Communication efficient and strongly secure secret sharing schemes based on algebraic geometry codes

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Abstract

Secret sharing schemes with optimal and universal communication overheads have been obtained independently by Bitar et al. and Huang et al. However, their constructions require a finite field of size \( q > n \), where \( n \) is the number of shares, and do not provide strong security. In this work, we give a general framework to construct communication efficient secret sharing schemes based on sequences of nested linear codes, which allows to use in particular algebraic geometry codes and allows to obtain strongly secure and communication efficient schemes. Using this framework, we obtain: 1) schemes with universal and close to optimal communication overheads for arbitrarily large lengths \( n \) and a fixed finite field, 2) the first construction of schemes with universal and optimal communication overheads and optimal strong security (for restricted lengths), which has the security advantages of perfect schemes and the storage efficiency of ramp schemes, and 3) schemes with universal and close to optimal communication overheads and close to optimal strong security defined for arbitrarily large lengths \( n \) and a fixed finite field.

Keywords: Secret sharing, algebraic geometry codes, communication efficiency, communication bandwidth, strong security, asymptotic secret sharing.

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1 Introduction

A secret sharing scheme is a procedure to encode a secret into \( n \) shares, given bijectively to \( n \) parties, in such a way that some specified collections of shares give all the information about the secret, while other collections of shares give no information at all. It is usual to specify such collections by threshold values \( t < r \): the secret can be recovered from any \( r \) shares, while no collection of \( t \) shares gives any information about the secret.

Traditionally, efforts have been made to obtain schemes where \( r \) is as low and \( t \) is as large as possible. The first construction, by Shamir [21], consists of a secret sharing scheme attaining information rate \( 1/n \) (perfect scheme) and any threshold values \( 0 \leq t < r \leq n \) with \( t = r - 1 \), which are optimal. Later, this construction was extended in [3, 30] to higher information rates \( \ell/n \) and threshold values \( t = r - \ell \), which are again optimal. Schemes where the information rate is \( \ell/n > 1/n \) (non-perfect schemes) are also called ramp schemes, and allow the shares to be smaller than the secret, hence allowing more flexibility and efficiency in terms of storage. However, all these previous optimal constructions require a finite field of size \( q > n \), which in
many situations requires performing computations over very large fields, and reduces the storage efficiency.

Later, in the works [6, 8, 15], more general frameworks for constructing secret sharing schemes are given in terms of linear (block) codes. Thanks to these approaches, Shamir’s scheme is extended to schemes based on algebraic geometry codes [4, 5, 6, 8, 19], which may have arbitrarily large length $n$ for any fixed finite field at the cost of achieving close to optimal parameters, instead of optimal, being the difference dependent on the field size.

On the other hand, as noticed first in [28] and then in [13], the overall amount of communication between the contacted parties and the user (decoding bandwidth), in a given secret sharing scheme, can be decreased by contacting more than $r$ shares. A lower bound on the communication overhead (the difference between the decoding bandwidth and the information contained in the secret) was first given in [28] for perfect schemes, and then in [13] for the general case.

A modification of Shamir’s secret sharing scheme was given independently in [1] and in [13] for any information rate, with optimal threshold values $r$ and $t$, and where optimal communication overheads are simultaneously achieved (universal) for any $d$ contacted shares and any $r \leq d \leq n$. However, as Shamir’s scheme, this construction requires a finite field of size $q > n$.

Therefore, the following question arises naturally: can we modify or extend the constructions in [1, 13] so that we may apply algebraic geometry codes and hence obtain near optimal thresholds and universal near optimal communication overheads for arbitrarily large lengths $n$ and a fixed finite field?

In this work, we answer the previous question positively, giving a general framework for constructing communication efficient secret sharing schemes based on sequences of nested linear codes in a similar way to that of [6, 8, 15]. Afterwards and using algebraic geometry codes, we obtain the desired communication efficient secret sharing schemes of arbitrary length and fixed finite field. More concretely, thanks to our general framework, we reduce the problem of finding communication efficient long secret sharing schemes to the well-known problem of finding algebraic curves with many rational points and low genus [14, 24, 25].

The introduced universal construction will have the same alphabet size as that of the universal constructions in [1, 13]. As in [1], we will also give a non-universal construction that has a significantly smaller alphabet.

On the other hand, our general framework allows us also to obtain schemes that are communication efficient and strongly secure at the same time. Strongly secure schemes were introduced in [20], and allow to keep parts of the secret safe even when more than $t$ shares are eavesdropped. In such a scheme, if the secret is constituted by $\ell$ components (each considered as a secret by itself), where $\ell/n$ is the information rate, then no information about any tuple of $\ell - \mu$ components is leaked if $t + \mu$ shares are eavesdropped. In particular, in an optimal strongly secure ramp scheme with information rate $\ell/n$ and $t = r - \ell$, no information about any component of the secret is leaked by eavesdropping any $r - 1$ shares, as in Shamir’s optimal perfect scheme, while having the storage advantages of optimal ramp schemes (since it actually is an optimal ramp scheme). In conclusion, an optimal strongly secure ramp scheme has the security advantages of perfect schemes and the storage efficiency of ramp schemes.

By means of the framework developed in this paper, we obtain the first construction of a secret sharing scheme with universal and optimal communication efficiency and optimal strong security at the same time. However, it requires a finite field of size $q > \ell + n$, as previous constructions of optimal strongly secure schemes [18]. As before, to relax this requirement, we give another construction based on algebraic geometry codes, whose communication efficiency and strong security are close to the optimal values, while having arbitrarily large lengths $n$ over a fixed finite field.

For convenience of the reader, a comparison between the obtained constructions in this paper
Throughout the paper, we aim at achieving strong security. By specializing the linear codes to MDS codes (such as Reed-Solomon codes), we obtain the first construction of schemes with universal communication efficiency and optimal strong security at the same time, at the cost of a larger alphabet than Construction 1. In Section 4, we specialize the previous linear codes to algebraic geometry codes, and see large field size and optimal strong security at the same time, at the cost of a larger alphabet than Construction 1. In Section 4, we specialize the previous linear codes to algebraic geometry codes, and see that the resulting schemes have close to optimal communication overheads, while their lengths are not bounded by the field size. In Section 5, we give special cases of Constructions 1 and 2 from Section 3 that aim at achieving strong security. By specializing the linear codes to MDS codes (such as Reed-Solomon codes), we obtain the first construction of schemes with universal communication efficiency and optimal strong security at the same time, at the cost of a large field size $q$ (but still of the order of $n$). On the other hand, by specializing the linear codes to algebraic geometry codes, we obtain schemes with close to optimal communication efficiency and strong security, while having lengths not bounded by the field size.

### 2 Definitions and preliminaries

#### 2.1 Notation

Throughout the paper, $q$ denotes a fixed prime power, and $\mathbb{F}_q$ denotes the finite field with $q$ elements. For positive integers $m$ and $n$, we denote by $\mathbb{F}_q^m$ the vector space of row vectors of length $n$ over $\mathbb{F}_q$, and we denote by $\mathbb{F}_q^{m \times n}$ the vector space of matrices with components over $\mathbb{F}_q$.

We also denote $[n] = \{1, 2, \ldots, n\}$ and $[m, n] = \{m, m + 1, \ldots, n - 1, n\}$ if $m \leq n$. For any vector $x \in \mathbb{F}_q^n$, any matrix $X \in \mathbb{F}_q^{m \times n}$ and any set $I \subseteq [n]$, we denote by $x_I$ and $X_I$ the vector in $\mathbb{F}_q^{|I|}$ and the matrix in $\mathbb{F}_q^{m \times |I|}$ obtained by restricting $x$ to the components indexed by $I$ and restricting $X$ to the columns indexed by $I$, respectively.

Sets $C \subseteq \mathbb{F}_q^n$ and $D \subseteq \mathbb{F}_q^{m \times n}$ are called linear codes if they are vector spaces over $\mathbb{F}_q$. We denote by $C^\perp$ the dual of the code $C$ with respect to the usual inner product in $\mathbb{F}_q^n$ and, for a set $I \subseteq [n]$, we define the restriction of $C$ and $D$ to $I$, respectively, as the linear codes

$$C_I = \{c_I \mid c \in C \} \subseteq \mathbb{F}_q^{|I|} \quad \text{and} \quad D_I = \{D_I \mid D \in D \} \subseteq \mathbb{F}_q^{m \times |I|}.$$ 

Finally, for random variables $X$ and $Y$, and a positive integer $a$, we denote by $H_a(X)$, $H_a(X|Y)$ and $I_a(X; Y)$ the entropy of $X$, conditional entropy of $X$ given $Y$, and the mutual information between $X$ and $Y$, respectively, all of them taking logarithms with base $a$. See [7, Chapter 2].
2.2 Communication efficient secret sharing

We will consider the general definition of secret sharing schemes from [13] Definition 1.

Definition 1 (Secret sharing schemes [13]). Let $\mathcal{A}$ be an alphabet of size $a$, and let $n, \ell, r$ and $t$ be positive integers. An $(n, \ell, r, t, \mathcal{A})$ secret sharing scheme is a randomized encoding function $F : \mathcal{A}^{\ell} \rightarrow \mathcal{A}^n$ (meaning that $F(s)$ may take several values in $\mathcal{A}^n$ with a given probability distribution, for each $s \in \mathcal{A}^{\ell}$) such that:

1. It has $r$-reconstruction: For any secret $s \in \mathcal{A}^{\ell}$, if $x = F(s)$ and $I \subseteq [n]$ is of size at least $r$, then
   \[ H_a(s|x_I) = 0. \]

2. It has $t$-privacy: For any secret $s \in \mathcal{A}^{\ell}$, if $x = F(s)$ and $I \subseteq [n]$ is of size at most $t$, then
   \[ H_a(s|x_I) = H_a(s). \]

We define the information rate of the scheme as $\ell/n$.

