Noncommutative Two Time Physics

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Abstract
We present a classical formalism describing two-time physics with Abelian canonical gauge field backgrounds. The formalism can be used as a starting point for the construction of an interacting quantized two-time physics theory in a noncommutative space-time.

1 Introduction
Two-Time Physics [1,2,3,4,5,6,7] is an approach that provides a new perspective for understanding ordinary one-time dynamics from a higher dimensional, more unified point of view including two time-like dimensions. This is achieved by introducing a new gauge symmetry that insure unitarity, causality and absence of ghosts. The new phenomenon in two-time physics is that the gauge symmetry can be used to obtain various one-time dynamical systems from the same simple action of two-time physics, through gauge fixing, thus uncovering a new layer of unification through higher dimensions [7].

An approach to the introduction of background gravitational and gauge fields in two-time physics was first presented in [7]. In [7], the linear realization of the $Sp(2, R)$ gauge algebra of two-time physics is required to be preserved when background gravitational and gauge fields come into play. To satisfy this requirement, the background gravitational field must satisfy a homothety condition [7], while in the absence of gravitational fields the gauge field must satisfy certain conditions [7] which were first proposed by Dirac [8] in 1936. Dirac proposed these conditions as subsidiary conditions to describe the usual 4-dimensional Maxwell theory of electromagnetism (in the Lorentz gauge) as a theory in 6 dimensions which automatically displays $SO(4, 2)$ symmetry. In the treatment of [7] the gauge field $A_M$ and the gravitational field $G_{MN}$ are explicit functions of position only.

Explicit dependence on position only for the gravitational and gauge fields may in some cases be interpreted as a certain restriction on the formalism, since two-time physics treats $X$ and $P$ as indistinguishable variables. In the most general situation, the gravitational and gauge fields in two-time physics must...
be described by a bi-local gravitational field $G_{MN}(X, P)$ and by a doublet of bi-local gauge fields, $A_i^M(X, P)$, $i = 1, 2$, as discussed in [2] and briefly reviewed in section three below. In this letter we follow an intermediate path between [2] and [7] and present a formalism for introducing a single bi-local background gauge field $A_{M}(X, P)$ in two-time physics. Our formalism reproduces an extended canonical version of Dirac’s first subsidiary condition found in [7,8], and can be used to construct an interacting quantized two-time physics theory in a noncommutative space-time. This last observation opens new research directions in two-time physics.

The paper is divided as follows. In the next section we review the basic formalism of two-time physics and show how the $SO(d,2)$ Lorentz generator for the free action can be obtained from a local scale invariance of the Hamiltonian. The presence of this local scale invariance implies that the free two-time physics theory can also be consistently formulated in terms of another set of phase space brackets, with the only difference being that the linear realization of the gauge algebra is replaced by a non-linear realization. This replacement does not introduce any inconsistencies into the formalism because in two-time physics the metric signature with two timelike dimensions is a requirement that comes from the constraint equations only, and not from a particular realization of the gauge algebra. We will see in section two that the linearity or nonlinearity of a gauge algebra depends on the fundamental set of canonical brackets that are being used to compute the algebra. In section three we extend these results to the interacting theory. Some concluding remarks appear in section four.

## 2 Two-time Physics

The central idea in two-time physics [1,2,3,4,5,6,7] is to introduce a new gauge invariance in phase space by gauging the duality of the quantum commutator $[X_M, P_N] = i\delta_{MN}$. This procedure leads to a symplectic $\text{Sp}(2,\mathbb{R})$ gauge theory. To remove the distinction between position and momenta we set $X^M_1 = X^M$ and $X^M_2 = P^M$ and define the doublet $X^M_i = (X^M_1, X^M_2)$. The local $\text{Sp}(2,\mathbb{R})$ acts as

$$\delta X^M_i(\tau) = \epsilon_{ik} \omega^{kl}(\tau) X^M_l(\tau)$$

(2.1)

$\omega^{ij}(\tau)$ is a symmetric matrix containing three local parameters and $\epsilon_{ij}$ is the Levi-Civita symbol that serves to raise or lower indices. The $\text{Sp}(2,\mathbb{R})$ gauge field $A^{ij}$ is symmetric in $(i,j)$ and transforms as

$$\delta A^{ij} = \partial_r \omega^{ij} + \omega^{jk} \epsilon_{kl} A^{lj} + \omega^{jk} \epsilon_{kl} A^{il}$$

