Crossing periodic orbits of nonsmooth Liénard systems and applications*

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Abstract
Continuing the investigation for the number of crossing periodic orbits of non-smooth Liénard systems in (2008 Nonlinearity 21 2121–42) for the case of a unique equilibrium, in this paper we allow the considered system to have one or multiple equilibria. By constructing two control functions that are decreasing in a much narrower interval than the one used in the above work in the estimate of divergence integrals, we overcome the difficulty of comparing the heights of orbital arcs caused by the multiplicity of equilibria and give results about the existence and uniqueness of crossing periodic orbits, which hold not only for a unique equilibrium but also for multiple equilibria. Moreover, we find a sufficient condition for the existence of periodic annuli formed by crossing periodic orbits. Applying our results to planar piecewise linear systems with a line of discontinuity and without sliding sets, we prove the uniqueness of crossing limit cycles and hence give positive answers to conjectures 1 and 2 of Freire et al’s work (2013 Planar Filippov Systems with Maximal Crossing Set and Piecewise Linear Focus Dynamics (Progress and Challenges in Dynamical Systems vol 54) (Heidelberg: Springer) pp 221–32).

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1. Introduction and main results

A class of important dynamical systems is the Liénard system, which is originated from physics and then is applied to engineering, biology, chemistry and more fields. As usual, the Liénard system can be written as

\[
\begin{align*}
\dot{x} &= F(x) - y, \\
\dot{y} &= g(x),
\end{align*}
\]

where \( F(x) := \int_0^x f(s) \, ds, \) \( f(x) \) and \( g(x) \) are two scalar functions. For system (1.1), a main and challenging subject is to study the existence, uniqueness and number of limit cycles, i.e., isolated periodic orbits in the phase space. The 13th Smale’s problem provided in [30] is about the maximum number of limit cycles in the polynomial system of form \( \dot{x} = F(x) - y, \dot{y} = x, \) i.e., the famous 16th Hilbert’s problem restricted to the polynomial system of form (1.1) with \( g(x) = x, \) and is still open. When system (1.1) is smooth, the investigation of the existence, uniqueness and number of limit cycles has a long history and many excellent results are obtained as given in the papers [9, 11–13, 25] and the monograph [34].

On the other hand, many models in practical problems are established by system (1.1) in a nonsmooth form such as

\[
\begin{align*}
\dot{x} &= ax + bx^3 - y, \\
\dot{y} &= x - \text{sgn}(x),
\end{align*}
\]

which is the limit case of a smooth oscillator (see [5, 6]). Another system

\[
\begin{align*}
\dot{x} &= Tx - y, \\
\dot{y} &= \begin{cases} 
1 - D \cdot X_{\text{ref}} & \text{if } x < 0, \\
-D \cdot X_{\text{ref}} & \text{if } x > 0
\end{cases}
\end{align*}
\]

is the Liénard form of a buck electronic converter (see [17]), where \( T, D, X_{\text{ref}} \) are parameters. Motivated by practical problems, many mathematicians and engineers have started to investigate the existence, uniqueness and number of limit cycles for the nonsmooth Liénard system (1.1). Usually, the nonsmoothness leads to much difficulty in the analysis of nonsmooth Liénard systems and much less results are obtained (see, e.g., [7, 23, 26, 28]), compared with the smooth case.

Consider Liénard system (1.1) with nonsmooth functions

\[
\begin{align*}
f(x) := \begin{cases} 
f^+(x) & \text{if } x > 0, \\
f^-(x) & \text{if } x < 0,
\end{cases}
g(x) := \begin{cases} 
g^+(x) & \text{if } x > 0, \\
g^-(x) & \text{if } x < 0,
\end{cases}
\end{align*}
\]

where \( f^+(x), g^+(x) : \mathbb{R} \to \mathbb{R} \) are \( C^1 \) functions. Hence, the nonsmoothness or even discontinuity of system (1.1) occurs only on \( y \)-axis, the switching line. Let \( F^\pm(x) := \int_0^x f^\pm(s) \, ds. \) We rewrite nonsmooth Liénard system (1.1) as

\[
(\dot{x}, \dot{y}) = \begin{cases} 
(F^+(x) - y, g^+(x)) & \text{if } x > 0, \\
(F^-(x) - y, g^-(x)) & \text{if } x < 0.
\end{cases}
\]

For convenience, we call the subsystem in \( x > 0 \) and \( x < 0 \) the right system and left system of (1.4), respectively. Since \( F^0(0) = 0, \) we define the \( x \)-component of the vector field
of (1.4) as \(-y\) on y-axis, i.e., \(\dot{x} = -y\). If \(g^+(0) = g^-(0)\), the y-component of the vector field of (1.4) is defined as \(g^+(0)\) on y-axis, i.e., \(\dot{y} = g^+(0)\). Then (1.4) is continuous. If \(g^+(0) \neq g^-(0)\), the y-component of the vector field of (1.4) has a jump discontinuity on y-axis, which implies that (1.4) is discontinuous. Thus we define the solution of (1.4) passing through a point in y-axis by the Filippov convention (see [14, 24]). In fact, the switching line consists of the origin \(O\) and two crossing sets, i.e., the positive y-axis and the negative y-axis. Let \(q\) be a point in y-axis. If \(q\) belongs to the positive (resp. negative) y-axis, then the solution of (1.4) passing through \(q\) crosses y-axis at \(q\) from right (resp. left) to left (resp. right). If \(q\) lies at \(O\), it is proved in [28, proposition 1] that \(q\) is a boundary equilibrium when \(g^+(0)g^-(0) = 0\), a pseudo-equilibrium when \(g^+(0)g^-(0) < 0\) and a regular point when \(g^+(0)g^-(0) > 0\). Thus an equilibrium of (1.4) has three types, including boundary equilibria and pseudo-equilibria lying in \(x = 0\), regular equilibria lying in \(x \neq 0\). In the third case, \(q\) is also called a parabolic fold-fold point (see [3]) and, more precisely, both orbits of the left and right systems passing through \(q\) are quadratically tangent to the y-axis from the left half plane when \(g^+(0) > 0, g^-(0) > 0\) and from the right half plane when \(g^+(0) < 0, g^-(0) < 0\).

A limit cycle of system (1.4) either totally lies in one of half planes \(x > 0\) and \(x < 0\), or presents intersections with both \(x > 0\) and \(x < 0\). Since the former one can be determined by one of the right and left systems, our attention is paid to the latter one, i.e., crossing limit cycle (see [24]). For (1.4), there exist a few results on the existence, uniqueness and number of crossing limit cycles, such as [7, 23, 26, 28]. In [28], a necessary condition of the existence of crossing limit cycles and a sufficient condition for the uniqueness are given. In [23], a sufficient condition of the existence and uniqueness of crossing limit cycles is presented. The number of crossing limit cycles is studied in [7, 26]. Unfortunately, we observe that almost in all publications (1.4) is required to satisfy

\[ g^+(x) > 0 \text{ for } x > 0, \quad g^-(x) < 0 \text{ for } x < 0. \tag{1.5} \]

Condition (1.5) implies that the origin \(O\) is the unique equilibrium of (1.4) and, hence, any crossing limit cycle surrounds \(O\) as indicated in [28]. However, in many practical problems system (1.4) may have multiple equilibria in the whole phase space, such as systems (1.2) and (1.3), and then a crossing limit cycle can surround multiple equilibria. On the existence, uniqueness and number of crossing limit cycles of (1.4) with multiple equilibria, as far as we know, the results are lacking and there are only a few results restricted to concrete models (see, e.g., [6]). For smooth Liénard systems with multiple equilibria, some results of limit cycles have been given in [11, 12, 33, 34].

The goal of this paper is to study the existence, nonexistence and uniqueness of crossing limit cycles for the nonsmooth Liénard system (1.4) allowing one or multiple equilibria. We particularly emphasize the case of multiple equilibria. In order to state our main results, we give some basic hypotheses for system (1.4) as the following (H1)–(H3).

(H1) There exists a constant \(x_0 \geq 0\) such that \(g^+(x)(x - x_0) > 0\) for all \(x > 0\) and \(x \neq x_0\).

(H2) \(f^+(x) > 0\) for \(x > 0\) and \(f^-(x) < 0\) for \(x < 0\).

Define

\[ p = p(x) := \begin{cases} F^+(x) & \text{if } x > 0, \\ F^-(x) & \text{if } x < 0. \end{cases} \tag{1.6} \]

Clearly, \(p(x)\) is continuous and \(p(0) = 0\) due to \(F^+(0) = 0\). Moreover, \(p'(x) = f^+(x) > 0\) for \(x > 0\) and \(p'(x) = f^-(x) < 0\) for \(x < 0\) by (H2). Thus \(p(x)\) is strictly increasing for \(x > 0\) and
strictly decreasing for $x < 0$, implying $p(x) ≥ 0$ for all $x ∈ \mathbb{R}$ and $p(x)$ has a strictly increasing inverse function

$$x^+(p) : [0, p^+) → [0, +∞)$$

and a strictly decreasing inverse function

$$x^-(p) : [0, p^-) → (−∞, 0),$$

where $p(x) → p^+(x → ±∞)$. It follows from the monotonicity that $p^+$ (resp. $p^-$) is either a constant or infinity.

