General Relativistic Scalar Field Models in the Large

Peter Hübner∗(pth@gravi.physik.uni-jena.de)
Max-Planck-Gesellschaft Arbeitsgruppe Gravitationstheorie
an der Universität Jena
Max-Wien-Platz 1
D-07743 Jena

June 2, 2021

Abstract

For a class of scalar fields including the massless Klein-Gordon field the general relativistic hyperboloidal initial value problems are equivalent in a certain sense. By using this equivalence and conformal techniques it is proven that the hyperboloidal initial value problem for those scalar fields has a unique solution which is weakly asymptotically flat. For data sufficiently close to data for flat spacetime there exist a smooth future null infinity and a regular future timelike infinity.

1 Introduction

1.1 General remarks

A large open problem of classical general relativity is the characterization of the structure of a spacetime by initial data. The flat case, Minkowski spacetime, is geodesically complete. To the other extreme the singularity theorems by R. Penrose and S. Hawking show that the spacetime cannot be geodesically complete if the data are large [20].

In the last years there has been remarkable progress in describing what happens if one goes from data for flat space to large data: The future of small data evolving in accordance with the Einstein equation with various matter models as sources, vacuum and Einstein-Maxwell-Yang-Mills, looks like the future of data for flat space [10, 18]. Nevertheless many problems are still unsolved.

∗This work is too a large extent part of my Ph. D. thesis which has been done at the Max-Planck-Institut für Astrophysik, Postfach 1523, D-85740 Garching
Those results were significantly improved by D. Christodoulou for spherically symmetric models with a massless Klein-Gordon scalar field as source. He was able to relate properties of the initial data to properties of singularities. But even in this case of high symmetry the questions left are still numerous as numerical simulations by M. Choptuik show [8]. He found very interesting properties, the so called echoing effect, for models which are in the parameter space of initial data near to the boundary which separates regular from singular spacetimes. In this paper conformal techniques are used to analyze the hyperboloidal initial value problem with scalar fields as matter models — for data near Minkowskian data the future of the initial value surface possesses a smooth future null infinity and a regular timelike infinity, for large data a smooth future null infinity exists for at least some time. In the second part of the introduction more about conformal techniques and their application for a mathematical description of asymptotically flat spacetimes will be said.

Although the primarily treated matter model is that of the conformally invariant scalar field, whose equations can be written as

\begin{equation}
\tilde{\Box} \tilde{\phi} - \frac{\tilde{R}}{6} \tilde{\phi} = 0
\end{equation}

\begin{equation}
(1 - \frac{1}{4} \tilde{\phi}^2) \tilde{R}_{ab} = \left( (\tilde{\nabla}_a \tilde{\phi})(\tilde{\nabla}_b \tilde{\phi}) - \frac{1}{2} \tilde{\phi} \tilde{\nabla}_a \tilde{\nabla}_b \tilde{\phi} - \frac{1}{4} \tilde{g}_{ab} (\tilde{\nabla}^c \tilde{\phi})(\tilde{\nabla}_c \tilde{\phi}) \right),
\end{equation}

the results obtained apply to a larger class of scalar field models, given by the class of actions (32), including the massless Klein-Gordon field, as shown in section 5. Note that an arbitrary factor can be absorbed into $\tilde{\phi}$ which changes the coefficients in (1b). My notational conventions are described in the appendix, the $\tilde{\cdot}$ marks quantities in the physical spacetime (see definition 1). The energy-momentum tensor for the conformally invariant scalar field can be written as

\begin{equation}
\tilde{T}_{ab} = (\tilde{\nabla}_a \tilde{\phi})(\tilde{\nabla}_b \tilde{\phi}) - \frac{1}{2} \tilde{\phi} \tilde{\nabla}_a \tilde{\nabla}_b \tilde{\phi} + \frac{1}{4} \tilde{\phi}^2 \tilde{R}_{ab} - \frac{1}{4} \tilde{g}_{ab} \left( (\tilde{\nabla}^c \tilde{\phi})(\tilde{\nabla}_c \tilde{\phi}) + \frac{1}{6} \tilde{\phi}^2 \tilde{R} \right).
\end{equation}

The analytic investigation presented in this paper show the well posedness of the initial value problem in unphysical spacetime, which is a technical construct to “compactified” asymptotically flat spacetimes in analogy to the compactification of the plane of complex numbers ($\mathbb{C}^2$) into the Riemann sphere ($S^2$).

One goal of this work was making myself familiar with the system in unphysical spacetime as a preparation for numerical work showing that the conformal techniques are well suited for a numerical investigation of global spacetime structure and gravitational radiation [21, 22]. To lower the computational resources required these calculations have been done for spherical symmetry. It is well known that spherically symmetric, uncharged vacuum models are Schwarzschild. The inclusion of matter removes that obstacle, the spacetime may evolve dynamically. Furthermore there is no gravitational radiation in spherically symmetric models. Therefore the matter model should also be a model for radiation. Scalar
fields with wave equations are choices for matter which model also radiation. The conformally invariant scalar field has been chosen since the matter equations are form invariant under rescalings of the metric and an appropriate transformation of the scalar field as the name already suggests. The scalar fields are interesting from the analytic viewpoint since for the first time conformal techniques could be used for matter models whose energy-momentum tensor has non-vanishing trace.

1.2 Asymptotically flat spacetimes

In this paper a geometrical, coordinate independent definition of asymptotical flatness along the lines suggested by R. Penrose will be used. A more thorough discussion of the ideas and the interpretation can be found at various places in the literature, e.g. [19, 26]. The definitions of asymptotical flatness given in the literature differ slightly. The following will be used here:

**Definition 1** A spacetime \( (\tilde{M}, \tilde{g}_{ab}) \) is called asymptotically flat if there is another “unphysical” spacetime \( (M, g_{ab}) \) with boundary \( \mathcal{J} \) and a smooth embedding by which \( \tilde{M} \) can be identified with \( M - \mathcal{J} \) such that:

1. There is a smooth function \( \Omega \) on \( M \) with \( \Omega |_{\tilde{M}} > 0 \) and \( g_{ab} |_{\tilde{M}} = \Omega^2 \tilde{g}_{ab} \).

2. On \( \mathcal{J} \) \( \Omega = 0 \) and \( \nabla_a \Omega \neq 0 \).

3. Each null geodesic in \( (\tilde{M}, \tilde{g}_{ab}) \) acquires a past and a future endpoint on \( \mathcal{J} \).

Because of item 3 null geodesically incomplete spacetimes like Schwarzschild are not asymptotically flat. The next definition includes those spacetimes which have only an asymptotically flat part:

**Definition 2** A spacetime is called weakly asymptotically flat if definition 1 with the exception of item 3 is fulfilled.

Definition 1 and 2 classify spacetimes, they do not require that Einstein’s equation is fulfilled. One would like to know:

Are they compatible with the Einstein equation with sources? Neither definition 1 nor 2 is in an initial value problem form: A given spacetime is or is not classified as asymptotically flat. But for a physical problem one would like to give “asymptotically flat data” and have guaranteed that they evolve into an at least weakly asymptotically flat spacetime.

Nevertheless the geometrically description was extremely helpful in analyzing asymptotically flat spacetimes and it can be successfully used as guideline to
construct a formalism which is better suited for analyzing initial value problems. This method has been developed and applied to various matter sources by H. Friedrich [11, 12, 13, 14, 15, 16, 17]. In this paper it will be applied to general relativistic scalar field models.

The idea is to choose a spacelike initial value surface in the unphysical spacetime \((M, g_{ab})\) and to evolve it. The problems to be faced are:

For Minkowski space the unphysical spacetime \((M, g_{ab})\) can be smoothly extended with three points, future \((i^+)\) and past \((i^-)\) timelike infinity, the end respectively the starting point of all timelike geodesics of \((\tilde{M}, \tilde{g}_{ab})\), and spacelike infinity \((i^0)\), the end point of all spacelike geodesics of \((\tilde{M}, \tilde{g}_{ab})\). The point \(i^0\) divides \(J\) into two disjunct parts, future \((J^+)\) and past \((J^-)\) null infinity. It is well known and has been discussed elsewhere that there are unsolved problems in smoothly extending a “normal” Cauchy hypersurface of \(\tilde{M}\) to \(i^0\) if the spacetime has non-vanishing ADM mass. Certain curvature quantities blow up at \(i^0\), reflecting the non-invariance of the mass under rescalings.

