On convergence of the distributions of random sequences with independent random indexes to variance–mean mixtures

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ABSTRACT
We prove a transfer theorem for random sequences with independent random indexes in the double array limit setting under relaxed conditions. We also prove its partial inverse providing the necessary and sufficient conditions for the convergence of randomly indexed random sequences. Special attention is paid to the case where the elements of the basic double array are formed as cumulative sums of independent not necessarily identically distributed random variables. Using simple moment-type conditions we prove the theorem on convergence of the distributions of such sums to normal variance–mean mixtures.

1. Introduction

Among all scale–location mixtures of normal laws, normal variance–mean mixtures specified by O. Barndorff-Nielsen and his colleagues[3] occupy a special position. A distribution function $F(x)$ is called a normal variance–mean mixture, if it has the form

$$F(x) = \int_0^\infty \Phi\left(\frac{x - \beta - \alpha z}{\sigma \sqrt{z}}\right) dG(z), \quad x \in \mathbb{R},$$

with some $\beta \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $\sigma \in (0, \infty)$, where $\Phi(x)$ is the standard normal distribution function, $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-z^2/2} dz$, $x \in \mathbb{R}$, and $G(z)$ is a distribution function such that $G(0) = 0$. The class of normal variance–mean mixtures is very wide and contains, say, generalized hyperbolic laws[1–3] and generalized variance gamma distributions[28] which proved to provide excellent fit to statistical data in various fields from atmospheric turbulence to financial markets, see, e.g., Refs.[4,7,18,19]. In normal variance–mean mixtures, mixing is performed with respect to both location and scale parameters. But since these parameters are tightly linked so that the location parameters (means) are proportional to the variances of the mixed laws, actually the mixing distribution is univariate. However, for a long time, these models did not have asymptotic grounds like limit theorems proved within simple asymptotic
settings (say, for sums or extrema of independent random variables) where normal variance–mean mixtures could appear as limit distributions. For example, necessary and sufficient conditions for the convergence of the distributions of random sums to normal variance–mean mixtures were found only recently for the case of identically distributed summands (see Refs. [9,16]).

At the same time, random sequences with independent random indexes play an important role in modeling real processes in many fields. Most popular examples of the application of these models usually deal with insurance and reliability theory [5,10], financial mathematics and queuing theory [5,8], and chaotic processes in plasma physics [14] where random sums are the principal mathematical models. More general randomly indexed random sequences arrive in the statistics of samples with random sizes. Indeed, very often the data to be analyzed are collected or registered during a certain period of time and the flow of informative events producing the observations forms a random point process, so that the number of available observations is unknown till the end of the process of their registration and also must be treated as a (random) observation.

The randomness of indexes usually leads to the fact that the limit distributions for the corresponding randomly indexed random sequences are heavy-tailed even in the situations where the distributions of nonrandomly indexed random sequences are asymptotically normal see, e.g., Refs. [6,8].

The literature on random sequences with random indexes is extensive, see, e.g., the references above and the references therein. The mathematical theory of random sequences with random indexes and, in particular, random sums, is well-developed. However, there still remain some unsolved problems. The criteria of convergence of the distributions of random sums of nonidentically distributed random summands and, moreover, more general statistics constructed from samples with random sizes have not been obtained yet. The aim of the present paper is to fill this gap.

The paper is organized as follows. Basic notation is introduced in Section 2. Here, an auxiliary result on the asymptotic rapprochement of the distributions of randomly indexed random sequences with special scale–location mixtures is proved. In Section 3, a transfer theorem is proved for random sequences with independent random indexes in the double array limit setting under relaxed conditions. Here, we also prove its partial inverse providing the necessary and sufficient conditions for the convergence of randomly indexed random sequences. Special attention is paid to the case where the elements of the basic double array are formed as cumulative sums of independent not necessarily identically distributed random variables. This case is considered in Section 4. To prove our results, we use simply tractable moment-type conditions which can be easily interpreted unlike general conditions providing the weak convergence of random sums of nonidentically distributed summands in Refs. [21,26] and Ref. [22]. In Section 5, we prove the theorem on convergence of the distributions of such sums to normal variance–mean mixtures. As a simple corollary of this result we can obtain some results of the recent paper [27]. That paper demonstrates that there is still a strong interest in geometric sums of nonidentically
distributed summands and to the application of the skew Laplace distribution which is a normal variance–mean mixture under exponential mixing distribution\cite{15}.

2. Notation: Auxiliary results

Assume that all the random variables considered in this paper are defined on one and the same probability space \((\Omega, \mathcal{F}, P)\). In what follows, the symbols \(\overset{d}{=}\) and \(\Longrightarrow\) will denote coincidence of distributions and weak convergence (convergence in distribution). A family \(\{X_j\}_{j\in\mathbb{N}}\) of random variables is said to be weakly relatively compact, if each sequence of its elements contains a weakly convergent subsequence. In the finite-dimensional case, the weak relative compactness of a family \(\{X_j\}_{j\in\mathbb{N}}\) is equivalent to its tightness:

\[
\lim_{R \to \infty} \sup_{n \in \mathbb{N}} P(|X_n| > R) = 0
\]
(see, e.g., Ref.\cite{23}).

Let \(\{S_n\}, n, k \in \mathbb{N}\), be a double array of random variables. For \(n, k \in \mathbb{N}\), let \(a_{n,k}\) and \(b_{n,k}\) be real numbers such that \(b_{n,k} > 0\). The purpose of the constants \(a_{n,k}\) and \(b_{n,k}\) is to provide weak relative compactness of the family of the random variables

\[
\left\{ Y_{n,k} \equiv \frac{S_{n,k} - a_{n,k}}{b_{n,k}} \right\}_{n,k \in \mathbb{N}}
\]

in the cases where it is required.

Consider a family \(\{N_n\}_{n\in\mathbb{N}}\) of nonnegative integer-valued random variables such that for each \(n, k \in \mathbb{N}\) the random variables \(N_n\) and \(S_{n,k}\) are independent. Especially note that we do not assume the row-wise independence of \(\{S_{n,k}\}_{k \geq 1}\). Let \(c_n\) and \(d_n\) be real numbers, \(n \in \mathbb{N}\), such that \(d_n > 0\). Our aim is to study the asymptotic behavior of the random variables

\[
Z_n \equiv \frac{S_{n,N_n} - c_n}{d_n}
\]
as \(n \to \infty\) and find rather simple conditions under which the limit laws for \(Z_n\) have the form of normal variance–mean mixtures. In order to do so, we first formulate a somewhat more general result following the lines of Ref.\cite{11}, removing superfluous assumptions, relaxing the conditions, and generalizing some of the results of that paper.

