Thermodynamic Black Di-Rings
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We show some physical properties of the systems of regular black di-ring (the regular gravitational systems with two $S^1$-rotational black rings arranged in a concentric way in five-dimensional asymptotically flat spacetimes). In particular the existence of isothermal systems of black di-ring is shown, in which both isothermality and isorotation between the inner black ring and the outer black ring are realized. We also give some properties of the thermodynamic black di-ring including discussion about thermodynamic stabilities of the system.

§1. Introduction

Since the two groundbreaking discoveries of Myers-Perry black holes (abbreviated to MP black holes)\textsuperscript{1) and $S^1$-rotating black rings by Emparan and Reall,\textsuperscript{2)} several nontrivial black hole systems have been obtained and used to clarify peculiar features of the higher dimensional gravity. In particular in five dimensions, solitonic solution-generation methods invented in the legendary era from the 1970s to the 1980s have been proved to be greatly powerful and applied to systematic reconstruction of known solutions including the MP black holes,\textsuperscript{3)} the $S^1$-rotating black rings\textsuperscript{5,6)} and also to generating of new solutions: $S^2$-rotating black rings\textsuperscript{7–9)} and doubly rotating black rings,\textsuperscript{10)} for example. As further trial, hunting new solutions has been attempted ambitiously to obtain novel black holes that have never been expected in four dimensions, like black hole systems with multi-horizons\textsuperscript{11–15)} or a topologically nontrivial horizon.\textsuperscript{16,17)}

The solutions of black di-ring among them correspond to concentric configurations composed of two independently $S^1$-rotating black rings. The authors first discovered the regular di-rings by using the solitonic method similar to the Bäcklund transformation (called di-ring I).\textsuperscript{12)} Successively, Evslin and Krishnan constructed another di-ring solution set (called di-ring II).\textsuperscript{13)} They used the inverse scattering method that was modified by Pomeransky to treat the higher dimensional case (hereafter abbreviated to PISM).\textsuperscript{3)} However, because of the complexity of their expressions further investigation of the physics of the di-ring systems was remaining to be done.

So first in this place, we briefly comment on the complete equivalence of these two different representations with the aid of the facts established by Hollands and Yazadjiev,\textsuperscript{4)} which concern the uniqueness of higher dimensional black holes. Then we shall show some properties of regular black di-rings, especially consider the thermodynamic regular black di-ring systems (the states in which both isothermality and
isorotation between the inner black ring and the outer black ring are realized). Actually one of the main purposes to study the higher dimensional gravity is to clarify the phase structure of thermal states of higher dimensional black objects and classify the properties into the universal ones independent of the dimensions of spacetimes and the peculiar ones depending on the number of dimensions. To accomplish this task the study of thermodynamic multi-horizon systems is clearly necessary because the multi-horizon systems appear naturally in higher dimensions. The thermodynamic black di-ring system may serve as the starting point for the study of thermodynamic multi-horizon systems.

For the systems of the black di-rings, we consider five dimensional spacetimes with three commuting Killing vector fields: a time-like Killing vector field and two axial-Killing vector fields. We assume further that one of the axial-Killing vector fields is orthogonal to the other. So the line-elements adopted here can be reduced to

\[
ds^2 = G_{tt}(dt)^2 + 2G_{t\psi}dt d\psi + G_{\psi\psi}(d\psi)^2 + G_{\phi\phi}(d\phi)^2 + e^{2\nu}(d\rho^2 + dz^2),
\]

(1.1)

where the metric coefficients are the functions of \((\rho, z)\) and \(\det G = -\rho^2\) is imposed. Based on this metric form, we can construct regular di-ring systems using either of the solitonic methods mentioned above. Here we omit the detail of the solution-generating methods, the expressions of the metric form and physical quantities of the di-ring solutions. It is noteworthy that the original representation of di-ring solutions which was first found by us (i.e., di-ring I) can be directly obtained by PISM.

In §2 we show how to confirm the equivalence of the two different solution sets of the black di-rings I and II. In §3 we first show the existence of regular thermodynamic black di-ring systems and next comment on some properties of the thermodynamic black di-rings. In §4 we give the discussion about thermodynamic stability (instability) of the systems.

§2. Equivalence of di-ring I and di-ring II

The strategy adopted here is the following. First, reconstructing the solution set of di-ring I with the PISM we show that the difference between the di-ring I and di-ring II comes from the difference of the corresponding seeds. Then we give the moduli-parameters and physical quantities to identify the solution sets respectively. Using these quantities we confirm the equivalence of these two solution sets with the aid of some mathematical facts. Once the equivalence is established, we can use the more convenient representation among the di-ring I and di-ring II according to problems we face.

