Gauge Dressing of 2D Field Theories

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Abstract

By using the gauge Ward identities, we study correlation functions of gauged WZNW models. We show that the gauge dressing of the correlation functions can be taken into account as a solution of the Knizhnik-Zamolodchikov equation. Our method is analogous to the analysis of the gravitational dressing of 2D field theories.

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1 Introduction

There are two dimensional quantum field theories whose correlation functions can be computed exactly. Solvability of the given models is based upon large symmetries which give rise to Ward identities in the form of differential equations. The celebrated examples are the Wess-Zumino-Novikov-Witten models whose correlators obey the Knizhnik-Zamolodchikov equation \[1\]. Remarkably, it turns out that correlation functions of CFT coupled to 2D gravity can be found as a solution to a differential equation using the Polyakov chiral gauge approach \[2\], \[3\] (see also \[4\]). This approach was used to study gravitationally dressed correlation functions \[5\], \[6\].

The aim of the present letter is to study a gauge analogue of the gravitational dressing and to analyze correlation functions of 2D field theories gauged with respect to some Lie group. Among these theories are coset constructions \[7\] which play an important role in string theory and statistical physics. We shall concentrate on this type of 2D CFT’s which are described as gauged WZNW models \[8\], \[9\], \[10\], \[11\] (such models were first discussed in the early days of string theory \[12\]). A recent discussion of these models is \[13\].

2 Gauge dressing of the Knizhnik-Zamolodchikov equation

A large class of 2D conformal field theories is described by gauged WZNW models with the following action (we use here the same normalization as in \[13\])

\[ S(g, A) = S_{WZNW}(g) + \frac{k}{2\pi} \int d^2z Tr \left[ Ag^{-1} \partial g - \bar{A} \partial g g^{-1} + A g^{-1} \bar{A} g - A \bar{A} \right], \tag{2.1} \]

where

\[ S_{WZNW}(g) = \frac{k}{8\pi} \int d^2z \ Tr g^{-1} \partial^\mu g g^{-1} \partial_\mu g + \frac{ik}{12\pi} \int d^3z \ Tr g^{-1} \partial^0 g g^{-1} \partial^0 g + \frac{ik}{12\pi} \int d^3z \ Tr g^{-1} \partial^1 g g^{-1} \partial^1 g \tag{2.2} \]

and \( g \in G, A, \bar{A} \) are the gauge fields taking values in the algebra \( \mathcal{H} \) of the diagonal group of the direct product \( H \times H, H \in G \).
It has become usual to study gauged WZNW models with the BRST method \cite{11}. However, this method is not very much of help in computing correlation functions, though, in principle, the free field realization of these theories allows one to calculate correlators of BRST invariant operators. We shall pursue a different approach which is parallel to the analysis of the gravitational dressing of 2D field theories.

Our starting point are the equations of motion of the gauged WZNW model:

\begin{equation}
\bar{\nabla}(\nabla gg^{-1}) = 0,
\end{equation}

\begin{equation}
\bar{\partial}A - \partial \bar{A} + [A, \bar{A}] = 0,
\end{equation}

where

\begin{equation}
\nabla = \bar{\partial} + \bar{A}, \quad \nabla = \partial + A.
\end{equation}

Under the gauge symmetry, the WZNW primary fields $\Phi_i$ and the gauge fields $A, \bar{A}$ transform respectively as follows

\begin{equation}
\delta \Phi_i = \epsilon^a (t^a_i + \bar{t}^a_i) \Phi_i, \quad 
\delta A = -\partial \epsilon - [\epsilon, A], \quad \delta \bar{A} = -\partial \bar{\epsilon} - [\epsilon, \bar{A}],
\end{equation}

where $t^a_i \in \mathcal{H}$.

