EPRL/FK asymptotics and the flatness problem

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Abstract
Spin foam models are an approach to quantum gravity based on the concept of sum over states, which aims to describe quantum spacetime dynamics in a way that its parent framework, loop quantum gravity, has not as of yet succeeded. Since these models’ relation to classical Einstein gravity is not explicit, an important test of their viability is the study of asymptotics—the classical theory should be obtained in a limit where quantum effects are negligible, taken to be the limit of large triangle areas in a triangulated manifold with boundary. In this paper we will briefly introduce the EPRL/FK spin foam model and known results about its asymptotics, proceeding then to describe a practical computation of spin foam and semiclassical geometric data for a simple triangulation with only one interior triangle. The results are used to comment on the ‘flatness problem’—a hypothesis raised by Bonzom (2009 Phys. Rev. D 80 064028) suggesting that EPRL/FK’s classical limit only describes flat geometries in vacuum.

Keywords: EPRL, quantum gravity, asymptotics, flatness, spin foams

Supplementary material for this article is available online
(Some figures may appear in colour only in the online journal)

1. Introduction
Spin foam models are an approach to quantum gravity heavily inspired by loop quantum gravity (LQG) [1]. It aims to address the parent theory’s issues with describing dynamics, while providing a clear picture of the quantum geometry of a general relativistic spacetime. LQG relies on a ‘canonical quantization’ of Einstein’s general relativity which uses its first
order formalism, and gives us a background-independent model with mostly well understood kinematics. However, information about dynamics is contained in the flow generated by the Hamiltonian constraint. Said flow generates time evolution—time being defined via a \(3+1\) ADM decomposition of spacetime [2] used to derive a Hamiltonian form for the Holst–Palatini action. Unfortunately, there are important technical difficulties in quantizing the Hamiltonian constraint and writing down the respective operator. While there are proposals for it, such as Thiemann’s [15], it remains as an open problem, especially because it has proven difficult to verify the viability of a given proposal.

The spin foam approach originated from an attempt to enunciate a path integral formulation of LQG. It uses the basis of spin network states, taking them as quantum states of a triangulated manifold which are summed over to form a partition function. Dynamics is determined by the probability amplitudes attributed to each state. Therefore, the problem to solve in the spin foam program is to define a set of amplitudes which is consistent with GR. Ponzano and Regge formulated a suitable model for 3D gravity [3], but the 4D problem is much more difficult in nature—3d general relativity in vacuum is purely topological, with no dynamical degrees of freedom, while the 4d theory is not [4].

The first concrete attempt at devising a spin foam model for 4d gravity was the Barrett–Crane model [23], which gave a set of bivector variables, obtainable from the spin foam parameters, and equivalent to a set of variables describing the Euclidean geometry of a triangulation. The model was later abandoned as it was found that the bivectors were over-constrained by the requirement of simplicity. The idea of enforcing that specific constraint only in a weak ‘expectation value’ sense instead of the strong sense led to two independent proposals (Engle/Pereira/Rovelli/Livine and Freidel/Krasnov), which turned out to be equivalent for Immirzi parameter \(0<\gamma<1\), and gave rise to the EPRL/FK model. Additionally, the Ooguri [5] and Crane–Yetter [6] models are often mentioned as triangulation-independent models that do not describe gravity.

Study of asymptotics was motivated by the need to determine if a given model reduces to classical general relativity in the \(\hbar \to 0\) limit. Since the relationship between GR and the currently proposed 4d spin foam models is not explicit from the amplitudes which define them, the ‘non-quantum’ limit serves as a necessary, albeit not sufficient, test to the validity of a given model. If a proposal does not reduce to GR in this limit, it can be immediately excluded from consideration.

In section 2, we briefly introduce the basic concepts of general spin foam models in four dimensions, and fully define the EPRL/FK model in the Euclidean setting with an Immirzi parameter \(0<\gamma<1\). Section 3 starts with a short review of past work and results on EPRL/FK asymptotics in an arbitrary simplicial complex with boundary. It then progresses to more detailed considerations on minute details of the formalism and the key tool used to derive a semiclassical limit, the stationary phase method. The ‘flatness problem’, as raised by Bonzom in 2009, is related to one of said details, the varying of the EPRL action over a discrete variable—trianglespins \(j_f\). We propose a different approach to this variation that fully acknowledges said discreteness, rather than pursuing the continuum approximation, which we see as problematic. Originally, said continuum approximation leads to the flatness problem via an equation of motion, which predicts all classical geometries generated by EPRL asymptotics to be flat. Since this is obviously at odds with GR and would invalidate the model if true, our alternative approach aims to shed further light on the issue.

Section 4 includes a thorough calculation of the classical geometry of a simplicial complex dubbed \(\Delta_3\), consisting of three 4-simplices, describing the methods used which apply to any Regge-like boundary data and presenting the results obtained from two examples with specified boundary. It is shown that for these two examples, the EPRL model has the correct asymptotic behaviour consistent with Regge calculus, the discrete formulation of GR. In particular, the flatness problem is not observed—but it is important to note that the results presented only apply to
particular cases, due to computational difficulties. They show, however, a need to investigate the issue further, and support our proposal that the original argument leading to the flatness problem is not valid. Section 5 is reserved for discussion of the results and possible routes of future research.

2. Spin foam models and EPRL/FK

Spin foams are constructed from arbitrary spin network states $\psi_\Gamma (\{ g_i \})$ over graphs $\Gamma$ embedded in a manifold $M$ (which corresponds to the spatial slice of spacetime), where $g_i$ are elements of a gauge group $G$. In gravity, the gauge group is the relativistic symmetry group of the theory (in general it could be any Lie group). The edges $l$ of $\Gamma$ have spins $j_l$ associated to them, corresponding to irreducible representations of $G$, while the graph’s vertices $v$ are labelled by intertwiners $i_v$. Now if we picture the extra time dimension and imagine the graph evolving into it, it will form a so-called 2-complex, where the edges are foliated into faces $f$ and the vertices into new edges $e$. The graph can change topologically with time, and there will be new vertices $v$, signalling points in spacetime where one edge breaks into several, or vice-versa with two or more edges joining into one. The ‘time-evolved’ graph is called the spin foam, and can be generally defined by

- an arbitrary 2-complex;
- representation spins $j_f$ for each face $f$ of the 2-complex;
- intertwiners $i_e$ for each edge $e$.

In four dimensions, the geometrical picture associated to spin foam gravity can be described intuitively with the existent duality between 2-complexes and triangulations of a 4D manifold. Indeed, a spin foam model in four dimensions can be defined as a state sum whose quantum states are configurations of a 4D simplicial complex $\Delta$ with its 4-simplices $\sigma_v$, tetrahedra $\tau_e$ and triangles $\delta_f$ coloured by a set of geometrical variables $c$ [7]. $\Delta$ can be associated with its dual 2-complex as shown in table 1:

| Simplicial complex | Dual 2-complex |
|-------------------|----------------|
| 4-simplex $\sigma_v$ | Vertex $v$ |
| Tetrahedron $\tau_e$ | Edge $e$ |
| Triangle $\delta_f$ | Face $f$ |

The state sum is defined for a given simplicial complex, and is a weighted sum over all possible colourings, with amplitudes attributed to each face, edge and vertex.

$$Z = \sum_{\text{colourings } c} \prod_f W_f(c) \prod_e W_e(c) \prod_v W_v(c)$$

(1)

$W_f, W_e, W_v$ are the face, edge and vertex amplitudes of each configuration, respectively. Defining a particular spin foam model corresponds to setting these amplitudes. We now state them for the EPRL/FK model [8, 9] in Euclidean signature.

Vertex amplitude $W_v$

We follow the construction of $W_v$ given in [14]. The colourings for the Euclidean EPRL/FK model are SU(2) quantum numbers $j_f$ for each face and SU(2) intertwiners $i_e$ for each edge, given by
\[ i_\varepsilon(k, n) = \int_{SU(2)} dh_v \bigotimes_{f \in e} h_v |k_f, n_f\rangle \]

where \(|k, n\rangle \equiv |k, \bar{n}, \theta_n\rangle\) are the Livine–Speziale coherent states \([10]\) in the spin-\(k\) representation of \(SU(2)\). They minimize the uncertainty \(\Delta(J^2) = \left| \langle J^2 \rangle - \langle J \rangle^2 \right|\) in the direction of angular momentum \(\bar{n}\), and their definition is

\[ |k, n\rangle \equiv G(\bar{n}) |k, k\rangle \]

where \(|k, k\rangle\) is the maximum angular momentum eigenstate of \(\hat{J}_z\) and \(G(\bar{n}) \in SU(2)\) rotates \(\bar{z}\) into \(\bar{n}\). There is a phase ambiguity in this definition that cannot be resolved in a canonical way, since the information about it is lost in the projection of the state vector \(|n\rangle \in S^3 \subset \mathbb{C}^2\) to \(S^2\) to obtain the rotation axis \(\bar{n}\). It will become apparent in a later section that this ambiguity is not reflected in any calculations, as all related phase factors cancel out.

For the intertwiner definition to make sense there must be an ordering of the faces in a tetrahedron \([11]\). Setting an ordering for the points in a 4-simplex, \(\sigma_v = (p_1, p_2, p_3, p_4, p_5) \equiv (1, 2, 3, 4, 5)\), is equivalent to doing the same for the tetrahedra in it, since the tetrahedron \(t_e\) can be defined as the one that does not contain the point \(i\). The operation\(^2\)

\[ \partial_1(v_1, ..., v_n) \equiv (-1)^i (v_1, ..., \hat{v}_i, ..., v_n) \]
\[ \partial_{n+1}(v_1, ..., v_n) \equiv \partial_n(v_1, ..., v_n) = (-1)^n (v_1, ..., v_{n-1}) \]

induces an ordering in a \((n-1)\)-simplex from that of a \(n\)-simplex. Using it, we can establish a coherent ordering of tetrahedra and triangles starting from what was defined for the 4-simplex. We can also define the orientation of a simplex——\((v_1, ..., v_n)\) is positively oriented if it is an even permutation of \((1, ..., n)\), and negatively oriented otherwise. Since \(\partial\) satisfies \(\partial_i \partial_j = -\partial_j \partial_i\), a consequence of the definition is that if \(f = t_{e_1} \cap t_{e_2}\), then the orientations of \(f\) induced by \(t_{e_1}\) and \(t_{e_2}\) are opposite. This has an intuitive explanation if one considers the normal vectors to each tetrahedron.

The construction of the 4-vertex amplitude is based on the spin network basis states of loop quantum gravity \([13]\), and it relies on defining a Spin(4) (that is, the Euclidean isometry group \(SO(4)\)) intertwiner \(\iota_\varepsilon\) from \(i_\varepsilon\), using the decomposition \(SU(2) \times SU(2) = Spin(4)\). First note that

\[ \iota_\varepsilon \in Hom_{SU(2)} \left( \mathbb{C}, \bigotimes_{f \in e} V_{k_f} \right) \]

since it is a \(SU(2)\)-invariant vector of \(\bigotimes_{f \in e} V_{k_f}\), where \(V_{k_f}\) is the vector space associated with the \(k_f\)-spin (irreducible unitary) representation of \(SU(2)\). One can construct an injection

\[ \phi : Hom_{SU(2)} \left( \mathbb{C}, \bigotimes_{f \in e} V_{k_f} \right) \rightarrow Hom_{Spin(4)} \left( \mathbb{C}, \bigotimes_{f \in e} V_{J^+_f} \right) \]

such that \(\phi(\iota_\varepsilon) = \iota_\varepsilon\) is the Spin(4) intertwiner. This is done by using the Clebsch–Gordan maps \(C_{k_f}^{J^+_f} : V_{k_f} \rightarrow V_{J^+_f} \otimes V_{J^+_f} \approx V_{J^+_f} \otimes V_{J^+_f}\) and constraining the values of \(J^+_f\) via the Immirzi

\(^2\) Note that a priori \(k_f \neq j_i\).

\(^3\) \((v_1, ..., v_1, ..., v_n) \equiv (v_1, ..., v_{n-1}, v_{n+1}, ..., v_n).\)
parameter: $j^\pm_f = \frac{1}{2} [1 \pm \gamma] j_f$ relates them to the original SU(2) quantum number (which is itself constrained by this relation, since $j^\pm_f \in \mathbb{N}_2$).

$$\iota_e(j_f, n_{ef}) \equiv \sum_{ke_f} \int_{\text{Spin}(4)} dg (\pi_{j_f^-} \otimes \pi_{j_f^+})(g) \circ \bigotimes_f c_{\epsilon_{kef}}^{j_f^+} \circ \iota_e(k_{ef}, n_{ef}),$$

where $g = (g^+, g^-)$, $g^\pm \in \text{SU}(2)$ and $\pi_{j_f^\pm} : \text{Spin}(4) \to V_{j_f^\pm}$, such that $(\pi_{j_f^-} \otimes \pi_{j_f^+})(g) : V_{j_f^-} \otimes V_{j_f^+} \to V_{j_f^-} \otimes V_{j_f^+}$. The integration over Spin(4) is there, once again, to ensure group invariance of the intertwiner.

