HOMOMORPHISMS ON GROUPS OF VOLUME-PRESERVING DIFFEOMORPHISMS VIA FUNDAMENTAL GROUPS

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Dedicated to Professor Takashi Tsuboi on the occasion of his 60-th birthday

Abstract. Let $M$ be a closed manifold. Polterovich constructed a linear map from the vector space of quasi-morphisms on the fundamental group $\pi_1(M)$ of $M$ to the space of quasi-morphisms on the identity component $\text{Diff}^\infty_0(M)_0$ of the group of volume-preserving diffeomorphisms of $M$. In this paper, the restriction $H^1(\pi_1(M); \mathbb{R}) \to H^1(\text{Diff}^\infty_0(M)_0; \mathbb{R})$ of the linear map is studied and its relationship with the flux homomorphism is described.

1. Introduction

Let $M$ be a closed connected Riemannian manifold and $\Omega$ a volume form on $M$. We denote by $\text{Diff}^\infty_0(M)_0$ the identity component of the group of volume-preserving $C^\infty$-diffeomorphisms of $M$. We assume that the center of the fundamental group $\pi_1(M)$ is finite. In [4], Gambaudo and Ghys constructed countably many quasi-morphisms on the group of area-preserving diffeomorphisms of the 2-disk from the signature quasi-morphism on the braid groups. After that, Polterovich introduced in [6] a similar construction of quasi-morphisms on $\text{Diff}^\infty_0(M)_0$ from quasi-morphisms on $\pi_1(M)$. Recently, Brandenbursky generalized these strategy and defined a homomorphism from the vector space of quasi-morphisms on the braid group or the fundamental group to the space of quasi-morphisms of volume-preserving diffeomorphisms [2][3].

Polterovich’s construction induces a linear map from the vector space of quasi-morphisms on $\pi_1(M)$ to the vector space of quasi-mor-phisms on $\text{Diff}^\infty_0(M)_0$. By restricting it on $H^1(\pi_1(M); \mathbb{R})$, we have the linear map $\Gamma: H^1(\pi_1(M); \mathbb{R}) \to H^1(\text{Diff}^\infty_0(M)_0; \mathbb{R})$, which is defined in section 2 of this paper. Studying the linear map $\Gamma: H^1(\pi_1(M); \mathbb{R}) \to H^1(\text{Diff}^\infty_0(M)_0; \mathbb{R})$, we have a sufficient condition for vanishing of the volume flux group which is first obtained by Kędra-Kotschick-Morita in another way.

Theorem 1.1 (Kędra-Kotschick-Morita[5]). If the center of $\pi_1(M)$ is finite, then the volume flux group of $M$ is trivial.

Let $\text{Flux}: \text{Diff}^\infty_0(M)_0 \to H^{-1}_d(M; \mathbb{R})$ be the $\Omega$-flux homomorphism. Let $I^k: H^k_{dR}(M; \mathbb{R}) \to H^k(M; \mathbb{R})$ be the isomorphism which gives the identification of the de Rham cohomology and the singular cohomology defined by

$$I^k([\eta])(\sigma) = \int_\sigma \eta$$

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2. PRELIMINARIES

In this section, we define a linear map

\[ \Gamma: H^1(\pi_1(M); \mathbb{R}) \to H^1(\text{Diff}^{\infty}_\Omega(M)_0; \mathbb{R}) \]

and recall a definition of the flux homomorphism.

Here and throughout this paper, we use functional notation. That is, for any homotopy classes \( \gamma_1 \) and \( \gamma_2 \) of loops with a fixed base point, the multiplication \( \gamma_1 \gamma_2 \) means that \( \gamma_2 \) is applied first.

Choose a base point \( x^0 \) of \( M \). For almost every \( x \in M \), we choose the shortest geodesic \( a_x: [0, 1] \to M \) connecting \( x^0 \) with \( x \) if it is uniquely determined. For any \( f \in \text{Diff}^{\infty}_\Omega(M)_0 \) and almost every \( x \in M \) for which both the geodesics \( a_x \) and \( a_{f(x)} \) is defined, we define the loop \( l(f; x): [0, 1] \to M \) by

\[
 l(f; x)(t) = \begin{cases} 
 a_x(3t) & (0 \leq t \leq \frac{1}{3}) \\
 f_{u-1}(x) & \left(\frac{1}{3} \leq t \leq \frac{2}{3}\right) \\
 a_{f(x)}(3 - 3t) & \left(\frac{2}{3} \leq t \leq 1\right)
\end{cases}
\]

where \( \{f_t\}_{t \in [0,1]} \) is an isotopy such that \( f_0 \) is the identity and \( f_1 = f \). Of course for some \( x \in M \) there exist two or more shortest geodesics connecting \( x^0 \) with \( x \). However for almost every \( x \in M \) the loop \( l(f; x) \) is well-defined. We denote by \( \gamma(f; x) \) the homotopy class represented by the loop \( l(f; x) \).

