OPTIMAL HARVESTING FOR AGE-STRUCTURED POPULATION DYNAMICS WITH SIZE-DEPENDENT CONTROL

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ABSTRACT. We investigate two optimal harvesting problems related to age-dependent population dynamics; namely we consider two problems of maximizing the profit for age-structured population dynamics with respect to a size-dependent harvesting effort. We evaluate the directional derivatives for the cost functionals. The structure of the harvesting effort is uniquely determined by its intensity (magnitude) and by its area of action/distribution. We derive an iterative algorithm to increase at each iteration the profit by changing the intensity of the harvesting effort and its distribution area. Some numerical tests are given to illustrate the effectiveness of the theoretical results for the first optimal harvesting problem.

1. Setting of the problems. There is an extensive mathematical literature devoted to the optimal harvesting of age-structured population dynamics. Various methods are used to search an optimal harvesting effort (control) in a certain space of functions depending on time and age, to derive its structure or to approximate it. A quite general harvested age-structured population dynamics is described by the following McKendrick model

\[
\begin{aligned}
\frac{\partial p(a,t)}{\partial t} + \frac{\partial p(a,t)}{\partial a} + \mu(a,t)p(a,t) + M(\int_0^A p(a,t)da)p(a,t) &= -u(a,t)p(a,t), & (a,t) &\in (0, A) \times (0, T) \\
p(0,t) &= \int_0^A \beta(a,t)p(a,t)da, & t &\in (0, T) \\
p(a,0) &= p_0(a), & a &\in (0, A),
\end{aligned}
\]

(1)

where \( A \in (0, +\infty) \) is the maximal age for the population species and \( T \in (0, +\infty) \) is the final moment of the harvesting process. Here \( \beta(a,t) \) is the fertility rate, \( \mu(a,t) \) is the mortality rate for individuals of age \( a \) at the moment \( t \), and \( p_0(a) \) is the initial density of the population of age \( a \). \( M(\int_0^A p(a,t)da)p(a,t) \) is a logistic term.

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\( \mathcal{M} \left( \int_0^A p(a,t) da \right) \) being an additional mortality rate, and is due to the competition for resources. Function \( u \) is a control and plays the role of an additional mortality rate, and is called “harvesting effort”; the quantity \( \int_0^T \int_0^A u(a,t)p(a,t) da \, dt \) represents the total harvest. When trying to implement a given harvesting effort \( u(a,t) \) we need to be able to establish the age of individuals. We, however are trying to do this by investigating the size of individuals. Actually, we notice that the size of an individual is a nondecreasing function of age (see Figure 1), strictly increasing on an age-interval \([0,a_0]\) and constant on \([a_0,A]\) \((a_0 \in (0,A))\).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{size_graph.png}
\caption{The size of an individual as a function of age \(a\).}
\end{figure}

It means that when the size of an individual is less than the size corresponding to age \(a_0\) (the maximal size) we may identify its age uniquely, and when the size is equal to the size corresponding to \(a_0\), then we only may say that its age is between \(a_0\) and \(A\). Hence, for individuals of age between \(a_0\) and \(A\) the harvesting effort has to be the same (because we cannot distinguish their age). What we can vary (in time) is the intensity of the harvesting effort.

Assume now that we are actually interested in the industrial fishing (fish harvesting). We may vary at any moment \(t\) the intensity of the harvesting \(w(t)\). For any given moment \(t\) we harvest at the same rate the individuals with a size superior to a certain value (which obviously corresponds to ages superior to a certain value \(\alpha(t) \in [0,a_0]\)). Hence, here is the form of a realistic harvesting effort

\[
 u(a,t) = w(t)H(a - \alpha(t)), \tag{2}
\]

where \(H\) is the Heaviside function. Obviously, \(\alpha : [0,T] \rightarrow [0,a_0]\) is a function (whose graph looks as in Figure 2) to be determined, and the intensity of the harvest is a function \(w : [0,T] \rightarrow [0,L]\) (where \(L \in (0,+\infty)\) is the maximal affordable intensity of the harvesting effort) and has to be determined as well.

Our goal is to maximize the total harvest/yield. Hence, we have to determine the optimal intensity of the harvesting effort \(w^*\), and the optimal \(\alpha^*\) (or the optimal area where the control should act; see Figure 2). We may view this as a
shape optimization problem. Here is the first optimal harvesting problem to be investigated

\[(OH) \quad \text{Maximize} \int_0^T \int_0^A w(t)H(a - \alpha(t))p(a, t)da \, dt,\]

subject to \(w \in W = \{v \in L^\infty(0, T); 0 \leq v(t) \leq L \text{ a.e.}\}\) (\(L \in (0, +\infty)\) is a constant) and \(\alpha \in A = \{\gamma \in L^\infty(0, T); 0 \leq \gamma(t) \leq a_0 \text{ a.e.}\}\), where \(p\) is the solution to (1) corresponding to \(u\) given by (2).

