ON POINCARÉ, FRIEDRICHS AND KORNS INEQUALITIES ON DOMAINS AND HYPERSURFACES

R. Duduchava

I. Javakhishvili Tbilisi State University, Andreas Razmadze Mathematical Institute, Tamarashvili str. 6, Tbilisi 0177, Georgia; roland.duduchava@tsu.ge

2010 Mathematics Subject Classification: Primary 35J57; Secondary 74J35, 58J32. Key words and phrases: Poincaré inequality, Friedrichs inequality, Poincaré-Korns inequality, Friedrichs-Korns inequality, Open mapping theorem, Bessel potential space, Hypersurface

Abstract

The celebrated Poincaré and Friedrichs inequalities estimate the $L^p$-norm of a function by the $L^p$-norm of the gradient. We prove the Poincarné inequality for a domain $\Omega \subset \mathbb{R}^n$ and for a hypersurface $\mathcal{C} \subset \mathbb{R}^n$ based on open mapping theorem of Banach only. For a cylinder which has a hypersurface as a base, is prove stronger inequality, involving only the surface derivatives. Similar inequalities for the uniform $C$-norm are proved as well. We also estimate $\mathbb{H}^m_p$-norm of functions prove inequalities for some generalizations of the mentioned inequalities.

We also prove Poincaré-Korns and Friedrichs-Korns inequalities for vector-functions estimating the $L^p$-norm of a function by the $L^p$-norm of the deformation tensor only on domains and on hypersurfaces. The proofs are based on the paper [Du10] of the author on Korns inequalities. And again, the norm of the function in a cylinder is estimated by is the deformation tensor on the base of the cylinder.

Introduction

Let $1 \leq p \leq \infty$ and $\Omega$ be a bounded connected open subset of the $n$-dimensional Euclidean space $\mathbb{R}^n$ with a Lipschitz boundary (a domain with the uniform cone property). Then there exists a constant $C$, depending only on $\Omega$ and $p$ such that for every function $\varphi$ in the Sobolev space $W^1_p(\Omega)$ the celebrated Poincaré inequality holds

$$\|\varphi - \varphi_\Omega\|_{L^p(\Omega)} \leq C\|\nabla \varphi\|_{L^p(\Omega)},$$

(1)

This work was supported by the Shota Rustaveli Georgian National Science Foundation No. GNSF/DI/10/5-101/12, Contract No. 13/14.
where
\[ \varphi_\Omega := \frac{1}{\text{mes } \Omega} \int_\Omega \varphi(y) dy \] (2)
is the average value of \( \varphi \) over \( \Omega \). Here \( \text{mes } \Omega \) stands for the Lebesgue measure of the domain \( \Omega \) and the constant \( C \) depends on \( \Omega \) and \( p \) only. When \( \Omega \) is a ball, the above inequality is called a Poincaré inequality, while for more general domains \( \Omega \) inequality (1) is known as a Sobolev inequality (cf., e.g., [DL90]).

Let \( M_0 \) be a subset of the closed domain \( M_0 \subset \overline{\Omega} \) of co-dimension 1 and have non-trivial measure \( \text{mes } M_0 \neq 0 \) (can be a non-trivial part of the boundary). Let \( \varphi^+ \) denote the trace of \( \varphi \) on \( M_0 \). The following
\[ \| \varphi \|_{L^p(\Omega)} \leq C \left( \| \nabla \varphi \|_{L^p(\Omega)}^p + \left| \int_{M_0} \varphi^+(x) d\sigma \right|^p \right)^{1/p}, \quad \varphi \in W^1_p(\Omega) \] (3)
is known as Friedrichs inequality for \( M_0 = \partial \Omega, p = 2 \) (see [Tr72, Theorem 6.28.2], [HW08, Theorem 4.1.7]).

If \( M_0 \) is the same as in (3), the next inequality
\[ \| \varphi \|_{L^p(\Omega)} \leq C \| \nabla \varphi \|_{L^p(\Omega)} \] (4)
for a function \( \varphi \in \tilde{W}^1_p(\Omega, M_0) \) which vanish on \( M_0 \), is a variant of inequalities (1), (3) (see [Tr72 Theorem 6.28.2], [HW08 Theorem 4.1.7] and [Wl87 Theorem 7.6, Theorem 7.7]).

The inequalities (3) and (4) hold, of course, if \( M_0 \) is a subdomain of \( \Omega \).

In contrast to (1), in inequalities (3) and (4) the domain \( \Omega \) can also be unbounded (might have an infinite measure), provided \( \text{mes } M_0 < \infty \) in (3).

Moreover, for a cylinder \( \Omega := C \times [a, b] \) with a base \( C \) which is a hypersurface in \( \mathbb{R}^n \), we prove a stronger inequality, namely the following
\[ \| \varphi \|_{L^p(\Omega)} \leq C \left( \| \nabla C \varphi \|_{L^p(\Omega)}^p + \left| \int_{M_0} \varphi^+(x) d\sigma \right|^p \right)^{1/p}, \quad \varphi \in \tilde{W}^1_p(C), \] (5)
\[ \| \varphi \|_{L^p(\Omega)} \leq C \| \nabla C \varphi \|_{L^p(\Omega)} \], \quad \varphi \in \tilde{W}^1_p(\Omega, M_0), \] (6)
where \( \nabla C = (D_1, \ldots, D_n)^\top \) is the surface gradient and \( D_1, \ldots, D_n \) are the Gunter’s derivatives (see § 1), and \( \varphi \in \tilde{W}^1_p(\Omega, M_0) \) vanishes on a \((n - 1)\)-dimensional strip \( M_0 := \Gamma_0 \times [a, b] \) with \( \Gamma_0 \subset \overline{C} \) a \((n - 2)\)-dimensional subset of \( \overline{C} \) (can be a piece of the boundary \( \partial C \)). The inequality (5) is remarkable, because contains only the surface derivatives and does not contains the derivative with respect to the variable \( t \in [a, b] \) transversal to the surface \( C \).
For a cylinder $I_\omega := \omega \times I$, $I := [a, b]$, with a flat base $\omega \subset \mathbb{R}^{n-1}$, the inequalities (5) and (6) have the form

