CI-GROUPS WITH RESPECT TO TERNARY RELATIONAL STRUCTURES:
NEW EXAMPLES

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Abstract. We find a sufficient condition to establish that certain abelian groups are not CI-groups with respect to ternary relational structures, and then show that the groups $\mathbb{Z}_3 \times \mathbb{Z}_2^2$, $\mathbb{Z}_7 \times \mathbb{Z}_2^3$, and $\mathbb{Z}_5 \times \mathbb{Z}_2^4$ satisfy this condition. Then we completely determine which groups $\mathbb{Z}_2^k \times \mathbb{Z}_p$, $p$ a prime, are CI-groups with respect to binary and ternary relational structures. Finally, we show that $\mathbb{Z}_2^5$ is not a CI-group with respect to ternary relational structures.

1. Introduction

In recent years, there has been considerable interest in which groups $G$ have the property that any two Cayley graphs of $G$ are isomorphic if and only if they are isomorphic by a group automorphism of $G$. Such a group is called a CI-group with respect to graphs, and this problem is often referred to as the Cayley isomorphism problem. The interested reader is referred to [10] for a survey on CI-groups with respect to graphs. Of course, the Cayley isomorphism problem can and has been considered for other types of combinatorial objects. Perhaps the most significant such result is a well-known theorem of Pálfy [12] which states that a group $G$ of order $n$ is a CI-group with respect to every class of combinatorial objects if and only if $n = 4$ or gcd$(n, \phi(n)) = 1$, where $\phi$ is the Euler phi function. In fact, in proving this result, Pálfy showed that if a group $G$ is not a CI-group with respect to some class of combinatorial objects, then $G$ is not a CI-group with respect to quaternary relational structures. As much work has been done on the case of binary relational structures (i.e., digraphs), until recently there was a “gap” in our knowledge of the Cayley isomorphism problem for $k$-ary relational structures with $k = 3$. As additional motivation to study this problem, we remark that a group $G$ that is a CI-group with respect to ternary relational structures is necessarily a CI-group with respect to binary relational structures.

Although Babai [1] showed in 1977 that the dihedral group of order $2p$ is a CI-group with respect to ternary relational structures, no additional work was done on this problem until the first author considered the problem in 2003 [5]. Indeed, in [5] a relatively short list of groups is given and it is proved that every CI-group with respect to ternary relational structures lies in this list (although not every group in this list is necessarily a CI-group with respect to ternary relational structures). Additionally, several groups in the list were shown to be CI-groups with respect to ternary relational structures. Recently, the second author [13] has shown that two groups given in [5] are not CI-groups with respect to ternary relational structures, namely $\mathbb{Z}_3 \times Q_8$ and $\mathbb{Z}_3 \times Q_8$. In this paper, we give a sufficient condition to ensure that certain abelian groups are not CI-groups with respect to ternary relational structures (Theorem 5), and then show that $\mathbb{Z}_3 \times \mathbb{Z}_2^2$, $\mathbb{Z}_7 \times \mathbb{Z}_2^3$, and $\mathbb{Z}_5 \times \mathbb{Z}_2^4$ satisfy this condition in Corollary 8 (and so are not CI-groups with respect to ternary relational structures). We then show that $\mathbb{Z}_5 \times \mathbb{Z}_2^3$ is a CI-group with respect to ternary relational structures. As the first author has shown [6] that $\mathbb{Z}_2^5 \times \mathbb{Z}_p$ is a CI-group with respect to ternary relational structures.
ternary relational structures. Theorem 5. The group $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ is a CI-group with respect to color ternary relational structures if and only if $p \neq 3$ and 7.

We will show that both $\mathbb{Z}_2^3 \times \mathbb{Z}_3$ and $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ are CI-groups with respect to binary relational structures. As it is already known that $\mathbb{Z}_2^4$ is a CI-group with respect to binary relational structures [10], we have the following result.

Corollary A. The group $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ is a CI-group with respect to color binary relational structures for all primes $p$.

We are then left in the situation of knowing whether or not any subgroup of $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ is a CI-group with respect to binary or ternary relational structures, with the exception of $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ with respect to ternary relational structures (as $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ is a CI-group with respect to binary relational structures [9]). We show that $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ is a CI-group with respect to ternary relational structures (which generalizes a special case of the main result of [9]) and we prove the following.

Corollary B. The group $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ is a CI-group with respect to color ternary relational structures if and only if $p \neq 3$.

Finally, using Magma [2] and GAP [3], we show that $\mathbb{Z}_2^5$ is not a CI-group with respect to ternary relational structures.

We conclude this introductory section with the formal definition of the objects we are interested in.

Definition 1. A k-ary relational structure is an ordered pair $X = (V, E)$, with $V$ a set and $E$ a subset of $V^k$. Furthermore, a color k-ary relational structure is an ordered pair $X = (V, (E_1, \ldots, E_c))$, with $V$ a set and $E_1, \ldots, E_c$ pairwise disjoint subsets of $V^k$. If $k = 2, 3, or 4$, we simply say that $X$ is a (color) binary, ternary, or quaternary relational structure.

The following two definitions are due to Babai [1].

Definition 2. For a group $G$, define $g_L : G \to G$ by $g_L(h) = gh$, and let $G_L = \{g_L : g \in G\}$. Then $G_L$ is a permutation group on $G$, called the left regular representation of $G$. We will say that a (color) k-ary relational structure $X$ is a Cayley (color) k-ary relational structure of $G$ if $G_L \leq \text{Aut}(X)$ (note that this implies $V = G$). In general, a combinatorial object $X$ will be called a Cayley object of $G$ if $G_L \leq \text{Aut}(X)$.

Definition 3. For a class $\mathcal{C}$ of Cayley objects of $G$, we say that $G$ is a CI-group with respect to $\mathcal{C}$ if whenever $X, Y \in \mathcal{C}$, then $X$ and $Y$ are isomorphic if and only if they are isomorphic by a group automorphism of $G$.

It is clear that if $G$ is a CI-group with respect to color k-ary relational structures, then $G$ is a CI-group with respect to k-ary relational structures.

Definition 4. For $g, h$ in $G$, we denote the commutator $g^{-1}h^{-1}gh$ of $g$ and $h$ by $[g, h]$.

2. The main ingredient and Theorem A

We start by proving the main ingredient for our proof of Theorem A.

Theorem 5. Let $G$ be an abelian group and $p$ an odd prime. Assume that there exists an automorphism $\alpha$ of $G$ of order $p$ fixing only the zero element of $G$. Then $\mathbb{Z}_p \times G$ is not a CI-group with respect to color ternary relational structures. Moreover, if there exists a ternary relational structure on $G$ with automorphism group $(G_L, \alpha)$, then $\mathbb{Z}_p \times G$ is not a CI-group with respect to ternary relational structures.
Proof. Since $\alpha$ fixes only the zero element of $G$, we have $|G| \equiv 1 \pmod{p}$ and so $\gcd(p, |G|) = 1$.

For each $g \in G$, define $\hat{g} : \mathbb{Z}_p \times G \to \mathbb{Z}_p \times G$ by $\hat{g}(i,j) = (i, j + g)$. Additionally, define $\tau, \gamma, \overline{\alpha}, \alpha : \mathbb{Z}_p \times G \to \mathbb{Z}_p \times G$ by $\tau(i, j) = (i + 1, j)$, $\gamma(i, j) = (i, \alpha(j))$, and $\overline{\alpha}(i, j) = (i, \alpha(j))$. Then $(\mathbb{Z}_p \times G)_L = \langle \tau, \gamma \rangle$.

