The Local Bifurcation of an Eco-Epidemiological Model in the Presence of Stage-Structured with Refuge

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ABSTRACT
In this paper, we establish the conditions of the occurrence of the local bifurcations, such as saddle node, transcritical and pitchfork, of all equilibrium points of an eco-epidemiological model consisting of a prey-predator model with SI (susceptible-infected) epidemic diseases in prey population only and a refuge-stage structure in the predators. It is observed that there is a transcritical bifurcation near the axial and free predator equilibrium points, near disease-free equilibrium point is a saddle-node bifurcation and near positive (coexistence) equilibrium point is a saddle-node bifurcation, a transcritical bifurcation and a pitchfork bifurcation. Further investigations for Hopf bifurcation near coexistence equilibrium point are carried out. Finally, numerical simulations are used to illustrate the occurrence of the local bifurcations of this model.

Keywords: Prey-predator, Eco-epidemiological, SI epidemic disease function, Local bifurcation model, Refuge, Stag-structure, Lyapunov

التفرع المحلي لنموذج بيئي وبائي بوجود مراحل عمرية وملجاً

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الخلاصه
في هذه البحث تم دراسة ظروف حدوث التفرع المحلي مثل (عند سرجية، تحول حرج، مزارة العشب) بالقرب من كل نقطة من نقاط التوازن لنموذج بيئي وبائي معروض لدى فرد فربة ومتزمر وجود مرض معد في مجتمعなし SI مع ملجاً للفربيسة ومراحل عمرية في مجتمع المفترس. لوحظ وجود تفرع من النوع (حول حرج) قرب نقطة التوازن $E_2$ و $E_1$ وبيتاً بالقرب من نقطة التوازن $E_3$ كفرع من النوع (عند سرجية). ولكن بالقرب من النقطة الموجبة $E_4$ تم حدوث تفرع من النوع (عند السرج)، تحول حرج، مزارة العشب. كذلك تم دراسة تفرع هبيف بالقرب من النقطة الموجبة $E_4$ فقط. وأخيراً تم استخدام المحاكاة العددية لتأكيد النتائج التحليلية.

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1. Introduction

Bifurcation theory is the mathematical study of changes in the qualitative asymptotic structure of a dynamical system [1, 2]. It attempts to explain various phenomena that have been discovered in the natural sciences over the centuries. Performing a bifurcation analysis is often a powerful way to analyze the properties of such systems. The prey and predator model is an important topic at present, as it is used to solve many problems in ecology, nature and other sciences. The prey system includes several interactions, including competition co-existence and stage-structured [3, 4]. The ecological models of the age stage are more logical than models that do not contain phase structure. In addition, there are several factors that affect the system, for example, refuge, disease, shelter and others. Sometimes, differences in any parameter in the system can lead to complex behaviors that lead to system instability, causing a bifurcation that is the main qualitative change in the behavior of a dynamic system as a result of changing one of its coefficients. The bifurcation is divided into two principal classes, local and global. Local bifurcation can be analyzed through changes in the local stability properties of equilibria or periodic orbits. While global bifurcation occurs when periodic orbits collide with equilibria. This causes changes in the topology of the trajectories in phase space which cannot be confined to a small neighborhood, as is the case with local bifurcation. These bifurcations happen when one varies a single parameter [5-9]. On the other hand, Perko [10] established the conditions of the occurrence of local bifurcation, such as saddle-node, transcritical, pitchfork, period -doubling, and Hopf bifurcation near coexistence equilibrium point. The Hopf bifurcation is a local bifurcation in which the equilibrium point of a dynamical system loses stability, as a pair of complex conjugate eigenvalues of the linearization around the equilibrium point cross the imaginary axis of complex plane. This type is also known as Poincare Andronov Hopf bifurcation.

In this paper, an application of Sotomayor’s theorem [11] for local bifurcation is used to study the occurrence of local bifurcation near the equilibria. Furthermore, Hopf bifurcation near positive equilibrium point conditions is established for a mathematical model previously proposed by Kafi and Majeed [12].

2. The mathematical models [12]

In this section, an eco-epidemiological model is proposed for study. The model consists of a prey, whose total population density at time $T$ is denoted by $N(T)$, interacting with a stag-structured predator. It is assumed that the prey population is infected by an infectious disease with the prey refuge. Now, the following assumptions are adopted in formulating the basic eco-epidemiology model:

1. There is an $SI$ epidemic disease in the prey population which divides the prey population into two classes, namely $S(T)$ that represents the density of susceptible prey at time $T$ and $I(T)$ which represents the density of infected prey at time $T$. Therefore, at any time $T$, we have $N(T) = S(T) + I(T)$. The predator population is divided into two classes, namely $S(T)$ that represents the density of immature predator at time $T$ and $W(T)$ which represents the density of mature predator at time $T$.

2. It is assumed that only susceptible prey $S$ is capable of reproducing in a logistic growth with a carrying capacity $K > 0$ and an intrinsic growth rate constant $r > 0$. The infected prey $I$ is removed before having the possibility of reproducing. However, the infected prey population $I$ still contribute with $S$ to population growth towards the carrying capacity.

3. The disease is transmitted within the same species by contact with an infected individual at infection rates of $\alpha > 0$ for the prey.

4. The mature predator $w(T)$ consumes the susceptible prey $S(T)$ and the infected prey $I(T)$, according to Holling type-II of functional responses with a maximum attack rate of $a > 0$, and a half saturation rate of $b > 0$ for the susceptible prey, as well as a maximum attack rate of $c > 0$ and a half saturation rate of $d > 0$ for the infected prey.

5. The disease may causes mortality with a constant mortality rate of $d_1 > 0$ for the infected prey.

6. The immature predator depends completely in its feeding on his parents, so that it feeds on the portion of the uptake food by the mature predator from the susceptible and infected prey, with portion rates of $0 < n_1 < 1$ and $0 < n_2 < 1$ associated with uptake rates of $0 < e_1 < 1$ and $0 < e_2 < $
The immature predator individuals grow up and become mature predator individuals with a grown up rate of $u > 0$.

7. There is a type of protection of the prey species from facing predation by the mature predator with refuge rate constants $m_1 \in (0,1)$ and $m_2 \in (0,1)$ for susceptible and infected prey, respectively.

8. Finally, in the absence of the predator facing death with natural death rates of $d_2 > 0$ and $d_3 > 0$ for immature and mature predators, respectively.

Therefore, the dynamics of the above proposed model can be represented by the following set of first order nonlinear differential equations.

$$\begin{align*}
\frac{dS}{dT} &= rs\left(1 - \frac{S + I}{k}\right) - \alpha SI - \frac{a(1-m_1)SW}{b + (1-m_1)S} \\
\frac{dI}{dT} &= \alpha SI - \frac{e_1(1-m_1)(1-m_2)SW}{b + (1-m_2)I} - d_1 I \\
\frac{dZ}{dT} &= e_1a(1-n_1)(1-m_1)SW + \frac{e_2c(1-n_2)(1-m_2)IW}{d + (1-m_2)I} - uZ - d_2 Z \\
\frac{dW}{dT} &= e_1a_n(1-m_1)SW + \frac{e_2cn_2(1-m_2)IW}{d + (1-m_2)I} + uZ - d_3 W
\end{align*}$$

For the simplicity of the above model, it is assumed that $m_1 = m_2 = m$. Then dimensional form of system (1) can be written as:

$$\begin{align*}
\frac{ds}{dt} &= s \left[ 1 - s - (1 + a_3) \frac{a_2w}{a_2w + (1-m)sw} \right] = f_1(s, i, z, w) \\
\frac{di}{dt} &= i \left[ a_1s - \frac{a_4 w}{a_5 + (1-m)i} - a_6 \right] = f_2(s, i, z, w) \\
\frac{dz}{dt} &= a_7sw + \frac{a_8iw}{a_5 + (1-m)i} - a_9z - a_{10}z = f_3(s, i, z, w) \\
\frac{dw}{dt} &= a_{11}sw + \frac{a_{12}iw}{a_5 + (1-m)i} + a_9z - a_{13}w = f_4(s, i, z, w)
\end{align*}$$

with $s(0) \geq 0$, $i(0) \geq 0$, $z(0) \geq 0$ and $w(0) \geq 0$. It is observed that the number of parameters was reduced from sixteen in system (1) to fourteen in system (2). Obviously, the interaction functions of system (2) are continuous and have continuous partial derivatives on the following positive four dimensional spaces.

$$R_4^+ = \{(s, i, z, w) \in R^4 : s(0) \geq 0, i(0) \geq 0, z(0) \geq 0, w(0) \geq 0\}$$

Therefore, these functions are Lipschitzian on $R_4^+$, and hence the existence and uniqueness of solutions for system (2) are guaranteed. Further, all the solutions for system (2) with non-negative initial conditions are uniformly bounded, as demonstrated in the following theorem.

**Theorem 1 [12]**

All the solutions of system (2) are uniformly bounded.

