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THE FUNDAMENTAL GROUP OF QUOTIENTS OF A
PRODUCT OF CURVES

THOMAS DEDIEU AND FABIO PERRONI

This article is dedicated to the memory of Fritz Grunewald

Abstract. We prove a structure theorem for the fundamental group of the quotient $X$ of a product of curves by the action of a finite group $G$, hence for that of any resolution of the singularities of $X$.

1. Introduction

The study of varieties isogenous to a product of curves was initiated by Catanese in [Cat00], inspired by a construction of Beauville. These varieties are quotients of a product of smooth projective curves $C_1 \times \cdots \times C_n$ by the free action of a finite group $G$.

Much of the work in this area has been focused in the $n=2$ case. Surfaces isogenous to a product of curves provide a wide class of surfaces quite manageable to work with, since they are determined by discrete combinatorial data. They were used successfully to address various questions (see e.g. the survey paper [BCP06]), and in particular to obtain substantial information about various moduli spaces of surfaces of general type (see e.g. [BC04, BCG08, BCGP09]).

In the case of a variety isogenous to a product, the action of $G$ is free, and $X := (C_1 \times \cdots \times C_n)/G$ is smooth. Furthermore, we have the following natural description of the fundamental group of $X$.

**Proposition 1.1.** [Cat00] If $X := (C_1 \times \cdots \times C_n)/G$ is the quotient of a product of curves by the free action of a finite group, then the fundamental group of $X$ sits in an exact sequence

$$1 \rightarrow \Pi_{g_1} \times \cdots \times \Pi_{g_n} \rightarrow \pi_1(X) \rightarrow G \rightarrow 1,$$

where each $\Pi_{g_i}$ is the fundamental group of $C_i$. This extension, in the unmixed case where each $\Pi_{g_i}$ is a normal subgroup, is determined by the associated maps $G \rightarrow \text{Out}(\Pi_{g_i})$ to the respective Teichmüller modular groups.

In the recent paper [BCGP09], Bauer, Catanese, Grunewald and Pignatelli prove that a similar statement still holds under weaker assumptions.

**Theorem 1.2.** [BCGP09, Thm. 0.10 and Thm. 4.1] Assume that $G$ acts faithfully on each curve $C_i$ as a group of automorphisms, and let $X := (C_1 \times \cdots \times C_n)/G$ be the (possibly singular) quotient by the diagonal action of $G$. Then the fundamental group $\pi_1(X)$ has a normal subgroup of finite index isomorphic to the product of $n$ surface groups. We call $G'$ the quotient group.
Here, by a surface group we mean a group isomorphic to the fundamental group of a compact Riemann surface. Note that, unlike in Proposition 1.1, the surface groups in Theorem 1.2 above are not necessarily isomorphic to the fundamental groups of the curves $C_1, \ldots, C_n$, and furthermore that the corresponding quotient $G'$ of $\pi_1(X)$ is not necessarily isomorphic to $G$.

The first step of the proof of Theorem 1.2 consists in showing that $\pi_1(X)$ is isomorphic to the quotient of the fibre product $T := T_1 \times_G \cdots \times_G T_n$ of $n$ orbifold surface groups (see Subsection 2.1) by its torsion subgroup $\text{Tors}(T)$. Whereas this first part rests upon geometrical considerations, the rest of the proof relies on an abstract group theoretic argument showing that this quotient necessarily contains a normal subgroup as described in Theorem 1.2. In particular, the relation occurring between the groups $G$ and $G'$ is not well understood.

Using a suitable resolution of the singularities of $X$, Bauer, Catanese, Grunewald and Pignatelli show in addition that the fundamental group of any resolution $Y$ of $X$ is isomorphic to the fundamental group of $X$, so that the same description holds for $\pi_1(Y)$.

In [BCGP09], as an important application of Theorem 1.2, many new families of algebraic surfaces $S$ of general type with $p_g(S) = 0$ are constructed, and several new examples of groups are realized as the fundamental group of an algebraic surface $S$ of general type with $p_g(S) = 0$. This increases notably our knowledge on algebraic surfaces. In fact the authors consider and classify all the surfaces whose canonical models arise as quotients $X := (C_1 \times C_2) / G$ of the product of two curves of genera $g(C_1), g(C_2) \geq 2$ by the action of a finite group $G$ such that $p_g(X) = q(X) = 0$.

In the present paper, we drop the assumption that the actions of $G$ on $C_1, \ldots, C_n$ are faithful. We obtain the following expected strengthening of Theorem 1.2.

**Theorem 1.3.** Let $C_1, \ldots, C_n$ be smooth projective curves, and let $G$ be a finite group acting on each $C_i$ as a group of automorphisms. Then the fundamental group of the quotient $X := C_1 \times \cdots \times C_n / G$ by the diagonal action of $G$ has a normal subgroup of finite index that is isomorphic to the product of $n$ surface groups.

