Mordell-Lang in positive characteristic

Paul Ziegler∗

October 11, 2018

Abstract

We give a new proof of the Mordell-Lang conjecture in positive characteristic for finitely generated subgroups. We also make some progress towards the full Mordell-Lang conjecture in positive characteristic.

1 Introduction

This article gives an algebro-geometric proof of the following result:

Theorem 1.1 (Hrushovski, [18]). Let $L$ be an algebraically closed field of characteristic $p > 0$. Let $A$ be a semiabelian variety over $L$, let $X \subset A$ be an irreducible subvariety and let $\Gamma \subset A(L)$ be a finitely generated subgroup. If $X(L) \cap \Gamma$ is Zariski dense in $X$, then there exist a semiabelian variety $B$ over $\overline{\mathbb{F}}_p$, a subvariety $Y$ of $B$ over $\overline{\mathbb{F}}_p$, a homomorphism $h: B_L \to A/\text{Stab}_A(X)$ with finite kernel and an element $a \in (A/\text{Stab}_A(X))(L)$ such that $X/\text{Stab}_A(X) = h(Y) + a$.

Here $\text{Stab}_A(X)$ denotes the translation stabilizer of $X$ in $A$. We call irreducible subvarieties $X$ satisfying the conclusion of the above theorem special.

An algebro-geometric proof of Theorem 1.1 has previously been given by Rössler [24]. We give an overview of the structure of this article. In Section 2 we collect a number of facts about a class of formal schemes which includes the closed formal subschemes of the completion of a semiabelian scheme over a discrete valuation ring along its zero section. In Section 3 we collect some facts about special subvarieties and give a criterion for special subvarieties (Theorem 3.10) based on the classification of subvarieties of semiabelian subvarieties which are invariant under an isogeny due to Pink and Rössler.

In Section 4 we set up our method for proving Theorem 1.1. In Subsection 4.1 we collect some facts about certain $p$-divisible groups. These are $p$-divisible groups which possess a filtration such that on each graded piece, a power of the relative Frobenius coincides, up to an isomorphism, with a power of the multiplication-by-$p$ morphism.

In Subsection 4.2 we construct our central tool, a certain Frobenius morphism $F$: Let $R$ be the valuation ring of a local field $K$ of positive characteristic, let $k$ be the residue field of $R$, let $\overline{K}$ be an algebraic closure of $K$ and let $R^{\text{per}} \subset \overline{K}$ be the perfection of $R$. Let $\mathcal{A}$ be a semiabelian scheme over $R$. Denote by $\widehat{\mathcal{A}}$ the completion of $\mathcal{A}$ along the zero section of the special fiber. This is a $p$-divisible group over $\text{Spf}(R)$. Assume that $\widehat{\mathcal{A}}$ is completely slope divisible. Then the facts from Subsection 4.1 yield a canonical isomorphism $(\widehat{\mathcal{A}}_k)_{\text{Spf}(R^{\text{per}})} \cong \widehat{\mathcal{A}}_{\text{Spf}(R^{\text{per}})}$. By transferring the Frobenius endomorphism of $\widehat{\mathcal{A}}_k$ with respect to $k$ via this isomorphism, we obtain an endomorphism $F$ of $\widehat{\mathcal{A}}_{R^{\text{per}}}$. Its significance lies in the following characterization of special subvarieties of $\widehat{\mathcal{A}}_k$, where for

∗Dept. of Mathematics, ETH Zürich CH-8092 Zürich, Switzerland, paul.ziegler@math.ethz.ch
Supported by the Swiss National Science Foundation
a subvariety $X$ of $A$ containing the zero section we denote by $\hat{X}$ its completion along the zero section:

**Theorem 1.2** (see Theorem 4.18). Let $X \subset A$ be an irreducible subvariety. Then the following are equivalent:

(i) The subvariety $X_K$ is special in $A_K$.

(ii) There exist $x \in X(K)$ and $n \geq 1$ such that $F^n_K(X-x) \subset \hat{X}-x$.

This can be considered as a formal analogue of the classification of subvarieties of semiabelian subvarieties which is invariant under an isogeny due to Pink and Rössler. Finally, in Subsection 4.3 we show that given a finitely generated field $L_0$ and a semiabelian variety $A$ over $L_0$, up to isogeny one can always spread out $A$ to an abelian scheme $\mathcal{A}$ as above.

In Section 5 using the methods from Section 4 we prove Theorem 1.1. First we reduce to the situation considered above which allows us to define $F$. Then using Theorem 1.2 we show that Theorem 1.1 follows from a variant of the following formal Mordell-Lang result:

**Theorem 1.3** (see Theorem 5.1). Let $\mathcal{G}$ be a formal group over $k$ which as a formal scheme is isomorphic to $\text{Spf}(k[[x_1, \ldots, x_n]])$. Let $\Gamma \subset \mathcal{G}(R)$ a finitely generated subgroup. Let $\mathcal{X} \subset \mathcal{G}_R$ be a closed formal subscheme. If $\mathcal{X}$ is the minimal closed formal subscheme containing $\mathcal{X}(R) \cap \Gamma$, then there exist closed formal subschemes $\mathcal{X}_1, \ldots, \mathcal{X}_m$ of $\mathcal{G}_R$ defined over a finite field extension of $k$ and elements $\gamma_1, \ldots, \gamma_m \in \mathcal{G}(R)$ such that $\mathcal{X} = \bigcup_j \mathcal{X}_j + \gamma_j$.

Theorem 1.3 is proven by the same method which was used by Abramovich and Voloch in [2] to prove Theorem 1.1 in the isotrivial case: First one reduces to the case that $\mathcal{X}$ is irreducible in a suitable sense. Then after a suitable translation one may assume that $\mathcal{X}(R) \cap p^i \Gamma$ is dense is $\mathcal{X}$ for all $i$. Denote by $R(p)$ the subring of $R$ consisting of all $p^i$-th powers. Since $p^i \Gamma \subset \mathcal{G}(R(p))$ it follows that $\mathcal{X}$ is defined over $R(p)$ for all $i \geq 0$. Hence $\mathcal{X}$ is defined over $k = \bigcap_{i \geq 0} R(p)$.

In Section 5 we consider the full Mordell-Lang conjecture, which is the statement obtained by allowing the group $\Gamma$ in Theorem 1.1 to be of finite rank. This conjecture is still open in general. We show that in case $A$ is ordinary by combining our method with a reduction due to Ghioca, Moosa and Scanlon, Conjecture 6.1 can be reduced to the following special case:

**Conjecture 1.4.** Let $L_0$ be a field which is finitely generated over $\mathbb{F}_p$, let $L$ an algebraic closure of $L_0$ and let $L_0^\text{per}$ be the perfect closure of $L_0$ in $L$. Let $A$ be a semiabelian variety over $L_0$ and $X \subset A_{L_0^\text{per}}$ an irreducible subvariety. Assume that the canonical morphism $\text{Tr}_{L/F_p}A \to A$ is defined over $L_0$, that there exists a finite subfield $\mathbb{F}_q$ of $L_0$ over which $\text{Tr}_{L/F_p}A$ can be defined and that $\text{Stab}_{A_{L_0^\text{per}}}(X)$ is finite. If $X(L_0^\text{per})$ is Zariski dense in $X_0$, then a translate of $X$ by an element of $A(L_0^\text{per})$ is defined over $L_0$.

This depends crucially on the fact that in case $A$ is ordinary, the endomorphism $F$ of $\hat{A}_{\text{Spf}(R^\text{per})}$ described above can already be defined over $R$. The argument proceeds similarly to the proof of Theorem 1.1 sketched above by reduction to an analogous statement (see Theorem 2.41) for formal group schemes.

**Acknowledgement** I am deeply grateful to Richard Pink for suggesting this topic to me and for his guidance. I thank Damian Rössler and Thomas Scanlon for pointing out a mistake in an earlier version of this article and for helpful conversations. I also thank Ambrus Pál for a helpful conversation.
2 Formal Schemes

Let \( R \) be the valuation ring of a local field \( K \) of characteristic \( p > 0 \). Denote by \( \mathfrak{m} \) the maximal ideal of \( R \) and by \( k \) the residue field of \( R \). Let \( \hat{K} \) be an algebraic closure, let \( \hat{R} \) be the valuation ring of the unique extension of the valuation of \( R \) to \( \hat{K} \) and let \( \hat{\mathfrak{m}} \) be the maximal ideal of \( \hat{R} \).

We denote by \( \hat{K} \) the completion of \( \hat{K} \). By a complete overfield \( K' \subset \hat{K} \) of \( K \) we mean a field which is complete with respect to the valuation induced from \( \hat{K} \). The valuation ring of such a \( K' \) will be denoted \( R' \) and the formal scheme associated to \( R' \) with its equipped with the valuation topology will be denoted by \( \text{Spf}(R') \).

By an adic ring we mean the same as in [15], that is a complete and separated topological ring whose topology is defined by an ideal \( J \) such that for each \( I \) the coefficients \( a_{I,J} \) converge to zero as \( J \) goes to infinity. We endow \( C'_{n,m} \) with the topology defined by the ideal \( J'_{n,m} \) generated by \( m \) and the variables \( x_1, \ldots, x_n \). This makes \( C'_{n,m} \) into an adic ring. For \( R' = R \) we let \( C_{n,m} := C'_{n,m} \) and \( J_{n,m} := J'_{n,m} \).

By formal schemes, we mean the same as in [15, Section 10]. In this section, we are concerned with affine formal schemes \( \mathcal{X} \) over \( \text{Spf}(R) \) defined by the following class of rings:

**Definition 2.1** (c.f. [19 Section 2.1] and [4 Section 1]). A topological \( R \)-algebra \( C \) is of formally finite type if it is adic and if for some ideal of definition \( J \) the quotients \( C/J^i \) are of finite type over \( R \) for all \( i \geq 0 \).

**Definition 2.2.** We denote by \( \text{AFS}_R \) the full subcategory of of the category of formal schemes over \( \text{Spf}(R) \) whose objects are the formal schemes of the form \( \text{Spf}(C) \) for \( C \) a topological \( R \)-algebra of formally finite type.

**Lemma 2.3** ([4 Lemma 1.2]). For a \( J \)-adic \( R \)-algebra \( C \) the following are equivalent:

(i) The ring \( C \) is of formally finite type.
(ii) The ring \( C/J^2 \) is finitely generated over \( R \).
(iii) The ring \( C \) is topologically isomorphic over \( R \) to a quotient of \( C_{n,m} \) for some \( n,m \geq 0 \).

**Remark 2.4.** Let \( \mathcal{X} = \text{Spf}(C) \in \text{AFS}_R \). By the remark after [15, Definition 10.14.2] closed formal subschemes of \( \mathcal{X} \) correspond to ideals of \( C \). Thus by Lemma 2.3 a formal scheme over \( \text{Spf}(R) \) is in \( \text{AFS}_R \) if and only if it admits a closed embedding into \( \text{Spf}(C_{n,m}) \) for some \( n,m \geq 0 \).

The following summarizes properties of topological \( R \)-algebras of formally finite type:

**Proposition 2.5.** Let \( C \) and \( C' \) be topological \( R \)-algebras of formally finite type.

(i) The Jacobson radical of \( C \) is an ideal of definition, in fact it is the largest ideal of definition. In particular there is a unique topology on the ring \( C \) which makes \( C \) into a topological \( R \)-algebra of formally finite type.
(ii) Every homomorphism \( C \to C' \) is continuous.
(iii) If \( C \to C' \) is a surjection and \( I \) an ideal of definition of \( C \), then the ideal generated by the image of \( I \) is an ideal of definition of \( C' \).
(iv) Each ideal of \( C \) is closed.
(v) The ring \( C \otimes_R K \) is Jacobson.
(vi) For each maximal ideal \( n \) of \( C \otimes_R K \) the quotient \( (C \otimes_R K)/n \) is a finite field extension of \( K \).

**Proof.** For (i) and (ii) see [19, Lemma 2.1]. For (iii) and (iv) see [4, Lemma 1.1]. For (v) see [19, Proposition 2.16] and for (vi) see [19, Lemma 2.3]. \( \square \)

We will also have to work with formal schemes \( \mathcal{X}_{Spf(R')} \) for \( \mathcal{X} \in AFS_R \) and \( R' \) the valuation ring of a complete overfield \( K' \subset \hat{K} \) and with closed formal subschemes of such formal schemes. However, in [15] the notion of a formal subscheme is only defined for locally Noetherian formal schemes, and valuation rings \( R' \) as above are in general not Noetherian. Thus we make the following definition:

**Definition 2.6.** A morphism \( Spf(C) \to Spf(C') \) of affine formal schemes is a **closed embedding** if the corresponding homomorphism \( C' \to C \) is surjective and the topology on \( C \) is the quotient topology induced from \( C' \). In this case we will say that \( Spf(C) \) is a **closed formal subscheme** of \( Spf(C') \).

Thus closed formal subschemes of \( Spf(C') \) correspond to closed ideals of \( C' \). In case \( C' \) is Noetherian, this definition coincides with the one from [15] by the remark after [15, Definition 10.14.2].

**Definition 2.7.** Let \( R' \) be the valuation ring of a complete overfield \( K' \subset \hat{K} \). We denote by \( AFS_{R'} \) the full subcategory of the category of formal schemes over \( Spf(R') \) whose objects are those formal schemes which admit a closed embedding into \( Spf(C'_{n,m}) \) for some \( n, m \geq 0 \).

**Lemma 2.8.** Let \( C \) be a Noetherian \( J \)-adic ring and \( C' \) a \( J' \)-adic ring. Let \( C \to C' \) be a ring homomorphism such that \( JC' \subset J' \) and such that for each \( i \geq 0 \) the induced homomorphism \( C/J^i \to C'/J'^i \) is faithfully flat.

Let \( 0 \to M' \to M \to M'' \to 0 \) a sequence of finitely generated \( C \)-modules. We endow these modules with the \( J \)-adic topology. Then the sequence \( 0 \to M' \otimes_C C' \to M' \otimes_C C' \to M'' \otimes_C C' \to 0 \) is exact if and only if \( 0 \to M' \otimes_C C' \to M' \otimes_C C' \to M'' \otimes_C C' \to 0 \) is exact.

**Proof.** Since \( JC' \subset J' \) the completed tensor product \( M' \otimes_C C' \) can be written as \( M' \otimes C C'/ (J')^i \), and analogously for \( M' \) and \( M'' \).

Assume that \( 0 \to M' \to M \to M'' \to 0 \) is exact.

For \( i \geq 0 \) let \( M'_i := J^i M \cap M' \). By [4, Theorem III.3.2.2] the topology on \( M' \) defined by the \( M'_i \) is the \( J \)-adic topology. This together with the fact that \( JC' \subset J' \) implies that \( (M'_i \otimes_C (J')^i)_{i \geq 0} \) is a fundamental system of neighborhoods of the identity in \( M' \otimes_C C' \). Thus \( M' \otimes_C C' \) can be written as \( M' \otimes C C'/ (J')^i \).

For \( i \geq 0 \) there is an exact sequence \( 0 \to M'/M'_i \to M/J^i M \to M''/J^i M'' \to 0 \) of \( C/J \)-modules. Since \( C/J \to C'/ (J')^i \) is flat, this induces an exact sequence

\[
0 \to M'/M'_i \otimes_{C/J} C'/ (J')^i \to M/J^i M \otimes_{C/J} C'/ (J')^i \to M''/J^i M'' \otimes_{C/J} C'/ (J')^i \to 0.
\]

This sequence can also be written as

\[
0 \to M'/M'_i \otimes_C C'/ (J')^i \to M \otimes_C C'/ (J')^i \to M'' \otimes_C C'/ (J')^i \to 0.
\]
The transition morphisms $M'/M'_i \otimes_C C'/(J')^i \to M'/M'_{i-1} \otimes_C C'/(J')^{i-1}$ are surjective. By [3 Proposition 10.2] this surjectivity implies that the sequence $0 \to M' \otimes_C C' \to M' \otimes_C C' \to M' \otimes_C C' \to 0$ obtained by taking the inverse limit of the above exact sequences is again exact.

To prove the other direction of the claim, by a direct verification it suffices to show that if $M$ is a non-zero finitely generated $C$-module endowed with the $J$-adic topology, the ring $M \otimes_C C'$ is non-zero. As above we can write $M \otimes_C C'$ as $\lim_i M/J^i M \otimes_C C'/(J')^i$ with surjective transition homomorphisms $M/J^i M \otimes_C C'/(J')^i \to M/J^{i-1} M \otimes_C C'/(J')^{i-1}$. As it is finitely generated over the complete Noetherian ring $C$, the module $M$ is complete. Hence the modules $M/J^i M$ are non-zero. Thus by the faithful flatness of $C/J^i \to C'/(J')^i$ the modules $M/J^i M \otimes_C C'/(J')^i$ are non-zero. Hence the surjectivity of the transition morphisms implies that $M \otimes_C C'$ is non-zero. 

**Lemma 2.9.** Let $R'$ be the valuation ring of a completely valued overfield $K' \subset \hat{K}$ of $K$. For $n, m, i \geq 0$ the ring homomorphism $C_{n,m}/J_{n,m} \to C'_{n,m}/(J'_{n,m})^i$ induced by the inclusion $C_{n,m} \hookrightarrow C'_{n,m}$ is faithfully flat.

**Proof.** The homomorphism in question is

$$R/m^i[x_1, \ldots, x_n, y_1, \ldots, y_n]/(x_1, \ldots, x_n)^i \to R'/m'R'/(mR')^i[x_1, \ldots, x_n, y_1, \ldots, y_n]/(x_1, \ldots, x_n)^i \cong R' \otimes_R R/m^i[x_1, \ldots, x_n, y_1, \ldots, y_n]/(x_1, \ldots, x_n)^i.$$ 

Thus the claim follows from the faithful flatness of $R \to R'$. 

**Lemma 2.10.** Let $\mathcal{X}' = \text{Spf}(C) \in AFS_R$ and $\mathcal{X}' = \text{Spf}(C')$ a closed formal subscheme of $\mathcal{X}$ defined by an ideal $I$ of $C$. Let $R'$ be the valuation ring of a complete overfield $K' \subset \hat{K}$ of $K$. Then $\mathcal{X}'_{\text{Spf}(R')}$ is the closed formal subscheme of $\mathcal{X}_{\text{Spf}(R')}$ defined by the ideal $I(C \otimes_R R')$ of $C \otimes_R R'$. This ideal is equal to $I \otimes_R R'$.

