1. Introduction

The list [1] of homogeneous symmetric special Kähler manifolds \( S \) [2–6] contains two infinite series \( CP_{n-1,1} = SU(1,n)/SU(n) \otimes U(1), \ SK(n+1) = SU(1,1)/U(1) \otimes SO(2,n)/SO(2) \otimes SO(n) \) and four exceptional cases including in particular the manifold:
\[
M_{3,3} = SU(3,3)/SU(3) \otimes SU(3) \otimes U(1).
\]

In a recent paper [7], the special geometry of the \( SK(n+1) \) manifolds and the associated \( SL(2,Z) \times SO(2,n,Z) \) automorphic superpotentials have been constructed for any \( n \). In the present letter we extend this analysis to the case of the manifold \( M_{3,3} \), that is the Teichmüller covering of the moduli space for the \( T^6/Z_3 \) orbifold. In this way we exhaust the analysis of automorphic superpotentials for the \( Z_3 \) orbifold compactification of superstrings. Indeed, comparing with the list appearing in ref. [8] we see that, with the exception of \( T^6/Z_3 \), all the other (untwisted) orbifold moduli spaces correspond to special values of \( n \) in the \( SK(n+1) \) series.

Our goal is to exhibit the special geometry of \( M_{3,3} \) and to construct the appropriate infinite sum defining the \( SU(3,3,Z) \) automorphic superpotential.

As for any other special manifold \( S \), the special geometry of \( M_{3,3} \) is encoded in a homogeneous of degree two holomorphic function \( F(X) \). More precisely, for any \( S \), we can construct a (holomorphic) section \( \Omega \equiv (X^\Lambda, i\partial_\Lambda F(X)) \) [4–6] of the symplectic \( Sp(2\dim S+2,R) \) bundle over \( S \), from which we extract the Kähler geometry of \( S \). In particular the Kähler potential is given by:
\[
G = -\log \| \Omega \|^2 \equiv -\log \left( -i \langle \bar{\Omega}|\Omega \rangle \right) \equiv -\log \left( \bar{X}^\Lambda \partial_\Lambda F + \bar{\partial}_\Lambda F X^\Lambda \right),
\]
(1.2)
where
\[
\langle \bar{\Omega}|\Omega \rangle = \Omega^\dagger \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \Omega
\]
is the norm of the symplectic section. The F function proposed in ref. [2], via supergravity considerations, for the case of \( M_{3,3} \) is of the form:
\[
F(X) \sim i \frac{\det X}{X^0},
\]
(1.3)
where \( X \) is a three by three matrix. Equation (1.3) is just a particular case of the general formula \( F(X) = id_{\Lambda\Sigma\Delta} \frac{X^\Lambda X^\Sigma X^\Delta}{X^0} \), where the \( d_{\Lambda\Sigma\Delta} \) are constant coefficients, valid for any \( S \) in the list of homogeneous symmetric special Kähler manifolds, except \( CP_{n-1,1} \).

As pointed out in ref. [7] one should be able to derive systematically the F function from the embedding of \( S \) into \( Sp(2\dim S+2,R) \). The action of the \( S \) isometry group \( G \) of \( S \) on the section \( \Omega \), under the appropriate symplectic embedding, induces the right duality transformations [9, 10] on the section, giving a recipe to calculate \( \partial_\Lambda F \) as a function of \( X \), and reducing the problem to the solution of an ordinary first–order differential equation. In this letter we show that, following this procedure, we get, in a rigorous way, a symplectic section \( \Omega \) correspondig to the F function (1.3).
Let $\Gamma$ denote the (target) space modular group of $S_{[11,12]}$. From the embedding of $S$ into $Sp(2 \dim S + 2, R)$ we retrieve the embedding of $\Gamma$ into $Sp(2 \dim S + 2, Z)$. Using the general formula proposed in ref. [8], we are able to construct the $\Gamma$ automorphic function (the superpotential) for the case in the title, writing it as a sum over integers describing a modular lattice. This formula is explicitly recalled in the next section (see eq. (2.11))

2. Construction of the section $\Omega$ for $M_{3,3}$

We start our programme by writing the coset representative of the manifold $M_{3,3} = \frac{G}{H}$ in projective coordinates [10]:

$$M = \begin{pmatrix} (1 - ZZ^\dagger)^{-\frac{1}{2}} & (1 - ZZ^\dagger)^{-\frac{1}{2}} Z \\ Z^\dagger (1 - ZZ^\dagger)^{-\frac{1}{2}} + Z^\dagger (1 - ZZ^\dagger)^{-1} Z \end{pmatrix}$$