In order to fix the gauge invariance, we impose the following condition

\begin{equation}
\bar{A} = 0.
\end{equation}

The given gauge fixing gives rise to the corresponding Faddeev-Popov ghosts with the action

\begin{equation}
S_{\text{ghost}} = \int d^2 z \ Tr(b\partial c).
\end{equation}

In the gauge \eqref{2.6}, the equations of motion take the following form

\begin{equation}
\bar{\partial}J = 0,
\end{equation}

\begin{equation}
\bar{\partial}A = 0,
\end{equation}

\begin{equation}
\bar{\partial}\bar{A} = 0.
\end{equation}
where

\[ J = -\frac{k}{2} \partial g g^{-1} - \frac{k}{2} g A g^{-1}. \]  

(2.9)

Thus, \( J \) is a holomorphic current in the gauge (2.6). Moreover, it has canonical commutation relations with the field \( g \) and itself:

\[ \{ J^a(w), g(z) \} = t^a g(z) \delta(w, z), \]  

(2.10)

\[ \{ J^a(w), J^b(z) \} = f^{abc} J^c(z) \delta(w, z) + k/2 \delta^{ab} \delta'(w, z). \]

The given commutators follow from the symplectic structure of the gauged WZNW model in the gauge (2.6). In this gauge, the field \( A \) plays a role of the parameter \( v_0 \) of the orbit of the affine group \( \hat{G} \) [14]. Therefore, the symplectic structure of the gauged WZNW model in the gauge (2.6) coincides with the symplectic structure of the original WZNW model [13].

There are residual symmetries which survive the gauge fixing (2.6). Under these symmetries the fields \( \Phi_i \) and the remaining gauge field \( A \) transform according to

\[ \tilde{\delta} \Phi_i = (\epsilon^A_L t^A_i + \epsilon^a_R \bar{t}^a) \Phi_i, \]  

(2.11)

\[ \tilde{\delta} A = -\partial \epsilon_R - [\epsilon_R, A], \]

where the parameters \( \epsilon_L \) and \( \epsilon_R \) are arbitrary holomorphic functions,

\[ \bar{\partial} \epsilon_{L,R} = 0. \]  

(2.12)

In eqs. (2.11) the generators \( t^A \) act on the left index of \( \Phi_i \), whereas \( \bar{t}^a \) act on the right index of \( \Phi_i \). One can notice that the left residual group is extended to the whole group \( G \), whereas the right residual group is still the subgroup \( H \).

Eq. (2.9) can be presented in the following form

\[ \frac{1}{2} \partial g + \eta g A + \frac{1}{\kappa} J g = 0. \]  

(2.13)

Here \( \eta \) and \( \kappa \) are some renormalization constants due to regularization of the singular products \( g A \) and \( J g \).
In order to compute $\eta$ and $\kappa$, we need to do a few things. First of all, we have to find how the gauge field $A$ acts on the fields $\Phi_i$. To this end, let us define dressed correlation functions
\[ \langle\langle \cdots \rangle\rangle \equiv \int \mathcal{D} \bar{A} \mathcal{D} A \langle \cdots \rangle \exp \left[ -\frac{k}{2\pi} \int d^2 z \text{Tr} \left\{ \bar{A} g^{-1} \partial g + A \bar{g} g^{-1} + A g^{-1} - A \bar{A} \right\} \right], \tag{2.14} \]
where $\langle \cdots \rangle$ is the correlation function before gauging. The latter is found as a solution to the KZ equation
\[ \begin{cases} \frac{1}{2} \frac{\partial}{\partial z_i} + \sum_{j \neq i}^{N} \frac{t^A_i t^A_j}{k + c_V(G)} \frac{1}{z_i - z_j} \\ \langle \Phi_1(z_1, \bar{z}_1) \Phi_2(z_2, \bar{z}_2) \cdots \Phi_N(z_N, \bar{z}_N) \rangle = 0. \end{cases} \tag{2.15} \]
Here $\Phi_i$ are the primary fields of the WZNW model (2.2), $t^A_i$ are the representations of the generators of $G$ for the fields $\Phi_i$, $c_V = \frac{f^{abc} f^{abc}}{\text{dim} G}$. (2.16)