The characteristic functions of the random variables \(Y_{n,k}\) and \(Z_n\) will be denoted by \(h_{n,k}(t)\) and \(f_n(t)\), respectively, \(t \in \mathbb{R}\).

Let \(Y\) be a random variable whose distribution function and characteristic function will be denoted by \(H(x)\) and \(h(t)\), respectively, \(x, t \in \mathbb{R}\). For \(n \in \mathbb{N}\), introduce the random variables

\[
U_n = \frac{b_{n,N_n}}{d_n}, \quad V_n = \frac{a_{n,N_n} - c_n}{d_n}
\]
and the function
\[ g_n(t) \equiv E h(t U_n) \exp \{ it V_n \} = \sum_{k=1}^{\infty} P(N_n = k) \exp \left\{ it \frac{a_{n,k} - c_n}{d_n} \right\} h \left( \frac{t b_{n,k}}{d_n} \right), \quad t \in \mathbb{R}. \]

It can be easily seen that \( g_n(t) \) is the characteristic function of the random variable \( Y \cdot U_n + V_n \) where the random variable \( Y \) is independent of the pair \((U_n, V_n)\). Therefore, the distribution function \( G_n(x) \) corresponding to the characteristic function \( g_n(t) \) is the scale–location mixture of the distribution function \( H(x) \):
\[
G_n(x) = E H \left( \frac{x - V_n}{U_n} \right), \quad x \in \mathbb{R}, \ n \in \mathbb{N}. \tag{1}
\]

In the double-array limit setting considered in this paper, to obtain nontrivial limit laws for \( Z_n \) we require the following additional coherency condition: for any \( T \in (0, \infty) \)
\[
\lim_{n \to \infty} E \sup_{|t| \leq T} \left| h_{n,N_n}(t) - h(t) \right| = 0. \tag{2}
\]

To clarify the sense of the coherency condition, note that if we had usual row-wise convergence of \( Y_{n,k} \) to \( Y \), then for any \( n \in \mathbb{N} \) and \( T \in [0, \infty) \)
\[
\lim_{k \to \infty} \sup_{|t| \leq T} |h_{n,k}(t) - h(t)| = 0. \tag{3}
\]

So we can say that coherency condition (2) means that “pure” row-wise convergence (3) takes place “on the average” so that the “row-wise convergence as \( k \to \infty \) is somehow coherent with the “principal convergence as \( n \to \infty \). It should be noted that in Ref.\cite{11} the coherency condition was used in a formally stronger and less tractable form.

In Section 5, we will deal with the case where the random variables \( S_{n,k} \) have the additive structure and are formed as sums of independent random variables. It turns out that under these additional “structural” assumptions the role of coherency condition is played by the “random Lindeberg condition.” Furthermore, if we deal not with a double array of random variables but simply with a sequence of random variables, then, as was shown in Refs.\cite{12,13}, the coherency condition turns out to be equivalent to the following set of conditions: the indexes \( N_n \) infinitely increase in probability, that is \( P(N_n \leq m) \to 0 \) as \( n \to \infty \) for any \( m \geq 0 \), and either \( d_n \to 0 \) or \( d_n \to \infty \) as \( n \to \infty \).

Remark 2.1. It can be easily verified that, since the values under the expectation sign in (2) are nonnegative and bounded (by two), hence the coherency condition (2) is equivalent to that of
\[
\sup_{|t| \leq T} \left| h_{n,N_n}(t) - h(t) \right| \longrightarrow 0 \quad \text{in probability as } n \to \infty \text{ for any } T \in (0, \infty).
\]
Lemma 2.1. Let the family of random variables \( \{U_n\}_{n \in \mathbb{N}} \) be weakly relatively compact. Assume that coherency condition (2) holds. Then, for any \( t \in \mathbb{R} \) we have

\[
\lim_{n \to \infty} |f_n(t) - g_n(t)| = 0. \tag{4}
\]

Proof. Let \( \gamma \in (0, \infty) \) be a real number to be specified later. For \( n \in \mathbb{N} \), denote

\[
K_{1,n} \equiv K_{1,n}(\gamma) = \{ k : b_{n,k} \leq \gamma d_n \}, \quad K_{2,n} \equiv K_{2,n}(\gamma) = \{ k : b_{n,k} > \gamma d_n \}.
\]

If \( t = 0 \), then the assertion of the lemma is trivial. Fix an arbitrary \( t \neq 0 \). By the formula of total probability, we have

\[
|f_n(t) - g_n(t)| = \left| \sum_{k=1}^{\infty} \mathbb{P}(N_n = k) \exp \left\{ it \frac{a_{n,k} - c_n}{d_n} \right\} \left[ h_{n,k} \left( \frac{tb_{n,k}}{d_n} \right) - h \left( \frac{tb_{n,k}}{d_n} \right) \right] \right|
\]

\leq \sum_{k \in K_{1,n}} \mathbb{P}(N_n = k) \left| h_{n,k} \left( \frac{tb_{n,k}}{d_n} \right) - h \left( \frac{tb_{n,k}}{d_n} \right) \right|

+ \sum_{k \in K_{2,n}} \mathbb{P}(N_n = k) \left| h_{n,k} \left( \frac{tb_{n,k}}{d_n} \right) - h \left( \frac{tb_{n,k}}{d_n} \right) \right| \equiv I_1 + I_2. \tag{5}

Choose an arbitrary \( \epsilon > 0 \).

First consider \( I_2 \). We obviously have

\[
I_2 \leq \sum_{k \in K_{2,n}(\gamma)} \mathbb{P}(N_n = k) = \mathbb{P}(U_n > \gamma). \tag{9}
\]

The weak relative compactness of the family \( \{U_n\}_{n \in \mathbb{N}} \) implies the existence of a \( \gamma_1 = \gamma_1(\epsilon) \) such that

\[
\sup_n \mathbb{P}(U_n > \gamma_1) < \epsilon.
\]

Therefore, setting \( \gamma = \gamma_1 \) from (9), we obtain

\[
I_2 < \epsilon. \tag{10}
\]

Now consider \( I_1 \) with \( \gamma \) chosen above. If \( k \in K_{1,n}(\gamma) \), then \( |tb_{n,k}/d_n| \leq \gamma |t| \) and we have

\[
I_1 \leq \sum_{k \in K_{1,n}(\delta, \gamma)} \mathbb{P}(N_n = k) \sup_{|\tau| \leq \gamma |t|} |h_{n,k}(\tau) - h(\tau)| \leq \mathbb{E} \sup_{|\tau| \leq \gamma |t|} |h_{n,N_n}(\tau) - h(\tau)|.
\]

Therefore, coherency condition (2) implies that there exists a number \( n_0 = n_0(\epsilon, \gamma) \) such that for all \( n \geq n_0 \)

\[
I_1 < \epsilon. \tag{11}
\]

Unifying (5), (10), and (11) we obtain that \( |f_n(t) - g_n(t)| < 2\epsilon \) for \( n \geq n_0 \). The arbitrariness of \( \epsilon \) proves (4). The lemma is proved. \( \square \)

Lemma 2.1 makes it possible to use the distribution function \( G_n(x) \) (see (1)) as an accompanying asymptotic approximation to \( F_n(x) \equiv \mathbb{P}(Z_n < x) \). In order to
obtain a limit approximation, in the next section we formulate and prove the transfer theorem.