(2.2)

The covariant derivative is

$$D_r X^M_i = \partial_r X^M_i - \epsilon_{ik} A^{kl} X^M_l$$

(2.3)

An action invariant under the $\text{Sp}(2,\mathbb{R})$ gauge symmetry is

$$S = \frac{1}{2} \int d\tau (D_r X^M_i) \epsilon^{ij} X^N_j \eta_{MN}$$

(2.4a)
After an integration by parts this action can be written as

\[ S = \int d\tau (\partial_\tau X^M_1 X^N_2 - \frac{1}{2} A^{ij} X^M_i X^N_j) \eta_{MN} \]

\[ = \int d\tau [\dot{X}.P - \frac{1}{2} \lambda_1 P^2 + \lambda_2 P.X + \frac{1}{2} \lambda_3 X^2] \] (2.4b)

where \( A^{11} = \lambda_3, A^{12} = A^{21} = \lambda_2, A^{22} = \lambda_1 \) and the canonical Hamiltonian is

\[ H = \frac{1}{2} \lambda_1 P^2 + \lambda_2 P.X + \frac{1}{2} \lambda_3 X^2 \] (2.5)

The equations of motion for the \( \lambda \)'s give the primary constraints

\[ \phi_1 = \frac{1}{2} P^2 \approx 0 \] (2.6)

\[ \phi_2 = P.X \approx 0 \] (2.7)

\[ \phi_3 = \frac{1}{2} X^2 \approx 0 \] (2.8)

and therefore we can not solve for the \( \lambda \)'s from their equations of motion. The values of the \( \lambda \)'s in action (2.4b) are arbitrary. Constraints (2.6)-(2.8), as well as evidences of two-time physics, were independently obtained in [9]. The notation \( \approx \) means that constraints (2.6)-(2.8) ”weakly vanish” [10]. Therefore, following Dirac’s convention [10] for systems with first-class constraints only, they are set strongly equal to zero only after all calculations have been performed.

If we consider the Euclidean, or the Minkowski metric as the background space-time, we find that the surface defined by the constraint equations (2.6)-(2.8) is trivial. The only metric giving a non-trivial surface and avoiding the ghost problem is the flat metric with two timelike dimensions [1,2,3,4,5,6,7]. We then must work in a \((d+2)\) dimensional Euclidean space-time. We emphasize here that this transition to a \((d+2)\) dimensional space-time is an imposition of the constraint equations (2.6)-(2.8).

We use the Poisson brackets

\[ \{ P_M, P_N \} = 0 \] (2.9a)

\[ \{ X_M, P_N \} = \delta_{MN} \] (2.9b)

\[ \{ X_M, X_N \} = 0 \] (2.9c)

where \( M, N = 1, ..., d+2 \), and verify that constraints (2.6)-(2.8) obey the algebra

\[ \{ \phi_1, \phi_2 \} = -2 \phi_1 \] (2.10a)
\{\phi_1, \phi_3\} = -\phi_2 \quad (2.10b)

\{\phi_2, \phi_3\} = -2\phi_3 \quad (2.10c)

These equations show that all constraints \(\phi\) are first-class constraints \([10]\). Equations (2.10) represent the linear symplectic \(\text{Sp}(2,\mathbb{R})\) gauge algebra.

Action (2.4) also has a global symmetry under Lorentz transformations \(\text{SO}(d,2)\) with generator \([1,2,3,4,5,6,7]\)

\[ L^{MN} = \epsilon^{ij} X^M_i X^N_j = X_M P_N - X_N P_M \quad (2.11) \]

It satisfies

\[ \{L_{MN}, L_{RS}\} = \delta_{MR} L_{NS} + \delta_{NS} L_{MR} - \delta_{MS} L_{NR} - \delta_{NR} L_{MS} \quad (2.12) \]

and is gauge invariant because it has identically vanishing brackets with the first-class constraints (2.6)- (2.8), \(\{L_{MN}, \phi_i\} = 0, i = 1,2,3. \) In \([7]\), the form (2.11) for the Lorentz generator of two-time physics is preserved when a background gauge field \(A_M(X)\) is introduced. In this letter we will show how, in a phase space with the usual Poisson brackets, the linear realization (2.10) of the gauge algebra and the form (2.11) for the Lorentz generator are preserved when a massless Abelian bi-local background gauge field \(A_M(X,P)\) is introduced. We will also show how, in a phase space with Snyder \([11]\) brackets, when the same gauge field is introduced, the form (2.11) for \(L_{MN}\) is preserved but the gauge algebra acquires a non-linear realization.

