Integrable scalar cosmologies
with matter and curvature

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Abstract

We show that some integrable (i.e., exactly solvable) scalar cosmologies considered by Fré, Sagnotti and Sorin (Nuclear Physics B \textbf{877}(3) (2013), 1028–1106) can be generalized to include cases where the spatial curvature is not zero and, besides a scalar field, matter or radiation are present with an equation of state $p^{(m)} = w \rho^{(m)}$; depending on the specific form of the self-interaction potential for the field, $w$ can be arbitrary or must be fixed suitably.

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1 Introduction

Scalar fields in cosmology. The consideration of scalar fields in cosmological models has a long story, and arises from different motivations. On one hand the inflaton, i.e., the entity driving inflation, is often modeled as scalar field. This approach originates from the work of some scholars at the beginning of the 1980’s: let us mention, in particular, Linde [17], Madsen and Coles [19]. On another hand, one can use a scalar field as a model for dark energy. This idea seems to have appeared in a 1988 paper by Ratra and Peebles [27]; Caldwell, Dave and Steinhardt [5] are credited for introducing, ten years later, the term “quintessence” to indicate a scalar model of dark energy. It is hardly the case to recall that the notion of dark energy was experimentally consolidated during the same years, thanks to the publication (between 1998 and 1999) of the observational results by the High-Z Supernova Search Team [28] and the Supernova Cosmology Project [23]. To our knowledge, Saini, Raychaudhury, Sahni and Starobinsky [31] were the first to set up a strictly quantitative connection between a scalar field model of dark energy and the observational data of [23, 28]. More precisely, in [31] an unspecified self-interaction potential for the dark energy scalar field is assumed, and the most probable shape of this potential fitting the data of [23, 28] on luminosity distance and redshift is determined for the epoch ranging from present time to the time when the spatial scale factor had half of its present value. Most of the papers cited before and in the sequel, as well as the present work, rely on a paradigm in which the universe is homogeneous and isotropic at each time, and the scalar field (modelling the inflaton or dark energy) is treated classically, so that its values are ordinary numbers rather than operators \(^1\). Due to the assumption of homogeneity and isotropy, the spacetime metric has the form of Friedmann-Lemaître-Robertson-Walker (FLRW), possibly with non zero spatial curvature. For the same reason, the scalar field only depends on time. These cosmological models might involve, besides the scalar field, some other form of matter described as a perfect fluid; here and in the sequel the term “matter” is used in a broad sense, including especially the case of a radiation gas. The presence of a matter fluid is typical of models where the scalar field represents dark energy; these encompass all the story of the universe except for the very early stages, and include epochs in which the matter contribution is dominant with respect to that of dark energy. On the contrary, models for the very early, inflationary stage of the universe typically ignore the role of matter and focus the attention on the inflaton scalar field. All the above models give rise to systems of ODEs, describing the time evolution of the main actors which include, especially, the scale factor in the FLRW metric and the scalar field. Let us point out some additional features of these cosmological models. It is commonly assumed that the scalar field is minimally coupled to gravity, and that it does not interact directly with matter if the latter is present; correspondingly, the stress energy tensors of the scalar field and of the matter fluid are separately conserved. In absence of different indications, all papers cited in the sequel fit into the scheme just outlined (spatial homogeneity and isotropy, minimal coupling of the scalar field with gravity, no direct interaction between the field and matter).

Integrable scalar cosmologies. Since the late 1980’s, the rising physical interest for cosmologies with scalar fields stimulated the search for integrable models, in which the evolution equations can be solved explicitly. It turned out that this is possible for models with certain features, like a special functional form for the self-interaction potential of the scalar field. Of course, the availability of exact solutions is an advantage with respect to numerical integration, since it allows to identify details and conceptual aspects that could be missed by a numerical approach. Since the very beginning of these investigations, it was understood that exact solutions can be obtained assuming an exponential form for the self-interaction potential of the scalar field; let us describe these results with the normalizations of the present work, borrowed from [13], which uses

\(^1\)Some of the cited references suggest that the “ordinary values” of the scalar field might arise as expectation values from an underlying quantum field theory, but we will not discuss this issue.
a suitable dimensionless version $\varphi$ of the scalar field (up to a purely numerical factor, $\varphi$ is the scalar field multiplied by the square root of the gravitational constant; for the precise definition see subsection 2.2, especially Eq. (2.2.9)).

In 1987 Barrow [1] assumed a potential of the form $V(\varphi) = \text{const. } e^{-\lambda \varphi}$ (with $\lambda$ another arbitrary constant), and a vanishing spatial curvature; he presented a particular exact solution of the evolution equations (but not the general solution) for the case of a scalar field alone. In the previously mentioned paper [27] of 1988, Ratra and Peebles considered the same exponential potential as in [1] (with $\lambda > 0$) and a vanishing spatial curvature; they presented some particular exact solutions of the evolution equations, both for the case of a scalar field alone and for a model with a scalar field and pressure-less matter (namely, dust) [3]. In the same year, Burd and Barrow [4] considered again the potential $V(\varphi) = \text{const. } e^{-\lambda \varphi}$ (with $\lambda > 0$), with possibly non-zero spatial curvature in arbitrary spacetime dimension $n + 1$; they proposed a detailed stability analysis of the model and presented some new exact solutions exhibiting the transition to power-law inflation at late times.

In 1990 de Ritis, Marmo, Platania, Rubano, Scudellaro and Stornaiolo [7] investigated a cosmology with a scalar field and no matter fluid in the case of zero spatial curvature, suggesting its use as an inflationary model. To analyze the evolution equations, they proposed a systematic use of the Lagrangian viewpoint. In this way they proved that the only potentials giving rise to a Noether symmetry for the system have the form (with the normalizations of the present work) $V(\varphi) = \text{const. } e^{\varphi} + \text{const. } e^{-\varphi} + \text{const.}$ Moreover, they constructed the general solution of the evolution equations for this class of potentials. The same authors extended these results to the case of a field non minimally coupled to gravity in [8].

In 1998 Chimento [6] investigated cosmological models driven by two scalar fields, one of them self-interacting with an exponential potential of the form $V(\varphi) = \text{const. } e^{-\lambda \varphi}$ (as in [1, 3, 27]) and the other one free and massive. Exact general solutions were obtained and examined in detail; in particular these solutions show the transition from expansion dominated by the free scalar field to that dominated by the self-interacting field, yielding a power-law inflation. The potential $V(\varphi) = \text{const. } e^{\varphi} + \text{const. } e^{-\varphi} + \text{const.}$ was reconsidered in 2002 by Rubano and Scudellaro [29], and in 2012 by Piedipalumbo, Scudellaro, Esposito and Rubano [25], again for zero spatial curvature but in presence of dust. These authors showed that the solvability of the evolution equations is preserved even with the addition of dust. They proposed this model for describing dark energy and dust up to the present time, and started an analysis of the physical significance of the solutions.

All papers [1, 6, 7, 8, 25, 27, 29] considered a spacetime with the “physical” dimension $3 + 1$.

The integrable cosmologies of Frè, Sagnotti and Sorin [13]. In 2013 these authors considered the FLRW cosmologies with a self-interacting scalar field, no matter fluid and zero spatial curvature, in arbitrary spacetime dimension $n + 1$. Their analysis was based on the Lagrangian formalism and on the possibility of gauge transformations for the time coordinate. More precisely, the approach of [13] describes a cosmology of the above type as a Lagrangian system with two degrees of freedom plus the constraint of zero energy; the Lagrangian coordinates are, basically, the instantaneous values of the scale factor and the scalar field. The Lagrangian depends on the scalar field self-potential and on a gauge function (describing the choice of the time coordinate), to be specified according to convenience in the investigation of integrable cases.

For nine classes of self-potentials individuated in [13], with a convenient choice of the gauge function and a suitable change of the Lagrangian coordinates, the Lagrange equations are solvable by quadratures (for arbitrary initial data) for one of the following reasons:

i) the Lagrangian is quadratic, so it gives rise to linear evolution equations;

ii) the Lagrange equations have a triangular structure, which essentially means that one of the equations involves only one of the (new) Lagrangian coordinates;

\[^{2}\text{In this connection, let us recall that the solution derived in presence of dust was regarded by the authors as too peculiar to be physically relevant.}\]
iii) the Lagrangian is separable, i.e., it is the sum of a Lagrangian depending only on the first coordinate and a Lagrangian depending only on the second one. In this case there are two independent subsystems, each one with one degree of freedom and a conserved energy, which can be used to reduce to quadratures the corresponding evolution equation.

iv) the Lagrangian can be represented as a suitable function of a complex coordinate (equivalent to a pair of real coordinates); the “holomorphic structure” underlying this representation ensues the conservation of a complex valued “energy” function, which allows to solve by quadratures the evolution equations.

With the notations of [13] and of the present work, the first one of the nine potential classes is formed by functions of the form $V(\phi) = \text{const.} e^{\phi} + \text{const.} e^{-\phi} + \text{const.}$; this case is solvable by the linearity of the Lagrange equations (see (i)), and this result extends to any spacetime dimension the previous results of [25, 29] on these potentials in dimension $3 + 1$.

The second potential class of [13] is formed by potentials of the form $V(\phi) = \text{const.} e^{2\gamma \phi} + \text{const.} e^{(1+\gamma)\phi}$, with $\gamma$ another arbitrary constant; this case is solvable due to the triangular structure of the Lagrange equations (see the previous item (ii)).

It is not the case to illustrate now the remaining seven classes of potentials described by [13]; we will describe each one of them in the sequel of this paper. Here we only say that such potentials are built using the exponential and some functions closely related to it (namely, hyperbolic trigonometric functions), together with their inverses.

Paper [13] subsequently passes from the Lagrangian to the Hamiltonian formalism for the FLRW cosmologies with a self-interacting scalar field and investigates the Liouville integrable cases, i.e., the cases in which there is a second constant of motion besides the Hamiltonian; this second constant of motion (when it exists) is automatically in involution with the Hamiltonian and, since the system under analysis has two degrees of freedom, the standard theories of Liouville and Hamilton-Jacobi allow to solve Hamilton’s equations by quadratures. In this investigation, the authors of [13] get a great advantage from the existing literature on Hamiltonian systems with two degrees of freedom which possess a second constant of motion. Furthermore, they extract from the previous literature 26 “sporadic” classes of Hamiltonian systems with such features; these correspond to cosmological models with 26 classes of self-interaction potentials $V(\phi)$, which are referred to as the “sporadic potentials”.

The last two integrable cosmologies introduced in [13] have self-potentials $V(\phi)$ of trigonometric type, and are related via suitable transformations to two Toda-type lattices. (At least one of these two potentials has a close relation with the ninth class of integrable potentials mentioned before.)

The present paper: adding matter or curvature to the Frè-Sagnotti-Sorin integrable cosmologies. Gravity and the scalar, self-interacting field are the only actors in the cosmologies of [13]. It is natural to wonder if the integrable models of [13] can be generalized adding a (homogeneous) matter fluid and/or removing the assumptions of zero spatial curvature. This is the subject addressed in the present work; here we extend a more limited analysis of the same issue performed in the PhD thesis of one of us (M.G.) [13], that was supervised by the other authors of the present work (D.F. and L.P.).

As for the matter fluid, we admit a standard equation of state of the form $p^{(m)} = w \rho^{(m)}$, where $p^{(m)}$ is the pressure, $\rho^{(m)}$ the density and $w$ a constant; an unspecified value $k$ could be considered for the spatial curvature.

We assume no direct interaction between the matter fluid and the scalar field so that, as already observed, there are separate conservation laws for the corresponding stress-energy tensors. Especially, the conservation equation for the matter fluid can be integrated, yielding the explicit dependence of the density $\rho^{(m)}$ on the scale factor. This information, as well as the presence of spatial curvature, can be implemented in the Lagrangian formalism; in the end, any cosmology of the type outlined above is described as a Lagrangian system with two degrees of freedom, in which the basic coordinates are (again) the instantaneous values of the scale factor and of the scalar field.
The Lagrangian derived in this way contains, as in [13], an unspecified “gauge function” related to the choice of the time coordinate; in comparison with the cited work, our Lagrangian has two additional terms depending on the scale factor and on the gauge function, which are traces of the matter fluid and of the spatial curvature. 

The next step in this construction is the reconsideration of the nine potential classes of [13], with the related choices of the gauge function and of new Lagrangian coordinates. It is natural to wonder if such transformations, allowing to solve the evolution equations in the purely scalar models of [13], do in fact ensure solvability also in presence of matter and/or spatial curvature, for one of the reasons (i-iv) listed in the previous paragraph.

This problem is addressed in the PhD thesis [15] for the first two potential classes of [13]. Concerning the first class potentials

$$V(\phi) = \text{const.} e^{\phi} + \text{const.} e^{-\phi} + \text{const.},$$

it is found that the model is still solvable with zero spatial curvature and the addition of matter with $w = 0$ (dust), due to the linearity of the evolution equation; indeed, the linearizability of this cosmological model in spacetime dimension $3 + 1$ had already been established in [25], so we are just extending the result of the cited papers to an arbitrary dimension $n + 1$.

Concerning the second class potentials

$$V(\phi) = \text{const.} e^{2\gamma \phi} + \text{const.} e^{(1+\gamma)\phi},$$

the thesis [15] finds that the system maintains its integrability features (of the type indicated in (ii)) for suitable values of the parameter $\gamma$ in presence of zero spatial curvature and matter with arbitrary $w$, or in presence of arbitrary spatial curvature and matter with $w = 1/n$ or $w = 2/(n-1)$ ($n + 1$ is as usual the spacetime dimension). It should be noted that the case $w = 1/n$, in which the stress-energy tensor of matter has zero trace, can be interpreted as a radiation gas. In all these cases, the value of $\gamma$ must be appropriately fixed (for example $\gamma = w$ in the first case, where $w$ is arbitrary).

The present work reports the above results from the thesis [15], completes the analysis of the second class potentials finding further integrable cases and then discusses the remaining seven classes of potentials listed by [13], showing that, for each one of these classes, there are several integrable extensions of the model with spatial curvature and/or matter.

Perhaps, other integrable extensions with curvature or matter could be associated to the 26 “sporadic” potentials and to the pair of Toda-type, trigonometric potentials considered in [13] after the nine classes; however, the analysis performed in the present work for the nine classes with the addition of matter or spatial curvature yields by itself a very long list of integrable cases, so we defer to future works the investigations related to sporadic or Toda-type potentials.

Qualitative and quantitative analysis of the solutions found in this paper. Of course, after discovering the mechanism ensuring integrability of a cosmological model it is essential to write explicitly its solution and to analyze it from a qualitative and quantitative viewpoint, so as to answer questions like the following: Does the model exhibit a Big Bang? Is there a Big Crunch, or does the universe exist forever (in terms of cosmic time)? What about the asymptotic behavior of the scale factor and of the energy densities of matter and of the scalar field near the Big Bang, near the Big Crunch, or in the infinitely far future? Which type of energy is dominating in these limits? Is there a particle horizon associated to the Big Bang? Is the model realistic for the whole story of universe, for most of it or at least for some stage of it? If so, can one fix the free parameters and/or the constants of integration in the solution of the model so as to fit the available observational data? We already mentioned that the integrable cases found in this paper adding matter or spatial curvature to models from [13] are a lot, so it is not possible to treat explicitly the above issues for all of them. Therefore, we perform the above mentioned qualitative and quantitative analyses just for some case studies.

Firstly, we consider the case of a first class self-interaction potential $V(\phi)$ with no spatial curvature and dust. If one identifies dust with ordinary matter and the scalar field with dark energy, this model describes with a good approximation the content of the universe for most of its story, from the end of the radiation dominated era to the very far future. The analysis presented here follows

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3The matter term also depends on an unspecified constant, related to the matter density at some reference time.
the thesis [15], and somehow refines the investigation of the authors who discovered this integrable case [25, 29]. The behavior of the solution of this model depends on the parameters in the potential $V(\phi)$ and on the integration constants; we show how to choose them so that the early universe is dominated by matter, the late universe by dark energy and the (dimensionless) energy densities of these two entities at present time have the values suggested by observational evidence (of course, in this computation we assume that the spacetime dimension is $3 + 1$).

Next, we consider a scalar field with a second class potential $V(\phi)$ to which, as already indicated, it is possible to add matter and curvature. Among the many integrable cases of this model, listed elsewhere in the paper, we choose that with zero spatial curvature and matter with arbitrary equation of state parameter $w$ (and with the parameter $\gamma$ in $V(\phi)$ fixed by the previously mentioned prescription $\gamma = w$, which ensures the triangular structure of the Lagrange equations). Again, there are many subcases of this model: we choose one with $0 < w < 1$ and suitable signs of the coefficients in the potential $V(\phi)$, which exhibits a Big Bang and no Big Crunch. The asymptotic behavior of the relevant observables near the Big Bang and in the very far future is determined for arbitrary $w \in (0, 1)$. Sticking to this subcase with a second class potential, we subsequently fix the spacetime dimension to be $3 + 1$ and set $w = 1/3$ (radiation gas). With these choices, we individuate a solution of the Lagrange equations that, although being entirely built with elementary functions, has a rather complicated structure implying a stage in which the scale factor grows exponentially with the cosmic time, preceded and followed by epochs in which the scale factor behaves like a power of the cosmic time. We show that the free parameters of the model and the constants of integration in this solution can be adjusted so that the exponential growth occurs in the very early universe and the scale factor is increased, say, by a factor $3 \times 10^{30}$ in a time interval between $(1/2) \times 10^{-34}$ sec and $10^{-34}$ sec after the Big Bang. This is the behavior postulated by inflationary theories; we think it can be of some interest to obtain such a behavior from an exact solution of Einstein’s equations with the simultaneous presence of radiation and of a scalar field; clearly, the latter ought to be interpreted the inflaton in this model.

The last case study considered in this paper is associated to the seventh class of potentials $V(\phi)$; the spatial curvature is zero and a type of matter is present with $w = (\ell - 1)/(\ell + 1)$, where $\ell \geq 2$ is an integer. This case is discussed since it provides a rather interesting example of separable system (see item (iii) in the previous paragraph). Indeed, upon introducing a suitable pair $(x, y)$ of Lagrangian coordinates, the Lagrangian is found to be the sum of two Lagrangians depending separately on $x, y$ (and their time derivatives). The first Lagrangian describes a non-linear repulser with potential energy proportional to $-x^{2\ell}$, the second one described a non-linear oscillator with potential energy proportional to $y^{2\ell}$. On account of energy conservation for these separate subsystems, we derive quadrature formulas for their motions and then return to the original variables of the model, i.e., the scale factor and the scalar field, ultimately performing a qualitative and quantitative analysis of their behavior. In this way we find, for example, that the system exhibits a Big Bang and an exponential growth of the scale factor (as a function of cosmic time) in the very far future; at intermediate times, there is a competition between the behaviors associated to the previously mentioned repulser and oscillator, whose effects depend on the parameters in the potential $V(\phi)$ and on the values assumed for the constants of integration.

**Organization of the paper.** Section 2 and the related Appendix A present some general facts on cosmologies with a scalar field minimally coupled to gravity and a matter fluid (not interacting directly with the scalar field, with a given equation of state $p^{(m)} = p^{(m)}(\rho^{(m)})$). After some generalities about the action functional and the stress-energy tensors of the field and of the matter fluid, we focus the attention on the homogeneous and isotropic case, in which the spacetime metric has the FRW form, and the equation of state for matter is assumed to have the form $p^{(m)} = w \rho^{(m)}$; this yields the Lagrangian setting with two degrees of freedom mentioned in the previous paragraphs. Section 3 considers the nine potential classes $V(\phi)$ of Fré, Sagnotti and Sorin, and lists the integrable cases that we have obtained adding matter or curvature. Section 4 and the related Appendices E, C

6
present the explicit solutions for some integrable cases of Section 3, accompanied by a qualitative and quantitative analysis. Here we discuss the results mentioned in the previous paragraph, i.e.: a review of the Rubano-Scudellaro-Pedipalumbo-Esposito model with dust \[25, 29\] (subsection 4.1); a general discussion of the class 2 potentials with the addition of matter (paragraph 4.2.3); an analysis of an integrable case with a class 7 potential and matter, yielding the previously mentioned model with a nonlinear repulsor and a nonlinear oscillator (subsection 4.3).

**Final remarks.** (a) One could wonder if the present integrability results (or those of \[13\]) could be extended to the case of non minimal coupling between gravity and the scalar field. We refer mainly to the case of a standard curvature coupling, in which the action functional for the system contains of a term proportional to \(R \phi^2\) (with \(R\) the scalar curvature). This problem certainly deserves further investigation. There is some hope to obtain these extensions for the purely scalar models of \[13\], using some formal transformations proposed in the literature \[14, 20\] to connect minimally coupled theories to systems with curvature coupling. However, the cited transformations refer to systems with no type of matter fluid, so they cannot be used for the cosmologies with matter of the present work.

(b) We already pointed out that no direct interaction between the matter fluid and the scalar field is ever considered throughout this paper. However, let us mention that some integrable FLRW cosmological models with such an interaction have appeared in the literature; we refer in particular, to the very recent work of Piedipalumbo, De Laurentis and Capozziello \[24\], where the scalar field represents dark energy and a possible interaction with (dark) matter is considered (see also the references cited therein).

(c) In most of the integrable cases presented in this work, a finite particle horizon appears; this fact can be checked by hand noting that the reciprocal of the scale factor, viewed as a function of cosmic time, diverges in a non-integrable way at the Big Bang. In the case of non-positive spatial curvature, the deep reason for this fact was pointed out in \[12\]; therein it was shown that the particle horizon is finite for all homogeneous and isotropic cosmologies with non positive spatial curvature, a self-interacting scalar field minimally coupled to gravity and a matter fluid with equation of state \(p^{(m)} = w \rho^{(m)}\), fulfilling the strong energy condition. As shown in \[12\], the particle horizon is absent if, instead of a canonical scalar field, one considers a *phantom* field whose action functional contains an anomalous term corresponding to a *negative* kinetic energy. It would be of some interest to search for FLRW integrable cosmologies with a phantom scalar field and matter; this subject is left to future investigations.

(d) In the present work, in \[13\] and in most of the other previously cited papers, the attention is focused on a “direct problem”: find for arbitrary initial data the solution of a cosmological model with a pre-assigned potential for the scalar field and, possibly, with matter having a suitable equation of state. On the other hand, there is also an “inverse problem”: find the scalar field potential producing a time evolution with a prescribed feature in a FLRW cosmologies with a purely scalar content, or including a matter fluid. To our knowledge, the first examples of this inverse approach date back to 1980’s and 1990’s: we will mention, in particular, the papers by Lucchin and Matarrese \[18\], Barrow \[2\], Ellis and Madsen \[11\], Eashter \[10\]. More recently, nice “inverse” results have been obtained by Dinakis, Karagiorgos, Zampeli, Paliathanasis, Christodoulakis and Terzis \[9\], and by Barrow and Paliathanasis \[8\]; the same approach is also partly employed in \[12\], for the case of a phantom field. The feature specified in the cited papers to determine the scalar field potential is, for example, the dependence on cosmic time of one of the following observables: the scale factor, the Hubble parameter, the ratio between the pressure and the density produced by the scalar field alone, or jointly by scalar field and matter. The distinction between the “direct” and “inverse” problems mentioned above is essential to understand the difference between the present work and the ones we have just mentioned.
2 The reference framework

2.1 A general cosmological model with matter and a scalar field

Throughout this paper we employ units in which the speed of light and the reduced Planck’s constant are \( c = 1 \) and \( \hbar = 1 \). In particular, indicating with \( L, T \) and \( M \) the spaces of lengths, times and masses we have \( L = T = M^{-1} \).

Let us introduce a cosmological model living in a spacetime of dimension

\[
d = n + 1, \quad \text{with} \quad n = 2, 3, 4, \ldots
\]  

(of course, \( n \) stands for the spatial dimension). Spacetime coordinates are typically indicated with \( (x^\mu) \), and the line element reads \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu \). The metric \( (g_{\mu\nu}) \) has signature \( (-, +, \ldots, +) \) and the corresponding covariant derivative, Ricci tensor and scalar curvature are respectively denoted with \( \nabla_\mu, R_{\mu\nu} \) and \( R \).

We assume that the content of the universe consists of:

(i) a scalar field \( \phi \) (of dimension \( \mathbb{L}^{-n+1} \)), minimally coupled to gravity and self-interacting with potential \( V(\phi) \) (of dimension \( \mathbb{L}^{n+1} \));

(ii) some kind of matter which can be described as a perfect fluid with mass-energy \( p^{(m)} \) and pressure \( p^{(m)} \), fulfilling an assigned equation of state \( p^{(m)} = p^{(m)}(\rho^{(m)}) \). Such matter does not interact directly with the scalar field. Besides, let us remark that here and in the sequel the term “matter” is used in a very generic sense (e.g., it possibly refers to a radiation gas).

The action functional \( SS \) for the above model depends on the spacetime metric, on the scalar field history and on the matter history (defined as in [16]) with the law

\[
SS := \int d^{n+1}x \sqrt{|g|} \left[ \frac{R}{2\kappa_n^2} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) - \rho^{(m)} \right],
\]

where \( g := \det(g_{\mu\nu}) \) and \( \kappa_n \) (of dimension \( \mathbb{L}^{(n-1)/2} \)) is, essentially, the square root of the universal gravitational constant. Note that \( SS \) is dimensionless in our units with \( \hbar = 1 \).

Demanding \( SS \) to be stationary under variations of the metric \( (g_{\mu\nu}) \) entails the Einstein’s equations

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa_n^2 (T^{(\phi)}_{\mu\nu} + T^{(m)}_{\mu\nu}),
\]

where the r.h.s. contains the stress-energy tensors of the scalar field and of the matter fluid:

\[
T^{(\phi)}_{\mu\nu} := \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi - g_{\mu\nu} V(\phi);
\]

\[
T^{(m)}_{\mu\nu} := (p^{(m)} + \rho^{(m)}) U_\mu U_\nu + p^{(m)} g_{\mu\nu},
\]

with \( U^\mu \) indicating the \( (n+1) \)-velocity of the fluid.

The stationary condition for \( SS \) with respect to variations of the field \( \phi \) gives the Klein-Gordon-type equation

\[
\Box \phi = V'(\phi),
\]

where \( \Box \phi := \nabla_\mu \nabla^\mu \phi = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu\nu} \partial_\nu \phi) \) (recall that \( g := \det(g_{\mu\nu}) \)).

Finally, the stationarity of \( SS \) under variations of the matter history gives the conservation law for stress-energy tensor of the matter fluid, namely

\[
\nabla_\mu T^{(m)}_{\mu\nu} = 0.
\]

Of course, the Einstein’s equations (2.1.3), along with the Bianchi identity, imply the conservation of the total stress-energy tensor \( T^{(\phi)}_{\mu\nu} + T^{(m)}_{\mu\nu} \). Combined with Eq. (2.1.7), this implies

\[
\nabla_\mu T^{(\phi)}_{\mu\nu} = 0.
\]
On the other hand, from the explicit expression (2.1.4) of $T^{(\nu)}_{\mu\nu}$ one gets
\[ \nabla_\mu T^{(\nu)}_{\mu\nu} = (\Box \phi - V'(\phi) ) \partial_\nu \phi. \]
(2.1.9)
Thus, Eqs. (2.1.3), (2.1.6) and (2.1.7) are not independent: in fact, one has the chain of implications (2.1.3) and (2.1.7) $\Rightarrow$ (2.1.6) $\Rightarrow$ (2.1.3) (at points where $\partial_\nu \phi \neq 0$). These considerations on Eqs. (2.1.3) (2.1.6) (2.1.7) have partial converses which are easily described for special geometries, such as a FLRW spacetime: in this case, to be addressed in the following, Eqs. (2.1.6) (2.1.7) imply that some of the Einstein’s equations (2.1.3) are actually constraints, holding at all times if and only if they are fulfilled at a particular time.

From here to the end of the paper, we assume that the equation of state for the matter fluid reads
\[ p^{(m)} = w \rho^{(m)}, \]
(2.1.10)
for some suitable real constant $w$, in principle arbitrary. When $w = 0$, the fluid is a dust; if $w = 1/n$ the trace $T^{(m) \mu}_{\mu}$ vanishes, as typical of a radiation gas. Besides, let us mention that the weak, dominant and strong energy conditions for $T^{(m) \mu}_{\mu}$ are respectively equivalent to (see, e.g., [16, 21])
\[ \rho^{(m)} \geq 0, \quad w \geq -1, \]
(2.1.11)
\[ \rho^{(m)} \geq 0, \quad -1 \leq w \leq 1, \]
(2.1.12)
\[ \rho^{(m)} \geq 0, \quad w \geq \frac{2}{n} - 1. \]
(2.1.13)

**Comparison with [13].** As already stressed, [13] considers a scalar field as the only content of the universe; thus, any statement of ours involving the matter fluid has no counterpart in the cited work.

Here and in the sequel, we employ notations as close as possible to those of [13]; however there are minor differences, to be pointed out step by step. For the moment, let us mention that our convention $(-,+,+,\ldots,+)$ for the metric signature is opposite to the convention $(+,-,-,\ldots)$ employed in [13]. Following [13], we indicate with $d$ the spacetime dimension; however, differently from [13] we often refer to the space dimension $n$ and write $d = n + 1$. In particular, our constant $\kappa_n$ coincides with the quantity $k_d$ of [13] (with $d = n + 1$).

In the sequel, as in [13] we restrict our attention to the case of a FLRW geometry that we describe using similar notations, apart from the symbol $\tau$ for cosmic time replacing the notation $t_c$ of [13]. In addition, let us stress that we admit arbitrary values for the constant, spatial sectional curvature, while [13] discusses only the case of zero curvature.

### 2.2 The homogeneous and isotropic case

From here to the end of the paper, the general model of the previous subsection is specialized to the case of a spatially homogeneous and isotropic universe.

**The spacetime and its metric.** To implement the above assumptions we consider a FLRW spacetime, given by the product of the time line and of a (simply connected) Riemannian manifold $M^n_k$ of constant sectional curvature $k$ (of dimension $\mathbb{L}^{-2}$). Using cosmic time $\tau$ and any system of coordinates $x = (x^i)_{i=1,\ldots,d}$ for $M^n_k$, we have
\[ ds^2 = -d\tau^2 + a^2(\tau) dl^2 = -d\tau^2 + a^2(\tau) h_{ij}(x) dx^i dx^j, \]
(2.2.1)
where $dl^2 = h_{ij}(x) dx^i dx^j$ is the line element of $M^n_k$ and $a(\tau) > 0$ is the dimensionless scale factor; typically, the latter is fixed so that $a(\tau_*) = 1$ at some reference time $\tau_*$. For our purposes it is convenient to use in place of $\tau$ a dimensionless “time” coordinate $t$, implicitly defined by the identity
\[ d\tau = \theta e^{B(t)} dt, \]
(2.2.2)
where $B(t)$ is a dimensionless “gauge function” to be determined and $\theta$ is a constant of dimension $T$. This re-parametrization of time is suggested in [13], where, however, the analogue of Eq. (2.2.2) contains no dimensional constant $\theta$ and reads $d\tau = e^{B(t)} dt$; due to this, the coordinate $t$ of [13] has dimension $T$.

Having $\theta$ at our disposal, we re-express the scalar curvature in terms of a dimensionless coefficient $k$, setting

$$k = \frac{k}{\theta^2};$$  \hspace{1cm} (2.2.3)

where $A$ is a dimensionless function. Substituting Eqs. (2.2.2) and (2.2.4) into Eq. (2.2.1), we obtain for the spacetime metric the representation

$$ds^2 = -\theta^2 e^{2B(t)} dt^2 + e^{2A(t)/n} \left( h_{ij}(x) dx^i dx^j \right),$$

which coincides with the one given in [13] apart from the presence of the constant $\theta$ and from the extension to non-zero values for the curvature $k$ of $dl^2$.

From here to the end of this work we will use a spacetime coordinate system $(x^\mu) \equiv (t, x^i) := (t, x^0, x^1, \ldots, x^n)$, where, as above, $x = (x^i)$ are coordinates on $M^n_k$; Greek indexes always range from 0 to $n$, Latin indexes from 1 to $n$. Moreover, we indicate derivatives with respect to $t$ with dots, namely,

$$\ddot{\cdot} := \frac{d}{dt}.$$  \hspace{1cm} (2.2.7)

In Appendix A we report the explicit expressions of the Ricci tensor $R_{\mu\nu}$ and of the scalar curvature $R$ for the metric (2.2.5).

Let us indicate with $U^\mu$ the $(n + 1)$-velocity of the FLRW frame (i.e., the future-directed, timelike vector field tangent to the lines with $x^\mu = \text{const.}$, normalized so that $U^\mu U_\mu = -1$); we have

$$U^\mu = \theta^{-1} e^{-B(t)} \delta^\mu_0, \hspace{0.5cm} U_\mu = -\theta e^{B(t)} \delta_{\mu 0}.$$  \hspace{1cm} (2.2.8)

**The scalar field and the matter content.** Let us now introduce rescaled versions $\varphi$, $\mathcal{V}$ of the field and of the potential, defined so that

$$\phi = \sqrt{n - 1 \frac{\varphi}{\kappa_n}}, \hspace{0.5cm} \mathcal{V}(\phi) = \frac{n - 1}{n} \frac{\mathcal{V}(\varphi)}{\kappa_n^2 \theta^2}.$$  \hspace{1cm} (2.2.9)

Both $\varphi$ and $\mathcal{V}(\varphi)$ are dimensionless. In the sequel, the terms “scalar field” and “potential” will be generally employed to indicate these rescaled quantities. Let us also remark that in [13] there are similar rescaled objects $\varphi_{[13]} = \varphi$ and $\mathcal{V}_{[13]}(\varphi_{[13]}) = \mathcal{V}(\varphi)/\theta^2$.

To comply with the hypothesis of spatial homogeneity, we assume that the field and the matter density depend only on time:

$$\varphi = \varphi(t), \hspace{1cm} \rho^{(m)} = \rho^{(m)}(t).$$  \hspace{1cm} (2.2.10)

In addition, we assume the matter fluid to be at rest in the FLRW frame, which entails that its $(n + 1)$-velocity $U^\mu$ is fixed as in Eq. (2.2.8). Let us also recall that we are considering an equation of state of the form (2.1.10), so that $p^{(m)} = w \rho^{(m)}$. 


In Appendix A we compute the stress-energy tensors of the scalar field and matter fluid starting from the general expressions (2.1.4) (2.1.5) and implementing the assumptions (2.1.10) (2.2.9) (2.2.10). Here we only mention that \( T^{(o)\mu\nu} \) has the form of the stress-energy tensor for a perfect fluid with the \((n+1)\)-velocity \( U^\mu \) of the FLRW frame, and with density and pressure respectively given by

\[
\rho^{(o)} := \frac{1}{\kappa_n^2 \theta^2} \frac{n-1}{n} \left( e^{-2B(t)} \frac{\dot{\varphi}^2}{2} + \mathcal{V}(\varphi) \right), \quad p^{(o)} := \frac{1}{\kappa_n^2 \theta^2} \frac{n-1}{n} \left( e^{-2B(t)} \frac{\dot{\varphi}^2}{2} - \mathcal{V}(\varphi) \right). \tag{2.2.11}
\]

In the sequel, we often refer to the “equation of state coefficient”

\[
w^{(o)} := \frac{p^{(o)}}{\rho^{(o)}}, \tag{2.2.12}
\]

depending on \( t \) and defined whenever \( \rho^{(o)}(t) \neq 0 \).

**The evolution equations.** We refer to Appendix A for all the statements reported in this paragraph. Let us first notice that the conservation law (2.1.7) for the stress-energy tensor of the matter fluid is fulfilled if and only if \( \rho^{(m)}(t) = \rho^{(m)}_s e^{-\left(w+w(1)A(t)\right)} \), where \( \rho^{(m)}_s \) is an integration constant with the dimension of \( \rho^{(m)} \), i.e., \( \mathcal{M}/\mathcal{L}^n = \mathcal{L}^{-(n+1)} \). For future convenience we set \( \rho^{(m)}_s = n(n-1) \Omega^{(m)}_s/(2 \kappa_n^2 \theta^2) \), with \( \Omega^{(m)}_s \) a dimensionless constant, so that

\[
\rho^{(m)} = n(n-1) \Omega^{(m)}_s/(2 \kappa_n^2 \theta^2) e^{-\left(w+w(1)A\right)}. \tag{2.2.13}
\]

Note that \( \text{sgn}(\rho^{(m)}(t)) = \text{sgn}(\Omega^{(m)}_s) \) at all times; unless otherwise stated, in the sequel we will typically assume \( \Omega^{(m)}_s \geq 0 \).

Next, let us consider the Einstein’s equations (2.1.9), that we re-write using the above expression for \( \rho^{(m)} \) and the related expression for \( p^{(m)} = w \rho^{(m)} \). There are just two independent equations, respectively corresponding to the group of indexes \((\mu, \nu) = (i, j) \in \{1, \ldots, n\}^2 \) and \((\mu, \nu) = (0, 0) \) in Eq. (2.1.9):

\[
\mathfrak{A} := \ddot{A} + \frac{\dot{A}^2}{2} - \dot{A} \dot{B} + \frac{\dot{\varphi}^2}{2} - e^{2B} \mathcal{V}(\varphi) + \frac{n^2 \Omega^{(m)}_s}{2} e^{2B-(w+1)A} + n(n-2) \frac{k}{2} e^{2B-2A/n};
\]

\[
\mathfrak{C} = 0,
\]

\[
\mathfrak{E} := \frac{\dot{A}^2}{2} - \frac{\dot{\varphi}^2}{2} - e^{2B} \mathcal{V}(\varphi) - \frac{n^2 \Omega^{(m)}_s}{2} e^{2B-(w+1)A} + n^2 \frac{k}{2} e^{2B-2A/n}. \tag{2.2.15}
\]

Finally, note that the field equation (2.1.6) becomes (with \( \mathcal{V} := d\mathcal{V}/d\varphi \))

\[
\ddot{\varphi} = 0, \quad \ddot{\varphi} := \ddot{\varphi} + (\dot{A} - \dot{B}) \dot{\varphi} + e^{2B} \mathcal{V}(\varphi). \tag{2.2.16}
\]

Before proceeding, let us point that a triple equivalent to the above set of equations Eqs. (2.2.14) (2.2.13) (2.2.16) is obviously given by \( \mathfrak{A} = \mathfrak{C} = 0, 2\mathfrak{E} = 0, \mathfrak{E} = 0 \); for \( \Omega^{(m)}_s = 0 \) and \( k = 0 \), the latter triple coincides with that reported in Eq. (2.12) of [13].

**An anticipation.** For the moment, \( \mathcal{B} \) is treated as an unspecified function of \( t \) (the same viewpoint is assumed in Appendix A). Starting from the forthcoming subsection 2.4 to the end of the paper, following [13] we will assume \( \mathcal{B}(t) = \mathcal{B}(A(t), \varphi(t)) \) for some assigned function \( \mathcal{B} \), and refer to this procedure as a gauge fixing. Thus, \( \mathcal{A} \) and \( \varphi \) will be ultimately recognized as the true degrees of freedom of the model.
Independence considerations. Regardless of the previously mentioned gauge fixing, Eqs. (2.2.14) [2.2.16] are not independent, as illustrated in the forthcoming items (i)(ii).

(i) We already pointed out that, in view of Eqs. (2.1.8) (2.1.9), the field equation \( \mathcal{H} = 0 \) (see (2.1.6)) is in fact a consequence of the other evolution equations \( \mathcal{A} = 0, \mathcal{E} = 0 \) (see Eqs. (2.1.3) (2.1.7)) in the spacetime region where the scalar field has a non-vanishing differential. As a matter of fact, in the present setting it can be checked by direct computations that \( \dot{\mathcal{H}} = \dot{A} \mathcal{A} - \dot{\mathcal{E}} - (\dot{A} - 2\dot{B}) \mathcal{E} \), yielding

\[
\mathcal{A} = 0, \quad \mathcal{E} = 0 \quad \Rightarrow \quad \mathcal{H} = 0 \quad \text{when} \quad \dot{\varphi} \neq 0.
\]  

(ii) As a partial converse, let us consider the relations \( \mathcal{A} = 0, \mathcal{H} = 0 \) supplemented with the initial condition \( \mathcal{E}(t_0) = 0 \) (namely, \( \mathcal{E} \) is required to vanish at some given time \( t_0 \)); we claim that

\[
\mathcal{A} = 0, \quad \mathcal{H} = 0, \quad \mathcal{E}(t_0) = 0 \quad \Rightarrow \quad \mathcal{E} = 0 \quad \text{at all times}.
\]  

To prove this, let us reconsider the identity in the text line before Eq. (2.2.17). If \( \mathcal{A} = 0, \mathcal{H} = 0 \) (at all times), this implies \( \mathcal{E} + (\dot{A} - 2\dot{B}) \mathcal{E} = 0 \) whence \( (d/dt)(e^{-A-2B}\mathcal{E}) = 0 \); the latter identity, supplemented with the initial condition \( \mathcal{E}(t_0) = 0 \), gives \( \mathcal{E} = 0 \) at all times. In the sequel we stick to the viewpoint expressed in item (ii): we regard \( \mathcal{A} = 0 \) and \( \mathcal{H} = 0 \) as the authentic evolution equations for the model, and \( \mathcal{E} = 0 \) as a constraint that is fulfilled at all times as soon as it is fulfilled by the initial data at some fixed time \( t_0 \).

Maximal domains for the solutions; Big Bang and Big Crunch. Of course, each solution \( (\mathcal{A}(t), \mathcal{B}(t), \varphi(t)) \) of the system \( \mathcal{A} = 0, \mathcal{H} = 0 \) (\( \mathcal{E} = 0 \)) is well defined for \( t \) in a suitable interval \( I \subset \mathbb{R} \). From now on, when we speak of a solution we always assume \( I \) to be maximal (i.e., that the solution cannot be extended to a larger interval). Let \( I = (t_{in}, t_{fin}) \), where \( -\infty \leq t_{in} < t_{fin} \leq +\infty \). Recall that \( a(t) := e^{A(t)/n} \) is the scale factor and that \( t, \tau \) are related by Eq. (2.2.2), which is equivalent to

\[
\tau(t) = \theta \int_{t_r}^{t} dt' e^{B(t')}
\]  

(here \( t_r \in I \) is chosen arbitrarily). If \( a(t) \to 0 \) (i.e., \( \mathcal{A}(t) \to -\infty \)) for \( t \to t_{in}^{+} \), and \( e^{B(t)} \) is integrable in a right neighborhood of \( t_{in} \) (initial singularity at a finite cosmic time), we say that the model has a Big Bang at \( t = t_{in} \). If \( a(t) = e^{A(t)/n} \) vanishes and \( e^{B(t)} \) is integrable for \( t \to t_{fin}^{-} \) (final singularity at a finite cosmic time), we say that the model has a Big Crunch at \( t = t_{fin} \).

