UNIVERSAL AUTOHOMEOMORPHISMS OF $\mathbb{N}^*$

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To the memory of Cor Baayen, who taught us many things

Abstract. We study the existence of universal autohomeomorphisms of $\mathbb{N}^*$. We prove that CH implies there is such an autohomeomorphism and show that there are none in any model where all autohomeomorphisms of $\mathbb{N}^*$ are trivial.

Introduction

This paper is concerned with universal autohomeomorphisms on $\mathbb{N}^*$, the Čech-Stone remainder of $\mathbb{N}$.

In very general terms we say that an autohomeomorphism $h$ on a space $X$ is universal for a class of pairs $(Y, g)$, where $Y$ is a space and $g$ is an autohomeomorphism of $Y$, if for every such pair there is an embedding $e : Y \to X$ such that $f \circ e = e \circ g$, that is, $h$ extends the copy of $g$ on $e[Y]$.

In [1, Section 3.4] one finds a general way of finding universal autohomeomorphisms. If $X$ is homeomorphic $X^\omega$ then the shift mapping $\sigma : X^\mathbb{Z} \to X^\mathbb{Z}$ defines a universal autohomeomorphism for the class of all pairs $(Y, g)$, where $Y$ is a subspace of $X$. One embeds $Y$ into $X^\mathbb{Z}$ by mapping each $y \in Y$ to the sequence $\langle g^n(y) : n \in \mathbb{Z} \rangle$.

Thus, the Hilbert cube carries an autohomeomorphism that is universal for all autohomeomorphisms of separable metrizable spaces and the Cantor set carries one for all autohomeomorphisms of zero-dimensional separable metrizable spaces. Likewise the Tychonoff cube $[0, 1]^{\kappa}$ carries an autohomeomorphism that is universal for all autohomeomorphisms of completely regular spaces of weight at most $\kappa$, and the Cantor cube $2^{\kappa}$ has a universal autohomeomorphism for all zero-dimensional such spaces.

Our goal is to have an autohomeomorphism $h$ on $\mathbb{N}^*$ that is universal for all autohomeomorphisms of all closed subspaces of $\mathbb{N}^*$. The first result of this paper is that there is no trivial universal autohomeomorphism of $\mathbb{N}^*$, and hence no universal autohomeomorphism at all in any model where all autohomeomorphisms of $\mathbb{N}^*$ are trivial. On the other hand, the Continuum Hypothesis implies that there is a universal autohomeomorphism of $\mathbb{N}^*$. The proof of this will have to be different from the results mentioned above because $\mathbb{N}^*$ is definitely not homeomorphic to its power $(\mathbb{N}^*)^\omega$; it will use group actions and a homeomorphism extension theorem.

We should mention the dual notion of universality where one requires the existence of a surjection $s : X \to Y$ such that $g \circ s = s \circ h$. For the space $\mathbb{N}^*$ this was investigated thoroughly in [2] for general group actions.

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1. Some preliminaries

Our notation is standard. For background information on $\mathbb{N}^*$ we refer to [5].

We denote by $\text{Aut}$ the automhomeomorphism group of $\mathbb{N}^*$. We call a member $h$ of $\text{Aut}$ trivial if there are cofinite subsets $A$ and $B$ of $\mathbb{N}$ and a bijection $b : A \to B$ such that $h$ is the restriction of $\beta b$ to $\mathbb{N}^*$.

In both sections we shall use the $G_\delta$-topology on a given space $(X, \tau)$; this is the topology $\tau_{\delta}$ on $X$ generated by the family of all $G_\delta$-subsets in the given space. It is well-known that $w(X, \tau_{\delta}) \leq w(X, \tau)^{\aleph_0}$; we shall need this estimate in Section 3.

2. What if all autohomeomorphisms are trivial?

To begin we observe that fixed-point sets of trivial autohomeomorphism of $\mathbb{N}^*$ are clopen. Therefore, to show that no trivial autohomeomorphism is universal it would suffice to construct a compact space that can be embedded into $\mathbb{N}^*$ and that has an autohomeomorphism whose fixed-point set is not clopen.

The example. We let $L$ be the ordinal $\omega_1 + 1$ endowed with its $G_\delta$-topology. Thus all points other than $\omega_1$ are isolated and the neighbourhoods of $\omega_1$ are exactly the co-countable sets that contain it. Then $L$ is a $P$-space of weight $\aleph_1$ and hence, by the methods in [4, Section 2], its Čech-Stone compactification $\beta L$ can be embedded into $\mathbb{N}^*$.

We define $f : L \to L$ such that $\omega_1$ is the only fixed point of $\beta f$. We put

\[
  f(\omega_1) = \omega_1,
  \]

\[
  f(2 \cdot \alpha) = 2 \cdot \alpha + 1, \quad \text{and}
\]

\[
  f(2 \cdot \alpha + 1) = 2 \cdot \alpha.
\]

This defines a continuous involution on $L$.