The vertex amplitude $W_v$ is then a closed spin network (more details on graphical calculus in [17] for the Lorentzian case) constructed by taking $\bigotimes_{e=1}^5 \iota_e$ and ‘joining the extremities’, for each face, of the two edges that share it, as illustrated in figure 1 (each face corresponds to $2 \times 2$ of the extremities, for a total of 40, since a 4-simplex has 10 tetrahedra) by using the so-called $\epsilon$-inner product

$$\epsilon_k : V_k \otimes V_k \to \mathbb{C}.$$  

The inner product is constructed by linearity from the $\epsilon_{1/2}$, given in our convention by the matrix $\epsilon_{1/2} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$. The spin network diagram can now be evaluated using the Kaufmann bracket [16] with parameter $A = -1$. In practice this means that each pair of crossing lines with spins $k_1, k_2$ adds a sign $(-1)^{4k_1 k_2}$. In the figure above, the $\epsilon$-inner product links correspond to crossing lines which result in an overall sign $(-1)^x$ in the amplitude.

Finally, $W_v$ takes the form (now introducing the dependence in $v$)

$$W_v = (-1)^x \sum_{\{ke_f\}} \int_{\text{Spin}(4)} \prod_{e \in v} dg_{ve}^+ dg_{ve}^- \int_{(S^3)^{10}} \prod_{ef} dn_{ef} \left( \bigotimes_f K_{ef} \right) \circ \left( \bigotimes_e \iota_e \right)$$

---

4 The sum over $ke_f$ is there because the edge amplitude has the practical effect of selecting these numbers. For a general $W_v$, they are summed over (as happens in the FK model for $\gamma > 1$).
where
\[
\mathcal{K}_{ef} = \left( e_j^- \otimes e_j^+ \right) \circ \left[ \left( \pi_j^-(K_{ef}) \otimes \pi_j^+(g_{ve}^+) \right) \circ C_{ef} \right] \otimes \left( \left( \pi_j^-(g_{ve}^-) \otimes \pi_j^+(g_{ve}^-) \right) \circ C_{ef} \right).
\]

In this expression, \( e, e' \) are the edges that share the face \( f \).

**Edge amplitude \( W_e \)**

The edge amplitude is taken in modern models to be a selection rule for the values of \( k_{ef} \), and is the only difference between the EPRL and FK models. Its choice depends on the value of the Immirzi parameter.

- For \( \gamma < 1 \), both EPRL and FK select the choice \( k_{ef} = j_f^+ + j_f^- \):
\[
W_e^{\gamma < 1} = d_e \prod_{f \in e} \delta_{k_{ef}, j_f^+ + j_f^-}.
\]

- For \( \gamma > 1 \), EPRL select \( k_{ef} = j_f^+ - j_f^- \),
\[
W_{\text{EPRL}, \gamma < 1} = d_e \prod_{f \in e} \delta_{k_{ef}, j_f^+ - j_f^-}.
\]

while FK’s amplitude is a weighed sum over all possible values of \( k_{ef} \), peaking at \( k_{ef} = j_f^+ - j_f^- \) (the expression in brackets is a squared 3j-symbol):
\[
W_{\text{EPRL}, \gamma < 1} = d_e \prod_{f \in e} \sum_{k_{ef}} d_{k_{ef}} \left[ \begin{pmatrix} j_f^+ & j_f^- & k_{ef} \\ j_f^+ & j_f^- & k_{ef} \end{pmatrix} \right] ^2.
\]

**Face amplitude \( W_f \)**

Fixing the face amplitude has been an open problem since the inception of spin foam models, since the structure of loop quantum gravity does not seem to impose any particular choice for it. It is often associated with the quantized area of a triangle (see for example [1]). While several choices have been proposed in the literature, the most common being simply the dimension of the SU(2) representation associated to the face, \( W_f = 2j_f + 1 \) (indeed, in [18] it is argued it is the correct choice), in the following we shall keep it as general as possible depending only on the face quantum numbers, \( W_f \equiv \mu(j_f) \).

For the rest of this study we will use the EPRL prescription, so that the partition function is (considering a manifold with boundary and fixed boundary data satisfying Regge-like conditions [14])
\[
Z(j_{ve}, g_{ve}, n_{ef}) = (-1)^{n_f} \sum_{j_f} \prod_j \mu(j_f) \int \prod_{ve} d_{g_{ve}} \int \prod_{ef} d_{n_{ef}} \int \prod_{e} d_{h_e} \left( \otimes_{f} \mathcal{K}_f \right) \circ \left( \otimes_{e} i_e(j_e^+ \pm j_e^-, n_{ef}) \right).
\]

where the chosen integration variables of the model are the face SU(2) quantum numbers \( j_f \), Spin(4) elements for each half-edge (ve) \( g_{ve} = (g_{ve}^+, g_{ve}^-) \), and the coherent state vectors \( |j_e^+ \pm j_e^-, n_{ef} \rangle \) for each edge connected to the vertex containing \( f \), for each \( f \).
2.1. Path integral formalism

In order to study the asymptotics of the model, we use the partition function written in a path integral form,

\[ Z = \sum_{c} e^{S[c]} . \]  

(15)

We will review the derivation of this form for the EPRL/FK model [21], but it is worth noting that Bonzom [19] has extended the process for any SFM under some general assumptions.

Introducing in (14) the expressions for \( \hat{\iota} \) and \( K_f \), \( \epsilon \)-inner products of coherent states appear. They can be written in terms of the standard Hilbert inner product by introducing the antilinear structure map \( J : V_k \to V_k \) defined by

\[ \epsilon_k(v_k, v'_k) = \langle J v_k | v'_k \rangle . \]  

(16)

\( J \) has several properties: it commutes with SU(2) group elements, satisfies \( J^2 = -1 \) and, since \( J(\vec{n} \cdot \vec{J}) = - (\vec{n} \cdot \vec{J}) J \), it takes a coherent state for the vector \( \vec{n} \) to one for \( -\vec{n} \). We should also notice that the orientation requirements described above (4) are the basis for a supplementary requirement on the \( n_{ef} \), which we will call here the weak gluing condition,

\[ |n_{ef}\rangle_v = J|n_{ef}\rangle_v \]  

(17)

for a tetrahedron that is shared by two vertices. Using this notation the partition function becomes

\[ Z = (-1)^X \sum_{j_f} \prod_f \mu(j_f) \int \prod_v d\theta_v^+ d\theta_v^- \int \prod_{ef} dn_{ef} \prod_v dh_v \prod_{vf} \Pi_{vf} \]  

(18)

where

\[ P_{vf} = (k_{ef}, J n_{ef} | \pi_{k_{ef}}(h_v^{-1}) C_{j_f j'_f} \pi_{j'_f} (g_{ev} g_{ve}^{-1}) \pi_{j_f} (g_{ev} g_{ve}^+) C_{k_{ef}}^* \pi_{k_{ef}}(h_{v'}) | k_{ef}, n_{ef} \rangle) \]  

(19)

can be interpreted as a propagator between two coherent states in the two edges sharing the face \( f \). Now the Clebsch–Gordan (C–G) maps are SU(2)-invariant, which means that the \( h_v \) can be commuted with the C’s into the Spin(4) terms, which take the form \( \pi_{j'_f} (h_v^{-1} g_{ev} g_{ve}^+ h_{v'}) \). The \( h_v \) can then be eliminated by a change of variables \( \tilde{g}_{ve}^\pm = g_{ve}^\pm h_v \), and the corresponding integrations over them add up to a prefactor \( \text{Vol}(\text{SU}(2))^\# \).

The action of the C–G maps is simple in the EPRL prescription. In particular for \( \gamma < 1 \) (the case \( \gamma > 1 \) is slightly more complicated in analysis but similar in result), we have \( k_{ef} = k_{e'f} = j_f + j'_f \): the C–G maps project to the highest spin subspace of \( V_{j_f} \otimes V_{j'_f} \). Remembering the property of coherent states that

\[ |k, n\rangle \sim \otimes 2^k \frac{1}{\sqrt{2}} n \equiv \otimes 2^k |n\rangle , \]  

(20)

which is a fully symmetric state and that the highest spin subspace is precisely the one obtained by full symmetrization, we conclude that

\[ C_{k_{ef}}^{j_f j'_f} |k_{ef}, n_{ef}\rangle = |k_{ef}, n_{ef}\rangle = \otimes 2^k |n_{ef}\rangle . \]  

(21)

Therefore the propagator simplifies to
\[ P_{\omega} = \langle J_n| g_{\omega}^* g_{\omega}^- | n_{\omega}\rangle^{2j_0^+} \langle J_n| g_{\omega}^+ g_{\omega}^- | n_{\omega}\rangle^{2j_0^−} \]  
and with some simple algebra we can now write

\[ Z = (-1)^{\nu} \sum_{j_f} \mu(j_f) \int \prod_v d\gamma_v^+ d\gamma_v^- \int \prod_{ef} d\gamma_{\omega} e^S, \]

where the ‘action’ is

\[ S = \sum_f \sum_v 2j_f^\pm \log \langle J_n| g_{v,\omega}^\pm g_{v,\omega}^\mp | n_{\omega}\rangle \equiv \sum_f S_f. \]  
Since, by the discussion above, the boundary data are considered to be fixed for the ‘path-integral’ approach, while only the interior data are dynamical, it is important to separate the action into its boundary and interior parts, \( S = S_I + S_B = \sum_f S_f + \sum_{\omega} S_f \). In section 3 we will see how the action here written can be related to that of Regge calculus in the large-\( j \) regime, the base point of the asymptotics discussion.

3. Asymptotics: general considerations and past work

The semiclassical limit in quantum gravity is commonly taken in the literature as the limit of large areas, since the discrete area spectrum of LQG is asymptotically indistinguishable from the continuous classical spectrum when the corresponding quantum number \( j_f \) is large (i.e. \( \Delta j \to 0 \)). Mathematically this is imposed by making the transformation \( j_f \to \lambda j_f, \forall f \) in the regime \( \lambda \to \infty \). For the EPRL model this means that its action is proportional to \( \lambda \), so that the partition function is (roughly) of the form

\[ I_{\lambda} = \int d^n z g(z) e^{\lambda F(z)}, \lambda \to \infty. \]  
This suggests the use of the stationary phase method to derive an approximation of \( I_{\lambda} \) in the large \( \lambda \) limit.

3.1. The stationary phase method

The main principle of the stationary phase method is that due to the large argument of the exponential in the integrand, the contributions to the integral near certain critical points are much larger than everywhere else, and the integral can be estimated by considering the function only near those points. Critical points are given by the following conditions:

- \( \Re(F(z)) \) is at its absolute maximum, so that \( e^{\lambda F(z)} \) is maximized;
- the oscillation is minimized, i.e. the variation of \( \arg(e^{\lambda F(z)}) \) in a neighbourhood of the point in question is the slowest. At a first order level this is obtained by extremizing the action, i.e. \( \partial F(z) = 0, \forall i \), so that the variation of \( \Im(F(z)) \) near a critical point \( z_0 \) is at least second order in \( z - z_0 \), rather than first.