This result should allow in the future the realization of interesting groups as fundamental groups of higher dimensional algebraic varieties, following the method developed in [BCGP09] for surfaces. Notice that, in the case where the $G$-actions are faithful, $X$ can only have isolated cyclic-quotient singularities, while if the actions are not faithful, then the singular locus of $X$ can have components of positive dimension, and the singularities are abelian-quotient singularities.

Again, one shows that any desingularization of the quotient $X$ has a fundamental group isomorphic to that of $X$. This time however, we have to rely on a strong result of Kollár [K93].

The proof follows then closely the one of Theorem 1.2 of [BCGP09]. The main new difficulty one has to overcome is to find a natural counterpart to the fibered product $T_1 \times_G \cdots \times_G T_n$, acting discontinuously on the product $\tilde{C}_1 \times \cdots \times \tilde{C}_n$ of the universal covers of $C_1, \ldots, C_n$. After that similar group theoretic arguments work with some slight modifications.

It has already been observed in [Cat00] that Theorem 1.3 follows directly from Theorem 1.2 when $n = 2$, by performing the quotient $(C_1 \times C_2) / G$ in successive steps. For $n > 2$ however, this procedure does not apply.
The paper is organized as follows. In Section 2, we fix notations and collect some basic facts about group actions on compact Riemann surfaces. Section 3 is devoted to the proof of Theorem 1.3: the proof itself is given in Subsection 3.1, using intermediate results proven in Subsections 3.2 and 3.3.

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2. NOTATIONS AND BASIC RESULTS

2.1. NOTATIONS. We work over the field of complex numbers \( \mathbb{C} \).

Let \( G \) be a group, and consider a subset \( H \subset G \). We write \( H \trianglelefteq G \) when \( H \) is a subgroup of \( G \), and \( H \triangleleft G \) when \( H \) is a normal subgroup of \( G \). If \( A \subset G \) is any subset, then \( \langle \langle A \rangle \rangle \) denotes the normal subgroup of \( G \) generated by \( A \).

Let \( g \) be a non-negative integer. We call \( \Pi_g \) the surface group of genus \( g \), defined as
\[
\Pi_g := \left\langle a_1, b_1, \ldots, a_g, b_g \middle| \prod_{i=1}^g [a_i, b_i] = 1 \right\rangle.
\]
This is the fundamental group of compact Riemann surfaces of genus \( g \). On the other hand, letting in addition \( m_1, \ldots, m_r \) be positive integers, we denote by \( \mathbf{T}(g; m_1, \ldots, m_r) \) the orbifold surface group of signature \( (g; m_1, \ldots, m_r) \), defined as
\[
\mathbf{T}(g; m_1, \ldots, m_r) := \left\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_r \middle| c_1^{m_1} \cdots c_r^{m_r} = \prod_{i=1}^g [a_i, b_i] \cdot c_1 \cdots c_r = 1 \right\rangle.
\]
It is obtained from the fundamental group of the complement of a set of \( r \) distinct points in a compact Riemann surface of genus \( g \), by quotienting by the normal subgroup generated by \( \gamma_1^{m_1}, \ldots, \gamma_r^{m_r} \), where each \( \gamma_i \) is a simple geometric counterclockwise loop around the \( i \)-th removed point.

Let \( G \) be a finite group. An appropriate orbifold homomorphism is a surjective homomorphism \( \varphi : \mathbf{T}(g; m_1, \ldots, m_r) \to G \) such that \( \varphi(c_i) \) has order \( m_i \) for \( i = 1, \ldots, r \).

The action of a group \( G \) as a group of homeomorphisms on a topological space \( X \) is said to be discontinuous if the following two conditions are satisfied: (i) the stabilizer of each point of \( X \) is finite; (ii) each point \( x \in X \) has a neighbourhood \( U \) such that \( g(U) \cap U = \emptyset \), for each \( g \in G \) such that \( gx \neq x \).
2.2. Basic results. The following result is essentially a reformulation of Riemann’s existence theorem (see [BCGP09, Thm. 2.1]).

**Theorem 2.1.** A finite group $G$ acts faithfully as a group of automorphisms on a compact Riemann surface of genus $g$ if and only if there are natural numbers $g', m_1, \ldots, m_r$ and an appropriate orbifold homomorphism

$$\varphi: T(g'; m_1, \ldots, m_r) \to G$$

such that the Riemann-Hurwitz relation holds:

$$2g - 2 = |G| \left( 2g' - 2 + \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) \right).$$

**Remark 2.2.** As already remarked in [BCGP09], under the above hypotheses, $g'$ is the geometric genus of $C' := C/G$, and $m_1, \ldots, m_r$ are the branching indices at the branching points of the $G$-cover $p: C \to C'$. The appropriate orbifold homomorphism $\varphi$ is induced by the monodromy of the Galois étale $G$-covering $p: C \to C'$, where $C'$ is the Riemann surface obtained from $C$ by removing the branch points of $p$, and $C' := p^{-1}(C^G)$. In particular, $\varphi(c_i)$ generates the stabilizer of the corresponding ramification point.