This definition includes most of the classical proposals for secret sharing schemes [1] [3] [4] [5] [6] [8] [13] [15] [17] [18] [19] [21] [28] [39]. As usual in the literature, we identify the $i$-th share $x_i$ with the $i$-th party, hence the set $[n]$ represents both the $n$ parties and their corresponding $n$ shares.

Traditional efforts have been made in order to obtain schemes where $r$ is as low as possible, while keeping $t$ and $\ell$ large with respect to $n$. The obvious benefit of a low value of $r$ is that less parties need to be contacted, hence it is expected that the amount of transmitted information to recover the secret is lower. However, as noticed first in [28] and later in [13], it is possible to recover the secret while transmitting less information, as long as more parties are contacted. Observe that, in the previous definition, when contacting $r$ parties to recover the secret, we use their whole shares. However if more parties are contacted, then we may preprocess their shares so that the overall amount of transmitted information is lower.

Formally, let $I \subseteq [n]$ be such that $\#I \geq r$, let $i \in I$, and let $E_{I,i} : \mathcal{A} \rightarrow \mathcal{B}_{I,i}$ be preprocessing functions, where $\#\mathcal{B}_{I,i} \leq \#\mathcal{A}$. The existence of decoding functions $D_I : \prod_{i \in I} \mathcal{B}_{I,i} \rightarrow \mathcal{A}^{\ell}$ such that $D_I((E_{I,i}(x_i))_{i \in I}) = s$, where $x = F(s)$, for all $s \in \mathcal{A}^{\ell}$, is equivalent to
\[
I((E_{I,i}(x_i))_{i \in I}; s) = 0, \tag{1}
\]
for all $s \in \mathcal{A}^{\ell}$. We may then define the communication overhead and decoding bandwidth as follows:

Definition 2 (Communication overhead and decoding bandwidth [13]). For a set $I \subseteq [n]$ and preprocessing functions $E_{I,i}$, $i \in I$, satisfying (1), we define the communication overhead and decoding bandwidth of $I$, respectively, as
\[
\text{CO}(I) = \sum_{i \in I} H_a(E_{I,i}(x_i)) - H_a(s), \quad \text{and} \quad \text{DB}(I) = \sum_{i \in I} H_a(E_{I,i}(x_i)).
\]

Observe that, after preprocessing the $i$-th share, the amount of information required to be transmitted by the $i$-th party is $H_a(E_{I,i}(x_i))$. Without assuming any processing of the overall information in the variables $E_{I,i}(x_i)$, $i \in I$, the total transmitted information by the contacted parties is just the sum of $H_a(E_{I,i}(x_i))$ for $i \in I$. Hence the decoding bandwidth is the total amount of information transmitted by the parties indexed by $I$. Observe that
\[
\text{DB}(I) \geq H_a((E_{I,i}(x_i))_{i \in I}) \geq H_a(s),
\]
where the first inequality is a particular case of [7 Theorem 2.6] and the second inequality is a particular case of [4 Exercise 2.4]. Therefore, the communication overhead measures the amount of overall extra information transmitted by the contacted parties.

From now on, we will assume that the secret $s$ is a uniform random variable on $\mathcal{A}^\ell$. To measure the quality of a scheme, we will use the following bounds given in [13 Proposition 1] and [13 Theorem 1]:

**Proposition 1** ([13]). For a set $I \subseteq [n]$ and preprocessing functions $E_{I,i}$, $i \in I$, the following bounds hold:

\[
\ell \leq r - t,
\]
\[
\text{CO}(I) \geq \frac{\ell t}{|I| - t},
\]
\[
\text{DB}(I) \geq \frac{t|I|}{|I| - t}.
\]

Observe that the definition of communication overhead in [13 Definition 2] takes the values $\log_q(\#E_{I,i})$ instead of the smaller values $H_q(E_{I,i}(x_i))$. However, the proofs of the previous bounds work in the same way.

### 2.3 Secret sharing schemes based on linear codes

A classical approach to constructing secret sharing schemes is by the so-called coset coding schemes. These were first considered by Wyner in [29] for the problem of reliable and secure communication over wire-tap channels, which may be seen as secret sharing. A particular case is obtained when choosing the family of cosets as the quotient vector space of two linear codes over $\mathbb{F}_q$, as considered in [6, 15, 17, 31], and which generalize Shamir’s secret sharing scheme [21]. The alphabet of these schemes is $\mathcal{A} = \mathbb{F}_q$.

**Definition 3** (Nested coset coding schemes [6, 15, 17, 31]). A nested linear code pair is a pair of linear codes $C_2 \supsetneq C_1 \subseteq \mathbb{F}_q^n$. Choose a vector space $W$ such that $C_1 = C_2 \oplus W$ and a vector space isomorphism $\psi : \mathbb{F}_q^n \rightarrow W$, where $\ell = \dim(C_1/C_2)$.

The nested coset coding scheme associated to $C_1$ and $C_2$ is the secret sharing scheme $F : \mathbb{F}_q^\ell \rightarrow \mathbb{F}_q^n$ defined by $F(s) = \psi(s) + c$, for $s \in \mathbb{F}_q^\ell$, where $c$ is the uniform random variable on $C_2$.

To evaluate the reconstruction and privacy thresholds $r$ and $t$ of a nested coset coding scheme, we need the concept of minimum Hamming distance of a nested linear code pair $C_2 \supsetneq C_1 \subseteq \mathbb{F}_q^n$ and of a linear code $C \subseteq \mathbb{F}_q^n$, defined respectively as follows:

\[
d_H(C_1, C_2) = \min\{\text{wt}_H(c) \mid c \in C_1, c \notin C_2\}, \quad \text{and}
\]
\[
d_H(C) = \min\{\text{wt}_H(c) \mid c \in C_1, c \neq \mathbf{0}\} = d_H(C, \{\mathbf{0}\}),
\]

where $\text{wt}_H(c) = \#\{i \in [n] \mid c_i \neq 0\}$ is the Hamming weight of the vector $c \in \mathbb{F}_q^n$.

The following result is proven in [8 Corollary 1.7]:

**Lemma 1** ([8]). The nested coset coding scheme based on the nested linear code pair $C_2 \supsetneq C_1 \subseteq \mathbb{F}_q^n$ is an $(n, \ell, r, t)_{\mathbb{F}_q}$ secret sharing scheme with $\ell = \dim(C_1/C_2)$ and

\[
r = n - d_H(C_1, C_2) + 1,
\]
\[
t = d_H(C_2^\perp, C_1^\perp) + 1.
\]

**Remark 1.** We will implicitly assume that $C_1$ is not degenerate. That is, $C_{1(i)} \neq \{\mathbf{0}\}$, for every $i \in [n]$. If $C_{1(i)} = \{\mathbf{0}\}$ for some $i \in [n]$, then the $i$-th share not only contains no information about the secret, but added to any set of shares, it does not increase the overall amount of information, hence it may be removed.
3 Communication efficient secret sharing schemes based on nested linear codes

We will now present a general construction of communication efficient secret sharing schemes based on linear codes that is inspired by the constructions obtained independently in [11] and [13, Section IV]. As we will show, our construction generalizes the previous two.

We will present two versions of it. The first reduces communication overheads by contacting any $d \geq r$ shares for one fixed value of $d$, while the second reduces communication overheads by contacting any $d \geq r$ shares for all values $r \leq d \leq n$, hence achieving more flexibility.

In the rest of the paper, the alphabet is $\mathcal{A} = \mathbb{F}_q^\alpha$, for a positive integer $\alpha$. The advantage of the first construction with respect to the second is a significant smaller parameter $\alpha$, hence a significant smaller alphabet.

We will measure information taking logarithms in base $q$ and will write $H$ and $I$ instead of $H_q$ and $I_q$, respectively. Recall from [7, Lemma 2.1.2] that $H_q = \alpha H_q^\alpha$ and $I_q = \alpha I_q^\alpha$.

3.1 Construction 1: Non-universal but small alphabet

Take linear codes $C_2 \supsetneq C_1 \subseteq C \subseteq \mathbb{F}_q^n$ and define positive integers

1. $\ell = k_1 - k_2$, where $k_1 = \dim(C_1)$ and $k_2 = \dim(C_2)$,
2. $t \leq d_H(C_2^\perp, C_2^\perp) - 1$ and $r \geq n - d_H(C_1, C_2) + 1$,
3. $k = \dim(C)$ and $d \geq n - d_H(C, C_2) + 1$ such that $d \geq r$.

Finally, define the positive integer $\alpha = k - k_2 \geq \ell$ and a generator matrix of $C$ of the form:

$$G = \begin{pmatrix} G_c \\ G_2 \\ G_3 \end{pmatrix} \in \mathbb{F}_q^{k \times n},$$

where $G_2 \in \mathbb{F}_q^{k_2 \times n}$ is a generator matrix of $C_2$ and $(G_c^T, G_2^T)^T \in \mathbb{F}_q^{k_1 \times n}$ is a generator matrix of $C_1$.

The secret space is the space of matrices $\mathcal{A}^\ell = \mathbb{F}_q^{\alpha \times \ell}$ (recall that $\mathcal{A} = \mathbb{F}_q^\alpha$). Now, for a secret $S \in \mathbb{F}_q^{\ell \times \ell}$, divide it into two subsecrets as follows:

$$S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}, \quad S_1 \in \mathbb{F}_q^{\ell \times \ell}, \quad S_2 \in \mathbb{F}_q^{(\alpha-\ell) \times \ell},$$

where $S = S_1$ in the case $\alpha = \ell$. Next, generate uniformly at random a matrix

$$R = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \in \mathbb{F}_q^{\alpha \times k_2}, \quad R_1 \in \mathbb{F}_q^{\ell \times k_2} \text{ and } R_2 \in \mathbb{F}_q^{(\alpha-\ell) \times k_2},$$

where $R = R_1$ in the case $\alpha = \ell$. Finally, the $i$-th share, $1 \leq i \leq n$, is the $i$-th column of the matrix

$$C = (c_1, c_2, \ldots, c_n) = \begin{pmatrix} S_1 \\ S_2 \\ R_1 \\ R_2 \end{pmatrix},$$

which is a column vector $c_i \in \mathbb{F}_q^{\ell \times 1}$, that is, a symbol in our alphabet $\mathcal{A} = \mathbb{F}_q^\alpha$.

