SIMULTANEOUS NORMALIZATION OF PERIOD MAP AND AFFINE STRUCTURES ON MODULI SPACES

KEFENG LIU AND YANG SHEN

ABSTRACT. We prove that the image of the lifted period map on the universal cover lies in a complex Euclidean space. We also prove that the Teichmüller spaces of a class of polarized manifolds have complex affine structures.

0. INTRODUCTION

Let \( \Phi : S \to \Gamma \setminus D \) be a period map which arises from geometry. This means that we have an algebraic family \( f : \mathcal{X} \to S \) of polarized algebraic manifolds over a quasi-projective manifold \( S \), such that for any \( q \in S \), the point \( \Phi(q) \), modulo the action of the monodromy group \( \Gamma \), represents the Hodge structure of the \( n \)-th primitive cohomology of the fiber \( f^{-1}(q) \).

Since period map is locally liftable, we can lift the period map to \( \tilde{\Phi} : \tilde{S} \to D \) by taking the universal cover \( \tilde{S} \) of \( S \) such that the diagram

\[
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\tilde{\Phi}} & D \\
\downarrow\pi & & \downarrow\pi \\
S & \xrightarrow{\Phi} & \Gamma \setminus D
\end{array}
\]

is commutative. Griffiths raised the following conjecture as Conjecture 10.1 in [9],

**Conjecture 1.** (Griffiths Conjecture) Given \( f : \mathcal{X} \to S \), there exists a simultaneous normalization of all the periods \( \Phi(X_s) \) (\( s \in S \)). More precisely, the image \( \tilde{\Phi}(\tilde{S}) \) lies in a bounded domain in a complex Euclidean space.

In this paper, we will prove that the image \( \tilde{\Phi}(\tilde{S}) \) lies in a complex Euclidean space \( N_+ \) to be defined below. We will also prove that for a large class of polarized manifolds, their Teichmüller spaces have complex affine structures. These results are crucial for us to prove the Torelli type theorems in our next paper [18]. The boundedness of \( \tilde{\Phi}(\tilde{S}) \) in the complex Euclidean space, which may depend on the choice of the adapted basis of the Hodge decomposition at the base point, will be studied in our forthcoming paper. One can refer to [19] for a special case of this phenomenon.

First recall that the period domain \( D \) can be realized as quotient of real Lie groups \( D = G_\mathbb{R} / V \), and its compact dual \( \check{D} \) as quotient of complex Lie groups
\( \mathring{D} = G_C/B, \) where \( V = B \cap G_R. \) The Hodge structure at a fixed point \( o \) in \( D \subseteq \mathring{D} \) induces a Hodge structure of weight zero on the Lie algebra \( \mathfrak{g} \) of \( G_C \) as

\[
\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^{k,-k} \quad \text{with} \quad \mathfrak{g}^{k,-k} = \{ X \in \mathfrak{g} \mid X H^{r,n-r} \subseteq H^{r+k,n-r-k}, \forall r \}.
\]

See Section 1 for the detail of the above notations. Then the Lie algebra of \( B \) is \( \mathfrak{b} = \bigoplus_{k \geq 0} \mathfrak{g}^{k,-k} \) and the holomorphic tangent space \( T_{1,0} D \) of \( D \) at the base point \( o \) is naturally isomorphic to

\[
\mathfrak{g}/\mathfrak{b} \cong \bigoplus_{k \geq 1} \mathfrak{g}^{-k,k} \cong \mathfrak{n}_+.
\]

We denote the corresponding unipotent group by

\[
N_+ = \exp(\mathfrak{n}_+)
\]

which is identified to \( N_+(o) \), its unipotent orbit of the base point \( o \), and is considered as a complex Euclidean space inside \( \mathring{D} \). We define

\[
\mathring{S}^\vee = \mathring{\Phi}^{-1}(N_+ \cap D)
\]

and prove that \( \mathring{S} \setminus \mathring{S}^\vee \) is an analytic subvariety of \( \mathring{S} \) with \( \text{codim}_C(\mathring{S} \setminus \mathring{S}^\vee) \geq 1. \)

Then we prove the first main result of our paper, Theorem 2, by proving that the finiteness of the Hodge distance implies the finiteness of the Euclidean distance.

**Theorem 2.** The image of the lifted period map \( \mathring{\Phi} : \mathring{S} \to D \) lies in \( N_+ \cap D \).

The key point of our proof is Lemma 2.1, which is an interpretation of the Griffiths transversality in terms of the block upper triangular matrix representations of the elements in \( N_+ \cap D \). Lemma 2.1 has many interesting applications. As the first application of the above result and our method, we will prove that there exist affine structures on the Teichmüller spaces of a large class of projective manifolds.

More precisely, we consider the moduli space of certain polarized manifolds with level \( m \) structure, and let \( Z_m \) be one of the irreducible components of the moduli space. Then the Teichmüller space \( \mathcal{T} \) is defined as the universal cover of \( Z_m \). Let \( m_0 \) be a positive integer. As a technical assumption we will require that \( Z_m \) is smooth and carries an analytic family

\[
f_m : U_m \to Z_m
\]

of polarized manifolds with level \( m \) structure for all \( m \geq m_0. \) It is easy to show that \( \mathcal{T} \) is independent of the levels as given in Lemma 3.2. Furthermore we introduce the notion of strong local Torelli in Definition 4.1. Our second main result is as follows.

**Theorem 3.** Assume that \( Z_m \) is smooth with an analytic family for all \( m \geq m_0 \), and strong local Torelli holds on the Teichmüller space \( \mathcal{T} \). Then there exists a complex affine structure on \( \mathcal{T} \).
Simultaneous normalization of the periods of algebraic manifolds and · · ·

See [17] and [18] for applications of the affine structure on the Teichmüller space, as well as the Torelli space, which is the moduli spaces of marked and polarized manifolds.

The idea of our proof of Theorem 3 goes as follows. Let \( a \subseteq n_+ \) be the abelian subalgebra of \( n_+ \) defined by the tangent map of the period map,

\[
a = d\Phi_p(T_1, 0_p, T_p)
\]

where \( p \) is a base point in \( T \) with \( \Phi(p) = a \), and let

\[A = \exp(a) \subseteq N_+.
\]

Then \( A \) can be considered as a complex Euclidean subspace of \( N_+ \). Let

\[P : N_+ \to A
\]

be the projection map, which is induced by the natural projection map of the complex Euclidean subspaces \( n_+ \to a \). Then the projection map \( P \) induces the holomorphic map

\[\Psi : T \to A
\]

with \( \Psi = P \circ \Phi \). We will prove that under the assumption of Theorem 3 the holomorphic map

\[\Psi : T \to A
\]

is an immersion into \( A \) which induces the affine structure on \( T \).

This paper is organized as follows. We review the basics of the period domain from Lie group point of view and introduce the open subset \( \bar{S}^\vee \) of \( \bar{S} \) in Section 1. In Section 2 we first prove the basic lemma, Lemma 2.1 under which we can understand Griffiths transversality in terms of the blocks of the matrices in \( N_+ \cap D \). Then the Hodge metric can be computed using certain blocks of the matrices in \( N_+ \cap D \). Finally we prove Theorem 2.

In Section 3, we apply our results to the moduli space \( Z_m \) with level \( m \) structure and the Teichmüller space \( T \). Assuming \( Z_m \) is smooth for \( m \geq m_0 \) for some positive integer \( m_0 \) and strong local Torelli holds for \( T \), we prove in Section 4 that there exists an affine structure on the Teichmüller space \( T \), which is induced by the holomorphic immersion \( \Psi : T \to A \).

Acknowledgement. We are very grateful to Professor Azniv Kasparian for her interest and useful comments.

1. Period domains and Lie groups

In this section we review the definitions and basic properties of period domains and period maps from Lie theory point of views.

Let \( H_\mathbb{Z} \) be a fixed lattice and \( H = H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C} \) its complexification. Let \( n \) be a positive integer, and \( Q \) a bilinear form on \( H_\mathbb{Z} \) which is symmetric if \( n \) is even.
and skew-symmetric if \( n \) is odd. Let \( h^{i,n-i}, 0 \leq i \leq n \), be integers such that \( \sum_{i=0}^{n} h^{i,n-i} = \dim_{\mathbb{C}} H \). The period domain \( D \) for the polarized Hodge structures of type \( \{H, Q, h^{i,n-i}\} \)

is the set of all the collections of the subspaces \( H^{i,n-i}, 0 \leq i \leq n \), of \( H \) such that

\[
H = \bigoplus_{0 \leq i \leq n} H^{i,n-i}, \quad H^{i,n-i} = H^{n-i,i}, \quad \dim_{\mathbb{C}} H^{i,n-i} = h^{i,n-i} \quad \text{for} \quad 0 \leq i \leq n,
\]

and on which \( Q \) satisfies the Hodge-Riemann bilinear relations,

(2) \( Q(H^{i,n-i}, H^{j,n-j}) = 0 \) unless \( i + j = n \);

(3) \( (\sqrt{-1})^{2k-n} Q(v, \overline{v}) > 0 \) for \( v \in H^{k,n-k} \setminus \{0\} \).

Alternatively, in terms of Hodge filtrations, the period domain \( D \) is the set of all the collections of the filtrations

\[
H = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^n,
\]

such that

(4) \( \dim_{\mathbb{C}} F^i = f^i \);

\[
H = F^i \oplus F^{n-i+1}, \quad \text{for} \quad 0 \leq i \leq n,
\]

where \( f^i = h^{n,0} + \cdots + h^{i,n-i} \), and on which \( Q \) satisfies the Hodge-Riemann bilinear relations in the form of Hodge filtrations

(5) \( Q(F^i, F^{n-i+1}) = 0 \);

(6) \( Q(Cv, \overline{v}) > 0 \) if \( v \neq 0 \),

where \( C \) is the Weil operator given by

\[
Cv = (\sqrt{-1})^{2k-n} v
\]

for \( v \in F^k \cap F^{n-k} \).

Let \((X, L)\) be a polarized manifold with \( \dim_{\mathbb{C}} X = n \), which means that \( X \) is a projective manifold and \( L \) is an ample line bundle on \( X \). For simplicity we use the same notation \( L \) to denote the first Chern class of \( L \) which acts on the cohomology groups by wedge product. Then the \( n \)-th primitive cohomology groups \( H^{n}_{pr}(X, \mathbb{C}) \) of \( X \) is defined by

\[
H^{n}_{pr}(X, \mathbb{C}) = \ker\{L : H^n(X, \mathbb{C}) \to H^{n+2}(X, \mathbb{C})\}.
\]

Let \( \Phi : S \to \Gamma \setminus D \) be the period map from geometry. More precisely we have an algebraic family

\[
f : \mathcal{X} \to S
\]
Simultaneous normalization of the periods of algebraic manifolds and \ldots

of polarized algebraic manifolds over a quasi-projective manifold $S$, such that for any $q \in S$, the point $\Phi(q)$, modulo certain action of the monodromy group $\Gamma$, represents the Hodge structure of the $n$-th primitive cohomology group $H^n_{pr}(X_{q}, \mathbb{C})$ of the fiber $X_q = f^{-1}(q)$. Here $H \simeq H^n_{pr}(X_{q}, \mathbb{C})$ for any $q \in S$.