(H3) There exist the limits

$$\lim_{p→0^+} \frac{g^+(x^+(p))}{f^+(x^+(p))} = η^+$$

and $−∞ < η^+ ≤ η^- < +∞$. Moreover, $g^+/f^+|_{x=x^+(p)} < g^-/f^-|_{x=x^-(p)}$ for all sufficiently small $p > 0$ if $η^+ = η^-$. Under hypothesis (H1), in the right half plane system (1.4) has no equilibria if $x_0 = 0$ and a unique equilibrium if $x_0 > 0$. Moreover, the unique equilibrium in the right half plane lies at $(x_0, F^+(x_0))$. Let $E$ be the point lying at $(x_0, F^+(x_0))$ if $x_0 > 0$ and $O$ if $x_0 = 0$. Since the equilibria in the left half plane are determined by the zeros of $g^-(x)$, the number of equilibria in the left half plane is not restricted by hypotheses (H1)–(H3), so that it is allowed for system (1.4) to have multiple equilibria in the left half plane. On the $y$-axis, system (1.4) has no equilibria if $g^+(0)g^-(0) > 0$ and a unique equilibrium $O$ if $g^+(0)g^-(0) ≤ 0$ as indicated in the third paragraph. As an example, consider system (1.4) satisfying

$$f^+(x) = 1, \quad f^-(x) = −1, \quad g^+(x) = x^2 + x + a$$

with $0 < a < 1/16$. It has four equilibria $O$, $(2a, 2a)$ and

$$\left(−\frac{1}{4} ± \left(\frac{1}{16} − a\right)^{1/2}, \frac{1}{4} ± \left(\frac{1}{16} − a\right)^{1/2}\right),$$

where the last two lie in the left half plane. A direct verification implies that this system satisfies hypotheses (H1)–(H3). Thus (H1)–(H3) are reasonable and system (1.4) may have multiple equilibria.

In the following, we state our main results.

**Theorem 1.1.** If system (1.4) with (H1)–(H3) has a crossing periodic orbit $Γ$, then

(a) $Γ$ surrounds $O$ and $E$ counterclockwise;

(b) the equations

$$F^−(x^−) = F^+(x^+), \quad \frac{g^−(x^−)}{f^−(x^−)} = \frac{g^+(x^+)}{f^+(x^+)}$$

have at least one solution $(x^−, x^+) = (x^−_e, x^+_e)$ with $x^−_e < x^+_e$ satisfying that $Γ$ transversally intersects both lines $x = x^−_e$ and $x = x^+_e$.

Theorem 1.1 provides two necessary conditions for the existence of crossing periodic orbits for system (1.4). These two necessary conditions help us determine the configuration of crossing periodic orbits. Any crossing periodic orbit neither lies in each side of the line $x = x_e$ nor lies in the strip $x^−_e < x < x^+_e$, where $(x^−_e, x^+_e)$ is the solution of (1.11) satisfying that
\( \dot{x}_0 < 0 < \dot{x}^+ \) is the narrowest strip. On the other hand, theorem 1.1 can be regarded as a generalization of [28, theorem 2] from one equilibrium to multiple equilibria. It is required in [28] that system (1.4) satisfies the condition (1.5), implying that \( O \) is the unique equilibrium of (1.4) and any crossing periodic orbit surrounds \( O \). On one hand, there is no any restrictions for \( g^- (x) \) in (H1) but there is in (1.5). On the other hand, when (1.5) holds, there exists a constant \( x_e := 0 \) such that \( g^- (x) (x - x_e) > 0 \) for \( x > 0 \) and \( x \neq x_e \). Thus, (H1) holds when (1.5) holds. On the contrary, we cannot get (1.5) by (H1) because there is no restrictions for \( g^- (x) \) but there is in (1.5) and there exist some functions \( g^+ (x) \) satisfying (H1) but not (1.5), such as the \( g^+ (x) \) given in (1.10). Thus (H1) is weaker than (1.5). In this paper we replace (1.5) by the weaker hypothesis (H1), which allows (1.4) to have multiple equilibria and crossing limit cycles surrounding multiple equilibria. For example, by theorem 1.1 the crossing periodic orbit \( \Gamma \) surrounds at least two equilibria (\( O \) and \( E \)) when \( x_e > 0 \) and \( g^- (0) g^- (0) \leq 0 \). Moreover, as it is indicated above theorem 1.1, the example (1.10) satisfies hypotheses (H1)–(H3). Therefore, our main results can be for a system with multiple equilibria.

**Theorem 1.2.** For system (1.4) with (H1)–(H3), assume that the equations in (1.11) have a unique solution \((x^-, x^+) = (x^-, x^+)\) with \( x_0 < 0 < x^+ \) and one of the following hypotheses holds:

(H4) \( x^+ \geq x_e \) and \( F^+ (x) \frac{f^+ (x)}{g^+ (x)} \) is increasing for \( x > x^+ \):

(H5) \( K^+ (x^+ (p_1)) < K^+ (x^+ (p_2)) \) for all \( p_1, p_2 \) satisfying \( p_2 \geq p_1 > F^+ (x^+) \), where

\[
K^+ (x) := \frac{(g^+ (x))^p f^+ (x) - (f^+ (x))^p g^+ (x)}{(f^+ (x))^3}
\]

and \( x^+ (p) \) are given in (1.7) and (1.8) respectively.

Then system (1.4) has at most one crossing periodic orbit, which is a stable and hyperbolic crossing limit cycle if it exists.

Theorem 1.2 is a result about the uniqueness of crossing limit cycles for system (1.4). In the aspect of the number of equilibria, theorem 1.2 can be regarded as a generalization of [28, theorem 3] from one equilibrium to multiple equilibria because system (1.4) is also allowed to have multiple equilibria under the assumptions of this theorem, such as the example (1.10), which has four equilibria and satisfies all assumptions of theorem 1.2. In the aspect of the smoothness of systems, theorem 1.2 can be regarded as a generalization of [11, theorems 2.1 and 2.2] from smooth Liénard systems to nonsmooth ones.

In the following theorem we give a sufficient condition for the existence of periodic annuli.

**Theorem 1.3.** For system (1.4) with (H2), assume that (1.9) holds and

\[
\frac{g^+ (x)}{f^+ (x)} \bigg|_{x = x^+ (p)} = \frac{g^- (x)}{f^- (x)} \bigg|_{x = x^- (p)}
\]

for all \( p > 0 \). If system (1.4) has a crossing periodic orbit, then there exists a periodic annulus including this crossing periodic orbit.

To apply our main results of system (1.4), we study the number of crossing limit cycles for the piecewise linear system

\[
\dot{z} = \begin{cases} 
A^+ z + b^+ & \text{if } x > 0, \\
A^- z + b^- & \text{if } x < 0,
\end{cases}
\]

(1.13)
where $z = (x, y)^\top \in \mathbb{R}^2$, 

$$A^\pm = \begin{pmatrix} a^\pm_1 & a^\pm_2 \\ a^\pm_{21} & a^\pm_{22} \end{pmatrix} \in \mathbb{R}^{2 \times 2}, \quad b^\pm = \begin{pmatrix} b^\pm_1 \\ b^\pm_2 \end{pmatrix} \in \mathbb{R}^2.$$ 

In this paper we always assume that system (1.13) is nondegenerate, i.e., $\det A^\pm \neq 0$. System (1.13) has been widely used as a model in engineering, physics and biology (see [1, 17, 31]), and many contributions have been made in recent years (see [4, 15, 16, 19, 21, 22, 32]). Although the two subsystems of (1.13) are linear, the switching of the vector fields in different regions leads to great complexity and difficulty in the research on the number of crossing limit cycles. When (1.13) is continuous, it is proved in [19] that there exists at most one crossing limit cycle and this number can be reached. When (1.13) is discontinuous, in many publications examples were provided for (1.13) to have three crossing limit cycles, such as [4, 15, 16, 22]. However, the problem of maximum number of crossing limit cycles for discontinuous system (1.13) is still open. On the other hand, we checked in all papers presenting examples with three crossing limit cycles and found that all these systems have sliding sets, namely $\{(0, y) : (a_{12}^+ y + b_1^+)(a_{12}^- y + b_1^-) < 0\} \neq \emptyset$. Therefore, a natural question is how about the maximum number of crossing limit cycles for discontinuous system (1.13) without sliding sets. This question was answered in [28, 29] for the case that $O$ is a $\Sigma$-monodromic singularity, i.e., all orbits in a small neighborhood of $O$ of (1.13) turn around $O$, and the maximum number is 1. Besides, it was proved in [20] (resp. [18]) that the maximum number is also 1 for focus–saddle type (resp. for focus–focus type with partial regions of parameters). However, the problem of the maximum number of crossing limit cycles for general discontinuous (1.13) without sliding sets is still open. Applying our main results for system (1.4), we completely answer this open problem in the following theorem.

**Theorem 1.4.** If discontinuous system (1.13) has no sliding sets, then there exists at most one crossing limit cycle and this number can be reached. Moreover, it is possible that the number of equilibria surrounded by this crossing limit cycle is exactly $k$ ($1 \leq k \leq 3$).

By theorem 1.4, we directly obtain a positive answer to conjecture 1 of [18, p 232] which is about the uniqueness of crossing limit cycles of system (1.13) with focus–focus dynamics and without sliding sets. On the other hand, conjecture 2 of [18, p 232] states the nonexistence of crossing limit cycles for some cases. By analyzing a Poincaré map associated to system (1.13) with focus–focus dynamics, in the end of section 3 we give a proof for nonexistence associated with theorem 1.4 and, hence, obtain a positive answer to conjecture 2 of [18, p 232].

The remainder of this paper is organized as follows. In section 2 we give proof of theorems 1.1–1.3 for nonsmooth Liénard system (1.4). In section 3 we apply the main results for system (1.4) to study the number of crossing limit cycles of system (1.13) and prove theorem 1.4.

### 2. Proof of theorems 1.1, 1.2 and 1.3

The purpose of this section is to provide proof of theorems 1.1–1.3. Firstly, we describe some geometrical properties of a crossing periodic orbit of system (1.4) with (H1).

