Casimir energy of a dilute dielectric ball in the mode summation method

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Abstract

In the \((\varepsilon_1 - \varepsilon_2)^2\)-approximation the Casimir energy of a dilute dielectric ball is derived using a simple and clear method of the mode summation. The addition theorem for the Bessel functions enables one to present in a closed form the sum over the angular momentum before the integration over the imaginary frequencies. The linear in \((\varepsilon_1 - \varepsilon_2)\) contribution into the vacuum energy is removed by an appropriate subtraction. The role of the contact terms used in other approaches to this problem is elucidated.

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I. INTRODUCTION

The progress in calculation of the Casimir energy is rather slow. In his pioneer paper [1] in 1948 H. B. G. Casimir calculated the vacuum electromagnetic energy for the most simple boundary conditions, for two parallel perfectly conducting plates placed in vacuum. Dielectric properties of the media separated by plane boundaries do not add new mathematical difficulties [2]. However the first result on the calculation of the Casimir energy for the non-flat boundaries was obtained only in 1968. By computer calculations, lasted 3 years, T. H. Boyer found the Casimir energy of a perfectly conducting spherical shell [3]. Account of dielectric and magnetic properties of the media in calculations of the vacuum energy for nonflat interface leads to new principal difficulties or, more precisely, to a new structure of divergencies.

The calculation of the Casimir energy in a special case, when both the material media have the same velocity of light, proves to be, from the mathematical standpoint, exactly the same as for perfectly conducting shells placed in vacuum and having the shape of the interface between these media [4–14].

The first attempt to calculate the Casimir energy of a dielectric compact ball has been undertaken by K. A. Milton in 1980 [15]. And only just recently the vacuum electromagnetic energy of a dilute dielectric ball was found [16–20]. The light velocity is discontinuous on the surface of such a ball. In Ref. [21] the analysis of the divergencies, which appear in calculation of the Casimir energy of a dielectric ball, has been carried out by determining the relevant heat kernel coefficients. The role of dispersion in this problem is now under study also [8,19,22–25].

Under these circumstances it is of indubitable interest to develop new methods for calculating the vacuum energy for non-flat boundaries with allowance for the material characteristics of the media. It is this aim that is pursued in the present paper.

Practically all the calculations of the Casimir energy for the boundaries with spherical or cylindrical symmetries use the uniform asymptotic expansion for the Bessel functions. In place of this we are employing the addition theorem for these functions that enables one to accomplish the summation over the values of the angular momentum exactly, i.e., in a closed form. In addition, the starting point in our calculation is a simple and clear representation of the vacuum energy as a half sum of the eigenfrequencies of electromagnetic oscillations connected with a dielectric ball (a global approach). This fact is also important due to the following consideration. From the mathematical standpoint, the most consistent method for treating the divergencies in calculations of the vacuum energy is the zeta regularization technique [26]. In this approach, one proceeds from the sum of the eigenfrequencies. In Refs. [18–21] the Casimir energy of a dielectric ball has been calculated by making use of the Green’s function method. An essential point there was the account of the so called contact terms. These terms encounter the expression for the vacuum energy being outside the logarithm function [16,18,27,28], therefore they do not appear when one proceeds from

\[^{1}\text{It is worth noting that the first right (but rough and not rigorous) estimation of the Casimir energy of a dilute dielectric ball has been done in the paper [4].}\]
the sum of the eigenfrequencies.

It is worth noting that the results of the Casimir energy calculation for a dilute dielectric ball, accomplished in the framework of the quantum field theory \[18,19\], coincide with those obtained by summing up the van der Waals interactions between individual molecules inside the ball \[16\] and by applying a special perturbation theory, where the dielectric ball is treated as a perturbation of the electromagnetic field in unbounded empty space \[17\].

The layout of the paper is as follows. In Sec. II the derivation of the integral representation for the vacuum energy is given by the mode sum and contour integration. The subtraction procedure that gives the renormalized Casimir energy in the \(\Delta n^2\)-approximation is discussed in detail as well as its physical justification. The addition theorem for the Bessel functions enables one to carry out the sum over the angular momentum in a closed form. It leads to an exact (in the \(\Delta n^2\)-approximation) value of the Casimir energy of a dilute dielectric ball. In the Conclusion (Sec. III) the method proposed here for calculating the Casimir energy is briefly discussed, as well as the implications of the obtained results concerning, specifically, the elucidation of the role of the contact terms used in other approaches to this problem. In the Appendix the analysis of the divergencies is accomplished revealing an important relation between the linear and quadratic in \(\Delta n\) contributions into the vacuum energy. It is this relation that provides a simple and effective scheme of calculations which is followed in this paper.

