θ-dependence and center symmetry in Yang-Mills theories

Claudio Bonati, Marco Cardinali, Massimo D’Elia, and Fabrizio Mazziotti

Dipartimento di Fisica dell’Università di Pisa and INFN - Sezione di Pisa, Largo Pontecorvo 3, I-56127 Pisa, Italy.

We investigate the relation between the realization of center symmetry and the dependence on the topological parameter θ in SU(N) Yang-Mills theories, exploiting trace deformations as a tool to regulate center symmetry breaking in a theory with a small compactified direction. We consider, in particular, SU(4) gauge theory, which admits two possible independent deformations, and study, as a first step, its phase diagram in the deformation plane for two values of the inverse compactified radius going up to \( L^{-1} \sim 500 \text{ MeV} \), comparing the predictions of the effective 1-loop potential of the Polyakov loop with lattice results. The θ-dependence of the various phases is then addressed, up to the fourth order in θ, by numerical simulations: results are found to coincide, within statistical errors, with those of the standard confined phase if center symmetry is completely restored and independently of the particular way this happens, i.e. either by local suppression of the Polyakov loop traces or by long range disorder.

I. INTRODUCTION

Pure gauge theories, defined on a space-time with one or more compactified direction, possess a symmetry under global transformations which can be classified as gauge transformations respecting the periodicity but for a global element of the center of the gauge group (e.g., \( \mathbb{Z}_N \) for SU(N) gauge theories): this is known as center symmetry. Such symmetry regulates most of the phase structure of the pure gauge theory, undergoing spontaneous symmetry breaking (SSB) for small enough compactification radii, and the Polyakov loop (holonomy) around the compactified direction is a proper order parameter for its realization. When the compactified direction is the thermal Euclidean direction, the transition is associated to deconfinement and the Polyakov loop is defined as

\[
P(\vec{x}) = \mathcal{P} \exp \left( i \int_0^L A_0(\vec{x}, \tau) d\tau \right);
\]

its trace vanishes in the confined phase (\( \langle \text{Tr} P \rangle = 0 \)), while it is different from zero for \( T > T_c \), where \( T_c \) is the deconfinement critical temperature (e.g., for SU(N), \( \langle \text{Tr} P \rangle = \alpha e^{2\pi n/N} \), with \( n \in \{0, 1, \ldots N-1\} \) and \( \alpha > 0 \)).

Yang-Mills theories are characterized by many other non-perturbative properties, whose relation to center symmetry is still not clear. Among them, a significant role is played by the dependence on the topological parameter θ, which enters the (Euclidean) Lagrangian as follows:

\[
\mathcal{L}_\theta = \frac{1}{4} F^a_{\mu \nu}(x) F^a_{\mu \nu}(x) - i \theta q(x),
\]

where \( q(x) \) is the topological charge defined by

\[
q(x) = \frac{g^2}{64\pi^2} \epsilon_{\mu \nu \rho \sigma} F^a_{\mu \nu}(x) F^a_{\rho \sigma}(x).
\]

A non-zero θ breaks CP symmetry explicitly, and a non-trivial dependence on θ is induced by gauge configurations with non-trivial winding number \( Q = \int d^4 x q(x) \) populating the path-integral of the theory. The relevant information is contained in the free energy density \( f(\theta) \), which around \( \theta = 0 \) can be usefully parametrized as a Taylor expansion as follows:

\[
f(\theta) = f(0) + \frac{1}{2} \chi \theta^2 (1 + b_2 \theta^2 + b_4 \theta^4 + \cdots)
\]

where the topological susceptibility \( \chi \) and the coefficients \( b_{2n} \) can be related to the cumulants of the topological charge distribution at \( \theta = 0 \) by the relations

\[
\chi = \frac{\langle Q^2 \rangle_{c, \theta=0}}{\mathcal{V}}, \quad b_{2n} = (-1)^n \frac{2}{(2n+2)!} \frac{\langle Q^{2n+2} \rangle_{c, \theta=0}}{\langle Q^2 \rangle_{c, \theta=0}}
\]

where \( \mathcal{V} \) is the four-dimensional volume.

General large-N arguments \[3, 4\] predict that, in the low temperature confined phase of the theory, the susceptibility stays finite in the large-N limit, while the \( b_{2n} \) are suppressed by increasing powers of \( 1/N \), as follows;

\[
\chi = \chi_\infty + O(N^{-2}), \quad b_{2n} = O(N^{-2j}).
\]

Such predictions have been checked successfully both for \( \chi \) \[3–8\], with \( \chi_\infty \) turning out to be compatible with the value predicted by the Witten-Veneziano solution to the \( U_A(1) \) problem \[3\] \[10\], and for the fourth order coefficient \( b_{2} \) \[3, 11\] \[12\].