Clearly, $(G_L, \alpha) = G_L \times \langle \alpha \rangle$ is a subgroup of $\text{Sym}(G)$ (where $G_L$ acts on $G$ by left multiplication and $\alpha$ acts as an automorphism). Note that the stabilizer of 0 in $(G_L, \alpha)$ is $\langle \alpha \rangle$. As $\alpha$ fixes only 0, we conclude that for every $g \in G$ with $g \neq 0$, the point-wise stabilizer of 0 and $g$ in $(G_L, \alpha)$ is 1. Therefore, by [14] Theorem 5.12, there exists a color Cayley ternary relational structure $Z$ of $G$ such that $\text{Aut}(Z) = \langle G_L, \alpha \rangle$. If there exists also a ternary relational structure with automorphism group $(G_L, \alpha)$, then we let $Z$ be one such ternary relational structure.

Let

$$U = \{((0_{\mathbb{Z}_p}, g), (0_{\mathbb{Z}_p}, h)) : (0, g, h) \in E(Z)\} \quad \text{and} \quad S = \{([\hat{g}, \gamma](1, 0), [\hat{g}, \gamma](2, 0)) : g \in G\} \cup U$$

and define a (color) ternary relational structure $X$ by

$$V(X) = \mathbb{Z}_p \times G \quad \text{and} \quad E(X) = \{k(0_{\mathbb{Z}_p} \times G, s_1, s_2) : (s_1, s_2) \in S, k \in (\mathbb{Z}_p \times G)_L\}.$$ 

If $Z$ is a color ternary relational structure, then we assign to the edge $k(0_{\mathbb{Z}_p} \times G, s_1, s_2)$ the color of the edge $(0, g, h)$ in $Z$ if $(s_1, s_2) \in U$ and $(s_1, s_2) = ((0_{\mathbb{Z}_p}, g), (0_{\mathbb{Z}_p}, h))$, and otherwise we assign a fixed color distinct from those used in $Z$. By definition of $X$ we have $(\mathbb{Z}_p \times G)_L \leq \text{Aut}(X)$ and so $X$ is a (color) Cayley ternary relational structure of $\mathbb{Z}_p \times G$.

We claim that $\overline{\alpha} \in \text{Aut}(X)$. As $\overline{\alpha}$ is an automorphism of $\mathbb{Z}_p \times G$, we have that $\overline{\alpha} \in \text{Aut}(X)$ if and only if $\overline{\alpha}(S) = S$ and $\overline{\alpha}$ preserves colors (if $X$ is a color ternary relational structure). By definition of $Z$ and $U$, we have $\overline{\alpha}(U) = U$ and $\overline{\alpha}$ preserves colors (if $X$ is a color ternary relational structure). So, it suffices to consider the case $s \in S - U$, i.e., $s = ([\hat{g}, \gamma](1, 0), [\hat{g}, \gamma](2, 0))$ for some $g \in G$. Note that now we need not consider colors as all the edges in $S - U$ are of the same color. Then $\overline{\alpha}g(i, j) = (i, \alpha(j) + \alpha(g)) = [\alpha(g)](i, j)$. Thus $\overline{\alpha}g = [\alpha(g)]\overline{\alpha}$. Similarly, $\overline{\alpha}^{-1}g = [\alpha(g)]^{-1}\overline{\alpha}$.

Clearly $\overline{\alpha}$ commutes with $\gamma$, and so $[\alpha(g)] = [\alpha(g)]\overline{\alpha}$. As $\overline{\alpha}$ fixes $(1, 0)$ and $(2, 0)$, we see that

$$\overline{\alpha}(s) = \overline{\alpha}([\hat{g}, \gamma](1, 0), [\hat{g}, \gamma](2, 0)) = ([\overline{\alpha}(g)], \gamma)(1, 0), [\overline{\alpha}(g)], \gamma)(2, 0)) = ([\alpha(g)](1, 0), [\alpha(g)], \gamma)(2, 0)) \in (S - U).$$

Thus $\overline{\alpha}(S) = S$, $\overline{\alpha}$ preserves colors (if $X$ is a color ternary relational structure) and $\overline{\alpha} \in \text{Aut}(X)$.

We claim that $\gamma^{-1}(\mathbb{Z}_p \times G)_L \gamma$ is a subgroup of $\text{Aut}(X)$. We set $\tau' = \gamma^{-1}\tau \gamma$ and $g' = \gamma^{-1}\gamma g$, for $g \in G$. Note that $\tau' = \gamma^{-1}\overline{\tau}^{-1}$. As $\overline{\alpha} \in \text{Aut}(X)$, we have that $\tau' \in \text{Aut}(X)$. Therefore it remains to prove that $\langle g' : g \in G \rangle$ is a subgroup of $\text{Aut}(X)$. Let $e \in E(X)$ and $g \in G$. Then $e = k((0, 0), s)$, where $s \in S$ and $k = \tau ax$, for some $a \in \mathbb{Z}_p$, $l \in G$. We have to prove that $g'(e) \in E(X)$ and has the same color of $e$ (if $X$ is a color ternary relational structure).

Assume that $s \in U$. As $g'(i, j) = (i, j + ax^{-1}(g))$, by definition of $U$, we have $g'[k((0, 0), s)] \in E(X)$ and has the same color of $e$ (if $X$ is a color ternary relational structure). So, it remains to consider the case $s \in S - U$, i.e., $s = ([\hat{x}, \gamma](1, 0), [\hat{x}, \gamma](2, 0))$ for some $x \in G$. As before, we need not concern ourselves with colors because all the edges in $S - U$ are of the same color.

Set $m = ka^{-x}(g)$. Since $\overline{\alpha}g = [\alpha(g)]\overline{\alpha}$ and $\alpha, \gamma$ commute, we get $\overline{\alpha}g' = ([\alpha(g)]' \overline{\alpha}$. Also observe that as $G$ is abelian, $g'$ commutes with $\hat{h}$ for every $g, h \in G$. Hence
\[ g'k = \gamma^{-1}g\gamma^a\gamma = \gamma^{-1}g\gamma^a\gamma = \gamma^{-1}g\gamma^a \gamma \]
\[ = \gamma^a\gamma\gamma^{-a}g^a\gamma = \gamma^a(\gamma^{-a}(g))' = \gamma^a(\gamma^{-a}(g))' = k\gamma^a(\gamma^{-a}(g))' \]
\[ = m(\gamma^{-a}(g)\gamma^{-a}(g))^{-1} \gamma^{-a}(g)\gamma = m(\gamma^{-a}(g), \gamma) \]

and
\[ g'[k((0,0), s)] = g'[k((0,0), [\tilde{x}, \gamma](1,0), [\tilde{x}, \gamma](2,0)) = m(\gamma^{-a}(g), \gamma)[((0,0), [\tilde{x}, \gamma](1,0), [\tilde{x}, \gamma](2,0)) = m(\gamma^{-a}(g), \gamma)[((0,0), [\gamma^{-a}(g), \gamma][\tilde{x}, \gamma](1,0), [\gamma^{-a}(g), \gamma][\tilde{x}, \gamma](2,0)) = m(\gamma^{-a}(g), \gamma)[((0,0), [\gamma^{-a}(g)\gamma x, \gamma](1,0), [\gamma^{-a}(g)\gamma x, \gamma](2,0)) \in E(X). \]

This proves that \( g' \in \text{Aut}(X) \). Since \( g \) is an arbitrary element of \( G \), we have \( \gamma^{-1}G_L\gamma \subseteq \text{Aut}(X) \).

As claimed, \( \gamma^{-1}(\mathbb{Z}_p \times G)_L\gamma \) is a regular subgroup of \( \text{Aut}(X) \) conjugate to \( (\mathbb{Z}_p \times G)_L \).

