Turbulent Filamentation in Lasers with High Fresnel Numbers

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Abstract

The theory of turbulent photon filamentation in lasers with high Fresnel numbers is presented. A survey of experimental observations of turbulent filamentation is given. Theoretical description is based on the method of eliminating field variables, which yields the pseudospin laser equations. These are treated by the scale separation approach, including the randomization of local fields and the method of stochastic averaging. The initial, as well as the transient and final stages of radiation dynamics are carefully analysed. The characteristics of photon filaments are obtained by involving the probabilistic approach to pattern selection.
1 Introduction

In nonlinear media interacting with electromagnetic fields there appear different spatiotemporal structures that are analogous to the structures arising in many other complex nonequilibrium systems [1–5]. The most known among such electromagnetic structures are the optical filaments which can be formed in passive nonlinear matter [3,4,6] and in active laser media [1,4,5]. Examples of filaments occurring in other nonequilibrium systems can be found in the reviews [2,5]. In addition, it is possible to mention the generation of filaments in hadron plasma [7] and in quark-gluon plasma formed under heavy nucleus collisions [8,9].

The present survey will be focused on the optical filaments developing in lasers. The behaviour and characteristics of these filaments essentially depend on the value of the Fresnel number $F \equiv R^2/\lambda L$, in which $R$ and $L$ are the internal radius, aperture radius, and effective length, respectively, of a cylindrical laser, and $\lambda$ is the optical wavelength. There are two types of optical filaments, regular and turbulent, corresponding to either low or high Fresnel numbers. The difference between these two filament types, their main properties, and the related experiments are discussed in Sec. 2.

Our concern here is the turbulent filaments arising in lasers with high Fresnel numbers. The transition from a regular filamentary structure to the turbulent filament behaviour is somewhat analogous to a crossover phase transition in statistical systems [9–11], or a more close analogy is the transition from the laminar to turbulent motion in liquids [1,2,12]. The description of the turbulent motion is notorious to be rather difficult and is usually done by invoking a kind of averaging procedure. Optical turbulent structures also require the usage of averaging techniques.

First, the turbulent filamentation in laser media was described on the basis of stationary models [13–16] invoking the notion of an effective time-averaged energy. A more elaborated approach, based on realistic evolution equations, was developed in Refs. [5,17–20]. It turned out that for this purpose it is convenient to work not with the standard Maxwell-Bloch equations but with the evolution equations for the pseudospin operators, resulting after the elimination of the field variables. Since this type of equations is less known for collective electromagnetic phenomena, these equations are derived in Sec. 3, and their specification for treating the turbulent filamentary structure is given in Sec. 4.

Turbulent filamentation in lasers is a self-organized process due to the photon exchange through the common radiation field. Coherent radiation in filaments develops in a self-organized way, even when there has been no coherence at the initial time. The triggering mechanism for the origination of the radiation coherence is the existence of transition dipolar waves caused by the dipolar part of effective atomic interactions. This is emphasized in Sec. 5.

The overall dynamics of radiating filaments is analysed in Sec. 6, starting from the quantum stage, through the transient flashing regime, to the stationary state. Generally, the filaments of different radii can emerge, whose classification is given by means of the probabilistic approach to pattern selection [18,21,22]. This allows us to define the typical filament radius and the number of filaments, which is done in Sec. 7.

The present work does not merely summarize and review the previous theoretical results on the turbulent filamentation in laser media but also contains several improvements of the theory, making the latter better grounded both mathematically and physically.


2 Experiments on Filamentation in Lasers

First of all, let us recall that there are two types of optical filaments, regular and turbulent, which is related to the value of the laser Fresnel number. The latter plays for optical systems the same role as the Reynolds number for moving fluids. When increasing the Reynolds number, a laminar fluid transforms to turbulent. In the similar manner, increasing the Fresnel number makes a regular filamentary structure turbulent. Optical turbulence implies, by analogy with the fluid turbulence, that the spatiotemporal dynamics is chaotic. This means that the radiating filaments are randomly distributed in space and are not correlated with each other.

Optical filaments are observed in the near-field cross-section of lasers. The typical picture, when varying the Fresnel number is as follows. At very small Fresnel numbers $F \ll 1$, there exists the sole transverse central mode uniformly filling the laser medium. When the Fresnel number is around $F \sim 1$, the laser cavity can house several transverse modes seen as a regular arrangement of bright spots in the transverse cross-section. Each mode corresponds to a filament extended through the cylindrical volume. This filamentary structure is regular in space, forming ordered geometric arrays, such as polygons. The transverse structure is imposed by the cavity geometry, being prescribed by the empty-cavity Gauss-Laguerre modes. Such regular structures are well understood theoretically, their description being based on the field expansion over the modal Gauss-Laguerre functions related to the cylindrical geometry [1]. For Fresnel numbers up to $F \approx 5$, the number of bright filaments follows the $F^2$ law as $F$ increases. The regular filamentary structures have been observed in several lasers, such as CO$_2$ and Na$_2$ lasers [1]. Similar structures also appear in many passive nonlinear media, e.g. in Kerr medium [3,4] and in active nonlinear media, as the photorefractive Bi$_{12}$SiO$_2$ crystal pumped by a laser [3,4].

As soon as the Fresnel number reaches $F \approx 10$, there occurs a qualitative change in the features of the filamentary structure: The regular filaments become turbulent. This transition goes gradually, as a crossover, with the intermittent behaviour in the region $5 < F < 15$. The character of this change is again common for lasers [23,24] and for active nonlinear media [3,4].

For Fresnel numbers $F > 15$, the arising filamentary structures are principally different from those existing at low Fresnel numbers. The spatial structures now have no relation to the empty-cavity modes. The modal expansion is no longer relevant and the boundary conditions have no importance. The laser medium houses a large number of parallel independent filaments exhibiting themselves as a set of bright spots randomly distributed in the transverse cross-section. The number of these random filaments is proportional to $F$, contrary to the case of low Fresnel numbers, when the number of filaments is proportional to $F^2$. The chaotic filaments, being randomly distributed is space, are not correlated with each other. Such a spatio-temporal chaotic behaviour is characteristic of hydrodynamic turbulence, because of which the similar phenomenon in optics is commonly called the optical turbulence.

In contrast to the regime of low $F$, where the regularity of spatial structures is prescribed by the cavity geometry and boundary conditions imposing their symmetry constraints, the turbulent optical filamentation is strictly self-organized, with its organization emerging from intrinsic properties of the medium. Since the optical turbulence is accompanied by the formation of bright filaments with a high density of photons, this phenomenon can be named
the turbulent photon filamentation. This phenomenon is common for lasers as well as for photorefractive crystals [3–5].

The first observations of the turbulent filamentary structures in lasers, to my knowledge, were accomplished in the series of experiments [25–29] with the resonatorless superluminous lasers on the vapours of Ne, Tl, Pb, N₂, and N₂⁺. In these experiments, the typical characteristics were as follows: \( \lambda \approx 5 \times 10^{-5} \text{ cm}, \ R \approx 0.1 – 0.3 \text{ cm}, \ L \approx 20 – 50 \text{ cm}, \) and \( F \approx 10 – 100. \) The number of filaments was \( N_f \sim 10^2 – 10^3, \) with the typical radius \( r_f \approx 0.01 \text{ cm}. \)

Then the filamentary structures in large-aperture optical devices have been observed in several lasers, as reviewed in [23,24], and in photorefractive crystals [3,4]. Numerical simulations have been accomplished [30]. Experimental works mainly dealt with the CO₂ lasers [23,24,31,32], dye lasers [33], and semiconductor lasers [34,35].

