Two analogs of Pleijel’s inequality

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1 Formulation of results

Let $N(\lambda)$ be a nondecreasing function defined on $\mathbb{R}_+ = (0, +\infty)$ such that $N(\lambda) = 0$ for small $\lambda$ and

$$
\int_0^\infty \lambda^{-1} dN(\lambda) < \infty. \quad (1)
$$

The Stieltjes transform of $N(\lambda)$ is defined as

$$
S(\zeta) = \int_0^\infty (\lambda - \zeta)^{-1} dN(\lambda), \quad \zeta \not\in \mathbb{R}_+. \quad (2)
$$

Fix a point $\zeta_0 = \lambda_0 + i\eta_0$ in the first quadrant of the complex plane $\mathbb{C}$. Denote by $\Gamma$ a contour that connects the point $\zeta_0$ to $\overline{\zeta}_0 = \lambda_0 - i\eta_0$ and does not cross the integration path $\mathbb{R}_+$ of (2).

Ake Pleijel in [1] obtained the inequality

$$
\left| N(\lambda_0) \right| - \frac{1}{2\pi i} \int_\Gamma S(\zeta) \, d\zeta \leq \eta_0 \sqrt{1 + \pi^{-2}} \left| S(\zeta_0) \right| \quad (3)
$$

and used it to give a short proof of Malliavin’s [2] Tauberian theorem. The inequality (3) found applications in spectral theory of differential and pseudo-differential operators (see e.g. [3, 4]). In the present paper two generalizations of this inequality are derived.

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*There exists an alternative convention according to which the Stieltjes transform of $f(t)$ is defined as $\int_0^\infty (\zeta + t)^{-1} f(t) \, dt$. 

1
For $\alpha > 0$, the Riesz mean of order $\alpha$ of $N(\lambda)$ is
\[ N^{(\alpha)}(\lambda) = \int_0^\lambda \left( 1 - \frac{x}{\lambda} \right)^\alpha dN(x), \quad \lambda > 0. \] (4)

Given a power asymptotics of the Stieltjes transform of $N(\lambda)$ along a certain parabola-like curve in $\mathbb{C}$ that avoids $\mathbb{R}_+$, the asymptotics of the Riesz means as $\lambda \to +\infty$ can be recovered using the following theorem.

**Theorem 1** Let the function $N(\lambda)$ be constant in a neighbourhood of $\lambda_0$. Then for any $\alpha > 0$
\[ |N^{(\alpha)}(\lambda_0)| - \frac{1}{2\pi i} \int_\Gamma S(\zeta) \left( 1 - \frac{\zeta}{\lambda_0} \right)^\alpha d\zeta \leq \frac{1}{\alpha \pi} \left( \frac{\eta_0}{\lambda_0} \right)^\alpha \eta_0 |S(\zeta_0)|. \] (5)

For $\alpha < 1$ the factor $(\alpha \pi)^{-1}$ in the right-hand side may be replaced by $\sqrt{\pi}^{-2} + 1/4$. *

Henceforth the branch $z^\alpha = \exp(\alpha \ln z)$ with $-\pi < \text{Im} \ln z \leq \pi$ is assumed.

If instead of (1) a weaker **condition with some integer $q > 1$**
\[ \int_0^\infty \lambda^{-q} dN(\lambda) < \infty \] (6)
holds, then the leading term of the asymptotics of $N(\lambda)$ can be recovered by means of the next theorem from the behaviour of its *generalized Stieltjes transform*
\[ S_q(\zeta) = \int_0^\infty (\lambda - \zeta)^{-q} dN(\lambda), \quad \zeta \notin \mathbb{R}_+. \] (7)

**Theorem 2** Let the function $N(\lambda)$ satisfy (6) and be constant in a neighbourhood of $\lambda_0$. There exist constants $C_0, C_1, \ldots, C_{q-2}$ (which depend only on $q$) such that
\[ \left| N(\lambda_0) - \frac{1}{2\pi i} \int_\Gamma S_q(\zeta)(\zeta - \lambda_0)^{q-1} d\zeta \right| \leq \sum_{m=0}^{q-2} C_m r_0^{q-1-m} \left| \int_\Gamma S_q(\zeta)(\lambda_0 - \zeta)^m d\zeta \right|. \] (8)

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* (Added in translation.) A continuous dependence of the constant on $\alpha$ can be achieved by replacing the term $1/4$ with $(1 - \alpha^{1+\epsilon})/4$, where $\epsilon \geq \epsilon_0 \approx 1/16$ (found numerically).

