Quantum-phase synchronization

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We study mechanisms that allow one to synchronize the quantum phase of two qubits relative to a fixed basis. Starting from one qubit in a fixed reference state and the other in an unknown state, we find that contrary to the impossibility of perfect quantum cloning, the quantum-phase can be synchronized perfectly through a joined unitary operation. When both qubits are initially in a pure unknown state, perfect quantum-phase synchronization through unitary operations becomes impossible. In this situation we determine the maximum average quantum-phase synchronization fidelity, the distribution of relative phases and fidelities, and identify optimal quantum circuits that achieve this maximum fidelity. A subset of these optimal quantum circuits enable perfect quantum-phase synchronization for a class of unknown initial states restricted to the equatorial plane of the Bloch sphere.

Introduction. The quantum mechanical phase marks arguably the most profound deviation of quantum mechanics from classical mechanics. It is at the base of all quantum mechanical interference as displayed e.g. in the double slit experiment, and, in the case of multi-particle systems, enhanced correlations compared to the classical world, as described by quantum mechanical entanglement. Some of the most spectacular quantum mechanical effects occur when phase coherence is established over a macroscopic number of constituents, as is the case e.g. for superconductivity [1], superfluidity, Bose-Einstein condensates [2], quantum magnets [3], or laser [4]. While the mechanisms that lead to synchronized quantum phases are well understood in these examples, one may ask what are the mechanisms in general that allow one to synchronize the quantum phases of different systems. Having an answer to that question might enable new types of macroscopic quantum effects that we are not aware of yet. In this paper we study quantum-phase synchronization (QPS) for the simplest possible example, namely two qubits and unitary propagation.

We emphasize that the effects sought here are very different from those in the field of quantum mechanics of systems that classically synchronize, also called quantum (stochastic) synchronization [5–7]. In those systems, one considers the periodic dynamics of (typically driven) oscillators with slightly different frequencies which under slight interaction give rise to a common dynamical mode in which all oscillators synchronize, and research has mainly examined the question to what extent quantum fluctuations affect that synchronization when the oscillators become microscopic. QPS, on the contrary, has no classical analog, as it concerns the quantum phase which is only defined in the quantum world. Recently, synchronization of an ensemble of interacting dipoles modeled as qubits was studied in [8]. However, a fixed dipole interaction was considered, whereas here we are interested in finding the SU(4) joint-evolution that leads to QPS. QPS is related to quantum cloning [9,10], and it has been proposed [11] and experimentally demonstrated [12] that quantum cloning can amplify entanglement to a macroscopic level. However, there are two crucial differences between QPS and quantum cloning: i.) We want to synchronize only the quantum phase of the state, not the full state itself. This implies a different target function (see below). ii.) While cloning aims at attaining concurrence of each output with the input state, QPS solely intends to achieve concurrence among the outputs but not with an initial state.

An unknown quantum state cannot be cloned perfectly [13,14]. This remains true even when restricting the set of input states, e.g. to states in the equatorial plane as in phase-covariant cloning (PCC) [15,16] or its generalizations [17,18]. But here we show that quantum phases can be perfectly synchronized in the standard situation of quantum cloning, where one has one qubit in a known initial (blank) pure state, and the qubit to be copied in an unknown state. We then ask for more and consider both initial states as unknown. We show that perfect QPS is not possible anymore in this situation, but find quantum circuits that achieve maximal average QPS fidelity.

Phase synchronization fidelity. A general pure state of a qubit (spin-1/2) can be written in a fixed computational basis \(\{|0\rangle, |1\rangle\}\) as

\[
|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi} \sin(\theta/2)|1\rangle,
\]

where \(\varphi \in [0, 2\pi]\) is the quantum phase of the state, i.e. the relative phase between the two basis states, and \(\theta \in [0, \pi]\) defines the relative weight of the two basis states. In the corresponding density matrix, \(\rho = (1 + \mathbf{n} \cdot \sigma)/2\), \(\varphi\) is coded in the azimuthal angle of the Bloch vector \(\mathbf{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \equiv (n_x, n_y, n_z)\), and \(\sigma = (\sigma_x, \sigma_y, \sigma_z)\) is the vector of Pauli matrices. For mixed states, we still consider \(\varphi\) as given by the Bloch vector the quantum phase of the state, as \(\varphi\) still determines the oscillatory behavior of expectation values, e.g. when \(\varphi\) evolves linearly with time and one measures \(\langle \sigma_x \rangle\). Only the contrast of the oscillations is reduced due to the admixture of the identity. Thus, two states have the same quantum mechanical phase if the \(xy\) components of their Bloch vectors are aligned. We therefore define quantum-phase fidelity between two states with...
where $||.||$ denotes the standard vector norm, $m_i$ the projection of the Bloch vector of qubit $i$ into the $xy$ plane ($m_i = (n_x, n_y)$), and $\Delta \varphi = \varphi_2 - \varphi_1$. If $||m_i|| = 0$, the phase of qubit $i$, and therefore also the relative phase and $F(p_1, p_2)$ are undefined in this basis. Consider a general linear quantum channel $\Phi$ on the 2 qubits, i.e., a completely positive map $\mathcal{M}_2(\mathbb{C}) \rightarrow \mathcal{M}_2(\mathbb{C})$ that maps density matrices to density matrices. Starting from an initial state $\rho = \rho_1 \otimes \rho_2$, we obtain a final state $\rho' = \Phi \rho$ and reduced states $\rho'_1 = \text{tr}_2 \rho'$, $\rho'_2 = \text{tr}_1 \rho'$. We define the phase synchronization fidelity (PSF) as

$$ F(\rho, \Phi) \equiv f(\rho'_1, \rho'_2). $$

With this definition, $F(\rho, \Phi) \in [-1, 1]$, and $F = 1$ ($F = -1$) corresponds to perfect synchronization (perfect anti-synchronization) of the quantum-phase for initial state $\rho$.

**Definition 1.** Quantum phase synchronization is said to be perfect for a two-qubit quantum channel $\Phi$ and a set of initial states $\mathcal{A}$, if $\forall \rho \in \mathcal{A}$ for which $F(\rho, \Phi)$ is defined, $F(\rho, \Phi) = 1$.

When allowing arbitrary channels, perfect QPS, i.e., $F(\rho, \Phi) = 1 \forall \rho$, can be achieved trivially by resetting both qubits to the same state. Therefore, optimizing QPS over arbitrary quantum channels is not very interesting. Another example, how non-unitary channels can achieve perfect QPS is the well-known optimal cloning machine of Bužek and Hillery [30]. It leads to perfect phase synchronization as both final reduced states are identical. Similarly, one easily sees that LOCC operations allow synchronized resetting of the two states. To avoid such trivial constructions, we therefore restrict ourselves to unitary channels $\Phi_U : \rho \rightarrow U^\dagger \rho U$, and write $F(\rho, U) \equiv F(\rho, \Phi_U)$. Note that unitary operations are also important from a practical perspective, when one tries to keep quantum coherence as long as possible (including, e.g., through error correction).

**One qubit in a known state.** First consider the standard initial state for quantum state cloning: $\rho = |\psi_1 \rangle \langle \psi_1 | \otimes |0 \rangle \langle 0 |$, i.e., the first qubit is in an unknown pure state $\rho_{1,p} \equiv |\psi_1 \rangle \langle \psi_1 |$, and the second in a known (blank) state $|0 \rangle$. No linear transformation exists that transforms $\rho$ such that at the output both qubits are in state $|\psi_1 \rangle$. However, one easily shows that perfect QPS can be achieved, $F(\rho, U) = 1$, for all $\rho$ of the above form. We use the following little lemma:

**Lemma 1.** Let $V_1, V_2$ be arbitrary single-qubit unitaries acting on an arbitrary (possibly entangled) two qubit state, $V_1 = R_{n_1}(\alpha)$, $V_2 = R_{m_2}(\beta)$ and $\rho_{n_1} \equiv e^{-i 2 \alpha \sigma_z}$. Then for an arbitrary density matrix $\rho \in \mathcal{D}_2$ (positive-semidefinite Hermitian matrices with trace one) and a local transformation $V_1 \otimes V_2$, the Bloch vectors of the reduced density matrices corresponding to the propagated state

$$ \rho' = V_1 \otimes V_2 \rho (V_1 \otimes V_2) \dagger $$

are given by the rotated initial Bloch vectors,

$$ n'_1 = \tilde{R}_{n_1}(\alpha) n_1, \quad n'_2 = \tilde{R}_{m_2}(\beta) n_2, $$