3. General transfer theorem and its inversion: The structure of limit laws

**Theorem 3.1.** Assume that coherency condition (2) holds. If there exist random variables \( U \) and \( V \) such that the joint distributions of the pairs \((U_n, V_n)\) converge to that of the pair \((U, V)\):

\[
(U_n, V_n) \xrightarrow{d} (U, V) \quad (n \to \infty),
\]

then

\[
Z_n \xrightarrow{d} Z = Y \cdot U + V \quad (n \to \infty),
\]

where the random variable \( Y \) is independent of the pair \((U, V)\).

**Proof.** Treating \( t \in \mathbb{R} \) as a fixed parameter, represent the function \( g_n(t) \) as

\[
g_n(t) = E_h(tU_n)e^{itV_n} \equiv E\varphi_t(U_n, V_n).
\]

Since for each \( t \in \mathbb{R} \) the function \( \varphi_t(x, y) \equiv h(tx)e^{ity}, x, y \in \mathbb{R} \), is bounded and continuous in \( x \) and \( y \), then by the definition of the weak convergence we have

\[
\lim_{n \to \infty} E\varphi_t(U_n, V_n) = E\varphi_t(U, V).
\]

Using the Fubini theorem it can be easily verified that the function on the right-hand side of (14) is the characteristic function of the random variable \( Y \cdot U + V \) where the random variable \( Y \) is independent of the pair \((U, V)\). Now the statement of the theorem follows from Lemma 2.1 by the triangle inequality. The theorem is proved.

It is easy to see that relation (13) is equivalent to the following relation between the distribution functions \( F(x) \) and \( H(x) \) of the random variables \( Z \) and \( Y \):

\[
F(x) = E H\left(\frac{x - V}{U}\right), \quad x \in \mathbb{R},
\]

that is, the limit law for normalized randomly indexed random variables \( Z_n \) is a scale–location mixture of the distributions which are limiting for normalized nonrandomly indexed random variables \( Y_n \). Among all scale–location mixtures, variance–mean mixtures attract a special interest (to be more precise, we should speak of normal variance–mean mixtures). Let us see how these mixtures can appear in the double-array setting under consideration.

Assume that the centering constants \( a_{n,k} \) and \( c_n \) are in some sense proportional to the scaling constants \( b_{n,k} \) and \( d_n \). Namely, assume that there exist \( \rho > 0, \alpha_n \in \mathbb{R} \), and \( \beta_n \in \mathbb{R} \) such that for all \( n, k \in \mathbb{N} \) we have

\[
a_{n,k} = \frac{b_{n,k}^{\rho + 1} \alpha_n}{d_n^\rho}, \quad c_n = d_n \beta_n,
\]

(16)
and there exist finite limits
\[ \alpha = \lim_{n \to \infty} \alpha_n, \quad \beta = \lim_{n \to \infty} \beta_n. \]

Then, under condition \((12)\)
\[ (U_n, V_n) = \left( \frac{b_{n,N_n}}{d_n}, \frac{a_{n,N_n} - c_n}{d_n} \right) = (U_n, a_n U^{\rho+1} + \beta) \quad (n \to \infty), \]
so that in accordance with Theorem 3.2 the limit law for \(Z_n\) takes the form
\[ P(Z < x) = eH\left( \frac{x - \beta - \alpha U^{\rho+1}}{U} \right), \quad x \in \mathbb{R}. \]

If \(\rho = 1\), then we obtain the “pure” variance–mean mixture
\[ P(Z < x) = eH\left( \frac{x - \beta - \alpha U^2}{U} \right), \quad x \in \mathbb{R}. \]

We will return to the discussion of convergence of randomly indexed sequences, more precisely, of random sums, to normal scale–location mixtures in Section 5.

In order to prove the result that is a partial inversion of Theorem 3.1, for fixed random variables \(Z\) and \(Y\) with the characteristic functions \(f(t)\) and \(h(t)\) introduce the set \(\mathcal{W}(Z|Y)\) containing all pairs of random variables \((U, V)\) such that the characteristic function \(f(t)\) can be represented as
\[ f(t) = Eh(tU)e^{\beta V}, \quad t \in \mathbb{R}, \quad (17) \]
and \(P(U \geq 0) = 1\). Whatever random variables \(Z\) and \(Y\) are, the set \(\mathcal{W}(Z|Y)\) is always nonempty since it trivially contains the pair \((0, Z)\). It is easy to see that representation \((17)\) is equivalent to relation \((15)\) between the distribution functions \(F(x)\) and \(H(x)\) of the random variables \(Z\) and \(Y\).

The set \(\mathcal{W}(Z|Y)\) may contain more that one element. For example, if \(Y\) is the standard normal random variable and \(Z \overset{d}{=} W_1 - W_2\) where \(W_1\) and \(W_2\) are independent random variables with the same standard exponential distribution, then along with the pair \((0, W_1 - W_2)\) the set \(\mathcal{W}(Z|Y)\) contains the pair \((\sqrt{W_1}, 0)\). In this case, \(F(x)\) is the symmetric Laplace distribution.