3. **Existence and stability of equilibrium points [12]**

System (2) has at most five equilibrium points, which are mentioned in the following:

- The equilibrium point $E_0 = (0,0,0,0)$, which is known as the vanishing point that always exists and unstable.
The axial equilibrium point \( E_1 = (1,0,0,0) \), which exists unconditionally. Also, it is a locally asymptotically stable if the following conditions hold:

\[
\frac{a_6}{a_1} > \frac{a_1}{a_2} > (a_{13} + a_9 + a_{10}),
\]
\[
a_{13}(a_9 + a_{10}) > \frac{a_{11}(a_9 + a_{10}) + a_9 a_7}{a_3 + (1 - m)}.
\]

The free predators’ equilibrium point \( E_2 = (s, i, 0, 0) \) which exists uniquely in the \( \text{Int. } R^2_x \) (Interior of \( R^2_x \)) of \( si - plane \), provided that:

\[
a_6 < a_1,
\]

where

\[
\begin{align*}
    \sigma &= \frac{a_6}{a_1}, \\
    \tilde{i} &= \frac{a_1 - a_6}{a_1(1 + a_1)}.
\end{align*}
\]

And it is a locally asymptotically stable if the following conditions hold:

\[
2\sigma + (1 + a_1) \tilde{i} > 1,
\]
\[
\sigma < \frac{a_6}{a_1},
\]
\[
a_{13} > \frac{a_{11} \sigma}{a_3 + (1 - m) \sigma} + \frac{a_{12} \tilde{i}}{a_5 + (1 - m) \tilde{i}},
\]
\[
\varphi_1 > \varphi_2,
\]

where:

\[
\varphi_1 = (a_9 + a_{10}) \left[ a_{13} - \left( \frac{a_{11} \sigma}{a_3 + (1 - m) \sigma} + \frac{a_{12} \tilde{i}}{a_5 + (1 - m) \tilde{i}} \right) \right],
\]
\[
\varphi_2 = a_9 \left[ a_{13} - \left( \frac{a_{11} \sigma}{a_3 + (1 - m) \sigma} + \frac{a_{12} \tilde{i}}{a_5 + (1 - m) \tilde{i}} \right) \right].
\]

The disease-free equilibrium point \( E_3 = (\bar{s}, 0, \bar{z}, \bar{w}) \) exists uniquely in the interior \( R^3_x \) of \( szw - space \) where,

\[
\bar{w} = \frac{(1 - \bar{s})[a_4 + (1 - m)\bar{s}]}{a_2},
\]
\[
\bar{z} = \frac{(a_9 + a_{10}) [a_3 + (1 - m)\bar{s}]}{a_3 a_{13}(a_9 + a_{10})},
\]
\[
\bar{s} = \frac{(a_9 + a_{10})[a_{11} - a_{13}(1 - m)] + a_7 a_9}{a_2 a_3},
\]

if the following conditions hold:

\[
\bar{s} < 1, \\
\hat{s} > a_{13}(1 - m).
\]

And it is a locally asymptotically stable if the following conditions hold:

\[
\bar{s} < \frac{a_2 a_3}{a_1 a_5},
\]
\[
2\bar{s} + \frac{a_2 a_3}{a_3 + (1 - m)\bar{s}} > 1,
\]
\[
a_{13} > \frac{a_{11} \sigma}{a_3 + (1 - m)\bar{s}},
\]
\[
n_{23} n_{44} > n_{43} n_{34},
\]
\[
n_{44}(n_{11} + n_{44}) > n_{31} n_{43}.
\]

Finally, the positive (coexistence) equilibrium point \( E_4 = (\bar{s}, \bar{i}, \bar{z}, \bar{w}) \) exists in the \( \text{Int. } R^4_x \) if the following conditions hold:

\[
a_2 a_6(1 - m) < a_3 a_4(1 + a_1),
\]
\[
a_{12} > a_{13}(1 - m).
\]
The characteristic equation of $f(E_0)$ is given by:
\[ \lambda^4 + H_1 \lambda^3 + H_2 \lambda^2 + H_3 \lambda + H_4 = 0, \]
where:
\[ H_1 = -(y_0 + y_1), \]
\[ H_2 = y_0y_1 + y_2 + y_3 - (y_4 + y_5 + y_6 + y_7), \]
\[ H_3 = -(y_0(y_2 - y_4) + y_1(y_3 - y_6) - m_{24}(y_8 - y_9 + y_{10}) + y_6y_{11} + m_{14}(y_{12} - y_{13} + y_{14})], \]
\[ H_4 = (y_2 - y_4)(y_3 - y_6) + (y_9 - y_{10})(y_{15} - y_{16}) + y_{17}(y_4 - y_{18}) + y_{19}(y_{13} - y_{14}). \]
with
\[ y_0 = m_{11} + m_{22}, \quad y_1 = m_{33} + m_{44}, \quad y_2 = m_{33}m_{44}, \quad y_3 = m_{11}m_{22}, \quad y_4 = m_{34}m_{43} > 0, \quad y_5 = m_{24}m_{42} < 0, \quad y_6 = m_{12}m_{21} < 0, \quad y_7 = m_{14}m_{41} < 0, \quad y_8 = m_{11}m_{42}, \quad y_9 = m_{32}m_{43} < 0, \quad y_{10} = m_{42}m_{33} < 0, \quad y_{11} = m_{14} + m_{41}, \quad y_{12} = m_{22}m_{41} < 0, \quad y_{13} = m_{31}m_{42} < 0, \quad y_{14} = m_{41}m_{33} < 0, \quad y_{15} = m_{11}m_{24} < 0, \quad y_{16} = m_{14}m_{21} < 0, \quad y_{17} = m_{12}m_{24} > 0, \quad y_{18} = m_{31}m_{43} > 0, \quad y_{19} = m_{14}m_{22} > 0. \]

Now by using Routh Hurwitz criterion all the eigenvalues, which represent the roots of eq. (24), have negative real parts if and only if $H_3 > 0$, $H_4 > 0$ and
\[ \Delta = (H_1H_2 - H_3)H_3 - H_2^2 H_4 > 0. \]
Now, $H_1 > 0$, $i = 1, 3$ and $4$, provided that:
\[ \min \left\{ W_1, W_2, W_3 \right\}, \]
\[ W_1 = \frac{a_4a_5w + a_6\left[a_5 + (1 - m)i\right]^2}{a_1\left[a_5 + (1 - m)i\right]^2}, \]
\[ W_2 = \frac{a_1s\left[a_5 + (1 - m)i\right] + a_2i\left[a_3 + (1 - m)s\right]}{a_3 + (1 - m)s + a_4i}, \]
\[ W_3 = \frac{a_8i}{a_7s + a_8i}, \]
\[ W_4 = \frac{a_5 + (1 - m)i}{a_3 + (1 - m)s + a_6i}, \]
\[ W_5 = \frac{a_7s}{a_3 + (1 - m)s + a_6i}, \]
where:
\[ W_1 = \frac{a_4a_1\left[1 - 2s - (1 + a_4)s\right]}{a_2a_3 + (1 - m)s}, \]
\[ W_2 = \frac{a_4\left[1 - 2s - (1 + a_4)s\right]}{a_1a_2\left[a_3 + (1 - m)s\right] + a_2a_3\left[a_5 + (1 - m)i\right]}, \]
\[ W_3 = \frac{a_3 + (1 - m)s}{a_3 + (1 - m)s + a_2a_3\left[a_5 + (1 - m)i\right]}, \]
\[ W_4 = \frac{a_9 + a_{10}}{a_13 - \frac{a_{11}s}{a_3 + (1 - m)s} + \frac{a_{12}i}{a_5 + (1 - m)i}}, \]
A straightforward computation shows that:
\[ \Delta = K_1 - K_2, \]
where:
\[ K_1 = (y_0y_1 - y_5 - y_7)(y_0 + y_1)(y_0y_2 - y_4) + y_1(y_3 - y_6) - m_{24}(y_8 - y_9 + y_{10}) + y_9 + y_{10} + y_6y_{11} - m_{14}(y_{12} - y_{13} + y_{14}) + y_{16}y_1(y_2 - y_4)^2 - m_{14}(y_2 - y_4)(y_{12} - y_{13} + y_{14}) + y_{17}(y_4 - y_{18}) + y_{19}(y_{13} - y_{14}), \]
\[ K_2 = m_{24}^2(y_8 - y_9 + y_{10})^2 + m_{14}^2(y_{12} - y_{13} + y_{14})^2 + y_6^2y_{11}^2 + (y_{12} - y_{13} + y_{14})^2. \]
Hence, $\Delta$ will be positive under conditions (25 – 28). Therefore, all the eigenvalues of $J_4$ have negative real parts under the given conditions and hence $E_4$ is locally asymptotically stable. However, it is unstable otherwise.

4. Local bifurcation analysis

In this section, the effects of varying the parameter values on the dynamical behavior of system (2) around each equilibrium point are studied. We recall that the existence of non-hyperbolic equilibrium point of system (2) is the necessary but not sufficient condition for bifurcation to occur. Therefore, in the following theorems, an application to the Sotomayor’s theorem for local bifurcation is appropriate.