**Proof.** We first show that the two partial traces that emerge from taking the partial state and calculating the Bloch vector can be combined to a trace over the whole system. For an arbitrary $\rho \in \mathcal{D}_4$ we find in the computational basis $\{|0_1 \rangle, |1_1 \rangle \}$ for the first qubit and $\{|0_2 \rangle, |1_2 \rangle \}$ for the second qubit

$$ \text{tr}(\sigma \otimes 1 \rho) = \sum_{i_1, j_2 = 0}^{1} \langle i_1 | \langle j_2 | \sigma \otimes 1 | i_1 \rangle | j_2 \rangle $$

$$ = \sum_{i_1 = 0}^{1} \langle i_1 | \langle j_2 | \sigma \otimes \sum_{k_2 = 0}^{1} | k_2 \rangle \langle k_2 | \rho | j_2 \rangle | i_1 \rangle $$

$$ = \sum_{i_1 = 0}^{1} \langle i_1 | \sigma \sum_{k_2 = 0}^{1} \langle k_2 | \rho | k_2 \rangle | i_1 \rangle $$

$$ = \text{tr}_1(\sigma \text{tr}_2(\rho)). $$

This is used in the following, where we prove the lemma for the first Bloch vector,

$$ n'_1 = \text{tr}_1(\sigma \text{tr}_2(R_{n_1}(\alpha) \otimes R_{m_2}(\beta) \rho (R_{n_1}(\alpha) \otimes R_{m_2}(\beta)) \dagger)) $$

$$ = \text{tr}(\sigma \otimes 1 R_{n_1}(\alpha) \otimes R_{m_2}(\beta) \rho (R_{n_1}(\alpha) \otimes R_{m_2}(\beta)) \dagger) $$

$$ = \text{tr}((R_{n_1}(\alpha) \otimes R_{m_2}(\beta))^\dagger \sigma \otimes 1 R_{n_1}(\alpha) \otimes R_{m_2}(\beta) \rho) $$

$$ = \text{tr}_1((R_{n_1}(\alpha) \sigma R_{n_1}(\alpha)) \otimes 1 \rho) $$

$$ = \text{tr}_1((R_{n_1}(\alpha) \sigma R_{n_1}(\alpha)) \text{tr}_2(\rho)) $$

$$ = \text{tr}_1(\sigma R_{n_1}(\alpha) \text{tr}_2(\rho) R_{n_1}^\dagger(\alpha)) $$

$$ = \tilde{R}_{n_1}(\alpha) n_1. $$

In lines [12] and [15] $\text{tr}_1(\sigma \text{tr}_2(\rho)) = \text{tr}(\sigma \otimes 1 \rho)$ is used, that was shown above. In lines [13] and [16] the cyclic property of the trace was used. Line [17] uses the rotation operator’s property

$$ n'_1 = \text{tr}(\sigma R_{n_1}(\alpha) \rho R_{n_1}^\dagger(\alpha)) = \tilde{R}_{n_1}(\alpha) n_1. $$

The proof for the second Bloch vector can be done in analogous fashion.

**Note:** Actually, this Lemma also holds for $n$ qubits, i.e., for arbitrary $\rho \in \mathcal{D}_{2^n}$, where we find

$$ \text{tr}(1_{2^{2l-1}} \otimes \sigma_l \otimes 1_{2^n - l} \rho) = \text{tr}(\sigma_l \text{tr}_{1,...,l-1,l+1,...,n}(\rho)). $$

(19)
where the trace operator’s subscripts indicate the qubits that are traced out. Then, the proof goes in a manner analogue to the lemma, and one finds for the $l$th Bloch vector

$$n'_l = \text{tr}_l(\sigma_l R_n(\alpha) \text{tr}_{1,l-1,l+1,n}(\rho) R_n^l(\alpha)),$$  \hspace{1cm} (20)

which, according to the definition of the rotation operator, implies that we may simply rotate the $l$th Bloch vector, $n'_l = R_n(\alpha)n_l$.

Now consider a propagation with a controlled-NOT (CNOT) operation with qubit 1 as control and qubit 2 as target \[31\]. We denote by $Cij$ a CNOT operation with $i$ as the controlling and $j$ as the controlled qubit, hence $U = C_{12}$. Then one easily obtains $n'_1 = n'_2 = \cos \theta_1 e_z$. Thus, the initial phase of qubit 1 is erased, but both Bloch vectors are aligned to the $z$-axis and identical to each other. Now act with an arbitrary local transformation that rotates the Bloch vectors away from the $z$-axis, $W \equiv R_n(\alpha) \otimes R_n(\alpha)$. Apart from the case where after $C_{12}$ both qubits ended up in the maximally mixed state, i.e. for $\theta_1 = \pi$, the PSF equals one, $F(\rho_{1,p} \otimes |00\rangle, W_{\text{sync}} C_{12}) = 1$. After the operation $W_{\text{sync}} C_{12}$, the two qubits are perfectly phase synchronized. Thus, since there is nothing inherently irreversible (different initial phases are mapped to the same final phase), tracing out a qubit introduces enough irreversibility to the unitary evolution for perfect QPS to be possible if one qubit is initially known. There is no contradiction to the result from PCC \[17\] that restriction to equatorial input states for $1 \rightarrow 2$ cloning yields a maximum fidelity of $1/2 + \sqrt{1/8} \approx 0.854$, as the fidelity maximized there is the overlap between initial and final states, not PSF. Our result can be extended to an initially mixed state of qubit 1. Furthermore, perfect QPS is easily extended from one to $n - 1$ qubits in blank states:

For initial states of the form $\rho = \rho_1 \otimes |00\rangle \langle 00| \otimes \cdots \otimes |00\rangle_n \langle 00| = \rho_1 \otimes |00\rangle \langle 00| \cdots \otimes |00\rangle_n \langle 00|$ where $\rho_1 = (1 - p)1/2 + p |\psi\rangle \langle \psi|$, with $0 \leq p \leq 1$ and an arbitrary pure state $|\psi\rangle$, see Eq. (1), the transformation $U = W_{\text{sync,n}} C_{12} \cdots C_{1n}$ achieves perfect QPS $f(\rho'_1, \rho'_2) = 1 \forall i \in \{2, \ldots, n\}$ as follows from direct calculation:

Applying $C \equiv C_{12} \cdots C_{1n}$ to the initial states one finds

$$\rho' = C \rho_1 \otimes |00\rangle \langle 00| C$$  \hspace{1cm} (21)

where $|\psi\rangle = \cos(\theta/2)|00\rangle + e^{i\varphi}\sin(\theta/2)|11\rangle$. Because Pauli matrices are traceless, calculation of the Bloch vectors gives zero for the first term in equation

$$\begin{align*}
\text{FIG. 1. (color online). Quantum circuit for the most general unitary transformation } U \in SU(4) \text{ on two qubits (see Fig.7 in [30]). The circuit } U_c \text{ is obtained by setting } V_1 = V_2 = W_1 = W_2 = I_2.
\end{align*}$$

For the second term we obtain identical Bloch vectors of all qubits in the final state,

$$n'_i = p \begin{pmatrix} 0 \\ 0 \\ \cos \theta_i \end{pmatrix}, i = 1, \ldots, n.$$  \hspace{1cm} (24)

$W_{\text{sync,n}} \equiv R_n(\alpha) \otimes \cdots \otimes R_n(\alpha)$ rotates all $n$ Bloch vectors away from the $z$-axis. This holds as Lemma 1 generalizes to $n$ qubits. Apart from the cases of maximally mixed states, $\theta_1 = \pi$ or $p = 0$, phases are defined and perfectly synchronized, $F(\rho_{1,p} \otimes |00\rangle, W_{\text{sync,n}} C) = 1$.

Both qubits in unknown states. We now attempt the more ambitious task of phase-synchronizing qubits that are both initially in unknown pure states, $\rho = \rho_{1,p} \otimes \rho_{2,p}$ with $\rho_{i,p} = |\psi_i\rangle \langle \psi_i|$, $i = 1, 2$, and $\psi_i$ of the form \[4\]. For this we look at the most general transformations $U$ of two qubits. The set of all unitaries $U \in SU(4)$ on two qubits can be broken down into CNOT operations and single qubit unitaries \[31,33\]. The parametrization of $SU(4)$ requires 15 real parameters. Khaneja et al. \[30\] as well as Kraus and Cirac \[37\], found a decomposition of an arbitrary $U \in SU(4)$ of the form

$$U = WVU^\dagger,$$  \hspace{1cm} (25)

where $V = V_1 \otimes V_2$, $W = W_1 \otimes W_2 \in SU(2) \times SU(2)$ are local unitary transformations exclusively acting on each qubit separately. $U_c$ is an element of the quotient space $SU(4)/SU(2) \times SU(2)$. Minimal circuits for $U_c$ were reported in \[38,40\]. We use the circuit from Vatan and Williams (theorem 5 in \[10\]) according to which a general unitary transformation can be written as

$$U = (W_1 \otimes W_2) U_c (V_1 \otimes V_2)$$  \hspace{1cm} (26)

where

$$U_c \equiv C_{21} (\mathbb{I}_2 \otimes R_\gamma(\beta)) C_{12} (R_\gamma(\gamma) \otimes R_\beta(\alpha)) C_{21}$$  \hspace{1cm} (27)

This leads to figure 1\[1\] for a general circuit for two qubits. We have three angles for $U_c$ and three angles for each local unitary, giving a total of 15 parameters. For all unitaries $U \in SU(4)$ we have the theorem:

**Theorem 1.** Perfect QPS by a unitary transformation is impossible for all initial pure product states of two qubits.

For the full proof we refer to appendix A. Here we give a short version that is valid if one neglects the sets of measure zero for which the PSF is undefined. The
proof proceeds by contradiction. Suppose there exists a
U ∈ SU(4) that perfectly quantum-phase synchronizes
all initial pure product states of two qubits. Consider
the two initial states with Bloch vectors defined by
θ_1 = 0, θ_2 = 0 for the first state and θ_1 = π, θ_2 = 0
for the second state. The Bloch vectors after the entangle-
gate U_c are $\mathbf{n}_1' = \cos(\alpha + \beta) \mathbf{e}_x, \mathbf{n}_2' = \cos(\alpha + \beta) \mathbf{e}_z$ for the
first state and $\mathbf{n}_1' = \cos(\alpha - \beta) \mathbf{e}_x, \mathbf{n}_2' = -\cos(\alpha - \beta) \mathbf{e}_z$
for the second. For a well defined PSF, i.e. non-
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z- and y-rotations, \( R_z(\sigma_i)R_y(\nu_i)R_z(\mu_i) \), we can restrict ourselves to one of the final z-rotations \( R_z(\sigma_i) \) without loss of generality as the PSF only measures the relative phase. We define the general transformation as

\[
U_g(\alpha, \beta, \gamma, \mu_1, \mu_2, \nu_1, \nu_2, \sigma_1) \equiv (R_z(\sigma_1)R_y(\nu_1)R_z(\mu_1)) \otimes (R_y(\nu_2)R_z(\mu_2)) U_c \text{ and look numerically for angles } \alpha, \beta, \gamma, \mu_1, \mu_2, \nu_1, \nu_2, \sigma_1 \text{ that maximize the mean PSF.}
\]

Without loss of generality, we can restrict all angles to \( \alpha, \beta, \gamma, \mu, \nu \) being \( x \)- and \( y \)-rotations, for initial pure product states.

