The 2-bridge knots of up to 16 crossings

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Abstract

For any given number of crossings $c$, there exists a formula to determine the number of 2-bridge knots of $c$ crossings, and indeed it is a simple matter to actually construct presentations of these knots. However, the determination of whether a given (prime) knot is a 2-bridge knot remains a nontrivial exercise, and we have no procedure to determine bridge numbers more generally. Herein, we identify the 2-bridge knots within the Hoste–Thistlethwaite–Weeks tables of prime knots of up to 16 crossings by an exhaustive search of a larger set of 2-bridge knots. As the unknot is the only knot with bridge number 1, this yields a lower bound of 3 for the bridge numbers of the remaining knots.

1 Introduction

Recall that the 2-bridge knots are a class of alternating prime knots. Most, but not all, are chiral, and as oriented knots, all are invertible. Although it is straightforward to determine a catalogue of 2-bridge knots, the determination of whether a given (prime) knot is a 2-bridge knot is nontrivial. In principle, one takes the 2-fold branched cover of the knot, then computes the characteristic (geometric) decomposition (by the orbifold theorem, we know such a decomposition exists). If the 2-fold cover is a lens space, then the knot is 2-bridge, otherwise not. Herein, we do not pursue this algorithm, rather, we identify the 2-bridge knots in the Hoste–Thistlethwaite–Weeks tables by exhaustively examining a larger catalogue of 2-bridge knots.

Adopting the notation of Ernst and Sumners [5], denote by $TK_c$ (respectively $TK^*_c$) the number of 2-bridge knots of $c$ crossings when a knot and its reflection are regarded as nondistinct (respectively distinct). Also let $ATK_c$ denote the number of achiral 2-bridge knots of $c$ crossings, so we have: $TK^*_c = 2TK_c - ATK_c$. As 2-bridge knots can be described purely by sequences of integers (see below), they may be counted combinatorially. Indeed, [5] presents the following formulae for $c \geq 3$, which in Table 1 we evaluate for $c \leq 16$. Note that

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there are no achiral alternating knots, 2-bridge or otherwise, with odd crossing numbers [6].

\[ T K_c^* = \begin{cases} \frac{1}{3}(2c-2 - 1) & c \text{ even} \\ \frac{1}{3}(2c-2 + 2(c-1)/2) & c \equiv 1 \pmod{4} \\ \frac{1}{3}(2c-2 + 2(c-1)/2 + 2) & c \equiv 3 \pmod{4} \end{cases} \]

\[ AT K_c = \begin{cases} \frac{1}{3}(2c-2)/2 + 1 & c \equiv 0 \pmod{4} \\ \frac{1}{3}(2c-2)/2 - 1 & c \equiv 2 \pmod{4} \\ 0 & c \text{ odd} \end{cases} \]

| c | $T K_c^*$ | $AT K_c$ |
|---|---|---|
| 3 | 2 | 1 |
| 4 | 4 | 1 |
| 5 | 5 | 1 |
| 6 | 7 | 3 |
| 7 | 8 | 5 |
| 8 | 9 | 11 |
| 9 | 10 | 21 |
| 10 | 11 | 43 |
| 12 | 13 | 85 |
| 14 | 15 | 11007 |
| 16 | 17 | 2774 |
| 18 | 19 | 5461 |
| 20 | 21 | 10007 |
| 22 | 23 | 15446 |
| 24 | 25 | 20017 |
| 26 | 27 | 24617 |
| 28 | 29 | 29217 |
| total | | 11007 |

Table 1: Numbers of 2-bridge and alternating prime knots.

The notation is compatible with the following conventions. Let $K_c$ denote the number of prime knots of $c$ crossings, and then denote by $K_c^A$ (respectively $K_c^N$) the number of alternating (respectively nonalternating) prime knots of $c$ crossings. Next, let $c_i^A$ (respectively $c_i^N$) denote the $i$th alternating (respectively nonalternating) prime knot (type) of $c$ crossings in the HTW tables of prime knots of up to 16 crossings [6]. Note that the convention is to use a superscript “A” to denote an alternating knot, whilst the “A” in $AT K_c$ denotes “achiral”.

