Comment on: “On the effects of the Lorentz symmetry violation yielded by a tensor field on the interaction of a scalar particle and a Coulomb-type field” Ann. Phys. 399 (2018) 117-123

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Abstract

We analyze the eigenvalues and eigenfunctions stemming from a recent study of the interaction of a scalar particle with a Coulomb potential in the presence of a background of the violation of the Lorentz symmetry

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established by a tensor field. We show, beyond any doubt, that the physical conclusions drawn by the authors from a truncation of a power series, coming from the application of the Frobenius method, are meaningless and nonsensical.

In a recent paper Vitória et al \cite{1} analyze the interaction of a scalar particle with a Coulomb-type potential in the presence of a background of the violation of the Lorentz symmetry established by a tensor field. The equation proposed by the authors is separable in cylindrical coordinates and the radial part is a solution to an eigenvalue equation with centrifugal-like ($r^{-2}$), Coulomb ($r^{-1}$) and harmonic ($r^2$) terms. The application of the Frobenius method leads to a three-term recurrence relation for the expansion coefficients and the authors force a truncation in order to obtain polynomial solutions. In this way they obtain analytical expressions for the energies of the system and conclude that there are permitted values of a parameter that characterizes the magnetic field.

The purpose of this Comment is the analysis of the effect of the truncation approach on the physical conclusions drawn by the authors.

The starting point of present discussion is the eigenvalue equation for the radial part of the Schrödinger equation

$$ F''(x) + \frac{1}{x} F'(x) - \frac{\gamma^2}{x^2} F(x) + \frac{\theta}{x} F(x) - x^2 F(x) + W F(x) = 0, $$

$$ W = \frac{\beta}{\tau}, \quad \beta = \mathcal{E}^2 - m^2 - p_z^2, \quad \gamma^2 = l^2 - \alpha^2, \quad \tau^2 = \frac{1}{2} gb\chi^2, \quad \theta = \frac{2\alpha E}{\sqrt{\tau}} \quad (1) $$

where $l = 0, \pm 1, \pm 2, \ldots$ is the rotational quantum number (restricted to $l^2 \geq \alpha^2$), $m$ the mass of the particle, $\alpha$ the strength of a Coulomb-type potential, $\mathcal{E}$ the energy, $b = -(K_{HB})_{zz} > 0$, $\chi$ comes from a magnetic field and $g$ is a constant. The constant $-\infty < p_z < \infty$ is the quantum number for the free motion along the $z$ direction. The authors simply set $\hbar = 1$, $c = 1$ though there are well known procedures for obtaining suitable dimensionless equations in a clearer and more rigorous way \cite{2}. In what follows we focus on the discrete values of $W$ that one obtains from the bound-state solutions of equation (1).
that satisfy
\[ \int_0^{\infty} |F(x)|^2 x \, dx < \infty. \] (2)

Notice that we have bound states for all \(-\infty < \theta < \infty\) and that the eigenvalues \(W\) satisfy
\[ \frac{\partial W}{\partial \theta} = -\left\langle \frac{1}{x} \right\rangle < 0, \] (3)
according to the Hellmann-Feynman theorem [3].

The eigenvalue equation (1) is an example of conditionally solvable (or quasi-exactly solvable) problems that have been widely studied by several authors and exhibit a hidden algebraic structure (see, for example, [4] and references therein).

In order to solve the eigenvalue equation (1) the authors proposed the ansatz
\[ F(x) = x^s \exp \left( -\frac{x^2}{2} \right) P(x), \quad P(x) = \sum_{j=0}^{\infty} a_j x^j, \quad s = |\gamma|, \] (4)
and derived the three-term recurrence relation
\[ a_{j+2} = -\frac{\theta}{(j+2)(j+2(s+1))} a_{j+1} + \frac{2j + 2s - W + 2}{(j+2)(j+2(s+1))} a_j, \]
\[ j = -1, 0, \ldots, a_{-1} = 0, \quad a_0 = 1. \] (5)