**Inequality (6) is weaker than (1) since we assume that $dN(\lambda) = 0$ near $\lambda = 0$.**

2
2 Proof of Theorem [1]

a. The left-hand side of (5) vanishes if one uses a closed contour of integration consisting of \( \Gamma \) and the segment \([\zeta, \bar{\zeta_0}]\). Indeed, since \( N(\lambda) \) is assumed constant in the vicinity of \( \lambda_0 \), one may change the order of integration:

\[
\frac{1}{2\pi i} \oint S(\zeta) \left( 1 - \frac{\zeta}{\lambda_0} \right)^\alpha d\zeta = \int_0^\infty dN(\lambda) \frac{1}{2\pi i} \oint \left( \frac{\lambda_0 - \zeta}{\lambda_0} \right)^\alpha \frac{d\zeta}{\lambda - \zeta}.
\]

The inner integral in the r.h.s. equals \( 2\pi i \left( 1 - \frac{\lambda}{\lambda_0} \right)^\alpha \) when \( \lambda < \lambda_0 \), and 0 when \( \lambda > \lambda_0 \). Thus the r.h.s. equals \( N(\alpha)(\lambda_0) \). In order to prove (5) we have to evaluate

\[
R_{\alpha}(\zeta_0) = \frac{1}{2\pi i} \int_{\zeta_0}^{\bar{\zeta_0}} S(\zeta) \left( 1 - \frac{\zeta}{\lambda_0} \right)^\alpha d\zeta \quad (9)
\]

(where the integration path is the vertical segment).

Set \( \zeta = \lambda_0 + i|\lambda - \lambda_0| \tau \), \( s(\lambda) = \text{sgn} (\lambda - \lambda_0) \), \( u(\lambda) = \eta_0 |\lambda - \lambda_0|^{-1} \). Changing the order of integration in (9), we get

\[
R_{\alpha}(\zeta_0) = \left( -i \right)^\alpha \frac{\eta_0}{2\pi} \left( \frac{\lambda_0}{\lambda} \right)^\alpha \int_0^\infty dN(\lambda) \frac{u(\lambda)}{u^\alpha(\lambda)} \int_{-u(\lambda)}^{u(\lambda)} \frac{\tau^\alpha \left( s(\lambda) + i\tau \right)}{1 + \tau^2} d\tau. \quad (10)
\]

We will find constants \( c_1 \in \mathbb{C} \) and \( c_2 > 0 \) so that for any \( u > 0 \) and \( s = \pm 1 \) the inequality

\[
\left| \frac{1}{u^\alpha} \int_{-u}^u \frac{\tau^\alpha (s + i\tau)}{1 + \tau^2} d\tau - c_1 \frac{su}{1 + u^2} \right| \leq c_2 \frac{u^2}{1 + u^2} \quad (11)
\]

will hold. Having (11) and the identities

\[
\int_{0}^{\infty} \frac{u(\lambda)s(\lambda)}{1 + u^2(\lambda)} dN(\lambda) = \eta_0 \text{Re} S(\zeta_0)
\]

and

\[
\int_{0}^{\infty} \frac{u^2(\lambda)}{1 + u^2(\lambda)} dN(\lambda) = \eta_0 \text{Im} S(\zeta_0)
\]
one can estimate the r.h.s. in (10) by means of the Schwarz inequality:

\[
|R_\alpha(\zeta_0)| \leq \left( \frac{\eta_0}{\lambda_0} \right)^\alpha \frac{\eta_0}{2\pi} \sqrt{|c_1|^2 + c_2^2} \sqrt{\text{Re}^2 S(\zeta_0) + \text{Im}^2 S(\zeta_0)}
\]

(12)

Our task is thus reduced to establishing the inequality (11) with constants \(c_1, c_2\) such that

\[
\sqrt{|c_1|^2 + c_2^2} \leq 2\alpha^{-1} \quad \text{and} \quad \leq \sqrt{\pi^2 + 4} \quad \text{if} \quad \alpha < 1.
\]

b. Using the change of variable \(\tau \mapsto -\tau\) on \([-u, 0]\), we get

\[
\int_{-u}^u \left( s + i\tau \right)^\alpha \left( 1 + \tau^2 \right)^{-1} d\tau = 2e^{i\pi\alpha/2} \int_0^u \left( a\tau^{\alpha+1} + sb\tau^\alpha \right) (1 + \tau^2)^{-1} \left( 1 + \tau^2 \right)^{-2} d\tau
\]

