Stability of differential-difference parabolic system with distributed parameters on the graph

V V Provotorov\textsuperscript{1} and A P Zhabko\textsuperscript{2}

\textsuperscript{1}Departament of Mathematics, Voronezh State University, 1, University Square, Voronezh, 394018, Russia
\textsuperscript{2}Department of Applied Mathematics and Control Processes, Saint-Petersburg State University, 7-9, University Quay, St.Petersburg, 199034, Russia

E-mail: vvprov@mail.ru

Abstract. The work presents the results of the analysis of the approximation of a differential system with distributed parameters on the network domain. This approximation consider in a space of summable functions with a compact bearer on network domain. The question is whether the differential system retains its main properties when its approximation – uniqueness weak solvability, continuity according to the input data, stability on Lyapunov and in what cases (under what conditions) it is possible to replace the analysis of the differential system by analysis of its differential-difference analogue. The study uses a simplified bearer of functions – graph, which is a private case of a network domain (the graph’s edges are parameterized by a one-dimensional spatial variable that changes on the graph). On the set of these functions study the differential-difference equation. This simplification only frees us from the unnecessary routine technical work, where the function bearer is an arbitrary the network domain of multi-dimensional Euclidean space, but as it does not affect on the transfer of ideas and results presented in the work to this multidimensional event. The transition from a differential system to a differential-difference analogue (differential-different system) is carried out using the method of finite difference on the time variable (Rote semi-digitization method). The conditions of weak uniquely solvability of differential-difference system and continuity of the input data are guaranteed the stability of a weak solution of differential-difference system. These conditions guarantee the stability of a weak solution of differential-difference system.

1. Introduction

The main results of the work be relate to the study of differential-difference system with distributed parameters on the graph. The universal method for this is the method of finite differences, but when the number of space variables increases this method is poorly applied due to the complexity of analysis of difference schemes. In the work shows under what conditions the differential-difference system inherits the properties of the initial differential system. The research tool for this are a priori estimate of weak solutions differential-difference system. The obtained results open the way to analyze distinct types of problems of optimal control network-like process (distributed, start, boundary controls), stabilization of weak solving and problems with a lag time for the initial differential system by means of consideration the same tasks for the differential-difference analogue of this system. It should be noted that the analysis of the stability of solutions of ordinary differential equations uses reduction to differential equations.
2. Materials and methods

Concepts and notations adopted in the works [4] are used. Denote by $\Gamma$ a bounded oriented geometric graph with edges $\gamma$ that are parameterized by a segment $[0, 1]$; $\partial \Gamma$, $J(\Gamma)$ are a sets boundary and interior nodes of the graph, respectively; $\Gamma_0$ is combination all edges that do not contain endpoints; $\Gamma_t = \Gamma_0 \times (0, t)$, $\partial \Gamma_t = \partial \Gamma \times (0, t)$, $t \leq T < \infty$, $T$ is fixed constant.

Standard notations are used for space of summable functions [5]: $L_p(\Gamma)$ ($p = 1, 2$) is the space measurable functions on $\Gamma_0$ summarized with a degree, the spaces $L_p(\Gamma_T)$ defined by analogy with $L_p(\Gamma)$; $L_{2,1}(\Gamma_T)$ is space functions $u(x, t)$ from $L_1(\Gamma_T)$, $\|u\|_{2,1,\Gamma_T} = \int_0^T (\int_\Gamma u^2(x, t)dx)^{1/2}dt$; $W^1_2(\Gamma)$ is the space functions from $L_2(\Gamma)$ having a generalized first-order derivative from $L_2(\Gamma)$; $W^2_1(\Gamma)$ is the space functions $u \in L_2(\Gamma_T)$ that have a generalized derivative $u_x$, also belonging to $L_2(\Gamma_T)$ (the space $W^1_2(\Gamma_T)$ is introduced similarly).

Let $V_2(\Gamma_T)$ is a set of all the functions $u(x, t)$ of space $W^{1,0}_2(\Gamma_T)$ that have the finite norm

$$\|u\|_{2,\Gamma_T} = \max_{0 \leq t \leq T} \|u(\cdot, t)\|_{L_2(\Gamma)} + \|u_x\|_{L_2(\Gamma_T)},$$

(1)

the functions $u(x, t)$ is continuous by $t$ in the norm space $L_2(\Gamma)$.

Let’s consider a differential equation regarding function $u(x, t)$ in a network domain $\Gamma_T$

$$\frac{\partial u(x, t)}{\partial t} - \frac{\partial}{\partial x} \left( a(x) \frac{\partial u(x, t)}{\partial x} \right) + b(x)u(x, t) = f(x, t), \quad x, t \in \Gamma_T. \quad (2)$$

For this equation below, the corresponding differential-difference analogue will be investigated. Here $a(x)$, $b(x)$ are fixed functions of the space $L_2(\Gamma)$:

$$0 < a_* \leq a(x) \leq a^*, \quad |b(x)| \leq \beta, \quad x \in \Gamma_0, \quad (3)$$

Ratio (2) is a system of differential parabolic equations with distributed parameters $a(x)$, $b(x)$ on each edge $\gamma$ of the graph $\Gamma$.

For the system (2) we will introduce the space of state $u(x, t)$ and auxiliary spaces. In the $W^2_1(\Gamma)$ space $W^2_1(\Gamma_T)$ consider the bilinear form

$$\ell(\mu, \nu) = \int_\Gamma \left( a(x) \frac{\partial u(x, t)}{\partial x} \frac{\partial \mu(x)}{\partial x} + b(x)\mu(x)\nu(x) \right) dx.$$

We will use the following statement [4, 6].