**II. MODE SUMMATION FOR VACUUM ELECTROMAGNETIC ENERGY OF A DILUTE DIELECTRIC BALL**

We shall consider a solid ball of radius \(a\) placed in an unbounded uniform medium. The nonmagnetic materials making up the ball and its surroundings are characterized by permittivity \(\varepsilon_1\) and \(\varepsilon_2\), respectively. It is assumed that the conductivity in both the media is zero. The natural system of units is used where \(c = \hbar = 1\).

We shall proceed from the standard definition of the vacuum energy as the sum over the eigenfrequencies of electromagnetic oscillations \[29\]

\[
E = \frac{1}{2} \sum_s (\omega_s - \overline{\omega}_s).
\]  

(2.1)

Here \(\omega_s\) are the classical frequencies of the electromagnetic field for the boundary conditions described above, and the frequencies \(\overline{\omega}_s\) correspond to a certain limiting boundary conditions that will be specified below.

The sum \((1/2) \sum_s \overline{\omega}_s\) in Eq. (2.1) plays the same role as the counter terms in the standard renormalization procedure in quantum field theory \[30\]. However in the renormalizable field models, considered in unbounded Minkowski space-time, the explicit form of these counter terms is known (at least, it is known the algorithm of their construction at each order of perturbation theory). Unlike this, there are no general rules for obtaining the terms that should be subtracted when calculating the vacuum energy. Therefore, in a new problem on calculating the Casimir energy it is necessary to specify the boundary conditions, determining the frequencies \(\overline{\omega}_s\), anew, appealing to some physical considerations.
In the case of the plane geometry of boundaries or when considering the Casimir effect for distinct bodies it is sufficient to subtract in Eq. (2.1) the contribution of the Minkowski space \[29,21\]. In the problem at hand it implies to take the limit \(a \to \infty\), i.e., that the medium 1 tends to fill the entire space. But it turns out that this subtraction is not sufficient because the linear in \(\varepsilon_1 - \varepsilon_2\) contribution into the vacuum energy retains. Further, we assume that the difference \(\varepsilon_1 - \varepsilon_2\) is small and content ourselves only with the \((\varepsilon_1 - \varepsilon_2)^2\)-terms.

The necessity to subtract the contributions into the vacuum energy linear in \(\varepsilon_1 - \varepsilon_2\) is justified by the following consideration. The Casimir energy of a dilute dielectric ball can be thought of as the net result of the van der Waals interactions between the molecules making up the ball \[19\]. These interactions are proportional to the dipole momenta of the molecules, i.e., to the quantity \((\varepsilon_1 - 1)^2\). Thus, when a dilute dielectric ball is placed in the vacuum, then its Casimir energy should be proportional to \((\varepsilon_1 - 1)^2\). It is natural to assume that when such a dielectric ball is surrounded by an infinite dielectric medium with permittivity \(\varepsilon_2\), then its Casimir energy should be proportional to \((\varepsilon_1 - \varepsilon_2)^2\). The physical content of the contribution into the vacuum energy linear in \(\varepsilon_1 - \varepsilon_2\) has been investigated in the framework of the microscopic model of the dielectric media (see Ref. \[31\] and references therein). It has been shown that this term represents the self-energy of the electromagnetic field attached to the polarizable particles or, in more detail, it is just the sum of the individual atomic Lamb shifts. Certainly this term in the vacuum energy should be disregarded when calculating the Casimir energy which is originated in the electromagnetic interaction between different polarizable particles or atoms \[17,18,32–34\].

