On isochronous analytic motions and the quantum spectrum

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Received 22 January 2019, revised 30 April 2019
Accepted for publication 2 May 2019
Published 2 October 2019

Abstract
The problem of the characterization of all analytic potentials which give rise to isochronous oscillatory motions is still open. However, there are several approaches to highlight motions with period \( T(E) \equiv T_0 \) independent of the energy. In this paper we propose to give the necessary and sufficient conditions for a center to be isochronous. The proofs produced here are self-contained. As corollaries we again find lots of known conditions previously established by different authors. This allows us to produce several classes of isochronous analytic motions. We are also interested in the quantum spectrum. We then use the WKB perturbation method to derive an expression for the corrections to the equally spaced spectra valid for analytic isochronous potentials.

Keywords: oscillatory motions, isochronicity, WKB method, quantum spectrum

1. Introduction

The origin of the existence problem of isochronous potentials dates back to Huygens in 1673 [1]. The literature on isochronous potentials (or more generally about isochronous centers) is particularly extensive and the interested reader is referred to [1–3] and references therein for more information.

However, it should be noted that this problem was initially posed by physicists. A tautochron curve is a curve in a vertical plane, where the time taken by a particle sliding along the curve under the uniform influence of gravity to its lowest point is independent of its point of departure.

The tautochron problem was solved by Huygens in the case where gravity alone acts. He proved geometrically in his Horologium Oscillatorium (1673) that the curve was a cycloid. This solution was later used to address the problem of the brachistochrone curve. This cycloid defines a sinusoidal oscillatory motion of constant pulsation. This observation thus establishes a link with the isochronous problem of the potential well. It is clear that the tautochron problem is in fact an isochronous one.

In the plane case with rational potentials it can be shown that the only isochronous potentials with a (constant) period \( T = 2\pi \) correspond either to the harmonic oscillator \( G(x) = \frac{1}{2}x^2 \) or to the isotonic potential of the form \( G(x) = \frac{1}{2}x^2 + \frac{a}{x} \) with a center at 0 [4].

This problem aroused a great deal of interest after the work of Landau and Lifschitz, who first produced an existence condition of isochronous potentials.

In quantum mechanics, the energy levels of a parabolic well are regularly spaced by a certain quantity. Moreover, it is possible to construct potentials, essentially different from the parabolic well, whose spectra are exactly harmonic. A link between these classical and quantum transformations was established by Eleonskii et al. [5]. They show that the classical limit of the isospectral transformation for the Schrodinger equation is precisely the isochronicity preserving the energy dependence of the oscillation frequency.

The semiclassical WKB method [6] is one of the most useful approximations for computing the energy eigenvalues of the Schrodinger equation. The main disadvantage of the WKB methods is that they easily work for a short time, while extensions to arbitrary times require significant technical efforts. It has a wider range of applicability than standard perturbation theory, which is restricted to perturbing potentials with small coupling constants. In particular, it permits us to write the quantization condition as a power series in \( \hbar \). Such series are generally non-convergent. The solvable potentials are those whose series can be explicitly summed. Other isoperiodic systems with equally spaced spectra have been studied, e.g. [7].

First, we recall some basic facts and introduce certain notations which will be useful in the rest of this paper. See [8] and [9] for details.
Consider the scalar equation with a center at the origin 0

$$\ddot{x} + g(x) = 0 \quad (1)$$

or its planar equivalent system

$$\dot{x} = y, \quad \dot{y} = -g(x) \quad (2)$$

where $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$ and $g(x) = \frac{dg(x)}{dx}$ is analytic on $R$ where $G(x)$ is the potential of $(1)$.

Suppose system $(2)$ admits a periodic orbit in the phase plane with energy $E$ and $g(x)$ has bounded period for real energies $E$. Given $G(x)$, let $T(E)$ denote the minimal period of this periodic orbit. Its expression is

$$T(E) = 2\int_a^b \frac{dx}{\sqrt{2E - 2G(x)}}. \quad (3)$$

$T(E)$ is well defined and there is a neighborhood of the real axis for which $T(E)$ is analytic.

We suppose that the potential $G(x)$ has one minimum value, which for convenience is located at the origin 0 and $\frac{d^2G(x)}{dx^2}(0) = 1$. The turning points $a$, $b$ of this orbit are solutions of $G(x) = E$.

Then the origin 0 is a center of $(2)$. This center is isochronous when the period of all orbits near 0 $\in R^2$ are constant $T = \frac{2\pi}{\sqrt{E(0)}} = 2\pi$. The corresponding potential $G(x)$ is also called isochronous.

Since the potential $G(x)$ has a local minimum at 0, then we may consider an involution $A$ by

$$G(A(x)) = G(x) \text{ and } A(x)x < 0$$

for all $x \in [a, b]$. So, any closed orbit is $A$-invariant and $A$ exchanges the turning points: $b = A(a)$.

In fact, $A(x)$ is well defined in the interval $[a, b]$. To see that, set the function

$$\rho(x) = \frac{x - A(x)}{2}.$$

This function is such that $\rho(A(x)) = -\rho(x)$ and $\rho'(x) = \frac{1 - A'(x)}{2}$. Since $A'(x) < 0$ we get $\rho'(x) > 0$ and therefore by the inverse function theorem $\rho$ is an analytic diffeomorphism on $[a, b]$. Then $A(x) = \rho^{-1}(-\rho(x))$ is an analytic well-defined function.

Conversely, by using this involution we may calculate for a prescribed period function $T(E)$ the distance between the turning points. Indeed, following Landau and Lifshitz [15, Chap. 3, 12.1] there is a corresponding distance between the turning points given by

$$T(E) = 2\int_0^E \left[ dA(a) \frac{dA(a)}{dG} - \frac{dA}{dG} \right] \frac{dG}{\sqrt{2E - 2G}}$$

which implies

$$A(a) - a = a - b = \frac{1}{\pi} \int_0^E \frac{T(\gamma)d\gamma}{\sqrt{2E - 2\gamma}}. \quad (4)$$

The potential is isochronous, which means the period is constant $(T = 2\pi)$, then for all orbits near zero one has necessarily

$$b - a = \frac{1}{\pi} \int_0^E \frac{T(\gamma)d\gamma}{\sqrt{2E - 2\gamma}} = 2\sqrt{2E}.$$

The present paper is an extended and improved version of a text published by the author as ArXiv 1109.4611. v4 [10].

In the first section we give new conditions for the potential to be isochronous and we precise his link with the older ones. Some of them were presented and commented in [10] as theorem B. We give here some refinements of these results. We show in addition that our condition is more general than all those previously introduced by Landau-Lifshitz, Koukles-Piskounov and Urabe.

2. Statement of results

The problem of determining whether a center is isochronous has attracted many researchers for a long time. This problem has been recently revived due to the advancement of computer algebra. New powerful algorithms have been discovered. We will propose an alternative approach in order to derive more directly new isochronous potentials. More precisely, we propose to give the necessary and sufficient conditions for an oscillatory motion to be isochronous. Our conditions seem more natural since they allow us to deduce all the others known before. Other equivalent characterizations will be considered. More exactly, we state the following:

**Theorem A.** Let $g(x)$ be an analytic function and $G(x) = \int_0^x g(s)ds$ and $A$ be the analytic involution defined by $G(A(x)) = G(x)$. Suppose that $x_0 = 0, x_0g(x) > 0$. Then equation $(1)$ has an isochronous center at $0$ if and only if the function

$$\frac{d}{dx}[G(x)/g^2(x)]$$

is $A$-invariant, i.e. $\frac{d}{dx}[G(x)/g^2(x)](x) = \frac{d}{dx}[G(x)/g^2(x)](A(x))$ in some neighborhood of $0$.

In order to study the period function $T(E)$ depending on the energy, it is sometimes convenient to study its derivatives. We need the following:

**Lemma 2-1.** The derivative of the period function (depending on the energy) $T'(E) = \frac{dT}{dE}$ may be written

$$T'(E) = \left( \frac{1}{E} \right) \int_0^E \frac{g^2(x) - 2G(x)g'(x)}{g^2(x)\sqrt{2E - 2G(x)}} dx. \quad (5)$$

This lemma was initially proved by Chow and Wang [11] but the proof given below is more direct.
Proof. Recall that
\[ T(E) = 2 \int_a^b \frac{dx}{\sqrt{2E - 2G(x)}}. \]
Consider the change \( G(x) = s^2 \), where \( s = u(x) \) is a function of \( x \) (this change has a sense because \( G(x) \) is positive in a neighborhood of 0 according to the hypothesis \( xg(x) > 0 \)). It yields
\[ T(E) = 4 \int_{-\sqrt{E}}^{\sqrt{E}} \frac{ds}{s^2} \sqrt{2E - 2s^2}. \]
By another change \( s = \sqrt{E} \sin \theta \) the period function may be expressed
\[ T(E) = 2 \int_{-\pi/2}^{\pi/2} d\theta \sqrt{2} \int_{-\pi/2}^{\pi/2} (u^{-1}(\sqrt{E} \sin \theta)) d\theta. \]
Then we get by another way the derivative of the period function
\[ T'(E) = \frac{2}{\sqrt{E}} \int_{-\pi/2}^{\pi/2} (u^{-1})''(\sqrt{E} \sin \theta) \sin \theta d\theta. \]
By splitting \( T'(E) = \frac{2}{\sqrt{E}} \int_{0}^{\pi/2} + \frac{2}{\sqrt{E}} \int_{\pi/2}^{\pi} \) and using the formula \( (u^{-1})''(u(x)) = -\frac{d^2}{dx^2}\) and again changing the variable \( s = u(x) = \sqrt{E} \sin \theta \) we then obtain
\[ T'(E) = \frac{1}{E} \int_0^b \frac{g^2(x) - 2G(x)g'(x)}{\sqrt{2E - 2G(x)}} dx + \frac{1}{E} \int_0^b \frac{g^2(x) - 2G(x)g'(x)}{\sqrt{2E - 2G(x)}} dx. \]
Since \( xg(x) > 0 \) the derivative can be written
\[ T'(E) = \left( \frac{1}{E} \right) \int_0^b \frac{g^2(x) - 2G(x)g'(x)}{\sqrt{2E - 2G(x)}} dx. \]
Thus, when \( (u^{-1})''(\sqrt{E} \sin \theta) \) is an even function then \( T'(E) \equiv 0 \).

The following result explicitly highlights the role and impact of involution on the behavior of the period \( T \equiv T(E) \).

Lemma 2.2. The derivative of the period function \( T(c) \) may also be written as
\[ T'(E) = \left( \frac{1}{E} \right) \int_0^b \frac{\frac{d}{dx} \left( \frac{G}{G'(x)} \right) - \frac{d}{dx} \left( \frac{G}{G'(x)} \right) g(x)}{\sqrt{2E - 2G(x)}} g(x) dx. \quad (6) \]
Moreover, we have
\[ \phi(G) = \frac{\sqrt{2}}{\pi} \int_0^\infty \frac{\gamma T'(\gamma)}{\sqrt{G - \gamma}} d\gamma \quad (7) \]
where \( \frac{d}{dx} \frac{G}{G'(x)} \) and \( \frac{d}{dx} \frac{G}{G'(x)} \).

Proof. Indeed, this derivative
\[ \frac{dT}{dE} = T'(E) = \left( \frac{1}{E} \right) \int_0^b \frac{g^2(x) - 2G(x)g'(x)}{g^3(x)\sqrt{2E - 2G(x)}} dx \]
can be expressed
\[ T'(E) = \left( \frac{1}{E} \right) \int_0^b \frac{d}{dx} \left( \frac{G}{G'(x)} \right) g(x) dx \quad (8) \]
for \( a < a < 0 \) and \( 0 < b < b \). By splitting the integral we get
\[ T'(E) = \left( \frac{1}{E} \right) \int_0^b \frac{g^2(x) - 2G(x)g'(x)}{g^3(x)\sqrt{2E - 2G(x)}} g(x) dx \]
and again changing the variable \( s = u(x) = \sqrt{E} \sin \theta \) we then obtain
\[ T'(E) = \left( \frac{1}{E} \right) \int_0^b \frac{g^2(x) - 2G(x)g'(x)}{g^3(x)\sqrt{2E - 2G(x)}} g(x) dx \]
On the other hand, considering the Laplace transformation (this method has been successfully applied for \( T(E) \), see e.g. [12]),
\[ \mathcal{L}\{f(E)\} = \int_0^\infty e^{-\gamma f(\gamma)} d\gamma \]
which verifies \( \mathcal{L}\{\lambda f(E)\} = \lambda \mathcal{L}\{f(E)\}, \lambda \in IR \) and \( s\mathcal{L}\{f(E)\} = \mathcal{L}\{f'(E)\} \) if \( f(0) = 0 \). A convolution theorem ensures
\[ \mathcal{L}\{f_1(E) \ast f_2(E)\} = \mathcal{L}\{f_1(E)\} \mathcal{L}\{f_2(E)\} \]
where \( f_1(E) \ast f_2(E) = \int_0^E f_1(E - \gamma) f_2(\gamma) d\gamma \). In our case we observe that the function \( ET'(E) \) may be written as a convolution
\[ \sqrt{2} ET'(E) = \frac{1}{\sqrt{E}} * \frac{dG}{dE}(E). \quad (9) \]
Using Laplace transformation, (9) may be written
\[ \sqrt{2} \mathcal{L}\{ET'(E)\} = \mathcal{L}\left\{ \frac{1}{\sqrt{E}} \right\} \mathcal{L}\left\{ \frac{dG}{dE}\right\} = \left[ \frac{\pi}{s} \right] s\mathcal{L}\{\Phi(E)\}. \]
Therefore,
\[ \mathcal{L}\{\Phi(E)\} = \sqrt{2} \left[ \frac{\pi}{s} \right] \mathcal{L}\{ET'(E)\} \]
\[ = \sqrt{2} \left[ \frac{\pi}{s} \right] \mathcal{L}\{ET'(E)\} \]
\[ = \sqrt{2} \left[ \frac{1}{\sqrt{E}} \right] \mathcal{L}\{ET'(E)\} \]
which after inverting gives
\[ \Phi(E) = \frac{\sqrt{\pi}}{\pi} \int_0^E \frac{\gamma T'(\gamma) \, d\gamma}{\sqrt{E - \gamma}}. \]

