Abstract

The assumption of normality in data has been considered in the field of statistical analysis for a long time. However, in many practical situations, this assumption is clearly unrealistic. It has recently been suggested that the use of distributions indexed by skewness/shape parameters produce more flexibility in the modelling of different applications. Consequently, the results show a more realistic interpretation for these problems. For these reasons, the aim of this paper is to investigate the effects of the generalisation of a discrimination function method through the class of multivariate extended skew-elliptical distributions, study in detail the multivariate extended skew-normal case and develop a quadratic approximation function for this family of distributions. A simulation study is reported to evaluate the adequacy of the proposed classification rule as well as the performance of the EM algorithm to estimate the model parameters.

Key words: classification; selection distributions; skew-elliptical; extended skew-normal; unobserved variable

1 Introduction

The goal of discriminant analysis is to obtain rules that describe the separation between groups of observations. Discriminant rules are often based on the empirical mean and the covariance matrix of the data (Hubert and Van Driessen, 2004). Several researchers have utilised assumptions of normality in the data for the classification of groups (McLachlan, 1992). However, these studies have prolonged this practice for many years without using the flexible and modern distributions that have been introduced recently. For example, the typical discriminant function method used is the linear discriminant function (LDF) obtained from the

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normality in the data (Bobrowski, 1986), and the quadratic discriminant function (QDF) (Hubert and Van der Veeken, 2010). Posteriorly, a new method was discovered by Azzalini and Capitanio (1999) using the multivariate skew-normal distribution (Azzalini and Dalla Valle, 1996; Azzalini, 2005; Contreras-Reyes and Arellano-Valle, 2012; Lee and McLachlan, 2013; Contreras-Reyes, 2014a,b) to obtain a non-linear discriminant function (NLDF). Hubert and Van Driessen (2004) propose a robust discriminant function obtained by inserting robust estimates into generalised maximum likelihood (ML) rules of normal distributions. In the case of high dimensional data, Hubert and Van der Veeken (2010) propose a robust discriminant method that is adjusted for skewness.

Extensions of the multivariate skew-normal distribution to the so-called skew-elliptical class of multivariate distributions have also been considered by different authors; see, e.g., Fang et al. (1990), Azzalini and Capitanio (1999), Branco and Dey (2001), Arellano-Valle and Genton (2005), Arellano-Valle and Azzalini (2006), Lee and McLachlan (2013), Arellano-Valle et al. (2013), Azzalini (2013) and Contreras-Reyes (2014a). De la Cruz (2008) considered a Bayesian non-linear regression model for longitudinal data to introduce a method of classification in which the residuals are skew-elliptically distributed in the sense defined in Sahu et al. (2003). More recently, Kim (2011) considered a discriminant function for screened data using the perturbed normal distributions on biomedical and psychological examples via ML estimation by an EM algorithm. Some other interesting applications have been released for the skew-elliptical distributions of a biomedical case by De la Cruz (2008), and Reza-Zadkarami and Rowhani (2010) have implemented this method to classify the pixels of satellite images, given the presence of skewness in the data, with the skew-normal distribution based on the approach Azzalini and Dalla-Valle (1996).

A useful method of multivariate classification analysis is generalised in this paper using the class of extended skew-elliptical (ESE) distributions defined by Arellano-Valle and Genton (2010a); see also Arellano-Valle and Genton (2010b) and Azzalini (2013). From this family, we studied the multivariate extended skew-normal (ESN) case (Azzalini and Capitanio, 1999; Capitanio et al., 2003; Contreras-Reyes, 2014b) created by generalising the skew-normal distribution and adding a fourth real parameter, \( \tau \). This last distribution is flexible enough to accommodate skewness and heavy tails. Capitanio et al. (2003) and Pacillo (2012) study the probabilistic properties of this distribution and its utility in the context of graphical models (Stanghellini, 2004). Canale (2011) analysed the likelihood function, the expected information matrix and the MLE parameter estimates. Arellano-Valle et al. (2006), Arellano-Valle and Genton (2010b) and Arellano-Valle and Azzalini (2006) placed this distribution into the more general classes of selection, the unified skew-elliptical (SUE) and unified skew-normal (SUN) distributions, respectively.

We start generalising the multivariate classification analysis to the general family of multivariate selection distributions, focalizing our fitting on the multivariate extended skew-elliptical subclass (Section 2). Then, we explore with special attention the extended skew-normal case, where an approximate discriminant function is derived, an EM algorithm is implemented to obtain the maximum likelihood estimates of the model parameters and a simulation study is developed to evaluated the performance of our finding (Section 3).
Classification rule for two groups after selection

Let $\mathbf{Y} \in \mathbb{R}^{d}$ be a random selection vector defined by $\mathbf{Y} = (\mathbf{X} \mid \mathbf{X}_0 \in C)$, where $\mathbf{X} \in \mathbb{R}^{d}$ and $\mathbf{X}_0 \in \mathbb{R}^{d_0}$ are two correlated random vectors with some known joint distribution and $C \subset \mathbb{R}^{d_0}$ is a proper selection set. If the random vector $\mathbf{X}$ has a probability density function (pdf) $p(\mathbf{x})$, then there exists a pdf for $\mathbf{Y}$ (Arellano-Valle et al., 2006) of the form

$$f(\mathbf{y}) = p(\mathbf{y}) \frac{\text{P}(\mathbf{X}_0 \in C \mid \mathbf{X} = \mathbf{y})}{\text{P}(\mathbf{X}_0 \in C)}.$$

Consider now two groups/populations $\Pi_1$ and $\Pi_2$ screened by a common selection mechanism $\mathbf{X}_0 \in C$. Thus, after selection, the pdf of population $\Pi_i$ is

$$f_i(\mathbf{y}) = p_i(\mathbf{y}) \frac{\text{P}(\mathbf{X}_0 \in C \mid \mathbf{X} = \mathbf{y}, \Pi_i)}{\text{P}(\mathbf{X}_0 \in C \mid \Pi_i)}, \quad i = 1, 2,$$

where $p_i(\mathbf{x}) = p(\mathbf{x} \mid \Pi_i)$ represents the pdf of $\mathbf{X}$ under the group $\Pi_i$, i.e., the pdf of the $i$th group before selection. Note that in (1), we can consider also the following assumption

$$\text{P}(\mathbf{X}_0 \in C \mid \Pi_i) = \text{P}(\mathbf{X}_0 \in C \mid \Pi_2).$$

An important consequence of this condition is that the prior probabilities $\pi_i = \text{P}(\Pi_i), i = 1, 2$, where $\pi_1 + \pi_2 = 1$, are unaffected by the selection mechanism. In fact, under assumption (2) and from Bayes’ theorem, $\text{P}(\Pi_i \mid \mathbf{X}_0 \in C) = \pi_i = \text{P}(\Pi_i), i = 1, 2$.

