Pathwise stationary solutions of stochastic Burgers equations with $L^2[0,1]$-noise and stochastic Burgers integral equations on infinite horizon

Yong Liu$^1,2$, Huaizhong Zhao$^1$

$^1$ Department of Mathematical Sciences, Loughborough University, Loughborough, LE11 3TU, UK.  
$^2$ LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, P.R. China  
liuyong@math.pku.edu.cn, H.Zhao@lboro.ac.uk

Summary. In this paper, we show the existence and uniqueness of the stationary solution $u(t, \omega)$ and stationary point $Y(\omega)$ of the differentiable random dynamical system $U : \mathbb{R} \times L^2[0,1] \times \Omega \rightarrow L^2[0,1]$ generated by the stochastic Burgers equation with $L^2[0,1]$-noise and large viscosity, especially, $u(t, \omega) = U(t, Y(\omega), \omega) = Y(\theta(t, \omega))$, and $Y(\omega) \in H^1[0,1]$ is the unique solution of the following equation in $L^2[0,1]$

$$Y(\omega) = \frac{1}{2} \int_{-\infty}^{0} T_\nu(-s) \frac{\partial(Y(\theta(s, \omega)))^2}{\partial x} ds + \int_{-\infty}^{0} T_\nu(-s)dW_s(\omega),$$

where $\theta$ is the group of $P$-preserving ergodic transformation on the canonical probability space $(\Omega, \mathcal{F}, P)$ such that $\theta(t, \omega)(s) = W(t + s) - W(t)$.

Keywords: Stochastic Burgers equations; random dynamical system; stationary solution, stochastic Burgers integral equations in infinite horizon.

1 Introduction

The stationary point (or stationary solution) is one of the fundamental concepts in dynamical systems. For example, for an autonomous ordinary differential equation (ODE): $\dot{X} = F(X)$, a stationary point is a point in the set $\{x : F(x) = 0\}$ in the phase space or the stationary solution is a trajectory (fixed in the autonomous case) satisfying $F(X(t)) = 0$ for any $t \in (-\infty, \infty)$. Roughly speaking, the behaviour of the solution near a stationary point describes the asymptotic properties of the dynamical systems and the stationary solution gives equilibrium state. For the infinite-dimensional dynamical systems generated by some partial differential equations (PDEs) of the following form (see [18, 22])
\[ \frac{\partial u(t,x)}{\partial t} = F(u, D_x u, D_x^2 u, \cdots), \]
a stationary point is a solution of the equation \( F(u(x), D_x u(x), D_x^2 u(x), \cdots) = 0 \), at least formally. The stationary point is a graph on the configuration space.

To extend the concept of the stationary point (or stationary solution) and establish its existence and the decomposition of stable and unstable manifolds on the tangent space of the stationary point to random dynamical systems (RDS) is a basic problem for RDS (\cite{1}). In recent years, the stable and unstable manifolds theorem has been established for finite-dimensional stochastic differential equations (\cite{14}, \cite{15}); and for the infinite-dimensional case, the stable manifold theorem for semilinear stochastic evolution equations and stochastic partial differential equations (SPDEs) was proved recently by Mohammed, Zhang and Zhao (\cite{17}); invariant manifolds for SPDE and smooth stable and unstable manifolds for stochastic evolution equations with one dimensional linear noise were studied by Duan, Lu and Schmalfuss (\cite{3}, \cite{4}).

To define the pathwise stationary solution of RDS, let \( (\Omega, \mathcal{F}, P) \) be a complete probability space, \( \theta : \mathbb{R} \times \Omega \to \Omega \) be a group of \( P \)-preserving ergodic transformations on \( (\Omega, \mathcal{F}, P) \), \( H \) be a separable Hilbert space with norm \( |\cdot| \) and Borel \( \sigma \)-algebra \( \mathcal{B}(H) \).

**Definition 1.1** (c.f. \cite{17}) A \( C^k \) perfect cocycle \( (U, \theta) \) on \( H \) is a \( (\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(H) \otimes \mathcal{F}, \mathcal{B}(H)) \)-measurable random field \( U : \mathbb{R}^+ \times H \times \Omega \to H \) with the following properties:

(i) For each \( \omega \in \Omega \), the map \( \mathbb{R}^+ \times H \ni (t, \xi) \mapsto U(t, \xi, \omega) \in H \) is continuous; for fixed \( (t, \omega) \in \mathbb{R}^+ \times \Omega \), the map \( H \ni \xi \mapsto U(t, \xi, \omega) \in H \) is \( C^k \).

(ii) \( U(t_1 + t_2, \cdot, \omega) = U(t_2, \cdot, \theta(t_1, \omega)) \circ U(t_1, \cdot, \omega) \) for all \( t_1, t_2 \in \mathbb{R}^+ \), all \( \omega \in \Omega \).

(iii) \( U(0, \xi, \omega) = \xi \) for all \( \xi \in H, \omega \in \Omega \).

**Definition 1.2** (c.f. \cite{17}) An \( \mathcal{F} \) measurable random variable \( Y : \Omega \to H \) is said to be a stationary random point for the cocycle \( (U, \theta) \) if it satisfies the following identity:

\[ U(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \]

for all \( (t, \omega) \in \mathbb{R}^+ \times \Omega \).

As pointed out by Mohammed, Zhang and Zhao in \cite{17}, this concept essentially gives a useful realization of the idea of an invariant measure for the stochastic dynamical system, and allows us to analyze the local almost sure or generic properties of the stochastic semiflow in a neighborhood of the stationary point. It was pointed out by Q. Zhang and Zhao in \cite{24}, this “one-force, one-solution” setting is a natural extension of equilibrium in deterministic systems to stochastic counterparts. Unlike the usual search for invariant measures, it describes the pathwise invariance of the stationary solution over time along the measurable and \( P \)-preserving transformation \( \theta_t : \Omega \to \Omega \). Only in
some special cases, an invariant measure can be realized as a stationary solution unless one considers an extended probability space (1). On the other hand, a stationary solution always generates an invariant measure. Therefore pathwise stationary solutions reveal more accurate information about random dynamical systems. We would like to point out that unlike deterministic cases, in general, the stationary solutions of the stochastic systems can not be given explicitly. Up to now, there has not been a method which can be applicable to SPDEs with great generalities. It was even unthinkable to represent them as solutions of certain differential or functional equations in general. The existence and construction of the stationary solutions for SPDEs is a subtle problem of great importance.

In this article, we will focus on the existence, uniqueness and representation of the stationary solution of the stochastic Burgers equation (SBE) with \( L^2[0,1] \) valued white noise as follows:

\[
\begin{align*}
  du(t,x) &= (\nu \Delta u(t,x) + \frac{1}{2} \partial_x^2 [u(t,x)^2])dt + dW(t), \quad t \geq 0; \ x \in (0,1) \\
  u(t,1) &= u(t,0) = 0 \\
  u(0,x) &= u_0(x).
\end{align*}
\] (1)

This equation has been studied intensively in the literature in the last ten years (see [7], [5], [8], [17], [19], [20] and the references therein), because of the interests stemmed from physics and mathematics. It is a more realistic simple model for turbulence comparing to deterministic Burgers equations. The latter do not display any chaotic phenomena as all solutions converge to a unique stationary solution (one graph on the configuration space). But for the stochastic Burgers equation, we know from definition that actually the stationary solution is a random moving graphs on the configuration space (so infinitely many graphs). In [19], [20], Sinai established the existence and uniqueness of stationary strong solution of Burgers equations perturbed by periodic forcing or random forcing. His main tools are Hopf-Cole transformation and Ito’s lemma, hence he required that the noise term which is a Brownian white noise in time has continuous 3rd-order derivative in spatial variable. Moreover, he discussed the stationary solution in the views of the statistical physics. Da Prato and Zabczyk studied the ergodicity of the SBE in [7] and [5]. In [17], Mohammed, Zhang and Zhao proved the \( C^1 \) cocycle property of (1). In [8], E, Khanin, Mazel and Sinai studied the pathwise stationary solution of the stochastic inviscid Burgers equation with periodic boundary condition.

The main results in this article is to prove under the condition \( \Lambda \) (large viscosity condition) which will be made precise in the beginning of Section 3, there is a unique stationary solution \( u \) or stationary random point \( Y(\omega) \) satisfying, for any \( t \geq 0 \) and a.e. \( \omega \in \Omega \),

\[ u(t,\omega) = u(t,Y(\omega),\omega) = Y(\theta(t,\omega)), \] (2)

and
\[ u(t) = \frac{1}{2} \int_{-\infty}^{t} T_{\nu}(t-s) \frac{\partial u^2(s)}{\partial x} ds + \int_{-\infty}^{t} T_{\nu}(t-s) dW_s, \quad (3) \]

or

\[ Y(\omega) = \frac{1}{2} \int_{-\infty}^{0} T_{\nu}(-s) \frac{\partial (Y(\theta(s,\omega)))^2}{\partial x} ds + \int_{-\infty}^{0} T_{\nu}(-s) dW_s(\omega). \quad (4) \]

Moreover, \( Y(\omega) \) is hyperbolic and whole \( L^2[0,1] \) is its stable manifold.

Here, we would like to stress the crucial role of the equation (3) or (4) and any stationary solution of the SBE is given by equation (3) or (4). We can only construct the stationary point under the large viscosity condition. The problem remains open without this condition. In fact, for SPDE, the essential difficulty of constructing the stationary point is due to the fact that the equation is non-autonomous for a.e. \( \omega \). But the cocycle property makes it possible to construct the stationary solution although it is difficult in general.

For Burgers equation (1), the stationary solution should satisfy equation (3), since it is easy to see from (3) that

\[ u(t) = T_{\nu}(t-t_1)u(t_1) + \frac{1}{2} \int_{t_1}^{t} T_{\nu}(t-s) \frac{\partial u^2(s)}{\partial x} ds + \int_{t_1}^{t} T_{\nu}(t-s) dW_s, \quad (5) \]

holds for any \( t_1 < t \). For other integral equations and backward doubly stochastic differential equations on infinite horizon, see [17] and [24] respectively.

To construct a solution of (3), we adopt the pull-back procedure used by Flandoli and Schmalfuss in [10] in the construction of the random attractors for 3D Navier-Stokes equations with non-regular force, and used by Mattingly in [13] in the construction of the unique invariant measure of 2D stochastic Navier-Stokes equation with large viscosity. The main and basic tools in [10] and [13] are the so-called Ladyzhenskaya's inequality (see [18] p244) and the weak (or weak*) compactness method. However, compactness method can not ensure the uniqueness of the solution of (3). If uniqueness of the solution of (3) does not hold, we cannot prove the stationary point satisfying (2). It seems that the uniqueness of the solution (3) is a crucial technical condition to obtain the stationarity of \( Y(\omega) \). Therefore, as in [13], we have to require the large viscosity to ensure the uniqueness of the solution of (3) and its large time stability.