$$\|\varphi\|_{L_p(I_\omega)} \leq C \left[ \|\nabla_\omega \varphi\|_{L_p(I_\omega)}^p + \|\varphi\|_{L_p(\Gamma_0 \times I)}^p \right]^{1/p}, \quad \varphi \in \mathbb{W}^1_p(I_\omega),$$

$$\|\varphi\|_{L_p(I_\omega)} \leq C \|\nabla_\omega \varphi\|_{L_p(I_\omega)}, \quad \varphi \in \mathbb{W}^1_p(I_\omega, \Gamma_0 \times I),$$

where $\nabla_\omega(U)$ is the gradient in $\omega$ (in $\mathbb{R}^{n-1}$) and contains only $(n - 1)$ derivatives.

Poincaré and Friedrichs inequalities also hold for a smooth surface

$$\|\varphi - \varphi_C \|_{L_p(C)} \leq C \|\nabla_C \varphi\|_{L_p(C)}, \quad \varphi \in \mathbb{W}^1_p(C),$$

$$\|\varphi\|_{L_p(C)} \leq C \left[ \|\nabla_C \varphi\|_{L_p(C)}^p + \left( \int_{\Gamma_0} \varphi^+(x) \, d\sigma \right) \right]^{1/p}, \quad \varphi \in \mathbb{W}^1_p(C)$$

$$\|\varphi\|_{L_p(C)} \leq C \|\nabla_C \varphi\|_{L_p(C)}, \quad \varphi \in \mathbb{W}^1_C(C, \Gamma_0),$$

where $\varphi_C$ denotes the average value of $\varphi$ over $C$:

$$\varphi_C := \frac{1}{\text{mes} C} \int_C \varphi(y) \, d\sigma.$$  

$\Gamma_0$ is a subset of the closed surface $\Gamma_0 \subset C$ of co-dimension 1 and has non-trivial measure $\text{mes} \Gamma_0 \neq 0$ ($\Gamma_0$ can be a non-trivial part of the boundary).

The inequalities (9) and (10) hold, of course, if $\Gamma_0$ is a subsurface of $C$.

The inequality (11) holds for surfaces of finite measure, while the inequality (12) does not needs such constraint and the surface $C$ might have infinite measure.

The following

$$\|\varphi\|_{W^l_p(\Omega)} \leq C \left[ \sum_{|\alpha|=m} \|\partial^\alpha \varphi\|_{L_p(\Omega)}^p \right]^{1/p} + \sum_{|\beta|<m} \left( \int_{\mathcal{M}_0} (\partial^\beta \varphi)^+(x) \, d\sigma \right)^{1/p}, \quad \varphi \in \mathbb{W}^m_p(\Omega),$$

$$\|\varphi\|_{W^l_p(\mathcal{M})} \leq C \left[ \sum_{|\alpha|=m} \|\partial^\alpha \varphi\|_{L_p(\mathcal{M})}^p \right]^{1/p} + \sum_{|\beta|<m} \left( \int_{\Gamma_0} (\mathcal{D}^\beta \varphi)^+(x) \, d\sigma \right)^{1/p}, \quad \varphi \in \mathbb{W}^m_p(\mathcal{M}),$$

$$\|\varphi\|_{W^l_p(\Omega)} \leq C \left[ \sum_{|\alpha|=m} \|\partial^\alpha \varphi\|_{L_p(\Omega)}^p \right], \quad \varphi \in \mathbb{W}^m_p(\Omega, \mathcal{M}_0),$$

$$\|\varphi\|_{W^l_p(\mathcal{M})} \leq C \left[ \sum_{|\alpha|=m} \|\partial^\alpha \varphi\|_{L_p(\mathcal{M})}^p \right], \quad \varphi \in \mathbb{W}^m_p(\Omega, \mathcal{M}_0),$$
\[ \| \varphi \|_{W^\ell_p(M)} \leq \| \varphi \|_{W^m_p(M)} \leq C \sum_{|\alpha| = m} \| \partial^\alpha \varphi \|_{L^p_p(M)} \quad \varphi \in \tilde{W}^m_p(M, \Gamma_0) \quad (16) \]

for \( \ell < m, m = 2, 3, \ldots \), generalize Poincaré inequalities \([1]\) and \([5]\), while the inequalities

\[ \| \varphi \|_{W^\ell_p(\Omega)} \leq \| \varphi \|_{W^m_p(\Omega)} \leq C \left[ \sum_{|\alpha| = m} \| \partial^\alpha \varphi \|_{L^p_p(\Omega)} \right]^{1/p} + \int_{\mathcal{M}_0} |\varphi^+(x)|^p d\sigma, \quad (17) \]

\[ \varphi \in W^m_p(\Omega), \]

\[ \| \varphi \|_{W^\ell_p(M)} \leq \| \varphi \|_{W^m_p(M)} \leq C \left[ \sum_{|\alpha| = m} \| \partial^\alpha \varphi \|_{L^p_p(M)} \right]^{1/p} + \int_{\Gamma_0} |\varphi^+(x)|^p d\sigma, \quad (18) \]

\[ \varphi \in W^m_p(M) \]

for \( \ell < m, m = 2, 3, \ldots \), generalize Friedrichs inequalities \([3]\) and \([10]\) (see \([13,2]\) Theorem 6.28.2], \([18] \) Theorem 4.1.7])