By choosing a spacelike (with respect to \(g_{ab}\)) hypersurface \(S\) not intersecting \(i^0\) but \(J^+ (J^-)\) we avoid the problems with \(i^0\). \(S\) is called a hyperboloidal hypersurface — the corresponding initial value problem is called a hyperboloidal initial value (a detailed definition for the scalar field models is given in section 4, definition 3). The domain of dependence \(D(S)\) of \(S\) will not contain the whole spacetime. The interior of \(S\) corresponds to an everywhere spacelike hypersurface in the physical spacetime which approaches a null hypersurface \(N\) asymptotically. If \(N\) is a light cone \(L\) then the domain of dependence of \(S\) is \(L\). Therefore the hyperboloidal initial value problem is well suited to describe the future (past) of data on the spacelike hypersurface \(S\), e. g. a stellar object and the gravitational radiation caused by its time evolution. It is not well suited to investigate the structure near \(i^0\).

But even for the hyperboloidal initial value problem there are “regularity” problems at \(J\): Transforming the Einstein equation from physical to unphysical spacetime an equation “singular” for \(\Omega = 0\) results. That problem is solved in this paper in analogy to H. Friedrich’s work. A new set of equations for the unphysical spacetime will be derived, its equivalence to the Einstein equation on \(\tilde{M}\) proven. This new set of equations is used to prove the consistency of the hyperboloidal initial value problem for scalar fields with (weakly) asymptotical flatness and the existence of a regular future (past) timelike infinity for data sufficiently close to data for Minkowski spacetime.

2 Regularizing the unphysical field equations

A first attempt for equations determining \((M, g_{ab})\) is the rescaled form of the field equation in physical spacetime. A closer look at the transformation of the
Einstein tensor under rescalings $g_{ab} = \Omega^2 \tilde{g}_{ab}$,

$$\tilde{G}_{ab} = G_{ab} + 2 \Omega^{-1} (\nabla_a \nabla_b \Omega - (\nabla^c \nabla_c \Omega) g_{ab}) + 3 \Omega^{-2} (\nabla^c \Omega)(\nabla_c \Omega) g_{ab},$$

(2)

shows that this first attempt fails. Either there are terms proportional to $\Omega^{-2}$ and $\Omega^{-1}$, which need special care on the set $\mathcal{I}$ of points where $\Omega = 0$, including $\mathcal{J}$, which is part of $M$. Or alternatively, the highest (second order) derivatives of the metric, hidden in the Einstein tensor, are multiplied by a factor of $\Omega^2$ and then the principal part of the second order equation for the metric components vanishes on $\mathcal{I}$. This behaviour of an equation will be called singular on $\mathcal{I}$.

In this section a system of equations without the singularity on $\mathcal{I}$ will be derived from the rescaled Einstein equation by introducing new variables and equations. The set of equations together with the equations for the matter variables may be a system with a very complicated principal part — as it is the case for a conformally invariant scalar field as matter model. A procedure is carried out to simplify the principal part to a form in which no equation contains both derivatives of matter variables as well as derivatives of geometry variables and the principal part of the subsystem for the geometry variables is the same as for the vacuum case ("standard form"). All variables already present in the vacuum case are called geometry variables.

It is shown that the procedure works for the conformally invariant scalar field. The procedure described does not use very restrictive assumptions — it is very general — and may work for most matter models, for which the unphysical matter equations can be regularized on $\mathcal{I}$.

### 2.1 The geometry part of the system

According to the definition of asymptotical flatness (definition 1) the unphysical spacetime is connected with the physical spacetime through the rescaling

$$g_{ab} \mid \tilde{M} = \Omega^2 \tilde{g}_{ab}.$$  

(3)

This rule also determines the transformation of the connection and the curvature. Additionally the transformation of the matter variables $\tilde{\Phi}$ under rescaling must be specified,

$$\Phi \mid \tilde{M} = \Phi[\tilde{g}_{ab}, \Omega, \tilde{\Phi}].$$  

(4)

It is assumed that $\Phi$ has a smooth limit on $\mathcal{J}$, the rescaled equations for the matter variables are regular on $\mathcal{I}$, and there exists a tensor $T_{ab}$, which is independent of derivatives of $\Omega$ and derivatives of curvature terms, fulfills

$$T_{ab} \mid \tilde{M} = \Omega^{-2} \tilde{T}_{ab},$$  

(5)

\footnote{In the general case it is not known how to achieve that.}
and has a limit on $J$. The conditions required may seem very restrictive but they can be fulfilled for Yang-Mills fields and for the conformally invariant scalar field.

The Riemann tensor will be split into its irreducible parts, the conformal Weyl tensor

$$C_{abc}^\ d =: \Omega d_{abc}^\ d,$$

the trace free part $\hat{R}_{ab}$ of the Ricci tensor $R_{ab}$ and the Ricci scalar $R$:

$$R_{abcd} = \Omega d_{abcd} + g_{c[a} \hat{R}_{b]d} - g_{d[a} \hat{R}_{b]c} + \frac{1}{6} g_{c[a} g_{b]d} R,$$  

A $\hat{}$ is used as an indication for trace free parts of tensors.

The irreducible decomposition of the energy-momentum tensor is

$$T_{ab} = \hat{T}_{ab} + \frac{1}{4} g_{ab} T.$$  

The irreducible parts transform under rescalings according to

$$\hat{T}_{ab} = \Omega^{-2} \tilde{T}_{ab}$$

and

$$T = \Omega^{-4} \tilde{T}.$$  

The vanishing of the divergence of $\tilde{T}_{ab}$ becomes

$$0 = \nabla^a \tilde{T}_{ab} = \Omega^4 \nabla^a \hat{T}_{ab} + \frac{1}{4} \Omega^4 \nabla_b T + \Omega^3 T \nabla_b \Omega.$$  

For energy-momentum tensors with non-vanishing trace equation (2) as an equation for the components of the irreducible parts of the energy-momentum tensor $T_{ab}$ is singular on $I$. Since (3) should be in same way part of the matter equations problems in regularizing the matter equations are to be expected.

### 2.1.1 A regular system

The part of (2) proportional to $\Omega^{-2}$ is a pure trace, thus the $\Omega^{-2}$ singularity is absent in the trace free equation. A decomposition into the trace and the trace free part moves the worst term into one equation.

From the rescaling rule for the Ricci scalar and tensor,

$$\tilde{R} = \Omega^2 R + 6 \Omega \nabla^a \nabla_a \Omega - 12 (\nabla^a \Omega) (\nabla_a \Omega),$$  

From the definition of asymptotical flatness and the Einstein equation it follows that $\Omega^{-1} T_{ab}$ has a limit on $J$. The faster fall off and the requirements on the form of $T_{ab}$ have technical reasons.
\[ \tilde{R}_{ab} := \tilde{R}_{ab} - \frac{1}{4} \tilde{g}_{ab} \tilde{R} \]
\[ = \tilde{R}_{ab} + 2 \Omega^{-1} \nabla_a \nabla_b \Omega - \frac{1}{2} \Omega^{-1} (\nabla^c \nabla_c \Omega) g_{ab}, \tag{11} \]
\[ \tilde{G}_{ab} = \tilde{T}_{ab}, \quad \tilde{G} = -\tilde{R}, \quad \text{and} \quad \tilde{T} = \tilde{G} \]

It follows
\[ \Omega R + 12 \nabla^a \nabla_a \Omega - 12 \Omega^{-1} (\nabla^a \Omega) (\nabla_a \Omega) = -\Omega^3 T \tag{12} \]
and
\[ \Omega \tilde{R}_{ab} + 2 \nabla_a \nabla_b \Omega - \frac{1}{2} (\nabla^c \nabla_c \Omega) g_{ab} = \Omega^3 T_{ab}. \tag{13} \]

Equation (12) can be dealt with by the following lemma:

**Lemma 1** From \( \tilde{R} + \tilde{T} = 0 \) (\( \equiv (12) \)) at one point, \( \tilde{G}_{ab} = \tilde{T}_{ab} \) (\( \equiv (13) \)), and \( \tilde{\nabla}^b \tilde{T}_{ab} = 0 \) \( \tilde{R} + \tilde{T} = 0 \) follows everywhere.

Proof:
\[ \tilde{\nabla}^a \tilde{T}_{ab} = \tilde{\nabla}^a \tilde{T}_{ab} + \frac{1}{4} \tilde{\nabla}_b \tilde{T} = 0. \]

Combined with
\[ 0 = \tilde{\nabla}^a \tilde{G}_{ab} \]
\[ = \tilde{\nabla}^a \tilde{G}_{ab} + \frac{1}{4} \tilde{\nabla}_b \tilde{G} \]
gives
\[ \tilde{\nabla}_b (\tilde{T} + \tilde{R}) = 0, \]

i.e. \( \tilde{T} + \tilde{R} \) is constant.