The distribution \( \text{Fig. 2} \) shows the optimal quantum circuit. The distribution \( \text{Fig. 3} \) gives the relative phase \( \Delta \varphi \) after the optimal quantum circuit \( U_{max} \) (radial coordinate, arbitrary units) as function of \( \Delta \varphi \in (-\pi, \pi) \) (azimuthal coordinate), i.e. the phase of the second Bloch vector measured relative to the first one, with \( \Delta \varphi = 0 \) corresponding to the \( x \)-axis (left panel). Corresponding distribution \( P(F) \) of the PSF, \( F = \cos(\Delta \varphi) \) (right panel). Both distributions are generated numerically.

\[
\text{Fig. 2. Quantum circuit } U_{max} \text{ that maximizes the mean PSF for initial pure product states.}
\]

\[
\text{Fig. 3. (color online). Distribution of the relative phase } P(\Delta \varphi) \text{ after the optimal quantum circuit } U_{max} \text{ (radial coordinate, arbitrary units) as function of } \Delta \varphi \in (-\pi, \pi) \text{ (azimuthal coordinate), i.e. the phase of the second Bloch vector measured relative to the first one, with } \Delta \varphi = 0 \text{ corresponding to the } x- \text{axis (left panel). Corresponding distribution } P(F) \text{ of the PSF, } F = \cos(\Delta \varphi) \text{ (right panel). Both distributions are generated numerically.}
\]

with \( \theta_1 = \theta_2 = \pi/2 \) and thus \( n_i = r_i(\cos \varphi_i, \sin \varphi_i, 0) \) where \( r_1 \) is the purity, \( i \in \{1,2\} \). These states are important in many applications, e.g., linearly polarized photons \([11]\) or the BB84 protocol \([12]\). The transformation \( U_{max} \) leads to the transformed Bloch vectors

\[
\mathbf{n}_1' = \frac{1}{2} \begin{pmatrix} r_1 \cos \varphi_1 - r_2 \sin \varphi_2 \\ -r_1 \sin \varphi_1 - r_2 \cos \varphi_2 \\ -r_1 r_2 \cos(\varphi_1 - \varphi_2) \end{pmatrix}, \\
\mathbf{n}_2' = \frac{1}{2} \begin{pmatrix} r_1 \cos \varphi_1 - r_2 \sin \varphi_2 \\ -r_1 \sin \varphi_1 - r_2 \cos \varphi_2 \\ r_1 r_2 \cos(\varphi_1 - \varphi_2) \end{pmatrix}.
\]

The resulting z-components are opposite, while the reduced Bloch vectors are perfectly synchronized,

\[
F \left( \rho_1(\theta_1 = \pi/2) \otimes \rho_2(\theta_2 = \pi/2), U_{max} \right) = 1,
\]

provided that they are well defined. This means that \( U_{max} \) achieves perfect QPS for the subset of equatorial initial states. The same is true for all transformations satisfying conditions \([11]\) and \([12]\). One may also wonder about the nature of the final two-qubit state created by \( U_{max} \) and in particular its entanglement. It turns out that the concurrence \([11]\) of the final state for initial pure equatorial states \( (r_i = 1) \) is \( C = \frac{1+\sin(\Delta \varphi)}{2} \). Thus, \( U_{max} \) directly encodes the initial relative phase in the final concurrence, such that \( C = 1/2 \) corresponds to \( \Delta \varphi \in \{0,\pi\} \) and deviations from \( C = 1/2 \) are proportional to \( \sin(\Delta \varphi) \). This is by itself an interesting property, with possible applications in quantum information theory. At the same time it implies that the final entanglement is irrelevant for perfect QPS.

To summarize, we have introduced the concept of quantum-phase synchronization at the example of two qubits. We have shown that in contrast to quantum cloning, perfect quantum-phase synchronization of one
qubit in an unknown state with \( n - 1 \) qubits in known fixed reference states is possible through joint unitary evolution. For the case of two qubits both initially in unknown states, perfect QPS for all initial states becomes impossible through unitary evolution. We have found quantum circuits that optimize the mean PSF (averaged over all pure initial product states), and the distribution of fidelities and final phase differences for one of the optimal quantum circuits. A discrete subset of the optimal quantum circuits can perfectly quantum-phase synchronize equatorial initial product states. Our work opens the road to investigations of quantum-phase synchronization for larger systems and may find interesting applications in quantum information processing. In particular it would be intriguing to see if phase synchronization can be achieved over distance, and explore applications in quantum key distribution.

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Appendix A: Proof of theorem 1

The set of initial angles parametrizing initial pure product states is restricted to \( \Omega \), i.e. \( 0 \leq \theta_i \leq \pi, 0 \leq \varphi_i \leq 2\pi \) with \( i \in \{1, 2\} \), and we simply use the notation \( \theta_i, \varphi_i, \varphi_i, \varphi_i \in \Omega \). Eight angles \( \sigma = (\alpha, \beta, \gamma, \mu_i, \nu_i, \sigma_1) \in \mathbb{R}^8 \) parametrize the unitary transformation \( U(\sigma) \). It follows a proof by contradiction.

Proof. Let \( U(\sigma_p), \sigma_p \in \mathbb{R}^8 \), be the unitary transformation that achieves \( F(\rho(\theta_i, \varphi_i), U(\sigma_p)) = 1 \) \( \forall \theta_i, \varphi_i \in \Omega \) for which \( \{|m_i(\theta_i, \varphi_i, \sigma)| \neq 0 \land \{|m_i(\theta_i, \varphi_i, \sigma)| = 0 \} \). This means that \( \forall \theta_i, \varphi_i \in \Omega \) the PSF may either be undefined, i.e. \( \{|m_i(\theta_i, \varphi_i, \sigma)| = 0 \land \{|m_i(\theta_i, \varphi_i, \sigma)| \neq 0 \} \), or well defined with \( F = 1 \). Let us consider discrete subsets \( \Omega_j \subset \Omega \) for \( \theta_i, \varphi_i \) and let \( \Sigma_j \) be the set of angles parametrizing the unitary transformation that achieves perfect QPS for \( \theta_i, \varphi_i \in \Omega_j \). Then it follows by assumption that \( \sigma_p \in \cap_j \Sigma_j \). In the following we find necessary conditions specifying different \( \Sigma_j, j = I, II...VI \), and obtain a contradiction by showing that conditions from different \( \Sigma_j \) are incompatible, i.e. \( \cap_j \Sigma_j = \emptyset \).

To find such conditions we do not consider Bloch vectors after the whole transformation \( U(\sigma) \), but we consider Bloch vectors after the entangling part \( U_c \) of \( U \), see equation (11) in the main text. It is worth recalling that according to lemma 1 final local transformations can be taken into account by directly rotating the Bloch vectors. As final local transformations are decomposed in \( z \)- and \( y \)-rotations, \( (R_z(\sigma_1)R_y(\nu_1)R_z(\mu_1)) \otimes (R_y(\nu_2)R_z(\mu_2)) \), their effect on the Bloch vectors can be taken easily into account.

Note, that oppositely directed Bloch vectors differ in their phase by \( \pi \), or their phases are undefined. Aligned Bloch vectors have the same phase, given it is well defined. Remarkably, this does not change after synchronous rotations. Thereby, two initial states leading to opposite Bloch vectors for qubit one and aligned Bloch vectors for qubit two (or vice versa) are particularly useful: \( F = 1 \) is impossible as first Bloch vectors exhibit identical rotations as well as second Bloch vectors. Thus, at least one of the initial states has to lead to an undefined PSF.

To simplify notation we define for an arbitrary angle \( \delta \) the corresponding set \( S_\delta \equiv \{\delta + n\pi | n \in \mathbb{Z}\} \) that contains all angles modulo \( \pi \). Further, we use \( S \equiv S_0 \cup S_{\pi/2} \).

Bloch vectors after \( U_c(\alpha, \beta, \gamma) \) are given by

\[
\mathbf{n}_1 = \begin{pmatrix}
\cos \gamma (\cos \alpha \sin \theta_2 \cos \varphi_2 + \sin \alpha \sin \theta_1 \cos \theta_2 \cos \varphi_1) - \sin \gamma (\cos \alpha \cos \theta_1 \sin \theta_2 \sin \varphi_2 + \sin \alpha \sin \theta_1 \sin \varphi_2) \\
\sin \gamma (\cos \beta \cos \theta_1 \sin \theta_2 \cos \varphi_2 - \sin \beta \sin \theta_1 \cos \varphi_1) + \cos \gamma (\cos \beta \sin \theta_2 \sin \varphi_2 - \sin \beta \sin \theta_1 \cos \varphi_1) \\
\cos \alpha (\cos \beta \cos \theta_2 + \sin \beta \sin \theta_1 \sin \theta_2 \sin \varphi_2) - \sin \alpha (\cos \beta \sin \theta_1 \sin \theta_2 \cos \varphi_2 + \sin \beta \cos \theta_2) \\
\end{pmatrix},
\]

\[
\mathbf{n}_2 = \begin{pmatrix}
\cos \gamma (\cos \alpha \sin \theta_1 \cos \varphi_1 + \sin \alpha \sin \theta_2 \cos \varphi_2) - \sin \gamma (\cos \alpha \cos \theta_1 \sin \theta_2 \sin \varphi_2 + \sin \alpha \sin \theta_1 \sin \varphi_2) \\
\sin \gamma (\cos \beta \cos \theta_1 \sin \theta_2 \cos \varphi_2 - \sin \beta \sin \theta_1 \cos \varphi_1) + \cos \gamma (\cos \beta \sin \theta_2 \sin \varphi_2 - \sin \beta \sin \theta_1 \cos \varphi_1) \\
\cos \alpha (\cos \beta \cos \theta_2 + \sin \beta \sin \theta_1 \sin \theta_2 \sin \varphi_2) - \sin \alpha (\cos \beta \sin \theta_1 \sin \theta_2 \cos \varphi_2 + \sin \beta \cos \theta_2) \\
\end{pmatrix}.
\]
I)