Let \(L_1(X_1, X_2)\) be the Lévy distance between the distributions of random variables \(X_1\) and \(X_2\): if \(F_1(x)\) and \(F_2(x)\) are the distribution functions of \(X_1\) and \(X_2\), respectively, then
\[ L_1(X_1, X_2) = \inf\{y \geq 0 : F_2(x - y) - y \leq F_1(x) \leq F_2(x + y) + y \text{ for all } x \in \mathbb{R} \}. \]

As is well known, the Lévy distance metrizes weak convergence. Let \(L_2((X_1, X_2), (Y_1, Y_2))\) be any probability metric which metrizes weak convergence in the space of two-dimensional random vectors. An example of such a metric is the Lévy–Prokhorov metric (see, e.g., Ref. [29]).
Theorem 3.2. Let the family of random variables $\{U_n\}_{n \in \mathbb{N}}$ be weakly relatively compact. Assume that coherency condition (2) holds. Then, a random variable $Z$ such that

$$Z_n \Rightarrow Z \quad (n \to \infty)$$

(18)

with some $c_n \in \mathbb{R}$ exists if and only if there exists a weakly relatively compact sequence of pairs $(U'_n, V'_n) \in \mathcal{W}(Z|Y)$, $n \in \mathbb{N}$, such that

$$\lim_{n \to \infty} L_2 ((U_n, V_n), (U'_n, V'_n)) = 0.$$  

(19)

Proof. “Only if” part. Prove that the sequence $\{V_n\}_{n \in \mathbb{N}}$ is weakly relatively compact. The indicator function of a set $A$ will be denoted by $\mathbb{I}(A)$. By the formula of total probability for an arbitrary $R > 0$, we have

$$P(|V_n| > R) = \sum_{k=1}^{\infty} P(N_n = k) \mathbb{I}\left(\left|\frac{a_n - c_n}{d_n}\right| > R\right)$$

$$= \sum_{k=1}^{\infty} P(N_n = k) P\left(\left|\frac{S_n - c_n}{d_n} - \frac{b_n}{b_n} \cdot \frac{S_n - a_n}{d_n}\right| > R\right)$$

$$\leq P\left(Z_n > \frac{R}{2}\right) + \sum_{k=1}^{\infty} P(N_n = k) P\left(\frac{b_n}{d_n} \cdot |Y_n| > \frac{R}{2}\right)$$

$$\equiv I_{1,n}(R) + I_{2,n}(R).$$

First consider $I_{2,n}(R)$. Using the set $K_{2,n} = K_{2,n}(\gamma)$ introduced in the preceding section, for an arbitrary $\gamma > 0$, we have

$$I_{2,n}(R) = \sum_{k \in K_{2,n}} P(N_n = k) P\left(|Y_{n,k}| > \frac{Rd_n}{2b_n}\right) + \sum_{k \in K_{2,n}} P(N_n = k) P\left(|Y_{n,k}| > \frac{Rd_n}{2b_n}\right)$$

$$\leq \sum_{k \in K_{2,n}} P(N_n = k) P\left(|Y_{n,k}| > \frac{R}{2\gamma}\right) + P(U_n > \gamma) \leq P\left(|Y_{n,N_n}| > \frac{R}{2\gamma}\right)$$

$$+ P(U_n > \gamma).$$

(20)

Fix an arbitrary $\epsilon > 0$. Choose $\gamma = \gamma(\epsilon)$ such that

$$P(U_n > \gamma(\epsilon)) < \epsilon$$

(21)

for all $n \in \mathbb{N}$. This is possible due to the weak relative compactness of the family $\{U_n\}_{n \in \mathbb{N}}$. Now choose $R' = R'(\epsilon)$ such that

$$P\left(|Y_{n,N_n}| > \frac{R'(\epsilon)}{2\gamma(\epsilon)}\right) < \epsilon.$$  

(22)

This is possible due to the weak relative compactness of the family $\{Y_{n,N_n}\}_{n \in \mathbb{N}}$ implied by coherency condition (2). Thus, from (20), (21), and (22), we obtain

$$I_{2,n}(R'(\epsilon)) < 2\epsilon$$

(23)
for all $n \in \mathbb{N}$. Now consider $I_{1,n}(R)$. From (18), it follows that there exists an $R'' = R''(\epsilon)$ such that

$$I_{1,n}(R''(\epsilon)) < \epsilon$$

(24)

for all $n \in \mathbb{N}$. From (23) and (24), it follows that if $R > \max\{R', R''\}$, then

$$\sup_{n} P(|V_n| > R) < 3\epsilon$$

and by virtue of the arbitrariness of $\epsilon > 0$, the family $\{V_n\}_{n \in \mathbb{N}}$ is weakly relatively compact. Hence, the family of pairs $\{(U_n, V_n)\}_{n \in \mathbb{N}}$ is weakly relatively compact.

Denote

$$\epsilon_n = \inf \{ L_2((U_n, V_n), (U, V)) : (U, V) \in \mathcal{W}(Z|Y), \quad n = 1, 2, \ldots \}$$

Prove that $\epsilon_n \to 0$ as $n \to \infty$. Assume the contrary. In this case $\epsilon_n \geq M$ for some $M > 0$ and all $n$ from some subsequence $N$ of natural numbers. Choose a subsequence $N_1 \subseteq N$ such that the sequence of pairs $\{(U_n, V_n)\}_{n \in N_1}$ weakly converges to some pair $(U, V)$. As this is so, for all $n \in N_1$ large enough, we will have $L_2((U_n, V_n), (U, V)) < M$. Applying Theorem 3.1 to the sequence $\{(U_n, V_n)\}_{n \in N_1}$, we make sure that $(U, V) \in \mathcal{W}(Z|Y)$ since condition (18) implies the coincidence of the limits of all convergent subsequences of $\{Z_n\}$. We arrive at the contradiction with the assumption that $\epsilon_n > M$ for all $n \in N_1$. Hence, $\epsilon_n \to 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, choose a pair $(U_n', V_n') \in \mathcal{W}(Z|Y)$ such that

$$L_2((U_n, V_n), (U_n', V_n')) \leq \epsilon_n + \frac{1}{n}.$$ 

The sequence $\{(U_n', V_n')\}_{n \in \mathbb{N}}$ obviously satisfies condition (19). Its weak relative compactness follows from (19) and the weak relative compactness of the sequence $\{(U_n, V_n)\}_{n \in \mathbb{N}}$ established above.

"If" part. Assume that the sequence $\{Z_n\}_{n \in \mathbb{N}}$ does not converge weakly to $Z$ as $n \to \infty$. In that case, the inequality $L_1(Z_n, Z) \geq M$ holds for some $M > 0$ and all $n$ from some subsequence $N$ of natural numbers. Choose a subsequence $N_1 \subseteq N$ so that the sequence of pairs $\{(U_n', V_n')\}_{n \in N_1}$ weakly converges to some pair $(U, V)$. Repeating the reasoning used to prove Theorem 3.1 we make sure that for any $t \in \mathbb{R}$

$$E e^{itZ} = E h(tU_n')e^{itV_n'} \longrightarrow E h(tU)e^{itV}$$

as $n \to \infty$, $n \in N_1$, that is, $(U, V) \in \mathcal{W}(Z|Y)$. From the triangle inequality

$$L_2((U_n, V_n), (U, V)) \leq L_2((U_n, V_n), (U_n', V_n')) + L_2((U_n', V_n'), (U, V))$$

and condition (19) it follows that $L_2((U_n, V_n), (U, V)) \to 0$ as $n \to \infty$, $n \in N_1$. Apply Theorem 3.1 to the double array $\{Y_{n,k}\}_{k \in \mathbb{N}, n \in N_1}$ and the sequence $\{(U_n, V_n)\}_{n \in N_1}$. As a result, we obtain that $L_1(Z_n, Z) \to 0$ as $n \to \infty$, $n \in N_1$, contradicting the assumption that $L_1(Z_n, Z) \geq M > 0$ for $n \in N_1$. Thus, the theorem is completely proved. □
Remark 3.1. It should be noted that in Ref.\textsuperscript{11} and some subsequent papers, a stronger and less convenient version of the coherency condition was used. Furthermore, in Ref.\textsuperscript{11} and the subsequent papers, the statements analogous to Lemma 2.1 and Theorems 3.1 and 3.2 were proved under the additional assumption of the weak relative compactness of the family \( \{Y_{n,k}\}_{n,k \in \mathbb{N}} \).