Recall that the monodromy group $\Gamma$, or global monodromy group, is the image of the representation of $\pi_1(S)$ in $\text{Aut}(H_{\mathbb{Z}}, \mathbb{Q})$, the group of automorphisms of $H_{\mathbb{Z}}$ preserving $\mathbb{Q}$.

By taking a finite index torsion-free subgroup of $\Gamma$, we can assume that $\Gamma$ is torsion-free, therefore $\Gamma \setminus D$ is smooth. This way we can just proceed on a finite cover of $S$ without loss of generality. We refer the reader to the proof of Lemma IV-A, pages 705 – 706 in [25] for such standard construction.

Since period map is locally liftable, we can lift the period map to $\tilde{\Phi} : \tilde{S} \to D$ by taking the universal cover $\tilde{S}$ of $S$ such that the diagram

$$
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\tilde{\Phi}} & D \\
\downarrow\pi & & \downarrow\pi \\
S & \xrightarrow{\Phi} & \Gamma \setminus D
\end{array}
$$

is commutative.

Let $q \in \tilde{S}$ be any point. We denote the Hodge filtration by

$$
H^n_{pr}(X_{q}, \mathbb{C}) = F^0_q \supseteq F^1_q \supseteq \cdots \supseteq F^n_q
$$

with $F^i_q = F^i H^n_{pr}(X_{q}, \mathbb{C}) = H^n_{pr,0}(X_{q}) \oplus \cdots \oplus H^n_{pr,i}(X_{q})$ for $0 \leq i \leq n$.

In this paper, all the vectors are taken as column vectors, since we will consider the left actions on the period domains. To simplify notations, we also write row of vectors and use the transpose sign $T$ to denote the corresponding column.

Let us introduce the notion of adapted basis for the given Hodge decomposition or Hodge filtration. We call a basis

$$
\xi = \{\xi_0, \ldots, \xi_{f_{n-1}}, \xi_{f_{n}}, \ldots, \xi_{f_{n-1}-1}, \ldots, \xi_{f_{k+1}}, \ldots, \xi_{f_{k-1}}, \ldots, \xi_{f_{1}}, \ldots, \xi_{f_0-1}\}^T
$$

of $H^n_{pr}(X_{q}, \mathbb{C})$ an adapted basis for the given Hodge decomposition if it satisfies

$$
H^n_{pr,k-n}(X_{q}) = \text{Span}_{\mathbb{C}} \{\xi_{f_{k+1}}, \ldots, \xi_{f_{k-1}}\}, \quad 0 \leq k \leq n.
$$

We call a basis

$$
\zeta = \{\zeta_0, \ldots, \zeta_{f_{n-1}}, \zeta_{f_{n}}, \ldots, \zeta_{f_{n-1}-1}, \ldots, \zeta_{f_{k+1}}, \ldots, \zeta_{f_{k-1}}, \ldots, \zeta_{f_{1}}, \ldots, \zeta_{f_0-1}\}^T
$$

of $H^n_{pr}(X_{q}, \mathbb{C})$ an adapted basis for the given filtration if it satisfies

$$
F^k_q = \text{Span}_{\mathbb{C}} \{\zeta_0, \ldots, \zeta_{f_{k-1}}\}, \quad 0 \leq k \leq n.
$$

For convenience, we set $f^{n+1} = 0$ and $m = f^0$. 

- 5 -
Remark 1.1. The adapted basis at the base point \( p \) can be chosen with respect to the given Hodge decomposition or the given Hodge filtration. While, in order that the period map is holomorphic, the adapted basis at any other point \( q \) can only be chosen with respect to the given Hodge filtration.

Definition 1.2. (1) Let
\[
\xi = \{ \xi_0, \cdots, \xi_{f^m-1}, \cdots, \xi_{f^k+1}, \cdots, \xi_{f^k-1}, \cdots, \xi_{f^1}, \cdots, \xi_{f^0-1} \}^T
\]
be the adapted basis with respect to the Hodge decomposition or the Hodge filtration at any point. The blocks of \( \xi \) are defined by
\[
\xi_\alpha = \{ \xi_{f^\alpha+n+1}, \cdots, \xi_{f^\alpha+n-1} \}^T,
\]
for \( 0 \leq \alpha \leq n \). Then
\[
\xi = \{ \xi_0^T, \cdots, \xi_n^T \}^T = \begin{pmatrix} \xi(0) \\ \vdots \\ \xi(n) \end{pmatrix}.
\]

(2) The blocks of an \( m \times m \) matrix \( \Psi = (\Psi_{ij})_{0 \leq i,j \leq m-1} \) are set as follows. For each \( 0 \leq \alpha, \beta \leq n \), the \((\alpha, \beta)\)-th block \( \Psi^{(\alpha, \beta)} \) is defined by
\[
\Psi^{(\alpha, \beta)} = (\Psi_{ij})_{f^\alpha+n+1 \leq i \leq f^\alpha+n-1, f^{\beta+n+1} \leq j \leq f^{\beta+n-1}}.
\]
In particular, \( \Psi = (\Psi^{(\alpha, \beta)})_{0 \leq \alpha, \beta \leq n} \) is called a block upper (lower resp.) triangular matrix if \( \Psi^{(\alpha, \beta)} = 0 \) whenever \( \alpha > \beta \) (\( \alpha < \beta \) resp.).

Let \( H_\mathbb{F} = H^m_{pr}(X, \mathbb{F}) \), where \( \mathbb{F} \) can be chosen as \( \mathbb{Z}, \mathbb{R}, \mathbb{C} \). Then \( H = H_\mathbb{C} \) under this notation. We define the complex Lie group
\[
G_\mathbb{C} = \{ g \in GL(H_\mathbb{C}) | Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_\mathbb{C} \},
\]
and the real one
\[
G_\mathbb{R} = \{ g \in GL(H_\mathbb{R}) | Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_\mathbb{R} \}.
\]
We also have
\[
G_\mathbb{Z} = \text{Aut}(H_\mathbb{Z}, Q) = \{ g \in GL(H_\mathbb{Z}) | Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_\mathbb{Z} \}.
\]

There is a left action of \( G_\mathbb{C} \) on \( \hat{D} \) such that the period domain \( D \) can be realized as the \( G_\mathbb{R} \)-orbit. Griffiths in [6] showed that \( G_\mathbb{C} \) acts on \( \hat{D} \) transitively, so does \( G_\mathbb{R} \) on \( D \). The stabilizer of \( G_\mathbb{C} \) on \( \hat{D} \) at the base point \( o \) is
\[
B = \{ g \in G_\mathbb{C} | gF_p^k = F_p^k, 0 \leq k \leq n \},
\]
and the one of \( G_\mathbb{R} \) on \( D \) is \( V = B \cap G_\mathbb{R} \). Thus we can realize \( \hat{D}, D \) as
\[
\hat{D} = G_\mathbb{C}/B, \text{ and } D = G_\mathbb{R}/V
\]
so that \( \hat{D} \) is an algebraic manifold and \( D \subseteq \hat{D} \) is an open complex submanifold.
The Lie algebra \( g \) of the complex Lie group \( G_C \) is
\[
    g = \{ X \in \text{End}(H_C) \mid Q(Xu, v) + Q(u, Xv) = 0, \text{ for all } u, v \in H_C \},
\]
and the real subalgebra
\[
    g_0 = \{ X \in g \mid XH_R \subseteq H_R \}
\]
is the Lie algebra of \( G_R \). Note that \( g \) is a simple complex Lie algebra and contains \( g_0 \) as a real form, i.e. \( g = g_0 \oplus \sqrt{-1} g_0 \).

On the linear space \( \text{Hom}(H_C, H_C) \) we can give a Hodge structure of weight zero by
\[
    g = \bigoplus_{k \in \mathbb{Z}} g_{k, -k} \quad \text{with} \quad g_{k, -k} = \{ X \in g \mid XH_{p, r, n-r} \subseteq H_{p, r+k, n-r-k}, \forall r\}.
\]

By the definition of \( B \), the Lie algebra \( b \) of \( B \) has the form
\[
    b = \bigoplus_{k \geq 0} g_{k, -k}.
\]
Then the Lie algebra \( v_0 \) of \( V \) is
\[
    v_0 = g_0 \cap b = g_0 \cap b \cap b = g_0 \cap g_{0, 0}.
\]

With the above isomorphisms, the holomorphic tangent space of \( \tilde{D} \) at the base point is naturally isomorphic to \( g/b \).

Let us consider the nilpotent Lie subalgebra \( n_+ := \bigoplus_{k \geq 1} g^{-k, k} \). Then one gets the isomorphism \( g/b \cong n_+ \). We denote the corresponding unipotent Lie group to be
\[
    N_+ = \exp(n_+).
\]

As \( \text{Ad}(g)(g_{k, -k}) \) is in \( \bigoplus_{k \geq 0} g^{i, -i} \) for each \( g \in B \), the subspace \( b \oplus g^{-1, 1}/b \subseteq g/b \) defines an \( \text{Ad}(B) \)-invariant subspace. By left translation via \( G_C \), \( b \oplus g^{-1, 1}/b \) gives rise to a \( G_C \)-invariant holomorphic subbundle of the holomorphic tangent bundle. It will be denoted by \( T_{h, 0}^{1, 0} \tilde{D} \), and will be referred to as the horizontal tangent subbundle. One can check that this construction does not depend on the choice of the base point.

The horizontal tangent subbundle, restricted to \( D \), determines a subbundle \( T_{h, 0}^{1, 0} D \) of the holomorphic tangent bundle \( T^{1, 0} D \) of \( D \). The \( G_C \)-invariance of \( T_{h, 0}^{1, 0} \tilde{D} \) implies the \( G_R \)-invariance of \( T_{h, 0}^{1, 0} D \). Note that the horizontal tangent subbundle \( T_{h, 0}^{1, 0} D \) can also be constructed as the associated bundle of the principle bundle \( V \to G_R \to D \) with the adjoint representation of \( V \) on the space \( b \oplus g^{-1, 1}/b \).

**Remark 1.3.** The \( G_C \)-invariance of \( T_{h, 0}^{1, 0} \tilde{D} \) is of great importance to our proof of our main theorems, which will be explained in Section 2. We also note that the subspace
\[
    p_b := \left( b \bigoplus \bigoplus_{k > 0, k \text{ odd}} g^{-k,k} \right) / b \subseteq g/b
\]
shares many similar properties to the subspace $\mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b}$ such as the curvature property. But $\mathfrak{p}_s$ is not $\text{Ad}(B)$-invariant, and hence can not be defined globally on $D$.

We also remark that both $\bigoplus_{k>0, k \text{ odd}} \mathfrak{g}^{-k,k}$ and $\mathfrak{g}^{-1,1}$ are $\text{Ad}(V)$-invariant. Hence they can be both globally defined on $D$.

