FIELDS OF MODULI AND FIELDS OF DEFINITION
OF ODD SIGNATURE CURVES

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Abstract. Let $X$ be a smooth projective curve of genus $g \geq 2$ defined over a field $K$. We show that $X$ can be defined over its field of moduli $K_X$ if the signature of the covering $X \to X/\text{Aut}(X)$ is of type $(0; c_1, \ldots, c_k)$, where some $c_i$ appears an odd number of times. This result is applied to $q$-gonal curves and to plane quartics.

Introduction

Let $X$ be a smooth projective curve of genus $g$ defined over a field $K$ and let $K_X$ be its field of moduli (see Section 1, Definition 1.1). It is well known that $X$ can be defined over $K_X$ if either $g = 0, 1$ or the automorphism group of $X$ is trivial. However, there are examples of curves which cannot be defined over $K_X$, as first observed by Earle and Shimura in [5, 14]. In [10] B. Huggins studied this problem for hyperelliptic curves in characteristic $p \neq 2$, proving that a hyperelliptic curve $X$ of genus $g \geq 2$ with hyperelliptic involution $\iota$ can be defined over $K_X$ provided that $\text{Aut}(X)/\langle \iota \rangle$ is not cyclic or is cyclic of order divisible by $p$.

The first examples of non-hyperelliptic curves not definable over their field of moduli have been given in [10] and [7].

Recently R. Hidalgo [6] considered complex curves $X$ such that the natural covering $\pi_X : X \to X/\text{Aut}(X)$ has signature of the form $(0; a, b, c, d)$, proving that $X$ can be defined over its field of moduli if $d \notin \{a, b, c\}$. In this paper we observe that such result can be extended to odd signature curves, i.e. curves such that the signature of $\pi_X$ is of the form $(0; c_1, \ldots, c_r)$ where some $c_i$ appears exactly an odd number of times. More precisely, we prove the following result, which is a consequence of [4, Theorem 3.1].

Theorem 0.1. Let $X$ be a smooth projective curve of genus $g \geq 2$ defined over a field $K$. If $X$ is an odd signature curve, then $K_X$ is a field of definition for $X$.

This result implies that non-normal $q$-gonal curves can be defined over their field of moduli and that plane quartics can be defined over their field of moduli if $|\text{Aut}(X)| > 4$. In the last section of the paper we construct examples of plane quartics with $\text{Aut}(X) \cong C_2$ which cannot be defined over their field of moduli and we prove that, in case $\text{Aut}(X) \cong C_2 \times C_2$, the field of moduli relative to the extension $\mathbb{C}/\mathbb{R}$ is always a field of definition. This implies the following.

Theorem 0.2. Let $X$ be a smooth plane quartic over $\mathbb{C}$ which is isomorphic to its conjugate. If $\text{Aut}(X)$ is not cyclic of order two, then $X$ can be defined over $\mathbb{R}$.

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1. Preliminaries

Let $X$ be a smooth projective curve defined over a field $K$. A subfield $N$ of $K$ is a field of definition of $X$ if there exists a curve $X'$ defined over $N$ such that $X'$ is isomorphic to $X$ over $K$. Moreover, we say that $X$ is definable over $N$ if there exists a curve $X'$ defined over $N$ such that $X'$ is isomorphic to $X$ over $\bar{K}$.

Definition 1.1. Let $K$ be a field, $\bar{K}$ be an algebraic closure of $K$ and $X$ be a curve defined over $K$. The field of moduli $K_X$ of $X$ is the intersection of all fields of definition of $X$, seen as a curve over $\bar{K}$.

Another definition for the field of moduli, relative to a given field extension $F/L$, is given as follows. If $P \in F[x_0, \cdots, x_n]$ and $\sigma \in \text{Aut}(F/L)$, then $P^\sigma$ denotes the polynomial obtained by applying $\sigma$ to the coefficients of $P$. If a curve $X$ is defined as the zero locus of the homogeneous polynomials $P_1, \cdots, P_s \in F[x_0, \cdots, x_n]$, then the polynomials $P_1^\sigma, \cdots, P_s^\sigma$ define a new smooth projective curve $X^\sigma$.

Definition 1.2. The field of moduli of $X$ relative to the extension $F/L$, denoted by $M_{F/L}(X)$, is the fixed field of the group

$$U_{F/L}(X) := \{ \sigma \in \text{Aut}(F/L) : X \text{ is isomorphic to } X^\sigma \text{ over } F \}.$$ 

Let $P$ be the prime field of $K$. By a theorem of Koizumi (see [11] and [9, Theorem 1.5.8]) the field of moduli $M_{K/P}(X)$ is a purely inseparable extension of the field of moduli $K_X$. In particular these two fields coincide if $K$ is a perfect field. For example, if $K = \mathbb{C}$, then $K_X = M_{\mathbb{C}/\mathbb{Q}}(X)$. The relationship between $K_X$ and the fields of moduli of $X$ relative to Galois extensions is given by the following result (see [9, Theorem 1.6.8]).

Theorem 1.3. Let $X$ be a smooth projective algebraic curve defined over a field $K$ and $K_X$ be the field of moduli of $X$. Then $X$ is definable over $K_X$ if and only if given any algebraically closed field $F \supseteq K$, and any subfield $L \subseteq F$ with $F/L$ Galois, $X$ (seen as a curve over $F$) can be defined over the field $M_{F/L}(X)$.