For a set $I \subseteq [n]$ and $i \in I$, the corresponding preprocessing function is

$$E_{I,i} : \mathbb{F}_q^n \longrightarrow \mathbb{F}_q^\ell.$$  

(7)
where $E_{I,i}(c_i)$ is obtained by restricting $c_i \in \mathbb{F}_q^{\alpha \times 1}$ to its first $\ell$ rows. We will need the following lemma, which is a direct consequence of [16, Lemma 1] together with [9, Lemma 2], as explained in the proof of [15, Theorem 4]. We recall the proof for convenience of the reader.

**Lemma 2** ([9, 15, 16]). If $I \subseteq [n]$ is such that $\#I < d_H(C_2^\perp, C_1^\perp)$, then $C_{2I} = C_I$.

**Proof.** Denoting by $C_I$ the shortened code of $C$ in $I$, that is,

$$C_I = \{c_I \mid c \in C, c_i = 0, i / \in I\} \subseteq \mathbb{F}_q^{\#I},$$

and similarly for $C_2$, [16] Lemma 1] states that

$$d_H(C_2^\perp, C_1^\perp) = \min\{\#I \mid \dim((C_2^\perp)^I / (C_1^\perp)^I) = 1\}.$$ 

On the other hand, [9, Lemma 2] implies that

$$\dim((C_2^\perp)^I / (C_1^\perp)^I) = \dim(C_I / C_{2I}),$$

hence the result follows.

Now we may prove the main result of this subsection:

**Theorem 1.** The previous secret sharing scheme has information rate $\ell/n$, reconstruction $r$, privacy $t$ and, for any $I \subseteq [n]$ with $\#I = d$, it holds that

$$\text{CO}(I) = \frac{\ell(d - k + k_2)}{k - k_2}, \text{ or } \text{DB}(I) = \frac{\ell d}{k - k_2}.$$ 

**Proof.** We prove each item separatedly.

1. Reconstruction $r$: Take $I \subseteq [n]$ of size at least $r$. From

$$(S_2 | R_2 | 0)G_I = (S_2 | R_2)G_{1I}$$

we obtain $S_2$ by [5], since $\#I \geq n - d_H(C_1, C_2) + 1$. On the other hand,

$$(S_1 | R_1 | S_2^T)G_I = (S_1 | R_1)G_{1I} + S_2^T G_{3I}.$$ 

By substracting $S_2^T G_{3I}$, we see that we only need to decode

$$(S_1 | R_1)G_{1I}.$$ 

Again, we obtain $S_1$ by [5], since $\#I \geq n - d_H(C_1, C_2) + 1$. Hence, we have obtained the whole secret $S$ and proven that $H(S | X_I) = 0$.

2. Privacy $t$: Take $I \subseteq [n]$ of size at most $t$. The eavesdropper obtains

$$W_I = \begin{pmatrix} S_1 & R_1 & S_2^T \end{pmatrix} G_I.$$ 

This random variable $W_I$ has support inside the linear code

$$C_I^T = \{X G_I \mid X \in \mathbb{F}_q^{\alpha \times k} \} \subseteq \mathbb{F}_q^{\alpha \times \#I}.$$
Recall from [7, Theorem 2.6.4] that, if a random variable $X$ has support in the set $\mathcal{X}$, then $H(X) \leq \log_q(\#\mathcal{X})$. Hence

$$H(W_I) \leq \log_q(\#C_{2I}^\alpha) = \dim(C_{2I}^\alpha) = \alpha \dim(C_{2I}).$$

On the other hand, using the analogous notation $C_{2I}^\alpha$ for $C_2$ instead of $C$, it holds that

$$H(W_I | S) = \log_q(\#C_{2I}^\alpha) = \dim(C_{2I}^\alpha) = \alpha \dim(C_{2I}),$$

since, given a value of $S$, the variable $W_I$ is a uniform random variable on an affine space obtained by translating the vector space $C_{2I}^\alpha$. Hence we obtain that

$$I(S; W_I) = H(W_I) - H(W_I | S) = \alpha(\dim(C_I) - \dim(C_{2I})) = 0,$$

where the last equality follows from the previous lemma since $\#I \leq d_H(C_1, C_2) + 1$.

3. Communication overhead and decoding bandwidth: Take $I \subseteq [n]$ of size $d$. By definition, $E_{I,i}(c_i)$ is the $i$-th column of the matrix

$$(S_1 | R_1 | S_2^T)G \in \mathbb{F}_q^{\ell \times n},$$

for each $i \in I$. Thus we may obtain $S_1$ and $S_2^T$ by $\ell$, since $\#I \geq n - d_H(C, C_2) + 1$, and hence we may obtain the whole secret $S$ or, in other words, $\ell$ is satisfied.

On the other hand, $E_{I,i}(c_i)$ is the uniform random variable on

$$C_\ell^I = \{Xg(i) \mid X \in \mathbb{F}_q^{\ell \times k}\}.$$

Since $C$ is not degenerate (see Remark 1), it follows that

$$H_q(\alpha)(E_{I,i}(c_i)) = \frac{\dim(C_\ell^I)}{\alpha} = \frac{\ell}{\alpha}.$$

Hence, we have that

$$\text{DB}(I) = \frac{\ell \#I}{\alpha} = \frac{\ell d}{k - k_2}, \quad \text{and} \quad \text{CO}(I) = \text{DB}(I) - \ell = \frac{\ell(d - k + k_2)}{k - k_2}.$$

\[\Box\]

**Remark 2.** Observe that, when taking $C = C_1$, our Construction 1 can be seen as an $\ell$-folded nested coset coding scheme as in Definition 3. Moreover, our Construction 1 specializes to the construction in [4, Section III] by choosing $C_2 \subsetneq C_1 \subseteq C$ as nested Reed-Solomon codes of dimensions $k_2$, $k_1$ and $k$, respectively, and where $t = k_2$, $r = k_1$ and $d = k$. The communication overhead for a set $I \subseteq [n]$ of size $d$ in the previous theorem coincides then with that in [4, Equation (4)], which is optimal due to [4].

**Remark 3.** Observe that $H(E_{I,i}(c_i)) = \ell/\alpha$ does not depend on $I$ or $i \in I$. Therefore, any choice of $d$ available parties may be contacted with the same communication costs and moreover, the information transmission from these parties may be performed in a balanced and parallel way.
Lemma 3

for convenience of the reader:

and the row vectors $g_j$ for $j = 1, 2, \ldots, h$.

Next assume that there exist a sequence of nested linear codes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ such that

$$r = \dim(\mathcal{C}_1, \mathcal{C}_2) + 1.$$ We call the parameter $r = \dim(\mathcal{C}_1, \mathcal{C}_2) + 1$ the parameter $r$.

Proof. We define also the positive integers $\alpha_j = k^{(j)} - k_2 - \ell - h - j$, $1 \leq j \leq h$, and $\alpha = \operatorname{LCM}(\alpha_1, \alpha_2, \ldots, \alpha_h)$ (observe that $\alpha_j = \alpha_j + 1$). We also set $\alpha_0 = 1$ for convenience. On the other hand, take a generator matrix of $\mathcal{C}_1$ of the form

$$G = \begin{pmatrix} G_1 & G_2 & \cdots & G_h \end{pmatrix} \in \mathbb{F}_q^{k \times n},$$

where $G_2 \in \mathbb{F}_q^{k_2 \times n}$ is a generator matrix of $\mathcal{C}_2$, $(G_1^T, G_2^T)^T \in \mathbb{F}_q^{k_1 \times n}$ is a generator matrix of $\mathcal{C}_1$, and the row vectors $g_1, g_2, \ldots, g_h$ span a complementary of $\mathcal{C}_1$ in $\mathcal{C}_j$, for $1 \leq j \leq h - 1$.

We will need the following calculations, which we take from [1]. We include the short proof for convenience of the reader:

Lemma 3 ([1]). For any $j = 1, 2, \ldots, h$, it holds that

$$\frac{\ell \alpha}{\alpha_j} = \frac{\ell \alpha}{\alpha_0 \alpha_1} + \frac{\ell \alpha}{\alpha_1 \alpha_2} + \cdots + \frac{\ell \alpha}{\alpha_{j-1} \alpha_j}. \quad (8)$$

Proof. Since $\alpha_j = \alpha_j + 1$, it follows that

$$\frac{\ell \alpha}{\alpha_{j-1} \alpha_j} = \ell \alpha \left( \frac{1}{\alpha_j} - \frac{1}{\alpha_{j-1}} \right).$$

Using this and $\alpha_0 = 1$, we see that

$$\frac{\ell \alpha}{\alpha_0 \alpha_1} + \frac{\ell \alpha}{\alpha_1 \alpha_2} + \cdots + \frac{\ell \alpha}{\alpha_{j-1} \alpha_j} = \ell \alpha \left( \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_{j-1}} \right) = \frac{\ell \alpha}{\alpha_j}.$$
The secret space is again $\mathcal{A}^t = \mathbb{F}_q^{n \times \ell}$ with alphabet $\mathcal{A} = \mathbb{F}_q^\alpha$, and we also generate uniformly at random a matrix $R \in \mathbb{F}_q^{\alpha \times k_2}$. In this case, we divide $S$ and $R$ as follows:

$$S = \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_h \end{pmatrix}, \quad R = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_h \end{pmatrix}, \quad S_j \in \mathbb{F}_q^{\ell \alpha/(\alpha_j-\alpha_j) \times \ell} \text{ and } R_j \in \mathbb{F}_q^{\ell \alpha/(\alpha_j-\alpha_j) \times k_2}, \quad 1 \leq j \leq h.$$  