(2.1)

where $Z$ is a complex $3 \times 3$ matrix. Let us denote by $A$ the $6 \times 3$ matrix given by

$$A = ((1 - ZZ^\dagger)^{-\frac{1}{2}}, (1 - ZZ^\dagger)^{-\frac{1}{2}} Z),$$

(2.2)

where the indices of $A^I_i$ run as follows: $I = (i, i^*)$; $i, i^* = 1, 2, 3$ ($i$ corresponds to the plus signs of the metric and $i^*$ to the minus signs). Following the general procedure discussed in [7, 10] we have to embed $G$ into the symplectic group of dimension $(9 + 1) \times 2 = 20$. If we consider the isometry group $G = SU(3,3)$, it is easily recognized that the three–index antisymmetric representation of $G$ has the required dimension. Hence let us define:

$$t^{IJK} = \epsilon^{ijk} A^I_i A^J_j A^K_K.$$

(2.3)

The three index antisymmetric tensor $t^{IJK}$ is acted on by the matrix $B = U^I_i U^J_j U^K_K$, where $U \in SU(3,3)$. One can verify that:

$$U^T C U = C,$$

(2.4)

where the matrix $C$ satisfies $C^T = -C$, $C^2 = -1$, and can be viewed as acting on the triplet $IJK$ as the Levi Civita symbol: $[C t]^{IJK} = \epsilon^{IJKLMN} t_{LMN}$. Moreover one has

$$U^\dagger E U = E,$$

(2.5)

where $E^2 = 1$, $E^\dagger = E$ and where $E$ acts on the three–index tensors as $E = \eta_{LM} \eta_{N'M'} \eta_{K'}$ ($\eta = (+, +, +, -,$ $-, -)$), antisymmetrized with respect to $L', M', N'$. Equations (2.4) and (2.5) show that $t^{IJK}$ realizes a symplectic embedding of $G$ into a symplectic group of dimension 20. However, due to the signature of the "metric" $E$ this group is $Usp(10,10)$ rather than $Sp(20,R)$. The use of a generalized Cayley matrix allows us to transform the $Usp(10,10)$ representation into the real symplectic one. Explicitly we set:

$$T^{LMN} = \epsilon^{ijk} A^I_i C^L_L A^J_j C^M_M A^K_K C^K_K = \hat{C} t,$$

(2.6)
\[ C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \times 3 \\ i \times 3 & i \times 3 \end{pmatrix} \]

is the “Cayley matrix” in the 3 + 3 space (i.e. in the fundamental representation of SU(3, 3)) and  \( \hat{C} \) as defined by equation (2.6), is the generalized Cayley matrix in the 20 space (i.e. in the three index antisymmetric representation of SU(3, 3)). Note that both \( C \) and \( \hat{C} \) are defined up to a phase. The three–index representation is now written as (if we drop an overall \( \frac{1}{\sqrt{2}} \) factor)

\[ T_{ijk} = \det \frac{1 + Z}{(1 - ZZ^\dagger)^{\frac{1}{2}}} \varepsilon_{ijk} \]

\[ T^i_{*jk} = \varepsilon^{ijk} \det \frac{1 + Z}{(1 - ZZ^\dagger)^{\frac{1}{2}}} \left( \frac{i(Z - 1)}{Z + 1} \right)_i \]

\[ T^{ij}_{*k} = \det \frac{1 + Z}{(1 - ZZ^\dagger)^{\frac{1}{2}}} \varepsilon^{irs} \left( \frac{i(Z - 1)}{Z + 1} \right)_i \left( \frac{i(Z - 1)}{Z + 1} \right)_r \]

\[ T^{*i}_{*j} = \det \frac{1 + Z}{(1 - ZZ^\dagger)^{\frac{1}{2}}} i(Z - 1) \]

where we adopt the convention: \( \det M_{3 \times 3} = \frac{1}{3!} \varepsilon_{ijk} \varepsilon_{i*_{j*} k*} M_i^{i*} M_j^{j*} M_k^{k*} \). Equation (2.7) gives the explicit form of the \( Sp(20, R) \) section \( \Omega \), and each line of such an equation has to be interpreted as \( X^A \) or as \( i \partial_A F(X) \). If we divide by the overall factor \( \det(1 + Z) \), which gives, toghether with its antiholomorphic counterpart, a real function contribution to the Kähler potential, we can read the first 10 components of \( \Omega \) as the elements of the matrix \( X^A_{i*} = [\frac{i(Z - 1)}{Z + 1}]_{i*} \) together with \( X^0 = 1 \). By recalling the definition \( L^A = e^\frac{i}{2} G X^A \), and recalling that for \( M_{3,3} \) the Kähler potential \( G \) is expressed by [10]

\[ G(Z, Z^\dagger) = - \log \det(1 - ZZ^\dagger), \]