**Proof.** Pick a surjection $C_{n,m} \to C$ for some $n, m > 0$. Note that $M \otimes_R R' = M \otimes_{C_{n,m}} C'_{n,m}$ for any topological $C'$-module $M$. By Proposition 2.5 the topology on $C'$ is the same as the topology defined by $J_{n,m}$. Hence by Lemma 2.8 applied to $C_{n,m} \to C_{n,m}$, which is possible by Lemma 2.9 there is an exact sequence $0 \to I \otimes_R R' \to C \otimes_R R' \to (C/I) \otimes_R R' \to 0$. Thus $\mathcal{X}'$ is the closed formal subscheme of $\mathcal{X}_{\text{Spf}(R')}$ defined by the ideal $I \otimes_R R'$ in $C \otimes_R R'$. Since $C$ is Noetherian, there is a surjective homomorphism of $C$-modules $C^{\otimes k} \to I$ for some $k \geq 0$. Again by Lemma 2.8 this induces a surjection $(C \otimes_R R')^{\otimes k} \cong C^{\otimes k} \otimes_R R' \to I \otimes_R R'$ which implies $I \otimes_R R' = I(C \otimes_R R')$. 

**Definition 2.11.** Let $\mathcal{X} = \text{Spf}(C)$ be an affine formal scheme and $\mathcal{X}_1, \ldots, \mathcal{X}_m$ be closed formal subschemes of $\mathcal{X}$ defined by closed ideals $I_1, \ldots, I_m$ of $C$. We say that $\mathcal{X}$ is the union of the formal subschemes $\mathcal{X}_i$ if the intersection of the ideals $I_i$ is the zero ideal of $C$.

**2.1 Points over $\hat{R}$**

**Definition 2.12.** Let $\mathcal{X} = \text{Spf}(C) \in AFS_R$. We define $\mathcal{X}(\hat{R})$ to be the set of homomorphisms $C \to \hat{R}$ of $R$-algebras.

**Lemma 2.13.** Let $C$ be a topological $R$-algebra of formally finite type and let $h : C \to \hat{R}$ be a homomorphism of $R$-algebras.

(i) The homomorphism $h$ factors through the valuation ring $R'$ of a finite field extension $K' \subset \hat{K}$ of $K$.

(ii) The homomorphism $h$ is continuous.
Definition 2.16. Let $\Gamma := \text{Aut}_{R}[R \otimes_{R} K \to \bar{K}]$ which associates to $h$ from the left by a bijection identifies $X$. This implies (i).

(ii): By (i) it suffices to show that if $R'$ is the valuation ring of a finite field extension $K' \subset \bar{K}$ of $K$ any homomorphism $h : C \to R'$ of $R$-algebras is continuous. Since $R'$ is of formally finite type this is a special case of Proposition 2.5 (ii).

Proof. (i): The homomorphism $h$ induces a homomorphism $C \otimes_{R} K \to \bar{R} \otimes_{R} K \to \bar{K}$ with the last homomorphism given by multiplication. Its kernel is a maximal ideal $n$ of $C \otimes_{R} K$. By Proposition 2.5 the quotient $(C \otimes_{R} K)/n$ is a finite field extension of $K$. This implies (i).

Remark 2.15. If $\pi_{i} : R \to K_{i}$ for a finite field extension $K_{i} \subset \bar{K}$ any homomorphism $h : C \to K_{i}$ of $R$-algebras exists uniquely. Thus associating to an element $(C_{n,m} \otimes \bar{K})$ the set of homomorphisms $\text{Spf}(C_{n,m})$ with respect to their maximal ideals. Then $\Gamma \cong \text{Spf}(R[[x_{1}, \ldots, x_{g}]]$ so that $X \in AFS_{R}$ and $X(\bar{R})$ is naturally identified with the set points in $X(\bar{R})$ which map to 0 in the special fiber (c.f. Subsection 2.6).

Note that the formation of $X(\bar{R})$ is functorial in $X$.

For a finite field extension $K' \subset \bar{K}$ of $K$ with valuation ring $R'$ and $X \in AFS_{R}$ we denote by $X(\bar{R})$ the set of homomorphisms $X \to \text{Spf}(R')$ over $\text{Spf}(R)$. There is a natural inclusion $X(\bar{R}) \to X(\bar{R})$ and Lemma 2.13 (i) implies:

Lemma 2.14. Let $X \in AFS_{R}$. The set $X(\bar{R})$ is the union $X(\bar{R}) = \cup_{K'} X(\bar{R}')$ where $K'$ varies over all finite field extensions of $K$ contained in $\bar{K}$.

Definition 2.15. Let $X \in AFS_{R}$. The set $X(\bar{R})$ can be described more concretely as follows: For any $r_{1}, \ldots, r_{n} \in \bar{m}$ and $s_{1}, \ldots, s_{m} \in \bar{R}$ there exists a unique continuous homomorphism $C_{n,m} \to \bar{R}$ which sends the $x_{i}$ to $r_{i}$ and the $y_{i}$ to $s_{i}$ and each homomorphism $C_{n,m} \to \bar{R}$ is of this form. Thus associating to an element $h \in \text{Spf}(C_{n,m})(\bar{R})$ the images of the $x_{i}$ and the $y_{i}$ gives a bijection $\text{Spf}(C_{n,m})(\bar{R}) \sim \bar{m}^{\otimes n} \otimes \bar{R}^{\otimes m}$. Any closed subscheme $X$ of $\text{Spf}(C_{n,m})$ is cut out by a family of formal power series $\{f_{i} \mid i \in I\} \subset C_{n,m}$. Each formal power series $f \in C_{n,m}$ induces a function $\bar{m}^{\otimes n} \otimes \bar{R}^{\otimes m} \to \bar{R}$. The above bijection identifies $X(\bar{R})$ with the set of points in $\bar{m}^{\otimes n} \otimes \bar{R}^{\otimes m}$ on which the $f_{i}$ are zero.

Definition 2.16. Let $\Gamma := \text{Aut}_{R}(\bar{R}) \cong \text{Aut}_{K}(\bar{K})$. For $X \in AFS_{R}$, we let $\Gamma$ act on $X(\bar{R})$ from the left by

$$
\Gamma \times X(\bar{R}) \to X(\bar{R})
$$

$$(\gamma, h) \mapsto \gamma \cdot h : \Gamma(\bar{R}, O_{\bar{R}}) \to \bar{R} \to \bar{R}.$$

Proof. (i): The map $h$ induces a homomorphism $C \otimes_{R} K \to \bar{R} \otimes_{R} K \to \bar{K}$ with the last homomorphism given by multiplication. Its kernel is a maximal ideal $n$ of $C \otimes_{R} K$. By Proposition 2.5 the quotient $(C \otimes_{R} K)/n$ is a finite field extension of $K$. This implies (i).

Proof. (ii): By (i) it suffices to show that if $R'$ is the valuation ring of a finite field extension $K' \subset \bar{K}$ of $K$ any homomorphism $h : C \to R'$ of $R$-algebras is continuous. Since $R'$ is of formally finite type this is a special case of Proposition 2.5 (ii).

Caution: The ring $\bar{R}$ is not complete with respect to the valuation topology. Thus there is no formal scheme $\text{Spf}(\bar{R})$ and $X(\bar{R})$ cannot be considered as $X(\text{Spf}(\bar{R}))$. The set $X(\bar{R})$ is also not the same as $X(\text{Spf}(\bar{R}))$.

Remark 2.15. For $X \in AFS_{R}$, the set $X(\bar{R})$ can be described more concretely as follows: For any $r_{1}, \ldots, r_{n} \in \bar{m}$ and $s_{1}, \ldots, s_{m} \in \bar{R}$ there exists a unique continuous homomorphism $C_{n,m} \to \bar{R}$ which sends the $x_{i}$ to $r_{i}$ and the $y_{i}$ to $s_{i}$ and each homomorphism $C_{n,m} \to \bar{R}$ is of this form. Thus associating to an element $h \in \text{Spf}(C_{n,m})(\bar{R})$ the images of the $x_{i}$ and the $y_{i}$ gives a bijection $\text{Spf}(C_{n,m})(\bar{R}) \sim \bar{m}^{\otimes n} \otimes \bar{R}^{\otimes m}$. Any closed subscheme $X$ of $\text{Spf}(C_{n,m})$ is cut out by a family of formal power series $\{f_{i} \mid i \in I\} \subset C_{n,m}$. Each formal power series $f \in C_{n,m}$ induces a function $\bar{m}^{\otimes n} \otimes \bar{R}^{\otimes m} \to \bar{R}$. The above bijection identifies $X(\bar{R})$ with the set of points in $\bar{m}^{\otimes n} \otimes \bar{R}^{\otimes m}$ on which the $f_{i}$ are zero.

Definition 2.16. Let $\Gamma := \text{Aut}_{R}(\bar{R}) \cong \text{Aut}_{K}(\bar{K})$. For $X \in AFS_{R}$, we let $\Gamma$ act on $X(\bar{R})$ from the left by

$$
\Gamma \times X(\bar{R}) \to X(\bar{R})
$$

$$(\gamma, h) \mapsto \gamma \cdot h : \Gamma(\bar{R}, O_{\bar{R}}) \to \bar{R} \to \bar{R}.$$

Proof. (i): The map $h$ induces a homomorphism $C \otimes_{R} K \to \bar{R} \otimes_{R} K \to \bar{K}$ with the last homomorphism given by multiplication. Its kernel is a maximal ideal $n$ of $C \otimes_{R} K$. By Proposition 2.5 the quotient $(C \otimes_{R} K)/n$ is a finite field extension of $K$. This implies (i).

Proof. (ii): By (i) it suffices to show that if $R'$ is the valuation ring of a finite field extension $K' \subset \bar{K}$ of $K$ any homomorphism $h : C \to R'$ of $R$-algebras is continuous. Since $R'$ is of formally finite type this is a special case of Proposition 2.5 (ii).

Caution: The ring $\bar{R}$ is not complete with respect to the valuation topology. Thus there is no formal scheme $\text{Spf}(\bar{R})$ and $X(\bar{R})$ cannot be considered as $X(\text{Spf}(\bar{R}))$. The set $X(\bar{R})$ is also not the same as $X(\text{Spf}(\bar{R}))$.
Proof. (i): It follows directly from the definition of $\psi$ that $\psi(h \cdot \gamma) = \psi(h)$ for all $\gamma \in \Gamma$ and $h \in \mathcal{X}(\bar{R})$. On the other hand let $h$ and $h'$ be two elements of $\mathcal{X}(\bar{R})$ which have the same image $n$ under $\psi$. By Proposition 2.5, the quotient $(C \otimes_R K)/n$ is a finite field extension $K'$ of $K$. The homomorphisms $h \otimes_R K$ and $h' \otimes_R K$ correspond to two different embeddings $i, i'$ of $K'$ into $K$ over $K$. There exists $\tilde{\gamma} \in \text{Aut}_K(K)$ such that $i' = \tilde{\gamma} \circ i$ and hence the restriction $\gamma \in \Gamma$ of $\tilde{\gamma}$ to $\bar{R}$ sends $h$ to $h'$. Thus the fibers of $\psi$ are exactly the $\Gamma$-orbits, which shows (i).

(ii) follows by a direct verification.

Definition 2.18. Let $R'$ be the valuation ring of a completely valued overfield $K'$ of $K$. Let $\mathcal{X} = \text{Spf}(C)$ be an affine formal scheme over $\text{Spf}(R')$.

(i) The formal scheme $\mathcal{X}$ is reduced if the ring $C$ is reduced.

(ii) The formal scheme $\mathcal{X}$ is flat over $R'$ if the ring $C$ is flat over $R'$.

(iii) We denote by $\text{AFS}_R^R$ the full subcategory of $\text{AFS}_R$ whose objects are those formal schemes which are reduced and flat over $R$.

Lemma 2.19. Let $\mathcal{X} = \text{Spf}(C) \in \text{AFS}_R$. Let $\mathcal{X}^\text{rf}$ be the closed formal subscheme of $\mathcal{X}$ defined by the ideal $\{ c \in C | \exists n \geq 0: (am)^n = 0 \}$. This formal scheme is reduced and flat over $R$ and the natural map $\mathcal{X}^\text{rf}(\bar{R}) \to \mathcal{X}(\bar{R})$ is a bijection.

Proof. Direct verification using the fact that an $R$-module is flat if and only if it has no $m$-torsion.

Proposition 2.20. Let $\mathcal{Y}_1, \mathcal{Y}_2 \in \text{AFS}_R^R$ be two closed formal subschemes of $\mathcal{X} \in \text{AFS}_R$. If $\mathcal{Y}_1(\bar{R}) \subset \mathcal{Y}_2(\bar{R})$, then $\mathcal{Y}_1 \subset \mathcal{Y}_2$.

Proof. Let $C$ be the ring of global sections of $\mathcal{X}$ and $I_1, I_2$ the ideals defining $\mathcal{Y}_1, \mathcal{Y}_2$. We want to prove $I_2 \subset I_1$. Let $\pi$ be a uniformizer of $R$. The fact that the $\mathcal{Y}_i$ are flat over $R$ means that $\pi$ is not a zero-divisor in $C/I_i$. This implies that it is enough to prove $I_2 \otimes_R K \subset I_1 \otimes_R K$ inside $C \otimes_R K$. Since by assumption the ideals $I_i$ are radical, so are the ideals $I_i \otimes_R K$. Since by Proposition 2.5 the ring $C \otimes_R K$ is Jacobson it suffices to prove that each maximal ideal of $C \otimes_R K$ which contains $I_1 \otimes_R K$ also contains $I_2 \otimes_R K$. This follows from the fact that $\mathcal{Y}_1(\bar{R}) \subset \mathcal{Y}_2(\bar{R})$ and Proposition 2.17.

For $i \geq 0$, we endow the ring $\bar{R}/(m\bar{R})^i$ with the quotient topology induced from the valuation topology on $\bar{R}$, with respect to which it is adic. Hence there is a formal scheme $\text{Spf}(\bar{R}/(m\bar{R})^i)$ and for $\mathcal{X} \in \text{AFS}_R$ we denote by $\mathcal{X}(\bar{R}/(m\bar{R})^i)$ the set of morphisms $\text{Spf}(\bar{R}/(m\bar{R})^i) \to \mathcal{X}$ over $\text{Spf}(R)$. There is a natural map $\mathcal{X}(\bar{R}) \to \mathcal{X}(\bar{R}/(m\bar{R})^i)$ for all $i \geq 0$.

Corollary 2.21. Let $\mathcal{X} \in \text{AFS}_R$. The set $\mathcal{X}(\bar{R})$ is nonempty if and only if for all $i \geq 0$ the set $\mathcal{X}(\bar{R}/(m\bar{R})^i)$ is nonempty.

Proof. The “only if” direction is clear. Conversely, assume that $\mathcal{X}(\bar{R})$ is empty. Let $\mathcal{X}^\text{rf} \in \text{AFS}_R^R$ be the closed formal subscheme given by Lemma 2.19. Since $\mathcal{X}^\text{rf}(\bar{R}) = \mathcal{X}(\bar{R})$ is empty, it follows from Proposition 2.20 that $\mathcal{X}^\text{rf}$ is the empty formal scheme. Hence, if we let $C := \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $\pi$ is a uniformizer of $R$, it follows from the definition of $\mathcal{X}^\text{rf}$ that for each $c \in C$ there exists $n \geq 0$ such that $(c\pi)^n = 0$. For $c = 1$ we get that there is an $n \geq 0$ such that the image $\pi^n$ in $C$ is zero. Hence for all $i > n$ there is no homomorphism $C \to \bar{R}/(m\bar{R})^i$ of $R$-algebras since for such $i$ the image of $\pi^n$ in $\bar{R}/(m\bar{R})^i$ is not zero. This proves the claim.
Lemma 2.22. Let $\mathcal{X} = \text{Spf}(C) \in \text{AFS}_R$ be the union of closed subschemes $\mathcal{X}_1, \ldots, \mathcal{X}_m$. Then $\mathcal{X}(\bar{R}) = \bigcup_i \mathcal{X}_i(\bar{R})$.

Proof. Let $I_1, \ldots, I_m \subset C$ be the ideals defining the $\mathcal{X}_i$. Let $h \colon C \to \bar{R}$. We want to show that $h(I_i) = 0$ for some $i$. If this is not the case we pick $0 \neq r_i \in h(I_i)$. Then the product $r$ of the $r_i$ is a non-zero element which lies in $h(I_i)$ for all $i$. For each $i$ pick $c_i \in I_i$ such that $h(c_i) = x$. Then the product of the $c_i$ lies in the intersection of the $I_i$ which by assumption is zero. Thus by applying $h$ to this product we get $x^m = 0$ and hence $x = 0$, which is a contradiction.

\[ \square \]

2.2 Irreducibility

Definition 2.23. Let $\mathcal{X} \in \text{AFS}_R^d$ be non-empty.

- The formal scheme $\mathcal{X}$ is irreducible if and only if $\text{Spec}(C)$ is irreducible.
- An irreducible component of $\mathcal{X}$ is a maximal irreducible closed formal subscheme.
- We call the formal scheme $\mathcal{X}$ is geometrically irreducible if and only if $\mathcal{X}_R$ is irreducible for all valuation rings $R'$ of finite field extensions $K'$ of $K$.

Note that the irreducible components of $\mathcal{X}$ correspond to the irreducible components of $\text{Spec}(C)$, that is to the minimal prime ideals of $C$. In particular there are finitely many such components. Also, since $C$ is reduced, the intersection of all its minimal prime ideals is the zero ideal. Thus $\mathcal{X}$ is the union of its irreducible components in the language of Definition 2.11.

Proposition 2.24. Let $\mathcal{X} \in \text{AFS}_R^d$ be non-empty. Each irreducible component of $\mathcal{X}$ is reduced and flat over $R$.

Proof. Let $C := \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. It suffices to show that each irreducible component of $\text{Spec}(C)$ is reduced and flat over $R$. Reducedness is clear. As $R$ is a discrete valuation ring, a scheme $X$ over $\text{Spec}(R)$ is flat if and only if its generic fiber is schematically dense in $X$. By a direct verification, if $\text{Spec}(C)$ satisfies this condition, then so does any irreducible component of $\text{Spec}(C)$.

\[ \square \]

Lemma 2.25. Let $\mathcal{X} = \text{Spf}(C) \in \text{AFS}_R^d$ be irreducible. If $\mathcal{X}_1, \ldots, \mathcal{X}_m$ are closed formal subschemes of $\mathcal{X}$ such that $\mathcal{X}$ is the union of the $\mathcal{X}_i$, then $\mathcal{X} = \mathcal{X}_i$ for some $i$.

