Bisections of Centrally Symmetric Planar Convex Bodies Minimizing the Maximum Relative Diameter

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Abstract. In this paper, we study the bisections of a centrally symmetric planar convex body which minimize the maximum relative diameter functional. We give a necessary and a sufficient condition for a minimizing bisection, as well as analyze the behavior of the so-called standard bisection.

Mathematics Subject Classification. Primary 52A40; Secondary 52A10.

Keywords. Centrally symmetric planar convex bodies, maximum relative diameter, minimizing bisections.

1. Introduction

Historically, the classical geometric functionals (perimeter, area, volume, inradius, circumradius, diameter and width), and the relations between them, have been intensively studied, yielding a great variety of optimization problems [10,19]. Possibly, the most relevant example is the isoperimetric problem, examining the relation between the surface area and the volume of sets in $\mathbb{R}^n$ [15–18]. Moreover, in the setting of Convex Geometry, these functionals play an important role and can be considered as the origin of this theory.

In this context, we will focus on a particular relative geometric problem concerning the diameter functional. This is one of the most natural magnitudes for measuring the size of a set, and has been thoroughly investigated in the literature. Some well-known and important results in $\mathbb{R}^n$ regarding this functional are, for instance, Jung’s theorem [11], establishing the inequality

Antonio Cañete is partially supported by the MICINN projects MTM2013-48371-C2-1-P and MTM2017-84851-C2-1-P, and by Junta de Andalucía grant FQM-325 (Consejería de Economía, Innovación, Ciencia y Empleo). Salvador Segura Gomis is partially supported by MINECO/FEDER project MTM2015-65430-P and “Programa de Ayudas a Grupos de Excelencia de la Región de Murcia”, Fundación Séneca, 19901/GERM/15.
between the diameter and the circumradius of a compact set, the isodiametric inequality \([1, 2]\), which asserts that the ball is the compact convex set of fixed volume with the minimum possible diameter, or Borsuk’s conjecture \([3]\), asking whether any compact set \(K\) can be divided into \(n+1\) subsets whose diameters are strictly less than the diameter of \(K\) (this conjecture does not hold for all sufficiently large \(n\), see \([12]\)). Additionally, more inequalities involving the diameter and other classical functionals for planar compact convex sets can be found in \([19]\).

In this work, we will consider the maximum relative diameter functional in \(\mathbb{R}^2\), which is defined in the following way: for a fixed planar compact convex set \(C\), a division of \(C\) into two connected subsets determined by a simple curve, with endpoints in the boundary of \(C\), but else in the interior of \(C\), will be called a bisection of \(C\). Then, for a bisection of \(C\) given by a curve \(P\), with subsets \(C_1, C_2\) (not necessarily of equal areas), the maximum relative diameter associated to \(P\) is

\[
d_M(P) = \max\{D(C_1), D(C_2)\},
\]

where \(D(S)\) denotes the Euclidean diameter of a planar set \(S\). In view of this definition, the maximum relative diameter clearly represents the largest distance in the subsets generated by the bisection. In this setting, we are interested in finding the optimal bisection for the maximum relative diameter functional. That is, among all the bisectons of \(C\), we look for the one attaining the minimum possible value for this functional, which can be considered as the division of \(C\) with both subsets as small as possible in terms of the diameter.

Throughout this paper, we will assume that the sets are centrally symmetric. This hypothesis provides enough geometric structure to deal with the problem, allowing us to obtain descriptive results for the minimizing bisectons (in the non-symmetric case, it does not seem possible to find similar properties for the optimal bisectons, due to the lack of symmetry). Moreover, considering divisions of the sets into two subsets is something naturally inherent to central symmetry. On the other hand, we point out that if we focus on the class of compact convex sets of given area, the minimum value for the maximum relative diameter when restricting to bisectons by line segments providing equal-area subsets is attained by a centrally symmetric set \([14, \text{Th. } 6]\). The same result has been recently proven in \([5]\) for general bisectons, which suggests, in some sense, that this kind of sets is certainly suitable for this functional.

The main results of this paper refer to the minimizing bisectons for the maximum relative diameter of centrally symmetric planar compact convex sets. We will see that we do not have uniqueness of solution for this problem, since proper slight modifications of a minimizing bisection are also minimizing (this is a common feature when working with the diameter functional). Moreover, to find a minimizing bisection, Proposition 3.1 assures that it is enough to focus on the bisectons given by a line segment passing through the center of symmetry of the set (which will always generate symmetric subsets of equal areas). This property agrees with the intuitive idea that the
corresponding subsets of an optimal bisection must be as balanced as possible. Proposition 3.4 shows a necessary condition for a bisection of this type to be minimizing (expressed in terms of the largest distances from the two endpoints of the segment determining the bisection), which will be complemented in Theorem 3.6, establishing a criterion to assert that a bisection (by a line segment passing through the center of symmetry) is minimizing.

This partitioning optimization problem has already been considered in [14], but with the additional restriction of bisections with equal-area subsets. In this setting, among other results, it is proved that a minimizing bisection is always given by a line segment passing through the center of symmetry of the set [14, Prop. 4], with no further description of the properties of the solutions. Our work is inspired by that paper, with the aim of extending the results therein to a more general situation (arbitrary bisections with non-equal area subsets), and describing the minimizing bisections in a more precise way.