Furthermore, the kernel of $\varphi$ is isomorphic to the fundamental group $\pi_1(C)$, and the action of $\pi_1(C)$ on the universal cover $\tilde{C}$ of $C$ extends to a discontinuous action of $T := T(g'; m_1, \ldots, m_r)$. Let $u: \tilde{C} \to C$ be the covering map. It is $\varphi$-equivariant, and $C/G \cong \tilde{C}/T$.

We now give two elementary facts that will be used in the following.

**Lemma 2.3.** (i) Let $x \in \tilde{C}$. Then the restriction of $\varphi$ to the stabilizer $St_x$ of $x$ (with respect to the action of $T$ on $\tilde{C}$) is injective.

(ii) Let $t \in St_x$. Then $t$ is conjugated to $c_i^m$, for some $i \in \{1, \ldots, r\}$ and $m \in \mathbb{N}$.

**Proof.** The $\pi_1(C)$-action on $\tilde{C}$ is free, so $\pi_1(C) \cap St_x = \{1\}$. This yields (i), because $\pi_1(C)$ is the kernel of $\varphi$.

To prove (ii), let $y = u(x)$. If $t = 1$, then the result is clear. Else, there exists an integer $i \in \{1, \ldots, m\}$ and a point $x' \in u^{-1}(y)$ that is fixed by $c_i$. It then follows from (i) that $St_{x'} = \langle c_i \rangle$, hence that $t$ is conjugated to a power of $c_i$.

3. Main Theorem

The main result of the paper is the following

**Theorem 3.1.** Let $C_1, \ldots, C_n$ be compact Riemann surfaces, and let $G$ be a finite group that acts as a group of automorphisms on each $C_i$. We consider the quotient of the product $C_1 \times \cdots \times C_n$ by the diagonal action of $G$. Then there is a normal subgroup of finite index $\Pi$ in the fundamental group

$$\pi_1 \left( \frac{C_1 \times \cdots \times C_n}{G} \right),$$

such that $\Pi$ is isomorphic to the product of $n$ surface groups.

Notice that, according to notations 2.1, a surface group is a group isomorphic to the fundamental group of a compact Riemann surface of genus a non negative integer $g$, in particular we admit also the “degenerate cases” where $g = 0, 1$.  

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The proof of this theorem follows closely that of [BCGP09, Thm. 4.1], and is given in the next subsections. Before we move on to this proof, let us give the following important consequence of Theorem 3.1.

**Corollary 3.2.** Let $C_1, \ldots, C_n$ and $G$ be as in the statement of Theorem 3.1, and let $Y$ be a resolution of the singularities of $X := (C_1 \times \cdots \times C_n)/G$. Then, the fundamental group of $Y$ is isomorphic to the fundamental group of $X$, and moreover it has a normal subgroup of finite index isomorphic to the product of $n$ surface groups.

**Proof.** The natural morphism

$$f_* : \pi_1(Y) \longrightarrow \pi_1(X)$$

induced by the resolution $f : Y \to X$ is an isomorphism. This follows directly from [K93, Sec. 7]: since $X$ is normal and only has quotient singularities, $Y$ is locally simply connected by [K93, Thm. 7.2], hence $f_*$ is an isomorphism by [K93, Lem. 7.2]. The second claim is now a direct consequence of Theorem 3.1. □

3.1. **Proof of the main theorem.** For $i = 1, \ldots, n$, we let

$$K_i = \ker (G \to \text{Aut}(C_i)) \quad \text{and} \quad H_i = G/K_i,$$

where $G \to \text{Aut}(C_i)$ is the morphism associated to the action of $G$ on $C_i$. We call $p_i$ the projection $G \to H_i$. Now $H_i$ acts faithfully on $C_i$, so we have (see Remark 2.2) a short exact sequence

$$1 \to \pi_1(C_i) \to T_i \overset{\varphi_i}{\longrightarrow} H_i \to 1,$$

(3.1)

where $T_i$ is an orbifold surface group, and $\varphi_i$ is an appropriate orbifold homomorphism. Let $\Sigma := G \times H_i$, $T_i$ be the fibered product corresponding to the Cartesian diagram

$$\begin{array}{cccccc}
\Sigma_i & \longrightarrow & T_i \\
\psi_i & \downarrow & \varphi_i \\
G & \overset{p_i}{\longrightarrow} & H_i.
\end{array}$$

We call $\psi_i : \Sigma_i \to G$ the projection on the first factor. Pulling-back (3.1) by $p_i : G \to H_i$, we obtain a short exact sequence

$$1 \to \pi_1(C_i) \to \Sigma_i \overset{\psi_i}{\longrightarrow} G \to 1,$$

(3.2)

where the left-hand side map is $\gamma \in \pi_1(C_i) \mapsto (1, \gamma) \in \Sigma_i$.