Hamiltonian (2.5) is invariant under the scale transformations

\[ X^M \rightarrow \tilde{X}^M = \exp\{\beta\} X^M \quad (2.13a) \]

\[ P_M \rightarrow \tilde{P}_M = \exp\{-\beta\} P_M \quad (2.13b) \]

\[ \lambda_1 \rightarrow \exp\{2\beta\} \lambda_1 \quad (2.13c) \]

\[ \lambda_2 \rightarrow \lambda_2 \quad (2.13d) \]

\[ \lambda_3 \rightarrow \exp\{-2\beta\} \lambda_3 \quad (2.13e) \]

where \(\beta\) is an arbitrary function of \(X\) and \(P\). Keeping only the linear terms in \(\beta\) in the transformation (2.13), after some algebra we arrive at the brackets

\[ \{\tilde{P}_M, \tilde{P}_N\} = (\beta - 1)[\{P_M, \beta\}P_N + \{\beta, P_N\}P_M] + \{\beta, \beta\} P_M P_N \quad (2.14a) \]

\[ \{\tilde{X}_M, \tilde{P}_N\} = (1 + \beta)[\delta_{MN}(1 - \beta) - \{X_M, \beta\} P_N] \]

\[ +(1 - \beta)X_M \{\beta, P_N\} - X_M X_N \{\beta, \beta\} \quad (2.14b) \]
\( \{ \tilde{X}_M, \tilde{X}_N \} = (1 + \beta)[X_M \{ \beta, X_N \} - X_N \{ \beta, X_M \}] + X_M X_N \{ \beta, \beta \} \) (2.14c)

for the transformed canonical variables. If we choose \( \beta = \phi_1 = \frac{1}{2} P^2 \approx 0 \) in equations (2.14), compute the brackets on the right hand sides, and impose the vanishing of \( \phi_1 \) at the end of the calculation, we arrive at the brackets

\[
\{ \tilde{P}_M, \tilde{P}_N \} = 0 \tag{2.15a}
\]

\[
\{ \tilde{X}_M, \tilde{P}_N \} = \delta_{MN} - P_M P_N \tag{2.15b}
\]

\[
\{ \tilde{X}_M, \tilde{X}_N \} = -(X_M P_N - X_N P_M) \tag{2.15c}
\]

Now, using again \( \beta = \phi_1 = \frac{1}{2} P^2 \approx 0 \) in transformations (2.13a) and (2.13b) and imposing the vanishing of \( \phi_1 \), these become the identity transformations

\[
\tilde{X}_M = X_M \tag{2.16a}
\]

\[
\tilde{P}_M = P_M \tag{2.16b}
\]

Substituting equations (2.16) in (2.15) we arrive at the brackets

\[
\{ P_M, P_N \} = 0 \tag{2.17a}
\]

\[
\{ X_M, P_N \} = \delta_{MN} - P_M P_N \tag{2.17b}
\]

\[
\{ X_M, X_N \} = -(X_M P_N - X_N P_M) \tag{2.17c}
\]

Brackets (2.17) are the classical equivalent of the Snyder commutators [11] which were proposed in 1947 as a way to solve the ultraviolet divergence problem in quantum field theory. In the canonical quantization procedure, where brackets are replaced by commutators according to the rule

\[
[\text{commutator}] = i \{ \text{bracket} \}
\]

the brackets (2.17) will lead directly to a noncommutative quantized [11] \( d + 2 \) dimensional space-time for two-time physics.