If $p \in \beta L \setminus L$ then $p \in \text{cl} \alpha$ for some $\alpha < \omega_1$ and then either $E = \{2 \cdot \beta : \beta < \alpha\}$ or $O = \{2 \cdot \beta + 1 : \beta < \alpha\}$ belongs to the ultrafilter $p$. But $f[E] \cap E = \emptyset = f[O] \cap O$, hence $\beta f(p) \neq p$.

Since $\omega_1$ is not an isolated point of $\beta L$, no matter how this space is embedded into $\mathbb{N}^*$ there is no trivial autohomeomorphism of $\mathbb{N}^*$ that would extend $\beta f$.

3. The Continuum Hypothesis

Under the Continuum Hypothesis the space $\mathbb{N}^*$ is generally very well-behaved and one would expect it to have a universal autohomeomorphism as well. We shall prove that this is indeed the case. We need some well-known facts about closed subspaces of $\mathbb{N}^*$.

First we have Theorem 1.4.4 from [5] which characterizes the closed subspaces of $\mathbb{N}^*$ under CH: they are the compact zero-dimensional $F$-spaces of weight $\mathfrak{c}$, and, in addition: every closed subset of $\mathbb{N}^*$ can be re-embedded as a nowhere dense closed $P$-set.

Second we have the homeomorphism extension theorem from [3]: CH implies that every homeomorphism between nowhere dense closed $P$-sets of $\mathbb{N}^*$ can be extended to an autohomeomorphism of $\mathbb{N}^*$. 

Step 1. We consider the natural action of \( \text{Aut} \) on \( \mathbb{N}^* \), that is the map \( \sigma : \text{Aut} \times \mathbb{N}^* \to \mathbb{N}^* \) given by \( \sigma(f, p) = f(p) \). This action is continuous when \( \text{Aut} \) carries the compact-open topology \( \tau \) and hence also when \( \text{Aut} \) carries the \( G_\delta \)-modification \( \tau_\delta \) of \( \tau \). For the rest of the construction we consider the topology \( \tau_\delta \).

Using this action we define an autohomeomorphism \( h : \text{Aut} \times \mathbb{N}^* \to \text{Aut} \times \mathbb{N}^* \) by \( h(f, p) = (f, f(p)) \). The map \( h \) is continuous because its two coordinates are and it is a homeomorphism because its inverse \( (f, f^{-1}(p)) \) is continuous as well.

Now if \( X \) is a closed subset of \( \mathbb{N}^* \) and \( g : X \to X \) is an autohomeomorphism then we can re-embed \( X \) as a nowhere dense closed \( P \)-set and we can then find an \( f \in \text{Aut} \) such that \( f \upharpoonright X = g \). We transfer this embedded copy of \( X \) to \( \{f\} \times \mathbb{N}^* \) in \( \text{Aut} \times \mathbb{N}^* \); for this copy of \( X \) we then have \( h \upharpoonright X = g \). It follows that \( h \) satisfies the universality condition.

Step 2. We embed \( \text{Aut} \times \mathbb{N}^* \) into \( \mathbb{N}^* \) in such a way that there is an autohomeomorphism \( H \) of \( \mathbb{N}^* \) such that \( H \upharpoonright (\text{Aut} \times \mathbb{N}^*) = h \). Then \( H \) is the desired universal autohomeomorphism of \( \mathbb{N}^* \).

To this end we list a few properties of this product.

Weight. The weight of the product is equal to \( c \), as both factors have weight \( c \). For \( \mathbb{N}^* \) this is clear and for \( \text{Aut} \) this follows because the topology \( \tau \) has weight \( c \) and one obtains a base for \( \tau_\delta \) by taking the intersections of all countable subfamilies of a base for \( \tau \).

Zero-dimensional and \( F \). The product is a zero-dimensional \( F \)-space as the product of the \( P \)-space \( \text{Aut} \) and the compact zero-dimensional \( F \)-space \( \mathbb{N}^* \), see [6, Theorem 6.1].

Strongly zero-dimensional. The product \( \text{Aut} \times \mathbb{N}^* \) is not compact, but we shall construct a compactification of it that is also a zero-dimensional \( F \)-space of weight \( c \).

For this we need to prove that \( \text{Aut} \times \mathbb{N}^* \) is actually strongly zero-dimensional. We prove more: the product is ultraparacompact, meaning that every open cover has a pairwise disjoint open refinement.

Let \( \mathcal{U} \) be an open cover of the product consisting of basic clopen rectangles.