We now have that \( Y = \gamma(X) \) is a Cayley (color) ternary relational structure of \( \mathbb{Z}_p \times G \) as \( \text{Aut}(Y) = \gamma \text{Aut}(X) \gamma^{-1} \). We will next show that \( Y \neq X \). Assume by way of contradiction that \( Y = X \). As \( \gamma(0, g) = (0, g) \) for every \( g \in G \), the permutation \( \gamma \) must map edges of \( U \) to themselves, so that \( \gamma(S - U) = S - U \). We will show that \( \gamma(S - U) \neq S - U \). Note that we need not concern ourselves with colors because as all the edges derived from \( S - U \) via translations of \( (\mathbb{Z}_p \times G)_L \) have the same color. Observing that
\[
([\hat{g}, \gamma](1,0), [\hat{g}, \gamma](2,0)) = ([\hat{g}^{-1}g\gamma(1,0), \hat{g}^{-1}g\gamma(2,0)) = ([\hat{g}^{-1}g\gamma(1,0), \hat{g}^{-1}g\gamma(2,0)) = ([\hat{g}^{-1}g\gamma(1,0), \hat{g}^{-1}g\gamma(2,0))
\]
\[
= \((\alpha^{-1} - \alpha)(g) = (\alpha^{-1} - \alpha)(h_g) = (\alpha^{-1} - \alpha)(h_g) \]
\[
\text{and} \quad (\alpha^{-2} - \alpha)(g) = (\alpha^{-2} - \alpha)(h_g) = (\alpha^{-2} - \alpha)(h_g) \]
\[
\text{Setting} \quad \iota : G \to G \text{ to be the identity permutation, we may rewrite the above equations as}
\]
\[
(\iota - \alpha)(g) = (\alpha^{-1} - \alpha)(h_g) \quad \text{and} \quad (\iota - \alpha^2)(g) = (\alpha^{-2} - \alpha)(h_g).
\]

Computing in the endomorphism ring of the abelian group \( G \), we see that \( (\alpha^{-2} - \alpha) = (\alpha^{-1} + \iota)(\alpha^{-1} - \iota) \). Applying the endomorphism \( \alpha^{-1} + \iota \) to the first equation above, we then have that
\[
(\alpha^{-1} + \iota)(\alpha^{-1} - \alpha)(g) = (\alpha^{-1} + \iota)(\alpha^{-1} - \alpha)(h_g) = (\alpha^{-2} - \alpha)(h_g) = (\iota - \alpha^2)(g).
\]

Hence \( (\alpha^{-1} + \iota)(\alpha^{-1} - \alpha) = \iota - \alpha^2 \), and so
\[
0 = (\alpha^{-1} + \iota)(\alpha^{-1} - \alpha) - (\iota - \alpha^2) = ((\alpha^{-1} + \iota) - (\iota + \alpha))(\alpha^{-1} - \alpha) = (\alpha^{-1} - \alpha)(\alpha^{-1} - \alpha),
\]
(here 0 is the endomorphism of \( G \) that maps each element of \( G \) to 0). As \( \alpha \) fixes only 0, the endomorphism \( \iota - \alpha \) is invertible, and so we see that \( \alpha^{-1} - \alpha = 0 \), and \( \alpha = \alpha^{-1} \). However, this implies that \( p = |\alpha| = 2 \), a contradiction. Thus \( \gamma(S - U) \neq S - U \) and so \( Y \neq X \).

We set \( T = \gamma(S) \), so that \( ((0,0), t) \in E(Y) \) for every \( t \in T \), where if \( X \) is a color ternary relational structure we assume that \( \gamma \) preserves colors. Now suppose that there exists \( \beta \in \text{Aut}(\mathbb{Z}_p \times G) \) such that \( \beta(X) = Y \). Since \( \gcd(p, |G|) = 1 \), we obtain that \( \mathbb{Z}_p \times 1_G \) and \( 1_{\mathbb{Z}_p} \times G \) are characteristic subgroups of \( \mathbb{Z}_p \times G \). Therefore \( \beta(i,j) = (\beta_1(i), \beta_2(j)) \), where \( \beta_1 \in \text{Aut}(\mathbb{Z}_p) \) and \( \beta_2 \in \text{Aut}(G) \). As \( \beta \) fixes \( (0,0) \), we must have that \( \beta(S) = T \). As there is no element of \( T \) of the form \((2, x_1, (1, y_1)), \)
we conclude that $\beta_1 = 1$ as $\beta_1(i) = i$ or $2i$. As $\alpha \in \text{Aut}(X)$ and $X \neq Y$, we have that $\beta_2 \not\in \langle \alpha \rangle$. Now observe that $\beta(U) = U$. Thus $\beta_2 \in \text{Aut}(Z) = \langle G_U, \alpha \rangle$. We conclude that $\beta_2 \in \langle \alpha \rangle$, a contradiction. Thus $X, Y$ are not isomorphic by a group automorphism of $\mathbb{Z}_p \times G$, and the result follows. \qed

The following two lemmas, which in our opinion are of independent interest, will be used (together with Theorem \ref{thm:transitive}) in the proof of Corollary \ref{cor:transitive}

**Lemma 6.** Let $G$ be a transitive permutation group on $\Omega$. If $x \in \Omega$ and $\text{Stab}_G(x)$ in its action on $\Omega - \{x\}$ is the automorphism group of a $k$-ary relational structure with vertex set $\Omega - \{x\}$, then $G$ is the automorphism group of a $(k+1)$-ary relational structure.

**Proof.** Let $Y$ be a $k$-ary relational structure with vertex set $\Omega - \{x\}$ and automorphism group $\text{Stab}_G(x)$ in its action on $\Omega - \{x\}$. Let $W = \{(x, v_1, \ldots, v_k) : (v_1, \ldots, v_k) \in E(Y)\}$, and define a $(k+1)$-ary relational structure $X$ by $V(X) = \Omega$ and $E(X) = \{g(w) : w \in W \text{ and } g \in G\}$. We claim that $\text{Aut}(X) = G$. First, observe that $\text{Stab}_G(x)$ maps $W$ to $W$. Also, if $e \in E(X)$ and $e = (x, v_1, \ldots, v_k)$ for some $v_1, \ldots, v_k \in \Omega$, then there exists $(x, u_1, \ldots, u_k) \in W$ and $g \in G$ with $g(x, u_1, \ldots, u_k) = (x, v_1, \ldots, v_k)$. We conclude that $g(x) = x$ and $g(u_1, \ldots, u_k) = (v_1, \ldots, v_k)$. Hence $g \in \text{Stab}_G(x)$ and $(v_1, \ldots, v_k) \in E(Y)$. Then $W$ is the set of all edges of $X$ with first coordinate $x$.

By construction, $G \leq \text{Aut}(X)$. For the reverse inclusion, let $h \in \text{Aut}(X)$. As $G$ is transitive, there exists $g \in G$ such that $g^{-1}h \in \text{Stab}_{\text{Aut}(X)}(x)$. Note that as $g \in G$, the element $g^{-1}h \in G$ if and only if $h \in G$. We may thus assume without loss of generality that $h(x) = x$. Then $h$ stabilizes set-wise the set of all edges of $X$ with first coordinate $x$, and so $h(W) = W$ and $h$ induces an automorphism of $Y$. As $\text{Aut}(Y) = \text{Stab}_G(x) \leq G$, the result follows. \qed

**Lemma 7.** Let $m \geq 2$ be an integer and $\rho \in \text{Sym}(\mathbb{Z}_{ms})$ be a semiregular element of order $m$ with $s$ orbits. Then there exists a digraph with vertex set $\mathbb{Z}_{ms}$ and with automorphism group $(\rho)$.

