Response functions of atom gravimeters

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Received: date / Accepted: date

Abstract Atom gravimeters are equivalent to non-multi-
level corner-cube gravimeters in translating the gravity
signal into the measurement result. This enables de-
scription of atom gravimeters as LTI systems. The sys-
tem’s impulse responses by acceleration, velocity, and
displacement are found to have the shape of triangle,
meander, and the Dirac comb resp. The effects of in-
homogeneous gravity field are studied for constant and
linear vertical gradients and self-attraction of the in-
strument. For the constant gradient the effective mea-
surement height is below the top of the trajectory at 1/6
and 7/24 of its length for the fountain and the release
types of the instruments resp. The analysis is expanded
to the gravimeters implementing the Bloch oscillations
at the apex of the trajectory. In filtering the vibrations
these instruments are equivalent to the first-order low-
pass filters, while other atom gravimeters are equivalent
to the second-order low-pass filters.

Keywords Atom gravimeter · LTI system · Impulse
response function · Vertical gravity gradient

1 Introduction

In 1991 Mark Kasevich noticed that the phase shift of
the atom interferometer is quadratically proportional
to the free-fall time of the atoms \[\text{[1, 2]}\], thus realiz-
ing the first gravimeter with cold atoms as test mass.
The following years saw rapid progress in this type
of instruments, which now approach in accuracy and
even exceed in sensitivity the best corner-cube gravime-
ters \[\text{[3–6]}\]. The atom and the corner-cube instruments
measure gravity acceleration in significantly different
ways, with almost non-overlapping sets of systematic ef-
teffects, so bringing both types of the instruments together
for comparisons is very beneficial for metrology. Similar
to corner-cube instruments, many systematic effects of
atom gravimeters can be estimated only by modelling
and computer simulation. To validate the corrections
for the effects it’s therefore important to compare re-
sults obtained with different models. In this paper we
outline atom gravimeters from the viewpoint of the the-
ory of linear systems. In particular, we find the impulse
response functions of atom gravimeters and apply them
to the analysis of the effects of the inhomogeneous grav-
ity field.

2 Atom gravimeter as LTI system

In atom gravimeters the positions of the free falling
atoms are related to the phases of the light field gener-
ated by two counter-propagating lasers. In the 3-pulse
gravimeters the phases \(\phi_1, \phi_2, \phi_3\) of the field correspond
to the positions of the atoms at the moments \(t_1, t_2, t_3\)
separated by the time interval \(T\). The interferometer
output in these gravimeters is \[\text{[7,8]}\]

\[\Delta\phi = kgT^2 + \phi_1 - 2\phi_2 + \phi_3,\]  \(1\)

where \(k\) is the wave vector of the light field. The quadratic
term of \(\text{[1]}\) is caused by the Doppler effect, as the ac-
celerating atoms see the static light wave as linearly
increasing frequency. The term can be cancelled by con-
trolling the frequencies of the lasers during the free fall.

\[\text{1 In later publications this parameter is often called the effective wave vector } k_{\text{eff}}\]
considered further in this paper.
The measurement of the acceleration in atom gravimeters consists in the experimental search of the frequency increase rate $\alpha$ that compensates the Doppler shift. The acceleration is obtained from the $\alpha$ as

$$g = 2\pi\alpha/k.$$  \hspace{1cm} (2)

The right $\alpha$ makes $\Delta\phi$ equal to 0 for any $T$ \cite{6}, so from (1) it also follows

$$\bar{g} = (\phi_1 - 2\phi_2 + \phi_3)/kT^2.$$ \hspace{1cm} (3)

As $z = \phi/k$, the same acceleration could also be found from the coordinates, if they were available \cite{10}:

$$\bar{g} = (z_3 - 2z_2 + z_1)/T^2.$$ \hspace{1cm} (4)

