Global existence and Hadamard differentiability of hysteresis-reaction-diffusion systems

Christian Münch*

September 18, 2018

Abstract

We consider a class of semilinear parabolic evolution equations subject to a hysteresis operator and a Bochner-Lebesgue integrable source term. The underlying spatial domain is allowed to have a very general boundary. In the first part of the paper, we apply semigroup theory to prove well-posedness and boundedness of the solution operator. Rate independence in reaction-diffusion systems complicates the analysis, since the reaction term acts no longer local in time. This demands careful estimates when working with semigroup methods. In the second part, we show Lipschitz continuity and Hadamard differentiability of the solution operator. We use fixed point arguments to derive a representation for the derivative in terms of the evolution system. Finally, we apply our results to an optimal control problem in which the source term acts as a control function and show existence of an optimal solution.

Keywords: Hysteresis operator, stop operator, global existence, semilinear parabolic evolution problem, solution operator, Hadamard differentiability, reaction-diffusion.

MSC subject class: 47J40, 35K51

1 Introduction

In this paper we analyze semilinear parabolic evolution equations of the form

\[
\frac{d}{dt} y(t) + (A_p y)(t) = (F[y])(t) + u(t) \quad \text{in } X \text{ for } t > 0, \\
y(0) = 0 \in X.
\]

In this context \( X \) is a product of dual spaces and \( A_p \) is an unbounded operator on \( X \).

The non-linearity \( F \) is a Nemytski operator, i.e. \( (F[y])(t) = f(y(t), W[Sy](t)) \). \( S \) is a linear operator which transforms the vector valued function \( y \) into a scalar valued map.

*Department of Mathematics - M6, Technical University of Munich, Boltzmannstr. 3, 85747 Garching, Germany. christian.muench@ma.tum.de
\( \mathcal{W} \) is a scalar stop operator. One way to represent the value of \( z = \mathcal{W}[v] \) is as the unique solution of the variational inequality

\[
(\dot{z}(t) - \dot{v}(t))(z(t) - \xi) \leq 0 \quad \text{for } \xi \in [a, b] \text{ and } t \in (0, T),
\]

\[
z(t) \in [a, b] \text{ for } t \in [0, T],
\]

\[
z(0) = z_0
\]

[4]. The forcing term \( u \in L^q(J_T; X) \) may for example serve as a control. Our choice for the notation in equation (1) is motivated by the application of our results to optimal control theory.

The major focus of this paper are well-posedness of (1) and Hadamard directional differentiability of the solution operator \( G \) which maps each \( u \) to the corresponding solution \( y \) of (1).

General semilinear parabolic problems with Lipschitz continuous non-linearities \( f(t, y(t)) \) and with a forcing term \( u(t) \) which is Bochner-Lebesgue integrable have, for instance, been analyzed in [10]. Differentiability of the solution mapping is discussed in [11]. Abstract evolution equations with (locally) Lipschitz continuous right-hand sides \( f(t, y(t)) \) and without an additional forcing term are for instance treated in [9, 12] and [10]. In these cases, the non-linearity \( f \) is local in time.

The main novelty of this paper comes from the hysteresis \( \mathcal{W} \), which is non-local in time. This adds a new challenge to the question of well-posedness since \( \mathcal{W}[Sy](t) \) depends not only on \( t \) but on the whole time history of \( y \) in \([0, t]\). Furthermore, \( \mathcal{W} \) is non-smooth so that differentiability of the solution operator to (1) is not clear at all. Because we can not expect Fréchet differentiability [5], we turn to the concept of Hadamard directional differentiability.

This work is organized as follows.

In Section 2 we collect results from the literature and state the main assumption. We do not consider product spaces of \( L^p(\Omega) \)-functions for \( X \) because we include very general domains \( \Omega \). The right side of equation (1) therefore takes its values only in a product of dual spaces. It is not easy to find a fully elaborated description of the functional setup for our problem. We do our best to provide a precise framework which includes all the required results.

In Section 3 we show well-posedness of equation (1) with \( u \in L^q((0, T); X) \). Theorem 3.1 is the first main result of this work.

After defining Hadamard directional differentiability, Section 4 contains a proof that the solution operator for (1) in \( u \) has this property. Theorem 4.7 is our second main result.

In Section 5 we apply Theorem 3.1 and Theorem 4.7 to an optimal control problem where the state equation takes the form of (1). Existence of an optimal control is shown in Theorem 5.4.

The results from Section 3 and Section 4 are also valid if \( A_p \) is replaced by a more general sectorial operator \( T_p \) which does not necessarily have to satisfy maximal parabolic Sobolev regularity. In this case \( y \) is a continuous function with values in a fractional power space. Equation (1) has to be interpreted in the sense of mild solutions then. The scalar stop operator \( \mathcal{W} \) can be replaced by a general hysteresis operator with appropriate properties, cf. Remark 4.5. In this paper, we focus on the operators \( A_p \) and \( \mathcal{W} \) in order to give an illustration right away.
We write $\mathcal{L}(X,Y)$ for the space of linear operators between spaces $X$ and $Y$ and $\mathcal{L}(X)$ for the space of linear operators on $X$. We also abbreviate the duality in $X$ by 

$$\langle x, y \rangle_{X^*,X} = \langle x, y \rangle_X.$$ 

## 2 Preliminaries and assumptions

### 2.1 Sobolev spaces including homogeneous Dirichlet boundary conditions

The setting and the theory of this section is strongly based on results from [8]. We recall several definitions, results and assumptions from this work. All Sobolev spaces are defined on a bounded domain $\Omega \subset \mathbb{R}^d$ with $d \geq 2$. The boundary regularity is defined in Assumption 2.2. 

We only consider real valued functions. For each component $j \in \{1, \ldots, m\}$ of the space of vector valued functions, see Definition 2.4, the boundary $\partial \Omega$ is decomposed into the corresponding Dirichlet part $\Gamma_{D_j}$ and the Neumann boundary $\Gamma_{N_j} := \partial \Omega \backslash \Gamma_{D_j}$, see Assumption 2.2. The cases $\Gamma_{D_j} = \emptyset$ and $\Gamma_{D_j} = \partial \Omega$ are not excluded [8, Comment after Definition 2.4] and [2, Remark 2.2 (iii)]. The assumed condition on $\Gamma_{D_j}$ requires the definition of an $I$-set where $I \in (0, d]$ [8, Definition 2.1].

**Definition 2.1.** For $0 < I \leq d$ and a closed set $M \subset \mathbb{R}^d$ let $\rho$ denote the restriction of the $I$-dimensional Hausdorff measure $H_I$ to $M$. Then we call $M$ an $I$-set if there are constants $c_1, c_2 > 0$ such that 

$$c_1 r^I \leq \rho(B_{\mathbb{R}^d}(x, r) \cap M) \leq c_2 r^I$$

for all $x$ in $M$ and $r \in [0, 1[$.

The assumption on the domain in our setting is the following [8, Assumption 2.3]:

**Assumption 2.2.** The domain $\Omega \subset \mathbb{R}^d$ is bounded and $\overline{\Omega}$ is a $d$-set. For $j \in \{1, \ldots, m\}$ the Neumann boundary part $\Gamma_{N_j} \subset \partial \Omega$ is open and $\Gamma_{D_j} = \partial \Omega \backslash \Gamma_{N_j}$ is a $(d-1)$-set.

**Remak 2.3.** As already mentioned in the beginning of this section, note that the cases $\Gamma_{D_j} = \emptyset$ and $\Gamma_{D_j} = \partial \Omega$ are not excluded [8, Comment after Definition 2.4] and [2, Remark 2.2 (iii)]. Assumption 2.2 allows for very general domains. For example, $\Omega$ may be a Lipschitz domain and for $j \in \{1, \ldots, m\}$, $\Gamma_{D_j}$ can be a $(d-1)$-dimensional manifold.

In the same manner as in [8, Definition 2.4] we define Sobolev spaces which include the Dirichlet boundary conditions for our state equation.

**Definition 2.4.** Let $U \subset \mathbb{R}^d$ be a domain and $p \in [1, \infty)$.

- $W^{1,p}(U)$ denotes the usual Sobolev space of functions $\psi \in L^p(U)$ whose weak partial derivatives exist in $L^p(U)$. The norm in $W^{1,p}(U)$ is 

$$\|\psi\|_{W^{1,p}(U)} = \left( \int_U \left( |\psi|^2 + \sum_{j=1}^d |\frac{\partial \psi}{\partial x_j}|^2 \right)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}}.$$
• For a closed subset $M$ of $\Omega$ we define

$$C_M^\infty(U) := \{ \psi|_U : \psi \in C_0^\infty(\mathbb{R}^d), \supp(\psi) \cap M = \emptyset \}$$

and denote by $W_{M}^{1,p}(U)$ the closure of $C_M^\infty(U)$ in $W^{1,p}(U)$.

• For $p > 1$ we write $p'$ for the Hölder conjugate of $p$.

The dual space $[W_{M}^{1,p'}(U)]^*$ of $W_{M}^{1,p'}(U)$ is called $W_{M}^{-1,p}(U)$.

**Remark 2.5.** We stick to the norm which is used in [8] which differs from the usual norm in Sobolev spaces. One reason for this choice is that it simplifies estimates concerning the duality between $W_{M}^{1,p}(U)$ and $W_{M}^{1,p'}(U)$. We may identify a function $\phi \in W_{M}^{1,p}(U)$ with an element in $W_{M}^{-1,p}(U)$ since for any $\psi \in W_{M}^{1,p'}(U)$ the Cauchy Schwarz inequality together with Hölder’s inequality yields

$$\int_U \left( \phi \psi + \sum_{j=1}^d \frac{\partial \phi}{\partial x_j} \frac{\partial \psi}{\partial x_j} \right) \, dx \leq \int_U \left( |\phi|^2 + \sum_{j=1}^d \left| \frac{\partial \phi}{\partial x_j} \right|^2 \right)^{\frac{p'}{2}} \left( |\psi|^2 + \sum_{j=1}^d \left| \frac{\partial \psi}{\partial x_j} \right|^2 \right)^{\frac{p}{2}} \, dx$$

$$\leq \left( \int_U \left( |\phi|^2 + \sum_{j=1}^d \left| \frac{\partial \phi}{\partial x_j} \right|^2 \right)^{\frac{p'}{2}} \, dx \right)^{\frac{1}{p}} \left( \int_U \left( |\psi|^2 + \sum_{j=1}^d \left| \frac{\partial \psi}{\partial x_j} \right|^2 \right)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p'}}.$$

We need the following assumption for each of the $m$ components [8, Assumption 4.11]:

**Assumption 2.6.** In the setting of Assumption 2.2 we suppose for all $j \in \{1, \cdots, m\}$ and any $x \in \Gamma_{N_j}$ that there is an open neighborhood $U_x$ of $x$ and a bi-Lipschitz mapping $\phi_x$ from $U_x$ onto a cube in $\mathbb{R}^d$ such that $\phi_x(\Omega \cap U_x)$ equals the lower half of the cube and such that $\partial \Omega \cap U_x$ is mapped onto the top surface of the lower half cube.