Now, according to Jacobian matrix of system (2) which is given in equation (4.1) in a previous work [12], it is easy to verify that, for any nonzero vector $\mathbf{H} = (\dot{H}_1, \dot{H}_2, \dot{H}_3, \dot{H}_4)^T$, we have:

$$D^2F_\mu(X, \mu)(\mathbf{H}, \dot{\mathbf{H}}) = \left[ t_{ij} \right]_{4 \times 1},$$

where:

$$t_{11} = 2 \left[ \frac{a_2 a_3 \dot{H}_1}{a_3 + (1 - m)s} \left( \frac{(1 - m)w \dot{H}_1}{a_3 + (1 - m)s} - \dot{H}_1^2 \right) - \dot{H}_1^2 - (1 + a_1) \dot{H}_1 \dot{H}_2 \right],$$

$$t_{21} = 2 \left[ \frac{a_4 a_5 \dot{H}_2}{a_5 + (1 - m)i} \left( \frac{(1 - m)w \dot{H}_2}{a_5 + (1 - m)i} + a_1 \dot{H}_1 \dot{H}_2 \right) \right],$$

$$t_{31} = 2 \left[ \frac{a_3 a_7 \dot{H}_1}{a_3 + (1 - m)s} \left( \frac{\dot{H}_1^2 - (1 - m)w \dot{H}_1}{a_3 + (1 - m)s} \right) \right],$$

$$t_{41} = 2 \left[ \frac{a_5 a_6 \dot{H}_2}{\dot{H}_2} \left( \frac{\dot{H}_2^2 - (1 - m)w \dot{H}_2}{a_5 + (1 - m)i} \right) \right],$$

and

$$D^3F_\mu(X, \mu)(\mathbf{H}, \dot{\mathbf{H}}, \ddot{\mathbf{H}}) = \left[ \bar{t}_{ij} \right]_{4 \times 1},$$

where:

$$\bar{t}_{11} = 6 \left[ \frac{a_2 a_3 (1 - m) \dot{H}_1^2}{a_3 + (1 - m)s} \left( \frac{\dot{H}_1^2 - (1 - m)w \dot{H}_1}{a_3 + (1 - m)s} \right) \right],$$

$$\bar{t}_{21} = 6 \left[ \frac{a_4 a_5 (1 - m) \dot{H}_2^2}{a_5 + (1 - m)i} \left( \frac{\dot{H}_2^2 - (1 - m)w \dot{H}_2}{a_5 + (1 - m)i} \right) \right],$$

$$\bar{t}_{31} = 6 \left[ \frac{a_3 a_7 (1 - m) \dot{H}_1^2}{a_3 + (1 - m)s} \left( \frac{\dot{H}_1^2 - (1 - m)w \dot{H}_1}{a_3 + (1 - m)s} - \dot{H}_1^2 \right) \right],$$

$$\bar{t}_{41} = 6 \left[ \frac{a_5 a_6 (1 - m) \dot{H}_2^2}{a_5 + (1 - m)i} \left( \frac{\dot{H}_2^2 - (1 - m)w \dot{H}_2}{a_5 + (1 - m)i} - \dot{H}_2^2 \right) \right],$$

where $X = (s, i, z, w)^T$ and $\mu$ is any bifurcation parameter.

In the following theorems, the local bifurcation conditions near equilibrium points are established.

4.1 Local bifurcation analysis near $E_1$

Theorem (2): If the parameter $a_6$ passes through the value $\tilde{a}_6 = a_1$ then system (2) at the axial equilibrium point $E_1 = (1, 0, 0, 0)$ possesses:
• No saddle-node bifurcation.
• Transcritical bifurcation.
• No pitchfork bifurcation.

Proof: According to the Jacobian matrix $J(E_1)$ given by eq. (4.3) [12], the system (2) at the equilibrium point $E_1 = (1,0,0,0)$ has a zero eigenvalue (say $\lambda_{1i} = 0$) at $a_6 = \bar{a}_6$, and the Jacobian matrix $J_1$ with $a_6 = \bar{a}_6$ becomes:

$$\dot{\bar{a}}_1 = J_1(a_6 = \bar{a}_6) = \left(\bar{r}_{ij}\right)_{4 \times 4},$$

where, $\bar{r}_{ij} = r_{ij}$ for all $i, j = 1, 2, 3, 4$ except $\bar{r}_{22} = a_1 - \bar{a}_6 = 0$.

Now, let $\dot{H}^{[1]} = \left(\dot{H}_1^{[1]}, \dot{H}_2^{[1]}, \dot{H}_3^{[1]}, \dot{H}_4^{[1]}\right)$ be the eigenvector corresponding to the eigenvalue $\lambda_{1i} = 0$. Thus $\left(\dot{J}_1 - \lambda_{1i} I\right)\dot{H}^{[1]} = 0$, which gives:

$$H^{[1]} = \left(\sigma_1 \dot{H}_2^{[1]}, \sigma_2 \dot{H}_3^{[1]}, \sigma_3, \sigma_4\right)$$

where $\dot{H}_2^{[1]}$ and $\dot{H}_3^{[1]}$ are any nonzero real numbers, where:

$$\sigma_1 = -(1 + a_1), \quad \sigma_2 = \frac{(a_9 + a_{10})[a_3 + (1 - m)]}{a_2}.$$ 

Let $\varnothing^{[1]} = \left(\varnothing_1^{[1]}, \varnothing_2^{[1]}, \varnothing_3^{[1]}, \varnothing_4^{[1]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{1i} = 0$ of the matrix $[\dot{J}_1]^{T}$. Then we have $\left(\dot{J}_1^{T} - \lambda_{1i} I\right)\varnothing^{[1]} = 0$. By solving this equation $\varnothing^{[1]}$, we obtain $\varnothing^{[1]} = (0, \varnothing_2^{[1]}, \sigma_3 \varnothing_4^{[1]}, \varnothing_4^{[1]})$ where $\varnothing_2^{[1]}$ and $\varnothing_4^{[1]}$ are any nonzero real numbers, where:

$$\sigma_3 = \frac{a_9}{a_9 + a_{10}}.$$ 

Now, 

$$\frac{\partial f}{\partial a_6} = f_{a_6}(X, a_6) = \left(\frac{\partial f_1}{\partial a_6}, \frac{\partial f_2}{\partial a_6}, \frac{\partial f_3}{\partial a_6}, \frac{\partial f_4}{\partial a_6}\right)^T = (0, - 1, 0, 0)^T.$$ 

So, $\frac{\partial f}{\partial a_6}(E_1, \bar{a}_6) = (0, 0, 0, 0)^T$ and hence $(\varnothing)^T \frac{\partial f}{\partial a_6}(E_1, \bar{a}_6) = 0$.

Therefore, according to Sotomayor’s theorem, the saddle-node bifurcation cannot occur. While the first condition of transcritical bifurcation is satisfied. Now, since:

$$Df_{a_6}(E_1, \bar{a}_6)\dot{H}^{[1]} = \left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}\right) \left(\begin{array}{c}
\sigma_1 \dot{H}_2^{[1]} \\
\dot{H}_2^{[1]} \\
\dot{H}_3^{[1]} \\
\sigma_2 \dot{H}_3^{[1]} \\
\end{array}\right) = \left(\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
\end{array}\right),$$

$$(\varnothing^{[1]})^T \left[Df_{a_6}(E_1, \bar{a}_6)\dot{H}^{[1]}\right] = - \dot{H}_2^{[1]} \varnothing_2^{[1]} \neq 0.$$ 

Now, by substituting $\dot{H}^{[1]}$ in (29), we get:

$$D^2 f(E_1, \bar{a}_6)(\dot{H}^{[1]}, \dot{H}^{[1]}) = \left(\begin{array}{c}
-2\sigma_4 \dot{H}_2^{[1]}[a_2 \sigma_3 \dot{H}_3^{[1]} + \dot{H}_2^{[1]}(\sigma_1 + 1 + a_1)] \\
2\dot{H}_2^{[1]}[\sigma_4 \dot{H}_3^{[1]} + \sigma_1 a_4 \dot{H}_3^{[1]} + \frac{a_9}{a_5}] \\
2\sigma_2 \dot{H}_2^{[1]} \dot{H}_3^{[1]}[a_2 \sigma_4 \sigma_1 \dot{H}_3^{[1]} + \frac{a_9}{a_5}] \\
2\sigma_2 \dot{H}_2^{[1]} \dot{H}_3^{[1]}[a_1 \sigma_4 \sigma_1 \dot{H}_3^{[1]} + \frac{a_{12}}{a_5}] \\
\end{array}\right),$$

where: $\sigma_4 = \frac{a_3}{[a_3 + (1 - m)]^2}$.

Hence, it is obtained that:

$$(\varnothing^{[1]})^T D^2 f(E_1, \bar{a}_6)(\dot{H}^{[1]}, \dot{H}^{[1]}) =$$

$$2\dot{H}_2^{[1]} \left(\varnothing_2^{[1]}[\sigma_1 a_4 \dot{H}_3^{[1]} - \frac{4 \sigma_2 \dot{H}_3^{[1]}}{a_5}] + \sigma_2 \dot{H}_3^{[1]} \varnothing_4^{[1]}[\sigma_4 \sigma_1 \dot{H}_3^{[1]}(\sigma_1 + a_1)] + \frac{(a_9 + a_{12})}{a_5}\right) \neq 0.$$
Thus, according to Sotomayor’s theorem, system (2) has transcritical bifurcation but does not experience a pitchfork bifurcation at $E_1$ with the parameter $a_6 = \bar{a}_6$ where $\bar{a}_6 = a_1$.

**4.2 Local bifurcation analysis near $E_2$**

**Theorem (3):** Suppose that the following conditions are satisfied:

$$\delta \neq 0,$$

$$\delta = -\mathcal{N}_2 \mathcal{N}_1 [a_3 \beta_1 + \mathcal{N}_1 + (a_3) \mathcal{N}_2] - \mathcal{N}_2 \mathcal{N}_4 [a_4 \beta_2 - a_1 \mathcal{N}_1] + \mathcal{N}_6 [a_7 \mathcal{N}_1 \beta_1 + a_4 \mathcal{N}_2 \beta_2] + a_5 \mathcal{N}_1 \beta_1 + a_5 ^2 \mathcal{N}_2 \beta_2,$$

$$\mathcal{N}_1 = -\frac{\bar{u}_{24}}{\bar{a}_{11}}, \quad \mathcal{N}_2 = \frac{\bar{u}_{12} u_{24} - \bar{u}_{14} \bar{u}_{21}}{\bar{u}_{12} \bar{u}_{12}} - \frac{\bar{u}_{34}}{\bar{u}_{43}}, \quad \mathcal{N}_3 = -\frac{\bar{u}_{11} (\bar{u}_{43} \bar{u}_{34} - \bar{u}_{44} \bar{u}_{33})}{\bar{u}_{24} (\bar{u}_{21} \bar{u}_{12} - \bar{u}_{24} \bar{u}_{11})},$$

$$\mathcal{N}_5 = -\frac{a_3}{\bar{a}_{11}}, \quad \mathcal{N}_6 = \frac{a_3}{(a_3 + (1-m)s)}, \quad \beta_1 = \frac{a_5}{(a_3 + (1-m)i)}, \quad \beta_2 = \frac{a_5}{(a_5 + (1-m)i)}.$$

Then system (2) at the equilibrium point $E_2 = \left( \frac{s}{s_i}, 0, 0 \right)$ with the parameter

$$\bar{a}_{13} = \frac{a_{11}s}{a_3 + (1-m)s} + \frac{a_{12}i}{a_5 + (1-m)i}$$

possesses:

- No saddle-node bifurcation.
- Transcritical bifurcation.
- No pitchfork bifurcation.