In fact, even in the case of the deterministic 2D Navier-Stokes equation with the given exterior force independent of time \( t \), the uniqueness of stationary solution (also called steady-state solution) is only obtained under the condition of large viscosity (see Section III.3 in [21] or theorem 1.3 in Chapter 2 in [23] for details, and our condition A is similar to those given in [21], [23]).

Let us define some notations used in the rest parts of this article:-

\[ L^2[0,1] \equiv \{ f : [0,1] \to R^1 \mid f(0) = f(1) = 0, \int_0^1 f^2(x) dx < \infty \}, \]
\[ H^1_0 \equiv \{ f \in L^2(0,1) | \int_0^1 (\frac{\partial}{\partial x} f(x))^2 dx < \infty \}, \]
\[ |f|_p \equiv (\int_0^1 |f(x)|^p dx)^{\frac{1}{p}}. \]

In Section 2, we will prove the existence and uniqueness of the mild solution of (1). Here, we emphasize that the solution of (1) under the \( L^2(0,1) \) valued noise is in \( L^2([0,T],H^1_0) \). So, combining this property and the proof of the \( C^1 \) perfect cocycle in [17], we have that the mild solution of (1) generates a \( C^1 \) perfect cocycle in \( H^1_0[0,1] \). As far as we know, it is not known that the solution of the 2D stochastic Navier-Stokes equation generates a \( C^1 \) perfect cocycle. In this paper we only consider stochastic Burgers equations. We hope to consider stochastic Navier-Stokes equations in future publications.

In the section 3, we will construct the stationary solution of SBE. Some of estimates similar to those of Mattingly ([10]) are needed. Although mattingly’s estimate is a key tool for our proof, our idea is different from Mattingly’s and equation (4) is new and plays important roles in our construction. We also proved the perfection version of (2) i.e. (2) holds for all \( t \) and a.e. \( \omega \in \Omega \). This plays an important role in our construction and does not seem obvious from Mattingly’s approach. In fact, we prove the identity (2) by using (3) and uniqueness of the mild solution of (1). Section 4 is an appendix, in which we will present some estimates needed in the previous sections.

2 The weak solution, mild solution and perfect cocycle

It is well known that \( \{ e_k(x) = \sqrt{\frac{2}{\pi}} \sin k\pi x, \ x \in [0,1], \ k = 1,2,\cdots \} \) is a normal orthogonal basis of \( L^2[0,1] \). Let \( W(t) \) be a \( L^2[0,1] \)-valued Brownian motion defined on the canonical filtered Wiener space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, P) \). That is, \( \Omega \) is the space of all continuous path \( \omega: \mathbb{R} \rightarrow L^2[0,1] \) such that \( \omega(0) = 0 \) with the compact open topology, \( \mathcal{F} \) is its Borel \( \sigma \)-field, \( P \) is the Wiener measure on \( \Omega \). The Brownian motion is given by:

\[ W(t,\omega) = \omega(t), \ \omega \in \Omega, \ t \in \mathbb{R}, \]

and may be represented by \( W(t) = \sum_{k=1}^{\infty} \sigma_k e_k B_k(t) \), where \( \sum_{k=1}^{\infty} \sigma_k^2 < \infty \), \( \{ B_k(t), \ t \in (-\infty, \infty) \ k = 1,2,\cdots \} \) are mutually independent real valued Brownian motions on \( (\Omega, \mathcal{F}, P) \), and

\[ \mathcal{F}_t = \sigma \{ \omega(s), s \leq t \} \vee \mathcal{N}, \] for any \( t \in \mathbb{R} \)

where \( \mathcal{N} \) are the null sets of \( \mathcal{F} \) (see [6] p86-87 and [1] p91).

Throughout this article, we denote by \( \theta: \mathbb{R} \times \Omega \rightarrow \Omega \) the standard \( P \)-preserving ergodic Wiener shift on \( \Omega \).
\[
\theta(t, \omega)(s) \equiv \omega(t+s) - \omega(t), \quad t, s \in \mathbb{R}.
\]

Hence \((W, \theta)\) is an helix:

\[
W(t_1 + t_2, \omega) - W(t_1, \omega) = W(t_2, \theta(t_1, \omega)), \quad t_1, t_2 \in \mathbb{R}, \quad \omega \in \Omega,
\]

and

\[
\theta^{-1}(u)F_t = F_{t+u}, \quad \forall \ t, u \in \mathbb{R}. \tag{6}
\]

Now, we denote \(W_{\nu\Delta}(t) = \int_0^t T_{\nu}(t-s)dW(s)\), and \(W_{\nu\Delta}(t_0, t) = \int_{t_0}^t T_{\nu}(t-s)dW(s)\), the so-called stochastic convolution with respect to the semigroup \(T_{\nu}(t) = e^{-t\nu\Delta}\). Some properties of the stochastic convolution \(W_{\nu\Delta}\) are presented in Appendix A. In particular, theorem A.2 shows that \(W_{\nu\Delta} \in L^2([0, t], H_0^1)\) a.s.

Now, consider the following PDE with random coefficients

\[
v(t) = T_{\nu}(t)v_0 + \frac{1}{2} \int_0^t T_{\nu}(t-s)\left[\frac{\partial}{\partial x}(v(s) + W_{\nu\Delta}(s))^2\right]ds,
\]

which can be considered as the mild form of the random Burgers equation

\[
\left\{ \begin{array}{l}
\frac{dv}{dt}(t) = \nu \Delta v(t) + \frac{1}{2} \frac{\partial}{\partial x}[v(t) + W_{\nu\Delta}(t)]^2, \\
v(0) = v_0(x).
\end{array} \right. \tag{7}
\]

Let’s consider the ODE used in Galerkin approximation

\[
\left\{ \begin{array}{l}
\frac{dv_n}{dt}(t) = \nu \Delta v_n(t) + \frac{1}{2} P_n \left[\frac{\partial}{\partial x}[v_n(t) + W_{\nu\Delta}(t)]^2\right], \\
v_n(t_0) = P_n(u_t_0),
\end{array} \right. \tag{8}
\]

where \(P_n\) is the projection operator in \(L^2[0,1]\) onto the space spanned by \(\{e_1, e_2, \ldots, e_n\}\), and \(W_{\nu\Delta}(t_0) = 0\).

**Lemma 2.1:** The global solution to (8) exists and is unique.

**Proof:** It’s easy to show the local existence and uniqueness by the standard fixed point theorem, and then

\[
\langle v_n, \frac{dv_n(t)}{dt} \rangle = \langle v_n(t), \Delta v_n(t) \rangle + \langle v_n(t), \frac{1}{2} P_n \left[\frac{\partial}{\partial x}[v_n(t) + W_{\nu\Delta}(t)]^2\right]\rangle.
\]

By the Dirichlet boundary conditions on \([0,1]\) and integration by parts

\[
\left(\frac{d}{dt}\right) |v_n(t)|^2 \leq \frac{1}{2} \left(\langle v_n(t), \frac{\partial}{\partial x}[v_n(t) + W_{\nu\Delta}(t)]^2\rangle\right)
\]

\[
= - \pi^2 \int_0^1 v_n(t, x)W_{\nu\Delta}(t, x)\frac{\partial}{\partial x}v_n(t, x)dx
\]

\[
= \frac{1}{2} \int_0^1 W_{\nu\Delta}^2(t, x)\frac{\partial}{\partial x}v_n(t, x)dx
\]

\[
= I + II. \tag{9}
\]
By the Hölder inequality and Sobolev embedding theorem, we have
\[ |II| = \left| \int_0^1 W_{v\Delta}(t, x) \frac{\partial}{\partial x} v_n(t, x) dx \right| \leq |W_{v\Delta}(t)|_4^2 \left| \frac{\partial}{\partial x} v(t) \right|_2^2; \]
and
\[ |I|^2 = \left| \int_0^1 v_n(t, x) W_{v\Delta}(t, x) \frac{\partial}{\partial x} v_n(t, x) dx \right|^2 \]
\[ \leq |v_n(t)|_4 |W_{v\Delta}(t)|_4 \left| \frac{\partial}{\partial x} v_n(t, \cdot) \right|_{L^2([0,1])} \]
\[ \leq \gamma^2 |\nabla v_n(t)|_2^4 |v_n(t)|_2^4 |W_{v\Delta}(t)|_4 \]
\[ = \gamma^2 |\nabla v_n(t)|_2^4 |v_n(t)|_2^4 |W_{v\Delta}(t)|_4. \]

Hence, due to Young’s inequality,
\[ \frac{1}{2} \frac{d}{dt} |v_n(t)|_4^2 + |\nabla v_n(t)|_2^2 \leq \varepsilon |\nabla v_n(t)|_2^2 + C_1(\varepsilon) |v_n(t)|_2^4 |W_{v\Delta}(t)|_4^4 + C_2(\varepsilon) |W_{v\Delta}(t)|_4^4. \]

Taking \( \varepsilon = \frac{1}{2}, \) we obtain
\[ \frac{d}{dt} |v_n(t)|_4^2 + |\nabla v_n(t)|_2^2 \leq C |v_n(t)|_2^2 |W_{v\Delta}(t)|_4^4 + |W_{v\Delta}(t)|_4^4 \]
for a constant \( C > 0. \) By the Gronwall inequality,
\[ |v_n(t)|_2^2 \leq |v_n(t_0)|_2^2 \exp \left\{ \int_{t_0}^t C |W_{v\Delta}(s)|_4^4 ds \right\} \]
\[ + \int_{t_0}^t \exp \left\{ \int_{s}^t C |W_{v\Delta}(r)|_4^4 dr \right\} |W_{v\Delta}(r)|_4^4 dr, \]
(10)
and
\[ \int_r^t |\nabla v_n(s)|_2^2 ds \leq |v_n(r)|_2^2 + C \int_r^t (|v_n(s)|_2^2 + |W_{v\Delta}(s)|_4^4 + |W_{v\Delta}(s)|_4^4) ds. \]
(11)

Using theorem 5.20 in [6] (p141) that says that \( W_{v\Delta}(\cdot) \in C([0,T];C[0,1]) \) and from (10), we have
\[ \sup_{t \in [0,T]} |v_n(t)|_2 \leq K_1(\omega) \quad \text{and} \quad \int_0^T |\nabla v_n(s)|_2^2 ds \leq K_2(\omega) \quad \text{uniformly in } n. \]
(12)

The first inequality implies that \( |v_n(t)|_2 \) is finite for all \( t > 0. \) Hence, by Lemma 2.4 in [15] (p18), we have the global solution to (5).