Proof. Let $I_1, \ldots, I_m \subset C$ be the ideals defining the $\mathcal{X}_i$. By assumption their intersection is zero. By assumption $C$ is integral. If all $I_i$ were non-zero, we could pick elements $0 \neq x_i \in I_i$ whose product would be zero. Thus one of the $I_i$ is zero, which is what we wanted.

\[ \square \]

In \cite{[9]} Section 7, de Jong gives a construction, due to Berthelot, of a “generic fiber” functor from $\text{AFS}_R$ to the category of quasi-separated rigid analytic spaces over $K$. We denote this functor by $\mathcal{X} \mapsto \mathcal{X}^\text{rig}$. It can be described as follows: The formal scheme $\text{Spf}(C_{n,m})$ is sent to the product of the open $n$-dimensional unit disc $D^m_K$ over $K$ and the closed $m$-dimensional unit disc $B^m_K$ over $K$. A closed formal subscheme $\mathcal{X}$ as above is cut out by a family of power series $\{ x_i \mid i \in I \} \subset C_{n,m}$. These $f_i$ induce global sections of $D^m_K \times B^m_K$, and $\mathcal{X}^\text{rig}$ is the closed rigid analytic subspace of $D^m_K \times B^m_K$ cut out by these global sections.

We will prove Proposition 2.28 by using results from \cite{[7]}. There Conrad introduces the notion of irreducibility of a quasi-separated rigid analytic space and that of an irreducible component of such a space. He also shows that this notion is well-behaved under the functor $\mathcal{X} \mapsto \mathcal{X}^\text{rig}$. The following is a slight reformulation of \cite{[7]} Theorem 2.3.1]:
Theorem 2.26. Let \( \mathcal{X} \in \text{AFS}_R^\text{rig} \) and \( \mathcal{X}_1, \ldots, \mathcal{X}_n \) be the irreducible components of \( \mathcal{X} \). The closed rigid analytic subvarieties \( \mathcal{X}_i^\text{rig} \) of \( \mathcal{X}^\text{rig} \) are the irreducible components of \( \mathcal{X}^\text{rig} \).

Proof. In fact, [1, Theorem 2.3.1] says the following: Let \( C := \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}) \) and let \( \hat{C} \) be the normalization of \( C \), that is the integral closure of \( C \) in its total ring of fractions. Let \( I_1, \ldots, I_n \) be the preimages in \( \hat{C} \) of the minimal prime ideals of \( C \). Let \( \mathcal{Y}_1, \ldots, \mathcal{Y}_n \) be the closed formal subschemes of \( \mathcal{X} \) defined by the ideals \( I_i \). Then \( \mathcal{Y}_i^\text{sch}, \ldots, \mathcal{Y}_n^\text{rig} \) are the irreducible components of \( \mathcal{Y}^\text{rig} \).

To show that this is the same statement as the one we want to prove, we need to show that the preimages of the minimal prime ideals of \( \hat{C} \) are exactly the minimal prime ideals of \( C \). This is true for any normalization homomorphism \( C \to \hat{C} \). \( \square \)

Corollary 2.27. A formal scheme \( \mathcal{X} \in \text{AFS}_R^\text{rig} \) is irreducible if and only if the rigid analytic space \( \mathcal{X}^\text{rig} \) is irreducible.

Proposition 2.28. Let \( \mathcal{X} \in \text{AFS}_R^\text{rig} \) be non-empty. There exists a finite field extension \( K' \subset K \) of \( K \) with valuation ring \( R' \) such that the irreducible components of \( \mathcal{X}^\text{rig}_{\text{Spf}(R')} \) are geometrically irreducible.

Proof of Proposition 2.28. In [1, Section 3.4], Conrad calls a quasi-separated rigid analytic space \( X \) over \( K \) geometrically irreducible if for all completely valued overfields \( K' \) of \( K \) the rigid analytic space \( X_{K'} \) is irreducible. By [1, Theorem 3.4.3], for any quasi-separated rigid analytic space \( X \) over \( K \) having finitely many irreducible components, there exists a finite field extension \( K' \subset K \) with valuation ring \( R' \) such that \( X^\text{rig}_{K'} \) has finitely many irreducible components which are geometrically irreducible. Using the compatibility of the functor \( \mathcal{X} \to \mathcal{X}^\text{rig} \) with base change to finite extensions of \( K \) (c.f. [9, 7.2.6]) the claim thus follows from Theorem 2.26. \( \square \)

2.2.1 Formal Schematic Image

Definition 2.29. Let \( f : \mathcal{X} \to \mathcal{X}' \) be a morphism of affine formal schemes. We define the formal schematic image \( f(\mathcal{X}) \) of \( f \) to be the intersection of all closed formal subschemes of \( \mathcal{X}' \) through which \( f \) factors.

Thus \( f(\mathcal{X}) \) is the smallest closed formal subscheme of \( \mathcal{X}' \) through which \( f \) factors. If \( \mathcal{X} = \text{Spf}(C) \) and \( \mathcal{X}' = \text{Spf}(C') \) then the ideal corresponding to \( f(\mathcal{X}) \) is the kernel of the homomorphism \( C' \to C \) corresponding to \( \mathcal{X} \to \mathcal{X}' \).

Lemma 2.30. Let \( \mathcal{G}, \mathcal{G}' \) be connected p-divisible groups over \( \text{Spf}(R) \) considered as formal schemes. Let \( f : \mathcal{G} \to \mathcal{G}' \) be an isogeny and \( \mathcal{V} \subset \mathcal{G}' \) be a closed formal subscheme. Let \( \mathcal{V} : = f^{-1}(\mathcal{V}) := \mathcal{V} \times_{\mathcal{G}'} \mathcal{G} \) be its preimage in \( \mathcal{G} \). Then \( \mathcal{V} \) is the formal schematic image of \( \mathcal{V} \) in \( \mathcal{V} \).

Proof. Let \( C := \Gamma(\mathcal{G}, O_\mathcal{G}) \) and \( C' := \Gamma(\mathcal{G}', O_{\mathcal{G}'}) \). By [20, Proposition 4.4], the rings \( C \) and \( C' \) are isomorphic to \( \text{Spf}(R[[x_1, \ldots, x_n]]) \) for some \( n \geq 0 \). First we want to show that \( C' \to C \) is flat. Since \( R[[x_1, \ldots, x_n]] \) is a regular local ring, by [12, Theorem 18.16] for this it suffices to show that \( \dim(C') = \dim(C/n'C) + \dim(C') \) where \( n' \) is the maximal ideal of \( C' \). This follows from the fact that \( C/n'C \) is finite over \( k \). Thus as a finite flat module over the local ring \( C' \), the ring \( C \) is finite free over \( C' \).

Let \( I' \subset C' \) be the ideal defining \( \mathcal{V} \). Then \( \mathcal{X} \) is the formal spectrum of \( C'/I' \otimes_{C'} C \). Since \( C \to C' \) is finite free we have \( C'/I' \otimes_{C'} C \cong C/I'C \).

Now let \( \mathcal{V}' \) be the formal schematic image of \( \mathcal{X} \) in \( \mathcal{V} \). It is contained in \( \mathcal{V} \). Thus it is defined by an ideal \( I' \) containing \( I' \). Its preimage in \( \mathcal{G} \) must coincide with \( \mathcal{X} \). Thus the
induced homomorphism $C'/I' \otimes_{C'} C \to C'/I \otimes_{C} C$ is an isomorphism. Since $C' \to C$ is finite free this implies that $C'/I' \to C'/I$ is an isomorphism. This means $\mathcal{Y}' = \mathcal{Y}$. \hfill \square

2.3 Base Change and Formal Schematic Closure

**Proposition 2.31.** Let $T \subset m^{\oplus n}$. Let $R'$ be the valuation ring of a complete overfield $K' \subset \bar{K}$ of $K$. Let $I \subset C_{n,0}$ and $I' \subset C'_{n,0}$ be the ideals consisting of those power series which vanish at all elements of $T$. Then $IC'_{n,0} = I'$.

**Proof.** The ideal $I$ is characterized by the left exact sequence

$$0 \to I \to C_{n,0} \xrightarrow{(ev_t)_{t \in T}} \prod_{t \in T} R$$

where for $t \in T$ we let $ev_t: C_{n,m} \to R$ be the function given by evaluation at $t$.

Let $D \cong C_{n,0}/I$ be the image of $C_{n,0}$ in $\prod_{t \in T} R$ endowed with the quotient topology from $C_{n,0}$. This is a topological $R$-algebra of formally finite type by Proposition 2.5. Lemma 2.10 yields an exact sequence

$$0 \to IC'_{n,0} \to C'_{n,0} \to D \otimes_{R} R' \to 0.$$  \hfill (2.32)

Let $D_i$ be the kernel of $D \to \prod_{t \in T} R \to \prod_{t \in T} R/m^t$. We claim that there exists a sequence $(j(i))_{i \geq 0}$ of positive integers going to infinity such that $D_i \subset J_{n,0}^{j(i)}$. If not there exists a sequence $(f_i)_{i \geq 0}$ in $D$ which do not converge to zero such that each $f_i$ lies in $D$. The ring $C_{n,0}$, being the inverse limit of the finite rings $C_{n,0}/J_{n,0}^i$, is compact. Hence so is the quotient $D$ of $C_{n,0}$. Hence after passing to a subsequence we may assume that the $f_i$ converge to an element $f \in D$. It follows that the images of the $f_i$ in $\prod_{t \in T} R$ converge to the image of $f$ with respect to the topology on $\prod_{t \in T} R$ defined by the ideals $\prod_{t \in T} m^t$. But by assumption the images of the $f_i$ converge to zero in $\prod_{t \in T} R$. Hence $f$ is zero, which is a contradiction. Thus there exists a sequence $(j(i))_{i \geq 0}$ as above. Since $J_{n,0}D \subset D_i$ this shows that the topology on $D$ defined by the $D_i$ is the $J_{n,0}$-adic topology. Thus $D \otimes_{R} R'$ can be written as $\lim_{\leftarrow i} D/D_i \otimes_{R} R'$.

The inclusion $D \hookrightarrow \prod_{t \in T} R$ induces injections $D/D_i \hookrightarrow (\prod_{t \in T} R/m^t)$ and hence injections $D/D_i \otimes_{R} R' \hookrightarrow (\prod_{t \in T} R/m^t) \otimes_{R} R'$. Hence we get a homomorphism

$$D \otimes_{R} R' = \lim_{\leftarrow i} (D/D_i \otimes_{R} R') \to \lim_{\leftarrow i} (\prod_{t \in T} R/m^t) \otimes_{R} R' \to \lim_{\leftarrow i} (\prod_{t \in T} R'/mR') \cong \prod_{t \in T} R'$$  \hfill (2.33)

and by a direct verification the composition of this homomorphism with the homomorphism $C'_{n,0} \to D \otimes_{R} R'$ from (2.32) is the homomorphism which evaluates a power series at elements of $T$.

We want to show that the homomorphism (2.33) is injective. That the first arrow in (2.33) is injective follows from the choice of the $D_i$, the flatness of $R \to R'$ and the left exactness of the inverse limit functor. By the left exactness of inverse limits, in order to prove injectivity of the second arrow in (2.33), it is enough to show that $\prod_{t \in T} R/m^t \otimes_{R} R' \to \prod_{t \in T} R'/mR'$ is injective for all $i$. Since any element of $\prod_{t \in T} R/m^t \otimes_{R} R'$ is contained in the image of $\prod_{t \in T} R/m^t \otimes_{R} R''$ for the valuation ring $R''$ of a finite field extension $K'' \subset K'$ for this step we may assume that $K'$ is finite over $K$. But then $R'$ is finite free over $R$ and hence $\prod_{t \in T} R/m^t \otimes_{R} R' \to \prod_{t \in T} R'/mR'$ is even an isomorphism in this situation.

Thus by combining (2.32) and (2.33) we get a left exact sequence

$$0 \to IC'_{n,0} \to C'_{n,0} \xrightarrow{(ev_t)_{t \in T}} \prod_{t \in T} R'$$

10
which shows $IC_{n,0}^{'} = I'$.

**Definition 2.34.** Let $R'$ be the valuation ring of a complete overfield $K' \subset \hat{K}$ of $K$. Let $\mathcal{X}$ be an affine formal scheme over $\text{Spf}(R')$ and $T \subset \mathcal{X}(R')$. We define the formal schematic closure of $T$ in $\mathcal{X}$ to be the intersection of all closed subschemes $\mathcal{Y}$ of $\mathcal{X}$ for which $T \subset \mathcal{Y}(R')$.

If $\mathcal{X}'$ is the formal schematic closure of $T$, then we say that $T$ is formal-schematically dense in $\mathcal{X}'$.

Thus the formal schematic closure of $T$ is the smallest closed subscheme of $\mathcal{X}'$ which contains $T$.

**Corollary 2.35.** Let $\mathcal{X} \in \text{AFS}_R$ and $T \subset \mathcal{X}(R)$. Let $\mathcal{Y}$ be the formal schematic closure of $T$ in $\mathcal{X}$. Let $R'$ be the valuation ring of a complete overfield $K' \subset \hat{K}$ of $K$. Then $\mathcal{Y}_{\text{Spf}(R')}$ is the formal schematic closure of $T \subset \mathcal{X}(R) \subset \mathcal{X}(R')$ inside $\mathcal{Y}_{\text{Spf}(R')}$.

**Proof.** It suffices to prove this for $\mathcal{X} = \text{Spf}(C_{n,m})$ in which case it is a reformulation of Proposition 2.31.

### 2.4 Transporters

Let $\mathcal{G}$ be a group object in the category $\text{AFS}_R$. We will need the existence of (strict) transporters in $\mathcal{G}$.

**Construction 2.36.** Let $\mathcal{X}, \mathcal{Y} \subset \mathcal{G}$ be two closed formal subschemes. For $i \geq 0$ let $\mathcal{G}_i$ be the $i$-th infinitesimal neighborhood of the zero section in $\mathcal{G}$ and let $\mathcal{X}_i, \mathcal{Y}_i := \mathcal{G}_i \cap \mathcal{X}, \mathcal{G}_i \cap \mathcal{Y}$. These are finite schemes over $R/m^i$. The group structure on $\mathcal{G}$ makes $\mathcal{G}_i$ into a group scheme over $R/m^i$ and $\mathcal{X}_i$ and $\mathcal{Y}_i$ are closed subschemes of $\mathcal{G}_i$. For $i \geq 0$ let $\text{Trans}_{\mathcal{G}_i}(\mathcal{X}_i, \mathcal{Y}_i)$ be the strict transporter in $\mathcal{G}_i$, that is the closed subscheme of $\mathcal{G}_i$ whose points are those points $g$ of $\mathcal{G}_i$ for which $\mathcal{X}_i + g = \mathcal{Y}_i$. It exists by [1, Exemple VI.6.4.2 e)]. Then for each $i$ one has a decreasing sequence $(\text{Trans}_{\mathcal{G}_{i+j}}(\mathcal{X}_{i+j}, \mathcal{Y}_{i+j}) \cap \mathcal{G}_i)_{j \geq 0}$ of subschemes of $\mathcal{G}_i$. By noetherianity this sequence stabilizes; let $\text{Trans}_{\mathcal{G}_i}(\mathcal{X}, \mathcal{Y})_i$ be its eventual value. Then $\text{Trans}_{\mathcal{G}_i}(\mathcal{X}, \mathcal{Y})_{i+1} \cap \mathcal{G}_i = \text{Trans}_{\mathcal{G}_i}(\mathcal{X}, \mathcal{Y})_i$ for all $i$. Hence the inductive limit of these schemes is a closed formal subscheme $\text{Trans}_{\mathcal{G}}(\mathcal{X}, \mathcal{Y})$ of $\mathcal{G}$.

**Proposition 2.37.** Let $\mathcal{X}, \mathcal{Y}$ be closed formal subschemes of $\mathcal{G}$ and let $R'$ be the valuation ring of a finite field extension $K'$ of $K$. Then $\text{Trans}_{\mathcal{G}}(\mathcal{X}, \mathcal{Y})(R') = \{g \in \mathcal{G}(R') \mid g + \mathcal{X}_{\text{Spf}(R')} = \mathcal{Y}_{\text{Spf}(R')}\}$.

**Proof.** Let $g \in \mathcal{G}(R')$ and let $m'$ be the maximal ideal of $R'$. For $j \geq 0$ we denote the ring $R'/(m')^j$ by $R'_j$ and let $g_j$ be the image of $g$ in $\mathcal{G}(R'_j)$. Then

$$g \in \text{Trans}_{\mathcal{G}}(\mathcal{X}, \mathcal{Y})(R') \iff \forall j \geq 0: g_j \in \text{Trans}_{\mathcal{G}}(\mathcal{X}, \mathcal{Y})(R'_j) \iff \forall j \geq 0: \forall i \gg j: g_j + (\mathcal{X}_i)_{R'_j} = (\mathcal{Y}_i)_{R'_j} \iff g + \mathcal{X}_{\text{Spf}(R')} = \mathcal{Y}_{\text{Spf}(R')}.$$

### 2.5 Descent

Let $R'$ be the valuation ring of a complete overfield $K' \subset \hat{K}$ of $K$. Let $\mathcal{X} \in \text{AFS}_R$ and $\mathcal{X}' \subset \mathcal{X}_{\text{Spf}(R')}$ a closed formal subscheme. If there exists a closed formal subscheme $\mathcal{X}''$ of $\mathcal{X}$ such that $\mathcal{X}''_{\text{Spf}(R')} = \mathcal{X}'$ then it follows from Lemmas 2.8, 2.9 and 2.10 that such an $\mathcal{X}$ is unique. In this case we will say that $\mathcal{X}'$ is defined over $R$. 


In the following we will always endow \( k[[x_1, \ldots, x_n]] \) with the \((x_1, \ldots, x_n)\)-adic topology. Let \( \mathcal{X} = \text{Spf}(C) \) be a formal scheme over \( \text{Spec}(k) \) which is isomorphic to \( \text{Spf}(k[[x_1, \ldots, x_n]]) \). We are interested in closed formal subschemes of \( \mathcal{X} \) and their base change to \( \text{Spf}(R) \), which lie in \( \text{AFS}_R \).

**Lemma 2.38.** Let \( \mathcal{X}' \) be a closed formal subscheme of \( \mathcal{X} \) defined by an ideal \( I \) of \( C \). Then \( \mathcal{X}'_{\text{Spf}(R)} \) is the closed formal subscheme of \( \mathcal{X}_{\text{Spf}(R)} \) defined by the ideal \( I(C \otimes_k R) \) of \( C \otimes_k R \). This ideal is equal to \( I \otimes_k R \).

**Proof.** Since the homomorphisms \( k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^i \to R[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^i \) are faithfully flat, this follows from Lemma 2.8 in the same way as Lemma 2.10.

Let \( \mathcal{X}' \subset \mathcal{X}_{\text{Spf}(R)} \) a closed formal subscheme. If there exists a closed formal subscheme \( \mathcal{X}'' \) of \( \mathcal{X} \) such that \( \mathcal{X}''_{\text{Spf}(R)} = \mathcal{X}' \) then it follows from Lemmas 2.8 and 2.38 such that such an \( \mathcal{X}' \) is unique. In this case we will say that \( \mathcal{X}' \) is defined over \( k \).

