Algebraic and geometric solutions of hyperbolic Dehn filling equations.

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Abstract

In this paper we study the difference between algebraic and geometric solutions of the hyperbolic Dehn filling equations (see Definition 3.6) for ideally triangulated 3-manifolds. We show that any geometric solution is an algebraic one, and we prove the uniqueness of the geometric solutions.

Then we do explicit calculations for three interesting examples. With the first two examples we see that not all algebraic solutions are geometric and that the algebraic solutions are not unique.

The third example is a non-hyperbolic manifold that admits a positive, partially flat solution of the compatibility and completeness equations.

1 Introduction

One of the most useful tools for studying the hyperbolic structures on 3-manifolds is the technique of ideal triangulations, introduced by Thurston in [T] to study the hyperbolic structure of the complement of the figure-eight knot. An ideal triangulation of an open 3-manifold $M$ is a description of $M$ as a disjoint union of copies of the standard tetrahedron with vertices removed (ideal tetrahedron), glued together by a given set of face-pairing maps.

Given an ideal triangulation on an open 3-manifold $M$ with toroidal ends (this is known to be necessary for hyperbolicity) the idea is to construct a hyperbolic structure on $M$ by defining it on each tetrahedron and then by requiring that such structures are compatible with a global one on $M$. See [T], [NZ], [BP] for more details.

A complete finite-volume hyperbolic structure with totally geodesic faces on an oriented ideal tetrahedron is described by a complex number with positive imaginary part, called modulus (see Section 2).

The compatibility conditions on the hyperbolic structures of the tetrahedra translate to algebraic equations on the moduli. These equations depend

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on the combinatorics of the triangulation and are called *compatibility equations* (see Section 2). When the moduli induce a structure on $M$, one can ask about the completeness of such a structure, and this translates to other algebraic equations on the moduli, called *completeness equations*. Once a complete structure on $M$ is found, one can study small perturbations by taking moduli which are near to the complete solution and which satisfy the compatibility equations. By studying the completions of the structures obtained in this way, one can see that in certain cases one obtains a Dehn filling of $M$ for suitable filling-parameters. Conversely, given a Dehn filling $N$ of $M$ with fixed filling-parameters, then there exists a system of equations on the moduli, which express the fact that the completion of the hyperbolic structure induced on $M$ by the moduli is exactly $N$. We call such equations *hyperbolic Dehn filling equations*. The completeness equations can be viewed as a particular case of hyperbolic Dehn filling equations, relative to the empty filling.

As the geometric conditions translate to algebraic equations, the problem can be studied from an algebraic point of view. The question that naturally arises is

**Question 1** *Does any solution of the algebraic equations have a geometric meaning?*

It is well-known that when all the moduli have a positive imaginary part, the solutions of the compatibility equations have a natural geometric interpretation as a decomposition of $M$ into geodesic ideal tetrahedra. In general it is not clear when an algebraic solution of the equations have a geometric interpretation leading to define a finite-volume hyperbolic structure on $M$. Moreover, it is still unknown whether a hyperbolic manifolds admits a decomposition into “positive” geodesic ideal tetrahedra.

Epstein and Penner [EP] have shown that a decomposition into convex geodesic ideal polyhedra always exists. By subdividing such a decomposition flat tetrahedra can appear. This translates to the fact that some modulus becomes real. Petronio and Weeks [PW] have shown that a partially flat solution of the compatibility and completeness equations leads to a complete hyperbolic structure on $M$, while a solution of the compatibility equations alone does not in general lead to an (even incomplete) hyperbolic structure (we notice that in [PW] the system of compatibility equations contains additional equations on the angles of the moduli). By perturbing a partially flat decomposition of $M$, negatively oriented tetrahedra appear. This translates to the fact that the imaginary part of some modulus becomes negative. The geometric meaning of a solution of the equations, that involves positive, neg-

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ative and flat tetrahedra is not clear. Petronio and Porti \[\text{PP}\] have shown that the results known in the case in which all the tetrahedra are positive extend near a partially flat triangulation obtained as a subdivision of an Epstein-Penner decomposition. This leads to the following problem.

**Question 2** To understand the geometric meaning of the mixed solutions.

Finally one can ask about uniqueness

**Question 3** Are the algebraic/geometric solutions unique?

In this paper we concentrate on the differences between algebraic and geometric solutions of hyperbolic Dehn filling equations. Our interpretation of mixed solutions will be in terms of developing maps and holonomy. Roughly speaking, we will call geometric solution a choice of moduli whose holonomy is well-defined and discrete (see Definition 3.6 for the exact definition).

In this setting, we show that algebraic is strictly bigger than geometric, in the sense that any geometric solution is algebraic but non all algebraic solutions are geometric, answering to Questions 1 and 2. We remark that the difference between algebraic and geometric solutions appears only with the presence of negative tetrahedra.

Then we show that the geometric solutions are unique while the algebraic ones are not, answering to Question 3.

The paper is structured as follows:

In Section 2 we define the notions of developing map and holonomy for a triangulation with moduli.

In Section 3 we describe the system of hyperbolic Dehn filling equations, we get the definition of geometric solution and we prove the following fact:

**Theorem 3.8** Each geometric solution of the hyperbolic Dehn filling equations is also an algebraic one.

In Section 4 we prove the uniqueness of the geometric solutions.

**Theorem 4.3** There exists at most one geometric solution of the hyperbolic Dehn filling equations.

In Section 5 we do explicit calculations for some interesting examples:

- First we study two one-cusped manifolds, that are bundles over $S^1$ with fiber a punctured torus and are called $LR^3$ and $L^2R^3$. These manifolds admit more algebraic solutions and the unique geometric one.

- Then we study a manifold with non-trivial JSJ decomposition, obtained by gluing a Seifert manifold to the complement of the figure-eight knot.
This manifold is not hyperbolic but it admits a partially flat solution of compatibility and completeness equations.

The manifold $LR^3$ (and $L^2R^3$) is interesting because, on one hand it shows that the algebraic solutions are not unique, on the other hand it provides an example of an algebraic solution which is not geometric (Proposition 5.1). We notice that such a “bad” solution does not involve flat tetrahedra and has a good behavior on the boundary. Namely, the boundary torus inherits a intrinsic Euclidean structure (up to scaling). This fact is surprising because the geometry of a finite-volume hyperbolic 3-manifold is strictly related to the geometry of its boundary.

Actually the equations on the moduli have an interpretation as conditions on the geometry of the boundary. More precisely, any ideal triangulation of $M$ induces a triangulation of the boundary tori, by considering the manifold with boundary obtained by chopping off an open regular neighborhood of the ideal vertices. A modulus for the hyperbolic structure of an ideal tetrahedron determines a modulus for the similarity structure of the triangles obtained by horospherical sections near the vertices. So an ideal triangulation with moduli of $M$ induces a triangulation with moduli of the boundary tori. The compatibility equations express the fact that the moduli for the triangles lead to similarity structures on the tori. The completeness equations express the fact that the structures of the boundary tori are Euclidean.

Moreover, when the imaginary part of the moduli is not negative, the control of what happens on the boundary it suffices to guarantee the structure on the whole $M$. For example, in order to have a complete finite-volume hyperbolic structure on $M$, it suffices to check that the boundary tori have Euclidean structures.

In [F1] it is shown that any algebraic solution of the compatibility and completeness equations for the similarity structure of a triangulated torus leads to a Euclidean structure, even if there are negative triangles, provided that the algebraic sum of the areas of the triangles is not zero. So the example of $LR^3$ shows that the Euclidean situation in dimension 2 and the hyperbolic one in dimension 3 become quite different when we allow the moduli to have negative imaginary part.

Finally, the manifold with non-trivial JSJ decomposition that we study in the last example is a manifold that admits a positive, partially flat triangulation of the compatibility and completeness equations. Such a solution cannot be geometric as the manifold is not hyperbolic. This seems to contradicts [PW]. Actually there is no contradictions because in our example the conditions on the angles are not satisfied. We notice that in general not any geometric solution satisfies the conditions on the angles. Nevertheless,
this example shows that such conditions play a central role for a solution to be geometric.

2 Ideal triangulations with moduli, developing maps and holonomy

Let $M$ be the interior of a compact 3-manifold $\overline{M}$ with boundary and let $\hat{M}$ be the space obtained from $\overline{M}$ by collapsing each component of its boundary to a point. The space $\hat{M}$ is homeomorphic to the space obtained from $\overline{M}$ by gluing to each boundary component $C$ the cone over $C$. In the sequel we will often identify $M$ with its image under the projection $\overline{M} \to \hat{M}$.

Let $\Delta$ be the standard 3-simplex and let $\Delta^*$ be the standard ideal 3-simplex, i.e. $\Delta$ with vertices removed.

