Some remarks on the asymmetric sum–product phenomenon

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Annotation.

Using some new observations connected to higher energies, we obtain quantitative lower bounds on
\[ \max \{|AB|, |A + C|\} \text{ and } \max \{|(A + \alpha)B|, |A + C|\}, \alpha \neq 0 \text{ in the regime when the sizes of finite subsets} \] \[ A, B, C \text{ of a field differ significantly.} \]

1 Introduction

Let \( p \) be a prime number and \( A, B \subset \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \) be finite sets. Define the sum set, the difference
set, the product set and the quotient set of \( A \) and \( B \) as
\[ A + B := \{a + b : a \in A, b \in B\}, \quad A - B := \{a - b : a \in A, b \in B\} \]
\[ AB := \{ab : a \in A, b \in B\}, \quad A/B := \{a/b : a \in A, b \in B, b \neq 0\}, \]
correspondingly. One of the central problems in arithmetic combinatorics [35] it is the sum–
product problem, which asks for estimates of the form
\[ \max\{|A + A|, |AA|\} \geq |A|^{1+c} \tag{1} \]
for some positive \( c \). This question was originally posed by Erdős and Szemerédi [13] for finite
sets of integers; they conjectured that (1) holds for all \( c < 1 \). The sum–product problem has
since been studied over a variety of fields and rings, see, e.g. [4], [6], [7], [12], [11], [13], [35] and
others. We focus on the case of \( \mathbb{F}_p \) (and sometimes consider \( \mathbb{R} \)), where the first estimate of the form (1) was proved by Bourgain, Katz, and Tao [11]. At the moment the best results in this
direction are contained in [23] and in [19].

In this article we study an asymmetric variant of the sum–product question ("the sum–
product theorem in \( \mathbb{F}_p \) for sets of distinct sizes") in the spirit of fundamental paper [3]. Let us
recall two results from here.

**Theorem 1** Given \( 0 < \varepsilon < 1/10 \), there is \( \delta > 0 \) such that the following holds. Let \( A \subset \mathbb{F}_p \) and
\[ p^{\delta} < |A| < p^{1-\varepsilon}. \]

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Then either
\[ |AB| > p^\delta |A| \quad \text{for all } B \subset \mathbb{F}_p, |B| > p^\varepsilon \]
or
\[ |A + C| > p^\delta |A| \quad \text{for all } C \subset \mathbb{F}_p, |C| > p^\varepsilon . \]

**Theorem 2** Given \( 0 < \varepsilon < 1/10 \), there is \( \delta > 0 \) such that the following holds. Let \( A \subset \mathbb{F}_p \) and \( p^\varepsilon < |A| < p^{1-\varepsilon} \).

Then for any \( x \neq 0 \) either
\[ |AB| > p^\delta |A| \quad \text{for all } B \subset \mathbb{F}_p, |B| > p^\varepsilon \]
or
\[ |(A + x)C| > p^\delta |A| \quad \text{for all } C \subset \mathbb{F}_p, |C| > p^\varepsilon . \]

Theorems [1,2] were derived in [3] from the following result of paper [5]. Given a set \( A \subseteq \mathbb{F}_p \) denote by \( \mathcal{T}^+_k(A) := \{|(a_1, \ldots, a_k, a'_1, \ldots, a'_k) \in A^{2k} : a_1 + \cdots + a_k = a'_1 + \cdots + a'_k\}| \). We write \( \mathcal{E}^+(A) \) for \( \mathcal{T}^+_{2k}(A) \).

**Theorem 3** For a positive integer \( Q \), there are a positive integer \( k \) and a real \( \tau > 0 \) such that if \( H \subseteq \mathbb{F}_p^* \) and
\[ |HH| < |H|^{1+\tau} \]
then
\[ \mathcal{T}^+_k(H) < |H|^{2k}(p^{-1+1/Q} + c_Q|H|^{-Q}) , \]
where \( c_Q > 0 \) depends on \( Q \) only.

The aim of this paper is to obtain explicit bounds in the theorems above. Our arguments are different and more elementary than in [3, 10] and [14]. In the proof we almost do not use the Fourier approach and hence the container group \( \mathbb{F}_p \). That is why we do not need in lower bounds for sizes of \( A, B, C \) in terms of the characteristic \( p \) but, of course, these sets must be comparable somehow. Also, the arguments work in \( \mathbb{R} \) as well and it differs this article from paper [3], say. Let us formulate our variant of Theorems [1,2] (see Corollary [33] below). One can show that Theorem [4] implies both of these results if \( |A| < p^{1/2-\varepsilon} \), say, see Remark [36] from section 5.

**Theorem 4** Let \( A, B, C \subseteq \mathbb{F}_p \) be arbitrary sets, and \( k \geq 1 \) be such that \( |A||B|^{1+(k+4)/2(k+4)} \leq p \) and
\[ |B|^{\frac{k}{k+4} + \frac{1}{2(k+4)}} \geq |A| \cdot C_*^{(k+4)/4} \log^k(|A||B|) . \tag{2} \]
where \( C_* > 0 \) is an absolute constant. Then
\[ \max\{|AB|, |A + C|\} \geq 2^{-3}|A| \cdot \min\{|C|, |B|^{\frac{1}{2(k+4)}2^{-k}}\} , \tag{3} \]
and for any $\alpha \neq 0$

$$\max\{|AB|, |(A+\alpha)C|\} \geq 2^{-3}|A| \cdot \min\{|C|, |B|^{\frac{1}{2(k+3)^{2^{-k}}}}\}. \tag{4}$$

Actually, we prove that the lower bounds for $|A+C|$, $|(A+\alpha)C|$ in (3), (4) could be replaced by similar upper bounds for the energies $\mathbb{E}^+(A,C)$, $\mathbb{E}^x(A+\alpha,C)$, see the second part of Corollary 33 from section 5. We call Theorem 4 an asymmetric sum–product result because $A$ can be much larger than $B$ and $C$ (say, $|A| > (|B|C|)^{100}$) in contrast with the usual quadratic restrictions which follow from the classical Szemerédi–Trotter Theorem, see [34], [35] for the real setting and see [11], [14], [24] for the prime fields. On the other hand, the roles of $B$, $C$ are not symmetric as well. The thing is that the method of the proof intensively uses the fact that if $|AB|$ is small comparable to $|A|$, then, roughly speaking, for any integer $k$ size of $(kA)B$ is small comparable to $kA$, roughly speaking (rigorous formulation can be found in section 5). Of course this observation is not true more in any sense if we replace $\times$ to $+$ and vice versa.

Also, we obtain a ”quantitative” version of Theorem 3.

**Theorem 5** Let $A, B \subseteq \mathbb{F}_p$ be sets, $M \geq 1$ be a real number and $|AB| \leq M|A|$. Then for any $k \geq 2$, $2^{16k}M^{2k+1}C_*^2 \log^8 |A| \leq |B|$, one has

$$T^+_}\leq 2^{4k+6}C_* \log^4 |A| \frac{M^{2k}|A|^{2k+1}}{p} + 16k^2 M^{2k+1}C_*(k-1) \log^4 |A| |A|^{2k+1} - |B|^{(k-1)} \mathbb{E}^+(A). \tag{5}$$

Here $C_* > 0$ is an absolute constant.

As a by-product we obtain the best constants in the problem of estimating of the exponential sums over multiplicative subgroups [5], [14] and relatively good bounds in the question on basis properties of multiplicative subgroups [15]. Also, we find a wide series of ”superquadratic” bounds in $\mathbb{R}$” [2] with four variables, see Corollary 35.

In contrast to paper [3] we prove Theorem 1 and Theorem 5 independently. We realise that Theorem 4 is equivalent to estimating another sort of energies, namely,

$$\mathbb{E}_k^+(A) := |\{(a_1, \ldots, a_k, a'_1, \ldots, a'_k) \in A^{2k} : a_1 - a_1' = \cdots = a_k - a_k'\}|$$

(see the definitions in section 2). Thus, a new feature of this paper is an upper bound for $\mathbb{E}_k^+(A)$ for sets $A$ with $|AB| \ll |A|$ for some large $B$, see Theorem 27 below. Such upper bound can be of independent interest.

**Theorem 6** Let $A, B \subseteq \mathbb{F}_p$ be two sets, $k \geq 0$ be an integer, and put $M := |AB^{k+1}|/|A|$. Then for any $k \geq 0$ such that

$$|B|^{k/8+1/2} \geq |A| \cdot M^{2k+1}2^{3k+1}C_*(k+4)^{4/3} \log^k |AB^k|,$$

where $C_* > 0$ is an absolute constant, we have

$$\mathbb{E}_k^+(A) \leq 2|AB^k|^{2k+1}. \tag{6}$$
Our approach develops the ideas from [3], [29] (see especially section 4 from here) and uses several sum–product observations of course. We avoid to repeat combinatorial arguments of Bourgain’s paper [3] (although we use a similar inductive strategy of the proof) but the method relies on recent geometrical sum–product bounds from Rudnev’s article [24] and further papers as [1], [21], [23], [31] and others. In some sense we introduce a new approach of estimating moments $M_k(f)$ (e.g., $T^+_k(H)$ in Theorem 3 or $E^+_k(A)$ in Theorem 6) of some specific functions $f$: instead of calculating $M_k(f)$ in terms of suitable norms of $f$, we comparing $M_k(f)$ and $M_{k/2}(f)$. If $M_k(f)$ is much less than $M_{k/2}(f)$, then we use induction and if not then thanks some special nature of the function $f$ we deriving from this fact that the additive energy $E^+$ of a level set of $f$ is huge and it gives a contradiction. Clearly, this process can be applied at most $O(\log k)$ number of times and that is why we usually have logarithmic savings (compare the index in $T^+_k(A)$ and the gain $|B| - (k-1)/2$ in estimate (5), say).

The paper is organized as follows. Section 2 contains all required definitions. In section 3 we give a list of the results, which will be further used in the text. In the next section we consider a particular case of multiplicative subgroups $\Gamma$ and obtain an upper estimate for $T^+_k(\Gamma)$. It allows us to obtain new upper bounds for the exponential sums over subgroups which are the best at the moment. This technique is developed in section 5 although we avoid to use the Fourier approach as was done in [3] and in the previous section 4. The last section 5 contains all main Theorems 4–6.

