Deformed 2d CFT:
Landau-Ginzburg Lagrangians and Toda theories

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Abstract

We consider the relation between affine Toda field theories (ATFT) and Landau-Ginzburg Lagrangians as alternative descriptions of deformed 2d CFT. First, we show that the two concrete implementations of the deformation are consistent once quantum corrections to the Landau-Ginzburg Lagrangian are taken into account. Second, inspired by Gepner’s fusion potentials, we explore the possibility of a direct connection between both types of Lagrangians; namely, whether they can be transformed one into another by a change of variables. This direct connection exists in the one-variable case, namely, for the sine-Gordon model, but cannot be established in general. Nevertheless, we show that both potentials exhibit the same structure of extrema.

1 Introduction

The study of 2d FT in the vicinity of a critical point has gained importance over the latter years. There are essentially two ways of investigation: The first is to analyse the critical behavior of 2d statistical models, in particular, of integrable type. The second begins with the critical model, some 2d CFT, and deforms it in a suitable manner such that the correlation length becomes finite. The most interesting deformations are again the so-called integrable, for which the non-critical model still has an infinite number of conservation laws that allow to calculate correlation functions. The simplest example is the thermal perturbation of the Ising model. It generalizes to the perturbation of
minimal models of the Virasoro algebra by the most relevant field, $\Phi_{(1,3)}$. It further generalizes to the perturbation of minimal models of $W$-algebras by the most relevant field of the thermal subalgebra, $\Phi(1, \cdots, 1 | \text{Adj})$. The integrability of these perturbations lies on the fact that the deformed theories can be realized as quantum affine Toda field theories (ATFT) with imaginary coupling constant $[1, 2]$.

Affine Toda field theories are defined in terms of a classical Lagrangian with a potential of exponential form. The coupling constant $\beta$ appears in the exponential; hence, when it takes imaginary values then the potential is periodic in the fundamental field and can therefore have soliton solutions. The description of quantum affine Toda field theories is best attained in terms of these solitons; due to integrability, their only interaction consists of elastic scattering. One can say that the structure of degenerate ground states of the classical potential is the dominant feature of these models for it determines the soliton types. The series $W_p^\beta$ of minimal models are obtained for $\beta^2 = -p/p + 1$. These are called restricted theories since the soliton spectrum has to be restricted for consistency. Then only a finite number of ground states and therefore a portion of the potential are allowed.

There is another Lagrangian description of deformed 2d CFT with $W$-symmetry, namely, the one given by Landau-Ginzburg (LG) Lagrangians. This description was introduced in a previous paper [4]. There was shown as well that those Landau-Ginzburg Lagrangians correspond to the universality classes of critical behavior of the integrable models of Jimbo et al [5]. To be precise, it was shown that there exists a perturbation of the Landau potential such that its ground states exactly correspond to the ground states of regime III of Jimbo et al models. Hence, this perturbed Landau potential possesses features similar to those of the portion of potential that characterizes restricted ATFT.

The question arises of whether a direct connection between these two types of classical descriptions of deformed 2d CFT can be established. The first problem that comes to one’s mind is that the deformations that produce the mentioned ground state structure seem to be different in either approach. In the case of the affine Toda field theory, it is $\Phi(1, \cdots, 1 | \text{Adj})$ whereas in the case of the Landau-Ginzburg theory not only the composite field corresponding to it intervenes but a combination of all the symmetry preserving relevant fields. This difference is due to the classical character of Landau-Ginzburg Lagrangians. The perturbed Lagrangian must include quantum corrections, bringing about all the other symmetry preserving relevant fields. We study this effect in the first section. We argue that for $n = 2$ the quantum effective potential is directly related to that of the corresponding ATFT, the sine-Gordon potential. In the second section we formalize the correspondence between $n = 2$ ATFT and the restricted sine-Gordon model. We generalize to other $n$ using Gepner’s ideas [6]. Finally, we study the ground states of affine Toda potentials and we compare them with those given by Landau potentials.

