Effective actions, relative cohomology and Chern Simons forms

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Abstract

The explicit expression of all the WZW effective actions for a simple group \(G\) broken down to a subgroup \(H\) is established in a simple and direct way, and the formal similarity of these actions to the Chern–Simons forms is explained. Applications are also discussed.

1 Introduction

Recently it has been shown by D’Hoker and Weinberg [1, 2] that the most general effective action of Wess–Zumino–Witten (WZW) type, with a compact symmetry group \(G\) broken down to a subgroup \(H\), is given by the non-trivial de Rham cocycles on the homogeneous coset manifold \(G/H\), and a cohomological descent-like procedure has been used in [2] to obtain explicit expressions for the lower order examples. Motivated by this work [1, 2] and our own on the properties of symmetric invariant tensors on simple algebras [3], we look here at the problem of finding all the invariant effective actions of WZW type in terms of the cohomology of the Lie algebra \(G\) relative to a subalgebra \(H\) [4]. By exploiting this (equivalent) point of view, we are able to find a general formula for WZW type actions on \(G/H\) for any compact, connected and simply connected simple Lie group \(G\) (the case of semisimple \(G\) may be reduced to it) and for arbitrary spacetime dimensions.

The structure of phenomenological Lagrangians and nonlinear realizations was elucidated thirty years ago [5]. Their relation to the standard Wigner little group construction, which is the result of parametrising the coset \(K \equiv G/H = \{gH|g \in G\}\) in terms of the Goldstone coordinates \(\varphi^a (a = 1, \ldots, \dim K)\), was emphasised in [6]. Indeed, for the left action of a global transformation \(g \in G = KH\) on the coset space \(K\), \(g : \varphi^a \mapsto \varphi'^a\), we find

\[ gu(\varphi) = u(\varphi')h(g, \varphi) , \]

\(1.1\)

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where \( h(g, \varphi) \in H \) is simply an element of the ‘little group’ of the coset reference point. Geometrically, \( u(\varphi) \) (e.g., \( \exp \varphi a_T a \)) may be viewed as a section of the bundle \( G(H, K) \); since \( gu(\varphi) \) and \( u(\varphi') \) belong to the same fibre, eq. (1.1) follows\(^1\).

The key idea of the standard nonlinear realizations in general is that we may write
\[
u^{-1}du = (u^{-1}du)_H + (u^{-1}du)_K \equiv \Omega_H^a T_a + \Omega_K^a T_a \quad .
\]

As is well known, \([7, I, p.103]\) the \( H \)-valued component of the canonical left invariant (LI) form \( \omega \) on \( G \) determines a LI connection on \( G \); the first term in (1.2) is its pullback to \( K \) by the section \( u(\varphi) \), and hence transforms as a connection. In contrast, \( \Omega_K \) transforms tensorially through \( h(g, \varphi) \in H \) under a left transformation of \( G \), its elements operating linearly for \( H \subset G \) and nonlinearly for those in \( G \setminus H \). Hence any \( H \)-invariant expression made out of \( \Omega_K \) will also be invariant under the whole \( G \) and is a candidate for an invariant nonlinear Lagrangian. However, as emphasised in [1], there are also invariant actions which do not come from an invariant Lagrangian density. These are generically referred to as WZW actions \([8]\), also discussed in the context of nonlinear sigma models in \([9]\).

In Witten’s derivation \([8]\) of the simplest WZW term for \( G = SU(2) \) \((H=e)\) and \( D=2 \) spacetime \( M \sim S^2 \), the fields \( g(x) \) are extended by means of an interpolating field \( g(x, \lambda) \) \((g(x, 0) = 0, g(x, 1) = g(x)) \) and the WZW action is given by
\[
I_{WZW} = \int dB d^2x d\lambda \epsilon^{\mu \nu} \text{Tr}(g^{-1} \partial g / \partial \lambda W_{\mu} W_{\nu})
\]

where \( W_{\mu} = g^{-1} \partial_{\mu} g \) and \( \partial B = M \); the construction uses \( \pi_2(G) = 0 \) and \( \pi_3(G) = \mathbb{Z} \) (which hold for any simple Lie group). Similar considerations can be made for higher \( D \), where the existence of a WZW term requires in particular \([10]\) that there is a non-trivial \((D + 1)\)-cocycle (form on \( G \)) for the Chevalley-Eilenberg \([4]\) (CE) cohomology.

When \( H \neq e \), the mappings \( \varphi^a : M \rightarrow G/H \) are the Goldstone fields, suitable extended to the analogous of \( B \) above \([1]\). The construction of D’Hoker and Weinberg shows that the WZW actions (i.e., invariant actions associated with non-invariant spacetime Lagrangian densities) on the coset \( K \) are given by non-trivial De Rham cocycles on \( K \), i.e., by closed non-exact forms on \( G/H \). The result of [1, 2] may be reformulated by stating that the WZW actions are classified by the non-trivial cocycles of the relative algebra cohomology \( H_0(G, H; \mathbb{R}) \) (for \( H = e, H_0(G, e; \mathbb{R}) = H_0(G; \mathbb{R}) \)), and this approach will lead us to a general expression for them. We shall restrict ourselves here to the ungauged case, and will not discuss the (related) problem of gauging the WZW actions \([9]\).