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## 2 Notation

In this paper $p$ is an odd prime number, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$. We denote the Fourier transform of a function $f : \mathbb{F}_p \to \mathbb{C}$ by $\hat{f}$,

$$\hat{f}(\xi) = \sum_{x \in \mathbb{F}_p} f(x) e(-\xi \cdot x),$$

(7)

where $e(x) = e^{2\pi ix/p}$. We rely on the following basic identities

$$\sum_{x \in \mathbb{F}_p} |f(x)|^2 = \frac{1}{p} \sum_{\xi \in \mathbb{F}_p} |\hat{f}(\xi)|^2,$$

(8)

$$\sum_{y \in \mathbb{F}_p} \left| \sum_{x \in \mathbb{F}_p} f(x) g(y - x) \right|^2 = \frac{1}{p} \sum_{\xi \in \mathbb{F}_p} |\hat{f}(\xi)|^2 |\hat{g}(\xi)|^2,$$

(9)

and

$$f(x) = \frac{1}{p} \sum_{\xi \in \mathbb{F}_p} \hat{f}(\xi) e(\xi \cdot x).$$

(10)

Let $f, g : \mathbb{F}_p \to \mathbb{C}$ be two functions. Put

$$(f * g)(x) := \sum_{y \in \mathbb{F}_p} f(y) g(x - y) \quad \text{and} \quad (f \circ g)(x) := \sum_{y \in \mathbb{F}_p} f(y) g(y + x).$$

(11)
Then
\[ \widehat{f * g} = \widehat{f} \widehat{g} \quad \text{and} \quad \widehat{f \circ g} = \widehat{f} \widehat{g}. \] (12)

Put \( E^+(A, B) \) for the common additive energy of two sets \( A, B \subseteq \mathbb{F}_p \) (see, e.g., [35]), that is,
\[ E^+(A, B) = |\{(a_1, a_2, b_1, b_2) \in A \times A \times B \times B : a_1 + b_1 = a_2 + b_2\}|. \]

If \( A = B \) we simply write \( E^+(A) \) instead of \( E^+(A, A) \) and \( E^+(A) \) is called the additive energy in this case. Clearly,
\[ E^+(A, B) = \sum_x (A * B)(x)^2 = \sum_x (A \circ B)(x)^2 = \sum_x (A \circ A)(x)(B \circ B)(x) \]
and by (9),
\[ E(A, B) = \frac{1}{p} \sum_{\xi} |\widehat{A}(\xi)|^2 |\widehat{B}(\xi)|^2. \] (13)

Also, notice that
\[ E^+(A, B) \leq \min\{|A|^2|B|, |B|^2|A|, |A|^{3/2}|B|^{3/2}\}. \] (14)

In the same way define the common multiplicative energy of two sets \( A, B \subseteq \mathbb{F}_p \)
\[ E^\times(A, B) = |\{(a_1, a_2, b_1, b_2) \in A \times A \times B \times B : a_1 b_1 = a_2 b_2\}|. \]

Certainly, the multiplicative energy \( E^\times(A, B) \) can be expressed in terms of multiplicative convolutions similar to (11).

Sometimes we use representation function notations like \( r_{AB}(x) \) or \( r_{A+B}(x) \), which counts the number of ways \( x \in \mathbb{F}_p \) can be expressed as a product \( ab \) or a sum \( a + b \) with \( a \in A \), \( b \in B \), respectively. For example, \( |A| = r_{A-A}(0) \) and \( E^+(A) = r_{A+A-A-A}(0) = \sum_x r_{A+A}^2(x) = \sum_x r_{A-A}^2(x) \). In this paper we use the same letter to denote a set \( A \subseteq \mathbb{F}_p \) and its characteristic function \( A : \mathbb{F}_p \to \{0, 1\} \). Thus, \( r_{A+B}(x) = (A * B)(x) \), say.

Now consider two families of higher energies. Firstly, let
\[ T^+_k(A) := |\{(a_1, \ldots, a_k, a'_1, \ldots, a'_k) \in A^{2k} : a_1 + \cdots + a_k = a'_1 + \cdots + a'_k\}| = \frac{1}{p} \sum_{\xi} |\widehat{A}(\xi)|^{2k}. \] (15)

Secondly, for \( k \geq 2 \), we put
\[ E^+_k(A) = \sum_{x \in \mathbb{F}_p} (A \circ A)(x)^k = \sum_{x \in \mathbb{F}_p} r_{A-A}^k(x) = E^+(\Delta_k(A), A^k), \] (16)
where
\[ \Delta_k(A) := \{(a, a, \ldots, a) \in A^k\}. \]

Thus, \( E^+_2(A) = T^+_2(A) = E^+(A) \). Also, notice that we always have \( E^+_k(A) \geq |A|^k \). Finally, let us remark that by definition (13) one has \( E^+_1(A) = |A|^2 \). Some results about the properties of the energies \( E^+_k \) can be found in [27]. Sometimes we use \( T^+_k(f) \) and \( E^+_k(f) \) for an arbitrary function \( f \) and the first formula from (16) allows to define \( E^+_k(A) \) for any positive \( k \). It was proved in
[31] Proposition 16] that \((E^+_k(f))^{1/2k}\) is a norm for even \(k\) and a real function \(f\). The fact that \((T^+_k(f))^{1/2k}\) is a norm is contained in [35].

Let \(A\) be a set. Put

\[ R[A] := \left\{ \frac{a_1 - a}{a_2 - a} : a, a_1, a_2 \in A, a_2 \neq a \right\} \]

and

\[ Q[A] := \left\{ \frac{a_1 - a_2}{a_3 - a_4} : a_1, a_2, a_3, a_4 \in A, a_3 \neq a_4 \right\}. \]

All logarithms are to base 2. The signs \(\ll\) and \(\gg\) are the usual Vinogradov symbols. When the constants in the signs depend on some parameter \(M\), we write \(\ll_M\) and \(\gg_M\). For a positive integer \(n\), we set \([n] = \{1, \ldots, n\}\).

3 Preliminaries

We begin with a variation on the famous Plünnecke–Ruzsa inequality, see [26, Chapter 1].

Lemma 7 Let \(G\) be a commutative group. Also, let \(A, B_1, \ldots, B_h \subseteq G\), \(|A + B_j| = \alpha_j|A|\), \(j \in [h]\). Then there is a non-empty set \(X \subseteq A\) such that

\[ |X + B_1 + \cdots + B_h| \leq \alpha_1 \cdots \alpha_h |X|. \]  

Further for any \(0 < \delta < 1\) there is \(X \subseteq A\) such that \(|X| \geq (1 - \delta)|A|\) and

\[ |X + B_1 + \cdots + B_h| \leq \delta^{-h} \alpha_1 \cdots \alpha_h |X|. \]

We need a result from [24] or see [21, Theorem 8]. By the number of point–planes incidences \(I(\mathcal{P}, \Pi)\) between a set of points \(\mathcal{P} \subseteq \mathbb{F}_p^3\) and a collection of planes \(\Pi\) in \(\mathbb{F}_p^3\), we mean

\[ I(\mathcal{P}, \Pi) := \left| \{(p, \pi) \in \mathcal{P} \times \Pi : p \in \pi\} \right|. \]

Theorem 8 Let \(p\) be an odd prime, \(\mathcal{P} \subseteq \mathbb{F}_p^3\) be a set of points and \(\Pi\) be a collection of planes in \(\mathbb{F}_p^3\). Suppose that \(|\mathcal{P}| = |\Pi|\) and that \(k\) is the maximum number of collinear points in \(\mathcal{P}\). Then the number of point–planes incidences satisfies

\[ I(\mathcal{P}, \Pi) \ll \frac{|\mathcal{P}|^2}{p} + |\mathcal{P}|^{3/2} + k|\mathcal{P}|. \]

Notice that in \(\mathbb{R}\) we do not need in the first term in estimate (19).

Let us derive a consequence of Theorem 8.
Lemma 9 Let $A, Q \subseteq \mathbb{F}_p$ be two sets, $A, Q \neq \{0\}$, $M \geq 1$ be a real number, and $|QA| \leq M|Q|$. Then

$$E^+(Q) \leq C^* \left( \frac{M^2|Q|^4}{p} + \frac{M^{3/2}|Q|^3}{|A|^{1/2}} \right),$$

where $C^* \geq 1$ is an absolute constant.

Proof. Put $A = A \setminus \{0\}$. We have

$$E^+(Q) = |\{q_1 + q_2 = q_3 + q_4 : q_1, q_2, q_3, q_4 \in Q\}| \leq |A|^2|Q|^2|QA|^2.$$

The number of the solutions to the last equation can be interpreted as the number of incidences between the set of points $P = Q \times QA \times A^2$ and planes $\Pi$ with $|P| = |\Pi| = |A|^3|Q||QA|$. Here $k = |QA|$ because $A, Q \neq \{0\}$. Using Theorem 8 and a trivial inequality $|QA| \leq |Q||A|$, we obtain

$$E^+(Q) \ll |A|^{-2} \left( \frac{|A|^2|Q|^2|QA|^2}{p} + |Q|^{3/2}|QA|^3/2|A|^{3/2} \right) \ll \frac{M^2|Q|^4}{p} + \frac{M^{3/2}|Q|^3}{|A|^{1/2}},$$

as required. □

Finally, we need a combinatorial

Lemma 10 Let $G$ be a finite abelian group, $A, P$ be subsets of $G$. Then for any $k \geq 1$ one has

$$\left( \sum_{x \in P} r_{A-A}(x) \right)^2 \leq |A|^k \sum_x r_{A-A}(x)r_{P-P}(x).$$

In particular,

$$\left( \sum_{x \in P} r_{A-A}(x) \right)^4 \leq |A|^{2k}E^+_2(A)E^+(P).$$

Proof. Clearly, inequality (22) follows from (21) by the Cauchy–Schwarz inequality. To prove estimate (21), we observe that

$$\left( \sum_{x \in P} r_{A-A}(x) \right)^2 \leq \left( \sum_{x_1, \ldots, x_k \in A} |P \cap (A - x_1) \cap \cdots \cap (A - x_k)| \right)^2 \leq |A|^k \sum_{x_1, \ldots, x_k} |P \cap (A - x_1) \cap \cdots \cap (A - x_k)|^2 = |A|^k \sum_x r_{P-P}(x)r_{A-A}(x)$$

as required. □

Combining Theorem 8 and Lemma 10, we obtain
Corollary 11  Let $A \subseteq \mathbb{F}_p$, and $B, P \subseteq \mathbb{F}_p^*$ be sets. Then for any $k \geq 1$ one has

$$
\left( \sum_{x \in P} r_{A^{-A}(x)}^k \right)^4 \leq C_* |A|^{2k} E_{2k}^+(AB) \left( \frac{|P|^4}{p} + \frac{|P|^3}{|B|^{1/2}} \right).
$$

(23)

Proof. By Lemma 10 we have

$$
\left( \sum_{x \in P} r_{A^{-A}(x)}^k \right)^2 \leq |A|^k \sum_{x} r_{A^{-A}(x)}^k r_{P-P}(x).
$$

Further clearly for any $b \in B$ the following holds

$$
r_{A^{-A}(x)} \leq r_{AB-AB}(xb).
$$

Hence

$$
\left( \sum_{x \in P} r_{A^{-A}(x)}^k \right)^2 \leq \frac{|A|^k}{|B|} \sum_{x \in B} \sum_{b \in B} r_{AB-AB}(xb) r_{P-P}(x) = \frac{|A|^k}{|B|} \sum_{x} r_{AB-AB}(x) r_{B(P-P)}(x).
$$

Using the Cauchy–Schwarz inequality, we obtain

$$
\left( \sum_{x \in P} r_{A^{-A}(x)}^k \right)^4 \leq \frac{|A|^{2k}}{|B|^2} E_{2k}^+(AB) \sum_{x} r_{B(P-P)}^2(x).
$$

To estimate the sum $\sum_x r_{B(P-P)}^2(x)$ we use Theorem 8 as in the proof of Lemma 9. We have

$$
\sum_x r_{B(P-P)}^2(x) \leq C_* \left( \frac{|B|^2 |P|^4}{p} + |B|^{3/2} |P|^3 \right).
$$

Thus,

$$
\left( \sum_{x \in P} r_{A^{-A}(x)}^k \right)^4 \leq C_* |A|^{2k} E_{2k}^+(AB) \left( \frac{|P|^4}{p} + \frac{|P|^3}{|B|^{1/2}} \right).
$$

This completes the proof. \hfill \Box

4 Multiplicative subgroups

In this section we obtain the best upper bounds for $T_k^+(\Gamma)$, $E_k^+(\Gamma)$ and for the exponential sums over multiplicative subgroups $\Gamma$. We begin with the quantity $T_k^+(\Gamma)$.