Next, we define $\tilde{G} := \Sigma_1 \times_G \cdots \times_G \Sigma_n$ as the fibered product corresponding to the Cartesian diagram below.

Let $\Delta : G \to G \times \cdots \times G$ be the diagonal morphism. Then $\tilde{G}$ can also be seen as the fibered product $G \times (G \times \cdots \times G) \to \Sigma_1 \times \cdots \times \Sigma_n \to G$ with respect to the two
morphisms $\Delta$ and $(\psi_1, \ldots, \psi_n)$. Therefore, the pull-back by $\Delta$ of the product of the $n$ exact sequences (3.2) for $i = 1, \ldots, n$ is a short exact sequence

$$1 \to \prod_{i=1}^{n} \pi_1(C_i) \to \tilde{G} \xrightarrow{\Psi} G \to 1,$$

where $\Psi$ is the first projection $G \times (G \times \cdots \times G) \to G$.

Now we have the following, coming from the fact that $\tilde{G}$ acts discontinuously on the universal cover of $C_1 \times \cdots \times C_n$.

**Proposition 3.3.** Let $G' \leq G$ be the normal subgroup of $\tilde{G}$ generated by those elements which have non-empty fixed-point set. Then

$$\pi_1 \left( \frac{C_1 \times \cdots \times C_n}{G} \right) \cong \frac{\tilde{G}}{G'}.$$

**Proof.** For $i = 1, \ldots, n$, the action of $T_i$ on the universal covering $\tilde{C}_i$ of $C_i$ (see Remark 2.2) induces an action of $\Sigma_i$ on $\tilde{C}_i$ via the projection of $\Sigma_i$ on its second factor $T_i$. We obtain in this way an action of $\tilde{G}$ on the product $\tilde{C}_1 \times \cdots \times \tilde{C}_n$.

This action is discontinuous: let $St_x$ be the stabilizer of a point $x \in \tilde{C}_1 \times \cdots \times \tilde{C}_n$ with respect to the action of $\tilde{G}$. Then the same argument as that in the proof of Lemma 2.3 shows that $\Psi_{St_x}$ is injective, from which it follows that $St_x$ is finite because $G$ is finite. On the other hand, condition (ii) in the definition of a discontinuous action is a consequence of the fact that the $T_i$-actions are themselves discontinuous.

Then, the main theorem in [Arm68] applies to our situation, and gives a group isomorphism

$$\pi_1 \left( \frac{\tilde{C}_1 \times \cdots \times \tilde{C}_n}{G} \right) \cong \frac{\tilde{G}}{G'}.$$

Eventually, since the universal covering $U: \tilde{C}_1 \times \cdots \times \tilde{C}_n \to C_1 \times \cdots \times C_n$ is $\Psi$-equivariant, we have an isomorphism

$$\frac{C_1 \times \cdots \times C_n}{G} \cong \frac{\tilde{C}_1 \times \cdots \times \tilde{C}_n}{G},$$

and the proposition follows. \qed

**Remark 3.4.** The elements of $\tilde{G}$ which have fixed-points are precisely those elements of finite order. Therefore $\tilde{G}'$ is the torsion subgroup of $\tilde{G}$.

Now the proof of Theorem 3.1 relies on the following result, the proof of which we postpone to Subsection 3.3.

**Proposition 3.5.** The quotient $\tilde{G}/\tilde{G}'$ is an extension

$$1 \to E \to \tilde{G}/\tilde{G}' \xrightarrow{\theta} T \to 1$$

of a finite group $E$ by a group $T$ that is a finite-index subgroup of a product of $n$ orbifold surface groups.

Using the results of [GJZ08], the latter fact enables one to show that there is a finite index normal subgroup $\Gamma \leq \tilde{G}/\tilde{G}'$ that injects in $T$.
Lemma 3.6. Let $S$ be a group sitting in an exact sequence

$$1 \to E \to S \to T \to 1,$$

where $E$ is a finite group, and $T$ is a finite index subgroup of a product of $n$ orbifold surface groups. Then $S$ is residually finite. In particular, there exists a finite index normal subgroup $\Gamma \trianglelefteq S$ such that $\Gamma \cap E = \{1\}$.

Proof. By [GJZ08, Prop. 6.1], an extension of a finite group by a group that is residually finite and good in the sense of [Ser94] is residually finite. It therefore suffices to show that $T$ enjoys the two aforementioned properties.

An orbifold surface group is residually finite. Therefore $T$ is itself residually finite, being a finite index subgroup of a product of orbifold surface groups.

By [GJZ08, Lem. 3.2], it is enough to show that a product of orbifold surface groups is good to prove that $T$ is good. But [GJZ08, Prop. 3.7] tells us that an orbifold surface group is good, and [GJZ08, Prop. 3.4] that a product of good groups is good.

\[\square\]

We are now in a position to complete the proof of our main theorem:

Proof of Theorem 3.1. Let $T_1 \times \cdots \times T_n$ be a product of $n$ orbifold surface groups containing $T$ as a finite index subgroup, and let us consider $\Gamma \trianglelefteq \tilde{G}/\tilde{G}'$ a normal subgroup of finite index such that $E \cap \Gamma = \{1\}$. Then $\theta(\Gamma) \leq T_1 \times \cdots \times T_n$ has finite index.