If the truncation condition \(a_{n+1} = a_{n+2} = 0\) has physically acceptable solutions then one obtains some exact eigenvalues and eigenfunctions. The reason is that \(a_j = 0\) for all \(j > n\) and the factor \(P(x)\) in equation (4) reduces to a polynomial of degree \(n\). This truncation condition is equivalent to \(W_s^{(n+1)} = 2(n+s+1)\) and \(a_{n+1} = 0\). The latter equation is a polynomial function of \(\theta\) of degree \(n+1\) and it can be proved that all the roots \(\theta^{(n,i)}_s\), \(i = 1, 2, \ldots, n+1\), \(\theta^{(n,i)}_s < \theta^{(n,i+1)}_s\), are real [5, 6]. If \(V(\theta, x) = -\theta/x + x^2\) denotes the parameter-dependent potential for the model discussed here, then it is clear that the truncation condition produces an eigenvalue \(W_s^{(n)}\) that is common to \(n+1\) different potential-energy functions \(V^{(n,i)}_s(x) = V(\theta^{(n,i)}_s, x)\). Notice that in this analysis we have deliberately omitted part of the interaction that has been absorbed into \(\gamma\) (or \(s\)) because it is not affected by the truncation approach. It is also worth noticing that the truncation condition only yields some particular eigenvalues and eigenfunctions because not all the solutions \(F(x)\) of (1) satisfying equation (2) have
polynomial factors $P(x)$. From now on we will refer to them as follows

\[ F_s^{(n,i)}(x) = x^s P_s^{(n,i)}(x) \exp \left( -\frac{x^2}{2} \right), \quad P_s^{(n,i)}(x) = \sum_{j=0}^{n} a^{(n,i)}_{j,s} x^j. \]  

(6)

We want to stress that the $n+1$ eigenfunctions $F_s^{(n,i)}(x)$, $i = 1, 2, \ldots, n+1$ share the same eigenvalue $W_s^{(n)}$, a point that was not taken into account by Vitória et al [1] and that is of utmost relevance, as shown below.

Let us consider the first cases as illustrative examples. When $n = 0$ we have $W_s^{(0)} = 2(s + 1)$, $\theta_s^{(0)} = 0$ and the eigenfunction $F_s^{(0)}(x)$ has no nodes. We may consider this case trivial because the problem reduces to the exactly solvable harmonic oscillator. Probably for this reason it was not explicitly considered by Vitória et al [1].

When $n = 1$ there are two roots $\theta_s^{(1,1)} = -\sqrt{4s+2}$ and $\theta_s^{(1,2)} = \sqrt{4s+2}$ and the corresponding non-zero coefficients are

\[ a^{(1,1)}_{1,s} = \frac{\sqrt{2}}{\sqrt{2s+1}}, \quad a^{(1,2)}_{1,s} = -\frac{\sqrt{2}}{\sqrt{2s+1}}, \]  

(7)

respectively. We appreciate that the eigenfunction $F_s^{(1,1)}(x)$ is nodeless and $F_s^{(1,2)}(x)$ has one node and that both correspond to the same eigenvalue $W_s^{(1)}$.

When $n = 2$ the results are

\[ \theta_s^{(2,1)} = -2\sqrt{4s+3}, \quad a^{(2,1)}_{1,s} = \frac{2\sqrt{4s+3}}{2s+1}, \quad a^{(2,1)}_{2,s} = \frac{2}{2s+1}, \]  

\[ \theta_s^{(2,2)} = 0, \quad a^{(2,2)}_{1,s} = 0, \quad a^{(2,2)}_{2,s} = -\frac{1}{s+1}, \]  

\[ \theta_s^{(2,3)} = 2\sqrt{4s+3}, \quad a^{(2,3)}_{1,s} = -\frac{2\sqrt{4s+3}}{2s+1}, \quad a^{(2,3)}_{2,s} = \frac{2}{2s+1}. \]  

(8)

Notice that $F_s^{(2,1)}(x)$, $F_s^{(2,2)}(x)$ and $F_s^{(2,3)}(x)$ have zero, one and two nodes, respectively, in the interval $0 < x < \infty$ and that the three eigenfunctions correspond to the same eigenvalue $W_s^{(2)}$.

From the truncation condition the authors derived

\[ \mathcal{E}_{n,l,p_s}^2 = m^2 + p_{z_s}^2 + 2\tau (n + |\gamma| + 1), \]  

(9)

as well as expressions for $\tau_{n,l,p_s}$ and $\chi_{n,l,p_s}$, $n = 1, 2$. They concluded that there are permitted values of $\chi$ that characterize the magnetic field. Since there are
square-integrable solutions to the eigenvalue equation (11) for all values of $\theta$ it is clear that such particular values of $\chi$ are just an artifact of the truncation method that yields particular solutions to the eigenvalue equation with polynomial factors $P_s^{(n,i)}(x)$. Besides, the allowed energies associated to the nodes $n = 1$ and $n = 2$ obtained by the authors have no physical meaning because they stem from different potentials $V_s^{(n,i)}(x)$. In what follows we discuss this point with more detail.