This paper is organised as follows: in Sec. 2 we review briefly the forms on coset manifolds and the relative Lie algebra cohomology. Sec. 3 is devoted to finding an explicit general formula for the non-trivial cocycles on \( G/H \) for a simple group, and in Sec. 4 we illustrate our result with applications. The formal connection between the cocycles on \( G/H \) and the expression for the Chern-Simons forms is exhibited in Sec. 5, where the relation between the two is clarified. The indices are as follows: \( i, j, \ldots \) refer to \( G \) (or its

\(^1\)Notice that the left action of \( G \) on \( G \) and on \( K \) (defined by the left cosets \( gH \)) and the right action of \( H \) on \( G \) are both compatible with the bundle projection.
algebra $\mathcal{G}$, $i = 1, \ldots, \dim G$; $\alpha, \beta, \ldots$ to the subgroup $H$ ($\mathcal{H}$), $\alpha = 1, \ldots, \dim H$, and the indices $a, b, \ldots$ parametrise the coset $K$ (or the vector space $\mathcal{K} = \mathcal{G}/\mathcal{H}$), $a = 1, \ldots, \dim K$.

### 2 Forms on cosets and relative algebra cohomology

From the point of view of physical applications, Lie algebra $\mathcal{G}$ cohomology groups are most conveniently described [4] in terms of forms on the associated simply connected group manifold $G$. For the trivial representation $\rho(\mathcal{G}) = 0$, the cohomology groups $H_q^0(\mathcal{G}, \mathbb{R})$ are characterised by a) closed and b) (say) left invariant (LI) $q$-forms on $G$ (the $q$-cocycles) modulo those which are the exterior derivative $d$ of a LI form (the coboundaries). Let $\omega^i$ be a basis of LI one-forms on $G$ so that

$$d\omega^i = -\frac{1}{2} C^i_{jk} \omega^j \wedge \omega^k \quad (i, j, k = 1, \ldots, \dim \mathcal{G})$$

$$(2.1)$$

($\omega$ is the $\mathcal{G}$-valued canonical form on $G$). Then the LI $q$-forms $\Omega$ on $G$ may be written as

$$\Omega = \frac{1}{q!} \Omega_{i_1 \ldots i_q} \omega^{i_1} \wedge \ldots \wedge \omega^{i_q} \quad ,$$

$$(2.2)$$

Then, those which determine Lie algebra $q$-cocycles satisfy the condition

$$s(\Omega)_{i_1 \ldots i_{q+1}} = -\frac{1}{2} \frac{1}{(q-1)!} C^i_{[i_1 i_2} \Omega_{i_3 \ldots i_q]} = 0 \quad ,$$

$$(2.3)$$

where, in the CE formulation, the coboundary operator $s$ may be identified with the exterior derivative $d$.

In the language of forms the **relative cohomology** with respect to a subalgebra $\mathcal{H} \subset \mathcal{G}$ is associated with the notion of projectability of forms on $G$ to the coset manifold $K = G/H$. This notion, which plays an essential rôle in the Chern-Weil theory of characteristic classes, actually means that there is a unique form $\bar{\Omega}$ on $G/H$ such that $\pi^* (\bar{\Omega}) = \Omega$, where $\pi^*$ is the pull-back of the canonical projection $\pi : G \rightarrow G/H$. A $q$-form $\Omega$ is projectable if (see, e.g. [7, II, p. 294])

$$\Omega(X_1, \ldots, X_q) = 0 \text{ if any } X \in \mathcal{H} \quad ,$$

$$(2.4)$$

and

$$(L_{X_\alpha} \Omega)(X_1, \ldots, X_q) = -\sum_{s=1}^{q} \Omega(X_1, \ldots, [X_\alpha, X_s], \ldots, X_q) = 0 \quad ,$$

$$(2.5)$$

i.e., if $\Omega$ is ‘orthogonal’ to $\mathcal{H}$ [(2.4)] and it is invariant under the right action of $H$ [(2.5)]. In (2.5), $L_X$ is the Lie derivative with respect to the LI vector field $X$; on LI forms, $L_{X_\alpha} \omega^i = -C^i_{jk} \omega^j$. As a result, a $q$-form $\Omega$ is a non-trivial $q$-cocycle for the relative Lie algebra cohomology $H^0_q(\mathcal{G}, \mathcal{H}; \mathbb{R})$ [4] if a) it is LI and closed, eq. (2.3) (i.e., it is a $q$-cocycle in $Z^0_q(\mathcal{G}; \mathbb{R})$); b) it is projectable and c) it is not the exterior derivative of a LI, projectable form (in which case it would be a coboundary). Our task is now to find the closed forms on the coset manifold $G/H$, parametrised by $\varphi^a$, which are non-trivial cocycles in $H^0_q(\mathcal{G}, \mathcal{H}; \mathbb{R})$. These will define effective actions of WZW type once they are pulled back to an enlarged spacetime manifold of the appropriate dimension.
An explicit formula for the cocycles on $G/H$