Equation (12) will not be used any longer since \( \tilde{\nabla}^b \tilde{T}_{ab} = 0 \) can be derived from the remaining equations, contract \((20g)\) or see the discussion following (11).

In the following the Ricci scalar \( R \) will be regarded as an arbitrary, given function. It fixes part of the gauge freedom on the transition from the physical to the unphysical spacetime as follows: The equations (12) and (13) are invariant under rescalings \((g_{ab}, \Omega) \mapsto (\bar{g}_{ab}, \bar{\Omega}) := (\Theta^2 g_{ab}, \Theta \Omega) \) with \( \Theta > 0 \). All the unphysical spacetimes \((M, \Theta^2 g_{ab}, \Theta \Omega)\) belong to the same physical spacetime \((\bar{M}, \bar{g}_{ab})\).

Under the rescaling \( \tilde{g}_{ab} = \Theta^2 g_{ab} \), \( R \) and \( \tilde{R} \) are connected by
\[ 6 \nabla^a \nabla_a \Theta = \Theta R - \Theta^3 \tilde{R}, \tag{14} \]

which is equation (16) where the covariant derivatives \( \nabla_a \) now corresponds to the unscaled metric. Solving (14) for a spacetime \((M, g_{ab})\) and data for \( \Theta \) and \( \Theta \) on a spacelike surface \( S \) we get at least locally a unphysical space time with
arbitrary Ricci scalar $\bar{R}$.
There is still conformal gauge freedom left as every rescaling with $\Theta > 0$ and
\[
\nabla^a \nabla_a \Theta = \frac{1}{6} \Theta R \left( 1 - \Theta^2 \right)
\] (15)
leaves the Ricci scalar unchanged.

Equation (13) serves as regular equation for $\Omega$. Substituting $\omega = \frac{1}{4} \nabla^c \nabla_c \Omega$ yields
\[
\nabla_a \nabla_b \Omega = -\frac{1}{2} \Omega \bar{R}_{ab} + \omega g_{ab} + \frac{1}{2} \Omega^2 \hat{T}_{ab},
\] (16)
which is a second order equation for $\Omega$.

The next step is to find equations for the metric and the quantities derived therefrom. Expressing the once contracted, second Bianchi identity $\left[ \nabla_a \bar{R}_{bc} \right] = 0$ in terms of $\hat{R}_{ab}$ and $d_{abc}^d$ results in
\[
\nabla_a \hat{R}_{b[c]} = -\frac{1}{12} \left( \nabla_a \bar{R} \right) g_{b[c]} - \left( \nabla_a \Omega \right) d_{abc}^d - \Omega \nabla_d d_{abc}^d .
\] (17)
The once contracted second Bianchi identity in the physical spacetime,
\[
\hat{\nabla}_a \hat{C}_{abc}^d = -\hat{\nabla}_a (\hat{R}_{b[c]} - \frac{1}{6} g_{b[c]} \bar{R}),
\]
一起 with
\[
\Omega^{-1} \hat{\nabla}_a \hat{C}_{abc}^d = \nabla_a (\Omega^{-1} C_{abc}^d),
\]
and the Einstein equation in physical spacetime provide us with an equation for $d_{abc}^d$:
\[
\nabla_d d_{abc}^d = -\Omega \nabla_a \hat{T}_{b[c]} - 3 \left( \nabla_a \Omega \right) \hat{T}_{b[c]} + g_{e[a} \hat{T}_{b]d} \left( \nabla^d \Omega \right)
+ \frac{1}{3} \left( \nabla_a \Omega \right) T g_{b[c]} + \frac{1}{12} \Omega \left( \nabla_a T \right) g_{b[c] =: t_{abc}}.
\] (18)

We can now derive the missing equation for $\omega$ from the integrability condition for (16) and by substituting (17):
\[
\nabla_a \omega = -\frac{1}{2} \bar{R}_{ab} \nabla^b \Omega - \frac{1}{12} \Omega \nabla_a \Omega - \frac{1}{24} \Omega \nabla_a R + \frac{1}{2} \Omega^2 \hat{T}_{ab} \nabla^b \Omega
- \frac{1}{6} \Omega^2 \left( \nabla_a \Omega \right) T - \frac{1}{24} \Omega^3 \nabla_a T .
\] (19)
In the following in addition to the abstract indices (small Latin letters) frame (underlined indices) and coordinate indices (Greek letters) are used. The used conventions are explained in the appendix in more detail.
Using $\Omega_a := \nabla_a \Omega$, the frame $e_\alpha^a$, and the Ricci rotation coefficients $\gamma^a_{ij}$ as further
variables, we get the following first order system of tensor equations for \( \Omega, \Omega_a, \omega, e^a_i, \gamma_{ijk}, \hat{R}_{ab}, \) and \( d_{abcd} \):

\[
N^{\Omega_a} = E^{\Omega_a} = \nabla_a \Omega - \Omega_a = 0 \tag{20a}
\]

\[
N_{\Omega}^{ab} = E_{\Omega}^{ab} = \nabla_a \Omega_b + \frac{1}{2} \hat{R}_{ab} - \omega g_{ab} - \frac{1}{2} \Omega^3 \hat{T}_{ab} = 0 \tag{20b}
\]

\[
N_{\omega}^{ab} = E_{\omega}^{ab} = \nabla_a \omega + \frac{1}{2} \hat{R}_{ab} \Omega^b + \frac{1}{12} R \Omega_a + \frac{1}{24} \Omega \nabla_a R - \frac{1}{2} \Omega^2 \hat{T}_{ab} \Omega^b + \frac{1}{6} \Omega^2 \Omega_a T + \frac{1}{24} \Omega^3 \nabla_a T = 0 \tag{20c}
\]

\[
N^{e_{a\ bc}} = E^{e_{a\ bc}} = T_{a\ bc} = 0 \tag{20d}
\]

\[
N^{\gamma_{abc\ d}} = E^{\gamma_{abc\ d}} = R_{\text{diff\ abc\ d}} - R_{\text{alg\ abc\ d}} = 0 \tag{20e}
\]

\[
E^{R\ abc} = \nabla_a \hat{R}_{bc} + \frac{1}{12} (\nabla_a R) g_{bc} + \Omega d_{abc\ d} + \Omega t_{abc} = 0 \tag{20f}
\]

\[
E^{d\ abc} = \nabla_d d_{abc\ d} - t_{abc} = 0 \tag{20g}
\]

where (20d) means vanishing torsion \( T_{a\ bc} \), expressed in frame index form,

\[
T^{i\ j\ k\ l} = (e_j^i (e_k^l) - e_k^i (e_j^l)) e_m^i + \gamma^i_{\ jk} - \gamma^i_{\ kj},
\]

and (20e) means that the curvature tensor in terms of the Ricci rotation coefficients, in frame index form

\[
R_{\text{diff\ ijk\ l}} = e_j^i (\gamma_{lk}^j) - e_j^i (\gamma_{lk}^j) - \gamma_{lm}^i \gamma_{jk}^m + \gamma_{jm}^i \gamma_{lk}^m + \gamma_{lm}^j \gamma_{ik}^m + \gamma_{jm}^j \gamma_{lk}^m - \gamma_{lk}^m \gamma_{jm}^i T_{mj\ li},
\]

should equal the combination

\[
\Omega d_{abcd} + g_{c[\ a} \hat{R}_{b\ d]} - g_{d[\ a} \hat{R}_{b\ c]} + \frac{1}{6} g_{c[\ a} g_{b\ d]} R =: R_{\text{alg\ abc\ d}},
\]

which is the irreducible decomposition of a tensor with the symmetry of the Riemann tensor (7). Hence (20d) and (20e) ensure that \( R_{\text{alg\ abc\ d}} \) is the curvature tensor corresponding to the connection given by the Ricci rotation coefficients which again is the torsion free connection coming from the metric (frame).