The turbulent nature of filamentation occurring in high Fresnel number lasers was carefully studied in a series of nice experiments [36–42] with CO₂ lasers and dye lasers. Irregular temporal behaviour was observed in local field measurements. It was found that the transverse correlation length was rather short. Randomly distributed transverse patterns generated in short times were observed, being shot-to-shot nonreproducible. For intermediate Fresnel numbers \( F \sim 10, \) instantaneous transverse structures were randomly distributed in space, but after being temporally averaged, they displayed a kind of regularity related to the geometrical boundary conditions. This type of combination of irregular instantaneous patterns with the averaged or stationary pattern, showing the remnant ordering, is understandable for the intermediate regime in the crossover region \( 5 < F < 15. \) Fully developed optical turbulence is reached as the Fresnel number increases up to \( F \sim 100. \)

The typical laser parameters are as follows [36–42]. The pulsed CO₂ laser, with the wavelength \( \lambda = 1.06 \times 10^{-3} \text{ cm} \) and frequency \( \omega = 1.78 \times 10^{14} \text{ s}^{-1}. \) emits the pulses of \( \tau_p \approx 0.7 \times 10^{-7} \text{ s or } 10^{-6} \text{ s}. \) The aperture radius \( R \approx 1 \text{ cm}, \) laser length \( L = 100 \text{ cm}. \) The inversion and polarization decay rates are \( \gamma_1 = 10^7 \text{ s}^{-1} \) and \( \gamma_2 = 3 \times 10^9 \text{ s}^{-1}. \) The CO₂ density is \( \rho = 2 \times 10^{18} \text{ cm}^{-3}. \) The Fresnel number is \( F \approx 10. \) The characteristic filament radius is \( r_f \approx 0.1 \text{ cm}. \)

The pulsed dye laser, with the wavelength \( \lambda = 0.6 \times 10^{-4} \text{ cm} \) and frequency \( \omega = 3.14 \times 10^{15} \text{ s}^{-1}. \) produces pulses of \( \tau_p \approx 0.5 \times 10^{-6} \text{ s}. \) The decay rates are \( \gamma_1 = 4 \times 10^8 \text{ s}^{-1} \) and \( \gamma_2 = 10^{12} \text{ s}^{-1}. \) The cavity length is \( L \approx 20 \text{ cm}. \) By varying the aperture radius between 0.3 cm and 0.8 cm, the Fresnel number can be changed by an order, between \( F = 15 \) and \( F = 110. \) The typical filaments radius is \( r_f \approx 0.01 \text{ cm}. \)

In a broad-aperture pulsed dye laser, two different spatial scales were noticed [33], one order of magnitude apart. The appearance of the second, larger, scale could be due to a nonvanishing interaction between filaments.

### 3 Pseudospin Laser Equations

Spatiotemporal laser dynamics is usually treated in the frame of the Maxwell-Bloch equations [43]. One commonly employs these equations for defining the points of stability of uniform solutions, which means the appearance of nonuniform structures. The linear stability analysis of the Maxwell-Bloch equations is the maximum one can do in the case of well developed
optical turbulence.

Another approach is based on the elimination of field variables and dealing solely with the equations for spin operators [5,44]. This approach possesses the following advantages: (i) It is microscopic, which allows us to better understand the underlying physics of collective effects. (ii) It is quantum, which allows for the description, when coherence has not yet been developed. (iii) It provides us the possibility not solely for finding instability points but for describing the whole dynamics under the condition of strong turbulence. Since this approach is not widely known, the related evolution equations are derived below.

Aiming at pursuing a microscopic picture, let us start with a realistic Hamiltonian

\[
\hat{H} = \hat{H}_a + \hat{H}_f + \hat{H}_{af} + \hat{H}_{mf}
\]

representing a system of resonant two-level atoms plus electrodynamic field. The atomic Hamiltonian is

\[
\hat{H}_a = \sum_{i=1}^{N} \omega_0 \left( \frac{1}{2} + S_z^i \right),
\]

where \(N\) is the number of atoms; \(\omega_0\), carrying transition frequency; \(S_z^i\), spin operator of an \(i\)-th atom. Wishing to be more rigorous in terminology, we should call the operators \(S_\alpha^i\), appearing here and in what follows, the pseudospin operators, since they correspond not to actual spins but to the population difference and dipole transition operators. But for brevity, one often calls them just spin operators, which should not bring confusion. The field Hamiltonian is

\[
\hat{H}_f = \frac{1}{8\pi} \int \left( E^2 + H^2 \right) \, d\mathbf{r}
\]

with electric field \(E\) and magnetic field \(H = \nabla \times \mathbf{A}\). The vector potential \(\mathbf{A}\) is assumed to satisfy the Coulomb calibration

\[
\nabla \cdot \mathbf{A} = 0.
\]

The atom-field interaction is described by the Hamiltonian

\[
\hat{H}_{af} = -\sum_{i=1}^{N} \left( \frac{1}{c} \mathbf{J}_i \cdot \mathbf{A}_i + \mathbf{P}_i \cdot \mathbf{E}_{0i} \right),
\]

in which the short-hand notation is used for the vector potential \(\mathbf{A}_i \equiv \mathbf{A}(\mathbf{r}_i, t)\) and the external electric field \(\mathbf{E}_{0i} \equiv \mathbf{E}_0(\mathbf{r}_i, t)\), and where the transition current is

\[
\mathbf{J}_i = i\omega_0 \left( \mathbf{d}S_i^+ - \mathbf{d}^*S_i^- \right)
\]

and the transition polarization is

\[
\mathbf{P}_i = \mathbf{d}S_i^+ + \mathbf{d}^*S_i^-,
\]

with the atomic transition dipole \(\mathbf{d}\) and the ladder operators \(S_\pm^i \equiv S_x^i \pm iS_y^i\). Except resonant atoms, the laser cavity can contain some additional filling matter [45] interacting with the electromagnetic field through the Hamiltonian

\[
\hat{H}_{mf} = -\frac{1}{c} \int \mathbf{j}_{mat}(\mathbf{r}, t) \cdot \mathbf{A}(\mathbf{r}, t) \, d\mathbf{r},
\]
where \( j_{\text{mat}} \) is the local density of current in the filling matter. Here and in what follows, we set \( \hbar \equiv 1 \).

The field operators satisfy the equal-time commutation relations
\[
\left[ E^\alpha (\mathbf{r}, t), A^\beta (\mathbf{r'}, t) \right] = 4\pi ic \delta_{\alpha\beta} (\mathbf{r} - \mathbf{r'}) ,
\]
\[
\left[ E^\alpha (\mathbf{r}, t), H^\beta (\mathbf{r'}, t) \right] = -4\pi ic \sum_\gamma \varepsilon_{\alpha\beta\gamma} \frac{\partial}{\partial r^\gamma} \delta(\mathbf{r} - \mathbf{r'}) ,
\]
(9)
in which \( \varepsilon_{\alpha\beta\gamma} \) is the unitary antisymmetric tensor [46] and the so-called transverse \( \delta \)-function is
\[
\delta_{\alpha\beta} (\mathbf{r}) \equiv \int \left( \delta_{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2} \right) \frac{d^3 k}{(2\pi)^3} = \delta_{\alpha\beta} \delta(\mathbf{r}) + \frac{1}{4\pi} D_{\alpha\beta} (\mathbf{r}) ,
\]
(10)
where in the dipolar tensor
\[
D_{\alpha\beta} (\mathbf{r}) \equiv \frac{\delta_{\alpha\beta} - 3 n^\alpha n^\beta}{r^3}
\]
(11)
one has \( n \equiv \mathbf{r}/r = \{ n^\alpha \} \) and \( r \equiv |\mathbf{r}| \).

The Heisenberg equations of motion for the field operators yield
\[
\frac{1}{c} \frac{\partial E}{\partial t} = \nabla \times \mathbf{H} - \frac{4\pi}{c} \mathbf{j} , \quad \frac{1}{c} \frac{\partial A}{\partial t} = -\mathbf{E} ,
\]
(12)
from where, with the Coulomb calibration (4), one has the equation for the vector potential
\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A} = -\frac{4\pi}{c} \mathbf{j} ,
\]
(13)
with the density of current
\[
j^\alpha (\mathbf{r}, t) = \sum_\beta \left[ \sum_{i=1}^N \delta_{\alpha\beta} (\mathbf{r} - \mathbf{r}_i) j^\beta_i (t) + \int \delta_{\alpha\beta} (\mathbf{r} - \mathbf{r'}) j_{\text{mat}}^\beta (\mathbf{r'}, t) \, d\mathbf{r'} \right] .
\]
(14)
The known solution to Eq. (13) is the sum
\[
\mathbf{A} (\mathbf{r}, t) = \mathbf{A}_{\text{vac}} (\mathbf{r}, t) + \frac{1}{c} \int \mathbf{j} \left( \mathbf{r'}, t - \frac{\mathbf{r} - \mathbf{r'}}{c} \right) \frac{d\mathbf{r'}}{|\mathbf{r} - \mathbf{r'|}}
\]
(15)
of the vacuum vector potential and the retarded potential.