Let $\Omega_i$ consist of two sets of initial angles $\theta_1 = \theta_2 = 0$ for state 1 and $\theta_1 = \pi, \theta_2 = 0$ for state 2 that (after the transformation $U_c(\alpha, \beta, \gamma)$) lead to Bloch vectors

| State 1: | State 2: |
|---------|---------|
| $\theta_1 = \theta_2 = 0: $ | $\theta_1 = \pi, \theta_2 = 0: $ |
| $n_1 = (0, 0, \cos(\alpha + \beta))$ | $n_1 = (0, 0, \cos(\alpha - \beta))$ |
| $n_2 = (0, 0, \cos(\alpha + \beta))$ | $n_2 = (0, 0, -\cos(\alpha - \beta))$ |

To derive conditions specifying $\Sigma_i$ it is appropriate to treat cases of vanishing Bloch vector components separately. For $\alpha + \beta \in S_{\pi/2}$ and $\alpha - \beta \notin S_{\pi/2}$, final Bloch vectors of state 1 are zero, corresponding to maximally mixed sub-states of qubit one and two. This leads independently from local transformations to an undefined PSF for state 1, while Bloch vectors of state 2 have non-vanishing, oppositely directed $z$-components. Undefined PSF is obtained when at least one Bloch vector remains on the $z$-axis. This allows arbitrary $z$-rotations ($\mu_i, \sigma_1$) while $\nu_1 \in S_0$ or $\nu_2 \in S_0$ for $y$-rotations. $F = 1$ requires to rotate Bloch vectors away from the $z$-axis by $\nu_1 \notin S_0$ and $\nu_2 \notin S_0$, ensuring that PSF is well defined. Then, after $y$-rotations, Bloch vectors of state 2 lie in the $x$-$z$-plane and, thus, having synchronized or anti-synchronized phases. Already synchronized phases require $\sigma_1 = 0$ modulo $2\pi$ to not destroy phase synchronization, while anti-synchronized phases require $\sigma_1 = \pi$ modulo $2\pi$. Altogether, this restricts $\sigma_1$ to $S_0$.

For $\alpha - \beta \in S_{\pi/2}$ and $\alpha + \beta \notin S_{\pi/2}$ Bloch vectors of state 2 are zero while Bloch vectors of state 1 have non-vanishing aligned $z$-components. Bloch vectors of state 1 lead to an undefined PSF if $\nu_1 \in S_0$ or $\nu_2 \in S_0$ while $\sigma_1 \in S_0$ is required to obtain $F = 1$ similar to above.

For $\alpha + \beta \in S_{\pi/2}$ and $\alpha - \beta \notin S_{\pi/2}$ Bloch vectors of both states are zero, which does not imply further conditions on $\nu_i, \mu_i$ or $\sigma_1$.

For $\alpha + \beta \notin S_{\pi/2}$ and $\alpha - \beta \notin S_{\pi/2}$ both states have Bloch vectors on the $z$-axis. Either first Bloch vectors are aligned and second opposite or vice versa. Thus, $F = 1$ is impossible for both states. Undefined PSF for at least one state is obtained from $\nu_1 \in S_0$ or $\nu_2 \in S_0$. This actually makes PSF undefined for both states.

In the following we summarize conditions in tables as the following I]. Different rows represent different cases while rows correspond to angles for which exist conditions. Note that for $\sigma_p \in \Sigma_1$ only one of the cases labeled by i)1., i)2., ...iv)2. in table II has to be fulfilled. The first part of labeling, i.e. i),ii)..., corresponds to the first column ($\alpha + \beta$ in table II), counting up if conditions for that angle change, while the second part of the labeling, 1..2......, simply counts the cases for each i),ii),... . $X$ refers to the complement of the set $X$. Blank table entries indicate that there is no condition for the corresponding angle and case.

### TABLE I.
Summarizing all possible cases with conditions on $\Sigma_1$. The first column labels the cases while other columns specify angles that are restricted to sets which are written in the rows (different cases).

| I | $\alpha + \beta$ | $\alpha - \beta$ | $\nu_1$ | $\nu_2$ | $\sigma_1$ |
|---|------------------|------------------|--------|--------|-----------|
| i) 1. | $S_{\pi/2}$ | $\overline{S}_{\pi/2}$ | $S_0$ |  |
| 2. | " | " | $S_0$ |  |
| 3. | " | " | $S_0$ |  |
| ii) 1. | $S_{\pi/2}$ | $S_{\pi/2}$ | $S_0$ |  |
| 2. | " | " | $S_0$ |  |
| 3. | " | " | $S_0$ |  |
| iii) | $S_{\pi/2}$ | $S_{\pi/2}$ |  |
| iv) 1. | $\overline{S}_{\pi/2}$ | $\overline{S}_{\pi/2}$ | $S_0$ |  |
| 2. | " | " | $S_0$ |  |

II)

Let $\Omega_{\Pi}$ consist of two sets of initial angles $\theta_1 = \theta_2 = \pi/2, \varphi_1 = 0, \varphi_2 = \pi/2$ for state 3 and $\theta_1 = \theta_2 = \pi/2, \varphi_1 = \pi, \varphi_2 = \pi/2$ for state 4 that lead to Bloch vectors
state 3: $\theta_1 = \theta_2 = \pi/2$, $\varphi_1 = 0$, $\varphi_2 = \pi/2$:
$n_1 = (0, \cos(\beta + \gamma), 0)$
$n_2 = (\cos(\beta + \gamma), 0, 0)$

state 4: $\theta_1 = \theta_2 = \pi/2$, $\varphi_1 = \pi$, $\varphi_2 = \pi/2$:
$n_1 = (0, \cos(\beta - \gamma), 0)$
$n_2 = (-\cos(\beta - \gamma), 0, 0)$

For $\beta + \gamma \in S_{\pi/2}$ and $\beta - \gamma \notin S_{\pi/2}$ as well as for $\beta + \gamma \notin S_{\pi/2}$ and $\beta - \gamma \in S_{\pi/2}$ we refrain from giving further conditions. For $\beta + \gamma \in S_{\pi/2}$ and $\beta - \gamma \in S_{\pi/2}$ both states have Bloch vectors with a non-vanishing component. PSF is undefined if the first or second Bloch vector is mapped onto the $z$-axis. To map the first Bloch vector onto the $z$-axis (which is identical for both first Bloch vectors), the first $z$-rotation has to rotate to the $x$-axis ($\mu_1 \in S_{\pi/2}$) such that the $y$-rotation can map the Bloch vector onto the $z$-axis ($\nu_1 \in S_{\pi/2}$). Similarly, one finds for the second Bloch vector $\mu_2 \in S_0$, $\nu_2 \in S_{\pi/2}$. Similar to I), it is impossible to achieve $F = 1$ for both states.

III)

state 5: $\theta_1 = \theta_2 = \pi/2$, $\varphi_1 = \pi/2$, $\varphi_2 = 0$:
$n_1 = (\cos(\alpha + \gamma), 0, 0)$
$n_2 = (0, \cos(\alpha + \gamma), 0)$

Exchanging $\alpha$ with $\beta$ and exchanging the first with the second Bloch vectors maps state 3 onto 5 and state 4 onto 6. Similarly, the conditions can be mapped by exchanging $\alpha$ with $\beta$ and by exchanging the local transformations of qubit one and two ($\mu_1 \leftrightarrow \mu_2$, $\nu_1 \leftrightarrow \nu_2$). Conditions from II and III are summarized in tables II

TABLE II. Summarizing cases with conditions on $\Sigma_{II}$ (left) and $\Sigma_{III}$ (right). The first column labels the cases while other columns specify angles that are restricted to sets which are written in the rows (different cases).

| II | $\beta + \gamma$ | $\beta - \gamma$ | $\mu_1$ | $\nu_1$ | $\mu_2$ | $\nu_2$ |
|----|------------------|------------------|--------|--------|--------|--------|
| i) | $S_{\pi/2}$ | $\overline{S}_{\pi/2}$ |        |        |        |        |
| ii) | $S_{\pi/2}$ | $S_{\pi/2}$ |        |        |        |        |
| iii) | $S_{\pi/2}$ | $S_{\pi/2}$ |        |        |        |        |
| iv) | $\overline{S}_{\pi/2}$ | $\overline{S}_{\pi/2}$ | $S_{\pi/2}$ |        |        |        |

| III | $\alpha + \gamma$ | $\alpha - \gamma$ | $\mu_1$ | $\nu_1$ | $\mu_2$ | $\nu_2$ |
|-----|------------------|------------------|--------|--------|--------|--------|
| i) | $S_{\pi/2}$ | $\overline{S}_{\pi/2}$ |        |        |        |        |
| ii) | $\overline{S}_{\pi/2}$ | $S_{\pi/2}$ |        |        |        |        |
| iii) | $S_{\pi/2}$ | $S_{\pi/2}$ |        |        |        |        |
| iv) | $\overline{S}_{\pi/2}$ | $\overline{S}_{\pi/2}$ | $S_{\pi/2}$ |        |        |        |

IV)

Let $\Omega_{IV}$ contain 16 sets of initial angles parametrizing states 7-22. We look at them as groups of four that are connected by maps allowing to infer conditions for further groups from conditions of the first group. The first group contains states 7-10,

state 7: $\theta_1 = \pi$, $\theta_2 = \pi/2$, $\varphi_2 = \gamma$:
$n_1 = (\cos \alpha, 0, \sin \alpha \sin \beta)$
$n_2 = (-\sin \beta, 0, -\cos \alpha \cos \beta)$

state 8: $\theta_1 = 0$, $\theta_2 = \pi/2$, $\varphi_2 = -\gamma$:
$n_1 = (\cos \alpha, 0, -\sin \alpha \sin \beta)$
$n_2 = (\sin \beta, 0, \cos \alpha \cos \beta)$

state 9: $\theta_1 = 0$, $\theta_2 = \pi/2$, $\varphi_2 = \pi - \gamma$:
$n_1 = (-\cos \alpha, 0, -\sin \alpha \sin \beta)$
$n_2 = (-\sin \beta, 0, \cos \alpha \cos \beta)$

state 10: $\theta_1 = \pi$, $\theta_2 = \pi/2$, $\varphi_2 = -\pi + \gamma$:
$n_1 = (-\cos \alpha, 0, \sin \alpha \sin \beta)$
$n_2 = (\sin \beta, 0, -\cos \alpha \cos \beta)$

Note that second Bloch vectors of state 7 and 8 as well as of state 9 and 10 are always directed oppositely (unless states are maximally mixed).