**Proof:** According to the Jacobian matrix $J_2$ given by eq.(4.4) [12], system (2) at the equilibrium point $E_2 = \left( \frac{s}{s_i}, 0, 0 \right)$ has zero eigenvalue (say $\lambda_2 = 0$) at $a_{13} = \bar{a}_{13}$, and the Jacobian matrix $J_2$ with $a_{13} = \bar{a}_{13}$ becomes:

$$\dot{J}_2 = J_2 (a_{13} = \bar{a}_{13}) = \left[ \dot{u}_{ij} \right]_{4 \times 4},$$

where, $\dot{u}_{ij} = u_{ij}$ for all $i, j = 1, 2, 3, 4$ except $\dot{u}_{44} = \frac{a_{11}x}{a_3 + (1-m)s} + \frac{a_{12}y}{a_5 + (1-m)i} - \bar{a}_{13} = 0$.

Let $\mathcal{H}^2 = \left( \mathcal{H}_1^2, \mathcal{H}_2^2, \mathcal{H}_3^2, \mathcal{H}_4^2 \right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_2 = 0$. Thus, $(J_2 - \lambda_2 I) \mathcal{H}^2 = 0$, which gives:

$$\mathcal{H}_1^2 = \mathcal{N}_1 \mathcal{H}_1^2, \quad \mathcal{H}_2^2 = \mathcal{N}_2 \mathcal{H}_2^2, \quad \mathcal{H}_3^2 = \mathcal{N}_3 \mathcal{H}_4^2, \quad \mathcal{H}_4^2 = \mathcal{N}_4 \mathcal{H}_4^2,$$

where $\mathcal{H}_4^2$ is any nonzero real number, with $\mathcal{N}_1$ and $\mathcal{N}_2$ which are mentioned in the state of the theorem.

Let $\mathcal{V}^2 = \left( \mathcal{V}_1^2, \mathcal{V}_2^2, \mathcal{V}_3^2, \mathcal{V}_4^2 \right)^T$ be the eigenvector associated with the eigenvalue $\lambda_2 = 0$ of the matrix $J_2$.

Then we have $(J_2 - \lambda_2 I) \mathcal{V}^2 = 0$. By solving this equation for $\mathcal{V}^2$, we obtain $\mathcal{V}^2 = \left( \mathcal{N}_5 \mathcal{V}_1^2, \mathcal{N}_4 \mathcal{V}_1^2, \mathcal{N}_6 \mathcal{V}_1^2, \mathcal{V}_4^2 \right)^T$, where $\mathcal{V}_4^2$ are any nonzero real numbers, with $\mathcal{N}_4, \mathcal{N}_5$ and $\mathcal{N}_6$ which are mentioned in the state of the theorem.

Now, consider that:

$$\frac{\partial f}{\partial a_{13}} (X, a_{13}) = f_{a_{13}} (X, a_{13}) = \left( \frac{\partial f_1}{\partial a_{13}}, \frac{\partial f_2}{\partial a_{13}}, \frac{\partial f_3}{\partial a_{13}}, \frac{\partial f_4}{\partial a_{13}} \right)^T = (0, 0, 0, -\mathcal{W})^T,$$

So,

$$\frac{\partial f}{\partial a_{13}} (E_2, a_{13}) = (0, 0, 0, 0)^T,$$

and hence $\left( \mathcal{V}^2 \right)^T f_{a_{13}} (E_2, \bar{a}_{13}) = 0$.

Therefore, according to Sotomayor’s theorem, the saddle-node bifurcation cannot occur. While the first condition of transcritical bifurcation is satisfied. Now, since

$$Df_{a_{13}} (E_2, \bar{a}_{13}) \mathcal{H}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \left( \begin{array}{c} \mathcal{H}_1^2 \\ \mathcal{H}_2^2 \\ \mathcal{H}_3^2 \\ \mathcal{H}_4^2 \end{array} \right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\mathcal{W}^2 \end{pmatrix},$$

$$\left( \mathcal{V}^2 \right)^T Df_{a_{13}} (E_2, \bar{a}_{13}) \mathcal{H}^2 = -\mathcal{W}^2 \mathcal{H}_4^2 \neq 0.$$
Moreover, by substituting $H^{[2]}$ in (29), we get:

$$D^2 f(E_2, \tilde{a}_{13})(\tilde{H}^{[2]}, \tilde{H}^{[2]}) = \begin{bmatrix}
-2\xi_1 \left( \tilde{H}_4^{[2]} \right)^2 \left[ a_2 \beta_1 + \xi_1 + (1 + a_1) \xi_2 \right] \\
-2\xi_2 \left( \tilde{H}_4^{[2]} \right)^2 \left[ a_4 \beta_2 - a_1 \xi_1 \right] \\
2 \left( \tilde{H}_4^{[2]} \right)^2 \left[ a_7 \xi_1 \beta_1 + a_8 \xi_2 \beta_2 \right] \\
2 \left( \tilde{H}_4^{[2]} \right)^2 \left[ a_{11} \xi_1 \beta_1 + a_{12} \xi_2 \beta_2 \right]
\end{bmatrix}. $$

Hence, it is obtained that:

$$\left( \tilde{\phi}^{[2]} \right)^T D^2 f(E_2, \tilde{a}_{13})(\tilde{H}^{[2]}, \tilde{H}^{[2]}) = 2\delta \tilde{H}_2^{[2]} \tilde{\phi}^{[2]},$$

where $\delta$ are mentioned in the state of the theorem.

So, by condition (31), we obtain that:

$$\left( \tilde{\phi}^{[2]} \right)^T D^2 f(E_2, \tilde{a}_{13})(\tilde{H}^{[2]}, \tilde{H}^{[2]}) \neq 0.$$

Thus, according to Sotomayor’s theorem, system (2) has a transcritical bifurcation at the equilibrium point $E_2 = (\tilde{s}, \tilde{i}, 0, 0)$ with the parameter $\tilde{a}_{13} = \frac{a_1 \tilde{s}}{a_3 + (1 - m) \tilde{s}} + \frac{a_2 \tilde{i}}{a_3 + (1 - m) \tilde{i}}$.

### 4.3 Local bifurcation analysis near $E_3 (\tilde{s}, 0, \tilde{z}, \tilde{w})$

**Theorem (4):** Suppose that the following conditions are satisfied:

$$2\xi + \frac{a_2 a_3 \tilde{w}}{a_3 + (1 - m) \tilde{w}} \xi^2 < 1, \quad (32)$$

$$\tilde{\bar{x}} < \frac{1}{2}, \quad (33)$$

$$\tilde{\bar{e}} \neq 0, \quad (34)$$

where:

$$\tilde{\bar{t}}_1 = \frac{-\tilde{n}_{31}}{\tilde{n}_{33}}, \quad \tilde{\bar{t}}_2 = \frac{\tilde{n}_{14} \tilde{n}_{31} - \tilde{n}_{34} \tilde{n}_{14}}{\tilde{n}_{31}}, \quad \tilde{\bar{t}}_3 = \frac{\tilde{n}_{42} \tilde{\bar{t}}_2}{\tilde{n}_{31}}, \quad \tilde{\bar{t}}_4 = -\frac{(\tilde{n}_{12} + \tilde{n}_{32} \tilde{\bar{t}}_4 + \tilde{n}_{42} \tilde{\bar{t}}_2)}{\tilde{n}_{22}},$$

$$\mu_1 = \frac{\tilde{\bar{t}}_2}{a_3 + (1 - m) \tilde{w}}, \quad \tilde{\bar{e}} = \mu_1 [a_2 - (a_7 \tilde{\bar{t}}_2 + \tilde{\bar{t}}_1 a_{11})].$$

Then system (2) at the equilibrium point $E_3 = (\tilde{s}, 0, \tilde{z}, \tilde{w})$ with the parameter value: $\tilde{a}_2 = \frac{(1 - 2s)(a_3 + (1 - m)\tilde{w})}{a_3 \tilde{w}}$ has a saddle-node bifurcation, but neither a transcritical nor a pitchfork bifurcation can occur at $E_3$.

**Proof:** By using the characteristic equation given by eq. (4.5) [12], system (2) at the equilibrium point $E_3$ has zero eigenvalue (say $\lambda_{3s} = 0$) at $\tilde{a}_2 = \tilde{a}_2$ and the Jacobian matrix $J_3$ with parameter $a_2 = \tilde{a}_2$ becomes:

$$\tilde{J}_3 = J_3(a_2 = \tilde{a}_2) = \left[ \tilde{n}_{ij} \right]_{4x4},$$

where,

$$\tilde{n}_{ij} = n_{ij} \text{ for all } i, j = 1, 2, 3, 4 \text{ except } \tilde{n}_{11} = 1 - \tilde{s} - \frac{a_2 a_3 \tilde{w}}{a_3 + (1 - m) \tilde{w}} = 0 \text{ and } \tilde{n}_{14} = \frac{a_2 \tilde{w}}{a_3 + (1 - m) \tilde{w}}.$$ 

Note that, $\tilde{a}_2 > 0$, which is provided by condition (33).