Further, we put for sake of symmetry

\[
\sqrt{\varepsilon_1} = n_1 = 1 + \frac{\Delta n}{2}, \quad \sqrt{\varepsilon_2} = n_2 = 1 - \frac{\Delta n}{2}.
\]

(2.2)

Here \(n_1\) and \(n_2\) are the refractive indices of the ball and of its surroundings, respectively, and it is assumed that \(\Delta n << 1\). From here it follows, in particular, that

\[
\varepsilon_1 - \varepsilon_2 = (n_1 + n_2)(n_1 - n_2) = 2\Delta n.
\]

(2.3)

Thus, using the definition (2.1) we shall keep in mind that really two subtractions should be done: first the contribution, obtained in the limit \(a \to \infty\), has to be subtracted and then all the terms linear in \(\Delta n\) should also be removed.

We present the vacuum energy defined by Eq. (2.1) in terms of the contour integral in the complex frequency plane. The details of this procedure can be found in Refs. \[3,35–37\]. Upon the contour deformation one gets

\[
E = -\frac{1}{2\pi} \sum_{l=1}^{\infty} (2l + 1) \int_0^\infty dy y \frac{d}{dy} \ln \frac{\Delta_l^{TE}(iy)\Delta_l^{TM}(iy)}{\Delta_l^{TE}(i\infty)\Delta_l^{TM}(i\infty)},
\]

(2.4)

where \(\Delta_l^{TE}(iy)\) and \(\Delta_l^{TM}(iy)\) are the left-hand sides of the equations determining the frequencies of the electromagnetic field

\[
\Delta_l^{TE}(a\omega) = 0, \quad \Delta_l^{TM}(a\omega) = 0.
\]

(2.5)

For pure imaginary values of the frequency variable \(\omega = iy\) (these values are needed in Eq. (2.4)), the expressions \(\Delta_l^{TE}(iy)\) and \(\Delta_l^{TM}(iy)\) are defined by
\[ \Delta_{l}^{\text{TE}}(iay) = \sqrt{\varepsilon_1 s'_1(k_1a)e_i(k_2a)} - \sqrt{\varepsilon_2 s'_1(k_1a)e'_i(k_2a)}, \]
\[ \Delta_{l}^{\text{TM}}(iay) = \sqrt{\varepsilon_2 s'_1(k_1a)e_i(k_2a)} - \sqrt{\varepsilon_1 s'_1(k_1a)e'_i(k_2a)}, \]

where \( k_i = \sqrt{\varepsilon_i y}, \) \( i = 1, 2, \) and \( s_l(x), e_l(x) \) are the modified Riccati–Bessel functions \[38\]

\[ s_l(x) = \sqrt{\frac{x}{2}} I_\nu(x), \quad e_l(x) = \sqrt{\frac{2x}{\pi}} K_\nu(x), \quad \nu = l + \frac{1}{2}. \]  

The prime in Eq. (2.4) stands for the differentiation with respect to the argument of the Riccati–Bessel functions.

In Eq. (2.4) we have introduced cutoff \( K \) in integration over the frequencies. This regularization is natural in the Casimir problem because physically it is clear that the photons of a very short wavelength do not contribute into the vacuum energy since they do not “feel” the boundary between the media with different permittivities \( \varepsilon_1 \) and \( \varepsilon_2. \) In the final expression the regularization parameter \( K \) should be put to tend to infinity, the divergencies, that may appear here, being canceled by appropriate counter terms.

The numerator (denominator) in the logarithm function in Eq. (2.4) is responsible for the first (second) term in the initial formula (2.1). For brevity we write in Eq. (2.4) simply \( \Delta_l(i\infty) \) instead of \( \lim_{a \to \infty} \Delta_l(iay). \) Taking into account the asymptotics of the Riccati–Bessel functions

\[ s_l(x) \simeq \frac{1}{2} e^x, \quad e_l(x) \simeq e^{-x}, \quad x \to \infty, \]

we obtain

\[ \Delta_l^{\text{TE}}(i\infty) \Delta_l^{\text{TM}}(i\infty) = -\frac{(n_1 + n_2)^2}{4} e^{2(n_1-n_2)y}. \]  