While not a rigorous proof (see [24, 25] for more detailed mathematical treatment), the essentials of the method can be understood with the following argument. That we need to
maximize the real part of $F(z)$ should be obvious in the large $\lambda$ regime, so assume in the following that $F(z) = \lambda f(z)$, $f \in \mathbb{R}$, and for simplicity $g(z) \equiv 1$ (the only condition on $g$ is that it allows for convergence of the integral, which will not be a problem in the cases we are interested in considering). Take a Taylor expansion of $f$ around an arbitrary point $z_0$:

$$
 f(z) \approx f(z_0) + \frac{\partial f}{\partial z} \bigg|_{z_0} (z - z_0)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial z^2} \bigg|_{z_0} (z - z_0)^2 (z - z_0)^2 + O(z^4)
$$

The stationary phase method assumes that when $z_0$ are critical points, the integral (25) is estimated by the formula

$$
 I_\lambda \approx \int d z_0 \int_{U(z_0)} d^n z \, e^{i \lambda f(z)}
$$

where $U(z_0)$ is a neighbourhood of $z_0$. Now suppose we only took the first order term in the Taylor expansion of $f$. Then

$$
 I_{\lambda}^1 \approx \int d z_0 \int_{U(z_0)} d^n z \exp[i \lambda(f(z_0) + D_z(z_0)(z - z_0))] = \int d z_0 \exp[i \lambda(f(z_0) - D_z(z_0))] \int_{U(z_0)} d^n z e^{i \lambda D_z(z)}.
$$

If we further assume that the contribution away from a critical point is (after taking the Taylor approximation) so small that the integral above can be extended to the whole $z$-space, the integral over $z$ is directly related to the delta ‘function’:

$$
 \int d^n z \, e^{i \lambda D_z(z)} = \frac{1}{2\pi \lambda} \delta(D_z(z_0))
$$

in this extremely crude approximation, divergences show up when $D_z(z_0) = 0$. While this points to the necessity of refining the method, which happens by taking the Taylor expansion to second order (enough in most applications), it also serves as a very simple justification that the contributions of points $z_0$ satisfying $D_z(z_0) = 0$ are dominant, justifying the definition of critical point above. Taking the second order expansion of $f$, then, we get the more accurate formula

$$
 I_{\lambda}^2 = \int d^4 z_0 \exp[i \lambda(f(z_0) - D_z(z_0))] \int d^n z \exp[i \lambda(D_z(z_0))] + H_{ij}(z_0)(z - z_0)^j (z - z_0)^j) \prod_i \delta(D_i(z_0)) \int_{\Sigma_C} d^n z e^{i \lambda H_{ij}(z(z - z_0)^j (z - z_0)^j)}
$$

where $\Sigma_C$, the critical surface, is the hypersurface$^5$ of $z$-space formed by all critical points. Using analytic continuation of the standard formula $\int d^n x \, e^{-i A \cdot x} \, x^2 = \sqrt{\frac{2\pi i}{\det A}}$ to complex $A$, we can solve the integral over $z$:

$^5$The critical surface is in fact a submanifold of $z$-space iff $\det H_z(z_0) \neq 0 \forall z_0 \in \Sigma_C$. 

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\[
\int d^\mu z e^{i \lambda H(z_0) (z - z_0) (z - z_0)'} = \left( \frac{2\pi}{i\lambda} \right)^{n/2} \frac{1}{\sqrt{\det H(z_0)}} \]

(31)

where \( H_r \) is the restriction of \( H \) to the orthogonal complement of its null space, as the conditions imposed on the \( z_0 \) constrain some degrees of freedom of \( H \).

3.2. EPRL asymptotics: the reconstruction theorem

In the context of state sum models the critical point equations can be interpreted as classical equations of motion for the interior variables of the simplicial complex (boundary data is fixed). Considering the action (24) for the Euclidean EPRL model with \( 0 < \gamma < 1 \), the equations of motion are

\[
\Re(S_I) = R_{\text{max}} \quad (32)
\]

\[
\delta_{g_{ve}} S_I = 0 \quad (33)
\]

\[
\delta_{n_{ef}} S_I = 0 \quad (34)
\]

\[
\delta_{j_f} S_I = 0 \quad (35)
\]

(35) is, however, a point of contention in existing literature. Unlike the other variables in play, the \( j_f \in \mathbb{Z}^2 \) are discrete, and it is unclear whether the stationary phase method can be extended to discrete sums. The only work in this direction that we are aware of is Lachaud’s [26] results for sums over finite fields, which is generally not the case of the \( j_f \) sums.

The other equations of motion can be written explicitly, and are as follows:

- (32) gives the gluing condition: \( R(g_{ve}) \vec{n}_{ef} = -R(g_{ve}') \vec{n}_{ef} \), where \( R(g) \) is the rotation matrix associated to \( g \) by the 2-1 surjective homomorphism \( \text{SU}(2) \to \text{SO}(3) \);
- (33) gives the closure condition: \( \sum_{f \in e} \sum_{\pm} 2j_f^\pm \epsilon_{ef}(v) R(g_{ve}) \vec{n}_{ef} = 0 \), where \( \epsilon_{ef}(v) \) is defined to be 1 if the orientation of \( f \) agrees with the one induced from \( e \) according to (4), and -1 otherwise. \( \epsilon_{ef}(v) \) are also subject to the orientation conditions, \( \epsilon_{ef}(v') = -\epsilon_{ef}(v) = -\epsilon_{ef}(v') \).
- if the previous two conditions are met, (34) is automatically satisfied.

The main existing result for EPRL asymptotics is the reconstruction theorem, proven originally by Barrett et al [14] for the case of one single 4-simplex, and more recently extended by Han and Zhang [11, 12]. It states that given a set of boundary data satisfying ‘Regge-like’ conditions guaranteeing their geometricity, and non-degenerate interior spin foam variables \( j_f, g_{ve}, n_{ef} \) satisfying the equations of motion, it is possible to construct a classical, non-degenerate geometry which matches them and is unique up to global symmetries. The proof is constructive and involves defining bivectors \( X_{ef}(j_f, n_{ef}) \) which are interpreted as area bivectors of the discrete geometry, while the \( g_{ve} \) are identified with the spin connection (in both cases up to sign factors). Additionally, the Regge deficit angles \( \Theta_f \) can be identified within the bivector formalism, such that the semiclassical action is found to be

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6 Han and Zhang developed their results for both the Euclidean and Lorentzian signature versions of the EPRL model. We will focus on Euclidean signature for this paper.
In the expression above, $N_j \in \mathbb{N}$ and $V_4$ is the 4-volume of the connected component of the discrete manifold that contains $f$. $V_4$’s sign depends on the orientation induced from spin foam variables. Since the first term is a half-integer times $i\pi$ and only gives a $\pm$ sign when exponentiated, it is mostly ignored. Therefore, the ‘classical’ form for $S$ bears an uncanny resemblance to the discrete Einstein–Hilbert action in Regge calculus [20]:

$$S_{\text{Regge}} = \sum f A_f \Theta_f$$

(37)

where $A_f$ is the area of the triangle $f$, which coincides with $\gamma_{jf}$ in the reconstructed geometry.

3.3. The j-equation and the flatness problem

Given (36), it is readily seen how the j-equation (35) was the original motivation to the ‘flatness problem’ mentioned by Freidel and Conrady [22] and later Bonzom [19]. The result shows that the EPRL action (24) can be written as

$$S = \sum_f j_f \Theta_f (g_{ve}, n_{ef})$$

where $\Theta_f$ is a quantity that is proportional, in the semiclassical limit, to the Regge-like deficit angle, $\Theta_f \rightarrow \pm \gamma \Theta_f$. If we were to ignore the discreteness of the $j_f$ and carry out the derivation as if it were continuous, the j-equation would be simply $\Theta_f = 0$, $\forall f$, therefore showing that the classical geometries reproduced by the model are restricted to be flat—a result that puts the model in question, since GR in four dimensions admits curved spacetime solutions. However, the applicability of this equation is questionable, not only because of the issues with the discreteness of $j_f$, but due to an ambiguity in the way the semiclassical limit is taken—taking the limit of large $j_f$, while at the same time summing over them. In the following we consider a slight reformulation.

Assume that in the semiclassical limit the boundary face quantum numbers are given by $j_f = \lambda j'_{fB}$, $\forall f_B$ where $j'_{fB} \in \mathbb{Z}$ and $\lambda \rightarrow \infty$. Then, define new interior variables $x_I = \frac{j_f}{\lambda} \in \mathbb{Z}$ and $x_{\pm I}$ accordingly. The partition function then takes the form

$$Z(\lambda j'_{fB}, g_{veB}, n_{efB}) = \sum_{x_I} \int \prod_v d g_{ve} \int \prod_{ef} d n_{ef} e^{i \lambda (S_I + S_B)}$$

(38)

with

$$S_I = -i \sum_{fI} \sum_{v \in f} \sum_{k} 2 x_I^k \log \left( \langle J n_{ef} | g_{ve}^k g_{ve}^k | n_{ef} \rangle \right)$$

and

$$S_B = -i \sum_{fB} \sum_{v \in f} \sum_{\pm} 2 j_{fB}^\pm \log \left( \langle J n_{ef} | g_{ve}^\pm g_{ve}^\pm | n_{ef} \rangle \right)$$

(39)

(we factor out $i$ to explicit the fact that the argument of the exponential becomes pure imaginary when the gluing condition is satisfied). With this prescription, we do not have to assume anything about the $x_I$’s, eliminating ambiguities, and the dependence of the partition function on $\lambda$ is completely explicit. Additionally, we can propose a workaround to the discreteness
issue, consisting of a continuum approximation for the $x_f$. Since the $\Delta x_f = \frac{1}{\lambda}$ tend to zero for large $\lambda$, it makes sense to consider replacing the sum over $x_f$ by an integral:

$$\frac{1}{\Delta x_f} \sum_{x_f} f(x_f) \Delta x_f \approx \frac{1}{\Delta x_f} \int_0^\infty f(x_f) \text{d}x_f$$

(40)

and therefore the ‘semiclassical’ partition function would be

$$Z_{SC}(\lambda x_f) = (2\lambda)^{\#f} \prod_{(ve)_I} \prod_{(ef)_I} \int \text{d}x_f \int \text{d}g_{ve} \int \text{d}n_{ef} e^{i\lambda(S_I + S_x)}. \quad (41)$$

Of course, one must be careful with the errors incurring from this approximation, which is essentially the rectangle method of numerical integration ‘done backwards’. It can be shown that the difference between the sum and the integral is of order $1/\lambda$, making the continuum approximation unreliable to compute any quantum corrections to the zero-order, $\lambda = \infty$ results. It could still be argued that that it can be used safely in the zero-order situation, but we will try to progress as much as possible without using it. The problem is to estimate the integral

$$\sum_{j_f} \mu(j_f) \int dY e^{\sum_{j_f} \lambda x_f \delta_\theta(Y)}$$

(42)

where we used $Y$ as short for the set of $g_{ve}$, $n_{ef}$ integration variables. Using the stationary phase method for the integral over $Y$, we obtain

$$\int dY e^{\sum_{j_f} \lambda x_f \delta_\theta(Y)} \approx \int_{\Sigma_c(x_f)} dY_C \prod_f e^{\lambda x_f \delta_\theta(Y_C)} \left( -\frac{2\pi i}{\lambda} \right)^{\#Y_C/2} \frac{1}{\sqrt{\det \left[ \sum_f x_f H_f(Y_C) \right]}}$$

(43)

where $Y_C$ are the critical points that solve the equations of motion, and $\Sigma_C$ the submanifold of $Y$-space they form. Ideally, if we use the continuum approximation, we could think of reversing order of integration and doing the $x$ integral first, but this is not possible for the general case because not only there is an $x$ dependence on the determinant factor, which is a priori arbitrary, but due to the closure condition the critical surface $\Sigma_C$ also depends on $x$. This makes the integral seemingly intractable without further assumptions. There are some heuristic considerations that can be made on this form of $Z$ that lead to something suggestive of the flatness problem, but the apparent ‘dead end’ we reach here leads us to consider a concrete example in which a full calculation is possible, the $\Delta_3$ manifold studied in section 4.

More recently, a different approach to asymptotics devised by Hellmann and Kaminski [29] derived a result similar to the flatness problem. Their main idea is to introduce the concept of wavefront sets for a distribution, which are designed with asymptotics in mind and represent the subspace of phase space where the distribution is peaked in the limit of large $\lambda$. The wavefront sets of partition functions of various models like BC and EPRL can be written using the holonomy (or operator) representation of spin foams [30] and their main result regarding asymptotics is an accidental curvature constraint acting on the deficit angles $\Theta_f$.

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7 Consider the difference $\int_{x_0}^{x_0+\Delta x} f(x) \text{d}x - f(x_0) \Delta x$. For $\Delta x = 1/2\lambda$ the difference is of order $1/\lambda^2$. In practical semiclassical calculations the integral will not extend to infinity because triangle inequalities limit the maximum value of $f$. The cutoff will be of order $\lambda$, so the error in approximating the sum by an integral is of order $1/\lambda$. 

which is not strictly flatness (the dependence on the Immirzi parameter is somewhat puzzling) but still a worrying result in terms of the accuracy of the theory’s asymptotics in respect to Einstein theory. It is noteworthy that for the BC model, which can essentially be obtained from EPRL by taking the limit $\gamma \to 0$, the wavefront approach leads to an exact flatness constraint.

4. An example: $\Delta_3$

In the following we will attempt to compute the asymptotic EPRL partition function for the case of the three 4-simplex manifold $\Delta_3$, which is represented in figure 2 below together with its 2-complex dual. This particular manifold is chosen as a simple example of a semiclassical calculation, since it has only one interior face $f_I$. Therefore, assuming the boundary data are fixed, Regge-like, and non-degenerate, the classical Euclidean geometry of $\Delta_3$ is completely determined by the area $j = \lambda x$ and the deficit angle $\Theta$ of $f_I$, two quantities that are easily seen to be completely determined by the boundary geometry. We will now define the EPRL model in this triangulation.