Theorem 12  Let $\Gamma \subseteq \mathbb{F}_p^*$ be a multiplicative subgroup. Then for any $k \geq 2$, $2^{6k} C_*^4 \leq |\Gamma|$ one has

$$
T_{2k}^+(\Gamma) \leq 2^{4k+6} C_* \log^4 |\Gamma| \cdot \frac{|\Gamma|^{2k+1}}{p} + 16^{k} C_*^{k-1} \log^{4(k-1)} |\Gamma| \cdot |\Gamma|^{2k+1 - \frac{k(k+2)}{2}} E^+(\Gamma),
$$

(24)

where $C_*$ is the absolute constant from Lemma 9.
In other words, by (26), we get

\[ T_{2s}(\Gamma) \leq 32C_4 s^4 \log^4 |\Gamma| \cdot \left( \frac{|\Gamma|^{4s}}{p} + |\Gamma|^{2s-1/2} T_s^+(\Gamma) \right). \]  

We have

\[ T_{2s}(\Gamma) = \sum_{x,y,z} r_{s\Gamma}(x) r_{s\Gamma}(y) r_{s\Gamma}(x+z) r_{s\Gamma}(y+z) \leq \frac{8}{5} \sum_{x,y,z}^t r_{s\Gamma}(x) r_{s\Gamma}(y) r_{s\Gamma}(x+z) r_{s\Gamma}(y+z) + \mathcal{E}, \]

where the sum above is taken over nonzero variables \( x \) with \( r(x) > T_{2s}(\Gamma)/(8|\Gamma|^{3s}) := \rho \) and

\[ \mathcal{E} \leq 4r_{s\Gamma}(0) \sum_{y,z} r_{s\Gamma}(y) r_{s\Gamma}(z) r_{s\Gamma}(y+z) \leq 4r_{s\Gamma}(0)|\Gamma|^s T_s^+(\Gamma) \leq 4|\Gamma|^{2s-1} T_s^+(\Gamma). \]  

Put \( P_j = \{ x : \rho 2^{j-1} < r_{s\Gamma}(x) \leq \rho 2^j \} \subseteq \mathbb{F}_p^* \). If (25) does not hold, then the possible number of sets \( P_j \) does not exceed \( L := s \log |\Gamma| \). By the Dirichlet principle there is \( \Delta = \Delta_{j_0} \), and a set \( P = P_{j_0} \) such that

\[ T_{2s}(\Gamma) \leq \frac{8}{5} L^4 (2\Delta)^4 \mathcal{E}^+(P) + \mathcal{E} = T^2_{2s}(\Gamma) + \mathcal{E}. \]

Indeed, putting \( f_i(x) = P_i(x) r_{s\Gamma}(x) \), and using the Hölder inequality, we get

\[ \sum_{x,y,z}^t r_{s\Gamma}(x) r_{s\Gamma}(y) r_{s\Gamma}(x+z) r_{s\Gamma}(y+z) \leq \sum_{x,y,z} L_{i,j,k,l} f_i(x) f_j(y) f_k(x+z) f_l(y+z) \leq \]

\[ \sum_{i,j,k,l} (\mathcal{E}^+(f_i) \mathcal{E}^+(f_j) \mathcal{E}^+(f_k) \mathcal{E}^+(f_l))^{1/4} = \left( \sum_{i=1}^L (\mathcal{E}^+(f_i))^{1/4} \right)^4 \leq L^3 \sum_{i=1}^L \mathcal{E}^+(f_i) \leq L^4 \max_i \mathcal{E}^+(f_i). \]

Moreover we always have \( |P| \Delta^2 \leq T_s^+(\Gamma) \) and \( |P| \Delta \leq |\Gamma|^s \). Using Lemma 9 we obtain

\[ \mathcal{E}^+(P) \leq C_5 \left( \frac{|P|^4}{p} + \frac{|P|^3}{|\Gamma|^{1/2}} \right). \]

Hence

\[ T_{2s}(\Gamma) \leq \frac{8}{5} (16C_4) L^4 \Delta^4 \left( \frac{|P|^4}{p} + \frac{|P|^3}{|\Gamma|^{1/2}} \right) \leq \frac{8}{5} (16C_4) L^4 \left( \frac{|\Gamma|^{4s}}{p} + \frac{|P|^3 \Delta^4}{|\Gamma|^{1/2}} \right). \]  

Let us consider the second term in (27). Then in view of \( |P| \Delta^2 \leq T_s^+(\Gamma) \) and \( |P| \Delta \leq |\Gamma|^s \), we have

\[ |P|^3 \Delta^4 = (P \Delta)^2 P \Delta^2 \leq |\Gamma|^{2s} T_s^+(\Gamma). \]

In other words, by (26), we get

\[ T_{2s}(\Gamma) \leq \frac{8}{5} (16C_4) L^4 \left( \frac{|\Gamma|^{4s}}{p} + |\Gamma|^{2s-1/2} T_s^+(\Gamma) \right) + 4|\Gamma|^{2s-1} T_s^+(\Gamma) \leq \]
\[ \leq 32C_4 s^4 \log^4 |\Gamma| \cdot \left( \frac{|\Gamma|^{4s}}{p} + |\Gamma|^{2s-1/2} T_s^+ (\Gamma) \right) \]

and inequality (25) is proved.

Now applying formula (25) successively \((k-1)\) times, we obtain

\[ T_{2k}^+ (\Gamma) \leq 2^{4k+6} C_4 \log^4 |\Gamma| \cdot \frac{|\Gamma|^{2k+1}}{p} + 16k^2 C_4^{k-1} \log^4 (k-1) |\Gamma| \cdot |\Gamma|^{2k+\cdots+4-k+1} E^+ (\Gamma) \leq \]

\[ \leq 2^{4k+6} C_4 \log^4 |\Gamma| \cdot \frac{|\Gamma|^{2k+1}}{p} + 16k^2 C_4^{k-1} \log^4 (k-1) |\Gamma| \cdot |\Gamma|^{2k+\cdots+4-k+1} E^+ (\Gamma) . \]  

(28)

To get the first term in the last formula we have used our condition \(2^{64} k C_4^4 \leq |\Gamma|\) to insure that \(|\Gamma|^{1/2} \geq 2^{4k+1} C_4 \log^4 |\Gamma|\). This completes the proof. □

**Remark 13** The condition \(2^{64} k C_4^4 \leq |\Gamma|\) can be dropped but then we will have the multiple \(16k^2 (C_4 \log |\Gamma|)^k\) in the first term of (24).

Splitting any \(\Gamma\)—invariant set onto cosets over \(\Gamma\) and applying the norm property of \(T_t^+\), we obtain

**Corollary 14** Let \(\Gamma \subseteq \mathbb{F}_p^*\) be a multiplicative subgroup, and \(Q \subseteq \mathbb{F}_p^*\) be a set with \(Q \Gamma = Q\). Then for any \(k \geq 2\), \(2^{64k} C_4^4 \leq |\Gamma|\) one has

\[ T_{2k}^+ (Q) \leq 2^{4k+6} C_4 \log^4 |\Gamma| \cdot \frac{|Q|^{2k+1}}{p} + 16k^2 C_4^{k-1} \log^4 (k-1) |\Gamma| \cdot |\Gamma|^{2k+\cdots+4-k+1} E^+ (\Gamma)|Q|^{2k+1} . \]  

(29)

Let \(\Gamma\) be a subgroup of size less than \(\sqrt{p}\). Considering a particular case \(k = 2\) of formula (12) of Theorem 12 and using \(E^+ (\Gamma) \ll |\Gamma|^{5/2-c}\), where \(c > 0\) is an absolute constant (see [28]), one has

**Corollary 15** Let \(\Gamma\) be a multiplicative subgroup, \(|\Gamma| \leq \sqrt{p}\). Then

\[ T_4^+ (\Gamma) \ll \frac{|\Gamma|^8 \log^4 |\Gamma|}{p} + |\Gamma|^{6-c} . \]

In particular, \(|4\Gamma| \gg |\Gamma|^{2+c}\).

Previous results on \(T_k^+ (\Gamma), |\Gamma| \leq \sqrt{p}\) with small \(k\) had the form \(T_k^+ (\Gamma) \ll |\Gamma|^{2k-2+c_k}\) with some \(c_k > 0\), see, e.g., [20]. The best upper bound for \(T_4^+ (\Gamma)\) can be found in [33].

Now we prove a corollary about exponential sums over subgroups which is parallel to results from [9], [10], [14]. The difference between the previous estimates and Corollary 16 is just slightly better constant \(C\) in [31].
Corollary 16 Let $\Gamma$ be a multiplicative subgroup, $|\Gamma| \geq p^\delta$, $\delta > 0$. Then
\[
\max_{\xi \neq 0} |\widehat{\Gamma}(\xi)| \ll |\Gamma| \cdot p^{-\frac{\delta}{2^{7+2k+1}}}.
\]
(30)
Further we have a nontrivial upper bound $o(|\Gamma|)$ for the maximum in (30) if
\[
\log |\Gamma| \geq \frac{C \log p}{\log \log p},
\]
(31)
where $C > 2$ is any constant.

Proof. We can assume that $|\Gamma| < \sqrt{p}$, say, because otherwise estimate (30) is known, see [20]. By $\rho$ denote the maximum in (30). Then by Theorem 12, a trivial bound $E_+^+(\Gamma) \leq |\Gamma|^3$ and formula (15), we obtain
\[
|\Gamma| \rho^{2k+1} \leq pT_{2k}(\Gamma) \leq 2^{4k+6}C_* \log^4 |\Gamma| \cdot |\Gamma|^{2k+1} + 16k^2 C_*^{k-1} \log^{4(k-1)} |\Gamma| \cdot |\Gamma|^{2k+1-k+1} p,
\]
(32)
provided $2^{64k}C_*^4 \leq |\Gamma|$. Put $k = \left\lfloor 2 \log p/\log |\Gamma| + 4 \right\rfloor \leq 2/\delta + 5$. Also, notice that
\[
\frac{p \log^{4(k-1)} |\Gamma|}{|\Gamma|^{k/2}} \leq 1
\]
(33)
because $k \geq 2 \log p/\log |\Gamma| + 4$ and $p$ is a sufficiently large number. Also, since $|\Gamma| \geq p^\delta$, it follows that $2^{64k}C_*^4 \leq |\Gamma|$ for sufficiently large $p$. Taking a power $1/2^{k+1}$ from both parts of (32), we see in view of (33) that
\[
\rho \ll |\Gamma| \left(\frac{\log |\Gamma|}{2^{k+1}} + \frac{1}{2^{k+1}}\right) \ll |\Gamma|^{1-\frac{1}{2^{k+1}}} \ll |\Gamma| \cdot p^{-\frac{\delta}{2^{7+2k+1}}}.
\]
To prove the second part of our corollary just notice that the same choice of $k$ gives something nontrivial if $2^{k+2} \leq \varepsilon \log |\Gamma|$ for any $\varepsilon > 0$. In other words, it is enough to have
\[
k + 2 \leq \frac{2 \log p}{\log |\Gamma|} + 7 \leq \log \log |\Gamma| - \log(1/\varepsilon).
\]
It means that the inequality $\log |\Gamma| \geq C \log p/(\log \log p)$ for any $C > 2$ is enough. This completes the proof. \hfill \Box

Remark 17 One can improve some constants in the proof (but not the constant $C$ in (31)), probably, but we did not make such calculations.

