BANACH ALGEBRAS, SAMELSON PRODUCTS, AND THE WANG DIFFERENTIAL

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ABSTRACT. Suppose given a principal $G$-bundle $\zeta : P \to S^k$ (with $k \geq 2$) and a Banach algebra $B$ upon which $G$ acts continuously. Let

$$\zeta \otimes B : P \times_G B \to S^k$$

denote the associated bundle and let

$$A_{\zeta \otimes B} = \Gamma(S^k, P \times_G B)$$

denote the associated Banach algebra of sections. Then $\pi_\ast \text{GL} A_{\zeta \otimes B}$ is determined by a mostly degenerate spectral sequence and by a Wang differential

$$d_k : \pi_\ast(\text{GL} B) \to \pi_{\ast + k - 1}(\text{GL} B).$$

We show that if $B$ is a $C^*$-algebra then the differential is given explicitly in terms of an enhanced Samelson product with the clutching map of the principal bundle. Analogous results hold after localization and in the setting of topological $K$-theory.

We illustrate our technique with a close analysis of the invariants associated to the $C^*$-algebra of sections of the bundle

$$\zeta \otimes M_2 : S^7 \times S^3 M_2 \to S^4$$

constructed from the Hopf bundle $\zeta : S^7 \to S^4$ and by the conjugation action of $S^3$ on $M_2 = M_2(C)$. We compare and contrast the information obtained from the homotopy groups $\pi_\ast(A_{\zeta \otimes M_2})$, the rational homotopy groups $\pi_\ast(A_{\zeta \otimes M_2}) \otimes \mathbb{Q}$ and the topological $K$-theory groups $K_\ast(A_{\zeta \otimes M_2})$.

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The group $GL_A$ of invertible elements of a Banach algebra $A$ has the homotopy type of a CW-complex, and hence its homotopy groups are in principle computable. We know that these groups hold a lot of information about $A$, since the topological $K$-theory groups $K_*(A)$ are given by the stabilized groups $\pi_*(GL(A \otimes K))$. The groups $\pi_*(GL_A)$ are far richer in information but also far more difficult to compute.

In joint work with Emmanuel Dror-Farjoun, we have developed a spectral sequence aimed at computing these groups in a wide variety of settings. Spectral sequences reveal and conceal. On the one hand, there are long lists of spectacular results in topology and algebra that have been obtained by spectral sequence techniques (cf. [15]). On the other hand, spectral sequence cognoscenti will testify that there is depressingly little known in general about differentials in spectral sequences, so that reducing a problem to a “spectral sequence calculation” may not in fact solve the problem at all. In order to put the results of this paper in context, we review briefly what is known about differentials in this genre of spectral sequence.

The classical Atiyah - Hirzebruch spectral sequence [2] takes the form

$$E_2 = H^*(X; K^*(pt)) \implies K^*(X).$$

Atiyah and Hirzebruch noted the following:

1. $d_2 n = 0$ for all $n$ because $K^{odd}(pt) = 0$.
2. $d_3$ is associated with the integral Steenrod operation $Sq^3 : H^k(X; \mathbb{Z}) \to H^{k+3}(X; \mathbb{Z})$.
3. For $k \geq 2$, each $d_k$ takes values in the torsion subgroup of $E_k$ and hence the spectral sequence collapses rationally:

$$E_2 \otimes \mathbb{Q} \cong E_\infty \otimes \mathbb{Q}.$$ 

Arlettaz [1] gives explicit integers governing the order of the torsion subgroups. In addition, there are always “dimension” arguments in particular cases. For example, if $H^*(X; \mathbb{Z}) = 0$ in all odd degrees, then $E_2 = 0$ in odd total degree, $d_j = 0$ for all $j \geq 2$, and so $E_2 \cong E_\infty$.

More generally, if $h^*$ is a generalized cohomology theory then there is a well-known spectral sequence

$$E_2 = H^*(X; h^*(pt)) \implies h^*(X).$$

In another direction there is the classical Federer spectral sequence [9]

$$E_2 = H^*(X; \pi_*(Y)) \implies H^*(F(X, Y)),$$

where $F(X, Y)$ denotes the function space of maps from $X$ to $Y$ with the compact-open topology. Sam Smith [22] shows that in the context of Quillen minimal models, differentials are related to Whitehead products. (Our results will echo this result in the integral situation.)

Moving to twisted $K$-theory $K^*_\Delta(X)$ associated to a principal bundle, the bundle is classified by its Dixmier-Douady invariant $\Delta$ and the spectral sequence takes the form

$$E_2 = H^*(X; K^*(pt)) \implies K^*_\Delta(X)$$

with the $d_3$ differential related to the Dixmier-Douady invariant by the result of J. Rosenberg [19]. Atiyah and Segal ([3] Prop. 7.5) show that if the base space is a
compact manifold then in the associated spectral sequence

$$E_2 = H^*(X; K^*(pt)) \otimes \mathbb{R} \Rightarrow K_\Delta^*(X) \otimes \mathbb{R},$$

all higher ($j \geq 4$) differentials are given by Massey products. They point out that this implies that all higher differentials vanish (over the reals) when the base space is a compact Kähler manifold, by the deep result of [7].

Let $GL_o B$ denote the path component of the identity of $GL B$. Similarly, for $C^*$-algebras, let $U_o B$ denote the path component of the identity of the unitary group $UB$. Recall that the inclusions $U_o B \subset GL_o B$ and $U B \subset GL B$ are deformation retractions and hence homotopy equivalences.

We consider the case of a fibre bundle over a sphere in order to isolate a new type of differential. Here are our primary results. Let $P$ denote a subring of the rational numbers. (We allow the cases $P = \mathbb{Z}$ and $P = \mathbb{Q}$ as well as intermediate rings.) We define a bundle of $C^*$-algebras precisely in §2.

**Theorem A.** Suppose that

$$\zeta \otimes B : \quad P \times_G B \to S^k$$

is a bundle of $C^*$-algebras over $S^k$ with $k \geq 2$. Let $A_{\zeta \otimes B}$ denote the associated $C^*$-algebra of continuous sections. Then there is a long exact Wang sequence

$$\cdots \to \pi_n(U_o A_{\zeta \otimes B}) \otimes \mathbb{P} \to \pi_n(U_o B) \otimes \mathbb{P} \xrightarrow{d_k} \pi_n(U_o B) \otimes \mathbb{P} \xrightarrow{\kappa} \pi_{n-1}(U_o A_{\zeta \otimes B}) \otimes \mathbb{P} \to \cdots$$

The differential $d_k$ is given by

$$d_k(a) = -g[\kappa, a]$$

where $g$ is the generator of $H^k(S^k; \mathbb{Z})$ and $[\kappa, a]$ is the enhanced Samelson product with the map $\kappa : S^{k-1} \to G$ that classifies the principal bundle.

The enhanced Samelson product referred to is a generalization of the classical Samelson product that we explain in §6.

Passing to limits, we also obtain the analogous sequence at the level of $K$-theory:

**Theorem B.** Suppose that

$$\zeta \otimes B : \quad P \times_G B \to S^k$$

is a bundle of $C^*$-algebras over $S^k$ with $k \geq 2$. Let $A_{\zeta \otimes B}$ denote the associated $C^*$-algebra of continuous sections. Then there is a long exact sequence

$$\cdots \to K_n(A_{\zeta \otimes B}) \otimes \mathbb{P} \xrightarrow{\rho} K_n(B) \otimes \mathbb{P} \xrightarrow{d_k} K_{n+k-1}(B) \otimes \mathbb{P} \xrightarrow{\kappa} K_{n-1}(A_{\zeta \otimes B}) \otimes \mathbb{P} \to \cdots$$

When $k = 3$ then $d_3$ is given by multiplication by $-\Delta_\zeta \beta$ where $\Delta_\zeta$ is the Dixmier-Douady integer and $\beta$ is Bott periodicity.

The paper is organized as follows. First we state the general spectral sequence results of [8], we specialize them to the case when the base is a sphere, and we derive the homological algebra version of the Wang sequence. We then make a preliminary identification of the differential. Then we make a detour to classical homotopy theory to define the enhanced Samelson product. Next, we put everything together to establish Theorems A and B.

