Mutually unbiased weighing matrices

D. Best · H. Kharaghani · H. Ramp

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Abstract Inspired by the many applications of mutually unbiased Hadamard matrices, we study mutually unbiased weighing matrices. These matrices are studied for small orders and weights in both the real and complex setting. Our results make use of and examine the sharpness of a very important existing upper bound for the number of mutually unbiased weighing matrices.

Keywords Weighing matrix · Unbiased weighing matrix · Hadamard matrix · Line sets

Mathematics Subject Classification 05B20

1 Introduction

A unit weighing matrix, $W$, with order $n$ and weight $w$, denoted by $UW(n, w)$, is an $n \times n$ matrix with entries whose absolute value falls in $\{0, 1\}$ and $WW^* = wI_n$, where $W^*$ is the usual conjugate transpose of $W$. This implies that the rows of $W$ are mutually orthogonal under the standard inner product in $\mathbb{C}^n$ and contain exactly $w$ nonzero entries in each row and column. When $n = w$ (i.e., no zeroes in the matrix), $W$ is a Hadamard matrix. A real weighing matrix is the one with entries in $\{0, \pm1\}$. Real weighing matrices have been well

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D. Best · H. Kharaghani (✉)
Department of Mathematics and Computer Science, University of Lethbridge, Lethbridge, AB T1K 3M4, Canada
e-mail: hadi@cs.uleth.ca

D. Best
e-mail: darcy.best@uleth.ca

H. Ramp
Department of Physics, University of Alberta, Edmonton, AB T6G 2R3, Canada
e-mail: ramp@ualberta.ca

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studied for small weights (see [7]) and large weights (see [8]). Note that there is an error in [7] which is corrected by Harada and Munemasa in [11, Sect. 4].

This article contains results for weighing matrices in both the real and complex setting. Motivated by the applications of real weighing matrices, we have studied unit weighing matrices in [4]. Our aim in this paper is to complement the work that started in [4], and to introduce a new concept of unbiasedness.

Two unit weighing matrices \( UW(n, w) \), \( H \) and \( K \), are unbiased if \( HK^* = \sqrt{w} L \), where \( L \) is a unit weighing matrix \( UW(n, w) \). A set of pairwise unbiased unit weighing matrices are called mutually unbiased unit weighing matrices. In the special case of \( n = w \), these are termed mutually unbiased Hadamard matrices (MUHM), which are of great interest to people working in areas related to the quantum information theory and as such, there is extensive literature on these matrices. We refer the reader to the most comprehensive survey paper [10] on MUHM. Mutually unbiased unit weighing matrices have also seen some application in quantum information science, specifically in the context of zero-error classical communication [13].

In [4], we concerned ourselves with the existence of certain unit weighing matrices; here, we are concerned about how many pairwise unbiased unit weighing matrices there are. In the general unimodular case, where nonzero entries have an absolute value of one, we lose much of structure, and thus, some of the restrictions, that can be found in the real case (see Lemma 1 for one such example). This makes it very challenging to locate complete sets.

If the entries of matrices in a set of mutually unbiased unit weighing matrices are limited to certain roots of unity, then a condition similar to Lemma 1 is found (e.g., see [2]), but very few concrete bounds exist in general. Section 2 will deal with the unit weighing matrices in general by giving the few known upper bounds and lower bounds on the size of these sets.

In Sect. 3, we will outline some of our computer searches for small orders of unit weighing matrices. As an extension to mutually unbiased unit weighing matrices, we will examine sets of Hadamard matrices whose pairwise products satisfy specific conditions in Sect. 4.

2 General restrictions

This section includes preparatory steps necessary for the vast amount of computations required in the next two sections. For the sake of completeness we include essential details of all that is needed. In doing so, some of the results of this section are easy to obtain.

We begin by reiterating a very well known result, (see [2]).

**Lemma 1** Let \( H \) and \( K \) be real unbiased weighing matrices with order \( n \) and weight \( w \). Then \( w \) must be a perfect square.

**Proof** Since \( H \) and \( K \) are integer matrices, \( HK^T = \sqrt{w} L \) must also be an integer matrix. \( \square \)

The next lemma will be derived through the utilization of the standard direct sum of matrices:

\[
W_1 \oplus W_2 \oplus \cdots \oplus W_k = \text{diag}(W_1, W_2, W_3, \ldots, W_k) = \begin{bmatrix}
W_1 & 0 & \cdots & 0 \\
0 & W_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & W_k
\end{bmatrix}
\]
Theorem 2 Let \( \{W_1, \ldots, W_k\} \) be a collection of sets of mutually unbiased weighing matrices of order \( n_i \) and weight \( w \). Then there exists at least

\[
\min_{1 \leq i \leq k} (|W_i|)
\]

mutually unbiased weighing matrices of order \( \sum_{i=1}^k n_i \) and weight \( w \).

Proof Let \( |W_i| = \ell_i \) and write \( W_i = \left\{ W_1^{(i)}, W_2^{(i)}, \ldots, W_{\ell_i}^{(i)} \right\} \) for each \( 1 \leq i \leq k \). Let

\[
m = \min_{1 \leq i \leq k} (|W_i|) = \min_{1 \leq i \leq k} (\ell_i).
\]

Then the set

\[
\left\{ \left( W_1^{(1)} \oplus W_1^{(2)} \oplus \cdots \oplus W_1^{(k)} \right), \left( W_2^{(1)} \oplus W_2^{(2)} \oplus \cdots \oplus W_2^{(k)} \right), \ldots, \left( W_m^{(1)} \oplus W_m^{(2)} \oplus \cdots \oplus W_m^{(k)} \right) \right\}
\]

gives the desired result by noting that \( (A \oplus B)(A \oplus B)^* = AA^* \oplus BB^* \). \( \square \)

Two weighing matrices, \( H \) and \( K \), are equivalent if \( H = PKQ \), where \( P \) and \( Q \) are unimodular permutation matrices (i.e., each row/column has exactly one nonzero unimodular entry). We use the notation \( H \cong K \).

Definition 3 Let \( W \) be a weighing matrix of order \( n \) and weight \( w \). If \( W = W_1 \oplus W_2 \) for some \( W_1 \) and \( W_2 \) of order strictly less than \( n \), then \( W \) is said to be decomposable\(^1\). We may write \( W \) in such a way that \( W = W_1 \oplus W_2 \oplus \cdots \oplus W_k \) where each \( W_i \) is indecomposable of order \( n_i \). The block structure of \( W \) is the \( k \)-tuple \( (n_1, n_2, \ldots, n_k) \).

When two weighing matrices have exactly the same block structure, we will be able to utilize the following proposition.

Proposition 4 If two weighing matrices (say \( H \) and \( K \)) of the same weight have the same block structure, then \( H \) is unbiased with \( K \) if and only if each indecomposable block of \( H \) is unbiased with the corresponding indecomposable block of \( K \).

Proof This is easily seen by noting that

\[
(\sum_{i=1}^m H_i \oplus \cdots \oplus H_m)(\sum_{i=1}^m K_i \oplus \cdots \oplus K_m)^* = (\sum_{i=1}^m H_i^* \oplus \cdots \oplus H_m^* K_m^*).
\]

\( \square \)

The block structures of matrices is repeatedly used in our proofs throughout the paper by applying the following proposition.