As is well known, the non-trivial primitive cocycles (i.e., that are not the product of other cocycles) on a simple group $G$ are all of odd order $[11]$ $(2m_s - 1)$, $s = 1, \ldots, l$ where $l$ is the rank of $G$. They are associated with the $l$ primitive invariant symmetric tensors $k_{i_1 \ldots i_{m_s}}$ of order $m_s$ (and Casimir operators of the same order) which may be constructed on $G[H]$ the properties of which have been studied recently [3]. Given such a tensor $k_{i_1 \ldots i_m}$, the $G/(2m-1)$-cocycle $\Omega(2m-1)$, or simply $\Omega$, is a form on $G \Omega \propto k_{i_1 \ldots i_{m-1}}d\omega^{i_1} \wedge \ldots \wedge d\omega^{i_{m-1}} \wedge \omega^s$, so that its coordinates are proportional to

$$\Omega_{i_1 \ldots i_{2m-2}s} \propto k_{i_1 \ldots i_{m-1}}[sC_{i_1 i_2 \ldots i_{2m-3} i_{2m-2}]}$$

(3.1)

(any non-primitive terms in $k$ do not contribute to (3.1); see Cor.3.1 in [3]). Due to the full antisymmetry of the structure constants, any semisimple group is reductive, $C_{\alpha \beta}^a = 0,$ $C_{\alpha \alpha}^\beta = 0$ ($\alpha, \beta$ in $H, a, b$ in $K$), i.e., $[H,H] \subset H, [H,K] \subset K, [K,K] \subset G$. The cocycles on $G/H$ are the projectable cocycles on $G$. To find general expressions for them for any simple $G$ consider the $(2m-1)$-form $\Omega(p)$ (cf. (3.1)),

$$\Omega(p) = k_{a_1 \ldots a_{p-1} p \ldots i_{m-1} b}C_{a_1 a_2}^{a_3} \ldots C_{a_{2p-3} a_{2p-2}}^{a_{2p-1} a_{2p-1}} \ldots C_{a_{2m-3} a_{2m-2}}^{a_{2m-1}}$$

$$\omega^a \wedge \ldots \wedge \omega^{a_{2p-1}} \wedge \omega^{a_{2m-2}} \wedge \omega^b.$$  

(3.2)

This form clearly satisfies the condition (2.4) $(i_X \alpha, \Omega(p)) = 0$ since $\omega^a (X_\alpha) = 0 \forall a \in K$ and $a$ in $H$. The proof that $L_{X_\alpha} \omega^a = -C_{\alpha \beta}^a \omega^b$ and the fact that the constants preceding $\omega^{a_1} \wedge \ldots \wedge \omega^{a_{2m-2}} \wedge \omega^b$ in (3.2) may be viewed as products of the invariant polynomials $k$, $C$ to which we may apply the following

Lemma 3.1

Let $k_{i_1 \ldots i_{n+1}}$ and $k'_{j_1 \ldots j_{m+1}}$ be two invariant tensors on $G$ (symmetry is not required here). Then,

$$L_\alpha (k_{a_1 \ldots a_n b} k_{b_1 \ldots b_m}^j \omega^{a_1} \otimes \ldots \otimes \omega^{a_n} \otimes \omega^{b_1} \otimes \ldots \otimes \omega^{b_m}) = 0$$

(3.3)

and

$$L_\alpha (k_{a_1 \ldots a_n j} k_{j b_1 \ldots b_m}^j \omega^{a_1} \otimes \ldots \otimes \omega^{a_n} \otimes \omega^{b_1} \otimes \ldots \otimes \omega^{b_m}) = 0$$

(3.4)

Proof: By using the $(G)$-invariance of $k$ we obtain

$$C_{\alpha a_1}^i k_{i a_2 \ldots a_n b} + \ldots + C_{\alpha a_n}^i k_{a_1 \ldots a_{n-1} b}^i + C_{\alpha \beta}^i k_{a_1 \ldots a_n}^i = 0.$$  

Now, using the fact that the coset is reductive, we get

$$C_{\alpha a_1}^a k_{a a_2 \ldots a_n b} + \ldots + C_{\alpha a_n}^a k_{a_1 \ldots a_{n-1} b}^a + C_{\alpha \beta}^a k_{a_1 \ldots a_n}^a = 0$$

and similarly for $k'$. Thus,

$$\left( C_{\alpha a_1}^a k_{a a_2 \ldots a_n b} + \ldots + C_{\alpha a_n}^a k_{a_1 \ldots a_{n-1} b}^a \right) k_{j b_1 \ldots b_m}^j + k_{a_1 \ldots a_n}^j (C_{\alpha b}^a k_{i b_2 \ldots b_m}^a + \ldots + C_{\alpha b}^a k_{j b_1 \ldots b_m}^j)$$

$$= -C_{\alpha \beta}^{ij} k_{a_1 \ldots a_n}^i k_{j b_1 \ldots b_m}^j - k_{a_1 \ldots a_n}^i C_{\alpha \beta}^{ij} k_{j b_1 \ldots b_m}^j = 0$$

(3.5)
from which (3.3) follows. Eq. (3.4) is deduced from the fact that \( k_{i_1 \ldots i_n j}^j \) is an invariant tensor on \( G \) so that
\[
\sum_{s=1}^{n} C_{\alpha i s}^k k_{i_1 \ldots i_n j}^j k_{j j_1 \ldots j m}^k + \sum_{t=1}^{m} C_{\alpha j t}^k k_{i_1 \ldots i_n j}^j k_{j j_1 \ldots j t k \ldots j m}^k = 0 .
\]

Setting \( i_s = a_s \) and \( j_t = b_t \) and using again the reductive property we obtain (3.4), q.e.d.