The Heisenberg equations of motion for the spin operators, satisfying the commutation relations
\[
\left[ S^+_i , S^-_j \right] = 2\delta_{ij} S^z_i , \quad \left[ S^z_i , S^\pm_j \right] = \pm\delta_{ij} S^\pm_i
\]
lead the equations
\[
\frac{dS^-_i}{dt} = -i\omega_0 S^-_i + 2S^z_i \left( k_0 \mathbf{d} \cdot \mathbf{A} - i \mathbf{d} \cdot \mathbf{E}_0 \right) ,
\]
of the vacuum vector potential and the retarded potential.
\[ \frac{dS_z^\alpha}{dt} = -S^\alpha_1 (k_0 \mathbf{d} \cdot \mathbf{A}_i - i \mathbf{d} \cdot \mathbf{E}_{0i}) - S^\alpha_\perp (k_0 \mathbf{d}^* \cdot \mathbf{A}_i + i \mathbf{d}^* \cdot \mathbf{E}_{0i}) . \] (16)

These are complimented by the retardation condition
\[ S^\alpha_i (t) = 0 \quad (t < 0) . \] (17)

In order that the notion of resonant atoms would have sense, one requires, as usual, that the atom-field interactions are small compared to the atomic transition energy, so that \( |\mathbf{d} \cdot \mathbf{E}| \ll \omega_0 \). Because of this, the retardation can be taken into account in the Born approximation
\[ S_j^-(t - \frac{r}{c}) = S_j^-(t) \Theta(ct - r) e^{ik_0r} , \quad S_j^z(t - \frac{r}{c}) = S_j^z(t) \Theta(ct - r) , \] (18)

where \( k_0 \equiv \omega_0/c \) and \( \Theta(t) \) is the unit-step function.

The idea of eliminating the field operators is based on the usage of the pseudospin equations (16), with the substituted there the vector potential (15). In this way, one meets the terms corresponding to the atomic self-action, which in the present approach can be treated as follows. The vector potential generated by a single atom is
\[ \mathbf{A}_s(r, t) = \frac{1}{c} \int \frac{\delta_{\alpha\beta}(r')}{|r - r'|} J \left( t - \frac{|r - r'|}{c} \right) dr' , \] (19)

with the current
\[ J \left( t - \frac{r}{c} \right) = i \omega_0 \left[ \mathbf{d} S^+(t) e^{-ik_0r} - \mathbf{d}^* S^-(t) e^{ik_0r} \right] \Theta(ct - r) , \]

where \( S^\alpha(t) \equiv S^\alpha(0, t) \). At small distance, such that \( k_0r \ll 1 \), one may write \( e^{ik_0r} \approx 1 + ik_0r \).

Substituting the transverse \( \delta \)-function (10) into the vector potential (19), we keep in mind that averaging the dipolar tensor (11) over spherical angles gives
\[ \int D_{\alpha\beta}(r) d\Omega(r) = 0 . \]

Then, for \( k_0r \ll 1 \), the vector potential (19) becomes
\[ \mathbf{A}_s(r, t) \approx \frac{2}{3} k_0^2 \left[ \mathbf{d} S^+(t) + \mathbf{d}^* S^-(t) \right] + i \frac{2k_0}{3r} \left[ \mathbf{d} S^+(t) - \mathbf{d}^* S^+(t) \right] . \] (20)

To avoid the divergence in the term \( 1/r \), let us average it between the electron wavelength \( \lambda_e = 2\pi \hbar/mc \), with \( m \) being the electron mass, and the radiation wavelength \( \lambda_0 = 2\pi/k_0 \). Taking into account that \( \lambda_e \ll \lambda_0 \), we have
\[ \frac{1}{\lambda_0 - \lambda_e} \int_{\lambda_e}^{\lambda_0} \frac{dr}{r} = \frac{k_0}{2\pi} \ln \left( \frac{mc^2}{\hbar} \right) \]

Then for the self-acting vector potential, we get
\[ \mathbf{A}_s(0, t) = \frac{2}{3} k_0^2 \left[ \mathbf{d} S^+(t) + \mathbf{d}^* S^-(t) \right] + i \frac{k_0}{3\lambda_0} \ln \left( \frac{mc^2}{\hbar \omega_0} \right) \left[ \mathbf{d} S^+(t) - \mathbf{d}^* S^-(t) \right] . \] (21)
Substituting this into Eqs. (16) for the case of a single atom, we employ the properties of operators of spin \( \frac{1}{2} \),
\[
S^- S^- = S^+ S^+ = 0, \quad S^z S^z = \frac{1}{4}, \quad S^- S^z = \frac{1}{2} S^-, \quad S^z S^- = -\frac{1}{2} S^-, \quad S^+ S^z = -\frac{1}{2} S^+.
\]
Then we come to the equations for a single atom
\[
dS^-/dt = -i (\omega_0 - \delta_L - i\gamma_0) S^- + \frac{d^2}{|d|^2} (\gamma_0 + i\delta_L) S^+, \quad dS^z/dt = -2\gamma_0 \left( \frac{1}{2} + S^z \right), \tag{22}
\]
in which the notation for the natural width
\[
\gamma_0 \equiv \frac{2}{3} |d|^2 k_0^3 \tag{23}
\]
and the Lamb shift
\[
\delta_L \equiv \frac{\gamma_0}{2\pi} \ln \left( \frac{mc^2}{\hbar\omega_0} \right) \tag{24}
\]
are introduced. The solutions to Eqs. (22), keeping in mind that \( \gamma_0 \ll \omega_0 \) and \( \delta_L \ll \omega_0 \), are
\[
S^- (t) = S^- (0) \exp \left\{ -i(\omega_0 - \delta_L) t - \gamma_0 t \right\},
\]
\[
S^z (t) = -\frac{1}{2} + \left[ \frac{1}{2} + S^z (0) \right] \exp (-2\gamma_0 t).
\]
Thus, the existence of self-action leads to the appearance of attenuation in the dynamics of the spin operators and to the Lamb frequency shift. The latter can always be included in the definition of the transition frequency \( \omega_0 \). Taking into consideration the attenuation, one usually generalizes the equations of motion by including \( \gamma_0 \), instead of \( \gamma_2 \), for \( S^- \) and inserting \( \gamma_1 \), instead of \( 2\gamma_0 \), for \( S^z \).

For a system of \( N \) radiating atoms, the vector potential (15) can be presented as a sum
\[
A = A_{\text{vac}} + A_{\text{rad}} + A_{\text{dip}} + A_{\text{mat}}. \tag{25}
\]
Here \( A_{\text{vac}} \) is caused by vacuum fluctuations. The vector potential
\[
A_{\text{rad}} (r, t) = \sum_j \frac{2}{3c|r - r_j|} J_j \left( t - \frac{|r - r_j|}{c} \right) \tag{26}
\]
is due to the spherical part of the potential (15), produced by radiating atoms, which in addition produce the dipolar part
\[
A^\alpha_{\text{dip}} (r, t) = -\sum_j \sum_{\beta} \int \frac{D_{\alpha\beta}(r' - r_j)}{4\pi c|r - r'|} J_j^\beta \left( t - \frac{|r - r'|}{c} \right) dr'. \tag{27}
\]
Finally, the action of matter, filling the cavity, creates the vector potential
\[
A^\alpha_{\text{mat}} (r, t) = \sum_{\beta} \int \frac{\delta_{\alpha\beta}(r'' - r''')}{c|r - r'|} J_{\text{mat}}^\beta \left( r''' - \frac{|r - r'|}{c} \right) dr' dr''. \tag{28}
\]
Let us combine the vacuum, dipole, and matter vector potentials into the sum
\[
\xi(r, t) \equiv 2k_0 \mathbf{d} \cdot (\mathbf{A}_{\text{vac}} + \mathbf{A}_{\text{dip}} + \mathbf{A}_{\text{mat}}) ,
\]
which describes local field fluctuations. Owing to the local nature of the fluctuating field (29), it can be treated as a random variable. Contrary to this, the radiation potential (26) is of long-range nature and can be responsible for collective effects. The existence of two types of variables, acting on different spatial scales, makes it possible to employ the scale separation approach [5,47–50]. Then \( \mathbf{A}_{\text{rad}} \) and \( \xi \) are considered as different operators. Since \( \mathbf{A}_{\text{rad}} \) is expressed through the spin operators, we may treat the set \( S_j \) of these operators as approximately commuting with \( \xi \). Thus, the total set of operators consists of two types of the operators, \( S \) and \( \xi \). For any operator function \( \hat{F} = \hat{F}(S, \xi) \), we may introduce two kinds of averages. One is the average over the spin variables,
\[
\langle \hat{F} \rangle \equiv \text{Tr}_S \hat{\rho} \hat{F}(S, \xi) ,
\]
with the trace over spins, and \( \hat{\rho} \) being a statistical operator. Another average is defined as
\[
\langle\langle \hat{F} \rangle\rangle \equiv \text{Tr}_\xi \hat{\rho} \hat{F}(S, \xi) ,
\]
with the trace over the stochastic field (29).