Proposition 5 Let \( \{W_1, \ldots, W_k\} \) be a set of mutually unbiased weighing matrices, of order \( n \) and weight \( w \), with the same block structure, say \( (n_1, \ldots, n_m) \). Then \( k \) is bounded above by the maximal size of a set of mutually unbiased weighing matrices of order \( n_i \) and weight \( w \), for \( 1 \leq i \leq k \).

Proof This follows from Proposition 4. \( \square \)

\(^1\) The term decomposable matrix is sometimes used to describe a reducible matrix. The reader is warned to not confuse the two terms in this manuscript.
When we examine an arbitrary set of mutually unbiased weighing matrices, they may not be in a form where Propositions 4 and 5 may be used. However, we may be able to apply appropriate row and column permutations in such a way that we may utilize those propositions. For example,

\[
H_1 = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & -0 & 0 \\
0 & 1 & 0 & -0
\end{pmatrix}
\quad \text{and} \quad
K_1 = \begin{pmatrix}
0 & 1 & 0 & i \\
1 & 0 & 0 & 0 \\
1 & 0 & -i & 0 \\
0 & 1 & 0 & -i
\end{pmatrix}
\]

are two indecomposable weighing matrices which are unbiased with one another. However, with appropriate row and column permutations\(^2\), we may examine

\[
H_2 = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & -0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & -0
\end{pmatrix}
\quad \text{and} \quad
K_2 = \begin{pmatrix}
1 & i & 0 & 0 \\
1 & i & 0 & 0 \\
0 & 0 & 1 & i \\
0 & 0 & 1 & -i
\end{pmatrix}
\]

which are also unbiased with one another, and where Propositions 4 and 5 may be used. We will call the block structure found in \(H_2\) and \(K_2\) suitable and the block structure found in \(H_1\) and \(K_1\) not suitable. Throughout the article, we will only concern ourselves with matrices that have a suitable block structure. To this end, we pose an algorithm to determine a matrix’s suitable block structure.

**Lemma 6** The suitable block structure of a weighing matrix of order \(n\) can be determined in \(O(n^3)\) steps.

**Proof** Let \(W\) be a weighing matrix of order \(n\) and \(W'\) be the equivalent weighing matrix that has a suitable block structure. We define \(G_W\) be the graph on \(n\) vertices with an edge between vertices \(i\) and \(j\) if and only if at least one nonzero entry in row \(i\) is in the same column as a nonzero entry in row \(j\) in \(W\). Two rows of \(W\) are in the same indecomposable block of \(W'\) if and only if there is a path between the corresponding nodes in \(G_W\). Thus, an indecomposable block of \(W'\) can be found by taking the rows corresponding to all vertices in any connected component of \(G_W\) and removing all columns that only have zeroes. The number of indecomposable blocks of \(W'\) is the number of connected components of \(G_W\).

In total, this process involves two parts. First, to build the graph, we look at all pairs of rows and examining each column, for a time of \(O(n^3)\). Then, we determine the number of connected components, which takes \(O(n^2)\) via depth first search for an overall complexity of \(O(n^3)\) steps. \(\square\)

It is noteworthy to point out that the asymptotic bound in Lemma 6 is not tight. The construction of \(G_W\) in the proof of Lemma 6 can be done by multiplying \(|W|\) by \(|W|^T\), where \(|W|\) is the matrix \(W\) having had the modulus operation to each of its entries. The nonzero entries in \(|W||W|^T\) signify an edge in \(G\). As of today, matrix multiplication can be done in \(O(n^{2.3727})\) steps [15], but in general, due to the fact that we are only concerned with the fact that an entry is nonzero, we can apply bit operations to make the \(O(n^3)\) algorithm significantly faster in practice.

The following two theorems were originally found by Delsarte et al. [9] using Jacobi polynomials, and later on by Calderbank et al. [6] using a completely different method. They are very important results that we will be using.

\(^2\) Note that the column permutations must be the same for both matrices to ensure they are still unbiased with one another.
Theorem 7 ([6, Eq. 5.9]) Let $V \subset \mathbb{C}^n$ be a set of unit vectors. If $|\langle v, w \rangle| \in \{0, \alpha\}$ for all $v, w \in V, v \neq w$, where $\alpha \in \mathbb{R}$ and $0 < \alpha < 1$, then

$$|V| \leq n \left(\frac{n + 1}{2}\right).$$

Moreover,

$$|V| \leq \frac{n(n + 1)(1 - \alpha^2)}{2 - (n + 1) \alpha^2} \quad (1)$$

if the denominator is positive.

Theorem 8 ([6, Eqs. 3.7 and 3.9]) If all of the entries of $V$ in Theorem 7 are real, then

$$|V| \leq \left(\frac{n + 2}{3}\right).$$

Moreover,

$$|V| \leq \frac{n(n + 2)(1 - \alpha^2)}{3 - (n + 2) \alpha^2} \quad (2)$$

if the denominator is positive.

It is important to note that in most cases, the second upper bound given in each theorem is smaller than the first, but not always. For example, if we are looking for real vectors with $n = 9$ and $\alpha = \frac{1}{2}$, the first bound gives us $|V| \leq 165$ whereas the second bound gives us $|V| \leq 297$.

The following are immediate corollaries to the previous two theorems.

Corollary 9 Let $W = \{W_1, \ldots, W_m\}$ be a set of mutually unbiased weighing matrices of order $n$ and weight $w$. Then we have that

$$m \leq \frac{(n - 1)(n + 2)}{2}. \quad (3)$$

Moreover, if $2w - (n + 1) > 0$, then

$$m \leq \frac{w(n - 1)}{2w - (n + 1)}. \quad (4)$$

Proof Define $V$ to be the set of all rows of $\frac{1}{\sqrt{w}}W_1, \ldots, \frac{1}{\sqrt{w}}W_m$ (note that $|V| = mn$). Since $W$ is a set of mutually unbiased weighing matrices, we may set $\alpha = \frac{1}{\sqrt{w}}$. Moreover, note that since all vectors in $V$ come from a weighing matrix of weight $w$, we may add the rows of the identity matrix to $V$ without disrupting the bi-angularity. By applying Theorem 7 to $V$ (with the rows of the identity matrix included), we obtain the desired results. \hfill \Box

Corollary 10 Let $W = \{W_1, \ldots, W_m\}$ be a set of real mutually unbiased weighing matrices of order $n$ and weight $w$. Then we have that

$$m \leq \frac{(n - 1)(n + 4)}{6}. \quad (5)$$

Moreover, if $3w - (n + 2) > 0$, then

$$m \leq \frac{w(n - 1)}{3w - (n + 2)}. \quad (6)$$

Proof Similar to Corollary 9. \hfill \Box
We compare the theoretic upper bound given in Corollary 9 to the results of both our computer searches and any improved (i.e., smaller) upper bounds we have found.