For each \( f \in \text{Aut} \) there is a finite subfamily \( \mathcal{U}_f \) of \( \mathcal{U} \) that covers \( \{f\} \times \mathbb{N}^* \), say \( \mathcal{U}_f = \{C_i \times D_i : i < k_f\} \). Let \( C_f = \bigcap_{i < k_f} C_i \) and \( D_{f,i} = D_i \setminus \bigcup_{j < i} D_j \) for \( i < k_f \). Then \( C_f = \{C_f \times D_{f,i} : i < k_f\} \) is a disjoint family of clopen rectangles that covers \( \{f\} \times \mathbb{N}^* \) and refines \( \mathcal{U} \).

Because \( \text{Aut} \) has weight \( c \), and we assume \( \text{CH} \), there is a sequence \( \{f_{\alpha} : \alpha \in \omega_1\} \) in \( \text{Aut} \) such that \( \{C_{f_{\alpha}} : \alpha \in \omega_1\} \) covers \( \text{Aut} \). Next we let \( V_{\alpha} = C_{f_{\alpha}} \setminus \bigcup_{\beta < \alpha} C_{f_{\beta}} \) for all \( \alpha \). Because \( \text{Aut} \) is a \( P \)-space the family \( \{V_{\alpha} : \alpha \in \omega_1\} \) is a disjoint open cover of \( \text{Aut} \).

The family \( \{V_{\alpha} \times D_{f_{\alpha}, i} : i < k_{f_{\alpha}}, \alpha \in \omega_1\} \) then is a disjoint open refinement of \( \mathcal{U} \).

A compactification. To complete Step 2 we construct a compactification of \( \text{Aut} \times \mathbb{N}^* \) that is a zero-dimensional \( F \)-space of weight \( c \) and that has an autohomeomorphism that extends \( h \). The Čech-Stone compactification would be the obvious candidate, were it not for the fact that its weight is equal to \( 2^c \). More precisely, using some continuous onto function from \( (\text{Aut}, \tau) \) onto [0,1] one obtains a clopen partition of \( (\text{Aut}, \tau_\delta) \) of cardinality \( c \). This shows that \( \beta(\text{Aut} \times \mathbb{N}^*) \) admits a continuous surjection onto the space \( \beta\mathbb{C} \) (where \( \mathbb{C} \) carries the discrete topology).
To create the desired compactification we build, either by transfinite recursion or by an application of the Löwenheim-Skolem theorem, a subalgebra \( B \) of the algebra of clopen subsets of \( \text{Aut} \times \mathbb{N}^* \) that is closed under \( h \) and \( h^{-1} \), of cardinality \( c \), and that has the property that for every pair of countable subsets \( A \) and \( B \) of \( B \) such that \( a \cap b = \emptyset \) whenever \( a \in A \) and \( b \in B \) there is a \( c \in B \) such that \( a \subseteq c \) and \( c \cap b = \emptyset \) for all \( a \in A \) and \( b \in B \). The latter condition can be fulfilled because \( \text{Aut} \times \mathbb{N}^* \) is an \( F \)-space — \( \bigcup A \) and \( \bigcup B \) have disjoint closures — and strongly zero-dimensional — the closures can be separated using a clopen set.

The Stone space \( \text{St}(B) \) of \( B \) is then a compactification of \( \text{Aut} \times \mathbb{N}^* \) that is a compact zero-dimensional \( F \)-space of weight \( c \), with an autohomeomorphism \( \tilde{h} \) that extends \( h \). We embed \( \text{St}(B) \) into \( \mathbb{N}^* \) as a nowhere dense \( P \)-set and extend \( \tilde{h} \) to an autohomeomorphism \( H \) of \( \mathbb{N}^* \).

References

[1] P. C. Baayen, *Universal morphisms*, Mathematical Centre Tracts, vol. 9, Mathematisch Centrum, Amsterdam, 1964. MR0172826

[2] Will Brian, *Universal flows and automorphisms of \( \mathcal{P}(\omega)/\text{fin} \)*, Israel J. Math. 233 (2019), no. 1, 453–500, DOI 10.1007/s11856-019-1913-3. MR4013982

[3] Eric K. van Douwen and Jan van Mill, *The homeomorphism extension theorem for \( \beta \omega \setminus \omega \)*, Papers on general topology and applications (Madison, WI, 1991), Ann. New York Acad. Sci., vol. 704, New York Acad. Sci., New York, 1993, pp. 345–350, DOI 10.1111/j.1749-6632.1993.tb52537.x. MR1277871

[4] Alan Dow and Jan van Mill, *An extremally disconnected Dowker space*, Proc. Amer. Math. Soc. 86 (1982), no. 4, 669–672, DOI 10.2307/2043007. MR674103

[5] Jan van Mill, *An introduction to \( \beta \omega \)*, Handbook of set-theoretic topology, North-Holland, Amsterdam, 1984, pp. 503–567. MR776630

[6] Stelios Negrepontis, *On the product of \( F \)-spaces*, Trans. Amer. Math. Soc. 136 (1969), 339–346, DOI 10.2307/1994718. MR234407