In order to make present discussion clear we write the actual eigenvalues of equation (11) as $W_{j,s}(\theta)$, $j = 0, 1, \ldots$, $W_{j,s} < W_{j+1,s}$. Given that there are square-integrable solutions for all $-\infty < \theta < \infty$, as indicated above, each eigenvalue can be considered to be a curve $W_{j,s}(\theta)$ in the $(\theta, W)$ plane. Therefore, the correct energies of the system should be

$$\mathcal{E}_{j,l,p_s} = m^2 + p_z^2 + \tau W_{j,s}. \quad (10)$$

Since the eigenvalue equation (11) is not exactly solvable, except for some particular values of $\theta$, we should resort to an approximate method in order to obtain the eigenvalues and eigenfunctions that are not given by the truncation condition. Here, we apply the well known Rayleigh-Ritz variational method that yields upper bounds to all the eigenvalues and choose the non-orthogonal basis set $\{x^{s+j} \exp\left(-\frac{x^2}{2}\right), \ j = 0, 1, \ldots\}$.

We arbitrarily choose $s = 0$ as a first illustrative example in order to facilitate the calculations. When $\theta = \theta_0^{(1,1)} = -\sqrt{2}$ the first four eigenvalues are $W_{0,0} = W_{1,1}^{(1)} = 4$, $W_{1,0} = 7.693978891$, $W_{2,0} = 11.50604238$, $W_{3,0} = 15.37592718$; on the other hand, when $\theta = \theta_0^{(1,2)} = \sqrt{2}$ we have $W_{0,0} = -1.459587134$, $W_{1,0} = W_{0,0}^{(1)} = 4$, $W_{2,0} = 8.344349427$, $W_{3,0} = 12.53290130$. Notice that the truncation condition yields only the ground state for the former model and the first excited state for the latter, missing all the other eigenvalues for each model potential.

As a second example we choose $s = 1$, again to facilitate the calculations. When $\theta = \theta_1^{(1,1)} = -\sqrt{6}$ the first four eigenvalues are $W_{0,0} = W_{1,1}^{(1)} = 6$, $W_{1,1} = 9.805784090$, $W_{2,1} = 13.66928892$, $W_{3,1} = 17.56601881$; on the other hand,
when $\theta = \theta_1^{(1,2)} = \sqrt{6}$ we have $W_{0,1} = 1.600357154$, $W_{1,1} = W_1^{(1)} = 6$, $W_{2,1} = 10.21072810$, $W_{3,1} = 14.35078474$. Notice that the truncation condition yields only the lowest state for the former model and the second-lowest one for the latter, missing all the other eigenvalues for each model potential.

In order to convince the reader about the accuracy of the variational method, tables 1 and 2 show how the approximate eigenvalues given by this approach converge from above towards the exact eigenvalues of equation (1) as the number $N$ of functions in the expansion increases. We appreciate that the variational method yields the exact eigenvalue $W_1^{(1)}$ for all $N$ because the corresponding eigenfunction is, in this case, a linear combination of only two basis functions.

From the analysis above one may draw the wrong conclusion that the truncation condition is utterly useless; however, it has been shown that one can extract valuable information about the spectrum of conditionally solvable models if one arranges and connects the roots $W_s^{(n)}$ properly [5, 6]. From the analysis outlined above we conclude that $(\theta_s^{(n,i)}, W_s^{(n)})$ is a point on the curve $W_{i-1,s}(\theta)$, $i = 1, 2, \ldots, n + 1$, so that we can easily construct some parts of such spectral curves. For example, Figure 1 shows several eigenvalues $W_0^{(n)}$ and $W_1^{(n)}$ given by the truncation condition (blue points) and red lines representing the variational calculations. Notice that the continuous variational curves $W_{j,s}(\theta)$ already connect the points $W_s^{(n)}$ corresponding to the truncation condition. In other words, the variational method yields all the eigenvalues $W_{j,s}(\theta)$ for any value of $\theta$ while the truncation results $W_s^{(n)}$ are just some particular points on the curves. Besides, it is clear that the variational curves $W_{j,s}(\theta)$ have negative slopes as predicted by the Hellmann-Feynman theorem (3). We clearly see that the allowed energies reported by Vitória et al [1] have no physical meaning because they correspond to many different problems instead of just one. In addition to it, the occurrence of discrete permitted values of the magnetic field parameter $\chi$ is a mere consequence of selecting particular points $(\theta_s^{(n,i)}, W_s^{(n)})$ on the curves $W_{j,s}(\theta)$. It should be clear from present analysis that such points (by themselves) do not exhibit any physical meaning. Notice that the truncation
method only yields the exact result in the trivial case $\theta = 0$. This fact is already discussed in many textbooks of quantum mechanics where it is shown that the coefficients of the power series expansions of the solutions to the exactly solvable quantum-mechanical models, like the harmonic oscillator, hydrogen atom, Morse oscillator, etc., satisfy two-term recurrence relations and not three-term ones like the quasi-exactly solvable problems [5, 6].