**Lemma.** Let the function $u(x) \in W^1_2(\Gamma)$ be such that $\ell(u, \nu) - \int_\Gamma f(x)\eta(x)dx = 0$ for any $\eta(x) \in W^2_1(\Gamma)$ ($f(x) \in L_2(\Gamma)$ is fixed function). Then narrowing $a(x)\frac{\partial u(x)}{\partial x} \rightarrow$ continuously at the endpoints of the arbitrary edge $\gamma \subset \Gamma$ ($f(\cdot)$, $\gamma$ is narrowing the function $f(\cdot)$ on the edge $\gamma$).

Let $\Omega_0(\Gamma)$ is a set of functions $u(x)$ that satisfy the conditions of Lemma and ratios

$$\sum_{\gamma \in R(\xi)} a(1)_{\gamma} \frac{\partial u(1}_{\partial x} = \sum_{\gamma \in r(\xi)} a(0)_{\gamma} \frac{\partial u(0)}{\partial x}$$

in all nodes $\xi \in J(\Gamma)$ (here $R(\xi)$ and $r(\xi)$ are sets of edges, respectively oriented “to node $\xi$” and “from node $\xi$”). Closing the set $\Omega_0(\Gamma)$ in the norm space $W^2_1(\Gamma)$ will designate through $W^1(\Gamma; \Gamma)$. At the same time, if the functions $u(x) \in \Omega_0(\Gamma)$ satisfy a homogeneous boundary condition $u(x)|_{\partial \Gamma} = 0$, then we get space $W^1_0(\Gamma)$. [1–3]. In the paper presented, similar ideas are applied to equations with private derivatives.
Let further $\Omega_a(\Gamma_T)$ is a set of functions $u(x, t) \in V_2(\Gamma_T)$ for which exit traces defined on sections of the domain $\Gamma_T$ by planes $t = t_0 \ (t_0 \in [0, T])$ as functions of space $W^1_0(a; \Gamma)$ and let these functions satisfy the ratios

$$\sum_{\gamma \in r(\xi)} a(1) \frac{\partial u(1, t)}{\partial x} = \sum_{\gamma \in r(\xi)} a(0) \frac{\partial u(0, t)}{\partial x} \quad (4)$$

for all nodes $\xi \in J(\Gamma)$. Closing the set $\Omega_a(\Gamma_T)$ by the norm (1) will denote by $V^{1,0}(a; \Gamma_T)$: $V^{1,0}(a; \Gamma_T) \subset W^{1,0}_2(\Gamma_T)$. Let’s introduce another subspace $W^{1,0}(a; \Gamma_T)$ of space $W^{2,0}_2(\Gamma_T)$: $W^{1,0}(a; \Gamma_T)$ is the closure is norm $W^{1,0}_2(\Gamma_T)$ by set continuously differentiable functions in domain $\Gamma_T$ that satisfy ratios (4) in all nodes $\xi \in J(\Gamma)$ at any $t \in [0, T]$ and zero on $\partial \Gamma_T$ (space $W^{1,0}(a; \Gamma_T)$ is introduced analogously): $W^{1,0}(a; \Gamma_T) \subset W^{1,0}_2(\Gamma_T)$.

Space $V^{1,0}(a; \Gamma_T)$ is the space of the state of the system (2), $W^{1,0}(a; \Gamma_T)$, $W^{1}(a; \Gamma_T)$ are auxiliary spaces. Note in addition, that the elements $u(x, t)$ of space $V^{1,0}(a; \Gamma_T)$ are continuous functions by variable $t$.

Next, let’s look at the parabolic system (2), the state of which $(x, t) \in \Gamma_T$ is determined by a weak solution $u(x, t)$ of the equation (2) et first in space $W^{1,0}(a; \Gamma_T)$, then in $V^{1,0}(a; \Gamma_T)$. In addition $u(x, t)$ satisfy the initial and boundary conditions

$$u |_{t=0} = \varphi(x), \quad u |_{x \in \partial \Gamma_T} = 0. \quad (5)$$

Here are the main statements in space $W^{1,0}(a; \Gamma_T)$, next in space $V^{1,0}(a; \Gamma_T)$, full proofs of these statements represent in the paper [4].

Let conditions take place (3) and $f(x, t) \in L_{2,1}(\Gamma_T)$, $\varphi(x) \in L_{2}(\Gamma)$. Let’s introduce a bilinear form

$$\ell_t(u, \eta) = \int_{\Gamma_T} \left( a(x) \frac{\partial u(x, t)}{\partial x} \frac{\partial \eta(x, t)}{\partial x} + b(x) u(x, t) \eta(x, t) \right) dx \ dt \quad (0 \leq t \leq T)$$

and identify a weak solution of the initial boundary value problem (2), (5) in the spaces $W^{1,0}(a; \Gamma_T)$ and $V^{1,0}(a; \Gamma_T)$ in the following.

**Definition 1.** A weak solution of a problem (2), (5) is a function $u(x, t)$, that belongs to space $W^{1,0}(a; \Gamma_T)$ and satisfies an integral identity

$$- \int_{\Gamma_T} u(x, t) \frac{\partial \eta(x, t)}{\partial t} dx \ dt + \ell_t(u, \eta) = \int_{\Gamma} \varphi(x) \eta(x, 0) dx + \int_{\Gamma_T} f(x, t) \eta(x, t) dx \ dt$$

for any function $\eta(x, t) \in W^{1}(a; \Gamma_T)$ that is zero at $t = T$.