For the spin averaging (30), we may use the decoupling
\[
\langle S_i^a S_j^\beta \rangle = \langle S_i^a \rangle \langle S_j^\beta \rangle \quad (i \neq j) ,
\]
keeping in mind the long-range nature of \( \mathbf{A}_{\text{rad}} \). This reminds us the mean-field approximation, which is valid for long-range forces [51]. However, there is a principal difference between the mean-field approximation and the decoupling (32). The latter involves only the spin degrees of freedom, not touching the stochastic variables \( \xi \), which are responsible for quantum effects. Employing a seemingly semiclassical form (32), at the same time preserving quantum features, associated with stochastic field \( \xi \), is close to the method of stochastic quantization used in quantum field theory [52]. Therefore the decoupling (32) can be called the \textit{stochastic mean-field approximation} or the \textit{quantum mean-field approximation}.

Let us now average the operator equations (16) over the spin variables, according to Eq. (30), with employing the following notation. The \textit{transition function}
\[
u_i(t) \equiv u(r_i, t) \equiv 2 \langle S_i^-(t) \rangle
\]
describes the local dipole transitions. The local characteristic of coherence is the \textit{coherence intensity}
\[
\nu_i(t) \equiv \nu(r_i, t) \equiv 2 \frac{1}{n_0} \sum_{<ij>} \left[ \langle S_i^+(t)S_j^-(t) \rangle + \langle S_j^+(t)S_i^-(t) \rangle \right] ,
\]
in which the sum is over the nearest neighbours and \( n_0 \) is the number of the latter. Finally, the local \textit{population difference} is
\[
\nu_i(t) \equiv s(r_i, t) \equiv 2 \langle S_i^z(t) \rangle .
\]
It is also convenient to pass to the continuous spatial representation, replacing the sums by the integrals as
\[
\sum_{j=1}^{N} \mapsto \rho \int d\mathbf{r} \quad (\rho \equiv \frac{N}{V}).
\] (36)

Let us introduce the notation for the effective field potential acting on atoms,
\[
f(\mathbf{r}, t) = f_0(\mathbf{r}, t) + f_{\text{rad}}(\mathbf{r}, t) + \xi(\mathbf{r}, t),
\] (37)
which consists of the part
\[
f_0(\mathbf{r}, t) \equiv -2i\mathbf{d} \cdot \mathbf{E}_0(\mathbf{r}, t),
\] (38)
due to an external electric field \(\mathbf{E}_0\), of the term
\[
f_{\text{rad}}(\mathbf{r}, t) \equiv 2k_0 \mathbf{d} \cdot \mathbf{A}_{\text{rad}}(\mathbf{r}, t),
\] (39)
caused by the radiating atoms, and of the fluctuating random field (29). The radiation potential (39), taking account of Eq. (26), can be written as
\[
f_{\text{rad}}(\mathbf{r}, t) = -i\gamma_0\rho \int \left[ G(\mathbf{r} - \mathbf{r}', t)u(\mathbf{r}', t) - \frac{d^2}{|d|^2} G^*(\mathbf{r} - \mathbf{r}', t)u^*(\mathbf{r}', t) \right] d\mathbf{r}',
\] with the transfer kernel
\[
G(\mathbf{r}, t) \equiv \frac{\exp(ik_0r)}{k_0r} \Theta(ct - r).
\]

In this way, from Eqs. (16), we obtain for the local functions (33) to (35) the evolution equations
\[
\frac{\partial u}{\partial t} = -(i\omega_0 + \gamma_2)u + fs, \quad \frac{\partial w}{\partial t} = -2\gamma_2w + (u^*f + f^*u)s, \quad \frac{\partial s}{\partial t} = -\frac{1}{2} (u^*f + f^*u) - \gamma_1(s - \zeta),
\] (40)
where \(\gamma_1\) and \(\gamma_2\) are the longitudinal and transverse attenuation rates and \(\zeta\) is the stationary pumping parameter. These stochastic differential equations are the basic equations to be used in what follows for describing the spatio-temporal evolution in lasers.

### 4 Turbulent Photon Filamentation

The external field \(\mathbf{E}_0\) is here the seed field
\[
\mathbf{E}_0(\mathbf{r}, t) = \frac{1}{2} E_1 e^{i(kz-\omega t)} + \frac{1}{2} E_1^* e^{-i(kz-\omega t)},
\] (41)
selecting a longitudinal mode of frequency \(\omega = kc\), but imposing no constraints on possible transverse modes. The propagation of the field (41) is along the axis \(z\), which is the axis of a cylindrical laser cavity. The frequency \(\omega\) has to be in resonance with the atomic transition frequency \(\omega_0\), so that the detuning be small,
\[
\frac{|\Delta|}{\omega_0} \ll 1 \quad (\Delta \equiv \omega - \omega_0).
\] (42)
The wavelength \( \lambda \equiv 2\pi c/\omega \) is usually much smaller than the effective laser radius \( R \) and length \( L \),
\[
\frac{\lambda}{R} \ll 1, \quad \frac{\lambda}{L} \ll 1.
\] (43)

The seed field (41) is very weak, such that
\[
\frac{|\nu_1|}{\omega_0} \ll 1 \quad (\nu_1 \equiv d \cdot E_1).
\] (44)

Keeping in mind the possibility of arising filamentary structures, we should look for the solution of Eqs. (40) in the form of the modal superposition
\[
u(r, t) = \sum_{n=1}^{N_f} u_n(r_\perp, t) e^{ikz}, \quad w(r, t) = \sum_{n=1}^{N_f} w_n(r_\perp, t),
\] (45)
in which \( N_f \) is the number of filaments and \( r_\perp \equiv \sqrt{x^2 + y^2} \). Note that the representation (45) is rather general and includes as well the case of no filamentary structure, when \( N_f = 1 \). The number of filaments \( N_f \) will be defined later in a self-consistent way. From the point of view of quantum field theory, the appearance of spatial structures corresponds to the existence of nonuniform field vacuum [53,54]. In the case when different filaments are not correlated with each other, one has
\[
u_m(r_\perp, t) s_n(r_\perp, t) = \delta_{mn} u_n(r_\perp, t) s_n(r_\perp, t).
\] (46)

This condition is typical of the turbulent regime, when the filaments are not mutually correlated [36–42]. Since filaments do not interact with each other, their location in the transverse cross-section is random. The radiation inside each filament is mainly concentrated along the filament axis, fading away at the distance much larger than the filament radius. In general, there can simultaneously exist the filaments of different radii.

Let us consider an \( n \)-th filament. And let the radiation in this filament be an order of magnitude weaker at the distance \( R_n \) from its axis than at the latter, so that the intensity function \( w_n(R_n, t) \) is an order of magnitude smaller at \( r_\perp = R_n \) than \( w_n(0, t) \) at \( r_\perp = 0 \). The effective radius of the filament, \( r_n \), can be defined by the averaging relation
\[
\frac{2}{R_n^2} \int_0^{R_n} w_n(r_\perp, t) r_\perp \, dr_\perp = w_n(r_n, t).
\] (47)
If the profile of the intensity function \( w_n \) is of normal law, that is, 
\[
w_n(r_\perp, t) = w_n(0, t) \exp \left( -\frac{r_\perp^2}{2r_n^2} \right),
\] (48)
where the filament radius \( r_n \) plays the role of the standard deviation, then the relation (47) yields
\[
r_n = \frac{R_n}{(4\pi)^{1/4}} = 0.55 \, R_n.
\] (49)
All radiation of a filament, with an effective radius \( r_n \), is concentrated inside the enveloping cylinder of radius \( R_n \). Let us define the averaged functions
\[
\begin{align*}
u(t) &\equiv \frac{1}{V_n} \int_{V_n} u_n(r_\perp, t) \, dr, \\
w(t) &\equiv \frac{1}{V_n} \int_{V_n} w_n(r_\perp, t) \, dr, \\
s(t) &\equiv \frac{1}{V_n} \int_{V_n} s_n(r_\perp, t) \, dr,
\end{align*}
\]
where the averaging is over the enveloping cylinder of volume \( V_n \equiv \pi R_n^2 L \), and the enumeration index \( n \) for the left-hand side functions is omitted in order to simplify the notation.