In the gauge (2.6), the dressed correlation functions (2.14) can be presented as follows
\[ \langle\langle \Phi_1(z_1, \bar{z}_1) \Phi_2(z_2, \bar{z}_2) \cdots \Phi_N(z_N, \bar{z}_N) \rangle\rangle = \int \mathcal{D} b \mathcal{D} c \exp(-S_{\text{ghost}}) \int \mathcal{D} A \exp[-S_{\text{eff}}(A)] \]
\[ \times \int \mathcal{D} g \ \Phi_1(z_1, \bar{z}_1) \Phi_2(z_2, \bar{z}_2) \cdots \Phi_N(z_N, \bar{z}_N) \exp[-\Gamma(g, A)], \tag{2.17} \]
where $S_{\text{eff}}(A)$ is the effective action of the field $A$ and $\Gamma(g, A)$ is formally identical to the original gauged WZNW action in the gauge (2.6). The action $S_{\text{eff}}$ is non-local and can be obtained by integration of the following variation (which follows from the Wess-Zumino anomaly condition)
\[ \partial \frac{\delta S_{\text{eff}}}{\delta A^a} + f^{abc} A^c \frac{\delta S_{\text{eff}}}{\delta A^b} = \tau \bar{\partial} A^a. \tag{2.18} \]
Here the constant $\tau$ is to be defined from the consistency condition of the gauge (2.6), which is
\[ J_{tot} \equiv \delta Z / \delta \bar{A}^a = 0, \quad a = 1, 2, \ldots, \text{dim} H, \tag{2.19} \]
at $\bar{A} = 0$. Here $Z$ is the partition function of the gauged WZNW model. Condition (2.19) amounts to the vanishing of the central charge of the affine current $J_{tot}$. This in turn means that $J_{tot}$ is a first class constraint [11]. In order to use this constraint, we need
to know the OPE of $A$ with itself. This can be derived as follows. Let us consider the identity
\[
\tau \langle\langle \hat{\partial} A(z) A(z_1) \cdots A(z_N) \rangle\rangle = \int D A A(z_1) \cdots A(z_N) \left[ \partial_x \frac{\delta S_{\text{eff}}}{\delta A^a(z)} + f^{abc} A^c(z) \frac{\delta S_{\text{eff}}}{\delta A^b(z)} \right] e^{-S_{\text{eff}}}.
\]
(2.20)

Here we used relation (2.18). Integrating by parts in the path integral, we arrive at the following formula
\[
\tau \langle\langle A^a(z) A^{a_1}(z_1) \cdots A^{a_N}(z_N) \rangle\rangle = \frac{1}{2\pi i} \sum_{k=1}^N \left\{ \frac{-\delta^{a_0 k}}{(z - z_k)^2} \left( \langle\langle A^{a_1}(z_1) \cdots \hat{A}_k^a \cdots A^{a_N}(z_N) \rangle\rangle \right) \right. \\
+ f^{a_0 b} \frac{z - z_k}{z} \left( \langle\langle A^{a_1}(z_1) \cdots A^b(z_k) \cdots A^{a_N}(z_N) \rangle\rangle \right),
\]
where $\hat{A}_k$ means that the field $A(z_k)$ is removed from the correlator. In the derivation of the last equation we used the following identity
\[
\bar{\partial}z \frac{1}{z - z_k} = 2\pi i \delta^{(2)}(z - z_k).
\]
(2.22)

From eq. (2.21) it follows that
\[
\tau A^a(z) A^b(0) = \frac{1}{2\pi i} \left[ \frac{-\delta^{ab}}{z^2} + \frac{f^{abc}}{z} A^c(0) \right] + \text{reg.}
\]
(2.23)

Along with condition (2.19), the equation (2.23) gives the expression for $\tau$
\[
\tau = \frac{i(k + 2c_V(H))}{4\pi}.
\]
(2.24)

We proceed to derive the Ward identity associated with the residual symmetry (2.11). The Ward identity comes from the variation of eq. (2.17) under transformations (2.11). Because it must be zero, we obtain the following relation
\[
\sum_{k=1}^N \bar{\partial}_z \delta(z, z_k) \langle\langle \Phi_1(z_1, \bar{z}_1) \Phi_2(z_2, \bar{z}_2) \cdots \Phi_N(z_N, \bar{z}_N) \rangle\rangle
\]
\[
+ \tau \langle\langle \hat{\partial}_z A^a(z) \Phi_1(z_1, \bar{z}_1) \Phi_2(z_2, \bar{z}_2) \cdots \Phi_N(z_N, \bar{z}_N) \rangle\rangle = 0,
\]
(2.25)

This yields
\[
2\pi \tau \langle\langle A^a(z) \Phi_1(z_1, \bar{z}_1) \Phi_2(z_2, \bar{z}_2) \cdots \Phi_N(z_N, \bar{z}_N) \rangle\rangle
\]
\[
= i \sum_{k=1}^N \frac{\bar{\partial}_k}{z - z_k} \langle\langle \Phi_1(z_1, \bar{z}_1) \Phi_2(z_2, \bar{z}_2) \cdots \Phi_N(z_N, \bar{z}_N) \rangle\rangle,
\]
(2.26)
which in turn gives rise to the OPE between the gauge field $A^a$ and $\Phi_i$

$$\frac{1}{2} A^a(z) \Phi_i(0) = \frac{1}{k + 2c_V(H)} \Phi_i(0).$$  \hspace{1cm} (2.27)