**Definition 2.** A weak solution of a problem (2), (5) is a function $u(x, t)$ that belongs to space $V^{1,0}(a; \Gamma_T)$ and satisfies an integral identity

$$\int_{\Gamma} u(x, t) \eta(x, t) dx - \int_{\Gamma} u(x, t) \frac{\partial \eta(x, t)}{\partial t} dx \ dt + \ell_t(u, \eta) = \int_{\Gamma} \varphi(x) \eta(x, 0) dx + \int_{\Gamma_T} f(x, t) \eta(x, t) dx \ dt \quad (6)$$

for any $t \in [0, T]$ and for any function $\eta(x, t) \in W^{1}(a; \Gamma_T)$.

In the work [4] presents a analysis of the stability on Lyapunov of the weak solution of equation (2) in the domain $\Gamma_\infty = \Gamma_0 \times (0, \infty)$. Using the Faedo-Galerkin method with a special basis, conditions were obtained to guarantee an uniqueness weak resolution of the problem (2), (5) in space $V^{1,0}(a; \Gamma_T)$ at any $T < \infty$. In the capacity a special basis was taken a set of generalized eigenfunctions of the elliptical operator $\Delta u = - \frac{d}{dx} \left( a(x) \frac{du}{dx} \right) + b(x) u(x)$ in the space $W^0_0(a; \Gamma)$. Under conditions (3) generalized eigenfunctions were nontrivial weak solutions.
of the spectral problem $\Lambda \phi = \lambda \phi$, $\phi|_{\partial \Gamma} = 0$ in a weak formulation. Namely, each eigenvalue $\lambda$ determines at least one weak solution $\phi(x) \in W_0^1(a; \Gamma)$ that satisfies the identity $\ell(\phi, \nu) = \lambda(\phi, \nu)$ in any function $\nu(x) \in W_0^1(a; \Gamma)$.

The operator $\Lambda$ has the following properties.

1. The spectrum of operator $\Lambda$ is discrete; the eigenvalues are real and have a finite multiplicity; eigenvalues can be numbered with regard to the multiplicity in ascending order their modules: $\{\lambda_i\}_{i \geq 1}$; generalized eigenfunctions are numbered respectively to their eigenvalues: $\{\phi_i(x)\}_{i \geq 1}$.

2. Numbers $\lambda_i$, $i = 1, 2, \ldots$, positive, with the exception for the finite number of the first; if $b(x) \geq 0$, then $\lambda_i > 0$, $i = 1, 2, \ldots$.

3. A set $\{\phi_i(x)\}_{i \geq 1}$ make up the orthogonal bases in space $W_0^1(a; \Gamma)$ and space $L_2(\Gamma)$, where $\|\phi_i\|_{L_2(\Gamma)} = 1$.

Theorem 1. Let $f(x) \in L_{2,1}(\Gamma_T)$, $\varphi(x) \in L_2(\Gamma)$, then the initial boundary value problem (2), (5) weak solvable in space $W^{1,0}(a; \Gamma_T)$ for any $0 < T < \infty$. Proof of theorem uses Faedo-Galerkin’s approximation on spectral bases $\{\phi_i(x)\}_{i \geq 1}$ [1]: approximate solutions $u^N(x, t)$ initial boundary value problem (2), (5) have a form

$$u^N(x, t) = \sum_{i=1}^{N} c_i^N(t) \phi_i(x),$$

where $N$ – fixed natural number, $c_i^N(t)$ are functions whose derivatives $\frac{dc_i^N(t)}{dt}$ belong $L_2(0, T)$ (functions that are absolutely continuous on $[0, T]$) and defined from a linear system

$$\left( \frac{\partial N}{\partial t}, \phi_i \right) + \int_\Gamma \left( a(x) \frac{\partial N(x, t)}{\partial x} \frac{\partial \phi_i}{\partial x} + b(x) u^N(x, t) \phi_i(x) \right) dx = (f, \phi_i),$$

$$c_i^N(0) = (\varphi, \phi_i), \ i = 1, N.$$

Further reasoning uses a priori estimates of the norms of approximate weak solutions $u^N(x, t)$ of the initial boundary value problem (2), (5) and their bounded for arbitrary $N$. Then it is possible to distinguish the sequence $\{u_i^N\}_{k \geq 1}$ from the sequence $\{u_i^N\}_{N \geq 1}$, which is weak convergent to solution $u(x, t) \in W^{1,0}(a, \Gamma_T)$ in the norm $W^{1,0}(\Gamma_T)$ (weak sequence compactness $\{u_i^N\}_{k \geq 1}$).

For simplicity, a few narrow space $L_{2,1}(\Gamma_T)$. Denote through $CL_{2,1}(\Gamma_T) \subset L_{2,1}(\Gamma_T)$ the space functions of $L_{2,1}(\Gamma_T)$, continuous on $t$ in the norm $L_2(\Gamma)$ and will assume that $f(x, t) \in CL_{2,1}(\Gamma_T)$ (this condition is meets in applications). Thus, the functions $f(x, t)$ are continuous by variable $t \in [0, T]$.

Theorem 2. If $f(x) \in CL_{2,1}(\Gamma_T)$ and $\varphi(x) \in L_2(\Gamma)$, then the initial boundary value problem (2), (5) weak solvable in space $V^{1,0}(a, \Gamma_T)$ for any $0 < T < \infty$.