Let $\mathcal{F}_s^k$, $0 \leq k \leq n$ be the Hodge bundles on $D$ with fibers $\mathcal{F}_s^k = F_s^k$ for any $s \in D$. As another interpretation of the horizontal bundle in terms of the Hodge bundles $\mathcal{F}_s^k \to D$, $0 \leq k \leq n$, one has

$$
T^{1,0}D \simeq T^{1,0}D \cap \bigoplus_{k=1}^n \text{Hom}(\mathcal{F}_s^k/\mathcal{F}_s^{k+1}, \mathcal{F}_s^{k-1}/\mathcal{F}_s^k).
$$

**Remark 1.4.** We remark that the elements in $N_+$ can be realized as nonsingular block upper triangular matrices with identity blocks in the diagonal; the elements in $B$ can be realized as nonsingular block lower triangular matrices. If $c, c' \in N_+$ such that $cB = c'B$ in $\tilde{D}$, then

$$
c'^{−1}c \in N_+ \cap B = \{1\},
$$

i.e. $c = c'$. This means that the matrix representation in $N_+$ of the unipotent orbit $N_+(o)$ is unique. Therefore with the fixed base point $o \in \tilde{D}$, we can identify $N_+$ with its unipotent orbit $N_+(o)$ in $\tilde{D}$ by identifying an element $c \in N_+$ with $[c] = cB$ in $\tilde{D}$. Therefore our notation $N_+ \subseteq \tilde{D}$ is meaningful. In particular, when the base point $o$ is in $D$, we have $N_+ \cap D \subseteq D$.

Now we define

$$
\tilde{S}' = \Phi^{-1}(N_+ \cap D).
$$

We first prove that $\tilde{S}' \setminus \tilde{S}'$ is an analytic subvariety of $\tilde{S}$ with $\text{codim}_C(\tilde{S}' \setminus \tilde{S}') \geq 1$.

**Lemma 1.5.** Let $p \in \tilde{S}$ be the base point with $\tilde{\Phi}(p) = \{F^0_p \subseteq F^{n-1}_p \subseteq \cdots \subseteq F^0_p\}$. Let $q \in \tilde{S}$ be any point with $\tilde{\Phi}(q) = \{F^n_q \subseteq F^{n-1}_q \subseteq \cdots \subseteq F^0_q\}$, then $\tilde{\Phi}(q) \in N_+$ if and only if $F^k_q$ is isomorphic to $F^k_p$ for all $0 \leq k \leq n$.

**Proof.** We fix $\eta = \{\eta_{(0)}, \cdots, \eta_{(n)}\}^T$ as the adapted basis for the Hodge filtration $\{F^0_p \subseteq F^{n-1}_p \subseteq \cdots \subseteq F^0_p\}$ at the base point $p$. For any $q \in \tilde{S}$, we choose an arbitrary adapted basis $\zeta = \{\zeta_{(0)}, \cdots, \zeta_{(n)}\}^T$ for the given Hodge filtration $\{F^n_q \subseteq F^{n-1}_q \subseteq \cdots \subseteq F^0_q\}$. Let

$$
A(q) = (A^{(\alpha\beta)}(q))_{0 \leq \alpha, \beta \leq n}
$$

be the transition matrix between the base $\zeta$ and $\eta$ for the same vector space $H_C$, where $A^{(\alpha\beta)}(q)$ are the corresponding blocks. Then

$$
\zeta = A(q) \cdot \eta.
$$
Therefore
\[ \tilde{\Phi}(q) \in N_+ = N_+ B/B \subseteq \tilde{D} \]
if and only if its matrix representation \( A(q) \) can be decomposed as \( L(q)U(q) \), where \( L(q) \in N_+ \) is a nonsingular block upper triangular matrix with identities in the diagonal blocks, and \( U(q) \in B \) is a nonsingular block lower triangular matrix.

By basic linear algebra, we know that \( (A^{(\alpha, \beta)}(q)) \) has such decomposition if and only if
\[ \det(A^{(\alpha, \beta)}(q))_{0 \leq \alpha, \beta \leq k} \neq 0 \]
for any \( 0 \leq k \leq n \). In particular, we know that \( (A^{(\alpha, \beta)}(q))_{0 \leq \alpha, \beta \leq k} \) is the transition map between the bases of \( F^k_p \) and \( F^k_q \). Therefore, \( \det((A^{(\alpha, \beta)}(q))_{0 \leq \alpha, \beta \leq k}) \neq 0 \) if and only if \( F^k_q \) is isomorphic to \( F^k_p \).

**Proposition 1.6.** The subset \( \tilde{S}^\vee \) is an open complex submanifold in \( \tilde{S} \), and \( \tilde{S} \setminus \tilde{S}^\vee \) is an analytic subvariety of \( \tilde{S} \) with codim\( C(\tilde{S} \setminus \tilde{S}^\vee) \geq 1 \).

**Proof.** From Lemma 1.5, one can see that \( \tilde{D} \setminus N_+ \subseteq \tilde{D} \) is defined as an analytic subvariety by the equations
\[ \{ q \in \tilde{D} : \det((A^{(\alpha, \beta)}(q))_{0 \leq \alpha, \beta \leq k}) = 0 \text{ for some } 0 \leq k \leq n \}. \]
Therefore \( N_+ \) is dense in \( \tilde{D} \), and that \( \tilde{D} \setminus N_+ \) is an analytic subvariety, which is closed in \( \tilde{D} \) and with codim\( C(\tilde{D} \setminus N_+) \geq 1 \).

We consider the period map \( \tilde{\Phi} : \tilde{S} \to \tilde{D} \) as a holomorphic map to \( \tilde{D} \), then
\[ \tilde{S} \setminus \tilde{S}^\vee = \tilde{\Phi}^{-1}(\tilde{D} \setminus N_+) \]
is the preimage of \( \tilde{D} \setminus N_+ \) of the holomorphic map \( \tilde{\Phi} \). Therefore \( \tilde{S} \setminus \tilde{S}^\vee \) is also an analytic subvariety and a closed set in \( \tilde{S} \). Because \( \tilde{S} \) is smooth and connected, \( \tilde{S} \) is irreducible. If \( \dim_C(\tilde{S} \setminus \tilde{S}^\vee) = \dim_C \tilde{S} \), then \( \tilde{S} \setminus \tilde{S}^\vee = \tilde{S} \) and \( \tilde{S}^\vee = \emptyset \), but this contradicts to the fact that the reference point \( p \) is in \( \tilde{S}^\vee \). Thus we conclude that
\[ \dim_C(\tilde{S} \setminus \tilde{S}^\vee) < \dim_C \tilde{S}, \]
and consequently codim\( C(\tilde{S} \setminus \tilde{S}^\vee) \geq 1 \). \( \square \)

**2. Proof of Theorem 2**

This section, consisting of three subsections, is devoted to the proof of our first main result, Theorem 2. In Section 2.1, we present the Griffiths transversality in terms of the blocks of the matrices in \( N_+ \cap D \). In Section 2.2, by using certain blocks of the matrices in \( N_+ \cap D \), we compute the Hodge metric on the period domain in the horizontal direction. Finally, in Section 2.3, we prove Theorem 2 by using the computations of the Hodge metrics in Section 2.2.
2.1. Basic lemmas. Let $\Phi : S \to \Gamma \backslash D$ be the period map, and $\tilde{\Phi} : \tilde{S} \to D$ be the lifted period map. In Definition 1.2, we have introduced the blocks of the adapted basis $\eta$ of the Hodge filtration at the base point $p$ as

$$\eta = \{\eta_{(0)}, \eta_{(1)}, \ldots, \eta_{(n)}\}^T,$$

where

$$\eta_{(\alpha)} = \{\eta_{f-\alpha+n+1}, \ldots, \eta_{f-\alpha+n-1}\}^T$$

is the basis of $H_{pr}^{n-\alpha,\alpha}(X_p)$ for $0 \leq \alpha \leq n$.

For any $q \in \tilde{S}^\vee$, we can choose the matrix representation of the image $\tilde{\Phi}(q)$ in $N_+$ by

$$\tilde{\Phi}(q) = (\Phi_{ij}(q))_{0 \leq i,j \leq m-1} \in N_+ \cap D.$$

Then, by Definition 1.2, the matrix $\tilde{\Phi}(q)$ is a block upper triangular matrix of the form,

$$\tilde{\Phi}(q) = \begin{bmatrix}
I & \Phi^{(0,1)}(q) & \Phi^{(0,2)}(q) & \ldots & \Phi^{(0,n-1)}(q) & \Phi^{(0,n)}(q) \\
0 & I & \Phi^{(1,2)}(q) & \ldots & \Phi^{(1,n-1)}(q) & \Phi^{(1,n)}(q) \\
0 & 0 & I & \ldots & \Phi^{(2,n-1)}(q) & \Phi^{(2,n)}(q) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & I & \Phi^{(n-1,n)}(q) \\
0 & 0 & 0 & \ldots & 0 & I
\end{bmatrix},$$

where, in the above notations, $0$ denotes zero block matrix and $I$ denotes identity block matrix. By using the matrix representation, we have the adapted basis

$$\Omega(q) = \{\Omega_{(0)}(q)^T, \Omega_{(1)}(q)^T, \ldots, \Omega_{(n)}(q)^T\}^T,$$

of the Hodge filtration at $q \in \tilde{S}^\vee$ as

$$\Omega_{(\alpha)}(q) = \sum_{\beta=0}^{n} \Phi^{(\alpha,\beta)}(q) \cdot \eta_{(\beta)}$$

$$= \eta_{(\alpha)} + \sum_{\beta \geq \alpha+1} \Phi^{(\alpha,\beta)}(q) \cdot \eta_{(\beta)},$$

where $\Omega_{(\alpha)}(q)$, together with $\Omega_{(0)}(q), \ldots, \Omega_{(\alpha-1)}(q)$, gives a basis of the Hodge filtration

$$F_q^{n-\alpha} = F^{n-\alpha}H^n_{pr}(X_q, \mathbb{C}).$$

Note that, in equation (11), the term $\Phi^{(\alpha,\beta)}(q) \cdot \eta_{(\beta)}$ is an element in $H_{pr}^{n-\beta,\beta}(X_p)$.

Let $q \in \tilde{S}^\vee$ be any point. Let $\{U; z = (z_1, \ldots, z_N)\}$ be any small holomorphic coordinate neighborhood around $q$ with

$$z_{\mu}(q) = 0, \ 1 \leq \mu \leq N,$$

such that $\tilde{\Phi}(U) \subset N_+ \cap D$ and

$$U = \{z \in \mathbb{C}^N : |z_{\mu}| < \epsilon, \ 1 \leq \mu \leq N\}.$$
Simultaneous normalization of the periods of algebraic manifolds and \ldots 11

For $0 \leq \alpha, \beta \leq n$, we define the matrix valued functions as

$$\Phi^{(\alpha,\beta)}(z) = \Phi^{(\alpha,\beta)}(q'), \text{ for } q' \in U \text{ and } z(q') = z.$$ 

In the following, the derivatives of the blocks,

$$\frac{\partial \Phi^{(\alpha,\beta)}}{\partial z_\mu}(z), \text{ for } 0 \leq \alpha, \beta \leq n, \ 1 \leq \mu \leq N,$$

will denote the blocks of derivatives of its entries,

$$\frac{\partial \Phi^{(\alpha,\beta)}}{\partial z_\mu}(z) = \left(\frac{\partial \Phi^{(\alpha,\beta)}}{\partial z_\mu}(z)\right)_{f^{-\alpha+n+1} \leq i \leq f^{-\alpha+n-1}, \ f^{-\beta+n+1} \leq j \leq f^{-\beta+n-1}}.$$

**Lemma 2.1.** Let the notations be as above. Then we have that

$$(12) \quad \frac{\partial \Phi^{(\alpha,\beta)}}{\partial z_\mu}(z) = \frac{\partial \Phi^{(\alpha,\alpha+1)}}{\partial z_\mu}(z) \cdot \Phi^{(\alpha+1,\beta)}(z),$$

for any $0 \leq \alpha < \beta \leq n$ and $1 \leq \mu \leq N$.

