FULLY DISCRETE FINITE ELEMENT APPROXIMATION OF THE 2D/3D UNSTEADY INCOMPRESSIBLE MAGNETOHYDRODYNAMIC-VOIGT REGULARIZATION FLOWS

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ABSTRACT. We devote the present paper to a fully discrete finite element scheme for the 2D/3D nonstationary incompressible magnetohydrodynamic-Voigt regularization model. This scheme is based on a finite element approximation for space discretization and the Crank-Nicolson-type scheme for time discretization, which is a two-step method. Moreover, we study stability and convergence of the fully discrete finite element scheme and obtain unconditional stability and error estimates of velocity and magnetic fields, respectively. Finally, several numerical experiments are investigated to confirm our theoretical findings.

1. Introduction. In the present paper, we mainly discuss stability and convergence of a fully discrete scheme for the unsteady incompressible magnetohydrodynamic (MHD)-Voigt regularization flows. Consider the Voigt regularization of evolution equations for the 2D/3D incompressible MHD flows in dimensionless form as follows [20, 23]:

\[
\begin{align*}
\mathbf{u}_t - \kappa_1 \Delta \mathbf{u}_t - Re^{-1} \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - s \mathbf{B} \cdot \nabla \mathbf{B} + \nabla p &= \mathbf{f}, \\
\nabla \cdot \mathbf{u} &= 0, \\
\mathbf{B}_t - \kappa_2 \Delta \mathbf{B}_t - Re_{m}^{-1} \Delta \mathbf{B} + \mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{u} - \nabla \lambda &= \nabla \times \mathbf{g}, \\
\nabla \cdot \mathbf{B} &= 0.
\end{align*}
\]

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Here, \( u \) represents the velocity field, \( B \) the magnetic field, \( p \) the hydrodynamic pressure, \( Re \) the hydrodynamic Reynolds number and \( Re_m \) the magnetic Reynolds number. We denote \( n \) a unit outward normal vector of \( \partial \Omega \), \( s \) is the coupling number, \( \kappa_1 \) and \( \kappa_2 \) are the retardation time, \( f \) a known source term and \( \nabla \times g \) is a given force on the magnetic field. Besides, we define \( \lambda := Re_m \nabla \cdot B (= 0) \) [6] which can be considered as a Lagrange multiplier corresponding to the solenoidal constraint of the magnetic field. For simplicity, as [20, 6], we consider the case of homogeneous Dirichlet conditions for both the velocity and the magnetic field.

As is known, the MHD equations are the Navier-Stokes equations of hydrodynamics coupled with Maxwell equations of electromagnetism via the Lorentz force and Ohm’s law. These equations can describe the interaction between a viscous, incompressible, electrically conducting fluid and an external magnetic field. Initiated by Alfvén in 1942 [1], MHD has been widely utilized in numerous branches of science [3, 5, 9, 11, 16, 32, 30, 35], including astrophysics and geophysics, as well as engineering. Besides, resources [13, 31] provide more physical background knowledge. In fact, understanding MHD flows is essential to many important applications, e.g., liquid metal cooling of nuclear reactors [2, 14, 34], process metallurgy [8] and sea water propulsion [27].

The MHD-Voigt regularization model is derived from MHD model by adding a Voigt regularization term to each of the momentum and magnetic field equations. Recently, a number of research works have focused on the Voigt-type regularizations in the context of various hydrodynamic models, see, e.g., [7, 10, 18, 19, 22, 26, 33, 28, 29]. An important aspect of the MHD-Voigt model is that when used as a regularization for the MHD equations the regularization is inviscid, which means it does not add artificial viscosity. Therefore, we refer to the Voigt-regularization as an inviscid regularization. Note that when \( \kappa_1 = \kappa_2 = 0 \), we formally retrieve MHD equations. MHD-Voigt model was first introduced and studied by Larios and Titi [23], where they have proved the global existence of solutions for the inviscid resistive MHD-Voigt regularization. Thereafter, they have obtained the uniqueness of strong solutions and the higher-order regularity of inviscid regularization of MHD equations in [24]. Kuberry et al. [20] have employed a second-order Crank-Nicolson method for time discretization and obtained unconditionally stability and optimal convergence of the MHD-Voigt. Furthermore, they have found that the Voigt regularization provides accurate reduced order models for MHD equations for fluid flow.

In this paper, inspired by [25, 17], we will study a fully discrete Crank-Nicolson-type scheme based on finite element approximation for numerically solving the transient incompressible MHD-Voigt equations. However, Layton et al. [25, 17] deal with linear equations while we focus on nonlinear equations. Hence, this paper can be considered as a sequel and a complement of the work of Layton et al. In addition, there are some differences between [20] and this paper. Firstly, the scheme is different. In this paper, the Crank-Nicolson-type scheme is applied and it is a three-time-level scheme. In order to avoid the value of level -1 and ensure more accurate solution, our process is divided into two steps. In Step I, we compute the values of \( u^1_h \) and \( B^1_h \) by Crank-Nicolson scheme. In Step II, we compute the values of \( u^n_h \) and \( B^n_h \) for \( n \geq 2 \) by Crank-Nicolson-type scheme. In [20], the common Crank-Nicolson scheme is used and it is a two-time-level scheme. Secondly, we prove unconditional stability of the fully discrete scheme by tracking the discrete energy in the scheme. In particular, the stability of the numerical solution in the first time
level is considered in this paper. Thirdly, in error estimate, we also consider the bound of the numerical error in the first time level. Besides, we consider the error estimates of \(\|u(t_n) - u_h^n\|, \|B(t_n) - B_h^n\|\) and \(\|p(t_n) - p_h^n\|\), which are missed in [20]. Fourth, in the numerical experiment section, we consider the convergence rates with Lemma 2.1.

In the usual notations which are commonly used in the mathematical study of fluids. We denote 2.

Preliminaries. In this section, we introduce some preliminary material and preliminary results and show a finite element approximation in spatial direction. In Section 3, we express the fully discrete scheme for the considered problem. Moreover, the unconditional stability and the convergence analysis of the fully discrete scheme are given. Section 4 deals with some numerical experiments, which confirm our theoretical findings. Finally, we end with a short conclusion.

2. Preliminaries. In this section, we introduce some preliminary material and notations which are commonly used in the mathematical study of fluids. We denote the usual \(L^2(\Omega)\) norm and its inner product by \(\|\cdot\|\) and \((\cdot, \cdot)\), respectively. The \(L^p(\Omega)\) norm and \(W^{m,p}(\Omega)\) norm are denoted by \(\|\cdot\|\) and \(\|\cdot\|_{W^{m,p}(\Omega)}\) respectively for \(m \in \mathbb{N}^+, 1 \leq p \leq \infty\). In particular, \(H^m(\Omega)\) is used to represent the space \(W^{m,2}(\Omega)\) and \(\|\cdot\|_{H^m}\) denotes the norm in \(H^m(\Omega)\).

For \(X\) being a normed function space in \(\Omega\), \(L^p(0,t;X)\) is the space of all functions defined on \(\Omega\) for which the norm

\[
\|\cdot\|_{L^p(0,t;X)} = \left( \int_0^t \|\cdot\|^p_X \, dt \right)^{\frac{1}{p}}, \quad p \in [1, \infty)
\]

is finite. For \(p = \infty\), the usual modification is used in the definition of this space.

The natural function spaces for our problem are

\[
X := H^1_0(\Omega)^d = \{v \in H^1(\Omega)^d : v|_{\partial \Omega} = 0\}, \quad X_0 := \{v \in X : \text{div}v = 0\},
\]

\[
Q := L^2_0(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q \, dx = 0\}.
\]

For \(f\) an element in the dual space of \(X\), its norm is defined by

\[
\|f\|_{-1} = \sup_{v \in X} \frac{|(f, v)|}{\|v\|}.
\]

Furthermore, in the rest of the paper, we adopt a bilinear form:

\[
a(u, v) := (\nabla u, \nabla v),
\]

and a skew-symmetrized trilinear form:

\[
b(u, v, w) := \frac{1}{2}(u \cdot \nabla v, w) - \frac{1}{2}(u \cdot \nabla w, v) = (u \cdot \nabla v, w) + \frac{1}{2}((\nabla \cdot u)v, w). \tag{2}
\]

Lemma 2.1. For \(u, v, w \in X\), we have

\[
|b(u, v, w)| \leq C\|\nabla u\|\|\nabla v\|\|\nabla w\|,
\]

where \(C > 0\) is a constant depending on \(\Omega\).

The proof of Lemma 2.1 can be found in [21]. Subsequently, \(C\) will denote a positive constant depending at most on the data \(\kappa_1, \kappa_2, Re, Re_m, \Omega, T, u, p\) and \(B\), and may stand for different values at its different occurrences. It is well known that the Gronwall’s inequality plays an important role in the study of differential systems of various kind. The following variation on the discrete Gronwall Lemma is given in [15].
Lemma 2.2. (Discrete Gronwall Lemma). For the integer \( m \geq 1 \), let \( \tau, C, a_m, b_m, c_m, r_m \), be nonnegative numbers such that
\[
a_n + \tau \sum_{m=1}^{n} b_m \leq \tau \sum_{m=1}^{n} r_m a_m + \tau \sum_{m=1}^{n} c_m + C.
\]
Suppose that \( \tau r_m < 1 \) for all \( m \) and set \( \sigma_m = (1 - \tau r_m)^{-1} \), then
\[
a_n + \tau \sum_{m=1}^{n} b_m \leq \exp \left( \tau \sum_{m=1}^{n} \sigma_m r_m \right) \left( \sum_{m=1}^{n} c_m + C \right).
\]

The following results can easily obtained by Taylor series expansion.

Lemma 2.3. Let \( \tau = t_{n+1} - t_n \) and \( \zeta(\cdot, t) \) be a function such that \( \zeta_{tt} \in L^2(0, T; L^2(\Omega)) \). Then there exists \( \theta_1 \in (0, 1) \) such that
\[
\left\| \frac{\zeta(t_{n+1}) + \zeta(t_{n-1})}{2} - \zeta(\cdot, t_n) \right\| \leq C \tau^2 \| \zeta_{tt}(\cdot, t_{n+\theta_1}) \|,
\]
Besides, if \( \zeta_{tt} \in L^2(0, T; L^2(\Omega)) \), then there exists \( \theta_2 \in (0, 1) \) such that
\[
\left\| \frac{\zeta(t_{n+1}) - \zeta(\cdot, t_{n-1})}{2\tau} - \zeta(\cdot, t_n) \right\| \leq C \tau^2 \| \zeta_{tt}(\cdot, t_{n+\theta_2}) \|.
\]

Then, by using the Green formulas, the variational formulation of problem (1) is to find \( (u, p, B, \lambda) \in (X, Q, X, Q) \) for all \( t \in (0, T) \) such that
\[
\begin{align*}
(u_t, v) + &\ k_1 a(u_t, v) + Re^{-1} a(u, v) + b(u, u, v) - sb(b, B, v) - (p, \nabla \cdot v) \\ + &\ (\nabla \cdot u, q) = (f, v), \\
(B_t, r) + &\ k_2 a(B_t, r) + Re^{-1} a(B, r) + b(u, B, r) - b(b, u, r) + (\lambda, \nabla \cdot r) \\ + &\ (\nabla \cdot B, \chi) = (\nabla \times g, r),
\end{align*}
\]
for all \( (v, q, r, \chi) \in (X, Q, X, Q) \).

From now on, let \( K_h = \{ K \} \) be a regular and quasi-uniform triangulation partition of \( \Omega \) with element diameters bounded by a real positive parameter \( h \) \((h \to 0)\). The conforming subspace pair \((X_h, Q^h, X^h, Q^h)\) of \((X, Q, X, Q)\) is constructed based on \( K_h \). Assume that the mixed finite element space pair \((X_h, Q^h, X^h, Q^h)\) satisfies the so-called discrete inf-sup condition:
\[
\inf_{q_h \in Q^h} \sup_{v_h \in X^h} \frac{(q_h, \nabla \cdot v_h)}{\| \nabla v_h \| q_h} \geq \beta > 0,
\]
where \( \beta \) is independent of \( h \). The discrete divergence free subspace of \( X^h \) is
\[
X^h_0 := \{ v_h \in X^h : (\nabla \cdot v_h, q_h) = 0, \forall q_h \in Q^h \}.
\]

Further, we assume that the pair of finite element space \((X^h, Q^h)\) has the following approximation properties [4]:
\[
\begin{align*}
\inf_{u_h \in X^h} \| u - u_h \| + h \| \nabla (u - u_h) \| &\leq Ch^{l+1} \| u \|_{k+1}, \ u \in H^{k+1}(\Omega)^d, \ 0 \leq k \leq l, \\
\inf_{p_h \in Q^h} \| p - p_h \| + h \| \nabla (p - p_h) \| &\leq Ch^{k_1+1} \| p \|_{k_1+1}, \ p \in H^{k_1+1}(\Omega), \ 0 \leq k_1 \leq l_1,
\end{align*}
\]
where \( l \) and \( l_1 \) are the degrees of the piecewise polynomials of \( X^h \) and \( Q^h \), respectively. For the Taylor-Hood element, \( l_1 = l - 1 \). Particularly, for the \( P_2-P_1 \) element, \( l = 2 \) and \( l_1 = 1 \). As for the mini-element, \( l_1 = 1 \), whereas the polynomials for the
velocity in an element \( K \) belong to \( P_1(K) \oplus \text{span} \prod_{i=1}^{d+1} \omega_i \), \( \{ \omega_i \} \) is the set of local basis functions for the element \( K \).

