The Light Quark Sector, \( CP \) Violation, and the Unitarity Triangle

Harald Fritzsch \(^*\) and Zhi-zhong Xing \(^{†}\)

Sektion Physik, Universität München, Theresienstrasse 37A, 80333 München, Germany

Abstract

In view of the observed strong hierarchy of quark masses, we propose a new description of flavor mixing which is particularly suited for models of quark mass matrices based on flavor symmetries. The necessary and sufficient conditions for \( CP \) violation are clarified. The emergence of \( CP \) violation is primarily linked to a large phase difference (near 90\(^{\circ}\)) in the light quark sector. The unitarity triangle is determined by the mass ratios of the light quarks. We conclude that the unitarity triangle should be close or identical to a rectangular triangle, and \( CP \) violation is maximal in this sense. Instructive predictions for flavor mixing angles and \( CP \)-violating parameters, which can directly be confronted with the forthcoming data from \( B \)-meson factories, are obtained from a specific texture of quark mass matrices.

\[ \text{[PACS number(s): 12.15.Ff, 11.30.Er, 11.30.Hv, 12.15.Hh]} \]
1 Introduction

A deeper understanding of flavor mixing and $CP$ violation, observed in the weak interactions, remains one of the major challenges in particle physics. In the standard electroweak theory with three quark families the phenomenon of flavor mixing is described by a $3 \times 3$ unitary matrix, which can be expressed in terms of four independent parameters, usually taken to be three rotation angles and one complex phase. There seems no way to obtain any further information about these parameters within the standard model. Any attempt to do so would require new physical inputs which are beyond the standard model.

At the present time it seems hopeless to find a complete solution to the fermion mass and flavor mixing problem by theoretical insight alone. One can hope, however, to detect a specific order in the tower of fermion masses and the four parameters of quark flavor mixing, especially in observing links between the parameters of the flavor mixing and the mass eigenvalues. That such links should exist, seems obvious to us. Like in any quantum mechanical system the mixing pattern of the states will influence the pattern of the mass eigenvalues, and vice versa. One possible way to make these links more transparent is to look for specific symmetry limits, e.g., by setting parameters, which are observed to be small, to zero and to study the situation in the symmetry limit first. Following such an approach, we shall demonstrate that (a) a specific description of quark flavor mixing can be derived, (b) two of the three flavor mixing angles are related directly to the quark mass ratios $m_u/m_c$ and $m_d/m_s$, and (c) the unitarity triangle of quark mixing related to $CP$ violation in $B$-meson decays is fixed in terms of these mass ratios and the modulus of the Cabibbo transition element $|V_{us}|$. Furthermore we shall give arguments why an inner angle of the unitarity triangle (angle $\alpha$) should be equal to $90^\circ$ or close to $90^\circ$.

The “standard” parametrization of the flavor mixing matrix (advocated by the Particle Data Group [1]) and the original Kobayashi-Maskawa parametrization [2] were introduced without taking possible links between the quark masses and the flavor mixing parameters into account. The parametrization introduced by us some time ago [3,4] is based on such a connection, although the specific relations between flavor mixing angles and quark masses might be more complicated than commonly envisaged. It is a parametrization which allows to interpret the phenomenon of flavor mixing as an evolutionary or tumbling process. In the limit in which the masses of the light quarks ($u, d$) and the medially light quarks ($c, s$) are set to zero, while the heavy quarks ($t, b$) acquire their masses, there is no flavor mixing [3]. Once the masses of the ($c, s$) quarks are introduced, while the ($u, d$) quarks remain massless, the flavor mixing is reduced to an admixture between two families, described by one angle $\theta$. As soon as the $u$- and $d$-quark masses are introduced as small perturbations, the full flavor mixing matrix involving a complex phase parameter and two more mixing angles ($\theta_u, \theta_d$) appears. These angles can be interpreted as rotations between the states ($u, c$) and ($d, s$), respectively. In either the “standard” parametrization or the Kobayashi-Maskawa representation, however, such specific limits are difficult to consider. For this reason we proceed to describe the flavor
mixing by use of the parametrization given in Ref. [3].

2 The flavor mixing matrix

In the standard model or those extensions which have no flavor-changing right-handed currents, it is always possible to choose a basis of flavor space in which the up- and down-type quark mass matrices are hermitian. Without loss of any generality the (1,3) and (3,1) elements of both mass matrices can further be arranged, through a common unitary transformation, to be zero [3]. Then one is left with hermitian quark mass matrices of the form

\[ M_q = \begin{pmatrix} E_q & D_q & 0 \\ D'_q^* & C_q & B_q \\ 0 & B'_q^* & A_q \end{pmatrix}, \]  

(2.1)

where \( q = u \) (up) or \( d \) (down), and the hierarchy \( |A_q| \gg |B_q|, |C_q| \gg |D_q|, |E_q| \) is generally expected. In this basis, there is no direct mixing between the heavy \( t \) (or \( b \)) quark and the light \( u \) (or \( d \)) quark in \( M_u \) (or \( M_d \)), i.e., the quark mass matrix is close to the well-known form of “nearest-neighbour” interactions [3].

A mass matrix of the type (2.1) can in the absence of complex phases be diagonalized by a 3 × 3 orthogonal matrix, described only by two rotation angles in the hierarchy limit of quark masses [4]. First, the off-diagonal element \( B_q \) is rotated away by a rotation matrix \( R_{23} \) between the second and third families. Then the element \( D_q \) is rotated away by a transformation \( R_{12} \) between the first and second families. No rotation between the first and third families is necessary in either the limit \( m_u \to 0, m_d \to 0 \) or the limit \( m_t \to \infty, m_b \to \infty \). Lifting such a hierarchy limit, which is not far from the reality, one needs an additional transformation \( R_{31} \) with a tiny rotation angle to fully diagonalize \( M_q \). Note, however, that the rotation sequence \((R_{12}R_{23})(R_{12}R_{23})^T\) is enough to describe the 3 × 3 real flavor mixing matrix, as the effects of \( R_{31}^u \) and \( R_{31}^d \) can always be absorbed into this sequence through redefining the relevant rotation angles. By introducing a complex phase angle into the rotation combination \((R_{23}^u)(R_{23}^d)^T\), we finally arrive at the following representation of quark flavor mixing [5]:

\[
V = \begin{pmatrix} c_u & s_u & 0 \\ -s_u & c_u & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\varphi} & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} \begin{pmatrix} c_d & -s_d & 0 \\ s_d & c_d & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} c_u s_d c + c_u c_d e^{-i\varphi} & s_u c_d c - c_u s_d e^{-i\varphi} & s_u s \\ c_u s_d c - s_u c_d e^{-i\varphi} & c_u c_d c + s_u s_d e^{-i\varphi} & c_u s \\ -s_d s & -c_d s & c \end{pmatrix}, \quad (2.2)
\]

where \( s_u \equiv \sin \theta_u, c_u \equiv \cos \theta_u \), etc. The three mixing angles can all be arranged to lie in the first quadrant, i.e., all \( s_u, s_d, s \) and \( c_u, c_d, c \) are positive. The phase \( \varphi \) may in general take all values between 0 and \( 2\pi \). Clearly \( CP \) violation is present, if \( \varphi \neq 0 \) or \( \pi \).
Although we have derived in a heuristic way the particular description of the flavor mixing matrix (2.2) from the hierarchical mass matrix (2.1), we should like to emphasize that (2.2) is a possible way to describe any mixing matrix, one out of nine inequivalent representations classified in Ref. [3].

If the phase \( \varphi \) in \( V \) is disregarded, the resulting rotation matrix (obtained from (2.2) for \( \varphi = 0 \)) is just the one used originally by Euler; i.e., the angles \( \theta, \theta_u \) and \( \theta_d \) correspond to the usual Euler angles [8]. Note that this is not the case for other representations of the flavor mixing matrix given in the literature [2, 9]. The representation given in (2.2) can be interpreted as follows. First, a rotation by the angle \( \theta_d \) takes place in the plane defined by the \( d \) and \( s \) quarks. It is followed by a rotation (angle \( \theta \)) in the \( b-s' \) plane, where \( s' \) denotes the superposition \( s' = d \sin \theta_d + s \cos \theta_d \). At the same time the orthogonal state \( d' = d \cos \theta_d - s \sin \theta_d \) is multiplied by the phase factor \( e^{-i\varphi} \). Finally a rotation (angle \( \theta_u \)) is applied in the \( 1-2 \) plane (about the new third axis).

The sequence of rotations corresponds just to the Euler sequence [8]: \( R_{12} R_{23} R_{12}^T \). On the other hand, the original Kobayashi-Maskawa representation [4] corresponds to the sequence \( R_{23} R_{12} R_{23}^T \), while the “standard” representation [1] corresponds to the sequence \( R_{23} R_{31} R_{12} \) (see also the classifications given in Ref. [3]). Although all descriptions of the flavor mixing matrix are mathematically equivalent, we emphasize that the Euler sequence \( R_{12} R_{23} R_{12}^T \) is physically of particular interest, as it involves the rotation matrices \( R_{12} \) and \( R_{12}^T \), which describe the rotations in the light quark sector, in a symmetric way. Since the flavor mixing matrix acts between the quark mass eigenstates \( \mathcal{U} = (u, c, t) \) and \( \mathcal{D} = (d, s, b) \), one could absorb the two \( R_{12} \) rotations in a redefinition of the quark fields. The charged weak transition term can be rewritten as follows:

\[
\mathcal{U}_L^* V \mathcal{D}_L = (u, c, t)_L V \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L = (u', c', t)_L \begin{pmatrix} e^{-i\varphi} & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} \begin{pmatrix} d' \\ s' \\ b \end{pmatrix}_L ,
\]

where \( u' = u \cos \theta_u - c \sin \theta_u \) and \( c' = c \cos \theta_u + u \sin \theta_u \). Thus the angles \( \theta_u \) and \( \theta_d \) describe the corresponding rotations in the \( (u, c) \) and \( (d, s) \) systems.

We should like to emphasize that the angles \( \theta_u \) and \( \theta_d \) can directly be measured from weak decays of \( B \) mesons and from \( B^0-\bar{B}^0 \) mixing. An analysis of the present experimental data yields [10]: \( \theta_u = 4.87^\circ \pm 0.86^\circ \) and \( \theta_d = 11.71^\circ \pm 1.09^\circ \). Taking the central values for illustration, one has

\[
\begin{align*}
    d' &= d \cos \theta_d - s \sin \theta_d \approx 0.979d - 0.203s , \\
    s' &= d \sin \theta_d + s \cos \theta_d \approx 0.203d + 0.979s , \\
    u' &= u \cos \theta_u - c \sin \theta_u \approx 0.996u - 0.085c , \\
    c' &= u \sin \theta_u + c \cos \theta_u \approx 0.085u + 0.996c .
\end{align*}
\]

The question, about whether these mixtures of mass eigenstates have a specific physical meaning, arises. This will be discussed in some more detail below. Due to the symmetric structure
of our mixing matrix (2.2), we are able to interpret the $\theta_d$ and $\theta_u$ rotations as specific transformations of the corresponding mass eigenstates. Such an interpretation is not possible for the third rotation given by $\theta$, measured to be $2.30^\circ \pm 0.09^\circ$ [10]. This rotation takes place between the third family of the massive quarks and the $c'$ and $s'$ states. One interpretation would be to associate the rotation of $\theta$ with a transformation among $b$ and $s'$. Another possibility is to describe the effect as a rotation among $t$ and $c'$. However, one could also write $\theta$ as a difference of two other angles, and describe the mixing effect as a combination of a rotation in the $(b, s')$ system and a rotation in the $(t, c')$ system. Thus a unique interpretation does not exist. We remark that the asymmetry between the $\theta$ rotation on the one hand and the $\theta_u$ and $\theta_d$ rotations on the other hand is a direct consequence of our flavor mixing matrix (which is in turn related to the hierarchical structure of the mass spectrum) and is linked to the fact that there exist three different quark families.