**Remark 2.7.** Assumption 2.6 has the following consequences:

1. Firstly, Assumption 2.6 is needed in order to assure the existence of continuous extension operators from $W_{\Gamma_{N_j}}^{1,p}(\Omega)$ to $W_{\Gamma_{N_j}}^{1,p}(\mathbb{R}^d)$ for all $j \in \{1, \cdots, m\}$ and $p \in (1, \infty)$. This in turn is required in [8, Section 3] to establish interpolation properties between the spaces $\{W_{\Gamma_{N_j}}^{1,p}(\Omega)\}_{p \in (1, \infty)}$ for fixed $j \in \{1, \cdots, m\}$. Secondly, the assumption is used in [8, Section 5] to prove elliptic and parabolic regularity results, see Theorem 2.10 below.

2. Under Assumption 2.6 it can be shown that the embeddings $W_{\Gamma_{N_j}}^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ are compact for $q \in [1, \frac{dp}{d-p})$ if $p \in (1, d)$ and for arbitrary $q \in [1, \infty)$ if $p \geq d$ [8, Remark 3.2]. The proof is almost equal to the proofs of [7, Part II, 5.6.1, Theorem 2] and [7, Part II, 5.7, Theorem 1].
2.2 Operators and their properties

In this subsection, we define the required Sobolev spaces of vector valued functions and introduce the operators $A_p$. Our notation differs from the one in [8]. This is done in order to provide a structured framework for the construction of $A_p$ and to highlight the spaces on which each particular operator acts. Results from the literature assure that $A_p$ satisfies the properties which we need for the analysis of (1) for particular values of $p$ to be chosen.

We begin with two definitions [8, Section 6]:

**Definition 2.8.** With Assumption 2.2 and Assumption 2.6 and $p \in [1, \infty)$ we define a Sobolev space of vector valued functions by the product space

$$\mathcal{W}^{1,p}_{\Gamma_D}(\Omega) := \prod_{j=1}^m \mathcal{W}^{1,p}_{\Gamma_D,j}(\Omega).$$

For $p \in (1, \infty)$ we denote its (componentwise) dual by $\mathcal{W}^{-1,p'}_{\Gamma_D}(\Omega)$.

We also define the operators

$$\mathcal{L}_p : \mathcal{W}^{1,p}_{\Gamma_D}(\Omega) \to L^p(\Omega, \mathbb{R}^{md}), \quad \mathcal{L}_p(u) := \text{vec}(\nabla u) = (\nabla u_1, \ldots, \nabla u_m)^T$$

and

$$I_p : \mathcal{W}^{1,p}_{\Gamma_D}(\Omega) \to \mathcal{W}^{-1,p'}_{\Gamma_D}(\Omega), \quad \langle I_p u, v \rangle_{\mathcal{W}^{-1,p'}_{\Gamma_D}(\Omega)} := \int_{\Omega} u \cdot v \, dx \quad \forall v \in \mathcal{W}^{1,p}_{\Gamma_D}(\Omega).$$

Now we can define the operators $A_p$ and state the associate properties:

**Definition 2.9.** Let the constants $d_1, \ldots, d_m > 0$ be given diffusion coefficients and

$$D = \text{diag}(d_1, \ldots, d_1, \ldots, d_m, \ldots, d_m) \in \mathbb{R}^{md \times md}.$$ 

For $p \in (1, \infty)$ we set

$$A_p : \mathcal{W}^{1,p}_{\Gamma_D}(\Omega) \to \mathcal{W}^{-1,p}_{\Gamma_D}(\Omega), \quad A_p := \mathcal{L}_p^* D \mathcal{L}_p.$$ 

We define the unbounded operator

$$A_p : \mathcal{W}^{-1,p}_{\Gamma_D}(\Omega) \to \mathcal{W}^{-1,p}_{\Gamma_D}(\Omega), \quad A_p := A_p I_p^{-1}$$

with domain

$$\text{dom}(A_p) = \text{ran}(I_p) \subset \mathcal{W}^{-1,p}_{\Gamma_D}(\Omega),$$

where $\text{ran}(I_p)$ stands for the range of $I_p$.

The following result is shown in [8, Theorem 5.6 and Theorem 5.12]:

**Theorem 2.10.** In the setting of Definition 2.8 and Definition 2.9 there exists an open interval $J$ around 2 such that for all $p \in J$ the operator $A_p + I_p$ is a topological isomorphism between $\mathcal{W}^{1,p}_{\Gamma_D}(\Omega)$ and $\mathcal{W}^{-1,p}_{\Gamma_D}(\Omega)$.

There is a constant $c > 0$ such that for all $p \in J$ and $\lambda \in \mathbb{C}_+ := \{ z \in \mathbb{C} : \text{Re} z \geq 0 \}$ the resolvent estimate

$$\|(A_p + 1 + \lambda)^{-1}\|_{\mathcal{L}(\mathcal{W}^{-1,p}_{\Gamma_D}(\Omega))} \leq \frac{c}{1 + |\lambda|}$$

holds true and $-A_p$ generates an analytic semigroup of operators on $\mathcal{W}^{-1,p}_{\Gamma_D}(\Omega)$. 

Remark 2.11. Let \( p \in J \) with \( J \) from Theorem 2.10. We equip \( \text{dom}(A_p) \) with the graph norm
\[
\|y\|_{\text{dom}(A_p)} = \|y\|_{\mathcal{W}^{-1,p}_D(\Omega)} + \|A_p y\|_{\mathcal{W}^{-1,p}_D(\Omega)}.
\]
Then \( A_p \) is densely defined and closed and \( \text{dom}(A_p) \) is topologically equivalent to \( \mathcal{W}^{-1,p}_D(\Omega) \). Remark 2.7 (ii) therefore implies that \( \text{dom}(A_p) \) is compactly embedded into \( \mathcal{W}^{-1,p}_D(\Omega) \).

Furthermore, for \( \theta \geq 0 \) the fractional power spaces \( X^\theta := \text{dom}([A_p + 1]^\theta) \subset \mathcal{W}^{-1,p}_D(\Omega) \) and the unbounded operators \([A_p + 1]^\theta\) are well-defined [9, Chapter 1]. Note that \( X^0 = \mathcal{W}^{-1,p}_D(\Omega) \). In \( X^\theta \) we use the norm
\[
\|y\|_{X^\theta} = \|(A_p + 1)^\theta y\|_{\mathcal{W}^{-1,p}_D(\Omega)}.
\]

Also for \( z \in \{ \zeta \in \mathbb{C} : \text{Re}(\zeta) > 0 \} \) one can define the fractional powers \([A_p + 1]^z\) by the inverse of the operators \([A_p + 1]^{-z}\) [14, Chapter 7]. For \( \theta \in \mathbb{R} \) and suitable \( y \in \mathcal{W}^{-1,p}_D(\Omega) \) one can further define \([A_p + 1]^{i\theta} y\) by the limit of \([A_p + 1]^{z} y\) for \( z \to i\theta \) with \( \text{Re}(z) > 0 \). This leads to the notion of bounded purely imaginary powers of an operator [14, Chapter 8].

We will not need the theory of purely imaginary powers in the rest of this paper. However, we will use the fact that \( A_p + 1 \) has bounded purely imaginary powers for \( p \in J \cap [2, \infty) \) in order apply an existing result, which allows us to represent the spaces \( X^\theta \) by complex interpolation spaces for \( \theta \in (0, 1) \), see Remark 2.13 below.

We introduce the notion of maximal parabolic regularity [11, Definition 2.7] or [2, Definition 11.2]. This property allows us to improve the regularity of the mild solution \( y \) of our evolution equation.

Definition 2.12. For \( p, q \in (1, \infty) \) and \((t_0, T) \subset \mathbb{R}\), we say that \( A_p \) satisfies maximal parabolic \( L^q((t_0, T); \mathcal{W}^{-1,p}_D(\Omega)) \)-regularity if for all \( g \in L^q((t_0, T); \mathcal{W}^{-1,p}_D(\Omega)) \) there is a unique solution \( y \in W^{1,q}((t_0, T); \mathcal{W}^{-1,p}_D(\Omega)) \cap L^q((t_0, T); \text{dom}(A_p)) \) of the equation
\[
\frac{d}{dt} y + A_p y = g, \quad y(t_0) = 0.
\]

The time derivative is taken in the sense of distributions [2, Definition 11.2]. We abbreviate
\[
Y_q := W^{1,q}((0, T); \mathcal{W}^{-1,p}_D(\Omega)) \cap L^q((0, T); \text{dom}(A_p)) \quad \text{and} \quad Y_{q,t} := \{ y \in Y_q : y(t) = 0 \} \text{ for } t \in [0, T].
\]

Remark 2.13. The following properties go along with maximal parabolic regularity:

1. Maximal parabolic regularity is independent of \( q \in (1, \infty) \) and of the interval \((t_0, T)\) so that we just say that \( A_p \) satisfies maximal parabolic regularity on \( \mathcal{W}^{-1,p}_D(\Omega) \) [2, Remark 11.3].

2. If \( A_p \) satisfies maximal parabolic regularity on \( \mathcal{W}^{-1,p}_D(\Omega) \) then
\[
\left( \frac{d}{dt} + A_p \right)^{-1} \text{ is bounded as an operator from } L^q((0, T); \mathcal{W}^{-1,p}_D(\Omega)) \text{ to } Y_{q,0} \quad [11, \text{Proof of Proposition 2.8}].
3. If \( p \in J \cap [2, \infty) \) with \( J \) from Theorem 2.10 then by [2, Theorem 11.5], \( A_p + 1 \) has bounded imaginary powers and satisfies maximal parabolic Sobolev regularity on \( W_{\Gamma_D}^{-1,p}(\Omega) \), see also Remark 2.11. This yields that for \( p \in J \cap [2, \infty) \) also \( A_p \) satisfies maximal parabolic Sobolev regularity on \( W_{\Gamma_D}^{-1,p}(\Omega) \) and with [6, Theorem 11.6.1] we conclude that we have the topological equivalences

\[
[W_{\Gamma_D}^{-1,p}(\Omega), W_{\Gamma_D}^{1,p}(\Omega)]_{\theta} \simeq [W_{\Gamma_D}^{-1,p}(\Omega), \text{dom}(A_p)]_{\theta} \simeq X^\theta
\]

for \( \theta \in (0, 1) \). By \([\cdot, \cdot]_{\theta}\) we mean complex interpolation.