With above statements, Galerkin mixed finite element approximation of problem (3)-(4) is to find \((u_h, p_h, B_h, \lambda_h) \in (X^h, Q^h, X^h, Q^h)\) for all \( t \in (0, T) \) such that

\[
\begin{align*}
(u_{ht}, v_h) + \kappa_1 a(u_{ht}, v_h) + \text{Re}^{-1} a(u_h, v_h) + b(u_h, u_h, v_h) - sb(B_h, B_h, v_h) \\
- (p_h, \nabla \cdot v_h) + (\nabla \cdot u_h, q_h) = (f, v_h),
\end{align*}
\]

(6)

\[
\begin{align*}
(B_{ht}, r_h) + \kappa_2 a(B_{ht}, r_h) + \text{Re}^{-1} a(B_h, r_h) + b(u_h, B_h, r_h) - b(B_h, u_h, r_h) \\
+ (\lambda_h, \nabla \cdot r_h) + (\nabla \cdot B_h, \chi_h) = (\nabla \times g, r_h),
\end{align*}
\]

(7)

for all \((v_h, q_h, r_h, \chi_h) \in (X^h, Q^h, X^h, Q^h)\).

3. Fully discrete scheme for MHD-Voigt equations. In this section, we consider a Crank-Nicolson-type scheme for time discretization and a mixed finite element method for spatial discretization of the problem (1). Let \( \{ t_n \}_{n=0}^{N+1} \) be a uniform partition of \([0, T]\) and \( t_n = n \tau \), where \( \tau > 0 \) is time step.

**Algorithm 3.1.** Step I: Find \((u^n_1, p^n_1, B^n_1, \lambda^n_1) \in (X^h, Q^h, X^h, Q^h)\) such that for all \((v_h, q_h, r_h, \chi_h) \in (X^h, Q^h, X^h, Q^h)\)

\[
\begin{align*}
\left( \frac{u^n_1 - u^n_0}{\tau}, v_h \right) + \kappa_1 a \left( \frac{u^n_1 - u^n_0}{\tau}, v_h \right) + \text{Re}^{-1} a \left( \frac{u^n_1 + u^n_0}{2}, v_h \right) - \left( \frac{p^n_1 + p^n_0}{2}, \nabla \cdot v_h \right) \\
+ b \left( \frac{u^n_1 + u^n_0}{2}, u^n_1 + u^n_0 \right) - sb \left( \frac{B^n_1 + B^n_0}{2}, B^n_1 + B^n_0 \right) + (\nabla \cdot u^n_1, q_h)
\end{align*}
\]

\[
= (f(t^n_0), v_h),
\]

(8)

\[
\begin{align*}
\left( \frac{B^n_1 - B^n_0}{\tau}, r_h \right) + \kappa_2 a \left( \frac{B^n_1 - B^n_0}{\tau}, r_h \right) + \text{Re}^{-1} a \left( \frac{B^n_1 + B^n_0}{2}, r_h \right) + (\frac{\lambda^n_1 + \lambda^n_0}{2}, \nabla \cdot r_h)
\end{align*}
\]

\[
- b \left( \frac{B^n_1 + B^n_0}{2}, u^n_1 + u^n_0 \right) + b \left( \frac{u^n_1 + u^n_0}{2}, B^n_1 + B^n_0 \right) + (\nabla \cdot B^n_1, \chi_h)
\]

\[
= (\nabla \times g(t^n_0), r_h),
\]

(9)

where \( \varrho(t^n_0) = \frac{1}{2} (\varrho(t_1) + \varrho(t_0)) \), \( \varrho = f \) and \( \nabla \times g \). For simplicity, we require the discrete initial conditions be point-wise divergence free, that is, \( u^n_0 = u_0 \) and \( B^n_0 = B_0 \) must be in \( X^h \).

Step II: For \( n \geq 1 \), given \((u^{n-1}_h, p^{n-1}_h, B^{n-1}_h, \lambda^{n-1}_h)\), \((u^n_h, p^n_h, B^n_h, \lambda^n_h) \in (X^h, Q^h, X^h, Q^h)\), find \((u^{n+1}_h, p^{n+1}_h, B^{n+1}_h, \lambda^{n+1}_h) \in (X^h, Q^h, X^h, Q^h)\) such that for all \((v_h, q_h, r_h, \chi_h) \in (X^h, Q^h, X^h, Q^h)\)

\[
\begin{align*}
\left( \frac{u^{n+1}_h - u^{n-1}_h}{2\tau}, v_h \right) + \kappa_1 a \left( \frac{u^{n+1}_h - u^{n-1}_h}{2\tau}, v_h \right) + \text{Re}^{-1} a \left( \frac{u^{n+1}_h + u^{n-1}_h}{2}, v_h \right) \\
+ b \left( \frac{u^{n+1}_h + u^{n-1}_h}{2}, u^{n+1}_h + u^{n-1}_h \right) - sb \left( \frac{B^{n+1}_h + B^{n-1}_h}{2}, B^{n+1}_h + B^{n-1}_h \right) + (\nabla \cdot u^{n+1}_h, q_h)
\end{align*}
\]

\[
= (f(t_n), v_h),
\]

(10)
Proof. In (10) and (11), set \( v_h = u_h^{n+1} + u_h^{n-1} \in X_h^0 \) and \( r_h = s(B_h^{n+1} + B_h^{n-1}) \in X_h^0 \), then add the results to obtain

\[
\begin{align*}
\frac{\|u_h^{n+1}\|^2 - \|u_h^{n-1}\|^2}{2\tau} + \kappa_1 \left( \|\nabla u_h^{n+1}\|^2 - \|\nabla u_h^{n-1}\|^2 \right) + \frac{\|E_h^n\|^2}{2\tau} + \kappa_2 \left( \|\nabla B_h^{n+1}\|^2 - \|\nabla B_h^{n-1}\|^2 \right) & = (f(t_n), u_h^{n+1} + u_h^{n-1}) + 2\tau (\nabla \times g(t_n), B_h^{n+1} + B_h^{n-1}),
\end{align*}
\]

since \( b(\varphi, \frac{1}{2} \varphi, \varphi) = 0 \), \( b(\frac{1}{2} \varphi, \frac{1}{2} \varphi, \varphi) = 0 \) and \( b(\varphi, \frac{1}{2} \varphi, \frac{1}{2} \varphi, \varphi) = 0 \). Here \( \varphi := u_h^{n+1} + u_h^{n-1} \) and \( \xi := B_h^{n+1} + B_h^{n-1} \). 

Multiply through by \( 2\tau \) and add \( s(\|u_h^n\|^2 + \kappa_1\|\nabla u_h^n\|^2 + s\|B_h^n\|^2 + \kappa_2\|\nabla B_h^n\|^2) \) to above equation to get

\[
\begin{align*}
\frac{\|u_h^{n+1}\|^2 - \|u_h^{n-1}\|^2}{2\tau} + \kappa_1 \left( \|\nabla u_h^{n+1}\|^2 - \|\nabla u_h^{n-1}\|^2 \right) + \frac{\|E_h^n\|^2}{2\tau} + \kappa_2 \left( \|\nabla B_h^{n+1}\|^2 - \|\nabla B_h^{n-1}\|^2 \right) & = 2\tau (f(t_n), u_h^{n+1} + u_h^{n-1}) + 2\tau s (\nabla \times g(t_n), B_h^{n+1} + B_h^{n-1}).
\end{align*}
\]

Denote the energy term by

\[
E_h^n + \frac{1}{2} := \frac{\|u_h^{n+1}\|^2 + \|u_h^n\|^2}{2} + \kappa_1 (\|\nabla u_h^{n+1}\|^2 + \|\nabla u_h^n\|^2) + s(\|B_h^{n+1}\|^2 + \|B_h^n\|^2) + \kappa_2 (\|\nabla B_h^{n+1}\|^2 + \|\nabla B_h^n\|^2).
\]

Then (12) becomes

\[
E_h^{n+1} - E_h^{n-1} + \tau Re^{-1} \left( \|\nabla (u_h^{n+1} + u_h^{n-1})\|^2 + \tau s Re^{-1} \|\nabla (B_h^{n+1} + B_h^{n-1})\|^2 \right) = 2\tau (f(t_n), u_h^{n+1} + u_h^{n-1}) + 2\tau s (\nabla \times g(t_n), B_h^{n+1} + B_h^{n-1}).
\]
Applying the Cauchy-Schwarz and Young’s inequalities, we obtain
\[
E^{n+\frac{1}{2}} - E^{n-\frac{1}{2}} + \frac{\tau}{2} \left( Re^{-1} \left\| \nabla (u_h^{n+1} + u_h^{n-1}) \right\|^2 + s Re_m^{-1} \left\| \nabla (B_h^{n+1} + B_h^{n-1}) \right\|^2 \right) \\
\leq 2\tau \left( Re \left\| f(t_n) \right\|^2_{-1} + s Re_m \left\| g(t_n) \right\|^2 \right).
\]

Sum up the above inequality from \( n = 1, \ldots, N \) to find
\[
E^{N+\frac{1}{2}} + \frac{\tau}{2} \sum_{n=1}^{N} \left( Re^{-1} \left\| \nabla (u_h^{n+1} + u_h^{n-1}) \right\|^2 + s Re_m^{-1} \left\| \nabla (B_h^{n+1} + B_h^{n-1}) \right\|^2 \right) \\
\leq E^{\frac{1}{2}} + 2\tau \sum_{n=1}^{N} \left( Re \left\| f(t_n) \right\|^2_{-1} + s Re_m \left\| g(t_n) \right\|^2 \right). \tag{14}
\]

For the first time level, take \( v_h = u_h^1 + u_h^0 \in X_h^0 \) in (8) and \( r_h = s(B_h^1 + B_h^0) \in X_h^0 \) in (9). Applying Cauchy-Schwarz and Young’s inequalities yields
\[
\left\| u_h^1 \right\|^2 + \left\| u_h^0 \right\|^2 + \kappa_1 \left\| \nabla u_h^1 \right\|^2 + \left\| \nabla u_h^0 \right\|^2 + \frac{Re^{-1}}{2} \left\| \nabla (u_h^1 + u_h^0) \right\|^2 \\
+ s \left\| B_h^1 \right\|^2 + \left\| B_h^0 \right\|^2 + s \kappa_2 \left\| \nabla B_h^1 \right\|^2 + \left\| \nabla B_h^0 \right\|^2 + \frac{s Re_m^{-1}}{2} \left\| \nabla (B_h^1 + B_h^0) \right\|^2 \\
\leq \frac{Re}{2} \left\| f(t_\frac{1}{2}) \right\|^2_{-1} + \frac{s Re_m}{2} \left\| g(t_\frac{1}{2}) \right\|^2.
\]

Multiplying the above inequality by \( \tau \), one can obtain that
\[
\left\| u_h^1 \right\|^2 + s \left\| B_h^1 \right\|^2 + \kappa_1 \left\| \nabla u_h^1 \right\|^2 + \left\| \nabla u_h^0 \right\|^2 + \frac{Re^{-1}}{2} \left\| \nabla (u_h^1 + u_h^0) \right\|^2 \\
+ \frac{s Re_m^{-1}}{2} \left\| \nabla (B_h^1 + B_h^0) \right\|^2 \\
\leq \left\| u_h^0 \right\|^2 + s \left\| B_h^0 \right\|^2 + \kappa_1 \left\| \nabla u_h^0 \right\|^2 + \left\| \nabla u_h^1 \right\|^2 + s \kappa_2 \left\| \nabla B_h^0 \right\|^2 + \left\| \nabla B_h^1 \right\|^2 \\
+ \frac{\tau}{2} \left( Re \left\| f(t_\frac{1}{2}) \right\|^2_{-1} + s Re_m \left\| g(t_\frac{1}{2}) \right\|^2 \right). \tag{15}
\]

Taking (15) into (14), we obtain the final form
\[
E^{N+\frac{1}{2}} + \frac{\tau}{2} \sum_{n=1}^{N} \left( Re^{-1} \left\| \nabla (u_h^{n+1} + u_h^{n-1}) \right\|^2 + s Re_m^{-1} \left\| \nabla (B_h^{n+1} + B_h^{n-1}) \right\|^2 \right) \\
\leq \left\| u_h^0 \right\|^2 + s \left\| B_h^0 \right\|^2 + \kappa_1 \left\| \nabla u_h^0 \right\|^2 + \left\| \nabla u_h^1 \right\|^2 + s \kappa_2 \left\| \nabla B_h^0 \right\|^2 + \left\| \nabla B_h^1 \right\|^2 \\
+ \frac{\tau}{2} \left( Re \left\| f(t_\frac{1}{2}) \right\|^2_{-1} + s Re_m \left\| g(t_\frac{1}{2}) \right\|^2 \right) \\
+ 2\tau \sum_{n=1}^{N} \left( Re \left\| f(t_n) \right\|^2_{-1} + s Re_m \left\| g(t_n) \right\|^2 \right).
\]

Then we achieve the desired unconditional stability. \( \square \)