Let \( F : \mathcal{X} \to \mathcal{X} \) be the Frobenius endomorphism of \( \mathcal{X} \) with respect to \( k \). If \( \mathcal{X}' \) is a closed formal subscheme of \( \mathcal{X}_{\text{Spf}(R)} \) defined by formal power series \( \{f_i \mid i \in I\} \subset R[[x_1, \ldots, x_n]] \), then the formal schematic image \( F(\mathcal{X}') \) of \( \mathcal{X}' \) under \( F \) is defined by the formal power series obtained by applying \( F \) to the coefficients of the \( f_i \).

**Lemma 2.39.** For \( i \geq 0 \) let \( R^{p^i} \) be the field consisting of \( p^i \)-th powers of elements of \( K \) and \( R^p \) the valuation ring of \( K^{p^i} \). Let \( \mathcal{X}' \) be a closed formal subscheme of \( \mathcal{X}_{\text{Spf}(R)} \). If \( \mathcal{X}' \) is defined over \( R^p \) for all \( i \geq 0 \), then \( \mathcal{X}' \) is defined over \( k \).

**Proof.** We may assume that \( \mathcal{X} = \text{Spf}(k[[x_1, \ldots, x_n]]) \). Let \( i \geq 0 \). As any element of \( R^{p^i} \) is congruent to an element of \( k \) modulo \( m^{p^i} \), the fact that \( \mathcal{X}' \) can be defined over \( R^{p^i} \) implies that the intersection of \( \mathcal{X}' \) with \( \text{Spec}((R/m^{p^i})[[x_1, \ldots, x_n]]/(x_1, \ldots, x_n)^i) \subset \mathcal{X} \) can be defined over \( k \). As \( \mathcal{X}' \) is the direct limit of these intersections, the claim follows by varying \( i \).

**Lemma 2.40.** A closed formal subscheme \( \mathcal{X}' \) of \( \mathcal{X}_{\text{Spf}(R)} \) is defined over \( k \) if and only if \( F(\mathcal{X}') = \mathcal{X}' \).

**Proof.** The “only if” direction is straightforward. For the other direction, the fact that \( F^i(\mathcal{X}') = \mathcal{X}' \) implies that \( \mathcal{X}' \) is defined over \( R^p \) for all \( i \geq 0 \). Thus we can conclude using Lemma 2.39.

**Proposition 2.41.** Let \( \mathcal{G} \) be a formal group scheme over \( k \) which as a formal scheme is isomorphism to \( \text{Spf}(k[[x_1, \ldots, x_n]]) \). Let \( \mathcal{X} \subset \mathcal{G}_{\text{Spf}(R)} \) be a closed formal subscheme. If for each \( i \geq 0 \) there exists a finite field extension \( K' \subset \bar{K} \) of \( K \) with valuation ring \( R' \) and \( g \in \mathcal{G}(R') \) such that \( \mathcal{X}_{\text{Spf}(R')} + g \) is defined over \( R^p \), then there exists a finite field extension \( K' \subset \bar{K} \) of \( K \) with valuation ring \( R' \) and \( g \in \mathcal{G}(R') \) such that \( \mathcal{X}_{\bar{R}'} + g \) is defined over \( k \).

**Proof.** We consider the closed formal subscheme \( \text{Trans}_{\mathcal{G}_{\text{Spf}(R)}}(\mathcal{X}, F(\mathcal{X})) \subset \mathcal{G}_{\text{Spf}(R)} \) given by Construction 2.36. Let \( i \geq 0 \) and pick \( R' \) as in the claim together with \( g \in \mathcal{G}(R') \) such that \( \mathcal{X}_{\text{Spf}(R')} + g \) is defined over \( R^p \). Let \( \pi \in R \) be a uniformizer. By identifying \( R \) with \( k[[\pi]] \) and considering defining equations for \( \mathcal{X}_{\text{Spf}(R')} + g \) with coefficients from \( k[[\pi^p]] \) one sees that \( \mathcal{X}_{\bar{R}'} + g \equiv F(\mathcal{X}_{\bar{R}'} + g) \pmod{\pi^{p^i}} \). Thus \( g - F(g) \in \text{Trans}_{\mathcal{G}}(\mathcal{X}, \mathcal{X})(\bar{R})/(m\bar{R})^{p^i} \). Thus Proposition 2.21 implies that there exists \( g' \in \text{Trans}_{\mathcal{G}}(\mathcal{X}, \mathcal{X})(\bar{R}) \).

The morphism \( \mathcal{G} \to \mathcal{G}, g \mapsto g - F(g) \) is the identity on the tangent space at zero and thus an isomorphism by [17, A.4.5]. Hence \( g' \) can be written as \( g'' - F(g'') \) for some \( g'' \in \mathcal{G} \). By Lemma 2.14 there exists \( R' \) as in the claim such that \( g'' \in \mathcal{G} \).
2.6 Formal schemes arising from schemes

Let \( \mathcal{X} \) be a scheme locally of finite type over \( R \) together with a \( k \)-valued point \( s : \text{Spec}(k) \to \mathcal{X} \) of the special fiber of \( \mathcal{X} \). We let \( \hat{\mathcal{X}} \) the the completion of \( \mathcal{X} \) along the closed subscheme \( s \). We denote by \( \mathcal{O}_{\mathcal{X},s} \) the stalk of \( \mathcal{O}_\mathcal{X} \) at the closed point in the image of \( s \). The formal scheme \( \hat{\mathcal{X}} \) is the formal spectrum of the completion \( \hat{\mathcal{O}}_{\mathcal{X},s} \) of the local ring \( \mathcal{O}_{\mathcal{X},s} \) with respect to its maximal ideal.

**Proposition 2.42.** (i) If \( \mathcal{X} \) is smooth over \( R \) at \( s \) of relative dimension \( n \) the formal scheme \( \hat{\mathcal{X}} \) is isomorphic to \( \text{Spf}(R[[x_1, \ldots, x_n]]) \).

(ii) The formal scheme \( \hat{\mathcal{X}} \) is in \( \text{AFS}_R \).

(iii) The set \( \hat{\mathcal{X}}(\mathcal{R}) \) can be naturally identified with the set of elements of \( \mathcal{X}(\mathcal{R}) \) which map the closed point of \( \text{Spec}(R) \) to the closed point in the image of \( s \).

**Proof.** (i) The fact that \( \mathcal{X} \) is smooth at \( s \) as implies (in fact is equivalent to) that there exist an affine neighbourhood \( U \) of \( s \) and an étale morphism \( U \to \text{Spec}(R[x_1, \ldots, x_n]) \) which maps the zero section of the special fiber of \( \text{Spec}(R[x_1, \ldots, x_n]) \) to \( s \) (c.f. [26], Tag 054L). Such an étale morphism induces a finite étale morphism \( \hat{\mathcal{O}}_{\mathcal{X},s} \to \mathcal{R}[x_1, \ldots, x_n] \). Since \( \hat{\mathcal{O}}_{\mathcal{X},s} \), being a complete local ring, is Henselian, the fact that both \( \mathcal{O}_{\mathcal{X},s} \) and \( \mathcal{R}[x_1, \ldots, x_n] \) have residue field \( k \) implies this is an isomorphism.

(ii) Using a closed embedding of an affine neighborhood of \( s \) into \( \text{Spec}(\mathcal{R}[x_1, \ldots, x_n]) \) for some \( n \geq 0 \) one gets a closed embedding of \( \hat{\mathcal{X}} \) into \( \text{Spf}(\mathcal{R}[x_1, \ldots, x_n]) \).

(iii) This follows from the fact that \( \hat{\mathcal{X}} = \text{Spf}(\hat{\mathcal{O}}_{\mathcal{X},s}) \). \( \Box \)

Let \( \mathcal{X}' \) be second scheme locally of finite type over \( R \) together with a \( k \)-valued point \( s' : \text{Spec}(k) \to \mathcal{X}' \). For a morphism \( h : \mathcal{X} \to \mathcal{X}' \) over \( R \) which is maps \( s \) to \( s' \) we denote by \( \hat{h} \) the induced morphism \( \hat{\mathcal{X}} \to \hat{\mathcal{X}'} \) of formal schemes over \( \text{Spf}(R) \).

**Proposition 2.43.** Let \( \mathcal{X} \) be a reduced scheme which is flat and of finite type over \( R \) and let \( s : \text{Spec}(k) \to \mathcal{X} \) a \( k \)-valued point of \( \mathcal{X} \).

(i) The formal scheme \( \hat{\mathcal{X}} \) is reduced and flat over \( R \).

(ii) Assume that \( \mathcal{X} \) is integral and let \( \mathcal{X}' \) be an irreducible component of \( \hat{\mathcal{X}} \). The set \( \mathcal{X}'(\mathcal{R}) \subset \mathcal{X}(\mathcal{R}) \) is schematically dense in \( \mathcal{X}(\mathcal{R}) \).

**Proof.** (i) Since \( \mathcal{X} \) is reduced, so is the local ring \( \mathcal{O}_{\mathcal{X},s_0} \). The ring \( \mathcal{O}_{\mathcal{X},s_0} \) is also excellent. Since the completion of any excellent reduced local ring is reduced (c.f. [16], Theorem 7.8.3) the formal scheme \( \hat{\mathcal{X}} \) is reduced. Flatness follows from the flatness of \( R \to \mathcal{O}_{\mathcal{X},s_0} \) and the flatness of \( \mathcal{O}_{\mathcal{X},s} \to \hat{\mathcal{O}}_{\mathcal{X},s} \).

(ii) Let \( \mathcal{Y} \subset \mathcal{X} \) be the schematic closure of \( \mathcal{X}'(\mathcal{R}) \subset \mathcal{X}(\mathcal{R}) \) and let \( \mathcal{I} \subset \mathcal{O}_\mathcal{X} \) be the sheaf of ideals defining \( \mathcal{Y} \).

As \( \mathcal{O}_{\mathcal{X},s} \) is a Noetherian local ring, the homomorphism \( \mathcal{O}_{\mathcal{X},s} \to \hat{\mathcal{O}}_{\mathcal{X},s} \) is faithfully flat and for any finitely generated \( \mathcal{O}_{\mathcal{X},s} \)-module \( M \), its completion with respect to the topology induced by the maximal ideal of \( \mathcal{O}_{\mathcal{X},s} \) is isomorphic to \( \hat{\mathcal{O}}_{\mathcal{X},s} \otimes_{\mathcal{O}_{\mathcal{X},s}} M \). By applying this to \( M = \mathcal{O}_\mathcal{Y} \), one sees that \( \mathcal{Y} \) is the formal closed subscheme of \( \hat{\mathcal{X}} \) corresponding to the ideal \( \mathcal{I} \mathcal{O}_{\mathcal{X},s} \). Since by construction \( \mathcal{X}'(\mathcal{R}) \subset \hat{\mathcal{Y}}(\mathcal{R}) \) Proposition 2.20 implies \( \mathcal{X}' \subset \hat{\mathcal{Y}} \). Thus \( \mathcal{I} \mathcal{O}_{\mathcal{X},s} \) is contained in a minimal prime ideal of \( \hat{\mathcal{O}}_{\mathcal{X},s} \) and hence the rings \( \hat{\mathcal{O}}_{\mathcal{X},s} \) and \( \hat{\mathcal{O}}_{\mathcal{Y},s} \) have the same dimension.

Using the flatness of \( \mathcal{O}_{\mathcal{X},s} \to \hat{\mathcal{O}}_{\mathcal{X},s} \) and the fact that the maximal ideal of \( \hat{\mathcal{O}}_{\mathcal{X},s} \) is generated by the image of the maximal ideal of \( \mathcal{O}_{\mathcal{X},s} \), Theorem 10.10 of [12] implies that...
\[ \dim(O_{X,s}) = \dim(\hat{O}_{X,s}). \] Analogously we get \( \dim(O_{Y,s}) = \dim(\hat{O}_{Y,s}). \) Thus \( \dim(Y) \geq \dim(O_{Y,s}) = \dim(\hat{O}_{Y,s}) = \dim(O_{X,s}) = \dim(\hat{O}_{X,s}). \) Since \( X \) is irreducible \( \dim(O_{X,s}) = \dim(X) \) and thus we get \( \dim(Y) = \dim(X) \) which using the irreducibility of \( X \) implies \( Y = X. \)

**Proposition 2.44.** Let \( X, X' \) be schemes locally of finite type over \( R. \) Let \( s, s' \) be \( k \)-valued points of \( X, X' \) and let \( f : X \rightarrow X' \) be morphism over \( R \) which maps \( s \) to \( s' \) and is flat at \( s. \) Let \( X' \) be the formal schematic image of the induced morphism \( f : \hat{X} \rightarrow \hat{X}'. \) Every irreducible component of \( X' \) is an irreducible component of \( \hat{X}'. \)

**Proof.** By the assumption of \( f \) the induced homomorphism \( O_{X',s'} \rightarrow O_{X,s} \) is flat. First we want to prove that the induced homomorphism \( \hat{O}_{X,s'} \rightarrow \hat{O}_{X,s} \) is also flat. Let \( n' \) be the maximal ideal of \( \hat{O}_{X',s'} \) and \( \hat{n}' \) its completion, which is the maximal ideal of \( \hat{O}_{X',s'} \). By the local criterion for flatness (see \([12, \text{Theorem 6.8}]\)) it is sufficient to show that \( \text{Tor}_1^{\hat{O}_{X',s'}}(\hat{O}_{X',s'}/\hat{n}', \hat{O}_{X,s}) = 0. \) Note that since \( \hat{n}' = m\hat{O}_{X',s'} \) we have \( \hat{O}_{X',s'}/\hat{n}' \cong O_{X',s'}/n' \otimes_{O_{X',s'}} \hat{O}_{X',s'}. \) Using this and the flatness of \( O_{X',s'} \rightarrow O_{X',s'} \) the Proposition 3.2 of \([27]\) on flat base change for \( \text{Tor} \) says

\[ \text{Tor}_1^{\hat{O}_{X',s'}}(\hat{O}_{X',s'}/\hat{n}', \hat{O}_{X,s}) = \text{Tor}_1^{O_{X',s'}}(O_{X',s'}/n', \hat{O}_{X,s}). \]

The second term of this equation is zero since the homomorphism \( O_{X',s'} \rightarrow O_{X,s} \) is flat, being the composition of the two flat homomorphisms \( O_{X',s'} \rightarrow \hat{O}_{X,s} \) and \( \hat{O}_{X,s} \rightarrow O_{X,s}. \) Thus \( \hat{O}_{X,s} \rightarrow O_{X,s} \) is also flat.

That the homomorphism \( O_{X',s'} \rightarrow O_{X,s} \) is flat implies by \([12, \text{Lemma 10.11}]\) that it has the going down property, that is for any prime ideal \( p \) of \( O_{X,s} \) and any prime ideal \( q \subset \hat{O}_{X,s} \cap p \) there exists a prime ideal \( q' \subset p \) of \( \hat{O}_{X,s} \) such that \( q = \hat{O}_{X',s'} \cap p'. \) By applying this to a minimal prime ideal \( p \) of \( \hat{O}_{X,s} \) one see that its pullback \( \hat{O}_{X',s'} \cap p \) is a minimal prime ideal in \( \hat{O}_{X',s'} \).

This means that the formal schematic image of any irreducible component of \( \hat{X} \) is an irreducible component of \( \hat{X}'. \) This implies the claim.

We will in particular apply the above to a smooth group scheme \( A \) over \( R \) with \( s \) the zero section of the special fiber. Then \( A \) is a formal group scheme, which, as a formal scheme, is isomorphic to \( \text{Spf}(R[[x_1, \ldots, x_n]]) \). Note that often one denotes by \( \hat{A} \) the completion of \( A \) along its zero section. This is a formal scheme over \( \text{Spec}(R) \), and \( \hat{A} \) as we define it is the base change of this formal scheme to \( \text{Spf}(R) \) along the morphism \( \text{Spf}(R) \rightarrow \text{Spec}(R) \) given by the identity homomorphism \( R \rightarrow R \).

If \( [p] : A \rightarrow \hat{A} \) is an epimorphism, then \( \hat{A} \) is a \( p \)-divisible group over \( \text{Spf}(R) \) (c.f. \([20, \text{Corollary 4.5}]\)). More precisely, it follow from \( \text{loc. cit.} \) that in this case \( \hat{A} \) is equal to the pullback to \( \text{Spf}(R) \) of the connected part \( A[p^\infty] \) of the \( p \)-divisible group \( A[p^\infty] \) of \( A \).

For a closed subscheme \( X \) of \( A \) containing the zero section, we denote by \( \hat{X} \) its completion along the zero section of the special fiber.

### 3 Special Subvarieties

For any group scheme \( A \) over a scheme \( S \) and any closed subscheme \( X \) of \( A \) we denote by \( \text{Stab}_A(X) \) the functor which associates to any scheme \( S' \) over \( S \) the set \( \{ a \in A(S') \mid a + X_{S'} = X_{S'} \} \) and acts on morphisms by pullbacks. In case \( S \) is a field or a valuation ring, which will be the only relevant cases for us, this functor is representable by a closed subscheme of \( A \), c.f. \([11, \text{Exp. VIII, Ex. 6.5(c)}] \).
For an extension $L \hookrightarrow L'$ of algebraically closed fields, recall the the notion of the $L'/L$-trace of an abelian variety $A$ over $L'$ (c.f. [5]): This is an abelian variety over $L$, which we will denote $\text{Tr}_{L'/L} A$, together with a homomorphism $\tau: (\text{Tr}_{L'/L} A)_{L'} \to A$ which satisfies the following universal property: For each abelian variety $B$ over $L$ together with a homomorphism $f: B_L \to A$, there exists a unique homomorphism $g: B \to \text{Tr}_{L'/L} A$ (defined over $L$) such that $f = \tau \circ g_{L'}$. For $L \subset L'$ algebraically closed this trace always exists (see [5] 6.2) and the map $\tau$ has finite kernel (see [5] 6.4]). Thus roughly speaking $\text{Tr}_{L'/L} A$ is the largest subobject of $A$ which can be defined over $L$. It is determined up to unique isomorphism and functorial in $A$.

Let $L$ be an algebraically closed field of characteristic $p > 0$ and $A$ a semiabelian variety over $L$.

**Definition 3.1.** We call a subvariety $X$ of $A$ special if $X$ is irreducible and there exist a semiabelian variety $B$ over $\bar{F}_p$, a subvariety $Y$ of $B$ over $\bar{F}_p$, a homomorphism $h: B_L \to A/\text{Stab}_A(X)$ with finite kernel and an element $a \in (A/\text{Stab}_A(X))(L)$ such that $X/\text{Stab}_A(X) = h(Y) + a$.

**Remark 3.2.** This notion of a special subvariety is equivalent to Hrushovski’s [18] notion of a special subvariety, as can be shown using Lemma 3.3 below.

**Lemma 3.3.** Let $B$ be a semiabelian variety over $L$, let $X \subset A$ an irreducible subvariety and let $f: A \to B$ a homomorphism. If $X$ is a special subvariety of $A$, then $f(X)$ is a special subvariety of $B$. If $f$ has finite kernel and $f(X)$ is a special subvariety of $B$, then $X$ is a special subvariety of $A$.