we obtain from (2.7) the following identifications:

\[ T^{i*}_{*j} = \varepsilon^{ijk} L^{0} \]

\[ T^{i*}_{*j} = \varepsilon^{ijk} L^{i*} \]

\[ T^{ij}_{*k} = \varepsilon^{ijk} \frac{\partial}{\partial L^i} \left( - \frac{\det L}{L^0} \right) = \varepsilon^{ijk} \frac{\partial}{\partial L^i} (iF(L)) \]  

\[ T^{i*}_{*j} = \varepsilon^{ijk} \frac{\partial}{\partial L^0} \left( - \frac{\det L}{L^0} \right) = \varepsilon^{ijk} \frac{\partial}{\partial L^0} (iF(L)). \]

The last two eq.s (2.9) have to be interpreted as differential equations satisfied by the \( F \) function. These are solved by:

\[ F(L) = i \frac{\det L}{L^0} \]

(2.10)
the same expression, of course, holds for \( F(X) \)). If we introduce the “special coordinates” \( S_i^* = L_i^*/L^0 = X_i^*/X^0 \), we immediately recover the standard expression of the Kähler potential \( G(S, \bar{S}) = -\log \det(S - \bar{S}) \), which is explicitly obtained from (1.2) and which coincides with the \(-\log \det(1 - ZZ^\dagger)\), while posing \( S = \frac{(Z - 1)}{Z + 1} \) (modulo the real part of a holomorphic function).

Our next step consists in searching an explicit formula for an automorphic superpotential in the case of \( T^6/Z_3 \). To this scope we briefly recall the general definition given in ref. \[8\]

\[
\log ||\Delta||^2 = \log \left[ ||\Delta||^2 e^G \right] = \left[ -\sum_{(M, \Sigma, N\Sigma) \in \Lambda} \log \left| \frac{M\Sigma X^\Sigma + iN\Sigma \partial\Sigma F}{X^\Sigma \partial\Sigma F + X^\Sigma \partial\Sigma F} \right|^2 \right]_{\text{reg}}, \tag{2.11}
\]

where the integers \((M, N)\) belong to a homogeneous lattice \( \Lambda_\Gamma \) associated with the target space modular group \( \Gamma \in Sp(2 + 2n, \mathbb{Z}) \), where \( \Gamma \) corresponds to a suitable definition of \( SU(3, 3, \mathbb{Z}) \) (see next section). The only difficult point, to make formula (2.11) explicit, is to find the explicit parametrization of the capital integers \((M, N)\) in terms of the small integers \( n \) spanning the Narain lattice for the \( T^6/Z_3 \) orbifold.

In particular, for our case, we need the formula relating 20 “integers” \( M^{IJK} \) in the three-times antisymmetric representation of \( Sp(2, R) \) (or equivalently \( Usp(10, 10) \)) to the “integers” \( l^I \) in the fundamental 6–dimensional representation of \( SU(3, 3) \). We write “integers” in quotes because both \( M^{IJK} \) and \( l^I \) are not integers, rather they are complex numbers parametrized by a double number of integers (real and imaginary parts). The solution of this problem is given by splitting the (complexified) momentum lattice of \( T^6/Z_3 \) into three conjugacy classes, and by constructing the symplectic integers in terms of the integers belonging to these classes. Let us give in some detail the analysis of the momentum lattice and its behaviour under the modular group \( SU(3, 3, \mathbb{Z}) \).

3. The momentum lattice of \( T^6/Z_3 \) orbifold and the modular group \( SU(3, 3, \mathbb{Z}) \)

Following a well–established literature we define a \( T^{2n}/Z_N \) orbifold via a two–step process \([13–15]\). First we introduce a 2n dimensional torus \( T^{2n} \) by identifying points in \( R^{2n} \) with respect to the action of a lattice group \( \Lambda_R \):

\[
X^\mu \sim X^\mu + v^\mu \; ; \; v^\mu \in \Lambda_R \tag{3.1}
\]

and we define \( T^{2n}/Z_N \) by identifying points in \( T^{2n} \) with respect to the action of a point group \( \mathcal{P} \sim Z_N \) that acts cristallographically on the lattice \( \Lambda_R \) and that is a subgroup of \( SO(2n) \):

\[
(\Theta X)^\mu \sim X^\mu \; ; \; \Theta \in SO(2n) \tag{3.2a}
\]

\[
(\Theta v)^\mu \in \Lambda_R \; \text{if} \; v^\mu \in \Lambda_R. \tag{3.2b}
\]
In the case of $T^6/Z_3$, the standard choice of $\Lambda_R$ corresponds to $\Lambda_R = RA_2 \otimes RA_2 \otimes RA_2$, where $RA_2$ is the root lattice of the simply laced Lie algebra $A_2$ [14]. In this way one easily obtains an $SO(6)$ rotation matrix $\Theta$, which maps $\Lambda_R$ into itself and such that $\Theta^3 = 1$.