3.2. Convergence of the fully discrete scheme.

**Theorem 3.2.** (Convergence analysis) Let the finite element spaces \((X^h, Q^h, X^h, Q^h)\) satisfy the discrete inf-sup condition and approximation property (5). Assume that \( C\tau \leq 1, u, B \in H^2(0, T; H^1(\Omega)^d) \cap L^2(0, T; H^{k+1}(\Omega)^d) \). Then the solution
\[(u^{n+1}_h, p^{n+1}_h, B^{n+1}_h, \lambda^{n+1}_h) \text{ to (8)-(11)} \text{ converges to the true solution with rate}
\[
\begin{align*}
&\|u(t_{N+1}) - u_h^{N+1}\|^2 + s \|B(t_{N+1}) - B_h^{N+1}\|^2 \\
&+ \tau \sum_{n=1}^{N} \left( \|\nabla \left( \frac{(u(t_{n+1}) - u_h^{n+1}) + (u(t_{n-1}) - u_h^{n-1})}{2} \right)\|^2 \\
&+ s \tau \sum_{n=1}^{N} \left( \|\nabla \left( \frac{(B(t_{n+1}) - B_h^{n+1}) + (B(t_{n-1}) - B_h^{n-1})}{2} \right)\|^2 \right) \right)
\end{align*}
\]
\[
\leq C(h^{2k} + \tau^4 + \kappa_1^2 \tau^4 + \kappa_2^2 \tau^4).
\]

**Proof.** Subtract (10)-(11) from (3)-(4) at \( t = t_n \) and thereby write
\[
\begin{align*}
&\left(u(t_n) - \frac{u_h^{n+1} - u_h^{n-1}}{2\tau}, v_h\right) + \kappa_1 a \left(u(t_n) - \frac{u_h^{n+1} - u_h^{n-1}}{2\tau}, v_h\right) \\
&+ Re^{-1} a \left(u(t_n) - \frac{u_h^{n+1} + u_h^{n-1}}{2}, v_h\right) + b \left(u(t_n), u(t_n), v_h\right) \\
&- b \left(\frac{u_h^{n+1} + u_h^{n-1}}{2}, v_h\right) - \frac{s}{2} \left(\frac{B(t_{n+1}) + B(t_{n-1})}{2}, v_h\right) = 0,
\end{align*}
\]
and
\[
\begin{align*}
&\left(B(t_n) - \frac{B_h^{n+1} - B_h^{n-1}}{2\tau}, r_h\right) + \kappa_2 a \left(B(t_n) - \frac{B_h^{n+1} - B_h^{n-1}}{2\tau}, r_h\right) \\
&+ Re^{-1} a \left(B(t_n) - \frac{B_h^{n+1} + B_h^{n-1}}{2}, r_h\right) + b \left(B(t_n), B(t_n), r_h\right) \\
&- b \left(\frac{B_h^{n+1} + B_h^{n-1}}{2}, r_h\right) - \frac{s}{2} \left(\frac{\lambda(t_{n+1}) + \lambda(t_{n-1})}{2}, r_h\right) = 0.
\end{align*}
\]
Then, we denote the corresponding errors by
\[
\begin{align*}
e^n_h &:= u(t_n) - u_h^n = (u(t_n) - U^n) - (u_h^n - U^n) =: \eta^n_h - \xi^n_h, \\
e_B^n &:= B(t_n) - B_h^n = (B(t_n) - B^n) - (B_h^n - B^n) =: \eta^n_B - \psi^n_h,
\end{align*}
\]
where \( U^n, B^n \in X_h \). Add
\[
\begin{align*}
&\pm \left(\frac{(u(t_{n+1}) - u(t_{n-1}), v_h)}{2\tau} + \kappa_1 a \left(\frac{(u(t_{n+1}) - u(t_{n-1}), v_h)}{2\tau}\right) \\
&+ Re^{-1} a \left(\frac{(u(t_{n+1}) + u(t_{n-1}), v_h)}{2}, \frac{p(t_{n+1}) + p(t_{n-1})}{2}, v_h\right) \\
&+ b \left(u(t_n) + \frac{u(t_{n+1}) + u(t_{n-1})}{2}, \frac{u_h^{n+1} + u_h^{n-1}}{2}, \frac{u(t_{n+1}) + u(t_{n-1}), v_h}{}\right) \\
&+ sb \left(B(t_n) + \frac{B(t_{n+1}) + B(t_{n-1})}{2}, \frac{B_h^{n+1} + B_h^{n-1}}{2}, \frac{B(t_{n+1}) + B(t_{n-1}), v_h}{}\right) \right)
\end{align*}
\]
to (17) to get the error equation

\[
\begin{align*}
\left( \frac{e_B^{n+1} - e_B^{n-1}}{2\tau}, r_h \right) &+ \kappa_2 a \left( \frac{e_B^{n+1} - e_B^{n-1}}{2\tau}, r_h \right) \\
+ Re^{-1} a \left( \frac{e_B^{n+1} + e_B^{n-1}}{2}, v_h \right) - \left( p(t_{n+1}) + p(t_{n-1}) - \frac{p_{h}^{n+1} + p_{h}^{n-1}}{2}, \nabla \cdot v_h \right) \\
&+ b \left( \frac{e_B^{n+1} + e_B^{n-1}}{2}, \frac{u(t_{n+1}) + u(t_{n-1})}{2}, v_h \right) + b \left( \frac{e_B^{n+1} + e_B^{n-1}}{2}, \frac{e_B^{n+1} + e_B^{n-1}}{2}, v_h \right) \\
- sb \left( \frac{e_B^{n+1} + e_B^{n-1}}{2}, B(t_{n+1}) + B(t_{n-1}), v_h \right) - sb \left( B_{h}^{n+1} + B_{h}^{n-1}, \frac{e_B^{n+1} + e_B^{n-1}}{2}, v_h \right) \\
&= \left( \frac{u(t_{n+1}) - u(t_{n-1})}{2\tau} - u(t_{n}), v_h \right) + \kappa_1 a \left( \frac{u(t_{n+1}) - u(t_{n-1})}{2\tau} - u(t_{n}), v_h \right) \\
+ Re^{-1} a \left( \frac{u(t_{n+1}) + u(t_{n-1})}{2} - u(t_{n}), v_h \right) + b \left( u(t_{n}), \frac{u(t_{n+1}) + u(t_{n-1})}{2} - u(t_{n}), v_h \right) \\
&+ b \left( B(t_{n+1}) + B(t_{n-1}) - B(t_{n}), \frac{B(t_{n+1}) + B(t_{n-1})}{2}, v_h \right) \\
&- sb \left( B(t_{n+1}) + B(t_{n-1}) - B(t_{n}), \frac{B(t_{n+1}) + B(t_{n-1})}{2}, v_h \right) \\
&= R_1(u, p, B, v_h).
\end{align*}
\]

Note that \( R_1(u, p, B, v_h) \) represents terms associated only with the true solution.

In light of Cauchy-Schwarz and Poincaré inequalities and Lemma 2.3, we have

\[
\begin{align*}
\left| \frac{u(t_{n+1}) - u(t_{n-1})}{2\tau} - u(t_{n}), v_h \right| &\leq C\tau^2 \| u_{ttt}(t_{n+\theta_2}) \| \| \nabla v_h \|, \\
\kappa_1 a \left( \frac{u(t_{n+1}) - u(t_{n-1})}{2\tau} - u(t_{n}), v_h \right) &\leq C\kappa_1 \tau^2 \| \nabla u_{ttt}(t_{n+\theta_2}) \| \| \nabla v_h \|, \\
Re^{-1} a \left( \frac{u(t_{n+1}) + u(t_{n-1})}{2} - u(t_{n}), v_h \right) &\leq CRe^{-1} \tau^2 \| \nabla u_{tt}(t_{n+\theta_1}) \| \| \nabla v_h \|, \\
b \left( u(t_{n}), \frac{u(t_{n+1}) + u(t_{n-1})}{2} - u(t_{n}), v_h \right) &\leq C\tau^2 \| \nabla u_{tt}(t_{n+\theta_1}) \| \| \nabla v_h \|, \\
b \left( B(t_{n}), \frac{B(t_{n+1}) + B(t_{n-1})}{2} - B(t_{n}), v_h \right) &\leq C\tau^2 \| \nabla B_{tt}(t_{n+\theta_1}) \| \| \nabla v_h \|, \\
b \left( \frac{u(t_{n+1}) + u(t_{n-1})}{2} - u(t_{n}), \frac{u(t_{n+1}) + u(t_{n-1})}{2}, v_h \right) &\leq C\tau^2 \| \nabla u_{tt}(t_{n+\theta_1}) \| \| \nabla v_h \|, \\
b \left( \frac{B(t_{n+1}) + B(t_{n-1})}{2} - B(t_{n}), \frac{B(t_{n+1}) + B(t_{n-1})}{2}, v_h \right) &\leq C\tau^2 \| \nabla B_{tt}(t_{n+\theta_1}) \| \| \nabla v_h \|. 
\end{align*}
\]

Similarly, for the magnetic field equation, we have

\[
\begin{align*}
\left( \frac{e_B^{n+1} - e_B^{n-1}}{2\tau}, r_h \right) &+ \kappa_2 a \left( \frac{e_B^{n+1} - e_B^{n-1}}{2\tau}, r_h \right) + Re^{-1} a \left( \frac{e_B^{n+1} + e_B^{n-1}}{2}, r_h \right) \\
- b \left( \frac{B_{h}^{n+1} + B_{h}^{n-1}}{2}, \frac{e_B^{n+1} + e_B^{n-1}}{2}, r_h \right) - b \left( \frac{e_B^{n+1} + e_B^{n-1}}{2}, \frac{u(t_{n+1}) + u(t_{n-1})}{2}, r_h \right) \\
+ b \left( \frac{u_{h}^{n+1} + u_{h}^{n-1}}{2}, \frac{e_B^{n+1} + e_B^{n-1}}{2}, r_h \right) + b \left( \frac{e_B^{n+1} + e_B^{n-1}}{2}, \frac{B(t_{n+1}) + B(t_{n-1})}{2}, r_h \right)
\end{align*}
\]
\[
\begin{align*}
= \left( \frac{B(t_{n+1}) - B(t_{n-1})}{2\tau} - B(t_n), r_h \right) - b \left( B(t_n), \frac{u(t_{n+1}) + u(t_{n-1})}{2}, u(t_n), r_h \right) \\
+ \kappa a \left( \frac{B(t_{n+1}) - B(t_{n-1})}{2\tau} - B(t_n), r_h \right) \\
+ Re^{-1} a \left( \frac{B(t_{n+1}) + B(t_{n-1})}{2} - B(t_n), r_h \right) \\
+ b \left( u(t_n), \frac{B(t_{n+1}) + B(t_{n-1})}{2} - B(t_n), r_h \right) \\
- b \left( \frac{B(t_{n+1}) + B(t_{n-1})}{2} - B(t_n), \frac{u(t_{n+1}) + u(t_{n-1})}{2}, r_h \right) \\
=: R_2(u, B, r_h).
\end{align*}
\]

Substituting (19)-(20) into (21) and (29) and choosing \( v_h, r_h \in X_0^h \), we have
\[
\begin{align*}
\left( \frac{\xi_{n+1} - \xi_{n-1}}{2\tau}, v_h \right) + \kappa a \left( \frac{\xi_{n+1} - \xi_{n-1}}{2\tau}, v_h \right) + Re^{-1} a \left( \frac{\xi_{n+1} + \xi_{n-1}}{2}, v_h \right) \\
+ b \left( \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2}, v_h \right) \\
+ b \left( \frac{U_h^{n+1} + U_h^{n-1}}{2} + \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2}, v_h \right) \\
- sb \left( \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2}, \frac{B(t_{n+1}) + B(t_{n-1})}{2}, v_h \right) \\
- sb \left( \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2}, \frac{B(t_{n+1}) + B(t_{n-1})}{2}, v_h \right) \\
= \left( \frac{\eta_{n+1} - \eta_{n-1}}{2\tau}, v_h \right) + \kappa a \left( \frac{\eta_{n+1} - \eta_{n-1}}{2\tau}, v_h \right) \\
+ Re^{-1} a \left( \frac{\eta_h^{n+1} + \eta_h^{n-1}}{2}, v_h \right) \\
+ b \left( \frac{\eta_h^{n+1} + \eta_h^{n-1}}{2}, \frac{\eta_h^{n+1} + \eta_h^{n-1}}{2}, v_h \right) \\
+ b \left( \frac{\eta_h^{n+1} + \eta_h^{n-1}}{2}, \frac{U(t_{n+1}) + U(t_{n-1})}{2}, v_h \right) \\
- sb \left( \frac{\eta_B^{n+1} + \eta_B^{n-1}}{2}, \frac{B(t_{n+1}) + B(t_{n-1})}{2}, v_h \right) \\
- sb \left( \frac{B_h^{n+1} + B_h^{n-1}}{2}, \frac{\eta_B^{n+1} + \eta_B^{n-1}}{2}, v_h \right) \\
+ R_1(u, B, v_h),
\end{align*}
\]

where \( R_1(u, B, v_h) \) contains the remaining items in \( R_1(u, p, B, v_h) \) after removing the pressure items. By combining with (22)-(28), we get
\[
|R_1(u, B, v_h)| \leq C \left( \tau^4 + \kappa_1^2 \tau^4 \right) + \frac{3Re^{-1}}{16} \| \nabla v_h \|^2.
\]
For magnetic field, we have

\[
\left(\psi_{h}^{n+1} - \psi_{h}^{n-1}, r_h\right) + \kappa_1 \left(\psi_{h}^{n+1} - \psi_{h}^{n-1}, r_h\right) + Re_m^{-1} a \left(\psi_{h}^{n+1} + \psi_{h}^{n-1}, r_h\right)
- \delta \left(\psi_{h}^{n+1} - \psi_{h}^{n-1}, \frac{u(t_{n+1}) + u(t_{n-1})}{2}, r_h\right) + b \left(u_{h}^{n+1} + u_{h}^{n-1}, \frac{u_{h}^{n+1} + u_{h}^{n-1}}{2}, r_h\right)
+ b \left(\frac{\xi_{h}^{n+1} + \xi_{h}^{n-1}}{2}, \frac{B(t_{n+1}) + B(t_{n-1})}{2}, r_h\right) - \delta \left(\frac{\xi_{h}^{n+1} + \xi_{h}^{n-1}}{2}, \frac{B(t_{n+1}) + B(t_{n-1})}{2}, r_h\right)
= \left(\frac{\eta_{B}^{n+1} - \eta_{B}^{n-1}}{2T}, \frac{\xi_{h}^{n+1} + \xi_{h}^{n-1}}{2}, r_h\right) + \kappa_1 \left(\frac{\eta_{B}^{n+1} - \eta_{B}^{n-1}}{2T}, \frac{\psi_{h}^{n+1} + \psi_{h}^{n-1}}{2}, r_h\right)
+ Re_m^{-1} a \left(\frac{\eta_{B}^{n+1} + \eta_{B}^{n-1}}{2T}, \frac{\xi_{h}^{n+1} + \xi_{h}^{n-1}}{2}, r_h\right) + R_2(u, B, r_h).
\]