Now we are in a position to define the product $[A^a, \Phi_i]$. Indeed, we can define this according to the following rule

$$A^a(z) \Phi_i(z, \bar{z}) = \oint \frac{d\zeta}{2\pi i} A^a(\zeta) \Phi_i(z, \bar{z}),$$  \hspace{1cm} (2.28)

where the nominator is understood as OPE (2.27). Formula (2.28) is a definition of normal ordering for the product of two operators.

Let us come back to eq. (2.13). Variation of (2.13) under the residual symmetry gives rise to the following relation

$$\left[ 1 - \eta \left( 1 - \frac{c_V(H)}{k + 2c_V(H)} \right) \right] \partial \epsilon_R(z) g(z) = 0.$$  \hspace{1cm} (2.29)

From this relation we find the renormalization constant $\eta$

$$\eta = \frac{k + 2c_V(H)}{k + c_V(H)}.$$  \hspace{1cm} (2.30)

In the classical limit $k \to \infty$, $\eta \to 1$.

With the given constant $\eta$ the equation (2.13) reads off

$$\left\{ \frac{\partial}{\partial z} + \frac{k + 2c_V(H)}{k + c_V(H)} A(z) + \frac{2}{\kappa} J(z) \right\} g(z) = 0,$$  \hspace{1cm} (2.31)

where $A(z)$ acts on $g$ from the right hand side. Now the constant $\kappa$ can be calculated from the condition that the combination $\partial + \eta A$ acted on $g$ as a Virasoro generator $L_{-1}$. For this to be the case, we have to satisfy

$$L_{-1} g = \frac{2 J^A_{-1} J^A_0}{k + c_V(G)} g.$$  \hspace{1cm} (2.32)

For eq. (2.31) to be consistent with eq. (2.32), the constant $\kappa$ has to be as follows

$$\kappa = \frac{1}{k + c_V(G)}.$$  \hspace{1cm} (2.33)

All in all, with the regularization given by eq. (2.28) and the Ward identity (2.26) the eq. (2.31) gives rise to the following differential equation

$$\left\{ \frac{1}{2} \frac{\partial}{\partial z_i} + \sum_{j \neq i}^N \left( \frac{t_i^A t_j^A}{k + c_V(G)} - \frac{\tilde{t}_i^a \tilde{t}_j^a}{k + c_V(H)} \right) \frac{1}{z_i - z_j} \right\} \langle \langle \Phi_1(z_1, \bar{z}_1) \Phi_2(z_2, \bar{z}_2) \cdots \Phi_N(z_N, \bar{z}_N) \rangle \rangle = 0,$$  \hspace{1cm} (2.34)
where $t_i^A \in G$ and $\bar{t}_i^a \in H$. The important comment to be made is that there does not exist a similar equation for the antiholomorphic coordinate $\bar{z}$. This is because we use the gauge (2.6) which is not symmetrical in $z$ and $\bar{z}$. In this aspect, our equation differs from the standard KZ equation which can be written both for the holomorphic and antiholomorphic coordinates. However, there is Lorentz symmetry which allows one to partially restore the dependence on $\bar{z}$.

Equation (2.34) is our main result. By solving it, one can find dressed correlation functions in the gauged WZNW model. The solutions can be expressed as products of the correlation functions in the WZNW model for the group $G$ at level $k$ and $H$ at level $-2c_V(H) - k$. In particular, for the two-point function the equation yields

$$\frac{1}{2} \partial \langle \langle \Phi_i(z, \bar{z}) \Phi_j(0) \rangle \rangle = \left[ \frac{t_i^A t_j^A}{k + c_V(G)} - \frac{\bar{t}_i^a \bar{t}_j^a}{k + c_V(H)} \right] \frac{1}{z} \langle \langle \Phi_i(z, \bar{z}) \Phi_j(0) \rangle \rangle. \quad (2.35)$$

