The Time-Dependent Von Kármán Shell Equation as a Limit of Three-Dimensional Nonlinear Elasticity

QIN Yizhao · YAO Peng-Fei

DOI: 10.1007/s11424-020-9146-4

Received: 28 April 2019 / Revised: 13 October 2019

Abstract The asymptotic behaviour of solutions of three-dimensional nonlinear elastodynamics in a thin shell is considered, as the thickness $h$ of the shell tends to zero. Given the appropriate scalings of the applied force and of the initial data in terms of $h$, it’s verified that three-dimesional solutions of the nonlinear elastodynamic equations converge to solutions of the time-dependent von Kármán equations or dynamic linear equations for shell of arbitrary geometry.

Keywords Nonlinear elasticity, thin shell, time-dependent von Kármán equations.

1 Introduction and Main Results

In this paper, we concern about the rigorous derivation of the two-dimensional dynamic models for a thin elastic shell starting from three-dimensional nonlinear elastodynamics. To be clear, we consider a thin elastic shell of reference configuration

$$S^h = \left\{ z = x + s\mathbf{n}(x) : x \in S, \quad -\frac{h}{2} < s < \frac{h}{2} \right\}, \quad 0 < h \leq h_0.$$ 

It’s a family of shells of small thickness $h$ around the middle surface $S$, where $S$ is a compact, connected, oriented 2d surface of the class $C^2$ embedded in $\mathbb{R}^3$ with a $C^2$ boundary $\partial S$. By $\mathbf{n}(x)$, we denote the unit normal to $S$ and $S_x$ stands for the tangent space at $x$. We suppose that the energy potential of this thin shell $W : \mathbb{R}^{1 \times 3} \rightarrow [0, \infty]$ is a continuous function with the following

QIN Yizhao
Department of Mathematical Science, Tsinghua University, Beijing 100084, China.
Email: qinyz@mail.tsinghua.edu.cn.

YAO Peng-Fei
Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China. Email: pfyao@iss.ac.cn.

This research was supported by the National Science Foundation of China under Grant Nos. 61473126 and 61573342, and Key Research Program of Frontier Sciences, CAS, under Grant No. QYZDJ-SSW-SYS011.

This paper was recommended for publication by Editor HU Xiaoming.
properties:

\[ W(RF) = W(F), \quad \forall F \in \mathbb{R}^{3 \times 3}, \quad R \in \text{SO}(3) \] (frame indifference); \quad (1)

\[ W(R) = 0, \quad \forall R \in \text{SO}(3); \] \quad (2)

\exists \text{ a positive constant } C \text{ such that }

\[ W(F) \geq C \text{dist}^2(F, \text{SO}(3)), \quad \forall F \in \mathbb{R}^{3 \times 3}; \] \quad (3)

\[ W \text{ is } C^2 \text{ in a neighbourhood of } \text{SO}(3); \] \quad (4)

\[ |DW(F)| \leq C(|F| + 1), \quad \forall F \in \mathbb{R}^{3 \times 3}. \] \quad (5)

Here, \text{SO}(3) denotes the group of proper rotations. The dynamic equations of nonlinear elasticity arise from the action functional

\[ E^h(u^h) = \frac{1}{h} \int_0^{\xi_h} \int_{S^h} \left[ \frac{|u^h_z|^2}{2} - W(\nabla u^h(z)) + \langle f^h, u^h(z) \rangle \right] d\xi dz \]

and by computing the Euler-Lagrange equations of the above energy, the equations of elastodynamic read as

\[ \partial^2\xi u^h - \text{div} DW(\nabla u^h) = f^h \quad \text{in} \quad (0, \xi_h) \times S^h, \] \quad (6)

where \( u^h : [0, \xi_h] \times S^h \to \mathbb{R}^3 \) is the deformation of the shell and \( f^h : [0, \xi_h] \times S^h \to \mathbb{R}^3 \) is an external body force applied to the shell. Equation (6) is supplemented by the initial data

\[ u^h|_{\xi=0} = \bar{u}^h, \quad \partial_\xi u^h|_{\xi=0} = \bar{\omega}^h, \] \quad (7)

and, respectively, by the mixed Neumann-clamped boundary conditions:

\[ u^h = z \quad \text{on} \quad (0, \xi_h) \times \left\{ z = x + s\mathbf{n}(x) : x \in \partial S, s \in \left( -\frac{h}{2}, \frac{h}{2} \right) \right\}, \] \quad (8)

\[ DW(\nabla u^h)\mathbf{n} = 0 \quad \text{on} \quad (0, \xi_h) \times \left\{ z = x \pm \frac{h}{2}\mathbf{n}(x) : x \in S \right\}. \] \quad (9)

Our purpose of this paper is to characterize the asymptotic behaviour of the solutions to (6), as the thickness \( h \) approaches to zero, by identifying the two-dimensional dynamic model for the thin elastic shell satisfied by their limit as \( h \to 0 \).

Lower dimensional models for thin bodies attract much attention in elasticity theory, as they are usually easier to handle from both analytical and numerical view than their three-dimensional counterparts. The problem of their rigorous derivation beginning from three-dimensional theory is one of the central issues in nonlinear elasticity. In the stationary case, the application of variational methods, especially the \( \Gamma \)-convergence, leads to the rigorous derivation of a hierarchy of limiting theories for thin plates and shells recently (see [1–7]). The \( \Gamma \)-convergence approach implies the convergence of minimizers of a sequence of functionals, to the minimizers of the limit. However, it doesn’t guarantees the convergence of the possibly non-minimizing critical points (the equilibria), which are the solutions of the Euler-Lagrange equations of the corresponding functionals. In this setting, Müller and Pakzad have first obtained convergence consequences in the von Kármán case for the thin plates in [8]. Then, the
results of convergence of equilibria have been generalized to the cases of rods, beams and shells (see [9–11]). Under the physical growth condition of energy density, similar results are also established in [12]. For more detailed survey in this direction, see [13].

As for the time-dependent cases, the model from 3d to 2d has only been established when the energy per unit volume decays like $h^4$ or stronger for thin plate so far (see [14, 15]). Here we shall combine [4] and [15] to obtain the time-dependent model for the thin shells in the von Kármán case.

Let $\Pi(x) = \nabla n(x)$ denote the negative second fundamental form of $S$ at $x$. Let $\pi$ be the projection onto $S$ along $n(x)$, that is, $\pi(z) = x$, for all $z = x + sn(x) \in S^h$. We assume that $0 < h \leq h_0$, with $h_0 > 0$ given sufficiently small to have $\pi$ well defined on each $S^h$ and $\frac{1}{2} < |1 + s\Pi(x)| < \frac{1}{2}$ for all $|s| < h_0/2$.

We recall some notations and results in the stationary case briefly. For an $H^1$ deformation $u$, we associate its elastic energy (scaled per unit thickness) with $I^h(u) = \frac{1}{h} \int_{S^h} W(\nabla u(z))dz$, where $W$ satisfy (1)–(4) as well. Furthermore, the total energy of thin shell in the stationary case is provided by $J^h(u) = I^h(u) - \frac{1}{h} \int_{S^h} \langle f^h, u(z) \rangle dz,$ (10) where the external force $f^h$, defined on $S^h$, is supposed to be $f^h(x + sn) = h\sqrt{c^h}f(x)\det(1 + s\Pi(x))^{-1}, \quad f(x) \in L^2(S, \mathbb{R}^3), \quad \int_S f(x) = 0. \quad (11)$

In (11), $e^h > 0$ is a given sequence obeying a prescribed scaling law. It’s shown that if $f^h$ scales like $h^\alpha$, then the minimizers $u^h$ of $J^h(u)$ satisfy $I^h(u^h) \sim h^\beta$ with $\beta = \alpha$ if $0 \leq \alpha < 2$ and $\beta = 2\alpha - 2$ if $\alpha > 2$. Throughout this note we shall assume that $\beta \geq 4$, or more generally

$$\lim_{h \to 0} \frac{e^h}{h^4} = \kappa < \infty. \quad (12)$$

In particular, the case that $S \subset \mathbb{R}^2$ corresponding to the von Kármán and purely linear theories of plates is derived rigorously in [2].

