Abstract

Worldsheet supersymmetric string action is written in duality invariant form for flat as well as curved backgrounds. First the action in flat backgrounds is written by introducing auxiliary fields. We also give the superfield form of these actions and obtain the off-shell supersymmetry algebra. The action has a modified Lorentz invariance and supersymmetry and reduces to the usual form when the auxiliary fields are eliminated using their equations of motion. Supersymmetric nonlinear sigma model in curved backgrounds is also written in manifestly duality invariant form when the background metric and tortion fields are independent of some of the coordinates.
1. Introduction

Symmetries of string theory play an important role in unravelling its basic structure. One such symmetry, namely target space duality, has received a great deal of attention recently [1]-[9]. It is a generalization of $R \rightarrow 1/R$ duality and interchanges the winding mode and Kluza-Klein excitations (see [10] for a comprehensive review.) In its usual form, at least for the curved backgrounds, target space duality has been formulated as a symmetry of the string effective action only [3]. To see whether it is a symmetry to all orders in $\alpha'$, it is useful to study this duality from worldsheet point of view. Recently an off-shell duality invariant formulation of worldsheet action was given in Ref.[4] by introducing extra ‘dual’ coordinate [10]. In this case local Lorentz invariance is not manifest. The action has a symmetry which reduces to the usual local Lorentz symmetry if the dual coordinates are integrated out using their equations of motion. This is possible since the dual coordinates are auxiliary fields on the worldsheet. Auxiliary nature of the dual coordinates is also evident from the fact that the Weyl anomaly is absent in $D = 26$. It has also been shown that one has to introduce the dual coordinates to make the duality invariance manifest even for the worldsheet equations of motion [3].

Similar developments have taken place in the case of closed NSR strings as well. Duality transformations of the worldsheet equations of motions were discussed in [6]. It was shown in [7] that, in superspace, the first order formalism adopted to write the bosonic string equations of motion can be extended to the fermionic case to implement the duality transformations.

Recently a worldsheet duality invariant action for the bosonic sector of the toroidally compactified four dimensional heterotic string in curved backgrounds was written by Schwarz and Sen [8]. Thus, in absence of anomalies, the toroidally compactified heterotic string theory should have the duality symmetry order by order in Newton’s constant, provided that at each order the full nonperturbative $\alpha'$ dependence is taken into account.

In this paper we address the case of superstrings both in flat backgrounds and nonlinear sigma model in curved gravitational and antisymmetric tensor field backgrounds when the backgrounds are independent of some of the coordinates.

The paper is organized in the following way. Section 2 reviews the requisite parts of Ref.[8]. Section 3 starts by fixing the notation for studying supersymmetric action in flat backgrounds. A duality invariant worldsheet supersymmetric action is obtained by defining a fermionic ‘vector’ field under duality transformations. Supersymmetric transformations are written in manifestly duality invariant form. Both these symmetries are modified in the compactified sector though they are equivalent to the usual symmetries on shell. Supercurrent is written in duality invariant form. Duality invariant supersymmetry is shown to be off-shell closed even in the absence of usual auxiliary fields of the supermultiplet. Therefore we can introduce our superfields as chiral multiplets. In Section 4 we start with a nonlinear supersymmetric sigma model in curved backgrounds when the backgrounds are independent
of some of the coordinates corresponding to the compact target coordinates of string theory. A duality invariant form is obtained for this action. Supersymmetric transformations are written in duality invariant form. We end with some discussions and conclusions in Section 5. Some definitions are collected in Appendix A.

2. Review of Duality Invariant Action for Bosonic Strings

A worldsheet action for the bosonic sector of the compactified heterotic string in curved backgrounds was written down by Schwarz and Sen [8]. In this paper, since we ignore the gauge fields, we are concerned with bosonic or fermionic strings only. The appropriately truncated action has the form

\[ S_B = -\frac{1}{2\pi} \int d^2\sigma \left\{ g_{\mu\nu} \eta^{\hat{\alpha}\hat{\beta}} \partial_{\hat{\alpha}} X^\mu \partial_{\hat{\beta}} X^\nu - b_{\mu\nu} \epsilon^{\hat{\alpha}\hat{\beta}} \partial_{\hat{\alpha}} X^\mu \partial_{\hat{\beta}} X^\nu \
- \partial_0 y^a L_{ab} \partial_1 y^b - \partial_1 y^a (LML)_{ab} \partial_0 y^b \right\}, \]  

(1)

where the uncompactified coordinates have Greek indices \( \mu, \nu, \cdots \) and Greek indices with a caret \( \hat{\alpha}, \hat{\beta}, \cdots \) are the worldsheet indices. Roman indices \( a, b, \cdots \) transform vectorially under duality transformations. The coordinates \( y^a \) have the internal coordinates \( y^i \) and its dual \( \tilde{y}_i \) as its components where the indices \( i, j, \cdots \) label internal (compactified) coordinates. The \( y^a \) equation of motion which follows from this action can be written in the following form

\[ \partial_0 y^a = -(ML)^a_y \partial_1 y^b. \]  

(2)

Here \( \partial_0 \) and \( \partial_1 \) are the worldsheet time and space derivatives respectively. \( L \) is a matrix which remains invariant under the \( O(d,d) \) transformations \( \Omega \), i.e. \( L \rightarrow \Omega^T L \Omega = L \). The matrix \( M \) transforms as \( M \rightarrow \Omega M \Omega^T \). Explicit forms of \( L \) and \( M \) are,

\[ L = \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix}, \quad M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}, \]  

