Fair Split of Profit generated by $n$ Parties

By
Elinor Mualem
Department of Mathematics Technion, Haifa, Israel
elinor.mualem@yashir.co.il
And
Abraham Zaks
Department of Mathematics Technion, Haifa, Israel
And, Actuarial Research Center, Haifa University, Israel
azaks@tx.technion.ac.il

Abstract

We consider $n$ parties with $n$ corresponding utility functions, denoted by $u_1, \ldots, u_n$. Given a positive amount of money $C$, a fair split of $C$ is a vector $(c_1, \ldots, c_n) \in \mathbb{R}^n$ such that $c_1 + \cdots + c_n = C$ and $u_1(c_1) = u_2(c_2) = \cdots = u_n(c_n)$. In this paper we show the existence and uniqueness of a fair split to any given amount of money $C$.

1 Introduction

In our context a utility function is a real function that is continuous and increasing. Also we will always assume that all utility functions map zero to zero. The positive real numbers (including zero) will be denoted by $\mathbb{R}_+$. Given a positive amount of money $C$, a split of $C$ is a vector $(c_1, \ldots, c_n) \in \mathbb{R}^n_+$ such that $c_1 + \cdots + c_n = C$. Given $n$ utility functions, $u_1, \ldots, u_n$, a fair split of $C$ is a split which for the following holds: $u_1(c_1) = u_2(c_2) = \cdots = u_n(c_n)$. Our aim in this paper is to show that a fair split exists and is unique, given the utility functions and the amount of money to split. The existence theorem (theorem 2) uses notations and a theorem that involve simplexes which we give now. For a finite set $S = \{1, 2, \ldots, n\} \subseteq \mathbb{R}^n$, the convex hull of $S$ is:

$$\text{Conv}(S) = \{\sum_{i=1}^{m} \lambda_i x_i | \sum_{i=1}^{m} \lambda_i = 1 \text{ and } \forall i \lambda_i \geq 0\}$$

We denote the standard basis for $\mathbb{R}^n$ by $\{e_1, \ldots, e_n\}$. The standard simplex $\Delta_{n,M}$ is the convex hull of $S = \{Me_1, \ldots, Me_n\}$ and a face of $\Delta_{n,M}$ is the convex hull of a subset of $S$. The faces of $\Delta_{n,M}$ will be denoted by $\Delta_{n,M}(T)$ when $T$ is a subset of $\{1, 2, \ldots, n\}$, so $\Delta_{n,M}(T)$ is the convex hull of $\{Me_i | i \in T\}$. The proof of theorem 2 uses the famous theorem from [KKM]: Let $A_1, \ldots, A_n$ be closed subsets of $\Delta_{n,M}$. If for all $T \subseteq \{1, 2, \ldots, n\}$ the following holds: $\Delta_{n,M}(T) \subseteq \bigcup_{i \in T} A_i$ then $\bigcap_{i=1}^{n} A_i \neq \emptyset$. 


2 Main Results

In this section we fix a set of \( n \) utility functions and a positive amount of money \( C \).

The main theorem: we prove as follows: There exist a unique fair split for the given utility functions and the amount of money. That is, there exist a unique vector \( (c_1, \ldots, c_n) \in \mathbb{R}_+^n \) such that \( u_i(c_i) = u_j(c_j) \) for all \( i, j \) and \( c_1 + \cdots + c_n = C \).

Proof. We prove the existence and uniqueness in two separate steps. First we start with the existence. We study a more general case.

Theorem 2: as the main theory but with the functions \( u_i \) are continuous, positive and map zero to zero (but not necessarily monotone).

The proof of Existence: Let \( u_1, \ldots, u_n \) be \( n \) real functions that are continuous, map zero to zero and are positive for all positive \( x \) (that is, \( u_i(x) > 0 \) for all \( x > 0 \)). For every positive \( C \) there is a vector \( (c_1, \ldots, c_n) \in \mathbb{R}_+^n \) such that \( u_i(c_i) = u_j(c_j) \) for all \( i, j \) and \( c_1 + \cdots + c_n = C \).

We observe that theorem 2 apply to our situation because the utility functions are increasing, map zero to zero and therefore are positive for positive values. Nevertheless, the monotonicity of the function is not used in the proof. In fact, there might be general cases, which are out of our context, when strict monotonicity or even monotonicity cannot be assumed. Given a positive amount of money \( C \), the set of splits of \( C \) is the standard simplex \( \Delta_{n:C} \), therefore a fair split is a vector in \( \Delta_{n:C} \). The proof we give here characterizes the set of fair splits as an intersection of specific subsets of \( \Delta_{n:C} \), and this intersection turns to be non-empty by using theorem 1. Proof of theorem 2. For \( 1 \leq i \leq n \) denote the following closed non-empty subsets of \( \Delta_{n:C} \):

\[
A_i = \left\{ (x_1, \ldots, x_n) \in \Delta_{n:C} \mid u_i(x_i) \geq u_j(x_j) \text{ for all } j \right\}
\]

