BEC From a Time-Dependent Variational Point of View

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Abstract

We use the time-dependent variational principle of Balian and Vénéroni to derive a set of equations governing the dynamics of a trapped Bose gas at finite temperature. We show that this dynamics generalizes the Gross-Pitaevskii equations in that it introduces a consistent dynamical coupling between the evolution of the condensate density, the thermal cloud and the "anomalous" density.

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1 Introduction

Bose-Einstein condensation (BEC) phenomenon has a long story since its discovery. Such a fascinating behavior of matter has intrigued many researchers[1]. Static as well as dynamic properties of BEC are now intensively studied, both experimentally and theoretically, in order to apprehend the way the BEC forms, develops and vanishes.

Various theoretical techniques have been used successfully, predicting correctly a great number of experimental results. Among these techniques, some are most widely used, such as the Bogoliubov[2], the Beliaev[3, 4] and the Hartree-Fock-Bogoliubov[5, 6, 7] approximations. Some other methods, such as the Monte-Carlo simulations[8] are rising interest since they tend to solve the exact quantum many-body problem.

Although being extremely precise in the static situations, the approximations mentioned above all adopt ad-hoc assumptions about the various quantities such as the order parameter $\Phi$ (or the condensate density $n_c$), the non-condensed density or thermal cloud $\tilde{n}$ or the anomalous density $\tilde{m}$. These assumptions lead inevitably to the fact that the approximations are no longer totally consistent in a sense which will become clearer soon.

In this paper, we rely on a different approach, based on the time-dependent variational principle of Balian and Vénéroni (BV) [9, 10]. Not only does this method retain the essential features of the physics behind the previous approximations, but it also allows one to bypass some (if not all) of the ad-hoc assumptions. Indeed, being variational, the formalism generates a consistent dynamics for the quantities $\Phi$, $n_c$, $\tilde{n}$ and $\tilde{m}$ by choosing a certain class of trial spaces.

This well-known advantage of this (and any) variational principle faces however a major difficulty related to the choice of the trial spaces. A (difficult) compromise must be found between a correct description of the physics on one hand, and the tractability of the calculations on the other. In this order of ideas, the BV variational principle requires to specify two objects: a density-like operator and a "measured" observable.

Our choice for the variational spaces consists of a gaussian time-dependent density-like operator and a single-particle operator for the observable. This last quantity turns out to be of no interest in our particular case but it may play a major role in other situations especially when one intends to calculate correlation functions[11, 12].

The machinery we are discussing has in fact already been used elsewhere[10], where we have recognized that the variational dynamical equations derived there are a generalization of the Gross-Pitaevskii equations[13], that takes into account the coupling between the order parameter and the various densities. We called this approach the "Time-Dependent Hartree-Fock-Bogoliubov" (TDHFB) approximation. The point is that the usual image of a collection of condensed atoms in a bath of thermal particles is not
totally true, since the bath has its own dynamics which is sensible to the condensate dynamics.

The purpose of the present paper is to go into further details in this variational approximation so as to make a closer connection with other well-known methods (such as the ones quoted above) used in the study of Bose-Einstein condensates. Among the problems that we intend to study, we can cite in particular the analysis of the static properties of the condensate (compared to what is known) as well as the small oscillations around the static solutions, when the effects of the coupling with the thermal cloud and the anomalous density are taken into account.

The important paper by A. K. Kerman and P. Tommasini [14] is closely related to ours. It uses however the Dirac variational principle which constrains the calculations to \( T = 0 \) (even if the authors give at the end of the paper a finite temperature prescription.) The authors also limit themselves to the uniform (that is non trapped) case. Therefore, according to what has been said above, we may consider our paper as a twofold generalization of [14].

The full TDHFB equations also deserve to be solved in order to study the large amplitude motion of the condensate and the thermal cloud. Despite its importance, we will postpone this study to a future publication which is in progress.

The paper is organized as follows. In section 2, we recall the major steps used in [10, 12] to derive the TDHFB equations and to write them down in the case of the BEC problem. Section 3 is devoted to the study of the static solutions, where we recover the results of [4] and generalize them to finite temperature. In section 4, we analyze the excitations of the condensate by using the RPA technique. Finally, we present some concluding remarks and perspectives in order to deal with more complicated situations.

