Bialgebras for Stanley symmetric functions

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Abstract

We construct a non-commutative, non-cocommutative, graded bialgebra $\Pi$ with a basis indexed by the permutations in all finite symmetric groups. Unlike the formally similar Malvenuto-Poirier-Reutenauer Hopf algebra, this bialgebra does not have finite graded dimension. After giving formulas for the product and coproduct, we show that there is a natural morphism from $\Pi$ to the algebra of quasi-symmetric functions, under which the image of a permutation is its associated Stanley symmetric function. As an application, we use this morphism to derive some new enumerative identities. We also describe analogues of $\Pi$ for the other classical types. In these cases, the relevant objects are module coalgebras rather than bialgebras, but there are again natural morphisms to the quasi-symmetric functions, under which the image of a signed permutation is the corresponding Stanley symmetric function of type B, C, or D.

Contents

1 Introduction 1
2 Preliminaries 4
  2.1 Algebras, coalgebras, and bialgebras ................................................... 4
  2.2 Shuffle algebra ................................................................. 5
3 Bialgebras of words 6
4 Bialgebras of permutations 9
5 Combinatorial bialgebras 14
6 Stanley symmetric functions 18
  6.1 Type A ................................................................. 18
  6.2 Types B, C, and D ................................................................. 19

1 Introduction

Fix a positive integer $n$ and let $S_n$ denote the symmetric group of permutations of $\{1, 2, \ldots, n\}$, which we write in one-line notation as words $\pi = \pi_1 \pi_2 \cdots \pi_n$ containing each $i \in \{1, 2, \ldots, n\}$ as a
The set $A$ is the partial order whose covering relations are

$$\pi_1 \cdots \pi_i \pi_{i+1} \cdots \pi_n \prec \pi_1 \cdots \pi_i+1 \pi_{i+1} \cdots \pi_n \quad \text{whenever} \quad \pi_i < \pi_i+1.$$ 

A reduced word for $\pi \in S_n$ corresponds to a maximal chain from the identity permutation $123 \cdots n$ to $\pi$ in this order, and the length $\ell(\pi)$ of a permutation $\pi$ is the number of covering relations in any such chain. We denote the number of reduced words for $\pi \in S_n$ by $r(\pi)$.

The original motivation for this paper comes from a sequence of identities relating counts of reduced words for certain permutations. Given a word $w = w_1 w_2 \cdots w_n$ with distinct integer letters, define the associated flattened word by $fl(w) = \phi(w_1)\phi(w_2)\cdots \phi(w_n) \in S_n$ where $\phi$ is the unique order-preserving bijection $\{w_1, w_2, \ldots, w_n\} \to \{1, 2, \ldots, n\}$. Now consider the subsets $A(n)$ and $B(m,n)$ of $S_n$ defined recursively as follows.

Let $A(n)$ be the set of permutations $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$ with $\pi_1 = \pi_n+1$ and $fl(\pi_2 \pi_3 \cdots \pi_{n-1}) \in A(n-2)$, where $A(1) = \{1\}$ and $A(2) = \{21\}$. For example:

$$A(3) = \{231, 312\},$$

$$A(4) = \{2431, 3412, 4213\},$$

$$A(5) = \{24531, 25341, 25431, 34512, 35142, 42513, 45213, 45132, 52314, 53124\}.$$

Next, let $B(1,n) = B(n,n) = \{n \cdots 321\}$ and define $B(m,n)$ for $1 < m < n$ to be the set of permutations $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$ with either $\pi_1 = m$ and $fl(\pi_2 \pi_3 \cdots \pi_n) \in B(m-1,n-1)$ or $\pi_n = m$ and $fl(\pi_1 \pi_2 \cdots \pi_{n-1}) \in B(m,n-1)$. For example:

$$B(2,3) = \{231, 312\},$$

$$B(2,4) = \{2431, 3412, 4132\},$$

$$B(3,4) = \{3241, 3421, 4213\},$$

$$B(2,5) = \{25431, 35412, 45132, 51432\},$$

$$B(3,5) = \{32541, 34512, 42513, 45213, 52143\},$$

$$B(4,5) = \{43251, 43512, 45213, 53214\}.$$

The set $A(n)$ has size $(n-1)!! = (n-1)(n-3)(n-5)\cdots$ while $B(m,n)$ has size $\frac{(n-1)!}{(m-1)!}$. One can show that all elements of $A(n)$ have length $\binom{n}{2} + \binom{q}{2}$ for $p = \left\lfloor \frac{n+1}{2} \right\rfloor$ and $q = \left\lceil \frac{n+1}{2} \right\rceil$, while all elements of $B(m,n)$ have length $\binom{n}{2} + \binom{n-m+1}{2}$.

On computing $r(\pi)$ for $\pi \in A(n)$ and $\pi \in B(m,n)$, one observes the following phenomenon:

**Proposition 1.1.** If $p \in \{\left\lfloor \frac{n+1}{2} \right\rfloor, \left\lceil \frac{n+1}{2} \right\rceil\}$ then $\sum_{\pi \in A(n)} r(\pi) = \sum_{\pi \in B(p,n)} r(\pi)$.

For example, it holds that

$$\sum_{\pi \in A(5)} r(\pi) = 9 + 10 + 5 + 16 + 16 + 5 + 10 + 9 = 19 + 5 + 16 + 16 + 5 + 19 = \sum_{\pi \in B(3,5)} r(\pi).$$

The only method we know to prove Proposition 1.1 is algebraic and indirect. We suspect that there should exist a natural bijection $\bigcup_{\pi \in A(n)} R(\pi) \leftrightarrow \bigcup_{\pi \in B(p,n)} R(\pi)$ for $p \in \{\left\lfloor \frac{n+1}{2} \right\rfloor, \left\lceil \frac{n+1}{2} \right\rceil\}$, where $R(\pi)$ denotes the set of reduced words for $\pi \in S_n$. Computer experiments indicate that such a bijection might be obtained from a variation of the Little map introduced in [18]. It remains an open problem to construct this.
A non-bijective proof of Proposition 1.1 goes as follows. For integers \( p, q > 0 \), let
\[
N(p, q) = N(q, p) = \binom{p+q}{p} \cdot r(p \cdots 321) \cdot r(q \cdots 321)
\]
where \( P = \binom{p}{q} \) and \( Q = \binom{q}{2} \). The following combines [7] Theorem 3.7 and [13] Theorem 1.4:

**Proposition 1.2** (See [7, 13]). If \( p = \lceil \frac{p+1}{2} \rceil \) and \( q = \lceil \frac{q+1}{2} \rceil \) then \( N(p, q) = \sum_{w \in A(n)} r(w) \).

The next statement, which is a corollary of our new results in this paper, implies Proposition 1.1:

**Proposition 1.3.** If \( 1 \leq p \leq n \) and \( q = n + 1 - p \) then \( N(p, q) = \sum_{w \in B(p, n)} r(w) \).

Write \( \pi = \pi' \pi'' \) to denote a length-additive factorization of \( \pi \in S_n \) as a product of \( \pi', \pi'' \in S_n \). To prove the last result, we interpret the two sides of Proposition 1.3 as the images of a product \( \Pi \cdot \Pi \) of the Malvenuto-Poirier-Reutenauer Hopf algebra [3, 20]. Part (b) is Corollary 4.2 and part (c) follows from Corollary 3.6. Given these results, we can quickly derive Proposition 1.3 as the images of a product of two elements of a certain bialgebra \( \Pi \) under a natural homomorphism \( \Pi \to \mathbb{Q} \). The following summarizes several of our main results, and gives the facts needed for this approach:

**Theorem 1.4.** Let \( k \) be a field. There exists a graded \( k \)-bialgebra \( \Pi \) with a basis given by the symbols \( [\pi] \), where \( \pi \) ranges over all elements of \( S_1 \sqcup S_2 \sqcup S_3 \sqcup \cdots \), with the following properties:

(a) The product of \( [m \cdots 321] \) and \( [(n - m + 1) \cdots 321] \) in \( \Pi \) is \( \sum_{\pi \in B(m, n)} [\pi] \).

(b) The coproduct of \( \Pi \) is the linear map with \( [\pi] \mapsto \sum_{\pi = \pi' \pi''} [\pi'] \otimes [\pi''] \).

(c) If \( \text{char}(k) = 0 \) then the linear map \( \Pi \to k \) with \( [\pi] \mapsto r(\pi) / s(\pi)! \) is an algebra morphism.

The bialgebra \( \Pi \) is non-commutative and non-cocommutative, and is a sub-object of a larger bialgebra of words \( \mathcal{W} \) constructed in Section 3. The bialgebra \( \mathcal{W} \) can be viewed as a generalization of the Malvenuto-Poirier-Reutenauer Hopf algebra [3] [20].

Theorem 4.1 asserts the existence of the bialgebra \( \Pi \). Part (a) of Theorem 1.4 is a special case of Theorem 4.6 which describes, more generally, the product in \( \Pi \) of any two basis elements \([\pi']\) and \([\pi'']\). Part (b) is Corollary 1.2 and part (c) follows from Corollary 3.6. Given these results, we can quickly derive Proposition 1.3 in the following way:

**Proof of Proposition 1.3**. Expressing the product in \( \Pi \) of \( [p \cdots 321] \) and \( [q \cdots 321] \) as in (a) and then applying the morphism in (c) gives \( N(p, q) = \sum_{\pi \in B(p, n)} r(\pi) / (P + Q)! \) for \( P = \binom{p}{q} \) and \( Q = \binom{q}{2} \). \( \square \)

The bialgebra of \( \Pi \) is of interest on its own, and can be used to give a simple construction of the Stanley symmetric function \( F_\pi \) of a permutation \( \pi \in S_n \). The precise definition of \( F_\pi \) is reviewed in Section 6.1.

Let \( \text{QSym} \) (see Section 5) denote the Hopf algebra of quasi-symmetric functions over \( k \) and write \( \zeta : \text{QSym} \to k \) for the algebra morphism that sets \( x_1 = 1 \) and \( x_2 = x_3 = \cdots = 0 \). Next let \( \zeta > : \Pi \to k \) be the linear map with

\[
\zeta > ([\pi]) = \begin{cases} 
1 & \text{if } \pi \text{ is such that } \pi_i = i - 1 \text{ whenever } \pi_i < i, \\
0 & \text{otherwise}.
\end{cases}
\]

Equivalently, \( \zeta > ([\pi]) = 1 \) if and only if \( \pi \) has a cycle decomposition in which every factor has the form \((b, b - 1, \cdots, a + 1, a)\) for some integers \( a < b \); this occurs precisely when \( \pi \) has a decreasing reduced word. The following combines Theorem 5.2 and Proposition 6.1.
Theorem 1.5. There is a unique graded bialgebra morphism \( \Psi : \Pi \to \text{QSym} \) that satisfies \( \zeta_\Pi = \zeta_{\text{QSym}} \circ \Psi_\Pi \). For this map, \( \Psi_\Pi([\pi]) = F_\pi \) is the Stanley symmetric function of \( \pi \in S_n \).

This result lets us lift enumerative identities like Proposition 1.3 to the level of symmetric functions; see, for example, Corollary 6.4.

There are also analogues of \( \Pi \) in types B/C and D, which can be used to give a simple algebraic construction of the Stanley symmetric functions in the other classical types. The relevant objects \( \Pi^B \) and \( \Pi^D \) are vector spaces spanned by signed permutations. These spaces are no longer bialgebras, but are naturally interpreted as graded \( \Pi \)-module coalgebras. It is an open problem to find an explicit formula for the \( \Pi \)-module action on these spaces, in the style of Theorem 4.6; see Problem 6.8.

The Hopf algebra of quasi-symmetric functions may be viewed as a \( \Pi \)-module coalgebra via (an analogue of) the morphism in Theorem 1.5. In Section 6.2, we show that there are canonical module coalgebra morphisms \( \Pi^B \to \text{QSym} \) and \( \Pi^D \to \text{QSym} \) under which the image of a signed permutation is precisely the associated Stanley symmetric function of type B, C, or D.

Here is a brief outline of what follows. After some preliminaries in Section 2, we construct the bialgebras \( \Pi \) and \( \Pi^B \) in Sections 3 and 4. In Section 5, we review and slightly extend some basic facts about combinatorial coalgebras and Hopf algebras from [1]. Finally, in Section 6 we discuss the relationship between our constructions and Stanley symmetric functions.

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2 Preliminaries

Let \( \mathbb{Z} \supset \mathbb{N} \supset \mathbb{P} \) denote the sets of all integers, nonnegative integers, and positive integers.

2.1 Algebras, coalgebras, and bialgebras

Throughout, we fix a field \( k \) and write \( \otimes = \otimes_k \) for the usual tensor product. We briefly review the notions of \( k \)-algebras, coalgebras, and bialgebras; for more background, see [8] or [12].

Definition 2.1. A \( k \)-algebra is a triple \( (A, \nabla, \iota) \) where \( A \) is a \( k \)-vector space and \( \nabla : A \otimes A \to A \) and \( \iota : k \to A \) are linear maps (the product and unit) making these diagrams commute:

\[
\begin{align*}
\begin{array}{ccc}
A \otimes A & \xrightarrow{\iota \otimes \iota} & A \otimes k \\
\nabla & \downarrow & \nabla \circ \iota \\
A & \xrightarrow{\iota \otimes \iota} & k \otimes A \\
\end{array}
\end{align*}
\]

The unit map \( \iota : k \to A \) of an algebra is completely determined by the unit element \( \iota(1) \in A \).