Inequalities in (12) imply that \( v_n \) is bounded in \( L^\infty(0,T;L^2[0,1]) \) and \( L^2(0,T;H_0^1) \) uniformly in \( n. \) These uniform bounds allow us to use the Alaoglu
compactness theorem to find a subsequence which we shall denote it by \( \{v_n\} \) such that
\[
v_n \rightharpoonup v \quad \text{in} \quad L^\infty(0, T; L^2[0, 1])
\]
i.e. \( v_n \) weak* converge to \( v \). We can extract a further subsequence, still denote by \( \{v_n\} \), such that
\[
v_n \rightarrow v \quad \text{in} \quad L^2(0, T; H^1_0)
\]
with
\[
v \in L^\infty(0, T; L^2[0, 1]) \cap L^2(0, T; H^1_0).
\]
From the discussion of Theorem 6.1 in [18], we know that the Laplacian operator \( \Delta \) is bounded from \( H^1_0 \) to \( H^1_0^* \) in the sense defined in Theorem 6.1 in [18]. Since \( v_n \) is bounded uniformly in \( L^2(0, T; H^1_0) \), \( \nu \Delta v_n \) is bounded uniformly in \( L^2(0, T; H^1_0^*) \).

Now, let us prove that \( P^n [\partial/\partial x v_n^2] \) are uniformly bounded in \( L^2(0, T; H^1_0^*) \).

First it is easy to show
\[
|\partial/\partial x v_n^2|_{H^1_0^*} \leq C |v_n|_{2} |\nabla v_n|_{2}.
\]
So Lemma 7.5 in [18] implies \( \|P_n B(v, v)\|_{L^2(0, T; H^1_0^*)} \leq \|B(v, v)\|_{L^2(0, T; H^1_0^*)} \). Therefore,
\[
\|P_n B(v_n, v_n)\|_{L^2(0, T; H^1_0^*)} \leq \int_0^T |B(v_n, v_n)|_{H^1_0^*}^2 \, ds \leq C \int_0^T |v_n(s)|^2 |\nabla v_n(s)|^2 \, ds \leq C \|v_n(s)\|_{L^\infty(0, T; L^2)}^2 \|v_n\|_{L^2(0, T; H^1_0^*)}^2.
\]

**Lemma 2.2.** The random perturbed \( P_n B(v_n + W_\Delta, v_n + W_\Delta) \) is uniformly bounded in \( L^2(0, T; H^1_0^*) \) a.s.

**Proof:** As \( v_n \) is uniformly bounded in \( L^\infty(0, T; L^2[0, 1]) \) and \( L^2(0, T; H^1_0^*) \), so does \( v_n + W_\Delta \). By [10], we finish the lemma. \( \square \)

**Lemma 2.3:** There exists a subsequence \( v_n \) such that \( \frac{dv_n}{dt} \rightharpoonup \frac{dv}{dt} \) in \( L^2(0, T; H^1_0^*) \).

**Proof:** Lemma 2.1 implies \( \frac{dv_n}{dt} \) is uniformly bounded in \( L^2(0, T; H^1_0^*) \). We can extract a further subsequence (relabelling again). Using the same argument as in [18] (p203-p204), we have
\[
\frac{dv_n}{dt} \rightharpoonup \frac{dv}{dt} \quad \text{in} \quad L^2(0, T; H^1_0^*).
\]
\( \square \)

**Lemma 2.4:** There exists a subsequence \( \{v_n\} \) such that
\[
P_n \left[ \frac{dv_n}{dt} \right] \rightharpoonup \left[ \frac{dv}{dt} \right] \quad \text{in} \quad L^2(0, T; H^1_0^*).
Proof: Step 1: Since $H^1_0 \subset \subset L^2[0,1]$, by Theorem 8.1 in [13], there is a subsequence $\{v_n\}$ (after relabelling) that converges to $v$ strongly in $L^2(0,T; L^2[0,1])$.

Step 2: Let $K = \{\sum_{i=1}^k \alpha_i(t) \phi_i; \phi_i \in H^1_0, \alpha_i \in C[0,T], k \in N\}$. We will show for any $\phi \in K$,

$$\int_0^T \int_0^1 v_n(\frac{\partial}{\partial x} \phi) \phi \, dx \, dt \to \int_0^T \int_0^1 v(\frac{\partial}{\partial x} v) \phi \, dx \, dt. \quad (17)$$

In fact, 

$$\left| \int_0^T \int_0^1 v_n(\frac{\partial}{\partial x} \phi) \phi \, dx \, dt - \int_0^T \int_0^1 v(\frac{\partial}{\partial x} v) \phi \, dx \, dt \right|$$

$$\leq \sum_{i=1}^k \int_0^T \alpha_i \left[ \int_0^1 v_n(\frac{\partial}{\partial x} \phi_i) \phi_i \, dx - \int_0^1 v(\frac{\partial}{\partial x} \phi_i) \phi_i \, dx \right] \, dt$$

$$\leq \frac{1}{2} \sum_{i=1}^k \left[ \int_0^T \alpha_i \left[ \int_0^1 (\frac{\partial}{\partial x} \phi_i) v_n^2 \, dx - \int_0^1 (\frac{\partial}{\partial x} \phi_i) v^2 \, dx \right] \, dt \right]$$

$$\leq \frac{1}{2} \sup_{t \in [0,T]} |\alpha_i(t)| \sum_{i=1}^k \int_0^T \int_0^1 \frac{\partial}{\partial x} \phi_i \left| v_n^2 - v^2 \right| \, dx \, dt. \quad (18)$$

Then by the Sobolev embedding theorem and Cauchy-Schwarz inequality,

$$\int_0^T \int_0^1 \left| \frac{\partial}{\partial x} \phi_i \right| v_n^2 - v^2 \, dx \, dt$$

$$\leq C \int_0^T |v_n + v|_{H^1} \int_0^1 \left| \frac{\partial}{\partial x} \phi_i \right| v_n - v \, dx \, dt$$

$$\leq C \int_0^T |v_n + v|_{H^1} \int_0^1 \left| \frac{\partial}{\partial x} \phi_i \right|^2 \, dx \frac{1}{2} |v_n - v|_2 \, dt$$

$$\leq C \left[ \int_0^1 \left| \frac{\partial}{\partial x} \phi_i \right|^2 \, dx \right]^\frac{1}{2} \left[ \int_0^T |v_n + v|_{H^1} \, dt \right]^\frac{1}{2} \left[ \int_0^T |v_n - v|_2^2 \, dt \right]^\frac{1}{2}. \quad (19)$$

It is easy to see from [13] and (19) that

$$\left| \int_0^T \int_0^1 v_n(\frac{\partial}{\partial x} \phi) \phi \, dx \, dt - \int_0^T \int_0^1 v(\frac{\partial}{\partial x} v) \phi \, dx \, dt \right| \leq C \left[ \int_0^T |v_n - v|_2^2 \, dt \right]^\frac{1}{2}.$$

Here $C$ is a generic positive constant. Since $v_n$ is uniformly bounded in $L^2(0,T; H^1_0)$, the convergence in (17) follows.

Step 3: It is obvious that

$$\left| \int_0^T \int_0^1 P_n v_n(\frac{\partial}{\partial x} \phi) \phi \, dx \, dt - \int_0^T \int_0^1 v(\frac{\partial}{\partial x} v) \phi \, dx \, dt \right|$$
\[ \int_0^T \sum_{i=1}^k \left( \int_0^1 \frac{\partial}{\partial x} v_n^2 \right) P_n \phi_i \alpha_i(t) dt - \int_0^T \int_0^1 \frac{\partial}{\partial x} \phi_i \alpha_i(t) dt \right| \\
+ \left| \int_0^T \sum_{i=1}^k \left( \int_0^1 \frac{\partial}{\partial x} v_n^2 \right) \phi_i \alpha_i(t) dt - \int_0^1 \frac{\partial}{\partial x} \phi_i \alpha_i(t) dt \right| \\
= I + II. \]

By Step 2, we know \( II \to 0 \) as \( n \to \infty \). To see \( I \to 0 \) as \( n \to \infty \), note

\[ \int_0^T \int_0^1 \left( \frac{\partial}{\partial x} \left( P_n \phi_i(t) - \phi_i(t) \right) \right)^2 dx dt \to 0, \]

noticing that Lemma 7.5 in [18] implies \( P_n \phi \to \phi \) in \( L^2(0,T;H^1_0) \). Moreover

\[ I \leq \| v_n \left( \frac{\partial}{\partial x} v_n \right) \|_{L^2(0,T;H^1_0)} \| P_n \phi - \phi \|_{L^2(0,T;H^1_0)}. \]

Since [18] means that \( v_n \left( \frac{\partial}{\partial x} v_n \right) \) is uniformly bounded in \( L^2(0,T;H^1_0) \), we have \( I \to 0 \) as \( n \to \infty \).

Step 4: It is well known that \( K \) is dense in \( L^2(0,T;H^1_0) \). Therefore, Step 2 and Step 3 show that for any for any \( \phi \in L^2(0,T;H^1_0) \),

\[ \int_0^T \int_0^1 P_n v_n \left( \frac{\partial}{\partial x} v_n \right) \phi dx dt \to \int_0^T \int_0^1 v \left( \frac{\partial}{\partial x} v \right) \phi dx dt. \quad (20) \]

This proves the lemma.

Using a similar argument as in [18] (p249), we have \( v_n(0) = P_n u_0 \to u_0 = u(0) \). Therefore, using the above lemmas, it is easy to deduce the following theorem.

**Theorem 2.5** There exists a solution \( v \in L^2(0,T;H^1_0) \cap L^\infty(0,T;L^2[0,1]) \) that satisfies

\[ \frac{dv}{dt} = \nu \Delta v + \frac{1}{2} \left( v + W \Delta \right) \]

as an equation in \( L^2(0,T;H^1_0) \).

**Theorem 2.6** There exists a unique mild solution \( u \in L^2(0,T;H^1_0) \cap C(0,T;L^2[0,1]) \) that satisfies

\[ u(t) = T_t u_0 + \frac{1}{2} \int_0^t T_t \left( t - s \right) \left( \frac{\partial}{\partial x} u^2(s) \right) ds + \int_0^t T_t (t - s) dW(s), \quad (22) \]

where \( u_0 \in L^2[0,1] \).
Before we prove theorem 2.6, we need some lemmas. Denote \( p(t, x, y) \) the heat kernel of \( \nu \Delta \) with the Dirichlet boundary conditions on \([0, 1]\). Note here \( p(\cdot, \cdot, y) \notin L^2(0, T; H^1_0) \).

Let \( f \in C^\infty_0([0, 1]) \) and \( K(t, y) = \int_0^1 p(t, x, y) f(x) dx \), then \( K(t, y) \in C^\infty((0, T) \times [0, 1]) \) and \( K(t, y) = \int_0^1 p(t, y, x) f(x) dx \). Obviously, the energy inequality implies

\[
K(t, y) \in L^2(0, T; H^1_0). \tag{23}
\]

**Lemma 2.7** Let \( v \in L^2(0, T; H^1_0) \cap L^\infty(0, T; L^2[0, 1]) \) be the solution of equation (21) in \( L^2(0, T; H^1_0) \), then \( \int_0^t \int_0^1 p(t-s, x, y) \frac{\partial}{\partial y} (v + W_{\nu \Delta})^2(s, y) dy ds \in L^2[0, 1] \).