### 2.1.2 Complications by the matter terms

The final goal is to use the terms \( \nabla_a \Omega, \nabla_a \Omega_b, \nabla_a \omega, (e_j^i (e_k^l) - e_k^i (e_j^l)) e_m^i, e_j^i (\gamma_{lk}^j) - e_j^i (\gamma_{lk}^j), \nabla_a \hat{R}_{bc}, \) and \( \nabla_d d_{abc\ d} \) in (20) as principal part for the geometry variables of the system. I will call these terms left side of the equations, the
remaining terms right side. The left side does not contain the complete principle part of the system yet as the energy momentum tensor $T_{ab}$ and its derivatives $\nabla_{[a}T_{b)c}$ may contain derivatives of the matter and geometry variables. In the case of the conformally invariant scalar field the field equation (1) remains invariant under the rescaling

$$\phi = \Omega^{-1} \tilde{\phi},$$

i.e.

$$\Box \phi - \frac{R}{6} \phi = 0.$$

The physical energy-momentum tensor $\tilde{T}_{ab}$ fulfills the assumed properties,

$$\tilde{T}_{ab} = \Omega^2 \left[ (\nabla_a \phi)(\nabla_b \phi) - \frac{1}{2} \phi \nabla_a \nabla_b \phi + \frac{1}{4} \phi^2 R_{ab} - \frac{1}{4} g_{ab} \left( (\nabla_c \phi)(\nabla_c \phi) + \frac{1}{6} \phi^2 R \right) \right] =: \Omega^2 T_{ab}.$$

The mentioned complications in (20) by the right sides are now obvious. Firstly $\nabla_{[a}T_{b)c}$ contains $\nabla_{[a} \nabla_b \nabla_c \phi$ terms which are eliminated with the identity $\nabla_{[a} \nabla_b \nabla_c \phi = \frac{1}{2} R_{abc} \nabla_d \phi$. To get rid of the second and first order derivatives of $\phi$ we use the first order system

$$\mathcal{N}_\phi a = \nabla_a \phi - \phi_a = 0 \quad (21a)$$

$$\mathcal{N}_{D\phi} ab = \nabla_a \phi_b - \hat{\phi}_{ab} - \frac{1}{4} \phi_c g_{ab} = 0 \quad (21b)$$

$$\mathcal{N}_{\Box \phi} = \phi_a^a - \frac{R}{6} \phi = 0 \quad (21c)$$

$$\mathcal{N}_{DD\phi} abc = \nabla_{[a} \hat{\phi}_{b]c} + \frac{1}{6} (\phi \nabla_a R + R \phi_a) g_{b]c} - \frac{1}{2} R_{abc} d \phi_d = 0 \quad (21d)$$

$$\mathcal{N}_{DD\Box \phi} a = \nabla_a \phi_b^b - \frac{1}{6} (\phi \nabla_a R + R \phi_a) = 0. \quad (21e)$$

for the variables $\phi$, $\phi_a$, the trace free symmetric tensor $\hat{\phi}_{ab}$ and the trace $\phi_a^a$. The system is derived from $\nabla_a \left( \Box \phi - \frac{R}{6} \phi \right) = 0$. System (21) also serves as matter part of the system for the unphysical spacetime. $t_{abc}$ is now written in a form which does not contain any derivatives of matter variables explicitly.

$\nabla_{[a}T_{b)c}$ and thus $t_{abc}$ still contain derivatives $\nabla_{[a} \hat{R}_{b)c}$ of the trace free Ricci tensor. By combining (20) and (20g) the derivatives of $\hat{R}_{ab}$ and $d_{abc} d$ can be decoupled. (20) and (20g) become

$$E'_R \grave{R}_{abc} = \nabla_{[a} \hat{R}_{b)c} + \frac{1}{12} (\nabla_{[a} R) g_{b]c} - \Omega d_{abc} d + \Omega m_{abc} = 0 \quad (22)$$

and

$$E'_{d \grave{d}_{abc}} = \nabla_d d_{abc} d - m_{abc} = 0, \quad (23)$$

9
with

\[ m_{abc} = \frac{1}{1 - \frac{1}{4} \Omega^2 \phi^2} \]

\[ \left( \frac{3}{2} \phi_{[a} \phi_{b]c} - \frac{1}{2} g_{[a} \phi_{b]d} \phi^d + \frac{1}{4} \phi \Omega d_{abc}^d \phi_d + \frac{1}{4} \phi \phi g_{[a} \hat{R}_{b]c}^d \phi_d - \frac{3}{4} \phi \phi [a \hat{R}_{b]c} \right. \]

\[ - \frac{1}{12} \phi \phi [a g_{b]c} R + \frac{1}{4} \phi \phi^2 d_{abc}^d \Omega_d \]

\[ - 3 \Omega [a \phi b_c + \frac{1}{4} \phi \phi^2 \hat{R}_{b]c} + \frac{1}{36} \phi^2 g_{b]c} R - \frac{1}{3} g_{b]c} \phi^d \phi_d \]

\[ + \Omega^d g_{[a} \phi b_d - \frac{1}{2} \phi \phi_{bd} + \frac{1}{4} \phi \phi^2 \hat{R}^b_d \right) \]

Note that \( m_{abc} \) may become singular for \( 1 - \frac{1}{4} \Omega^2 \phi^2 = 1 - \frac{1}{4} \tilde{\phi}^2 = 0 \). In the Einstein equations for the physical spacetime \( \tilde{R}_{ab} \) carries a factor \( 1 - \frac{1}{4} \tilde{\phi}^2 \) too. We will need later that

\[ N_{m_{abc}} := t_{abc} - m_{abc} = -\frac{1}{4} \Omega \phi^2 \left( N_{R_{abc}} + \frac{2}{3} \Omega m_{[a|d]d g_{b]c} \right) \]

where \( N_{R_{abc}} \) is the null quantity representing the final form of the equation for \( \hat{R}_{ab} \) \(^{(25)}\). The final form of the equation for \( d_{abc}^d \) is obtained from \((23)\) by replacing \( E'_{d_{abc}} = 0 \) with

\[ N_{d_{abc}} := E'_{d_{abc}} + \frac{2}{3} m_{[a|d]d g_{b]c} = 0 \]

(24)

This gives \( N_{d_{abc}} \) the same index symmetry properties as the Weyl tensor. That replacement does not change the equation since \( m_{ab}^b = 0 \) as will be seen later. Analogously we replace \((22)\) with

\[ N_{R_{abc}} := E'_{R_{abc}} - \frac{2}{3} \Omega m_{[a|d]d g_{b]c} = 0 \]

(25)

the contraction \( N_{R_{ab}}^b = 0 \) is then the contracted second Bianchi identity.

### 3 Evolution equations and constraints

In the following I will assume a system \( \mathcal{N} = 0 \) of the form \((20a) - (20e), (24), (25), \) the geometry part, and a matter part, in the case of the scalar field model system \((21)\). \( m_{abc} \) and \( t_{abc} \) are assumed to differ only by terms expressible as null quantities. The energy-momentum tensor \( T_{ab} \), its derivatives \( \nabla_a T_{bc} \), \( m_{abc} \), and \( t_{abc} \) are assumed to be expressed in variables and thus do not contain any explicit derivative of variables. By these assumptions the principal part of the system has block form, the geometry block and the matter block. The two blocks are
coupled through the right sides.

In this chapter the system $\mathcal{N} = 0$ will be reduced to a system of symmetric hyperbolic time evolution equations, the subsidiary system. Sufficient conditions for the equivalence of the subsidiary system and $\mathcal{N} = 0$ are given as conditions on $m_{abc}$. If the system can be put into the described block form there do not arise any more conditions from the geometry part of the system for any matter. The explicit carry out is technical and lengthy, the idea can be summarized as follows: All the equations of the system are regarded as null quantities. By requiring the vanishing of some of these null quantities and by choosing an appropriate gauge condition for the coordinates and the frame we get a symmetric hyperbolic subsidiary system of evolution equations. Which null quantities to choose can best be seen by a decomposition into the irreducible parts in the spinor calculus as performed in \cite{[17]}. The solution of this symmetric hyperbolic subsystem exists and is unique. To complete the proof we must show that the solution obtained in this way is consistent with the rest of the equations, i.e. that all null quantities remain zero if they are initially zero ("propagation of the constraints"). For that purpose a symmetric hyperbolic system of time evolution equations for the remaining null quantities is derived. Sufficient conditions for the propagation of the constraints are firstly the homogeneity of the evolution equations for the remaining null quantities in the null quantities since then the unique solution of these evolution equations is the vanishing of all null quantities for all times if they vanish on the initial surface and secondly that the domain of dependence of $S$ with respect to the equations for the propagation of the constraints is a superset of the domain of dependence of $S$ with respect to the subsidiary system.

3.1 A symmetric hyperbolic subsystem of evolution equations

Introducing a timelike vector field $t^a$ not necessarily hypersurface orthogonal and its orthogonal projection tensor $h_{ab} := g_{ab} - t_a t_b / (t_c t^c)$ allows to split the system of equations into two categories, the equations containing time derivatives and those containing no time derivatives (the constraints). The equations with time derivatives provide a under/overdetermined system of evolution equations. The system is overdetermined since for some quantities there are too many time evolution equations, e.g. there are 12 time evolution equations from $\mathcal{N}_{R\,abc} = 0$ for 9 independent tensor components. 3 equations are a linear combination of the other 9 equations and the constraints. An irreducible decomposition of the tensors $\mathcal{N}$ is a systematic way to analyze these dependencies. Since all the types of tensor index symmetries appearing in the system have been thoroughly investigated in \cite{[17]} I will only state which combinations are needed.