2 The TDHFB Equations

The General TDHFB equations were derived in [10] using the BV variational principle. For technical reasons, they were written in terms of the creation and annihilation operators \( a^+ \) and \( a \). They may however easily be translated in a more appropriate notation for the BEC problem using the boson field operator \( \hat{\Psi}(r) \) in the Schrödinger picture, in exactly the same way as our previous work[12]. Let us first recall some of our notations. We introduced the gaussian density operator \( \mathcal{D}(t) \) (with variational parameters \( \mathcal{N}(t), \lambda(t) \) and \( S(t) \)):

\[
\mathcal{D}(t) = \mathcal{N}(t) \exp(\lambda(t) \tau \alpha) \exp\left(\frac{1}{2} \alpha \tau S(t) \alpha\right),
\]

(2.1)
where $\alpha = (a^+, a)$ and $\tau$ is a symplectic constant matrix. Then, we defined the one-boson expectation value $\langle \alpha \rangle$ and the single-particle density matrix $\rho$ by:

$$
\langle \alpha \rangle = \begin{pmatrix} \langle a \rangle \\ \langle a^+ \rangle \end{pmatrix}, \quad 1 + 2\rho = \begin{pmatrix} \langle aa^+ + a^+a \rangle & -2\langle \bar{a}\bar{a} \rangle \\ 2\langle a^+a^+ \rangle & -\langle \bar{a}\bar{a}^+ + \bar{a}^+\bar{a} \rangle \end{pmatrix},
$$

which are directly related to $\lambda(t)$ and $S(t)$ respectively (see Eq. A.6 of ref.[12]). Operators such as $\bar{a}$ are simply the centered operators $a - \langle a \rangle$. The expectation values are to be taken with respect to the density operator (2.1).

Introducing (2.1) in the BV variational action-like leads, beside the conservation of the partition function $\mathcal{Z} = \text{Tr} \mathcal{D}(t)$, to what we called the TDHFB equations:

$$
i\hbar \frac{d\langle \alpha \rangle}{dt} = \tau \frac{\partial \langle H \rangle}{\partial \langle \alpha \rangle},
$$

$$i\hbar \frac{d\rho}{dt} = -2 \left[ \rho, \frac{\partial \langle H \rangle}{\partial \rho} \right],
$$

in which $\langle H \rangle$ is the energy. Some interesting properties are discussed in ([12, 10]).

In order to make connection with the BEC phenomenon, we introduce first the Hamiltonian for trapped bosons[4]:

$$
H = \int_r a^+(r) \left[ -\frac{\hbar^2}{2m} \Delta + V_{\text{ext}}(r) - \mu \right] a(r) + \frac{g}{2} \int_r a^+(r)a(r)a(r)a(r),
$$

where $V_{\text{ext}}(r)$ is the trapping potential, $\mu$ is the chemical potential and $g$ is the coupling constant. The energy $\mathcal{E} = \langle H \rangle$ is easily computed yielding:

$$
\mathcal{E} = \int_r \left[ -\frac{\hbar^2}{2m} \Phi^* \Delta \Phi - \frac{\hbar^2}{2m} \Phi \Delta \Phi^* + (V_{\text{ext}} - \mu + 2g\bar{n})|\Phi|^2 + \frac{g}{2} |\Phi|^4 - \frac{\hbar^2}{2m} \bar{n} \Delta \right.
$$

$$
+ (V_{\text{ext}} - \mu)\bar{n} + g\bar{n}^2 + \frac{g}{2} (|\bar{m}|^2 + \bar{m}^*\Phi^2 + \bar{m} \Phi^*) \right],
$$

where the condensate density $n_c = |\Phi|^2$, the non-condensed density $\bar{n}$ and the anomalous density $\bar{m}$ are identified respectively with $|\langle a \rangle|^2$, $\langle a^+\bar{a} \rangle$ and $\langle \bar{a}\bar{a} \rangle$. This is simply because we identify in our formalism the boson field operator $\hat{\Psi}$ with the destruction operator $a$ and its fluctuation $\hat{\Psi}$ with $\bar{a}$.