**Proof.** For each $i \in \mathbb{Z}_s$, set

$$\rho_i = (0, 1, \ldots, m - 1) \cdots (im, im + 1, \ldots, im + m - 1) \quad \text{and} \quad V_i = \{im + j : j \in \mathbb{Z}_m\}.$$

We inductively define a sequence of graphs $\Gamma_0, \ldots, \Gamma_{s-1} = \Gamma$ such that the subgraph of $\Gamma$ induced by $\mathbb{Z}_{(i+1)m}$ is $\Gamma_i$, the indegree of $\Gamma$ at each vertex in $V_i$ is $i+1$, and $\text{Aut}(\Gamma_i) = \langle \rho_i \rangle$, for each $i \in \mathbb{Z}_s$.

We set $\Gamma_0$ to be the directed cycle of length $m$ with edges $\{(j, j+1) : j \in \mathbb{Z}_m\}$ and with automorphism group $\langle \rho_0 \rangle$. Inductively assume that $\Gamma_{s-2}$, with the above properties, has been constructed. We construct $\Gamma_{s-1}$ as follows. First, the subgraph of $\Gamma_{s-1}$ induced by $\mathbb{Z}_{(s-1)m}$ is $\Gamma_{s-2}$. Then we place the directed $m$ cycle $\{(s-1)m + j, (s-1)m + j + 1 : j \in \mathbb{Z}_m\}$ whose automorphism group is $\langle ((s-1)m, (s-1)m + 1, \ldots, (s-1)m + m - 1) \rangle$ on the vertices in $V_{s-1}$. Additionally, we declare the vertex $(s-1)m$ to be outadjacent to $(s-2)m$ and to every vertex that $(s-2)m$ is outadjacent to that is not contained in $V_{s-2}$. Finally, we add to $\Gamma_{s-1}$ every image of one of these edges under an element of $\langle \rho_{s-1} \rangle$.

By construction, $\rho_{s-1}$ is an automorphism of $\Gamma_{s-1}$ and the subgraph of $\Gamma_{s-1}$ induced by $\mathbb{Z}_{(s-1)m}$ is $\Gamma_{s-2}$. Then each vertex in $\Gamma_{s-1} \cap V_i$ has indegree $i + 1$ for $0 \leq i \leq s - 2$, while it is easy to see that each vertex of $V_{s-1}$ has indegree $s$. Finally, if $\delta \in \text{Aut}(\Gamma_{s-1})$, then $\delta$ maps vertices of indegree $i + 1$ to vertices of indegree $i + 1$, and so $\delta$ fixes set-wise $V_i$, for every $i \in \mathbb{Z}_s$. Additionally, the action induced by $\delta$ on $V_{s-1}$ is necessarily $\langle ((s-1)m, (s-1)m + 1, \ldots, (s-1)m + m - 1) \rangle$ as this is the automorphism group of the subgraph of $\Gamma_{s-1}$ induced by $V_{s-1}$. Moreover, arguing by induction, we may assume that the action induced by $\delta$ on $V(\Gamma_{s-1}) - V_{s-1}$ is given by an element of $\langle \rho_{s-2} \rangle$. If $\delta \notin \langle \rho_{s-1} \rangle$, then $\text{Aut}(\Gamma_{s-1})$ has order at least $m^2$, and there is some element of $\text{Aut}(\Gamma_{s-1})$ that is the identity on $V(\Gamma_{s-2})$ but not on $V_{s-1}$ and vice versa. This however is not possible as each vertex of $V_{s-2}$ is outadjacent to exactly one vertex of $V_{s-1}$. Then $\text{Aut}(\Gamma_{s-1}) = \langle \rho_{s-1} \rangle$ and the result follows. \qed
and we say that $g$ block for every block for subsets are always blocks for. Observe that a complete block system is a partition of $\mathbb{Z}_15$ points. Then $\langle \mathbb{Z}_1 \times \mathbb{Z}_2 \rangle$, settles the question of which groups $G$ are CI-groups with respect to color ternary relational structures. From a computational point of view, the number of points is too large to enable a computer to determine the answer without some additional information. Lemma 6.1 in [6] is the only result that uses the hypothesis $p \geq 11$. For convenience, we report [6, Lemma 6.1].

**Lemma 9.** Let $p \geq 11$ be a prime and write $H = \mathbb{Z}_2^3 \times \mathbb{Z}_p$. For every $\phi \in \text{Sym}(H)$, there exists $\delta \in \langle H_L, \phi^{-1}H_L\phi \rangle$ such that $\langle H_L, \delta^{-1} \phi^{-1}H_L\phi \delta \rangle$ admits a complete block system consisting of 8 blocks of size $p$.

In particular, to prove that $\mathbb{Z}_2^3 \times \mathbb{Z}_5$ is a CI-group with respect to color ternary relational structures, it suffices to prove that Lemma 9 holds true also for the prime $p = 5$. We begin with some intermediate results which accidentally will also help us to prove that $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ is a CI-group with respect to color binary relational structures. (Here we denote by $\text{Alt}(X)$ the alternating group on the set $X$ and by $\text{Alt}(n)$ the alternating group on $\{1, \ldots, n\}$.)

**Lemma 10.** Let $P_1$ and $P_2$ be partitions of $\mathbb{Z}_n$ where each block in $P_1$ and $P_2$ has order $p \geq 2$. Then there exists $\phi \in \text{Alt}(\mathbb{Z}_n)$ such that $\phi(P_1) = P_2$.

**Proof.** Let $P_1 = \{\Delta_1, \ldots, \Delta_{n/p}\}$ and $P_2 = \{\Omega_1, \ldots, \Omega_{n/p}\}$. As $\text{Alt}(n)$ is $(n-2)$-transitive, there exists $\phi \in \text{Alt}(n)$ such that $\phi(\Delta_i) = \Omega_i$, for $i \in \{1, \ldots, n/p-1\}$. As both $P_1$ and $P_2$ are partitions, we see that $\phi(\Delta_{n/p}) = \Omega_{n/p}$ as well. \hfill \Box

**Lemma 11.** Let $n = 8p$, $G = (\mathbb{Z}_2^3 \times \mathbb{Z}_p)_L$ and $\delta \in \text{Sym}(n)$. Suppose that $\langle G, \delta^{-1}G\delta \rangle$ admits a complete block system $\mathcal{C}$ with $p$ blocks of size 8 such that $\text{Alt}(C) \leq \text{Stab}_{(G, \delta^{-1}G\delta)}(C)_{|C}$, where $C \in \mathcal{C}$.
Then there exists $\gamma \in \langle G, \delta^{-1}G\delta \rangle$ such that $\langle G, \gamma^{-1}\delta^{-1}G\delta\gamma \rangle$ admits a complete block system $\mathcal{B}$ with $4p$ blocks of size 2.