**Proof.** The main idea of the proof is to rewrite the Griffiths transversality in terms of the matrix representations of the image of the period map in $N_+ \cap D$. The rest only uses basic linear algebra.

Let us consider the adapted basis

$$\Omega(q) = \left\{\Omega^{(0)}(q)^T, \Omega^{(1)}(q)^T, \cdots, \Omega^{(n)}(q)^T\right\}^T,$$

at $q \in \tilde{S}^\nu$ as given in (11). We denote

$$F_n^{z-\alpha} = F_q^{z-\alpha} = F_n^{z-\alpha} H_{pr}(X_q, \mathbb{C})$$

for $z(q) = z \in U$.

By Griffiths transversality, especially the computations in Page 813 of [7], we have that

$$\frac{\partial \Omega^{(o)}}{\partial z_\mu}(z)$$

lies in $F_n^{z-\alpha-1}$. By the constructions of the basis in (11), we know that $F_n^{z-\alpha-1}$ has basis

$$\left\{\Omega^{(0)}(z)^T, \Omega^{(1)}(z)^T, \cdots, \Omega^{(\alpha+1)}(z)^T\right\}^T,$$
which gives local holomorphic sections of the Hodge bundle $\mathcal{F}_{n-a-1}$. Therefore there exist matrices $A(0)(z), A(1)(z), \cdots, A(a+1)(z)$ such that

\[
\frac{\partial \Omega_{(a)}(z)}{\partial z_{\mu}} = A(0)(z) \cdot \Omega_{(0)}(z) + \cdots + A(a+1)(z) \cdot \Omega_{(a+1)}(z)
\]

(13) \hspace{1cm} \equiv \hspace{1cm}

\[
A(0)(z) \cdot \Omega_{(0)}(z) + \cdots + A(a)(z) \cdot \Omega_{(a)}(z) \mod \bigoplus_{\gamma \geq 1} H^{n-a-\gamma,a+\gamma}_{pr}(X_p)
\]

(14) \hspace{1cm} \equiv \hspace{1cm}

\[
\frac{\partial \Omega_{(a)}(z)}{\partial z_{\mu}} = \sum_{\beta \geq a+1} \frac{\partial \Phi^{(a+1)}_{(\beta)}}{\partial z_{\mu}}(z) \cdot \eta(\beta)
\]

\[
= \frac{\partial \Phi^{(a,a+1)}}{\partial z_{\mu}}(z) \cdot \eta(a+1) + \sum_{\beta \geq a+2} \frac{\partial \Phi^{(a,\beta)}}{\partial z_{\mu}}(z) \cdot \eta(\beta)
\]

(15) \hspace{1cm} \equiv \hspace{1cm}

\[
0 \mod \bigoplus_{\gamma \geq 1} H^{n-a-\gamma,a+\gamma}_{pr}(X_p).
\]

By comparing equations (14) and (15), we see that

\[
\begin{cases}
A(0)(z) = 0 \\
A(0)(z)\Phi^{(0,1)}(z) + A(1)(z) = 0 \\
\cdots \\
A(0)(z)\Phi^{(0,a)}(z) + \cdots + A(a-1)(z)\Phi^{(a-1,a)}(z) + A(a)(z) = 0,
\end{cases}
\]

which implies inductively that $A(0)(z) = 0, \cdots, A(a)(z) = 0$. Therefore

\[
\frac{\partial \Omega_{(a)}(z)}{\partial z_{\mu}} = A(a+1)(z) \Omega_{(a+1)}(z)
\]

\[
= A(a+1)(z) \cdot \left( \eta(a+1) + \sum_{\beta \geq a+2} \Phi^{(a+1,\beta)}(z) \cdot \eta(\beta) \right)
\]

\[
= A(a+1)(z) \cdot \eta(a+1) + \sum_{\beta \geq a+2} A(a+1)(z)\Phi^{(a+1,\beta)}(z) \cdot \eta(\beta).
\]

By comparing types again, we have that

\[
A(a+1)(z) = \frac{\partial \Phi^{(a,a+1)}}{\partial z_{\mu}}(z)
\]
Simultaneous normalization of the periods of algebraic manifolds and · · · 13

and

\[
\frac{\partial \Phi^{(\alpha, \beta)}}{\partial z_\mu}(z) = A_{(\alpha+1)}(z) \cdot \Phi^{(\alpha+1, \beta)}(z)
\]

\[
= \frac{\partial \Phi^{(\alpha, \alpha+1)}}{\partial z_\mu}(z) \cdot \Phi^{(\alpha+1, \beta)}(z),
\]

for \( \beta \geq \alpha + 2 \). Since \( \Phi^{(\alpha+1, \alpha+1)}(z) \) is identity matrix, (16) is satisfied trivially for \( \beta = \alpha + 1 \). Therefore, we have proved equation (12).

\[\square\]

Remark 2.2. By the proof of Lemma 2.1, we have that

\[
(d\tilde{\Phi})_q \left( \frac{\partial}{\partial z_\mu} \right) = \bigoplus_{0 \leq \alpha \leq n-1} \frac{\partial \Phi^{(\alpha, \alpha+1)}}{\partial z_\mu}(0)
\]

as elements in

\[
\bigoplus_{0 \leq \alpha \leq n-1} \text{Hom}(F_{q^\alpha} / F_{q^{\alpha+1}}, F_{q^\alpha-1} / F_{q^\alpha})
\]

which maps \( \Omega_{(\alpha)}(q) \) to

\[
A_{(\alpha+1)}(0) \cdot \Omega_{(\alpha+1)}(q) = \frac{\partial \Phi^{(\alpha, \alpha+1)}}{\partial z_\mu}(0) \cdot \Omega_{(\alpha+1)}(q).
\]

Note that, by construction in (11), each \( \Omega_{(\alpha)}(q) \) can be considered as a basis of \( F_{q^\alpha} / F_{q^{\alpha+1}} \).

2.2. Hodge metric on the period domain. Let \( D \) be the period domain. Since \( D \) can be realized as the quotient \( G_R/V \) of real Lie groups, the Killing form on the Lie algebra

\[
\mathfrak{g} = \mathfrak{g}_0 \otimes_R \mathbb{C}
\]

induces a metric on the subspace \( n_+ \) of the quotient space

\[
(\mathfrak{g}_0/\mathfrak{v}_0) \otimes_R \mathbb{C} \simeq \left( \bigoplus_{k>0} \mathfrak{g}^{-k,k} \right) \oplus \left( \bigoplus_{k<0} \mathfrak{g}^{-k,k} \right) \cong n_+ \oplus n_-.
\]

This gives a \( G_R \)-invariant metric on the period domain \( D \) via \( G_R \)-translation, which is called the Hodge metric on the period domain \( D \).

On the other hand, the bilinear form \( Q \) on the complex vector space \( H \) also induces a non-degenerate bilinear form on the the Lie algebra \( \mathfrak{g} \). Since \( \mathfrak{g} \) is simple, this non-degenerate bilinear form is equal to the Killing form on \( \mathfrak{g} \) up to a constant. Hence the Hodge metric on \( D \) can also be induced by the bilinear form \( Q \).

For later convenience, we give the explicit formula for the Hodge metric in the direction of the horizontal tangent space

\[
\mathcal{T}^h_{\Phi(q)} D = \bigoplus_{1 \leq k \leq n} \text{Hom}(F_{q^k} / F_{q^{k+1}}, F_{q^{k-1}} / F_{q^k})
\]

for any point \( q \in \tilde{S} \).
Let $\tilde{Q}$ be the Hermitian form on $H$, which is defined by
$$\tilde{Q}(u, v) = Q(C \cdot u, v),$$
where $C$ is the Weil operator. The following lemma is well-known which follows from linear algebra.

**Lemma 2.3.** (1). The restriction of the Hermitian form $\tilde{Q}$ to each component $H^k_{pr,n-k}(X_q)$ is well-defined for $0 \leq k \leq n$. Furthermore, the Hodge decomposition
$$H = \bigoplus_{0 \leq k \leq n} H^k_{pr,n-k}(X_q)$$
is an orthogonal decomposition with respect to the Hermitian form $\tilde{Q}$.

(2). Let $\zeta(q) = \{\zeta(0)(q)^T, \cdots, \zeta(n)(q)^T\}^T$ be the adapted basis for the Hodge filtration at the point $\Phi(q)$. Let $v \in T^h_{\Phi(q)} D$ be the matrix $v = \oplus_{0 \leq \alpha \leq n-1} V_\alpha$ in
$$\bigoplus_{0 \leq \alpha \leq n-1} \text{Hom}(F_n^{m-\alpha}/F_n^{m-\alpha+1}, F_n^{m-n-\alpha}/F_n^{m-n-\alpha-1})$$
such that $V_\alpha$ maps $\zeta(\alpha)(q)$ to $V_\alpha \cdot \zeta(\alpha+1)(q)$. Then the Hodge metric $\|v\|_{\text{Hod}}$ of $v \in T^h_{\Phi(q)} D$ is
$$\|v\|_{\text{Hod}} = \sum_{0 \leq \alpha \leq n-1} \|V_\alpha\|_{\text{Mat}} \frac{\sqrt{\tilde{Q}(\zeta(\alpha+1)(q), \zeta(\alpha+1)(q))}}{\sqrt{\tilde{Q}(\zeta(\alpha)(q), \zeta(\alpha)(q))}}.$$ 

Here, $\|V\|_{\text{Mat}}$ denotes the norm of the matrix $V = (V_{ij})$ which is defined by
$$\|V\|_{\text{Mat}} = \sqrt{\sum_{ij} |V_{ij}|^2}.$$

By applying Lemma 2.1, we have the following proposition.

**Proposition 2.4.** Let $\tilde{\Phi} : \tilde{S}^\vee \to N_+ \cap D$ be the restricted period map. Let $\zeta(q) = \{\zeta(0)(q)^T, \cdots, \zeta(n)(q)^T\}^T$ be any adapted basis for the Hodge filtration at the point $\tilde{\Phi}(q)$ for $q \in \tilde{S}^\vee$. Then the Hodge metric on $(d\tilde{\Phi})_q \left( \frac{\partial}{\partial z_\mu} \right)$ is given by
$$\left\| (d\tilde{\Phi})_q \left( \frac{\partial}{\partial z_\mu} \right) \right\|_{\text{Hod}} = \sum_{0 \leq \alpha \leq n-1} \frac{\|\partial \Phi^{(\alpha, \alpha+1)} / \partial z_\mu(0)\|_{\text{Mat}} \sqrt{\tilde{Q}(\zeta(\alpha+1)(q), \zeta(\alpha+1)(q))}}{\sqrt{\tilde{Q}(\zeta(\alpha)(q), \zeta(\alpha)(q))}},$$

where $(U; z)$ is a local coordinate around $q \in \tilde{S}^\vee$ such that $z(q) = 0$.