The particle horizon. Suppose the model has a Big Bang at \( \tau_{in} = \tau(t_{in}) \). The lapse of conformal time that has passed from the Big Bang to any cosmic time \( \tau = \tau(t) \) is

\[
\Theta(\tau) := \int_{\tau_{in}}^{\tau} \frac{d\tau'}{a(\tau')} = \theta \int_{t_{in}}^{t} dt' e^{B(t')-A(t')/n}.
\]  

The above integral can be finite or infinite. The interpretation of \( \Theta(\tau) \) is well known, and can be summarized as follows, writing \( \mathbf{p}_0, \mathbf{p}, \) etc. for the points of \( \mathcal{M}_k^n \) and dist for the distance on \( \mathcal{M}_k^n \) corresponding to the metric \( dt^2 \) (see Eq. (2.2.1)); correspondingly, for each \( \mathbf{p} \in \mathcal{M}_k^n \), the ball \( B(\mathbf{p}, \tau) := \{ \mathbf{p}_0 \in \mathcal{M}_k^n \mid \text{dist}(\mathbf{p}_0, \mathbf{p}) \leq \Theta(\tau) \} \) is the subset of \( \mathcal{M}_k^n \) formed by the points \( \mathbf{p}_0 \) which had the time to interact causally with \( \mathbf{p} \) from the Big Bang up to \( \tau \). This subset is the whole \( \mathcal{M}_k^n \) if and only if \( \Theta(\tau) \geq \delta_k \), where \( \delta_k := \sup(\text{dist}(\mathbf{p}_0, \mathbf{p}) \mid \mathbf{p}_0 \in \mathcal{M}_k^n) \) is the diameter of \( \mathcal{M}_k^n \) and is in fact independent of \( \mathbf{p} \). One has \( \delta_k = +\infty \) if \( k < 0 \) and \( \delta_k = \pi/\sqrt{k} = \theta \pi/\sqrt{k} \) if \( k > 0 \). Of course the situation where \( \Theta(\tau) \geq \delta_k \) is of special interest, since it allows to explain the homogeneity of the universe at time \( \tau \). As well known, many FLRW cosmologies violate this condition; this will happen, in particular, for many of the integrable cosmologies presented in this work.

\footnote{in fact, one shows that there exists a causal curve starting from \( \mathbf{p}_0 \) at a time \( \tau_0 \in (\tau_{in}, \tau) \) and ending at \( \mathbf{p} \) at time \( \tau \) if and only if \( \text{dist}(\mathbf{p}_0, \mathbf{p}) \leq \Theta(\tau) \).}
Cosmological constant behavior for matter. Let us specialize our considerations to the case where the parameter in the equation of state \( w \) for matter is

\[
  w = -1. \tag{2.2.21}
\]

With this position, Eq. (2.2.13) and the equation of state itself reduce to

\[
  \rho^{(m)} = \text{const.} = \frac{n(n-1)\Omega_s^{(m)}}{2\kappa_n^2\theta^2}, \quad p^{(m)} = \text{const.} = -\rho^{(m)}, \tag{2.2.22}
\]

which entail for the matter stress-energy tensor the expression

\[
  T^{(m)}_{\mu\nu} := -\frac{n(n-1)\Omega_s^{(m)}}{2\kappa_n^2\theta^2} \theta \rho^{(m)} g_{\mu\nu}. \tag{2.2.23}
\]

Moving this term from the left-hand side to the right-hand side of the Einstein’s equations (2.1.3) we get

\[
  R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{n(n-1)\Omega_s^{(m)}}{2n\theta^2} g_{\mu\nu} = \kappa_n^2 T^{(\phi)}_{\mu\nu}, \tag{2.2.24}
\]

corresponding to a model with cosmological constant \( \Lambda = n(n-1)\Omega_s^{(m)}/(2n\theta^2) \) (of dimension \( \mathbb{L}^{-2} \)).

Cosmological constant behavior for the field. Let us now search for a solution of the model with

\[
  \varphi(t) = \text{const.} = \varphi_0. \tag{2.2.25}
\]

Eq. (2.2.16) entails that the above condition can be fulfilled if and only if

\[
  V'(\varphi_0) = 0. \tag{2.2.26}
\]

In this case, the field stress-energy tensor becomes (see Appendix A)

\[
  T^{(\phi)}_{\mu\nu} = -\frac{n-1}{n\kappa_n^2\theta^2} V(\varphi_0) g_{\mu\nu}. \tag{2.2.27}
\]

Bringing this term to the right-hand side of the Einstein’s equations (2.1.3) we obtain

\[
  R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{(n-1)\Omega^{(\phi)}}{2n\theta^2} g_{\mu\nu} = \kappa_n^2 T^{(m)}_{\mu\nu}, \quad \Omega^{(\Lambda)} := 2 V(\varphi_0), \tag{2.2.28}
\]

corresponding to a model with a cosmological constant \( \Lambda = (n-1)\Omega^{(\phi)}/(2n\theta^2) \) (note that \( \Omega^{(\Lambda)} \) is dimensionless while \( \Lambda \) has dimension \( \mathbb{L}^{-2} \), as expected).

Let us also mention that, according to Eqs. (2.2.11) (2.2.12)

\[
  \varphi = \text{const.} = \varphi_0 \iff p^{(\phi)} = -\rho^{(\phi)} \iff w^{(\phi)} = -1, \tag{2.2.29}
\]

where the second equivalence holds under the complementary assumption \( \rho^{(\phi)} \neq 0 \), or \( V(\varphi_0) \neq 0 \).

The Hubble parameter. The time-dependent Hubble parameter is given by

\[
  H := \frac{1}{a} \frac{da}{d\tau} = \frac{e^{-B\hat{A}}}{n \theta}. \tag{2.2.30}
\]

Here, the first equality is the standard definition in terms of the scale factor \( a \) and cosmic time \( \tau \), while the second identity follows from Eqs. (2.2.22) (2.2.4).
The dimensionless density parameters. These are the time-dependent quantities

\[ \Omega^{(m)} := \frac{2 \kappa_n^2}{n(n-1)} \frac{\rho^{(m)}}{H^2}, \quad \Omega^{(e)} := \frac{2 \kappa_n^2}{n(n-1)} \frac{\rho^{(e)}}{H^2}, \quad \Omega^{(k)} := -\frac{k}{g^2 H^2 a^2}. \]  

From Eqs. (2.2.31), (2.2.11), (2.2.13) and (2.2.30) we get

\[ \Omega^{(m)} = n^2 \Omega^{(m, e^{2B-(w+1)A})}/A^2, \quad \Omega^{(e)} = \frac{\dot{\varphi}^2 + 2 e^{2B} V(\varphi)}{A^2}, \quad \Omega^{(k)} = -\frac{n^2 k e^{2B-2A/n}}{A^2}. \]  

By comparison with Eq. (2.2.15), we see that

\[ \mathcal{C} = 0 \iff \Omega^{(m)} + \Omega^{(e)} + \Omega^{(k)} = 1. \]  

The parameters \( \Omega^{(m)} \) and \( \Omega^{(k)} \) are standard objects in cosmology (see, e.g., [33]). \( \Omega^{(e)} \) plays a role similar to the dimensionless parameter \( \Omega^{(X)} := 2 \Lambda/(n(n-1)H^2) \) usually considered when a cosmological term \( \Lambda g_{\mu\nu} \) is present in Einstein’s equations. In agreement with the remark after Eq. (2.2.1), in the sequel we often set \( a(t_*) = 1 \), i.e. \( A(t_*) = 0 \), at some reference time \( t_* \). Moreover, setting \( \theta := 1/|H(t_*)| \), from Eq. (2.2.30) we obtain \( |A(t_*)| = n e^{B(t_*)} \). By comparison with the first relation in Eq. (2.2.32), these facts entail the identity

\[ \Omega^{(m)}(t_*) = \Omega^{(m)}_* . \]  

2.3 A Lagrangian viewpoint

Let us return to the general expression (2.1.2) for the action functional, and evaluate it on a history of the type considered in subsection 2.2. A computation sketched in Appendix A yields

\[ SS = \frac{1}{\kappa_n^2 \theta} \int d^nx \sqrt{h(x)} \int dt \left[ \frac{n-1}{n} \mathcal{L}(A, \varphi, B, \dot{\varphi}) + \frac{d}{dt} \left( e^{A-B} \dot{\mathcal{A}} \right) \right], \]  

where \( h(x) := \det(h_{ij}(x)) \) and

\[ \mathcal{L}(A, \varphi, B, \dot{\varphi}) := e^{A-B} \left( -\frac{\dot{A}^2}{2} + \frac{\dot{\varphi}^2}{2} \right) - e^{A+B} V(\varphi) - \frac{n^2 \Omega^{(m)}_*}{2} e^{-wA+B} + \frac{n^2 k}{2} e^{w_2A+B}. \]  

In Eq. (2.3.1), the integral \( \int d^nx \sqrt{h(x)} \) is an irrelevant multiplicative factor (although, infinite if \( k \leq 0 \)); the total \( t \)-derivative in the integral is also irrelevant. In conclusion, \( SS \) is related to the (dimensionless) Lagrangian function \( \mathcal{L} \) written in Eq. (2.3.2), which is degenerate since it does not depend on \( \dot{B} \).

Independently of the previous considerations, it can be checked by direct computations that the Lagrange equations induced by \( \mathcal{L} \) are equivalent to the evolution equations of the model under analysis. In fact, the Lagrange derivatives

\[ \frac{\delta \mathcal{L}}{\delta q} := -\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) + \frac{\partial \mathcal{L}}{\partial q} \quad (q = A, \varphi, B) \]  

are such that

\[ \frac{\delta \mathcal{L}}{\delta A} = e^{A-B} \mathfrak{A}, \quad \frac{\delta \mathcal{L}}{\delta \varphi} = -e^{A-B} \mathfrak{F}, \quad \frac{\delta \mathcal{L}}{\delta B} = e^{A-B} \mathfrak{C}, \]  

which ensures the equivalence between the Lagrange equations \( \delta \mathcal{L}/\delta q = 0 \) (\( q = A, \varphi, B \)) and the evolution equations \( \mathfrak{A} = 0, \mathfrak{F} = 0, \mathfrak{C} = 0 \) (see Eqs. (2.2.14) (2.2.15) (2.2.16)). We already noted that such evolution equations are not independent; from the present Lagrangian viewpoint, this is a consequence of the degeneracy of \( \mathcal{L} \).

Finally, let us mention that for \( \Omega^{(m)}_* = 0 \) and \( k = 0 \) the Lagrangian (2.3.2) coincides with the one appearing in Eq. (2.11) of [13].
2.4 Gauge fixing and the energy constraint

From here to the end of the work it is assumed that

\[ B = \mathcal{B}(A, \varphi), \]  

(2.4.1)

where \( \mathcal{B} : \mathbb{R}^2 \to \mathbb{R} \) is a suitable function, referred to as the gauge function in the sequel (the same attitude is proposed in [13] for the special case \( \Omega^{(m)}_\ast = k = 0 \)).

Of course, the evolution equations are still \( \mathfrak{A} = 0, \mathfrak{F} = 0, \mathcal{E} = 0 \). Besides, the results of the previous paragraphs continue to hold, with \( B \) fixed according to Eq. (2.4.1) and

\[ \dot{B} = \partial_A \mathcal{B}(A, \varphi) A + \partial_\varphi \mathcal{B}(A, \varphi) \dot{\varphi}. \]  

(2.4.2)

Under the same gauge fixing, the Lagrangian \( \mathcal{L} \) of Eq. (2.3.2) becomes

\[ e^{A-B(A,\varphi)} \left( -\frac{\dot{A}^2}{2} + \frac{\dot{\varphi}^2}{2} \right) - e^{A+B(A,\varphi)} \mathcal{V}(\varphi) - \frac{n^2 \Omega^{(m)}_\ast}{2} e^{-wA+B(A,\varphi)} + \frac{n^2 k}{2} e^{\frac{n-2}{n}A+B(A,\varphi)}. \]

Moreover, it can be easily checked that

\[ \delta \mathcal{L} / \delta q = -\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) + \frac{\partial \mathcal{L}}{\partial q} \quad (q = A, \varphi) \]

(2.4.4)

and the energy function

\[ E := \sum_{q=A,\varphi} \dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} = e^{A-B(A,\varphi)} \left( -\frac{\dot{A}^2}{2} + \frac{\dot{\varphi}^2}{2} \right) + e^{A+B(A,\varphi)} \mathcal{V}(\varphi) + \frac{n^2 \Omega^{(m)}_\ast}{2} e^{-wA+B(A,\varphi)} - \frac{n^2 k}{2} e^{\frac{n-2}{n}A+B(A,\varphi)}. \]

(2.4.5)

Of course, \( E \) is a constant of motion for the Lagrange equations \( \delta \mathcal{L} / \delta q = 0 \) \( (q = A, \varphi) \). Moreover, it can be easily checked that

\[ \begin{pmatrix} \delta \mathcal{L} / \delta A \\ \delta \mathcal{L} / \delta \varphi \\ E \end{pmatrix} = e^{A-B} \begin{pmatrix} 1 & 0 & \partial_A \mathcal{B} \\ 0 & -1 & \partial_\varphi \mathcal{B} \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mathfrak{A} \\ \mathfrak{F} \\ \mathcal{E} \end{pmatrix}, \]

(2.4.6)

where \( \mathfrak{A}, \mathfrak{F}, \mathcal{E} \) are evaluated with \( B, \dot{B} \) as in Eqs. (2.4.1), (2.4.2).

Summing up: after gauge fixing, the evolution equations \( \mathfrak{A} = 0, \mathfrak{F} = 0, \mathcal{E} = 0 \) are equivalent to the Lagrange equations \( \delta \mathcal{L} / \delta q = 0 \) \( (q = A, \varphi) \), supplemented with the condition \( E = 0 \) (the latter condition is satisfied at all times if and only if it is fulfilled by the initial datum \( (A(t_0), \dot{A}(t_0), \varphi(t_0), \dot{\varphi}(t_0)) \)).

From now on, to analyze the dynamics of our cosmological model we systematically refer to the Lagrangian \( \mathcal{L} \) of Eq. (2.4.3) and to the energy constraint \( E = 0 \). Whenever we speak of a solution of (one or all) these equations, we always tacitly assume that the interval of definition is maximal; this convention is consistent with the domain prescriptions of subsection 2.2 and will be applied also to the solutions obtained using Lagrangian coordinates different from \( (A, \varphi) \) (say, the coordinates \( (x, y) \) of the next sections). The plan for the sequel is to consider specific choices for \( \mathcal{V} \), allowing to solve explicitly the corresponding Lagrange equations.
### 3 Adding matter and curvature to the integrable models of Fré, Sagnotti and Sorin

Let us repeat once more that [13] considers purely scalar, spatially flat cosmologies, i.e., models with no matter content and zero spatial curvature. Referring to this framework, Fré, Sagnotti and Sorin identified nine classes of self-interaction potentials \( V(\phi) \) for the scalar field that, after an appropriate gauge fixing \( B = B(A, \phi) \) and a suitable coordinate transformation for the Lagrangian \( \mathcal{L}(A, \phi, \dot{A}, \dot{\phi}) \), produce solvable Lagrange equations. The gauge function \( B(A, \phi) \) and the coordinate transformation just mentioned are given explicitly in [13] (together with the energy constraint) for each one of the nine potential classes; these results are summarized in Table 1 on page 1048 of [13].

In this section we show that, for all classes of potentials in the cited Table 1, extended cosmological models including matter and possibly curvature can be introduced and solved explicitly using the same coordinate transformations employed in [13] for the corresponding, purely scalar cosmologies.

In these extended cosmologies the matter fluid has an equation of state of the form \( p(m) = w \rho(m) \), where the coefficient \( w \) has some fixed specific value or remains arbitrary; in the cases with arbitrary \( w \) (occurring for three of the nine potential classes), some free parameter \( \gamma \) labeling the potentials becomes a prescribed function of \( w \).

To the best of our knowledge, the possibility to build integrable extensions with matter or curvature was previously unknown for all the cosmologies in Table 1 of [13], with the notable exception of the first class of potentials which was analyzed in [25] a short time before the publication of [13] in the case of matter with \( w = 0 \) (dust), zero curvature and space dimension \( n = 3 \).

The following subsections 3.1-3.9 present extended cosmologies for the nine potential classes in Table 1 of [13], starting from the case of [25] (here generalized to an arbitrary space dimension). In each subsection we indicate how the Lagrangian function can be reduced by a proper gauge fixing and a suitable coordinates transformation to one of the canonical forms analyzed in the forthcoming paragraph preceding subsections 3.1-3.9. Following the strategies outlined in the said paragraph, the Lagrange equations can be systematically reduced to quadratures in all cases of interest; in particular, explicit expressions for the corresponding solutions can always be derived. These expressions can be used to investigate the chief qualitative features of each specific model: presence of a Big Bang and corresponding asymptotic behavior; presence of a Big Crunch or, in absence of it, long time behavior of the system; comparison of the density parameters. We will exemplify these issues for some subcases of the nine classes in the forthcoming Section 4.

To conclude this prologue to the nine cases, let us repeat what we already stated in the Introduction: after the nine potential classes of Table 1, [13] considers other 26 “sporadic” potentials, also producing purely scalar integrable cosmologies; the possibility to maintain the integrability features of these sporadic models with the addition of matter or curvature is not discussed in this work, and left as an open problem for future investigations.

**Solvable Lagrangian systems met in the analysis of the nine potential classes.** In the subsequent developments we will replace the Lagrangian coordinates \( A, \phi \) with either a new pair of real coordinates or with a complex one, with the rationale of obtaining simple canonical forms for the Lagrange equations. Under these coordinate transformations, the Lagrangian function \( \mathcal{L} \) assumes the form of one of the types described below (which are actually the same types occurring in [13] in the case of zero spatial curvature and no matter content).

Let us point out a fact that will never be mentioned again in the sequel: like the quantities \( A, \phi, V(\phi) \), the new Lagrangian coordinates \( x, y, \xi, \eta \), etc. introduced in the sequel to treat the nine potential classes are all dimensionless.

**a) Quadratic Lagrangians.** Assume that there exist two real Lagrangian coordinates \( x, y \) such that \( \mathcal{L}(x, y, \dot{x}, \dot{y}) \) is the difference between two quadratic functions of the variables \((\dot{x}, \dot{y})\) and \((x, y)\). In this case the Lagrange equations are linear and can be decoupled via further, linear coordinate transformations. It is unnecessary to give further details on this elementary case, that will appear in subsection 3.1.
b) Triangular Lagrangians. Assume that there exist two real Lagrangian coordinates $x, y$ such that

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = -\mu \dot{x} \dot{y} + u(x) y - h(x), \quad (3.0.1)$$

for some $\mu \in \mathbb{R} \setminus \{0\}$ and some pair of smooth functions $u, h$. The corresponding energy function is

$$E(x, y, \dot{x}, \dot{y}) = -\mu \dot{x} \dot{y} - u(x) y + h(x), \quad (3.0.2)$$

while the Lagrange equations $\delta \mathcal{L}/\delta y = 0$ and $\delta \mathcal{L}/\delta x = 0$ are, respectively,

$$\mu \dot{x} + u(x) = 0, \quad (3.0.3)$$

$$\mu \dot{y} + u'(x) y = h'(x) \quad (3.0.4)$$

($u', h'$ are the derivatives of $u, h$). The system \[3.0.3\] \[3.0.4\] is patently triangular and can be reduced to quadratures following the procedure described hereafter. Firstly, note that Eq. \[3.0.3\] involves only the unknown function $x(t)$; in fact, it describes a one-dimensional conservative system, admitting as a constant of motion the energy

$$F(x, \dot{x}) := \frac{1}{2}\mu \dot{x}^2 + U(x), \quad \text{with } U \text{ s.t. } U' = u. \quad (3.0.5)$$

Any solution $t \mapsto x(t)$ of Eq. \[3.0.3\] with energy $F(x(t), \dot{x}(t)) \equiv F$ fulfills $(2/\mu)(F-U(x(t))) = \dot{x}^2(t) \geq 0$, and thus it takes values within a connected component of the region $\{x \mid (2/\mu)(F-U(x)) \geq 0\}$. For any such solution, let $t_0 < t_1$ be fixed instants in its domain of definition and assume that

$$x(t_0) = x_0, \quad \text{sign} \dot{x}(t) = \text{const.} \equiv \sigma \in \{\pm 1\} \text{ for } t \in (t_0, t_1). \quad (3.0.6)$$

Then,

$$\dot{x}(t) = \sigma \sqrt{\frac{2}{\mu} (F-U(x(t)))} \text{ for } t \in (t_0, t_1), \quad (3.0.7)$$

which entails

$$\sqrt{\frac{\mu}{2}} \int_{t_0}^{x(t)} \frac{dx}{\sqrt{F-U(x)}} = \sigma (t-t_0) \text{ for } t \in [t_0, t_1]. \quad (3.0.8)$$

Next, let $t \mapsto y(t)$ be a map forming, together with the previous function $t \mapsto x(t)$, a solution of the system \[3.0.3\] \[3.0.4\] and consider the total energy $E(x(t), y(t), \dot{x}(t), \dot{y}(t)) \equiv E$. Since $\dot{x}(t)$ does not vanish and has constant sign for $t \in (t_0, t_1)$, there exists a smooth function

$$Y : I \to \mathbb{R}, \quad I := \{x(t) \mid t \in (t_0, t_1)\}, \quad (3.0.9)$$

such that

$$y(t) = Y(x(t)) \quad \text{for } t \in (t_0, t_1). \quad (3.0.10)$$

Correspondingly we get $\dot{y} = Y'(x) \dot{x}$, whence $\dot{x} \dot{y} = Y'(x) \dot{x}^2 = (2/\mu)(F-U(x))Y'(x)$. Inserting the latter expression for $\dot{x} \dot{y}$ and the relation \[3.0.10\] for $y$ into Eq. \[3.0.2\] we obtain that, for $x = x(t) \in I$,

$$E = -2(F-U(x))Y'(x) - u(x)Y(x) + h(x); \quad (3.0.11)$$

equivalently, recalling that $u = U'$, we have

$$Y'(x) = -\frac{U'(x)}{2(F-U(x))} Y(x) - \frac{E-h(x)}{2(F-U(x))} \text{ for } x \in I. \quad (3.0.12)$$

Noting that the latter is a linear inhomogeneous ODE for $Y$, by elementary manipulations we get

$$Y(x) = \sqrt{F-U(x)} \left(P(E,F;x) + K\right) \quad \text{for } x \in I; \quad (3.0.13)$$

$$K \in \mathbb{R}, \quad P(E,F;\cdot) \text{ s.t. } \partial_x P(E,F;x) = -\frac{E-h(x)}{2(F-U(x))^{3/2}}. \quad (3.0.14)$$
Eqs. (3.0.7), (3.0.10) and (3.0.13) give the desired reduction to quadratures of the system (3.0.3) (on any time interval where \( \dot{x} \) has constant sign).

Of course, a Lagrangian of the form

\[
\mathcal{L}(x, y, \dot{x}, \dot{y}) = -\mu \dot{x} \dot{y} + u(y) x - h(y)
\]

(3.0.14)
can be treated in a similar way, interchanging the roles of \( x \) and \( y \).

c) Special triangular Lagrangian. A very simple subcase of the previous framework occurs if the Lagrangian has the triangular form (3.0.1) with \( \mu \lambda \) depending on the sign of \( \mu \).

Depending on the sign of \( \lambda/\mu \), Eq. (3.0.16) (3.0.17) are elementary.

Regarding \( x \), always use the term “oscillator” to indicate a system of any one of the three kinds just mentioned.

Regarding \( x \), \( \lambda/\mu > 0 \) and set

\[
\omega := \sqrt{\frac{\lambda}{\mu}};
\]

(3.0.18)
then, up to a time shift \( t \mapsto t + \text{const.} \), the general solution of Eq. (3.0.16) is of the form

\[
x(t) = A \sin(\omega t) + \frac{\nu}{\lambda},
\]

(3.0.19)
where \( A \in \mathbb{R} \) is an arbitrary constant. If \( t_0 \in \mathbb{R} \) and \( J \) is any open real interval such that \( t_0 \in J \) and the integral appearing in the forthcoming Eq. (3.0.20) exists for all \( t \in J \); then, the general solution on \( J \) of the evolution equation obtained inserting the expression (3.0.19) for \( x(t) \) into Eq. (3.0.17) is given by

\[
y(t) = B \cos(\omega t) + C \sin(\omega t) + \frac{1}{\mu \omega} \int_{t_0}^{t} ds \sin(\omega(t-s)) h'(A \sin(\omega s) + \frac{\nu}{\lambda}),
\]

(3.0.20)
where \( B, C \in \mathbb{R} \) are arbitrary constants.

Of course, also in this case we obtain a system with similar solvability features interchanging the roles of \( x \) and \( y \) in Eq. (3.0.15).

d) Separable Lagrangian. Assume there exist two real Lagrangian coordinates \( x_1, x_2 \) such that (5)

\[
\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2) = \mathcal{L}_1(x_1, \dot{x}_1) + \mathcal{L}_2(x_2, \dot{x}_2) - C, \quad \mathcal{L}_i(x_i, \dot{x}_i) := \frac{1}{2} \mu_i \dot{x}_i^2 - U_i(x_i)
\]

(3.0.21)

\(^5\) Additive constants appearing in a Lagrangian, like the term \(-C\) in Eq. (3.0.21), are usually regarded as irrelevant. However, in the applications considered in this paper we will always be interested in solutions of the Lagrange equations fulfilling the energy constraint \( E = 0 \) (see subsection 2.4); in this connection, additive constants appearing in the definition of the Lagrangian \( \mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2) \) are relevant, since they contribute to the energy function \( E := \sum_{i=1}^{2} \dot{x}_i (\partial \mathcal{L}/\partial \dot{x}_i) - \mathcal{L} \) (see, e.g., Eq. (3.0.20)).
where $C \in \mathbb{R}$, $\mu_i \in \mathbb{R}\setminus\{0\}$ and $U_i$ is a smooth real function for $i = 1, 2$. In this case the Lagrange equations $0 = \delta L / \delta x_i$ ($i = 1, 2$) describe two decoupled subsystems admitting as constants of motion the energies

$$E_i(x_i, \dot{x}_i) := \frac{1}{2} \mu_i \dot{x}_i^2 + U_i(x_i) ;$$

(3.0.22)
of course, the total energy corresponding to the Lagrangian (3.0.21) is

$$E(x_1, x_2, \dot{x}_1, \dot{x}_2) = E_1(x_1, \dot{x}_1) + E_2(x_2, \dot{x}_2) + C .$$

(3.0.23)

Any pair of motions $x_i(t)$ ($i = 1, 2$) of the separate subsystems with energies $E_i$ are confined within connected regions where $\text{sign}(\mu_i)(E_i - U(x_i)) \geq 0$, and can be reduced to quadratures via the relations

$$\sqrt{\frac{\mu_i}{2}} \int_{x_i(t_0)}^{x_i(t)} \frac{dx_i}{E_i - U_i(x_i)} = \sigma_i(t - t_0) ,$$

(3.0.24)

for any $t$ such that $\sigma_i := \text{sign} \dot{x}_i(t) = \text{const.} \in \{\pm 1\}$ on $(t_0, t)$.

e) One-dimensional, holomorphic conservative system. From here to the end of the paper, $\mathbb{R}z, \mathbb{Z}z$ and $\overline{z}$ indicate the real part, the imaginary part and the conjugate of any complex number $z$. Assume that there exist an open subset $D \subset C$ and a complex Lagrangian coordinate $z \in D$ (with $\dot{z} \in C$) such that $[\overline{z}]

$$L(z, \dot{z}) = -\Im \left( \frac{1}{2} \mu \dot{z}^2 - U(z) \right) - C ,$$

(3.0.25)

where $\mu \in \mathbb{C}\setminus\{0\}, C \in \mathbb{R}$ and $U : D \to C$ is a holomorphic function. Let us point out that $L(z, \dot{z}) = -\frac{1}{2\tau} \left( \frac{1}{2} \mu \dot{z}^2 - U(z) \right) + \frac{1}{\overline{\tau}} \left( \frac{1}{2} \bar{\mu} \bar{z}^2 - U(\overline{z}) \right)$ (where $\dot{\overline{z}} := \overline{\tau}$); so the Lagrange equations $\delta L / \delta z = 0$ and $\delta L / \delta \overline{z} = 0$ are, respectively,

$$\mu \dot{z} = -U'(z) ,$$

(3.0.26)

$$\bar{\mu} \bar{\dot{z}} = -\overline{U'(\overline{z})} ,$$

(3.0.27)

($U'$ is the complex derivative of $U$) $[\overline{z}].$ It appears that Eq. (3.0.27) is the complex conjugate of Eq. (3.0.26), therefore the cited equations are fully equivalent.

From now on we fix the attention on Eq. (3.0.26). This possesses as a constant of motion the “complexified” energy

$$\mathcal{E}(z, \dot{z}) = \frac{1}{2} \mu \dot{z}^2 + U(z) \in \mathbb{C} ;$$

(3.0.28)

starting from here, we derive a quadrature formula for real, one natural adaptation to the complex framework of the approach usually employed for real, one dimensional conservative systems. More precisely, consider an open set $P \subset \mathbb{C}\setminus\{0\}$ such that the mapping $P \to P^2 := \{p^2 \mid p \in P\}, p \mapsto p^2$ is biholomorphic; denote with $\sqrt{\cdot} : P^2 \to P$ its inverse. Let $z$ be any solution of Eq. (3.0.26) (hence, of Eq. (3.0.27)) with complex energy $\mathcal{E}(z(t), \dot{z}(t)) \equiv \mathcal{E}$; moreover, let $\mathcal{O} \subset \mathbb{C}$ be an open, simply connected subset such that $\frac{1}{\mu} \left( \mathcal{E} - U(0) \right) \subset P^2$ and assume that $z(t') \in \mathcal{O}, \dot{z}(t') \in P$ for all $t' \in [t_0, t]$. Then,

$$\sqrt{\frac{\mu}{2}} \int_{z(t_0)}^{z(t)} \frac{dz}{\mathcal{E} - U(z)} = t - t_0 ,$$

(3.0.29)

where $\int_{z(t_0)}^{z(t)}$ indicates the integration along any path in $\mathcal{O}$ with initial point $z(t_0)$ and final point $z(t)$ (the integral is independent of the chosen path).

\textsuperscript{6}The constant $C$ in Eq. (3.0.25) is not irrelevant for our purposes: see the footnote on page [13]

\textsuperscript{7}Note that $\partial_z \overline{U}(z) = 0$ and $\partial_z U(z) = U'(z)$. 

19
To proceed, let us remark that the usual energy function $E := \dot{z} \frac{\partial L}{\partial \dot{z}} + \ddot{z} \frac{\partial L}{\partial z} - L$ associated to the Lagrangian (3.0.26) is

$$E(z, \dot{z}) = -\Re \left( \frac{1}{2} \mu \dot{z}^2 + U(z) \right) + C = -\Re E(z, \dot{z}) + C.$$ (3.0.30)

Of course, there are subcases in which Eqs. (3.0.26) (3.0.27) can be integrated by elementary means without even referring to Eq. (3.0.29). In particular, if $U(z) = \frac{1}{2} k z^2$ $(k \in \mathbb{C})$ (3.0.31), Eq. (3.0.26) has the elementary form

$$\mu \ddot{z} + k z = 0 \quad (3.0.32)$$

and we refer to this system as a “complex oscillator”.

### 3.1 Class 1 potentials

The first class of potentials in Table 1 of [13] has the form

$$V(\varphi) := V_1 e^\varphi + V_2 e^{-\varphi} + 2 V_0 \quad (V_0, V_1, V_2 \in \mathbb{R}) .$$ (3.1.1)

For these potentials, the cited reference suggests to introduce the gauge function $B$ and the coordinates $x, y$, defined by

$$B(A, \varphi) := 0 ; \quad A = \log(xy) , \quad \varphi = \log(x/y) \quad (x, y > 0) .$$ (3.1.2) (3.1.3)

In the case of no matter and zero curvature ($\Omega^\text{m} = 0, k = 0$), the above positions give rise to a quadratic Lagrangian and, consequently, to linear evolution equations for $x$ and $y$. Let us implement the same positions in our framework with matter and curvature, searching for additional cases with a quadratic Lagrangian. Eqs. (3.1.1), (3.1.2) and (3.1.3) give the following expressions for the Lagrangian (2.4.3) and the energy function (2.4.5):

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = -2 \ddot{x} \dot{y} - V_1 x^2 - V_2 y^2 - 2 V_0 x y - \frac{n^2}{2} \Omega^\text{m} (x) \frac{\dot{x}^2}{x} + \frac{n^2}{2} k (xy) \frac{n^2}{n^2} ;$$ (3.1.4)

$$E(x, y, \dot{x}, \dot{y}) = -2 \dot{x} \dot{y} + V_1 x^2 + V_2 y^2 + 2 V_0 x y + \frac{n^2}{2} \Omega^\text{m} (x) \frac{\dot{y}^2}{y} - \frac{n^2}{2} k (xy) \frac{n^2}{n^2} .$$ (3.1.5)

The Lagrangian (3.1.4) is still quadratic, up to additive constants, in the following cases with matter or curvature (i.e., with $(\Omega^\text{m}, k)$ not constrained to be $(0,0)$):

i) $k = 0, w = 0$ (dust);

ii) $k = 0, w = -1$ (cosmological constant);

iii) $n = 2, w = 0$ (dust);

iv) $n = 2, w = -1$ (cosmological constant);

v) $\Omega^\text{m} = 0, n = 2$.

In each one of the above cases, the Lagrange equations in the coordinates $x, y$ form a linear system, and can be decoupled via further coordinate changes of linear type. Correspondingly, let us stress that the admissible solutions must fulfill the energy constraint $E = 0$, as well as the conditions $x(t), y(t) > 0$ (cf. Eq. (3.1.3)).
As an example, let us consider case (i), providing (at least if \( n = 3 \)) a rather realistic model of the physical universe for most of its history; this is just the case considered (for \( n = 3 \)) in [23]. The Lagrangian (3.1.4) and the energy (3.1.5) reduce, respectively, to

\[
\mathcal{L}(x, y, \dot{x}, \dot{y}) = -2 \dot{x} \dot{y} - V_1 x^2 - V_2 y^2 - 2 V_0 x y - \frac{n^2}{2} \Omega^{(m)}_x, \tag{3.1.6}
\]

\[
E(x, y, \dot{x}, \dot{y}) = -2 \dot{x} \dot{y} + V_1 x^2 + V_2 y^2 + 2 V_0 x y + \frac{n^2}{2} \Omega^{(m)}_x. \tag{3.1.7}
\]

The Lagrange equations decouple under a further, linear change of coordinates (3.1.8) whose solutions can be determined by elementary means. Let us point out that in the present case, the admissible solutions are those fulfilling \( u \in \Omega^{(m)}_x \).

\[
\mathcal{L}(u, v, \dot{u}, \dot{v}) = \mathcal{L}_1(u, \dot{u}) + \mathcal{L}_2(v, \dot{v}) - \frac{n^2}{2} \Omega^{(m)}_x, \tag{3.1.9}
\]

\[
\mathcal{L}_1(u, \dot{u}) := -\frac{1}{2} \dot{u}^2 - \sqrt{V_1 V_2 + V_0} u^2, \quad \mathcal{L}_2(v, \dot{v}) := \frac{1}{2} \dot{v}^2 - \sqrt{V_1 V_2 - V_0} v^2; \tag{3.1.10}
\]

\[
E_1(u, \dot{u}) := -\frac{1}{2} \dot{u}^2 + \sqrt{V_1 V_2 + V_0} u^2, \quad E_2(v, \dot{v}) := \frac{1}{2} \dot{v}^2 + \sqrt{V_1 V_2 - V_0} v^2. \tag{3.1.11}
\]

The (separable) Lagrangian (3.1.9) gives rise to the system of uncoupled equations

\[
\ddot{u} - \left( \sqrt{V_1 V_2 + V_0} \right) u = 0, \quad \ddot{v} + \left( \sqrt{V_1 V_2 - V_0} \right) v = 0, \tag{3.1.11}
\]

whose solutions can be determined by elementary means. Let us point out that in the present case, the admissible solutions are those fulfilling \( u(t) > |v(t)| \) (a relation which is equivalent to \( x(t), y(t) > 0 \)). More details about the qualitative behavior of such solutions will be given in subsection 4.1 of Section 4.

### 3.2 Class 2 potentials

The second class of potentials in Table 1 of [13] is formed by the functions

\[
\mathcal{V}(\varphi) := V_1 e^{2\gamma \varphi} + V_2 e^{(1+\gamma) \varphi}, \quad (V_1, V_2 \in \mathbb{R}, \gamma \in \mathbb{R} \setminus \{\pm 1\}). \tag{3.2.1}
\]

Ref. [13] suggests to study these potentials fixing the gauge function \( \mathcal{B} \) and introducing new coordinates \( x, y \) as follows:

\[
\mathcal{B}(A, \varphi) := -\gamma \varphi; \tag{3.2.2}
\]

\[
A = \log(x^{1+\gamma} y^{-1-\gamma}), \quad \varphi = \log(x^{1+\gamma} y^{-1-\gamma}) \quad (x, y > 0). \tag{3.2.3}
\]

In the case of no matter and zero curvature, the Lagrangian obtained via these prescriptions is of the special triangular kind (3.0.15).

Let us now apply the same prescriptions (3.2.2) (3.2.3) in our framework with matter and curvature, and search for additional triangular cases. The Lagrangian (2.4.3) and the energy (2.4.5) become, respectively,

\[
\mathcal{L}(x, y, \dot{x}, \dot{y}) = -\frac{2 \dot{x} \dot{y}}{1-\gamma^2} - V_1 x y - V_2 x^{1+\gamma} - \frac{n^2}{2} \Omega^{(m)}_x \left( \frac{w+\gamma}{1+\gamma} x^{1+\gamma} - \frac{w-\gamma}{1+\gamma} y^{1+\gamma} + \frac{n^2}{2} k x \frac{n(1+\gamma)-2}{(m+1)(1+\gamma)} \frac{n(1+\gamma)-2}{m(1+\gamma)} \right), \tag{3.2.4}
\]

\[
E(x, y, \dot{x}, \dot{y}) = -\frac{2 \dot{x} \dot{y}}{1-\gamma^2} + V_1 x y + V_2 x^{1+\gamma} + \frac{n^2}{2} \Omega^{(m)}_x \left( \frac{w+\gamma}{1+\gamma} x^{1+\gamma} - \frac{w-\gamma}{1-\gamma} y^{1-\gamma} - \frac{n^2}{2} k x \frac{n(1+\gamma)-2}{m(1+\gamma)} \frac{n(1+\gamma)-2}{m(1-\gamma)} \right). \tag{3.2.5}
\]
The Lagrangian (3.2.4) has a triangular structure in the cases with matter or curvature listed below. In each one of these cases, the admissible solutions are those fulfilling the energy constraint $E = 0$ and the conditions $x(t), y(t) > 0$ (cf. Eq. (3.2.3)).

i) $k = 0$, $\gamma = w \neq \pm 1$. The Lagrangian (3.2.4) and the energy (3.2.5) become, respectively,

$$\mathcal{L}(x, \dot{x}, y, \dot{y}) = \frac{2}{1-w^2} \dot{x} \dot{y} - V_1 x y - V_2 x^2 \frac{\Omega^m}{2} x^{-\frac{2w}{1+w}} , \quad (3.2.6)$$

$$E(x, \dot{x}, y, \dot{y}) = \frac{2}{1-w^2} \dot{x} \dot{y} + V_1 x y + V_2 x^2 \frac{\Omega^m}{2} x^{-\frac{2w}{1+w}} . \quad (3.2.7)$$

The Lagrangian (3.2.6) has the special triangular structure (3.0.15) and the related equations $\delta \mathcal{L} / \delta y = 0$, $\delta \mathcal{L} / \delta x = 0$ read, respectively,

$$\ddot{x} - \frac{1}{2} (1-w^2) V_1 x = 0 ,$$
$$\ddot{y} - \frac{1}{2} (1-w^2) V_1 y = (1-w)V_2 x^{1-w} - \frac{n^2}{2} (1-w) w \Omega^m x^{-\frac{1+3w}{1+w}} . \quad (3.2.8)$$

The relation in the first line of Eq. (3.2.8) describes an oscillator [8]; when its general solution is substituted into the relation in the second line of Eq. (3.2.8), the latter can be interpreted in terms of a forced oscillator. Let us remark that the system (3.2.8) can be treated by elementary means.

ii) $\gamma = w = \frac{1}{n}$ (radiation gas). The Lagrangian (3.2.4) and the energy (3.2.5) become

$$\mathcal{L}(x, \dot{x}, y, \dot{y}) = \frac{2}{n^2-1} \dot{x} \dot{y} - \left( V_1 x - \frac{n^2}{2} k x^\frac{n-3}{n+1} \right) y - \left( V_2 x^\frac{2n}{n+1} + \frac{n^2}{2} \Omega^m x^{-\frac{3}{n+1}} \right) , \quad (3.2.9)$$

$$E(x, \dot{x}, y, \dot{y}) = \frac{2}{n^2-1} \dot{x} \dot{y} + \left( V_1 x - \frac{n^2}{2} k x^\frac{n-3}{n+1} \right) y + \left( V_2 x^\frac{2n}{n+1} + \frac{n^2}{2} \Omega^m x^{-\frac{3}{n+1}} \right) . \quad (3.2.10)$$

The Lagrangian (3.2.9) has the triangular form (3.0.1), and can be treated with the methods described contextually; in particular, note that Eqs. (3.0.1) (3.0.5) are fulfilled in the present case with $u(x) = -V_1 x + \frac{n^2}{2} k x^\frac{n-3}{n+1}$ and $U(x) = -\frac{1}{2} V_1 x^2 + \frac{n^2(n+1)}{4(n-1)} k x^\frac{2(n-1)}{n+1}$.

ii) $n = 3$, $\gamma = w = \frac{1}{3}$. In this particular subcase of case (ii), the Lagrangian (3.2.4) and the energy function (3.2.5) read

$$\mathcal{L}(x, \dot{x}, y, \dot{y}) = -\frac{9}{4} \dot{x} \dot{y} - \left( V_1 x - \frac{9}{2} k \right) y - V_2 x^\frac{3}{2} - \frac{9}{2} \Omega^m x^{-\frac{1}{2}} , \quad (3.2.11)$$

$$E(x, \dot{x}, y, \dot{y}) = -\frac{9}{4} \dot{x} \dot{y} + \left( V_1 x - \frac{9}{2} k \right) y + V_2 x^\frac{3}{2} + \frac{9}{2} \Omega^m x^{-\frac{1}{2}} . \quad (3.2.12)$$

The Lagrangian (3.2.11) is of the special triangular form (3.0.15) and the related equations $\delta \mathcal{L} / \delta y = 0$, $\delta \mathcal{L} / \delta x = 0$ entail, respectively,

$$\ddot{x} - \frac{4}{9} V_1 x = -2k , \quad \ddot{y} - \frac{4}{9} V_1 y = \frac{2}{3} V_2 x^{1/2} - \Omega^m x^{-3/2} . \quad (3.2.13)$$

The first relation in Eq. (3.2.13) describes an oscillator with a constant “curvature force”; once $x(t)$ has been determined, the second relation in Eq. (3.2.13) describes another forced oscillator. Also in this case, we have a system of equations which can be solved by elementary means.