A key mathematical fact to establish the equivalence of the di-ring systems is in the work by Hollands and Yazadjiev, which has discussed the uniqueness of five dimensional stationary black holes with asymptotic flatness and axial \(U(1)^2\)-symmetry.\(^4\) Originally they considered the systems of a single black hole, but their proof can be applied to the systems of multiple black holes so that their theorem is still valid with some modification. From the mathematical fact deduced from their
Fig. 1. Rod-structures describing the seeds of the di-rings. The upper and lower rod-diagrams correspond to the di-ring I and di-ring II respectively. Black rods interpreted to have $1/2$ line mass density, while gray rods (holes) correspond to $-1/2$ line mass density. Two solitons are removed and recovered at the positions $a_1$ and $a_4$.

In works, we can infer that two different systems of black di-ring become isometric when all the lengths of rods and the Komar angular momenta coincide with each other. The Komar angular momenta can be replaced with other two independent physical quantities, though some auxiliary conditions are usually needed to resolve some residual discrete degeneracy. In the case of the di-rings, using the lengths of rods in place of the interval structures is enough for the mathematical fact to be valid, because the rod vectors that correspond to the axes are trivial. Furthermore, it should be noticed that the procedure of the proof seems to be independent of the existence of conical singularities on the axes. So the above statement is generalized to the case with conical singularities. For the case of black di-rings, the Komar angular momentum corresponding to the $\phi$-rotation is automatically zero so that another Komar angular momentum corresponding to the $\psi$-rotation is essential to determine the solution, once the rod-lengths are fixed.

The di-ring I and di-ring II are generated by PISM from the corresponding seeds respectively. The rod structures described in Fig. 1 show the seeds that are used to generate di-ring I and di-ring II. The parameters $(u, v)$ and $(p, q)$ inscribed on the Fig. 1 correspond to lengths of the gray rods (holes) and have the following range:

\begin{align}
0 &\leq u, \quad 0 \leq v \leq d_2, \\
0 &\leq p, \quad 0 \leq q \leq d_2,
\end{align}

respectively. Following the procedure of PISM, first two solitons with trivial BZ-parameters are removed at the positions $a_1$ and $a_4$ and then the solitons are added with nontrivial BZ-parameters at the same positions. After adjusting the BZ-parameters in the following way

\[
\left\{ b_{L}^{(I)} = \pm \left( \frac{2a_{21}a_{61}a_{71}}{a_{31}a_{51}} \right)^{1/2}, \quad b_{R}^{(I)} = \pm \left( \frac{2a_{42}a_{64}a_{74}}{a_{43}a_{54}} \right)^{1/2} \right\}
\]
\[
\begin{align*}
\left\{ b_L^{(II)} &= \pm \left( \frac{2a_{31}a_{61}a_{71}}{a_{21}a_{51}} \right)^{1/2}, \\
b_R^{(II)} &= \pm \left( \frac{2a_{43}a_{64}a_{74}}{a_{42}a_{54}} \right)^{1/2} \right\}, \\
\end{align*}
\] (2.4)

we obtain the regular di-ring systems respectively up to conical singularities. Here \(\{ b_L^{(I)}, b_R^{(I)} \} \) and \(\{ b_L^{(II)}, b_R^{(II)} \} \) are assigned to BZ-parameters of di-ring I and II respectively, where the subscripts ‘L’ and ‘R’ means ‘outer’ and ‘inner’ so that \(b_L\) and \(b_R\) correspond to the solitons at \(a_1\) and \(a_4\) respectively. The symbols \(a_{ij}\) is defined as \(a_i - a_j\) and also can be described with the rod lengths and hole lengths. So the sets of moduli-parameters \(\{ u, v, d_1, d_2, d_3, d_4 \}^I\) and \(\{ p, q, d_1, d_2, d_3, d_4 \}^II\) provide the solution sets of di-ring I and II, respectively. Furthermore we can say that other physical quantities of the di-rings can be represented with these parameters. The following expressions of ADM masses \(M^I\) and \(M^{II}\) and periodic angles \(\Delta\phi^{(I)}_R\) and \(\Delta\phi^{(II)}_R\) that are needed to keep regularity on the \(\phi\)-axis \(d_4\) are useful for the next study,