With these identifications, the Eqs.(2.3) now take the form

$$
i\hbar \dot{\Phi} = \frac{\partial \mathcal{E}}{\partial \Phi^*},
$$

$$
i\hbar \dot{\bar{n}} = 2 \left( \bar{m}^* \frac{\partial \mathcal{E}}{\partial \bar{m}} - \bar{m} \frac{\partial \mathcal{E}}{\partial \bar{m}} \right),
$$

$$
i\hbar \dot{\bar{m}} = 4(\bar{n} + \frac{1}{2}) \frac{\partial \mathcal{E}}{\partial \bar{m}} + 4 \frac{\partial \mathcal{E}}{\partial \bar{m}} \bar{m},
$$

which constitutes a closed self-consistent system. The coupling between the order parameter $\Phi$, the non-condensed density and the anomalous density occurs via the derivatives of $\mathcal{E}$ which still contain $\bar{n}$ and $\bar{m}$.
Beside the conservation of the energy, the equations (2.6) exhibit the unitary evolution of the density matrix (already visible in (2.3)) by means of the conservation of the "Heisenberg parameter" $I$ defined as $\rho(\rho + 1) = \frac{I - 1}{4}$, or equivalently

$$I = (2\tilde{n} + 1)^2 - 4|\tilde{m}|^2.$$  \hfill (2.7)

We recall the reader that $I = \coth^2(\hbar \omega / kT)$ for a thermal distribution.

The expression (2.5) for the energy allows us to write down the Eqs.(2.6) more explicitly. They indeed read

$$i\hbar \dot{\Phi} = \left(-\frac{\hbar^2}{2m}\Delta + V_{\text{ext}} - \mu + gn_c + 2g\tilde{n}\right) \Phi + g\tilde{m}\Phi^*,$$

$$i\hbar \dot{\tilde{n}} = g\left(\tilde{m}^*\Phi^2 - \tilde{m}\Phi^*\right),$$

$$i\hbar \dot{\tilde{m}} = g(2\tilde{n} + 1)\Phi^2 + 4\left(-\frac{\hbar^2}{2m}\Delta + V_{\text{ext}} - \mu + 2gn + \frac{g}{4}(2\tilde{n} + 1)\right)\tilde{m}.$$  \hfill (2.8)

The consistency of our derivation mentioned in the introduction is now clear. Indeed, we obtain in Eqs.(2.8) a self-consistent dynamics of the order parameter, the thermal cloud and the anomalous density. The equation governing the evolution of $\Phi$ has been obtained elsewhere [4, 15, 16, 17, 18] as an extension of the Gross-Pitaevskii equation, but to our knowledge, the two last equations in (2.8), governing the evolution of $\tilde{n}$ and $\tilde{m}$, were never written down before at finite temperature. The exception is ref.[14], where a zero temperature (that is $I = 1$) and uniform ($V_{\text{ext}} = 0$) version was derived. Indeed, taking these limits in (2.7) and (2.8) provides the equations obtained in ref.[14]. Therefore, our equations are more general. They describe not only a self-consistent dynamics of the non-condensed and anomalous densities but also a feedback effect on the order parameter and therefore on the condensate density. The coupling is however intimately related to the two-body interactions and completely disappears for noninteracting systems, justifying therefore the Popov and the Beliaev approximations for weakly interacting atomic gases.

It is worth noticing that this dynamics is also number conserving since the total density $n = n_c + \tilde{n}$ is preserved during the evolution.

As a final remark, we may note that the information on the temperature is encoded in the equation (2.7) which is a property of the density operator (2.1). Indeed, if $T$ is specified, then the Heisenberg parameter $I$ (which we recall is a conserved quantity) is calculable and this in turn allows the computation of $\tilde{n}$ (respectively $\tilde{m}$) once $\tilde{m}$ (respectively $\tilde{n}$) is known. The last two equations in (2.8) are therefore not totally independent.

### 3 The Static Solutions

The static properties of the BEC at $T = 0$ are well known[4]. Indeed, at zero temperature, all the atoms are condensed. Therefore, $\tilde{n}0 = 0$ and $\tilde{m}0 = 0$ and $n_c0$ equals
the total density of the gas \( n \). The last two equations in (2.8) become meaningless, and the first one gives:

\[
\left( -\frac{\hbar^2}{2m} \Delta + V_{\text{ext}} - \mu + gn_0 \right) \Phi_0 = 0.
\]

This provides the density profile in the Thomas-Fermi (TF) approximation[19]:

\[
n_0(r) = \frac{1}{g} (\mu - V_{\text{ext}}(r)).
\]

Upon defining the "size" of the fundamental state \( a_{H_0} = (\hbar/m\omega_0)^{1/2} \) and the s-wave scattering length \( a = mg/4\pi\hbar^2 \), we obtain for a spherical potential \( V_{\text{ext}}(r) = \frac{1}{2}m\omega_0^2 r^2 \), the condensate radius \( R_{TF} \) and the chemical potential \( \mu \) for a gas of \( N_0 \) bosons as

\[
R_{TF} = a_{H_0} \left( 15N_0 \frac{a}{a_{H_0}} \right)^{1/5},
\]

\[
\mu = \frac{1}{2} \hbar \omega_0 \left( 15N_0 \frac{a}{a_{H_0}} \right)^{2/5},
\]

which are well-known expressions in the literature[4, 7].