All above inequalities hold also for the space of 1-smooth functions \( C^1(\Omega) \)-just replace the \( L^p \)-norm by \( \| \varphi \|_{C^1(\Omega)} := \max_{x \in \Omega} |\varphi(x)| \) and \( W^1 \)-norm by \( \| \varphi \|_{C^1(\Omega)} := \max_{x \in \Omega} |\varphi(x)| + \max_{x \in \Omega} |\nabla \varphi(x)| \). For example, the inequality \([1]\) acquires the form

\[ \max_{x \in \Omega} |\varphi(x) - \varphi_\Omega(x)| \leq C \max_{x \in \Omega} |\nabla \varphi(x)|. \quad (19) \]

There is only one essential difference: in analogues of inequalities \([1]\), \([12]\), \([9]\), \([13]\) and \([17]\) the sets \( \mathcal{M}_0 \) and \( \Gamma_0 \) can be one point sets.

It turned out, that for vector-functions \( U(x) = (U_1(x), \ldots, U_n(x))^\top \) on a domain \( \Omega \subset \mathbb{R}^n \) even gradient is superfluous in the inequalities \([1]\), \([4]\) and it suffices to take the deformation tensor:

\[ \| U \|_{L^p_p(\Omega)} \leq C \left[ \| \text{Def} U \|_{L^p_p(\Omega)} \right]^{1/p} + \| U \|_{Q_p^1(\Omega_0)} \], \quad U \in W_p^1(\Omega), (20) \]

\[ \| U \|_{L^p_p(\mathcal{C})} \leq C \left[ \| \text{Def}_c U \|_{L^p_p(\mathcal{C})} \right]^{1/p} + \| U \|_{Q_p^1(\Gamma_0)} \], \quad U \in W_p^1(\mathcal{C}), (21) \]

\[ \| U \|_{L^p_p(\Omega)} \leq C \| \text{Def} U \|_{L^p_p(\Omega)}, \quad U \in \tilde{W}_p^1(\Omega, \mathcal{M}_0), (22) \]

\[ \| U \|_{L^p_p(\mathcal{C})} \leq C \| \text{Def}_c U \|_{L^p_p(\mathcal{C})}, \quad U \in \tilde{W}_p^1(\mathcal{C}, \Gamma_0), (23) \]

where \( \mathcal{M}_0 \) and \( \Gamma_0 \) are the same as in \([3]\) and \([10]\), respectively. \( \text{Def}(U) \) and \( \text{Def}_c(U) \) are the domain and the surface deformation tensors, respectively (see \([9]\) and \([10]\)), and only \( \frac{n(n+1)}{2} < n^2 \) different linear combinations of the \( n^2 \) derivatives \( \partial_j U_k \) (of derivatives \( D_j U_k \), respectively; \( j, k = 1, \ldots, n \)) are involved.
For a cylinder $\Omega := C \times [a,b]$ with a base $C$ which is a hypersurface in $\mathbb{R}^n$ and a vector-function $U = (U_1, \ldots, U_n)^\top$, we prove a stronger inequality, namely the following

$$\|U\|_{L^p(\Omega)} \leq C \left\| \text{Def}_C(U) \right\|_{L^p(\Omega)}^p + \|U\|_{L^p(M_0)}^p \right\|^{1/p}, \quad U \in W^1_p(\Omega),$$

(24)

$$\|U\|_{L^p(\Omega)} \leq C \left\| \text{Def}_C(U) \right\|_{L^p(\Omega)}^p,$$

(25)

where $M_0 := \Gamma_0 \times [a,b]$ is a strip, $\Gamma_0 \subset \overline{C}$. The inequality (24) is remarkable, because estimates the vector-function $U$, instead of $n(n+1)$ derivatives $D_j U_k$, $j = 1, \ldots, n+1, k = 1, \ldots, n$ including the transversal derivatives $D_{n+1} U_k$, $k = 1, \ldots, n$, by only surface deformation tensor $\text{Def}_C(U)$.

For a cylinder $I_\omega := \omega \times I$, $I := [a,b]$, with a flat base $\omega \subset \mathbb{R}^{n-1}$, the inequalities (24) and (25) have the form

$$\|U\|_{L^p(I_\omega)} \leq C \left\| \text{Def}_\omega(U) \right\|_{L^p(I_\omega)}^p + \|U\|_{L^p(\Gamma_0 \times I)}^p \right\|^{1/p},$$

(26)

$$\|U\|_{L^p(I_\omega)} \leq C \left\| \text{Def}_\omega(U) \right\|_{L^p(I_\omega)}^p,$$

(27)

where $\text{Def}_\omega(U)$ is the deformation tensor in $\omega$ (in $\mathbb{R}^{n-1}$) and contains only $\frac{n(n-1)}{2}$ derivatives.

The inequalities (20)-(23) follow from Korns inequalities and we call: (20)-(21) Friedrichs-Korns inequalities and (22)-(23) Poincaré-Korns inequalities.

1 Auxiliaries

Throughout the present paper we will assume that $C$ be a sufficiently smooth hypersurface in $\mathbb{R}^n$ with the Lipschitz boundary $\Gamma := \partial C$ (a surface with the uniform cone property), defined by a real valued smooth function

$$C = \left\{ x \in \Omega : \Psi(x) = 0 \right\},$$

(1)

which is regular $\nabla \Psi(x) \neq 0$. The normalized gradient

$$\nu(x) := \frac{\nabla \Psi(x)}{|\nabla \Psi(x)|}, \quad x \in C$$

(2)

defines the unit normal vector field on $C$.