By the projective symmetry, the two-point function has the following expression

$$\langle \langle \Phi_i(z, \bar{z}) \Phi_j(0) \rangle \rangle = G_{ij} |z|^{4\Delta_i}, \quad (2.36)$$

where $\Delta_i$ is the anomalous conformal dimension of $\Phi_i$ after the gauge dressing and $G_{ij}$ is the Zamolodchikov metric which can be diagonalized. After substitution of expression (2.36) into eq. (2.35), and using the fact that, as a consequence of the residual symmetry (2.11), the dressed correlation functions must be singlets of both the left residual group $G$ and the right residual group $H$, we find

$$\Delta_i = c_i(G) \frac{k}{k + c_V(G)} - c_i(H) \frac{k}{k + c_V(H)}, \quad (2.37)$$

where $c_i(G) = t_i^A t_i^A$, $c_i(H) = \bar{t}_i^a \bar{t}_i^a$.

### 3 SL(2)/U(1) coset construction

In what follows we shall focus on the $SL(2)/U(1)$ coset construction. This model is particularly interesting as it provides a description of 2D black holes [16]. The central charge of the theory is

$$c_{SL(2)/U(1)} = \frac{3k}{k + 2} - 1 \quad (3.38)$$
where our convention for the sign of $k$ is opposite to \cite{16}. So it is $k = -9/4$ which will give the central charge $c = 26$. The $U(1)$ subgroup can be either compact or non-compact. In the case when $U(1)$ is compact, our equation takes the following form

$$
\left\{ \frac{\partial}{\partial z_i} + 2 \sum_{j \neq i} \frac{t_i^A t_j^A}{k + 2} - \frac{\bar{t}_i^3 \bar{t}_j^3}{k} \right\} \langle \langle \Phi_1(z_1, \bar{z}_1) \Phi_2(z_2, \bar{z}_2) \cdots \Phi_N(z_N, \bar{z}_N) \rangle \rangle = 0,
$$

(3.39)

where $t_i^A \in SL(2)$. While in the non-compact case, the equation is

$$
\left\{ \frac{\partial}{\partial z_i} + 2 \sum_{j \neq i} \frac{t_i^A t_j^A}{k + 2} + \frac{\bar{t}_i^3 \bar{t}_j^3}{k} \right\} \langle \langle \Phi_1(z_1, \bar{z}_1) \Phi_2(z_2, \bar{z}_2) \cdots \Phi_N(z_N, \bar{z}_N) \rangle \rangle = 0,
$$

(3.40)

The solutions for the two-point functions of fields $\Phi^m_l$ in the representation $l$ of $SL(2)$, with $\bar{t}_l \Phi^m_l = \bar{m} \Phi^m_l$ are

$$
\langle \langle \Phi^m_l(z, \bar{z}) \Phi^\bar{m}_l(0) \rangle \rangle = \frac{1}{|z|^{4\Delta_l}},
$$

(3.41)

Where equation (2.34) becomes

$$
\Delta_l^\bar{m} = \frac{l(l+1)}{k+2} - g_{33} \frac{\bar{m}^2}{k}.
$$

(3.42)

In the case of a compact $U(1)$, $g_{33} = +1$, and in the non-compact case $g_{33} = -1$. This reproduces the spectrum of dimensions found in \cite{17}.

As an example of the use of equation (2.13) for higher multi-point correlation functions, we consider the four-point function for the field $g(z, \bar{z})$ in the fundamental representation of $SL(2)$ ($l = 1/2$, $\bar{m} = \pm 1/2$). Because of the projective invariance and the residual symmetry (2.11), this has the form:

$$
\langle\langle g_{\epsilon_1}(1) g_{\epsilon_2}(2) g_{\epsilon_3}(3) g_{\epsilon_4}(4) \rangle\rangle = |(z_1 - z_4)(z_2 - z_3)|^{4\Delta_1^{1/2}} \sum_{A=1,2} I_A G^{\bar{m}_1 \bar{m}_2 \bar{m}_3 \bar{m}_4}_A (x, \bar{x}),
$$

(3.43)

where:

$$
x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}, \quad I_1 = \delta_{\epsilon_1 \epsilon_2} \delta_{\epsilon_3 \epsilon_4}, \quad I_2 = \delta_{\epsilon_1 \epsilon_4} \delta_{\epsilon_2 \epsilon_3}, \quad \bar{m}_1 + \bar{m}_2 + \bar{m}_3 + \bar{m}_4 = 0.
$$

(3.44)

For the correlation function (3.43), solving equations (3.39) and (3.40) gives the holomorphic conformal blocks as products of the solutions of the KZ equations for $SL(2)$ and

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Taking the limits as $x \rightarrow 0, 1, \infty$ gives the dimensions of the primary fields that appear in the operator product expansions of $g(z)g'(z')$ and $g(z)g(z')$. The result is $\Delta = \Delta_0^0, \Delta_0^1, \Delta_1^0$ or $\Delta_1^1$.