4
Definition 2.2. A \( k \)-coalgebra is a triple \((A, \Delta, \epsilon)\) where \(A\) is a \(k\)-vector space and \(\Delta : A \to A \otimes A\) and \(\epsilon : A \to k\) are linear maps (the coproduct and counit) making these diagrams commute:

\[
\begin{array}{ccc}
A & \xrightarrow{\Delta} & A \otimes A \\
\downarrow & & \downarrow \\
k \otimes A & \xrightarrow{\epsilon \otimes \text{id}} & A \otimes k
\end{array}
\quad \quad \quad \quad
\begin{array}{ccc}
A \otimes k & \xleftarrow{\text{id} \otimes A} & A \otimes A \\
\downarrow & & \downarrow \\
A & \xleftarrow{\Delta} & A \otimes A
\end{array}
\]

Write \( \beta : A \otimes B \xrightarrow{\sim} B \otimes A \) for the linear isomorphism with \(a \otimes b \mapsto b \otimes a\). An algebra is commutative if \(\nabla \circ \beta = \nabla\). A coalgebra is cocommutative if \(\beta \circ \Delta = \Delta\).

Definition 2.3. A \( k \)-bialgebra is a tuple \((A, \nabla, \iota, \Delta, \epsilon)\) where \((A, \nabla, \iota)\) is a \(k\)-algebra, \((A, \Delta, \epsilon)\) is a \(k\)-coalgebra, the composition \(\epsilon \circ \iota\) is the identity map \(k \to k\), and these diagrams commute:

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{\nabla} & A \\
\downarrow & & \downarrow \\
A & \xrightarrow{\Delta} & A \otimes A
\end{array}
\quad \quad \quad \quad
\begin{array}{ccc}
A & \xrightarrow{\epsilon} & A \otimes k \\
\downarrow & & \downarrow \\
A & \xrightarrow{\epsilon} & k \otimes A
\end{array}
\]

Going forward, we often refer to \(k\)-vector spaces, \(k\)-algebras, \(k\)-coalgebras, and \(k\)-bialgebras simply as vector spaces, algebras, coalgebras, and bialgebras.

A morphism of \((\text{bi, co})\) algebras is a linear map that commutes with the relevant \((\text{co})\)unit and \((\text{co})\)product maps. If \(A\) is an algebra then \(A \otimes A\) is an algebra with product \((\nabla \otimes \nabla) \circ (\text{id} \otimes \beta \otimes \text{id})\) and unit \((\iota \otimes \iota) \circ (k \xrightarrow{\sim} k \otimes k)\). If \(A\) is a coalgebra then \(A \otimes A\) becomes a coalgebra in a similar way. The diagrams \((2.1)\) express that the coproduct and counit of a bialgebra are algebra morphisms, and that the product and unit are coalgebra morphisms.

Given a bialgebra \((H, \nabla, \iota, \Delta, \epsilon)\) and linear maps \(f, g : H \to H\), define \(f \ast g = \nabla \circ (f \otimes g) \circ \Delta\). The operation \(\ast\), called the convolution product, makes the vector space \(\text{End}(H)\) of linear maps \(H \to H\) into a \(k\)-algebra with unit element \(\iota \circ \epsilon\), called the convolution algebra of \(H\). The bialgebra \(H\) is a Hopf algebra if the identity map \(\text{id} : H \to H\) has a left and right inverse \(S : H \to H\) in the convolution algebra. The morphism \(S\) is called the antipode of \(H\); if it exists, then it is the unique linear map \(H \to H\) with \(\nabla \circ (\text{id} \otimes S) \circ \Delta = \nabla \circ (S \otimes \text{id}) \circ \Delta = \iota \circ \epsilon\).

A vector space \(V\) is graded if it has a direct sum decomposition \(V = \bigoplus_{n \in \mathbb{N}} V_n\). A linear map \(f : U \to V\) between graded vector spaces is graded if it has the form \(f = \bigoplus_{n \in \mathbb{N}} f_n\) where each \(f_n : U_n \to V_n\) is linear. An algebra \((V, \nabla, \iota)\) is graded if \(V\) is graded and the product and unit are graded linear maps, where \(V \otimes V\) is identified with the graded vector space \(\bigoplus_{n \in \mathbb{N}} \left( \bigoplus_{i+j=n} V_i \otimes V_j \right)\) and \(k\) is viewed as the graded vector space in which all elements have degree zero. Similarly, a coalgebra \((V, \Delta, \epsilon)\) is graded if \(V\) is graded and the coproduct and counit are graded linear maps.

A bialgebra is graded if it is graded as both an algebra and a coalgebra. A morphism of graded \((\text{bi, co})\) algebras is a morphism of graded \((\text{bi, co})\) algebras that is a graded linear map.

### 2.2 Shuffle algebra

Throughout, we use the term word to mean a finite sequence of positive integers. For each \(n \in \mathbb{N}\), let \(\text{Shuffle}_n\) be the \(k\)-vector space whose basis is the set of all \(n\)-letter words, so that \(\text{Shuffle}_0\) is the 1-dimensional vector space spanned by the unique empty word \(\emptyset\). Write \(\text{Shuffle} = \bigoplus_{n \in \mathbb{N}} \text{Shuffle}_n\) for the corresponding graded vector space.
When \( w = w_1 w_2 \cdots w_n \) is a word and \( I = \{ i_1 < i_2 < \cdots < i_k \} \subset \{ 1, 2, \ldots, n \} \), we set \( w|_I = w_{i_1} w_{i_2} \cdots w_{i_k} \). Given words \( u = u_1 u_2 \cdots u_m \) and \( v = v_1 v_2 \cdots v_n \), define

\[
    u \uplus v = \sum_{I \subseteq \{ 1, 2, \ldots, m+n \}} \uplus_I(u, v)
\]

where \( w = \uplus_I(u, v) \) is the unique \((m+n)\)-letter word with \( w|_I = u \) and \( w|_I^c = v \). Multiplicities may result in this sum; for example, \( 12 \uplus 21 = 2 \cdot 1221 + 1212 + 2121 + 2 \cdot 2112 \). Let \( u \circ v \) denote the concatenation of \( u \) and \( v \). Both operations \( \uplus \) and \( \circ \) extend to graded linear maps \( \text{Shuffle} \otimes \text{Shuffle} \to \text{Shuffle} \). If \( u \) and \( v \) are nonempty, and \( u' \) and \( v' \) are the subwords given by omitting the first letters, then \( u \uplus v = u_1 \circ (u' \uplus v) + v_1 \circ (u \uplus v') \). We typically suppress the symbol \( \circ \) and write \( uv \) for \( u \circ v \).

Define \( \iota : k \to \text{Shuffle} \) and \( \epsilon : \text{Shuffle} \to k \) to be the linear maps with \( \iota(1) = \emptyset \) and \( \epsilon(\emptyset) = 1 \) and \( \epsilon(w) = 0 \) for all words \( w \neq \emptyset \). Define \( \Delta : \text{Shuffle} \to \text{Shuffle} \otimes \text{Shuffle} \) to be the linear map with

\[
    \Delta(w) = \sum_{i=0}^{n} w_1 \cdots w_i \otimes w_{i+1} \cdots w_n
\]

for each \( n \)-letter word \( w \), so that \( \Delta(\emptyset) = \emptyset \otimes \emptyset \). The tuple \( (\text{Shuffle}, \uplus, \iota, \epsilon, \Delta, \epsilon) \) is a graded Hopf algebra, called the \textit{shuffle algebra} \cite{26} \S 1.4, which is commutative but not cocommutative. Its antipode is the linear map with \( S(w) = (-1)^{n} w_n \cdots w_2 w_1 \) for words \( w = w_1 w_2 \cdots w_n \).

3 Bialgebras of words

Let \( \text{End}(\text{Shuffle}) \) denote the vector space of \( k \)-linear maps \( \text{Shuffle} \to \text{Shuffle} \), viewed as a \( k \)-algebra with respect to the convolution product \( f \circ g \mapsto f \ast g := \uplus \circ (f \otimes g) \circ \Delta \). One can sometimes construct interesting bialgebras by pairing subalgebras of convolution algebras with a compatible coproduct. This is our approach here, mimicking the description of the Malvenuto-Poirier-Reutenauer algebra in \cite{15, 20}.

Let \( w = w_1 w_2 \cdots w_m \) be a word. When \( m > 0 \), let \( \max(w) = \max\{ w_1, w_2, \ldots, w_m \} \), and define \( \max(\emptyset) = 0 \). If we view \( w \) as a map \( \{ 1, 2, \ldots, m \} \to \mathbb{P} \), and if \( v \) is a word with at least \( \max(w) \) letters, then the composition \( v \circ w = v_{w_1} v_{w_2} \cdots v_{w_m} \) is another \( m \)-letter word. We interpret \( v \circ \emptyset \) as \( \emptyset \). For each word \( w \) with \( \max(w) \leq n \in \mathbb{N} \), define \( [w, n] \in \text{End}(\text{Shuffle}) \) to be the linear map with

\[
    [w, n](v) = \begin{cases} v \circ w = v_{w_1} v_{w_2} \cdots v_{w_m} & \text{if } v \text{ is a word with length } n \\ 0 & \text{for all other words } v. \end{cases}
\]

Observe that \( [\emptyset, 0] = \iota \circ \epsilon \) is the unit element of \( \text{End}(\text{Shuffle}) \). The following is evident:

\textbf{Lemma 3.1.} If \( w \) is a word with \( \max(w) \leq n \in \mathbb{N} \), then \( [w, n](123 \cdots n) = w \).

Thus \( [v, m] = [w, n] \) in \( \text{End}(\text{Shuffle}) \) if and only if \( v = w \) and \( m = n \). Let \( W_n \) for \( n \in \mathbb{N} \) be the set of endomorphisms \( [w, n] \) where \( w \) is a word with \( \max(w) \leq n \). Define \( W = \bigcup_{n \in \mathbb{N}} W_n \). Let \( W \) (respectively, \( W_n \)) be the subspace of \( \text{End}(\text{Shuffle}) \) spanned by \( W \) (respectively, \( W_n \)). Since the set of words is a basis for \( \text{Shuffle} \), \textbf{Lemma 3.1} implies that \( W \) is linearly independent. Therefore:

\textbf{Corollary 3.2.} The set \( W \) is a basis for \( W = \bigoplus_{n \in \mathbb{N}} W_n \).
When \( w = w_1 w_2 \cdots w_m \) is a word and \( n \in \mathbb{N} \), define \( w \uparrow n = (w_1 + n)(w_2 + n) \cdots (w_m + n) \) to be the word formed by incrementing each letter of \( w \) by \( n \). When \( w^1, w^2, \ldots, w^l \) is a finite sequence of words with \( \max(w^i) \leq n \) and \( a_1, a_2, \ldots, a_l \in \mathbb{k} \), let \( \sum_i a_i w^i, n = \sum_i a_i [w^i, n] \in \mathbb{W}_n \). Finally, define \( \nabla_{\uparrow} : \mathbb{W} \otimes \mathbb{W} \rightarrow \mathbb{W} \) to be the linear map with

\[
\nabla_{\uparrow}([v, m] \otimes [w, n]) = [v \uplus (w \uparrow m), n + m] \in \mathbb{W}_{m+n}
\]

for \( [v, m] \in \mathbb{W}_m \) and \([w, n] \in \mathbb{W}_n \). For example, \( \nabla_{\uparrow}([123] \otimes [22]) = [125, 5] + [152, 5] + [512, 5] \). Note that \( v \uplus (w \uparrow m) \) is the multiplicity-free sum of all words \( u \) with \( u \cap \{1, 2, \ldots, m\} = v \) and \( u \cap (m + n) = w \uparrow m \), where \( u \cap S \) is the subword formed by removing all letters not in \( S \).

**Proposition 3.3.** If \( \alpha, \beta \in \mathbb{W} \) then \( \nabla_{\uparrow}(\alpha \otimes \beta) = \alpha \ast \beta \).

Thus, \( \nabla_{\uparrow} \) is associative and \( \mathbb{W} \) is a subalgebra of the convolution algebra \( \text{End}(\text{Shuffle}) \).

**Proof.** Let \( [v, m] \in \mathbb{W}_m \) and \([w, n] \in \mathbb{W}_n \). If \( u = u_1 u_2 \cdots u_l \) is a word then \( ([v, m] * [w, n])(u) \) is the sum over \( i \in \{0, 1, \ldots, l\} \) of \( [v, m](u_1 u_2 \cdots u_i) \uplus [w, n](u_{i+1} u_{i+2} \cdots u_l) \), which is precisely \((u \circ v) \uplus (u \circ w \uparrow m) = [v \uplus (w \uparrow m), n + m](u) \) if \( l = m + n \) and zero otherwise.

The coalgebra structure of Shuffle induces a coalgebra structure on \( \mathbb{W} \). For \( n \in \mathbb{N} \), let \( \rho_n : \text{Shuffle} \rightarrow \mathbb{W}_n \) be the surjective linear map with \( \rho_n(w) = [w, n] \) if \( w \) is a word with \( \max(w) \leq n \) and with \( \rho_n(w) = 0 \) if \( \max(w) > n \). Write \( \epsilon \) and \( \Delta \) for the counit and coproduct of Shuffle.

**Lemma 3.4.** For each \( n \in \mathbb{N} \), it holds that \( \ker(\rho_n) \subseteq \ker(\epsilon) \cap \ker((\rho_n \otimes \rho_n) \circ \Delta) \).

**Proof.** A basis for \( ker(\rho_n) \) is the set of words \( w \in \text{Shuffle} \) with \( \max(w) > n \). Such a word \( w = w_1 w_2 \cdots w_m \) is nonempty and \( \max(w_1 \cdots w_i) > n \) or \( \max(w_{i+1} \cdots w_m) > n \) for each \( 0 \leq i \leq m \).

