CAN WE CONSTRAIN THE NEUTRON-STAR EQUATION OF STATE FROM QPO OBSERVATIONS?

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ABSTRACT

Using an accurate general-relativistic solution for the external spacetime of a spinning neutron star, we assess for the first time the consistency and reliability of geodesic models of quasiperiodic oscillations (QPOs), which are observed in the X-ray flux emitted by accreting neutron stars. We analyze three sources: 4U1608-52, 4U0614+09, and 4U1728-34. Our analysis shows that geodesic models based on pairs of high-frequency QPOs, identified with the azimuthal and periastron precession frequencies, provide consistent results. We discuss how observations of QPO doublets can be used to constrain the equation of state of neutron stars in a way complementary to other observations. A combined analysis of the three sources favours neutron stars with mass above two solar masses and relatively stiff equations of state.

Subject headings: gravitation - neutron stars - accretion, accretion disks - X-rays: binaries

1. INTRODUCTION

Neutron stars (NSs) are the most extreme stellar objects in the universe, providing an excellent laboratory of fundamental physics. Due to their strong gravitational field and to the supra-nuclear densities reached in their inner core, they are a natural laboratory to probe the theory of gravity in the large-curvature regime, and the “ground state” of matter in the non-perturbative regime of quantum chromo-dynamics. Observations of NSs have long been limited by relatively poor data and the complex properties of these objects. Two observational windows on NSs are now available, which provide complementary information. On the one hand, the recent detection of the gravitational wave signal emitted in the coalescence of a NS binary allowed to measure the tidal deformability of these objects, thus allowing an estimate of the NS radius to be obtained in a model-independent way, and constraints on the equation of state (EoS) of NS matter to be placed ([Abbott et al. 2017; Bauswein et al. 2017; [Abbott et al. 2018; Harry & Hendler 2018; Most et al. 2018]. On the other hand, RXTE and some of the present large-area, space-borne X-ray telescopes (such as NICER and AstroSAT/LAXPC) have yielded observations the high-energy emission from accreting NSs in low-mass X-ray binaries (LMXBs) with high time resolution (down to $\mu$s) and throughput, coupled with low to medium spectral resolution. Based on these observations different techniques has been developed and to some extent exploited to derive constraints on the mass-radius relation of NSs and thus their EoS (for a review see [Watts et al. 2016]). The next generation of X-ray missions, such as eXTP ([De Rosa et al. 2019] and Athena ([Barcons et al. 2015]), will feature significantly higher effective area, in turn afford higher precision NS mass-radius measurements.

Quasi-periodic oscillations (QPOs) in the X-ray flux emitted by accreting NSs are a promising diagnostic of the NS structure, since they are believed to originate in the innermost region of the accretion flow. Two kinds of oscillations are observed: the high-frequency (HF) QPOs, reaching frequencies of the order of the kHz and often observed in pairs, and low-frequency (LF) QPOs, with frequency between $\approx 0.01$ to $\approx 50$ Hz. Several models have been proposed to explain such oscillations, which belong to two main classes. In the geodesic models, the observed frequencies are associated to the geodesic motion of the fluid elements in the accretion disk. In these models, the QPO frequencies are associated to combinations of the orbital and epicyclic frequencies of geodesics in the NS spacetime (see e.g. [Stella et al. 1999]; [Stella et al. 1999]; [Abramowicz & Kluzniak 2001]; for a review see [van der Klis 2006]). In other models instead, the QPOs are due to the oscillatory motion of the entire disk (see e.g. [Rezzolla et al. 2003]).

In the relativistic precession model (RPM) ([Stella et al. 1999], the upper and lower HF QPOs coincide with the azimuthal frequency $\nu_\phi$, and the periastron precession frequency, $\nu_{\text{per}} = \nu_\phi - \nu_r$, whereas the LF QPO mode is identified with the nodal precession frequency, $\nu_{\text{nod}} = \nu_\phi - \nu_\theta$. These three QPO signals ($\nu_\phi, \nu_{\text{per}}, \nu_{\text{nod}}$) are assumed to be generated at the same orbital radius. Correlated QPO frequency variations that are observed from individual NSs at different times are well reproduced in the RPM as a result of variations in the radius at which the QPOs are emitted.