Let $\tilde{H}^{[3]} = \left( \tilde{H}_1^{[3]}, \tilde{H}_2^{[3]}, \tilde{H}_3^{[3]}, \tilde{H}_4^{[3]} \right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{3s} = 0$.

Thus $\left( \tilde{J}_3 - \lambda_{3s} I \right) \tilde{H}^{[3]} = 0$, which gives:

$$\tilde{H}^{[3]} = \left( \tilde{H}_1^{[3]}, 0, \tilde{H}_3^{[3]}, 0 \right)^T,$$

where $\tilde{H}_1^{[3]}$ is any nonzero number with $\tilde{I}_4$, which is mentioned in the state of the theorem.

Let $\phi^{[3]} = \left( \phi_1^{[3]}, \phi_2^{[3]}, \phi_3^{[3]}, \phi_4^{[3]} \right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{3s} = 0$ of the matrix $\tilde{J}_3$. Then we have $\left( \tilde{J}_3^T - \lambda_{3s} I \right) \phi^{[3]} = 0$.

By solving this equation for $\phi^{[3]}$, we obtain:

$$\phi^{[3]} = \left( \phi_1^{[3]}, \phi_4^{[3]}, \tilde{I}_3 \phi_2^{[3]}, \tilde{I}_4 \phi_1^{[3]} \right)^T,$$

where $\phi_4^{[3]}$ is any nonzero number with $\tilde{I}_2$, $\tilde{I}_3$, and $\tilde{I}_4$, which is mentioned in the state of the theorem.
Now, 
\[ \frac{\partial f}{\partial a_2} = f_a(X, a_2) = \left( \frac{\partial f_1}{\partial a_2}, \frac{\partial f_2}{\partial a_2}, \frac{\partial f_3}{\partial a_2}, \frac{\partial f_4}{\partial a_2} \right)^T = \left( -\frac{a_3w + s[a_3 + (1-m)s]}{[a_3 + (1-m)s]^2} s, 0, 0, 0 \right)^T, \]
So, \[ f_a\left(E_3, \hat{a}_2\right) = \left( -\frac{a_2\tilde{\theta} + s[a_2 + (1-m)s][\tilde{s}]^2}{[a_2 + (1-m)s]^2} s, 0, 0, 0 \right)^T, \]
and hence \[ \left( \phi^{[3]} \right)^T f_a\left(E_3, \hat{a}_2\right) = -\phi_i^{[3]} \frac{a_2\tilde{\theta} + s[a_2 + (1-m)s][\tilde{s}]^2}{[a_2 + (1-m)s]^2} s \neq 0. \]

Therefore, according to Sotomayor’s theorem, neither a transcritical nor a pitchfork bifurcation can occur at \( E_3 \), while the first condition of a saddle-node bifurcation is satisfied.
Moreover, by substituting \( H^{[3]} \) in (29), we get:
\[ D^2 f(E_3, \hat{a}_2)(H^{[3]}, H^{[3]}) = [c^T]_{4 \times 1} = \begin{bmatrix} 2a_2\tilde{\mu}_i \tilde{H}_i^{[3]} - \tilde{\mu}_i^2 \\ 0 \\ -2(\tilde{H}_i^{[4]} - \tilde{\mu}_i^2) a_7 \mu_1 \\ -2(\tilde{H}_i^{[4]} - \tilde{\mu}_i^2) a_{11} \mu_1 \end{bmatrix}. \]
Hence, it is obtained that:
\[ \left( \phi^{[3]} \right)^T D^2 f(E_3, \hat{a}_2)(H^{[3]}, H^{[3]}) = 2\phi_i^{[3]} \tilde{H}_i^{[3]} \tilde{\dot{\epsilon}}. \]
So, according to the condition (34) we obtain that:
\[ \left( \phi^{[3]} \right)^T D^2 f(E_4, \hat{a}_2)(H^{[3]}, H^{[3]}) \neq 0. \]
Thus, by using Sotomayor’s theorem, system (2) has a saddle-node bifurcation at \( E_5 = (\tilde{s}, \tilde{z}, \tilde{w}) \) at the parameter:
\[ \hat{a}_2 = \frac{(1-28)[a_2 + (1-m)s]^2}{a_2\tilde{\theta}}. \]

### 4.4 Local bifurcation analysis near \( E_4 \)

**Theorem (5):** Suppose that the following conditions are satisfied:
\[ s > \frac{1 - (1 - a_1) i}{1 - (1 - a_1) \tilde{s}} \left( \frac{a_3 + (1-m)s}{2} - a_2\tilde{\delta} \right), \]
\[ w > \frac{a_0 [a_3 + (1-m)s]}{a_4 a_5} \]
\[ D_1 \neq 1, \]
\[ D_2 \left( \tilde{s} + D_2 \tilde{i} \right) \neq i, \]
\[ \tilde{\epsilon} \neq 0, \]
\[ \tilde{\epsilon} \neq 0, \]
where:
\[ D_1 = \frac{\tilde{m}_{24} \tilde{m}_{12}}{\tilde{m}_{11} \tilde{m}_{24} - \tilde{m}_{14} \tilde{m}_{12}}, \]
\[ D_2 = -\frac{\tilde{m}_{14} \tilde{D}_1}{\tilde{m}_{12}}, \]
\[ D_3 = -\frac{(\tilde{m}_{32} + \tilde{m}_{31} \tilde{D}_2 + \tilde{m}_{34} \tilde{D}_1)}{\tilde{m}_{33}}, \]
\[ \tilde{D}_4 = \frac{\tilde{m}_{12} (\tilde{m}_{43} \tilde{m}_{31} - \tilde{m}_{33} \tilde{m}_{21}) - \tilde{m}_{11} (\tilde{m}_{43} \tilde{m}_{32} - \tilde{m}_{42} \tilde{m}_{33})}{(\tilde{m}_{43} \tilde{m}_{32} - \tilde{m}_{21} \tilde{m}_{33}) \tilde{D}_4}, \]
\[ \tilde{D}_5 = \frac{\tilde{m}_{33}}{\tilde{m}_{33}}, \]
\[ \tilde{D}_6 = \frac{\tilde{m}_{33} \tilde{D}_4}{\tilde{D}_4}, \]
\[ \tilde{Y}_1 = \frac{a_3 \tilde{D}_2}{[a_3 + (1-m)s]^2} \left( \frac{(1-m)\tilde{w}}{a_3 + (1-m)s} - \tilde{D}_2 \right), \]
\[ \tilde{Y}_2 = \frac{a_5}{[a_5 + (1-m)i]^2} \left( \frac{(1-m)\tilde{w}}{a_5 + (1-m)i} - \tilde{D}_1 \right), \]
\[ \tilde{Y}_3 = \frac{a_3(1-m)\tilde{D}_2^2}{[a_3 + (1-m)s]^3} \left( \frac{a_3 - (1-m)\tilde{s}}{a_3 + (1-m)s} - \tilde{D}_2 \right), \]
\[ \tilde{Y}_4 = \frac{a_5}{[a_5 + (1-m)i]^3} \left( \frac{a_5(1-m)\tilde{w}}{a_5 + (1-m)i} - \tilde{D}_1 \right), \]
\[ \tilde{\epsilon} = \tilde{D}_5 \tilde{D}_2 [a_2 \tilde{Y}_1 + \tilde{D}_2 - (1 + a_1)] + a_2 \tilde{Y}_2 + a_5 \tilde{D}_1 - \tilde{D}_6 [a_7 \tilde{Y}_1 + a_8 \tilde{Y}_2] - \tilde{D}_4 [a_{11} \tilde{Y}_1 + a_{12} \tilde{Y}_2], \]
\[ \tilde{\epsilon} = a_2 \tilde{D}_5 \tilde{Y}_3 + a_4 \tilde{Y}_4 - [\tilde{D}_6 (a_7 \tilde{Y}_3 + a_8 \tilde{Y}_4) + \tilde{D}_4 (a_{11} \tilde{Y}_3 + a_{12} \tilde{Y}_4)]. \]
Then system (2) at the equilibrium point $E_4 = \left(\frac{s}{\bar{s}}, \frac{i}{\bar{i}}, \frac{z}{\bar{z}}, \frac{w}{\bar{w}}\right)$ with the parameter value:

$$\bar{a}_1 = \frac{a_4 a_5 w - a_6 [a_5 + (1 - m) i]^2}{s [a_5 + (1 - m) i]^2},$$

has a saddle-node bifurcation, a transcritical and a pitchfork bifurcation at $E_4$.

**Proof:** The characteristic equation given by eq. (4.6) [12] of system (2) having zero eigenvalue (say $\lambda_4i = 0$) if and only if $H_4 = 0$, and then $E_4$ becomes a non-hyperbolic equilibrium point. Clearly, the Jacobian matrix of system (2) at the equilibrium point $E_4$ with parameter $a_4 = \bar{a}_1$ becomes:

$$J_4 = \begin{bmatrix} \bar{m}_{11} & \bar{m}_{12} \\ \bar{m}_{21} & \bar{m}_{22} \end{bmatrix},$$

where, $\bar{m}_{ij} = m_{ij}$ for all $i, j = 1, 2, 3, 4$ except $m_{ij}$ which is given by:

$$\bar{m}_{11} = 1 - 2\bar{s} - (1 + \bar{a}_1)\bar{i} - \frac{a_2 a_3 w}{[a_3 + (1 - m)\bar{s}]^2}, \quad \bar{m}_{12} = -(1 + \bar{a}_1)\bar{s}$$

and $\bar{m}_{21} = \bar{a}_1 \bar{i}$. Note that, $\bar{a}_1 > 0$ provided the condition (36).