It is worthwhile to point out the similarity and difference between our new parametrization and the Kobayashi-Maskawa parametrization, which both result from rotations in the 1–2 and 2–3 planes (i.e., $R_{12}$ and $R_{23}$), in the description of quark flavor mixing. To make a comparison, we write out the latter as follows:

$$V_{KM} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & s_2 \\ 0 & -s_2 & c_2 \end{pmatrix} \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & e^{-i\delta} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & -s_3 \\ 0 & s_3 & c_3 \end{pmatrix},$$

where $s_1 \equiv \sin \theta_1$, $c_1 \equiv \cos \theta_1$, etc. The mixing angles $(\theta_u, \theta_d, \theta)$ are related to $(\theta_1, \theta_2, \theta_3)$ simply through the product of $|V_{ub}|$ and $|V_{ud}|$, i.e., $s_u s_d s^2 = s_1^2 s_2 s_3$ holds. One can also link the phase parameter $\varphi$ to $\delta$ with the help of the common rephasing-invariant measure of $CP$ violation [11]; i.e.,

$$J = s_u c_u s_d c_d s^2 c \sin \varphi = \frac{s_1^2}{c_1} c_2 s_2 c_3 \sin \delta.$$

With no fine-tuning of the relevant mixing angles, we arrive at the equality between $\varphi$ and $\delta$ to an excellent degree of accuracy:

$$\frac{\sin \varphi}{\sin \delta} = \frac{c_1}{s_1} \frac{c_2}{c_u} \frac{c_3}{c_d} c = 1 - O(\lambda^2),$$

where $\lambda \approx s_d \approx s_1 \approx 0.2$. Therefore a large $CP$-violating phase (close to $90^\circ$), as required either phenomenologically [11] or in a specific dynamical scheme [12, 13], must manifest itself in both (2.2) and (2.5). The difference between these two representations is however significant. For example, the Kobayashi-Maskawa parametrization starts from the second largest matrix element of $V$ (i.e., $|V_{ud}|$ instead of $|V_{ub}|$) and leads to quite complicated results for the ratios...
\[ |V_{ub}/V_{cb}| \text{ and } |V_{td}/V_{ts}| \]. As summarized in Refs. [3, 4], the new parametrization (2.2) has a number of advantages over all the others in the study of heavy flavor decays and quark mass matrices. Its usefulness will be seen more clearly in the present work.

As an example we explore the interesting connection between our parametrization (2.2) and the unitarity triangle of quark mixing defined by the orthogonality relation

\[ V_{ub}^* V_{ud} + V_{cb}^* V_{cd} + V_{tb}^* V_{td} = 0 \tag{2.8} \]

in the complex plane. The inner angles of this triangle, usually denoted as

\[ \alpha = \arg \left( -\frac{V_{tb}^* V_{td}}{V_{ub}^* V_{ud}} \right), \]
\[ \beta = \arg \left( -\frac{V_{cb}^* V_{cd}}{V_{tb}^* V_{td}} \right), \]
\[ \gamma = \arg \left( -\frac{V_{ub}^* V_{ud}}{V_{cb}^* V_{cd}} \right), \tag{2.9} \]

can be determined from some \( CP \)-violating asymmetries in \( B \)-meson decays [14]. Current data indicate that the unitarity triangle (2.8) is congruent, to a good degree of accuracy, with another unitarity triangle defined by the orthogonality relation \( V_{td}^* V_{ud} + V_{ts}^* V_{us} + V_{tb}^* V_{ub} = 0 \) in the complex plane [15]. In view of the approximate congruency between two unitarity triangles and the smallness of three mixing angles, we find that the parametrization (2.2) takes an instructive leading-order form:

\[ V \approx \begin{pmatrix} e^{-i\alpha} & s_C e^{i\gamma} & s_u s_d \\ s_C e^{i\beta} & 1 & s \\ -s_d s & -s & 1 \end{pmatrix}, \tag{2.10} \]

where \( s_C \equiv \sin \theta_C \approx |s_u - s_d e^{-i\varphi}| \) with \( \theta_C \) denoting the Cabibbo rotation angle [16]. Clearly \( \alpha \approx \varphi \) holds as a straightforward result of (2.10). In this approximation \( |V_{ub}^* V_{ud}|, |V_{cb}^* V_{cd}| \) and \( |V_{tb}^* V_{td}| \), three sides of the unitarity triangle (2.8), are rescaled to \( s_u, s_d \) and \( s_C \) respectively. The latter are three sides of a new triangle with smaller area \( (\approx s_u s_d \sin \alpha/2) \), which will subsequently be referred to as the “light-quark triangle” in the heavy quark limit \( (m_t \to \infty, m_b \to \infty) \). The values of \( \alpha, \beta \) and \( \gamma \) can therefore be given in terms of \( s_u, s_d \) and \( s_C \) with the help of the cosine theorem. In particular, relations like [3]

\[ \sin \alpha : \sin \beta : \sin \gamma \approx s_C : s_u : s_d \tag{2.11} \]

may directly be confronted with the upcoming data on \( CP \) asymmetries in \( B \) decays [17]. Motivated by these interesting results, we shall investigate the role that the light quark sector plays in \( CP \) violation for a variety of realistic textures of quark mass matrices.

### 3 Symmetry limits of quark masses

Going farther from the previous discussions [3, 4], we remark two useful limits of quark masses and analyze their corresponding consequences on flavor mixing.
3.1 The chiral limit of quark masses

In the limit \( m_u \to 0, m_d \to 0 \) ("chiral limit"), where both the up and down quark mass matrices have zeros in the positions \( (1, 1), (1, 2), (2, 1), (1, 3) \) and \( (3, 1) \) (see also Ref. [1]), the flavor mixing angles \( \theta_u \) and \( \theta_d \) vanish. Only the \( \theta \) rotation affecting the heavy quark sector remains, i.e., the flavor mixing matrix effectively takes the form

\[
\hat{V} = \begin{pmatrix} \cos \hat{\theta} & \sin \hat{\theta} \\ -\sin \hat{\theta} & \cos \hat{\theta} \end{pmatrix},
\]

where \( \hat{\theta} \) denotes the value of \( \theta \) which one obtains in the limit \( \theta_u \to 0, \theta_d \to 0 \). We see that \( \hat{V} \) is a real orthogonal matrix, arising naturally from \( V \) in the chiral limit.

The flavor mixing angle \( \hat{\theta} \) can be derived from hermitian quark mass matrices of the following general form (in the limit \( m_u \to 0, m_d \to 0 \)):

\[
\hat{M}_q = \begin{pmatrix} \hat{C}_q & \hat{B}_q \\ \hat{B}_q^* & \hat{A}_q \end{pmatrix},
\]

where \( |\hat{A}_q| \gg |\hat{B}_q|, |\hat{C}_q| \); and \( q = u \) (up) or \( d \) (down). Note that the phase difference between \( \hat{B}_u \) and \( \hat{B}_d \), denoted as \( \kappa \equiv \arg(\hat{B}_u) - \arg(\hat{B}_d) \), has no effect on \( CP \) symmetry in the chiral limit, but it may affect the magnitude of \( \hat{\theta} \). It is known that current data on the top-quark mass and the \( B \)-meson lifetime disfavor the special case \( \hat{C}_u = \hat{C}_d = 0 \) for \( \hat{M}_u \) and \( \hat{M}_d \) (see, e.g., Ref. [2]), hence we take \( \hat{C}_q \neq 0 \) and define a ratio \( \hat{r}_q \equiv |\hat{B}_q|/|\hat{C}_q| \) for convenience. Then we can obtain the flavor mixing angle \( \hat{\theta} \), in terms of the quark mass ratios \( m_c/m_t, m_s/m_b \) and the parameters \( \hat{r}_u, \hat{r}_d \), by diagonalizing the mass matrices in (3.2). In the next-to-leading order approximation, \( \sin \hat{\theta} \) reads

\[
\sin \hat{\theta} = \left| \hat{r}_d \frac{m_s}{m_b} (1 - \hat{\delta}_d) - \hat{r}_u \frac{m_c}{m_t} (1 - \hat{\delta}_u) e^{i\kappa} \right|, \tag{3.3}
\]

where two correction terms are given by

\[
\hat{\delta}_u = (1 + \hat{r}_u^2) \frac{m_c}{m_t}, \quad \hat{\delta}_d = (1 + \hat{r}_d^2) \frac{m_s}{m_b}. \tag{3.4}
\]

In view of the fact \( m_u/m_b \sim O(10) m_c/m_t \) from current data [1, 8], we find that the flavor mixing angle \( \hat{\theta} \) is primarily linked to \( m_s/m_b \) provided \( |\hat{r}_u| \approx |\hat{r}_d| \). Note that in specific models, e.g., those describing the mixing between the second and third families as an effect related to the breaking of an underlying “democratic symmetry” [9, 20], the ratios \( \hat{r}_u \) and \( \hat{r}_d \) are purely algebraic numbers (such as \( |\hat{r}_u| = |\hat{r}_d| = 1/\sqrt{2} \) or \( \sqrt{2} \)).

For illustration, we take \( \hat{r}_u = \hat{r}_d \equiv \hat{r} \) to fit the experimental result \( \sin \hat{\theta} = 0.040 \pm 0.002 \) with the typical inputs \( m_b/m_s = 26 - 36 \) and \( m_t/m_c \sim 250 \). It is found that the favored value of \( |\hat{r}| \) varies in the range \( 1.0 - 2.5 \), dependent weakly on the phase parameter \( \kappa \).

Note that both \( m_s/m_b \) and \( m_c/m_t \) evolve with the energy scale (e.g., from the weak scale \( \mu \sim 10^2 \) GeV to a superhigh scale \( \mu \sim 10^{16} \) GeV, or vice versa), therefore \( \hat{\theta} \) itself is also scale-dependent.
3.2 The heavy quark limit

The limit \(m_t \to \infty, m_b \to \infty\) is subsequently referred to as the “heavy quark limit”. In this limit, in which the \((3,3)\) elements of the up and down mass matrices formally approach infinity but all other matrix elements are fixed, the angle \(\theta\) vanishes. The flavor mixing matrix, which is nontrivial only in the light quark sector, takes the form:

\[
\tilde{V} = \left( \begin{array}{cc} \tilde{c}_u & \tilde{s}_u \\ -\tilde{s}_u & \tilde{c}_u \end{array} \right) \left( \begin{array}{cc} e^{-i\tilde{\phi}} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \tilde{c}_d & -\tilde{s}_d \\ \tilde{s}_d & \tilde{c}_d \end{array} \right)
\]

\[
= \left( \begin{array}{cc} \tilde{s}_u \tilde{s}_d + \tilde{c}_u \tilde{c}_d e^{-i\tilde{\phi}} & \tilde{s}_u \tilde{c}_d - \tilde{c}_u \tilde{s}_d e^{-i\tilde{\phi}} \\ \tilde{c}_u \tilde{s}_d - \tilde{s}_u \tilde{c}_d e^{-i\tilde{\phi}} & \tilde{c}_u \tilde{c}_d + \tilde{s}_u \tilde{s}_d e^{-i\tilde{\phi}} \end{array} \right),
\]

where \(\tilde{s}_u = \sin \tilde{\theta}_u, \tilde{c}_u = \cos \tilde{\theta}_u\), etc. The angles \(\tilde{\theta}_u\) and \(\tilde{\theta}_d\) are the values for \(\theta_u\) and \(\theta_d\) obtained in the heavy quark limit. Since the \((t,b)\) system is decoupled from the \((c,s)\) and \((u,d)\) systems, the flavor mixing can be described as in the case of two families. Therefore the mixing matrix \(\tilde{V}\) is effectively given in terms of only a single rotation angle, the Cabbibo angle \(\theta_C\) [16]:

\[
\sin \theta_C = |\tilde{s}_u \tilde{c}_d - \tilde{c}_u \tilde{s}_d e^{-i\tilde{\phi}}| .
\]

Of course \(\tilde{V}(\theta_C)\) is essentially a real matrix, because its complex phases can always be rotated away by redefining the quark fields.