\[
M^I = \frac{3\pi}{4} \left( a_{65} + a_{41} \frac{b^{(I)}_R - b^{(I)}_L}{(b^{(I)}_R - b^{(I)}_L)^2} \right), \\
M^{II} = \frac{3\pi}{4} (a_{31} + a_{64}), \\
\left( \frac{\Delta\phi^{(I)}_R}{2\pi} \right)^2 = \frac{a_{73}a_{76}(a_{74}b^{(I)}_L - a_{71}b^{(I)}_R)^2}{a_{71}a_{72}a_{74}a_{75}(b^{(I)}_L - b^{(I)}_R)^2}, \\
\left( \frac{\Delta\phi^{(II)}_R}{2\pi} \right)^2 = \frac{a_{71}a_{74}a_{73}a_{76}}{a_{72}^2a_{75}^2},
\] (2.5)

(2.6)

(2.7)

(2.8)

where Eqs. (2.3) and (2.4) are used to describe the BZ-parameters with moduli-parameters.

Owing to the mathematical fact mentioned above, we can conclude that two di-rings are isometrically equivalent when the conditions \(d_i^{(I)} = d_i^{(II)} (i = 1, \cdots, 4)\), \(M^I = M^{II}\) and \(\Delta\phi^{(I)}_R = \Delta\phi^{(II)}_R\) are satisfied. From these conditions we can also extract a simple mapping function between \(u, v\) and \(p, q\) explicitly, \((p, q) = (p(u, v), q(u, v))\). Examining the mapping function with the aid of numerical methods we can confirm that the correspondence between \((u, v)\)-space and \((p, q)\)-space is onto and one-to-one if infinities on the spaces are included. Finally we can say that the systems of di-ring I and di-ring II are completely equivalent.

\section*{§3. Existence of regular thermodynamic black di-rings}

In this place we show the existence of thermodynamic black di-rings (i.e., systems of regular black di-ring in thermodynamic equilibrium) and give some peculiar properties of the thermodynamic black di-rings.\(^1\) In the rest the following normalized

\(^1\) According to Emparan’s comment at YKIS 2010, similar results were found by Emparan and Figueras.
quantities corresponding to ADM angular momentum, area of horizon, temperature of horizon and angular velocity of horizon are used:

\[ j^2 = \frac{27\pi}{32G M^3}, \quad a_h = \frac{3}{16}\sqrt{\frac{3}{\pi}} \frac{A_h}{(GM)^{3/2}}, \]

\[ \tau_h = \sqrt{\frac{32\pi}{3}(GM)^{1/2}T_h}, \quad \omega_h = \sqrt{\frac{8}{3\pi}(GM)^{1/2}\Omega_h}. \]

First we must impose the following balance conditions between gravitational attractions and centrifugal forces on the di-ring solution to eliminate conical singularities on the axes:

\[ \Delta\phi_L = 2\pi, \quad \Delta\phi_R = 2\pi. \]

Here \( \Delta\phi_L \) and \( \Delta\phi_R \) are the periodic angles to keep regularity on the outer and inner \( \phi \)-axes respectively as depicted in Fig. 2.

To ensure that a system of regular black di-ring becomes a thermodynamic equilibrium state, the temperatures \( (\tau_L, \tau_R) \) and angular velocities \( (\omega_L, \omega_R) \) of the outer (L) and inner (R) black rings of the system must satisfy the following condition,

\[ \tau_L = \tau_R, \quad \omega_L = \omega_R. \]

It should be noticed that in contrast with the case of black Saturn, construction of a thermodynamic black di-ring with two black rings is not so trivial as pointed out in Ref. 18). Actually black rings with the same temperature and angular velocity have the same shape and size,\(^{19}\) so that we must arrange two identical rings into a thermodynamic di-ring keeping the conditions (3·1) and (3·2). That is, it seems rather natural that thermodynamic di-rings do not appear in the phase space. If there remains any possibility of thermodynamic di-rings, some non-linearity may play a key role.