Similarly, we have

\[
|R_2(u, B, r_h)| \leq C \left(T^4 + \kappa_2^2 T^4\right) + \frac{3Re_m^{-1}}{16} \|\nabla r_h\|^2.
\]

Taking \(v_h = \frac{1}{2} (\xi_{h}^{n+1} + \xi_{h}^{n-1})\) in (30) and \(r_h = \frac{1}{2} (\psi_{h}^{n+1} + \psi_{h}^{n-1})\) in (32), then simplifying yields the equations

\[
\frac{1}{4T} \left(\|\xi_{h}^{n+1}\|^2 - \|\xi_{h}^{n-1}\|^2\right) + \kappa_1 \left(\|\nabla \xi_{h}^{n+1}\|^2 - \|\nabla \xi_{h}^{n-1}\|^2\right)
+ Re^{-1} \left\|\nabla \left(\frac{\xi_{h}^{n+1} + \xi_{h}^{n-1}}{2}\right)\right\|^2
= \left(\frac{\eta_{B}^{n+1} - \eta_{B}^{n-1}}{2T}, \frac{\xi_{h}^{n+1} + \xi_{h}^{n-1}}{2}, r_h\right) + \kappa_1 \left(\frac{\eta_{B}^{n+1} - \eta_{B}^{n-1}}{2T}, \frac{\psi_{h}^{n+1} + \psi_{h}^{n-1}}{2}, r_h\right)
+ Re^{-1} a \left(\frac{\eta_{h}^{n+1} + \eta_{h}^{n-1}}{2}, \frac{\xi_{h}^{n+1} + \xi_{h}^{n-1}}{2}, r_h\right) + R_1 \left(u, B, \frac{\xi_{h}^{n+1} + \xi_{h}^{n-1}}{2}\right)
+ b \left(\frac{\eta_{h}^{n+1} + \eta_{h}^{n-1}}{2}, \frac{u(t_{n+1}) + u(t_{n-1})}{2}, \frac{\xi_{h}^{n+1} + \xi_{h}^{n-1}}{2}\right)
+ b \left(\frac{\eta_{h}^{n+1} + \eta_{h}^{n-1}}{2}, \frac{u(t_{n+1}) + u(t_{n-1})}{2}, \frac{\xi_{h}^{n+1} + \xi_{h}^{n-1}}{2}\right)
- sb \left(\frac{\eta_{B}^{n+1} + \eta_{B}^{n-1}}{2}, B(t_{n+1}) + B(t_{n-1}), \frac{\xi_{h}^{n+1} + \xi_{h}^{n-1}}{2}\right)
- sb \left(\frac{\eta_{B}^{n+1} + \eta_{B}^{n-1}}{2}, B(t_{n+1}) + B(t_{n-1}), \frac{\xi_{h}^{n+1} + \xi_{h}^{n-1}}{2}\right)
- b \left(\frac{\xi_{h}^{n+1} + \xi_{h}^{n-1}}{2}, \frac{u(t_{n+1}) + u(t_{n-1})}{2}, \frac{\xi_{h}^{n+1} + \xi_{h}^{n-1}}{2}\right)
+ sb \left(\frac{\psi_{h}^{n+1} + \psi_{h}^{n-1}}{2}, B(t_{n+1}) + B(t_{n-1}), \frac{\xi_{h}^{n+1} + \xi_{h}^{n-1}}{2}\right)
+ sb \left(\frac{\psi_{h}^{n+1} + \psi_{h}^{n-1}}{2}, B(t_{n+1}) + B(t_{n-1}), \frac{\xi_{h}^{n+1} + \xi_{h}^{n-1}}{2}\right),
\]

\[\text{(34)}\]
and

\[
\frac{1}{4\tau} \left( \| \psi_h^{n+1} \|^2 - \| \psi_h^{n-1} \|^2 \right) + \frac{\kappa_2}{4\tau} \left( \| \nabla \psi_h^{n+1} \|^2 - \| \nabla \psi_h^{n-1} \|^2 \right) \\
+ \mathcal{R}_{m}^{-1} \left\| \nabla \left( \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right) \right\|^2 \\
= \left( \frac{\eta_B^{n+1} - \eta_B^{n-1}}{2\tau}, \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right) + \kappa_2 a \left( \frac{\eta_B^{n+1} - \eta_B^{n-1}}{2\tau}, \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right) \\
+ \mathcal{R}_{m}^{-1} a \left( \frac{\eta_B^{n+1} + \eta_B^{n-1}}{2}, \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right) + R_2 \left( u, B, \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right) \\
+ b \left( \frac{u_h^{n+1} + u_h^{n-1}}{2}, \frac{\eta_B^{n+1} + \eta_B^{n-1}}{2}, \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right) \\
- b \left( \frac{\eta_B^{n+1} + \eta_B^{n-1}}{2}, \frac{u(t_{n+1}) + u(t_{n-1})}{2}, \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right) \\
+ b \left( \frac{\eta_u^{n+1} + \eta_u^{n-1}}{2}, \frac{B(t_{n+1}) + B(t_{n-1})}{2}, \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right) \\
- b \left( \frac{B_h^{n+1} + B_h^{n-1}}{2}, \frac{\eta_u^{n+1} + \eta_u^{n-1}}{2}, \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right) \\
+ b \left( \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2}, \frac{u(t_{n+1}) + u(t_{n-1})}{2}, \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right) \\
- b \left( \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2}, \frac{B(t_{n+1}) + B(t_{n-1})}{2}, \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right) \\
+ b \left( \frac{B_h^{n+1} + B_h^{n-1}}{2}, \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2}, \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right).
\]

Invoking the inequalities, we get

\[
\left( \frac{\eta_u^{n+1} - \eta_u^{n-1}}{2\tau}, \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2} \right) \leq \frac{1}{2\tau} \left\| \eta_u^{n+1} - \eta_u^{n-1} \right\| \left\| \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2} \right\|^2 \\
\leq \frac{1}{4\tau} \int_{t_{n-1}}^{t_{n+1}} \left\| \frac{\partial \eta_u}{\partial t} \right\|^2 dt + \frac{1}{2} \left\| \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2} \right\|^2,
\]

\[
a \left( \frac{\eta_u^{n+1} - \eta_u^{n-1}}{2\tau}, \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2} \right) \leq \frac{1}{2\tau} \left\| \nabla (\eta_u^{n+1} - \eta_u^{n-1}) \right\| \left\| \nabla \left( \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2} \right) \right\|^2 \\
\leq \frac{1}{4\tau} \int_{t_{n-1}}^{t_{n+1}} \left\| \frac{\partial (\nabla \eta_u)}{\partial t} \right\|^2 dt + \frac{1}{2} \left\| \nabla \left( \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2} \right) \right\|^2.
\]

which also hold for $\eta_B$. 

Each of the trilinear terms except the last one on the RHS of (34) can be bounded by the Cauchy-Schwarz, Hölder’s and Young’s inequalities. Together with the estimate in Lemma 2.1 and the assumptions on $u, B$, we have

\[
\begin{align*}
&|b\left(\frac{\psi_h^{n+1} + \psi_h^{n-1}}{2}, B(t_{n+1}) + B(t_{n-1}), \xi_h^{n+1} + \xi_h^{n-1}\right)| \\
&\leq C\left\|\frac{\psi_h^{n+1} + \psi_h^{n-1}}{2}\right\|^2 + C\left\|\frac{\xi_h^{n+1} + \xi_h^{n-1}}{2}\right\|^2, \\
&|b\left(\frac{\eta_B^{n+1} + \eta_B^{n-1}}{2}, B(t_{n+1}) + B(t_{n-1}), \xi_h^{n+1} + \xi_h^{n-1}\right)| \\
&\leq C\left\|\frac{\eta_B^{n+1} + \eta_B^{n-1}}{2}\right\|^2 + C\left\|\frac{\xi_h^{n+1} + \xi_h^{n-1}}{2}\right\|^2, \\
&|b\left(\frac{u_h^{n+1} + u_h^{n-1}}{2}, B(t_{n+1}) + B(t_{n-1}), \xi_h^{n+1} + \xi_h^{n-1}\right)| \\
&\leq C\left\|\frac{u_h^{n+1} + u_h^{n-1}}{2}\right\|^2 + C\left\|\frac{\xi_h^{n+1} + \xi_h^{n-1}}{2}\right\|^2, \\
&|b\left(\frac{\xi_h^{n+1} + \xi_h^{n-1}}{2}, u(t_{n+1}) + u(t_{n-1}), \xi_h^{n+1} + \xi_h^{n-1}\right)| \leq C \left\|\frac{\xi_h^{n+1} + \xi_h^{n-1}}{2}\right\|^2, \\
&|b\left(\frac{u_h^{n+1} + u_h^{n-1}}{2}, \eta_B^{n+1} + \eta_B^{n-1}, \xi_h^{n+1} + \xi_h^{n-1}\right)| \\
&\leq C\left\|\nabla \left(\frac{u_h^{n+1} + u_h^{n-1}}{2}\right)\right\|^2 \left\|\nabla \left(\frac{\eta_B^{n+1} + \eta_B^{n-1}}{2}\right)\right\|^2 + \frac{3Re^{-1}}{16} \left\|\nabla \left(\frac{\xi_h^{n+1} + \xi_h^{n-1}}{2}\right)\right\|^2, \\
&|b\left(\frac{B_h^{n+1} + B_h^{n-1}}{2}, \eta_B^{n+1} + \eta_B^{n-1}, \xi_h^{n+1} + \xi_h^{n-1}\right)| \\
&\leq C\left\|\nabla \left(\frac{B_h^{n+1} + B_h^{n-1}}{2}\right)\right\|^2 \left\|\nabla \left(\frac{\eta_B^{n+1} + \eta_B^{n-1}}{2}\right)\right\|^2 + \frac{3Re^{-1}}{16} \left\|\nabla \left(\frac{\xi_h^{n+1} + \xi_h^{n-1}}{2}\right)\right\|^2.
\end{align*}
\]

In conjunction with (31), (36), (37) and above estimates yields

\[
\begin{align*}
\frac{1}{4r} \left(\|\xi_h^{n+1}\|^2 - \|\xi_h^{n-1}\|^2\right) + \frac{\kappa_1}{4r} \left(\|\nabla \xi_h^{n+1}\|^2 - \|\nabla \xi_h^{n-1}\|^2\right) + \frac{Re^{-1}}{4} \left\|\nabla \left(\frac{\xi_h^{n+1} + \xi_h^{n-1}}{2}\right)\right\|^2 \\
\leq \frac{1}{4r} \int_{t_{n-1}}^{t_{n+1}} \left(\frac{\partial \eta_h}{\partial t}\right)^2 + \frac{\partial \nabla \eta_h}{\partial t} \left\|\nabla \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2}\right\|^2 dt + \frac{\kappa_1}{2} \left\|\nabla \left(\frac{\xi_h^{n+1} + \xi_h^{n-1}}{2}\right)\right\|^2 \\
+ C \left\|\nabla \left(\frac{u_h^{n+1} + u_h^{n-1}}{2}\right)\right\|^2 + \left\|\nabla \left(\frac{\eta_B^{n+1} + \eta_B^{n-1}}{2}\right)\right\|^2 + \left\|\nabla \left(\frac{\xi_h^{n+1} + \xi_h^{n-1}}{2}\right)\right\|^2 \\
+ C \left\|\nabla \left(\frac{B_h^{n+1} + B_h^{n-1}}{2}\right)\right\|^2 + \left\|\nabla \left(\frac{\eta_B^{n+1} + \eta_B^{n-1}}{2}\right)\right\|^2 + \left\|\nabla \left(\frac{\xi_h^{n+1} + \xi_h^{n-1}}{2}\right)\right\|^2 \\
+ \left\|\nabla \left(\frac{\xi_h^{n+1} + \xi_h^{n-1}}{2}\right)\right\|^2 + \left\|\nabla \left(\frac{\eta_B^{n+1} + \eta_B^{n-1}}{2}\right)\right\|^2 + \left\|\nabla \left(\frac{\xi_h^{n+1} + \xi_h^{n-1}}{2}\right)\right\|^2
\end{align*}
\]
We now step back from (38) and return to (35), which can be majorized in a similar way to the momentum system, yielding

\[
\frac{1}{4r} \left( \| \psi_h^{n+1} \|^2 - \| \psi_h^{n-1} \|^2 \right) + \frac{\kappa_2}{4r} \left( \| \nabla \psi_h^{n+1} \|^2 - \| \nabla \psi_h^{n-1} \|^2 \right) + \frac{Rc_n}{4r} \left\| \nabla \left( \phi_h^{n+1} + \phi_h^{n-1} \right) \right\|^2 \\
\leq \frac{1}{4r} \int_{t_{n-1}}^{t_{n+1}} \left( \left\| \frac{\partial \eta_B}{\partial t} \right\|^2 + \left\| \frac{\partial (\nabla \eta_B)}{\partial t} \right\|^2 \right) dt + \frac{\kappa_2}{2} \left\| \nabla \left( \psi_h^{n+1} + \psi_h^{n-1} \right) \right\|^2 \\
+ C \left\| \nabla \left( \frac{\eta_B^{n+1} + \eta_B^{n-1}}{2} \right) \right\|^2 + \frac{\kappa_1}{4r} \left\| \nabla \left( \phi_h^{n+1} + \phi_h^{n-1} \right) \right\|^2 \\
+ C \left( \left\| \frac{\eta_B^{n+1} + \eta_B^{n-1}}{2} \right\|^2 + \left\| \frac{\eta_B^{n+1} + \eta_B^{n-1}}{2} \right\|^2 \right) \\
+ C \left( \left\| \frac{u_h^{n+1} + u_h^{n-1}}{2} \right\|^2 + \left\| \frac{u_h^{n+1} + u_h^{n-1}}{2} \right\|^2 \right) \\
+ b \left( \frac{B_h^{n+1} + B_h^{n-1}}{2}, \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2}, \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right). \tag{39}
\]