Now we estimate a ”dual” quantity $E_+^+(Q)$ for $\Gamma$–invariant set $Q$ (about duality of $T_{k/2}(A)$ and $E_k^+(A)$, see [27] and formulae (36)–(39)). We give even two bounds and both of them use the Fourier approach.
Theorem 18 Let $\Gamma \subseteq \mathbb{F}_p^*$ be a multiplicative subgroup, and $Q \subseteq \mathbb{F}_p^*$ be a set with $Q\Gamma = Q$ and $|Q|^2 |\Gamma| \leq p^2$. Then for $0 \leq k$, $2^{64k} C_8^4 \leq |\Gamma|$ one has

\[
E_{2k+1}^+(Q) \leq 2^{2k+2+3}(\log |Q|)^{2k+1}|Q|^{2k+1} \times \\
\times \left(2^{4k+6}\log^4 |Q| + 16k^2 C_{k-1} (\log |Q|)^{4(k-1)} \cdot |\Gamma|^{-\frac{(k+1)}{2}} p \right).
\]  

Further let $k \geq 1$ be such that $|\Gamma|^\frac{k+2}{2} \geq |Q| \log 4k |Q|$. Then

\[
E_{2k+1}^+(Q) \leq (2^8 C_8)^{k+1}|Q|^{2k+1} |\Gamma|^\frac{1}{2}.
\]

Proof. We begin with (31) and we prove this inequality by induction. For $k = 0$ the result is trivial in view of our condition $|Q|^2 |\Gamma| \leq p^2$. Put $s = 2^k$, $k \geq 1$. By the Parseval identity and formula (12), we have

\[
E_{2s}^+(Q) = \frac{1}{p^{2s-1}} \sum_{x_1+\cdots+x_{2s}=0} |\hat{Q}(x_1)|^2 \cdots |\hat{Q}(x_{2s})|^2 \leq 
\]

\[
\leq 2s|Q|^2 E_{2s-1}^+(Q) \cdot \frac{1}{p^{2s-1}} \sum_{x_1+\cdots+x_{2s}=0 \setminus \forall j x_j \neq 0} |\hat{Q}(x_1)|^2 \cdots |\hat{Q}(x_{2s})|^2 = 
\]

\[
= \frac{2s|Q|^2 E_{2s-1}^+(Q)}{p} + \mathcal{E}_s(Q).
\]

Put $L = \log |Q|$. By the Parseval identity

\[
\frac{1}{p^{2s-1}} \sum_{x_1+\cdots+x_{2s}=0 \setminus \forall j x_j \neq 0} |\hat{Q}(x_1)|^2 \cdots |\hat{Q}(x_{2s})|^2 \leq 
\]

\[
\leq \max_{x \neq 0} |\hat{Q}(x)|^2 \cdot \frac{1}{p^{2s-1}} \sum_{x_1+\cdots+x_{2s}=0 \setminus \forall j x_j \neq 0} |\hat{Q}(x_1)|^2 \cdots |\hat{Q}(x_{2s})|^2 \leq \max_{x \neq 0} |\hat{Q}(x)|^2 \cdot |Q|^{2s-1}.
\]

Hence as in the proof of Theorem 12 consider $\rho^2 = E_{2s}^+(Q)/(8|Q|^{2s-1})$, further, the sets $P_j = \{x : \rho 2^{j-1} \leq |\hat{Q}(x)| \leq \rho 2^j\} \subseteq \mathbb{F}_p^*$ and using the Dirichlet principle, we find $\Delta = \Delta_{j_0} \geq \rho$ and $P = P_{j_0}$ such that

\[
\mathcal{E}_s(Q) \leq \frac{4L^{2s}(2\Delta)^{4s}}{p^{2s-1}} T^+_s(P).
\]

Clearly, $P \Gamma = P$ (and this is the crucial point of the proof, actually). Applying Corollary 14 we get

\[
\mathcal{E}_s(Q) \leq \frac{2^{4s+2}L^{2s} \Delta^{4s}}{p^{2s-1}} \cdot \left(2^{4k+6}\log^4 |\Gamma| \cdot \frac{|P|^{2s}}{p} + 16k^2 C_{k-1} \log^{4(k-1)} |\Gamma| \cdot |\Gamma|^{-\frac{(k+1)}{2}} E^+(\Gamma)|P|^{2s} \right).
\]

(40)
By the Parseval identity, we see that
\[ \Delta^2 |P| \leq |Q|p. \]  
(41)

Whence
\[ E'_{2s}(Q) \leq 2^{4s+2}L^{2s}|Q|^{2s} \cdot \left( 2^{4k+6}L^4 + 16k^2C_s^{-1}L^{4(k-1)} \cdot |\Gamma| \right)^{\frac{(k+1)}{2}}E^+(\Gamma)p. \]  
(42)

Using a trivial bound $E^+(\Gamma) \leq |\Gamma|^3$, we get
\[ E'_{2s}(Q) \leq 2^{4s+2}L^{2s}|Q|^{2s} \cdot \left( 2^{4k+6}L^4 + 16k^2C_s^{-1}L^{4(k-1)} \cdot |\Gamma| \right)^{\frac{(k+1)}{2}}p. \]  
(43)

Applying a crude bound $E^+_{2s-1}(Q) \leq |Q|^{s-1}E^+_s(Q)$, the condition $|Q|^2|\Gamma| \leq p^2$, and induction assumption, we get
\[ \frac{2s|Q|^2E^+_{2s-1}(Q)}{p} \leq \frac{2s|Q|^{s+1}E^+_s(Q)}{p} \leq \]  
\[ \leq \frac{2s|Q|^{s+1}}{p} \cdot L^s|Q|^s \cdot 2^{2s+3} \left( 2^{4k+6}L^4 + 16k^2C_s^{-1}L^{4(k-2)} \cdot |\Gamma| \right)^{\frac{1}{2}} \leq \]  
\[ \leq 2^{4s+2}L^{2s}|Q|^{2s} \cdot \left( 2^{4k+6}L^4 + 16k^2C_s^{-1}L^{4(k-1)} \cdot |\Gamma| \right)^{\frac{(k+1)}{2}}p. \]

Hence combining the last estimate with (43), we derive
\[ E^+_{2k+1}(Q) \leq 2^{2k+2}L^2|Q|^{2k+1} \cdot \left( 2^{4k+6}L^4 + 16k^2C_s^{-1}L^{4(k-1)} \cdot |\Gamma| \right)^{\frac{(k+1)}{2}} \]

and thus we have obtained (44).

To get (45), put $l = 2^{k-1}$, $k \geq 1$ and consider $E^+_l(Q)$. Further define $g(x) = r^l_{Q-Q}(x)$ and notice that $\hat{g}(\xi) \geq 0$, $\hat{g}(0) = E^+_l(Q)$. Moreover, taking the Fourier transform and using the Dirichlet principle, we get
\[ E^+_l(Q) = \frac{1}{p^3} \sum_{x,y,z} \hat{g}(x)\hat{g}(y)\hat{g}(x+y+z) = \frac{E^+_l(Q)E^+_l(Q)}{p^3} \leq \frac{4L^4(2\omega)^4}{p^3}E^+(G), \]  
(44)

where $G = \{ \xi : \omega < \hat{g}(\xi) \leq 2\omega \} \subseteq \mathbb{R}^n$, and $\omega \geq 2^{-3}E^+_l(Q)|Q|^{-3l} := \rho_s$ because the sum over $\hat{g}(\xi) < \rho_s$ by formula (10) does not exceed
\[ \frac{4\rho_s}{p^3} \cdot \sum_{x,y,z} \hat{g}(y)\hat{g}(x+y+z) = 4\rho_s^3(0) = 4\rho_s|Q|^{3l}. \]

Further in view of the Parseval identity, we see that
\[ \omega^2|G| \leq \sum_{\xi \in G} \hat{g}(\xi)^2 \leq pE^+_l(Q), \]  
(45)