It follows that there are unique linear maps \( \epsilon_{\otimes} : \mathbb{W} \rightarrow \mathbb{k} \) and \( \Delta_{\otimes} : \mathbb{W} \rightarrow \mathbb{W} \otimes \mathbb{W} \) satisfying \( \epsilon_{\otimes}(\rho_n(w)) = \epsilon(w) \) and \( \Delta_{\otimes}(\rho_n(w)) \) for each word \( w \) and integer \( n \in \mathbb{N} \). If \([w, n] \in \mathbb{W}_n \) and \( w = w_1 w_2 \cdots w_m \), then these maps have the explicit formulas

\[
\epsilon_{\otimes}([w, n]) = \begin{cases} 1 & \text{if } w = \emptyset \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \Delta_{\otimes}([w, n]) = \sum_{i=0}^{n} [w_1 \cdots w_i, n] \otimes [w_{i+1} \cdots w_m, n].
\]

Write \( \iota_{\uparrow} \) for the linear map \( \mathbb{k} \rightarrow \mathbb{W} \) with \( \iota_{\uparrow}(1) = [0, 0] \). We consider \( \mathbb{W} \) to be a graded vector space in which \([w, n] \in \mathbb{W}_n \) is homogeneous with degree \( \ell(w) \), the length of the word \( w \).

**Theorem 3.5.** \( (\mathbb{W}, \nabla_{\uparrow}, \iota_{\uparrow}, \Delta_{\otimes}, \epsilon_{\otimes}) \) is a graded bialgebra, but not a Hopf algebra.

**Proof.** Lemma 3.4 implies that \( (\mathbb{W}, \nabla_{\uparrow}, \iota_{\uparrow}) \) is a graded algebra. For each \( n \in \mathbb{N} \), the subspace \( ker(\rho_n) \) is a coideal of \( \text{Shuffle}, \Delta, \epsilon \) by Lemma 3.4 and \( (\mathbb{W}_n, \Delta_{\otimes}, \epsilon_{\otimes}) \) is the graded coalgebra obtained by transferring the structure maps of the quotient \( \text{Shuffle}/\ker(\rho_n) \) via the map \( \rho_n \). Direct sums of coalgebras are coalgebras, so \( (\mathbb{W}, \Delta_{\otimes}, \epsilon_{\otimes}) \) is a graded coalgebra.

We have \( \epsilon_{\otimes} \circ \iota_{\uparrow} = \text{id} \) and the counit (respectively, unit) is obviously an algebra (respectively, coalgebra) morphism. Let \( \beta \) denote the standard isomorphism \( U \otimes V \cong V \otimes U \) and write \( \nabla \) for the product \( v \otimes w \rightarrow v \uplus w \) of Shuffle. Let \([v, m], [w, n] \in \mathbb{W} \). It follows from the definitions that

\[
\Delta_{\otimes} \circ \nabla_{\uparrow}([v, m] \otimes [w, n]) = \Delta_{\otimes} \circ \rho_{m+n} \circ \nabla(v \otimes (w \uparrow m)) = (\rho_{m+n})^{\otimes 2} \circ \Delta \circ \nabla(v \otimes (w \uparrow m)).
\]
Similarly, we have
\[(\nabla_w)^{\otimes 2} \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\Delta_{\otimes})^{\otimes 2}([v,m] \otimes [w,n])\]
\[= (\nabla_w)^{\otimes 2} \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\rho_m \otimes \rho_m \otimes \rho_n \otimes \rho_n) \circ \Delta^{\otimes 2}(v \otimes w)\]
\[= (\rho_{m+n})^{\otimes 2} \circ \nabla^{\otimes 2} \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ \Delta^{\otimes 2}(v \otimes (w \uparrow m)).\]

It holds that \(\Delta \circ \nabla = \nabla^{\otimes 2} \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ \Delta^{\otimes 2}\) since \text{Shuffle} is a Hopf algebra, so the final expressions in the two equations are equal. Thus \(\Delta\) is an algebra morphism so \((\mathcal{W}, \nabla_w, \iota_w, \Delta_{\otimes}, \epsilon_{\otimes})\) is a graded bialgebra.

This bialgebra is not a Hopf algebra since if \(S : \mathcal{W} \to \mathcal{W}\) is a linear map and \(n \in \mathbb{P}\), then \(\nabla_w \circ (\mathrm{id} \otimes S) \circ \Delta_{\otimes}([0,n]) = \nabla_w([0,n] \otimes S([0,n])) \in \bigoplus_{m \in \mathbb{N}} \mathcal{W}_{m+n}\), which means that
\[\nabla_w \circ (\mathrm{id} \otimes S) \circ \Delta_{\otimes}([0,n]) \neq \iota_w \circ \epsilon_{\otimes}([0,n]) = [0,0] \in \mathcal{W}_0.\]

The map \(\iota_w \circ \epsilon_{\otimes}\) is therefore not invertible in the convolution algebra \(\text{End}(\mathcal{W})\).

**Corollary 3.6.** When \(\text{char}(k) = 0\), the linear map \(\mathcal{W} \to k\) with \([w,n] \mapsto \frac{1}{n!}\) for \([w,n] \in \mathcal{W}\) is an algebra morphism.

**Proof.** If \(\lambda\) is this map, then \(\lambda \circ \nabla_w([v,m] \otimes [w,n]) = \frac{\ell(v)+\ell(w)}{\ell(v)+\ell(w)} \lambda([v,m]) \lambda([w,n])\).

The bialgebra \(\mathcal{W}\) has a well-known quotient. Let \(w = w_1w_2 \cdots w_n\) be a word. Suppose the set \(S = \{w_1, w_2, \ldots, w_n\}\) has \(m\) distinct elements. If \(\phi\) is the unique order-preserving bijection \(S \to \{1,2,\ldots,m\}\), then we define \(\text{fl}(w) = \phi(w_1)\phi(w_2) \cdots \phi(w_n)\).

A **packed word** is a word \(w\) with \(w = \text{fl}(w)\). Such a word is just a surjective map \(\{1,2,\ldots,n\} \to \{1,2,\ldots,m\}\) for some \(m, n \in \mathbb{N}\). Packed words are also referred to in the literature as **surjective words** [15], **Fubini words** [25], and **initial words** [21]. Define \(\mathcal{I}_P\) to be the subspace of \(\mathcal{W}\) spanned by all differences \([v,m] - [w,n]\) where \([v,m], [w,n] \in \mathcal{W}\) have \(\text{fl}(v) = \text{fl}(w)\).

**Proposition 3.7.** The subspace \(\mathcal{I}_P\) is a homogeneous bi-ideal of \((\mathcal{W}, \nabla_w, \iota_w, \Delta_{\otimes}, \epsilon_{\otimes})\). The quotient bialgebra \(\mathcal{W}_P = \mathcal{W}/\mathcal{I}_P\) is a graded Hopf algebra.

**Proof.** The subspace \(\mathcal{I}_P\) is homogeneous since two words \(v\) and \(w\) must have the same length if \(\text{fl}(v) = \text{fl}(w)\). It is easy to check that \(\mathcal{I}_P\) is a bi-ideal. The quotient bialgebra \(\mathcal{W}_P = \mathcal{W}/\mathcal{I}_P\) is connected and therefore a graded Hopf algebra [2, §2.3.2].

Let \(\mathcal{W}_P\) be the set of all packed words. If \([w,n] \in \mathcal{W}\) and \(w\) is a word with \(m\) distinct letters then \(v = \text{fl}(w)\) is the unique packed word such that \([w,n] + \mathcal{I}_P = [v,m] + \mathcal{I}_P\). Identify \(v \in \mathcal{W}_P\) with the coset \([v,m] + \mathcal{I}_P\) so that we can view \(\mathcal{W}_P\) as a basis for \(\mathcal{W}_P\). It is not hard to work out formulas for the product and coproduct of \(\mathcal{W}_P\) in the basis \(\mathcal{W}_P\), but we omit these details here. The subspace of \(\mathcal{W}_P\) spanned by the words in \(\mathcal{W}_P\) that have no repeated letters is a Hopf subalgebra, namely, the **Malvenuto-Poirier-Reutenauer Hopf algebra** of permutations [3, 20], sometimes also called the Hopf algebra of **free quasi-symmetric functions** \(\mathcal{FQSym}\) [9].
4 Bialgebras of permutations

Recall that $S_n$ denotes the group of permutations of $\{1, 2, \ldots, n\}$. For each $1 \leq i \leq n - 1$, let $s_i \in S_n$ be the simple transposition given in cycle notation by $(i, i + 1)$. Then $S_n$ is the finite Coxeter group of type $A_{n-1}$ relative to the generating set $\{s_1, s_2, \ldots, s_{n-1}\}$.

A reduced word for $\pi \in S_n$ is a word $i_1 i_2 \cdots i_l$ of minimal length such that $\pi = s_{i_1} s_{i_2} \cdots s_{i_l}$. Let $R(\pi)$ be the set of such words and define $\ell(\pi)$ to be their common length. For $n \in \mathbb{N}$ and $\pi \in S_{n+1}$, define $[\pi] = \sum_{w \in R(\pi)} [w, n] \in W_n$. Let $\Pi_n = \mathbb{K}$-span\{ $[\pi] : \pi \in S_{n+1}$\} and $\Pi = \bigoplus_{n \in \mathbb{N}} \Pi_n$.

**Theorem 4.1.** The subspace $\Pi$ is a graded sub-bialgebra of $(W, \nabla, \epsilon, \Delta, \epsilon_\odot)$.

This is a special case of more general results in [22, §5]. We include a self-contained proof.

**Proof.** The word property for Coxeter groups [6, Theorem 3.3.1] asserts that for each $\pi \in S_n$, the set $R(\pi)$ is an equivalence class under the strongest relation with $vw \sim v'w'$ whenever $v \sim v'$ and $w \sim w'$, such that $ij \sim ji$ and $i(i+1)i \sim (i+1)i(i+1)$ for $i, j \in \mathbb{P}$ with $|j-i| > 1$. Moreover, an equivalence class under this relation is equal to $R(\pi)$ for some permutation $\pi$ if and only if it contains no words with adjacent repeated letters. It is clear from these observations that the coproduct $\Delta_\odot$ satisfies $\Delta_\odot(\Pi) \subset \Pi \otimes \Pi$ and has the formula in Corollary 1.2.

The unit element of $W$ is $[\emptyset, 0] = [\pi]$ for $\pi = 1 \in S_1$. To show that $\nabla_\odot(\Pi \otimes \Pi) \subset \Pi$, let $\pi' \in S_{m+1}$ and $\pi'' \in S_{n+1}$. No word formed by shuffling $v$ and $w \uparrow m$ for $v \in R(\pi')$ and $w \in R(\pi'')$ contains $m(m+1)m$ or $(m+1)m(m+1)$ as a consecutive subword, since this would imply that two adjacent letters of $v$ or $w$ are equal. It follows that the set of all words formed by shuffling $v$ and $w \uparrow m$ for some $v \in R(\pi')$ and $w \in R(\pi'')$ is a union of $\sim$-equivalence classes. No words in this set may contain adjacent repeated letters since $u \cap \{1, 2, \ldots, m\} \in R(\pi')$ and $u \cap (m+\mathbb{P}) \in R(\pi'')$, so $\nabla_\odot([\pi] \otimes [\pi'']) \in \Pi$. Each $[\pi]$ is homogeneous of degree $\ell(\pi)$, so $\Pi$ is a graded sub-bialgebra. \[\square\]

If $\pi \in S_n$ then $\ell(\pi)$ may be computed as the number of integer pairs $(i, j)$ with $1 \leq i < j \leq n$ and $\pi(i) > \pi(j)$. We write $\pi = \pi' \pi''$ if $\pi, \pi', \pi'' \in S_n$ and $\pi = \pi' \pi''$ and $\ell(\pi) = \ell(\pi') + \ell(\pi'')$.

**Corollary 4.2.** If $\pi \in S_n$ then $\Delta_\odot([\pi]) = \sum_{\pi = \pi' \pi''} [\pi'] \otimes [\pi'']$ and $\epsilon_\odot([\pi]) = \begin{cases} 1 & \text{if } \ell(\pi) = 0 \\ 0 & \text{if } \ell(\pi) \neq 0. \end{cases}$

The product of $\Pi$ takes more work to describe. In the following definitions, we consider $S_n$ and $S_n$ to be disjoint for all $m \neq n$. We represent elements of $S_n$ in one-line notation, that is, by writing the word $\pi_1 \pi_2 \cdots \pi_n$ to mean the permutation $\pi \in S_n$ with $\pi(i) = \pi_i$.

**Definition 4.3.** Suppose $a = a_1 a_2 \cdots a_k$ and $b = b_1 b_2 \cdots b_l$ are words with no repeated letters. Let $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_l\}$ and $n = k + l$, and assume $A$ and $B$ are both contained in $\{1, 2, \ldots, n\}$. Write $\phi$ and $\psi$ for the order-preserving bijections $A \to \{1, 2, \ldots, n\} - B$ and $B \to \{1, 2, \ldots, n\} - A$. Finally, define $a \triangleright b = \phi(a) \in S_n$ and $a \triangleright b = \phi(a) b \in S_n$.

For example, $a \triangleright \emptyset = \emptyset \triangleright a = a$ and $1357 \triangleright 6543 = 135786543$ and $1357 \triangleright 6543 = 12786543$.