The system is underdetermined since there are 10 time evolution equations for
the frame and the Ricci rotation coefficients missing. By adding
\[ e^a_i g^{jb} e^j_k \left( \nabla_a e^b_k \right) = -F_i^a = \gamma^{ikb} \]  
and
\[ \partial_k \gamma^{ikj} + \gamma^{ikj} F_k^l + \gamma^{lkj} \gamma^{ikl} - \gamma^{lkj} \gamma^{lki} = F^{ij}, \]
the system becomes complete. The freedom of giving ten functions corresponds
to the freedom of giving the lapse and the shift to determine the coordinates and
the six parameters of the Lorentz group to determine the frame on every point.
The gauge freedom is discussed in full detail in [13, 17].

A choice which makes the system especially simple for analytic considerations is
a Gaussian coordinate and frame system defined as follows. Give on the spacelike
initial value surface \( S \) coordinates \( x^\mu, \mu = 1..3, \) and 3 orthonormal vector fields
\( e^a_i, i = 1..3 \) in this hypersurface. The affine parameter of the geodesics of the
hypersurface orthonormal, timelike vector field \( e_0^a \) defines the time coordinate
\( x^0 = t. \) The spacelike coordinates are transported by these geodesics into a
neighbourhood of the initial surface. By geodesic transport of \( e^a_i, i = 1..3, \)
and \( e_0^a \) a frame is obtained in this neighbourhood. By construction we have
\[ e_0^0 = 1, \quad e_0^\mu = 0 \text{ for } \mu = 1..3 \]
and
\[ \gamma^{i0k} = 0. \]

It is well known that Gaussian coordinates develop caustics if the energy momen-
tum tensor fulfills certain energy conditions, see e.g. [28, lemma 9.2.1]. In the un-
physical spacetime the \( \Omega \) terms provide a kind of unphysical energy-momentum
tensor. Whether this energy-momentum tensor fulfills the energy-momentum
conditions is a difficult question and not known to the author. Nevertheless the
coordinates develop caustics as has been shown by numerical calculations [22].

The following combinations give a symmetric hyperbolic system for the remaining
variables as can be deduced from the considerations in [12]:

\[ N_{\Omega_0} = 0, \]  
\[ N_{D\Omega_0} = 0, \]  
\[ N_{\omega_0} = 0, \]  
\[ N_{e^a_i 0} = 0, \]  
\[ N_{\gamma^{01c}d} = 0, \]  
\[ g^{ab} N_{R_{2ab}} = 0, \quad \hat{i} = 1, 2, 3, \]  
\[ N_{R_{0i0}} = 0, \quad \hat{i} = 1, 2, 3, \]  
\[ N_{R_{0i}} + N_{R_{0i}} = 0, \quad (\hat{i}, \hat{j}) = (1, 2), (1, 3), (2, 3), \]  
\[ N_{d_{212}} - N_{d_{313}} + N_{d_{202}} - N_{d_{303}} = 0, \]
\[-N_d^{102} + N_d^{121} = 0, \quad (28j)\]
\[N_d^{101} = 0, \quad (28k)\]
\[N_d^{102} + N_d^{121} = 0, \quad (28l)\]
\[-N_d^{213} + N_d^{313} + N_d^{202} - N_d^{303} = 0, \quad (28m)\]
\[N_d^{213} + N_d^{313} + N_d^{203} + N_d^{302} = 0, \quad (28n)\]
\[-N_d^{103} + N_d^{121} = 0, \quad (28o)\]
\[N_d^{123} = 0, \quad (28p)\]
\[-N_d^{103} - N_d^{121} = 0, \quad (28q)\]
\[N_d^{213} + N_d^{312} - N_d^{203} - N_d^{302} = 0, \quad (28r)\]
\[N_{\phi i}^{\bar{0}} = 0, \quad (28s)\]
\[N_{D\phi}^{\bar{0} i} = 0, \quad (28t)\]
\[g^{ab} N_{D D\phi}^{\bar{0} i} = 0, \quad (28u)\]
\[N_{D D\phi}^{\bar{0} i} = 0, \quad \bar{i} = 1, 2, 3, \quad (28v)\]
\[N_{D D\phi}^{\bar{0} i j} + N_{D D\phi}^{\bar{0} j i} = 0, \quad (\bar{i}, \bar{j}) = (1, 2), (1, 3), (2, 3), \quad (28w)\]
\[N_{D \phi}^{\bar{0} i} = 0. \quad (28x)\]

To see that the system is really symmetric hyperbolic one has to write down the system explicitly. By an appropriate, in the explicit form of the system obvious definition of new variables, the system has the structure

\[
A_i \partial_t f + \sum_{i=1}^{3} A_{xi} \partial_x^i f + b(f, x^\mu) = 0, \quad (29)
\]

with a diagonal matrix $A_i$, which is positive definite for $1 - \frac{1}{4} \Omega^2 \phi^2 > 0$, and symmetric matrices $A_{xi}$. $f$ is the vector build from the variables. All the remaining equations are linear combinations of (28) and constraints. Since the explicit form of the constraints is not needed I do not list them.

As the entries in $A_i$ coming from $N_R^{\bar{0}} = 0$ and $N_d^{\bar{0}} = 0$ vanish for $1 - \frac{1}{4} \Omega^2 \phi^2 = 0$ the following results apply only if $\Omega^2 \phi^2 < 4$ everywhere on the initial value surface $S$ and thus in a neighbourhood of $S$. The physical Einstein equations have a corresponding singularity for $1 - \frac{1}{4} \phi^2 = 0$ (see equation (14)).

### 3.2 A sufficient condition for the propagation of the constraints

According to the analysis in [17], involving the left hand side of the following identities, a symmetric hyperbolic system of evolution equations for the remaining null quantities can be extracted from:

\[
\nabla_{[a} N_{b \Omega]} = -\frac{1}{2} T_{cd} \nabla_c \Omega - N_{D \Omega}^{[ab]} \quad (30a)
\]
\[ \nabla_{[a} \mathcal{D}_b \Omega_{c]} = \]
\[ \frac{1}{2} \mathcal{N}_{\gamma [ab} d \Omega_c - \frac{1}{2} T^d_{ab} \nabla_d \Omega_c + \frac{1}{2} \Omega \mathcal{N}_R_{abc} - \frac{1}{2} \mathcal{R}_{[c]} \nabla_{a} \Omega_d - \frac{1}{2} \Omega^2 \mathcal{N}_b \Omega_{[a} \nabla_{b]} - \frac{3}{2} \Omega^2 \mathcal{N}_a \nabla_{b}] \]
\[ - \frac{1}{2} \mathcal{N}_\omega [ag]c + \frac{1}{3} \Omega^2 m_{[a]d} g_{b]c} + \frac{1}{3} \Omega^2 \mathcal{N}_m [a]d g_{b]c} \]  
(30b)

\[ \nabla_{[a} \mathcal{N}_\omega b] = \]
\[ - \frac{1}{2} T^c_{ab} \nabla_c \omega - \frac{1}{24} \Omega T^c_{ab} \nabla_c T + \frac{1}{24} \Omega \mathcal{N}_a \nabla_b R + \frac{1}{8} \Omega^2 \mathcal{N}_b \nabla_a T \]
\[ + \frac{1}{2} (\mathcal{R}_{[c]} - \Omega^2 \mathcal{T}_{[c]} \mathcal{D}_b \Omega) ^c + (\frac{1}{24} R + \frac{1}{6} \Omega^2 T) \mathcal{N}_{D \Omega} [a b] + \frac{1}{2} \Omega^2 \mathcal{N}_R_{abc} \]
\[ + \frac{1}{3} \Omega \Omega [m]c + \frac{1}{2} \Omega^2 \mathcal{N}_m abc \]  
(30c)

