Calculation of an inhomogeneous polymer thick-walled cylindrical shell taking into account creep under the action of temperature load

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Abstract. The article presents the calculation of a polymer thick-walled cylindrical shell taking into account creep under the action of an uneven temperature field. The relevance of the work lies in the widespread introduction of polymer pipes in the repair and construction of heating pipelines, sewerage systems, water supply systems. The problem is solved in an axisymmetric setting under conditions of plane deformation. The calculation is based on the nonlinear Maxwell-Gurevich equation, which is widely used in the calculations of polymer structures.

1. Introduction
In the problem under consideration, the cylindrical shell is in a plane deformed state. Temperature is considered to be a function of radius and time. Relaxation and elastic characteristics, which are strongly dependent on temperature, will be functions of the coordinate and time on the inner and outer surfaces of the shell, and temperatures. In the problem under consideration, the cylindrical shell is in a plane deformed state. Temperature is also considered to be a function of radius and time. The relaxation and elastic characteristics, which are strongly dependent on temperature, will be functions of the coordinate and time on the inner and outer surfaces of the shell, where temperatures \( T(a) = T_a \) and \( T(b) = T_b \) act. Here \( a \) and \( b \) are, respectively, the radii of the inner and outer surfaces of the shell. Taking into account the axial symmetry, the problem in the geometric formulation is one-dimensional (all functions depend only on \( r \)), and with regard to the influence of temperature, which in some cases can change in time, it is considered as quasi-stationary.

2. Physical relations for a viscoelastic material
The problems of viscoelasticity of continuously inhomogeneous bodies considered in this article are related to problems of polymer mechanics. Below are physical relations in differential form that are valid for polymers and composites [1], which can be considered quite general and in particular cases applicable to other materials [2, 3].

Consider the equations describing the proper viscoelastic behavior of the material, i.e. defining term \( \varepsilon_{jk}^* \). Due to the micro inhomogeneity of the structure at the molecular and supramolecular level, the polymer can have a spectrum of relaxation times; therefore, highly elastic deformations in the general case are the sums of individual components, each of which corresponds to a certain member of the spectrum:
Usually, when solving problems of mechanics in relation to rigid polymers in quasi-static regimes, we will consider it sufficient to take into account one or two components of highly elastic deformation. For the rate of the components of highly elastic deformation, there is an unambiguous dependence on the parameters of the deformation process. The generalized nonlinear Maxwell equation is valid as analytical expressions for these dependences (constraint equations). This equation was derived in [4] based on molecular concepts of the behavior of materials. Later, it was repeatedly tested in solving various problems in the mechanics of polymers and composites [2].

\[
\dot{\varepsilon}^*_jk = \sum_s (\varepsilon^*_jk)_s
\]

(1)

2. Derivation of resolving equations

As in the problems of the theory of elasticity, the problems of the theory of creep can be solved in stresses and in displacements. Below is the way to deal with the displacements. The basic formulas of the theory of creep formally coincide with the equations of the theory of elasticity. For deformations we have

\[
\varepsilon^*_z = \varepsilon^0_z + \varepsilon^*_z + \alpha T
\]

Assuming from the plane deformation condition \( \varepsilon^*_z = 0 \), we obtain

\[
\varepsilon^0_z = -\varepsilon^*_z - \alpha_T T
\]

Then the elastic component of the average deformation will be equal to

\[
\varepsilon^0_{cp} = \frac{1}{3}(\varepsilon^0_r + \varepsilon^0_\theta + \varepsilon^0_z) = \frac{1}{3}\left(\frac{\partial u}{\partial r} + \frac{u}{r} - (\varepsilon^*_r + \varepsilon^*_\theta + \varepsilon^*_z) - 3\alpha_T T\right) = \frac{1}{3}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) - \alpha_T T
\]

(6)

The last transformation is made on the basis of the hypothesis that the volumetric creep strain is zero.
Using the Cauchy relations and Hooke's law for elastic deformations

\[
\varepsilon^0_r = \frac{1}{E} (\sigma_r - \nu \sigma_0) ; \quad \varepsilon^0_0 = \frac{1}{E} (\sigma_0 - \nu \sigma_0)
\]

we get expressions:

\[
\varepsilon^0_r = \frac{\partial u}{\partial r} - \varepsilon^*_r - \alpha_T r ; \quad \varepsilon^0_0 = \frac{u}{r} - \varepsilon^*_0 - \alpha_T r
\]

(7)

Substituting (6) and (7) into Hooke's law in the Lamé form, we obtain

\[
\sigma_r = \lambda \left( \frac{\partial u}{\partial r} + \frac{u}{r} \right) + 2\mu \frac{\partial u}{\partial r} - 2\mu \varepsilon^*_r - 3K\alpha_T T
\]

(8)

\[
\sigma_0 = \lambda \left( \frac{\partial u}{\partial r} + \frac{u}{r} \right) + 2\mu \frac{u}{r} - 2\mu \varepsilon^*_0 - 3K\alpha_T T
\]