This construction has been discussed in the literature [13, 14] but we need to recall it here. Indeed we have to illustrate the properties of the momentum lattice we shall utilize in the derivation of the $SU(3,3,\mathbb{Z})$ modular group and of the coefficients $M^{IJK}$.

We begin by introducing a complex structure in $R^6$. This is done by substituting three complex coordinates $Z^i$ to the six real coordinates $X^\mu$ via the relation:

$$X^\mu = Z^i e_i^\mu + \bar{Z}^{i^*} e_{i^*}^\mu$$

(3.3)

$(i, i^* = 1, 2, 3)$, where $\{e_i^\mu, e_{i^*}^\mu\}$ is a basis of six complex, linear, independent vectors fulfilling the conditions [16]:

$$e_i^{\mu*} = e_i^\mu, \quad \quad (3.4a)$$

$$(e_i, e_j) = (e_{i^*}, e_{j^*}) = 0 \quad (e_i, e_{j^*}) = g_{ij^*} \quad (3.4b)$$

In (3.4) the scalar product $(,)$ is defined with respect to some constant metric $g_{\mu\nu}$ with $(+,+,+,-,+,-)$ signature:

$$(v, w) = v^\mu w^\nu g_{\mu\nu} \quad (3.5)$$

The Hermitian form:

$$g_{ij^*} = (e_i, e_{j^*}) = g_{j^*i} \quad (3.6)$$

defines a Hermitian metric in $R^6$ equipped with the complex structure (3.3):

$$g_{\mu\nu}X^\mu X^\nu = 2Z^i \bar{Z}^{i^*} g_{ij^*} \quad (3.7)$$

The torus $T^6$ is obtained by setting the following identification of points in $R^6$ [14]:

$$Z^i = Z^i + (n^i + \Theta m^i)\sqrt{2} \quad (3.8)$$

where $n^i, m^i \in \mathbb{Z}$

$$\Theta = e^{2\pi i/3} \quad (3.9)$$

Equation (3.8) corresponds to the modding by a lattice $\Lambda_R = RA_2 \otimes RA_2 \otimes RA_2$, as claimed at the beginning. Indeed for the algebra $A_2$ a system of simple roots is given by the two-dimensional vectors:

$$\alpha_1 = (\sqrt{2}, 0) = \sqrt{2} \quad \alpha_2 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right) = \sqrt{2}e^{2\pi i/3} \quad (3.10)$$

so that an element of the root lattice $RA_2$ can be represented by the following complex number:

$$n\alpha_1 + m\alpha_2 = \sqrt{2}(n + m\Theta) \quad (n, m \in \mathbb{Z}) \quad (3.11)$$
The dual-weight lattice $W A_2$ is spanned by the simple weights:

$$\lambda_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\right) = \sqrt{\frac{2}{3}} e^{i \pi / 3}$$  \hspace{1cm} (3.12a)

$$\lambda_2 = \left(0, \sqrt{\frac{2}{3}}\right) = \sqrt{\frac{2}{3}} e^{i \pi / 2}$$  \hspace{1cm} (3.12b)

and a generic element of this lattice is represented by the following complex number:

$$p \lambda_1 + q \lambda_2 = \sqrt{\frac{2}{3}} (p \omega_1 + q \omega_2),$$  \hspace{1cm} (3.13)

where $\omega_1 = e^{i \pi / 3}$ and $\omega_2 = e^{i \pi / 2}$.

The metric $g_{ij^*}$ defined by eq. (3.6) enters, together with an antisymmetric two-form $B_{ij^*}$, the two-dimensional $\sigma$ model action on the $T^6$ torus [12,15]:

$$S = \int d^2 \xi \partial_\alpha Z^i \partial^{\alpha} \bar{Z}^{j^*} (g_{ij^*} + B_{ij^*}).$$  \hspace{1cm} (3.14)

The nine complex parameters encoded in the complex $3 \times 3$ matrix

$$M_{ij^*} = g_{ij^*} + B_{ij^*},$$  \hspace{1cm} (3.15)

parametrize the orbifold $T^6 / Z_3$ moduli space whose special Kähler geometry we have described in the previous section. For a generic $T^6$ torus we would have 36 moduli corresponding to an arbitrary $g_{\mu\nu}$ metric and an arbitrary $B_{\mu\nu}$ two-form. On the contrary, for the orbifold, we just have the freedom of choosing $g_{ij^*}$ and $B_{ij^*}$, since the complex structure (3.3) cannot be deformed. Indeed in addition to the identification (3.8) under the lattice group $\Lambda_R$ we also have the identification under the point group $Z_3$. The generator $\Theta$ of $Z_3$ acts on the complex coordinates $Z^i$ as a multiplication by $\Theta$:

$$Z^i \sim \Theta Z^i = e^{2 i \pi / 3} Z^i.$$  \hspace{1cm} (3.16)