A fair split \( (c_1, \ldots, c_n) \) of \( C \) is a vector in \( \Delta_{n:C} \) such that \( u_i(c_i) = u_j(c_j) \) for all \( i, j \). Therefore fair splits are exactly the vectors which belong to the intersection of the sets \( A_1, \ldots, A_n \). Hence, we prove that this intersection is not empty and we do so by showing that the hypothesis of theorem 1 hold. Let \( T \) be a subset of \( \{1, 2, \ldots, n\} \) and let \( v = (x_1, x_2, \ldots, x_n) \) be a vector in the face \( F = \Delta_{n:C}(T) \). We show that \( v \) is contained in \( \bigcup_{i \in T} A_i \) and hence show that the hypothesis hold. Denote the index \( k \) for which \( u_k(x_k) \) is maximal among \( u_1(x_1), \ldots, u_n(x_n) \). Then \( v \in A_k \). For indexes \( j \notin T \) we have \( x_j = 0 \) thus \( u_j(x_j) = u_j(0) = 0 \). For indexes \( i \in T \) we have \( x_i \geq 0 \) and there is an index \( r \in T \) such that \( x_r > 0 \). For this index \( u_r(x_r) > 0 \). Therefore, the index \( k \) (where the maximum is attained) is in \( T \) and the following holds:

\[
v \in A_k \subseteq \bigcup_{i \in T} A_i
\]

and therefore \( F \subseteq \bigcup_{i \in T} A_i \). Next we prove the uniqueness. The uniqueness depend on the strictly monotonicity property of the utility functions, a fact that we will exploit in the proof. [Uniqueness] Assuming the utility function are strictly monotone, a fair split is unique. Let \( v = (c_1, c_2, \ldots, c_n) \in \Delta_{n:C} \) be a fair split and let \( w = (x_1, x_2, \ldots, x_n) \) be any
other vector in $\Delta_{n:C}$. We show that $w$ is not a fair split. Since $w \neq v$ there is an index $i$ for which $x_i \neq c_i$, and we assume without lose of generality that $x_i < c_i$. The sum of coordinates of both vectors is $C$ and therefore the following holds:

$$\sum_{j \neq i} x_j > \sum_{j \neq i} c_j$$

In particular there is an index $k \neq i$ such that $x_k > c_k$. Now, since the utility functions are strictly increasing we have:

$$u_i(x_i) < u_i(c_i) = u_k(c_k) < u_k(x_k)$$

hence $u_i(x_i) \neq u_k(x_k)$ so in particular $w$ is not a fair split. By a remark of Professor Baruch Granovski, the assumption of theorem 2 and assuming that $u_i(C) = 1$ for all $i$ can be used to give much simpler proof of theorems 2 as follows: Let $u_i$, $i = 1,...,n$ be strictly increasing and continuous functions mapping $[0,C]$ to $[0,1]$. Then there is a vector $(x_1,...,x_n) \in \Delta_{n:C}$ such that $u_i(x_i) = u_j(x_j)$ for $1 \leq i,j \leq n$. Since $u_i$ are bijective the inverse functions $u_i^{-1}$ are also continuous and strictly increasing. Also, $u_i^{-1}(0) = 0$ and $u_i^{-1}(1) = C$. Define the function

$$g(x) = u_1^{-1}(x) + \cdots + u_n^{-1}(x)$$

Then, $g$ is continuous, $g(0) = 0$ and $g(1) = nC$. Hence, by the mean value theorem there is a $0 \leq z \leq 1$ such that

$$g(z) = u_1^{-1}(z) + \cdots + u_n^{-1}(z) = C$$

Consequently, if we define $x_i = u_i^{-1}(z)$ then $(x_1,...,x_n)$ is in $\Delta_{n:C}$ and $u_i(x_i) = z$ for all $i = 1,...,n$ so the theorem is proved.

### 3 Example

Assume we have 3 traders that wish to split between them the sum of $C = 148,847.86$. The traders have the following corresponding utility functions: $u_1(x) = \ln(1 + x)$, $u_2(x) = \sqrt{x}$ and $u_3(x) = x$. To be fair, we normalize the 3 functions such that $u_i(C) = 1$ for $i = 1,2,3$, that is our functions are:

$$u_1(x) = \frac{\ln(1+x)}{\ln(1+C)}, u_2(x) = \frac{\sqrt{x}}{\sqrt{C}}, u_3(x) = \frac{x}{C}$$

Theorem 2 insure the existence of a vector $(c_1, c_2, c_3)$ with positive entries that satisfy the following equations:

$$u_1(c_1) = u_2(c_2) = u_3(c_3)$$
$$c_1 + c_2 + c_3 = C$$

that is,

$$\frac{\ln(1+c_1)}{1+C} = \frac{\sqrt{c_2}}{\sqrt{C}} = \frac{c_3}{C}$$

Using MATLAB, we find the following solution:

$$c_1 = 1490.82 \quad c_2 = 56,032.06 \quad c_3 = 91,324.98$$
4 BIBLIOGRAPHY

[Sha72] L.S. Shapley, Cores of Convex Games, International Journal of Game Theory, Volume 1, (1929), pp. 11-26.

[KT79] D. Kahneman and A. Tversky, Prospect Theory: An Analysis of Decision Under Risk, Econometrica, Volume 47, Number 2,(1979). pp. 263-291.

[Bor74] K. Borch, The Mathematical Theory of Insurance, D.C. Heath and Company, 1974.

[MZ] E. Mualem A. Zaks, Joining insured groups: How to split the emerging profit SSRN 2929290 (2017)

[KKM] B. Knaster, C. Kuratowski, and C. Mazurkiewicz, Ein Beweis des Fixpunktsatzes fuer n-dimensionale Simplexe, Fundamenta Mathematicae 14 (1929), 132-137.

[BGH+97] N.L. Brower, H.U. Gerber, J.C. Hickman, D.A. Jones & C.J. Nesbitt, Actuarial Mathematics, The Society of Actuaries, 1997.

[Fis94] P.C. Fishburn, Utility and Subjective Probability, Handbook of Game Theory, Vol. 2, Elsevier Science B.V., pp. 1397-1435, 1994;