The connection between ATFT and Landau-Ginzburg Lagrangians for CFT with

*See [3] for a study of these solitons.
$N = 2$ supersymmetry has been studied in [7]. Since these Landau-Ginzburg Lagrangians are not related to those of the bosonic theories, there is no apparent relation with our work.

2 Quantum corrections to Landau-Ginzburg Lagrangians

Let us introduce some terminology of thermodynamics to better describe the features of the phase diagrams in which we are interested. The ground states of regime III of Jimbo et al models are ordered phases that coexist along a manifold ending at the multicritical point. We are interested in the particular line of multiple coexistence given by perturbation with $\Phi(1, \cdots, 1 \mid \text{Adj})$. It is not difficult to find the form of the equation for multiple coexistence in terms of the coefficients of the Landau potential. To be definite, let us consider the case with one order parameter $\varphi$. The potential $V = \varphi^{2k}$ has $k$ minima which unfold under perturbation. A generic perturbation is given by a polynomial in $\varphi$ of degree $2k - 2$, which is thus the codimension of the multicritical point. Since the heights of the minima with respect to a given one are specified by $k-1$ numbers, this is the codimension of the multiple coexistence manifold. If we only admit $Z_2$ symmetric perturbations, the codimension of the multicritical point is $k-1$ and the codimension of the multiple coexistence manifold $(k-1)/2$ or $(k/2) - 1$ for $k$ odd or even, respectively.

For example, let $k = 4$. The perturbed symmetric potential is

$$V = \varphi^8 + u \varphi^6 + v \varphi^4 + w \varphi^2. \quad (1)$$

It can also be written as

$$V = (\varphi^2 - b_1^2)^2 \left((\varphi^2 - b_2^2)^2 + d\right) \quad (2)$$

with another set of parameters more related to its shape. The condition for quadruple coexistence is $d = 0$. Then the minima occur for $\varphi = \pm b_1, \pm b_2$. That condition is expressed in $uvw$-space as a surface (codimension-one submanifold) with parametric equations

$$u = -2(b_1^2 + b_2^2), \quad (3)$$
$$v = b_1^4 + 4b_1^2b_2^2 + b_2^4, \quad (4)$$
$$w = -2(b_1^2b_2 + b_2^2b_1). \quad (5)$$

Since the position of the minima with positive coordinates, $b_1$ and $b_2$, is still arbitrary, we can impose one further condition on their ratio, $b_2 = \alpha b_1$, thus fixing the shape of the potential up to overall rescaling. This restricted multiple coexistence occurs along lines which end at the tetracritical point. Their parametric dependence is given by dimensional analysis, while the coefficients, as functions of $\alpha$, specify a particular line. The important point is that $v$ and $w$ take non-null values along these lines. The line
$v = w = 0$ and $u \leq 0$ is not of coexistence but a critical metastable line on which the two middle minima disappear and is not of interest. In contrast, we know that the exact quantum perturbation that produces multiple coexistence is $\Phi_{(1,3)}$, corresponding to $\varphi^{2k-2}$; hence only $u$ being non-null. However, the identification of primary fields with composite fields in the Landau-Ginzburg approach is only valid at the critical point. In the perturbed theory there are quantum effects that induce field mixing. The least relevant perturbation $\varphi^{2k-2}$ has the coupling of lowest dimension (in space dimension $d > 2$). Therefore, a perturbation with it or, in other words, a non-null value of its coupling constant will contribute to all other coupling constants, of higher dimension.

To analyse quantum effects, let us consider the simplest case, $k = 3$, with potential

$$V = u \varphi^6 + v \varphi^4 + w \varphi^2. \quad (6)$$

At second order perturbation theory in $v$ there is a contribution to the two-point vertex, $w = cv^2$, given by the setting-sun Feynman diagram

![Diagram](7)

The value of the coupling $u$ can be taken to be its fixed-point value $u^*$. It is obtained as the non-trivial solution of $\beta(u) = 0$. The $\beta$-function is calculated from the 6-point vertex, given at second order by the diagram

![Diagram](8)