Although we do not have a differential equation for the anti-holomorphic part of the correlation function, there are a number of conditions which it must satisfy. The full correlation function (3.43) must be single valued in the euclidean domain $\bar{x} = x^*$, and it can have no singularities except at the points $x = 0, x = 1$ and $x = \infty$. We can also insist that the spectrum of dimensions for the fields in the OPE should be the same for the right and left dimensions $\bar{\Delta}$ and $\Delta$. Also, the full four-point function should have the crossing symmetry

$$G_{A}^{\bar{m}_{1}\bar{m}_{2}\bar{m}_{3}\bar{m}_{4}}(x, \bar{x}) = \sum_{p=0,1} F_{A(p)}^{m_{1}m_{2}m_{3}m_{4}}(x)C_{(p)}^{\bar{m}_{1}\bar{m}_{2}\bar{m}_{3}\bar{m}_{4}}(\bar{x})$$

$$F_{A(p)}^{m_{1}m_{2}m_{3}m_{4}}(x) = F_{A}^{(p)}(x)\bar{F}_{A}^{(p)(x)}$$

$$F_{1}^{(0)}(x) = x^{-\frac{3}{2(2+k)}}(1-x)^{\frac{1}{2(2+k)}}F(\frac{1}{2+k}, -\frac{1}{2+k}; \frac{k}{2+k}; x)$$

$$F_{2}^{(0)}(x) = \frac{1}{k}x^{\frac{1+2k}{2(2+k)}}(1-x)^{\frac{1}{2(2+k)}}F(\frac{1}{2+k}, \frac{3+k}{2+k}; \frac{2+2k}{2+k}; x)$$

$$F_{1}^{(1)}(x) = x^{\frac{1}{2(2+k)}}(1-x)^{\frac{1}{2(2+k)}}F(\frac{1}{2+k}, \frac{3}{2+k}; \frac{4+k}{2+k}; x)$$

$$F_{2}^{(1)}(x) = -2x^{\frac{1}{2(2+k)}}(1-x)^{\frac{1}{2(2+k)}}F(\frac{1}{2+k}, \frac{3}{2+k}; \frac{2}{2+k}; x)$$

$$f^{-++} = f^{++++} = \left(\frac{x}{1-x}\right)^{-\frac{g_4}{2k}}$$

$$f^{---} = f^{+++} = \left(\frac{1-x}{x}\right)^{-\frac{g_4}{2k}}$$

$$f^{--} = f^{+-} = \left[(x)(1-x)\right]^{\frac{g_4}{2k}}.$$  (3.45)

Taking the limits as $x \rightarrow 0, 1, \infty$ gives the dimensions of the primary fields that appear in the operator product expansions of $g(z)g'(z')$ and $g(z)g(z')$. The result is $\Delta = \Delta_0^0, \Delta_0^1, \Delta_1^0$ or $\Delta_1^1$.

Although we do not have a differential equation for the anti-holomorphic part of the correlation function, there are a number of conditions which it must satisfy. The full correlation function (3.43) must be single valued in the euclidean domain $\bar{x} = x^*$, and it can have no singularities except at the points $x = 0, x = 1$ and $x = \infty$. We can also insist that the spectrum of dimensions for the fields in the OPE should be the same for the right and left dimensions $\bar{\Delta}$ and $\Delta$. Also, the full four-point function should have the crossing symmetry

$$G_{A}^{\bar{m}_{1}\bar{m}_{2}\bar{m}_{3}\bar{m}_{4}}(x, \bar{x}) = \sum_{B=1,2} E_{AB}G_{B}^{\bar{m}_{1}\bar{m}_{2}\bar{m}_{3}\bar{m}_{4}}(1-x, 1-\bar{x})$$

$$= \sum_{B=1,2} E_{AB}G_{B}^{\bar{m}_{4}\bar{m}_{2}\bar{m}_{3}\bar{m}_{1}}(1-x, 1-\bar{x})$$  (3.46)