**Proof.** Let $\zeta(q) = \{\zeta(0)(q)^T, \cdots, \zeta(n)(q)^T\}^T$ be any adapted basis for the Hodge filtration at the point $\tilde{\Phi}(q)$ for $q \in \tilde{S}^\vee$. By Lemma 2.3, we only need to prove that
$$(d\tilde{\Phi})_q \left( \frac{\partial}{\partial z_\mu} \right) = \bigoplus_{0 \leq \alpha \leq n-1} \frac{\partial \Phi^{(\alpha, \alpha+1)}}{\partial z_\mu}(0).$$
as elements in
\[ \bigoplus_{0 \leq \alpha \leq n-1} \text{Hom}(F_q^{n-\alpha}/F_q^{n-\alpha+1}, F_q^{n-\alpha-1}/F_q^{n-\alpha}), \]
which maps \( \zeta(\alpha) \) to
\[ \frac{\partial \Phi^{(\alpha,\alpha+1)}}{\partial z_\mu}(0) \cdot \zeta(\alpha+1). \]

For \( q \in \tilde{S}^\vee \), by the proof of Lemma 1.5, we have that
\[ \zeta(q) = L(q)U(q) \cdot \eta \triangleq L(q) \cdot \eta(q), \]
where \( L(q) \in N_+ \) and \( U(q) \in B \). Recall that \( \eta \) is the adapted basis for the Hodge decomposition at the point \( o \in D \). Then \( \eta(q) = U(q) \cdot \eta \) is an adapted basis for the Hodge filtration at the base point \( o \in D \), which together with \( L(q) \in N_+ \) gives the adapted basis \( \zeta(q) \) for the Hodge filtration at \( q \).

Under the local coordinate \((U; z)\) around \( q \in \tilde{S}^\vee \) with \( z(q) = 0 \), we have that \( \eta(z) = U(z) \cdot \eta \) and
\[
\zeta(\alpha)(z) = \sum_{\beta=0}^{n} \Phi^{(\alpha,\beta)}(z) \cdot \eta(\beta)(z)
= \eta(\alpha)(z) + \sum_{\beta \geq \alpha+1} \Phi^{(\alpha,\beta)}(z) \cdot \eta(\beta)(z).
\]
Here \( \eta(0) = \eta(q) \). Hence
\[
\frac{\partial \zeta(\alpha)}{\partial z_\mu}(0) = \sum_{\beta=0}^{n} \Phi^{(\alpha,\beta)}(0) \cdot \frac{\partial \eta(\beta)}{\partial z_\mu}(0) + \sum_{\beta \geq \alpha+1} \Phi^{(\alpha,\beta)}(0) \cdot \eta(\beta)(0).
\]
Note that
\[
\frac{\partial \eta(\beta)}{\partial z_\mu}(0) \in F_o^{n-\beta},
\]
and by the definition of the blocks \( \Phi^{(\alpha,\beta)}, \beta \geq \alpha \), of the elements in \( N_+ \), we have that
\[
\sum_{\beta=0}^{n} \Phi^{(\alpha,\beta)}(0) \cdot \frac{\partial \eta(\beta)}{\partial z_\mu}(0)
\]
lies in \( F_q^{n-\alpha} \), which implies that
\[
\frac{\partial \zeta(\alpha)}{\partial z_\mu}(0) \equiv \sum_{\beta \geq \alpha+1} \frac{\partial \Phi^{(\alpha,\beta)}}{\partial z_\mu}(0) \cdot \eta(\beta)(0) \mod (F_q^{n-\alpha}).
\]
By Lemma 2.1, we also have that
\[
\sum_{\beta \geq \alpha + 1} \frac{\partial \Phi^{(\alpha, \beta)}}{\partial z_\mu}(0) \cdot \eta(\beta)(0) = \frac{\partial \Phi^{(\alpha, \alpha + 1)}}{\partial z_\mu}(0) \cdot \eta(\alpha + 1)(0) + \sum_{\beta \geq \alpha + 2} \frac{\partial \Phi^{(\alpha, \beta)}}{\partial z_\mu}(0) \cdot \eta(\beta)(0)
\]
\[
= \frac{\partial \Phi^{(\alpha, \alpha + 1)}}{\partial z_\mu}(0) \left( \eta(\alpha + 1)(0) + \sum_{\beta \geq \alpha + 2} \Phi^{(\alpha + 1, \beta)}(0) \cdot \eta(\beta)(0) \right)
\]
\[
= \frac{\partial \Phi^{(\alpha, \alpha + 1)}}{\partial z_\mu}(0) \cdot \zeta(\alpha + 1)(0).
\]

Therefore we have proved that
\[
\frac{\partial \zeta(\alpha)}{\partial z_\mu}(0) \equiv \frac{\partial \Phi^{(\alpha, \alpha + 1)}}{\partial z_\mu}(0) \cdot \zeta(\alpha + 1)(0) \mod (F_q^{n-a}),
\]
which finishes the proof of the lemma. \qed

2.3. **Hodge distance finiteness implies Euclidean distance finiteness.** Let \( \tilde{\Phi} : \tilde{S} \to D \) be the period map with base point \( p \in \tilde{S} \). For any points \( s_1, s_2 \in D \), let
\[
d_{\text{Hod}}(s_1, s_2)
\]
be the distance defined by the Hodge metric on \( D \).

Recall that, from [11] we know that the Riemannian sectional curvature of the period domain \( D \) is nonpositive in the horizontal direction, while the Euclidean metric on \( N_+ \) has zero curvature. Therefore the next proposition, which implies that the Hodge length of a geodesic in the horizontal direction should be larger than the Euclidean length, is an analogue of the comparison theorem in Riemannian geometry as given in Corollary 1.35 in page 25 of [4].

We remark that our proof of the following proposition only uses Hodge theory, which can be considered as a substantial application of the Griffiths transversality in terms of the blocks of the matrices in \( N_+ \cap D \).

**Proposition 2.5.** Let \( \tilde{\Phi} : \tilde{S}^\vee \to N_+ \cap D \) be the restricted period map. Let \( \gamma : [0, T] \to \tilde{S} \) be a smooth curve such that \( \gamma(0) = p \) and \( \gamma([0, T]) \subset \tilde{S}^\vee \). Then \( \gamma(T) \in \tilde{S}^\vee \), which is equivalent to that \( \Phi(\gamma(T)) \in N_+ \cap D \).

**Proof.** By the definition of \( \tilde{S}^\vee \), we have that \( \tilde{\Phi}(\gamma(t)) \in N_+ \cap D \) for any \( t \in [0, T] \).

We denote
\[
\tilde{\Phi}(\gamma(t)) = (\Phi^{(\alpha, \beta)}(t))_{0 \leq \alpha, \beta \leq n} \in N_+, \ t \in [0, T)
\]
Simultaneous normalization of the periods of algebraic manifolds and ... 17

In the following blocks of $\sim\Phi(\gamma(t))$ as $t \to T$,

$$\Phi(\gamma(t)) = \begin{bmatrix}
I & \Phi^{(0,1)}(t) & \Phi^{(0,2)}(t) & \ldots & \Phi^{(0,n-1)}(t) & \Phi^{(0,n)}(t) \\
0 & I & \Phi^{(1,2)}(t) & \ldots & \Phi^{(1,n-1)}(t) & \Phi^{(1,n)}(t) \\
0 & 0 & I & \ldots & \Phi^{(2,n-1)}(t) & \Phi^{(2,n)}(t) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & I & \Phi^{(n-1,n)}(t) \\
0 & 0 & 0 & \ldots & 0 & I
\end{bmatrix},$$

we need to show that

$$(17) \quad \|\Phi^{(\alpha,\beta)}(T)\|_{\text{Mat}} < \infty \text{ for } 0 \leq \alpha, \beta \leq n.$$  

We first prove $$(17)$$ for $\beta = \alpha + 1$ and $0 \leq \alpha \leq n - 1$ in Lemma 2.6. Then we finish the proof by induction based on the results in Lemma 2.1 and Lemma 2.6. \qed

Lemma 2.6. $\|\Phi^{(\alpha,\alpha+1)}(T)\|_{\text{Mat}} < \infty$ for $0 \leq \alpha \leq n - 1$.

Proof. For any $t \in [0, T]$, let $\zeta(t)$ be the adapted basis for the Hodge filtration at the point $\sim\Phi(\gamma(t))$. Then the Hodge metric $\sim Q(\zeta(\alpha)(t), \zeta(\alpha)(t))$ is positive for $t \in [0, T]$ and $0 \leq \alpha \leq n$. By the compactness of $[0, T]$, there exists $m, M > 0$ such that

$$m \leq \sqrt{\sim Q(\zeta(\alpha)(t), \zeta(\alpha)(t))} \leq M$$

for any $t \in [0, T]$ and $0 \leq \alpha \leq n$.

Then, by applying Proposition 2.4, we have that

$$d_{\text{Hod}}(\Phi(p), \sim\Phi(\gamma(T))) = \int_0^T \left\|d(\Phi \circ \gamma) \left( \frac{d}{dt} \right) \right\|_{\text{Hod}} dt$$

$$= \int_0^T \sum_{0 \leq \alpha \leq n-1} \|\Phi^{(\alpha,\alpha+1)}(t)\|_{\text{Mat}} \frac{\sqrt{\sim Q(\zeta(\alpha+1)(t), \zeta(\alpha+1)(t))}}{\sqrt{\sim Q(\zeta(\alpha)(t), \zeta(\alpha)(t))}} dt$$

$$\geq \sum_{0 \leq \alpha \leq n-1} \int_0^T \frac{m}{M} \|\Phi^{(\alpha,\alpha+1)}(t)\|_{\text{Mat}} dt.$$

Therefore

$$\|\Phi^{(\alpha,\alpha+1)}(T)\|_{\text{Mat}} = \left\|\int_0^T \Phi^{(\alpha,\alpha+1)}(t) dt \right\|_{\text{Mat}}$$

$$(18) \quad \leq \int_0^T \|\Phi^{(\alpha,\alpha+1)}(t)\|_{\text{Mat}} dt$$

$$\leq C d_{\text{Hod}}(\Phi(p), \sim\Phi(\gamma(T))) < \infty,$$

for some constant $C$ depending only on $m, M$.

Therefore we have proved that $\|\Phi^{(\alpha,\alpha+1)}(T)\|_{\text{Mat}} < \infty$ for $0 \leq \alpha \leq n - 1$. \qed
Continued proof of Proposition 2.5 We prove (17) by induction. Choose a constant $C > 0$ such that

$$
\|\Phi^{(n-1,n)}(T)\|_{\text{Mat}} \leq C.
$$

We assume that

$$
\|\Phi^{(\alpha,\beta)}(T)\|_{\text{Mat}} < C \text{ for fixed } \alpha \text{ and any } \beta > \alpha.
$$

Then inequality (19) implies that inequality $(\ast_\alpha)$ holds for $\alpha = n - 1$.