Next, we define the matrices

- $M_1 = (S_1 \ R_1 \ d_{1,1} \ d_{1,2} \ \ldots \ d_{1,h-2} \ d_{1,h-1}) \downarrow \ell \alpha/(\alpha_0 \alpha_1)$
- $M_2 = (S_2 \ R_2 \ d_{2,1} \ d_{2,2} \ \ldots \ d_{2,h-2} \ o) \downarrow \ell \alpha/(\alpha_1 \alpha_2)$
- $M_3 = (S_3 \ R_3 \ d_{3,1} \ d_{3,2} \ \ldots \ o \ o) \downarrow \ell \alpha/(\alpha_2 \alpha_3)$
- $\vdots$
- $M_{h-1} = (S_{h-1} \ R_{h-1} \ d_{h-1,1} \ 0 \ \ldots \ 0 \ 0) \downarrow \ell \alpha/(\alpha_{h-2} \alpha_{h-1})$
- $M_h = (S_h \ R_h \ 0 \ 0 \ \ldots \ 0 \ 0) \downarrow \ell \alpha/(\alpha_{h-1} \alpha_h)$

where the column vectors $d_{u,v} \in \mathbb{F}_q^{\ell \alpha/(\alpha_u-\alpha_v) \times 1}$ are defined column-wise iteratively as follows:

- for $v = 1, 2, \ldots, h - 1$, the components of the $v$-th column, which is the matrix

$$\begin{pmatrix} d_{1,v} \\ d_{2,v} \\ \vdots \\ d_{h,v} \end{pmatrix} \in \mathbb{F}_q^{\ell \alpha/\alpha_h \times 1},$$

are the components (after some fixed rearrangement) of the matrix

$$(S_{h-v+1} | d_{h-v+1,1} | d_{h-v+1,2} | \ldots | d_{h-v+1,v-1}) \in \mathbb{F}_q^{\ell \alpha/\alpha_h \times \alpha_h \times v-1}.$$  

Observe that the sizes of all matrices in this construction fit due to (5). For convenience, we define the matrices

$$M'_j = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_j \end{pmatrix} \in \mathbb{F}_q^{\ell \alpha/\alpha_j \times k_{(j)}}$$  

for $1 \leq j \leq h$. Observe also that $M'_h \in \mathbb{F}_q^{\alpha \times k_{(h)}$, since $\ell \alpha/\alpha_h = \alpha$. Finally, the $i$-th share, $1 \leq i \leq n$, is the $i$-th column of the matrix

$$C = (c_1, c_2, \ldots, c_n) = M'_h G \in \mathbb{F}_q^{\alpha \times n},$$

which is a column vector $c_i \in \mathbb{F}_q^{\alpha \times 1}$, that is, a symbol in our alphabet $\mathcal{A} = \mathbb{F}_q^\alpha$.

For a set $I \subseteq [n]$ of size $d_j$ and $i \in I$, the corresponding preprocessing function is

$$E_{i,j}^{(j)} : \mathbb{F}_q^{\alpha \times 1} \rightarrow \mathbb{F}_q^{\ell \alpha/\alpha_j},$$

where $E_{i,j}^{(j)}(c_i)$ is obtained by restricting $c_i \in \mathbb{F}_q^{\alpha \times 1}$ to its first $\ell \alpha/\alpha_j$ rows.

We may now establish the main result of this subsection:
Theorem 2. The previous secret sharing scheme has information rate $\ell/n = (k_1 - k_2)/n$, reconstruction $r$, privacy $t = d_H(C^+_2) - 1$ and, for any $r \leq d \leq n$ and any $I \subseteq [n]$ with $\#I = d$, it holds that

$$\text{CO}(I) = \frac{\ell(d_j - k^{(j)} + k_2)}{k^{(j)} - k_2}, \quad \text{or} \quad \text{DB}(I) = \frac{\ell d_j}{k^{(j)} - k_2},$$

where $d = d_j$ and $j = n + 1 - d$.

Proof. We proceed as in the proof of Theorem 1:

1. Reconstruction $r$: Take $I \subseteq [n]$ of size at least $r$. From $(S_h|R_h|0)G_I = (S_h|R_h)G_{II}$ we obtain $S_h$ by (3), since $\#I \geq n - d_H(C_1, C_2) + 1$. By definition, we have obtained $d_{u,1}$, $1 \leq u \leq h - 1$. Hence, subtracting $d_{h-1,1}g^{(h-1)}_I$ from $(S_{h-1}|R_{h-1}|0)G_I$ as in the proof of Theorem 1, we may obtain $(S_{h-1}|R_{h-1})G_{II}$, and thus we obtain $S_{h-1}$ by (3), since $\#I \geq n - d_H(C_1, C_2) + 1$. Now, we have also obtained $d_{u,2}$, $1 \leq u \leq h - 2$. Proceeding iteratively in the same way, we see that we may obtain all the matrices $S_j$, for $j = 1, 2, \ldots, h$, and thus we obtain the whole secret $S$. In particular, we have shown that $H(S|X_I) = 0$.

2. Privacy $t$: Take $I \subseteq [n]$ of size at most $t$, and assume that the eavesdropper obtains $W_I = C_I = M'_I G_I$. As in the proof of Theorem 1, we see that

$$I(S; W_I) = H(W_I) - H(W_I \mid S) \leq \alpha(\dim(F_q^{\#I}) - \dim(C_2t)) = 0,$$

where the last equality follows from Lemma 2 since

$$\#I \leq d_H(C^+_2) - 1 = d_H(C^+_2, \{0\}) - 1 = d_H(C^+_2, (F_q^{\#I})^\perp) - 1.$$

3. Communication overhead and decoding bandwidth: Take $I \subseteq [n]$ of size $d_j$ for some $1 \leq j \leq h$. By definition, $E^{(j)}_{1,I}(c_i)$ is the $i$-th column of the matrix

$$M'_j G \in F_{q}^{\alpha/\alpha_j \times n},$$

for each $i \in I$. As in the proof of Theorem 1, we may obtain the matrix $M'_j$ by (3), since $\#I \geq n - d_H(C^{(j)}, C_2) + 1$. By definition, the matrices $S_1, S_2, \ldots, S_j$ are contained in $M'_j$. On the other hand, the vectors $d_{1,h-j}, d_{2,h-j}, \ldots, d_{j,h-j}$ are also contained in $M'_j$, and from them we obtain by definition $S_{j+1}$ and $d_{j+1,1}, d_{j+1,2}, \ldots, d_{j+1,h-j-1}$. Now, the vectors $d_{1,h-j-1}, d_{2,h-j-1}, \ldots, d_{j,h-j-1}$ are contained in $M'_j$ and we also have $d_{j+1,h-j-1}$, hence we may obtain by definition $S_{j+2}$ and $d_{j+2,1}, d_{j+2,2}, \ldots, d_{j+2,h-j-2}$.

Continuing iteratively in this way, we may obtain all $S_1, S_2, \ldots, S_h$ and hence the secret matrix $S$. In other words, we have proven that the preprocessing functions satisfy (1).

Finally, $E^{(j)}_{1,I}(c_i)$ is the uniform random variable on

$$C^{(j)\alpha/\alpha_j} = \{XG^{(j)}_{1,I} \mid X \in F_{q}^{\alpha/\alpha_j \times k^{(j)}}\},$$
where

\[ G^{(j)} = \begin{pmatrix} G_c \\ G_2 \\ g_{h-1} \\ g_{h-2} \\ \vdots \\ g_j \end{pmatrix} \in \mathbb{F}_q^{k \times n}. \]

Since \( C^{(j)} \) is not degenerate (see Remark 4), it follows that

\[ H_{q^\alpha}(E^{(j)}_{I,i}(c_i)) = \frac{\dim(C^{(j)}_{\alpha/\alpha_j})}{\alpha_j} = \frac{\ell}{\alpha_j} \cdot \frac{1}{\alpha_j} = \frac{\ell}{\alpha_j}. \]

Hence, we have that

\[ \text{DB}(I) = \frac{\ell \#I}{\alpha_j} = \frac{\ell d_j}{k^{(j)} - k_2}, \quad \text{and} \quad \text{CO}(I) = \text{DB}(I) - \ell = \frac{\ell(d_j - k^{(j)} + k_2)}{k^{(j)} - k_2}. \]



Remark 4. As in the previous subsection, our Construction 2 specializes to the construction in \cite[Section IV]{1} by choosing \( C_2 \subset C_1 \) and \( C^{(j)} \) as nested Reed-Solomon codes of dimensions \( k_2, k_1 \) and \( k^{(j)} \), respectively, and where \( t = k_2, r = k_1 \) and \( d_j = k^{(j)} \) (observe that, in this case, the MDS gap is \( \gamma = r - k_1 = 0 \)). The communication overhead for a set \( I \subseteq [n] \) of size \( d_j \) in the previous theorem coincides then with that in \cite[Equation (4)]{1}, which is optimal due to \cite{3}.

Remark 5. Observe, as in the previous subsection, that \( H(E^{(j)}_{I,i}(c_i)) = \ell/\alpha_j \) depends on \( j \) (that is, on \( d = d_j \)) but does not depend on \( I \) or \( i \in I \). Therefore, for a fixed \( r \leq d \leq n \), any \( d \) available parties may be contacted with the same communication costs, and the information transmission from these parties may be performed in a balanced and parallel way.

4 Communication efficient secret sharing schemes based on algebraic geometry codes

In this section, we will see that algebraic geometry codes fit into the previous two constructions of communication efficient secret sharing schemes, and allow to obtain schemes with arbitrarily large length \( n \) while keeping the field size \( q \) fixed or bounded and keeping low communication overhead, although not necessarily optimal.

Algebraic geometry codes fit in both Construction 1 and Construction 2 since it is easy to find sequences of nested algebraic geometry codes contained and containing a given one, as we will see. However, we will only focus on Construction 2 for brevity, since the arguments for Construction 1 are similar and simpler.

We refer to \cite{12, 19, Section 12.7} and \cite{24} for general references on algebraic geometry codes.