| Type       | Upper bounds | Examples found |
|------------|--------------|----------------|
| UW(2,2)    | 2            | 2, 4           |
| UW(3,2)    | 5            | 0 (See [4])    |
| UW(3,3)    | 3            | 3, 3           |
| UW(4,2)    | 9            | 2 (Lemma 12)   |
| UW(4,3)    | 9            | 9, 6           |
| UW(4,4)    | 4            | 4, 4           |
| UW(5,2)    | 14           | 0 (See [4])    |
| UW(5,3)    | 14           | 0 (See [4])    |
| UW(5,4)    | 8            | 5 (Theorem 17) |
| UW(5,5)    | 5            | 5, 5           |
| UW(6,2)    | 20           | 2 (Lemma 12)   |
| UW(6,3)    | 20           | 3 (Theorem 14) |
| UW(6,4)    | 20           | 20, 6          |
| UW(6,5)a   | 25a          | 8a             |
| UW(6,6)a   | 6a           | 2a             |
| UW(7,2)    | 27           | 0 (See [4])    |
| UW(7,3)    | 27           | 3 (Theorem 14) |
| UW(7,4)    | 27           | 8 (Theorem 21) |
| UW(7,5)    | 15           | 0 (See [4])    |
| UW(7,6)a   | 9a           | 0a             |
| UW(7,7)    | 7            | 7, 7           |

*Signify cases where the smallest upper bound and largest lower bound do not meet. Note that UW(6, 6) is the most highly sought after set of matrices [1].

3 Mutually unbiased weighing matrices

3.1 Computer search

In this section, we will be searching for sets of mutually unbiased weighing matrices. To aid in this search, we will be using the ideas explored in the previous section. In particular, each set that we look at will have a suitable block structure.

With unit weighing matrices, an exhaustive computer search is impractical, if not impossible, to perform since each nonzero entry in each matrix has infinitely many choices. To this end, we restricted the entries to small roots of unity in our computer searches. For each type of matrix, we searched for matrices over the $m$th roots of unity, with $m \leq 24$. Searches with higher $m$ become increasingly impractical due to the algorithmic complexity. As one observes from Table 1, the 12th roots of unity seem to be the largest group needed to find some maximal sets. Many of the maximal sets that we found do not match the upper bound given in Corollary 9. For many cases, we prove smaller upper bounds.

Mutually unbiased unit Hadamard matrices have been extensively studied for prime power orders. In fact, at each prime power, a set can be constructed that meets the upper bound given in Corollary 9. A proof of the following Theorem can be found in [10].

**Theorem 11** For any prime power $q$, there exists a set of $q$ mutually unbiased (Butson) Hadamard matrices $UW(q, q)$. 
3.2 Upper bound for mutually unbiased weighing matrices of weight 2

In [4, Theorem 10], we proved that $UW(n, 2)$ do not exist for odd orders. For $n$ even, we have the following.

**Lemma 12** Let $n$ be even. Then there are at most two mutually unbiased weighing matrices of order $n$ and weight 2.

**Proof** Assume that we have a set of mutually unbiased weighing matrices of the appropriate order and weight. From [4], we know that one of the matrices may be transformed into

$$
\left( \begin{array}{cc} 1 & 1 \\ 1 & - \\
\end{array} \right) \otimes I_{n/2},
$$

where “−” denotes “−1”. Permute the rows of the second matrix so that there is a nonzero in the top-left entry. The second entry in the top row must be nonzero, otherwise the inner product of the top row of the first and second matrices will be neither 0 nor $\sqrt{2}$. Continue this argument so that the block structure is the same between all matrices in the set of unbiased weighing matrices. By applying Corollary 9 to Proposition 5, we have our result. \(\square\)

3.3 Upper bound for mutually unbiased weighing matrices of weight 3

**Lemma 13** A $UW(n, 3)$, $H$, is unbiased with $K$ if and only if $K$ has the same block structure as $H$.

**Proof** From [4, Theorem 12], we know that $H$ may be transformed into a matrix of the following form:

$$
\begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \bar{\omega} \\
1 & \bar{\omega} & \omega
\end{pmatrix} \oplus \cdots \oplus
\begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \bar{\omega} \\
1 & \bar{\omega} & \omega
\end{pmatrix} \oplus
\begin{pmatrix}
1 & 1 & 1 & 0 \\
1 & - & 0 & 1 \\
1 & 0 & - & - \\
0 & 1 & - & 1
\end{pmatrix} \oplus \cdots \oplus
\begin{pmatrix}
1 & 1 & 1 & 0 \\
1 & - & 0 & 1 \\
1 & 0 & - & - \\
0 & 1 & - & 1
\end{pmatrix},
$$

where $\omega = e^{i \frac{2\pi}{3}}$.

We may assume that the first 3 rows of $K$ have a 1 in the first column by means of normalization by a unit number, and appropriate row and column permutations.

Assume that the top left block in $H$ is a $UW(3, 3)$. In the first row of $K$, if the first three entries are $(1, 0, 0)$, then the inner product of this row and the first row of $H$ can obviously not be of the desired form. Moreover, if there are two nonzero entries (i.e., either $(1, a, 0)$ or $(1, 0, a)$), then there must be a third entry in columns 4 through $n$. The inner product of this row and three different rows in $H$ will simply be a unimodular number (this is true by the structure of $H$), and thus, not in the desired form. This means that the first three entries must all be nonzero. This argument can be made for the second and third row of $K$, and thus, the top left corner of $K$ is a $UW(3, 3)$, as desired.

Now assume that the top left block in $H$ is a $UW(4, 3)$. If columns 2, 3 and 4 are all zero in any of the first 3 rows, then the inner product of row 1 in $H$ and that row will give us a unimodular number. If there is exactly 1 nonzero in columns 2, 3 and 4, then the inner product of that row and the fourth row of $H$ will be unimodular. Thus, we know that in the first 3 rows of $K$, all 3 nonzero entries must appear in the first four columns.

We will now show that the first zero in these rows will not be in the same column. Assume that one column has at least two zeroes. This means that at least one of columns 2, 3 and
will be complete (i.e., no more nonzero entries may go into that column). Column 1 is already complete, so in our fourth row, there is either 1 or 2 nonzeroes in the first 3 columns. By taking the inner product of the fourth row of $K$ by the appropriate row in $H$, we will get a unimodular number. Thus, the first zero in the first 4 rows must be in different columns (note that the first zero in row 4 must be in column 1). Furthermore, through appropriate row permutations and negations, the second entry in row 4 must be a 1. The next two entries are clearly nonzero or there is 1-orthogonality (see [4]) within $K$. Thus, in the first 4 rows of $K$, the three nonzero entries must appear in the first 4 rows, with the first zeroes of the rows in different columns (i.e., a $UW(4, 3)$).