(13)

where

\[
a = \sin \frac{\pi\alpha}{2}, \quad b = \cos \frac{\pi\alpha}{2}.
\]

(14)

We will need the integral representations

\[
\frac{u^{\alpha+1}}{1 + u^2} = \left. \frac{\tau^{\alpha+1}}{1 + \tau^2} \right|_0^u = \int_0^u \left( \alpha - 1 \right)\tau^{\alpha+2} + \left( \alpha + 1 \right)\tau^\alpha \left( 1 + \tau^2 \right)^{-2} d\tau
\]

(15)

and

\[
\frac{u^{\alpha+2}}{1 + u^2} = \int_0^u \frac{\alpha\tau^{\alpha+3} + (\alpha + 2)\tau^{\alpha+1}}{(1 + \tau^2)^2} d\tau.
\]

(16)

Multiplying the inequality (11) by \(u^\alpha\) and making the substitutions (13), (15), (16), we transform (11) to the equivalent form

\[
\left| \int_0^u \frac{a\tau^{\alpha+3} + sk^-\tau^{\alpha+2} + a\tau^{\alpha+1} + sk^+\tau^\alpha}{(1 + \tau^2)^2} d\tau \right|
\]

(17)

\[
\leq \frac{1}{2} c_2 \int_0^u \frac{\alpha\tau^{\alpha+3} + (\alpha + 2)\tau^{\alpha+1}}{(1 + \tau^2)^2} d\tau,
\]

where

\[
k_\pm = b - \frac{c_1}{2} e^{-i\pi\alpha/2} (\alpha \pm 1).
\]

In order for (17) to hold for small positive \(u\), the coefficient of \(\tau^\alpha\) must equal 0, so we set

\[
c_1 = \frac{2b}{\alpha + 1} e^{i\pi\alpha/2}.
\]

(18)
With this value of $c_1$, the numerator in the l.h.s. of (17) is real. Clearing the absolute value notation, we rewrite (17) as a system of two inequalities, in which we leave the least favorable sign of $s$ (so as to make the coefficient of $\tau^{\alpha+2}$ negative):

$$\int_0^u (c_2\alpha + 2a)\tau^{\alpha+3} - \frac{4|b|}{\alpha + 1} \tau^{\alpha+2} + (c_2\alpha + 2c_2 \mp 2a)\tau^{\alpha+1} \frac{d\tau}{(1 + \tau^2)^2} \geq 0.$$ 

Taking the least favorable sign in front of $2a$, we get

$$\int_0^u P_2(\tau) \tau^{\alpha+1} \frac{d\tau}{(1 + \tau^2)^2} \geq 0,$$

where

$$P_2(\tau) = (c_2\alpha - 2|a|)\tau^2 - \frac{4|b|}{\alpha + 1} \tau + (c_2\alpha - 2|a| + 2c_2).$$

The inequality (19) with $c_1$ defined by (18) implies (11). The rest of the proof amounts to finding an appropriate value of $c_2$.

**c.** It suffices to ensure that the quadratic polynomial $P_2(\tau)$ is nonnegative. Let us choose the value of $c_2$ that makes its discriminant equal to zero:

$$c_2 = 2 \frac{|a|\alpha + 1)^2 + \sqrt{a^2 + \alpha(\alpha + 2)}}{\alpha(\alpha + 1)(\alpha + 2)}.$$ 

In view of (14) we have $|a|, |b| \leq 1$, $a^2 + b^2 = 1$, and the following estimates readily follow:

$$\frac{|a|}{\alpha} \leq c_2 \leq \frac{|a|\alpha + 1 + 1}{\alpha(\alpha + 2)}.$$ 

The left estimate shows that the coefficient of $\tau^2$ in $P_2(\tau)$ is nonnegative. By our choice of $c_2$, this leads to (19) and hence to (11).

Using the right inequality in (21) together with (14) and (18), we find $|c_1|^2 + c_2^2 \leq 4\alpha^{-2}$, as required.