Interchanging the variables \( E \) and \( G \) we obtain
\[ \Phi(G) = \frac{\sqrt{\pi}}{\pi} \int_0^G \frac{\gamma T'(\gamma) \, d\gamma}{\sqrt{G - \gamma}}. \]

**Proof of theorem A.** Supposing the hypothesis of theorem A that \( \frac{d}{dx}[G(x)/g^2(x)] \) is \( A \)-invariant holds, then by lemma 2-2
\[
T'(E) = \left( \frac{1}{E} \right) \int_0^b \frac{d}{dx} \left( \frac{G(x)}{g^2(x)} \right) g(x) \, dx
\]
\[ = \left( \frac{1}{E} \right) \int_0^b \frac{d}{dx} \left( \frac{G^2(x)}{g^2(x)} \right) g(x) \, dx. \]

This obviously implies that \( T'(E) \equiv 0 \) and the center 0 of equation (1) is isochronous. Conversely, let \( T'(E) \equiv 0 \), then by lemma 2-2
\[
T'(E) = \left( \frac{1}{E} \right) \int_0^b \frac{d}{dx} \left( \frac{G(x)}{g^2(x)} \right) g(x) \, dx = 0.
\]

Since \( b > 0 \) and the function \( \frac{d}{dx} \left( \frac{G(x)}{g^2(x)} \right) = \frac{d}{dx} \left( \frac{G^2(x)}{g^2(x)} \right) \) is analytic, then by lemma 2-2, \( T'(\gamma) \equiv 0 \) implies \( \Phi(G) \equiv 0 \) and
\[
\frac{d\Phi}{dG} = \left( \frac{d}{dx} \left( \frac{G(x)}{g^2(x)} \right) \right) - \frac{d}{dx} \left( \frac{G^2(x)}{g^2(x)} \right) \equiv 0.
\]

This means the function \( \frac{d}{dx} \left( \frac{G(x)}{g^2(x)} \right) \) is \( A \)-invariant.

We may also prove the following:

**Theorem B.** Let \( G(x) = \int_0^x g(x) \, ds \) be an analytic potential. Suppose that \( x > 0 \), \( xg(x) > 0 \). Then equation (1) has an isochronous center at 0 if and only if
\[ x - \frac{2G}{g} = F(G) \] (10)

where \( F \) is an analytic function defined in some neighborhood of 0.

**Corollary 2-3.** Under the hypotheses of theorem B, equation (1) has an isochronous center at 0 if and only if
\[ \frac{d}{dx} [G(x)/g^2(x)] = f(G) \] (11)

where \( f \) is an analytic function defined in some neighborhood of 0.

**Proof of corollary 2-3.** Condition \( x - \frac{2G}{g} = F(G) \) implies by theorem B that
\[ \frac{d}{dx} \left[ x - \frac{2G}{g} \right] = \frac{d}{dx} F(G) = 1 - \frac{2G}{g} + 2G' \frac{g}{g^2} \]
\[ = -f(G) = -1 + 2G' \frac{g}{g^2}. \]

Conversely, let us consider \( F \) the primitive of \( f \) such that \( F(0) = 0 \). Then
\[ \frac{d}{dx} \left[ G(x)/g^2(x) \right] = f(G(x)) \iff g(x) \frac{d}{dx} [G(x)/g^2(x)] \]
\[ = g(x)f(G(x)). \]

Integration by parts yields \( G(1)/g^2(1) - x = F(G(x)) \). Thus, it is equivalent to the condition \( \frac{d}{dx} [G/g^2](x) = f(G) \). Thus, one gets another equivalent condition of isochronicity.

**Proof of theorem B.** We start from the derivative
\[ T'(E) = \left( \frac{1}{E} \right) \int_0^b \frac{d}{dx} \left( \frac{G(x)}{g^2(x)} \right) g(x) \, dx \]
which may be expressed after integration by parts as
\[ T'(E) = \left( \frac{1}{E} \right) \int_0^b \frac{1}{3 (\sqrt{2E - 2G(x)})^3} \Phi(x) \, dx \]
where
\[ \Phi(x) = \int_0^x \frac{d}{dx} \left( \frac{G(x)}{g^2(x)} \right) g(x) \, dx = \frac{2G}{g} - x. \]

Thus,
\[ T'(E) = \left( \frac{1}{3E} \right) \int_a^b \frac{\left( \frac{2G}{g} - x \right)}{(\sqrt{2E - 2G(x)})^3} \, g(x) \, dx. \]

When \( T'(E) \equiv 0 \) therefore \( x - \frac{2G}{g} \) is \( A \)-invariant and \( x - \frac{2G}{g} \) is an analytic function dependent on \( G \) by the following lemma.

**Lemma 2-4.** Consider the function \( \phi(G, x) = \frac{d}{dx} [G(x)/g^2(x)] \). Suppose in addition \( \phi(G, x) \) is \( A \)-invariant then its derivatives with respect to \( G \) are all \( A \)-invariant for \( x \) in a neighborhood of 0 (i.e. \( \frac{d}{dx} \left( \frac{\phi(G, A(x))}{g^2} \right) = \frac{d}{dx} \left( \frac{\phi(G, x)}{g^2} \right) \) for any integer \( p > 0 \)).

**Proof.** We will proceed by recurrence. Notice that
\[ \frac{d\phi(G, x)}{dG} = \frac{d}{dG} \left[ G(x)/g^2(x) \right] = \frac{d}{dx} \frac{d}{dG} [G(x)/g^2(x)] \]
since the functions \( G \) and \( g \) are analytic. Moreover, one gets
\[ \frac{d}{dx} \frac{d}{dG} [G(x)/g^2(x)] = \frac{1}{g(x)} \frac{d^2}{dx^2} \left[ \frac{G(x)}{g^2(x)} \right]. \]

On the other hand, we have seen that \( G(A(x)) = G(x) \) implies \( g(A(x)) \frac{d}{dx} \left( \frac{A(x)}{g^2(x)} \right) = g(x) \). Since \( \phi(G, x) \) is \( A \)-invariant,
then
\[ \frac{d}{dx} [G/g^2](x) = \frac{d}{dx} [G/g^2](A(x)) \]

which implies deriving once more
\[ \frac{d^2}{dx^2} [G/g^2](x) = \frac{d^2}{dx^2} [G/g^2](A(x)) \frac{dA(x)}{dx}. \]

Therefore
\[ \frac{1}{g(A(x))} \frac{d^2}{dx^2} \left[ \frac{G(A(x))}{g^2} \right] = \frac{1}{g(A(x))} \frac{d^2}{dx^2} \left[ \frac{G(A(x))}{g^2} \right] \frac{dA(x)}{dx} \]
\[ \times \frac{dA(x)}{dx} = \frac{1}{g(x)} \frac{d^2}{dx} [G/g^2](x) \]

which means \( \frac{d}{dx} \phi_n(G, x) \) is A-invariant.

Suppose now the following derivative \( \phi_n(G, x) = \frac{d^n\phi(G, x)}{dx^n} \) is A-invariant for any integer \( n \leq p \). Then
\[ \frac{d^{n+1}\phi(G, x)}{dx^{n+1}} = \frac{d}{dx} \phi_n(G, x) = \frac{1}{g(A(x))} \frac{d}{dx} \phi_n(G, A(x)) \]
\[ \times \frac{dA(x)}{dx} = \frac{1}{g(x)} \frac{d}{dx} \phi_n(G, x). \]

Since \( \phi_n(G, A(x)) = \frac{d^n\phi(G, A(x))}{dx^n} = \frac{d^n\phi(G, x)}{dx^n} = \phi_n(G, x) \) then deriving this expression one obtains
\[ \frac{d}{dx} \phi_n(G, x) = \frac{d}{dx} \phi_n(G, A(x)) \frac{dA(x)}{dx}. \]

Thus,
\[ \frac{1}{g(A(x))} \frac{d}{dx} \phi_n(G, A(x)) = \frac{1}{g(A(x))} \frac{d}{dx} \phi_n(G, A(x)) \]
\[ \times \frac{dA(x)}{dx} = \frac{1}{g(x)} \frac{d}{dx} \phi_n(G, x). \]

That means
\[ \phi_{n+1}(G, x) = \frac{d^{n+1}\phi(G, x)}{dx^{n+1}} = \frac{d^{n+1}\phi(G, A(x))}{dx^{n+1}} \]
\[ = \phi_{n+1}(G, A(x)). \]

Some other equivalent conditions may also be deduced [2].

**Corollary 2-5.** Under the hypothesis above, 0 is an isochronous center of (1) if and only if \( x = x(G) \) is an analytic solution of the linear ordinary differential equation (ODE)
\[ 2G \frac{d^2x}{dG^2} + \frac{dx}{dG} = f(G), \]
where \( f \) is an analytic function.

Moreover, this solution must satisfy the conditions:
\[ x(0) = 0, \quad \lim_{G \to 0} \left( \frac{\sqrt{2}x}{2G} \right) = 1. \]

**Proof of corollary 2-5.** Let us again consider that \( f(t) \) is the integral of \( f(t) \). We have seen that condition \( \frac{d}{dx} [G/g^2] = f(G) \) is equivalent to
\[ 2G \frac{dx}{dG} = x + f(G). \quad (13) \]

We derive (10) with respect to the variable \( G \), we then obtain (9).

So, for any analytic function \( f \) the linear equation (9) admits a unique solution \( x = x(G) \) according to the initial conditions. More precisely, consider the change \( x = \sqrt{2G} + y \). Then \( y = y(G) \) is a solution of
\[ 2G \frac{dy}{dG} + \frac{dy}{dG} = f(G) \]
with initial points \( y(0) = 0, y'(0) = 1 \). A resolution of the last equation yields
\[ y(G) = \sqrt{2G} \int_0^G \frac{F(\nu)}{(2\nu)^{3/2}} d\nu. \]

According to the initial conditions, this solution \( y(G) \) is analytic. Thus, a solution of (9) may be written
\[ x(G) = \sqrt{2G} \left( 1 + \int_0^G \frac{F(\nu)}{(2\nu)^{3/2}} d\nu \right). \]

**Corollary 2-6.** Let \( G(x) = \int_0^x g(s) ds \) be an analytic potential of the scalar equation (1). Let \( A(x) \) be an analytic involution defined by \( G(A(x)) = G(x) \) and \( A(x) < 0 \).

Then 0 is an isochronous center of (1) if and only if
\[ \frac{d}{dx} [G(x)/g^2(x)] = \frac{4A''(x)}{(1 - A'(x))^3}. \]

Moreover, the last expression is A-invariant.

**Proof of corollary 2-6.** Let \( A \) be an analytic involution, then \( G(A(x)) = G(x) \) implies
\[ \frac{dG}{dx} = g(x) = \frac{dA}{dx} g(A(x)) \]
and
\[ \frac{G}{g^2}(A(x)) = \left( \frac{dA}{dx} \right)^2 \frac{G}{g^2}(x). \]

Deriving the last expression we then obtain
\[ A'(x) \frac{d}{dx} \frac{G}{g^2}(A(x)) = 2A'(x)A''(x) \frac{G}{g^2}(x) + A'^2(x) \frac{d}{dx} \frac{G}{g^2}(x). \]
Since $A'(x) = 0$, then after simplification we get the differential equation
\[
\frac{d}{dx}(A(x)) = 2A''(x)\frac{G}{g'(x)} + A'(x)\frac{d}{dx}\left(\frac{g}{g'}(x)\right). \tag{14}
\]
By theorem B, \(\frac{d}{dx}[G(x)/g^2(x)] = f(G)\) implies \(\frac{d}{dx}[G(x)/g^2(x)] = \frac{d}{dx}[G(x)/g(A(x))]\).
Therefore, the solution of equation (17) is
\[
\frac{G}{g^2(x)} = \frac{2}{(1 - A'(x))^2} \tag{15}
\]
since $A'(0) = -1$. Thus, by deriving one gets \(\frac{d}{dx}[G(x)/g^2(x)] = \frac{d}{dx}[G(x)/g(A(x))]\).