**Proof.** The homomorphism $f$ induces a homomorphism $\bar{f}: A/\text{Stab}_A(X) \to B/\text{Stab}_B(f(X))$.

If $X$ is special, then there exist a semiabelian variety $C$ over $\bar{F}_p$, a subvariety $Y$ of $C$ over $\bar{F}_p$, a homomorphism $h: C_L \to A/\text{Stab}_A(X)$ with finite kernel and an element $a \in (A/\text{Stab}_A(X))(L)$ such that $X/\text{Stab}_A(X) = h(Y) + a$. Let $D$ be the connected component of the identity of the kernel of $f \circ h$ equipped with the reduced scheme structure. This is a semiabelian subvariety of $B_L$. Thus it is defined over $\bar{F}_p$. The homomorphism $\bar{f} \circ h$ induces a homomorphism $h': C/D \to B/\text{Stab}_B(f(X))$ with finite kernel such that $f(X)/\text{Stab}_B(f(X)) = h'(Y)/D + \bar{f}(a)$. Thus $f(X)$ is special in $B$.

Now let $f$ have finite kernel and $f(X)$ be special in $B$. After replacing $B$ by the image of $f$, we may assume that $f$ is an isogeny. Then $\bar{f}$ is also an isogeny. Pick an isogeny $g: B/\text{Stab}_B(f(X)) \to A/\text{Stab}_A(X)$ such that $g \circ \bar{f} = [n]$ for some $n \in \mathbb{Z}^{\neq 0}$. There exist a semiabelian variety $C$ over $\bar{F}_p$, a subvariety $Y$ of $C$ over $\bar{F}_p$, a homomorphism $h: C_L \to B/\text{Stab}_X(f(X))$ with finite kernel and an element $b \in (B/\text{Stab}_B(f(X)))(L)$ such that $f(X)/\text{Stab}_B(f(X)) = h(Y) + b$. Then $[n](X/\text{Stab}_A(X)) = g(h(Y)) + g(b)$. Let $Y'$ be an irreducible component of $[n]^{-1}(Y)$. Then $g(h(Y')) + g(b)$ is an irreducible component of $[n]^{-1}([n](X/\text{Stab}_A(X)))$. Since the $n$-torsion points of $(A/\text{Stab}_A(X))(L)$ act transitively on the irreducible components of $[n]^{-1}([n](X/\text{Stab}_A(X)))$, it follows that $X/\text{Stab}_A(X)$ is a translate of $g(h(Y'))$. This shows that $X$ is special in $A$.

**Lemma 3.4.** Let $L'$ be an algebraically closed overfield of $L$ and $X \subset L$ a subvariety. Then $X$ is special in $A$ if and only if $X_{L'}$ is special in $A_{L'}$.

**Proof.** The “only if” direction follows directly from the definition of a special subvariety. For the other direction see the proof of [23] Lemma 1.2].

**Lemma 3.5.** Let $L_0$ be an arbitrary field, let $A_0$ be a semiabelian variety over $L_0$ and let $X_0 \subset A_0$ be an irreducible subvariety. Let $\Gamma \subset A_0(L_0)$ be a finitely generated subgroup such that $X_0(L_0) \cap \Gamma$ is Zariski dense in $X_0$. For any semiabelian variety $A_0'$ over $L_0$ which is isogenous to $A_0$, there exist a finitely generated subgroup $\Gamma'$ of $A_0'(L_0)$ and an irreducible subvariety $X_0'$ of $A_0'$ such that $X_0(L_0) \cap \Gamma'$ is Zariski dense in $X_0'$ and such that $X_0$ is special in $A_0$ if and only if $X_0'$ is special in $A_0$. 

\[15\]
Proof. Let \( f : A_0 \to A_0' \) be an isogeny. Take \( \Gamma' := f(\Gamma) \) and \( X_0' := f(X_0) \). Lemma 3.3 implies that \( \Gamma' \) and \( X_0' \) have the required properties.

In the following lemma, which gives an equivalent description of special subvarieties, we denote by \( \tau \) the canonical morphism \( \text{Tr}_{L/\bar{\mathbb{F}}_p} A \to A \).

**Lemma 3.6.** An irreducible subvariety \( X \subset A \) is special in \( A \) if and only if there exists a closed subvariety \( Y \subset \text{Tr}_{L/\bar{\mathbb{F}}_p} A \) defined over \( \bar{\mathbb{F}}_p \) and \( a \in A(L) \) such that \( X + a = \tau(Y) + \text{Stab}_A(X) \).

**Proof.** Because the formation of \( \text{Tr}_{L/\bar{\mathbb{F}}_p} A \) is functorial in \( A \), there is a commutative diagram

\[
\begin{array}{ccc}
\text{Tr}_{L/\bar{\mathbb{F}}_p} A & \xrightarrow{\tau} & A \\
\downarrow & & \downarrow \\
\text{Tr}_{L/\bar{\mathbb{F}}_p} (A/\text{Stab}_A(X)) & \xrightarrow{\tau'} & A/\text{Stab}_A(X)
\end{array}
\]

with both semiabelian varieties on the left as well as the morphism between them defined over \( \bar{\mathbb{F}}_p \) and with \( \tau \) and \( \tau' \) having finite kernel. The “if” direction follows directly from this. For the “only if” direction, we assume that \( X \) is special. Then there exist \( Y \subset B, h \) and \( a \) as in Definition 3.1. Since the homomorphism \( h \) factors through \( \tau' \), we may replace \( Y \) by its image in \( \text{Tr}_{L/\bar{\mathbb{F}}_p} (A/\text{Stab}_A(X)) \) and assume \( B = \text{Tr}_{L/\bar{\mathbb{F}}_p} (A/\text{Stab}_A(X)) \). Let \( Y' \) be the inverse image of \( Y \) in \( \text{Tr}_{L/\bar{\mathbb{F}}_p} A \). Note that \( Y' \) is defined over \( \bar{\mathbb{F}}_p \). If \( a' \in A(L) \) is a lift of \( a \) the identity \( \tau'(Y) = X/\text{Stab}_A(X) + a \) implies \( \tau(Y') + \text{Stab}_A(X) = X + a' \). \( \square \)

**Specialness Criteria**

**Theorem 3.7** (Pink-Rössler, see [23 Theorem 3.1]). Let \( \varphi : A \to A \) be an isogeny. Let \( X \subset A \) be an irreducible subvariety such that \( \varphi(X) = X + a \) for some \( a \in A(L) \). Then \( \varphi(\text{Stab}_A(X)) = \text{Stab}_A(X) \) and we denote the isogeny \( A/\text{Stab}_A(X) \to A/\text{Stab}_A(X) \) induced by \( \varphi \) by \( \varphi \).

There exist finitely many homomorphisms \( h_\alpha : A_\alpha \to A/\text{Stab}_A(X) \) for certain \( \alpha \in \mathbb{Q}_{\geq 0} \), where the \( A_\alpha \) are semiabelian varieties endowed with isogenies \( \varphi_\alpha : A_\alpha \to A_\alpha \) satisfying \( \varphi \circ h_\alpha = h_\alpha \circ \varphi_\alpha \) and irreducible subvarieties \( X_\alpha \subset A_\alpha \) satisfying \( \varphi_\alpha(X_\alpha) = X_\alpha + a_\alpha \) for some \( a_\alpha \in A_\alpha(L) \) such that:

- If \( \alpha = 0 \), then \( \varphi_\alpha \) is an automorphisms of finite order of \( A_\alpha \).
- If \( \alpha > 0 \), then there exist positive integers \( r \) and \( s \) such that \( \alpha = r/s \) and \( \varphi_\alpha^s = \text{Frob}_{p^r} \) for some model of \( A_\alpha \) over \( \bar{\mathbb{F}}_p \).
- The morphism

\[
h := \sum_\alpha h_\alpha : \prod_i A_\alpha \to A/\text{Stab}_A(X)
\]

has finite kernel and, for some point \( \bar{a} \in (A/\text{Stab}_A(X))(L) \),

\[
X/\text{Stab}_A(X) = \bar{a} + h(\prod_i X_\alpha).
\]

We will only need the following consequence of Theorem 3.7

**Corollary 3.8.** Let \( \varphi : A \to A \) an isogeny whose minimal polynomial does not have any complex roots which are roots of unity. Let \( X \subset A \) be an irreducible variety such that \( \varphi(X) = X + a \) for some \( a \in A(L) \). Then \( X \) is a special subvariety of \( A \).
Proof. The condition on the minimal polynomial of $\varphi$ ensures that $\varphi$ cannot act as an automorphism of finite order on any subquotient of $A$. Hence the term $A_0$ in Theorem 3.7 does not appear, and it follows using Lemma 3.3 that $X$ is special in $A$. \hfill $\square$

**Definition 3.9.** We call a polynomial $f \in \mathbb{Z}[t]$ good if it is monic, if $f(0) \neq 0$ and if no complex root of $f$ is a root of unity.

**Theorem 3.10.** Let $G \subset A_{\ell}(L)$ be a subgroup and $\Phi: G \to G$ an endomorphism such that there exists a good polynomial $f \in \mathbb{Z}[t]$ which annihilates $\Phi$.

Let $X \subset A$ be an irreducible subvariety. If there exists a subset $T$ of $G \cap X(L)$ which is Zariski dense in $X$ and which satisfies $\Phi(T) \subset T$, then $X$ is special in $A$.

**Proof.** Write $f(t) = t^n + \sum_{i=0}^{n-1} a_i t^i$ with $a_i \in \mathbb{Z}$. Let $\varphi$ be the endomorphism of $A^n$ defined by the matrix

$$
\begin{bmatrix}
0 & 0 & \cdots & 0 & -a_0 \\
1 & 0 & \cdots & 0 & -a_1 \\
0 & 1 & \ddots & & \vdots \\
0 & \ddots & 1 & 0 & \vdots \\
0 & \cdots & \cdots & \cdots & -a_{n-1}
\end{bmatrix},
$$

which satisfies $f(\varphi) = 0$ and $\varphi(a, \Phi(a), \ldots, \Phi^{n-1}(a)) = (\Phi(a), \Phi^2(a), \ldots, \Phi^n(a))$ for $a \in G$. Let $X'$ be the Zariski closure of the set $\{(x, \Phi(x), \ldots, \Phi^{n-1}(x)) \mid x \in T\}$ in $A^n$. The fact that $\Phi(T) \subset T$ implies that $\varphi(X') \subset X'$. Since $a_0 = f(0) \neq 0$, the above matrix is invertible over $\mathbb{Q}$ and hence $\varphi$ is an isogeny. Hence for each irreducible component $Z$ of $X'$, its image $\varphi(Z)$ is also an irreducible component of $X'$. Thus every irreducible component of $X'$ is invariant under some power of $\varphi$. Hence by the assumption on $f$ and Corollary 3.8 each irreducible component of $X'$ is special in $A^n$.

Let $\pi: A^n \to A$ be the projection to the first factor. The fact that $T$ is Zariski dense in $X$ implies $\pi(X') = X$. Since $X$ is irreducible, some irreducible component of $X'$ maps onto $X$ under $\pi$. Hence Lemma 3.3 implies that $X$ is special in $A$. \hfill $\square$

### 4 The General Setup

#### 4.1 Completely slope divisible $p$-divisible groups

First we collect some terminology and facts from [22].

Let $S$ be a scheme over $\overline{\mathbb{F}}_p$. Let $\text{Frob}: S \to S$ be the absolute Frobenius morphism $x \mapsto x^p$. For a scheme $G$ over $S$ and $s \geq 1$ we write $G^{(p^s)} = G \times_{S, \text{Frob}^s} S$. We denote by $F^s: G \to G^{(p^s)}$ the Frobenius morphism relative to $S$.

Let $L$ be an algebraically closed field of characteristic $p > 0$ and $\mathcal{G}$ a $p$-divisible group over $L$. For a rational number $\lambda \geq 0$, one calls $\mathcal{G}$ **isoclinic of slope** $\lambda$ if there exist integers $r \geq 0$ and $s \geq 0$ such that $\lambda = r/s$ and a $p$-divisible group $\mathcal{G}'$ over $L$ which is isogenous to $\mathcal{G}$ and for which there exists an isomorphism $\psi: \mathcal{G}(p^r) \to \mathcal{G}$ making the following diagram commute (c.f. [22 Section 1]):

$$
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{F^r} & \mathcal{G}(p^r) \\
\downarrow{\psi} & & \downarrow{\psi} \\
\mathcal{G} & \xrightarrow{\phi} & \mathcal{G}
\end{array}
$$

Every $p$-divisible group $\mathcal{G}$ over $L$ is isogenous to a direct sum of isoclinic $p$-divisible group and the slopes appearing in such a direct sum determine $\mathcal{G}$ up to isogeny. They are called the slopes of $\mathcal{G}$ and are assembled into the Newton polyon of $\mathcal{G}$ (see e.g. [10 IV.5]).
Definition 4.1 (c.f. [22 Definition 1.2]). Let $s \geq 1$ and $r_1, \ldots, r_m$ be integers such that $s \geq r_1 > r_2 > \ldots > r_m \geq 0$. A $p$-divisible group $G$ over a scheme $S$ is said to be completely slope divisible with respect to these integers if $G$ has a filtration by $p$-divisible subgroups

$$0 = G_0 \subset G_1 \subset \ldots \subset G_m = G$$

such that the following properties hold:

(i) For $i = 1, \ldots, m$ the kernel of $[p^r]: G_i \to G_i$ is contained in the kernel of $F^s: G_i \to G_i^{(p^r)}$.

(ii) For $i = 1, \ldots, m$ the kernel of $[p^r]: G_i/G_{i-1} \to G_i/G_{i-1}$ is equal to the kernel of $F^s: G_i/G_{i-1} \to (G_i/G_{i-1})^{(p^r)}$.

A $p$-divisible group $G$ is completely slope divisible if there exist integers $s \geq r_1 > r_2 > \ldots > r_m \geq 0$ such that $G$ is completely slope divisible with respect to these integers.

Remark 4.2. Let $G$ be a $p$-divisible group which is completely slope divisible with respect to $s \geq r_1 > r_2 > \ldots > r_m \geq 0$. Note that condition $(ii)$ is equivalent to the existence of isomorphisms $(G_i/G_{i+1})^{(p^r)} \sim G_i/G_{i+1}$ such that the following diagram commutes:

$$\begin{array}{ccc}
G_i/G_{i+1} & \stackrel{F^s}{\longrightarrow} & (G_i/G_{i+1})^{(p^r)} \\
[p^r] \downarrow & & \downarrow \cong \\
G_i/G_{i+1} & & 
\end{array}$$

Thus all geometric fibers of the subquotients $G_i/G_{i+1}$ are isoclinic of slope $r_i/s$ and in particular the Newton polygon of $G$ is constant on $S$.

Remark 4.3. Let $G$ be a $p$-divisible group which is completely slope divisible with respect to $s \geq r_1 > r_2 > \ldots > r_m \geq 0$. By the remark after Definition 1.2 of [22] there exists a unique filtration $(G_i)_{i=0,\ldots,m}$ satisfying the conditions above for the given integers $s \geq r_1 > r_2 > \ldots > r_m \geq 0$.

Theorem 4.4 ([22 Theorem 2.1]). Let $G$ be a $p$-divisible group over a integral normal Noetherian scheme $S$ with constant Newton polygon. There exists a completely slope $p$-divisible group over $S$ which is isogenous to $G$.

We call a scheme $S$ of characteristic $p > 0$ perfect if for each open set $U$ of $S$ the endomorphism $x \mapsto x^p$ of the ring $O_S(U)$ is an isomorphism.

Proposition 4.5 (Oort-Zink). Let $G$ be a $p$-divisible group over a perfect scheme $S$ which is completely slope divisible with respect to integers $s \geq r_1 > r_2 > \ldots > r_m \geq 0$. Let $(G_i)_{i}$ be a filtration as in Definition 4.1 with respect to these integers. The filtration $(G_i)_{i}$ splits uniquely, that is there are unique sections $G_i/G_{i+1} \to G_i$ of the quotient maps $G_i \to G_i/G_{i+1}$.

Proof. This is [22 Proposition 1.3]. Although the uniqueness of the splittings is not part of the statement there, it is shown in the proof given there.

Proposition 4.6. A $p$-divisible group $G$ over a scheme $S$ is completely slope divisible if it is so fpqc-locally on $S$.

Proof. Let $S' \to S$ be an fpqc covering of $S$ such that $G_{S'}$ is completely slope divisible with respect to integers $s \geq r_1 > r_2 > \ldots > r_m \geq 0$. By Remark 4.3 there are unique subgroups $G'_i$ of $G_{S'}$ satisfying the conditions of Definition 4.1. The pullbacks of $(G'_i)_{i}$ along the two morphisms $S' \times_S S' \to S'$ both satisfy the conditions of Definition 4.1 over
has the required property. Hence by fpqc descent the subgroups \(G'_i\) of \(\mathcal{G}_S'\) arise by base change from subgroups \(G_i\) of \(\mathcal{G}\). By fpqc descent the conditions of Definition [4.1] hold for \((\mathcal{G})\) if they hold fpqc-locally on \(S\). Thus \(\mathcal{G}\) is completely slope divisible.

**Proposition 4.7.** Let \(\mathcal{G}\) be a \(p\)-divisible group over a scheme \(S\) which is completely slope divisible. Let \(\mathcal{H}\) be a \(p\)-divisible subgroup of \(\mathcal{G}\). Then \(\mathcal{H}\) and \(\mathcal{G}/\mathcal{H}\) are completely slope divisible.

**Proof.** Let \((\mathcal{G}_i)\) be a filtration as in Definition [4.1. Let \(\mathcal{H}_i := \mathcal{H} \cap \mathcal{G}_i\). Then it follows by a direct verification that \((\mathcal{H}_i)\) and \((\mathcal{G}_i/\mathcal{H}_i)\) have the required properties.

**Lemma 4.8** (see [22, Corollary 1.10]). Let \(G \to S\) be a finite flat group scheme over a connected base scheme \(S\). Let \(\psi: G \xrightarrow{\sim} G^{(p^r)}\) be an isomorphism. Then there exists a finite étale morphism \(T \to S\) and a morphism \(T \to \text{Spec}(\mathbb{F}_{p^r})\) such that \(G_T\) is obtained by base change from a finite group scheme \(H\) over \(\text{Spec}(\mathbb{F}_{p^r})\)

\[H \times_{\text{Spec}(\mathbb{F}_{p^r})} T \xrightarrow{\sim} G_T\]

and \(\psi\) is induced from the identity on \(H\).

The argument in the proof of the following proposition is taken from the proof of Proposition 3.1 of [22].