(3)

where \( G_{ij} \) and \( B_{ij} \) are the internal parts of the metric and antisymmetric tensor backgrounds. We are taking the total metric and antisymmetric tensor fields to be in block diagonal form in the target space and internal coordinates:

\[ g_{AB} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & G_{ij} \end{pmatrix}; \quad b_{AB} = \begin{pmatrix} b_{\mu\nu} & 0 \\ 0 & B_{ij} \end{pmatrix} \]  

(4)

The equation of motion (2) has the component form

\[ \partial_0 y^i = G^{ij} B_{jk} \partial_1 y^k - G_{ij} \partial_1 \tilde{y}_j, \quad \partial_0 \tilde{y}_i = B_{ij} G^{jk} B_{kl} \partial_1 y^l - G_{ij} \partial_1 y^i - B_{ij} G^{jk} \partial_1 \tilde{y}_k \]  

(5) \hspace{1cm} (6)

where (5) and (6) are the \( \tilde{y}_i \) and \( y^i \) equations of motion respectively.
The modified reparametrization under which the internal part of (1) is invariant has the form
\[ \delta y^a = \xi^1 \partial_1 y^a - \xi^0 (ML)_b^a y^b. \] (7)

The symmetric and traceless energy-momentum tensor for (1) can be found either by Noether method or by varying its generalization obtained by coupling it to the worldsheet metric \( h_{\tilde{\alpha} \tilde{\beta}} \) which gives Weyl and reparametrization invariant action. Such action was also written in \( [8] \). By varying it with respect to the metric \( h_{\tilde{\alpha} \tilde{\beta}} \) we get the following expressions for the compactified sector
\[ T_{00} = -T_{11} = -\frac{1}{2} \partial_1 y^a (LML)_{ab} \partial_1 y^b, \]
\[ T_{01} = T_{10} = -\frac{1}{2} \partial_1 y^a L_{ab} \partial_1 y^b. \] (8)

Using (6) to eliminate \( \tilde{y}_i \), we find that the eqs.(8) reduce to the usual form of the energy momentum tensor.

3. Superstrings in Flat Backgrounds

In this section we give a manifestly duality invariant worldsheet supersymmetric string action in flat backgrounds and discuss some of its aspects. The two component Majorana-Weyl fermions in the uncompactified sector are denoted by \( \psi^\mu \) and the ones in the compactified sector by \( \psi^i \). We use the following representation of \( \gamma \)-matrices in two dimensions.
\[ \gamma^0 = \sigma^2; \ \gamma^1 = i\sigma^1; \ \gamma^5 = \gamma^0 \gamma^1 = \sigma^3. \] (9)

In this section we restrict ourself to flat backgrounds and also put the antisymmetric tensor field \( B_{ij} \) equal to zero. In this case \( M \) becomes \( M = \text{diag}(G^{ij}, G^{ij}) \).

The fermionic action which gives the supersymmetric action when combined with the bosonic counterpart (whose duality invariant form is given by (1) is
\[ S_F = \frac{i}{2\pi} \int d^2 \sigma (G_{ij} \bar{\psi}^i \gamma^\alpha \partial_\alpha \psi^j + G_{\mu \nu} \bar{\psi}^\mu \gamma^\alpha \partial_\alpha \psi^\nu). \] (10)

We have to put this action in duality invariant form such that, when combined with \( S_B \) given by eqn.(1), it gives the desired duality invariant superstring action. To write this action in duality invariant form we have to complete the fermionic dual multiplet
\[ \psi^a \equiv \left( \begin{array}{c} \psi^i \\ \bar{\psi}^i \end{array} \right). \] (11)

We now show that the fermionic action is duality invariant without introducing new fields. This is due to the fact that the fermionic action is already of the first order form. Similar results were used in Ref.[11] to write the supersymmetric version of the duality invariant Maxwell action. To implement it in the present case, we start with the usual supersymmetry transformation
\[ \delta_S X^\mu = \bar{\epsilon} \chi^\mu \ ; \delta_S \chi^\mu = -i \gamma^\alpha \partial_\alpha X^\mu \epsilon, \] (12)
\[ \delta_S y^i = \bar{\epsilon} \psi^i; \delta_S \psi^i = -i \gamma^{\hat{\alpha}} \partial^{\hat{\alpha}} y^i \epsilon. \]  
(13)

The transformations in eq.(12) remain unchanged under duality and we have to concentrate on eq.(13). We generalize the first transformation in (13) as

\[ \delta_S y^a = \bar{\epsilon} \psi^a. \]  
(14)

To generalize the second transformation of eq.(13) we write

\[ \delta_S \psi^a = -i \gamma^1 \partial_1 y^a \epsilon + i (ML)^a_b \gamma^0 \partial_1 y^b \epsilon. \]  
(15)

This equation has two components

\[ \delta_S \psi^i = -i \gamma^1 \partial_1 y^i \epsilon + i \gamma^0 G^{ij} \partial_1 \tilde{y}^j \]  
(16)

and

\[ \delta_S \tilde{\psi}_i = -i \gamma^1 \partial_1 \tilde{y}_i \epsilon + i \gamma^0 G_{ij} \partial_1 y^j. \]  
(17)

Eqn.(16) reduces to (13) by using equation of motion (2). We also find that the choice

\[ \tilde{\psi}_i = G_{ij} \gamma^5 \psi^j. \]  
(18)

is consistent with eq.(15). This explicit form for \( \tilde{\psi}_i \) and above duality invariant form of the susy transformation can now be used to cast the fermionic action in appropriate form. From eqn.(18)

\[ \tilde{\psi}_i = -G_{ij} \psi^j \gamma^5. \]  
(19)