**Proof:** For every \( f \in L^2[0, 1] \), by Cauchy-Schwartz inequality

\[
\int_{[0, T] \times [0, 1]} |p(t-s, x, y) f(x) \frac{\partial}{\partial y} (v + W_{\nu \Delta})^2(s, y)| dxdyds \\
\leq 2 \left[ \int_{[0, T] \times [0, 1]} p(t-s, x, y) f^2(x) (v + W_{\nu \Delta})^2(s, y) dxdyds \right]^{\frac{1}{2}} \\
\times \left[ \int_{[0, T] \times [0, 1]} p(t-s, x, y) \left( \frac{\partial}{\partial y} (v + W_{\nu \Delta}) \right)^2(s, y) dxdyds \right]^{\frac{1}{2}} \\
\leq 2C \left( \int_0^t \frac{1}{\sqrt{T-s}} ds \right)^{\frac{1}{2}} \left( \int_0^1 f^2(x) dx \right)^{\frac{1}{2}} \left[ ||v + W_{\nu \Delta}||_{L^\infty(0, T; L^2[0, 1])} \right]^{\frac{1}{2}} \\
\times \left[ \int_0^t \int_0^1 \left( \int_0^1 p(t-s, x, y) dx \right) \left( \frac{\partial}{\partial y} (v + W_{\nu \Delta}) \right)^2 dy ds \right]^{\frac{1}{2}} \\
\leq C(t) ||v + W_{\nu \Delta}||_{L^\infty(0, T; L^2[0, 1])}^{\frac{1}{2}} \cdot ||v + W_{\nu \Delta}||_{L^2(0, T; H^1_0)} \cdot ||f||_{L^2[0, 1]}^{\frac{1}{2}}.
\]

This means that the linear functional \( L : f \mapsto \int_0^t f(x) \int_0^1 \int_0^1 p(t-s, x, y) \left( \frac{\partial}{\partial y} (v + W_{\nu \Delta}) \right)^2 dy ds dx ds \) is bounded in \( L^2[0, 1] \). Then by the Rieze representation theorem, we have

\[
\int_0^t \int_0^1 p(t-s, x, y) \left( \frac{\partial}{\partial y} (v + W_{\nu \Delta}) \right)^2 dy ds \in L^2[0, 1].
\]

□

**Lemma 2.8** The following equality holds in \( L^2[0, 1] \)

\[
v(t) = T_\nu(t)v(0) + \frac{1}{2} \int_0^t T_\nu(t-s) \frac{\partial}{\partial x} (v + W_{\nu \Delta})^2 ds \tag{24}
\]

**Proof:** From (23) and Theorem 2.5, we know
\[
\int_0^1 \int_0^t K(t-s,x) \frac{\partial v(s,x)}{\partial s} dxds = - \int_0^1 \int_0^1 \frac{\partial}{\partial x} K(t-s,x) \frac{\partial}{\partial x} v(s,x) dxds + \frac{1}{2} \int_0^1 \int_0^1 K(t-s,x) \frac{\partial}{\partial x} (v + W_{\nu \Delta})^2 dxds.
\]

It follows from the integration by parts formula and the fact that
\[
\begin{cases}
\frac{\partial}{\partial t} K(t,y) = \frac{1}{4} \Delta K(t,y), \\
K(t,0) = K(t,1) = 0, \\
K(0,y) = f(y),
\end{cases}
\]
then
\[
\int_0^1 f(y)v(t,y) dy = \int_0^1 K(t,y)v(0,y) dy + \frac{1}{2} \int_0^t \int_0^1 \int_0^1 p(t-s,y,x)f(y) dy \frac{\partial}{\partial x} (v + W_{\nu \Delta})^2 dxds + \frac{1}{2} \int_0^1 f(y) \int_0^t \int_0^1 p(t-s,y,x) \frac{\partial}{\partial x} (v + W_{\nu \Delta})^2 dxds dy.
\]

Due to Lemma 2.7 and \(C_0^\infty([0,1])\) is dense in \(L^2[0,1]\), we get the following equality in \(L^2[0,1]\),
\[
v(t) = T_{\nu}(t)v(0) + \frac{1}{2} \int_0^t T_{\nu}(t-s) \frac{\partial}{\partial x} (v + W_{\nu \Delta})^2 ds.
\]

The proof of theorem 2.6: Lemma 2.8 and the definition of \(W_{\nu \Delta}\) imply the existence of a mild solution. Now, we only need to show that \(u \in C(0,T;L^2[0,1])\) and the uniqueness.

By (15), we know \(v \in L^2(0,T;H^1_0)\) and by Lemma 2.3 we know \(\frac{dv}{dt}\) in \(L^2(0,T;H^1_0)^{\ast}\). Thus Theorem 7.2 in [18] implies that \(v \in C(0,T;L^2[0,1])\). At the same time, by Theorem 5.20 in [9], \(W_{\nu \Delta} \in C([0,T];C[0,1])\). It then follows that \(u \in C(0,T;L^2[0,1])\).

Let \(u_1\) and \(u_2\) be two solutions of (22) with the same initial data, then using Proposition A.3 in the Appendix, we have
\[
|u_1(t) - u_2(t)|^2 = \int_0^1 \left( u_1(t,x) - u_2(t,x) \right)^2 dx
\]
\[
= \int_0^1 \int_0^t T_{\nu}(t-s) \left( \frac{\partial u_1^2}{\partial y} - \frac{\partial u_2^2}{\partial y} \right) ds^2 dx
\]
\[
\leq C \int_0^1 dx \left( \frac{1}{2} \int_0^t ds \int_0^1 \frac{1}{\sqrt{t-s} \sqrt{t-s}} e^{-\frac{(x-y)^2}{2(t-s)}} |u_1^2(s,y) - u_2^2(s,y)| dy \right)^2
\]
Then equation (1) has a unique mild solution with the following properties:

\[ u \]

Iterating it, and using the elementary estimate again, we get

\[ \int_0^t \frac{1}{(t-s)^\frac{\alpha}{2}} ds \int_0^t \frac{1}{(t-s)^\frac{\alpha}{2}} \left( \int_0^1 \frac{c_1}{\sqrt{t-s}} e^{- \frac{c_1^2 (x-y)^2}{2(t-s)}} |u_1(s,y) - u_2(s,y)|^2 dy \right)^2 ds dx \]

\[ \leq C \sup_{0 \leq s \leq t} |u_1(s) + u_2(s)|^2 \int_0^t \frac{1}{(t-s)^\frac{\alpha}{2}} \left( |u_1(s) - u_2(s)|^2 ds \right) \]

\[ \leq C(\omega) \int_0^t \frac{1}{(t-s)^\frac{\alpha}{2}} |u_1(s) - u_2(s)|^2 ds, \]

where \( C \) is a generic constant that may change from one line to another. We iterate the above computation,

\[ |u_1(t) - u_2(t)|_2^2 \leq C(\omega)^2 \int_0^t \frac{1}{(t-s)^\frac{\alpha}{2}} \int_0^s \frac{1}{(s-r)^\frac{\alpha}{2}} |u_1(r) - u_2(r)|_2^2 dr ds. \]

Consider now the elementary estimate

\[ \int_r^t \frac{s^\alpha}{(t-s)^\frac{\alpha}{2} (s-r)^\frac{\alpha}{2}} ds = \int_0^{t-r} \frac{(s+r)^\alpha}{(t-r-s)^\frac{\alpha}{2} s^\gamma} ds \leq \frac{C}{(t-r)^{\frac{\alpha}{2} + \gamma - 1}}, \quad t \geq r > 0, \]

we have

\[ |u_1(t) - u_2(t)|_2^2 \leq C \int_0^t \frac{1}{(t-s)^\frac{\alpha}{2}} |u_1(r) - u_2(r)|_2^2 dr. \]

Iterating it, and using the elementary estimate again, we get

\[ |u_1(t) - u_2(t)|_2^2 \leq C \int_0^t |u_1(s) - u_2(s)|_2^2 ds. \]

So, the Gronwall inequality implies that \( u_1 = u_2 \) in \( L^2[0,1] \). We complete the uniqueness.

Moreover, the mild inequality implies that \( u_1 = u_2 \) in \( L^2[0,1] \). We complete the uniqueness.

\[ \square \]

**Theorem 2.9** (see also Theorem 1.4.3 [17]) Consider stochastic Burgers equation (4). Then equation (4) has a unique mild solution with a \((\mathcal{B}(R^+) \otimes \mathcal{B}(L^2[0,1]) \otimes F, \mathcal{B}(L^2[0,1]))\) measurable version \( u : \mathbb{R}^+ \times L^2[0,1] \times \Omega \to L^2[0,1] \) having the following properties:

(i) For each \( \psi \in L^2[0,1] \), \( u(\cdot, \psi, \cdot) : \mathbb{R}^+ \times \Omega \to L^2[0,1] \) is \((\mathcal{F}_t)_{t \geq 0}\) adapted;

(ii) \( u, \theta \) is a \( C^1 \) perfect cocycle on \( L^2[0,1] \);

(iii) For each \( (t, \omega) \in (0, \infty) \times \Omega \), the map \( L^2(0,1) \ni \psi \mapsto u(t, \psi, \omega) \in L^2[0,1] \) takes bounded set into relatively compact sets;
(iv) For each \((t, \psi, \omega) \in (0, \infty) \times L^2[0,1] \times \Omega\), the Fréchet derivative 
\(Du(t, \psi, \omega) \in L(L^2[0,1])\) is compact. Furthermore, the map
\[ [0, \infty) \times L^2[0,1] \times \Omega \ni (t, \psi, \omega) \mapsto Du(t, \psi, \omega) \in L(L^2[0,1]) \]
is strong measurable and for each \((t, \omega) \in (0, \infty) \times \Omega\), the map \(L^2[0,1] \ni \psi \mapsto Du(t, \psi, \omega) \in L^2[0,1]\) takes bounded sets into relative compact sets;

(v) For any positive \(\alpha, \rho\)
\[ E \log^+ \sup_{0 \leq t \leq a, |\psi| \leq \rho} \left\{ |u(t, \psi, \cdot)|_2 + \|Du(t, \psi, \cdot)\|_{L(L^2[0,1])} \right\} < \infty. \]

Proof: The uniqueness of the solution implies that the Galerkin approximation sequences \(v_n\) converge to \(v\) for a.e. \(\omega\) in weak* topology \(L^\infty((0,T), L^2[0,1])\). This shows that \(v = (B(R^+) \otimes B[L^2[0,1]]) \otimes F, B(L^2[0,1]))\) measurable, where \(B_w(L^2[0,1]))\) is the \(\sigma\)-algebra generated by the weak topology in \(L^2[0,1]\). Because \(L^2[0,1]\) is a separable Hilbert space, it is well-known that \(v = (B(R^+) \otimes B[L^2[0,1]]) \otimes F, B(L^2[0,1]))\) measurable. Since \(u = v - W_{\nu,\Delta}\) and \(W_{\nu,\Delta}\) is \(B(R^+) \otimes B[L^2[0,1]]) \otimes F, B(L^2[0,1]))\) measurable, we know that \(u\) is \(B(R^+) \otimes B[L^2[0,1]]) \otimes F, B(L^2[0,1]))\).