For a homomorphism \( \phi \in H^1(\pi_1(M); \mathbb{R}) \), we define the homomorphism \( \Gamma(\phi) \in H^1(\text{Diff}^{\infty}_\Omega(M)_0; \mathbb{R}) \) by

\[ \Gamma(\phi)(f) = \int_{x \in M} \phi(\gamma(f; x)) \Omega. \]

For almost every \( x \in M \), the homotopy class \( \gamma(f; x) \) is well-defined and is unique up to elements of the center of \( \pi_1(M) \). Since the center of \( \pi_1(M) \) is finite, the image of \( \gamma(f; x) \) by the homomorphism \( \phi: \pi_1(M; x^0) \to \mathbb{R} \) is independent of the choice of the flow \( \{f_t\} \). Since the manifold \( M \) is compact, the loops \( l(f; x) \) have uniformly bounded length for fixed \( \{f_t\} \). Hence the map \( \gamma(f; x): M \to \pi_1(M; x^0) \) has a finite image and the value \( \Gamma(\phi)(f) \) is well-defined.

Let \( \text{Diff}^{\infty}_\Omega(M)_0 \) be the universal cover of \( \text{Diff}^{\infty}_\Omega(M)_0 \). Consider a path \( \{f_t\}_{t \in [0,1]} \) in \( \text{Diff}^{\infty}_\Omega(M)_0 \) such that \( f_0 \) is the identity. Let \( X_t \) be the corresponding vector field. Then the map \( \overline{\text{Flux}}: \text{Diff}^{\infty}_\Omega(M)_0 \to H^{n-1}_{\text{dr}}(M; \mathbb{R}) \) is defined by

\[
 \overline{\text{Flux}}(\{f_t\}) = \left[ \int_0^1 \iota_{X_t}(\Omega) dt \right],
\]

where \( \iota_{X_t} \) is the interior product by \( X_t \). The map \( \overline{\text{Flux}}: \text{Diff}^{\infty}_\Omega(M)_0 \to H^{n-1}_{\text{dr}}(M; \mathbb{R}) \) is a well-defined homomorphism and called the \( \Omega \)-flux homomorphism. The fundamental group \( \pi_1(\text{Diff}^{\infty}_\Omega(M)_0) \) is contained in \( \text{Diff}^{\infty}_\Omega(M)_0 \) as a subgroup of deck transformations. The image \( G_\Omega = \overline{\text{Flux}}(\pi_1(\text{Diff}^{\infty}_\Omega(M)_0)) \) of \( \pi_1(\text{Diff}^{\infty}_\Omega(M)_0) \) by the
The homotopy classes $\Omega$-flux homomorphism $\tilde{\text{Flux}} : \text{Diff}_\Omega^\infty(M)_0 \to H_{dR}^{n-1}(M; \mathbb{R})$ is called the volume flux group of $M$ and the homomorphism $\text{Flux} : \text{Diff}_\Omega(M)_0 \to H_{dR}^{n-1}(M; \mathbb{R})$ descends to the homomorphism $\text{Flux} : \text{Diff}_\Omega^\infty(M)_0 \to H_{dR}^{n-1}(M; \mathbb{R})/G_\Omega$, which is also called the $\Omega$-flux homomorphism.

3. Proofs

In this section, we give proofs of Theorems 1.1 and 1.2. The following theorem is mentioned in [6] without proof.

**Theorem 3.1.** The linear map
\[
\Gamma : H^1(\pi_1(M); \mathbb{R}) \to H^1(\text{Diff}_\Omega^\infty(M)_0; \mathbb{R})
\]
is injective.

We give a proof of Theorem 3.1. Let $\beta \in \pi_1(M; x^0)$. Suppose that we can choose a loop $l$ representing $\beta$ without self-intersection. Choose a tubular neighborhood $N \subset M$ of $l$ and a diffeomorphism $\varphi : N \to D^{n-1} \times S^1$. Let $(z, s)$ be the coordinate on $D^{n-1} \times S^1$. We may assume that there exists $\Omega' \in A^{n-1}(D^{n-1}; \mathbb{R})$ such that $\varphi^*(\Omega' ds) = \Omega|_N$ by changing the neighborhood $N$ and diffeomorphism $\varphi$ if necessary. Let $\omega : D^{n-1} \to \mathbb{R}$ be a function such that $\omega(z) = 0$ in a neighborhood of the boundary. We define the volume-preserving diffeomorphism $f_\omega$ of $D^{n-1} \times S^1$ by
\[
f_\omega(z, s) = (z, s + \omega(z)).
\]
and define $F_\omega \in \text{Diff}_\Omega^\infty(M)_0$ to be the identity outside $N$ and $F_\omega = \varphi^{-1} f_\omega \varphi$ on $N$.