The presence of the superscript decorating the crossing number identifies the knot as from the HTW tables, and facilitates the concurrent use of the classical undecorated Alexander–Briggs notation; for instance 5$_1^A$ and 5$_2$ denote the same knot type. Dowker–Thistlethwaite codes and many other data associated with the knots in the HTW tables may be accessed via the program Knotscape (version 1.01), as mentioned in [6].

Now, each 2-bridge knot may be expressed via a presentation known as Conway’s normal form. By restricting the degrees of freedom used in choosing this presentation, each 2-bridge knot may be uniquely expressed, indeed this remains true when a knot and its reflection are regarded as distinct. Importantly however, the canonical presentation is not generally minimal, that is, the presentation will usually have more crossings than the prime knot type to which it corresponds. So, by exhaustively enumerating and identifying sufficiently many canonical presentations, we may identify all the 2-bridge knots within the HTW tables. This approach to enumerating the 2-bridge knots is less mathematically elegant than that of [6], but is easy to implement.

Herein, we compile a list of the 2-bridge knots of up to 16 crossings by first enumerating all such canonical Conway presentations of up to 28 crossings. To each presentation we assign an orientation, and then use this to deduce an appropriate DT code. This process is automated using Mathematica. We then
use the Locate (that is, the “Locate in Table”) function of Knotscape to identify which elements of our list of DT codes correspond to which (alternating) prime knots within the HTW tables.

Now let $P$ be a Conway presentation identified as being a knot of type $T$ described by a DT code within the tables. In general, $T$ will be chiral, and it is of interest to ask to which chirality of $T$ our presentation $P$ corresponds. However, any DT code necessarily corresponds to both a knot and its reflection (the book by Adams [1, p38] shows how). That is, the HTW tables do not provide us with canonical representatives of knot types in the way that classical pictorial knot tables do, so we may not identify $P$ with a particular chiral knot, only a knot type. In contrast, the DT codes of the HTW tables do prescribe orientations whereas classical tables do not. However, 2-bridge knots are invertible, so we may ignore this issue. All that said, Knotscape happily evaluates various polynomial invariants of knot types described by DT codes; so its routines make an opaque choice which determines a chirality.

Apart from its intrinsic interest, we have compiled our list of 2-bridge knots to facilitate our enquiries into the properties of the Links–Gould invariant $LG^{2,1}$. Currently, skein relations sufficient to determine the value of this invariant for any arbitrary link remain unknown, and the only general method for its evaluation is a computationally-expensive state model method. This method is currently computationally feasible for links for which braid presentations of at most 5 strings are available: to date we have been able to evaluate $LG^{2,1}$ this way only for 37, 547 of the 1,701,936 knots in the HTW tables. However, recent work by Ishii [5] has determined a formula for $LG^{2,1}$ for 2-bridge links described by canonical Conway presentations. So, by identifying 2-bridge knots within the HTW tables, we may evaluate $LG^{2,1}$ for many prime knots for which the state model method remains infeasible. Braids for the HTW knots may be obtained via the use of the program K2K by Imafuji and Ochiai [7]; this program is a Mathematica interface to the C program KnotTheoryByComputer. We observe that the string indices of K2K-generated braids corresponding to our 2-bridge knots are generally greater than 5; indeed this reflects their inherent complexity. The results of the application of Ishii’s formula to evaluations of $LG^{2,1}$ for 2-bridge knots from the HTW tables, together with further material on the current state of evaluations of $LG^{2,1}$, appear in [4].

Our intensive use of Knotscape in this manner demonstrates the robustness of its algorithms for reducing Dowker–Thistlethwaite codes. This is significant as it is known that these algorithms are necessarily incomplete.