The following fundamental fact is noted in the proof: with each fixed $t \in [0, T]$ trace of a weak solution of problem (2), (5) belonging to space $W^{1,0}(a; \Gamma_T)$ is a element of space $W_0^1(a; \Gamma)$ and this trace is continuously depending on $t$ in norm $W^1_0(\Gamma)$. This means that a weak solution to the initial boundary value problem (2), (5) can be presented as a series

$$u(x, t) = \sum_{i=1}^{\infty} \left( \varphi_i e^{-\lambda_i t} + \int_0^t f_i(\tau) e^{-\lambda_i (t-\tau)} d\tau \right) \phi_i(x),$$

(7)

on the basis $\{\phi_i(x)\}_{i \geq 1}$ of space $W_0^1(a; \Gamma)$, granting the decompositions $\varphi(x) = \sum_{i=1}^{\infty} \varphi_i \phi_i(x)$,
standard way establishes uniformity convergence on \( t \in [\epsilon, T] \) a series (7) and a series obtained by its differentiation term-by-term by \( t \) in norm \( W_2^1(\Gamma) \) at any \( \epsilon > 0, T < \infty \). Beginning from in order that the sum of the first \( N \) members of the series (7) satisfies the identity (6) of definition 2 with the initial function \( \varphi^N(x) = \sum_{i=1}^{N} \varphi_i \phi_i(x) \), passage to the limit at \( N \to \infty \) establishes the existence of a weak solution \( u(x, t) \in V^{1,0}(a, \Gamma_T) \) of the initial boundary value problem (2), (5) at any \( T < \infty \).

The following statement is set in a standard way.

**Theorem 3.** The initial boundary value problem (2), (5) has the only weak solution belonging to the space \( V^{1,0}(a, \Gamma_T) \) for any \( 0 < T < \infty \).

When analyzing stability on Lyapunov of the solution \( u(x, t) \in V^{1,0}(a, \Gamma_T) \) of the initial boundary value problem (2), (5) for the arbitrary \( T > 0 \) you need to know the behavior of the function \( u(x, t) \) at unlimited increase variable \( t \). Sufficient conditions for this are presented in the next statement.

**Theorem 4.** Let \( f(x) \in CL_{2,1}(\Gamma_T) \), \( \varphi(x) \in L_2(\Gamma) \) and the condition

\[
\int_{t}^{t+1} \|f(\cdot, \xi)\|^2_{L_2(\Gamma)} d\xi \leq A \quad (8)
\]

has been fulfilled for any \( t \geq 0 \), then for a weak solution \( u(x, t) \in V^{1,0}(a, \Gamma_{\infty}) \) of the problem (2), (5) occur estimates

\[
\int_{t}^{t+1} \|u(\cdot, \xi)\|^2_{W_2^1(\Gamma)} d\xi \leq C \quad \text{for any} \quad t \geq 0, \quad \|u(\cdot, t)\|_{L_2(\Gamma)} \leq C \quad \text{in} \quad t \to +\infty,
\]

where \( C \) is a positive constant.

Detailed proof is presented in the work [4].

Further research assumes a fulfilled condition \( 0 \leq b(x) \leq \beta < \infty, x \in \Gamma \), from which it should be \( \lambda_i > 0 \) for all \( i \geq 1 \).

Let the function \( \overline{\varphi}(x, t) \in V^{1,0}(a, \Gamma_{\infty}) \) is a weak solution of the equation (2) and satisfies the conditions (5) where \( \varphi(x) = \overline{\varphi}(x) \). The solution \( \overline{\varphi}(x, t) \) will be called unperturbed by the state of the system (2), then the \( u(x, t) \) is perturbed by the state of the system (2), satisfying the conditions (5).

**Definition 3.** The unflappable state of the system (2) is called stability on Lyapunov state, if for any \( t_0 > 0 \) and \( \epsilon > 0 \) there is such \( \delta(t_0, \epsilon) > 0 \) that out of inequality \( \|\varphi - \overline{\varphi}\|_{L_2(\Gamma)} < \delta(t_0, \epsilon) \) follows inequality \( \|u(\cdot, t) - \overline{\varphi}(\cdot, t)\|_{W_2^1(a, \Gamma)} < \epsilon \) when \( t \geq t_0 \).

**Theorem 5.** Let \( f(x) \in CL_{2,1}(\Gamma_T), \varphi(x) \in L_2(\Gamma) \), the conditions (8) and \( 0 \leq b(x) \leq \beta < \infty, x \in \Gamma \), then the unperturbed state of the system (2) in the domain \( \Gamma_{\infty} \) is stable.

Proof of the Theorem is a direct consequence of representations \( \overline{\varphi}(x, t) \) and \( u(x, t) \) in the form of a series (7).

3. Results and discussion

We will make the transition from a differential system (2), (5) to a differential-difference analogue using the method of finite difference by variable \( t \) (Rothe semi-digitization method).

On the segment \([0, T]\) we will construct a difference net \( \omega_T = \{t_k = k\tau, k = 0, 1, 2, ..., M\} \), \( \tau = T/M \) (\( M \) is fixed positive natural number). We cut the domain \( \Gamma_T \) with the planes \( t = t_k \), \( k = 0, 1, 2, ..., M \); sections \( \Gamma_T \) in any \( k \) are \( \Gamma \). Equation (2) will replace the differential-difference equation

\[
\frac{1}{\tau}(u(k) - u(k - 1)) - \frac{d}{dx} \left( a(x) \frac{du(k)}{dx} \right) + b(x)u(k) = f(\cdot, k), \quad (9)
\]
where \( u(x) = u(x,k) \in W_0^1(a; \Gamma) \), \( f_r(k) = \frac{1}{\tau} \int_{(k-1)\tau}^{k\tau} f(x,t)dt \in L_2(\Gamma), \) \( k = 1, 2, ..., M. \)

Functions \( u(k) (k = 1, 2, ..., M) \) are solutions for (9) if they meet the conditions

\[
\begin{align*}
    u(0) &= \varphi(x), \quad \left. u(k) \right|_{x = \partial T} = 0. \tag{10}
\end{align*}
\]

This means that for each fixed number \( k \in \{1, 2, ..., M\} \) the system (9), (10) is the boundary value problem for the elliptical equation (9) relative \( u(k) \).