For convenience, we introduce the following moduli-parameters composed of the rods \( d_i \) \((i = 1 \sim 4)\)

\[ h_1 = d_1 + d_2 + d_3 + d_4 = (a_{72}), \quad h_3 = d_3 + d_4 = (a_{75}), \]
\[ h_2 = d_2 + d_3 + d_4 = (a_{73}), \quad h_4 = d_4 = (a_{76}). \]
Fortunately the conditions (3.1) and (3.2) can be nearly solved in an analytical way. In fact the four equations in (3.1) and (3.2) are reduced to

\[ 0 = \left( 4h_1^2 - 7h_3h_4 + 4h_4^2 \right) h_3^3 - \left( 8h_1^3 - 8h_1^2h_4 - h_1h_4^2 + 4h_4^3 \right) h_3^2 + (5h_4^4 - 2h_1^3h_4 - 2h_1^2h_4^2 + h_1h_4^3 + h_4^4) h_3 - h_1^5 \]  
for I and II, \hspace{1cm} (3.4)

\[ h_2 = h_1 - h_3 + h_4 \]  
and also

\[ u - v = \frac{(h_1 - h_3)(h_2^2 + 2h_1h_3 - h_3^2)}{h_3h_4}, \quad (u + h_1)(h_2 - v) = \frac{h_1^2h_2^2}{h_3h_4} \]  
for I \hspace{1cm} (3.5)

or

\[ p + q = \frac{(h_1 - h_2)(h_3^2 + 2h_1h_2 - h_2^2)}{h_2h_4}, \quad (p + h_1)(q + h_3) = \frac{h_1^2h_3^2}{h_2h_4} \]  
for II. \hspace{1cm} (3.6)

The procedure to determine the moduli-parameters using these equations is the following: (i) one of the parameters are fixed using the arbitrariness of global scaling, for example \( h_1 \) is fixed to 1; (ii) choose \( h_4 \) as a free parameter; (iii) several candidates of the pair \((h_2, h_3)\) are given by solving the equation (3.4) and then using the equation (3.5); (iv) select the pair satisfying \( h_1 = 1 > h_2 > h_3 > h_4 > 0 \); (v) from (3.6) or (3.7), we compute \((u, v)\) or \((p, q)\).

Following the above procedure we can easily show the existence of the thermodynamic black di-ring. In Fig. 3 the states of thermodynamic black di-ring appear as a continuous thick curve with a cusp. Other thermodynamic black objects are also shown for comparison. The behavior of thermodynamic di-ring is similar to other thermodynamic objects. The phase of the di-ring has a ‘fat ring’ branch and a
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Fig. 4. Total area $a_h$ vs squared angular momentum $j^2$. The behavior of the black objects in the thin branch is shown.

Fig. 5. Ratio of horizon area of the central black hole or inner black ring to that of the outer black ring ($R_h$). The dashed curve and thick curve are for black Saturn (BS) and black di-ring (BD) respectively.

‘thin ring’ branch. the state of the di-ring is less dominant than others with respect to entropy (i.e., area $a_h$) in both branches. In the fat branch, the curves of black ring, black Saturn and black di-ring approach that of MP black hole respectively as depicted in Fig. 3. More detailed behavior of the four black objects is shown in the upper right diagram magnified near the extremal point. As an interesting fact, each pair (MP black hole and black ring, black Saturn and black di-ring) has similar behavior near the extremal point. From the graphs in Fig. 4, in the thin branch the curve of black Saturn immediately asymptotes to that of black ring while the black di-ring acts independently. Actually we can confirm that as $j^2$ increases infinitely, the area of black Saturn approaches that of black ring, while that of black di-ring approaches a half of black ring. Figure 5 also contrasts the black Saturn and black di-ring clearly, which shows the behavior of the ratio of the horizon area of inner black ring or central black hole to the horizon area of outer black ring ($R_h$). The ratio of the di-ring holds a constant value, unity, while that of the black Saturn quickly vanishes as the angular momentum $j^2$ increases. That is, the inner ring and outer ring of the black di-ring always have the same entropy. This fact can be analytically confirmed from the reduced constraint (3.5) and the isothermality.

§4. Thermal stability (instability) of thermodynamic black di-rings

Finally, as one of important properties of thermodynamic di-rings we discuss whether thermodynamic local stability is realized or not. We follow the approach that was introduced by Evslin and Krishnan to treat the case of black Saturn. They found the existence of meta-stable states of black Saturn. To do this, under the condition of fixed mass and angular momentum we shall search for local maxima of the corresponding entropy function, which is a function of appropriate moduli-parameters. If we find that some maximal eigenvalues of Hessian of the entropy
function become negative, we can say that meta-stability occurs at this point. The results of the survey for the black di-ring and black Saturn are shown in Figs. 6 and 7, respectively. Any stable parameter region does not appear in the di-ring case, while the window of meta-stability opens in the black Saturn case as pointed out in Ref. 20).

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