Since the cocycles on the coset manifold are LI closed forms on \( K \), we look now for a closed form. Since \( L_\alpha \Omega_{(p)} = (i_{X_\alpha} d + d i_{X_\alpha}) \Omega_{(p)} = i_{X_\alpha} d \Omega_{(p)} = 0 \), when computing \( d \Omega_{(p)} \) we may ignore the \( \omega^a \) components. A straightforward if somewhat lengthy calculation, which uses the Jacobi identity and the fact that the coset is reductive shows that
\[
d \Omega_{(p)} = -\frac{1}{2} \Pi_{(p-1)} + \frac{2m - p}{2p} \Pi_{(p)} ,
\]
where the \( 2m \)-forms \( \Pi_{(p-1)} \) and \( \Pi_{(p)} \) are given by
\[
\Pi_{(p)} = k_{\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_m} C_{\alpha_1 \beta_2}^{\alpha_2} \ldots C_{\alpha_p \beta_{p+1}}^{\alpha_{p+1}} C_{\beta_{p+2} \beta_{p+2}}^{\alpha_{p+2}} \ldots C_{\beta_{2m-1} \beta_{2m}}^{\alpha_{2m}} \omega^1 \wedge \ldots \wedge \omega^{2m} .
\]

For \( p = 0 \) we find that that \( \Pi_{(0)} = 0 \) due to the \( G \)-invariance of \( k_{i_1 \ldots i_m} \). Let us now define a new \((2m-1)\)-form \( \bar{\Omega} \) by
\[
\bar{\Omega} = \sum_{s=1}^{m} \alpha_m(s) \Omega_{(s)} ,
\]
\[
\alpha_m(s) \equiv \prod_{r=1}^{s-1} \frac{2m - r}{r} , \quad \alpha_m(1) \equiv 1 , \quad \alpha_m(m) = \frac{(2m-1)!}{(m-1)!m!} .
\]

Then we find from (3.5)
\[
d \bar{\Omega} = \sum_{s=1}^{m} \alpha_m(s) \left( -\frac{1}{2} \Pi_{(s-1)} + \frac{2m - s}{2s} \Pi_{(s)} \right)
\]
\[
= -\frac{1}{2} \left( \sum_{s=0}^{m-1} \alpha_m(s+1) \Pi_{(s)} - \sum_{s=1}^{m} \alpha_m(s) \frac{2m - s}{s} \Pi_{(s)} \right)
\]
\[
= -\frac{1}{2} \left( \sum_{s=1}^{m-1} \alpha_m(s+1) - \alpha_m(s) \frac{2m - s}{s} \right) \Pi_{(s)} + \frac{1}{2} \alpha_m(m) \Pi_{(m)} = \frac{1}{2} \alpha_m(m) \Pi_{(m)}
\]

Hence \( d \bar{\Omega} \) will be zero if \( \Pi_{(m)} = 0 \) i.e., if \( k_{\alpha_1 \ldots \alpha_m} = 0 \). Thus, we have proven the following

**Theorem 3.1**

Let \( G \) be a simple, simply connected group and \( H \) a closed subgroup. A primitive non-trivial \((2m-1)\)-cocycle in \( H^{(2m-1)}(G, \mathcal{H}; \mathbb{R}) \) is represented by the closed \((2m-1)\)-form \( \bar{\Omega} \) on the coset \( G/H \) given in (3.7). This form is defined through (3.6) by a polynomial of order \( m \) on \( G \) which vanishes on \( \mathcal{H} \), i.e., when all its indices take values in \( \mathcal{H} \).
4 Applications

As an application of our general formula (3.7) let us find the expression for the three- ($m = 2$) and five- ($m = 3$) cocycles. Eq. (3.7) gives

$$\bar{\Omega}^{(3)} = \Omega_{(1)} + \alpha_2(2)\Omega_{(2)} = (k_{i_1a_3} C_{a_1a_2}^{i_1} \omega^{a_1} \wedge \omega^{a_2} \wedge \omega^{a_3}) \quad (4.1)$$

$$\bar{\Omega}^{(5)} = \Omega_{(1)} + \alpha_3(2)\Omega_{(2)} + \alpha_3(3)\Omega_{(3)} = (k_{i_1i_2a_5} C_{a_1a_2}^{i_1} C_{a_3a_4}^{i_2} + 5k_{a_1i_2a_5} C_{a_1a_2}^{a_1} C_{a_3a_4}^{i_2} + 10k_{a_1a_2a_5} C_{a_1a_2}^{a_1} C_{a_3a_4}^{a_2} ) \omega^{a_1} \wedge \ldots \wedge \omega^{a_5} \quad (4.2)$$