Now every orbifold surface group contains a surface group as a finite index subgroup (see e.g. [Bea95]), so let $\Pi_i$ be a finite index surface group in $T_i$ for each $i = 1, \ldots, n$.

For each $i$, we consider

$$\theta(\Gamma)_i := \theta(\Gamma) \cap (\{1\} \times \cdots \times T_i \times \cdots \times \{1\}$$

as a subgroup $\theta(\Gamma)_i \leq T_i$, and set

$$\Pi'_i := \bigcap_{g \in T_i} g(\theta(\Gamma)_i \cap \Pi_i) g^{-1},$$

the biggest normal subgroup of $T_i$ contained in $\theta(\Gamma)_i \cap \Pi_i$. Then $\Pi'_i$ has finite index in $\Pi_i$, and thus is itself a surface group. Eventually, $\Pi := \Pi'_1 \times \cdots \times \Pi'_n$ is a subgroup of $\theta(\Gamma)$, which is normal and of finite index in $T$. Therefore, $\theta^{-1}(\Pi) \cap \Gamma$ is a normal subgroup of $\tilde{G}/\tilde{G}'$, with finite index, and isomorphic to $\Pi$.

\[\square\]

3.2. Results in group theory. In this subsection, we prove some technical results that are needed for the proof of Proposition 3.5.

Let $\Sigma$ be any group, $R \trianglelefteq \Sigma$ be a normal subgroup, and $L \subset \Sigma$ be a subset. We define

$$N(R, L) := \langle\langle hkh^{-1}k^{-1} | h \in L, k \in R \rangle\rangle_{\Sigma} \quad (3.4)$$

and

$$\hat{\Sigma} := \hat{\Sigma}(R, L) := \Sigma/N(R, L). \quad (3.5)$$

We call $\hat{R}$ and $\hat{L}$ the images of $R$ and $L$ respectively by the projection $\Sigma \to \hat{\Sigma}$. There is an isomorphism: $\hat{\Sigma}/\langle\langle L \rangle\rangle_{\Sigma} \cong \Sigma/\langle\langle L \rangle\rangle_{\Sigma}$. Notice also that $N(R, L) \trianglelefteq R$ and $N(R, L) \trianglelefteq \langle\langle L \rangle\rangle_{\Sigma}$, which implies that $\hat{R}$ is a normal subgroup of $\hat{\Sigma}$.
Lemma 3.7. If $R \subseteq \Sigma$ has finite index, and if $L \subset \Sigma$ is a finite subset consisting of elements of finite order, then $(\langle L \rangle)_\Sigma$ is finite.

Proof. The subgroup $(\langle L \rangle)_\Sigma$ is the image of $(\langle L \rangle)_\Sigma$ under the projection $\Sigma \to \hat{\Sigma}$. Since $R$ has finite index in $\Sigma$, and $L$ is finite, it follows that $(\langle L \rangle)_\Sigma$ is generated by finitely many elements which are conjugated to those of $\hat{L}$. Since the elements of $L$ have finite order, these generators have finite order as well.

The center $Z(\langle L \rangle)_\Sigma$ of $(\langle L \rangle)_\Sigma$ contains $R \cap (\langle L \rangle)_\Sigma$, and hence has finite index in $(\langle L \rangle)_\Sigma$. Now, by [BCGP09, Lem. 4.6], if a group $S$ is generated by finitely many elements of finite order, and if its centre has finite index in $S$, then $S$ is finite. From this we conclude that $(\langle L \rangle)_\Sigma$ is finite.

We now consider the particular case when $\Sigma$ is a group constructed as in Subsection 3.1: $\Sigma = G \times_H T$, where $G$ is a finite group, $H$ is a quotient of $G$, and $T$ is any group coming with a surjective morphism $\varphi : T \to H$.

Lemma 3.8. The projection on the second factor $q : \Sigma \to T$ induces a morphism

$$\tilde{q} : \frac{\Sigma}{\langle\langle L \rangle\rangle_{\Sigma}} \to \frac{T}{\langle\langle q(L) \rangle\rangle_{T}}$$

(3.6)
in a natural way. It is surjective, and has finite kernel.