*In the generalized sense stipulated for this term in the discussion after Eqs. (3.0.10) (3.0.17); the specific kind of this oscillator depends on the sign of $(1 - w^2) V_1$. Similar explanations will never be repeated in the sequel.*
iii) $\gamma = w = \frac{2}{n} - 1$. The Lagrangian (3.2.4) and the energy (3.2.5) reduce to
\[ \mathcal{L}(x, \dot{x}, y, \dot{y}) = -\frac{n^2}{2(n-1)} \dot{x} \dot{y} - V_1 x y - V_2 x^n - \frac{n^2}{2} \Omega_*^{(m)} k x^{n-2}, \] (3.2.14)
\[ E(x, \dot{x}, y, \dot{y}) = -\frac{n^2}{2(n-1)} \dot{x} \dot{y} + V_1 x y + V_2 x^n + \frac{n^2}{2} \Omega_*^{(m)} k x^{n-2}. \] (3.2.15)
The Lagrangian (3.2.19) has the special triangular form (3.0.15) and the equations $\delta \mathcal{L}/\delta y = 0$, $\delta \mathcal{L}/\delta x = 0$ read
\[ \ddot{x} - \frac{2(n-1)}{n^2} V_1 x = 0, \] \[ \ddot{y} - \frac{2(n-1)}{n^2} V_1 y = \frac{2(n-1)}{n} V_2 x^{n-1} + (n-2)(n-1) \Omega_*^{(m)} k x^{n-3}. \] (3.2.16)
Again, the equation for $x(t)$ describes a free oscillator and, once this function has been determined, the equation for $y(t)$ describes a forced oscillator.

iv) $\gamma = \frac{2}{n} - 1, w = \frac{4}{n} - 3$. The Lagrangian (3.2.4) and the energy (3.2.5) are, respectively,
\[ \mathcal{L}(x, \dot{x}, y, \dot{y}) = -\frac{n^2}{2(n-1)} \dot{x} \dot{y} - \left( V_1 x + \frac{n^2}{2} \Omega_*^{(m)} x^{2n-3} \right) y - \left( V_2 x^n - \frac{n^2}{2} k x^{2n-2} \right), \] (3.2.17)
\[ E(x, \dot{x}, y, \dot{y}) = -\frac{n^2}{2(n-1)} \dot{x} \dot{y} + \left( V_1 x + \frac{n^2}{2} \Omega_*^{(m)} x^{2n-3} \right) y + \left( V_2 x^n - \frac{n^2}{2} k x^{2n-2} \right). \] (3.2.18)
The Lagrangian (3.2.17) has the triangular structure (3.0.1), an can be treated with the corresponding methods; in particular, Eqs. (3.0.11) (3.0.15) are fulfilled in the present case with $u(x) = -V_1 x - \frac{n^2}{2} \Omega_*^{(m)} x^{2n-3}$ and $U(x) = -\frac{V_1}{2} x^2 - \frac{n^2}{4(n-1)} \Omega_*^{(m)} x^{2n-2}$.

iv0) $n = 2, \gamma = 0, w = -1$ (cosmological constant). This is the subcase of case (iv) corresponding to $n = 2$. The Lagrangian (3.2.4) and the energy function (3.2.5) reduce to
\[ \mathcal{L}(x, \dot{x}, y, \dot{y}) = -2 \dot{x} \dot{y} - \left( V_1 + 2 \Omega_*^{(m)} \right) x y - V_2 x^2 + 2 k, \] (3.2.19)
\[ E(x, \dot{x}, y, \dot{y}) = -2 \dot{x} \dot{y} + \left( V_1 + 2 \Omega_*^{(m)} \right) x y + V_2 x^2 - 2 k. \] (3.2.20)
The Lagrangian (3.2.19) has the special triangular form (3.0.15). The related equations $\delta \mathcal{L}/\delta y = 0$, $\delta \mathcal{L}/\delta x = 0$ read, respectively,
\[ \ddot{x} - \left( \frac{1}{2} V_1 + \Omega_*^{(m)} \right) x = 0, \] \[ \ddot{y} - \left( \frac{1}{2} V_1 + \Omega_*^{(m)} \right) y = V_2 x \] (3.2.21)
respectively describe a free oscillator and another oscillator with an external force proportional to $x(t)$. Let us remark that, up to a constant, the Lagrangian (3.2.19) also belongs to the class of the quadratic Lagrangians.

v) $k = 0, \gamma = w + 1 \neq \pm 1$. The Lagrangian (3.2.4) and the energy (3.2.5) become
\[ \mathcal{L}(x, \dot{x}, y, \dot{y}) = -\frac{8}{(1-w)(3+w)} \dot{x} \dot{y} - \left( V_1 x + \frac{n^2}{2} \Omega_*^{(m)} x^{-\frac{1+4w}{3+w}} \right) y - V_2 x^{\frac{4}{3+w}}, \] (3.2.22)
\[ E(x, \dot{x}, y, \dot{y}) = -\frac{8}{(1-w)(3+w)} \dot{x} \dot{y} + \left( V_1 x + \frac{n^2}{2} \Omega_*^{(m)} x^{-\frac{1+4w}{3+w}} \right) y + V_2 x^{\frac{4}{3+w}}. \] (3.2.23)
The Lagrangian (3.2.22) has the triangular form (3.0.1), an can be treated with the corresponding methods; in particular, Eqs. (3.0.11) (3.0.15) are fulfilled in the present case setting $u(x) = -V_1 x - \frac{n^2}{2} \Omega_*^{(m)} x^{\frac{1+4w}{3+w}}$ and $U(x) = -\frac{1}{2} V_1 x^2 - \frac{n^2}{3+w} \Omega_*^{(m)} x^{\frac{2(1-w)}{3+w}}$. 

9The position $w = 2/n - 1$ makes this case perhaps less interesting than the previous ones, since it gives $w < 0$ for $n \geq 3$ (for $n = 2$ one has $w = 0$, typical of a dust fluid). Nonetheless, if $w = 2/n - 1$ and we assume $\Omega_*^{(m)} \geq 0$ in Eq. (2.2.13) (non-negative matter density), the requirements (2.1.11) (2.1.12) corresponding to the weak and dominant energy conditions are both fulfilled for any $n \geq 2$ (in fact, for $n \geq 1$).
\[ v_0, k = 0, \gamma = \frac{1}{3}, w = -\frac{1}{3}. \] This is a subcase of case (v), corresponding to \( w = -1/3 \). The Lagrangian (3.2.24) and the energy (3.2.3) reduce to

\[
\mathcal{L}(x, \dot{x}, y, \dot{y}) = -\frac{9}{4} \dot{x} \dot{y} - \left( V_1 x + \frac{n^2}{2} \Omega^{(m)}_v \right) y - V_2 x^\frac{3}{2}, \tag{3.2.24}
\]

\[
E(x, \dot{x}, y, \dot{y}) = -\frac{9}{4} \dot{x} \dot{y} + \left( V_1 x + \frac{n^2}{2} \Omega^{(m)}_v \right) y + V_2 x^\frac{3}{2}. \tag{3.2.25}
\]

The Lagrangian (3.2.24) has the special triangular form (3.0.15). The equations \( \delta \mathcal{L}/\delta y = 0, \delta \mathcal{L}/\delta x = 0 \) entail

\[
\ddot{x} - \frac{4}{9} V_1 x = \frac{2n^2}{9} \Omega^{(m)}_v, \quad \ddot{y} - \frac{4}{9} V_1 y = \frac{2}{3} V_2 x^{1/2} \tag{3.2.26}
\]

and describe two forced oscillators, the first one with a constant external force and the second one with an external force depending on \( x(t) \).

\( \text{vi)} \ \gamma = \frac{1}{n}, w = \frac{2}{n} - 1. \] The Lagrangian (3.2.27) and the energy (3.2.5) read

\[
\mathcal{L}(x, \dot{x}, y, \dot{y}) = -\frac{2n^2}{n^2-1} \dot{x} \dot{y} - \left( V_1 x + \frac{n^2}{2} (\Omega^{(m)}_v - k) x^{n-3} \right) y - V_2 x^{\frac{2n}{n-1}}, \tag{3.2.27}
\]

\[
E(x, \dot{x}, y, \dot{y}) = -\frac{2n^2}{n^2-1} \dot{x} \dot{y} + \left( V_1 x + \frac{n^2}{2} (\Omega^{(m)}_v - k) x^{n-3} \right) y + V_2 x^{\frac{2n}{n-1}}. \tag{3.2.28}
\]

The Lagrangian (3.2.27) has the triangular structure (3.0.1), and can be treated with the corresponding methods; in particular, Eqs. (3.0.1) (3.0.3) are fulfilled in the present case with \( u(x) = -V_1 x - \frac{n^2}{2} (\Omega^{(m)}_v - k) x^{n-3} \) and \( U(x) = -\frac{1}{2} V_1 x^2 - \frac{n^2}{4(n-1)} (\Omega^{(m)}_v - k) x^{\frac{2n}{n-1}} \).

\( \text{vi0)} \ n = 3, \gamma = \frac{1}{3}, w = -\frac{1}{3}. \) This is the subcase of case (vi) corresponding to \( n = 3 \). The Lagrangian (3.2.24) and the energy (3.2.3) reduce to

\[
\mathcal{L}(x, \dot{x}, y, \dot{y}) = -\frac{9}{4} \dot{x} \dot{y} - \left( V_1 x + \frac{9}{2} (\Omega^{(m)}_v - k) \right) y - \frac{3}{2} V_2 x^{\frac{3}{2}}, \tag{3.2.29}
\]

\[
E(x, \dot{x}, y, \dot{y}) = -\frac{9}{4} \dot{x} \dot{y} + \left( V_1 x + \frac{9}{2} (\Omega^{(m)}_v - k) \right) y + V_2 x^\frac{3}{2}. \tag{3.2.30}
\]

The Lagrangian (3.2.29) has the special triangular form (3.0.15) and the equations \( \delta \mathcal{L}/\delta y = 0, \delta \mathcal{L}/\delta x = 0 \) imply

\[
\ddot{x} - \frac{4}{9} V_1 x = 2 (\Omega^{(m)}_v - k), \quad \ddot{y} - \frac{4}{9} V_1 y = \frac{2}{3} V_2 x^{1/2}. \tag{3.2.31}
\]

Again, we have an oscillator with a constant external force and another oscillator with an external force depending on \( x(t) \).

### 3.3 Class 3 potentials

We consider potentials of the form

\[
\mathcal{V}(\varphi) = V_1 e^{2\varphi} + V_2 \quad (V_1, V_2 \in \mathbb{R}). \tag{3.3.1}
\]

Ref. [13] treats these potentials fixing the gauge function \( \mathcal{B} \) and introducing the Lagrangian coordinates \( x, y \), defined as follows:

\[
\mathcal{B}(\mathcal{A}, \varphi) = -\varphi, \tag{3.3.2}
\]

\[
\mathcal{A} = \frac{1}{2} \log x + y; \quad \varphi = \frac{1}{2} \log x - y \quad (x > 0, \ y \in \mathbb{R}). \tag{3.3.3}
\]

\[
\text{Ref. [13] treats these potentials fixing the gauge function } \mathcal{B} \text{ and introducing the Lagrangian coordinates } x, y, \text{ defined as follows:}
\]

\[
\mathcal{B}(\mathcal{A}, \varphi) = -\varphi, \tag{3.3.2}
\]

\[
\mathcal{A} = \frac{1}{2} \log x + y; \quad \varphi = \frac{1}{2} \log x - y \quad (x > 0, \ y \in \mathbb{R}). \tag{3.3.3}
\]
In the case of no matter and zero curvature, the Lagrangian obtained with the above positions has the special triangular form (3.3.15). Let us make the same positions in our framework with matter and curvature, and search for other triangular cases. Eqs. (3.3.1), (3.3.2) and (3.3.3) yield for the Lagrangian (2.4.3) and for the energy (2.4.5) the following expressions:

\[
\mathcal{L}(x, \dot{x}, y, \dot{y}) = -\dot{x} \ddot{y} - V_1 x - V_2 e^{2y} - \frac{n^2}{2} \Omega^{(m)}_x x^{-\frac{1+\omega}{2}} e^{(1-w)y} + \frac{n^2}{2} k x^{-1/n} e^{\frac{2(n-1)}{n} y} ;
\]

and

\[
E(x, \dot{x}, y, \dot{y}) = -\dot{x} \ddot{y} + V_1 x + V_2 e^{2y} + \frac{n^2}{2} \Omega^{(m)}_x x^{-\frac{1+\omega}{2}} e^{(1-w)y} - \frac{n^2}{2} k x^{-1/n} e^{\frac{2(n-1)}{n} y} .
\]

We aim to classify the cases with matter or curvature in which the Lagrangian (3.3.4) has a triangular form. It is evident that a triangular structure cannot be attained when \( k \neq 0 \); thus, we set \( k = 0 \) and search for triangular cases with matter (i.e., with \( \Omega^{(m)} \neq 0 \)). Below we report a list of such cases; in each one of these cases, the admissible solutions are those fulfilling the energy constraint \( E = 0 \) and the condition \( x(t) > 0 \) (cf. Eq. (3.3.3)).

i) \( k = 0, \ w = -1 \) (cosmological constant). The Lagrangian (3.3.4) and the energy (3.3.5) read

\[
\mathcal{L}(x, y, \dot{x}, \dot{y}) = -\dot{x} \ddot{y} - V_1 x - \left( V_2 + \frac{n^2}{2} \Omega^{(m)}_x \right) e^{2y} ,
\]

and

\[
E(x, y, \dot{x}, \dot{y}) = -\dot{x} \ddot{y} + V_1 x + \left( V_2 + \frac{n^2}{2} \Omega^{(m)}_x \right) e^{2y} .
\]

The Lagrangian (3.3.6) has the special triangular form (3.0.14). The related equations \( \delta \mathcal{L}/\delta x = 0, \ \delta \mathcal{L}/\delta y = 0 \) imply, respectively,

\[
\ddot{y} = V_1 , \quad \ddot{x} = 2 \left( V_2 + \frac{n^2}{2} \Omega^{(m)}_x \right) e^{2y} ;
\]

their general solution is readily found to be

\[
x(t) = \left( V_2 + \frac{n^2}{2} \Omega^{(m)}_x \right) \frac{e^{2\beta - \frac{n^2}{4} \Omega^{(m)}_x}}{V_1} \left( \frac{V_1 t + \alpha}{\sqrt{V_1}} \right) \left( \frac{\sqrt{\pi} \text{ Erfi} \left( \frac{V_1 t + \alpha}{\sqrt{V_1}} \right) - e^{\frac{(V_1 t + \alpha)^2}{V_1}}}{\sqrt{V_1}} \right) + \gamma t + \delta ,
\]

\[
y(t) = \frac{1}{2} V_1 t^2 + \alpha t + \beta ,
\]

where \( \alpha, \beta, \gamma, \delta \) are arbitrary integration constants and \( \text{Erfi} \) is the imaginary error function.

As an example, let us mention that Eqs. (3.3.7) and (3.3.9) imply \( E = V_1 \delta - \alpha \gamma \), so the energy constraint \( E = 0 \) holds if and only if \( V_1 \delta = \alpha \gamma \). The issue of finding a (maximal) interval where \( x(t) > 0 \) is strictly related to the choice of the integration constants in Eq. (3.3.9).

ii) \( k = 0, \ w = -3 \). The Lagrangian (3.3.4) and the energy (3.3.5) become, respectively,

\[
\mathcal{L}(x, y, \dot{x}, \dot{y}) = -\dot{x} \ddot{y} - \left( V_1 + \frac{n^2}{2} \Omega^{(m)}_x e^{4y} \right) x - V_2 e^{2y} ,
\]

and

\[
E(x, y, \dot{x}, \dot{y}) = -\dot{x} \ddot{y} + \left( V_1 + \frac{n^2}{2} \Omega^{(m)}_x e^{4y} \right) x + V_2 e^{2y} .
\]

The Lagrangian (3.3.10) has the triangular structure (3.0.14), an can be treated with the corresponding methods; in particular, Eq. (3.0.14) and the analogue of Eq. (3.0.5) are fulfilled in the present case with \( u(y) = -V_1 - \frac{n^2}{2} \Omega^{(m)}_x e^{4y} \) and \( U(y) = -V_1 y - \frac{n^2}{8} \Omega^{(m)}_x e^{4y} \).
iii) \( k = 0, V_2 = 0, w = 1 \). Let us first remark that in this case the potential does also belong to the class discussed in subsection 3.2 (since \( \mathcal{V}(\varphi) \) is of the form (3.2.1) with \( V_2 = 0 \) and \( \gamma = 1 \)). The Lagrangian (3.3.4) and the energy (3.3.5) read, respectively,\
\[\mathcal{L}(x, y, \dot{x}, \dot{y}) = -\dot{x} \dot{y} - V_1 x - \frac{n^2}{2} \Omega_s^{(m)} x^{-1}, \quad (3.3.12)\]
\[E(x, y, \dot{x}, \dot{y}) = -\dot{x} \dot{y} + V_1 x + \frac{n^2}{2} \Omega_s^{(m)} x^{-1}. \quad (3.3.13)\]
We have a special triangular Lagrangian, of the form (3.0.15) with \( \lambda = \sigma = 0 \). The corresponding equations \( \delta \mathcal{L}/\delta y = 0, \delta \mathcal{L}/\delta x = 0 \) entail\
\[\ddot{x} = 0, \quad \ddot{y} = V_1 - \frac{n^2}{2} \Omega_s^{(m)} x^{-2}. \quad (3.3.14)\]
Again, the solutions can be easily derived and read\
\[x(t) = \alpha t + \beta, \quad y(t) = \frac{1}{2 \alpha^2} \left(V_1 (\alpha t + \beta)^2 + \frac{n^2}{2} \Omega_s^{(m)} \log(\alpha t + \beta)\right) + \gamma t + \delta, \quad (3.3.15)\]
where \( \alpha, \beta, \gamma, \delta \) are integration constants. Let us notice that Eqs. (3.3.13) (3.3.15) imply \( E = -\alpha \gamma \), showing that the energy constraint \( E = 0 \) holds only if \( \alpha = 0 \). Finding a (maximal) interval where \( x(t) > 0 \) is a trivial task, once the integration constants in Eq. (3.3.15) have been assigned.

3.4 Class 4 potentials

Let us consider the potential\
\[\mathcal{V}(\varphi) = V \varphi e^{2 \varphi} \quad (V \in \mathbb{R}). \quad (3.4.1)\]
Ref. [13] suggests to treat these potentials using the gauge function \( \mathcal{B} \) and the coordinates \( x, y \), defined by\
\[\mathcal{B}(\mathcal{A}, \varphi) = - (\mathcal{A} + 2 \varphi), \quad (3.4.2)\]
\[\mathcal{A} = \frac{1}{4} \log x + y, \quad \varphi = \frac{1}{4} \log x - y \quad (x > 0, \ y \in \mathbb{R}). \quad (3.4.3)\]
In the case with no matter and zero curvature, these prescriptions yield again a special triangular Lagrangian of the form (3.0.15).

Also in this case, we try to extend the treatment of [13] to our framework with matter and curvature and search for additional triangular cases. With the positions (3.4.1), (3.3.2) and (3.4.3), the Lagrangian (2.4.3) and the energy (2.4.5) become, respectively,
\[\mathcal{L}(x, \dot{x}, y, \dot{y}) = -\frac{1}{2} \dot{x} \dot{y} - V \left(\frac{1}{4} \log x - y\right) - \frac{n^2}{2} \Omega_s^{(m)} x^{-1/w} e^{(1-w)y} + \frac{n^2}{2} k x^{-\frac{n+1}{2n}} e^{\frac{2(n-1)}{n} y}, \quad (3.4.4)\]
\[E(x, \dot{x}, y, \dot{y}) = -\frac{1}{2} \dot{x} \dot{y} + V \left(\frac{1}{4} \log x - y\right) + \frac{n^2}{2} \Omega_s^{(m)} x^{-1/w} e^{(1-w)y} - \frac{n^2}{2} k x^{-\frac{n+1}{2n}} e^{\frac{2(n-1)}{n} y}. \quad (3.4.5)\]
In presence of matter (\( \Omega_s^{(m)} > 0 \)), the only case where the Lagrangian (3.4.4) has a triangular structure is the following.

i) \( k = 0, w = 1 \). The Lagrangian (3.4.4) and the energy (3.4.5) reduce to\
\[\mathcal{L}(x, y, \dot{x}, \dot{y}) = -\frac{1}{2} \dot{x} \dot{y} + V y - \left(\frac{1}{4} V \log x + \frac{n^2}{2} \Omega_s^{(m)} x^{-1}\right), \quad (3.4.6)\]
\[E(x, y, \dot{x}, \dot{y}) = -\frac{1}{2} \dot{x} \dot{y} - V y + \left(\frac{1}{4} V \log x + \frac{n^2}{2} \Omega_s^{(m)} x^{-1}\right). \quad (3.4.7)\]
The Lagrangian (3.4.6) has the special triangular form (3.0.15). The related equations \( \delta \mathcal{L} / \delta y = 0, \delta \mathcal{L} / \delta x = 0 \) entail, respectively,

\[
\dot{x} = -2V, \quad \dot{y} = \frac{1}{2} V x^{-1} - n^2 \Omega^{(m)} x^{-2} ;
\]

their general solution is given by

\[
x(t) = -V t^2 + \alpha t + \beta , \quad y(t) = \gamma t + \delta + \frac{1}{4} \log(-V t^2 + \alpha t + \beta) - \frac{(4 n^2 \Omega^{(m)} - \Delta) (\alpha - 2V t)}{4 \Delta^{3/2}} \log \left( \frac{\sqrt{\Delta} + \alpha - 2V t}{\sqrt{\Delta} - \alpha + 2V t} \right),
\]

where \( \alpha, \beta, \gamma, \delta \) are integration constants and \( \Delta := \alpha^2 + 4V \beta \). From Eqs. (3.4.5) (3.4.9) it follows that \( E = \frac{4 n^2 \Omega^{(m)} V - \Delta (\alpha^2 + 2\delta V)}{\Delta} \), which shows that the energy constraint \( E = 0 \) holds if and only if \( \alpha \gamma = \frac{2V}{\Delta} \left(2 n^2 \Omega^{(m)} - \Delta \right) \). Again, finding a (maximal) interval where \( x(t) > 0 \) is elementary, once the integration constants in Eq. (3.4.9) have been assigned.

### 3.5 Class 5 potentials

This class is formed by potentials of the form

\[
\mathcal{V}(\varphi) = V_1 \log(\coth \varphi) + V_2 \quad (V_1, V_2 \in \mathbb{R}) .
\]

It should be noted that \( \mathcal{V}(\varphi) \) is well defined only for \( \varphi \in (0, \infty) \) (apart from the trivial case where \( V_1 = 0 \), entailing \( \mathcal{V}(\varphi) = \text{constant} = V_2 \)).

Ref. [13] suggests to analyze these potentials by means of the gauge function \( \mathcal{B} \) and of the new coordinates \( x, y \) defined by

\[
\mathcal{B}(\mathcal{A}, \varphi) = -\mathcal{A}, \quad \mathcal{A} = \frac{1}{2} \log \left( \frac{x^2 - y^2}{2} \right), \quad \varphi = \frac{1}{2} \log \left( \frac{x + y}{x - y} \right) \quad (x > y > 0).
\]

In absence of matter and curvature, the Lagrangian \( \mathcal{L}(x, \dot{x}, y, \dot{y}) \) obtained via these prescriptions is separable.

Following our general approach, let us implement the prescriptions of [13] in our framework with matter and curvature, and search for additional separable cases. Eqs. (3.5.1), (3.5.2) and (3.5.3) yield for the Lagrangian (2.4.3) and the energy (2.4.5) the expressions

\[
\mathcal{L}(x, \dot{x}, y, \dot{y}) = \frac{\dot{y}^2 - \dot{x}^2}{4} - V_1 (\log x - \log y) - V_2 - \frac{n^2}{2} \Omega^{(m)} \left( \frac{x^2 - y^2}{2} \right)^{-\frac{1+w}{2}} + \frac{n^2}{2} k \left( \frac{x^2 - y^2}{2} \right)^{-\frac{1}{n}},
\]

\[
E(x, \dot{x}, y, \dot{y}) = \frac{\dot{y}^2 - \dot{x}^2}{4} + V_1 (\log x - \log y) + V_2 + \frac{n^2}{2} \Omega^{(m)} \left( \frac{x^2 - y^2}{2} \right)^{-\frac{1+w}{2}} - \frac{n^2}{2} k \left( \frac{x^2 - y^2}{2} \right)^{-\frac{1}{n}}.
\]

The Lagrangian (3.5.4) is given by the sum of two functions depending separately on \((x, \dot{x})\) and \((y, \dot{y})\), plus two additional terms proportional to \( \Omega^{(m)} \) and \( k \), respectively, both consisting of suitable powers of \( x^2 - y^2 \). The only cases where the latter additional terms disappear or are themselves separable, yielding again a separable Lagrangian of the form (3.0.21), are those where \( k = 0 \) and the exponent \(- (1+w)/2 \) equals 0 or 1. We discuss these two cases in the sequel, keeping in mind that the corresponding Lagrange equations can be reduced to quadratures as indicated in Eq. (3.0.21); correspondingly, let us also repeat that admissible solutions must also fulfill the energy constraint \( E = 0 \) and the conditions \( x(t) > y(t) > 0 \) (cf. Eq. (3.5.3)).
i) $k = 0, w = -1$. The Lagrangian (3.5.4) and the energy (3.5.5) become, respectively,

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \mathcal{L}_1(x, \dot{x}) + \mathcal{L}_2(y, \dot{y}) - V_2 + 2 n^2 \Omega_i(m),$$

$$\mathcal{L}_1(x, \dot{x}) = -\frac{1}{4} \dot{x}^2 - V_1 \log x, \quad \mathcal{L}_2(y, \dot{y}) = \frac{1}{4} \dot{y}^2 + V_1 \log y;$$

$$E(x, y, \dot{x}, \dot{y}) = E_1(x, \dot{x}) + E_2(y, \dot{y}) + V_2 - 2 n^2 \Omega_i(m),$$

$$E_1(x, \dot{x}) = -\frac{1}{4} \dot{x}^2 + V_1 \log x, \quad E_2(y, \dot{y}) = \frac{1}{4} \dot{y}^2 - V_1 \log y.$$

(3.5.6)

(3.5.7)

ii) $k = 0, w = -3$. The Lagrangian (3.5.4) and the energy (3.5.5) become

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \mathcal{L}_1(x, \dot{x}) + \mathcal{L}_2(y, \dot{y}) - V_2,$$

$$\mathcal{L}_1(x, \dot{x}) = -\frac{1}{4} \dot{x}^2 - V_1 \log x + 4 n^2 \Omega_i(m) x^2, \quad \mathcal{L}_2(y, \dot{y}) = \frac{1}{4} \dot{y}^2 + V_1 \log y - 4 n^2 \Omega_i(m) y^2;$$

$$E(x, y, \dot{x}, \dot{y}) = E_1(x, \dot{x}) + E_2(y, \dot{y}) + V_2,$$

$$E_1(x, \dot{x}) = -\frac{1}{4} \dot{x}^2 + V_1 \log x - 4 n^2 \Omega_i(m) x^2, \quad E_2(y, \dot{y}) = \frac{1}{4} \dot{y}^2 - V_1 \log y + 4 n^2 \Omega_i(m) y^2.$$

(3.5.8)

(3.5.9)

3.6 Class 6 potentials

Let us consider potentials of the form

$$\mathcal{V}(\varphi) = V_1 \arctan e^{-\varphi} + V_2 \quad (V_1, V_2 \in \mathbb{R}).$$

(3.6.1)

In connection with this class, [13] suggests to consider the gauge function

$$\mathcal{B}(\mathcal{A}, \varphi) = -\mathcal{A}$$

(3.6.2)

and to replace the Lagrangian coordinates $(\mathcal{A}, \varphi)$ with a conveniently defined, complex variable $z$. Regarding this complex setting, it can be useful to introduce the following conventions, somehow implicit in the cited reference:

- Let $\mathbb{C}_\times := \mathbb{C}\setminus(-\infty, 0]$ be the open region in the complex plane $\mathbb{C}$ obtained removing the negative real semi-axis. Correspondingly, consider the determination of the argument function given by

$$\arg: \mathbb{C}_\times \setminus(-\infty, 0] \to (-\pi, \pi), \quad z \mapsto \arg z.$$  

(3.6.3)

This entails, in particular, that $\arg z = 0$ for $z \in (0, \infty)$ and that $\arg(\bar{z}) = -\arg z$ for all $z \in \mathbb{C}_\times$.

- The usual, natural logarithm $\log: (0, \infty) \to \mathbb{R}$ possesses the extension

$$\log: \mathbb{C}_\times \to \mathbb{R} + i(-\pi, \pi), \quad z \mapsto \log z := \log |z| + i \arg z,$$

(3.6.4)

with $\arg$ as in Eq. (3.6.3). Such an extension is a holomorphic function fulfilling $e^{\log z} = z$ and $\log \bar{z} = \log |z| - i \arg z = \log z$ for all $z \in \mathbb{C}_\times$.

Keeping these premises in mind, the complex formalism of [13] can be described as follows: the coordinates $(\mathcal{A}, \varphi) \in \mathbb{R}^2$ are replaced by a complex coordinate $z \in \mathcal{D}$ with

$$\mathcal{D} := \{z \in \mathbb{C} \mid \Re z, \Im z > 0\} \subset \mathbb{C}_\times,$$

(3.6.5)

related to $(\mathcal{A}, \varphi)$ by

$$\mathcal{A} = \frac{1}{2} \log \left(\frac{z^2 - \bar{z}^2}{2i}\right) = \frac{1}{2} \log (2 \Re z \Im z), \quad \varphi = \frac{1}{2} \log \left(i \frac{z + \bar{z}}{z - \bar{z}}\right) = \frac{1}{2} \log \left(\frac{\Re z}{\Im z}\right).$$

(3.6.6)
The correspondence $z \mapsto (\mathcal{A}, \varphi)$ defined above is one-to-one between $\mathcal{D}$ and $\mathbb{R}^2$, with inverse

$$z = \frac{1}{\sqrt{2}} \left( e^{\mathcal{A} + \varphi} + i e^{\mathcal{A} - \varphi} \right).$$

(3.6.7)

Let us note that the second relation in Eq. (3.6.6) implies $e^{-2\varphi} = \Im z/\Re z$, which allows to express the potential (3.6.1) as

$$\mathcal{V}(\varphi) = V_1 \arctan \frac{\Im z}{\Re z} + V_2 = V_1 \arg z + V_2 = V_1 \Im \log z + V_2.$$

(3.6.8)

In absence of matter and curvature, (13) expresses the Lagrangian function associated to a potential of the form (3.6.1) fixing the gauge and introducing a complex coordinate $z$ as above; the Lagrangian $\mathcal{L}(z, \dot{z})$ obtained in this way is of the holomorphic type (3.0.25), so the related Lagrange equations can be integrated by quadratures.

Following our general philosophy, hereafter we try to generalize the results of (13) to cases with matter or curvature. To this purpose, first note that the prescriptions (3.6.2) (3.6.6) yield the following expressions for the Lagrangian (2.4.3) and for the energy (2.4.5):

$$\mathcal{L}(z, \dot{z}) = -\Im \left( \frac{1}{2} \dot{z}^2 + V_1 \log z \right) - \frac{n^2}{2} \Omega^{(m)} (\Im z^2)^{1/2} + \frac{n^2}{2} k (\Im z^2)^{-1/2} - V_2,$$

(3.6.9)

$$E(z, \dot{z}) = -\Im \left( \frac{1}{2} \dot{z}^2 - V_1 \log z \right) + \frac{n^2}{2} \Omega^{(m)} (\Im z^2)^{1/2} - \frac{n^2}{2} k (\Im z^2)^{-1/2} + V_2.$$

(3.6.10)

(here and in the sequel, $\Im z^2$ stands for $\Im (z^2)$). In passing, let us point out that Eqs. (3.6.9) (3.6.10) have the same structure of Eqs. (3.5.4) (3.5.5) in the previous section, a fact which becomes evident if one considers the replacement $(z, \dot{z}) \rightarrow (x, y)$.

We are interested in cases in which the Lagrangian (3.6.9) maintains the holomorphic structure (3.0.25) even in presence of matter or curvature. Clearly, this structure occurs only if $k = 0$ and the exponent $-(1 + w)/2$ equals $0$ or $1$, which yields the cases discussed below. As usual, let us remark that admissible solutions must fulfill the energy constraint $E = 0$, as well as the condition $z(t) \in \mathcal{D}$.

**i) $k = 0, w = -1$.** The Lagrangian (3.6.9) and the energy (3.6.10) become, respectively,

$$\mathcal{L}(z, \dot{z}) = -\Im \left( \frac{1}{2} \dot{z}^2 + V_1 \log z \right) - \frac{n^2}{2} \Omega^{(m)} - V_2,$$

(3.6.11)

$$E(z, \dot{z}) = -\Im \left( \frac{1}{2} \dot{z}^2 - V_1 \log z \right) + \frac{n^2}{2} \Omega^{(m)} + V_2.$$

(3.6.12)

One can apply the methods described below Eq. (3.0.25) to solve the Lagrange equations by quadratures; in the present case the holomorphic function $U$ and the complexified energy $\mathcal{E}$ of Eqs. (3.0.25) (3.0.28) are $U(z) = -V_1 \log z$, $\mathcal{E}(z, \dot{z}) = \frac{1}{2} \dot{z}^2 - V_1 \log z$.

**ii) $k = 0, w = -3$.** The Lagrangian (3.6.9) and the energy (3.6.10) reduce to

$$\mathcal{L}(z, \dot{z}) = -\Im \left( \frac{\dot{z}^2}{2} + V_1 \log z + \frac{n^2}{2} \Omega^{(m)} z^2 \right) - V_2,$$

(3.6.13)

$$E(z, \dot{z}) = -\Im \left( \frac{\dot{z}^2}{2} - V_1 \log z - \frac{n^2}{2} \Omega^{(m)} z^2 \right) + V_2.$$

(3.6.14)

Again, one should refer to the methods reported below Eq. (3.0.25); in the present case the holomorphic function $U$ and the complexified energy function $\mathcal{E}$ of Eqs. (3.0.25) (3.0.28) are $U(z) = -V_1 \log z - \frac{n^2}{2} \Omega^{(m)} z^2$, $\mathcal{E}(z, \dot{z}) = \frac{1}{2} \dot{z}^2 - V_1 \log z - \frac{n^2}{2} \Omega^{(m)} z^2$. 

29
3.7 Class 7 potentials

Let us consider a potential of the form

\[ V(\varphi) = V_1 \cosh(\gamma \varphi) + V_2 \sinh(\gamma \varphi) \quad (V_1, V_2 \in \mathbb{R}, \; \gamma \in \mathbb{R} \setminus \{0\}) \; ; \]  

(3.7.1)

here and in the sequel we assume

\[ \varphi \in I_{\gamma, V_2}, \quad I_{\gamma, V_2} := \left\{ \begin{array}{ll}
(\infty, +\infty) & \text{if } \frac{2}{\gamma} - 2 \in \{0, 1, 2, \ldots\} \text{ or } V_2 = 0,

\frac{1}{\gamma} (0, +\infty) & \text{if } \frac{2}{\gamma} - 2 \not\in \{0, 1, 2, \ldots\} \text{ and } V_2 \neq 0.
\end{array} \right. \]  

(3.7.2)

(In the above \( \frac{1}{\gamma} (0, +\infty) := \{ \psi / \gamma \mid \psi \in (0, +\infty) \} \); this set equals \((0, +\infty) \) if \( \gamma > 0 \), and \((\infty, 0) \) if \( \gamma < 0 \).

In any case, \( I_{\gamma, V_2} \) is a maximal interval where the function \( V \) in Eq. (3.7.1) is well defined and \( C^\infty \).

Let us also note that

\[ \frac{2}{\gamma} - 2 = h \in \{0, 1, 2, \ldots\} \Leftrightarrow \gamma = \frac{2}{h + 2}, \quad h \in \{0, 1, 2, \ldots\} . \]  

(3.7.3)

To treat a potential of the above form, \([13]\) suggests to use the gauge function \( B \) and the real coordinates \( x, y \), determined as follows:

\[ B(A, \varphi) = (1 - 2\gamma) A ; \]  

\[ A = \frac{1}{2\gamma} \log(x^2 - y^2), \quad \varphi = \frac{1}{2\gamma} \log \left( \frac{x + y}{x - y} \right), \quad (x, y) \in D_{\gamma, V_2} \subset \mathbb{R}^2. \]  

(3.7.4)

(3.7.5)

The domain \( D_{\gamma, V_2} \) is not indicated explicitly in \([13]\), but it is evident that we can put

\[ D_{\gamma, V_2} := \left\{ \begin{array}{ll}
(x, y) \in \mathbb{R}^2 & \text{if } \frac{2}{\gamma} - 2 \in \{0, 1, 2, \ldots\} \text{ or } V_2 = 0,

(x, y) \in \mathbb{R}^2 & \text{if } \frac{2}{\gamma} - 2 \not\in \{0, 1, 2, \ldots\} \text{ and } V_2 \neq 0.
\end{array} \right. \]  

(3.7.6)

In fact, it can be readily checked that the map \((x, y) \mapsto (A, \varphi)\) described in Eq. (3.7.6) is a smooth diffeomorphism between the open sets \( D_{\gamma, V_2} \) and \( \mathbb{R} \times I_{\gamma, V_2} \), with inverse function given by

\[ x = \frac{1}{2} \left( e^{\gamma(A+\varphi)} + e^{\gamma(A-\varphi)} \right), \quad y = \frac{1}{2} \left( e^{\gamma(A+\varphi)} - e^{\gamma(A-\varphi)} \right). \]  

(3.7.7)

In the case of no matter and zero curvature, the Lagrangian \( L(x, \dot{x}, y, \dot{y}) \) obtained with the above prescriptions is separable.

As usual, let us try to use the same prescriptions adding matter and curvature. Eqs. (3.7.1) (3.7.4) (3.7.5) allow to express the Lagrangian (2.4.3) and the energy (2.4.5), respectively, as

\[ L(x, \dot{x}, y, \dot{y}) = \frac{\dot{y}^2 - \dot{x}^2}{2\gamma^2} - V_1 x^{2/\gamma - 2} - V_2 y^{2/\gamma - 2} - \frac{n^2}{2} \Omega^{(m)}_\gamma (x^2 - y^2)^{\frac{1-2\gamma-w}{n\gamma}} + \frac{n^2}{2} k (x^2 - y^2)^{\frac{n(1-\gamma)-1}{n\gamma}} , \]  

(3.7.8)

\[ E(x, \dot{x}, y, \dot{y}) = \frac{\dot{y}^2 - \dot{x}^2}{2\gamma^2} + V_1 x^{2/\gamma - 2} + V_2 y^{2/\gamma - 2} + \frac{n^2}{2} \Omega^{(m)}_\gamma (x^2 - y^2)^{\frac{1-2\gamma-w}{n\gamma}} - \frac{n^2}{2} k (x^2 - y^2)^{\frac{n(1-\gamma)-1}{n\gamma}}. \]  

(3.7.9)

Similarly to the case of potentials of the form (3.5.1), the Lagrangian (3.7.8) is given by the sum of two functions depending separately on \((x, \dot{x})\) and \((y, \dot{y})\), plus two extra terms proportional to \( \Omega^{(m)}_\gamma \) and \( k \), respectively, which consist of powers of \( x^2 - y^2 \) with exponents \( \frac{1-2\gamma-w}{n\gamma} \) and \( \frac{n(1-\gamma)-1}{n\gamma} \). The only situations in which these extra terms disappear or are themselves separable, yielding a separable Lagrangian of the form (3.0.21), are listed in the following. The resulting (decoupled) Lagrange equations for \( x(t), y(t) \) can be reduced to quadratures as indicated in Eq. (3.0.24); the admissible solutions must also fulfill the energy constraint \( E = 0 \) and the condition \((x(t), y(t)) \in D_{\gamma, V_2} \) (cf. Eqs. (3.7.5) (3.7.6)).