These two expressions have also been derived by D'Hoker by a lengthier cohomological descent-like procedure [2], involving the consideration of non-trivial representations $\rho$ of $G$. To exhibit the computational convenience of the general formula (3.7), we give one further example, the seven-cocycle,

$$\bar{\Omega}^{(7)} = \Omega_{(1)} + \alpha_4(2)\Omega_{(2)} + \alpha_4(3)\Omega_{(3)} + \alpha_4(4)\Omega_{(4)} = (k_{i_1i_2i_3a_7} C_{a_1a_2}^{i_1} C_{a_3a_4}^{i_2} C_{a_5a_6}^{i_3} + 7k_{a_1i_2i_3a_7} C_{a_1a_2}^{a_1} C_{a_3a_4}^{i_2} C_{a_5a_6}^{i_3} + 21k_{a_1a_2i_3a_7} C_{a_1a_2}^{a_1} C_{a_3a_4}^{a_2} C_{a_5a_6}^{i_3} + 35k_{a_1a_2a_3a_7} C_{a_1a_2}^{a_1} C_{a_3a_4}^{a_2} C_{a_5a_6}^{a_3} ) \omega^{a_1} \wedge \ldots \wedge \omega^{a_7} \quad (4.3)$$

Of course, the existence of these cocycles depends on the existence of invariant polynomials of the appropriate degree which are zero on $H$. These are all known for all simple algebras (see also [3] in this respect); in particular all three exist for $G = su(n), n \geq m$.

Let us consider now some specific examples.

a) ($SU(n)/SU(m)$ cosets)

Consider the case $K = SU(3)/SU(2) \sim S^5$. To construct a 5-cocycle on $S^5$ we need a 3rd-order polynomial vanishing on $SU(2)$. This is provided by the $d_{a_1a_2a_3}$ polynomial which satisfies $d_{a_1a_2a_3} = 0 \forall \alpha \in SU(2)$. Similarly for the case $K = SU(4)/SU(2)$ we have two invariant polynomials of (3rd and 4th order) vanishing on $SU(2)$, which give rise to the 5- and 7-cocycles respectively, etc. For the case $m > 2$, we can always construct symmetric invariant polynomials on $su(n)$ which are zero on $su(m)$ (see [3]).

b) (Symmetric cosets)

The simplest cases in which formula (3.7) gives rise to a non-trivial result are furnished by symmetric cosets $G/H$ ($[K,K] \subset H$). Such examples, when they exist, are simple because then all terms in (3.7) have the same structure since $C_{ab}^c = 0$. They require $k_{a_1\ldots a_m} = 0$ and that the components $k_{a_1\ldots a_m} a$ do not all vanish. For instance, for $G = SU(n), m = 3$ and $d_{a_1a_2a_3} = 0$, the five-cocycle becomes proportional to

$$d_{a_1a_2a_3} C_{a_1a_2}^{a_1} C_{a_2a_3}^{a_2} \omega^{a_1} \wedge \ldots \wedge \omega^{a_5} \quad (4.4)$$

Symmetric spaces in which the $d_{a_1a_2a_3}$ vanish and the $d_{a_1a_2a_4}$ do not, are provided by the families [13, p. 518] $SU(n)/SO(n)$ and $SU(2n)/Sp(2n)$.

The simplest case, one which leads directly to a Wess-Zumino term in a four dimensional field theory, is that of $SU(3)/SO(3)$. To clarify this, we work with the generators $T_A = \frac{1}{2} \lambda_A$
of $\text{su}(3)$, where the $\lambda_A$ are the set of standard Gell-Mann matrices. For the $\text{so}(3)$ generators, we take $\alpha \in \{2, 5, 7\}$ and, since $C_7^2 = \frac{1}{2}$, $[\lambda_\alpha, \lambda_\beta] = i\varepsilon_{\alpha\beta\gamma} \lambda_\gamma$. Then the coset indices $\alpha \in \{1, 3, 4, 6, 8\}$. To see that it is correct to write $SU(3)/SO(3)$, we note that reduction of the octet $SU(3)$ with respect to $SO(3)$ produces $j = 1$ and $j = 2$ $SO(3)$-multiplets, and we can argue that integral $j$ values only arise in the reduction of triality zero $SU(3)$ representations. Explicitly, we can show that, for $c \in \mathbb{R}$

$$i\frac{1}{\sqrt{2}}c(\lambda_1 - i\lambda_3), \quad i\frac{1}{\sqrt{2}}c(\lambda_4 + i\lambda_6), \quad c\lambda_8, \quad i\frac{1}{\sqrt{2}}c(\lambda_4 - i\lambda_6), \quad -i\frac{1}{\sqrt{2}}c(\lambda_1 + i\lambda_3), \quad (4.5)$$

are the standard Racah components $T_q$, $q = (2, 1, 0, -1, -2)$ of a rank 2 tensor operator of $SO(3)$. In this example, one can see by inspection of the $d$-tensor of $SU(3)$, that the $d_{a_1a_2a_3}$ all vanish, but there are eight non-zero triples for which $d_{a_1a_2a_3} \neq 0$. It is in fact easy to see without explicit calculation that (4.4) is a multiple of $\omega^1 \wedge \omega^3 \wedge \omega^4 \wedge \omega^6 \wedge \omega^8$.