The following embedding properties will be used several times [1, Theorem 3]:

**Remark 2.14.** Let \( p \in J \) with \( J \) from Theorem 2.10. With \( q \in (1, \infty) \) one has

\[
Y_q \hookrightarrow C^\beta((0,T); (W_{\Gamma_D}^{-1,p}(\Omega), \text{dom}(A_p))_{\eta,1}) \hookrightarrow C^\beta((0,T); [W_{\Gamma_D}^{-1,p}(\Omega), \text{dom}(A_p)]_{\theta}) \hookrightarrow \text{C}([0,T]; (W_{\Gamma_D}^{-1,p}(\Omega), \text{dom}(A_p))_{\eta,q}) \hookrightarrow \text{C}([0,T]; [W_{\Gamma_D}^{-1,p}(\Omega), \text{dom}(A_p)]_{\theta})
\]

for every \( 0 < \theta < \eta < 1 - 1/q \) and \( 0 \leq \beta < 1 - 1/q - \eta \). \((\cdot, \cdot)_{\eta,1}\) or \((\cdot, \cdot)_{\eta,q}\) respectively means real interpolation here. Compactness of the first embeddings follows because \( \text{dom}(A_p) \) is compactly embedded into \( W_{\Gamma_D}^{-1,p}(\Omega) \), see Remark 2.11.

**Remark 2.15.** For \( p \in J \) with \( J \) from Theorem 2.10 we collect several estimates for the operator \((A_p + 1)^\theta\) and the analytic semigroup \( \exp(-A_p t) \): For \( t > 0 \) and arbitrary \( 0 < \gamma < 1 \) it is shown in [9, Theorem 1.3.4] that for some \( C > 0 \) one can estimate

\[
\|\exp(-A_p t)\|_{L(W_{\Gamma_D}^{-1,p}(\Omega))} \leq C \exp((1 - \gamma) t) \quad \text{and} \quad \|(A_p + 1)^\theta \exp(-A_p t)\|_{L(W_{\Gamma_D}^{-1,p}(\Omega))} \leq C t^\theta \exp((1 - \gamma) t).
\]

Moreover for each \( \theta \geq 0 \), according to [9, Theorem 1.4.3], there is some \( C_\theta \in (0, \infty) \) such that

\[
\|(A_p + 1)^\theta \exp(-A_p t)\|_{L(W_{\Gamma_D}^{-1,p}(\Omega))} \leq C_\theta t^{-\theta} \exp((1 - \gamma) t). \tag{5}
\]

The constants \( C_\theta \) are bounded if \( \theta \) is contained in any compact subinterval of \((0, \infty)\) and also for \( \theta \downarrow 0 \).

**2.3 Main assumption and notation**

We collect several assumptions and introduce some short notation for the spaces and functions.

**Assumption 2.16.** We always suppose that Assumption 2.2 and Assumption 2.6 hold. Moreover we assume:

- \( d \geq 2 \) and with \( J \) from Theorem 2.10 there holds \( p \in J \cap [2, \infty) \) and \( 2 \geq p \left( 1 - \frac{1}{d} \right) \).
• For some \( w \in \mathcal{W}^{1,p'}_{\Gamma_D}(\Omega) \simeq \mathcal{W}^{-1,p'}_{\Gamma_D}(\Omega)^* \) the operator \( S \in \mathcal{W}^{-1,p'}_{\Gamma_D}(\Omega)^* \) from equation (1) is given by
\[
S y = \langle y, w \rangle \mathcal{W}^{1,p'}_{\Gamma_D}(\Omega) \quad \forall y \in \mathcal{W}^{-1,p}_{\Gamma_D}(\Omega).
\]
Note that \( S \) belongs to \( [X^\theta]^* \) for all \( \theta \geq 0 \) because of the embedding \( X^\theta \hookrightarrow \mathcal{W}^{-1,p}_{\Gamma_D}(\Omega) \). We assume \( S \neq 0 \).

• We will need a fractional power space with exponent strictly smaller than one. This fact is highlighted by a new parameter \( \alpha \) instead of \( \theta \in [0, \infty) \) from above. Assume that for some \( \alpha \in (0, 1) \) the function \( f : X^\alpha \times \mathbb{R} \rightarrow \mathcal{W}^{-1,p}_{\Gamma_D}(\Omega) \) is locally Lipschitz continuous with respect to the \( X^\alpha \)-norm.
This means that for every \( y_0 \in X^\alpha \) there is a constant \( L(y_0) \) and a neighbourhood \( V(y_0) = \{ y \in X^\alpha : \| y - y_0 \|_{X^\alpha} \leq \delta \} \) of \( y_0 \) such that
\[
\| f(y_1, x_1) - f(y_2, x_2) \|_X \leq L(y_0) (\| y_1 - y_2 \|_\alpha + |x_1 - x_2|)
\]
for every \( y_1, y_2 \in V(y_0) \) and all \( x_1, x_2 \in \mathbb{R} \).
Moreover, \( f \) is assumed to have at most linear growth along solutions, i.e.
\[
\| f(y, x) \|_{\mathcal{W}^{-1,p}_{\Gamma_D}(\Omega)} \leq M (1 + \| y \|_\alpha + |x|)
\]
for some constant \( M > 0 \).

In the setting of Assumption 2.16 we collect the notation for the rest of the work:

• For the particular \( p \) from Assumption 2.16 we set
\[
X := \mathcal{W}^{-1,p}_{\Gamma_D}(\Omega)
\]
with \( \mathcal{W}^{-1,p}_{\Gamma_D}(\Omega) \) from Definition 2.9. We sometimes identify elements \( v \in X^* \) with their Riesz representation in \( \mathcal{W}^{1,p'}_{\Gamma_D}(\Omega) \), i.e.
\[
\langle v, y \rangle_X = \langle y, v \rangle \mathcal{W}^{1,p'}_{\Gamma_D}(\Omega) \quad \forall y \in X.
\]

• The operators \( A_p \) and the spaces \( X^\theta = \text{dom}([A_p+1]^\theta) \) are defined as in Definition 2.9 and Remark 2.11.

• The spaces \( Y_q \) and \( Y_{q,t} \) are defined as in Definition 2.12.

• \( \mathcal{W} \) is the scalar stop operator. This operator is represented by (2)-(4). Other representations can for example be found in [13, Chapter III.3].

• We abbreviate \( J_T = (0, T) \).
2.4 Regularity of the stop operator

The stop operator $\mathcal{W}$ which is represented by (2)-(4) is Lipschitz continuous as an operator on $C(\overline{J_T})$ according to [13] Part 1, Chapter III Lemma 2.1, Theorem 3.2 and Theorem 3.3 with

$$|\mathcal{W}[v_1](t) - \mathcal{W}[v_2](t)| \leq 2 \sup_{0 \leq \tau \leq t} |v_1(\tau) - v_2(\tau)| \quad \text{and} \quad \mathcal{W}[v](t) \leq 2 \sup_{0 \leq \tau \leq t} |v(\tau)| + z_0 \quad (6)$$

for all $v, v_1, v_2 \in C(\overline{J_T})$ and $t \in [0, T]$. We have to add $z_0$ in (6) because, by (1), $\mathcal{W}[v](0) = z_0$ for any $v \in C(\overline{J_T})$.

$\mathcal{W}$ is also bounded and weakly continuous on $W^{1,q}(J_T)$ for $q \in [1, \infty)$ [13] Part 1, Chapter III., Theorem 3.2.

3 Well-posedness of the evolution equation

We recap equation (11) from the introduction which is

$$\frac{d}{dt} y(t) + (A_p y)(t) = (F[y])(t) + u(t) \quad \text{in} \quad X = W^{-1,q}_T(\Omega) \quad \text{for} \quad t > 0,$$

$$y(0) = 0 \quad \text{in} \quad X,$$

where $(F[y])(t) := f(y(t), \mathcal{W}[Sy](t))$. We recall that $X$ is a product of dual spaces. In this section we show well-posedness of the problem. The first aim is to show that for every $u \in L^q(J_T; X)$ with $q \in (\frac{1}{1-\alpha}, \infty]$ problem (11) has a unique mild solution $y \in C(\overline{J_T}; X^\alpha)$, where $\alpha$ is fixed by Assumption 2.16. In particular, this means that $(F[y]) + u$ is contained in $L^1(J_T; X)$ and that $y$ solves the integral equation

$$y(t) = \int_0^t \exp(-A_p(t-s))[(F[y])(s) + u(s)] \, ds, \quad t \in J_T \quad (7)$$

[13] Definition 7.0.2]. Afterwards we prove that the unique mild solution even belongs to $Y_{s,0}$ where $s = q$ if $q < \infty$ and with $s \in (1, \infty)$ arbitrary if $q = \infty$.

**Theorem 3.1.** Let Assumption 2.16 hold.

Then for all $u \in L^q(J_T; X)$ with $q \in (\frac{1}{1-\alpha}, \infty]$ problem (11) has a unique mild solution $y = y(u)$ in $C(\overline{J_T}; X^\alpha)$. Note that $X^\alpha \subset X$ since $\alpha \in (0, 1)$.

The solution mapping

$$G : u \mapsto y(u), \quad L^q(J_T; X) \rightarrow C(\overline{J_T}; X^\alpha)$$

is locally Lipschitz continuous.

$G$ is linearly bounded with values in $C(\overline{J_T}; X^\alpha)$, i.e. for some $C = C(T) > 0$ there holds

$$\|G(u)\|_{C(\overline{J_T}; X^\alpha)} \leq C(T) (1 + \|u\|_{L^q(J_T; X)}) \quad (8)$$

for all $u \in L^q(J_T; X)$ and $C$ is independent of $u$. All statements remain valid if $C(\overline{J_T}; X^\alpha)$ is replaced by $Y_{s,0}$ where $s = q$ if $q < \infty$ and $s \in (1, \infty)$ arbitrary if $q = \infty$. 


Proof. We prove the theorem as in [10, Theorem 7.1.3] by a fixed point argument. Several estimates can be found in [11, Appendix A] in a similar form. We extend the results in [10] and [11] by allowing for non-linearities which are only locally Lipschitz continuous and not Lipschitz continuous on bounded sets. Moreover, non-locality of the hysteresis operator in time requires additional work in several steps. We prove the theorem directly for $u \in L^q(J_T; X)$ as it is done in [10, Theorem 7.1.3]. In [11, Appendix A] the corresponding statement is first shown for smooth right hand sides and afterwards extended by a density argument.

In the following, $c$ always denotes a generic constant which is adapted during the proof. Note that for $\beta > -1$ there holds
\[
\int_0^t (t-s)^\beta \, ds = \frac{t^{1+\beta}}{1+\beta}.
\] (9)

The proof is divided into five steps.