Once we know that the top left block of $H$ and $K$ are the same, if we examine the bottom right $(n - 3) \times (n - 3)$ or $(n - 4) \times (n - 4)$ block, we have a $UW(n - 3, 3)$ or $UW(n - 4, 3)$, and we can recursively use the same argument to obtain the desired result.

\[ \begin{align*}
\text{Theorem 14} & \quad \text{The upper bound on the number of MUWM of the form } UW(n, 3) \text{ is:} \\
& = \begin{cases} 
3 & \text{if } n \not\equiv 0 \mod 4 \\
9 & \text{if } n \equiv 0 \mod 4 
\end{cases} \\
\text{where } n \in \{3, 4\} \cup \{k : k \geq 6\}. 
\end{align*} \]

\[ \text{Proof} \quad \text{Using Lemma 13 with Proposition 5 and the fact that the upper bound for } \text{UW}(3, 3) \text{ is 3 and } \text{UW}(4, 3) \text{ is 9 via Corollary 9, we have that if the matrix contains a } \text{UW}(3, 3) \text{ in its block structure, then it acts as a limiting factor, causing the upper bound to be 3. Otherwise, it is 9, which can only occur when } n \text{ is a multiple of 4. Noting that there is no } \text{UW}(5, 3) \text{ (Table 1), we have the upper bound for all values of } n. \]

\[ \text{Corollary 15} \quad \text{The upper bound given in Theorem 14 is tight for all } n \in \{3, 4\} \cup \{k : k \geq 6\}. \]

\[ \text{Proof} \quad \text{A computer search has shown the bounds to be tight for } \text{UW}(4, 3) \text{ (see Appendix 1, Table 3) and the bound for } \text{UW}(3, 3) \text{ is attained through Theorem 11. We may construct the } \text{UW}(n, 3) \text{ by adjoining the appropriate amount of } \text{UW}(4, 3) \text{ and } \text{UW}(3, 3) \text{ together along the main diagonals. If } n \text{ is a multiple of 4, use only } \text{UW}(4, 3)s \text{ along the main diagonal. Otherwise, it does not matter which blocks are used. A simple induction will show that every integer larger than 5 may be written in the form of } 3m + 4l. \]

3.4 Upper bound for mutually unbiased weighing matrices of weight 4

3.4.1 $UW(5,4)$

\[ \text{Lemma 16} \quad \text{Let } W \text{ be a unit weighing matrix that is unbiased with} \\
W_5 = \begin{pmatrix}
1 & 1 & 1 & 0 \\
1 & \omega & \overline{\omega} & 0 & 1 \\
1 & \overline{\omega} & 0 & \omega & \overline{\omega} \\
1 & 0 & \omega & \overline{\omega} & \omega \\
0 & 1 & \overline{\omega} & \omega & \omega
\end{pmatrix},
\]

where $\omega = e^{i \frac{2\pi}{5}}$. Then every nonzero entry in $W$ is a sixth root of unity, up to equivalence.

\[ \text{Proof} \quad \text{Since } W_5W^* = 2L \text{ for some weighing matrix } L, \text{ we know that each row of } W \text{ must be orthogonal with exactly one row of } W_5. \text{ Moreover, we may permute the rows of } W \text{ so that row } i \text{ is orthogonal with row } i \text{ of } W_5. \text{ We know that the first nonzero entry in each row of } W\]

\[ \text{Springer} \]
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may be a one. Using the definition of \( m \)-orthogonality and the results given in [4, Sect. 3], we can determine that there are at most 11 different rows that are orthogonal to each of the rows of \( W_5 \), each with exactly one free variable.

Let \( b \) be an arbitrary unimodular number and \( \alpha \) a primitive third root of unity (either \( \omega \) or \( \bar{\omega} \)). The four main observations that are used in each line of the proof are:

\[
\begin{align*}
(O1) & \quad |1 - \alpha + b| = 2 \implies b \in \{\pm \alpha\}, \\
(O2) & \quad |1 + \alpha + b| = 2 \implies b = -\alpha, \\
(O3) & \quad |3 + b| = 2 \implies b = -1, \\
(O4) & \quad 1 + \alpha + \alpha = 0.
\end{align*}
\]

We will examine all candidates for row 1 of \( W \). There are only 11 different candidates (up to a free variable), they are:

\[
\begin{align*}
(A) & \quad 1 - b - b 0 \\
(B) & \quad 1 b - -b 0 \\
(C) & \quad 1 b -b 0 \\
(D) & \quad 1 \omega \bar{\omega} 0 b \\
(E) & \quad 1 \bar{\omega} \omega 0 b \\
(F) & \quad 1 \omega 0 \bar{\omega} b \\
(G) & \quad 1 \bar{\omega} 0 \omega b \\
(H) & \quad 1 0 \omega \bar{\omega} b \\
(I) & \quad 1 0 \bar{\omega} \omega b \\
(J) & \quad 0 1 \omega \bar{\omega} b \\
(K) & \quad 0 1 \bar{\omega} \omega b
\end{align*}
\]

For each candidate, we will show that in order to be unbiased with the other four rows of \( W_5 \), the free variable must be a sixth root of unity. In some cases, we will show that the row cannot be unbiased with a specific row of \( W_5 \). To avoid a lengthy proof, we only give three examples.

(A) By taking the complex inner product with row 2 of \( W_5 \), we have that \( |1 - \omega + \bar{\omega} b| = 2 \).

By using \((O1)\), we have that \( \omega b = \pm \bar{\omega} \) which implies that \( b \in \{\pm 1\} \). Thus, all entries in the candidate row are sixth roots of unity.

(G) By taking the complex inner product with row 3 of \( W_5 \), we have that \( |1 + 1 + 1 + \bar{\omega} b| = 2 \).

By using \((O3)\), we have that \( \omega b = -1 \) which implies that \( b = -\bar{\omega} \). Thus, all entries in the candidate row are sixth roots of unity.

(J) By taking the complex inner product with row 5 of \( W_5 \), we have that \( |1 + \omega + \bar{\omega} + \omega b| = 2 \).

By using \((O4)\), we have that \( |\omega \bar{\omega}| = 2 \) which implies that \( |b| = 2 \), which is a contradiction since \( b \) is a unimodular number. Thus, \((J)\) cannot be unbiased with row 5, so it may not be the row that is orthogonal with row 1 of \( W_5 \).

For each of the five rows of \( W_5 \), there are 11 different candidates for per row (each with exactly one free variable). In every case, the free variable is shown to be a sixth root of unity or have absolute value 2 (as in the examples above).

\[\square\]

**Theorem 17** The largest number of mutually unbiased weighing matrices of the form \( UW(5, 4) \) is 5.

**Proof** In [4, Lemma 15], it is proven that all \( UW(5, 4) \) are equivalent to \( W_5 \) given in Lemma 16. Thus, given a set of mutually unbiased weighing matrices, we may permute and multiply by a unit number the rows and columns of the matrices in such a way that one
of them is $W_5$. By Lemma 16, we know that any matrix that is unbiased with $W_5$ must only contain 0 and the sixth roots of unity, leaving just $5 \times 6^3 = 1,080$ rows to check (there are five locations for the zero, and we may set the first nonzero entry to be 1). An exhaustive computer search was done over these rows, which revealed that the maximal set of mutually unbiased weighing matrices contains five elements. One collection of these matrices is included in Table 4 in Appendix 1.

\[ \square \]

3.4.2 UW(6,4)

This is the first case where the upper bound given in Corollary 9 seems unattainably high (20 mutually unbiased weighing matrices). However, relatively quickly, our computer program gave us the following.