**Proof.** Clearly both $G$ and $\delta^{-1}G\delta$ are regular, and so both $\text{fix}_G(\mathcal{C})$ and $\text{fix}_{\delta^{-1}G\delta}(\mathcal{C})$ are semiregular of order 8. As $\text{Alt} (8)$ is simple and as $\text{fix}_{\langle G, \delta^{-1}G\delta \rangle}(\mathcal{C}) \triangleleft \text{Stab}_{\langle G, \delta^{-1}G\delta \rangle}(\mathcal{C})|_{\mathcal{C}}$, we have that $\text{Alt}(\mathcal{C}) \leq \text{fix}_{\langle G, \delta^{-1}G\delta \rangle}(\mathcal{C})|_{\mathcal{C}}$, for every $\mathcal{C} \in \mathcal{C}$. Let $J \leq \text{fix}_G(\mathcal{C})$ and $K \leq \text{fix}_{\delta^{-1}G\delta}(\mathcal{C})$ be both of order 2. Fix $C_0 \in \mathcal{C}$, and let $O_1, \ldots, O_4$ be the orbits of $J|_{C_0}$, and $O'_1, \ldots, O'_4$ be the orbits of $K|_{C_0}$. By Lemma [10] there exists $\gamma_0 \in \text{fix}_{\langle G, \delta^{-1}G\delta \rangle}(\mathcal{C})$ such that $\gamma_0^{-1}(O'_i) = O_i$, for each $i \in \{1, \ldots, 4\}$. Hence the orbits of $J|_{C_0}$ and $(\gamma_0^{-1}K\gamma_0)|_{C_0}$ are identical.

Recall that two transitive actions are equivalent if and only if the stabilizer of a point in one action is the same as the stabilizer of a point in the other [3, Lemma 1.6B]. Suppose now that the action of $\text{fix}_{\langle G, \delta^{-1}G\delta \rangle}(\mathcal{C})$ on $C_0$ is equivalent to the action of $\text{fix}_{\langle G, \delta^{-1}G\delta \rangle}(\mathcal{C})$ on $C$. Let $\omega_J$ generate $J$ and let $\omega_K$ generate $K$. As the orbits of $J|_{C_0}$ and $(\gamma_0^{-1}K\gamma_0)|_{C_0}$ are identical and $|\omega_J| = |\omega_K| = 2$, we see that $\omega_J|_{C_0} = \omega_K|_{C_0}$ and $(\gamma_0^{-1}K\gamma_0)|_{C_0}$. Hence $(\omega_J\gamma_0^{-1}\omega_K\gamma_0)|_{C_0} = 1$, and so $(\omega_J\gamma_0^{-1}\omega_K\gamma_0)|_{C} = 1$. Therefore the orbits of $J|_{C}$ and $(\gamma_0^{-1}K\gamma_0)|_{C}$ are identical.

Define an equivalence relation $\equiv$ on $\mathcal{C}$ by $C \equiv C'$ if and only if the action of $\text{fix}_{\langle G, \delta^{-1}G\delta \rangle}(\mathcal{C})$ on $C$ is equivalent to the action of $\text{fix}_{\langle G, \delta^{-1}G\delta \rangle}(\mathcal{C})$ on $C'$. Since $\text{Alt}(8)$ has only one permutation representation of degree 8 [3, Theorem 5.3], we obtain that $C \not\equiv C'$ if and only if the action of $\text{fix}_{\langle G, \delta^{-1}G\delta \rangle}(\mathcal{C})|_{C, \mathcal{C}'}$ on $C'$ is not faithful. Thus $C \not\equiv C'$ if and only if there exists $\alpha \in \text{fix}_{\langle G, \delta^{-1}G\delta \rangle}(\mathcal{C})$ such that $\alpha|_{C} = 1$ but $\alpha|_{C'} \neq 1$.

Let $E_0$ be the $\equiv$-equivalence class containing $C_0$ and set $L_1 = \{ \alpha \in \text{fix}_{\langle G, \delta^{-1}G\delta \rangle}(\mathcal{C}) : \alpha|_{C} = 1 \text{ for every } C \in E_0 \}$. Let $C_1$ be in $\mathcal{C}$ with $C_1 \neq C_0$ and let $E_1$ be the $\equiv$-equivalence class containing $C_1$. Then there exists $\omega \in \text{fix}_{\langle G, \delta^{-1}G\delta \rangle}(\mathcal{C})$ with $\omega|_{C_0} = 1$ and $\omega|_{C_1} \neq 1$. From the definition of $\equiv$, we see that $\omega|_{C} = 1$, for every $C \in E_0$, that is, $\omega \in L_1$ and $L_1 \neq 1$. As $L_1 \triangleleft \text{fix}_{\langle G, \delta^{-1}G\delta \rangle}(\mathcal{C})$ and $\text{Alt}(8)$ is simple, we conclude that $\text{Alt}(C_1) \leq L_1|_{C_1}$.

As both $J$ and $K$ are semiregular of order 2, the groups $J|_{C_1}$ and $(\gamma_0^{-1}K\gamma_0)|_{C_1}$ are generated by even permutations. So $J|_{C_1} \leq L_1|_{C_1}$ and $(\gamma_0^{-1}K\gamma_0)|_{C_1} \leq L_1|_{C_1}$. By Lemma [10] there exists $\gamma_1 \in L_1$ such that the orbits of $J|_{C_1}$ and $(\gamma_0^{-1}K\gamma_0)|_{C_1}$ are identical. In particular, the orbits of $J|_{C}$ and $(\gamma_0^{-1}K\gamma_0)|_{C}$ are identical, for every $C \in E_1$. Furthermore, as $L_1|_{C} = 1$ for every $C \in E_0$, we have that the orbits $J|_{C}$ and $(\gamma_0^{-1}K\gamma_0)|_{C}$ are identical for every $C \in E_0 \cup E_1$.

Applying inductively the previous two paragraphs to the various $\equiv$-equivalence classes, we find $\gamma \in \langle G, \delta^{-1}G\delta \rangle$ such that the orbits of $J$ and $(\gamma^{-1}\delta^{-1}K\delta\gamma)$ are identical. Since $|J| = 2$, we get $J = \gamma^{-1}\delta^{-1}K\delta\gamma$. As $J \triangleleft G$ and $\gamma^{-1}\delta^{-1}K\delta\gamma \triangleleft \gamma^{-1}\delta^{-1}G\delta\gamma$, we obtain $J \triangleleft \langle G, \gamma^{-1}\delta^{-1}G\delta\gamma \rangle$ and the orbits of $J$ form a complete block system for $\langle G, \gamma^{-1}\delta^{-1}G\delta\gamma \rangle$ of $4p$ blocks of size 2.

The proof of the following result is analogous to the proof of [3, Lemma 6.1].

**Lemma 12.** Let $H$ be an abelian group of order $\ell p$, where $\ell < p$ and $p$ is prime. Let $\phi \in \text{Sym}(H)$. Then there exists $\delta \in \langle H_L, \phi^{-1}H_L\phi \rangle$ such that $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi \delta \rangle$ admits a complete block system with blocks of size $p$.

**Lemma 13.** Let $p \geq 5$, $H = \mathbb{Z}_p^3 \times \mathbb{Z}_p$, and $\phi \in \text{Sym}(H)$. Then either there exists $\delta \in \langle H_L, \phi^{-1}H_L\phi \rangle$ such that $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi \delta \rangle$ admits a complete block system with blocks of size $p$ or $\langle H_L, \phi^{-1}H_L\phi \rangle$ admits a complete block system $\mathcal{B}$ with blocks of size 8 and $\text{fix}_K(\mathcal{B})|_{\mathcal{B}}$ is isomorphic to a primitive subgroup of $\text{AGL}(3,2)$, for $B \in B$.

**Proof.** Set $K = \langle H_L, \phi^{-1}H_L\phi \rangle$. As $H$ has a cyclic Sylow $p$-subgroup, we have by [3, Theorem 3.5A] that $K$ is doubly-transitive or imprimitive. If $K$ is doubly-transitive, then by [11] Theorem 1.1 we
have that Alt$(H) \leq K$. Now Lemma 10 reduces this case to the imprimitive case. Thus we may assume that $K$ is isomorphic with a complete block system $C$.