1. We show the existence of local solutions of problem (1).

Consider $v_u(t) := \int_0^t e^{-A_p(t-s)}u(s) \, ds$. $v_u$ belongs to $C(J_T; X^\alpha)$ for arbitrary $T > 0$. Moreover, since $q' < \alpha - 1$, we have by (5) and (9)
\[
\|v_u\|_{C(J_T; X^\alpha)} \leq \left( \int_0^T \| e^{-A_p(t-s)}\|_{L^q(X^\alpha, X^\alpha)} \, ds \right)^{1/q'} \|v_u\|_{L^q(J_T; X)} \leq ce^{(1-\gamma)\frac{T}{q'-\alpha}} \|v_u\|_{L^q(J_T; X)} < \infty.
\] (10)

Let $\delta > 0$ be small enough so that $f$ is Lipschitz continuous in $B_{X^\alpha}(0, \delta) \times \mathbb{R}$ with a constant $L(0) > 0$.

We apply Assumption 2.16 and (6) to estimate
\[
\|(F[y_1])(t) - (F[y_2])(t)\|_X \\
\leq L(0) \left( \|y_1(t) - y_2(t)\|_{X^\alpha} + 2\|S\|_{[X^\alpha], \infty} \sup_{0 \leq \tau \leq t} \|y_1(\tau) - y_2(\tau)\|_{X^\alpha} \right) \\
\leq c \sup_{0 \leq \tau \leq t} \|y_1(\tau) - y_2(\tau)\|_{X^\alpha}
\] (11)

for all $y_1, y_2 \in B_{C(J_T; X^\alpha)}(0, \delta)$ and $t \in J_T$.

The mapping
\[
\Phi_u(y)(t) := \int_0^t e^{-A_p(t-s)} \left[ f(y(s), W[Sy](s)) + u(s) \right] \, ds
\]
is well defined on $C(J_T; X^\alpha)$. This is shown as in [111 Appendix A (ii)].
For \( y_1, y_2 \in B_{C(J_T; X^\alpha)}(0, \delta) \) we have by (5), (9) and (11) that
\[
\|\Phi_u(y_1) - \Phi_u(y_2)\|_{C(J_T; X^\alpha)} \leq \int_0^T \|e^{-A_p(t-s)}\|_{L(X, X^\alpha)} \|F(y_1) - F(y_2)\|_{C(J_T; X)} \, ds
\leq c e^{(1-\gamma)T} T^{1-\alpha} \|y_1 - y_2\|_{C(J_T; X)} < \frac{1}{2} \|y_1 - y_2\|_{C(J_T; X)}
\]
for \( T \) small enough.

Consequently, in this case \( \Phi_u \) is a \( \frac{1}{2} \)-contraction.

Using this result together with (5) and (9) we obtain for \( y \in B_{C(J_T; X^\alpha)}(0, \delta) \)
\[
\|\Phi_u(y)(t)\|_{X^\alpha} \leq \|\Phi_u(y)(t) - \Phi_u(0)(t)\|_{X^\alpha} + \|\Phi_u(0)(t)\|_{X^\alpha}
\leq \frac{\delta}{2} + \left( \int_0^T \|e^{-A_p(t-s)}\|_{L(X, X^\alpha)}^{q'} \, ds \right)^{1/q'} \|F[0] + u\|_{L^q(J_T; X)}
\leq \frac{\delta}{2} + c e^{(1-\gamma)T} T^{1/q'-\alpha} \|f(0, z_0) + u\|_{L^q(J_T; X)} \leq \delta
\]
if \( T \) is small enough.

Because \( \Phi_u \) then maps \( B_{C(J_T; X^\alpha)}(0, \delta) \) into itself and since \( B_{C(J_T; X^\alpha)}(0, \delta) \) is a closed subset of \( C(J_T; X^\alpha) \), Banach’s fixed point theorem yields a unique fixed point \( y \) of \( \Phi_u \) in \( B_{C(J_T; X^\alpha)}(0, \delta) \).

This fixed point defines a (local) mild solution of problem (1) in \( J_T \) [10, Definition 7.0.2].

2. We show that global mild solutions for problem (11) exist and boundedness of the solution mapping \( G \).

This part requires some caution since the hysteresis operator is non-local in time.

Remember that the local mild solution \( y \) of (11) takes the form (7). With Assumption 2.15 and the second estimate in (6) we estimate
\[
|W[S y](t)| \leq 2 \|S\|_{L^{q'}} \sup_{0 \leq \tau \leq t} \|y(\tau)\|_{X^\alpha} + |z_0| \quad \text{for all } t \in J_T.
\]
Moreover, by (5) there holds
\[
\|((A_p + 1)^\alpha \exp(-A_p t))\|_{L(W_0^{1,p}(\Omega))} \leq C_0 t^{-\alpha} \exp((1 - \gamma) t).
\]

Equation (9) yields
\[
\left( \int_0^t (t-s)^{-\alpha q'} \, ds \right)^{1/q'} = \left( \frac{t^{1-\alpha q'}}{1-\alpha q'} \right)^{1/q'} = \frac{t^{1/q'-\alpha}}{(1-\alpha q')^{1/q'-\alpha}}.
\]

We combine those three observations to obtain a bound for the norm of \( y(t) \) for all \( t \in J_T \) in the form
\[
\|y(t)\|_{X^\alpha} \leq c e^{(1-\gamma)T} \left[ \int_0^t (t-s)^{-\alpha} \left( 1 + 3 \sup_{0 \leq \tau \leq s} \|y(\tau)\|_{X^\alpha} + |z_0| \right) \, ds + t^{1/q'-\alpha} \|u\|_{L^q(J_T; X)} \right]
\leq c_0(T) \int_0^t (t-s)^{-\alpha} \sup_{0 \leq \tau \leq s} \|y(\tau)\|_{X^\alpha} \, ds + c_1(T) [1 + \|u\|_{L^q(J_T; X)}], \tag{12}
\]
where $c_0(T), c_1(T) > 0$ are constants which depend on $T$ (and on $q'$ and the fixed value $\alpha$). As in the proof of [12, Theorem 6.3.3] note that if the solution of $(1)$ exists on $[0, T]$ it can be continued as long as $\|y(t)\|_{X^\alpha}$ remains bounded with $t \uparrow T$.

Clearly this is the case if
\[
\sup_{0 \leq \tau < T} \|y(\tau)\|_{X^\alpha} \leq C(T)
\]
for some $C(T) > 0$.

It is not hard to show that the function $t \mapsto \sup_{0 \leq \tau < t} \|y(\tau)\|_{X^\alpha}$ is continuous on $[0, T]$.

We prove that for $t \in J_T$ the function
\[
g : \tau \mapsto \int_0^\tau (\tau - s)^{-\alpha} \sup_{0 \leq \tau' \leq s} \|y(\tau')\|_{X^\alpha} \, ds, \quad \tau \in J_t
\]
is monotone increasing.

Let $t_0 \in J_t$ and $\delta > 0$ be given. Then by a shift of the integration interval we obtain
\[
g(t_0 + \delta) - g(t_0) = \int_0^{t_0 + \delta} (t_0 + \delta - s)^{-\alpha} \sup_{0 \leq \tau' \leq s} \|y(\tau')\|_{X^\alpha} \, ds - \int_0^{t_0} (t_0 - s)^{-\alpha} \sup_{0 \leq \tau' \leq s} \|y(\tau')\|_{X^\alpha} \, ds
\]
\[
= \int_0^{t_0} (t_0 - s)^{-\alpha} \left( \sup_{0 \leq \tau' \leq s + \delta} \|y(\tau')\|_{X^\alpha} - \sup_{0 \leq \tau' \leq s} \|y(\tau')\|_{X^\alpha} \right) \, ds
\]
\[
+ \int_0^{\delta} (t_0 + \delta - s)^{-\alpha} \sup_{0 \leq \tau' \leq s} \|y(\tau')\|_{X^\alpha} \, ds \geq 0.
\]

Because $g$ is monotone increasing we can take the supremum in $[12]$ on both sides to get
\[
\sup_{0 \leq \tau \leq t} \|y(\tau)\|_{X^\alpha} \leq c_0(T) \int_0^t (t - s)^{-\alpha} \sup_{0 \leq \tau' \leq s} \|y(\tau')\|_{X^\alpha} \, ds + c_1(T)[1 + \|u\|_{L^q(J_T; X)}].
\]

By Gronwall’s Lemma this implies
\[
\sup_{0 \leq \tau \leq t} \|y(\tau)\|_{X^\alpha} \leq C(T)(1 + \|u\|_{L^q(J_T; X)})
\]
for $C(T) > 0$ and for all $t \in J_T$ [12, Lemma 6.7], which proves $(13)$.

3. Local Lipschitz continuity of the solution mapping is shown in a similar way as global existence but we have to be careful because $f$ is only locally Lipschitz continuous and not Lipschitz continuous on bounded sets as is the case in [11].
The function \((y(\cdot), v) \mapsto f(y(\cdot), v)\) is locally Lipschitz continuous from \(C(\mathcal{T}; X^\alpha) \times \mathbb{R}\) to \(C(\mathcal{T}; X)\) with respect to the \(C(\mathcal{T}; X^\alpha)\)-norm. To see this, note first that the set \(y(\mathcal{T}) \subset X^\alpha\) is compact for given \(y \in C(\mathcal{T}; X^\alpha)\) by continuity of \(y\) and since the interval \(\mathcal{T} \subset \mathbb{R}\) is a compact set. Moreover, \(y(\mathcal{T})\) equipped with the subspace topology in \(X^\alpha\) is separable, again because \(y\) is continuous and since \(\mathcal{T}\) is separable. Let \(\{x_i\}_{i \in \mathbb{N}} \subset X^\alpha \cap y(\mathcal{T})\) be a dense subset of \(y(\mathcal{T})\). The function \((\tilde{y}, v) \mapsto f(\tilde{y}, v)\) is locally Lipschitz continuous from \(X^\alpha \times \mathbb{R}\) to \(X\). So one can find constants \(\varepsilon(x_i) > 0\) such that \((\tilde{y}, v) \mapsto f(\tilde{y}, v)\) is Lipschitz continuous on \(B_{X^\alpha}(x_i, \varepsilon(x_i)) \times \mathbb{R}\). Because \(\{x_i\}_{i \in \mathbb{N}}\) is dense in \(y(\mathcal{T})\), it follows that the set \(y(\mathcal{T})\) is contained in \(\bigcup_{i \in \mathbb{N}} B_{X^\alpha}(x_i, \varepsilon(x_i))\). Since \(y(\mathcal{T})\) is compact in \(X^\alpha\), one can find a finite subcover \(\bigcup_{k=1}^K B_{X^\alpha}(x_i, \varepsilon(x_i))\) which still contains \(y(\mathcal{T})\). Now the function \((\tilde{y}, v) \mapsto f(\tilde{y}, v)\) is Lipschitz continuous on \(\bigcup_{i=1}^K B_{X^\alpha}(x_i, \varepsilon(x_i)) \times \mathbb{R}\) with a modulus given by the maximum of the Lipschitz constants on \(B_{X^\alpha}(x_i, \varepsilon(x_i)) \times \mathbb{R}\) over all \(i \in \{1, \ldots, k\}\). Since \(V_\nu := \{\tilde{y} \in C(\mathcal{T}; X^\alpha) : y(t) \in B_{X^\alpha}(x_i, \varepsilon(x_i)) \forall t \in \mathcal{T}\}\) is a neighbourhood of \(y\) in \(C(\mathcal{T}; X^\alpha)\), this proves that \((\tilde{y}(\cdot), v) \mapsto f(\tilde{y}(\cdot), v)\) is Lipschitz continuous from \(B_{X^\alpha}(x_i, \varepsilon(x_i)) \times \mathbb{R} \subset C(\mathcal{T}; X^\alpha) \times \mathbb{R}\) to \(C(\mathcal{T}; X)\), i.e. \((y(\cdot), v) \mapsto f(y(\cdot), v)\) is locally Lipschitz continuous from \(C(\mathcal{T}; X^\alpha) \times \mathbb{R}\) to \(C(\mathcal{T}; X)\) with respect to the \(C(\mathcal{T}; X^\alpha)\)-norm. Moreover, there even holds the pointwise estimate

\[
\|f(y_1(t), v_1) - f(y_2(t), v_2)\|_X \leq L(y)(\|y_1(\tau) - y_2(\tau)\|_{X^\alpha} + |v_1 - v_2|)
\]

for all \(y_1, y_2 \in V_y, v_1, v_2 \in \mathbb{R}\) and \(t \in \mathcal{T}\) and for some \(L(y) > 0\).