Boundary faces are notated $f_{ij}$, $i, j \in \{1, ..., 5\}$ where $f_{ij}$ is the triangle that does not contain the points $i, j$ of the 4-simplex $v$ it belongs to, and has the area variable $x_{ij}$. Edges are labelled $e^k_v$, $k \in \{1, ..., 5\}$ and $e^k_v$ is the tetrahedron that does not contain the point $k$ of $v$. We will call the $n_{ef}$ as $[n_{ef}]$, $v \in \{A, B, C\}$ for clarity, while the interior $g_{ev}$ are labelled $g_A^5, g_B^6, g_C^5$ accordingly to the figure. The partition function is (proportional to, with extra pre-factors not being of importance in the analysis)

$$Z = \sum_{x=\lambda/\lambda} \frac{\mu(\lambda x)}{x^{n_Y}} \int_{\Sigma_C(s)} dY_c \frac{e^{i\lambda x\Theta(Y_c)}}{\sqrt{\det H_c(Y_c)}}$$

(45)

noting that the dimension $\# Y$ of $Y$-space is that of 12 copies of $S^3$ associated to the interior $g_{ev}$ and other 6 copies associated to the interior $n_{ef}$. The dimension $\# Y_C$ of the critical surface is the number of degrees of freedom unconstrained by the equations of motion.

4.1. Solving the equations of motion

We will now study the equations of motion for $\Delta_3$. For starters, $n_{ef}$ and $n_{ef}'$ are related by the weak gluing equations (17):

$$\gamma \Theta_f = 0 \mod 2\pi,$$

(44)
\[ |n_{6,56}\rangle = J |n_{5,45}\rangle \]
\[ |n_{5,45}\rangle = J |n_{6,46}\rangle \]
\[ |n_{4,46}\rangle = J |n_{5,56}\rangle . \tag{46} \]

We can choose a simpler notation for the interior \( n_{ij} \) so that (46) reads
\[
|n_{AC}\rangle = J |n_{CA}\rangle \\
|n_{CB}\rangle = J |n_{BC}\rangle \\
|n_{BA}\rangle = J |n_{AB}\rangle . \tag{47} \]

Stationary phase computation on the \( g, n \) integrals results in 6 interior gluing conditions,
\[
R(g_{CS}^{\pm}) \triangleright \vec{n}_{CA} = -R(g_{CS}^{\pm}) \triangleright \vec{n}_{CB} \\
R(g_{BA}^{\pm}) \triangleright \vec{n}_{BC} = -R(g_{BA}^{\pm}) \triangleright \vec{n}_{BA} \\
R(g_{AS}^{\pm}) \triangleright \vec{n}_{AB} = -R(g_{AS}^{\pm}) \triangleright \vec{n}_{AC} \tag{48} \]

36 interior-boundary gluing conditions,
\[
R(g_{AS}^{\pm}) \triangleright \vec{n}_{3,15} = -R(g_{AS}^{\pm}) \triangleright \vec{n}_{1,15} \\
R(g_{AS}^{\pm}) \triangleright \vec{n}_{6,16} = -R(g_{AS}^{\pm}) \triangleright \vec{n}_{1,16} \\
R(g_{AS}^{\pm}) \triangleright \vec{n}_{6,16} = -R(g_{AS}^{\pm}) \triangleright \vec{n}_{1,16} \\
R(g_{AS}^{\pm}) \triangleright \vec{n}_{6,16} = -R(g_{AS}^{\pm}) \triangleright \vec{n}_{1,16} \\
R(g_{AS}^{\pm}) \triangleright \vec{n}_{4,14} = -R(g_{AS}^{\pm}) \triangleright \vec{n}_{1,14} \\
R(g_{AS}^{\pm}) \triangleright \vec{n}_{6,16} = -R(g_{AS}^{\pm}) \triangleright \vec{n}_{1,16} \\
R(g_{AS}^{\pm}) \triangleright \vec{n}_{6,16} = -R(g_{AS}^{\pm}) \triangleright \vec{n}_{1,16} \tag{49} \]

and 6 closure conditions,
\[
x \left[ (1 + \gamma)R(g_{CS}^{+}) + (1 - \gamma)R(g_{CS}^{-}) \right] \triangleright \vec{n}_{CA} + b.t.(C+) = 0 \\
x \left[ (1 + \gamma)R(g_{AS}^{+}) + (1 - \gamma)R(g_{AS}^{-}) \right] \triangleright \vec{n}_{AC} + b.t.(A+) = 0 \\
x \left[ (1 + \gamma)R(g_{BA}^{+}) + (1 - \gamma)R(g_{BA}^{-}) \right] \triangleright \vec{n}_{BC} + b.t.(B+) = 0 \tag{50} \\
-x \left[ (1 + \gamma)R(g_{CS}^{+}) + (1 - \gamma)R(g_{CS}^{-}) \right] \triangleright \vec{n}_{CB} + b.t.(C-) = 0 \\
-x \left[ (1 + \gamma)R(g_{AS}^{+}) + (1 - \gamma)R(g_{AS}^{-}) \right] \triangleright \vec{n}_{AB} + b.t.(A-) = 0 \\
-x \left[ (1 + \gamma)R(g_{BA}^{+}) + (1 - \gamma)R(g_{BA}^{-}) \right] \triangleright \vec{n}_{BA} + b.t.(B-) = 0 \tag{51} \]

where the \( b.t. \) represents terms depending exclusively on boundary variables. Indeed, the closure conditions contain sums over edges in each vertex, so each of them contains exactly one term corresponding to the interior edge, and the rest of the sum depends on the boundary edge variables. The boundary terms are labelled by the edges they pertain to.

First off, we will note that equations (49) determine all the interior \( g_{uv} \) uniquely in terms of boundary data. Indeed, consider the first equation referring to \( g_{AS}^{\pm} \). The only term in this equation that is not a boundary variable is \( R(g_{AS}^{\pm}) \), and the indices 1, 2, 3 can be grouped in a matrix form equation:
\[
R(g_{AS}^{\pm}) \triangleright \begin{bmatrix} \vec{n}_{1,15}^{\pm} \\ \vec{n}_{2,25}^{\pm} \\ \vec{n}_{3,35}^{\pm} \end{bmatrix} = - \begin{bmatrix} R(g_{AS}^{\pm}) \triangleright \vec{n}_{1,15}^{\mp} & R(g_{AS}^{\pm}) \triangleright \vec{n}_{2,25}^{\pm} & R(g_{AS}^{\pm}) \triangleright \vec{n}_{3,35}^{\pm} \end{bmatrix} . \tag{52} \]
Note that the non-degeneracy assumption on the boundary data implies that, since all tetrahedra are non-degenerate, any set of three out of the four \( \vec{n}_d \) that define a tetrahedron must be linearly independent. This means that \( N_{A5} \) is invertible in the equation above, which can then immediately be solved:

\[
R(g_{A5}^\pm) = -N_{A5}^{-1} V_{A5}^\pm
\]

and similar solutions are derived for the remaining \( g_{i\nu} \). This result means that the purely interior gluing conditions (48), if consistent (consistency should be guaranteed by the boundary data being Regge-like), are redundant, however we will analyse them together with the closure conditions in the following, as they have valuable physical content for the problem.

It is possible to eliminate three of the closure equations by using the gluing ones: indeed, substituting (48) on (51), we obtain (50) while being forced to impose that \( h.t.(A+) = -h.t.(A-) \) (and similar for the \( B\pm \) and \( C\pm \) boundary terms). Conditions on boundary variables are not problematic if they can be related to the equations for Regge-like data. To elaborate on this and to properly solve the closure conditions we need to specify the boundary data. The equations (50) in their full form are

\[
\begin{align*}
[(1 + \gamma) R(g_{A5}^+)]^T & (1 + \gamma) R(g_{A5}^-) \left( x_{C} + x_{A}^2 \vec{n}_{5,441} + x_{A}^2 \vec{n}_{5,442} + x_{A}^2 \vec{n}_{5,443} \right) = 0 \\
[(1 + \gamma) R(g_{B6}^+)]^T & (1 + \gamma) R(g_{B6}^-) \left( x_{C} + x_{A}^2 \vec{n}_{6,61} + x_{A}^2 \vec{n}_{6,62} + x_{A}^2 \vec{n}_{6,63} \right) = 0 \\
[(1 + \gamma) R(g_{C4}^+)]^T & (1 + \gamma) R(g_{C4}^-) \left( x_{A} + x_{C}^2 \vec{n}_{4,51} + x_{C}^2 \vec{n}_{4,52} + x_{C}^2 \vec{n}_{4,53} \right) = 0.
\end{align*}
\]

The solution of these equations is simple to obtain, noting that they are of the form \( M \triangleright \vec{v} = 0 \), a condition satisfied if and only if \( \vec{v} = 0 \) or \( M \) has a vanishing determinant. The second possibility can be ruled out, though, by proving that \( M = (1 + \gamma) G + (1 - \gamma) H \) has nonzero determinant for all \( G, H \in SO(3) \) and \( 0 < \gamma < 1 \). Proof starts with noting that \( (\det M)^2 = \det(M^t M) \). It is possible to get a general expression for \( \det(M^t M) \):

\[
M^t M = [(1 + \gamma) G + (1 - \gamma) H]^T [(1 + \gamma) G + (1 - \gamma) H] = 2(1 + \gamma^2) I + (1 - \gamma^2) (G^T H + H^T G)
\]

\[
= 2(1 + \gamma^2) I + (1 - \gamma^2)(A + A')
\]

defining \( A \equiv G^T H \in SO(3) \). We can compute the determinant in a basis where \( A + A' \) is diagonal—note that the identity matrix is basis-invariant and \( A + A' \) is a symmetric real matrix, hence diagonalizable. To do so we need its eigenvalues, which can be found using one of the several possible parameterizations of \( SO(3) \). Here we use a parameterization by Janaki and Rangarajan [27]:

\[
A = \begin{bmatrix}
\cos \theta_1 \cos \theta_2 & \sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2 \sin \theta_3 & \sin \theta_1 \sin \theta_2 + \cos \theta_1 \sin \theta_2 \cos \theta_3 \\
-\sin \theta_1 \cos \theta_2 & \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \sin \theta_3 & \cos \theta_1 \sin \theta_2 - \sin \theta_1 \sin \theta_2 \cos \theta_3 \\
-\sin \theta_2 & -\cos \theta_2 \sin \theta_3 & \cos \theta_2 \cos \theta_3
\end{bmatrix}
\]

where \( \theta_i \in [0, 2\pi] \) are angles for simple rotations. \( A + A' \) can then be diagonalized, being a symmetric real matrix. There is a basis in which \( A + A' \) is symmetric, being a matrix

\[
A + A' = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \quad a = 2, \quad b = c = \sin \theta_1 \sin \theta_2 \sin \theta_3 + \cos \theta_1 (\cos \theta_2 + \cos \theta_3) + \cos \theta_2 \cos \theta_3 - 1
\]

(57)
are its eigenvalues. In this basis,
\[
M' M = 2(1 + \gamma^2) \begin{bmatrix} 1 & 2 \\ 1 & b \end{bmatrix} + (1 - \gamma^2) \begin{bmatrix} 1 \\ b \end{bmatrix}
\]
\[
= \begin{bmatrix} 2(1 + \gamma^2) + b(1 - \gamma^2) & 0 \\ 2(1 + \gamma^2) + b(1 - \gamma^2) & b \end{bmatrix}
\]
so that \((\det M)^2 = 4 \left[ 2(1 + \gamma^2) + b(1 - \gamma^2) \right]^2\). Therefore,
\[
\det M = 0 \iff b = -2 \frac{1 + \gamma^2}{1 - \gamma^2}.
\]
It is straightforward to verify that \(-2 \leq b \leq 2\) for all values of \(\theta_i\), which makes the above condition impossible in the \(0 < \gamma < 1\) range we are working on. Hence, \(M\) is always invertible in the conditions of our study, and the closure conditions are simplified:
\[
x\vec{n}_{CA} + x_{31}^C \vec{n}_{4,41}^C + x_{42}^C \vec{n}_{4,42}^C + x_{43}^C \vec{n}_{4,43}^C = 0 \\
x\vec{n}_{BC} + x_{61}^B \vec{n}_{6,61}^B + x_{62}^B \vec{n}_{6,62}^B + x_{63}^B \vec{n}_{6,63}^B = 0 \\
x\vec{n}_{AB} + x_{51}^A \vec{n}_{5,51}^A + x_{52}^A \vec{n}_{5,52}^A + x_{53}^A \vec{n}_{5,53}^A = 0.
\]
Notice that these are precisely the necessary and sufficient conditions for the 3 tetrahedra of \(\Delta_3\) that contain the interior face \(f\) to be geometrical in the Euclidean sense, which shows that the large areas limit for this manifold imposes a discrete classical geometry on it. Also, the partition function is considerably simplified, since \(x\) and all the interior \(\vec{n}_{ij}\) are fixed:
\[
x = \sqrt{[x_{41}^C \vec{n}_{4,41}^C + x_{42}^C \vec{n}_{4,42}^C + x_{43}^C \vec{n}_{4,43}^C]}
\]
\[
\vec{n}_{BA} = -\frac{x_{41}^C \vec{n}_{4,41}^C + x_{42}^C \vec{n}_{4,42}^C + x_{43}^C \vec{n}_{4,43}^C}{[x_{41}^C \vec{n}_{4,41}^C + x_{42}^C \vec{n}_{4,42}^C + x_{43}^C \vec{n}_{4,43}^C]}
\]
and similarly for \(\vec{n}_{AC}\) and \(\vec{n}_{CB}\). In particular, note that \(x\) is fixed in terms of boundary data by the gluing/closure conditions. Study of the partition function’s distribution over \(j\) is necessary to confirm it peaks around this critical point, but having it fully determined by boundary data is consistent with the classical geometry of \(\Delta_3\).