We mention that [5] also includes formulae for the numbers $TL_c^*$, $TL_c$ and $ATL_c$ of 2-component 2-bridge links. We have not pursued the identification of these links here as the existing tables of multicomponent prime links are not as extensive as those of the true knots. More generally, the methodology applied to identify the 2-bridge knots can be applied to identify the torus and pretzel knots. Although we currently have no formulae for $LG^{2,1}$ for knots of these classes, relatively efficient versions of state model algorithms can be implemented for them. However, these knots, particularly the pretzels, are comparatively rare in the HTW tables, and we reach a point of diminishing returns.
2 Conway presentations for 2-bridge links

Abstracting material from the excellent books by Kawauchi [9] pp21–26 and Murasugi [10] pp171–196, we describe the 2-bridge links using the plait presentation known as Conway’s normal form, as depicted in Figure 1. In this form, any 2-bridge link may be described in terms of a presentation \( C(a) \), where \( a \) is a finite sequence \( a_1, \ldots, a_n \) of nonzero integers \( a_i \). In the present context, we need not define the meaning of “2-bridge”; we are only interested in the 2-bridge links as that class of links for which there exist presentations of the normal form. Note the convention relating the signs of the \( a_i \) to those of the crossings, and observe that such a presentation has either 1 or 2 components.

Figure 1: The 2-bridge link presentation \( C(a) \equiv C(a_1, \ldots, a_n) \), illustrated for positive coefficients \( a_i \). Reflect the crossings for negative \( a_i \).

Now choose integers \( \alpha \) and \( \beta \) such that \( \alpha > 0 \) and \( \gcd(\alpha, \beta) = 1 \) in the following continued fraction, referred to as the slope of the presentation \( C(a) \):

\[
\frac{\alpha}{\beta} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{a_n}}}
\]

The presentation \( C(a) \) is then said to be of 2-bridge link type \( S(\alpha, \beta) \). After Murasugi [10] p180], we denote the continued fraction by \([a_1, \ldots, a_n]\).

The 2-bridge knots are in fact precisely the rational knots. Conway presentations are equal (that is, they correspond to the same link) when their slopes are equal; for example \( C(2, -3) = C(1, 1, 2) = S(5, 3) \), as \([2, -3] = [1, 1, 2] = \frac{5}{3} \).

More generally, 2-component 2-bridge links \( S(\alpha, \beta) \) and \( S(\alpha', \beta') \) are equal if and only if \( \alpha = \alpha' \) and \( \beta \equiv \beta' \pmod{\alpha} \), and 2-bridge knots \( S(\alpha, \beta) \) and \( S(\alpha', \beta') \) are equal if and only if \( \alpha = \alpha' \) and \( \beta \equiv \beta' \pmod{\alpha} \). Next, observe that \( S(\alpha, \beta)^* = S(\alpha, -\beta) \); in fact, \( S(\alpha, \beta) \) is achiral if and only if \( \beta^2 = -1 \pmod{\alpha} \). Also observe that \( C(a)^* = C(-a) \), where by \(-a\) we intend the sequence \(-a_1, \ldots, -a_n\).

Now, if for some \( S(\alpha, \beta) \), we seek a representative presentation \( C(a) \), we may choose all the \( a_i \) to be of the same sign as \( \beta \). Such a \( C(a) \) is thus an alternating presentation, so \( S(\alpha, \beta) \) is necessarily an alternating link. Moreover, any \( S(\alpha, \beta) \)
always has a presentation $C(a)$ with only even components $a_i$ (although the $a_i$ are not necessarily all of the same sign). For example, $C(2,-3) = S(5,3) = S(5,2) = C(2,2)$. Knowing that any 2-bridge link may be expressed via such a presentation $C(a)$ with all $a_i$ even, we observe that it is a knot if $n$ is even and a 2-component link if $n$ is odd.