**Definition 4.4.** Let $u \in S_{m+1}$ and $v \in S_{n+1}$. Set $w_i = v_i + m$ and suppose $j$ and $k$ are the indices with $u_j = w_{k+1} = m + 1$. Define a subset $S_{\odot}(u, v) \subset S_{m+n+1}$ inductively as follows:

(a) If $m = 1$ then let $S_{\odot}(u, v) = \{u_1 u_2 \cdots u_m w_1 w_2 \cdots w_n + 1\}$.
(b) If \( w_1 = m + 1 \) then let \( S_{\omega}(u, v) = \{ u_1 u_2 \cdots u_m u_{m+1} w_2 \cdots w_{n+1} \} \).

(c) Otherwise, define \( \tilde{u} = \text{fl}(u_{j+1} u_{j+2} \cdots u_{m+1}) \) and \( \tilde{v} = \text{fl}(v_1 v_2 \cdots v_k) \) and let

\[
S_{\omega}(u, v) = \left\{ u_1 u_2 \cdots u_j \parallel \sigma : \sigma \in S_{\omega}(\tilde{u}, v) \right\} \sqcup \left\{ \sigma \setminus w_{k+1} w_{k+2} \cdots w_{n+1} : \sigma \in S_{\omega}(u, \tilde{v}) \right\}.
\]

**Example 4.5.** If \( u = 231 \) and \( v = 312 \), then \( w = 534, j = k + 1 = 2, \) and \( \tilde{u} = \tilde{v} = 1, \) so \( S_{\omega}(\tilde{u}, v) = \{312\} \) and \( S_{\omega}(u, \tilde{v}) = \{231\} \) and

\[
S_{\omega}(231, 312) = \{23 \parallel 312\} \sqcup \{231 \setminus 34\} = \{23514, 25134\}.
\]

On the other hand, if \( u = 312 \) and \( v = 231 \), then \( w = 453, j = 1 \) and \( k + 1 = 3, \) and \( \tilde{u} = \tilde{v} = 12, \) so \( S_{\omega}(\tilde{u}, v) = \{1342\} \) and \( S_{\omega}(u, \tilde{v}) = \{3124\} \) and

\[
S_{\omega}(312, 231) = \{3 \parallel 1342\} \sqcup \{3124 \setminus 3\} = \{31452, 41253\}.
\]

For a more complicated example, one can check that

\[
S_{\omega}(4213, 4132) = \{4217365, 7213465\},
\]

\[
S_{\omega}(4132, 4213) = \{4137526, 4157236, 4172536, 5137246, 5172346, 7132546\}.
\]

If \( p, q \in \mathbb{P} \) are such that \( p + q = n + 1 \) then \( S_{\omega}(p \cdots 321, q \cdots 321) \) is just the set \( B(p, n) \) defined in the introduction. The following therefore implies Theorem 4.4(a):

**Theorem 4.6.** If \( u \in S_{m+1} \) and \( v \in S_{n+1} \) then \( \nabla_{\omega}([u] \otimes [v]) = \sum_{\pi \in S_{\omega}(u, v)} [\pi] \).

For the proof, we will need some facts about wiring diagrams. Fix \( n, N \in \mathbb{P} \) and let \( I = \{ x \in \mathbb{R} : 0 \leq x \leq N \} \). Let \( f_1, f_2, \ldots, f_n : I \rightarrow \mathbb{R} \) be continuous functions. The tuple \( D = (f_1, f_2, \ldots, f_n) \) is a *wiring diagram* if \( f_1(0) < f_2(0) < \cdots < f_n(0) \) and for each integer \( 1 \leq i \leq N \), we have:

1. The numbers \( f_1(i), f_2(i), \ldots, f_n(i) \) are all distinct.
2. For each \( 1 \leq j \leq n \), the restriction of \( f_j \) to \( (i-1, i) := \{ x \in \mathbb{R} : i - 1 < x < i \} \) is a line.
3. At most two functions \( f_j \) and \( f_k \) intersect in the open interval \( (i-1, i) \), and an intersection occurs only if \( f_j(i) \) and \( f_k(i) \) are consecutive elements of \( \{f_1(i), f_2(i), \ldots, f_n(i)\} \).

Properties (1)-(3) imply that if \( f_j \) and \( f_k \) intersect in the open interval \( (i-1, i) \), then \( f_j(i-1) \) and \( f_k(i-1) \) are consecutive elements in \( \{f_1(i-1), f_2(i-1), \ldots, f_n(i-1)\} \). The function \( f_j \) is the \( j \)th wire of \( D \). We say that the wires \( f_j \) and \( f_k \) cross if \( f_j(x) = f_k(x) \) for some \( x \in I \). For example,
are wiring diagrams with \( N = 5 \) and \( n = 6, 6, 4, 4 \), respectively.

The wiring diagram \( D \) is reduced if all distinct wires \( f_j \) and \( f_k \) cross at most once. For \( 0 \leq i \leq N \), define \( \pi^i = \text{fl}(f_1(i)f_2(i)\cdots f_n(i)) \in S_n \) and say that \( D \) is a wiring diagram for \( \pi = \pi^N \). Then \( \pi^0 = 1 \in S_n \), and if \( 0 < i \leq N \) then either \( \pi^i = \pi^{i-1} \) or \( \pi^i = s_j \pi^{i-1} \) for some \( j \). Consider the word \( \nu \) of length \( N \) whose \( i \)th letter is 0 if \( \pi^i = \pi^{i-1} \) or otherwise the index \( j \) such that \( \pi^i = s_j \pi^{i-1} \).

Define \( \text{word}(D) \) to be the sequence formed by removing all zeros from \( \nu \) and then reversing the resulting subword. We refer to \( \text{word}(D) \) as the word associated to \( D \). If \( i_1 \cdots i_2 i_1 = \text{word}(D) \) then \( \pi = s_{i_1} \cdots s_{i_2} s_{i_1} \), and it holds that \( \text{word}(D) \in \mathcal{R}(\pi) \) if and only if \( D \) is reduced. The examples above are reduced wiring diagrams for \( \pi = 315264, 512364, 3142 = \text{fl}(3152), \) and \( 1234 = \text{fl}(2364) \) with words 24351, 43521, 231, and 3.

Any word \( i_1 \cdots i_2 i_1 \) such that \( \pi = s_{i_1} \cdots s_{i_2} s_{i_1} \in S_n \) is the word of the wiring diagram for \( \pi \) given as follows: take \( N = l \), define \( \pi^0 = 1 \in S_n \) and \( \pi^j = s_{i_1} \cdots s_{i_2} s_{i_1} \in S_n \), and let \( D = (f_1, f_2, \ldots, f_n) \) where \( f_i \) is the piecewise linear function \( I \to \mathbb{R} \) connecting the points \((x, y) = (j, \pi^j(i))\) for \( j = 0, 1, \ldots, l \). We refer to this as the standard wiring diagram of the word \( i_1 \cdots i_2 i_1 \). The first two examples above are the standard wiring diagrams for 24351 and 435231.

Suppose \( D = (f_1, f_2, \ldots, f_n) \) is a wiring diagram for \( \pi \in S_n \). Let \( i \in \{1, 2, \ldots, n\} \). Then \( E = (f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n) \) is a wiring diagram for \( \text{fl}(\pi_1 \cdots \pi_{i-1} \pi_{i+1} \cdots \pi_n) \in S_{n-1} \), and if \( D \) is reduced then \( E \) is also reduced. In turn, \((f_{i+1}, f_{i+2}, \ldots, f_n) \) and \((f_1, f_2, \ldots, f_{i-1}) \) are wiring diagrams for some \( x \in S_{n-i} \) and \( y \in S_{i-1} \), and we have \( \pi = \pi_1 \pi_2 \cdots \pi_{i-1} / x = y \). This is illustrated by the examples above, since 315264 = 3142464 and 512364 = 5142463.

Proof of Theorem 4.6. The reader may find it helpful to consult the example following the proof, which works through our argument in a concrete case.

Let \( u \in S_{m+1} \), \( v \in S_{n+1} \), and \( w_{i} = v_{i} + m \). If \( u_{m+1} = m + 1 \) then no reduced word for \( u \) involves the letter \( m \), so every word obtained by shuffling \( a \in \mathcal{R}(u) \) and \( b \uparrow m \) for \( b \in \mathcal{R}(v) \) is equivalent under the Coxeter relation \( \sim \) described in the proof of Theorem 4.4 to \( a(b \uparrow m) \), which is a reduced word for \( \pi = u_1 u_2 \cdots u_m w_1 w_2 \cdots w_{n+1} \in S_{m+n+1} \). Since \( \nabla_\omega([u] \otimes [v]) \) is a multiplicity-free sum of elements \([\sigma] \) with \( \sigma \in S_{m+n+1} \), it follows that \( \nabla_\omega([u] \otimes [v]) = [\pi] \) as claimed. If \( w_1 = m + 1 \) then no reduced word for \( v \) involves the letter 1, and it follows similarly that \( \nabla_\omega([u] \otimes [v]) = [\pi] \) for \( \pi = u_1 u_2 \cdots u_{m+1} w_2 w_3 \cdots w_{n+1} \in S_{m+n+1} \).

Suppose \( u_j = u_{k+1} = m + 1 \) where \( 1 \leq j < m+1 \) and \( 0 < k \leq n \). Then every word \( a \in \mathcal{R}(u) \) contains the letter \( m \) and every word \( b \in \mathcal{R}(v) \) contains the letter 1. Write \( \tilde{a} = \text{fl}(u_{j+1}u_{j+2} \cdots u_{m+1}) \) and \( \tilde{b} = \text{fl}(v_{k}v_{k+1} \cdots v_{n}) \). Assume by induction that if \( a' \in S_{m+1} \) and \( b' \in S_{n+1} \) then \( \nabla_\omega([a'] \otimes [b']) = \sum_{\pi \in S_{r}([a'], [b'])} [\pi] \). Fix \( a \in \mathcal{R}(u) \) and \( b \in \mathcal{R}(v) \). Since no two wires in a reduced wiring diagram cross twice, the \( j \)th wire of the standard wiring diagram of \( a \) is monotonically increasing and equal to \( m + 1 \) at \( x = N \), and is an upper bound for wires \( 1, 2, \ldots, j - 1 \). Similarly, the \( (k+1) \)th wire of the standard wiring diagram of \( b \) is monotonically decreasing and is equal to \( 1 \) at \( x = N \), and is a lower bound for wires \( k+1, k+2, \ldots, n+1 \).

Let \( c \) be a word obtained by shuffling \( a \) and \( b \uparrow m \), i.e., assume \( c \cap \{1, 2, \ldots, m\} = a \) and \( c \cap \{m+1\} = b \uparrow m \). Let \( D = (f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_{n+1}) \) be the standard (reduced) wiring diagram of \( c \) and suppose this is a wiring diagram for the permutation \( \pi \in S_{m+n+1} \). Let \( N = \ell(a) + \ell(b) = \ell(c) \) and note that \( c \) contains both \( m \) and \( m + 1 \) as letters. If \( c \cap \{m, m+1\} \) begins with \( m \), then the wire \( f_j \) is monotonically increasing with \( f_j(N) = m + 1 \), each of the wires \( f_1, f_2, \ldots, f_{j-1} \) is bounded above by \( f_j \), and we have \( \pi_1 \pi_2 \cdots \pi_{j} = u_1 u_2 \cdots u_j \). It follows in this case
that if \( E = (f_{j+1}, f_{j+2}, \ldots, f_m, g_1, g_2, \ldots, g_{n+1}) \) then

\[
\text{word}(E) \cap (m - j + \mathbb{P}) = b \uparrow (m - j) \quad \text{and} \quad \text{word}(E) \cap \{1, 2, \ldots, m - j\} \in \mathcal{R}(\tilde{u}),
\]

so \( \pi = u_1 u_2 \cdots u_j \| \sigma \) for some \( \sigma \in \mathcal{S}_w(\tilde{u}, v) \) by induction. Alternatively, if \( c \cap \{m, m + 1\} \) begins with \( m + 1 \), then the wire \( g_{k+1} \) is monotonically decreasing with \( g_{k+1}(N) = m + 1 \), each of the wires \( g_{k+2}, g_{k+3}, \ldots, g_{n+1} \) is bounded below by \( g_{k+1} \), and we have

\[
\pi_{m+k+1} \pi_{m+k+2} \cdots \pi_{m+n+1} = w_{k+1} w_{k+2} \cdots w_{n+1}.
\]

It follows in this case that if \( F = (f_1, f_2, \ldots, f_m, g_1, g_2, \ldots, g_k) \) then

\[
\text{word}(F) \cap \{1, 2, \ldots, m\} = a \quad \text{and} \quad \text{word}(F) \cap (m + \mathbb{P}) \in \mathcal{R}(\tilde{v}),
\]

so \( \pi = a \| w_{k+1} w_{k+2} \cdots w_{n+1} \) for some \( \sigma \in \mathcal{S}_w(u, \tilde{v}) \) by induction.

Since we know that the set of all shuffles of \( a \in \mathcal{R}(u) \) and \( b \uparrow m \) for \( b \in \mathcal{R}(v) \) decomposes as a disjoint union of the sets \( \mathcal{R}(\pi) \) for certain permutations \( \pi \in \mathcal{S}_{m+n+1} \), the preceding argument shows \( \nabla_{\mathcal{S}}([u] \otimes [v]) \) is a multiplicity-free sum of terms \([\pi]\) where \( \pi \) ranges over a subset of \( \mathcal{S}_w(u, v) \).