Upon calculation of these diagrams one obtains $c$ and $u^*$. It is convenient to normalize the potential as before, dividing by $u^*$ and redefining the two relevant couplings, $\tilde{v} = v/u^*$ and $\tilde{w} = w/u^*$. They are related through

$$\tilde{w} = cu^* \tilde{v}^2. \quad (9)$$

We have to compare this relation with the equation of the triple line. This equation can be simply obtained by writing the potential as

$$V = (\varphi^2 - b^2)^2 \varphi^2 \quad (10)$$

in the parametric form

$$v = -2b^2, \quad (11)$$

$$w = b^4 \quad (12)$$

or

$$w - \frac{1}{4} v^2 = 0. \quad (13)$$

\footnote{It actually belongs to the critical manifold that separates regimes III and IV.}
It has the same form as (9). To compare the coefficients of \( v^2 \) we have to evaluate \( c u^* \). Fortunately, we do not need to evaluate both diagrams independently but its quotient, since \( u^* \) is in inverse proportion to the value of (8). Furthermore, we note that both diagrams are essentially the same, namely, the propagator cubed \((\Delta^3)\), except for combinatorial factors. These factors can be found for any \( k \) in \([9]\). Thus we obtain the dimension-independent value

\[
c u^* = \frac{(4!)^2 \Delta^3}{6 (6!)^2 \Delta^3} = 0.08. \tag{14}
\]

It is substantially smaller than 1/4 in (13). The difference is due to higher order corrections, which are known to be non-negligible \([9]\).

A similar argument can be made in favor of the fact that a multiple coexistence line calculated in perturbation theory within the Landau-Ginzburg approach converges to the exact line provided by the \( \Phi(1, \cdots, 1 \mid \text{Adj}) \) perturbation. However, given the computational restrictions, we must content ourselves by saying that any W-algebra Landau-Ginzburg Lagrangian perturbed by the least relevant field develops due to quantum effects non-null values for the coupling constants of the other (symmetry-preserving) relevant fields such that the renormalized equation for that line is compatible with the exact one.

The problem that we face now is that, in general, there is an entire manifold of coexistence, as we saw for \( k = 4 \) before. In absence of an accurate perturbative solution, we have to consider further conditions to define a unique line that corresponds to the ATFT. The difference between coexistence lines can be assumed to represent the shape of the potential while the parameter along lines is a scale variable representing its size. The essential feature of the shape of the potential is the height of the walls that separate its ground states. This height measures how close the respective phases are in a thermodynamical sense. In field theory it determines the energy of the kinks that interpolate between ground states. A reasonable condition may be to take all those heights equal. In so doing, the generic potential for the Virasoro series, with one order parameter, takes a particularly simple form, namely, of a Chebyshev polynomial of the first kind.\(^\dagger\) This type of polynomial is defined by

\[
T_n(\cos \phi) = \cos(n \phi), \tag{15}
\]

where the degree is \( n = 2k \) in our case. Obviously it has the set of maxima and the set of minima at the same height, which is 1 or \(-1\), respectively. Furthermore, with \( \phi \) as the fundamental field it is the potential of the sine-Gordon model, the simplest ATFT. In this light, we see that the condition of equal heights above is, besides natural, the only one that leads to this correspondence.

\(^\dagger\)The theories with \( N = 2 \) supersymmetry are much more constrained, to the extent that it is possible to establish recursion relations among perturbed composite fields which can be solved to give the Chebyshev polynomial potential \([8]\).
The Chebyshev polynomial ansatz provides definite values for the coupling constants of the quantum potential. It also gives the field mixing structure. In particular, the least relevant composite field, $\varphi^{2k-2}$, becomes
\begin{equation}
\left[ \varphi^{2k-2} \right] \equiv \frac{\partial V}{\partial u} = \varphi^{2k-2} + \cdots.
\end{equation}
(16)

An examination of the coefficients in this expression shows that it can be identified as Chebyshev polynomial of the second kind, $U_{2k-2}(\varphi)$. The appearance of this particular type of polynomial is not coincidental, according to the ideas introduced in the next section.