The collection of the tangential Günter’s derivatives are defined as follows (cf. [Gu53, KGBB79, DMM06, Du10, Du11])

$$D_j := \partial_j - \nu_j(x) \partial_\nu = \partial_{d_j}, \quad \nu_j(x) := \frac{\partial_j \Psi(x)}{|\nabla \Psi(x)|} \quad j = 1, \ldots, n,$$

(3)
where
\[ e^1 = (1, 0, \ldots, 0)^T, \ldots, e^n = (0, \ldots, 0, 1)^T \] (4)
is the natural basis in \( \mathbb{R}^n \) and \( \partial_\nu := \sum_{j=1}^n \nu_j \partial_j \) denotes the normal derivative. For each \( 1 \leq j \leq n \), the first-order differential operator \( D_j = \partial_{d^j} \) is the directional derivative along the tangential vector
\[ d^j := \pi_S e^j, \quad \langle \nu(x), d^j(x) \rangle \equiv 0, \quad \sum_{j=1}^n \nu_k d^k = 0, \quad j = 1, \ldots, n, \] (5)
the projection of \( e^j \) on the space of tangential vector fields to \( S \).

The surface gradient \( \nabla_S \varphi \) is the collection of the Günter’s derivatives
\[ \nabla_S \varphi := (D_1 \varphi, \ldots, D_n \varphi)^T \] (6)
and is an equivalent form of the surface gradient defined in the differential geometry by means of covariant metric tensor (see [DMM06, Du10, Du11]). The next Lemma 1.1 was proved in [Du10, Lemma 1.2].

**Lemma 1.1** For \( \varphi \in C^1(\mathcal{S}) \) the surface gradient vanishes \( \nabla_S \varphi \equiv 0 \) if and only if \( \varphi(x) \equiv \text{const.} \)

\( W^{1,p}_p(\Omega) \) and \( W^{1,p}_p(\mathcal{C}) \), \( 1 < p < \infty \), denote the Sobolev spaces on a domain \( \Omega \subset \mathbb{R}^n \) and the surface \( \mathcal{C} \) endowed with the norm:
\[ \| \varphi \|_{W^{1,p}_p(\Omega)} := \left[ \| \varphi \|_{L_p(\Omega)} + \sum_{j=1}^n \| \partial_j \varphi \|_{L_p(\Omega)} \right]^{1/p} \] (7)
and, respectively,
\[ \| \varphi \|_{W^{m,p}_p(\mathcal{C})} := \left[ \| \varphi \|_{L_p(\Omega)} + \sum_{j=1}^n \| D_j \varphi \|_{L_p(\mathcal{C})} \right]^{1/p} . \] (8)

Let us define the space \( \widetilde{W}^{1,p}_p(\Omega, \mathcal{M}_0) \) for a domain \( \Omega \subset \mathbb{R}^n \) with a Lipshitz boundary \( \mathcal{M} := \partial \Omega \) and a subsurface \( \mathcal{M}_0 \subset \mathcal{M} \) of non-zero measure as the closure in \( \mathcal{W}^{1,p}_p(\Omega) \) of the set \( C^\infty(\Omega, \mathcal{M}_0) \) of smooth functions \( \varphi(x) \) which have vanishing traces on \( \mathcal{M}_0 \), i.e. \( \varphi^+(x) = 0 \) for all \( x \in \mathcal{M}_0 \). The space \( \widetilde{W}^{1,p}_p(\Omega, \mathcal{M}_0) \) inherits the standard norm \( \| \varphi \|_{W^{1,p}_p(\Omega)} \) from the space \( W^{1,p}_p(\Omega) \) (see (7)).

If \( \mathcal{C} \) is a subsurface of a closed surface \( \mathcal{S} \) without boundary, \( \widetilde{W}^{1,p}_p(\mathcal{C}) \) denotes the space of functions \( \varphi \in W^{1,p}_p(\mathcal{S}) \), supported in \( \overline{\mathcal{C}} \). Let \( \mathcal{C}^c = \mathcal{S} \setminus \mathcal{C} \) be the complemented surface with the common boundary \( \partial \mathcal{C} = \partial \mathcal{C}^c = \Gamma \); The notation \( \widetilde{W}^{1,p}_p(\mathcal{C}) \) is used for
the factor space $W^1_p(C)/\overline{W}^1_p(C^c)$. The space $W^s_p(C)$ can also be interpreted as the space of restrictions $r_C\varphi := \varphi|_C$ of all functions $\varphi \in W^1_p(S)$ to the subsurface $C$.

Similarly are defined the spaces $\overline{W}^1_p(\Omega)$ and $W^1_p(\Omega)$ for a domain $\Omega \subset \mathbb{R}^n$.

We refer to [?, Du10, Du11] for details about these spaces.

For an $n$-vector-function $U(x) = (U_1(x), \ldots, U_n(x))^\top$ on a domain in the Euclidean space $\Omega \subset \mathbb{R}^n$ the deformation tensor reads

$$\text{Def}(U) = [D_{jk}(U)]_{n \times n}, \quad D_{jk}(U) := \frac{1}{2} [\partial_j U_k + \partial_k U_j].$$

The following form of the important deformation (strain) tensor on a surface $\mathcal{C}$ was identified in [DMM06]:

$$\text{Def}_C(U) = [\mathcal{D}_{jk}(U)]_{n \times n}, \quad U = \sum_{j=1}^n U_j \mathbf{d}^j \in \mathcal{V}(\mathcal{C}), \quad j, k = 1, \ldots, n,$$



\begin{equation}
\mathcal{D}_{jk}(U) := \frac{1}{2} \left( (D^S_{jk} U)_k + (D^S_{kj} U)_j \right) = \frac{1}{2} \left[ D_k U_j + D_j U_k + \sum_{m=1}^n U_m D_m (\nu_j \nu_k) \right],
\end{equation}

where $(D^S_{jk} U)_k := \langle D^S_{jk} U, \mathbf{e}^k \rangle$ and $\mathcal{V}(\mathcal{C})$ is the linear space of all tangential vector-functions to the surface $\mathcal{C}$.