The proof of (i)–(v) is given in [17] in details.

3 The construction of the stationary solution

We define \(u(t, \omega; t_0, u_0)\) the value of the mild solution of \(\mathbb{H}\) at time \(t\) with an initial data \(u_0\) at time \(t_0\). We define \(\delta_0 = \lambda_1 \nu - \frac{\sigma^2}{2\lambda_1}\), where \(\lambda_1 = \pi^2\), the first non-zero eigenvalue of \(\Delta_1\); \(\varepsilon_0 \equiv E(W_1 - W_0^0)^2 = \sum_{k=1}^{\infty} \sigma^2_k\); \(\gamma\) is the minimal constant such that the inequality, \(\max_{x \in [0,1]} |u(x)| \leq \gamma\|u\|_{H^1[0,1]}\) if \(u \in H^1([0,1])\), holds.

In this section, we assume

**Condition A:** \(\frac{\varepsilon_0^3}{\delta_0^2} > \frac{\pi^2}{2\lambda_1}\) i.e. \(\delta_0 > 0\).

Then we can prove the following lemma.

**Lemma 3.1:** Assume that \(u(t)\) satisfies the following equation in \(L^2[0,1]\) for any \(t \in \mathbb{R}\)
\[ u(t) = \frac{1}{2} \int_{-\infty}^{t} T_\nu(t-s) \frac{\partial u^2(s)}{\partial x} ds + \int_{-\infty}^{t} T_\nu(t-s) dW_s \] (26)
and
\[ \sup_{t \in \mathbb{R}} E|u(t)|_{L^p}^2 < \infty, \quad p \geq 1, \] (27)
then \(u(t)\) is unique.

Proof: For any \(t_1 < t\), by Fubini Theorem, we have
This shows that \( u(\delta) \)
for any \( n > |l| \) and \( t \) for any \( r \) for any \( n \) with initial data \( u \) with initial data \( n < t \), we have, for any \( n \) are two solutions of (26) under condition (27), we have, for any \( n \), we have that the equation
\[
\frac{\partial}{\partial t} T_\nu(t, l, \omega) = 1
\]
where \( T_\nu(t, l, \omega) \) is a constant. So,
\[
|u(t_1 + \tau, t_1 - n, \omega) - l(t_1 + \tau, t_1 - n, \omega)| \leq c \delta^2 |n| e^{-\delta(n + \tau)},
\]
and \( \delta \in (0, \delta_0) \) is a constant. So,
\[
|u(t_1 + \tau, \omega) - l(t_1 + \tau, \omega)|^2 < \varepsilon, \quad \text{for any } \tau > 0.
\]
This shows that \( u(\cdot) = l(\cdot) \) for a.e. \( \omega \).

**Lemma 3.2:** Let \( u \) satisfy (26) and (27) in Lemma 3.1, then we have, for any \( r \in \mathbb{R} \)
\[
\langle u(\cdot, \theta(r, \omega)) = u(\cdot + r, \omega) \rangle \quad \text{for a.e. } \omega.
\]

**Proof:** Because \( \theta \) is a \( P \)-preserving on the probability space \((\Omega, \mathcal{F}, P)\), for any \( r \), we have that the equation
\[
u(t(\cdot) + s, \theta(r, \omega))
\]
\[
\frac{\partial}{\partial t} T_\nu(t, \cdot, \omega) = \frac{1}{2} \int_{-\infty}^{t} T_\nu(t - s) \frac{\partial^2 u^2(s, \theta(r, \omega))}{\partial x^2} ds + \int_{-\infty}^{t} T_\nu(t - s) dW_s(\theta(r, \omega))
\]
\[
\frac{u(t) = \frac{1}{2} \int_{-\infty}^{t} T_\nu(t - s) \frac{\partial u(s, \theta(r, \omega))}{\partial x} ds + \int_{-\infty}^{t} T_\nu(t - s) dW_s(\theta(r, \omega))}{u(t) = \frac{1}{2} \int_{-\infty}^{t} T_\nu(t - s) \frac{\partial u(s, \theta(r, \omega))}{\partial x} ds + \int_{-\infty}^{t} T_\nu(t - s) dW_s(\theta(r, \omega))}
\]
\[
\frac{u(t) = \frac{1}{2} \int_{-\infty}^{t} T_\nu(t - s) \frac{\partial u(s, \theta(r, \omega))}{\partial x} ds + \int_{-\infty}^{t} T_\nu(t - s) dW_s(\theta(r, \omega))}{u(t) = \frac{1}{2} \int_{-\infty}^{t} T_\nu(t - s) \frac{\partial u(s, \theta(r, \omega))}{\partial x} ds + \int_{-\infty}^{t} T_\nu(t - s) dW_s(\theta(r, \omega))}
\]
\[
\frac{u(t) = \frac{1}{2} \int_{-\infty}^{t} T_\nu(t - s) \frac{\partial u(s, \theta(r, \omega))}{\partial x} ds + \int_{-\infty}^{t} T_\nu(t - s) dW_s(\theta(r, \omega))}{u(t) = \frac{1}{2} \int_{-\infty}^{t} T_\nu(t - s) \frac{\partial u(s, \theta(r, \omega))}{\partial x} ds + \int_{-\infty}^{t} T_\nu(t - s) dW_s(\theta(r, \omega))}
\]
On the other hand, let $s' = s + r$, then

\[
u(t, \theta(r, \omega)) = \frac{1}{2} \int_{-\infty}^{t+r} T_{\nu}(t + r - s') \frac{\partial u^{2}(s' - r, \theta(r, \omega))}{\partial x} ds' \\
+ \int_{-\infty}^{t+r} T_{\nu}(t + r - s') dW_{s'-r}(\theta(r, \omega)) \\
= \frac{1}{2} \int_{-\infty}^{t} T_{\nu}(t + r - s') \frac{\partial u^{2}(s' - r, \theta(r, \omega))}{\partial x} ds' \\
+ \int_{-\infty}^{t} T_{\nu}(t + r - s') dW_{s'-r}(\theta(r, \omega)) \\
+ \frac{1}{2} \int_{t}^{t+r} T_{\nu}(t + r - s') \frac{\partial u^{2}(s' - r, \theta(r, \omega))}{\partial x} ds' \\
+ \int_{t}^{t+r} T_{\nu}(t + r - s') dW_{s'-r}(\theta(r, \omega)) = T_{\nu}(r) \left[ \frac{1}{2} \int_{-\infty}^{t} T_{\nu}(t - s') \frac{\partial u^{2}(s' - r, \theta(r, \omega))}{\partial x} ds' \\
+ \int_{-\infty}^{t} T_{\nu}(t - s') dW_{s'-r}(\theta(r, \omega)) \right] \\
+ \frac{1}{2} \int_{t}^{t+r} T_{\nu}(t + r - s') \frac{\partial u^{2}(s' - r, \theta(r, \omega))}{\partial x} ds' \\
+ \int_{t}^{t+r} T_{\nu}(t + r - s') dW_{s'-r}(\theta(r, \omega)).
\]

On the other hand, let $y(\cdot, \omega) = u(\cdot - r, \theta(r, \omega))$, then we have

\[
y(t + r, \omega) = T_{\nu}(r) \left[ \frac{1}{2} \int_{-\infty}^{t} T_{\nu}(t - s) \frac{\partial y^{2}(s, \omega)}{\partial x} ds + \int_{-\infty}^{t} T_{\nu}(t - s) dW_{s}(\omega) \right] \\
+ \frac{1}{2} \int_{t}^{t+r} T_{\nu}(t + r - s) \frac{\partial y^{2}(s, \omega)}{\partial x} ds + \int_{t}^{t+r} T_{\nu}(t + r - s) dW_{s}(\omega) \\
= T_{\nu}(r)y(t, \omega) + \frac{1}{2} \int_{t}^{t+r} T_{\nu}(t + r - s) \frac{\partial y^{2}(s, \omega)}{\partial x} ds \\
+ \int_{t}^{t+r} T_{\nu}(t + r - s) dW_{s}(\omega).
\]

By the uniqueness of equation (11) (see Theorem 2.6), we know $y(t + r, \omega) = u(t + r, \omega)$ a.s. So, $u(\cdot, \theta(r, \omega)) = u(\cdot + r, \omega)$ a.s.

Using a similar argument, for $r < 0$,

\[
u(t, \theta(r, \omega)) = \frac{1}{2} \int_{-\infty}^{t+r} T_{\nu}(t + r - s') \frac{\partial u^{2}(s' - r, \theta(r, \omega))}{\partial x} ds' \\
+ \int_{-\infty}^{t+r} T_{\nu}(t + r - s') dW_{s'-r}(\theta(r, \omega)). \tag{31}
\]
Let \( y(\cdot, \omega) = u(\cdot - r, \theta(r, \omega)) \), then (31) becomes

\[
y(t + r, \omega) = \frac{1}{2} \int_{-\infty}^{t+r} T_{\nu}(t + r - s') \frac{\partial y^2(s', \omega)}{\partial x} ds' + \int_{-\infty}^{t+r} T_{\nu}(t + r - s') dW_s.
\]

The uniqueness of (26) proved in Lemma 3.1 shows that \( y(t+r, \omega) = u(t+r, \omega) \), a.s. i.e. \( u(\cdot, \theta(r, \omega)) = u(\cdot + r, \omega) \) a.s. \( \square \)

Now, let’s construct the solution of equation (26). We denote for \( n \in \mathbb{Z}^+ \)

\[
u_n(t, x, \omega) = \begin{cases} u(t, -n, 0, \omega), & \text{for } t > -n, \\ 0, & \text{for } t \leq -n. \end{cases}
\]

Then we can prove the following lemma.

**Lemma 3.3** For a.e.\( \omega \), and any \( N \in \mathbb{Z}^+ \), \( u_n(t, x, \omega) \rightarrow u^*(t, x, \omega) \) in \( C([-N,N], L^2[0,1]) \) as \( n \rightarrow \infty \) under the norm \( |u(t)|_{\infty, L^2, N} \equiv \sup_{t\in[-N,N]} |u(t)|_2 \).

Moreover \( u^* \) satisfies (20), (27) and (31).

**Proof:** At first, following Theorem A.7, we know that \( u_n \) is a Cauchy sequence in \( C([-N,N], L^2[0,1]) \) under the norm \( |u|_{\infty, L^2, N} \). Because the space \( C([-N,N], L^2) \) is complete, there is a \( u^* \) such that \( \lim_{n \rightarrow \infty} u_n = u^* \) in \( C([-N,N], L^2[0,1]) \). Since \( N \) is arbitrary, \( u^*(t, \omega) \) is defined for all time.