The classical Snyder brackets (2.17) are associated to a non-linear realization of the gauge algebra of two-time physics. Computing the algebra of constraints (2.6)-(2.8) using these brackets we arrive at the expressions

\[
\{ \phi_1, \phi_2 \} = -2\phi_1 + 4\phi_1^2 \tag{2.18a}
\]

\[
\{ \phi_1, \phi_3 \} = -\phi_2 + 2\phi_1 \phi_2 \tag{2.18b}
\]
\( \{\phi_2, \phi_3\} = -2\phi_3 + \phi_2^2 \)  

(2.18c)

Since the space-time metric with two timelike dimensions is a consequence of the constraint equations (2.6)-(2.8) only, the above non-linear gauge algebra exactly corresponds to the same expression (2.11) for the \( SO(d,2) \) generator. In fact, \( L_{MN} \) explicitly appears with a minus sign in the right hand side of the Snyder bracket (2.17c), giving an interesting connection of the Lorentz invariance of action (2.4) with the scale invariance (2.13) of Hamiltonian (2.5).

The conclusion at this point is that the free two-time physics theory can also be consistently formulated in a phase space where the Snyder brackets (2.17) are valid. The only difference in this alternative formulation is that the linear realization (2.10) of the gauge algebra is substituted by the non-linear realization (2.18). In the next section we will see that this remains true when an Abelian massless bi-local gauge field \( A_M(X,P) \) that satisfies extended Dirac’s subsidiary conditions is introduced.

### 3 2T Physics with Abelian Gauge Fields

In two-time physics, interactions with gravitational fields \( G_{MN}(X,P) \) and gauge fields \( A^M(X,P) \) in a way that respects the \( Sp(2,R) \) gauge symmetry is also possible. In the presence of these interactions the free action (2.4a) is modified as [2]

\[
S_{G,A} = \frac{1}{2} \int d\tau [(D_\tau X_M^i)\epsilon^{ij}X_j^NG_{MN}(X_1,X_2) + (D_\tau X_M^i)\epsilon^{ij}A_{jM}(X_1,X_2)] 
\]

(3.1)

\( G_{MN} \) is a scalar under \( Sp(2,R) \) and a symmetric traceless tensor in \( d+2 \) dimensions. \( A^M \) is a doublet under \( Sp(2,R) \) and a vector in \( d+2 \) dimensions. For the local \( Sp(2,R) \) invariance to hold, there must be restrictions on the functional forms of both \( G_{MN}(X_1,X_2) \) and \( A^M(X_1,X_2) \) since the arguments \( (X_1,X_2) \) also transform under \( Sp(2,R) \). For consistency with local symmetry, gravity and gauge interactions are more conveniently expressed in terms of bi-local fields \( G_{MN}(X_1,X_2) \) and \( A^M(X_1,X_2) \) in \( d+2 \) dimensions [2]. Bi-local fields were advocated in [12,13] as a means of extending supergravity and super Yang-Mills theory to (10,2) dimensions based on clues from the BPS solutions of extended supersymmetry. The use of bi-local gravitational and gauge fields in two-time physics was, however, apparently not further motivated beyond [2,12,13].

In this letter we study another possibility of introducing background gauge fields in two-time physics. Here we study the problem of introducing a single bi-local background gauge field \( A_M(X_1,X_2) \) in two-time physics. We are able to obtain the necessary conditions for the local \( Sp(2,R) \) and global \( SO(d,2) \) symmetries to hold in this case. In our treatment, when only one background
bi-local gauge field \( A_M(X(\tau), P(\tau)) \) is introduced, the free 2T action (2.4b) becomes

\[
S = \int d\tau \{ \dot{X}.P - \left[ \frac{1}{2} \lambda_1 (P - A)^2 + \lambda_2 (P - A).X + \frac{1}{2} \lambda_3 X^2 \right] \}
\]

(3.2)

where the Hamiltonian is

\[
H = \frac{1}{2} \lambda_1 (P - A)^2 + \lambda_2 (P - A).X + \frac{1}{2} \lambda_3 X^2
\]

(3.3)

The equations of motion for the multipliers now give the constraints

\[
\phi_1 = \frac{1}{2} (P - A)^2 \approx 0 \quad (3.4)
\]

\[
\phi_2 = (P - A).X \approx 0 \quad (3.5)
\]

\[
\phi_3 = \frac{1}{2} X^2 \approx 0 \quad (3.6)
\]