A vector $U \in \overline{W}^1_p(\Omega)$ is called a rigid motion if $\text{Def}(U) = 0$ and a vector $V \in W^1_p(\mathcal{C})$ is called a Killings vector field on the surface $\mathcal{C}$ if $\text{Def}_C(V) = 0$.

The next Theorem 1.2 (Korns I inequality for domains “without boundary condition”) is well known (see [Ci00] for a simple proof when $p = 2, m = 1$ and see [Du10] Theorem 2.3 for a general case).

Theorem 1.3 is proved in [Du10] Theorem 2.3]. P. Ciarlet proved it in [Ci00] for the case $p = 2, m = 1$, manifold without boundary, for curvilinear coordinates and covariant derivatives.

**Theorem 1.2** Let $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ be a domain with the Lipshitz boundary and

$$\|\text{Def}(U)\|_{L^p(\Omega)} := \left[ \sum_{j,k=1}^n \|D_{jk}(U)\|_{L^p(\Omega)}^p \right]^{1/p}, \quad U \in \overline{W}^1_p(\Omega).$$

Then the inequality

$$\|U\|_{\overline{W}^1_p(\Omega)} \leq M \left[ \|U\|_{L^p(\Omega)}^p + \|\text{Def}(U)\|_{L^p(\Omega)}^p \right]^{1/p}$$

holds with some constant $M > 0$ or, equivalently, the equality

$$\|U\|_{\overline{W}^1_p(\Omega)}_0 := \left[ \|U\|_{L^p(\Omega)}^p + \|\text{Def}(U)\|_{L^p(\Omega)}^p \right]^{1/p}$$

defines an equivalent norm on the space $\overline{W}^1_p(\Omega)$.

A rigid motion $U$, $\text{Def}(U) = 0$, has the unique continuation property: if $U(x) = 0$ on a set $\mathcal{M}_0$ described in [33], than $U(x) = 0$ everywhere on $\Omega$. 7
Theorem 1.3 Let $1 < p < \infty$, $C \subset \mathbb{R}^n$ be a Lipshitz hypersurface with or without boundary and (see (10) for the deformation tensor $\text{Def}_C(V)$)

$$\|\text{Def}_C(V)\|_{L^p(C)} := \left[ \sum_{j,k=1}^n \|\mathcal{D}_{jk}(V)\|_{L^p(C)}^p \right]^{1/p}, \quad V \in W^1_p(C).$$

Then the inequality

$$\|V\|_{W^1_p(C)} \leq M \left[ \|V\|_{L^p(C)}^p + \|\text{Def}_C(V)\|_{L^p(C)}^p \right]^{1/p}$$

holds with some constant $M > 0$ or, equivalently, the equality

$$\|V\|_{W^1_p(C)} := \left[ \|V\|_{L^p(C)}^p + \|\text{Def}_C(V)\|_{L^p(C)}^p \right]^{1/p}$$

defines an equivalent norm on the space $W^1_p(S)$.

A Killing vector field $V$, $\text{Def}_C(V) = 0$, has the unique continuation property: if $V(x) = 0$ on a set $\Gamma_0$ described in (10), then $V(x) = 0$ everywhere on $C$.

For the proofs of the next Theorem 1.4 and Theorem 1.5 (Korns II inequality for domains “with boundary condition”) we refer to the same sources [Ci00, Du10] mentioned above.

Theorem 1.4 Let $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ be a domain with the Lipshitz boundary. Then the inequality

$$\|U\|_{W^1_p(\Omega)} \leq M \|\text{Def}(U)\|_{L^p(\Omega)}$$

holds with some constant $M > 0$ or, equivalently, the equality

$$\|U\|_{W^1_p(\Omega)} := \|\text{Def}(U)\|_{L^p(\Omega)}$$

defines an equivalent norm on the space $W^1_p(\Omega)$.

Theorem 1.5 Let $1 < p < \infty$, $C \subset \mathbb{R}^n$ be a Lipshitz hypersurface with boundary. Then the inequality

$$\|V\|_{W^1_p(C)} \leq M \|\text{Def}_C(V)\|_{L^p(C)}$$

holds with some constant $M > 0$ or, equivalently, the equality

$$\|V\|_{W^1_p(C)} := \|\text{Def}_C(V)\|_{L^p(C)}$$

defines an equivalent norm on the space $W^1_p(C)$. 
Remark 1.6 A remarkable consequences of the foregoing theorems are the facts that the spaces $W^1_p(\Omega)$ and $\hat{W}^1_p(\Omega)$ (as well as the spaces $W^1_p(C)$ and $\hat{W}^1_p(C)$), where

$$\hat{W}^1_p(\Omega) := \left\{ \mathbf{U} = (U_1 \ldots, U_n)^\top : U_j, D_{jk}(\mathbf{U}) \in L^p(\Omega) \text{ for all } j, k = 1, \ldots n \right\}$$

$$\hat{W}^1_p(C) := \left\{ \mathbf{V} = (V_1 \ldots, V_n)^\top : V_j, D_{jk}(\mathbf{V}) \in L^p(C) \text{ for all } j, k = 1, \ldots n \right\}$$

are isomorphic (i.e., can be identified), although only $\frac{n(n+1)}{2} < n^2$ linear combinations of the $n^2$ derivatives $\partial_j U_k$ (of derivatives $D_j U_k$, respectively), $j, k = 1, \ldots n$ are involved in the definition of the equivalent norms in (13) and (13) (of the norms in (18) and (20), respectively).

