ON DOUBLY MINIMAL SYSTEMS AND A QUESTION REGARDING PRODUCT RECURRENCE

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Abstract. We show that a doubly minimal system $X$ has the property that for every minimal system $Y$ the orbit closure of any pair $(y, x) \in Y \times X$ is either $Y \times X$ or it has the form $\Gamma_\pi = \{(\pi(x), x) : x \in X\}$ for some factor map $\pi : X \to Y$. As a corollary we resolve a problem of Haddad and Ott from 2008 regarding product recurrence.

In this note a dynamical system is a pair $(X, T)$ where $X$ is a compact metric space and $T$ a self homeomorphism. The reader is referred to [F] for most of the notions used below and for the necessary background.

In [F, Theorem 9.11, p. 181] Furstenberg has shown that a point $x$ of a dynamical system $(X, T)$ is product-recurrent (i.e. has the property that for every dynamical system $(Y, S)$ and a recurrent point $y \in Y$, the pair $(x, y)$ is a recurrent point of the product system $X \times Y$) if and only if it is a distal point (i.e. a point which is proximal only to itself). In [AF] Auslander and Furstenberg posed the following question: if $(x, y)$ is recurrent for all minimal points $y$, is $x$ necessarily a distal point? Such a point $x$ is called a weakly product recurrent point. This question is answered in the negative in [HO].

It turns out (see also [DSY, Theorem 4.3]) that a negative answer was already at hand for Harry Furstenberg when he and Joe Auslander posed this question. In fact, many years before he proved a theorem according to which an F-flow is disjoint from every minimal system [F]. As a direct consequence of this theorem, if $X$ is an F-flow, $x$ a transitive point in $X$, $Y$ any minimal system and $y$ any point in $Y$, then the pair $(x, y)$ has a dense orbit in $X \times Y$. In particular $(x, y)$ is a recurrent point of the product system $X \times Y$. Thus a transitive point $x$ in an F-flow is weakly product recurrent. Since such a point is never distal, one concludes that $x$ is indeed weakly product recurrent but not distal.

In [HO, Question 5.3] the authors pose the following natural question:

0.1. Problem. Is every minimal weakly product recurrent point a distal point?

(This was also repeated in [DSY, Question 9.2].)

In this note we show that, here again, the answer is negative. The example is based on a result of [FKS] concerning POD systems and on the existence of doubly minimal systems (see [K] and [W]). A minimal dynamical system $(X, T)$ is called proximal orbit dense (POD) if it is totally minimal and for any distinct points $u$ and...
v in X, there exists an 0 ≠ n ∈ ℤ such that \( \Gamma_n = \{(T^nx, x) : x ∈ X\} \) is contained in \( \overline{O_{T×T}}(u, v) \), the orbit closure of \((u, v)\) in the product system \( X × X \).

A minimal \((X, T)\) is called \textit{doubly minimal} [W] (or a system having \textit{topologically minimal self joinings} in the sense of del Junco [K]) if the only orbit closures of \( T × T \) in \( X × X \) are the graphs \( \Gamma_m = \{(T^mx, x) : x ∈ X\} \), \( m ∈ ℤ \) and all of \( X × X \). Clearly a doubly minimal system is POD. In [FKS] the authors prove the following striking property of POD systems:

0.2. **Theorem.** If \((Y, S)\) is POD then any minimal \((X, T)\) that is not an extension of \((Y, S)\) is disjoint from it.

For the reader’s convenience we reproduce the short proof:

\[Proof.\] Suppose \( Y \) is not a factor of \( X \) and let \( M \) be a minimal subset of \( Y × X \). Since \( X \) is not an extension of \( Y \), there exist \( y, y' ∈ Y \) with \( y \neq y' \) and \( x ∈ X \) such that \((y, x), (y', x) ∈ M\). From the POD property it follows that for some \( z ∈ X \) and \( n \neq 0 \) the points \((y, z)\) and \((T^n y, z)\) are both in \( M \). This implies that \((T^n × id_X)M \cap M \neq \emptyset \) and, as \( M \) is minimal, it follows that \((T^n × id_X)M = M\). Finally, since \( Y \) is totally minimal we deduce that \( M = Y × X \), as required. \( \square \)

We will strengthen this property for doubly minimal systems as follows:

0.3. **Theorem.** If \((Y, S)\) is doubly minimal and \((X, T)\) is any minimal system then the orbit closure of any point \((y, x) ∈ Y × X\) is either all of \( Y × X \) or it is the graph \( \Gamma_π = \{((π(x), x) : x ∈ X\} \) of some factor map \( π : X → Y \).

\[Proof.\] Let \( Y \) be a doubly minimal system. In particular \( Y \) is weakly mixing and has the POD property. Let \( X \) be a minimal system. By [FKS] either \( X \) and \( Y \) are disjoint, or \( Y \) is a factor of \( X \). In the first case the product system \( Y × X \) is minimal.