Consider an irreducible projective curve \( \mathcal{X} \) over \( \mathbb{F}_q \) (which in this paper means irreducible over the algebraic closure of \( \mathbb{F}_q \)) with algebraic function field \( \mathcal{F} \), and let \( g = g(\mathcal{X}) = g(\mathcal{F}) \) be its genus. Points in \( \mathcal{X} \) correspond with places in \( \mathcal{F} \) and we say that they are rational if they are rational over \( \mathbb{F}_q \) (have coordinates over \( \mathbb{F}_q \)). A divisor over \( \mathcal{X} \) is a formal sum \( D = \sum_{P \in \mathcal{X}} \mu_P P \), for integers \( \mu_P \in \mathbb{Z} \) which are all zero except for a finite number. The support of \( D \) is defined
as \( \{ P \in \mathcal{X} \mid \mu_P \neq 0 \} \), and \( D \) is called rational if all points in its support are rational. We define the degree of the rational divisor \( D \) as \( \deg(D) = \sum_{P \in \mathcal{X}} \mu_P \in \mathbb{Z} \).

On the other hand, for divisors \( D = \sum_{P \in \mathcal{X}} \mu_P P \) and \( E = \sum_{P \in \mathcal{X}} \lambda_P P \), we write \( D \leq E \) if \( \mu_P \leq \lambda_P \), for all \( P \in \mathcal{X} \), and for an algebraic function \( f \in \mathcal{F} \), we define its divisor as \( (f) = \sum_{P \in \mathcal{X}} \nu_P(f)P \), where \( \nu_P \) is the valuation at the point \( P \) (see [24, Definition 1.1.12] or [12, Definition 2.36]). Hence we may define the Riemann-Roch space (see [24, Definition 1.1.4] or [12, Definition 2.15]) of a divisor \( D \) as the vector space over \( \mathbb{F}_q \) given by:

\[
\mathcal{L}(D) = \{ f \in \mathcal{F} \mid (f) + D \geq 0 \}.
\]

Finally, for rational divisors \( D = P_1 + P_2 + \cdots + P_n \) and \( G \) over \( \mathcal{X} \) with disjoint supports and where the points \( P_i \) are pairwise distinct, we define the corresponding algebraic geometry code (see [24, Equation (2.3)] or [12, Definition 2.64]), or AG code for short, as the following code:

\[
\mathcal{C}_L(D,G) = \{ (f(P_1), f(P_2), \ldots, f(P_n)) \mid f \in \mathcal{L}(G) \} \subseteq \mathbb{F}_q^n.
\]

We will need the following two well-known results on the parameters of algebraic geometry codes. The first is the well-known Goppa-bound [24, Corollary 2.2.3 (a)] (see also [12, Theorem 2.65]), together with its dual statement (see [24, Theorem 2.2.7] and [24, Theorem 2.2.8]):

**Lemma 4 (Goppa bound)**. If \( \deg(G) < n \) and \( \mathcal{C} = \mathcal{C}_L(D,G) \), or if \( 2g - 2 < \deg(G) < n \) and \( \mathcal{C} = \mathcal{C}_L(D,G)^\perp \), then

\[
d_H(C) \geq n - \dim(C) - g + 1.
\]

The following lemma is [24, Corollary 2.2.3(b)] (see also [12, Theorem 2.65]):

**Lemma 5**. If \( 2g - 2 < \deg(G) < n \), then

\[
\dim(\mathcal{C}_L(D,G)) = \deg(G) - g + 1.
\]

### 4.1 Algebraic geometry codes for Construction 2

Let the notation be as in the beginning of this section. In the following result, we gather the parameters obtained in Construction 2 when using AG codes. The corresponding result for Construction 1 can be obtained in an analogous way.

**Proposition 2**. Let \( \mathcal{X} \) be an irreducible projective curve over \( \mathbb{F}_q \), let \( P_1, P_2, \ldots, P_n \) be pairwise distinct rational points on \( \mathcal{X} \), and let \( G_2, G_1 = G^{(h)} \), \( G^{(h-1)} \), \ldots, \( G^{(2)} \), \( G^{(1)} \) be rational divisors on \( \mathcal{X} \) whose supports do not contain the previous points \( P_i, 1 \leq i \leq n \), and where \( h = n - \deg(G_1) > 0 \). Assume also that

\[
G_2 \preceq G_1 = G^{(h)} \preceq G^{(h-1)} \preceq \cdots \preceq G^{(2)} \preceq G^{(1)},
\]

\[
2g - 2 < \deg(G_2) < \deg(G_1) < n \quad \text{and} \quad \deg(G^{(j-1)}) = \deg(G^{(j)}) + 1, \quad \text{for} \quad j = 2, 3, \ldots, h \quad \text{(observe that} \quad \deg(G^{(1)}) = n - 1 \).
\]

The secret sharing scheme in Construction 2 using the linear codes

\[
\mathcal{C}_2 = \mathcal{C}_L(D,G_2), \quad \mathcal{C}_1 = \mathcal{C}_L(D,G_1), \quad \mathcal{C}^{(j)} = \mathcal{C}_L(D,G^{(j)}),
\]

of dimensions \( k_2, k_1 \) and \( k^{(j)} \), respectively, for \( 1 \leq j \leq h \), has information rate \( \ell/n = (k_1 - k_2)/n = (\deg(G_1) - \deg(G_2))/n \), reconstruction \( r = k_1 + g = \deg(G_1) + 1 \), privacy \( t = k_2 - g = \deg(G_2) - 2g + 1 \), and, for any \( r \leq d \leq n \) and any \( I \subseteq [n] \) of size \( d \), it holds that

\[
\text{CO}(I) = \frac{\ell(d_j - k^{(j)} + k_2)}{k^{(j)} - k_2} = \frac{\ell(t + 2g)}{d - t - 2g}, \quad \text{or} \quad \text{DB}(I) = \frac{\ell d_j}{k^{(j)} - k_2} = \frac{\ell d}{d - t - 2g},
\]

where \( d = d_j = k^{(j)} + g = \deg(G^{(j)}) + 1 \) and \( j = n+1-d \).
Proof. It follows from combining Theorem 2, Lemma 4 and Lemma 5. Observe that, since \( G_{2} \leq G_{1} \) and \( G^{(j)} \leq G^{(j-1)} \), then \( L(G_{2}) \leq L(G_{1}) \) and \( L(G^{(j)}) \leq L(G^{(j-1)}) \) and hence \( C_{2} \leq C_{1} \) and \( C^{(j)} \leq C^{(j-1)} \), \( 2 \leq j \leq h \). Moreover, \( C_{2} \neq C_{1} \) and \( \dim(C^{(j-1)}) = \dim(C^{(j)}) + 1, 2 \leq j \leq h \), by Lemma 6 since \( 2g - 2 < \deg(G_{2}) < \deg(G_{1}) \leq \deg(G^{(1)}) < n \) and \( \deg(G^{(j-1)}) = \deg(G^{(j)}) + 1, 2 \leq j \leq h \).

Observe that, if \( r \leq d < d' \leq n \), then
\[
\frac{\ell(t+2g)}{d-t-2g} > \frac{\ell(t+2g)}{d'-t-2g},
\]
and therefore the communication overhead decreases as the number of contacted shares \( d \) increases, as expected.

For more concrete constructions and existential proofs and to obtain the largest possible lengths, it is usual to consider the so-called one-point algebraic geometry codes. These are \( \text{AG} \) codes where the support of the divisor \( G \) is constituted by a single point. That is, \( G = \mu Q \) for some rational point \( Q \in \mathcal{X} \) and integer \( \mu \in \mathbb{Z} \). It holds that \( \deg(G) = \mu \). The next consequence follows:

**Corollary 1.** If there exists an irreducible projective curve \( \mathcal{X} \) over \( \mathbb{F}_{q} \) with \( N \geq n + 1 \) rational points and genus \( g \), then for any \( 0 \leq t < r \leq n \) with \( r - t > 2g \), there exists a secret sharing scheme with information rate \( \ell/n = (r - t - 2g)/n \), reconstruction \( r \), privacy \( t \) and, for any \( r \leq d \leq n \) and any \( I \subseteq [n] \) of size \( d \), it holds that
\[
\text{CO}(I) = \frac{\ell(t + 2g)}{d - t - 2g}, \quad \text{or} \quad \text{DB}(I) = \frac{\ell d}{d - t - 2g}.
\]

**Proof.** Take pairwise distinct rational points \( Q, P_{1}, P_{2}, \ldots, P_{n} \in \mathcal{X} \), and define \( \mu_{2} = t + 2g - 1, \mu_{1} = r - 1 \) and \( \mu^{(j)} = d_{j} - 1 \), where \( d_{j-1} = d_{j} + 1 \), for \( j = h, h-1, \ldots, 2, 1 \) and \( h = n - r + 1 \). Observe that \( t \geq 0 \) implies that \( \mu_{2} > 2g - 2, r > t + 2g \) implies that \( \mu_{1} > \mu_{2} \), and \( r \leq n \) implies that \( \mu_{1} < n \). Defining \( G_{2} = \mu_{2}Q, G_{1} = \mu_{1}Q \) and \( G^{(j)} = \mu^{(j)}Q, 1 \leq j \leq h \), the assumptions in the previous proposition are satisfied and thus the result follows.

**Remark 6.** Observe that the parameters in the previous corollary are close to the optimal values, in view of Lemma 7, whenever the genus \( g \) is close to 0. As a particular case, taking curves of genus \( g = 0 \) (such as the projective line, which is the case of Reed-Solomon codes), we obtain schemes with optimal parameters, which are the ones obtained in [1] and [7].

In this way, we have reduced the problem of finding secret sharing schemes with low decoding bandwidth to the well studied problem of finding irreducible projective curves with many rational points and low genus [14, 24, 23].

In the next subsection, we will give a well-known explicit family of algebraic geometry codes and apply it to the previous corollary.