We define now what we mean by ideal triangulation. Roughly speaking, an ideal triangulation of $M$ is a presentation of $M$ as a union of ideal tetrahedra. By a face-pairing rule $r : F_1 \to F_2$ between two-dimensional faces of two tetrahedra (possibly the same tetrahedron) we mean a bijective correspondence $r$ from the vertices of $F_1$ to the ones of $F_2$. A realization of a face-pairing rule is a homeomorphism that extends the rule.

Definition 2.1 Let $\{\Delta_i, i \in I\}$ be a set of copies of $\Delta$ and let $\{r_j : F_{j_1} \to F_{j_2}, j \in J\}$ be a set of face-pairing rules between two-dimensional faces of the $\Delta_i$’s. We say that $\tau = (\{\Delta_i\}, \{r_j\})$ is an ideal triangulation of $M$ if for each $j \in J$ there exists a set $\{f_j : F_{j_1} \to F_{j_2}, j \in J\}$ of simplicial maps such that each $f_j$ is an extension of the rule $r_j$ and such that there exists a homeomorphism $\varphi : (\bigcup_i \Delta_i^*)/\{f_j\} \to M$. If $M$ is oriented, we fix an orientation for $\Delta$ and we require the $r_j$’s to be orientation reversing and $\varphi$ to be orientation preserving. We say that $\mathcal{R} = (\{f_j\}, \varphi)$ is a realization of $\tau$.

Given a realization of $\tau$, for each $i \in I$ we set $\varphi_{\Delta_i}$ to be the composition of $\varphi$ with the inclusion $\Delta_i^* \to \bigcup_i (\Delta_i^*)/\{f_j\}$.

To require $\varphi$ to be a homeomorphism from $(\bigcup_i \Delta_i^*)/\{f_j\}$ to $M$ is equivalent to require $\varphi$ to extend to a homeomorphism from $(\bigcup_i \Delta_i)/\{f_j\}$ to $\hat{M}$ such that it is one-to-one from $(\bigcup_i \Delta_i/\{f_j\}) \setminus (\bigcup_i \Delta_i^*/\{f_j\})$ to $\hat{M} \setminus M$. Hence the vertices of $\bigcup_i \Delta_i/\{f_j\}$ are in one-to-one correspondence with the connected components of $\partial \overline{M}$.

Let $U$ be an open regular neighborhood of the 0-skeleton of a tetrahedron $\Delta$, we call $\Delta^-$ the truncated tetrahedron $\Delta \setminus U$ and we call $\partial^- \Delta = \partial U$.

Given an ideal triangulation $\tau$ it is possible to truncate each $\Delta_i$ in such a way that $(\bigcup_i \Delta^-_i/\{f_j\}, \bigcup_i \partial^- \Delta_i/\{f_j\})$ is homeomorphic to $(\overline{M}, \partial \overline{M})$. In
other words, each ideal triangulation induces a triangulation of $\partial \overline{M}$ (here triangulation means possibly with auto-adjacencies).

We introduce now the notion of modulus of a hyperbolic ideal tetrahedron. Let $A$ be a straight ideal tetrahedron in $\mathbb{H}^3$, i.e. a geodesic tetrahedron in $\mathbb{H}^3$ such that $\partial \mathbb{H}^3 \cap A$ is the 0-skeleton of $A$. Since such a tetrahedron is the convex hull of its vertices, it is completely determined by them.

The hyperbolic ideal tetrahedra that we consider in the sequel can be flat, but not degenerate. This means that a tetrahedron can be completely contained in a 2-plane, but we always require that its vertices are four distinct points.

An orientation of an abstract tetrahedron is an ordering of its vertices up to even permutations. When $A$ is fat (i.e. it is not contained in a 2-plane), the orientations of $A$ as a subset of $\mathbb{H}^3$ are in correspondence with its orientations as an abstract tetrahedron.

Consider now the half-space model $\mathbb{C} \times \mathbb{R}^+$ of $\mathbb{H}^3$ and let $(v_1, v_2, v_3, v_4)$ be an ordering of the vertices of $A$. Then the Isom$^+(\mathbb{H}^3)$-class of $A$ is determined by a complex number $z$ by mapping $(v_1, v_2, v_3, v_4)$ to $(0, 1, \infty, z)$ via an isometry. Since Isom$^+(\mathbb{H}^3)$ acts transitively on the set of triples of points at infinity, it is always possible to map $(v_1, v_2, v_3)$ to $(0, 1, \infty)$ via an element of Isom$^+(\mathbb{H}^3)$. Moreover such an isometry is unique and the number $z$ is the cross-ratio $[v_1 : v_2 : v_3 : v_4]$.

Since the vertices of $A$ are four distinct points in $\mathbb{C} \cup \{\infty\}$, it follows that $z \not\in \{0, 1, \infty\}$. By slicing $A$ with a horosphere $\mathbb{C} \times \{t\}$, for a sufficiently large $t$ we obtain a triangle with a Euclidean structure. Up to scaling, this structure is exactly the one of the triangle in $\mathbb{C}$ with vertices in $\{0, 1, z\}$. It follows that the hyperbolic structure of $A$ is determined by the similarity structure (Euclidean up to scaling) of a horospherical triangle at a vertex of $A$.

It is easily checked that as the ordering of the vertices varies in the same orientation class, then $z$ varies on the set $\{z, \frac{1}{1-z}, 1 - \frac{1}{z}\}$. This ambiguity can be avoided by fixing a preferred edge $e$ of $A$ and arranging the vertices $(v_1, v_2, v_3, v_4)$ in such a way that $e$ joins $v_1$ and $v_3$ (note that $[v_1 : v_2 : v_3 : v_4] = [v_2 : v_4 : v_1 : v_2]$). The number $z$ is called modulus of $A$ associated to $e$. It is easy to see that at opposite edges is associated the same modulus.

**Remark 2.2** In the sequel we tacitly assume that an orientation and an edge for each tetrahedra have been fixed.

From now on we consider $M$ to be oriented.
Definition 2.3 Let $\tau$ be an ideal triangulation of $M$. A choice of moduli $z = \{z_i, \ i \in I\}$ for $\tau$, is a choice of a complex number $z_i \neq 0, 1$ for each tetrahedron $\Delta_i$ of $\tau$. We write $(\tau, z)$ to mean an ideal triangulation $\tau$ with a choice of moduli $z$ for $\tau$.

Definition 2.4 Let $\Delta$ be the standard tetrahedron. We say that a continuous map $f : \Delta \to \mathbb{H}^3$ is straight if

- $f$ maps the vertices of $\Delta$ to $\partial \mathbb{H}^3$;
- for each sub-simplex $F$ of $\Delta$, the image of $F$ is contained in the convex hull of the image of its vertices;
- for each sub-simplex $F$ of $\Delta$, the map $f|_F$ is a homeomorphism whenever $f|_{\partial F}$ is a homeomorphism.

Definition 2.5 Let $z$ be a choice of a modulus for a tetrahedron $\Delta$ and $\xi : \Delta \to \mathbb{H}^3$ be a continuous map.

We say that $\xi$ is compatible with $z$ if it is straight and its image is a geodesic ideal tetrahedron of modulus $z$.

Definition 2.6 Let $R = (\{f_j\}, \varphi)$ be a realization of an ideal triangulation with moduli $(\tau, z)$ of $M$. Let $\{g_i : \Delta_i \to \mathbb{H}^3\}$ be a set of maps, each one compatible with the relative $z_i$. For each $j \in J$ let $\psi_j$ be the unique orientation-preserving isometry which realizes the face-pairing rule $r_j$ between the corresponding faces of the hyperbolic tetrahedra $g_i(\Delta_i)$.

We say that $\{g_i\}$ is compatible with $R$ if for each $j$, called $\Delta_{i_1}$ and $\Delta_{i_2}$ the tetrahedra glued by the rule $r_j$, we have that

$$f_j = g_{i_2}^{-1} \circ \psi_j \circ g_{i_1}. \quad (1)$$

Remark 2.7 Each time we have some covering and some object $o$ which can be lifted in some natural way, as usual we call $\tilde{o}$ a lift of $o$. When it is clear, we leave to the reader the proofs that what we do is independent from the chosen lift. In the following $\tilde{M}$ will be the universal covering of $M$ and $\tilde{\tau}$ the lift of $\tau$ on $\tilde{M}$.

Given $(\tau, z)$, the idea is to define a hyperbolic structure on $M$ by taking, for each $i$, an ideal hyperbolic straight tetrahedron of modulus $z_i$ and then by gluing them realizing the rules $r_j$ via isometries.

In order to succeed in this construction it is easy to see that a necessary condition is that for each edge $e$ of $\tau$ the product of moduli around $e$ is 1. As mentioned above, if $z_i$ is the modulus of $\Delta_i$ associated to an edge, then
changing edge, the modulus change in the set \( \{ z_i, \frac{1}{1-z_i}, 1 - \frac{1}{z_i} \} \). It follows that the condition of the product of moduli around the edges can be written as a system \( C \) of algebraic equations on the moduli, having the form

\[
\pm \prod_i z_i^{\alpha_i} (1 - z_i)^{\beta_i} = 1
\]

where \( \alpha_i, \beta_i \in \mathbb{Z} \) depend on the triangulation and on the chosen preferred edges of the tetrahedra. These equations are called compatibility equations.