Here, we are only interested in knots, so let us regard the 2-bridge knots as those $S(\alpha, \beta)$ with presentations $C(a_1, \ldots, a_n)$ where both $n$ and each $a_i$ are even (and nonzero!). We shall refer to such a presentation as an even (Conway) presentation. In fact, all 2-bridge knots $S(\alpha, \beta)$ are necessarily prime, and it turns out that they have odd $\alpha$. We assign an orientation to an even presentation $C(a)$ by giving that of the upper arc, as depicted in Figure 2.

![Figure 2: The oriented 2-bridge knot even presentation $C(a) \equiv C(a_1, \ldots, a_n)$, where both $n$ and each $a_i$ are nonzero even integers. The illustration depicts positive coefficients $a_i$; reflect the crossings for negative $a_i$.](image)

Next, note the equality: $C(a_1, \ldots, a_n) = C(a_1, \ldots, a_n \pm 1)$. More generally, if none of the first and last components of sequences $a$ and $b$ are of unit magnitude, the oriented 2-bridge links with presentations $C(a)$ and $C(b)$ are equal if and only if $a$ and $b$ are of the same length $n$ and $b = a$ or $b = (-)^{n+1}a^r$, where by $a^r$ we intend the sequence $a_n, \ldots, a_1$. This fact means that an even presentation $C(a)$, corresponding to some 2-bridge knot type $S(\alpha, \beta)$ is in fact a unique even presentation for $S(\alpha, \beta)$, modulo only the consideration that $C(a) \ast = C(-a)$. If we consider $C(a)$ as oriented, then by inspection, the inverse is: $-C(a) = C(-a^r)$. However 2-bridge links are known to be invertible, so $C(a) = C(-a^r)$ and $C(-a) = C(a^r)$; and if $a$ is palindromic, that is if $a = a^r$, then $C(a)$ is achiral and in fact $C(a) = C(a^r) = C(-a) = C(-a^r)$.

### 3 DT codes for Conway presentations

We now describe how to construct a Dowker–Thistlethwaite code corresponding to an even Conway presentation $C(a_1, \ldots, a_n)$. A fluent description of the algorithm used to construct such codes is available in [6]. The algorithm begins by walking once around an oriented diagram, numbering crossings from 1 onwards, and multiplying by $-1$ any even numbers written down whilst travelling over a crossing. So, each crossing of an $m$-crossing presentation bears one unsigned odd and one signed even number, drawn from the set $\{1, \ldots, 2m\}$. The DT code is then formed as the sequence of the even numbers corresponding to ordered odd crossing numbers $1, 3, \ldots, 2m - 1$. 

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To apply this procedure to an even Conway presentation \( C(a_1, \ldots, a_n) \), we initially number the crossings increasing from the upper left of Figure 2. Then, for each \( i = 1, \ldots, n \), let \( m_i \) be the (left-to-right) sequence associated with the leftwards-travelling middle strand and \( o_i \) the sequence associated with the rightwards-travelling outer strand (which alternates with \( i \) between the upper and lower strands). Then, the DT code associated with \( C(a) \) describes the pairings \((m_i, o_i)\) for \( i = 1, \ldots, n \), by inspection, the components of the sequences are:

\[
(m_i)_k = (M_i + 1 - k)s^k_i \quad \text{for} \quad k = 1, \ldots, A_i,
\]

\[
(o_i)_k = (O_i + k)s^{k+1}_i \quad \text{for} \quad k = 1, \ldots, A_i,
\]

where the offsets \( M_i \) and \( O_i \) are:

\[
M_i = O_{n-1} + \sum_{j=i}^n A_j
\]

\[
O_i = \begin{cases} 
\sum_{j=1}^{i-2, \text{odd}} A_j, & \text{odd } i \\
M_1 + \sum_{j=2}^{i-2, \text{even}} A_j, & \text{even } i.
\end{cases}
\]

For example, for \( C(+2, -2) \), which corresponds to the (negative, left-handed) trefoil knot \( 3^1_A \), we obtain the DT code \((6, -8, 2, -4)\); note that the DT code may be regarded as corresponding to either chirality of \( 3^1_A \). In contrast the palindromic presentation \( C(+2, +2) \) is the achiral \( 4^1_A \) with DT code \((6, 8, 2, 4)\).