4. Limit theorems for random sums of independent random variables

Let \( \{X_{n,j}\}_{j \geq 1}, n \in \mathbb{N} \), be a double array of row-wise independent not necessarily identically distributed random variables. For \( n, k \in \mathbb{N} \), denote

\[
S_{n,k} = X_{n,1} + \cdots + X_{n,k}.
\]

If \( S_{n,k} \) is a sum of independent random variables, then the condition of weak relative compactness of the sequence \( \{U_n\}_{n \in \mathbb{N}} \) used in the preceding section can be replaced by the condition of weak relative compactness of the family \( \{Y_{n,k}\}_{n,k \in \mathbb{N}} \) which is, in fact, considerably less restrictive. Indeed, let, for example, the random variables \( S_{n,k} \) possess moments of some order \( \delta > 0 \). Then, if we choose \( b_{n,k} = (E|S_{n,k} - a_{n,k}|^\delta)^{1/\delta} \), then by the Markov inequality

\[
\lim_{R \to \infty} \sup_{n,k \in \mathbb{N}} \mathbb{P}(|Y_{n,k}| > R) \leq \lim_{R \to \infty} \frac{1}{R^\delta} = 0,
\]

that is, the family \( \{Y_{n,k}\}_{n,k \in \mathbb{N}} \) is weakly relatively compact.

Theorem 4.1. Assume that the random variables \( S_{n,k} \) have the form (25). Let the family of random variables \( \{Y_{n,k}\}_{n,k \in \mathbb{N}} \) be weakly relatively compact. Assume that condition (2) holds. Then, convergence (18) of normalized random sums \( Z_n \) to some random variable \( Z \) takes place with some \( c_n \in \mathbb{R} \) if and only if there exists a weakly relatively compact sequence of pairs \( (U'_n, V'_n) \in \mathcal{W}(Z|Y) \), \( n \in \mathbb{N} \), such that condition (19) holds.

Proof. It suffices to prove that in the case under consideration condition (18) implies the weak relative compactness of the family \( \{U_n\}_{n \in \mathbb{N}} \). In what follows, the symmetrization of a random variable \( X \) will be denoted by \( X^{(s)} \), \( X^{(s)} = X - X' \) where \( X' \) is a random variable independent of \( X \) such that \( X' \overset{d}{=} X \). For \( q \in (0, 1) \), let \( \ell_n(q) \) be the greatest lower bound of \( q \)-quantiles of the random variable \( N_n \), \( n \in \mathbb{N} \). Assume that for each \( n \), the random variables \( N_n, X^{(s)}_{n,1}, X^{(s)}_{n,2}, \ldots \) are jointly independent and introduce the random variables

\[
Q_n = \frac{1}{d_n} \sum_{j=1}^{N_n} X^{(s)}_{n,j}, \quad n \in \mathbb{N}.
\]

Using the symmetrization inequality

\[
\mathbb{P}(X^{(s)} \geq R) \leq 2 \mathbb{P}\left( |X - a| \geq \frac{R}{2} \right),
\]
which is valid for any random variable $X$, any $a \in \mathbb{R}$ and $R > 0$ (see, e.g., Ref.\cite{23}), we obtain

$$P(|Q_n| \geq R) = \sum_{k=1}^{\infty} P(N_n = k) P \left( \left| \frac{1}{d_n} \sum_{j=1}^{k} X_{n,j}^{(s)} \right| \geq R \right)$$

$$\leq 2 \sum_{k=1}^{\infty} P(N_n = k) P \left( \left| \frac{1}{d_n} \left( \sum_{j=1}^{k} X_{n,j} - c_n \right) \right| \geq \frac{R}{2} \right)$$

$$= 2 P \left( \left| \frac{1}{d_n} \left( \sum_{j=1}^{N_n} X_{n,j} - c_n \right) \right| \geq \frac{R}{2} \right)$$

for any $R > 0$ and $n \in \mathbb{N}$. Hence,

$$\lim_{R \to \infty} \sup_n P(|Q_n| \geq R) \leq 2 \lim_{R \to \infty} \sup_n P \left( \left| \frac{1}{d_n} \left( \sum_{j=1}^{N_n} X_{n,j} - c_n \right) \right| \geq \frac{R}{2} \right) = 0$$

by virtue of (18). Hence, the sequence $\{Q_n\}_{n \in \mathbb{N}}$ is weakly relatively compact.

Now prove that

$$C(q) \equiv \sup_n \frac{b_{n,\ell_n(q)}}{d_n} < \infty$$

(26)

for each $q \in (0, 1)$. For this purpose, we use the Lévy inequality

$$P \left( \max_{1 \leq m \leq k} \left| \sum_{j=1}^{m} X_j^{(s)} \right| \geq R \right) \leq 2 P \left( \left| \sum_{j=1}^{k} X_j^{(s)} \right| \geq R \right),$$

which is valid for any independent random variables $X_1, \ldots, X_k$ and any $R > 0$ and for an arbitrary $q \in (0, 1)$ obtain the following chain of inequalities:

$$2 P(|Q_n| \geq R) = 2 \sum_{k=1}^{\infty} P(N_n = k) P \left( \left| \frac{1}{d_n} \sum_{j=1}^{k} X_{n,j}^{(s)} \right| \geq R \right)$$

$$\geq 2 \sum_{k \geq \ell_n(q)} P(N_n = k) P \left( \left| \frac{1}{d_n} \sum_{j=1}^{\ell_n(q)} X_{n,j}^{(s)} \right| \geq R \right)$$

$$\geq \sum_{k \geq \ell_n(q)} P(N_n = k) P \left( \left| \frac{\ell_n(q)}{d_n} \sum_{j=1}^{\ell_n(q)} X_{n,j}^{(s)} \right| \geq R \right)$$

$$= P \left( N_n \geq \ell_n(q) \right) P \left( \left| \frac{\ell_n(q)}{d_n} \sum_{j=1}^{\ell_n(q)} X_{n,j}^{(s)} \right| \geq R \right)$$

$$= (1 - q) P \left( \left| \frac{\ell_n(q)}{d_n} \sum_{j=1}^{\ell_n(q)} X_{n,j}^{(s)} \right| \geq R \right).$$
Hence, the weak relative compactness of the family \( \{Q_n\}_{n \in \mathbb{N}} \) established above, for each \( q \in (0, 1) \) implies the weak relative compactness of the family \( \{Q_n^{(q)}\}_{n \in \mathbb{N}} \) where

\[
Q_n^{(q)} = \frac{1}{d_n} \sum_{j=1}^{\ell_a(q)} X_{n,j}, \quad n \in \mathbb{N}.
\]