**Lemma 2.1.** If system (1.4) with (H1) has a crossing periodic orbit $\Gamma$, then

(a) $\Gamma$ intersects the curve $y = F^+(x)$ [resp. $y = F^-(x)$] at a unique point in $x > 0$ (resp. $x < 0$), denoted by $A$ (resp. $C$) as in figure 1;
then it satisfies the geometrical properties in lemma 2.1 and we still use the denotations in the proof of lemma 2.1, see figure 2(a). Conclusion (a) is obtained directly from lemma 2.1 (c).

Proof of theorem 1.1.

According to the third paragraph in section 1, $O$ is an equilibrium when $g^+(0)g^-(0) \leq 0$ and a parabolic fold-fold point when $g^+(0)g^-(0) > 0$. This implies that $\Gamma$ cannot pass through $O$. Moreover, $x > 0$ (resp. $< 0$) for all $(x, y)$ in the region below (resp. above) $y = p(x)$, where $p(x)$ is defined in (1.6). Therefore, conclusions (a) and (b) hold and $\Gamma$ surrounds $O$ counterclockwise. Let $B$ (resp. $D$) be the intersection of $\Gamma$ and the positive y-axis (resp. the negative y-axis) and $\Gamma := \Gamma_{AB} \cup \Gamma_{BC} \cup \Gamma_{CD} \cup \Gamma_{DA}$, see figure 1. If $x_e = 0$, conclusion (c) obviously holds because $E$ lies at $O$. If $x_e > 0$, it follows from (H1) that the curve corresponding to $\Gamma_{DA}$ (resp. $\Gamma_{AB}$) goes down for $0 < x < x_e$ and goes up for $x_e < x < x_A$ as $t$ increases. Thus $\Gamma$ also surrounds $E$ counterclockwise, implying that conclusion (c) holds. □

Proof of theorem 1.1. Under hypotheses, if system (1.4) has a crossing periodic orbit $\Gamma$, then it satisfies the geometrical properties in lemma 2.1 and we still use the denotations in the proof of lemma 2.1, see figure 2(a). Conclusion (a) is obtained directly from lemma 2.1 (c).

In order to prove conclusion (b), we apply the change $p = p(x)$ to system (1.4), where $p(x)$ is defined in (1.6). Thus the right and left systems of (1.4) are transformed into

\[
\begin{aligned}
\dot{p} &= f^+(x^+(p))(p - y), \\
\dot{y} &= g^+(x^+(p)),
\end{aligned}
\]

\[
\begin{aligned}
\dot{p} &= f^-(x^-)(p - y), \\
\dot{y} &= g^-(x^-),
\end{aligned}
\]

(2.1)

respectively, from which we get

\[
\frac{dy}{dp} = \frac{\varphi^+(p)}{p - y} := \frac{g^+(x^+(p))}{f^+(x^+(p))(p - y)}, \quad \frac{dy}{dp} = \frac{\varphi^-(p)}{p - y} := \frac{g^-(x^-)(p))}{f^-(x^-)(p)(p - y)}
\]

(2.2)

for $p > 0$, respectively. Here $x^+(p)$ (resp. $x^-(p)$) given in (1.7) [resp. (1.8)] is the inverse function of $p = p(x)$ for $x \geq 0$ (resp. $x < 0$). Clearly, (2.1) and (2.2) can be continuously extended to $p = 0$ due to the continuieties of $f^+, g^+$ at $x = 0$ and (H3). Thus (2.1) and (2.2) are well defined for $p \geq 0$. Moreover, under the change $p = p(x)$, the crossing periodic orbit $\Gamma$ becomes the orbit $\gamma := \gamma_{AB} \cup \gamma_{BC} \cup \gamma_{CD} \cup \gamma_{DA}$ in $py$-plane as shown in figure 2(b), where
respectively.

Thus \( \gamma_{DA} \cup \gamma_{AB} \) are the orbits of the first system and the second one in (2.1), respectively.

Let \( \Delta^+ \) (resp. \( \Delta^- \)) be the region surrounded by \( y \)-axis and \( \Gamma_{DA} \cup \Gamma_{AB} \) (resp. \( \Gamma_{BC} \cup \Gamma_{CD} \)), \( \Omega^+ \) (resp. \( \Omega^- \)) be the region surrounded by \( y \)-axis and \( \gamma_{DA} \cup \gamma_{AB} \) (resp. \( \gamma_{BC} \cup \gamma_{CD} \)), \( S(\Omega^\pm) \) be the areas of \( \Omega^\pm \). By Green’s formula we have

\[
0 = \int_{\Gamma_{DA} \cup \Gamma_{AB}} -g^+(x)dx + (F^+(x) - y)dy + \int_{\Gamma_{BC} \cup \Gamma_{CD}} -g^-(x)dx + (F^-(x) - y)dy
\]

\[
= \int_\Delta^+ f^+(x)dx + \int_\Delta^- f^-(x)dx - \int_\Delta^+ g^+(x)dx + (F^+(x) - y)dy - \int_\Delta^- g^-(x)dx + (F^-(x) - y)dy
\]

\[
= \int_\Delta^+ f^+(x)dx + \int_\Delta^- f^-(x)dx + \int_\Delta^+ g^+(x)dx - \int_\Delta^- g^-(x)dx + (F^+(x) - y)dy + (F^-(x) - y)dy
\]

\[
= \int_\Delta^+ d\phi dx - \int_\Delta^- d\phi dy
\]

Thus \( \gamma_{BC} \cup \gamma_{CD} \) crosses \( \gamma_{DA} \cup \gamma_{AB} \) at some \( p \in (0, \min\{p_A, p_C\}) \), where \( p_A \) and \( p_C \) are the abscissas of points \( A \) and \( C \) in \( py \)-plane. Otherwise, from (H3) we get that \( \gamma_{BC} \) (resp. \( \gamma_{DA} \)) always lies below \( \gamma_{AB} \) (resp. \( \gamma_{CD} \)). This means that \( S(\Omega^+) - S(\Omega^-) > 0 \), contradicting (2.3).

Suppose that \( \varphi^+(p) < \varphi^-(p) \) for \( 0 < p < \min\{p_A, p_C\} \), then \( y_{AB}(p) > y_{BC}(p) \) and \( y_{CD}(p) > y_{DA}(p) \). By applying the theory of differential inequalities to systems in (2.2), where \( y = y_{AB}(p) \), \( y = y_{BC}(p) \), \( y = y_{CD}(p) \) and \( y = y_{DA}(p) \) describe the orbits \( \gamma_{AB} \), \( \gamma_{BC} \), \( \gamma_{CD} \) and \( \gamma_{DA} \), respectively. Thus \( \gamma_{BC} \cup \gamma_{CD} \) does not cross \( \gamma_{DA} \cup \gamma_{AB} \), implying a contradiction. Consequently, \( \varphi^+(p) = \varphi^-(p) \) has at least one solution in \( 0 < p < \min\{p_A, p_C\} \), denoted by \( p_\ast \). Choosing \( x^+_\ast := x^+(p_\ast) \) and \( x^-_\ast := x^-(p_\ast) \), we finally obtain that the equation (1.11) have at least one solution \( (x^-, x^+) = (x^-_\ast, x^+_\ast) \) with \( x^-_\ast < 0 < x^+_\ast \) satisfying that \( \Gamma \) transversally intersects both the verticals \( x = x^\pm_\ast \), i.e., conclusion (b) is proved.

\( \square \)
**Proof of theorem 1.2.** The essential idea of this proof comes from [11, 28] and it is accomplished by two steps. Assume that system (1.4) has a crossing periodic orbit \( \Gamma := (x(t), y(t)) \) and define
\[
\lambda_{\Gamma} := \int_{\Gamma^-} f^-(x(t)) \, dt + \int_{\Gamma^+} f^+(x(t)) \, dt,
\]
where \( \Gamma^+ := \Gamma \cap \{(x, y) : x \geq 0\} \) and \( \Gamma^- := \Gamma \cap \{(x, y) : x \leq 0\} \). In the first step, we prove \( \lambda_{\Gamma} < 0 \), which implies that \( \Gamma \) is a stable and hyperbolic crossing limit cycle by [10, theorem 2.1]. In the second step, we prove that system (1.4) cannot have two stable and hyperbolic limit cycles in succession, which implies the uniqueness of crossing limit cycles associated with the result of the first step.

**Step 1.** We prove \( \lambda_{\Gamma} < 0 \).

Following the denotations and geometric properties of \( \Gamma \) in lemma 2.1 and theorem 1.1, we firstly prove \( p_C > p_A \). In fact, under the assumption of theorem, \( \varphi^+(p) = \varphi^-(p) \) has a unique solution \( p_1 \in (0, \min\{p_A, p_C\}) \). Moreover, \( \varphi^+(p) < \varphi^-(p) \) for \( 0 < p < p_C \), and \( \varphi^+(p) > \varphi^-(p) \) for \( p > p_C \). Applying the theory of differential inequalities to systems in (2.2), we obtain
\[
y_{AB}(p) > y_{BC}(p), \quad y_{CD}(p) > y_{DA}(p) \quad \text{for} \quad 0 < p < p_C. \tag{2.4}
\]

From the proof of theorem 1.1 \( \gamma_{BC} \cup \gamma_{CD} \) must cross \( \gamma_{DA} \cup \gamma_{AB} \). Without loss of generality, assume that \( \gamma_{BC} \cup \gamma_{CD} \) and \( \gamma_{DA} \cup \gamma_{AB} \) have a crossing point at a value \( p_1 \) for which \( \gamma_{AB} \) crosses \( \gamma_{BC} \). Then, it follows from (2.4) and \( \varphi^+(p) > \varphi^-(p) \) for \( p > p_1 \) that \( p_1 < p_C \) and
\[
y_{AB}(p) > y_{BC}(p) \quad \text{for} \quad 0 < p < p_1, \tag{2.5}
y_{AB}(p) < y_{BC}(p) \quad \text{for} \quad p_1 < p < \min\{p_A, p_C\}.
\]

Here the theory of differential inequalities is applied in the second inequality. Moreover, since the number of crossing points of \( \gamma_{BC} \cup \gamma_{CD} \) and \( \gamma_{DA} \cup \gamma_{AB} \) must be even from (2.4), we further obtain that there exists a value \( p_2 \) with \( p_1 < p_2 < p_C \) such that \( \gamma_{CD} \) crosses \( \gamma_{DA} \) at \( p_2 \). Similarly,
\[
y_{CD}(p) > y_{DA}(p) \quad \text{for} \quad 0 < p < p_2, \tag{2.6}
y_{CD}(p) < y_{DA}(p) \quad \text{for} \quad p_2 < p < \min\{p_A, p_C\}.
\]

Hence, combining with (2.5) and (2.6), we get \( p_C > p_A \).