We should like to stress that the heavy quark limit, which carries the flavor mixing matrix \(V\) to its simplified form \(\tilde{V}\), is not far from the reality, since \(1-c \approx 0.1\%\) holds [3]. Therefore \(\theta_u, \theta_d\) and \(\phi\) are expected to approach \(\tilde{\theta}_u, \tilde{\theta}_d\) and \(\tilde{\phi}\) rapidly, as \(\theta \to 0\), corresponding to \(m_t \to \infty\) and \(m_b \to \infty\). However, the concrete limiting behavior depends on the specific algebraic structure of the up and down mass matrices. If two hermitian mass matrices have the parallel hierarchy with texture zeros in the \((1,1)\) \((2,2)\), \((1,3)\) and \((3,1)\) elements, for example, the magnitude of \(\theta\) is suppressed by the terms proportional to \(m_t^{-1/2}\) and \(m_b^{-1/2}\) [7]; and if the \((2,2)\) elements are kept nonvanishing and comparable in magnitude with the \((2,3)\) and \((3,2)\) elements, then \(\theta\) is dependent on \(m_t^{-1}\) and \(m_b^{-1}\) [19, 20].

The angles \(\tilde{\theta}_u\) and \(\tilde{\theta}_d\) as well as the phase \(\tilde{\phi}\) are well-defined quantities in the heavy quark limit. The physical meaning of these quantities can be seen more clearly, if we take into account a specific and realistic model for the Cabibbo-type mixing in the light quark sector. It is well known that in the absence of the \(u\)-quark mass a relation between the Cabibbo angle \(\theta_C\) and the mass ratio \(m_d/m_s\) follows, if the quark mass matrices have the structure:

\[
\tilde{M}_u = \left( \begin{array}{cc} 0 & 0 \\ 0 & m_c \end{array} \right),
\]

\[
\tilde{M}_d = \left( \begin{array}{cc} 0 & \tilde{B}_d \\ \tilde{B}_d^* & \tilde{A}_d \end{array} \right).
\]

The diagonalization of \(\tilde{M}_d\) leads to the relation \(\tan \theta_C = \sqrt{m_d/m_s}\). The texture-zero pattern of \(\tilde{M}_d\), i.e., the vanishing of its \((1,1)\) element, is already present in certain classes of models.
Figure 1: The light-quark triangle (LT) in the complex plane.

(see, e.g., Refs. [21, 22]). The relation for the Cabibbo angle is known to agree very well with the experimental observation. For numerical discussions, we make use of the quark masses 

\[ \begin{align*} m_u &= (5.1 \pm 0.9) \text{ MeV}, \\
m_d &= (9.3 \pm 1.4) \text{ MeV}, \\
m_s &= (175 \pm 25) \text{ MeV} \quad \text{and} \\
m_c &= (1.35 \pm 0.05) \text{ GeV} \end{align*} \]

at the scale \( \mu = 1 \text{ GeV} \) [18]. Then one finds \( \theta_C = 13.0^\circ \pm 1.8^\circ \) or \( \sin \theta_C = 0.225 \pm 0.031 \), consistent with the observed value of \( |V_{us}| \) (i.e., \( 0.217 \leq |V_{us}| \leq 0.224 \) [1]).

The situation will change once \( m_u \) is introduced, i.e., \( \tilde{M}_u \) takes the same form as \( \tilde{M}_d \) given in (3.7). In this case the mass matrices result in the following relation [7]:

\[
\sin \theta_C = | R_u - R_d e^{-i \psi} | , \tag{3.8}
\]

where

\[
\begin{align*}
R_u &= \sqrt{\frac{m_u}{m_u + m_c}} \sqrt{\frac{m_s}{m_d + m_s}}, \\
R_d &= \sqrt{\frac{m_c}{m_u + m_c}} \sqrt{\frac{m_d}{m_d + m_s}}, \tag{3.9}
\end{align*}
\]

and \( \psi \equiv \arg(\tilde{B}_u) - \arg(\tilde{B}_d) \) denotes the relative phase between the off-diagonal elements \( \tilde{B}_u \) and \( \tilde{B}_d \) (in the limit \( m_u \to 0 \) this phase can be absorbed through a redefinition of the quark fields). We find that the same structure for the Cabibbo-type mixing matrix has been obtained as in the decoupling limit discussed above. If we set

\[
\begin{align*}
\tan \tilde{\theta}_u &= \frac{m_u}{m_c}, \\
\tan \tilde{\theta}_d &= \frac{m_d}{m_s}, \tag{3.10}
\end{align*}
\]

and \( \tilde{\varphi} = \psi \) for (3.6), then the result in (3.8) and (3.9) can exactly be reproduced.

Indeed the relation in (3.6) or (3.8) defines a triangle in the complex plane (see Fig. 1 for illustration), which will be denoted as the “light-quark triangle” (LT). Taking into account the central values of the Cabibbo angle (\( \sin \theta_C = |V_{us}| = 0.2205 \)) and the light quark mass ratios (\( m_s/m_d = 18.8 \) and \( m_c/m_u = 265 \)), we can calculate the phase parameter from (3.8) and obtain \( \tilde{\varphi} = \psi \approx 79^\circ \). If we allow the mass ratios and \( \theta_C \) to vary in their ranges given above, then \( \tilde{\varphi} \) may vary in the range \( 38^\circ - 115^\circ \). We find that \( \tilde{\varphi} \) has a good chance to be around \( 90^\circ \) (see also Ref. [12]). The case \( \tilde{\varphi} \approx 90^\circ \) (i.e., the LT is rectangular) is of special interest, as
we shall see later, since it implies that the area of the unitarity triangle of flavor mixing takes its maximum value for the fixed quark mass ratios – in this sense, the $CP$ symmetry of weak interactions would be maximally violated.

It is worth remarking that the quark mass ratios $m_d/m_s$ and $m_u/m_c$ are essentially independent of the renormalization-group effect from the weak scale to a superhigh scale (or vice versa), so is the Cabibbo angle $\theta_C$. As a result the sides and angles of the LT are to a very good degree of accuracy scale-independent. This interesting feature of the light quark sector implies that the prediction for $\tilde{\theta}_u$ and $\tilde{\theta}_d$ from quark mass matrices at any high scale (e.g., $\mu \sim 10^{16}$ GeV) can directly be confronted with the low-scale experimental data.

The two symmetry limits discussed above are both not far from the reality, in which the strong hierarchy of quark masses ($m_u \ll m_c \ll m_t$ and $m_d \ll m_s \ll m_b$) has been observed. They will serve as a guide in the subsequent discussions about generic quark mass matrices and their consequences on flavor mixing.

4 Analysis of generic mass matrices

Now we return to the case of three quark families. In the standard model or its extensions which have no flavor-changing right-handed currents, one can always adopt a basis of flavor space in which both the up- and down-type quark mass matrices are hermitian and have vanishing (1,3) and (3,1) elements, as shown in (2.1). Such a basis is of special interest in case of a strong mass hierarchy (as realized by nature), since no explicit mixing between the very massive $t$ (or $b$) quark and the very light $u$ (or $d$) quark is introduced. The mixing can then be regarded as of the “nearest neighbour” type [4]. Thus without loss of generality one may discuss the model-independent properties of flavor mixing and $CP$ violation on the basis of the mass matrices (2.1), i.e.,

$$
M_u = \begin{pmatrix} E_u & D_u & 0 \\
D^*_u & C_u & B_u \\
0 & B^*_u & A_u \end{pmatrix},
$$

$$
M_d = \begin{pmatrix} E_d & D_d & 0 \\
D^*_d & C_d & B_d \\
0 & B^*_d & A_d \end{pmatrix}.
$$

(4.1)

The phases of $D_{u,d}$ and $B_{u,d}$ elements are denoted as $\phi_{D_{u,d}}$ and $\phi_{B_{u,d}}$, respectively. The phase differences

$$
\phi_1 = \phi_{D_u} - \phi_{D_d},
$$

$$
\phi_2 = \phi_{B_u} - \phi_{B_d}
$$

(4.2)

are the source of $CP$ violation in weak interactions of quarks. It is clear that $M_u$ and $M_d$ consist totally of twelve parameters.
If the hermiticity is not imposed on the arbitrary up and down mass matrices in the standard model, then they can be taken as the full “nearest-neighbor” mixing form with texture zeros in the (1,1), (2,2), (1,3) and (3,1) positions [23]:

\[
\mathcal{M}_q = \begin{pmatrix}
0 & \mathcal{X}_q & 0 \\
\mathcal{X}_q' & 0 & \mathcal{Y}_q \\
0 & \mathcal{Y}_q' & \mathcal{Z}_q
\end{pmatrix},
\]

(4.3)

where \(\arg(\mathcal{X}_q') = \arg(\mathcal{X}_q)\) and \(\arg(\mathcal{Y}_q') = \arg(\mathcal{Y}_q)\) (for \(q = u\) and \(d\)). In this special basis the light quarks are assumed to acquire masses through an interaction with their nearest neighbors. It is straightforward to find that the non-hermitian mass matrices \(M_u, M_d\) have the same number of free parameters as the hermitian mass matrices \(M_{u,d}\), therefore one could be transformed to the other \(^\dagger\). In our point of view the hermitian basis (4.1) is more natural and will be adopted in the subsequent discussions.

### 4.1 Conditions for CP violation

We first discuss the necessary and sufficient conditions for CP violation in the standard electroweak model and clarify some ambiguity associated with this problem in the literature. As the flavor mixing matrix \(V\) is obtained from the diagonalization of the mass matrices \(M_u\) and \(M_d\), there must be some kind of relation between the parameters of \(V\) and \(M_u, M_d\). The conditions for CP violation can be counted either at the level of quark mass matrices or at the level of the flavor mixing matrix. One must distinguish between these two different levels.

At the level of quark mass matrices it is obvious that CP symmetry will be violated, if and only if there is at least one nontrivial phase difference between \(M_u\) and \(M_d\). In other words, \(\text{Im}(M_{uij}M_{dij}^*) \neq 0\) (for \(i, j = 1, 2, 3\) and \(i \neq j\)) is the necessary and sufficient condition for CP violation in the standard model. If one defines a commutator for \(M_u\) and \(M_d\), \([M_u, M_d] = i\mathcal{C}\), then it is easy to find

\[
\frac{C_{ii}}{2} = \text{Im}(M_{uij}M_{dij}^*) + \text{Im}(M_{uik}M_{dik}^*),
\]

(4.4)

for \(i, j, k = 1, 2, 3\) but \(i \neq j \neq k\). Clearly \(C_{ii} \neq 0\), if CP symmetry is not conserved.