Given a smooth projective algebraic curve $Y$ defined over $L$, a branched covering $\phi : X \to Y$ defined over $F$ and $\sigma \in \text{Aut}(F/L)$, we denote by $\phi^\sigma : X^\sigma \to Y^\sigma$ the branched covering obtained by applying $\sigma$ to the defining polynomials of $\phi$.

Assume now that a curve $L$ is a field of definition of a curve $X$ over $F$, i.e. there exists an isomorphism $g : X \to Y$, where $Y$ is a curve defined over $L$. If $\sigma \in \text{Aut}(F/L)$, then $f_\sigma := (g^\sigma)^{-1} \circ g : X \to X^\sigma$ is an isomorphism (observe that $Y = Y^\sigma$) and $f_{\tau \sigma} = f^\tau_\sigma \circ f_\sigma$ holds for all $\sigma, \tau \in \text{Aut}(F/L)$. The following theorem by A. Weil shows that the latter condition is also sufficient for the field $L$ to be a field of definition for $X$.

Theorem 1.4 (Weil [15]). Let $X$ be a smooth projective algebraic curve defined over a field $F$ and let $F/L$ be a Galois extension. If for every $\sigma \in \text{Aut}(F/L)$ there is an isomorphism $f_\sigma : X \to X^\sigma$ defined over $F$ such that the compatibility condition $f_{\tau \sigma} = f^\tau_\sigma \circ f_\sigma$ holds for all $\sigma, \tau \in \text{Aut}(F/L)$, then there exist a smooth projective algebraic curve $Y$ defined over $L$ and an isomorphism $g : X \to Y$ defined over $F$ such that $g^\sigma \circ f_\sigma = g$. 
The following result by Dèbes-Emsalem is a consequence of Weil’s theorem and provides a sufficient condition for the curve \( X \) to be defined over the field \( M_{F/L}(X) \) (see [3, §2.4] for the definition of field of moduli of a covering).

**Theorem 1.5** (Dèbes-Emsalem [4]). Let \( F/L \) be a Galois extension and \( X \) be a smooth projective curve of genus \( g \geq 2 \) defined over \( F \) with \( L := M_{F/L}(X) \). Then there exist a smooth projective curve \( B \) defined over \( L \) and a Galois branched covering \( \phi : X \to B \) defined over \( F \), with \( \text{Aut}(X) \) as its deck group, so that \( M_{F/L}(\phi) = L \). Moreover, if \( B \) contains at least one \( L \)-rational point outside of the branch locus of \( \phi \), then \( L \) is also a field of definition of \( X \).

**Remark 1.6.** The condition \( L := M_{F/L}(X) \) in Theorem 1.5 is not restrictive since by [4, Proposition 2.1] the field of moduli relative to the extension \( F/M_{F/L}(X) \) is \( M_{F/L}(X) \).

2. **Proof of the theorem**

Let \( \phi : X \to X/G \) be a branched Galois covering between smooth projective curves and let \( q_1, \ldots, q_r \) be its branch points. The *signature* of \( \phi \) is defined as \((g_0; c_1, \ldots, c_r)\), where \( g_0 \) is the genus of \( X/G \) and \( c_i \) is the ramification index of any point in \( \phi^{-1}(q_i) \). The *branch divisor* of \( \phi \), denoted by \( D(\phi) \), is the divisor of \( X/G \) defined by \( D(\phi) = \sum_{i=1}^{r} c_i q_i \).

**Definition 2.1.** A smooth projective curve \( X \) of genus \( g \geq 2 \) has odd signature if the signature of the covering \( \pi_X : X \to X/\text{Aut}(X) \) is of the form \((0; c_1, \ldots, c_r)\) where some \( c_i \) appears exactly an odd number of times.

**Definition 2.2.** Let \( B \) be a smooth projective curve defined over a field \( L \). A divisor \( D = p_1 + \cdots + p_r \) of \( B \) is called \( L \)-rational if for each \( \sigma \in \text{Aut}(L/L) \) we have that \( D^\sigma := \sigma(p_1) + \cdots + \sigma(p_r) = D \).

The following is an easy consequence of Riemann-Roch theorem and the fact that a curve of genus zero with a \( L \)-rational point is isomorphic to \( \mathbb{P}^1(L) \) (see also [9, Lemma 4.0.4]).

**Lemma 2.3.** Let \( B \) be a smooth projective curve of genus 0 defined over an infinite field \( L \) and suppose that \( B \) has an \( L \)-rational divisor \( D \) of odd degree. Then \( B \) has infinitely many \( L \)-rational points.

**Lemma 2.4.** Given a Galois branched covering \( \phi : X \to X/G \) as before defined over \( F \), we have \( D(\phi^\sigma) = D(\phi)^\sigma \) for any \( \sigma \in \text{Aut}(F/L) \).

**Proof.** Observe that \( \sigma \circ \phi = \phi^\sigma \circ \sigma \), where we denote by \( \sigma \) the bijection acting as \( \sigma \) on the coordinates of the points of \( X \) and \( X/G \). Thus \( q_i \) belongs to the support of \( D(\phi) \) if and only if \( \sigma(q_i) \) is in the support of \( D(\phi^\sigma) \) and the fibers over the two points have the same cardinality. \( \square \)

The proof of Theorem 0.1 follows from Theorem 1.3 and the following result.