At \( T > T_{\text{BEC}} \), there is no condensate so that \( n_0 = 0 \) and \( \tilde{n}0 = n \). The static TDHFB equations reduce to a single equation which becomes in the TF approximation

\[
(V_{\text{ext}}(r) - \mu + 10gn + g) \tilde{n}0 = 0,
\]

and leads to either a uniform density profile (for \( \tilde{n}0 = 0 \)), which seems unreasonable in the presence of a confining field, or to a non-uniform density of the form

\[
n = \frac{1}{10} \left( \frac{\mu - V_{\text{ext}}(r)}{g} - 1 \right),
\]

giving rise to the anomalous density

\[
|\tilde{n}0|^2 = \left( n - \frac{\sqrt{T} - 1}{2} \right) \left( n + \frac{\sqrt{T} + 1}{2} \right).
\]

The maximum spreading of the trapped gas \( R \) as well as the chemical potential \( \mu \) can be computed from (3.5). We obtain the results

\[
R = a_{H_0} \left( 15N_0 \frac{a}{a_{H_0}} \right)^{1/5},
\]

\[
\mu = g + \frac{1}{2} \hbar \omega_0 \left( 15N_0 \frac{a}{a_{H_0}} \right)^{2/5},
\]

where we can notice the similarities with the totally condensed phase. In the two cases, the spreading of the condensate (or of the global system for \( T > T_{\text{BEC}} \)) depends essentially on the balance between the self-interactions and the confining potential. Furthermore, we note on (3.6) that at a maximum distance \( R_{\text{max}} \) from the center of the trap

\[
R_{\text{max}} = R \sqrt{1 - \frac{10g}{m\omega_0^2R^2}(\sqrt{T} - 1)},
\]
the anomalous density vanishes.

When $0 < T < T_{\text{BEC}}$, we have of course $n_c0 \neq 0$ and $\tilde{n}0 \neq 0$. In the TF approximation, we see that the value $\tilde{m}0 = 0$ can no longer be retained. Therefore, the anomalous density, although (maybe) small, is an essential ingredient in the resolution of the static equations.

It is important to notice at this stage that the TF approximation is a somewhat hazardous hypothesis for the thermal cloud[20]. Indeed, the traditional image of a condensate surrounded by a smooth thermal cloud is a rather simplified picture. Fortunately, we see on Eqs.(2.8) that $\tilde{n}$ can be eliminated in favor of the "relevant" variables $n_c$ and $\tilde{m}$. We will henceforth mean by TF approximation the neglect of the kinetic terms in the equations of $n_c$ and $\tilde{m}$.

Let us set $\xi = \frac{1}{g}(V_{\text{ext}}(r) - \mu)$ and introduce the parametrization $\tilde{m}0 = |\tilde{m}0| \exp(i\alpha)$ and $\Phi0 = \sqrt{n_c0} \exp(i\phi)$. We then obtain the implicit solutions:

$$
|\tilde{m}0| = \frac{1}{4} \left[ \frac{I}{1 - 2q} - (1 - 2q) \right],
$$

$$
\tilde{n}0 = -\frac{1}{2} + \frac{1}{4} \left[ \frac{I}{1 - 2q} + (1 - 2q) \right],
$$

$$
n_c0 = 1 - \xi - \frac{1}{4} \left[ \frac{I}{1 - 2q} + 3(1 - 2q) \right],
$$

(3.9)

with $q = \xi + n$. This is obtained by setting $\alpha = 2\phi + \pi$ which is compatible with the Hugenholtz-Pines theorem [21] expressed in our context by the identity $|\tilde{m}0| = \xi + n + \tilde{n}0$.

The equations (3.9), together with the third static equation in (2.8) may be combined to yield a quartic equation for $q$ alone which can then be solved numerically to provide temperature and position-dependent density profiles. An important preliminary result is that the anomalous density is always very small compared to $n_c0$ or $\tilde{n}0$ whatever the conditions are, therefore, justifying a posteriori, the TF approximation used above. We shall discuss this and other numerical results in a separate work[22].