The previous line of reasoning is in principle applicable to the W-series, with potentials with several order parameters. The effect of quantum corrections is essentially the same. The definition of a unique line of coexistence is however more delicate, since the top of the wall which separates two minima may not correspond to a maximum but to a saddle point. We can try, regardless of this problem, to generalize the previous form in terms of Chebyshev polynomials. For example, in the $W_3$ case it is convenient a triangular coordinate system which consists of 3 coordinates, $x_1$, $x_2$ and $x_3$, subject to the constraint $x_1 + x_2 + x_3 = 0$. A simple potential with the correct symmetry that generalizes the one-variable case is the sum
\begin{equation}
V = \sum_{i=1}^{3} T_{2k}(x_i).
\end{equation}
(17)

Although the polynomial degree is correct, we do not achieve the potentials obtained in [4]. The first non-trivial case (3-state Potts model) occurs when $k = 2$. Now, it is straightforward to verify that for $k = 2$ the potential (17) has no third order term and hence it has full rotational symmetry, contrary to the correct $D_3$ symmetry. A more subtle way to generalize the Chebyshev polynomials to higher dimension is required. We concern ourselves with this problem in the next section.

3 Landau-Ginzburg Lagrangians as restricted affine Toda theories

We have seen above that the Landau potential of the Virasoro minimal models for the required type of phase coexistence is closely related to the potential of the sine-Gordon model. The latter has infinitely many ground states, all placed periodically, while the former has only a finite number. Thus the Landau potential can only represent a strecht of the sine-Gordon model. Besides, it is not a periodic function but a polynomial. Of course, these differences come from the different variables used to define them; they are related by
\begin{equation}
\varphi \simeq \cos \phi.
\end{equation}
(18)

\footnote{In general, the symmetry of (17) is $D_6$, as a consequence of the absence of odd powers of the third order term, nevertheless higher than the required $D_3$ symmetry.}
Hence, the only relevant values of the sine-Gordon field are \(0 < \phi < \pi\), for which \(-1 < \varphi < 1\). The number of extrema that fit in this interval is \(2k\), the degree of the Chebyshev polynomial \([13]\). The deep reason for the restriction in the number of minima is the truncation of the soliton spectrum that occurs at certain values of the coupling constant: The soliton and antisoliton which this model possess belong to a representation of a quantum group. For certain representations (\(q\) is a root of unity) there is truncation and only a finite number of solitons and antisolitons can be composed \([10]\). These solitons precisely correspond to the kinks of the Landau potential. The details of this truncation are not essential in the present context.

The picture based on the identification \([18]\) is of a qualitative nature yet. The reader may have noticed that our actual sine-Gordon potential is

\[
V(\varphi) = \cos (\beta \phi),
\]

with

\[
\beta^2 = \frac{p}{p + 1}.
\]

Hence the value of \(\beta\) is not integer and does not fit a Chebyshev polynomial \([15]\). However, the existence of a finite number of kinks related to \(k\) supports the idea. Perhaps we must scale \(\phi\) somehow. We present now a tentative argument in this line.

First, we must recall that primary fields of the form \(\Phi_{(1,m)}\) or \(\Phi_{(n,1)}\) are identified in the Liouville theory with exponentials of the field corresponding to vertex operators of the Dotsenko-Fateev construction of minimal models \([1]\). Since the remaining primary fields can be generated as products of fields of those two types, we may think that the identification with the Dotsenko-Fateev construction holds throughout. The elementary field is then represented as

\[
\varphi \equiv \Phi_{(2,2)} = \exp \left( i \alpha_{(2,2)} \phi \right).
\]

This is not quite of the form \([18]\). Neither is it real. It is then reasonable to take

\[
\varphi = \cos \left( \alpha_{(2,2)} \phi \right),
\]

which amounts to a rescaling of \(\phi\) in \([18]\). Now, we should have an even integer for the quotient of the argument of \([19]\) by the argument of \([22]\),

\[
\frac{\beta}{\alpha_{(2,2)}} = \frac{\sqrt{\frac{p}{p+1}}}{2 \sqrt{\frac{p}{p+1}}} = -2p.
\]

It is indeed an even integer, though negative and of absolute value larger than we need, producing extra ground states. It is easy to see that the required value is \(2 (p - 1)\), for example, using the Ising model with \(p = 3\) and 2 ground states. In fact, it is possible