For what follows, we shall need the definition of the coupling functions
\[
\begin{align*}
\alpha(t) &\equiv \gamma_0 \rho \int_{V_n} \Theta(ct - r) \frac{\sin(k_0 r - k z)}{k_0 r} \, dr, \\
\beta(t) &\equiv \gamma_0 \rho \int_{V_n} \Theta(ct - r) \frac{\cos(k_0 r - k z)}{k_0 r} \, dr.
\end{align*}
\]
Introduce also the averaged stochastic field
\[
\xi(t) \equiv \frac{1}{V_n} \int \xi(r, t)e^{-i k z} \, dr
\]
and the field potential
\[
f_1(t) \equiv -i \mathbf{d} \cdot \mathbf{E}_1 e^{-i \omega t} + \xi(t).
\]
Then, substituting the representation (45) into the evolution equations (40) and averaging according to Eq. (50), we come to the equations
\[
\begin{align*}
\frac{du}{dt} &= -i(\omega_0 + \beta s)u - (\gamma_2 - \alpha s)u + f_1 s, \\
\frac{dw}{dt} &= -2(\gamma_2 - \alpha s)w + (u^* f_1 + f_1^* u)s, \\
\frac{ds}{dt} &= -\alpha w - \frac{1}{2}(u^* f_1 + f_1^* u)s - \gamma_1 (s - \zeta),
\end{align*}
\]
describing the dynamics of each of the filaments.

In this way, the representation (45), together with the averaging procedure (50), have made it possible to pass from the equations (40) in partial derivatives to the ordinary differential equations (55). Following further the scale separation approach [5,47–50], we can more simplify Eqs. (55) by taking into account the existence of small parameters related to the standard situation, when the attenuation rates are essentially smaller than the transition frequency,
\[
\frac{\gamma_0}{\omega_0} \ll 1, \quad \frac{\gamma_1}{\omega_0} \ll 1, \quad \frac{\gamma_2}{\omega_0} \ll 1.
\]
This tells us that the function \( u \) in Eqs. (55) is fast, as compared to the slow functions \( w \) and \( s \), which are the temporal quasi-invariants of motion. The collective width
\[
\Gamma \equiv \gamma_2 - \alpha s,
\]
collective frequency
\[
\Omega \equiv \omega_0 + \beta s,
\]
and effective detuning
\[ \delta \equiv \omega - \Omega = \Delta - \beta s \quad (59) \]
are also slow functions in time, as compared to the fast function \( u \). The latter can be found from the first of Eqs. (55), which gives
\[
u_1 \frac{u_0}{\delta + i\Gamma} e^{-i(\Omega + \Gamma)t} + \frac{\nu_1 s}{\delta + i\Gamma} e^{-iw}\]
\[ + s \int_0^t \xi(t') e^{-i(\Omega + \Gamma)(t-t')} dt' . \quad (60) \]

Without the loss of generality, it is possible to choose the phase of the external field \( E_1 \) so that \( u_0^* d \cdot E_1 \) be real, that is,
\[ u_0^* \nu_1 = u_0 \nu_1^* , \quad u_0 \equiv u(0) . \quad (61) \]

The solution (60) has to be substituted into the second and third of Eqs. (55), which are the equations for the slow functions. Then the right-hand sides of these equations are to be averaged over time and over the stochastic variables according to the rule
\[ \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \ll \ldots \gg \ dt , \]
keeping fixed all temporal quasi-invariants. The stochastic variable \( \xi \), describing local fluctuations, is assumed to be such that
\[ \ll \xi(t) \gg = 0 . \quad (62) \]

Define the effective attenuation
\[ \tilde{\Gamma} \equiv \gamma_3 + \frac{|\nu_1|^2 \Gamma}{\delta^2 + \Gamma^2} \left(1 - e^{-\Gamma t}\right) , \quad (63) \]
in which \(|\delta| < |\Gamma|\) and the first term is caused by the quantum local fluctuations resulting in the quantum attenuation
\[ \gamma_3 \equiv \text{Re} \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dt \int_0^t \ll \xi(t') \xi(t) \gg e^{-(i(\Omega + \Gamma)(t-t')} dt' . \quad (64) \]

Thus we obtain the equations for the guiding centers
\[ \frac{dw}{dt} = -2(\gamma_2 - \alpha s)w + 2\tilde{\Gamma}s^2 , \quad \frac{ds}{dt} = -\alpha w - \tilde{\Gamma}s - \gamma_1(s - \zeta) . \quad (65) \]

Here the fast field fluctuations have been averaged out, and only the slow dynamics of filaments is left.
5 Transition Dipolar Waves

An essential part of the quantum local fluctuations, leading to the quantum attenuation (64), is due to the dipolar vector potential (27) entering the local field (29). In order to better understand the physical origin of these fluctuations, let us consider the dipolar part of the local field (29) having the form

\[ 2k_0 d \cdot A_{\text{dip}}(r_i, t) = i \sum_{j(\neq i)} \left[ b_{ij}^* S_j^-(t) - c_{ij} S_j^+(t) \right], \quad (66) \]

where we use the notation

\[ b_{ij}^* \equiv \frac{k_0^2}{2\pi} \sum_{\alpha\beta} d^{\alpha} \left( D_{ij}^{\alpha\beta} d^{\beta} \right)^*, \quad c_{ij} \equiv \frac{k_0^2}{2\pi} \sum_{\alpha\beta} d^{\alpha} D_{ij}^{\alpha\beta} d^{\beta}, \]

\[ D_{ij}^{\alpha\beta} \equiv \int \Theta(ct - |r_i - r'|) \frac{D_{ij}^\alpha(r' - r_j)}{|r_i - r'|} \exp \left( -i k_0 |r_i - r'| \right) \, dr'. \]

Leaving in the pseudospin equations (16) only the terms related to the dipolar part of the vector potential, we have

\[ \frac{dS_i^-}{dt} = -i \omega_0 S_i^- + i S_i^z \sum_{j(\neq i)} \left( b_{ij}^* S_j^- - c_{ij} S_j^+ \right), \]

\[ \frac{dS_i^z}{dt} = -\frac{i}{2} S_i^+ \sum_{j(\neq i)} \left( b_{ij}^* S_j^- - c_{ij} S_j^+ \right). \quad (67) \]

These equations describe the pseudospin fluctuations, which can be characterized by the deviations

\[ \delta S_i^- = S_i^- - < S_i^- >, \quad \delta S_i^z = S_i^z - < S_i^z > \]

from the corresponding average values. Linearizing Eqs. (67) with respect to the deviations (68), under the condition \(< S_i^- >= 0\), shows that

\[ S_i^- = \delta S_i^- , \quad S_i^z = \text{const}. \]

Employing the Fourier transforms for the pseudospin operators

\[ S_j^- = \sum_k S_k^- \exp (i\mathbf{k} \cdot r_j) , \quad S_k^- = \frac{1}{N} \sum_j S_j^- \exp (-i\mathbf{k} \cdot r_j) \]

and, similarly, for the coefficients

\[ b_{ij} = \sum_k b_k \exp (i\mathbf{k} \cdot r_j) , \quad c_{ij} = \sum_k c_k \exp (i\mathbf{k} \cdot r_j) , \]

where \( r_{ij} \equiv r_i - r_j \), we get the equations for \( S_k^- \) and \( S_k^+ \) which read

\[ \frac{dS_k^-}{dt} = -i \mu_k S_k^- - i \lambda_k S_k^+ , \quad \frac{dS_k^+}{dt} = i \mu_k S_k^+ + i \lambda_k S_k^- , \quad (70) \]
where
\[ \mu_k \equiv \omega_0 - b_k^* < S_i^z >, \quad \lambda_k \equiv c_k < S_i^z >. \] (71)
The solution to Eqs. (70) has the form
\[ S_k^- = u_k e^{-i\omega_k t} + v_k^* e^{i\omega_k t}, \] (72)
with the spectrum
\[ \omega_k = \sqrt{|\mu_k|^2 - |\lambda_k|^2}. \] (73)
This means that the dipolar part of the vector potential generates local fluctuations realized as a kind of transition dipolar waves. Such waves are analogous to the dipolar spin waves in magnets [55]. The spectrum (73) is always positive, since \(|b_k|^2 = |c_k|^2 \ll \omega_0\). This implies that the transition dipolar waves are stable. In the long-wave limit \(k \to 0\), one has \(b_0 = c_0\), because of which
\[ \lim_{k \to 0} \omega_k = \omega_0. \] (74)

The transition dipolar waves are responsible only for local fluctuations, but they do not participate in collective effects which are related to the coupling functions (51) and (52). These functions are zero at the initial time \(t = 0\), though they grow very fast. Collective effects come to play after the interaction time \(\tau_{int} = a/c\), where \(a\) is the nearest-neighbour distance and \(c\) light velocity. For \(a \sim 10^{-8}\) cm and \(c \sim 10^{10}\) cm/s, this time is very short, \(\tau_{int} \sim 10^{-18}\) s. After this time, the coupling function (51) quickly reaches its maximal value \(g'\gamma_2\) and (52) grows to \(g'\gamma_2\). Here the coupling parameters
\[ g \equiv \rho \gamma_0 \frac{\gamma_2}{\gamma_2} \int_{V_n} \frac{\sin(k_0 r - k z)}{k_0 r} dr \] (75)
and, respectively,
\[ g' \equiv \rho \gamma_0 \frac{\gamma_2}{\gamma_2} \int_{V_n} \frac{\cos(k_0 r - k z)}{k_0 r} dr \] (76)
are introduced. Recall that for each filament there are its own coupling parameters, whose values depend on the filament characteristics and the number of atoms in the filament. Strictly speaking, collective effects appear already for two atoms [56,57]. However noticeable radiation coherence develops only when a large number of atoms is involved.