**Proposition 4.9.** Let \(R\) be a perfect strictly henselian local ring over \(\mathbb{F}_p\) and \(\mathcal{G}\) a completely slope divisible \(p\)-divisible group over \(R\). Then there exists a \(p\)-divisible group \(\mathcal{G}_0\) over \(\mathbb{F}_p\) such that \(\mathcal{G}_0\) is isomorphic to \(\mathcal{G}\). In case that \(\mathcal{G}\) has a single slope, it suffices that \(R\) be strictly henselian.

**Proof.** In case \(\mathcal{G}\) has multiple slopes, by Proposition [4.5] we can write \(\mathcal{G}\) as a direct sum of completely slope divisible groups having a single slope. Thus it suffices to treat this case. Then there exist \(s \geq r \geq 0\) and an isomorphism \(\psi: \mathcal{G}^{(p^r)} \to \mathcal{G}\) such that \(\psi \circ F^s = [p^r]\).

For \(n \geq 0\) denote by \(\mathcal{G}(n)\) the kernel of \([p^n]: \mathcal{G} \to \mathcal{G}\). Applying Lemma [4.8] to \(\mathcal{G}(n)\) and \(\psi^{-1}\) we obtain finite group schemes \(\mathcal{G}_0(n)\) over \(\text{Spec}(\mathbb{F}_{p^r})\) and isomorphisms

\[\mathcal{G}(n) \cong \mathcal{G}_0(n) \times_{\text{Spec}(\mathbb{F}_{p^r})} R.\]

The inductive limit of the group schemes \(\mathcal{G}_0(n)\) is a \(p\)-divisible group \(\mathcal{G}_0\) over \(\mathbb{F}_{p^r}\) which has the required property.

The following result is probably not new, but we could not find a reference.

**Proposition 4.10.** Let \(k\) be an algebraically closed field of positive characteristic. Let \(\mathcal{G}\) and \(\mathcal{H}\) be \(p\)-divisible groups over \(k\). For any integral scheme \(S\) over \(k\) the base change map

\[\text{Hom}_k(\mathcal{G}, \mathcal{H}) \to \text{Hom}_S(\mathcal{G}_S, \mathcal{H}_S)\]

is an isomorphism of \(\mathbb{Z}_p\)-modules.

**Proof.** Let \(k'\) be the function field of \(S\). First we claim that for any finite flat group schemes \(G\) and \(H\) over \(S\) the natural map \(\text{Hom}_S(G, H) \to \text{Hom}_{k'}(G_{k'}, H_{k'})\) is injective. We may assume that \(S\) is affine, say \(S = \text{Spec}(R)\). Then \(G\) and \(H\) are the spectrum of finite flat \(R\)-algebra \(A_G\) and \(A_H\). Since these are flat over \(R\), the homomorphisms \(A_G \to A_G \otimes_R k'\) and \(A_H \to A_G \otimes_R k'\) are injective. Hence any homomorphism \(A_H \to A_G\) of \(R\) algebras is determined by its generic fiber \(A_H \otimes_R k' \to A_G \otimes k'\). This shows that \(\text{Hom}_S(G, H) \to \text{Hom}_{k'}(G_{k'}, H_{k'})\) is injective. By applying this to group schemes \(\mathcal{G}[p^n]\) and \(\mathcal{H}[p^n]\) for \(n \geq 0\) one gets that the homomorphism \(\text{Hom}_S(\mathcal{G}_S, \mathcal{H}_S) \to \text{Hom}_{k'}(\mathcal{G}_{k'}, \mathcal{H}_{k'})\) is an isomorphism.
is injective. Thus we may assume that $S = \text{Spec}(k')$. We may also assume that $k'$ is algebraically closed.

We use the theory of Dieudonné modules. Denote by $W(k)$ (resp. $W(k')$) the ring of Witt vectors of $k$ (resp. $k'$), by $\sigma$ the lift of Frobenius to these rings and by $B(k)$ (resp. $B(k')$) their quotient field. Let $M(\mathcal{G})$ and $M(\mathcal{H})$ be the contravariant Dieudonné modules associated to $\mathcal{G}$ and $\mathcal{H}$. They are free $W(k)$-modules endowed with a $\sigma$-linear self-map $F$ and a $\sigma^{-1}$-linear self-map $V$.

A homomorphism $\mathcal{G}_k' \to \mathcal{H}_k'$ corresponds to a $W(k)$-linear homomorphism $M(\mathcal{H}_k') = M(\mathcal{H}) \otimes_{W(k)} W(k') \to M(\mathcal{G}_k') = M(\mathcal{G}) \otimes_{W(k)} W(k')$ compatible with $V$ and $F$. We need to show that any such homomorphism arises from a homomorphism $M(\mathcal{H}) \to M(\mathcal{G})$. Since we are dealing with free $W(k)$-modules, it suffices to prove that the induced homomorphism $M(\mathcal{H}) \otimes_{W(k)} B(k') \to M(\mathcal{G}) \otimes_{W(k)} B(k')$ arises from a homomorphism $M(\mathcal{H}) \otimes_{W(k)} B(k) \to M(\mathcal{G}) \otimes_{W(k)} B(k)$. The $(B(k))$-vector spaces $M(\mathcal{H}) \otimes_{W(k)} B(k')$ and $M(\mathcal{G}) \otimes_{W(k)} B(k')$ together with the $\sigma$-linear endomorphism induced by $F$ are what is called an $F$-space in [10, Chapter IV]. By the theorem in [10, Section IV.4], each such $F$-space is a direct sum of certain simple $F$-spaces denoted $E^\lambda_k$ for $\lambda \in \mathbb{Q}^\geq$. Furthermore, by a proposition in [10, Section IV.3], if $\lambda \neq \lambda'$, any homomorphism $E^\lambda_k \to E^{\lambda'}_k$ of $F$-spaces is zero. Hence it suffices to prove that any endomorphism of $E^\lambda_k \otimes_{B(k)} B(k')$ of $F$-spaces arises from an endomorphism of $E^\lambda_k$. This follows from the description of such endomorphisms given by a proposition in [10, Section IV.3].

**Proposition 4.11.** Let $R$ be a discrete valuation ring of characteristic $p$ with perfect residue field $k$ and perfection $R \mapsto R^{\text{per}}$. Let $\mathcal{G}$ be a completely slope divisible group over $R$. There exists a unique isomorphism $\mathcal{G}_{R^{\text{per}}} \cong (\mathcal{G}_k)_{R^{\text{per}}}$ which is the identity in the fiber over $k$. In case $\mathcal{G}$ has a single slope, this isomorphism is already defined over $R$.

**Proof.** Let $R^{\text{per}} \hookrightarrow R^{\text{psh}}$ be a strict henselization of $R^{\text{per}}$ and $\bar{k}$ the residue field of $R^{\text{psh}}$. Note that $k \mapsto \bar{k}$ is an algebraic closure of $k$. Let $R^{\text{per}} \hookrightarrow R'$ be a finite etale extension of $R^{\text{per}}$. Using the fact that the relative Frobenius morphism of $R'$ over $R^{\text{per}}$ is an isomorphism one sees that $R'$ is again perfect. Hence $R^{\text{psh}}$ is perfect. Hence by Proposition 4.9 there exists a $p$-divisible group $\mathcal{G}'$ over $\bar{k}$ such that $\mathcal{G}_{R^{\text{psh}}} \cong \mathcal{G}'_{R^{\text{psh}}}$. By taking the special fiber of this isomorphism, we get $\mathcal{G}_k \cong \mathcal{G}'$, so that we may take $\mathcal{G}' = \mathcal{G}_k$. Then Proposition 4.10 implies that there exists a unique isomorphism $\psi: (\mathcal{G}_k)_{R^{\text{psh}}} \cong \mathcal{G}_{R^{\text{psh}}}$ which is the identity in the special fiber. For any $\sigma \in \text{Aut}(R^{\text{psh}}/R^{\text{per}})$, the conjugate of $\psi$ by $\sigma$ is again the identity in the special fiber and thus is equal to $\psi$. Thus $\psi$ is defined over $R^{\text{per}}$ by Galois descent.

In case $\mathcal{G}$ has a single slope, one does not need to pass to $R^{\text{per}}$ to split the slope filtration. Thus with the same argument as above one obtains $\psi$ over a strict henselization of $R$ and sees that it is defined over $R$ by Galois descent.

Now let $R$ be a discrete valuation ring as in Section 2. The results in this subsection are formulated for $p$-divisible groups over $\text{Spec}(R)$, however below we will work with $p$-divisible groups over $\text{Spf}(R)$. Thus we will need the following:

**Proposition 4.12 ([20, Lemma 4.16]).** Let $R'$ be the valuation ring of a complete overfield $K' \subset \hat{K}$ of $K$. The base change functor $\mathcal{G} \mapsto \mathcal{G}_{\text{Spf}(R')}$ from the category of $p$-divisible groups over $\text{Spec}(R')$ to the category of $p$-divisible groups over $\text{Spf}(R')$ is an equivalence.

Accordingly we define:

**Definition 4.13.** Let $R'$ be the valuation ring of a complete overfield $K' \subset \hat{K}$ of $K$. A $p$-divisible group over $\text{Spf}(R')$ is completely slope divisible if and only if the corresponding group over $\text{Spec}(R')$ is completely slope divisible.
4.2 Nice Semiabelian Schemes

Let $K$ be a local field of characteristic $p > 0$ with valuation ring $R$. Let $\bar{K}$ be an algebraic closure of $K$ and $\bar{R}$ the valuation ring of $\bar{K}$. Let $R^{\text{per}} \subset \bar{R}$ be the perfection of $R$. Denote by $k$ (resp. $\bar{k}$) the residue field of $R$ (resp. $\bar{R}$).

**Definition 4.14.** We call a semiabelian scheme $A$ over $\text{Spec}(R)$ nice if $A$ is an extension of an abelian scheme by a torus over $\text{Spec}(R)$ and the $p$-divisible group $\hat{A}$ over $\text{Spf}(R)$ is completely slope divisible.

Let $A$ be a nice semiabelian scheme over $R$.

**Lemma 4.15.** Let $B \subset A$ be a semiabelian subgroup scheme. Then $B$ and $A/B$ are nice.

**Proof.** The ranks of the toric parts of $B$ and $A/B$ are constant since they have constant sum and can only go up upon specialization. Thus by [13, Corollary 2.11] both $B$ and $A/B$ are extensions of an abelian scheme by a torus over $S$. Proposition 4.17 shows that the formal completions of both group schemes are again completely slope divisible. \(\square\)

**Construction 4.16.** We construct an isogeny $F_A : \hat{A}_{\text{Spf}(R^{\text{per}})} \to \hat{A}_{\text{Spf}(R^{\text{per}})}$ as follows: Propositions 4.11 and 4.12 yield a unique isomorphism $(\hat{A}_k)_{\text{Spf}(R^{\text{per}})} \cong \hat{A}_{\text{Spf}(R^{\text{per}})}$ which is the identity on the special fiber. The $p$-divisible group $\hat{A}_k$, being defined over the finite field $k$, has a Frobenius endomorphism with respect to $k$. Transferring the base change of this Frobenius endomorphism to $\text{Spf}(R^{\text{per}})$ to an endomorphism of $\hat{A}_{\text{Spf}(R^{\text{per}})}$ via the above isomorphism yields $F_A$.

The following summarizes the relevant properties of $F_A$:

**Proposition 4.17.** (i) There exists a good polynomial which annihilates $F_A$.

(ii) The endomorphism $F_A$ is the Frobenius endomorphism with respect to a suitable model of $\hat{A}_{\text{Spf}(R^{\text{per}})}$ over the finite field $k$.

(iii) Let $B$ be another nice semiabelian scheme over $R$. For any homomorphism $f : A \to B$, the induced homomorphism $\hat{f}_R : \hat{A}_{\text{Spf}(R^{\text{per}})} \to \hat{B}_{\text{Spf}(R^{\text{per}})}$ satisfies $F_B \circ \hat{f}_{\text{Spf}(R^{\text{per}})} = \hat{f}_{\text{Spf}(R^{\text{per}})} \circ F_A$.

(iv) In case $A$ has a model over $k$, the endomorphism $F_A$ is the one induced by the Frobenius endomorphism of such a model.

(v) If one replaces $R$ by a finite extension $R'$ contained in $R$, then $F_A$ is replaced by $F_A^N$, where $N$ is the degree of the extension of the residue fields of $R$ and $R'$.

**Proof.** (i) By the construction of $F_A$ it suffices to show that there exists a good polynomial which annihilates the Frobenius endomorphism of $\hat{A}_k$. This follows from the Riemann hypothesis for abelian varieties, see for example [13, Fact 3.1].

(ii) This follows directly from the construction.

(iii) Pick isomorphisms $A_{R^{\text{per}}} \cong (A_k)_{R^{\text{per}}}$ and $B_{R^{\text{per}}} \cong (B_k)_{R^{\text{per}}}$ as in Construction 4.16. Under these identifications by Proposition 4.10 the homomorphism $\hat{f}_{R^{\text{per}}} : \hat{A}_{R^{\text{per}}} \to \hat{B}_{R^{\text{per}}}$ arises by base change from its special fiber $f_k : A_k \to B_k$. Since the latter is defined over $k$, it is compatible with the Frobenius endomorphisms of $\hat{B}_k$ and $\hat{A}_k$. This implies $F_B \circ \hat{f}_{R^{\text{per}}} = \hat{f}_{R^{\text{per}}} \circ F_A$.

(iv) This follows directly from the construction.

(v) This follows directly from the construction. \(\square\)

**Theorem 4.18.** Let $X \subset A_K$ be an irreducible subvariety and $X$ its schematic closure in $A$. Then the following are equivalent:
4.3 Choice of a nice valuation

Let \( k \) be an algebraically closed field equipped with a non-archimedean valuation \( v \) with valuation ring \( R \). Let \( A \) be a semiabelian scheme over \( R \) which is an extension of an abelian scheme over \( R \) by a torus over \( R \). For \( a \in A(L) \), if \( na \in A(R) \) for some \( n \in \mathbb{Z} \neq 0 \), then \( a \in A(R) \).

Proof. Let

\[
\begin{array}{ccc}
0 & \longrightarrow & T \\
& \longrightarrow & A \\
& \longrightarrow & B \\
& \longrightarrow & 0
\end{array}
\]

be exact with \( T \) a torus over \( R \) and \( B \) an abelian scheme over \( R \). Then \( \pi(a) \in B(L) = B(R) \).

Since \( R \) is strictly Henselian the flat cohomology group \( H^1(R, T) \) is zero. Thus the point \( \pi(a) \in B(R) \) lifts to a point \( a' \in A(R) \). Since \( R \) is strictly Henselian, the torus \( T \) is split. Thus for \( t = a - a' \in T(L) \) we have \( t^n \in (R^*)^r \) which implies that \( t \in (R^*)^r \). Hence \( a \) lies in \( A(R) \).  

Proposition 4.20. Let \( L_0 \) be a field which is finitely generated over \( \mathbb{F}_p \) and let \( A \) be a semiabelian variety over \( L_0 \). There exists an embedding of \( L_0 \) into a local field \( K \) such

(i) The subvariety \( X_\mathbb{A} \) is special in \( A_\mathbb{A} \).
(ii) There exist a finite field extension \( \bar{K} \subset K \) with valuation ring \( \bar{R} \), \( x \in X(\bar{K}) \) and \( n \geq 1 \) such that \( F^n_\mathbb{A}(X_\mathbb{A} - x)_{\text{Spf}(\mathbb{F}_p)} \subset (X_\mathbb{A} - x)_{\text{Spf}(\mathbb{F}_p)} \).
(iii) There exist \( x \in X(\bar{K}) \), \( n \geq 1 \) and subset \( T \subset \hat{A}(\bar{R}) \cap (X - x)(\bar{K}) \) which is Zariski dense in \( X_\mathbb{A} - x \) and which satisfies \( F^n_\mathbb{A}(T) \subset T \).

Proof. (i) \( \Rightarrow \) (ii): Using Proposition 4.17 (iii) we see that if (ii) holds for \( X/\text{Stab}(X) \subset A_\mathbb{A}/\text{Stab}(X) \), then it holds for \( X \subset A_\mathbb{A} \). Hence we may assume that \( \text{Stab}(X) = 0 \). Then there exist a semiabelian variety \( B \) defined over a finite field \( k' \) containing \( k \), a subvariety \( Y \) of \( B \), a homomorphism \( h: B_R \rightarrow A_R \) with finite kernel and \( a \in A(\bar{K}) \) such that \( X_\mathbb{A} = h(Y_\mathbb{A}) + a \). Note that it suffices to prove (ii) after replacing \( K \) by a finite field extension contained in \( \bar{K} \). After doing so we may assume \( a \in A(K) \) and \( k = k' \). After suitably translating we may assume that \( 0 \in Y(k') \). Then \( a \in X(K) \), so that after translating \( X \) by \( -a \) we may assume \( a = 0 \).

As \( Y \) is defined over \( k' \), the set \( h(Y(k')) \subset X(\bar{K}) \) is Zariski dense in \( X(\bar{K}) \). Thus there exists \( y \in Y(\bar{K}) \) such that \( \mathcal{X} \) is smooth at \( h(x) \) and such that \( h \) is flat at \( y \). After possibly replacing \( k' \) and \( k \) by finite field extensions we may assume that \( y \in Y(k') \) and \( h(y) \in X(K) \). Then after replacing \( Y \) by \( Y - y \) and \( X \) by \( X - h(x) \) we may assume that \( \mathcal{X} \) is smooth over \( \bar{R} \) at 0. By Proposition 2.7 the homomorphism \( h \) extends to a homomorphism \( h: B_R \rightarrow A_R \). After replacing \( K \) by a finite field extension contained in \( \bar{K} \) we may assume that \( h \) is defined over \( R \). By Proposition 2.42 the formal scheme \( \hat{X} \) is isomorphic to \( \text{Spf}(R[[x_1, \ldots, x_n]]) \) for some \( n \) and hence is irreducible. Thus it follows from Proposition 2.44 that \( \hat{X} \) is the formal schematic image of \( Y_{\text{Spf}(R)} \) under \( h \).

We have \( F_\mathbb{A} \circ \hat{h} = \hat{h} \circ F_{B_R} \) by Proposition 4.17. This together with the fact that \( \hat{Y}_{\text{Spf}(R)} \) is invariant under a suitable power of \( F_{B_R} \) implies that \( \hat{X} = h(Y_{\text{Spf}(R)}) \) is invariant under a suitable power of \( F_\mathbb{A} \). This shows (ii).

(ii) \( \Rightarrow \) (iii): By Proposition 2.43 set \( T := \hat{A}(\bar{R}) \cap (X_\mathbb{A} - x)(\bar{K}) \) is Zariski dense in \( X_\mathbb{A} - x \).