On the other hand, we have \( \varphi^+(p) > \varphi^-(p) > 0 \) and \( 0 > p - y_{AB}(p) > p - y_{BC}(p) \) for \( p_1 < p < p_C \). Thus
\[
\frac{dy_{AB}(p) - dy_{BC}(p)}{dp} = \frac{\varphi^+(p)}{p - y_{AB}(p)} - \frac{\varphi^-(p)}{p - y_{BC}(p)} < \frac{\varphi^+(p) - \varphi^-(p)}{p - y_{BC}(p)} < 0
\]
for all \( p \) with \( p_1 < p < p_C \), so that \( y_{AB}(p) - y_{BC}(p) \) is strictly decreasing in \( p_1 < p < p_C \) and then \( p_C > p_A \), i.e., \( p_C > p_A \).

Let \( \mathcal{M} \) and \( \mathcal{N} \) (resp. \( \mathcal{P} \) and \( \mathcal{Q} \)) be the intersections of \( \Gamma \) and the vertical \( x = x_+ \) (resp. \( x = x_- \)). Then we denote \( \Gamma \) by
\[
\Gamma = \Gamma_{DP} \cup \Gamma_{PA} \cup \Gamma_{AQ} \cup \Gamma_{QB} \cup \Gamma_{DN} \cup \Gamma_{NC} \cup \Gamma_{CN} \cup \Gamma_{CM} \cup \Gamma_{MD}
\]
as shown in Figure 3(a) and γ which corresponds with Γ under the change \( p = p(x) \) by

\[
\gamma = \gamma_{DP} \cup \gamma_{PA} \cup \gamma_{AQ} \cup \gamma_{QB} \cup \gamma_{BN} \cup \gamma_{NC} \cup \gamma_{CM} \cup \gamma_{MD}
\]
as shown in Figure 3(b), where \( \gamma_{DP} \cup \gamma_{PA} \cup \gamma_{AQ} \cup \gamma_{QB} \) and \( \gamma_{BN} \cup \gamma_{NC} \cup \gamma_{CM} \cup \gamma_{MD} \) are the orbits of the first system and the second one in (2.1), respectively. Therefore,

\[
\lambda_{\Gamma} = \int_{\Gamma_{DP} \cup \Gamma_{QB}} f^+(x) \, dx + \int_{\Gamma_{BN} \cup \Gamma_{MD}} f^-(x) \, dx + \int_{\Gamma_{PA} \cup \Gamma_{AQ}} f^+(x) \, dx + \int_{\Gamma_{NC} \cup \Gamma_{CM}} f^-(x) \, dx
\]

\[
= \int_{\gamma_{DP} \cup \gamma_{QB}} \frac{dp}{p-y} + \int_{\gamma_{BN} \cup \gamma_{MD}} \frac{dp}{p-y} + \int_{\gamma_{PA} \cup \gamma_{AQ}} \frac{dp}{p-y} + \int_{\gamma_{NC} \cup \gamma_{CM}} \frac{dp}{p-y}
\]

(2.7)
due to \( p = f^+(x^\pm(p)(p-y) \) in (2.1). For brevity, we neglect the variable \( t \) of \( x, y \) and \( p \) in (2.7) and the last of this proof if confusion does not arise.

Firstly, we prove

\[
J_1 := \int_{\gamma_{DP} \cup \gamma_{QB}} \frac{dp}{p-y} + \int_{\gamma_{BN} \cup \gamma_{MD}} \frac{dp}{p-y} < 0.
\]

(2.8)
Let \( y = y_{DP}(p), y = y_{QB}(p), y = y_{BN}(p) \) and \( y = y_{MD}(p) \) for \( 0 < p < p_\ast \) describe \( \gamma_{DP}, \gamma_{QB}, \gamma_{BN} \) and \( \gamma_{MD} \), respectively. Then

\[
p - y_{DP}(p) > 0, \quad p - y_{MD}(p) > 0, \quad p - y_{BN}(p) < 0, \quad p - y_{QB}(p) < 0.
\]

Moreover, from (2.5) and (2.6) we have

\[
y_{BN}(p) < y_{QB}(p), \quad y_{MD}(p) > y_{DP}(p)
\]

(2.9)
for \( 0 < p < p_\ast \). Hence,

\[
J_1 = \int_0^{p_\ast} \frac{dp}{p-y_{DP}(p)} + \int_0^{p_\ast} \frac{dp}{p-y_{QB}(p)} + \int_0^{p_\ast} \frac{dp}{p-y_{BN}(p)} + \int_0^{p_\ast} \frac{dp}{p-y_{MD}(p)}
\]

\[
= \int_0^{p_\ast} \frac{y_{DP}(p)-y_{MD}(p)}{(p-y_{DP}(p))(p-y_{MD}(p))} \, dp + \int_0^{p_\ast} \frac{y_{BN}(p)-y_{QB}(p)}{(p-y_{BN}(p))(p-y_{QB}(p))} \, dp
\]

\[
< 0,
\]
i.e., (2.8) holds.

Secondly, we prove

\[
J_2 := \int_{\gamma_{PA} \cup \gamma_{AQ}} \frac{dp}{p-y} + \int_{\gamma_{NC} \cup \gamma_{CM}} \frac{dp}{p-y} < 0,
\]
implying \( \lambda_{\Gamma} = J_1 + J_2 < 0 \) from (2.7) and (2.8). To do this, we define

\[
\mu := \frac{p_A - p_\ast}{p_C - p_\ast}, \quad \eta := \frac{(p_C - p_A)p_\ast}{p_C - p_\ast}
\]

as in [11]. Clearly, \( 0 < \mu < 1 \) due to \( p_C > p_A > p_\ast \) and \( \eta = (1 - \mu)p_\ast \). By the linear transformation

\[
\begin{align*}
\tilde{p} &= \mu p + \eta := \psi(p), \\
\tilde{y} &= \mu y + \eta := \phi(y),
\end{align*}
\]

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the second system in (2.1) is transformed into
\[
\begin{aligned}
\dot{\tilde{p}} &= f^+ \left( x - \left( \frac{\tilde{p} - \eta}{\mu} \right) \right) (\tilde{p} - \tilde{y}), \\
\dot{\tilde{y}} &= \mu g^+ \left( x - \left( \frac{\tilde{p} - \eta}{\mu} \right) \right).
\end{aligned}
\]
(2.10)

Denote the orbit of (2.10) corresponding with \( \gamma_{NC} \cup \gamma_{CM} \) by \( \tilde{\gamma}_{NC} \cup \tilde{\gamma}_{CM} \). Since \( \psi(p_c) = p_A \), \( \phi(p_c) = p_A \), \( \psi(p_c) = p_* \) and \( \phi(p_c) = p_* \), the orbit \( \tilde{\gamma}_{NC} \cup \tilde{\gamma}_{CM} \) is from \( (p_*, \mu y_{NC}(p_*) + \eta) \) to \( (p_*, \mu y_{CM}(p_*) + \eta) \) after passing through \( A \), see figure 3(b). Thus
\[
J_2 = \int_{\gamma_{PA} \cup \gamma_{AQ}} \frac{dp}{p - y} + \int_{\tilde{\gamma}_{NC} \cup \tilde{\gamma}_{CM}} \frac{d\tilde{p}}{\tilde{p} - \tilde{y}}
\]
(2.11)
\[
= \int_{p_c}^{p_A} \frac{dp}{p - y_{PA}(p)} + \int_{p_c}^{p_A} \frac{dp}{p - y_{AQ}(p)} + \int_{p_c}^{p_*} \frac{dp}{p - y_{NC}(p)} + \int_{p_c}^{p_*} \frac{d\tilde{p}}{\tilde{p} - y_{CM}(\tilde{p})}
\]
where \( y = \tilde{y}_{NC}(p) \) and \( y = \tilde{y}_{CM}(\tilde{p}) \) describe \( \tilde{\gamma}_{NC} \) and \( \tilde{\gamma}_{CM} \), respectively. In order to prove \( J_2 < 0 \), from (2.11) it is sufficient to prove
\[
y_{PA}(p) - \tilde{y}_{CM}(\tilde{p}) < 0, \quad \tilde{y}_{NC}(p) - y_{AQ}(p) < 0
\]
(2.12)
for \( p_* \leq p < p_A \) because \( p - y_{PA}(p) > 0 \), \( p - \tilde{y}_{CM}(\tilde{p}) > 0 \), \( p - y_{AQ}(p) < 0 \), \( p - \tilde{y}_{NC}(p) < 0 \). Clearly, \( \phi(y) - y = (1 - \mu)(p_* - y) \), \( p_* - y_{CM}(p_*) > 0 \) and \( p_* - y_{NC}(p_*) < 0 \). Thus
\[
\tilde{y}_{CM}(p_*) - y_{CM}(p_*) = \phi(y_{CM}(p_*)) - y_{CM}(p_*) = (1 - \mu)(p_* - y_{CM}(p_*)) > 0, \\
\tilde{y}_{NC}(p_*) - y_{NC}(p_*) = \phi(y_{NC}(p_*)) - y_{NC}(p_*) = (1 - \mu)(p_* - y_{NC}(p_*)) < 0
\]
(2.13)
due to \( 0 < \mu < 1 \). Using (2.9) and (2.13), we get
\[
y_{PA}(p_*) = y_{DP}(p_*) \leq y_{AQ}(p_*) = y_{CM}(p_*) < \tilde{y}_{CM}(p_*),
\]
\[
y_{AQ}(p_*) = y_{QA}(p_*) \geq y_{BA}(p_*) = y_{NC}(p_*) > \tilde{y}_{NC}(p_*),
\]
(2.14)
i.e., (2.12) holds for \( p = p_* \).