The next Lemma 1.7 is a slight generalization of [Tr72, Theorem 6.28.2] proved there for $p = 2$.

Lemma 1.7 Let $\Omega$ be a bounded domain with the Lipschitz boundary (a surface with the uniform cone property), $m = 1, 2, \ldots$, $1 \leq p < \infty$ and let $F(\varphi)$ be a non-negative continuous functional on the Sobolev space $W^m(\Omega)$:

i. $F : W^m(\Omega) \to \mathbb{R}$ and $F(\lambda \varphi) = |\lambda| F(\varphi)$ for all complex $\lambda \in \mathbb{C}$ and all functions $\varphi \in H^m(\Omega)$;

ii. $0 \leq F(\varphi) \leq C \|\varphi\|_{W^m_p(\Omega)}$ for some constant $C > 0$ and $F(P) \neq 0$ for all polynomials of degree less than $m$.

Then the formula

$$\|\varphi\|_{W^m_p(\Omega)} := \left[ \sum_{|\alpha| = m} \|\partial^\alpha \varphi\|_{L^p(\Omega)}^p + F^p(\varphi) \right]^{1/p}$$

(21)

defines an equivalent norm on the Sobolev space $W^m(\Omega)$.

Lemma is valid if we replace $\Omega$ by a hypersurface $C$ and partial derivatives $\partial^\alpha$ by Günter derivatives $D^\alpha$.

Proof: Let us note that $\|\varphi\|_{W^m_p(\Omega)}$ in (21) defines a norm on $W^m_p(\Omega)$ indeed. Since other properties are trivial to check, we will only check that $\|\varphi\|_{W^m_p(\Omega)} = 0$ implies $\varphi = 0$. Then $F(\varphi) = 0$ and all derivatives of order $m$ vanish: $\partial^\alpha \varphi = 0$ for all $|\alpha| = m$. The latter means that the corresponding function is polynomial of order less than $m$, i.e., $\varphi(x) = \sum_{|\beta| < m} c_\beta x^\beta$. Since $F(\varphi) = 0$, we get $\varphi \equiv 0$ due to the property (ii).
Due to the condition (ii) holds the inequality
\[
\|\psi\|_{W^m_p(\Omega)} \leq \left[ \sum_{\alpha=m} \|\partial^\alpha \varphi\|_{L^p(\Omega)} \right]^p + \|\psi\|_{W^m_p(\Omega)}^{1/p} \leq 2^{1/p} \|\psi\|_{W^m_p(\Omega)}.
\]

Therefore the embedding of the spaces \(W^1_p(\Omega) \subset W^1_{p,F}(\Omega)\), where \(W^1_{p,F}(\Omega)\) is the closure of \(C^m(\Omega)\) with respect to the norm \(\|\psi\|_{W^m_p(\Omega)}\), is continuous.

If we apply the open mapping theorem of Banach (see [Ru73, Theorem 2.11, Corollary 2.12.b]), we conclude that the inverse inequality
\[
\|\psi\|_{W^1_p(\Omega)} \leq C\|\psi\|_{W^1_{p,F}(\Omega)} := C\|\nabla \psi\|_{L^p(\Omega)}
\]
holds and accomplishes the proof. \(\square\)

2 Proofs of the basic inequalities

Proof of inequality (1): Let \(W^1_{p,\#}(\Omega)\) denote the subspace of \(W^1_{p,\#}(\Omega)\), consisting of functions with mean value zero:
\[
\varphi_{\Omega} := \frac{1}{\text{mes } \Omega} \int_\Omega \varphi(y)dy = 0.
\]

The formula
\[
\|\varphi\|_{W^1_{p,\#}(\Omega)} := \|\nabla \varphi\|_{L^p(\Omega)}
\]
defines an equivalent norm in the space \(W^1_{p,\#}(\Omega)\). Since other properties are trivial to check, we only have to check that \(\|\varphi\|_{W^1_{p,\#}(\Omega)} = \|\nabla \varphi\|_{L^p(\Omega)} = 0\) implies \(\varphi = 0\). Indeed, the trivial norm implies that the gradient vanishes \(\nabla \varphi = 0\), which means that the corresponding function is constant \(\varphi = C_0 = \text{const}\); since the mean value is zero \(\varphi_{\Omega} = C_0 = 0\) and \(\varphi \equiv 0\).

The inequality \(\|\psi\|_{W^1_{p,\#}(\Omega)} \leq \|\psi\|_{W^1_p(\Omega)}\), where
\[
\|\psi\|_{W^1_p(\Omega)} := \left[ \|\psi\|_{L^p(\Omega)}^p + \|\nabla \psi\|_{L^p(\Omega)}^p \right]^{1/p}
\]
is the standard subspace norm on \(W^1_{p,\#}(\Omega)\) is trivial. Therefore the embedding \(W^1_{p,\#}(\Omega) \subset W^1_p(\Omega)\) with the appropriate norms is continuous and proper, since constants belong to \(W^1_p(\Omega)\) but not to \(W^1_{p,\#}(\Omega)\).

If we apply the open mapping theorem of Banach (see [Ru73, Theorem 2.11, Corollary 2.12.b]), we conclude that the inverse inequality
\[
\|\psi\|_{W^1_p(\Omega)} \leq C_1\|\psi\|_{W^1_{p,\#}(\Omega)} = C_1\|\nabla \psi\|_{L^p(\Omega)}
\]
holds with some constant $C_1 < \infty$ for all $\psi \in W^1_{p,\#}(\Omega)$ (see [1172 Theorem 6.28.2] for a similar proof).