On the other hand, at asymptotically large T, i.e. small compactification radius, the theory becomes weakly coupled and one expects that instanton calculus can be safely applied, leading to the validity of the dilute instanton gas approximation (DIGA) \[13, 18\]

\[
f(\theta) - f(0) \sim \chi(T) (1 - \cos \theta)
\]

\[
\chi(T) \sim T^4 \exp[-8\pi^2/g^2(T)] \sim T^{-\frac{13}{2}}N^4.
\]
which predicts that the topological susceptibility vanishes exponentially fast with $N$, while the $b_{2n}$ coefficients stay constant (for instance $b_2 = -1/12$), contrary to the large-$N$ low-$T$ scaling. The asymptotically large temperature at which DIGA should set in is not known a priori; moreover, while the prediction for $\chi(T)$ comes from a 1-loop computation, the $(1 - \cos \theta)$ dependence expresses the fact that instantons and anti-instantons can be treated as independent, non-interacting objects, which is the essential feature of DIGA, and this could be true far before perturbative estimates become reliable.

In fact, various theoretical arguments support the idea that the change of regime should take place right after $T_c$, and faster and faster as $N$ increases. This scenario is strongly supported by lattice computations: the topological susceptibility drops at $T_c$ [7, 22, 26], and it does so faster and faster as $N$ increases, pointing to a vanishing of $\chi$ right after $T_c$ in the large-$N$ limit [7, 22]. The vanishing of $\chi$ might not be enough to prove that DIGA sets in, so that a stronger and definite evidence comes from studies of the coefficient $b_2$, proving that it reaches its DIGA value right after $T_c$, and faster and faster as $N$ increases [20, 27].

As a consequence of the drastic change in the $\theta$-dependent part of the free energy around $T_c$, the critical temperature itself is affected by the introduction of a non-zero $\theta$, in particular $T_c$ turns out to be a decreasing function of $\theta$ [28].

The facts summarized above point to a strict relation between the realization of center symmetry and the $\theta$-dependence of SU($N$) Yang-Mills theories, which one would like to investigate more closely. A powerful tool, in this respect, is represented by trace deformed Yang-Mills theories, which have been introduced in Ref. [37], although already explored by lattice simulations in Ref. [52]. The idea, which is inspired by the perturbative form of the Polyakov loop effective action at high temperature [14], is to introduce one or more (depending on the gauge group) center symmetric couplings to the Polyakov loop and its powers, so as to inhibit the spontaneous breaking of center symmetry even in the presence of an arbitrarily small compactification radius. In this way, one can approach the weak coupling regime, where semiclassical approaches are available, while keeping center symmetry intact, so that the relation with $\theta$-dependence can be investigated more systematically.

Several works have already considered the use of trace deformed theories and also possible alternatives, like the introduction of adjoint fermions or the use of non-thermal boundary conditions [39, 50]. There are actually already well definite semiclassical predictions regarding $\theta$-dependence in the center-symmetric phase [37, 60, 62], which come essentially from the fact that in the limit of small compactification radius the deformed theory can be described in terms of non-interacting objects with topological charge $1/N$ (a sort of Dilute Fractional Instanton Gas Approximation, or DFIGA). This leads to predict $f(\theta) - f(0) \propto 1 - \cos(\theta/N)$, hence for instance $b_2 = -1/(12N^2)$. While these predictions are in agreement with general large-$N$ scaling for the confined phase exposed above, they are not in quantitative agreement with the lattice results for the confined phase, which yield instead $b_2 = -0.23(3)/N^2$ [10]; in addition, also the topological susceptibility itself is predicted to show significant deviations, for large $N$ and small compactification radius [60], from the behavior showed in the standard confined phase.

It is therefore quite remarkable that, instead, lattice results obtained for SU(3), which have been reported for the first time in Ref. [63], show that one recovers exactly the same $\theta$-dependence as in the confined phase (i.e. the same value, within errors, for both $\chi$ and $b_2$) as soon as the trace deformation is strong enough to inhibit the breaking of center symmetry. The disagreement with semiclassical predictions is not a surprise, since the values of the compactification radius $L$ explored in Ref. [63] go up to $L^{-1} \equiv T \approx 500$ MeV, while the condition for the validity of the semiclassical approximation is $T \gg N\Lambda$ where $\Lambda$ is the non-perturbative scale of the theory, so that $T \sim 500$ MeV is a scale where non-perturbative corrections can still be important. What is a surprise, claiming for further investigations, is the fact that such non-perturbative corrections are exactly the same as in the standard confined phase, leading to the same $\theta$-dependence also from a quantitative point of view.

The purpose of the present study is to make progress along this line of investigation, by extending the results of Ref. [63] to larger SU($N$) gauge groups, considering in particular the case $N = 4$. There are various reasons to expect that the study of SU(4) may lead to new non-trivial insights. Apart from the fact that the space of trace deformations extends to two independent couplings, we have that the possible breaking patterns of the center symmetry group $Z_4$ are more complex, including also a partial $Z_4 \rightarrow Z_2$ breaking which corresponds to a phase differing from both the standard confined and the deconfined phase of the undeformed theory.