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1 Wang’s original 1949 paper [28] gave a direct and elementary proof of a homology version of this sequence. Serre ([21] p. 471) put the result into the spectral sequence setting. Since then it has appeared in many contexts.
Finally, we illustrate our results with a close analysis of the invariants associated to the $C^*$-algebra of sections of the bundle

$$\zeta \otimes M_2 : \quad S^7 \times S^3 \to S^4$$

constructed from the Hopf bundle $\zeta : S^7 \to S^4$ and by the conjugation action of $S^3$ on $M_2 = M_2(\mathbb{C})$. We explicitly compute the homotopy groups $\pi_n(U_0 A\zeta \otimes M_2)$ for $n \leq 8$ and contrast these results with the computation of the rational homotopy groups $\pi_n(U_0 A\zeta \otimes M_2) \otimes \mathbb{Q}$ and with the topological $K$-theory groups $K_n(A\zeta \otimes M_2)$ by explicit examination of the relevant groups and the maps

$$\pi_n(U_0 A\zeta \otimes M_2) \longrightarrow \pi_n(U_0 A\zeta \otimes M_2) \otimes \mathbb{Q}$$

$$\downarrow$$

$$K_{n+1}(A\zeta \otimes M_2).$$

The following table summarizes our calculation.

The homotopy, rational homotopy and $K$-theory of $A\zeta \otimes M_2$ in low degrees

| $n$ | $\pi_n(U_0 A\zeta \otimes M_2)$ | $\pi_n(U_0 A\zeta \otimes M_2) \otimes \mathbb{Q}$ | $K_{n+1}(A\zeta \otimes M_2)$ |
|-----|-------------------|-------------------|-------------------|
| 1   | $\mathbb{Z} \oplus \mathbb{Z}/2$ | $\mathbb{Q}$ | $\mathbb{Z}$ |
| 2   | 0                 | 0                 | 0                 |
| 3   | $\mathbb{Z}$     | $\mathbb{Q}$     | $\mathbb{Z}$     |
| 4   | 0                 | 0                 | 0                 |
| 5   | 0                 | 0                 | $\mathbb{Z}$     |
| 6   | $\mathbb{Z}/60$  | 0                 | 0                 |
| 7   | $\mathbb{Z}/4$ or $(\mathbb{Z}/2)^2$ | 0 | $\mathbb{Z}$ |
| 8   | $\mathbb{Z}/4 \oplus \mathbb{Z}/2$ or $(\mathbb{Z}/2)^3$ | 0 | 0 |

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2. THE SPECTRAL SEQUENCES

We recall for reference the main results of Farjoun-Schochet [8].

Suppose that $X$ is a finite dimensional compact metric space and $\zeta : P \to X$ is a standard principal $G$ bundle for some topological group $G$ that acts continuously on a Banach algebra $B$ via $\alpha : G \to \text{Aut}(B)$. Let

$$\zeta \otimes B : \quad P \times_G B \to X$$

be the associated fibre bundle. (We refer to this set-up as a standard bundle of Banach algebras.) Let

$$A_{\zeta \otimes B} = \Gamma(X, P \times_G B)$$

A principal $G$-bundle is standard if $X$ is a finite complex or if the bundle is a pullback of a principal $G$-bundle over some CW-complex. See [8] for details.
denote the set of continuous sections of the bundle with pointwise operations. This has a natural structure of a Banach algebra, and if $B$ is a $C^*$-algebra then so is $A_{\zeta \otimes B}$. If $B$ is unital, then $A_{\zeta \otimes B}$ is also unital, with identity the canonical section that to each point $x \in X$ assigns the identity in $(P \times_G B)_x$.

We are interested in $GL_{A_{\zeta \otimes B}}$, the group of invertible elements in $A_{\zeta \otimes B}$. (If $A_{\zeta \otimes B}$ is not unital, then we understand this to mean the kernel of the natural map $GL (A_{\zeta \otimes B}) \rightarrow GL (\mathbb{C})$.) This is a space of the homotopy type of a CW-complex, second countable if $B$ is separable. It may have many (homeomorphic) path components; let $GL_{o A_{\zeta \otimes B}}$ denote the path component of the identity.

If $A$ is a unital $C^*$-algebra then denote the group of unitary elements of $A$ by $U_A$ and its identity path component by $U_o A$. If it is not unital then we define $U_A$ to be the kernel of the natural map $U(A^+) \rightarrow U \mathbb{C}$ and similarly for $U_o A$. The inclusion $UA \rightarrow GL A$ is a homotopy equivalence.

Let $P$ denote a subring of the rational numbers. (We allow the cases $P = \mathbb{Z}$ and $P = \mathbb{Q}$ as well as intermediate rings.)

**Theorem 2.1.** Suppose that $X$ is a finite dimensional compact metric space and that $\zeta \otimes B : P \times_G B \rightarrow X$ is a standard bundle of Banach algebras. Let $A_{\zeta \otimes B}$ denote the associated algebra of sections. Then:

1. There is a second quadrant spectral sequence converging to
   $$
   \pi_* (GL_{o A_{\zeta \otimes B}}) \otimes P
   $$
   with
   $$
   E^2_{p,q} \cong H^p (X; \pi_q (GL_{o B}) \otimes P)
   $$
   and
   $$
   d^r : E^r_{p,q} \rightarrow E^r_{p-r,q+r-1}.
   $$

2. If $X$ has dimension at most $n$, then $E^{n+1} = E^\infty$.

3. The spectral sequence is natural with respect to pullback diagrams

   \[
   \begin{array}{ccc}
   f^* P \times_G B & \xrightarrow{f \times 1} & P \times_G B \\
   \downarrow f^* \zeta \otimes B & & \downarrow \zeta \otimes B \\
   X' & \xrightarrow{f} & X
   \end{array}
   \]

   and associated map $f^* : A_{\zeta \otimes B} \rightarrow A_{f^* \zeta \otimes B}$.

4. The spectral sequence is natural with respect to $G$-equivariant maps

   $$
   \phi : B \rightarrow B'
   $$
   of Banach algebras.

Generally, this spectral sequence does not collapse, even rationally.

Note that in many cases of interest, for instance $B = M_n (\mathbb{C})$, the groups $\pi_* (GL_{o B})$ are unknown, and so the integral version of the spectral sequence cannot be used directly to compute $\pi_* (GL_{o A_{\zeta \otimes B}})$. However, frequently the groups $\pi_* (GL_{o B}) \otimes \mathbb{Q}$ are known and hence the rational form of the spectral sequence will be practical.

Using the version of Bott periodicity established by R. Wood [32] and M. Karoubi [12] and taking limits of spectral sequences, we derive the following.
Theorem 2.2. Suppose that $X$ is a finite dimensional compact metric space, $B$ is a Banach algebra, and $\zeta \otimes B : P \times_G B \to X$ is a standard bundle of Banach algebras. Then there is a second quadrant spectral sequence

$$E_{p,q}^2 \cong H^p(X; K_{q+1}(B) \otimes \mathbb{P}) \implies K_{p+1}(A_\zeta) \otimes \mathbb{P}$$

which is the direct limit over $t$ of the corresponding spectral sequences converging to

$$\pi_*(\text{GL}_o(A_\zeta \otimes B \otimes M_t) \otimes \mathbb{P}).$$

If $X$ has dimension at most $n$ then $E_{n+1}^\infty = E^\infty$.

This result is due to J. Rosenberg [19] when $B = K$ and $A_\zeta \otimes K$ is a continuous trace $C^*$-algebra over a finite complex $X$.

3. The spectral sequences when $X$ is a sphere

Suppose that $X = S^k$ with $k \geq 2$. Then several things simplify radically. First of all, bundles are automatically standard and so we will simply say bundle of Banach algebras in this setting. Second, $X$ is simply connected and hence the local coefficients in the spectral sequences trivialize. Third, the $E^2$ term vanishes except in columns $p = 0$ and $p = -k$. This implies that $E^2 = E^k$ and $E^{k+1} = E^\infty$ in the spectral sequences, so that the only possible non-zero higher differential is $d_k$. Combining these elementary observations we have the following versions of Theorems 2.1 and 2.2.