Since $\varphi_0 := \varphi - \varphi_\Omega \in W^1_{p,\#}(\Omega)$, we have

$$\|\varphi - \varphi_\Omega \| W^1_p(\Omega) = \|\varphi - \varphi_\Omega \| L_p(\Omega) + \|\nabla\varphi \| L_p(\Omega) \leq C_1^p \|\nabla\varphi \| L_p(\Omega).$$

The claimed inequality (1) follows with the constant $C := (C_1^p - 1)^{1/p}$. □

**Proof of inequalities (3), (4), (13) and (16):** Inequalities (1) and (15) are particular cases of (13). Inequality (13) follows from (21) if the functional $F$ is chosen as follows:

$$F(\varphi) := \left[ \sum_{|\beta| < m} \left| \int_{\mathcal{M}_0} (\partial^\beta \varphi)^+(x) \, d\sigma \right|^p \right]^{1/p}.$$

The condition $F(\varphi) \leq C \|\varphi \| W^m_p(\Omega)$ (see Lemma [7] ii) holds due to the Sobolev’s continuous embeddings $W^m_{p-k-1/p}(\mathcal{M}_0) \subset L_p(\mathcal{M}_0)$, $k = 0, 1, \ldots, m - 1$, and the trace theorem

$$\|(\partial^\beta \varphi)^+ \| W^m_{p-k-1/p}(\mathcal{M}_0) \leq C_1 \|\varphi \| W^m_p(\Omega), \quad |\beta| < m. \quad \Box$$

**Proof of inequalities (9), (10), (11), (14) and (16):** Inequality (9) is proved verbatim to (1).

Inequalities (10), (11) and (16) follow from (14) (are particular cases). Inequality (14) follows from (21) for surfaces if the functional $F$ is chosen as follows:

$$F(\varphi) := \left[ \sum_{|\beta| < m} \left| \int_{\Gamma_0} (D^\beta \varphi)^+(x) \, d\sigma \right|^p \right]^{1/p}.$$

The condition $F(\varphi) \leq C \|\varphi \| W^m_p(\mathcal{C})$ (see Lemma [7] ii) holds due to the Sobolev’s continuous embeddings $W^m_{p-k-1/p}(\Gamma_0) \subset L_p(\Gamma_0)$, $k = 0, 1, \ldots, m - 1$, and the trace theorem

$$\|(D^\beta \varphi)^+ \| W^m_{p-k-1/p}(\Gamma_0) \leq C_1 \|\varphi \| W^m_p(\mathcal{C}), \quad |\beta| < m. \quad \Box$$

**Proof of inequalities (5) and (6):** Let $\Omega := C \times [a, b]$ and $\mathcal{M}_0 := \Gamma_0 \times [a, b]$. To prove the inequality (5) we proceed similarly: the formula

$$\|\varphi \| W^1_p(\Omega) := \left[ \|\nabla\varphi \| L_p(\Omega) \right]^p + \int_{\mathcal{M}_0} |\varphi^+(x)|^p \, d\sigma \right]^{1/p}.$$
defines a norm in the space $W^1_p(C)$. Indeed, we have to check that $||\varphi||_{W^1_p(\Omega)} = 0$, which implies
\[
\nabla C \varphi(x, t) = 0 \quad \forall x \in C, \quad t \in [a, b], \quad \int_{M_0} |\varphi^+(\tau, t)|^p d\sigma = 0, \tag{3}
\]
gives $\varphi = 0$. But from the first equality in (3), due to Lemma 1.1, follows $\varphi(x, t) = \varphi(t)$ is independent of the surface variable. But since $\varphi(\tau, t) = 0$ on $M_0$ (see the second equality in (3)), the function vanishes on the entire level surface $C \times \{t\}$ for all $t \in [a, b]$. Then $\varphi = 0$ in $\Omega$.

Due to Sobolev’s continuous embedding $W^{1-1/p}(\mathcal{M}_0) \subset L^p(\mathcal{M}_0)$ and the trace theorem
\[
||\varphi||_{W^{1/p}(\mathcal{M}_0)} \leq C_1 ||\varphi||_{W^1_p(\Omega)} \tag{4}
\]
(see [?]), the initial norm in the space $W^1_p(\Omega)$
\[
||\varphi||_{W^1_p(\Omega)} := [||\varphi||_{L^p(\Omega)} + ||\partial_t \varphi||_{L^p(\Omega)} + \|\nabla C \varphi||_{L^p(\Omega)}]^2]^{1/p}
\]
(see [Du10, Du11]) estimates, obviously, the introduced norm
\[
\|\nabla C \varphi||_{W^1_p(\Omega)} \leq C_2 ||\varphi||_{W^1_p(\Omega)}.
\]
Then from open mapping theorem of Banach follows the inverse inequality
\[
||\varphi||_{L^p(\Omega)} \leq ||\varphi||_{W^1_p(\Omega)} \leq C \|\varphi||_{W^1_p(\Omega)} = 0
\]
and this accomplishes the proof of (5).