To show that every term indexed by \( \pi \in \mathcal{S}_w(u, v) \) appears in the product, suppose \( \sigma \in \mathcal{S}_w(u, v) \) and \( \tilde{c} \in \mathcal{R}(\sigma) \). By induction, \( \tilde{c} \) is a shuffle of \( \tilde{a} \) and \( b \uparrow (m - j) \) for some \( \tilde{a} \in \mathcal{R}(\tilde{u}) \) and \( b \in \mathcal{R}(v) \). Let \( d \) be any reduced word for \( \tau := \text{fl}(u_1 u_2 \cdots u_j) \) and let \( e \) be the word formed by concatenating the sequences \( (u_i - 1)(u_i - 2) \cdots \tau(i) \) as \( i \in \{1, 2, \ldots, j\} \) varies in the order that makes the values \( u_i \) increasing. Then \( a := ed(\tilde{a} \uparrow j) \) is a reduced word for \( u = u_1 u_2 \cdots u_j \| \tilde{u} \), while \( c := ed(\tilde{c} \uparrow j) \) is a reduced word for \( \sigma \| w_{k+1} w_{k+2} \cdots w_{n+1} \| \sqrt{A}_1 \) and also a shuffle of \( a \) and \( b \uparrow m \). We conclude that \( [u_1 u_2 \cdots u_j \| \sigma] \) appears in the product \( \nabla_{\mathcal{S}}([u] \otimes [v]) \).

Similarly, suppose \( \sigma \in \mathcal{S}_w(u, \tilde{v}) \) and \( \tilde{c} \in \mathcal{R}(\sigma) \). Then \( \tilde{c} \) is a shuffle of \( \tilde{a} \) and \( b \uparrow m \) for some \( \tilde{a} \in \mathcal{R}(u) \) and \( b \in \mathcal{R}(\tilde{v}) \). Let \( d \) be any reduced word for \( \tau := \text{fl}(v_{k+1} v_{k+2} \cdots v_{n+1}) \) and let \( e \) be the word formed by concatenating the sequences \( v_i(v_i + 1) \cdots (\tau(i - k) + k) \) as \( i \in \{k+1, k+2, \ldots, n+1\} \) varies in the order that makes the values \( v_i \) decreasing. Then \( b := e(d \uparrow k) \) is a reduced word for \( v = v_k v_{k+2} \cdots v_{n+1} \), while \( c := (e \uparrow m)(d \uparrow (m + k)) \) is a reduced word for \( \sigma \| w_{k+1} w_{k+2} \cdots w_{n+1} \) and also a shuffle of \( a \) and \( b \uparrow m \). Hence \( [\sigma \| w_{k+1} w_{k+2} \cdots w_{n+1}] \) appears in the product \( \nabla_{\mathcal{S}}([u] \otimes [v]) \).

This completes our proof that \( \nabla_{\mathcal{S}}([u] \otimes [v]) = \sum_{\pi \in \mathcal{S}_w(u,v)} \mathbb{P}([\pi]) \).

**Example 4.7.** We illustrate our proof strategy by computing \( \nabla_{\mathcal{S}}([231] \otimes [312]) \). Let \( u = 231 \) and \( v = 312 \) so that \( m = n = 2 \) and \( w = 534 \). The permutations \( u \) and \( v \) each have one reduced word:

\( \mathcal{R}(u) = \{12\} \) and \( \mathcal{R}(v) = \{21\} \).

These words have the following wiring diagrams:

![Diagram](image.png)

We have \( u_j = w_{k+1} = m + 1 \) for \( j = k + 1 = 2 \). Let \( a = 12 \in \mathcal{R}(u) \) and \( b = 21 \in \mathcal{R}(v) \). There are \( 6 = \binom{2}{1} \) different shuffles \( c \) of \( a \) and \( b \uparrow m \). The three wiring diagrams \( D = (f_1, f_2, g_1, g_2, g_3) \) of the shuffles \( c \) such that \( c \cap \{m, m + 1\} \) begins with \( m = 2 \) are shown below:

![Diagrams](image.png)
In these pictures, the wire $f_j$ is shown as a dashed line. As described in the proof, this wire is monotonically increasing and eventually equal to $m+1$, and is an upper bound for $f_1, f_2, \ldots, f_{j-1}$. Moreover, if $D$ is a wiring diagram for $\pi \in S_{m+n-1}$ then $\pi_1 \cdots \pi_j = u_1 \cdots u_j = 23$ as claimed. Removing the wires $f_1, f_2, \ldots, f_j$ from each diagram produces a wiring diagram for $v$, which is the unique element in $S_u(\tilde{u}, v)$ since $\tilde{v} = \text{fl}(u_{j+1}u_{j+2} \cdots u_{m+1}) = 1$:

\[
\begin{array}{ccc}
X & X & X \\
\end{array}
\]

The word 1243 corresponds to the reduced word $c = ed(\hat{c} \uparrow j)$ described in the second to last paragraph of the proof of Theorem 4.6.

Next, consider the wiring diagrams $D = (f_1, f_2, g_1, g_2, g_3)$ for the three words $c$ obtained by shuffling $a = 12$ and $b \uparrow m = 43$ such that $c \cap \{m, m+1\}$ begins with $m + 1 = 3$:

\[
\begin{array}{ccc}
\hline
\text{4312} & \text{1432} & \text{4132} \\
\hline
\end{array}
\]

In these pictures, the wire $g_{k+1}$ is shown as a dashed line. This wire is monotonically decreasing and eventually equal to $m+1$, and is a lower bound for $g_{k+2}, \ldots, g_{n+1}$. In turn, if $D$ is a wiring diagram for $\pi \in S_{m+n-1}$ then $\pi_{m+k+1} \cdots \pi_{m+n+1} = u_{k+1} \cdots u_{n+1} = 34$ as claimed, and it is easy to see that removing the wires $g_{k+1}, g_{k+2}, \ldots, g_{n+1}$ from each diagram produces a wiring diagram for $u$, which is the unique element of $S_u(u, \tilde{v})$ since $\tilde{v} = \text{fl}(v_1v_2 \cdots v_k) = 1$:

\[
\begin{array}{ccc}
\hline
\text{4312} & \text{1432} & \text{4132} \\
\hline
\end{array}
\]

The word 4312 corresponds to the reduced word $c = (e \uparrow m)(d \uparrow (m+k))\hat{c}$ described in the last paragraph of the proof of Theorem 4.6. We conclude that

$$
\nabla_u([231] \otimes [312]) = \sum_{\pi \in S_u(u, \tilde{v})} [\pi] + \sum_{\pi \in S_u(u, \hat{v})} [\pi] = \sum_{\pi \in S_u(u, \tilde{v})} [\pi] = [23514] + [25134].
$$

Let $X \subset S_{m+1}$ and $Y \subset S_{n+1}$. Since we can recover $(u, v) \in S_{m+1} \times S_{n+1}$ from $\pi \in S_u(u, v)$ by intersecting its reduced words with $\{1, 2, \ldots, m\}$ and $m + \{1, 2, \ldots, n\}$, the sets $S_u(u, v)$ are disjoint for all $u \in X$ and $v \in Y$. Define $S_u(X, Y) = \bigsqcup_{(u, v) \in X \times Y} S_u(u, v) \subset S_{m+n+1}$ and $S_u(u, Y) = S_u(\{u\}, Y)$ and $S_u(X, v) = S_u(X, \{v\})$. The associativity of $\nabla_u$ implies the following:

**Corollary 4.8.** If $(\pi^1, \pi^2, \pi^3) \in S_{l+1} \times S_{m+1} \times S_{n+1}$ then $S_u(\pi^1, S_u(\pi^2, \pi^3)) = S_u(S_u(\pi^1, \pi^2), \pi^3)$. 


For permutations \( \pi^1, \pi^2, \ldots, \pi^k \), let \( S_\omega(\pi^1, \pi^2, \ldots, \pi^k) = S_\omega(\pi^1, S_\omega(\pi^2, \ldots, \pi^k)) \) where \( S_\omega(\pi) = \{ \pi \} \). Using this notation, we can answer a natural question about what happens when we apply the flattening map \( f \) to the reduced words of a permutation.

A permutation \( \pi = \pi_1 \pi_2 \cdots \pi_n \in S_n \) is irreducible if \( \{ \pi_1, \pi_2, \ldots, \pi_m \} \neq \{1, 2, \ldots, m\} \) for all \( 1 \leq m < n \). If \( u \in S_m \) and \( v \in S_n \), then define \( u \circ v = u(v \uparrow m) \in S_{m+n} \). Clearly \( \pi \circ (\pi' \circ \pi'') = (\pi \circ \pi') \circ \pi'' \), so we can omit all parentheses in expressions using \( \circ \). Moreover, every \( \pi \in S_n \) has a unique factorization \( \pi = \pi^1 \circ \pi^2 \circ \cdots \circ \pi^k \) where each \( \pi^i \in S_1 \sqcup S_2 \sqcup S_3 \sqcup \cdots \) is irreducible.

**Corollary 4.9.** Suppose \( \pi = \pi^1 \circ \pi^2 \circ \cdots \circ \pi^k \in S_n \) where each \( \pi^i \) is an irreducible permutation. The flattening map \( f \) is then a bijection \( R(\pi) \rightarrow \bigcup_{\sigma \in S_\omega(\pi^1, \pi^2, \ldots, \pi^k)} R(\sigma) \).

**Proof.** The map \( f \) is certainly injective, and if \( n_i \in \mathbb{N} \) is such that \( \pi^i \in S_{n_i+1} \), then \( f(R(\pi)) \) is the disjoint union of the reduced words of the permutations \( \sigma \in S_{n_1+n_2+\cdots+n_k+1} \) for which \( [\sigma] \) is a term in the product \( \nabla^{(k-1)}([\pi^1] \otimes [\pi^2] \otimes \cdots \otimes [\pi^k]) \), which by Theorem 4.16 is \( \sum_{\sigma \in S_\omega(\pi^1, \pi^2, \ldots, \pi^k)} [\sigma] \).

**Example 4.10.** We have \( R(231645) = R(231 \oplus 312) = \{1254, 1524, 1542, 5124, 5142, 5412\} \) and \( \{1243, 1423, 1432, 4123, 4132, 4312\} = R(2514) \sqcup R(25134) = \bigcup_{\sigma \in S_\omega(231, 312)} R(\sigma) \).

A permutation is fully commutative if none of its reduced words contains a consecutive subword of the form \( i(i+1)i \). It is well-known [3, Theorem 2.1] that \( \pi \in S_n \) is fully commutative if and only if \( \pi \) is 321-avoiding in the sense that no indices \( i < j < k \) have \( \pi_i > \pi_j > \pi_k \).

**Corollary 4.11.** Let \( u \in S_m \) and \( v \in S_n \) be fully commutative. Set \( w = v \uparrow (m-1) \). Form \( u' \in S_{m+n-1} \) from \( uw \) by replacing the second occurrence of \( m \) by \( u_m \) and then removing the \( m \)th letter. Form \( v' \in S_{m+n-1} \) from \( uw \) by replacing the first occurrence of \( m \) by \( w_1 \) and then removing the \( (m+1) \)th letter. If \( u' = v' \) then \( \nabla_\omega([u] \otimes [v]) = [u'] = [v'] \); otherwise \( \nabla_\omega([u] \otimes [v]) = [u'] + [v'] \).

For example, \( \nabla_\omega([4123] \otimes [2341]) = [4125673] + [5123674] \).

**Proof.** This follows from Theorem 4.16 since if \( u \) and \( v \) are 321-avoiding, then the elements \( \hat{u} \) and \( \hat{v} \) in Definition 4.4 must both be identity permutations.

Let \( \Pi^{FC} \) be the space spanned by \( [\pi] \) for all fully commutative elements \( \pi \in S_n \) and \( n \in \mathbb{P} \).

**Corollary 4.12.** The subspace \( \Pi^{FC} \) is a graded sub-bialgebra of \( (\Pi, \nabla_\omega, \epsilon, \Delta_\circ) \).

**Proof.** Clearly \( \Delta_\circ(\Pi^{FC}) \subset \Pi^{FC} \otimes \Pi^{FC} \), and \( \nabla_\omega(\Pi^{FC} \otimes \Pi^{FC}) \subset \Pi^{FC} \) holds by Corollary 4.11.

## 5 Combinatorial bialgebras

A composition \( \alpha \) of \( n \in \mathbb{N} \), written \( \alpha \vdash n \), is a sequence of positive integers \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \) with \( \alpha_1 + \alpha_2 + \cdots + \alpha_l = n \). The unique composition of \( n = 0 \) is the empty word \( \emptyset \). Let \( \mathbb{k}[[x_1, x_2, \ldots]] \) be the algebra of formal power series with coefficients in \( \mathbb{k} \) in a countable set of commuting variables. The monomial quasi-symmetric function \( M_{\alpha} \) indexed by a composition \( \alpha \vdash n \) with \( l \) parts is

\[
M_{\alpha} = \sum_{i_1 < i_2 < \cdots < i_l} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_l}^{\alpha_l} \in \mathbb{k}[[x_1, x_2, \ldots]].
\]

When \( \alpha \) is the empty composition, set \( M_{\emptyset} = 1 \). For each \( n \in \mathbb{N} \), the set \( \{ M_{\alpha} : \alpha \vdash n \} \) is a basis for a subspace \( \mathbb{Q} \text{Sym}_n \subset \mathbb{k}[[x_1, x_2, \ldots]] \). The vector space of quasi-symmetric functions
QSym = \bigoplus_{n \in \mathbb{N}} QSym_n is a commutative subalgebra of \mathbb{k}[[x_1, x_2, \ldots]]. This algebra is a graded Hopf algebra whose coproduct and counit are the linear maps with \(\Delta(M_\alpha) = \sum_{\alpha = \beta \gamma} M_\beta \otimes M_\gamma\) and with \(\epsilon(M_\emptyset) = 1\) and \(\epsilon(M_\alpha) = 0\) for all compositions \(\alpha \neq \emptyset\); see [1, \S 3].