where $E_{12} = E_{21} = 1, E_{11} = E_{22} = 0$. An ansatz which satisfies all these requirements is:

$$C_{(0)}^{---}(\bar{x}) = C_{(0)}^{+++}(\bar{x}) = f^{+++}(\bar{x}) \left\{ \alpha F_{1}^{(0)}(\bar{x}) + \beta F_{2}^{(0)}(\bar{x}) \right\}$$

$$C_{(1)}^{---}(\bar{x}) = C_{(1)}^{+++}(\bar{x}) = hf^{+++}(\bar{x}) \left\{ \alpha F_{1}^{(1)}(\bar{x}) + \beta F_{2}^{(1)}(\bar{x}) \right\}$$
functions as
\[ C_{(0)}^{++-}(\bar{x}) = C_{(0)}^{+- -}(\bar{x}) = f^{++-}(\bar{x}) \left\{ \alpha F_2^{(0)}(\bar{x}) + \beta F_1^{(0)}(\bar{x}) \right\} \]
\[ C_{(1)}^{++-}(\bar{x}) = C_{(1)}^{+- -}(\bar{x}) = h f^{++-}(\bar{x}) \left\{ \alpha F_2^{(1)}(\bar{x}) + \beta F_1^{(1)}(\bar{x}) \right\} \]
\[ C_{(0)}^{++-}(\bar{x}) = \delta C_{(0)}^{+- -}(\bar{x}) = \gamma f^{++-}(\bar{x}) \left\{ F_1^{(0)}(\bar{x}) + F_2^{(0)}(\bar{x}) \right\} \]
\[ C_{(1)}^{++-}(\bar{x}) = \delta C_{(1)}^{+- -}(\bar{x}) = h \gamma f^{++-}(\bar{x}) \left\{ F_1^{(1)}(\bar{x}) + F_2^{(1)}(\bar{x}) \right\} \]
\[ h = \frac{1}{4} \frac{\Gamma(\frac{1}{2+k}) \Gamma(\frac{3}{2+k}) \Gamma^2(\frac{k}{2+k})}{\Gamma(\frac{1+k}{2+k}) \Gamma^2(\frac{2+k}{2+k})}. \] (3.47)

To determine the constants \( \alpha, \beta, \gamma, \delta \), we consider the OPE which can be deduced from equations (3.43) and (3.47). In the region close to \( x = 0 \), we have in particular (with \( \Delta = \Delta_1^{1/2} \))

\[ G_1^{++-}(x, \bar{x}) = \delta G_1^{+- -}(x, \bar{x}) = \gamma |x|^{-4\Delta} (1 + O(x, \bar{x}) + \ldots) \]
\[ -h \gamma |x|^{2\Delta_0^0 - 4\Delta} (1 + O(x, \bar{x}) + \ldots). \] (3.48)

Substituting eq. (3.48) into eq. (3.43), we can find the OPEs:

\[ g_{\epsilon_1}^-(z, \bar{z}) g_{\epsilon_2}^{+0}(0) = \sqrt{\gamma} |z|^{-4\Delta} (\delta_{\epsilon_1 \epsilon_2} I + 2 \sqrt{h} \epsilon_1 |z|^{2\Delta_0^0} t^{A \epsilon_1 \epsilon_2} (\Phi_0^0) A(z, \bar{z}) + \ldots) \] (3.49)

\[ g_{\epsilon_1}^+(z, \bar{z}) g_{\epsilon_2}^{-0}(0) = \pm \sqrt{\gamma} |z|^{-4\Delta} (\delta_{\epsilon_1 \epsilon_2} I + 2 \epsilon \sqrt{h} \epsilon_1 |z|^{2\Delta_0^0} t^{A \epsilon_1 \epsilon_2} (\Phi_0^0) A(z, \bar{z}) + \ldots), \]

where \( \epsilon = \pm 1 \). Equations (3.48) and (3.49) imply that we have normalized the two-point functions as

\[ \langle \langle g_{\epsilon_1}^-(z, \bar{z}) g_{\epsilon_2}^{+0}(0) \rangle \rangle = \sqrt{\gamma} |z|^{4\Delta_0^0} \delta_{\epsilon_1 \epsilon_2} \]
\[ \langle \langle g_{\epsilon_1}^+(z, \bar{z}) g_{\epsilon_2}^{-0}(0) \rangle \rangle = \pm \sqrt{\gamma} |z|^{-4\Delta_0^0} \delta_{\epsilon_1 \epsilon_2} \]
\[ \langle \langle (\Phi_0^0) A(z, \bar{z}) (\Phi_0^0) B(0) \rangle \rangle = \delta_{AB} |z|^{4\Delta_0^0} \] (3.50)