Let $\mathcal{V}(S, \mathbb{R}^3)$ be the space of all $H^2$ infinitesimal isometries on $S$. For each $V \in \mathcal{V}(S, \mathbb{R}^3)$, there exists a matrix field $A \in H^1(S, \mathbb{R}^{3 \times 3})$ such that $\partial_t V(x) = A(x)\tau$ and $A(x)^T = -A(x) \quad \forall x \in S, \quad \text{a.e.,} \quad \tau \in S_x. \quad (13)$

For $F \in L^2(S, \mathbb{R}^{3 \times 3})$, let $F_{\tau}(x) = \{(F(x)\tau, \eta)\}_{\tau \in S_x}$. The quadratic forms $Q_2(x, \cdot)$ are given by

$$Q_2(x, F_{\tau}) = \min\{Q_3(\tilde{F}) \colon (\tilde{F} - F)_{\tau} = 0\}, \quad Q_3(F) = D^2W(\text{id}) (F, F).$$

The form $Q_3$ is defined for all $F \in \mathbb{R}^{3 \times 3}$, while $Q_2(x, \cdot)$ for a given $x \in S$, is defined on tangential minors $F_{\tau}$ of $F \in \mathbb{R}^{3 \times 3}$. Both forms depend only on the symmetric parts of their arguments and are positive definite on the space of symmetric matrices (see [1]).
We define the linear operators \( \mathcal{L}_3 : \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3} \) and \( \mathcal{L}_2(x, .) : \mathbb{R}^{2 \times 2} \to \mathbb{R}^{2 \times 2} \) by
\[
Q_3(F) = \mathcal{L}_3 F : F \quad \text{and} \quad Q_2(x, F_{\tan}) = \mathcal{L}_2(x, F_{\tan}) : F_{\tan} \quad \forall F \in \mathbb{R}^{3 \times 3},
\]
respectively, \( F_1 : F_2 = \text{tr}(F_1^T F_2) \) for two matrices \( F_1 \) and \( F_2 \).

If \( \kappa = 0 \) in (12), the \( \Gamma \)-limiting of (10) is given by
\[
J(V, \overline{Q}) = \frac{1}{24} \int_S Q_2(x, (\nabla (A n) - A \Pi)_{\tan}) dx - \int_S \langle f, \overline{Q} \rangle dx, \quad \forall V \in \mathcal{V}, \quad \overline{Q} \in \text{SO}(3). \tag{14}
\]
For \( \kappa > 0 \), the \( \Gamma \)-limit of \( J^h \) is
\[
J(V, B_{\tan}, \overline{Q}) = \frac{1}{2} \int_S Q_2 \left( x, B_{\tan} - \frac{\sqrt{\kappa}}{2} (A^2)_{\tan} \right) dx + \frac{1}{24} \int_S Q_2(x, (\nabla (A n) - A \Pi)_{\tan}) dx
\]
\[
- \int_S \langle f, \overline{Q} \rangle dx, \tag{15}
\]
where \( B_{\tan} \) on \( S \) belongs to the finite strain space \( \mathcal{B} \) which is defined as follows. Given a vector field \( u \in H^1(S, \mathbb{R}^3) \), by \( \text{sym} \nabla u \) we mean the bilinear form on \( S_x \), given by \( \text{sym} \nabla u(\tau, \eta) = \frac{1}{2} \left[ (\partial_u u(x, \eta) + (\partial_\eta u(x, \tau)) \right] \) for all \( \tau, \eta \in S_x \). Then the finite strain space is given by
\[
\mathcal{B} = \{ \text{sym} \nabla u^h : \ u^h \in H^1(S, \mathbb{R}^3) \}^{L^2(S)}
\]
with the \( L^2 \) norm.

Next, we consider the time-dependent case. Let the external force be given by
\[
f^h(\xi, x + sn) = h \sqrt{e^h} f(h \xi, x), \quad f(\xi, x) \in L^2((0, \infty); L^2(S, \mathbb{R}^3)). \tag{16}
\]
We assume that the initial data \( \overline{w}^h \) and \( \hat{w}^h \) have the following scaling conditions in terms of \( h \)
\[
\frac{1}{2} \int_{S^h} | \overline{w}^h(z) |^2 dz + \int_{S^h} W(\nabla \overline{w}^h(z)) dz \leq C h e^h, \tag{17}
\]
where \( C > 0 \) is a uniform constant independent of \( h \).

Let \( u^h \) be a sequence of solutions to (6) on \( [0, T/h] \times S^h \). As usual, we rescale \( S^h \) to the fixed domain \( S^{ho} \) and the time to \( t = h \xi \). We set
\[
y^h(t, x + shn(x)) \equiv u^h \left( \frac{t}{h}, x + \frac{sh}{h_0} n(x) \right), \quad \text{on} \quad (0, T) \times S^{ho}. \tag{18}
\]
It follows from (7) that
\[
y^h(0, x + shn(x)) = \overline{w}^h \left( x + \frac{sh}{h_0} n(x) \right), \quad \partial_t y^h(0, x + shn(x)) = \frac{1}{h} \hat{w}^h \left( x + \frac{sh}{h_0} n(x) \right), \tag{19}
\]
for \( x + shn \in S^{ho} \).

We have the following.
Theorem 1.1 Let the assumptions (1)–(5) and (12) hold. Let \((\hat{w}^h) \subset L^2(S^h, \mathbb{R}^3)\) and \((\hat{w}^h) \subset H^1(S^h, \mathbb{R}^3)\) satisfying the boundary conditions (8) and (9) be the sequences of initial data of (6) with the scaling assumption (17). Let \(h_0 > 0\) be given small and \(T > 0\). Let the external force \(f^h\) have the property (16). For all \(h \in (0, h_0)\), let \(y^h \in L^2((0, T); H^1(S^h_0, \mathbb{R}^3))\) with
\[
\partial_t y^h \in L^2((0, T); L^2(S^h_0, \mathbb{R}^3)) \quad \text{and} \quad \partial_t^2 y^h \in L^2((0, T); H^{-1}(S^h_0, \mathbb{R}^3))
\]
be weak solutions to (6) in \((0, T) \times S^h_0\) with initial data (19), the boundary conditions (46), (47), and the energy inequalities
\[
\begin{align*}
\int_{S^h_0} \frac{h^2}{2} | \partial_t y^h(t, x + sn(x)) |^2 + W(\nabla_h y^h(t, x + sn(x))) \, dz \\
\leq \int_{S^h_0} \frac{1}{2} \hat{w}^h \left( x + \frac{sh}{h_0}n(x) \right)^2 + W \left( \nabla_h \hat{w}^h \left( x + \frac{sh}{h_0}n(x) \right) \right) \det \frac{F(s)}{F(s)} \, dz \\
+ \frac{h^2}{2} \int_0^T \int_{S^h_0} (f(t, x), \partial_t y^h) \, dt \, dz,
\end{align*}
\]
for \(t \in (0, T)\), where \(F(s)\) is given by
\[
F(s) = \text{id} + s\Pi.
\]
Then for \(y^h(t, x + sn(x))\), defined on the common domain \((0, T) \times S^h_0\), we have:
(i) \(y^h\) converges in \(L^\infty((0, T), H^1(S^h_0, \mathbb{R}^3))\) to \(\pi\).
(ii) The scaled average displacements:
\[
V^h(t, x) = \frac{h}{\sqrt{e^h}} \int_{\frac{h_0}{2}}^{h_0} y^h(t, x + sn(x)) - xs \, ds,
\]
converges (up to a subsequence) in \(L^q((0, T), H^1(S, \mathbb{R}^3))\) to some \(V \in L^\infty((0, T), V)\) for \(1 \leq q < \infty\). Besides, \(\partial_t V^h\) converges weakly-star in \(L^\infty((0, T), L^2(S, \mathbb{R}^3))\) to \(\partial_t V\) and \(V^h\) converges to \(V\) in \(L^\infty((0, T), L^2(S, \mathbb{R}^3))\), respectively. Then \(V \in W^{1,\infty}((0, T), L^2(S, \mathbb{R}^3)) \cap L^\infty((0, T), V)\).
(iii) \(\frac{1}{h} \text{sym} \nabla V^h\) converges weakly in \(L^2((0, T), L^2(S))\) to some \(B_{\tan} \in L^2((0, T), B)\).
(iv) The couple \((V, B_{\tan})\) satisfies the following two variational dynamical equations. If \(\kappa > 0\), for all \(\tilde{V} \in L^2((0, T); V \cap H^2_0(S, \mathbb{R}^3)) \cap H^1_0((0, T); H^1_0(S, \mathbb{R}^3))\) with \(\tilde{A} = \nabla \tilde{V}\) given as in (13) and all \(B_{\tan} \in L^2((0, T), B)\) there hold:
\[
\begin{align*}
\int_0^T \int_S L_2 \left( x, \left( B - \frac{\sqrt{\kappa}}{2} A^2 \right)_{\tan} \right) : \tilde{B}_{\tan} \, dx \, dt &= 0, \quad \text{(23)} \\
\int_0^T \int_S \langle f, \tilde{V} \rangle \, dx \, dt + \int_0^T \int_S \langle \mathcal{V}_t, \tilde{V}_t \rangle \, dx \, dt &= -\sqrt{\kappa} \int_0^T \int_S L_2 \left( x, \left( B - \frac{\sqrt{\kappa}}{2} A^2 \right)_{\tan} \right) : (A\tilde{A})_{\tan} \, dx \, dt \\
+ \frac{1}{12} \int_0^T \int_S L_2 \left( \nabla (A\mathcal{N}) - A\Pi \right)_{\tan} : [(\nabla (A\tilde{A}))_{\tan} - (A\Pi)_{\tan}] \, dx \, dt.
\end{align*}
\]
\(\Box\) Springer
If $\kappa = 0$, then (24) is still true where the first term in the right hand side of (24) equals the zero. Moreover, the initial data $V'(0, x) = \overline{\theta}(x) \in V$ with $\overline{\theta}(x) = 0$ on $\partial S$ and $V_t(0, x) = \overline{u}(x) \in L^2(S)$ in the both cases, where $\overline{\theta}(x)$ and $\overline{u}(x)$ are the limits of $V^h(0, x)$ and $V^h_t(0, x)$ in a certain sense, respectively. The boundary values of $V$ satisfy that $V(t, x) = 0$ and $(\nabla V(t, x))^T n = 0$ for a.e. $(t, x) \in (0, T) \times \partial S$.