To prove the converse we again require lemma 2-2:
\[
T'(c) = \left(\frac{1}{c\sqrt{2}}\right) \int_0^b \frac{d}{dx}(G/g^2(x)) - \frac{d}{dx}(\frac{g}{g'}(x)) \sqrt{c - G(x)} - g(x)dx. \tag{16}
\]

Proof. Indeed, we have seen that
\[
T'(c) = \left(\frac{1}{c\sqrt{2}}\right) \int_a^b \frac{d}{dx}(G/g^2(x)) - \frac{d}{dx}(\frac{g}{g'}(x)) \sqrt{c - G(x)} - g(x)dx
\]
for $a < c < 0$ and $0 < c < b$.

By splitting the integral we get
\[
T'(c) = \left(\frac{1}{c\sqrt{2}}\right) \int_0^a \frac{d}{dx}(G/g^2(x)) - \frac{d}{dx}(\frac{g}{g'}(x)) \sqrt{c - G(x)} - g(x)dx
\]
\[
+ \left(\frac{1}{c\sqrt{2}}\right) \int_0^b \frac{d}{dx}(G/g^2(x)) - \frac{d}{dx}(\frac{g}{g'}(x)) \sqrt{c - G(x)} - g(x)dx
\]
\[
= \left(\frac{1}{c\sqrt{2}}\right) \int_0^a \frac{d}{dx}(G/g^2(x)) - \frac{d}{dx}(\frac{g}{g'}(x)) \sqrt{c - G(x)} - g(x)dx
\]
\[
+ \left(\frac{1}{c\sqrt{2}}\right) \int_0^b \frac{d}{dx}(G/g^2(x)) - \frac{d}{dx}(\frac{g}{g'}(x)) \sqrt{c - G(x)} - g(x)dx
\]
\[
= \left(\frac{1}{c\sqrt{2}}\right) \int_0^a \frac{d}{dx}(G/g^2(x)) - \frac{d}{dx}(\frac{g}{g'}(x)) \sqrt{c - G(x)} - g(x)dx.
\]

Thus, $T'(c) \equiv 0$ implies necessarily \(\frac{d}{dx}[G(x)/g^2(x)] = \frac{d}{dx}[G(x)/g(A(x))]\) and conversely \(\frac{d}{dx}[G(x)/g^2(x)] = \frac{d}{dx}[G(x)/g(A(x))]\) implies $T'(c) \equiv 0$.

This involution $A(x)$ may also be defined as a solution of a linear ODE. The following result is analogous to corollary 2-5.

**Corollary 2-7.** Under the hypotheses of theorem A, equation (1) admits an isochronous center at 0 if and only if the involution $A = G(x)$ is a solution of
\[
2G \frac{dA}{dG} = A(G) + F(G) \tag{16}
\]
where $F(x) = \int_0^x f(u)du$ is the analytic integral function.
Moreover, this solution must satisfy the conditions
\[
A(0) = 0, \quad \lim_{G \to 0} \frac{A^2}{2G} = 1.
\]

**Proof of corollary 2-7.** To see that, we start from $A(\bar{A}(x)) = x$ and $G(A(x)) = G(x)$. So equation (11) is equivalent to $T'(E) \equiv 0$. We show that it is the same for the higher corrections for potentials. This allows summing the series for this type of potential:
\[
2G(A(x)) = \frac{dG(A(x))}{dx}[A(x) + F(G(A(x))] = \frac{dG(A(x))}{dx}(x + F(G)).
\]
On the other hand, deriving $G(A(x)) = G(x)$ with respect to $x$ we get \(\frac{dG(A(x))}{dx} = \frac{dG(x)}{dx}(x + F(G))\). We then obtain
\[
A(x) + F(G(x)) = [x + F(G(x))]\frac{dA(x)}{dx}.
\]
Hence we deduce the ODE (16).

### 3. Links with previous results

The preceding results have several consequences and corollaries. We are able to deduce a lot of necessary and sufficient conditions established by different authors in order to have isochronous motions, among them Urabe [13], Landau and Lifschitz [14], and Koukles and Piskounov [15].

#### 3.1. Condition of Landau and Lifschitz

These authors provided an excellent starting point by deducing the constraint placed on the form of the potential by a prescribed energy dependence of the period. They proved the following:

**Proposition 3-1** [15, Chap. 3]. Let $G(x) = \int_0^x g(s)ds$ be an analytic potential. Suppose that for $x \neq 0$, $xg(x) > 0$. Equation (1) has an isochronous center at the origin 0 if and only if
\[
x - A(x) = 2\sqrt{2G(x)} \tag{17}
\]
for all $0 < x < b$ where $A$ is the involution such that $G(A(x) = G(x)$ and $A(x) < 0$ for $x \neq 0$.

**Proof of proposition 3-1.** We need the following lemmata:

**Lemma 3-2.** For any analytic involution $A(x)$ defined for all $x \in [a, b]$, the following expression
\[
\frac{4A''(x)}{(1 - A'(x))^3} = \frac{4A''(x)}{(1 - A(x))^3} \tag{18}
\]
is $A$-invariant:
\[
\frac{4A''(x)}{(1 - A'(x))^3} = \frac{4A''(x)}{(1 - A(x))^3}.
\]
Proof. Indeed, to prove this lemma we start from \( A(A(x)) = x \). By deriving \( A'(A(x))A'(x) = 1 \) and \( A''(A(x))A'(x) + A'(A(x))A''(x) = 0 \), we yield
\[
A'(A(x)) = \frac{1}{A'(x)} \quad \text{and} \quad A''(A(x)) = -\frac{A''(x)}{A'^3(x)}. 
\]
Replacing \( x \) by \( A(x) \) in expression (15), one obtains
\[
\frac{A''(A(x))}{(1 - A'(A(x)))^3} = -\frac{A''(x)}{A'^3(x)}\left(1 - \frac{1}{A'(x)}\right)^3 = -\frac{A''(x)}{A'^3(0)}(A'(x))^3 - 1. 
\]
So, since \( A'(x) \neq 0 \) one gets after simplification the \( A \)-invariance of (18):
\[
\frac{4A''(A(x))}{(1 - A'(A(x)))^3} = \frac{4A''(x)}{(1 - A'(x))^3}. 
\]

Lemma 3-3. When the function \( \frac{d}{dx}\left[\frac{G(x)}{g^2(x)}\right] \) is \( A \)-invariant then the following holds:
\[
\frac{d}{dx}\left[\frac{G(x)}{g^2(x)}\right] = \frac{2}{(1 - A'(x))^3} = \frac{4A''(x)}{(1 - A'(x))^3}. \tag{19}
\]

Proof. Since \( G(A(x)) = G(x) \) a direct calculation yields
\[
\frac{d}{dx}\left[\frac{G(x)}{g^2(A(x))}\right] = \frac{d}{dx}\left[\frac{G(x)}{g^2(A(x))}\right] = \frac{g^2 - 2G'}{g^3}(A(x)). 
\]
Since \( g(A(x))A'(x) = g(x) \) and \( g'(A(x))A''(x) + g(A(x))A''(x) = g'(x) \) one gets
\[
\left[\frac{g^2 - 2G'}{g^3}\right](A(x)) = \frac{g^2A'' - 2G(gA'' - 2gA'A'' - gA''A)\left(\frac{g}{A'(x)}\right)^3}{g^2A'' - 2GgA''\left(\frac{g}{A'(x)}\right)^2}. 
\]
It implies
\[
\left(\frac{g^2 - 2G'}{g^3}\right)(x) (1 - A'(x)) = \left(\frac{2GA''}{g^2}\right)(x). 
\]
Therefore
\[
\frac{G(x)}{g^2(x)} = \frac{2}{(1 - A'(x))^3}. 
\]

On the other hand we may deduce the following differential equation
\[
1 - A'(x) = \frac{g(x) \sqrt{2}}{\sqrt{G(x)}}. 
\]
By integration we obtain
\[
x - A(x) = 2\sqrt{2G(x)} \quad \text{for all} \quad 0 < x < b. \tag{20}
\]
These lemmata imply that if \( G(x) \) is an isochronous potential with \( G(0) = 0 \) then \( x - A(x) = 2\sqrt{2G(x)} \).

Conversely, starting from \( x - A(x) = 2\sqrt{2G(x)} \) and deriving twice one gets
\[
\frac{d}{dx}\left[\frac{G(x)}{g^2(x)}\right] = \frac{4A''(x)}{(1 - A'(x))^3}. 
\]
Then by theorem B, proposition 3-1 is proved.

3.2. Theorems of Koukles and Piskounov

When \( g(x) \) is a continuous function, with \( g(x) \) and \( x \) having the same sign, Koukles and Piskounov [15] produced necessary and sufficient conditions so that the center of the system (2) is isochronous.

Proposition 3-4 [13, Th 5]. A set of necessary and sufficient conditions for the period of every solution of (1) near 0 to be equal to a constant \( T_0 \) is:
\[
1.-g(x) \text{ is continuous and positive for small positive } x. \\
2.-\lim_{x \to 0} \frac{g(x)}{x} = 0. \\
3.-T_0 \geq \limsup_{x \to 0} \frac{2g}{g^2} \frac{\pi}{\sqrt{2G} \psi(x) - \Theta[\psi(x)]} = x. \\
4.-Let \Theta(x) \text{ and } \psi(x) \text{ be defined as follows} \\
\int_0^{\psi(x)} g(u) du = x \quad \text{and} \quad \frac{T_0}{\pi} \sqrt{2\psi(x) - \Theta[\psi(x)]} = x.
\]
then \( g(-x) = -\psi(x) \).

Although this result appears better than proposition 3-1, it remains difficult to apply except in certain specific situations.

When \( g(x) \) is an analytic function, with \( g(x) \) and \( x \) having the same sign, Koukles and Piskounov [15] produced a necessary and sufficient condition so that the center of the system (2) is isochronous.

Proposition 3-5 [13, Th 6]. Let \( g(x) \) be a real analytic function. Then the center 0 of equation (1) \( \ddot{x} + g(x) = 0 \) is isochronous if and only if the inverse function \( x = \Theta(z) \)
\[
\left[\int_0^z g(\xi)d\xi\right]^{-1} = \Theta(z).
\]
is of the form
\[ x = \Theta(z) = \sqrt{z} + P(z) \]  
where \( P \) is a real analytic function such that \( P(0) = 0 \).

**Proof of proposition 3-5.** This result is a direct consequence of corollary 3-3. Indeed, considering the change \( x = \sqrt{2G} + y \) we have proved that \( y = P(G) \) is a unique solution of
\[
2G \frac{d^2y}{dG^2} + \frac{dy}{dG} = f(G)
\]
with initial points \( y(0) = 0, y'(0) = 1 \). A resolution of the last equation yields
\[
y(G) = \sqrt{2G} \int_0^G \frac{F(\nu)}{(2\nu)^{3/2}} d\nu.
\]
Conditions \( y(0) = 0, y'(0) = 1 \) imply this solution \( y = P(G) \) is necessarily analytic. Thus, a solution of (9) may be written
\[
x(G) = \sqrt{2G} (1 + \int_0^G \frac{F(\nu)}{(2\nu)^{3/2}} d\nu) = \sqrt{2G} + H(G)
\]
where \( P(G) \) is analytic in \( G \). The proof is achieved.

### 3.3. Urabe theorem

Later, Urabe proposed some refinements of proposition 3-5 by considering the assumption of the differentiability of \( g(x) \) at 0 and proved the following result, which is used more than proposition 3-5 because it considers less rigid hypotheses.

**Proposition 3-6 [13].** Let \( g(x) \) be an analytic function defined in \( V_0 \) a neighborhood of 0 verifying \( xg(x) > 0 \) in \( V_0 \). Then the system (2) has an isochronous center at the origin 0 if and only if \( g(x) \) may be written
\[
g(x) = \frac{X}{1 + h(X)}
\]
where \( h(X) \) is a \( C^1 \) odd function and \( X = \sqrt{2G(x)} \), \( \frac{X}{h} > 0 \) for \( x \neq 0 \).

**Proof of proposition 3-6.** In the analytic case, theorem B and corollary 3-3 together imply that
\[
x = \frac{2G}{g} + F(G) = \sqrt{2G} + P(G).
\]
Thus,
\[
\frac{2G}{g} = \sqrt{2G} + \psi(G)
\]
(where \( \psi(G) \) is an analytic function in \( G \)) is another necessary and sufficient condition for the potential of equation (1) to be isochronous. Simplifying by \( \sqrt{2G} \) yields
\[
\frac{\sqrt{2G}}{g} = 1 + \frac{\psi(G)}{\sqrt{2G}}
\]
which is equivalent to the Urabe condition
\[
g(x) = \frac{\sqrt{2G}}{1 + \frac{\psi(G)}{\sqrt{2G}}} = \frac{X}{1 + h(X)}
\]
where \( X = \sqrt{2G} \).

**Remark 3-7.** Notice that the method of Urabe used an intermediary function \( h \) that is not in general explicitly known. Our (equivalent) conditions are simply relations between an isochronous potential \( G \) and its first derivative \( g \). More precisely, any analytic solution \( G = G(x) \) of equation (6)
\[
2G(x) - xg(x) = g(x)F(G(x)) \quad \text{or equivalently of equation (7)}
\]
\[
\frac{d}{dx}[G(x)/g^2(x)] = f(G(x))
\]
provides an isochronous potential verifying \( G(0) = G'(0) = 0, G''(0) = 1 \). This fact will be illustrated by several multi-parameter families of isochronous potentials.