Let us prove inequality $(\ast_\alpha)$ for $\alpha$ from inequality $(\ast_{\alpha+1})$ for $\alpha + 1$. By Lemma 2.1, we have that

$$
\dot{\Phi}^{(\alpha,\beta)}(t) = \dot{\Phi}^{(\alpha,\alpha+1)}(t) \cdot \Phi^{(\alpha+1,\beta)}(t), \beta > \alpha.
$$

Then by inequalities (18) and $(\ast_\alpha)$, we have that

$$
\|\Phi^{(\alpha,\beta)}(T)\|_{\text{Mat}} \leq \int_0^T \|\dot{\Phi}^{(\alpha,\beta)}(t)\|_{\text{Mat}} dt
\leq C \int_0^T \|\dot{\Phi}^{(\alpha,\alpha+1)}(t)\|_{\text{Mat}} dt
\leq Cd_{\text{Hod}}(\tilde{\Phi}(p), \tilde{\Phi}(\gamma(T))) \leq C'.
$$

Now we have proved (17), which implies that

$$
\tilde{\Phi}(\gamma(T)) = \lim_{t \to T} \tilde{\Phi}(\gamma(t)) \in N_+ \cap D.
$$

This is equivalent to say that $\gamma(T) \in \tilde{S}^\vee$. □

Remark 2.7. Let $\gamma : [0, 1] \to N_+ \cap D$ be a smooth curve. We call $\gamma$ a horizontal curve, provided that the tangent vector $\dot{\gamma}(t)$, $t \in [0, 1]$, lies in the underlying real tangent bundle corresponding to the horizontal subbundle $T^{1,0}_n D$. The proof of Proposition 2.5 actually shows that for any horizontal curve $\gamma$ in $N_+ \cap D$, the length of $\gamma$ with respect to the Euclidean metric on $N_+$ is smaller than the length of $\gamma$ with respect to the Hodge metric on $D$.

Now we give an example to illustrate the simple idea behind the proof of Proposition 2.5

Example 2.8. Let us consider the case of weight 2. Then the element of $\tilde{\Phi}(\gamma(t)) \in N_+ \cap D$ as $t \to T$ is as follows

$$
\tilde{\Phi}(\gamma(t)) = \begin{bmatrix}
I & \Phi^{(0,1)}(t) & \Phi^{(0,2)}(t) \\
0 & I & \Phi^{(1,2)}(t) \\
0 & 0 & I
\end{bmatrix}.
$$
Simultaneous normalization of the periods of algebraic manifolds and · · · 19

From Lemma 2.6, we have that \( \| \Phi^{(0,1)}(T) \|_{\text{Mat}} < \infty \) and \( \| \Phi^{(1,2)}(T) \|_{\text{Mat}} < \infty \).

From Lemma 2.1 which follows from Griffiths transversality, we get that

\[
\dot{\Phi}^{(0,2)}(t) = \dot{\Phi}^{(0,1)}(t) \cdot \Phi^{(1,2)}(t)
\]

for \( t \in [0, T) \). Therefore

\[
\| \Phi^{(0,2)}(T) \|_{\text{Mat}} \leq \int_0^T \| \dot{\Phi}^{(0,2)}(t) \|_{\text{Mat}} dt
\]

\[
\leq \int_0^T \| \dot{\Phi}^{(0,1)}(t) \cdot \Phi^{(1,2)}(t) \|_{\text{Mat}} dt
\]

\[
\leq C \int_0^T \| \dot{\Phi}^{(0,1)}(t) \|_{\text{Mat}} dt < \infty.
\]

Now we can prove our first main result.

**Theorem 2.9.** The image \( \tilde{\Phi}(\tilde{S}) \) of the period map \( \tilde{\Phi} : \tilde{S} \to D \) lies in \( N_+ \cap D \).

**Proof.** Let \( q \in \tilde{S} \) be any point. Since \( \tilde{S}^\vee \subseteq \tilde{S} \) is open dense in \( \tilde{S} \) with \( \tilde{S} \setminus \tilde{S}^\vee \) as an analytic subset of \( \tilde{S} \), we can find a curve \( \gamma : [0, T] \to \tilde{S} \) such that \( \gamma(0) = p \), \( \gamma(T) = q \) and \( \gamma([0, T)) \subseteq \tilde{S}^\vee \). By Proposition 2.5, we conclude that

\[
q = \gamma(T) \in \tilde{S}^\vee,
\]

which implies that \( \tilde{S} = \tilde{S}^\vee \) and \( \tilde{\Phi}(\tilde{S}) \subseteq N_+ \cap D \). \( \square \)

3. Moduli spaces and Teichmüller spaces

In this section, we introduce the definitions of moduli space with level \( m \) structure and Teichmüller space of polarized manifolds. Most results in this section are standard and well-known. For example, one can refer to [22] for the knowledge of moduli space, and [14], [24] for the knowledge of deformation theory.

Let \( (X, L) \) be a polarized manifold. The moduli space \( \mathcal{M} \) of polarized manifolds is the complex analytic space parameterizing the isomorphism classes of polarized manifolds with the isomorphism defined by

\[
(X, L) \sim (X', L') \iff \exists \text{ biholomorphic map } f : X \to X' \text{ s.t. } f^*L' = L.
\]

We fix a lattice \( \Lambda \) with a pairing \( Q_0 \), where \( \Lambda \) is isomorphic to \( H^n(X_0, \mathbb{Z})/\text{Tor} \) for some \((X_0, L_0) \in \mathcal{M} \) and \( Q_0 \) is defined by the cup-product. For a polarized manifold \((X, L)\), we define a marking \( \gamma \) as an isometry of the lattices

\[
\gamma : (\Lambda, Q_0) \to (H^n(X, \mathbb{Z})/\text{Tor}, Q).
\]
For any integer \( m \geq 3 \), we define an \( m \)-equivalent relation of two markings on \((X, L)\) by
\[
\gamma \sim_m \gamma' \text{ if and only if } \gamma' \circ \gamma^{-1} - \text{Id} \in m \cdot \text{End}(H^n(X, \mathbb{Z})/\text{Tor}),
\]
and denote by \([\gamma]_m\) the set of all the \( m \)-equivalent classes of \( \gamma \). Then we call \([\gamma]_m\) a level \( m \) structure on the polarized manifold \((X, L)\). Two polarized manifolds with level \( m \) structure \((X, L, [\gamma]_m)\) and \((X', L', [\gamma']_m)\) are said to be isomorphic if there exists a biholomorphic map \( f : X \to X' \) such that \( f^*L' = L \) and
\[
f^*\gamma' \sim_m \gamma,
\]
where \( f^*\gamma' \) is given by \( \gamma' : (A, Q_0) \to (H^n(X', \mathbb{Z})/\text{Tor}, Q) \) composed with the induced map
\[
f^* : (H^n(X', \mathbb{Z})/\text{Tor}, Q) \to (H^n(X, \mathbb{Z})/\text{Tor}, Q).
\]

Let \( Z_m \) be one of the irreducible components of the moduli space of polarized manifolds with level \( m \) structure, which parameterizes the isomorphism classes of polarized manifolds with level \( m \) structure.

From now on, we will assume that the irreducible component \( Z_m \) defined as above is a smooth complex manifold with an analytic family \( f_m : U_m \to Z_m \) of polarized manifolds with level \( m \) structure for all \( m \geq m_0 \), where \( m_0 \geq 3 \) is some fixed integer. For simplicity, we can also assume that \( m_0 = 3 \).

Let \( T_m \) be the universal cover of \( Z_m \) with covering map \( \pi_m : T_m \to Z_m \). Then we have an analytic family \( g_m : \mathcal{V}_m \to T_m \) such that the following diagram is cartesian
\[
\begin{array}{ccc}
\mathcal{V}_m & \longrightarrow & U_m \\
\downarrow^{g_m} & & \downarrow^{f_m} \\
T_m & \longrightarrow & Z_m
\end{array}
\]
i.e. \( \mathcal{V}_m = U_m \times_{Z_m} T_m \). Such a family is called the pull-back family. We call \( T_m \) the Teichmüller space of polarized manifolds with level \( m \) structure.

The proof of the following lemma is obvious.

**Lemma 3.1.** Assume that \( Z_m \) is smooth for \( m \geq 3 \), then the Teichmüller space \( T_m \) is smooth and the pull-back family \( g_m : \mathcal{V}_m \to T_m \) is an analytic family.

We will give two proofs of the following lemma, which allows us to simply denote \( T_m \) by \( T \), the analytic family by \( g : \mathcal{V} \to T \) and the covering map by \( \pi_m : T \to Z_m \).

**Lemma 3.2.** The Teichmüller space \( T_m \) does not depend on the choice of \( m \).
Proof. The first proof uses the construction of moduli space with level \( m \) structure, see Lecture 10 of [22], or pages 692 – 693 of [26]. Let \( m_1 \) and \( m_2 \) be two different integers, and

\[
U_{m_1} \rightarrow Z_{m_1}, \ U_{m_2} \rightarrow Z_{m_2}
\]

be two analytic families with level \( m_1 \) structure and level \( m_2 \) structure respectively. Let \( T_{m_1} \) and \( T_{m_2} \) be the universal covering space of \( Z_{m_1} \) and \( Z_{m_2} \) respectively. Let \( m = m_1 m_2 \) and consider the analytic family \( U_m \rightarrow Z_m \). From the discussion in page 130 of [22] or pages 692 – 693 of [26], we know that \( Z_m \) is a covering space of both \( Z_{m_1} \) and \( Z_{m_2} \). Let \( T \) be the universal covering space of \( Z_m \). Since \( Z_m \) is a covering space of both \( Z_{m_1} \) and \( Z_{m_2} \), we conclude that \( T \) is the universal cover of both \( Z_{m_1} \) and \( Z_{m_2} \), i.e.

\[
T_{m_1} \cong T_{m_2} \cong T.
\]

If the analytic family \( f_m : U_m \rightarrow Z_m \) is universal as defined in page 9 of [24], then we have a second proof. Let \( m_1, m_2 \) be two different integers \( \geq 3 \), and let \( T_{m_1} \) and \( T_{m_2} \) be the corresponding Teichmüller space with the universal families

\[
g_{m_1} : V_{m_1} \rightarrow T_{m_1}, \ g_{m_2} : V_{m_2} \rightarrow T_{m_2}
\]

respectively. Then for any point \( p \in T_{m_1} \) and the fiber \( X_p = g_{m_1}^{-1}(p) \) over \( p \), there exists \( q \in T_{m_2} \) such that \( Y_q = g_{m_2}^{-1}(q) \) is biholomorphic to \( X_p \). By the definition of universal family, we can find a local neighborhood \( U_p \) of \( p \) and a holomorphic map

\[
h_p : U_p \rightarrow T_{m_2}, \ p \mapsto q
\]

such that the map \( h_p \) is uniquely determined. Since \( T_{m_1} \) is simply-connected, all the local holomorphic maps

\[
\{h_p : U_p \rightarrow T_{m_2}, \ p \in T_{m_1}\}
\]

patches together to give a global holomorphic map \( h : T_{m_1} \rightarrow T_{m_2} \) which is well-defined. Moreover \( h \) is unique since it is unique on each local neighborhood of \( T_{m_1} \). Similarly we have a holomorphic map \( h' : T_{m_2} \rightarrow T_{m_1} \) which is also unique. Then \( h \) and \( h' \) are inverse to each other by the uniqueness of \( h \) and \( h' \). Therefore \( T_{m_1} \) and \( T_{m_2} \) are biholomorphic. \( \square \)

From now on we will denote \( T = T_m \) for any \( m \geq 3 \) and call \( T \) the Teichmüller space of polarized manifolds.