**Lemma 2.8** Let \( (\tau, z) \) be an ideal triangulation with moduli on \( M \) such that equations \( C \) are satisfied.

Then for each realization \( \mathcal{R} = (\{f_j\}, \varphi) \) of \( \tau \) there exists a set of maps \( \{ g_i : \Delta_i \to \mathbb{H}^3 \} \) compatible with \( \mathcal{R} \).

**Proof.** We define the \( g_i \)'s recursively on the \( n \)-skeleta of \( \tau \). On the 0-skeleton we define the maps simply looking at the compatibility with the moduli. Then a set of maps \( \psi_j \in \text{Isom}^+(\mathbb{H}^3) \) as in Definition 2.6 is well-defined.

Let \( e \) be an edge of a tetrahedron \( \Delta_{i0} \) with vertices \( e_0 \) and \( e_1 \). We set \( g_{i0} \) on \( e \) to be an homeomorphism onto the geodesic between \( g_{i0}(e_0) \) and \( g_{i0}(e_1) \).

Now we define the \( g_i \)'s on the edges glued to \( e \) by the maps \( f_j \) using the formula 1 of Definition 2.6. Note that since \( C \) holds this is a good definition.

We define the \( g_i \)'s on the others edges in a similar way. Once the \( g_i \)'s are defined on the 1-skeleton there are no problems to use again the formula 1 to define them on the 2-skeleton and there are no obstructions to extend such maps to the 3-cells.

\[\square\]

**Remark 2.9** From now on when we speak about an ideal triangulation, we suppose that a realization \( \mathcal{R} \) has been fixed and that each set of maps compatible with the moduli is also compatible with \( \mathcal{R} \).

**Definition 2.10** Let \( (\tau, z) \) be an ideal triangulation with moduli of \( M \). A developing map for \( (\tau, z) \) is a continuous map \( D : \tilde{M} \to \mathbb{H}^3 \) such that for each \( i \), the map \( \varphi_{\Delta_i} \) lifts to a map \( \tilde{\varphi}_{\Delta_i} \) such that \( D \circ \tilde{\varphi}_{\Delta_i} \) extends to a map from \( \Delta_i \) to \( \mathbb{H}^3 \) which is compatible with \( z_i \). Moreover, if \( \tilde{\varphi}_{\Delta_i}^1 \) and \( \tilde{\varphi}_{\Delta_i}^2 \) are two lifts of the same map, then we require that there exists an element \( \Phi \) of \( \text{Isom}^+(\mathbb{H}^3) \) such that \( D \circ \tilde{\varphi}_{\Delta_i}^1 = \Phi \circ D \circ \tilde{\varphi}_{\Delta_i}^2 \).

Let \( (\tau, z) \) be a triangulation with moduli of \( M \) and suppose \( C \) holds. Then there exists a representation \( h : \pi_1(M) \to \text{Isom}^+(\mathbb{H}^3) \) which is well-defined up to conjugation, defined as follows:
Let $\Delta_{i_1}$ be a base-tetrahedron. A path of tetrahedra is a sequence $\gamma = \Delta_{i_1} \xrightarrow{r_{j_1}} \Delta_{i_2} \cdots \xrightarrow{r_{j_s}} \Delta_{i_{s+1}}$ such that the $\Delta_{i_k}$’s are tetrahedra of $\tau$ and $r_{j_k}$ is a face pairing rule of $\tau$ from a face of $\Delta_{i_k}$ to one of $\Delta_{i_{k+1}}$. A loop is a path in which $i_{s+1} = i_1$. For each $i$ let $g_i : \Delta_i \to \mathbb{H}^3$ be a map compatible with $z_i$ and for each $j$ let $\psi_j$ the only orientation preserving isometry which realizes the rule $r_j$ between the corresponding faces of the geodesic tetrahedra $g_i(\Delta_i)$. We define $h$ first on the set of paths of tetrahedra by setting

$$h(\gamma) = \psi_{j_1} \circ \cdots \circ \psi_{j_s}.$$ 

Let $P$ be the space of loops of tetrahedra. It is a standard fact that $P$ projects to $\pi_1(M,x_0)$, where $x_0$ is any point of the image of $\varphi_{\Delta_{i_1}}$. The fact that $\mathcal{C}$ holds imply that $h(\gamma) = h(\gamma')$ if $[\gamma] = [\gamma']$ in $\pi_1(M,x_0)$. So $h$ induces a map $h : \pi_1(M,x_0) \to \text{Isom}^+(\mathbb{H}^3)$. It is easily checked that $h$ is a representation and that its conjugacy class does not depend on the choices of the maps $g_i$ and of the base-tetrahedron.

The representation $h$ is called holonomy of $z$.

**Proposition 2.11** Let $(\tau,z)$ be a triangulation with moduli of $M$ such that $z$ is a solution of equations $\mathcal{C}$. Then there exist a developing map $D$ and a representative of the holonomy such that $D$ is $\pi_1(M)$-equivariant, where $\pi_1(M)$ acts on $\tilde{M}$ by deck transformations and on $\mathbb{H}^3$ via $h$.

**Proof.** Let $\tilde{\tau}$ be the triangulation of $\tilde{M}$ and let $\Delta_{i_1}$ be a base tetrahedron of $\tilde{\tau}$. We do the same construction used to define the holonomy. As above we fix maps $g_i$ (the same $g_i$ for all lifts of the same tetrahedron of $\tau$) and the isometries $\psi_j$, and as above we define a map $\tilde{h}$ from the space of paths of tetrahedra of $\tilde{\tau}$ to $\text{Isom}^+(\mathbb{H}^3)$.

Now let $\Delta_k$ be a tetrahedron of $\tilde{\tau}$ and let $\gamma$ be a path from $\Delta_{i_1}$ to $\Delta_k$. We define $D_k : \Delta_k \subset \tilde{\Delta_k} \to \mathbb{H}^3$ by $D_k = \tilde{h}(\gamma) \circ g_k$. Since $\tilde{M}$ is simply connected, then the definition of $D_k$ does not depend on the choice of $\gamma$. Moreover, since the $g_i$’s are compatible with the moduli, the union of the $D_k$ projects to a developing map $D : \tilde{M} \to \mathbb{H}^3$.

If we fix a base point in $\tilde{M}$ in the base-tetrahedron, then a representative $h$ of the holonomy is fixed; and the $\pi_1(M)$-equivariance of $D$ follows by construction. \qed

**Remark 2.12** It is easy to see that if a developing map exists, then $\mathcal{C}$ holds and so the holonomy is defined. Moreover, every developing map can be
obtained via the above construction. It follows that for each developing map $D$ the choices of a base point and its lift determine a representative $h$ of the holonomy such that $D$ is $\pi_1(M)$-equivariant.

**Definition 2.13** Let $(\tau, z)$ be a triangulation with moduli of $M$ such that $z$ is a solution of equations $C$. Let $N$ be an oriented, complete hyperbolic 3-manifold (hence $\tilde{N} = \mathbb{H}^3$). We call a map $f : M \to N$ hyperbolic if its lift $\tilde{f} : \tilde{M} \to \mathbb{H}^3$ is a developing map.

**Remark 2.14** Let $M, \tau, z, N, f$ as in Definition 2.13. Let $V(\Delta_i)$ denote the 0-skeleton of $\Delta_i \in \tau$. The maps $\varphi_{\Delta_i}$, and so also $f \circ \varphi_{\Delta_i}$, are defined only on $\Delta_i^*$. Nevertheless, since $\tilde{f}$ is a developing map, then the maps $\tilde{f} \circ \tilde{\varphi}_{\Delta_i}$ extend to the whole $\Delta_i$. Thus $\tilde{f} \circ \tilde{\varphi}_{\Delta_i}(V(\Delta_i))$ is well-defined.

**Definition 2.15** Let $M, \tau, z, N, f$ be as in Definition 2.13. Let $\gamma$ be an oriented geodesic in $N$. Let $v$ be a vertex of $\tau$. We say that $f$ spirals around $\gamma$ near $v$ if, in a suitable half-space model of $\mathbb{H}^3$ in which $\tilde{\gamma}$ is the oriented line $(0, \infty)$, for each tetrahedron $\Delta_i$ having $v$ as a vertex there exists a lift $\tilde{f} \circ \tilde{\varphi}_{\Delta_i}$ which maps $v$ to $\infty$.

**Proposition 2.16** Let $M, \tau, z, N, f$ be as in definition 2.13. Let $\Gamma$ be the subgroup of $\text{Isom}(\mathbb{H}^3)$ such that $N = \mathbb{H}^3 / \Gamma$. Then the image of a holonomy representation relative to $\tilde{f}$ consists of elements of $\Gamma$. Moreover, called $h_N$ the holonomy of $N$, i.e. the isomorphism $h_N : \pi_1(N) \to \Gamma$, we have that $h = h_N \circ f_*$. 