and by formula (10)
\[ \omega|G| = \sum_{\xi \in G} \hat{g}(\xi) = pg(0) = p|Q|^l. \]  
(46)
Clearly, $G$ is $\Gamma$–invariant set (again it is the crucial point of the proof). Further returning to (44) and applying Lemma 9, we see that
\[
E_{d}^{+}(Q) = \frac{4E_{l}^{+}(Q)E_{\gamma l}^{+}(Q)}{p} + \frac{2^{6}L^{4}\omega^{4}}{p^{3}}E^{+}(G) \leq \frac{4E_{l}^{+}(Q)E_{\gamma l}^{+}(Q)}{p} + \frac{2^{6}C_{s}L^{4}\omega^{4}}{p^{3}} \left( \frac{|G|^{4}}{p} + \frac{|G|^{3}}{|\Gamma|^{1/2}} \right) = \frac{4E_{l}^{+}(Q)E_{\gamma l}^{+}(Q)}{p} + E_{d}^{+}(Q).
\]
Applying (45), (46), we get
\[
E_{d}^{+}(Q) \leq 2^{6}C_{s}L^{4}|Q|^{4l} + \frac{2^{6}C_{s}L^{4}(\omega|G|)^{2}2^{2}|G|}{|\Gamma|^{1/2}p^{3}} \leq 2^{6}C_{s}L^{4}|Q|^{4l} + 2^{6}C_{s}L^{4}|Q|^{2l}E_{2l}^{+}(Q)|\Gamma|^{-1/2}.
\]
It follows that
\[
E_{d}^{+}(Q) \leq 4E_{l}^{+}(Q)E_{\gamma l}^{+}(Q) + 2^{6}C_{s}L^{4}|Q|^{2l}E_{2l}^{+}(Q) \left( \frac{|Q|^{2l}}{E_{2l}^{+}(Q)} + \frac{1}{|\Gamma|^{1/2}} \right) . \tag{47}
\]
Further estimating the first term of (47) very roughly as
\[
\frac{E_{l}^{+}(Q)E_{\gamma l}^{+}(Q)}{p} \leq \frac{|Q|^{r+1}E_{\gamma l}^{+}(Q)}{p} \leq \frac{|Q|^{2l+1}E_{\gamma l}^{+}(Q)}{p},
\]
we get in view of our condition $|Q|^2|\Gamma| \leq p^2$ that this term is less than $L^4|Q|^{2l}E_{2l}^{+}(Q)|\Gamma|^{-1/2}$. Whence
\[
E_{d}^{+}(Q) \leq 2^{7}C_{s}L^{4}|Q|^{2l}E_{2l}^{+}(Q) \left( \frac{|Q|^{2l}}{E_{2l}^{+}(Q)} + \frac{1}{|\Gamma|^{1/2}} \right) . \tag{48}
\]
Notice that the term $\frac{|Q|^{2l}}{E_{2l}^{+}(Q)} + \frac{1}{|\Gamma|^{1/2}} \leq 2$. Applying bound (47) exactly 0 $\leq s \leq k$ times, where $s$ is the maximal number (if it exists) such that the second term $\frac{1}{|\Gamma|^{1/2}}$ in formula (48) dominates, we obtain
\[
E_{2k+1}^{+}(Q) \leq (2^{s}C_{s})^{s}L^{4s}|\Gamma|^{-s/2}|Q|^{2s+\ldots+2k-s+1}E_{2k+s+1}^{+}(Q) \left( \frac{|Q|^{2k-s+1}}{E_{2k+s+1}^{+}(Q)} + \frac{1}{|\Gamma|^{1/2}} \right) . \tag{49}
\]
Now by the definition of $s$, we see that the first term in (49) dominates. Whence, using (47), (48) one more time (if $s < k$), we get
\[
E_{2k+1}^{+}(Q) \leq 2(2^{s}C_{s})^{s}L^{4s}|\Gamma|^{-s/2}|Q|^{2k+1-2k-s+1} \cdot |Q|^{2k-s+1} = 2(2^{s}C_{s})^{s}L^{4s}|\Gamma|^{-s/2}|Q|^{2k+1} . \tag{50}
\]
From the assumption $|\Gamma|^{4k+2} \geq |Q|\log^{4k}Q|$, it follows that $|\Gamma| \geq |Q|^{2/(k+2)} \log^{8k/(k+2)}|Q|$. Hence bound (50) is much better than (35) if $s < k$. If $s = k$, then by the same calculations, we derive
\[
E_{2k+1}^{+}(Q) \leq (2^{8}C_{s})^{k}L^{4k}|\Gamma|^{-k/2}E_{2}^{+}(Q)|Q|^{2k+1-2} .
\]
Since $|Q|^2|\Gamma| \leq p^2$ by Lemma [9] it follows that $E^+(Q) \leq 2C_s|Q|^3/|\Gamma|^{1/2}$ and hence
\[ E^+_{2k+1}(Q) \leq (2^8C_s)^{k+1}L^{4k}|\Gamma|^{-(k+1)/2}|Q|^{2k+1+1}. \]

Further by the choice of $k$, namely, $|\Gamma|^{k+2} \geq |Q|\log^{4k}|Q|$ we see that the last bound is better than (53). Finally, if $s = 0$, then by definition $E^+_{2k}(Q) \leq |Q|^{2k}|\Gamma|^{1/2}$ and hence $E^+_{2k+1}(Q) \leq |Q|^{2k+1}|\Gamma|^{1/2}$. This completes the proof. \hfill \Box

**Remark 19** From the second part of the arguments above one can derive explicit bounds for the energies $E^+_s(Q)$ for small $s$. For example,
\[ E_4(Q) \ll \frac{|Q|^2E_3(Q)}{p} + (\log |\Gamma|)^4|Q|^4 + (\log |\Gamma|)^4|Q|^2E(Q)|\Gamma|^{-1/2}. \]

Now we obtain an uniform upper bound for size of the intersection of an additive shift of any $\Gamma$–invariant set. Our bound (52) is especially effective if sizes of $Q_1,Q_2$ are comparable with size of $\Gamma$, namely, $|Q_1|, |Q_2| \ll |\Gamma|^C$. $C$ is an absolute constant (which can be large). In this case the number $k$ below is a constant as well.

**Corollary 20** Let $\Gamma \subseteq \mathbb{F}_p^*$ be a multiplicative subgroup, $|\Gamma| \geq p^\delta$, $\delta > 0$, and $Q_1,Q_2 \subseteq \mathbb{F}_p^*$ be two sets with $Q_1\Gamma = Q_1$, $Q_2\Gamma = Q_2$, $|Q_1|^2|\Gamma| \leq p^2$, $|Q_2|^2|\Gamma| \leq p^2$. Put $Q = \max\{|Q_1|, |Q_2|\}$. Then for any $x \neq 0$, one has
\[ |Q_1 \cap (Q_2 + x)| \ll \sqrt{|Q_1||Q_2|} \log Q \cdot p^{-\frac{4}{27+2\delta}}. \]

Further choose $k \geq 1$ such that $|\Gamma|^{\frac{k+2}{2}} \geq Q\log^{4k}Q$. Then for an arbitrary $x \neq 0$ the following holds
\[ |Q_1 \cap (Q_2 + x)| \ll \sqrt{|Q_1||Q_2|} \cdot |\Gamma|^{-\frac{4}{2}2^{-k}}. \]

**Proof.** From the conditions $|Q_1|^2|\Gamma| \leq p^2$, $|Q_2|^2|\Gamma| \leq p^2$, it follows that $|\Gamma| \leq p^{2/3}$. Put $L = \log Q$. On the one hand, applying the Cauchy–Schwarz inequality, we obtain
\[ \sum_y r_{Q_1-Q_2}(y)^2 \leq (E^+_{2k+1}(Q_1))^{1/2}(E^+_{2k+1}(Q_2))^{1/2}. \]

On the other hand, by formula (54) of Theorem 13 and $\Gamma$–invariance of $Q_1$, $Q_2$, we have
\[ |\Gamma||Q_1 \cap (Q_2 + x)|^{2k+1} \leq \sum_y r_{Q_1-Q_2}(y)^2 \leq 2^{2k+3}L^{2k+1}\left(|Q_1||Q_2|\right)^2 \log^4 |Q| + 16k^2C_s^{-1}L^{4(k-1)}|\Gamma|^{-(k+1)/2}p, \]

provided $2^{64k}C_s^4 \leq |\Gamma|$. As in Corollary 16 choosing $k = \lceil 2\log p/\log |\Gamma| + 4 \rceil \leq 2/\delta + 5$ and applying an analogue of (33) which holds for large $p$, namely,
\[ \frac{p\log^{4(k-1)}|Q|}{|\Gamma|^{k/2}} \ll 1. \]
we obtain

\[ |Q_1 \cap (Q_2 + x)| \ll L \sqrt{|Q_1||Q_2|} \cdot (|\Gamma|^{-1/2} + |\Gamma|^{1/2}) \ll L \sqrt{|Q_1||Q_2|}|\Gamma|^{-1/2} \ll \]

\[ \ll L \sqrt{|Q_1||Q_2|} p^{-\frac{\delta}{2^{7+25}}} \]

and it easy to insure that inequality \( 2^{64k} \Gamma^4 \leq |\Gamma| \) takes place for sufficiently large \( p \).

To derive (52), we just use the second formula (35) of Theorem 18 and the previous calculations. This completes the proof. \( \Box \)

**Remark 21** It is known, see, e.g., [20] that if \( \Gamma \subseteq \mathbb{F}_p^* \) is a multiplicative subgroup with \( |\Gamma| < p^{3/4} \), then for any \( x \neq 0 \) one has \( |\Gamma \cap (\Gamma + x)| \ll |\Gamma|^{2/3} \) and this bound it tight in some regimes. One can extend this to larger \( \Gamma \)-invariant sets and obtain a lower bound of a comparable quality. It gives a lower estimate in (51).

Indeed, let \( \Gamma \subseteq \mathbb{F}_p^* \) be a multiplicative subgroup with \( |\Gamma| < p^{1/2} \). Consider \( R = R[\Gamma] \) and \( Q = Q[\Gamma] \). It was proved in [30] that \( |R| \gg |\Gamma|^2/\log |\Gamma| \) and one can check that \( R = 1 - R \), see, e.g., [21]. Finally, the set \( Q \) is \( \Gamma \)-invariant and it is easy to check [32] that \( |Q| \leq |\Gamma|^3 \). Whence

\[ |Q \cap (1 - Q)| \geq |R| \gg \frac{|\Gamma|^2}{\log |\Gamma|} \gg \frac{|Q|^{2/3}}{\log |Q|} \cdot \]

Also, notice that if \( |\Gamma| < p^{1/2} \) and \( |Q[\Gamma]|^2 |\Gamma| \leq p^2 \), then \( |Q[\Gamma]| \gg |\Gamma|^{2+c} \) for some \( c > 0 \), see the first part of Corollary 35 from the next section.

Corollary 20 gives a nontrivial upper bound for the common additive energy of an arbitrary invariant set and any subset of \( \mathbb{F}_p \).

**Corollary 22** Let \( \Gamma \subseteq \mathbb{F}_p^* \) be a multiplicative subgroup, \( |\Gamma| \geq p^\alpha \), \( \delta > 0 \), and \( Q \subseteq \mathbb{F}_p^* \) be a set with \( Q\Gamma = Q \), \( |Q|^2 |\Gamma| \leq p^2 \). Then for any set \( A \subseteq \mathbb{F}_p \), one has

\[ E^+(A, Q) \ll |Q||A|^2 \cdot p^{-\frac{\delta}{2^{7+25}}} \log |Q| + |A||Q|. \quad (53) \]

Further for an arbitrary \( \alpha \neq 0 \) the following holds

\[ E^x(A, Q + \alpha) \ll |Q||A|^2 \cdot p^{-\frac{\delta}{2^{7+25} - 1}} \log |Q| + |A||Q|. \quad (54) \]

In particular,

\[ |A + Q| \gg |Q| \cdot \min\{|A|, p^{\frac{\delta}{2^{7+25} - 1}} \log^{-1} |Q|\}, \quad (55) \]

and

\[ |A(Q + \alpha)| \gg |Q| \cdot \min\{|A|, p^{\frac{\delta}{2^{7+25} - 1}} \log^{-1} |Q|\}. \quad (56) \]

Further if \( k \geq 1 \) is chosen as \( |\Gamma|^{\frac{\delta}{2 + 12}} \geq |Q| \log^k |Q| \), then one can replace the quantity \( p^{\frac{\delta}{2^{7+25} - 1}} \log^{-1} |Q| \) above by \( |\Gamma|^{-\frac{1}{2} \cdot 2^{-k}} \).
where \( B \) is a sufficiently large set. From the last result we derive our quantitative asymmetric sum–product Theorem 5 from the introduction. Let us begin with an upper bound for \( T \).



In this section we obtain an upper bound for \( T \). We have

\[
E^+(A, Q) = \sum_x r_{A-A}(x)Q-Q(x) = |A||Q| + \sum_{x \neq 0} r_{A-A}(x)Q-Q(x) \ll |A||Q| + |Q|^2 \cdot p^{-\frac{d}{2^d+2^d-1}} \log |Q|
\]

as required. Similarly

\[
E^x(A, Q + \alpha) \ll |A||Q| + \sum_{x \neq 0, 1} r_{A/A}(x)r_{(Q+\alpha)/(Q+\alpha)}(x) \ll |A||Q| + |Q|^2 \cdot p^{-\frac{d}{2^d+2^d-1}} \log |Q|
\]

because in view of Corollary 20 one has

\[
r_{(Q+\alpha)/(Q+\alpha)}(x) = |Q \cap (Q + \alpha(x - 1))| \ll |Q| \cdot p^{-\frac{d}{2^d+2^d-1}} \log |Q|.
\]

So, we have obtained bounds (53)–(56) with \( p^{-\frac{d}{2^d+2^d-1}} \log^{-1} |Q| \) and to replace it by \( |\Gamma|^{-\frac{1}{2} 2^{-k}} \) one should use the second part of Corollary 20. This completes the proof.