### 4.2 Using Hermitian codes

In this subsection, we apply Corollary 1 to the case where \( \mathcal{X} \) is a Hermitian curve. These curves give the so-called Hermitian codes (see [23, 26, 27] Section VI or [11] Section VI), which are not MDS but are considered in many cases in practice to be better than Reed-Solomon codes for the same information rate, since they are considerably longer and still have a large minimum Hamming distance (See [27] Section VI for a discussion).
N = curve X defined by \( x^{u+1} - y^u z - y z^u = 0 \). This curve is called a Hermitian curve and has \( N = u^3 + 1 = \sqrt{q^3} + 1 \) rational points, and genus

\[
g = g(X) = \frac{u(u-1)}{2} = \frac{q-\sqrt{q}}{2}.
\]

Specializing Corollary 1 to these curves, we obtain the following result:

**Proposition 3.** Let \( X \) be a Hermitian curve over \( \mathbb{F}_q \), where \( q = u^2 \). Let \( Q, P_1, P_2, \ldots, P_u \) be its \( N = u^3 + 1 \) rational points and define \( D = P_1 + P_2 + \cdots + P_u \). Let \( n = u^3 \) and \( 0 \leq t < r \leq n \) be such that \( r - t > 2g+1 = q - \sqrt{q} + 1 \). The corresponding secret sharing scheme as in Corollary 1 constructed using \( X \) has information rate \( \ell/n = (r-t-q+\sqrt{q})/(\sqrt{q}) \), reconstruction \( r \), privacy \( t \) and, for any \( r \leq d \leq n \) and any \( I \subseteq [n] \) of size \( d \), it holds that

\[
CO(I) = \frac{\ell(t+q-\sqrt{q})}{d-t-q+\sqrt{q}}, \text{ or } DB(I) = \frac{\ell d}{d-t-q+\sqrt{q}}.
\]

These schemes can be explicitly constructed using explicit descriptions of the generator matrices of the involved Hermitian codes, which can be found in [23][20].

### 4.3 Asymptotic behaviour for a fixed finite field

In this subsection, we use the previous secret sharing schemes based on one-point AG codes to obtain sequences of schemes that are communication efficient and whose lengths go to infinity while being defined over a fixed finite field \( \mathbb{F}_q \). We also treat only Construction 2 for brevity, being Construction 1 analogous.

As observed in Remark 6, Corollary 1 states the existence of communication efficient schemes depending on the existence of irreducible projective curves \( X \) over \( \mathbb{F}_q \) with as many rational points and small genus as possible. Therefore, it will be essential to make use of Ihara’s constant

\[
A(q) = \limsup_{g(X) \to \infty} \frac{N(X)}{g(X)}
\]  

(14)

where the limit is taken over all irreducible projective curves \( X \) over \( \mathbb{F}_q \) of genus \( g(X) > 0 \), and where \( N(X) \) denotes the number of rational points in \( X \). Serre’s lower bound [20] and the Drinfeld-Vlăduț upper bound [25] state that

\[
c \log(q) \leq A(q) \leq \sqrt{q} - 1,
\]  

(15)

for a constant \( c > 0 \) that does not depend on \( q \), and where the equality \( A(q) = \sqrt{q} - 1 \) holds if \( q \) is a perfect square [14]. See also [12] Section 2.9], [19] Section 12.7.7] and [24] Chapter 7] for more details on Ihara’s constant and asymptotic behaviour of AG codes.

We will consider sequences of irreducible projective curves \( (X_i)_{i=1}^{\infty} \) such that \( N(X_i) \to \infty \) and

\[
\lim_{i \to \infty} \frac{N(X_i)}{g(X_i)} = A(q).
\]  

(16)

We may now state the following asymptotic consequence of Corollary 1:

**Proposition 4.** For any \( 0 \leq T < R < 1 \) with \( R - T > 2/A(q) \), there exist strictly increasing sequences of positive integers \( (n_i)_{i=1}^{\infty}, (\ell_i)_{i=1}^{\infty}, (r_i)_{i=1}^{\infty} \) and \( (t_i)_{i=1}^{\infty} \), and a sequence of secret sharing schemes such that, for large enough \( i \), it holds that \( 0 < r_i - t_i \leq \ell_i \leq n_i \), the \( i \)-th scheme has
length $n_i$, information rate $\ell_i/n_i$, reconstruction $r_i$, privacy $t_i$ and, for any $R \leq D \leq 1$ and any $I \subseteq [n_i]$ of size $d_i = [Dn_i]$, the corresponding decoding bandwidth is $DB(I) = DB_i$, where

$$L = \lim_{i \to \infty} \frac{\ell_i}{n_i} = R - T - \frac{2}{A(q)}, \quad R = \lim_{i \to \infty} \frac{r_i}{n_i}, \quad T = \lim_{i \to \infty} \frac{t_i}{n_i}, \quad D = \lim_{i \to \infty} \frac{d_i}{n_i},$$

and

$$LD \leq \limsup_{i \to \infty} \frac{DB_i}{n_i} \leq \frac{LD}{D - T - 2/A(q)},$$

being the first term in (17) the optimal value for $\limsup_{i \to \infty} DB_i/n_i$.

**Proof.** Take a sequence of irreducible projective curves $(X_i)_{i=1}^{\infty}$ satisfying (15) and define $n_i = N(X_i) - 1$, $r_i = \lceil Rn_i \rceil$ and $t_i = \lceil Tn_i \rceil$, which satisfy $r_i - t_i > 2g(X_i)$ for $i$ large enough. The result then follows by defining divisors and one-point AG codes as in the proof of Corollary 11.

**Remark 7.** Observe that (15) implies that, as $q \to \infty$, it holds that $2/A(q) \to 0$, and thus by (17) it follows that the asymptotic decoding bandwidth can be as close to the optimal value as wanted for large enough finite fields.

### 4.4 Using García-Stichtenoth’s second tower

In this subsection, we apply Proposition 4 to the so-called second tower of projective curves $(X_i)_{i=1}^{\infty}$ introduced by García and Stichtenoth in [10]. This sequence of projective curves is defined over a field whose size $q$ is a perfect square and satisfies that $A(q) = \sqrt{q} - 1$, hence being asymptotically optimal.

Moreover, they are explicitly defined in [10], and there are efficient algorithms to construct the corresponding algebraic geometry codes, as shown in [22] (the $i$-th algorithm has complexity $O(n_i^2 \log(n_i)^3)$, begin $n_i$ the length of the $i$-th code). We recall here that this sequence of codes is one of the few known sequences that has better asymptotic parameters than the existencial Gilbert-Varshamov bound.

Assume again that the field size is a perfect square $q = u^2$, and consider García and Stichtenoth’s second tower of projective curves $(X_i)_{i=1}^{\infty}$ from [10]. The $i$-th curve $X_i$ has $N_i > u^r(u-1)$ rational points, and its genus is given by

$$g_i = g(X_i) = \begin{cases} (u^r - 1)^2 & \text{if } i \text{ is even}, \\ (u^{i/2} - 1)(u^{i/2} - 1) & \text{if } i \text{ is odd}. \end{cases}$$

Specializing Proposition 4 to this tower of curves, we obtain the following result:

**Proposition 5.** For any finite field $\mathbb{F}_q$ whose size $q$ is a perfect square, and for any $0 \leq T < R \leq 1$ with $R - T > 2/\sqrt{q} - 1$, there exist strictly increasing sequences of positive integers $(n_i)_{i=1}^{\infty}$, $(\ell_i)_{i=1}^{\infty}$, $(r_i)_{i=1}^{\infty}$ and $(t_i)_{i=1}^{\infty}$, and a sequence of secret sharing schemes defined over $\mathbb{F}_q$ such that, for large enough $i$, it holds that $0 \leq r_i - t_i \leq \ell_i \leq n_i$, the $i$-th scheme has length $n_i$, information rate $\ell_i/n_i$, reconstruction $r_i$, privacy $t_i$ and, for any $R \leq D \leq 1$ and any $I \subseteq [n_i]$ of size $d_i = [Dn_i]$, the corresponding decoding bandwidth is $DB(I) = DB_i$, where

$$L = \lim_{i \to \infty} \frac{\ell_i}{n_i} = R - T - \frac{2}{\sqrt{q} - 1}, \quad R = \lim_{i \to \infty} \frac{r_i}{n_i}, \quad T = \lim_{i \to \infty} \frac{t_i}{n_i}, \quad D = \lim_{i \to \infty} \frac{d_i}{n_i},$$

and

$$LD \leq \limsup_{i \to \infty} \frac{DB_i}{n_i} \leq \frac{LD}{D - T - 2/\sqrt{q} - 1},$$

being the first term in (18) the optimal value for $\limsup_{i \to \infty} DB_i/n_i$. Moreover, the $i$-th scheme can be constructed explicitly with complexity $O(n_i^2 \log(n_i)^3)$.
5 Strongly secure and communication efficient schemes

In this section, we see how to obtain strongly secure secret sharing schemes as in [13] that are also communication efficient at the same time. As explained in Section 4, strongly secure schemes were introduced in [30] and allow to keep parts of the secret safe even when more than $t$ shares are eavesdropped. This feature makes them behave like perfect schemes in terms of security, while having the efficiency in storage of ramp schemes. Hence, they are preferable than both usual perfect and ramp schemes.

Although strongly secure secret sharing schemes where introduced in [30], we will use the extended definition from [15, Definition 18]:

**Definition 4** ([15, 50]). We say that a secret sharing scheme $F : A^t \rightarrow A^n$ is $\sigma$-strongly secure if, for all $I \subseteq [\ell]$ and all $J \subseteq [n]$ with $\#I + \#J \leq \sigma + 1$, it holds that

$$I(s_I; x_J) = 0,$$

where $s$ denotes the random variable corresponding to the secret in $A^t$, and $x = F(s)$ denotes the random variable corresponding to the shares in $A^n$.