**Remark 1.1** In Theorem 1.1 we have made the regularity assumption (20). In the case of the thin plates such regularities have been established in [14].

**Remark 1.2** By the scaling conditions (17) for the initial data and from [4], we have that

$$
\frac{1}{\sqrt{e^h}} \int_{\partial S} \overline{u}^h \left(x + \frac{sh}{h_0} n\right) ds \to \overline{u}(x) \text{ in } L^2(S) \tag{25}
$$

and

$$
\frac{h}{\sqrt{e^h}} \int_{\partial S} \overline{\theta}^h \left(x + \frac{sh}{h_0} n\right) - x ds \to \overline{\theta}(x) \text{ in } H^1(S). \tag{26}
$$

Moreover, we also obtain that $\overline{\theta}(x) \in V$ with $\overline{\theta}(x) = 0$ on $\partial S$. For more detail, see [4, 15].

## 2 Some Modifications in the Stationary Shell Theory

We here list the some results in [4].

**Theorem 2.1** (see [4]) Let $u^h \in H^1(S^h, \mathbb{R}^3)$ be a sequence of deformations of the thin shell $S^h$. Assume (12) and let the scaled energy $\frac{1}{\sqrt{e^h}} \int u^h$ be uniformly bounded. Then there exists a sequence of matrix fields $R^h \in H^1(S, \mathbb{R}^{3 \times 3})$ with $R^h(x) \in SO(3)$ for a.e. $x \in S$, such that:

$$
\| \nabla u^h - R^h \pi \|_{L^2(S^h)} \leq Ch^\frac{1}{2} \sqrt{e^h} \quad \text{and} \quad \| \nabla R^h \|_{L^2(S)} \leq Ch^{-1} \sqrt{e^h}
$$

and another sequence of matrices $Q^h \in SO(3)$ such that

(i) $\| (Q^h)^T R^h - R^h \|_{L^p(S)} \leq C \sqrt{e^h}$, for $p \in [1, \infty]$;

(ii) $\frac{1}{\sqrt{e^h}} ((Q^h)^T R^h - R^h) \text{ converges (up to a subsequence) to a skew-symmetric matrix field } \tilde{A}, \text{ weakly in } H^1(S) \text{ and strongly in } L^p(S)$, where $p \in [1, \infty]$.

Moreover, there is a sequence $c^h \in \mathbb{R}^3$ such that for the normalized rescaled deformations:

$$
\tilde{y}^h(x + s n) = (Q^h)^T y^h(x + s n) - c^h, \text{ where } y^h(x + s n) \triangleq u^h \left(x + \frac{sh}{h_0} n(x)\right)
$$

defined on the common domain $S^{h_0}$, the following holds:

(iii) $\| \nabla y^h - R^h \pi \|_{L^2(S^{h_0})} \leq C \sqrt{e^h}$ and $\tilde{y}^h$ converge in $H^1(S^{h_0})$ to $\pi$;

(iv) The scaled average displacements $\overline{\tilde{V}}^h$, defined as $\overline{\tilde{V}}^h(x) = \frac{h}{\sqrt{e^h}} \int_{\partial S} \tilde{y}^h(x + s n(x)) - x ds$, converge (up to a subsequence) in $H^1(S)$ to some $\tilde{V} \in V$, whose gradient is given by $\tilde{A}$, as in (12) and

$$
\lim_{h \to 0} \frac{h}{\sqrt{e^h}} ((Q^h)^T \nabla y^h - \text{id}) = \tilde{A} \pi,
$$

in $L^2(S^{h_0})$ up to a subsequence.
Therefore, by (iii) in Theorem 2.1, we obtain a value problem.

where

Thus, we have

Further, the tangential minor of $G$ satisfies that

As in [2], we set

Proof

Let $G^h = \frac{1}{\sqrt{h}}((R^h)^T \nabla h^h - \text{id})$. Then $G^h$ has a subsequence converging weakly in $L^2(S^{ho})$ to a matrix field $G$. Further, the tangential minor of $G$ satisfies that

where $G_0(x) = \int_{h_0}^{h_1} (x + s \mathbf{n}) ds$.

We observe that there is a byproduct $Q^h$ in Theorem 2.1. The construction of $Q^h$ in the appendices in [4] implies that it only depends on $h$ in the stationary case while in the time dependent case, it may depend on the time $t$ and be not differentiable on $t$, which makes it more complicated in our analysis. In order to cope with it, we eliminate $Q^h$ by some idea in [16, Lemma 13].

We define the first moment by

\[
\tilde{z}^h(x) = \int_{h_0}^{h_1} s \left[ \tilde{y}^h(x + s \mathbf{n}) - \left( x + \frac{sh}{h_0} \mathbf{n} \right) \right] ds \]

to determine the limit of $\frac{1}{\sqrt{h}}\tilde{z}^h$ as $h \to 0$, which is useful for dealing with the related boundary value problem.