So we now assume that there is a factor map \( π : X → Y \). We consider an arbitrary point \((y_0, x_1) ∈ Y × X\) and denote \( y_1 = π(x_1) \). We will denote the acting transformation on both \( X \) and \( Y \) by the letter \( T \).

**Case 1:** \( y_1 = T^ny_0 \) for some \( n ∈ ℤ \).

In this case the orbit closure \( \overline{O_{T×T}}(y_0, x_1) \) has the form

\[ \Gamma_πT^n = \{(π(x), T^nx) : x ∈ X\}, \]

and is isomorphic to \( X \).

**Case 2:** \( y_1 \not∈ O(y_0) \).

Recall that by double minimality we have in this case that

\[ \overline{O_{T×T}}(y_0, y_1) = Y × X. \]

Also note that, as the union of the graphs \( \bigcup_{n ∈ ℤ} Γ_n \), where \( Γ_n = \{(T^nx, x) : x ∈ X\} \), is dense in \( X × X \), the union of the graphs \( \bigcup Γ_πT^n \) is dense in \( Y × X \).

Let \((u, v)\) be an arbitrary point in \( Y × X \) and fix an \( ε > 0 \).

(i) Choose a point \( w ∈ X \) and \( m ∈ ℤ \) such that \((π(w), T^mw) \sim (u, v)\).

(ii) Choose a sequence \( n_i ∈ ℤ \) such that for some point \( z ∈ X \)

\[ T^{n_i}(y_0, x_1) → (π(w), z), \quad \text{with} \quad π(z) = T^{n_i}π(w). \]
(iii) Choose a sequence \( k_j \in \mathbb{Z} \) such that 
\[
T^{k_j}z \to T^mw, \quad \text{whence} \quad T^{k_j}\pi(z) = T^{k_j}T^m\pi(w) \to T^m\pi(w),
\]
and 
\[
T^{k_j}\pi(w) \to \pi(w).
\]

Now
\[
\lim_{j} \lim_{i} T^{k_j}T^n_i(y_0, x_1) = \lim_{j} T^{k_j}(\pi(w), z) = (\pi(w), T^m w) \sim (u, v).
\]

Since \( \epsilon > 0 \) is arbitrary we conclude that \((u, v) \in \overline{O_T \times T(y_0, x_1)}\), hence \( \overline{O_T \times T(y_0, x_1)} = Y \times X \). □

As a corollary of this theorem and the fact that there are weakly mixing doubly minimal systems ([K] and [W]) we get a negative answer to Problem 0.1.

First note that a minimal weakly mixing system does not admit a distal point. One way to see this is via the fact that in a minimal weakly mixing system \( X \), for every point \( x \in X \) there is a dense \( G_\delta \) subset \( X_0 \subset X \) such that for every \( x' \in X_0 \) the pair \((x, x')\) is proximal; see [F, Theorem 9.12], or [AK] for an even stronger statement.

0.4. Theorem. **There exists a minimal dynamical system \( Y \) which is weakly mixing (hence in particular does not have distal points) yet it has the property that for every minimal system \( X \) every pair \((y, x)\) is recurrent.**

Proof. Let \( Y \) be a weakly mixing doubly minimal system and \( X \) a minimal system. By [FKS] either \( X \) and \( Y \) are disjoint, or \( Y \) is a factor of \( X \). In the first case the product system \( Y \times X \) is minimal and, in particular, every pair \((y, x)\) is recurrent.

In the second case we have, by Theorem 0.3,
\[
\overline{O_{T \times T}(y, x)} = \Gamma_\pi = \{ (\pi(z), z) : z \in X \},
\]
for a factor map \( \pi : X \to Y \). Again \((y, x) = (\pi(x), x)\) is recurrent and the proof is complete. □

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