Upon substituting Eqs. (2.6) and (2.8) into Eq. (2.4) and changing the integration variable \( ay \to y, \) we cast Eq. (2.4) into the form (see Eq. (tefE2) in Ref. [4])

\[ E = -\frac{1}{2\pi a} \sum_{l=1}^{\infty} (2l+1) \int_0^{y_0} dy y \frac{d}{dy} \ln \left\{ \frac{4 e^{-2(n_1-n_2)y}}{(n_1+n_2)^2} \left[ n_1 n_2 (s'_l e_l)^2 + (s_l e'_l)^2 \right] - (n_1^2 + n_2^2) s_l s'_l e_l e'_l \right\}, \]  

where \( s_l \equiv s_l(n_1y), \) \( e_l \equiv e_l(n_2y), \) \( y_0 = aK. \)

It should be noted here that in Eq. (2.9) only the first subtraction is accomplished, which removes the contribution into the vacuum energy obtained when \( a \to \infty. \) As noted above, for obtaining the final result all the terms linear in \( \Delta n \) should be discarded also.

Further it is convenient to rewrite Eq. (2.9) in the form

\[ E = E_1 + E_2 \]  

with

\[ E_1 = \frac{\Delta n}{2\pi a} \sum_{l=1}^{\infty} (2l+1) \int_0^{y_0} y dy, \]  

\[ E_2 = -\frac{1}{2\pi a} \sum_{l=1}^{\infty} (2l+1) \int_0^{y_0} dy y \frac{d}{dy} \ln \left[ W_l^2(n_1y,n_2y) - \frac{\Delta n^2}{4} P_l^2(n_1y,n_2y) \right], \]  

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where
\[
W_l(n_1y, n_2y) = s_l(n_1y)e_l'(n_2y) - s_l'(n_1y)e_l(n_2y),
\]
\[
P_l(n_1y, n_2y) = s_l(n_1y)e_l'(n_2y) + s_l'(n_1y)e_l(n_2y).
\]

The term $E_1$ accounts for only the expression $\exp(-2\Delta n y)$ in the argument of the logarithm function in Eq. (2.9) and it appears as a result of subtracting the Minkowski space contribution into the Casimir energy (the sum with $\bar{m}_s$ in Eq. (2.1) and the denominator in Eq. (2.4)).

It is worth noting that the term $E_1$ is exactly the Casimir energy considered by Schwinger in his attempt to explain the sonoluminescence [39]. Really, introducing the cutoff $K$ for frequency integration and the cutoff $y = \omega/a$ for the angular momentum summation we arrive at the result
\[
E_1 = \frac{\Delta n}{\pi a} \int_0^a y dy \sum_{l=1}^{\infty} \left( l + \frac{1}{2} \right) \sim \frac{\Delta n}{2\pi a} \int_0^{aK} y^3 dy = \Delta n \frac{K^4a^3}{8\pi}.
\] (2.15)

We have substituted here the summation over $l$ by integration. Up to the multiplier $(-2/3)$ it is exactly the Schwinger value for the Casimir energy of a ball ($\epsilon_1 = 1$) in water ($\sqrt{\epsilon_2} \approx 4/3$) [27]. The term linear in $\Delta n$ and of the same structure was also derived in Refs. [17,32,33].

As it was explained above the energy $E_1$ should be discarded.

In our calculation, we content ourselves with the $\Delta n^2$-approximation. Hence, in Eq. (2.12) one can put $P_l^2(n_1y, n_2y) \approx P_l^2(y, y)$ and keep in expansion of the logarithm function only the terms proportional to $\Delta n^2$. In this approximation, the contributions of $W_l^2$ and $P_l^2$ into the vacuum energy are additive
\[
E_{\text{ren}} = E_{W_{\text{ren}}} + E_{P_{\text{ren}}}.
\] (2.16)

In the Appendix it is shown that for obtaining the $\Delta n^2$–contribution into the Casimir energy of the function $W_l^2$ in the argument of the logarithm in Eq. (2.12), it is sufficient to calculate the $\Delta n^2$–contribution of the function $W_l^2$ alone but changing the sign of this contribution to the opposite one (see Eq. (A20)). Hence,
\[
E_W = \frac{1}{2\pi a} \sum_{l=1}^{\infty} (2l+1) \int_0^{\pi/2} dy y \frac{d}{dy} W_l^2(n_1y, n_2y),
\] (2.17)

and only the $\Delta n^2$-term being preserved in this expression.