Equations (3.8) and (3.16) are compatible just because $\Theta$ acts crystallographically on the lattice $\Lambda_R$. Indeed:

$$\Theta \sqrt{2} (n^i + m^i) = \sqrt{2} (n^{i'} + \Theta m^{i'}),$$  \hspace{1cm} (3.17)

where $n^{i'} = -m^i$, $m^{i'} = n^i - m^i$, which follows from

$$\Theta^2 = e^{4 i \pi / 3} = -1 - \Theta.$$  \hspace{1cm} (3.18)

The momentum lattice is introduced in the usual way by considering the plane waves $\exp(i P_i X^i)$ and demanding that they are single-valued on the torus $T^6$ with the complex structure (3.3). This implies:

$$P_\mu = g_{\mu\nu} (P_i e^{i \nu} + \bar{P}_i e^{i \nu})$$  \hspace{1cm} (3.19a)

$$P_i = \sqrt{\frac{2}{3}} (p_i \omega_1 + q_i \omega_2) \hspace{1cm} (p_i, q_i \in \mathbb{Z}),$$  \hspace{1cm} (3.19b)
where \( \{ e^{i\nu}, e^{i^*\nu}\} \) \( i, i^* = 1, 2, 3 \) form the dual basis to the basis (3.4). Following a standard procedure the winding modes can be included into the momentum lattice, which becomes the Lorentzian 12 dimensional Narain lattice \( \Lambda_W \) with signature \( g_{\bar{\mu}\bar{\nu}} = \text{diag}(+,-,+,-,+,-,-,−,-,−,−) \). In complete analogy to Eqs. (3.19), one writes:

\[
P_{\bar{\mu}} = g_{\bar{\mu}\bar{\nu}}(P_I e^{I\bar{\nu}} + \bar{P}_I e^{I^*\bar{\nu}}) \quad (I = 1, \cdots 6)
\]

\[
P_I = \sqrt{\frac{2}{3}}(p_I \omega_1 + q_I \omega_2) \quad (p_I, q_I \in \mathbb{Z}),
\]

where \( e^{I\bar{\nu}}, e^{I^*\bar{\nu}} \) are the basis vectors of the Narain lattice \( \Lambda_W \), and \( P_{\bar{\mu}} \) its elements. We have:

\[
(e^I, e^J) = 0 \quad (e^{I^*}, e^{J^*}) = 0
\]

\[
(e^I, e^{J^*}) = g^{IJ^*}.
\]

The metric \( g^{IJ^*} \) is a Hermitian metric with signature \( \text{diag}(+,-,+,-,−,-,−) \); hence the sesquilinear form \( v^\dagger g w \) is invariant against the transformations of a group isomorphic to \( SU(3,3) \). This is the origin of the \( SU(3,3) \) symmetry discussed in the previous section. Its role is clarified by considering the level–matching condition in the Narain lattice [13] (i.e. the equality of the left and right masses):

\[
0 = P_{\bar{\mu}} P_{\bar{\nu}} g_{\bar{\mu}\bar{\nu}} = \frac{2}{3} g^{IJ^*} (p_I p_J + q_I q_J) + p_I q_J).
\]

Equation (3.22) follows upon straightforward substitution of eq. (3.19) and (3.13) into \( P_{\bar{\mu}} P_{\bar{\nu}} g_{\bar{\mu}\bar{\nu}} \). By means of a similarity transformation, the metric \( g^{IJ^*} \) could be reduced to the standard \( SU(3,3) \) metric \( \eta^{IJ^*} = \text{diag}(+,-,+,-,−,-,−) \). Indeed there exists a non–singular \( 6 \times 6 \) matrix \( \Omega \) such that:

\[
g^{IJ^*} = (\Omega^\dagger \eta \Omega)^{IJ^*}.
\]

Consider now the matrix \( S \) given by:

\[
S = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 1 \\
\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}
\end{pmatrix},
\]

where \( 1 \) is the unit matrix in the six dimensions. Equation (3.22) can be rewritten as follows:

\[
0 = u^T g u + v^T g v
\]

where

\[
\begin{pmatrix}
u \\
 u
\end{pmatrix} = S \begin{pmatrix}
p \\
 q
\end{pmatrix}.
\]
The quadratic form (3.25) is the standard $SO(6, 6)$ invariant form. The elements of $SO(6, 6)$ have the generic form:

$$A = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$  \hspace{1cm} (3.27)$$

where the $6 \times 6$ blocks fulfil the following conditions:

$$A^T g A + C^T g C = g$$
$$A^T g B + C^T g D = 0$$
$$B^T g B + D^T g D = g.$$  \hspace{1cm} (3.28)