**Lemma 3.2.** For any $\phi \in H^1(\pi_1(M); \mathbb{R})$,
\[
\Gamma(\phi)(F_\omega) = \phi(\beta) \int_{x \in D^{n-1}} \omega(z) \Omega'.
\]

**Proof.** Note that the base point $x^0$ of $M$ is in $N$. Let us denote $\varphi(x^0)$ by $(z^0, s^0)$ and $\varphi(x)$ by $(z^1, s^1)$. Let $v$ be the smallest non-negative number such that $s^1 + v = s^0$. For each $x \in N$ we define the paths $l_1, l_2, l_3 : [0, 1] \to D^{n-1} \times S^1$ by
\[
l_1(t) = (tz^0 + (1-t)s^1, s^1),
\]
\[
l_2(t) = (z^0, s^1 + tv),
\]
\[
l_3(t) = (z^1, s^1 + t(\omega(z^1) - [\omega(z^1)])).
\]
We define the homotopy classes $\zeta_x, \eta_x$ of loops in $M$ by
\[
\zeta_x = [(\varphi^{-1})_*(l_2l_1)a_x], \quad \eta_x = [a_{F_\omega(x)}(\varphi^{-1})_*(l_3l_1)a_x].
\]

Since the path $\{F_\omega\}$ connects the identity and $F_\omega$ in $\text{Diff}_\Omega^\infty(M)_0$, the homotopy class $\gamma(F_\omega; x)$ is trivial if $x \notin N$. On the other hand, $\gamma(F_\omega; x)$ can be written as
\[
\gamma(F_\omega; x) = \eta_x \zeta_x^{-1} \beta|\omega(z^1)| \zeta_x
\]
if $x \in N$. Therefore,
\[
\Gamma(\phi)(F_\omega) = \int_{x \in N} \phi(\gamma(F_\omega; x)) \Omega
\]
\[
= \phi(\beta) \int_{x \in N} |\omega(z^1)| \Omega + \int_{x \in N} \phi(\eta_x) \Omega.
\]
Since $F_k^\omega = F_{k\omega}$ for any $k \in \mathbb{Z}$,
\[
\Gamma(\phi)(F_\omega) = \lim_{k \to \infty} \frac{1}{k} \Gamma(\phi)(\gamma(F_{k\omega}; x))\Omega.
\]
Since the domain $N$ is compact, the value $\phi(\eta_x)$ is bounded and thus we have
\[
\Gamma(\phi)(F_\omega) = \phi(\beta) \int_{x \in N} \omega(z)\Omega = \phi(\beta) \int_{z \in D^{n-1}} \omega(z)\Omega.
\]

Proof of Theorem 1.1. Suppose a homomorphism $\phi \in H^1(\pi_1(M); \mathbb{R})$ is non-trivial. Then there exists a homotopy class $\beta$ of a loop without self-intersection in $M$ such that $\phi(\beta) \neq 0$. It is sufficient to prove that there exists $g \in \text{Diff}_\Omega^\infty(M)_0$ such that $\Gamma(\phi)(g) \neq 0$. If we choose a function $\omega : D^{n-1} \to \mathbb{R}$ such that
\[
\int_{z \in D^{n-1}} \omega(z)\Omega' \neq 0,
\]
then by Lemma 3.2 we have $\Gamma(\phi)(F_\omega) \neq 0$. \qed