To discuss this and other examples involving $G = SU(4)$, it is most convenient to generalise the Gell-Mann $\lambda$-matrices from $SU(3)$ to $SU(4)$ in a fashion different from that in [3, 14] and to use the $d$ and $f$ tensors that follow from this new set. Thus, set

$$\lambda_i = \left( \begin{array}{cc} \sigma_i & 0 \\ 0 & 0 \end{array} \right), \quad \lambda_{i+12} = \left( \begin{array}{cc} 0 & 0 \\ 0 & \sigma_i \end{array} \right), \quad \lambda_8 = \sqrt{\frac{1}{2}} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right),$$

where $\sigma_i$ are the three Pauli matrices. For $i = 4$ to 7, we retain the $\lambda_i$ of $SU(3)$ so that for $i=4$ to 7 and 9 to 12, the $\lambda_i$ of [14] are used. In particular, $\lambda_3$, $\lambda_8$, $\lambda_{15}$ are diagonal, all $\lambda$’s are hermitian and $\lambda_i$ for $i \in \{2, 5, 7, 10, 12, 14\}$ are antisymmetric. In fact we have only changed our choices of $\lambda_8$ and $\lambda_{15}$, so that very little further evaluation of $d$ and $f$ tensors is needed.

Consider first $G/H = SU(4)/SO(4)$ in which $SO(4)$ is generated by the set of six antisymmetric $\lambda$’s just mentioned, while the $SU(4)$ $d$-tensors are as tabulated in [3]. The $d_{a_1a_2a_3}$ do vanish, since they correspond to the trace of products of three antisymmetric matrices, while for many triples the $d_{a_1a_2a_3} \neq 0$. Thus a simple five-cocycle is allowed in this model with nine Goldstone fields. We may contrast this with the non-symmetric reductive model $SU(4)/[SU(2) \times SU(2)]$ in which the subgroup generators are $\lambda_i$ for $i \in \{1, 2, 3, 13, 14, 15\}$. The relevant $d_{a_1a_2a_3}$ obviously vanish but, since the $C_{ab}^c$ are not all zero, all the three terms of (4.1) survive giving a much more complicated Wess-Zumino term for this model with 15-6=9 Goldstone fields.

Consider next the symmetric coset $SU(4)/S[U(2) \times U(2)]$, where $\mathcal{H}$ is larger than in the previous example, with the extra generator $\lambda_8$. The $d_{a_1a_2a_3}$ do not all vanish now, and hence there are no five-cocycles of type (3.7) [1]. Finally, consider the five-dimensional symmetric coset $SU(4)/Sp(4, \mathbb{R})$. A presentation of $C_2 = sp(4, \mathbb{R})$ in Cartan-Weyl form with positive roots $r_1 = (1, -1), r_2 = (0, 2), r_3 = (1, 1), r_4 = (2, 0)$ can be given in terms of the above $4 \times 4$ $\lambda$-matrices of $SU(4)$. Writing $\sqrt{2}E_{\pm \mu} = X_\mu \pm iY_\mu$ for $\mu = 1, 2, 3, 4$ for the raising and lowering operators associated with the roots, the realisation is

$$H_1 = \lambda_3, \quad X_4 = \lambda_1, \quad Y_4 = \lambda_2,$$
$$H_2 = \lambda_{15}, \quad X_2 = \lambda_{13}, \quad Y_2 = \lambda_{14},$$
$$\sqrt{2}X_1 = \lambda_4 - \lambda_{11}, \quad \sqrt{2}Y_1 = \lambda_5 + \lambda_{12},$$
$$\sqrt{2}X_3 = \lambda_6 + \lambda_9, \quad \sqrt{2}Y_3 = -\lambda_7 + \lambda_{10}. \quad (4.6)$$

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This enables an explicit check that the corresponding $d_{a_1 a_2 a_3}$ are indeed all zero, an easy tabulation of $C_2$ structure constants $C_{\alpha\beta}$, $1 \leq \alpha, \beta, \gamma \leq 10$, and the evaluation of the five-cocycle of the model.

There are no seven-dimensional symmetric cosets. Consider then the case of the 9-cocycle determining a Wess-Zumino term in $D = 8$ spacetime. For $G/H$ symmetric, all the coordinates of $\bar{\Omega}^{(g)} = \bar{\Omega}_{a_1 \ldots a_9}^{(g)} \omega^{a_1} \wedge \ldots \wedge \omega^{a_9}$ become proportional to

$$k^{(5)}_{a_1 a_2 a_3 a_4 [a_5} C_{a_6 a_7 C_{a_8 a_9)]}$$

(4.7)

Let $G/H = SU(n)/SO(n)$ with $so(n)$ generated by the $\frac{1}{2}n(n-1)$ $n \geq 4$ imaginary antisymmetric $\lambda_{a}$ matrices, the coset generators $\lambda_{a}$ being real symmetric and traceless. Then, if we take $k_{i_1 \ldots i_5} \sim \text{sTr}(\lambda_{i_1} \ldots \lambda_{i_5})$ it is obvious that $\text{sTr}(\lambda_{a_1} \ldots \lambda_{a_3})$ is zero but that $\text{sTr}(\lambda_{a_1} \ldots \lambda_{a_4} \lambda_{a_5})$ is not. Hence we will get a Wess-Zumino term from

$$\bar{\Omega}_{a_1 \ldots a_9}^{(g)} \omega^{a_1} \wedge \ldots \wedge \omega^{a_9} \quad , \quad \bar{\Omega}_{a_1 \ldots a_9}^{(g)} \propto k^{(5)}_{a_1 a_2 a_3 a_4 a_5} C_{a_6 a_7 C_{a_8 a_9]}$$

(4.8)

In (4.8) we may use equivalently $d^{(5)}$ for the $su(n)$ polynomial, (see [3]).