Note that the other two closure conditions also give expressions for \(x\), leading to additional constraints on boundary data:
\[
|x_{13}^A \vec{n}_{1,13}^A + x_{14}^A \vec{n}_{1,14}^A + x_{15}^A \vec{n}_{1,15}^A| = |x_{13}^B \vec{n}_{1,13}^B + x_{14}^B \vec{n}_{1,14}^B + x_{15}^B \vec{n}_{1,15}^B| = |x_{13}^C \vec{n}_{1,13}^C + x_{14}^C \vec{n}_{1,14}^C + x_{15}^C \vec{n}_{1,15}^C|.
\]
Additionally, the relations between (50) and (51) make it so that
\[
\vec{n}_{CA} = -\vec{n}_{CB} \\
\vec{n}_{BC} = -\vec{n}_{BA} \\
\vec{n}_{AC} = -\vec{n}_{AB}
\]
and together with weak gluing, we obtain that \(\vec{n}_{AB} = \vec{n}_{BC} = \vec{n}_{CA} \equiv \vec{n}\). The partition function is now reduced to
\[
Z = \frac{\mu(\lambda x)}{x^3} \int_{\Sigma_c} dY_c \frac{e^{i\lambda \theta(Y_c)}}{\sqrt{\det H(Y_c)}}
\]

where, with \( x \) and \( \vec{n}_f \) fixed, the only integrations remaining are over group elements and the phases \( \alpha_{\alpha f} \), and the face amplitude \( \mu \) becomes no more than a pre-factor. The critical surface \( \Sigma_C \) in this new expression is \( S^2 \times U(1)^3 \), corresponding to the one free vector \( \vec{n} \in S^2 \) and the three free phases \( \alpha_{0 R}, \alpha_{BC}, \alpha_{CA} \) necessary to define the respective coherent states.

The results obtained in this section seem positive towards the consistency of EPRL/FK asymptotics with Regge calculus, and in contradiction with the flatness problem. Indeed, we are able to obtain geometrically consistent values for the key quantities in this problem, the area \( \gamma_f \) and the deficit angle \( \Theta \) of the only interior triangle in the manifold. In fact, a similar result has been claimed by Perini and Magliaro [28], although the paper in question does not treat the problem in detail and fails to address one important difficulty—the behaviour of the state contributions when \( j \) is varied. This is a problem because \( j \) is discrete and it is unclear how to vary the action over it. While we get equations of motion that guarantee the nonexistence of a critical point when \( j \) is different from the unique critical value \( j_0 \), it has not been properly justified that this point’s contribution dominates over non-critical configurations with different values of \( j \). Additionally, the value of \( j \) that solves exactly the closure conditions will in general be a non-integer, leading to an uncertainty in this calculation which is important to address. The closure conditions will, in general, not be exactly satisfied, because of the discreteness feature.

4.2. Variation over \( j \)

To address the issue, we will use results from chapter 7 of [25] related to the stationary phase method. In particular we are interested in the following theorem about the study of the stationary phase integral when the functions that define it depend on free parameters.

**Theorem.** Let \( f(x,y) \) be a complex valued \( C^\infty \) function in a neighbourhood \( K \) of \((0,0) \in \mathbb{R}^{n+m} \), such that \( \Im(f) \geq 0, \Im(f(0,0)) = 0, D_x f(0,0) = 0 \) and \( \det D^2_x f(0,0) \neq 0 \). Let \( u \) be a \( C^\infty \) function with compact support in \( K \). Then

\[
\int u(x,y) e^{i\lambda f(x,y)} dx \sim e^{i\lambda f(0,0)} \sqrt{\frac{1}{\det D^2_x f(0,0)^{n/2}}} \left( \frac{2\pi i}{\lambda} \right)^{n/2} \det D^2_x f(0,0)^{n/2},
\]

where the superscript 0 in front of the determinant signals that the corresponding function is specified modulo the ideal I of functions generated by the derivatives \( D_x f(x,y) \).

Roughly, the theorem states that if \( x = 0 \) is a critical point of \( f \) when \( y = 0 \), then when \( y \neq 0 \) the point is ‘moved’ (and in general not a critical point), with its contribution to the full integral being approximated by the formula above. The key point is that if \( f^0 \) has an imaginary part, this contribution is suppressed by a factor \( e^{-\lambda \Im(f^0)} \). We are interested in this suppression factor for the integral we are studying, where the free parameter \( y \) is taken to be \( x - x_0, x_0 \) being the critical value of \( x \). It is only left to choose \( f^0 \). The proof of the theorem above uses the Malgrange preparation theorem, also explained in chapter 7 of [25]. Malgrange’s theorem states the existence of a set of functions \( X'(y) \) satisfying \( X'(0) = 0 \) such that the ideal I of functions generated by \( \frac{\partial f}{\partial \overline{y}} \) is also generated by \( \{ x' - X'(y) \} \). It can be used to write the following expansion for \( f(x,y) \) near the critical point:

\[
f(x,y) \approx \sum_{|\alpha| < N} \frac{f^{(\alpha)}(y)}{\alpha !} (x - X(y))^{\alpha} \mod I^{N}, \quad \forall N
\]
\( f^0 \) is the term independent of \( x \) in this expansion. It is also noted that the \( f^1_\ell(y) \) belong to \( f^N \) for any \( N \), so that they can be chosen to vanish. This is an intuitive result when compared to a Taylor expansion around a critical point. Since we are only looking for the leading term of \( f^0 \) to obtain the suppression factor, we will consider an expansion to second order \( (N = 2) \), and use \( f^\prime \)’s Taylor series to compute the relevant functions:

\[
\begin{align*}
  f(x, y) & \approx f(0, 0) + \left. \frac{\partial f}{\partial x} \right|_{(0,0)} x^l + \left. \frac{\partial f}{\partial y} \right|_{(0,0)} y \\
  & \quad + \frac{1}{2} \left. \frac{\partial^2 f}{\partial y \partial x} \right|_{(0,0)} x^l x^j.
\end{align*}
\] (67)

The second order Malgrange expansion for \( f(x, y) \) is (setting \( f^1 = 0 \))

\[
\begin{align*}
  f(x, y) & \approx f^0(y) + \frac{1}{2} f^2_{\ell j}(y)(x^l - X^l(y))(x^j - X^j(y)).
\end{align*}
\] (68)

Equating both expansions and gathering terms independent, linear and quadratic in \( x \), we get

\[
\begin{align*}
  f^0 & = f(0, 0) + \delta_1 y + \frac{1}{2} \delta_2 y^2 = f^0 + \frac{1}{2} f^2_{\ell} X^l \\
  \frac{1}{2} K_{,l} x^l y & = -\frac{1}{2} (f^2_{\ell j} + f^2_{j \ell}) x^l X^j \\
  \frac{1}{2} H_{ij} x^i x^j & = \frac{1}{2} f^2_{\ell} X^l x^j
\end{align*}
\] (70)

which we solve to obtain \( (H^{-1})^\ell_j \) is the inverse matrix of \( H_{ij} \). Remember we assumed \( \det H \neq 0 \)

\[
\begin{align*}
  f^0 & = f(0, 0) + \delta_1 y + \frac{1}{2} \delta_2 y^2 - \frac{1}{2} K_{,l} H^{-1}_{ij} K_{,j} y^2 \\
  -H^{-1}_{ij} K_{,l} y & = X^j \\
  f^2_{ij} & = H_{ij}.
\end{align*}
\] (71)

Applying to the \( \Delta_3 \) case, remembering that we chose \( y = x - x_0 \), we see that \( f(0, 0) \) is the action at the critical point \( S_C \), \( \delta_1 = -i \tilde{\Theta}_C \sim \pm \gamma \Theta_{\text{Regge}} \) and \( \delta_2 = 0 \). Note that \( \delta_1 \) is real. We are only interested in the imaginary part of \( f^0 \), which is quadratic in \( (x - x_0) \), and gives us the suppressing factor as

\[
\exp \left( -\frac{\lambda}{2} \Im (K_{,l} H^{-1}_{ij} K_{,j}) (x - x_0)^2 \right).
\] (72)

Note that the variation of \( x \) has to be discrete. We would set \( j = j_0 + \frac{n}{\lambda}, n \in \mathbb{Z} \), so that \( x - x_0 = \frac{n}{\lambda} \). This allows us to write the partition function as a sum over \( n \) in terms of the term corresponding to \( n = 0 \), the critical term:

\[
Z = Z_C \sum_n \exp \left( -\frac{A n^2}{4 \lambda} \right)
\] (73)

where \( A = -\Im (K_{,l} H^{-1}_{ij} K_{,j}) \). If \( x \) is thought of as an approximately continuous variable, the distribution of \( x \) values follows a Gaussian curve with standard deviation \( \sigma = \sqrt{\frac{1}{\lambda}} \). This
is a sufficiently small deviation, assuming $A$ finite, to conclude that the distribution of the $(j_f, g_{ve}, n_{ef})$ variables is sufficiently peaked around the critical surface. Since $A$ does not have any $\lambda$ dependence, the positive result should be guaranteed simply by $A \neq 0$. However, the most rigorous approach to this problem is to compute the sum of the series in (73) and obtain the statistics of the discrete variable $n$ (note, in particular, that $j_0$ as given by the closure equations might not be a semi-integer, so the dominant contribution would come from the semi-integer closest to it). The EPRL/FK action

$$S = -2i \sum_f \sum_{v \in f} \sum_{\pm} j_f (1 \pm \gamma) \log \langle J_{n_{ef}} \mid (g_{ve}^\pm)^f \mid (g_{ve}') \mid n_{ef} \rangle$$

(74)

can be interpreted in terms of this stationary phase method by setting $j_f \equiv y$ as the free parameter, and $x_i \equiv (\{g_{ve}\}_a, \{n_{ef}\}_b)$ as the dependent variables, where $a, b$ signal a coordinate system in which to express the interior $g_{ve}, n_{ef}$ (which can be, for example, the parameterizations of SU(2) and $H^{1/2}$ specified in the appendix). The quantities necessary to compute the approximate partition function (73) are

$$K_i = \left. \frac{\partial^2 S}{\partial j_f \partial x_i} \right|_{\text{critical}}$$

(75)

$$H_{ij} = \left. \frac{\partial^2 S}{\partial x_i \partial x_j} \right|_{\text{critical}}$$

(76)

where ‘critical’ means the derivatives are computed at the unique critical point for $\Delta_3$ determined in section 4.1. $K_i$ is simplified due to the action being linear in $j$, being reduced to first derivatives of the quantum deficit angle of the interior face $\theta_f$. The conditions of theorem (65) require that $\det H \neq 0$ for the stationary phase method to be applicable. However, explicit algebraic computation of this determinant proves to be a bit too cumbersome, because of the dependence of the derivatives in question on a high number of $a$ priori arbitrary boundary variables, $\{g_{ve}, n_{ef}\}_B$. Even though it is possible to compute $\det H$ explicitly in terms of them, and obtain a numeric answer if numeric data are introduced for the EPRL variables, it is not clear whether it is nonzero for all possible values. For that reason, we will analyse the determination of EPRL boundary data from geometric constructions, in order to obtain values for $H$ in concrete cases.

While showing consistency of the EPRL behaviour with Einstein theory in such examples is not a proof for the general case even within $\Delta_3$, it would nevertheless be an interesting result, and on the flipside, an inconsistency would be a significant negative result on its own. To summarize the possible outcomes:

- $\det H = 0$: then the stationary phase method is not valid (in particular the quantity $A$ is not defined), and we must find a different method to evaluate the asymptotics;
- $\det H \neq 0$ and $A = 0$: in that case the Gaussian distribution (73) has infinite standard deviation and as such will not specify the semiclassical value of $x$, failing to reproduce the expected classical result;
- $\det H \neq 0$ and $A \neq 0$: the Gaussian distribution around the semiclassical value of $x$ should guarantee reproduction of the expected geometric values. In particular, if this happens for a certain boundary configuration, continuity conditions assure that the EPRL asymptotics match the expected classic solutions in an open neighbourhood of the con-
figuration, which would give us some confidence that the semiclassical limit is correct for a significant range of boundary data. It does not, however, discard the possibility of there existing isolated points in the critical surface for which one of the two situations above happen. It is unclear how this would affect the overall statistics.