\[ \nabla_{[a} \mathcal{N}_R_{bc}]d = \]
\[ \frac{1}{2} \mathcal{N}_\gamma [ab] d \mathcal{D}_f - \frac{1}{2} T^f_{ab} \nabla_f \mathcal{D}_d - \frac{1}{2} \Omega \mathcal{N}_a \nabla_b d \mathcal{D}_f - \frac{1}{2} \Omega^2 \mathcal{N}_b \nabla_a d \mathcal{D}_f \]
\[ + \mathcal{N}_a \Omega [a]d \mathcal{D}_f + \mathcal{D}_f \mathcal{N}_a [a]d \mathcal{D}_f + \frac{1}{12} \Omega \mathcal{N}_a \nabla_b d \mathcal{D}_f \]
\[ + \frac{1}{12} \Omega T^f_{ab} \nabla_f g_{cd} + 2 \Omega \mathcal{N}_a \nabla m_{bc]d} + \mathcal{D}_f \mathcal{N}_m [a]d \mathcal{D}_f \]
\[ - \Omega \nabla_{[a} \mathcal{N}_m bc]d + 2 \Omega \mathcal{N}_a \nabla m_{bc]d} - \frac{2}{3} \Omega \nabla_{[a} \nabla m_{bc]d} \]  
(30d)

\[ \nabla^c \mathcal{N}_d_{abc} = \]
\[ \frac{1}{2} \mathcal{N}_\gamma d_{[a} f d b c} d + \frac{1}{2} \mathcal{N}_\gamma d_{a f} d b c d + \frac{1}{2} \mathcal{N}_\gamma d_{c f} d a b c d + \frac{1}{2} \mathcal{N}_\gamma d_{d f} d a b c f \]
\[ - \frac{1}{2} T^f_{abcd} \nabla_d m_{abc} + \frac{2}{3} \nabla [m]c \]  
(30e)

\[ \nabla_{[a} \mathcal{N}_e d_{bc]} = \mathcal{N}_\gamma [abc] d + \mathcal{N}_e [a]d \mathcal{N}_e d_{bc]f} \]  
(30f)

\[ \nabla_{[a} \mathcal{N}_e d_{bc]} = \mathcal{N}_\gamma [abc] d + \mathcal{N}_e [a]d \mathcal{N}_e d_{bc]f} \]  
(30f)
\[ \nabla_{[\alpha} \mathcal{N}_{\phi \beta]} = -\frac{1}{2} T^{c}_{ab} \nabla_{c} \phi - \mathcal{N}_{D\phi \ [ab]} \]  

(30h)

\[ \nabla_{[\alpha} \mathcal{N}_{D\phi \beta]c} = \frac{1}{2} \gamma_{abc} \phi_{d} - \frac{1}{2} T^{d}_{ab} \nabla_{d} \phi_{c} - \mathcal{N}_{DD\phi \ abc} - \frac{1}{4} \mathcal{N}_{D\square\phi \ [ab]g]c} \]  

(30i)

\[ \nabla_{a} \mathcal{N}_{\square\phi} = \mathcal{N}_{D\square\phi \ a} - \frac{1}{6} R \mathcal{N}_{\phi \ a} \]  

(30j)

\[ \nabla_{[\alpha} \mathcal{N}_{DD\phi \ bc]d} = \frac{1}{2} \gamma_{abc} \phi_{d} + \frac{1}{2} \gamma_{[ab]d} \phi_{c} - \frac{1}{2} T^{f}_{[ab} \nabla_{f]} \phi_{c]d} + \frac{1}{6} \mathcal{N}_{\phi \ [a} (\nabla_{b} R) g_{c]d} \]  

- \frac{1}{12} \phi T^{f}_{[abg]c]d} \nabla_{f]} R + \frac{1}{6} R \mathcal{N}_{D\phi \ [ab]g]c]d} + \frac{1}{2} T^{g}_{[ab} R \text{diff}_{c]g]d} \phi_{f] - \frac{1}{2} R_{[bc]d} \phi_{f] \mathcal{N}_{D\phi \ a]}f} \]  

+ \frac{1}{2} (\nabla_{[a} \gamma_{bc]d} \phi_{f] \mathcal{N}_{D\phi \ bc]d}) \phi_{f], \quad (30k) \]  

\[ \nabla_{[\alpha} \mathcal{N}_{D\square\phi \ bc]d} = \frac{1}{2} T^{d}_{ab} \nabla_{d} \phi_{c]d} - \frac{1}{6} \mathcal{N}_{\phi \ [a} \nabla_{b]} R - \frac{1}{12} \phi T^{c}_{ab} \nabla_{c} R - \frac{1}{6} R \mathcal{N}_{D\phi \ [ab]} \]  

(30l)

The last term in (30k) is homogeneous in null quantities as can be seen from (30g). The deviation of these equalities is even more lengthy than the equalities itself, but the essential ideas behind it can already be seen in the deviation of the first:

\[ \nabla_{[\alpha} \Omega_{\beta]} = \nabla_{[\alpha} \Omega_{\beta]} \]  

- \frac{1}{2} T^{d}_{ab} \nabla_{d} \phi_{c]d} - \frac{1}{6} \mathcal{N}_{\phi \ [a} \nabla_{b]} R = \frac{1}{2} T^{c}_{ab} \nabla_{c} R - \frac{1}{6} R \mathcal{N}_{D\phi \ [ab]} \]  

(31a)

\[ \nabla^{c} \mathcal{N}_{m ab} + \frac{2}{3} \nabla_{[a} \mathcal{N}_{m b],c}^{c} = 0 \mod \mathcal{N} \]  

(31b)

\[ \nabla_{[a} \mathcal{N}_{m b],c}^{c} = 0 \mod \mathcal{N} \]  

(31c)

and

\[ \Omega \nabla_{[a} \mathcal{N}_{m bc],d} = f \nabla_{[a} \mathcal{N}_{R bc],d} \mod \mathcal{N}, \quad f \neq -1. \]  

(31d)
A straightforward but long calculation shows that these conditions are fulfilled by the conformally invariant scalar field with \( f = -\frac{1}{4} \Omega^2 \phi^2 \). Equation (30d) becomes singular for \( 1 - \frac{1}{4} \Omega^2 \phi^2 = 0 \).

The very technical integrability conditions (31) have a very simple interpretation. Replacing \( m_{abc} \) with \( t_{abc} \) — they only differ by null quantities — the conditions (31a–31c) reduce to \( \tilde{\nabla}_b \tilde{T}_{ab} = 0 \) and \( \tilde{\nabla}_b \tilde{\nabla}^c \tilde{T}_{ac} = 0 \). Condition (31d) is only of technical nature, it gives the principal part a simple block form.

From the considerations in [17] also follows that the domain of dependence of \( S \) with respect to the evolution equation of the constraints includes the domain of dependence of \( S \) with respect to the subsidiary system.

4 The hyperboloidal initial value problem

So far a system of equations (\( \mathcal{N} = 0 \)) has been derived which contains for at least one choice of gauge a symmetric hyperbolic subsystem of evolution equations. The remaining equations in the system — either constraints or a combination of constraints and time evolution equations — will be satisfied for a solution of the evolution equations, if the constraints are satisfied by the initial data. If both, the time evolution and the constraints, are fulfilled, \( (\tilde{M}, \tilde{g}_{ab}, \tilde{\phi}) \) is a weakly asymptotically flat solution of the Einstein equation. This follows from the way the system \( \mathcal{N} = 0 \) for the unphysical spacetime has been derived.

The essential points in the proofs of the theorems in [17, chapter 10] are the symmetric hyperbolicity of the subsidiary system and the form (20b) of the equations for \( \Omega \). Therefore the same techniques can be used and the proofs will not be repeated. The difference to the model treated here lies in the derivation of the subsidiary system and the proof of the propagation of the constraints, which has been done in the previous chapters.

4.1 The initial value problem

We consider the following initial value problem:

Definition 3 A “hyperboloidal initial data set for the conformally invariant scalar field” consists of a pair \((\bar{S}, f_0)\) such that:

1. \( \bar{S} = S \cap \partial S \) is a smooth manifold with boundary \( \partial S \) diffeomorphic to the closed unit ball in \( \mathbb{R}^3 \). As coordinates on \( S \) the pull backs of the natural coordinates on \( \mathbb{R}^3 \) are used.

2. \( f_0 \) is the vector \( f \) of functions in system (28) written in the form (29) at initial time \( t_0 \).
3. The fields provided by $f_0$ have uniformly continuous derivatives with respect to the coordinates of $S$ to all orders.\footnote{The assumption about the smoothness of the data can certainly be weakened from $C^\infty$ to $C^n$ for sufficiently large $n$ but then more technical effort would be needed in the proofs.}

4. On $S$: $\Omega > 0$. On $\partial S$: $\Omega = 0$ and $\nabla_a \Omega$ is a future directed null vector.

5. The fields provided by $f_0$ satisfy the constraints following from $\mathcal{N} = 0$ \cite{22} and \cite{23} and the gauge conditions.