The way one can move across the different phases by tuning the two deformation couplings can be predicted based on the 1-loop Polyakov loop effective potential. However, as we will discuss, numerical simulations show the presence of non-trivial corrections induced by fluctuations, which lead to complete center symmetry restoration also when this is not expected. Moreover, one has

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1 There are various examples of quantum field theories with non-trivial $\theta$-dependence where $\chi$ is predicted to vanish in some limit, while the $b_{2n}$ coefficients do not reach their DIGA values, like CP$^{N-1}$ models in two dimensions and in the large-$N$ limit [10, 31, 53] or QCD with dynamical fermions in the chiral limit [31, 36].

2 Of course, this offers the possibility to investigate connection of center symmetry to many other non-perturbative features of Yang-Mills theory, although in the present study we are exclusively concerned with $\theta$-dependence.
the possibility to check whether the \( \theta \)-dependence of the standard confined phase is achieved just for complete or also after partial restoration of center symmetry.

The paper is organized as follows. In Section II we review the definition of \( SU(N) \) pure gauge theories in the presence of trace deformations, our lattice implementations and the numerical strategies adopted to investigate \( \theta \)-dependence; in Section III we first compare the predictions of 1-loop computations of the phase diagram with numerical results, then discuss the \( \theta \)-dependence observed for the various phases; finally, in Section IV we draw our conclusions.

II. TECHNICAL AND NUMERICAL SETUP

To investigate the relation between center symmetry and \( \theta \)-dependence we will use, as already anticipated in Section II, trace deformed Yang-Mills theories. In order to inhibit the spontaneous breaking of center symmetry when the theory is defined on a manifold with a compactified dimension, new terms (the trace deformations) are added to the standard Yang-Mills action, which are directly related to traces of powers of Polyakov loops along the compactified direction.

The action of the trace deformed \( SU(N) \) Yang-Mills theory is thus

\[
S_{\text{def}} = S_{YM} + \sum_{\vec{n}} \left( \sum_{j=1}^{[N/2]} h_j |\text{Tr} P^j(\vec{n})|^2 \right) ,
\]

where \( \vec{n} \) denotes a generic point on a surface orthogonal to the compactified direction, the \( h_j \) are new coupling constants, \( P(\vec{n}) \) is the Polyakov loop associated to the compactified direction and \( [ \ ] \) denotes the floor function. The number of possible trace deformations is equal to the number of independent, center-symmetric functions of the Polyakov loop; in general, for \( N > 3 \), more than one deformation could be needed, in order to prevent the possibility of a partial breaking of the center symmetry, with a nontrivial subgroup of \( \mathbb{Z}_N \) left unbroken.

In order to clarify this point, let us specialize to the case \( N = 4 \), which is the one that will be thoroughly investigated in the following, and it is the simplest case in which a partial breaking of center symmetry can take place. For \( N = 4 \) the action in Eq. (8) reduces to

\[
S_{\text{def}} = S_{YM} + h_1 \sum_{\vec{n}} |\text{Tr} P(\vec{n})|^2 + h_2 \sum_{\vec{n}} |\text{Tr} P^2(\vec{n})|^2
\]

and complete restoration of \( \mathbb{Z}_4 \) requires the vanishing of the expectation values of the two traces, \( \text{Tr} P \) and \( \text{Tr} P^2 \).

A priori none of the two new terms in the action is sufficient to guarantee complete center symmetry restoration: for instance, \( M = \text{diag}(1,1,-1,-1) \) has \( \text{Tr} M = 0 \) but \( \text{Tr} M^2 \neq 0 \), while \( M = \text{diag}(1,1,i,-i) \) has \( \text{Tr} M^2 = 0 \) but \( \text{Tr} M \neq 0 \). If \( \langle \text{Tr} P \rangle = 0 \) and \( \langle \text{Tr} P^2 \rangle \neq 0 \) (a possibility which is forbidden if \( N \leq 3 \)) center symmetry is spontaneously broken with the breaking pattern \( \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \), which corresponds to the fact that single quarks are confined but couples of quarks are not.

It thus seems that the term \( |\text{Tr} P(\vec{n})|^2 \) in the action is needed to force \( \langle \text{Tr} P \rangle = 0 \) and the term \( |\text{Tr} P^2(\vec{n})|^2 \) to force \( \langle \text{Tr} P^2 \rangle = 0 \), but one should also take into account the following fact. Trace deformations are spatially local quantities, i.e. they tend to suppress \( \text{Tr} P(\vec{n}) \) and \( \text{Tr} P^2(\vec{n}) \) pointwise. However, the restoration of a global symmetry can also be induced by disorder, since order parameters are spatially averaged quantities, and this is what actually happens in many well known cases, just like ordinary Yang-Mills theory (see, e.g., the discussion on the adjoint Polyakov loop in Ref. [62]). This will be particularly important in the following, when we will present an analysis of the predicted phase diagram of the deformed \( SU(4) \) gauge theory based on the 1-loop effective potential of the Polyakov loop: this kind of analysis assumes a spatially uniform Polyakov loop, hence neglects the possibility of long-distance disorder. This is a possible explanation of the fact that numerical results will show sometimes deviations from the 1-loop effective potential prediction, so that, for instance, center symmetry can be restored completely in some cases by adding just one trace.