30
i) $k = 0$, $\gamma = \frac{1 - w}{2} \neq 0$. The Lagrangian (3.7.8) and the energy (3.7.9) take the form

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \mathcal{L}_1(x, \dot{x}) + \mathcal{L}_2(y, \dot{y}) - \frac{n^2}{2} \Omega^{(m)}_x,$$

$$\mathcal{L}_1(x, \dot{x}) = -\frac{2}{(1-w)^2} \dot{x}^2 - V_1 x^{\frac{2(1+w)}{1-w}}, \quad \mathcal{L}_2(y, \dot{y}) = \frac{2}{(1-w)^2} \dot{y}^2 - V_2 y^{\frac{2(1+w)}{1-w}}; \quad (3.7.10)$$

$$E(x, y, \dot{x}, \dot{y}) = E_1(x, \dot{x}) + E_2(y, \dot{y}) + \frac{n^2}{2} \Omega^{(m)}_y,$$

$$E_1(x, \dot{x}) = -\frac{2}{(1-w)^2} \dot{x}^2 + V_1 x^{\frac{2(1+w)}{1-w}}, \quad E_2(y, \dot{y}) = \frac{2}{(1-w)^2} \dot{y}^2 + V_2 y^{\frac{2(1+w)}{1-w}}. \quad (3.7.11)$$

Let us mention that in the subcases $w = -1$, $w = -1/3$ and $w = 0$ (dust), the exponents of $x$ in $\mathcal{L}_1(x, \dot{x})$ and of $y$ in $\mathcal{L}_2(y, \dot{y})$ become, respectively, $0, 1$ and $2$, so that the Lagrange equations are elementary.

ii) $k = 0$, $\gamma = \frac{1 - w}{4} \neq 0$. The Lagrangian (3.7.8) and the energy (3.7.9) become

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \mathcal{L}_1(x, \dot{x}) + \mathcal{L}_2(y, \dot{y}), \quad (3.7.12)$$

$$\mathcal{L}_1(x, \dot{x}) = -\frac{8}{(1-w)^2} \dot{x}^2 - V_1 x^{\frac{2(3+w)}{1-w}} - \frac{n^2}{2} \Omega^{(m)}_x x^2,$$

$$\mathcal{L}_2(y, \dot{y}) = \frac{8}{(1-w)^2} \dot{y}^2 - V_2 y^{\frac{2(3+w)}{1-w}} - \frac{n^2}{2} \Omega^{(m)}_y y^2;$$

$$E(x, y, \dot{x}, \dot{y}) = E_1(x, \dot{x}) + E_2(y, \dot{y}), \quad (3.7.13)$$

$$E_1(x, \dot{x}) = -\frac{8}{(1-w)^2} \dot{x}^2 + V_1 x^{\frac{2(3+w)}{1-w}} + \frac{n^2}{2} \Omega^{(m)}_x x^2,$$

$$E_2(y, \dot{y}) = \frac{8\dot{y}^2}{(1-w)^2} + V_2 y^{\frac{2(3+w)}{1-w}} + \frac{n^2}{2} \Omega^{(m)}_y y^2.$$  

Let the subcases $w = -3, -5/3, -1$ are elementary, since $x$ and $y$ appear in $\mathcal{L}_1(x, \dot{x})$ and $\mathcal{L}_2(y, \dot{y})$ with exponents $0, 1$ or $2$.

iii) $\Omega^{(m)}_x = 0$, $\gamma = \frac{n-1}{n}$. The Lagrangian (3.7.8) and the energy (3.7.9) reduce to

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \mathcal{L}_1(x, \dot{x}) + \mathcal{L}_2(y, \dot{y}) + \frac{n^2}{2} k,$$

$$\mathcal{L}_1(x, \dot{x}) = -\frac{n^2}{2(n-1)^2} \dot{x}^2 - V_1 x^{\frac{2}{n-1}}, \quad \mathcal{L}_2(y, \dot{y}) = \frac{n^2}{(n-1)^2} \dot{y}^2 - V_2 y^{\frac{2}{n-1}}; \quad (3.7.14)$$

$$E(x, y, \dot{x}, \dot{y}) = E_1(x, \dot{x}) + E_2(y, \dot{y}) - \frac{n^2}{2} k,$$

$$E_1(x, \dot{x}) = -\frac{n^2}{2(n-1)^2} \dot{x}^2 + V_1 x^{\frac{2}{n-1}}, \quad E_2(y, \dot{y}) = \frac{n^2}{(n-1)^2} \dot{y}^2 + V_2 y^{\frac{2}{n-1}}. \quad (3.7.15)$$

In the subcases $n = 2$ and $n = 3$ the exponents of $x$ in $\mathcal{L}_1(x, \dot{x})$ and of $y$ in $\mathcal{L}_2(y, \dot{y})$ become, respectively, $2$ and $1$, so the Lagrange equations are elementary.

iv) $\Omega^{(m)}_y = 0$, $\gamma = \frac{n-1}{2n}$. The Lagrangian (3.7.8) and the energy (3.7.9) become

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \mathcal{L}_1(x, \dot{x}) + \mathcal{L}_2(y, \dot{y}), \quad (3.7.16)$$

$$\mathcal{L}_1(x, \dot{x}) = -\frac{2n^2}{2(n-1)^2} \dot{x}^2 - V_1 x^{\frac{2(n+1)}{n-1}} + \frac{n^2}{2} k x^2, \quad \mathcal{L}_2(y, \dot{y}) = \frac{2n^2}{2(n-1)^2} \dot{y}^2 - V_2 y^{\frac{2(n+1)}{n-1}} - \frac{n^2}{2} k y^2; \quad (3.7.17)$$

$$E(x, y, \dot{x}, \dot{y}) = E_1(x, \dot{x}) + E_2(y, \dot{y}),$$

$$E_1(x, \dot{x}) = -\frac{2n^2}{2(n-1)^2} \dot{x}^2 + V_1 x^{\frac{2(n+1)}{n-1}} - \frac{n^2}{2} k x^2, \quad E_2(y, \dot{y}) = \frac{2n^2}{2(n-1)^2} \dot{y}^2 + V_2 y^{\frac{2(n+1)}{n-1}} + \frac{n^2}{2} k y^2.$$
\textbf{v}) \gamma = -\frac{n-1}{n}, \ w = -\frac{n-2}{n} 

The Lagrangian (3.7.8) and the energy (3.7.9) reduce to

\begin{align*}
\mathcal{L}(x, y, \dot{x}, \dot{y}) &= \mathcal{L}_1(x, \dot{x}) + \mathcal{L}_2(y, \dot{y}) - \frac{n^2}{2} (\Omega^{(m)} - k), \\
\mathcal{L}_1(x, \dot{x}) &= -\frac{n^2}{2(n-1)^2} \ddot{x}^2 - V_1 x^{\frac{2}{n-1}}, \\
\mathcal{L}_2(y, \dot{y}) &= \frac{n^2}{(n-1)^2} \dot{y}^2 - V_2 y^{\frac{2}{n-1}}, \\
E(x, y, \dot{x}, \dot{y}) &= E_1(x, \dot{x}) + E_2(y, \dot{y}) + \frac{n^2}{2} (\Omega^{(m)} - k), \\
E_1(x, \dot{x}) &= -\frac{n^2}{2(n-1)^2} \ddot{x}^2 + V_1 x^{\frac{2}{n-1}}, \\
E_2(y, \dot{y}) &= \frac{n^2}{(n-1)^2} \dot{y}^2 + V_2 y^{\frac{2}{n-1}}.
\end{align*}

(3.7.18)

(3.7.19)

Let us note the close similarities with subcase (iii): \mathcal{L}_1 and \mathcal{L}_2 coincide with the homonymous functions in (3.7.14), and \mathcal{L} is like the homonymous Lagrangian in the same equation with \(k\) replaced by \(k - \Omega^{(m)}\). The subcases \(n = 2, 3\) are elementary, for the same reasons indicated in subcase (iii).

\textbf{vi}) \gamma = -\frac{n-1}{n}, \ w = -\frac{3n-4}{n} 

The Lagrangian (3.7.8) and the energy (3.7.9) become

\begin{align*}
\mathcal{L}(x, y, \dot{x}, \dot{y}) &= \mathcal{L}_1(x, \dot{x}) + \mathcal{L}_2(y, \dot{y}) + \frac{n^2}{2} k, \\
\mathcal{L}_1(x, \dot{x}) &= -\frac{n^2}{2(n-1)^2} \ddot{x}^2 - V_1 x^{\frac{2}{n-1}} - \frac{n^2}{2} \Omega^{(m)} x^2, \\
\mathcal{L}_2(y, \dot{y}) &= \frac{n^2}{(n-1)^2} \dot{y}^2 - V_2 y^{\frac{2}{n-1}} + \frac{n^2}{2} \Omega^{(m)} y^2; \\
E(x, y, \dot{x}, \dot{y}) &= E_1(x, \dot{x}) + E_2(y, \dot{y}) - \frac{n^2}{2} k, \\
E_1(x, \dot{x}) &= -\frac{n^2}{2(n-1)^2} \ddot{x}^2 + V_1 x^{\frac{2}{n-1}} + \frac{n^2}{2} \Omega^{(m)} x^2, \\
E_2(y, \dot{y}) &= \frac{n^2}{(n-1)^2} \dot{y}^2 + V_2 y^{\frac{2}{n-1}} - \frac{n^2}{2} \Omega^{(m)} y^2.
\end{align*}

(3.7.20)

(3.7.21)

The subcases \(n = 2, 3\) are elementary, since \(x, y\) appear in \(\mathcal{L}_1(x, \dot{x}), \mathcal{L}_2(y, \dot{y})\) with exponents 2 or 1.

\textbf{vii}) \gamma = -\frac{n-1}{2n}, \ w = \frac{1}{n} 

The Lagrangian (3.7.8) and the energy (3.7.9) take the forms

\begin{align*}
\mathcal{L}(x, y, \dot{x}, \dot{y}) &= \mathcal{L}_1(x, \dot{x}) + \mathcal{L}_2(y, \dot{y}) - \frac{n^2}{2} \Omega^{(m)}, \\
\mathcal{L}_1(x, \dot{x}) &= -\frac{2n^2}{2(n-1)^2} \ddot{x}^2 - V_1 x^{\frac{2(n+1)}{n-1}} + \frac{n^2}{2} k x^2; \\
\mathcal{L}_2(y, \dot{y}) &= \frac{2n^2}{2(n-1)^2} \dot{y}^2 - V_2 y^{\frac{2(n+1)}{n-1}} - \frac{n^2}{2} k y^2; \\
E(x, y, \dot{x}, \dot{y}) &= E_1(x, \dot{x}) + E_2(y, \dot{y}) + \frac{n^2}{2} \Omega^{(m)}, \\
E_1(x, \dot{x}) &= -\frac{2n^2}{2(n-1)^2} \ddot{x}^2 + V_1 x^{\frac{2(n+1)}{n-1}} - \frac{n^2}{2} k x^2; \\
E_2(y, \dot{y}) &= \frac{2n^2}{2(n-1)^2} \dot{y}^2 + V_2 y^{\frac{2(n+1)}{n-1}} + \frac{n^2}{2} k y^2.
\end{align*}

(3.7.22)

(3.7.23)

Note the strong similarities with subcase (iv): \mathcal{L}_1 and \mathcal{L}_2 are as in Eq. (3.7.10), while \mathcal{L} is like the homonymous Lagrangian in the same equation with the additional constant \(\frac{n^2}{2} \Omega^{(m)}\).

\textbf{viii}) \gamma = \frac{n-1}{2n}, \ w = -\frac{n-2}{n} 

The Lagrangian (3.7.8) and the energy (3.7.9) reduce to

\begin{align*}
\mathcal{L}(x, y, \dot{x}, \dot{y}) &= \mathcal{L}_1(x, \dot{x}) + \mathcal{L}_2(y, \dot{y}) - \frac{n^2}{2} \Omega^{(m)} k, \\
\mathcal{L}_1(x, \dot{x}) &= -\frac{2n^2}{2(n-1)^2} \ddot{x}^2 - V_1 x^{\frac{2(n+1)}{n-1}} - \frac{n^2}{2} \Omega^{(m)} k x^2, \\
\mathcal{L}_2(y, \dot{y}) &= \frac{2n^2}{2(n-1)^2} \dot{y}^2 - V_2 y^{\frac{2(n+1)}{n-1}} + \frac{n^2}{2} \Omega^{(m)} k y^2.
\end{align*}

(3.7.24)
\[ E(x, y, \dot{x}, \dot{y}) = E_1(x, \dot{x}) + E_2(y, \dot{y}), \]
\[ E_1(x, \dot{x}) = -\frac{2n^2}{2(n-1)^2} \dot{x}^2 + V_1(x) \left( \frac{(\Omega^m) - k}{2} \right) x^2, \]
\[ E_2(y, \dot{y}) = \frac{2n^2}{2(n-1)^2} \dot{y}^2 + V_2(y) \left( \frac{(\Omega^m) - k}{2} \right) y^2; \]

Note that \( L_1, L_2 \) and \( L \) are like the homonymous Lagrangians in Eq. \( [3.7.16] \), with \( k \) replaced by \( k - \Omega^m \).

### 3.8 Class 8 potentials

Let us consider potentials of the form
\[ V(\varphi) = C (\cosh(2\gamma \varphi))^{\frac{1}{\gamma} - 1} \sin \left[ \left( \frac{1}{\gamma} - 1 \right) \arctan \frac{1}{\sinh(2\gamma \varphi)} + \vartheta \right] \quad (C \in [0, \infty), \vartheta \in [0, 2\pi], \gamma \in \mathbb{R}\{0\}). \]

These make sense for \( \varphi \in \mathbb{R}\{0\} \). However it is natural to require that \( \varphi \) ranges within a connected domain; so, we assume \( \gamma \varphi \in (0, \infty) \), i.e., \( \varphi \in (0, \infty) \) if \( \gamma > 0 \) and \( \varphi \in (-\infty, 0) \) if \( \gamma < 0 \). The alternative choice \( \gamma \varphi \in (-\infty, 0) \) could be treated similarly.

Like the class of potentials addressed in subsection 3.6, the present class can be treated using a complex formalism. To this purpose, let \( C_\gamma \), \( \arg \) and \( \log \) be defined as in Eqs. \( (3.6.3), (3.6.4) \) of subsection 3.6 (see also the related comments); in addition, let us put
\[ z^\lambda := e^{\lambda \log z} \quad \text{for } z \in C_\gamma, \lambda \in \mathbb{R}. \]

For any \( z, \lambda \) as above, the map \( z \mapsto z^\lambda \) is holomorphic on \( C_\gamma \) and we have \( z^\lambda = |z|^\lambda e^{i\lambda \arg z}, z^{-\lambda} = \overline{z}^\lambda \).

Potentials of the form \( (3.8.1) \) were treated in \([13]\) fixing the gauge function \( B \) and replacing the Lagrangian coordinates \((A, \varphi)\) with a complex variable \( z \), defined as follows:
\[ B(A, \varphi) = (1 - 2\gamma)A; \]
\[ A = \frac{1}{2\gamma} \log \left( \frac{z^2 - \bar{z}^2}{2i} \right) = \frac{1}{2\gamma} \log (2Rz\Im z), \quad \varphi = \frac{1}{2\gamma} \log \left( \frac{z + \bar{z}}{z - \bar{z}} \right) = \frac{1}{2\gamma} \log \left( \frac{Rz}{\Im z} \right) \]

(here and in the sequel \( Rz, \Im z \) stand for \( \Re(z^2), \Im(z^2) \)). In the above, \( z \) is tacitly assumed to belong to a suitable domain \( D \subset \mathbb{C} \), which is not described explicitly in \([13]\); in any case, the coordinate transformation \( (3.8.4) \) is one-to-one between \( D \) and the set \( \{ (A, \varphi) \mid A \in \mathbb{R}, \gamma \varphi \in (0, \infty) \} \) if and only if
\[ D := \{ z \in \mathbb{C} \mid \Re z > \Im z > 0 \} = \{ z \in C_\gamma \mid 0 < \arg z < \pi/4 \}. \]

From here to the end of this subsection, we stick to \( (3.8.5) \). Let us also point out that the inverse of the transformation \((A, \varphi) \mapsto z \in D\) described in Eq. \( (3.8.4) \) is given by
\[ z = \frac{1}{\sqrt{2}} \left( e^{\gamma(A+\varphi)} + e^{i\gamma(A-\varphi)} \right). \]

To proceed, we claim that Eqs. \( (3.8.1), (3.8.4) \) entail the following identities \([10]\):
\[ e^A = (3z^2)^{\frac{1}{3\gamma}}, \quad V(\varphi) = \frac{\Im\left(Vz^{\gamma-2}\right)}{(3z^2)^{\frac{1}{3\gamma} - 1}} \quad \text{with} \quad V := C e^{i\theta}. \]

\[10\]The first identity in Eq. \( (3.8.2) \) follows trivially from the expression for \( A \) in Eq. \( (3.8.4) \) and from the basic equality \( \frac{x^2 - \bar{x}^2}{2} = \Im x^2 \). To obtain the second identity in Eq. \( (3.8.7) \), first notice that the expression for \( \varphi \) in Eq. \( (3.8.4) \) entails \( e^{\gamma \varphi} = \frac{\Re(z^2)}{\Im z^2} \); writing “cosh” and “sinh” in terms of exponentials, from the latter identity we infer
\[ \cosh(2\gamma \varphi) = \frac{(Rz^2 + (\Im z)^2)}{2 \Re z \Im z} = \frac{|z|^2}{\Im z^2}, \quad \sinh(2\gamma \varphi) = \frac{(Rz^2 - (\Im z)^2)}{2 \Re z \Im z} = \frac{\Re z^2}{\Im z^2}. \]
The main result of [13] about potentials of the form (3.8.1) with no matter and zero curvature is that the Lagrangian $\mathcal{L}(z, \dot{z})$ arising from the gauge choice (3.8.3) and the coordinate change (3.8.4) is of the holomorphic type (3.0.25). As usual, we try to generalize this result using the prescriptions of [13] in presence of matter and curvature. Using Eqs. (3.8.1), (3.8.3) and (3.8.4) (and some related identities, especially Eq. (3.8.7)), the Lagrangian (2.4.3) and the corresponding energy (2.4.5) become

$$\mathcal{L}(z, \dot{z}) = -\Im \left( \frac{1}{2\gamma^2} \frac{\dot{z}^2}{z^2} + V \frac{z^2}{\gamma^2} \right) - \frac{n^2}{2} \Omega_\ast^{(m)} \left( \frac{1-2w}{2\gamma} \right) \left( \frac{1}{z^2} \right) + \frac{n^2}{2} k \left( \frac{1}{z^2} \right) \left( \frac{n(1-\gamma)}{2\gamma} - \frac{1}{2\gamma} \right)$$,

$$E(z, \dot{z}) = -\Im \left( \frac{1}{2\gamma^2} \frac{\dot{z}^2}{z^2} - V \frac{z^2}{\gamma^2} \right) + \frac{n^2}{2} \Omega_\ast^{(m)} \left( \frac{1-2w}{2\gamma} \right) \left( \frac{1}{z^2} \right) - \frac{n^2}{2} k \left( \frac{1}{z^2} \right) \left( \frac{n(1-\gamma)}{2\gamma} - \frac{1}{2\gamma} \right).$$

Let us note the close analogies between the present Lagrangian (3.8.8) and the Lagrangian (3.7.8); in fact, writing $3z^2 = \frac{z^2 z^2}{2\gamma}$ and using similar relations (in particular for $3z^2$) we see that the role played in Eq. (3.8.8) by the complex pair $(z, \dot{z})$ is similar to the role played in Eq. (3.7.8) by the real pair $(x, y)$.

We now search for cases with matter or curvature, in which the Lagrangian (3.8.8) has the holomorphic structure (3.0.25). The problem is similar to that of finding the separable cases for the Lagrangian (3.7.8); it can be treated fixing the attention on the terms in Eq. (3.8.8) containing $3z^2$, which have coefficients proportional to $\Omega_\ast^{(m)}$ or $k$ and exponents $\frac{1-2w}{2\gamma} = \frac{n(1-\gamma)}{2\gamma}$. It appears that the Lagrangian (3.8.8) has the holomorphic structure (3.0.25) if $k = 0$ and $\frac{1-2w}{2\gamma} \in \{0, 1\}$, or $\Omega_\ast^{(m)} = 0$ and $\frac{n(1-\gamma)}{2\gamma} \in \{0, 1\}$ and $\frac{n(1-\gamma)}{2\gamma} - \frac{1}{2\gamma} \in \{0, 1\}$. This yields the following 8 cases, which can be all reduced to quadratures following the prescriptions below Eq. (3.0.25).

**i)** $k = 0, \gamma = \frac{1-w}{2} \neq 0$. The Lagrangian (3.8.8) and the energy (3.8.9) reduce to

$$\mathcal{L}(z, \dot{z}) = -\Im \left( \frac{2}{(1-w)^2} \frac{\dot{z}^2}{z^2} + V \frac{z^2}{2(1-w)} \right) - \frac{n^2}{2} \Omega_\ast^{(m)} z^2,$n

$$E(z, \dot{z}) = -\Im \left( \frac{2}{(1-w)^2} \frac{\dot{z}^2}{z^2} - V \frac{z^2}{2(1-w)} \right) + \frac{n^2}{2} \Omega_\ast^{(m)} z^2.$$  

Let us mention that in the subcases $w = -1$, $w = -1/3$ and $w = 0$ (dust), the exponent of $z$ in $\mathcal{L}$ becomes, respectively, $0, 1, 2$, so that the Lagrange equations are elementary. For example, if $w = 0$ we have $\mathcal{L}(z, \dot{z}) = -\Im (2\frac{\dot{z}^2}{z^2} + V \frac{z^2}{2}) - \frac{n^2}{2} \Omega_\ast^{(m)}$ and the Lagrange equations reduce to $\ddot{z} = \frac{V}{2} z$, thus describing a complex oscillator.

**ii)** $k = 0, \gamma = \frac{1-w}{4} \neq 0$. The Lagrangian (3.8.8) and the energy (3.8.9) become

$$\mathcal{L}(z, \dot{z}) = -\Im \left( \frac{2}{(1-w)^2} \frac{\dot{z}^2}{z^2} + V \frac{z^2}{2(1-w)} \right) + \frac{n^2}{2} \Omega_\ast^{(m)} z^2,$n

$$E(z, \dot{z}) = -\Im \left( \frac{2}{(1-w)^2} \frac{\dot{z}^2}{z^2} - V \frac{z^2}{2(1-w)} \right) - \frac{n^2}{2} \Omega_\ast^{(m)} z^2.$$  

The subcases $w = -3, -5/3, -1$ are elementary, since $z$ appears in $\mathcal{L}(z, \dot{z})$ with exponents $0, 1$ or 2.

In view of the above relations, starting from Eq. (3.8.1) and setting $C := V e^{-i\theta}$ we infer the following chain of identities:

$$V(\varphi) = (\cosh(2\gamma \varphi))^{1/2} \Im \left( V e^{i(\frac{\varphi}{\gamma} - 1)} \frac{1}{\left( \frac{\varphi}{\gamma} \right)^{1/2}} \right) = (\frac{|z|^2}{(3z^2)^{1/2}}) \Im \left( V e^{i(\frac{\varphi}{\gamma} - 1)} \frac{1}{\left( \frac{\varphi}{\gamma} \right)^{1/2}} \right)$$

$$= (\frac{|z|^2}{(3z^2)^{1/2}}) \Im \left( V e^{i(\frac{\varphi}{\gamma} - 1)} \frac{1}{\left( \frac{\varphi}{\gamma} \right)^{1/2}} \right).$$
iii) $\Omega^{(m)} = 0$, $\gamma = \frac{n-1}{n}$. The Lagrangian (3.8.8) and the energy (3.8.9) become

$$L(z, \dot{z}) = -\Im \left( \frac{n^2}{2(n-1)^2} \dot{z}^2 + V z^\frac{2(n+1)}{n-1} \right) + \frac{n^2}{2} \Omega^{(m)} k,$$

$$E(z, \dot{z}) = -\Im \left( \frac{n^2}{2(n-1)^2} \dot{z}^2 - V z^\frac{2(n+1)}{n-1} \right) - \frac{n^2}{2} \Omega^{(m)} k.$$  

(3.8.14) (3.8.15)

The subcases $n = 2$ and $n = 3$ are elementary, since $z$ appears in $L(z, \dot{z})$ with exponents 2 and 1.

iv) $\Omega^{(m)} = 0$, $\gamma = \frac{n-1}{2n}$. The Lagrangian (3.8.8) and the energy (3.8.9) become

$$L(z, \dot{z}) = -\Im \left( \frac{2n^2}{(n-1)^2} \dot{z}^2 + V z^\frac{2(n+1)}{n-1} - \frac{n^2}{2} \Omega^{(m)} k \right),$$

$$E(z, \dot{z}) = -\Im \left( \frac{2n^2}{(n-1)^2} \dot{z}^2 - V z^\frac{2(n+1)}{n-1} + \frac{n^2}{2} \Omega^{(m)} k \right).$$

(3.8.16) (3.8.17)

The Lagrangian (3.8.18) is like the Lagrangian (3.8.14), with $k$ replaced by $k - \Omega^{(m)}$. Besides, let us mention that the subcases $n = 2, 3$ are elementary, for the same reasons indicated in (iii).

v) $\gamma = -\frac{n-1}{n}$, $w = -\frac{n-2}{n}$. The Lagrangian (3.8.8) and the energy (3.8.9) become

$$L(z, \dot{z}) = -\Im \left( \frac{n^2}{2(n-1)^2} \dot{z}^2 + V z^\frac{2}{n-1} \right) - \frac{n^2}{2} \left( \Omega^{(m)} - k \right),$$

$$E(z, \dot{z}) = -\Im \left( \frac{n^2}{2(n-1)^2} \dot{z}^2 - V z^\frac{2}{n-1} \right) + \frac{n^2}{2} \left( \Omega^{(m)} - k \right).$$

(3.8.18) (3.8.19)

vi) $\gamma = \frac{n-1}{n}$, $w = -\frac{3n-4}{n}$. The Lagrangian (3.8.8) and the energy (3.8.9) become

$$L(z, \dot{z}) = -\Im \left( \frac{2n^2}{(n-1)^2} \dot{z}^2 + V z^\frac{2(n+1)}{n-1} + \frac{n^2}{2} \Omega^{(m)} z^2 \right) + \frac{n^2}{2} k,$$

$$E(z, \dot{z}) = -\Im \left( \frac{2n^2}{(n-1)^2} \dot{z}^2 - V z^\frac{2(n+1)}{n-1} - \frac{n^2}{2} \Omega^{(m)} z^2 \right) - \frac{n^2}{2} k.$$  

(3.8.20) (3.8.21)

The subcases $n = 2$ and $n = 3$ are elementary, since $z$ appears in $L(z, \dot{z})$ with exponents 2 and 1.

vii) $\gamma = -\frac{n-1}{2n}$, $w = \frac{1}{n}$ (radiation). The Lagrangian (3.8.8) and the energy (3.8.9) become

$$L(z, \dot{z}) = -\Im \left( \frac{2n^2}{(n-1)^2} \dot{z}^2 + V z^\frac{2(n+1)}{n-1} - \frac{n^2}{2} \Omega^{(m)} z^2 \right) - \frac{n^2}{2} \Omega^{(m)} k,$$

$$E(z, \dot{z}) = -\Im \left( \frac{2n^2}{(n-1)^2} \dot{z}^2 + V z^\frac{2(n+1)}{n-1} - \frac{n^2}{2} \Omega^{(m)} z^2 \right) + \frac{n^2}{2} \Omega^{(m)} k.$$  

(3.8.22) (3.8.23)

The Lagrangian (3.8.22) is like the Lagrangian (3.8.10), with the additional constant $-\frac{n^2}{2} \Omega^{(m)} k$.

viii) $\gamma = \frac{n-1}{2n}$, $w = -\frac{n-2}{n}$. The Lagrangian (3.8.8) and the energy (3.8.9) become

$$L(z, \dot{z}) = -\Im \left( \frac{2n^2}{(n-1)^2} \dot{z}^2 + V z^\frac{2(n+1)}{n-1} - \frac{n^2}{2} \left( \Omega^{(m)} - k \right) z^2 \right),$$

$$E(z, \dot{z}) = -\Im \left( \frac{2n^2}{(n-1)^2} \dot{z}^2 - V z^\frac{2(n+1)}{n-1} + \frac{n^2}{2} \left( \Omega^{(m)} - k \right) z^2 \right).$$

(3.8.24) (3.8.25)

Note that the Lagrangian (3.8.24) is like the Lagrangian (3.8.10), with $k$ replaced by $k - \Omega^{(m)}$. 

35
3.9 Class 9 potentials

Let us consider potentials of the form

\[ \mathcal{V}(\varphi) = V_1 e^{2\gamma \varphi} + V_2 e^{\frac{2}{\gamma} \varphi} \quad (V_1, V_2 \in \mathbb{R}, \gamma \in (-1,1) \setminus \{0\}) . \]  

(3.9.1)

[13] treats this class of potentials fixing the gauge function \( \mathcal{B} \) and introducing new Lagrangian coordinates \((x, y)\) related to \((A, \varphi)\) by a “Lorentz transformation”, as follows:

\[ \mathcal{B}(A, \varphi) = A; \quad A = \frac{x - \gamma y}{\sqrt{1 - \gamma^2}}, \quad \varphi = \frac{y - \gamma x}{\sqrt{1 - \gamma^2}} \quad (x, y \in \mathbb{R}) . \]  

(3.9.2)  

(3.9.3)

In absence of matter and curvature, the Lagrangian \( \mathcal{L} \) and the energy \( \mathcal{E} \) yield the following expressions for the separable.

We now add matter and curvature, and use again the above prescriptions trying to find new separable cases. Eqs. (3.9.1), (3.9.2) and (3.9.3) yield the following expressions for the Lagrangian (2.4.3) and the energy (2.4.5):

\[ \mathcal{L}(x, \dot{x}, y, \dot{y}) = \frac{y^2 - x^2}{2} - V_1 e^{2\sqrt{1-\gamma^2}x} - V_2 e^{\frac{2\sqrt{1-\gamma^2}}{\gamma} y} - \frac{n^2}{2} \Omega_m e^{\frac{1-m}{\sqrt{1-\gamma^2}} (x - \gamma y)} + \frac{n^2}{2} k e^{\frac{2(1-m)}{\sqrt{1-\gamma^2}} (x - \gamma y)} ; \]  

(3.9.4)

\[ E(x, \dot{x}, y, \dot{y}) = \frac{y^2 - x^2}{2} + V_1 e^{2\sqrt{1-\gamma^2}x} + V_2 e^{\frac{2\sqrt{1-\gamma^2}}{\gamma} y} + \frac{n^2}{2} \Omega_m e^{\frac{1-m}{\sqrt{1-\gamma^2}} (x - \gamma y)} - \frac{n^2}{2} k e^{\frac{2(1-m)}{\sqrt{1-\gamma^2}} (x - \gamma y)} . \]  

(3.9.5)

The only situation with matter or curvature where the Lagrangian (3.9.1) is separable, is the one described hereafter.

i) \( k = 0, \, w = 1 \). The Lagrangian (3.9.4) and the energy (3.9.5) reduce to

\[ \mathcal{L}(x, y, \dot{x}, \dot{y}) = \mathcal{L}_1(x, \dot{x}) + \mathcal{L}_2(y, \dot{y}) - \frac{n^2}{2} \Omega_m^k , \]  

\[ \mathcal{L}_1(x, \dot{x}) := -\frac{1}{2} \dot{x}^2 - V_1 e^{2\sqrt{1-\gamma^2}x} , \quad \mathcal{L}_2(y, \dot{y}) := -\frac{1}{2} \dot{y}^2 - V_2 e^{\frac{2\sqrt{1-\gamma^2}}{\gamma} y} ; \]  

(3.9.6)

\[ E(x, y, \dot{x}, \dot{y}) = E_1(x, \dot{x}) + E_2(y, \dot{y}) + \frac{n^2}{2} \Omega_m^k , \]  

\[ E_1(x, \dot{x}) := -\frac{1}{2} \dot{x}^2 + V_1 e^{2\sqrt{1-\gamma^2}x} , \quad E_2(y, \dot{y}) := -\frac{1}{2} \dot{y}^2 + V_2 e^{\frac{2\sqrt{1-\gamma^2}}{\gamma} y} . \]  

(3.9.7)

4 Explicit form and detailed analysis of some solutions

In the previous section, we provided a list of integrable cases with matter or space curvature associated to the nine potential classes of Fré-Sagnotti-Sorin; let us recall that each one of these cases is solvable for one of the reasons (a-e) indicated at the beginning of section 3 (linearity of the Lagrange equations, triangular or special triangular Lagrangian, separable Lagrangian, one-dimensional holomorphic and conservative system).

Of course, after indicating a reason for the solvability of the Lagrange equations one should give the explicit form of the (general) solution and analyze it both qualitatively and quantitatively. In particular, one should investigate the occurrence of an initial Big Bang singularity and the related presence of a particle horizon (see Eqs. (2.2.19) and (2.2.20) and the associated comments), as well as the possible development of a Big Crunch or, in absence of it, the evolution of the system for long times. In connection with these issues, it is essential to determine the asymptotic behavior of the main elements of the model: the scale factor \( a \), the scalar field \( \varphi \) and the related equation of state parameter \( w^{(\phi)} \), together with the density parameters \( \Omega^{(m)}, \Omega^{(\phi)}, \Omega^{(k)} \). If the model turns out to
be physically plausible, at least for some epoch in the evolution of the universe, one should also choose sensible values for the parameters in the potential \( V(\varphi) \) and for the constants of integration of the solution, so as to make contact with the available observational data. In the forthcoming subsections 4.1-4.2 we discuss the above issues (or some of them) for some specific cases, taken as examples.

All the cases to be discussed in the sequel have vanishing scalar curvature, i.e.,

\[ k = 0 \quad \text{(4.0.1)} \]

and possess a Big Bang (to which we devote most our attention) at

\[ t = t_m = 0 \quad \text{(4.0.2)} \]

Following Eq. (2.2.19), we define the cosmic time as

\[ \tau(t) := \theta \int_0^t dt' e^{B(t')} \quad \text{(4.0.3)} \]

keeping in mind that the integrability of \( e^B \) in a right neighborhood of \( t = 0 \) is required by the very definition of Big Bang. Of course, \( \tau(t) \to 0^+ \) for \( t \to 0^+ \) and we can speak of the inverse function \( t = t(\tau) \). In each one of the cases analyzed in the sequel, regarding the presence of a particle horizon it is important to ascertain whether the following integral is finite or infinite at any time \( \tau_1 = \tau(t_1) \) (see Eq. (2.2.20) and the related comments):

\[ \Theta(\tau_1) := \int_0^{\tau_1} \frac{d\tau}{a(\tau)} = \theta \int_0^{t_1} dt e^{B(t)} - A(t)/n \quad \text{(4.0.4)} \]

On the other hand, on account of the physically plausible assumption (4.0.1), from Eqs. (2.2.32) (2.2.33) we infer \( \Omega^{(k)} = 0 \) and, consequently,

\[ \Omega^{(m)} + \Omega^{(\varphi)} = 1 \quad \text{(4.0.5)} \]

Making reference to the above relation, we will say that matter dominates at the Big Bang if \( \Omega^{(m)}(t) \to 1 \) (or equivalently, \( \Omega^{(\varphi)}(t) \to 0 \)) for \( t \to 0^+ \); conversely, we will say that the scalar field (or the dark energy) dominates at the Big Bang if \( \Omega^{(\varphi)}(t) \to 0 \) (or equivalently, \( \Omega^{(m)}(t) \to 1 \)) in the same limit. (One can define similarly the cases where matter or the scalar field dominate at the Big Crunch, if this exists.) In addition, since a negative matter density cannot be related to any realistic physical model, we further require the parameter \( \Omega^{(m)} \) to be positive or null, i.e.,

\[ \Omega^{(m)} \geq 0 \quad \text{(4.0.6)} \]

Finally let us point out that, in agreement with the general results of [12], in all cosmological models with a scalar field and a matter fluid to be discussed in the sequel there is a particle horizon at any time after the Big Bang whenever the matter fluid fulfills as a strict inequality the strong (whence, the weak) energy condition, which in the present context means (cf. Eq. (2.1.13))

\[ w > \frac{2}{n} - 1 \quad \text{(4.0.7)} \]

### 4.1 Spatially flat, dust solutions for class 1 potentials

Let us consider an \((n + 1)\)-dimensional, spatially flat cosmology with matter content described by a dust fluid; accordingly, besides Eq. (4.0.1), we posit

\[ w = 0 \quad \text{(4.1.1)} \]
Moreover, we assume that the self-interaction potential for the field is given by
\[ V(\varphi) = V_1 e^{\varphi} + V_2 e^{-\varphi}, \quad \text{with} \quad V_1, V_2 > 0; \] (4.1.2)
for notational convenience, in the sequel we shall put
\[ V := \sqrt{V_1 V_2} > 0. \] (4.1.3)

The model depicted above was previously studied by Rubano and Scudellaro [29], and by Piedipalumbo, Scudellaro, Esposito and Rubano [23], in the physically most relevant case with spatial dimension \( n = 3 \). Hereafter, we review within our framework the results of [23, 29], generalizing them to the case of arbitrary \( n \geq 2 \); in addition, we enrich the analysis discussing the asymptotic behavior of the density parameters \( \Omega^{(m)}, \Omega^{(s)} \) near the Big Bang.

To begin with, let us notice that the potential \( (4.1.2) \) is clearly of the form \( (3.1.1) \) (with \( V_0 = 0 \) and the conditions stated above on \( V_1, V_2 \)); to be more precise, as a consequence of Eqs. \( (4.0.1)-(4.1.1) \), the cosmological model under analysis belongs to the integrable subcase (i) of class 1, discussed previously in subsection 3.1.

In this connection, let us recall that it is convenient to fix the gauge function \( B(\mathcal{A}, \varphi) \) as in Eq. \( (3.1.2) \), which gives \( B = 0 \). In view of Eq. \( (2.2.2) \), this implies that the cosmic time \( \tau \) and the coordinate time \( t \) are linearly related:
\[ t = \tau/\theta. \] (4.1.4)

It was shown in subsection 3.1 that the Lagrangian function for the model that we are considering is separable and can be reduced to quadratures introducing a new pair of coordinates \( u, v \), related to \( \mathcal{A}, \varphi \) via (cf. Eqs. \( 3.1.3 \) \( 3.1.8 \))
\[ \mathcal{A} = \log \left( \frac{u^2 - v^2}{4} \right), \quad \varphi = \log \left( \sqrt{\frac{V_2}{V_1}} \frac{u - v}{u + v} \right). \] (4.1.5)

With the previously stated assumptions, the Lagrange equations \( (3.1.11) \) for \( u, v \) reduce to
\[ \ddot{u} - V u = 0, \quad \ddot{v} + V v = 0, \] (4.1.6)
and the corresponding solutions are readily found to be
\[ u(t) = A \cosh (\sqrt{V} t) + B \sinh (\sqrt{V} t), \quad v(t) = C \cos (\sqrt{V} t) + D \sin (\sqrt{V} t), \] (4.1.7)
where \( A, B, C, D \in \mathbb{R} \) are suitable integration constants.

From Eqs. \( (3.1.10)-(4.1.7) \), by elementary computations we infer the following expression for the energy \( E \equiv E(u, v, \dot{u}, \dot{v}) \) of the system:
\[ E = \frac{1}{2} V \left( A^2 - B^2 + C^2 + D^2 \right) + \frac{n^2}{2} \Omega^{(m)} \] (4.1.8)

Taking the above relation into account, to fulfill the energy constraint \( E = 0 \) we set
\[ \Omega^{(m)} = \frac{1}{n^2} V \left( B^2 - A^2 - C^2 - D^2 \right). \] (4.1.9)

Furthermore, for fixed values of the parameters we take as a domain for the solutions \( 4.1.7 \) the maximal interval \( I \subset \mathbb{R} \) such that (cf. the last comment at the end of subsection 3.1)
\[ t_{\text{in}} \equiv 0 \in I \quad \text{and} \quad u(t) > |v(t)| \quad \text{for all} \ t \in I. \] (4.1.10)

Finally let us mention that Eqs. \( 2.2.11 \ 2.2.12 \ 2.2.32 \ 4.1.2 \ 4.1.5 \) (recalling as well that \( B = 0 \)) give the following representations for the coefficient \( u^{(s)} \) in the field equation of state and for the matter density parameter \( \Omega^{(m)} \):
\[ u^{(s)} = \frac{(u - v)\dot{u}^2 - V(u^4 - v^4)}{(u - v)\dot{u}^2 + V(u^4 - v^4)}, \] (4.1.11)
\[ \Omega^{(m)} = \frac{n^2 \Omega^{(m)}}{(u - v)^2 \dot{u}^2}. \] (4.1.12)
4.1.1 Big Bang analysis

Let us wonder under which conditions the solution (4.1.7) produces a Big Bang at some instant, that we conventionally choose as the time origin \( t = 0 \) (cf. Eq. (4.0.2)). Such conditions entail that \( a(t) \to 0 \) (i.e., \( A(t) \to -\infty \)) for \( t \to 0^+ \) and, even prior to this, that \( a(t) \) (hence, \( A(t) \)) is well defined in a right neighborhood of \( t = 0 \); in terms of the functions \( u(t) \), \( v(t) \), this amounts to require

\[
u^2(t) - v^2(t) \to 0 \quad \text{and} \quad u(t) > |v(t)| \quad \text{for} \quad t \to 0^+ \tag{4.1.13}
\]

(here and in the sequel, an expression of the form “\( f(t) > 0 \) for \( t \to 0^+ \)” means that there exits some \( \epsilon > 0 \) such that \( f(t) > 0 \) for all \( t \in (0, \epsilon) \)).

From the explicit expressions reported in Eq. (4.1.7), it follows straightforwardly that

\[
u(t) = A + B\sqrt{V}t + \frac{1}{2}VA t^2 + O(t^3), \quad v(t) = C + D\sqrt{V}t - \frac{1}{2}VC t^2 + O(t^3) \quad \text{for} \quad t \to 0^+. \tag{4.1.14}
\]

The above relations show that the first condition in Eq. (4.1.13) is fulfilled if and only if \( A^2 - C^2 = 0 \), while it is necessary to assume that \( A \geq 0 \) in order to satisfy the second condition in the same equation; thus, we require

\[
A = |C| \geq 0. \tag{4.1.15}
\]

In the sequel we will distinguish three subcases of the latter constraint.

i) \( A = C > 0 \). In this case the second condition in Eq. (4.1.13) holds if and only if \( B \geq D \).

It should be noticed that for \( B = D \) the energy constraint written in the form (4.1.9) entails \( \Omega^m_\ast = -\frac{2}{n^2}VA^2 < 0 \); since this contradicts our general assumption (4.0.6), from now we assume

\[
B > D. \tag{4.1.16}
\]

To go on, we note that the previously mentioned expressions for \( A \), \( \varphi \), \( w^{(o)} \) and \( \Omega^{(m)} \) (see Eqs. (4.1.4) (4.1.9) (4.1.11) (4.1.12)) imply the following, for \( t \to 0^+ \):

\[
A(t) = \log t + \log \left( \frac{A(B - D)}{2} \sqrt{V} \right) + O(t); \tag{4.1.17}
\]

\[
\varphi(t) = \log t + \log \left( \frac{B - D}{2A} V^{3/4}_2 V^{1/4}_1 \right) + O(t); \tag{4.1.18}
\]

\[
w^{(o)}(t) = 1 - \frac{8A}{B - D} \sqrt{V} t + O(t^2); \tag{4.1.19}
\]

\[
\Omega^{(m)}(t) = \frac{2(B^2 - D^2 - 2A^2) V^1_2}{A(B - D)} t + O(t^2). \tag{4.1.20}
\]

Of course, similar expansions in terms of the cosmic time \( \tau \) can be obtained simply recalling that \( t = \tau/\theta \) (see Eq. (4.1.4)); in particular, from Eqs. (2.2.4) (4.1.17) we obtain

\[
a(\tau) = \left( \frac{A(B - D)}{2} \sqrt{V} \right)^{1/n} (\tau/\theta)^{1/n} + O((\tau/\theta)^{(1/n)+1}) \quad \text{for} \quad \tau/\theta \to 0^+. \tag{4.1.21}
\]

From Eqs. (4.0.4) (4.1.21), noting that \( 1/n < 1 \) for any \( n \geq 2 \) we infer that there exists a particle horizon at any finite time \( \tau \). Eq. (4.1.20) shows that \( \Omega^{(m)}(t) \to 0 \), indicating that the scalar field dominates close to the Big Bang.