Note that CP symmetry would be conserved, if two quarks with the same charge were degenerate in mass eigenvalues. This is well known, but we shall give a proof here. We assume the \(i\) and \(j\) quarks in the up sector to be degenerate, then they would not be distinguished from each other by any quantum number. Hence any linear combination of the mass eigenstates \(|i\rangle\) and \(|j\rangle\), e.g., \(|i'\rangle\) or \(|j'\rangle\) in the form

\[
\begin{pmatrix}
i' \\
j'
\end{pmatrix} = \begin{pmatrix}
\cos \vartheta e^{+i\xi} & \sin \vartheta e^{+i\zeta} \\
-\sin \vartheta e^{-i\xi} & \cos \vartheta e^{-i\zeta}
\end{pmatrix}\begin{pmatrix}
i \\
j
\end{pmatrix},
\]

(4.5)

\(^\dagger\)If one imposes the hermiticity on (4.3) or the nearest-neighbor mixing on (4.1), then the resultant mass matrices take the particularly simple form which was first proposed and discussed by one of the authors about twenty years ago [21].
remains a mass eigenstate. Without loss of any physical content, the elements of \( M_u \) in the \( i \)-th and \( j \)-th lines and rows can be rearranged by three arbitrary (real) parameters \( \vartheta, \xi \) and \( \zeta \). This, together with other known freedoms, allows one to remove all possible phase differences between \( M_u \) and \( M_d \), i.e., \( \text{Im} \left( M_{uij} M_{dij}^* \right) = 0 \) (for \( i \neq j \)) appears.

A similar proof is valid for the down quark sector. We then conclude that a non-degeneracy between the quarks with the same charge is a necessary (but not sufficient) condition for \( CP \) violation in the standard model. This condition can be more explicitly written, in terms of the determinant of \( C \), as follows [24]:

\[
\text{Det} \ C = -2J \prod_{i<j} (\lambda_i - \lambda_j) \prod_{\alpha<\beta} (\lambda_\alpha - \lambda_\beta),
\]

(4.6)

where \( J \) can be found in (2.6), \( \lambda_i \) and \( \lambda_\alpha \) denote the quark mass eigenvalues, and the subscripts \((i, j)\) and \((\alpha, \beta)\) run over \((u, c, t)\) and \((d, s, b)\) respectively. However, it should be noted that the parameter \( J \) itself does depend on the product of two mass-eigenvalue differences \((\lambda_i - \lambda_j)\) and \((\lambda_\alpha - \lambda_\beta)\), as we shall prove in the next subsection. In this sense \( J \) and \( \text{Det} \ C \) contain the same information about \( CP \) violation; i.e., the latter is not more fundamental than the former, contrary to popular belief.

Now we discuss the condition for \( CP \) violation at the level of the flavor mixing matrix. Of course \( CP \) symmetry is violated, if \( V \) contains a nontrivial complex phase which cannot be removed through the redefinition of quark-field phases. The most appropriate measure of \( CP \) violation (due to the unitarity of \( V \)) is the rephasing-invariant parameter \( J \), whose relation with three mixing angles and the \( CP \)-violating phase has been given in (2.6). Obviously \( J \) vanishes if \( \varphi = 0 \) or \( \pi \). Note that for \( \theta_u = 0 \) or \( \pi/2 \) the phase parameter \( \varphi \) can be removed from \( V \). Therefore the resultant flavor mixing matrix is a real \( 3 \times 3 \) matrix described by only two rotation angles (\( \theta_d \) and \( \theta \)). A similar situation will appear if \( \theta_d = 0, \pi/2 \) or \( \theta = 0, \pi/2 \). The necessary and sufficient condition for \( CP \) violation in the standard model is then \( J \neq 0 \) or \( \varphi \neq 0, \pi \). Since \( s_u = 0 \) or \( c_u = 0 \) will definitely (though indirectly) lead to \( \varphi = 0 \) or \( \pi \), it is unnecessary to count the condition \( \theta_u \neq 0 \) or \( \pi/2 \) together with \( \varphi \neq 0 \) or \( \pi \). So is the situation for \( \theta_d \) and \( \theta \).

In reality quark masses of each sector have been found to perform a clear hierarchy, and all elements of the flavor mixing matrix are nonvanishing [11]. Therefore the realistic condition for \( CP \) violation is only associated with the existence of one nontrivial phase parameter in \( V \), which in turn requires (at least) one nontrivial phase difference between \( M_u \) and \( M_d \).

### 4.2 Exact analytical result for \( J \)

Let us derive the exact analytical relation between the \( CP \)-violating parameter \( J \) and the quark mass-eigenvalue differences. Without loss of generality, we just adopt the basis of flavor space which accommodates the hermitian quark mass matrices \( M_u \) and \( M_d \) in (4.1). A basis-independent proof can similarly be carried out for two arbitrary mass matrices \( M'_u \) and \( M'_d \),
if one starts from the hermitian products \( H_u \equiv M_u^\dagger M_u^\prime \) and \( H_d \equiv M_d^\dagger M_d^\prime \) and arranges them to be of the same form as \( M_u \) and \( M_d \). This can always be done by appropriately adjusting the fields of right-handed quarks, which are iso-singlets in the standard model.

For convenience we decompose \( M_q \) into \( M_q = P_q^\dagger \overline{M}_q P_q \), where

\[
\overline{M}_q = \begin{pmatrix}
E_q & D_q & 0 \\
D_q & C_q & B_q \\
0 & B_q & A_q
\end{pmatrix}
\]  

(4.7)

is a real symmetric matrix, and \( P_q = \text{Diag}\{1, e^{i\phi_{D_q}}, e^{i(\phi_{B_q} + \phi_{D_q})}\} \) is a diagonal phase matrix. In the following we shall neglect the subscript “q”, only if there is no necessity to distinguish between the up and down quark sectors. \( \overline{M} \) can be diagonalized by use of the orthogonal transformation \( O^\dagger \overline{M} O = \text{Diag}\{\lambda_1, \lambda_2, \lambda_3\} \), where \( \lambda_i \) (for \( i = 1, 2, 3 \)) are quark mass eigenvalues and may be either positive or negative. As a result, we have

\[
\sum_{i=1}^{3} \lambda_i = A + C + E ,
\]

\[
\prod_{i=1}^{3} \lambda_i = ACE - A|D|^2 - E|B|^2 ,
\]

\[
\sum_{i=1}^{3} \lambda_i^2 = A^2 + 2|B|^2 + C^2 + 2|D|^2 + E^2 .
\]

(4.8)

It is a simple exercise to solve the nine matrix elements of \( O \) in terms of the parameters of quark mass matrices. Explicitly, three diagonal elements of \( O \) read:

\[
O_{11} = \left[ 1 + \left( \frac{\lambda_1 - E}{|D|} \right)^2 + \left( \frac{|B|}{|D|} \cdot \frac{\lambda_1 - E}{\lambda_1 - A} \right)^2 \right]^{-1/2} ,
\]

\[
O_{22} = \left[ 1 + \left( \frac{|D|}{\lambda_2 - E} \right)^2 + \left( \frac{|B|}{\lambda_2 - A} \right)^2 \right]^{-1/2} ,
\]

\[
O_{33} = \left[ 1 + \left( \frac{\lambda_3 - A}{|B|} \right)^2 + \left( \frac{|D|}{|B|} \cdot \frac{\lambda_3 - A}{\lambda_3 - E} \right)^2 \right]^{-1/2} ;
\]

(4.9)

and then six off-diagonal elements of \( O \) can be obtained from the relations

\[
O_{2i} = \frac{\lambda_i - E}{|D|} O_{1i} ,
\]

\[
O_{3i} = \frac{|B|}{\lambda_i - A} O_{2i} .
\]

(4.10)

The flavor mixing matrix turns out to be \( V \equiv O^\dagger (P_u P_d^\dagger) O_d \). More specifically, we have

\[
V_{i\alpha} = O_{i1}^u O_{1\alpha}^d + O_{i2}^u O_{2\alpha}^d e^{i\phi_1} + O_{i3}^u O_{3\alpha}^d e^{i(\phi_1 + \phi_2)} ,
\]

(4.11)

\(^1\)Here and hereafter, the off-diagonal elements \( B \) and \( D \) are both taken to be nonvanishing. The relevant calculations will somehow be simplified if one of them vanishes.
where the Latin subscript $i$ and the Greek subscript $\alpha$ run over $(u, c, t)$ and $(d, s, b)$ respectively, and the phase differences $\phi_{1,2}$ have been defined in (4.2).

The $CP$-violating parameter $J$ can be calculated from the common imaginary part of nine rephasing invariants of $V$, i.e., $J = |\text{Im}(V_{i\alpha}V_{j\beta}V_{i\alpha}V_{j\beta}^*)|$ for $i \neq j$ and $\alpha \neq \beta$ [11]. With the help of (4.10) and (4.11), one may express $J$ in terms of the parameters of quark mass matrices. After a lengthy calculation, we arrive at the following exact and rephasing-invariant result:

$$J = (\lambda_i - \lambda_j)(\lambda_\alpha - \lambda_\beta) f^{ij}_{\alpha\beta},$$

(4.12)

where

$$f^{ij}_{\alpha\beta} = \left(\frac{O_{1i}^u O_{1j}^d O_{1\alpha}^d O_{1\beta}^u}{|D_u D_d|}\right)^2 \left[ T_1 \sin \phi_1 + T_2 \sin \phi_2 + T_3 \sin(\phi_1 + \phi_2) \
+ T_4 \sin(\phi_1 - \phi_2) + T_5 \sin(2\phi_1 + \phi_2) + T_6 \sin(\phi_1 + 2\phi_2) \right].$$

(4.13)

The expressions of $T_i$ (for $i = 1, 2, \cdots, 6$) are listed in Appendix A. One can see that $J$ depends definitely on the mass-eigenvalue differences $(\lambda_i - \lambda_j)$ of the up sector and $(\lambda_\alpha - \lambda_\beta)$ of the down sector. Since the subscripts $(i, j)$ and $(\alpha, \beta)$ run over the corresponding quarks $(u, c, t)$ and $(d, s, b)$, $J$ would vanish if any two quarks with the same charge were degenerate in mass eigenvalues. Therefore $J$ carries the same information about $CP$ violation as $\text{Det} \ C$ in (4.6). Two remarks are in order.

(a) Note that a phase combination in the form of $\sin(2\phi_1)$, $\sin(2\phi_2)$ or $\sin(\phi_1 + \phi_2)$ has no contribution to $J$. The reason is simply that in $J$ the terms associated with $e^{i2\phi_1}$ and $e^{-i2\phi_1}$ have the same magnitude and cancel each other. So it the case for the terms associated with $e^{i2\phi_2}$ and $e^{-i2(\phi_1 + \phi_2)}$. Once the hierarchy of $M_u$ and $M_d$ is taken into account, the magnitude of $J$ is expected to be dominated by the term proportional to $\sin \phi_1$ (see the next subsection).