Multiply (39) by \( s \) and add it to (38), and use that \( b(B, \psi, \xi) = -b(B, \xi, \psi) \). Along with Poincaré inequality, we obtain

\[
\frac{1}{4r} \left( \| \xi_h^{n+1} \|^2 - \| \xi_h^{n-1} \|^2 \right) + \frac{s}{4r} \left( \| \psi_h^{n+1} \|^2 - \| \psi_h^{n-1} \|^2 \right) + \frac{Rc_n}{4r} \left\| \nabla \left( \phi_h^{n+1} + \phi_h^{n-1} \right) \right\|^2 \\
+ \frac{\kappa_2}{4r} \left( \| \nabla \psi_h^{n+1} \|^2 - \| \nabla \psi_h^{n-1} \|^2 \right) + \frac{Rc_n}{4r} \left\| \nabla \left( \xi_h^{n+1} + \xi_h^{n-1} \right) \right\|^2 \\
+ \frac{Rc_n}{4r} \left\| \nabla \left( \phi_h^{n+1} + \phi_h^{n-1} \right) \right\|^2 \\
\leq \frac{1}{4r} \int_{t_{n-1}}^{t_{n+1}} \left( \left\| \frac{\partial \eta_B}{\partial t} \right\|^2 + \left\| \frac{\partial (\nabla \eta_B)}{\partial t} \right\|^2 \right) dt + \frac{s \kappa_1}{2} \left\| \nabla \left( \psi_h^{n+1} + \psi_h^{n-1} \right) \right\|^2 \\
+ \frac{s \kappa_2}{2} \left\| \nabla \left( \psi_h^{n+1} + \psi_h^{n-1} \right) \right\|^2 + \frac{s \kappa_2}{2} \left\| \nabla \left( \psi_h^{n+1} + \psi_h^{n-1} \right) \right\|^2 + C \left( \left\| \frac{\eta_B^{n+1} + \eta_B^{n-1}}{2} \right\|^2 + \left\| \frac{\eta_B^{n+1} + \eta_B^{n-1}}{2} \right\|^2 \right) \\
+ C \left( \left\| \frac{\eta_B^{n+1} + \eta_B^{n-1}}{2} \right\|^2 + \left\| \frac{\eta_B^{n+1} + \eta_B^{n-1}}{2} \right\|^2 \right) \\
+ C \left( \left\| \frac{u_h^{n+1} + u_h^{n-1}}{2} \right\|^2 + \left\| \frac{u_h^{n+1} + u_h^{n-1}}{2} \right\|^2 \right) \\
+ C \left( \left\| \frac{u_h^{n+1} + u_h^{n-1}}{2} \right\|^2 + \left\| \frac{u_h^{n+1} + u_h^{n-1}}{2} \right\|^2 \right). \tag{40}
\]
Multiplying by $4\tau$ and summing over time steps now give

$$
\|\xi_h^{N+1}\|^2 + s\|\psi_h^{N+1}\|^2 + \kappa_1 \|\nabla \xi_h^{N+1}\|^2 + s\kappa_2 \|\nabla \psi_h^{N+1}\|^2 \\
+ \tau \sum_{n=1}^{N} \left( Re^{-1} \left\| \nabla \left( \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2} \right) \right\|^2 + s Re^{-1} \left\| \nabla \left( \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right) \right\|^2 \right)
\leq \|\xi_h^{n}\|^2 + s\|\psi_h^{n}\|^2 + \kappa_1 \|\nabla \xi_h^{n}\|^2 + s\kappa_2 \|\nabla \psi_h^{n}\|^2 + C \left( \tau^4 + \kappa_1^2 \tau^4 + \kappa_2^2 \tau^4 \right)
+ C \int_{t}^{T} \left( \|\partial_t(\xi_u)\|^2 + \kappa_1 \|\partial_t(\psi_u)\|^2 + s \|\partial_t(\psi_u)\|^2 + s \kappa_2 \|\partial_t(\psi_u)\|^2 \right) dt
$$

$$
+ C \tau \sum_{n=1}^{N} \left( \left\| \nabla \left( \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2} \right) \right\|^2 + \left\| \nabla \left( \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right) \right\|^2 \right)
+ C \tau \sum_{n=1}^{N} \left( \left\| \nabla \left( \frac{\xi_h^{B,n+1} + \xi_h^{B,n-1}}{2} \right) \right\|^2 + \left\| \nabla \left( \frac{\psi_h^{B,n+1} + \psi_h^{B,n-1}}{2} \right) \right\|^2 \right)
+ C \tau \sum_{n=1}^{N} \left( \left\| \nabla \left( \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2} \right) \right\|^2 + \left\| \nabla \left( \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right) \right\|^2 \right)
+ C \tau \sum_{n=1}^{N} \left( \left\| \nabla \left( \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2} \right) \right\|^2 + \left\| \nabla \left( \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right) \right\|^2 \right)
\tag{41}
$$

Next we use approximation properties of the spaces and the stability estimate, which reduces (41) to

$$
\|\psi_h^{N+1}\|^2 + s\|\psi_h^{N+1}\|^2 + \kappa_1 \|\nabla \psi_h^{N+1}\|^2 + s\kappa_2 \|\nabla \psi_h^{N+1}\|^2 \\
+ \sum_{n=1}^{N} \left( Re^{-1} \left\| \nabla \left( \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2} \right) \right\|^2 + s Re^{-1} \left\| \nabla \left( \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right) \right\|^2 \right)
\leq C \tau^4 + \kappa_1^2 \tau^4 + \kappa_2^2 \tau^4 + C \tau \sum_{n=1}^{N} \left( \|\xi_h^{n}\|^2 + s\|\psi_h^{n}\|^2 + \kappa_1 \|\nabla \xi_h^{n}\|^2 + s\kappa_2 \|\nabla \psi_h^{n}\|^2 \right)
+ \|\xi_h^{n}\|^2 + s\|\psi_h^{n}\|^2 + \kappa_1 \|\nabla \xi_h^{n}\|^2 + s\kappa_2 \|\nabla \psi_h^{n}\|^2.
\tag{42}
$$

Furthermore, subtract (8)-(9) from (3)-(4) at $t = t_\frac{1}{2}$ to obtain the following results

$$
b \left( u(t_\frac{1}{2}), u(t_\frac{1}{2}), \nu_h \right) - b \left( \frac{u_h^1 + u_h^0}{2}, \frac{u_h^1 + u_h^0}{2}, \nu_h \right)
= b \left( u(t_\frac{1}{2}), u(t_\frac{1}{2}) \right) - b \left( \frac{u(t_1) + u(t_0)}{2}, \nu_h \right) + b \left( \frac{u_h^1 + u_h^0}{2}, \frac{e_h^1 + e_h^0}{2}, \nu_h \right)
+ b \left( \frac{e_h^1 + e_h^0}{2}, u(t_1) + u(t_0) , \nu_h \right) + b \left( \frac{u(t_1) + u(t_0)}{2}, \frac{u(t_1) + u(t_0)}{2}, \nu_h \right),
\tag{43}
$$
\[
\begin{align*}
&b \left( B(t_2), B(t_2), v_h \right) + b \left( \frac{B_1^0 + B_2^0}{2}, \frac{B_1^0 + B_2^0}{2}, v_h \right) \\
&= b \left( B(t_2), B(t_2), \frac{B(t_1) + B(t_0)}{2}, v_h \right) + b \left( \frac{B_1^0 + B_2^0}{2}, \frac{e_1^0 + e_2^0}{2}, v_h \right) \\
&+ b \left( \frac{e_1^0 + e_2^0}{2}, B(t_1) + B(t_0), v_h \right) + b \left( B(t_1) - \frac{B(t_1) + B(t_0)}{2}, \frac{B(t_1) + B(t_0)}{2}, v_h \right), \\
&\quad \text{(44)}
\end{align*}
\]

\[
\begin{align*}
&b \left( u(t_2), B(t_2), r_h \right) - b \left( \frac{u_1^0 + u_2^0}{2}, \frac{B_1^0 + B_2^0}{2}, r_h \right) \\
&= b \left( u(t_2), B(t_2), \frac{B(t_1) + B(t_0)}{2}, r_h \right) + b \left( \frac{u_1^0 + u_2^0}{2}, \frac{e_1^0 + e_2^0}{2}, r_h \right) \\
&+ b \left( \frac{e_1^0 + e_2^0}{2}, u(t_1) + u(t_0), r_h \right) + b \left( u(t_1) - \frac{u(t_1) + u(t_0)}{2}, \frac{u(t_1) + u(t_0)}{2}, r_h \right). \\
&\quad \text{(45)}
\end{align*}
\]

\[
\begin{align*}
&b \left( B(t_2), u(t_2), r_h \right) + b \left( \frac{B_1^0 + B_2^0}{2}, \frac{u_1^0 + u_2^0}{2}, r_h \right) \\
&= b \left( B(t_2), u(t_2), \frac{B(t_1) + B(t_0)}{2}, r_h \right) + b \left( \frac{B_1^0 + B_2^0}{2}, \frac{e_1^0 + e_2^0}{2}, r_h \right) \\
&+ b \left( \frac{e_1^0 + e_2^0}{2}, u(t_1) + u(t_0), r_h \right) + b \left( u(t_1) - \frac{u(t_1) + u(t_0)}{2}, \frac{u(t_1) + u(t_0)}{2}, r_h \right). \\
&\quad \text{(46)}
\end{align*}
\]

Taking \( v_h = \frac{1}{2} (\xi_h^1 + \xi_h^0) \) and \( r_h = \frac{1}{2} (\psi_h^1 + \psi_h^0) \), the trilinear terms on the RHS of (43)-(46) can be treated exactly as in (21) and (29). This leads to the upper bound

\[
\begin{align*}
\| \xi_h^1 \|^2 + s\| \psi_h^1 \|^2 + \kappa_1 \| \nabla \xi_h^1 \|^2 &+ s\kappa_2 \| \nabla \psi_h^1 \|^2 \\
&+ \tau Re^{-1} \left\| \nabla \left( \frac{\xi_h^1 + \xi_h^0}{2} \right) \right\|^2 + \tau Re_m^{-1} \left\| \nabla \left( \frac{\psi_h^1 + \psi_h^0}{2} \right) \right\|^2 \\
&\leq C \left( \tau^4 + \kappa_1^2 \tau^4 + \kappa_2^2 \tau^4 + h^{2k} \right) + C \tau \left( \| \xi_h^1 \|^2 + \| \psi_h^1 \|^2 \right).
\end{align*}
\]

Inserting above estimate into (42) yields

\[
\begin{align*}
\| \xi_h^{N+1} \|^2 + s\| \psi_h^{N+1} \|^2 + \kappa_1 \| \nabla \xi_h^{N+1} \|^2 &+ s\kappa_2 \| \nabla \psi_h^{N+1} \|^2 \\
&+ \tau \sum_{n=1}^{N} \left( Re^{-1} \left\| \nabla \left( \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2} \right) \right\|^2 + sRe_m^{-1} \left\| \nabla \left( \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right) \right\|^2 \right) \\
&\leq C \left( \tau^4 + \kappa_1^2 \tau^4 + \kappa_2^2 \tau^4 + h^{2k} \right) + C \tau \sum_{n=1}^{N} \left( \| \xi_h^n \|^2 + s\| \psi_h^n \|^2 + \kappa_1 \| \nabla \xi_h^n \|^2 + s\kappa_2 \| \nabla \psi_h^n \|^2 \right).
\end{align*}
\]

Hence, arguing as in discrete Gronwall Lemma, if there holds \( C\tau \leq 1 \), then for any \( n \)

\[
\begin{align*}
\| \xi_h^{N+1} \|^2 + s\| \psi_h^{N+1} \|^2 &+ \kappa_1 \| \nabla \xi_h^{N+1} \|^2 + s\kappa_2 \| \nabla \psi_h^{N+1} \|^2 \\
&+ \tau \sum_{n=1}^{N} \left( Re^{-1} \left\| \nabla \left( \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2} \right) \right\|^2 + sRe_m^{-1} \left\| \nabla \left( \frac{\psi_h^{n+1} + \psi_h^{n-1}}{2} \right) \right\|^2 \right) \\
&\leq C \left( \tau^4 + \kappa_1^2 \tau^4 + \kappa_2^2 \tau^4 + h^{2k} \right).
\end{align*}
\]

If we apply triangle inequality, then we arrive at (16) and complete the proof. \( \square \)
We now estimate the error of pressure. Before this, we should give the estimate of the following velocity errors.