The discretization of the action in Eq. (9) does not present particular difficulties: for the Yang-Mills action \( S_{YM} \) we adopt the standard Wilson action [65] (in the following \( \beta \) will denote the bare coupling \( \beta = 6/g^2 \)) and trace deformations can be rewritten straightforwardly in terms of the lattice variables. The update of the links directed along spatial directions can be performed by using heatbath and overrelaxation algorithms [67–69] implemented à la Cabibbo-Marinari [70], while for the temporal links (which do not enter linearly in the action) we have to resort to a Metropolis update [66].

The procedure we used to assign an integer topological charge value \( Q \) to a given configuration is the following [3]: first of all we reduced the ultraviolet noise present in the configuration by using cooling [71, 72] (the numerical equivalence of different smoothing algorithms was shown in several studies, see Refs. [73, 81]), then we computed on the smoothed configurations the quantity

\[
Q_{n_i} = \sum_x q_L(x), \quad \text{where } q_L(x) = \text{the discretization of the topological charge density introduced in Refs. [82, 83]}
\]

\[
q_L(x) = \frac{1}{2\pi^2} \sum_{\mu
\\nu\alpha\beta}^{\pm4} \epsilon_{\mu\nu\alpha\beta} \text{Tr} (\Pi_{\mu\nu}(x)\Pi_{\alpha\beta}(x)) .
\]

In this expression \( \Pi_{\mu\nu} \) is the plaquette operator and the modified Levi-Civita tensor \( \epsilon_{\mu\nu\alpha\beta} \) coincides with the standard one for positive indices, while its value for negative indices is completely determined by \( \epsilon_{\mu\nu\alpha\beta} = -\epsilon_{(-\mu)\nu\alpha\beta} \) and complete antisymmetry. The integer value of the topological charge \( Q \) is finally related to \( Q_{n_i} \).
where “round” stands for the rounding to the closest integer and the constant $\alpha$ was fixed in such a way as to make $\langle (Q - \alpha Q_{ni})^2 \rangle$ as small as possible (see Refs. \[3,15\] for more details).

From the Monte-Carlo history of $Q$ it is straightforward to estimate the topological susceptibility by using Eq. (10). This is a priori possible also for the coefficient $b_2$, however this is known not to be the most efficient way of extracting it: a $b_2$ estimator with a more favorable signal-to-noise ratio (especially for large volumes) can be obtained by performing simulations at non-vanishing (imaginary, to avoid the sign problem) values of the topological SUSY

In practice, if a $\theta$-term of the form $-\theta_L q_L(x)$ is added to the lattice action, $b_2$, $\chi$ and the finite lattice renormalization constant of $q_L(x)$ \[6\] can be extracted from the cumulants of the topological charge distribution at $\theta_L \neq 0$. This approach, although apparently more computationally demanding than the standard one at $\theta_L = 0$, turns out in fact to be much more efficient to obtain reliable estimates of $b_2$. For more details we refer to Ref. \[15\], where the same method used in the present study was adopted and explained at length.

We finally note that, despite the advantages of the imaginary-$\theta$ method, a determination of $b_2$ is still significantly more challenging than a determination of the topological susceptibility. For this reason in Section III B we will use the topological susceptibility when performing a broad scan of the $\theta$-dependence across the phase diagram, while $b_2$ will be measured only for some specific points.

III. RESULTS

The description of our numerical results is divided in two steps. First, we will discuss the phase structure of the deformed $SU(4)$ gauge theory in the $h_1$-$h_2$ plane and for values of the compactification radius (temperature) for which center symmetry is broken at $h_1 = h_2 = 0$: we will make use of predictions coming from the 1-loop effective potential, and compare them with results from numerical simulations. In the second part, the $\theta$-dependence which is found in the different phases will be presented and discussed.

A. Phase diagram in the deformation space: 1-loop predictions confront numerical results

In the perturbative regime, the effective potential of a translation invariant $SU(4)$ configuration (with $P(\vec{n}) \equiv P$) assumes the form \[5\]

\[ V[P] = \mathcal{E}(P) + h_1 |\text{Tr}P|^2 + h_2 |\text{Tr}P^2|^2 , \]

where $\mathcal{E}(P)$ is the 1-loop effective potential of the standard Yang-Mills theory computed in Ref. \[17\]:

\[ \mathcal{E}(P) = \sum_{k=1}^{\infty} \frac{1}{k^4} |\text{Tr}P^k|^2 . \]

Since Eq. (12) is an $SU(4)$ invariant function, the effective potential can be conveniently rewritten as a function of the three independent eigenvalues of $P$.