Suppose that the blocks of $C$ have size $\ell p$, where $\ell = 2$ or 4. Notice that $p > \ell$. As $H$ is abelian, fix$_{H_L}(C)$ is a semiregular group of order $\ell p$ and fix$_{\phi^{-1}H_L}(C)$ is also a semiregular group of order $\ell p$. Then, for $C \in C$, both fix$_{H_L}(C)|_C$ and fix$_{\phi^{-1}H_L}(C)|_C$ are regular groups of order $\ell p$. Let $C \in C$. By Lemma 12 there exists $\delta \in \langle\text{fix}_{H_L}(C), \text{fix}_{\phi^{-1}H_L}(C)\rangle$ such that $\langle\text{fix}_{H_L}(C), \text{fix}_{\phi^{-1}H_L}(C)\rangle|_C$ admits a complete block system $\mathcal{B}_C$ consisting of blocks of size $p$. Let $C' \in C$ with $C' \neq C$. Arguing as above, there exists $\delta' \in \langle\text{fix}_{H_L}(C), \text{fix}_{\phi^{-1}H_L}(C)\rangle$ such that $\langle\text{fix}_{H_L}(C), \text{fix}_{\phi^{-1}H_L}(C)\rangle|_{C'}$ admits a complete block system $\mathcal{B}_{C'}$ consisting of blocks of size $p$. Note that $\delta'H_L, \delta' \in \langle\text{fix}_{H_L}(C), \text{fix}_{\phi^{-1}H_L}(C)\rangle|_C$ and so $\langle\text{fix}_{H_L}(C), \text{fix}_{\phi^{-1}H_L}(C)\rangle|_C$ admits $\mathcal{B}_C$ as a complete block system. Repeating this argument for every block in $C$, we find $\delta \in \langle\text{fix}_{H_L}(C), \text{fix}_{\phi^{-1}H_L}(C)\rangle$ such that $\langle\text{fix}_{H_L}(C), \text{fix}_{\phi^{-1}H_L}(C)\rangle|_C$ admits a complete block system $\mathcal{B}_C$ consisting of blocks of size $p$. Let $\mathcal{B} = \cup_C \mathcal{B}_C$. We claim that $\mathcal{B}$ is a complete block system for $\langle H_L, \phi^{-1}H_L \phi \rangle$, which will complete the argument in this case.

Let $\rho \in H_L$ be of order $p$. By construction, $\rho \in \text{fix}_{H_L}(\mathcal{B})$. As $H$ is abelian, $\text{fix}_{H_L}(\mathcal{B})|_C$ is abelian, for every $C \in C$. Then $\mathcal{B}C$ is formed by the orbits of some subgroup of $\text{fix}_{H_L}(\mathcal{B})|_C$ of order $p$, and as $(\rho)|_C$ is the unique subgroup of $\text{fix}_{H_L}(\mathcal{B})|_C$ of order $p$, we obtain that $\mathcal{B}C$ is formed by the orbits of $\langle\rho\rangle$. Then $\mathcal{B}$ is formed by the orbits of $\langle\rho\rangle \triangleleft H_L$ and $\mathcal{B}$ is a complete block system for $H_L$. An analogous argument for $\phi^{-1}(\rho)\phi\phi^{-1}$ gives that $\mathcal{B}$ is a complete block system for $\phi^{-1}H_L\phi\phi^{-1}$. Then $\mathcal{B}$ is a complete block system for $\langle H_L, \phi^{-1}H_L\phi\phi^{-1} \rangle$ with blocks of size $p$, as required.

Suppose that the blocks of $C$ have size 8. Now $H_L / \mathcal{C}$ and $\phi^{-1}H_L\phi / \mathcal{C}$ are cyclic of order $p$, and as $Z_p$ is a CI-group [1 Theorem 2.3], replacing $\phi^{-1}H_L\phi$ by a suitable conjugate, we may assume that $\langle H_L, \phi^{-1}H_L\phi / \mathcal{C} = H_L / \mathcal{C}$. Then $K / \mathcal{C}$ is regular and $\text{Stab}_K(C) = \text{fix}_K(C)$, for every $C \in C$.

Suppose that $\text{Stab}_K(C)|_C$ is isomorphic, for $C \in C$. By [4 Exercise 1.5.10], the group $K$ admits a complete block system $\mathcal{D}$ with blocks of size 2 or 4. Then $K / \mathcal{D}$ has degree $2p$ or $4p$ and, by Lemma 13 there exists $\delta \in K$ such that $\langle H_L, \phi^{-1}H_L\phi / \mathcal{D} \rangle$ admits a complete block system $\mathcal{B}'$ with blocks of size $p$. In particular, $\mathcal{B}'$ induces a complete block system $\mathcal{B}'$ for $\langle H_L, \phi^{-1}H_L\phi \rangle$ with blocks of size $2p$ or $4p$, and we conclude by the case previously considered applied with $\mathcal{C} = \mathcal{B}'$. Suppose that $\text{Stab}_K(C)|_C$ is primitive, for $C \in C$. If $\text{Stab}_K(C)|_C \geq \text{Alt}(C)$, then the result follows by Lemma 11 and so we may assume that this is not the case. By [4 Theorem 1.1], we see that $\text{Stab}_K(C)|_C \leq \text{AGL}(3,2)$. The result now follows with $\mathcal{B} = C$.

Corollary 14. Let $H = Z_2^3 \times Z_5$ and $\phi \in \text{Sym}(H)$. Then there exists $\delta \in \langle H_L, \phi^{-1}H_L\phi \rangle$ such that $\langle H_L, \phi^{-1}H_L\phi \rangle$ admits a complete block system with blocks of size 5.

Proof. Set $K = \langle H_L, \phi^{-1}H_L\phi \rangle$. By Lemma 13 we may assume that $K$ admits a complete block system $\mathcal{B}$ with blocks of size 8 and with $\text{Stab}_K(\mathcal{B})|_B \leq \text{AGL}(3,2)$, for $B \in \mathcal{B}$. As $|\text{AGL}(3,2)| = 8 \cdot 7 \cdot 6 \cdot 4$, we see that a Sylow 5-subgroup of $K$ has order 5. Let $\langle\rho\rangle$ be the subgroup of $H_L$ of order 5. So $\langle\rho\rangle$ is a Sylow 5-subgroup of $K$. Then $\phi^{-1}(\rho)\phi$ is also a Sylow 5-subgroup of $K$, and by a Sylow theorem there exists $\delta \in K$ such that $\phi^{-1}(\rho)\phi\phi^{-1} = \langle\rho\rangle$. We then have that $\langle H_L, \phi^{-1}H_L\phi \rangle$ has a unique Sylow 5-subgroup, whose orbits form the required complete block system $\mathcal{B}$.

We are finally ready to prove Theorem A.

Proof of Theorem A. If $p$ is odd, then the paragraph following the proof of Corollary 8 shows that it suffices to prove that Lemma 9 holds for the prime $p = 5$. This is done in Corollary 14. If $p = 2$, then the result can be verified using GAP or Magma.

3. Proof of Corollaries A and B

Before proceeding to our next result we will need the following definitions.
Definition 15. Let $G$ be a permutation group on $\Omega$ and $k \geq 1$. A permutation $\sigma \in \text{Sym}(\Omega)$ lies in the $k$-closure $G^{(k)}$ of $G$ if for every $k$-tuple $t \in \Omega^k$ there exists $g_t \in G$ (depending on $t$) such that $\sigma(t) = g_t(t)$. We say that $G$ is $k$-closed if the permutations lying in the $k$-closure of $G$ are the elements of $G$, that is, $G^{(k)} = G$. The group $G$ is $k$-closed if and only if there exists a color $k$-ary relational structure $X$ on $\Omega$ with $G = \text{Aut}(X)$, see [14].