**Lemma 3.3.** Under the assumptions of Theorem 3.2, there exists a constant $C$ such that

$$
Re^{-1} \tau^2 \left\| \nabla \left( \frac{e_u^{n+1} - e_u^{n-1}}{2\tau} \right) \right\|^2 + Re^{-1} \left\| \nabla \left( \frac{e_u^{n+1} + e_u^{n-1}}{2} \right) \right\|^2 + \tau \sum_{n=1}^{N} \left\| \frac{e_u^{n+1} - e_u^{n-1}}{2\tau} \right\|^2
$$

$$+ \kappa_1 \tau \sum_{n=1}^{N} \left\| \nabla \left( \frac{e_u^{n+1} - e_u^{n-1}}{2\tau} \right) \right\|^2 \leq C \left( \tau^4 + \kappa_1^2 + \kappa_2 + h^{2k} \right).
$$

**Proof.** Invoke the error decomposition (19)-(20). We begin this proof by setting $v_h = \frac{1}{2\tau}(\xi_h^{n+1} - \xi_h^{n-1}) \in X_0^\perp$ in (21) to obtain

$$
\left\| \frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau} \right\|^2 + \kappa_1 \left\| \nabla \left( \frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau} \right) \right\|^2 + Re^{-1} \left\| \nabla \xi_h^{n+1} \right\|^2 - \left\| \nabla \xi_h^{n-1} \right\|^2
$$

$$= \left( \frac{\eta_u^{n+1} - \eta_u^{n-1}}{2\tau}, \frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau} \right) + \kappa_1 a \left( \frac{\eta_u^{n+1} - \eta_u^{n-1}}{2\tau}, \frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau} \right)
$$

$$+ Re^{-1} a \left( \frac{\eta_u^{n+1} + \eta_u^{n-1}}{2}, \frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau} \right) - R_1 \left( u, B, \frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau} \right)
$$

$$+ b \left( \frac{u_h^{n+1} + u_h^{n-1}}{2}, \frac{e_u^{n+1} + e_u^{n-1}}{2}, \frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau} \right)
$$

$$+ b \left( \frac{e_u^{n+1} + e_u^{n-1}}{2}, \frac{u(t_{n+1}) + u(t_{n-1})}{2}, \frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau} \right)
$$

$$- sb \left( \frac{e_B^{n+1} + e_B^{n-1}}{2}, \frac{B(t_{n+1}) + B(t_{n-1})}{2}, \frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau} \right)
$$

$$- sb \left( \frac{B_h^{n+1} + B_h^{n-1}}{2}, \frac{e_B^{n+1} + e_B^{n-1}}{2}, \frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau} \right).
$$

Combining with (22)-(28) and replacing $v_h$ with $\frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau}$, we get

$$R_1 \left( u, B, \frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau} \right) \leq \frac{3\kappa_1}{20} \left\| \nabla \left( \frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau} \right) \right\|^2 + C \tau^4.
$$

The first two terms on the RHS of (49) can be calculated in the same way as (36) and (37), we have

$$
\left\| \left( \frac{\eta_u^{n+1} - \eta_u^{n-1}}{2\tau}, \frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau} \right) \right\| + \kappa_1 \left\| a \left( \frac{\eta_u^{n+1} - \eta_u^{n-1}}{2\tau}, \frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau} \right) \right\|
$$

$$\leq C h^{2k} + \frac{3\kappa_1}{20} \left\| \nabla \left( \frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau} \right) \right\|^2.
$$
For the third term, we obtain
\[
\begin{align*}
Re^{-1} & \left| a \left( \frac{\eta_{n+1}^u + \eta_{n-1}^u}{2}, \frac{\xi_{h+1}^n - \xi_{h-1}^n}{2\tau} \right) \right| \\
& \leq C \left| \nabla \left( \frac{\eta_{n+1}^u + \eta_{n-1}^u}{2} \right) \right|^2 + \left| \nabla \left( \frac{\xi_{h+1}^n - \xi_{h-1}^n}{2\tau} \right) \right|^2 \\
& \leq Ch^{2k} + \frac{3\kappa_1}{20} \left| \nabla \left( \frac{\xi_{h+1}^n - \xi_{h-1}^n}{2\tau} \right) \right|^2 .
\end{align*}
\]

Then, adding \( \pm b \left( \frac{1}{2}(e_{u}^{n+1} + e_{u}^{n-1}), \frac{1}{2}(e_{u}^{n+1} + e_{u}^{n-1}), \frac{1}{2\tau}(\xi_{h}^{n+1} - \xi_{h}^{n-1}) \right) \) to the first nonlinear term in (49), integrating by parts and applying Hölder’s inequality we can obtain
\[
\begin{align*}
& \left| b \left( \frac{u_{h+1}^n + u_{h-1}^n}{2}, \frac{e_{u}^{n+1} + e_{u}^{n-1}}{2}, \frac{\xi_{h+1}^n - \xi_{h-1}^n}{2\tau} \right) \right| \\
& \leq \left| b \left( \frac{u_{n+1} + u_{n-1}}{2}, \frac{e_{u}^{n+1} + e_{u}^{n-1}}{2}, \frac{\xi_{h+1}^n - \xi_{h-1}^n}{2\tau} \right) \right| \\
& + \left| b \left( \frac{e_{u}^{n+1} + e_{u}^{n-1}}{2}, \frac{e_{u}^{n+1} + e_{u}^{n-1}}{2}, \frac{\xi_{h+1}^n - \xi_{h-1}^n}{2\tau} \right) \right| \\
& + \left| b \left( \frac{\xi_{h+1}^n - \xi_{h-1}^n}{2}, \frac{e_{u}^{n+1} + e_{u}^{n-1}}{2}, \frac{\xi_{h+1}^n - \xi_{h-1}^n}{2\tau} \right) \right| .
\end{align*}
\]

Now, we bound three nonlinear terms on the RHS of (50) separately. For the first term, with the help of the Cauchy-Schwarz and Young’s inequalities we can obtain
\[
\begin{align*}
& \left| b \left( \frac{u_{n+1} + u_{n-1}}{2}, \frac{e_{u}^{n+1} + e_{u}^{n-1}}{2}, \frac{\xi_{h+1}^n - \xi_{h-1}^n}{2\tau} \right) \right| \\
& \leq \frac{3}{10} \left| \nabla \left( \frac{e_{u}^{n+1} + e_{u}^{n-1}}{2} \right) \right|^2 + C \left| \nabla \left( \frac{e_{u}^{n+1} + e_{u}^{n-1}}{2} \right) \right|^2 .
\end{align*}
\]

We next bound the second and third nonlinear terms using Lemma 2.1, approximation property of \( u \) and Young’s inequality.
\[
\begin{align*}
& \left| b \left( \frac{\eta_{h+1}^n + \eta_{h-1}^n}{2}, \frac{e_{u}^{n+1} + e_{u}^{n-1}}{2}, \frac{\xi_{h+1}^n - \xi_{h-1}^n}{2\tau} \right) \right| \\
& \leq \frac{3\kappa_1}{40} \left| \nabla \left( \frac{\xi_{h+1}^n - \xi_{h-1}^n}{2} \right) \right|^2 + Ch^{2k} \left| \nabla \left( \frac{e_{u}^{n+1} + e_{u}^{n-1}}{2} \right) \right|^2 .
\end{align*}
\]

Similarly, we arrive at
\[
\begin{align*}
& \left| b \left( \frac{\xi_{h+1}^n - \xi_{h-1}^n}{2}, \frac{e_{u}^{n+1} + e_{u}^{n-1}}{2}, \frac{\xi_{h+1}^n - \xi_{h-1}^n}{2\tau} \right) \right| \\
& \leq \frac{3\kappa_1}{40} \left| \nabla \left( \frac{\xi_{h+1}^n - \xi_{h-1}^n}{2} \right) \right|^2 + C \left| \nabla \left( \frac{\xi_{h+1}^n - \xi_{h-1}^n}{2} \right) \right|^2 .
\end{align*}
\]

Finally, we consider the second nonlinear term in (49). Using Cauchy-Schwarz and Young’s inequalities, the regularity assumption of \( u \) and the approximation
property of $u$, we get

$$\left| b \left( \frac{e^{n+1} + e^{n-1}}{2}, u(t_{n+1}) + u(t_{n-1}), \frac{\xi^{n+1} - \xi^{n-1}}{2\tau} \right) \right| \leq b \left( \frac{\eta^{n+1} + \eta^{n-1}}{2}, u(t_{n+1}) + u(t_{n-1}), \frac{\xi^{n+1} - \xi^{n-1}}{2\tau} \right)$$

$$+ b \left( \frac{\xi^{n+1} + \xi^{n-1}}{2}, u(t_{n+1}) + u(t_{n-1}), \frac{\xi^{n+1} - \xi^{n-1}}{2\tau} \right)$$

$$\leq C \left\| \nabla \left( \frac{\eta^{n+1} + \eta^{n-1}}{2} \right) \right\| \left\| \frac{\xi^{n+1} - \xi^{n-1}}{2\tau} \right\| + C \left\| \nabla \left( \frac{\xi^{n+1} + \xi^{n-1}}{2} \right) \right\| \left\| \frac{\xi^{n+1} - \xi^{n-1}}{2\tau} \right\|$$

$$\leq \frac{3}{16} \left\| \frac{\xi^{n+1} - \xi^{n-1}}{2\tau} \right\|^2 + C \left\| \nabla \left( \frac{\psi^{n+1} + \psi^{n-1}}{2} \right) \right\|^2 + Ch^{2k}.$$

Similarly, we can obtain

$$\left| b \left( \frac{e^{n+1} + e^{n-1}}{2}, B(t_{n+1}) + B(t_{n-1}), \frac{\xi^{n+1} - \xi^{n-1}}{2\tau} \right) \right| \leq \frac{3}{16} \left\| \frac{\xi^{n+1} - \xi^{n-1}}{2\tau} \right\|^2 + C \left\| \nabla \left( \frac{\psi^{n+1} + \psi^{n-1}}{2} \right) \right\|^2 + Ch^{2k},$$

and

$$\left| b \left( \frac{B^{n+1} + B^{n-1}}{2}, e^{n+1} + e^{n-1}, \frac{\xi^{n+1} - \xi^{n-1}}{2\tau} \right) \right| \leq \frac{3}{16} \left\| \frac{\xi^{n+1} - \xi^{n-1}}{2\tau} \right\|^2 + Ch^{2k} \left\| \nabla \left( \frac{e^{n+1} + e^{n-1}}{2} \right) \right\|^2 + C \left\| \nabla \left( \frac{e^{n+1} + e^{n-1}}{2} \right) \right\|^2$$

$$+ \frac{3\kappa_1}{20} \left\| \nabla \left( \frac{\xi^{n+1} - \xi^{n-1}}{2\tau} \right) \right\|^2 + C \left\| \nabla \left( \frac{\psi^{n+1} + \psi^{n-1}}{2} \right) \right\|^2 \left\| \nabla \left( \frac{e^{n+1} + e^{n-1}}{2} \right) \right\|^2.$$
At the $t = t^2$ time level, choosing $v_h = \frac{1}{\tau}(\xi_1^{1} - \xi_h^{0})$ and $U^0 = u_0^h$ in the initial error decomposition gives $\xi_h^{1}$. Due to (47), we obtain

\[
\frac{1}{2}\left\| \left( \frac{\xi_1^{1} - \xi_h^{0}}{\tau} \right) \right\|^2 + \frac{\kappa_1}{2}\left\| \nabla \left( \frac{\xi_1^{1} - \xi_h^{0}}{\tau} \right) \right\|^2 + R e^{-1}\left\| \nabla \xi_h^{1} \right\|^2 - \left\| \nabla \xi_h^{0} \right\|^2
\leq C \left( \tau^4 + \kappa_1^2 \tau^4 + \kappa_2^2 \tau^4 + h^{2k} \right) + \frac{\tau^2}{2} \left| b \left( u_{\tau\tau}(t_\theta), \frac{u(t_1) + u(t_0)}{2}, \frac{\xi_1^{1} - \xi_h^{0}}{\tau} \right) \right| ,
\]

(52)

Sum (51) over the time levels $N \geq 1$ and add to (52). Multiply by $4\tau$ and due to (48), we have

\[
\tau \sum_{n=1}^{N} \left\| \frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau} \right\|^2 + \kappa_1 \tau \sum_{n=1}^{N} \left\| \nabla \left( \frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau} \right) \right\|^2 + R e^{-1}\left\| \nabla \xi_h^{N+1} \right\|^2
\leq C \tau \sum_{n=1}^{N} \left\| \nabla \left( \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2\tau} \right) \right\|^2 + C \tau^3 \left| b \left( u_{\tau\tau}(t_\theta), \frac{u(t_1) + u(t_0)}{2}, \frac{\xi_1^{1} - \xi_h^{0}}{\tau} \right) \right| ,
\]

(53)