Lipschitz continuity of \(W\), see Subsection 2.4 together with Assumption 2.16 yields that also \(y \mapsto F[y]\) is locally Lipschitz continuous from \(C(\mathcal{T}; X^\alpha)\) to \(C(\mathcal{T}; X)\) and for \(y \in C(\mathcal{T}; X^\alpha)\) there exists a neighbourhood \(V_y\) of \(y\) and a constant \(L(y) > 0\) such that the pointwise estimate

\[
\|F(y_1)(t) - F(y_2)(t)\|_X \leq L(y) \sup_{0 \leq \tau \leq t} \|y_1(\tau) - y_2(\tau)\|_{X^\alpha}
\]

holds for all \(y_1, y_2 \in V\) and \(t \in \mathcal{T}\). Let \(y = G(u)\) be the solution of problem (1) corresponding to \(u\). Moreover, let \(\delta > 0\) be small enough so that \(F\) is Lipschitz continuous in \(B_{C(\mathcal{T}; X^\alpha)}(y, \delta)\) with modulus \(L(y)\).

For \(R > 0\) to be chosen let \(\tilde{u} \in B_{L^2(\mathcal{T}; X)}(u, R)\) be arbitrary. There holds \(y(0) = G(\tilde{u})(0) = 0\). Continuity of \(y\) and \(G(\tilde{u})\) that we can find some \(\tau > 0\) such that

\[
\sup_{0 \leq t \leq \tau} \|y(t) - G(\tilde{u})(t)\|_{X^\alpha} < \delta.
\]

With (13), (12), (9), (15) and Assumption 2.16 we obtain

\[
\|y(t) - G(\tilde{u})(t)\|_{X^\alpha} \\
\leq ce^{(1-\gamma)T} \int_0^t (t-s)^{-\alpha} \|((F[y])(s) - (F[G(\tilde{u})])(s))\|_X + \|u - \tilde{u}\|_X\| ds \\
\leq c(T, y) \int_0^t (t-s)^{-\alpha} \sup_{0 \leq \tau \leq s} \|y(\tau) - G(\tilde{u})(\tau)\|_X ds + c\|u - \tilde{u}\|_{L^2(\mathcal{T}; X)}
\]
for $t \in [0, \tau)$ and constants $c(T, y), c > 0$.

Similar as in Step 2 one can use Gronwall’s Lemma to prove that there is some $C(T, y) > 0$ such that

$$\sup_{0 \leq t \leq \tau} \|y(t) - G(\tilde{u})(t)\|_{X^\alpha} \leq C(T, y)\|u - \tilde{u}\|_{L^q(J_T; X)} < \delta$$

if $R$ is chosen small enough, since $\tilde{u} \in \overline{B_{L^q(J_T; X)}(u, R)}$. Repeating the argument shows that

$$\sup_{0 \leq t \leq T} \|y(t) - G(\tilde{u})(t)\|_{X^\alpha} \leq C(T, y)\|u - \tilde{u}\|_{L^q(J_T; X)} < \delta$$

for some appropriate $R > 0$ and all $\tilde{u} \in \overline{B_{L^q(J_T; X)}(u, R)}$. This implies that $G$ maps

$\overline{B_{L^q(J_T; X)}(u, R)}$ into $\overline{B_{C(J_T; X^\alpha)}(y, \delta)}$ and $F$ is Lipschitz continuous on this set. A similar computation yields a constant $C(T, y) > 0$ such that for arbitrary $u_1, u_2 \in \overline{B_{L^q(J_T; X)}(u, R)}$ there holds

$$\sup_{0 \leq t \leq T} \|G(u_1)(t) - G(u_2)(t)\|_{X^\alpha} \leq C(T, y)\|u_1 - u_2\|_{L^q(J_T; X)}.$$

This proves that $G$ is Lipschitz continuous in $\overline{B_{L^q(J_T; X)}(u, R)}$.

So we have shown local Lipschitz continuity of $G$ from $L^q(J_T; X)$ to $C(J_T; X^\alpha)$.

4. Uniqueness of the mild solution follows by local Lipschitz continuity of $G$ if one inserts $u_1 = u_2$.

5. The last statement of the theorem follows from maximal parabolic Sobolev regularity of $A_p$, cf. Remark 2.13. One applies $(\frac{d}{dt} + A_p)^{-1}$ to $F[y] + u \in L^q(J_T; X)$ or to $F[y_1] - F[y_2] + u_1 - u_2 \in L^q(J_T; X)$ respectively, see also [11, Proposition 2.8]. Note that this is the only step of the proof in which we must assume $p \in J \cap [2, \infty)$ with $J$ from Theorem 2.10. In the previous steps also $p \in J$ would have been sufficient.

\[ \square \]

4 Hadamard directional differentiability

4.1 Definition and properties

We want to show differentiability of the solution mapping $G$ for problem (11). Because the hysteresis operator is not smooth we can not expect a Fréchet derivative. Therefore we consider a weaker form of differentiability, the Hadamard directional derivative [3, 5]. To start with, we define directional differentiability of a mapping $g : U \subset X \to Y$ from an open set $U \subset X$ of a normed vector space $X$ into a normed vector space $Y$ [3 Definition 2.44]:

**Definition 4.1.** Let $X, Y$ be normed vector spaces. We call $g$ directionally differentiable at $x \in U \subset X$ in the direction $h \in X$ if

$$g'[x; h] := \lim_{\lambda \downarrow 0} \frac{g(x + \lambda h) - g(x)}{\lambda}$$

exists in $Y$. If $g$ is directionally differentiable at $x$ in every direction $h$ we call $g$ directionally differentiable at $x$. 

\[ 14 \]
Using this definition we introduce the concept of the Hadamard directional derivative:

**Definition 4.2.** If \( g \) is directionally differentiable at \( x \in U \) and if in addition for all functions \( r : [0, \lambda_0) \to X \) with \( \lim_{\lambda \to 0} \frac{r(\lambda)}{\lambda} = 0 \)

\[
g'[x; h] = \lim_{\lambda \to 0} \frac{g(x + \lambda h + r(\lambda)) - g(x)}{\lambda}
\]

for all directions \( h \in X \), we call \( g'[x; h] \) the Hadamard directional derivative of \( g \) at \( x \) in the direction \( h \).

Note that \( g(x + \lambda h + r(\lambda)) \) is only well defined if \( \lambda \) is already small enough so that \( x + \lambda h + r(\lambda) \in U \).

We will frequently use the following properties of the concept of Hadamard directional differentiability:

**Lemma 4.3.** [3] Proposition 2.47] Suppose that \( g : U \subset X \to Y \) is Hadamard directionally differentiable at \( x \in U \) and that \( f : V \subset g(U) \to Z \) is Hadamard directionally differentiable at \( g(x) \in V \). Then \( f \circ g : U \to Z \) is Hadamard directionally differentiable at \( x \) and

\[
(f \circ g)'[x; h] = f'[g(x); g'[x; h]].
\]

**Lemma 4.4.** [3] Proposition 2.49] Suppose that \( g : U \subset X \to Y \) is directionally differentiable at \( x \in U \) and in addition Lipschitz continuous with modulus \( c(x) \) in a neighbourhood of \( x \). Then \( g \) is Hadamard directionally differentiable at \( x \) and \( g'[x; \cdot] \) is Lipschitz continuous on \( X \) with modulus \( c(x) \).

### 4.2 Hadamard differentiability of the stop operator

The stop operator \( \mathcal{W} \) from Subsection 2.4 is Hadamard directionally differentiable as a mapping \( C[0, T] \to L^p(0, T) \). This follows from the corresponding result for the play operator [5 Proposition 5.5] and because

\[
\mathcal{P} + \mathcal{W} = \text{Id}
\]
defines a scalar play operator \( \mathcal{P} \) [5 Part 1 Chapter III Proposition 3.3].

In [5] Hadamard directional differentiability is only proved for the case when \( [a, b] = [-r, r] \) for some \( r > 0 \) and for the corresponding symmetrical play \( \mathcal{P}_r \).

One can generalize this result by concatenating \( \mathcal{P}_r \) with \( r = \frac{b-a}{2} \) with the affine linear transformation

\[
\mathcal{T} : [-r, r] \to [a, b], \quad \mathcal{T} : x \mapsto x + \frac{b + a}{2}.
\]

Then for piecewise monotone input functions \( v \) with a monotonicity partition \( 0 = t_0 \leq \cdots \leq t_k = T \) and for \( z(t_i) := \mathcal{P}_r(\mathcal{T}(v))(t_i) \), \( z(0) = z_0 \) we obtain inductively

\[
\mathcal{P}_r(\mathcal{T}(v))(t) = \max\{\mathcal{T}(v(t)) - r, \min\{\mathcal{T}(v(t)) + r, z(t_i)\}\}
= \max\{v(t) + a, \min\{v(t) + b, v(t_i)\}\} = \mathcal{P}(v)(t)
\]

for \( t \in [t_i, t_i + 1] \). By extension to more general input functions it follows \( \mathcal{P}_r \circ \mathcal{T} = \mathcal{P} \).

Differentiability of \( \mathcal{T} \) and Hadamard directional differentiability of \( \mathcal{P}_r \) together with the chain rule yield Hadamard directional differentiability for \( \mathcal{P} \) and then also for \( \mathcal{W} \).
Remark 4.5. As already mentioned in the introduction, all our results hold if we replace the stop operator by \( P \) or by another hysteresis operator with appropriate properties. The main reason why we decided for \( W \) is the following: In Section 5 we apply our results to an optimal control problem in which (1) is the state equation. We will derive an adjoint system for this problem in a forthcoming paper. This is achieved by a regularization of (2)-(4).