For \(t \gg \tau_{int}\), the collective width (57) becomes
\[ \Gamma = \gamma_2 (1 - gs) \] (77)
and the collective frequency (58) is
\[ \Omega = \omega_0 + g'\gamma_2 s. \] (78)
When the seed field is negligibly small, so that \(|\nu_1| \ll \gamma_2\), then the effective attenuation (63) simplifies to
\[ \tilde{\Gamma} \simeq \gamma_3. \] (79)
Therefore, for times after \(\tau_{int}\), the evolution equations (65) rearrange to the equations
\[ \frac{dw}{dt} = -2\gamma_2(1 - gs)w + 2\gamma_3 s^2, \quad \frac{ds}{dt} = -g\gamma_2 w - \gamma_3 s - \gamma_1(s - \zeta), \] (80)
characterizing the filament dynamics. Recall that Eqs. (80) describe the slow dynamics, since the uncorrelated fast oscillations have been averaged out in the process of derivation of Eqs. (80). The random fast fluctuations are of less physical importance, being, in addition, much smaller than the correlated slow functions [36–42].

6 Temporal Dynamics of Filaments

Equations (80), describing the filament dynamics, are to be complemented by the initial conditions \( w_0 = w(0) \) and \( s_0 = s(0) \). The first dynamic stage in the interval \( 0 < t < \tau_{\text{int}} \) can be ignored because of a very short interaction time \( \tau_{\text{int}} \). In order that at the initial time \( t = 0 \) the coherence intensity would be an increasing function of time, such that \( dw/dt > 0 \), it is necessary and sufficient that the inequality

\[
\gamma_2(g s_0 - 1) w_0 + \gamma_3 s_0^2 > 0 \tag{81}
\]

be valid, which is a criterion of laser generation. Since the second term in Eq. (81) is always non-negative, a sufficient condition for laser generation is \( gs_0 > 1 \).

A necessary condition for the formation of filaments is the appearance of coherence between radiating atoms. At the beginning, there exists a quantum stage of spontaneously radiating atoms, before coherence develops to a noticeable amount. The incoherent quantum stage lasts till the crossover time \( t_c \), after which coherent effects become dominant, and filaments are being formed. During the quantum stage \( 0 < t < t_c \), the development of coherence is caused by the quantum fluctuations related to the term \( 2 \gamma_3 s^2 \) in the first of Eqs. (80). At this quantum stage, the system is yet uniform. Filaments are well formed after the crossover time \( t_c \). The formation of filaments inside a uniform atomic system, due to local quantum fluctuations, is somewhat analogous to the formation of galaxies from a uniform matter, due to local fluctuations [58], or to the stratification in quantum systems [59,60]. But the principal feature of laser filamentation is that the atomic matter in the laser cavity does not stratify as such. Nonuniformity happens for the distribution of radiating atoms, that is, it occurs on the level of photons, but not atoms themselves. This is why the filamentation in lasers can be called the photon filamentation.

To estimate the crossover time \( t_c \), we need, first, to consider the quantum stage of evolution, when there is no yet self-organized coherence. Assuming that at the initial time there is no coherence imposed by external fields, so that \( w_0 = 0 \), we get the equations for the quantum stage,

\[
\frac{d w}{d t} = 2 \gamma_3 s^2 , \quad \frac{d s}{d t} = -(\gamma_1 + \gamma_3) s + \gamma_2 \zeta . \tag{82}
\]

The solution for the population difference is

\[
s = \left( s_0 - \frac{\gamma_1 \zeta}{\gamma_1 + \gamma_3} \right) \exp\left\{-(\gamma_1 + \gamma_3) t\right\} + \frac{\gamma_1 \zeta}{\gamma_1 + \gamma_3} .
\]

At short time, when \( (\gamma_1 + \gamma_3) t \ll 1 \), this yields

\[
s \simeq s_0 - \left( s_0 - \frac{\gamma_1 \zeta}{\gamma_1 + \gamma_3} \right) (\gamma_1 + \gamma_3) t + \frac{1}{2} \left( s_0 - \frac{\gamma_1 \zeta}{\gamma_1 + \gamma_3} \right) (\gamma_1 + \gamma_3)^2 t^2 . \tag{83}
\]
The latter form, since usually $\gamma_1 < \gamma_3$, is valid if $\gamma_3 t_c \ll 1$. At this quantum stage, the solution for the coherence intensity, given by the first of Eqs. (82), essentially depends on the initial value $s_0$ of the population difference. For an arbitrary $s_0$, we have

$$w \simeq 2\gamma_3 s_0^2 t - 2\gamma_3 s_0 (\gamma_1 + \gamma_3) \left(s_0 - \frac{\gamma_1 \zeta}{\gamma_1 + \gamma_3}\right) t^2 +$$

$$\frac{2}{3} \gamma_3 (\gamma_1 + \gamma_3)^2 \left(s_0 - \frac{\gamma_1 \zeta}{\gamma_1 + \gamma_3}\right) \left(2s_0 - \frac{\gamma_1 \zeta}{\gamma_1 + \gamma_3}\right) t^3.$$  \hspace{1cm} (84)

When $s_0 \neq 0$, then the leading term in the coherence intensity is linear in time,

$$w \simeq 2\gamma_3 s_0^2 t \quad (s_0 \neq 0).$$

But if $s_0 = 0$, then Eq. (84) gives the cubic dependence

$$w \simeq \frac{2}{3} \gamma_1^2 \gamma_3 \zeta^2 t^3 \quad (s_0 = 0).$$

During the quantum stage, the evolution of coherence is mainly due to the term $2\gamma_3 s_0^2$ in the first of Eqs. (80). But at the following coherent stage, the term $2\gamma_2 (gs - 1) w$ in this equation becomes dominant. Therefore, the crossover time $t_c$ can be defined as that one, where the two terms of different nature coincide,

$$\gamma_2 (gs - 1) w = \gamma_3 s_0^2 \quad (t = t_c).$$  \hspace{1cm} (85)

Note that under the initial condition $s_0 = 0$, no noticeable coherence can evolve. This follows from Eq. (85) which, together with Eqs. (83) and (84), yields $t_c \sim T_2 \equiv 1/\gamma_2$. The rising of coherence should occur during the crossover time $t_c \ll T_2$.