Proposition 2.1 Under the assumptions of Theorem 2.1, we have

\[
\frac{1}{\sqrt{e^{ch}}} \tilde{z}^h \to \frac{h_0}{12} \tilde{\mathbf{n}}, \quad \text{in} \quad H^1(S, \mathbb{R}^3), \quad \text{as} \quad h \to 0. \tag{27}
\]

Proof As in [2], we set

\[
Y^h = \tilde{y}^h(x + s \mathbf{n}) - \left( x + \frac{sh}{h_0} \mathbf{n} \right), \quad \overline{Y}^h = \int_{h_0}^{h_1} Y^h ds, \quad Z^h = Y^h - \overline{Y}^h.
\]

Thus, we have

\[
\frac{h_0}{h} \partial_n Z^h = \nabla h \tilde{y}^h(x + s \mathbf{n})(x) - \mathbf{n}(x).
\]

Therefore, by (iii) in Theorem 2.1, we obtain

\[
\left\| \frac{h_0}{h} \partial_n Z^h - [(Q^h)^T R^h - \text{id}] \mathbf{n} \right\|_{L^2(S^{ho})} \leq C \sqrt{e^{ch}}.
\]

Since $\int_{h_0}^{h_1} Z^h = 0$ and $\int_{h_0}^{h_1} s [(Q^h)^T R^h - \text{id}] ds = 0$, by Poincâre’s inequality,

\[
\left\| \frac{h_0}{h} \partial_n Z^h - s [(Q^h)^T R^h - \text{id}] \mathbf{n} \right\|_{L^2(S^{ho})} \leq C \sqrt{e^{ch}}.
\]
Multiply the quantity inside the above norm by \( \frac{h_s}{\sqrt{e_h}} \) and integrate with respect to \( s \) over \((-h_0/2, h_0/2)\) to lead to

\[
\left\| \frac{1}{\sqrt{e_h}} \tilde{\zeta}_h - \frac{h_0}{12} \frac{h}{\sqrt{e_h}} (Q_h)^T R^h - \text{id} \right\|_{L^2(S)} \leq C h.
\]

Thus, we have

\[
\frac{1}{\sqrt{e_h}} \tilde{\zeta}_h - \frac{h_0}{12} \frac{h}{\sqrt{e_h}} \nabla \tilde{\zeta}_h \to 0 \quad \text{in} \quad L^2(S).
\]

Moreover, by straightforward calculation, we have for any \( \tau \in S_x \),

\[
\partial_\tau \tilde{\zeta}_h(x) = \int_{-h_0/2}^{h_0/2} s [\nabla_b \tilde{y}_h(x + s n(x)) - \text{id}] F \left( \frac{sh}{h_0} \right) \tau ds.
\]

Using (ii) and (iii) in Theorem 2.1, we conclude that \( \frac{\sqrt{e_h}}{e_h} \tilde{\zeta}_h \) are bounded in \( H^1(S, \mathbb{R}^3) \). The proof is complete.

Let

\[
V^h(x) = V^h[y^h](x) = \frac{h}{\sqrt{e_h}} \int_{-h_0}^{h_0} y^h(x + s n(x)) - x ds
\]

and

\[
\zeta^h(x) = \int_{-h_0}^{h_0} s \left[ y^h(x + s n(x)) - \left( x + \frac{hs}{h_0} n(x) \right) \right] ds.
\]

Now we consider some assumptions on the boundary value of \( V^h \) and \( \frac{\sqrt{e_h}}{e_h} \zeta^h \) and by these boundary value conditions, we will have that \( Q_h = \text{id} \) and \( e^h = 0 \).

**Lemma 2.2** Let the assumptions in Theorem 2.1 hold and let the boundary conditions in (46) be true. Then the following asymptotic identities

\[
Q^h = \text{id} + O \left( \frac{\sqrt{e_h}}{h} \right), \quad \text{sym} (Q^h - \text{id}) = O \left( \frac{e^h}{h^2} \right), \quad e^h = O \left( \frac{\sqrt{e_h}}{h} \right).
\]

hold.

**Proof** First, we shall show that there is an open segment \( \Gamma \) of \( \partial S \) such that, for \( x \in \Gamma \),

\[
x = x_{\tan} + \langle x, n(x) \rangle n(x), \quad \int_{\Gamma} xd\Gamma = 0, \quad \int_{\Gamma} |x_{\tan}| d\Gamma > 0.
\]

In fact, we may assume that

\[
\int_{\partial S} xd\Gamma = 0.
\]

Otherwise, we can translate \( S \). If \( \int_{\partial S} |x_{\tan}| d\Gamma > 0 \), then we can let \( \Gamma = \partial S \). Let

\[
x_{\tan} = 0 \quad \text{for} \quad x \in \partial S.
\]

Then \( \partial S \) is a curve on a sphere centered at the origin. Let \( \Gamma \subset \partial S \) be a segment such that

\[
\int_{\Gamma} xd\Gamma \neq 0.
\]
We translate $S$ such that

$$\int_{\Gamma} x \, d\Gamma = 0.$$  

Since $\Gamma$ is on a sphere not centered at the origin, we have

$$\int_{\Gamma} |x_{\text{tan}}| \, d\Gamma > 0.$$  

Then comparing the definitions of $V^h$ with $\tilde{V}^h$ and $\zeta^h$ with $\tilde{\zeta}^h$, respectively, we obtain that

$$\frac{\sqrt{e^h}}{h} V^h = \frac{\sqrt{e^h}}{h} Q^h \tilde{V}^h + (Q^h - \text{id}) x + c^h, \quad (30)$$

$$\zeta^h = Q^h \tilde{\zeta}^h + \frac{h^2}{12} (Q^h - \text{id}) n(x), \quad (31)$$

where we still denote $Q^h c^h$ by $c^h$.

Using (27), (31), the embedding $H^1(S) \hookrightarrow L^2(\Gamma)$, and $\zeta^h = 0$ in $L^2(\Gamma)$, we see that

$$\| (Q^h - \text{id}) n \|_{L^2(\Gamma)} \leq C \sqrt{e^h} \frac{\sqrt{e^h}}{h}, \quad (32)$$

which yield by $Q^h \in \text{SO}(3)$ that

$$\| [(Q^h)^T - \text{id}] n \|_{L^2(\Gamma)} \leq C \sqrt{e^h} \frac{\sqrt{e^h}}{h}. \quad (33)$$

We fix a point $x_0 \in \Gamma$ such that

$$x_{0\text{tan}} \neq 0.$$  

Let $\tau_1(x), \tau_2(x), n(x)$ be a local frame at $x_0$ on $\overline{S}$ with the positive orientation, where $n(x) = \tau_1(x) \wedge \tau_2(x)$. Let $\Gamma_0 \subset \Gamma$ be an open neighborhood of $x_0$ in $\Gamma$ such that the frame is well defined on $\Gamma_0$. Let

$$Q_0(x) = (\tau_1(x), \tau_2(x), n(x)) \quad \text{for} \quad x \in \Gamma_0.$$  

Obviously, we have $Q_0 \in \text{SO}(3)$. Let $Q^h_{\text{tan}}$ denote the $2 \times 2$ submatrix $(\langle Q^h \tau_i, \tau_j \rangle)_{i,j=1,2}$ of $Q_0^T Q^h Q_0$. Via (32) and (33), there exists a matrix $\hat{Q}^h(x) \in \text{SO}(2)$ such that

$$|Q^h_{\text{tan}}(x) - \hat{Q}^h(x)| \leq C \sqrt{e^h} \frac{\sqrt{e^h}}{h} \quad \text{for} \quad x \in \Gamma_0, \quad (34)$$

where the constant $C$ is independent of $x \in \Gamma$.

From (30), (32), and $V^h = 0$ on $\Gamma$ it follows that

$$\| (Q^h - \text{id}) x + c^h \|_{L^2(\Gamma)} \leq C \sqrt{e^h} \frac{\sqrt{e^h}}{h}. \quad (35)$$

It follows from (29), (33), and (35) that

$$|c^h| \leq C \sqrt{e^h} \frac{\sqrt{e^h}}{h}, \quad \| (Q^h - \text{id}) x_{\text{tan}} \|_{L^2(\Gamma)} \leq C \sqrt{e^h} \frac{\sqrt{e^h}}{h}. \quad (36)$$
By (34) and (36), we have
\[ \| (\hat{Q}^h(x) - \text{id}) \|_{L^2(\Gamma_0)} \leq C \frac{\sqrt{e^h h}}{h}. \] (37)

Now, since $\hat{Q}^h(x) \in SO(2)$, $2|\hat{Q}^h - \text{id}|x_{\text{tan}}| = |\hat{Q}^h(x) - \text{id}|x_{\text{tan}}|^2$. It follows from (37) that
\[ \| Q^h_{\text{tan}} - \text{id} \|_{L^2(\Gamma_0)} \leq C \frac{\sqrt{e^h}}{h}. \] (38)

From (34) and (38), we obtain
\[ \| Q^h - \text{id} \|_{L^2(\Gamma_0)} \leq C \frac{\sqrt{e^h}}{h}. \] (39)

Moreover, from the relation
\[ 2 \text{sym} (Q^h - \text{id}) = -((Q^h)^T - \text{id})(Q^h - \text{id}), \]
we have the second asymptotic identity in (28).