4.3 Hadamard differentiability of the solution operator for the evolution equation

We want to prove Hadamard directional differentiability of the solution operator for problem (1).

Assumption 4.6. In addition to Assumption 2.16 we assume that \( f \) is directionally differentiable and therefore Hadamard directionally differentiable.

The statement of the following theorem is almost equal to [11, Theorem 3.2], but the proof is different due to the hysteresis operator and because our function \( f \) is only locally Lipschitz continuous.

Also, in Step 2 of the following proof we show a statement which is very similar to [11, Lemma 3.1], but again the proof has to be different in our setting.

Theorem 4.7. Let Assumption 4.6 hold.

For any \( q \in \left( \frac{1}{1-\alpha}, \infty \right) \) the solution operator \( G : \text{L}^q(J_T; X) \to \text{C}(\overline{J_T}; X^\alpha) \) of problem (1) is Hadamard directionally differentiable.

Its derivative \( y^{u,h} := G'[u; h] \) at \( u \in \text{L}^q(J_T; X) \) in direction \( h \in \text{L}^q(J_T; X) \) is given by the unique mild solution \( \zeta \in \text{C}(\overline{J_T}; X^\alpha) \) of

\[
\dot{\zeta}(t) + (A_p\zeta)(t) = F'[y; \zeta](t) + h(t) \quad \text{in} \quad J_T,
\]

\( \zeta(0) = 0, \)

where \( F'[y; \zeta](t) = f'(y(t), W[Sy](t)); (y(t), W'[Sy; S\zeta](t)) \) and \( y = G(u) \), see Theorem 3.2.

Moreover, \( G'[u; h] \in Y_{q,0} \) is the Hadamard directional derivative of \( G : \text{L}^q(J_T; X) \to Y_{q,0} \).

The mapping \( h \mapsto G'[u; h] \) is Lipschitz continuous from \( \text{L}^q(J_T; X) \) to \( \text{C}(\overline{J_T}; X^\alpha) \) and to \( Y_{q,0} \) with a modulus of continuity \( c = C(G(u), T) \).

Proof. We show the theorem in five steps.

1. First we prove that the function \( \tilde{F} : \text{C}(\overline{J_T}; X^\alpha) \times \text{L}^q(J_T) \to \text{L}^q(J_T; X) \),

\[
\tilde{F} : (y, v) \mapsto [t \mapsto f(y(t), v(t))]
\]

is Hadamard directionally differentiable. We want to use Lemma 4.4.

(a) We show that \( \tilde{F} \) is well-defined.

Since \( q > 1 \) we have for \( x_1, x_2 \in \mathbb{R}_+ \)

\[
(x_1 + x_2)^q \leq 2^{q-1}(x_1^q + x_2^q).
\]
Let \((y, v) \in \mathcal{C}(\overline{J_T}; X^\alpha) \times L^q(J_T)\) be given. Measurability of \(\tilde{F}(y, v)\) follows from measurability of \(y\) and \(v\) and from continuity of \(f\) in both components.

Furthermore, for a.e. \(s \in J_T\) we estimate
\[
\|f(y(s), v(s))\|_X^q \leq M^q(\|y(s)\|_{X^\alpha} + |v(s)| + 1)^q \leq M^q 2^{q-1}(\|y(s)\|_{X^\alpha} + 1)^q + |v(s)|^q
\]
with \(M\) from Assumption 2.16 so that \(\tilde{F}(y, v) \in L^q(J_T; X)\).

(b) We show that \(\tilde{F}\) is locally Lipschitz continuous with respect to the \(C(\overline{J_T}; X^\alpha)\)-norm. As in Step 3 in the proof of Theorem 3.1 note that \((y(\cdot), v) \mapsto f(y(\cdot), v)\) is locally Lipschitz continuous from \(C(\overline{J_T}; X^\alpha) \times \mathbb{R} \rightarrow C(\overline{J_T}; X)\) with respect to the \(C(\overline{J_T}; X^\alpha)\)-norm.

For \(y \in C(\overline{J_T}; X^\alpha)\) let \(B_{C(\overline{J_T}; X^\alpha)}(y, \delta) \times \mathbb{R}\) be given such that this function is Lipschitz continuous with modulus \(L(y)\).

Consider any \(y_1, y_2 \in B_{C(\overline{J_T}; X^\alpha)}(y, \delta)\) and \(v_1, v_2 \in L^q(J_T)\).

By (14) we obtain for a.e. \(s \in J_T\)
\[
\|\tilde{F}(y_1, v_1)(s) - \tilde{F}(y_2, v_2)(s)\|_X \leq L(y) \|y_1(s) - y_2(s)\|_{X^\alpha} + |v_1(s) - v_2(s)|.
\]

Minkowski’s inequality and \(\|y_1 - y_2\|_{L^q(J_T; X^\alpha)} \leq T^{1/q}\|y_1 - y_2\|_{C(\overline{J_T}; X^\alpha)}\) yields
\[
\|\tilde{F}(y_1, v_1) - \tilde{F}(y_2, v_2)\|_{L^q(J_T; X)} \leq L(y) \left[T^{1/q}\|y_1 - y_2\|_{C(\overline{J_T}; X^\alpha)} + |v_1 - v_2|\right]
\leq L(y)(1 + T^{1/q}) \left[\|y_1 - y_2\|_{C(\overline{J_T}; X^\alpha)} + |v_1 - v_2|\right].
\]

(c) We show that \(\tilde{F}\) is directionally differentiable.

To this aim, consider \(y \in C(\overline{J_T}; X^\alpha)\) from Step 1 (b) and any \(v \in L^q(J_T)\).

Let \((h, l) \in C(\overline{J_T}; X^\alpha) \times L^q(J_T)\) be arbitrary and \(\lambda_0 > 0\) small enough so that \(y + \lambda h \in B_{C(\overline{J_T}; X^\alpha)}(y, \delta)\) for all \(\lambda \in (0, \lambda_0]\).

For each \(\lambda \in (0, \lambda_0]\) we define the differential quotient
\[
\tilde{F}_\lambda := \frac{1}{\lambda}[\tilde{F}(y + \lambda h, v + \lambda l) - \tilde{F}(y, v)].
\]

For a.e. \(s \in J_T\) we have that
\[
\lim_{\lambda \to 0} \tilde{F}_\lambda(s) = f'(([y(s), v(s)]; (h(s), l(s)))] \in X
\]
because \(f\) is directionally differentiable by Assumption 2.16.

We can also estimate for a.e. \(s \in J_T\) and \(\lambda_0\) small enough
\[
\|\tilde{F}_\lambda(s)\|_X \leq L(y) \|h(s)\|_{X^\alpha} + |l(s)|
\]
and the right side is contained in \(L^q(J_T)\).

It follows by Lebesgue’s dominated convergence theorem that \(\tilde{F}_\lambda\) converges to the function
\[
s \mapsto f'(([y(s), v(s)]; (h(s), l(s)))]
\]
in \(L^q(J_T; X)\) as \(\lambda \to 0\), which implies directionally differentiability of \(\tilde{F}\).

This step is actually analogous to the proof of [11 Lemma 3.1]. The other steps needed some additional work.
(d) By Lemma 4.4, Steps 1(b) and 1(c) imply that $\tilde{F}$ is Hadamard directionally differentiable and that $(h,l) \mapsto \tilde{F}'[(y,v);(h,l)]$ is Lipschitz continuous.

2. Let $F : C(\mathcal{J}_T; X^\alpha) \to L^q(J_T; X)$, $(F[y])(t) := f(y(t), \mathcal{W}[Sy](t))$ be defined as in Theorem 3.1. We show that $F$ is Hadamard directionally differentiable [11] Lemma 3.1]. Because the identity mapping $\text{Id}$ on $C(\mathcal{J}_T; X^\alpha)$ and $S : C(\mathcal{J}_T; X^\alpha) \to C(\mathcal{J}_T)$ are linear and continuous they are Fréchet differentiable with derivatives $\text{Id}$ and $S$. Lemma 4.3 together with Subsection 4.2 yields that the mapping

$$y \mapsto (y, \mathcal{W}[Sy])$$

is Hadamard directionally differentiable from $C(\mathcal{J}_T; X^\alpha)$ into $C(\mathcal{J}_T; X^\alpha) \times L^q(J_T)$ with derivative

$$h \mapsto (h, \mathcal{W}'[Sy; Sh]).$$

Applying Lemma 4.3 another time and using Step 1 we conclude that $F$ is Hadamard directionally differentiable with

$$F'[y; h](t) = f'[y(t), \mathcal{W}[Sy](t)); (h(t), \mathcal{W}'[Sy; Sh](t))$$

for $y, h \in C(\mathcal{J}_T; X^\alpha)$ and a.e. $t \in J_T$.

From Step 3 in the proof of Theorem 3.1 we know that $F$ is locally Lipschitz continuous from $C(\mathcal{J}_T; X^\alpha)$ to $C(\mathcal{J}_T; X)$ and therefore also from $C(\mathcal{J}_T; X^\alpha)$ to $L^q(J_T; X)$. Lemma 4.3 implies that for any $y \in C(\mathcal{J}_T; X^\alpha)$ the mapping $h \mapsto F'[y; h]$ is Lipschitz continuous from $C(\mathcal{J}_T; X^\alpha)$ to $L^q(J_T; X)$.

3. We have seen in the end of Step 2 that $F$ is locally Lipschitz continuous from $C(\mathcal{J}_T; X^\alpha)$ to $C(\mathcal{J}_T, X)$ with a pointwise estimate of the form [15].

4. We show that for any $y \in C(\mathcal{J}_T; X^\alpha)$ and $h \in L^q(J_T; X)$ the integral equation

$$\zeta(t) = \int_0^t e^{-Aq(t-s)}[F'[y; \zeta](s) + h(s)] \, ds$$

has a unique solution $\zeta(h)$ in $C(\mathcal{J}_T; X^\alpha)$ and that for fixed $y$ the mapping $h \mapsto \zeta(h)$ is Lipschitz continuous with a modulus $C = C(y, T)$.

By Step 2 the function $\zeta \mapsto F'[y; \zeta]$, where $F'[y; \zeta]$ is given by

$$t \mapsto f'[y(t), \mathcal{W}[Sy](t)); (\zeta(t), \mathcal{W}'[Sy; S\zeta](t)),$$

is Lipschitz continuous from $C(\mathcal{J}_T; X^\alpha)$ to $L^q(J_T; X)$.