A point which deserves special notice is the existence of a hyperboloidal initial data set. The proof that those data exist has to overcome two problems. Firstly, the regularity of the solution on $\partial S$ which is the consistency of the data with asymptotical flatness. For scalar field data with compact support regularity conditions are given in \cite{2, 3}, which are sufficient for the existence of a solution of the constraints near $\partial S$.

Secondly there is a problem with a possible singularity of equations in $\mathcal{N} = 0$ at $1 - \frac{1}{4} \Omega^2 \phi^2 = 0$.

L. Anderson and P. Chrusciel are preparing a paper analyzing both problems \cite{2}.

4.2 Theorems

The “theorems” will be given in a form not containing every technical detail, since these technical details would make them lengthy and can be easily deduced from the theorems in \cite{17} by replacing the Yang-Mills matter with the (conformally invariant) scalar field.

Since the constraints of $\mathcal{N} = 0$ propagate we have:

**Theorem 2** Any (sufficiently smooth) solution of the subsidiary system satisfying the constraints on a spacelike hypersurface $\bar{S}$ and $1 - \frac{1}{4} \Omega^2 \phi^2 > 0$ defines in the domain of dependence with respect to $g_{ab}$ of $\bar{S}$ a solution to the unphysical system. Thus $(\tilde{M}, \tilde{g}_{ab}, \tilde{\phi})$ is a weakly asymptotically flat solution of the Einstein equation.

Since the evolution equations are symmetric hyperbolic a unique solution of the initial value problem exists for a finite time. From the combination with theorem (2) follows:

**Corollary 3** For every regular solution of the constraints on $\bar{S}$ with $1 - \frac{1}{4} \Omega^2 \phi^2 |_{\bar{S}} > 0$ exists locally a unique, weakly asymptotically flat solution of the Einstein equation.

For the Minkowski space we can extent $\bar{S}$ and the solution of the constraints beyond $\partial S$ to $S'$ and get a solution in the unphysical spacetime which extents beyond $i^+$. The continuous dependence of the solution of symmetric hyperbolic
systems on the data and the form of (20a), (20b) and (20c) (see the proof of theorem (10.2) in [17]) guarantees that there is a solution covering the whole domain of dependence of $\bar{S}$. Furthermore the proof there shows that $\{ p \mid \Omega(p) = 0 \}$ has an isolated critical point $i^+$, where all future directed timelike geodesics of $(\bar{M}, \bar{g}_{ab})$ end, thus:

**Theorem 4** For a sufficient small deviation of the data from Minkowskian data the solution of theorem 3 possesses a regular future null infinity and a regular future timelike infinity.

5 The conformal equivalence of the scalar fields

This section shortly reviews the equivalence transformation between spacetime models with scalar matter under the viewpoint of solving hyperboloidal initial value problems. Other aspects of this equivalence transformation, especially the generation of exact solutions, have been studied in [1, 5, 6, 23, 24, 25, 29].

5.1 Local equivalence of solutions

Spacetime models $(\bar{M}, \bar{g}_{ab}, \bar{\phi})$ with scalar matter $\bar{\phi}$ described by the action

$$\tilde{S} = \int_{\bar{M}} \left[ A(\bar{\phi}) R - B(\bar{\phi}) (\bar{\nabla}_a \bar{\phi}) (\bar{\nabla}^a \bar{\phi}) \right] \left( -\bar{g} \right)^{\frac{1}{2}} d^4 \bar{x}$$

(32)

will be considered. Boundary terms in the action have been omitted, $\bar{g}$ is the determinant of $\bar{g}_{\mu\nu}$.

By varying the action $\tilde{S}$ with respect to $\bar{\phi}$ and $\bar{g}_{ab}$ the following field equations result:

$$B(\bar{\phi}) \Box \bar{\phi} + \frac{1}{2} \frac{dB}{d\bar{\phi}} \left( \bar{\nabla}^a \bar{\phi} \right) \left( \bar{\nabla}_a \bar{\phi} \right) + \frac{1}{2} \frac{dA}{d\bar{\phi}} \bar{R} = 0$$

(33a)

$$A(\bar{\phi}) \left( \bar{R}_{ab} - \frac{1}{2} \bar{R} \bar{g}_{ab} \right) + B(\bar{\phi}) \left( \frac{1}{2} \left( \bar{\nabla}^c \bar{\phi} \right) \left( \bar{\nabla}_c \bar{\phi} \right) \bar{g}_{ab} - \left( \bar{\nabla}_a \bar{\phi} \right) \left( \bar{\nabla}_b \bar{\phi} \right) \right) - \left( \bar{\nabla}_a \bar{\nabla}_b A(\bar{\phi}) \right) + \left( \bar{\nabla}^c \bar{\nabla}_c A(\bar{\phi}) \right) \bar{g}_{ab} = 0.$$  

(33b)

$A$ and $B$ are assumed to be $C^\infty$ functions. For $B \neq 0$ the principal part of (33a) does not vanish and thus (33a) is a wave equation. For that reason I assume $B(\bar{\phi}) > \epsilon > 0$ for every $\bar{\phi}$. (33b) is a second order equation for the metric if $A(\bar{\phi}) \neq 0$.

In the spacetime region $\bar{H} := \{ x \in \bar{M} \mid \text{sign}(A(\bar{\Phi})) > 0 \ \forall \bar{\Phi} \in [\bar{\phi}_0, \bar{\phi}(x)] \}$ the
transformation

\[ \tilde{\phi} = \int_{\tilde{\phi}_0}^{\tilde{\phi}} \frac{1}{A} \sqrt{\frac{3}{2} \left( \frac{dA}{d\tilde{\phi}} \right)^2 + A B} \, d\tilde{\phi} \quad (34a) \]

\[ \tilde{g}_{ab} = A \tilde{g}_{ab} \quad (34b) \]

gives a solution of the system (33) with a massless Klein-Gordon field \( \tilde{\phi} \) as matter model corresponding to the choice \( (A, B) = (1, 1) \) and the equations

\[ \Box \tilde{\phi} = 0 \quad (35a) \]

\[ \tilde{R}_{ab} - \frac{1}{2} \tilde{R} \tilde{g}_{ab} = \tilde{T}_{ab}[\tilde{\phi}] \quad (35b) \]

with energy momentum tensor

\[ \tilde{T}_{ab}[\tilde{\phi}] = (\tilde{\nabla}_a \tilde{\phi})(\tilde{\nabla}_b \tilde{\phi}) - \frac{1}{2} (\tilde{\nabla}_c \tilde{\phi})(\tilde{\nabla}^c \tilde{\phi}) \tilde{g}_{ab}. \quad (35c) \]

From the assumptions about \( A \) and \( B \) follows that the corresponding Klein-Gordon field will be unbounded approaching the part of the boundary of \( \tilde{H} \) where \( A(\tilde{\phi}) \to 0 \). The singularity in the Klein-Gordon field shows up at least in a singularity of the equations for \( (\tilde{M}, \tilde{g}_{ab}, \tilde{\phi}) \).

For two of the scalar fields in the above class the field equations are very special, the already mention massless Klein-Gordon field \( \tilde{\phi} \) (33) and the conformally invariant scalar field \( \tilde{\phi}, (A, B) = (1 - \frac{1}{4} \phi^2, \frac{3}{2}) \) (\( \tilde{\phi} \) can be rescaled by an arbitrary factor).

The first, because the set of equations in the physical spacetime becomes remarkable simple and has been analyzed intensely with analytical (e.g. [9]) and numerical (e.g. [7]) methods for spacetimes with spherical symmetry.

The second, yielding the equations (1), because the matter equations are invariant under rescalings \( g_{ab} = \Omega^2 \tilde{g}_{ab} \) and \( \phi = \Omega^{-1} \tilde{\phi} \).

The transformation between the two special cases is

\[ \tilde{\phi} = \sqrt{6} \arctanh \frac{\tilde{\phi}}{2} \quad (36a) \]

\[ \tilde{g}_{ab} = (1 - \frac{1}{4} \tilde{\phi}^2) \tilde{g}_{ab} \quad (36b) \]

which is a bijective mapping from \( \tilde{\phi} \in ] - 2, 2[ \) to \( \tilde{\phi} \in ] - \infty, \infty[ \).

Due to the following diagram, illustrating the above described relations, it is evident that there is a variable transformation regularizing the unphysical equations for the Klein-Gordon field:

\[ \text{The choice of the parameter } \tilde{\phi}_0 \text{ reflects gauge freedom. Models where there is no } \tilde{\phi}_0 \text{ with } A(\tilde{\phi}_0) > 0 \text{ will not be considered.} \]
By mapping an arbitrary scalar field $\tilde{\phi}$ with action (32) to the Klein-Gordon field $\tilde{\phi}$ and then to the conformally invariant scalar field $\check{\phi}$ regular equations for $\bar{\phi}$ are obtained.