In the $(u, v)$ basis the $Z_3$ generator $\Theta$ is given by the matrix:

$$\Theta_{(u, v)} = \begin{pmatrix} \cos \left( \frac{2\pi}{3} \right) & -\sin \left( \frac{2\pi}{3} \right) \\ \sin \left( \frac{2\pi}{3} \right) & \cos \left( \frac{2\pi}{3} \right) \end{pmatrix}.$$  \hspace{1cm} (3.29)

The normalizer of $Z_3$ in $SO(6, 6)$ is the group $SU(3, 3)$. It is composed of those matrices that have the special form:

$$A = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$  \hspace{1cm} (3.30a)

with:

$$A^T g A + B^T g B = 0 \quad ; \quad A^T g B = B^T g A.$$  \hspace{1cm} (3.30b)

In the $(p, q)$ basis the $Z_3$ generator is integer–valued:

$$\Theta_{(p, q)} = S^{-1} \Theta_{(u, v)} S = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},$$  \hspace{1cm} (3.31)

a proof that $Z_3$ acts crystallographically also on the Narain lattice. In the torus compactification the modular group is $\Gamma(T^6) = SO(6, 6, Z)$, namely the subgroup of $SO(6, 6)$ that maps the Narain lattice into itself. In the orbifold case the modular group $\Gamma(T^6/Z_3)$ is the subgroup of of $SU(3, 3)$ that maps the Narain lattice into itself. We name this group $SU(3, 3, Z)$ and we easily identify its elements. In the $(p, q)$ basis an $SU(3, 3)$ element is obtained from eq (3.30) via conjugation with the matrix $S$. We get:

$$A_{(p, q)} = S^{-1} A S = \begin{pmatrix} A - \frac{B}{\sqrt{3}} & -2 \frac{B}{\sqrt{3}} \\ 2 \frac{B}{\sqrt{3}} & A + \frac{B}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} H & H - K \\ -H + K & K \end{pmatrix},$$  \hspace{1cm} (3.32)

where we have set:

$$H = A - \frac{B}{\sqrt{3}} \quad ; \quad K = A + \frac{B}{\sqrt{3}}.$$  \hspace{1cm} (3.33)
In terms of the blocks $H, K$ the conditions (3.30b) become:

$$K^T gK = H^T gH$$

(3.34)

$$H^T gH + K^T gK - \frac{1}{2} K^T gH - \frac{1}{2} H^T gK = g .$$

(3.35)

The group $SU(3, 3, Z)$ is obtained by demanding that $H, K$ should be integer–valued: $H_{IJ}, K_{IJ} \in Z$. Since $\det A(p,q) = \det A = 1$, this condition is compatible with the group structure and $SU(3, 3, Z)$ is well defined. Equivalently we can say that the group $SU(3, 3, Z)$ is composed by all the pseudo–unitary $6 \times 6$ matrices $\mathcal{U}$:

$$\mathcal{U}^T g \mathcal{U} = g ; \quad \det \mathcal{U} = 1$$

(3.36)

that have the special form:

$$\mathcal{U} = \frac{i}{2}(K + H) + \frac{i}{\sqrt{3}} \frac{\sqrt{3}}{2}(K - H),$$

(3.37)

$K, H$ being integer–valued matrices. In this case eq.s (3.34) and (3.35) follow from insertion of (3.37) into (3.36). The matrices $\mathcal{U}$ have the property that acting on complex vectors of the form:

$$l^I = \frac{1}{\sqrt{2}} p^I + \frac{i}{\sqrt{6}} (p^I + 2q^I) p^I, q^I \in Z$$

(3.38)

map them into complex vectors of the same form. Equations (3.37), (3.38) are the final parametrization of the $SU(3, 3, Z)$ modular group and of the Narain lattice for the $T^6/Z_3$ orbifold. They are the starting point for the construction of the $M^{IJK}$ coefficients appearing in the automorphic superpotential formula.