Theorem 3.1. Suppose that $\zeta \otimes B : P \times_G B \to S^k$ is a bundle of Banach algebras with $k \geq 2$. Let $A_\zeta \otimes B$ denote the associated section algebra. Then there is a second quadrant spectral sequence converging to

$$\pi_*(\text{GL}_o(A_\zeta \otimes B) \otimes \mathbb{P})$$

with $E^2 = 0$ except for

$$E_{0,q}^2 = E_{0,q}^k = H^0(S^k; \mathbb{Z}) \otimes \pi_q(\text{GL}_o B) \otimes \mathbb{P}$$

$$E_{-k,q}^2 = E_{-k,q}^k = H^k(S^k; \mathbb{Z}) \otimes \pi_q(\text{GL}_o B) \otimes \mathbb{P}$$

and the only higher possibly non-zero differential is

$$d^k : E_{0,q}^k \to E_{-k,q+k-1}^k.$$ 

Thus $E^{k+1} = E^\infty$.

Theorem 3.2. Suppose that $\zeta \otimes B : P \times_G B \to S^k$ is a bundle of Banach algebras with $k \geq 2$. Then there is a second quadrant spectral sequence converging to

$$K_{p+1}(A_\zeta \otimes B) \otimes \mathbb{P}$$

with $E^2 = 0$ except for

$$E_{0,q}^2 = E_{0,q}^k = H^0(S^k; \mathbb{Z}) \otimes K_{q+1}(B) \otimes \mathbb{P}$$

$$E_{-k,q}^2 = E_{-k,q}^k = H^k(S^k; \mathbb{Z}) \otimes K_{q+1}(B) \otimes \mathbb{P}$$

and the only possibly non-zero higher differential is

$$d^k : E_{0,q}^k \to E_{-k,q+k-1}^k.$$ 

Thus $E^{k+1} = E^\infty$. 
4. Deriving the Wang sequence

We may rephrase the conclusion of Theorem 3.1 as asserting the existence of a long exact sequence

\[ 0 \to E_{0,q}^\infty \to E_{0,q}^k \xrightarrow{d_k} E_{-k,q+k-1}^\infty \to E_{-k,q+k-1}^\infty \to 0 \]

and after identifications we obtain the exact sequence

\[(\ast) \quad 0 \to E_{0,q}^\infty \to \pi_q(GL_\infty A) \otimes \mathbb{P} \xrightarrow{d_k} \pi_{q+k-1}(GL_\infty B) \otimes \mathbb{P} \to E_{-k,q+r-1}^\infty \to 0.\]

On the other hand, the filtration that creates the spectral sequence comes from the cell filtration of \(X = S^k\) and hence simplifies dramatically to become

\[(\ast\ast) \quad 0 \to E_{-k,n+k}^\infty \to \pi_n(GL_\infty A_\otimes B) \otimes \mathbb{P} \to E_{0,n}^\infty \to 0 \]

with

\[E_{-k,n+k}^\infty \cong H^k(S^k) \otimes \pi_{n+k}(GL_\infty B) \otimes \mathbb{P}\]

and

\[E_{0,n}^\infty \cong H^0(S^k) \otimes \pi_n(GL_\infty B) \otimes \mathbb{P}.\]

Splice the two sequences \(\{\ast\}\) and \(\{\ast\ast\}\) together as follows. Splicing the sequences at \(E_{0,n}^\infty\) gives the composite

\[r : \pi_n(GL_\infty A_\otimes B) \otimes \mathbb{P} \to E_{0,n}^\infty \to E_{0,n}^k \cong \pi_n(GL_\infty B) \otimes \mathbb{P}.\]

It is easy to see that the map \(r\) is induced by the evaluation map

\[r : GL_\infty A_\otimes B \to GL_\infty B\]

that takes a section and restricts it to the basepoint \(x_0 \in S^k\).

Splicing the sequence at \(E_{-k,n+k-1}^\infty\) gives the composite

\[s : \pi_{n+k-1}(GL_\infty B) \otimes \mathbb{P} \cong E_{-k,n+k-1}^\infty \to E_{-k,n+k-1}^\infty \to \pi_{n-1}(GL_\infty A_\otimes B) \otimes \mathbb{P}\]

where the map \(s\) corresponds to the inclusion of a pointed section into the space of all sections.

We obtain the following generalization of the Wang sequence [13].

**Theorem 4.1.** Suppose that \(X = S^k\) with \(k \geq 2\) and that

\[\zeta \otimes B : \quad P \times_G B \to S^k\]

is a bundle of Banach algebras. Let \(A_{\zeta \otimes B}\) denote the associated section algebra. Then there is a long exact sequence

\[\cdots \to \pi_n(GL_\infty A_{\zeta \otimes B}) \otimes \mathbb{P} \xrightarrow{r} \pi_n(GL_\infty B) \otimes \mathbb{P} \xrightarrow{d_k} \pi_{n+k-1}(GL_\infty B) \otimes \mathbb{P} \xrightarrow{s} \pi_{n-1}(GL_\infty A_{\zeta \otimes B}) \otimes \mathbb{P} \to \cdots\]

Passing to limits, we also obtain the analogous sequence at the level of \(K\)-theory:

**Theorem 4.2.** Suppose that \(X = S^k\) with \(k \geq 2\) and that

\[\zeta \otimes B : \quad P \times_G B \to S^k\]

is a bundle of Banach algebras. Let \(A_{\zeta \otimes B}\) denote the associated section algebra. Then there is a long exact sequence

\[\cdots \to K_n(A_{\zeta \otimes B}) \otimes \mathbb{P} \xrightarrow{r} K_n(B) \otimes \mathbb{P} \xrightarrow{d_k} K_{n+k-1}(B) \otimes \mathbb{P} \xrightarrow{s} K_{n-1}(A_{\zeta \otimes B}) \otimes \mathbb{P} \to \cdots\]

Remark 4.3. This result agrees with the result of Rosenberg [19] on continuous trace algebras over $S^3$. He shows there that if $d_3$ is an isomorphism then $K_*(A_{\zeta \otimes K}) = 0$. If $d_3 = 0$ then $K_0(A_{\zeta \otimes K}) = \mathbb{Z}$ and $K_1(A_{\zeta \otimes K}) = 0$. If $d_3$ is multiplication by $s \neq 0, \pm 1$ then $K_0(A_{\zeta \otimes K}) = 0$ and $K_1(A_{\zeta \otimes K}) = \mathbb{Z}/s$.

To illustrate the use of the Wang sequence for those not so familiar with spectral sequence arguments, we calculate a simple example.

**Theorem 4.4.** Suppose that $\zeta \otimes B : P \times_G B \to S^k$ is a bundle of $C^*$-algebras. Assume that $k$ is even and that $K_1(B) = 0$. Then there is a short exact sequence

$$0 \to K_k(B) \xrightarrow{r} K_0(A_{\zeta \otimes B}) \xrightarrow{r} K_0(B) \to 0.$$ 

and $K_1(A_{\zeta \otimes B}) = 0$.

**Proof.** The differential in the Wang sequence is a map $d_k : K_n(B) \to K_{n+k-1}(B)$ and since $k$ is even it changes parity. The fact that $K_1(B) = 0$ then implies that $d_k$ is always the zero map. Thus the long exact Wang sequence degenerates as shown. \hfill $\square$

**Remark 4.5.** We believe that Theorem 4.4 is a hint at a non-commutative Thom isomorphism theorem, generalizing the classical Thom isomorphism theorem that for a complex vector bundle $V \to X$ relates $K^*(X)$ with the $K$-theory of the Thom space of the bundle. The map $r$ corresponds to the zero section of the bundle, and so it is not unreasonable to define

$$\tilde{K}_*(A_{\zeta \otimes B}) \cong \text{Ker} [K_*(A_{\zeta \otimes B}) \xrightarrow{r} K_*(B)]$$

so that there is a Thom-type isomorphism

$$K_{k+s}(B) \cong \tilde{K}_s(A_{\zeta \otimes B})$$

induced by the map $s$. Note that the map $s$ is locally in effect the product with the $K$-theory fundamental class of the even-dimensional sphere $S^k$ so this has the right flavor as well.