From (55) one can obtain that for any multiplicative subgroup \( \Gamma \subseteq \mathbb{F}_p^* \) there is \( N \) such that \( N\Gamma = \mathbb{F}_p^* \) and \( N \ll \delta^{-1} \delta^{-1} \). The results of comparable quality were obtained in [15].

5 The proof of the main result

In this section we obtain an upper bound for \( T_k^+(A) \) (see Theorem 23) and an upper bound for \( E_k^+(A) \) (see Theorem 27) in the case when size of the product set \( AB \) is small comparable to \( A \), where \( B \) is a sufficiently large set. From the last result we derive our quantitative asymmetric sum–product Theorem 5 from the introduction. Let us begin with an upper bound for \( T_k^+(A) \).

**Theorem 23** Let \( A, B \subseteq \mathbb{F}_p \) be sets, \( M \geq 1 \) be a real number and \( |AB| \leq M|A| \). Then for any \( k \geq 2, 2^{16k} M^{2k+1} C_s^2 \log^8 |A| \leq |B| \), one has

\[
T_{2k}^+(A) \leq 2^{4k+6} C_s \log^4 |A| \cdot \frac{M^{2k} |A|^{2k+1}}{p} + 16k^2 C_s^{k-1} M^{2k+1} \log^{4(k-1)} |A| \cdot |A|^{2k+1} - 4|B|^{-\frac{1}{2}} E^+(A).
\]

**Proof.** We have \( B \neq \{0\} \) by the condition \( 2^{16k} M^{2k+1} C_s^2 \log^8 |A| \leq |B| \), say. We apply the arguments and the notation of the proof of Theorem 12. Fix any \( s \geq 2 \) and put \( L := s \log |A| \). Our intermediate aim is to prove

\[
T_{2s}^+(A) \leq C_s^4 M^{2s} \log^4 |A| \cdot \left( \frac{|A|^{2s}}{p} + \frac{|A|^{2s}}{\sqrt{|B|} T_s^+(A)} \right),
\]

(58)
where $C = 2^5 C_\ast$. As in the proof of Theorem 12 we get

$$T_{2s}^+(A) \leq \frac{8}{5} L^4 (2\Delta)^4 E^+(P) + \mathcal{E},$$

where

$$\mathcal{E} \leq 4|A|^{2s-1} T_s^+(A)$$

(59)

further $\Delta > T_{2s}^+(A)/(8|A|^{3s})$ is a real number and $P = \{x : \Delta < r_{sA}(x) \leq 2\Delta\} \subseteq \mathbb{P}$. Moreover, we always have $|P|\Delta^2 \leq T_s^+(A)$.

To proceed as in the proof of Theorem 12 we need to estimate $|PB|$. Observe that for any $x \in PB$ the following holds

$$|PB| \Delta \leq \sum_{x \in PB} r_{sAB}(x) \leq |AB|^s \leq M^s|A|^s.$$  (60)

Hence using Lemma 9, we obtain

$$E^+(P) \leq C_* \left( \frac{M^{2s}|A|^{2s}|P|^2}{\Delta^2 p} + \frac{M^{3s/2}|A|^{3s/2}|P|^{3/2}}{\Delta^{5/2}|B|^{1/2}} \right).$$

Hence in view of estimate (59), combining with $|P|\Delta \leq |A|^s$ and $|P|\Delta^2 \leq T_s^+(A)$, we get

$$T_{2s}^+(A) \leq \frac{8}{5} (16C_\ast)L^4 \Delta^4 \left( \frac{M^{2s}|A|^{2s}|P|^2}{\Delta^2 p} + \frac{M^{3s/2}|A|^{3s/2}|P|^{3/2}}{\Delta^{5/2}|B|^{1/2}} \right) + 4|A|^{2s-1} T_s^+(A) =$$

$$= \frac{8}{5} (16C_\ast)L^4 \left( \frac{M^{2s}|A|^{2s}|P|^2}{p} + \frac{M^{3s/2}|A|^{3s/2}|P|^{3/2}}{|B|^{1/2}} \right) + 4|A|^{2s-1} T_s^+(A) \leq$$

$$\leq \frac{8}{5} (16C_\ast)L^4 \left( \frac{M^{2s}|A|^{4s}}{p} + \frac{M^{3s/2}|A|^{3s/2} T_s^+(A)}{|B|^{1/2}} \right) + 4|A|^{2s-1} T_s^+(A) \leq$$

$$\leq 32C_\ast L^4 \left( \frac{M^{2s}|A|^{4s}}{p} + \frac{M^{3s/2}|A|^{2s} T_s^+(A)}{|B|^{1/2}} \right)$$

and inequality (58) is proved. Here we have used a trivial inequality $|B|^{1/2} \leq |A|$ which follows from $|B| \leq |AB| \leq M|A| \leq |B|^{1/2} |A|$ because $M^2 \leq 2^{16k} M^{2k+1} C_\ast^2 \log^8 |A| \leq |B|$.

Now applying formula (58) successively $(k-1)$ times, we obtain

$$T_{2k}(A) \leq 2^{4k+6} C_\ast \log^4 |A| \cdot \frac{M^{2k}|A|^{2k+1}}{p} + 16^{k^2} M^{2k+1} C_\ast^{k-1} \log^{4(k-1)} |A| \cdot |A|^{2k-4} \times 4|B|^{-(k-1)} E^+(A) \leq$$

$$\leq 2^{4k+6} C_\ast \log^4 |A| \cdot \frac{M^{2k}|A|^{2k+1}}{p} + 16^{k^2} M^{2k+1} C_\ast^{k-1} \log^{4(k-1)} |A| \cdot |A|^{2k+1-4} |B|^{-(k-1)} E^+(A).$$  (61)

To get the first term in we have used our condition $2^{16k} M^{2k+1} C_\ast^2 \leq |B|$ to insure that $|B|^{1/2} \geq 2^{4k+1} C_\ast M^{2k} \log^4 |A|$. This completes the proof. $\square$
Remark 24 It is easy to see that instead of the assumption $|AB| \ll |A|$ we can assume a weaker condition $|A^s \cdot \Delta_s(B)| \ll |A|^s$, $1 < s \leq 2^{k-1}$, see formula (60).

The same arguments work in the case of real numbers. In this situation we have no characteristic $p$ and hence we have no any restrictions on the parameter $k$.

Theorem 25 Let $A, B \subset \mathbb{R}$ be finite sets, $M \geq 1$ be a real number and $|AB| \leq M|A|$. Then for any $k \geq 2$, one has

$$T_{2^k}^+(A) \leq 16^{k^2} C_s^{k-1} M \frac{2}{3}(2^{k-1}) \log^{4(k-1)} |A| \cdot |A|^{2k+1} |B|^{-\frac{k}{3}} .$$

(62)

Corollary 26 Let $A \subset \mathbb{R}$ be a finite set, $M \geq 1$ be a real number and $|AA| \leq M|A|$ or $|A/A| \leq M|A|$. Then for any $k \geq 2$, one has

$$|2^k A| \gg_k |A|^{1+\frac{3}{2}k} M^{-\frac{3}{2}(2^{k-1})} \cdot \log^{-4(k-1)} |A| .$$

(63)

Bounds of such a sort were obtained in [18] by another method. The best results concerning lower bounds for multiple sumsets $kA$, $k \to \infty$ of sets $A$ with small product/quotient set can be found in [12].

To obtain an analogue of Theorem 18 for sets with $|AA| \ll |A|$ we cannot use the same arguments as in section 4 because the spectrum is not an invariant set in this case. Moreover, in $\mathbb{R}$ there is an additional difficulty with using Fourier transform: the dual group of $\mathbb{R}$ does not coincide with $\mathbb{R}$ of course. That is why we suggest another method which works in "physical space" but not in the dual group.

To formulate our main result about $E_{2^k}^+(Q)$ for sets $Q$ with small product $Q\Gamma$ for some relatively large set $\Gamma$ we need some notation. Let us write $Q^{(k)} = |Q\Gamma^{k-1}|$ for $k \geq 1$ and $Q^{(k)} = |Q|$ for $k = -1$.

Theorem 27 Let $\Gamma, Q \subseteq \mathbb{F}_p$ be two sets, and $k \geq 0$ be an integer. Suppose that $|Q\Gamma^{k+1}||Q\Gamma^k||\Gamma| \leq p^2$, further $Q^{(k)}|\Gamma| \leq p$, and $M = |Q\Gamma^{k+1}|/|Q|$. Then either

$$E_{2^{k+1}}^+(Q) \leq (M^{2k+1}2^{3k+1} C_s^{(k+4)/4} \log^k Q^{(k)}) \cdot |Q|^{2k+1} |\Gamma|^{-k/8-1/2}$$

or

$$E_{2^{k+1}}^+(Q) \leq 2(Q^{(k)})^{2k+1} .$$

(64)

In particular, if we choose $k$ such that $|\Gamma|^{k/8+1/2} \geq |Q| \cdot M^{2k+1}2^{3k+1} C_s^{(k+4)/4} \log^k Q^{(k)}$, then

$$E_{2^{k+1}}^+(Q) \leq 2(Q^{(k)})^{2k+1} .$$

(65)
Applying Corollary 11, we obtain
\[ E_{5l/2}^+(Q) \leq 8C^*_1 \log |Q| \cdot |Q|^{1/2}E_{5l}^+(Q \Gamma)|\Gamma|^{-1/8} \] (66)
or
\[ E_{5l/2}^+(Q) \leq 2|Q|^{5l/2}. \] (67)

Put \( g(x) = r_{Q-Q}(x), L = \log |Q| \) and \( E_{5l/2}^+(Q) = E_{5l/2}^+(Q) - |Q|^{5l/2} \geq 0 \). We will assume below that \( E_{5l/2}^+(Q) \geq 2^{-1}E_{5l/2}^+(Q) \) because otherwise we obtain (67) immediately. Using the Dirichlet principle, we find a set \( P \) and a positive number \( \Delta \) such that \( P = \{ x : \Delta < g(x) \leq 2\Delta \} \subseteq \mathbb{F}_p \) and
\[ E_{5l/2}^+(Q) \leq L \sum_{x \in P} r_{Q-Q}^5(x). \]

Applying Corollary 11, we obtain
\[ E_{5l/2}^+(Q) \leq L(2\Delta)^{3/2} \sum_{x \in P} r_{Q-Q}^l(x) \leq 3C^{1/4}_1 L \Delta^{3/2} |Q|^{l/2}(E_{2l}(Q \Gamma))^{1/4} \left( \frac{|P|^4}{p} + \frac{|P|^3}{|\Gamma|^{1/2}} \right)^{1/4} \]
\[ \leq 3C^{1/4}_1 L |Q|^{l/2}(E_{2l}(Q \Gamma))^{1/4} \left( \frac{\Delta^6|P|^4}{p} + \frac{\Delta^6|P|^3}{|\Gamma|^{1/2}} \right)^{1/4}. \]