**Theorem 18** There are 20 mutually unbiased weighing matrices of order 6 and weight 4.

**Proof** A set of matrices attaining this bound can be found in Appendix 1, Table 5.

Each of the elements in the set of matrices given are over the sixth root of unity. One special feature of this set of matrices is that it attains the upper bounds given in both (3) and (4).

The first four matrices given in Table 5 are real matrices, which falls just short of the upper bound given in Corollary 10. This turns out to be an optimal set of real weighing matrices.

**Theorem 19** There are no more than four mutually unbiased real weighing matrices of order 6 and weight 4.

**Proof** An exhaustive computer search over real weighing matrices was performed and found that there were no sets of mutually unbiased real weighing matrices of order 6 and weight 4.

\[ \square \]

3.4.3 UW(7,4)

**Lemma 20** Let $W$ be a unit weighing matrix that is unbiased with

\[
W_7 = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0
\end{pmatrix},
\]

Then every nonzero entry in $W$ is either 1 or $-1$, up to equivalence.

**Proof** We can easily see that there are only $\binom{7}{3} = 35$ possible zero placements that are valid in a row of $W$. Similar to the proof of Lemma 16, we will only show a couple cases, as the rest follow similarly. Let $a, b, c$ be arbitrary unimodular numbers.

(A) \( \begin{pmatrix} 1 & a & b & c & 0 & 0 & 0 \end{pmatrix} \)

- Taking the complex inner product with row 2 of $W_7$, we have that $|1 + a| \in \{0, 2\}$ which implies $a \in \{\pm1\}$. 

\[ \square \]
have the same zero pattern as $W$.

Of particular note, the only rows that do not cause a contradiction are those seven rows which have the real weighing matrix $W$.

Similarly to the proof of Theorem 17, one matrix in the set may be transformed into the real weighing matrix $W_7$ given in Lemma 20, since every $UW(7, 4)$ is equivalent to this matrix (see [4, Sect. 3.4]). By Lemma 20, every weighing matrix equivalent to $W_7$ must also be real, so we may use Corollary 10 to provide us with this bound.

Using a computer search, we find eight real mutually unbiased weighing matrices $W(7, 4)$ given in Table 6 in Appendix 1. This achieves the real upper bound given by Corollary 10. By Theorem 21, this is also the maximal set of $UW(7, 4)$, despite not achieving the upper bound of 24 given by Corollary 9.

### 3.4.4 $UW(8,4)$

**Theorem 22** The maximum number of real mutually unbiased weighing matrices of order 8 and weight 4 is 14.

**Proof** A set of 14 $W(8, 4)$ has been generated by a computer search and can be found in Table 7 in Appendix 1. This meets the upper bound given by Corollary 10. This set of matrices attain both the upper bounds in (5) and (6).
Table 3  Nine mutually unbiased weighing matrices of order 4 and weight 3, $UW(4, 3)$

$$
\begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 0 & \omega \\
1 & 0 & \omega & 1 \\
1 & 0 & \omega & 1 \\
0 & 1 & \omega & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 0 & \bar{\omega} \\
1 & 0 & \bar{\omega} & 1 \\
1 & 0 & \bar{\omega} & 1 \\
0 & 1 & \bar{\omega} & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 0 & \omega \\
1 & 0 & \bar{\omega} & 1 \\
1 & 0 & \bar{\omega} & 1 \\
0 & 1 & \bar{\omega} & 1
\end{bmatrix}
$$

Table 4  Five mutually unbiased weighing matrices of order 5 and weight 4, $UW(5, 4)$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & \bar{\omega} & 0 & 1 & 0 \\
1 & 0 & \bar{\omega} & \bar{\omega} & 0 \\
0 & 1 & \bar{\omega} & \omega & 0
\end{bmatrix}
\begin{bmatrix}
1 & \omega & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 \\
0 & 1 & \omega & \bar{\omega} & \omega
\end{bmatrix}
\begin{bmatrix}
1 & \bar{\omega} & 0 & 0 & \omega \\
1 & 1 & 0 & 0 & \omega \\
1 & 0 & 1 & 1 & \omega \\
0 & 1 & \omega & \bar{\omega} & \bar{\omega}
\end{bmatrix}
\begin{bmatrix}
1 & 0 & \bar{\omega} & 0 & \omega \\
1 & 1 & \omega & \omega & 0 \\
1 & \omega & 1 & \omega & 0 \\
0 & \omega & \omega & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 0 & \omega & \bar{\omega} \\
1 & 0 & \bar{\omega} & \omega & 1 \\
1 & 0 & \omega & \bar{\omega} & 1 \\
0 & 1 & \omega & \bar{\omega} & 1
\end{bmatrix}
$$

Further investigations into $UW(8, 4)$ using large roots of unity have proven fruitless. Odd roots of unity produce maximal sets smaller than that of the real case, and even roots of unity become computationally unfeasible after the fourth root of unity, which returns the set of $W(8, 4)$ as the maximal set of mutually unbiased weighing matrices.

### 4 Unbiased hadamard matrices

So far, we have only examined a very special case of unbiasedness. Our selection of the values of $n$ and $\alpha$ in (1) and (2), as well as imposing a certain structure to our matrices, make it possible to append the identity to the set of weighing matrices. More precisely, considering each row of all weighing matrices in a set of mutually unbiased weighing matrices of order $n$ and the rows of the identity matrix of order $n$ as vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$, they form a class of bi-angular vectors. We now make a different selection for the value of $\alpha$ in such a way that it is no longer possible to add the identity matrix and preserve the bi-angularity. In doing so, we are introducing a new concept of unbiasedness. Below, in Table 2, we give an example of a set of eight Hadamard matrices of order 8 that form a bi-angular set of vectors in $\mathbb{R}^8$, but no rows of the identity matrix can be added to the set and preserve bi-angularity. In the following set, $\alpha = \frac{1}{2}$, but if the identity is added, it would introduce the inner product of $\frac{1}{\sqrt{8}}$ (up to absolute value) and the bi-angularity of the lines would disappear.

The rows of these matrices are generated from the BCH-code [5,12] of length 7 with weight distribution $\{(0, 1), (2, 21), (4, 35), (6, 7)\}$ (see [14] for more information about BCH-codes). Once the codewords are generated, we append a column of zeroes, then perform the following operation onto each entry of the codewords:

$$f(i) = \begin{cases} 
1 & \text{if } i = 0 \\
-1 & \text{if } i = 1.
\end{cases}$$

We were also able to generate 32 Hadamard matrices of order 32 which have inner products in $\{0, \pm 8\}$ through a similar process. The weight distribution of the order 32 matrices is...
We believe that this set of vectors contains the needed ingredients to make the Hadamard matrices required. Moreover, we pose the following questions in Tables 8, 9, 10 and 11 in Appendix 2.