\footnote{Making reference to the comments related to Eq. (4.0.7), let us remark that in the present case the strong energy condition is in fact fulfilled (not only as a strict inequality), since \( w = 0 \geq \frac{2}{n} - 1 \) for all \( n \geq 2 \).}
ii) \( \mathbf{A} = - \mathbf{C} > 0 \). This is qualitatively very similar to the previous case. The second condition in Eq. (4.1.13) holds only if \( B \geq -D \); yet, for \( B = -D \) the energy constraint (4.1.33) yields a negative matter density \( \Omega_\ast^{(m)} < 0 \), thus violating our hypothesis (4.0.6). So, we assume

\[
B > -D .
\]

Then, for \( t \to 0^+ \) we have

\[
\mathcal{A}(t) = \log t + \log \left( \frac{A(B + D) \sqrt{V}}{2} \right) + O(t) ;
\]

\[
\varphi(t) = - \log t - \log \left( \frac{B + D}{2A} \frac{V_1^{3/4}}{V_2^{3/4}} \right) + O(t) ;
\]

\[
w^{(o)}(t) = 1 - \frac{8A \sqrt{V}}{B + D} t + O(t^2) ;
\]

\[
\Omega^{(m)}(t) = \frac{2(B^2 - D^2 - 2A^2 \sqrt{V})}{A(B + D)} t + O(t^2) .
\]

Correspondingly, from Eqs. (2.2.4) (4.1.4) (4.1.29) we get

\[
a(\tau) = e^{A(\tau/\theta)/n} = \left( \frac{A(B + D) \sqrt{V}}{2} \right)^{1/n} \left( \frac{\tau}{\theta} \right)^{1/n} + O\left( \left( \frac{\tau}{\theta} \right)^{(1/n)+1} \right) \quad \text{for } \tau/\theta \to 0^+ ,
\]

which allows us to infer that there is a particle horizon. Moreover, Eq. (4.1.26) shows that the scalar field dominates near the Big Bang.

iii) \( \mathbf{A} = \mathbf{C} = 0 \). In this setting the second condition in Eq. (4.1.13) holds if and only if

\[
B > |D| \geq 0 \quad (D \in \mathbb{R}) .
\]

As a consequence, Eq. (4.1.9) implies a strictly positive matter density \( \Omega_\ast^{(m)} = \frac{1}{n} V(B^2 - D^2) > 0 \), in agreement with the general condition stated in Eq. (4.0.6).

To investigate the asymptotic behavior of the system near the Big Bang, note that for \( t \to 0^+ \) we have

\[
\mathcal{A}(t) = 2 \log t + \log \left( \frac{(B^2 - D^2) V}{4} \right) + O(t^2) ;
\]

\[
\varphi(t) = \log \left( \frac{B - D}{B + D} \frac{V_2}{V_1} \right) + O(t^2) ;
\]

\[
w^{(o)}(t) = -1 + \frac{8B^2D^2 V}{9(B^4 - D^4)} t^2 + O(t^4) ;
\]

\[
\Omega^{(m)}(t) = 1 - \frac{(B^2 + D^2) V}{B^2 - D^2} t^2 + O(t^4) .
\]

Furthermore, Eqs. (2.2.4) (4.1.4) (4.1.29) entail

\[
a(\tau) = e^{A(\tau/\theta)/n} = \left( \frac{(B^2 - D^2) V}{4} \right)^{1/n} \left( \frac{\tau}{\theta} \right)^{2/n} + O\left( \left( \frac{\tau}{\theta} \right)^{(2/n)+2} \right) \quad \text{for } \tau/\theta \to 0^+ .
\]

From Eqs. (4.0.4) (4.1.33) we infer that a particle horizon is present if \( n \geq 3 \), and absent if \( n = 2 \) (in the latter case the integral in Eq. (4.0.4) diverges logarithmically). Eq. (4.1.31) indicates a field equation of state close to that of a cosmological constant (recall Eq. (2.2.29)). On the other hand, Eq. (4.1.32) shows that \( \Omega^{(m)}(t) \to 1 \); thus, differently from the previous cases, here matter is dominant near the Big Bang.

\[\text{Since we are assuming } w = 0, \text{ for } n = 2 \text{ the strong energy condition is not fulfilled as a strict inequality (see Eq. (4.0.7)). Due to this, the hypothesis in [12] Eq. (34) is not satisfied, which explains why the general conclusions of [12] Prop. 1 do not hold in this case. This suggests that the hypotheses underlying [12] Prop. 1 are somehow optimal.}\]
4.1.2 Behavior of the model in the far future

First of all, let us remark that the bare solutions written in Eq. (4.1.7) make sense for any \( t \in \mathbb{R} \). However, one should not forget that the second condition in Eq. (4.1.10) puts severe restrictions on the maximal admissible domain \( I \subset \mathbb{R} \) for such solutions. In presence of a Big Bang at \( t = 0 \), the most enticing scenarios are those corresponding to an endless evolution of the universe, namely,

\[
I = (0, +\infty) \, .
\]  

(4.1.34)

In the sequel we restrict the attention to cases where the integration constants \( A, B, C, D \) characterizing the solutions (4.1.7) are such that the condition in Eq. (4.1.31) occurs \n\cite{13}, and proceed to investigate the asymptotic behavior of the corresponding cosmological model for \( t \to +\infty \).

From the explicit expressions (4.1.7) for \( u(t), v(t) \) we easily infer the following: \( u(t) = \frac{A+B}{2} e^{\sqrt{V} t} + O(e^{-\sqrt{V} t}) \) for \( t \to +\infty \); \( v(t) \) is oscillatory, with \( |v(t)| \leq |C| + |D| \) for all \( t \in (0, +\infty) \). Thus, we see a posteriori that the second condition in Eq. (4.1.10) is fulfilled in a neighborhood of \( +\infty \) if and only if

\[
A + B > 0 \, ,
\]  

(4.1.35)

which we assume from now on. With this assumption the solutions (4.1.7) are admissible, at least, in neighborhood of infinity and we have the following asymptotics for \( t \to +\infty \) (recall Eqs. (4.1.9) (4.1.10) (4.1.11) (4.1.12)):

\[
A(t) = 2\sqrt{V} t + 2\log \left( \frac{A+B}{4} \right) + O(e^{-2\sqrt{V} t}) \; ;
\]  

(4.1.36)

\[
\varphi(t) = \log \sqrt{\frac{V}{V_1}} + O(e^{-\sqrt{V} t}) \; ;
\]  

(4.1.37)

\[
w^{(\varphi)}(t) = -1 + O(e^{-2\sqrt{V} t}) \; ;
\]  

(4.1.38)

\[
\Omega^{(m)}(t) = \frac{4(B^2 - A^2 - C^2 - D^2)V}{(A+B)^2} e^{-2\sqrt{V} t} + O(e^{-4\sqrt{V} t}) \, .
\]  

(4.1.39)

From the above relations we infer, especially, that the field \( \varphi \) eventually behaves as a cosmological constant and becomes the dominant contribution (since \( \Omega^{(\varphi)} = 1 - \Omega^{(m)} \to 1 \) for \( t \to +\infty \); see Eq. (4.1.31)).

It is straightforward to derive similar expansions in terms of the cosmic time \( \tau \), recalling that Eq. (4.1.31) gives \( t = \tau/\theta \); in particular, from Eqs. (2.2.3) (4.1.30) we obtain

\[
a(\tau) = \left( \frac{A+B}{4} \right)^{2/n} e^{(2/n)\sqrt{V}(\tau/\theta)} + O(e^{-\left(\frac{n-1}{2n}\right)\sqrt{V}(\tau/\theta)}) \quad \text{for} \quad \tau/\theta \to +\infty \, .
\]  

(4.1.40)

Thus, for large times the scale factor diverges, the field equation of state resembles the cosmological constant case (2.2.29) and the scalar field is dominant \( (\Omega^{(\varphi)}(t) \to 1 \text{ since } \Omega^{(m)}(t) \to 0) \). All these features are attained with exponential speed.

4.1.3 Quantitative analysis of one of the previous cases

Hereafter we reconsider the general model described at the beginning of the present subsection 4.1 and show how to fix all the (so far unspecified) associated parameters \( n, \theta, \Omega^{(m)}, V_1, V_2, A, B, C, D \) so as to provide a physically plausible scenario.

\footnote{As a matter of fact, there do exist such admissible choices of \( A, B, C, D \). For example, making reference to the cases analyzed in the previous subsection 4.1.1 it can be checked by direct inspection that Eq. (4.1.31) certainly holds if

\[ A = C > 0, \quad B > D > 0 \quad \text{or} \quad A = -C > 0, \quad B > -D > 0 \quad \text{or} \quad A = C = 0, \quad B > |D| > 0 \, . \]

In all the cases mentioned above, to infer that \( u(t) > |v(t)| \) for all \( t \in (0, \infty) \) it suffices to notice that \( \cosh(z) > |\cos(z)| \) and \( \sinh(z) > |\sin(z)| \) for any \( z > 0 \).}
To this purpose, we restrict the attention to the case of space dimension and spatial curvature respectively given by (see Eq. (4.1.1))

\[ n = 3 , \quad k = 0 . \] (4.1.41)

Furthermore, we require that a Big Bang singularity occurs at \( t = 0 \); correspondingly, we assume matter to be dominant near the Big Bang, i.e., \( \Omega^{(m)}(t) \to 1 \) for \( t \to 0^+ \). The analysis of subsection 4.1.1 (see, especially, case (iii) therein) indicates that the above conditions can be realized only if

\[ A = 0 \quad \text{and} \quad C = 0 . \] (4.1.42)

To proceed let us remark that, on account of the gauge invariance \( \varphi \mapsto \varphi + \text{const.} \), without any loss of generality we can assume that (see Eq. (4.1.3))

\[ V_1 = V_2 = V > 0 . \] (4.1.43)

Then, the potential (4.1.2) reduces to

\[ V(\varphi) = 2V \cosh \varphi , \] (4.1.44)

and the associated solution (4.1.7) reads (see also Eq. (4.1.28))

\[ u(t) = B \sinh(\sqrt{V} t) , \quad v(t) = D \sin(\sqrt{V} t) \quad \text{with} \quad B > |D| \geq 0 . \] (4.1.45)

Furthermore, note that the zero-energy constraint (4.1.9) becomes

\[ \Omega^*(t) = \frac{V}{9} (B^2 - D^2) > 0 . \] (4.1.46)

Next, let introduce a reference time \( t_* > 0 \) and set

\[
\begin{cases}
a(t_*) = 1 , \\
\varphi(t_*) = \varphi_* , \\
H(t_*) = 1/\theta
\end{cases}
\] (4.1.47)

where \( \theta \) is the usual time constant (see Eq. (4.1.4)) and \( \varphi_* \in \mathbb{R} \) is an arbitrary parameter; in the sequel we shall discuss as examples a couple of sensible choices of \( \varphi_* \). Expressing \( a \equiv e^{A/3} \), \( \varphi \) and \( H \equiv \dot{A}/(3 \theta) \) (see Eqs. (2.2.31) (2.2.30) and recall that here \( B \equiv 0 \)) in terms of the Lagrangian variables \( u,v \) (see Eq. (4.1.5)), the above conditions (4.1.47) imply

\[
\begin{cases}
\frac{u^2(t_*) - v^2(t_*)}{u(t_*) + v(t_*)} = e^{\varphi_*} , \\
u(t_*) \dot{u}(t_*) - v(t_*) \dot{v}(t_*) = \frac{3}{2} .
\end{cases}
\] (4.1.48)

By simple algebraic manipulations (recalling the constraint \( u > |v| \geq 0 \)), these allow us to infer that

\[
\begin{cases}
u(t_*) = 2 \cosh(\varphi_* / 2) , \\
v(t_*) = -2 \sinh(\varphi_* / 2) , \\
\cosh(\varphi_* / 2) \dot{u}(t_*) + \sinh(\varphi_* / 2) \dot{v}(t_*) = 3 .
\end{cases}
\] (4.1.49)

Taking into account the explicit expressions for \( u(t_*) \) and \( v(t_*) \) (see Eq. (4.1.45)), the first two relations in Eq. (4.1.49) can be trivially solved in terms of the two unknown parameters \( B,D \); more precisely, introducing the short-hand notation

\[ s_* := \sqrt{V} t_* , \] (4.1.50)
we get
\[ B = \frac{2 \cosh(\varphi_*/2)}{\sinh(s_*)}, \quad D = -\frac{2 \sinh(\varphi_*/2)}{\sin(s_*)}. \]  
(4.1.51)

Substituting the above expressions for \( B, D \) in the zero-energy constraint (4.1.46) and solving for \( V \), we obtain
\[ V = \frac{9 \Omega_*^{(m)} \sinh^2(s_*) \sin^2(s_*)}{4 \left( \cosh^2(\varphi_*/2) \sin^2(s_*) - \sinh^2(\varphi_*/2) \sin^2(s_*) \right)}. \]  
(4.1.52)

Finally, the above relations (4.1.51), (4.1.52) and the last identity in Eq. (4.1.49) give
\[ \sqrt{\cosh^2(\varphi_*/2) \sin^2(s_*) - \sinh^2(\varphi_*/2) \sin^2(s_*)} = \sqrt{\Omega_*^{(m)}} \left[ \cosh^2(\varphi_*/2) \cosh(s_*) \sin(s_*) - \sinh^2(\varphi_*/2) \sin(s_*) \cos(s_*) \right]. \]  
(4.1.53)

For assigned values of \( \varphi_* \) and \( \Omega_*^{(m)} \), one can look for a solution \( s_* \) of the above equation by numerical methods (assuming that the said solution exists). In the following we analyze in more detail a couple of examples corresponding, respectively, to the choices \( \varphi_* = 0 \) and \( \varphi_* = 1/2 \).

**The case \( \varphi_* = 0 \).** First of all let us remark that this particular choice of \( \varphi_* \) corresponds to the (unique) minimum of the potential (4.1.44). To begin with, note that Eq. (4.1.53) reduces to
\[ \cosh(s_*) = \frac{1}{\sqrt{\Omega_*^{(m)}}}. \]  
(4.1.54)

Assuming \( \Omega_*^{(m)} < 1 \), the above equation can be solved analytically, which yields
\[ s_* = \arccosh\left(\frac{1}{\sqrt{\Omega_*^{(m)}}}\right). \]  
(4.1.55)

Substituting this solution in the expressions (4.1.51), (4.1.52) for \( B, D, V \) we obtain
\[ B = 2 \sqrt{\frac{\Omega_*^{(m)}}{1 - \Omega_*^{(m)}}}, \quad D = 0, \quad V = \frac{9}{4} \left( 1 - \Omega_*^{(m)} \right); \]  
(4.1.56)

besides, from Eq. (4.1.50) we infer
\[ t_* = \frac{2}{3 \sqrt{1 - \Omega_*^{(m)}}} \arccosh\left(\frac{1}{\sqrt{\Omega_*^{(m)}}}\right). \]  
(4.1.57)

On account of the exact identity \( D = 0 \) in Eq. (4.1.56) and of Eqs. (4.1.7), (4.1.42), we have \( v(t) = 0 \) for all \( t \in (0, \infty) \): recalling the expression for \( \varphi \) in terms of the Lagrangian coordinates \( u, v \) (see Eq. (4.1.5)), this implies
\[ \varphi = \text{const.} = 0 \equiv \varphi_* . \]  
(4.1.58)

Therefore, making reference to Eqs. (2.2.25), (2.2.29) and to the related comments, we can say that the field \( \varphi \) plays the role of a cosmological constant in the present setting. Correspondingly, it can be checked by direct computations that (see Eqs. (4.1.11), (4.1.12); cf. also Eq. (2.2.29))
\[ w^{(s)}(t) = \text{const.} = -1, \]  
(4.1.59)
\[ \Omega^{(m)}(t) = \frac{9 \Omega_*^{(m)} \sinh^2(3 t)}{u^2} = \cosh^{-2}\left(\frac{3}{2} \sqrt{1 - \Omega_*^{(m)}} \ t\right). \]  
(4.1.60)
To say more, notice that the previous relations \((4.1.55)-(4.1.57)\) allow to express each of the parameters \(s_*, B, D, V, t_*\) in terms of the sole matter density \(\Omega_*^{(m)}\); choosing for the latter the accepted value (see, e.g., [32, p. 128]; see also [26])

\[
\Omega_*^{(m)} = 0.308 ,
\]

the said relations give

\[
s_* = 1.194... , \quad B = 1.334... , \quad D = 0 , \quad V = 1.557 , \quad t_* = 0.957... .
\]

Notably, setting (see, e.g., [32, p. 128]; see also [26])

\[
H(t_*) \equiv H_* = 67.89 \frac{km}{s \cdot Mpc} \simeq 2.20017 \times 10^{-18} s^{-1}
\]

one finds that the age of the universe in the cosmological model under analysis is

\[
\tau_* := \theta t_* = \frac{t_*}{H_*} \simeq 4.34975 \times 10^{17} s \simeq 13.793 \times 10^9 \text{years} ,
\]

in agreement with the accepted value for this quantity (cf., e.g., [32, p. 129]).

**The case \(\varphi_* = 1/2\).** In this case, once we have fixed \(\Omega_*^{(m)} = 0.308\) as in Eq. \((4.1.61)\), solving Eq. \((4.1.53)\) for \(s_*\) yields

\[
s_* = 1.09829... .
\]

Substituting the above solution in the explicit expressions \((4.1.51)-(4.1.52)\) we obtain

\[
B = 1.54775... , \quad \quad D = -0.56739... , \quad \quad V = 1.33682... ,
\]

and from Eq. \((4.1.50)\) it follows that

\[
t_* = 0.949905... .
\]

Fixing \(H_*\) as in Eq. \((4.1.63)\), the latter value for \(t_*\) corresponds to an age of the universe of about

\[
\tau_* := \theta t_* = \frac{t_*}{H_*} \simeq 4.31743 \times 10^{17} s \simeq 13.6905 \times 10^9 \text{years} .
\]

**Further considerations on the previous cases with \(\varphi_* = 0\) and \(\varphi_* = 1/2\).** Figs. 1 and 2 show the plot of \(a(\tau)\) as a function of the dimensionless ratio \(\tau/\theta\) on two different intervals (namely, for \(\tau/\theta \in (0,1)\) and \(\tau/\theta \in (0,10)\)), in the two cases with \(\varphi_* = 0\) and \(\varphi_* = 1/2\) analyzed previously. By close inspection of these figures it appears that after the Big Bang at \(\tau = 0\) \((a(\tau) \to 0^+\) for \(\tau/\theta \to 0^+)\) the universe experiences an initial phase of decelerated expansion (see Fig. 1), followed by an endless accelerated expansion (see Fig. 2).

Figure 3 shows the plot of the (rescaled) scalar field \(\varphi(\tau)\) for \(\tau/\theta \in (0,1)\) in the two cases with \(\varphi_* = 0\) and \(\varphi_* = 1/2\). Let us recall that for \(\varphi_* = 0\) we get \(\varphi = \text{const.} = 0\) by purely analytic means (see Eq. \((4.1.58)\)). On the other hand when \(\varphi_* = 1/2\) we have, in particular, \(\varphi(\tau) \to \log \left( \frac{B-D}{\tau-D} \right) = 0.76896... \) for \(\tau/\theta \to 0^+\) and \(\tau \to 0\) for \(\tau/\theta \to \infty\). Figure 4 shows the plot of the equation of state coefficient \(w^{(s)}(\tau)\) for the field (see, especially, Eq. \((4.1.11)\)), for \(\tau/\theta \in (0,10)\) (with \(\varphi_* = 0\) and \(\varphi_* = 1/2\)). We already mentioned that \(w^{(s)} = \text{const.} = -1\) when \(\varphi_* = 0\) (see Eq. \((4.1.59)\)). In the case with \(\varphi_* = 1/2\), we have the following: \(w^{(s)} \to -1\) for \(\tau/\theta \to 0^+\); \(w^{(s)}(\tau_*) = -0.93632...\) at the present cosmic time \(\tau_*\) given in Eq. \((4.1.68)\); \(w^{(s)} \to -1\) for \(\tau/\theta \to \infty\). In particular, the latter relation indicates that in the future the scalar field ultimately behaves as a cosmological constant (see the discussion after Eq. \((2.2.12)\)).

Figures 5 and 6 show the plot of the matter density parameter \(\Omega^{(m)}(\tau)\) (see Eq. \((4.1.12)\)), respectively for \(\tau/\theta \in (0,1)\) and \(\tau/\theta \in (0,10)\) (with \(\varphi_* = 0\) and \(\varphi_* = 1/2\)). From these figures we infer that
the universe is initially filled almost exclusively with matter ($\Omega^{(m)}(\tau) \to 1^-$, $\Omega^{(\phi)}(\tau) \to 0^+$ for $\tau/\theta \to 0^+$). Afterwards, matter continues to dominate over the scalar field (which is supposed to model dark energy) until the cosmic time $\bar{\tau}$ implicitly defined by the equality

$$\Omega^{(m)}(\bar{\tau}) = \Omega^{(\phi)}(\bar{\tau}) = 1/2 ; \quad (4.1.69)$$

more precisely, from Eqs. (4.1.4), (4.1.47), (4.1.60) we deduce

$$\bar{\tau} = \frac{2 \arccosh \sqrt{2}}{3H_* \sqrt{1 - \Omega^{(m)}_*}} \approx 10.18 \times 10^9 \text{ years} \quad \text{for } \varphi_* = 0 , \quad (4.1.70)$$
while by numerical methods from Eqs. (4.1.4) (4.1.7) (4.1.12) we get

$$\bar{\tau} \simeq 9.90505 \times 10^9 \text{ years} \quad \text{for } \varphi_* = 1/2 \ .$$

(4.1.71)

It is worth noting that the above results are in good agreement with those obtained in the benchmark model (see, e.g., [30]). To proceed, let us mention that \( \Omega^{(m)}(\tau_*) = 0.308 \) by construction (see Eqs. (2.2.34) (4.1.61)); therefore, since \( \Omega^{(\phi)}(\tau_*) = 1 - \Omega^{(m)}(\tau_*) = 0.692 \) (see Eq. (4.1.5)), we see that the scalar field (namely, dark energy) is the dominant contribution at the present time. In the future, the field continues to be dominant and eventually fills the whole universe (\( \Omega^{(m)}(\tau) \rightarrow 0^+, \Omega^{(\phi)}(\tau) \rightarrow 1^- \) for \( \tau/\theta \rightarrow \infty \)).

As a final remark let us mention that, in both cases with \( \varphi_* = 0 \) and \( \varphi_* = 1/2 \), the potential \( V(\varphi) \) is qualitatively similar to the phenomenological self-interaction potential reconstructed by Saini, Raychaudhury, Sahni and Starobinsky in [31] (see, especially, FIG. 3 therein), starting from the empirical redshift-luminosity distance curve [23] (14).

### 4.2 Spatially flat solutions for class 2 potentials with \( \gamma = w \neq \pm 1 \)

In this subsection we analyze \((n+1)\)-dimensional, spatially flat \((k = 0; \text{see Eq. (4.0.1)})\) cosmological models whose matter content consists of a perfect fluid with an arbitrary equation of state parameter

$$w \in \mathbb{R} \setminus \{ \pm 1 \} ,$$

and a scalar field with self-interaction potential

$$V(\varphi) = V_1 e^{2w\varphi} + V_2 e^{(1+w)\varphi} \quad (V_1, V_2 \in \mathbb{R}) .$$

(4.2.2)

For notational convenience, in the sequel we put

$$\varepsilon := - \text{sgn}\left((1 - w^2) V_1 \right) , \quad \omega := \sqrt{\frac{|(1 - w^2) V_1|^2}{2}}$$

(4.2.3)

(here and below, \( \text{sgn}(z) \) is such that \( \text{sgn}(z) = -1 \) for \( z < 0 \), \( \text{sgn}(z) = 0 \) for \( z = 0 \) and \( \text{sgn}(z) = +1 \) for \( z > 0 \)).

The potential (4.2.2) is of the form (3.2.1) with \( \gamma = w \neq \pm 1 \); thus, the model under analysis fits into the integrable subcase (i) of class 2, discussed previously in subsection 3.2. Correspondingly, we fix the gauge function \( B(A, \varphi) \) as in Eq. (3.2.2) and introduce the pair of Lagrangian coordinates \( x, y \) related to \( A, \varphi \) via Eq. (3.2.3); let us remark that these coordinates must fulfill the condition

$$x > 0 \quad \text{and} \quad y > 0 .$$

(4.2.4)

To proceed, recall that with the above positions the Lagrangian function has the special triangular structure (3.0.15), and the related Lagrange equations (3.2.8) become

$$\ddot{x} + \varepsilon \omega^2 x = 0 ,$$

$$\ddot{y} + \varepsilon \omega^2 y = (1 - w) V_2 x^{\frac{1+w}{1+w}} - \frac{n^2}{2} (1 - w) w \Omega^{(m)}(\tau) x^{\frac{1+w}{1+w}} .$$

(4.2.5)  (4.2.6)

The corresponding solutions can be determined explicitly, treating the cases \( \varepsilon = -1, 0, +1 \) separately.

Before providing a detailed analysis of these cases, let us point out that Eqs. (2.2.11) (2.2.12) (2.2.32)14To make a quantitative comparison with the results of [31], it should be noted that the dimensionless scalar field \( \varphi \) and the potential \( V(\varphi) \) of the present work (with \( n = 3 \) and \( c = 1 \)) are related as follows to the analogues \( \phi_{[31]} \), \( V_{[31]}(\phi_{[31]}) \) employed in [31] (see [13, App. G] for more details):

\[
\phi_{[31]} = \frac{1}{\sqrt{12\pi}} \varphi , \quad V_{[31]}(\phi_{[31]}) = \frac{2}{9} V(\varphi) .
\]
yield the following expression for the coefficient $w^{(s)}$ and for the matter density parameter $\Omega^{(m)}$:

$$w^{(s)} = \frac{e^{2w} \varphi^2 - 2V(\varphi)}{e^{2w} \varphi^2 + 2V(\varphi)} = \frac{(1-w) \dot{x} - (1+w) x \dot{y})^2 - 2(1-w^2)^2 (V_1 x^2 y^2 + V_2 x \frac{i+w}{1+w} y)}{(1-w) \dot{x} + (1+w) x \dot{y})^2 + 2(1-w^2)^2 (V_1 x^2 y^2 + V_2 x \frac{i+w}{1+w} y)}, \quad (4.2.7)$$

$$\Omega^{(m)} = \frac{n^2 \Omega_{s}^{(m)} e^{-2w} \varphi^{-(1+w)} A}{A^2} = \frac{n^2 (1-w^2)^2 \Omega_{s}^{(m)} x \frac{i+w}{1+w} y}{((1-w) \dot{x} + (1+w) x \dot{y})^2} \quad (4.2.8)$$

In the forthcoming paragraphs we present analytic expressions for the solutions $x(t), y(t)$ of the Lagrange equations (4.2.5), (4.2.6). Afterwards, in subsections 4.2.1, 4.2.2 we investigate the presence of a Big Bang and the long time behavior in an exemplary case, specified below (see Eq. (4.2.43)).

\[\text{i) } \varepsilon = -1. \] In this case we have

$$(1-w^2) V_1 > 0 \quad (4.2.9)$$

and Eqs. (4.2.5), (4.2.6) read

\[\dot{x} - w^2 x = 0 \quad (4.2.10)\]

$$\dot{y} - w^2 y = (1-w) V_2 x \frac{i+w}{1+w} - \frac{n^2}{2} (1-w) w \Omega_{s}^{(m)} x \frac{i+w}{1+w}. \quad (4.2.11)$$

After possibly a time translation $t \rightarrow t + \text{const.}$ and a time reflection $t \rightarrow -t$, any admissible solution of Eqs. (4.2.10) (4.2.11) can be written in one of the following forms, for the values of $w$ indicated contextually (see Appendix B for the derivation of the following expressions):

\[x(t) = A \sinh(\omega t), \quad y(t) = C \cosh(\omega t) + D \sinh(\omega t)\]

$$+ \frac{V_2}{V_1} \frac{i+w}{1+w} \sinh \frac{2}{1+w} (\omega t) \left[ 1 - \frac{2}{3+w} \cosh(\omega t) \right. \left. _2F_1 \left( \frac{1}{2}, \frac{3+w}{2}, \frac{5+w}{2}; \sinh^2(\omega t) \right) \right]$$

$$+ \frac{n^2 \Omega_{s}^{(m)}}{2V_1} A^{-\frac{i+w}{1+w}} \sinh \frac{i+w}{1+w} (\omega t) \left[ 1 + \frac{2}{w} \cosh(\omega t) \right. \left. _2F_1 \left( \frac{1}{2}, \frac{1-w}{2}, \frac{3+w}{2}; \sinh^2(\omega t) \right) \right]$$

for $A > 0$ and $w \in \mathbb{R} \setminus \{\pm 1\}, \quad w \neq -\frac{3+2h}{1+2h}$ for all $h \in \{0,1,2,...\}$ ; \quad (4.2.12)

\[x(t) = A \cosh(\omega t), \quad y(t) = C \cosh(\omega t) + D \sinh(\omega t)\]

$$+ \frac{V_2}{V_1} \frac{i+w}{1+w} \left[ \cosh(\omega t) \left( 1 - \cosh \frac{2}{1+w} (\omega t) \right) + \frac{2}{1+w} \sinh(\omega t) \right. \left. _2F_1 \left( \frac{1}{2}, \frac{1-w}{2}, \frac{3}{2}; -\sinh^2(\omega t) \right) \right]$$

$$+ \frac{n^2 \Omega_{s}^{(m)}}{2V_1} A^{-\frac{i+3w}{1+w}} \cosh(\omega t) \left[ 1 - \frac{2}{1+w} \sinh(\omega t) \right. \left. _2F_1 \left( \frac{1}{2}, \frac{1+3w}{2}, \frac{3}{2}; \sinh^2(\omega t) \right) \right]$$

for $A > 0$ and $w \in \mathbb{R} \setminus \{\pm 1\}$ ; \quad (4.2.13)

\[x(t) = A e^{\omega t}, \quad y(t) = C e^{\omega t} + D \sinh(\omega t)\]

$$+ \frac{V_2}{V_1} \frac{1+w}{2} \left[ \cosh(\omega t) + \frac{1-w}{1+w} \sinh(\omega t) - e^{\frac{i+3w}{1+w} \omega t} \right]$$

$$+ \frac{n^2 \Omega_{s}^{(m)}}{4V_1} A^{-\frac{i+3w}{1+w}} \left[ 1 + \frac{2}{1+w} \cosh(\omega t) - \frac{1+3w}{1+w} \sinh(\omega t) - e^{\frac{i+3w}{1+w} \omega t} \right]$$

for $A > 0$ and $w \in \mathbb{R} \setminus \{\pm 1\}, \quad w \neq 0, -1/2$ . \quad (4.2.14)
In the above Eqs. (4.2.12) (4.2.13) (4.2.14), \( \omega \) is defined as in Eq. (4.2.3), \( A, C, D \) are real integration constants and \( _2F_1 \) is the ordinary, Gaussian hypergeometric function.

The common constraint \( A > 0 \) grants that \( x(t) > 0 \) as required in Eq. (4.2.4), and even accounts for the fractional powers of \( A \) appearing in the expressions for \( y(t) \). Clearly, we understand the above solutions to be defined on a maximal interval (see, again, Eq. (4.2.4))

\[
I \subset \mathbb{R} \quad \text{such that} \quad y(t) > 0 \quad \text{for all} \quad t \in I .
\] (4.2.15)

Before proceeding, let us highlight that the derivation of Eq. (4.2.12) under the general condition \( w \neq \frac{1}{x} \frac{1}{2} (h = 0, 1, 2, \ldots) \) comprises some subtleties, related to considerations on analytic continuations (see Appendix B). On the other hand, the condition \( w \neq 0, -1/2 \) in Eq. (4.2.14) can be removed intending this solution via a suitable limiting procedure; more precisely, in view of the basic identities (see Eqs. (B.12) (B.13) in Appendix B)

\[
\lim_{w \to -1/2} \left[ \frac{1 + w}{1 + 2w} \left( \cosh(\omega t) - \frac{1 + w}{1 + 2w} \sinh(\omega t) - e^{-\frac{1 + 3w}{1 + w} \omega t} \right) \right] = 2(\omega t e^{\omega t} - \sinh(\omega t)) \quad \text{(4.2.16)}
\]

\[
\lim_{w \to 0} \left[ \frac{1 + w}{2w} \left( \cosh(\omega t) + \frac{1 - w}{1 + w} \sinh(\omega t) - e^{\frac{1 - w}{1 + w} \omega t} \right) \right] = \omega t e^{\omega t} - \sinh(\omega t) \quad \text{(4.2.17)}
\]

and of elementary trigonometric identities, the expression for \( y(t) \) in Eq. (4.2.14) reduces to

\[
y(t) = C \cosh(\omega t) + D \sinh(\omega t) + \frac{V_2 A}{V_1} \left( \omega t e^{\omega t} - \sinh(\omega t) \right) \quad \text{for} \quad w = 0 ; \quad \text{(4.2.18)}
\]

\[
y(t) = C \cosh(\omega t) + D \sinh(\omega t) + \frac{2 V_2 A^3}{V_1} e^{\omega t} \sin^2(\omega t) + \frac{n^2 \Omega_s^{(m)}}{2 V_1} \left( \omega t e^{\omega t} - \sinh(\omega t) \right) \quad \text{for} \quad w = -1/2 . \quad \text{(4.2.19)}
\]

To go on, let us recall the expression (3.2.7) for the energy \( E \); from here and from Eqs. (4.2.12) (4.2.13) (4.2.14) we obtain, respectively, \(^{15}\)

\[
E = - V_1 A D , \quad \text{(4.2.20)}
\]

\[
E = A \left( \frac{n^2 \Omega_s^{(m)}}{2} - A^{-\frac{1 + 3w}{1 + w}} + \frac{V_2 A^{1 + w}}{V_1} + V_1 C \right) , \quad \text{(4.2.21)}
\]

\[
E = A \left( \frac{n^2 \Omega_s^{(m)}}{2} A^{-\frac{1 + 3w}{1 + w}} + \frac{V_2 A^{1 + w}}{V_1} + V_1 (C - D) \right) . \quad \text{(4.2.22)}
\]

Taking the above relations into account and recalling that in the present case \( V_1 \neq 0 \) (see Eq. (4.2.9)), we infer that to fulfill the zero-energy constraint \( E = 0 \) we must put in Eqs. (4.2.12) (4.2.13) (4.2.14), respectively,

\[
D = 0 , \quad \text{(4.2.23)}
\]

\[
C = \frac{V_2}{V_1} A^{1 + w} - \frac{n^2 \Omega_s^{(m)}}{2 V_1} A^{-\frac{1 + 3w}{1 + w}} , \quad \text{(4.2.24)}
\]

\[
C = D - \frac{V_2}{V_1} A^{1 + w} - \frac{n^2 \Omega_s^{(m)}}{2 V_1} A^{-\frac{1 + 3w}{1 + w}} . \quad \text{(4.2.25)}
\]

\(^{15}\) For this computation, it is convenient to recall that \( E \) is a constant of motion. Especially, it suffices to compute \( E \equiv E(x(t), \dot{x}(t), y(t), \dot{y}(t)) \) for any given \( t \) or in a suitable limit, e.g. for \( t \to 0^+ \). The facts mentioned in the present footnote apply to all the subsequent energy computations in this work, but it will never be repeated.
ii) \( \epsilon = 0 \). Since we are assuming \( w \neq \pm 1 \), this case uniquely corresponds to the choice (cf. Eq. (4.2.23))
\[
V_1 = 0 .
\] (4.2.26)
Keeping this in mind, Eqs. (4.2.5)–(4.2.6) reduce to
\[
\ddot{x} = 0 ,
\] (4.2.27)
\[
\ddot{y} = (1 - w) V_2 x \frac{1}{1+w} - \frac{n^2}{2} (1 - w) w \Omega_s^{(m)} x^{-\frac{1+3w}{1+w}} .
\] (4.2.28)
After a time translation \( t \to t + \text{const.} \) and possibly a time reflection \( t \to -t \), any solution of
Eqs. (4.2.27)–(4.2.28) can be written in one of the following forms (see Appendix B for more details):
\[
x(t) = A t ,
y(t) = C + D t + \frac{V_2 (1 + w)^2 (1 - w)}{2 (3 + w)} A^{-\frac{1}{1+w}} t^{\frac{3+w}{1+w}} + \frac{n^2 (1 + w)^2}{4} \Omega_s^{(m)} A^{-\frac{1+3w}{1+w}} t^{\frac{1-w}{1+w}}
\]
for \( A > 0 \) and \( w \in \mathbb{R} \setminus \{\pm 1\} \); (4.2.29)
\[
x(t) = A ,
y(t) = C + D t + \frac{V_2 (1 - w) A^{-\frac{1}{1+w}} - \frac{n^2 (1 - w) w}{2} \Omega_s^{(m)} A^{-\frac{1+3w}{1+w}}}{2}
\]
for \( A > 0 \) and \( w \in \mathbb{R} \setminus \{\pm 1\} , w \neq -3 \). (4.2.30)
In both Eqs. (4.2.29)–(4.2.30), \( \omega \) is defined as in Eq. (4.2.23), while \( A, C, D \) are real integration constants. Again, the restriction \( A > 0 \) grants that \( x(t) > 0 \) (in accordance with Eq. (4.2.4)), and further accounts for the fractional powers of \( A \) in the expressions for \( y(t) \). The above solutions are tacitly understood to be defined on a maximal interval (see, again, Eq. (4.2.4))
\[
I \subset \mathbb{R} \text{ such that } x(t), y(t) > 0 \text{ for all } t \in I .
\] (4.2.31)
Especially, on account of the fact that \( A > 0 \), for the solution (4.2.29) on has \( I \subset (0, \infty) \). Furthermore, let us remark that the exceptional case \( w = -3 \) for the solution (4.2.30) could be treated separately (see, in particular, Eq. (B.5) in Appendix B).
Recalling once more Eq. (3.2.7) for the energy \( E \), from Eqs. (4.2.27)–(4.2.30) we obtain, respectively,
\[
E = -\frac{2}{1-w^2} AD ;
\] (4.2.32)
\[
E = A^{-\frac{1}{1+w}} \left( V_2 A^2 + \frac{n^2 \Omega_s^{(m)}}{2} \right) .
\] (4.2.33)
Thus, to fulfill the zero-energy constraint \( E = 0 \) we must require, respectively,
\[
D = 0 ;
\] (4.2.34)
\[
V_2 < 0 \text{ and } A = \sqrt{\frac{n^2 \Omega_s^{(m)}}{2 |V_2|}} \text{ or } V_2 = 0 \text{ and } \Omega_s^{(m)} = 0 .
\] (4.2.35)
It should be noted that the second configuration in Eq. (4.2.35) \( (V_2 = \Omega_s^{(m)} = 0) \) entails \( \mathcal{V}(\varphi) = 0 \); thus, it corresponds to a cosmological model where the universe contains only a non-interacting scalar field \( \varphi \).
iii) $\varepsilon = +1$. According to the definitions in Eq. (4.2.3), in this case we have

\[ V_1 (1 - w^2) < 0, \quad (4.2.36) \]

and Eqs. (4.2.5) (4.2.6) are equivalent to

\begin{align*}
\ddot{x} + \omega^2 x &= 0, \quad (4.2.37) \\
\dot{y} + \omega^2 y &= (1 - w) V_2 x \frac{2}{1 + w} - \frac{n^2 (1 - w) w}{2} \Omega_\ast^{(m)} x^{-\frac{1 + 3 w}{1 + w}}. \quad (4.2.38)
\end{align*}

After a time translation $t \to t + \text{const.}$, any solution of Eqs. (4.2.37) (4.2.38) can be written as follows, for the indicated values of $w$ (see Appendix B):

\[ x(t) = A \sin(\omega t), \]
\[ y(t) = C \cos(\omega t) + D \sin(\omega t) + V_2 \frac{1}{V_1} \frac{2}{1 + w} \sin^\frac{1 + w}{1 + w} (\omega t) \]
\[ + \frac{V_2}{V_1} A^{\frac{1}{1 + w}} \sin^\frac{1 + w}{1 + w} (\omega t) \left[ 1 - \frac{2 \cos(\omega t)}{3 + w} \right] 2F_1 \left( \frac{1}{2}, \frac{1}{2} + \frac{w}{3 + w}, \frac{3 + w}{2 + 2w}, \sin^2(\omega t) \right) \]
\[ - \frac{d^2 \Omega_\ast^{(m)}}{2 V_1} A^{-\frac{1}{1 + w}} \sin^{-\frac{1}{1 + w}} (\omega t) \left[ 1 + \frac{2 w \cos(\omega t)}{1 - w} \right] 2F_1 \left( \frac{1}{2}, \frac{1}{2} - \frac{w}{2 + 2w}, \frac{3 + w}{2 + 2w}, \sin^2(\omega t) \right) \]

for $A > 0$ and $w \in \mathbb{R} \setminus \{\pm 1\}$, $w \neq -\frac{3 + 2h}{1 + 2h}$ for all $h \in \{0, 1, 2, \ldots\}$. \quad (4.2.39)

Also in this case, $\omega$ is defined as in Eq. (4.2.3), $A, C, D$ are real integration constants and $2F_1$ denotes the Gaussian hypergeometric function. The well-posedness of the fractional powers of $A$ in the expressions for $y(t)$ is ensured by the condition $A > 0$.