It follows from (28), (30), and (31) that

**Lemma 2.3** Suppose that
\[ I^h(y^h) = \int_{S_{h_0}} W(\nabla_y^h)dy^h \leq Ce^h, \quad \lim_{h \to 0} e^h = 0. \] (40)

Moreover, let (46) and (29) hold. Then
\[ V^h \to V \quad \text{in} \quad H^1(S, \mathbb{R}^3) \quad \text{with} \quad V \in \mathcal{V}, \]
\[ \frac{1}{\sqrt{e^h h}} \to \frac{h_0}{12} A \text{ in } H^1(S, \mathbb{R}^3). \] (41)

Moreover, there is $A_0 \in SO(3)$ such that
\[ V = \tilde{V} + A_0 x + c_0, \quad A = \tilde{A} + A_0 \in \text{so}(3), \]
where so(3) is the set of all the $3 \times 3$ anti-symmetric matrices.

**Remark 2.2** In Lemma 2.2,
\[ A_0 = \lim_{h \to 0} \frac{h}{\sqrt{e^h h}}(Q^h - \text{id}), \quad c_0 = \lim_{h \to 0} \frac{h}{\sqrt{e^h h}}. \]

Now, by applying Lemmas 2.2 and 2.3, Theorem 2.1 can be rewritten as the following.

**Theorem 2.4** (see [4]) Let $u^h \in H^1(S^h, \mathbb{R}^3)$ be a sequence of deformations of thin shell $S^h$. Suppose that (12) and all the assumptions in Lemma 2.2 hold true. Then there is a sequence of matrix fields $R^h \in H^1(S, \mathbb{R}^3)$ with $R^h(x) \in SO(3)$ for a.e. $x \in S$, satisfying:
\[ \| \nabla u^h - R^h \pi \|_{L^2(S^h)} \leq C e^h \sqrt{e^h}, \quad \| \nabla R^h \|_{L^2(S)} \leq Ch^{-1} \sqrt{e^h}; \]
\[ i) \| R^h - \text{id} \|_{H^1(S)} \leq C \frac{\sqrt{e^h}}{h}; \]
\[ \text{ Springer} \]
(ii) $A^h \triangleq \frac{h}{\sqrt{c h}} (R^h - \text{id})$ converges (up to a subsequence) to a skew-symmetric matrix field $A = \tilde{A} + A_0$, weakly in $H^1(S)$ and strongly in $L^p(S)$.

Moreover, for the rescaled deformations

$$y^h(x + sn) \triangleq u^h \left( x + \frac{sh}{h_0} n(x) \right)$$

defined on the common domain $S^{h_0}$, the following holds:

(iii) $\| \nabla_h y^h - R^h \pi \|_{L^2(S^{h_0})} \leq C \sqrt{h}$ and $y^h$ converges in $H^1(S^{h_0})$ to $\pi$;

(iv) The scaled average displacements $V^h$, defined as $V^h(x) = \frac{h}{\sqrt{c h}} \int_{S^{h_0}} y^h(x + sn(x)) - x ds$

converges (up to a subsequence) in $H^1(S)$ to $V = \bar{V} + A_0 x + c_0 \in V$, and

$$\lim_{h \to 0} \frac{h}{\sqrt{c h}} (\nabla_h y^h - \text{id}) = A \pi \quad \text{in} \quad L^2(S^{h_0}). \quad (44)$$

(v) $\frac{1}{n} \text{sym} \nabla V^h$ converges (up to a subsequence) in $L^2(S)$ to some $B_{tan} \in B$;

(vi) $\lim_{h \to 0} \frac{h^2}{\sqrt{c h}} (R^h - \text{id}) = \frac{1}{2} A^2 \quad \text{in} \quad L^p(S), \quad \text{where} \quad p \in [1, \infty)$;

(vii) Let $G^h = \frac{1}{\sqrt{c h}}((R^h)^T \nabla_h y^h - \text{id})$. Then $G^h$ has a subsequence converging weakly in $L^2(S^{h_0})$ to a matrix field $G$. Further,

$$G(x + sn)\tau = G_0(x)\tau + \frac{t}{h_0} (\nabla (An) - A\tau)\tau, \quad \forall \tau \in S_z,$$

where $G_0(x) = \int_{S^{h_0}} G(x + sn)ds$.

3 The Proof of Theorem 1.1

We need to make some preparations for deriving the two-dimensional evolutionary nonlinear shell model from the corresponding three-dimensional elastodynamic system.

Let $y^h$ be given in (18). Define

$$\nabla_h y^h(t, x + sn(x)) = \nabla u^h \left( \frac{t}{h}, x + \frac{sh}{h_0} n(x) \right).$$

A straightforward calculation yields, for all $x \in S$, $s \in (-\frac{h_0}{2}, \frac{h_0}{2})$, and $\tau \in S_z$,

$$\partial_\tau y^h(t, x + sn(x)) = \nabla_h y^h(t, x + sn(x)) F \left( \frac{sh}{h_0} \right) F^{-1}(s) \tau, \quad (45)$$

$$\partial_n y^h(t, x + sn(x)) = \frac{h}{h_0} \nabla_h y^h(t, x + sn(x)) n(x),$$

where $F(s)$ is given in (22). Therefore, the boundary conditions in (8) and (9) become

$$y^h(t, x + sn(x)) = x + \frac{sh}{h_0} n(x) \quad \text{on} \quad \left\{ x + sn(x) : x \in \partial S, s \in \left(-\frac{h_0}{2}, \frac{h_0}{2}\right) \right\} \times (0, T), \quad (46)$$

$$DW(\nabla_h y^h)|n = 0 \quad \text{on} \quad \left\{ x \pm \frac{h_0}{2} n(x) : x \in S \right\} \times (0, T). \quad (47)$$
The conditions (17) are
\[
\int_{S_{h_0}} \left[ \frac{h^2}{2} |\partial_t y^h(0, x + s n(x))|^2 + W(\nabla_h y^h(0, x + s n(x))) \right] \frac{\det F(\frac{sh}{h_0})}{\det F(s)} \, dz \leq C_{h_0} e^h. \tag{48}
\]

For each \( \varphi \in C_0^\infty ((0, \infty) \times S_{h_0}) \), we consider the test function
\[
\psi^h(\xi, x + s n) = \varphi \left( h\xi, x + \frac{sh_0}{h} n \right) \quad \text{for} \quad (\xi, x + s n) \in (0, \infty) \times S^h.
\]

We have the following Euler-Lagrange equations
\[
\int_0^{T/h} \int_{S^h} \frac{[u^h_\xi, \psi^h]}{\det F} - DW(\nabla u^h) : \nabla \psi^h + (f^h, \psi^h) \, d\xi dz = 0. \tag{49}
\]

Let \( \tau_1, \tau_2 \) be a local form on \( S \). In (49) \( \nabla \psi^h \) is given by
\[
\nabla \psi^h(\xi, x + s n) \tau_i = \nabla \varphi \left( h\xi, x + \frac{sh_0}{h} n \right) F\left( \frac{sh}{h_0} \right) F^{-1}(s) \tau_i \quad \text{for} \quad i = 1, 2,
\]
\[
\nabla_n \psi^h(\xi, x + s n) = \frac{h_0}{h} \nabla_n \varphi \left( h\xi, x + \frac{sh_0}{h} n \right),
\]
where \( F(s) \) is given in (22). It is easy to check that (49) can be rewritten as
\[
\int_{T,S,h_0} \frac{[(h^h_\xi, h\varphi)] - DW(\nabla h^h) : \nabla \varphi h + h\sqrt{e^h(f, \varphi)]} \, ds dx dt = 0, \tag{50}
\]
where
\[
\int_{T,S,h_0} = \int_0^{T} \int_{S} \int_{h_0}^{yh} P_h = \left( F^{-1}\left( \frac{sh}{h_0} \right) F(s) \tau_1, F^{-1}\left( \frac{sh}{h_0} \right) F(s) \tau_2, \frac{h_0}{h} n \right) (\tau_1, \tau_2, n)^T.
\]

By similar arguments as in [4] and [15], we have Lemma 3.1 below.