Secondly, for any \( t \) and \( t_0 < t \), we will show that \( u^* \) satisfies,

\[
u^*(t) = T_{\nu}(t - t_0) u^*(t_0) + \frac{1}{2} \int_{t_0}^{t} T_{\nu}(t - s) \frac{\partial (u^*(s, \omega))^2}{\partial x} ds + \int_{t_0}^{t} T_{\nu}(t - s) dW_s.
\]

For this, similar to the proof of formula (4.13) in [9], for any \( N \), any sufficiently large \( n \) and a.e.\( \omega \), we have

\[
\int_{-N}^{N} \left| \frac{\partial u_n(s)}{\partial x} \right|^2 ds < \xi_N(\omega) < \infty.
\]

This means that we can find a subsequence, still denoted by \( u_n \), weakly converge to \( u^* \) in \( L([-N,N], H^1_0) \), by Alaoglu Compactness Theorem. Therefore for a.e.\( \omega \), for any \( N, u^* \in L([-N,N], H^1_0) \). Moreover, using the same estimate as in Lemma 2.7, we get

\[
\frac{1}{2} \int_{t_0}^{t} T_{\nu}(t - s) \frac{\partial (u^*(s, \omega))^2}{\partial x} ds \in L^2[0,1].
\]

Thus

\[
\left| \int_{t_0}^{t} T_{\nu}(t - s) \frac{\partial (u^*(s, \omega))^2}{\partial x} ds - \int_{t_0}^{t} T_{\nu}(t - s) \frac{\partial (u_n(s, \omega))^2}{\partial x} ds \right|_2^2
\]

\[
= \int_{0}^{1} \left[ \int_{t_0}^{t} \int_{0}^{1} p(t - s, y, x) \left( \frac{\partial (u^*(s, \omega))^2}{\partial x} - \frac{\partial (u_n(s, \omega))^2}{\partial x} \right) dyds \right] dy
\]

\[
\leq C \sup_{t_0 \leq s \leq t} \left( \| u_n(s) + u^*(s) \|_2^2 \right) \int_{t_0}^{t} \frac{1}{(t - s)^2} \| u_n(s) - u^*(s) \|_2^2 ds,
\]
using the same estimate technique in the proof of the uniqueness of Theorem 2.6. Since \( u_n \to u^* \) in \( C([-N,N],L^2[0,1]) \) under norm \( |u(t)|_{\infty,L^2,N} \), it is easy to know that
\[
\left| \int_0^t T_\nu(t-s) \frac{\partial (u^*(s,\omega))^2}{\partial x} ds - \int_0^t T_\nu(t-s) \frac{\partial (u_n(s,\omega))^2}{\partial x} ds \right|_2 \to 0.
\]

At the same time, obviously \( u_n(t_0), T_\nu(t-t_0)u_n(t_0) \) strongly converge to \( u^*(t) \) and \( T_\nu(t-t_0)u^*(t_0) \) in \( L^2[0,1] \) respectively, hence (33) holds.

Due to Theorem A.4, \( \sup_{n} \sup_{t \in \mathbb{R}} E|u_n(t)|^{2p} < \infty \), for any \( p \geq 1 \). This implies that
\[
\sup_{t \in \mathbb{R}} E|u^*(t)|^{2p} < \infty, \quad p \geq 1.
\]

Finally, let’s prove that \( u^* \) satisfies (20). From (33), it is easy to know, for any \( 0 < m < n, \)
\[
\frac{1}{2} \int_{-n}^{-m} T_\nu(-s) \frac{\partial (u^*(s,\omega))^2}{\partial x} ds = - \int_{-n}^{-m} T_\nu(-s)dW_s - T_\nu(n)u^*(-n) + T_\nu(m)u^*(-m).
\]

Thus
\[
E \left| \int_{-n}^{-m} T_\nu(-s) \frac{\partial (u^*(s,\omega))^2}{\partial x} ds \right|_2 \leq C \left[ \sup E \left| \int_{-n}^{-m} T_\nu(-s)dW_s \right|_2 \right] + E \left| T_\nu(n)u^*(-n) \right|_2 + E \left| T_\nu(m)u^*(m) \right|_2 = I + II + III.
\]

It is easy to know that \( I \to 0 \) as \( n, m \to \infty \). Moreover, by Poincaré’s inequality (see [2]), we know \( II \leq e^{-\nu\lambda_n} E|u^*(-n)|^2 \) and \( III \leq e^{-\nu\lambda_m} E|u^*(-m)|^2 \), then from (34) we know that \( II \) and \( III \) converge to 0 as \( n, m \to \infty \). So, there is a subsequence \( n' \) such that, for a.e. \( \omega, \int_0^{n'} T_\nu(-s) \frac{\partial (u^*(s,\omega))^2}{\partial x} ds \to \int_{-\infty}^0 T_\nu(-s) \frac{\partial (u^*(s,\omega))^2}{\partial x} ds \) and \( \int_0^{n'} T_\nu(-s)dW_s \to \int_{-\infty}^0 T_\nu(-s)dW_s \) in \( L^2[0,1] \) as \( n' \to \infty \), and \( T(n)u^*(-n) \to 0 \) in \( L^2[0,1] \) as \( n \to \infty \). Therefore, \( u^* \) satisfies equations (20) and (27) and Lemma 3.2 implies (30) holds.

**Remark:** Assume that \( \tilde{u} \) is another stationary solution for (11), since \( \theta(t) \) is a \( P \)-preserving ergodic Wiener shift on \( \Omega, \tilde{u}(t) \) have the same law for any \( t \), and its law is the invariant measure for (11). Similar to the proof of ergodicity of stochastic Burgers equation in Chapter 14 of [7], we can conclude that the invariant measure of (11) is unique. Therefore, the law of \( \tilde{u}(t) \) identifies that of \( u^*(t) \). So, \( \tilde{u} \) satisfies the moment estimate (27). Because \( \tilde{u} \) is the solution for (11) for any \( t \in \mathbb{R} \), using the same reasoning as inequalities of (25), (29) in the proof Lemma 3.1, we know that \( u^* = \tilde{u} \) so there exists unique stationary solution for (11).
Now we have proved that equation $26$ has a unique solution $u^*$ in $C((-\infty, +\infty), L^2([0, 1]))$ and for any $r$, $30$ holds almost surely. On the other hand, the stationary solution of $1$ is unique so must satisfy $26$ and $27$. It remains to prove $30$ holds for all $r$ almost surely by a perfection argument.

**Theorem 3.4** Under the Condition A, there exists a unique stationary solution $\hat{u}^*(\cdot)$ for the stochastic Burgers equation $7$ satisfying:

(i) $\hat{u}^*$ is $(B(\mathbb{R}) \otimes \mathcal{F}, B(L^2[0, 1]))$ measurable;

(ii) $\hat{u}^*(\cdot, \omega): \mathbb{R} \to L^2[0, 1]$ is continuous for all $\omega \in \Omega$;

(iii) For all $r, t \in \mathbb{R}$ and $\omega \in \Omega$, $\hat{u}^*(t, \theta(r, \omega)) = \hat{u}^*(t + r, \omega)$;

(iv) Let $\tilde{N} \equiv \{\omega : \hat{u}^*(t, \omega) \neq u^*(t, \omega) \text{ for some } t \in \mathbb{R}\}$, then $\tilde{N} \in \mathcal{F}$ and $P(\tilde{N}) = 0$. This implies that $\hat{u}^*$ and $u^*$ are indistinguishable and satisfies $26$ and $27$;

(v) $\hat{u}^*$ is $(\mathcal{F}_t, t \in \mathbb{R})$ adapted.

Moreover, let $Y(\omega) = \hat{u}^*(0, \omega)$, which is $\mathcal{F}_0$ measurable. Then for all $\omega \in \Omega$,

$$Y(\omega) = \frac{1}{2} \int_{-\infty}^{0} T_\nu(-s) \frac{\partial(Y(\theta(s, \omega)))^2}{\partial x} ds + \int_{-\infty}^{0} T_\nu(-s) dW_s(\omega), \quad \text{(35)}$$

thus, $Y(\omega)$ is the stationary point and $\hat{u}^*(t, 0, Y(\omega), \omega) = Y(\theta(t, \omega))$ for $t \geq 0$.

**Proof:** The above remark implies the uniqueness. Now, we use a similar argument of the perfection of crude cocycle in $11$ (see p15) or $11$ to prove (i)–(v) and $35$ hold. First note due to Lemmas 3.2 and 3.3, we have for any $r \in \mathbb{R}$, there is a $N_r \in \mathcal{F}$ such that $P(N_r) = 0$ and for any $\omega \in N_r$,

$$u^*(t, \theta(r, \omega)) = u^*(t + r, \omega), \text{ for any } t.$$

Denote

$$M \equiv \{(r, \omega) \in \mathbb{R} \times \Omega, \ u^*(t, \theta(r, \omega)) = u^*(t + r, \omega) \text{ for any } t\};$$

$$\Omega_0 \equiv \{\omega \in \Omega, (s, \omega) \in M, \lambda - a.e. \ s \in \mathbb{R}\};$$

$$\Omega_1 \equiv \{\omega \in \Omega, \theta(s, \omega) \in \Omega_0, \lambda - a.e. \ s \in \mathbb{R}\}.$$

Here $\lambda(dx)$ is the Lebesgue measure on $\mathbb{R}$. We will prove the theorem in the following 7 steps.

Step 1. We should show that $M$ is a measurable set in $B(\mathbb{R}) \otimes \mathcal{F}$. Since $u^*(t, \omega)$ is continuous in $C((-\infty, +\infty), L^2)$, we have

$$M = \bigcap_{q \in \mathbb{Q}} \{(r, \omega) \in \mathbb{R} \times \Omega, \ u^*(t, \theta(r, \omega)) = u^*(t + r, \omega)\},$$

where $\mathbb{Q}$ is the set of rational number in $\mathbb{R}$. By the construction of $u^*$, we know that $u^*(t, \omega)$ is $(B(\mathbb{R}) \otimes \mathcal{F}, B(L^2[0, 1]))$ measurable. Because $\theta(r, \omega)$ is $B(\mathbb{R}) \otimes \mathcal{F}$ measurable, these imply that $u^*(t + r, \omega)$ and $u^*(t, \theta(r, \omega))$ both are $(B(\mathbb{R}) \otimes \mathcal{F}, B(L^2[0, 1]))$ measurable. Furthermore, since $L^2[0, 1]$ is an Hausdorff
and second countable space, we have that \( \{(r, \omega) \in \mathbb{R} \times \Omega, \ u^*(t, \theta(r, \omega)) = u^*(t + r, \omega)\} \) is \((\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{B}(L^2[0, 1]))\) measurable for every \( t \in \mathbb{Q} \), therefore, \( M \) is \((\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{B}(L^2[0, 1]))\) measurable.

Moreover, using Fubini’s theorem, we get \( \lambda \otimes P(\mathbb{R} \times \Omega \setminus M) = 0 \).