Observe that, if a scheme is $\sigma$-strongly secure and $\sigma' \leq \sigma$, then it is also $\sigma'$-strongly secure. Hence we may define the maximum strength of a scheme $F : A^t \rightarrow A^n$ as

$$\sigma_{\text{max}}(F) = \max\{\sigma \mid F \text{ is } \sigma\text{-strongly secure}\}.$$

Observe that the reconstruction and privacy threshold values are also monotonous. Hence, we define for convenience the minimum reconstruction and maximum privacy, respectively, of a scheme $F : A^t \rightarrow A^n$ as

$$r_{\text{min}}(F) = \min\{r \mid F \text{ has } r\text{-reconstruction}\},$$

$$t_{\text{max}}(F) = \max\{t \mid F \text{ has } t\text{-privacy}\}.$$

We next give upper bounds on $\sigma_{\text{max}}(F)$ in terms of $r_{\text{min}}(F)$ and $t_{\text{max}}(F)$ that follow easily from the definitions. They will allow us to claim the optimality of the construction in Corollary 18 which we will present in Subsection 5.2.

**Lemma 6.** For a secret sharing scheme $F : A^t \rightarrow A^n$, it holds that

$$\sigma_{\text{max}}(F) \leq t_{\text{max}}(F) + \ell - 1 \leq r_{\text{min}}(F) - 1. \tag{19}$$

**Proof.** Take $I = [\ell]$. By the definition of $\sigma$-strong security, it holds that $I(s_I; x_J) = 0$ for any $J \subseteq [n]$ of size $\sigma + 1 - \ell$. Hence the scheme $F$ has $\sigma + 1 - \ell$-privacy. Since $t_{\text{max}}(F)$ is the maximum privacy threshold, it holds by definition that $\sigma_{\text{max}}(F) + 1 - \ell \leq t_{\text{max}}(F)$, hence the first bound follows. The second bound follows directly from (2).

Observe that a strongly secure secret sharing scheme not only satisfies that $\sigma_{\text{max}}(F) - \ell + 1$ is a privacy threshold but, in addition, if more than $\sigma_{\text{max}}(F) - \ell + 1$ shares are leaked and some information about the secret is obtained by an eavesdropper, we are guaranteed that no information about any collection of $\mu$ components of the secret $s$ are leaked if the number of leaked shares is at most $\sigma_{\text{max}}(F) - \mu + 1$, $\mu \leq \ell$. In particular, no information about any single component $s_i \in A$ of the secret is leaked if at most $\sigma_{\text{max}}(F)$ shares are obtained by the eavesdropper.

In the optimal case $\sigma_{\text{max}}(F) = t_{\text{max}}(F) + \ell - 1 = r_{\text{min}}(F) - 1$, if $t_{\text{max}}(F) + \mu$ shares are eavesdropped, for some $\mu > 0$, then no information is leaked about any collection of $\ell - \mu$
components of \(s, \mu \leq \ell\). Observe that in particular, no information is leaked about any component \(s_i \in \mathcal{A}\) if \(r_{\min}(F) - 1\) shares are eavesdropped, which is the same amount as in an optimal perfect scheme. However, the scheme \(F: \mathcal{A}^\ell \rightarrow \mathcal{A}^u\) is also an optimal ramp scheme, hence having the security advantages of perfect schemes and the efficiency in storage of ramp schemes.

### 5.1 Massey-type secret sharing schemes

We will consider Construction 2 combined with Massey-type secret sharing schemes, which are a modification of the schemes in [17]. As in previous sections, we omit the details regarding Construction 1 for brevity, since they are analogous.

We need two definitions first. For a set \(I \subseteq [n]\) and a linear code \(C \subseteq \mathbb{F}_q^n\), we say that \(I\) is an information set for \(C\) if \(\dim(C_I) = \dim(C)\) and, on the other hand, we define the shortened code of \(C\) in \(I\) as the linear code

\[
C^I = \{c_I | c \in C, c_i = 0, \forall i \notin I\} \subseteq \mathbb{F}_q^{|I|}.
\]

**Definition 5.** Let \(\mathcal{D}_1 = \mathcal{D}^{(h)} \supsetneq \mathcal{D}^{(h-1)} \supsetneq \ldots \supsetneq \mathcal{D}^{(2)} \supsetneq \mathcal{D}^{(1)} \subseteq \mathbb{F}_q^{\ell+n}\) be linear codes where \(\mathcal{D}_{[\ell]} = \mathbb{F}_q^\ell\) and \([\ell + 1, \ell + n]\) is information set for \(\mathcal{D}^{(1)}\). Assume also that \(k^{(j)} = \dim(\mathcal{D}^{(j)})\) and \(k^{(j)} = k^{(j+1)} + 1\), for \(j = 1, 2, \ldots, h - 1\), where \(k^{(h)} = \dim(\mathcal{D}_1)\), and define the codes

\[
\mathcal{C}_2 = \mathcal{D}^{(1)}_{[\ell+1, \ell+n]} \supsetneq \mathcal{C}_1 = \mathcal{D}^{(2)}_{[\ell+1, \ell+n]}, \quad \text{and} \quad \mathcal{C}^{(j)} = \mathcal{D}^{(j)}_{[\ell+1, \ell+n]}, \quad 1 \leq j \leq h.
\]

We say that the secret sharing scheme in Construction 2 is of Massey-type if it is constructed using linear codes \(\mathcal{C}_2 \supsetneq \mathcal{C}_1 \supset \mathcal{C}^{(h)} \supsetneq \mathcal{C}^{(h-1)} \supsetneq \ldots \supset \mathcal{C}^{(2)} \supset \mathcal{C}^{(1)} \subseteq \mathbb{F}_q^n\) as in (21).

**Remark 8.** Observe that the assumption that \([\ell + 1, \ell + n]\) is an information set for \(\mathcal{D}^{(1)}\) implies that the projection from any \(\mathcal{D}^{(j)}\) onto the coordinates in \([\ell + 1, \ell + n]\) is one to one (hence \(\dim(\mathcal{C}^{(j)}) = k^{(j)}\)), and together with the assumption \(\mathcal{D}_{[\ell]} = \mathbb{F}_q^\ell\), it implies moreover that \(\ell = \dim(\mathcal{C}_1) - \dim(\mathcal{C}_2)\).

In Construction 2, we will see the \(i\)-th component of the secret \(S \in \mathbb{F}_q^{\alpha \times \ell}\) as its \(i\)-th column \(S_i, 1 \leq i \leq \ell\). Since we are assuming that the random variable \(S\) is uniform on \(\mathbb{F}_q^{\alpha \times \ell}\), it follows that its columns are uniformly distributed in the alphabet \(\mathcal{A} = \mathbb{F}_q^\alpha\) and statistically independent. Hence we may state the following lemma, which is proven in [15] Theorem 19. We recall the proof for convenience of the reader.

**Lemma 7 ([15]).** If the columns of the secret \(S \in \mathbb{F}_q^{\alpha \times \ell}\) are statistically independent, it holds that

\[
I(S_I; X_J) = \sum_{j=1}^u I(S_{i_j}; X_J, S_{i_1}, S_{i_2}, \ldots, S_{i_{j-1}}),
\]

for all \(I = \{i_1, i_2, \ldots, i_u\} \subseteq [\ell]\) and all \(J \subseteq [n]\).

**Proof.** Since the columns in \(i\) are statistically independent, the following equalities follow from the chain rule of conditional entropy (see [7] Theorem 2.5.1]):

\[
I(S_I; X_J) = \sum_{j=1}^u H(S_{i_j}, S_{i_1}, S_{i_2}, \ldots, S_{i_u} \mid X_J)
\]

\[
= \sum_{j=1}^u H(S_{i_j} \mid X_J, S_{i_1}, S_{i_2}, \ldots, S_{i_{j-1}})
\]

\[
= \sum_{j=1}^u I(S_{i_j}; X_J, S_{i_1}, S_{i_2}, \ldots, S_{i_{j-1}}).
\]

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Now we are ready to prove the following result:

**Theorem 3.** With notation as in Definition 5 define the linear codes

\[ G_{2,i} = D_1^{[1|+n}\setminus\{i\}, \quad 1 \leq i \leq \ell. \]

With parameters \(d_1, d_2, \ldots, d_h\) as in Construction 2, the Massey-type secret sharing scheme \(F: \mathcal{A}^\ell \rightarrow \mathcal{A}^n\) in Definition 5 has parameters and communication efficiency as in Theorem 2, and moreover it holds that

\[ \sigma_{\text{max}}(F) \geq \min\{d_H(G_{2,i}^+) \mid 1 \leq i \leq \ell\} - 1. \]

**Proof.** It holds that \(\ell = \dim(C_1) - \dim(C_2)\) and \(k^{(j)} = \dim(C^{(j)}), 1 \leq j \leq h\), as in Remark 8, hence all the results in Theorem 2 hold.

On the other hand, let \(F^{(1)} \in \mathbb{F}_q^{k^{(1)} \times (\ell+n)}\) be a generator matrix of \(D^{(1)}\) such that the upper leftmost \(\ell \times \ell\) submatrix is the identity (recall that \(D_{1|\ell} = \mathbb{F}_q^\ell\)), and the first \(k_j^{(1)}\) rows form a generator matrix of \(D^{(j)}, 1 \leq j \leq h\).

For a fixed \(1 \leq i \leq \ell\), it holds that \(D_{1|i} = \mathbb{F}_q\) since \(D_{1|i} = \mathbb{F}_q^\ell\), and it holds that \([\ell+n] \setminus \{i\}\) is an information set for \(D^{(1)}\) since \([\ell+1, \ell+n]\) is an information set for that same code. Thus \(1 = \dim(G_{1,i}) - \dim(G_{2,i})\), where \(G_{1,i} = D_{1|[\ell+n]\setminus\{i\}}\). Hence we may see Construction 2 using the Massey-type scheme from Definition 5 as a scheme where the secret is the \(i\)-th column of \(S_i \in \mathbb{F}_q^{n \times 1}\) and the random keys are \(S_1, S_2, \ldots, \tilde{S}_i, \ldots, S_\ell, R_1, R_2, \ldots, R_{h_k} \in \mathbb{F}_q^{\alpha \times 1}\), where there are \(\ell+n-1\) shares, given by the columns of

\[ D = (d_1, d_2, \ldots, d_{\ell+n-1}) = M_{h_1}'F^{(1)}_{[\ell+n]\setminus\{i\}} \in \mathbb{F}_q^{\alpha \times (\ell+n-1)}, \]

where \(M_{h_1}'\) is as in 9. By a similar argument to that of the proof of Theorem 1 and Theorem 2, it holds that

\[ I(S_i; X_J, S_J) = 0, \quad (22) \]

where \(I \subseteq [\ell] \setminus \{i\}, J \subseteq [n]\) and \(#I + #J \leq d_H(G_{2,i}^+)\).