As before, we can define the period map \( \Phi_{Z_m} : Z_m \rightarrow \Gamma \backslash D \) and the lifted period map \( \tilde{\Phi} : T \rightarrow D \) such that the diagram

\[
\begin{array}{ccc}
T & \xrightarrow{\tilde{\Phi}} & D \\
\downarrow{\pi_m} & & \downarrow{\pi} \\
Z_m & \xrightarrow{\Phi_{Z_m}} & \Gamma \backslash D
\end{array}
\]
is commutative.

From now on, we will also assume that the local Torelli theorem holds, i.e. the tangent map of the lifted period map \( \tilde{\Phi} \) is injective at every point of the Teichmüller space \( \mathcal{T} \).

Now we prove a lemma concerning the monodromy group \( \Gamma \) on \( Z_m \) with level \( m \geq 3 \), which will be used in the following discussion.

**Lemma 3.3.** Let \( \gamma \) be the image of some element of \( \pi_1(Z_m) \) in \( \Gamma \) under the monodromy representation. Suppose that \( \gamma \) is of finite order, then \( \gamma \) is trivial. Therefore we can assume that \( \Gamma \) is torsion-free and \( \Gamma \setminus D \) is smooth.

**Proof.** Look at the period map locally as \( \Phi_{Z_m} : \Delta^* \to \Gamma \setminus D \). Assume that \( \gamma \) is the monodromy action corresponding to the generator of the fundamental group of \( \Delta^* \). We lift the period map to \( \tilde{\Phi} : \mathbb{H} \to D \), where \( \mathbb{H} \) is the upper half plane and the covering map from \( \mathbb{H} \) to \( \Delta^* \) is

\[
\tilde{\Phi}(z + 1) = \gamma \tilde{\Phi}(z) \quad \text{for any } z \in \mathbb{H}.
\]

But \( \tilde{\Phi}(z + 1) \) and \( \tilde{\Phi}(z) \) correspond to the same point when descending onto \( Z_m \), therefore by the definition of \( Z_m \) we have

\[
\gamma \equiv I \mod (m).
\]

But \( \gamma \) is also in \( \text{Aut}(H_z) \), applying Serre’s lemma [23] or Lemma 2.4 in [26], we have \( \gamma = I \). \( \square \)

### 4. Affine structures on Teichmüller spaces

In this section, we first apply Theorem 2.9 to the Teichmüller spaces defined in the previous section. Then we construct a holomorphic map \( \Psi : \mathcal{T} \to A \) where \( A \) is a Euclidean subspace of \( N_+ \), and introduce the notion of strong local Torelli. We then prove that this holomorphic map is an immersion if strong local Torelli holds. This immersion induces a global complex affine structure on \( \mathcal{T} \).

Let us consider

\[
a = d\Phi_p(T^{1,0}_p \mathcal{T}) \subseteq T^{1,0}_o D \simeq n_+
\]

where \( p \) is the base point in \( \mathcal{T} \) with \( \Phi(p) = o \). By Griffiths transversality, \( a \subseteq g^{-1,1} \) is an abelian subspace, therefore \( a \subseteq n_+ \) is an abelian subalgebra of \( n_+ \) determined by the tangent map of the period map

\[
d\tilde{\Phi} : T^{1,0} \mathcal{T} \to T^{1,0} D.
\]

Consider the corresponding Lie group

\[
A \triangleq \exp(a) \subseteq N_+.
\]

Then \( A \) can be considered as a complex Euclidean subspace of \( N_+ \) with the induced Euclidean metric from \( N_+ \).
Simultaneous normalization of the periods of algebraic manifolds and · · · 23

Define the projection map $P : N_+ \to A$ by

$$P = \exp \circ p \circ \exp^{-1}$$

where $\exp^{-1} : N_+ \to \mathfrak{n}_+$ is the inverse of the exponential map $\exp : \mathfrak{n}_+ \to N_+$, and

$$p : \mathfrak{n}_+ \to a$$

is the orthogonal projection map from the complex Euclidean space $\mathfrak{n}_+$ to its Euclidean subspace $a$ with respect to the Hodge metric at the base point.

The period map $\tilde{\Phi} : T \to N_+ \cap D$ composed with the projection $P$ gives a holomorphic map, $\Psi = P \circ \tilde{\Phi}$,

(20) $\Psi : T \to A$.

Let us recall the definition of complex affine structure on a complex manifold.

**Definition 4.1.** Let $M$ be a complex manifold of complex dimension $n$. If there is a coordinate cover $\{(U_i, \varphi_i); i \in I\}$ of $M$ such that $\varphi_{ik} = \varphi_i \circ \varphi_k^{-1}$ is a holomorphic affine transformation on $\mathbb{C}^n$ whenever $U_i \cap U_k$ is not empty, then $\{(U_i, \varphi_i); i \in I\}$ is called a complex affine coordinate cover on $M$ and it defines a holomorphic affine structure on $M$.

We will prove that the holomorphic map

$$\Psi : T \to A$$

defines a global affine structure on the Teichmüller space $T$, under the condition introduced in the following definition.

**Definition 4.2.** The period map $\tilde{\Phi} : T \to D$ is said to satisfy strong local Torelli, if the local Torelli holds, i.e. the tangent map of the period map $\tilde{\Phi}$ is injective at each point of $T$, and furthermore, there exists a holomorphic subbundle $\mathcal{H}$ of the Hodge bundle

$$\bigoplus_{k=1}^{n} \text{Hom}(\mathcal{F}^k / \mathcal{F}^{k+1}, \mathcal{F}^{k-1} / \mathcal{F}^k),$$

such that the tangent map of the period map induces an isomorphism on $T$ of the tangent bundle $T^{1,0}T$ of $T$ to the Hodge subbundle $\mathcal{H}$,

(21) $d\tilde{\Phi} : T^{1,0}T \sim \to \mathcal{H}$.

Recall that $\mathcal{F}^k$, $0 \leq k \leq n$ are Hodge bundles with fibers given by the corresponding linear subspaces in the Hodge filtration $F_q^k$ for any $q \in T$. Note that for variation of Hodge structure from geometry, the Hodge bundles naturally exist on $T$ which are the same as pull-backs of the corresponding Hodge bundles on $D$. Therefore in the above definition, for convenience we have identified the Hodge bundles on $T$ to the corresponding Hodge bundles on $D$. 

- 23 -
The most famous example for which strong local Torelli holds are Calabi–Yau manifolds with the Hodge subbundle $H$ given by $\text{Hom}(\mathcal{F}_n, \mathcal{F}_{n-1}/\mathcal{F}_n)$. For more examples for which strong local Torelli holds, please see Section 2 of [18].

**Theorem 4.3.** Assume that the irreducible component $Z_m$ of the moduli space with level $m$ structure is smooth with an analytic family for $m \geq m_0$ and the strong local Torelli holds for $\mathcal{T}$. Then there exists a complex affine structure on the Teim"uller space $\mathcal{T}$.

**Proof.** Let $\mathcal{H}$ be the Hodge subbundle in Definition 4.2 with the isomorphism

$$d\tilde{\Phi} : T^{1,0} \mathcal{T} \sim \mathcal{H}.$$

Let $\mathcal{H}_A$ be the restriction of $\mathcal{H}$ to $A$. Since $\mathcal{H}_A$ is a Hodge subbundle, we have the natural commutative diagram

$$\begin{array}{ccc}
\mathcal{H}_A \mid_o & \sim & \mathcal{H}_A \mid_s \\
\llap{\approx} & & \llap{\approx} \\
T^{1,0} \mathcal{A} \mid_o & \sim & T^{1,0} \mathcal{A} \mid_s
\end{array}$$

at any point $s = \exp(X)$ in $A$ with $X \in a$. Here $d\exp(X)$ is the tangent map of the left translation by $\exp(X)$. Here note that the action $d\exp(X)$ on $\mathcal{H}_A$ is induced from the action on the homogeneous tangent bundle $T^{1,0} \mathcal{D}$, since $\mathcal{H}$ is a homogeneous subbundle of $T^{1,0} \mathcal{D}$. The subbundle

$$G_C \times_B ((a \oplus b)/b) \subseteq G_C \times_B ((g^{-1,1} \oplus b)/b) = T^{1,0} \mathcal{D}$$

is also a homogeneous subbundle of $T^{1,0} \mathcal{D}$, such that the restriction

$$G_C \times_B ((a \oplus b)/b) \mid_A = T^{1,0} A.$$

See [5] for the basic properties of homogeneous vector bundles.

Hence we have the isomorphism of tangent spaces $T^{1,0} A \simeq \mathcal{H}_A \mid_s$ for any $s \in A$, which implies that

$$T^{1,0} A \simeq \mathcal{H}$$

as homogeneous holomorphic bundles on $A$.

Recall that, for any $x = \exp(X) \cdot o \in N_+, X \in n_+$, our definition of the projection $P$ is given by

$$P(x) = \exp(p(X)) \cdot o \in A$$

where $p : n_+ \to a$ is the natural projection map. A direct computation implies that

$$dP_x = d\exp(X) \circ dp \circ d\exp(-X).$$

From this and the translation invariance of Hodge bundles, it follows that the tangent map of the projection map

$$P : N_+ \to A$$

- 24 -
Simultaneous normalization of the periods of algebraic manifolds and · · · 25

maps the tangent bundle of $D$,$$ T^{1,0}D \subseteq \bigoplus_{k=1}^{n} \bigoplus_{l=1}^{k} \text{Hom}(F^{k}/F^{k+1}, F^{k-l}/F^{k-l+1}) $$onto its subbundle $T^{1,0}A \simeq \mathcal{H}_A$.

Therefore by using the translation invariance of the Hodge subbundle, we deduce that the tangent map$$ d\Psi = dP \circ d\tilde{\Phi} : T^{1,0}\mathcal{T} \to T^{1,0}A $$at any point $q \in \mathcal{T}$ composed with the projection map onto the Hodge subbundle $\mathcal{H}_A$ is explicitly given by

$$ T_q^{1,0}\mathcal{T} \to \bigoplus_{k=1}^{n} \text{Hom}(F^{k}_q/F^{k+1}_q, F^{k-1}_q/F^{k}_q) \to T_{\Psi(q)}^{1,0}A \simeq \mathcal{H}_A|_{\Psi(q)} $$

which is an isomorphism by (22). This implies that $d\Psi_q$ is also an isomorphism for any $q \in \mathcal{T}$. So we have proved that the holomorphic map $\Psi : \mathcal{T} \to A$ is nondegenerate which induces an affine structure on $\mathcal{T}$ from $A$. □

It is well-known that the Teichmüller spaces of Riemann surfaces and hyperkähler manifolds have complex affine structures. There are many more examples of projective manifolds that satisfy the conditions in our theorem. In Section 2 of [18], we have verified that the moduli and Teichmüller spaces from the following examples satisfy the conditions of Theorem 4.3: K3 surfaces; Calabi-Yau manifolds; hyperkähler manifolds; smooth hypersurface of degree $d$ in $\mathbb{P}^{n+1}$ satisfying $d|(n+2)$ and $d \geq 3$; arrangement of $m$ hyperplanes in $\mathbb{P}^{n}$ with $m \geq n$; smooth cubic surface and cubic threefold. Hence their Teichmüller spaces all have complex affine structures.