5. Identifying the Differential

In order to identify the unknown differential in these theorems, we must look at the exact couple that gives rise to the spectral sequence as constructed in [8] §4. Suppose that $X$ is a finite complex. The space of invertible sections of the bundle

$$P \times_G \text{GL}_\infty B \to X$$

is filtered up to homotopy by a descending filtration

$$\cdots \mathcal{F}_{p+1}X \hookrightarrow \mathcal{F}_pX \cdots$$

(See [8] for details.) The resulting exact couple is given by

$$D^1_{-p,q} \cong \pi_{q-p}(\mathcal{F}_pX) \otimes \mathbb{P}$$

and

$$E^1_{-p,q} = \pi_{q-p}(\mathcal{F}_pX/(\mathcal{F}_{p+1}X)) \otimes \mathbb{P}.$$ 

The structural maps are given as follows:
(1) The map $i^1: D^1_{p,q} \to D^1_{p+1,q-1}$ is given by the natural map induced by the filtration:

$$
\begin{array}{ccc}
D^1_{p,q} & \xrightarrow{i^1} & D^1_{p+1,q-1} \\
\downarrow \cong & & \downarrow \cong \\
\pi_{q-p}(F_pX) & \longrightarrow & \pi_{q-p}(F_{p-1}X)
\end{array}
$$

(2) The map $j^1: D^1_{p,q} \to E^1_{p,q}$ is given by

$$
\begin{array}{ccc}
D^1_{p,q} & \xrightarrow{j^1} & E^1_{p,q} \\
\downarrow \cong & & \downarrow \cong \\
\pi_{q-p}(F_pX) & \longrightarrow & \pi_{q-p}((F_pX)/(F_{p+1}X)) \otimes \mathbb{P}
\end{array}
$$

(3) The map $\delta^1: E^1_{p,q} \to D^1_{p-1,q}$ is given by

$$
\begin{array}{ccc}
E^1_{p,q} & \xrightarrow{\delta^1} & D^1_{p-1,q} \\
\downarrow \cong & & \downarrow \cong \\
\pi_{q-p}((F_pX)/(F_{p+1}X)) \otimes \mathbb{P} & \longrightarrow & \pi_{q-p-1}((F_{p+1}X)) \otimes \mathbb{P}
\end{array}
$$

with differential $d^1 = j^1 \delta^1: E^1_{p,q} \longrightarrow E^1_{p-1,q}$.

We may identify the $E^1$ term by noting that

$$(F_pX)/(F_{p+1}X) \cong \bigvee \alpha S^p$$

a wedge of spheres, and hence

$$E^1_{p,q} \cong \pi_{q-p}(\Gamma(\bigvee \alpha S^p, (\text{GL}_\alpha A_{\mathbb{C} \otimes B})|_{S^p})) \otimes \mathbb{P} \cong \pi_{q-p}(F_* (\bigvee \alpha S^p, \text{GL}_\alpha B)) \otimes \mathbb{P} \cong \cong \bigoplus \alpha \pi_{q-p}(\Omega^p \text{GL}_\alpha B) \otimes \mathbb{P} \cong \bigoplus \pi_{q}(\text{GL}_\alpha B) \otimes \mathbb{P} \cong C^p(X; \pi_{q}(\text{GL}_\alpha B) \otimes \mathbb{P})$$

so that

$$E^1_{p,q} \cong C^p(X; \pi_{q}(\text{GL}_\alpha B) \otimes \mathbb{P}),$$

the cellular cochains of $X$ with coefficients in $\pi_{q}(\text{GL}_\alpha B) \otimes \mathbb{P}$. The $d^1$ differential is the usual cellular differential and so

$$E^2_{p,q} \cong \tilde{H}^p(X; \pi_{q}(\text{GL}_\alpha B) \otimes \mathbb{P}).$$

However, when $X = S^k$ then the matter becomes a lot simpler- the $E^2$ term vanishes except for $p = 0, k$. The differentials $d^s$ vanish for $s < k$. Internally, the derived exact couples have the property that the maps

$$D^s_{-u,v} \xrightarrow{\iota^1} D^s_{-u+1,v-1} \xrightarrow{\iota^2} \cdots \longrightarrow D^s_{-1,v-u+1}$$

are isomorphisms for $s \leq k$. Thus we may identify the $d^k$ differential as the composite

$$E^k_{0,q} \xrightarrow{\delta^k} D^k_{-1,q} \xrightarrow{\cong} \cdots \xrightarrow{\cong} D^k_{-k,q+k-1} \xrightarrow{j^k} E^k_{-k,q+k-1}$$

We summarize:
Proposition 5.1. In the case of Theorem 4.1 where \( X = S^k \), the \( d^k \) differential is given as the composite

\[
\delta^k_{0,q} : E^k_{0,q} \rightarrow E^k_{0,q} \leftarrow E^k_{0,q} \rightarrow E^k_{-k,q+k-1} \rightarrow E^k_{-k,q+k-1}
\]

where

\[
E^k_{0,q} = E^k_{0,q} = H^0(S^k; \mathbb{Z}) \otimes \pi_q(\text{GL}_o B) \otimes \mathbb{P}
\]

\[
E^k_{-k,q+k-1} = E^k_{-k,q+k-1} = H^k(S^k; \mathbb{Z}) \otimes \pi_{q+k-1}(\text{GL}_o B) \otimes \mathbb{P}.
\]

So what is this map? Its identification requires a detour. We must generalize the classical Samelson product.

6. Enhanced Samelson Products

Let \( G \) be a topological group and let \( Y_1 \) and \( Y_2 \) be topological spaces with distinguished basepoint. We take the identity as the basepoint for topological groups. Let \([\cdot,\cdot]\) denote based homotopy classes of maps. The traditional Samelson product (cf. [20], [29] p. 467, [17] §6.3) is a pairing

\[
[Y_1, G] \times [Y_2, G] \to [Y_1 \wedge Y_2, G]
\]

defined by

\[
[[\phi], [\psi]] = [\eta] \quad \text{where} \quad \eta(w \wedge y) = \phi(w)\psi(y)\phi(w)^{-1}\psi(y)^{-1}.
\]

If \( Y_1 = S^r \) and \( Y_2 = S^s \) this induces a pairing

\[
[Y_1, G] \times [Y_2, GL B] \to [Y_1 \wedge Y_2, GL B]
\]

defined by

\[
[[\phi], [\psi]] = [\eta] \quad \text{where} \quad \eta(w \wedge y) = \phi(w)(\psi(y))\psi(y)^{-1}.
\]

Taking \( Y_1 = S^r \) and \( Y_2 = S^s \) gives an enhanced Samelson product

\[
[Y_1, G] \times [Y_2, GL B] \to [Y_1 \wedge Y_2, GL B]
\]

by

\[
[[\phi], [\psi]] = [\eta] \quad \text{where} \quad \eta(w \wedge y) = \alpha_{\phi(w)}(\psi(y))\psi(y)^{-1}.
\]

If \( B \) is a \( C^* \)-algebra then the same formula induces an enhanced Samelson product

\[
[Y_1, G] \times [Y_2, U_o B] \to [Y_1 \wedge Y_2, U_o B]
\]

It is elementary to show that a morphism of \( G \)-algebra \( B \to B' \) induces a commuting diagram

\[
\pi_r(G) \times \pi_s(\text{GL} B) \xrightarrow{\cdot \cdot} \pi_{r+s}(\text{GL} B)
\]

\[
\downarrow \quad \downarrow
\]

\[
\pi_r(G) \times \pi_s(\text{GL} B') \xrightarrow{\cdot \cdot} \pi_{r+s}(\text{GL} B')
\]

and hence induces a pairing

\[
[Y_1, G] \xrightarrow{\cdot \cdot} [K_s(B) \to K_{s+r}(B)].
\]
Remark 6.2. If the action of $G$ on $B$ is inner then we write the action of $G$ on $GL\ B$ as 
\[(g, b) \rightarrow gb^{-1}.\]
Then
\[\alpha_{\phi(w)}(\psi(y)) = \phi(w)\psi(y)\phi(w)^{-1}\]
and hence
\[\llbracket \phi, \psi \rrbracket = [\eta]\] where
\[\eta(w \wedge y) = \phi(w)\psi(y)\phi(w)^{-1}\psi(y)^{-1}\]
which is the traditional Samelson formula. So the enhanced Samelson product is a true generalization of the classical Samelson product.