4.3. Constructing EPRL spin foam variables from geometrical data

To obtain the EPRL spin foam variables \( g_{ve}, \vec{n}_{ef}, j_f \) for a given example, we will carry the procedure of the reconstruction theorem backwards, and determine how they are related to the geometrical data which defines the classical triangulation \( \Delta \). Obtaining the spins \( j_f \) is straightforward. Indeed, it has already been established that \( j_f \) are directly related to the triangle areas via \( A_f = \gamma j_f \) (within our semiclassical approximation of \( \gamma \) being large).

The Livine–Speziale coherent states \( |n_{ef}\rangle \) are expressed in terms of \( \vec{n}_{ef} \in S^2 \), the normal vectors to the tetrahedron faces’ Euclidean images in the tangent spaces \( T_e \Delta = \mathbb{R}^3 \), and phases \( \alpha_{ef} \in U(1) \). The latter can be consistently defined by imposing Regge boundary conditions but do not affect the model’s dynamics, and can be ignored. The one difficulty in correctly identifying the \( \vec{n}_{ef} \) is that computing the norms of the geometrical tetrahedra in \( \mathbb{R}^3 \) does not immediately tell you which \( n_{ef} \) is which within a certain tetrahedron. A solution to this issue is to consider gluing matrices. Indeed, considering a gluing equation

\[
R(g_{ve}^+)\vec{n}_{ef} = -R(g_{ve}^-)\vec{n}_{ef},
\]

notice that the + and − equations contained in it can both be manipulated to give the value of \( \vec{n}_{ef} \), and therefore

\[
(R^{-1}(g_{ve}^+)R(g_{ve}^-) - R^{-1}(g_{ve}^-)R(g_{ve}^+))\vec{n}_{ef} = 0.
\]

Defining the matrix in brackets as the gluing matrix between two tetrahedra, \( R_{ve} \), \( \vec{n}_{ef} \) must lie in its null space, and furthermore, if the tetrahedron is non-degenerate (which we are assuming it is) such null space must have dimension 1. Comparing the resultant null spaces with the normals of the geometric tetrahedra then gives the correct answer for \( \vec{n}_{ef} \).

Obtaining the \( g_{ve} \) is somewhat less trivial. The first step is to identify what they represent geometrically. Indeed, \( g_{ve} \) are \( SO(4) \) group elements related to the triangulated equivalent of the spin connection, which in the geometrical setup translates to mapping the geometrical tetrahedron \( e \in v \) to its image in the tangent space \( T_e \Delta \). We have to define what this means, though.

Consider a 4-simplex \( v \in \Delta \) and a tetrahedron \( e \in v \) defined by points \( p_1, ..., p_4 \). Note that for a general triangulation each 4-simplex lives on its own copy of \( \mathbb{R}^4 \); if the entire triangulation can be embedded isometrically in \( \mathbb{R}^4 \) that implies all the deficit angles are zero and the triangulation is flat. We will define the tetrahedron’s geometric matrix \( M_{ve} \) and projected matrix \( M_{ve}^{(3)} \):

• to construct \( M_{ve} \), consider an oriented trivector \( \tau_{ve} = \{ \tau_{ve}^1, \tau_{ve}^2, \tau_{ve}^3 \} \) consisting of the three edge vectors coming out of a previously defined pivot point. For example, if \( p_1 \) is chosen as the pivot, a possible trivector is \( \{ p_2 - p_1, p_3 - p_1, p_4 - p_1 \} \). If \( e \) is non-degenerate, the trivector defines a (non-orthonormal) basis of the 3D hyperplane \( e \) lives on, which can be equated to \( T_e \Delta \). Compute the normal to this hyperplane, \( \vec{N}_{ve} \), which is the normal to the tetrahedron. Note that there are two possible orientations for this normal, so we will

\footnote{It is still necessary to consider the geometric tetrahedra with this procedure since simply solving (78) gives the correct normals up to a minus sign, which must be fixed in accordance with geometric consistency.}
establish as a convention that the orientation to choose is the one that makes \( \det M_{ve} > 0 \). The full matrix is then

\[
M_{ve} = \{N_{ve}, \tau_{ve}^1, \tau_{ve}^2, \tau_{ve}^3\}. \tag{79}
\]

Note that this matrix is, by construction, invertible, since its 4 columns are linearly independent.

- for \( M_{ve}^{(3)} \), write down an orthonormal basis of \( T_e \Delta \) as defined above, for example using the Gram–Schmidt orthonormalization algorithm, and determine the coordinates of the vectors in \( \tau_{ve} \) on that basis. Call them \( \tau_{ve}^{(3)} \). We will regard \( T_e \Delta \) as a subspace of \( \mathbb{R}^4 \) normal to \( (1, 0, 0, 0) \), since it will help with decomposing \( g_{ve} \) into its SU(2) components \( g_{ve}^\pm \). The projected tetrahedron matrix is then

\[
M_{ve}^{(3)} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & (\tau_{ve}^{(3)}_1)^3 & (\tau_{ve}^{(3)}_2)^3 \\
0 & (\tau_{ve}^{(3)}_3)^3 & (\tau_{ve}^{(3)}_4)^3 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \tag{80}
\]

which is also invertible by the same reasons as above. Note that \( M_{ve} \) is not unique to a tetrahedron, but the \( g_{ve} \) rotation will be well defined provided that the orientations of both are consistent with respect to the considerations of section 2, that is, deriving the orientation of each tetrahedron from the 4-simplex \( v \) by (4) and permuting the edge vectors in \( \tau_{ve} \) to guarantee the same sign for all \( M_{ve} \) associated with \( v \). With these definitions in place, \( g_{ve} \) is the SO(4) matrix that rotates the projected matrix into the geometric matrix, i.e.

\[
g_{ve} \cdot M_{ve}^{(3)} = M_{ve} \iff g_{ve} = M_{ve} (M_{ve}^{(3)})^{-1}. \tag{81}
\]

Next step is to find \( g_{ve} \)'s SU(2) components. To do this we will use a result of van Elfrinkhof [31] which gives an algorithm for decomposition of a SO(4) rotation into left- and right-iso-clinic rotations, which can each be associated to SU(2) elements. Given a matrix \( g \in \text{SO}(4) \), define the associate matrix

\[
\text{Asc}(g) = \frac{1}{4} \begin{bmatrix}
\mathbf{g}_{00} + \mathbf{g}_{11} + \mathbf{g}_{22} + \mathbf{g}_{33} & \mathbf{g}_{10} - \mathbf{g}_{01} - \mathbf{g}_{23} + \mathbf{g}_{32} & \mathbf{g}_{20} + \mathbf{g}_{31} - \mathbf{g}_{02} - \mathbf{g}_{13} & \mathbf{g}_{30} - \mathbf{g}_{12} + \mathbf{g}_{01} - \mathbf{g}_{23} \\
\mathbf{g}_{10} - \mathbf{g}_{01} + \mathbf{g}_{23} - \mathbf{g}_{32} & \mathbf{g}_{11} - \mathbf{g}_{22} - \mathbf{g}_{33} & \mathbf{g}_{21} - \mathbf{g}_{32} + \mathbf{g}_{03} & \mathbf{g}_{31} + \mathbf{g}_{02} - \mathbf{g}_{13} - \mathbf{g}_{23} \\
\mathbf{g}_{20} - \mathbf{g}_{31} + \mathbf{g}_{02} - \mathbf{g}_{13} & \mathbf{g}_{21} - \mathbf{g}_{32} + \mathbf{g}_{03} & \mathbf{g}_{30} - \mathbf{g}_{01} - \mathbf{g}_{23} + \mathbf{g}_{13} & \mathbf{g}_{01} + \mathbf{g}_{22} - \mathbf{g}_{33} - \mathbf{g}_{13} \\
\mathbf{g}_{30} + \mathbf{g}_{12} - \mathbf{g}_{03} & \mathbf{g}_{31} + \mathbf{g}_{02} - \mathbf{g}_{13} & \mathbf{g}_{01} + \mathbf{g}_{22} - \mathbf{g}_{33} - \mathbf{g}_{13} & \mathbf{g}_{10} + \mathbf{g}_{31} - \mathbf{g}_{03} - \mathbf{g}_{23}
\end{bmatrix},
\tag{82}
\]

van Elfrinkhof’ s theorem states that \( \text{Asc}(g) \) has rank one and is normalized under the Euclidean norm, \( \sum_q (\text{Asc}(g)_q)^2 = 1 \), and that there exists a duo of vectors \( (a, b, c, d) \) and \( (p, q, r, s) \) in \( S^3 \times S^3 \) such that

\[
\text{Asc}(g) = \begin{bmatrix}
ap & aq & ar & as \\
pb & bq & br & bs \\
cp & cq & cr & cs \\
dp & dq & dr & ds
\end{bmatrix}.
\tag{83}
\]
More precisely, there are exactly two vector pairs in $S^3 \times S^3$ that satisfy this, since for a given $\{(a, b, c, d), (p, q, r, s)\}$, their opposites $\{(-a, -b, -c, -d), (-p, -q, -r, -s)\}$ also constitute a solution. Since there is a group isomorphism between $S^3$ and SU(2) given by

$$\phi: S^3 \rightarrow SU(2)$$

$$(a, b, c, d) \rightarrow a \mathbf{1} + i (b\sigma_1 + c\sigma_2 + d\sigma_3),$$

(84)

where $\sigma_i$ are the Pauli matrices and $\mathbf{1}$ is the identity matrix in SU(2), the aforementioned vector duos are directly mapped to SU(2) group elements. The decomposition is made explicit within SO(4) by the formula

$$g = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ -c & -d & b & a \end{pmatrix} \cdot \begin{pmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{pmatrix},$$

(85)

where the left and right matrices are left-isoclinic and right-isoclinic, respectively. (85) can also be specified neatly in quaternion notation. Consider the set of quaternions $\mathbb{H} \approx \mathbb{R}^4$ with the basis vectors $1, i, j, k$. $\mathbb{H}$ can also be defined in $\mathbb{C}^{2 \times 2}$ by extending the domain of the map $\phi$ in (84) to all of $\mathbb{R}^4$. Using the latter formulation, the SU(2) × SU(2) action on a vector $v \in \mathbb{H}$ is neatly written as

$$(g^+, g^-) \cdot v = g^+ v (g^-)^{-1}$$

(86)

and translates to the action of the SO(4) matrix with $(g^+, g^-)$ as its left and right isoclinic components according to the van Elfrinkhof formula. We will use these results to establish the correspondence

$$g_{ve}^+ = \phi(a, b, c, d)$$

$$g_{ve}^- = |\phi(p, q, r, s)|^{-1}.$$ (87)

Now there is an issue with this definition, which is the ambiguity between which of the two vector pairs that solve the van Elfrinkhof theorem to choose for each $g_{ve}$ in order to maintain consistency, since SU(2) × SU(2) double covers SO(4). We will address this problem by establishing an algorithm. For notation simplicity write $M \equiv \text{Asc}(g)$. First analyze cases where $M_{11} \neq 0$ (resulting that $a, p \neq 0$). Define

$$K = \sqrt{M_{11}^2 + M_{12}^2 + M_{13}^2 + M_{14}^2}.$$ (88)

Since, using (83),

$$p = \frac{M_{11}}{a}; q = \frac{M_{12}}{a}; r = \frac{M_{13}}{a}; s = \frac{M_{14}}{a}$$

(89)

and $p^2 + q^2 + r^2 + s^2 = 1$, it follows that $a = \pm \sqrt{M_{11}^2 + M_{12}^2 + M_{13}^2 + M_{14}^2} = \pm K$. For the sake of consistency we will always take the positive root $a = K$. It is then straightforward to obtain

$$p = M_{11} / K; q = M_{12} / K; r = M_{13} / K; s = M_{14} / K$$

$$a = K; b = K \frac{M_{21}}{M_{11}}; c = K \frac{M_{31}}{M_{11}}; d = K \frac{M_{41}}{M_{11}}.$$ (90)
Whenever \( M_{11} \neq 0 \) this algorithm provides a consistent definition of the \( g^+ \) and \( g^- \), but when \( M_{11} = 0 \) a similar process can be carried out by choosing a non-zero entry \( M_{ij} \) (it exists since both parameter vectors are non-zero) and defining

\[
K = \sqrt{\sum_{i=1}^{4} M_{ij}^2}.
\]  

(91)

If we use the notation \((a, b, c, d) \equiv (x_1, x_2, x_3, x_4)\) and \((p, q, r, s) = (y_1, y_2, y_3, y_4)\) then we can define a solution for them as follows:

\[
x_i = K
\]
\[
y_l = \frac{M_{il}}{K}, \quad l \in \{1, 2, 3, 4\}
\]
\[
x_i = K \frac{M_{ij}}{M_{ij}}, l \neq i.
\]

(92)

To finalize this section we will mention the two geometrical examples considered for this study. Given the circumstances of the flatness problem, it was deemed appropriate to consider a flat and a non-flat version of \( \Delta_3 \) in calculations. As mentioned above, a flat triangulation is easily defined by considering an embedding of it in \( \mathbb{R}^4 \), but it’s somewhat less trivial to define a non-flat one. For the latter we will consider a figure analogous to a triangulation of \( S^4 \) by taking an embedding of \( \Delta_3 \) into \( \mathbb{R}^5 \) given by an equilateral 5-simplex centered at the origin. This embedding is defined by assigning the 6 points of \( \Delta_3 \) into the 6 points of the 5-simplex.