Finally we prove that the image of the affine holomorphic map

$$ \Psi : \mathcal{T} \to A $$

lies in $A \cap D$. We also prove that $A \cap D$ is a complete space with respect to the metric induced from the Hodge metric on $D$. These results will be important for our next paper [18], in which we prove the global Torelli type theorems.

Recall that under strong local Torelli, the tangent map of the period map

$$ \Phi : \mathcal{T} \to N_+ \cap D $$

is injective. Hence the Hodge metric on $D$ induces a metric on $\mathcal{T}$, which is still called the Hodge metric on $\mathcal{T}$. The inclusion of $i : A \cap D \to D$ as a submanifold of $D$ induces a metric from the Hodge metric on $D$, which we will call as the Hodge metric on $A \cap D$. We denote the norm of the corresponding metrics on $\mathcal{T}$ and $A \cap D$ by $|| \cdot ||_{\text{Hod}}$.

**Proposition 4.4.** The space $A \cap D$ is a complete space with respect to the Hodge metric.
Proof. Recall that the holomorphic tangent bundle $T_{1,0}A$,
\[
A \times a \simeq AB \times_B (a \oplus b)/b \subseteq G_C \times_B (g^{-1,1} \oplus b)/b = T_{h,0}\tilde{D},
\]
which is a horizontal subbundle with restriction
\[
T_{1,0}(A \cap D) = (AB \times_B (a \oplus b))|_{A \cap D} \subseteq T_{h,0}\tilde{D}.
\]
From this we deduce that the inclusion map
\[i : A \cap D \to D\]
is a horizontal map, so it can be considered as a period map.

Let $\{x_n\}$ be a Cauchy sequence in $A \cap D$ with the Hodge metric. Then $\{x_n\}$ has a limit $x_\infty$ in $D$. By the proof of Proposition 2.5 and Remark 2.7, we get that the finiteness of the Hodge distance $d_{\text{Hod}}(x_1, x_\infty)$ implies the finiteness of the Euclidean distance $d_{\text{Euc}}(x_1, x_\infty)$ in $A$. Hence the limit $x_\infty$ also lies in $A$. Therefore $A \cap D$ is complete with respect to the Hodge metric. □

Lemma 4.5. Let $q \in T$ such that $s = \Psi(q) \in A \cap D$, then the tangent map
\[
(d\Psi)_q : T_{1,0}T \to T_{1,0}(A \cap D)
\]
has norm $\| (d\Psi)_q \|_{\text{Hod}} = 1$ with respect to the Hodge metric on $T_{1,0}T$ and the Hodge metric on $T_{1,0}(A \cap D)$.

Proof. As given in the proof of Theorem 4.3 we have the isomorphisms of bundles in (23) derived by identifying $T_{1,0}A$ with the Hodge subbundle $H_A$ in the strong local Torelli condition. From this we see that the tangent map
\[
d\Psi_q = dP_{\tilde{\Phi}(q)} \circ d\tilde{\Phi}_q : T_{1,0}T \to T_{1,0}(A \cap D)
\]
is an isomorphism from the subspace
\[
T_{1,0}T \simeq d\tilde{\Phi}_q(T_{1,0}T) \subseteq \oplus_{k=1}^n \text{Hom}(\mathcal{F}_q^{k+1}/\mathcal{F}_q^k, \mathcal{F}_q^k/\mathcal{F}_q^{k+1})
\]
to the subspace
\[
T_{1,0}A \simeq T_{1,0}(A \cap D) \subseteq \oplus_{k=1}^n \text{Hom}(\mathcal{F}_s^{k+1}/\mathcal{F}_s^k, \mathcal{F}_s^k/\mathcal{F}_s^{k+1})
\]
of the horizontal tangent bundle
\[
(24) T_hD = \oplus_{k=1}^n \text{Hom}(\mathcal{F}_s^{k+1}/\mathcal{F}_s^k, \mathcal{F}_s^k/\mathcal{F}_s^{k+1})|_D
\]
at $q$ and $s$ respectively. Both of the Hodge metrics on $T_{1,0}T$ and $T_{1,0}(A \cap D)$ are induced from the Hodge metric on the Hodge bundle (24), which is induced by the bilinear form $Q$. From this we deduce that the tangent map $d\Psi_q$ is an isometry, which implies that $\| (d\Psi)_q \|_{\text{Hod}} = 1$. □

Proposition 4.6. The image of $\Psi$ lies in $A \cap D$. 

- 26 -
Simultaneous normalization of the periods of algebraic manifolds and \cdots

**Proof.** Let \( p \in T \) be the base point and \( \tau = \Psi(p) = P(o) \). Let \( q \in T \) be any point and \( s = \Psi(q) \in A \). Let \( \gamma : [0,1] \rightarrow T \) be the geodesic curve with respect to the Hodge metric on \( T \) such that \( \gamma(0) = p, \gamma(1) = q \) and

\[
d_{\text{Hod}}(p,q) = \int_0^1 ||\dot{\gamma}(t)||_{\text{Hod}} dt < \infty.
\]

We need to show that \( \Psi(\gamma(t)) \in D \) for \( t \in [0,1] \).

Since \( D \) is open and the base point \( \tau = \Psi(p) \in D \), there exists \( \epsilon \in (0,1] \), such that \( \Psi(\gamma(t)) \in D \) for \( t \in [0,\epsilon) \). Let \( \epsilon_0 \in (0,1] \) such that

\[
\epsilon_0 = \max \{ \epsilon : \Psi(\gamma(t)) \in D \text{ for } t \in [0,\epsilon) \}.
\]

Suppose that \( \epsilon_0 \leq 1 \). Then \( \Psi(\gamma(t)) \in D \), for \( t \in [0,\epsilon_0) \), and \( \Psi(\gamma(\epsilon_0)) \notin D \) by the maximality of \( \epsilon_0 \).

Then, by Lemma [4.5] we have that

\[
d_{\text{Hod}}(\tau, \Psi(\gamma(\epsilon_0))) \leq \text{length}(\Psi \circ (\gamma|_{[0,\epsilon_0]}))
\]

\[
= \int_0^{\epsilon_0} ||d\Psi(\dot{\gamma}(t))||_{\text{Hod}} dt
\]

\[
= \int_0^{\epsilon_0} ||\dot{\gamma}(t)||_{\text{Hod}} dt \leq \int_0^1 ||\dot{\gamma}(t)||_{\text{Hod}} dt < \infty.
\]

This implies that \( \Psi(\gamma(\epsilon_0)) \in D \), which is a contradiction. Hence \( \epsilon_0 > 1 \) and \( s = \Psi(\gamma(1)) = \Psi(q) \in D \). \( \Box \)

**References**

[1] J. Carlson, A. Kasparian, and D. Toledo, Variations of Hodge structure of maximal dimension, Duke Journal of Math, 58 (1989), pp. 669-694.

[2] J. Carlson, S. Muller-Stach, and C. Peters, Period Mappings and Period Domains, Cambridge University Press, (2003).

[3] E. Cattani, F. El Zein, P. A. Griffiths, and L. D. Trang, Hodge Theory, Mathematical Notes, 49, Princeton University Press, (2014).

[4] J. Cheeger and D. Ebin, Comparison theorems in Riemannian geometry, American Mathematical Soc., Vol. 365, (2008).

[5] P. Griffiths, On the differential geometry of homogeneous vector bundles, Transactions of the American Mathematical Society, 109(1) (1963), pp. 1-34.

[6] P. Griffiths, Periods of integrals on algebraic manifolds I, Amer. J. Math., 90 (1968), pp. 568-626.

[7] P. Griffiths, Periods of integrals on algebraic manifolds II, Amer. J. Math., 90 (1968), pp. 805-865.

[8] P. Griffiths, Periods of integrals on algebraic manifolds III, Publ. Math. IHES., 38 (1970), pp. 125-180.

[9] P. Griffiths, Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems, Bull. Amer. Math. Soc., 76, no.2 (1970), pp. 228-296.

[10] P. Griffiths, Topics in transcendental algebraic geometry, Annals of Mathematics Studies, Volume 106, Princeton University Press, Princeton, NJ, (1984).

[11] P. Griffiths and W. Schmid, Locally homogeneous complex manifolds, Acta Math., 123 (1969), pp. 253-302.
[12] P. Griffiths and J. Wolf, Complete maps and differentiable coverings, *Michigan Math. J.*, Volume 10, Issue 3 (1963), pp. 253–255.

[13] K. Kato and S. Usui, *Classifying spaces of degenerating polarized Hodge structures*, Annals of Mathematics Studies, 169, Princeton University Press, Princeton, NJ, (2009).

[14] K. Kodaira and J. Morrow, *Complex manifolds*, AMS Chelsea Publishing, Providence, RI, (2006), Reprint of the 1971 edition with errata.

[15] K. Konno, *Infinitesimal Torelli theorem for complete intersections in certain homogeneous Kähler manifolds*, II, *Tohoku Math. J.*, 42, no.3 (1990), pp. 333-338.

[16] K. Liu and Y. Shen, *Hodge metric completion of the Teichmüller space of Calabi–Yau manifolds*, arXiv:1305.0231.

[17] K. Liu and Y. Shen, and X. Chen, *Applications of the affine structures on the Teichmüller spaces*, *Geometry and Topology of Manifolds*, 10th China-Japan Conference (2014).

[18] K. Liu and Y. Shen, *From local Torelli to global Torelli*, arXiv:1512.08384, (2015).

[19] K. Liu and Y. Shen, *Moduli spaces as ball quotients I, local theory*, arXiv:1810.10892, (2018).

[20] R. Mayer, *Coupled contact systems and rigidity of maximal variations of Hodge structure*, *Trans. AMS*, 352, no.5 (2000), pp. 2121-2144.

[21] C. Peters, *The local Torelli theorem I*, *Complete Intersections, Mathematische Annalen*, 217, Issue 1 (1975), pp. 1-16.

[22] H. Popp, *Moduli theory and classification theory of algebraic varieties*, Lecture Notes in Mathematics, 620, Springer-Verlag, Berlin-New York, (1977).

[23] J.-P. Serre, *Rigidité du foncteur de Jacobi d’échelon n ≥ 3*, *Sém. Henri Cartan*, 13, no.17 (1960/61), Appendix.

[24] Y. Shimizu and K. Ueno, *Advances in moduli theory*, Translation of Mathematical Monographs, 206, American Mathematics Society, Providence, Rhode Island, (2002).

[25] A. Sommese, On the rationality of the period mapping, *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 5, Issue 4 (1978), pp. 683-717.

[26] B. Szendrői, *Some finiteness results for Calabi-Yau threefolds*, *J. London Math. Soc.*, (2) 60 (1999), pp. 689-699.