Definition 16. For color digraphs $\Gamma_1$ and $\Gamma_2$, we define the \textit{wreath product} of $\Gamma_1$ and $\Gamma_2$, denoted $\Gamma_1 \wr \Gamma_2$, to be the color digraph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and edge set $E_1 \cup E_2$, where $E_1 = \{(x_1, y_1), (x_1, y_2) : x_1 \in V(\Gamma_1), (y_1, y_2) \in E(\Gamma_2)\}$ and the edge $((x_1, y_1), (x_2, y_2)) \in E_1$ is colored with the same color as $(y_1, y_2)$ in $\Gamma_2$, and $E_2 = \{((x_1, y_1), (x_2, y_2)) : (x_1, x_2) \in E(\Gamma_1), y_1, y_2 \in V(\Gamma_2)\}$ and the edge $((x_1, y_1), (x_2, y_2)) \in E_2$ is colored with the same color as $(x_1, x_2)$ in $\Gamma_1$.

Definition 17. For permutation groups $G \leq \text{Sym}(X)$ and $H \leq \text{Sym}(Y)$, we define the \textit{wreath product} of $G$ and $H$, denoted $G \wr H$, to be the permutation group $G \wr H \leq \text{Sym}(X \times Y)$ consisting of all permutations of the form $(x, y) \mapsto (g(x), h_x(y))$, $g \in G$, $h_x \in H$.

The following very useful result (see [1, Lemma 3.1]) characterizes CI-groups with respect to a class of combinatorial objects.

Lemma 18. Let $H$ be a group and let $\mathcal{K}$ be a class of combinatorial objects. The following are equivalent.

1. $H$ is a CI-group with respect to $\mathcal{K}$,
2. whenever $X$ is a Cayley object of $H$ in $\mathcal{K}$ and $\phi \in \text{Sym}(H)$ such that $\phi^{-1}H_L\phi \leq \text{Aut}(X)$, then $H_L$ and $\phi^{-1}H_L\phi$ are conjugate in $\text{Aut}(X)$.

Proof of Corollary A. From Theorem A, it suffices to show that $\mathbb{Z}_2^3 \times \mathbb{Z}_3$ and $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ are CI-groups with respect to color binary relational structures. As the transitive permutation groups of degree 24 are readily available in GAP or Magma, it can be shown using a computer that $\mathbb{Z}_2^3 \times \mathbb{Z}_3$ is a CI-group with respect to color binary relational structures. It remains to consider $H = \mathbb{Z}_2^3 \times \mathbb{Z}_7$.

Fix $\phi \in \text{Sym}(H)$ and set $K = \langle H_L, \phi^{-1}H_L\phi \rangle$. Assume that there exists $\delta \in K$ such that $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta \rangle$ admits a complete block system with blocks of size 7. Now, it follows by [9] (see the two paragraphs following the proof of Corollary [S]) that $H_L$ and $\delta^{-1}\phi^{-1}H_L\phi\delta$ are conjugate in $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta \rangle^3$. Since $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta \rangle^3 \leq \langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta \rangle^2$, the corollary follows from Lemma [13] (and from Definition [15]).

Assume that there exists no $\delta \in K$ such that $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta \rangle$ admits a complete block system with blocks of size 7. By Lemma [13], the group $\mathcal{K}$ admits a complete block system $\mathcal{C}$ with blocks of size 8 and $\text{fix}_\mathcal{K}(\mathcal{C})$ is isomorphic to a primitive subgroup of $\text{AGL}(3, 2)$, for $C \in \mathcal{C}$. Suppose that 7 and $\text{fix}_\mathcal{K}(\mathcal{C})$ are relatively prime. So, a Sylow 7-subgroup of $\mathcal{K}$ has order 7. We are now in the position to apply the argument in the proof of Corollary [14] Let $\langle \rho \rangle$ be the subgroup of $H_L$ of order 7. Then $\phi^{-1}(\rho)$ is a Sylow 7-subgroup of $\mathcal{K}$, and by a Sylow theorem there exists $\delta \in K$ such that $\delta^{-1}\phi^{-1}(\rho)\phi\delta = \langle \rho \rangle$. We then have that $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta \rangle$ has a unique Sylow 7-subgroup, whose orbits form a complete block system with blocks of size 7, contradicting our hypothesis on $\mathcal{K}$. We thus assume that 7 divides $\text{fix}_\mathcal{K}(\mathcal{C})$ and so $\text{fix}_\mathcal{K}(\mathcal{C})$ acts doubly-transitively on $C$, for $C \in \mathcal{C}$.

Fix $C \in \mathcal{C}$ and let $L$ be the point-wise stabilizer of $C$ in $\text{fix}_\mathcal{K}(\mathcal{C})$. Assume that $L \neq 1$. Now, we compute $K^{(2)}$ and we deduce that $H_L$ and $\phi^{-1}H_L\phi$ are conjugate in $K^{(2)}$, from which the corollary will follow from Lemma [13]. As $L \lhd \text{fix}_\mathcal{K}(\mathcal{C})$, we have $L|C \lhd \text{fix}_\mathcal{K}(\mathcal{C})|C|$, for every $C \in \mathcal{C}$. As a nontrivial normal subgroup of a primitive group is transitive [15, Theorem 8.8], either $L|C$ is transitive or $L|C = 1$. Let $\Gamma$ be a Cayley color digraph on $H$ with $K^{(2)} = \text{Aut}(\Gamma)$. Let $C = \{C_i : i \in \mathbb{Z}_7\}$ where $C_i = \{(x_1, x_2, x_3, i) : x_1, x_2, x_3 \in \mathbb{Z}_2\}$, and assume without loss of generality that $C = C_0$. Suppose that there is an edge of color $\kappa$ from some vertex of $C_i$ to some vertex of $C_j$, where $i \neq j$. Then there is an edge of color $\kappa$ from some vertex of $C_0$ to some vertex of
C_{j-i}. Additionally, \( j - i \) generates \( \mathbb{Z}_7 \), so there is a smallest integer \( s \) such that \( L|_{C_{s(j-i)}} = 1 \) while \( L|_{C_{(s+1)(j-i)}} \) is transitive. As there is an edge of color \( \kappa \) from some vertex of \( C_{s(j-i)} \) to some vertex of \( C_{(s+1)(j-i)} \), we conclude that there is an edge of color \( \kappa \) from every vertex of \( C_{s(j-i)} \) to every vertex of \( C_{(s+1)(j-i)} \). This implies that there is an edge of color \( \kappa \) from every vertex of \( C_i \) to every vertex of \( C_j \), and then \( \Gamma \) is the wreath product of a Cayley color digraph \( \Gamma_1 \) on \( \mathbb{Z}_7 \) and a Cayley color digraph \( \Gamma_2 \) on \( \mathbb{Z}_2^2 \). Since \( \text{fix}_K(C) \) is doubly-transitive on \( C \), we have \( \text{Aut}(\Gamma_2) \cong \text{Sym}(8) \). Therefore \( K^{(2)} = \text{Aut}(\Gamma_1) \cap \text{Aut}(\Gamma_2) \cong \text{Aut}(\Gamma_1) \cap \text{Sym}(8) \). By [7, Corollary 6.8] and Lemma [18], \( H_L \) and \( \phi^{-1}H_L\phi \) are conjugate in \( K^{(2)} \). We henceforth assume that \( L = 1 \), that is, \( \text{fix}_K(C) \) acts faithfully on \( C \), for each \( C \in \mathcal{C} \).