Despite the apparent simplicity of Eq. (12), it is far from trivial to obtain a closed analytical expression for the position of its absolute minimum. It is nevertheless possible to gain some analytical insight into the breaking of center symmetry and the structure of the phase diagram of the $SU(4)$ deformed Yang-Mills theory. Every matrix $M \in SU(4)$ satisfying $TrM = TrM^2 = 0$ is equivalent to the diagonal matrix with eigenvalues $\lambda_k = e^{i\alpha_k}$ ($k = 0, \ldots, 3$), with $\alpha_k = \frac{\pi}{4} + k\frac{\pi}{4}$. If we denote by $\mathcal{R}$ the region of the $(h_1, h_2)$ plane corresponding to points for which $\{\lambda_k\}$ is a local minimum of Eq. (12), the parameter region in which center symmetry is not broken is necessarily a subset of $\mathcal{R}$ and $\mathbb{Z}_4$ is surely broken for all the values $(h_1, h_2)$ outside $\mathcal{R}$. The region $\mathcal{R}$ can be analytically determined and it can be seen that

\[ \mathcal{R} = \left\{ h_1 > \frac{5}{24} \right\} \cap \left\{ h_2 > \frac{1}{96} \right\} , \]

FIG. 1: Graphical representation, in the plane $(h_1, h_2)$, of the region $\mathcal{R}$ corresponding to points for which $\lambda_k = e^{i\alpha_k}$ ($k = 0, \ldots, 3$), with $\alpha_k = \frac{\pi}{4} + k\frac{\pi}{4}$, is a local minimum of the 1-loop effective potential.

FIG. 2: Phase diagram obtained from simulations performed at bare coupling $\beta = 11.15$ on a $6 \times 32^4$ lattice, corresponding to an inverse compactification radius $L^{-1} = T \simeq 393$ MeV.
as shown in Fig. 1. In particular, as anticipated, we see that a single deformation is not sufficient to ensure the absence of center symmetry breaking in the 1-loop effective action: the axes $h_1 = 0$ and $h_2 = 0$ lay outside $\mathcal{R}$ and $\mathbb{Z}_4$ has to be broken there.

To test the effectiveness of the 1-loop potential in predicting the phase diagram, we also numerically investigate the phase diagram of the lattice deformed Yang-Mills theory, using a $6 \times 32^3$ lattice and two values of the lattice coupling larger than the critical value $\beta_c \simeq 10.79$ (see Ref. [64]). More in detail, we considered $\beta = 11.15$ (corresponding to an inverse compactification radius $T \approx 393$ MeV) and $\beta = 11.40$ ($T \approx 482$ MeV), then performed a scan of the plane $(h_1, h_2)$ in the range $[0, 2] \times [0, 2]$ with a step $\Delta = 0.1$, for a total of 441 simulation points for each $\beta$ value. The scale has been fixed using the determination of Ref. [64] (see in particular Eq. (35) therein) and fixing the string tension to be $\sigma = (440 \text{ MeV})^2$.

The phase diagram obtained from numerical simulations performed at $\beta = 11.15$ is shown in Fig. 2 in a small region around the origin $\mathbb{Z}_4$ is completely broken, while outside there is no breaking at all, apart from a region at large values of $h_1$, where $\mathbb{Z}_4$ breaks partially.

![FIG. 3](image1.png)

FIG. 3: An example of complete breaking of center symmetry ($\mathbb{Z}_4 \rightarrow \text{Id}$) at $\beta = 11.15$. We report the Monte-Carlo histories of $\text{Re}(\text{Tr}P)$, $\text{Im}(\text{Tr}P)$, $\text{Re}(\text{Tr}P^2)$, $\text{Im}(\text{Tr}P^2)$ for $h_1 = 0.0$, $h_2 = 0.1$. Both $\text{Re}(\text{Tr}P)$ and $\text{Re}(\text{Tr}P^2)$ are non-zero.

![FIG. 4](image2.png)

FIG. 4: An example of complete restoration of center symmetry for $\beta = 11.15$, $h_1 = 0.0$ and $h_2 = 1.7$. The Monte-Carlo histories of all quantities, $\text{Re}(\text{Tr}P)$, $\text{Im}(\text{Tr}P)$, $\text{Re}(\text{Tr}P^2)$, and $\text{Im}(\text{Tr}P^2)$, fluctuate around their zero average values. It is interesting to notice that the fluctuations of $\text{Tr}P$ are significantly larger than those of $\text{Tr}P^2$: indeed $\langle P \rangle$ should not be zero according to the 1-loop effective potential, and vanishes because of long range disorder.

![FIG. 5](image3.png)

FIG. 5: The histogram of $\text{Re}(\text{Tr}P^2)$ computed using a $6 \times 32^3$ lattice at bare coupling $\beta = 11.15$ for three different values of $h_1$ along the $h_2 = 0$ axis.
FIG. 6: $\langle |\text{Tr}P| \rangle$ and $\langle |\text{Tr}P^2| \rangle$ computed using a $6 \times 32^3$ lattice at bare coupling $\beta = 11.15$ for different values of the deformation parameters. Different datasets correspond to deformations of the form $(h_1 \neq 0, h_2 = 0)$, $(h_1 = 0, h_2 \neq 0)$ and $(h_1 = h_2)$.