**d.** To finish the proof, let us show that if $\alpha < 1$ then one may use the value $c_2 = 2|a|\alpha^{-1}$ instead of (20). With this new choice of $c_2$, the constant $c_3$ in the r.h.s. of (12) becomes

$$c_3 = 2 \sqrt{\left(\frac{\cos(\pi\alpha/2)}{\alpha + 1}\right)^2 + \left(\frac{\sin(\pi\alpha/2)}{\alpha}\right)^2} \leq \sqrt{4 + \pi^2},$$
as the theorem claims. We have to verify the inequality

\[ 4 \left| a \right| \alpha \int_0^u \frac{\tau^{\alpha+1}}{(1 + \tau^2)^2} d\tau - 4 \left| b \right| \frac{\alpha + 1}{\alpha + 1} \int_0^u \frac{\tau^{\alpha+2}}{(1 + \tau^2)^2} d\tau \geq 0 \]

in the interval \( 0 < \alpha < 1 \). The left-hand side, as a function of \( u \), is positive for small \( u \) and has a unique critical point (maximum) on \( \mathbb{R}_+ \). It remains to check that

\[ \frac{\left| a \right|}{\alpha} \int_0^\infty \frac{\tau^{\alpha+1}}{(1 + \tau^2)^2} d\tau \geq \frac{\left| b \right|}{\alpha + 1} \int_0^\infty \frac{\tau^{\alpha+2}}{(1 + \tau^2)^2} d\tau. \]

Using the substitution \( \tau^2 = t \) we express the integrals in terms of Euler’s Beta function and the last inequality takes the form

\[ \frac{\left| a \right|}{\alpha} B \left( \frac{\alpha + 2}{2}, \frac{2 - \alpha}{2} \right) \geq \frac{\left| b \right|}{\alpha + 1} B \left( \frac{\alpha + 3}{2}, \frac{1 - \alpha}{2} \right). \]

The right and left sides are in fact equal: it follows from (14) and the identities

\[ B \left( \frac{\alpha + 2}{2}, \frac{2 - \alpha}{2} \right) = \frac{\pi \alpha}{2} \sin(\pi \alpha/2), \quad B \left( \frac{\alpha + 3}{2}, \frac{1 - \alpha}{2} \right) = \frac{\pi (\alpha + 1)/2}{\cos(\pi \alpha/2)}. \]

The proof is complete.

3 Proof of Theorem 2

a. The left-hand side of the inequality (8), likewise the l.h.s. of the inequality (9) in Theorem 1, vanishes if the closed contour of integration consisting of \( \Gamma \) and the segment \([\zeta_0, \zeta_0]\) is used. In the right-hand side of (8), integration over \( \Gamma \) can be replaced by integration over \([\zeta_0, \zeta_0]\) since

\[ \int_\Gamma S_q(\zeta)(\lambda_0 - \zeta)^m d\zeta = \int_0^\infty dN(\lambda) \int_\Gamma \frac{(\lambda_0 - \zeta)^m}{(\lambda - \zeta)^q} d\zeta, \]

and the residue of the integrand at \( \zeta = \lambda \) equals 0 as \( m \leq q - 2 \). Therefore (8) is equivalent to the inequality

\[ \left| \int_0^\infty V_{q,q-1}(\lambda) dN(\lambda) \right| \leq \sum_{m=0}^{q-2} C_m \left| \int_0^\infty V_{q,m}(\lambda) dN(\lambda) \right|, \quad (22) \]
where

\[ V_{q,m}(\lambda) = \eta_0^{q-1-m} \int_0^\infty \frac{(\lambda_0 - \zeta)^m}{(\lambda - \zeta)^q} \, d\zeta, \quad m = 0, 1, \ldots, q - 1. \]  

(23)

The substitutions \( \zeta = \lambda_0 + i\eta_0 \tau, \mu = (\lambda - \lambda_0)\eta_0^{-1} \) bring (23) to the form

\[ V_{q,m}(\lambda) = (-i)^{m-1} T_{q,m}(\mu), \]  

(24)

where

\[ T_{q,m}(\mu) = \int_{-1}^{1} \frac{\tau^m \, d\tau}{(\mu - i\tau)^q}. \]  

(25)

b. Let us study properties of the functions \( T_{q,m}(\mu) \).

1\textsuperscript{o}. The function \( T_{q,m}(\mu) \) is even if \( q - m \) is even, and odd if \( q - m \) is odd. This is verified by changing \( \mu \) into \( -\mu \) in (25) and \( \tau \) into \( -\tau \).