**Remark 1.** Relations (9), (10) \((k = \frac{T}{\tau}, M)\) can be understand as a implicit difference scheme of first-order approximation on variable \( t \) for the initial boundary value problem (2), (5) with an elliptical operator \( \Lambda; \) \( \tau = \frac{T}{M} \) is the approximation step. Difference quotient \( \frac{1}{\tau}[u(k) - u(k - 1)] \) is left approximation of derivative function \( u(x,t) \) with respect to \( t \) at point \( t = t_k \).

**Remark 2.** System (9), (10) is one of the approximation methods applied to the parabolic system (2), (5). This avoids many technical difficulties in approximating the input problem (2), (5) on both variables \( t \) and \( x \), above all, when \( x \) is a multidimensional variable.

**Definition 4.** A weak solution of the differential-difference equation (9) with conditions (10) are called functions \( u(k) (k = \frac{T}{\tau}, M) \), belonging to space \( W^1_0(a; \Gamma) \) and satisfying integral identity

\[
\begin{align*}
    \int_{\Gamma} u(k)t \nu(x)dx &+ \ell(u(k), \nu) = \int_{\Gamma} f_r(k)\nu(x)dx, \tag{11}
\end{align*}
\]

for any function \( \nu(x) \in W^1_0(a; \Gamma) \); equality \( u(0) = \varphi(x) \) in relation (1) be understood almost everywhere, \( u(k)t = \frac{1}{\tau}[u(k) - u(k - 1)] \), \( \ell(u(k), \nu) \) is the differential form

\[
\ell(u(k), \nu) = \int_{\Gamma} \left( a(x)\frac{d\varphi(x)}{dx}\nu(x) + b(x)\nu(x)\right) dx \quad (u(x;k) = u(k)).
\]

Here is the situation under which the differential-difference system (9), (10) inherits the properties of the differential system (2), (5). This is primarily a fundamental property of continuity on the original data, as the stability of any difference scheme (and its convergence) is determined by the presence of this property. The research tool for this are a priori estimate of weak solutions differential-difference system (9), (10).

**Theorem 6.** Let \( f(x,t) \in L_{2,1}(\Gamma_T); \varphi(x) \in L_2(\Gamma) \). Under \( \tau \leq \tau_0 < \frac{1}{\sqrt{T}} \) and any \( k = 1, 2, ..., M \) for functions \( u(k) \in W^1_0(a; \Gamma) \) correctly estimates

\[
\|u(k)\|_{2, \Gamma} \leq e^{2q\tau} (\|\varphi\|_{2, \Gamma} + 2\|f_r(k)\|_{2,1, \Gamma}), \quad \|f_r(k)\|_{2,1, \Gamma} = \tau \sum_{s=1}^{k} \|f_r(s)\|_{2, \Gamma}, \tag{12}
\]

and

\[
\|u(m)\|_{2, \Gamma}^2 + 2a_\tau \sum_{k=1}^{m} \|\frac{du(k)}{dx}\|^2 + \tau^2 \sum_{k=1}^{m} \|u(k)t\|_{2, \Gamma}^2 \leq C \left( \|\varphi\|_{2, \Gamma}^2 + \|f_r(m)\|_{2,1, \Gamma}^2 \right), \tag{13}
\]

where the constant \( C \) depends only on \( a_\tau, \beta, T \) and does not depend on the step \( \tau \).

**Proof.** Obvious transformations in equality

\[
u(k - 1)^2 \leq u(k)^2 + \tau^2 u(k)^2 - 2\tau u(k)u(k)t,
\]

reduce to a relation

\[
2\tau u(k)u(k)t = u(k)^2 + \tau^2 u(k)^2 - u(k - 1)^2. \tag{14}
\]
Put in the identity (11) and granting the ratio (14), get inequality
\[
\int_{\Gamma} u(k)^2 dx - \int_{\Gamma} u(k-1)^2 dx + \tau^2 \int_{\Gamma} (u(k))_t^2 dx + 2a_\tau \int_{\Gamma} \left(\frac{du(k)}{dx}\right)^2 dx \leq -2\tau \int_{\Gamma} b(x) u(k)^2 dx + 2\tau \int_{\Gamma} f_\tau(k) u(k) dx,
\]
whence follows inequality \(|| \cdot ||_{2,\Gamma} = || \cdot ||_{L^2(\Gamma)}||
\[
\begin{align*}
||u(k)||^2_{2,\Gamma} - ||u(k-1)||^2_{2,\Gamma} + \tau^2 ||u(k)_t||^2_{2,\Gamma} + 2a_\tau ||\frac{du(k)}{dx}||^2 &\leq -2\tau \int_{\Gamma} b(x) u(k)^2 dx + 2\tau \int_{\Gamma} f_\tau(k) u(k) dx \leq 2\beta \tau ||u(k)||^2_{2,\Gamma} + 2\tau ||f_\tau(k)||_{2,\Gamma} ||u(k)||_{2,\Gamma}. \\
\end{align*}
\]
Thus under \(k = 1, 2, \ldots, M\)
\[
||u(k)||^2_{2,\Gamma} - ||u(k-1)||^2_{2,\Gamma} + \tau^2 ||u(k)_t||^2_{2,\Gamma} + 2a_\tau ||\frac{du(k)}{dx}||^2 \leq 0 \tau ||u(k)||^2_{2,\Gamma} + 2\tau ||f_\tau(k)||_{2,\Gamma} ||u(k)||_{2,\Gamma}. \\
\]
for \(\varrho = 2\beta\). From here it should be
\[
||u(k)||^2_{2,\Gamma} - ||u(k-1)||^2_{2,\Gamma} \leq 0 \tau ||u(k)||^2_{2,\Gamma} + 2\tau ||f_\tau(k)||_{2,\Gamma} ||u(k)||_{2,\Gamma}. \\
\]
Let’s say that \(||u(k)||_{2,\Gamma} + ||u(k-1)||_{2,\Gamma} > 0\). Divide inequality (16) by \(||u(k)||_{2,\Gamma} + ||u(k-1)||_{2,\Gamma}\) and because
\[
\frac{||u(k)||^2_{2,\Gamma}}{||u(k)||^2_{2,\Gamma} + ||u(k-1)||^2_{2,\Gamma}} \leq 1,
\]
we get a estimate of norm \(||u(k)||_{2,\Gamma}\):
\[
||u(k)||_{2,\Gamma} \leq \frac{1}{1-\varrho^2} ||u(k-1)||_{2,\Gamma} + \frac{2\varrho}{1-\varrho^2} ||f_\tau(k)||_{2,\Gamma}, \\
\]
at \(\tau \leq \tau_0 < \frac{1}{2\varrho}\).