We have \( \Delta|P| \leq E_l^+(Q), \Delta^2|P| \leq E_{2l}^+(Q) \) and hence \( \Delta^6|P|^4 \leq (E_{2l}^+(Q))^2(E_l^+(Q))^2 \). It follows that
\[ E_{5l/2}^+(Q) \leq 3C^{1/4}_1 L |Q|^{l/2}(E_{2l}(Q \Gamma))^{1/4} \left( \frac{(E_{2l}^+(Q))^2(E_l^+(Q))^2}{p} + \frac{(E_l^+(Q))^3}{|\Gamma|^{1/2}} \right)^{1/4}. \]

To prove that the first term \( \frac{(E_{2l}(Q))^2(E_l^+(Q))^2}{p} \) is less than \( \frac{(E_{2l}^+(Q))^3}{|\Gamma|^{1/2}} \), we need to check that
\[ (E_l^+(Q))^2|\Gamma|^{1/2} \leq E_{2l}^+(Q)p. \]

But using the Hölder inequality, we see that the required estimate follows from
\[ (E_l^+(Q))^2|\Gamma|^{1/2} \leq (E_{2l}(Q))^2 \left( \frac{2l-1}{2l-1} \right) |Q|^{2l-1} |\Gamma|^{1/2} \leq E_{2l}^+(Q)p \]
or, in other words, from
\[ |Q|^{2l} |\Gamma|^{(2l-1)/2} \leq E_{2l}^+(Q)p^{2l-1}. \] (68)

Finally, we can suppose that for any \( s \geq 2 \) one has, say,
\[ E_{s}^+(Q) \geq |Q|^{s+1} |\Gamma|^{-1/2} \log s^{-1/2} \]
because otherwise estimate (67) follows easily. In view of our assumption \(|Q||\Gamma| \leq p\), we obtain
\[ |Q|^{2l-1} |\Gamma|^{1/2} \log 2l+1 \leq p^{2l-1} |\Gamma|^{1/2} \log 2l-1 \leq p^{2l-1} \]
and hence (68) takes place for \( l \geq 2 \). For \( l = 1 \) see calculations below. Hence under this assumption and the inequality \( E_{5l/2}^+(Q) \geq 2^{-1}E_{5l/2}^+(Q) \), we have
\[
E_{5l/2}^+(Q) \leq 8C_{s}^{1/4} \log |Q| \cdot |Q|^{l/2} E_{2l}^+(Q \Gamma |\Gamma|^{-1/8})
\]
and we have proved (69). Trivially, it implies that
\[
E_{4l}^+(Q) \leq 8C_{s}^{1/4} \log |Q| \cdot |Q|^{2l} E_{2l}^+(Q \Gamma |\Gamma|^{-1/8})
\]
and subsequently using this bound, we obtain
\[
E_{2k+1}^+(Q) \leq (2^{3k}C_{s}^{k/4} \log k |Q \Gamma^{k-1} |) \cdot M^{2k-1+\ldots+2} |Q|^{2k+\ldots+2} E^+(Q \Gamma^k) |\Gamma|^{-k/8} = \\
(2^{3k}M^{2k-2}C_{s}^{k/4} \log k |Q \Gamma^{k-1} |) \cdot |Q|^{2k+1-2} E^+(Q \Gamma^k) |\Gamma|^{-k/8}.
\]
Now recalling the assumption \(|Q \Gamma^{k+1}||\Gamma|^{k} |\Gamma| \leq p^2\) and applying Lemma 9 we get
\[
E_{2k+1}^+(Q) \leq (M^{2k+1}2^{3k+1}C_{s}^{(k+4)/4} |Q \Gamma^{k-1} |) \cdot |Q|^{2k+1} |\Gamma|^{-k/8-1/2}.
\]
In particular, this final step covers the remaining case \( l = 1 \) above. This completes the proof. \( \square \)

**Remark 28** Let \( \Gamma \) be a multiplicative subgroup and \( Q \Gamma = Q \). Then by Theorem 27 if \(|Q ||\Gamma| \leq p\) and a number \( k_1 \) is chosen as \(|\Gamma|^{k_1} 8/\sqrt{\pi} \geq |Q| \log k_1 |Q|\), then \( E_{2k+1}^+(Q) \ll_{k_1} |Q|^{2k+1} \). Let us compare this with Theorem 18. By this result, choosing \( k_2 \) such that \(|\Gamma|^{k_2} 8/\sqrt{\pi} \geq |Q| \log 4k_2 |Q|\), we get \( E_{2k+1}^+(Q) \ll_{k_2} |Q|^{2k+1} |\Gamma|^{1/2} \). After that applying the second part of Corollary 27 other \( n := 2k_2+1 \) times, we obtain
\[
E_{2k_2+2}^+(Q) \ll_{k_2} |Q|^{2k_2+2} + E_{2k+1}^+(Q) (|Q| |\Gamma|^{-1/2} 2^{-k_2})^n \ll_{k_2} |Q|^{2k_2+2} + |Q|^{2k_2+1} |\Gamma|^{1/2} |Q|^{n} |\Gamma|^{-1/2} \ll \ll |Q|^{2k_2+2}.
\]
Thus, Theorem 18 gives slightly better bound (in the case of multiplicative subgroups) but of the same form.

**Remark 29** From formula (70), it follows that for any \( l \) one has \( E_{l}^+(Q) \geq |Q|^{2l} \). Hence upper bound (65) can has place just for small sets \( Q \). For example, taking the smallest possible \( l = 2 \) and comparing \(|Q|^2 \) with \(|Q|^4/p\) we see that the condition \(|Q| < \sqrt{p}\) is enough. If \( Q = \Gamma \), where \( \Gamma \) is a multiplicative subgroup, then it is possible to refine this condition because in the proof of Theorem 18 another method (the Fourier approach) was used. We did not make such calculations.

Now we can obtain analogues of Corollaries 20 and 22.

**Corollary 30** Let \( \Gamma, Q_1, Q_2 \subseteq \mathbb{F}_p \) be sets. Take \( k \geq 0 \) such that for \( j = 1, 2 \) one has
\[
|Q_j \Gamma^{k+2}||Q_j \Gamma^{k+1}||\Gamma| \leq p^2, |Q_j \Gamma^k| |\Gamma| \leq p, |Q_j \Gamma| \leq M_s |Q_j|, |Q_j \Gamma^{k+2}| \leq M |Q_j|, \text{ and}
\]
\[
|\Gamma|^{k/8+1/2} \geq |Q_j| \cdot M_s M^{2k+1} 2^{3k+1} C_{s}^{(k+4)/4} \log |Q_j \Gamma^k|.
\]
Then for any \( x \neq 0 \) the following holds
\[
|Q_1 \cap (Q_2+x)| \leq 2M_s M \sqrt{|Q_1| |Q_2|} \cdot |\Gamma|^{-1/2-k}.
\]
Proof. Denote by \( \rho \) the quantity \(|Q_1 \cap (Q_2 + x)|\). On the one hand, applying the Cauchy–Schwarz inequality and the second part of Theorem 27, we obtain

\[
\sum_{y} r_{Q_1 \cap rQ_2}(y) \leq (E_{2k+1}(\Gamma Q_1))^1/2(E_{2k+1}(\Gamma Q_2))^1/2 \leq 2^{3k+2}M^{k+1}(|Q_1||Q_2|)^{2k} \leq 2^{3k+2}M^{k+1}M_*^{k+1}(|Q_1||Q_2|)^{2k}.
\]

On the other hand, it is easy to see that for any \( y \in \Gamma x \) one has \( r_{Q_1 \cap rQ_2}(y) \geq \rho \). Thus,

\[
\rho^{2k+1}\Gamma \leq 2^{3k+2}M^{k+1}M_*^{k+1}(|Q_1||Q_2|)^{2k}.
\]

and hence

\[
\rho \leq 2M_*M\sqrt{|Q_1||Q_2|} \cdot |\Gamma|^{-\frac{1}{2} - k}.
\]

Here we have used inequality \( k \geq 5 \) which easily follows from \(|\Gamma| \leq |Q_1\Gamma| \leq M|Q_1|\) and condition (69). This completes the proof.

In the next two corollaries we show how to replace the condition \(|Q\Gamma^k| \ll |Q|\) to a condition with a single multiplication, namely, \(|Q\Gamma| \ll |Q|\).

Corollary 31 Let \( \Gamma, Q \) be subsets of \( \mathbb{F}_p \), \( M \geq 1 \) be a real number, \(|Q\Gamma| \leq M|Q|\). Suppose that for \( k \geq 1 \) one has \((2M)^{k+1}|Q||\Gamma| \leq p\), and

\[
|\Gamma|^{k/8+1/2} \geq |Q| \cdot (2M)^{(k+3)/2} C_s^{(k+4)/4} \log^k((2M)^k|Q|).
\]

Then for any \( A \subseteq \mathbb{F}_p \) the following holds

\[
|A + Q| \geq 2^{-3}|Q| \cdot \min\{|A|, 2^{-(4+k)}M^{-(k+3)}|\Gamma|^{\frac{1}{2} - k}\},
\]

and for any \( \alpha \neq 0 \)

\[
|A(Q + \alpha)| \geq 2^{-3}|Q| \cdot \min\{|A|, 2^{-(4+k)}M^{-(k+3)}|\Gamma|^{\frac{1}{2} - k}\}.
\]

Proof. Using Lemma 7 find a set \( X \subseteq Q_1, |X| \geq |Q|/2 \) such that for any \( l \) the following holds \(|XT^l| \leq (2M)^l|X|\). Also, notice that \(|XT| \leq |QT| \leq 2M|X|\). We apply Corollary 30 with \( M = (2M)^{k+2}, M_* = 2M \) and see that for any \( x \neq 0 \) the following holds

\[
|Q_1 \cap (Q_2 + x)| \leq 2^{k+4}M^{k+3}\sqrt{|Q_1||Q_2|} \cdot |\Gamma|^{-\frac{1}{2} - k}.
\]

Here \( Q_1 = X \) and \( Q_2 = X \) or \( Q_2 = \alpha X \). Using the arguments from Corollary 22, we estimate the energies \( E^+(A, X), E^+(A, X + \alpha) \). In particular, we obtain lower bounds for the sumset from (72) and the product set from (73). It remains to check condition \((2M)^{2k+3}|Q||\Gamma| \leq p^2\). But it follows from \((2M)^{k+1}|Q||\Gamma| \leq p\) if \( M \leq |\Gamma|/2 \). The last inequality is a simple consequence of (71). This completes the proof.