**Theorem 2.5.** Let \( X \) be a smooth projective curve of genus \( g \geq 2 \) defined over an algebraically closed field \( F \) and let \( L \subset F \) be a subfield such that \( F/L \) is Galois. If \( X \) is an odd signature curve, then \( M_{F/L}(X) \) is a field of definition for \( X \).
Proof. By Remark 1.6 we can assume that $M_{F/L}(X) = L$. By Theorem 1.5 there exists a canonical $L$-model $B$ of $X/\text{Aut}(X)$ and a commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f_{\sigma}} & X^\sigma \\
\pi_X & \downarrow & \downarrow \pi_X^\sigma \\
X/\text{Aut}(X) & \xrightarrow{h_{\sigma}} & (X/\text{Aut}(X))^\sigma \\
\downarrow g & & \downarrow g^\sigma \\
B & \xrightarrow{\sigma} & B^\sigma
\end{array}
\]

where $\sigma \in \text{Aut}(F/L)$ (this coincides with $U_{F/L}$ by [4, Proposition 2.1]) and $f_{\sigma}, h_{\sigma}, g$ are isomorphisms. Let $\phi = g \circ \pi_X$. The fact that $f_{\sigma}$ is an isomorphism and Lemma 2.4 imply that $D(\phi) = D(\phi^\sigma) = D(\phi)^\sigma$, i.e. $D(\phi)$ is an $L$-rational divisor. Also, as $g$ is an isomorphism, $D(\phi) = g(D(\pi_X))$ and $\phi$ has the same signature of $\pi_X$. If $q_1, \cdots, q_{2k+1}$ are the points in the support of $D(\phi)$ with the same coefficient $c_i$, then the divisor $q_1 + \cdots + q_{2k+1}$ is an $L$-rational divisor of odd degree.

If $L$ is infinite this implies, by Lemma 2.3, that $B$ has an $L$-rational point outside of the branch locus of $\phi$, thus $X$ can be defined over $L$ by Theorem 1.5. In case $L$ is finite the result follows from [10, Corollary 2.11].

\[\square\]

3. Cyclic $q$-gonal curves

Let $F$ be an algebraically closed field of characteristic $p \neq 2$ and let $X$ be an algebraic curve of genus $g \geq 2$ defined over $F$. If the automorphism group of $X$ contains a cyclic subgroup $C_q$, where $q$ is a prime number, such that $X/C_q$ has genus zero, then the curve is called a cyclic $q$-gonal curve. If in addition $C_q$ is normal in $\text{Aut}(X)$, then $X$ is called a normal cyclic $q$-gonal curve. In this case the reduced automorphism group $\overline{\text{Aut}(X)} := \text{Aut}(X)/C_q$ is isomorphic to a finite subgroup of $\text{PGL}_2(F)$.

In case $\overline{\text{Aut}(X)}$ is not cyclic B. Huggins [10, Theorem 5.3] and A. Kontogeorgis [12, Proposition 3.2] proved the following theorem.

**Theorem 3.1.** Let $K$ be a perfect field of characteristic $p \neq 2$ and let $F$ be an algebraic closure of $K$. Let $X$ be a normal cyclic $q$-gonal curve over $F$ such that $\overline{\text{Aut}(X)}$ is not cyclic or that $\overline{\text{Aut}(X)}$ is cyclic of order divisible by $p$. Then $X$ can be defined over its field of moduli relative to the extension $F/K$.

In case $\overline{\text{Aut}(X)}$ is cyclic of order $n$ and $p = 0$, then $X$ is isomorphic to a curve with equation $y^q = f(x)$, where $f$ is as given in Table 3. Observe that $\overline{\text{Aut}(X)}$ is generated by $\nu(x) = \zeta_n x$, where $\zeta_n$ is a primitive $n$-th root of unity. The three cases in Table 3 differ by the number $N$ of branch points of the cover $X \to X/C_q$ fixed by $\nu$.

**Corollary 3.2.** Let $X$ be a normal cyclic $q$-gonal curve of genus $g \geq 2$ defined over a field $K$ of characteristic zero such that $\overline{\text{Aut}(X)}$ is cyclic of order $n \geq 2$ and let $N$ be as above. If either $N = 1$, or $N = 0$ and $\frac{2q-2+2q}{n(q-1)}$ is odd, or $N = 2$ and $\frac{2q-2+2q}{n(q-1)}$ is odd, then $X$ is definable over $K_X$.