Let $\mathbf{y}$ be an observed value of a random selection vector $\mathbf{Y}$. A binary classification rule partitions the feature space $\mathbb{R}^{d}$ into disjoint regions $R_1$ and $R_2$. If $\mathbf{y}$ falls into region $R_1$, it is classified as belong to $\Pi_1$, whereas if $\mathbf{y}$ falls into region $R_2$, it is classified into $\Pi_2$. Misclassification occurs either if $\mathbf{y}$ is assigned to $\Pi_2$, but actually belongs to $\Pi_1$, or if $\mathbf{y}$ is assigned to $\Pi_1$, but actually belongs to $\Pi_2$. The total probability of misclassification (TPM) is thus defined by

$$\text{TPM} = \pi_1 \text{P}(\mathbf{Y} \in R_1 \mid \Pi_2) + \pi_2 \text{P}(\mathbf{Y} \in R_2 \mid \Pi_2).$$

Following Welch (1939), McLachlan (1992) and Timm (2002), the optimal classification rule (or Bayes rule) for two groups that minimises the TPM is to allocate $\mathbf{y}$ to $\Pi_1$ if

$$\frac{f_1(\mathbf{y})}{f_2(\mathbf{y})} > \frac{\pi_2 c(2|1)}{\pi_1 c(1|2)},$$

and to assign $\mathbf{y}$ to $\Pi_2$ otherwise, where $c(i|k)$ denotes the cost associated with classifying $\mathbf{y}$ into $\Pi_i$ when, in fact, the correct decision is to classify $\mathbf{y}$ into $\Pi_k, k = 1, 2$. As is well known, (4) is equivalent to assigning
an observation to the population with the largest posterior probability

\[ P(\Pi_i \mid Y = y) = \frac{\pi_i f_i(y)}{\sum_{i=1}^{2} \pi_i f_i(y)}, \quad i = 1, 2. \]

In addition, under the selection pdfs (1), the optimal rule (4) is equivalent to considering the region of classification into \( \Pi_1 \) as defined by the set of \( y \in \mathbb{R}^d \), for which

\[ \frac{p_1(y)}{p_2(y)} > \frac{\pi_2(y)}{\pi_1(y)} \frac{c(2|1)}{c(1|2)} \frac{P(X_0 \in C \mid \Pi_1)}{P(X_0 \in C \mid \Pi_2)}, \quad (5) \]

where

\[ \pi_i(y) = \frac{\pi_i P(X_0 \in C \mid X = y, \Pi_i)}{\sum_{i=1}^{2} \pi_i P(X_0 \in C \mid X = y, \Pi_i)}, \quad i = 1, 2. \]

Moreover, under assumption (2), the optimal rule (5) simplifies to

\[ \frac{p_1(y)}{p_2(y)} > \frac{\pi_2(y)}{\pi_1(y)} \frac{c(2|1)}{c(1|2)}, \quad (6) \]

which is equivalent to assigning \( y \) to the population with the largest posterior selection probability \( \pi_i(y \mid C) = P(\Pi_i \mid X_0 \in C, X = y) \), where

\[ \pi_i(y \mid C) = \frac{\pi_i p_i(y) P(X_0 \in C \mid X = y, \Pi_i)}{\sum_{i=1}^{2} \pi_i p_i(y) P(X_0 \in C \mid X = y, \Pi_i)} = \frac{\pi_i(y) p_i(y)}{\sum_{i=1}^{2} \pi_i(y) p_i(y)}, \quad i = 1, 2. \]

The extension of the classification rule (6) for \( K \geq 2 \) groups is straightforward, and we consider this rule next for a special class of elliptical selection distributions, where \( X_0 \) and \( X \) have a multivariate elliptical joint distribution (Arellano-Valle et al., 2006).

The most well-known class of selection distributions is obtained when we consider a multivariate elliptical joint distribution for \( X_0 \) and \( X \) (Arellano-Valle et al., 2006). In such a case, we obtain the so-called selection elliptical distributions, in which the specification of the selection set \( C \) has an important role in introducing skewness in the selection distribution.

### 2.1 Extended skew-elliptical discriminant functions

We consider the classification rule (6) for which \( d_0 = 1 \), i.e., a classification process when an input vector \( X \) is perturbed by a (latent) screening mechanism \( X_0 + \tau > 0 \) for some constant \( \tau \), where \( X_0 \) is a standardised unity random variable. More specifically, we consider the case where the joint distribution of \( X_0 \) and \( X \) belong to the multivariate elliptical family (Fang et al., 1990), denoted by

\[ X_* = \begin{pmatrix} X_0 \\ X \end{pmatrix} \sim El_{1+d} \begin{pmatrix} 0 \\ \xi_* \end{pmatrix}, \quad \Omega_* = \begin{pmatrix} 1 & \delta^\top \\ \delta & \Omega \end{pmatrix}, \quad h(1+d), \quad (7) \]
where \( \xi \in \mathbb{R}^d \), \( \delta \in \mathbb{R}^d \) and \( \Omega \in \mathbb{R}^{d \times d} \) are such that \( 1 - \delta^\top \Omega \delta > 0 \) and \( \Omega > 0 \) (i.e., positive definite). In addition, \( h^{(d+1)} \) is a \((d+1)\)-variate generator density function, such that

\[
g(w) = \frac{\pi^{(d+1)/2}}{\Gamma((d+1)/2)} w^{(d+1)/2 - 1} h^{(d+1)}(w), \quad w > 0,
\]

is a density on \((0, \infty)\). In other words, in (7) we are assuming that \( X_* = (X_0, X^\top) \) has an elliptical density defined on \( \mathbb{R}^{d+1} \) of the form \( p_*(x_*) = |\Omega_*|^{-1/2} h^{(d+1)}((x_* - \xi_*)^\top \Omega_*^{-1} (x_* - \xi_*)) \).

Under (7), we have \( X_0 \sim El_1(0, 1, h^{(1)}) \), \( X \sim El_d(\mathbf{X}, \Omega, h^{(d)}) \) and \( X_0 | \mathbf{X} = y \sim El_1(\delta^\top \Omega^{-1}(y - \xi), 1 - \delta^\top \Omega^{-1} \delta, h^{(1)}_Q) \), where \( Q = (y - \xi)^\top \Omega^{-1} (y - \xi) \). Hence, the distribution of \( Y \equiv (\mathbf{X} | \mathbf{X_0} + \tau > 0) \) belongs to the class of ESE distributions, with pdf given by

\[
f(y) = \frac{|\Omega|^{-1/2}}{f(\tau; h^{(1)}_Q)} h^{(d)}(Q) F \left( \eta^\top (y - \xi) + \tau; h^{(1)}_Q \right), \quad y \in \mathbb{R}^d, \tag{8}
\]

where \( \eta = \Omega^{-\frac{1}{2}} \sqrt{1 - \delta^\top \Omega^{-1} \delta} \), \( \tau = \tau / \sqrt{1 - \delta^\top \Omega^{-1} \delta} = \tau \sqrt{1 + \eta^\top \Omega \eta} \), and \( h^{(k)} \) \((1 \leq k \leq d)\) is the \(k\)-variate marginal density generator induced by \( h^{(d+1)} \), \( F(x; h^{(1)}) = \int_{-\infty}^x h^{(1)}(y) dy \) and \( F(x; h^{(1)}_Q) = \int_{-\infty}^x h^{(1)}_Q(y) dy \) are the univariate distribution functions induced by the marginal and conditional generators \( h^{(1)} \) and \( h^{(1)}_Q(w) = h^{(d+1)}(w + Q)/h^{(d)}(Q) \), respectively. We write \( Y \sim ESE_d(\xi, \Omega, \eta, \tau, h^{(d)}) \) to indicate that a random vector \( Y \) has pdf (8). For \( \tau = 0 \), we obtain the important subclass of skew-elliptical (SE) distributions, with pdf

\[
f(y) = 2|\Omega|^{-1/2} h^{(d)}(Q) F \left( \eta^\top (y - \xi); h^{(1)}_Q \right), \quad y \in \mathbb{R}^d, \tag{9}
\]

and denoted by \( Y \sim SE_d(\xi, \Omega, \eta, h^{(d)}) \). See Genton (2004), Azzalini (2005), Arellano-Valle and Azzalini (2006) and Arellano-Valle and Genton (2010a, b) for a review of these models.