It is worth mentioning that the analogous question for divisions into a larger number of subsets has also been studied: for a given $k$-rotationally symmetric planar compact convex set $C$, with $k \in \mathbb{N}$, $k \geq 3$, a $k$-partition of $C$ is a decomposition of $C$ into $k$ connected subsets $C_1, \ldots, C_k$, determined by $k$ simple curves starting in the boundary of $C$, all of them ending in an interior point of $C$. And similarly, for a given $k$-partition of $C$, we can define its associated maximum relative diameter as

$$\max\{D(C_i) : i = 1, \ldots, k\}.$$ 

In this context, we can investigate the $k$-partitions of $C$ attaining the minimum value for $d_M$. This was treated in [7] (see also [6]), where it is proved that the so-called standard $k$-partition (constructed by using $k$ inradius segments symmetrically placed, see Fig. 1) is a solution for this problem, for any $k \geq 3$.

We will see in Sect. 4 that the previous result does not hold in our setting (which would correspond to $k = 2$): a standard bisection (provided by two symmetric inradius segments) is not minimizing in general (for instance, see Example 2). In fact, the standard bisection of a given centrally symmetric planar compact convex set could not even be uniquely defined, as shown in Example 4. These are two remarkable differences to the case of $k$-rotationally symmetric planar compact convex sets, with $k \geq 3$.

We have organized this paper as follows. In Sect. 2, we give the precise definitions and statement of the problem. Section 3 contains the main results. We prove in Proposition 3.1 that for any minimizing bisection, we can

![Figure 1. Some standard $k$-partitions for $k \geq 3$](image-url)
find another bisection given by a line segment passing through the center of symmetry of the set with the same value for the maximum relative diameter, and, therefore, we can focus on this type of bisections in the search of a minimizing one. Taking this into account, Proposition 3.4 gives a necessary condition for minimizing bisections, suggesting, in some sense, that there is needed the existence of a certain symmetry, related to the farthest distances with respect to the endpoints of the segment. Finally, Theorem 3.6 states some sufficient conditions to assure that a given bisection (by a line segment passing through the center of symmetry) is minimizing. Unfortunately, this result is not completely sharp, since there are examples of minimizing bisections for which Theorem 3.6 is not applicable, see Example 1. In Sect. 4 we discuss the main properties of standard bisections, showing that they are not minimizing in general (see Examples 2 and 3), and that it may not be uniquely defined (see Sect. 4.1). Section 5 contains several examples, showing some minimizing bisections in each case by using Theorem 3.6. We conclude these notes with some comments of interest in Sect. 6.

1.1. Notations

For two distinct points \( a, b \in \mathbb{R}^2 \), the closed or open segment with endpoints \( a, b \) will be denoted by \( \overline{ab} \), or \( (a, b) \), respectively. Moreover, for \( a \in \mathbb{R}^2 \) and \( r > 0 \) we will write \( B(a, r) \) for the open ball of center \( a \) and radius \( r \). Also, for \( A \subset \mathbb{R}^2 \), we will denote by \( \overline{A} \) and \( \partial A \) the closure and the boundary of \( A \), respectively. In particular, \( \overline{B}(a, r) \) is the closed ball of center \( a \) and radius \( r \). For \( a, b, c \in \mathbb{R}^2 \) not collinear, we will write \( \Delta abc \) for the triangle with vertices \( a, b, c \), and \( \angle abc \) for the angle of that triangle at \( b \).

2. Preliminaries

Let us denote by \( \mathcal{C}_2 \) the class of centrally symmetric planar convex bodies (recall that a body is, as usual, a compact set). The central symmetry of a set \( C \in \mathcal{C}_2 \) means that there exists a point \( p \in C \) (called the center of symmetry of \( C \)) such that \( C \) is invariant under the action of the rotation of angle \( \pi \) centered at \( p \). Some examples of sets of this class are depicted in Fig. 2.

Throughout this paper, we will focus on some particular divisions of the sets, called bisections. In Remark 2.5, we will justify that these are the most convenient divisions for our problem. Note that the following definition can also be phrased in a more general setting.

Figure 2. Some centrally symmetric planar convex bodies
**Definition 2.1.** Let \( C \in \mathcal{C}_2 \). A bisection of \( C \) is a decomposition of \( C \) into two connected subsets, into which a simple curve \( P \), with endpoints in \( \partial C \), but else in the interior of \( C \), divides \( C \).

**Remark 2.2.** We point out that the curve determining a given bisection does not contain, in general, the center of symmetry of the set, and moreover, the corresponding subsets do not have necessarily equal areas, as shown in Fig. 3.

![Figure 3](image-url)  
**Figure 3.** Three different bisections for an ellipse

We now proceed to define the geometric functional considered in this work, previously introduced in [14].

**Definition 2.3.** Let \( C \in \mathcal{C}_2 \), and let \( P \) give a bisection of \( C \), with associated subsets \( C_1, C_2 \). The maximum relative diameter of \( P \) is defined as

\[
d_M(P) = \max\{D(C_1), D(C_2)\},
\]

where \( D(S) \) denotes the Euclidean diameter of \( S \).

**Remark 2.4.** Recall that the diameter of a planar compact set is always attained by a pair of points lying in the boundary of the set, and in the case of a polygon, by two of its vertices.

The purpose of these notes is to investigate the bisections of a fixed centrally symmetric planar convex body \( C \) that minimize the maximum relative diameter functional, in the same spirit as in [6, 7], see also [14]: we plan to determine these bisections precisely or, at least, describe some of their geometrical properties. These bisections will be called minimizing in the whole paper. In this direction, some partial results have been obtained in the case of bisections providing equal-area subsets: in this more restrictive setting, it has been proved that, for any set \( C \in \mathcal{C}_2 \), there always exists a minimizing bisection given by a line segment passing through the center of symmetry of the set [14, Prop. 4]. However, no additional details have been outlined for the solutions, and nothing else is known. We will consider this problem in the most general setting (that is, for bisections generating subsets which have not necessarily equal areas), progressing in the description of these optimal bisections.