Proof. We have $q(\langle\langle L \rangle\rangle_{\Sigma}) = \langle\langle q(L) \rangle\rangle_{T}$. The map $\tilde{q}$ is therefore induced by the composition $\Sigma \xrightarrow{\tilde{q}} T \to T/\langle\langle q(L) \rangle\rangle_{T}$, which is clearly surjective. To prove the finiteness of its kernel, notice that for any $(g, t) \in q^{-1}(\langle\langle q(L) \rangle\rangle_{T})$, there exists $h \in G$ with $(h, t) \in \langle\langle L \rangle\rangle_{\Sigma}$, hence $gh^{-1} \in K := \ker(G \to H)$. It follows that $q^{-1}(\langle\langle q(L) \rangle\rangle_{T}) = K/\langle\langle L \rangle\rangle_{\Sigma}$, where $K$ is seen as a subgroup in $\Sigma$ via the injection $k \in K \mapsto (k, 1) \in \Sigma$. Eventually, $\ker(\tilde{q}) \cong K \subset G$, which is finite.

3.3. Realization of the fundamental group as a suitable extension. In this subsection, we give a full proof of Proposition 3.5. We use the basic results in group theory established in Subsection 3.2 above.

For $i = 1, \ldots, n$, we fix the following presentation for the orbifold groups $T_i$ in (3.1):

$$T_i = \left\langle a_{i1}, b_{i1}, \ldots, a_{ig_i}, b_{g_i}, c_{i1}, \ldots, c_{ir_i} \right|$$

$$c_{i1}^{a_{i1}} = \cdots = c_{ir_i}^{a_{ir_i}} = \prod_{j=1}^{g_i} [a_{ij}, b_{ij}] \cdot c_{i1} \cdot \cdots \cdot c_{ir_i} = 1 \right\rangle,$$

and set $R_i = \pi_1(C_i)$. We write the elements of $\tilde{G}$ as $(g, z_1, \ldots, z_n)$, with $(g, z_i) \in \Sigma_i$ for $i = 1, \ldots, n$. Then we have:

Lemma 3.9. For each $i = 1, \ldots, n$, there exists a finite subset $\mathcal{N}_i \subset \tilde{G}$, such that

$$\langle\langle \mathcal{N}_i \rangle\rangle_{\tilde{G}} = G'',$

and whose elements are of the form

$$(g, z_1d_1^1z_1^{-1}, \ldots, d_1^1, \ldots, d_n^1, \ldots, z_n\ell_n, \ell_n^{-1})$$

for some $g \in G$, some $d_j \in \{c_{j1}, \ldots, c_{jr_j}\}$ and $\ell_j \in \mathbb{N}$ for $j = 1, \ldots, n$, and some $z_j \in T_j$ for $j \neq i$. 

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Remark 3.10. As a direct consequence of Lemma 3.9, if
\[(g, z_1d_1^i z_1^{-1}, \ldots, d_i^i, \ldots, z_n d_n^i z_n^{-1}) \in N_i\]
for some \(i\), then for any \(j \neq i\), there exists
\[(h, y_1, \ldots, y_j, \ldots, y_n) \in \Sigma_1 \times_G \cdots \times_G \Sigma_j \times_G \cdots \times_G \Sigma_n\]
(where a hat means that the corresponding factor is omitted), such that
\[(h g^{-1}, y_1 z_1 d_1^i z_1^{-1} y_1^{-1}, \ldots, y_i d_i^i y_i^{-1}, \ldots, d_j^i, \ldots, y_n d_n^i z_n^{-1} y_n^{-1}) \in N_j.\]

Proof of Lemma 3.9. Let \(s \in \tilde{G}\) be an element with non empty fixed-point set, and let us fix \(i \in \{1, \ldots, n\}\). By Lemma 2.3 (ii), \(s\) writes
\[s = (g, z_1 d_1^i z_1^{-1}, \ldots, z_n d_n^i z_n^{-1}),\]
with notations as in the statement of the Lemma. Obviously, one can find \(h \in G, \zeta \in G\) for each \(j \neq i\), such that \((h, \zeta_1, z_1, \ldots, z_i, \ldots, z_n) \in \tilde{G}\), and therefore \(s\) is conjugated in \(\tilde{G}\) to an element of type
\[(g', y_1 d_1^i y_1^{-1}, \ldots, d_i^i, \ldots, y_n d_n^i y_n^{-1}).\]  \(\) (3.7)

Now we claim that there exists finite sets \(A_j \subset T_j, j = 1, \ldots, i, \ldots, n\), such that each element of \(\tilde{G}\) as in (3.7) is conjugated in \(\tilde{G}\) to some
\[(g'', x_1 d_1^i x_1^{-1}, \ldots, d_i^i, \ldots, x_n d_n^i x_n^{-1}),\]
with \(x_j \in A_j\) for each \(j \neq i\). Then it is clear that one can build \(N_i\) as required.

To prove our claim, first note that if \((g, z_1 d_1^i z_1^{-1}, \ldots, d_i^i, \ldots, z_n d_n^i z_n^{-1}) \in \tilde{G}\), then an \((n + 1)\)-uple \((g, \zeta_1 d_1^i \zeta_1^{-1}, \ldots, d_i^i, \ldots, \zeta_n d_n^i \zeta_n^{-1})\) corresponds to an element of the fibered product \(\hat{G}\) if and only if for each \(j \neq i\), \(\varphi_j(z_j^{-1} \zeta_j)\) belongs to the centralizer \(C_{H_j}(\varphi_j(d_j^i))\) of \(\varphi_j(d_j^i)\) in \(H_j\).