Substituting the distances traveled by the atoms from the coordinate $z_1$: $S_1 = z_2 - z_1$, $S_2 = z_3 - z_1$ and the corresponding time intervals $T_1 = T$, $T_2 = 2T$ into (4) reveals the equivalence of the 3-pulse atom gravimeters with the 3-level measurement schemas (fig.3) used in some corner-cube instruments, e.g. \cite{11}:

$$\bar{g} = \left(\frac{S_2}{T_2} - \frac{S_1}{T_1}\right) \frac{2}{T_2 - T_1}.$$ \hspace{1cm} (5)

Let the acceleration of atoms during the free fall change like $g(t)$. The distance relates to the acceleration via double integration, so

$$\bar{g} = \left(\int_0^{2T} \int_0^t g(\tau)d\tau dt - \int_0^T \int_0^t g(\tau)d\tau dt\right) \frac{2}{T}.$$ \hspace{1cm} (6)

The above formula relates the acceleration of atoms $g(t)$ to the measured acceleration $\bar{g}$ linearly, i.e. linear combination of partial accelerations

$$g(t) = ag_1(t) + bg_2(t),$$ \hspace{1cm} (7)

translates into

$$\bar{g} = a\bar{g}_1 + b\bar{g}_2,$$ \hspace{1cm} (8)

where $\bar{g}_1$ and $\bar{g}_2$ are the results of independent measurements of $g_1(t)$ and $g_2(t)$. This linearity plus the time invariance of (6) (meaning that $T_1$ and $T_2$ are pre-defined for a drop) enables treatment of atom gravimeter as linear time-invariant (LTI) system. In such a system the input and the output signals are connected by the convolution operation:

$$\bar{g}(t) = \int_{-\infty}^{+\infty} g(\tau)h(t-\tau)d\tau,$$ \hspace{1cm} (9)

where $h(t)$ is the impulse response function of the system. Applied to absolute gravimeters, the convolution enables presenting the measurement result in the form

$$\bar{g} = \int_0^{2T} g(t)w_g(t)dt,$$ \hspace{1cm} (10)

i.e. as the acceleration of the atoms weighted over the measurement interval. Here we have $\bar{g} = \bar{g}(2T)$ and $w_g(t) = h(2T-t)$, as the measured gravity is attributed to the end of the measurement interval. Therefore, the gravimeter’s weighting function is its impulse response function turned backwards with respect to the end of the measurement interval. The impulse response can be found as reaction of the system on the Dirac delta function $\delta(t)$ \cite{12}. As \cite{5} has two double integrators, we first find the reaction of a double integrator on $\delta(t)$. The first integral of $\delta(t)$ is the Heaviside step function $u(t)$, the second integral is the unit ramp function $r(x)$ (fig.1):

$$r(t) = \begin{cases} 0 & t < 0, \\ t & t \geq 0. \end{cases}$$ \hspace{1cm} (11)

The equivalence of the double integration to the convolution with the unit ramp \cite{11} follows from the integration by parts rule $\int_0^T udv = [uv]_0^T - \int_0^T vdu$:

$$\int_0^T \int_0^T g(\tau)d\tau dt = \int_0^T \int_0^T g(t)dt h(t-\tau) d\tau dt = \int_0^T \int_0^T g(t)(T-t) dt.$$ \hspace{1cm} (12)

Two ramp functions combined according to the formula \cite{6} yield the the gravimeter’s weighting function by acceleration $w_g(t)$:

$$w_g(t) = \begin{cases} \frac{r(2T-t) - r(T-t)}{T} \frac{2}{T} & 0 \leq t \leq T, \\ \frac{t}{T} & T \leq t \leq 2T. \end{cases}$$ \hspace{1cm} (13)

Some systematic effects are more convenient to analyze in terms of the atoms’ velocity or displacement rather than acceleration. Applying again integration by parts
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Fig. 2 Weighting functions of atom gravimeters (top to bottom): by acceleration, by velocity, by displacement.