**Remark 2.5.** We point out that a given set \( C \in \mathcal{C}_2 \) can be decomposed into two connected subsets by means of divisions which are not bisections. This can be done using a simple closed curve entirely contained in the interior of \( C \). In general, these decompositions are not good candidates for our problem since, in view of Remark 2.4, all of them have maximum relative diameter equal to \( D(C) \), which is an immediate upper bound for our functional. Therefore, they will not be taken into account in these notes, and we will focus on the notion of bisection from Definition 2.1.
Remark 2.6. The uniqueness of solution is not expected for this optimization problem, as it is usually the case for questions involving the diameter functional. In fact, if we have a minimizing bisection given by a simple curve $P$, of a centrally symmetric planar convex body $C$, slight modifications of $P$ can be done preserving the value of the maximum relative diameter, hence being minimizing as well (see Proposition 3.1, where we can slightly modify the chord preserving its endpoints). This property suggests that a complete description of all the minimizing bisections of $C$ is not a feasible task.

2.1. Bisections by a Line Segment Passing Through the Center of Symmetry

Let $C \in \mathcal{C}_2$, and let $p$ be the center of symmetry of $C$. The bisections of $C$ given by a line segment passing through $p$ possess some special properties and will play an important role for our problem. If a bisection of $C$ is determined by a line segment $P$ passing through $p$, the corresponding subsets $C_1$, $C_2$ will be congruent due to the existing symmetry (they will coincide up to the rotation of angle $\pi$ about $p$), and so both of them will have the same area, and also $D(C_1) = D(C_2)$. Denoting by $v_1$, $v_2 \in \partial C$ the endpoints of $P$, [14, Prop. 3] leads to

$$d_M(P) = \max\{d(v_1, x) : x \in \partial C\},$$

where $d$ stands for the Euclidean distance in the plane. Equality (2.1) implies that the maximum relative diameter associated to this bisection will be given by the distance between an endpoint of $P$ and any of its corresponding farthest points in $\partial C$.

Apart from this, for a bisection determined by a line segment $P$ passing through $p$, there is another equivalent expression for computing $d_M(P)$, which will be useful in this work. For any $x \in \partial C_1$, we will denote by $F_{C_1}(x)$ the set of farthest points from $x$ in $\partial C_1$ (notice that $F_{C_1}(x)$ is non-empty due to compactness, and it may reduce to a single point). Since $D(C_1) = D(C_2)$, we can just focus on one of the subsets provided by $P$, and using again [14, Prop. 3], we will have that

$$d_M(P) = D(C_1) = \max\{d(v_1, \phi_{C_1}(v_1)), d(v_2, \phi_{C_1}(v_2))\},$$

where $v_1$, $v_2$ are the endpoints of $P$, and $\phi_{C_1}(v_i) \in F_{C_1}(v_i), i = 1, 2$.

Remark 2.7. We note that it is easy to check that equalities (2.1) and (2.2) are not true for bisections given by a general planar curve. In fact, they do not hold even for bisections by a line segment which does not pass through the center of symmetry, see Fig. 4.
Remark 2.8. For a given $C \in \mathcal{C}_2$, and a bisection of $C$ determined by an arbitrary simple curve $P$ (not necessarily a line segment), with endpoints $v_1, v_2 \in \partial C$, but else in the interior of $C$, and with subsets $C_1, C_2$, it may happen that $v_1 \in F_{C_1}(v_2)$ and $v_2 \in F_{C_1}(v_1)$. In that case, it turns out that the associated maximum relative diameter equals $d(v_1, v_2)$ in view of (2.2), and moreover, $C$ will be contained in the symmetric lens $B(v_1, d_M(P)) \cap B(v_2, d_M(P))$.

3. Main Results

In this section we obtain the main results of this paper. Proposition 3.1, which extends [14, Prop. 4] to the case of bisections with subsets which have not necessarily equal areas, shows that there is always a minimizing bisection given by a line segment passing through the center of symmetry of the set. Proposition 3.4 states a necessary condition for a bisection to be minimizing, and Theorem 3.6 establishes some conditions which are sufficient to guarantee that a given bisection is minimizing. These last two results (which are proved for bisections given by a line segment passing through the center of symmetry) reveal some of the geometric restrictions for optimality.

Proposition 3.1. Let $C \in \mathcal{C}_2$, and let $p$ be the center of symmetry of $C$. Let $P$ give a minimizing bisection for $d_M$ (the respective subsets are not necessarily of equal areas). Then there exists a line segment $P'$ passing through $p$ providing a bisection such that $d_M(P) = d_M(P')$.

Proof. Let $C_1, C_2$ be the subsets determined by $P$, and let $v_1, v_2$ be the endpoints of $P$ (notice that $v_1 \in \overline{C_1} \cap \overline{C_2}$). We can assume that $d_M(P) = D(C_1) \geq D(C_2)$. Let $v'_1 \in \partial C$ be the symmetric image of $v_1$ with respect to $p$, and consider the bisection $P'$ given by the segment $v_1 v'_1$ (which passes through $p$).

Taking into account (2.1), we have that $d_M(P') = d(v_1, z)$, for certain $z \in \partial C$. If $z \in \partial C_1$, then $d(v_1, z) \leq D(C_1) = d_M(P)$. And if $z \in \partial C_2$, then $d(v_1, z) \leq D(C_2) \leq D(C_1) = d_M(P)$. Thus $d_M(P') = d(v_1, z) \leq d_M(P)$, which implies that $d_M(P') = d_M(P)$ since $P$ is minimizing. □

Remark 3.2. A consequence of Proposition 3.1 is that, to find a minimizing bisection for a centrally symmetric planar convex body, we can focus on bisections given by line segments passing through the center of symmetry. Note that for each one of these bisections, the endpoints of the corresponding segment are always symmetric with respect to the center of symmetry of the set.