If two groups \( \Pi_1 \) and \( \Pi_2 \) have ESE distributions satisfying the condition (7), we then have \( \Pi_i : ESE_d(\xi_i, \Omega_i, \eta_i, \tau, h^{(d)}) \), \( i = 1, 2 \). Hence, by applying (9) to each group we conclude that the optimal rule (9) for these ESE groups yields the region of classification into \( \Pi_1 \) defined by the set of \( y \in \mathbb{R}^d \), for which

\[
\frac{h^{(d)}(Q_1)}{h^{(d)}(Q_2)} > \frac{\pi_2 F \left( \eta_2^\top (y - \xi_2) + \bar{\tau}_2; h^{(1)}_Q \right)}{\pi_1 F \left( \eta_1^\top (y - \xi_1) + \bar{\tau}_1; h^{(1)}_Q \right)},
\]

where \( Q_i = (y - \xi_i)^\top \Omega_i^{-1} (y - \xi_i) \) and \( \bar{\tau}_i = \tau \sqrt{1 + \eta_i^\top \Omega_i \eta_i} \), \( i = 1, 2 \). This is equivalent to assigning \( y \) to population with largest posterior selection probability,

\[
\pi_i(y | \tau) = \frac{\pi_i h^{(d)}(Q_i) F \left( \eta_i^\top (y - \xi_i) + \bar{\tau}_i; h^{(1)}_Q \right)}{\sum_{i=1}^2 \pi_i h^{(d)}(Q_i) F \left( \eta_i^\top (y - \xi_i) + \bar{\tau}_i; h^{(1)}_Q \right)}, \quad i = 1, 2.
\]

For \( \tau = 0 \), (9) corresponds to the optimal rule to classify an observation \( y \) in two SE groups \( \Pi_i : SE_d(\xi_i, \Omega_i, \eta_i, h^{(d)}) \), \( i = 1, 2 \). For \( \tau = 0 \) and \( \eta = 0 \), (9) reduces to the optimal classification rule of
two (symmetric) elliptical populations $\Pi_i : ESE_d(\xi_i, \Omega_i, h^{(d)}), i = 1, 2$, which consists of assigning $y$ to $\Pi_1 : E_{\lambda}(\xi_1, \Omega_1, h^{(d)})$ if
\[
\frac{h^{(d)}(Q_1)}{h^{(d)}(Q_2)} > \pi_2/\pi_1,
\]
or to $\Pi_2 : E_{\lambda}(\xi_2, \Omega_2, h^{(d)})$ otherwise.

All of these rules depend on the choice of the generator $h^{(d+1)}$. In discriminant analysis, one of the most convenient and popular choices corresponds to the normal multivariate distribution, for which $h^{(m)}_a(u) = h^{(m)}_a(u) = (2\pi)^{-m/2}e^{-u/2}$ for all $a, u > 0$ and $m \geq 1$. The multivariate normal scale mixture class is another important family of elliptical distributions, in which we find the multivariate $t$ distribution (Arellano-Valle and Bolfarine, 1995) with density generator
\[
h^{(m)}(u) = \frac{\Gamma\left(\frac{m+\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^{m/2}} \left(1 + \frac{u}{\nu}\right)^{-(m+\nu)/2},
\]
where $u > 0$ and the parameter $\nu > 0$ denotes the degrees of freedom.

3 Multivariate extended skew-normal case

The multivariate ESN distribution was introduced in Azzalini and Capitanio (1999) as a first extension of the multivariate skew-normal distribution that was introduced by Azzalini and Dalla Valle (1996) and, was later analysed in detail by Capitanio et al. (2003), Canale (2011), Pacillo (2012) and Azzalini (2013). Here, we consider a slight variant proposed by Capitanio et al. (2003). Let $Y \sim ESN_d(\xi, \Omega, \eta, \tau)$ denote a $d \times 1$-dimensional ESN random vector, with location vector $\xi \in \mathbb{R}^d$, positive definite dispersion matrix $\Omega \in \mathbb{R}^{d \times d}$, shape/skewness parameter $\eta \in \mathbb{R}^d$, extended parameter $\tau \in \mathbb{R}$, and with pdf given by
\[
p(y) = \phi_d(y; \xi, \Omega)\Phi(\eta^\top(y - \xi) + \bar{\tau}) / \Phi(\tau),
\]
where $y \in \mathbb{R}^d$ and, as was defined above, $\bar{\tau} = \tau \sqrt{1 + \eta^\top\Omega\eta}$. Here $\phi_d(y; \xi, \Omega)$ is the probability density function of $N_d(\xi, \Omega)$, the $d$-variate distribution, and $\Phi$ is the univariate $N(0, 1)$ cumulative distribution function. Note that $\phi_d(y; \xi, \Omega) = |\Omega|^{-1/2}\phi_d\left(\Omega^{-1/2}(y - \xi)\right)$, where $\phi_d(z)$ is the probability density function of $N_k(0, I_d)$, the unit $d$-variate normal distribution.

The ESN random vector $Y \sim ESN_d(\xi, \Omega, \eta, \tau)$ has selection representation $Y \overset{d}{\sim} (X_0 \mid X_0 + \tau > 0)$, where from $\begin{bmatrix} X_0 \\ X_0^\top \end{bmatrix} \sim N_{d+1}(\xi_s, \Omega_s)$. Thus, its distribution function can be computed as $F_{ESN}(y) = P(Y \leq y) = P(-X_0 < \tau, X \leq y)/P(-X_0 < \tau), y \in \mathbb{R}^d$, that is $F_{ESN}(y) = \Phi_{1+d}(y; \xi_s, \Omega_{ss})/\Phi(\tau)$, where $\Phi_{1+d}(y; \xi_s, \Omega_{ss})$ is the $N_{1+d}(\xi_s, \Omega_{ss})$-distribution function at $y_\tau = (\tau, y^\top)\top$, with mean vector...
\( \xi_\ast = (0, \xi^\top)^\top \) and variance-covariance matrix
\[
\Omega_{\ast\ast} = \begin{pmatrix}
1 & -\delta^\top \\
-\delta & \Omega
\end{pmatrix}.
\]