Let $\dot{H}^{[4]} = \left(\dot{H}_1^{[4]}, \dot{H}_2^{[4]}, \dot{H}_3^{[4]}, \dot{H}_4^{[4]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_4i = 0$.

Thus $J_4 - \lambda_4i I \dot{H}^{[4]} = 0$, which gives:

$$\dot{H}^{[4]} = \left(D_2 \dot{H}_2^{[4]}, D_3 \dot{H}_3^{[4]}, D_4 \dot{H}_4^{[4]}\right)^T,$$

where $\dot{H}_2^{[4]}$ is any nonzero number with $D_1, D_2$ and $D_3$ which is mentioned in the text of the theorem.

Let $\varphi^{[4]} = (\varphi_1^{[4]}, \varphi_2^{[4]}, \varphi_3^{[4]}, \varphi_4^{[4]})^T$ be the eigenvector associated with the eigenvalue $\lambda_4i = 0$ of the matrix $J_4$. Then we have $(J_4 - \lambda_4i I) \varphi^{[4]} = 0$.

By solving this equation for $\varphi^{[4]}$, we obtain:

$$\varphi^{[4]} = \left(D_2 \varphi_2^{[4]}, D_3 \varphi_3^{[4]}, D_4 \varphi_4^{[4]}\right)^T,$$

where $\varphi_2^{[4]}$ is any nonzero number with $D_4, D_5$ and $D_6$ which is mentioned in the state of the theorem.

Now,

$$\frac{\partial f}{\partial a_1} f_{a_1}(X, a_1) = \begin{bmatrix} \frac{\partial f_1}{\partial a_1} & \frac{\partial f_2}{\partial a_1} & \frac{\partial f_3}{\partial a_1} & \frac{\partial f_4}{\partial a_1} \end{bmatrix}^T = (-s \bar{i}, \bar{s} i, 0, 0)^T,$$

So, $f_{a_1}(E_4, \bar{a}_1) = \left(-s \bar{i}, \bar{s} i, 0, 0\right)^T$ and hence it is obtained that:

$$(\varphi^{[4]})^T f_{a_1}(E_4, \bar{a}_1) = s \bar{i} \varphi_2^{[4]}(1 - \bar{D}_5).$$

So, according to the condition (37a) we obtain that:

$$(\varphi^{[4]})^T f_{a_1}(E_4, \bar{a}_1) \neq 0.$$

Therefore, according to Sotomayor’s theorem 4, neither a transcritical nor a pitchfork bifurcation can occur at $E_4$, while the first condition of a saddle-node 4 bifurcation is satisfied. Moreover, by substituting $\dot{H}^{[4]}$ in (29), we get:

$$D^2 f(E_4, \bar{a}_1) \left(\dot{H}^{[4]}, \ddot{H}^{[4]}\right) = \begin{bmatrix} 2(\dot{H}_2^{[4]})^2 D_2 [a_2 \ddot{Y}_1 + D_2 - (1 + a_1)] \\ -2 (\dot{H}_1^{[4]})^2 [a_7 \ddot{Y}_2 + a_8 \ddot{Y}_2] \\ -2 (\dot{H}_1^{[4]})^2 [a_13 \ddot{Y}_1 + a_12 \ddot{Y}_2] \end{bmatrix}. $$

Hence, it is obtained that:

$$\left(\varphi^{[4]}\right)^T D^2 f(E_4, \bar{a}_1) \left(\dot{H}^{[4]}, \ddot{H}^{[4]}\right) = 2 \varphi_2^{[4]} \left(\ddot{H}_2^{[4]}\right)^2 \dddot{Y}_1 E.$$

So, according to the condition (38a) we obtain that:

$$\left(\varphi^{[4]}\right)^T D^2 f(E_4, \bar{a}_1) \left(\dot{H}^{[4]}, \ddot{H}^{[4]}\right) \neq 0.$$
Thus, by using Sotomayor’s theorem, system (2) has a saddle-node bifurcation at \( E_4 = \left( \frac{s}{i}, \frac{z}{w} \right) \) at \( \tilde{a}_1 \).

Now, if the condition (37a) is not satisfied, we obtain that:

\[
\left( \varphi^{[4]} \right)^T f_{\tilde{a}_1}(E_4, \tilde{a}_1) = 0.
\]

Therefore, according to Sotomayor’s theorem, the saddle-node bifurcation cannot occur. While the first condition of transcritical bifurcation is satisfied. Now, since

\[
Df_{\tilde{a}_1}(E_4, \tilde{a}_1) \dot{H}^{[4]} = \begin{pmatrix}
-\tilde{i} & -\tilde{s} & 0 & 0 \\
\tilde{i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
\tilde{D}_2 \dot{H}_2^{[4]} \\
\dot{H}_2^{[4]} \\
\tilde{D}_3 \dot{H}_2^{[4]} \\
\dot{H}_2^{[4]} \\
\end{pmatrix} = \begin{pmatrix}
- \tilde{h}_2^{[4]} (\tilde{s} + \tilde{D}_2 i) \\
\tilde{h}_2^{[4]} \\
0 \\
0
\end{pmatrix},
\]

So, according to the condition (37b) we obtain that:

\[
\left( \varphi^{[4]} \right)^T Df(E_4, \tilde{a}_1)(\dot{H}^{[4]}, \dot{H}^{[4]}) = 0.
\]

Now, by substituting \( \dot{H}^{[4]} \) in (29), we get:

\[
D^2 f(E_4 , \tilde{a}_1)(\dot{H}^{[4]}, \dot{H}^{[4]}),
\]

Hence, it is obtained that:

\[
\left( \varphi^{[4]} \right)^T D^2 f(E_4 , \tilde{a}_1)(\dot{H}^{[4]}, \dot{H}^{[4]}) = 2 \varphi^{[4]} \left( \dot{H}^{[4]} \right)^2 \mathcal{E}.
\]

So, according to the condition (38a) we obtain that:

\[
\left( \varphi^{[4]} \right)^T D^2 f(E_4 , \tilde{a}_1)(\dot{H}^{[4]}, \dot{H}^{[4]}) = 0.
\]

Now, if the condition (38a) is not satisfied, by substituting \( \dot{H}^{[4]} \) in (30), we obtain that:

\[
D^3 f(E_4 , \tilde{a}_1)(\dot{H}^{[4]}, \dot{H}^{[4]}, \dot{H}^{[4]}),
\]

Thus, by using Sotomayor’s theorem, system (2) has a transcritical bifurcation and pitch fork bifurcation at \( E_4 = \left( s, \frac{z}{w} \right) \) by the conditions (37a) and (38a), respectively, which are not satisfied at \( \tilde{a}_1 \).

5. Hopf bifurcation analysis

In this section, the possibility of the occurrence of a Hopf bifurcation near the positive equilibrium point of the system (2) is investigated, as shown in the following theorem.

**Theorem 5:** Suppose that the local conditions (21 - 23) with the following conditions are satisfied:

\[
\hat{A}_1 > \hat{A}_2, \quad (39)
\]
\[
\hat{A}_3 > \hat{A}_4, \quad (40)
\]
\[
\hat{A}_5 > \hat{A}_6, \quad (41)
\]
\[
\hat{A}_7 < \hat{A}_8, \quad (42)
\]

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where:

\[
\begin{align*}
\mathcal{A}_1 &= [y_0 + \gamma - \gamma_6[y_0^2 - \gamma - 2y_0 - y_0m_24(y_0 - y_0 + y_9 + y_10) + 3y_1(y_0^2 + y_0(y_3 - y_6)] + \\
& 3m_33 - y_0m_{14}(y_2 - y_3 + y_14) + y_0y_6y_{11} + y_17y_{18} + \gamma_0 - y_16 \\
\mathcal{A}_2 &= -y_{15} - y_{14}y_{17} - y_{19}(y_{13} - y_{14}) - y_0^2y_4 - y_9, \\
\mathcal{A}_3 &= [2m_{33}(y_3 - y_6)2y_0y_9 + 2E(y_3 - y_6) + m_{24}(y_8 - y_9 + y_10) - y_0y_4 - y_6y_{11}] + y_0y_4^2 + \\
& y_0^3y_4y_6y_{11}(y_{15} + y_{16}) + y_0^2(3E^2m_{33} + m_{33}^3) - [y_0^2y_4 + (y_9 - y_{10})] - \\
&2(y_4 - y_{18})m_{33}y_{17} - 2Ey_19(y_{13} - y_{14}), \\
\mathcal{A}_4 &= (y_0y_5 + y_1y_5 + y_0y_7)[(y_3 - y_6) + y_0m_{33}] + m_{14}(y_12 - y_{13} + y_14)(y_0^2 - y_7) + \\
&2y_0y_17(y_{14} - y_{18}) + m_{33}[(y_{12} - y_{13} - y_{14})^2m_{14}y_4 - m_{14}(y_{12} - y_{13} + y_{14})] - \\
m_{14}(y_{12} - y_{13} + y_{14})[y_4 + 2E(y_0 - 2m_{33}y_9) + (y_3 - y_6)m_{14}(y_{12} - y_{13} + y_{14}) + \\
2y_0y_4] - 2m_{33}(y_9 - y_0) + y_0(y_3 - y_6)(3E^2m_{33} + m_{33}^3) + 3E^2m_{33}^2 + 2E(- \\
&2y_0(y_4 + y_7 - y_6) - y_0y_24(y_9 - y_0 + y_{10}) + y_0y_6y_{11} + (y_{15} + y_{16})(2y_0 - \\
&2E - 2E_17(y_{14} - y_{18}) - 2y_{19}(y_{13} - y_{14})m_{33}, \\
\mathcal{A}_5 &= [(y_0 + y_1)y_5 + (y_0y_7)][m_{24}(y_8 - y_9 + y_{10}) + \mathcal{E}(y_3 - y_6)y_0^2 - y_7] - y_6y_{11} + \\
&y_0m_{33} + m_{14}(y_{12} - y_{13} + y_{14}) + y_0y_4 + y_0(y_3 - y_6)[m_{33}^3 + m_{33}(2 + \mathcal{E}^3 - y_4) - \\
&1 - m_{14}(y_{12} - y_{13} + y_{14} - 1) + y_19(y_{13} - y_{14})[1 - \mathcal{E}^2m_{33} + 2y_0y_33 - 2E^{m_{33} + \\
m_{33}(2y_0 - 2E - \mathcal{E}^3)(y_9 - y_{10}) - m_{24}m_{33}(y_0^2 - y_7) + (y_3 - y_6)^2(y_0m_{33} + \\
2Em_{33}y_9) + \mathcal{E}_6y_{11}(2m_{14} - y_0^2 + y_7) + \mathcal{E}_3m_{33}^2 + Em_{33}(E^2 - 2y_0)[y_0y_4 + y_6y_{11} - y_4^2] + \\
\mathcal{E}_3m_{33}(E^2 - 2y_0)[y_0y_4 + y_6y_{11} - y_4^2 + m_{33}^2(y_0^2 - y_7)(y_3 - y_6) - y_6m_{24}(y_8 - y_9 + \\
y_{10})(y_0 - \mathcal{E} + 2Em_{14} - (y_{15} + y_{16}) + \mathcal{E}m_{14}m_{33}(y_4 - \mathcal{E}^2y_0 - y_0^2 + y_7)(y_{12} - y_{13} + y_{14}) - \\
y_17(y_{14} - y_{18})(\mathcal{E} + m_{33}^2). \\
\end{align*}
\]

And

\[
\mathcal{E} = \frac{a_{11}^3}{a_3 + (1 - m)s} + \frac{a_{12}^i}{a_5 + (1 - m)i}.
\]

Then at the parameter value \( a_{13} = a_{13}^* \), system (2) has a Hopf bifurcation near the point \( E_4 \).

**Proof:** Consider the characteristic equation of system (2) at \( E_4 \) which is given by eq. (24), then by using the Hopf bifurcation theorem, for \( n=4 \), we need to find a parameter, say \( (a_{13})^* \), to verify that

the necessary and sufficient conditions for the Hopf bifurcation are satisfied, that is:

\[
A_i \left( a_{13} \right) > 0; \quad i = 1, 3, 4, \quad \Delta_i \left( a_{13} \right) > 0, \quad H_i^2 \left( a_{13} \right) - 4 \Delta_i \left( a_{13} \right) > 0 \quad \text{and} \quad \Delta_3 \left( a_{13} \right) = 0,
\]

where \( A_i; \ i = 1, 3, 4 \) represents the coefficients of the characteristic eq. (24).

Straightforward computation gives that:

\[
A_i \left( a_{13} \right) > 0; \quad i = 1, 3, 4 \quad \text{and} \quad \Delta_1 \left( a_{13} \right) > 0 \quad \text{under the local conditions (25-28),}
\]

while \( A_i^3 \left( a_{13} \right) - 4 \Delta_i \left( a_{13} \right) > 0, \) provided that the condition (43) holds.

On the other hand, it is observed that \( \Delta_3 = 0 \) gives that:

\[
H_3 \left( H_1H_2 - H_3 \right) - H_2^2H_4 = 0.
\]

By a straightforward computation, we get:

\[
M_1a_{13}^3 + M_2a_{13}^2 + M_3a_{13} + M_4 = 0,
\]

where:

\[
(43)
\]

\[
(44)
\]
\[ M_1 = -m_{33} [\gamma_0^2 + m_{33}^2 + \gamma_0 (\gamma_3 - \gamma_0)], \]
\[ M_2 = \mathbf{A}_1 + \mathbf{A}_2, \]
\[ M_3 = \mathbf{A}_3 + \mathbf{A}_4, \]
\[ M_4 = \mathbf{A}_5 + \mathbf{A}_6, \]

where \( \mathbf{A}_i; \ i = 1,2,3,4,5,6 \) which are mentioned in the text of the theorem.

Clearly, \( M_i > 0, \ i = 2,3,4 \) provided that in addition to the conditions (25 – 28), the conditions (39 – 42) holds.

Note that, by using Descarte rule of sign, eq.(45) has a unique positive root \( a_{13}^*. \)

Now, at \( a_{13} = a_{13}^* \) the characteristic equation given by eq. (24) can be written as:

\[
\left( \lambda^2 + \frac{H_3}{H_1} \right) \left( \lambda^2 + H_1 \lambda + \frac{\Delta_1}{H_1} \right) = 0, \text{ which has four roots:}
\]

\[ \lambda_{1,2} = \pm i \frac{H_3}{H_1}, \quad \text{and} \quad \lambda_{3,4} = \frac{1}{2} \left( -H_1 \pm \sqrt{H_1^2 - 4 \frac{\Delta_1}{H_1}} \right). \]

Clearly, at \( a_{13} = a_{13}^* \) there are two pure imaginary eigenvalues (\( \lambda_1 \) and \( \lambda_2 \)) and two eigenvalues which are real and negative. Now for all values of \( a_{13} \) in the neighborhood of \( a_{13}^* \), the roots are in general of the following form:

\[ \lambda_1 = \varepsilon_1 + i \varepsilon_2, \quad \lambda_2 = \varepsilon_1 - i \varepsilon_2, \quad \lambda_{3,4} = \frac{1}{2} \left( -H_1 \pm \sqrt{H_1^2 - 4 \frac{\Delta_1}{H_1}} \right). \]

Clearly, \( Re(\lambda_N(a_{13})) \bigg|_{a_{13}=a_{13}^*} = \varepsilon_1 (a_{13}^*) = 0, N = 1,2 \), which implies that the first condition of the necessary and sufficient conditions for Hopf bifurcation are satisfied at \( a_{13} = a_{13}^* \). Now, according to the verification of the transversality condition, we must prove that:

\[ \dot{\Theta}(a_{13}) \dot{\Psi}(a_{13}) + \Gamma(a_{13}) \Phi(a_{13}) \neq 0, \]

where \( \dot{\Theta}, \dot{\Psi}, \dot{\Gamma} \) and \( \Phi \) are given in lemma (1) of a previous work [11]. Note that for \( a_{13} = a_{13}^* \) we have \( \varepsilon_1(a_{13}^*) = 0 \) and \( \varepsilon_2(a_{13}^*) = \frac{H_3}{H_1} \), thus it gives the following simplifications:

\[ \dot{\Psi}(a_{13}^*) = -2 H_3 (a_{13}^*), \]
\[ \dot{\Phi}(a_{13}^*) = 2 \frac{\varepsilon_2(a_{13}^*)}{H_3} (H_1 H_2 - 2 H_3), \]
\[ \dot{\Theta}(a_{13}^*) = H'_3 (a_{13}^*) - H_1 H'_2 (a_{13}^*), \]
\[ \dot{\Gamma}(a_{13}^*) = \varepsilon_2 (a_{13}^*) \left( H_3 (a_{13}^*) - \frac{H_3}{H_1} H_1 (a_{13}^*) \right), \]

where:

\[ H'_1 = \frac{dH_1}{da_{13}} \bigg|_{a_{13}=a_{13}^*} = 1, \]
\[ H'_2 = \frac{dH_2}{da_{13}} \bigg|_{a_{13}=a_{13}^*} = -(m_{11} + m_{22} + m_{33}), \]
\[ H'_3 = \frac{dH_3}{da_{13}} \bigg|_{a_{13}=a_{13}^*} = m_{11} + m_{22} + m_{11} m_{22}, \]
\[ H'_4 = \frac{dH_4}{da_{13}} \bigg|_{a_{13}=a_{13}^*} = -m_{33} (\gamma_3 - \gamma_6). \]

Then, we get that:

\[ \dot{\Theta}(a_{13}) \dot{\Psi}(a_{13}) + \dot{\Gamma}(a_{13}) \Phi(a_{13}) = \mathbf{A}_7 + \mathbf{A}_8 \neq 0, \text{ where:} \]
Now, according to condition (45), we have:
\[
\dot{\Theta}(a_{i3}) \dot{\Psi}(a_{i3}) + \dot{\Gamma}(a_{i3}) \Phi(a_{i3}) \neq 0.
\]
So, we obtain that the Hopf bifurcation occurs around the equilibrium point \( E_4 \) at the parameter \( a_{i3} = a_{i3}^* \).