Now we prove an analogue of Corollary 30 where we require just \(|Q_j\Gamma|, j = 1, 2\) are small comparable to \(|Q_j|\). For simplicity we formulate the next corollary in the situation \(|Q'| = |Q|\) but of course general bound takes place as well.
Corollary 32 Let $\Gamma, Q, Q'$ be subsets of $\mathbb{F}_p$, $|Q'| = |Q|$, $M \geq 1$ be a real number, $|Q\Gamma|, |Q'T| \leq M|Q|$. Suppose that for $k \geq 1$ one has $(2M)^{k+1}|Q||\Gamma| \leq p$, and

$$|\Gamma|^{\frac{k}{8} + \frac{1}{2(k+4)}} \geq |Q| \cdot M^{(k+3)2k} C_s^{(k+4)/4} \log^k (|\Gamma|^{\frac{k}{2(k+4)}2^{-k}} |Q|)$$

Then for any $x \neq 0$ one has

$$|Q \cap (Q' + x)| \leq 4M|Q| \cdot |\Gamma|^{-\frac{1}{2(k+4)}2^{-k}}.$$  \hfill (74)

Proof. Let $\tilde{Q} = Q \cap (Q' + x)$. Then $|\tilde{Q}\Gamma| \leq |Q\Gamma| \leq M|Q| = M|Q|/|\tilde{Q}| \cdot |\tilde{Q}| = M|\tilde{Q}|$. Similarly, $|Q - x\Gamma| \leq |Q\Gamma| \leq M|Q|$. Applying the second part of Corollary 31 with $\alpha = x$, $Q = \tilde{Q}$, $A = \Gamma$ and $M = \tilde{M}$, we get

$$M|Q| \geq |(\tilde{Q} - x)\Gamma| \geq 2^{-(7+k)}|\tilde{Q}|\tilde{M}^{-(k+3)}|\Gamma|^{\frac{1}{2}2^{-k}} = 2^{-(7+k)}M^{-(k+3)}|\tilde{Q}|^{k+4}|Q|^{-(k+3)}|\Gamma|^{\frac{1}{2}2^{-k}}$$

provided

$$|\Gamma|^{\frac{k}{8} + \frac{1}{2(k+4)}} \geq |Q| \cdot (2\tilde{M})^{(k+3)2k} C_s^{(k+4)/4} \log^k ((2\tilde{M})^k |Q|) \geq |\tilde{Q}| \cdot (2\tilde{M})^{(k+3)2k} C_s^{(k+4)/4} \log^k ((2\tilde{M})^k |\tilde{Q}|).$$

It gives us

$$|\tilde{Q}| \leq 4M|Q| \cdot |\Gamma|^{-\frac{1}{2(k+4)}2^{-k}}.$$ \hfill (75)

Now if inequality does not hold, then $\tilde{M} \leq |\Gamma|^{\frac{1}{2(k+4)}2^{-k}}/4$. Hence the condition

$$|\Gamma|^{\frac{k}{8} + \frac{1}{2(k+4)}} \geq |Q| \cdot M^{(k+3)2k} C_s^{(k+4)/4} \log^k (|\Gamma|^{\frac{k}{2(k+4)}2^{-k}} |Q|)$$

is enough. This completes the proof. \hfill \Box

Now we are ready to prove the main asymmetric sum–product result of this section.

Corollary 33 Let $A, B, C \subseteq \mathbb{F}_p$ be arbitrary sets, and $k \geq 1$ be such that $|A||B|^{1+\frac{(k+1)}{2(k+4)}}2^{-k} \leq p$ and

$$|B|^{\frac{k}{8} + \frac{1}{2(k+4)}} \geq |A| \cdot C_s^{(k+4)/4} \log^k (|A||B|).$$ \hfill (76)

Then

$$\max\{|AB|, |A + C|\} \geq 2^{-3}|A| \cdot \min\{|C|, |B|^{\frac{1}{2(k+4)}2^{-k}}\},$$ \hfill (77)

and for any $\alpha \neq 0$

$$\max\{|AB|, |(A + \alpha)C|\} \geq 2^{-3}|A| \cdot \min\{|C|, |B|^{\frac{1}{2(k+4)}2^{-k}}\}.$$ \hfill (78)

Moreover,

$$|AB| + \frac{|A|^2|C|^2}{\mathbb{E}^+(A,C)} \geq 2^{-4}|A| \cdot \min\{|C|, |B|^{\frac{1}{2(k+4)}2^{-k}}\},$$ \hfill (79)
and for any $\alpha \neq 0$ the following holds

\[
|AB| + \frac{|A|^2|C|^2}{E^\times(A + \alpha, C)} \geq 2^{-4}|A| \cdot \min\{|C|, |B|^{\frac{1}{2(k+4)^2-k}}\},
\]

provided

\[
|B|^{\frac{k}{2} - \frac{1}{2} + \frac{1}{2(k+4)}} \geq |A| \cdot C_s^{(k+4)/4} \log^k(|A||B|).
\]

**Proof.** We will prove just (77) because the same arguments hold for (78). Put $|AB| = M|A|, M \geq 1$. Applying Corollary 31 with $Q = A$, $\Gamma = B$, $A = C$ and choosing $k$ such that

\[
|B|^{k/8+1/2} \geq |A| \cdot 2^{(k+3)2^k} M^{(k+3)2^k} C_s^{(k+4)/4} \log^k((2M)^k|A|),
\]

we obtain

\[
|A + C| \geq 2^{-3}|A| \cdot \min\{|C|, 2^{-(k+4)} M^{-(k+3)} |B|^{1/2-k}\}.
\]

Thus, by small calculations (which correspond to the optimal choice of the parameter $M$, namely, $M = M_0 = 2^{-2} |B|^{\frac{1}{2(k+4)^2-k}}$), we get

\[
\max\{|AB|, |A + C|\} \geq 2^{-3}|A| \cdot \min\{|C|, |B|^{\frac{1}{2(k+4)^2-k}}\}.
\]

Substituting $M = M_0$ into (81), we obtain our condition (76). The condition $(2M)^{k+1} |A||B| \leq p$ gives us $|A||B|^{1+\frac{k+1}{2(k+4)^2-k}} \leq p$.

To prove (79), (80) we use Corollary 32 instead of Corollary 31 and apply the arguments of the proof of Corollary 22. We obtain

\[
E^+(A, C), E^\times(A + \alpha, C) \leq 2|A||C| + 4|A||C|^2 \cdot M \cdot |B|^{-\frac{1}{2(k+4)^2-k}}
\]

After that it remains to choose $M$ optimally, $M = 2^{-1}|B|^{\frac{1}{2(k+4)^2-k}}$. This completes the proof. \(\square\)

Notice that one cannot obtain any nontrivial bounds for $\min\{E^\times(A, B), E^+(A, C)\}$. Just take $B$ equals a geometric progression, $C$ equals an arithmetic progression, $|B| = |C|$ and $A = B \cup C$.

**Remark 34** The results of this section take place in $\mathbb{R}$. In this case we do not need in any conditions containing the characteristic $p$.

Corollary 33 gives us a series of examples of "superquadratic expanders" with four variables. The first example of such an expander with four variables was given in [25].

**Corollary 35** Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an injective function. Then for any $\kappa < \frac{1}{40} 2^{-16}$ and an arbitrary finite set $A \subset \mathbb{R}$ one has $|R[A]| \varphi(A) \gg |A|^{2+\kappa}$. In particular,

\[
R[A]A = \left\{ \frac{(y-x)w}{z-x} : x, y, z, w \in A, x \neq z \right\}
\]
is a superquadratic expander with four variables. Moreover, for any finite sets \(A, B, C, D\) of equal sizes one has

\[
\left| \left\{ \frac{(y-x)w}{z-x} : x \in A, y \in B, z \in C, w \in D, x \neq z \right\} \right| \gg |A|^{2+k}.
\]

**Proof.** By a result from [17], [22], we have \(|R[A]| \gg |A|^2/\log |A|\). Further \(R[A] = 1 - R[A]\) and \(R^{-1}[A] = R[A]\), see Remark 21. Hence applying estimate (78) of Corollary 33 with \(A = R[A]\), \(B = C = \varphi(A)\) and \(\alpha = -1\), we obtain

\[
|R[A]| \gg |A| \cdot |A|^{\frac{1}{2(k+4)^2-k}},
\]

provided

\[
|A|^{\frac{1}{2(k+4)^2-k}} \geq |R[A]| \cdot 2^{2k} C_s^{(k+4)/4} \log |A| \geq |R[A]| \cdot C_s^{(k+4)/4} \log |A| \varphi(A).
\]

Put \(|R[A]| = C |A|^{2+c}/\log |A|, c \geq 0\) and \(C > 0\) is an absolute constant. Then taking \(k = 16+8c\), say, we satisfy (83) for large \(A\). It follows that

\[
|R[A]| \varphi(A) \gg |A|^{2+c+\frac{1}{2(20+8c)^2-16-8c}} \log^{-1} |A|.
\]

One can check that the optimal choice of \(c\) is \(c = 0\). Finally, to prove (82) just notice that from the method of [17], [22] it follows that \(\left| \left\{ \frac{b-a}{c-b} : a \in A, b \in B, c \in C, c \neq a \right\} \right| \gg |A|^2/\log |A|\) for any sets \(A, B, C\) of equal cardinality. After that repeat the arguments above. This completes the proof. \(\square\)

**Remark 36** Let us show quickly how Corollary 33 implies both Theorems 1, 2 for sets \(A\) with \(|A| < p^{1/2-\varepsilon}\) (the appearance \(\sqrt{p}\) bound was discussed in Remark 29).

Let \(B, C\) be some sets of sizes greater than \(p^\delta\) such that max\(|AB|, |A + C|\) \(\leq p^\delta |A|\) or max\(|(A + \alpha)B|, |A + C|\) \(\leq p^\delta |A|\) for some \(\alpha \neq 0\). We can find sufficiently large \(k = k(\varepsilon)\) such that condition (76) takes place for \(B\) because \(|A| < p^{1/2-\varepsilon} \leq p\) and \(|B| \geq p^\delta\). Applying Corollary 33 for \(A, B, C\), we arrive to a contradiction. Finally, to insure that \(|A||B|^{1+\frac{(k+1)}{2(k+4)^2-k}} \leq p\) just use the assumption \(|A| < p^{1/2-\varepsilon}\), inequality \(|B| \leq |AB| \leq p^\delta |A|\) and take sufficiently small \(\delta = \delta(\varepsilon)\) and sufficiently large \(k = k(\varepsilon)\).

Let \(A \subset \mathbb{R}\) be a finite set. Consider a characteristic of \(A\) (see, e.g., [32]), which generalize the notion of small multiplicative doubling of \(A\). Namely, put

\[
d^+(A) := \inf_{f} \min_{B \neq \emptyset} \frac{|f(A) + B|^2}{|A||B|},
\]

where the infimum is taken over convex/concave functions \(f\).

**Problem.** Suppose that \(d^+(A) \leq |A|^\varepsilon, \varepsilon > 0\) is a small number. Is it true that there is \(k = k(\varepsilon)\) such that \(E_k(A) \ll |A|^k\)?
Notice that one cannot obtain a similar bound for $T_k^+(A)$. Indeed, let $A = \{1^2, 2^2, \ldots, n^2\}$. Then one can show that for such $A$ the quantity $d^+(A)$ is $O(1)$ (see, e.g., [32]) but, clearly, $|kA| \ll_k |A|^2$. It means that it is not possible to obtain any upper bound for $T_k^+(A)$ of the form $T_k^+(A) \ll |A|^{2k-2-c}$, $c > 0$ and hence any analogues of Theorems 23, 25 for sets $A$ with small $d^+(A)$.

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