From Arellano-Valle and Azzalini (2006) and Arellano-Valle and Genton (2010a,b), a stochastic representation of the ESN distribution is
\[
Y^d = W + \delta U, \tag{11}
\]
where \( \delta = \Omega \eta / \sqrt{1 + \eta^\top \Omega \eta} \). \( U \sim LN_{(-\tau, \infty)}(0, 1) \), which is independent of \( W \sim N_d(\xi, \Sigma) \), where \( LN_{(-\tau, \infty)}(0, 1) \) represents the unit normal distribution truncated below the point \(-\tau\) and \( \Sigma = \Omega - \delta \delta^\top > 0 \). Because, assuming \( \Omega > 0 \), we have \( \| \delta \| < 1 \), where \( \delta = \Omega^{1/2} \delta \); thus, the matrix \( \Sigma > 0 \). The stochastic representation (11) is equivalent to the hierarchical representation
\[
Y | U = u \sim N_d(\xi + \delta u, \Sigma), \tag{12}
U \sim LN_{(-\tau, \infty)}(0, 1). \tag{13}
\]

It is worth noting here that for \( i = 1, 2 \) the above representations lead to the reparametrization of \( \Omega_i \) and \( \eta_i \) as
\[
\Omega_i = \Sigma_i + \delta_i \delta_i^\top, \quad \eta_i = \frac{\Sigma_i^{-1} \delta_i}{\sqrt{1 + \delta_i^\top \Sigma_i^{-1} \delta_i}}, \tag{14}
\]
under which \( \bar{\tau}_i = \tau \sqrt{1 + \delta_i^\top \Sigma_i^{-1} \delta_i} \). An advantage of this parameterization is that the \( \delta \)'s parameters reflect in a more genuine way the actual degree of asymmetry present in the model. In fact, the components of these vectors correspond precisely to the marginal skewness parameters (Azzalini & Capitanio, 1999). As will be seen later in Subsection 3.2, this parameterization is also useful for the implementation of the EM algorithm.

The above representations are useful to generate random samples from the ESN distribution as well as to study its moments and further probabilistic properties. For instance, considering that \( E(U) = \zeta_1(\tau) \) and \( E(U^2) = 1 - \tau \zeta_1(\tau) \), where \( \zeta_1(z) = \phi(z)/\Phi(z) \), we find easily from (14) that
\[
E[Y] = \xi + \zeta_1(\tau) \delta \quad \text{and} \quad \text{Var}[Y] = \Omega + \zeta_2(\tau) \delta \delta^\top, \tag{15}
\]
where \( \zeta_2(\tau) = -\zeta_1(\tau)\{\tau + \zeta_1(\tau)\} \). Also, for every \( a \in \mathbb{R}^d \) and \( b \in \mathbb{R} \) it follows from (14) that
\[
a^\top Y + b \sim ESN_1(\xi_a, \Omega_a, \eta_a, \tau), \tag{16}
\]
where \( \xi_a = a^\top \xi + b, \Omega_a = a^\top \Omega a \) and \( \eta_a = \Sigma_a^{-1} \delta_a / \sqrt{1 + \Sigma_a^{-1} \delta_a^2} = \Omega_a^{-1} \delta_a / \sqrt{1 - \Omega_a^{-1} \delta_a^2} \), where \( \Sigma_a = \Omega_a - \delta_a^2 \) and \( \delta_a = a^\top \delta \).
On the other hand, from (12)-(13), it is straightforward to show that, conditionally on \( Y = y \), the random variable \( U \) has a left-truncated normal distribution, namely

\[
U \mid Y = y \sim LT N_{(-\tau, \infty)} (\alpha, \beta^2),
\]

i.e., with pdf \( p(u\mid y) = \phi_1(u; \alpha, \beta^2) 1_{(-\infty, \infty)} / \Phi(\theta) \), where \( 1_A \) is the indicator function of a subset \( A \), and the parameters \( \alpha = \alpha(y) \), \( \beta^2 \) and \( \theta \) are given by

\[
\alpha = \delta^\top \Omega^{-1} (y - \xi) = \frac{\delta^\top \Sigma^{-1} (y - \xi)}{1 + \delta^\top \Sigma^{-1} \delta}, \quad \beta^2 = 1 - \delta^\top \Omega^{-1} \delta = \frac{1}{1 + \delta^\top \Sigma^{-1} \delta}, \quad \theta = \frac{\alpha + \tau}{\beta}.
\]

By Johnson et al. (1994; pp. 156, 158), the first and second moments of (17) are

\[
E [U \mid Y = y] = \alpha + \beta \zeta_1 (\theta),
\]
\[
E [U^2 \mid Y = y] = \alpha^2 + \beta^2 + (\alpha - \tau) \beta \zeta_1 (\theta).
\]

Note that for the limit case as \( \tau \to \infty \) we have \( E[U \mid Y = y] = \alpha \) and \( E[U^2 \mid Y = y] = \alpha^2 + \beta^2 \).

### 3.1 A linear approximation of the ESN classification rule

As Kim (2011), we consider in this section an approximate classification rule for the ESN case. Consider two multivariate ESN groups \( \Pi_i : ESN_d(\xi_i, \Omega_i, \eta_i, \tau), i = 1, 2 \), which satisfy condition (2). In this case, the ESE optimal rule described by (9) reduces to the decision to allocate \( y \) to group 1 if

\[
\Psi_{ESN}(y) = \log \left\{ \frac{\phi_k(y; \xi_1, \Omega_1)}{\phi_k(y; \xi_2, \Omega_2)} \right\} + \log \left\{ \frac{\Phi(\eta_1^\top (y - \xi_1) + \tilde{\tau}_1)}{\Phi(\eta_2^\top (y - \xi_2) + \tilde{\tau}_2)} \right\} > \log \left\{ \frac{\pi_2}{\pi_1} \right\},
\]

and \( y \) is assigned to group 2 otherwise, where \( \tilde{\tau}_i = \tau \sqrt{1 + \eta_i^\top \Omega_i \eta_i}, i = 1, 2 \). As byproducts, we have for \( \tau = 0 \) the skew-normal rule, and for \( \eta_1 = \eta_2 = 0 \) (or \( \tau = \infty \)) the heteroscedastic normal rule.

The ESN discriminant function \( \Psi_{ESN}(y) \) defined in (20) can be rewritten as

\[
\Psi_{ESN}(y) = \Psi_N(y) + \log \Phi(\eta_1^\top (y - \xi_1) + \tilde{\tau}_1) - \log \Phi(\eta_2^\top (y - \xi_2) + \tilde{\tau}_2),
\]

where

\[
\Psi_N(y) = \frac{1}{2} \{ (y - \xi_2)^\top \Omega_2^{-1} (y - \xi_2) - (y - \xi_1)^\top \Omega_1^{-1} (y - \xi_1) \} + \frac{1}{2} \log \left\{ \frac{\vert \Omega_2 \vert}{\vert \Omega_1 \vert} \right\}.
\]

Note that \( \Psi_N(y) \) is the discriminant function that classifies a given vector \( y \in \mathbb{R}^d \) in two normal population \( N_d(\xi_i, \Omega_i), i = 1, 2 \). As is well-known, if \( \Omega_1 = \Omega_2 = \Omega \), then this function reduces to the linear function \( \Psi_L(y) = (\xi_1 - \xi_2)^\top \Omega^{-1} (y - \bar{\xi}), \) where \( \bar{\xi} = (\xi_1 + \xi_2)/2 \).