Similar to Theorem 3.1, this together with [5] and [9] implies that for any $0 < \tilde{T} \leq T$ the function

$$g : \zeta \mapsto \left[ t \mapsto \int_0^t e^{-Aq(t-s)}[F'[y; \zeta](s) + h(s)] \, ds \right]$$

has a unique solution $g(h)$ in $C(\mathcal{J}_{\tilde{T}}; X^\alpha)$ and that for fixed $y$ the mapping $h \mapsto g(h)$ is Lipschitz continuous with a modulus $C = C(y, T, \tilde{T})$. 

By Step 2 the function $g \mapsto F'[y; g]$, where $F'[y; g]$ is given by

$$t \mapsto f'[y(t), \mathcal{W}[Sy](t)); (g(t), \mathcal{W}'[Sy; Sg](t)),$$

is Lipschitz continuous from $C(\mathcal{J}_{\tilde{T}}; X^\alpha)$ to $L^q(J_{\tilde{T}}; X)$. 

Similar to Theorem 3.1, this together with [5] and [9] implies that for any $0 < \tilde{T} \leq T$ the function
is well-defined on $C([0, \tilde{T}]; X^\alpha)$ and Lipschitz continuous with a modulus of the form

$$L(\tilde{T}) = C(y) e^{(1-\gamma)t} \tilde{T}^{1/q-\alpha}.$$ 

This observation together with Gronwall’s Lemma already implies the statement about Lipschitz continuity for fixed $y$, provided that the fixed point mapping $h \mapsto \zeta(h)$ is well-defined.

We show by induction that $g$ has a fixed point in $C(J_{\tilde{T}}; X^\alpha)$. Let $k \in \mathbb{N}$ be large enough so that $L \left( \frac{4}{k} \right) = C(y) e^{(1-\gamma)t} \left( \frac{4}{k} \right)^{1/q-\alpha} \leq \frac{1}{2}$ and set $t_j := \frac{4j}{k}$ for $1 \leq j \leq k$.

We prove that $g$ has a fixed point in $C(J_{\tilde{T}}; X^\alpha) = C([0, t_j]; X^\alpha)$.

To this aim we define $H(t) := \int_0^t e^{-A_{p'}(t-s)} h(s) \, ds$ and $N_0 := \|H\|_{C(J_{\tilde{T}}; X^\alpha)}$ and consider $g$ on $\overline{B_{C(J_{\tilde{T}}; X^\alpha)}(H, N_0)}$. Note that $g$ is a contraction on $C(J_{\tilde{T}}; X^\alpha)$ so that we can apply Banach’s fixed point theorem if $g$ maps $\overline{B_{C(J_{\tilde{T}}; X^\alpha)}(H, N_0)}$ into itself.

By definition we have $g(0) = H$.

Because $L \left( \frac{4}{k} \right) < \frac{1}{2}$, for $\zeta \in \overline{B_{C(J_{\tilde{T}}; X^\alpha)}(H, N_0)}$ there holds

$$\|g(\zeta)(t) - H(t)\|_{X^\alpha} = \|g(\zeta)(t) - g(0)(t)\|_{X^\alpha} \leq \frac{1}{2}\|\zeta(t)\|_{X^\alpha} \leq \frac{1}{2}\|\zeta(t)\|_{X^\alpha} + \frac{1}{2}\|H(t)\|_{X^\alpha} \leq N_0$$

so that indeed $g$ maps $\overline{B_{C(J_{\tilde{T}}; X^\alpha)}(H, N_0)}$ into itself.

We obtain a unique fixed point $\zeta_1 \in \overline{B_{C(J_{\tilde{T}}; X^\alpha)}(H, N_0)}$ of $g_1 := g : C(J_{\tilde{T}}; X^\alpha) \rightarrow C(J_{\tilde{T}}; X^\alpha)$.

Inductively, we set $N_j := 2N_{j-1} + N_0$ for $2 \leq j \leq k$ and define

$g_j : C(J_{\tilde{T}}; X^\alpha) \rightarrow C(J_{\tilde{T}}; X^\alpha)$ as

$$g_j(\zeta)(t) := \begin{cases} 
\zeta_{j-1}(t) & \text{if } t \in [0, t_{j-1}], \\
\zeta_{j-1}(t_{j-1}) + \int_{t_{j-1}}^t e^{-A_{p'}(t-s)} [F'(y; \zeta)(s) + h(s)] \, ds & \text{if } t \in [t_{j-1}, t_j],
\end{cases}$$

assuming that the unique fixed point $\zeta_{j-1}$ of $g_{j-1}$ exists from the previous step. We show that $g_j$ has a fixed point.

Note that $g_j(0) = \zeta_{j-1}(t_{j-1}) + H - H(t_{j-1}) \in C(J_{\tilde{T}}; X^\alpha)$ and that $g_j$ is a $\frac{1}{2}$-contraction on $C(J_{\tilde{T}}; X^\alpha)$. So we are left to show that $g_j$ maps $\overline{B_{C(J_{\tilde{T}}; X^\alpha)}(H, N_j)}$ into itself.

Let $\zeta \in \overline{B_{C(J_{\tilde{T}}; X^\alpha)}(H, N_j)}$ be given. On $[0, t_{j-1}]$ we can estimate

$$\|g_j(\zeta)(t) - H(t)\|_{X^\alpha} = \|\zeta_{j-1}(t) - H(t)\| \leq N_{j-1} \leq N_j$$

19
by induction. For $t \in [t_{j-1}, t_j]$ we can estimate
\[
\|g_j(\zeta)(t) - H(t)\|_{X^\alpha} = \|\zeta_{j-1}(t_{j-1}) - H(t_{j-1}) + g_j(\zeta)(t) - g_j(0)(t)\|_{X^\alpha}
\leq \|\zeta_{j-1}(t_{j-1}) - H(t_{j-1})\|_{X^\alpha} + \frac{1}{2}\|\zeta(t)\|_{X^\alpha}
\leq N_{j-1} + \frac{1}{2}\|\zeta(t) - H(t)\|_{X^\alpha} + \frac{1}{2}\|H(t)\|_{X^\alpha}
\leq N_{j-1} + \frac{N_j}{2} + \frac{1}{2}\|H\|_{C(J, X^\alpha)}
\leq N_{j-1} + \frac{2N_{j-1}}{2} + \frac{1}{2}\|H\|_{C(J, X^\alpha)}
\leq 2N_{j-1} + \|H\|_{C(J, X^\alpha)} = N_j.
\]

So indeed $g_j$ maps $B_{C(J, X^\alpha)}(H, N_j)$ into itself and we obtain a unique fixed point
\[
\zeta_j \in B_{C(J, X^\alpha)}(H, N_j)
\]
of $g_j$.

We have
\[
\zeta_2(t) = \zeta_1(t) = g(\zeta_1)(t)
\]
for $t \in \mathcal{T}_{t_1}$ and
\[
\zeta_2(t) = \int_0^{t_1} e^{-A_p(t-s)}[F'[y; \zeta_1](s) + h(s)] \, ds + \int_{t_1}^t e^{-A_p(t-s)}[F'[y_0; \zeta_2](s) + h(s)] \, ds
\]
for $t \in [t_1, t_2]$ which implies
\[
\zeta_2(t) = \int_0^t e^{-A_p(t-s)}[F'[y; \zeta_2](s) + h(s)] \, ds = g(\zeta_2)(t)
\]
on $[0, t_2]$.

Inductively, it follows $\zeta_j = g(\zeta_j)$ for all $j \in \{1, \cdots k\}$ which shows that $\zeta = \zeta(h) := \zeta_k$ is the unique solution of the integral equation
\[
\zeta(t) = \int_0^t e^{-A_p(t-s)}[F'[y; \zeta](s) + h(s)] \, ds
\]
in $C(J, X^\alpha)$, i.e. a fixed point of $g$.

5. We now come to the proof of the statement of the theorem.

Let any $u \in L^q(J; X)$ be given and $y = G(u)$. For $h \in L^q(J; X)$ and $\lambda > 0$ we denote
\[
y_{\lambda} := G(u + \lambda h).
\]
Let $\zeta = \zeta(h) \in C(J, X^\alpha)$ be the function from Step 4.
Similar as in \cite[Theorem 3.2]{11} we estimate with (5) and (9), Step 2 and Step 3 and for \( \lambda > 0 \) small enough

\[
\left\| \frac{y_\lambda(t) - y(t)}{\lambda} - \zeta(t) \right\|_{X^\alpha} \leq C_\alpha e^{(1-\gamma)T} \int_0^t (t - s)^{-\alpha} \left( \left\| \frac{(F[y + \lambda\zeta])(s) - (F[y])(s)}{\lambda} - F'[y; \zeta](s) \right\|_X \right.
\]

\[
+ \left. \left\| \frac{(F[y + \lambda\zeta])(s) - (F[y\lambda])(s)}{\lambda} \right\|_{L^q(J_T; X)} \right) \, ds
\]

\[
\leq c e^{(1-\gamma)T} \left( t^{1/q-\alpha} \left\| \frac{F[y + \lambda\zeta] - F[y]}{\lambda} - F'[y; \zeta] \right\|_{L^q(J_T; X)} \right.
\]

\[
+ L(y) \int_0^t (t - s)^{-\alpha} \sup_{0 \leq \tau' \leq s} \left\| \frac{y_\lambda(\tau') - y(\tau')}{\lambda} - \zeta(\tau') \right\|_{X^\alpha} \, ds \right) .
\]

The first term converges to zero with \( \lambda \to 0 \) by Step 2. The estimate of the second term holds because of (15) in Step 3, which was the local Lipschitz continuity of \( F \), and by local Lipschitz continuity of \( G \) from \( L^q(J_T; X) \) to \( C(J_T; X^\alpha) \) (see Theorem 3.1).

We take the supremum \( \sup_{0 \leq \tau' \leq t} \) on both sides and apply Gronwall’s Lemma to see that \( \frac{y_\lambda - y}{\lambda} \) converges to \( \zeta \) in \( C(J_T; X^\alpha) \). So we find that \( \zeta \) is the directional derivative of \( G \) at \( u \) in direction \( h \).

Local Lipschitz continuity of \( G \) and Lemma 4.4 imply that the solution mapping for problem (11) is Hadamard directionally differentiable from \( L^q(J_T; X) \) to \( C(J_T; X^\alpha) \).