Nevertheless, one can gain further insights into the static properties at $0 < T < T_{\text{BEC}}$ by choosing the parametrization

$$
1 + 2\tilde{n}0 = \sqrt{T} \cosh \sigma
$$

$$
2|\tilde{m}0| = \sqrt{T} \sinh \sigma,
$$

(3.10)

which automatically satisfies (2.7). Indeed, since we know that the $T = 0$ case is given by $\sigma = 0$, we may approximately solve the static problem to first order in $\sigma$. This is equivalent to a low temperature expansion (but far from the transition). After some
algebra, we obtain

\[ n_{c0} = -\xi - (\sqrt{T} - 1) + \sqrt{T} \eta \]

\[ \tilde{n}0 = \frac{\sqrt{T} - 1}{2} + \sqrt{T} \eta^2 \]

\[ |\tilde{m}0| = -\sqrt{T} \eta (1 + 2\eta) \]

(3.11)

\( \eta \) being a small expansion parameter. What we observe on (3.11) is that it is a natural extension to finite temperature of an expression like (3.2). To lowest order (\( \eta = 0 \)), the result is a temperature-dependent shift of the condensate density with respect to the \( T = 0 \) case.

Let us see what happens to the condensate radius and the chemical potential which were given by (3.3). We define the condensate radius in the TF approximation by the point where \( n_{c0} \) vanishes. This gives

\[ V_{\text{ext}}(R_{\text{TF}}) = \mu - (\sqrt{T} - 1)g. \]

(3.12)

The number of condensed atoms \( N_c \), which is a measurable quantity, writes

\[ N_c = 4\pi \int_0^{R_{\text{TF}}} n_{c0}(r)r^2dr. \]

(3.13)

After integrating and using (3.12), we obtain the following remarkable expressions:

\[ R_{\text{TF}} = a_{H0} \left( 15N_c \frac{a}{a_{H0}} \right)^{1/5}, \]

\[ \mu = g(\sqrt{T} - 1) + \frac{1}{2} \hbar \omega_0 \left( 15N_c \frac{a}{a_{H0}} \right)^{2/5}. \]

(3.14)

The condensate radius is thus given by the same formula as its zero temperature counterpart, the sole difference lying in the appearance of \( N_c \) instead of the total number \( N0 \). For the chemical potential, we observe, as for \( n_{c0} \), a temperature-dependent shift with respect to the \( T = 0 \) case, but here also, it is the number of condensed atoms which is involved and not the total number of atoms.

### 4 Excitations of the Condensate

The small excitations of the condensate (collective modes) are well studied with the RPA technique. Indeed, one may first set

\[ \Phi = \Phi_0 + \delta\Phi \]
\[ \tilde{n} = \tilde{n}0 + \delta\tilde{n} \]
\[ \tilde{m} = \tilde{m}0 + \delta\tilde{m}, \]

(4.1)

\( \Phi_0, \tilde{n}0 \) and \( \tilde{m}0 \) being the static solutions satisfying (3.1), (3.4) or (3.9) according to the phase the system is in and \( \delta\Phi, \delta\tilde{n} \) and \( \delta\tilde{m} \) are small deviations from local equilibrium.
Then, one may expand the dynamical equations (2.8) dropping terms up to second order, but keeping the kinetic terms since they are crucial for the excitation spectrum.

At zero temperature, we recall that the sole meaningful equation is the first one in the eqs.(2.8). Its expansion around the static solution (3.1) leads to the following equation:

\[
\frac{i}{\hbar} \delta \dot{\Phi} = \left( -\frac{\hbar^2}{2gm} \Delta + \xi + 2n_c0 \right) \delta \Phi + \Phi0^2 \delta \Phi^*. \tag{4.2}
\]

Upon setting \( \epsilon_k = \hbar^2 k^2 / 2m \), we can solve for the modes \( \delta \Phi(\mathbf{r}, t) = e^{-i\omega t} u(\mathbf{r}) + e^{i\omega t} v(\mathbf{r}) \) to obtain the dispersion relation for a uniform gas

\[
\hbar \omega_k = \sqrt{\epsilon_k (\epsilon_k + 2g n_c0)}, \tag{4.3}
\]

which is the Bogoliubov spectrum at \( T = 0 \) [7]. For instance, at small momenta, one obtains the phonon spectrum \( \omega_k = \sqrt{\frac{\epsilon_n0}{m} k} \).