Each \(\alpha \vdash n\) can be rearranged to form a partition of \(n\), denoted sort(\(\alpha\)). The **monomial symmetric function** indexed by a partition \(\lambda \vdash n\) is \(m_\lambda = \sum_{\text{sort}(\alpha) = \lambda} M_\alpha\). Write \(\lambda \vdash n\) when \(\lambda\) is a partition of \(n\) and let \(\text{Sym}_n = \mathbb{k}\text{-span}\{m_\lambda : \lambda \vdash n\}\). The subspace \(\text{Sym} = \bigoplus_{n \in \mathbb{N}} \text{Sym}_n \subset \text{QSym}\) is the familiar graded Hopf subalgebra of symmetric functions.

**Definition 5.1.** A pair \((A, \zeta)\) is a **combinatorial coalgebra** if \((A, \Delta, \epsilon)\) is a graded coalgebra and \(\zeta : A \to \mathbb{k}\) is a linear map with \(\zeta(a) = \epsilon(a)\) whenever \(a \in A\) is homogeneous of degree zero. A morphism of combinatorial coalgebras \((A, \zeta) \to (A', \zeta')\) is a graded coalgebra morphism \(\Phi : A \to A'\) with \(\zeta = \zeta' \circ \Phi\). A combinatorial coalgebra \((A, \zeta)\) in which \(A\) is a graded bialgebra and \(\zeta\) is an algebra morphism is a **combinatorial bialgebra**. Morphisms of combinatorial bialgebras are morphisms of combinatorial coalgebras that are bialgebra morphisms.

The objects in this definition are mild generalizations of combinatorial coalgebras and Hopf algebras as introduced in [1], which restricts to the case when \(A\) is connected and has finite graded dimension. For similar definitions of “combinatorial” monoidal structures in other categories, see, for example, [21, \S 5.4] and [22]. Let \(\zeta_{\text{QSym}} : \text{QSym} \to \mathbb{k}\) be the linear map with

\[
\zeta_{\text{QSym}}(M_\alpha) = \begin{cases} 
1 & \text{if } \alpha = \emptyset \text{ or } \alpha = (n) \text{ for some } n \in \mathbb{N} \\
0 & \text{if } \alpha \text{ is any other composition.}
\end{cases}
\]

Then \(\zeta_{\text{QSym}}\) is the restriction of the algebra morphism \(\mathbb{k}[[x_1, x_2, \ldots]] \to \mathbb{k}\) setting \(x_1 = 1\) and \(x_2 = x_3 = \cdots = 0\), so \((\text{QSym}, \zeta_{\text{QSym}})\) is a combinatorial bialgebra.

Let \(X(W)\) denote the set of linear maps \(\zeta : W \to \mathbb{k}\) for which \((W, \zeta)\) is a combinatorial bialgebra, i.e., with \(\zeta([\emptyset, n]) = 1\) for all \(n \in \mathbb{N}\) and \(\zeta \circ \nabla_{w} = \nabla_{k} \circ (\zeta \otimes \zeta)\). If \(\zeta \in X(W)\) then \((II, \zeta)\) is also a combinatorial bialgebra.

For \(l \in \mathbb{N}\), let \(\pi_l : W \to W\) be the natural projection onto the subspace of degree \(l\) elements, that is, the linear map with \(\pi_l([w, n]) = [w, n]\) if \(\ell(w) = l\) and \(\pi_l([w, n]) = 0\) otherwise. Given a linear map \(\zeta : W \to \mathbb{k}\) and a composition \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \neq \emptyset\), define \(\zeta_\alpha\) to be the map

\[
W \xrightarrow{\Delta^{(m-1)}_\otimes} W \otimes^m \xrightarrow{\pi_{\alpha_1} \otimes \cdots \otimes \pi_{\alpha_m}} W \otimes^m \xrightarrow{\zeta \circ^m} \mathbb{k} \otimes^m \xrightarrow{\nabla_k^{(m-1)}} \mathbb{k}.
\]

For the empty composition, let \(\zeta_{\emptyset} = \epsilon_\otimes\).

**Theorem 5.2.** Suppose \(\zeta \in X(W)\). The linear map \(\Psi : W \to \text{QSym}\) with

\[
\Psi(x) = \sum_\alpha \zeta_\alpha(x) M_\alpha
\]

for \(x \in W\), where the sum is over all compositions, is then the unique morphism of combinatorial bialgebras \((W, \zeta) \to (\text{QSym}, \zeta_{\text{QSym}})\) and \((II, \zeta) \to (\text{QSym}, \zeta_{\text{QSym}})\).

The sum defining \(\Psi\) has only finitely many nonzero terms since \(\zeta_\alpha([w, n]) \neq 0\) implies \(\alpha \vdash \ell(w)\).

**Proof.** Both \((W_n, \zeta)\) and \((II_n, \zeta)\) are combinatorial coalgebras in the sense of [1], which is slightly more specific than our definition. By [1, Theorem 4.1], \((\text{QSym}, \zeta_{\text{QSym}})\) is the terminal object in the
category of combinatorial coalgebras and \((5.2)\) is the unique morphism \((W_n, \zeta) \to (QSym, \zeta_{QSym})\); this is a bialgebra morphism \(W \to QSym\) since the maps

\[
W_m \otimes W_n \xrightarrow{\Psi \otimes \Psi} QSym \otimes QSym \xrightarrow{\nabla} QSym \quad \text{and} \quad W_m \otimes W_n \xrightarrow{\nabla w} W_{m+n} \xrightarrow{\Psi} QSym
\]

are both morphisms \((W_m \otimes W_n, \nabla \circ (\zeta \otimes \zeta)) \to (QSym, \zeta_{QSym})\). The restriction of \(\Psi\) is a morphism of combinatorial bialgebras \((\Pi, \zeta) \to (QSym, \zeta_{QSym})\), so must be the unique such morphism. \(\square\)

Theorem 5.2 is a special case of the following more general fact:

**Theorem 5.3** (\cite{1}; see \cite{22}). Suppose \((B, \zeta)\) is a combinatorial bialgebra. Define \(\Psi : B \to QSym\) by the formula \((5.2)\), where \(\zeta_{\alpha} : B \to k\) is the map \((5.1)\) with \(B\) and \(\Delta_{\otimes}\) replaced by \(B\) and its coproduct. Then \(\Psi\) is the unique morphism of combinatorial bialgebras \((B, \zeta) \to (QSym, \zeta_{QSym})\).

This is a minor extension of \cite[Theorem 4.1]{1} and \cite[Remark 4.2]{1}, and holds by essentially the same arguments; a detailed proof is included in \cite[§7]{22}.

The are four particularly natural elements \(\zeta \in X(W)\), for which we can describe the map \(\Psi\) explicitly. First, let \(\zeta_{\leq} : W \to k\) be the linear map with

\[
\zeta_{\leq}(\{w, n\}) = \begin{cases} 1 & \text{if } w \text{ is weakly increasing} \\ 0 & \text{otherwise} \end{cases} \quad \text{for } \{w, n\} \in W. \quad (5.3)
\]

Define \(\zeta_{\geq}, \zeta_<, \zeta_>\) to be the linear maps \(W \to k\) given by the same formula but with “weakly increasing” replaced by “weakly decreasing,” “strictly increasing,” and “strictly decreasing.”

**Proposition 5.4.** For each symbol \(\bullet \in \{\leq, \geq, <, >\}\), it holds that \(\zeta_\bullet \in X(W)\).

**Proof.** It suffices to check that \(\zeta_\bullet \circ \nabla_{\omega}(\{v, m\} \otimes \{w, n\}) = \zeta_\bullet(\{v, m\})\zeta_\bullet(\{w, n\})\), which is routine. \(\square\)

Given \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \vdash n\), let \(I(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{l-1}\}\). The map \(\alpha \mapsto I(\alpha)\) is a bijection from compositions of \(n\) to subsets of \([n-1]\). Write \(\alpha \leq \beta\) if \(\alpha, \beta \vdash n\) and \(I(\alpha) \subseteq I(\beta)\). The **fundamental quasi-symmetric function** associated to \(\alpha \vdash n\) is

\[
L_\alpha = \sum_{\alpha \leq \beta} M_\beta = \sum_{\substack{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in I(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n} \in QSym_n.
\]

The set \(\{L_\alpha : \alpha \vdash n\}\) is another basis of \(QSym_n\). Given \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \vdash n\), let \(\beta \vdash n\) be such that \(I(\beta) = \{1, 2, \ldots, n-1\} \setminus I(\alpha)\) and define the **reversal**, **complement**, and **transpose** of \(\alpha\) to be

\[
\alpha^r = (\alpha_l, \alpha_{l-1}, \alpha_1), \quad \alpha^c = \beta, \quad \text{and} \quad \alpha^t = (\alpha^r)^c = (\alpha^c)^r.
\]

For a word \(w = w_1 w_2 \cdots w_n\), define \(w^r = w_n \cdots w_2 w_1\) and \(\text{Des}(w) = \{i \in [n-1] : w_i > w_{i+1}\}\). Finally, for each \(\bullet \in \{\leq, \geq, <, >\}\), let \(\Psi_\bullet\) be the map \((5.2)\) defined relative to \(\zeta = \zeta_\bullet\).

**Proposition 5.5.** If \(\{w, n\} \in \mathbb{W}\) and \(\alpha \vdash \ell(w)\) are such that \(\text{Des}(w) = I(\alpha)\) then

\[
\Psi_{\leq}(\{w, n\}) = L_\alpha, \quad \Psi_{\geq}(\{w, n\}) = L_{\alpha^c}, \quad \Psi_{\leq}(\{w^r, n\}) = L_{\alpha^r}, \quad \text{and} \quad \Psi_{\geq}(\{w^r, n\}) = L_{\alpha^t}.
\]
Proof. The first identity is well-known and follows by inspecting (5.2) and (5.3). In detail, equations (5.2) and (5.3) show that \( \Psi \leq ([w, n]) = \sum_{\alpha} M_{\alpha} \) where the sum is over all compositions whose parts record the lengths of the factors in a decomposition of \( w \) into weakly increasing subwords, that is, the compositions \( \alpha \vdash \ell(w) \) with \( \text{Des}(w) \subset I(\alpha) \). One derives the other identities similarly. \( \square \)

The set \( \mathcal{X}(W) \) is a monoid with unit element \( t_{\omega} \circ c_\circ \) and product \( \zeta' := \nabla_k \circ (\zeta \otimes \zeta') \circ \Delta_\circ \). For any symbols \( \bullet, \circ \in \{\leq, \geq, <, >\} \) define \( \zeta_{\bullet \circ} := \zeta \circ \zeta_\circ \in \mathcal{X}(W) \) and let \( \Psi_{\bullet \circ} \) be the unique morphism \( (W, \zeta_{\bullet \circ}) \to (QSym, \Psi_{QSym}) \) given by (5.2) for \( \zeta = \zeta_{\bullet \circ} \). For example, if \( [w, n] \in \mathcal{W} \) then

\[
\zeta_{>\leq}([w, n]) = \begin{cases} 
1 & \text{if } w = \emptyset, \\
2 & \text{if } w_1 > \cdots > w_i \leq w_{i+1} \leq \cdots \leq w_m \text{ where } 1 \leq i \leq m = \ell(w) \\
0 & \text{otherwise.}
\end{cases}
\]

Similar formulas hold for the other possibilities of \( \zeta_{\bullet \circ} \).

One calls \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_t) \vdash n \) a peak composition if \( \alpha_i \geq 2 \) for \( 1 \leq i < l \), i.e., if \( 1 \notin I(\alpha) \) and \( i \in I(\alpha) \Rightarrow i + 1 \notin I(\alpha) \). The number of peak compositions of \( n \) is the \( n \)th Fibonacci number. The peak quasi-symmetric function [28, Proposition 2.2] of a peak composition \( \alpha \vdash n \) is

\[
K_\alpha = \sum_{\beta \vdash n \atop I(\alpha) \subset I(\beta) \cup (I(\beta) + 1)} 2^{\ell(\beta)} M_\beta = \frac{1}{2} \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} 2^{\{i_1, i_2, \ldots, i_n\}} x_{i_1} x_{i_2} \cdots x_{i_n} \in QSym_n.
\]

If \( \alpha = (2, 2) \vdash 4 \), for example, then \( I(\alpha) = \{2\} \) and the subsets \( I(\beta) \subseteq \{1, 2, 3\} \) that satisfy \( I(\alpha) \subset I(\beta) \cup (I(\beta) + 1) \) are \( \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \), so

\[
K_{(2,2)} = 4M_{(1,3)} + 4M_{(2,2)} + 8M_{(1,1,2)} + 8M_{(1,2,1)} + 8M_{(2,1,1)} + 16M_{(1,1,1,1)}.
\]

If \( k \) does not have characteristic two, then the functions \( K_\alpha \) (with \( \alpha \) a peak composition) are a basis for a graded Hopf subalgebra of \( QSym \) [1, Proposition 6.5], which we denote by

\[
OQSym = \mathcal{O} - \text{span}\{K_\alpha : \alpha \text{ is a peak composition}\}.
\]

Given a peak composition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_t) \), define \( \alpha^o = (\alpha_1 + 1, \alpha_{l-1}, \ldots, \alpha_2, \alpha_1 - 1) \).