In particular, if \(I = \{i_1, i_2, \ldots, i_u\} \subseteq [\ell], J \subseteq [n]\) and \(#I + #J \leq \min\{d_H(G_{2,i}^+) \mid 1 \leq i \leq \ell\}\), then it follows from (22) that

\[ I(S_{i_j}; X_J, S_{i_1}, S_{i_2}, \ldots, S_{i_{j-1}}) = 0, \quad (23) \]

for \(j = 1, 2, \ldots, u\). Therefore, the result follows from (23) and Lemma 7.

**5.2 Strongly secure and communication efficient schemes based on MDS codes**

In this subsection, we give the first construction of a ramp secret sharing scheme that has optimal threshold parameters with respect to its information rate, optimal strong security and optimal communication efficiency.

We have the following consequence of Theorem 3:

**Corollary 2.** Let the notation be as in Definition 5 and assume that all the nested linear codes \(D^{(j)}\) are MDS codes, for \(j = 1, 2, \ldots, h\). Set also \(k_1 = \dim(C_1) = k^{(h)}\) and \(k_2 = \dim(C_2) = k_1 - \ell\). Then the Massey-type secret sharing scheme \(F: \mathcal{A}^\ell \rightarrow \mathcal{A}^n\) in Definition 5 has information rate \(\ell/n = (k_1 - k_2)/n\), reconstruction \(r = r_{\min}(F) = k_1\), privacy \(t = t_{\max}(F) = k_2\) (hence \(\ell = r-t\) is
optimal by \(20\)), optimal decoding bandwidth (meaning equality in \(21\) and \(22\)) for any \(r \leq d \leq n\) and any set \(I \subseteq [n]\) of size \(d\), and in addition it has maximum strength

\[\sigma_{\text{max}}(F) = k_1 - 1 = t + \ell - 1 = r - 1,\]

which is optimal by \(19\). In particular, it holds that

\[I(S_I; X_J) = 0,\]

for all \(I \subseteq [\ell]\) and all \(J \subseteq [n]\) with \(#I + #J \leq k_1\).

Such sequence of nested MDS linear codes \(D_1 = D^{(h)} \subseteq D^{(h-1)} \subseteq \cdots \subseteq D^{(2)} \subseteq D^{(1)} \subseteq \mathbb{F}_q^{\ell+n}\)
exists if \(\ell + n \leq q\).

**Proof.** We just need to show that \(d_H(G_{2,i}^{(h)}) \geq k_1\), for \(i = 1, 2, \ldots, \ell\). Since \(D_1\) is MDS of dimension \(k_1^{(h)} = k_1\), the linear code \(G_{2,i}^{(h)}\) is again MDS, which implies in turn that \(G_{2,i}^{(h)}\) is MDS. The length of this latter code is \(\ell + n - 1\) and its dimension is \(\ell + n - k_1\), therefore \(d_H(G_{2,i}^{(h)}) = k_1\), and we are done.

To show the existence of such sequence of nested MDS linear codes when \(\ell + n \leq q\), we just need to take a sequence of Reed-Solomon codes defined by evaluating polynomials of increasing degrees and over the same \(\ell + n\) points in \(\mathbb{F}_q\). In the terminology of AG codes, we would choose \(X\) as the projective line over \(\mathbb{F}_q\), \(Q = \infty, P_1, P_2, \ldots, P_{\ell+n}\) pairwise distinct points in \(\mathbb{F}_q\), \(D = P_1 + P_2 + \cdots + P_{\ell+n}\), and \(D^{(j)} = C_{\ell}(D, (k^{(j)} - 1)(Q)), 1 \leq j \leq h\) (recall that \(g(X) = 0\)). \(\Box\)

Observe that the requirement on the field size \((q > \ell + n)\) is the same as in previous optimal strongly secure schemes \(18\), and the requirement on the alphabet size is the same as in previous optimal communication efficient schemes \(11\) \(13\).

For each component of the secret \(S_i \in A\), the schemes behaves as an optimal perfect scheme (as Shamir’s scheme), and for the entire secret \(S \in A^\ell\), it behaves as an optimal ramp scheme. Therefore, it has the security advantages of the first and the storage efficiency of the second, while being communication efficient at the same time.

### 5.3 Strongly secure and communication efficient schemes based on algebraic geometry codes

In this subsection, we extend Corollary\(2\) by using general AG codes. In this way, we overcome the limitation on the field size in the previous construction, at the cost of near optimality. We give a construction analogous to the one in Corollary\(1\).

**Corollary 3.** If there exists an irreducible projective curve \(X\) over \(\mathbb{F}_q\) with \(N \geq \ell + n + 1\) rational points and genus \(g\), then for any \(0 \leq t < r \leq n\) with \(r - t > 2g\), there exists a secret sharing scheme \(F : A^\ell \rightarrow A^n\) with information rate \(\ell/n = (r - t - 2g)/n\), reconstruction \(r\), privacy \(t\) and, for any \(r \leq d \leq n\) and any \(I \subseteq [n]\) of size \(d\), it holds that

\[\text{CO}(I) = \frac{\ell(t + 2g)}{d - t - 2g}, \quad \text{or} \quad \text{DB}(I) = \frac{\ell d}{d - t - 2g}.\]

In addition, the scheme has maximum strength

\[\sigma_{\text{max}}(F) = t + \ell - 1 = r - 2g - 2.\]

In particular, it holds that

\[I(S_I; X_J) = 0,\]

for all \(I \subseteq [\ell]\) and all \(J \subseteq [n]\) with \(#I + #J \leq t + \ell = r - 2g - 1\).
Proof. As in the proof of Corollary 1, take pairwise distinct rational points \( Q, P_1, P_2, \ldots, P_{t+n} \in \mathcal{X} \), where \( \ell = r-t-2g \), and define \( \mu_2 = t+2g-1, \mu_1 = r-1 \) and \( \mu^{(j)} = d_j-1 \), where \( d_j = d_j+1 \), for \( j = h, h-1, \ldots, 3, 2, d_h = r \) and \( h = n-r+1 \). Define \( E = P_1 + P_2 + \cdots + P_{t+n} \), and the AG codes

\[
D_1 = C_{\mathcal{L}}(E, \mu_1 Q), \quad \text{and} \quad D^{(j)} = C_{\mathcal{L}}(E, \mu^{(j)} Q),
\]

for \( 1 \leq j \leq h \). Since \( 2g-2 < \mu_1 < n \), it holds that \( \dim(D_1) = \mu_1 - g + 1 \geq \ell \) by Lemma 5. Hence by rearranging the points \( P_i \) if necessary, we may assume that \( D_1[\ell] = \mathbb{F}_q^\ell \). On the other hand, since \( 2g-2 < \mu^{(j)} = n-1 < n \leq \ell + n \), we have that \( [\ell + 1, \ell + n] \) is an information space for \( D^{(j)} \) by Lemma 5. Therefore the assumptions in Definition 5 are satisfied.

On the other hand, if \( D = P_{t+1} + P_{t+2} + \cdots + P_{t+n} \), then

\[
C_2 = C_{\mathcal{L}}(D, \mu_1 Q - P_1 - P_2 - \cdots - P_t) \subseteq C_1 = C_{\mathcal{L}}(D, \mu_1 Q), \quad \text{and} \quad C^{(j)} = C_{\mathcal{L}}(D, \mu^{(j)} Q),
\]

where \( \mu_2 = \mu_1 - \ell = \deg(\mu_1 Q - P_1 - P_2 - \cdots - P_t) \). Therefore, the assumptions in Proposition 2 are satisfied and hence the result follows, except for the claim on the maximum strength, which we now prove. For a fixed \( i \in [\ell] \), observe that

\[
G_{2,i} = C_{\mathcal{L}}(P_1 + P_2 + \cdots + P_i + \cdots + P_{t+n}, \mu_1 Q - P_i).
\]

Since \( 2g-2 < \mu_1 - 1 < \ell + n - 1 \), it follows that \( \dim(G_{2,i}) = \mu_1 - g \) by Lemma 5. Therefore, it holds that

\[
d_H(G_{2,i}) \geq \mu_1 - 2g + 1
\]

by Lemma 5. Hence the scheme has maximum strength

\[
\sigma_{\max}(F) \geq \mu_1 - 2g = r - 2g - 1 = t + \ell - 1
\]

by Theorem 5 and we are done.

\[ \square \]

6 Conclusion

In this paper, we have given a new framework to construct communication efficient secret sharing schemes based on sequences of nested linear codes, extending the previous Shamir-type communication efficient schemes from the literature. We have given two general constructions, one with small alphabet but non-universal low decoding bandwidths, and one with large alphabet but universal low decoding bandwidths.

By specializing the codes to algebraic geometry codes, we have obtained communication efficient secret sharing schemes with low decoding bandwidths and large lengths for a fixed finite field, in contrast with previous works.

We have also obtained constructions of secret sharing schemes that are communication efficient and strongly secure at the same time. In particular, we have obtained the first secret sharing schemes with optimal communication efficiency and optimal strong security, which has the security advantages of optimal perfect schemes and the storage efficiency of optimal ramp schemes. Their field sizes are however lower bounded by (but still linear on) the lengths. We have then given a construction based on algebraic geometry codes with large lengths for a fixed finite field, at the cost of near optimal communication efficiency and near optimal strong security.

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