(a) – 3-pulse atom gravimeter
(b) – atom gravimeter implementing Bloch oscillations

Explanations:

1. The function similar to $w_v(t)$ is known in atom interferometry as sensitivity function $^{[14]}$.

2. The function similar to $w_v(t)$ is known in atom interferometry as sensitivity function $^{[14]}$.

3. Analysis of systematic effects using response functions

If the atoms’ acceleration changes during the free fall like

$$g(t) = g_0 + \Delta g(t),$$  \hspace{1cm} (19)

the additional component caused by the term $\Delta g(t)$ can be found as

$$\Delta g = \int_0^{2T} \Delta g(t)w_g(t)dt,$$  \hspace{1cm} (20)

where $w_g(t)$ is the weighting function of the gravimeter by acceleration. If the disturbance is expressed in terms of velocity or displacement, the additional component can be found using the corresponding weighting functions $w_v(t)$ or $w_z(t)$ as

$$\Delta g = T^{-2} \left( \int_0^{2T} \Delta V(t)dt - \int_0^T \Delta V(t)dt \right),$$  \hspace{1cm} (21)

$$\Delta g = T^{-2} \int_0^{2T} \Delta z(t) \left( \delta(t) - 2\delta(t-T) + \delta(t-2T) \right)dt$$

$$= \left( \Delta z(0) - 2\Delta z(T) + \Delta z(2T) \right)T^{-2}. \hspace{1cm} (22)$$

It’s interesting to observe that the formulas (22) and (4) are alike, as the gravimeter’s weighting function by displacement is just the Dirac comb sampling the continuously changing coordinate in three points.

If the disturbance is expanded like

$$\Delta g(t) = \sum_{n=1}^{N} a_n t^n,$$  \hspace{1cm} (23)

its influence on the measured gravity, according to (10), can be found as

$$\Delta g = \sum_{n=1}^{N} a_n C_n,$$  \hspace{1cm} (24)

where $C_n$ is the $n$-th moment of the $w_g(t)$:

$$C_n = \int_0^{2T} t^n w_g(t)dt = \frac{2n+2}{(n+1)(n+2)} T^n. \hspace{1cm} (25)$$

Similar formulas can be derived also for $w_v(t)$ and $w_z(t)$. 

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$w_g(t)$

$w_v(t)$

$w_z(t)$

$w_{\delta}(t)$
which according to (24) and (25) translates into the following measured gravity:

$$\overline{g} = g_0 \pm \gamma \left( z_1 + V_0 T \pm \frac{7}{12} g_0 T^2 \right).$$  \hspace{1cm} (28)

This agrees with the result obtained also in \[10, 15\]. At the effective measurement height $h_{eff}$ the measured gravity equals actual gravity:

$$\overline{g} = g_0 \pm \gamma h_{eff}. \hspace{1cm} (29)$$

Comparison of (29) and (28) leads to the conclusion that the effective measurement height of atom gravimeters is

$$h_{eff} = z_1 + V_0 T \pm \frac{7}{12} g_0 T^2. \hspace{1cm} (30)$$

For the release gravimeters, let $h$ be the length of the idle part of the trajectory, and $H$ be the total trajectory length. Obviously, $h = z_1$, $H = z_3$. As $V_0 = g_0 z_1$, $T = (t_3 - t_1)/2$, $t_1^2 = 2 z_1/g_0$, $t_3^2 = 2 z_3/g_0$, we get

$$h_{eff} = \frac{7}{24} (H + h) \pm \frac{5}{12} \sqrt{H h}. \hspace{1cm} (31)$$

For the fountain gravimeters, the initial velocity $V_0$ equals $g_0 T$, as the atoms reach the apex in time $T$. The total trajectory height $H = g_0 T^2/2$ leads to the following location of $h_{eff}$ above the initial position:

$$h_{eff} = \frac{5}{6} H + z_1. \hspace{1cm} (32)$$

As the gravity at the apex of the trajectory is $g_0 - \gamma (H + z_1)$, the same point is at the following distance below the apex:

$$h'_{eff} = \frac{1}{6} H. \hspace{1cm} (33)$$

The gravity at the effective measurement height corresponds to the measurement with no vertical gradient correction.