*Proof.* The group $\pi_1(M)$ acts on $\tilde{M}$ via deck transformations and $\pi_N$ acts on $\tilde{N} = \mathbb{H}^3$ via $h_N$. It is possible to chose the base-points in such a way that for all $\alpha \in \pi_1(M)$ and for all $x \in \tilde{M}$ we have $\tilde{f}(\alpha(x)) = h_N f_*(\alpha)(\tilde{f}(x))$. Since $\tilde{f}$ is a developing map, then $f(\alpha(x)) = h(\alpha)(\tilde{f}(x))$. It follows that $h_N f_*(\alpha)$ and $h(\alpha)$ coincide on the image of $\tilde{f}$. Since the dimension of $\text{Im}(...)$ is at least two and since both $h(\alpha)$ and $f_*(\alpha)$ are orientation preserving isometries, then they coincide on the whole $\mathbb{H}^3$.

3 Hyperbolic Dehn filling equations

First of all, we recall the definition of Dehn filling of a manifold.
Definition 3.1 Let $M$ be the interior of an oriented 3-manifold $\overline{M}$ such that $\partial \overline{M}$ is a union of tori, $\partial \overline{M} = \sqcup_1 T_n$. For each $T_n$ let $(\mu_n, \lambda_n)$ be a basis for $H_1(T_n, \mathbb{Z})$. Let $(p, q) = \{(p_n, q_n)\}$ where $(p_n, q_n)$ is either a pair of coprime integers or the symbol $\infty$. For each $n$ such that $(p_n, q_n) \neq \infty$, let $L_n$ be an oriented solid torus, $m_n$ be a meridian of $T_n' = \partial L_n$, $l_n$ be a loop in $T_n$ such that $[l_n] = p_n \mu_n + q_n \lambda_n$ and $\varphi_n : T_n \to T_n'$ be an orientation reversing homeomorphism such that $\varphi_n(l_n) = m_i$.

The Dehn filling of $M$ with parameters $(p, q)$ is the manifold

$$M(p, q) = \overline{M} \sqcup \{L_n\}/\{\varphi_n\}$$

The tori $L_n$ are called filling tori.

Remark 3.2 The resulting manifold $M(p, q)$ actually depends only on the coefficients $(p, q)$ and not on the maps $\varphi_n$’s.

Remark 3.3 Not all the boundary tori are filled in $M(p, q)$. Namely, a torus $T_n$ is filled if and only if $(p_n, q_n) \neq \infty$. If $(p_n, q_n) = \infty$ for all $n$, then $M(p, q) = M$.

Suppose that we have an ideal triangulations of a manifold $M$ with a choice of moduli that satisfy $\mathcal{C}$. We recall that if each $z_n$ has positive imaginary part then the moduli define an (incomplete) hyperbolic structure on $M$. In this section we introduce some equations on the moduli, which we call hyperbolic Dehn filling equations. When the moduli have positive imaginary part, such equations imply that the completion of the hyperbolic structure defined by the moduli on $M$ is a fixed Dehn filling of $M$.

More generally, these equations can be written down even without restrictions on the imaginary part of the moduli. In that case, in general, there is not an obvious geometric interpretation of the solutions of the equations. For this reason we distinguish between algebraic and geometric solutions of the hyperbolic Dehn filling equations.

The principal condition expressed by the equations is that if $m$ is a loop in a boundary torus killed in homology by the filling, then $h(m) = 1$.

From now on, let $M$ be the interior of an oriented compact 3-manifold $\overline{M}$ such that $\partial \overline{M}$ is the disjoint union of $k$ tori, let $\tau$ be an ideal triangulation of $M$ and let $z$ be a choice of moduli for $\tau$ satisfying the compatibility equations $\mathcal{C}$.

Let $T \subset \partial \overline{M}$ be a boundary torus. We push $T$ a little bit in $M$ and we consider the image of the natural map $\pi_1(T) \to \pi_1(M)$. We consider the half-space model $\mathbb{C} \times \mathbb{R}^+$ of $\mathbb{H}^3$ and a developing map $D$ such that the vertex relative to $T$ is lifted to a vertex mapped to $\infty$ via $D$. Then there exists
a choice of the base-points such that the image of the holonomy \( h(\pi_1(T)) \) consists of maps which fix \( \infty \). It follows that by considering the restriction to \( \partial \mathbb{H}^3 \equiv \mathbb{C}P^1 \) of the maps in \( h(\pi_1(T)) \), we obtain a representation \( h_T \) of \( \pi_1(T) \) in the automorphism of \( \mathbb{C} \). Moreover, since the restriction to \( \partial \mathbb{H}^3 \) of a positive isometry is a Moebius transformation, then \( h_T \) actually is a representation \( h_T : \pi_1(T) \to \text{Aff}(\mathbb{C}) \). Since \( h \) is well-defined up to conjugation, then the dilation component of \( h_T \) is well-defined, which is a representation \( \rho_T : \pi_1(T) \to \mathbb{C}^* \).

Since \( \pi_1(T) \) is Abelian, it follows that \( h_T(\pi_1(T)) \) consists of maps which commute with each other. Then it is easy to see that either they are all translations, or they have a common fixed point. In the former case we have \( \rho_T \equiv 1 \). In the latter case, up to conjugation, we can suppose that the fixed point is 0. Thus we get \( h_T = \rho_T \), in the sense that for all \( \alpha \in \pi_1(T) \) and \( \zeta \in \mathbb{C} \), \( h_T(\alpha)[\zeta] = \rho_T(\alpha) \cdot \zeta \).

**Remark 3.4** In the following, if there are no ambiguities, by writing \( \rho_T \equiv 1 \) we mean that \( h_T(\pi_1(T)) \) consists of translations and by \( h_T = \rho_T \) we mean that \( h_T(\pi_1(T)) \) consists of maps which fix 0. We notice that \( h \), \( h_T \) and \( \rho_T \) depend on \( z \). When we need to emphasize that, we write \( \rho(z) \) and so on.

To write the equations, we need to work with \( \log(\rho_T) \). In the following definition we fix a suitable determination of the logarithm of \( \rho_T \). Formally, a boundary torus \( T \) of \( \overline{M} \) is not contained in \( M \), so a regular neighborhood \( U \) of \( T \) is not a sub set of \( M \). If there are no ambiguities, we does not distinguish between \( U \) and \( U \cap M \).

**Definition 3.5** Let \( M, \tau, z \) be as above and let \( D \) be a developing map. Let \( T \) be a boundary component of \( \overline{M} \) and \( \widetilde{T} \) be a lift of \( T \). Consider the model \( \mathbb{C} \times \mathbb{R}^+ \) of \( \mathbb{H}^3 \) such that the vertex relative to \( \widetilde{T} \) is mapped to \( \infty \). Suppose that \( h_T = \rho_T \) and suppose that the following condition holds:

There exists regular neighborhood \( \widetilde{U} \subset \overline{M} \) of \( \widetilde{T} \) such that the developed image of \( \widetilde{U} \) does not intersect the line \( (0, \infty) \).

Then we choose a determination of \( \log(\rho_T) \) as follows: let \( H \) be the universal covering of \( \mathbb{H}^3 \setminus (0, \infty) \) made by using the covering \( \exp : \mathbb{C} \to \mathbb{C}^* \).

Let \( x_0 \) and \( \widetilde{x}_0 \) be base points in \( T \) and \( \widetilde{T} \). Let \( \gamma : [0, 1] \to T \) be a loop at \( x_0 \) and \( \widetilde{\gamma} \) be its lift starting from \( \widetilde{x}_0 \). After pushing a little \( T \) inside \( M \), let \( \widetilde{\alpha} : [0, 1] \to \mathbb{C}^* \) be the complex component of \( D \circ \widetilde{\gamma} \). As \( D \circ \widetilde{\gamma} \) lifts to \( H \), the path \( \widetilde{\alpha} \) lifts to a path \( \overline{\alpha} : [0, 1] \to \mathbb{C} \). Since \( h_T = \rho_T \), then \( \widetilde{\alpha}(1) = \rho_T(\gamma)[\overline{\alpha}(0)] \), and then \( \alpha(1) = \log(\rho_T(\gamma)) + \overline{\alpha}(0) \).
The points $\alpha(0)$ and $\alpha(1)$ depend only on the homotopy class of $\gamma$ and on the choice of the base-points. If we change the base-points, then the determination of $\log(\rho_T([\gamma]))$ changes by a conjugation by translations and so it is well-defined.