For $\alpha \in S_0$ the first Bloch vectors are identical for state 7 and 8 (9 and 10) only having a non-vanishing $x$-component. Thus $F = 1$ is impossible. Mapping the first Bloch vectors onto the $z$-axis requires $\mu_1 \in S_0$ in order to keep the $y$-component zero, and $\nu_1 \in S_{\pi/2}$ in order to map the $x$-component onto the $z$-axis. On the other hand, mapping the second Bloch vectors onto the $z$-axis requires the following, dependent on values of $\beta$: For $\beta \in S_0$ second Bloch vectors lie on the $z$-axis and stay there if $\nu_2 \in S_0$. For $\beta \in S_{\pi/2}$ second Bloch vectors lie on the $x$-axis and are mapped onto the $z$-axis by $\mu_2 \in S_0$ (keeping the $y$-component zero) and $\nu_2 \in S_{\pi/2}$ (rotating $x$-component onto the $z$-axis). For $\beta \notin S$ second Bloch vectors have non-vanishing $x$- and $z$-components, and to map them onto the $z$-axis we would
need $\mu_2 \in S_0$ (keeping the $y$-component zero) and $\nu_2 \in S_3$ with $\delta \equiv \arctan \left( \frac{n_2 \cdot \nu_2}{n_2 \cdot \nu_2} \right) = \pm \arctan \left( \frac{\sin \beta}{\cos \alpha \cos \beta} \right) \notin S$, where the minus sign of $\delta$ belongs to states 7 and 8 and the plus to states 9 and 10. Thus, for $\beta \notin S$ second Bloch vectors can not all be mapped onto the $z$-axis.

For $\alpha \in S_{\pi/2}$ and $\beta \in S_0$ Bloch vectors are zero. For $\alpha \in S_{\pi/2}$ and $\beta \notin S_0$ first Bloch vectors have a non-vanishing $z$-component while second Bloch vectors have a non-vanishing $x$-component. Comparing states 7 and 9 first Bloch vectors are oppositely directed while second Bloch vectors are aligned, implying that $F = 1$ is impossible. PSF is undefined if $\nu_1 \in S_0$, keeping the first Bloch vectors on the $z$-axis, or if second Bloch vectors are rotated to the $z$-axis, $\mu_2 \in S_0$, $\nu_2 \in S_{\pi/2}$.

For $\alpha \notin S$ and $\beta \in S_0$ first Bloch vectors have a non-vanishing $x$-component and second Bloch vectors have a non-vanishing $z$-component. For states 7 and 8 first Bloch vectors are aligned while second Bloch vectors are directed oppositely. Then, $F = 1$ is impossible, and PSF is undefined if for the first Bloch vectors $\mu_1 \in S_0$, $\nu_1 \in S_{\pi/2}$ or if for the second Bloch vector $\nu_2 \in S_0$.

For $\alpha \notin S$ and $\beta \in S_{\pi/2}$, looking at states 7 and 9 second Bloch vectors are identical while first Bloch vectors are directed oppositely. Then, $F = 1$ is impossible and an undefined PSF is obtained by $\mu_1 \in S_0$, $\nu_1 \in S_{\delta_2}$ with $\delta_2 \equiv -\arctan \left( \frac{\cos \alpha}{\sin \alpha \sin \beta} \right) \notin S$ for first Bloch vectors or by $\mu_2 \in S_0$, $\nu_2 \in S_{\pi/2}$ for second Bloch vectors. Equally looking at states 8 and 10, second Bloch vectors are identical while first Bloch vectors are directed oppositely. Then, $F = 1$ is impossible and an undefined PSF is obtained by $\mu_1 \in S_0$, $\nu_1 \in S_{-\delta_2}$ for first Bloch vectors or by $\mu_2 \in S_0$, $\nu_2 \in S_{\pi/2}$ for second Bloch vectors. Thus, as conditions for first Bloch vectors can not be true at the same time, conditions for second Bloch vectors have to be true for $\alpha \notin S$ and $\beta \in S_{\pi/2}$.

For $\alpha \notin S$ and $\beta \notin S$ all Bloch vectors have non-vanishing $x$- and $y$-components. In the following table we compare states pairwise to find necessary conditions for $F = 1$ or an undefined PSF. We use the angles $\delta_1 \equiv -\arctan \left( \frac{\sin \beta}{\cos \alpha \cos \beta} \right) \notin S$ and $\delta_2$ defined as above.

### TABLE III. Summarizing conditions for states 7-10 for the case that $\alpha, \beta \notin S$.

| states | $\mu_1$ | $\nu_1$ | $\mu_2$ | $\nu_2$ |
|--------|--------|--------|--------|--------|
| 7 & 8   | $F = 1$: $S_0$ | $\overline{S}_0$ | $S_0$ | $S_{\delta_1}$ |
|        | undefined: $S_0$ | $\overline{S}_0$ | $S_0$ | $S_{-\delta_2}$ |
| 9 & 10  | $F = 1$: $S_0$ | $\overline{S}_0$ | undefined: $S_0$ | $S_{\delta_2}$ |
| 7 & 9   | $F = 1$: $S_0$ | $\overline{S}_0$ | undefined: $S_0$ | $S_{-\delta_2}$ |
| 8 & 10  | $F = 1$: $S_0$ | $\overline{S}_0$ | undefined: $S_0$ | $S_{\delta_2}$ |

It can be seen from table III that it is not possible to achieve an undefined PSF for all states. It follows by this that the conditions $\mu_1, \mu_2 \in S_0$ and $\nu_1, \nu_2 \notin S_0$ for $F = 1$ need to be true for $\alpha, \beta \notin S$.

Group two consists of states 11-14,

**state 11:** $\theta_1 = \pi, \theta_2 = \pi/2, \varphi_2 = \pi/2 + \gamma$:

$\begin{align*}
\mu_1 &= (0, \cos \beta, \sin \alpha \sin \beta) \\
n_1 &= (0, \cos \alpha, -\cos \alpha \cos \beta)
\end{align*}$

**state 13:** $\theta_1 = 0, \theta_2 = \pi/2, \varphi_2 = -\pi/2 - \gamma$:

$\begin{align*}
\mu_1 &= (0, -\cos \beta, -\sin \alpha \sin \beta) \\
n_1 &= (0, \sin \alpha, \cos \alpha \cos \beta)
\end{align*}$

**state 12:** $\theta_1 = 0, \theta_2 = \pi/2, \varphi_2 = \pi/2 - \gamma$:

$\begin{align*}
\mu_1 &= (0, \cos \beta, -\sin \alpha \sin \beta) \\
n_1 &= (0, -\sin \alpha, \cos \alpha \cos \beta)
\end{align*}$

**state 14:** $\theta_1 = \pi, \theta_2 = \pi/2, \varphi_2 = -\pi/2 + \gamma$:

$\begin{align*}
\mu_1 &= (0, -\cos \beta, \sin \alpha \sin \beta) \\
n_1 &= (0, -\sin \alpha, -\cos \alpha \cos \beta)
\end{align*}$

To map states 7 onto 11, 8 onto 12, 9 onto 13 and 10 onto 14 we can exchange $\alpha$ with $\beta$ and rotate after the transformation $U_c$ first Bloch vectors by $R_z(\pi/2)$ and second Bloch vectors by $R_z(-\pi/2)$. Thus, conditions from IV can be mapped onto conditions from V by exchanging $\alpha \leftrightarrow \beta$ and by shifting $\mu_1 \rightarrow \mu_1 + \frac{\pi}{2}$ and $\mu_2 \rightarrow \mu_2 - \frac{\pi}{2}$.

Group three consists of states 15-18,
state 15: $\theta_1 = \pi/2, \theta_2 = \pi, \varphi_1 = \gamma$:
$n_1 = (\sin \alpha, 0, -\cos \alpha \cos \beta)$
$n_2 = (\cos \beta, 0, \sin \alpha \sin \beta)$

state 17: $\theta_1 = \pi/2, \theta_2 = 0, \varphi_1 = \pi - \gamma$:
$n_1 = (\sin \alpha, 0, \cos \alpha \cos \beta)$
$n_2 = (\cos \beta, 0, -\sin \alpha \sin \beta)$

As for group two, states of group one are mapped onto states of group three if we exchange $\alpha$ with $\beta$ and exchange the Bloch vectors. Thus, conditions from IV can be mapped onto conditions from VI by the following exchange operations $\alpha \leftrightarrow \beta$, $\mu_1 \leftrightarrow \mu_2$, $\nu_1 \leftrightarrow \nu_2$ and by mapping $\sigma_1 \rightarrow -\sigma_1$.