Remark 6.3. If $B$ is a $C^*$-algebra and $G$ is locally compact then $(B, G)$ form what G. Pedersen \cite{Pedersen} calls a $C^*$-dynamical system. Pedersen shows \cite[p. 257]{Pedersen} that there is a Hilbert space $\mathcal{H}$ upon which $G$ acts by the regular representation $\lambda$ and a faithful covariant representation $\rho$ of $(B, G)$ of the dynamical system. Thus up to isomorphism we may replace $(B, G)$ by $(\rho(B), G)$. Then
\[\rho(\alpha_g(b)) = \lambda_g\rho(b)\lambda_g^{-1}.\]
So in this case too the enhanced Samelson product reduces down to a commutator of the form $U_gbU_g^{-1}b^{-1}$ as well, even though $U_g$ is not in $GL\ B$.

7. Identifying the Differential More Precisely

We introduce some notation in order to analyze the situation over spheres. Let $G$ be a topological group and let $\zeta : P \rightarrow S^k$ denote a principal $G$-bundle. It is classified by its clutching map $\kappa : S^{k-1} \rightarrow G$ which we realize explicitly as follows.

Let $D^n$ denote the $n$-ball, regarded as the cone on $S^{n-1} :$
\[D^n = [0, 1] \times S^{n-1}/\{0\} \times S^{n-1}\]
and we write $(t, x) \rightarrow tx$. We decompose the base space $S^k$ as the disjoint union $S^k = H^+ \cup H^-$ of upper and lower closed hemispheres, with equator $S^{k-1} = H^+ \cap H^-$. The restriction of the principal bundle $\zeta : P \rightarrow S^k$ to each closed hemisphere trivializes, and so there are sections
\[\sigma^\pm : H^\pm \rightarrow H^\pm \times G \quad \sigma^\pm(x) = (x, s^\pm(x))\]
and a clutching map $\kappa : S^{k-1} \rightarrow G$ satisfying
\[s^+(x) = \kappa(x)s^-(x)\kappa(x)^{-1} \quad \forall x \in S^{k-1}\]
that determine the principal bundle $\zeta$ up to equivalence. The triviality of $\zeta$ over each hemisphere implies that the associated bundle
\[P \times_G GL_0 \rightarrow S^k\]
is also trivial over each hemisphere. Thus we may alternately describe $GL_0 \ (A_{\zeta \otimes B})$ as
\[GL_0 \ A_{\zeta \otimes B} = \{(s^+, s^-) : H^+ \cup H^- \rightarrow G : s^+(x) = s^-(x)\kappa(x)^{-1} \quad \forall x \in S^{k-1}\}\]
where we identify $GL_0 \ (A_{\zeta \otimes B}) \cong GL_0 \ (A_{\zeta \otimes B})$ by
\[s \rightarrow (s\sigma^+, s\sigma^-).\]
Taking the south pole $x_o$ as basepoint of $S^k$, the evaluation map
\[r : \Gamma(S^k, P \times_G GL_0 \rightarrow GL_0 \ B) \rightarrow GL_0 \ B\]
is given in this picture by \( r(s^+, s^-) = s^-(x_0) \).

Define the set of based maps
\[
F_\bullet(S^k, GL_o B) = \{ s \in F(S^k, GL_o B) : s(x_0) = e \}
\]
and similarly for based spaces of sections \( GL_o \bullet \) and \( U_o \bullet \).

The following proposition would seem to be folklore.

**Proposition 7.1.** There is a natural identification
\[
\pi_n(GL_o \bullet A_\zeta \otimes B) \cong \pi_n(F_\bullet(S^k, GL_o B)) \cong \pi_{n+k}(GL_o B).
\]

**Proof.** Let
\[
GL_o''A_\zeta \otimes B = \{ (s^+, s^-) \in GL_o' A_\zeta \otimes B : s^-(x) = e \ \forall x \in H^- \}.
\]

It is an exercise (cf. Wockel [30] Lemma 4.1.6) to show that the natural inclusion
\[
GL_o''A_\zeta \otimes B \rightarrow GL_o' A_\zeta \otimes B
\]
is a homotopy equivalence. But then it is easy to see that
\[
GL_o''A_\zeta \otimes B = \{ s^+ : H^+ \rightarrow GL_o B : s^+(x) = e \ \forall x \in \partial(H^+) \} \cong F_\bullet(S^k, GL_o B)
\]
so the proposition is immediate. \( \square \)

The first two parts of the following result are due to Thomsen ([26], Theorem 1.9). The third part generalizes Wockel [31], Theorem 2.3 and we have adapted his proof as well.

**Theorem 7.2.** Let \( \zeta : P \rightarrow S^k \) be a principal \( G \)-bundle with clutching map \( \kappa : S^{k-1} \rightarrow G \). Let \( B \) be a \( C^* \)-algebra upon which \( G \) acts and let
\[
A_\zeta \otimes B = \Gamma(S^k, P \times_G B)
\]
denote the associated \( C^* \)-algebra. Then:

1. There is an associated exact sequence of topological groups
\[
(\phi) \quad U_o \bullet A_\zeta \otimes B \rightarrow U_o A_\zeta \otimes B \xrightarrow{r} U_o B.
\]
2. The evaluation map \( r \) admits continuous local sections, and so \( (\phi) \) is a fibre bundle over a paracompact space, hence a fibration.
3. Let \( \partial \) denote the boundary homomorphism in the long exact homotopy sequence associated to the evaluation fibration and let \( \delta_n \) denote the composition
\[
\pi_n(U_o B) \xrightarrow{\partial} \pi_{n-1}(U_o \bullet A_\zeta \otimes B) \cong \pi_{n-1}(F_\bullet(S^k, U_o B)) \cong \pi_{n+k-1}(GL_o B).
\]

Then \( \delta_n \) is given by
\[
\delta_n(a) = -\alpha_{\kappa,a}^{-1} = -[\kappa,a]
\]
where \([\kappa,a]\) is the enhanced Samelson product.

**Proof.** We roughly sketch Thomsen’s proof of (1) and (2). We are in the classical situation with a topological group, a closed subgroup, and a quotient group. Steenrod’s Bundle Structure Theorem ([24], p. 30) shows that the existence of continuous local sections is necessary and sufficient for \( \phi \) to be a fibre bundle. Thomsen produces these sections. The base space \( U_o B \) is metric, hence paracompact, and it is a standard fact (cf. Spanier [23], Theorem 7.14, page 96) that a fibre bundle over a paracompact space is a fibration. This establishes (1) and (2).
Our proof of (3) follows Wockel [31, Theorem 2.3] in spirit. Represent \( a \) by 
\[
\begin{align*}
& a : [0, 1] \times S^{n-1} \to U_o B \\
& \text{with } a \text{ trivial on } \{0, 1\} \times S^{n-1}. \text{ Without loss of generality we may also assume that}
& \text{it is trivial on } [0, 1] \times \{x_o\}. \text{ Define maps } A^\pm \text{ as follows:}
\end{align*}
\]
\[
\begin{align*}
& A^+ : D^n \times H^+ \to U_o B \\
& A^+ (d, tx) = \alpha_{n(x)}(a(t(d))) \\
& A^- : D^n \times H^- \to U_o B \\
& A^- (d, y) = a(d).
\end{align*}
\]
If \( t = 1 \) then 
\[
A^+ (d, x) = \alpha_{n(x)}(a(d)) \quad \text{and} \quad A^- (d, y) = a(d)
\]
as desired, and so the maps patch together to form (after taking adjoints) a map 
\[
A : D^n \to U_o A_{\xi \otimes B}
\]
with the property that 
\[
ev(A(d)) = A^- (d, 0) = a(d).
\]
Now collapse all of \( H^- \) to the south pole, the basepoint of \( S^k \). The result is another copy of \( S^k \), of course and by definition \( \delta_n \) is given by 
\[
\delta_n (a) = \left[ A^+ \right]_{\partial D^n \times D^k} \in \left[ \partial D^n \times S^k, U_o B \right].
\]
Define \( \tilde{A} : D^n \times D^k \to \text{GL}_o B \) by 
\[
\tilde{A}(d, x) = A^+ (d, x) a(d)^{-1}
\]
Then:
\[
\begin{align*}
& (1) \quad \tilde{A} = \ast \text{ on } \partial D^n \times \partial D^k. \\
& (2) \quad \tilde{A} = \ast \text{ on } D^n \lor D^k. \\
& (3) \quad \tilde{A} = A \text{ on } \partial D^n \times D^k \text{ because } a \text{ is trivial there.} \\
& (4) \quad \tilde{A}(d, x) = [\kappa, a] \text{ on } D^n \times \partial D^k \text{ since } t = 1 \text{ there.}
\end{align*}
\]
Thus 
\[
\delta_n (a) = \left[ A^+ \right]_{\partial D^n \times D^k} = \left[ \tilde{A} \right]_{\partial D^n \times D^k} = - \left[ \tilde{A} \right]_{D^n \times \partial D^k} = -[\kappa, a].
\]
To complete the proof we note that the entire spectral sequence is natural under localization.