\footnote{Do not confuse this truncation in the spectrum with the truncation in the OPE alluded to before.}
to obtain this precise value. We must recall that in the Dotsenko-Fateev construction primary fields are defined up to a conjugation. It is thus possible to use instead of $\beta = \alpha_{(1,3)}$ its conjugate, $\alpha_{(p-1,p-2)}$. We then have

$$\frac{\alpha_{(p-1,p-2)}}{\alpha_{(2,2)}} = \frac{-\frac{p-1}{\sqrt{p(p+1)}}}{\frac{1}{2\sqrt{p(p+1)}}} = 2(p-1),$$

as desired.

We believe that this argument shows that the equivalence between both descriptions can be made rigorous. However, we must emphasize that the sine-Gordon theory or, in general, ATFT are exact quantum theories whereas Landau-Ginzburg theories are classical Lagrangians. Any discrepancy between them is to be explained within this philosophy. So it happens for the kinetic term, which we have neglected so far. The sine-Gordon kinetic term is written as an infinite series expansion under the identification (18) or (22),

$$\left(\partial \varphi\right)^2 = \frac{1}{\sin^2 \varphi} \left(\partial \varphi\right)^2 = (1 + \varphi^2 + \varphi^4 + \cdots) \left(\partial \varphi\right)^2.$$

All the terms but the first are irrelevant fields. Hence it is possible to keep just the ordinary kinetic term in the Landau-Ginzburg Lagrangian.

Field identifications similar to (18) have in fact been proposed in a slightly different context, that of fusion rings. A fusion ring is the algebraic structure of primary fields of some rational conformal field theory. It was discovered by Gepner that the fusion rings of $SU(n)$ Wess-Zumino-Witten models can be expressed in terms of potentials, called fusion potentials [6]. Well, the fusion potentials for $SU(2)$ are the Chebyshev polynomials of the first kind. Here also the most relevant field is to be identified with $\cos \varphi$. The other primary fields are Chebyshev polynomials of the second kind and reproduce the fusion rules. We must remark that one should not think of the fusion potential as the same as the Landau potential. Both produce rings of perturbations (actually algebras) by quotienting the ring of polynomials by the ideal generated by the derivatives of $V$ (redundant fields). However, the ring of perturbations of the fusion potential includes all primary fields whereas that of the Landau potential only includes relevant primary fields. Nevertheless, there is some resemblance between both structures, as is manifested by the form (16) found for the renormalized field $\left[\varphi^{2k-2}\right]$ as a Chebyshev polynomial of the second kind.

It is tempting to consider the Gepner’s potentials for $SU(n)$ as a generalization of Chebyshev polynomials suitable for our purposes. Before proceeding, we need to recall how Gepner’s potentials are constructed [6]. The primary fields of the Wess-Zumino-Witten model with Kac-Moody symmetry $SU(n)_l$ correspond to the representations of $SU(n)$ with dominant weights up to level $l$. Gepner associates to each primary field the character of its corresponding representation, which can be calculated, for example, with the Weyl character formula. The characters are functions of as many variables as the rank of the group and are invariant under the Weyl group. This is clearly exhibited by the Weyl formula, which involves summation over the elements of the Weyl group.
Gepner’s fusion potential is

\[ V = \frac{1}{m} \sum_{k=1}^{n} e^{i m 2\pi \phi_k}, \]  

(26)

where \(2\pi\phi_k\) are angular variables, constrained by \(\sum \phi_k = 0\), that parametrize the Cartan subalgebra and \(m\) is some integer. Since \(V\) is a completely symmetric polynomial of

\[ q_k = e^{i 2\pi \phi_k}, \]

it can be in turn expressed as a polynomial of the elementary symmetric monomials in \(q_k\),

\[ \sigma_r = \sum_{i_1, \ldots, i_r} q_{i_1} \cdots q_{i_r}, \quad r = 1, \ldots, n - 1. \]  

(27)