For $E_P$ we have
\[
E_P = \frac{\Delta n^2}{8\pi a} \sum_{l=1}^{\infty} (2l+1) \int_0^{\pi/2} dy y \frac{d}{dy} P_l^2(n_1y, n_2y).
\] (2.18)

Usually, when calculating the vacuum energy in the problem with spherical symmetry, the uniform asymptotic expansion of the Bessel functions is used [38]. As a result, an approximate value of the Casimir energy can be derived, the accuracy of which depends on the number of terms preserved in the asymptotic expansion.
We shall persist in another way employing the technique of the paper \[12\]. By making use of the addition theorem for the Bessel functions \[38\], we first do the summation over the angular momentum \(l\) in Eq. (2.12) and only after that we will integrate over the imaginary frequency \(y\). As a result, we obtain an exact (in the \(\Delta n^2\)–approximation) value of the Casimir energy in the problem involved.

Further the following addition theorem for the Bessel functions \[38\] will be used

\[
\sum_{l=0}^{\infty} (2l + 1) s_l(\lambda r) e_l(\lambda \rho) P_l(\cos \theta) = \frac{\lambda r \rho}{R} e^{-\lambda R} \equiv D,
\]

(2.19)

where

\[
R = \sqrt{r^2 + \rho^2 - 2r \rho \cos \theta}.
\]

(2.20)

Differentiating the both sides of Eq. (2.13) with respect to \(\lambda r\) and squaring the result we deduce

\[
\sum_{l=0}^{\infty} (2l + 1)[s'_l(\lambda r)e_l(\lambda \rho)]^2 = \frac{1}{2r \rho} \int_{r-\rho}^{r+\rho} \left( \frac{1}{\lambda} \frac{\partial D}{\partial r} \right)^2 R dR.
\]

(2.21)

Here the orthogonality relation for the Legendre polynomials

\[
\int_{-1}^{+1} P_l(x) P_m(x) dx = \frac{2\delta_{lm}}{2l + 1}
\]

has been taken into account. Now we put

\[
\lambda = y, \quad r = n_1 = 1 + \frac{\Delta n}{2}, \quad \rho = n_2 = 1 - \frac{\Delta n}{2}.
\]

(2.22)

Applying Eq. (2.21) and analogous ones, we derive

\[
\sum_{i=1}^{\infty} (2l + 1)W_i^2(n_1 y, n_2 y) = \frac{1}{2r \rho \lambda^2} \int_{r-\rho}^{r+\rho} R dR (D_r - D_\rho)^2 - e^{2\Delta ny}
\]

\[
= \frac{\Delta n^2}{8} \int_{\Delta n}^{2} \frac{e^{-2yR}}{R^5} \left( 4 + R^2 + 4y R - yR^3 \right)^2 dR - e^{2\Delta ny},
\]

(2.23)

\[
\sum_{i=1}^{\infty} (2l + 1)P_i^2(y, y) = \frac{1}{2} \int_{0}^{2} \left[ \frac{\partial}{\partial y} \left( \frac{y}{R} e^{-yR} \right) \right]^2 R dR - e^{-4y}.
\]

(2.24)

Here \(D_r\) and \(D_\rho\) stand for the results of the partial differentiation of the function \(D\) in Eq. (2.19) with respect to the corresponding variables and with the subsequent substitution of (2.22). The last terms in Eqs. (2.23) and (2.24) are \(W_0^2(n_1 y, n_2 y)\) and \(P_0^2(y, y)\), respectively. As it was stipulated before, in Eq. (2.23) we have to keep only the terms proportional to \(\Delta n^2\) and in Eq. (2.24) we have put \(\Delta n = 0\).