The statement for \( Y_{q,0} \) follows with Remark 2.13 just as in the proof of \cite[Theorem 3.2]{11}.

\[ \square \]

5 Application to an optimal control problem

In this section, we apply the results from Theorem 3.1 and Theorem 4.7 to an optimal control problem. We consider either distributed controls in

\[
U_1 := L^2(J_T; \tilde{U}_1) := L^2(J_T; [L^2(\Omega)]^m)
\]

or Neumann boundary controls in

\[
U_2 := L^2(J_T; \tilde{U}_2) := L^2(J_T; \prod_{i=1}^m L^2(\Gamma_{N_i}, \mathcal{H}_{d-1})) .
\]

Moreover, we will define continuous operators \( B_i : \tilde{U}_i \to X \) for \( i \in \{1, 2\} \), see Assumption 5.1.
With $u \in U_i$, Theorem 4.1 implies well-posedness of the following state equation:

\[
\begin{align*}
\dot{z}(t) &= A_p y(t) + f(y(t), z(t)) + B_i u(t) \quad \text{in } W^{-1,p}_\Gamma(\Omega) \text{ for } t \in (0, T), \\
y(0) &= 0 \quad \text{in } W^{-1,p}_\Gamma(\Omega), \\
(\dot{z}(t) - S\dot{y}(t))(z(t) - \xi) &\leq 0 \quad \text{for } \xi \in [a, b] \text{ and } t \in (0, T), \\
z(t) &\in [a, b] \quad \text{for } t \in [0, T], \\
z(0) &= z_0.
\end{align*}
\]

Note that (18) implies $z = W[Sy]$. For $i \in \{1, 2\}$ and given $\kappa > 0$ consider the optimal control problem

\[
\min_{u \in U_i} J(y, u) := \frac{1}{2} \|y - y_d\|_{U_i}^2 + \frac{\kappa}{2} \|u\|_{U_i}^2 \\
= \frac{1}{2} \int_0^T \|y(s) - y_d(s)\|_{W^1(\Omega)}^2 \, ds + \frac{\kappa}{2} \int_0^T \|u(s)\|_{U_i}^2 \, ds
\]

subject to (17), (18).

**Assumption 5.1.** In addition to Assumption 4.1, we assume:

- $\alpha \in (0, \frac{1}{2})$. This assumption is needed in the proof of Lemma 5.3.
- $B_1$ is defined by

\[
B_1 : [L^2(\Omega)]^m \to X, \quad \langle B_1 u, v \rangle_{W^{-1,p}_\Gamma(\Omega)} := \int_\Omega u \cdot v \, dx, \quad v \in W^{1,p}_\Gamma(\Omega).
\]

Since $2 \geq p \left(1 - \frac{1}{d}\right)$ the embeddings $L^2(\Gamma_{N_j}, H_{d-1}) \hookrightarrow W^{-1,p}_\Gamma(\Omega)$ are continuous for $j \in \{1, \ldots , m\}$ [8 Remark 5.11].

Therefore also

\[
B_2 : \prod_{j=1}^m L^2(\Gamma_{N_j}, H_{d-1}) \to X, \quad \langle B_2 y, v \rangle_{W^{1,p}_\Gamma(\Omega)} = \sum_{j=1}^m \int_{\Gamma_{N_j}} y_j v_j \, dH_{d-1}, \quad v \in W^{1,p}_\Gamma(\Omega)
\]

is continuous.

- The desired state $y_d$ in (19) is in $U_1$ and $\kappa > 0$ is given.

**Remark 5.2.** Theorem 4.7 yields Hadamard directional differentiability of $G \circ B_i : U_i \to Y_{2,0}$ for $i \in \{1, 2\}$ and $(y, z) = (G(B_i u), W[SG(B_i u)])$ solves (17), (18) for $u \in U_i$. Therefore the reduced cost function $J : U_i \to \mathbb{R}$, $J(u) = J(G(B_i u), u)$ is Hadamard directionally differentiable.

**Lemma 5.3.** Let Assumption 5.1 hold.

Suppose that for $\{u_n\}_{n \in \mathbb{N}} \subset U_i$ it holds $u_n \to u$ in $U_i$ with $i \in \{1, 2\}$.

Then $y_n = G(B_i u_n) \to G(B_i u)$ weakly in $Y_{2,0}$ and strongly in $C(T; X^\alpha)$ and $z_n = W[Sy_n] \to W[SG(B_i u)]$ weakly in $H^1(J_T)$ and strongly in $C(J_T)$ [4, Lemma 2.3].

If the convergence of $u_n$ is strong then $y_n \to G(B_i u)$ in $Y_{2,0}$ strongly.
Proof. The proof is a combination of the proofs for \cite{11} Lemma 2.10 and \cite{1} Lemma 2.3. Let $u_n \rightarrow u$ in $U_i$.

By Assumption 5.1 we have $\alpha \in (0, \frac{1}{2})$ so that $\frac{1}{1-\alpha} < 2 = q$. We can therefore use Theorem 5.1 and Theorem 4.7 with $u$ and $h$ replaced by $B_i u$ and $B_i h$ and with $L^2(J_T; X)$ replaced by $U_i$. By Remark 2.13 and (8) there exists some $c > 0$ such that

$$
\|y_n\|_{Y_{2,0}} \leq \left\| \left( \frac{d}{dt} + A_p \right)^{-1} \right\|_{\mathcal{L}(L^2(J_T;X), Y_{2,0})} \left( \|B_i u_n\|_{L^2(J_T;X)} + \|F[y_n]\|_{L^2(J_T;X)} \right)
$$

so that a subsequence $y_{n_k}$ weakly converges in $Y_{2,0}$ to some $y$. By Remark 2.14 we know that $Y_{2,0}$ is compactly embedded into $C(J_T; X^\alpha)$ so that the convergence is strong in this space. We also have that $S y_{n_k}$ converges weakly to $S y$ in $H^1(J_T)$ because $S \in X^\alpha$.

From Subsection 2.4 we know that $W$ is weakly continuous on $H^1(J_T)$ so that weak convergence of $S y_{n_k}$ implies weak convergence of $z_{n_k}$ to $W[S y] = z$ in $H^1(J_T)$ and then also strong convergence in $C(J_T)$.

Weak continuity of $\frac{d}{dt}, A_p$ and $B_i$ yields

$$
\frac{d}{dt} y_{n_k} + A_p y_{n_k} \rightarrow \frac{d}{dt} y + A_p y \quad \text{and} \quad B_i y_{n_k} \rightarrow B_i y \quad \text{in} \quad L^2(J_T; X)
$$

\cite{11} Lemma 2.10.

For $n_k$ large enough we obtain by strong convergence of $y_{n_k}$ in $C(J_T; X^\alpha)$ and by local Lipschitz continuity of $f$

$$
\|f(y_{n_k}; z_{n_k}) - f(y, z)\|_{C(J_T;X)} \leq L(y)(\|y_{n_k} - y\|_{C(J_T;X^\alpha)} + \|z_{n_k} - z\|_{C(J_T)})
$$

so that $f(y_{n_k} (\cdot), z_{n_k} (\cdot))$ converges to $f(y (\cdot), z (\cdot))$ in $C(J_T; X)$.

We pass to the limit in (11) and conclude that $y = G(B_i u)$ and $z = W[S y]$. Uniqueness of the limit implies (weak) convergence of the whole sequence. The statement about strong convergence if $\{u_n\}$ converges to $u$ strongly in $U_i$ follows because in this case

$$
\|y_n - y\|_{Y_{2,0}} \leq \left\| \left( \frac{d}{dt} + A_p \right)^{-1} \right\|_{\mathcal{L}(L^2(J_T;X), Y_{2,0})} \left( \|B_i(u_n - u)\|_{L^2(J_T;X)} + \|F[y_n] - F[y]\|_{L^2(J_T;X)} \right)
$$

(20)

and since the right side then converges to zero.

\[\square\]

**Theorem 5.4.** Let Assumption 5.7 hold. Then for $i \in \{1, 2\}$, there exists an optimal control $\overline{\pi} \in U_i$ for the optimal control problem (17) - (19). This means that $\overline{\pi}$, together with the optimal state $\overline{y} = G(\overline{\pi})$, which solves (17), are a solution of the minimization problem (19). The solution of (18) is given by $\overline{z} = W[\overline{S} \overline{y}]$.

**Proof.** The proof uses Lemma 5.3 and is analogous to the proof of \cite{11} Proposition 2.11.

\[\square\]
Remark 5.5. In a forthcoming paper we derive an adjoint system and optimality conditions for problem (17)-(19). The differences between the control problem for $U_1$ and $U_2$ will become obvious during this analysis. We will first derive optimality conditions for problem (17)-(19) with either distributed or boundary controls, i.e. $i \in \{1, 2\}$. Since $B_1$ has dense range we are able to improve those for $i = 1$. We can also show uniqueness of the adjoint system for the case of distributed controls.

Acknowledgement

The author is supported by the DFG through the International Research Training Group IGDK 1754 „Optimization and Numerical Analysis for Partial Differential Equations with Nonsmooth Structures”. The author would like to thank Prof. Brokate from the Technical University of Munich and Prof. Fellner from the Karl-Franzens University of Graz for thoroughly proofreading the manuscript, as well as Dr. Joachim Rehberg from the Weierstrass Institute in Berlin for the helpful discussions.
References

[1] Herbert Amann. Linear parabolic problems involving measures. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A: Matemáticas (RACSAM)*, 95(1):85–120, 2001.

[2] Pascal Auscher, Nadine Badr, Robert Haller-Dintelmann, and Joachim Rehberg. The square root problem for second-order, divergence form operators with mixed boundary conditions on $L^p$. *Journal of Evolution Equations*, 15(1):165–208, 2014.

[3] J.F. Bonnans and A. Shapiro. *Perturbation Analysis of Optimization Problems*. Springer Series in Operations Research. Springer, 2000.

[4] Martin Brokate and Pavel Krejčí. Optimal control of ode systems involving a rate independent variational inequality. *Discrete and continuous dynamical systems*, 18:331–348, 2013.

[5] Martin Brokate and Pavel Krejčí. Weak differentiability of scalar hysteresis operators. *Discrete and Continuous Dynamical Systems*, 35(6):2405–2421, 2015.

[6] C.M. Carracedo and M. Sanz Alix. *The Theory of Fractional Powers of Operators*. North-Holland Mathematics Studies. Elsevier, 2001.

[7] L.C. Evans. *Partial Differential Equations*. American Mathematical Society, 2 edition, 2010.

[8] Robert Haller-Dintelmann, Alf Jonsson, Dorothee Knees, and Joachim Rehberg. Elliptic and parabolic regularity for second-order divergence operators with mixed boundary conditions. *Mathematical Methods in the Applied Sciences*, 2015.

[9] Daniel Henry. *Geometric theory of semilinear parabolic equations*. Lecture Notes in Mathematics. Springer, 1981.

[10] A. Lunardi. *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, volume 16 of *Progress in Nonlinear Differential Equations and Their Applications*. Springer Science and Business Media, 1995.

[11] C. Meyer and L. Susu. Optimal control of nonsmooth, semilinear parabolic equations. Technical report, Fakultät für Mathematik, TU Dortmund, 9 2015. Ergebnisberichte des Instituts für Angewandte Mathematik, Nummer 524.

[12] Amnon Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Applied Mathematical Sciences. Springer, 1983.

[13] Augusto Visintin. *Differential models of hysteresis*, volume 111. Springer Science & Business Media, 2013.

[14] Atsushi Yagi. *Abstract parabolic evolution equations and their applications*. Springer Science & Business Media, 2009.