We must now define a set of brackets between the canonical variables and the gauge field. A convenient set is

\[
\{ X_M, A_N \} = \frac{\partial A_N}{\partial P_M}
\]

(3.7a)

\[
\{ P_M, A_N \} = -\frac{\partial A_N}{\partial X_M}
\]

(3.7b)

\[
\{ A_M, A_N \} = 0
\]

(3.7c)

Brackets (3.7a) and (3.7b) are the usual Poisson brackets for a vector function \( A_M(X, P) \). Bracket (3.7c) is imposed as an initial simplifying restriction on the possible functional forms of the gauge field \( A_M(X, P) \). It is a restriction to Abelian gauge fields.

Computing the algebra of constraints (3.4)-(3.6) using the brackets (2.9) and (3.7) we obtain the equations

\[
\{ \phi_1, \phi_2 \} = -2\phi_1 + (P^M - A^M) \frac{\partial}{\partial X_M}(X.A) - 2(P - A).A
\]

\[-X^M \frac{\partial}{\partial X_M}[(P - A).A] - X^M \frac{\partial}{\partial X_M}(\frac{1}{2} A^2) + (P^M - A^M) \frac{\partial}{\partial P_M}[(P - A).A] + (P^M - A^M) \frac{\partial}{\partial P_M}(\frac{1}{2} A^2) \] (3.8a)

\[
\{ \phi_1, \phi_3 \} = -\phi_2 + X^M \frac{\partial}{\partial P_M}[(P - A).A]
\]
\[-X.A + X^M \frac{\partial}{\partial P^M} \left( \frac{1}{2} A^2 \right) \]  

(3.8b)

\[
\{ \phi_2, \phi_3 \} = -2\phi_3 + X^M \frac{\partial}{\partial P^M} (X.A) 
\]

(3.8c)

Equations (3.8) exactly reproduce the linear gauge algebra (2.10) when the conditions

\[ X.A = 0 \]  

(3.9a)

\[ (P - A).A = 0 \]  

(3.9b)

\[ \frac{1}{2} A^2 = 0 \]  

(3.9c)

hold. Condition (3.9a) is the first of Dirac’s subsidiary conditions [7,8] on the gauge field. Condition (3.9b) is not an independent condition. It is the canonical conjugate to condition (3.9a), but incorporating the minimal coupling prescription

\[ P_M \rightarrow P_M - A_M \]  

(3.10)

to gauge fields. Condition (3.9c) implies that the canonical vector field \( A_M(X, P) \) is a massless gauge field, describing infinite-range interactions. When conditions (3.9) hold, the linear gauge algebra (2.10) is reproduced by constraints (3.4)-(3.6). Thus, when (3.9) holds, the only possible space-time metric associated with constraints (3.4)-(3.6) giving a non-trivial surface and avoiding the ghost problem is the flat metric with two timelike dimensions.

The interacting Hamiltonian (3.3) will be invariant under transformations (2.13) when the gauge field effectively transforms as

\[ A_M \rightarrow \tilde{A}_M = \exp(-\beta) A_M \]  

(3.11)

which is consistent with transformation (2.13b) and with the minimal coupling prescription (3.10). Choosing now \( \beta = \phi_1 = \frac{1}{2}(P - A)^2 \approx 0 \), and performing the same steps as in the free theory, we obtain the brackets

\[
\{ P_M, P_N \} = 0
\]

(3.12a)

\[
\{ X_M, P_N \} = \delta_{MN} - P_M P_N + P_N \frac{\partial}{\partial P_M} [(P - A).A] 
\]

(3.12b)

\[
\{ X_M, X_N \} = -(X_M P_N - X_N P_M) + X_M \frac{\partial}{\partial P_N} [(P - A).A] 
\]

(3.12c)
As can be verified, the above brackets reduce to the Snyder brackets \((2.17)\) when conditions \((3.9)\) hold. In other words, when conditions \((3.9)\) are valid, the Lorentz \(SO(d, 2)\) generator for the two-time physics model with a bi-local gauge field \(A_M(X, P)\) described by action \((3.2)\) is identical to \(L_{MN}\) in \((2.11)\).