Under the assumptions of the geodesic model QPOs produced around accreting NSs are determined by the spacetime metric, which can be characterized—assuming the NS to be stationary—by the NS mass, rotation rate, and multipole moments. Although the multipole moments of NSs form an infinite tower, they are related, to a good approximation, by some “three-hair relations” which are approximately independent of the...
the spacetime around a NS can be accurately described in terms of just three quantities: its mass, its angular momentum, and its quadrupole moment (an explicit form of the spacetime metric in terms of these quantities has been derived in Pappas (2017)). The QPO frequencies in geodesic models can be computed for any stationary and axisymmetric spacetime parametrised in terms of the first five relativistic multipole moments (Ge- roch 1970a,b; Hansen 1974; Ford et al. 1989), i.e., the mass $M$, the angular momentum $J$, the mass quadrupole $M_2$, the spin octupole $S_3$, and the mass hexadecapole $M_4$ (Pappas 2017). The line element is (Papapetrou 1953),

$$ds^2 = f^{-1} \left[ c^2 (d\rho^2 + dz^2) + \rho^2 d\phi^2 \right] - f (dt - \omega d\phi)^2,$$

where $(\rho, z)$ are the Weyl-Papapetrou coordinates and the metric components $f$, $\omega$, and $\zeta$, shown in Appendix A are functions of the multipole moments and of the coordinates $(\rho, z)$.

To fully describe the stellar structure, the next step is to prescribe the right set of multipole moments. Recent work (Pappas & Apostolatos 2014; Yagi et al. 2014) has shown that for NSs the first few relativistic multipole moments can be expressed as

$$M_2 = -\alpha j^2 M^3, \quad S_3 = -\beta j^3 M^4, \quad M_4 = \gamma j^4 M^5,$$

where $M$ is the mass and $j = J/M^2$ is the spin parameter, with $J$ being the intrinsic angular momentum of the star. For NSs the coefficients $\alpha$, $\beta$, and $\gamma$ can be much larger than 1 (see Doneva & Pappas (2018) for a review). Furthermore it has been shown that for realistic EoS the higher order NS multipole moments (higher than $M_2$) can be expressed in terms of the quadrupole, the angular momentum and the mass (Pappas & Apostolatos 2014; Stein et al. 2014; Yagi et al. 2014), we refer the reader to Pappas (2017) for a more detailed discussion. The spin octupole and the mass hexadecapole of a NS are related to the quadrupole by the relations

$$y_1 = -0.36 + 1.48 x^{0.65},$$
$$y_2 = -4.749 + 0.27613 x^{1.5146} + 5.5168 x^{0.2229},$$

where $y_1 = \sqrt{-S_3} = \sqrt{\beta}, \quad y_2 = \sqrt{M_4} = \sqrt{\gamma}, \quad x = \sqrt{-M_2} = \sqrt{\alpha}, \quad M_n = \frac{M_n}{M^{n+1}}$, $S_n = \frac{S_n}{M^{n+2}}$ are the (dimensionless) reduced moments. Therefore, the description of the spacetime and the various geodetic frequencies will only depend on three parameters: the mass $M$ (that we express in units of kilometers in $G = c = 1$ units), the dimensionless spin parameter $j$, and the dimensionless reduced quadrupole $\alpha = -M_2$. This is particularly relevant for our analysis, as it reduces the number of intrinsic parameters to be constrained.

The spacetime metric considered in this work is equivalent to that introduced in Pappas (2017), with the further feature that it reduces to the Schwarzschild metric in the limit $j \rightarrow 0$ (see Appendix A for technical details).

The epicyclic frequencies of the Hartle-Thorne model for slowly-rotating NSs up to second-order in the rotation rate, have been recently analysed in detail by Urbanova et al. (2019). Our metric extends that used in Urbanova et al. (2019) for any spin and quadrupole moment.
have a subdominant effect on inferred values of the source parameters.

We sample the posterior distribution [5] using a Markov Chain Monte Carlo (MCMC) approach, based on the Metropolis-Hastings algorithm [Gilks et al. 1996]. The random jump within the parameter space is chosen according to a multivariate Gaussian distribution, whose covariance matrix is continuously updated through a Gaussian adaptation scheme [Miller & Sbalzarini 2010], which increases the mixing of the chains and boost the convergence to the target distribution. For each set of data, we run 4 independent chains of $2 \times 10^6$ samples, generally discarding the first 10% of the simulation as a burn in. The convergence of the processes is then assessed by a standard Rubin test.