Remark 7.3. If the analog of Theorem 7.2 holds for Banach algebras then the main theorems of this paper would similarly generalize. The obstacle would seem to be to generalize Thomsen’s Theorem 1.9 of [26] to this context; we believe that this is possible.

8. Proofs of the Main Theorems

We have already done the heavy lifting for Theorem A. Here is the conclusion:

Proof of Theorem A
Proof. By naturality we may restrict to the case $\mathbb{P} = \mathbb{Z}$. Theorem \[Theorem1\] gives us a long exact sequence

$$
\cdots \to \pi_n(U_0 A_{\xi \otimes B}) \xrightarrow{r} \pi_n(U_0 B) \xrightarrow{d_k} \pi_{n+k-1}(U_0 B) \xrightarrow{s} \pi_{n-1}(U_0 A_{\xi \otimes B}) \to \cdots
$$

and so it suffices to show that $d_k(1 \otimes a) = -g[\kappa, a]$. Proposition \[Proposition5\] identifies the differential $d_k$ as the composite

$$
E_{0,q}^2 \xrightarrow{\delta_k^k} D_{-1,q}^k \xleftarrow{\sim} \cdots \xleftarrow{\sim} D_{-k,q+k-1}^k \xrightarrow{j_k} E_{-k,q+k-1}^k
$$

where

$$
E_{0,q}^2 = E_{0,q}^k \cong H^0(S^k; \mathbb{Z}) \otimes \pi_q(\text{GL}_k \mathbb{B})
$$

$$
E_{-k,q+k-1}^2 = E_{-k,q+k-1}^k \cong H^k(S^k; \mathbb{Z}) \otimes \pi_{q+k-1}(U_0 B).
$$

Since $D_{-k,q+k-1}^k = D_{-k,q+k-1}^1$ in all intermediate parts of the filtration we can see that the differential is really induced by the connecting homomorphism $\delta_n$ in the filtration associated with the fiberation

$$
U_0 \cdot A_{\xi \otimes B} \to U_0 A_{\xi \otimes B} \xrightarrow{r} U_0 B.
$$

Theorem \[Theorem7\] identifies this connecting homomorphism as an enhanced Samelson product $\delta_n(a) = -[\kappa, a]$ and from there the proof is immediate.

In order to prove Theorem \[Theorem12\] we need some more information about the special case $k = 3$ and we turn our attention to that case now.

Let $k = 3$, so that we focus on principal bundles over $S^3$. Let $B = \mathcal{K}$, the compact operators on the standard Hilbert space $\mathcal{H}$. In this case the (contractible) unitary group $\mathcal{U} = \mathcal{U}(\mathcal{H})$ acts on $\mathcal{K}$ by conjugation, its center $S^1$ acts trivially, of course, and hence the action descends to an action of the projective unitary group $\mathcal{PU}$ on $\mathcal{K}$. Note that $\mathcal{PU} \simeq BS^1 \simeq K(\mathbb{Z}, 2)$.

Let

$$
\mathcal{UK} = \{ u \in \mathcal{U}(\mathcal{H}) : u - 1 \in \mathcal{K}\},
$$

the unitary group of $\mathcal{K}$. Recall that $\Omega^2 \mathcal{UK} \simeq \mathcal{UK}$ is one way of stating Bott periodicity.

Suppose that $\xi : P \to S^3$ is a principal $\mathcal{PU}$-bundle. This bundle is classified by a clutching map $\kappa : S^2 \to \mathcal{PU}$ as per the notation above, and the homotopy class of $\kappa$ in $[S^2, \mathcal{PU}] \cong H^2(S^2; \mathbb{Z})$ has the form $\Delta_\xi g$ (where $g$ is a generic term for canonical generator) and so determines an integer $\Delta_\xi$ which is essentially the Dixmier-Douady class of the principal bundle.

Let $A_{\xi \otimes \mathcal{K}}$ denote the associated $C^*$-algebra. Then a consequence of the Theorem is that $\pi_n(UA_{\xi \otimes \mathcal{K}})$ is determined up to group extension by the differential

$$
\delta_3 : \pi_n(\mathcal{UK}) \to \pi_{n+2}(\mathcal{UK}).
$$

These groups are zero for $n$ even and $\mathbb{Z}$ for $n$ odd by Bott periodicity. If we see this in terms of $E_2$ it corresponds to

$$
d_3 : H^0(S^3) \otimes \pi_n(\mathcal{UK}) \to H^3(S^3) \otimes \pi_{n+2}(\mathcal{UK})
$$

and we have proved that

$$
d_3(1 \otimes a) = -g[\kappa, a].\]
**Proposition 8.1.** With the notation above, for $n$ odd,

$$[\kappa, a] = \Delta_\zeta \hat{\beta}(a)$$

where $\hat{\beta}(a)$ is the composite

$$S^{n+2} \simeq S^2 \wedge S^n \xrightarrow{1\wedge a} S^2 \wedge UK \xrightarrow{\hat{\beta}} UK$$

and $\hat{\beta} : S^2 \wedge UK \rightarrow UK$ is the adjoint of the Bott periodicity identification $\beta : UK \simeq \Omega^2 UK$.

**Proof.** Regard $\kappa : S^2 \rightarrow PU \simeq BS^1$ as a line bundle $L \rightarrow S^2$ with first Chern class $c_1(L) = \Delta_\zeta g$. Regard $a : S^n \rightarrow UK$ as the clutching map of a $K$-bundle $F \rightarrow S^{n+1}$. Then the bundle $L \otimes F \rightarrow S^2 \wedge S^{n+1} \cong S^{n+3}$ is represented by the clutching map

$$\kappa(x)a(y)\kappa(x)^{-1} : S^2 \wedge S^n \rightarrow UK.$$

Then we appeal to the argument of Proposition 2.1 of Atiyah-Segal [3]. They note that $L \otimes F$ is a sub-bundle of the trivial bundle $\mathcal{H} \otimes F$ and hence

$$\kappa(x)a(y)\kappa(x)^{-1}a(y)^{-1} = \Delta_\zeta (1 \wedge a(y)^{-1}).$$

Using notation introduced previously, we rewrite this as

$$[\kappa, a] = \Delta_\zeta \hat{\beta}(a)$$

and this proves the proposition.

□

**Proof of Theorem [B]**

**Proof.** There is a natural isomorphism (or definition, depending upon how you set up $K_*$ )

$$K_{n+1}(A) \cong \pi_n(U(A \otimes \mathcal{K})) \cong \lim_{j} \pi_n(U(A \otimes M_j)). \quad (n \geq 0)$$