As functions of \(\phi_k\), these are the characters of the fundamental representations of \(SU(n)\). They correspond to the elementary fields or order parameters. This identification generalizes (18). Since \(\sigma_r\) are symmetric in \(\phi_k\), we can restrict the domain of these variables to a fundamental region of the Weyl group. Furthermore, as functions of \(q_k\) they are periodic with \(\phi_k\). One can see that the fundamental domain of \(\phi_k\) is the \(n-1\)-simplex formed by the fundamental weights with \(D_n\) symmetry. Its geometrical center is at \(\phi = \rho/n\), where \(\rho\) is the Weyl vector. It can be shown that all the elementary fields (27) vanish on this point. The values of the elementary fields on the vertex of the fundamental domain corresponding to \(\omega_k\) are

\[ \sigma_r = \binom{n}{r} e^{i 2\pi \frac{k_r}{n}}, \quad r = 1, \ldots, n - 1. \]  

(28)

One can see that the mapping from \(\phi\) to \(\sigma_r\) amounts to a deformation of the simplex that preserves its symmetry properties.

The fusion potential (26) is certainly of the ATFT type. However, if we take \(\phi\) to stand for the ATFT field, we should consider the slightly different form

\[ V = -\sum_{k=1}^{n} \exp (i m 2\pi \alpha_k \cdot \phi), \]  

(29)

where \(\alpha_k\) are the set of positive roots plus minus the highest root \(\alpha_n \equiv \alpha_0\). This potential possesses two types of symmetries, rotations-reflections and translations. The first type is the symmetry of the extended root system, in this case \(D_n\). The translational symmetry

\[ \phi \rightarrow \phi + \frac{l}{m} \omega_k, \quad l \in \mathbb{Z}, \quad k = 1, \ldots, n - 1, \]  

(30)

is due to its exponential form and is especially interesting since it is the cause of the existence of solitons. When \(l = m\), the transformation (30) maps the fundamental domain of the elementary fields (27) onto another domain, producing an automorphism of these fields,

\[ \sigma_r = e^{i 2\pi \frac{r}{n}} \sigma_r. \]  

(31)
It corresponds to the $Z_n$ symmetry of the $W_n^P$ models.

The AFTF potential is not real (like Gepner’s) but we can take its real part. This real potential is able to be related with a Landau potential. It can be easily plotted for the $W_3$ case and looks like a 2d generalization of sine-Gordon.\footnote{This fact was noted in $[11]$.} There are valleys placed on a triangular array and small mountains on a hexagonal array (Kagomé lattice) interlaced with the former. (See fig. 1.) Now, we are to compare the structure of extrema inside a fundamental domain of $\sigma$ with the structure of extrema of $W_3^P$ Landau potentials $[4]$. It has been observed before $[3]$ that the ATFT with $W_3$ symmetry in the restricted case has solitons that match the kinks between ground states of the corresponding IRF models of Jimbo et al. We showed in $[4]$ that the Landau potential reproduces these ground states. These ground states appear as a consequence of the symmetry under translations $(30)$. A translation $(30)$ produces new ground states only for $l \leq m$. Hence there must be as many as fit in a triangle of side $m$. This is the maximum number of minima found before for the Landau potential with $p = m + 3$ $[4, 14]$. It is also important to check that the other extrema, maxima and saddle points, coincide as well. It can be done by inspection on fig. 1 and comparison with the results in $[4, 14]$.

In the case $n = 3$, the generic symmetry of ATFT potentials, $D_n$, coincides with the Weyl symmetry $S_n$ of the expression (27) for the elementary fields. Therefore, we can express the potential as a polynomial in the elementary fields, $\sigma$ and $\bar{\sigma}$, according to Gepner’s idea. However, the polynomials thus constructed do not have the correct structure of extrema. This is because the minima situated on the border of the fundamental domain are pushed to infinity after expressing the potential in terms of $\sigma$ and $\bar{\sigma}$ (The jacobian of this change of variables vanishes on the border.) Therefore, the Landau potential misses them. We could enlarge the fundamental domain by shifting the border outwards up to the nearest maxima. Unfortunately, this would spoil the Weyl symmetry of the potential, which could not then be expressed in terms of $\sigma$ and $\bar{\sigma}$. There seems to be no way to obtain a suitable polynomial to be identified with the Landau potential.