If $s_0 \neq 0$, then Eq. (85) gives the crossover time

$$t_c = \frac{s_0/2}{\gamma_1 (s_0 - \zeta) + \gamma_2 (gs_0 - 1)s_0 + \gamma_3 s_0}.$$  \hspace{1cm} (86)

Usually, one has $\gamma_1 \ll \gamma_2 \sim \gamma_3$, and, as is discussed above, one should have $gs_0 > 1$ in order that collective effects could be important. Then the crossover time (86) can be reduced to

$$t_c = \frac{T_2}{2(gs_0 - 1)}.$$  \hspace{1cm} (87)

For $gs_0 \gg 1$, it is evident that $t_c \ll T_2$. At the crossover time (87), solutions (83) and (84) can be approximated as

$$w(t_c) \simeq 2\gamma_3 t_c s_0^2, \quad s(t_c) \simeq s_0.$$  \hspace{1cm} (88)

After the crossover time $t_c$, coherent effects become important. For the coupling parameter $g \gg 1$, keeping in mind that $\gamma_2 \sim \gamma_3$, one has $g\gamma_2 \gg \gamma_3$. In the standard situation, $\gamma_1 \ll \gamma_2$, because of which in the temporal interval $t_c < t \ll T_1 \equiv 1/\gamma_1$ equations (80) can be simplified to the evolution equations

$$\frac{dw}{dt} = -2\gamma_2 (1 - gs) w, \quad \frac{ds}{dt} = -g\gamma_2 w.$$  \hspace{1cm} (89)
describing the transient coherent stage of filament dynamics. These equations can be solved exactly, resulting in the solution

\[ w = \left( \frac{\gamma_p}{g\gamma_2} \right)^2 \text{sech}^2 \left( \frac{t - t_0}{\tau_p} \right), \quad s = \frac{1}{g} - \frac{\gamma_p}{g\gamma_2} \tanh \left( \frac{t - t_0}{\tau_p} \right), \]  
(90)

whose integration parameters are obtained from the initial conditions (88). The pulse width \( \gamma_p \) is defined by the relations

\[ \gamma_p^2 \equiv \gamma_g^2 + 2(g\gamma_2)^2\gamma_3 t_c s_0^2, \quad \gamma_g \equiv (gs_0 - 1)\gamma_2, \quad \gamma_p \tau_p \equiv 1. \]  
(91)

The pulse time, taking into account that \( \gamma_3 t_c \ll 1 \), reads as

\[ \tau_p = \frac{T_2}{gs_0 - 1} \left[ 1 - \frac{\gamma_3 t_c g^2 s_0^2}{(gs_0 - 1)^2} \right]. \]  
(92)

And the delay time is

\[ t_0 = t_c + \frac{\tau_p}{2} \ln \left| \frac{\gamma_p + \gamma_g}{\gamma_p - \gamma_g} \right|. \]  
(93)

Recall that solution (90) is valid only if the pulse time \( \tau_p \) is much less than \( T_1 \). When \( \tau_p \) is of order or much longer than \( T_1 \), as in many experiments [36–42], than one needs to deal with the total Eqs. (80).

In view of the fact that \( \gamma_3 t_c \ll 1 \), the pulse width can be written as

\[ \gamma_p \simeq (gs_0 - 1)\gamma_2 + \frac{g^2\gamma_2 \gamma_3 t_c s_0^2}{gs_0 - 1}, \]

where it is assumed that \( gs_0 > 1 \). Then the delay time (93) becomes

\[ t_0 = t_c + \frac{\tau_p}{2} \ln \left| \frac{2(gs_0 - 1)^2}{g^2 \gamma_3 t_c s_0^2} \right|. \]  
(94)

The latter, under strong coupling, when \( gs_0 \gg 1 \) and \( \tau_p \simeq 2 t_c \), rearranges to

\[ t_0 \simeq t_c + t_c \ln \left| \frac{2}{\gamma_3 t_c} \right|, \]

with \( t_c \simeq T_2/2gs_0 \).

At the final dynamic stage, when \( t \gg T_1 \), we have to consider all terms in Eqs. (80). These can be written in the form

\[ \frac{dw}{dt} = v_1, \quad \frac{ds}{dt} = v_2, \]  
(95)

with the notation

\[ v_1 = -2\gamma_2(1 - gs)w + 2\gamma_3 s^2, \quad v_2 = -g\gamma_2 w - \gamma_3 s - \gamma_1(s - \zeta). \]
The stability properties of the solutions to Eq. (95) are characterized by the Jacobian matrix 
\[ \hat{J}(t) = [J_{ij}(t)] \], with the elements
\[ J_{11} \equiv \frac{\partial v_1}{\partial w} = 2\gamma_2(gs - 1) , \quad J_{12} \equiv \frac{\partial v_1}{\partial s} = 2\gamma_2gw + 4\gamma_3s , \]
\[ J_{21} \equiv \frac{\partial v_2}{\partial w} = -g\gamma_2 , \quad J_{22} \equiv \frac{\partial v_2}{\partial s} = -\gamma_1 - \gamma_3 . \] (96)

Considering the stationary solutions to Eqs. (95), given by the equations \( v_1 = v_2 = 0 \), we will present only those of them that correspond to stable fixed points. It is possible to separate out three cases, depending on the values of the coupling \( g \) and pumping parameter \( \zeta \).

When \( g\zeta \ll -1 \), then the stationary solutions
\[ w^* \simeq \frac{\gamma_3|\zeta|}{\gamma_2|g|} , \quad s^* \simeq \zeta \left( 1 - \frac{\gamma_3}{\gamma_1|g\zeta|} \right) , \] (97)
present a stable node, since the eigenvalues of the Jacobian matrix, that is, the characteristic exponents, are
\[ J_1 \simeq -\gamma_1 - \gamma_3 , \quad J_2 \simeq -2\gamma_2|g\zeta| . \]

Then the sole transient coherent pulse occurs, after which the radiation intensity diminishes to a value proportional to \( w^* \).

For \( |g\zeta| \ll 1 \), the stationary solutions
\[ w^* \simeq \left( \frac{\gamma_1\zeta}{\gamma_1 + \gamma_3} \right)^2 \frac{\gamma_3}{\gamma_2} \left[ 1 + \frac{\gamma_1(\gamma_1 - \gamma_3)|g\zeta|}{(\gamma_1 + \gamma_3)^2} \right] , \quad s^* \simeq \frac{\gamma_1\zeta}{\gamma_1 + \gamma_3} \left( 1 - \frac{\gamma_1\gamma_3|g\zeta|}{(\gamma_1 + \gamma_3)^2} \right) \] (98)
also corresponds to a stable node, with the characteristic exponents
\[ J_1 \simeq -\gamma_1 - \gamma_3 , \quad J_2 \simeq -2\gamma_2 . \]

Hence again, only a sole transient pulse can appear.

Finally, when both the coupling and pumping are strong, so that \( g\zeta \gg 1 \), the stationary solutions
\[ w^* \simeq \frac{\gamma_1\zeta}{\gamma_2g} , \quad s^* \simeq \frac{1}{g} \left( 1 - \frac{\gamma_3}{\gamma_1|g\zeta|} \right) \] (99)
represent a stable focus, with the characteristic exponents
\[ J_{1,2} \simeq -\frac{1}{2} (\gamma_1 + \gamma_3) \pm i\omega_\infty , \]
where the effective asymptotic frequency
\[ \omega_\infty \equiv \sqrt{2g\zeta\gamma_1\gamma_2} . \]

This means that the strong pumping supports the pulsing regime of radiation, when a set of bursts arise, being, at long times, separated from each other by the asymptotic period
\[ T_\infty \equiv \frac{2\pi}{\omega_\infty} = \pi \sqrt{\frac{2T_1T_2}{g\zeta}} . \] (100)
The pulsing regime of radiation resembles pulsating stationary states happening in some statistical systems [61].

In this way, the flashing dynamics of each radiating filament depends on the related coupling parameter \( g \) and the level of stationary pumping defined by the pumping parameter \( \zeta \). In general, there are the following stages of evolution. The interaction stage \( 0 < t < \tau_{\text{int}} \), when each atom radiates independently, before the signal reaches its nearest neighbours. This stage is very short and usually can be neglected. The quantum stage \( \tau_{\text{int}} < t < t_c \), when the radiation dynamics is mainly governed by local quantum fluctuations. After the crossover time \( t_c \), coherent collective effects become important, signifying the presence of the coherent stage. At this latter stage, depending on the value of the pumping parameter \( \zeta \), there occurs either a sole coherent pulse or a series of coherent bursts. If a stationary pumping is absent, then the coherent stage lasts in the interval \( t_c < t < t_0 + \tau_p \), and for \( t \gg t_0 \) it changes to the relaxation stage, when the coherence intensity exponentially diminishes to the low level (97).