Second, note that if \(k_j \in R_j\) for some \(j \neq i\), then \((1, 1, \ldots, k_j, \ldots, 1) \in \tilde{G}\), and therefore any element \((g, \ldots, z_j d_j^i z_j^{-1}, \ldots) \in \tilde{G}\) is conjugated to
\[(g, \ldots, (k_j z_j) d_j^i (k_j z_j)^{-1}, \ldots) \in \tilde{G}.\]

Then our claim follows from the fact that for each \(j \neq i\), \(R_j \leq \varphi_j^{-1}(C_{H_j}(\varphi_j(d_j^i)))\) has finite index.

From now on, we let \(N_1, \ldots, N_n\) be as in Lemma 3.9.

Lemma 3.11. For \(i = 1, \ldots, n\), if
\[(g, z_1 d_1^i z_1^{-1}, \ldots, d_i^i, \ldots, z_n d_n^i z_n^{-1}) \in N_i,\]
then for all \(k_i \in R_i\), we have
\[(1, 1, \ldots, d_i^i k_i d_i^{-i} k_i^{-1}, \ldots, 1) \in \tilde{G}'.\]

Proof. Let \(k_i \in R_i\). Then \(\tilde{k_i} := (1, 1, \ldots, k_i, \ldots, 1) \in \tilde{G}\), and our result follows from the equality
\[(1, 1, \ldots, d_i^i k_i d_i^{-i} k_i^{-1}, \ldots, 1) = \]
\[(g, z_1 d_1^i z_1^{-1}, \ldots, d_i^i, \ldots, z_n d_n^i z_n^{-1}) \tilde{k_i} (g, z_1 d_1^i z_1^{-1}, \ldots, d_i^i, \ldots, z_n d_n^i z_n^{-1})^{-1} \tilde{k_i}^{-1},\]
and the fact that \(\langle\langle N_i\rangle\rangle_{\hat{G}} = \hat{G}'\). \(\square\)

For \(i = 1, \ldots, n\), we let \(L_i \subset \Sigma_i\) be the image of \(N_i\) by the projection \(\hat{G} \to \Sigma_i\). The first projection \(\psi_i : \Sigma_i \to G\) then induces an epimorphism \(\hat{\psi}_i : \hat{\Sigma}_i := \hat{\Sigma}_i(R_i, L_i) \to G\) (see Subsection 3.2 for a definition of \(\hat{\Sigma}_i\)).

Eventually, we let \(\hat{G}\) be the fibered product

\[
\hat{\Sigma}_1 \times_G \cdots \times_G \hat{\Sigma}_n \cong G \times_{(G \times \cdots \times G)} (\hat{\Sigma}_1 \times \cdots \times \hat{\Sigma}_n), \tag{3.8}
\]

and we define a map \(\Phi : \hat{G} \to \hat{G}/\hat{G}'\) by the formula

\[
\Phi([s_1], \ldots, [s_n]) = [(s_1, \ldots, s_n)], \tag{3.9}
\]

where \(s_i \in \Sigma_i\) for each \(i\) (here we see \(\hat{G}\) as contained in \(\hat{\Sigma}_1 \times \cdots \times \hat{\Sigma}_n\), using its description by the left-hand side of (3.8) rather than by its right-hand side, and similarly we see \(\hat{G}\) as contained in \(\Sigma_1 \times \cdots \times \Sigma_n\)). It is a consequence of Lemma 3.11 that \(\Phi\) is well-defined by (3.9). Now we have:

**Lemma 3.12.** The morphism \(\Phi\) is surjective, and has finite kernel.

**Proof.** The surjectivity follows at once from (3.9). On the other hand, an element \(((s_1), \ldots, [s_n]) \in \hat{G}\) lies in \(\ker \Phi\) if and only if \((s_1, \ldots, s_n) \in \hat{G}'\). This implies for \(i = 1, \ldots, n\) that \(s_i \in \langle\langle L_i\rangle\rangle_{\Sigma_i}\), because \(\hat{G}' = \langle\langle N_i\rangle\rangle_{\hat{G}}\). We thus see that \(\ker \Phi\), seen as contained in \(\hat{\Sigma}_1 \times \cdots \times \hat{\Sigma}_n\), is contained in \(\langle\langle \hat{L}_1\rangle\rangle_{\hat{\Sigma}_1} \times \cdots \times \langle\langle \hat{L}_n\rangle\rangle_{\hat{\Sigma}_n}\), which is finite by Lemma 3.7.