Assume that (26) does not hold. In that case, there exist a \( q^* \in (0, 1) \) and a sequence \( \mathcal{N} \) of natural numbers such that

\[
\frac{b_{n,\ell_a(q^*)}}{d_n} \to \infty, \quad n \to \infty, \quad n \in \mathcal{N}. \tag{27}
\]

According to the conditions of the theorem, the family of random variables \( \{Y_{n,k} = (S_{n,k} - a_{n,k})/b_{n,k}\}_{n,k \in \mathbb{N}} \) is weakly relatively compact. Therefore, a subsequence \( \mathcal{N}_1 \subseteq \mathcal{N} \) can be chosen so that

\[
Y_{n,\ell_a(q^*)} = \frac{1}{b_{n,\ell_a(q^*)}} \left( \sum_{j=1}^{\ell_a(q^*)} X_{n,j} - a_{n,\ell_a(q^*)} \right) \Rightarrow Y, \quad n \to \infty, \quad n \in \mathcal{N}_1, \tag{28}
\]

where \( Y \) is some random variable. From (27) and (28), it follows that for any \( R \in \mathbb{R} \)

\[
P(\ell_a(q^*) \leq R) = P \left( \frac{d_n R}{b_{n,\ell_a(q^*)}} \right) \to P(Y < 0) \geq \frac{1}{2}, \quad n \to \infty, \quad n \in \mathcal{N}_1,
\]

contradicting the weak relative compactness of the family \( \{Q_n^{(q^*)}\} \) established above. So, (26) holds for any \( q \in (0, 1) \).

It is easy to make sure that \( N_n \overset{d}{=} \ell_n(W) \), where \( W \) is a random variable with the uniform distribution on \((0, 1)\). Therefore, with the account of (26) for any \( R \geq 0 \) and \( n \in \mathbb{N} \), we have

\[
P(U_n \geq R) = P \left( \frac{b_{n,\ell_a(W)}}{d_n} \geq R \right) = \int_0^1 \mathbb{I} \left( \frac{b_{n,\ell_a(q)}}{d_n} \geq R \right) dq \leq \int_0^1 \mathbb{I} (C(q) \geq R) dq = P(C(W) \geq R)
\]

so that

\[
\lim_{R \to \infty} \sup_n P(U_n \geq R) = \lim_{R \to \infty} P(C(W) \geq R) = 0,
\]

that is, the sequence \( \{U_n\}_{n \in \mathbb{N}} \) is weakly relatively compact.

The rest of the proof of Theorem 4.1 repeats that of Theorem 3.2 word-for-word. The theorem is proved. \( \Box \)
5. A version of the central limit theorem for random sums with a normal variance–mean mixture as the limiting law

Let \( \{X_{n,j}\}_{j \geq 1}, n \in \mathbb{N} \), be a double array of row-wise independent not necessarily identically distributed random variables. As in the preceding section, let

\[
S_{n,k} = X_{n,1} + \cdots + X_{n,k}, \quad n, k \in \mathbb{N}.
\]

The distribution function and the characteristic function of the random variable \( X_{n,j} \) will be denoted by \( F_{n,j}(x) \) and \( f_{n,j}(t) \), respectively,

\[
f_{n,j}(t) = \int_{-\infty}^{\infty} e^{itx} dF_{n,j}(x), \quad t \in \mathbb{R}.
\]

It is easy to see that in this case

\[
h_{n,k}(t) \equiv \mathbb{E} \exp \{itY_{n,k}\} = \exp \left\{-\frac{it}{b_{n,k}} \right\} \prod_{j=1}^k f_{n,j} \left(\frac{t}{b_{n,k}}\right), \quad t \in \mathbb{R}.
\]

Denote \( \mu_{n,j} = \mathbb{E}X_{n,j}, \quad \sigma_{n,j}^2 = \text{DX}_{n,j} \) and assume that \( 0 < \sigma_{n,j}^2 < \infty, \quad n, j \in \mathbb{N} \). Denote

\[
A_{n,k} = \mu_{n,1} + \cdots + \mu_{n,k} \quad (= \mathbb{E}S_{n,k}), \quad B_{n,k}^2 = \sigma_{n,1}^2 + \cdots + \sigma_{n,k}^2 \quad (= \text{DS}_{n,k})
\]

It is easy to make sure that \( \mathbb{E}S_{n,N_n} = \mathbb{E}A_{n,N_n}, \quad \text{DS}_{n,N_n} = \mathbb{E}B_{n,N_n}^2 + \text{DA}_{n,N_n}, \quad n \in \mathbb{N} \). In order to formulate a version of the central limit theorem for random sums with the limiting distribution being a normal variance–mean mixture, assume that non-random sums, as usual, are centered by their expectations and normalized by by their mean square deviations and put \( a_{n,k} = A_{n,k}, \quad b_{n,k} = \sqrt{B_{n,k}^2}, \quad n, k \in \mathbb{N} \). Although it would have been quite natural to normalize random sums by their mean square deviations as well, for simplicity we will use slightly different normalizing constants and put \( d_n = \sqrt{\mathbb{E}B_{n,N_n}^2} \). Recall that we use the notation \( \Phi(x) \) for the standard normal distribution function.