We now write (10) using (18) or (19) as

\[ S_F = \frac{i}{4 \pi} \int d^2 \sigma (\bar{\psi}^{\hat{\alpha}} \gamma^\hat{\alpha} \gamma^5 \partial_{\hat{\alpha}} \psi_i + \bar{\psi}_i \gamma^{\hat{\alpha}} \gamma^5 \partial_{\hat{\alpha}} \psi^i + 2 G_{\mu \nu} \bar{\psi}^{\mu} \gamma^{\hat{\alpha}} \partial_{\hat{\alpha}} \psi^\nu). \]  
(20)

Here we have also used the fact that the gravitational background is flat. Using the matrix \( L \) from eqn.(3) we can write this action in the manifestly duality invariant form as

\[ S_{F1} = \frac{i}{2 \pi} \int d^2 \sigma \left( \frac{1}{2} L_{ab} \bar{\psi}^a \gamma^{\hat{\alpha}} \gamma^5 \partial^{\hat{\alpha}} \psi^b + G_{\mu \nu} \bar{\psi}^{\mu} \gamma^{\hat{\alpha}} \partial_{\hat{\alpha}} \psi^\nu \right). \]  
(21)

One could also write, using (18) and (19),

\[ S_F = \frac{i}{2 \pi} \int d^2 \sigma (G^{ij} \bar{\psi}_i \gamma^{\hat{\alpha}} \partial_{\hat{\alpha}} \psi_j + G_{\mu \nu} \bar{\psi}^{\mu} \gamma^{\hat{\alpha}} \partial_{\hat{\alpha}} \psi^\nu) \]  
(22)

which, when added with (10), has the duality invariant form

\[ S_{F2} = \frac{i}{2 \pi} \int d^2 \sigma \left( \frac{1}{2} (LML)_{ab} \bar{\psi}^a \gamma^{\hat{\alpha}} \partial_{\hat{\alpha}} \psi^b + G_{\mu \nu} \bar{\psi}^{\mu} \gamma^{\hat{\alpha}} \partial_{\hat{\alpha}} \psi^\nu \right). \]  
(23)

One notices the both of the forms, eqs.(21) and (23), of the fermionic action have usual Lorentz invariance of the original action.
We can also write the supercurrent, 

\[ J^\alpha = \frac{1}{2} \gamma^\beta \gamma^\alpha \psi^A \partial_\beta X^B G_{AB} \]

\[ = \frac{1}{2} \gamma^\beta \gamma^\alpha \psi^\mu \partial_\beta X^\nu g_{\mu\nu} + \frac{1}{2} \gamma^\beta \gamma^\alpha \psi^i \partial_\beta y^j G_{ij}, \]

in the duality invariant form (upto equations of motion) as,

\[ J^\alpha = \frac{1}{2} \gamma^\beta \gamma^\alpha \psi^\mu \partial_\beta X^\nu g_{\mu\nu} + \frac{1}{2} \gamma^\beta \gamma^\alpha \psi^a (LML)_{ab} \partial_\beta y^b. \]

(24)

Next we give a superfield formulation of the action. Since there is no explicit reparametrization invariance in the bosonic sector, it is absent in the superfield form as well. With the help of the following superderivatives,

\[ D_0 = \frac{\partial}{\partial \bar{\theta}} - i \gamma^0 \theta \partial_0; \quad D_1 = \frac{\partial}{\partial \theta} - i \gamma^1 \theta \partial_1; \]

(26)

and the chiral superfield

\[ Y^a = y^a + i \theta \psi^a \]

(27)

an appropriate superfield action can be written in the following form:

\[ S_F = -\frac{i}{4\pi} \int d^2 \sigma d^2 \theta [\bar{D}_1 Y^a (LML)_{ab} D_1 Y^b + L_{ab} \bar{D}_0 Y^a \gamma^5 D_1 Y^b]. \]

(28)

Notice that the superfield in this case is chiral, i.e., it does not contain the auxiliary field of the supersymmetry multiplet. The supersymmetry algebra closes off-shell and we have

\[ \{ \delta_{S1}, \delta_{S2} \} y^a = 2i \bar{\epsilon}_1 \gamma^1 \epsilon_2 \partial_1 y^a - 2i (ML)^a_b \bar{\epsilon}_1 \gamma^0 \epsilon_2 \partial_1 y^b. \]

(29)

4. Nonlinear \( \sigma \)-Models in Curved Backgrounds

In this section we start with a supersymmetric nonlinear sigma model in gravitational and antisymmetric tensor field backgrounds which are independent of some of the coordinates and discuss how to write down a manifestly duality invariant form of the action. We take the action given in Ref.[12, 13]

\[ S_{NSM} = -\frac{1}{2\pi} \int d^2 \sigma [ \quad g_{AB} \partial_\alpha X^A \partial_\hat{\beta} X^B - i g_{AB} \bar{\psi}^A (\Phi \psi)^B \]

\[ + b_{AB} \bar{\epsilon}_\beta \partial_\alpha X^A \partial_\beta X^B - i \partial_\alpha (b_{AB} \bar{\psi}^A \gamma^5 \hat{\psi}^B) \]

\[ + \frac{1}{8} R_{ABCD} \bar{\psi}^A (1 + \gamma^5) \psi^C \bar{\psi}^B (1 + \gamma^5) \psi^D \]