Remark 3.3. In fact, Proposition 3.1 shows that, if $C \in \mathcal{C}_2$ and $p$ denotes its center of symmetry, for any bisection of $C$ given by an arbitrary curve $P$, we can find another bisection determined by a line segment $P'$ passing through $p$, with $d_M(P') \leq d_M(P)$.

The next Proposition 3.4 states a necessary condition for a bisection to be minimizing, if it is determined by a line segment passing through the center
of symmetry, in view of Remark 3.2. This condition is expressed by means of the farthest points to the endpoints of the segment. In some sense, this result suggests a certain balance for the optimal divisions: the distances between each endpoint and its corresponding farthest point must coincide (being also equal to the value of the maximum relative diameter, due to equality (2.2)).

**Proposition 3.4** (Necessary condition). Let $C \in \mathcal{C}_2$, and let $p$ be the center of symmetry of $C$. Let $P$ give a bisection of $C$ by a line segment passing through $p$, with endpoints $v_1$, $v_2 \in \partial C$, and subsets $C_1$, $C_2$. If $P$ is a minimizing bisection for $d_M$, then

$$d_M(P) = d(v_1, \phi_{C_1}(v_1)) = d(v_2, \phi_{C_1}(v_2)), \quad (3.1)$$

where $\phi_{C_1}(v_i) \in F_{C_1}(v_i)$, $i = 1, 2$.

Proof. First, if $d_M(P) = d(v_1, v_2)$, then $v_2 \in F_{C_1}(v_1)$ and $v_1 \in F_{C_1}(v_2)$, in view of (2.2) and the definition of farthest point, yielding the desired equality (3.1). So we can suppose that $d_M(P) > d(v_1, v_2)$.

Assume now that (3.1) does not hold. Then, e.g.,

$$d_M(P) = D(C_1) = d(v_1, \phi_{C_1}(v_1)) > d(v_2, \phi_{C_1}(v_2)), \quad (3.2)$$

and, in particular, $d(v_1, \phi_{C_1}(v_1)) > d(v_1, v_2)$. This necessarily implies that $\phi_{C_1}(v_1) \neq v_2$. In this case, we will see that we can find a bisection of $C$ with strictly smaller maximum relative diameter, obtaining a contradiction. Our idea is, in some sense, to decrease the longer distance and to increase the shorter distance from (3.2).

We may suppose that $p = (0, 0)$, $v_1 = (0, 1)$ and $v_2 = (0, -1)$, and $C_1$ (respectively, $C_2$) lies to the left (respectively, right) hand side of $v_1v_2$. By applying a slight rotation, of angle $\theta > 0$ and centered at $p$, to the line spanned by the segment $v_1v_2$, we can consider a new bisection given by a different line segment $P^\theta$, with new endpoints $v_1^\theta$, $v_2^\theta \in \partial C$ and subsets $C_{1}^\theta$, $C_{2}^\theta$, satisfying that $v_i^\theta \in \partial C_1$. Note that, when $\theta$ goes to zero, the sequence $\{v_{1i}\}_\theta$ tends to $v_1$, and by continuity, the sequence $\{\phi_{C_{1i}}(v_i^\theta)\}_\theta$ tends to $\phi_{C_1}(v_1)$. Thus, since $\phi_{C_1}(v_1) \neq v_2$ and $\phi_{C_1}(v_1) \in \partial C_1$, we can assume that $\phi_{C_1}(v_i^\theta) \in \partial C_1 \setminus \{v_2\}$, for $\theta$ small enough. Therefore, $d(v_1^\theta, \phi_{C_1}(v_i^\theta)) \leq D(C_1)$, because both points belong to $C_1$. But the equality cannot occur, in view of [14, Prop. 3] (recall that $v_1^\theta, \phi_{C_1}(v_i^\theta) \notin \{v_1, v_2\}$), so we have that $d(v_1^\theta, \phi_{C_1}(v_i^\theta)) < D(C_1)$.

Finally, since $\{v_i^\theta\}_\theta$ tends to $v_i$ when $\theta$ goes to zero, $i = 1, 2$, it follows, due to the continuity of the Euclidean distance, that the inequality

$$d(v_i^\theta, \phi_{C_1}(v_i^\theta)) > d(v_1^\theta, \phi_{C_1}(v_1^\theta))$$

will be preserved for $\theta$ small enough. This implies $d_M(P^\theta) = d(v_1^\theta, \phi_{C_1}(v_1^\theta))$, by using (2.2). Thus,

$$d_M(P^\theta) = d(v_1^\theta, \phi_{C_1}(v_1^\theta)) < D(C_1) = d_M(P),$$

which contradicts the minimizing property of the bisection given by $P$. \(\Box\)

**Remark 3.5.** We now mention some brief comments concerning Proposition 3.4.
(i) The reverse of Proposition 3.4 does not hold in general: this can be seen by considering, for instance, a rectangle and the bisection given by the orthogonal line segment to the shortest edges passing through the center of symmetry, which satisfies (3.1) but it is clearly not minimizing. Therefore, some additional hypotheses are needed for an eventual sufficient condition.

(ii) A geometric interpretation of this result is that the maximum relative diameter of a minimizing bisection (given by a line segment passing through the center of symmetry) is necessarily provided by at least two different segments in each congruent subset, unless it is uniquely achieved by the distance between the endpoints of the corresponding line segment determining the bisection.