The calculation of the contribution \(E_P\) to the Casimir energy is straightforward. Upon differentiation of the right-hand side of Eq. (2.24) with respect to \(y\), the integration over \(dR\) can be done here. Substitution of this result into Eq. (2.18) gives
\[ E_P = -\frac{\Delta n^2}{2\pi a} \left( -\frac{1}{4} \right) \int_0^{y_0} dy \left[ e^{-4y} \left( 2y^2 + 2y + \frac{1}{2} \right) - \frac{1}{2} \right]. \] (2.25)

The term \((-1/2)\) in the square brackets in Eq. (2.23) gives rise to the divergence when \(y_0 \to \infty\)

\[ E^\text{div}_P = -\frac{\Delta n^2}{16\pi a} y_0. \] (2.26)

Therefore we have to subtract it with the result

\[ E^\text{ren}_P = E_P - E^\text{div}_P = \frac{5}{128} \frac{\Delta n^2}{\pi a}. \] (2.27)

As far as the expression (2.23), it is convenient to substitute it into Eq. (2.17), to do the integration over \(y\) and only after that to address the integration over \(dR\)

\[
\frac{\Delta n^2}{8} \int_{\Delta n}^{2} dR \int_0^{\infty} dy y \frac{d}{dy} \left[ e^{-2yR} \left( 4 + R^2 + 4yR - yR^3 \right)^2 \right] = \\
= -\frac{\Delta n^2}{4} \int_{\Delta n}^{2} \left( \frac{10}{R^6} + \frac{1}{R^4} + \frac{1}{8R^2} \right) dR \\
= \frac{1}{8} \left( \frac{\Delta n^2}{3} - \frac{4}{\Delta n^3} - \frac{2}{3\Delta n} - \frac{\Delta n}{4} \right). \] (2.28)

We have put here \(y_0 = \infty\) without getting the divergencies. As it is explained in the Appendix, in Eq. (2.28) we have to pick up only the term proportional to \(\Delta n^2\). Remarkably that this term is finite. It is an essential advantage of our approach. The rest of the terms in this equation are irrelevant to our consideration. Thus the counter term for \(E_W\) vanishes due to the regularizations employed (see the Appendix). In view of this we have

\[ E^\text{ren}_W = E_W = \frac{1}{2\pi a} \frac{1}{8} \frac{\Delta n^2}{3} = \frac{1}{48} \frac{\Delta n^2}{\pi a}. \] (2.29)

Finally we arrive at the following result for the Casimir energy of a dilute dielectric ball

\[ E^\text{ren} = E^\text{ren}_W + E^\text{ren}_P = \frac{\Delta n^2}{\pi a} \left( \frac{1}{48} + \frac{5}{128} \right) = \frac{23}{384} \frac{\Delta n^2}{\pi a}. \] (2.30)

Taking into account the relation (2.3) between \(\varepsilon_i\) and \(n_i, i = 1, 2\), we can write

\[ E^\text{ren} = \frac{23}{1536} \frac{(\varepsilon_1 - \varepsilon_2)^2}{\pi a}. \] (2.31)

\(^2\)This divergence has the same origin as those arising in summation over \(l\) when the uniform asymptotic expansions of the Bessel functions are used [18,19]. The technique employed here is close to the multiple scattering expansion [10], where these divergencies are also subtracted.
At the first time, this value for the Casimir energy of a dilute dielectric ball has been derived in Ref. [16] by summing up the van der Waals interactions between individual molecules making up the ball ($\varepsilon_2 = 1$). The result (2.31) was obtained also by treating a dilute dielectric ball as a perturbation in the complete Hamiltonian of the electromagnetic field for relevant configuration [17]. In papers [18,19], the value close to the exact one (2.31) has been obtained by employing the uniform asymptotic expansion of the Bessel functions.

In Ref. [4] the estimation of the Casimir energy of a dilute dielectric ball has been done taking into account, as it is clear now, only the second term in Eq. (2.30). And nevertheless it was not so bad having the accuracy about 35%.

III. CONCLUSION

In this paper the exact (in the $\Delta n^2$–approximation) value of the Casimir energy of a dilute dielectric ball is derived in the framework of the quantum field theory. The starting point is the mode summation by making use of the contour integration in the complex frequency plane. Unlike the other approaches to this problem, we do not use the uniform asymptotic expansion of the Bessel functions.

The key point in our consideration is employment of the addition theorem for the Bessel functions which enables us to do the summation over the angular momentum values in a closed form. As a by-product, it is shown that the role of the contact terms, at least in the $\Delta n^2$–approximation, consists only in removing the linear in $\Delta n$ contributions to the Casimir energy. They do not contribute to the finite value of this energy.

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APPENDIX: ANALYSIS OF THE DIVERGENCIES GENERATED BY $W_l^2$

Here we reveal an important relation between linear and quadratic in $\Delta n$ terms in $W_l^2$ defined in Eq. (2.13).