(30)

where

\[ (D_\hat{\beta} \psi)^A \equiv \partial_\alpha \psi^A + (\eta_{\hat{\alpha}} \Gamma^A_{BC} - \epsilon_{\hat{\alpha} \beta} S^A_{BC}) \partial_\beta X^B \psi^C, \]

(31)

and \( R_{ABCD}, \Gamma^A_{BC}, \) and \( S^A_{BC} \) are defined in Appendix A.
In the case when the backgrounds are independent of some of the coordinates \( y^i \)’s and depend only on the rest of the coordinates \( X^{\mu} \)’s, the background metric \( g_{AB} \) and the antisymmetric tensor field can be written in a block diagonal form (4). Also \( S_{ABC} \) and \( \Gamma^A_{BC} \) break into various parts out of which the only nonzero ones are,

\[
S_{\mu\nu\lambda} = \frac{1}{2} [\partial_{\mu} b_{\nu\lambda} + \partial_{\nu} b_{\lambda\mu} + \partial_{\lambda} b_{\mu\nu}]; \quad (32)
\]

\[
S_{\mu ij} = \frac{1}{2} \partial_{\mu} B_{ij}; \quad (33)
\]

\[
\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} [\partial_{\mu} g_{\sigma\nu} + \partial_{\nu} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\nu}]; \quad (34)
\]

\[
\Gamma_{ij}^{\mu} = -\frac{1}{2} g^{\mu\nu} \partial_{\nu} G_{ij}; \quad (35)
\]

\[
\Gamma_{i\mu j} = \frac{1}{2} G^{il} \partial_{\mu} G_{lj}. \quad (36)
\]

For this form of the background fields the total action can be written in a form which shows the internal and target space coordinate indices explicitly. This is necessary to find the duality invariant form. We start with the four fermion term in the following subsection.

4.1 FOUR FERMION TERMS

The four fermion terms of the action \( S_{ff} \) are given by

\[
S_{ff} \equiv \frac{1}{16\pi} \int d^2\sigma R_{ABCD}[\bar{\psi}^A (1 + \gamma^5) \psi^B][\bar{\psi}^B (1 + \gamma^5) \psi^D]. \quad (37)
\]

In general \( R_{ABCD} \) could give sixteen different terms because any of the four indices could take two values \( \mu \) or \( i \). It turns out that for our particular form of the background fields \( R_{ABCD} \) vanishes unless the number of internal (or space-time) indices is even. This means we get only the following eight different terms

\[
S_{ff} = \Sigma_{i=1}^{8} S_{ff}^{(i)} \quad (38)
\]

where \( S_{ff}^{(i)} \)’s are again given in Appendix A. We now consider each term of this action. First term is already duality invariant. We take the last term containing only internal fermions:

\[
S_{ff}^{(8)} = \frac{1}{16\pi} \int d^2\sigma \langle R_{ijkl} [\bar{\psi}^i (1 + \gamma^5) \psi^k][\bar{\psi}^j (1 + \gamma^5) \psi^l] \rangle. \quad (39)
\]

This can be written as

\[
S_{ff}^{(8)} = \Sigma_{i=1}^{8} \tilde{S}_i \quad (40)
\]

where explicit form of various \( \tilde{S} \)’s are given in Appendix A. Using the identity

\[
[\bar{\psi}^i (1 + \gamma^5) \psi^k][\bar{\psi}^j (1 + \gamma^5) \psi^l] = -[\bar{\psi}^i (1 + \gamma^5) \psi^l][\bar{\psi}^j (1 + \gamma^5) \psi^k] \quad (41)
\]

we find that

\[
\tilde{S}_1 = \tilde{S}_2, \tilde{S}_3 = \tilde{S}_4, \tilde{S}_5 = \tilde{S}_8, \tilde{S}_7 = \tilde{S}_6. \quad (42)
\]
Therefore we get
\[ S_{ff}^{(8)} = 2(\tilde{S}_1 + \tilde{S}_3 + \tilde{S}_5 + \tilde{S}_7). \] (43)

We now generalize eq. (18) for curved backgrounds,
\[ \bar{\tilde{\psi}}_i = G_{ij} \gamma^5 \psi^j + B_{ij} \psi^j. \] (44)

Also
\[ \bar{\tilde{\psi}}_i = -G_{ij} \bar{\psi}^j \gamma^5 + B_{ij} \bar{\psi}^j. \] (45)

Then after using the Fierz identities
\[ \bar{\psi} \chi = \bar{\chi} \psi, \]
\[ \bar{\psi} \gamma^5 \chi = -\bar{\chi} \gamma^5 \psi, \] (46)
we get,
\[ \tilde{S}_1 + \tilde{S}_3 + \tilde{S}_5 = \frac{1}{64\pi} \int d^2 \sigma \{ \partial \mu G_{ik}(\bar{\psi}^i \psi^k)[L_{ab} \bar{\psi}^a \gamma^5 \partial^\mu \psi_b]. \} \] (47)

Similarly
\[ \tilde{S}_3 + \tilde{S}_7 = \frac{1}{64\pi} \int d^2 \sigma \{ \partial \mu B_{ik}(\bar{\psi}^i \gamma^5 \psi^k)[L_{ab} \bar{\psi}^a \gamma^5 \partial^\mu \psi_b]. \} \] (48)