Fix some topological group $G$, a principal $G$-bundle $P \rightarrow S^k$ and let $G$ act on a Banach algebra $B$ as usual. Then $G$ acts on $B \otimes M_j$ by acting trivially on the second factor and the (non-unital) inclusions $M_j \rightarrow M_{j+1}$ that insert the matrix ring into the top left of the next matrix ring induce $G$-equivariant maps $B \otimes M_j \rightarrow B \otimes M_{j+1}$. For brevity in this proof we write $B_j = B \otimes M_j$. This induces a sequence of morphisms

$$A_\zeta \otimes B \rightarrow A_\zeta \otimes B_2 \rightarrow A_\zeta \otimes B_3 \rightarrow \ldots A_\zeta \otimes B_j \rightarrow \ldots$$
Each of these algebras has an associated Wang sequence $[A]$ and the naturality of the spectral sequence that gave them birth yields a commuting diagram

\[
\begin{array}{cccccccc}
\pi_n(U_0 A_{\zeta \otimes B}) & \rightarrow & \pi_n(U_0 B) & \rightarrow & \pi_{n+k-1}(U_0 B) & \rightarrow & \pi_{n-1}(U_0 A_{\zeta \otimes B}) & \rightarrow \\
\rightarrow & & \rightarrow & & \rightarrow & & \rightarrow & \\
\pi_n(U_0 A_{\zeta \otimes B_2}) & \rightarrow & \pi_n(U_0 B_2) & \rightarrow & \pi_{n+k-1}(U_0 B_2) & \rightarrow & \pi_{n-1}(U_0 A_{\zeta \otimes B_2}) & \rightarrow \\
\rightarrow & & \rightarrow & & \rightarrow & & \rightarrow & \\
\pi_n(U_0 A_{\zeta \otimes B_3}) & \rightarrow & \pi_n(U_0 B_3) & \rightarrow & \pi_{n+k-1}(U_0 B_3) & \rightarrow & \pi_{n-1}(U_0 A_{\zeta \otimes B_3}) & \rightarrow \\
\rightarrow & & \rightarrow & & \rightarrow & & \rightarrow & \\
\pi_n(U_0 A_{\zeta \otimes B_j}) & \rightarrow & \pi_n(U_0 B_j) & \rightarrow & \pi_{n+k-1}(U_0 B_j) & \rightarrow & \pi_{n-1}(U_0 A_{\zeta \otimes B_j}) & \rightarrow \\
\rightarrow & & \rightarrow & & \rightarrow & & \rightarrow & \\
\end{array}
\]

Taking direct limits over $j$ preserves exactness, and thus there is a long exact sequence

\[\cdots \rightarrow K_n(A_{\zeta \otimes B}) \xrightarrow{\partial} K_n(B) \xrightarrow{d_k} K_{n+k-1}(B) \xrightarrow{\sigma} K_{n-1}(A_{\zeta \otimes B}) \rightarrow \cdots\]

If $B$ is a $C^*$-algebra then the differential $d_k$ in the homotopy sequences is given by

\[d_k(1 \otimes a) = -g \otimes [\kappa, a]\]

where $g$ is the generator of $H^k(S^k; \mathbb{Z})$ and $[\kappa, a]$ is the enhanced Samelson product with the class $\kappa : S^{k-1} \rightarrow G$ that classifies the principal bundle and this passes to direct limits as well. Similarly we may apply $(-) \otimes \mathbb{P}$ to obtain the full result with coefficients.

The only remaining issue is the explicit identification of the differential in the case $k = 3$ and that is done in the previous proposition. \qed

9. An Example

In this section we illustrate our result in a very concrete case. Take the principal bundle to be the Hopf bundle

\[\zeta : S^7 \rightarrow S^4\]

obtained from the multiplicative structure of the quaternions, with group $G = S^3 = SU_2$. Take $B = M_2$ (the $2 \times 2$ complex matrices) with $S^3$ acting upon $M_2$ by conjugation. Then there is an associated bundle of $C^*$-algebras

\[\zeta \otimes M_2 : S^7 \times S^3 M_2 \rightarrow S^4\]

and as usual we denote by $A_{\zeta \otimes M_2}$ the associated $C^*$-algebra of continuous sections. We want to compute $\pi_n(UA_{\zeta \otimes M_2})$. The Wang sequence then takes the form

\[\pi_n(U_0 A_{\zeta \otimes M_2}) \rightarrow \pi_n(U_2) \xrightarrow{d_1} \pi_{n+3}(U_2) \rightarrow \pi_{n-1}(UA_{\zeta \otimes M_2}) \rightarrow \]

where $U_n = U(M_n)$. Recall that $U_2 \cong S^1 \times S^3$ as topological spaces, though not as groups. Serre’s classical results on homotopy imply that $\pi_n(U_2)$ is a finite group.
for each \( n > 3 \) and that these are groups are non-zero for infinitely many values of \( n \).

We record for reference the first twelve homotopy groups of \( U_2 \). Let \( \eta \in \pi_3(S^2) \) denote the Hopf generator and by abuse of notation its various suspensions, so for instance we write \( \eta^2 \in \pi_5(S^3) \) for the composition
\[
S^5 \xrightarrow{\eta^2} S^4 \xrightarrow{\eta} S^3.
\]

We use \( a_n \) as labels for classes when there don’t seem to be standard names; the subscript denotes the homotopy group.

- \( \pi_1(U_2) \cong \pi_1(S^1) \cong \mathbb{Z} \) on the class of the upper left corner inclusion \( S^1 \to U_2 \) which we denote \( a_1 \). In all higher degrees the natural inclusion \( S^3 \cong SU_2 \to U_2 \) induces an isomorphism in homotopy.
- \( \pi_2(U_2) = 0 \).
- \( \pi_3(U_2) \cong \mathbb{Z} \). The generator is given by the natural inclusion.
- \( \iota : S^3 \cong SU_2 \to U_2 \).
- \( \pi_4(U_2) \cong \mathbb{Z}/2 \) on the class \( \eta \).
- \( \pi_5(U_2) \cong \mathbb{Z}/2 \) on the class \( \eta^2 \).
- \( \pi_6(U_2) \cong \mathbb{Z}/12 \) on the class \( a_6 \). The 2-primary part is generated by a class \( \nu' \) (in Toda’s [27] notation).
- \( \pi_7(U_2) \cong \mathbb{Z}/2 \) on the class \( \nu' \eta \).
- \( \pi_8(U_2) \cong \mathbb{Z}/2 \) on the class \( \nu' \eta^2 \).
- \( \pi_9(U_2) \cong \mathbb{Z}/3 \) on the class \( a_9 \). (J. C. Moore [16], Theorem 5.3.)
- \( \pi_{10}(U_2) \cong \mathbb{Z}/15 \) on the class \( a_{10} \). (J. C. Moore [16], Theorem 5.3 and Lemma 5.1.)
- \( \pi_{11}(U_2) \cong \mathbb{Z}/2 \) (Toda [27] Theorem 7.2).
- \( \pi_{12}(U_2) \cong (\mathbb{Z}/2)^2 \) (Toda [27] Theorem 7.2).

Recall that we have shown that the differential \( d_4 \) is given by
\[
d_4(a) = -g[\kappa, a]
\]
where \( \kappa \) is the clutching map of the principal bundle. In this example, we have:

**Proposition 9.1.** The clutching map of the Hopf bundle \( S^7 \to S^4 \) is the identity map \( \iota : S^3 \to S^3 \).

We are indebted to John Klein for the following proof of this fact.

**Proof.** This fact is a general characteristic of the Hopf construction, in the case \( G = S^3 \) with its standard multiplication. If \( G \) is a topological group with multiplication \( G \times G \to G \), one has a Hopf construction
\[
G \ast G \to SG
\]
where ⋆ denotes join and SG is the unreduced suspension of G. This is a fibre bundle with fibre at the basepoint G. The bundle projection is given by
\[ t g + (1 - t) h \rightarrow t (g h) \quad t \in [0,1], \; g, h \in G. \]
The clutching map in this case is given by the map \[ G \rightarrow \text{homeo}(G) \] which is adjoint to left multiplication. This factors through the identity map of G considered as acting by left multiplication on itself which shows that the fibration has a reduction of structure group to G and has clutching map \( \iota : G \rightarrow G. \)

In light of the Proposition, we see that in our example the differential is given by
\[ d_4(a) = -g[a, a] \]
where \( \iota : S^3 \rightarrow S^3 \) is the identity map. So we must calculate the Samelson product
\[ [\iota, -] : \pi_n(U_2) \rightarrow \pi_{n+3}(U_2). \]
Here is the result. Note that each entry that is non-zero corresponds to a non-zero \( d_4 \) differential.