(b) The dependence of $J$ on the product of two mass-eigenvalue differences $(\lambda_i - \lambda_j)$ and $(\lambda_\alpha - \lambda_\beta)$ is indeed a basis-independent result, although we have obtained it in a specific basis of flavor space for the quark mass matrices. The basis-independent calculation of $J$ is straightforward, as we have mentioned above, if one starts from $H_u = M_u^\dagger M_u$ and $H_d = M_d^\dagger M_d$ for arbitrary $M_u$ and $M_d$. In this case it is easy to find

$$J = \left(\lambda_i^2 - \lambda_j^2\right) \left(\lambda_\alpha^2 - \lambda_\beta^2\right) F^{ij}_{\alpha\beta},$$

(4.14)

where $F^{ij}_{\alpha\beta}$ can be read off from $f^{ij}_{\alpha\beta}$ through the replacements of matrix elements from $M_{u,d}$ to $H_{u,d}$. Of course the results in (4.12) and (4.14) essentially have the same physical meaning.

In reality it is known that quark masses show a strong hierarchy in either sector. Therefore $J$ vanishes if and only if both $\phi_1 = 0$ (or $\pi$) and $\phi_2 = 0$ (or $\pi$) hold. The necessary and sufficient condition for $CP$ violation in the standard model is trivially $\phi \neq 0$ or $\pi$. 


4.3 Flavor mixing angles and the CP-violating phase

Now let us take the hierarchy of quark masses ($|\lambda_1| \ll |\lambda_2| \ll |\lambda_3|$) into account for the hermitian mass matrices $M_u$ and $M_d$ in (4.1). This implies $|A_q| \gg |B_q|, |C_q| \gg |D_q|, |E_q|$ for both sectors. Our purpose is to calculate the mixing angles ($\theta_u$, $\theta_d$ and $\theta$) and the CP-violating phase ($\phi$) in an analytically exact way.

Certainly the orthogonal matrix $O$ used to diagonalize $\mathbf{M}$ in (4.7) can further be written as a product of three matrices $R_{12}$, $R_{23}$ and $R_{31}$, which describe simple rotations in the 1–2, 2–3 and 3–1 planes respectively:

\[
R_{12}(\omega) = \begin{pmatrix}
  c_\omega & s_\omega & 0 \\
  -s_\omega & c_\omega & 0 \\
  0 & 0 & 1
\end{pmatrix},
\]

\[
R_{23}(\sigma) = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & c_\sigma & s_\sigma \\
  0 & -s_\sigma & c_\sigma
\end{pmatrix},
\]

\[
R_{31}(\tau) = \begin{pmatrix}
  c_\tau & 0 & s_\tau \\
  0 & 1 & 0 \\
  -s_\tau & 0 & c_\tau
\end{pmatrix},
\]

(4.15)

where $s_\omega \equiv \sin \omega$, $c_\omega \equiv \cos \omega$, etc. Taking $O = R_{13}R_{23}R_{12}$ for example, we arrive at

\[
\tan \tau = \frac{|D|}{|B|} \cdot \frac{(E - \lambda_1) + (C - \lambda_2)}{\lambda_3 - E},
\]

\[
\tan \sigma = \frac{|D|}{|B|} \cdot \frac{(E - \lambda_1) + (C - \lambda_2)}{\sqrt{|D|^2 + |B|^2 \tan^2 \tau}},
\]

\[
\tan \omega = \frac{|D|}{\lambda_2 - E} \cdot \frac{c_\sigma}{c_\tau} + s_\sigma \tan \tau.
\]

(4.16)

In view of the hierarchy of quark mass matrices, we find that the magnitude of $\tan \tau$ is highly suppressed, leading to an excellent approximation $\tau \approx 0^\circ$. Thus the matrix $O$ is dominantly described by only two rotation angles, $\omega$ and $\sigma$. This is naturally expected, since in lowest order the diagonalization of the mass matrices is provided by a rotation in the 2–3 plane and a rotation in the 1–2 plane. Due to the vanishing of the (1,3) and (3,1) matrix elements, a rotation in the 3–1 plane is essentially unnecessary. This approximation has been used to derive an interesting parametrization of the flavor mixing matrix [7, 23, 26, 27], whose form is quite similar to that given in (2.2). Note, however, that the exact parametrization (2.2) is indeed independent of the above approximation, because the contribution from rotation matrices $R_{13}^{u,d}$ (up) and $R_{13}^{d}$ (down) to the flavor mixing matrix can always be absorbed by redefining its three overall mixing angles. Since the concrete calculation of those mixing angles from $R_{12}^{u,d}$, $R_{23}^{u,d}$ and $R_{31}^{d}$ is rather complicated and less instructive (see, e.g., Ref. [27]), we shall subsequently follow a different and more straightforward procedure towards the same goal.
We make use of the expression of $V$ given in (4.11). The parametrization of $V$ in terms of three mixing angles ($\theta_u$, $\theta_d$, $\theta$) and one $CP$-violating phase ($\varphi$) has been shown in (2.2). To link these four parameters with the parameters of quark mass matrices in a concise way, we first define four dimensionless quantities:

\[
X_u \equiv \frac{|D_u|}{\lambda_u^2 - E_u} \cdot \frac{|D_d|}{|B_d|} \left( \frac{\lambda_u^3 - A_u}{\lambda_u^3 - E_u} \right) + \frac{\lambda_u^3 - A_u}{|B_d|} e^{i\varphi_1} + \frac{|B_u|}{\lambda_u^3 - A_u} e^{i(\varphi_1 + \phi_2)},
\]

\[
Y_u \equiv \frac{|D_u|}{\lambda_u^2 - E_u} \cdot \frac{|D_d|}{|B_d|} \left( \frac{\lambda_u^3 - A_u}{\lambda_u^3 - E_u} \right) + \frac{\lambda_u^3 - A_u}{|B_d|} e^{i\varphi_1} + \frac{|B_u|}{\lambda_u^3 - A_u} e^{i(\varphi_1 + \phi_2)};
\]

and $(X_d, Y_d)$ can directly be obtained from $(X_u, Y_u)$ through the subscript exchange $u \leftrightarrow d$ in (4.17). After a lengthy but straightforward calculation, we arrive at

\[
\tan \theta_u = \frac{O_{21}^u}{O_{22}^u} \cdot \frac{X_u}{Y_u},
\]

\[
\tan \theta_d = \frac{O_{21}^d}{O_{22}^d} \cdot \frac{X_d}{Y_d},
\]

and

\[
\sin \theta = \left[ (O_{21}^u)^2 X_u^2 + (O_{22}^u)^2 Y_u^2 \right]^{1/2} O_{33}^u,
\]

\[
= \left[ (O_{21}^d)^2 X_d^2 + (O_{22}^d)^2 Y_d^2 \right]^{1/2} O_{33}^d,
\]

where $O_{21}, O_{22}$ and $O_{33}$ for up and down sectors have been given in (4.9) and (4.10). Also an indirect relation between $\varphi$ and $\phi_{1,2}$ can be obtained as follows:

\[
\cos \varphi = \frac{s_u^2 c_d c^2 + c_u^2 s_d^2 - |V_{us}|^2}{2 s_u c_u s_d c_d e},
\]

where

\[
|V_{us}| = O_{11}^u O_{22}^d \left| \frac{|D_d|}{\lambda_d^2 - E_d} \right| + \frac{\lambda_d^2 - E_d}{|D_u|} e^{i\varphi_1} \left( 1 + \frac{|B_u|}{\lambda_u^3 - A_u} \cdot \frac{|B_d|}{\lambda_d^3 - A_d} e^{i\phi_2} \right).
\]

If the hierarchies of the matrix elements and mass eigenvalues of $M_{u,d}$ are taken into account, one can see that the effect of $\phi_2$ on $|V_{us}|$ is strongly suppressed and thus negligible. Fitting $|V_{us}|$ with current data should essentially determine the magnitude of $\phi_1$. Note also that the terms associated with $\phi_1$ and $\phi_2$ may primarily be cancelled in the ratios $X_u/Y_u$ and $X_d/Y_d$ due to the hierarchical structures of $M_u$ and $M_d$, hence the dependence of $\theta_u$ and $\theta_d$ on $\phi_{1,2}$ could be negligible in the leading order approximation. Although the mixing angle $\theta$ may be sensitive to $\phi_1$ and $\phi_2$ (or one of them), its smallness indicated by current data makes the factor $\cos \theta$ in the denominator of $\cos \varphi$ completely negligible. As a result, (4.20) and (4.21) imply that the $CP$-violating phase $\varphi$ depends dominantly on $\phi_1$ through $|V_{us}|$, unless the magnitude of $\phi_1$ is very small. Without fine-tuning, we find that a delicate numerical analysis does support the argument made here, i.e., $\phi_2$ plays a negligible role for $CP$ violation in $V$, because of the hierarchy of quark masses. The observed $CP$ violation is linked primarily to the phases in the $(1,2)$ and $(2,1)$ elements of the quark mass matrices.
5 A realistic texture of mass matrices

In order to get definite predictions for the flavor mixing angles and $CP$ violation, we proceed to specify the general hermitian mass matrices in (2.1) or (4.1) by taking $E_q = 0$:

$$M_q = \begin{pmatrix} 0 & D_q & 0 \\ D_q^* & C_q & B_q \\ 0 & B_q^* & A_q \end{pmatrix}. \quad (5.1)$$

In case of two quark families, this is just the form taken for $\tilde{M}_d$ in (3.7). As remarked above, the texture zeros in (1,3) and (3,1) positions can always be arranged. Thus the physical constraint is as follows: in the flavor basis in which (1,3) and (3,1) elements of $M_{u,d}$ vanish, the (1,1) element of $M_{u,d}$ vanishes as well. This can strictly be true only at a particular energy scale. The vanishing of the (1,1) element can be viewed as a result of an underlying flavor symmetry, which may either be discrete or continuous. In the literature a number of such possibilities have been discussed (see, e.g., Refs. [7] – [30]). Here we shall not discuss further details in this respect, but concentrate on the phenomenological consequences of such a texture pattern. It is particularly interesting that some predictions of this ansatz for the mixing angles and the unitarity triangle are approximately independent of the renormalization-group effects, therefore a specification of the energy scale at which the texture of $M_{u,d}$ holds is unnecessary for our purpose. We believe that $M_q$ given in (5.1) is a realistic candidate for the quark mass matrices of a (yet unknown) fundamental theory responsible for fermion mass generation and $CP$ violation, and we shall make some further speculations about this point at the end of this paper.