In two earlier papers on this journal Bakke [7] and Bakke and Furtado [8] discussed physical systems with different interactions, arrived at the same eigenvalue equation, applied the same approach and, consequently, draw somewhat similar wrong physical conclusions.

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Addendum

According to the reviewer: “Another point to be observed is the dependence of the Rayleigh-Ritz variational method on the choice of the wave function. Despite not being mentioned by the authors of this comment, the wave function used in the Rayleigh-Ritz variational method is obtained from the asymptotic analysis made by Vitória et al. If one uses another wave function that differs from the wave function obtained from the asymptotic analysis made by Vitória et al, therefore, the results will be different. In addition, no mathematical proof has been shown in this comment that clarifies the relation of the approximate solutions to the biconfluent Heun equation.”

This comment is surprising. In order to apply the Ritz variational method it is mandatory that the basis functions satisfy the correct boundary conditions at $x = 0$ and $x \to \infty$. Therefore, we have chosen the simplest basis set that satisfy such boundary conditions. The set of Gaussian functions chosen here is complete.
and, for this reason it should give the actual eigenvalues of the problem at hand. This fact is clearly revealed in the convergence of the approximate eigenvalues shown in Tables 1 and 2. The Ritz variational method is well known and has been widely used for the study of many quantum-mechanical problems.

References

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### Table 1: Eigenvalues $W_{j,0}$ for $\gamma = 0$ and $\theta = -\sqrt{2}$

| $N$ | $W_{00}$         | $W_{10}$         | $W_{20}$         | $W_{30}$         |
|-----|-----------------|-----------------|-----------------|-----------------|
| 2   | 4.000000000     | 10.49997602     |                 |                 |
| 3   | 4.000000000     | 7.751061995     | 19.88102859     |                 |
| 4   | 4.000000000     | 7.694010921     | 11.97562584     | 33.92039998     |
| 5   | 4.000000000     | 7.693979367     | 11.51212379     | 17.05520450     |
| 6   | 4.000000000     | 7.693978905     | 11.50604696     | 15.46896992     |
| 7   | 4.000000000     | 7.693978892     | 11.50604243     | 15.37652840     |
| 8   | 4.000000000     | 7.693978891     | 11.50604238     | 15.37592761     |
| 9   | 4.000000000     | 7.693978891     | 11.50604238     | 15.37592718     |
| 10  | 4.000000000     | 7.693978891     | 11.50604238     | 15.37592718     |

### Table 2: Eigenvalues $W_{j,0}$ for $\gamma = 0$ and $\theta = \sqrt{2}$

| $N$ | $W_{00}$         | $W_{10}$         | $W_{20}$         | $W_{30}$         |
|-----|-----------------|-----------------|-----------------|-----------------|
| 2   | -1.180391283    | 4.000000000     |                 |                 |
| 3   | -1.401182256    | 4.000000000     | 9.284143096     |                 |
| 4   | -1.449885589    | 4.000000000     | 8.345259771     | 17.66452696     |
| 5   | -1.458156835    | 4.000000000     | 8.344361267     | 12.69095166     |
| 6   | -1.459389344    | 4.000000000     | 8.344349784     | 12.53313315     |
| 7   | -1.459560848    | 4.000000000     | 8.344349442     | 12.53290257     |
| 8   | -1.459583736    | 4.000000000     | 8.344349427     | 12.53290132     |
| 9   | -1.459586704    | 4.000000000     | 8.344349427     | 12.53290130     |
| 10  | -1.459587081    | 4.000000000     | 8.344349427     | 12.53290130     |
| 11  | -1.459587128    | 4.000000000     | 8.344349427     | 12.53290130     |
| 12  | -1.459587134    | 4.000000000     | 8.344349427     | 12.53290130     |
| 13  | -1.459587134    | 4.000000000     | 8.344349427     | 12.53290130     |
Figure 1: Eigenvalues $W_{j,0}$ (upper panel) and $W_{j,1}$ (lower panel) obtained from the truncation condition (blue points) and from the variational method (red lines)