A new approach to the derivation of the law of universal gravitation from Kepler’s laws

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Abstract. Everyone knows that the inverse square law follows from Kepler's third law. Let us prove more: the law of universal gravitation follows from Kepler's third law.

1. Kepler’s third law
German mathematician and astronomer Johannes Kepler (1571-1630) discovered three empirical laws of planetary motion in the Solar System. These laws are true provided that the masses of the planets are negligible compared to the mass of the Sun. The English mathematician, physicist and astronomer Isaac Newton (1643-1727) established a close connection between Kepler's laws and the law of universal gravitation discovered by him.

Let us show that the law of universal gravitation (discovered in 1666, published in 1687) is a consequence of Kepler's third law (discovered in 1618, published in 1619). Kepler's first and second laws (published in 1609) also follow from Kepler's third law, since these laws are derived from the law of universal gravitation.

Kepler’s third law

$$\frac{a^3}{T^2} = C$$  \hspace{1cm} (1)

connects for each planet the semi-major axis $a$ of the elliptical orbit with the orbital period $T$. Constant $C$ is the same for all planets. Modern measurements show that

$$C = 2509.45482 \times 10^{16} \text{ km}^3 \text{ day}^{-2},$$

if the mass $m$ of the planet and the mass $M$ of the Sun are related by the inequality

$$0 < m \ll M.$$

The law will also be true for other celestial systems, for example, for the Jupiter-satellites system and the Mars-satellites system. The corresponding values of Kepler’s constant will already be different:

$$C = 2.3955 \times 10^{16} \text{ km}^3 \text{ day}^{-2},$$

$$C = 0.0008098 \times 10^{16} \text{ km}^3 \text{ day}^{-2}.$$

We see that the central body of the system (Sun, Jupiter, Mars) significantly affects the value of Kepler’s constant.
2. Inverse square law

In the case of a circular orbit, the distance $R$ of the planet from the Sun remains constant, then

$$a = R.$$  

(2)

Dutch mathematician, physicist and astronomer Christiaan Huygens (1629-1695) and Newton independently found a formula for centripetal force, as well as for centrifugal force. For the case of a circular orbit, Huygens and Newton equated the force $F$ of the planet's attraction to the Sun and the centripetal force of the planet:

$$F = m\omega^2 R.$$  

(3)

The planet's circular velocity $\omega$ is related to its period $T$ by a simple formula

$$\omega = \frac{2\pi}{T}.$$  

(4)

Four equalities (1), (2), (3), (4) give the formula

$$F = \frac{4\pi^2 Cm}{R^2}.$$  

(5)

known as the inverse square law. Newton argued that Wren, Hooke and Halley also came to this law independently of each other [2].

3. Kepler's constant depends only on the mass of the Sun

Let us formulate a number of statements concerning the Kepler constant.

The constant $C$ does not depend on the parameters of the planet and on the distance $R$.

The force $F$ of the planet's attraction to the Sun depends only on one parameter of the planet, on the mass $m$.

The methodological principle of symmetry allows us to assert that the Sun attracts the planet with the same force with which the planet attracts the Sun. Newton's third law of mechanics confirms this statement.

From the same principle it follows that the force $F$ of the Sun's attraction to the planet depends only on one parameter of the Sun, on the mass $M$.

The constant $C$ depends on only one parameter, the mass of the Sun $M$.

Considering the above, formula (5) takes the form:

$$F = \frac{4\pi^2 C (M)m}{R^2}.$$  

(6)

4. Functional equation for Kepler's constant

The properties of homogeneity and isotropy of three-dimensional Euclidean space make it possible to assert that the force of attraction will not change if the Sun and the planet are mutually swapped. In other words, formula (6) remains valid if the masses $M$ and $m$ are interchanged:

$$F = \frac{4\pi^2 C (M)m}{R^2} = \frac{4\pi^2 C (m)M}{R^2}.$$ 

For the unknown function $C \cdot$, we find the functional equation

$$C (M)m = C (m)M \text{ or } \frac{C(M)}{M} = \frac{C(m)}{m}.$$ 

Simple functional equation

$$f (x) = f (y)$$

has a solution $f(x) = \text{const}$. Hence,

$$C (x) = kx, k = \text{const}.$$ 

As a result, we get the formula
The gravitational constant $G$ is equal to the force of attraction of two unit masses located at a unit distance:

$$M = 1, \ m = 1, \ R = 1 \Rightarrow G = F.$$ 

Hence,

$$G = 4\pi^2 k.$$ 

Finally, we get Newton's law of universal gravitation \[2\]

$$F = \frac{GMm}{R^2}.$$ \hspace{1cm} (7)

5. What is Kepler's constant equal to?

The found formulas make it possible to simultaneously refine Kepler's law for a circular orbit:

$$\frac{R^3}{T^2} = \frac{GM}{4\pi^2}.$$ 

According to the methodological principle of correspondence, Kepler's law for an elliptical orbit takes the form:

$$\frac{a^3}{T^2} = \frac{GM}{4\pi^2}.$$ 

A well-known result of Newton for a pair of bodies (Sun and planet),

$$\frac{a^3}{T^2} = \frac{G(M+m)}{4\pi^2},$$

goes beyond Kepler's third law, since the right part of the last formula is not a constant value and depends on the planet's mass $m$. As a result, the natural condition,

$$0 < m \ll M,$$

necessary for a rigorous formulation of Kepler's three laws is violated.

6. The average distance of a planet to the Sun is equal to the semi-major axis of the planet's orbit

This statement first appears as a hypothesis when we come across formula (2). According to Kepler's first law, the planet moves along an elliptical trajectory. The proof of the statement is related to the central symmetry in the plane of the orbit relative to the central point of the orbit. Two points $P$ (planet) and $S$ (Sun – the first focus) are translated by symmetry into points $P'$ (a point in the orbit) and $S'$ (the second focus), respectively. Points $P, S, P', S'$ form a parallelogram. The averaging of the planet-Sun distances is performed for all pairs of distances $PS$ and $P'S$. The parallelogram gives the averaging of each pair equal to $a$, from which our statement follows.

7. The uniqueness of the law of universal gravitation

Four variables $F, M, m, R$ and constant $G$ can be connected together by only one formula (7). Let us prove this statement. The so-called pi-theorem and the definition of the gravitational constant in the case of attraction of two equal masses $m$ give the formula

$$F = \frac{Gm^2}{R^2}.$$ 

The principle of superposition of gravitational forces, formulated by Newton [2], and a simple identity
\[ Mm + mM = (M + m)^2 - M^2 - m^2 , \]

applied to the masses \( M \) and \( m \), finally give the formula (7).

The uniqueness of the law means that numerous old attempts to present the law of universal gravitation in a "corrected" form

\[ F = \frac{GMm}{R^{2+\varepsilon}} , \quad -1 \ll \varepsilon \ll 1 , \]

lead to only one conclusion: \( \varepsilon = 0 \).

The results of the article presented in paragraphs 4, 6 and 7 belong to the author.

References
[1] Kepler I 1619 Harmonices mundi libri V (Linz)
[2] Newton I 1687 Philosophiae naturalis principia mathematica (Londini)