### 5.2 The hyperboloidal initial value problem

Since $A(\phi) > 0$ on $\tilde{S}$ all scalar field models connected by transformation (34) to a hyperboloidal initial value problem with a conformally invariant scalar field as matter source are weakly asymptotically flat.

For a massless KG model there is a one parameter gauge freedom in the scalar field. If $\check{\phi}$ is a solution, then so is $\check{\phi} + \check{\phi}_0$ with $\check{\phi}_0 = \text{const}$, as the energy momentum tensor depends on derivatives of $\check{\phi}$ only. This can also be seen by mapping a Klein-Gordon model to a Klein-Gordon model with $\check{\phi}_0 \neq 0$ and (34). The analogue holds for every considered scalar field model. For the hyperboloidal initial value problem $\phi = \check{\phi}/\Omega$, therefore $\phi$ vanishes at $J$, fixing the gauge in $\check{\phi}$.

In definition (3) $1 - \frac{1}{4} \Omega^2 \phi^2 |_S > 0$ was assumed. But with the Bekenstein black hole [6] a weakly asymptotically flat solution is known where $A(\check{\phi})$ vanishes on a regular part of the spacetime. In this case the transformation gives a possible extension of a massless Klein-Gordon scalar field solution beyond a singularity – the Klein-Gordon field $\check{\phi}$ and the metric $\bar{g}_{ab}$ degenerate there. It is a pleasure for me to thank Helmut Friedrich, Bernd Schmidt, and Jürgen Ehlers for the very helpful discussions during the grow of this work which is part of my Ph. D. thesis.
A Notation

The signature of the Lorentzian metric $g_{ab}$ is $(−, +, +, +)$. Whenever possible I use abstract indices as described in [27, chapter 2]. Small Latin letters denote abstract indices, underlined small Latin letters are frame indices. For the components of a tensor with respect to coordinates small Greek letters are used. The frame $\left( \frac{\partial}{\partial x^\mu} \right)_a$ is constructed from the coordinates $x^\mu$, $e_i^a$ denotes an arbitrary frame. In this notation $v_a$ is a covector, $v_i$ a scalar, namely $v_a e_i^a$.

$v(f)$ is defined to be the action of the vector $v^a$ on the function $f$, i.e. for every covariant derivative $\nabla_a$: $t(f) = t^a \nabla_a f$.

The transformation between abstract, coordinate, and frame indices is done by contracting with $e_i^a$ and $e^a_\mu$. All indices may be raised and lowered with the metric $g_{AB}$ and the inverse $g^{AB}$. $g^{AC} g_{CB} = \delta^A_B$, $A$ and $B$ are arbitrary indices, e.g. $e_a^a = g_{ab} e_b^b$ and $e_a^\mu = g^{\mu i} e_i^a$.

For a frame $e_i^a$ and a covariant derivative $\nabla_a$ the Ricci rotation coefficients are defined as

$$\gamma_i^a_j := e_i^b \nabla_b e_j^a.$$  

From this definition follows

$$e_i^a e_j^b (\nabla_a t^b) = e_i^a (t^j) + \gamma_j^i k \t^k.$$  

With respect to a coordinate frame $e_\mu^a \equiv \left( \frac{\partial}{\partial x^\mu} \right)^a$ the components $\gamma^a_{\mu \nu}$ are the Christoffel symbols $\Gamma^a_{\mu \nu}$.

The torsion $T_{a bc}$ is defined by

$$\nabla_a \nabla_b f - \nabla_b \nabla_a f = -T_{c ab} \nabla_c f,$$

the Riemann tensor $R_{abc}^d$ by

$$\nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c = R_{abc}^d \omega_d - T_{ab}^d \nabla_d \omega_c.$$  

Contraction gives the Ricci tensor,

$$R_{ab} = R_{acb}^c,$$

and the Ricci scalar

$$R = R_{ab} g^{ab}.$$  

The Einstein tensor is given by

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}.$$  

The speed of light $c$ is set to 1 as the gravitational constant $\kappa$ in $G_{ab} = \kappa T_{ab}$.
References

[1] A. J. Accioly, U. F. Wichoski, S. F. Kwok, and N. L. P. Pereira da Silva. Classical equivalence of $\lambda r\phi^2$ theories. Class. Quantum Grav., 10:L215–L219, 1993.

[2] L. Andersson and P. T. Chrusciel, in preparation.

[3] L. Andersson, P. T. Chrusciel, and H. Friedrich. On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einstein’s field equations. Comm. Math. Phys., 149:587, 1992.

[4] A. Ashtekar and B. G. Schmidt. Null infinity and Killing fields. J. Math. Phys., 21(4):862–867, 1980.

[5] J. D. Bekenstein. Exact solutions of Einstein-conformal scalar equations. Ann. Phys., 82:535–547, 1974.

[6] J. D. Bekenstein. Black holes with scalar charge. Ann. Phys, 91:75–82, 1975.

[7] M. W. Choptuik. “Critical” behaviour in massless scalar field collapse. In Ray d’Inverno, editor, Approaches to Numerical Relativity, pages 202–222. Proceedings of the International Workshop on Numerical Relativity, Cambridge University Press, 1992.

[8] M. W. Choptuik. Universality and scaling in gravitational collapse of a massless scalar field. Phys. Rev. Lett., 70(1):9–12, January 1993.

[9] D. Christodoulou. The formation of black holes and singularities in spherically symmetric gravitational collapse. Communications on Pure and Applied Mathematics, XLIV:339–373, 1991.

[10] D. Christodoulou and S. Klainerman. The Global Nonlinear Stability of the Minkowski Space. Princeton University Press, 1993.

[11] H. Friedrich. On the regular and the asymptotic characteristic initial value problem for Einstein’s vacuum field equations. Proc. R. Soc. London, A 375:169–184, 1981.

[12] H. Friedrich. Cauchy problems for the conformal vacuum field equations in general relativity. Commun. Math. Phys., 91:445–472, 1983.

[13] H. Friedrich. On the hyperbolicity of Einstein’s and other gauge field equations. Commun. Math. Phys., 100:445–472, 1985.

[14] H. Friedrich. On purely radiative space-times. Commun. Math. Phys., 103:35–65, 1986.
[15] H. Friedrich. On the existence of n-geodesically complete or future complete solutions of Einstein’s field equations with smooth asymptotic structure. *Commun. Math. Phys.*, 107:587–609, 1986.

[16] H. Friedrich. On static and radiative space-times. *Commun. Math. Phys.*, 119:51–73, 1988.

[17] H. Friedrich. On the global existence and the asymptotic behavior of solutions to the Einstein-Maxwell-Yang-Mills equations. *J. Differential Geometry*, 34:275–345, 1991.

[18] H. Friedrich. Asymptotic structure of space-time. In *Recent Advances in General Relativity: Essays in Honour of E. T. Newman*. Birkhäuser Inc., 1993.

[19] R. Geroch. Asymptotic structure of space-time. In F. R. Esposito and L. Witten, editors, *Asymptotic Structure of Space-Time*, pages 1–105. Plenum Press, 1976.

[20] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1973.

[21] P. Hübner, in preparation.

[22] P. Hübner. *Numerische und analytische Untersuchungen von (singulären, asymptotisch flachen Raumzeiten mit konformen Techniken*. PhD thesis, Ludwig-Maximilians-Universität München, 1993.

[23] C. Klimčík. Search for the conformal scalar hair at arbitrary d. *J. Math. Phys.*, 34(5):1914–1926, 1993.

[24] C. Klimčík and P. Kolnýk. Interacting Einstein-conformal scalar waves. *Phys. Rev. D*, 48(2):616–621, July 1993.

[25] D. N. Page. Minisuperspaces with conformally and minimally coupled scalar fields. *J. Math. Phys.*, 32(12), 1991.

[26] R. Penrose. Conformal treatment of infinity. In C. Dewitt and B. DeWitt, editors, *Relativity, Groups and Topology*, pages 565–584. Gordon and Breach, 1964.

[27] R. Penrose and W. Rindler. *Spinors and space-time*, volume I, II. Cambridge University Press, 1984.

[28] R. M. Wald. *General Relativity*. University of Chicago Press, 1984.
[29] B. C. Xanthopoulos and T. E. Dialynas. Einstein gravity coupled to a massless conformal scalar field in arbitrary space-time dimensions. *J. Math. Phys.*, 33(4):1463–1471, 1992.