We obtain the resolving equation in displacements by substituting the expression for stresses (8) into the equilibrium equation

\[
\frac{\partial^2 u}{\partial r^2} + \varphi(r,t) \frac{\partial u}{\partial r} + \psi(r,t) \frac{u}{r} = f(r,t)
\]

(9)

In this equation

\[
\varphi(r,t) = \frac{1}{r} + \frac{1}{\lambda + 2\mu} \frac{\partial (\lambda + 2\mu)}{\partial r} ; \quad \psi(r,t) = \frac{1}{r^2} + \frac{1}{\lambda + 2\mu} \frac{\partial \lambda}{\partial r}
\]

\[
f(r,t) = \frac{1}{\lambda + 2\mu} \left[ 3\alpha_T \frac{\partial (KT)}{\partial r} + 2\mu \left( \frac{\partial \varepsilon^*_r}{\partial r} + \frac{\varepsilon^*_r - \varepsilon^*_0}{r} \right) + 2 \frac{\partial \varepsilon^*_r}{\partial r} \right]
\]

(10)

3. Solution method

Resolving equation (9) is a second order partial differential equation with variable coefficients. Due to the nonlinearity and complexity of the coefficients, an analytical solution of these equations cannot be found even with significant simplifications. If we assume that the temperature field and force loads change slowly in time, then the creep problems can be considered as quasi-stationary. One of the first works in which a “layer by layer” method was proposed for solving quasistationary creep problems was [5]. Later this method was used in [6, 7] (such methods for solving creep problems are also called stepwise, incremental, etc).

Let us explain the essence of the “layer-by-layer integration” method by the example of solving equation (9) under the action of a temperature field. At the zero stage (at \( t = 0 \)), the problem of determining the temperature field \( T = T_0(r) \) is first solved and the dependences of the elastic and relaxation parameters of the material on temperature and coordinates are found

\[
E = E_0[T_0(r)] , \quad \nu = \nu_0[T_0(r)] , \quad E_{es,b} = E_{es,b}[T_0(r)],...
\]

If the loading is considered instantaneous, then at time \( t = 0 \) the initial conditions will be valid:

\[
t = 0; \quad \varepsilon^*_{r,0} = \varepsilon^*_{\varphi,0} = 0
\]

(11)

Thus, at the zero stage, we arrive at an elastic problem. In this case, in equation (9), the partial derivatives with respect to the radius can be replaced by ordinary ones (in the subsequent formulas, a prime is used to denote such derivatives):

\[
u^0 + \varphi(0,0) u^0 + \psi(0,0) u_0 = f(r,0)
\]

(12)

where

\[
\varphi(0,0) = \frac{1}{r} + \frac{\lambda_0 + 2\mu_0}{\lambda_0 + 2\mu_0} ; \quad \psi(0,0) = \frac{1}{r^2} - \frac{\lambda_0}{r(\lambda_0 + 2\mu_0)}
\]

\[
f(r,0) = \frac{3\alpha_T}{\lambda_0 + 2\mu_0} (K_0 T_0)^\prime
\]

(13)
Here, the dependences of the mechanical characteristics on the radius are due to the initial (in this case, temperature) inhomogeneity of the material. Equation (12) with boundary conditions

\[
\begin{align*}
t = 0; & \quad \varepsilon_r^* = \varepsilon_0^* = 0 \\
\end{align*}
\]

is a two-point boundary value problem, which, due to the complexity of the coefficients of the equation, must be solved numerically. One of the effective methods for solving such boundary value problems is the run-through method [8, 9].

Having determined at the zero stage all the necessary quantities (displacements, deformations and stresses), from the corresponding physical equations, for example (2), one can find the creep strain rates

\[
\begin{align*}
\left( \frac{\partial \varepsilon_r^*}{\partial t} \right)_0, \left( \frac{\partial \varepsilon_0^*}{\partial t} \right)_0
\end{align*}
\]

Assuming that the time step $\Delta t$ can be arbitrarily small, one can carry out a linear approximation in time and calculate the creep deformations at the next "time layer" $t = \Delta t$:

\[
\begin{align*}
t = t_1 = \Delta t; & \quad \varepsilon_{r,1}^* = \left( \frac{\partial \varepsilon_r^*}{\partial t} \right)_0 \cdot \Delta t; \quad \varepsilon_{0,1}^* = \left( \frac{\partial \varepsilon_0^*}{\partial t} \right)_0 \cdot \Delta t
\end{align*}
\]

By numerical differentiation, the derivatives of the creep deformations along the radius are also found in the right-hand side of Eq. (9). In the case of a non-stationary thermal process, one should also analytically or by linear approximation in time to determine a new temperature distribution and new dependences of mechanical characteristics on temperature, and, consequently, on radius. Having thus formed the right-hand side of equation (9), we again come to the elastic problem with a new function $f(r)$ and, in the general case, with new coefficients $\varphi(r)$ and $\psi(r)$. Also solving this problem numerically, we obtain a solution at the first stage. Continuing the process to an arbitrary point in time, it is possible to determine stresses, deformations and displacements at any point in the body.