**Proof.** The signature of the covering $\pi_X : X \to X/\text{Aut}(X)$ is given in Table 1. If $N = 1$ then clearly $X$ has odd signature. Otherwise, if $N = 0$, the number of branch points with ramification index $q$ equals $\frac{2q-2+2q}{n(q-1)}$ by the Riemann-Hurwitz
formula, thus again $X$ has odd signature. Similarly for $N = 2$. Thus the result follows from Theorem 0.1. □

| $N$ | signature of $\pi_X$ | $f(x)$ |
|-----|---------------------|-------|
| 0   | $(0; n, n, q, \ldots, q)$ | $x^{n^t} + \cdots + a_i x^{n(t-i)} + \cdots + a_{t-1} x^n + 1$ where $q \mid nt$ |
| 1   | $(0; nq, q, \ldots, q)$ | $x^{n^t} + \cdots + a_i x^{n(t-i)} + \cdots + a_{t-1} x^n + 1$ where $q \nmid nt$ |
| 2   | $(0; nq, nq, q, \ldots, q)$ | $x^{n^t} + \cdots + a_i x^{n(t-i)} + \cdots + a_{t-1} x^n + 1$ where $q \nmid nt + 1$ |

**Table 1.** Cyclic $q$-gonal curves with $\text{Aut}(X) = C_n$

We will now construct examples of cyclic $q$-gonal curves not definable over their field of moduli following [9, 10]. Let $m, n > 1$ be two integers, $a_1, \ldots, a_m \in \mathbb{C}$ and consider the polynomial

$$f(x) := \prod_{1 \leq i \leq m} (x^n - a_i)(x^n + 1/a_i).$$

We will look for such an $f$ with the following properties: $|a_i| \neq |a_j|$ if $i \neq j$, $a_i/a_i \neq a_j/a_j$ if $i \neq j$, $|a_i| \neq 1/a_j$ for all $i, j$, $f(0) = -1$. Moreover, if $n = 3$ we ask that the following automorphism does not map the zero set of $f$ into itself:

$$\tau : x \mapsto \frac{-x - \sqrt{3} - 1}{x(\sqrt{3} - 1) + 1}.$$ 

We observe that such polynomials exist for any $m, n$: for $n \neq 3$ we can consider

$$f(x) = \prod_{1 \leq i \leq m} (x^n - (l + 1)\kappa^l)(x^n + \frac{\kappa^l}{l+1}),$$

and for $n = 3$ the polynomial:

$$f(x) = (x^3 - \alpha^3)(x^3 + \frac{1}{\alpha^3}) \prod_{1 \leq l \leq m-1} (x^3 - (l + 1)\kappa^l)(x^3 + \frac{\kappa^l}{l+1}),$$

where $\kappa$ is a primitive $m$-th root of $(-1)^{m-1}$ and $\alpha = -(2 + \sqrt{3})$ (observe that $\tau(\alpha) = \alpha$).

**Lemma 3.3.** Let $X$ be a cyclic $q$-gonal curve over $\mathbb{C}$ given by $y^q = f(x)$, where $f$ is as in (1) and satisfies the properties mentioned above. Then:

i) $\text{Aut}(X)$ is generated by $(x, y) = (x, \zeta_q y)$ and $\nu(x, y) = (\zeta_q x, y)$;

ii) the signature of $\pi_X$ is $(0; q, \ldots, q, n, n)$ if $q \mid 2mn$ and $(0; q, \ldots, q, n, qn)$ otherwise, where $q$ appears $2m$-times.

**Proof.** Observe that ii) is obvious by Table 1. If $n \neq 3$, then i) follows from [10, Lemma 6.1] and its proof (which does not depend on the fact that $m$ is odd). For $n = 3$ we need to exclude the missing case $(\overline{\nu}) < \overline{\text{Aut}}(X) \cong A_4$, where $\overline{\nu}$ is the image of $\nu$ in $\overline{\text{Aut}}(X)$. Suppose we are in this case, then by [2, Corollary 3.2] $\tau$ would be an automorphism of $f(x)$, giving a contradiction. □
The following generalizes [9, Proposition 5.0.5] and [10, Proposition 6.2]. Observe that if \( q \) does not divide \( mn \), then \( X \) is an odd signature curve by the previous Lemma, thus it can be defined over its field of moduli relative to the extension \( \mathbb{C}/\mathbb{R} \).

**Proposition 3.4.** Let \( X \) be a cyclic q-gonal curve over \( \mathbb{C} \) given by \( y^q = f(x) \), where \( q > 2 \), \( f \) is as in (1) and satisfies the properties mentioned above, \( m, n > 1 \) and \( q \mid mn \). The field of moduli of \( X \) relative to the extension \( \mathbb{C}/\mathbb{R} \) is \( \mathbb{R} \) and is a field of definition of \( X \) if and only if \( n \) is odd.

**Proof.** Observe that \( X \) is isomorphic to the conjugate curve 

\[
\tilde{X} : y^q = \prod_{1 \leq i \leq m} (x^n - \bar{a}_i)(x^n + 1/a_i)
\]

by the isomorphism

\[
\mu(x, y) = \left( \frac{1}{\zeta_{2nm}}, \frac{\zeta_{2q} y}{x^{2mn/q}} \right).
\]

By Lemma 3.3 the automorphism group of \( X \) is generated by \( \iota \) and \( \nu \), thus any isomorphism between \( X \) and \( \tilde{X} \) is of the form \( \mu^j \nu^k \), where \( 0 \leq j \leq q - 1 \) and \( 0 \leq k \leq n - 1 \). An easy computation shows that

\[
(\mu^j \nu^k)_{\mu^j \nu^k} = (\iota')^{2k+1} \nu^{2k+1},
\]

where \( \iota'(x, y) = (x, \zeta_{m}^{n/q}) \). Moreover, since \( \iota \) commutes with \( \mu \) and \( \nu \):

\[
(\mu^j \nu^k)_{\mu^j \nu^k} = \mu^{-3j} \nu^{3k} = \mu^k \nu^k = (\mu^k \nu^k)_{\mu^k \nu^k}.
\]