Adding (47) and (48) and using the procedure we adopted to obtain these equations we get the following completely duality invariant expression
\[ \tilde{S}_1 + \tilde{S}_3 + \tilde{S}_5 + \tilde{S}_7 = \frac{1}{64\pi} \int d^2 \sigma [L_{ab} \bar{\psi}^a \gamma^5 \partial^\mu \psi_b][L_{cd} \bar{\psi}^c \gamma^5 \partial^\mu \psi_d]. \] (49)

Thus the total contribution of \( S_{ff}^{(8)} \) is
\[ S_{ff}^{(8)} = \frac{1}{32\pi} \int d^2 \sigma \{ [L_{ab} \bar{\psi}^a \gamma^5 \partial^\mu \psi_b][L_{cd} \bar{\psi}^c \gamma^5 \partial^\mu \psi_d]. \}. \] (50)

We now return to other terms in the action \( S_{ff} \). Using relations similar to eq. (11) we combine the following terms
\[ S_{ff}^{(3)} + S_{ff}^{(4)} + S_{ff}^{(6)} + S_{ff}^{(7)} \]
\[ = \frac{1}{16\pi} \int d^2 \sigma (\mathcal{R}_{\mu ij} - \mathcal{R}_{ij\mu} + \mathcal{R}_{ij\nu} - \mathcal{R}_{ij\nu}) \]
\[ [\bar{\psi}^\mu (1 + \gamma^5) \psi^\nu][\bar{\psi}^\nu (1 + \gamma^5) \psi^\mu]. \] (51)

The background dependent coefficients can be computed using the expressions (32)-(10) and (A.24). These are given in eqs. (A.24) of Appendix A. Using these it is straightforward to show that
\[ S_{ff}^{(3)} + S_{ff}^{(4)} + S_{ff}^{(6)} + S_{ff}^{(7)} \]
\[ = \frac{1}{16\pi} \int d^2 \sigma \{ [\bar{\psi}^\mu (1 + \gamma^5) \psi^\nu][\partial \nu (LML)_{ab} \bar{\psi}^a \gamma^5 \partial^\mu \psi_b] \]
\[ + \frac{1}{2} \Gamma_{\mu \nu \rho} [\bar{\psi}^\mu \psi^\nu][L_{ab} \bar{\psi}^a \gamma^5 \partial_\alpha \psi_b] \]
\[ - \frac{1}{2} S_{\alpha \mu \nu}[\bar{\psi}^{\mu \gamma^5 \nu}] [L_{ab} \bar{\psi}^a \gamma^5 \partial_\alpha \psi_b] \]
\[ - \frac{1}{2}[\bar{\psi}^\mu (1 + \gamma^5) \psi^\nu][L_{ab} \bar{\psi}^a \gamma^5 \partial_\alpha \partial_\nu \psi_b]. \] (52)
Finally we have to consider the remaining two four fermion terms

\[
S_{ff}^{(2)} + S_{ff}^{(5)} = - \frac{1}{16\pi} \int d^2\sigma \{ R_{\mu\nu ij} [\bar{\psi}^i(1 + \gamma^5)\psi^j][\bar{\psi}^i(1 + \gamma^5)\psi^j] \\
+ R_{ij\mu\nu} [\bar{\psi}^i(1 + \gamma^5)\psi^j][\bar{\psi}^i(1 + \gamma^5)\psi^j]\}
\]  

(53)

where \( R_{\mu\nu ij} \) and \( R_{ij\mu\nu} \) can be calculated using \((A.23)\). Again repeating the familiar steps we arrive at

\[
S_{ff}^{(2)} + S_{ff}^{(5)} = \frac{1}{32\pi} \int d^2\sigma (LML)_{ab} [ (\bar{\psi}^\mu \gamma^5 \partial_\nu \psi^a) (\bar{\psi}^\mu \gamma^5 \partial_\nu \psi^b) - (\bar{\psi}^\mu \gamma^5 \partial_\nu \psi^a) (\bar{\psi}^\mu \gamma^5 \partial_\nu \psi^b) \\
+(\bar{\psi}^\mu \partial_\nu \psi^a)(\bar{\psi}^\mu \partial_\nu \psi^b) - (\bar{\psi}^\mu \partial_\nu \psi^a)(\bar{\psi}^\mu \partial_\nu \psi^b)].
\]

(54)

We have thus shown that all the four fermion terms can be written in duality invariant form.

4.2 KINETIC TERM FOR Fermions

We now discuss the kinetic term for fermions

\[
S_{KE}^{f} = - \frac{i}{2\pi} \int d^2\sigma \{ G_{\mu\nu} \bar{\psi}^\mu (\mathcal{D}\psi)^\nu + G_{ij} \bar{\psi}^i (\mathcal{D}\psi)^j \},
\]

(55)

where,

\[
(D_\alpha \psi)^\mu = \partial_\alpha \psi^\mu + \eta_{\alpha\beta} \Gamma_{\mu\nu}^\alpha \partial^\beta X^\nu \psi^\mu + \eta_{\alpha\beta} \Gamma_{ij}^\mu \partial^\beta \bar{\psi}^i \psi^j \\
-\epsilon_{\alpha\beta} S_{\nu\lambda}^\mu \partial^\beta X^\nu \psi^\lambda - \epsilon_{\alpha\beta} S_{ij}^\mu \partial^\beta \bar{\psi}^i \psi^j
\]

and

\[
(D_\alpha \psi)^i = \partial_\alpha \psi^i + \eta_{\alpha\beta} \Gamma_{ij}^\alpha \partial^\beta X^\mu \psi^j + \eta_{\alpha\beta} \Gamma_{ij}^\mu \partial^\beta \bar{\psi}^i \psi^j \\
-\epsilon_{\alpha\beta} G^{ij} S_{\mu\nu}^i \partial^\beta y^\mu \psi^\nu - \epsilon_{\alpha\beta} G^{ij} S_{\mu\nu}^i \partial^\beta X^\mu \psi^\nu
\]