An important special case of the ESN discriminant rule (20) occurs when we assume the same dispersion
and skewness for the both groups, i.e., $\Omega_1 = \Omega_2$ and $\eta_1 = \eta_2$. Under these assumptions, the ESN groups are different because $\xi_1 \neq \xi_2$, but they are homoscedastic. Thus, if $\Pi_i$ is the ESN$_d(\xi_i, \Omega, \eta, \tau)$ population, $i = 1, 2$, we then have

$$\Psi_{ESN}(y) = \Psi_L(y) + \log \Phi (\eta^\top (y - \xi_1) + \tilde{\tau}) - \log \Phi (\eta^\top (y - \xi_2) + \tilde{\tau})$$

(21)

where $\tilde{\tau} = \tau \sqrt{1 + \eta^\top \Omega \eta}$. As before, the resulting ESN-region of classification into $\Pi_1$ is defined by the set of $y \in \mathbb{R}^d$ for which $\Psi_{ESN}(y) > \log(\pi_2/\pi_1)$; otherwise, we allocate $y$ into $\Pi_2$.

Unlike the homoscedastic normal case, the classification function $\Psi_{ESN}(y)$ defined in (21) is non-linear in the observed vector $y$. However, as in Kim (2011), we can approximate it by using a linear classification rule. To do this, we need the second-order Taylor expansion given by $\log \Phi(x + a) \approx \log \Phi(a) + \zeta_1(a)x + (1/2)\zeta_2(a)x^2$, where $\zeta_2(x) = \zeta_1'(x) = -\zeta_1(x)\{x + \zeta_1(x)\}$. Applying this expansion to each of the last two terms of (21), we obtain the following linear approximation of the ESN rule

$$\hat{\Psi}_{ESN}(y) = (\xi_1 - \xi_2)^\top \{\Omega^{-1} - \zeta_2(\tilde{\tau}) \eta \eta^\top\} \{y - \hat{\xi} - \zeta_1(\tilde{\tau})\} (\xi_1 - \xi_2)^\top \eta.$$  

(22)

This result allows us to obtain the following approximate ESN classification rule

Assign $y$ to $\Pi_1$ if $\hat{\Psi}_{ESN}(y) > \gamma$,
Assign $y$ to $\Pi_2$ if $\hat{\Psi}_{ESN}(y) \leq \gamma$,

where $\gamma$ is chosen so that the TPM of $\hat{\Psi}_{ESN}(y)$ is minimized.

If $\eta = 0$ (or $\tau = \infty$), the ESN linear approximate rule (22) reduces to the normal linear classification rule $\Psi_L(y) = (\xi_1 - \xi_2)^\top \Omega^{-1} (y - \hat{\xi})$ whenever the value of $\tau$. If $\tau = 0$, then $\tilde{\tau} = 0$, $\zeta_1(0) = -\sqrt{2/\pi}$ and $\zeta_2(0) = [-\zeta_1(0)]^2 = -2/\pi$. In this case, we obtain in (22) an approximate classification rule for the multivariate skew-normal case.

From (22) we have $\hat{\Psi}_{ESN}(Y) = a^\top Y + b$, with

$$a = \{\Omega^{-1} - \zeta_2(\tilde{\tau}) \eta \eta^\top\} (\xi_1 - \xi_2), \quad b = -a^\top \hat{\xi} - \zeta_1(\tilde{\tau}) \eta^\top (\xi_1 - \xi_2).$$

Hence, from (16) we find $\hat{\Psi}(Y) | \Pi_i \sim ESN_1(\xi_{ai}, \Omega_a, \eta_a, \tau), i = 1, 2$, with

$$\xi_{ai} = a^\top \xi_i + b, \quad \Omega_a = a^\top \Omega a, \quad \eta_a = \frac{\Omega_a^{-1} \delta_a}{\sqrt{1 - \Omega_a^{-1} \delta_a^2}} \quad \text{and} \quad \delta_a = a^\top \delta.$$  

(23)

In particular, from (15) we obtain for $i = 1, 2$ that

$$E[\hat{\Psi}_{ESN}(Y) | \Pi_i] = \xi_{ai} + \zeta_1(\tau) \delta_a \quad \text{and} \quad \text{Var}[\hat{\Psi}_{ESN}(Y) | \Pi_i] = \Omega_a + \zeta_2(\tau) \delta_a^2.$$
Note here that

\[ D_{12} = E[\hat{\Psi}_{ESN}(Y) \mid \Pi_1] - E[\hat{\Psi}_{ESN}(Y) \mid \Pi_2] = \xi_{a1} - \xi_{a2} = \Delta^2 - \zeta_2(\eta^\top(\xi_1 - \xi_2))^2, \]

where \( \Delta^2 = (\xi_1 - \xi_2)^\top \Sigma^{-1}(\xi_1 - \xi_2) \) is the squared Mahalanobis distance between two \( d \)-variate normal populations, \( N_d(\xi_1, \Sigma) \) and \( N_d(\xi_2, \Sigma) \) say. Clearly, \( D_{12} = \Delta^2 \) if \( \eta = 0 \) (or \( \tau = \infty \)), and \( D_{12} = 0 \) if \( \xi_1 = \xi_2 \). Therefore, \( D_{12} \) could be used as a discrepancy index between two \( d \)-variate ESN population, \( \Pi_1 \) and \( \Pi_2 \).

Finally, from (3) the TPM induced by \( \hat{\Psi}_{ESN}(Y) \) is

\[
\text{TPM}(\hat{\Psi}_{ESN}) = \pi_1 P \left\{ \hat{\Psi}_{ESN}(Y) \leq \gamma \mid \Pi_1 \right\} + \pi_2 P \left\{ \hat{\Psi}_{ESN}(Y) > \gamma \mid \Pi_2 \right\}
\]

\[ = \pi_1 \frac{\Phi_2(c; \xi_{a1}, \Omega_a)}{\Phi(\tau)} + \pi_2 \left\{ 1 - \frac{\Phi_2(c; \xi_{a2}, \Omega_a)}{\Phi(\tau)} \right\}, \quad (24) \]

where

\[ c = \begin{pmatrix} \tau \\ \gamma \end{pmatrix}, \quad \xi_{a1} = \begin{pmatrix} 0 \\ \xi_{a1} \end{pmatrix}, \quad \xi_{a2} = \begin{pmatrix} 0 \\ \xi_{a2} \end{pmatrix} \quad \text{and} \quad \Omega_a = \begin{pmatrix} 1 & -\delta_a \\ -\delta_a & \Sigma_a \end{pmatrix}. \]