For $n > 3$ the structure of the extrema of the real part of the ATFT potential in the fundamental domain of $\sigma_r$ also agrees with the structure of the extrema of the corresponding Landau potential. However, now it is not even possible to express the real part of the ATFT potential in terms of $\sigma_r$, since the former has only $D_n$ symmetry and the latter has full Weyl symmetry. It is known that ATFTs only possess the symmetry of the extended Dynkin diagram, which can be identified with a subgroup of the Weyl group isomorphic to the semidirect product of the center of the Lie group and complex conjugation $[12]$. This group is $D_n$ for the Lie algebra $A_{n-1}$, while the full Weyl group is $S_n$. This fact may seem an undesirable feature of ATFT potentials in regard to their relation with Landau potentials. On the contrary, it turns out to be a necessary property for the soliton spectrum of restricted ATFT to fit the kink structure provided by Landau potentials; namely, the kinks that interpolate between
ground states belonging to a diagram of dominant weights. The kinks that actually correspond to solitons (indecomposable into others more elementary) must have weights belonging to some fundamental representation as topological charges. However, not all the weights of each fundamental representation can appear, as will be seen in a concrete example below.

It is not difficult to show for \( n > 3 \) that the ground state structure of the restricted ATFT potential is the same as that of the Landau potential. We shall do it for \( n = 4 \), with potential

\[
V(\phi) = -\sum_{k=0}^{3} \exp(i 2\pi \alpha_k \cdot \phi). \tag{32}
\]

This potential is real along the directions of the weights of the fundamental representations and the horizontal roots of \( SU(4) \). For the directions along the fundamental weight of the representation 4, \( \omega_1 \),

\[
\phi = \phi \frac{\omega_1}{|\omega_1|} \Rightarrow \alpha_2 \cdot \phi = \alpha_3 \cdot \phi = 0,
\]

and along the fundamental weight of \( \bar{4}, \omega_3 \),

\[
\phi = \phi \frac{\omega_3}{|\omega_3|} \Rightarrow \alpha_2 \cdot \phi = \alpha_1 \cdot \phi = 0,
\]

the potential is

\[
V(\phi) = -2 - 2 \cos \frac{2\pi \phi}{\sqrt{3}/2}. \tag{33}
\]

Due to the \( D_4 \) symmetry, this form of the potential holds for the other weights of the representations 4 and \( \bar{4} \), completing the diagonals of a square.

Along the fundamental weight of the representation 6, \( \omega_2 \), the potential is

\[
V(\phi) = -2 - 2 \cos(2\pi \phi). \tag{34}
\]

We know again that this form holds for those weights of the representation 6 related to \( \omega_2 \) by the \( D_4 \) symmetry, namely, the horizontal weights. However, the potential along the two vertical weights is now different. Since

\[
\alpha_1 \cdot \phi = \alpha_3 \cdot \phi = -\alpha_2 \cdot \phi = -\alpha_0 \cdot \phi
\]

the potential is

\[
V(\phi) = -4 \cos(2\pi \phi). \tag{35}
\]

A similar form is found for the horizontal roots. With these results it is easy to figure out the overall structure of the potential. There are minima \( (V = -4) \) situated on a body centered cubic (bcc) lattice and maxima \( (V = 4) \) situated on a similar lattice interlaced with the previous one (fig. 2). Let us remark once more the absence of Weyl

**We omit the coupling constant \( \beta \) (or number \( m \) in the restricted case).
symmetry. It is reflected in the fact that the potential for the two vertical weights of 6 is different from that for the four horizontal ones. In particular, the solitons that link the minima corresponding to the two vertical weights of 6 go over a higher wall (twice as high) and are therefore more energetic than those that link the minima corresponding to the horizontal weights of 6.