In an attempt to continue this, we have generated the 128² codewords from the BCH-code of order 127, but were not able to partition them into the 128 Hadamard matrices needed due to computer memory restrictions. The inner products between the vectors are all in \{0, \pm 16\}. We do believe that this set of vectors contains the needed ingredients to make the Hadamard matrices required. Moreover, we pose the following

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
\hline
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
\hline
1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
\hline
1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
\hline
1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 \\
\hline
\end{tabular}
\caption{20 mutually unbiased weighing matrices of order 6 and weight 4, \textit{UW}(6, 4)\label{tab:uw6_4}}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
\hline
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
\hline
1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
\hline
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
\hline
\end{tabular}
\caption{Eight mutually unbiased real weighing matrices of order 7 and weight 4, \textit{W}(7, 4)\label{tab:w7_4}}
\end{table}
It is important to note that the number of vectors found through Conjecture 23 is usually less than the bound given in Theorem 8. We believe that the upper bound is too high in this case because the vectors are all flat (i.e., all the entries of the vectors have the same absolute value). In fact, we feel that the upper bounds given in Theorems 7 and 8 are rarely obtained if $V$ is a set of flat vectors. It seems that finding a general bound for flat vectors is a challenging problem.

Using the terminology from [2], these matrices form a set of weakly unbiased Hadamard matrices. However, it is important to note that the matrices formed here are a very special kind of unbiased Hadamard matrices since the entire set of vectors forms a set of bi-angular lines (whereas the vectors from [2] give tri-angular lines). These matrices seem to form very nice combinatorial objects, which are discussed in further detail in a forthcoming paper [3].

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Mutually unbiased weighing matrices

Note added during proof

It seems that Conjecture 23 has been resolved since submission. See H. Nozaki and S. Suda: “Association schemes related to weighing matrices”. arXiv:1309.3892v1 [math.CO], 2013.

Appendix 1: Sets attaining the smallest upper bound

This section includes a library of sets of weighing matrices whose size equal the smallest upper bound that is known. To save space, we define \( \omega := e^{2\pi i/3} \) and \( \overline{\omega} := -\omega \).

Appendix 2: Hadamard matrices of order 32

In Tables 8, 9, 10 and 11, we show the partition of the 32^2 vectors into 32 Hadamard matrices of order 32 (denoted by \( H_1, H_2, \ldots, H_{32} \)). Each section represents one Hadamard matrix, and each hexadecimal number represents one row of the matrix (where each digit represents four entries). The most significant binary digit represents the left-most entry of the 4-tuple and the least significant binary digit represents the right-most digit. For example, 4259F1BA represents 0100 0101 1111 0001 1011 1010. We then convert the binary string to a \( \pm 1 \) string using the function \( f \) defined by formula (7).
| Table 8 | $H_1$ through $H_8$ |
|--------|------------------|
| 00000000 | 4259F1BA 203AEB5 50967C6E 59F1BA84 47FC04A7 4E9BC24D 4B3E3750 |
| 62631F0F | 7C6EA12C 1E0DBE23 259F1BA8 32F56361 750967C6 176A78C9 67C6EA12 |
| 62A12CF8 | 70AC92DB 55339793 0CC233F7 12CF8DD4 05A5F51D 3B92A58B 79CB5431 |
| 1B8A4B3E | 5C544F99 6B04D9E5 3E375096 2CF8DD42 0967C6EA 3750967C 29D5285F |
| 6EF49ECD | 2D2FA8E1 755CD5F3 017C11CA 5F047280 236AD3AC 07770F92 |
| 727B6854 | 4DCBF0F5 5663B46A 7819AEBE 129A3FE1 0055B235 24486E0B 44AC39BE |
| 0E19C978 | 38C29892 4AE94F23 15B88246 438E8419 514109CD 67935827 2AD51546 |
| 69D6236A | 3687E3DF 093274DF 1CDF44AC 60B1E580 5F047280 236AD3AC 07770F92 |
| 47DBB9D5 | 1F0EC4C7 6A07A301 182C7960 0384325E 5CD5F2EB 5BF74F4C 529089A6 |
| 636065EB | 114B8FFA 7F8A9372 405F0472 71AEP83F 4E1AP73F 2AP799E60 |
| 0AE3F4B4 | 0DC14913 35563B4B 20BB53C7 78C82ED5 29DC952D 768D5598 327DFE13 |
| 1669022D | 3C31A55E 4938C298 04A68FF9 6D251EA6 64428D4C 55B23401 3B13118F |
| 050E9174 | 10E3A107 191D05D8 475760CE 7935826D 20C438E9 6BAFBD8C 629DC953 |
| 2E8143A4 | 3529089A 46514117 7770F920 52BA50BD 02799EE6 3C17C45 4026F5C |
| 5CFF2BF0 | 7EB428FF 27B3377B 29F64C36 65EA5C61 1E6ADA4A 084BEA93 325E0708 |
| 3B6C73D7 | 5B824624 7007F6B2 49121B83 6CD8B21E 1794AB95 0C3CE5AB 55CD5F2F |
| 7357CBAB | 66ABF9E8 7475760C 017C11CA 5F070DC7 2A67112F 3F342A7D 5FAF16E9 |
| 14E4A9A7 | 3F4B143C 61B72DS1 4DLFF013 62C17C45 381697D5 4A3D4DB4 149128B9 |
| 7328A085 | 740A12D2 0103A4E 66C590F6 2D84CC88 58F2C060 65EAE6C 4A42269A |
| 13CCF730 | 2DFBA7A6 3869FCFB 4D60B59D 2AD91A01 588DaB4E 1B39C1BE 619846F |
| 76595ADF | 03503D19 5C641771 1CF8DD8 0515465B 0E3A0163 4CE6DE19 1DFADCB8 |
| 288A5DFC | 5EAC6COD 372FFD52 4109CC81 12B0E6FA 6496D70B 119FBCD4 4F487EE |
| 396A861F | 007E62B7 50E91740 69FCA771 781C2192 5D833A3A 3400AB65 42629A49 |
| 250E7086 | 75760CCE 3A450D28 2BA50BCB 26CF62B1 7B3777A5 67B9813C 6AD34CAE |
| 7A4F666F | 0F4601A9 5195068A 7A300D41 56B7BB2D 238EDEB 4407DD7 3171513F |
| 24E30A62 | 56C8D003 0864BC0E 435AB5E 1DF6E753 1AA1B3DA 23C1B5C5 6880EBBB |
| 0F396A87 | 7D6DBC8 7D12B0E6 447836F9 4325E070 310E3A11 68FF0895 362CB876 |
| 1A45F4A4 | 3653EC98 6FD3D32 249C614C 1089C7D 51E6DA41 081BD720 6FA2561C |
| 0BCA574B | 621C7421 70864BC0 4993A6F1 723728A9 20B0FE6D 5B3EDCFC 32DBA7A |
| 5C79E682 | 7EC3308D 77F14452 3C9AC137 171513E7 10621C75 4EE4A963 1E276738 |
| 058F2C06 | 195068AA 02F82394 3BEDCEA5 0CBD58D9 2045859B 5B099910 79B431F |
| 47D6DDBC | 6567B7B3 554CE25D 6C590F6C 35AB5E8 62BE00FE 40A1D22E 297FF144 |
| Table 9 | $H_9$ through $H_{16}$ |
|---------|----------------------|
| 09B3C9AD 7B9B13CC 7DC6BFA1 | 2C2CD205 2A727E68 513E62E3 74A1794B 13679359 |
| 00D40F47 254B14EF | 2315B882 3F9F4E1B 4DB4947A 44D35290 0FED65C0 1A0055B3 |
| 614C93F8 5859A409 72FD0526 | 6712E555 068A32A 6E7523BF 36F888F1 1C5E9F0E |
| 30A6249C | 5760CE8E 682B8FD2 428DFE9D 5E070864 4BEA3817 39C1E276 15393F34 |
| 185132B | 16166903 146FF7E5 4680156D 7B8745FB 0FC7B5DB 241DDC3E 1A2A8CA8 |
| 266442D8 2A58A773 | 6F23EB6E 631F0EC5 3DCC09E6 3FBB5970 48C56E20 4ABFC70F |
| 61669023 | 6DA75788 0182C593 5D028748 3AC46D5A 1F5B76F2 6DB8E7AC 46ACAC76 |
| 7623EB6E | 03FB5970 5D858535 338972AB 31F0EC4D 7ACE9DBF 748B0A50 44F9B8B8 |
| 1F5A50AE 2F78B75 | 117E00BF 347FC04B 5347FC05 789D9C09 46541A2A 04D0EB90 |
| 48116167 | 216C2664 6D70AC93 5D028748 3AC46D5A 1F5B76F2 6DB8E7AC 46ACAC76 |
| 0AB646B1 | 0A4890DD 539B2A59 48EF7B3B 2F295D29 76D8E7AD 3A3ABB06 6335D7DE |
| 2192F038 | 762631F1 04F33DCC 78634ABC 34811617 63CB0182 5DFC5114 11E0DEBE |
| 6C0CBD59 | 3BB87C90 1740A1D2 2E554CE3 47836F89 05DA9E33 35FD07DD 40F4601B |
| 5265E5FFA | 201037AE 77A4F667 1E72D50D 2767383C 29224371 49C614C4 6249C614 |
| 4E8B11B6 | 3CCF7702 5C2B2487 1037AE40 7E96828B 5B5C2B25 6B7BB2BC 02AD91A1 |
| 0B9F57E | 0CE8EAE3 653EC986 79E18D2A 70D3F9F5 1905DA9F 328A084F 55195068 |
| 5017C11C | 3D32DFBA 11CA02F8 7CEF15CE 415C7E96 784993A7 042732BE 37D12B0E |
| 6902C22D | 4F1905DB 5E52B5A1 63B1D989 76D8E7AD 3A3ABB06 6335D7DE |
| 28DFEC9 | 223C1B7D 1B29F64C 269A9484 45FAF16F 0EC46C3F 2C796030 54B14EE5 |
| 72AA6713 | 4BB8A222 67475760 3377A4F7 156CD9D0 6DA43A3D 0A6249C6 1F8F79B5 |
| 7022DF9A | 3DB362C8 0524486F 33F6198B 68564985 6CA7D930 41DCC3DE 34DA4A22 |
| 261B29F6 | 0206F5C8 0B133220 0C438E85 1E8C0351 794AE943 53124E30 62E2A27D |
| 4F9B8B9A | 770F9200 2F73E61C 5430F937 17EBC5BB 3A91DF6F 46FF7E43 65C01F0A |
| 7E6854E4 | 48BA0508 23199451 285E5BBB 19AEB6F6 10C9781C 5D73575D 5A7A58B0 |
| 50C3CE5B | 01D7753A 1A7F3E9D 37052449 08B03B49 3B62E2A3 39405F04 7DB9D4F8 |
| 66119FB1 | 4C4938C2 2BBFD2D0 25CA99D9 6F76595B 74D1265 06F5C804 61332216 |
| 302799EE | 6854E4FC 4B6B5656 1318F877 73FCAFC2 7A969289 0F920EEE 1D5D833A |
| 452FEF28 | 59A408B1 22BE143A 5E86B516 143A5D0 57E173FC 420C438F 2CA66F77 |
| 2DA21593 | 24C9D379 1DCC3E48 66902C3 61B29F64 442D844C 2ABCA834 58A77255 |
| 430F396B | 745FAP7 1AAE83EF 36065EAD 08310E3B 737D12B0 01568C00 0F13B39C |
| 3124E3A0 | 06745756 38432508 56E20918 23BE6EDE 68D5598E 7A1AD45A 13945050 |
| 7D3869FD | 14BBF8A2 | 51C0B4BF 4D4A4226 6FF7E429 4A68F8F1 3F619847 5F85CFF2 |
Table 10  $H_{17}$ through $H_{24}$