Next, we define a morphism

\[
\Theta : \prod_{i=1}^n \hat{\Sigma}_i \to \prod_{i=1}^n \frac{\Sigma_i}{\langle\langle L_i\rangle\rangle_{\Sigma_i}} \to \prod_{i=1}^n \frac{T_i}{\langle\langle q_i(L_i)\rangle\rangle_{\Sigma_i}}, \tag{3.10}
\]

as in Subsection 3.2: the left-hand side map in (3.10) is the product of the projections

\[
\hat{\Sigma}_i \to \frac{\hat{\Sigma}_i}{\langle\langle L_i\rangle\rangle_{\Sigma_i}} \cong \frac{\Sigma_i}{\langle\langle L_i\rangle\rangle_{\Sigma_i}},
\]

and the \(q_i\)'s are induced by the second projections \(q_i : \Sigma_i \to T_i\) as in Lemma 3.8.

We have

\[
\ker \Phi \subset \langle\langle \hat{L}_1\rangle\rangle_{\hat{\Sigma}_1} \times \cdots \times \langle\langle \hat{L}_n\rangle\rangle_{\hat{\Sigma}_n} \subset \ker \Theta, \tag{3.11}
\]

and \(\ker \Theta\) is finite by both Lemmas 3.7 and 3.8.

Let us set

\[
T := \Theta(\hat{G}).
\]

Notice that \(\hat{G}\) has finite index in \(\prod_{i=1}^n \hat{\Sigma}_i\) because \(G\) is finite, and therefore that \(T\) has finite index in \(\prod_{i=1}^n T_i/\langle\langle q_i(L_i)\rangle\rangle\). We have a short exact sequence

\[
1 \to E_1 \to \hat{G} \xrightarrow{\Theta} T \to 1. \tag{3.12}
\]


Clearly, $\ker \Phi \subset E_1$. Therefore, setting $E := E_1 / \ker \Phi$, we obtain the following commutative diagram

$$
\begin{array}{ccccccccc}
1 & 1 & 1 \\
1 & \longrightarrow & \ker \Phi & \longrightarrow & \ker \Phi & \longrightarrow & 1 \\
1 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & E_1 & \longrightarrow & \tilde{G} & \longrightarrow & T & \longrightarrow & 1 \\
1 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & E & \longrightarrow & \tilde{G} / \tilde{G} & \longrightarrow & T & \longrightarrow & 1 \\
1 & 1 & 1 \\
\end{array}
$$

(3.13)

where $\theta$ is the morphism induced by $\Theta|_{\tilde{G}}$ which makes the diagram commutative.

We then claim that the lower row of the diagram (3.13) is the short exact sequence we are looking for: exactness follows from an easy diagram chase; the finiteness of $E$ follows from that of $E_1$; eventually, each $T_i / \langle \langle q_i(L_i) \rangle \rangle$ is an orbifold surface group, because $q_i(L_i)$ consists of finite order elements (see e.g. [BCGP09, Lem. 4.7]), so that $T$ is a finite index subgroup in a product of orbifold surface groups. This concludes the proof of Proposition 3.5.

References

[Arm65] M. A. Armstrong. On the fundamental group of an orbit space, Proc. Cambridge Philos. Soc. 61 (1965) 639–646.
[Arm68] M. A. Armstrong. The fundamental group of the orbit space of a discontinuous group, Proc. Cambridge Philos. Soc. 64 (1968) 299–301.
[BC04] I. Bauer and F. Catanese, Some new surfaces with $p_g = q = 0$, in The Fano Conference (2004), 123–142, Univ. Torino, Turin.
[BCG08] I. Bauer, F. Catanese, F. Grunewald. The classification of surfaces with $p_g = q = 0$ isogenous to a product of curves, Pure Appl. Math. Q. 4 (2008), 547–586.
[BCGP09] I. Bauer, F. Catanese, F. Grunewald, R. Pignatelli. Quotients of products of curves, new surfaces with $p_g = 0$ and their fundamental groups, arXiv:0809.3420.
[BCP06] I. Bauer, F. Catanese, and R. Pignatelli, Complex surfaces of general type: some recent progress, in Global aspects of complex geometry (2006), 1–58, Springer, Berlin.
[Bea95] A. F. Beardon. The geometry of discrete groups, volume 91 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1995. Corrected reprint of the 1983 original.
[Cat00] F. Catanese. Fibred surfaces, varieties isogenous to a product and related moduli spaces, Amer. J. Math. 122 (2000), no. 1, 1–44.
[Cat03] F. Catanese. Moduli spaces of surfaces and real structures, Ann. of Math. 158 (2003) 577–592.
[GJZ08] F. Grunewald, A. Jaikin-Zapirain, P.A. Zaleskii. Cohomological goodness and the profinite completion of Bianchi groups, Duke Math. J. 144 (2008), no. 1, 53–72.
[K93] J. Kollár. Shafarevich maps and plurigenera of algebraic varieties, Invent. Math. 113 (1993), no. 1, 177–215.
[Ser94] J.-P. Serre. Cohomologie galoisienne, volume 5 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, fifth edition, 1994.