**Lemma 3.1** (i), (ii), and (iii) in Theorem 1.1 hold.

**Lemma 3.2** (i) \( V^h(t, x) = \frac{h}{\sqrt{e^h}} \int_{\partial S} y^h(t, x + s n(x)) - x ds \) converges to \( V \in L^\infty((0, T), V) \) weakly-star in \( L^\infty((0, T), H^1(S, R^3)) \), where
\[
V(t, x) = 0, \quad (\nabla V)^T n = 0 \quad \text{for} \quad (t, x) \in (0, T) \times \partial S,
\]
\[
V(0, x) = \overline{\eta}(x), \quad V_t(0, x) = \hat{\eta}(x) \quad \text{for} \quad x \in S.
\]

(ii) \( A^h(t, x) = A \in L^\infty((0, T), H^1(S, R^{3\times 3})) \) weakly-star in \( L^\infty((0, T), H^1(S, R^{3\times 3})) \), \( \text{sym} \ A^h \rightarrow 0 \) strongly in \( L^\infty((0, T), L^p(S, R^{3\times 3})) \), for \( 1 \leq p < \infty \), and \( (A^h \tau) \) is compact in \( L^q((0, T), L^p(S, R^3)) \) for all \( 1 \leq q < \infty, 2 \leq p < \infty \) and \( \tau \in X(S) \), where
\[
A^h(t, x) = \frac{h}{\sqrt{e^h}} (R^h(t, x) - \text{id}), \quad A_{\text{tan}} = (\nabla V)_{\text{tan}}.
\]
and

\[ \frac{h}{\sqrt{e^h}}(\text{sym } A^h)_{\text{tan}} \rightarrow \frac{1}{2} A^2_{\text{tan}} \text{ strongly in } L^2((0,T), L^2(S)). \]

Moreover, \( A \) is a skew-symmetric matrix and the above \( A \) satisfies

\[ A(t, x) n = 0 \quad \text{for } (t, x) \in (0, T) \times \partial S. \]

(iii) \( G^h = \frac{1}{\sqrt{e^h}}(R^h)^T \nabla_h y^h - \text{id} \) \( \rightarrow G \) \( \text{weakly-star in } L^\infty((0,T), L^2(S^\text{n}, \mathbb{R}^{3 \times 3})), \)

where

\[ \text{sym } G(t, x + s n(x))_{\text{tan}} = \left( B - \frac{\sqrt{\kappa}}{2} A^2 \right)_{\text{tan}} + \frac{s}{h_0} (\nabla(A n) - A \Pi)_{\text{tan}}, \tag{52} \]

\[ B_{\text{tan}} = \lim_{h \to 0} \frac{1}{h} \text{sym } \nabla V^h \text{ weakly in } L^2((0,T), L^2(S)). \]

Proof By a similar argument as in [15] and [4], we obtain (i)–(iii), where

\[ G(t, x + s n) \tau = G_0(t, x) \tau + \frac{t}{h_0} (\nabla(A n) - A \Pi) \tau, \quad \forall \tau \in S_x, \quad G_0(t,x) = \int_0^{\frac{h}{h_0}} G(t,x + s n) ds. \]

Next, we compute \( G_0(t, x) \). We have

\[ \frac{1}{h} \text{sym } \nabla_{\text{tan}} V^h(t, x) = \frac{1}{\sqrt{e^h}} \int_0^{\frac{h}{h_0}} \text{sym } [\nabla_{\text{tan}} y^h(t, x + s n(x)) F(s) - \text{id}] ds \\
= \frac{1}{\sqrt{e^h}} \int_0^{\frac{h}{h_0}} \text{sym } [\nabla_{\text{tan}} y^h(t, x + s n(x)) F(s) - (R^h)_{\text{tan}}] ds \\
+ \frac{1}{\sqrt{e^h}} \text{sym } (R^h - \text{id})_{\text{tan}}. \tag{53} \]

It follows from (ii) that

\[ \frac{1}{\sqrt{e^h}} \text{sym } (R^h - \text{id})_{\text{tan}} = \frac{\sqrt{\kappa}}{h^2} \frac{h}{\sqrt{e^h}} \text{sym } A^h_{\text{tan}} \rightarrow \frac{\sqrt{\kappa}}{2} A^2_{\text{tan}} \text{ strongly in } L^2((0,T), L^2(S)). \]

To treat the first term in the right hand side of (53), we observe that

\[ \frac{1}{\sqrt{e^h}} [\nabla y^h F(s) - R^h] \tau = R^h G^h \tau + \frac{sh}{h_0 \sqrt{e^h}} \nabla_h y^h \Pi \tau \quad \text{for } \tau \in S_x. \]

Using the above formulas in (53) and letting \( h \to 0 \), we obtain

\[ B_{\text{tan}} = \text{sym } [G_0(t, x)]_{\text{tan}} + \frac{\sqrt{\kappa}}{2} A^2_{\text{tan}}, \]

which yields the formula (52).

Lemma 3.3 Let

\[ E^h = \frac{1}{\sqrt{e^h}} DW(\text{id} + \sqrt{e^h} G^h). \]

Then
(i) $E^h, R^h E^h \to E = \mathcal{L}G$ weakly-star in $L^\infty((0, T), L^2(S^h, \mathbb{R}^{3 \times 3}))$, where $E$ is symmetric.

(ii) $E_{\tan}(t, x + s n(x)) = L_2(x, G_{\tan}(t, x + s n(x)))$.

(iii) 
$$
\lim_{h \to 0} \frac{1}{h} \| \text{skew } E^h \|_{L^\infty((0, T), L^p(S^h, \mathbb{R}^{3 \times 3}))} = 0 \quad \text{for } p \in (1, 4/3).
$$

Moreover, let
$$
\mathcal{E}(t, x) = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} E(t, x + s n(x)) ds, \quad \mathcal{E}(t, x) = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} s E(t, x + s n(x)) ds.
$$

Then
(iv) $\mathcal{E}_{\tan}(t, x) = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} L_2(x, G_{\tan}(t, x + s n(x))) ds = L_2(x, (B - \frac{\varepsilon}{2} A^2)_{\tan})$.

(v) $\mathcal{E}_{\tan}(t, x) = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} s L_2(x, G_{\tan}(t, x + s n(x))) ds = \frac{h_0}{12} L_2(x, (\nabla (A n) - A H)_{\tan})$.

Proof (i) From (5), $\{E^h\}$ is bounded in $L^\infty((0, T), L^2(S^h, \mathbb{R}^{3 \times 3}))$, and thus
$$
E^h \to E = \mathcal{L}G \quad \text{weakly-star in } L^\infty((0, T), L^2(S^h, \mathbb{R}^{3 \times 3})),
$$
arguing as in [8, Proposition 2.2]. By (i) in Theorem 2.4 and the weakly-star convergence of $E^h$, we also have
$$
R^h E^h \to E \quad \text{weakly-star in } L^\infty((0, T), L^2(S^h, \mathbb{R}^{3 \times 3})),
$$

(ii) Follows from an argument as in [11, Lemma 2.3].

(iii) Since $DW(F)F^T$ is symmetric for all $F \in \mathbb{R}^{3 \times 3}$ (see [15, p.257]), we have
$$
E^h - (E^h)^T + \sqrt{e^h}[E^h(G^h)^T - G^h(E^h)^T] = 0.
$$

It follows from (iii) in Lemma 3.2 and (i) that
$$
\sup_{t \in (0, T)} \| \text{skew } E^h \|_{L^2(S^h, \mathbb{R}^{3 \times 3})} \leq C, \quad \sup_{t \in (0, T)} \| \text{skew } E^h \|_{L^1(S^h, \mathbb{R}^{3 \times 3})} \leq C \sqrt{e^h}.
$$

By the interpolation inequality, we have for $p \in (1, 2)$,
$$
\frac{1}{h} \sup_{[0, T]} \| \text{skew } E^h \|_{L^p(S^h, \mathbb{R}^{3 \times 3})} \leq \frac{1}{h} \sup_{[0, T]} \| \text{skew } E^h \|_{L^1} \sup_{[0, T]} \| \text{skew } E^h \|_{L^2}^{1-\theta} \leq \frac{C}{h^{\theta}} (e^h)^{\frac{\theta}{2}}.
$$

where $\frac{1}{p} = \theta + \frac{1-\theta}{2}$ and $\theta \in (1/2, 1)$. Thus, (iii) follows.