(iii) The reader may compare Proposition 3.4 with [14, Prop. 3]: in the case of a minimizing bisection given by a line segment passing through the center of symmetry, the farthest distances from both endpoints must coincide, providing the value of the maximum relative diameter.

We will now prove the main Theorem 3.6, which establishes some sufficient conditions for a given bisection to be minimizing. We will apply it later to find minimizing bifactions for several sets in $C_2$.

**Theorem 3.6.** Let $C \in C_2$, and let $p$ be the center of symmetry of $C$. Let $P$ give a bisection of $C$ by a line segment passing through $p$, with endpoints $v_1$, $v_2$, and subsets $C_1$, $C_2$. If there exist $\phi_{C_1}(v_1) \in F_{C_1}(v_1)$, $\phi_{C_1}(v_2) \in F_{C_1}(v_2)$ such that

(i) $d(v_1, \phi_{C_1}(v_1)) = d(v_2, \phi_{C_1}(v_2))$, and

(ii) $(\partial C_2) \setminus (v_1, v_2) \subset A_1 \cup A_2$, where $A_i$ is the complement of the open ball $B_i = B(\phi_{C_1}(v_i), d(v_i, \phi_{C_1}(v_i)))$, for $i = 1, 2$,

then $P$ is a minimizing bisection for $d_M$.

**Proof.** Notice that $d_M(P) = d(v_1, \phi_{C_1}(v_1)) = d(v_2, \phi_{C_1}(v_2))$, in view of (2.2) and the assumed hypothesis. Consider now any bisection of $C$ determined by a line segment $\widetilde{P}$ passing through $p$, with endpoints $\widetilde{v}_1$, $\widetilde{v}_2$. One of these endpoints, say $\widetilde{v}_2$, will necessarily lie in $\partial C_2 \setminus (v_1, v_2)$, and so $\widetilde{v}_2 \in A_1 \cup A_2$. Without loss of generality, we may assume that $\widetilde{v}_2 \in A_1$. Then,

$d(\widetilde{v}_2, \phi_{C_1}(v_1)) \geq d(v_1, \phi_{C_1}(v_1))$, and so

$d_M(\widetilde{P}) \geq d(\widetilde{v}_2, \phi_{C_1}(v_1)) \geq d(v_1, \phi_{C_1}(v_1)) = d_M(P)$,

which yields the minimizing character of the bisection determined by $P$, taking into account Proposition 3.1. \qed

**Remark 3.7.** We note that the second hypothesis in Theorem 3.6 is equivalent to $(\partial C_2 \setminus (v_1, v_2)) \cap (B_1 \cap B_2) = \emptyset$, with the notation therein.

**Remark 3.8.** We emphasize that, to apply Theorem 3.6, we need to find appropriate farthest points from the endpoints of the line segment which determines the bisection. That is, the hypotheses of Theorem 3.6 may not hold for all possible choices of the corresponding farthest points, as shown in the following example. Consider a rhomb $C$ formed by joining two congruent
equilateral triangles, and the bisection given by the common edges, with endpoints \( v_1, v_2 \in \partial C \), see Fig. 5. It is clear that \( v_1 \in F_{C_1}(v_2) \) and \( v_2 \in F_{C_1}(v_1) \), but Theorem 3.6 cannot be used with those choices. However, the vertex \( q \) of \( C \) belongs to \( F_{C_1}(v_1) \cap F_{C_1}(v_2) \), and it is possible to apply the result for that farthest point.

![Figure 5. A rhomb \( C \) formed by joining two congruent equilateral triangles, and a minimizing bisection of \( C \)](image)

Theorem 3.6 provides only sufficient and not necessary conditions as there exist minimizing bisections which cannot be identified by means of this result. This can be easily seen for a circle: any bisection given by a diameter is optimal, but the second hypothesis is not verified. We will also illustrate this fact in the following Example 1.

**Example 1.** Let \( C \) be the centrally symmetric hexagon depicted in Fig. 6, obtained by cutting symmetrically two opposite corners of a square of edge length 8, so that \( C \) is circumscribed about the incircle of this square (the lengths of the resulting edges are 3.31... and 5.66... units, with non-right angles equal to \( 3\pi/4 \)). Consider the bisection determined by the segment \( P \) joining the midpoints of the shortest edges of \( C \). It can be checked that \( d_M(P) = 8.17... \), provided by the distance between the endpoint \( v_1 \) and \( \phi_{C_1}(v_1) \), see Fig. 7. For any other bisection given by a line segment \( P' \) passing through the center of symmetry, it follows that \( d_M(P') > d_M(P) \), since one of the corresponding subsets will contain either the segment \( v_1 \phi_{C_1}(v_1) \) or \( v_2 \phi_{C_1}(v_2) \), both with length equal to \( d_M(P) \), or the segment \( x \phi_{C_2}(v_2) \) or \( x \phi_{C_2}(v_1) \), both with length equal to 8.34... (where \( \phi_{C_2}(v_i) \) is the farthest point from \( v_i \) in \( \partial C_2 \), \( i = 1, 2 \)). Therefore, \( P \) is a minimizing bisection, but Theorem 3.6 cannot be applied because the second condition is not satisfied, as shown in Fig. 7.