\[\begin{array}{c}
\begin{array}{c}
\text{Figure 2. } \alpha \text{ as a function of time } t. \text{ The hashed region is the area where the control acts.}
\end{array}
\end{array}\]

Here are the hypotheses to be used

\[(H1) \quad \beta \in C([0, A] \times [0, T]), \mu \in C([0, A] \times [0, T]),\]

\[\beta(a, t) \geq 0, \quad \forall (a, t) \in [0, A] \times [0, T], \quad \mu(a, t) \geq 0, \quad \forall (a, t) \in [0, A] \times [0, T];\]

\[(H2) \quad p_0 \in C([0, A]), p_0(a) \geq 0, \text{ for any } a \in [0, A];\]

\[(H3) \quad \mathcal{M} : [0, +\infty) \to [0, +\infty) \text{ is continuously differentiable, } \mathcal{M}(0) = 0, \mathcal{M}'(r) \geq 0, \text{ for any } r \geq 0,\]

\[\lim_{r \to +\infty} \mathcal{M}(r) = +\infty.\]

We denote by

\[J(w, \alpha) = \int_0^T \int_0^A w(t)H(a - \alpha(t))p^{w, \alpha}(a, t)da \, dt\]

(the cost functional at \((w, \alpha) \in W \times A\), where \(p^{w, \alpha}\) is the solution to (1), with \(u\) given by (2)). We shall say indistinctly that \((w, \alpha)\) or \(u(a, t) = w(t)H(a - \alpha(t))\) is the control. For any \((w, \alpha) \in W \times A\) there exists a unique solution \(p^{w, \alpha}\) to (1), where \(u(a, t) = w(t)H(a - \alpha(t))\). By a solution to (1) we mean a function \(p \in L^\infty((0, A) \times (0, T))\), absolutely continuous along almost any characteristic line.
(of equation \(a - t = \text{const.}\), and such that

\[
\begin{align*}
Dp(a,t) + \mu(a,t)p(a,t) + \mathcal{M}(\int_0^A p(a,t)da)p(a,t) &= -u(a,t)p(a,t), \quad \text{a.e.} \ (a,t) \in (0, A) \times (0,T) \\
\lim_{\varepsilon \to 0^+} p(\varepsilon, t + \varepsilon) &= \int_0^A \beta(a,t)p(a,t)da, \quad \text{a.e.} \ t \in (0,T) \\
\lim_{\varepsilon \to 0^+} p(a + \varepsilon, \varepsilon) &= p_0(a), \quad \text{a.e.} \ a \in (0,A).
\end{align*}
\]

Here \(Dp\) is given by

\[
Dp(a,t) = \lim_{\varepsilon \to 0} \frac{p(a + \varepsilon, t + \varepsilon) - p(a,t)}{\varepsilon}
\]

(\(Dp\) is a directional derivative). Since the solution to (1) is absolutely continuous along almost any characteristic line, the conditions

\[
\lim_{\varepsilon \to 0^+} p(\varepsilon, t + \varepsilon) = \int_0^A \beta(a,t)p(a,t)da
\]
a.e. \(t \in (0,T)\) and

\[
\lim_{\varepsilon \to 0^+} p(a + \varepsilon, \varepsilon) = p_0(a)
\]
a.e. \(a \in (0,A)\), are meaningful. It is important to notice that the solution \(p\) satisfies the first equation in (1) along the characteristic lines. In order to prove the existence and uniqueness of a solution to (1) we integrate along the characteristic lines and use the Banach fixed point result. For the definition of a solution to (1) and for other age-dependent systems, and for the existence, uniqueness, and other basic properties of such solutions we refer to Chapter 2 in [2].

Sometimes, we modify as time increases the value of \(\alpha\) (recalibrate the harvesting devices) and we have to pay a certain cost which is proportional to the length of \(\Gamma_\alpha\), where \(\Gamma_\alpha = \{(t, \alpha(t)); \ t \in [0,T]\}\). If we assume that \(\alpha\) is smooth, then the cost to be paid for controlling/acting in the hashed area (see Figure 2) is

\[
k \cdot \text{length}(\Gamma_\alpha) = k \int_0^T (1 + \alpha'(t)^2)^{\frac{1}{2}} dt,
\]

where \(k \in (0, +\infty)\) is a constant. In this situation a reasonable optimal harvesting problem is the following one

\((\text{OH2})\) Maximize \(\int_0^T \int_0^A w(t)H(a - \alpha(t))p(a,t)da \ dt - k \int_0^T (1 + \alpha'(t)^2)^{\frac{1}{2}} dt\),

subject to \(w \in W = \{v \in L^\infty(0,T); \ 0 \leq v(t) \leq L \text{ a.e.}\} \) and \(\alpha \in \mathcal{A}_2 = \{\gamma : [0,T] \to [0,a_0]; \ \gamma \text{ is a smooth function}, \ 0 \leq \gamma(t) \leq a_0, \ \forall t \in [0,T]\}\), where \(p\) is the solution to (1) corresponding to \(u\) given by (2).