**Theorem 5.1.** Assume that the following conditions hold:

(i) for every \( n \in \mathbb{N} \), the ratio \( \frac{\mu_{n,j}}{\sigma_{n,j}^2} \) does not depend on \( j \) and, as \( n \to \infty \),

\[
\alpha_n \equiv \frac{\mu_{n,j}}{\sigma_{n,j}^2} \sqrt{\mathbb{E}B_{n,N_n}^2} \longrightarrow \alpha \quad 0 < |\alpha| < \infty; \quad (29)
\]

(ii) (the random Lindeberg condition) for any \( \epsilon > 0 \)

\[
\lim_{n \to \infty} \mathbb{E} \frac{1}{B_{n,N_n}^2} \sum_{j=1}^{N_n} \int_{|x-\mu_{n,j}| > \epsilon B_{n,N_n}} (x - \mu_{n,j})^2 dF_{n,j}(x) = 0. \quad (30)
\]

Then, the convergence of the normalized random sums

\[
\frac{S_{n,N_n}}{\sqrt{\mathbb{E}B_{n,N_n}^2}} \longrightarrow Z \quad (31)
\]
to some random variable $Z$ as $n \to \infty$ takes place if and only if there exists a random variable $U$ such that

$$P(Z < x) = E\Phi\left(\frac{x - \alpha U}{\sqrt{U}}\right), \quad x \in \mathbb{R}, \quad (32)$$

and

$$\frac{B_{n,N_n}^2}{EB_{n,N_n}^2} \Longrightarrow U \quad (n \to \infty). \quad (33)$$

Proof. We will deduce Theorem 5.1 as a corollary of Theorem 4.1.

First, let $a_{n,k} = A_{n,k}, b_{n,k} = B_{n,k}, n, k \in \mathbb{N}$. Then, the family of the random variables $\{Y_{n,k}\}_{n,k} \in \mathbb{N}$ is weakly relatively compact, since by the Chebyshev inequality

$$\lim_{R \to \infty} \sup_{n,k} P\left(|Y_{n,k}| > R\right) = \lim_{R \to \infty} \sup_{n,k} P\left(\left|\frac{S_{n,k} - A_{n,k}}{B_{n,k}}\right| > R\right) \leq \lim_{R \to \infty} \frac{1}{R^2} = 0. (34)$$

Second, prove that under the conditions of the theorem the coherency condition (2) holds with $h(t) = e^{-t^2/2}, t \in \mathbb{R}$. Denote $\Delta_{n,k}(x) = |H_{n,k}(x) - \Phi(x)|$, where $H_{n,k}(x) = P(S_{n,k} - A_{n,k} < B_{n,k}x)$. By integration by parts, for any $t \in \mathbb{R}$, we have

$$|h_{n,k}(t) - e^{-t^2/2}| = \left|t \int_{-\infty}^{\infty} e^{tx} \left[|H_{n,k}(x) - \Phi(x)|\right] dx\right| \leq |t| \int_{-\infty}^{\infty} \Delta_{n,k}(x) dx. \quad (35)$$

To estimate the integrand on the right-hand side of (34), we will use the following result of V. V. Petrov\[^{24}\]. Let $G$ be the class of real-valued functions $g(x)$ of the argument $x \in \mathbb{R}$ such that the function $g(x)$ is even, nonnegative for all $x$ and positive for $x > 0$; the functions $g(x)$ and $x/g(x)$ are nondecreasing for $x > 0$. In Ref.\[^{24}\], it was proved that, whatever a function $g \in G$ is, if $E(Y_{n,j}^2 g(X_{n,j}) < \infty, n, j \in \mathbb{N}$, then there exists a positive finite absolute constant $C$ such that for any $x \in \mathbb{R}$

$$\Delta_{n,k}(x) \leq \frac{C}{B_{n,k}^2(1 + |x|)^2g(B_{n,k}(1 + |x|))} \sum_{j=1}^{k} E(X_{n,j} - \mu_{n,j})^2g(X_{n,j} - \mu_{n,j}). \quad (36)$$

Hence, it is easy to see that the properties of the function $g \in G$ guarantee that

$$\Delta_{n,k}(x) \leq \frac{C}{(1 + |x|)^2} \cdot \frac{1}{B_{n,k}^2g(B_{n,k})} \sum_{j=1}^{k} E(X_{n,j} - \mu_{n,j})^2g(X_{n,j} - \mu_{n,j}). \quad (37)$$

Now choosing $g(x) = \min\{|x|, B_{n,k}\} \in G$ and repeating the reasoning used to prove Theorem 5.5 in Secton 3, Chapt. V of ref.\[^{25}\], and relation (3.8) there, from (35) we obtain that for all $n, k \in \mathbb{N}$ and an arbitrary $\epsilon > 0$

$$\Delta_{n,k}(x) \leq \frac{2C}{(1 + |x|)^2} \left\{\epsilon + \frac{1}{B_{n,k}^2} \sum_{j=1}^{k} E\left[(X_{n,j} - \mu_{n,j})^2 I(\{|X_{n,j} - \mu_{n,j}| > \epsilon B_{n,k}\})\right]\right\}. \quad (36)$$
Using (34) and (36), we obtain that for arbitrary $\epsilon > 0$ and $T \in (0, \infty)$
\[
E \sup_{|t| \leq T} |h_{n,N_n}(t) - e^{-t^2/2}| \leq |T|E \int_{-\infty}^{\infty} |H_{n,N_n}(x) - \Phi(x)| \, dx
\leq 4C|T| \left\{ \epsilon + E \frac{1}{B_{n,N_n}^2} \sum_{j=1}^{N_n} \int_{|x-\mu_{n,j}| > \epsilon B_{n,N_n}} (x - \mu_{n,j})^2 dF_{n,j}(x) \right\}.
\]
Hence, from (30), it follows that for an arbitrary $\epsilon > 0$
\[
\lim_{n \to \infty} E \sup_{|t| \leq T} |h_{n,N_n}(t) - e^{-t^2/2}| \leq 4C|T|\epsilon,
\]
and since $\epsilon > 0$ can be taken arbitrarily small, the coherency condition (2) holds.

Third, let $d_n = \sqrt{EB_{n,N_n}^2}$, $c_n = 0$, $n \in \mathbb{N}$. Then, $\mu_{n,j} = \alpha_n \sigma_{n,j}^2 / \sqrt{EB_{n,N_n}^2}$, $n$, $j \in \mathbb{N}$, so that $A_{n,N_n} = \alpha_n B_{n,N_n}^2 / \sqrt{EB_{n,N_n}^2}$ and relations (29) and (30) guarantee that relation (16) holds with $\rho = 1$, $\beta_n = \beta = 0$, $n \in \mathbb{N}$, so that if (33) holds along some subsequence $N$ of natural numbers, then
\[
(U_n, V_n) = \left( \sqrt{\frac{B_{n,N_n}^2}{EB_{n,N_n}^2}}, \alpha_n \frac{B_{n,N_n}^2}{EB_{n,N_n}^2} \right) \Rightarrow \left( \sqrt{U}, \alpha U \right), \quad n \to \infty, \quad n \in N,
\]
so that the limit law has the form of normal variance–mean mixture (32).