We consider flat prior distributions for all the parameters within the following ranges $M \in [0.7, 3]M_\odot$, $\chi \in [0, 1]$, $\alpha \in [1.5, 15]$, $r_i \in [R_{\text{NS}}, 15M]$. Note that we set the prior on $r_1$ such that it is always larger than the stellar circumferential radius $R_{\text{NS}}$. The latter can be expressed with very good accuracy (within a few percent) as a function of $M, j$, and $\alpha$:

$$R_{\text{NS}}/M = \sum_{i=0}^3 [B_i j^i + A_i j^i \alpha N_i^2/2 + C_i j^i \alpha N_2^2/2],$$

(7)

were $(B_i, A_i, C_i, N_i, N_2)$ are numerical coefficients that can be found in [Pappas 2015].

Once the $P(\vec{\theta}|\vec{O})$ is sampled by the MCMC, we derive the probability distribution of the intrinsic source parameters, by marginalising the joint posterior distribution over the emission radii:

$$P(M, \chi, \alpha) = \int P(\vec{\theta}|\vec{O})d\theta_1 \ldots d\theta_n. \quad (8)$$

In the following sections we will apply this analysis to the three astrophysical sources ($4U1608-52$, $4U0614+09$, and $4U1728-34$), and compute $P(M, \chi, \alpha)$ using various combinations of the QPO frequencies shown in Table 2. This step is a crucial point of our analysis as it provides a consistency test for the RPM using an accurate, fully-relativistic model of the epicyclic frequencies. Indeed, if the assumptions of the method are reliable, different sets of doublets for a given source must provide probability distributions for masses, spin and quadrupole moments which are consistent with each other within their uncertainties. On the contrary, an inconsistency of the different probability distributions might signal systematics in the geodesic model.

4. RESULTS

We analyse HF QPOs doublets, i.e. those QPOs that correspond to the periastron and the azimuthal frequencies. In this case the MCMC requires at least three sets of observations, i.e. $(\nu_r, \nu_\phi)_{i=1,2,3}$, to be solved in terms of $(M, \alpha, j)$ and three emission radii $(r_1, r_2, r_3)$. As a representative case, we first focus on 4U1608-52. We show the posterior probabilities in Fig. 1 for different combinations of the doublets (see Table 4 for their numerical values).

Our analysis shows good agreement between the posteriors obtained using different datasets. While the mass distributions shows the largest shift between the peaks of the marginal distributions, the spin parameter distributions show the best agreement. We note, however, that the quadrupole moment of the star remains essentially unconstrained by the MCMC, its probability distributions being almost flat between $\alpha \sim 2.5$ and $\alpha \sim 4$. This is expected for two reasons: first, the quadrupole moment gives a subleading contribution to the spacetime metric relative to the mass and spin; furthermore, its effect on the QPO frequencies is larger on $\nu_\phi$ (which we do not include in our analysis), while $\nu_r$ and $\nu_\perp$ are only weakly affected by its variation.

The box plot shown in Fig. 2 supports the reliability of the geodesic model: it can be clearly seen how the median of the distributions are all consistent with each other, and the interquartile ranges overlap within good accuracy.

The above findings motivate a more general study of this source in which we take into account at the same time all the doublets listed in Table 2, i.e. setting $N_{\text{obs}} = 7$ in Eq. (6). We show the results of this analysis in the triangle plot of Fig. 3. The diagonal and off-diagonal panels show the marginalised and the 2D joint distributions of $M, j$, and $\alpha$, respectively. The mass is the parameter that we determine with the best accuracy: at 90% confidence level we find $M \sim [1.92, 2.32]M_\odot$, with a median of $M = 2.07M_\odot$. The bottom row of Fig. 3 shows that the inclusion of all the datasets does not modify quantitatively the constraints on $\alpha$, with the flat posterior distribution within $\sim [1.50, 3.30]$ at 90%. Our analysis also suggests a relatively low spin value for $4U1608-52$, compatible with zero, and within the interval $j \sim [0, 0.26]$ and a median $j \sim 0.1$. As shown by the 2D distribution $P(M, j)$ the spin parameter is also strongly correlated with the stellar mass.