For both optimal harvesting problems we shall evaluate the directional derivatives. For the first optimal harvesting problem we will derive an iterative algorithm to improve at each step the intensity of the harvesting effort \(w\) and the area where the control acts (the hashed region in Figure 2).

Note that some particular harvesting problems related to structured population dynamics have been studied by several authors; see [1]–[5], [7], [8], [10], [11], [14]–[18], [20], [24]–[26], [28], [29]. For a large class of optimal control problems related to biological models see [22], [23]. For basic results and methods in age-dependent population dynamics we refer to [2], [21], [27]. The reader may find some other non-standard control problems (e.g. controllability problems) related to age-dependent
population dynamics in [19] (see also the references therein). We have also to mention some interesting approaches for optimal harvesting problems with measures in [9], [12], [13].

Our present paper is organized as follows. Section 2 is devoted to the evaluation of the directional derivative of the cost functional for problem (OH). A numerical iterative algorithm to improve at each step the harvesting effort is derived in section 3. Some numerical tests are given as well. A separable case for (OH) is treated in section 4. The evaluation of the directional derivative of the cost functional for problem (OH2) is the main goal of section 5. Some final comments are made.

2. The directional derivative of the cost functional for problem (OH). The main goal of this section is to calculate the directional derivative of J. Consider some arbitrary \( v \in L^\infty(0,T) \) and \( \zeta \in L^\infty(0,T) \) such that for any \( \theta > 0 \), sufficiently small, \( w(t) + \theta v(t) \in [0,L] \) and \( \alpha(t) + \theta \zeta(t) \in [0,a_0] \) a.e. \( t \in (0,T) \). We have that

\[
\frac{1}{\theta}[J(w + \theta v, \alpha + \theta \zeta) - J(w, \alpha)]
\]

\[
= \frac{1}{\theta} \int_0^T \int_0^A [(w(t) + \theta v(t))H(a - \alpha(t) - \theta \zeta(t))p^{w,\alpha}(a,t) - w(t)H(a - \alpha(t))p^{w,\alpha}(a,t)]da \, dt
\]

\[
\to \int_0^T \int_0^A w(t)H(a - \alpha(t))z(a,t)da \, dt + \int_0^T \int_0^A v(t)H(a - \alpha(t))p^{w,\alpha}(a,t)dt - \int_0^T w(t)\zeta(t)p^{w,\alpha}(a(t),t)dt,
\]

as \( \theta \to 0 \) (since \( p^{w,\alpha} \) is absolutely continuous along almost any characteristic line, \( p^{w,\alpha}(\alpha(t),t) \) in the last integral is meaningful), where \( z \) is the solution to

\[
\begin{cases}
\partial_t z(a,t) + \partial_a z(a,t) + \mu(a,t)z(a,t) + M\int_0^a p^{w,\alpha}(a,t)da z(a,t) \\
+ M'\int_0^a p^{w,\alpha}(a,t)da(\int_0^a z(a,t)da)p^{w,\alpha}(a,t) \\
- w(t)H(a - \alpha(t))z(a,t) - v(t)H(a - \alpha(t))p^{w,\alpha}(a,t) \\
+ w(t)\zeta(t)p^{w,\alpha}(a(t),t)\delta(a - \alpha(t)), \quad (a,t) \in (0,A) \times (0,T) \\
z(0,t) = \int_0^A \beta(a,t)z(a,t)da, \quad t \in (0,T) \\
z(a,0) = 0, \quad a \in (0,A).
\end{cases}
\]

(3)

Here \( \delta(a - \rho) \) denotes the Dirac mass at \( \rho \). By a solution to (3) we mean a function \( z \in L^\infty((0,A) \times (0,T)) \), that satisfies the first equation in (3) along almost any characteristic line (of equation \( a - t = \text{const.} \)), and such that

\[
\begin{align*}
\lim_{\varepsilon \to 0^+} z(\varepsilon, t + \varepsilon) &= \int_0^A \beta(a,t)z(a,t)da, \quad a.e. \ t \in (0,T) \\
\lim_{\varepsilon \to 0^+} z(a + \varepsilon, \varepsilon) &= 0, \quad a.e. \ a \in (0,A).
\end{align*}
\]