Proof of Theorem 3.1. It is known that the flux homomorphism gives the abelianization of the group $\text{Diff}_\Omega^\infty(M)_0$ [1]. Hence for any homomorphism $\phi \in H^1(\pi_1(M); \mathbb{R})$ there exists a homomorphism
\[
A_\phi : H^{n-1}_\text{dR}(M; \mathbb{R})/G_\Omega \to \mathbb{R}
\]
such that the homomorphism $\Gamma(\phi) \in H^1(\text{Diff}_\Omega^\infty(M)_0; \mathbb{R})$ can be represented by the composition of homomorphisms $\text{Flux} : \text{Diff}_\Omega^\infty(M)_0 \to H^{n-1}_\text{dR}(M; \mathbb{R})/G_\Omega$ and $A_\phi : H^{n-1}_\text{dR}(M; \mathbb{R})/G_\Omega \to \mathbb{R}$. That is,
\[
\Gamma(\phi) = A(\phi) \circ \text{Flux} : \text{Diff}_\Omega^\infty(M)_0 \to \mathbb{R}.
\]
Since the diffeomorphism $F_\omega$ is the time 1-map of the time independent vector field
\[
X_x = \left\{ \begin{array}{ll}
(\varphi^{-1})_*(\omega(z) \frac{d}{dz}) & \text{if } x \in N \\
0 & \text{if } x \notin N
\end{array} \right.,
\]
we have
\[
\text{Flux}(F_\omega) = \iota_{X}\Omega = \varphi^*[\omega(z)\Omega'].
\]
In particular,
\[
\text{Flux}(F_{s\omega}) = s\text{Flux}(F_\omega)
\]
for any $\beta \in \pi_1(M)$, any function $\omega : D^{n-1} \to \mathbb{R}$ and any $s \in \mathbb{R}$. On the other hand by Lemma 3.2
\[
\Gamma(\phi)(F_{t\omega}) = t\Gamma(\phi)(F_\omega)
\]
for any $t \in \mathbb{R}$. Choose elements $\beta_1, \ldots, \beta_m \in \pi_1(M, x^0)$ whose images by the projection $\pi_1(M, x^0) \to H_1(M; \mathbb{Z})$ form a basis of $H_1(M; \mathbb{R})$. If we replace $\beta$ with $\beta_1, \ldots, \beta_m$, then $(n-1)$-classes $\varphi^*[\omega(z)\Omega']$'s form a basis of $H^{n-1}_\text{dR}(M; \mathbb{R})$. Hence if there exists a non-trivial element $\xi \in G_\Omega$, then $A_\phi(\xi) = 0$ for any $t \in \mathbb{R}$. The map $A_\phi$ descends to the linear map $A_\phi' : H^{n-1}_\text{dR}(M; \mathbb{R})/\langle G_\Omega \rangle \to \mathbb{R}$, where $\langle G_\Omega \rangle$ means the vector subspace of $H^{n-1}_\text{dR}(M; \mathbb{R})$ spanned by elements of $G_\Omega$.

By Theorem 3.1
\[
\text{rank}_\mathbb{R} H^1(M; \mathbb{R}) = \text{rank}_\mathbb{R} \text{Im} \Gamma \leq \text{rank}_\mathbb{R} \text{Hom}(H^{n-1}_\text{dR}(M; \mathbb{R})/\langle G_\Omega \rangle, \mathbb{R}).
\]
If there exists a non-trivial element $\xi \in G$, then
\[ \text{rank}_R \text{Hom}(H^{n-1}_{dR}(M; \mathbb{R})/\langle G \rangle, \mathbb{R}) < \text{rank}_R H^{n-1}(M; \mathbb{R}). \]
while by the Poincaré duality
\[ \text{rank}_R H^1(M; \mathbb{R}) = \text{rank}_R H^{n-1}(M; \mathbb{R}). \]
This contradiction shows that there’s no non-trivial element in $G$. □

**Proof of Theorem 1.2.** The statement is that $A_{\phi} = \phi \circ PD \circ I^{n-1} : H^{n-1}_{dR}(M; \mathbb{R}) \to \mathbb{R}$. Since $A_{\phi} : H^{n-1}_{dR}(M; \mathbb{R}) \to \mathbb{R}$ is a linear map, it is sufficient to choose $\eta_1, \ldots, \eta_m$ generating $H^{n-1}_{dR}(M; \mathbb{R})$ and prove that $A_{\phi}(\eta_i) = \phi \circ PD \circ I^{n-1}(\eta_i)$ for $1 \leq i \leq m$.

Since
\[ \text{Flux}(F_\omega) = \iota_X \Omega = \varphi^*[\omega(z)\Omega'], \]
we have
\[ I^{n-1} \circ \text{Flux}(F_\omega)(\sigma) = \int_{\varphi^*\sigma} \omega(z)\Omega'. \]
Therefore,
\[ PD \circ I^{n-1} \circ \text{Flux}(F_\omega) = \left( \int_{z \in \partial^{n-1}D} \omega(z)\Omega' \right) \beta. \]
Comparing this equation with Lemma 3.2, we have
\[ \Gamma(\phi)(F_\omega) = \phi \circ PD \circ I^{n-1} \circ \text{Flux}(F_\omega) \]
for any $\phi \in H^1(M; \mathbb{R})$.

As in the proof of Theorem 1.1 choose homotopy classes $\beta_1, \ldots, \beta_m \in \pi_1(M, x^0)$ whose images by the projection $\pi_1(M, x^0) \to H_1(M; \mathbb{Z})$ form a basis of $H_1(M; \mathbb{R})$. If we replace $\beta$ with $\beta_1, \ldots, \beta_m$, then $\text{Flux}(F_\omega)$’s form a basis of $H^{n-1}_{dR}(M; \mathbb{R})$ and hence this completes the proof. □

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