Let \(||u(k)||_{2,\Gamma} + ||u(k-1)||_{2,\Gamma} = 0\), then from (16) follow the relation \(0 \leq \varrho \tau ||u(k)||_{2,\Gamma} + 2\tau ||f_\tau(k)||_{2,\Gamma}\) and inequality
\[
||u(k)||_{2,\Gamma} \leq \varrho \tau ||u(k)||_{2,\Gamma} - ||u(k-1)||_{2,\Gamma} + 2\tau ||f_\tau(k)||_{2,\Gamma},
\]
which give a estimate (17).

Next, from the estimate (17) follows
\[
\begin{align*}
||u(k)||^2_{2,\Gamma} &\leq \frac{1}{1-\varrho^2} ||u(k-1)||^2_{2,\Gamma} + \frac{2\varrho}{1-\varrho^2} ||f_\tau(k)||^2_{2,\Gamma} \leq \\
&\leq \frac{1}{(1-\varrho^2)^k} ||u(0)||^2_{2,\Gamma} + 2\tau \sum_{s=1}^{k} \frac{1}{(1-\varrho^2)^s} ||f_\tau(s)||^2_{2,\Gamma} \leq \\
&\leq \frac{1}{(1-\varrho^2)^k} \left( ||u(0)||^2_{2,\Gamma} + 2\tau \sum_{s=1}^{k} ||f_\tau(s)||^2_{2,\Gamma} \right) \leq \\
&\leq e^{2\varrho T} \left( ||u(0)||^2_{2,\Gamma} + 2||f_\tau(k)||^2_{2,\Gamma} \right),
\end{align*}
\]
where \(||f_\tau(k)||^2_{2,\Gamma} = \tau \sum_{s=1}^{k} ||f_\tau(s)||^2_{2,\Gamma}\) and the relations are taken into account \(\frac{\varrho^2}{1-\varrho^2} k \leq \frac{\varrho T}{1-\varrho^2} \leq 2\varrho T\) (\(\tau < \frac{1}{2\varrho}\), \(\frac{1}{(1-\varrho^2)^k} \leq e^{2\varrho T}\). Thus, a estimate (12) for the norm was obtained \(||u(k)||_{2,\Gamma}\):
\[
||u(k)||^2_{2,\Gamma} \leq e^{2\varrho T} \left( ||\varphi||^2_{2,\Gamma} + 2||f_\tau(k)||^2_{2,\Gamma} \right).
\]
It should be noted that when \( m = 1, 2, ..., M \) the relations
\[
\sum_{k=1}^{m} \| f_r(k) \|_{2,1,Γ} = \sum_{k=1}^{m} \left( \tau \sum_{s=1}^{k} \| f_r(s) \|_{2,Γ} \right) =
\]
\[
= \tau \sum_{s=1}^{1} \| f_r(s) \|_{2,Γ} + \tau \sum_{s=1}^{2} \| f_r(s) \|_{2,Γ} + ... + \tau \sum_{s=1}^{m} \| f_r(s) \|_{2,Γ} \leq
\]
\[
\leq m \tau \sum_{s=1}^{m} \| f_r(s) \|_{2,Γ} = m \| f_r(m) \|_{2,1,Γ}.
\]
are correctly. This and the estimate (18) give inequality
\[
\sum_{k=1}^{m} \| u(k) \|_{2,Γ} \leq e^{2C \tau} \sum_{k=1}^{m} \left( \| \varphi \|_{2,Γ} + 2 \| f_r(k) \|_{2,1,Γ} \right) \leq
\]
\[
\leq m e^{2C \tau} \left( \| \varphi \|_{2,Γ} + 2 \| f_r(m) \|_{2,1,Γ} \right).
\]

Summing up the relations (15) by \( k \) from 1 to and considering the inequality (18), (19), we get
\[
\| u(m) \|_{2,Γ} \leq 2a_{\ast} \tau \sum_{k=1}^{m} \| u(k) \|_{2,Γ} \leq
\]
\[
\leq \sum_{k=1}^{m} \left( \beta \tau \| u(k) \|_{2,Γ} + 2\tau \| f_r(k) \|_{2,1,Γ} \right) \leq
\]
\[
\leq \sum_{k=1}^{m} \| f_r(k) \|_{2,1,Γ} \leq C \left( \| \varphi \|_{2,Γ} + \| f_r(m) \|_{2,1,Γ} \right),
\]
where the constant \( C \) is expressed only through constant \( a_{\ast}, \beta, T \). Thus we get inequality (13), in addition \( \| f_r(m) \|_{2,1,Γ} \leq \| f \|_{2,1,Γ} \). This follows from the relations
\[
\| f_r(k) \|_{2,1,Γ} = \tau \sum_{s=1}^{k} \| f_r(s) \|_{2,Γ} = \tau \sum_{s=1}^{k} \| f_r(\cdot, s) \|_{2,Γ} \leq \int_{0}^{k\tau} \| f(\cdot, t) \|_{2,Γ} dt = \| f \|_{2,1,Γ_{k\tau}},
\]
here \( \Gamma_{k\tau} = \Gamma_0 \times (0, k\tau) \). Theorem is proven.