The existence and uniqueness of a solution \( z \) to (3) follow as in Chapter 2 in [2].
By a solution to (5) we mean a function \( q \) that solves the following problem

\[
\begin{align*}
\partial_t q(a, t) + \partial_a q(a, t) - \mu(a, t) q(a, t) &= -\beta(a, t) q(0, t) \\
+ M(\int_0^A p^{w, \alpha}(a, t) da) q(a, t) \\
+ M'(\int_0^A p^{w, \alpha}(a, t) da) \int_0^A p^{w, \alpha}(a, t) q(a, t) da \\
+ w(t) H(a - \alpha(t))(1 + q(a, t)), & \quad \text{a.e. } (a, t) \in (0, A) \times (0, T) \\
q(A, t) = 0, & \quad t \in (0, T) \\
q(a, T) = 0, & \quad a \in (0, A).
\end{align*}
\] (5)

By a solution to (5) we mean a function \( q \in L^\infty((0, A) \times (0, T)) \), absolutely continuous along almost any characteristic line, and such that

\[
Dq(a, t) - \mu(a, t) q(a, t) = -\beta(a, t) q(0, t) \\
+ M(\int_0^A p^{w, \alpha}(a, t) da) q(a, t) \\
+ M'(\int_0^A p^{w, \alpha}(a, t) da) \int_0^A p^{w, \alpha}(a, t) q(a, t) da \\
+ w(t) H(a - \alpha(t))(1 + q(a, t)), & \quad \text{a.e. } (a, t) \in (0, A) \times (0, T) \\
q(A, t) = 0, & \quad \text{a.e. } t \in (0, T) \\
q(a, T) = 0, & \quad \text{a.e. } a \in (0, A).
\]

By \( q(0, t) \) we mean

\[
q(0, t) = \lim_{\varepsilon \to 0^+} q(\varepsilon, t + \varepsilon)
\]

a.e. \( t \in (0, T) \). The existence and uniqueness of a solution to (5) follow as in Chapter 2 in [2].

If we multiply the first equation in (5) by \( z(a, t) \) and integrate over \((0, T) \times (0, A)\) we get that

\[
\begin{align*}
\int_0^T \int_0^A z(a, t)[Dq(a, t) - \mu(a, t) q(a, t)] da \, dt &= -\int_0^T \int_0^A \beta(a, t) q(0, t) z(a, t) da \, dt \\
+ \int_0^T \int_0^A z(a, t)[M(\int_0^A p^{w, \alpha}(a', t) da') q(a, t) \\
+ M'(\int_0^A p^{w, \alpha}(a', t) da') \int_0^A p^{w, \alpha}(a', t) q(a', t) da' + w(t) H(a - \alpha(t))(1 + q(a, t)))] da \, dt.
\end{align*}
\]

Integrating by parts and using (3) and (5) we obtain that

\[
\begin{align*}
-\int_0^T z(0, t) q(0, t) dt - \int_0^T \int_0^A q(a, t)[Dz(a, t) + \mu(a, t) z(a, t)] da \, dt \\
= -\int_0^T z(0, t) q(0, t) dt + \int_0^T \int_0^A z(a, t)[M(\int_0^A p^{w, \alpha}(a', t) da') q(a, t)
\end{align*}
\]

And...
By (3) we obtain after an easy calculation that

\[ M(\int_0^A p^{w,\alpha}(a',t)da') \int_0^A p^{w,\alpha}(a',t)q(a',t)da' + w(t)H(a-\alpha(t))(1+q(a,t))da \ dt. \]

By (4) and (6) we conclude that

\[ dJ(w,\alpha)(v,\zeta) = \int_0^T v(t) \int_\alpha(t) p^{w,\alpha}(a,t)(1+q(a,t))da \ dt 
- \int_0^T w(t)\zeta(t)p^{w,\alpha}(\alpha(t),t)(1+q(\alpha(t),t))dt. \]

As in [1] it follows that problem (OH) has at least one optimal control \((w^*,\alpha^*)\). Using (7) we may conclude that

**Theorem 2.1.** If \((w^*,\alpha^*)\) is an optimal control for (OH), and if \(p^* := p^{w^*,\alpha^*}\) and \(q^*\) is the solution to (5) corresponding to \(w := w^*\) and \(\alpha := \alpha^*\), then

\[
    w^*(t) = \begin{cases} 
    0, & \text{if } \int_0^A p^*(a,t)(1+q^*(a,t))da < 0 \\
    L, & \text{if } \int_0^A p^*(a,t)(1+q^*(a,t))da > 0
    \end{cases}
\]

and

\[
    \alpha^*(t) = \begin{cases} 
    0, & \text{if } w^*(t) > 0 \text{ and } p^*(\alpha^*(t),t)(1+q^*(\alpha^*(t),t)) > 0 \\
    a_0, & \text{if } w^*(t) > 0 \text{ and } p^*(\alpha^*(t),t)(1+q^*(\alpha^*(t),t)) < 0.
    \end{cases}
\]

Remark that if \(p^{w,\alpha}(a,t) > 0\) a.e. \((a,t) \in (0,A) \times (0,T)\), then (9) may be rewritten as

\[
    \alpha^*(t) = \begin{cases} 
    0, & \text{if } w^*(t) > 0 \text{ and } q^*(\alpha^*(t),t) > -1 \\
    a_0, & \text{if } w^*(t) > 0 \text{ and } q^*(\alpha^*(t),t) < -1.
    \end{cases}
\]

Notice that the time intervals when \(w(t) = 0\) correspond to prohibition periods for fishing.