If \( \eta = 0 \), then \( \delta_a = 0, \xi_{a1} = (0, \Delta^2/2)^\top, \xi_{a2} = (0, -\Delta^2/2)^\top, \Omega_a = \Delta^2 \) and \( \Omega_a = \text{diag}(1, \Sigma_a). \) Also, \( \Phi_2(c; \xi_{a1}, \Omega_a) = \Phi(\tau)\Phi\left(-\frac{\Delta}{2} + \frac{\gamma}{\Delta}\right) \) and \( \Phi_2(c; \xi_{a2}, \Omega_a) = \Phi(\tau)\Phi\left(-\frac{\Delta}{2} - \frac{\gamma}{\Delta}\right). \) Therefore, the TPM(\( \hat{\Psi}_{ESN} \)) becomes the TPM of the normal linear rule \( \Psi_L(Y) \), namely

\[ \text{TPM}(\Psi_L) = \pi_1 \Phi\left(-\frac{\Delta}{2} + \frac{\gamma}{\Delta}\right) + \pi_2 \Phi\left(-\frac{\Delta}{2} - \frac{\gamma}{\Delta}\right). \]

### 3.2 A conditional normal classification rule

According to [12], we could consider the complete random vector \((Y, U)\) and then define the classification rule

\[ \Psi_{CN}(y, u) = \log \left\{ \frac{f_1(y \mid u)f_1(u)}{f_2(y \mid u)f_2(u)} \right\} = \log \left\{ \phi_d(y; \xi_1 + \delta_1 u, \Sigma_1) \right\} - \log \left\{ \phi_d(y; \xi_2 + \delta_2 u, \Sigma_2) \right\}, \]

where we have used that \( f_1(u) = f_2(u) \) since the distribution of \( U \) only depends on the parameter \( \tau \), which is being assumed equal for both populations. That is, this rule corresponds to one that compares the conditional normal populations \( N_d(\xi_i + \delta_i u, \Sigma_i), i = 1, 2 \), and is given by

\[ \Psi_{CN}(y; u) = \Psi_0(y) - \{ \delta_2^\top \Sigma_2^{-1} (y - \xi_2) - \delta_1^\top \Sigma_1^{-1} (y - \xi_1) \} u + \frac{1}{2} \{ \delta_2^\top \Sigma_2^{-1} \delta_2 - \delta_1^\top \Sigma_1^{-1} \delta_1 \} u^2, \]

where

\[ \Psi_0(y) = \frac{1}{2} \{ (y - \xi_2)^\top \Sigma_2^{-1} (y - \xi_2) - (y - \xi_1)^\top \Sigma_1^{-1} (y - \xi_1) \} + \frac{1}{2} \log \left\{ \left| \Sigma_2 \right| \right\}, \]

Let \( \Psi_{CN}(y) = E[\Psi_{CN}(Y; U) \mid Y = y] = \pi_1 E[\Psi_{CN}(y; U) \mid y \in \Pi_1] + \pi_2 E[\Psi_{CN}(y; U) \mid y \in \Pi_2]. \) By
3.3 ML estimation by the EM algorithm

To estimate the maximum likelihood ESN discriminant functions, we proceed with the EM algorithm proposed by Dempster et al. (1977). Based on (12)-(13), it is better to work with the EM algorithm based on a multivariate normal distribution to perform the ML estimation for the population parameters, instead of maximising the complex likelihood function of the ESN distribution. For a comprehensive account of the
EM algorithm, see McLachlan and Krishnan (1997).

Let \( Y_{ij}, j = 1, \ldots, n_i \), be a random sample from population \( \Pi_i : ESN_d(\xi_i, \Omega, \eta, \tau), \ i = 1, 2 \). Then, we have the following hierarchical representation from (12)-(13):

\[
Y_{ij} \mid (U_{ij}, \Pi_i) \sim N_d(\xi_i + \delta U_{ij}, \Sigma),
\]

\[
U_{ij} \mid \Pi_i \sim LTN_{(-\tau, \infty)}(0, 1),
\]

\( i = 1, 2 \) and \( j = 1, \ldots, n_i \), where \( \Sigma = \Omega - \delta \delta^T \). For \( i = 1, 2 \), we define the latent and observed vectors \( U_i = (U_{in_1}, \ldots, U_{in_n})^T \) and \( Y_i = (Y_{i1}^T, \ldots, Y_{in_i}^T)^T \), respectively. Therefore, when the parameter \( \tau \) is assumed to be known, the log-likelihood function for \( \Theta = (\xi_1, \xi_2, \Sigma, \delta) \) based on the complete data \( (Y_i, U_i, i = 1, 2) \) is

\[
\ell(\Theta \mid Y_i, U_i, i = 1, 2) = -\frac{n_1 + n_2}{2} \left\{ (d + 1) \log(2\pi) + 2 \log \Phi(\tau) + \log |\Sigma| \right\}
- \frac{1}{2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_i} (Y_{ij} - \xi_i)^T \Sigma^{-1} (Y_{ij} - \xi_i) + \delta^T \Sigma^{-1} \sum_{i=1}^{n_1} \sum_{j=1}^{n_i} (Y_{ij} - \xi_i) U_{ij}
- \frac{1}{2} (1 + \delta^T \Sigma^{-1} \delta) \sum_{i=1}^{n_1} \sum_{j=1}^{n_i} U_{ij}^2.
\]

(28)

Thus, we can proceed to implement the EM algorithm for the \( k \)th iteration as follows:

**E-step:** Assume that after the \( k \)th iteration, the current estimate for \( \Theta \) is given by \( \hat{\Theta}_{(k)} \). By (28), the \( Q \)-function is defined by

\[
Q(\Theta \mid \hat{\Theta}_{(k)}) = E \left[ \ell(\Theta \mid Y_i, U_i, i = 1, 2) \mid \hat{\Theta}_{(k)}, Y_i, i = 1, 2 \right]
= -\frac{n_1 + n_2}{2} \left\{ (d + 1) \log(2\pi) + 2 \log \Phi(\tau) + \log |\Sigma| \right\}
- \frac{1}{2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_i} (Y_{ij} - \hat{\xi}_{(k)}^i)^T \Sigma^{-1} (Y_{ij} - \hat{\xi}_{(k)}^i) + \hat{\delta}_{(k)}^T \Sigma^{-1} \sum_{i=1}^{n_1} \sum_{j=1}^{n_i} (Y_{ij} - \hat{\xi}_{(k)}^i) \hat{U}_{ij(k)}
- \frac{1}{2} (1 + \hat{\delta}_{(k)}^T \Sigma^{-1} \hat{\delta}_{(k)}) \sum_{i=1}^{n_1} \sum_{j=1}^{n_i} \hat{U}_{ij(k)}^2,
\]