### 7 Spatial Structure of Filaments

Filaments are randomly distributed in the transverse cross-section of the laser cavity, evolving in space and time independently of each other. The characteristics of each filament essentially depend on the value of the related coupling parameter (75). For cylindric symmetry the latter can be presented in the form

\[
g = 2\pi \rho \frac{\gamma_0}{\gamma_2} \int_0^{R_n} r_\perp dr_\perp \int_{-L/2}^{L/2} \frac{\sin(k_0 \sqrt{r_\perp^2 + z^2} - kz)}{k_0 \sqrt{r_\perp^2 + z^2}} dz . \tag{101}
\]

Keeping in mind the resonance condition \( k_0 \approx k \) and introducing the variable \( x = k(\sqrt{r_\perp^2 + z^2} - z) \), we have

\[
g = 2\pi \rho \frac{\gamma_0}{k\gamma_2} \int_0^{R_n} r_\perp dr_\perp \int_{kL}^{kL/kr_\perp/L} \frac{\sin x}{x} dx . \tag{102}
\]

Since, according to the inequality (43), we have \( \lambda \ll L \), then the upper limit \( kL \) in the integral (102) can be replaced by \( kL \to \infty \). This gives

\[
g = 2\pi \rho \frac{\gamma_0}{k\gamma_2} \int_0^{R_n} \left[ \frac{\pi}{2} - \text{Si} \left( \frac{kr_\perp^2}{L} \right) \right] r_\perp dr_\perp , \tag{103}
\]

with the integral sine

\[
\text{Si}(x) \equiv \int_0^x \frac{\sin u}{u} du = \frac{\pi}{2} + \int_x^\infty \frac{\sin u}{u} du .
\]

Introducing the notation

\[
\varphi \equiv \frac{\pi R_n^2}{\lambda L} , \tag{104}
\]

varying in the interval \( 0 \leq \varphi \leq \pi F \) and playing the role of an effective Fresnel number for a given filament, we transform Eq. (103) to

\[
g(\varphi) = \pi \frac{\rho \gamma_0 L}{k^2 \gamma_2} \left[ \pi \varphi - \int_0^{2\varphi} \text{Si}(x) dx \right] . \tag{105}
\]
In the same manner, the coupling (76) can be reduced to

\[ g'(\varphi) = -\pi \frac{\rho \gamma_0 L}{k^2 \gamma_2} \int_0^{2\varphi} \text{Ci}(x) \, dx , \tag{106} \]

with the integral cosine

\[ \text{Ci}(x) \equiv \int_x^{\infty} \frac{\cos u}{u} \, du . \]

Performing the integration in Eqs. (105) and (106), we may write

\[ g(\varphi) = \pi \frac{\rho \gamma_0 L}{k^2 \gamma_2} \left[ \pi \varphi - 2\varphi \text{Si}(2\varphi) + 1 - \cos(2\varphi) \right] , \]

\[ g'(\varphi) = \pi \frac{\rho \gamma_0 L}{k^2 \gamma_2} \left[ \sin(2\varphi) - 2\varphi \text{Ci}(2\varphi) \right] . \]

Thus, the coupling parameters are functions of the effective variable (104), which, in turn, depends on the enveloping radius \( R_n \) related to the effective filament radius \( r_n \) by Eq. (49). In general, the filaments of different radii can arise. However, some of them are more stable than other, because of which the overwhelming majority of filaments possess the radii close to a typical value.

The distribution of filaments with respect to their radii and, hence, the typical radius, can be found by invoking the general method of probabilistic pattern selection \[18,21,22\]. Following this approach, we define the probability distribution

\[ p(\varphi, t) = \frac{1}{Z(t)} \exp\{-X(\varphi, t)\} \tag{107} \]

for a filament characterized by the variable \( \varphi \) at the moment of time \( t \). Here

\[ X(\varphi, t) = \text{Re} \int_0^t \text{Tr} \hat{J}(\varphi, t') \, dt' \tag{108} \]

is the \textit{expansion exponent}, expressed through the Jacobian matrix \( \hat{J} \) of the evolution equations, and

\[ Z(t) = \int \exp\{-X(\varphi, t)\} \, d\varphi \]

is the normalizing factor. The expansion exponent (108) defines the \textit{local expansion rate}

\[ \Lambda(\varphi, t) \equiv \frac{1}{t} X(\varphi, t) . \tag{109} \]

The latter can be represented as the sum of the local Lyapunov exponents \[18,21,22\]. The partial sum of only positive Lyapunov exponents defines the entropy production rate \[62,63\], hence, the latter does not coincide with the local expansion rate (109).

Thus, the probability for the appearance of filaments, characterized by the parameter \( \varphi \), is given by the probability distribution (107). As is evident, the most probable is the filament with a typical \( \varphi \) satisfying the \textit{principle of minimal expansion} \[18,21,22\]

\[ \max_{\varphi} p(\varphi, t) \iff \min_{\varphi} X(\varphi, t) \iff \min_{\varphi} \Lambda(\varphi, t) . \tag{110} \]
This general principle follows from the minimization of the pattern information \[22\] and can be employed for arbitrary dynamical systems.

The problem of turbulent photon filamentation is described by the evolution equations (80) or (95). The corresponding Jacobian matrix is given by Eqs. (96) from where

\[
\text{Tr} \hat{J}(\varphi, t) = -\gamma_1 - \gamma_3 - 2\gamma_2(1 - gs),
\]

with \(g = g(\varphi)\) and \(s = s(t)\). For \(t \gg T_1\), the expansion rate (109) can be presented as

\[
\Lambda(\varphi, t) \simeq -\gamma_1 - \gamma_3 - 2\gamma_2(1 - gs^*). \tag{111}
\]

Using Eqs. (97) to (99), we get

\[
\Lambda(\varphi, t) \simeq -\gamma_1 - \gamma_3 - 2\gamma_2 \left(1 - \frac{\gamma_1 g\zeta}{\gamma_1 + \gamma_3}\right) \quad \left(|g\zeta| \ll 1\right),
\]

\[
\Lambda(\varphi, t) \simeq -\gamma_1 - \gamma_3 - 2\gamma_2 \gamma_3 \gamma_1 g\zeta \quad \left(g\zeta \gg 1\right). \tag{112}
\]

The stationary pumping parameter \(\zeta\) is in the interval \(-1 \leq \zeta \leq 1\), depending on the level of pumping. When there is no stationary pumping, \(\zeta = -1\). One says that the pumping is weak, if \(-1 < \zeta < 0\), and it is strong, if \(0 < \zeta < 1\). Keeping in mind that the coupling parameter \(g\) is positive, we see that there exist two different cases, when the stationary pumping is weak or absent, \(\zeta < 0\), and when it is strong, \(\zeta > 0\). According to the principle of minimal expansion (110), the minimum of the expansion rate corresponds to the maximum of \(g(\varphi)\) if \(\zeta < 0\), and to the minimum of \(g(\varphi)\) if \(\zeta > 0\). The extrema of \(g(\varphi)\), as follows from Eq. (105), are given by the equation

\[
\text{Si}(2\varphi) = \frac{\pi}{2}. \tag{113}
\]

In the standard situation of absent or weak pumping, \(\zeta < 0\), we have to look for the absolute maximum of \(g(\varphi)\). Then Eq. (113) gives \(\varphi = 0.96\). From relation (104), we have \(R_n = 0.55\sqrt{\lambda L}\), and from Eq. (49), we find the typical filament radius

\[
r_f = 0.3\sqrt{\lambda L}. \tag{114}
\]

The number of filaments can be estimated as \(N_f \approx R^2/R_n^2\), which yields

\[
N_f \approx 3.3F. \tag{115}
\]

The linear dependence of the filament number on the Fresnel number is characteristic of the turbulent photon filamentation.

Note that under strong stationary pumping \((\zeta > 0)\), when we need to look for the minimum of \(g(\varphi)\), we would have \(\varphi = 2.45\), hence, \(R_n = 0.88\sqrt{\lambda L}\) and \(r_f = 0.5\sqrt{\lambda L}\).

The formula (114) for the typical filaments can be compared with the radii observed in experiments. Thus, in different vapour lasers [25–29], one has \(r_f \approx 0.01\) cm. For the CO$_2$
laser and dye lasers, it was found [36–42] that $r_f \approx 0.1$ cm and $r_f \approx 0.01$ cm, respectively. All these data are in good agreement with formula (114).

It is also worth mentioning that a similar kind of turbulent photon filamentation could arise in another type of matter, called photon band-gap materials. These materials possess a prohibited band gap, where light cannot propagate. The spontaneous radiation of atoms, with a frequency inside the prohibited band gap, is strongly suppressed [64]. However, if the density of doped atoms is sufficiently high, coherent interactions may develop (see review [5]). Then atoms can start radiating even inside the prohibited band gap. An unrealistic model, with the radiation length $\lambda \gg L$ much larger than the system size was considered [65], where atoms with the resonance frequency at the band edge could produce collective spontaneous emission. Collective phenomena in the true atomic radiation of wavelength $\lambda \ll L$, with the atomic frequency inside the prohibited band gap were also considered and the effect of collective liberation of light was predicted [66–70]. Coherent radiation, accompanying this effect, should be realized by means of a bunch of turbulent filaments.

In conclusion, the theory of turbulent photon filamentation in large-aperture lasers has been presented. This theory makes it possible to describe the spatial filamentary stricture as well as its dynamics. The typical filament radius, predicted by the theory, is in good agreement with experiments for different lasers.

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