In case \( n \) is even the cocycle condition in Theorem 1.4 does not hold since \( \nu^{2k+1} \neq \text{id} \) for any \( k \), thus \( X \) cannot be defined over \( \mathbb{R} \). Otherwise, if \( n \) is odd, we have \((\mu^k \nu^k)_{\mu^k \nu^k} = \text{id} \) with \( k = (n-1)/2 \), so that \( X \) can be defined over \( \mathbb{R} \). \( \Box \)

**Corollary 3.5.** Let \( X \) be a non-normal q-gonal curve defined over a field \( K \) of characteristic zero. Then \( X \) is definable over \( K_X \).

**Proof.** By [16, Theorem 8.1] the signature of \( \pi_X \) is given in Table 2. In any case \( X \) has odd signature, thus the result follows from Theorem 0.1. \( \Box \)

### 4. Plane quartics

In this section \( X \) will be a smooth plane quartic defined over an algebraically closed field of characteristic zero. Table 3 lists all possible automorphism groups of smooth plane quartics. Moreover, for each group, it gives the equation of a plane quartic. 

| \( q \) | signature of \( \pi_X \) | \( g \) | \( \text{Aut}(X) \) |
|------|----------------|------|------------|
| 3    | (0; 2, 3, 8)  | 2    | \text{GL}(2, 3) |
| 3    | (0; 2, 3, 12) | 3    | \text{SL}(2, 3)/\text{CD} |
| 5    | (0; 2, 4, 5)  | 4    | \text{S}_5 |
| \( \ell \) | (0; 2, 3, \ell) | 3    | \text{PSL}(2, \ell) |
| \( q \geq 5 \) | (0; 2, 3, 2q) | \( (\ell-1)(q-2) \) | \( (\text{C}_q \times \text{C}_q) \times \text{S}_3 \) |
| \( q \geq 3 \) | (0; 2, 2, 2, q) | \( (q-1)^2 \) | \( (\text{C}_q \times \text{C}_q) \rtimes \text{V}_4 \) |
| \( q \geq 3 \) | (0; 2, 4, 2q) | \( (q-1)^2 \) | \( (\text{C}_q \times \text{C}_q) \rtimes \text{D}_4 \) |

**Table 2.** Non-normal q-gonal curves.
quartic having this group as automorphism group (n.a. means “not above”, i.e. not isomorphic to other models above it in the table) and the signature of the covering $\pi_X$ (see [1, Theorem 16 and §2.3]).

| $G$                          | equation                                                                 | signature  |
|------------------------------|--------------------------------------------------------------------------|------------|
| $\text{PSL}_2(7)$            | $z^4y + y^3x + x^4z$                                                     | $(0; 2, 3, 7)$ |
| $S_3$                        | $z^4 + az^2yx + z(y^3 + x^3) + by^2x^2$                                  | $(0; 2, 2, 2, 2, 3)$ |
| $C_2 \times C_2$            | $x^4 + y^4 + z^4 + ax^2y^2 + bx^2z^2 + cy^2z^2$                          | $(0; 2, 2, 2, 2, 2)$ |
| $D_4$                        | $x^4 + y^4 + z^4 + az^2(y^2 + x^2) + by^2x^2$                             | $(0; 2, 2, 2, 2, 2)$ |
| $S_4$                        | $x^4 + y^4 + z^4 + (az^2y^2 + z^2x^2 + y^2x^2)$                          | $(0; 2, 2, 2, 3)$ |
| $C_4 \times S_3$            | $z^4 + y^4 + x^4$                                                        | $(0; 2, 3, 8)$ |
| $C_4 \oplus (C_2)^4$        | $z^4 + y^4 + x^4 + az^2y^2$                                              | $(0; 2, 2, 2, 4)$ |
| $C_4 \oplus A_4$            | $x^4 + y^4 + xz^3$                                                       | $(0; 2, 3, 12)$ |
| $C_6$                        | $z^4 + az^2y^2 + y^3 + yx^3$                                             | $(0; 2, 3, 3, 6)$ |
| $C_9$                        | $z^4 + zy^4 + yx^3$                                                      | $(0; 3, 9, 9)$ |
| $C_{13}$                     | $z^4L_1(y, x) + L_4(y, x)$ (n.a.)                                         | $(0; 3, 3, 3, 3)$ |
| $C_2$                        | $z^4 + z^2L_2(y, x) + L_4(y, x)$ (n.a.)                                  | $(1; 2, 2, 2)$ |

Table 3. Automorphisms of plane quartics.

Table 3 and Theorem 2.5 imply the following result.

**Corollary 4.1.** Let $X$ be a smooth plane quartic defined over an algebraically closed field $K$ of characteristic zero. If either $\text{Aut}(X)$ is trivial or $|\text{Aut}(X)| > 4$, then $X$ is definable over $K_X$.