(57)

We write \((53)\) as

\[
S_{KE}^{f} = S_{KE}^{f0} + S_{KE}^{f1} + S_{KE}^{f2} + S_{KE}^{f3} + S_{KE}^{f4}
\]

(58)

such that \( S_{KE}^{f0} \) is the part unaffected by duality transformations. The other terms are written as follows.

\[
S_{KE}^{f1} = - \frac{i}{2\pi} \int d^2\sigma \{ G_{\mu\nu} \bar{\psi}^\mu \gamma^\alpha (\eta_{\alpha\beta} \Gamma_{ij}^\nu \partial^\beta y^i \psi^j - \epsilon_{\alpha\beta} G^{\nu\lambda} S_{\lambda ij} \partial^\beta y^i \psi^j)\}
\]

(59)

\[
S_{KE}^{f2} = - \frac{i}{2\pi} \int d^2\sigma \{ G_{ij} \bar{\psi}^i \gamma^\alpha (\eta_{\alpha\beta} \Gamma_{\mu k}^j \partial^\beta X^\mu \psi^k - \epsilon_{\alpha\beta} G^{ij} S_{k\mu} \partial^\beta X^\mu \psi^k)\}
\]

(60)

\[
S_{KE}^{f3} = - \frac{i}{2\pi} \int d^2\sigma \{ \psi^i \gamma^\alpha G_{ij} (\eta_{\alpha\beta} \Gamma_{\mu j}^i \partial^\beta y^i - \epsilon_{\alpha\beta} G^{ij} S_{\mu ij} \partial^\beta y^i)\}
\]

(61)

\[
S_{KE}^{f4} = - \frac{i}{2\pi} \int d^2\sigma \{ G_{ij} \bar{\psi}^i \gamma^\alpha \partial_\alpha \psi^j \}
\]

(62)
We start with $S_{KE}^{f_1}$ for which we find the following duality invariant form

$$S_{KE}^{f_1} = -\frac{i}{4\pi} \int d^2\sigma \{ \bar{\psi}^{\mu} \gamma^5 \gamma^\alpha \eta_{\alpha\beta} L_{ab} \partial^\beta y^a \partial_\mu \psi^b \}.$$

(63)

Next we take $S_{KE}^{f_2}$. Using the Fierz identities one finds that first term of eq. (61) vanishes. We combine its second term with $S_{KE}^{f_4}$ to write the following duality invariant expression

$$S_{KE}^{f_2} + S_{KE}^{f_4} = -\frac{i}{4\pi} \int d^2\sigma \{ L_{ab} \bar{\psi}^a \gamma^5 \gamma^\alpha \partial_\alpha \psi^b \}.$$

(64)

Another equivalent duality invariant form for these terms is

$$S_{KE}^{f_2} + S_{KE}^{f_4} = -\frac{i}{4\pi} \int d^2\sigma \{ (LML)_{ab} \bar{\psi}^a \gamma^5 \gamma^\alpha \partial_\alpha \psi^b \}.$$

(65)

Finally the term $S_{KE}^{f_3}$ can be put into the following form

$$S_{KE}^{f_3} = -\frac{i}{4\pi} \int d^2\sigma \{ \eta_{\alpha\beta} L_{ab} \bar{\psi}^a \gamma^5 \gamma^\alpha \partial_\beta y^b \gamma^\mu \psi^b \}.$$

(66)

Thus we have obtained the duality invariant form of the total action.

2.3 SUPERSYMMETRY TRANSFORMATIONS

Finally we discuss some issues related to supersymmetry transformations. The usual supersymmetric transformations for the nonlinear sigma models in arbitrary backgrounds are

$$\delta_S X^A = \bar{\epsilon} \psi^A,$$

$$\delta_S \psi^A = -i \gamma^\alpha \partial_\alpha X^A \epsilon + \frac{1}{2} \epsilon (\Gamma^A_{BC} \bar{\psi}^B \psi^C + S^A_{BC} \bar{\psi}^B \gamma^5 \psi^C) \ .$$

(67)

These transformations are separated into internal and external coordinates and for the bosonic part we have, as before

$$\delta_S X^\mu = \bar{\epsilon} \psi^\mu; \quad \delta_S y^a = \bar{\epsilon} \psi^a$$

(68)

and for $\psi^\mu$ we have

$$\delta_S \psi^\mu = -i \gamma^\alpha \partial_\alpha X^\mu \epsilon + \frac{1}{2} \epsilon (\Gamma^\mu_{\nu\lambda} \bar{\psi}^{\nu \lambda} \psi^\mu + S^\mu_{\nu\lambda} \bar{\psi}^{\nu \lambda} \gamma^5 \psi^\mu)
- \frac{1}{4} G^{\mu\nu} (\bar{\psi}^i \psi^j \partial_\nu G_{ij} + \bar{\psi}^i \gamma^5 \psi^j \partial_\nu B_{ij})$$