The MCMC also leads to constrain the emission radius corresponding to each series of frequencies. For the analysed configurations we find that the orbital radii are located deep within the gravitational field of the NS, with $r_j/M \lesssim 6$ for all the frequency doublets. Figure 4 shows the distribution of the 90% confidence intervals for the 7 values. Note that these constraints also provide a first upper bound on the stellar radius. Requiring that the oscillations within the disk occur at orbital distances larger than $R_{\text{NS}}$ gives the limit $R_{\text{NS}} \lesssim 4.5M$ at 90% confidence level.

The analysis of $4U0614+09$ and $4U1728-34$ yields similar results compared to the first source. Figure 5 shows that for each system a general agreement exists when the Bayesian inference is performed using different series of doublets. $4U1728-34$ seems to show the largest spread in the posterior distribution of the intrinsic parameters when the full dataset of frequencies is taken into account. Both $4U0614+09$ and $4U1728-34$ are characterised by a mass distribution which peaks around $\sim 2M_\odot$. The posteriors of such parameter are shown in Fig. 6, which also shows how different combinations of doublets seem to be consistent with the same physical systems, and con-
Fig. 1.— Marginalised posterior probabilities of \((M, j, \alpha)\) for 4U1608-52 computed using different series of high frequency QPO doublets.

Fig. 2.— Box and whiskers plots for \((M, j, \alpha)\), corresponding to the data shown in Fig. 1. White vertical lines in each coloured box mark the median of the parameters. The edges of the box identify the upper and lower quartiles, while the ends of the whiskers yield the maximum and minimum observation for the dataset.

Fig. 3.— Triangle plot for the posterior of the intrinsic parameters of 4U1608-52. Diagonal and off-diagonal panels refer to marginalised and 2D joint posterior distributions, respectively. Dashed and solid curves identify contours at 68% and 90% confidence intervals, while coloured dots represent the actual points sampled by the MCMC.

Fig. 4.— 90% credible interval for the posterior distribution of the radius of emission for 4U1608-52.

Fig. 5.— Same as in Fig. 2 but for the sources 4U0614+09 (top) and 4U1728-34 (bottom).

4U1608-52. The median of all the intrinsic parameters of the sources, together with their 90% uncertainties, are shown in Table 4.

Relation (7) allows to estimate the NS radius using the inferred values of \((M, j, \alpha)\). We use the numerical results obtained through the analysis of the full series of doublets of 4U1608-52, 4U0614+09, and 4U1728-34 to build the joint 2D distribution of \(P(M, R_{NS})\). The contour plot of Fig. 7 shows the 90% interval of the mass-radius poste-
1.8 2.0 2.2 2.4 2.6 2.8 3.0
M[M⊙]

Fig. 6.— Posterior distributions for the mass of the binary systems 4U0614+09 and 4U1728–34. Dashed curves refer to different series of doublets, while the solid black line identify the result obtained by fitting all the available datasets.

| source     | $M/M_\odot$ | $j$  | $\alpha$ |
|------------|-------------|------|----------|
| 4U1608–52  | $2.07^{+0.29}_{-0.45}$ | $0.10^{+0.16}_{-0.20}$ | $2.56^{+0.74}_{-1.04}$ |
| 4U0614+09  | $2.10^{+0.35}_{-0.15}$  | $0.20^{+0.24}_{-0.04}$  | $2.22^{+0.04}_{-1.18}$  |
| 4U1728–34  | $2.11^{+0.34}_{-0.27}$  | $0.27^{+0.24}_{-0.27}$  | $2.00^{+0.46}_{-0.46}$  |

TABLE 1
MEDIAN AND 90% VALUES FOR THE INTRINSIC PARAMETERS OF THE LMXB SOURCES ANALYSED.

Fig. 7.— 90% 2D credible interval and marginalised distributions of the mass and radius for the 3 sources analysed. Coloured curves represent the $M$-$R$ profiles for some representative EoS still compatible with the gravitational-wave event GW170817 (Abbott et al. 2018), with the existence of NSs with $\sim 2M_\odot$ (Antoniadis et al. 2013).

5. CONCLUSIONS

By using an accurate description of the spacetime around a NS in terms of three independent parameters (mass, spin, and quadrupole moment), we have developed a novel method to assess the consistency and the reliability of geodesic models of QPOs observed in the X-ray flow from accreting NSs.