To prove (2.12) for \( p_* < p < p_A \), we consider system (2.10) without tildes and the first system in (2.2) for \( p_* < p < p_A \). We rewrite them as
\[
\frac{dy}{dp} = h(p, y) := \frac{\varphi^+ \left( (p - \eta)/\mu \right)}{(p - \eta)/\mu}, \quad \frac{p - \eta}{p - y},
\]
(2.15)
\[
\frac{dy}{dp} = H(p, y) := \frac{\varphi^+(p)}{p}, \quad \frac{p}{p - y},
\]
(2.16)
respectively. We only prove the first inequality in (2.12) when (H4) or (H5) holds, and the second one can be treated analogously. Thus \( p - y > 0 \) is always assumed in the following.

Assume that system (1.4) satisfies (H4). Consider function
\[
\frac{\varphi^+(p)}{p} \quad \text{for} \quad p_* < p < p_A,
\]
(2.17)
and we call it a control function. Then $\varphi^+(p)/p$ is decreasing in $p_* < p < p_A$. Moreover, since $0 < \mu < 1$ and $\eta = (1 - \mu)p_* > 0$, we have $(p - \eta)/\mu > p$ for $p > p_*$. Thus

$$\frac{\varphi^-(p - \eta)/\mu}{(p - \eta)/\mu} < \frac{\varphi^+(p - \eta)/\mu}{(p - \eta)/\mu} < \frac{\varphi^+(p)}{p}$$

(2.18)
due to $\varphi^-(p) < \varphi^+(p)$ for $p > p_*$. According to (H4), we get $x^+_x > x_*$, so that $g^+(x)/f^+(x) > 0$ for $x > x^+_x$ by (H1) and (H2), i.e., $\varphi^+(p) > 0$ for $p_* < p < p_A$. Hence, for $p_* < p < p_A$ and $p - y > 0$, $H(p, y) > 0$ and then

$$h(p, y) < \frac{\varphi^+(p)}{p} \frac{p - \eta}{p - y} = H(p, y) \frac{p - \eta}{p} < H(p, y),$$

(2.19)

where (2.18) and the fact that $p > \eta > 0$ are used. Since $y(p_A) < \tilde{y}_{CM}(p_A)$ as in (2.14), if there exists $p$ with $p_* < p < p_A$ such that $y(p_A) = \tilde{y}_{CM}(p_A)$, we obtain from (2.19) that $y(p_A) > \tilde{y}_{CM}(p)$ for $p < p < p_A$ by applying the theory of differential inequalities to systems (2.15) and (2.16). This contradicts the fact that $y(p_A) = \tilde{y}_{CM}(p_A)$, and consequently, $y(p_A) < \tilde{y}_{CM}(p)$ for $p_* < p < p_A$. That is, the first inequality of (2.12) holds under (H4).

Now assume that system (1.4) satisfies (H5). Letting

$$G(p) := \mu \varphi^\prime\left(\frac{p - \eta}{\mu}\right) - \varphi^+(p) \quad \text{for} \quad p_* < p < p_A,$$

(2.20)

we also call $G(p)$ a control function and obtain

$$G(p) = K\left(x^\prime\left(\frac{p - \eta}{\mu}\right)\right) - K^+(x^+(p)) < 0$$

by (H5) and $(p - \eta)/\mu > p$. Moreover, since $\eta = (1 - \mu)p_*$ and $\varphi^+(p_* \pm 0) = \varphi^-(p_*)$,

$$G(p_*) = \mu \varphi^\prime\left(\frac{p_* - \eta}{\mu}\right) - \varphi^-(p_*) = \mu \varphi^\prime(p_*) - \varphi^+(p_*) = (\mu - 1)\varphi^+(p_*)$$

by (H5) and $(p - \eta)/\mu > p$. Moreover, since $\eta = (1 - \mu)p_*$ and $\varphi^+(p_* \pm 0) = \varphi^-(p_*)$,

$$G(p_*) = \mu \varphi^\prime\left(\frac{p_* - \eta}{\mu}\right) - \varphi^-(p_*) = \mu \varphi^\prime(p_*) - \varphi^+(p_*) = (\mu - 1)\varphi^+(p_*)$$

Combining with $\varphi^+(p_*) = g^+(x^+_x)/f^+(x^+_x)$ and $0 < \mu < 1$, we further have $G(p_*) \leq 0$ if $x^+_x > x_*$ and $G(p_*) > 0$ if $0 < x^+_x < x_*$ by (H1) and (H2). In the first case, $G(p) < 0$ and then

$$h(p, y) - H(p, y) = \frac{G(p)}{p - y} < 0$$

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{figure3a}
\caption{(a)}
\end{subfigure}
\hfill
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{figure3b}
\caption{(b)}
\end{subfigure}
\caption{The eight orbit arcs of $\Gamma$ in $xy$-plane and the ones of $\gamma$ in $py$-plane.}
\end{figure}
for \( p_s < p < p_A \) and \( p - y > 0 \), implying that \( y_{PA}(p) - \tilde{y}_{CM}(p) < 0 \) for \( p_s < p < p_A \) by a same analysis with the last paragraph. In the second case, \( G(p) \) has at most one zero point in \( p_s < p < p_A \). When \( G(p) \) has no zero points, \( G(p) > 0 \) for \( p_s < p < p_A \), so that \( h(p,y) - H(p,y) > 0 \) due to \( p - y > 0 \). By the theory of differential inequalities, it directly follows from \( y_{PA}(p) < \tilde{y}_{CM}(p) \) that \( y_{PA}(p) - \tilde{y}_{CM}(p) < 0 \) for \( p_s < p < p_A \). When \( G(p) \) has a zero point, denoted by \( q \), we get \( h(p,y) - H(p,y) > 0 \) for \( p_s < p < q \) and \( h(p,y) - H(p,y) < 0 \) for \( q < p < p_A \). Thus \( y_{PA}(p) - \tilde{y}_{CM}(p) < 0 \) for \( q < p < p_A \). The first inequality of (2.12) also holds under (H5).

**Step 2.** We prove the uniqueness of crossing periodic orbits.

Assume that system (1.4) has two adjacent crossing periodic orbits \( \Gamma_1 \) and \( \Gamma_2 \). By theorem 1.1, both \( \Gamma_1 \) and \( \Gamma_2 \) surround \( O \) and \( E \). Moreover, it follows from step 1 and [10, theorem 2.1] that both \( \Gamma_1 \) and \( \Gamma_2 \) are stable and hyperbolic crossing limit cycles. Let \( \mathcal{A} \) be the open region surrounded by \( \Gamma_1 \) and \( \Gamma_2 \). Consider the \( \alpha \)-limit set \( L \) of the orbit of (1.4) with some initial value \( (x_0, y_0) \in \mathcal{A} \) having \( \Gamma_1 \) as the \( \omega \)-limit set. Similarly to the smooth case [11], by the Poincaré–Bendixson theorem in nonsmooth dynamical systems (see [2]) and the special structure of (1.4), \( L \) consists of an equilibrium \( (\bar{x}, p(\bar{x})) \) in \( \mathcal{A} \) and an unstable homoclinic orbit to \((\bar{x}, p(\bar{x})) \) which intersects the switching line \( x \)-axis. On the other hand, since (H1) holds and all crossing periodic orbits surround \( E \), we get \( \bar{x} < 0 \). That is, \((\bar{x}, p(\bar{x})) \) is an equilibrium of the left system. Thus, from (H2) we have \( \text{div}(F'(x) - y, g^-(x))|_{x=\bar{x}} = f'(\bar{x}) < 0 \). This contradicts that \( L \) is unstable by [8, theorem 1], which holds not only for hyperbolic saddles but also for semi-hyperbolic ones. That is, \( L \) cannot be \( \alpha \)-limit set of the orbit of (1.4) with the initial value \((x_0, y_0) \). Finally, we conclude that (1.4) has at most one crossing periodic orbit.

Combining with steps 1 and 2, we complete the proof. That is, system (1.4) has at most one crossing periodic orbit, which is stable and hyperbolic if it exists.

In the proof of theorem 1.2 a key technique for estimating the divergence integrals along the crossing limit cycle \( \Gamma \) is to construct two control functions, i.e., functions \( \varphi^+(p)/p \) and \( G(p) \) defined in (2.17) and (2.20). The first function is also used in [28, p 2126], where the limit cycles of the nonsmooth Liénard system (1.4) with a unique equilibrium were studied. But we only need that it is decreasing in the narrower interval \( 0 < p^* < p < p_A \) than the interval \( 0 < p < p_A \) required in [28], which helps us overcome the difficulty of comparing the heights of orbital arcs caused by the multiplicity of equilibria. The second function is new and we also only need that it is decreasing for \( p^* < p < p_A \), instead of \( 0 < p < p_A \).