Lemma 5.6. The linear map \( QSym \to \) with \( L_\beta \mapsto L_{\beta^o} \) for all compositions \( \beta \) has \( K_\alpha \mapsto K_{\alpha^o} \) for each composition \( \alpha \).

Proof. The given map has \( M_\beta \mapsto M_{\beta^o} \) for all \( \beta \), so the result holds since \( I(\alpha) \subset I(\beta) \cup (I(\beta) + 1) \) if and only if \( I(\alpha^o) \subset I(\beta^o) \cup (I(\beta^o) + 1) \) for all \( \alpha \vdash n \) where \( \alpha \) is a peak composition. \( \square \)

For a word \( w = w_1w_2 \cdots w_n \), define the sets \( \text{Peak}(w) = \{i : 1 < i < n, w_{i-1} = w_i > w_{i+1}\} \) and \( \text{Val}(w) = \{i : 1 < i < n, w_{i-1} \geq w_i < w_{i+1}\} \). For \( \alpha \vdash n \), let \( \Lambda(\alpha) \vdash n \) be the peak composition with \( I(\Lambda(\alpha)) = \{i \geq 2 : i \in I(\alpha) \text{, } i - 1 \notin I(\alpha)\} \).

If \( w \) is a word and \( \alpha \vdash \ell(w) \) and \( \text{Des}(w) = I(\alpha) \), then \( \text{Peak}(w) = I(\Lambda(\alpha)) \).

Proposition 5.7. If \( [w, n] \in \mathcal{W} \) and \( \alpha, \beta \vdash \ell(w) \) have \( \text{Peak}(w) = I(\alpha) \) and \( \text{Val}(w) = I(\beta) \) then

\[
\Psi_{>\leq}([w, n]) = K_\alpha, \quad \Psi_{\leq>}([w, n]) = K_\beta, \quad \Psi_{\leq<}([w, n]) = K_{\alpha^o}, \quad \text{and} \quad \Psi_{\geq>}([w^r, n]) = K_{\beta^o}.
\]
Proof. Let $\Theta : \text{QSym} \to \text{OQSym}$ be the linear map with $\Theta(L_\alpha) = K_{\Lambda(\alpha)}$ for all compositions $\alpha$. The linear map $\nu_{\text{QSym}} : \text{QSym} \to \k$ whose value at $L_\alpha$ is 1 if $\alpha = \emptyset$, 2 if $\alpha$ has the form $(1, \ldots, 1, k)$, and 0 otherwise is an algebra morphism, and $\Theta$ is the unique morphism of combinatorial bialgebras $(\text{QSym}, \nu_{\text{QSym}}) \to (\text{QSym}, \zeta_{\text{QSym}})$ [1] Example 4.9.

Using Proposition 5.5 it is easy to check that $\zeta_{\text{QSym}} \circ \Theta \circ \Psi < = \nu_{\text{QSym}} \circ \Psi < = \zeta_{\geq}$, so $\Theta \circ \Psi <$ is a morphism of combinatorial bialgebras $(\mathcal{W}, \zeta_{\geq}) \to (\text{QSym}, \zeta_{\text{QSym}})$. Since there is only one such morphism, we have $\Psi_{\geq} = \Theta \circ \Psi <$ which implies the first identity. Similarly, as $\zeta_{\text{QSym}} \circ \Theta \circ \Psi > = \zeta_{<\geq}$, we have $\Psi_{<\geq} = \Theta \circ \Psi > (w,n) = K_{\Lambda(\alpha)}$ for the composition $\alpha \vdash \ell(w)$ such that $I(\alpha) = \text{Des}(w^T)$. Tracing through the definitions, one finds that $\Lambda(\alpha)$ is precisely the composition $\beta \vdash \ell(w)$ with $\text{Val}(w) = I(\beta)$. The other identities follow by Lemma 5.11. 

6 Stanley symmetric functions

In this section, we assume $\k$ has characteristic zero, so that $\k$ contains the rational numbers $\mathbb{Q}$.

6.1 Type A

The Stanley symmetric function of a permutation $\pi \in S_n$ with $l = \ell(\pi)$ is the power series

$$F_\pi = \sum_{w \in R(\pi)} \sum_{i_1 \leq i_2 \leq \ldots \leq i_l \text{ if } w_j < w_{j+1}} x_{i_1} x_{i_2} \cdots x_{i_l} \in \k[[x_1, x_2, \ldots]]. \quad (6.1)$$

This definition differs from Stanley’s original construction in [27] by an inversion of indices, a common convention in subsequent literature. As the name suggests, one has $F_\pi \in \text{Sym}$ [27]; more strongly, each $F_\pi \in \mathbb{N}\text{-span}\{s_\lambda : \lambda \vdash \ell(\pi)\}$ is Schur positive [10]. Among other reasons, these functions are of interest since they are the stable limits of the Schubert polynomials $S_\pi$, and since the coefficient of any square-free monomial in $F_\pi$ gives the size of $R(\pi)$.

The definition (6.1) is more canonical than it first appears: $F_\pi$ is the image of $[\pi] \in \Pi$ under the unique morphism of combinatorial bialgebras $(\Pi, \zeta_>) \to (\text{QSym}, \zeta_{\text{QSym}})$. To be precise, let $\omega : \text{Sym} \to \text{Sym}$ be the linear map with $\omega(s_\lambda) = s_{\lambda^T}$, where $\lambda^T$ denotes the usual transpose of a partition. This map is the restriction of the linear map $\text{QSym} \to \text{QSym}$ with $L_\alpha \to L_{\alpha^T}$ for all $\alpha$.

Proposition 6.1. If $\pi \in S_n$ then $\Psi_>([\pi]) = \Psi_>([\pi]) = F_\pi$ and $\Psi_<([\pi]) = \Psi_<([\pi]) = \omega(F_\pi) = F_{\pi^{-1}}$.

The last equality is [19] Corollary 7.22.

Proof. Let $\pi \in S_n$. Since every weakly increasing (respectively, decreasing) consecutive subsequence of a reduced word for $\pi$ is strictly increasing (respectively, decreasing), it is clear from (5.2) that $\Psi_>([\pi]) = \Psi_>([\pi])$ and $\Psi_>([\pi]) = \Psi<(\pi)$. The inner sum in (6.1) is $L_\alpha$ for the composition $\alpha \vdash \ell(w)$ with $I(\alpha) = \text{Des}(w)$, so Proposition 5.5 implies that $F_\pi = \Psi_>([\pi])$. Since $F_\pi \in \text{Sym}$, it follows that $F_{\pi^{-1}} = \Psi_>([\pi])$ and $\omega \circ \Psi_>([\pi]) = \Psi_<(\pi)$, which implies the other equalities.

Applying $\Psi_>$ to Corollary 4.2 and Theorem 6.6 gives the following identities:

Corollary 6.2. If $\pi \in S_n$ then $\Delta(F_\pi) = \sum_{\pi = \pi' \cdot \pi''} F_{\pi'} \otimes F_{\pi''}$.

Corollary 6.3. If $\pi' \in S_m$ and $\pi'' \in S_n$ then $F_{\pi'} F_{\pi''} = \sum_{\pi \in S_{\omega(\pi', \pi'')}} F_\pi$. 

18
Let $w_0^{(n)} = n \cdots 321 \in S_n$. It is well-known that $F_{w_0^{(n)}} = s_{\delta_n}$ for $\delta_n = (n-1, \ldots, 3, 2, 1)$ [27]. Moreover, if $n \in \mathbb{P}$ and $p = \left\lceil \frac{n+1}{2} \right\rceil$ and $q = \left\lfloor \frac{n+1}{2} \right\rfloor$, then $s_{\delta_p}s_{\delta_q} = P_{(n-1,n-3,n-5,\ldots)}$ [14 Corollary 1.14], where $P_{\lambda} \in \text{Sym}$ is the Schur $P$-function indexed by a strict partition $\lambda$ (see [28 §A.3]).

**Corollary 6.4.** If $n \in \mathbb{P}$ and $p \in \{\left\lceil \frac{n+1}{2} \right\rceil, \left\lfloor \frac{n+1}{2} \right\rfloor\}$ then $P_{(n-1,n-3,n-5,\ldots)} = \sum_{\pi \in B(p,n)} F_{\pi}$.

**Proof.** This follows by applying $\Psi_>$ to Theorem 1.3(a) with $m = p \in \{\left\lceil \frac{n+1}{2} \right\rceil, \left\lfloor \frac{n+1}{2} \right\rfloor\}$. \hfill $\Box$

**Example 6.5.** For $n = 5$, we have $P_{(4,2)} = F_{32541} + F_{34512} + F_{35142} + F_{42513} + F_{43512} + F_{52143}$.

### 6.2 Types B, C, and D

In a similar way, we can realize the Stanley symmetric functions of types B, C, and D from [4,11,16,17] as the images of certain canonical morphisms from algebraic structures on signed permutations.

Given $n \in \mathbb{P}$, let $B_n$ denote the group of signed permutations of $\{\pm 1, \pm 2, \ldots, \pm n\}$, that is, permutations $\pi$ such that $\pi(-i) = -\pi(i)$ for all $i$. Let $s_1^B, s_2^B, s_3^B, \ldots, s_n^B \in B_n$ be the elements

$$s_1^B = (-1,1) \quad \text{and} \quad s_i^B = (-i,-i+1)(i-1,i) \quad \text{for} \quad 2 \leq i \leq n.$$ 

Let $s_1^D, s_2^D, s_3^D, \ldots, s_n^D \in B_n$ be the elements

$$s_1^D = s_1^B s_2^B s_1^B = (-2,1)(-1,2) \quad \text{and} \quad s_i^D = s_i^B \quad \text{for} \quad 2 \leq i \leq n,$$

The normal subgroup $D_n := \langle s_1^D, s_2^D, \ldots, s_n^D \rangle \subset B_n = \langle s_1^B, s_2^B, \ldots, s_n^B \rangle$ consists of the signed permutations $\pi$ for which the number of integers $1 \leq i \leq n$ with $\pi(i) < 0$ is even.

It is well-known that $B_n$ and $D_n$ are the finite Coxeter group of types $B_n/C_n$ and $D_n$ relative to the generating sets $\{s_1^B, s_2^B, \ldots, s_n^B\}$ and $\{s_1^D, s_2^D, \ldots, s_n^D\}$. Given $\pi \in B_n$, write $R^B(\pi)$ and $R^D(\pi)$ for the sets of words $i_1 i_2 \cdots i_l$ of minimal length such that $\pi = s_{i_1}^B s_{i_2}^B \cdots s_{i_l}^B$ and $\pi = s_{i_1}^D s_{i_2}^D \cdots s_{i_l}^D$, respectively. We refer to elements of these sets as reduced words. Evidently $R^D(\pi)$ is empty unless $\pi \in D_n$. Write $\ell^B(\pi)$ for the common value of $\ell(w)$ for all $w \in R^B(\pi)$, and define $\ell^D(\pi)$ for $\pi \in D_n$ analogously. Explicit formulas for these length functions appear in [6 Chapter 8].

For $\pi \in B_n$, define $[\pi]_B = \sum_{w \in R^B(\pi)} [w,n] \in W_n$ and $[\pi]_D = \sum_{w \in R^D(\pi)} [w,n] \in W_n$. Let

$$\Pi_n^B = \langle [\pi]_B : \pi \in B_n \rangle \quad \text{and} \quad \Pi_n^D = \langle [\pi]_D : \pi \in D_n \rangle$$

and define

$$\Pi^B = \bigoplus_{n \geq 1} \Pi_n^B \quad \text{and} \quad \Pi^D = \bigoplus_{n \geq 2} \Pi_n^D.$$ 

Observe that $[\pi]_D = 0$ if $\pi \in B_n - D_n$. The nonzero elements $[\pi]_B \in \Pi^B$ and $[\pi]_D \in \Pi^D$ are homogeneous of degree $\ell^B(\pi)$ and $\ell^D(\pi)$, respectively.

Suppose $B$ is a (graded) bialgebra and $M$ is a (graded) coalgebra. One says that $M$ is a (graded) right $B$-module coalgebra if $M$ is a (graded) right $B$-module such that the multiplication map $M \otimes B \to M$ is a (graded) coalgebra morphism. Since $\Pi$, $\Pi^B$, and $\Pi^D$ are all subspaces of the bialgebra $W$, we can multiply their elements together using $\nabla_{\omega}$. 

**Theorem 6.6.** The subspaces $\Pi^B$ and $\Pi^D$ are graded sub-coalgebras of $(W, \Delta_\circ, \epsilon_\circ)$ and graded right module coalgebras for $(\Pi, \nabla_{\omega}, \epsilon_{\omega}, \Delta_\circ, \epsilon_\circ)$. 


Proof. By [6, Theorem 3.3.1], each set $\mathcal{R}^B(\pi)$ for $\pi \in B_n$ is an equivalence class under the strongest relation with $vw \sim_B v'w'$ whenever $v \sim_B v'$ and $w \sim_B w'$, and such that $ij \sim_B ji$ if $|j - i| > 1$, $i(i + 1)j \sim_B (i + 1)ji$ if $i \geq 2$, and $1212 \sim_B 2121$. Likewise, each set $\mathcal{R}^D(\pi)$ for $\pi \in D_n$ is an equivalence class under the strongest relation with $vw \sim_D v'w'$ whenever $v \sim_D v'$ and $w \sim_D w'$, and such that $ij \sim_D ji$ if $|j - i| > 1$ and $\{i, j\} \neq \{1, 3\}$, $i(i + 1)j \sim_D (i + 1)i(j + 1)$ if $i \geq 2$, and $12 \sim_D 21$ and $131 \sim_D 31$. An equivalence class under the corresponding relation is equal to $\mathcal{R}^B(\pi)$ or $\mathcal{R}^D(\pi)$ for some signed permutation $\pi$ if and only if it contains no words with adjacent repeated letters. It follows that the coproduct $\Delta_\circ$ satisfies $\Delta_\circ(\Pi^B) \subset \Pi^B \otimes \Pi^B$ and $\Delta_\circ(\Pi^D) \subset \Pi^D \otimes \Pi^D$.