Observe that the hypothesis in the Corollary is equivalent to ask that $\text{Aut}(X)$ is not isomorphic to either $C_2$ or $C_2 \times C_2$. We will now construct a plane quartic $X$ with $\text{Aut}(X) \cong C_2$ and of field of moduli $\mathbb{R}$ but not definable over $\mathbb{R}$. Consider the family $X_{a_1, a_2, a_3}$ of plane quartics defined by

$$y^4 + y^2(x - a_1z)(x + \frac{1}{a_1}z) + (x - a_2z)(x + \frac{1}{a_2}z)(x - a_3z)(x + \frac{1}{a_3}z) = 0,$$

where $a_1 \in \mathbb{R}$ and $a_2a_3 \in \mathbb{R}$. The following Lemma implies that the generic curve in the family is smooth and has automorphism group of order two.

**Lemma 4.2.** The plane quartic $X_{a_1, a_2, a_3}$ with $a_1 = 1, a_2 = 1 - i$ and $a_3 = 2(i - 1)$ is smooth and its automorphism group is generated by $\nu(x : y : z) = (x : -y : z)$.

**Proof.** We recall that any automorphism of a smooth plane quartic is induced by an element of $\text{PGL}(3, \mathbb{C})$. If $\text{Aut}(X)$ properly contains the cyclic group generated by $\nu$, then it contains a subgroup isomorphic to either $C_2 \times C_2, C_6$ or $S_3$ by [1, pag.26]. We will now exclude each of these cases.
The first case can be excluded because an explicit computation shows that there is no involution, except \( \nu \), which preserves the four fixed points of \( \nu \).

Now suppose that \( \text{Aut}(X) \) contains a cyclic subgroup of order 6 generated by \( \alpha \) with \( \nu = \alpha^3 \). The automorphism \( \tau := \alpha^2 \) induces an order three automorphism \( \varpi \) on the elliptic curve \( E := X/(\nu) \) having fixed points. This is a contradiction since the curve \( E \) (whose equation can be obtained replacing \( y^2 \) with \( y \) in the equation of \( X \)) has \( j \)-invariant distinct from zero.

Finally, suppose that \( \text{Aut}(X) \) contains a subgroup \( \langle \nu, \gamma \rangle \) isomorphic to \( S_3 \). Here we will apply a method suggested by F. Bars [1]. By [1, Theorem 29], up to a change of coordinates the equation of \( X \) takes the following form:

\[
(u^3 + v^3)w + u^2v^2 + auvw^2 + bw^4 = 0.
\]

and the generators of \( S_3 \) with respect to the coordinates \( (u,v,w) \) are

\[
\alpha := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta := \begin{pmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Thus there exists \( A \in \text{PGL}(3, \mathbb{C}) \) such that \( A\alpha A^{-1} = \nu, A\beta A^{-1} = \gamma \). The first condition implies that \( A \) is an invertible matrix of the following form

\[
A = \begin{pmatrix} a & a & c \\ d & -d & 0 \\ g & g & l \end{pmatrix}.
\]

Note that \( X \) has exactly four bitangents \( x = s_jz, \ j = 1, 2, 3, 4 \) invariant under the action of the involution \( \nu \), where \( s_j \) are the zeros of

\[
\Delta = (x^2 - 1)^2 - 4(x + (1 + i))(x + \frac{1}{1 - i})(x - 2(-1 + i))(x - \frac{1}{2(1 + i)}).
\]

Let \( b_{j1} = (s_j, q_j, 1), b_{j2} = (s_j, -q_j, 1) \) be the two tangency points of the line \( x = s_jz \). On the other hand, observe that the line \( w = 0 \) is invariant for \( \alpha \) and it is bitangent to \( X \) at \( p_1 = (1 : 0 : 0), p_2 = (0 : 1 : 0) \). Thus for some \( j \) we have \( \{Ap_1, Ap_2\} = \{b_{j1}, b_{j2}\} \), from which we get \( a = s_jg, d = \pm q_jg \). By means of these remarks and using the Magma [13] code available at this webpage

https://sites.google.com/site/squispeme/home/fieldsofmoduli

we proved that \( \gamma = A\beta A^{-1} \) is not an automorphism of \( X \).

\[\square\]

**Proposition 4.3.** Let \( X_{a_1,a_2,a_3} \) as defined previously with \( \text{Aut}(X_{a_1,a_2,a_3}) \cong C_2 \). Then the field of moduli of \( X_{a_1,a_2,a_3} \) relative to the extension \( \mathbb{C}/\mathbb{R} \) is \( \mathbb{R} \) and is not a field of definition for \( X \).

**Proof.** Observe that the following is an isomorphism between \( X := X_{a_1,a_2,a_3} \) and its conjugate \( \bar{X} \):

\[
\mu(x : y : z) = (-z : iy : x).
\]

Since \( \text{Aut}(X) \) is generated by \( \nu(x : y : z) = (x : -y : z) \), the only isomorphisms between \( X \) and \( \bar{X} \) are \( \mu \) and \( \mu\nu \). Observe that \( \bar{\mu}\mu = \nu \) and \( (\mu\nu)\mu\nu = \nu \). Therefore Weil’s cocycle condition from Theorem 1.4 does not hold, so \( X \) cannot be defined over \( \mathbb{R} \).