- \( n = 1 \) : \( [\iota, a_1] = 0 \)
- \( n = 3 \) : \( [\iota, \iota] = a_6 \) by the result of I. M. James [11], p. 176.
- \( n = 4 \) : \( [\iota, \eta] = \nu' \eta \) since (working 2-primary)
  \[ [\iota, \eta] = [\iota, \iota] \circ \eta = \nu' \eta. \]
- \( n = 5 \) : \( [\iota, \eta^2] = \nu' \eta^2 \) by the same argument.
- \( n = 6 \) : \( [\iota, a_6] = [\iota, [\iota, \iota]] = a_9 \) by I. M. James [10], §3.
- \( n = 7 \) : \( [\iota, \nu' \eta] = [\iota, \nu'] \circ \eta = a_9 \circ \eta = 0 \) (since \( a_9 \) has order 3 and \( \eta \) has order 2.)
- \( n = 8 \) : \( [\iota, \nu' \eta^2] = 0 \) by same argument.
- \( n = 9 \) : \( [\iota, a_9] = 0 \) by nilpotency.

Feeding this information into the Wang long exact sequence produces the following results:

**Theorem 9.2.** Let
\[ \zeta : S^7 \rightarrow S^4 \]
denote the Hopf bundle. Form the associated bundle of C*-algebras
\[ S^7 \times_{S^3} M_2 \rightarrow S^4 \]
and let \( A_{\zeta \otimes M_2} \) denote the associated C*-algebra of continuous sections. Then \( \pi_* (U_* A_{\zeta \otimes M_2}) \) is given as follows:
- \( \pi_1(U_0 A_{\zeta \otimes M_2}) \) fits into a split short exact sequence
  \[ 0 \rightarrow \pi_5(U_2) \rightarrow \pi_1(U_0 A_{\zeta \otimes M_2}) \rightarrow \pi_1(U_2) \rightarrow 0 \]
  with \( \pi_5(U_2) \cong \mathbb{Z}/2 \) and \( \pi_1(U_2) \cong \mathbb{Z} \) and so \( \pi_1(U_0 A_{\zeta \otimes M_2}) \cong \mathbb{Z} \oplus \mathbb{Z}/2. \)
- \( \pi_2(U_0 A_{\zeta \otimes M_2}) = 0. \)
• \( \pi_3(U_0 A_{\zeta \otimes M_2}) \) fits in a short exact sequence

\[
0 \to \pi_3(U_0 A_{\zeta \otimes M_2}) \to \pi_3(U_2) \to \pi_6(U_2) \to 0
\]

with \( \pi_3(U_2) \cong \mathbb{Z} \) and \( \pi_6(U_2) \cong \mathbb{Z}/12 \), and so \( \pi_3(U_0 A_{\zeta \otimes M_2}) \cong \mathbb{Z} \).

• \( \pi_4(U_0 A_{\zeta \otimes M_2}) = 0 \).

• \( \pi_5(U_0 A_{\zeta \otimes M_2}) = 0 \).

• \( \pi_6(U_0 A_{\zeta \otimes M_2}) \) fits in a short exact sequence

\[
0 \to \pi_{10}(U_2) \to \pi_6(U_0 A_{\zeta \otimes M_2}) \to \mathbb{Z}/4 \to 0
\]

with \( \pi_{10}(U_2) \cong \mathbb{Z}/15 \) and \( \mathbb{Z}/4 \) the 2-primary component of \( \pi_6(U_2) \), so that \( \pi_6(U_0 A_{\zeta \otimes M_2}) \cong \mathbb{Z}/60 \).

• \( \pi_7(U_0 A_{\zeta \otimes M_2}) \) fits in a short exact sequence

\[
0 \to \pi_{11}(U_2) \to \pi_7(U_0 A_{\zeta \otimes M_2}) \to \pi_7(U_2) \to 0
\]

with \( \pi_{11}(U_2) \cong \mathbb{Z}/2 \) and \( \pi_7(U_2) \cong \mathbb{Z}/2 \). So \( \pi_7(U_0 A_{\zeta \otimes M_2}) \) is either \( \mathbb{Z}/4 \) or \( (\mathbb{Z}/2)^2 \).

• \( \pi_8(U_0 A_{\zeta \otimes M_2}) \) fits in a short exact sequence

\[
0 \to \pi_{12}(U_2) \to \pi_8(U_0 A_{\zeta \otimes M_2}) \to \pi_8(U_2) \to 0
\]

with \( \pi_{12}(U_2) \cong (\mathbb{Z}/2)^2 \) and \( \pi_8(U_2) \cong \mathbb{Z}/2 \). So \( \pi_8(U_0 A_{\zeta \otimes M_2}) \) is either \( \mathbb{Z}/4 \oplus \mathbb{Z}/2 \) or \( (\mathbb{Z}/2)^3 \).

We contrast this with the analogous computation in rational homotopy and in \( K \)-theory.

**Theorem 9.3.** With the notation above,

1. The rational homotopy groups of \( U_0 A_{\zeta \otimes M_2} \) are zero except for

\[
\pi_j(U_0 A_{\zeta \otimes M_2}) \otimes \mathbb{Q} \cong \mathbb{Q} \quad j = 1 \text{ and } 3.
\]

2. The (matrix) stable homotopy groups are zero in even degrees and

\[
\pi_j(U_0 (A_{\zeta \otimes M_2} \otimes K)) \cong \mathbb{Z} \quad j \text{ odd}
\]

3. The \( K \)-theory groups are given by \( K_0(A_{\zeta \otimes M_2}) \cong \mathbb{Z} \) and \( K_1(A_{\zeta \otimes M_2}) = 0 \).

**Proof.** In the rational homotopy case the only non-zero homotopy groups of \( U_2 \) are

\[
\pi_1(U_2) \otimes \mathbb{Q} \cong \pi_3(U_2) \otimes \mathbb{Q} \cong \mathbb{Q}.
\]

Since the Wang differential changes degree by three, the differential must be identically zero, the spectral sequence collapses, and the long exact Wang sequence turns into many short exact sequences. As \( \pi_j(U_2) = 0 \) except for \( j = 1, 3 \), the result is as stated.

In the stable case the situation is similar, since stably \( \pi_4(U_0 (M_2 \otimes K)) \cong \mathbb{Z} \) for \( * \) odd and zero for \( * \) even by Bott periodicity. Again, the Wang differential is identically zero and the result follows. Part (3) follows from (2) essentially by definition. \( \square \)
The following table summarizes the calculations in this section.

| n  | $\pi_n(U_\circ A_{\zeta \otimes M_2})$ | $\pi_n(U_\circ A_{\zeta \otimes M_2} \otimes \mathbb{Q})$ | $K_{n+1}(A_{\zeta \otimes M_2})$ |
|----|----------------------------------|------------------------------------------------|---------------------------------|
| 1  | $\mathbb{Z} \oplus \mathbb{Z}/2$ | $\mathbb{Q}$                                    | $\mathbb{Z}$                    |
| 2  | 0                                | 0                                              | 0                               |
| 3  | $\mathbb{Z}$                     | $\mathbb{Q}$                                    | $\mathbb{Z}$                    |
| 4  | 0                                | 0                                              | 0                               |
| 5  | 0                                | 0                                              | $\mathbb{Z}$                    |
| 6  | $\mathbb{Z}/60$                  | 0                                              | 0                               |
| 7  | $\mathbb{Z}/4 \text{ or } (\mathbb{Z}/2)^2$ | 0                                              | $\mathbb{Z}$                    |
| 8  | $\mathbb{Z}/4 \oplus \mathbb{Z}/2 \text{ or } (\mathbb{Z}/2)^3$ | 0                                              | 0                               |

We note several features of the table:

1. The right column is periodic, reflecting Bott periodicity.
2. The center column shows the beginning of periodicity - entries in degrees 1 and 3 will persist to the direct limit. However in degree 5 the rational homotopy vanishes, reflecting the fact that the matrix ring is not large enough to pick up the classes that will eventually generate the stable periodic elements.
3. The calculations suggest that perhaps $\pi_n(U_\circ A_{\zeta \otimes M_2} \otimes \mathbb{Q}) = 0$ for $n > 3$. In fact this is true, by [13]. This implies that $\pi_n(U_\circ A_{\zeta \otimes M_2})$ is finite for each $n > 3$.
4. As $\pi_n(U_2) \neq 0$ for infinitely many values of $n$, we would suppose that the same is true for $\pi_n(U_\circ A_{\zeta \otimes M_2})$.

We regard this example as an excellent illustration of what is lost by focusing attention only upon $K_\ast(A_{\zeta \otimes B})$. The richness of detail that is evident while studying the individual homotopy groups is completely lost upon matrix stabilization and passage to $K$-theory.

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