### 3.4. Other consequences

The next result is natural, although unexpected. However, it will be useful for the sequel, in particular in order to perform the WKB expansion to all orders for any isochronous potential.

**Proposition 3-8.** Let \( G(x) = \int_0^x g(s) ds \) be an analytic potential defined in a neighborhood of 0. Suppose equation (1) has an isochronous center at 0. Let \( g^{(n)}(x) \) be the \( n \)th derivative of the potential (with respect to \( x \)):
\[
g^{(n)}(x) = \frac{d^n}{dx^n}G(x), n \geq 1, \quad \text{then these derivatives may be expressed under the form}
\]
\[
g^{(n)}(x) = a_n(G)\sqrt{2G} + b_n, n \geq 0
\]
where \( a_n \) and \( b_n \) are analytic functions with respect to \( G \).

In fact, as we will see in the sequel, the functions \( a_n \) and \( b_n \) are only dependent on \( G_1 \) for the odd part of \( G = G(x) \).

**Proof of proposition 3-8.** By proposition 3-4, condition \( x(G) = \sqrt{2G} + P(G) \) with \( P = P(G) \) is a non-zero analytic function implying that equation (1) has an isochronous center at 0. Deriving with respect to \( G \) one obtains
\[
\frac{dx}{dG} = \frac{1}{\sqrt{2G}} + P'(G) = \frac{1}{g}
\]
or equivalently
\[
\frac{g}{\sqrt{2G}} = \frac{1}{1 + \sqrt{2G}P'(G)} = a_1(G) + \frac{b_1(G)}{\sqrt{2G}}
\]
with
\[
a_1(G) = \frac{-1}{2GP'' - 1} \quad \text{and} \quad b_1(G) = \frac{2GP'}{2GP'' - 1}.
\]
Notice that by hypothesis \( G \) is defined in a neighborhood of 0 then \( 2GP'' - 1 \) is necessarily non-zero.
The functions $a_1(G)$ and $b_1(G)$ are analytic since $P$ and $P'$ are too. Deriving $g'(x)$ it yields

$$g'(x) = \frac{dg}{dx} = \frac{d}{dx}a_1(G)\sqrt{G} + \frac{1}{2} a_1(G)g\frac{\sqrt{G}}{G} + \frac{d}{dx}b_1(G)$$

$$= \frac{d}{dG}a_1(G)g\sqrt{2}G + \frac{1}{2} a_1(G)g\frac{\sqrt{2}G}{G} + \frac{d}{dG}b_1(G),$$

where the symbol prime ' means $\frac{d}{dG}$ and $a_1$ or $b_1$ stands for $a_1(G)$ or $b_1(G)$.

After replacing $g(x) = a_1(G)\sqrt{2}G + b_1(G)$ one obtains

$$2Ga'_1a_1 + a'_1\sqrt{2}G b_1 + a^2_1 + \frac{1}{2} \frac{a_1\sqrt{2}b_1}{G}$$

$$+ b'_1a_1\sqrt{2}G + b'_1b_1.$$  

By simplifying one finds the expression of $g'(x) = a_2(G)\sqrt{2}G + b_2(G)$ with

$$a_2(G) = a'_1b_1 + \frac{a_1b_1}{2G} + b'_1a_1$$

$$b_2(G) = 2Ga'_1a_1' + a^2_1 + b'_1b_1'$$

where

$$\frac{a_1b_1}{2G} = \frac{P'}{(2G^2 - 1)^2}$$

which is analytic. Then the functions $a_2(G)$ and $b_2(G)$ are analytically dependent on the functions $a_1(G)$, $b_1(G)$ and their derivatives.

Suppose now that until order p one has

$$g^{(p)}(x) = a_p(G) + b_p(G)$$

where the functions $a_p(G)$ and $b_p(G)$ are analytic with respect to $G$. Thanks to Maple we are able to carry out the calculations. Deriving this expression with respect to x yields

$$g^{(p+1)}(x) = \frac{d}{dx}a_p(G)\sqrt{2}G + \frac{1}{2} a_p(G)g\frac{\sqrt{2}G}{G}$$

$$+ \frac{d}{dx}b_p(G) = a'_p(G)g\sqrt{2}G + \frac{1}{2} a_p(G)g\frac{\sqrt{2}G}{G}$$

$$+ b'_p(G)g$$

$$= (a'_p(G))(a_1\sqrt{2}G + b_1)\sqrt{2}G$$

$$+ \frac{1}{2} a_p(G)(a_1\sqrt{2}G + b_1)\sqrt{2}G$$

$$+ b'_p(G)g.$$  

By simplifying one obtains

$$g^{(p+1)}(x) = 2Ga'a_p + a_1a_p + b_1\sqrt{2}b'_p$$

$$+ \sqrt{G}b_1a'_p + \frac{1}{2} \frac{a_1\sqrt{2}b_1}{G} + b'_1b'_p$$

where the symbol prime ' means $\frac{d}{dG}$ and $a_p$ or $b_p$ stands for $a_p(G)$ or $b_p(G)$.

By simplifying one finds the expression of

$$g^{(p+1)}(x) = a_{p+1}(G)\sqrt{2}G + b_{p+1}(G)$$

with

$$a_{p+1}(G) = a'_p b_1 + \frac{a_1b_1}{2G} + b'_p a_1$$

$$b_{p+1}(G) = 2Ga'a_p + a_1a_p + b_1b'_p.$$  

Moreover, since

$$a_p\frac{b_1}{2G} = \frac{a'_p P'}{2GP^2 - 1}$$

then the function $\frac{a_1b_1}{2G}$ is analytic with respect to $G$. Thus, $a_{p+1}(G)$ and $b_{p+1}(G)$ are also analytic.

Notice that the coefficients $a_p(G)$ et $b_p(G)$ depend analytically on $P$ and its derivatives. It should be mentioned that their complexity increases rapidly with the order $n$ except obviously for the first terms. Namely, after replacing $a_1$ and $b_1$ in the expression of $a_2$ and $b_2$ and after simplifying, we get

$$a_2 = a_1 b_1 + \frac{a_1b_1}{2G} + b'_1 a_1$$

$$= 2GP^3 + 12G^2pP^2 + 3P + 2GP^p$$

$$b_2 = 2Ga'_1 a_1' + a_1^2 + b_1 b_1'$$

$$= 6G^2P^2 + 12G^2pP^p + 1 + 8G^3pP^p.$$  

The next one is already complicated enough

$$a_3 = a_1 b_1^2 + 2a_1 b'_1 b_1 + 2b'_1 a_1 b_1 + 4a_1^2 a_1'$$

$$+ 2Ga'_1 a_1' a_1'' + b_1^2 a_1$$

$$+ 4a_1 b_1^2 + 4b'_1 a_1 b_1 - a_1 b_1^2$$

$$+ \frac{4G}{4G}$$

$$b_3 = 4Ga''_1 b_1 a_1 + 4Ga'_1 b_1 a_1' - a_1 b_1^2$$

$$+ 2Gb_1^2 a_1^2 + 2a_1^2 b_1'$$

$$+ 2Ga'_1 b_1 + b_1^2 b_1' + b_1^2 b_1.$$  

**Remark 3-8-1.** It seems very likely that we can show the following result: let $G(x)$ be an analytic potential such that $xG(x) > 0$, $x \neq 0$. Suppose the nth derivative may be expressed as $g^{(n-1)}(x) = a_n(G)\sqrt{2}G + b_n(G)$, $n \geq 0$ where $a_n$ and $b_n$ are analytic functions, then the potential is isochronous.

For that, we need to solve the differential system for a given $g^{(1)}(x) = g'(x)$:

$$a_2 = a_1 b_1 + \frac{a_1b_1}{2G} + b'_1 a_1$$

$$b_2 = 2Ga'_1 a_1' + a_1^2 + b_1 b_1'$$

where the symbol prime ' means $\frac{d}{dG}$ and $a_p$ or $b_p$ stands for $a_p(G)$ or $b_p(G)$. Moreover, the initial conditions are $a_1(0) = 1$, $b_1(0) = 0$. This means that the class of isochronous potentials is rigid.
Remark 3-8-2. In fact, in the case of isochronism knowing 

$$a_1(G) = \frac{-1}{2GP^3-1}, \quad b_1(G) = \frac{2GP}{2GP^3-1}$$

we then obtain a simple relation between them after eliminating $P'$:

$$b_1^2 = 2G(a_1^2 - a_1).$$  \ (25)

This means that given an analytic potential $G$, and two functions $a_1(G)$ and $b_1(G)$ verifying relation (13) so that $g(x) = \frac{2G}{dx}$ can be expressed $g(x) = a_1(G)\sqrt{2G} + b_1(G)$, then $G$ is an isochronous potential.

This result is also interesting. On the one hand it is more general than lemma 2-2 and on the other hand it will be very useful for the sequel in the WKB approximation for isochronous potentials.

Proposition 3-9. Let $G(x) = \int_a^x g(s)ds$ be an analytic potential and $\phi(x)$ a function defined in a neighborhood of 0. Let $A$ be the analytic involution defined by $G(A(x)) = G(x)$. Then for $a < 0 < b = A(a)$ and $G(a) = G(b) = E$ the following integral holds:

$$\int_a^b \frac{\phi(x)}{\sqrt{E - G(x)}} g(x)dx = \int_0^b \frac{\phi(x) - \phi(A(x))}{\sqrt{E - G(x)}} g(x)dx$$

In particular, if we may express $\phi(x) = u(G)\sqrt{2G} + v(G)$ then

$$\int_a^b \frac{\phi(x)}{\sqrt{E - G(x)}} g(x)dx = \int_0^b 2u(G)\frac{\sqrt{2G}}{\sqrt{E - G(x)}} g(x)dx$$

Proof of proposition 3-9. It suffices to split the integral

$$\int_a^b \frac{\phi(x)}{\sqrt{E - G(x)}} g(x)dx = \int_0^a \frac{\phi(x)}{\sqrt{E - G(x)}} g(x)dx + \int_0^b \frac{\phi(x)}{\sqrt{E - G(x)}} g(x)dx.$$ 

Recall that $a < 0 < b$. By definition when $x \in [a, 0]$ then $A(x) \in [0, b]$, and conversely by a change of variable $x = A(y)$ the integral becomes

$$\int_a^b \frac{\phi(x)}{\sqrt{E - G(x)}} g(x)dx = -\int_0^b \frac{\phi(A(y))}{\sqrt{E - G(y)}} g((A(y)))A'(y)dy$$

since $g((A(y)))A'(y) = g(y)$. Therefore

$$\int_a^b \frac{\phi(x)}{\sqrt{E - G(x)}} g(x)dx = \int_0^b \frac{\phi(x)}{\sqrt{E - G(x)}} g(x)dx - \int_0^b \frac{\phi(A(y))}{\sqrt{E - G(y)}} g(y)dy = .$$

On the other hand, since $\phi(x) = u(G)\sqrt{2G} + v(G)$, then the following integral may be written

$$\int_a^b \frac{\phi(x)}{\sqrt{E - G(x)}} g(x)dx = \int_a^b \frac{u(G)\sqrt{2G} + v(G)}{\sqrt{E - G(x)}} g(x)dx$$

$$= \int_a^b \frac{u(G)\sqrt{2G}}{\sqrt{E - G(x)}} g(x)dx + \int_a^b \frac{v(G)}{\sqrt{E - G(x)}} g(x)dx.$$ 

The last integral can be written

$$\int_a^b \frac{v(G)}{\sqrt{E - G(x)}} g(x)dx = \int_0^E \frac{v(G)}{\sqrt{E - G}} dG = 0$$

since $v(G)$ is analytic. The other integral can be written

$$\int_a^b \frac{u(G)\sqrt{2G}}{\sqrt{E - G(x)}} g(x)dx = \int_a^b \frac{u(G)\sqrt{2G}}{\sqrt{E - G(x)}} g(x)dx$$

$$+ \int_0^b \frac{2u(G)\sqrt{2G}}{\sqrt{E - G(x)}} g(x)dx$$

$$= \int_0^b \frac{u(G)\sqrt{2G}A(y)}{\sqrt{E - G(y)}} g(A(y))A'(y)dy$$

$$= \int_0^b \frac{u(G)\sqrt{2G}(y)}{\sqrt{E - G(y)}} g(y)dy$$

$$= \int_0^b \frac{u(G)\sqrt{2G}(y)}{\sqrt{E - G(y)}} g(y)dy$$

since $2G(A(y)) = -\sqrt{2G}$. Finally,

$$\int_a^b \frac{\phi(x)}{\sqrt{E - G(x)}} g(x)dx = \int_0^b \frac{u(G)\sqrt{2G}}{\sqrt{E - G(x)}} g(x)dx$$

$$+ \int_0^b \frac{u(G)\sqrt{2G}}{\sqrt{E - G(x)}} g(x)dx.$$ 

Corollary 3-10. Under the hypotheses of proposition 3-7, consider the derivatives of $g$: $g^{(n)}(x) = \frac{d^n g}{dx^n}$. Then the analytic function

$$V_{m,v}(x) = \prod_{j=1}^m \left( \frac{d^j g}{dx^j} \right)^{\nu}$$

may be expressed under the form

$$V_{m,v}(x) = u_{m,v}(G)\sqrt{2G} + v_{m,v}(G) \quad (26)$$

where $\nu = (\nu_1, \nu_2, \ldots, \nu_m)$ and $u_{m,v}$ and $v_{m,v}$ are analytic functions with respect to $G$.