Using data from the LMXBs 4U1608–52, 4U0614+09, and 4U1728–34, we found that a geodesic model in which the twin HF QPOs are identified with the azimuthal and precession frequencies provides consistent results for all three sources considered in this work. Further analysis on more sources and larger HF QPOs datasets are required to test our model further, and confirm our results.

We were able to infer consistent measurements of the mass, spin, and mass quadrupole moment of the three sources we considered. These quantities can be translated in a measurement of the NS radius, which can be used to constrain the NS EoS with QPO observations. The constraints obtained in the mass-radius diagram (Fig. 7), based on observational errors on the QPOs from RTXE are not very stringent, but are already indicative of NSs with mass above two solar masses and with a relatively stiff EoS. While these bounds are not in tension with those from GW170817 (Abbott et al. 2018), they tend to support stiffer EoS, i.e. larger NS radii. We remark that a measurement of the mass quadrupole moment can also be used to directly constrain the EoS, without an explicit determination of the radius (Pappas 2012).

We have focused on a geodesic model, in particular the RPM and a modified version thereof which uses only the HF QPOs. A full exploitation of the RPM would also require to include a low frequency component which is associated with the nodal precession frequency (similarly to what has been done by Stella & Vietri (1999) and Motta et al. (2014)). The latter depends strongly on the stellar spin and the quadrupole moment and therefore could provide further information on the NS structure. An detailed analysis based on QPOs triplets from different LMXBs is already ongoing, and will be presented in a follow-up publication.

Our method can be directly applied to any geodesic model, for instance the epicyclic resonance model (Abramowicz et al. 2004). It can also be applied to diskoseismic models related to geodesics (see Tsang & Pappas 2016). Finally, it might be also applied to non-geodesic models (Rezzolla et al. 2003). In the latter, the QPOs are interpreted as fluid oscillations in the disk, whose geometry is nonetheless affected by the spacetime of the NS. Although in this case the effects of the multipole moments are expected to be less important, using an accurate spacetime geometry might improve also non-geodesics models.

In summary, we have assessed the reliability of the geodesic model of NS QPOs based on the doublets of HF QPOs (Stella & Vietri 1999). We argue that the consistency and reliability of geodesic models for QPOs can be directly tested with current observations, and that the prospect for these tests and for constraints on the NS EoS with QPOs will dramatically improve with the upcoming X-ray observatories. The improved capabilities of the next generation X-ray missions such as eXTP (De Rosa et al. 2019) and Athena (Barcons et al. 2015), in particular their significantly higher sensitivity, will allow us to detect a significantly higher number of HFQPOs. This will allow us to test our model further and to place more stringent constraints to the mass-radius diagram, which will be then compared with the bounds set by other di-
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APPENDIX

A. METRIC COMPONENTS OF ROTATING NEUTRON STARS

In this appendix we present the explicit form of the metric functions introduced in Sec. 2 for the line element given in Eq. (1), which describes the spacetime around a rotating NS. We refer the reader to Pappas (2017) for further details. The components of the metric are given as functions of the coordinates (ρ, z), and of the multipole moments, and are given by the following expressions:

\[
f(\rho, z) = 1 - \frac{2M}{\sqrt{\rho^2 + z^2}} + \frac{2M^2}{\rho^2 + z^2} + \frac{(M_2 - M^3)}{(\rho^2 + z^2)^{3/2}} + \frac{2z^2}{(\rho^2 + z^2)^3} - \frac{2}{M} \frac{\rho^2}{2(\rho^2 + z^2)^{9/2}} + \frac{B(\rho, z)}{28(\rho^2 + z^2)^{9/2}} + \frac{A(\rho, z)}{14(\rho^2 + z^2)^3},
\]

\[
\omega(\rho, z) = \frac{2J_2 \rho^2}{(\rho^2 + z^2)^{3/2}} - \frac{2J_1 M \rho^2}{(\rho^2 + z^2)^{7/2}} + \frac{F(\rho, z)}{2(\rho^2 + z^2)^{11/2}} + \frac{H(\rho, z)}{4(\rho^2 + z^2)^{11/2}} + \frac{G(\rho, z)}{4(\rho^2 + z^2)^{11/2}}.
\]

\[
\zeta(\rho, z) = \frac{\rho^2}{4(\rho^2 + z^2)^4} \left( J_2 (\rho^2 - 2z^2) + M (M_3 + 3M_2) (\rho^2 - 4z^2) + M_2 \rho^2 \right),
\]