## 4 Enumerating even Conway presentations

For even positive integers \( m \) and \( n \), let \( U^{m,n} \) be the set of even Conway presentations \( C(a_1, \ldots, a_n) \) of \( m = \sum_{i=1}^n |a_i| \) crossings (so that \( n \leq m/2 \)), restricted such that \( U^{m,n} \) contains only one element of each knot type equivalence class \( \{C(a), C(a'), C(-a), C(-a')\} \). Denote by \( TK^{m,n} \) the size of \( U^{m,n} \) and set \( TK^{c,n}_n \) as the number of \( c \)-crossing knots (not presentations) in \( U^{m,n} \). Then we have \( TK^{m,n} = \sum_c TK^{c,n}_n \), where the index \( c \) runs over some finite set of crossing numbers depending on \( m \) and \( n \), and \( TK_c = \sum_{m,n} TK^{m,n}_c \), where the indices \( m, n \) run over some finite set depending on \( c \).

An efficient procedure (which we have implemented in Mathematica) to compile the sequences underlying \( U^{m,n} \) is to:

1. find all ordered sets \( \{a_1, \ldots, a_n\} \) of even positive integers \( a_i \) such that \( \sum_{i=1}^n a_i = m \), then
2. find all permutations of each such set, and
3. for each such permutation, find all \( 2^n \) variants obtained by multiplying each component by \( \pm 1 \), and finally
4. identify all cliques of the form \( \{a, a', -a, -a'\} \), and eliminate all but one element from each (for good measure, select one with \( a_1 > 0 \) ).
Here we are interested in even presentations $C(a)$ which are prime knots of at most 16 crossings, and by inspection (see below) we find that for all even $m$ between 4 and 28 there exist some even $n \leq 14$ such that $U^{m,n}$ contains some such $C(a)$. Table 2 lists $TK^{m,n}$ for various $m$ and $n$ of interest, together with their sums $TK^m \triangleq \sum_n TK^{m,n}$, where the sum runs over all even positive $n$ such that $2n < m$.

| $m$ | 2 | 4 | 6 | 8 | 10 | 12 | 14 | $TK^m$ |
|-----|---|---|---|---|----|-----|-----|--------|
| 4   | 2 |   |   |   |    |     |     |        |
| 6   | 2 |   |   |   |    |     |     | 2      |
| 8   | 4 | 6 |   |   |    |     |     | 10     |
| 10  | 4 | 16|   |   |    |     |     | 20     |
| 12  | 6 | 44| 20|   |    |     |     | 70     |
| 14  | 6 | 80| 96|   |    |     |     | 182    |
| 16  | 8 | 146| 348| 72|    |     |     | 574    |
| 18  | 8 | 224| 896| 512|    |     |     | 1640   |
| 20  | 10| 344| 2040| 2336| 272|    |     | 5002   |
| 22  | 10| 480| 4032| 7592| 2560|    |     | 14762  |
| 24  | 12| 670| 7432| 21290| 14160| 1056|    | 44530  |
| 26  | 12| 880| 12672| 50688| 56320| 12288|    | 132860 |
| 28  | 14| 1156| 20652| 109984| 183280| 80064| 4160| 598964 |
| total| 98| 4046| 48188| 192472| 256592| 93408| 4160| 598964 |

Table 2: The numbers $TK^{m,n}$ of 2-bridge knot types defined by $m$-crossing even presentations $C(a_1, \ldots, a_n)$, together with their sums $TK^m$. Above the zigzag are $U^{m,n}$ which contribute to our census of 2-bridge knots of up to 16 crossings.

We use MATHEMATICA to compute DT codes for all elements of $U^{m,n}$ for the various $m$ and $n$ described in Table 2. These DT codes are then fed to the Locate function of KNOTSCAPE for identification as alternating prime knots within the HTW tables. For the vast majority of the presentations with $m > 16$, KNOTSCAPE reports that the knot is of more than 16 crossings, and instead of returning an identification, it returns a reduced DT code and associated crossing number.