4. Construction of the $M^{IJK}$ coefficients

Naïvely $l^I$ are in the six of $SU(3, 3)$ while $M^{IJK}$ are in the three–times antisymmetric representation, where we are considering the $Usp(10, 10)$ symplectic group. The only possibility of constructing $M^{IJK}$ out of the $l^I$ momenta would be:

$$M^{IJK} = [l^I l^J l^K]$$

which unfortunately is zero! The way out of this riddle results from the properties of the Narain weight lattice $\Lambda_W$ which, while modded with respect to its root sublattice $\Lambda_R \subset \Lambda_W$, splits into three conjugacy classes that are separately invariant under the action of $SU(3, 3, Z)$. Working in the $(p, q)$ basis (related to the actual momenta $l^I$ via eq. (3.38)) we define $\Lambda_R$ as the sublattice, where $p, q$ have the form:

$$p^I = 2n^I - m^I$$

$$q^I = 2m^I - n^I ,$$

(4.1)
where \( n^I, m^I \in \mathbb{Z} \). This definition is inspired by the relation between simple roots and simple weights in the \( A_2 \) case:

\[
\alpha_1 = 2\lambda_1 + \lambda_2 \quad ; \quad \alpha_2 = -\lambda_1 + 2\lambda_2 ,
\]

so that

\[
n\alpha_1 + m\alpha_2 = (2n - m)\lambda_1 + (2m - n)\lambda_2 .
\]

Equation (4.1) is equivalent to the condition:

\[
\frac{1}{3}(p^I - q^I) \in \mathbb{Z} \quad (4.4)
\]

or

\[
n^I = \frac{1}{3}(2p^I + q^I) \in \mathbb{Z} \quad (4.5a)
\]

\[
m^I = \frac{1}{3}(p^I + 2q^I) \in \mathbb{Z} \quad (4.5b)
\]

An important result is the following:

**Lemma:** The modular group \( SU(3,3,\mathbb{Z}) \) maps the root sublattice \( \Lambda_R \) into itself.

This follows straightforwardly from eq.(3.32)

\[
\frac{1}{3}(p' - q') = \frac{1}{3}[Hp + (H - K)q + (K - H)p - Kq] = H\frac{1}{3}(2p + q) - K\frac{1}{3}(2q + p) \quad (4.6)
\]

If condition (4.4) (implying (4.5)) is fulfilled by \((p,q)\), the same condition is fulfilled by the transformed \((p',q')\). We can now write the complete Narain lattice \( \Lambda_W \) as the sum of three sublattices

\[
\Lambda_W = \Lambda_0 + \Lambda_1 + \Lambda_2 , \quad (4.7)
\]

where \( \Lambda_0 = \Lambda_R \) is the already defined root lattice while \( \Lambda_1 \) and \( \Lambda_2 \) are defined below:

**Definition:** Let \( \alpha = 1, 2 \). A vector \((p,q) \in \Lambda_W\) belongs to \( \Lambda_\alpha \subset \Lambda_W \) if and only if there exists an integer–valued non–zero six–vector \( x^I \in \mathbb{Z} \) such that:

\[
\left( \frac{1}{3}(p^I - q^I) + \frac{\alpha}{3}x^I \right) \in \mathbb{Z} . \quad (4.8)
\]

The reason why the above is a good definition and why (4.7) is a good decomposition is the following. For each value of the index \( I \) the difference \( p^I - q^I \) can be 0, 1, 2 mod 3. The root lattice is that sublattice such that \( p^I - q^I = 0 \) mod 3 for all the values of \( I \). \( \Lambda_1 \) is composed by those vectors such that \( p^I - q^I = 0 \) mod 3 for some values \( I \) (at least one value) and \( p^I - q^I = 0 \) mod 3 in all the other cases. An analogous definition is given for \( \Lambda_2 \). Since we have exhausted all the possibilities, any vector \((p,q) \in \Lambda_W\) can be written as the sum of a vector in \( \Lambda_0 \) plus a vector in \( \Lambda_1 \), plus a vector in \( \Lambda_2 \). We have now the following:
Theorem: The lattices $\Lambda_\alpha$ are invariant under the action of the modular group $SU(3, 3, Z)$.

Proof: Using the definitions (4.5) for each $(p, q) \in \Lambda_\alpha$ we can write:

$$n^I = \frac{1}{3}(2p^I + q^I) = \bar{n}^I + \frac{\alpha}{3}x^I \quad (4.9a)$$

$$m^I = \frac{1}{3}(2q^I + p^I) = \bar{m}^I - \frac{\alpha}{3}x^I, \quad (4.9b)$$

where $\bar{n}^I \bar{m}^I \in Z$. Under the action of $SU(3, 3, Z)$ we get:

$$\frac{1}{3}(p' - q') = Hn - Km = H\bar{n} - \bar{m} + \frac{\alpha}{3}(H + K)x; \quad (4.10)$$

since $H\bar{n} - K\bar{m} \in Z$ it follows that:

$$\frac{1}{3}(p' - q') + \frac{\alpha}{3}x' \quad (4.11)$$

where

$$x' = -(H + K)x. \quad (4.12)$$

Therefore, provided $x' \neq 0$, the image of a vector in $\Lambda_\alpha$ is still in $\Lambda_\alpha$. On the other hand $x'$ cannot be zero. Indeed if $x'$ were zero then the image of $(p, q) \in \Lambda_\alpha$, under the $SU(3, 3, Z)$ group element $\gamma$ we consider, would be in $\Lambda_0$. Consider now the inverse group element $\gamma^{-1}$: we obtain $\gamma^{-1}\gamma(p, q) = (p, q) \in \Lambda_\alpha$, with $\alpha = 1, 2$. This would imply that the image of the $\Lambda_0$ element $\gamma(p, q)$ under the $SU(3, 3, Z)$ transformation $\gamma^{-1}$ is not in $\Lambda_0$, contrary to the lemma we have shown. Hence $x' \neq 0$ and the theorem is proved.

Relying on this theorem we can now conclude the construction of the $M^{IJK}$ coefficients. Extending the index $\alpha$ to the value $\alpha = 0$ corresponding to the root sublattice we can set:

$$M^{IJK} = \epsilon^{\alpha\beta\gamma}l^I_\alpha l^J_\beta l^K_\gamma, \quad (4.13)$$

where the $l^I_\alpha \in \Lambda_\alpha$ is given in terms of $p^I, q^I$ by eq. (3.38). The final formula for the automorphic superpotential (where we are considering the $Usp(10, 10)$ representation) is encoded in the following $\zeta$–function regularization:

$$\log{|\Delta|^2} e^G = -\lim_{s \to 0} \frac{d}{ds}\zeta(s)$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \sum_{l^I_\alpha \in \Lambda_\alpha} e^{-it|M^{IJK}t_{IJK}|^2} \quad (4.14)$$

As can be seen, by summing independently on the three sublattices $\Lambda_\alpha$ we are actually summing on $\Lambda_W$. The coefficients (4.13) transform as a three–index antisymmetric representation of $SU(3, 3, Z)$ because of our theorem. Utilizing the embedding of $SU(3, 3)$ into $Usp(10, 10)$ (or via Cayley in $Sp(20, R)$) we discussed in section 2 we also
see that $SU(3, 3, Z)$ is a suitable discrete subgroup of $Usp(10, 10)$ and that $M^{IJK}$ span the corresponding 20 dimensional symplectic representation of this modular group.

5. Conclusions

In this paper we studied the special Kähler geometry of the manifold $T^6/Z_3 = SU(3, 3)/SU(3) \otimes SU(3) \otimes U(1)$ utilizing the symplectic embedding technique introduced in a previous publication [7]. In this way we retrieved the cubic prepotential $F(X)/(X^0)^2$ already discussed in the literature. Next we studied the target space duality group of the $T^6/Z_3$ orbifold. This group acts as a discrete isometry on the special Kähler moduli space and is isomorphic to a discrete subgroup of $SU(3, 3)$ which we called $SU(3, 3, Z)$. Therefore the moduli space of the $T^6/Z_3$ orbifold is the special Kähler orbifold $SU(3, 3)/SU(3) \otimes SU(3) \otimes U(1)/SU(3, 3, Z)$.

The group $SU(3, 3, Z)$ acts in a $Usp(10, 10)$ symplectic way on the orbifold lattice, whose momenta and winding numbers can be classified in the threefold antisymmetric representation of $SU(3, 3, Z)$ where the three different conjugacy classes of this discrete group are involved in the tensor product. This symplectic action is crucial to define a duality–invariant automorphic function via a $\zeta$ function regularization of the determinant of a “mass operator” on the $T^6/Z_3$ orbifold [7,8].
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ON THE MODULI SPACE OF THE $T^6/Z_3$ ORBIFOLD
AND ITS MODULAR GROUP *

Sergio Ferrara
CERN - Theory Division
CH - 1211 Geneva 23, Switzerland

Pietro Frè and Paolo Soriani
SISSA - International School for Advanced Studies
Via Beirut 2, I-34100 Trieste, Italy
and
I.N.F.N. sezione di Trieste
Area di Ricerca
Padriciano 99, 34012, Trieste

ABSTRACT

We describe the duality group $\Gamma = SU(3,3,\mathbb{Z})$ for the Narain lattice of the $T^6/Z_3$ orbifold and its action on the corresponding moduli space $M_{3,3}/\Gamma$, where $M_{3,3} = SU(3,3)/SU(3)\otimes SU(3)\otimes U(1)$. A symplectic embedding of the momenta and winding numbers allows us to connect the orbifold lattice to the special geometry of $M_{3,3}$. As an application, a formal expression for an automorphic function, which is a candidate for a non-perturbative superpotential, is given.

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