## 5 Relative cohomology and Chern-Simons forms

Let us consider $\Omega_{(p)}$ in (3.2) further. First we introduce

$$k_{a_1 \ldots a_p-1 b \ldots a_m-1 b} = \frac{1}{m!} \text{sTr}(\text{ad}X_{a_1} \ldots \text{ad}X_{a_p-1} \text{ad}X_{a_p} \ldots \text{ad}X_{a_m-1} \text{ad}X_{b})$$

(5.1)

which arises by restricting the indices of the invariant symmetric polynomial of order $m$ on $\mathcal{G}$, given by a symmetric trace, to the appropriate values. Next, we make the identifications

$$W^\alpha = -\frac{1}{2} C^a_{ab} \omega^a \wedge \omega^b \quad , \quad U = \omega|_\mathcal{H} \quad , \quad (U \wedge U)^i = \frac{1}{2} C^i_{ab} \omega^a \wedge \omega^b$$

(5.2)

and $V = \omega|_\mathcal{H}$, where $\omega = u^{-1} du$. Thus, $V$ determines the LI invariant $\mathcal{H}$-connection and $W$ the associated curvature, $W = dV + V \wedge V$ (which leads to $W^\alpha$ in (5.2) using the Maurer-Cartan eqs.). Then, the form in (3.2) may be rewritten as

$$\Omega_{(p)} = (-1)^{p-1} \frac{2^{m-1}}{m!} \text{sTr}(W^{p-1}(U^2)^{m-p}U)$$

(5.3)

If we replace the $m!$ terms in sTr by the sum $\mathcal{S}$ over all possible products (‘words’) which contain a total power $(p-1) [(m-p)]$ of the curvature $W$ [component $U$], eq. (5.3) may be written as

$$\Omega_{(p)} = \frac{2^{m-1}}{(m-1)!} (-1)^{p-1} (p-1)! (m-p)! \text{Tr}\{\mathcal{S} [(W^{p-1}(U^2)^{m-p})] U] \}$$

(5.4)
As a result, the general expression (3.7) for the \((2m-1)\)-cocycle on the coset \(K\) may be rewritten as
\[
\bar{\Omega}^{(2m-1)} = \frac{2^{m-1}}{(m-1)!} \sum_{p=0}^{m-1} (-1)^p(p)!(m-p-1)! \alpha_m(p+1) \text{Tr}\left\{ \mathcal{S} \left[ (\mathcal{W}^p(\mathcal{U}^2)^{m-p-1}) \mathcal{U} \right] \right\}, \quad (5.5)
\]
which leads to
\[
\bar{\Omega}^{(2m-1)} = \frac{2^{m-1}}{(m-1)!} \sum_{p=0}^{m-1} (-1)^p(m-p-1)! (2m-1) \cdots (2m-p) \text{Tr}\left\{ \mathcal{S} \left[ (\mathcal{W}^p(\mathcal{U}^2)^{m-p-1}) \mathcal{U} \right] \right\}. \quad (5.6)
\]

Now, recalling the expression of the Beta function,
\[
B(l, s) = \int_0^1 dt t^{l-1}(1-t)^{s-1} = \frac{(l-1)!(s-1)!}{(l+s-1)!}, \quad (5.7)
\]
we see that, renaming \(p \to (m-p-1)\), \(\Omega\) may be written in the form
\[
\bar{\Omega}^{(2m-1)} = \frac{2^{m-1}}{(m-1)!} \sum_{p=0}^{m-1} (-1)^{m-p-1}(p)!(2m-1) \cdots (m+p+1) \text{Tr}\left\{ \mathcal{S} \left[ W^{m-p-1}(\mathcal{U}^{2})^{p} \right] \mathcal{U} \right\}
\]
\[
= 2^{m-1} \frac{(2m-1)!}{(m-1)! m!} \sum_{p=0}^{m-1} (-1)^{m-p-1} \frac{p! m!}{(m+p)!} \text{Tr}\left\{ \mathcal{S} \left[ W^{m-p-1}(\mathcal{U}^{2})^{p} \right] \mathcal{U} \right\}
\]
\[
= (-1)^{m-1} 2^{m-1} \frac{(2m-1)!}{(m-1)! m!} \sum_{p=0}^{m-1} \int_0^1 dt m t^{m-1} (1-t)^p \text{Tr}\left\{ \mathcal{S} \left[ W^{m-p-1}(\mathcal{U}^{2})^{p} \right] \mathcal{U} \right\}
\]
\[
= (-1)^{m-1} 2^{m-1} \frac{(2m-1)!}{(m-1)! m!} \sum_{p=0}^{m-1} \int_0^1 dt m t^{m-1} (1-t)^p \text{Tr}\left\{ \mathcal{S} \left[ W^{m-p-1}(\mathcal{U}^{2})^{p} \right] \mathcal{U} \right\}. \quad (5.8)
\]