(69)

which can be written in the following duality invariant form

$$\delta_S \psi^\mu = -i \gamma^\alpha \partial_\alpha X^\mu \epsilon + \frac{1}{2} \epsilon (\Gamma^\mu_{\nu\lambda} \bar{\psi}^{\nu \lambda} \psi^\mu + S^\mu_{\nu\lambda} \bar{\psi}^{\nu \lambda} \gamma^5 \psi^\mu) - \frac{1}{4} G^{\mu\nu} L_{ab} \bar{\psi}^a \gamma^5 \partial_\nu \psi^b.$$

(70)
Next we consider the supersymmetry transformations for the internal fermions. The usual transformation, which follows from eq. (67), is

$$\delta S \psi^i = -i \gamma^\alpha \partial_\alpha y^i \epsilon + \frac{1}{2} G^{ij} \bar{\psi}^\mu \gamma^5 \partial_\mu \psi^j.$$  

(71)

Again the bosonic term in this transformation can be modified using $\bar{y}$ equation of motion (5). When combined with the corresponding transformation for $\bar{\psi}$ the modified supersymmetry transformation for internal fermions can be written in the following duality invariant form:

$$\delta S \psi^a = -i \gamma^1 \partial_1 y^a \epsilon + i \gamma^0 (ML)^a b \partial_1 y^b \epsilon + \frac{1}{2} (ML)^{ab} \bar{\psi}^\mu \gamma^5 \partial_\mu \bar{\psi}^b.$$  

(72)

5. Conclusions

In this article we gave duality invariant formulations of worldsheet actions for superstrings in flat as well as curved backgrounds. We found that, in both cases, the actions have a modified Lorentz invariance and supersymmetry. On-shell these are equivalent to to the usual symmetries. We encountered a novel situation in which the supersymmetry is off-shell closed. We also found the duality invariant forms for the supercurrent, supersymmetric transformations and the equations of motion.

It will be very interesting to discuss the quantization of these duality invariant actions. In particular, it should be verified that the critical dimensions remains ten even though we have introduced new ‘auxiliary’ fields as in case of bosonic strings.

Another problem is to write the duality invariant action for the more interesting case of heterotic strings including the fermions. We hope to return to these questions in near future.

ACKNOWLEDGEMENTS

I am thankful to Alok Kumar for suggesting the problem, discussions, and reading the manuscript. I also thank Ashoke Sen for numerous discussions and suggestions.
Appendix A

In this Appendix we collect some of the notations and definitions. First of all we need the following definitions for the four fermion term of the supersymmetric nonlinear sigma model:

\[ \mathcal{R}_{ABCD} = \mathcal{R}_{ABCD} + \mathcal{S}_{ABCD}, \]  
\[ \mathcal{S}_{ABCD} = \Sigma_{ABCD} + T_{ABCD}, \]
\[ R_{ABCD} = g_{AE}[\partial_C \Gamma^E_{BD} - \partial_D \Gamma^E_{BC} + \Gamma^E_{BD} \Gamma^F_{FC} - \Gamma^E_{BC} \Gamma^F_{FD}], \]
\[ \Gamma^A_{BC} = \frac{1}{2} g^{AD}[\partial_B g_{DC} + \partial_D g_{BC} - \partial_D g_{BC}], \]
\[ S_{ABC} = \frac{1}{2} [\partial_A b_{BC} + \partial_C b_{AB} + \partial_B b_{CA}], \]
\[ \Sigma_{ABCD} = S_{AEC} S^E_{BD} - S_{AED} S^E_{BC}, \]
\[ T_{ABCD} = g_{AE}[S^E_{BC,D} - S^E_{BD,C}]. \]

Next when separated into compact and noncompact indices we get the following eight different terms in the four fermion part of the action

\[ S^{(1)}_{ff} = -\frac{1}{16\pi} \int d^2 \sigma \{ \mathcal{R}_{\mu \nu \alpha \beta} [\bar{\psi}^\mu (1 + \gamma^5) \psi^\alpha][\bar{\psi}^\nu (1 + \gamma^5) \psi^\beta] \}, \]
\[ S^{(2)}_{ff} = -\frac{1}{16\pi} \int d^2 \sigma \{ \mathcal{R}_{\mu \nu \rho \sigma} [\bar{\psi}^\mu (1 + \gamma^5) \psi^\rho][\bar{\psi}^\nu (1 + \gamma^5) \psi^\sigma] \}, \]
\[ S^{(3)}_{ff} = -\frac{1}{16\pi} \int d^2 \sigma \{ \mathcal{R}_{\mu \nu \rho \sigma} [\bar{\psi}^\mu (1 + \gamma^5) \psi^\rho][\bar{\psi}^\mu (1 + \gamma^5) \psi^\sigma] \}, \]
\[ S^{(4)}_{ff} = -\frac{1}{16\pi} \int d^2 \sigma \{ \mathcal{R}_{\mu \nu \rho \sigma} [\bar{\psi}^\mu (1 + \gamma^5) \psi^\rho][\bar{\psi}^\mu (1 + \gamma^5) \psi^\sigma] \}, \]
\[ S^{(5)}_{ff} = -\frac{1}{16\pi} \int d^2 \sigma \{ \mathcal{R}_{\mu \nu \rho \sigma} [\bar{\psi}^\mu (1 + \gamma^5) \psi^\rho][\bar{\psi}^\mu (1 + \gamma^5) \psi^\sigma] \}, \]
\[ S^{(6)}_{ff} = -\frac{1}{16\pi} \int d^2 \sigma \{ \mathcal{R}_{\mu \nu \rho \sigma} [\bar{\psi}^\mu (1 + \gamma^5) \psi^\rho][\bar{\psi}^\mu (1 + \gamma^5) \psi^\sigma] \}, \]
\[ S^{(7)}_{ff} = -\frac{1}{16\pi} \int d^2 \sigma \{ \mathcal{R}_{\mu \nu \rho \sigma} [\bar{\psi}^\mu (1 + \gamma^5) \psi^\rho][\bar{\psi}^\mu (1 + \gamma^5) \psi^\sigma] \}, \]
\[ S^{(8)}_{ff} = -\frac{1}{16\pi} \int d^2 \sigma \{ \mathcal{R}_{\mu \nu \rho \sigma} [\bar{\psi}^\mu (1 + \gamma^5) \psi^\rho][\bar{\psi}^\mu (1 + \gamma^5) \psi^\sigma] \}. \]