Let us define the equilateral 5-simplex by building it ‘from the ground up’ from an equilateral triangle centered at the origin. A triangle in \( \mathbb{R}^2 \) with the desired characteristics is given by\[
\{ A_2, B_2, C_2 \} = \left\{ \left( -\frac{1}{2} - \frac{1}{2\sqrt{3}} \right), \left( \frac{1}{2}, \frac{1}{2\sqrt{3}} \right), \left( 0, \frac{1}{\sqrt{3}} \right) \right\}.
\]

(93)

Adding the third axis \( x^2 \) we see that if a fourth point is \( D_5 = (0, 0, a_5) \), then the tetrahedron formed by

\[
\{ A_3, B_3, C_3, D_3 \} = \left\{ \left( -\frac{1}{2} - \frac{1}{2\sqrt{3}}, -\frac{a_3}{3}, \frac{1}{2\sqrt{3}} \right), \left( \frac{1}{2}, -\frac{1}{2\sqrt{3}}, -\frac{a_3}{3} \right), \left( 0, \frac{1}{\sqrt{3}}, -\frac{a_3}{3} \right), (0, 0, a_3) \right\}
\]

(94)

is centered in the origin and \( a_3 \) can be fixed to make it equilateral by forcing \( C_3D_5 = 1 \). (Note that if \( O_3 \) is the centre of the triangle \( A_3B_3C_3 \) then \( O_3D_5 \) is normal to said triangle and therefore \( A_3D_5 = B_3D_5 = C_3D_5 \).) Solving that constraint gives \( a_3 = \sqrt{\frac{3}{2}} \).

Similarly, we construct a 4-simplex under the same conditions by adding the axis \( x^3 \), defining the point \( E_4 = (0, 0, 0, a_4) \) and considering the 4-simplex

\[
\{ A_4, B_4, C_4, D_4, E_4 \} = \left\{ \left( -\frac{1}{2} - \frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, -\frac{a_4}{4}, \frac{\sqrt{3}}{2} \right), \left( \frac{1}{2}, -\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, -\frac{a_4}{4} \right), \left( 0, \frac{1}{\sqrt{3}}, -\frac{a_4}{4} \right), (0, 0, 0, a_4) \right\}.
\]

(95)

By analogous argument to what we used for the tetrahedron, this 4-simplex is centered in the origin and will be equilateral if \( D_4E_4 = 1 \), which is solved to give \( a_4 = \sqrt{\frac{3}{2}} \).
Finally, we add the axis \( x^4 \), define \( F_5 = (0, 0, 0, 0, \alpha_5) \) and consider the 5-simplex
\[
\{A_5, B_5, C_5, D_5, E_5, F_5\} = \left\{ \left( -\frac{1}{3} - \frac{1}{3} \sqrt{5} - \frac{1}{3} \sqrt{5} \right), \left( \frac{1}{3} - \frac{1}{3} \sqrt{5} - \frac{1}{3} \sqrt{5} \right), \left( 0, 0, 0, \frac{1}{5} \sqrt{5} \right), \left( 0, 0, 0, \frac{1}{5} \sqrt{5} \right) \right\}. 
\]

The 5-simplex has the characteristics we need if \( \mathcal{E}_5 F_5 = 1 \), which is satisfied when \( \alpha_5 = \sqrt{\frac{5}{12}} \).

The coordinates of the equilateral 5-simplex to be used are therefore
\[
\{A_5, B_5, C_5, D_5, E_5, F_5\} = \left\{ \left( -\frac{1}{3} - \frac{1}{3} \sqrt{5} - \frac{1}{3} \sqrt{5} \right), \left( \frac{1}{3} - \frac{1}{3} \sqrt{5} - \frac{1}{3} \sqrt{5} \right), \left( 0, 0, 0, \frac{1}{5} \sqrt{5} \right), \left( 0, 0, 0, \frac{1}{5} \sqrt{5} \right) \right\}. 
\]

This example is particularly simple in numeric terms since the construction implies that all triangles have the same area \( A_f = \sqrt{3}/4 \), and the normal vectors \( \vec{n}_{ef} \) can all be derived from the same equilateral tetrahedron in \( \mathbb{R}^3 \), only taking care to match their orientations correctly.

For the flat example, we considered an embedding of \( \Delta_3 \) in \( \mathbb{R}^4 \) using the coordinates
\[
a = \left( -\frac{1}{2}, -\frac{1}{2} \sqrt{3}, 0, 0 \right) 
b = \left( \frac{1}{2}, -\frac{1}{2} \sqrt{3}, 0, 0 \right) 
c = \left( 0, \frac{1}{3}, 0, 0 \right) 
d = \left( 0, 0, \frac{1}{2}, -\frac{1}{2} \sqrt{3} \right) 
e = \left( 0, 0, \frac{1}{2}, -\frac{1}{2} \sqrt{3} \right) 
f = \left( 0, 0, 0, \frac{1}{3} \sqrt{3} \right). 
\]

The ancillary files available online as supplementary data (stacks.iop.org/CQG/35/095003/media) include detailed Mathematica code for computing the spin foam variables \( g_{ef}, \vec{n}_{ef}, j_f \) of both geometrical configurations, and then using them to determine the relevant derivatives \( K_e \) and \( H_{jp} \) as well as the decay parameter \( \Lambda = -3 \left( K,K \right) \). Here we will only state the results, which unfortunately could only be obtained in numeric form for the coordinates chosen and a given value of the Immirzi parameter. Note that the Immirzi parameter must be consistent with triangle areas to ensure that the values of \( j_f \) are half-integer, and according to the EPRL prescription \( 0 < \gamma < 1 \). The results were
\[ \Delta_3^{(\text{curved})} : \text{used } \gamma = \frac{\sqrt{3}}{2}, A = 6.62021 \]
\[ \Delta_3^{(\text{flat})} : \text{used } \gamma = \frac{1}{2} \sqrt{\frac{5}{3}}, A = 14.4389. \]

(99)

The significant finding here is that they are both nonzero within numerical error, and therefore in the examples considered the asymptotic spin foam analysis matches what is expected from general relativity.

5. Conclusions and future work

The main result to be taken from this paper is that, for the two example \( \Delta_3 \) configurations considered, the zero-order semiclassical limit of the EPRL/FK model is consistent with Regge calculus, and therefore general relativity. By continuity arguments, the same is valid in an open neighbourhood of each critical point—and in particular, the flatness problem is shown not to happen for this subset of triangulations. While this is an incomplete result, it is nevertheless a positive step towards proving the model’s asymptotic consistency with GR. There are two key observations which lead to this result:

- Firstly, varying the asymptotic EPRL action with respect to the discrete spins \( j_f \) is a delicate issue. We have decided not to follow the original idea of simply ignoring discreteness and taking an ad hoc continuum approximation to perform the variation. Instead we attempted to explicitly acknowledge the discreteness of \( j \) and study its effects on the statistics of the partition function. For the \( \Delta_3 \) triangulation in particular, since there is only one internal face spin \( j \), the dependence on it can be expressed as a sum over \( j \) of integrals on the continuous variables \( g_{ve}, n_{ef} \). The Malgrange preparation theorem and its corollaries can then be used to apply the stationary phase method, and make explicit the distribution with respect to \( j \) in a neighbourhood of the critical point. However, the validity of this method depends on the \( A \) quantity defined in section 4.2 being finite and mathematically meaningful. Finding \( A \) in the general case requires computing the Hessian determinant of the action at the (singular) critical point for any possible boundary data, a calculation which so far appears too cumbersome to be feasible. Since we have not been able to solve the general case yet, we have settled with choosing two sets of example boundary data—one curved and one flat geometry, so they could be compared in light of the flatness problem. \( A \) was then numerically computed for both, and the result was non-zero and finite in both cases.

- Secondly, at the classical level, the spin foam data for \( \Delta_3 \)'s interior triangle \( f \) are entirely specified by the boundary data, and it was shown that the singular critical point corresponds to the area and deficit angle of \( f \) consistent with classical geometricity of the triangulation. We developed a process to perform the converse of the reconstruction theorem and recover EPRL variables from geometric variables. This process allows for direct association of a classical configuration with the semiclassical EPRL one, simply by specifying geometric boundary data. It is then possible to verify consistency of EPRL with Regge calculus for given boundary data sets, which was done in both examples considered.

A few further remarks. We note that Perini and Magliaro [28] have already made the claim that the critical point for a given boundary configuration is unique and corresponds to the expected classical geometry. However, the subtleties regarding the partition function \( Z \)'s
distribution over \(j\) are not addressed in their work. In particular, the fact that the classical \(j_0\) may not be an integer and there is a range of \(j\) near \(j_0\) that contributes significantly to the partition function is not noted. It is just assumed that non-critical configurations are exponentially suppressed. In our analysis, when \(A \neq 0\), the distribution of \(Z\) near the critical point is indeed dictated by a Gaussian distribution, but its width increases with \(\lambda\), although the relative uncertainty \(\Delta j / j \approx \Delta j / \lambda\) is suppressed for large \(\lambda\). We believe further analysis in a broader set of examples is necessary to clarify these issues.

On the flatness problem, we have stated a counter-argument to its original formulation based on the continuum \(j\)-equation, and our results hint that it might not be present (although it must be reiterated that we do not have a general proof). However, Hellmann/Kaminski seem to have recovered it under a more rigorous approach with their holonomy spin foam formalism. It would be important to investigate any possible incompatibilities between both formalisms.

It is also a priority for future work to explore the behaviour of \(A\) in different configurations. Obtaining a full expression for all possible boundaries is quite challenging and we are not certain if it is possible, but studying algebraic properties of the boundary data could lead to exact expressions for \(A\) in certain subsets of possible configurations. This could be useful in further generalizing the results presented in this paper.

Appendix. Geometric interpretation of the quantum deficit angle

We will attempt to find a compact expression for the deficit angle \(\tilde{\Theta}\) using the data of section 4.1. The ‘quantum deficit angle’ for \(A_3\) is

\[
\tilde{\Theta} = \pm 2i \sum_x (1 + \gamma) \left[ \log \langle \mathcal{J} n_{a|c} \rangle \langle g^{\pm}_{\alpha} \rangle \gamma^{\pm}_{\beta} |n_{ab}\rangle + \log \langle \mathcal{J} n_{a|c} \rangle \langle g^{\pm}_{\alpha} \rangle \gamma^{\pm}_{\beta} |n_{ac}\rangle \right] + \log \langle \mathcal{J} n_{a|b} \rangle \langle g^{\pm}_{\alpha} \rangle \gamma^{\pm}_{\beta} |n_{ab}\rangle + \log \langle \mathcal{J} n_{a|b} \rangle \langle g^{\pm}_{\alpha} \rangle \gamma^{\pm}_{\beta} |n_{bc}\rangle .
\]  

(A.1)

We will focus on the first of the three matrix elements in the above expression. The results for the other two can be easily extrapolated by symmetry. In order to perform the necessary computations, we will use the following parameterizations of \(SU(2)\) and the Hilbert space \(\mathcal{H}^{1/2}\) of spin \(\frac{1}{2}\) states:

- For the \(SU(2)\) variables, we use the decomposition

\[
\forall g \in SU(2), \quad g = z^\alpha \Sigma_{\alpha}, \quad (z^0)^2 + (z^1)^2 + (z^2)^2 + (z^3)^2 = 1
\]  

(A.2)

where \(\Sigma_0 = 1\) and \(\Sigma_i = i \sigma_i\), for \(i = 1, 2, 3\) (\(\sigma_i\) are the Pauli matrices). \(SU(2)\) is therefore diffeomorphic to \(S^3\), and considering the change of variables

\[
\begin{align*}
\zeta^0 & = \cos \gamma \cos \beta^1 \\
\zeta^3 & = \cos \gamma \sin \beta^1 \\
\zeta^1 & = \sin \gamma \cos \beta^2 \\
\zeta^2 & = \sin \gamma \sin \beta^2,
\end{align*}
\]  

(A.3)

with Jacobian \(\frac{\sin(2\gamma)}{2}\), where \(0 < \beta^i < 2\pi\) and \(0 < \gamma < \frac{\pi}{2}\), it follows that a general \(SU(2)\) matrix can be written as

\[
g = \begin{bmatrix}
\cos \gamma e^{i\beta^1} & i \sin \gamma e^{-i\beta^2} \\
-i \sin \gamma e^{i\beta^2} & \cos \gamma e^{-i\beta^1}
\end{bmatrix}.
\]  