FIG. 7: $\langle |\text{Tr}P^2| \rangle$ computed using a $6 \times 32^3$ lattice at bare coupling $\beta = 11.15$ for different values of $h_1$ along the $h_2 = 0$ axis.

FIG. 8: Phase diagram obtained from simulations performed at bare coupling $\beta = 11.40$ on a $6 \times 32^3$ lattice, corresponding to an inverse compactification radius $L^{-1} = T \simeq 482$ MeV.

The picture that emerges is in striking contrast with the expectations based on the 1-loop effective potential: even a single deformation is capable of completely stabilizing center symmetry ($0.2 < h_1 \lesssim 4$ for $h_2 = 0$, or $h_2 > 1.1$ for $h_1 = 0$). This can be noticed by looking at Fig. 5 and Fig. 7.

Moving to the larger value of $\beta$ that we have explored (corresponding to a smaller compactification radius), one may expect that predictions based on the 1-loop effective potential get more reliable. The phase diagram obtained for $\beta = 11.40$ is shown in Fig. 8. We can see that indeed the new partially broken phase becomes more manifest, so that center symmetry is now broken along the whole $h_1$ axis, as predicted in terms of the 1-loop potential; however, along the $h_1 = 0$ axis the discrepancy persists, with center symmetry being protected just by the $|\text{Tr}P^2(\vec{n})|^2$ deformation.

Notice that in sketching Fig. 8 we have not made any statement about the order of the various transition lines. This is an issue that should be considered in future studies and by now we can just make some general statements: direct transitions from the completely broken phase to the completely restored phase are expected to be first order, as for the standard deconfining phase transition of $SU(4)$, while transition from the partially restored phase should be in the universality class of the 3D Ising model.

Note that $\langle \text{Tr}P \rangle$ and $\langle \text{Tr}P^2 \rangle$ identically vanish on finite lattices, apart from possible numerical issues related to ergodicity breaking for large volumes.
if they are second order, however they can still be first order, this depends on the dynamics of the system and should be checked by more extensive numerical simulations.

To further investigate the origin of the inconsistencies between the prediction of the 1-loop effective potential and the phase diagram observed in numerical simulations, we studied the quantities

\[ \langle |\text{Tr} P_{\text{loc}}| \rangle \equiv \frac{1}{V} \sum_{\vec{n}} \langle |\text{Tr} P(\vec{n})| \rangle \]  

(16)

\[ \langle |\text{Tr} P_{\text{loc}}^2| \rangle \equiv \frac{1}{V} \sum_{\vec{n}} \langle |\text{Tr} P^2(\vec{n})| \rangle . \]  

(17)

Since the squared modulus in this case is taken over local, rather than spatially averaged, quantities, such observables should be less sensitive to long range disorder and follow more closely the prediction of the 1-loop effective potential.

Our results have been obtained by performing simulations using three different setups for the deformation parameters \( h_1 \) and \( h_2 \) in Eq. (12): the first two setups are the ones in which only a single deformation is present, i.e. \( h_1 \neq 0 \) and \( h_2 = 0 \) or \( h_1 = 0 \) and \( h_2 \neq 0 \). The third setup is the one in which both deformations are active and, for the sake of the simplicity, we restricted to the “diagonal” configuration \( h_1 = h_2 \). We show in particular results obtained for \( \beta = 11.15 \) on the \( 6 \times 32^3 \) lattice (which is one of the two setups already discussed above), which are reported in Figs. 9 and 10 and there compared to reference values obtained on the same lattice and without any deformation at \( \beta = 10.50 \), which is deep into the confined phase. The corresponding quantities, for which the squared modulus is taken after the spatial average, have been already shown in Fig. 6.

The general lesson we can learn by comparing the different behaviors is the following. On one hand, it is clear that the local quantities, \( \langle |\text{Tr} P_{\text{loc}}| \rangle \) and \( \langle |\text{Tr} P_{\text{loc}}^2| \rangle \), are significantly more suppressed, with respect to their values in the standard confined phase, when a direct coupling to the relevant deformation is present (i.e., respectively, \( h_1 \neq 0 \) or \( h_2 \neq 0 \); this fact was already noticed and discussed in Ref. [63], pointing out to a different kind (from a dynamical point of view) of center symmetry restoration in the trace deformed theory, with respect to the standard confined phase.