It should be noted that the requirement $x(t) > 0$ in Eq. (4.2.4) can be fulfilled only for $t \in (0, \pi/\omega)$. Accordingly, we understand that the solution (4.2.39) is defined on a maximal interval

\[ I \subset (0, \pi/\omega) \quad \text{such that} \quad y(t) > 0 \quad \text{for all} \quad t \in I. \quad (4.2.40) \]

To proceed, let us mention that Eqs. (3.2.7) (4.2.39) imply

\[ E = V_1 A D; \quad (4.2.41) \]

therefore, to fulfill the energy constraint $E = 0$ we must require

\[ D = 0. \quad (4.2.42) \]

4.2.1 Big Bang analysis

In the present subsection we proceed to investigate the presence of an initial Big Bang singularity and the asymptotic behavior close to it for one of the previously described solutions, taken as an example. More precisely, let us assume that

\[ V_1 > 0 \quad \text{and} \quad -1 < w < 1; \quad (4.2.43) \]

correspondingly, we restrict the attention to the solution described in Eqs. (4.2.12) (4.2.23) \footnote{Of course, the conditions in Eq. (12.33) correspond to a special case of Eq. (12.41) and certainly fulfill the restrictions on $w$ stated in Eq. (12.12).} and proceed to examine the circumstances in which a Big Bang occurs at $t = 0$ (cf. Eq. (14.0.2)). To this
proves that let us further remark that Eqs. (4.2.47) and (4.2.50) imply arguments related to Eq. (2.2.19), we can properly speak of a Big Bang at purpose, let us first remark that the expressions for \( x(t) \) and \( y(t) \) in Eq. (4.2.12) and the constraint in Eq. (4.2.20) ensue the following asymptotics, for \( t \to 0^+ \) [17]:

\[
x(t) = A\omega t + \frac{1}{6} A (\omega t)^3 + O(t^5),
\]

\[
y(t) = C + \frac{1}{2} C (\omega t)^2 + \frac{n^2 \Omega_s^{(m)}}{2V_1} \frac{1+w}{1-w} A^{\frac{1-3w}{1+w}}(\omega t)^{\frac{1-w}{1+w}} + O\left(t^{\min\left\{4, \frac{1+w}{1+w}\right\}}\right).
\]

(4.2.44)

(4.2.45)

While the hypothesis \( A > 0 \) in Eq. (4.2.12) grants that \( x(t) > 0 \) for all \( t \in (0, \infty) \), the above expansion for \( y(t) \) makes evident that, under the assumptions in Eq. (4.2.33), the analogous condition \( y(t) > 0 \) (cf. Eq. (4.2.4)) can be fulfilled in a right neighborhood of \( t = 0 \) if and only if

\[
C > 0 \quad \text{or} \quad C = 0 \quad \text{and} \quad \Omega_s^{(m)} > 0.
\]

(4.2.46)

Hereafter we proceed to analyze these two cases separately.

i) \( C > 0 \). From Eqs. (4.2.3) (4.2.4) (4.2.5) we infer, for \( t \to 0^+ \),

\[
A(t) = \frac{1}{1+w} \log t + \log \left( (A\omega)^{\frac{1}{1+w}} C^{\frac{1}{1-w}} \right) + O\left(t^{\min\left\{2, \frac{1-w}{1+w}\right\}}\right),
\]

\[
\varphi(t) = \frac{1}{1+w} \log t + \log \left( (A\omega)^{\frac{1}{1+w}} C^{\frac{1}{1-w}} \right) + O\left(t^{\min\left\{2, \frac{1-w}{1+w}\right\}}\right).
\]

(4.2.47)

(4.2.48)

In view of Eqs. (2.2.4) (3.2.2) and of the above expansions, we obtain

\[
a(t) = (A\omega)^{-\frac{w}{n(1+w)}} C^{-\frac{w}{n(1-w)}} t^{\frac{1}{n(1+w)}} + O\left(t^{\min\left\{\frac{2n(1+w)+1}{n(1+w)}, \frac{n(1-w)+1}{n(1+w)}\right\}}\right),
\]

\[
e^B(t) = (A\omega)^{-\frac{w}{1+w}} C^{\frac{w}{1-w}} t^{\frac{w}{1+w}} + O\left(t^{\min\left\{\frac{2w}{1+w}, \frac{1-2w}{1+w}\right\}}\right).
\]

(4.2.49)

(4.2.50)

Recalling again that \(-1 < w < 1\), Eq. (4.2.49) shows that \( a(t) \to 0 \) for \( t \to 0^+ \), while Eq. (4.2.50) proves that \( e^B(t) \) is integrable in a right neighborhood of \( t = 0 \). Therefore, making reference to the arguments related to Eq. (2.2.19), we can properly speak of a Big Bang at \( t = 0 \). In this connection, let us further remark that Eqs. (4.2.47) and (4.2.50) imply

\[
e^{B(t)-A(t)/n} = (A\omega)^{-\frac{1+nw}{n(1+w)}} C^{-\frac{1+nw}{n(1-w)}} t^{-\frac{1+nw}{n(1+w)}} + O\left(t^{\min\left\{\frac{n(2+w)+1}{n(1+w)}, \frac{n(1-w)+1}{n(1+w)}\right\}}\right) \quad \text{for} \quad t \to 0^+,
\]

(4.2.51)

which, on account of Eq. (2.2.20) and of the fact that \( \frac{1+nw}{n(1+w)} < 1 \) in our case with \( n \geq 2 \) and \(-1 < w < 1\), ensures the finiteness of the particle horizon at each time.

Next, let us note that Eqs. (4.2.7) (4.2.8) and the asymptotic expansions in Eqs. (4.2.44) (4.2.45) (see also Eq. (4.2.3)) give

\[
w^{(a)}(t) = 1 - 8 \left(\frac{1+w}{1-w}\right) (\omega t)^2 + O\left(t^{\min\left\{4, \frac{1+w}{1+w}\right\}}\right),
\]

\[
\Omega^{(m)}(t) = \frac{n^2 \Omega_s^{(m)} (1+w)^2}{C (A\omega)^{\frac{1-3w}{1+w}} t^{\frac{1-w}{1+w}}} + O\left(t^{\min\left\{\frac{4+2w}{1+w}, \frac{2(1-w)}{1+w}\right\}}\right).
\]

(4.2.52)

(4.2.53)

Eq. (4.2.52) suggests that the scalar field behaves as stiff matter close to the Big Bang. On the other hand, Eq. (4.2.53) indicates that \( \Omega^{(m)}(t) \to 0 \) for \( t \to 0^+ \); in view of Eq. (4.0.5) this implies \( \Omega^{(s)}(t) \to 1 \) for \( t \to 0^+ \), thus showing that the scalar field is dominant at the Big Bang.

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17To derive the expansions (4.2.44) (4.2.45) it is useful to recall that the Gauss series for \( _2F_1 \) (see, e.g., [22, Eq. 15.2.1]) yields \( _2F_1(a, b; c; z) = 1 + O(z) \) for \( z \to 0 \) and any \( a, b, c \in \mathbb{R} \). Besides, the condition \(-1 < w < 1\) implies \( \frac{1+w}{1-w} > 2 \).
Regarding the cosmic time \( \tau \equiv \tau(t) \), let us notice that Eqs. (4.2.49) and (4.2.50) imply
\[
\tau(t)/\theta = \frac{(1+w)}{(A\omega)^{1+w}} t^{1+w} + O\left(t^{\min\left\{\frac{1+2w}{1+w}, \frac{2w}{1+w}\right\}}\right) \quad \text{for } t \to 0^+, \quad (4.2.54)
\]
which can be locally inverted to give
\[
t(\tau) = \frac{(A\omega)^w}{(1+w)^{1+w} C^{\frac{w}{1+w}}} (\tau/\theta)^{1+w} + O\left((\tau/\theta)^{\min\{2,3(1+w)\}}\right) \quad \text{for } \tau/\theta \to 0^+. \quad (4.2.55)
\]

The expansions (4.2.49)-(4.2.53) can be reformulated in terms of the cosmic time \( \tau \), using Eq. (4.2.55) for \( t = t(\tau) \); for example, Eq. (4.2.49) is equivalent to
\[
a(\tau) = \left(C A\omega\right)^{1/n} (\tau/\theta)^{1/n} + O\left((\tau/\theta)^{\min\left\{\frac{2n(1+w)+1}{n}, \frac{n(1-w)+1}{n}\right\}}\right) \quad \text{for } \tau/\theta \to 0^+. \quad (4.2.56)
\]

**ii) \( C = 0, \Omega^{(m)}_* > 0 \).** First notice that Eqs. (3.2.3) (4.2.49) (4.2.55) imply, for \( t \to 0^+ \),
\[
A(t) = \frac{2}{1+w} \log t + \frac{1}{1-w} \log \left(\frac{n^2 \Omega^{(m)}_* (1+w)^2}{4 \left(A\omega\right)^{1+w}}\right) + O(t^2), \quad (4.2.57)
\]
\[
\varphi(t) = -\frac{1}{1-w} \log \left(\frac{n^2 \Omega^{(m)}_* (1+w)^2}{4 \left(A\omega\right)^2}\right) + O(t^2). \quad (4.2.58)
\]

On account of Eqs. (2.2.4) (3.2.2), the above expansions allow us to infer that
\[
a(t) = \left(\frac{n^2 \Omega^{(m)}_* (1+w)^2}{4}\right)^{\frac{1}{n(1-w)}} (A\omega)^{-\frac{4w}{n(1-w)^2}} t^{\frac{2}{n(1+w)}} + O\left(t^{\frac{2+2n(1+w)}{n(1+w)}}\right), \quad (4.2.59)
\]
\[
e^{\mathcal{B}(t)} = \left(\frac{n^2 \Omega^{(m)}_* (1+w)^2}{4 \left(A\omega\right)^2}\right)^{\frac{w}{1-w}} + O(t^2). \quad (4.2.60)
\]

Similarly to the case with \( C > 0 \) discussed in the previous paragraph, the above relations show that \( a(t) \to 0 \) for \( t \to 0^+ \) and imply the integrability of \( e^{\mathcal{B}(t)} \) in a right neighborhood of \( t = 0 \) (for \(-1 < w < 1\)); so, we have a Big Bang at \( t = 0 \). To say more, Eqs. (4.2.57) (4.2.60) give
\[
e^{\mathcal{B}(t)-A(t)/n} = \left(\frac{n^2 \Omega^{(m)}_* (1+w)^2}{4}\right)^{\frac{n-1}{n(1-w)}} (A\omega)^{\frac{2w(2-n(1+w))}{n(1-w^2)}} t^{\frac{2}{n(1+w)}} + O\left(t^{\frac{2n(1+w)-2}{n(1+w)}}\right) \quad \text{for } t \to 0^+, \quad (4.2.61)
\]
which, together with Eq. (2.2.20), indicates that the particle horizon is finite whenever \( \frac{2}{n(1+w)} < 1 \).

In our case with \( n \geq 2 \) and \(-1 < w < 1\), this is equivalent to \( w > (2/n) - 1 \) (cf. Eq. (4.10.7) and the related comments); especially, let us point out that the latter condition is fulfilled in the case of radiation where \( w = 1/n \).

Concerning the coefficient in the field equation of state and the matter density parameter \( \Omega^{(m)} \), from Eqs. (4.2.7) (4.2.8) (4.2.41) (4.2.45) we obtain
\[
w^{(w)}(t) = -1 + \frac{2}{\left(\frac{4+w}{3+w}\right)^2} \left(\frac{2w}{1-w} + \frac{2V_2 A^2}{n^2 \Omega^{(m)}_*}\right)^2 (\omega t)^2 + O(t^4), \quad (4.2.62)
\]
\[
\Omega^{(m)}(t) = 1 - \left(\frac{1+w}{1-w} + \frac{2V_2 A^2}{n^2 \Omega^{(m)}_*}\right) (\omega t)^2 + O(t^4). \quad (4.2.63)
\]
52
The above relations indicate, respectively, that close to the Big Bang the scalar field \( \varphi \) behaves as a cosmological constant whereas the dominant contribution comes from the matter fluid. To conclude, from Eqs. (2.2.19) and (4.2.60) we readily infer

\[
\tau(t)/\theta = \left( \frac{n^2 \Omega_\star^{(m)} (1+w)^2}{4 (A \omega)^2} \right)^{\frac{w}{1-w}} t + O(t^2) \quad \text{for } t \to 0^+, \tag{4.2.64}
\]

which entails, by inversion,

\[
t(\tau) = \left( \frac{n^2 \Omega_\star^{(m)} (1+w)^2}{4 (A \omega)^2} \right)^{\frac{1-w}{w}} (\tau/\theta) + O((\tau/\theta)^2) \quad \text{for } \tau/\theta \to 0^+, \tag{4.2.65}
\]

Of course, the previous expansions (4.2.59)-(4.2.65) can be rephrased in terms of the cosmic time \( \tau \), using Eq. (4.2.65); for example, we have

\[
a(\tau) = \left( \frac{n^2 \Omega_\star^{(m)} (1+w)^2}{4} \right)^{\frac{1}{n(1+w)}} (\tau/\theta) \frac{2}{n(1+w) + O\left( \left( \tau/\theta \right)^{2+n(1+w)} \right)} \quad \text{for } \tau/\theta \to 0^+. \tag{4.2.66}
\]

### 4.2.2 Behavior of the model in the far future

The qualitative behavior on large time scales of the model under analysis depends sensibly on the choice of the parameters which characterize the solutions \( x(t), y(t) \) of the Lagrange equations (4.2.5) (4.2.6). In the sequel we account (at least partially) for this rather predictable fact, referring once more to the exemplary case whose Big Bang phenomenology was examined in the previous paragraph.

Correspondingly, we assume again that \( V_1 > 0 \) and \(-1 < w < 1\) (see Eq. (4.2.43)), and consider the solution given in Eqs. (4.2.12)-(4.2.23); in addition we suppose that either \( C > 0 \), or \( C = 0 \) and \( \Omega_\star^{(m)} > 0 \) (see Eq. (4.2.50)), which grants the occurrence of a Big Bang \( t = 0 \).

Next, let us recall that the maximal admissible domain of definition \( I \subset (0, +\infty) \) for the said solutions \( x(t), y(t) \) must be such that \( x(t) > 0 \) and \( y(t) > 0 \) for all \( t \in I \) (see Eq. (4.2.41)). Of course, the expression \( x(t) = A \sinh(\omega t) \) with \( A > 0 \) (see Eq. (4.2.12)) ensures \( x(t) > 0 \) on the whole positive half-line \((0, +\infty)\); therefore, any restriction on the domain \( I \) comes from the requirement \( y(t) > 0 \) for \( t \in I \).

In general, the maximal interval where the expression for \( y(t) \) in Eq. (4.2.12) gives \( y(t) > 0 \) cannot be determined by purely analytical means and one must perform a numerical evaluation. On the other hand, let us notice that for \( t \to +\infty \) we have \(^{18}\)

\[
y(t) = \begin{cases} 
\left[ C + \frac{A^{1+w}}{w} \Gamma\left(\frac{1+2w}{1+w}\right) \Gamma\left(\frac{1-w}{2(1+w)}\right) \left(\frac{1-w}{w} V_2 + \frac{n^2 \Omega_\star^{(m)}}{A^2} \right) \right] e^{\omega t} + O\left(e^{1+w\omega t}\right) & \text{for } 0 < w < 1, \\
\frac{(1-w^2)^2 A V_2}{2 \omega^2} t e^{\omega t} + O(e^{\omega t}) & \text{for } w = 0, \\
\frac{-(1-w)(1+w)^2}{w} V_2 A^{1+w} \frac{e^{1+w\omega t}}{4 \omega^2} + O\left(e^{(1+w)\omega t}\right) & \text{for } -1 < w < 0.
\end{cases} \tag{4.2.67}
\]

---

\(^{18}\)To derive the expansions in Eq. (4.2.67) one should recall that for \( z \to -\infty \) (see, e.g., \(^{22}\) Eqs. (15.2.2)-(15.8.2))

\[
_{2}F_{1}(a,b,c;z) = \frac{\Gamma(c) \Gamma(b-a) \Gamma(1-(b-a))}{\Gamma(b) \Gamma(c-a) \Gamma(a-b+1)} (-z)^{-a} - \frac{\Gamma(c) \Gamma(b-a) \Gamma(1-(b-a))}{\Gamma(a) \Gamma(b-c) \Gamma(b-a+1)} (-z)^{-b} + O\left((-z)^{-\min(a+b+1)}\right).
\]

53
Recalling the previous assumptions on the parameters, the above asymptotics show that in order to fulfill the inequality \( y(t) > 0 \) in the limit \( t \to \infty \), it is necessary to demand \[ V_2 > -\frac{w}{1-w} \left[ \frac{2\sqrt{\pi}}{\Gamma \left( \frac{1+2w}{2(1+w)} \right)} \frac{C}{A^{1+w}} + \frac{n^2 \Omega^{(m)}}{A^2} \right] \quad \text{for } 0 < w < 1 , \\
V_2 > 0 \quad \text{for } -1 < w \leq 0 . \tag{4.2.68} \]

Whenever the conditions in Eq. (4.2.68) are violated, \( y(t) \) eventually becomes negative; since \( y(t) \) is positive close to the Big Bang (for \( t \to 0^+ \)), it follows that \( y(t) \) must vanish at some finite time, namely,

\[ \exists t_\ast \in (0, +\infty) \quad \text{such that } \quad y(t_\ast) = 0 . \tag{4.2.70} \]

In this case the maximal admissible interval \( I \) is a (finite) subset of \((0,t_\ast)\). Besides, given that \( x(t) = A \sinh(\omega t) \) is strictly positive and finite for all \( t \in (0,t_\ast) \), from Eqs. (2.2.2, 3.2.3) we see that

\[ a(t) = e^{A(t)/n} = x(t) \frac{1}{n(1+w)} y(t) \frac{1}{n(1+w)} \to 0 \quad \text{for } t \to t_\ast^- . \tag{4.2.71} \]

The above relation suggests that a Big Crunch could occur at \( t = t_\ast \); in this regard, it should be recalled that the very definition of Big Crunch also requires that \( e^{B(t)} \) is integrable in a left neighborhood of \( t_\ast \). Noting that Eqs. (3.2.2, 3.2.3) give

\[ e^{B(t)} = e^{-w \varphi(t)} = x(t)^{\frac{1}{1-w}} y(t)^{\frac{1}{1-w}} , \tag{4.2.72} \]

we see that \( e^{B(t)} \) is certainly integrable for \( t \to t_\ast^- \) if \( 0 < w < 1 \), while a finer analysis is needed when \(-1 < w < 0 \).

On the other side, let us stress that the fulfillment of the conditions in Eq. (4.2.68) is certainly not sufficient to ensure \( y(t) > 0 \) for all \( t \in (0, +\infty) \). Notwithstanding, in the upcoming subsection 4.2.3 we are going to show that this condition is actually attained at least for a specific choice of the parameters. By continuity arguments, this fact indicates that there also exist other values of the parameters for which the maximal admissible domain is in fact

\[ I = (0, +\infty) . \tag{4.2.73} \]

Assuming the maximal domain \( I \) to be as above and restricting the attention to the case

\[ 0 < w < 1 , \tag{4.2.74} \]

of interest for the subsequent applications, for \( t \to +\infty \) we obtain

\[ A(t) = \frac{2}{1-w^2} \omega t \tag{4.2.75} \]

\[ + \frac{1}{1-w} \log \left\{ C + \frac{A^{1+w}}{2 \sqrt{\pi} \Gamma \left( \frac{1+2w}{2(1+w)} \right)} \frac{1}{1+w} \Gamma \left( \frac{1-w}{2(1+w)} \right) \left( 1 - \frac{w}{V_2} + \frac{n^2 \Omega^{(m)}}{A^2} \right) \right\} + O \left( e^{-\frac{2w \omega t}{1+w}} \right) , \]

\[ \varphi(t) = -\frac{2w}{1-w^2} \omega t \tag{4.2.76} \]

\[ - \frac{1}{1-w} \log \left\{ C + \frac{A^{1+w}}{2 \sqrt{\pi} \Gamma \left( \frac{1+2w}{2(1+w)} \right)} \frac{1}{1+w} \Gamma \left( \frac{1-w}{2(1+w)} \right) \left( 1 - \frac{w}{V_2} + \frac{n^2 \Omega^{(m)}}{A^2} \right) \right\} + O \left( e^{-\frac{2w \omega t}{1+w}} \right) , \]

\[ ^{19}\text{In this connection, note that } \Gamma \left( \frac{1+2w}{2(1+w)} \right) > 0 \text{ for all } 0 < w < 1. \]
$w^{(s)}(t) = -1 + 2w^2 + O\left(e^{-\frac{2w}{1+w} \omega t}\right) \quad (4.2.77)$

$\Omega^{(m)}(t) = \frac{2n^2 \Omega^{(m)}_s (1-w^2)^2 (A/2)^{\frac{1}{1+w}}}{(A \omega)^2 \left(C + \frac{A^{\frac{1}{1+w}}}{2\sqrt{\pi}} \Gamma\left(\frac{1+2w}{1+w}\right) \Gamma\left(\frac{1-w}{2(1+w)}\right) \left(\frac{1-w}{w} V_2 + \frac{n^2 \Omega^{(m)}_s}{A^2}\right)\right)} e^{-\frac{2(1+2w)}{1+w} \omega t} + O\left(e^{-\frac{2(1+3w)}{1+w} \omega t}\right). \quad (4.2.78)$

In particular, note that the field $\varphi$ is the dominant contribution for $t \to +\infty$ (since, $\Omega^{(s)} = 1 - \Omega^{(m)} \to 1$ for $t \to \infty$; see Eq. (4.0.5)).

Also in this case, the previous asymptotic expansions can be rephrased in terms of the cosmic time $\tau$; to this purpose, one should first notice that Eqs. (2.2.2) (3.2.2) (4.2.76) entail, for $t \to +\infty$,

$$
\tau(t)/\theta = \frac{1-w^2}{2^{1-w} w^2 A^{\frac{1}{1+w}} \omega} \left(C + \frac{A^{\frac{1}{1+w}}}{2\sqrt{\pi}} \Gamma\left(\frac{1+2w}{1+w}\right) \Gamma\left(\frac{1-w}{2(1+w)}\right) \left(\frac{1-w}{w} V_2 + \frac{n^2 \Omega^{(m)}_s}{A^2}\right)\right)^{-\frac{1}{w}} e^{\frac{2w}{1+w} \omega t} + O\left(e^{-\frac{2w(2w-1)}{1+w} \omega t+1}\right). \quad (4.2.79)
$$

As an example, let us mention that by inverting the above relation, from Eqs. (2.2.4) (4.2.75) we get the following, for $\tau/\theta \to +\infty$:

$$
a(\tau) = \left(\frac{A}{C + \frac{A^{\frac{1-w}{1+w}}}{2\sqrt{\pi}} \Gamma\left(\frac{1+2w}{1+w}\right) \Gamma\left(\frac{1-w}{2(1+w)}\right) \left(\frac{1-w}{w} V_2 + \frac{n^2 \Omega^{(m)}_s}{A^2}\right)}\right)^{\frac{1}{w}} \left(\frac{\tau/\theta}{\theta}\right)^{\frac{1}{n w^2}} + O\left(\left(\frac{\tau/\theta}{\theta}\right)^{\max\left\{\frac{1-n w^2}{n w^2}, \frac{1-n w(1-w)}{n w^2}\right\}}\right). \quad (4.2.80)
$$

4.2.3 Qualitative analysis of one of the previous cases. A model for inflation

We now proceed to examine in more detail a particular case of the cosmological model analyzed before in the present subsection 4.2, selecting specific values for the associated free parameters. Let us anticipate that the rationale behind the said choice of parameters is to realize an inflationary scenario, where an early stage inflation occurs during the radiation dominated era. This scenario would allow, among else, to resolve the flatness, horizon and monopole problems. The above considerations and the arguments to be presented in the sequel are largely inspired by the model portrayed in [30, Sec. 11.4], where inflation is triggered by a true cosmological constant; on the contrary, here we plan to mimic this cosmological constant contribution in terms of the scalar field $\varphi$ with self-interaction potential as in Eq. (4.2.2).

To begin with, let us fix the space dimension and the spatial curvature as (see Eq. (4.0.1))

$$
n = 3, \quad k = 0. \quad (4.2.81)
$$

We further suppose that the ordinary matter content of the universe can be described by means of a perfect fluid of radiation type, i.e., we posit

$$
w = 1/3. \quad (4.2.82)
$$

To proceed, let us refer to the considerations related to Eqs. (2.2.25) (2.2.29); these indicate that the field $\varphi$ can effectively reproduce a cosmological constant contribution whenever the self-interaction potential $V(\varphi)$ possesses a stationary point (see Eq. (2.2.29)). Taking this fact and the gauge invariance $\varphi \mapsto \varphi + \text{const.}$ into account, we require the potential in Eq. (4.2.2) to attain a maximum at $\varphi = 0$; accordingly, we set

$$
V_1 = 2V, \quad V_2 = -V \quad \text{for some } V > 0. \quad (4.2.83)
$$
With the above choices, the potential reduces to

\[ V(\varphi) = V \left( 2e^{\varphi} - e^{\varphi} \right) \]  

(see Fig. 7 for the plot of the map \( \varphi \in \mathbb{R} \mapsto V(\varphi)/V \)).

Since the condition in Eq. (4.2.9) is certainly fulfilled in the case under analysis, we can refer to the Lagrange equations (4.2.10)-(4.2.11) and assume that the corresponding solutions \( x(t), y(t) \) are as in Eq. (4.2.12); taking also into account the associated zero-energy constraint (4.2.23) and using some known relations for the hypergeometric functions \( _2F_1 \) appearing in Eq. (4.2.12), we obtain

\[ x(t) = A \sinh(\omega t) \]  

\[ y(t) = C \cosh(\omega t) + \frac{\sqrt{A}}{2} \left( 1 + \frac{9\Omega^0(m)}{2A^2V} \right) \sqrt{\sinh(\omega t)} - \frac{\sqrt{A}}{2} \left( 1 - \frac{9\Omega^0(m)}{2A^2V} \right) \sqrt{\sinh(\omega t)} \cosh(\omega t) \ _2F_1 \left( \frac{1}{2}, \frac{1}{4}, \frac{5}{4}; -\sinh^2(\omega t) \right) , \]  

where, according to Eq. (4.2.3),

\[ \omega = \frac{2\sqrt{2V}}{3} . \]  

Quite understandably, the analysis of the model at issue becomes significantly simpler if we arbitrarily fix the parameter \( A \) (left unspecified until now) so as to get rid of the hypergeometric function \( _2F_1 \) appearing in the expression for \( y(t) \) in Eq. (4.2.85); to this purpose, from now on we set

\[ \Omega^0(m) > 0 \]  

\[ A = \frac{\sqrt{9\Omega^0(m)}}{2V} \equiv \frac{2\sqrt{\Omega^0(m)}}{\omega} . \]  

Next, let us require a Big Bang to occur at \( t = 0 \) and recall that, for this to happen, it is necessary to assume that \( C \geq 0 \) (see Eq. (4.2.46)); accordingly, for later convenience we put

\[ C = \sqrt{A} \zeta \equiv \left( \frac{4\Omega^0(m)}{\omega^2} \right)^{1/4} \zeta \quad \text{for} \quad \zeta \geq 0 . \]  

\footnote{In particular, the derivation of the expression for \( y(t) \) in Eq. (4.2.85) requires the use of the identity

\[ _2F_1 \left( \frac{1}{2}, \frac{1}{4}, \frac{5}{4}; z \right) = \frac{5}{3} \left[ _2F_1 \left( \frac{1}{2}, \frac{1}{4}, \frac{5}{4}; z \right) - \sqrt{1-z} \right] \quad \text{for} \quad z \in \mathbb{R} . \]

This identity can be derived with some elementary computations starting from the Gauss series representation of \( _2F_1 \) (see, e.g., [22, Eq. 15.2.1]) and using some known relations for the Euler gamma functions \( \Gamma \) appearing therein. Let us mention that the same identity could also be derived (with some more effort) from the relations for contiguous hypergeometric functions (see, e.g., [22 § 15.5(ii)]).}
With the above choices (4.2.87) (4.2.88), Eq. (4.2.85) reduces to
\[ x(t) = 2\sqrt{\Omega_s^{(m)}} \frac{\sinh(\omega t)}{\omega}\sinh(\omega t), \quad y(t) = \left(\frac{4\Omega_s^{(m)}}{\omega^2}\right)^{1/4} \frac{\sqrt{\sinh(\omega t)}}{\sinh(\omega t)} \left[ 1 + \zeta \frac{\cosh(\omega t)}{\sqrt{\sinh(\omega t)}} \right]. \tag{4.2.89} \]

It is evident that the above expressions fulfill \( x(t) > 0 \) and \( y(t) > 0 \) for all \( t > 0 \) (cf. Eq. (4.2.4)); so, we can understand the solution (4.2.89) to be defined on the maximal admissible domain
\[ I = (0, +\infty) \). \tag{4.2.90} \]

To go on, let us note that Eqs. (2.2.2) (3.2.2) (3.2.3) and Eq. (4.2.89) give the following expression for the cosmic time:
\[ \tau(t)/\theta = \int_0^t dt' x^{-1/4}(t') y^{1/2}(t') = \int_0^t dt' \sqrt{1 + \zeta \frac{\cosh(\omega t')}{\sqrt{\sinh(\omega t')}}}. \tag{4.2.91} \]

Concerning the scale factor and the field, from Eq. (2.2.4), Eq. (3.2.3) (with \( \gamma = w = 1/3 \)) and Eqs. (4.2.89) we infer
\[ a(t) = x^{1/4}(t) y^{1/2}(t) = \left(\frac{4\Omega_s^{(m)}}{\omega^2}\right)^{1/4} \frac{\sqrt{\sinh(\omega t)}}{\sinh(\omega t)} \left[ 1 + \zeta \frac{\cosh(\omega t)}{\sqrt{\sinh(\omega t)}} \right]. \tag{4.2.92} \]
\[ \varphi(t) = \log(x^{3/4}(t) y^{-3/2}(t)) = -\frac{3}{2} \log \left[ 1 + \zeta \frac{\cosh(\omega t)}{\sqrt{\sinh(\omega t)}} \right]. \tag{4.2.93} \]

Finally, the equation of state coefficient for the scalar field and the density pressure of radiation can be determined using Eqs. (1.2.7) (1.2.8) and (4.2.89), which yield
\[ w^{(\phi)}(t) = -1 + \frac{2\zeta^2 (1 - \sinh^2(\omega t))^2}{4 \sinh^3(\omega t) + 12 \zeta \cosh(\omega t) \sinh^{5/2}(\omega t) + \zeta^2 (1 + 3 \sinh^2(\omega t))^2}, \tag{4.2.94} \]
\[ \Omega^{(m)}(t) = \frac{\sinh(\omega t) + \zeta \cosh(\omega t) \sqrt{\sinh(\omega t)}}{(\cosh(\omega t) \sqrt{\sinh(\omega t)} + \frac{\zeta}{2} (1 + 3 \sinh^2(\omega t)))^2}. \tag{4.2.95} \]

In the sequel we first discuss the exceptional case \( \zeta = 0 \), and then proceed to analyze the more generic configuration with \( \zeta > 0 \).

**The case \( \zeta = 0 \).** This case deserves a special mention because it corresponds to a scenario where the field \( \varphi \) behaves exactly as a cosmological constant. Eqs. (4.2.91) (4.2.92) with \( \zeta = 0 \) give the following results, for \( t \in (0, +\infty) \):
\[ \tau(t)/\theta = t, \tag{4.2.96} \]
and \[ a(t) = \left(\frac{4\Omega_s^{(m)}}{\omega^2}\right)^{1/4} \frac{\sqrt{\sinh(\omega t)}}{\sinh(\omega t)}, \quad \varphi(t) = \text{const.} = 0, \quad w^{(\phi)}(t) = \text{const.} = -1, \quad \Omega^{(m)}(t) = 1/\cosh^2(\omega t). \tag{4.2.97} \]

Equivalently, viewing these observables as function of the cosmic time \( \tau \in (0, +\infty) \):
\[ a(\tau) = \left(\frac{4\Omega_s^{(m)}}{\omega^2}\right)^{1/4} \frac{\sqrt{\sinh(\omega \tau/\theta)}}{\sinh(\omega \tau/\theta)}, \quad \varphi(\tau) = \text{const.} = 0, \tag{4.2.97} \]
\[ w^{(\phi)}(\tau) = \text{const.} = -1, \quad \Omega^{(m)}(\tau) = \frac{1}{\cosh^2(\omega \tau/\theta)}. \tag{4.2.97} \]

In particular, from the above explicit expression for \( a(\tau) \) we infer \[^{21}\]
\[ a(\tau) = \left\{ \begin{array}{ll} \left(\frac{4\Omega_s^{(m)}}{\omega^2}\right)^{1/4} (\tau/\theta)^{1/2} + O(\tau^{5/2}) & \text{for } \tau/\theta \to 0^+, \\
\left(\frac{\Omega_s^{(m)}}{\omega^2}\right)^{1/4} e^{\frac{\zeta}{2} (\tau/\theta)} + O(e^{-\frac{3\zeta}{2} \tau/\theta}) & \text{for } \tau/\theta \to +\infty. \end{array} \right. \tag{4.2.98} \]

\[^{21}\]Let us point out that in the setting under analysis one cannot naively refer to the asymptotic expansions in Eqs. (1.2.7) (1.2.8) for \( \zeta = 0 \), since in this case the dominant contribution written in Eq. (4.2.67) for \( 0 < w < 1 \) vanishes identically.
The case $\zeta > 0$. Let us return to Eqs. (4.2.91)-(4.2.95), that we now use with $\zeta > 0$. We first derive the asymptotic expansions of $\tau(t)/\theta$, $a(t)$, $w(\phi)(t)$, $\Omega^{(m)}(t)$ in the limit of small and large $t$. The behaviour of $\tau(t)/\theta$ in these limits can be derived from the general asymptotic expansions (4.2.54)-(4.2.79), which in the present setting reduce to (22)

$$
\tau(t)/\theta = \begin{cases}
\frac{4}{3} \left( \frac{\zeta^2}{\omega} \right) t^{1/4} + O(t^{5/4}) & \text{for } t \to 0^+,
2^{7/4} \epsilon^{1/2} \frac{1}{\omega} \epsilon^{1/4} t + O(1) & \text{for } t \to +\infty.
\end{cases}
$$

(4.2.99)

The behavior of $a(t)$, $w(\phi)(t)$, $\Omega^{(m)}(t)$ for small and large $t$ is obtained by direct inspection of Eqs. (4.2.91)-(4.2.95) which give, respectively:

$$
a(t) = \begin{cases}
\sqrt{\zeta} \left( \frac{4 \Omega^{(m)}}{\omega} \right)^{1/4} t^{1/4} + O(t^{3/4}) & \text{for } t \to 0^+,
\sqrt{\zeta} \left( \frac{\Omega^{(m)}}{2 \omega^2} \right)^{1/4} e^{t/\omega} t + O(e^{t/\omega}) & \text{for } t \to +\infty.
\end{cases}
$$

(4.2.100)

$$
w(\phi)(t) = \begin{cases}
1 - 16 \omega^2 t^2 + O(t^{5/2}) & \text{for } t \to 0^+,
- \frac{7}{9} + O(e^{-3/4} \omega t) & \text{for } t \to +\infty.
\end{cases}
$$

(4.2.101)

$$
\Omega^{(m)}(t) = \begin{cases}
\frac{4 \sqrt{\omega}}{\zeta} t^{1/2} + O(t) & \text{for } t \to 0^+,
\frac{32}{9 \sqrt{2} \zeta} e^{-3/4 \omega t} + O(e^{-3/4 \omega t}) & \text{for } t \to +\infty.
\end{cases}
$$

(4.2.102)

For our purposes it is also important to consider the behavior of the above observables when $t$ ranges in a compact interval, and $\zeta$ is sent to zero. Indeed, let us fix any two times $0 < t_1 < t_2$; then, from Eqs. (4.2.91)-(4.2.95) it readily follows that

$$
\tau(t)/\theta = t + O(\zeta), \quad a(t) = \left( \frac{4 \Omega^{(m)}}{\omega^2} \right)^{1/4} \sqrt{\sinh(\omega t)} + O(\zeta),
$$

$$
w(\phi)(t) = -1 + O(\zeta^2), \quad \Omega^{(m)}(t) = \frac{1}{\cosh^2(\omega t)} + O(\zeta),
$$

(4.2.103)

for $\zeta \to 0^+$, uniformly in $t \in [t_1, t_2]$. Let us rephrase the previous results viewing the above observables as functions of cosmic time. From Eqs. (4.2.99) and (4.2.100)-(4.2.102) we get

$$
a(\tau) = \begin{cases}
\left( \frac{3 \zeta}{\sqrt{2} \omega} \right)^{1/3} \left( \Omega^{(m)}_* \right)^{1/4} (\tau/\theta)^{1/3} + O(\tau/\theta) & \text{for } \tau/\theta \to 0^+,
\frac{1}{3} \sqrt{\frac{\omega^5}{2048}} \left( \Omega^{(m)}_* \right)^{1/4} (\tau/\theta)^3 + O((\tau/\theta)^2) & \text{for } \tau/\theta \to +\infty.
\end{cases}
$$

(4.2.104)

More precisely, the asymptotics in Eq. (4.2.99) follow from the cited Eqs. (4.2.51) and (4.2.79) fixing $n = 3$, $w = 1/3$, $V_1 = 2V$ and $V_2 = -V$ (with $V > 0$), $A = (9 \Omega^{(m)}_*/2V)^{1/2}$ and $C = (9 \Omega^{(m)}_*/2V)^{1/4} \zeta$ (with $\Omega^{(m)}_* > 0$), in agreement with Eqs. (4.2.81)-(4.2.82) (4.2.83)-(4.2.88).
Let us briefly comment the above results. According to Eq. (4.2.107), on each compact interval \( t \in [\tau_1, \tau_2] \), with \( \tau_1 > 0 \) so as to ensure a strict separation from the Big Bang, for \( \zeta \) sufficiently small the scale factor \( a(\tau) \) grows exponentially and \( w^{(\zeta)}(\tau) \) is close to \(-1\), indicating that the field behaves approximately like a cosmological constant. As a consequence, the behavior of the system for \( \tau \in [\tau_1, \tau_2] \) and small \( \zeta \) is similar to that described in the previous paragraph for \( \zeta = 0 \) and all \( \tau \in (0, +\infty) \). The situation is completely different if we approach the Big Bang, or we consider the very far future; for example, Eq. (4.2.101) shows that \( a(\tau) \) has a power law dependence on \( \tau/\theta \) with exponents \( 1/3 \) and 3, respectively, for \( \tau/\theta \to 0^+ \) and \( \tau/\theta \to +\infty \).

The presence of an epoch of exponential growth for \( a(\tau) \), preceded and followed by periods with slower growth, is typical of inflationary models. In the sequel we will show that one can adjust the parameters of the system so as to obtain a rather realistic model for inflation, even from a quantitative viewpoint.

**An interlude on the quantitative determination of cosmic time.** Let us recall that \( \tau(t)/\theta \) is expressed via Eq. (4.2.91) as a nontrivial integral over the interval \( (0, t) \). The numerical computation of this integral (for specified values of all parameters) is problematic, especially in the situation of greatest interest for us. In fact, for \( t' \to 0^+ \) the integrand function in Eq. (4.2.91) behaves like \( \sqrt{\zeta}(\omega t')^{-1/4} \), the product of the divergent factor \( (\omega t')^{-1/4} \) by the parameter \( \sqrt{\zeta} \). To make things worse, in the sequel we are mostly interested in a case where \( \zeta \) is very small.

Fortunately, the problem that we have just outlined can be overcome. In fact, starting from the integral representation (4.2.91) it is possible to determine analytically two elementary functions \( T^{\pm}_\zeta \) such that \( T^{-}_\zeta(t) \leq \tau(t)/\theta \leq T^{+}_\zeta(t) \) for arbitrary \( \zeta, t > 0 \); we refer to Appendix C for a detailed description of such functions. The same Appendix shows that, in the application with small \( \zeta \) considered in next paragraph, \( T^{+}_\zeta(t) \) and \( T^{-}_\zeta(t) \) are very close for all values of \( t \) taken in account, so that the mean \((1/2)(T^{-}_\zeta + T^{+}_\zeta)(t)\) is a very accurate approximant for \( \tau(t)/\theta \). In the calculations mentioned in the next paragraph, \( \tau(t)/\theta \) has always been approximated with the previous mean.

**A plausible scenario with inflation.** Let us now present a reasonable choice of the parameters left unspecified for the model under analysis, which can in fact lead to a physically plausible inflationary scenario. The key idea that we are going to pursue in the sequel is that the scale factor grows exponentially in a compact interval of cosmic time, at least for very small values of \( \zeta \) (a fact made evident by the asymptotic expansion written in Eq. (4.2.92)).

Inspired by the arguments suggested in [30, Sec. 11.4], we set the time parameter \( \theta \) equal to the alleged time of Grand Unified Theory (GUT), namely,

\[
\theta = 10^{-36} \text{ sec} ,
\]  

(4.2.108)
and presume that the universe undergoes an inflationary expansion at least during the interval of cosmic time approximately comprised between \( \tau \simeq \theta \) and \( \tau \simeq N \theta \), for some given \( N \) large enough. In the sequel we will refer to the case where

\[
N = 100 , \tag{4.2.109}
\]

even though analogous results can be derived also for other values of \( N \). Correspondingly, let us assume that the dimensionless parameter \( \zeta \) introduced in Eq. (4.2.88) is exponentially small with respect to \( N \); more precisely, we put

\[
\zeta = e^{-N} \simeq 3.72008 \ldots \times 10^{-44} . \tag{4.2.110}
\]

On the contrary, let \( \Omega_\pi^{(m)} \) and \( V \) be independent of \( N \) and comparable to unity, so that the same holds true for \( \omega \) (due to Eq. (4.2.86)); as an example, let us fix

\[
\Omega_\pi^{(m)} = 0.9 , \quad V = 1 , \quad \omega = \frac{2 \sqrt{2 V}}{3} \simeq 0.9428 \ldots . \tag{4.2.111}
\]

Having fixed all the involved parameters, for any given \( t \) we can calculate the numerical values of the quantities \( \tau(t)/\theta \), \( a(t) \), \( w^{(\phi)}(t) \), \( \Omega^{(m)}(t) \). For \( \tau(t)/\theta \) we use, in place of the integral representation (4.2.91), the very accurate approximation method mentioned in the previous paragraph and described in Appendix C for \( a(t) \), \( w^{(\phi)}(t) \) and \( \Omega^{(m)}(t) \) we use Eqs. (4.2.92) (4.2.93) (4.2.94).

Figs. 8-11 give \( a(\tau) \) as a function of the dimensionless quantity \( \tau/\theta \), for different ranges of the latter; these figures were obtained drawing the curve \( t \rightarrow \left( \tau(t)/\theta , a(\tau(t)) \right) \) for \( t \) within different intervals
(namely, for $t \in (0, 10^{-90})$, $t \in (0, 200)$, $t \in (0, 240)$ and $t \in (0, 400)$). Logarithmic scales are used for both $\tau/\theta$ and $a(\tau)$ in Fig.11; this makes evident that there is a comparatively short interval of cosmic time where the scale factor increases abruptly. Just to give an idea of the orders of magnitude involved in these arguments, let us mention that $\tau_1 = 50 \theta$, $\tau_2 = 150 \theta \Rightarrow \frac{a(\tau_2)}{a(\tau_1)} = 2.97056 \ldots \times 10^{30}$.

Fig. 12 and Fig. 13 refer to the equation of state parameter for the field; more precisely they give $w(\phi)(\tau)$ as a function of $\tau/\theta$, for the latter variable ranging in two different intervals. Again, these graphs were obtained as parametric plots (the curve $t \mapsto (\tau(t)/\theta, w(\phi)(\tau(t)))$ was plotted for $t \in (0, 10^{-29})$ and $t \in (0, 230)$, respectively). In particular, Fig.12 suggests $w(\phi)(\tau) \to 1^-$ for $\tau/\theta \to 0^+$, in agreement with Eq. (4.2.105). On the other hand, Fig.13 exhibits a sharp transition from $w(\phi) = -1$ to $w(\phi) = -7/9 \approx -0.777 \ldots$; this behavior corresponds to the general features previously pointed out in Eqs. (4.2.103)-(4.2.105).