Group four consists of states 19-22,

state 19: $\theta_1 = \pi/2, \theta_2 = \pi, \varphi_1 = \pi/2 + \gamma$:
$n_1 = (0, \sin \beta, -\cos \alpha \cos \beta)$
$n_2 = (0, \cos \alpha, \sin \alpha \sin \beta)$

state 21: $\theta_1 = \pi/2, \theta_2 = 0, \varphi_1 = -\pi/2 - \gamma$:
$n_1 = (0, \sin \beta, \cos \alpha \cos \beta)$
$n_2 = (0, -\cos \alpha, -\sin \alpha \sin \beta)$

To map states from group one onto states of group four we rotate after the transformation $U_c$ first Bloch vectors by $R_z(\pi/2)$ and second Bloch vectors by $R_z(-\pi/2)$, and then we exchange Bloch vectors. Thus, conditions from IV can be mapped onto conditions from VII by the following exchange operations $\mu_1 \leftrightarrow \mu_2 + \frac{\pi}{2}$, $\nu_1 \leftrightarrow \nu_2$ and by mapping $\sigma_1 \rightarrow -\sigma_1$.

Conditions from groups 1-4 are given in the tables summarizing conditions of $\Sigma_{IV}$.

V)

$\Omega_{IV}$ refers to states 23 and 24. We do not derive conditions for $\Sigma_{IV}$ yet but will refer to these states later, as we then have a certain set of conditions simplifying the handling of states 23 and 24.

state 23: $\theta_1 = \pi/2, \theta_2 = \pi/2, \varphi_1 = 3\pi/4, \varphi_2 = 5\pi/4$:
$n_1 = \left(\frac{\cos(\alpha + \gamma) + \cos(\beta + \gamma)}{\sqrt{2}}, \frac{\sin(\alpha - \beta)}{2}\right)$
$n_2 = \left(\frac{\cos(\beta + \gamma) + \cos(\alpha + \gamma)}{\sqrt{2}}, \frac{\sin(\alpha - \beta)}{2}\right)$

VI)

$\Omega_{VI}$ refers to states 25 and 26. We do not derive conditions for $\Sigma_{VI}$ yet but will refer to these states later, as we then have a certain set of conditions simplifying the handling of states 25 and 26.

state 25: $\theta_1 = \pi/4, \theta_2 = \pi/4, \varphi_1 = 0, \varphi_2 = 0$:
$n_1 = \frac{1}{2} (\cos \gamma (\sqrt{2} \cos \alpha + \sin \alpha) - \sin \gamma (\cos \beta - \sqrt{2} \sin \beta), \sqrt{2} \cos (\alpha + \beta) - \cos \beta \sin \alpha)$
$n_2 = \frac{1}{2} (\cos \gamma (\sqrt{2} \cos \beta + \sin \beta), \sin \gamma (\cos \alpha - \sqrt{2} \sin \alpha), \sqrt{2} \cos (\alpha + \beta) - \cos \alpha \sin \beta)$

state 26: $\theta_1 = \pi/4, \theta_2 = \pi/4, \varphi_1 = \pi, \varphi_2 = 0$:
$n_1 = \frac{1}{2} (\cos \gamma (\sqrt{2} \cos \alpha + \sin \alpha), -\sin \gamma (\cos \beta - \sqrt{2} \sin \beta), -\sqrt{2} \cos (\alpha + \beta) + \cos \beta \sin \alpha)$
$n_2 = \frac{1}{2} (-\cos \gamma (\sqrt{2} \cos \beta + \sin \beta), \sin \gamma (\cos \alpha - \sqrt{2} \sin \alpha), -\sqrt{2} \cos (\alpha + \beta) + \cos \alpha \sin \beta)$

The proof proceeds by considering cases of I) one by one. Proving that each case contradicts other conditions proves a contradiction to the assumption.
TABLE IV. Summarizing conditions for various cases of groups 1-4. Combining tables for groups 1-4 one obtains the last table summarizing all possible cases with conditions on $\Sigma_{IV}$. The first column labels the cases while other columns specify angles that are restricted to sets which are written in the rows (different cases).

| group 1 | $\alpha$ | $\beta$ | $\mu_1$ | $\nu_1$ | $\mu_2$ | $\nu_2$ |
|---------|---------|---------|---------|---------|---------|---------|
| i) 1.   | $S_0$   | $S_0$   | $S_0$   | $S_{\pi/2}$ |
|         | "       | "       | S_0     |          |
| ii) 1.  | $S_{\pi/2}$ | S_0   |          |          |
|         | "       | "       | S_0     |          |
| iii) 1. | $S_{\pi/2}$ | S_0   |          |          |
|         | "       | "       | S_0     |          |

| group 2 | $\alpha$ | $\beta$ | $\mu_1$ | $\nu_1$ | $\mu_2$ | $\nu_2$ |
|---------|---------|---------|---------|---------|---------|---------|
| i) 1.   | $S_0$   | $S_0$   | $S_{\pi/2}$ | $S_{\pi/2}$ |
|         | "       | "       | S_0     |          |
| ii) 1.  | $S_{\pi/2}$ | S_0   |          |          |
|         | "       | "       | S_0     |          |
| iii) 1. | $S_{\pi/2}$ | S_0   |          |          |
|         | "       | "       | S_0     |          |

| group 3 | $\alpha$ | $\beta$ | $\mu_1$ | $\nu_1$ | $\mu_2$ | $\nu_2$ |
|---------|---------|---------|---------|---------|---------|---------|
| i) 1.   | $S_0$   | $S_0$   | $S_{\pi/2}$ |
|         | "       | "       | S_0     |          |
| ii) 1.  | $S_{\pi/2}$ | S_0   |          |          |
|         | "       | "       | S_0     |          |
| iii) 1. | $S_{\pi/2}$ | S_0   |          |          |
|         | "       | "       | S_0     |          |

| group 4 | $\alpha$ | $\beta$ | $\mu_1$ | $\nu_1$ | $\mu_2$ | $\nu_2$ |
|---------|---------|---------|---------|---------|---------|---------|
| i) 1.   | $S_0$   | $S_0$   | $S_{\pi/2}$ |
|         | "       | "       | S_0     |          |
| ii) 1.  | $S_{\pi/2}$ | S_0   |          |          |
|         | "       | "       | S_0     |          |
| iii) 1. | $S_{\pi/2}$ | S_0   |          |          |
|         | "       | "       | S_0     |          |

| IV      | $\alpha$ | $\beta$ | $\mu_1$ | $\nu_1$ | $\mu_2$ | $\nu_2$ |
|---------|---------|---------|---------|---------|---------|---------|
| i) 1.   | $S_0$   | $S_0$   | $S_0$   |
|         | "       | "       | S_0     |          |
| ii) 1.  | $S_{\pi/2}$ | S_0   |          |          |
|         | "       | "       | S_0     |          |
| iii) 1. | $S_{\pi/2}$ | S_0   |          |          |
|         | "       | "       | S_0     |          |
First let us consider one of cases I)i) or I)ii) to be true. From conditions for \( \alpha \) and \( \beta \) it follows that
\[
\alpha + \beta \in S_{\pi/2}, \quad \alpha - \beta \notin S_{\pi/2}
\]
\[
\Rightarrow 2\alpha, 2\beta \notin S_0
\]
\[
\Rightarrow \alpha, \beta \notin S.
\] (A3)
\[
\alpha, \beta \notin S_{\pi/2}, \quad \alpha - \beta \in S_{\pi/2}
\]
Analogously, \( \alpha + \beta \notin S_{\pi/2}, \quad \alpha - \beta \in S_{\pi/2} \) leads to the same conclusion. This already contradicts conditions from IV indicating \( \alpha, \beta \in S \).

Second let us consider I)iii) to be true. This implies
\[
\alpha \pm \beta \in S_{\pi/2} \Rightarrow 2\alpha, 2\beta \in S_0 \Rightarrow \alpha, \beta \in S
\]
and \( \alpha \in S_0 \Leftrightarrow \beta \in S_{\pi/2} \).
\[
\text{as well as } \alpha \in S_{\pi/2} \Leftrightarrow \beta \in S_0.
\] (A5)

It follows that one of cases IV)i)2., IV)i)3., IV)ii)1. or X)ii)2. has to be true. This corresponds to two possible cases for the local transformations denoted by A1 and A2,

\[
\text{A1: } \mu_1 \in S_0, \nu_1 \in S_{\pi/2} \text{ and } \mu_2, \nu_2 \in S_{\pi/2}
\]
\[
\text{A2: } \mu_2 \in S_0, \nu_2 \in S_{\pi/2} \text{ and } \mu_1, \nu_1 \in S_{\pi/2}.
\] (A6)
(A7)

Alternatives i) and ii) from II) and III) lead, similarly to (A3) to \( \alpha \notin S \) or \( \beta \notin S \), thus, contradicting conditions (A5). The combination of I)iii), II)iii) and III)iii) leads to a contradiction for the conditions for \( \alpha, \beta, \gamma \), while the combination of II)iv) and III)iv) contradicts the conditions (A6) and (A7), respectively. There remain two combinations, namely II)iii), III)iv), that agree with A1, and II)iv), III)iii), that agree with A2. For \( \alpha \in S_0, \beta \in S_{\pi/2} \) II)iii) \( (\beta + \gamma \in S_{\pi/2}) \) implies \( \gamma \in S_0 \) (A1.1) while III)iii) implies \( \gamma \in S_{\pi/2} \) (A2.1), and for \( \alpha \in S_{\pi/2}, \beta \in S_0 \) II)ii) implies \( \gamma \in S_{\pi/2} \) (A1.2) while III)ii) implies \( \gamma \in S_0 \) (A2.2).