On the other hand, when no direct coupling to the relevant deformation is present (i.e., along the \( (0, h) \) axis for \( \langle |\text{Tr} P_{\text{loc}}| \rangle \) and along the \( (h, 0) \) axis for \( \langle |\text{Tr} P_{\text{loc}}^2| \rangle \)), the local quantities are not significantly suppressed or remain almost constant, in agreement with the predictions of the 1-loop effective potential, meaning that in this case the complete restoration of center symmetry takes place because of long range disorder. This is also appreciable from Fig. 4 where the Monte-Carlo histories of the spatially averaged quantities are shown for the same \( \beta \) value and for a point along the \( (0, h) \) axis where \( Z_4 \) is completely restored: \( \text{Tr} P \), which is not coupled to any deformation, averages to zero, but with much larger fluctuations with respect to \( \text{Tr} P^2 \); we interpret this as a manifestation of the fact that \( \text{Tr} P \) is locally non-zero, but fails to reach an ordered phase at large scales.

\[ \text{FIG. 9: Mean value of the local quantity } \langle |\text{Tr} P_{\text{loc}}| \rangle \text{. The black line indicates the value of the undeformed theory for } \beta = 10.50. \text{ The lattice used is } 6 \times 32^3 \text{ and the bare coupling } \beta = 11.15. \]

\[ \text{FIG. 10: Mean value of the local quantity } \langle |\text{Tr} P_{\text{loc}}^2| \rangle \text{. The black line indicates the value of the undeformed theory for } \beta = 10.50. \text{ The lattice used is } 6 \times 32^3 \text{ and the bare coupling } \beta = 11.15. \]

B. \( \theta \)-dependence of the various phases

We are now going to discuss the \( \theta \)-dependence of the different phases identified previously for the deformed \( SU(4) \) theory. It is interesting, in particular, to ask whether the different ways in which \( Z_4 \) can be restored manifest themselves also in a different \( \theta \)-dependence or not. Let us start from the case of the \( 6 \times 32^3 \) lattice at bare coupling \( \beta = 11.15 \) (\( T \approx 393 \text{ MeV} \)), whose phase diagram was shown in Fig. 2. In Fig. 11 we report the behaviour of the topological susceptibility \( \chi \) as a function of the deformation parameters \( h_1 \) and \( h_2 \), for the three
deformation setups introduced above. In order to have a direct comparison with the $T = 0$ result, we plot the ratio between the topological susceptibility $\chi$ in the deformed theory and the one at $T = 0$ continuum value computed in ordinary Yang-Mills theory in Ref. [16]. We are using here the fact, explicitly verified in Ref. [63], that the lattice spacing can be considered to be independent of the deformation parameters. From data in Fig. 11 we see that this is indeed the case for all the deformation setups studied. Moreover, the topological susceptibility always reaches a plateau for large deformations, at a value which is consistent with that of $\chi$ measured at $T = 0$ in ordinary Yang-Mills theory. This asymptotic value is however approached differently in the different deformation setups: when using $b_2 = 0$ or $h_1 = h_2$ the plateau starts from $h \approx 0.2$, while in the setup with $h_1 = 0$ it starts from $h \approx 1.2$. The reason for this behaviour is clear from the phase diagram shown in Fig. 2; these values of the deformation parameters are the one that are needed to reach the $\mathbb{Z}_4$-symmetric phase when moving along the axes or along the diagonal of the phase diagram.

Using the same lattice setting we computed also the coefficient $b_2$ related to the fourth power of $\theta$ in the expansion of the free energy, see Eq. (11). As explained in Sec. IIIA the estimation of $b_2$ is computationally much more demanding than that of $\chi$; for this reason we decided to compute $b_2$ just for three values of the deformations deep in the plateau region, one for each of the three deformation setups previously adopted (with $h = 1.5$ in all the cases). We computed $b_2$ by means of the imaginary $\theta$ method discussed in Sec. IIIA using 7 values of $\theta_L$ in the range $[0, 12]$. The outcome of this analysis is reported in Fig. 12 also for $b_2$ there is a nice agreement between the values computed in the deformed theory in the $\mathbb{Z}_4$ restored phase and the one obtained in the $T = 0$ Yang-Mills case, for all the deformation setups.

It is interesting to compare the results obtained for $b_2$ with the values predicted by using two well known approximation schemes. The first one is the Diga, which is expected to be reliable in ordinary Yang-Mills theory for a small value of the compactification radius. In this approximation the system is supposed to be well approximated by a gas of weakly interacting degrees of freedom, carrying an unit of topological charge ($\pm 1$), and the coefficient $b_2$ is predicted to be $-1/12$. The second approximation scheme is the DFIGA, which is expected to be valid in the center symmetric phase of the deformed theory for small values of the compactification length. In this case the degrees of freedom are still expected to be weakly interacting, but now they carry a fractional topological charge, quantized in units of $1/N$. In this scenario the predicted value is $b_2 = -1/(12N^2)$, i.e. $b_2 = -1/192$ for $SU(4)$. Both these values are shown in Fig. 12 and they are clearly not compatible with numerical data, indicating that the compactification length used is still too large for the interactions between the fractional degrees of freedom to be negligible.

Let us now repeat the same analysis for the second value of the bare coupling constant $\beta$ studied in Section IIIA i.e. $\beta = 11.40$ (corresponding to $T \approx 482$ MeV). The values of the deformations used are $0 \leq h_1 \leq 2$ and $0 \leq h_2 \leq 2$. Three different phases are present, see Fig. 8 and one could expect that also the $\theta$-dependence shows some signal of the presence of the phase with $\mathbb{Z}_4$ broken to $\mathbb{Z}_2$.