Finally, let us consider the density parameter $\Omega^{(m)}$ for the radiation content of the universe. Fig.14 and Fig.15 represent $\Omega^{(m)}(\tau)$ as a function of $\tau/\theta$ (and were obtained plotting the curve $t \mapsto (\tau(t)/\theta, \Omega^{(m)}(\tau(t)))$ for $t \in (0, 10^{-86})$ and $t \in (0, 7)$, respectively). Fig.14 makes evident that $\Omega^{(m)}(\tau) \to 0^+$ for $\tau/\theta \to 0^+$ (see Eq. (4.2.106) for the leading order in the corresponding asymptotic expansion). Fig.15 shows instead that $\Omega^{(m)}(\tau) \approx 1$ for small (though, not too small) values of $\tau/\theta$ and $\Omega^{(m)}(\tau)$ rapidly vanishes for larger values of the cosmic time, in accordance with the general features mentioned in Eqs. (4.2.103)-(4.2.105).
4.3 An integrable case with a class 7 potential. The nonlinear repulsor/oscillator model

In this section, we refer to the integrable subcase (i) in the analysis of class 7 potentials (see page 31), with an additional prescription: the exponent $2/\gamma - 2$ in the potential $V$ of Eq. (3.7.1), in the Lagrangian $\mathcal{L}$ of Eq. (3.7.8) and in the energy $E$ of Eq. (3.7.9) is required to be an even integer $2\ell \geq 4$. Here are some motivations for this additional requirement:

i) The map $\xi \rightarrow \xi^{2/\gamma - 2}$, appearing in Eqs. (3.7.1) (3.7.8) (3.7.9), is well defined, smooth and bounded from below for $\xi$ ranging throughout the whole real axis if and only if $2/\gamma - 2 = 2\ell \in \{0, 2, 4, \ldots\}$.

ii) For $2/\gamma - 2 = 0$ (i.e., $\gamma = 1$), the potential $V$ of Eq. (3.7.1) is constant. For $2/\gamma - 2 = 2$ (i.e. $\gamma = 1/2$), Eq. (3.7.1) gives $V(\varphi) = (1/2)(V_1 - V_2) + (1/2)(V_1 + V_2)\cosh \varphi$: thus, $V$ becomes a class 1 potential (cf. Eq. (3.1.1)), to be treated with the simpler methods already described for that class. Summing up, the cases $2/\gamma - 2 = 0, 2$ can be regarded as trivial; excluding them from the considerations in the previous item (i), we are left with the condition $2/\gamma - 2 = 2\ell \in \{4, 6, 8, \ldots\}$.

To proceed, let us recall that the subcase (i) of page 31 requires $k = 0$, $\gamma = (1 - w)/2$; on top of that, we assume $V_1$ and $V_2$ to be positive. Thus, the complete list of our choices is the following:

$$V_1 > 0, \quad V_2 > 0, \quad k = 0, \quad \gamma = \frac{1}{\ell + 1}, \quad w = \frac{\ell - 1}{\ell + 1}, \quad \ell \in \{2, 3, 4, \ldots\}$$

(4.3.1)

In particular, note that $1/3 < w < 1$ for all $\ell \in \{2, 3, 4, \ldots\}$ and $w = 1/3$ if and only if $\ell = 2$.

With the above choices (4.3.1), Eqs. (3.7.1) (3.7.8) and (3.7.9) for the potential $V(\varphi)$, the gauge function $B \equiv B$ and the coordinate change $(x, y) \rightarrow (A, \varphi)$ respectively reduce to:

$$V(\varphi) = V \left[ \left( \cosh \left( \frac{\varphi}{\ell + 1} \right) \right)^{2\ell} + \left( \sinh \left( \frac{\varphi}{\ell + 1} \right) \right)^{2\ell} \right], \quad \varphi \in I_{\gamma, V_2} = (-\infty, +\infty);$$

(4.3.2)

$$B = \frac{\ell - 1}{2} \log (x^2 - y^2);$$

(4.3.3)

$$A = \frac{\ell + 1}{2} \log (x^2 - y^2), \quad \varphi = \frac{\ell + 1}{2} \log \left( \frac{x + y}{x - y} \right);$$

(4.3.4)

$$(x, y) \in D := \{(x, y) \in \mathbb{R}^2 \mid x > 0, \ -x < y < x\}.$$  

(4.3.5)

Note that, for $A$ as above one has $A \rightarrow -\infty$ when $x^2 - y^2 \rightarrow 0^+$. Correspondingly, noting that the scale factor for the cosmology under analysis is given by (see Eq. (2.2.4))

$$a = e^{A/n} = (x^2 - y^2)^{\frac{\ell + 1}{2n}},$$  

(4.3.6)

we have $a \rightarrow 0^+$ in the limit $x^2 - y^2 \rightarrow 0^+$; this fact is especially relevant for the presence of a Big Bang. To say more, assuming that a Big Bang does actually occur at $t = 0$ in agreement with Eq. (4.0.2) and adding the conventional prescription $\tau(t) \rightarrow 0^+$ for $t \rightarrow 0^+$, from Eqs. (4.0.3) (4.3.3) we derive the following expression for the cosmic time coordinate:

$$\tau(t)/\theta = \int_0^t dt' \left( x^2(t') - y^2(t') \right)^{\frac{\ell - 1}{2}}.$$

(4.3.7)

Next let us recall that, in the subcase (i) for the potential class 7, the Lagrangian (3.7.8) and the corresponding energy (3.7.9) take the separable forms (3.7.10) (3.7.11). With the prescriptions

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23We will show in the sequel that this condition can in fact be attained.
stated in Eq. (4.3.1), the cited Eqs. (3.7.10) (3.7.11) give:

\[
L(x, y, \dot{x}, \dot{y}) = L_1(x, \dot{x}) + L_2(y, \dot{y}) - \frac{n^2}{2} \Omega^{(m)}_x,
\]

\[
L_1(x, \dot{x}) = -\frac{(\ell + 1)^2}{2} \dot{x}^2 - V_1 x^{2\ell}, \quad L_2(y, \dot{y}) = \frac{(\ell + 1)^2}{2} \dot{y}^2 - V_2 y^{2\ell};
\]

\[
E(x, y, \dot{x}, \dot{y}) = E_1(x, \dot{x}) + E_2(y, \dot{y}) + \frac{n^2}{2} \Omega^{(m)}_x,
\]

\[
E_1(x, \dot{x}) = -\frac{(\ell + 1)^2}{2} \dot{x}^2 + V_1 x^{2\ell}, \quad E_2(y, \dot{y}) = \frac{(\ell + 1)^2}{2} \dot{y}^2 + V_2 y^{2\ell}.
\]

In the sequel we discuss the solutions of the Lagrange equations corresponding to the above 1-dimensional Lagrangians \(L_1, L_2\), providing explicit quadrature formulas for them and discussing their asymptotic behaviors in different regimes of interest for the applications to be discussed in the sequel.

Before proceeding with this analysis, let us express in terms of the coordinates \(x, y\) two other relevant observables: namely, the coefficient \(w^{(o)}\) in the equation of state for the field and the dimensionless density parameter for matter \(\Omega^{(m)}\). The general relations (2.2.12), combined with Eqs. (4.3.1), (4.3.3), (4.3.4) for the case under analysis, yield the following results:

\[
w^{(o)} = \frac{(\ell + 1)^2 (x \dot{y} - \dot{x} y)^2 - 2 (x^2 - y^2)(V_1 x^{2\ell} + V_2 y^{2\ell})}{(\ell + 1)^2 (x \dot{y} - \dot{x} y)^2 + 2 (x^2 - y^2)(V_1 x^{2\ell} + V_2 y^{2\ell})},
\]

\[
\Omega^{(m)} = \frac{n^2 \Omega^{(m)}_x (x^2 - y^2)}{(\ell + 1)^2 (x \dot{y} - \dot{x} y)^2}.
\]

### 4.3.1 Constants of motion. Qualitative analysis and quadrature formulas

From here to the end of the present subsection [4.3] we stick to the configuration described by Eq. (4.3.1) and refer to Eqs. (4.3.8), (4.3.9) for the Lagrangian \(L\) and the corresponding energy \(E\).

The Lagrangian system described by \(L\) can be analyzed in terms of the separate 1-dimensional subsystems with Lagrangians \(L_1, L_2\), whose energies \(E_1, E_2\) are constants of motion. Of course, the Lagrangians \(L_1, L_2\) (as well as the energies \(E_1, E_2\)) are well defined and smooth for any real \(x\) and \(y\). Keeping this in mind, in the sequel we shall first study separately the subsystems with Lagrangians \(L_1, L_2\) assuming \(x, y \in (\infty, +\infty)\), and reserve to a second step the implementation of the condition \((x, y) \in \mathcal{D}\) (see Eq. (4.3.3)).

From the expression (4.3.9) for the total energy \(E\), we see that the constraint \(E = 0\) is fulfilled if and only if \(E_1\) and \(E_2\) are expressed as follows:

\[
E_1 = -\mathcal{E}, \quad E_2 = \mathcal{F}, \quad \mathcal{E} := \mathcal{F} + \frac{n^2}{2} \Omega^{(m)}_x.
\]

The expression for \(E_2\) in Eq. (4.3.9) makes evident that \(\mathcal{F} \geq 0\), and that \(\mathcal{F} = 0\) only along motions with \(y(t) = 0\) and \(\dot{y}(t) = 0\) for all \(t\); in the sequel we exclude such motions, fixing

\[
\mathcal{F} \in (0, +\infty).
\]

Since we are assuming \(\Omega^{(m)}_x \geq 0\) (see Eq. (4.0.6)), the above condition also grants that \(\mathcal{E} \in (0, +\infty)\). Let us now consider two motions \(t \mapsto x(t)\) and \(t \mapsto y(t)\) fulfilling the Lagrange equations, with energies fixed according to the above prescriptions (and \(t\) ranging within suitable intervals). Then, from Eqs. (4.3.9), (4.3.12) we infer

\[
\frac{(\ell + 1)^2}{2} \dot{x}^2 - V_1 x^{2\ell} = \mathcal{E}, \quad \frac{(\ell + 1)^2}{2} \dot{y}^2 + V_2 y^{2\ell} = \mathcal{F}.
\]

63
The above equations can be interpreted as the conservation laws for the energies of two fictitious mechanical systems with kinetic energies \( \frac{\ell+1}{2} x^2 \), \( \frac{\ell+1}{2} \dot{y}^2 \) and potential energies \(-V_1 x^{2\ell}, V_2 y^{2\ell}\), that we can denominate, respectively, as a non-linear repulsor and a non-linear oscillator.

From the second equality in Eq. (4.3.14) we see that \( t \mapsto y(t) \) is an oscillatory motion such that

\[
y(t) \in \left[ -\left( \frac{\mathcal{F}}{V_2} \right)^{1/(2\ell)}, \left( \frac{\mathcal{F}}{V_2} \right)^{1/(2\ell)} \right] \quad \text{for all } t
\]

(4.3.15)

(the above interval is the set \( \{y \in \mathbb{R} \mid V_2 y^{2\ell} \leq \mathcal{F} \} \) and the times \( t \) such that \( y(t) = \pm \left( \frac{\mathcal{F}}{V_2} \right)^{1/(2\ell)} \) are inversion times for the motion). Since \( V_1 x(t)^{2\ell} \geq 0 \) and \( \mathcal{E} > 0 \), from the first equality in Eq. (4.3.14) we infer \( \dot{x}(t)^2 > 0 \), i.e. \( \dot{x}(t) \neq 0 \) for all \( t \). Thus \( \dot{x}(t) \) has a constant sign, and the function \( t \mapsto x(t) \) is strictly monotonic.

From Eq. (4.3.14) we also infer quadrature formulas containing the hypergeometric-type function

\[
F_\ell(z) := \, _2F_1\left( \frac{1}{2}, \frac{1}{2\ell}; \frac{1}{2\ell} + 1, z \right).
\]

(4.3.16)

More precisely, we have the following \(^{24}\):

\[
x(t) \text{ is well defined for } t \in [t_1, t_2] \text{ and sign } \dot{x}(t) = \xi \in \{\pm 1\} \text{ for all } t \in (t_1, t_2)
\]

\[
\Rightarrow \xi (t_2 - t_1) = \frac{\ell + 1}{\sqrt{2\ell}} \left[ \int_{t_1}^{t_2} F_\ell \left( -\left( V_1/\mathcal{E} \right) x^{2\ell} \right) - \int_{t_1}^{t_2} F_\ell \left( -\left( V_1/\mathcal{E} \right) x^{2\ell} \left( t_1 \right) \right) \right]
\]

\[
y(t) \text{ is well defined for } t \in [t_1, t_2] \text{ and sign } \dot{y}(t) = \sigma \in \{\pm 1\} \text{ for all } t \in (t_1, t_2)
\]

\[
\Rightarrow \sigma (t_2 - t_1) = \frac{\ell + 1}{\sqrt{2\ell}} \left[ \int_{t_1}^{t_2} F_\ell \left( \left( V_2/\mathcal{F} \right) y^{2\ell} \right) - \int_{t_1}^{t_2} F_\ell \left( \left( V_2/\mathcal{F} \right) y^{2\ell} \left( t_1 \right) \right) \right]
\]

For future use, let us mention the asymptotics \(^{25}\)

\[
F_\ell(-z) = \frac{C_\ell}{z^{1/(2\ell)}} - \frac{1}{(\ell - 1) \sqrt{z}} + O \left( \frac{1}{z^{3/2}} \right) \text{ for } z \to +\infty,
\]

(4.3.19)

\[
C_\ell := \frac{\Gamma \left( \frac{\ell - 1}{2\ell} \right) \Gamma \left( \frac{2\ell + 1}{2\ell} \right)}{\Gamma \left( \frac{\ell + 1}{2\ell} \right)};
\]

here the dominating term is \( C_\ell / z^{1/(2\ell)} \), since \( 0 < \frac{1}{2\ell} \leq \frac{1}{4} \). Again for future use, let us also mention the special value \(^{26}\)

\[
F_\ell(1) = \frac{\sqrt{\pi} \Gamma \left( \frac{2\ell + 1}{2\ell} \right)}{\Gamma \left( \frac{\ell + 1}{2\ell} \right)}.
\]

---

\(^{24}\)Let us give a few details on the derivation of Eq. (4.3.17). To express the integral in Eq. (4.3.17) in terms of \( F_\ell \) write \( \int_{x(t_1)}^{x(t_2)} dx / \sqrt{\mathcal{E} + V_1 x^{2\ell}} = \frac{1}{\sqrt{\mathcal{E}}} \int_{0}^{x(t_2) - x(t_1)} dx / \sqrt{1 + V_1 x^{2\ell}} \) and then use for \( i = 1, 2 \) the following relations, based on the change of variable \( x = x(t) s^{1/(2\ell)} \) with \( s \in [0, 1] \):

\[
\int_{x(t_i)}^{x(t)} dx / \sqrt{1 + V_1 x^{2\ell}} = \frac{\pi}{2\ell} \int_{0}^{\mathcal{E}} ds \, s^{-1/2\ell} / \sqrt{1 + (V_1/\mathcal{E}) x^{2\ell}(s)} = x(t_1) F_\ell \left( 1 - \left( V_1/\mathcal{E} \right) x^{2\ell}(t_1) \right).
\]

One proceeds similarly to express via \( F_\ell \) the integral in Eq. (4.3.18).

\(^{25}\)This asymptotics follows from the definition (4.3.16) of \( F_\ell \) and from some known identities about hypergeometric functions, namely: a Kummer transformation (see, e.g., [22] Eq. 15.8.2] and the elementary relations \( _2F_1 \left( a, 0, \zeta \right) = 1 + O(\zeta) \) for \( \zeta \to 0 \).

\(^{26}\)Eq. (4.3.20) follows the definition (4.3.16) of \( F_\ell \) and from a general result about \( _2F_1 \left( a, b, c, 1 \right) \) (see, e.g., [22] Eq. 15.4.20]).
4.3.2 Choosing the initial data. More on the qualitative and quantitative analysis

We now fix the attention on the solutions \( t \mapsto x(t) \) and \( t \mapsto y(t) \) of the Lagrange equations for \( L_1, L_2 \) with energies as in Eqs. (4.3.12) (4.3.13) and with the following initial data, specified at time \( t = 0 \) by convention:

\[
x(0) = y(0) = Y \in \left(0, (F/V_2)^{1/2(2\ell)}\right), \quad \dot{x}(0) = u > 0, \quad \dot{y}(0) = v > 0.
\]  

(4.3.21)

The upper bound \((F/V_2)^{1/2}\) prescribed here for \( Y \) is motivated by Eq. (4.3.15). The equality \( x(0) = y(0) \) will be employed in the sequel to infer that the scale factor \( a(t) \) vanishes for \( t \to 0^+ \) (see the considerations after Eqs. (4.3.20)-(4.3.21)), a fact related to the occurrence of a Big Bang. Let us also point out that the assumptions in Eq. (4.3.21) and Eq. (4.3.14) for the energies, here employed at time \( t = 0 \), give

\[
u = \frac{\sqrt{2}}{\ell + 1} \sqrt{E + V_1 Y^{2\ell}}, \quad v = \frac{\sqrt{2}}{\ell + 1} \sqrt{F - V_2 Y^{2\ell}}.
\]  

(4.3.22)

Each one of the solutions \( t \mapsto x(t) \) and \( t \mapsto y(t) \) is intended to be defined on the maximal admissible domain, that is on the largest interval containing \( t = 0 \) on which the solution is well defined.

The discussion of subsection 4.3.1 combined with the present assumptions, ensures that \( t \mapsto x(t) \) is a strictly increasing function, while \( t \mapsto y(t) \) oscillates.

The map \( t \mapsto x(t) \) has a bounded domain of the form

\[
(t_{\min}, t_{\max}) \quad \text{with} \quad -\infty < t_{\min} < 0 < t_{\max} < +\infty.
\]  

(4.3.23)

The finite times \( t_{\min}, t_{\max} \) are characterized by the fact that

\[
x(t) \to -\infty \quad \text{for} \quad t \to t_{\min}^+, \quad x(t) \to +\infty \quad \text{for} \quad t \to t_{\max}^-
\]  

(4.3.24)

(with \(( \cdots )^\pm \) indicating the limit from above or below). To determine \( t_{\max} \), it suffices to employ the quadrature formula (4.3.17) with \( \xi = +1 \), \( t_1 = 0 \), \( t_2 = t_{\max} \) and \( x(t_1), x(t_2) \) replaced by \( Y, +\infty \), respectively; in this way we obtain

\[
t_{\max} = \frac{\ell + 1}{\sqrt{2}} \int_Y^{+\infty} \frac{dx}{\sqrt{E + V_1 x^{2\ell}}} = \frac{\ell + 1}{\sqrt{2} E} \left[ \lim_{x \to +\infty} x F_1\left(-\left(V_1/E\right)x^{2\ell}\right) - Y F_1\left(-\left(V_1/E\right)Y^{2\ell}\right) \right]
\]  

(4.3.25)

\((\text{the limit } x \to +\infty \text{ indicated above is computed using the asymptotic expansion written in Eq. (4.3.19); the cited equation also defines the constant } C_t).\)

The time \( t_{\min} \) could be determined by similar computations, but is irrelevant for the subsequent applications.

Let us now pass to the function \( t \mapsto y(t) \), oscillating in the range indicated by Eq. (4.3.15). This function is well defined for all \( t \in (-\infty, +\infty) \), and periodic:

\[
y(t + T) = y(t) \quad \text{.}
\]  

(4.3.26)

The period \( T \) is twice the time needed by \( y(t) \) to pass from the minimum to the maximum of the interval in Eq. (4.3.15), and this time can be computed using the quadrature formula (4.3.18); this gives

\[
T = 2 \frac{\ell + 1}{\sqrt{2}} \int_{-(F/V_2)^{1/2(2\ell)}}^{(F/V_2)^{1/2(2\ell)}} \frac{dy}{\sqrt{F - V_2 y^{2\ell}}} = \frac{2\sqrt{2}(\ell + 1)F_1(1)}{\sqrt{F}(V_2/F)^{1/2(2\ell)}},
\]  

(4.3.27)

with \( F_1(1) \) given by Eq. (4.3.20). From here to the end of the present Section 4.3, \( x(t) \) and \( y(t) \) are the functions considered herein, i.e., the solutions of maximal domain of the Lagrange equations with initial data (4.3.21) and energies (4.3.12) (4.3.13).
Behavior of $x(t)$ for positive times. The limits $t \to 0^+$ and $t \to t_{\text{max}}^-$. For $t \in (0, t_{\text{max}})$ the function $x(t)$ increases, starting from the initial value $x(0) = Y > 0$ and ultimately diverging. The small $t$ behavior of $x(t)$ is determined by the smoothness of these functions and by the initial data (4.3.21), which of course imply

$$x(t) = Y + u t + O(t^2) \quad \text{and} \quad \dot{x}(t) = u + O(t) \quad \text{for} \ t \to 0^+. \quad (4.3.28)$$

On the other hand, the quadrature formula (4.3.17) with $\xi = +1$, $t_1 = 0$, $t_2 = t \in (0, t_{\text{max}})$ and $x(t_1) = Y$ gives

$$t = \frac{\ell + 1}{\sqrt{2}} \int_Y^{x(t)} \frac{dx}{\sqrt{E + V_1 x^{2\ell}}} = \frac{\ell + 1}{\sqrt{2} E} \left[ x(t) F_\ell \left( -\frac{V_1}{E} x^{2\ell} \right) - Y F_\ell \left( -\frac{V_1}{E} Y^{2\ell} \right) \right]. \quad (4.3.29)$$

From here and from Eq. (4.3.25) for $t_{\text{max}}$, we obtain

$$t_{\text{max}} - t = \frac{\ell + 1}{\sqrt{2} E} \left[ C_\ell \left( \frac{V_1}{E} \right)^{1/2t} x^{2\ell} \right] - x(t) F_\ell \left( -\frac{V_1}{E} x^{2\ell} \right) \quad \text{for all} \ t \in (0, t_{\text{max}}). \quad (4.3.30)$$

We know that $x(t) \to +\infty$ for $t \to t_{\text{max}}^-$; this fact, together with the asymptotics (4.3.19), entails

$$t_{\text{max}} - t = \frac{\ell + 1}{(\ell - 1) \sqrt{2} V_1} \frac{1}{x^{2\ell - 1}(t)} + O \left( \frac{1}{x^{2\ell - 1}(t)} \right), \quad (4.3.31)$$

whence

$$x(t) = \left( \frac{\ell + 1}{(\ell - 1) \sqrt{2} V_1} \right)^{\frac{1}{\ell - 1}} \frac{1}{(t_{\text{max}} - t)^{\frac{1}{\ell - 1}}} + O \left( \frac{1}{(t_{\text{max}} - t)^{\frac{2\ell - 1}{\ell - 1}}} \right) \quad \text{for} \ t \to t_{\text{max}}^-. \quad (4.3.32)$$

The above result also allows to determine the behavior of $\dot{x}(t)$ as $t$ approaches $t_{\text{max}}$. In fact, from Eq. (4.3.14) and from the positivity of $\dot{x}(t)$ we infer

$$\dot{x}(t) = \sqrt{2} \left( \frac{V_1}{E} \right)^{1/2t} x^{2\ell} \left( \frac{V_2}{V_1} \right), \quad (4.3.33)$$

Behavior of $y(t)$ for positive times. The limit $t \to 0^+$. Since $y(0) > 0$ (see Eq. (4.3.21)), $\dot{y}(t)$ will be positive from $t = 0$ up to the first positive inversion time

$$t_* := \min \left\{ t \in (0, +\infty) \mid \dot{y}(t) = 0 \right\} \quad (4.3.34)$$

at which $y$ attains its maximum value $y(t_*) = (\mathcal{F}/V_2)^{1/(2\ell)}$. The small $t$ behavior of $y(t)$ is determined by the same smoothness of considerations that, combined with the initial data (4.3.21), give

$$y(t) = Y + v t + O(t^2) \quad \text{and} \quad \dot{y}(t) = v + O(t) \quad \text{for} \ t \to 0^+. \quad (4.3.35)$$

On the other hand, using the quadrature formula (4.3.18) with $\sigma = +1$, $t_1 = 0$, $t_2 = t \in (0, t_*)$, $y(t_1) = Y$ we get

$$t = \frac{\ell + 1}{\sqrt{2}} \int_Y^{y(t)} \frac{dy}{\sqrt{3 - V_2 y^{2\ell}}} = \frac{\ell + 1}{\sqrt{2} \sqrt{3}} \left[ y(t) F_{\ell} \left( \frac{V_2}{V_1} y^{2\ell} \right) - Y F_{\ell} \left( \frac{V_2}{V_1} Y^{2\ell} \right) \right]. \quad (4.3.36)$$

To invert the asymptotic relation between $t_{\text{max}} - t$ and $x(t)$ one should note that, since $t_{\text{max}} - t = \frac{\text{const.}}{x^{2\ell}(t)} \left( 1 + O \left( x^{-2\ell}(t) \right) \right)$ for some nonzero constant, it follows $x(t) = \frac{\text{const.}}{(t_{\text{max}} - t)^{1/2\ell - 1}} \left( 1 + O \left( (t_{\text{max}} - t)^{2\ell/(\ell - 1)} \right) \right)$. 

66
Taking the limit \( t \to t_* \) in the last equation we get

\[
t_* = \frac{\ell + 1}{\sqrt{2}} \int_{Y}^{(V/V_2)^{1/2(\ell)}} \frac{dy}{\sqrt{F-V_2 y^{2\ell}}} = \frac{\ell + 1}{\sqrt{2} \ell} \left[ \frac{V}{V_2} \right]^{1/2(\ell)} F_{\ell}(1) - Y F_{\ell} \left( \left( \frac{V_2}{V} \right) Y^{2\ell} \right) .
\] (4.3.37)

After the inversion time \( t_* \), \( y(t) \) decreases until it reaches the subsequent inversion time, and so on. Note that, depending on the choices of the parameters, it can be \( t_* < t_{\text{max}} \), or \( t_{\text{max}} > t_* \), or even \( t_* = t_{\text{max}} \); in the sequel we will give explicit examples of the last two alternatives (see Eqs. (4.3.63), (4.3.64) and the related comments). When \( t_* < t_{\text{max}} \), \( y(t) \) has enough time to invert (at least once) its motion before the explosion of \( x(t) \).

In any case, the smoothness of \( y(t) \) for all \( t \in (-\infty, +\infty) \) ensures the finiteness of \( y(t_{\text{max}}), \dot{y}(t_{\text{max}}) \)

and gives the obvious relations

\[
y(t) = y(t_{\text{max}}) + O(t - t_{\text{max}}), \quad \dot{y}(t) = \dot{y}(t_{\text{max}}) + O(t - t_{\text{max}}) \quad \text{for} \ t \to t_{\text{max}} .
\] (4.3.38)

The condition \((x(t), y(t)) \in D\). We now claim that the functions \( t \mapsto x(t), y(t) \) fulfill

\[ x(t) > 0 \ \text{and} \ -x(t) < y(t) < x(t) \quad \text{for all} \ t \in (0, t_{\text{max}}) . \] (4.3.39)

This is means that, \((x(t), y(t)) \in D\) (see Eq. (4.3.30)) for all \( t \in (0, t_{\text{max}}) \), a mandatory requirement for associating to the motion \( t \mapsto (x(t), y(t)) \) a cosmological model via the transformation (4.3.3)-(4.3.4). For the proof of Eq. (4.3.39), we refer to Appendix D.

### 4.3.3 Big Bang analysis

From now on we consider the cosmology corresponding to the motion \( t \in (0, t_{\text{max}}) \mapsto (x(t), y(t)) \in D \) described in the preceding subsection.

The asymptotic behavior of \( x(t) \) and \( y(t) \) for \( t \to 0^+ \) is obviously determined by Eqs. (4.3.28), (4.3.35), which involve the initial conditions \( Y, u, v \) introduced in Eqs. (4.3.21)-(4.3.22); in the sequel we will frequently use the related quantity

\[
Z := (u - v) Y > 0 .
\] (4.3.40)

From Eqs. (4.3.6), (4.3.10), (4.3.11) we obtain the following, for \( t \to 0^+ \):

\[
a(t) = \left( 2 Z \right)^{\frac{\ell + 1}{4}} t^{\frac{\ell + 1}{2}} + O \left( t^{\frac{\ell + 1}{2}} \right) ;
\] (4.3.41)

\[
w^{(\phi)}(t) = 1 - \frac{8 (V_1 + V_2) X_{2\ell}}{(\ell + 1)^2 Z} t + O(t^2) ;
\] (4.3.42)

\[
\Omega^{(m)}(t) = \frac{2 n^2 \Omega^{(m)}_{\ell + 1} (t)}{(\ell + 1)^2 Z} t + O(t^2) .
\] (4.3.43)

It is convenient to describe the limit \( t \to 0^+ \) in terms of the cosmic time \( \tau \); this will be done starting from Eq. (3.3.7), that gives an integral representation for \( \tau(t) \). From here we obtain

\[
\tau(t)/\theta = \int_0^t dt' \left[ \left( 2 Z \right)^{\frac{\ell + 1}{4}} (t')^{\frac{\ell + 1}{2}} + O \left( (t')^{\frac{\ell + 1}{2}} \right) \right] = \frac{\left( 2 Z \right)^{\frac{\ell + 1}{4}}}{(\ell + 1)^2 Z} t^{\frac{\ell + 1}{2}} + O \left( \left( \frac{\ell + 1}{4} \right) \right) \quad \text{for} \ t \to 0^+ .
\] (4.3.44)

The above relation shows, in particular, that \( \tau(t) \to 0^+ \) for \( t \to 0^+ \). Considering the inverse function \( t \mapsto t(\tau) \), one readily checks that Eq. (4.3.44) implies

\[
t(\tau) = \frac{(\ell + 1)^{\frac{\ell + 1}{4}}}{2 Z^{\frac{\ell + 1}{4}}} \left( \tau/\theta \right)^{\frac{\ell + 1}{4}} + O \left( \left( \tau/\theta \right)^{\frac{\ell + 1}{4}} \right) \quad \text{for} \ \tau \to 0^+ .
\] (4.3.45)
that Eq. (4.3.52) implies the following, for \( \tau \) asymptotics (4.3.45) into Eqs. (4.3.41)-(4.3.43), we find the following for dominants on matter density.

\[
\Omega^{(m)}(\tau) = \frac{n^2 \Omega^{(m)}_\ell}{((\ell + 1)Z)^{2\over \ell+1}} (\tau/\theta)^{2\over \ell+1} + O((\tau/\theta)^{2\ell+1}) .
\]

From the above asymptotic expansions it appears that, for \( \tau \to 0^+ \): the scale factor \( a(\tau) \) vanishes, which indicates the occurrence of a Big Bang; the reciprocal \( 1/a(\tau) \) diverges in a non-integrable way if \( n = 2 \), indicating the absence of a particle horizon in the case of space dimension 2, while it diverges in an integrable way if \( n > 2 \), showing that the model has finite particle horizon when the space dimension is equal to 3 or greater; \( \Omega^{(m)}(\tau) \to 0 \), which on account of Eq. (4.3.5) proves that the field energy density dominates on matter density close to the Big Bang.

### 4.3.4 Behavior of the model in the far future

From Eqs. (4.3.6) (4.3.10) (4.3.11) and (4.3.12) (4.3.38) we infer that the scale factor, the equation of state coefficient for the field and the matter density parameter behave, respectively, as follows for \( t \to t_{max}^\tau \):

\[
a(t) = \left( \frac{\ell + 1}{(\ell - 1)\sqrt{2V_1}} \right) \frac{t_{max} - t}{t_{max}}^{\ell+1} \frac{(\ell+1)}{n(\ell+1)} + O(\frac{t_{max} - t}{t_{max}}) ;
\]

\[
w^{(\ell)}(t) = -1 + O(\frac{t_{max} - t}{t_{max}}) ;
\]

\[
\Omega^{(m)}(t) = n^2 \Omega^{(m)}_\ell (2V_1)^{\ell+1} \frac{(\ell+1)}{\ell+1} (t_{max} - t)^{-\ell+1} + O((t_{max} - t)^{-\ell+1}) .
\]

It is convenient to describe the limit \( t \to t_{max}^\tau \) in terms of the cosmic time. To this purpose, we must first determine the asymptotics of \( \tau(t) \) in this limit starting from the integral representation (4.3.7); since it is not so obvious how to proceed, we have given some detail on this computation in Appendix [D]. Here we only report the final result, which is

\[
\tau(t)/\theta = \log\frac{t_{max}}{t_{max} - t} + P + O\left((t_{max} - t)^{2\ell+1}\right) 
\]

for \( t \to t_{max}^\tau \),

\[
P := \int_0^{t_{max}} dt' \left[ (x^2(t') - y^2(t'))^{\ell+1} - \frac{\ell+1}{(\ell-1)\sqrt{2V_1}} (t_{max} - t')^{-1} \right].
\]

Thus \( \tau(t) \to +\infty \) for \( t \to t_{max}^\tau \). Considering the inverse function \( t \to t(\tau) \), it can be readily checked that Eq. (4.3.52) implies the following, for \( \tau \to +\infty \):

\[
t_{max} - t(\tau) = Q e^{-(\ell+1)\sqrt{2V_1}(\tau/\theta)} + O\left(e^{-\sqrt{2V_1}(\tau/\theta)}\right), \quad Q := t_{max} e^{-(\ell+1)\sqrt{2V_1}P} .
\]

Insert the above asymptotic relation into Eqs. (4.3.49)-(4.3.51), we find the following for \( \tau \to +\infty \):

\[
a(\tau) = \left( \frac{\ell + 1}{(\ell - 1)\sqrt{2V_1}Q} \right)^{\ell+1} \frac{(\ell+1)}{n(\ell+1)} e^{\sqrt{2V_1}(\tau/\theta)} + O\left(e^{-(\ell+1-2n)\sqrt{2V_1}Q}/n(\ell+1)\right) ;
\]

\[
w^{(\ell)}(\tau) = -1 + O\left(e^{-2\sqrt{2V_1}(\tau/\theta)}\right) ;
\]

\[
\Omega^{(m)}(\tau) = n^2 \Omega^{(m)}_\ell (2V_1)^{\ell+1} \frac{(\ell+1)}{\ell+1} Q e^{-2\sqrt{2V_1}(\tau/\theta)} + O\left(e^{-2\sqrt{2V_1}(\tau/\theta)}\right) .
\]

As we can see, for \( \tau \to +\infty \) the following phenomena occur with exponential speed: the scale factor diverges, the field behaves like a cosmological constant \( w^{(\ell)}(\tau) \to -1 \) and the field energy density dominates on matter density.

68
4.3.5 Some numerical examples

Let us now restrict the attention to physically realistic, 3-dimensional scenarios; to this purpose, we put (cf. Eq. (4.1.61))

\[ n = 3, \quad \Omega^m = 0.308. \tag{4.3.57} \]

In the sequel we consider two exemplary configurations corresponding, respectively, to the following choices of the parameters \( \ell, V_1, V_2 \) which characterize the potential \( V(\varphi) \) of Eq. (4.3.2):

\[
\ell = 2, \quad V_1 = 1, \quad V_2 = 0.1; \tag{4.3.58}
\]

\[
\ell = 2, \quad V_1 = 1, \quad V_2 = 10^3. \tag{4.3.59}
\]

Making reference to Eq. (4.3.12), we specify the energies of the two Lagrangian subsystems setting

\[
\mathcal{F} = 10, \quad \mathcal{E} = \mathcal{F} + \frac{n^2 \Omega^m}{2} = 11.386. \tag{4.3.60}
\]

Furthermore, keeping in mind the upper bound in Eq. (4.3.21) we choose

\[ Y = 0.1. \tag{4.3.61} \]

Let us remark that on account of Eqs. (4.3.21)–(4.3.22), the above choices (4.3.57)–(4.3.61) completely determine the initial data:

\[
x(0) = y(0) = Y = 0.1, \tag{4.3.62}
\]

\[
x'(0) = \sqrt{\frac{2}{\ell+1}} \sqrt{\mathcal{E} + V_1 Y^2} = 1.5906\ldots, \quad y'(0) = \sqrt{\frac{2}{\ell+1}} \sqrt{\mathcal{F} - V_2 Y^2} = \begin{cases} 1.4907\ldots & \text{for } V_2 = 0.1, \\ 1.4832\ldots & \text{for } V_2 = 10^3. \end{cases}
\]

Concerning the final time \( t_{\text{max}} \) and the inversion time \( t_s \) described, respectively, by Eqs. (4.3.25) and (4.3.31)–(4.3.37), let us point out that the two cases (4.3.58)–(4.3.59) (together with the other choices specified above) describe qualitatively different scenarios. In fact, \( t_{\text{max}} = 2.0782\ldots \) for \( V_1 = 1 \) (and any \( V_2 \)), while \( t_s = 2.7140\ldots \) for \( V_2 = 0.1 \) and \( t_s = 0.2109\ldots \) for \( V_2 = 10^3 \) (independently of \( V_1 \)); thus, we have

\[
t_{\text{max}} < t_s \quad \text{for } V_1 = 1, \quad V_2 = 0.1, \tag{4.3.63}
\]

\[
t_{\text{max}} > t_s \quad \text{for } V_1 = 1, \quad V_2 = 10^3. \tag{4.3.64}
\]

Figs. 16 and 17 show the plot of the Lagrangian coordinates \( x(t), y(t) \) for \( t \in (0, t_{\text{max}}) \). Especially, Fig. 17 makes evident that \( y(t) \) is strictly increasing for \( t \in (0, t_{\text{max}}) \) if \( \ell, V_1, V_2 \) are fixed as in Eq. (4.3.58), while \( y(t) \) oscillates if \( \ell, V_1, V_2 \) are as in Eq. (4.3.59). This fact is in agreement with the previous considerations related to Eqs. (4.3.63)–(4.3.64).
Figs. 18-29 represent the observables $a(\tau), w^{(\phi)}(\tau), \Omega^{(m)}(\tau)$ as functions of $\tau/\theta$. For each one of these observables we consider the choices \eqref{eq:4.3.57} \eqref{eq:4.3.60} \eqref{eq:4.3.61} and both choices \eqref{eq:4.3.58} \eqref{eq:4.3.59} for the involved parameters; in addition, for each one of the previous choices we consider two possible ranges for $\tau/\theta$, corresponding in terms of the coordinate time $t$ to the intervals $(0, t_{\text{max}}/2)$ (figures with an odd numbering) and $(0, \vartheta t_{\text{max}})$ (figures with an even numbering), with $\vartheta = 0.999$ so that $\tau/\theta \in (0, 12.8673 ...)$ in the case \eqref{eq:4.3.58} and $\tau/\theta \in (0, 14.0804 ...)$ in the case \eqref{eq:4.3.59}. (In fact, all these figures were obtained plotting the curves $t \mapsto (\tau(t)/\theta, a(t)), (\tau(t)/\theta, w^{(\phi)}(t)), (\tau(t)/\theta, \Omega^{(m)}(t))$ for $t \in (0, t_{\text{max}}/2)$ or $t \in (0, \vartheta t_{\text{max}})$).
Figure 24. $a(\tau)$ as a function of $\tau/\theta$, with parameters fixed as in Eqs. (4.3.57)-(4.3.60) and (4.3.59).

Figure 25. $a(\tau)$ as a function of $\tau/\theta$, with parameters fixed as in Eqs. (4.3.57)-(4.3.60) and (4.3.59).

Figure 26. $w^{(0)}(\tau)$ as a function of $\tau/\theta$, with parameters fixed as in Eqs. (4.3.57)-(4.3.60) and (4.3.59).

Figure 27. $w^{(0)}(\tau)$ as a function of $\tau/\theta$, with parameters fixed as in Eqs. (4.3.57)-(4.3.60) and (4.3.59).

Figure 28. $\Omega^{(m)}(\tau)$ as a function of $\tau/\theta$, with parameters fixed as in Eqs. (4.3.57)-(4.3.60) and (4.3.59).

Figure 29. $\Omega^{(m)}(\tau)$ as a function of $\tau/\theta$, with parameters fixed as in Eqs. (4.3.57)-(4.3.60) and (4.3.59).

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A Appendix. On the calculations of Section 2

Let us consider the spacetime metric (2.2.5) and the coordinate system (2.2.6) (recalling that Greek indexes range from 0 to n, while Latin indexes range from 1 to n). Moreover, we make all the assumptions stated in Section 2 about the scalar field and the matter fluid.

The metric $g_{\mu\nu}$. From Eq. (2.2.5) we infer (for $i,j = 1, \ldots, n$)

$$g_{00} = -\theta^2 e^{2B}, \quad g_{0i} = g_{i0} = 0, \quad g_{ij} = e^{2A/n} h_{ij}; \quad (A.1)$$

recall that $A,B$ are functions of $x^0 \equiv t$, while the $h_{ij}$’s are functions of the space coordinates $(x^i) \equiv \mathbf{x}$. For future use, let us mention that $g := \det(g_{\mu\nu})$ can be expressed as follows in terms of $h := \det(h_{ij}) > 0$:

$$g = -\theta^2 e^{2A+2B} h. \quad (A.2)$$

The Ricci tensor $R_{\mu\nu}$ and scalar curvature $R$. Given the metric (A.1), these are $(i,j = 1, \ldots, n)$

$$R_{00} = -\ddot{A} + \ddot{A}B - \frac{1}{n} \dot{A}^2, \quad R_{ij} = \left[ \frac{1}{n} e^{2A/n-2B} (\ddot{A} + \ddot{A}B) + (n-1) k \right] \frac{\dot{h}_{ij}}{\theta^2}, \quad (A.3)$$

$$R_{0i} = R_{i0} = 0; \quad R = \frac{e^{-2B}}{\theta^2} \left( 2\ddot{A} - 2\dot{A}B + \frac{n+1}{n} \dot{A}^2 \right) + \frac{n(n-1)k}{\theta^2} e^{-2A/n}. \quad$$

Note that $(n-1)k \dot{h}_{ij}/\theta^2 = (n-1)k \dot{h}_{ij}$ is the Ricci tensor of the Riemannian manifold $(M^\nu, h_{ij})$.

The stress-energy tensor for the scalar field. Eqs. (2.2.4), (2.2.9) imply

$$T^{(s)}_{\mu\nu} = \frac{n-1}{n \kappa^2_n} \left[ \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu} \partial_\alpha \varphi \partial^\alpha \varphi - \frac{1}{\theta^2} g_{\mu\nu} \mathcal{V}(\varphi) \right]. \quad (A.4)$$

Since $\varphi$ depends only on $t$ as indicated in (2.2.10) and $g_{\mu\nu}$ is as in (A.1), for $i,j = 1, \ldots, n$ we have

$$T^{(s)}_{00} = \frac{n-1}{n \kappa^2_n} \left( \frac{1}{2} \dot{\varphi}^2 + e^{2B} \mathcal{V}(\varphi) \right), \quad T^{(s)}_{0i} = T^{(s)}_{i0} = 0, \quad (A.5)$$

Comparing this result with Eq. (2.2.8) for $U_\mu$ and Eq. (A.1) for $g_{\mu\nu}$, it follows that $T^{(s)}_{\mu\nu}$ can be written in the fluid-like form

$$T^{(s)}_{\mu\nu} = (p^{(s)} + \rho^{(s)}) U_\mu U_\nu + \rho^{(s)} g_{\mu\nu}, \quad (A.6)$$

with $p^{(s)}$, $\rho^{(s)}$ as in Eq. (2.2.11).