Proof. By proposition 3-8 when $G$ is isochronous any derivative of $g$ may be written

$$g^{(n)}(x) = a_n(G)\sqrt{2G} + b_n(G), \quad n \geq 0$$

where $a_n$ and $b_n$ are analytic functions. It is easy to realize that it is the same for any power of any derivative $(g^{(n)}(x))^\nu_n$. We may prove that by recurrence

$$(g^{(n)}(x))^\nu_n = a_{n,v}(G)\sqrt{2G} + b_{n,v}(G).$$
More generally, we may also prove by recurrence that a product of power of derivatives has a similar expression
\[ (g^{(n)}(x))^{(p)}(x) \] for any order
\[ n, p \]
so that the error of the leading-order expansion to any order study the spectra of quantum systems and to calculate the expansion. This method due to Dunham is often employed to
\[ V_{m,p}(x) = \prod_{j=1}^{m} \left( \frac{d^j g}{dx^j} \right)^p = u_{m,p}(G) \sqrt{2G} + v_{m,p}(G). \]
Thus we may write for any product
\[ V_{m,p}(x) \]
where the mass
\[ m \]
and from the results of section 3 we will give an estimate of the nth WKB correction.

4. The quantum spectrum and WKB quantization

The semiclassical method of torus quantization is the first term of a certain h-expansion, usually called the WKB expansion. This method due to Dunham is often employed to study the spectra of quantum systems and to calculate the semiclassical spectrum is perfectly regularly spaced. The semiclassical approximation should converge to zero as the limit
\[ \hbar \to 0 \]
where the mass
\[ m \]
and from the results of section 3 we will give an estimate of the nth WKB correction.

4.1. WBK series to all orders

Following Robnik and Romanovski [16], who considered the Schrodinger equation,
\[ \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + G(x) \right] \psi(x) = E\psi(x). \] (27)
The Hamiltonian of the system is given by
\[ H = \frac{\hbar^2}{2m} + G(x) \]
where the mass
\[ m \]
and from the results of section 3 we will give an estimate of the nth WKB correction.

This Hamiltonian is a constant of motion, whose value is equal to the total energy
\[ E. \]
To calculate all the terms of the WKB expansion one observes that the wave function can always be written as
\[ \psi(x) = \exp \left( \frac{i}{\hbar} \sigma(x) \right) \]
where the phase
\[ \sigma(x) \]
is a complex function verifying the Riccati differential equation
\[ \sigma''(x) + \left( \frac{1}{2} \right) \sigma''(x) = 2(E - G(x)). \]
The WKB expansion for the phase is a power series in \( \hbar \):
\[ \sigma(x) = \sum_{k=0}^{\infty} \left( \frac{\hbar}{k} \right)^k \sigma_k(x). \]
After replacing and comparing like powers of \( \hbar \) we obtain the recursion relations
\[ \sigma_0^2 = 2(E - G) \]
and
\[ \sum_{k=0}^{l} \sigma_k' \sigma_{n-k} + \sigma_{n-1} = 0, n \geq 1. \]
The quantization condition is obtained by requiring the uniqueness of the wave function
\[ \int_{\gamma} d\sigma = \sum_{k=0}^{\infty} \left( \frac{\hbar}{k} \right)^k \int_{\gamma} d\sigma_k = 2\pi m, \quad n \in N \] (28)
where the integration contour \( \gamma \) surrounds the two turning points of \( G(x) \) at energy \( E \).

The first term of this series is the Bohr–Sommerfeld formula. It is proportional to the action
\[ I_0(E): \]
\[ \int_{\gamma} d\sigma_0 = 2 \int_{a}^{b} \sqrt{2(E - G(x))} dx = 2\pi I_0(E). \]
The second term is the Maslov correction. It can be shown to be equal to
\[ \left( \frac{\hbar}{l} \right)^k \int_{\gamma} d\sigma_l = -\pi l. \]
All the other odd terms vanish. These terms \( \sigma_{2k+1} \), \( k \geq 1 \), are total derivatives and so \( \int_{\gamma} \sigma'_{2k+1} = 0 \). This permits us to rewrite the quantization condition as
\[ \sum_{k=0}^{\infty} I_{2k}(E) = \left( n + \frac{1}{2} \right) l, \quad n \in N \]
where
\[ I_{2k}(E) \frac{1}{\hbar} \left( \frac{\hbar}{l} \right)^{2k} \int_{a}^{b} d\sigma_{2k}, \quad k \in N. \] (29)
When \( G(x) \) is analytic and \( \chi_{G}(x) > 0 \), it has been proved in [17] that the contour integrals can be replaced by equivalent Riemann integrals between the two turning points. More precisely, they proved the following:
\[ I_{2k}(E) = -\hbar^2 \frac{\partial^2}{24\sqrt{2}\pi \cdot \partial E^2} \int_{a}^{b} \frac{\chi^2(x)}{\sqrt{E - G(x)}} dx \]
and

\[ I_4(E) = -\frac{\hbar^4}{4\sqrt{2}\pi} \left[ \frac{1}{120 \partial E^3} \int_a^b \frac{g^2(x)}{\sqrt{E - G(x)}} \, dx \right. \]

\[ \left. - \frac{1}{288 \partial E^3} \int_a^b \frac{g^2(x)g'(x)}{\sqrt{E - G(x)}} \, dx \right]. \]

Robnik and Salasnich [6] elaborated an efficient algorithm in order to simplify the function \( d\nu \), and produced the following formula for even \( m \)

\[ \sigma' = \sum_{L(v) = m} \frac{2^n - 1 + |
u|}{(m - 3 + 2|
u|)!} \frac{\partial E^2}{\partial E} \frac{U_\nu G^{(\nu)}}{E - G}, \]

where \( \nu = (\nu_1, \nu_2, \ldots, \nu_m) \), \( \nu_j \in N \), \( L(\nu) = \sum_{j=1}^m j \nu_j \), and \( |
u| = \sum_{j=1}^m \nu_j \).

Moreover, they proved

\[ \int d\sigma' = 2 \sum_{L(v) = m} \frac{2^n - 1 + |
u|}{(m - 3 + 2|
u|)!} \frac{\partial E^2}{\partial E} \frac{U_\nu G^{(\nu)}}{E - G} \]

\[ \int_a^b \frac{U_\nu G^{(\nu)}}{E - G} \, dx \]

where

\[ G^{(\nu)}(x) = \prod_{j=1}^m \left( \frac{dG}{dx^j} \right)^{\nu_j}. \]

Coefficients \( U_\nu \) are defined by a recurrence equation but do not play a role in deriving the result.

Notice that higher-order corrections quickly increase in complexity and we know only a few cases where a WKB approximation in order to simplify the function \( d\nu \) and produced the following formula for even \( m \).

By proposition 3-9, we may express

\[ I_2(E) = -\frac{\hbar^2}{24\sqrt{2}\pi} \int_a^b \frac{2a(G) \sqrt{2G}}{E - G} g(x) \, dx \]

\[ = -\frac{\hbar^2}{24\sqrt{2}\pi} \partial^0 \int_a^b \frac{2a(v) \sqrt{2v}}{E - v} \, dv. \]

Then making the change of variables \( u = \frac{v}{E} \) (we suppose here \( \omega = 1 \)),

\[ I_2(E) = -\frac{\hbar^2}{24\pi} \frac{\partial^0}{\partial E^2} \int_0^1 \frac{2a(E) \sqrt{E}}{\sqrt{1 - u}} \, du. \]

Another formulation is given by an Abel-integral type

\[ I_2(E) = -\frac{\hbar^2}{24\pi} \frac{\partial^0}{\partial E^2} \int_0^E \frac{(\sqrt{2v})^3}{\sqrt{E - v}} \, dv \left[ \frac{2da}{dG} (v) + v \frac{d^2a}{dG^2} (v) \right]. \]

A similar calculation gives the fourth-order correction (see appendix A for details)

\[ I_4(E) = -\frac{\hbar^4}{4\sqrt{2}\pi} \int_a^b \frac{1}{120 \partial E^3} \frac{g^2(x)}{\sqrt{E - G(x)}} \, dx \]

\[ - \frac{1}{288 \partial E^3} \int_a^b \frac{g^2(x)g'(x)}{\sqrt{E - G(x)}} \, dx \]

By proposition 3-9 one gets \( g = \frac{dG}{dx} = a(G)\sqrt{2G} + b(G) \).

We now use the preceding results of section 3 in order to express \( I_2(E) \) and \( I_4(E) \) as well as the \( n \)th correction \( I_n(E) \) in terms related to the isochronous potential \( G(x) \) and its derivatives.

By proposition 3-8, we may write \( g = \frac{dG}{dx} = a(G)\sqrt{2G} + b(G) \). Writing

\[ I_2(E) = -\frac{\hbar^2}{24\sqrt{2}\pi} \frac{\partial^0}{\partial E^2} \int_a^b \frac{2a(G) \sqrt{2G}}{E - G} g(x) \, dx \]

\[ = -\frac{\hbar^2}{24\sqrt{2}\pi} \frac{\partial^0}{\partial E^2} \int_a^b g(x) \, dx. \]

and by proposition 3-9 we may express

\[ I_4(E) = -\frac{\hbar^2}{24\pi} \frac{\partial^0}{\partial E^2} \int_0^1 \frac{2a(E) \sqrt{E}}{\sqrt{1 - u}} \, du. \]

By proposition 3-8 one gets \( g = \frac{dG}{dx} = a(G)\sqrt{2G} + b(G) \).

We may easily prove

\[ \frac{g'^2}{g} = a_{12}(G)\sqrt{2G} + b_{12}(G) \]

where \( a_{12}(G) \) and \( b_{12}(G) \) are analytic functions, as well as

\[ g(x)g'(x) = c_{12}(G)\sqrt{2G} + d_{12}(G) \]

where \( c_{12}(G) \) and \( c_{12}(G) \) are analytic functions. Similar to \( I_2, I_4 \) may be expressed through Abel integrals:

\[ I_4(E) = -\frac{\hbar^4}{4\sqrt{2}\pi} \frac{\partial^0}{\partial E^2} \int_a^b \frac{g^2(x)}{\sqrt{E - G(x)}} \, dx \]

\[ - \frac{1}{288 \partial E^3} \int_a^b \frac{g^2(x)g'(x)}{\sqrt{E - G(x)}} \, dx \]

\[ = -\frac{\hbar^4}{4\sqrt{2}\pi} \frac{\partial^0}{\partial E^2} \int_0^E \frac{(\sqrt{2v})^3}{\sqrt{E - v}} \, dv \left[ \frac{2da}{dG} (v) + v \frac{d^2a}{dG^2} (v) \right]. \]
Starting from the expression for \( I_2(E) \) and using the properties of Abel transforms, as noted in [18], it is possible to invert the problem and to calculate the general expression of the functions \( a_j(G) \) and \( b_j(G) \) corresponding to a prescribed function \( I_2(E) \). One can choose \( I_2(E) \) (and deduce the corresponding analytic isochronous potential) such that its asymptotic decay is faster than the asymptotic decay of \( I_2(E) \). Therefore, \( I_2(E) \) and \( I_2(E) \) grow exponentially fast as \( E \) grows to \( \infty \).

As we will show below, these conclusions are similar to higher-order corrections.

We turn now to the upper-order WKB correction. The explicit expression for \( I_2(E) \) is given by

\[
I_{2n}(E) = -\frac{\sqrt{\pi}}{\pi} \int_0^{2n} \frac{2^{[\nu]} L_{(\nu)}=2n (2n - 3 + 2[\nu])!! J_\nu(E)}{\left( E - G \right)} \quad (31)
\]

where

\[
J_\nu(E) = \frac{\partial^n \frac{1 - |\nu|}{E} \int_a^b \frac{U_\nu G^{(\nu)}}{\sqrt{E - G}} dx}{\partial E^{n-1+|\nu|}}
\]

and where \( \nu = (\nu_1, \nu_2, \ldots, \nu_{2n}) \), \( \nu_j \in N \), \( L(\nu) = \sum_{j=1}^{2n} \nu_j \) and \( |\nu| = \sum_{j=1}^{2n} \nu_j \). The coefficients \( U_\nu \) satisfy a certain recurrence relation not useful for the sequel.