(29)

which is the conditional expectation of (28) with respect to the conditional distribution of the missing data \( (U_i, i = 1, 2) \), given the current estimate \( \hat{\Theta}_{(k)} \) and the observed data \( (Y_i, i = 1, 2) \). Here, \( \hat{U}_{ij(k)} = E \left[ U_{ij} \mid (\hat{\Theta}_{(k)}, Y_i) \right] \) and \( \hat{U}_{ij(k)}^2 = E \left[ U_{ij}^2 \mid (\hat{\Theta}_{(k)}, Y_i) \right] \). To compute these conditional moments, we note first by (17) and (27) that

\[
U_{ij} \mid (\hat{\Theta}_{(k)}, Y_i) \sim LTN_{(-\tau, \infty)} \left( \hat{\alpha}_{ij(k)}, \hat{\beta}_{ij(k)}^2 \right),
\]
where \( \hat{\alpha}_{ij(k)} = \hat{\beta}_{(k)}^2 \delta_{(k)}^T \Sigma_{(k)}^{-1} (y_{ij} - \hat{\xi}_{i(k)}) \) and \( \hat{\beta}_{(k)}^2 = (1 + \delta_{(k)}^T \Sigma_{(k)}^{-1} \delta_{(k)})^{-1}. \) Hence, by applying (18)-(19) we then obtain

\[
\begin{align*}
\hat{U}_{ij(k)} &= \hat{\alpha}_{ij(k)} + \hat{\beta}_{(k)} \zeta_1 (\hat{\alpha}_{ij(k)}), \\
\hat{U}^2_{ij(k)} &= \hat{\alpha}_{ij(k)}^2 + \hat{\beta}^2_{(k)} + (\hat{\alpha}_{ij(k)} - \tau) \hat{\beta}_{(k)} \zeta_1 (\hat{\alpha}_{ij(k)}),
\end{align*}
\]

(30), (31)

where \( \hat{\theta}_{ij(k)} = (\tau + \hat{\alpha}_{ij(k)}) / \hat{\beta}_{(k)} \)

**M-step:** Update the estimate \( \hat{\Theta}_{(k)} \) by \( \hat{\Theta}_{(k+1)} = (\hat{\xi}_{1(k+1)}, \hat{\xi}_{2(k+1)}, \hat{\Sigma}_{(k+1)}, \hat{\delta}_{(k+1)}) \) with

\[
\begin{align*}
\hat{\xi}_{i(k+1)} &= \bar{Y}_i - \hat{\delta}_{(k+1)} \hat{U}_{i(k)}, \quad i = 1, 2, \\
\hat{\Sigma}_{(k+1)} &= \frac{1}{n_1 + n_2} \sum_{i=1}^{n_2} \sum_{j=1}^{n_i} \{(Y_{ij} - \hat{\xi}_{i(k+1)})(Y_{ij} - \hat{\xi}_{i(k+1)})^T \}
\end{align*}
\]

(32)

\[
\begin{align*}
&- 2 \hat{U}_{ij(k)} (Y_{ij} - \hat{\xi}_{i(k+1)}) \hat{\delta}_{(k+1)}^T \left\{ U_{ij(k)}^2 \hat{\delta}_{(k+1)} \hat{\delta}_{(k+1)}^T \right\}, \\
\hat{\delta}_{(k+1)} &= \frac{\sum_{i=1}^{n_1} \sum_{j=1}^{n_i} \hat{U}_{ij(k)} Y_{ij} - \sum_{i=1}^{n_1} \sum_{j=1}^{n_i} \hat{U}_{ij(k)} \bar{Y}_i}{\sum_{i=1}^{n_1} \sum_{j=1}^{n_i} \hat{U}^2_{ij(k)} - \sum_{i=1}^{n_1} \hat{U}^2_{i(k)}},
\end{align*}
\]

(33), (34)

where

\[
\begin{align*}
\bar{Y}_i &= \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij} \quad \text{and} \quad \hat{U}_{i(k)} = \frac{1}{n_i} \sum_{j=1}^{n_i} \hat{U}_{ij(k)}, \quad i = 1, 2.
\end{align*}
\]

Note by replacing (32) in (33) we have for each iteration that

\[
\hat{\Sigma}_{(k+1)} = \frac{1}{n_1 + n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_i)(Y_{ij} - \bar{Y}_i)^T,
\]

i.e., the ML of \( \Sigma \) do not depend on \( k \)th iteration but, only depend on the sample.

Taking into account that the EM algorithm proposed in this work to estimate the ESN model parameters assumes a known value for the selection parameter \( \tau \), we then have that the equation (25) corresponds to a profile log-likelihood function of the location, scale and shape parameters for a given \( \tau \). In this sense, Capitanio et al. (2003) concludes that a direct maximisation of the ESN log-likelihood function with respect to all its parameters simultaneously appeared troublesome, while the construction of the profile log-likelihood was much more stable and numerically satisfactory (Arellano-Valle and Genton, 2010). However, simultaneously Canale (2011) estimates the four parameters and concludes that a disadvantage of this approach is the singularity produced in the Fisher information matrix when \( \eta = 0 \), as \( |\tau| \to \infty \). Capitanio et al. (2003) notice that \( \tau \) is effectively removed from the Fisher information matrix when \( \eta = 0 \). Hence, the above discussion applies to the case where it is known that \( \eta \neq 0 \).

Finally, given the MLEs of \( \Sigma \) and \( \delta \), the MLEs of the original parameters \( \Omega \) and \( \eta \) are obtained easily from the relations given (14). Thus, we proceed to classify a new observation \( y_0 \) to \( \Pi_1 \) if \( \hat{\Psi}(y_0) > \log (\pi_2 / \pi_1) \).
or, to otherwise classify $y_0$ to $H_2$, where $\hat{\Psi}(y)$ is a ESN discriminant function estimated by ML.

### 3.4 Monte-Carlo simulations

We proceed to simulate and verify the performance of the EM algorithm and the ESN discriminant function according to Reza-Zadkarami and Rowhani (2010) and Kim (2011), for which we use a Monte-Carlo framework. Specifically, we proceed as follows by considering the bivariate case ($d = 2$):

1. For $i = 1, 2$, simulated randomly a training samples of size $n = 100, 250$ and $500$ from $Y_i \sim ESN_d(\xi_i, \Omega, \eta, \tau)$, using the stochastic representation (11). By Capitanio et al. (2003) and Arellano-Valle and Genton (2010), the ESN data generation proceeds in the following steps:

   a. Given the parameter set $(\xi_i, \Omega, \eta, \tau)$ associated to the ESN distribution of $Y_i$ for the $i$th associated group, compute the auxiliary parameters $\delta = \eta \Omega / \sqrt{1 + \eta \Omega \eta}$, $\Sigma = \Omega - \delta \delta^T$ and $\tau = \tau \sqrt{1 + \eta \Omega \eta}$.

   b. From the stochastic representation (11) it follows that $Y_i \overset{d}{=} X_i + \delta X_0i$, where $X_0i \sim \mathcal{N}_{1+d}(0, ((\sigma_{rs})))$, $r, s = 1, \ldots, d$. Note that $X_0i$ and $X_i$ are independent. Therefore, from this multivariate normal distribution, generate $X_0i$ and $X_i$;

   c. If $X_0i + \tau > 0$, then generate $Y_i = X_i + \delta X_0i$.