3. **A numerical algorithm for problem (OH).** Based on (7) and Theorem 2.1, we develop a conceptual iterative algorithm to improve at each step the control \((w,\alpha)\), i.e. the intensity of the harvesting effort and the region where the control acts (in order to obtain a higher value for \(J\)).

**Step 0:** Set \(k := 0\) and \(J^{(0)}\) to a small value;
Initialize \(w^{(0)}(t), \alpha^{(0)}(t)\) and set \(u^{(0)}(a,t) = w^{(0)}(t)H(a-\alpha^{(0)}(t))\).

**Step 1:** Compute \(p^{(k+1)}\) the solution of (1) corresponding to \(u^{(k)}\);
Compute \(q^{(k+1)}\) the solution of (5) corresponding to \(w^{(k)}\) and \(\alpha^{(k)}\);
Evaluate the integral

\[ J^{(k+1)} = \int_0^T \int_0^A w^{(k)}(t)H(a-\alpha^{(k)}(t))p^{(k+1)}(a,t)da \ dt. \]

**Step 2:** If \(|J^{(k+1)} - J^{(k)}| < \varepsilon_1\) or \(J^{(k+1)} \leq J^{(k)}\) then **STOP**;
Step 5: Compute the new $\alpha^{(k+1)}$.

Evaluate $I(t) = \int_{a}^{A} p^{(k+1)}(a, t)(1 + q^{(k+1)}(a, t)) da$;

Compute $\hat{w}^{(k+1)}$ according to the formula (see (8))

$$\hat{w}^{(k+1)}(t) = \begin{cases} 0, & \text{if } I(t) < 0 \\ L, & \text{if } I(t) > 0 \\ w^{(k)}(t), & \text{if } I(t) = 0. \end{cases}$$

Define $\hat{w}_{\gamma}^{(k+1)}(t) = \gamma w^{(k)}(t) + (1 - \gamma) \hat{w}^{(k+1)}(t)$ for $\gamma \in [0, 1]$.

Compute $\gamma_* \in [0, 1]$ and respectively $\lambda_* \in [0, 1]$ the solution of the maximization problem

$$\text{Maximize } J(\hat{w}_{\gamma}^{(k+1)}, \alpha^{(k)}) \text{ subject to } \gamma \in [0, 1];$$

Set $w^{(k+1)}(t) = \hat{w}_{\gamma_*}^{(k+1)}(t)$.

Step 4: Compute $p^{(k+1)}$ the solution of (1) corresponding to $u(a, t) = w^{(k+1)}(t)H(a - \alpha^{(k)}(t))$;

Compute $q^{(k+1)}$ the solution of (5) corresponding to $w^{(k+1)}$ and $\alpha^{(k)}$.

Step 5: Compute the new $\alpha^{(k+1)}$.

Compute $\tilde{\alpha}^{(k+1)}$ according to the formula (see (9), (10))

$$\tilde{\alpha}^{(k+1)}(t) = \begin{cases} 0, & \text{if } w^{(k+1)}(t)p^{(k+1)}(\alpha^{(k)}(t), t)(1 + q^{(k+1)}(\alpha^{(k)}(t), t)) > 0 \\ a_0, & \text{if } w^{(k+1)}(t)p^{(k+1)}(\alpha^{(k)}(t), t)(1 + q^{(k+1)}(\alpha^{(k)}(t), t)) < 0 \\ \alpha^{(k)}(t), & \text{otherwise}. \end{cases}$$

Define $\tilde{\alpha}_{\lambda}^{(k+1)}(t) = \lambda \alpha^{(k)}(t) + (1 - \lambda) \tilde{\alpha}^{(k+1)}(t)$ for $\lambda \in [0, 1]$.

Compute $\lambda_* \in [0, 1]$ and respectively $\alpha_* \in [0, 1]$ the solution of the maximization problem

$$\text{Maximize } J(w^{(k+1)}, \tilde{\alpha}_{\lambda}^{(k+1)}) \text{ subject to } \lambda \in [0, 1];$$

Set $\alpha^{(k+1)}(t) = \tilde{\alpha}_{\lambda_*}^{(k+1)}(t)$.

Step 6: Compute the new control $u^{(k+1)}(a, t) = w^{(k+1)}(t)H(a - \alpha^{(k+1)}(t))$.

Step 7: If $\|u^{(k+1)} - u^{(k)}\| < \varepsilon_2$ then STOP.

Else $k := k + 1$; go to Step 1.