(iv) and (v) follow from (iii) in Lemma 3.2 and (ii), respectively.

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1

1) Proof of (23)

Using the formulas $DW(F) = Q^T DW(QF)$ for $F \in \mathbb{R}^{3 \times 3}$, $Q \in SO(3)$, we have
$$
DW(\nabla h y^h) = R^h DW(\text{id} + \sqrt{e^h} G^h) = \sqrt{e^h} R^h E^h.
$$
In addition, from (17) and (21), we have
\[
\frac{h^2}{\sqrt{e^h}} \sup_{t \in [0,T]} \|y^h_t\|_{L^2(S^{h_0})}^2 \leq C(1 + \|f\|_{L^2((0,T) \times S)}^2)\sqrt{e^h}. \tag{54}
\]

For any \( \phi \in L^2((0,T), L^2(S^{h_0}, \mathbb{R}^3)) \), let
\[
\varphi(t, x + s n) = \int_{-h_n/2}^s \phi(t, x + \eta n) d\eta.
\]
Then
\[
\nabla_n \varphi = \phi, \quad \nabla \varphi P^h_n = \frac{h_0}{h} \varphi.
\]
Using this \( \varphi \) in (50), we obtain
\[
h_0 \int_{T,S,h_0} (R^h E^h_n, \varphi) ds dx dt = \int_{T,S,h_0} \left[ \left( \frac{h^2}{\sqrt{e^h}} y^h_t, h \varphi_t \right) - h \sum_{i=1}^2 \langle R^h E^h \tau_i, \nabla \varphi P^h_{\tau_i} \rangle \right.
+ h^2 \langle f, \varphi \rangle \right] \det F \left( \frac{sh}{h_0} \right) ds dx dt,
\]
which yields, by letting \( h \to 0 \),
\[
\int_{T,S,h_0} \langle E_n, \varphi \rangle ds dx dt = 0 \quad \text{for any} \quad \phi \in L^2((0,T), L^2(S^{h_0}, \mathbb{R}^3)),
\]
that is,
\[
E_n = 0 \quad \text{a.e. on} \quad (0,T) \times S^{h_0}. \tag{55}
\]

For \( \phi(t, x) \in L^2((0,T); H^1(S, \mathbb{R}^3)) \cap H^1_0((0,T); L^2(S, \mathbb{R}^3)) \) with \( \phi = 0 \) on \( (0,T) \times \partial S \), this time we let
\[
\varphi(t, x + s n) = \phi(t, x).
\]
Then
\[
\nabla_n \varphi = 0, \quad \nabla \varphi P_{\tau_i} = \nabla \phi F^{-1} \left( \frac{sh}{h_0} \right) \tau_i, \quad i = 1, 2.
\]
Thus, (50) can be written as
\[
\int_{T,S,h_0} \left[ \left( \frac{h^2}{\sqrt{e^h}} y^h_t, \varphi_t \right) - \sum_{i=1}^2 \langle R^h E^h \tau_i, \nabla \phi F^{-1} \left( \frac{sh}{h_0} \right) \tau_i \rangle \right] + h^2 \langle f, \varphi \rangle \right] \det F \left( \frac{sh}{h_0} \right) ds dx dt = 0. \tag{56}
\]
From (i) in Theorem 2.4, (i) and (ii) in Lemma 3.3, we obtain
\[
\lim_{h \to 0} \sum_{i=1}^2 \left( \langle R^h E^h \tau_i, \nabla \phi F^{-1} \left( \frac{sh}{h_0} \right) \tau_i \rangle \right) = \sum_{i=1}^2 \langle E \tau_i, \nabla \phi \tau_i \rangle = E \tan : \text{sym} \nabla \tan \phi
= \mathcal{L}_2(x, G \tan (t, x + s n(x))) : \text{sym} \nabla \tan \phi, \tag{57}
\]
where the convergence is weakly-star in \( L^\infty((0,T), L^2(S^{h_0}, \mathbb{R})) \).

Letting \( h \to 0 \) in (56) after using (54) and (57), we obtain (23) by (iv) in Lemma 3.3.
2) Proof of (24)
Let $\tilde{V} \in L^2((0, T), \mathcal{V} \cap H^2_0(S, \mathbb{R}^3) \cap H^1_T((0, T), H^1(S, \mathbb{R}^3))$ and let $\varphi(t, x + sn(x)) = s\tilde{A}(t, x)n(x)$, where $\tilde{A}$ is skew-symmetric such that $\partial_s\tilde{V} = \tilde{A}\tau$ for all $\tau \in S_x$. Then

$$
\nabla n\varphi = \tilde{A}n, \quad \nabla \varphi F^{-1}\left(\frac{sh}{h_0}\right) F(s)\tau = sn\nabla (\tilde{A}n)F^{-1}\left(\frac{sh}{h}\right) \tau \quad \text{for} \quad \tau \in S_x.
$$

Using (50), (54), (ii), (iv) and (v) in Lemma 3.3, we obtain

$$
\lim_{h \rightarrow 0} \frac{1}{h} \int_{T, S, h_0} \langle Rh^h n, \tilde{A} n \rangle F\left(\frac{sh}{h_0}\right) dsdxdt
$$

$$
= \lim_{h \rightarrow 0} \frac{1}{h} \int_{T, S, h_0} s\left[\left(\frac{h^2}{\sqrt{e^h}} \partial_i, \partial_i \tilde{A} n\right) - \sum_{i=1}^2 \left\langle Rh^h \tau_i, \nabla \nabla (\tilde{A}n)F^{-1}\left(\frac{sh}{h}\right) \tau_i \right\rangle + h(f, \tilde{A}n)\right] \det F\left(\frac{sh}{h_0}\right) dsdxdt
$$

$$
= - \int_{T, S, h_0} \sum_{i=1}^2 \left\langle E\tau_i, \nabla (\tilde{A}n)\tau_i \right\rangle dsdxdt
$$

$$
= - \frac{h_0}{12} \int_0^T \int_S E_{2}(x, (\nabla (\tilde{A}n) - A\Pi)_{tan}) : \nabla n\tilde{A}n dsdxdt.
$$

Next, for $\tilde{V}(t, x) \in L^2((0, T), \mathcal{V} \cap H^2_0(S, \mathbb{R}^3) \cap H^1_T((0, T), H^1(S, \mathbb{R}^3))$, let

$$
\varphi(t, x + sn) = \tilde{V}(t, x).
$$

Let $\tilde{A}$ be the skew-symmetric matrix such that $\partial_s\tilde{V} = \tilde{A}\tau$ for all $\tau \in S_x$. Then

$$
\nabla n\varphi = \nabla \tilde{V}n = 0, \quad \nabla \varphi\tau = \tilde{A}F^{-1}(s)\tau \quad \text{for} \quad \tau \in S_x.
$$

It follows from (50) that

$$
\int_{T, S, h_0} \left[\left(\frac{h^2}{\sqrt{e^h}} \partial_i, \tilde{V}_i\right) + \langle f, \tilde{V} \rangle\right] \det F\left(\frac{sh}{h_0}\right) dsdxdt
$$

$$
= \frac{1}{h} \int_{T, S, h_0} \sum_{i=1}^2 \left\langle Rh^h \tau_i, \tilde{A}F^{-1}\left(\frac{sh}{h_0}\right) \tau_i \right\rangle \det F\left(\frac{sh}{h_0}\right) dsdxdt
$$

$$
= \frac{h^2}{h^2} \int_{T, S, h_0} \sum_{i=1}^2 \left\langle A^h \tau_i, \tilde{A}F^{-1}\left(\frac{sh}{h_0}\right) \tau_i \right\rangle \det F\left(\frac{sh}{h_0}\right) dsdxdt
$$

$$
+ \frac{1}{h} \int_{T, S, h_0} \sum_{i=1}^2 \left\langle E^h \tau_i, \tilde{A}F^{-1}\left(\frac{sh}{h_0}\right) \tau_i \right\rangle \det F\left(\frac{sh}{h_0}\right) dsdxdt.
$$