**Proof of theorem 1.3.** As in the proof of theorem 1.1, by change \( p = p(x) \) we transform the right and left systems of (1.4) into the systems in (2.1) and then get differential equations in (2.2) for \( p > 0 \). The equations in (2.2) can be continuously extended to \( p = 0 \) by defining \( \varphi^+(0) = \varphi^- \) due to (1.9). Additionally, from (1.12) we have \( \varphi^+(p) \equiv \varphi^- \) for all \( p \geq 0 \), implying that the two equations in (2.2) coincide for \( p \geq 0 \). Hence, any orbit of the first differential equations in (2.2) going from a point in the negative \( y \)-axis to a point in the positive \( y \)-axis corresponds with a crossing periodic orbit of (1.4). Conversely, the existence of crossing periodic orbits of (1.4) ensures that the first differential equations in (2.2) have an orbit going from a point in the negative \( y \)-axis to a point in the positive \( y \)-axis. Consequently, if (1.4) has a crossing periodic orbit \( \Gamma \), then all orbits in the neighborhood of \( \Gamma \) are crossing periodic orbits by the continuous dependence of solutions on initial values.
3. Application to discontinuous piecewise linear systems

In this section we apply theorems 1.1–1.3 to study the number of crossing limit cycles for discontinuous system (1.13). In particular, the proof of theorem 1.4 will be presented.

According to [17], system (1.13) has no crossing limit cycles for \( a_2^+a_{12}^- \leq 0 \) because the \( x \)-component of both vector fields has same sign on crossing sets. For \( a_2^+a_{12}^+ > 0 \), it is proved in [17, proposition 3.1] that (1.13) is \( C^\infty \)-homeomorphic to the Liénard canonical form

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix}
t_R & -1 \\
d_R & 0
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} - \begin{pmatrix}
-b \\
a_R
\end{pmatrix} \quad \text{if } x > 0,
\begin{pmatrix}
t_L & -1 \\
d_L & 0
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} - \begin{pmatrix}
0 \\
a_L
\end{pmatrix} \quad \text{if } x < 0,
\tag{3.1}
\end{equation}

where \( t_{(R,0)} \) and \( d_{(R,0)} \) are the traces and determinants of \( A^\pm \),

\[
b = \frac{a_{12}^+}{a_{12}^-}b_1^+ - b_1^-,
\quad a_L = a_{12}^-b_2^- - a_{22}^-b_1^-,
\quad a_R = \frac{a_{12}^+}{a_{12}^-}(a_{12}^-b_2^+ - a_{22}^-b_1^+).
\]

Although (1.13) and (3.1) are not \( \Sigma \)-equivalent, there exists a topological equivalence for all their orbits without sliding segments as indicated in [17]. This means that crossing limit cycles of (1.13) are transformed into crossing limit cycles of (3.1) in a homeomorphic way. Therefore, we only need to consider (3.1) for studying the existence, uniqueness and number of crossing limit cycles of (1.13). It is worth mentioning that if system (3.1) without sliding sets, namely \( b = 0 \), has no equilibria, then it cannot have crossing periodic orbits because a crossing periodic orbit surrounds at least one equilibrium by [2].

**Theorem 3.1.** Assume that system (3.1) satisfies

\[
b = 0, \quad t_L < 0, \quad t_R > 0, \quad d_L > 0, \quad d_R > 0. \tag{3.2}
\]

(a) If \( a_R/t_R > a_L/t_L \), then a necessary condition for the existence of crossing periodic orbits is \( d_R/t_R^2 > d_L/t_L^2 \). In addition, if there exists a crossing periodic orbit, then it is unique and stable.

(b) If \( a_R/t_R < a_L/t_L \), then a necessary condition for the existence of crossing periodic orbits is \( d_R/t_R^2 < d_L/t_L^2 \). In addition, if there exists a crossing periodic orbit, then it is unique and unstable.

(c) If \( a_R/t_R = a_L/t_L \), then a necessary condition for the existence of crossing periodic orbits is \( d_R/t_R^2 = d_L/t_L^2 \). In addition, if there exists a crossing periodic orbit, then there exists a periodic annulus including this crossing periodic orbit.

**Proof.** Since \( b = 0 \), system (3.1) is exactly the nonsmooth Liénard system (1.4) satisfying

\[
F^+(x) = t_Rx, \quad f^+(x) = t_R, \quad g^+(x) = d_Rx - a_R,
\]
\[
F^-(x) = t_Lx, \quad f^-(x) = t_L, \quad g^-(x) = d_Lx - a_L. \tag{3.3}
\]

Clearly, it follows from (3.3) and \( d_R > 0 \) in (3.2) that (H1) holds by choosing \( x_e := 0 \) if \( a_R \leq 0 \) and \( x_e := a_R/d_R \) if \( a_R > 0 \). Moreover, (H2) holds because of (3.3) and \( t_R > 0 > t_L \) in (3.2). By the definitions of \( x^+(p) \) and \( x^-(p) \) given below (H2), we get \( x^+(p) = p/t_R \) and \( x^-(p) = p/t_L \). Moreover, for system (3.1) the equation (1.11) become
\[ t_L x^- = t_R x^+, \quad \frac{d_L x^- - a_L}{t_L} = \frac{d_R x^+ - a_R}{t_R}, \]  

(3.4)

If \( a_R / t_R > a_L / t_L \), then

\[
\lim_{p \to 0^+} \frac{g^+(x^+(p))}{f^+(x^+(p))} = \frac{a_R}{t_R} \quad \frac{a_L}{t_L} = \frac{g^-(x^-(p))}{f^-(x^-(p))},
\]

i.e., (H3) holds. Thus, by theorem 1.1 a necessary condition for the existence of crossing periodic orbits is that the equation (3.4) have solutions with \( x^- < 0 < x^+ \), which is equivalent to \( d_R / t_R^2 > d_L / t_L^2 \), because \( a_R / t_R > a_L / t_L \) and \( t_R > 0 \). On the other hand, if (3.1) has a crossing periodic orbit, then

\[
K^-(x^- (p_2)) = \frac{d_L}{t_L} < \frac{d_R}{t_R} = K^+(x^+ (p_1))
\]

for all \( p_1, p_2 \) satisfying \( p_2 > p_1 > 0 \). Thus, (H5) holds, where \( K^+(x^+(p)) \) are defined in (H5). By theorem 1.2, (3.1) has a unique crossing periodic orbit, which is stable. Therefore, conclusion (a) is proved.

If \( a_R / t_R < a_L / t_L \), applying the change

\[
(t, x, y, t_L, d_L, a_L, t_R, d_R, a_R) \rightarrow (-t, -x, y, -t_L, d_R, -a_R, -t_R, d_L, -a_L)
\]

(3.5)

to (3.1) we find that the form of (3.1) is invariant. Thus conclusion (b) is directly obtained from conclusion (a).

If \( a_R / t_R = a_L / t_L \), then

\[
\lim_{p \to 0^+} \frac{g^+(x^+(p))}{f^+(x^+(p))} = \frac{a_R}{t_R} = \frac{a_L}{t_L} = \lim_{p \to 0^+} \frac{g^-(x^-(p))}{f^-(x^-(p))}.
\]

Define

\[
\Lambda(p) := \frac{g^+(x^+(p))}{f^+(x^+(p))} - \frac{g^-(x^-(p))}{f^-(x^-(p))}
\]

for \( p > 0 \). For system (3.1), we get

\[
\Lambda(p) = \left( \frac{d_R}{t_R^2} - \frac{a_R}{t_R} \right) - \left( \frac{d_L}{t_L^2} - \frac{a_L}{t_L} \right) = \left( \frac{d_R}{t_R^2} - \frac{d_L}{t_L^2} \right) p.
\]

When \( d_R / t_R^2 < d_L / t_L^2 \), we have \( \Lambda(p) < 0 \) for \( p > 0 \) i.e., (H3) holds. Moreover, \( (0, 0) \) is the unique solution of equation (3.4). Hence, (3.1) has no crossing periodic orbits by theorem 1.1. When \( d_R / t_R^2 > d_L / t_L^2 \), by the changes \((t, x, y) \rightarrow (-t, -x, y)\) and (3.5), the nonexistence of crossing periodic orbits is directly obtained from the case of \( d_R / t_R^2 < d_L / t_L^2 \). Thus, \( d_R / t_R^2 = d_L / t_L^2 \) is a necessary condition for the existence of crossing periodic orbits. Then, if there exists a crossing periodic orbit, we have \( \Lambda(p) \equiv 0 \) for all \( p > 0 \), i.e., condition (1.12) of theorem 1.3 is satisfied. Therefore, there exists a periodic annulus including this crossing periodic orbit by theorem 1.3. Conclusion (c) is proved. \( \square \)

We remark that a similar result to theorem 3.1 is given in [28, theorem 4] for system (3.1) satisfying (3.2) and \( a_L > 0 > a_R \), which is not required in our theorem 3.1. So theorem 3.1
generalizes [28, theorem 4] and this generalization is crucial for us to prove theorem 1.4. We will see this in the proof of theorem 1.4 later.

**Lemma 3.1.** Assume that $b = 0$, $d_L, d_R \neq 0$ in system (3.1). Then there exist no crossing limit cycles if $t_L, t_R \geq 0$.

**Proof.** If $t_L, t_R \geq 0$ and $t_L + t_R \neq 0$, the result of no crossing limit cycles is obtained directly from [17, proposition 3.7]. If $t_L, t_R \geq 0$ and $t_L + t_R = 0$, i.e., $t_L = t_R = 0$, the equilibrium of the left (resp. right) system of (3.1) is either a center when $d_L > 0$ (resp. $d_R > 0$) or a weak saddle (the sum of two eigenvalues is zero) when $d_L < 0$ (resp. $d_R < 0$). By [27, theorems 2 and 4], system (3.1) has no crossing limit cycles.