### 4. Standard Bisection

In this section we will introduce a particular bisection for a centrally symmetric planar convex body, called **standard bisection**. Its construction is analogous to the one described in [7, Sect. 3], which concerns the standard \( k \)-partitions of \( k \)-rotationally symmetric planar convex bodies (for \( k \in \mathbb{N}, k \geq 3 \)). We will emphasize here the different behavior of the standard bisections in our setting with respect to those ones, in terms of optimality and uniqueness.
Figure 6. Centrally symmetric hexagon obtained by cutting a square

Figure 7. Theorem 3.6 cannot be applied since two pieces of $\partial C_2 \setminus (v_1, v_2)$ are not contained in $A_1 \cup A_2$

Figure 8. Standard bisections for an ellipse and a rectangle

**Definition 4.1.** Let $C \in \mathcal{C}_2$, and let $p$ be the center of symmetry of $C$. A standard bisection of $C$ is a decomposition of $C$ determined by a diameter of the incircle of $C$ centered at $p$, with endpoints in $\partial C$. We will denote the corresponding diameter segment by $P_2(C)$, or simply $P_2$.

Note that, for a given set in $\mathcal{C}_2$, it is always possible to construct an associated standard bisection (due to the central symmetry), which will be given by any shortest chord of the set passing through the center of symmetry, see Fig. 8.

As noted in the Introduction, it is known [7, Th. 4.5] that, for any $k$-rotationally symmetric planar convex body, its corresponding standard $k$-partition (defined by means of $k$ inradius segments symmetrically placed) is always minimizing for the maximum relative diameter functional, when
$k \geq 3$, see Fig. 1. It is then natural to ask whether the standard bisection is minimizing in the centrally symmetric case (which corresponds to $k = 2$). This holds for a wide variety of sets of this class, but it is not true in general, as shown in the following Example 2.

**Example 2.** Let $C$ be a rhomb with one angle at most $\pi/3$ and consider an associated standard bisection of $C$, depicted in the left hand side of Fig. 9. It is clear that this bisection is not minimizing, since the bisection determined by the vertical segment $P$ passing through the center of symmetry (right hand side of Fig. 9) has smaller value for $d_M$. In fact, the standard bisection does not satisfy the necessary condition from Proposition 3.4, and Theorem 3.6 yields that the bisection given by $P$ is minimizing for $d_M$.

![Figure 9. The standard bisection of the rhomb is not minimizing](image)

One might think that the standard bisection from Example 2 is not minimizing essentially because the necessary condition from Proposition 3.4 does not hold. The following example shows that even when this necessary condition is satisfied, we cannot assure the minimizing character of a given standard bisection.

**Example 3.** Let $S$ be a square, and call $v_1, v_2$ the midpoints of the upper and lower edges, and $w_1, w_2$ the midpoints of the other two edges, see Fig. 10. Consider $C = S \cap B(v_1, d(v_1, v_2)) \cap B(v_2, d(v_1, v_2))$, which is a centrally symmetric planar convex body. It is clear that the bisection of $C$ provided by the segment $P_2 = v_1 v_2$ is standard, as well as the bisection given by $P'_2 = w_1 w_2$.

![Figure 10. The standard bisection of $C$ determined by $P_2$ is minimizing, while the standard one given by $P'_2$ is not](image)
Both of them satisfy the necessary condition from Proposition 3.4, but we have that \( d_M(P'_2) = d(w_1, x) > d(v_1, v_2) = d_M(P_2) \), where \( x \in \partial C \) is the point indicated in Fig. 10, and so the bisection given by \( P'_2 \) is not minimizing. Moreover, Theorem 3.6 implies that the bisection determined by \( P_2 \) is minimizing for \( d_M \).

Examples 2 and 3 reveal that a standard bisection is not optimal in general. Although some partial results can be obtained in some restrictive situations, we will refer to Theorem 3.6 to determine whether a given bisection is minimizing.

**Remark 4.2.** Consider \( C \in \mathcal{C}_2 \) and a standard bisection of \( C \) given by \( P_2 \), with endpoints \( v_1, v_2 \in \partial C \). If \( d_M(P_2) = d(v_1, v_2) \), then this bisection is necessarily minimizing. The reason is that for any other bisection determined by a line segment \( P \) passing through the center of symmetry of \( C \), with endpoints \( w_1, w_2 \in \partial C \), it follows that \( d(w_1, w_2) \geq d(v_1, v_2) \), since \( v_1 v_2 \) is a diameter of an incircle of \( C \), and so \( d_M(P) \geq d(w_1, w_2) \geq d(v_1, v_2) = d_M(P_2) \). This property does not hold for bisections which are not standard: if we consider an ellipse \( C \), and the bisection determined by the segment \( P = v_1 v_2 \), where \( D(C) = d(v_1, v_2) \), then we clearly have that \( d_M(P) = d(v_1, v_2) \), but the bisection is not minimizing.

### 4.1. Uniqueness of the Standard Bisection

In general, the standard bisection of a centrally symmetric planar convex body is not uniquely defined: we have two different ones for a given square (joining the midpoints of each pair of opposite edges), and an infinite number of them for a circle (provided by the diameter segments). In these two cases, the maximum relative diameters of the different standard bisections coincide, and so this fact is not relevant for our optimization problem. However, the lack of uniqueness may also refer to the values of the maximum relative diameter, as shown in the following Example 4.

**Example 4.** Let \( C \) be a planar cap body, that is, the convex hull of a circle and two exterior points symmetric with respect to the center (which will be called the vertices of \( C \)). This centrally symmetric planar convex body possesses infinitely many associated standard bisections, determined by each pair of symmetric points lying in the circular arcs of \( \partial C \). In this setting, if the vertices of \( C \) have a distance at least \( \sqrt{3} \) times the diameter of the circle, then all the standard bisections of \( C \) will have different values for the maximum relative diameter. For instance, for the two standard bisections from Fig. 11, the maximum relative diameter equals the distance between an endpoint of the corresponding segment and a vertex of the cap body, thus attaining distinct values. We point out that the same happens for the standard bisections of the set from Example 1, as indicated in Sect. 5 below.
Remark 4.3. The behavior described in Example 4 is another remarkable peculiarity of our problem with respect to the analogous one for $k$-rotationally symmetric planar convex bodies ($k \geq 3$), where different standard $k$-partitions always yield the same value for the maximum relative diameter, due to [7, Lemma 3.2].