5.1 Flavor mixing angles

We take $C_q \neq 0$ and define $|B_q|/C_q \equiv r_q$ for each quark sector [1]. The magnitude of $r_q$ is expected to be of $O(1)$. The parameters $A_q$, $|B_q|$, $C_q$ and $|D_q|$ in (5.1) can be expressed in terms of the quark mass eigenvalues and $r_q$. Applying such results to the general formulas listed in (4.17) – (4.19), we get three mixing angles of $V$ as follows:

$$\tan \theta_u = \sqrt{\frac{m_u}{m_c}} (1 + \Delta_u),$$
$$\tan \theta_d = \sqrt{\frac{m_d}{m_s}} (1 + \Delta_d),$$
$$\sin \theta = \left| r_d \frac{m_u}{m_b} (1 - \delta_d) - r_u \frac{m_c}{m_t} (1 - \delta_u) e^{i\phi_2} \right|, \quad (5.2)$$

Note that the special condition $C_q/A_q = |D_q/B_q|^2$ has been imposed on $M_q$ in a recent paper [31]. It leads to vanishing flavor mixing among all three quark families in the chiral limit of $u$- and $d$-quark masses, and in turn requires a kind of correlation between the flavor mixing angles $\theta_u$, $\theta_d$ and $\theta$. This unusual feature is apparently in conflict with our arguments made in section 3 (see also Ref. [3]). Beyond the chiral symmetry limit the aforementioned condition is equivalent to taking $|r_u| \approx \sqrt{m_u m_t} / m_c$ and $|r_d| \approx \sqrt{m_d m_b} / m_s$ in our case. Clearly both $r_u$ and $r_d$ are of $O(1)$ in magnitude, consistent with the common expectation.
where the next-to-leading order corrections read

\[ \Delta_u = \sqrt{\frac{m_c m_d}{m_u m_s} m_b} \left| \Re \left[ e^{i\phi_1} - \frac{r_u}{r_d} \cdot \frac{m_c m_b}{m_t} e^{i(\phi_1 + \phi_2)} \right] \right|^{-1}, \]

\[ \Delta_d = \sqrt{\frac{m_u m_s}{m_c m_d} m_t} \left| \Re \left[ e^{i\phi_1} - \frac{r_d}{r_u} \cdot \frac{m_u m_s}{m_c m_b} e^{i(\phi_1 + \phi_2)} \right] \right|^{-1}; \] (5.3)

and

\[ \delta_u = \frac{m_u}{m_c} + \left(1 + r_u^2\right) \frac{m_c}{m_t}, \]

\[ \delta_d = \frac{m_d}{m_s} + \left(1 + r_d^2\right) \frac{m_s}{m_b}. \] (5.4)

Clearly the result for \( \delta_{u,d} \) in (3.4) can be reproduced from \( \delta_{u,d} \) in (5.4), if one takes the chiral limit \( m_u \to 0, m_d \to 0 \). From (5.2) we also observe that the phase \( \phi_2 \) is only associated with the small quantity \( m_c / m_t \) in \( \sin \theta \). To get the relationship between \( \varphi \) and \( \phi_1 \) or \( \phi_2 \), we first calculate \( |V_{us}| \) from the quark mass matrices by use of (4.21). It turns out that

\[ |V_{us}| = \left(1 - \frac{m_u}{2 m_c} - \frac{m_d}{2 m_s}\right) \left| \sqrt{\frac{m_d}{m_s}} - \sqrt{\frac{m_u}{m_c}} e^{i\phi_1} \right| \] (5.5)

in the next-to-leading order approximation. Note that this result can also be achieved from (3.8) and (3.9), which were obtained in the heavy quark limit. Confronting (5.5) with current data on \( |V_{us}| \) leads to the result \( \phi_1 \sim 90^\circ \), as we have discussed before. Therefore \( \cos \phi_1 \) is expected to be a small quantity. Then we use (4.20) together with (5.2) and (5.5) to calculate \( \cos \varphi \). In the same order approximation, we arrive at

\[ \cos \varphi = \sqrt{\frac{m_u m_s}{m_c m_d}} \Delta_u + \sqrt{\frac{m_c m_d}{m_u m_s}} \Delta_d + (1 - \Delta_u - \Delta_d) \cos \phi_1. \] (5.6)

The contribution of \( \phi_2 \) to \( \varphi \) is substantially suppressed at this level of accuracy.

For simplicity, we proceed by taking \( r_u = r_d \equiv r \), which holds in some models with natural flavor symmetries [19]. Then \( \sin \theta \) becomes proportional to a universal parameter \( |r| \). In view of the fact \( m_s / m_b \sim O(10) \) \( m_c / m_t \), we find that the result in (5.3) can be simplified as

\[ \Delta_u = \sqrt{\frac{m_c m_d}{m_u m_s} m_b} \cos \phi_1, \]

\[ \Delta_d = 0. \] (5.7)

Also the relation between \( \varphi \) and \( \phi_1 \) in (5.6) is simplified to

\[ \cos \varphi = \left(1 + \frac{m_s}{m_b}\right) \cos \phi_1. \] (5.8)

As \( m_s / m_b \sim 4\% \), it becomes apparent that \( \varphi \approx \phi_1 \) is a good approximation. Note that \( \phi_1 = \varphi \) holds exactly in the heavy quark limit, in which \( \varphi \) has been denoted as \( \tilde{\varphi} \) (see (3.5) as well.
Figure 2: The rescaled unitarity triangle (UT) in the complex plane.

as Fig. 1). The equality $\phi_1 = \tilde{\phi}$ follows, i.e., both stand for the phase difference between the mass matrix elements $D_u$ and $D_d$.

Following (4.12) we evaluate the dependence of the $CP$-violating measurable $J$ on $\phi_1, \phi_2$ and their various combinations. The results for six coefficients of $f_{\alpha\beta}^{ij}$ are listed in Appendix B. We confirm that the magnitude of $J$ is dominated by the $\sin \phi_1$ term and receives one-order smaller corrections from the $\sin(\phi_1 \pm \phi_2)$ terms. As a result,

$$J \approx |r|^2 \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{m_d}{m_s}} \left( \frac{m_s}{m_b} \right)^2 \sin \phi_1$$

(5.9)

holds to a good degree of accuracy. Clearly $J \sim O(10^{-5}) \times \sin \phi_1$ with $\sin \phi_1 \sim 1$ is favored by current data.

The result of $J$ in (5.9) might give the impression that $CP$ violation is absent if either $m_u$ or $m_d$ vanishes. This is not exactly true, however. If we set $m_u = 0$, $J$ is not zero, but it becomes dependent on $\sin \phi_2$ with a factor which is about two orders of magnitude smaller (i.e., of order $10^{-7}$):

$$J \approx |r|^2 \frac{m_c}{m_t} \cdot \frac{m_d}{m_s} \left( \frac{m_s}{m_b} \right)^2 \sin \phi_2 .$$

(5.10)

Certainly this possibility is already ruled out by experimental data.

Note also that the model predicts

$$\tan \theta_u = \frac{|V_{ub}|}{V_{cb}} = \sqrt{\frac{m_u}{m_c}} \left( 1 + \Delta_u \right) ,$$

$$\tan \theta_d = \frac{|V_{td}|}{V_{ts}} = \sqrt{\frac{m_d}{m_s}} \left( 1 + \Delta_d \right) ,$$

(5.11)

a result obtained first by one of us from a more specific pattern of quark mass matrices [3]. In $B$-meson physics, $|V_{ub}/V_{cb}|$ can be determined from the ratio of the decay rate of $B \to (\pi, \rho)\nu_\ell$ to that of $B \to D^* \nu_\ell$; and $|V_{td}/V_{ts}|$ can be extracted from the ratio of the rate of $B_d^0 - \bar{B}_d^0$ mixing to that of $B_s^0 - \bar{B}_s^0$ mixing.

### 5.2 The unitarity triangle

We are now in a position to calculate the unitarity triangle (UT) of quark flavor mixing defined in (2.8), whose three inner angles are denoted as $\alpha, \beta$ and $\gamma$ in (2.9). Note that three sides
of the unitarity triangle can be rescaled by $V_{cb}^*$ (see Fig. 2 for illustration). The resultant triangle reads

$$|V_{cd}| = |S_d - S_u e^{-i\alpha}|,$$  \hfill (5.12)

where $S_u = |V_{ub}V_{ud}/V_{cb}|$ and $S_d = |V_{tb}V_{td}/V_{cb}|$. After some calculations $S_u$, $S_d$ and $\alpha$ are obtained from the above quark mass texture in the next-to-leading order approximation:

$$S_u = \sqrt{\frac{m_u}{m_c}} \left( 1 - \frac{1}{2} \frac{m_u}{m_d} - \frac{1}{2} \frac{m_d}{m_s} \right) + \sqrt{\frac{m_c m_d}{m_u m_b}} \frac{m_s}{m_c m_s} \cos \phi_1 + \sqrt{\frac{m_u m_d}{m_c m_s}} \cos \phi_1,$$

$$S_d = \sqrt{\frac{m_d}{m_s}} \left( 1 + \frac{1}{2} \frac{m_u}{m_d} - \frac{1}{2} \frac{m_d}{m_s} \right);$$  \hfill (5.13)

and

$$\sin \alpha = \left( 1 - \sqrt{\frac{m_u m_d}{m_c m_s}} \cos \phi_1 \right) \sin \phi_1.$$

A comparison of the rescaled UT in Fig. 2 with the LT in Fig. 1, which is obtained in the heavy quark limit, is interesting. We find $\alpha \approx 0.0056$, $m_d/m_s = 0.045$ and $m_u/m_b = 0.033$ to calculate these three $CP$-violating parameters from the LT and from the rescaled UT separately. Both approaches lead to $\alpha \approx 90^\circ$, $\beta \approx 20^\circ$ and $\gamma \approx 70^\circ$, which are in good agreement with the results obtained from the standard analysis of current data on $|V_{ub}/V_{cb}|$, $|\epsilon_K|$, $B_d^0 - \bar{B}_d^0$ mixing and $B_s^0 - \bar{B}_s^0$ mixing [10]. Note that among three $CP$-violating observables only $\sin(2\beta)$ is remarkably sensitive to the value of $m_u/m_c$, which
involves quite large uncertainty (e.g., \sin(2\beta) may change from 0.4 to 0.8 if \( m_u/m_c \) varies in the range 0.002 – 0.01). For this reason we emphasize again that the numbers given above can only serve as an illustration. A more reliable determination of the quark mass values is crucial, in order to test the ansätze of quark mass matrices in a numerically decisive way \( \parallel \).

It is also worth mentioning that the result \( \tan \theta_d = \sqrt{m_d/m_s} \) is particularly interesting for the mixing rates of \( B_d^0-\overline{B_d^0} \) and \( B_s^0-\overline{B_s^0} \) systems, measured by \( x_d \) and \( x_s \) respectively \( \parallel \). As \( x_d = 0.723 \pm 0.032 \) has been well determined \( \parallel \), the prediction for the value of \( x_s \) is

\[
x_s = x_d \chi_{\text{su}(3)} \frac{m_s}{m_d} = 19.8 \pm 3.5 ,
\]

where \( m_s/m_d = 18.9 \pm 0.8 \), obtained from the chiral perturbation theory \( \parallel \), has been used. This result is certainly consistent with the present experimental bound on \( x_s \), i.e., \( x_s > 14.0 \) at the 95\% confidence level \( \parallel \). A measurement of \( x_s \sim 20 \) may be realized at the forthcoming HERA-B and LHC-B experiments.

### 5.3 Comparison with the Ramond-Roberts-Ross patterns

The quark mass matrices \( M_u \) and \( M_d \) given in (5.1) have parallel structures with four texture zeros (here a pair of off-diagonal texture zeros are counted as one zero due to the hermiticity of \( M_u \) and \( M_d \)). Giving up the parallelism between the structures of \( M_u \) and \( M_d \), Ramond, Roberts and Ross (RRR) have found that there exist five phenomenologically allowed patterns of quark mass matrices – each of them has five texture zeros \( \parallel \), as listed in Table 1. The RRR patterns I, II or IV can be formally regarded as a special case of our four-texture-zero ansatz (5.1), with \( B_u = 0 \), \( C_u = 0 \) or \( B_d = 0 \), respectively. Note that \( M_u \) of the RRR pattern III or V has nonvanishing (1,3) and (3,1) elements, therefore these two patterns are essentially different from the mass matrices assumed in (4.1) or (5.1). As a comparison, here we make some brief comments on consequences of the RRR patterns on the flavor mixing angles \( \theta \), \( \theta_u \) and \( \theta_d \).