In the following we look at states 23 and 24 from V), doing a case-by-case analysis for A1.1, A1.2, A2.1 and A2.2:

* A1.1:
\( \alpha, \gamma \in S_0, \beta \in S_{\pi/2} \)
\( \mu_1 \in S_0, \nu_1 \in S_{\pi/2} \text{ and } \mu_2, \nu_2 \in S_{\pi/2} \)
These conditions allow us to give the reduced Bloch vectors for state 15 and state 16 after the transformation \( (R_y(\nu_2)R_z(\mu_1)) \otimes (R_y(\nu_2)R_z(\mu_2)) U_c \). This means that up to a remaining z-rotation \( \sigma_1 \) all rotations are already applied to the Bloch vectors.

state 23:
\[
m_1 = \left( \pm \frac{1}{2} \sin(\alpha - \beta), 0 \right), \quad m_2 = \left( \pm \frac{1}{2} \sin(\alpha + \beta), 0 \right)
\]
state 24:
\[
m'_1 = \left( \pm \frac{1}{2} \sin(\alpha + \beta), 0 \right), \quad m'_2 = \left( \pm \frac{1}{2} \sin(\alpha + \beta), 0 \right)
\]
The \( \pm \) signs are consistent for each component of the first reduced Bloch vectors as well as for each component of the second reduced Bloch vectors. Conditions for \( \alpha, \beta, \gamma \) give
\[
\sin(\alpha - \beta) = -\sin(\alpha + \beta) \Rightarrow m_1 = m'_1 \text{ and } m_2 = -m'_2.
\] (A8)

This means that the second reduced Bloch vectors are opposite while the first reduced Bloch vectors are identical. It follows that the remaining z-rotation \( \sigma_1 \) neither leads to an undefined PSF nor synchronizes both states.

* A1.2:
\( \alpha \in S_0, \beta, \gamma \in S_{\pi/2} \)
\( \mu_1, \nu_1 \in S_{\pi/2} \text{ and } \mu_2 \in S_0, \nu_2 \in S_{\pi/2} \)
A1.2 leads analogously to A1.1 to the same conclusion as A1.1.
• A2.1: 
\[ \alpha, \gamma \in S_{\pi/2}, \beta \in S_0 \]
\[ \mu_1 \in S_0, \nu_1 \in S_{\pi/2} \text{ and } \mu_2, \nu_2 \in S_{\pi/2} \]
Again, these conditions allow us to give the reduced Bloch vectors for case 15 and case 16 after the transformation \((R_y(\nu_1)R_z(\mu_1)) \otimes (R_y(\nu_2)R_z(\mu_2))U_c.\)

state 23:
\[ m_1 = \left( \pm \frac{1}{2} \sin(\alpha - \beta), 0 \right), m_2 = \left( \pm \frac{1}{2} \sin(\alpha - \beta), 0 \right) \]

state 24:
\[ m_1' = \left( \pm \frac{1}{2} \sin(\alpha + \beta), 0 \right), m_2' = \left( \pm \frac{1}{2} \sin(\alpha + \beta), 0 \right) \]

Conditions for \(\alpha, \beta, \gamma\) give
\[ \sin(\alpha - \beta) = \sin(\alpha + \beta) \Rightarrow m_1 = -m_1' \text{ and } m_2 = m_2'. \] (A9)

As the first reduced Bloch vectors are opposite while the second reduced Bloch vectors are identical, the remaining \(z\)-rotation \((\sigma_1)\) neither leads to an undefined PSF nor synchronizes both states.

• A2.2: 
\[ \alpha \in S_{\pi/2}, \beta, \gamma \in S_0 \]
\[ \mu_1, \nu_1 \in S_{\pi/2} \text{ and } \mu_2, \nu_2 \in S_{\pi/2} \]
A2.2 leads analogously to A2.1 to the same conclusion as A2.1.

Thus, I)iii) leads to a contradiction.

Let us consider I)iv) to be true. \(\alpha \pm \beta \notin S_{\pi/2}\) already contradicts most cases of IV). Still possible are the cases IV)i)1. and IV)i)3.. Both imply that \(\nu_1, \nu_2 \in S_0\). I)iv)1. saying \(\nu_1 \in S_0\) and I)iv)2. saying \(\nu_2 \in S_0\) do not add new conditions and can thus be handled together. IV)i)1. says \(\alpha, \beta \in S_0\) which we denote by B1, while IV)i)3. says \(\alpha, \beta \in S_{\pi/2}\) which we denote by B2. Similarly to (A3), each of the cases II)i), II)ii), III)i) and III)ii) implies \(\alpha \notin S\) or \(\beta \notin S\), thus contradicting B1 as well as B2.

Let us consider II)iii) and III)iv) to be true: Given B1, it follows from II)iii) that \(\gamma \in S_{\pi/2}\), which contradicts \(\alpha \pm \gamma \notin S_{\pi/2}\) from III)iv). Given B2, it follows from II)iii) that \(\gamma \in S_0\), again contradicting III)iv).

In analogue fashion one finds a contradiction for II)iv) and III)iii). Thus, there solely remains II)iii) and III)iii) that imply \(\gamma \in S_{\pi/2}\) for B1 and \(\gamma \in S_0\) for B2.

In the following we look separately for B1 and B2 at states 25 and 26 from VI).

• For B1, we have \(\alpha, \beta \in S_0, \gamma \in S_{\pi/2}\) and \(\nu_1, \nu_2 \in S_0\). Remarkably, \(y\)-rotations \((\nu_1, \nu_2)\) are restricted such that they may change the sign of the \(x\)- and \(z\)-component or do nothing. Thus, it suffices to look at the reduced Bloch vectors after \(U_c\). We find

state 25:
\[ m_1 = \frac{1}{2} (0, \sin \gamma \cos \beta), m_2 = \frac{1}{2} (0, \sin \gamma \cos \alpha) \]

state 26:
\[ m_1' = \frac{1}{2} (0, -\sin \gamma \cos \beta), m_2' = \frac{1}{2} (0, \sin \gamma \cos \alpha) = m_2. \]

As first reduced Bloch vectors are oppositely directed and second reduced Bloch vectors aligned it is always possible to replace the \(y\)-rotations (that can change the sign of the \(x\)-component) by some rotation around the \(z\)-axis. Thus, final local rotations can be seen as one \(z\)-rotation for each qubit. Due to the orientation of reduced Bloch vectors \(F = 1\) is impossible for both states but PSF is well defined.

• For B2, \(\alpha, \beta \in S_{\pi/2}, \gamma \in S_0\) and \(\nu_1, \nu_2 \in S_0\) we find

state 25:
\[ m_1 = \frac{1}{2} (\cos \gamma \sin \alpha, 0), m_2 = \frac{1}{2} (\cos \gamma \sin \beta, 0) \]

state 26:
\[ m_1' = \frac{1}{2} (\cos \gamma \sin \beta, 0) = m_1, m_2' = \frac{1}{2} (-\cos \gamma \sin \beta, 0) = -m_2. \]
and, analogously to B1, \( F = 1 \) is impossible for both states but PSF is well defined.

By this it follows that I)iv) leads to a contradiction, too. As all cases of I) lead to a contradiction this contradicts the assumption. Perfect QPS by unitary transformations is impossible.

\[ \square \]

**Appendix B: Proof of theorem 2**

As well as for the proof of theorem 1, we define the range of initial angles \( \theta_i, \varphi_i \in \Omega \) by \( 0 \leq \theta_i \leq \pi, 0 \leq \varphi_i \leq 2\pi \) with \( i \in \{1, 2\} \) and denote the set of angles parametrizing \( U \) by \( \sigma \equiv (\alpha, \beta, \gamma, \mu_i, \nu_i, \sigma_1) \in \mathbb{R}^8 \). We say that a subset \( \Omega_0 \subset \Omega \) has measure zero if it can be parametrized by less than four parameters. The PSF is undefined if and only if \( \{||m_1(\theta_i, \varphi_i, \sigma)|| = 0 \lor ||m_2(\theta_i, \varphi_i, \sigma)|| = 0\} \).

**Lemma 2.** For any \( \sigma \) with \( ||m_1(\theta_i, \varphi_i, \sigma)|| = 0 \forall \theta_i, \varphi_i \in \Omega_{0,1} \Rightarrow \mu(\Omega_{0,1}) = 0 \).

**Proof.**

\[
||m_1|| = 0 \Rightarrow m_{1,x}(\theta_i, \varphi_i, \sigma) = 0
\]

\[
\Leftrightarrow \sum_{j=1}^{12} c_j(\sigma)t_j(\theta_i, \varphi_i) = 0, \tag{B1}
\]

with simple trigonometric expressions \( t_j(\theta_i, \varphi_i) \) and coefficients \( c_j(\sigma) \) given by

\[
\begin{align*}
t_1 &= \sin \theta_1 \sin \varphi_1 & c_1 &= -a \sin \gamma \sin \alpha \\
t_2 &= \cos \theta_1 \sin \theta_2 \sin \varphi_2 & c_2 &= -a \sin \gamma \cos \alpha \\
t_3 &= \sin \theta_1 \cos \theta_2 \cos \varphi_1 & c_3 &= a \cos \gamma \sin \alpha \\
t_4 &= \sin \theta_2 \cos \varphi_2 & c_4 &= a \cos \gamma \cos \alpha \\
t_5 &= \sin \theta_1 \cos \varphi_1 & c_5 &= b \sin \gamma \sin \beta \\
t_6 &= \cos \theta_1 \sin \theta_2 \cos \varphi_2 & c_6 &= -b \sin \gamma \cos \beta \\
t_7 &= \sin \theta_1 \cos \theta_2 \sin \varphi_1 & c_7 &= b \cos \gamma \sin \beta \\
t_8 &= \sin \theta_2 \sin \varphi_2 & c_8 &= -b \cos \gamma \cos \beta \\
t_9 &= \cos \theta_1 & c_9 &= -c \sin \alpha \sin \beta \\
t_{10} &= \sin \theta_1 \sin \theta_2 \cos \varphi_1 \cos \varphi_2 & c_{10} &= -c \sin \alpha \cos \beta \\
t_{11} &= \sin \theta_1 \sin \theta_2 \sin \varphi_1 \sin \varphi_2 & c_{11} &= c \cos \alpha \sin \beta \\
t_{12} &= \cos \theta_2 & c_{12} &= c \cos \alpha \cos \beta,
\end{align*}
\]

where the the angles \( \mu_1, \nu_1, \sigma_1 \), parametrizing the local rotations after \( U_c \), are part of coefficients \( a, b, c \) defined by

\[
\begin{align*}
a &= \cos \mu_1 \cos \nu_1 \cos \sigma_1 - \sin \mu_1 \sin \sigma_1 \\
b &= \sin \mu_1 \cos \nu_1 \cos \sigma_1 + \cos \mu_1 \sin \sigma_1 \\
c &= \cos \sigma_1 \sin \nu_1.
\end{align*}
\]