If we assume that \( |\langle \langle g^-(z, \bar{z}) g^{+0}(0) \rangle \rangle| = |\langle \langle g^+(z, \bar{z}) g^{-0}(0) \rangle \rangle| \), then we must have \( \delta = 1 \). Now we can substitute the OPEs (3.49) back into the four-point function (3.43), and find the leading terms in the expansion of \( G_1^{++-}(x, \bar{x}) \). We already have an expression for this function in equations (3.43) and (3.47), and for the two to be consistent we must have

\[ \beta = \pm \gamma, \quad \beta - 2\alpha = \epsilon \beta. \] (3.51)
The only remaining ambiguity, apart from the normalization of $g(z, \bar{z})$, is in the relative sign of the two point functions $\langle g^{-}(z, \bar{z})g^{+}(0) \rangle$ and $\langle g^{+}(z, \bar{z})g^{-}(0) \rangle$, and the sign of $\epsilon$. The final expression for the full four-point function is

$$G_{\bar{m}_1 \bar{m}_2 \bar{m}_3 \bar{m}_4}(x, \bar{x}) = \sum_{B=1,2} C_{\bar{m}_1 \bar{m}_2 \bar{m}_3 \bar{m}_4}^B f_{\bar{m}_1 \bar{m}_2 \bar{m}_3 \bar{m}_4}(x)|^2 \left( F_A^{(0)}(x)F_B^{(0)}(\bar{x}) + hF_A^{(1)}(x)F_B^{(1)}(\bar{x}) \right)$$

(3.52)

where if $\epsilon = -1$,

$$C_B^{+-+} = C_B^{-+-} = 1, \quad C_B^{++-} = C_B^{-++} = C_B^{+-+} = C_B^{-++} = \pm 1, \quad B = 1, 2$$

(3.53)

and if $\epsilon = +1$,

$$C_B^{+-+} = C_B^{-++} = 1, \quad B = 1, 2$$

$$C_2^{++-} = C_2^{-++} = C_1^{+-+} = C_1^{-++} = \pm 1$$

(3.54)

and all other $C_{\bar{m}_1 \bar{m}_2 \bar{m}_3 \bar{m}_4}^B$ are 0.

4 Conclusion

For the gauged WZNW models, which form a particular class of 2D gauged theories, we have derived a differential equation which, in principle, allows one to find correlation functions of these models. We are aware of an attempt to obtain a generalized KZ equation for coset constructions [18]. We think that our equation is different from the one obtained in [18]. However, we do not know what the relation is between these two differential equations.

We have used our equation to show that the $SL(2)/U(1)$ conformal blocks can be expressed as products of $SL(2)$ and $U(1)$ conformal blocks. We can use this fact to shed some light on a suggestion in [19] that logarithmic operators will arise in the spectrum of 2D black holes. This now implies that the logarithmic operators must also be present in the spectrum of the non-unitary $SL(2)$ WZNW model at $k = -9/4$. There are no logarithmic singularities in the conformal blocks for fields in finite dimensional representations of $SL(2)$ (except at some special values of $k$, which do not include $k = -9/4$). However,
in the context of $2D$ black holes, the spectrum includes the infinite dimensional representations [17]. Unfortunately, we do not know how to solve the KZ equations for these infinite dimensional representations, but we can consider the free-field representation for them [20]. The vertex operator in the representation $l$, $V_l$ and the “reflected” operator $V_{-l-1}$, have the same dimension. In the case of $l = -1/2$, these become degenerate and we might expect the second operator to become a logarithmic operator. This is similar to the way a logarithmic operator (the puncture operator) appears in the Liouville model [6], [21]. Logarithmic operators are expected to appear in the spectrum of a theory only when they have integer dimensions, which is the case for the $l = -1/2$ operator at $k = -9/4$. The fact that we can see the appearance of logarithmic operators in the spectrum of $SL(2,R)/U(1)$ coset construction suggests that we can use our method for further systematic studying of these operators in different models, for example non-unitary minimal models [22].

We hope that this approach gives us new opportunities to study coset models.

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