Remark 4.4. For a given set $C \in \mathcal{C}_2$, the standard bisection of $C$ is uniquely defined if and only if the associated incircle touches $\partial C$ at exactly two points.

5. Some Examples

In this section we collect several examples of centrally symmetric planar convex bodies, indicating one of the minimizing bisections in each case.

The corresponding standard bisections are minimizing for the square and the rectangle, by direct application of Theorem 3.6.

The case of the circle is special, since the maximum relative diameter functional is constant for any bisection (such a constant is the diameter of the circle). Therefore, any bisection of the circle can be considered minimizing (Fig. 12).

Figure 12. Minimizing bisections for the rectangle and the circle

For the hexagon treated in Example 1, which is depicted in Fig. 6, we have already described a minimizing bisection. We point out that such a bisection is standard, and that there are two other standard bisections for this set (joining each pair of longer opposite symmetric edges), which are not minimizing (it can be checked that the necessary condition from Proposition 3.4 does not hold). Recall that Theorem 3.6 cannot be applied for this optimal bisection.
We have studied the rhomb with one angle at most $\pi/3$, in Example 2: the standard bisection is not minimizing, and the bisection given by a vertical line segment passing through the center of symmetry minimizes $d_M$, see Fig. 9.

For the cap body of Example 4, with the radius of the initial circle $\mathcal{C}$ being 1, we have already indicated that there are infinitely many associated standard bisections, each of them with a different value for the maximum relative diameter when the vertices are sufficiently far from the center of symmetry, see Fig. 11. In these situations, among all of them, Proposition 3.4 gives that the unique minimizing bisection is the one determined by a vertical line segment passing through $p$, where $p$ is the center of symmetry. In fact, this bisection is the unique minimizing one given by a line segment passing through $p$. Namely, let us consider a minimizing bisection determined by a line segment $v_1v_2$ passing through $p$, with subsets $C_1$, $C_2$, taking into account Proposition 3.1. Let $q$ be the vertex of the left hand side cap, and assume that $q$ belongs to $C_1$ (possibly also to $C_2$). There are two cases. If $v_1$, $v_2$ lie on the circular arcs of $\partial C$, we have that $q$ is a farthest point both from $v_1$ and from $v_2$ in $\partial C_1$. Then, unless we have the bisection from Fig. 13 (given by a vertical segment), it can be checked that the necessary condition from Proposition 3.4 is not satisfied.

Suppose now that the above $v_1$, $v_2$ lie in opposite segments included in $\partial C$, with $v_1$ (respectively, $v_2$) on the left (respectively, right) hand side of the vertical chord of $C$ in Fig. 13. If $v_1 = q$ then the bisection is given by a horizontal line segment, with maximum relative diameter strictly greater than the one associated to the bisection by a vertical line segment. Therefore, suppose $v_1 \neq q$. Since $v_1$ does not lie on a circular arc included in $\partial C$, therefore $v_1$ is a relative inner point of the segment included in $\partial C$, and containing $v_1$. We may suppose that $v_1$ lies in the open upper half (and also in the open left half) of Fig. 13.

We are going to prove that

$$\phi_{C_1}(v_2) = q.$$ 

Observe that $C_1$ is bounded by four segments and one circular arc $A_1$. The “vertices” of $C_1$ are, in the positive sense, $v_2$, $v_1$, $q$, and the two endpoints of $A_1$. Clearly $\phi_{C_1}(v_2)$ cannot lie in the relative interiors of any of the four segments bounding $C_1$. However, also $\phi_{C_1}(v_2) \in A_1$ is impossible. Namely, the maximal distance of $v_2$ and a point of the boundary of $\mathcal{C}$ is attained at the intersection point of $v_1v_2$ and $\mathcal{C}$, closer to $v_1$. But this distance is strictly
smaller than \( d(v_1, v_2) \). These imply that

\[
\phi_{C_1}(v_2) \in \{v_1, q\}.
\]

We are going to show that \( d(v_1, v_2) < d(q, v_2) \). Note that it suffices to show that \( \pi/2 < \angle v_2 v_1 q = \angle pv_1 q \), which is equivalent to \( \angle tv_1 p < \pi/2 \), where \( t \) is the point of tangency of \( C \) and the line determined by the segment \( q \overline{v_1} \). However, this last angle is a non-right angle of the right triangle with hypotenuse \( \overline{pt} \), and legs \( \overline{tv_1} \) and \( \overline{v_1 t} \). This ends the proof of \( \phi_{C_1}(v_2) = q \).

Next, we are going to determine \( \phi_{C_1}(v_1) \). Analogously as for \( \phi_{C_1}(v_2) \), also \( \phi_{C_1}(v_1) \) cannot lie in the relative interior of any of the four segments bounding \( C_1 \), and also \( \phi_{C_1}(v_1) \notin A_1 \). This gives

\[
\phi_{C_1}(v_1) \in \{v_2, q\}.
\]

The case \( \phi_{C_1}(v_1) = q = \phi_{C_1}(v_2) \) contradicts the necessary condition of Proposition 3.4. Therefore,

\[
\phi_{C_1}(v_1) = v_2,
\]

and since the bisection is minimizing, Proposition 3.4 implies that

\[
d(v_1, v_2) = d(v_1, \phi_{C_1}(v_1)) = d(v_2, \phi_{C_1}(v_2)) = d(v_2, q).
\]

That is, \( \Delta v_1 v_2 q \) is an isosceles triangle, with base \( v_1 v_2 \). But this is impossible, since we have already checked that the angle \( \angle v_2 v_1 q \) is greater than \( \pi/2 \). This contradiction implies that \( v_1 \) and \( v_2 \) cannot lie in opposite segments of \( \partial C \), and so we conclude that the unique minimizing bisection is the one determined by the vertical segment. Notice that Theorem 3.6 also yields the minimizing property of that bisection, but not the corresponding uniqueness.