| $H_{17}$ | $H_{18}$ | $H_{19}$ | $H_{20}$ | $H_{21}$ | $H_{22}$ | $H_{23}$ | $H_{24}$ |
|---------|---------|---------|---------|---------|---------|---------|---------|
| 22977F14 | 1CA02F82 | 7AB1B033 | 3FCAPC2E | 2CD20459 | 580C163C | 01FDAC8B | 3E1D89B |
| 7F4F4CB7E | 132216C | 4B14EE4B | 0E6FA256 | 59DB639F | 1D775A21 | 3058F2C0 | 579E18D2 |
| 0FB8D7F5 | 4486E0A5 | 67EC3309 | 23400A87 | 002AD91B | 687E3DE7 | 12E554CF | 2D0571FA |
| 45519506 | 4AC79BE8 | 663B46A6 | 7523B8DD | 69A94844 | 7B66C590 | 56496D71 | 31878673 |
| 25B5C2B3 | 60CE8EAE | 7201037A | 2A27CC5D | 06DF111F | 2462B710 | 377A4F67 | 427328A1 |
| 5F2EB899 | 2FB0FBBF | 6E8BF5E3 | 094D1FF1 | 73D67D9 | 6F5C084D | 1A557E86 | 36AD3AC4 |
| 089A6A52 | 1B822925 | 601BC57A | 50BA75C9 | 516B0D06 | 43A45D02 | 7C478783 | 070864B3 |
| 14109CCB | 15C7E968 | 4DE1264F | 7D930D94 | 38E84189 | 4C3653EC | 5E9FDE38 | 393F342A |
| 13E62E2B | 39BE958B | 1DA35566 | 503D1807 | 0129A3FF | 7A6B7F74 | 2216C266 | 571AFA50 |
| 4CB7EE9E | 25347FC1 | 7D4702D3 | 37FBF215 | 45D02875 | 14C4938C | 6F88F8F7 | 061C7DF4 |
| 2C53B92B | 7302799E | 0F6CD8B2 | 595ADEED | 5E78634A | 3E9273FF | 4B955339 | 06B01E58 |
| 2B71048C | 66EF49ED | 1A818EC1 | 084E6515 | 68A3A320 | 30D94F5B | 7420C439 | 42F295D3 |
| 028748BA | 104C56E | 0E457B4D | 49B7F8FA | 722BDA61 | 3CB0182C | 0722B7A7 | 2E7F9F58 |
| 2BDA6E0E | 35D7DEC6 | 7EE9E996 | 40DE9900 | 778E2F7C | 6983915F | 307220DB | 3915E3D3 |
| 7B4C1C8B | 5ED30723 | 60E457B5 | 6541A2A8 | 6C266AED | 15ED0373 | 22BDA60F | 27185312 |
| 4C1C8AF7 | 19F20384 | 0BE08E50 | 1C8AF699 | 457B4C1D | 57B4C1C9 | 521134D4 | 5B76F23E |
| 4B44D352 | 11B569D6 | 715156E3 | 0D3F9F4F | 4F666E5F | 64BC0E10 | 4601A81F | 3D4D8494 |
| 18D2A5FC | 21C7420D | 2F823940 | 5DA9B321 | 697D471 | 7836F890 | 037A4E08 |
| 5A8B5E86 | 045859A5 | 6DDBC8FA | 6AF7955D | 767383C4 | 412315B8 | 54CE25CB | 1FF0129B |
| 7F14452E | 26E5FFAA | 3308CF9D | 3A6F0933 | 342A772E | 53EC869C | 28A084E7 | 0A1D228E |
| 2F563607 | 4CE25CB | 39EB3B6D | 30BCFD87 | 48CB0E10 | 3EC986CA | 75F7B19A | 11661619 |
| 64680157 | 00BBAD11 | 42A727E6 | 1806A07B | 099910B6 | 541A2A8C | 07DC6BFB | 7C907770 |
| 28748BA0 | 2631FE0D | 5D7DEC66 | 1F241DDC | 1643DB36 | 21134D4A | 72D50C3D | 6DF0CB8D |
| 6A2D7A1A | 00FED65C | 7BB2CAD7 | 634ABC5F | 5A5F3C1 | 45859A41 | 37AE402E | 533892F5 |
| 78E2F7CE | 1B569D62 | 4FB261B2 | 41F71AFF | 22437053 | 76A7C838 | 2561CDF4 | 5E2DD17F |
| 67383C4E | 151362E2 | 2B428B69 | 697D4703 | 590F6CD8 | 035E93B5 | 4890DC15 | 574A1795 |
| 46D5A758 | 34FE7D39 | 0AC92DAF | 12315B88 | 71853124 | 20C60B1E | 6E5FFF4A | 0DEB9008 |
| 601A81E9 | 048C56E2 | 3D989BD3 | 5068AA32 | 7FC04A69 | 1C7420C5 | 3ABB0674 | 33CC9C0E |
| 075DD689 | 598ED1AA | 332216C2 | 7F35EC35 | 0A37BF3C | 0918AD4C | 7254B14F | 5A1A879D |
| 609B3C9B | 3D676D8F | 4B415CC7 | 717BE778 | 15925B5D | 21ED9B16 | 462B7104 | 7C11CA02 |
| 6DEB74D6 | 3E483BB8 | 22C2CD21 | 54E4FDCC | 2C87B66C | 4860EA49 | 18F87627 | 16BD066A |
| 1BD72010 | 300D40F5 | 63B46AAC | 57CBAE87 | 6DF111E1 | 45042733 | 047280BE | 2F8A8E05B |
### Table 11