From Fig. 12, where the results for the topological susceptibility are reported, we see that this is indeed the case: errors are larger than for $\beta = 11.15$ but it is quite
clear that the values of \( \chi \) approach \( \chi_{T=0} \) only for two of the three deformation setups adopted, namely the one in which \( h_1 = 0 \) and the one in which \( h_1 = h_2 \). By looking at the phase diagram in Fig. \([8]\) we see that these are the only two setups in which the deformations induce a complete restoration of the center symmetry, and that the values of the deformation at which the plateaux are reached are consistent with the boundaries of the region with broken center symmetry. In the remaining deformation setup, in which \( h_2 = 0 \), center symmetry is not completely restored by increasing the value of \( h_1 \), and the system enters the phase in which center symmetry is broken to its \( \mathbb{Z}_2 \) subgroup. While it is not clear why in this phase the susceptibility seems to approach zero as we increase \( h_1 \), it is tempting to interpret the peak at \( h \approx 0.3 \) (at which point \( \chi \approx \chi_T \)) as a proximity effect due to the closeness of the completely restored phase in the phase diagram (see Fig. \([8]\)). In order to investigate this hypothesis we computed the value of the topological susceptibility also using a different setup, i.e. varying \( h_1 \) and keeping \( h_2 = 0.25 \), because from the phase diagram of Fig. \([8]\) we see that in this setup the system passes across all the symmetry breaking patterns. Results are shown in Fig. \([13]\). We can clearly see that the case \((h_1, h_2 = 0.25)\) is in between the “diagonal” case and the one with only the \( h_1 \) deformation: the values of the deformation parameter at which the topological susceptibility is compatible with the one at \( T = 0 \) correspond to the region in which center symmetry is completely restored, this can be appreciated comparing with the phase diagram shown in Fig. \([8]\).

The presence of the partially broken phase is evident also from the values of \( b_2 \) computed at \( \beta = 11.40 \), which are shown in Fig. \([15]\). The values of \( b_2 \) in the phase with completely restored center symmetry are again compatible with the results obtained at \( T = 0 \) in \([16]\), while the values corresponding to the deformation parameters \( h_1 = 1.5, h_2 = 0 \) and \( h_1 = 3.0, h_2 = 0 \) are incompatible with \( b_2(T = 0) \), and lay in the middle between the DIGA prediction \( -1/12 \) and the DFIGA prediction \( -1/192 \).

Altogether lattice data indicate that the \( \theta \)-dependence of the deformed theory coincides with the one of ordinary Yang-Mills theory at \( T = 0 \) only when center symmetry is completely recovered, and this happens independently of the specific way the restoration takes place, i.e. either by local suppression of Tr P and Tr P^2, or by long range disorder. Instead, in the phase in which center symmetry is only partially restored both the topological susceptibility and \( b_2 \) do not reach a clear plateau as a function of the deformation parameter, and they assume values somewhere in between the deconfined and the confined case.
IV. CONCLUSIONS

In this paper we have investigated the relation between center symmetry and $\theta$-dependence in Yang-Mills theories, exploiting trace deformations in order to control the realization of center symmetry breaking in a theory with a small compactified direction. Extending previous results presented in Ref. [63] for the SU(3) pure gauge theory, we have considered SU(4), which is particularly interesting since, apart from allowing a larger space of independent trace deformations, is also the first SU($N$) gauge group for which the center group admits various patterns of symmetry breaking.

As a first step, we have investigated the phase diagram of the theory in the deformation space and for various values of the inverse compactified radius, reaching values up to $L^{-1} \sim 500$ MeV. We have considered predictions from the 1-loop effective potential of the Polyakov loop and compared them to results of numerical lattice simulations, in which the fate of center symmetry breaking has been studied both by global (i.e. averaged over all directions orthogonal to the compactified direction) and local quantities. We have shown that center symmetry in the deformed theory can be completely restored in a way which is sometimes qualitatively different from that of the standard confined phase, as evinced from the expectation value of local quantities directly coupled to the deconfined phase and that of the deconfined phase, interpolating in some way between them.

The failure to reproduce predictions for the $\theta$-dependence coming from semiclassical computations (in particular those equivalent to a sort of DFIGA) can be ascribed, as for the SU(3) results reported in Ref. [63], to the fact that our inverse compactifications radius is still not large. On the other hand, the striking agreement with results from the standard confined phase confirms and reinforces the evidence, already shown for SU(3), for a strict relation between the realization of center symmetry and other relevant non-perturbative features of Yang-Mills theories.

Future studies could extend the present investigation in various directions. Considering other relevant non-perturbative properties, such as the spectrum of glueball masses, is a first non-trivial goal that should be pursued. The extension to large SU($N$) gauge groups is of course another interesting direction.

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