In Step 2 and Step 7, $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are prescribed convergence parameters, and $\| \cdot \|$ is the $L^2$-norm. The algorithm we have used is a gradient-type one. For details about the gradient methods see [6, §2.3].

The age interval $[0, A]$ and the time interval $[0, T]$ are approximated by two grids of $M$, and respectively $N$ equidistant nodes

$$a_i = (i - 1)h, \quad i = 1, 2, \ldots, M, \quad M = 1 + A/h$$

and

$$t_j = (j - 1)h, \quad j = 1, 2, \ldots, N, \quad N = 1 + T/h,$$

where $h > 0$ is the grid step.

System (1) (Step 1 and Step 4) is approximated by an Euler-type scheme, ascending with respect to time levels. Using the same idea we approximate system (5) (Step 1 and Step 4), but descending with respect to time levels. The integrals
from Step 1, Step 3, and Step 5 are numerically computed using the trapezoidal formula corresponding to the discrete grid.

For the numerical tests we use the following data: \( \beta(a,t) = Ba^2(A - a)(1 + \sin(\pi/Aa))(\sin(10t))^\gamma \), \( \mu(a,t) = \exp(-a)/(A - a) \), and \( p_0(a) = c_1(A - a) \), with \( c_1 > 0 \), \( a \in [0, A] \) and \( t \in [0, T] \). We also consider \( M(r) = c_2r \), with \( c_2 > 0 \). The fertility and mortality rates are represented in Figure 3.

We also take \( A = 1 \), \( T = 0.8 \), \( h = 0.005 \), \( L = 10 \), \( a_0 = 0.3 \), \( c_1 = c_2 = 1 \), and \( \varepsilon_1 = \varepsilon_2 = 0.000001 \).

For \( w^{(0)} = L/2 \), \( \alpha^{(0)} = 0.2 \) and \( B = 30 \), the algorithm ends in 14 iterations, when the second condition in Step 2 is fulfilled.

In Table 1 and in Figure 4 below, it can be seen that the algorithm provides a higher value for \( J \) at each iteration.

The control \( u \) corresponding to the last iteration is given in Figure 5(a). The section of the control \( u \) corresponding to the time level \( t = 0.6 \) can be seen in Figure 5(b).

We take another value for the constant \( B \) for the function \( \beta \), \( B = 75 \), \( w^{(0)} = 1 \), and \( \alpha^{(0)} = 0.2 \). The algorithm ends when the condition in Step 7 is fulfilled. The
Table 1. The value of $J$ at each iteration

| Iteration | $J$          |
|-----------|--------------|
| 1         | 0.464525126289841 |
| 2         | 0.533792098410522 |
| 3         | 0.545212800519842 |
| 4         | 0.552867826650825 |
| 5         | 0.556828710910306 |
| 6         | 0.558793583997659 |
| 7         | 0.559787107591890 |
| 8         | 0.560285396165302 |
| 9         | 0.560534749191512 |
| 10        | 0.560659455111737 |
| 11        | 0.560721812501533 |
| 12        | 0.560752991933830 |
| 13        | 0.560768581787776 |
| 14        | 0.560776376743360 |

Figure 4. The representation of $J$ as a function of iteration.

The control $u$ corresponding to the last iteration can be seen in Figure 6(a) and the section for the time level $t = 0.1$ in Figure 6(b). In this case, the highest value of $J$ is 0.7143.

It can be observed that the harvesting effort takes the maximum value $L$ when the fertility rate $\beta$ is approximatively 0. In the first test, at the beginning of the time interval, the harvest effort is 0. In second test, at the beginning of the time interval, the harvest effort is 0, for $a \leq a_0$, and $L$, for $a > a_0$. A similar behaviour can be seen at the end of the time interval for the both tests: $u$ is 0, for $a \leq 0.2$, and $L$, for $a > 0.2$ (0.2 is the initialization value for $\alpha$).
4. A separable case for (OH). An important particular case is obtained when we consider that $\alpha \equiv 0$, meaning that the harvesting effort is $w(t)$ (the intensity of fishing) and this shows that for $w(t) = 0$ we have prohibition for fishing and when $w(t) > 0$ the intensity of fishing does not depend on age (is the same for all ages).