\[\square\]
Finally we study plane quartics with automorphism group isomorphic to \(C_2 \times C_2\), which belong to the following family:

\[
X_{a,b,c} : x^4 + y^4 + z^4 + ax^2y^2 + bx^2z^2 + cy^2z^2 = 0,
\]

where \(a, b, c \in \mathbb{C}\). It can be easily checked that \(X_{a,b,c}\) is smooth unless \(a^2 + b^2 + c^2 - abc = 4\) or some of \(a^2, b^2, c^2\) is equal to 4. A subgroup of \(\text{Aut}(X_{a,b,c})\) isomorphic to \(C_2 \times C_2\) is generated by the involutions:

\[
\iota_1(x:y:z) = (-x:y:z), \quad \iota_2(x:y:z) = (x:-y:z).
\]

We will denote by \(G \cong S_3 \times (C_2 \times C_2)\) the group acting on the triples \((a, b, c) \in \mathbb{C}^3\) generated by

\[
g_1(a, b, c) = (b, a, c), \quad g_2(a, b, c) = (b, c, a), \quad g_3(a, b, c) = (-a, -b, c), \quad g_4(a, b, c) = (a, -b, c).
\]

The following comes from a result by E.W. Howe [8, Proposition 2], observing that any isomorphism between \(X_{a,b,c}\) and \(X_{g(a,b,c)}\), \(g \in G\), is defined over \(\mathbb{Q}(i)\).

**Proposition 4.4.** If \(a^2, b^2, c^2\) are pairwise distinct, then \(\text{Aut}(X_{a,b,c}) \cong C_2 \times C_2\). Moreover, if \(F\) is a field containing \(\mathbb{Q}(i)\), then a plane quartic \(X_{a',b',c'}\) is isomorphic to \(X_{a,b,c}\) over \(F\) if and only if \(g(a, b, c) = (a', b', c')\) for some \(g \in G\).

The following result and Corollary 4.1 prove Theorem 0.2.

**Corollary 4.5.** Let \(X_{a,b,c}\) as before with \(a^2, b^2, c^2\) pairwise distinct. If the field of moduli of \(X_{a,b,c}\) relative to the extension \(\mathbb{C}/\mathbb{R}\) is \(\mathbb{R}\), then it is a field of definition for \(X_{a,b,c}\).

**Proof.** By Proposition 4.4, the curve \(X_{a,b,c}\) and its conjugate \(X_{\bar{a}, \bar{b}, \bar{c}}\) are isomorphic over \(\mathbb{C}\) if and only if \(g(a, b, c) = (\bar{a}, \bar{b}, \bar{c})\) for some \(g \in G\). It is enough to consider the generators of \(G\).

i) If \((\bar{a}, \bar{b}, \bar{c}) = g_1(a, b, c) = (b, a, c)\) then \(\mu : X_{a,b,c} \to X_{b,a,c}, \mu(x : y : z) = (x : z : y)\) is an isomorphism and \(\bar{\mu} = \mu\).

ii) If \((\bar{a}, \bar{b}, \bar{c}) = g_2(a, b, c) = (b, c, a)\), i.e., \(\bar{a} = b, \bar{b} = c, \bar{c} = a\), then \(a = b = c \in \mathbb{R}\), contradicting the hypothesis on \(a, b, c\). So this case does not appear.

iii) If \((\bar{a}, \bar{b}, \bar{c}) = g_3(a, b, c) = (-a, -b, c)\) then \(\mu : X_{a,b,c} \to X_{-a,-b,c}, \mu(x : y : z) = (ix : y : z)\) is an isomorphism and \(\bar{\mu} = \mu\).

iv) If \((\bar{a}, \bar{b}, \bar{c}) = g_4(a, b, c) = (a, -b, c)\) then \(\mu : X_{a,b,c} \to X_{a,-b,-c}, \mu(x : y : z) = (x : y : iz)\) is an isomorphism and \(\bar{\mu} = \mu\).

Therefore by Weil’s Theorem we conclude that \(X_{a,b,c}\) can be defined over \(\mathbb{R}\). \(\Box\)

We now determine the field of moduli of a plane quartic in the family. Consider the following polynomials invariant for \(G\):

\[
j_1(a, b, c) = abc, \quad j_2(a, b, c) = a^2 + b^2 + c^2, \quad j_3(a, b, c) = a^4 + b^4 + c^4;
\]

**Proposition 4.6.** Let \(F/K\) be a general Galois extension with \(\mathbb{Q}(i) \subset F \subset \mathbb{C}\) and let \(a, b, c \in F\) such that \(a^2, b^2, c^2\) are pairwise distinct and \(X_{a,b,c}\) is smooth. The field of moduli of \(X_{a,b,c}\) relative to the extension \(F/K\) equals \(K(j_1, j_2, j_3)\).