By corollary 3.10, \( G^{(\nu)} \) may be expressed under the form

\[
G^{(\nu)}(x) = \prod_{j=1}^{2n} \left( \frac{dG}{dx} \right)^{\nu_j} = u_{\nu,0}(G) \sqrt{2G} + v_{\nu,0}(G)
\]

where \( u_{\nu,0} \) and \( v_{\nu,0} \) are analytic functions with respect to \( G \). Therefore,

\[
J_\nu(E) = \frac{\partial^n \frac{1 - |\nu|}{E} \int_a^b \frac{U_\nu u_{\nu,0}(G) \sqrt{2G}}{\sqrt{E - G}} g(x) dx}{\partial E^{n-1+|\nu|}}
\]

By corollary 3.9, we can write

\[
J_\nu(E) = \frac{\partial^n \frac{1 - |\nu|}{E} \int_0^b \frac{2 U_\nu u_{\nu,0}(G) \sqrt{2G}}{\sqrt{E - G}}} {\partial E^{n-1+|\nu|}} g(x) dx
\]

and

\[
J_{\nu,0}(E) = \frac{\partial^n \frac{1 - |\nu|}{E} \int_0^E \frac{2 U_\nu u_{\nu,0}(\nu v) \sqrt{2v}}{\sqrt{E - v}}}{\partial E^{n-1+|\nu|}} dv.
\]

Another equivalent formulation via Abel integrals is

\[
J_\nu(E) = 2 U_\nu E^{-n+1-|\nu|} \int_0^E \frac{(2v)^n - 1 + |\nu|}{\sqrt{E - v}} \frac{\partial^n \frac{1 - |\nu|}{E} \int_0^b u_{\nu,0}(\nu v) dv}{\partial E^{n-1+|\nu|}} dv.
\]

Similar to \( I_2 \) and \( I_4 \), the \( n \)th correction \( I_{2n} \) seems to be expressed through Abel integrals:

\[
I_{2n}(E) = -\frac{\sqrt{\pi}}{\pi} \int_0^{2n} \frac{2^{[\nu]} L_{(\nu)}=2n (2n - 3 + 2[\nu])!! J_\nu(E)}{\left( E - G \right)} \quad (31)
\]

Thus, the WKB corrections \( I_2(E) \) grow exponentially fast as \( E \) grows to \( \infty \). The calculation above suggests the entire WKB series should be summed for any isochronous potential and would be finite as \( E \) grows to \( \infty \), although such series are generally non-convergent for any Newtonian potential. The isochronous potentials are then solvable.

5. On the parameterization of isochronous potentials

The purpose of this section is to highlight natural conditions so that an analytic potential \( G(x) \) is isochronous. To that end let us write

\[
G(x) = \frac{1}{2} x^2 + G_1(x) + G_2(x)
\]

where the function \( G_1(x) = \sum_{k>2} \frac{a_{2k-1}}{2k-1} x_{2k-1} \) is odd and \( G_2(x) = \sum_{k>2} \frac{a_{2k}}{2k} x_{2k} \) is even. Below we shall prove that once the odd part \( G_1 \) of an analytic isochronous potential is specified then the even part \( G_2 \) is determined. In this context as we have seen foregoing the successive derivatives of \( G(x) \) can be written \( g^{(n)}(x) = a_n(G) \sqrt{2G} + b_n(G) \), \( n > 0 \). This means that the analytic functions may only depend on the odd part \( G_1 \), \( a_n(G) = a_n(G_1) \) and \( b_n(G) = b_n(G_1) \). This can significantly simplify the problem. This remark is also valid for all the powers of derivative products

\[
V_{m,n}(x) = u_{m,n}(G_1) \sqrt{2G} + v_{m,n}(G_1)
\]

where \( \nu = (\nu_1, \nu_2, \ldots, \nu_m) \) and \( u_{m,n} \) and \( v_{m,n} \) are analytic functions with respect to \( G_1 \).

Now let us write

\[
g(x) = x + \sum_{n>2} \frac{a_n x^n}{n+1}
\]

and suppose that \( r_0 \) is the radius of convergence of these power series. By the Cauchy–Hadamard formula, \( \frac{1}{r_0} = \lim_{n \to \infty} |a_n|^{1/n} \).

We are looking for general conditions of coefficients \( a_n \) ensuring the isochronicity of the center 0 of equation (1) \( \dot{x} + g(x) = 0 \). Recall that only the harmonic potential \( G(x) = \frac{1}{2} x^2 \) is an even isochronous potential \( (G_1(x) \equiv 0) \). That means the odd coefficients \( a_{2k+1} \) cannot all be zero when the isochronous potential is non-harmonic. In fact, we will prove little more.

Starting from theorem \( B \) and its corollaries we will show that there are infinitely many necessary conditions verified by the coefficients so that the potential \( G(x) \) is isochronous.

Theorem C. Let the analytic potential

\[
G(x) = \frac{1}{2} x^2 + \sum_{n>3} \frac{a_{n-1}}{n} x^n = \frac{1}{2} x^2 + G_1(x) + G_2(x)
\]
from equation (1). When equation (1) has an isochronous center at 0 then the odd coefficients of the expansion of $g(x)$ can be expressed in terms of rational polynomials involving the even coefficients:

$$a_{2k+1} = f(a_{2k}, a_{2k-2}, \ldots, a_2).$$  \tag{34}

In particular, when the potential $G(x)$ is isochronous then $G_2(x) \equiv 0$ is equivalent to $G_1(x) \equiv 0$, i.e. $G(x) = \frac{1}{2}x^2$ is harmonic.

Thanks to Maple we are able to calculate the first terms:

$$a_1 = \frac{10}{9}a_2^2, \quad a_5 = \frac{14}{5}a_2a_4 - \frac{56}{27}a_2^4,$$

$$a_7 = -\frac{592}{45}a_2^4a_3 + \frac{848}{81}a_2^6 + \frac{24}{7}a_2a_6 + \frac{36}{25}a_2^2,$$

$$a_9 = \frac{110}{27}a_2a_8 - \frac{440}{21}a_2^3a_6 + \frac{27808}{243}a_2^5a_4 - \frac{536800}{6561}a_2^7a_4a_6 + \frac{52}{15}a_4a_8 + \frac{78}{49}a_6^2,$$

$$a_{11} = \frac{52}{17}a_2^3a_9 + \frac{57616}{5681080}a_2^4a_7 + \frac{2600}{324}a_2^3a_8 + \frac{12508}{567}a_2^5a_6 + \frac{52}{15}a_4a_8 + \frac{78}{49}a_6^2,$$

$$a_{13} = -\frac{72}{27}a_2^3a_6^2 + \frac{2632}{27}a_2^3a_4^2a_8 + \frac{38176}{27}a_2^4a_4^2a_6 + \frac{70}{17}a_2a_1 + \frac{42}{17}a_2a_4 + \frac{10}{3}a_6 + \frac{943064}{1215}a_4^2 + \frac{37576908}{1400}a_4^2a_2 + \frac{166544}{225}a_2^3a_3 + \frac{920}{729}a_2^3a_6^2 + \frac{3190080}{21900}a_2^3a_6 + \frac{19683}{297}a_2^4a_2 + \frac{300944}{729}a_2^4a_2a_8 - \frac{7468100}{6561}a_2^4a_2^2 - \frac{616}{729}a_4^2.$$

**Proof of theorem C.** Let the analytic function

$$g(x) = x + \sum_{k=2}^{\infty} a_kx^k.$$

By theorem A the potential $G(x) = \frac{1}{2}x^2 + \sum_{n=1}^{\infty} a_n x^n$ is isochronous if and only if the following equality holds (8):

$$\frac{d}{dx} \left( \frac{G(x)}{x^{n+1}}, \right) = f(G),$$

where $f$ is an analytic function. Set

$$f(G) = b_0 + b_1G + b_2G^2 + b_3G^3 + \ldots$$

After replacing and equaling the two sides of (8):

$$\left(x + \sum_{k=2}^{\infty} a_kx^k \right)^{-1} = \left(x + \sum_{k=2}^{\infty} \frac{2a_k}{a_{k-2}}x^{k-1} \right) \left(1 + \sum_{k=2}^{\infty} \frac{a_kx^{k-1}}{x} \right).$$

Then we identify the analytic expansions of the two expressions. The unknown coefficients will then be determined by comparing powers in $x$:

$$b_0 = -\frac{2}{3}a_2, \quad a_3 = \frac{10a_2}{9}, \quad b_1 = 11a_2a_3 - 6a_2^2 - \frac{24}{5}a_4$$

$$- (10/3)a_5 - (1/3)b_1a_2 + \frac{116}{15}a_2a_4 + (14/3)a_4^2 + 4a_3^2 - 13a_2a_2^2 = 0,$$

$$- \frac{30a_6}{7} - \frac{b_2}{4} - \frac{b_1a_3}{4} + 10a_2a_5 - \frac{20a_5^2}{3} + \frac{21a_1a_5}{2} - 17a_4a_2^2 - \frac{35a_2a_2^2}{2} + 25a_3a_2^2 = 0, \ldots$$

From these recursion formulae the coefficients can be easily determined. After eliminating $b_0, b_1, b_2, \ldots$ we then deduce the expressions of coefficients $a_3, a_5, a_7, \ldots$

The last part of theorem C will be proved by recurrence. Suppose that all coefficients $a_n = 0$ for any $n < 2p$ where $p$ is a positive integer and $b_0 = b_1 = b_2 = \ldots = b_{p-1} = 0$. So, we may write

$$g(x) = x + a_{2p}x^{2p} + a_{2p+1}x^{2p+1} + a_{2p+2}x^{2p+2} + \ldots$$

and

$$G(x) = \frac{1}{2}x^2 + \frac{a_{2p}}{2p+1}x^{2p+1} + \frac{a_{2p+1}}{2p+2}x^{2p+2} + \frac{a_{2p+2}}{2p+3}x^{2p+3} + \ldots$$

We calculate

$$\frac{d}{dx} \left( \frac{G}{x^p} \right) = -2p a_{2p}x^{2p-1} - a_{2p+1}x^{2p}(2p+1) + a_{2p+2}x^{2p+2} + \ldots$$

and equaling with

$$f(G) = b_pG^p + b_{p+1}G^{p+1} + \ldots = \frac{b_p}{2p}x^{2p} + \frac{b_{p+1}}{2p+1}x^{2p+2} + \frac{b_{p+2}}{2p+3}x^{2p+4} + \ldots$$

Thus, necessarily $a_{2p} = 0$.

**Remarks 1.** Consider now condition (10) of theorem B

$$x - \frac{2G}{g} = F(G)$$

$$= b_0 + b_1G + b_2G^2 + b_3G^3 + b_4G^4 + \ldots$$

(where $F(G)$ is analytic). It ensures the isochronicity of the potential $G$.

It is also possible to express coefficients $a_n$ of the expansion of $g(x) = \frac{dG(x)}{dx}$ in terms of $b_0, b_1, b_3, \ldots$

Indeed, after replacing the expressions of $g$ and $G$ in

$$xg(x) - 2G(x) = g(x)F(G)$$

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and by identifying
\[ x^2 + \sum_{n \geq 2} a_n x^{n+1} - 2G = \left[ x + \sum_{n \geq 2} a_n x^n \right] \times [b_0 + b_1 G + b_2 G^2 + b_3 G^3 + b_4 G^4 + ...]. \]
we then find
\[ G(x) = \frac{1}{2} x^2 - \frac{1}{2} b_0 x^3 + \frac{5}{8} b_1 x^4 - \left( \frac{1}{24} b_1 + \frac{7}{8} b_0 \right) x^5 + \frac{7 b_0}{270} \left( \frac{405}{8} b_0 + \frac{45}{8} b_1 \right) x^6 + ... \]
It is also possible to express coefficients \( a_n \) of the expansion of \( g(x) = \frac{dx}{dt} \) in terms of \( b_0, b_1, b_2, ... \)
Indeed, \( g'(0) = 1, \ g''(0) = -3b_0, \ g''(0) = 15b_0^2, \ g^{(4)}(0) = -105b_0^3 - 30b_1, \ g^{(5)}(0) = 945b_0^5 + 630b_0 b_1, \ g^{(6)}(0) = -10395b_0^7 - 11340b_0^2 b_1 - 840b_2, \ g^{(7)}(0) = 207900 b_0^3 b_1 + 302406b_0^2 b_2 + 1351356b_0 b_3 + 11340b_3 + ... \) \( g \) may also be written \( g(x) = x - \frac{1}{2} b_0 x^2 + \frac{5}{8} b_1 x^3 - \left( \frac{1}{24} b_1 + \frac{7}{8} b_0 \right) x^4 + \frac{7}{24} b_0 \left( \frac{405}{8} b_0 + \frac{45}{8} b_1 \right) x^5 + \left( \frac{7}{120} b_2 - \frac{21}{8} b_0 b_1 - \frac{231}{16} b_0^2 \right) x^6 + \left( \frac{125}{8} b_0^3 b_1 + \frac{240}{16} b_0^2 b_2 + \frac{3}{16} b_0 b_3 + \frac{4}{16} b_3 \right) x^7 + ... \) and the equation \( x + g(x) = 0 \) has an isochronous center at 0 for parameter values \( b_0, b_1, b_2, ... \)
Thus, any isochronous potential \( G \) is a multi-parameter analytic function which may be expressed under the general form
\[ G(x) = \frac{1}{2} x^2 - \frac{1}{2} b_0 x^3 + \frac{5}{8} b_1 x^4 - \left( \frac{1}{24} b_1 + \frac{7}{8} b_0 \right) x^5 + \frac{7}{270} b_0 \left( \frac{405}{8} b_0 + \frac{45}{8} b_1 \right) x^6 + \frac{7}{120} b_2 - \frac{21}{8} b_0 b_1 - \frac{231}{16} b_0^2 \right) x^7 + ... \] In fact, only the odd part plays a role:
\[ G_1(x) = -\frac{1}{2} b_0 x^3 + \left( \frac{-1}{24} b_1 + \frac{7}{8} b_0 \right) x^5 + ... \] A general expression of \( a_{2p} \) as a polynomial of \( b_0, b_1, b_2, ... \) seems difficult to build. Nevertheless, it is possible to know the first coefficient
\[ a_{2p+1} = \frac{-1}{2^{p-1}} \frac{2p+1}{(2p)(2p-1)} a_{2p-1} + ... \]
and only if
\[ G(x) = \frac{1}{8a^2} \left[ \alpha x + 1 - \frac{1}{\alpha x + 1} \right]^2 x > \frac{-1}{\alpha} \]

The proof we give below is different from that given in [4]. We will use a simple property of involutions introduced by Kuczma [17].