For any $N \geq 1$, add the inequality (53) to itself at the previous time level. Using the identity

\[
\left\| \nabla \xi_h^{N+1} \right\|^2 + \left\| \nabla \xi_h^{N-1} \right\|^2 = \frac{1}{2}\tau^2 \left\| \nabla \left( \frac{\xi_h^{N+1} + \xi_h^{N-1}}{2\tau} \right) \right\|^2 + 2 \left\| \nabla \left( \frac{\xi_h^{N+1} + \xi_h^{N-1}}{2\tau} \right) \right\|^2 ,
\]

we obtain

\[
\frac{1}{2}R e^{-1}\tau^2 \left\| \nabla \left( \frac{\xi_h^{N+1} + \xi_h^{N-1}}{2\tau} \right) \right\|^2 + 2R e^{-1}\left\| \nabla \left( \frac{\xi_h^{N+1} + \xi_h^{N-1}}{2\tau} \right) \right\|^2 + \tau \sum_{n=1}^{N} \left\| \frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau} \right\|^2
\leq C \tau \sum_{n=1}^{N} 2R e^{-1}\left\| \nabla \left( \frac{\xi_h^{n+1} + \xi_h^{n-1}}{2\tau} \right) \right\|^2 + C \left( \tau^4 + \kappa_1^2 \tau^4 + \kappa_2^2 \tau^4 + h^{2k} \right) + C \tau^3 \left| b \left( u_{\tau\tau}(t_\theta), \frac{u(t_1) + u(t_0)}{2}, \frac{\xi_1^{1} - \xi_h^{0}}{\tau} \right) \right| + C \tau^3 \left| b \left( B_{\tau\tau}(t_\theta), \frac{B(t_1) + B(t_0)}{2}, \frac{\xi_1^{1} - \xi_h^{0}}{\tau} \right) \right| .
\]

Using the Cauchy-Schwarz and Young’s inequalities yields

\[
C \tau^3 \left| b \left( u_{\tau\tau}(t_\theta), \frac{u(t_1) + u(t_0)}{2}, \frac{\xi_1^{1} - \xi_h^{0}}{\tau} \right) \right| \leq \frac{\tau}{4} \left\| \frac{\xi_1^{1} - \xi_h^{0}}{\tau} \right\|^2 + C \tau^5 ,
\]

\[
C \tau^3 \left| b \left( B_{\tau\tau}(t_\theta), \frac{B(t_1) + B(t_0)}{2}, \frac{\xi_1^{1} - \xi_h^{0}}{\tau} \right) \right| \leq \frac{\tau}{4} \left\| \frac{\xi_1^{1} - \xi_h^{0}}{\tau} \right\|^2 + C \tau^5 .
\]

With the help of discrete Gronwall Lemma and (47), we arrive that

\[
R e^{-1}\tau^2 \left\| \nabla \left( \frac{\xi_h^{N+1} + \xi_h^{N-1}}{2\tau} \right) \right\|^2 + R e^{-1}\left\| \nabla \left( \frac{\xi_h^{N+1} + \xi_h^{N-1}}{2\tau} \right) \right\|^2 + \tau \sum_{n=1}^{N} \left\| \frac{\xi_h^{n+1} - \xi_h^{n-1}}{2\tau} \right\|^2
\leq C \left( \tau^4 + \kappa_1^2 \tau^4 + \kappa_2^2 \tau^4 + h^{2k} \right) .
\]

The proof of the lemma is now concluded by the triangle inequality.  

□
We finish this section by deriving error estimate of the pressure.

**Theorem 3.4.** Let \((u_{n+1}^h, p_{n+1}^h, B_{n+1}^h, \lambda_{n+1}^h)\) satisfy (10)-(11) for \(n \geq 1\). Let \((u_1^h, p_1^h, B_1^h, \lambda_1^h)\) satisfy (8)-(9). Then, under the assumptions of Lemma 3.3, we have

\[
\frac{\sqrt{\sum_{n=1}^{N} \left| p(t_{n+1}) - p_{n+1}^h + (p(t_{n-1}) - p_{n-1}^h) \right|^2}}{\tau} \leq C \left( \tau^4 + \kappa_1^2 \tau^4 + \kappa_2^2 \tau^4 + h^2k \right).
\]

**Proof.** Considering (21) and applying the discrete inf-sup condition to obtain for any \(n \geq 1\),

\[
\begin{align*}
\beta & \left\| \frac{(p(t_{n+1}) - p_{n+1}^h) + (p(t_{n-1}) - p_{n-1}^h)}{2} \right\|_2 \\
& \leq C \left\| \frac{e_{u_{n+1}} - e_{u_{n-1}}}{2\tau} \right\| + \kappa_1 \left\| \nabla \left( \frac{e_{u_{n+1}} - e_{u_{n-1}}}{2\tau} \right) \right\| \\
& \leq + C(1 + Re^{-1}) \left\| \nabla \left( \frac{e_{u_{n+1}} + e_{u_{n-1}}}{2} \right) \right\| \\
& \quad + C \left\| \nabla \left( \frac{e_{B_{n+1}} + e_{B_{n-1}}}{2} \right) \right\|_2^2 + C \left\| \nabla \left( \frac{e_{u_{n+1}} + e_{u_{n-1}}}{2} \right) \right\|_2^2 \\
& \leq + C \left\| \nabla \left( \frac{e_{B_{n+1}} + e_{B_{n-1}}}{2} \right) \right\|_2 + C \tau^2. \quad (54)
\end{align*}
\]

Then sum over the time levels from 1 to \(N\) and multiply it by \(\tau\). From Theorem 3.2 and Lemma 3.3, we obtain

\[
\frac{\tau}{\sum_{n=1}^{N} \left| p(t_{n+1}) - p_{n+1}^h + (p(t_{n-1}) - p_{n-1}^h) \right|^2} \leq C \left( \tau^4 + \kappa_1^2 \tau^4 + \kappa_2^2 \tau^4 + h^2k \right). \quad (55)
\]

\(\square\)

4. **Numerical experiments.** In this section, we conduct some numerical examples to exhibit the unconditional stability and convergence rate of the proposed algorithm for 2D/3D unsteady MHD-Voigt model (1).

4.1. **Stability verification.** The prescribed solutions are in \(\Omega = [0, 1]^d\), and \(d = 2, 3\). Choose the source terms \(f\) and \(\nabla \times g\) with equation parameters \(Re = 1\), \(Re_m = 1\), \(s = 0.1\), \(\kappa_1 = \kappa_2 = 0.01\), \(\alpha = 1\) and the final time \(T = 1\) such that the exact solutions are

\[
\begin{align*}
u_1 &= \alpha \pi \sin^2(\pi x) \sin(\pi y) \cos(\pi y) e^{-t}, \\
u_2 &= - \alpha \pi \sin(\pi x) \sin^2(\pi y) \cos(\pi x) e^{-t}, \\
p &= \alpha \cos(\pi x) \cos(\pi y) e^{-t}, \\
B_1 &= \alpha \sin(\pi x) \cos(\pi y) e^{-t}, \\
B_2 &= - \alpha \cos(\pi x) \sin(\pi y) e^{-t},
\end{align*}
\]
for $d = 2$ and
\[
\begin{align*}
    u_1 &= -\frac{1}{2} \alpha \pi \sin^2(\pi x) \sin(\pi y) \cos(\pi y) \sin(\pi z) \cos(\pi z)e^{-t}, \\
    u_2 &= \alpha \pi \sin(\pi x) \cos(\pi x) \sin^2(\pi y) \sin(\pi z) \cos(\pi z)e^{-t}, \\
    u_3 &= -\frac{1}{2} \alpha \pi \sin(\pi x) \cos(\pi x) \sin(\pi y) \cos(\pi y) \sin^2(\pi z) e^{-t}, \\
    p &= \alpha \cos(\pi x) \cos(\pi y) \cos(\pi z)e^{-t}, \\
    B_1 &= -\frac{1}{2} \alpha \pi \sin(\pi x) \cos(\pi x) \cos(\pi y) \sin(\pi z) e^{-t}, \\
    B_2 &= \alpha \pi \cos(\pi x) \sin(\pi y) \cos(\pi z)e^{-t}, \\
    B_3 &= -\frac{1}{2} \alpha \pi \cos(\pi x) \cos(\pi y) \sin(\pi z)e^{-t},
\end{align*}
\]
for $d = 3$.

In order to validate Theorem 3.1, we use $P_2$-$P_1$ finite element pair is used for the velocity-pressure and magnetic field-Lagrange multiplier systems to compute the values of $\|u_n^h\|_0$, $\|\nabla u_n^h\|_0$, $\|B_n^h\|_0$ and $\|\nabla B_n^h\|_0$ for 2D and 3D problems with different time steps, which are listed in Table 1-8. By comparing the values with different space meshes under the same time step, we can observe from these tables that the value of $\|u_n^h\|_0$, $\|\nabla u_n^h\|_0$, $\|B_n^h\|_0$ and $\|\nabla B_n^h\|_0$ tends to be a constant, which shows that no time-step restriction is needed.

**Table 1.** $\|u_n^h\|_0$ of the considered scheme for the 2D problem.

| $\frac{1}{\tau}$ | $2^6$ | $2^5$ | $2^4$ | $2^3$ |
|-----------------|-------|-------|-------|-------|
| $2^2$           | 0.34080 | 0.34067 | 0.34014 | 0.32677 |
| $2^3$           | 0.35282 | 0.35269 | 0.35215 | 0.33852 |
| $2^4$           | 0.35376 | 0.35363 | 0.35308 | 0.33944 |

**Table 2.** $\|\nabla u_n^h\|_0$ of the considered scheme for the 2D problem.

| $\frac{1}{\tau}$ | $2^6$ | $2^5$ | $2^4$ | $2^3$ |
|-----------------|-------|-------|-------|-------|
| $2^2$           | 2.49610 | 2.49517 | 2.49125 | 2.39290 |
| $2^3$           | 2.56163 | 2.56067 | 2.55668 | 2.45729 |
| $2^4$           | 2.56670 | 2.56574 | 2.56174 | 2.46228 |

**Table 3.** $\|B_n^h\|_0$ of the considered scheme for the 2D problem.

| $\frac{1}{\tau}$ | $2^6$ | $2^5$ | $2^4$ | $2^3$ |
|-----------------|-------|-------|-------|-------|
| $2^2$           | 0.25873 | 0.25863 | 0.25824 | 0.25554 |
| $2^3$           | 0.26001 | 0.25991 | 0.25951 | 0.25682 |
| $2^4$           | 0.26001 | 0.25999 | 0.25960 | 0.25690 |
Table 4. $\|\nabla B^n_h\|_0$ of the considered scheme for the 2D problem.

| $h$ | $2^0$ | $2^1$ | $2^2$ | $2^3$ |
|-----|-------|-------|-------|-------|
| $2^2$ | 1.15137 | 1.15093 | 1.14917 | 1.13716 |
| $2^3$ | 1.15530 | 1.15486 | 1.15310 | 1.14113 |
| $2^4$ | 1.15556 | 1.15512 | 1.15336 | 1.14140 |

Table 5. $\|u^n_h\|_0$ of the considered scheme for the 3D problem.

| $h$ | $2^0$ | $2^1$ | $2^2$ | $2^3$ |
|-----|-------|-------|-------|-------|
| $2^2$ | 1.15137 | 1.15093 | 1.14917 | 1.13716 |
| $2^3$ | 1.15530 | 1.15486 | 1.15310 | 1.14113 |
| $2^4$ | 1.15556 | 1.15512 | 1.15336 | 1.14140 |

Table 6. $\|\nabla u^n_h\|_0$ of the considered scheme for the 3D problem.

| $h$ | $2^0$ | $2^1$ | $2^2$ | $2^3$ |
|-----|-------|-------|-------|-------|
| $2^2$ | 1.15137 | 1.15093 | 1.14917 | 1.13716 |
| $2^3$ | 1.15530 | 1.15486 | 1.15310 | 1.14113 |
| $2^4$ | 1.15556 | 1.15512 | 1.15336 | 1.14140 |

Table 7. $\|B^n_h\|_0$ of the considered scheme for the 3D problem.

| $h$ | $2^0$ | $2^1$ | $2^2$ | $2^3$ |
|-----|-------|-------|-------|-------|
| $2^2$ | 1.15137 | 1.15093 | 1.14917 | 1.13716 |
| $2^3$ | 1.15530 | 1.15486 | 1.15310 | 1.14113 |
| $2^4$ | 1.15556 | 1.15512 | 1.15336 | 1.14140 |

Table 8. $\|\nabla B^n_h\|_0$ of the considered scheme for the 3D problem.

| $h$ | $2^0$ | $2^1$ | $2^2$ | $2^3$ |
|-----|-------|-------|-------|-------|
| $2^2$ | 1.15137 | 1.15093 | 1.14917 | 1.13716 |
| $2^3$ | 1.15530 | 1.15486 | 1.15310 | 1.14113 |
| $2^4$ | 1.15556 | 1.15512 | 1.15336 | 1.14140 |

4.2. Convergence rate verification. Example 1: Our first example is designed to test the predicted rates of convergence with $(P_2-P_1-P_2-P_1)$ finite element pair for $(u, p, B, \lambda)$.

Here we denote

$$\|E(\sigma)\| := \left( \tau \sum_{n=1}^{N} \|\nabla ((\sigma(t_{n+1}) - \sigma_h^{n+1} + (\sigma(t_{n-1}) - \sigma_h^{n-1}))\|^2 \right)^{\frac{1}{2}},$$

$\sigma = u$ and $B$, and

$$\|E(p)\| := \left( \tau \sum_{n=1}^{N} \|(p(t_{n+1}) - p_h^{n+1}) + (p(t_{n-1}) - p_h^{n-1})\|^2 \right)^{\frac{1}{2}}.$$
The prescribed solution in $\Omega = [0,1] \times [0,1]$ has the same form as the solution in stability verification for 2D problem.