| Mutual unbiased weighing matrices 255 |
|--------------------------------------|

| $H_{25}$ through $H_{32}$ |
|---------------------------|
| 16E8BF5F 597007F6 483BB87C 712E554D 2FFDF526E 5393F342 3A10621D 42D84CC8 |
| 251EA6DA 0D40F461 4C9D3785 3EB6EDE4 467EC331 7BCDA1F9 5DD6880F 6E20918A |
| 6A861E73 57357CB9 12E30A6A 64C3653E 07A300D5 7F6B2E00 7588DB48 03058F2C |
| 30F396E9 21B82923 1C0B4BE8 34551950 18ADC412 09E67B98 6065EAC7 2B5BDD97 |
| 47A9B692 27CC5C55 20E1F2E2 62B71048 3BC717BE 173FCAFE 02D2FA8F |
| 775A203B 799E6D04 32216C26 4E4FC6DA 29089A6A 6C73D677 524A86E1 56563BF4 |
| OC163CB0 0F40728A 1D775BFB 19F80B3B 034B227D 1ED9B164 5B404D0E |
| 58737D12 69A155C6 4F7E7387 7F95F85C 3972A9A7 7DEC66BA 1F71AF9E 6A78C82F |
| 4D9E4D61 2F038432 544F988B 71D83111 1AE2A8CA 02A07C31 0A9C9F9A |
| 1B4D4A44 1134DA44 34ABC0F0 04D9E4D7 5636065F 2146FF7F |
| 73A91DF7 36D251EA 43DB362C 0E85017C 233F6199 4A0E3F84 643BDB3B 2D71A1AD |
| 2EAB9ABF 109CA29 274DE127 791F5B7E 4ECE7078 50C1F0AC 70F920EE 32A01514 |
| 55E78634 6CF26B05 0571FA5A 77DB9D49 49ECCDDF 47280B39 1E5B0C16 6BDD6A2A |
| 206FC9B0 3B4A4ACC 5B23400B 6236AD3A 7E3E6D61 0C9781C2 6514109D 025347FD |
| 29892715 52C53B93 197A1B1B 17BE778E 0BB53C65 400A6B47 3C64176B 35826CF3 |
| 4947A9B6 29A3F603 08163132 6BFA0FB9 27E6854E 5BD69657 320BB53D 1E3F687F |
| 20918ADC 0222C2D3 4075D96D 62C87B66 5C9AA9C0 34CE7078 657F4F04 5598ED1A |
| 70524487 79603058 7E173FCA 52EFE288 198467ED 3B39C1E2 0B1E580C 6C8D002B |
| 357CBAAF 77254B15 055B23A1 2ED4F191 17C11CA0 4702D2FB 0C69579E 4E3A0624 |
| 0EE1EF24 239405F0 1C2192F0 6EA9A481 1546541A 4A179A4F 7E87E782 75A203AF |
| 51BDF91 078D9C6E 3F1EF369 2AF3C31A 3158824 4542EFE2 6928F536 6D76D8E7 |
| 7CC55C54 09CCA283 2DD17EBD 7280B3E0 58D819B7 43706254 4D352908 2486BB57 |
| 604F3D3C 1B032F57 1264E989 36793583 00AB684E 5FFAA3DC 39C4E4CE 569D6236 |
| 5A203AEF 163CB018 64E9BC25 1879CB55 33A3ABB0 6AACC768 71048C56 06D6283A |
| 5ADEECB3 7FBF2147 0D9F42B6 7F41F71B 4FCD0A9C 280BE0E8 71FAP0A0 4F33DC00 |
| 18871D09 26B04D9F 28F536D2 546541A2 031D806B 549B97FE 418871D0 023F5637 |
| 335D7DEC 31D806A1 26E9BC3C 16C26644 64176A79 417A678D 6A521134 3DBED60F |
| 24370525 4A9629D9 08CFD867 3EE35FD1 50427329 561CDF44 3784993B 01A81E8D |
| 2D50C3CF 6F759E18 75DD6811 7CBAEE6B 121B8293 73834EC4 0E91740A 1B7C4747 |
| 0F66B2E0 31DA5556 1D22E814 4CC8B580 6F093275 666E4F9F 592B5BC3 38BDF3BC |
| 45AF435A 14452EFE 603058F2 43F1EF37 2269A948 5F7B19A6 7A40206 2B06F6A2 |

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