(iii) \( \Rightarrow \) (i): Using Proposition 4.17 (i), Theorem 3.10 applied to \( G = \hat{A}(\bar{R}) \) and \( \Phi = F^n_\mathbb{A} \) shows that \( X_\mathbb{A} - x \) is special in \( A_\mathbb{A} \).  

\[ \square \]
that the semiabelian variety \( A_K \) extends to a semiabelian scheme \( A \) over the valuation ring \( R \) which is isogenous to a nice semiabelian scheme over \( R \).

If we are given a finitely generated subgroup \( \Gamma \subset A(L_0) \) (resp. a finite rank subgroup \( \Gamma' \subset A(L_0^{\text{per}}) \)) we can pick \( v \) such that \( \Gamma \subset A(R) \) (resp. such that \( \Gamma' \subset A(R^{\text{per}}) \), where \( R^{\text{per}} \) denotes the valuation ring of the unique extension of \( v \) to \( L_0^{\text{per}} \)).

**Proof.** There exists a ring \( R_0 \subset L_0 \) with quotient field \( L_0 \) which is finitely generated over \( \mathbb{F}_p \) such that \( A \) extends to a semiabelian scheme \( A \) over \( R_0 \). Since the Newton polygon of \( \hat{A} \) is generically constant, after localizing \( R_0 \) we may assume that \( \hat{A} \) has constant Newton polygon. Since the rank of the toric part of \( A \) is generically constant, after further localization we may assume that this rank is constant. Then \( A \) is globally an extension of an abelian scheme by a torus by [13, Corollary 2.11].

If we are given a finitely generated subgroup \( \Gamma \) as above, after further localization we may assume that a finite generating set of \( \Gamma \), and thus all of \( \Gamma \), is contained in \( A \) by [13, Corollary 2.11].

If we are given a finite rank subgroup \( \Gamma' \) as above, after further localization we may assume that \( \Gamma' \) consists of divison points of \( \Gamma \). Then as before after further localization we may assume that \( \Gamma' \subset A(R_0) \).

By [23, Lemma 3.1] there exists an embedding \( R_0 \hookrightarrow R := \mathbb{F}_q[[t]] \) for a suitable power \( q \) of \( p \). We pick such an embedding and let \( K := \mathbb{F}_q((t)) \). By Theorem 4.4, the \( p \)-divisible group \( \hat{A}_{\text{Spf}(R)} \) is isogenous to a completely slope divisible group \( \mathcal{G} \) over \( R \). Since an isogeny \( \hat{A}_{\text{Spf}(R)} \to \mathcal{G} \) is given by the quotient by a finite group scheme, this shows that we can find a nice semiabelian scheme \( A' \) over \( R \) which is isogenous to \( A_{\text{Spf}(R)} \). In case we are given \( \Gamma' \) as above, Lemma 4.19 ensures \( \Gamma' \subset A(R^{\text{per}}) \). Thus the embedding has the required properties. \( \square \)

## 5 Proof of Mordell-Lang for finitely generated groups

### 5.1 A formal Mordell-Lang theorem

Let \( R \) be the valuation ring of a local field \( K \) of characteristic \( p > 0 \) and let \( m \) be its maximal ideal. Let \( \bar{R} \) be the valuation ring of an algebraic closure \( \bar{K} \) of \( K \) and let \( \bar{m} \) be the maximal ideal of \( \bar{R} \). Let \( \mathcal{G} \) be a formal group over \( \text{Spf}(\bar{R}) \) which as a formal scheme is isomorphic to \( \text{Spf}(\bar{R}[[x_1, \ldots, x_n]]) \). From an isomorphism \( \mathcal{G} \cong \text{Spf}(\bar{R}[[x_1, \ldots, x_n]]) \) one gets a bijection \( \mathcal{G}(\bar{R}) \cong \bar{m}^{\bar{m}} \) as in Remark 2.15. This endows \( \mathcal{G}(\bar{R}) \) with a valuation topology which is independent of the chosen isomorphism. The fact that \( \prod(p) : \mathcal{G} \to \mathcal{G} \) acts by zero on the tangent space of \( \mathcal{G} \) implies that for all \( g \in \mathcal{G}(\bar{R}) \) the sequence \( (p^n g)_n \geq 0 \) converges to zero with respect to the valuation topology. This implies that the \( \mathbb{Z} \)-module structure on \( \mathcal{G}(\bar{R}) \) can be uniquely extended to \( \mathbb{Z}_p \)-module structure which is continuous with respect to the valuation topology.

First we prove the following Mordell-Lang statement for formal schemes in positive characteristic:

**Theorem 5.1.** Let \( K \) be a local field of characteristic \( p \) with valuation ring \( R \) and residue field \( k \). Let \( \mathcal{G} \) be a formal group over \( k \) which as a formal scheme is isomorphic to \( \text{Spf}(k[[x_1, \ldots, x_n]]) \). Let \( \bar{\mathcal{G}} \subset \mathcal{G}(\text{Spf}(\bar{R}^{\text{per}})) \) be a finite rank \( \mathbb{Z}_p \)-module. Let \( \mathcal{X} \subset \mathcal{G}(\text{Spf}(\bar{R}^{\text{per}})) \) be a closed formal subscheme.

If \( \mathcal{X}(\bar{R}^{\text{per}}) \cap \bar{\mathcal{G}} \) is formal-schematically dense in \( \mathcal{X} \), then there exist a finite field extension \( K' \) of \( K \) with valuation ring \( R' \), closed formal substructures \( \mathcal{X}_1, \ldots, \mathcal{X}_m \) of \( \mathcal{G}(\text{Spf}(R')) \) and elements \( \gamma_1, \ldots, \gamma_m \in \bar{\mathcal{G}} \) such that \( \mathcal{X}_j + \gamma_j \) is defined over the residue field of \( R' \) and such that \( \mathcal{X}(\text{Spf}(R'^{\text{per}})) = \cup_j (\mathcal{X}_j)\text{Spf}(R'^{\text{per}}) \).

For the proof of Theorem 5.1 we need the following lemma:
Lemma 5.2. Let $K$ be a local field of characteristic $p$ with valuation ring $R$ and residue field $k$ with $q$ elements. For $i \geq 0$ let $K^{q^i}$ be the field consisting of $q^i$-th powers of elements of $K$ and $R^{q^i}$ the valuation ring of $K^{q^i}$. Let $\mathcal{G}$ be a formal group scheme over $k$. Then $q^i\mathcal{G}(R) \subset \mathcal{G}(R^{q^i})$.

Proof. Let $F: \mathcal{G} \to \mathcal{G}$ be the Frobenius endomorphism of $\mathcal{G}$ with respect to $k$. By [1] Section VII A.1 there exists a “Verschiebung” endomorphism $V: \mathcal{G} \to \mathcal{G}$ such that $[q] = F \circ V = V \circ F$. Using the fact that $F^i(G(R)) \subset G(R^{q^i})$ this implies the claim.

Proof of Theorem 5.3. Since $\widehat{\Gamma}$ is a finite rank $\mathbb{Z}_p$-module, after possibly replacing $K$ by a finite extension we may assume that $\widehat{\Gamma} \subset \mathcal{G}(R)$. Since then $\mathcal{G}^a(R^{q^i}) \cap \widehat{\Gamma} \subset \mathcal{G}(R)$, Proposition 2.31 implies that $\mathcal{G}^a$ is defined over $R$. Let $\mathcal{G}^a_1, \ldots, \mathcal{G}^a_m$ be the irreducible components of $\mathcal{G}^a$. Then $\mathcal{G}^a(R) \cap \widehat{\Gamma}$ is formally schematically dense in $\mathcal{G}^a_i$ for each $i$. It suffices to prove the claim for each $\mathcal{G}^a_i$, that is we may assume that $\mathcal{G}^a$ is irreducible.

Since the group $\Gamma/\gamma$ is finite for all $i \geq 0$ and since $\mathcal{G}^a$ is irreducible, it follows using Lemma 2.25 that we can choose $(\gamma_i)_{i \geq 0} \in \Gamma^2$ such that $p^i\Gamma \cap (\mathcal{G}^a + \gamma_i)(R)$ is formally schematically dense in $\mathcal{G}^a + \gamma_i$ and such that $\gamma_{i+1} \equiv \gamma_i \pmod{p^i\Gamma}$ for all $i \geq 0$. Since the finite rank $\mathbb{Z}_p$-module $\Gamma$ is complete for the $p$-adic topology, there exists $\gamma \in \Gamma$ such that $\gamma \equiv \gamma_i \pmod{p^i\Gamma}$ for $i \geq 0$. Then $\mathcal{G}^a + \gamma$ is the formal schematic closure of $p^i\Gamma$ for all $i \geq 0$. Let $q$ be the number of elements of $k$. For $i \geq 0$ let $K^{q^i}$ be the field consisting of $q^i$-th powers of elements of $K$ and $R^{q^i}$ the valuation ring of $K^{q^i}$. Since $q^i\Gamma \subset \mathcal{G}(R^{q^i})$ by Lemma 5.2 Proposition 2.31 implies that $\mathcal{G}^a + \gamma$ is defined over $R^{q^i}$. Thus Lemma 2.39 implies that $\mathcal{G}^a + \gamma$ is defined over $k$.

5.2 Proof of Mordell-Lang for finitely generated groups

Using Theorem 5.1 we can now give an algebraic proof of Theorem 1.1

Theorem 5.3 (Hrushovski). Let $L$ be an algebraically closed field of positive characteristic. Let $A$ be a semiabelian variety over $L$, let $X \subset A$ an irreducible subvariety and $\Gamma \subset A(L)$ a finitely generated subgroup. If $X(L) \cap \Gamma$ is Zariski dense in $X$, then $X$ is a special subvariety of $A$.

Proof. Let $L_0 \subset L$ be a field which is finitely generated over $\mathbb{F}_p$, such that $A$ arises by base change from an abelian variety $A_0$ over $L_0$, such that $X$ arises by base change from a subvariety $X_0$ defined over $L_0$ and such that $\Gamma \subset A_0(L_0)$. By Proposition 4.20 there exists an embedding $L_0 \to K$ into a local field $K$ and a semiabelian scheme $\mathcal{A}$ over $R$ which has generic fiber $A_K$, is isogenous to a nice semiabelian scheme over $R$ and satisfies $\Gamma \subset A(R)$. Let $K$ an algebraic closure of $K$ and denote $\hat{R}$ the valuation ring of $K$, by $k$ the finite residue field of $R$ and by $\mathcal{X}$ the schematic closure of $X_0$, $\gamma$ inside $\mathcal{A}$.

By Lemma 5.3 it suffices to prove that $X^{\gamma}$ is special in $A^{\hat{R}}$. Using Lemma 5.3 we can replace $\mathcal{A}$ by an isogenous semiabelian variety, so that we may assume that $\mathcal{A}$ is nice.

Since we have an exact sequence

$$0 \to \hat{A}(R) \to A(R) \to A(k) \to 0$$

with $A(k)$ finite, after replacing $X$ by a suitable translate we may assume that $(\Gamma \cap \hat{A}(R)) \cap X(R)$ is schematically dense in $X$. After replacing $\Gamma$ by $\Gamma \cap \hat{A}(R)$ we may thus assume $\Gamma \subset \hat{A}(R)$.

Let $\hat{F}_A: \hat{A}_{\text{Spf}(R^{q^i})} \to \hat{A}_{\text{Spf}(R^{q^i})}$ be the endomorphism given by Construction 4.16. By Proposition 4.17 (ii) there exist a $p$-divisible group $\mathcal{G}$ over $k$ and an isomorphism $\Psi: \mathcal{G}_{\text{Spf}(R^{q^i})} \cong \hat{A}_{\text{Spf}(R^{q^i})}$ under which $F_A$ corresponds to the Frobenius endomorphism of $\mathcal{G}$ with respect to $k$. 

24
Let \( \bar{\Gamma} \) be the closure of \( \Gamma \) with respect to the valuation topology on \( \hat{\mathcal{A}}(\bar{R}) \). Since \( \Gamma \) is finitely generated this is a finite rank \( \mathbb{Z}_p \)-module. Let \( \mathcal{X} \subset \mathcal{A} \) be the formal schematic closure of \( \Gamma \cap \mathcal{X}(R) \) inside \( \mathcal{A} \). Then \( \mathcal{X} \) is the formal schematic closure of \( \mathcal{X}(R) \cap \bar{\Gamma} \) in \( \mathcal{A} \). Thus by Corollary 2.35 the formal scheme \( \mathcal{X}(\text{Spf}(R_{\text{per}})) \) is the formal schematic closure of \( \mathcal{X}(R) \cap \bar{\Gamma} \) in \( \mathcal{A}(\text{Spf}(R_{\text{per}})) \). Thus by Theorem 5.1 applied to \( \Psi(\mathcal{X}(\text{Spf}(R_{\text{per}}))) \) and \( \Phi(\bar{\Gamma}) \), there exist closed formal subschemes \( \mathcal{X}_1, \ldots, \mathcal{X}_n \) of \( \mathcal{X}(R) \), elements \( \gamma_1, \ldots, \gamma_n \in \bar{\Gamma} \) and \( n \geq 0 \) such that \( F_n^p(\mathcal{X}_j + \gamma_j) \subset \mathcal{X}_j + \gamma_j \) and \( \mathcal{X} = \cup_i \mathcal{X}_i \). Since \( \Gamma \cap \mathcal{X}(R) \subset \mathcal{X}(R) \), the set \( \mathcal{X}(\bar{R}) \) is schematically dense in \( \mathcal{X}(R) \). By Proposition 4.18 (iii) holds for \( T := \mathcal{X}_i(R) + \gamma_i \). By Proposition 4.18 this implies that \( \mathcal{X}(\bar{R}) \) is special in \( \hat{\mathcal{A}}(\bar{R}) \).

6 Towards full Mordell-Lang

The full Mordell-Lang conjecture in positive characteristic is the following conjecture:

**Conjecture 6.1.** Let \( L \) be an algebraically closed field of positive characteristic. Let \( A \) be a semiabelian variety over \( L \), let \( X \subset A \) be an irreducible subvariety and let \( \Gamma \subset A(L) \) be a subgroup of finite rank. If \( X(L) \cap \Gamma \) is Zariski dense in \( X \), then \( X \) is a special subvariety of \( A \).

In this section we show that in case \( A \) is an ordinary or supersingular abelian variety by combining our method with a reduction due to Ghioca, Moosa and Scanlon, Conjecture 6.1 can be reduced to the following special case:

**Conjecture 6.2.** Let \( L_0 \) be a field which is finitely generated over \( \mathbb{F}_p \), let \( L \) be an algebraic closure of \( L_0 \) and let \( L_{0,\text{per}} \) the perfect closure of \( L_0 \) in \( L \). Let \( A \) be a semiabelian variety over \( L_0 \) and \( X \subset A(L_{0,\text{per}}) \) an irreducible subvariety. Assume that the canonical morphism \( \text{Tr}_{L/F_p} A \to A \) is defined over \( L_0 \), that there exists a finite subfield \( F_q \) of \( L_0 \) over which \( \text{Tr}_{L/F_p} A \) can be defined and that \( \text{Stab}_{A(L_{0,\text{per}})}(X) \) is finite. If \( X(L_{0,\text{per}}) \) is Zariski dense in \( X_{L_0} \), then a translate of \( X \) by an element of \( A(L_{0,\text{per}}) \) is defined over \( L_0 \).

**Remark 6.3.** We expect that Conjecture 6.2 holds without the condition on the field of definition of the morphism \( \text{Tr}_{L/F_p} A \to A \), the existence of \( F_q \) as above and the condition on \( \text{Stab}_{A(L_{0,\text{per}})}(X) \). However we were unable to deduce Conjecture 6.2 from Conjecture 6.1 without these conditions. Note that they can always be achieved by dividing by \( \text{Stab}_{A(L_{0,\text{per}})}(X) \) and by replacing \( L_0 \) by a suitable finite extension.

**Remark 6.4.** Consider the case of a supersingular abelian variety \( A \). Then \( A \) is isogenous to a power of a supersingular elliptic curve (c.f. [21, Theorem 4.2]), and thus in order to prove Conjecture 6.1 for \( A \), it suffices to prove it for powers of supersingular elliptic curves. But these can be defined over a finite field and for such abelian varieties, Conjecture 6.1 is proven in [14]. Thus the results in this section are only interesting in case \( A \) is ordinary.

**Lemma 6.5.** Let \( L_0 \) be a finitely generated field of characteristic \( p > 0 \) with perfection \( L_0 \hookrightarrow L_{0,\text{per}} \) and let \( A \) be a semiabelian variety over \( L_0 \).

(i) The group \( A(L_{0,\text{per}}) \) has finite rank.

(ii) Let \( \Gamma \subset A(L_{0,\text{per}}) \) be a subgroup and \( n \geq 0 \). The group \( \Gamma/p^n\Gamma \) is finite.

**Proof.** (i) For \( n \geq 0 \) the group \( p^n A(L_{0,\text{per}}) \) is contained in \( A(L_0) \). This together with the fact that \( A(L_0) \) is finitely generated implies that \( A(L_{0,\text{per}}) \) has finite rank.

(ii) By Claim 1 in the proof of Theorem 2.2 of [14] the torsion subgroup of \( A(L_{0,\text{per}}) \) is finite. Hence both the rank and the size of the torsion subgroup of the finitely generated
groups $\Gamma_i := \Gamma \cap A(L_0^{p^{-i}})$ are bounded as $i \geq 0$ varies. Thus the size of the groups $\Gamma_i/p^n\Gamma_i$ is bounded as $i$ varies. Let $\Gamma_i$ be the image of $\Gamma_i/p^n\Gamma_i$ in $\Gamma/p^n\Gamma$. The groups $\Gamma_i$ form an ascending sequence of finite groups of bounded size, thus for all $i \gg 0$ the $\Gamma_i$ coincide. Since $\Gamma/p^n\Gamma$ is the union of the $\Gamma_i$ this shows (iii).

**Proposition 6.6.** Conjecture [6.7] implies Conjecture [6.2]

**Proof.** After translating $X$ by an element of $X(L_0^{per})$ we may assume that $0 \in X$. By Lemma [6.3] the group $A(L_0^{per})$ has finite rank. Hence by Conjecture [6.1] the subvariety $X_L$ is special in $A_L$. Thus by Lemma [3.6] there exist a subvariety $Y \subset \text{Tr}_{L/[q]} A$ over $\mathbb{F}_p$ and $a \in A(L)$ such that $X_L = \tau(Y_L) + \text{Stab}_A(X)_L + a$, where $\tau$ is the natural homomorphism $\text{Tr}_{L/[q]} A \to A$ which by assumption is defined over $L_0$. Since by assumption $\text{Stab}_A(X)$ is finite and $X$ is irreducible, there exists $a' \in A(L)$ such that $\tau(Y_L) = X_L + a'$.