Let us put

\[ x_1 = y \left(1 + \frac{\Delta n}{2}\right), \quad x_2 = y \left(1 - \frac{\Delta n}{2}\right), \quad \Delta x = \Delta n y. \tag{A1} \]

The Taylor expansion yields
\[ W_l(x_1, x_2) = s_t(x_1)e'_t(x_2) - s'_t(x_1)e_t(x_2) = -1 + (2s'_t e'_t - s''_t e_t) \frac{\Delta x}{2} + \left[ \frac{1}{2} (s''_t e'_t - s''_t e_t) + \frac{3}{2} (s''_t e'_t - s''_t e_t) \right] \frac{\Delta x^2}{4} + O(\Delta x^3). \] (A2)

For brevity we have dropped the argument \( y \) of the function \( s_l \) and \( e_l \), and have used the value of the Wronskian
\[ W \{ s_l(y), e_l(y) \} = s_l e'_l - s'_l e_l = -1. \] (A3)

By making use of the equation for the Riccati–Bessel functions
\[ w''_l(y) - L(l, y) w_l(y) = 0, \quad L(l, y) \equiv 1 + \frac{l(l + 1)}{y^2}, \] (A4)
we obtain
\[ s'''_l e_l - s'_l e'''_l = L(l, y), \]
\[ s''_l e'_l - s''_l e''_l = -L(l, y). \] (A5)

Substitution of (A5) into (A2) gives
\[ W_l(x_1, x_2) = -1 + [s'_l e'_l - L(l, y)s_l e_l] \Delta x - \frac{1}{2} L(l, y) \Delta x^2 + O(\Delta x^3). \] (A6)

Squaring Eq. (A6) one gets
\[ W^2_l(x_1, x_2) = 1 + A_l \Delta n + B_l \Delta n^2 + O(\Delta n^3), \] (A7)
where
\[ A_l = y(s''_l e_t + s''_l e_t - 2s'_l e'_t) = 2y \left[ 2L(l, y)s_l e_l - \frac{1}{2} (s'_l e'_t)'' \right], \] (A8)
\[ B_l = y^2 L(l, y) + \frac{1}{4} A_l^2. \] (A9)

In terms of these notations we can write
\[ \ln \left( W^2_l - \frac{\Delta n^2}{4} P_l^2 \right) = A_l \Delta n + \left( B_l - \frac{A_l^2}{2} \right) \Delta n^2 - \frac{\Delta n^2}{4} P_l^2 + O(\Delta n^3). \] (A10)

The terms quadratic in \( \Delta n \) in Eq. (A10) exactly reproduce the function \( F_l(y) \) in Eq. (9) of the paper [18]. It is this function that affords the whole finite value of the Casimir energy in the problem under consideration. Unlike the papers [18–20] we didn’t introduce the contact terms in the definition of the Casimir energy and nevertheless we have reproduced the key function \( F_l(y) \). It implies that the contact terms do not give a contribution into the finite part of the Casimir energy in the problem under consideration. They merely cancel the terms \( A_l \Delta n \) in Eq. (A10).
Now we show, without invoking the contact terms, that the \( A_l \) terms in Eq. (A10) do not contribute into the vacuum energy.

Using Eq. (2.19) with \( \theta = 0 \) we introduce the notation

\[
\sum_{l=1}^{\infty} (2l + 1) s_l(yr) e_l(y\rho) = \frac{yr \rho}{|r - \rho|} e^{-y|r-\rho|} \equiv \mathcal{D}(r, \rho, y).
\]  

Taking into account the explicit form of the coefficients \( A_l \) defined in Eq. (A8) one can write

\[
\Delta n \sum_{l=1}^{\infty} (2l + 1) A_l = y\Delta n \left( \frac{\partial^2}{\partial r^2} - 2 \frac{\partial^2}{\partial r \partial \rho} + \frac{\partial^2}{\partial \rho^2} \right) \mathcal{D}(r, \rho, y) \bigg|_{r=\rho=1} + 1.
\]  