2. Compute the maximum likelihood of $(\xi_1, \xi_2, \Sigma, \delta, \tau)$ through the EM algorithm described in Section 3.2 from the training samples obtained in step (1), and estimate the ESN discriminant rules.

3. The procedure related to steps 1-2 is repeated $B = 1000$ times.

4. Then, the indicators $\text{BIAS}(\theta) = \bar{\theta} - \theta$ and $\sqrt{\text{MCE}(\theta)} = \sqrt{\sum_{i=1}^{B} B^{-1}(\hat{\theta}_i - \theta)^2}$ of the ML estimates are summarized, where $\theta$ is the true parameter, $\bar{\theta} = \sum_{i=1}^{B} \hat{\theta}_i / B$ and $\hat{\theta}_i$ is the $i$-sample estimate.

5. Two additional random samples $Y^*_1$ and $Y^*_2$ of size $n^* = 500$ with the same parameters of the step (1) are generated as test samples.

6. For these test samples, the individuals are classified using the ESN discriminant rules of step (2).
Table 1: BIAS and $\sqrt{\text{MCE}}$ of the ML estimates obtained by EM algorithm for each simulation.

| $\tau$ | N  | values | $\xi_{11}$ | $\xi_{12}$ | $\xi_{21}$ | $\xi_{22}$ | $\sigma_{11}$ | $\sigma_{12}$ | $\sigma_{22}$ | $\delta_1$ | $\delta_2$ |
|-------|----|--------|------------|------------|------------|------------|--------------|--------------|--------------|------------|------------|
| 5     | 100| BIAS   | 0.113      | 0.014      | 0.038      | 0.005      | 0.031        | 0.013        | 0.014        | 0.081      | 0.045      |
|       |    | $\sqrt{\text{MCE}}$ | 0.333      | 0.150      | 0.183      | 0.121      | 0.241        | 0.140        | 0.150        | 0.152      | 0.112      |
| 250   |    | BIAS   | 0.090      | 0.010      | 0.033      | 0.004      | 0.008        | 0.003        | 0.008        | 0.067      | 0.043      |
|       |    | $\sqrt{\text{MCE}}$ | 0.177      | 0.096      | 0.093      | 0.076      | 0.157        | 0.092        | 0.096        | 0.093      | 0.064      |
| 500   |    | BIAS   | 0.080      | 0.007      | 0.001      | 0.003      | 0.007        | 0.002        | 0.002        | 0.064      | 0.037      |
|       |    | $\sqrt{\text{MCE}}$ | 0.087      | 0.065      | 0.057      | 0.053      | 0.112        | 0.067        | 0.068        | 0.064      | 0.042      |
| 50    | 100| BIAS   | 0.102      | 0.010      | 0.001      | 0.002      | 0.025        | 0.005        | 0.010        | 0.090      | 0.043      |
|       |    | $\sqrt{\text{MCE}}$ | 0.259      | 0.147      | 0.173      | 0.116      | 0.249        | 0.151        | 0.156        | 0.160      | 0.103      |
| 250   |    | BIAS   | 0.081      | 0.012      | 0.032      | 0.006      | 0.009        | 0.003        | 0.002        | 0.072      | 0.047      |
|       |    | $\sqrt{\text{MCE}}$ | 0.125      | 0.092      | 0.082      | 0.074      | 0.167        | 0.094        | 0.094        | 0.093      | 0.058      |
| 500   |    | BIAS   | 0.081      | 0.008      | 0.037      | 0.003      | 0.002        | 0.001        | 0.003        | 0.064      | 0.047      |
|       |    | $\sqrt{\text{MCE}}$ | 0.086      | 0.067      | 0.058      | 0.053      | 0.110        | 0.065        | 0.064        | 0.066      | 0.039      |

Training samples of step 1 are randomly simulated using R software (R Development Core Team, 2013). Table 1 summarises the results of EM algorithm for a set of parameters given by

$$\xi_1 = \begin{pmatrix} 0 \\ 4.5 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 2 \\ 1.5 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 2.5 & 1.5 \\ 1.5 & 0.8 \end{pmatrix}, \quad \eta = \begin{pmatrix} 2.5 \\ 1.5 \end{pmatrix}, \quad \tau = \{5, 50\}$$

We see from Table 1 that the BIAS and $\sqrt{\text{MCE}}$ indicators tend to decrease when $N$ increase, indicating that its performs is well in estimating the $ESN_2(\xi_i, \Omega, \eta, \tau)$, $i = 1, 2$, distributions. Table 2 shows a high classification accuracy. In fact, the overall classification accuracy for the both-simulations classification tends to increase when $N$ increase. Comparing both values of $\tau$, the method is slightly better for $\tau = 5$ than for $\tau = 50$ (97.8% of accuracy versus 96.8% for $N = 500$, respectively).

4 Conclusions

This paper considers a new classification method for non-gaussian data. We obtain a region to classify multivariate observations, considering a classification rule derived from the multivariate extended skew-normal distribution. In particular, we have as byproduct the classical linear classification rule due the properties of the class of distributions considered. Although the material in this paper focuses on an extended skew-normal model, it can be extended to numerous potential distributions of the skew-elliptical class as well.
Table 2: Number of individuals classified by the approximate ESN rule in each simulation. The diagonal shows the number of correctly classified sample units for each group.

| $\tau$ | N  | Allocated   | Original      | Group 1 | Group 2 | Total | Total (%) |
|-------|----|-------------|---------------|--------|--------|-------|-----------|
|       |    | Group 1     | Group 2       |        |        |       |           |
| 5     | 100| 482         | 25            | 507    | 95.1   |       |           |
|       |    | 18          | 475           | 493    | 96.4   |       |           |
|       |    | 500         | 500           | 1000   | -      |       |           |
|       |    | 96.4        | 95.0          | -      | 95.7   |       |           |
| 250   | 500| 485         | 20            | 505    | 96.5   |       |           |
|       |    | 15          | 480           | 495    | 97.0   |       |           |
|       |    | 500         | 500           | 1000   | -      |       |           |
|       |    | 97.0        | 96.0          | -      | 96.5   |       |           |
| 50    | 100| 471         | 31            | 502    | 93.8   |       |           |
|       |    | 29          | 469           | 498    | 94.2   |       |           |
|       |    | 500         | 500           | 1000   | -      |       |           |
|       |    | 94.2        | 93.8          | -      | 94.0   |       |           |
| 250   | 500| 474         | 15            | 489    | 96.9   |       |           |
|       |    | 26          | 485           | 511    | 94.9   |       |           |
|       |    | 500         | 500           | 1000   | -      |       |           |
|       |    | 94.8        | 97.0          | -      | 95.9   |       |           |
| 500   |    | 485         | 17            | 502    | 96.61  |       |           |
|       |    | 15          | 483           | 498    | 96.98  |       |           |
|       |    | 500         | 500           | 1000   | -      |       |           |
|       |    | 97.0        | 96.6          | -      | 96.8   |       |           |

Acknowledgment

Arellano-Valle’s research was partially supported by grant FONDECYT (Chile) 1120121. Contreras-Reyes’s research was supported by Instituto de Fomento Pesquero (IFOP), Valparaíso, Chile.

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