where,

\[
A(\rho, z) = 8\rho^2 z^2 \left( 24J_2 M + 17M_2^2 M + 21M_4 + \rho^4 (-10J_2 M + 7M_5 + 32M_2 M^2 - 21M_4) + 8\rho^4 (20J_2 M - 7M_5 - 2M_2 M^2 - 7M_4) \right),
\]

\[
B(\rho, z) = \rho^4 \left( 10J_2 M^2 + 10M_2 M^3 + 21M_4 M + 7M_2^2 \right) + 4z^4 \left( -40J_2 M^2 - 14JS_3 + 7M_5 + 30M_2 M^3 + 14M_4 M + 7M_2^2 \right),
\]

\[
H(\rho, z) = \rho^2 z^2 \left( J (M_2 - 2M^3) - 3MS_3 \right) + \rho^4 \left( J M_2 + 3MS_3 \right),
\]

\[
G(\rho, z) = \rho^2 \left[ -j^3 (\rho^4 + 8\rho^4 - 12\rho^2 z^2) + JM \left( (M_3 + 2M_2) \rho^4 - 8(3M_3 + 2M_2) \rho^4 \right) + 4 \left( M_3 + 10M_2 \right) \rho^2 z^2 + M_2 S_3 \left( 3\rho^4 - 40z^4 + 12\rho^2 z^2 \right) \right],
\]

\[
F(\rho, z) = \rho^4 \left( S_3 - JM^2 \right) - 4\rho^2 z^2 \left( JM^2 + S_3 \right).
\]

The spacetime defined above can be given in an even more convenient form so as to have the right Schwarzschild limit when the rotation goes to zero, i.e., \( j \to 0 \). To this aim, we resum the expansions of \( f(\rho, z) \) and \( \zeta(\rho, z) \), using the variable \( r = \sqrt{\rho^2 + z^2} \), such that when the rotation vanishes the metric coincides with the exact Schwarzschild solution. With this procedure, we obtain:

\[
f(\rho, z) = 1 - \frac{4M}{r} + \frac{\alpha j^4 M^6 \left( \rho^2 - 2z^2 \right)^2}{r^{10}} + \frac{2\beta j^4 M^6 z^2 \left( 2z^2 - 3\rho^2 \right)}{r^{10}} - \frac{\gamma j^4 M^5 \left( 3\rho^4 + 8z^4 - 24\rho^2 z^2 \right) \left( -2Mr + r^2 \right)}{4r^{11}} - \frac{j^2 M^4}{14r^{11}} \left[ 2M^2 \left( -5\rho^4 + 80z^4 + 54\rho^2 z^2 \right) r \right] - M \left( 20z^2 - \rho^2 \right) \left( 5\rho^2 + 4z^2 \right) r^2 + 28z^2 r^5 \right] + \frac{\alpha j^2 M^3}{7r^{11}} \left[ M^3 \left( -5\rho^4 - 60z^4 + 96\rho^2 z^2 \right) + 2M^2 \rho^2 \left( -4\rho^4 + 22z^2 - 17\rho^2 z^2 \right) + 14M_4 \rho^6 - 2z^2 - 3\rho^2 z^2 + 7 \left( 2z^2 - \rho^2 \right)^2 \right) r^5 \right] + \frac{\beta^2 M^5 \rho^2 \left( 1 - 3\alpha \right) \rho^2 + 4(3\alpha - 2)z^2}{4r^{8}} \right].
\]

\[
\zeta(\rho, z) = \frac{1}{2} \log \left( \frac{r^2 - M^2 + r_+ r_-}{2r_+ r_-} \right) + \frac{j^2 M^4 \rho^2 \left( 1 - 3\alpha \right) \rho^2 + 4(3\alpha - 2)z^2}{4r^8}.
\]
where \( r_\pm = \sqrt{(M \pm z)^2 + \rho^2} \) and we have used the definitions (2) for the moments. When \( j \rightarrow 0 \), from Eq. (A2) it follows that \( \omega \rightarrow 0 \) and the functions \( f \) and \( \zeta \) take their Schwarzschild form. This metric, as the previous one, is accurate up to \( M_j \) in the moments and up to order \( O(M^0/r^0) \) with respect to the vacuum field equations.