When \( r = \rho = 1 \) the derivatives of the function \( \mathcal{D} \) in Eq. (A12) tend to infinity. Therefore a preliminary regularization should be introduced here in order to put our consideration on a rigorous mathematical footing. To this end we define the right-hand side of Eq. (A12) in the following way

\[
\Delta n \sum_{l=1}^{\infty} (2l + 1) A_l = \Delta n \lim_{\varepsilon \to 0} \left( \mathcal{D}_{rr} - 2\mathcal{D}_{r\rho} + \mathcal{D}_{\rho\rho} \right) \bigg|_{r=1+\varepsilon/2, \rho=1-\varepsilon/2} + 1,
\]  

where the positive constant \( \varepsilon \) is a regularization parameter. From the explicit form of the function \( \mathcal{D}(r, \rho, y) \) (see Eq. (A11)) it follows immediately

\[
\lim_{\varepsilon \to 0} \left( \mathcal{D}_{rr} - \mathcal{D}_{\rho\rho} \right) \bigg|_{r=1+\varepsilon/2, \rho=1-\varepsilon/2} = 0.
\]  

The analogous limit for the differences

\[
\mathcal{D}_{rr} - \mathcal{D}_{r\rho} \quad \text{and} \quad \mathcal{D}_{\rho\rho} - \mathcal{D}_{r\rho}
\]  

also vanishes. Hence in the regularization introduced above the sum under consideration has the following value

\[
\Delta n \sum_{l=1}^{\infty} (2l + 1) A_l = 1.
\]  

It implies immediately that the term linear in \( \Delta n \), which encounters Eq. (A10) does not contribute into the vacuum energy \( E_2 \) defined in Eq. (2.12).

Now we show that the contributions into the Casimir energy given by \( \sum_l B_l \) and by \( (1/4) \sum_l A_l^2 \) are the same. In other words, \( y^2 L(l, y) \) in Eq. (A9) does not give any finite contribution into the vacuum energy. In order to prove this, we consider the expression

\[
I = \sum_{l=1}^{\infty} \nu \int_0^{\nu} y^2 \, dy, \quad \nu = l + \frac{1}{2}.
\]  

Instead of the cutoff regularization we shall use here the analytical regularization presenting (A17) in the following form
\[
I = \lim_{s \to 0} \sum_{l=1}^{\infty} \nu \int_{0}^{\infty} y^{2-s} dy = \lim_{s \to 0} \sum_{l=1}^{\infty} \nu^{4-s} \int_{0}^{\infty} z^{2-s} dz \\
= \lim_{s \to 0} \lim_{\mu^2 \to 0} \sum_{l=1}^{\infty} \nu^{4-s} \int_{0}^{\infty} (z^2 + \mu^2)^{1-s/2} dz . 
\] (A18)

Here the change of integration variable \( y = \nu z \) is done and the photon mass \( \mu \) is introduced. Further we have

\[
I = \lim_{s \to 0} \lim_{\mu^2 \to 0} \left[ (2^{-4+s} - 1) \zeta(s-4) - 2^{-4+s} \right] \frac{\mu^{3-s}}{2} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(-\frac{3}{2} + \frac{s}{2}\right)}{\Gamma\left(\frac{s}{2} - 1\right)}
\]

\[
= -\frac{\pi}{24} \lim_{s \to 0} \lim_{\mu^2 \to 0} \frac{\mu^2}{\Gamma\left(\frac{s}{2} - 1\right)} \to 0 . 
\] (A19)

In view of all this, we are left with the following scheme for calculating the Casimir energy in the \( \Delta n^2 \)-approximation in the problem under consideration. First, the \( \Delta n^2 \)-contribution should be find, which is given by the sum \( \sum_l W_l^2 \). Upon changing its sign to the opposite one, we obtain the contribution generated by \( W_l^2 \), when this function is in the argument of the logarithm. Obviously, this result would be deduced directly if one could find in a closed form the sum \( \sum_l W_l^2 W_l^2 \) [12]. This assertion can be explained by a symbolic formula

\[
\ln \left( W_l^2 - \frac{\Delta n^2}{4} P_l^2 \right) \sim -\Delta n^2 B_l - \frac{\Delta n^2}{4} P_l^2 + O(\Delta n^3) . 
\] (A20)

The sign \( \sim \) means here the equality subject to the regularizations described above are employed.
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