Remark that the control is $u(a,t) = w(t)$ and the unique solution to (1) is separable and may be written as

$$p(a,t) = h(t)g(a,t), \quad (a,t) \in [0,A] \times [0,T],$$

where $g$ is the unique solution to

$$\begin{align*}
\partial_t g(a,t) + \partial_a g(a,t) + \mu(a,t)g(a,t) &= 0, \quad (a,t) \in (0,A) \times (0,T) \\
g(0,t) &= \int_0^A \beta(a,t)g(a,t)da, \quad t \in (0,T) \\
g(a,0) &= p_0(a), \quad a \in (0,A),
\end{align*}$$

and $h$ is the unique solution to

$$\begin{align*}
h'(t) + \mathcal{M}(G(t)h(t))h(t) &= -w(t)h(t), \quad t \in (0,T) \\
h(0) &= 1,
\end{align*}$$

where $G(t) = \int_0^A g(a,t)da$, for any $t \in [0,T]$. 

\begin{figure}[h]
\centering
\begin{miniroenv}{c}
\begin{miniroenv}{c}
\begin{miniroenv}{c}
(a) $u(a,t)$
\end{miniroenv}
\begin{miniroenv}{c}
(b) $u(a,t)$ for $t = 0.6$
\end{miniroenv}
\end{miniroenv}
\begin{miniroenv}{c}
\begin{miniroenv}{c}
\begin{miniroenv}{c}
(a) $u(a,t)$
\end{miniroenv}
\begin{miniroenv}{c}
(b) $u(a,t)$ for $t = 0.1$
\end{miniroenv}
\end{miniroenv}
\end{miniroenv}
\caption{The harvesting effort for Test 1.}
\end{figure}
We rediscover an optimal harvesting problem investigated in [4]:

\[(OHs) \quad \text{Maximize } \int_0^T w(t)h^w(t)G(t)dt,\]

subject to \(w \in W\). Here \(h^w\) is the solution to \((11)\). The cost functional \(\Phi\) is defined on \(W\) by

\[\Phi(w) = \int_0^T w(t)h^w(t)G(t)dt.\]

Let us evaluate the directional derivative for this function. Consider an arbitrary \(v \in L^\infty(0,T)\) such that for sufficiently small \(\theta > 0\) we have \(w(t) + \theta v(t) \in [0,L]\) a.e. \(t \in (0,T)\).

This yields

\[\frac{1}{\theta}[\Phi(w + \theta v) - \Phi(w)] = \frac{1}{\theta}\int_0^T ([w(t) + \theta v(t)]h^{w + \theta v}(t) - w(t)h^w(t))G(t)dt\]

\[\rightarrow \int_0^T w(t)y(t)G(t)dt + \int_0^T v(t)h^w(t)G(t)dt = d\Phi(w)(v),\]

as \(\theta \to 0\), where \(y\) is the solution to

\[\begin{align*}
y'(t) + M(G(t)h^w(t))y(t) + M'(G(t)h^w(t))G(t)y(t)h^w(t) \\
= -w(t)y(t) - v(t)h^w(t), & \quad t \in (0,T) \\
y(0) = 0.
\end{align*}\]

Let \(r\) be the solution to the following problem

\[\begin{align*}
r'(t) = M(G(t)h^w(t))r(t) + M'(G(t)h^w(t))G(t)r(t)h^w(t) \\
+ w(t)r(t) + G(t), & \quad t \in (0,T) \\
r(T) = 0.
\end{align*}\]

If we multiply the first equation in \((13)\) by \(y(t)\) and integrate over \([0,T]\) we get after an easy calculation (and using \((12)\)) that

\[\int_0^T w(t)y(t)G(t)dt = \int_0^T v(t)r(t)h^w(t)dt.\]

Using the form of \(d\Phi(w)(v)\) we obtain that

\[d\Phi(w)(v) = \int_0^T v(t)h^w(t)(r(t) + G(t))dt.\]

The existence of an optimal control \(w^*\) for (OHs) follows in a standard manner.

**Theorem 4.1.** If \(w^*\) is an optimal control for (OHs), and if \(h^* := h^{w^*}\) and \(r^*\) is the solution to \((13)\) corresponding to \(w := w^*\), then

\[w^*(t) = \begin{cases} 0, & \text{if } G(t) + r^*(t) < 0 \\ L, & \text{if } G(t) + r^*(t) > 0. \end{cases}\]

This implies that actually \(r^*\) is the solution to the following problem

\[\begin{align*}
r'(t) = M(G(t)h^*(t))r(t) + M'(G(t)h^*(t))G(t)r(t)h^*(t) \\
+ L(G(t) + r(t))^+, & \quad t \in (0,T) \\
r(T) = 0.
\end{align*}\]
5. The directional derivative of the cost functional for problem (OH2). Let

\[ J_2(w, \alpha) = \int_0^T \int_0^A w(t)H(a - \alpha(t))p^{w,\alpha}(a,t)da \ dt - k \int_0^T (1 + \alpha'(t)^2) \frac{1}{2} dt \]