**Proof.** The morphism \(\varphi(a, b, c) = (abc, a^2 + b^2 + c^2, a^4 + b^4 + c^4)\) has degree 24 = \(|G|\) and clearly \(\varphi(g(a, b, c)) = \varphi(a, b, c)\) for any \(g \in G\). Thus, by Proposition 4.4, \(X_{a,b,c}\) is isomorphic to \(X_{a',b',c'}\) over \(F\) if and only if \(j_k(a, b, c) = j_k(a', b', c')\) for \(k = 1, 2, 3\).
Observe that $X_{a,b,c}^\sigma = X_{\sigma(a),\sigma(b),\sigma(c)}$ is isomorphic to $X_{a,b,c}$ over $F$ if and only if for $k = 1, 2, 3$ we have
\[ j_k := j_k(a, b, c) = j_k(\sigma(a), \sigma(b), \sigma(c)) = \sigma(j_k(a, b, c)). \]
Thus $U_{F/K}(X_{a,b,c}) = \{ \sigma \in \text{Aut}(F/K) : X_{a,b,c}^\sigma \cong X_{a,b,c} \} = \text{Aut}(F/K(j_1, j_2, j_3))$.
Since $L/K$ is a general Galois extension we deduce that
\[ M_{F/K}(X_{a,b,c}) = \text{Fix}(U_{F/K}(X_{a,b,c})) = K(j_1, j_2, j_3). \]

\[ \square \]

**Remark 4.7.** Proposition 4.4 can be generalized to the case when $F$ does not contain $\mathbb{Q}(i)$. In this case $X_{a',b',c'}$ is isomorphic to $X_{a,b,c}$ over $F$ if and only if $g(a, b, c) = (a', b', c')$ for some $g \in \langle g_1, g_2 \rangle$ and the field of moduli relative to a general Galois extension $F/K$ equals $K(j_2, j_3, j_5)$ where $j_4(a, b, c) = a + b + c$, $j_5(a, b, c) = a^3 + b^3 + c^3$.

We now consider the Galois extension $\mathbb{Q}(a, b, c)/\mathbb{Q}(j_1, j_2, j_3)$, assuming that $\mathbb{Q}(i) \subset \mathbb{Q}(a, b, c)$. If $\sigma$ acts on $\mathbb{Q}(a, b, c)$ and $\text{Im}(\sigma)$ is a CM field, then $X_{a,b,c}^\sigma \cong X_{a,b,c}$ and $\sigma$ acts on $(a, b, c)$ as some $g_\sigma \in G$ by Proposition 4.4. Thus we can define a natural injective group homomorphism
\[ \psi : \text{Aut}(\mathbb{Q}(a, b, c)/\mathbb{Q}(j_1, j_2, j_3)) \to G, \quad \sigma \mapsto g_\sigma. \]
Observe that, if $a, b, c \in \mathbb{C}$ are generic, then $\psi$ is an isomorphism since the degree of the extension $\mathbb{Q}(a, b, c)/\mathbb{Q}(j_1, j_2, j_3)$ is $24 = |G|$.

**Proposition 4.8.** Let $a, b, c \in \mathbb{C}$ such that $a^2, b^2, c^2$ are pairwise distinct, $X_{a,b,c}$ is smooth and $\mathbb{Q}(i) \subset \mathbb{Q}(a, b, c)$. If $\text{Im}(\psi) \subset \langle g_1, g_2 \rangle$, then $X_{a,b,c}$ can be defined over $\mathbb{Q}(j_1, j_2, j_3)$.

**Proof.** According to Weil’s Theorem 1.4 we need to choose an isomorphism $f_\sigma : X_{a,b,c} \to X_{\sigma(a),\sigma(b),\sigma(c)}$ for any $\sigma \in \text{Aut}(\mathbb{Q}(a, b, c)/\mathbb{Q}(j_1, j_2, j_3))$ such that the following condition holds for all $\sigma, \tau$:
\[ (2) \quad f_{\sigma \tau} = f_\sigma^\tau \circ f_\sigma. \]
We assume that $\text{Im}(\psi) = \langle g_1, g_2 \rangle$, the case when there is just an inclusion is similar. Let $\sigma_1 = \psi^{-1}(g_1)$ and $\sigma_2 = \psi^{-1}(g_2)$. We choose $f_\sigma(x : y : z) = (z : x : y)$, $f_\sigma(x : y : z) := (x : z : y)$ and $f_\sigma := f_{\sigma_2}^\sigma \circ f_{\sigma_1}^\sigma$, if $\sigma = \sigma_1^r \circ \sigma_2^s$. Observe that $f_\sigma$ is always defined over $\mathbb{Q}$, so that $f_\sigma^\tau = f_{\sigma \tau}$. Thus condition (2) clearly holds. \[ \square \]

**Example 4.9.** Consider a plane quartic $X = X_{a,b,c}$ where $a = \alpha, b = \bar{a}$ with $\alpha \in \mathbb{Q}(i)$ and $c \in \mathbb{Q}$ such that $a^2, b^2, c^2$ are pairwise distinct and the curve is smooth. By Proposition 4.6 the field of moduli of the curve relative to the extension $\mathbb{Q} \subset \mathbb{Q}(a, b, c) = \mathbb{Q}(i)$ is $\mathbb{Q}$. The Galois group $\text{Aut}(\mathbb{Q}(i)/\mathbb{Q})$ is generated by the complex conjugation $\sigma(z) = \bar{z}$ and $\psi(\sigma) = g_1$. An isomorphism between $X$ and $X^{\sigma}$ is given by $f_\sigma(x : y : z) = (x : z : y)$. Since $\text{id} = f_{\sigma^2} = f_{\sigma}^2 \circ f_{\sigma} = (f_{\sigma})^2$, then $X$ can be defined over $\mathbb{Q}$.

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