Step 2. Taking \( \nu(dt) \equiv \frac{1}{\sqrt{2 \pi}} e^{-\frac{t^2}{2}} \lambda(dt) \), it is easy to know that \( \omega \in \Omega_0 \) iff \( \int_{\mathbb{R}} 1_M(s, \omega) \nu(ds) = 1 \). Fubini’s theorem implies \( \Omega_0 \in \mathcal{F} \), and moreover \( \lambda \otimes (\mathbb{R} \times \Omega \setminus M) = 0 \), one can see that

\[
\int_{\mathbb{R}} \int_{\Omega} 1_M(s, \omega) P(d\omega) \nu(ds) = 1 \quad \text{i.e.} \quad \int_{\Omega} \int_{\mathbb{R}} 1_M(s, \omega) \nu(ds) P(d\omega) = 1,
\]

so, \( \int_{\mathbb{R}} 1_M(s, \omega) \nu(ds) = 1 \) \( \text{P.a.s.} \). This shows that \( P(\Omega_0) = 1 \). Similarly, one can show \( \Omega_1 \in \mathcal{F} \) and \( P(\Omega_1) = 1 \).

Now, we show that \( \Omega_1 \) is invariant under \( \theta, \text{i.e.}, \omega \in \Omega_1 \) implies \( \theta(t, \omega) \in \Omega_1 \) for all \( t \in \mathbb{R} \). For this, for \( \omega \in \Omega_1 \), there exists a \( A_\omega \in \mathcal{B}(\mathbb{R}) \) such that \( \lambda(A_\omega) = 0 \) and for any \( s \in A_\omega, \theta(s, \omega) \in \Omega_0 \). For any \( t \in \mathbb{R} \), noticing \( A_{\theta(t, \omega)} = A_\omega - t \), it is obvious \( \lambda(A_{\theta(t, \omega)}) = 0 \) and

\[
\theta(s, \omega) = \theta(s-t+t, \omega) = \theta(s-t, \theta(t, \omega)) \in \Omega_0.
\]

This implies that \( \theta(t, \omega) \in \Omega_1 \).

Step 3. Define \( \hat{u}^* \),

\[
\hat{u}^*(t, \omega) = \begin{cases} u^*(t - s, \theta(s, \omega)) & \text{if } \omega \in \Omega_1, \theta(s, \omega) \in \Omega_0 \\ 0 & \text{if } \omega \in \Omega_1 \end{cases}
\]

where 0 is the constant function 0 in \( L^2[0, 1] \).

Firstly, we should show \( \hat{u}^* \) is well-defined. Assume that \( \omega \in \Omega_1, \theta(s, \omega) \in \Omega_0 \) and \( \theta(u, \omega) \in \Omega_0 \). Note there exist \( \Lambda'_{\theta(s, \omega)} \) and \( \Lambda'_{\theta(u, \omega)} \in \mathcal{B}(\mathbb{R}) \) such that \( \lambda(\Lambda'_{\theta(s, \omega)}) = \lambda(\Lambda'_{\theta(u, \omega)}) = 0 \), and for any \( \alpha \in \Lambda'_{\theta(s, \omega)}, \beta \in \Lambda'_{\theta(u, \omega)} \), and for all \( t \in \mathbb{R} \),

\[
\begin{align*}
 u^*(t + \alpha, \theta(s, \omega)) &= u^*(t, \theta(\alpha, \theta(s, \omega))) = u^*(t, \theta(\alpha + s, \omega)), \\
 u^*(t + \beta, \theta(u, \omega)) &= u^*(t, \theta(\beta, \theta(u, \omega))) = u^*(t, \theta(\beta + u, \omega)).
\end{align*}
\]

We can always take \( \alpha \in \Lambda'_{\theta(s, \omega)}, \beta \in \Lambda'_{\theta(u, \omega)} \) such that \( \alpha + s = \beta + u \), so we have, for any \( t \)

\[
\begin{align*}
 u^*(t + \alpha, \theta(s, \omega)) &= u^*(t + \beta, \theta(u, \omega)) = u^*(t + \alpha + s - s, \theta(s, \omega)) \\
 &= u^*(t + \beta + u - u, \theta(u, \omega)).
\end{align*}
\]

This implies that \( u^*(t - s, \theta(s, \omega)) = u^*(t - u, \theta(u, \omega)) \) for any \( t \in \mathbb{R} \).

Step 4. Obviously, \( \hat{u}^*(., \omega) : \mathbb{R} \to L^2[0, 1] \) is continuous for all \( \omega \in \Omega \).

Step 5. We should show that \( \hat{u}^* \) satisfies for all \( r, t \in \mathbb{R} \), \( \hat{u}^*(t, \theta(r, \omega)) = \hat{u}^*(t + r, \omega) \), for all \( \omega \in \Omega \). The assertion is clear for \( \omega \notin \Omega_1 \). We assume \( \omega \in \Omega_1 \), thus \( \theta(r, \omega) \in \Omega_1 \) for any \( r \in \mathbb{R} \). By the definition of \( \hat{u}^* \), we have
\[ \hat{u}^*(t + r, \omega) = u^*(t + r - s, \theta(s, \omega)) \text{ for all } s \in R \setminus A'_\omega, \ t, r \in R, \]
and
\[ \hat{u}^*(t, \theta(r, \omega)) = u^*(t - u, \theta(u, \theta, \omega)) = u^*(t - u, \theta(u + r, \omega)) \]
for all \( u \in R \setminus A'_{\theta(r, \omega)}, \ t, r \in R. \)

On the other hand, note for any \( r \), there exists \( s \in R \setminus A'_\omega, \ u \in R \setminus A'_{\theta(r, \omega)} \) satisfying \( r + u = s \). For if this is not true, then there exists a \( r \in R \) such that for any \( s \in R \setminus A'_\omega, \ u \in R \setminus A'_{\theta(r, \omega)}, \ r + u = s \) cannot be true. That is to say that for any \( u \in R \setminus A'_{\theta(r, \omega)} \), \( r + u \notin R \setminus A'_\omega \) so \( r + u \in A'_\omega \). This is certainly not true since \( \nu(R \setminus A'_{\theta(r, \omega)}) = 1 \) and \( \nu(A'_{\omega}) = 0 \). Thus the assertion holds for any \( t \) and \( r \).

Step 6. Let
\[
B(s, t, \omega) \equiv \begin{cases} u^*(t - s, \theta(s, \omega)) & \omega \in \Omega_1, \ \theta(s, \omega) \in \Omega_0 \\ 0 & \omega \in \Omega_1 \end{cases} \quad (37)
\]

Using the same reasoning as the step 6 on p20 of [1], we know that \( \hat{u}^*(t, \omega) \) is \( \mathcal{B}(R) \otimes \mathcal{F} \) measurable.

Because \( \hat{u}^* \) and \( u^* \) both are continuous and \( \mathcal{B}(R) \otimes \mathcal{F} \) measurable,
\[
\{ \omega : \hat{u}^*(t, \omega) = u^*(t, \omega), \text{ for all } t \in R \} = \cap_{t \in \mathbb{Q}} \{ \omega : \hat{u}^*(t, \omega) = u^*(t, \omega) \} \in \mathcal{F}.
\]

Moreover, for any \( t, \omega \in \Omega_0 \cap \Omega_1 \), we obtain \( \hat{u}^*(t, \omega) = u^*(t, \omega) \). Since \( P(\Omega_0 \cap \Omega_1) = 1 \), we know \( P(\{ \omega : \hat{u}^*(t, \omega) = u^*(t, \omega) \}) = 1 \). All of these imply that \( \hat{u}^* \) and \( u^* \) are indistinguishable.

Step 7. We should prove \( \hat{u}^* \) is \( (\mathcal{F}_t)_{t \in \mathbb{R}} \) adapted. Due to the construction of \( u^* \), we know that \( u^* \) is adapted i.e. \( u^*(t, \cdot) \in \mathcal{F}_t \) (see the beginning of section 2 for the definition of \( \mathcal{F}_t \)). It is easy to know that \( B(s, t, \omega) \in \mathcal{F}_t \) for any \( s, t \).

By (38) and \( P(\Omega_0) = 1 \), \( P(\Omega_1) = 1 \), this means that \( \hat{u}^* \) is adapted to \( (\mathcal{F}_t)_{t \in \mathbb{R}} \).

Furthermore, due to Theorem 2.10, Theorem A.5 in this paper and Theorem 2.1.1, 2.1.2 and 2.2.1 in [17], we have the following dynamical behaviour near \( Y(\omega) \).

**Theorem 3.5** If condition A holds, then for solution of (37), \( u(t, 0, u_0, \omega) \) stating at time 0, with initial data \( u_0 \in L^2[0, 1] \), we have,
\[
\lim_{t \to -\infty} \frac{1}{t} \log |u(t, 0, u_0, \omega) - Y(\theta(t, \omega))|_2 \leq -\delta, \text{ P.a.s.} \quad (38)
\]
and
\[
\lim_{t \to -\infty} \log \|Du(t, Y(\omega), \omega)\|_{L(L^2[0, 1])} \leq -\delta, \ \forall \omega \in \Omega, \quad (39)
\]
where \( L(L^2[0, 1]) \) is the Hilbert space of all bounded linear operators with operator norm, and \( \delta \in (0, \delta_0) \).
The theorem says that \((u, \theta)\) is a \(C^1\) perfect cocycle on \(L^2[0, 1]\), and the stationary point \(Y(\omega)\) is hyperbolic, especially, its the largest Lyaponov spectrum not larger than \(-\delta(< 0)\) and \(L^2[0, 1]\) is the stable manifold of \(Y(\omega)\) for all \(\omega \in \Omega^*\).

Appendix A

Although we believe that experts in this field are familiar with some properties here and some of them can be found in [6], we would like to include them here for completeness.

It is easy to see from Theorem 5.4 in [6] that \(W_{\nu \Delta}\) is the unique weak solution in \(L^2[0, 1]\) of the following SPDE,

\[
\begin{align*}
    dX(x, t) &= \nu \Delta X(x, t) dt + dW(t), \quad t \geq 0; \quad x \in (0, 1), \\
    X(0, t) &= X(1, t) = 0, \\
    X(x, 0) &= 0, \quad x \in (0, 1).
\end{align*}
\]  

(40)

Therefore for any \(e_k\), by integration by parts

\[
\langle W_{\nu \Delta}(t), e_k \rangle = \int_0^t \langle W_{\nu \Delta}(s), \nu \Delta e_k \rangle ds + \sigma_k (B_k(t) - B_k(0)).
\]

(41)

Let \(W_{\nu \Delta, k} = \langle W_{\nu \Delta}, e_k \rangle\). Obviously, \(W_{\nu \Delta, k}\) satisfies the following O-U equation in 1 dimension:

\[
W_{\nu \Delta, k}(t) = -\nu \pi^2 k^2 \int_0^t W_{\nu \Delta, k}(s) ds + \sigma_k (B_k(t) - B_k(0)),
\]

(42)

so

\[
W_{\nu \Delta, k}(t) = \sigma_k \int_0^t e^{-\nu \pi^2 k^2 (t-s)} dB_k(s).
\]

(43)

Let \(W_{\nu \Delta(n)}(t) \equiv P_n W_{\nu \Delta}(t) = \sum_{k=1}^n W_{\nu \Delta, k}(t)e_k\), then by Itô’s formula, we have

\[
|W_{\nu \Delta(n)}(t)|^2 + \nu \int_0^t \left| \frac{\partial}{\partial x} W_{\nu \Delta(n)}(s) \right|^2 ds = \sum_{k=1}^n \int_0^t W_{\nu \Delta, k}(s) \sigma_k dB_k(s) + t \sum_{k=1}^n \sigma_k^2.
\]

(44)

\textbf{Lemma A.1:} \(\sum_{k=1}^n \int_0^t W_{\nu \Delta, k}(s) \sigma_k dB_k(s)\) is a martingale.