Let $x \in S$ be fixed. For simplicity, we select an orthonormal basis $\tau_1(x), \tau_2(x)$ in $S_x$ such that

$$
\Pi \tau_1 = \nabla n, n = \lambda_1 \tau_1 \quad \text{for} \quad i = 1, 2,
$$

where $\lambda_1 \lambda_2$ is the Gaussian curvature. Then

$$
F^{-1}\left(\frac{sh}{h_0}\right) \tau_i = \frac{h_0 \tau_i}{h_0 + sh\lambda_i} \quad \text{for} \quad i = 1, 2.
$$

@ Springer
Thus we have
\[
\sum_{i=1}^{2} \left( E^h \tau_i, \tilde{A} F^{-1} \left( \frac{sh}{h_0} \right) \tau_i \right) = \sum_{i=1}^{2} \langle E^h \tau_i, \tilde{A} \tau_i \rangle - h \sum_{i=1}^{2} \frac{\lambda_i}{h_0 + sh \lambda_i} \langle s E^h \tau_i, \tilde{A} \tau_i \rangle
\]
\[
= E^h_{\tan} : \tilde{A}_{\tan} + \sum_{i=1}^{2} \langle E^h \tau_i, n \rangle \langle \tilde{A} \tau_i, n \rangle - h \sum_{i=1}^{2} \frac{\lambda_i}{h_0 + sh \lambda_i} \langle s E^h \tau_i, \tilde{A} \tau_i \rangle
\]
\[
= \text{skew } E^h_{\tan} : \tilde{A}_{\tan} - \langle E^h T_n, \tilde{A} n \rangle - h \sum_{i=1}^{2} \frac{\lambda_i}{h_0 + sh \lambda_i} \langle s E^h \tau_i, \tilde{A} \tau_i \rangle
\]
\[
= \text{skew } E^h_{\tan} : \tilde{A}_{\tan} + 2 \langle \text{skew } E^h n, \tilde{A} n \rangle + \frac{\sqrt{h}}{h} \langle A^h E^h n, \tilde{A} n \rangle - \langle R^h E^h n, \tilde{A} n \rangle
\]
\[
- h \sum_{i=1}^{2} \frac{\lambda_i}{h_0 + sh \lambda_i} \langle s E^h \tau_i, \tilde{A} \tau_i \rangle
\]

since \( \tilde{A}^T = -\tilde{A} \) and \( \langle \tilde{A} n, n \rangle = 0 \). Using (iv) and (iii) in Lemma 3.3, (58), and (12), we obtain
\[
\lim_{h \to 0} \frac{1}{h} \int_{T, S, h_0} \sum_{i=1}^{2} \left( E^h \tau_i, \tilde{A} F^{-1} \left( \frac{sh}{h_0} \right) \tau_i \right) \det F \left( \frac{sh}{h_0} \right) ds dx dt
\]
\[
= \frac{1}{12} \int_{0}^{T} \int_{S} \mathcal{L}_2(x, (\nabla (\tilde{A} n) - A \tilde{A})_{\tan}) : [\nabla (\tilde{A} n) - A \tilde{A}]_{\tan} ds dx dt. \quad (60)
\]

Moreover, from (55), (iv) and (i) in Lemma 3.3 and (ii) in Lemma 3.2 including the compactness of \( (A^h) \) and the strong convergence of \( \text{sym } A^h \), we have
\[
\lim_{h \to 0} \int_{T, S, h_0} \sum_{i=1}^{2} \left( A^h E^h \tau_i, \tilde{A} F^{-1} \left( \frac{sh}{h_0} \right) \tau_i \right) \det F \left( \frac{sh}{h_0} \right) ds dx dt
\]
\[
= - \int_{0}^{T} \int_{S} \sum_{i=1}^{2} \langle E \tau_i, A \tilde{A} \tau_i \rangle dx dt = - \int_{0}^{T} \int_{S} \mathcal{E}_{\tan} (A \tilde{A})_{\tan} dx dt
\]
\[
= - \int_{0}^{T} \int_{S} \mathcal{L}_2\left(x, \left( B - \frac{\sqrt{K}}{2} A^2 \right)_{\tan} \right) : (A \tilde{A})_{\tan} dx dt. \quad (61)
\]

Finally, using (54), (60) and (61) in (59), we let \( h \to 0 \) to obtain (24).

References

[1] Friesecke G, James R D, and Müller S, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three dimesional elasticity, *Comm. Pure Appl. Math.*, 2002, 55: 1461–1506.
[2] Friesecke G, James R D, and Müller S, A hierarchy of plate models derived from nonlinear elasticity by gamma-convergence, *Arch. Ration. Mech. Anal.*, 2006, **180**: 183–236.

[3] Lewicka M, Mora M G, and Pakzad M R, A nonlinear theory for shells with slowly varying thickness, *C. R. Acad. Sci. Paris, Sér. I*, 2009, **347**: 211–216.

[4] Lewicka M, Mora M G, and Pakzad M R, Shell theory arising as low energy Γ-limit of 3d nonlinear elasticity, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 2010, **IX**: 253–295.

[5] Lewicka M, Mora M G, and Pakzad M R, The matching property of infinitesimal isometries on elliptic surfaces and elasticity of thin shells, *Arch. Ration. Mech. Anal.*, 2011, **200**: 1023–1050.

[6] Lewicka M and Pakzad M R, The infinite hierarchy of elastic shell models: Some recent results and a conjecture, *Infinite Dimensional Dynamical Systems, Fields Institute Communications*, Springer, New York, 2013, **64**: 407–420.

[7] Yao P F, Linear strain tensors on hyperbolic surfaces and asymptotic theories for thin shells, *SIAM J. Math. Anal.*, 2019, **51**: 1387–1435.

[8] Müller S and Pakzad M R, Convergence of equilibria of thin elastic plates-the von Kármán case, *Comm. Part. Differ. Equ.*, 2008, **33**: 1018–1032.

[9] Mora M G and Müller S, Convergence of equilibria of three-dimensional thin elastic beams, *Proc. Roy. Soc. Edinburgh Sect. A. Math.*, 2008, **138**: 873–896.

[10] Mora M G, Müller S, and Schultz M G, Convergence of equilibria of planar thin elastic beams, *Indiana Univ. Math. J.*, 2007, **56**: 2413–2438.

[11] Lewicka M, A note on convergence of low energy critical points of nonlinear elasticity functionals, for thin shells of arbitrary geometry, *ESAIM: COCV*, 2011, **17**: 493–505.

[12] Mora M G and Scardia L, Convergence of equilibria of thin elastic plates under physical growth conditions for the energy density, *J. Diff. Equ.*, 2012, **252**: 35–55.

[13] Müller S, Mathematical problems in thin elastic sheets: Scaling limits, packing, crumpling and singularities, *Vector-Valued Partial Differential Equations and Applications*, 125–193, LNM 2179, Springer, Cham., 2017.

[14] Abels H, Mora M G, and Müller S, Large time existence for thin vibrating plates, *Comm. Part. Differ. Equ.*, 2011, **36**: 2062–2102.

[15] Abels H, Mora M G, and Müller S, The time-dependent von Kármán plate equation as a limit of 3d nonlinear elasticity, *Calc. Var.*, 2011, **41**: 241–259.

[16] Lecumberry M and Müller S, Stability of slender bodies under compression and validity of the von Kármán theory, *Arch. Ration. Mech. Anal.*, 2009, **193**: 255–310.

[17] Simon J, Compact sets in the space $L^p(0, T; B)$, *Ann. Mat. Pura Appl.*, 1987, **146**: 65–96.

[18] Spivak M, *A Comprehensive Introduction to Differential Geometry*, vol V, Second Edition, Publish or Perish Inc. Australia, 1979.

[19] Taylor M, *Partial Differential Equations I: Basic Theory*, Second Edition, Springer, 2011.