Define an equivalence relation on \( H \) by \( h \equiv k \) if and only if \( \text{Stab}_{\text{fix}_K(C)}(h) = \text{Stab}_{\text{fix}_K(C)}(k) \). The equivalence classes of \( \equiv \) form a complete block system \( \mathcal{D} \) for \( K \). As \( \text{fix}_K(C) \) is primitive and not regular, each equivalence class of \( \equiv \) contains at most one element from each block of \( \mathcal{C} \). We conclude that \( \mathcal{D} \) either consists of 8 blocks of size 7 or each block is a singleton. Since we are assuming that \( K \) has no block system with blocks of size 7, we have that each block of \( \mathcal{D} \) is a singleton.

Fix \( C \) and \( D \) in \( \mathcal{C} \) with \( C \neq D \) and \( h \in C \). Now, \( \text{Stab}_{\text{fix}_K(C)}(h) \) is isomorphic to a subgroup of \( \text{GL}(3,2) \) and acts with no fixed points on \( D \). From [4, Appendix B], we see that \( \text{AGL}(3,2) \) is the only doubly-transitive permutation group of degree 8 whose point stabilizer admits a fixed-point-free action of degree 8. Therefore \( \text{fix}_K(C) \cong \text{AGL}(3,2) \). Additionally, \( \text{Stab}_{\text{fix}_K(C)}(h)|D \) is transitive on \( D \).

Suppose that \( \Gamma \) is a color digraph with \( K^{(2)} = \text{Aut}(\Gamma) \) and suppose that there is an edge of color \( \kappa \) from \( \ell \in E \), with \( E \in \mathcal{C} \) and \( E \neq D \). Then \( \text{Stab}_{\text{fix}_K(C)}(h)|E \) is transitive, and so there is an edge of color \( \kappa \) from \( h \) to every vertex of \( E \). As \( \text{fix}_K(C) \) is transitive on both \( C \) and \( E \), we see that there is an edge of color \( \kappa \) from every vertex of \( C \) to every vertex of \( D \). We conclude that \( \Gamma \) is a wreath product of two color digraphs \( \Gamma_1 \) and \( \Gamma_2 \), where \( \Gamma_1 \) is a Cayley color digraph on \( \mathbb{Z}_7 \) and \( \Gamma_2 \) is either complete or the complement of a complete graph, and \( K^{(2)} = \text{Aut}(\Gamma_1) \cap \text{Sym}(8) \). The result then follows by the same arguments as above.

\textbf{Proof of Corollary B.} From Corollary [3] and Theorem A, it suffices to show that \( \mathbb{Z}_2^2 \times \mathbb{Z}_7 \) is a CI-group with respect to color ternary relational structures. As the transitive permutation groups of degree 28 are readily available in GAP or Magma, it can be shown using a computer that \( \mathbb{Z}_2^2 \times \mathbb{Z}_7 \) is a CI-group with respect to color ternary relational structures. (We note that a detailed analysis similar to the proof of Corollary A for the group \( \mathbb{Z}_2^2 \times \mathbb{Z}_7 \) also gives a proof of this theorem.)

\textbf{4. Concluding remarks}

In the rest of this paper, we discuss the relevance of Theorem A to the study of CI-groups with respect to ternary relational structures. Using the software packages [2] and [8], we have determined that \( \mathbb{Z}_2^5 \) is not a CI-group with respect to ternary relational structures. Here we report an example witnessing this fact: the group \( G \) has order 2048, \( V \) and \( W \) are two nonconjugate elementary abelian regular subgroups of \( G \), and \( X = (\{1,\ldots,32\},E) \) is a ternary relational structure with
$G = \text{Aut}(X)$. 

\[
V = \langle (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)(25,26)(27,28)(29,30)(31,32), \\
(1,3)(2,4)(5,7)(6,8)(9,11)(10,12)(13,15)(14,16)(17,19)(18,20)(21,23)(22,24)(25,27)(26,28)(29,31)(30,32), \\
(1,5)(2,6)(3,7)(4,8)(9,13)(10,14)(11,15)(12,16)(17,21)(18,22)(19,23)(20,24)(25,29)(26,30)(27,31)(28,32), \\
(1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16)(17,25)(18,26)(19,27)(20,28)(21,29)(22,30)(23,31)(24,32), \\
(1,17)(2,18)(3,19)(4,20)(5,21)(6,22)(7,23)(8,24)(9,25)(10,26)(11,27)(12,28)(13,29)(14,30)(15,31)(16,32) \rangle,
\]

\[
W = \langle (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)(25,26)(27,28)(29,30)(31,32), \\
(1,3)(2,4)(5,7)(6,8)(9,11)(10,12)(13,15)(14,16)(17,20)(18,19)(21,24)(22,23)(25,27)(26,29)(27,30)(31,32), \\
(1,5)(2,6)(3,7)(4,8)(9,14)(10,13)(11,16)(12,15)(17,22)(18,21)(19,24)(20,23)(25,29)(26,30)(27,31)(28,32), \\
(1,9)(2,10)(3,11)(4,12)(5,14)(6,13)(7,16)(8,15)(17,27)(18,26)(19,25)(20,26)(21,23)(22,31)(23,30)(24,29), \\
(1,17)(2,18)(3,20)(4,19)(5,22)(6,21)(7,23)(8,24)(9,27)(10,28)(11,26)(12,25)(13,32)(14,31)(15,29)(16,30) \rangle,
\]

\[
G = \langle V, W \rangle, \quad (25,26)(27,28)(29,30)(31,32), (1,11)(2,12)(3,9)(4,10)(5,13)(6,14)(7,15)(8,16)(17,19)(18,20)(25,27)(26,28) \rangle,
\]

\[
E = \{ g((1,3,9)), g((1,5,25)) : g \in G \}.
\]

**Definition 19.** For a cyclic group $M = \langle g \rangle$ of order $m$ and a cyclic group $\langle z \rangle$ of order $2^d$, $d \geq 1$, we denote by $D(m, 2^d)$ the group $\langle z \rangle \rtimes M$ with $g^z = g^{-1}$.

Combining Theorem A with [5, Theorem 9], [5, Lemma 6], the construction given in [13] and the previous paragraph, we have the following result which lists every group that can be a CI-group with respect to ternary relational structures (although not every group on the list needs to be a CI-group with respect to ternary relational structures).

**Theorem 20.** If $G$ is a CI-group with respect to ternary relational structures, then all Sylow subgroups of $G$ are of prime order or isomorphic to $\mathbb{Z}_4$, $\mathbb{Z}_2^d$, $1 \leq d \leq 4$, or $Q_8$. Moreover, $G = U \times V$, where $\gcd(|U|, |V|) = 1$, $U$ is cyclic of order $n$, with $\gcd(n, \varphi(n)) = 1$, and $V$ is one of the following:

1. $\mathbb{Z}_4^d$, $1 \leq d \leq 4$, $D(m, 2)$, or $D(m, 4)$, where $m$ is odd and $\gcd(mn, \varphi(mn)) = 1$,
2. $\mathbb{Z}_4$, $Q_8$.

Furthermore,

(a) if $V = \mathbb{Z}_4$, $Q_8$, or $D(m, 4)$ and $p \mid n$ is prime, then $4 \nmid (p - 1)$,
(b) if $V = \mathbb{Z}_2^d$, $d \geq 2$, or $Q_8$, then $3 \nmid n$,
(c) if $V = \mathbb{Z}_2^d$, $d \geq 3$, then $7 \nmid n$,
(d) if $V = \mathbb{Z}_2^2$, then $5 \nmid n$.

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