**Remark 3.** The estimate (12) guarantee the stability of the difference scheme (differential-difference system) (9), (10), that is, the continuous dependence of solution by the differential-difference equation (9) from the input data \( f_r(k) = \frac{1}{\tau} \int_{(k-1)\tau}^{k\tau} f(x, t) dt \) \( (k = 1, 2, ..., M) \) and \( \varphi(x) \) (see boundary conditions (10)). From the same estimate there is a weak uniqueness solvability for each fixed \( k \) of elliptical equation (9) with boundary conditions (10) at sufficiently small \( \tau \).

The estimate (13) is a discrete analogue of energy inequality for weak solutions of problem (2), (6) and, as shown in the proof below Theorem 7, establishes a weak convergence in the norm of space \( W^{1,0}_0(a; \Gamma_T) \) of the sequence of piecewise constant function
\[
\| u_M(x, t) = u(k), t \in ((k-1)\tau, k\tau), k = 1, 2, ..., M \quad (M = 1, 2, ...),
\]
to a weak solution \( u(x, t) \in W^{1,0}_0(a; \Gamma_T) \) of problem (2), (5).

**Theorem 7.** Weak solution to the initial boundary value problem (2), (5) in space \( W^{1,0}_0(a; \Gamma_T) \) is the limit of functions \( u_M(x, t) \) under \( \tau \rightarrow 0 \) \( (M \rightarrow \infty) \).

**Proof.** It is obvious that for each \( M \) the function \( u_M(x, t) \), defined by a ratio (20), will belong to space \( W^{1,0}_0(a, \Gamma_T) \) and satisfy inequality (13), in addition
\[
\| u_M \|_{2,Γ_T} + \| \frac{\partial u_M}{\partial x} \|_{2,Γ_T} \leq C^{*},
\]
where \( C^* \) is constant, independent of \( \tau \); below \( \| \cdot \|_{2, \Gamma_T} \) is the standard norm of space \( L_2(\Gamma_T) \). Let’s introduce a function \( f_M(x, t) = f_\tau(k), t \in ((k - 1)\tau, k\tau] \), \( k = 1, 2, ..., M \) \( (M = 1, 2, ...) \).

Let \( M \to \infty \), then \( \tau \to 0 \). Granting the inequality (21), the sequence \( u_M(x, t) \) contains a subsequence that is weakly converge to \( u(x, t) \in W_{0}^{1, 0}(a, \Gamma_T) \) in norm \( W_{2}^{1, 0}(\Gamma_T) \). Let’s show that function \( u(x, t) \) is a weak solution of a problem (2), (5), i. e. satisfy the conditions (4) under any \( t \in (0, T) \) and the condition \( u|_{t=T} = 0 \). In addition we take that \( u(x, t) \in C^4(\Gamma_{T+\tau}) \), it satisfy the conditions (4) under any \( t \in (0, T) \) and the condition \( u|_{t=T+\tau} = 0 \). The functions \( u(k) \) will be defined by equality \( \eta(k) = \eta(x, k\tau) \), \( k = 1, 2, ..., M \), and \( \eta(k)_t = \frac{1}{\tau}[\eta(k + 1) - \eta(k)] \) (difference quotients \( \eta(k)_t, \eta(k)_t = \frac{1}{\tau}[\eta(k) - \eta(k - 1)] \) define right and left approximations for \( \frac{\partial \eta}{\partial t} \) at the point \( t = k\tau \), respectively).

Let’s introduce a function \( \eta \) defined by equality (22), we will get an identity of definition 1. This means that the function \( u(x, t) \) is a weak solution of a problem (2), (5) (see the statement of Theorem 3) and the estimate (20) the sequence \( \{u_M(x, t)\} \) is weakly convergent to \( u(x, t) \). Theorem is proven.

Put in a ratio (11) and we will sum the obtained identities by \( k \) from 1 until \( M \), then granting equality (22), we will get an identity

\[
-\tau \sum_{k=1}^{M} \int_{\Gamma} u(k)\eta(k)_t dx - \int_{\Gamma} \varphi(x)\eta(1) dx + \tau \sum_{k=1}^{M} \ell_T(u(k), \eta(k)) =
\]

where we’ll present as

\[
- \int_{\Gamma_T} u_M(x, t)\frac{\partial \eta_M(x, t, s)}{\partial t} dx dt + \ell_T(u_M, \eta_M) - \int_{\Gamma} \varphi(x)\eta(1) dx =
\]

which we’ll present as

\[
- \int_{\Gamma_T} u_M(x, t)\frac{\partial \eta_M(x, t, s)}{\partial t} dx dt + \ell_T(u_M, \eta_M) - \int_{\Gamma} \varphi(x)\eta(1) dx =
\]

Going over to the limit in the ratio (23) on the chosen above weakly convergent sequence on the norm of space \( W_{0}^{1, 0}(a, \Gamma_T) \) (here \( u(x, t) \in W_{0}^{1, 0}(a, \Gamma_T) \) is the limit function), we get an integral identity of definition 1. This means that the function \( u(x, t) \) is a weak solution to the initial boundary value problem (2), (5) and \( u(x, t) \in W_{0}^{1, 0}(a, \Gamma_T) \). By virtue of the uniqueness solution of the problem (2), (5) (see the statement of Theorem 3) and the estimate (20) the sequence \( \{u_M(x, t)\} \) is weakly convergent to \( u(x, t) \). Theorem is proven.