Now we give a proof of theorem 1.4.

**Proof of theorem 1.4.** As indicated in the second paragraph of this section, we can equivalently consider system (3.1) to investigate the existence, uniqueness and number of crossing limit cycles of discontinuous system (1.13). Furthermore, it is easy to verify that (1.13) has no sliding sets if and only if (3.1) has no ones and that (1.13) is nondegenerate if and only if (3.1) is nondegenerate. Therefore, we only need to consider nondegenerate (3.1) without sliding sets. By the nonexistence of sliding sets and nondegeneracy, (3.1) satisfies $b = 0$ and $d_R d_L \neq 0$.

Totally there are 7 cases

(C1) $a_L = a_R = 0$, (C2) $a_L > 0 \geq a_R$,

(C3) $a_L < 0 \leq a_R$, (C4) $a_L > 0, a_R > 0$

and

(C5) $a_L = 0, a_R < 0$, (C6) $a_L = 0, a_R > 0$, (C7) $a_L < 0, a_R < 0$.

By the change

$$(x, y, t, t_L, d_L, a_L, t_R, d_R, a_R) \rightarrow (-x, -y, t, t_R, d_L, -a_R, t_L, d_L, -a_L),$$

(C5), (C6) and (C7) are transformed into (C2), (C3) and (C4), respectively. Thus, we only need to consider (C1), ..., (C4).

Assume that (3.1) satisfies (C1). Then (3.1) is continuous, where the definition of continuity is given below (1.4). It is proved in [19, corollary 3] that continuous (1.13) has at most one crossing limit cycle, so does (3.1).

Assume that (3.1) satisfies (C2). When $a_R = 0$ and $t_R^2 - 4d_R \geq 0$, the equilibrium of the right system lies in the switching line $y$-axis and it is neither focus nor center, implying that (3.1) cannot have crossing limit cycles. When either $a_R = 0, t_R^2 - 4d_R < 0$ or $a_L > 0 \geq a_R$, the origin $O$ is a $\Sigma$-monodromic singularity (see [29]), i.e., all orbits in a small neighborhood of $O$ turn around $O$. Thus (3.1) also has at most one crossing limit cycle by [29, theorem 1.1].

Assume that (3.1) satisfies (C3). When $d_L < 0$ or $d_R < 0$, at least one of the left and right systems is a saddle. Moreover, this saddle lies in $x > 0$ if it is of the left system and $x < 0$ if it is of the right one, so that (3.1) has no crossing limit cycles. When $d_R > 0, d_L > 0$ and $t_L, t_R \geq 0$, (3.1) also has no crossing limit cycles by lemma 3.1. When $d_R > 0, d_L > 0$ and $t_R < 0 < t_L$, (3.1) satisfies condition (3.2) in theorem 3.1. Thus (3.1) has at most one crossing limit cycle by theorem 3.1. When $d_R > 0, d_L > 0$ and $t_L < 0 < t_R$, by the change

$$(x, y, t, t_L, d_L, a_L, t_R, d_R, a_R) \rightarrow (x, -y, -t, -t_L, d_L, a_L, -t_R, d_R, a_R)$$

(3.6)
we obtain the uniqueness of crossing limit cycles from the case \( d_R > 0, d_L > 0, t_L < 0 < t_R \).

Assume that (3.1) satisfies (C4). Applying the change

\[
(t, x, y) \rightarrow \begin{cases} 
(t/a_R, x/a_R, y) & \text{for } x > 0, \\
(t/a_L, x/a_L, y) & \text{for } x \leq 0
\end{cases}
\]  

(3.7)

to (3.1), we obtain

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{cases}
\begin{pmatrix}
\frac{t_R}{a_R} - 1 & x \\
0 & 1
\end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } x > 0, \\
\begin{pmatrix}
\frac{t_L}{a_L} - 1 & x \\
0 & 1
\end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } x < 0.
\end{cases}
\]  

(3.8)

Observing that (3.8) is continuous, we know that (3.8) has at most one crossing limit cycle by [19, corollary 3] again. Finally, we conclude that (3.1) has at most one crossing limit cycle because (3.7) is a homeomorphism. In conclusion, nondegenerate and discontinuous (1.13) without sliding sets has at most one crossing limit cycle.

In the following we consider the number of equilibria in this unique crossing limit cycle. Suppose that (1.13) without sliding sets has a unique crossing limit cycle \( \Psi \). We denote the number of equilibria inside \( \Psi \) by \( N_{\text{inside}} \), the total number of admissible equilibria outside \( y \)-axis and pseudo-equilibria on \( y \)-axis by \( N_{\text{total}} \), the number of admissible saddles by \( N_{\text{saddle}} \). Here an equilibrium of (1.13) is admissible if it is either an equilibrium of the left system lying in \( x < 0 \) or an equilibrium of the right system lying in \( x > 0 \) as defined in [1, definition 5.1]. We claim that

\[
N_{\text{inside}} = N_{\text{total}} - N_{\text{saddle}}.
\]  

(3.9)

In fact, the pseudo-equilibrium lies at \( O \) if it exists and, hence, \( \Psi \) surrounds it. If there is an admissible equilibrium in \( x > 0 \), it is either a saddle or a focus or a center because system (1.13) with an admissible node has no crossing limit cycles as indicated in [27]. In the case of saddle, it is impossible for \( \Psi \) to surround this saddle, otherwise, \( \Psi \) intersects the separatrices of this saddle. In the case of focus or center, we easily get that \( \Psi \) surrounds this equilibrium by the orbit property of linear systems. Similarly, when there is an admissible equilibrium in \( x < 0 \), \( \Psi \) also surrounds this equilibrium if it is a focus or a center and excludes it if it is a saddle. Thus, (3.9) holds. Since \( N_{\text{total}} \) is at most 3 (corresponds to the case that one left equilibrium, \( O \) and one right equilibrium), we get that \( N_{\text{inside}} \) is at most 3.

In order to show the reachability of the unique crossing limit cycle and that \( N_{\text{inside}} \) maybe \( k \) (\( 1 \leq k \leq 3 \)), we consider the following example

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{cases}
\begin{pmatrix}
2 & -1 \\
2 & 0
\end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} & \text{if } x > 0, \\
\begin{pmatrix}
-4 & -1 \\
5 & 0
\end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 5\epsilon \end{pmatrix} & \text{if } x < 0.
\end{cases}
\]  

(3.10)

where \( \epsilon \in \mathbb{R} \) is a parameter. For the right system, its equilibrium \( E_1 := (1, 2) \) is an unstable focus of (3.10) and \( O \) is a visible tangency point, i.e., the orbit of the right system passing through \( O \) is tangent to \( x = 0 \) at \( O \) from the right side. Then the forward orbit of the right system starting from \((0, y_0)\) with \( y_0 \leq 0 \) evolves in the right half plane until it reaches again \( y \)-axis at a point \((0, y_1)\) with \( y_1 > 0 \) after a finite time \( t^+ > 0 \). Hence we define the right
Poincaré map $P_R(y_0) := y_1$. As completed in [15, 17], the parametric representation of $P_R$ is given by

\[
y_0(t^+) = \frac{e^{-t^+} - \cos t^+}{\sin t^+} + 1, \quad y_1(t^+) = -\frac{e^{-t^+} - \cos t^+}{\sin t^+} + 1
\]

for $t^+ \in (\pi, i^+]$ and

\[
P_R(0) = -2e^{i^+} \sin i^+ > 0, \quad \lim_{y_0 \to +\infty} P_R'(y_0) = -e^{i^+}, \quad (3.11)
\]

where $i^+ \in (\pi, 2\pi)$ satisfies $y_0(i^+) = 0$, i.e., the time for the orbit passing from $(0, 0)$ to $(0, P_R(0))$ in the right half plane.

For the left system, its equilibrium $E_2 := (\epsilon, -4\epsilon)$ is a stable focus. Moreover, $O$ is a boundary equilibrium if $\epsilon = 0$ and an invisible (resp. visible) tangency point if $\epsilon > 0$ (resp. $\epsilon < 0$), i.e., the orbit of the left system passing through $O$ is tangent to $x = 0$ at $O$ from right (resp. left) side. Then there exists $z_0 \geq 0$ such that the forward orbit of the left system starting from $(0, z_0)$ with $z_0 \geq z_0$ evolves in the left half plane and reaches again $y$-axis at a point of form $(0, z_1)$ with $z_1 \leq 0$ after a finite time $i^- > 0$. Choosing $z_0$ as the minimum one, we define the left Poincaré map $P_L(z_0) := z_1$. From [15, 17] again, the expression of $P_L$ is given by

\[
z_0(i^-) = \epsilon \left(\frac{e^{2i^-} - \cos i^- - 2 \sin i^-}{\sin i^-}\right), \quad z_1(i^-) = -\epsilon \left(\frac{e^{-2i^-} - \cos i^- + 2 \sin i^-}{\sin i^-}\right)
\]

if $\epsilon \neq 0$ and $P_L(z_0) = -e^{-2\pi z_0}$ if $\epsilon = 0$, where $i^- \in (\pi, i^-]$ (resp. $[0, \pi)$) for $\epsilon < 0$ (resp. $> 0$) and $i^- \in (\pi, 2\pi)$ satisfies $z_1(i^-) = 0$. In addition, we have

\[
\begin{align*}
\tilde{z}_0 &= \begin{cases} 
0 & \text{if } \epsilon > 0, \\
5\epsilon e^{2i^-} \sin i^- & \text{if } \epsilon < 0,
\end{cases} \\
\lim_{y_0 \to +\infty} P_L'(\tilde{z}_0) &= -e^{-2i^-}. \quad (3.12)
\end{align*}
\]