**Proof of proposition 6-1.** Let us recall that for an involution \( A \) we get \( G(A(x)) = G(x) \) and \( A(x) < 0 \) for all \( x \in [a, b] \).
By proposition 3-1 this potential is isochronous if
\[ x - A(x) = 2\sqrt{2G(x)} \text{ for all } 0 < x < b \]
or equivalently it verifies the functional equation
\[ G(x) = G(x - 2\sqrt{2G(x)}) \text{ for all } 0 < x < b. \]
When the potential is rational it is required that \( G(x) \) has to be the square of a rational function of the form
\[ G(x) = \left( \frac{1}{2} x \frac{dP(x)}{Q(x)} \right)^2 \]
where \( P(x) \) and \( Q(x) \) are polynomials without common zeros.

This means the involution has to be meromorphic and have the same poles as \( G(x) \). It can be written
\[ A(x) = x - \left( \frac{2P(x)}{Q(x)} \right) = \frac{xQ(x) - 2P(x)}{Q(x)}. \]

But, it is known (see for example theorem (15.4) p. 296 of [17]) that the only meromorphic functions verifying \( A^2(x) = x \) are \( A(x) = L^{-1} \frac{\alpha x + b}{\alpha x + 1} \) where \( L(x) = ax + b \) is an affine function since other meromorphic functions are not invertible. \( A \) is then a homographic function.
Thus, by hypothesis and since \( A(0) = 0 \) and \( A'(0) = -1 \) we get
\[ A(x) = -\frac{x}{\alpha x + 1} \]
implying
\[ \frac{2P(x)}{Q(x)} = 1 + \frac{1}{\alpha x + 1} = \frac{\alpha x + 2}{\alpha x + 1}. \]

**6. Applications to family of isochronous potentials**

**6.1. Rational potentials**

It is well known that the harmonic potential \( G = \frac{1}{2} x^2 \) is the only polynomial potential which is isochronous. Concerning the rational case, Chalykh and Veselov [4] proved the following:

**Proposition 6-1 ([4]).** Let \( G(x) = \int_0^x g(s) ds \) be an analytic potential. Suppose that for \( x = 0, xg(x) > 0 \), a rational potential \( G(x) \) (which is not a polynomial) is isochronous if
\[ \frac{2P(x)}{Q(x)} = 1 + \frac{1}{\alpha x + 1} = \frac{\alpha x + 2}{\alpha x + 1}. \]

6.2. A three-parameter family of isochronous potentials

We now apply the above considerations to the determination of isochronous potentials whose analytical expression can be given explicitly. We will use theorem A or B and their corollaries.

To be concrete, we first derive a three-parameter family of potentials which appears a little more general than the one given by Dorignac [18].
Let us consider
\[
\frac{2G}{g} - x = F(G) = 2a \frac{(-1 + \sqrt{1 + bG})}{b \sqrt{1 + bG}}.
\]
where \(a\) and \(b\) are real parameters such that \(2a^2 \leq b\).

We have seen that \(F\) and \(P\) are relied on by \(F(G) = 2G \frac{dP(G)}{dG} - P(G)\). We may deduce the analytic function \(P\):
\[
P(G) = -2a + \sqrt{a^4 + 4Gba^2}.
\]
Thus
\[
x = \sqrt{2G} + P(G) = \frac{-2a + \sqrt{2Gb} + \sqrt{a^4 + 4Gba^2}}{b}.
\]
A resolution of these equations yields
\[
G(x) = \frac{8a^2 + (b + 2a^2)(4ax + bx^2) - (4a^2 + 2abx) \sqrt{2(2 + bx^2 + 4ax)}}{2(b - 2a^2)^2}.
\]

Then, the above potential is isochronous according to theorem \(B\). Applying the scaling property of isochronous potentials, the potentials \(G(x)\) and \(\frac{1}{\gamma}G(\gamma x)\) have the same period. That means the following three-parameter potential family is isochronous
\[
G(x) = \frac{1}{2e^2} X^2(\text{cx})
\]
\[
= \frac{[2a + bcx - a \sqrt{2(2 + bc^2x^2 + 4acx)}]^2}{2e^2(b - 2a^2)}.
\] (39)

Many special cases of this three-parameter family will be described below. Moreover, the associated involution is
\[
A(x) = \frac{2e^2 x - 2a - bcx + a \sqrt{2(2 + bc^2x^2 + 4acx)}}{2e^2(b - 2a^2^2)}.
\]

1 - For any \(\gamma \neq 0\) and \(\beta = 0\) we obtain the two-parameter family of isochronous potentials (see (24) of [18]) introduced for the first time by Stillinger and Stillinger [12]. Indeed, after replacing in (21) \(\beta\) by \(2\alpha\) and simplifying by \(2\alpha\) it yields
\[
G(x) = \frac{[1 + \gamma x - \sqrt{1 + \alpha^2 \gamma^2 x^2 + 2\alpha x^2 \gamma x}]^2}{2(1 - \alpha^2)}.
\] (40)

2 - The case \(\alpha = \beta = 0\) and \(\gamma = 1\) yields the harmonic potential: \(G(x) = \frac{1}{2} x^2\).

3 - The case \(\beta = 0\) and \(\gamma = 1\) gives the Urabe potential (see corollary 2-2):
\[
G(x) = \frac{4}{\alpha^2} - \frac{2}{\alpha} \left( x + 2 \sqrt{1 - \frac{\alpha x}{\alpha}} \right).
\]

4 - The case \(2\alpha = \beta\) and \(\gamma = 1\) yields the Bolotin–MacKay potential (see [19] or [20]). Indeed,
\[
\frac{d}{dx} \left( \frac{G}{g^2} \right) = \frac{\alpha}{(1 + 2\alpha x)\sqrt{2}}
\]
implies
\[
G(x) = \frac{1 + x - \sqrt{1 + \alpha^2 x^2 + 2\alpha x}}{2(1 - \alpha^2)}.
\]

We then deduce the function \(h\) of proposition 3-4:
\[
h(X) = \frac{\alpha X}{\sqrt{1 + X^2 \alpha^2}}.
\] (41)

5 - The case \(2\alpha^2 = \beta\) and \(\gamma = 1\) yields the isotonic potential (see [18]). Thanks to Maple a resolution of
\[
\frac{d}{dx} \left( \frac{G}{g^2} \right) = \frac{\alpha}{(1 + \alpha^2 x)\sqrt{2}}
\]
gives
\[
G = \frac{1}{4} \alpha^2 \left(\frac{\alpha}{1 + \alpha x}\right). \frac{1}{\alpha^2}
\]

The function \(h\) of proposition 3-4 is
\[
h(X) = \frac{\alpha X}{\sqrt{1 + X^2 \alpha^2}}
\]
and its integral is
\[
H(X) = \frac{\sqrt{1 + X^2 \alpha^2}}{\alpha} - \frac{1}{\alpha}.
\]

Then
\[
x = \sqrt{2G} + \sqrt{\frac{1 + 2\alpha x^2}{\alpha}} - \frac{1}{\alpha} = \sqrt{2G} + P(G).
\]

We then obtain by another way
\[
G = \frac{1}{4} \alpha^2 \left(\frac{\alpha}{1 + \alpha x}\right)^2 - \frac{1}{4} \alpha^2 \left[\frac{1}{\alpha x + 1} - \frac{1}{\alpha x + 1}\right]2.
\] (42)

6.3. Other isochronous potentials

We will give here one-parameter families of potentials with constant period. Using the scaling property in multiplying \(x\) by a non-zero real \(\gamma\), the potentials \(G(x)\) and \(\frac{1}{\gamma}G(\gamma x)\) have
the same period. To our knowledge, some families of potentials derived in this subsection have never before appeared in the literature. However, we could not highlight families with more than one parameter. It would be interesting to find a family with at least two parameters including all the examples below.

\[
u(x) = \sqrt[4]{1 - 15a^2x^2 + 39a^4x^4 + 1 + a^6x^6 + 6\sqrt{3} - 1 + 11a^2x^2 + a^4x^4a^3x^3}
\]

1 - Let us consider the following example

\[
\frac{2G}{g} - x = F(G) = -\frac{c(6G + 1)}{(1 + 2G)^2}
\]

where \(c \neq 0\) is a parameter. Since we have seen \(F(G) = 2G\frac{\partial P(G)}{\partial G} - P(G)\) it yields

\[
x = \sqrt{2G} + P(G) = \sqrt{2G} - \frac{c}{1 + 2G}
\]

and

\[
\frac{1}{g} = \frac{1}{\sqrt{2G}} + P'(G) = \frac{1}{\sqrt{2G}} + \frac{2c}{(1 + 2G)^2}
\]

Solving the equation \(x = \sqrt{2G} - \frac{c}{1 + 2G}\) and writing

\[
u = \frac{u(x)}{u(x)} = \frac{-10a^2x^2 + 1 + a^4x^4}{6a^2x\sqrt{u(x)}} + \frac{1 + a^2x^2}{6a^2x}\]

then we find the isochronous potential

\[
P(G) = 2\frac{aG}{\sqrt{1 + 2a^2G}}.
\]

If we write

\[
u(x) = \frac{1}{6}x^2 - \frac{1}{3}x^2 + \frac{\sqrt{2x}}{6}
\]

its integral which verifies \(x = \sqrt{2G} + P(G)\) has the following form:

\[
\frac{1}{2} u(x) + \frac{-10a^2x^2 + 1 + a^4x^4}{6a^2x\sqrt{u(x)}} + \frac{1 + a^2x^2}{6a^2x}
\]

The involution associated with this potential is

\[
A(x) = \frac{-u(x)}{3a^2x} = \frac{-10a^2x^2 + 1 + a^4x^4}{3a^2x\sqrt{u(x)}} + \frac{-1 + 2a^2x^2}{3a^2x}
\]

2 - Let

\[
\frac{2G}{g} - x = F(G) = -2\frac{-2aG}{(1 + 2a^2G)^{3/2}}
\]

where the parameter \(a \neq 0\). The function \(P'\) defined by

\[
P'(G) = 2\frac{a(1 + a^2G)}{(1 + 2a^2G)^{3/2}}
\]

its involution \(A\) is

\[
A(x) = \frac{x}{3} - 2\sqrt{2}\left\{\frac{1}{3}u(x) - \frac{6 - 2x^2}{3u(x)}\right\}
\]

3 - Let

\[
\frac{2G}{g} - x = F(G) = \frac{a - 2G + 4Ga}{(1 + 2G)^{3/2}}
\]

where the parameter \(a \neq 0\). By \(F(G) = 2G\frac{\partial P(G)}{\partial G} - P(G)\) one gets the function \(P'\)

\[
P'(G) = \frac{2 + 2G - a}{(1 + 2G)^{3/2}}.
\]

Its integral \(P\) is

\[
P(G) = \frac{a + 2G}{\sqrt{1 + 2G}} - a.
\]

Let us denote

\[
v(x) = -6x^6a + 6x^4a^3 - x^6a^2 - a^2 + x^8 - 12a^4 + 6a^3 + 23x^2a^2 - 2x^4a
\]

\[-x^4 + 11x^6 + 15a^2x^2 + 8a^5
\]

\[-32a^3x^2 - 27a^2x^4 + 28ax^4
\]

\[u(x) = 1 - 15x^2 + 39x^4 + 24x^2a - 42x^2a^2 + 1 - 6a + 12a^2 + x^6 - 6ax^4 - 8a^3 + 6\sqrt{3}\sqrt{v(x)x^3}.
\]
Then the associated isochronous potential is
\[
G(x) = \frac{1}{2} \left[ \frac{u(x)}{6x} + \frac{-10x^2 + 1 - 4a + x^4 - 4x^2a + 4a^2}{6u(x)} \right].
\]
Moreover,
\[
A(x) = -\frac{u(x)}{3x} - \frac{-10x^2 + 1 - 4a + x^4 - 4x^2a + 4a^2}{3xu(x)} + \frac{-1 + 2x^2 + 2a}{3x}.
\]

4 - Let us consider
\[
\frac{2G}{g} = x = F(G) = \frac{\alpha}{(1 + 2\beta^2G)^{3/2}}.
\]
Then the function \( P' \) which verifies \( \frac{1}{g} = \frac{1}{2G} + P'(G) \) may be written
\[
P'(G) = \frac{(3 + 4 b^2)a b^2}{(1 + 2 b^2G)^{3/2}}.
\]
Its integral is
\[
P(G) = \frac{\alpha(1 + 2(2G))^{3/2}}{\sqrt{1 + (2G)^{3/2}}} - \alpha.
\]
Taking for example \( \alpha = 12 \) and thanks to Maple we find another isochronous potential.