(the cost functional at \((w, \alpha) \in W \times A\)). The main goal of this section is to calculate the directional derivative of \(J_2\). Consider some arbitrary \(v \in L^\infty(0, T)\) and smooth \(\zeta : [0, T] \to [0, a_0]\) such that for any \(\theta > 0\), sufficiently small, \(w + \theta v \in W\) and \(\alpha + \theta \zeta \in A\). This yields

\[
\frac{1}{\theta} [J_2(w + \theta v, \alpha + \theta \zeta) - J_2(w, \alpha)]
\]

\[
= \frac{1}{\theta} \left\{ \int_0^T \int_0^A \left[ (w(t) + \theta v(t))H(a - \alpha(t) - \theta \zeta(t))p^{w+\theta v,\alpha+\theta \zeta}(a,t) - w(t)H(a - \alpha(t))p^{w,\alpha}(a,t) \right] da \ dt \right\}
\]

\[
- \frac{k}{\theta} \int_0^T \left[ (1 + (\alpha'(t) + \theta \zeta'(t))^2) \frac{1}{2} - (1 + \alpha'(t)^2) \frac{1}{2} \right] dt
\]

\[
\rightarrow \int_0^T \int_0^A w(t)H(a - \alpha(t))z(a,t)da \ dt + \int_0^T \int_0^A v(t)H(a - \alpha(t))p^{w,\alpha}(a,t)dt
\]

\[
- \int_0^T w(t)\zeta(t)p^{w,\alpha}(\alpha(t), t)dt - k \int_0^T \frac{\alpha'(t)}{(1 + \alpha'(t)^2)^{\frac{3}{2}}} \zeta'(t)dt,
\]

as \(\theta \to 0\), where \(z\) is the solution to (3).

Hence,

\[
dJ_2(w, \alpha)(v, \zeta) = \int_0^T \int_0^A w(t)H(a - \alpha(t))z(a,t)da \ dt
\]

\[
+ \int_0^T \int_0^A v(t)H(a - \alpha(t))p^{w,\alpha}(a,t)da \ dt
\]

\[
- \int_0^T w(t)\zeta(t)p^{w,\alpha}(\alpha(t), t)dt
\]

\[
- k \int_0^T \frac{\alpha'(t)}{(1 + \alpha'(t)^2)^{\frac{3}{2}}} \zeta'(t)dt.
\]

Let \(q\) be the solution to (5). If we multiply the first equation in (5) by \(z(a,t)\) and integrate over \((0, T) \times (0, A)\) we get after a calculation similar to the one in section 2, and by an integration by parts, that

\[
dJ_2(w, \alpha)(v, \zeta) = \int_0^T v(t) \int_0^A p^{w,\alpha}(a,t)(1 + q(a,t))da \ dt
\]

\[
- \int_0^T w(t)\zeta(t)p^{w,\alpha}(\alpha(t), t)(1 + q(\alpha(t), t))dt
\]

\[
- k \int_0^T \frac{\alpha'(T)}{(1 + \alpha'(T)^2)^{\frac{3}{2}}} \zeta(T) + k \frac{\alpha'(0)}{(1 + \alpha'(0)^2)^{\frac{3}{2}}} \zeta(0)
\]

\[
+ k \int_0^T \frac{\alpha''(t)}{(1 + \alpha'(t)^2)^{\frac{3}{2}}} dt.
\]

(14)
As in [1] it follows that problem (OH2) has at least one optimal control \((w^*, \alpha^*)\). Let \(p^* := p^{w^*, \alpha^*}\) and \(q^*\) is the solution to (5) corresponding to \(w := w^*\) and \(\alpha := \alpha^*\). Using (14) we may conclude that

\[
    w^*(t) = \begin{cases} 
        0, & \text{if } \int_{A}^{0} p^*(a, t)(1 + q^*(a, t))da < 0 \\
        L, & \text{if } \int_{A}^{0} p^*(a, t)(1 + q^*(a, t))da > 0.
    \end{cases}
\]

We also get that the gradient ascendent with respect to \(\alpha\) (as in [5]) is

\[
    \frac{\partial \Psi(t, \theta)}{\partial \alpha} = k \frac{\partial_{\alpha} \alpha(t, \theta)}{(1 + \partial_{\alpha} \alpha(t, \theta))^{\frac{3}{2}}}.
\]

\[
    -w^*(t)p^*(\alpha(t, \theta), t)(1 + q^*(\alpha(t, \theta), t)) - \psi(\alpha(t, \theta)), \quad t \in (0, T), \theta > 0
\]

\[
    \partial_{\alpha} \alpha(T, \theta) + \psi(\alpha(T, \theta)) = \partial_{\alpha} \alpha(0, \theta) - \psi(\alpha(0, \theta)) = 0, \quad \theta > 0.
\]

Here \(\psi = \partial \Psi\) is the subdifferential of \(\Psi\), where

\[
    \Psi(r) = \begin{cases} 
        0, & \text{if } r \in [0, a_0] \\
        +\infty, & \text{if } r \in \mathbb{R} \setminus [0, a_0].
    \end{cases}
\]

**Remark** \(\alpha\) satisfies a Signorini boundary condition.

This allows to derive a conceptual algorithm to improve at any step \(w\) and \(\alpha\) in order to obtain a bigger value for \(J_2\) (as in [5]).

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