Inspection of the results of a range of searches of various $U^{m,n}$ indicates that $T^{m,n}_c$ is nonzero exactly for $m - n + 1 \leq c \leq m$. This shows that for a given $m$, presentations with higher values of $n$ have greater opportunities for reduction. For our purposes, we need not prove that $T^{m,n}_c = 0$ for $c < m - n + 1$ (see below), however it appears that a closer reading of [5] should indicate its origin. Demanding that the crossing numbers $c$ be at most 16 thus means demanding $m - n + 1 \leq 16$. As both $m$ and $n$ are even and $2n \leq m$, we then have $m \leq 28$ and $n \geq m - 14$; that is, only choices of $m$ and $n$ above the zigzag line in Table 2 contribute to our collection of 2-bridge knots of up to 16 crossings.

5 The results

In this manner, we identify as 2-bridge knots a total of 5546 alternating prime knots within the HTW tables. That is, corresponding to each such knot type,
as defined by its DT code, we have an oriented 2-bridge knot defined by an even Conway presentation. As mentioned above, the Locate function identifies a given DT code as corresponding to some knot type in the tables, and it can do no more as a DT code corresponds to both a knot and its reflection (although it does determine their orientations). The 5546 knots we identify become 11007 when we count knots and their reflections as distinct; that is, there are 85 achiral knots which correspond precisely to the even presentations $C(a)$ with palindromic sequences $a$.

Tables $3$ and $4$ describe $TK^c_{m,n}, TK^c_m = \sum_n TK^c_{m,n}$ and $TK^c_c = \sum_m TK^c_m$ for $3 \leq c \leq 16$ for the $U_{m,n}$ mentioned in Table $2$. The numbers in the last row of Table $3$ sum to 598, 964, the number of 2-bridge knot types listed in Table $2$. For $m \leq 16$ (only), the final column of Table $4$ reconstructs $TK^m = \sum_c TK^m_c$, in agreement with the final column of Table $2$. Similarly, for $m \leq 16$ (only), the final row of Table $3$ reconstructs $TK^c = \sum_{m,n} TK^{m,n}_c$, in agreement with the final row of Table $4$ (For $m > 16$, we do not have enough data to recover these numbers.)

Importantly, our list of 5546 knots is complete, as our observed $TK^c_c$ are in agreement with those of Table $1$. (For this reason, we need not prove the experimental observation that $T^c_{m,n} = 0$ for $c < m - n + 1$.) This fact, together with the symmetries of Table $3$, indicates that the Locate function is successful in reducing to minimal presentations all 598, 964 of the 2-bridge knot types with even Conway presentations of up to 28 crossings. This success is a remarkable credit to the ability of the Locate function to reduce an input $m$-term DT code corresponding to a $c$-crossing alternating knot to a $c$-term DT code (where we intend $c \leq 16$). The Locate function (of course!) cannot be guaranteed to always succeed in this if $m > 16$.

To illustrate the results, Table $5$ lists the 2-bridge knots of up to 8 crossings, together with the sequences $a$ of their even Conway presentations $C(a)$, and associated DT codes. Tables $6$-$9$ then list the indices of all the 2-bridge knots of between 3 and 16 crossings. Sequences $C(a)$ for these knots are available on request from the author.

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Table 3: The numbers $TK_{m,n}^c$ of $c$-crossing alternating prime knots identified as $m$-crossing 2-bridge knot presentations $C(a_1, \ldots, a_n)$ within the $U_{m,n}^c$ of Table 2.
Table 4: Observed $TK^m_c = \sum_n TK^{m,n}_c$, and $TK_c = \sum_m TK^m_c$.

Table 5: The 2-bridge knots of up to 8 crossings, with even Conway presentations $C(a) \in U^{m,n}$, and associated DT codes.
Table 8: Identified 2-bridge alternating prime knots $10^4$, part 1/2.
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