As mentioned above, the stability of the difference scheme mean the continuous dependence of solution of the difference scheme on the initial data.

**Definition 5.** Differential-difference system (9), (10) is stable by initial data \( f_\tau(k), \varphi(x) \), if at sufficiently small \( \tau \leq \tau_0 < \frac{1}{M} \) there is inequality

\[
\|u(k)\|_{2, \Gamma} \leq C_1\|\varphi\|_{2, \Gamma} + C_2\|f_\tau(k)\|_{2, 1, \Gamma}, \quad k = 1, 2, ..., \]

where \( C_1 \) and \( C_2 \) are positive constants, independent of \( \tau \).

By definition 3 the stability of the differential-difference system (9), (10) (i. e. the stability of the difference scheme (9), (10)) is a direct consequence of the first statement of the Theorem 7 (relation (12)).
By analogy with the unperturbed and perturbed solutions of system (2), (5) are introduced the unperturbed and perturbed solutions of differential-difference system (9), (10). We will designate through $\bar{\varphi}(x)$ and $\varphi(x)$ the unperturbed and perturbed solutions of the differential-difference equation (9), respectively, in addition the conditions (10) for $\bar{\varphi}(x)$ be fulfilled at $\varphi(x) = \varphi(x)$.

**Definition 6.** The unperturbed solution $\bar{\varphi}(x)$ of the differential-difference system (9), (10) is stable on Lyapunov, if for any $k_0 > 0$ and $\epsilon > 0$ it is possible to specify $\delta(k_0, \epsilon) > 0$ that in the fulfilment of inequality $\|\varphi - \bar{\varphi}\|_{L_2(\Gamma)} < \delta(k_0, \epsilon)$ is performed $\|u(k) - \bar{u}(k)\|_{W^1(\alpha, \Gamma)} < \epsilon$ for all $k \geq k_0$.

By virtue of the definition 6 out of the second statement of the theorem 6 it should be the stability on Lyapunov of the unperturbed solution $u(k)$ of the differential-difference system (9), (10), if inequality (13) remains true for the constant $C$, that depends only on $\alpha_\ast$, $\beta$ and not depends on $T$ and steps $\tau$.

**4. Conclusion**

The main results of the work be relate to the study of differential-difference system (9), (10). The motive for this study was purely practical questions – the construction of approximations of weak solutions for the problem (2), (5) and analysis of their properties. The universal method for this is the method of finite differences, but when the number of space variables increases (when this number becomes 2, 3 and more) this method is poorly applied due to the complexity of analysis of difference schemes. However, when the system of generalized eigenfunctions of the elliptical equation operator (2) is known, these complexity are overcome by partial approximation of the system (2), (5) by the temporary variable, i.e. the transition to a differential-difference system (9), (10). In the work shows under what conditions the differential-difference system (9), (10) inherits the properties of the differential system (2), (5). First of all, the fundamental property of continuity according to the initial data, as the stability of any difference scheme (and its convergence) is a consequence of the presence of this property. Other properties (primarily resolution, uniqueness) are either a direct consequence of the indicated property or a necessary condition. The research tool for this are a priori estimate of weak solutions differential-difference system (9), (10) and subsequent construction of approximations of a weak problem solution (2), (5) in the form of piecewise constant approximations (20) on a time variable. The obtained results open the way to analyze distinct types of problems of optimal control network-like process (distributed, start, boundary controls), stabilization of weak solving and problems with a lag time for the initial differential system (2), (5) (see, for example [7]), work by means of consideration the same tasks for the differential-difference analogue (9), (10) of this system. It should also be noted that the results obtained above are apply extend to a multidimensional case similar to the one in the work [8-11].

**References**

[1] Aleksandrov A Yu and Platonov A V 2009 On stability and dissipativity of some classes of complex systems Automation and Remote Control 8 pp. 1265–1280.

[2] Kamachkin A M and Yevstafyeva V V 2000 Oscillations in a relay control system at an external disturbance Control Applications of Optimization: Proceedings of the 11th IFAC Workshop 2 pp. 459–462.

[3] Kamachkin A M, Potapov D K and Yevstafyeva V. V 2017 Existence of subharmonic solutions to a hysteresis system with sinusoidal external influence. Electronic Journal of Differential Equations 140 pp. 1–10.

[4] Zhukho A P, Shindyapin A I and Provotorov V V 2019 Stability of weak solutions of parabolic systems with distributed parameters on the graph Vestnik of Saint Petersburg University Applied Mathematics Computer Science Control Processes 4 457–471.

[5] Adams R A and Fournier J J F 2003 Sobolev Spaces (Amsterdam: Elsevier/Academic Press)

[6] Nicas J 2012 Direct Methods in the Theory of Elliptic Equations (Heidelberg: Springer)

[7] Provotorov V V and Provotorova E N 2017 Optimal control of the linearized Navier-Stokes system in a netlike domain Vestnik Sankt-Peterburgskogo Universiteta Prikladnaya Matematika Informatika Processy Upravleniya 4 428-441.
[8] Baranovskii E S 2017 Global solutions for a model of polymeric flows with wall slip *Mathematical Methods in the Applied Sciences* **14** 5035–5043.

[9] Baranovskii E S 2019 Steady flows of an Oldroyd fluid with threshold slip *Communications on Pure and Applied Analysis* **2** 735–750.

[10] Artemov M A, Baranovskii E S 2019 Solvability of the boussinesq approximation for water polymer solutions *Mathematics* **7** Article ID 611

[11] Provotorov V V, Ryazhskikh V I, Gnilitskaya Yu A 2017 Unique weak solvability of nonlinear initial boundary value problem with distributed parameters in the netlike domain *Vestnik of Saint Petersburg University Applied Mathematics Computer Science Control Processes* **3** 264–277.