Let us now fix a basis $(\mu, \lambda)$ for $H_1(T, \mathbb{Z})$ and let $(a, b)$ be a pair of coprime integers. Consider the Dehn filling of $M$ with parameters $(a, b)$, i.e. the filling in which the meridian $m$ of the solid torus is mapped to an oriented loop homotopic to $a\mu + b\lambda$. So the coefficient $(a, b)$ induces an orientation of $m$. Since the gluing map inverts the orientations of the boundary tori, then the core $\gamma$ of the filling tours is canonically oriented by requiring that $m$ turns around $\gamma$ by following the right-hand rule in the oriented solid tours.

We are now ready to give the hyperbolic Dehn filling equations.

**Definition 3.6** Let $M$ be the interior of an oriented compact 3-manifold $\overline{M}$ such that $\partial \overline{M}$ is the disjoint union of $k$ tori, let $\tau$ be an ideal triangulation of $M$ and let $z$ be a choice of moduli for $\tau$ satisfying the compatibility equations $C$.

For each boundary torus $T_n$ let $(\mu_n, \lambda_n)$ be a basis for $H_1(T_n, \mathbb{Z})$. Let $(p, q) = \{(p_n, q_n), \; n = 1, \ldots, k\}$ be such that $(p_n, q_n)$ is either a pair of coprime integers or the symbol $\infty$. Let $\rho_n(z)$ be the dilation component of the holonomy of $T_n$ when $z$ varies on the space of solutions of the compatibility equations.

We say that $z$ is an algebraic solution of the $(p, q)$-equations if for each $n = 1, \ldots, k$ we have:

- If $(p_n, q_n) = \infty$ then $\rho_n(z) \equiv 1$.
- If $(p_n, q_n) \neq \infty$ then $h_{T_n}(z) = \rho_n(z)$, the condition of Definition 3.5 holds, and

\[
p_n \log(\rho_n(z)[\mu_n]) + q_n \log(\rho_n(z)[\lambda_n]) = 2\pi i.
\]

We say that $z$ is a geometric solution of the $(p, q)$-equations if, called $N = M_{(p, q)}$ the Dehn filling of $M$ with parameters $(p_n, q_n)$, we have:

a) $N$ is complete hyperbolic and the cores of filling tori are disjoint geodesics $\{\gamma_n\}$.

b) There exists a proper map $f : M \to N \setminus \{\gamma_n\} \subset N$ of degree 1, which is hyperbolic w.r.t. $z$. 

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c) For each boundary torus $T_n$ with $(p_n, q_n) \neq \infty$, called $v_n$ the vertex correspondent to $T_n$, $f$ spirals around the relative $\gamma_n$ near $v_n$ ($\gamma_n$ has the orientation induced by the Dehn filling coefficient $(p_n, q_n)$).

**Remark 3.7** When all the coefficients $(p_n, q_n)$ are $\infty$, then the system of the $(p, q)$-equations is exactly the classical system $M$ of the so-called completeness equations. When the moduli have positive imaginary part the equations $M$ imply that the hyperbolic structure defined by the moduli on $M$ is complete (of finite volume).

**Theorem 3.8** Let $M, \tau, \{(\mu_n, \lambda_n)\}, (p, q)$ be as in Definition 3.6. Then each geometric solution of the $(p, q)$-equations is also algebraic.

**Proof.** Let $z$ be a geometric solution of the $(p, q)$-equation.

Let $\Gamma$ be the subgroup of $\text{Isom}^+(\mathbb{H}^3)$ such that $N = \mathbb{H}^3/\Gamma$. So the holonomy of $N$, as a hyperbolic manifold, is an isomorphism $h_N : \pi_1(N) \to \Gamma$. By Proposition 2.16 the holonomy of $M$ is obtained by composing the homomorphism $f_\ast : \pi_1(M) \to \pi_1(N)$ with $h_N$.

Each geodesic $\gamma_n$ (the cores of the filling tori) can be viewed as an element of $\pi_1(N)$. For each $n$, let $\Gamma_n \subset \Gamma$ be the set of all conjugates of $h_N(\gamma_n)$ and let $P$ be the set of all parabolic elements of $\Gamma$. Note that $P$ is exactly the image of all boundary elements of $\pi_1(N)$. Since $f$ is proper, $\tilde{f}$ maps each vertex of $\tilde{\tau}$ to a fixed point of an element either of $\bigcup_n \Gamma_n$ or of $P$.

Moreover $f$ is surjective on $N \setminus \{\gamma_n\}$ because it has degree 1. Since $f$ spirals around $\gamma_n$ near $v_n$, it maps the unfilled cusps of $M$ to the cusps of $N$. This implies that for $(p_n, q_n) = \infty$ the holonomy of $T_n$ consists of parabolic elements and so $\rho_n \equiv 1$.

If $(p_n, q_n) \neq \infty$ then the image of $h_{T_n}$ is contained in the subgroup of $\Gamma$ generated by $\gamma_n$, so $h_{T_n} = \rho_n$. The fact that $\text{Im}(f) = N \setminus \{\gamma_n\}$ implies the condition of Definition 3.5 and the fact that $N = M_{(p, q)}$ implies that $p_n \log(\rho_n(z)[\mu_n]) + q_n \log(\rho_n(z)[\lambda_n]) = 2\pi i$.

For each subgroup $\Gamma$ of $\text{Isom}(\mathbb{H}^3)$, let $\text{Fix}(\Gamma)$ denote the set of fixed points of all the elements of $\Gamma$.

**Remark 3.9** In the proof of Theorem 3.8 we showed that for each $\Delta_n$ of $\tau$, we have $\tilde{f} \circ \tilde{\varphi}_{\Delta_n}(V(\Delta_n)) \subset \text{Fix}(\Gamma)$.

**Remark 3.10** It is well-known that if each $z_n$ has positive imaginary part, then an algebraic solution is also a geometric solution.

**Remark 3.11** In Section 5 we give examples of an algebraic solutions that are not geometric.
4 Uniqueness

In this section we prove the uniqueness of the geometric solutions. We'll need the following version of the rigidity theorem for complete hyperbolic 3-manifolds of finite volume, which can be found in [BCS] and in [F2].

Theorem 4.1 (Strong statement of Mostow’s rigidity) Let $M_1$ and $M_2$ be two complete connected hyperbolic 3-manifolds of finite volume. Let $f : M_1 \to M_2$ be a continuous proper map such that $\text{vol}(M_1) = |\text{deg}(f)| \text{vol}(M_2)$. Then $f$ is proper homotopic to a locally isometric covering of degree $\text{deg}(f)$ of $M_1$ onto $M_2$.

This theorem in particular implies that a 3-manifold carries at most one hyperbolic structure up to isometries. In the following when we speak about a hyperbolic 3-manifold $M$, we mean that $M$ is equipped with its unique hyperbolic structure.

In the following we fix an oriented 3-manifold $M$ with $k$ toroidal cusps equipped with an ideal triangulation $\tau = (\{\Delta_i\}, \{r_j\})$ and a realization $\mathcal{R} = (\{f_j\}, \varphi)$. For each boundary torus $T_n$ we fix a basis $(\mu_n, \lambda_n)$ for $H_1(T_n, \mathbb{Z})$. We fix the Dehn filling coefficients $(p, q) = \{(p_n, q_n)\}$ and a geometric solution $z = \{z_n\}$ of the $(p, q)$-equation.

Let $N = M_{(p,q)}$ equipped with its hyperbolic structure $N = \mathbb{H}^3/\Gamma$ with $\Gamma < \text{Isom}(\mathbb{H}^3)$ and let $\{\gamma_n\}$ be the set of the geodesic cores of the filling tori.

The following proposition shows that the set of the geodesics $\gamma_n$ is uniquely determined by $(p, q)$.

Proposition 4.2 Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two finite-volume, complete hyperbolic structure on $N$ such that the $\gamma_n$’s are geodesics w.r.t. both $\mathcal{G}_1$ and $\mathcal{G}_2$.

Then there exists an isometry $\alpha : (N, \mathcal{G}_1) \to (N, \mathcal{G}_2)$ such that for each $n \alpha(\gamma_n) = \gamma_n$.

Proof. By rigidity, the identity $\text{Id} : (N, \mathcal{G}_1) \to (N, \mathcal{G}_2)$ is homotopic to an isometry $\alpha$. Thus for each $n$ the loop $\gamma_n$ is homotopic to $\alpha(\gamma_n)$.

By hypothesis $\gamma_n$ is geodesic w.r.t. both $\mathcal{G}_1$ and $\mathcal{G}_2$. Since $\alpha$ is an isometry it follows that $\alpha(\gamma_n)$ is a geodesics w.r.t. $\mathcal{G}_2$. Hence $\gamma_n$ and $\alpha(\gamma_n)$ are geodesics w.r.t. $\mathcal{G}_2$ and are homotopic, and so they must coincide. \qed

The remaining part of the section is devoted to prove the following

Theorem 4.3 In the hypotheses fixed at the beginning of this section, the moduli $z_i$’s are uniquely determined by the coefficients $(p, q)$. 

