to a finite number of curves that describe the bathymetry.

The solution described in this article can be used for elongated structures such as rivers, well-differentiated watercourses, elongated ports or ports with a narrow and long mouth. For shapes such as bays, structures with a square or rectangular floor plan, it is necessary to consider a box-type model, which considers the width of a channel as a function of its length, ($y = w(x)$). This last consideration results in a more complicated form for the analytical solution, so power series may not be suitable for modeling velocity and elevation.

It was found that the solutions converge numerically from a $N > 1000$, which opens a field of research to derive the analytical convergence of the solutions. The implemented methodology makes its use as an excellent teaching exercise for students of the basic professional training courses and easy to implement in the programming courses. The use of complex mathematical functions is not necessary.
Figure 5. Intersection of both convergences, elevation and velocity, in the unit channel.

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