To prove the lemma we show the opposite direction negated,

\[ \mu(\Omega_{0,1}) \neq 0 \Rightarrow \nexists \sigma \text{ with } \sum_{j=1}^{12} c_j(\sigma)t_j(\theta_i, \varphi_i) = 0 \forall \theta_i, \varphi_i \in \Omega_{0,1}. \]
The trivial solution is impossible, i.e. \( \sigma \) such that \( c_j = 0 \ \forall j \):

Suppose that \( c_j = 0 \ \forall j \)

\[
\begin{align*}
c_j &= 0, j = 1, 2, 3, 4 \Rightarrow a = 0 \\
c_j &= 0, j = 5, 6, 7, 8 \Rightarrow b = 0 \\
c_j &= 0, j = 9, 10, 11, 12 \Rightarrow c = 0.
\end{align*}
\]

However, \( c = 0 \Rightarrow \)

\[
\begin{align*}
i) & \cos \sigma_1 = 0 \Rightarrow a = \mp \sin \mu_1, b = \pm \cos \mu_1 \Rightarrow \frac{2}{\pi} \mu_1 \text{ such that } a = b = 0 \\
ii) & \sin \nu_1 = 0 \Rightarrow a = \pm \cos \mu_1 \cos \sigma_1 - \sin \mu_1 \sin \sigma_1 \Rightarrow \\
& 1) 0 = \cos \mu_1 = \sin \sigma_1 \Rightarrow b \neq 0 \\
& 2) 0 = \sin \mu_1 = \cos \sigma_1 \Rightarrow b \neq 0 \\
& 3) a = 0 \Rightarrow 1 = \tan \mu_1 \tan \sigma_1\end{align*}
\]

\[
\Rightarrow 1 = -\tan^2 \mu_1 \text{ which is impossible.}
\]

As \( \mu(\Omega_{0,1}) \neq 0 \), \( \Omega_{0,1} \) provides a range for each parameter that generally consists of arbitrary unions of subintervals (with respect to the intervals of \( \Omega \)) up to an arbitrary zero set on \( \mathbb{R} \). This means that within these subintervals (i.e. for almost all points of \( \Omega_{0,1} \)) the derivatives by the parameters \( \theta_i, \varphi_i \) are well defined.

Using the fact that \( \sin(x) \) and \( \cos(x) \) are reproduced by taking four times the derivation by \( x \) we find the following set of equations

\[
\begin{align*}
\sum_j c_j t_j &= 0 & (I) \\
d^4/d\theta_1^4 \sum_j c_j t_j &= t_4, t_8, t_{12} \to 0 & (II) \\
d^4/d\theta_2^4 \sum_j c_j t_j &= t_1, t_5, t_9 \to 0 & (III) \\
d^4/d\varphi_1^4 \sum_j c_j t_j &= t_2, t_4, t_6, t_8, t_9, t_{12} \to 0 & (IV) \\
d^4/d\varphi_2^4 \sum_j c_j t_j &= t_1, t_3, t_5, t_7, t_9, t_{12} \to 0. & (V)
\end{align*}
\]

Note that a concatenation of the derivations in equations II-V simply combines the corresponding conditions for the \( t_j \). In the following we use the notation "(I,II,V)" referring to the concatenation of the derivations in I,II, and V, and we write "I±II" referring to addition/subtraction of the corresponding equations for the \( c_j, t_j \),

\[
\begin{align*}
IV-(III,IV) & : c_1 t_1 + c_3 t_5 = 0 & (B2) \\
(II,V)-(IV,V) & : c_2 t_2 + c_6 t_6 = 0 & (B3) \\
(III,IV)-(IV,V) & : c_3 t_3 + c_7 t_7 = 0 & (B4) \\
V-(II,V) & : c_4 t_4 + c_8 t_8 = 0 & (B5) \\
(IV,V) & : c_{10} t_{10} + c_{11} t_{11} = 0 & (B6) \\
I-III-IV+(IV,V) & : c_9 t_9 = 0 & (B7) \\
I-II-V+(II,V) & : c_2 t_{12} = 0. & (B8)
\end{align*}
\]

We showed that the trivial solution \( c_j = 0 \ \forall j \) is impossible. Thus, it can be easily shown that at least one of the equations \([B2,B8]\) is a non-trivial equation for the \( t_j(\theta_i, \varphi_i) \) that is not true \( \forall \theta_i, \varphi_i \in \Omega_{0,1} \).

As we showed that for \( \mu(\Omega_{0,1}) \neq 0 \) there do not exist coefficients \( c_j(\sigma) \) that satisfy \( ||m_1|| = 0 \ \forall \theta_i, \varphi_i \in \Omega_{0,1} \Rightarrow \mu(\Omega_{0,1}) = 0 \), which proves the lemma.

\[\square\]

**Lemma 3.** For any \( \sigma \) with \( ||m_2(\theta_i, \varphi_i, \sigma)|| = 0 \ \forall \theta_i, \varphi_i \in \Omega_{0,2} \Rightarrow \mu(\Omega_{0,2}) = 0.\)
Proof.

\[ ||m_2|| = 0 \Rightarrow m_{2,y}(\theta_i, \varphi_i, \sigma) = 0 \]
\[ \Leftrightarrow \sum_{j=1}^{8} d_j(\sigma) s_j(\theta_i, \varphi_i) = 0. \quad (B9) \]

The proof goes analogue to lemma 2 as the following substitutions are true,

\[
\begin{align*}
d_1 &\equiv c_4, d_2 \equiv -c_3, d_3 \equiv -c_2, d_4 \equiv c_1 \\
d_5 &\equiv -c_8, d_6 \equiv c_7, d_7 \equiv c_6, d_8 \equiv -c_5 \\
s_j &\equiv t_j,
\end{align*}
\]

with

\[
\begin{align*}
a &\equiv \cos \mu_2 \\
b &\equiv \sin \mu_2 \\
c &\equiv 0.
\end{align*}
\]

Proof. According to the lemma 2 \(||m_1|| = 0\) is only true for a set of of initial angles \(\Omega_{0,1}\) with measure zero, and according to lemma 3 \(||m_2|| = 0\) is only true for a set of of initial angles \(\Omega_{0,2}\) with measure zero. As \(||m_1|| = 0 \lor ||m_2|| = 0\) implies that \(\Omega_0 \subset \Omega_{0,1} \cup \Omega_{0,2}\), and as unions of zero sets remain zero sets, it follows that \(\Omega_0\) has measure zero.

\[ \square \]

Appendix C: Gradients

One may verify analytically that at the position of the maxima and minima the gradient of the average PSF is zero for any transformation \(U_\theta\) evaluated at an extremal set \(\Sigma_{\text{ext}}\) for the parameters according to conditions (14) in the main text (with \(\sigma_1 = \mu_2 = \nu_2 = 0\),

\[
\nabla \langle F(\rho_1, p \otimes \rho_2, p, U_{c,z}(\Sigma)) \rangle \big|_{\Sigma=\Sigma_{\text{ext}}} = \langle \nabla F(\rho_1, p \otimes \rho_2, p, U_{c,z}(\Sigma))(\varphi_1, \varphi_2, \theta_1, \theta_2) \rangle \big|_{\Sigma=\Sigma_{\text{ext}}} 
\]
\[
= \langle \nabla F(\rho_1, p \otimes \rho_2, p, U_{c,z}(\Sigma))(\varphi_1', \varphi_2', \theta_1', \theta_2') \rangle \big|_{\Sigma=\Sigma_{\text{ext}}} 
\]
\[
= -\langle \nabla F(\rho_1, p \otimes \rho_2, p, U_{c,z}(\Sigma))(\varphi_1, \varphi_2, \theta_1, \theta_2) \rangle \big|_{\Sigma=\Sigma_{\text{ext}}} 
\]
\[
\Rightarrow \nabla \langle F(\rho_1, p \otimes \rho_2, p, U_{c,z}(\Sigma)) \rangle \big|_{\Sigma=\Sigma_{\text{ext}}} = 0, 
\]

where \(\nabla = (\partial/\partial \alpha, \partial/\partial \beta, \partial/\partial \gamma, \partial/\partial \mu_1, \partial/\partial \nu_1, \partial/\partial \sigma_1, \partial/\partial \mu_2, \partial/\partial \nu_2)\). In line (C2) the symmetry transformation

\[
\begin{align*}
\theta_1 &\rightarrow \varphi_1' = \pi - \theta_1 \\
\theta_2 &\rightarrow \varphi_2' = \pi - \theta_2
\end{align*}
\]

was applied for \(\partial/\partial \alpha, \partial/\partial \beta, \partial/\partial \gamma, \partial/\partial \mu_1, \partial/\partial \sigma_1, \partial/\partial \mu_2\)-components while the symmetry transformation

\[
\begin{align*}
\varphi_1 &\rightarrow \varphi_1' = \pi + \varphi_1 \\
\varphi_2 &\rightarrow \varphi_2' = \pi + \varphi_2
\end{align*}
\]

was applied for \(\partial/\partial \nu_1\)- and \(\partial/\partial \nu_2\)-components. Calculation gives line (C3). To analytically verify that we have a local maximum (minimum), the Hessian matrix of second derivatives needs to be shown negative definite (positive definite), which could not be achieved analytically.

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