It is worth remarking that while the spacetime is given in Weyl-Papapetrou coordinates, which differ from the usual Schwarzschild-like or quasi-isotropic ones, the various geodesic frequencies are coordinate independent quantities, while the relevant radii on the equatorial plane can be expressed in terms of the circumferential radius which is also a geometric and coordinate independent quantity.

## B. QPO FREQUENCIES

In this appendix we show the QPO frequencies, and the corresponding relative uncertainties, for the three LMXB systems considered in this paper. Frequencies come from RXTE observations and were taken from [van Doesburgh & van der Klis (2017)](#).

### 4U1608-52

| # | \( \nu_\phi \) | \( \sigma^{(\phi)}_{\nu_\phi} \) | \( \sigma^{(-)}_{\nu_\phi} \) | \( \nu_{per} \) | \( \sigma^{(\perp)}_{\nu_{per}} \) | \( \sigma^{(-)}_{\nu_{per}} \) |
|---|---|---|---|---|---|---|
| 1 | 849.92 | 6.94 | 6.53 | 535.32 | 15.4 | 23.1 |
| 2 | 940.93 | 12.1 | 12.5 | 655.78 | 2.15 | 2.07 |
| 3 | 958.61 | 8.19 | 8.36 | 654.7 | 0.23 | 0.23 |
| 4 | 976.6 | 6.89 | 7.00 | 674.76 | 1.26 | 1.24 |
| 5 | 1034.6 | 10.6 | 10.3 | 769.32 | 0.83 | 0.79 |
| 6 | 1041.1 | 7.04 | 7.32 | 774.82 | 0.83 | 0.81 |
| 7 | 1053.1 | 11.2 | 13.6 | 740.61 | 0.59 | 0.54 |

### 4U0614+09

| # | \( \nu_\phi \) | \( \sigma^{(\phi)}_{\nu_\phi} \) | \( \sigma^{(-)}_{\nu_\phi} \) | \( \nu_{per} \) | \( \sigma^{(\perp)}_{\nu_{per}} \) | \( \sigma^{(-)}_{\nu_{per}} \) |
|---|---|---|---|---|---|---|
| 1 | 957.11 | 8.97 | 9.24 | 636.61 | 1.98 | 2.1 |
| 2 | 959.41 | 7.06 | 7.73 | 649.9 | 1.61 | 1.8 |
| 3 | 1076.4 | 11.2 | 14.4 | 749.84 | 1.77 | 1.68 |
| 4 | 1103.8 | 10.7 | 11.1 | 761.02 | 1.21 | 1.29 |
| 5 | 1116.7 | 16.9 | 21.7 | 753.15 | 5.67 | 5.23 |

### 4U1728-34

| # | \( \nu_\phi \) | \( \sigma^{(\phi)}_{\nu_\phi} \) | \( \sigma^{(-)}_{\nu_\phi} \) | \( \nu_{per} \) | \( \sigma^{(\perp)}_{\nu_{per}} \) | \( \sigma^{(-)}_{\nu_{per}} \) |
|---|---|---|---|---|---|---|
| 1 | 717.9 | 5.09 | 5.04 | 377 | 18.6 | 15 |
| 2 | 873.25 | 3.36 | 3.3 | 538.38 | 37.4 | 37.1 |
| 3 | 972.49 | 5.68 | 5.51 | 641.5 | 3.66 | 4.2 |
| 4 | 1089.2 | 3.85 | 3.97 | 752.42 | 0.67 | 0.66 |
| 5 | 1091.4 | 10.6 | 10.8 | 740.48 | 0.84 | 0.87 |
| 6 | 1107.3 | 9.99 | 9.72 | 778.22 | 2.85 | 2.64 |
| 7 | 1118.8 | 7.29 | 7.53 | 801.78 | 10.8 | 11 |
| 8 | 1149.9 | 1.58 | 1.16 | 816.36 | 1.08 | 1.21 |

### TABLE 2

QPO FREQUENCIES (WITH EXPERIMENTAL ERRORS \( \sigma^{(\pm)} \)) OBSERVED FOR THE THREE SOURCES ANALYSED IN THIS PAPER, 4U1608-52, 4U0614+09, AND 4U1728-34. ACCORDING THE RPM, \((\nu_\phi, \nu_{per})\) CORRESPOND TO THE HF QPO DOUBLETS.

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