(a) For the RRR pattern I, the magnitude of \( \sin \theta \) is governed by the (2,3) and (3,2) elements of \( M_d \). Therefore \( |B_d| \sim |C_d| \) is expected, to result in \( \sin \theta \sim m_s/m_b \). In the leading order approximation, \( \tan \theta_u = \sqrt{m_u/m_c} \) and \( \tan \theta_d = \sqrt{m_d/m_s} \) hold. The next-to-leading order corrections to these two quantities are almost indistinguishable from those obtained in (5.3) and (5.7) for our four-texture-zero ansatz.

(b) The mass matrix \( M_u \) of the RRR pattern II takes the well-known form suggested originally by one of us in Refs. \( \parallel \). To reproduce the experimental value of \( \sin \theta \), the possibility \( |B_d| \gg |C_d| \) has been abandoned and the condition \( |B_d| \sim |C_d| \) is required. However, significant cancellation between the term proportional to \( \sqrt{m_c/m_t} \) (from \( M_u \) \( \parallel \) and

\[\parallel\]

\[\parallel\]
Table 1: Five RRR patterns of quark mass matrices.

| Pattern | I          | II         | III        | IV          | V          |
|---------|------------|------------|------------|-------------|------------|
| $M_u$   | ($D_u^0$, $C_u$, $B_u$) | ($D_u^0$, $C_u$, $B_u$) | ($D_u^0$, $C_u$, $B_u$) | ($D_u^0$, $C_u$, $B_u$) | ($D_u^0$, $C_u$, $B_u$) |
| $M_d$   | ($D_d^0$, $C_d$, $B_d$) | ($D_d^0$, $C_d$, $B_d$) | ($D_d^0$, $C_d$, $B_d$) | ($D_d^0$, $C_d$, $B_d$) | ($D_d^0$, $C_d$, $B_d$) |

that proportional to $m_s/m_b$ (from $M_d$) in $\sin \theta$ may take place, if the phase difference between $B_u$ and $B_d$ is vanishing or very small. The leading order results $\tan \theta_u = \sqrt{m_u/m_c}$ and $\tan \theta_d = \sqrt{m_d/m_s}$ can still be obtained here, but their next-to-leading order corrections may deviate somehow from those obtained in the above subsections.

(c) From the RRR pattern IV one can arrive at $\sin \theta \sim \sqrt{m_c/m_t}$ with the necessary condition $|C_u| \ll |B_u|$, since the (2,3) and (3,2) elements of $M_d$ vanish. The results for $\tan \theta_u$ and $\tan \theta_d$ are similar to those obtained from the RRR pattern I.

(d) The nonvanishing (1,3) and (3,1) elements of $M_u$ in the RRR pattern III make its prediction for the mixing angles $\theta_u$ and $\theta_d$ quite different from all patterns discussed above. Analytically one can find $\tan \theta_u \sim (m_b/m_s)\sqrt{m_u/m_t}$, while $\tan \theta_d$ is a complicated combination of the terms $\sqrt{m_d/m_s}$ and $(m_b/m_s)\sqrt{m_u/m_t}$ with a relative phase. In addition, $\sin \theta \sim m_s/m_b$ holds under the condition $|B_d| \sim |C_d|$, similar to the RRR pattern I.

(e) For the RRR pattern V, the necessary condition $|B_u| \gg |C_u|$ is required in order to reproduce $\sin \theta \sim \sqrt{m_c/m_t}$. Here again the nonvanishing (1,3) and (3,1) elements of $M_u$ result in very complicated expressions for $\tan \theta_u$ and $\tan \theta_d$ (even more complicated than those in the RRR pattern III [34]).

For reasons of naturalness and simplicity, we argue that the RRR patterns III and V are unlikely to be good candidates for the quark mass matrices in an underlying theory of fermion mass generation.

6 Discussions and conclusion

We have studied the phenomena of quark flavor mixing and $CP$ violation in the context of generic hermitian mass matrices. The necessary and sufficient conditions for $CP$ violation in the standard model have been clarified at both the level of quark mass matrices and that of the flavor mixing matrix. Our particular observation is that $CP$ violation is primarily linked to a phase difference of about $90^\circ$ in the light quark sector, and this property becomes most
apparent in the new parametrization (2.2). To be more specific, we have analyzed a realistic pattern of quark mass matrices with four texture zeros and given predictions for the flavor mixing and \(CP\)-violating parameters. The approximate congruency between the light-quark triangle (LT) and the rescaled unitarity triangle (UT), which provides an intuitive and scale-independent connection of \(CP\)-violating observables to quark mass ratios, is particularly worth mentioning.

Let us make some further comments on the quark mass matrix (5.1), its phenomenological hints and its theoretical prospects.

Naively one might not expect any prediction from the four-texture-zero mass matrices in (5.1), since they totally consist of ten free parameters (two of them are the phase differences between \(M_u\) and \(M_d\)). This is not true, however, as we have seen. We find that two predictions, 
\[
\tan \theta_u \approx \sqrt{m_u/m_c}, \quad \text{and} \quad \tan \theta_d \approx \sqrt{m_d/m_s},
\]
can be obtained in the leading order approximation. In some cases the latter may even hold in the next-to-leading order approximation, as shown in (5.2) and (5.7). Note again that these two relations, as a consequence of the hierarchy and texture zeros of our quark mass matrices, are essentially independent of the renormalization-group effects. This interesting scale-independent feature can also be seen from the LT and the rescaled UT as well as their inner angles \((\alpha, \beta, \gamma)\).

It remains to be seen whether the interesting possibility \(\varphi \approx \phi_1 \approx 90^\circ\), indicated by current data of quark masses and flavor mixing, could arise from an underlying flavor symmetry or a dynamical symmetry breaking scheme. Some speculations about this problem have been made (see, e.g., Refs. \[12, 13\] and Refs. \[13, 20\]). However, no final conclusion has been reached thus far. It is remarkable, nevertheless, that we have at least observed a useful relation between the area of the UT \(A_{\text{UT}}\) and that of the LT \(A_{\text{LT}}\) to a good degree of accuracy:

\[
A_{\text{UT}} \approx |V_{cb}|^2 A_{\text{LT}} \approx \sin^2 \theta A_{\text{LT}}. \tag{6.1}
\]

Since \(A_{\text{UT}} = J/2\) measures the magnitude of \(CP\) violation in the standard model, we conclude that \(CP\) violation is primarily linked to the light quark sector. This is a natural consequence of the strong hierarchy between the heavy and light quark masses, which is on the other hand responsible for the smallness of \(J\) or \(A_{\text{UT}}\).

Is it possible to derive the quark mass matrix (5.1) in some theoretical frameworks? To answer this question we first specify the hierarchical structure of \(M_q\) in terms of the mixing angle \(\theta_q\) (for \(q = d\) or \(s\)). Adopting the radiant unit for the mixing angles (i.e., \(\theta_u \approx 0.085\), \(\theta_d \approx 0.204\) and \(\theta \approx 0.040\)), we have

\[
\begin{align*}
\frac{m_u}{m_c} & \sim \frac{m_c}{m_t} \sim \theta_u^2, \\
\frac{m_d}{m_s} & \sim \frac{m_s}{m_b} \sim \theta_d^2.
\end{align*}
\tag{6.2}
\]

Then the mass matrices \(M_u\) and \(M_d\), which have the mass scales \(m_t\) and \(m_b\) respectively, take
the following parallel hierarchies:

\[
M_u \sim m_t \begin{pmatrix} 0 & \theta^3_u & 0 \\ \theta^3_u & \theta^2_u & \theta^2_u \\ 0 & \theta^2_u & 1 \end{pmatrix},
\]

\[
M_d \sim m_b \begin{pmatrix} 0 & \theta^3_d & 0 \\ \theta^3_d & \theta^2_d & \theta^2_d \\ 0 & \theta^2_d & 1 \end{pmatrix},
\]

where the relevant complex phases have been neglected. Clearly all three flavor mixing angles can properly be reproduced from (6.3), once one takes \( \theta \approx \theta^3_d \gg \theta^3_u \) into account. The CP-violating phase \( \varphi \) in \( V \) comes essentially from the phase difference between the \( \theta^3_u \) and \( \theta^3_d \) terms.

Of course \( \theta_u \) and \( \theta_d \), which are more fundamental than the Cabibbo angle \( \theta_C \) in our point of view, denote perturbative corrections to the rank-one limits of \( M_u \) and \( M_d \) respectively. They are responsible for the generation of light quark masses as well as the flavor mixing. They might also be responsible for CP violation in a specific theoretical framework (e.g., the pure real \( \theta_u \) and the pure imaginary \( \theta_d \) might lead to a phase difference of about 90° between \( M_u \) and \( M_d \), which is just the source of CP violation favored by current data). The small parameter \( \theta_q \) could get its physical meaning in the Yukawa coupling of an underlying superstring theory: \( \theta_q = \langle \Theta_q \rangle / \Omega_q \), where \( \langle \Theta_q \rangle \) denotes the vacuum expectation value of the singlet field \( \Theta_q \), and \( \Omega_q \) represents the unification (or string) mass scale which governs higher dimension operators (see, e.g., Refs. [25, 34, 35]). The quark mass matrices of the form (6.3) could then be obtained by introducing an extra (horizontal) \( U(1) \) gauge symmetry or assigning the matter fields appropriately.

A detailed study of possible dynamical models responsible for the quark mass matrices (5.1) or (6.3) is certainly desirable but beyond the scope of this work. However, we believe that the texture zeros and parallel hierarchies of up and down quark mass matrices do imply specific symmetries, perhaps at a superhigh scale, and have instructive consequences on flavor mixing and CP-violating phenomena. The new parametrization of the flavor mixing matrix that we advocated is particularly useful in studying the quark mass generation, flavor mixing and CP violation.
Appendices

A  Coefficients $T_i$ for generic hermitian mass matrices

The six coefficients of $f_{ij}^{\alpha\beta}$ in (4.12) can all be expressed in terms of the parameters of quark mass matrices $M_u$ and $M_d$. For simplicity we define

$$Z_E \equiv (\lambda_i - E_u)(\lambda_j - E_u)(\lambda_{\alpha} - E_d)(\lambda_{\beta} - E_d),$$

$$Z_A \equiv (\lambda_i - A_u)(\lambda_j - A_u)(\lambda_{\alpha} - A_d)(\lambda_{\beta} - A_d).$$

Then $T_i$ (for $i = 1, 2, \cdots, 6$) are found to be:

$$T_1 = \left(1 - \frac{Z_E}{|D_uD_d|^2}\right) + \frac{Z_E}{Z_A} \cdot \frac{|B_uB_d|^2}{|D_uD_d|^2} \left[\frac{\lambda_i\lambda_j + 2A_uE_u - A^2_u - E_u(\lambda_i + \lambda_j)}{(\lambda_i - A_u)(\lambda_j - A_u)}ight] + \frac{\lambda_{\alpha}\lambda_{\beta} + 2A_dE_d - A^2_d - E_d(\lambda_{\alpha} + \lambda_{\beta})}{(\lambda_{\alpha} - A_d)(\lambda_{\beta} - A_d)},$$

(A.2)

$$T_2 = \frac{Z_E^2}{Z_A} \cdot \frac{|B_uB_d|}{|D_uD_d|^2} \left(1 - \frac{|B_uB_d|^2}{Z_A^{2}}\right) - \frac{Z_E}{Z_A} \cdot \frac{|B_uB_d|}{|D_uD_d|} \left[\frac{\lambda_i\lambda_j + 2A_uE_u - A^2_u - A_u(\lambda_i + \lambda_j)}{(\lambda_i - E_u)(\lambda_j - E_u)}ight] + \frac{\lambda_{\alpha}\lambda_{\beta} + 2A_dE_d - E^2_d - A_d(\lambda_{\alpha} + \lambda_{\beta})}{(\lambda_{\alpha} - E_d)(\lambda_{\beta} - E_d)},$$

(A.3)

$$T_3 = \frac{Z_E}{Z_A} \cdot \frac{|B_uB_d|}{|D_uD_d|} \cdot (\lambda_i + \lambda_j)(A_d - E_d) + (\lambda_{\alpha} + \lambda_{\beta})(A_u - E_u) - 2A_uA_d + 2E_uE_d$$

$$+ \left(1 - \frac{Z_E}{Z_A} \cdot \frac{|B_uB_d|^2}{|D_uD_d|^2}\right) \frac{|B_uB_d|}{|D_uD_d|} (A_u - E_u)(A_d - E_d),$$

(A.4)

$$T_4 = -\frac{Z_E}{Z_A} \cdot \frac{|B_uB_d|}{|D_uD_d|} \cdot (A_u - E_u)(A_d - E_d),$$

(A.5)

$$T_5 = \frac{Z_E}{Z_A} \cdot \frac{|B_uB_d|}{|D_uD_d|},$$

(A.6)

$$T_6 = -\frac{Z_E}{Z_A} \cdot \frac{|B_uB_d|^2}{|D_uD_d|^2}.$$  

(A.7)

The relative magnitude of these coefficients can be seen, after the hierarchy and texture zeros of $M_u$ and $M_d$ are specified.