In step-by-step methods, the question of their convergence remains open. One way to analyse convergence is to compare the results obtained at different values $\Delta t$. For long-term processes, when the creep rate decreases, an uneven (increasing) time step can be used. There are some other possibilities for analysing and improving the convergence of the method, based, for example, on comparing the results at two successive stages of the iterative process with the subsequent adjustment of the previous step in time.

4. Creep of an unevenly heated cylinder

A characteristic feature of temperature problems for structural elements made of polymeric materials is a strong dependence of all mechanical characteristics on temperature. Thus, even with relatively small gradients of the temperature field, it is necessary to solve the problem of mechanics taking into account the inhomogeneity of the material.

Consider the problem of calculating a thick-walled cylinder in an axisymmetric temperature field determined by the following boundary and initial conditions

\[
\begin{align*}
t = 0, & \quad T(r,0) = T_0 = \text{const} \\
0 < t < t_1, & \quad T(a,t) = T_0 + \beta t; \quad T(b,t) = T_0 \\
t > t_1, & \quad T(a,t) = T_0 + \beta t_1; \quad T(b,t) = T_0
\end{align*}
\]

(14)

In accordance with (14), a constant temperature $T_0$ is maintained on the outer surface of the cylinder throughout the entire time, and the inner surface is first heated (over time $t_1$), and then its temperature
is also maintained constant. If you set the final temperature $T_1$ of the inner surface of the cylinder, then the heating time can be determined by the formula $t_1 = (T_1 - T_0) / \beta$, where $\beta$ is the heating rate. The process is unsteady until a certain moment of time $t_2 > t_1$, until the final distribution of temperature along the radius is established. In Figure 1 shows the results of the numerical solution of the equation

$$\lambda \nabla^2 T - c \rho \frac{\partial T}{\partial t} + W = 0$$

obtained under the assumption of constancy of the temperature conductivity coefficient with the following initial data: $a = 8\text{mm}; b = 28\text{mm}; T_0 = 28\degree\text{C}; T_1 = 100\degree\text{C}; \beta = 60\degree\text{C/h}$. From the given dependences, it can be concluded that the stabilization of the temperature field occurs at a time of 3.6 hours.

Assuming the cylinder is sufficiently long, we will assume that a plane deformed state occurs in it, for which the resolving equation in displacements (9) is valid. Considering a short-term process (up to 100 hours), we will restrict ourselves in calculations only to the “senior” component of highly elastic deformation.

Below are the results of solving the problem of creep for a cylinder made of EDT-10 epoxy resin. For this material, the dependences of mechanical characteristics on temperature were studied in the fundamental work [10]. Taking into account the insignificant change in the considered temperature range of Poisson’s ratio $\nu$ and coefficient of linear thermal expansion $\alpha$, we will assume that they are constant and equal to $\nu = 0.3; \alpha = 8 \cdot 10^{-5}$/deg. In the aforementioned work, for EDT-10, the following empirical dependences on the temperature of the elastic modulus and relaxation characteristics corresponding to the leading component of the highly elastic deformation are given:

$$E = (8302 - 1.75T_K) \text{ [MPa]};\quad E_{\sigma_1} = (11340 - 30T_K) \text{ [MPa]};$$

$$m^*_1 = (7.75 - 0.011T_K) \text{ [MPa]};\quad \eta^*_{01} = 685 \cdot 10^8 \exp(-0.0275T_K) \text{ [MPa \cdot sec]}.$$

Here $T_K$ is the temperature in Kalvin scale, K.

Figures 2 and 3 show the variation along the radius of these characteristics for some points in time. Numerical designations in these and subsequent figures are the same as in figure 1. It should be noted that, with the exception of the modulus of elasticity, all other characteristics of the material stabilize already by the end of heating of the inner surface of the cylinder at $t = 1.2$ hours.
As will be shown below, it is essential to take into account in the calculations a relatively short heating time compared to the entire process, since the increase in temperature occurring during this period significantly accelerates the creep process from the very beginning. The calculation was carried out by the numerical method described in Section 3.

Figure 4 shows the stress plots $A$ for some points in time.

For comparison, the same figure shows the stress diagram obtained from the elastic solution and corresponding to the heating end time ($t = 1.2$ hours). The following results should be noted. In the initial period, during the heating process, the stresses increase, this is natural, since the temperature loads
increase. Then, during the creep of the cylinder, a significant relaxation of stresses occurs in both the stretched and compressed zones. In this case, if at $t > 3.6$ hours the temperature distribution along the radius remains unchanged (see figure 1), then the relaxation process continues, which leads to an even greater decrease in stresses.

**Conclusion**

The article shows that the solution of viscoelasticity problems based on the nonlinear Maxwell-Gurevich equation using differential equations is quite simple. Several papers on a related topic have been published [11 - 13]. Note that the method of sequential loading was used to solve the nonlinear problem in the article. You can also use the method of successive approximations [14], which may be a development of the topic under consideration.

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