\textbf{Proof:} In fact, we only need to show that

\[
E \int_0^t \left( \frac{W_{\nu \Delta, k}(s) \sigma_k}{\sigma_k} \right)^2 ds < \infty.
\]

(45)

In fact from [12], we have
\[ E \int_0^t \left( W_{\nu,k}(s) \sigma_k \right)^2 ds = \sigma_k^4 E \int_0^t \left( \int_0^s e^{-\nu \pi^2 k^2 (s-r)} dB_k(r) \right)^2 ds \\
= \sigma_k^4 \int_0^t E \left( \int_0^s e^{-\nu \pi^2 k^2 (s-r)} dB_k(r) \right)^2 ds \\
= \sigma_k^4 \int_0^t \int_0^s e^{-2\nu \pi^2 k^2 (s-r)} dr ds \\
= \frac{\sigma_k^4}{2 \nu \pi^2 k^2} \left( t + \frac{1}{2 \nu \pi^2 k^2} (e^{-2\nu \pi^2 k^2 t} - 1) \right) < \infty \]

Using Lemma A.1, \[E\] and \( \sum_{k=1}^{\infty} \sigma_k^2 < \infty \), one can get it immediately

**Theorem A.2:** \( E \left( \int_0^t \left| \frac{\partial}{\partial x} W_{\nu,k}(s) \right|^2 ds \right) < \infty \), therefore, for any \( t \), \( W_{\nu,k} \in L^2([0,t], H_0^1) \) a.s.

By theorem 16.2 in [12] (p413), one easily deduces that

**Proposition A.3:** There exist positive constants \( c_1, c_2 \) such that

\[ \left| \frac{\partial p(t,x,y)}{\partial y} \right| \leq \frac{c_1}{t} e^{-\frac{(x-y)^2}{2s^2t}}, \]

for all \( t > 0, x,y \in [0,1] \). Then one can pick a positive constant \( c_3 \) such that

\[ \int_0^1 \frac{c_1}{t} e^{-\frac{y^2}{2s^2t}} dy \leq 1. \]

The following results are from [13]. Although he dealt with the case of 2D stochastic Navier-Stokes equation, all of his techniques and reasonings can be used to 1D stochastic Burgers equations with Dirichlet boundary condition. So, we omit the proof of the theorems.

**Theorem A.4** (see Lemma 2 in [13] for details) Assume the initial condition satisfies \( E|u(t_0)|^{2p} < \infty \) for some \( p \geq 1 \), then there is a constant \( c \) such that

\[ E|u(t,t_0,u(t_0))|^{2p} \leq c^p (p-1)!. \]

**Theorem A.5** (see Theorem 1 in [13]) Assume Condition A holds. Fix a \( \delta \in (0, \delta_0) \) and a time \( t_0 \). Let \( u_0 \in L^2 \) be an initial condition, measurable with respect to \( F_0 \), such that \( E|u_0|^2 < \infty \) for some \( p > 1 \). Let \( \overline{u} \equiv \overline{u}(t,t_0,\overline{u}_0) \) denote the solution starting from some other arbitrary initial data \( \overline{u}_0 \in L^2 \).

Then, there is a positive integer-valued random time \( \tau(\delta,t_0,u_0) \), independent of \( \overline{u} \), such that

\[ |u(t) - \overline{u}(t)|^2 \leq |u_0 - \overline{u}_0|^2 e^{-2\delta(t-t_0)} \]
for all $t > t_0 + \tau$. In addition, $E(\tau^q) < \infty$ for any $q \in (0, p - 1)$.

Theorem A.6 (see Theorem 2 in [13]) Assume Condition A holds. Fix a
$\delta \in (0, \delta_0)$ and a $t \in \mathbb{R}$. Let $\{u_0(n)\}$ be a sequence of random variable
with $n \in \mathbb{Z}^+$. Assume that the $u_0(n)$ are measurable with respect to $\mathcal{F}_{t-n}$ and that
$E|u_0(n)|_{2}^{2p}$ is uniformly bounded in $n$ for some $p > 2$. Then the following hold:

1. With probability one, there exists a random $\mathbb{Z}$-valued time $\Pi(\epsilon, \delta, t, \omega) > 0$
such that for real $s > 0$ and all $n \in \mathbb{Z}$ with $n > \Pi$ we have

$$\sup_{u_0^* \in A_n} |u(t + s, t - n, u_0(n), \omega) - u(t + s, t - n, u_0^*, \omega)|_2 \leq \epsilon \delta^2 n |e^{-\delta(n + s)}|.$$

Here $A_n$ is the set $\{u_0^* : |u_0^*|^2 \leq \frac{\epsilon \delta^2}{2} |n|\}$. In addition, $E(\Pi^q) < \infty$ for any
$q \in (0, p - 2)$.

2. Let $\{y_0(n)\}$ be a second sequence of random variables with $n \in \mathbb{Z}^+$ measurable
with respect to $\mathcal{F}_{t-n}$ and $E|y_0(n)|_{2}^{2p}$ uniformly bounded in $n$ for
some $p > 2$. Then with probability one, there exists another $\mathbb{Z}$-valued random
time $n'$ such that for real $s > 0$ and all $n \in \mathbb{Z}$ with $n > n'$ we have

$$|u(t + s, t - n, u_0(n)) - u(t + s, t - n, y_0(n))|_2 \leq \epsilon \delta^2 n |e^{-\delta(n + s)}|.$$

Again, $E(\Pi^q) < \infty$ for any $q \in (0, p - 2)$.

Theorem A.7 (see Corollary 1 in [13]) Under Condition A, fix $t \in \mathbb{Z}$
and a $\delta \in (0, \delta_0)$. Given any $\epsilon > 0$, with probability one, there is a positive
$\mathbb{Z}$-valued random time $n^*(\epsilon, \delta, t_1)$ such that for all $\tau \leq 0$ and all $n_1, n_2 \in \mathbb{Z}$,
if $n_1, n_2 < t - 1 - n^*$, we have

$$|u(t + \tau, n_1, 0) - u(t + \tau, n_2, 0)|_2 \leq \epsilon e^{-\delta \tau}.$$

Furthermore, $n^*(\omega)$ is a stationary random variable with all moments finite.

Acknowledgements One of the authors, YL, would like to acknowledge the financial support of EPSRC GR/R69518 and partial supports of NSFC (No.10531070, 10171101, 10101002), SRF for ROCS. Both authors would like to thank Prof. D. Elworthy for his invitation to visit the University of Warwick, and Prof. S. Peng for his invitation to visit Shandong University.

References

1. Arnold, L. Random dynamical systems, Springer (1998).
2. Chen, M.F., Eigenvalues, Inequalities, and Ergodic Theory, Springer (2004).
3. Duan, J. Lu, K., Schmalfuss, B. Smooth stable and unstable manifolds for stochastic
evolutionary equations. J. Dynam. Differential Equations 16 (2004), no. 4, 949–972.
4. Duan, J. Lu, K., Schmalfuss, B. Invariant manifolds for stochastic partial differential equations. Ann. Probab. 31 (2003), no. 4, 2109–2155.
5. Da Prato, G., *Kolmogorov Equations for stochastic PDEs*. Birkhäuser (2004)
6. Da Prato, G., and Zabczyk, J., *Stochastic equations in infinite dimensions*, Cambridge University Press (1992).
7. Da Prato, G., and Zabczyk, J., *Ergodicity for Infinite Dimensional Systems*, Cambridge University Press (1996).
8. E.W., Khanin, K., Mazel, A., Sinai, Ya. *Invariant measures for the Burgers equations with stochastic forcing*, Annals of Math. 151 (2000), 877-960
9. Flandoli, F., *Dissipativity and invariant measures for stochastic Navier-Stokes Equations*, NoDEA 1: 403-426(1994)
10. Flandoli, F., Schmalfuss, B. *Weak solution and attractors for three-dimensional Navier-Stokes equations with nonregular force*. J. Dynam. Differential Equations vol.11 No.2 355-398 (1999)
11. Kager, G., Scheutzow, M., *Generation of one-sided random dynamical systems by stochastic differential equations*. Electronic J.Prob., 2:1-17 (1997)
12. Ladyzenskaja, O.A., Solonnikov, V.A., Uralceva, N.N., *Linear and quasi-linear equations of parabolic type*. Translation of Mathematical Monographs, vol. 23, American Mathematical Society (1968)
13. Mattingly, J.C., *Ergodicity of 2D Navier-Stokes equations with random forcing and large viscosity*, Commum. Math. Phys. 206; 273-288 (1999)
14. Mohammed, S.-E.A, Scheutzow, M.K.R. *The stable manifold theorem for nonlinear stochastic systems with memory. I. Existence of the semiflow*. J. Funct. Anal. 205 (2003), no. 2, 271–305.
15. Mohammed, S.-E.A, Scheutzow, M.K.R. *The stable manifold theorem for nonlinear stochastic systems with memory. II. The local stable manifold theorem*. J. Funct. Anal. 206 (2004), no. 2, 253–306.
16. Mohammed, S.-E.A, Scheutzow, M.K.R. *The stable manifold theorem for stochastic differential equations*. Ann. Probab. 27 (1999), no. 2, 615–652.
17. Mohammed, S., Zhang, T.S., Zhao, H.Z. *The stable manifold theorem for semilinear stochastic evolution equations and stochastic partial differential equations*, Memoirs of the American Mathematical Society (to appear).
18. Robinson, J., *Infinite-Dimensional Dynamical Systems*, Cambridge University Press (2001).
19. Sinai, Ya. Burgers system driven by a periodic stochastic flow. In: *Itô’s stochastic calculus and probability theory*, 347–353, Springer, Tokyo, 1996
20. Sinai, Ya. *Two results concerning asymptotic behaviour of solutions of the Burgers equation with force*. J. Statist. Phys. 64 (1991), no. 1-2, 1–12.
21. Sohr., H., *The Navier-Stokes equations: an elementary functional analytic approach*. Birkhäuser, (2001)
22. Temam, R., *Infinite-Dimensional Dynamical Systems in Mechanics and physics, 2nd Edition* Applied Mathematical Sciences, vol 68. Springer-Verlag (1997)
23. Temam, R., *Navier-Stokes equations: Theory and numerical analysis*. AMS Chelsea publishing, (2001)
24. Zhang, Q., Zhao, H.Z., *Pathwise stationary solution of stochastic partial differential equations and backward doubly stochastic differential equations on infinite horizon*, Preprint (2006) available at arXiv.org/math/0602054