The fact that $0 \in X$ implies $a' \in \text{Im}(\tau)(L)$. Thus $X \subset \text{Im}(\tau)$. After replacing $A$ by $\text{Im}(\tau)$ we may assume that $\tau$ is an isogeny. We fix a model of $\text{Tr}_{L/[q]} A$ over a finite field $\mathbb{F}_q$ contained in $L_0$ and let $F: \text{Tr}_{L/[q]} A \to \text{Tr}_{L/[q]} A$ the Frobenius endomorphism of $\text{Tr}_{L/[q]} A$ with respect to $\mathbb{F}_q$. Since by [8, Theorem 6.12] the homomorphism $\tau$ is purely inseparable, the induced map $\tau: \text{Tr}_{L/[q]} A(L_{0}^{per}) \to A(L_{0}^{per})$ is surjective. Let $Y'$ be an irreducible component of the Zariski closure of $\tau^{-1}(X(L_{0}^{per})) \subset \text{Tr}_{L/[q]} A(L_{0}^{per})$. Let $\Gamma := \text{Tr}_{L/[q]} A(L_0)$. This is a finitely generated $F$-invariant subgroup of $\text{Tr}_{L/[q]} A(L)$ and $Y'(L) \cap (\cup_{n \geq 0} F^{-n}\Gamma) = Y'(L) \cap \text{Tr}_{L/[q]} A(L_{0}^{per})$ is Zariski dense in $Y'$. Thus by [14, Proposition 3.9] for some $n \geq 0$ the set $Y'(L) \cap F^{-n}\Gamma \subset Y'(L_0^{q^{-n}})$ is Zariski dense in $Y'$. Thus $X(L_0^{q^{-n}})$ is Zariski dense in $X$. Since the group $A(L_0^{q^{-n}})/q^nA(L_0^{q^{-n}})$ is finite, there thus exists $a'' \in A(L_0^{q^{-n}})$ such that $(a'' + q^nA(L_0^{q^{-n}})) \cap X(L_0^{q^{-n}})$ is Zariski dense in $X$. Since $q^nA(L_0^{q^{-n}}) \subset A(L_0)$ and since Zariski closure commutes with base change this implies that $X - a''$ is defined over $L_0$. 

**6.1 A specialness criterion**

Let $K$ be a local field of characteristic $p > 0$ with valuation ring $R$ and residue field $k$. Let $\bar{K}$ be an algebraic closure of $K$ and $\bar{R}$ the valuation ring of $\bar{K}$. Let $A$ be an abelian scheme over $R$ such that the Newton polygon of $\bar{A}$ is constant. Denote $A_{\bar{K}}$ by $A$.

The following definition is somewhat ad hoc and adapted to our present needs:

**Definition 6.7.** Let $i \geq 0$ and $X$ a subvariety of $A$. Denote the schematic closure of $X$ in $A$ by $X$. We say that $X$ is $K/K^p$-special in $A$ if $X$ is irreducible and there exists an abelian variety $B$ over $K^p$, a subvariety $Y$ of $B$ over $K^p$, a homomorphism $h: B_K \to A/\text{Stab}_A(X)$ with finite kernel and an element $a \in (A/\text{Stab}_A(X))(\bar{R})$ such that $(X/\text{Stab}_A(X))(\bar{K}) = h(Y_{\bar{K}}) + a$.

**Theorem 6.8.** Assume that $A$ is ordinary or supersingular. Let $X$ be an irreducible subvariety of $A$ containing 0. If $X$ is $K/K^p$-special in $A$ for all $i \geq 0$, then $X_K$ is special in $A_K$.

**Proof.** Since the Newton polygon of $\bar{A}$ is constant, by Proposition [4.4] the $p$-divisible group $\bar{A}$ is isogenous to a completely slope divisible $p$-divisible group. Since isogenies of $p$-divisible groups are quotients by finite subgroup schemes, this implies that there exists a nice abelian scheme $A'$ over $\bar{R}$ together with an isogeny $f: A \to A'$. It follows directly from Definition 6.7 that $f(X)$ is again $K/K^p$-special in $A'_{\bar{K}}$ for all $i \geq 0$. By Lemma [3.3] the subvariety $X_K$ of $A_K$ is special if and only if $f(X_K)$ is special in $A'_{\bar{K}}$. Thus we may replace $A$ by $A'$ and can assume that $A$ is nice.
Let $\mathcal{X}$ be the schematic closure of $X$ in $\mathcal{A}$. After dividing by $\text{Stab}_A(\mathcal{X})$ we may assume that $\text{Stab}_A(X) = 0$. By Proposition 2.28 after replacing $K$ by a finite field extension we may assume that each irreducible component of $\mathcal{X}$ is geometrically irreducible.

Let $i \geq 0$. By assumption there exists an abelian variety $B$ over $K^p$, a subvariety $Y \subset B$, a homomorphism $h: B_K \to A$ and $a \in \mathcal{A}(\bar{R})$ such that $X_K = h(Y_K) + a$. Let $\mathcal{B}$ be the Néron model of $B$ over $R^p$. Since $\mathcal{A}$ is the Néron model of $A_K$, the homomorphism $h$ extends to a homomorphism $\hat{h}: \mathcal{B}_R \to \mathcal{A}$. Since the generic fiber of the completion $\hat{h}: \hat{\mathcal{B}}_R \to \hat{\mathcal{A}}$ is an isogeny, Tate’s conjecture implies that $\hat{h}$ is an isogeny. Since the Newton polygon of $\hat{\mathcal{A}}$ is constant, the existence of this isogeny implies that the Newton polygon of $\hat{\mathcal{B}}$ is constant. Thus, by Theorem 6.9 and Proposition 4.12, there exists a completely slope divisible $p$-divisible group $\mathcal{G}$ over $\text{Spf}(R^p)$ together with an isogeny $h^*: \mathcal{G} \to \hat{\mathcal{B}}$.

Let $\mathcal{Y}$ be the schematic closure of $Y$ in $\mathcal{B}$. Pick a finite field extension $K' \subset \bar{K}$ of $K$ with valuation ring $R'$ such that $a \in \mathcal{A}(R')$. Then $h(\mathcal{Y}_{R'}) + a = X_{R'}$. Let $\mathcal{Y} := (h')^{-1}(\mathcal{Y}) \subset \mathcal{G}$. Then $X_i := \hat{h}(h'(\mathcal{Y}_{R'})) + a \subset \mathcal{A}(\text{Spf}(R'))$, where $\hat{h}(h'(\mathcal{Y}_{R'}))$ is the formal schematic image of $\mathcal{Y}_{R'}$. By Lemma 2.30 we have $h'(\mathcal{Y}_{R'}) = \mathcal{Y}_{R'}$. Thus by Proposition 2.44 each irreducible component of $\mathcal{X}_i$ is an irreducible component of $\mathcal{A}(\text{Spf}(R'))$. Since each irreducible component of $\mathcal{X}$ is geometrically irreducible, it follows that $\mathcal{X}_i$ is the union of some of the irreducible components of $\mathcal{X}$.

Using the fact that $\mathcal{A}$, and hence $\mathcal{G}$, has a single slope, Propositions 4.11 and 4.12 yield unique isomorphisms $\psi: \mathcal{G} \xrightarrow{\sim} (\mathcal{A}_k)_{\text{Spf}(R')} = \hat{\mathcal{A}}_{\text{Spf}(R')}$ and $\psi': \hat{\mathcal{A}} \xrightarrow{\sim} (\mathcal{A}_k)_{\text{Spf}(R')}$ which are the identity in the special fiber. Under these identifications, by Proposition 4.10 the homomorphism $\hat{h} \circ h^*: \mathcal{G}_R \to \hat{\mathcal{A}}$ arises by base change from its special fiber. Thus the identity $\mathcal{X}_i = \hat{h}(h'(\mathcal{Y})) + a$ shows that a translate of $\psi'(\mathcal{X}_i)$ by an element of $\mathcal{A}_k(\bar{R})$ is defined over $R^p$.

As we saw above, each $\mathcal{X}_i$ is the union of some of the irreducible component of $\mathcal{Y}$. Since there are only finitely many such components it follows that there exists $\mathcal{X} \subset \mathcal{X}_i$ such that $\mathcal{X} = \mathcal{X}_i$ for infinitely many $i$. Theorem 2.41 implies that there exists a finite field extension $K' \subset \bar{K}$ with valuation ring $R'$ and $x \in \mathcal{X}(R')$ such that $\psi'(\mathcal{X}_{\text{Spf}(R')} - x)$ is defined over $k$. By construction, the endomorphism $F_A$ corresponds to the Frobenius endomorphism of $\mathcal{A}_k$ under $\psi'$. Thus $T := \mathcal{X}(\bar{R}) - x$ satisfies $F_A(T) \subset T$. By Proposition 2.43 (ii) the set $T$ is Zariski dense in $X_{\bar{R}} - x$. Thus condition (iii) of Theorem 4.18 is satisfied and Theorem 4.18 implies that $X_{\bar{R}}$ is special in $A_{\bar{R}}$.

6.2 Proof of the reduction

**Theorem 6.9.** Conjecture 6.1 and Conjecture 6.2 are equivalent for abelian varieties which are ordinary or supersingular.

**Proof.** One direction was already proved in Proposition 6.6 above. Now we prove that Conjecture 6.2 implies Conjecture 6.1.

By [13, Theorem 2.2], Conjecture 6.1 is implied by:

**Conjecture 6.10.** Let $L_0$ be a field which is finitely generated over $\bar{F}_p$, $L$ an algebraic closure of $L_0$ and $L_0^\text{per}$ the perfect closure of $L_0$ in $L$. Let $A$ be a semiabelian variety over $L_0$, let $X \subset A$ an irreducible subvariety and let $\Gamma \subset A(L_0^\text{per})$ a subgroup of finite rank. If $X(L_0^\text{per}) \cap \Gamma$ is Zariski dense in $X_L$, then $X_L$ is a special subvariety of $A_L$.

Furthermore, from the proof of [13, Theorem 2.2], one sees that if one only wants to prove Conjecture 6.1 for abelian varieties which are ordinary or supersingular, then it suffices to prove Conjecture 6.10 for such abelian varieties. This is what we do now.
After replacing $L_0$ by a finite extension we may assume that the canonical morphism $\tau: \text{Tr}_{\bar{L}/\bar{F}_q} A \to A$ is defined over $L_0$ and that there exists a finite subfield $\bar{F}_q$ of $L_0$ over which $\text{Tr}_{\bar{L}/\bar{F}_q} A$ can be defined. After dividing by $\text{Stab}_A(X)$ we may assume that $\text{Stab}_A(X) = 0$.

By Proposition 4.20 there exists an embedding of $L_0$ into a local field $K$ such that the abelian variety $A_K$ extends to an abelian scheme $\mathcal{A}$ over the valuation ring $R$ of $K$ which is isogenous to a nice abelian scheme over $R$ and such that $\Gamma \subset A(L_0^{\text{per}}) \subset \mathcal{A}(R^{\text{per}})$.

Let $X$ be the schematic closure of $X_K$ inside $\mathcal{A}$. By Lemma 3.4 it suffices to prove that $X_K$ is special in $\mathcal{A}_K$.

Since there is an exact sequence

$$0 \to \hat{A}(R^{\text{per}}) \to \mathcal{A}(R^{\text{per}}) \to A(k) \to 0$$

with $A(k)$ finite there exists $\gamma \in \Gamma \cap X(L_0^{\text{per}})$ such that $X(\bar{K}) \cap (\gamma + (\Gamma \cap \hat{A}(R^{\text{per}})))$ is Zariski dense in $X_K$. After replacing $L_0$ by a finite extension we may assume that $\gamma \in A(L_0)$. Then after translating $X$ by $X - \gamma$ and replacing $\Gamma$ by $\Gamma \cap \hat{A}(R^{\text{per}})$ we may assume that $\Gamma \subset \hat{A}(R^{\text{per}})$ and $0 \in X$.

Fix a finite subfield $\bar{F}_q$ of $L_0$ over which $\text{Tr}_{\bar{L}/\bar{F}_q} A_L$ can be defined. By Theorem 6.8 it now suffices to show that $X_K$ is $K/K^{q^i}$-special in $A$ for all $i \geq 0$. Thus we fix such an $i$. We will work with the abelian variety $A^{(q^i)}$, which is naturally defined over $L_0^{q^i}$, together with the Verschiebung homomorphism $V: A^{(q^i)}_L \to A$. By Lemma 6.5 the group $\Gamma/q^i\Gamma$ is finite. Thus there exists $\gamma \in \Gamma$ such that $X(L_0^{q^i}) \cap (\gamma + q^i\Gamma)$ is Zariski dense in $X_K$. Let $T \subset \Gamma$ such that $\gamma + q^iT = X(L_0^{q^i}) \cap (\gamma + q^i\Gamma)$. Let $F: A \to A^{(q^i)}$ be the relative $q^i$-Frobenius and $Y \subset A^{(q^i)}_{L_0^{q^i}}$ an irreducible component of the Zariski closure of $F(T) \subset A^{(q^i)}(L_0^{q^i})$. The fact that $V \circ F = [q^i]$ implies $V(Y) + \gamma = X$.

By Theorem 6.4 the formation of $\text{Tr}_{\bar{L}/\bar{F}_q} A_L$ commutes with purely inseparable base change. Thus $\tau^{(q^i)}: \text{Tr}_{\bar{L}/\bar{F}_q} A_L \cong (\text{Tr}_{\bar{L}/\bar{F}_q} A_L)^{(q^i)} \to A^{(q^i)}_L$ is the $L/\bar{F}_q$-trace of $A^{(q^i)}_L$. This morphism is defined over $L_0^{q^i}$. Since $\text{Stab}_A(X) = 0$ the stabilizer $\text{Stab}_{A^{(q^i)}_{L_0^{q^i}}}(Y)$ is contained in the kernel of $V$ and thus is finite. Thus Conjecture 6.2 applied to to $Y \subset A^{(q^i)}_{L_0^{q^i}}$ gives an element $a \in A^{(q^i)}(L_0^{q^i})$ such that $Y + a$ is defined over $L_0^{q^i}$.

We consider $a$ as an element of $A^{(q^i)}(K^{q^i})$. By the choice of embedding $L_0 \hookrightarrow K$, the element $V(a) \in A(L_0^{q^i})$ lies in $\mathcal{A}(R^{\text{per}})$. Hence $q^ia = F(V(a)) \in A^{(q^i)}(R^{\text{per}})$. Thus $a \in A^{(q^i)}(R^{\text{per}})$ by Lemma 4.19. Since the natural map $A^{(q^i)}(R^{q^i}) \to A^{(q^i)}(k)$ is surjective, there exists $a' \in A^{(q^i)}(R^{q^i})$ such that $a - a' \in A^{(q^i)}(R^{\text{per}})$. Then $Y + a - a'$ is defined over $K^{q^i}$ and $V(Y + a - a') + V(a' - a) + \gamma = X$ with $V(a' - a) + \gamma \in \hat{A}(R^{\text{per}})$. Thus $X_K$ is $K/K^{q^i}$-special and we are done.

References

[1] Schémas en groupes. I: Propriétés générales des schémas en groupes. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 151. Springer-Verlag, Berlin, 1970.

[2] Dan Abramovich and José Felipe Voloch. Toward a proof of the Mordell-Lang conjecture in characteristic $p$. Internat. Math. Res. Notices, (5):103–115, 1992.

[3] M. F. Atiyah and I. G. Macdonald. Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.
[4] Vladimir G. Berkovich. Vanishing cycles for formal schemes. II. *Invent. Math.*, 125(2):367–390, 1996.

[5] Siegfried Bosch, Werner Lütkemhord, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)].* Springer-Verlag, Berlin, 1990.

[6] Nicolas Bourbaki. *Commutative algebra. Chapters 1–7.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1989. Translated from the French, Reprint of the 1972 edition.

[7] Brian Conrad. Irreducible components of rigid spaces. *Ann. Inst. Fourier (Grenoble)*, 49(2):473–541, 1999.

[8] Brian Conrad. Chow’s $K/k$-image and $K/k$-trace, and the Lang-Néron theorem. *Enseign. Math. (2)*, 52(1-2):37–108, 2006.

[9] A. J. de Jong. Crystalline Dieudonné module theory via formal and rigid geometry. *Inst. Hautes Études Sci. Publ. Math.*, (82):5–96 (1996), 1995.

[10] Michel Demazure. *Lectures on p-divisible groups.* Lecture Notes in Mathematics, Vol. 302. Springer-Verlag, Berlin, 1972.

[11] Michel Demazure and Alexander Grothendieck (ed.). *Schémas en groupes II (SGA 3)*, volume 152 of *Lecture Notes in Mathematics.* Springer-Verlag, 1970.

[12] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[13] Gerd Faltings and Ching-Li Chai. *Degeneration of abelian varieties*, volume 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)].* Springer-Verlag, Berlin, 1990. With an appendix by David Mumford.

[14] Dragos Ghioca and Rahim Moosa. Division points on subvarieties of isotrivial semiabelian varieties. *Int. Math. Res. Not.*, pages Art. ID 65437, 23, 2006.

[15] A. Grothendieck. Éléments de géométrie algébrique. I. Le langage des schémas. *Inst. Hautes Études Sci. Publ. Math.*, (4):228, 1960.

[16] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II. *Inst. Hautes Études Sci. Publ. Math.*, (24):231, 1965.

[17] Michiel Hazewinkel. *Formal groups and applications*, volume 78 of *Pure and Applied Mathematics.* Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978.

[18] Ehud Hrushovski. The Mordell-Lang conjecture for function fields. *J. Amer. Math. Soc.*, 9(3):667–690, 1996.

[19] Christian Kappen. Uniformly rigid spaces. [http://arxiv.org/abs/1009.1056](http://arxiv.org/abs/1009.1056)

[20] William Messing. *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes.* Lecture Notes in Mathematics, Vol. 264. Springer-Verlag, Berlin, 1972.

[21] Frans Oort. Subvarieties of moduli spaces. *Invent. Math.*, 24:95–119, 1974.

[22] Frans Oort and Thomas Zink. Families of $p$-divisible groups with constant Newton polygon. *Doc. Math.*, 7:183–201 (electronic), 2002.

[23] Richard Pink and Damian Rössler. On $\psi$-invariant subvarieties of semiabelian varieties and the Manin-Mumford conjecture. *J. Algebraic Geom.*, 13(4):771–798, 2004.

[24] Damian Rössler. On the Manin-Mumford and Mordell-Lang conjectures in positive characteristic. [http://www.math.univ-toulouse.fr/~rossler/mypage/pdf-files/ml4.pdf](http://www.math.univ-toulouse.fr/~rossler/mypage/pdf-files/ml4.pdf)
[25] Thomas Scanlon. A positive characteristic Manin-Mumford theorem. Compos. Math., 141(6):1351–1364, 2005.

[26] The Stacks Project Authors. stacks project. http://stacks.math.columbia.edu, 2013.

[27] Charles A. Weibel. An introduction to homological algebra, volume 38 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994.