The results for the MHD-Voigt model are presented in Table 9. From the table, it can be easily seen that the presented method works well and keeps the convergence rates just like the theoretical analysis in the previous sections. Further, the order of convergence for velocity and magnetic terms with varying $\kappa_1, \kappa_2$ are given through Table 10 and Table 11.

**Table 9.** Error and convergence rates for the considered scheme with $\tau = O(h)$ for the 2D problem.

| $h$     | $\|E(u)\|$  | Rate | $\|E(B)\|$  | Rate | $\|E(p)\|$  | Rate |
|---------|-------------|------|-------------|------|-------------|------|
| 1/10    | 0.062879    | —    | 0.009524    | —    | 0.005226    | —    |
| 1/20    | 0.015703    | 2.001| 0.002346    | 2.021| 0.001348    | 1.955|
| 1/40    | 0.003903    | 2.008| 0.000581    | 2.014| 0.000303    | 2.153|

**Table 10.** Numerical convergence rates for velocity in $H^1$-norm with variation in $\kappa_1$ and $\kappa_2$.

| $h$     | $\kappa_1 = 1E-2, \kappa_2 = 1E-2$ | Rate | $\kappa_1 = 1E-4, \kappa_2 = 1E-4$ | Rate | $\kappa_1 = 1E-8, \kappa_2 = 1E-8$ | Rate |
|---------|----------------------------------|------|----------------------------------|------|----------------------------------|------|
| 1/10    | —                                | —    | —                                | —    | —                                | —    |
| 1/20    | 2.001                            | 2.008| 2.008                            | 2.001| 2.008                            | 2.008|
| 1/40    | 2.008                            | 2.010| 2.011                            | 2.008| 2.011                            | 2.011|

**Table 11.** Numerical convergence rates for magnetic in $H^1$-norm with variation in $\kappa_1$ and $\kappa_2$.

| $h$     | $\kappa_1 = 1E-2, \kappa_2 = 1E-2$ | Rate | $\kappa_1 = 1E-4, \kappa_2 = 1E-4$ | Rate | $\kappa_1 = 1E-8, \kappa_2 = 1E-8$ | Rate |
|---------|----------------------------------|------|----------------------------------|------|----------------------------------|------|
| 1/10    | 2.021                            | 2.027| 2.026                            | 2.026| 2.021                            | 2.021|
| 1/20    | 2.016                            | 2.016| 2.016                            | 2.016| 2.013                            | 2.013|

Example 2: This example we test the predicted rates of convergence for 3D problem using $(P_1-P_1-P_1-P_3)$ finite element pair. We choose the forcing function $f$ and $\nabla \times g$ in such a way that the exact solution $(u, p, B, \lambda)$ is

$u_1 = \alpha (y^4 + z)e^{-t}, \quad u_2 = \alpha (x + z^3)e^{-t}, \quad u_3 = \alpha (x^2 + y^2)e^{-t},$

$p = \alpha (2x - 1)(2y - 1)(2z - 1)e^{-t},$

$B_1 = \alpha \sin(yz)e^{-t}, \quad B_2 = \alpha \sin(x + z)e^{-t}, \quad B_3 = \alpha y \sin(x^2)e^{-t}.$

Here, we take $Re = 1, Re_m = 1, s = 0.1, \kappa_1 = \kappa_2 = 0.01, T = 1$ and $\alpha = 1.$

The results can be found in Table 12 and we also compute the order of convergence with varying $\kappa_1, \kappa_2$ in Table 13 and Table 14, respectively.
### Table 12. Error and convergence rates for the considered scheme with $\tau = \Theta(h)$ for the 3D problem.

| $h$ | $\|E(\mathbf{u})\|$ | Rate | $\|E(\mathbf{B})\|$ | Rate | $\|E(p)\|$ | Rate |
|-----|-------------------|------|-------------------|------|-------------------|------|
| 1/2 | 0.690006          | —    | 0.325897          | —    | 0.256334          | —    |
| 1/4 | 0.273947          | 1.333| 0.135307          | 1.268| 0.077825          | 1.720|
| 1/6 | 0.163001          | 1.280| 0.086166          | 1.113| 0.040745          | 1.596|
| 1/8 | 0.117710          | 1.132| 0.066413          | 0.905| 0.026852          | 1.449|

### Table 13. Numerical convergence rates for velocity in $H^1$-norm with variation in $\kappa_1$ and $\kappa_2$.

| $h$ | Rate | Rate | Rate | Rate | Rate | Rate |
|-----|------|------|------|------|------|------|
| $\kappa_1 = \kappa_2 = 1E-2$ | $\kappa_1 = \kappa_2 = 1E-4$ | $\kappa_1 = \kappa_2 = 1E-8$ | $\kappa_1 = 1E-8$, $\kappa_2 = 1E-2$ | $\kappa_1 = 1E-8$, $\kappa_2 = 1E-2$ |
| 1/2 | —    | —    | —    | —    | —    | —    |
| 1/4 | 1.333| 1.058| 1.049| 1.333| 1.049| —    |
| 1/6 | 1.280| 1.038| 1.016| 1.280| 1.016| —    |
| 1/8 | 1.132| 1.013| 0.974| 1.132| 0.974| —    |

### Table 14. Numerical convergence rates for magnetic in $H^1$-norm with variation in $\kappa_1$ and $\kappa_2$.

| $h$ | Rate | Rate | Rate | Rate | Rate | Rate |
|-----|------|------|------|------|------|------|
| $\kappa_1 = \kappa_2 = 1E-2$ | $\kappa_1 = \kappa_2 = 1E-4$ | $\kappa_1 = \kappa_2 = 1E-8$ | $\kappa_1 = 1E-8$, $\kappa_2 = 1E-2$ | $\kappa_1 = 1E-8$, $\kappa_2 = 1E-2$ |
| 1/2 | —    | —    | —    | —    | —    | —    |
| 1/4 | 1.268| 1.055| 1.048| 1.048| 1.268| —    |
| 1/6 | 1.113| 0.960| 0.948| 0.948| 1.113| —    |
| 1/8 | 0.905| 0.889| 0.864| 0.864| 0.905| —    |

4.3. Hartmann flows. Hartmann flow is a classical benchmark problem for the MHD model which involves a steady unidirectional flow. It describes the flow of a liquid metal through a channel under an external transverse magnetic field \(^{12}\) and has been investigated in \(^{39, 37, 36}\). Based on the physical parameters $Re$, $Re_m$ and the coupling coefficient $s$, we consider both 2D and 3D Hartmann flows with Hartmann number $H_a = \sqrt{ReRe_m}s$.

Firstly, we consider the domain $\Omega = (0, 2) \times (-1,1)$ under the influence of the transverse magnetic field $B^d = (0, 1)$. The exact solutions to the 2D MHD problem are shown by \(^{12}\)

\[
\mathbf{u}(x, y) = \left( \frac{GRe}{H_a \tanh(H_a)} \left( 1 - \frac{\cosh(yH_a)}{\cosh(H_a)} \right), 0 \right), \\
p(x, y) = -Gx - \frac{G^2}{2s} \left( \frac{\sinh(yH_a)}{\sinh(H_a)} - y \right)^2 + 10, \\
\mathbf{B}(x, y) = \left( \frac{G}{s} \left( \frac{\sinh(yH_a)}{\sinh(H_a)} - y \right), 1 \right). 
\]
We impose no-slip boundary conditions on the wall and Neumann boundary conditions on the inlet and the outlet:

\[ \mathbf{u} = 0 \quad \text{on} \quad y = \pm 1, \]
\[ (p I - Re^{-1} \nabla \mathbf{u}) \mathbf{n} = p \mathbf{n} \quad \text{on} \quad x = 0, 2 \]
\[ \mathbf{n} \times \mathbf{B} = \mathbf{n} \times \mathbf{B}^d \quad \text{on} \quad \partial \Omega, \]

where \( I \) is the identity matrix.

For the 2D problem, we set \( G = 1 \) and consider following four cases:

Case 1 : \( H_a = 0.5, \ Re = Re_m = 0.1, \ s = 25; \)
Case 2 : \( H_a = 5, \ Re = Re_m = 1, \ s = 25; \)
Case 3 : \( H_a = 50, \ Re = Re_m = 10, \ s = 25; \)
Case 4 : \( H_a = 150, \ Re = Re_m = 30, \ s = 25. \)

The analytical solutions \( \mathbf{u}(1, y) \) and \( \mathbf{B}(1, y) \) are presented in Figure 1-4 alongside the numerical ones \( \mathbf{u}(1, y_k) \) and \( \mathbf{B}(1, y_k) \) \((y_k = -1 + 0.1k, k = 0, ..., 20)\) obtained by Algorithm 3.1. From the figures, we find that the numerical solutions of our scheme are consistent with the exact solutions of Hartmann flow.

Secondly, we consider the 3D Hartmann flow with the rectangular duct \( \Omega = (0, 2) \times (-y_0, y_0) \times (-z_0, z_0) \) under the influence of the uniform magnetic field \( B^d = (0, 1, 0) \). The solutions of this model are as follows \([12, 39, 36]\):

\[ \mathbf{u}(x, y, z) = (H_u(y, z), 0, 0), \quad \mathbf{B}(x, y, z) = (H_B(y, z), 1, 0), \]
\[ p(x, y, z) = -Gx - sH_B^2(y, z)/2 + 10, \]

with

\[ H_u(y, z) = -\frac{1}{2} GRe(z^2 - z_0^2) + \sum_{i=0}^{+\infty} a_i(y) \cos(\lambda_i z), \quad H_B(y, z) = \sum_{i=0}^{+\infty} b_i(y) \cos(\lambda_i z), \]

where

\[ a_i = A_i \cosh(p_1 y) + B_i \cosh(p_2 y), \]
\[ b_i = \frac{1}{sRe} \left( A_i \frac{\lambda_i^2 - p_1^2}{p_1} \sinh(p_1 y) + B_i \frac{\lambda_i^2 - p_2^2}{p_2} \sinh(p_2 y) \right), \]
\[ A_i = -\frac{p_1(\lambda_i^2 - p_2^2)}{\gamma_i} u_i(y_0) \sinh(p_2 y_0), \quad B_i = \frac{p_2(\lambda_i^2 - p_1^2)}{\gamma_i} u_i(y_0) \sinh(p_1 y_0), \]
\[ \lambda_i = \frac{(2i + 1)\pi}{2z_0}, \quad p_{1,2} = \frac{\lambda_i^2 + H_a^2/2 \pm H_a \sqrt{\lambda_i^2 + H_a^2/4}}{2}, \quad u_i(y_0) = -\frac{2GRe}{\lambda_i^2 z_0} \sin(\lambda_i z_0), \]
\[ \gamma_i = p_2(\lambda_i^2 - p_1^2) \sinh(p_1 y_0) \cosh(p_2 y_0) - p_1(\lambda_i^2 - p_2^2) \sinh(p_2 y_0) \cosh(p_1 y_0). \]

The boundary conditions are imposed by

\[ \mathbf{u} = 0 \quad \text{on} \quad y = \pm y_0 \quad \text{and} \quad z = \pm z_0, \]
\[ (p I - Re^{-1} \nabla \mathbf{u}) \mathbf{n} = p \mathbf{n} \quad \text{on} \quad x = 0 \quad \text{and} \quad x = 2, \]
\[ \mathbf{n} \times \mathbf{B} = \mathbf{n} \times \mathbf{B}^d \quad \text{on} \quad \partial \Omega. \]

We take \( G = 1, \ y_0 = \frac{1}{2} \) and \( z_0 = \frac{1}{4} \). Besides, we design this investigation with the parameters \( Re = Re_m = 30, \ s = 1. \) The numerical results of the presented method, Zhang’s method \([39]\) and the Linearized Crank-Nicolson method at final
time $T = 10$ are shown in Table 15. From this table, we can find that the presented method has the best accuracy among these methods.

5. Conclusions. In this work, we have presented a fully discrete scheme in solving the 2D/3D MHD-Voigt problem. We obtain unconditional stability and convergence analysis of the velocity and magnetic. Moreover, we discuss the convergence rate of the velocity and magnetic with varying $\kappa_1, \kappa_2$. All computational results support the theoretical analysis and demonstrate the effectiveness of the method.

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Figure 2. $H_a = 5$, $Re = Re_m = 1$ (left: velocity; right: magnetic field).

Table 15. Errors for the different methods of 3D Hartmann flow at $T = 10$.

| Methods               | $\tau = h$ | $\|u(T) - u_h\|_{0,2}$ | $\|B(T) - B_h\|_{0,2}$ |
|-----------------------|------------|------------------------|------------------------|
| Algorithm 3.1         | 1/4        | 8.20E-02                | 3.47E-02                |
| Zhang’s algorithm [39] | 1/4        | 9.49E-02                | 7.22E-02                |
| Linearized Crank-Nicolson [39] | 1/4        | 9.50E-02                | 7.22E-02                |
| Algorithm 3.1         | 1/8        | 2.44E-02                | 1.27E-02                |
| Zhang’s algorithm [39] | 1/8        | 3.58E-02                | 3.24E-02                |
| Linearized Crank-Nicolson [39] | 1/8        | 3.58E-02                | 3.24E-02                |
| Algorithm 3.1         | 1/16       | 1.09E-02                | 9.38E-03                |
| Zhang’s algorithm [39] | 1/16       | 1.15E-02                | 1.08E-02                |
| Linearized Crank-Nicolson [39] | 1/16       | 1.15E-02                | 1.08E-02                |

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Figure 3. $H_a = 50$, $Re = Re_m = 10$ (left: velocity; right: magnetic field).

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Figure 4. $H_a = 150$, $Re = Re_m = 30$ (left: velocity; right: magnetic field).

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