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Proof. Let $f$ be a hyperbolic map as in Definition 3.6. By definition of hyperbolic map, the lift $\tilde{f}$ is a developing map. By Proposition 2.16 the holonomy of $z$ is the composition $h = h_N \circ f_*$, where $h_N$ is the holonomy of the hyperbolic manifold $N$.

Let $v_n$ be the vertex of $\tau$ relative to the $n$th cusp of $M$ and let $\tilde{v}_n$ be one of its lifts. Let $\text{Stab}(\tilde{v}_n)$ be the stabilizer of $\tilde{v}_n$ in $\pi_1(M)$ (acting on $\tilde{M}$ via deck transformations). Then $\tilde{f}(\tilde{v}_n)$ is fixed by $h(\text{Stab}(\tilde{v}_n))$. If $T_n$ is the boundary torus relative to $v_n$, then $\text{Stab}(\tilde{v}_n)$ is conjugate to $\pi_1(T_n)$. It follows that $h(\text{Stab}(\tilde{v}_n))$ is either parabolic (if $(p_n, q_n) = \infty$) or generated by a conjugate of $h_N(\gamma_i)$. In the former case $h(\text{Stab}(\tilde{v}_n))$ consists of exactly one point and so $\tilde{f}(\tilde{v}_n)$ is completely determined. In the latter case $h(\text{Stab}(\tilde{v}_n))$ consists of two points, but the condition c) of Definition 3.6 allows us to determine $\tilde{f}(\tilde{v}_n)$.

Since $\tilde{f}$ is a developing map, then the modulus of a tetrahedron is completely determined by the $\tilde{f}$-image of its vertices. It follows that the moduli $z_i$’s depend only on $h_N \circ f_*$. We now prove that the moduli does not depend on $f$.

Let $g$ be another hyperbolic map as in Definition 3.6. Let us call $h_f$ and $h_g$ the holonomy relative respectively to $f$ and $g$. From the properties of the holonomy (see Section 2) it follows that there exists an element $\varphi \in \text{PSL}(2, \mathbb{C})$ such that $h_g = \varphi h_f \varphi^{-1}$. Thus if a point $p$ is fixed by $h_f(\text{Stab}(\tilde{v}_n))$, then the point $\varphi(p)$ is fixed by $\varphi h_f(\text{Stab}(\tilde{v}_n))\varphi^{-1} = h_g(\text{Stab}(\tilde{v}_n))$. By using again the condition c) of Definition 3.6 we see that $\tilde{g}(v) = \varphi \tilde{f}(v)$ for each vertex $v$ of $\tilde{\tau}$. Since $\varphi$ is an isometry, then the modulus of a tetrahedron of $\tilde{\tau}$ depends only on $h_N$. In other words the moduli $z_i$’s depend only on the hyperbolic structure of $N$, which is unique by Theorem 4.1 and the next Lemma.

\begin{lemma}

The manifold $N$ has finite volume.

\end{lemma}

Proof. Let $\text{vol}(\Delta_i)$ denote the hyperbolic volume of the ideal tetrahedron of modulus $z_i$, where $\text{vol}(\Delta_i)$ is taken negative if $\text{Im}(z_i) < 0$. Since $f$ is a hyperbolic map, then the volume of its image satisfies $\text{vol}(\text{Im}(f)) \leq \sum |\text{vol}(\Delta_i)| < \infty$.

Moreover $f$ is a degree 1 map from $N \setminus \{\gamma_n\}$ to $N \setminus \{\gamma_n\}$, hence $\text{vol}(N) = \text{vol}(N \setminus \{\gamma_n\}) = \text{vol}(\text{Im}(f)) < \infty$.

\end{proof}

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5 Examples

In this section we do explicit calculations of the solutions of the compatibility and completeness equation for some particular one-cusped 3-manifold.

We now fix some notations. Let $L$ and $R$ be the following matrix of $\text{SL}(2,\mathbb{Z})$:

\[
L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}
\]

Each element of $A \in \text{SL}(2,\mathbb{Z})$ can be written as a product $A = \prod A_i^{n_i}$, with $A_i \in \{L, R\}$ and $n_i \in \mathbb{N}$.

Let $S$ be the punctured torus $\left( \mathbb{R}^2 \setminus \mathbb{Z}^2 \right)/\mathbb{Z}^2$. Then each element $A \in \text{SL}(2,\mathbb{Z})$ induce an homeomorphism $\varphi_A$ of $S$. Given $A = \prod A_i^{n_i}$, we call $\prod A_i^{n_i}$ the manifold obtained from $S \times [0,1]$ by gluing $(x,0)$ to $(\varphi_A(x),1)$.

For such manifolds, using the algorithm described in [FH], one easily obtain an ideal triangulation with $\sum n_i$ tetrahedra.

We notice that the complement of the figure-eight knot is the manifold $LR$, and its standard ideal triangulation with two tetrahedra is the one obtained according with [FH].

We use the following notations for labeling the simplices. For each vertex $v$ of a tetrahedron $X$, we call $X_v$ the triangle obtained by chopping off the vertex $v$ from $X$ and $X^v$ the face of $X$ opposite to $v$. Given a tetrahedron $X$ and two vertices $v, w$ of $X$, by abusing notation, we use the label $w$ also for the edge of the triangle $X_v$ corresponding to the face $X^w$. A modulus for a tetrahedron $X$ is named $z_X$ and we will specify the edge to which is referred.

5.1 The manifold $LR^3$

Let $M$ be the manifold $LR^3$, i.e. the manifold obtained as described above by using the element $LR^3 = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ of $\text{SL}(2,\mathbb{Z})$.

Using the algorithm described in [FH], we get the ideal triangulation $\tau$ of $M$ with four tetrahedra, labeled $A, B, C, D$, pictured in Figure 1.

We label the vertices of the tetrahedra as in Figure 1 (we use such labels because they are natural using the algorithm of [FH]). The moduli are referred to the edge $0 \frac{1}{1}$ (note that such edge is common to all the tetrahedra).

The face-pairing rules of $\tau$ are, according with the arrows of the picture:

\[
\begin{align*}
A^\sharp & \leftrightarrow B^\sharp \\
A^\circ & \leftrightarrow B^0 \\
B^\sharp & \leftrightarrow C^\sharp \\
B^\circ & \leftrightarrow C^0 \\
C^\sharp & \leftrightarrow D^\sharp \\
C^\circ & \leftrightarrow D^0 \\
D^\sharp & \leftrightarrow A^\circ \\
D^\circ & \leftrightarrow A^0
\end{align*}
\]
Figure 1: Ideal triangulation of $M$

The induced triangulation on the boundary torus is the one of Figure 2.

We can write down the compatibility and completeness equations. It is easy to check that $\mathcal{C} + \mathcal{M}$ is equivalent to the following system:

$$
\begin{align*}
\mathcal{C} & = \\
\mathcal{C}_1 & = z_A\left(1 - \frac{1}{z_A}\right)^2z_D^2z_C^2z_B^2\frac{1}{1 - z_B} = 1 \\
\mathcal{C}_2 & = \left(1 - \frac{1}{z_A}\right)^2\frac{1}{1 - z_D}(1 - \frac{1}{z_B})^2\frac{1}{1 - z_C} = 1 \\
\mathcal{C}_3 & = (1 - \frac{1}{z_D})^2\frac{1}{1 - z_C}z_A = 1 \\
\mathcal{C}_4 & = (1 - \frac{1}{z_C})^2\frac{1}{1 - z_D}\frac{1}{1 - z_B} = 1
\end{align*}
$$

$$
\mathcal{M} = z_Dz_Cz_B(1 - z_A) = 1
$$

Moreover, the product of the four equations $\mathcal{C}$ is exactly the square of the product of all the moduli, and so it is 1. So if three equations are satisfied, then also the remaining one must be satisfied. It follows that we can discard one of the $\mathcal{C}$’s.
Figure 2: The triangulation of the boundary torus
We discard $C_2$. By using $M$ in $C_1$ and then $C_1$ in $C_4$ and $M$ we obtain the following system of equations, equivalent to $C + M$:

\[
\begin{align*}
M. & \quad \frac{z_D z_C (1 - z_A)^2}{z_A} = -1 \\
C_1. & \quad z_A = \frac{1}{1 - z_B} \\
C_3. & \quad \left(\frac{z_D - 1}{z_D}\right)^2 \frac{z_A}{1 - z_C} = 1 \\
C_4. & \quad \left(\frac{z_C - 1}{z_C}\right)^2 \frac{z_A}{1 - z_D} = 1
\end{align*}
\]

By solving the system, we found four non-degenerate solutions; one completely positive, giving the hyperbolic structure of $M$, and other with two negative tetrahedra, and their conjugates (i.e. the same situations but with inverted orientations). The following table contains numerical approximations of the solutions. Note that even if the modulus $z_B$ is different from the modulus $z_A$, the equation $C_1$ implies that the geometric versions of $A$ and $B$ are isometric.

| Solution 1 | Volumes |
|------------|---------|
| $z_A$      | 0.4275047 + i 1.5755666 | 0.9158907 |
| $z_B$      | 0.8395957 + i 0.5911691 | 0.9158907 |
| $z_C$      | 0.7271548 + i 0.2284421 | 0.5786694 |
| $z_D$      | 0.7271548 + i 0.2284421 | 0.5786694 |

| Solution 2 | Volumes |
|------------|---------|
| $z_A$      | 1.0724942 + i 0.5921114 | 0.8144270 |
| $z_B$      | 0.2854042 + i 0.3945194 | 0.8144270 |
| $z_C$      | -1.7271548 - i 0.6779619 | -0.2398640 |
| $z_D$      | -1.7271548 - i 0.6779619 | -0.2398640 |

Note that in the case 2, the total volume is particularly small, this imply that, even if the identification space is defined, it cannot be a hyperbolic manifold.