\( S^{(8)}_{ff} \) has the following parts

\[ \tilde{S}_1 = \frac{1}{16\pi} \int d^2 \sigma \{ G_{ip} \Gamma^p_{jl} \Gamma^p_{jk} [\bar{\psi}^i (1 + \gamma^5) \psi^j][\bar{\psi}^j (1 + \gamma^5) \psi^l] \}, \]
\[ \tilde{S}_2 = -\frac{1}{16\pi} \int d^2 \sigma \{ G_{ip} \Gamma^p_{jl} [\bar{\psi}^i (1 + \gamma^5) \psi^j][\bar{\psi}^j (1 + \gamma^5) \psi^l] \}, \]
\[ \tilde{S}_3 = -\frac{1}{16\pi} \int d^2 \sigma \{ S_{ipkl} S^p_{jl} [\bar{\psi}^i (1 + \gamma^5) \psi^j][\bar{\psi}^j (1 + \gamma^5) \psi^l] \}. \]
\[ \tilde{S}_4 = \frac{1}{16\pi} \int d^2\sigma \{ S_{\mu\nu} S_{\rho\sigma} \} \{ \bar{\psi}^i (1 + \gamma^5) \psi^j \} \}, \quad (A.19) \]
\[ \tilde{S}_5 = -\frac{1}{16\pi} \int d^2\sigma \{ G_{\mu\nu} \Gamma_{\rho\sigma} \} \{ \bar{\psi}^i (1 + \gamma^5) \psi^j \} \}, \quad (A.20) \]
\[ \tilde{S}_6 = -\frac{1}{16\pi} \int d^2\sigma \{ G_{\mu\nu} \Gamma_{\rho\sigma} \} \{ \bar{\psi}^i (1 + \gamma^5) \psi^j \} \}, \quad (A.21) \]
\[ \tilde{S}_7 = \frac{1}{16\pi} \int d^2\sigma \{ G_{\mu\nu} \Gamma_{\rho\sigma} \} \{ \bar{\psi}^i (1 + \gamma^5) \psi^j \} \}, \quad (A.22) \]
\[ \tilde{S}_8 = \frac{1}{16\pi} \int d^2\sigma \{ G_{\mu\nu} \Gamma_{\rho\sigma} \} \{ \bar{\psi}^i (1 + \gamma^5) \psi^j \} \}, \quad (A.23) \]

We need the following expressions to write (51) in duality invariant form

\[ R_{\mu \nu ij} = -R_{\mu ij \nu} = R_{ij \mu \nu} = -R_{ij \nu \mu} = \]
\[ \frac{1}{2} \Gamma_{\mu \nu} \partial \alpha G_{ij} - \frac{1}{2} \partial \mu \partial \nu G_{ij} + \frac{1}{4} G_{lm} \partial \mu G_{mj} \partial \nu G_{il} \]
\[ \Sigma_{\mu \nu ij} = -\Sigma_{\mu ij \nu} = \Sigma_{ij \mu \nu} = -\Sigma_{ij \nu \mu} = \]
\[ \frac{1}{2} G_{lm} \partial \mu B_{mj} \partial \nu B_{il} - \frac{1}{2} S_{\mu \nu} \partial \alpha B_{ij} \]
\[ T_{\mu \nu ij} = -T_{\mu ij \nu} = T_{ij \mu \nu} = -T_{ij \nu \mu} = \]
\[ -\frac{1}{2} \partial \mu \partial \nu B_{ij} - \frac{1}{2} S_{\mu \nu} \partial \alpha G_{ij} \]
\[ + \frac{1}{2} \Gamma_{\mu \nu} \partial \alpha B_{ij} + \frac{1}{4} G_{lm} (\partial \mu G_{mj} \partial \nu B_{il} + \partial \nu G_{il} \partial \mu B_{mj})). \quad (A.24) \]

To put (53) in duality invariant form we need

\[ R_{\mu \nu ij} = R_{ij \mu \nu} = \frac{1}{4} G_{lm} (\partial \mu G_{mj} \partial \nu G_{il} - \partial \mu G_{im} \partial \nu G_{lj}), \]
\[ \Sigma_{\mu \nu ij} = \Sigma_{ij \mu \nu} = \frac{1}{4} G_{lm} (\partial \mu B_{mj} \partial \nu B_{il} - \partial \nu B_{im} \partial \mu B_{lj}), \]
\[ T_{\mu \nu ij} = -T_{ij \mu \nu} = \frac{1}{4} G_{lm} (-\partial \mu G_{mj} \partial \nu B_{il} + \partial \nu G_{im} \partial \mu B_{lj} \]
\[ - \partial \nu B_{im} \partial \mu G_{lj} + \partial \mu B_{im} \partial \nu G_{lj})). \quad (A.25) \]
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