Let the following functions
\[
v(x) = \sqrt{1024 x^6 + 4608 x^5 a^3 + 26496 x^4 a^6 + 62208 x^3 a^9 + 101412 x^2 a^{12} + 86022 x a^{15} + 459278},
\]
\[
u(x) = -5751 + 31536 \alpha + 21600 x^2 a^2 + 768 x^3 a^3 + 256 x^4 a^4 + 72 \sqrt{3} v(x).
\]
Solving \( x = \sqrt{2G} + P(G) \) and using the scaling property the isochronous potential is
\[
G(x, \gamma) = \left( \frac{1}{2} \right) \left[ \frac{u(\gamma x)}{24 \alpha(3 + 4 \gamma \alpha x)} + \frac{(16 \gamma^2 x^2 a^2 + 24 \gamma \alpha x^3 - 423)(3 + 4 \gamma \alpha x)}{24 \alpha [u(\gamma x)]^{1/3}} \right]
\[
+ \frac{3 + 4 \gamma \alpha x}{24 \alpha} \right]^{1/3}.
\]
Its involution is (for \( \gamma = 1 \))
\[
A(x) = -\frac{\left[ u(x) \right]^{1/3}}{12 \alpha(3 + 4 \alpha x)} + \frac{(16x^2 \alpha^2 + 24 x \alpha - 423)(3 + 4 \alpha x)}{12 \alpha [u(x)]^{1/3}}
\[
+ \frac{8 \alpha x - 3}{12 \alpha}.
\]

7. Discussion and conclusion

In this paper we were first interested in the different isochronous conditions of the analytical potentials \( G(x) \). It turns out that the condition (6) of theorem B
\[
x = \frac{2G}{g} = F(G), \quad F \text{ analytic}
\]
seems the finest since it allows us (among other properties) to produce new examples of isochronous oscillatory motions.

Moreover, proposition 3-8 makes it possible to characterize in a simple way all the derivatives of an isochronous potential (23) as well as their product (26) where \( \nu = (\nu_1, \nu_2, \ldots, \nu_m) \) and \( u_{m,\nu} \) and \( u_{m,\nu} \) are analytic functions with respect to \( G \) (see corollary 3-10).

This last point will be useful to study the WKB series and their convergence in case the sum is identical to the spectrum. The results of section 3 permit us to express the nth correction
\[
I_{2n}(E) = \left( \frac{\hbar}{2\pi} \right)^{2n} \int \gamma d\gamma_{2k}, k \in N
\]
in terms related to the isochronous potential \( G(x) \) and its derivative products. It may be expressed through Abel integrals (see (21))
\[
I_{2n}(E) = -\hbar^{2n} \sum_{L(\nu) = 2n} \frac{2^{v \nu + 1} U_L E^{-n + 1 - |\nu|}}{\pi (2n - 3 + 2 |\nu|)!!}
\times \int_{0}^{\sqrt{E - \nu}} \frac{v^{n - 1 + |\nu|}}{\partial E^{n - 1 + |\nu|}} u_{m,\nu}(v) dv
\]
where \( u_{m,\nu}(G) \) is such that
\[
G^{(\nu)}(x) = \sum_{j=1}^{m} \left( \frac{d G^{(\nu)}}{d x} \right)^{j} = u_{m,\nu}(G) \sqrt{2G} + v_{m,\nu}(G).
\]
As is already known, the spectrum of an isochronous potential is generally not strictly regularly spaced, in contrast to the harmonic one. In [18] the author asserts (Claim 9): The most general family of analytic isochronous potentials, such that all the terms of the WKB series, \( I_{2n}(E) \), \( n \geq 1 \), defined above, are constant (energy-independent), is the family of the isotonic oscillator with potential (35). Here a calculation yields
\[
\frac{d G^{(\nu)}}{d x^j} = \frac{1}{(\nu + 1)!} \left( \frac{d G}{dx} \right)^{j}, j \geq 3.
\]
\[
\text{Thus} \quad \Gamma^{2n}_{j=1} \left( \frac{d G^{(\nu)}}{d x} \right)^{j} = \frac{k_\nu}{(\alpha x + 1)^{\nu}},
\]
which equals \( u_{\nu}(G) \sqrt{2G} + v_{\nu}(G) \) where \( \sqrt{2G} = \frac{1}{2\hbar^2} \left[ (\alpha x + 1 - \frac{1}{\alpha x + 1}) \right] \). The interesting question is whether this family is the only one (among other isochronous potentials) to be both classical and quantum harmonic. Moreover, although the condition (31) is quite a natural way to ensure that the spectrum is strictly equally spaced, the function \( \Sigma_{\nu} I_{2n}(E) \) could be energy-dependent and the quantization condition
It turns out that an exact expression may be available. We shall justify the expressions of
\[
\frac{g'^2}{g} = a_{1,2}(G) \sqrt{2G} + b_{1,2}(G), \quad g(x)g'(x) = c_{1,2}(G) \sqrt{2G} + d_{1,2}(G)
\]
where \( a_{1,2}(G), b_{1,2}(G), c_{1,2}(G) \) and \( c_{1,2}(G) \) are analytic functions. We have seen that the isochronicity conditions are
\[
x = \sqrt{2G} + P(G) \quad \text{and} \quad \frac{1}{\sqrt{2G}} + P'(G) = \frac{1}{g}.
\]
It implies
\[
g' = \frac{dg}{dx} = 2 \frac{G(\sqrt{2} \sqrt{G} + \phi(G) - 2 Gf(G))}{(\sqrt{2} \sqrt{G} + \phi(G))^3}
\]
where \( \phi = 2Gp'(G) \) and \( f = P'(G) + 2GP''(G) \). Let \( A \) be the involution defined above by \( G(A(x)) = G(x) \). Recall that \( \sqrt{2G(A(x))} = -\sqrt{2G(x)} \). Thus
\[
gg'(x) - gg'(A(x)) = 4 \frac{G^2 (-\sqrt{2} \sqrt{G} + \phi(G) - 2 Gf(G))}{(\sqrt{2} \sqrt{G} + \phi(G))^4} \]
\[
- 4 \frac{G^2 (\sqrt{2} \sqrt{G} + \phi(G) - 2 Gf(G))}{(\sqrt{2} \sqrt{G} + \phi(G))^2}.
\]
After simplification one has
\[
\frac{gg'(x) - gg'(A(x))}{2G} = 8 \frac{G^5 \sqrt{2} (-4 G^2 - 4 G(\phi(G))^2 + 3(\phi(G))^2 - 16 G \phi(G)(\phi(G))^2 - 8 (\phi(G))^3 Gf(G))}{(\sqrt{2} \sqrt{G} - \phi(G))^2(\sqrt{2} \sqrt{G} + \phi(G))^2} = 2c_{1,2}(G) \sqrt{2G}.
\]

By the same method one obtains
\[
\frac{gg'(x) + gg'(A(x))}{2G} = 8G^2(-12 G^2 \phi(G) + 4 G(\phi(G))^3 + (\phi(G))^5 - 8 G^3 \phi(G) - 24 G^2 \phi(G)(\phi(G))^2 - 2 Gf(G)(\phi(G))^4)}{(\sqrt{2} \sqrt{G} - \phi(G))^2(\sqrt{2} \sqrt{G} + \phi(G))^4} = 2d_{1,2}(G) = \frac{64G^5 (-3P' + 4GP^3 + 4G^3 P5 - G' - 12GP'P^2 - 4G^2 P'4 - G^3 P'P''}{(\sqrt{2} \sqrt{G} - \phi(G))^2(\sqrt{2} \sqrt{G} + \phi(G))^4}.
\]

The fourth WKB correction \( I_4 \) is
\[
I_4(E) = -\frac{\hbar^2}{4\sqrt{2} \pi} \left[ \frac{1}{120} \frac{\partial^3}{\partial E^3} \int_a^b g'^2(x) \frac{g(x)}{\sqrt{E - G(x)}} g(x)dx \right.
\]
\[
- \left. \frac{1}{288} \frac{\partial^4}{\partial E^4} \int_a^b \frac{g(x)g'(x)}{\sqrt{E - G(x)}} g(x)dx \right].
\]
\[
\frac{g'^2}{g} = 2a_{1,2}(G) \sqrt{2G}
\]
\[
= -128 \frac{G^{9/2} \sqrt{2} (-P' - 2GP'^3 + GP'' + 12G^2 P''^2 P'^2 + 4G^3 P''^3 P'^4 - 8G^3 P'P''^2 - 16G^3 P''^2 P'^3)}{(\sqrt{2} \sqrt{G} - 2GP')^4(\sqrt{2} \sqrt{G} + 2GP')^4}.
\]
In the same way, one obtains
\[ \frac{g'^2}{g} + \frac{g'^2}{\sin(\sqrt{2G})} = 2b_{1,2}(G)(\sqrt{2G} - 2GP')^{-5} \]
\[ \times (\sqrt{2G} + 2GP')^{-5} \]
\[ = 64G^3(-4GP^n + 4G^2P^5) \]
\[ + 20GP^{n+3} + 5P' \]
\[ - 80G^2P^nP'^2 - 80G^3P^4P'' + 160G^4P^n3P'^2 \]
\[ + 40G^3P'^2P'' + 32G^4P^n2P'^5) \]

Notice that the denominator is a power of \(\sqrt{2G} - 2GP'\), which is equal to \(2G - 4G^2P^2 = 2G(1 - 2GP^2)\). We may simplify by \(G\) and \(I - 2GP^2(G)\) is always non-zero. This means that all the functions \(a_{1,2}(G)\), \(b_{1,2}(G)\), \(c_{1,2}(G)\) and \(d_{1,2}(G)\) are analytic.

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**References**

1. Calogero F 2008 *Isochronous Systems* (Oxford: Oxford University Press)
2. Chavarriga J and Grau M 2003 Some open problems related to 16th Hilbert problem *Sci. Ser. A Math. Sci. (N. S.)* 9 1–26
3. Chavarriga J and Sabatini M 1999 A survey of isochronous centers *Qual. Theo. Dyn. Syst.* 1 1
4. Chalykh O and Veselov A 2005 A remark on rational isochronous potentials *J. Nonlinear Math. Phys.* 12-1 179–83
5. Eleonskii V M, Korolev V G and Kulagin N E 1997 On a classical analog of the isospectral Schrodinger problem *JETP Lett.* 65 889
6. Robnik M and Salasnich L 1997 WKB to all orders and the accuracy of the semiclassical quantization *J. Phys. A: Math. Gen.* 30 1711
7. Carinena J F, Perelomov A M and Ranada M F 2007 Isochronous classical systems and quantum systems with equally spaced spectra *J. Phys. 87* 012007
8. Chouikha A R 2005 Monotonicity of the period function for some planar differential systems, I. Conservative and quadratic systems *Appl. Math. 32* 305–25
9. Rothe F 1993 Remarks on periods of planar Hamiltonian systems *SIAM J. Math. Anal.* 24 129–54
10. Chouikha R 2011 Period function and characterizations of isochronous potentials arXiv:1109.4611
11. Chow S N and Wang D 1986 On the monotonicity of the period function of some second order equations *Casopis Pest. Mat.* 111 14–25
12. Osypowski E T and Olsson M G 1987 Isochronous motion in classical mechanic *Amer. J. Phys.* 55 720–5
13. Urabe M 1962 The potential force yielding a periodic motion whose period is an arbitrary continuous function of the amplitude of the velocity *Arch. Ration. Mech. Anal.* 11 27–33
14. Landau L D and Lifschitz E M 1960 *Mechanics, Course of Theoretical Physics* vol 1 (Oxford: Pergamon)
15. Kóukles I and Piskounov N 1937 Sur les vibrations tautochrones dans les systèmes conservatifs et non conservatifs *C. R. Acad. Sci., URSS* 17 417–75
16. Robnik M and Romanovski V G 2000 Some properties of WKB series *J. Phys. A: Math. Gen.* 33 5093
17. Kuczma M 1968 Functional equations in a single variable *Monografie Matematyczne* (Warsaw: Tom 46)
18. Dorignac J 2005 On the quantum spectrum of isochronous potentials *J. Phys. A: Math. Gen.* 38 6183–210
19. Bolotin S and MacKay R S 2003 Isochronous potentials *Proc. 3rd Conf. on Localization and Energy Transfer in Nonlinear Systems* pp 217–24
20. Stillerger F H and Stillinger D K 1989 Pseudoharmonic oscillators and inadequacy of semiclassical quantization *J. Phys. Chim.* 93 6890
21. Erdélyi A, Magnus W, Oberhettinger F and Tricomi F G 1954 *Tables of Integral Transforms* vol II (New York: McGraw-Hill)