From the first equality of (3.11) and (3.12), there exists

\[
\epsilon_0 := \frac{2e^{i^+} \sin i^+}{5e^{2i^-} \sin i^-} < 0
\]

such that $\tilde{z}_0 = P_R(0)$ for $\epsilon = \epsilon_0$, $\tilde{z}_0 < P_R(0)$ for $\epsilon > \epsilon_0$ and $\tilde{z}_0 > P_R(0)$ for $\epsilon < \epsilon_0$. Let $P(y_0) := P_L(P_R(y_0))$. If $\epsilon > \epsilon_0$, then $P(y_0)$ is well defined for $y_0 \leq 0$, and $P(0) < 0$ due to $P_R(0) > 0$ and $P_L(\tilde{z}_0) = z_1(i^-) = 0$. If $\epsilon < \epsilon_0$, $P(y_0)$ is well defined for $y_0 \leq \tilde{y}_0$, where $\tilde{y}_0 < 0$ satisfies $P(\tilde{y}_0) = 0$. On the other hand, from the second equality of (3.11) and (3.12) we have

\[
\lim_{y_0 \to +\infty} P'(y_0) = \lim_{y_0 \to +\infty} P_L'(P_R(y_0)) \cdot P_R'(y_0) = e^{-i^+} < 1,
\]

implying that $P(y_0) > y_0$ for $y_0$ closed to $-\infty$. Therefore, $P(y_0)$ has a fixed point in $y_0 < 0$ if $\epsilon > \epsilon_0$, together with the first part of this proof, which corresponds to a unique crossing limit cycle of system (3.10), i.e., the reachability is proved. Notice that $y_0 = 0$ is also a fixed point of $P(y_0)$ for $\epsilon = \epsilon_0$, but it corresponds to a homoclinic orbit to $O$ because $O$ is a pseudo-saddle for $\epsilon < 0$, see figure 4(b). Moreover, it is worth mentioning that system (3.10) goes through a homoclinic bifurcation at $\epsilon = \epsilon_0$ as shown in figure 4.

For system (3.10) equilibrium $E_3$ of the left system is a boundary equilibrium if $\epsilon = 0$ and an admissible equilibrium if $\epsilon < 0$, but it is not an equilibrium if $\epsilon > 0$. Moreover, $O$ is a
regular point if $\epsilon > 0$, a boundary equilibrium if $\epsilon = 0$ and a pseudo-saddle if $\epsilon < 0$. Thus system (3.10) exactly has one equilibrium $E_1$ if $\epsilon > 0$, two equilibria $O(E_2)$ and $E_1$ if $\epsilon = 0$, and three equilibria $E_1$, $O$ and $E_2$ if $\epsilon < 0$. Joining the last paragraph, we eventually obtain that $N_{\text{inside}} = 1$ (resp. 2, 3) if $\epsilon > 0$ (resp. $\epsilon = 0$, $\epsilon_0 < \epsilon < 0$). The proof is completed. □

For system (3.1) with $b = 0$ and $d_L d_R \neq 0$ we give some information on the location of the unique crossing limit cycle in the following, where $d_L d_R \neq 0$ means that (3.1) is nondegenerate. Since there exist no crossing limit cycles in the case of $t_L t_R \geq 0$ by lemma 3.1, using (3.6) we only need to consider $t_L < 0 < t_R$.

(a) If $d_R > 0$ and $d_L > 0$, i.e., there is no saddle, it follows from theorem 3.1 that a necessary condition for the existence of crossing limit cycles is $\Delta_1 \Delta_2 > 0$, where

$$\Delta_1 := \frac{a_R}{t_R} - \frac{a_L}{t_L}, \quad \Delta_2 := \frac{d_R}{t_R} - \frac{d_L}{t_L}.$$ 

In the case of $\Delta_1 > 0$, hypotheses (H1)--(H3) hold as it was verified in the proof of theorem 3.1. Thus, by theorem 1.1 we obtain that the crossing limit cycle of system (3.1) intersects the two verticals $x = x^+_1$ and $x = x^-_1$, where

$$x^+_1 = \frac{1}{t_R} \Delta_1 > 0, \quad x^-_1 = \frac{1}{t_L} \Delta_1 < 0.$$ 

This provides a lower bound for the amplitude of the crossing limit cycle. In the case of $\Delta_1 < 0$, using (3.5) we obtain the same result. That is, the crossing limit cycle of system (3.1) intersects the two verticals $x = x^+_1$ and $x = x^-_1$.

(b) If $d_R < 0$, the equilibrium of the right system of (3.1) is a saddle, which lies in $x < 0$ for $a_R > 0$ and $x > 0$ for $a_R < 0$. Then (3.1) has no crossing limit cycles for $a_R > 0$ as said in [27], while for $a_R < 0$, the crossing limit cycle of (3.1) must lie in the half plane $x < a_R / d_R$ if it exists, where $a_R / d_R > 0$ is the abscissa of this saddle. Similarly, the crossing limit cycle of (3.1) lies in the half plane $x > a_L / d_L$ with $a_L / d_L < 0$ if $d_L < 0$. When $d_R < 0$ and $d_L < 0$, the crossing limit cycle of (3.1) lies in the band $a_L / d_L < x < a_R / d_R$ with $a_R / d_R > a_L / d_L$.

In conjecture 2 of [18, p 232], the investigated system is
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{cases}
\begin{pmatrix} 2\gamma_R & -1 \\ 1 + \gamma_R^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ a_R \end{pmatrix} & \text{if } x > 0, \\
\begin{pmatrix} 2\gamma_L & -1 \\ 1 + \gamma_L^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ a_L \end{pmatrix} & \text{if } x < 0
\end{cases}
\]  
(3.13)

with

\[
\gamma_L \gamma_R < 0, \quad a_L < 0 < a_R.
\]  
(3.14)

As completed in [18] (also in [15, 17]), we can define a left Poincaré map \( P_L : [y_0, +\infty) \) associated to (3.13) with (3.14). The parametric presentation can be given by

\[
y = \frac{a_L}{1 + \gamma_L^2} \frac{e^{-\gamma_L t} \varphi_L(t)}{\sin t}, \quad P_L(y) = -\frac{a_L}{1 + \gamma_L^2} \frac{e^{\gamma_L t} \varphi_L(-t)}{\sin t},
\]

where \( \varphi_L(t) = 1 - e^{\gamma_L t} \sin t \), \( y_0 = 0 \) if \( \gamma_L > 0 \) and \( y_0 = \hat{y} > 0 \) satisfies \( P_L(\hat{y}) = 0 \) if \( \gamma_L < 0 \). Moreover, we also can define a right Poincaré map \( P_R : (-\infty, z_0] \) and the parametric presentation is given by

\[
z = \frac{a_R}{1 + \gamma_R^2} \frac{e^{-\gamma_R t} \varphi_R(t)}{\sin t}, \quad P_R(z) = -\frac{a_R}{1 + \gamma_R^2} \frac{e^{\gamma_R t} \varphi_R(-t)}{\sin t},
\]

where \( z_0 = 0 \) if \( \gamma_R > 0 \) and \( z_0 = \hat{z} < 0 \) satisfies \( P_R(\hat{z}) = 0 \) if \( \gamma_R < 0 \). Define \( y^* := P_L(0) \) for \( \gamma_L > 0 \) and \( z^* := P_R(0) \) for \( \gamma_R > 0 \). Then conjecture 2 of [18, p 232] is exactly stated as follows.

For system (3.13) with (3.14), the following statements hold.

(a) If \( \gamma_L < 0 \) and \( (\gamma_L + \gamma_R)(\hat{y} - z^*) < 0 \), then there exist no crossing periodic orbits.

(b) If \( \gamma_L > 0 \) and \( (\gamma_L + \gamma_R)(\hat{z} - y^*) > 0 \), then there exist no crossing periodic orbits.

In the following we give a positive answer to conjecture 2 of [18, p 232]. In fact, we only prove statement (a) because statement (b) can be proved similarly. Let \( P(y) = P_R(P_L(y)) \). If \( \gamma_L < 0 \), then \( \gamma_R > 0 \) and \( P \) is well defined for \( y \in [\hat{y}, +\infty) \). In the case \( \hat{y} - z^* > 0 \), we have \( P(\hat{y}) = P_R(P_L(\hat{y})) = P_R(0) = z^* < \hat{y} \). Moreover, it follows from \( \hat{y} - z^* > 0 \) that \( \gamma_L + \gamma_R < 0 \), so that

\[
\lim_{y \to +\infty} P'(y) = e^{(\gamma_L + \gamma_R)y} < 1
\]

as obtained in [18, p 231]. Thus \( P(y) < y \) in a neighborhood of the point at infinity, which implies that \( P(y) \leq y \) for all \( y \in [\hat{y}, +\infty) \) in the case \( \hat{y} - z^* > 0 \) because of the uniqueness of crossing limit cycles given in theorem 1.4. Similarly, in the case \( \hat{y} - z^* < 0 \), we can get \( P(\hat{y}) = P_R(P_L(\hat{y})) = P_R(0) = z^* > \hat{y} \) and \( P(y) > y \) in a neighborhood of the point at infinity, so that \( P(y) \geq y \) for all \( y \in [\hat{y}, +\infty) \) by theorem 1.4 again. Hence, if \( P(y) \) has a fixed point in \( [\hat{y}, +\infty) \), then it must be semi-stable whatever \( \hat{y} - z^* > 0 \) or \( \hat{y} - z^* < 0 \). Then the crossing periodic orbit associated to this fixed point is also semi-stable, which implies a contradiction because the unique crossing limit cycle of (3.13) is either stable or unstable from theorem 3.1. Thus, statement (a) holds.

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