For finite temperatures and above the transition, we linearize the TDHFB eqs.(2.8) without using the TF approximation in order to keep the kinetic term \( \epsilon'_k = \epsilon_k / g \). The procedure provides a \((4 \times 4)\) RPA system of the form: \( i\hbar \dot{V} = MV \), where \( V \) is the column vector \((\delta \Phi, \delta \Phi^*, \delta \bar{m}, \delta \bar{m}^*)\) and the RPA matrix is given by:

\[
M = \begin{pmatrix}
M_1 & M_2 & M_3^* & 0 \\
-M_2^* & -M_1 & 0 & -M_3 \\
M_4 & -M_5 & M_6 & 0 \\
M_5^* & -M_4^* & 0 & -M_6
\end{pmatrix}, \tag{4.4}
\]

with

\[
\begin{align*}
M_1 &= \epsilon'_k + \xi + 2\bar{n}0 \\
M_2 &= \bar{m}0 - \Phi0^2 \\
M_3 &= \Phi0 \\
M_4 &= 2\Phi0(1 + 2\bar{n}0 - n_c0) - 2\bar{m}0\Phi0^* \\
M_5 &= 2\Phi0(\bar{m}0 + \Phi0^2) \\
M_6 &= 4(\epsilon'_k + \xi + 2n) + 1 + 2\bar{n}0.
\end{align*} \tag{4.5}
\]

After some algebra, we obtain the expressions for the eigenfrequencies:

\[
\omega_{\pm} = \frac{g}{\hbar} \left\{ \frac{1}{2} \left( B \pm \sqrt{B^2 - 4E} \right) \right\}^{1/2}, \tag{4.6}
\]

where

\[
B = 4n_c0(1 + 2\bar{n}0) + [4(M_1 + 2n_c0) + 1 + 2n][4(M_1 + n_c0) + 1 + 2n],
\]

and

\[
E = -8n_c0[1 + 2\bar{n}0 + 3M_1 + 4n_c0][M_1(1 + 2\bar{n}0) + 4(M_1 + n_c0)(M_1 + 2n_c0)].
\]
The static solutions are computed from (3.9) with $\xi$ replaced by $\epsilon'_k + \xi$.

The spectrum (4.6) clearly exhibits a departure from the Bogoliubov spectrum (4.3). It is indeed temperature and position-dependent. The two modes $\omega_{\pm}$ reduce respectively to $\epsilon_k + V_{\text{ext}} - \mu$ and $4(\epsilon_k + V_{\text{ext}} - \mu)$ in the noninteracting case ($g = 0$). Therefore, one may say in the interacting case that $\omega_-$ and $\omega_+$ describe the coupled oscillations of the condensate fraction and the anomalous density. What remains is a comparison between these modes and existing data. But this requires a detailed knowledge of the static solution as shown by (4.6). Indeed, the temperature dependence of the static solution complicates the expressions for the eigenfrequencies and one therefore cannot use the simplifications that appear in the $T = 0$ case, as was performed in [6, 14].

The detailed study of the static solution as well as the eigenmodes is in progress and we will postpone the results to a future paper[22].

5 Conclusions and Perspectives

We have been concerned in this paper with a dynamical variational generalization of the Gross-Pitaevskii equation, which takes into account the coupling of the condensate with the thermal cloud and with the anomalous density. We show that our derivation is consistent with all known approximations which go beyond the Gross-Pitaevskii approximation, namely, the Popov, the Beliaev and the HFB approximations.

The equations that we obtain are fully self-consistent, mainly because they not only introduce a dynamics of the thermal cloud and the anomalous density, but they allow also for a consistent feedback effect of these densities on the condensate fraction.

Instead of solving the full dynamical equations, we choose to focus first on the static situation, where theoretical works as well as experimental data do exist.

At zero temperature, we obtain familiar expressions for the chemical potential and the condensate radius. But for finite temperatures and above the transition, the situation is much more complicated and requires a numerical study.

The preliminary results show a good qualitative agreement with what is known; in particular, the anomalous density is always extremely small which is compatible with the Thomas-Fermi approximation.

We then turn to the small amplitude motion and derive RPA-like equations which provide two coupled modes of oscillations; the well-known breathing modes of the condensate. A direct comparison with the results of [6, 14] is unfortunately a little bit delicate since both the $T = 0$ and the $V_{\text{ext}} = 0$ cases were considered there. We have shown in
particular that, owing to a temperature dependence from the beginning, makes the $T=0$ limit a rather subtle question.

This paper is dedicated to the memory of Dominique Vautherin (Arthur), an active member of the Division de Physique Théorique, Institut de Physique Nucléaire (IPN), Orsay-France.

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