The reader will recognise that the integral in (5.8) is formally identical to that giving the expression of the Chern-Simons form \([15]\) \(\Omega^{(2m-1)}\) of the Chern character \(c_{h_m}\) which are relevant in the theory of non-abelian anomalies (see \([16]\) and references therein). This means that if we know the coefficients which determine the terms for the Chern-Simons forms of various orders (see, e.g. \([18, \S 10.13]\)), we also know the cocycles \(\bar{\Omega}\) in (5.5), (5.6) or (5.8) and vice versa. This explains the similarity between the two types of \((2m-1)\)-forms.

In general, the \((2m)\)-form \(d\bar{\Omega}^{(2m-1)}\) gives the Chern character \(c_{h_m}\); in contrast, the cocycle form \(\bar{\Omega}^{(2m-1)}\) is closed, \(d\bar{\Omega}^{(2m-1)} = 0\), since \(\Pi_{(m)}\) in eq. (3.8), (3.6) is zero precisely for the polynomials which vanish on \(\mathcal{H}\).

We may now use (5.8) to recast from it the expression of the WZW terms calculated previously. For \(m = 2, 3\), eq. (5.8) gives
\[
\bar{\Omega}^{(3)} = -3! \left[ \text{Tr}(WU) - \frac{1}{3} \text{Tr}(U^3) \right] = [3k_{\alpha_1 \alpha_2 \alpha_3} C_{\alpha_1 \alpha_2} \alpha_3 + k_{\alpha_5 \alpha_3} C_{\alpha_5 \alpha_3}] \omega_{\alpha_1} \wedge \omega_{\alpha_2} \wedge \omega_{\alpha_3}; \quad (5.9)
\]
\[
\bar{\Omega}^{(5)} = \frac{2^5 5!}{2! 3!} \left[ \text{Tr}(W^2U) - \frac{1}{4} \text{Tr}(WU^3 + U^2W)U + \frac{1}{10} \text{Tr}(U^5) \right]
\]
\[
= (10k_{\alpha_1 \alpha_2 \alpha_5} C_{\alpha_1 \alpha_2} \alpha_3 C_{\alpha_5 \alpha_3} + 5k_{\alpha_1 \alpha_5 \alpha_3} C_{\alpha_1 \alpha_5} \alpha_3 C_{\alpha_5 \alpha_3} + k_{\alpha_1 \alpha_2 \alpha_5} C_{\alpha_1 \alpha_2} \alpha_3 C_{\alpha_5 \alpha_3}) \omega_{\alpha_1} \wedge \ldots \wedge \omega_{\alpha_5}, \quad (5.10)
\]
and (4.2) which have been reordered to show the origin of their terms. Similarly, for $m = 4$ we obtain (cf. (4.3)),

\[
\Omega^{(7)} = -\frac{2^3 \cdot 7!}{3! \cdot 4!} \left[ \text{Tr}(W^3 U) - \frac{1}{5} \text{Tr}(2W^2 U^3 + WU^2 WU) + \frac{1}{15} \text{Tr}(3WU^5) - \frac{1}{35} \text{Tr}(U^7) \right] \\
= (35k_{\alpha_1 \alpha_2 \alpha_3 \alpha_7}C_{\alpha_1 \alpha_2}C_{\alpha_3 \alpha_4}C_{\alpha_5 \alpha_6} + 21k_{\alpha_1 \alpha_2 \alpha_3 \alpha_7}C_{\alpha_1 \alpha_2}C_{\alpha_3 \alpha_4}C_{\alpha_5 \alpha_6} \\
+ 7k_{\alpha_1 \alpha_2 \alpha_3 \alpha_7}C_{\alpha_1 \alpha_2}C_{\alpha_3 \alpha_4}C_{\alpha_5 \alpha_6} + 7k_{\alpha_1 \alpha_2 \alpha_3 \alpha_7}C_{\alpha_1 \alpha_2}C_{\alpha_3 \alpha_4}C_{\alpha_5 \alpha_6}) \omega^{a_1} \wedge \ldots \wedge \omega^{a_7}.
\]  

(5.11)

The (global) $(-)^{m-1}2^{m-1} \frac{(2m-1)!}{(m-1)! m!}$ factor in (5.8) came from the definition of $\Omega_{(p)}$ in (3.2). By replacing this factor by $\left(\frac{i}{2\pi}\right)^m \frac{1}{m!} (2\pi)$ we may adjust it so that the $U^{2m-1}$ term has the standard factor $(-1)^{m-1} \left(\frac{i}{2\pi}\right)^m \frac{(2m-1)!}{(m-1)! m!} (2\pi)$ (i.e., $\frac{i}{12\pi}$, $\frac{-i}{240\pi^2}$, $\frac{-i}{6720\pi^3}$ for the 3, 5, 7 cocycles in (5.9), (5.10) and (5.11)).

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