Fourth, recently in ref. [16], it was proved that normal variance–mean mixtures are identifiable, that is, if $\mathbb{P}(Y < x) = \Phi(x)$, then the set $\mathcal{W}(Z|Y)$ contains at most one pair of the form $(\sqrt{U}, \alpha U)$. This means that in the case under consideration, condition (19) reduces to (33). The theorem is proved. \hfill \square

**Remark 5.1.** To explain the meaning of the conditions of Theorem 5.1 and thus make some comments concerning the range of its applicability, we should say that the essence of condition $(i)$ is that although the summands may have different distributions, their expectations must be proportional to their variances. This is a good hint when an explanation is sought for a high adequacy of normal variance–mean mixtures as models of statistical regularities observed in practice. For example, the property of proportionality of the expectations of increments to their variances is inherent in the Wiener-type processes with nonzero drift which describe, say, the process of Brownian motion in a moving medium. By the way, the requirement of proportionality of the expectations of elementary summands to their variances is the circumstance which makes it impossible to obtain nontrivial (skewed) normal variance–mean mixtures as limit laws for random sums in the asymptotic setting of “cumulative sums” where only one basic sequence of summands is available, see ref. [16]. As concerns the random Lindeberg condition $(ii)$, we should note that it is a kind of a necessary condition at least in the case of nonrandom indexes.

**Remark 5.2.** Concerning the randomness of indexes, we should say that, as it has has already been mentioned in the introduction, the data to be analyzed is collected or
registered during a certain period of time and the flow of informative events producing the observations forms a random point process, so that the number of informative events that occur within a certain time unit is random. It substantially depends on the intensity of the flow which is very often random itself, as it happens, say, in high-frequency financial applications, see, e.g., ref.\textsuperscript{[17]}. Therefore, the “if and only if” character of Theorem 5.1 makes it possible to reconstruct the distribution of the intensity of the flow of informative events (jumps of the registered process) from that of the registered process\textsuperscript{[17]}.

Remark 5.3. In accordance with what has been said in Remark 3.1, the random Lindeberg condition (ii) can be used in the following form: for any $\epsilon > 0$

$$
\frac{1}{B_{n,N_n}^2} \sum_{j=1}^{N_n} \int_{|x-\mu_{n,j}|>\epsilon B_{n,N_n}} (x-\mu_{n,j})^2 dF_{n,j}(x) \longrightarrow 0
$$

in probability as $n \rightarrow \infty$.

Remark 5.4. For $n, j, k \in \mathbb{N}$ denote $v_{n,j}^3 = E[X_{n,j} - \mu_{n,j}]^3$, $M_{n,k}^3 = v_{n,1}^3 + \ldots + v_{n,k}^3$, $l_{n,k}^3 = M_{n,k}^3 - n^{-3}$. It is easy to see that for each $n \in \mathbb{N}$

$$
\sum_{k=1}^{\infty} P(N_n = k) \frac{1}{B_{n,k}^3} \sum_{j=1}^{k} \int_{|x-\mu_{n,j}|>\epsilon B_{n,k}} |x-\mu_{n,j}|^3 dF_{n,j}(x)
$$

$$
\leq \frac{1}{\epsilon} \sum_{k=1}^{\infty} P(N_n = k) \frac{M_{n,k}^3}{(l_{n,k})^3} = \frac{1}{\epsilon} E L_{n,N_n}^3.
$$

Therefore, if the third absolute moments of the summands exist, then the random Lindeberg condition (ii) follows from the random Lyapunov condition

$$
\lim_{n \rightarrow \infty} E L_{n,N_n}^3 = 0
$$

which seems to be more easily verifiable than the random Lindeberg condition. For example, let for each $n \in \mathbb{N}$ the random variables $X_{n,1}, X_{n,2}, \ldots$ be identically distributed. Denote the generating function of the random variable $N_n$ by $\psi_n(s)$,

$$
\psi_n(s) = E s^{N_n} = \sum_{k=1}^{\infty} s^k P(N_n = k), \quad 0 \leq s \leq 1.
$$

Then

$$
E L_{n,N_n} = \frac{v_{n,1}^3}{\sigma_{n,1}^3} E \frac{1}{\sqrt{N_n}} \leq \frac{v_{n,1}^3}{\sigma_{n,1}^3} \sqrt{\frac{1}{N_n}} = \frac{v_{n,1}^3}{\sigma_{n,1}^3} \left( \int_0^1 \frac{\psi_n(s)}{s} ds \right)^{1/2},
$$
and the random Lyapunov condition holds, if
\[
\lim_{n \to \infty} \frac{V^3_{n,1}}{\sigma^3_{n,1}} \left( \int_0^1 \psi_n(s) ds \right)^{1/2} = 0
\]
implies the random Lindeberg condition and hence, the coherency condition.

**Remark 5.5.** The class of normal variance–mean mixtures is very wide and, in particular, contains the class of generalized hyperbolic distributions which, in turn, contains (a) symmetric and skew Student distributions (including the Cauchy distribution) with inverse gamma mixing distributions; (b) variance gamma distributions (including symmetric and non-symmetric Laplace distributions) with gamma mixing distributions; (c) normal/inverse Gaussian distributions with inverse Gaussian mixing distributions including symmetric stable laws. By variance–mean mixing, many other initially symmetric types represented as pure scale mixtures of normal laws can be skewed, e.g., as was done to obtain nonsymmetric exponential power distributions in ref. [9] or nonsymmetric two-sided Weibull distributions in ref. [20].

According to Theorem 5.1, all these laws can be limiting for random sums of independent nonidentically distributed random variables. For example, to obtain the skew Student distribution for \(Z\), it is necessary and sufficient that in (32) and (33) the random variable \(U\) has the inverse gamma distribution \([15]\). To obtain the variance gamma distribution for \(Z\), it is necessary and sufficient that in (32) and (33) the random variable \(U\) has the gamma distribution \([15]\). In particular, for \(Z\) to have the asymmetric Laplace distribution, it is necessary and sufficient that \(U\) has the exponential distribution.

**Remark 5.6.** Note that the nonrandom sums in the coherency condition are centered, whereas in (31) the random sums are not centered, and if \(\alpha \neq 0\), then the limit distribution for random sums becomes skew unlike usual nonrandom summation, where the presence of the systematic bias of the summands results in that the limit distribution becomes just shifted. So, if noncentered random sums are used as models of some real phenomena and the limit variance–mean mixture is skew, then it can be suspected that the summands are actually biased.

**Remark 5.7.** In limit theorems of probability theory and mathematical statistics, the centering and normalization of random variables are used to obtain nontrivial asymptotic distributions. It should be especially noted that to obtain reasonable approximation to the distribution of the basic random variables (in our case, \(S_{N_n, N_0}\)), both centering and normalizing values should be nonrandom. Otherwise, the approximate distribution becomes random itself and, say, the problem of evaluation of quantiles becomes senseless.

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