**Example 2. Linear vertical gravity gradient**

At some gravimetric sites the vertical gravity gradient varies significantly over the free-fall trajectory and cannot be considered a constant. The linear gradient $\gamma_1 + \gamma_2 z$ changes gravity like

$$g(z) = g_0 \pm (\gamma_1 + \gamma_2 z) z. \hspace{1cm} (34)$$

Unlike the constant gradient, the parameters $\gamma_1$ and $\gamma_2$ depend on the coordinate system used to analyze the correction. Expressing (34) in terms of time gives us

$$g(t) = g_0 + A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4, \hspace{1cm} (35)$$
The self-attraction of absolute gravimeter can produce a complex-shaped disturbance, for which the approximations by low-degree polynomials, like in the previous examples, are not sufficient. The figure 4 shows the self-attraction of atom gravimeter along the atoms’ trajectory and its approximation by the 6-th degree polynomial. To evaluate the effect of this disturbance, for which the approximation does not exceed the maximum error of the approximation. In our case the approximation led to the error of less than 0.05 µGal. For atom gravimeters the approximation can be performed so that the correction uncertainty would not exceed max |ε(t)|/4. The detailed analysis of this issue is out of the scope of the present publication and will be presented elsewhere.

4 Response functions of atom gravimeter implementing Bloch oscillations

In this type of gravimeters the atoms are thrown up vertically and travel to the apex of the trajectory during the time T, where they hover for the time T’ tossed by the Bloch oscillations, and then drop back to the original position. The formula (4) for this instrument becomes

$$\gamma = (\phi_1 - \phi_2 - \phi_3 + \phi_4) / [kT(T + T')]$$.

As per section 2 this gravimeter can be modeled by three double integrators with integration times of T, T + T’, and 2T + T’. Impulse response functions of this gravimeter are shown on the fig.2b. As T ≪ T’, the gravimeter performs near-uniform averaging of the atoms’ acceleration. The instrument described in [17] has the trajectory length of only 0.8 mm, so the corrections for the constant and the linear gravity gradients are minuscule, and the correction for the self-attraction can be taken as the disturbance in any point of the trajectory taken with the opposite sign.

The gravimeter’s frequency response is Fourier transform of its impulse response function [18]. Due to its almost-uniform impulse response, the atom gravimeter with Bloch oscillations is equivalent to the first-order low-pass filter. By comparison (fig.5), the 3-pulse atom gravimeter is equivalent to the second order, while the corner-cube gravimeter is equivalent to the third-order low-pass filter [18] (fig.6). These characteristics follow from the gravimeters’ logic in translating the input gravity to the measurement result and do not include any additional vibration shielding that instruments may possess.

5 Conclusions

We analyzed atom gravimeters as LTI systems, found their impulse response functions and applied them to evaluation of certain disturbances. The following conclusions sum up the analysis.
1. Atom gravimeters are equivalent to non-multi-level corner-cube gravimeters in translating the gravity signal into the measurement result.

2. Weighting function of atom gravimeter by acceleration can be determined as impulse response of the LTI system consisting of several double integrators. The acceleration, velocity, and displacement weighting functions are successive derivatives changing sign on every succession.

3. The effect of a disturbance on atom gravimeter can be found by replacing the time powers in the disturbance expansion with corresponding moments of the weighting function.

4. The effective measurement height of atom gravimeters is located below the apex of the trajectory on 1/6 of its total length for the fountain, and on about 7/24 ditto for the release type gravimeters.

5. Error in the analysis of a disturbance arising from the disturbance approximation does not exceed the maximum error of the approximation.

6. With respect to the vibration disturbances the atom gravimeter with Bloch oscillations is equivalent to the first-order low-pass filter, while the 3-pulse atom gravimeter is equivalent to the second order low-pass filter.

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