2\textsuperscript{o}. If \( 0 \leq m \leq q - 2 \), then we can write

\[ T_{q,m}(\mu) = \frac{P_{q,m}(\mu)}{(\mu^2 + 1)^{q-1}}, \]  

(26)

where \( P_{q,m}(\mu) \) is a polynomial (even or odd depending on the evenness of \( q - m \)). Indeed, expanding \( \tau^m \) in powers of \( \mu - i\tau \), integrating the resulting linear combination of the functions \( (\mu - i\tau)^{n-q} \) \( (n = 0, \ldots, m) \) with respect to \( \tau \), and taking the common denominator, we obtain (26).

3\textsuperscript{o}. From (25) it is easy to find the asymptotics of \( T_{q,m} \) as \( \mu \to +\infty \):

\[ T_{q,m}(\mu) = b_{q,m} \mu^{-q} + O(\mu^{-q-2}) \quad \text{for } m \text{ even}, \]

\[ T_{q,m}(\mu) = b_{q,m} \mu^{-q-1} \quad \text{for } m \text{ odd}, \]  

(27)

where \( b_{q,m} \neq 0 \).

Comparing to (26), we see that the polynomial \( P_{q,m} \) \( (m \leq q - 2) \) is of exact degree \( q - 2 \) for \( m \) even, and of exact degree \( q - 3 \) for \( m \) odd.

c. Let us show that \( \{P_{q,m}\}, \; m = 0, \ldots, q - 2 \), is a basis in the space of polynomials of degree at most \( q-2 \). It suffices to verify that the corresponding functions \( T_{q,m} \) are linearly independent. Suppose, to the contrary, that some

their linear combination is zero:

\[ L(\mu) = \int_{-1}^{1} \frac{U(\tau)}{(\mu - i\tau)^q} \, d\tau = 0, \]
where $U(\tau)$ is a polynomial of degree at most $q - 2$. Consider $L$ as an analytic function of complex variable $\mu$. It is regular outside the segment $[-i, i]$, therefore, it equals zero identically. Let $\gamma$ be a closed contour around the segment $[-i, i]$. For all integer $n \geq q - 1$ we have

$$0 = \int_{\gamma} L(z) z^n \, dz = \int_{-1}^{1} U(\tau) \, d\tau \int_{\gamma} \frac{z^n \, dz}{(z - i\tau)^q} = \beta_n \int_{-1}^{1} U(\tau) \, \tau^{n-q+1} \, d\tau,$$

and $\beta_n \neq 0$. Hence for any polynomial $\tilde{U}(\tau)$

$$\int_{-1}^{1} U(\tau) \, \tilde{U}(\tau) \, d\tau = 0.$$

Taking $\tilde{U}$ to be the complex-conjugate of $U$ leads to the conclusion $U \equiv 0$, which proves the linear independence of the functions $T_{q,m}$.

d. Consider first the case of even $q$. The result of (c) shows that for some $C_0', \ldots, C_{q-2}'$

$$\sum_{m=0}^{q-2} C_m' P_{q,m}(\mu) = 1 + \mu^{q-2},$$

or — cf. (26) — that

$$\sum_{m=0}^{q-2} C_m' T_{q,m}(\mu) = \frac{1 + \mu^{q-2}}{(1 + \mu^2)^{q-1}} =: H_q(\mu). \quad (28)$$

As $|\mu| \to \infty$, we have

$$|T_{q,q-1}(\mu)| \sim |b_{q,q-1}| |\mu|^{-q-1} = o(H_q(\mu)).$$

The function $T_{q,q-1}(\mu)$ is bounded, while $\min H_q(\mu) > 0$ on every finite interval. Therefore there exists a positive $C$ such that $|T_{q,q-1}(\mu)| \leq C H_q(\mu)$ for all real $\mu \neq 0$. Using (24) and passing from $T_{q,m}$ to $V_{q,m}$, then integrating with $dN(\lambda)$, we get (22).