B  Coefficients $T_i$ for the ansatz with 4 texture zeros

Here we estimate the coefficients of $f_{ij}^{\alpha\beta}$ for the ansatz of quark mass matrices discussed in section 5.2. Choosing $i = u, j = c$ and $\alpha = d, \beta = s$, we arrive approximately at

$$Z_E \approx m_u m_c m_d m_s,$$

$$Z_A \approx m^2_t m^2_b.$$  

(B.1)
Then $T_i$ can straightforwardly be obtained from the exact analytical results (A.1) – (A.7):

$$T_1 \approx |r|^2 \left[ \left( \frac{m_c}{m_t} \right)^2 + \left( \frac{m_s}{m_b} \right)^2 \right] \sim 1.4 \times |r|^2 \times 10^{-3},$$  \hspace{1cm} (B.2)

$$T_2 \approx |r|^2 \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{m_d}{m_s}} \left( \frac{m_c}{m_t} \right)^2 \left( \frac{m_s}{m_b} \right)^2 \left( 1 + \frac{m_t}{m_u} + \frac{m_b}{m_d} \right) \sim 2 \times |r|^2 \times 10^{-5},$$  \hspace{1cm} (B.3)

$$T_3 \approx -|r|^2 \frac{m_c m_s}{m_t m_b} \sim -1.5 \times |r|^2 \times 10^{-4},$$  \hspace{1cm} (B.4)

$$T_4 \approx -|r|^2 \frac{m_u}{m_c} \cdot \frac{m_s}{m_b} \sim -1.5 \times |r|^2 \times 10^{-4},$$  \hspace{1cm} (B.5)

$$T_5 \approx |r|^2 \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{m_d}{m_s}} \left( \frac{m_c}{m_t} \right)^2 \left( \frac{m_s}{m_b} \right)^2 \sim 3 \times |r|^2 \times 10^{-10},$$  \hspace{1cm} (B.6)

$$T_6 \approx -|r|^4 \left( \frac{m_c}{m_t} \right)^2 \left( \frac{m_s}{m_b} \right)^2 \sim -2 \times |r|^4 \times 10^{-8},$$  \hspace{1cm} (B.7)

in the leading order approximation. We see that $T_1$ is dominant in $f_{ij}^{\alpha\beta}$, and the contribution of $T_3$ and $T_4$ to $f_{ij}^{\alpha\beta}$ can be treated as the next-to-leading order corrections. This result shows again that the phase parameter $\phi_1 \approx \varphi$ dominates the magnitude of $CP$ violation in the flavor mixing matrix.

To the same degree of accuracy, one gets

$$\left( \lambda_u - \lambda_c \right) \left( \lambda_d - \lambda_s \right) \approx m_c m_s,$$

$$\frac{\left( O_{1u}^a O_{1c}^u O_{1d}^d O_{1s}^d \right)^2}{|D_u D_d|} \approx \frac{1}{m_c m_s} \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{m_d}{m_s}}$$  \hspace{1cm} (B.8)

from the mass matrices in (5.1). Applying (B.2) and (B.8) to (4.12), we then obtain the magnitude of the $CP$-violating parameter $\mathcal{J}$, as shown in (5.9).
References

[1] Particle Data Group, C. Caso et al., Eur. Phys. J. C 3 (1998) 1.

[2] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49 (1973) 652.

[3] H. Fritzsch and Z.Z. Xing, Phys. Lett. B 413 (1997) 396.

[4] H. Fritzsch and Z.Z. Xing, Phys. Rev. D 57 (1998) 594.

[5] H. Fritzsch, Phys. Lett. B 184 (1987) 391.

[6] H. Fritzsch, Phys. Lett. B 73 (1978) 317.

[7] H. Fritzsch, Nucl. Phys. B 155 (1979) 189.

[8] L. Euler, in Memoires de l’Academie de Sciences de Berlin, 16 (1760) 176; part of the English translation can be found in: R. Dugas, A History of Mechanics (Dover Publications, 1988), p. 276.

[9] See, e.g., L. Maiani, in Proceedings of the International Symposium on Lepton and Photon Interactions at High Energies (DESY, Hamburg, 1977), p. 867; L.L. Chau and W.Y. Keung, Phys. Rev. Lett. 53 (1984) 1802; H. Harari and M. Leurer, Phys. Lett. B 181 (1986) 123; H. Fritzsch and J. Plankl, Phys. Rev. D 35 (1987) 1732.

[10] F. Parodi, P. Roudeau, and A. Stocchi, hep-ph/9802289; hep-ex/9903063.

[11] C. Jarlskog, Phys. Rev. Lett. 55 (1985) 1039.

[12] H. Fritzsch and Z.Z. Xing, Phys. Lett. B 353 (1995) 114.

[13] D. Delépine, J.M. Gérard, R. G. Felipe, and J. Weyers, Phys. Lett. B 411 (1997) 167.

[14] For a recent review, see the BABAR Physics Book, edited by P.F. Harrison and H.R. Quinn, SLAC-R-504 (1998).

[15] Z.Z. Xing, Nucl. Phys. B (Proc. Suppl.) 50 (1996) 24.

[16] N. Cabibbo, Phys. Rev. Lett. 10 (1963) 531.

[17] An updated direct measurement of $\sin 2\beta$ has recently been reported by the CDF Collaboration (see, CDF/PUB/BOTTOM/CDF/4855, February 1998). The preliminary result $\sin 2\beta = 0.79^{+0.41}_{-0.44}$ is consistent very well with the standard-model expectation.

[18] H. Leutwyler, Phys. Lett. B 378 (1996) 313; J. Gasser and H. Leutwyler, Phys. Rep. C 87 (1982) 77.

[19] H. Harari, H. Haut, and J. Weyers, Phys. Lett. B 78 (1978) 459; Y. Chikashige, G. Gelmini, R.P. Peccei, and M. Roncadelli, Phys. Lett. B 94 (1980) 499; Y. Koide, Phys. Rev. D 28 (1983) 252; H. Fritzsch, in Proceedings of the Europhysics Topical Conference on Flavor Mixing in Weak Interactions, Erice, 1984, edited by L.L. Chau (Plenum, New York), p. 717; C. Jarlskog,
in Proceedings of the International Symposium on Production and Decay of Heavy Hadrons, Heidelberg, 1986, edited by K.R. Schubert and R. Waldi (DESY, Hamburg), p. 331; M. Tanimoto, Phys. Rev. D 41 (1990) 1586; P. Kaus and S. Meshkov, Phys. Rev. D 42 (1990) 180; H. Fritzsch and J. Plankl, Phys. Lett. B 237 (1990) 451; H. Fritzsch and D. Holtmanspötter, Phys. Lett. B 338 (1994) 290; G.C. Branco and J.I. Silva-Marcos, Phys. Lett. B 359 (1995) 166; H. Fritzsch and Z.Z. Xing, Phys. Lett. B 372 (1996) 265; H. Lehmann, C. Newton, and T.T. Wu, Phys. Lett. B 384 (1996) 249; Z.Z. Xing, J. Phys. G 23 (1997) 1563; K. Kang and S.K. Kang, Phys. Rev. D 56 (1997) 1511.

[20] For recent works, see, H. Fritzsch and Z.Z. Xing, Phys. Lett. B 440 (1998) 313; J.I. Silva-Marcos, Phys. Lett. B 443 (1998) 276; T. Shinozaki, H. Tanaka, and I.S. Sogami, Prog. Theor. Phys. 100 (1998) 615; S.L. Adler, Phys. Rev. D 59 (1999) 015012; V.V. Kiselev, [hep-ph/9806523]; A. Mondragon and E. Rodriguez-Jaurregui, [hep-ph/9807214].

[21] H. Fritzsch, Phys. Lett. B 70 (1977) 436.

[22] S. Weinberg, in Transactions of the New York Academy of Sciences, 38 (1977) 185; F. Wilczek and A. Zee, Phys. Lett. B 70 (1977) 418.

[23] G.C. Branco, L. Lavoura, and F. Mota, Phys. Rev. D 39 (1989) 3443.

[24] C. Jarlskog, in CP Violation, edited by C. Jarlskog (World Scientific, Singapore, 1989), p. 3.

[25] C.D. Froggatt and H.B. Nielsen, Nucl. Phys. B 147 (1979) 277.

[26] S. Dimopoulos, L.J. Hall, and S. Raby, Phys. Rev. Lett. 68 (1992) 1984; See, also, H. Lehmann, C. Newton, and T.T. Wu, in Ref. [19].

[27] A. Rasin, [hep-ph/9708216] (unpublished).

[28] H. Georgi and D.V. Nanopoulos, Nucl. Phys. B 155 (1979) 52.

[29] P. Ramond, R.G. Roberts, and G.G. Ross, Nucl. Phys. B 406 (1993) 19.

[30] S.N. Gupta and J.M. Johnson, Phys. Rev. D 44 (1991) 2110; S. Rajpoot, Mod. Phys. Lett. A 7 (1992) 309; D. Du and Z.Z. Xing, Phys. Rev. D 48 (1993) 2349; P.S. Gill and M. Gupta, Phys. Rev. D 56 (1997) 3143; M. Randhawa, V. Bhatnagar, P.S. Gill, and M. Gupta, [hep-ph/9903428].

[31] J.L. Chkareuli and C.D. Froggatt, Phys. Lett. B 450 (1999) 158.

[32] R. Barbieri, L.J. Hall, and A. Romanino, [hep-ph/9812384]; Phys. Lett. B 401 (1997) 47.

[33] V. Giménez and G. Martinelli, Phys. Lett. B 398 (1997) 135.

[34] T. Kobayashi and Z.Z. Xing, Int. J. Mod. Phys. A 13 (1998) 2201; Mod. Phys. Lett. A 12 (1997) 561.

[35] L. Ibáñez and G.G. Ross, Phys. Lett. B 332 (1994) 100.