6. Numerical simulation
In this section, the dynamical behavior of system (2) is studied numerically for a set of parameters and different sets of initial points. The objectives of this part are:
1- Investigating the effect of varying the value of each parameter on the dynamical behavior of system (2).
2- Confirming the obtained analytical results.
It is observed that, for the following set of hypothetical parameters that satisfies stability conditions of the positive equilibrium point, system (2) has a globally asymptotically stable positive equilibrium point, as shown in Figure- (1) (a – d).

\[
\begin{align*}
    a_1 &= 2, a_2 = 0.3, m = 0.7, a_3 = 0.5, a_4 = 0.2, \\
    a_5 &= 0.5, a_6 = 0.01, a_7 = 0.1, a_8 = 0.1, a_9 = 0.5, \\
    a_{10} &= 0.01, a_{11} = 0.1, a_{12} = 0.1, a_{13} = 0.1
\end{align*}
\]

**Figure 1** - The time series of the solution of system (2) starting from the three different initial points \((0.4, 0.5, 0.6, 0.6), (0.4, 0.5, 0.7, 0.9), \) and \((0.8, 0.9, 1.5, 0.5)\) for the data given by eq. (6.1). (a) the trajectories of s as a function of time, (b) the trajectories of i as a function of time, (c) trajectories of z as a function of time, (d) the trajectories of w as a function of time.
Clearly, Figure – (1) shows that system (2) is globally asymptotically stable as the solution of system (2) approaches asymptotically to the positive equilibrium point $E_4 = (0.233, 0.050, 0.116, 1.173)$, starting from three different initial points, which confirms our obtained analytical results.

Now, in order to discuss the effects of the parameters’ values of system (2) on the dynamical behavior of the system, the system is solved numerically for the data given in eq. (6.1) with varying one parameter at each time, The obtained results are given in Table-1, while more details are provided elsewhere [4].

| Range of parameter | Numerical behavior of system (2) | Bifurcation point |
|--------------------|---------------------------------|-------------------|
| $0.1 \leq a_1 < 1.9$ | Approaches to the infected prey free equilibrium point $E_3$ | $a_1 = 1.9$ |
| $1.9 \leq a_1 < 2.2$ | Approaches to the positive equilibrium point $E_4$ | |
| $0.1 \leq a_2 < 0.29$ | $0.29 \leq a_2 < 1$ | Approaches to the infected prey free equilibrium point $E_3$ |
| $0.25 < a_3 \leq 0.4$ | $0.4 < a_3 \leq 1$ | Approaches to the positive equilibrium point $E_4$ |
| $0.1 \leq a_4 < 0.22$ | $0.22 \leq a_4 < 1$ | Approaches to the positive equilibrium point $E_4$ |
| $0.45 \leq a_5 < 0.6$ | Approaches to the infected prey free equilibrium point $E_5$ | $a_5 = 0.45$ |
| $0 \leq a_6 < 0.3$ | $0.3 \leq a_6 < 1$ | Approaches to the positive equilibrium point $E_4$ |
| $0.0001 \leq a_7 < 0.11$ | $0.11 \leq a_7 < 0.15$ | Approaches to the positive equilibrium point $E_4$ |
| $0.01 \leq a_{13} < 0.97$ | $0.97 \leq a_{13} < 0.13$ | Approaches to the positive equilibrium point $E_4$ |
| $a_7 = 0.11$ | | $a_{13} = 0.97$ |

The effects of varying the predation rate on susceptible prey $a_2$ in the range of $0.1 \leq a_2 < 0.29$ while keeping the other parameters as the data given in eq.(6.1), causes extinction in the infected prey and the system will approach to the infected prey free equilibrium point $E_3$, as shown in Figure-(2) a, for a typical value of $a_2 = 0.2$ . In the range of $0.29 \leq a_2 < 1$, it is observed that the solution of system (2) approaches asymptotically to the positive equilibrium point $E_4$, as shown in Figure- (2) b for a typical value of $a_2 = 0.35$. 

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Figure 2 (a): Time series of the solution of system (2) which approaches asymptotically to the infected prey free equilibrium point $E_3 = (0.298, 0, 0.205, 2.070)$ in the positive quadrant of $szw$ space, for the data given in eq. (6.1) with $a_2 = 0.2$. (b): Time series of the solution of system (2) approaches asymptotically to the positive equilibrium point in the int. $0$ of $R^4$. For the data given in eq. (6.1) with $a_2 = 0.35$.

By varying of the parameter $a_7$, which represents the conversion rate from the susceptible prey to the immature predator in the range $0.0001 \leq a_7 < 0.11$, and keeping the rest of parameters values as in the data given in eq. (6.1), the solution of system (2) still approaches asymptotically to the positive equilibrium point $E_4 = (0130, 0.144, 0.067, 0.677)$ in the int. of $R^4$. However, by increasing this parameter further to $0.11 \leq a_7 \leq 0.15$, it is observed that system (2) still approaches the infected prey free equilibrium point $E_3$, as shown in Figure 3 (b), for a typical value of $a_7 = 0.12$.

Figure 3 (a) Time series of the solution of system (2) for the data given by (6.1) with $a_7 = 0.08$, which approaches to $E_4 = (0.150, 0.139, 0.074, 0.785)$. (b) Time series of the solution of system (2) for the data given by eq. (6.1) with $a_7 = 0.12$, which approaches to $E_3 = (0.267, 0, 0.153, 1.460)$.

7. Conclusions and discussion

In this paper, we proposed and analyzed an eco-epidemiological mathematical model consisting of a prey-predator model involving an SI infectious disease in a prey-stage structured predator species with a prey refuge. Further, in this model, we used Holling type II of functional responses for the predation of susceptible and infected preys which are outside refuge, as well as a linear incidence rate for describing the transition of disease. Our aim is to study the role of infectious diseases on the dynamics. Also, system (2) was solved numerically for different sets of initial points and different sets of parameters, starting with the hypothetical set of data given by eq. System (6.1). The following observations were obtained.

- System (2) has only one type of attractor in Int. $R^4$ approaches to a globally stable point.
- For the set of hypothetical parameters' values given in eq. (6.1), system (2) approaches asymptotically to a globally stable positive point $E_4 = (0.234, 0.049, 0.117, 1.178)$. 

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As the infection rate of prey $a_1$ increases to 1.9, with keeping the rest of parameters as in eq. (6.1), the solution of system (2) approaches to the infected free equilibrium point $E_3$. However if $1.9 \leq a_1 < 2.2$, then the infected prey will grow again and then the trajectory is transferred from the infected prey free equilibrium point to the positive equilibrium point $E_4$. Thus, the parameter $a_1 = 1.9$ is a bifurcation point.

As the attack rate of the mature predator on the susceptible prey $a_2$ increases to 0.29, with keeping the rest of parameters as in eq. (6.1), the solution of system (2) approaches the infected free equilibrium point $E_3$. However, if $0.29 \leq a_2 < 1$, then the infected prey will grow again and then the trajectory is transferred from the infected prey free equilibrium point to the positive equilibrium point $E_4$. Thus, the parameter $a_2 = 0.29$ is a bifurcation point.

As the half saturation rate of the predator upon the susceptible prey increases to 0.4, with keeping the rest of parameters as in eq. (6.1), the solution of system (2) approaches to the infected free equilibrium point $E_3$. In the range $0.4 < a_3 < 1$, the trajectory is transferred from the infected free equilibrium point $E_3$ to the positive equilibrium point $E_4$. Thus, $a_3 = 0.4$ is a bifurcation point.

As the attack rate of the mature predator on the infected prey $a_4$ increases to 0.22, with keeping the rest of parameters as in eq. (6.1), the solution of system (2) approaches to the positive equilibrium point $E_4$. However, if $0.22 < a_4 < 1$, the trajectory is transferred from the positive equilibrium point $E_4$ to the disease free equilibrium point $E_3$. Thus, $a_4 = 0.22$ is a bifurcation point.

As the half saturation rate of the predator upon the infected prey increases to 0.45, with keeping the rest of parameters as in eq. (6.1), the solution of system (2) approaches to the disease free equilibrium point $E_3$. In the range $0.45 < a_5 < 0.6$, the trajectory is transferred from the disease free equilibrium point $E_3$ to the positive equilibrium point $E_4$. Thus, $a_5 = 0.45$ is a bifurcation point.

Moreover, increasing the parameter $a_6$, which represents the death rate of the infected prey due to the disease, to 0.3, and keeping the rest of parameter values as data given in eq. (6.1), the solution of system (2) approaches to the positive equilibrium point $E_4$. But, for $0.3 < a_6 < 1$, the trajectory is transferred from the positive equilibrium point $E_4$ to the infected prey free equilibrium point $E_3$ and, thus, the parameter $a_6 = 0.3$ is bifurcation point.

Now, increasing the conversion rate of food from susceptible prey to immature predator $a_7$ to 0.11, with keeping the rest of parameters as in eq. (6.1), the solution of system (2) approaches to the positive equilibrium point $E_4$. For $0.11 < a_7 < 0.15$, the trajectory is transferred from the infected prey free equilibrium point $E_3$ and, thus, the parameter $a_7 = 0.11$ is a bifurcation point.

The parameters $a_9, a_{10}, a_{11}, a_{12}$ and $m$, which represent the rate of food from infected prey to immature predator, the growth rate of immature predator onto mature predator, the natural death rate of immature predator, the conversion rate of food from susceptible prey onto mature predator, the conversion rate of food from infected prey onto mature predator, and the number of prey inside the refuge parameter respectively, are not bifurcation points of system (2), with keeping the rest of parameters as in eq. (6.1).

As the natural death rate of mature predators $a_{13}$ increases to 0.097, with keeping the rest of parameters as in eq. (6.1), the solution of system (2) approaches to the infected prey free equilibrium point $E_3$. In the range $0.097 < a_{13} < 0.15$, the trajectory is transferred from the infected prey free equilibrium point $E_3$ to the positive equilibrium point $E_4$ and, thus, $a_{13} = 0.097$ is a bifurcation point.

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