e. Now consider the case of odd $q$. There exist constants $C_0'', \ldots, C_{q-3}''$ such that

$$\sum_{m=0}^{q-3} C_m'' T_{q,m}(\mu) = \frac{1 + \mu^{q-3}}{(1 + \mu^2)^{q-1}} =: H_q(\mu). \quad (29)$$
Now $T_{q, q-1}(\mu)$ decays at infinity slower than $H_q(\mu)$. However, as seen from (27),
\[
\left| T_{q, q-1}(\mu) - \frac{b_{q, q-1}}{b_{q, 0}} T_{q, 0}(\mu) \right| = O(\mu^{-q-2}) = o(H_q(\mu)), \quad |\mu| \to \infty.
\]
Here, like previously in (d), the left-hand side is a bounded function, while \( \min H_q(\mu) \) > 0 on every finite interval. Therefore there exists a positive \( C \) such that for all real \( \mu \neq 0 \),
\[
\left| T_{q, q-1}(\mu) - \frac{b_{q, q-1}}{b_{q, 0}} T_{q, 0}(\mu) \right| \leq C \sum_{m=0}^{q-3} C''_m T_{q, m}(\mu).
\]
Passing from $T_{q, m}$ to $V_{q, m}$, integrating with $dN(\lambda)$, and bringing the term \( \text{const} \int_0^\infty V_{q, 0}(\lambda) dN(\lambda) \) over to the right-hand side, we get (22).

The proof is finished.

**Remark.** Note that due to (b 1°) the coefficients of odd functions $T_{q, m}$ in (28), (29) are equal to 0. Hence in the right-hand side of (8) the actual summation is carried over the values of index \( m \) for which \( q - m \) is even; if \( q \) is odd, the value \( m = 0 \) is also included.

In conclusion I would like to thank M.S. Agranovich for suggesting the problem and for attention to this work.

**References**

[1] Pleijel Å. On a theorem by P. Malliavin. Israel J. Math. 1, 166–168 (1963).

[2] Malliavin P. Un théorème taubérien avec reste pour la transformation de Stieltjes, C.R. Acad. Sci. Paris, 255, 2351–2352 (1962).

[3] Agmon S. Asymptotic formulas with remainder estimates for eigenvalues of elliptic operators. Arch. Rational Mech. Anal. 28, 166–183 (1968).

[4] Pham The Laï. Estimation du reste dans la theorie spectrale d’une classe d’operateurs elliptiques degeneres, Seminaire Goulaouic-Schwartz 1975-76, expose n° X.

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Remarks added in translation

1. In both Theorems 1 and 2, the function $N(\lambda)$ does not have to be continuous at $\lambda_0$. If $\lambda_0$ is a point of discontinuity of $N(\lambda)$, Theorem 1 remains valid without change, while in Theorem 2 the value $N(\lambda_0)$ in the left-hand side can be replaced by any value between $N(\lambda_0 - 0)$ and $N(\lambda_0 + 0)$.

2. The following theorem stands in the same relation to Theorem 2 as Theorem 1 to Pleijel’s inequality (3).

**Theorem 3** Let function $N(\lambda)$ defined on $\mathbb{R}_+$ be nondecreasing, equal zero near $\lambda = 0$, and satisfy condition (6). For any $\alpha > 0$ and any integer $q = 2, 3, \ldots$ there exist nonnegative constants $C_0, \ldots, C_{q-2}$, depending on $q$ and $\alpha$, such that for any $\lambda > 0$

$$\left| N^\alpha(\lambda_0) - \frac{\alpha B(q, \alpha)}{2\pi i} \int_{\Gamma} S_q(\zeta) (\zeta - \lambda_0)^{q-1} \left( 1 - \frac{\zeta}{\lambda_0} \right)^\alpha d\zeta \right|$$

$$\leq \sum_{m=0}^{q-2} C_m \left( \frac{\eta_0}{\lambda_0} \right)^\alpha \cdot \eta_0^{q-1-m} \left| \int_{\Gamma} S_q(\zeta) (\zeta - \lambda_0)^m d\zeta \right|,$$

where

$$B(q, \alpha) = \frac{\Gamma(q)\Gamma(\alpha)}{\Gamma(q + \alpha)}.$$

**Proof**: Repeat the proof of Theorem 2 replacing the function $T_{q,q-1}(\mu)$ by the function

$$T_{q,q-1+\alpha}(\mu) = \int_{-1}^{1} (\mu - i\tau)^{q-1+\alpha} d\tau.$$

3. An application of Theorem 1 can be found in:

Agranovich M.S. Elliptic operators on closed manifolds. Encycl. Math. Sci., 63 (Partial Differential Equations VI), Springer-Verlag, 1994, Theorem 6.1.6.