As $W$ is a bialgebra, to show that $\Pi^B$ and $\Pi^D$ are graded right $\Pi$-module coalgebras it suffices to check that $\nabla_\omega(\Pi^B \otimes \Pi) \subset \Pi^B$ and $\nabla_\omega(\Pi^D \otimes \Pi) \subset \Pi^D$. These inclusions follow by arguments similar to the proof of Theorem 4.6.

We write $\pi =_B \pi' \pi''$ to mean that $\pi, \pi', \pi'' \in B_n$ and $\pi = \pi' \pi''$ and $\ell^B(\pi) = \ell^B(\pi') + \ell^B(\pi'')$. Define the operator $\pi =_D \pi' \pi''$ similarly. The following is clear from the preceding proof:

**Corollary 6.7.** If $\pi \in B_n$ then 

$$\Delta_\circ([\pi]_B) = \sum_{\pi =_B \pi' \pi''} [\pi']_B \otimes [\pi'']_B \quad \text{and} \quad \Delta_\circ([\pi]_D) = \sum_{\pi =_D \pi' \pi''} [\pi']_D \otimes [\pi'']_D.$$ 

The one-line representation of $\pi \in B_n$ is the word $\pi_1 \pi_2 \cdots \pi_n$ where $\pi_i = \pi(i)$, with negative letters indicated as barred entries $\bar{i}$ rather than $-i$. For example, the eight elements of $B_2$ are $12, \bar{1}2, 1\bar{2}, 21, \bar{2}1, 2\bar{1}$, and $\bar{1}\bar{2}$. There should be an analogue of Theorem 4.6 describing the right action of $\Pi$ on $\Pi^B$ and $\Pi^D$, but this remains to be found. For example, we have

\[
\begin{align*}
[123]_B[213] & = [14325]_B + [12435]_B + [12435]_B + [14325]_B + [42315]_B, \\
[123]_D[213] & = [14325]_D + [12435]_D + [12435]_D + [14325]_D + [42315]_D, \\
[123]_B[231] & = [14352]_B + [12453]_B + [12453]_B + [14352]_B + [42351]_B, \\
[123]_D[231] & = [14352]_D + [12453]_D + [12453]_D + [14352]_D + [42351]_D, \\
[132]_B[312] & = [13524]_B + [15234]_B + [53214]_B, \\
[132]_D[312] & = [13524]_D + [15234]_D + [53214]_D + [53214]_D, \\
[132]_B[321] & = [13542]_B + [14523]_B + [15243]_B + [43521]_B + [53241]_B + [54213]_B, \\
[132]_D[321] & = [13542]_D + [14523]_D + [15243]_D + [43521]_D + [53241]_D + [54213]_D + [54321]_D + [54321]_D + [54321]_D,
\end{align*}
\]

using infix notation $[\pi']_B \otimes [\pi'']$ in place of $\nabla_\omega([\pi']_B \otimes [\pi''])$.

**Problem 6.8.** Describe the products $\nabla_\omega([\pi']_B \otimes [\pi''])$ and $\nabla_\omega([\pi']_D \otimes [\pi''])$.

There is a module coalgebra version of Definition 5.1

**Definition 6.9.** If $(B, \zeta)$ is a combinatorial bialgebra and $(M, \xi)$ is a combinatorial coalgebra such that $M$ is a graded right $B$-module coalgebra with $\xi(mb) = \xi(m)\zeta(b)$ for all $m \in M$ and $b \in B$, then we say that $(M, \xi)$ is a combinatorial $(B, \zeta)$-module coalgebra. A morphism of combinatorial $(B, \zeta)$-module coalgebras is a morphism of combinatorial coalgebras that is a $B$-module map.
For any combinatorial bialgebra \( (B, \zeta) \), the pair \( (\text{QSym}, \zeta_{\text{QSym}}) \) is a combinatorial \( (B, \zeta) \)-module coalgebra for the \( B \)-module structure given by \( \text{QSym} \otimes B \xrightarrow{\text{id} \otimes \Psi} \text{QSym} \otimes \text{QSym} \xrightarrow{\nabla} \text{QSym} \) where \( \Psi \) is the unique morphism of combinatorial bialgebras \( (B, \zeta) \rightarrow (\text{QSym}, \zeta_{\text{QSym}}) \). This observation relies in general on Theorem 5.3, but certainly holds for \( B = \Pi \) without that result.

**Theorem 6.10.** Suppose \( \zeta^A : \Pi \rightarrow k \) and \( \zeta^B : \Pi^B \rightarrow k \) and \( \zeta^D : \Pi^D \rightarrow k \) are maps such that \( (\Pi^A, \zeta^A) \) is a combinatorial bialgebra and \( (\Pi^B, \zeta^B) \) and \( (\Pi^D, \zeta^D) \) are combinatorial \( (\Pi, \zeta^A) \)-module coalgebras. There are then unique morphisms of combinatorial \( (\Pi^A, \zeta^A) \)-module coalgebras \( (\Pi^B, \zeta) \rightarrow (\text{QSym}, \zeta_{\text{QSym}}) \) and \( (\Pi^D, \zeta) \rightarrow (\text{QSym}, \zeta_{\text{QSym}}) \).

**Proof.** Since \( (\text{QSym}, \zeta_{\text{QSym}}) \) is the terminal object in the category of combinatorial coalgebras \([1, \text{Theorem } 4.1]\), which contains \( (\Pi^B, \zeta^B) \) for all \( n \in \mathbb{P} \), there exists a unique morphism of combinatorial coalgebras \( \Psi : (\Pi^B, \zeta^B) \rightarrow (\text{QSym}, \zeta_{\text{QSym}}) \). This map is a \( \Pi \)-module morphism since

\[
\Pi^B_m \otimes \Pi^B_n \xrightarrow{\Psi \otimes \Psi} \text{QSym} \otimes \text{QSym} \xrightarrow{\nabla} \text{QSym}
\]

are both morphisms of combinatorial coalgebras \( (\Pi^B_m \otimes \Pi^B_n, \nabla \circ (\zeta^B \otimes \zeta^A)) \rightarrow (\text{QSym}, \zeta_{\text{QSym}}) \), and therefore must coincide. The analogous statement for \( (\Pi^D, \zeta^D) \) follows by the same argument. \( \square \)

As with Theorem 5.2, the preceding result may be realized as a special case of a more general principle. The proof of the following statement requires a slight extension of \([1, \text{Theorem } 4.1]\), applicable to our more general definition of combinatorial coalgebras. This is given in \([22, \S 7]\). The result then follows by repeating the preceding proof with \( \Pi \) and \( \Pi^B \) replaced by \( B \) and \( M \).

**Theorem 6.11.** Suppose \( (B, \zeta) \) is a combinatorial bialgebra and \( (M, \xi) \) is a combinatorial \( (B, \zeta) \)-module coalgebra. The unique morphism of combinatorial coalgebras \( (M, \xi) \rightarrow (\text{QSym}, \zeta_{\text{QSym}}) \) is a \( B \)-module morphism, and hence also the unique morphism of \((B, \zeta)\)-module coalgebras.

The type \( B \) Stanley symmetric function of \( \pi \in B_n \) with \( l = \ell_B(\pi) \) is the power series

\[
F^B_\pi = \sum_{w \in \mathcal{R}^B(\pi)} \sum_{j \notin \text{Peak}(w) \text{ if } i_j = i_{j+1}} 2^{|\{i_1, i_2, \ldots, i_l\}| - \ell_0(\pi)} x_{i_1} x_{i_2} \cdots x_{i_l}
\]

where \( \ell_0(\pi) = |\{1 \leq i \leq n : \pi(i) < 0\}| \), which is the number of 1’s in every \( w \in \mathcal{R}^B(\pi) \); see \([10, \text{Theorem } 2.7]\). The type \( C \) Stanley symmetric function of \( \pi \in B_n \) is

\[
F^C_\pi = 2^{\ell_0(\pi)} F^B_\pi.
\]

(This definition is \([4, \text{Proposition } 3.4]\); however, the authors in \([4]\) refer to \( F^C_\pi \) as a “\( B_n \) Stanley symmetric function.”) Finally, the type \( D \) Stanley symmetric function of \( \pi \in D_n \) with \( l = \ell_D(\pi) \) is

\[
F^D_\pi = \sum_{w \in \mathcal{R}^D(\pi)} \sum_{j \notin \text{Peak}(w) \text{ if } i_j = i_{j+1}} 2^{|\{i_1, i_2, \ldots, i_l\}| - \ell_0(\pi)} x_{i_1} x_{i_2} \cdots x_{i_l}
\]

where \( \ell_0(\pi) = |\{1 \leq i \leq n : \pi(i) < 0\}| \). The formula for \( F^D_\pi \) also makes sense when \( \pi \in B_n - D_n \), but gives \( F^D_\pi = 0 \) since \( \mathcal{R}^D(\pi) = \emptyset \).

The Hopf subalgebra \( \mathcal{O}\text{Sym} = \text{Sym} \cap \mathcal{O}\text{QSym} \) has two distinguished bases indexed by strict partitions: the Schur \( P \)-functions \( P_\lambda \) mentioned earlier, and the Schur \( Q \)-functions given by \( Q_\lambda = 2^{\ell(\lambda)} P_\lambda \).
Proof. We have $(\Pi^B, \zeta^B, \zeta^D) \in \mathfrak{X}(\mathbb{W})$. Part (c) holds since $\Psi^C$ is a morphism of combinatorial bialgebras $(\mathbb{W}, \zeta^C) \to (\text{QSym}, \zeta_{\text{QSym}})$ by definition. Given these observations and Theorem 6.10, the other claims follow since the linear maps $\kappa^B, \kappa^D : \mathbb{W} \to \mathbb{k}$ with $\kappa^B([w,n]) = 2^{-o_B(w)}\kappa^C([w,n]) = 2^{-o_D(w)}\kappa^D([w,n]) = 2^{-o_D(w)}\kappa^D([w,n])$ have $\kappa^B \circ \nabla_w([\pi]^B_1 \otimes [\pi]^B_2) = \kappa^B([\pi]^B_3)$ if $\pi^1 \in B_m$ and $\pi^2 \in S_n$ and $m \geq 1$, and $\kappa^D \circ \nabla_w([\pi]^D_1 \otimes [\pi]^D_2) = \kappa^D([\pi]^D_3)$ if $\pi^1 \in D_m$ and $\pi^2 \in S_n$ and $m \geq 2$.

Finally, we observe that the type B, C, and D Stanley symmetric functions occur as the images of the unique morphisms of combinatorial module coalgebras described in the previous result.

**Proposition 6.13.** If $\pi \in B_n$ then $\Psi^B([\pi]^B) = F^B_{\pi}$ and $\Psi^C([\pi]^C) = F^C_{\pi}$ and $\Psi^D([\pi]^D) = F^D_{\pi}$.

Of course, if $\pi \in B_n - D_n$ then $\mathcal{R}^D(\pi) = \emptyset$ so $[\pi]^D = 0$ and $F^D_{\pi} = 0$.

**Proof.** These identities are immediate from the definitions and Proposition 5.7.

This interpretation may be used to recover [3] Corollaries 3.5 and 3.11:

**Corollary 6.14** (See [3]). Let $\pi \in B_n$. Then $F^B_{\pi} = F^B_{\pi^{-1}}$ and $F^C_{\pi} = F^C_{\pi^{-1}}$ and $F^D_{\pi} = F^D_{\pi^{-1}}$.

**Proof.** No reduced word contains adjacent repeated letters, so $F^C_{\pi} = \Psi^C_{\pi^1 \cdot \pi^2} = \Psi^C_{\pi^1 \cdot \pi^2}$. By Lemma 5.6 and Proposition 5.7, $F^C_{\pi^{-1}} = \Psi^C_{\pi^1 \cdot \pi^2}$ is the image of $\Psi^C_{\pi^1 \cdot \pi^2}$ under the linear involution of $\text{QSym}$ with $L_\beta \mapsto L_{\beta^t}$. Since $F^C_{\pi} \in \text{Sym}$, this involution has no effect so $F^B_{\pi} = F^B_{\pi^{-1}}$ and $F^C_{\pi} = F^C_{\pi^{-1}}$. The same argument, mutatis mutandis, shows that $F^D_{\pi} = F^D_{\pi^{-1}}$.

Applying $\Psi_B$, $\Psi_C$, and $\Psi_D$ to Corollary 6.7 gives the following:
Corollary 6.15. If $\pi \in B_n$ then the following coproduct formulas hold:

$$\Delta(F^B_\pi) = \sum_{\pi = \pi' \cdot \pi''} F^B_{\pi'} \otimes F^B_{\pi''}, \quad \Delta(F^C_\pi) = \sum_{\pi = \pi' \cdot \pi''} F^C_{\pi'} \otimes F^C_{\pi''}, \quad \Delta(F^D_\pi) = \sum_{\pi = \pi' \cdot \pi''} F^D_{\pi'} \otimes F^D_{\pi''}.$$  

An analogue of Corollary 6.3 in types B, C, and D would follow from this result and a solution to Problem 6.8.

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