On the other hand, we point out that considering the convex hull of the union of the rhomb in Example 2 (see Fig. 9) and the inflation of its incircle from its center in ratio \( 1 + \varepsilon \), with \( \varepsilon > 0 \) small enough (with the four vertices of the rhomb lying outside this inflation), provides examples with infinitely many standard bisections, none of them being minimizing.

Another example is the following: let \( C \) be a centrally symmetric planar convex body contained in the rectangle with vertices \((\pm r, \pm 1)\), where \( r \geq 2 \), satisfying also that \( C \) contains the rhomb with vertices \( v_1 = (0, 1) \), \( v_2 = (0, -1) \) and \( (\pm r, 0) \). The idea is trying to apply Theorem 3.6 to prove that the chord \( v_1 v_2 \) provides an optimal bisection. We let \( C_1 = \{(x, y) \in C : x \leq 0\} \). Any farthest point \( \phi_{C_1}(v_1) \) from \( v_1 \) in \( \partial C_1 \) lies necessarily in the region \( \{(x, y) \in C_1 : x^2 + (y - 1)^2 \geq r^2 + 1\} \), which is bounded by the segment \((-\sqrt{r^2 - 3}, -1)(-r, -1)(-r, 0)(-\sqrt{r^2 - 3}, -1)\), and the smaller circular arc joining \((-r, 0)\) with \((-\sqrt{r^2 - 3}, -1)\), and having center \((0, 0)\). The tangent line \( L \) of the circle of center \( \phi_{C_1}(v_1) \) at its boundary point \( v_1 \) is orthogonal to \( \phi_{C_1}(v_1) \overline{v_1} \). It is clear that if \( L \cap \{x, y) \in C : x \geq 0\} \) lies in the angular domain determined by the points \((0, 0), (0, 1)\) and \((r, 0)\), and if the analogous property holds also for \( v_2 \), then hypothesis ii) from Theorem 3.6 is satisfied, and so, if also the necessary condition holds (which is hypothesis i) from Theorem 3.6), then \( v_1 v_2 \) provides a minimizing bisection. Note that it suffices to check the above condition only for \( \phi_{C_1}(v_1) = (-\sqrt{r^2 - 3}, -1)\),
which is equivalent to have $\alpha \geq \pi/2$, where $\alpha$ is the angle of the angular domain determined by $(-\sqrt{r^2 - 3}, -1), (0, 1)$ and $(r, 0)$. This holds since $r \geq 2$ and so the scalar product of $(-\sqrt{r^2 - 3}, -1) - (0, 1)$ and $(r, 0) - (0, 1)$ is non-positive. Figure 14 shows an example of this statement, namely an ellipse cut down with two symmetrical chords joining endpoints of the two axes.

6. Some Concluding Remarks

We finish these notes with some comments related with this optimization problem.

6.1. Optimal Set

Another interesting question for this problem is searching for the optimal sets, that is, the centrally symmetric planar convex bodies of unit area with the minimum possible value for the maximum relative diameter functional. The unit-area condition here is required just as a normalization for the sets of the class. In this setting, the optimal set is unique and has been obtained in [14, Example 2.3 and Th. 5]: it consists of the intersection of a symmetric circular lens and a certain strip (bounded by two parallel lines), the strip being orthogonal to the segment connecting the centers of the circles.

6.2. Dual Problems

There are optimization problems dual to the one discussed in this paper, but their solutions are trivial. For instance, if we search for the bisections attaining the maximum possible value for $d_M$, it is clear that we can consider a bisection determined by a diameter segment of the set (and so, the diameter will be such a maximum value). In fact, this will happen for any bisection with a subset containing two points whose distance equals the diameter of the set.

On the other hand, we can consider the minimum relative diameter functional, defined as

$$d_m(P) = \min\{D(C_1), D(C_2)\},$$

where $P$ is a simple curve determining a bisection of $C \in \mathcal{C}_2$, with subsets $C_1, C_2$. This functional has been already studied in some previous works, see [8,9]. It is easy to check that $d_m$ tends to zero for bisections with one of its associated subsets being reduced to a point, and that its maximum value will be attained again by a bisection given by a diameter segment.
6.3. Relation with Borsuk’s Conjecture

For a given $C \in \mathcal{C}_2$, the optimization problem for the maximum relative diameter functional treated in this paper is not interesting when that functional is constant over all the bisections of $C$ (in this case, any bisection is minimizing). This situation only happens when $C$ is a circle, and it is related with the classical Borsuk’s conjecture in the following way. In $\mathbb{R}^2$, this conjecture is true and states that any planar convex body $C$ can be divided into three or less subsets with strictly smaller diameters [3]. If $C$ is, in addition, centrally symmetric, it has been proved that the unique set requiring three subsets in the above statement is the circle [4,13], which is equivalent, in our setting, to the fact that $C$ is a circle if and only if any bisection of $C$ has constant maximum relative diameter.

Acknowledgments

The authors would like to thank the referees for useful comments which have improved these notes.

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Received: March 7, 2018.
Revised: April 25, 2019.
Accepted: October 9, 2019.