Up to now we have proved that, when conditions \((3.9)\) are valid, action \((3.2)\) has a gauge algebra identical to the linear algebra \((2.10)\) and a Lorentz generator identical to \((2.11)\). To complete the proof of the consistency of action \((3.2)\), we must verify the gauge invariance of \(L_{MN}\) under gauge transformations generated by constraints \((3.4)-(3.6)\). Using brackets \((2.9)\) and \((3.7)\) we find the equations

\[
\{L_{MN}, \phi_1\} = X_M \frac{\partial}{\partial X_N} [(P - A).A] + X_M \frac{\partial}{\partial X_N} \left(\frac{1}{2} A^2\right)
\]

\[-X_N \frac{\partial}{\partial X_M} [(P - A).A] - X_N \frac{\partial}{\partial X_M} \left(\frac{1}{2} A^2\right) + P_M \frac{\partial}{\partial P_N} [(P - A).A]
\]

\[+P_M \frac{\partial}{\partial P_N} \left(\frac{1}{2} A^2\right) - P_N \frac{\partial}{\partial P_M} [(P - A).A] - P_N \frac{\partial}{\partial P_M} \left(\frac{1}{2} A^2\right) \tag{3.13a}\]

\[
\{L_{MN}, \phi_2\} = X_M \frac{\partial}{\partial X_N} (X.A) - X_N \frac{\partial}{\partial X_M} (X.A)
\]

\[+P_M \frac{\partial}{\partial P_N} (X.A) - P_N \frac{\partial}{\partial P_M} (X.A) \tag{3.13b}\]

\[\{L_{MN}, \phi_3\} = 0 \tag{3.13c}\]

We see from the above equations that \(L_{MN}\) is gauge invariant, \(\{L_{MN}, \phi_i\} = 0\), when conditions \((3.9)\) are valid. Action \((3.2)\), complemented with the extended Dirac’s subsidiary conditions \((3.9)\), gives therefore a consistent description of two-time physics with background canonical gauge fields in a phase space with the Poisson brackets \((2.9)\) and \((3.7)\).

Action \((3.2)\) can also be used to describe two-time physics with background canonical gauge fields in a phase space with Snyder brackets \((2.17)\) together with brackets \((3.7)\). This can be seen as follows. First, if we compute the algebra of constraints \((3.4)-(3.6)\) using these brackets we will find that the resulting gauge algebra reduces to the nonlinear algebra \((2.18)\) when conditions \((3.9)\) hold. Second, as we saw in this section, the Lorentz generator for action \((3.2)\) is identical to \(L_{MN}\) when conditions \((3.9)\) hold. Third, if we compute the brackets \(\{L_{MN}, \phi_i\}\) with \(\phi_i\) given by \((3.4)-(3.6)\), using these brackets we will find that \(L_{MN}\) is gauge invariant when conditions \((3.9)\) hold. The conclusion of this is that Dirac’s conditions \((3.9)\) are also the necessary subsidiary conditions for the consistency of the formalism in a phase space with Snyder brackets \((2.17)\) together with brackets \((3.7)\).
4 Concluding remarks

In this work we presented two distinct ways to study two-time physics with background Abelian canonical gauge fields. In the first way, which will correspond to a commutative $d+2$ dimensional space-time in the quantized theory, the fundamental brackets are the Poisson brackets (2.9) together with (3.7), the $SO(d,2)$ Lorentz generator $L_{MN}$ is given by (2.11), and the gauge algebra is the linear algebra (2.10). In the second way, which will correspond to a noncommutative $d+2$ dimensional space-time in the quantized theory, the fundamental brackets are the Snyder brackets (2.17) together with (3.7), the $SO(d,2)$ Lorentz generator is the same $L_{MN}$ given by (2.11), and the gauge algebra is the nonlinear algebra (2.18). The equivalence or not of these two approaches is a subject for future investigations.

As a final observation, notice that the subsidiary conditions (3.9) appear in equations (3.8), (3.12) and (3.13) in derivatives with respect to $X_M$ and with respect to $P_M$. Therefore conditions (3.9), and the formalism we presented above, remain valid in the more restrictive case when $A_M = A_M(X)$. This case is useful to make contact with the one-time dynamics, but for formal theoretical investigations in two-time physics the case with $A_M = A_M(X,P)$ seems to be more reliable.

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