Let us see what happens in the restricted case. The kinks corresponding to the two vertical weights of 6 are in the spectrum above those corresponding to the horizontal ones. Thus the simplest model contains the kinks associated to the weights of 4 and \( \bar{4} \) and only the horizontal weights of 6.\footnote{This fact has already been noted in the context of the computation of exact soliton solutions [15].} In other words, we have only the kinks associated to the dominant weight diagram at lowest level. This diagram occurs in the fundamental domain of \( \sigma_r \) when \( m = 1 \). We further find in this domain one maximum at the center and six saddle points situated between minima. This is the ground state structure given by the Landau potential of \( W_{(4)}^5 \). The next restricted model allows double kinks in the directions of 4 and \( \bar{4} \) and in the horizontal of 6. It allows as well the single kink (and antikink) in the vertical direction of 6. They all fit in the dominant weight diagram at the next level. Once more, one can see that also the maxima and saddle points correspond to the Landau potential of \( W_{(4)}^6 \).

4 Conclusions

The role of Landau-Ginzburg and ATFT Lagrangians as classical descriptions of deformed 2d CFT has been studied to find out to what extent they are equivalent. The first essential observation is that it is necessary to consider quantum corrections to the Landau-Ginzburg Lagrangian to reproduce the multiple phase coexistence given by the ATFT at the outset. The lowest order corrections for the tricritical Ising model \((M_4)\) have been obtained and shown to produce the adequate type of renormalization.

For more complicated models of the minimal Virasoro series, it has been argued that the renormalized potential coincides with a Chebyshev polynomial of the first kind. Since this polynomial is formally the fusion potential obtained by Gepner for \( SU(2) \), we have studied whether Gepner's construction of potentials for \( SU(n) \) can be adapted to ATFT. We have seen that his identification of elementary LG fields in terms of (the Toda field) \( \phi \) as Lie group characters \([27]\) is adequate with regard to their expected symmetry properties. We have also shown that the fundamental domain of definition of elementary fields contains the correct structure of minima according to the spectrum of solitons in restricted ATFT. Furthermore, all the extrema (minima, maxima and saddle points) agree with those obtained from Landau potentials.

Unfortunately, it is crucial in Gepner's construction that the potential be a completely symmetric function of \( \phi \), or, in other words, that it have Weyl symmetry. This condition is not satisfied by the ATFT potential. Moreover, in the \( W_{(3)} \) case, in which it is indeed satisfied, even though a Landau like potential can be obtained by Gepner's
method, it is not the correct Landau potential, known from [4]: It misses the ground states on the boundary. We can therefore conclude that the relation between both types of classical Lagrangians is indirect: They yield the same extrema but cannot be related by a change of variables.

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References

[1] T.J. Hollowood and P. Mansfield, Phys. Lett. B 226 (1989) 73
[2] T. Eguchi and S.K. Yang, Phys. Lett. B 224 (1989) 373
[3] T.J. Hollowood, Nucl. Phys. B 384 (1992) 523
[4] J. Gaite, Nucl. Phys. B 411 (1994) 321
[5] Jimbo et al, Nucl. Phys. B 300 (1988) 74
[6] D. Gepner, Com. Math. Phys. 141 (1991) 381
[7] P. Fendley, W. Lerche, S.D. Mathur, N.P. Warner, Nucl. Phys. B 348 (1991) 66
[8] R. Dijgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B 352 (1991) 59
[9] C. Itzykson and J.-M. Drouffe, Théorie statistique des champs (InterEditions, Paris, 1989), page 312;
P.S. Howe and P.C. West, Phys. Lett. B 223 (1989) 371
[10] A. LeClair, Phys. Lett. B 230 (1989) 103
[11] P. Dorey and F. Ravanini, Int. Jour. Mod. Phys. A 8 (1993) 873
[12] D.I. Olive and N. Turok, Nucl. Phys. B 215 (1983) 470
[13] T. Nakatsu, Nucl. Phys. B 356 (1991) 499
[14] J. Gaite, Phys. Lett. B 325 (1994) 51
[15] N.J. MacKay and W.A. McGhee, Int. Jour. Mod. Phys. A8 (1993) 2791
Fig. 1  Structure of extrema (contour plot) for the real part of the $n = 3$ ATFT potential with the dominant weight diagram of level 3.
Fig. 2  Unit cell of the lattice of extrema for the real part of the $n = 4$ ATFT potential. Minima $V = -4$ are displayed in black and maxima $V = 4$ in grey.