The inequality (3) is a direct consequence of (5). □

**Proof of inequalities (9):** The proof is verbatim to the proof of inequality (11), using the standard norm (8) and the equivalent norm $||\nabla C \varphi||_{L^p(\Omega)}$ on the space $W^{1,p}(C)$. We also have to apply Lemma 1.1 to conclude that $\nabla C \varphi = 0$ for $\varphi \in W^1_p(C, \Gamma_0)$ implies $\varphi \equiv 0$. □

**Proof of inequalities (10), (11) and (14):** Inequality (11) is a particular case of (10) (and of (9)), while (10) is, in its turn, a particular case, $m = 1$, of (14). Inequality (14) follows from (21) if the functional $F$ is chosen as follows:
\[
F(\varphi) := \left[ \sum_{|\beta| < m} \int_{M_0} (\partial^\beta \varphi)^+(x) d\sigma \right]^{1/p}
\]
The condition $F(\varphi) \leq C ||\varphi||_{W^m_p(\Omega)}$ (see Lemma 1.7) holds due to the Sobolev’s continuous embeddings $W^{m-1/p}(\mathcal{M}_0) \subset L^p(\mathcal{M}_0)$ and the trace theorem
\[
||\varphi^+||_{W^{m-1/p}(\mathcal{M}_0)} \leq C_1 ||\varphi||_{W^m_p(\Omega)}.
\]

12
Proof of inequalities (17) and (18): These inequalities follow from (21) if the functional $F$ is chosen as follows

$$F(\varphi) := \left[ \int_{\mathcal{M}_0} |\varphi^+(x)|^p d\sigma \right]^{1/p}$$

for a domain $\Omega$ and

$$F(\varphi) := \left[ \int_{\Gamma_0} |\varphi^+(x)|^p d\sigma \right]^{1/p}$$

for a hypersurface $\mathcal{C}$ (see (4) for the justification of the condition $F(\varphi) \leq C\|\varphi\|_{H^m}$ in Lemma 1.7.ii).

Proof of inequality (19) (and of similar ones): For the space of smooth functions $C^1(\Omega)$ the proof is verbatim to the cases of the space $W^{1,p}(\Omega)$.

Proof of inequalities (20) and (21): Based on the unique continuation property (see Theorem 1.2 and Theorem 1.3), we prove easily that

$$\|U\|_{W^{1,p}_p(\Omega)} := \left[ \|\text{Def} U\|_{L_p(\Omega)}^p + \|U^+\|_{L_p(\mathcal{M}_0)}^p \right]^{1/p},$$

$$\|U\|_{W^{1,p}_p(\mathcal{C})} := \left[ \|\text{Def}_C U\|_{L_p(\mathcal{C})}^p + \|U^+\|_{L_p(\Gamma_0)}^p \right]^{1/p}$$

define norms in the spaces $W^{1,p}_p(\Omega)$ and in $W^{1,p}_p(\mathcal{C})$, respectively. Then the obvious inequalities

$$\|U\|_{W^{1,p}_p(\Omega)} \leq \|U\|_{W^{1,p}_p(\Omega)} \quad \text{and} \quad \|U\|_{W^{1,p}_p(\mathcal{C})} \leq \|U\|_{W^{1,p}_p(\mathcal{C})},$$

with the equivalent norms on the spaces $W^{1,p}_p(\Omega)$ and in $W^{1,p}_p(\mathcal{C})$ defined in (13) and (16) (see Theorem 1.4 and Theorem 1.5) and the open mapping theorem of Banach ensure that the inverse inequalities

$$\|U\|_{L_p(\Omega)} \leq C_1\|U\|_{W^{1,p}_p(\Omega)} \quad \text{and} \quad \|U\|_{L_p(\mathcal{C})} \leq C_1\|U\|_{W^{1,p}_p(\mathcal{C})},$$

hold and accomplish the proof.

Proof of inequalities (22) and (23): These inequalities are obvious consequences of (20) and (21).

Proof of inequalities (24) and (25): Inequality (24) is proved verbatim to inequality (3) by using, instead of Lemma 1.1, the unique continuation property of Killing’s vector fields, solutions to the equations system $\text{Def}_C U = 0$ (see Theorem 1.3).

Inequality (25) is an obvious consequence of (24).
References

[Ca00] P.G. Ciarlet, *Mathematical Elasticity, Vol. III: Theory of Shells*, Studies in Mathematics and Applications, 29, Elsevier, North-Holland, Amsterdam, 2000.

[DL90] R. Dautray, J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Springer-Verlag, Berlin 1990.

[Du10] R. Duduchava, Lions’s lemma, Korns inequalities and Lamé operator on hypersurfaces, *Operator Theory: Advances and Applications*, Vol. 210, 43-77, 2010 Springer AG, Basel.

[Du11] R. Duduchava, A revised asymptotic model of a shell. *Memoirs on Differential Equations and Mathematical Physics* 52, 2011, 65-108.

[DMM06] R. Duduchava, D. Mitrea, M. Mitrea, Differential operators and boundary value problems on surfaces. *Mathematische Nachrichten* 279, No. 9-10 (2006), 996-1023.

[Gu53] N. Günter, *Potential Theory and its Application to the Basic Problems of Mathematical Physics*, Fizmatgiz, Moscow 1953 (Russian. Translation in French: Gauthier-Villars, Paris 1994).

[HW08] G. C. Hsiao & W. L. Weendland, *Boundary Integral Equations*, Applied Mathematical Sciences, Springer-Verlag Berlin Heidelberg, 2008.

[KGBB79] V. Kupradze, T. Gegelia, M. Basheleishvili, T. Burchuladze, *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoe-lasticity*, North-Holland, Amsterdam 1979 (Russian edition: Nauka, Moscow 1976).

[Ru73] W. Rudin, *Functional Analysis*, McGraw-Hill Company. New York 1973.

[Tr95] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, 2nd edition, Johann Ambrosius Barth Verlag, Heidelberg-Leipzig 1995.

[Tr72] H. Triebel, *Höhere Analysis*, Dt. Verlag d. Wissenschaften, Berlin, 1972 (English translation: *Higher analysis*, Huthig Pub Limited, 1992).

[Wl87] J. Wloka, *Partial Differential Equations*, Cambridge University Press, Cambridge, 1987.