In Figures 3 and 4 we draw how the triangulations of the boundary torus of $M$ looks like when we choose the moduli as in the solution 2. There are two types of triangles, the positive ones, relative to the tetrahedra $A$ and $B$ and the negative ones, relative to $C$ and $D$. In Figure 3 are pictured the four triangles of the top strip of the triangulation of Figure 2.
Figure 3: The triangles $D_0$, $C_0$, $B_0$, $A_0$ with the moduli of the solution 2.

The two parts of Figure 4 are the top and bottom part of the triangulation of Figure 2.

Figure 4: Geometric triangulation of the boundary torus, solution 2.

Now we look at the algebraic expression of the solution of 2. A simple calculation shows that the moduli can be expressed by the following equations:

\[
\begin{align*}
    z_C &= z_D = w \\
    z_A &= \frac{w^2}{1 - w} \\
    z_B &= 1 - \frac{1}{z_A} = \frac{w^2 + w - 1}{w^2} \\
    w^4 + 2w^3 - w^2 - 3w + 2 &= 0
\end{align*}
\]

The four solutions correspond to the four roots $w_1, \overline{w}_1, w_2, \overline{w}_2$ of the polynomial $P(w) = w^4 + 2w^3 - w^2 - 3w + 2$. Looking at the reduction (mod 2)
of \( P \), we see that \( P \) is irreducible over \( \mathbb{Z} \), and then also over \( \mathbb{Q} \).

The holonomy representation can be explicitly calculated as a function of \( w \). Let us fix a fundamental domain \( F \) for \( M \) obtained by taking one copy of each tetrahedron and then performing the gluing:

\[
A^\frac{3}{2} \longleftrightarrow B^0 \quad B^1 \longleftrightarrow C^0 \quad C^\frac{3}{2} \longleftrightarrow D^0
\]

Consider now the geometric version of \( F \), i.e. a developed image of \( F \). The holonomy is generated by the isometries corresponding to the remaining face-pairing rules. We consider the upper half-space model of \( \mathbb{H}^3 \) with the coordinates in which the points \( 0, 1, \infty \) of \( \partial \mathbb{H}^3 \) are the vertices of \( D \) labeled respectively \( \frac{3}{2}, 0, \frac{4}{3} \). Calculations show that in this model the holonomy is generated by the elements of \( PSL(2, \mathbb{C}) \) represented by the matrices:

\[
\begin{pmatrix}
1 & \frac{w^2}{-w^2+w+1} \\
0 & w
\end{pmatrix}, \quad
\begin{pmatrix}
0 & -w \\
\frac{1}{w} & -w-1
\end{pmatrix}, \quad
\begin{pmatrix}
1 & -w^2 \\
-1 & w^2+w-1
\end{pmatrix}
\]

that respectively correspond to the face-pairing rules

\[
A^0 \rightarrow D^\frac{3}{2} \quad C^\frac{3}{2} \rightarrow D^\frac{4}{3} \quad B^2 \rightarrow A^0
\]

What is important is that the entries of such matrices are numbers belonging to \( \mathbb{Q}(w) \) (and this can be proved even without the explicit calculations).

**Proposition 5.1** The solution 2 is not geometric.

**Proof.** This obviously follows from the uniqueness of geometric solutions, nevertheless we give an alternative proof. Let \( w_1 \) (resp. \( w_2 \)) be the root of \( P \) relative to the solution 1 (resp. 2) of \( C + M \). So \( w_1 \) gives the hyperbolic structure of \( M \). Let \( h_j : \pi_1(M) \rightarrow PSL(2, \mathbb{C}) \) be the holonomy representation relative to \( w_j, j = 1, 2 \). Since \( P \) is irreducible and since the entries of the holonomy-matrices are in \( \mathbb{Q}(w) \), it follows that a relation between elements holds for \( h_1 \) if and only if it holds for \( h_2 \). Since \( h_1 \) is the holonomy of the complete hyperbolic structure of \( M \), then it is faith-full, and it follows that also \( h_2 \) is faith-full.

The image of \( h_2 \) cannot be discrete because otherwise \( \mathbb{H}^3/h_2 \) will be a hyperbolic manifold \( M' \) with a volume too small (actually, to obtain an absurd it suffices that \( vol(h_2) \neq vol(h_1) \)).

By Proposition 2.16 the holonomy of any geometric solution is discrete, so the solution 2 cannot be geometric.

\( \square \)
Similarly, from the fact that $h_2$ is not discrete and Proposition 2.16 it follows that there is no map, which is hyperbolic w.r.t. the solution 2, from $LR^3$ to any hyperbolic manifold.

Finally, we show that the image of $h_2$ is dense in $PSL(2, \mathbb{C})$. We’ll need the following standard fact about $PSL(2, \mathbb{C})$ (see for example [K] or [G]).

**Lemma 5.2** Let $G$ be a non-elementary subgroup of $PSL(2, \mathbb{C})$ and suppose that $G$ is not discrete. Then the closure of $G$ is either $PSL(2, \mathbb{C})$ or it is conjugate to $PSL(2, \mathbb{R})$ or to a $\mathbb{Z}_2$-extension of $PSL(2, \mathbb{R})$.

**Proposition 5.3** The image of the holonomy relative to the solution 2 is dense in $PSL(2, \mathbb{C})$.

**Proof.** It is easy to check that the image of $h_2$ is a non-elementary subgroup of $PSL(2, \mathbb{C})$. Suppose that its closure is conjugate to $PSL(2, \mathbb{R})$ or to a $\mathbb{Z}_2$-extension of $PSL(2, \mathbb{R})$. Then there exist a line in $\mathbb{C} \cup \{\infty\} = \partial \mathbb{H}^3$ which is $h_2$-invariant. By looking at the parabolic elements of $h_2$, it is easy to see that such a line does not exist. The thesis follows by Lemma 5.2.

This example is interesting for several reasons. On one hand it shows that an algebraic solution of $C + M$ can be non geometric. On the other hand it shows that there is no uniqueness of the algebraic solutions.

Moreover this example does not involve flat tetrahedra, so it is quite “regular”. Finally, the bad solution of $C + M$ of $LR^3$ has the property that “everything works OK at the boundary”, in the meaning that the triangulation with moduli induced on the boundary torus defines on it a Euclidean structure (up to scaling) with non-zero area. Roughly speaking, this means that the cusp of $LR^3$ would like to have a complete hyperbolic structure of finite volume according to the bad solution of $C + M$, but the rest of the manifold does not agree.

### 5.2 The manifold $L^2R^3$

The case of $LR^3$ is not at all isolated in the family of fiber bundle with fiber a punctured torus. In this section we do calculations for the manifold $L^2R^3$.

$$L^2R^3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix}.$$

Using the algorithm described in [FH], we get the ideal triangulation $\tau$ of $M$ with five tetrahedra, labeled $A, B, C, D, E$, pictured in Figure 5.
We label the vertices of the tetrahedra as in Figure 5. The moduli $z_A$ and $z_B$ are referred to the edge $0 \frac{1}{1}$, while $z_C$, $z_D$, $z_E$ to the edge $0 \frac{2}{2}$.

The induced triangulation on the boundary torus is the one of Figure 6.