(A.4)
For the $H^{1/2}$ variables, naively, one could parametrize them as follows:

$$\forall |n\rangle \in H^{1/2}, |n\rangle = \begin{bmatrix} w^0 + i w^1 \\ w^2 + i w^3 \end{bmatrix}, (w^0)^2 + (w^1)^2 + (w^2)^2 + (w^3)^2 = 1$$  \hspace{1cm} (A.5)$$

obtaining $\int_{H^{1/2}} dn = \int_S dw$. However, it is advantageous to consider a change of variables that reflects the construction of a coherent state. Recall that

$$|n\rangle = e^{i\alpha} G(\bar{n}) |+\rangle$$  \hspace{1cm} (A.6)$$

where $\bar{n} \in S^2$, $\alpha$ is an undetermined phase and $|+\rangle = (1, 0)$ is the eigenstate of $J_z$ with eigenvalue $\frac{1}{2}$. The SU(2) element $G(\bar{n})$ is the rotation that takes $\vec{z}$ to $\vec{n}$ and is readily calculated. Consider the parameterization of $S^2$ in spherical coordinates

$$\bar{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).$$  \hspace{1cm} (A.7)$$

To go from $\vec{z}$ to $\vec{n}$ we perform a rotation of angle $\theta$ around the axis $\vec{n}_\perp = (-\sin \phi, \cos \phi, 0)$. From this we get

$$G(\bar{n}) = \exp \left( \frac{i \theta}{2} \vec{\sigma} \cdot \vec{n}_\perp \right)$$

$$= \exp \left( \frac{i \theta}{2} (\cos \phi \sigma_y - \sin \phi \sigma_x) \right)$$

$$= \begin{bmatrix} \cos \frac{\theta}{2} & e^{-i\phi} \sin \frac{\theta}{2} \\ -e^{i\phi} \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{bmatrix}$$

(A.8)

and therefore

$$|n\rangle = e^{i\alpha} \begin{bmatrix} \cos \frac{\theta}{2} \\ -e^{i\phi} \sin \frac{\theta}{2} \end{bmatrix}. \hspace{1cm} (A.9)$$

The Jacobian of the change of coordinates from $\vec{w}$ to $(\theta, \phi, \alpha)$ is $\frac{\sin(\theta)}{2}$.

Since the matrix element $\langle n_{AC} | (g_{\pm C}^4 \dagger g_{\pm C}^5 | n_{CB}\rangle$ is a scalar, it does not depend on the choice of basis in $H^{1/2}$. Since the vector part for each of the coherent states present is the same, we will choose a basis in which $\vec{n}_{AB} = \bar{n} = (0, 0, 1)$ to carry out computations$^9$. This translates to

$$|n_i\rangle = e^{i\alpha_i} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \hspace{1cm} (A.10)$$

for $i \in \{BA, CB, AC\}$. Notice that due to each of the coherent states appearing exactly once as a bra and a ket in (A.15), the contribution of the phases $\alpha_i$ will cancel out and we can just consider $|n\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ from now on. With the coherent states taken care of, we can move on to $g_{C4}^\pm$ and $g_{C5}^\pm$. We need to use the gluing conditions (48) to relate the two in order to exhaust the constraints incurring from them, so we will also need an expression for $R(g)$ for $g \in SU(2)$.

Westra$^{10}$ gives us a parameterization for $g = \begin{bmatrix} x & y \\ -\bar{y} & \bar{x} \end{bmatrix}$, $|x|^2 + |y|^2 = 1$:

$^9$There appears to be an ambiguity with this choice, coming from the parameterization of $S^2$ in spherical co-
ordinates—$\bar{n} = (0, 0, 1)$ is obtained when $\theta = 0$, which makes $\phi$ undefined. But it is evident from (A.8) that

$G(0,0,1) = I$.

$^{10}$www.mat.univie.ac.at/~westra/so3su2.pdf
\[ R(g) = \begin{bmatrix} \Re(x^2 - y^2) & \Im(x^2 + y^2) & -2\Re(xy) \\ -\Im(x^2 - y^2) & \Re(x^2 + y^2) & 2\Im(xy) \\ 2\Re(xy) & 2\Im(xy) & |x|^2 - |y|^2 \end{bmatrix}. \] (A.11)

In our set of coordinates for SU(2), \( x = \cos \gamma e^{i\beta} \) and \( y = i \sin \gamma e^{-i\beta} \), hence we can write

\[ R(g) = \begin{bmatrix} \cos^2 \gamma \cos(2\beta) + \sin^2 \gamma \cos(2\beta) & \cos^2 \gamma \sin(2\beta) + \sin^2 \gamma \sin(2\beta) & \sin(2\gamma) \sin(\beta + \beta) \\ -\cos \gamma \sin(2\beta) + \sin \gamma \sin(2\beta) & -\cos \gamma \cos(2\beta) - \sin \gamma \cos(2\beta) & \sin(2\gamma) \cos(\beta + \beta) \\ \sin(2\gamma) \sin(\beta + \beta) & -\sin(2\gamma) \cos(\beta + \beta) & \cos(2\gamma) \end{bmatrix}. \] (A.12)

While daunting at first, this expression becomes more tractable within the context of the gluing condition and the basis choice we made for \( \bar{\eta}_{\text{AB}} \). The gluing condition is reduced to

\[ \begin{bmatrix} \sin(2\gamma_A) \sin(\beta^+_A - \beta^+_B) \\ \sin(2\gamma_A) \cos(\beta^+_A - \beta^+_B) \\ \cos(2\gamma_A) \end{bmatrix} = \begin{bmatrix} \sin(2\gamma_B) \sin(\beta^-_B - \beta^-_A) \\ \sin(2\gamma_B) \cos(\beta^-_B - \beta^-_A) \\ \cos(2\gamma_B) \end{bmatrix}. \] (A.13)

where the variables labelled \( A \) pertain to \( g_{A_1} \) and the ones labelled \( B \) pertain to \( g_{B_1} \), and we omit the \( \pm \) index for simplicity. It is clear that the gluing condition does not fix \( g_{A_2} \) completely given \( g_{B_1} \), since they only depend on the differences \( \beta^+_A - \beta^+_B \equiv \delta_{A,B} \). Analysing the equations,

- the third equation implies \( \gamma_A = \gamma_B = \gamma \), since \( 2\gamma_{A,B} \in [0, \pi] \) and the cosine function is injective in this domain;
- given that \( \gamma_A = \gamma_B \), the first and second equations read \( \sin \delta_A = \sin \delta_B \) and \( \cos \delta_A = \cos \delta_B \), which for \( \delta_{A,B} \in [0, 2\pi] \) is enough to infer \( \delta_A = \delta_B \).

Hence, we have that, in our chosen basis for \( \mathcal{H}/2 \), if \( g_{A_1}^\pm \) is given by the coordinates \((\gamma_\pm, \beta^+_\pm, \beta^2_\pm)\), then \( g_{B_1}^\pm \) is given by \( (\gamma_\pm, \beta^+_\pm + \epsilon_\pm, \beta^2_\pm + \epsilon_\pm) \) where \( \epsilon_\pm \in [0, 2\pi] \). We can now compute \( \langle n | \left(g_{A_1}^\pm \right)^+ | g_{B_1}^\pm | n \rangle \):

\[ \langle n | \left(g_{A_1}^\pm \right)^+ | g_{B_1}^\pm | n \rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \cos \gamma e^{-i\beta} & -i \sin \gamma e^{-i\beta} \\ -i \sin \gamma e^{i\beta} & \cos \gamma e^{i\beta} \end{bmatrix} \begin{bmatrix} \cos \gamma e^{i(\beta^2 + \epsilon)} & i \sin \gamma e^{i(\beta^2 + \epsilon)} \\ i \sin \gamma e^{-i(\beta^2 + \epsilon)} & \cos \gamma e^{-i(\beta^2 + \epsilon)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \gamma e^{i(\beta^2 + \epsilon)} & i \sin \gamma e^{i(\beta^2 + \epsilon)} \\ i \sin \gamma e^{-i(\beta^2 + \epsilon)} & \cos \gamma e^{-i(\beta^2 + \epsilon)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \] (A.14)

Taking logarithms, we get simply \( i \epsilon \), and substituting (with proper labels) on the expression for \( \Theta \) and repeating the process for the other two inner products in \( \Theta \) (we shall identify the variables pertaining to each of these terms with an index \( i \in \{1, 2, 3\} \)), we obtain

\[ \Theta = \pm 2 \sum_{i=1}^{3} (1 + \gamma_i) \sum_{i=1}^{3} \epsilon_i^\pm. \] (A.15)

Remember that all \( g_{\nu} \) have been determined earlier using the interior-boundary conditions. Therefore, the \( \epsilon_i^\pm \) can be expressed in terms of the boundary data through some simple algebra. We give an example. \( R(g_{A_1}^\pm) \) and \( R(g_{B_1}^\pm) \) are known. Let us call them \( A, B \) for simplicity.
Using the parameterization (A.12), we want to find either $\beta_1$ or $\beta_2$ for each matrix, and take their difference to obtain $\epsilon$. Step by step:

- $\gamma$ is obtained through $\cos(2\gamma) = A_{33}$. Since $2\gamma \in [0, \pi]$, the cosine function is injective in this domain and we can write $\gamma = \frac{1}{2} \cos^{-1}(A_{33})$. There will be three cases to consider due to the possibility of $\sin(2\gamma)$ being zero.

- If $0 < \gamma < \pi/2$, it’s easy to extract the sine and cosine of $\beta_1 \pm \beta_2$ through $A_{31}$, $A_{32}$ and $A_{12}$, $A_{13}$ respectively. The angles can then be obtained using the angle function $A_1(x, y) \equiv 2 \tan^{-1}\left(\frac{y}{x}\right)$. The result for $\beta_1$ is

  $$\beta_1 = \frac{1}{2} \left[ A_1 \left( \frac{A_{13}}{\sqrt{1 - A_{33}^2}}, \frac{A_{23}}{\sqrt{1 - A_{33}^2}} \right) + A_1 \left( \frac{A_{31}}{\sqrt{1 - A_{33}^2}}, \frac{A_{32}}{\sqrt{1 - A_{33}^2}} \right) \right].$$

  \hspace{1cm} (A.16)

- If $\gamma = 0$, it is readily seen that $R(g)$ does not depend on $\beta_2$ but $\beta_1$ has a simple expression

  $$\beta_1 = \frac{1}{2} A_1 \left( A_{12}, A_{11} \right).$$

  \hspace{1cm} (A.17)

- If $\gamma = \pi/2$, $R(g)$ does not depend on $\beta_1$ instead. $\beta_2$ is found to be

  $$\beta_2 = \frac{1}{2} A_1 \left( A_{12}, A_{11} \right)$$

  \hspace{1cm} (A.18)

so we can combine the two extremal cases into one, as they give the same formal expression for $\epsilon$.

We will now explain the relevance of the $\epsilon_\pm$. As seen in (A.15), the deficit angle $\tilde{\Theta}$ has a very simple expression in terms of them, and they can be interpreted geometrically. Indeed, note that the expression for $\tilde{\Theta}_f$ in a general face can be written as a sum over vertices, $\tilde{\Theta}_f = \sum_{v \in f} \tilde{\Theta}_v$. We know from Han/Zhang’s work (among others) that the action is interpreted as a holonomy around a certain face, going through all the vertices it belongs to. And in the expression for $\tilde{\Theta}_v$,

$$\tilde{\Theta}_v = \sum_{\pm} 2(1 \pm \gamma) \log \langle \mathcal{J} n_v | g_v^\pm g_{v'}^\pm | n_{v'} \rangle$$

\hspace{1cm} (A.19)
\[
\sim \sum_{\pm} 2(1 \pm \gamma)\epsilon_i^\pm
\]
(A.20)

the inner product clearly illustrates the parallel transport between the two tetrahedra in \(v\) which contain \(f\). Therefore \(\tilde{\Theta}_f\) can be associated to the internal angle \(\angle(e, e')_f\), as illustrated in figure 3.

The sum of all internal angles is equal to \(2\pi\) minus the deficit angle \(\Theta_{\text{Regge}}\), while the sum of all the \(\tilde{\Theta}_f\) should tend asymptotically to a sign factor times \(i\gamma\Theta_{\text{Regge}}\). Hence, the correct identification which relates the \(\epsilon\) to the internal angles is

\[
\pm \frac{i}{\gamma} \tilde{\Theta}_f = \pm \frac{2i}{\gamma} \sum_{\pm} \sum_i (1 \pm \gamma)\epsilon_i^\pm \sim \angle(e, e')_f.
\]
(A.21)

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