It is easy to see that the system of compatibility and completeness equations $C + M$ is equivalent to the following one:

\[
\begin{align*}
z_A z_B &= z_C z_D z_E \\
z_C &= \frac{1}{1 - z_A} \\
(1 - z_D)^2 z_E^2 &= (1 - z_E)^2 z_D^2 \\
(1 - \frac{1}{1 - z_A})^2 &= (1 - z_B)^2 \\
(1 - \frac{1}{z_E})^2 \frac{1}{1 - z_D} (1 - \frac{1}{z_A}) &= 1
\end{align*}
\]

By solving such a system (we have done that with a computer), we found eight solutions. The following tables contain numerical approximations of the solutions. Note that even if the modulus $z_A$ is different from the modulus $z_C$, the second equation implies that the geometric versions of $A$ and $C$ are isometric.

| Solutions 1 | volume | Solutions 2 | volume |
|-------------|--------|-------------|--------|
| $z_A$ 0.75 + i0.6614378 | 0.9626730 | 0.75 - i0.6614378 | -0.9626730 |
| $z_B$ 1.25 + i0.6614378 | 0.7413987 | 1.25 - i0.6614378 | -0.7413987 |
| $z_C$ 0.5 + i1.3228756 | 0.9626730 | 0.5 - i1.3228756 | -0.9626730 |
| $z_D$ 1 | * | 1 | * |
| $z_E$ 1 | * | 1 | * |
Figure 6: The triangulation of the boundary torus

| Solutions 3       | volume | Solutions 4       | volume |
|-------------------|--------|-------------------|--------|
| $z_A$             | 1.588633261 | 0                 | 1.127804076 | 0 |
| $z_B$             | 1.370528159 | 0                 | 1.113321168 | 0 |
| $z_C$             | -1.69885025 | 0                 | -7.824476637 | 0 |
| $z_D$             | 0.3783840018 | 0                 | 0.2518509745 | 0 |
| $z_E$             | -3.387066549 | 0                 | -0.6371698130 | 0 |

| Solutions 5       | volume | Solutions 6       | volume |
|-------------------|--------|-------------------|--------|
| $z_A$             | 0.4950484 + i0.3298695 | 0.7399514 | 0.4950484 - i0.3298695 | -0.7399514 |
| $z_B$             | 0.6011109 + i0.9321327 | 1.0089809 | 0.6011109 - i0.9321327 | -1.0089809 |
| $z_C$             | 1.3880304 + i0.9067580 | 0.7399514 | 1.3880304 - i0.9067580 | -0.7399514 |
| $z_D$             | 0.5022247 + i0.2691269 | 0.6433681 | 0.5022247 - i0.2691269 | -0.6433681 |
| $z_E$             | 0.6077815 + i0.3441339 | 0.7596486 | 0.6077815 - i0.3441339 | -0.7596486 |
The Solutions 1 and 2 contain degenerated tetrahedra. We notice that the non-degenerate moduli of such solutions are exactly the ones that give the hyperbolic structure on the manifold obtained by removing the tetrahedra $D$ and $E$ and adding the gluing rules:

\[
C^\frac{1}{2} \leftrightarrow A^\frac{1}{2} \quad \text{via} \quad (0, \frac{3}{1}, \frac{0}{1}) \leftrightarrow (0, \frac{1}{0}, \frac{1}{1}) \\
C^\frac{1}{4} \leftrightarrow A^0 \quad \text{via} \quad (0, \frac{1}{0}, \frac{3}{1}) \leftrightarrow (0, \frac{1}{0}, \frac{1}{1})
\]

Now we look at the algebraic expression of the Solutions 3-8. Let $P(x) = x^6 + 4x^5 + 3x^4 + 3x^3 - 4x^2 + 2$. A simple calculation shows that the moduli can be expressed in terms of roots of $P$ by the following expressions:

\[
\begin{align*}
z_A &= \frac{1}{22}(5w^5 + 19w^4 + 9w^3 + 6w^2 - 8w + 17) \\
z_B &= \frac{1}{44}(10w^5 + 49w^4 + 62w^3 + 34w^2 - 16w + 34) \\
z_C &= \frac{1}{11}(-12w^5 - 39w^4 - 4w^3 - 10w^2 + 72w - 32) \\
z_D &= \frac{1}{22}(-4w^5 - 13w^4 + 6w^3 + 15w^2 + 2w + 4) \\
z_E &= w \\
P(w) &= 0
\end{align*}
\]

As in the case of $LR^3$ the polynomial $P$ is irreducible, and the solutions 3, 4, 7, 8 are not geometric.

### 5.3 A manifold with non-trivial JSJ decomposition

The manifold we consider in this section is obtained by gluing to boundary torus of the complement of the figure-eight knot a Seifert manifold. The resulting manifold, which we call $M$, clearly is not hyperbolic because it contains an incompressible tours (the old boundary torus).

This example is interesting because the manifold $M$ admits an ideal triangulation with four tetrahedra such that there exists a positive, partially flat solution of $C + M$. Obviously such a solution cannot be geometric, as $M$ is not hyperbolic.
At a first look, this seems to contradict the result of [PW], but fortunately there are no contradictions. The point is that in the present example the moduli do not satisfy the equations on the angles. Namely, when a modulus is positive then it is well defined the angle associated to a modulus, in such a way that the sum of angles of any horospherical triangle is always $2\pi$. Then in addition to equations $C$ one can require that the sum of the angles around any edge is exactly $2\pi$. Such equations are called $C^*$.

In [PW] is proved that a partially flat solution of $C^* + M$ leads to a hyperbolic structure. Here we show a partially flat solution of $C + M$ that does not satisfy $C^*$.

This shows that the equations $C^*$ play a fundamental role in order to have hyperbolicity. Nevertheless, we notice that equations $C^*$ are not necessary. Namely, there exist examples of ideal triangulations of the complement of the figure-eight knot whose unique geometric solution does not satisfy $C^*$.

We describe now our manifold $M$. Let $A$ be the following subset of $\mathbb{C}$:

$$A = \{z \in \mathbb{C} : |z| \leq 4, |z - 2| > 1, |z + 2| > 1\}.$$  

$A$ is a disc with two holes. Let $I \subset A$ be the set of the point with zero real part. Let $S$ be the space obtained from $A \times [0,1]$ by gluing $(z,0)$ to $(-z,1)$ and let $L$ be the Möbius strip coming from $I$. This is the piece we want to glue to complement of the figure-eight knot. We call $C_e$ and $C_i$ the external and internal components of $\partial S$. Note that $\partial L \subset C_e$.

We will glue $C_e$ to the boundary torus of the complement of the figure-eight knot. To do this, we specify where we glue the boundary of the Möbius strip. Using the classical triangulation of the complement of the figure-eight knot we get the following picture when looking from the cusp:

![Figure 7: The boundary of the complement of the figure-eight knot](image)

There are pictured the eight equilateral triangles of the boundary. The dash lines represent the standard spine dual of the ideal triangulation. Finally the marked line is where we glue $\partial L$. 


Since $S$ retracts to $C_e \cup L$, then a spine of $M$ is obtained simply by gluing a Möbius strip to the spine of the complement of the figure-eight knot as in Figure 7. Such a spine as a vertex more than the old one, but is not standard. We perform a lune move along the Möbius strip and we obtain a standard spine (see [M] for details about the moves on the spines) with five vertices. As the new spine is standard, its dual is an ideal triangulation with five tetrahedra. Such a triangulation can be simplified with a $MP$ move, replacing the three new tetrahedra with an equivalent pair of tetrahedra.

At the end, we get the triangulation of $M$ sketched in Figure 8.

Figure 8: The ideal triangulation of $M$

The tetrahedra labeled $A, B$ are the old ones of the complement of the figure-eight knot. The gluing rules are the following:

$A^0_0 \leftrightarrow B^1_1 : (0, \frac{1}{0}, 1) \leftrightarrow (0, \frac{1}{1}, 1)$

$A^1_1 \leftrightarrow B^0_0 : (0, \frac{1}{0}, 1) \leftrightarrow (0, \frac{1}{1}, 1)$

$B^1_1 \leftrightarrow G^0_0 : (0, \frac{1}{0}, \frac{2}{2}) \leftrightarrow (b, \beta, \alpha)$

$F^\alpha \leftrightarrow G^\beta : (\beta, \gamma, t) \leftrightarrow (\gamma, \alpha, b)$

$F^\alpha \leftrightarrow G^\beta : (\beta, \gamma, t) \leftrightarrow (\gamma, \alpha, b)$

We call the moduli $z_A, z_B$ are referred to the edge $0_0$ and $z_F, z_G$ to $\alpha\beta$. The triangulation of the boundary torus is the one of Figure 9.

We see that the system of compatibility and completeness equations is equivalent to the following one:

$$
\begin{cases}
1 - z_A \cdot z_B \cdot \frac{z_F}{z_G} = 1 \\
z_G z_F = 1
\end{cases}
\begin{cases}
(1 - Z_A)^2 \cdot \frac{z_B^2}{z_A} = 1 \\
z_B(1 - z_A) = 1
\end{cases}
$$
Figure 9: Triangulation with moduli of the boundary torus

From that easily we get $z_G = z_F$ and $z^2_F = 1$. Since we are looking for non degenerate solutions, we have $z_F = z_G = -1$. Using that we get $z_A = z_B$ and

$$z^2_A - z_A + 1 = 0$$

and then $z_A = z_B = \frac{1 \pm i \sqrt{3}}{2}$. That is, the ideal tetrahedra $F$ and $G$ are flat but not degenerate, while $A$ and $B$ are regular, exactly as in the complement of the figure-eight knot. We notice that the space obtained by gluing together the geometric versions of the tetrahedra $A, B, F, G$ is not a manifold.

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