Let us mention that, in the special case $\varphi(t) = \text{const.} = \varphi_0$, Eqs. (A.1), (A.5) (or, equivalently, Eqs. (2.2.11), (A.6)) entail $T^{(s)}_{\mu\nu} = -\frac{n-1}{n \kappa^2_n \theta^2} \mathcal{V}(\varphi_0) g_{\mu\nu}$, as stated in Eq. (2.2.27).

The stress-energy tensor for the matter fluid. This has the form (2.1.10), where the pressure and density are related by the equation of state $p^{(m)} = w \rho^{(m)}$ (see Eq. (2.1.10)). Using Eq. (2.2.8) for $U_\mu$ and Eq. (A.1) for $g_{\mu\nu}$, we obtain

$$T^{(m)}_{00} = \theta^2 e^{2B} \rho^{(m)}, \quad T^{(m)}_{0i} = T^{(m)}_{i0} = 0, \quad T^{(m)}_{ij} = w e^{2A/n} \rho^{(m)} h_{ij}; \quad (A.7)$$

recall that, according to (2.2.10), $\rho^{(m)}$ depends only on $t$.

The conservation law for the stress-energy tensor of the matter fluid. Let $\nabla_\mu$ be the covariant derivative associated to the metric (A.1); then, from Eq. (A.7) for $T^{(m)}_{\mu\nu}$ we get the following:

$$\nabla_\nu T^{(m)}_{\mu\nu} = -\mathcal{E}^{(m)}, \quad \nabla_\mu T^{(m)}_{\mu\nu} = 0; \quad \mathcal{E}^{(m)} := \dot{\rho}^{(m)} + (w + 1)A(t) \dot{A}. \quad (A.8)$$

So, the conservation law $\nabla_\mu T^{(m)}_{\mu\nu} = 0$ is equivalent to $\mathcal{E}^{(m)} = 0$; clearly, the latter relation holds if and only if $\rho^{(m)}(t) = \rho^{(m)}_0 e^{-(w+1)A(t)}$ for some constant $\rho^{(m)}_0$, which can be written in the form (2.2.13).
Einstein’s equations. Let us consider the Eistein equations (2.1.3), i.e.,
\[ E_{\mu\nu} := 0, \quad E_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \kappa^2 (T^{(\phi)}_{\mu\nu} + T^{(m)}_{\mu\nu}). \] (A.9)
From Eqs. (A.3), (A.5), (A.7) for \( R_{\mu\nu}, R, T^{(\phi)}_{\mu\nu}, T^{(m)}_{\mu\nu} \) and (2.2.13) for \( \rho^{(m)} \), we infer \( i,j = 1, \ldots, n \)
\[ E_{00} = \frac{n-1}{n} \mathcal{E}, \quad E_{00} = E_{i0} = 0, \quad E_{ij} = \frac{1}{n \theta^2} e^{2A/n-2B} \mathfrak{A} h_{ij}, \] (A.10)
with \( \mathfrak{A}, \mathcal{E} \) as in Eqs. (2.2.14), (2.2.15). Thus, if we assume Eq. (2.2.13) for \( \rho^{(m)} \) (in agreement with the conservation law for \( T^{(m)}_{\mu\nu} \)), Einstein’s equations are equivalent to \( \mathfrak{A} = 0, \mathcal{E} = 0 \).

The evolution equation for the scalar field. According to Eq. (2.1.6), this is \( \square \phi - V'(\phi) = 0 \). Let us recall that \( \square \phi = \frac{1}{\sqrt{|g|}} \partial_{\mu}(\sqrt{|g|} g^{\mu\nu} \partial_{\nu} \phi) \) with \( g := \det(g_{\mu\nu}) \). Expressing the metric and the corresponding determinant as in Eqs. (A.1), (A.2), the field \( \phi \) with the potential \( V \) as in Eq. (2.2.9) (involving the dimensionless equivalents \( \varphi, V \)) and finally recalling that \( \varphi \) depends only on \( t \), we obtain
\[ \square \phi - V'(\phi) = -\sqrt{\frac{n-1}{n}} \frac{\phi e^{-2B}}{\kappa^2 \theta^2} \mathfrak{F}, \] (A.11)
with \( \mathfrak{F} \) as in Eq. (2.2.16). Thus, the field equation (2.1.6) is equivalent to \( \mathfrak{F} = 0 \).

Evaluating the action functional on a history as above. Let us consider the action functional \( SS \) defined by Eq. (2.1.2), and evaluate it on a history of the type considered in the previous paragraphs, so that: the spacetime metric \( g_{\mu\nu} \) is given by Eq. (A.1) (implying Eqs. (A.2) (A.3) for the determinant \( g \) and the scalar curvature \( R \)); the scalar field \( \phi \) depends only on \( t \) and is represented with the related potential as in Eq. (2.2.9); the matter density is given by Eq. (2.2.13). In this way we obtain (with \( h := \det(h_{ij}) \))
\[ SS = \frac{1}{\kappa^2 \theta} \int dt \, d^n x \sqrt{h(x)} \left[ e^{A-B} \left( \dot{\varphi} - \dot{A} \varphi + \frac{n+1}{2n} \dot{A}^2 + \frac{n-1}{2n} \dot{\varphi}^2 \right) \right. \\
\left. - \frac{n-1}{n} e^{A+B} V(\varphi) - \frac{n(n-1)}{2} \Omega^{(\omega)} e^{-\omega A+B} + \frac{n(n-1)}{2} k e^{n-2 A+B} \right]; \] (A.12)
and it is straightforward to put the above result in the form (2.3.2).

B Appendix. On the explicit solutions of subsection 4.2

This appendix contains some details on the derivation of the explicit expressions reported in subsection 4.2 for the solutions \( x(t), y(t) \) of the Lagrange equations (4.2.5) (4.2.6). As in the cited subsection, we proceed to discuss separately the cases where the parameter \( \varepsilon \) defined according to Eq. (4.2.3) takes the values \(-1, 0, +1\).

B.1 The case \( \varepsilon = -1 \)

To begin with, let us recall that in this case Eqs. (4.2.5) (4.2.6) reduce, respectively, to Eqs. (4.2.10) (4.2.11). By elementary arguments one infers that any positive solution of Eq. (4.2.10) can be written in one of the following forms, after possibly a time translation \( t \to t + \text{const.} \) and a time reflection \( t \to -t \):
\[ x(t) = A \sinh(\omega t) \quad \text{for } A > 0 \text{ and } t \in (0, +\infty); \quad (B.1) \]
\[ x(t) = A \cosh(\omega t) \quad \text{for } A > 0 \text{ and } t \in \mathbb{R}; \quad (B.2) \]
\[ x(t) = A e^{\omega t} \quad \text{for } A > 0 \text{ and } t \in \mathbb{R}. \quad (B.3) \]
Supposing that the time coordinate $t$ has been fixed so that the solution of Eq. (4.2.10) takes one of the forms (B.1) (B.2) (B.3), we proceed to determine the corresponding solution of Eq. (4.2.11) (on the intervals mentioned above) starting from the familiar representation

$$y(t) = C \cosh(\omega t) + D \sinh(\omega t)$$

$$+ \frac{1}{\omega} \int_0^t ds \ \sinh(\omega(t-s)) \left[(1-w)V_2 x(s) \frac{w}{1-w} - \frac{1+3w}{2} \Omega_1 x(s) \right], \quad \text{(B.4)}$$

where $C, D \in \mathbb{R}$ are arbitrary constants. This representation is understood to hold for all values of $w \in \mathbb{R}\{\pm1\}$ granting the convergence of the integral over $s \in (0,t)$; nonetheless, we shall see later that the final result can be extended to other values of $w$ by analytic continuation (see, especially, the remark at the end of this subsection).

Of course, the evaluation of the integral in Eq. (B.4) with $x$ given by one of Eqs. (B.1) (B.2) (B.3) can be reduced to the computation of the following integrals, for $\eta = \frac{1-w}{1+w}$ or $\eta = -\frac{1+3w}{1+w}$:

$$\frac{1}{\omega} \int_0^t ds \ \sinh(\omega(t-s)) \sinh^\eta(\omega t)$$

$$= \frac{1}{\omega} \int_0^t ds \ \cosh(\omega t) \sinh^\eta(\omega t)$$

$$= \frac{1}{\omega^2} \left( \eta + 1 \right) - \frac{\cosh(\omega t) \sinh^\eta(\omega t) }{2 \omega^2} \int_0^1 dv \ \frac{v^{\eta/2}}{\sqrt{1 + \sinh^2(\omega t) v}}; \quad \text{(B.5)}$$

On the one hand, by means of basic trigonometric identities, for any $\omega, t > 0$ and $\eta > -1$ we infer

$$\frac{1}{\omega} \int_0^t ds \ \sinh(\omega(t-s)) \ cosp^\eta(\omega t)$$

$$= -\frac{\cosh(\omega t) }{\omega} \int_0^t ds \ \sinh(\omega t) \ \cosin^\eta(\omega t) + \frac{\sinh(\omega t) }{\omega} \int_0^t ds \ \cosin^\eta(\omega t)$$

$$= \frac{\cosh(\omega t) }{\omega^2} \left(1 - \cosin^\eta(\omega t) \right) + \frac{\sinh^2(\omega t) }{2 \omega^2} \int_0^1 dv \ \frac{(1 + \sinh^2(\omega t) v)^{\eta/2}}{\sqrt{v}}. \quad \text{(B.7)}$$

The integrals over $v \in (0,1)$ appearing in Eqs. (B.6) (B.7) can be expressed in terms of hypergeometric functions $\text{F}1(\alpha,\beta,\gamma; z)$, recalling the integral identity (see, e.g., [22] Eqs. 15.1.2 and 15.6.1):

$$\int_0^1 dv \ \frac{v^{\beta-1}(1-v)^{\gamma-1}}{(1-z v)^{\alpha}} = \frac{\Gamma(\beta) \Gamma(\gamma-\beta)}{\Gamma(\gamma)} \text{F}1(\alpha,\beta,\gamma; z) \quad \text{for } \alpha, \beta, \gamma, z \in \mathbb{R} \text{ with } \gamma > \beta > 0. \quad \text{(B.8)}$$

Using the above identity with $\alpha = 1/2$, $\beta = 1 + \eta/2$, $\gamma = 2 + \eta/2$ and $z = -\sinh^2(\omega t)$ (along with the basic relations $\Gamma(1) = 1$, $\Gamma(2 + \eta/2) = (1 + \eta/2) \Gamma(1 + \eta/2)$), from Eq. (B.6) we infer

$$\frac{1}{\omega} \int_0^t ds \ \sinh(\omega(t-s)) \ \cosin^\eta(\omega t)$$

$$= \frac{\sinh^\eta(\omega t)}{\omega^2} \left[ \frac{1}{\eta + 1} - \frac{\cosh(\omega t)}{\eta + 2} \right] \text{F}1(\frac{1}{2}, \frac{1 + \eta}{2}, 2 + \eta; -\sinh^2(\omega t)) \quad \text{. (B.9)}$$

The condition $\eta > -1$ is necessary to ensure the convergence of the integrals in Eq. (B.6). Besides, let us mention that the last identity in the cited equation follows making the change of variable $s = \frac{1}{\eta} \arcsinh(\sinh(\omega t) \sqrt{\gamma})$ in the second integral of the preceding expression. The same change of variable is used to derive the last identity in Eq. (B.7).
likewise, using the identity Eq. (B.8) with \( \alpha = -\eta/2, \beta = 1/2, \gamma = 3/2 \) and \( z = -\sinh^2(\omega t) \) (along with the relations \( \Gamma(1/2)/\Gamma(3/2) = 2 \) and \( 2F_1(\beta, \alpha, \gamma; z) = 2F_1(\alpha, \beta, \gamma; z) \)), from Eq. (B.7) we infer

\[
\frac{1}{\omega} \int_0^t ds \frac{\sinh(\omega (t-s)) \cosh(\eta (\omega s))}{(\cosh(\omega t) - \cosh^{\eta+1}(\omega t))} + \frac{\sinh^2(\omega t) \ 2F_1\left(\frac{1}{2}, -\eta, \frac{3}{2}; -\sinh^2(\omega t)\right)}{\eta + 1} = \frac{1}{\omega^2} \left[ \cosh(\omega t) \left(1 - \cosh^\eta(\omega t)\right) \right]. \tag{B.10}
\]

On the other hand, for any \( \omega, t > 0 \) and \( \eta \in \mathbb{R} \), by direct computations we get

\[
\frac{1}{\omega} \int_0^t ds \sinh(\omega (t-s)) e^{\eta \omega s} = \frac{\cosh(\omega t) + \eta \sinh(\omega t) - e^{\eta \omega t}}{(1 - \eta^2)\omega^2}. \tag{B.11}
\]

Let us remark that the right-hand sides of Eqs. (B.10) (B.11) must be intended in a natural limit sense for \( \eta = \pm 1 \); more precisely, we understand that

\[
\left. \frac{1 - \cosh^{-1}(\omega t)}{1 + \eta} \right|_{\eta = 1} := \lim_{\eta \to 1} \frac{1 - \cosh^{-1}(\omega t)}{1 + \eta} = -\log \left( \cosh(\omega t) \right), \tag{B.12}
\]

\[
\left. \frac{\cosh(\omega t) + \eta \sinh(\omega t) - e^{\eta \omega t}}{1 - \eta^2} \right|_{\eta = \pm 1} := \lim_{\eta \to \pm 1} \frac{\cosh(\omega t) + \eta \sinh(\omega t) - e^{\eta \omega t}}{1 - \eta^2} = \frac{\omega t e^{\pm \omega t} - \sinh(\omega t)}{2}. \tag{B.13}
\]

Summing up, Eqs. (B.1) (B.2) (B.3) for \( x(t) \) and the corresponding expressions for \( y(t) \) descending from Eqs. (B.4) and (B.9) (B.10) (B.11) give rise to the explicit solutions \( \{1.2.12\} \{1.2.13\} \{1.2.14\} \) reported in the main text. Correspondingly, Eqs. (B.12) (B.13) account for Eqs. \( \{4.2.16\} \{4.2.17\} \).

**A final remark on the solution \( \{4.2.12\} \).** The arguments presented above for the derivation of the solution \( \{4.2.12\} \) rely on the use of identity (B.9) for \( \eta = \pm \frac{\pm 3w}{2+2h} \) and \( \eta = -\frac{\pm 3w}{2+2h} \). As noted previously, the integral in the cited identity converges only for \( \eta > -1 \); consequently, the expression \( \{4.2.12\} \) for \( y(t) \) would hold in principle only for \( \frac{\pm w}{1+2h} > -1 \) and \( -\frac{\pm w}{1+2h} > -1 \), which happens if and only if \( -1 < w < 0 \). However for our purposes, it suffices that the expression \( \{4.2.12\} \) for \( y(t) \) provides a solution of Eq. \( \{4.2.11\} \); having proven that this occurs for \( -1 < w < 0 \), by elementary consideration based on analytic continuation we can infer that the same holds on the entire region of analyticity w.r.t. \( w \). Regarding this region, let us recall that for any fixed \( z \in (\infty, 1) \), \( 2F_1(\alpha, \beta, c; z) \) is analytic w.r.t. the parameters \( \alpha, \beta, c \) for \( \alpha, \beta, c \in \mathbb{R} \) and \( c \in \mathbb{R} \setminus \{0, -1, -2, \ldots\} \) (see, e.g., [22]). In Eq. \( \{4.2.12\} \) there appear two hypergeometric terms with \( c = \frac{3+3w}{2+2h} \) and \( c = \frac{3+3w}{2+2h} + 1 \), which are both different from 0, -1, -2, ... if and only if there holds the condition stated in Eq. \( \{4.2.12\} \), namely,

\[
w \neq -\frac{3+2h}{1+2h} \quad \text{for all } h \in \{0, 1, 2, \ldots\}. \tag{B.14}
\]

**B.2 The case \( \epsilon = 0 \)**

Recall that in this case \( V_1 = 0 \) and Eqs. \( \{4.2.5\} \{4.2.6\} \) are respectively equivalent to Eqs. \( \{4.2.7\} \{4.2.8\} \); the corresponding solutions can be derived by elementary means, paying due attention to some critical value of the parameter \( w \).

It appears that, after a time translation \( t \to t + \text{const.} \) and possibly a time reflection \( t \to -t \), any positive solution of Eq. \( \{4.2.7\} \) can be written in one of the following ways:

\[
x(t) = A t \quad \text{for } A > 0 \text{ and } t \in (0, +\infty); \tag{B.1}
\]

\[
x(t) = A \quad \text{for } A > 0 \text{ and } t \in (-\infty, +\infty). \tag{B.2}
\]
The related solutions of Eq. (4.2.28) can be easily derived via the following integral representation, evaluating the basic integrals which result upon substitution of the expressions (B.1)–(B.2) for $x(t)$:

$$y(t) = C_0 + D_0 + \frac{1}{t_0} \int_0^t ds \int_0^s ds' \left[ (1 - w) V_2 x(s')^{1 + w} - \frac{n^2}{2} (1 - w) \omega \Omega_s^{(m)} x(s')^{1 + w} \right], \quad (B.3)$$

where $C_0, D_0 \in \mathbb{R}$ are integration constants and $t_0 \in (0, +\infty)$ is fixed arbitrarily.

On the one hand, Eqs. (B.1)–(B.3) imply, for any $w \in \mathbb{R} \setminus \{ \pm 1 \}$ such that $w \neq -3$,

$$y(t) = C + D t + \frac{V_2 (1 + w) (1 - w)}{2 (3 + w)} A^{1 + w} t_0^{1 + w} + \frac{n^2 (1 + w)^2 \Omega_s^{(m)}}{4} A^{-1 + 3 w} t_0^{1 + w},$$

$$C := C_0 + \frac{V_2 (1 - w)}{3 + w} A^{1 + w} t_0^{1 + w} - \frac{2 n^2 (1 + w) \omega \Omega_s^{(m)}}{4} A^{-1 + 3 w} t_0^{1 + w},$$

$$D := D_0 - \frac{V_2 (1 - w) ^2}{2} A^{1 + w} t_0^{1 + w} - \frac{n^2 (1 - w)^2 \Omega_s^{(m)}}{4} A^{-1 + 3 w} t_0^{1 + w}; \quad (B.4)$$

the exceptional case $w = -3$ (not so relevant for physically realistic applications) must be treated separately and gives

$$y(t) = C + D t - \frac{4 V_2}{A^2} (\log t_0 - 1) - \frac{3 n^2 \Omega_s^{(m)}}{A^4 t_0^2} + \frac{4 V_2}{A^2 t_0} + \frac{2 n^2 \Omega_s^{(m)}}{A^4 t_0^2}. \quad (B.5)$$

On the other hand, Eqs. (B.2)–(B.3) imply, for any $w \in \mathbb{R} \setminus \{ \pm 1 \}$,

$$y(t) = C + D t + \frac{t^2}{2} \left( (1 - w) V_2 A^{1 + w} - \frac{n^2}{2} (1 - w) \omega \Omega_s^{(m)} A^{-1 + 3 w} \right),$$

$$C := C_0 + \frac{t^2}{2} \left( (1 - w) V_2 A^{1 + w} - \frac{n^2}{2} (1 - w) \omega \Omega_s^{(m)} A^{-1 + 3 w} \right),$$

$$D := D_0 - t_0 \left( (1 - w) V_2 A^{1 + w} - \frac{n^2}{2} (1 - w) \omega \Omega_s^{(m)} A^{-1 + 3 w} \right). \quad (B.6)$$

The expressions (B.1)–(B.2) for $x(t)$ and (B.4)–(B.5) for $y(t)$ are patently equivalent to the explicit solutions (4.2.29)–(4.2.30) reported in the main text.

### B.3 The case $\varepsilon = +1$

Recall that in this case Eqs. (4.2.31)–(4.2.36) reduce, respectively, to Eqs. (4.2.37)–(4.2.42). By direct inspection it appears that any positive solution of Eq. (4.2.37) can be written as follows, after a time translation $t \rightarrow t + \text{const.}$:

$$x(t) = A \sin(\omega t) \quad \text{for } A > 0 \quad \text{and } t \in (0, \pi/\omega). \quad (B.1)$$

For the general solution of Eq. (4.2.38) (on the interval mentioned above), we have the familiar representation

$$y(t) = C \cos(\omega t) + D \sin(\omega t) + \frac{1}{\omega} \int_0^t ds \sin (\omega (t-s)) \left[ V_2 (1 - w) x(s)^{1 + w} - \frac{n^2 w (1 - w)}{2} \Omega_s^{(m)} x(s)^{1 + w} \right], \quad (B.2)$$

where $C, D \in \mathbb{R}$ are integration constants. This representation is understood to hold for all values of $w \in \mathbb{R} \setminus \{ \pm 1 \}$ granting the convergence of the integral over $s \in (0, t)$; however, also in this case the final result can be extended to other values of $w$ (see the remark at the end of this subsection).
The calculation of the integral in Eq. (B.2) with \( x \) as in Eq. (B.1) is reduced to the evaluation of the subsequent integral, for \( \eta = \frac{1-w}{1+w} \) or \( \eta = -\frac{1+3w}{1+w} \):

\[
\frac{1}{\omega} \int_0^t ds \sin(\omega (t-s)) \sin^\eta(\omega s) .
\] (B.3)

By arguments similar to those mentioned in subsection B.1, for any \( \omega > 0 \), \( 0 < t < \pi/\omega \) and \( \eta > -1 \) we have \(^{29}\)

\[
\frac{1}{\omega} \int_0^t ds \sin(\omega (t-s)) \sin^\eta(\omega s) \\
= \frac{\sin(\omega t)}{\omega} \int_0^t ds \cos(\omega s) \sin^\eta(\omega s) - \frac{\cos(\omega t)}{\omega} \int_0^t ds \sin^{\eta+1}(\omega s) \\
= \frac{\sin^{\eta+2}(\omega t)}{\omega^2 (\eta+1)} - \frac{\cos(\omega t) \sin^{\eta+2}(\omega t)}{2 \omega^2} \int_0^1 dv \frac{v^{\eta/2}}{\sqrt{1 - \sin^2(\omega t) v}} .
\] (B.4)

Also in this case, the remaining integral can be expressed in terms of the hypergeometric function \(_2F_1\), resorting to the identity (B.8). More precisely, employing the cited identity with \( \alpha = 1/2, \beta = 1 + \eta/2, \gamma = 2 + \eta/2 \) and \( z = \sin^2(\omega t) \) (along with the basic relations \( \Gamma(1) = 1, \Gamma(2 + \eta/2) = (1 + \eta/2) \Gamma(1 + \eta/2) \)), we obtain

\[
\frac{1}{\omega} \int_0^t ds \sin(\omega (t-s)) \sin^\eta(\omega s) \\
= \frac{\sin^{\eta+2}(\omega t)}{\omega^2} \left[ \frac{1}{\eta+1} - \frac{\cos(\omega t)}{\eta+2} \, _2F_1 \left( \frac{1}{2} ; \frac{1}{\eta+2} ; 2 + \frac{\eta}{2} ; \sin^2(\omega t) \right) \right] .
\] (B.5)

Eq. (B.1) for \( x(t) \) and the corresponding expression for \( y(t) \) deduced from Eqs. (B.2) and (B.5) give rise to the explicit solution (4.2.39) reported in the main text.

**A final remark on the solution (4.2.39).** Considerations analogous to those reported in the concluding remark of subsection B.1 allow us to infer that, although in principle the expression (4.2.39) for \( y(t) \) would hold only for \( -1 < w < 0 \), a posteriori it can understood to hold for any \( w \neq \frac{-1+3h}{1+2h} \) (\( h = 0, 1, 2, \ldots \)) (cf. Eq. (B.1))

**C Appendix. Derivation of upper and lower bounds for \( \tau(t)/\theta \) from the integral representation (4.2.39)**

Let us refer to the framework of subsection 4.2.3 and consider the expression for cosmic time given in Eq. (4.2.39). With an obvious change of integration variable, this can be re-written as

\[
\frac{\tau(t)}{\theta} = \frac{1}{\omega} \int_0^\omega ds \sqrt{1 + \zeta \cosh s / \sinh s} \text{ for } t \in (0, +\infty) .
\] (C.1)

From here to the end of this Appendix we assume \( \zeta > 0 \). Our goal is to derive from Eq. (C.1) upper and lower bounds \( T_\zeta^\pm(t) \) for \( \tau(t)/\theta \), expressed via elementary functions. To this purpose let us introduce a pair of functions, defined as follows for \( z \in (0, +\infty) \):

\[
P(z) := \sqrt{\sqrt{z} + z} \left( \frac{1}{2} + \sqrt{z} \right) - \frac{1}{4} \log \left( 1 + 2 \sqrt{z} + 2 \sqrt{\sqrt{z} + z} \right) ; \quad (C.2)
Q(z) := 2 \sqrt{z} + 1 + \log \frac{\sqrt{z} + 1 - 1}{\sqrt{z} + 1 + 1} . \quad (C.3)
\]

\(^{29}\)The condition \( \eta > -1 \) is required to ensure the convergence of the integral in Eq. (B.3). Correspondingly, let us note that the last identity in the cited equation can be derived making the change of variable \( s = \frac{1}{2} \arccos(\sqrt{1 - \sin^2(\omega t) v}) \)
To go on, let us fix two real parameters $\ell, L$ such that

$$0 < \ell < \log \left(1 + \sqrt{2}\right) < L < \infty,$$  \hspace{1cm} (C.4)

and set

$$M_{\ell,L} := \max \left\{ \frac{\cosh \ell}{\sqrt{\sinh \ell}}, \frac{\cosh L}{\sqrt{\sinh L}} \right\}.$$  \hspace{1cm} (C.5)

Finally let us define two continuous, piecewise smooth functions $T_{\zeta,\ell}^\pm$ on $(0, +\infty)$, setting:

$$T_{\zeta,\ell}^- (t) := \begin{cases} \frac{\zeta^2}{\ell} P_{\ell, t} \left( \frac{\omega \sinh \ell}{\zeta^2 \ell \cosh^2 \ell} t \right) & \text{for } 0 < t \leq \ell/\omega, \\ \frac{\zeta^2}{\ell} P_{\ell, t} \left( \frac{\sinh \ell}{\ell \cosh^2 \ell} t \right) & \text{for } \ell/\omega < t \leq L/\omega, \\ \frac{\zeta^2}{\ell} P_{\ell, t} \left( \frac{\ell}{\ell \cosh^2 \ell} t \right) - \frac{1}{\omega^2} \left[ Q \left( \frac{\zeta}{\sqrt{2}} e^{t/2} \right) - Q \left( \frac{\zeta}{\sqrt{2}} e^{L/2} \right) \right] & \text{for } L/\omega < t < +\infty, \end{cases}$$  \hspace{1cm} (C.6)

$$T_{\zeta,\ell}^+ (t) := \begin{cases} \frac{\zeta^2}{\ell} P_{\ell, t} \left( \frac{\omega \sinh \ell}{\zeta^2 \ell \cosh^2 \ell} t \right) & \text{for } 0 < t \leq \ell/\omega, \\ \frac{\zeta^2}{\ell} P_{\ell, t} \left( \frac{\sinh \ell}{\ell \cosh^2 \ell} t \right) & \text{for } \ell/\omega < t \leq L/\omega, \\ \frac{\zeta^2}{\ell} P_{\ell, t} \left( \frac{\ell}{\ell \cosh^2 \ell} t \right) - \frac{1}{\omega^2} \left[ Q \left( \frac{\zeta \cosh L}{\sqrt{\sinh L}} e^{(\ell t - L)/2} \right) - Q \left( \frac{\zeta \cosh L}{\sqrt{\sinh L}} e^{L/2} \right) \right] & \text{for } L/\omega < t < +\infty. \end{cases}$$  \hspace{1cm} (C.7)

We now claim that

$$T_{\zeta,\ell}^- (t) \leq \tau(t)/\theta \leq T_{\zeta,\ell}^+ (t) \quad \text{for } t \in (0, +\infty).$$  \hspace{1cm} (C.8)

Most of this Appendix is devoted to the proof of this statement. After the end of the proof, in the last two paragraphs of the Appendix we discuss the asymptotics of $T_{\zeta,\ell}^\pm (t)$ for small and large $t$, and present a numerical appreciation of these upper and lower bounds (related to the inflationary model discussed in the final paragraph of subsection 12.2.3).

**Preliminaries for the proof of (C.8).** Let us consider the function appearing under the square root in Eq. (C.1), namely

$$J : (0, +\infty) \to (0, +\infty), \hspace{1cm} J(s) := \frac{\cosh s}{\sqrt{\sinh s}}.$$  \hspace{1cm} (C.9)

It can be easily checked that $J$ is a convex function, attaining its global minimum at a point $s_\ast \in (0, +\infty)$; more precisely, we have (asinh is the inverse hyperbolic sine function)

$$s_\ast := \text{asinh}(1) = \log \left(1 + \sqrt{2}\right), \hspace{1cm} J(s_\ast) = \min_{s \in (0, +\infty)} J(s) = \sqrt{2}. \hspace{1cm} (C.10)$$

Let us note that $s_\ast$ appears in the inequalities (C.4) about $\ell, L$. The following bounds can be deduced by elementary arguments \[^{30}\]:

$$\frac{1}{\sqrt{s}} \leq J(s) \leq \sqrt{\frac{\ell}{\sinh \ell}} \frac{1}{\sqrt{s}}, \hspace{1cm} \text{for } 0 < s \leq \ell; \hspace{1cm} (C.11)$$

$$\sqrt{2} \leq J(s) \leq M_{\ell,L}, \hspace{1cm} \text{for } \ell \leq s \leq L; \hspace{1cm} (C.12)$$

$$\frac{1}{\sqrt{2}} e^{s/2} \leq J(s) \leq \frac{\cosh L}{e^{L/2} \sqrt{\sinh L}} e^{s/2}, \hspace{1cm} \text{for } L \leq s < +\infty. \hspace{1cm} (C.13)$$

[^{30}]: Let us give a few more details on the derivation of Eqs. (C.11), (C.12), (C.13). To prove Eq. (C.11) it suffices to note that the map $s \in (0, +\infty) \to \sqrt{s} J(s)$ is strictly increasing and further fulfills $\sqrt{s} J(s) \to 1^+$ for $s \to 0^+$. Eq. (C.12) follows straightforwardly from the previously mentioned features of $J(s)$. Finally, to infer Eq. (C.13) just notice that the map $s \in (0, +\infty) \to e^{-s/2} J(s)$ is strictly decreasing and such that $e^{-s/2} J(s) \to (1/\sqrt{2})^+ s \to +\infty$. 

78
We now proceed to prove Eq. (C.8), analyzing separately the cases \(0 < t \leq \ell/\omega\), \(\ell/\omega < t \leq L/\omega\) and \(L < t \leq +\infty\).

**Proof of (C.8) for \(0 < t \leq \ell/\omega\).** For the said values of \(t\), from Eqs. (C.1) (C.11) we readily infer

\[
\frac{1}{\omega} \int_0^{\omega t} ds \sqrt{1 + \frac{\zeta}{\sqrt{s}}} \leq \tau(t)/\theta \leq \frac{1}{\omega} \int_0^{\omega t} ds \sqrt{1 + \frac{\ell \cosh^2 \ell}{\sinh \ell} \frac{\zeta}{\sqrt{s}}} ,
\]

which by obvious changes of the integration variables can be rephrased as

\[
\frac{\zeta^2}{\omega} \int_0^{\omega t/\omega^2} d\sigma \sqrt{1 + \frac{1}{\sqrt{\sigma}}} \leq \tau(t)/\theta \leq \frac{\zeta^2 \ell \cosh^2 \ell}{\omega \sinh \ell} \int_0^{(\omega t \sinh \ell)/(\zeta^2 \ell \cosh^2 \ell)} d\sigma \sqrt{1 + \frac{1}{\sqrt{\sigma}}} .
\]

Then, noting the basic identity

\[
\int_0^{\sqrt{z}} d\sigma \sqrt{1 + \frac{1}{\sqrt{\sigma}}} = \sqrt{\sqrt{z} + z} \left(\frac{1}{2} + \sqrt{\frac{1}{2}}\right) - \frac{1}{4} \log \left(1 + 2 \sqrt{\sqrt{z} + z} + \sqrt{\sqrt{z} + z}\right) \quad \text{for any \(z > 0\)} ,
\]

and recalling the definition of \(P\) given in Eq. (C.2), we obtain

\[
\frac{\zeta^2}{\omega} P\left(\frac{\omega}{\zeta^2} t\right) \leq \tau(t)/\theta \leq \frac{\zeta^2 \ell \cosh^2 \ell}{\omega \sinh \ell} P\left(\frac{\omega \sinh \ell}{\zeta^2 \ell \cosh^2 \ell} t\right) \quad \text{for \(0 < t \leq \ell/\omega\)} .
\]

The upper and lower bounds in Eq. (C.17) are just \(T^\pm_\zeta(t)\).

**Proof of (C.8) for \(\ell/\omega < t \leq L/\omega\).** Splitting the integral in the representation (C.1) for \(\tau(t)/\theta\) at \(s = \ell\) and taking into account the bounds (C.11) (C.12), for the values of \(t\) under analysis we obtain

\[
\frac{1}{\omega} \int_0^{\ell} ds \sqrt{1 + \frac{\zeta}{\sqrt{s}}} + \frac{1}{\omega} \int_\ell^{\omega t} ds \sqrt{1 + \zeta \sqrt{2}} \leq \tau(t)/\theta \leq \frac{1}{\omega} \int_0^{\ell} ds \sqrt{1 + \frac{\ell \cosh^2 \ell}{\sinh \ell} \frac{\zeta}{\sqrt{s}}} + \frac{1}{\omega} \int_\ell^{\omega t} ds \sqrt{1 + \zeta M_{t,L}} .
\]

Now, evaluating the integrals for \(s \in (0, \ell)\) with the same methods presented in the previous paragraph, and computing explicitly the trivial integrals for \(s \in (\ell, \omega t)\) we readily get

\[
\frac{\zeta^2}{\omega} P\left(\frac{\ell}{\zeta^2} t\right) + \sqrt{1 + \zeta \sqrt{2}} \left(1 - \frac{\ell}{\omega}\right) \leq \tau(t)/\theta \leq \frac{\zeta^2 \ell \cosh^2 \ell}{\omega \sinh \ell} P\left(\frac{\sinh \ell}{\zeta^2 \ell \cosh^2 \ell} t\right) + \sqrt{1 + \zeta M_{t,L}} \left(1 - \frac{\ell}{\omega}\right) \quad \text{for \(\ell/\omega \leq t \leq L/\omega\)} .
\]

The upper and lower bounds in Eq. (C.19) are just \(T^\pm_\zeta(t)\).

**Proof of (C.8) for \(L/\omega < t < +\infty\).** In this case, let us separate the integral in Eq. (C.1) at \(s = \ell\) and at \(s = L\); then, recalling the bounds (C.11) (C.12) (C.13), for the considered values of \(t\) we obtain

\[
\frac{1}{\omega} \int_0^{\ell} ds \sqrt{1 + \frac{\zeta}{\sqrt{s}}} + \frac{1}{\omega} \int_\ell^{L} ds \sqrt{1 + \zeta \sqrt{2}} + \frac{1}{\omega} \int_L^{\omega t} ds \sqrt{1 + \frac{\zeta}{\sqrt{2}}} e^{s/2} \leq \tau(t)/\theta \leq \frac{1}{\omega} \int_0^{\ell} ds \sqrt{1 + \frac{\ell \cosh^2 \ell}{\sinh \ell} \frac{\zeta}{\sqrt{s}}} + \frac{1}{\omega} \int_\ell^{L} ds \sqrt{1 + \zeta M_{t,L}} + \frac{1}{\omega} \int_L^{\omega t} ds \sqrt{1 + \frac{\ell \cosh L}{e^{L/2} \sinh L} \frac{\zeta}{e^{s/2}}} .
\]
The integrals for \(s \in (0, \ell)\) and for \(s \in (\ell, L)\) can be treated as described in the previous paragraph. On the other hand, the integrals for \(s \in (L, \omega t)\) can both be recast in the following form, performing the change of the integration variable \(\sigma := \eta e^{s/2}\) (for \(\eta = \frac{\zeta}{\sqrt{2}}\) and \(\eta = \frac{\zeta \cosh L}{e^{L/2}\sqrt{\sinh L}}\), respectively):

\[
\int_{\omega t}^{\omega t} ds \sqrt{1 + \eta e^{s/2}} = 2 \int_{\eta e^{\ell/2}}^{\eta e^{t/2}} d\sigma \frac{\sqrt{1 + \sigma}}{\sigma} = 2 \left[ Q(\eta e^{t/2}) - Q(\eta e^{L/2}) \right],
\]

(C.21)

where \(Q\) is defined as in Eq. (C.3). Summing up, the above arguments allow us to infer that

\[
\frac{\zeta^2}{\omega} P\left(\frac{\ell}{\zeta} e^{\sigma t/2}\right) + \sqrt{1 + \zeta^2} \frac{L - \ell}{\omega} + 2 \frac{\omega}{\eta} \left[ Q\left(\frac{\zeta}{\sqrt{2}} e^{\omega t/2}\right) - Q\left(\frac{\zeta}{\sqrt{2}} e^{L/2}\right) \right]
\leq \tau(t)/\theta \leq \frac{\zeta^2 \ell \cosh^2 \ell}{\omega \sinh \ell} P\left(\frac{\sinh \ell}{\zeta^2 \cosh^2 \ell}\right) + \sqrt{1 + \zeta M_{\ell, \omega}} \frac{L - \ell}{\omega} + 2 \frac{\omega}{\eta} \left[ Q\left(\frac{\zeta \cosh L}{\sqrt{\sinh L}} e^{(\omega t - L)/2}\right) - Q\left(\frac{\zeta \cosh L}{\sqrt{\sinh L}}\right) \right],
\]

(C.22)

for \(L/\omega < t < +\infty\). The obtained upper and lower bounds are just \(T^\pm(t)\).

The arguments described in the previous two paragraphs prove Eq. (C.8) for all \(t \in (0, +\infty)\).

**Asymptotics of \(T^\pm(\ell)\) for small and large \(t\).** The asymptotic behavior of \(P(z), Q(z)\) for \(z \to 0^+\) and \(z \to +\infty\) is readily derived from the definitions (C.2) (C.3). Especially, note that

\[
P(z) = \frac{4}{3} z^{3/4} + O(z^{5/4}) \quad \text{for } z \to 0^+, \quad Q(z) = 2 \sqrt{z} + O(z^{-1/2}) \quad \text{for } z \to +\infty.
\]

(C.23)

From here and from Eqs. (C.6) (C.7) one infers

\[
T^-_\ell(t) = \begin{cases} 4 \left(\frac{\zeta^2}{\omega}\right)^{1/4} t^{3/4} + O(t^{5/4}) & \text{for } t \to 0^+ , \\ \frac{2^{7/4} \ell \cosh^2 \ell e^{\ell/2}}{\omega} - \frac{1}{4} \omega t + O(1) & \text{for } t \to +\infty ; \end{cases}
\]

(C.24)

\[
T^+_\ell(t) = \begin{cases} 4 \left(\frac{\zeta^2 \ell \cosh^2 \ell}{\omega \sinh \ell}\right)^{1/4} t^{3/4} + O(t^{5/4}) & \text{for } t \to 0^+ , \\ \frac{4 \left(\frac{\zeta \cosh L}{e^{L/2} \sqrt{\sinh L}}\right)^{1/2}}{\omega} e^{\ell/2} + O(1) & \text{for } t \to +\infty . \end{cases}
\]

(C.25)

By comparison with Eq. (1.2.39) about \(\tau(t)/\theta\), we see that \(T^-_\ell(t)\) has just the same asymptotics as \(\tau(t)/\theta\) both for small and large \(t\); on the other hand, the asymptotics of \(T^+_\ell(t)\) and \(\tau(t)/\theta\) are very similar in both limits.

**A numerical test.** Let us consider the following prescriptions for the parameters of the model, which are used in the final paragraph of subsection 4.2.3 to get a realistic model of inflation:

\[
\zeta = e^{-100} \simeq 3.72008 \times 10^{-44}, \quad \Omega^{(m)} = 0.9, \quad V = 1, \quad \omega = \frac{2 \sqrt{2} V}{3} \simeq 0.9428... .
\]

(C.26)

To build the upper and lower bounds \(T^\pm_\ell\) of Eqs. (C.20) (C.21), let us fix as follows the parameters \(\ell, L\) appearing therein (and in Eq. (C.4)):

\[
\ell = \frac{1}{2} \log (1 + \sqrt{2}) \simeq 0.4406... , \quad L = 2 \log (1 + \sqrt{2}) \simeq 1.7627... .
\]

(C.27)

Fig. 30 and 31 are plots of the functions \(T^\pm_\ell\) over different time intervals (namely: for \(t \in (0, 10^{-90})\) and for \(t \in (0, 240)\), respectively). The graphs of \(T^\pm_\ell\) are very close in Fig. 30, and practically
coincide in Fig. 31. Let us recall that, according to Eq. (C.8), we have $T^\tau(t) \leq \tau(t)/\theta \leq T(t)$ for all $t > 0$. For $t$ in the ranges of the two figures these bounds, being very close, determine $\tau(t)/\theta$ up to a very small uncertainty; in particular, $\tau(t)/\theta$ is approximated with excellent accuracy by the mean value $(1/2)(T^\tau(t) + T(t))$. For all the computations in the final paragraph of subsection 4.2.3 (and in particular, to construct Figs. 8-15 therein) we have used this approximation for $\tau(t)/\theta$ (with $\ell, L$ as in (C.27)). To conclude, let us remark that Fig. 31 exhibits the approximate linear dependence $\tau(t)/\theta \simeq t$ for positive (not too large) values of the coordinate time $t$, in accordance with the expansion given in Eq. (4.2.103).

D Appendix. On the model of subsection 4.3

Let us keep all the assumptions and notations of the cited subsection; in particular, we consider a motion $t \mapsto x(t), y(t)$ with initial conditions as in (4.3.21), (4.3.22). Hereafter we justify some technical statements appearing without proof in subsection 4.3.

D.1 Proof of Eq. (4.3.39): $x(t) > 0$ and $x(t) < y(t) < x(t)$ for all $t \in (0, t_{\text{max}})$

The statement in (4.3.39) about $x(t)$ is obvious, since $x(0) = Y > 0$ and $t \mapsto x(t)$ is a strictly increasing function. In the rest of this subsection we show how to derive the inequalities $-x(t) < y(t) < x(t)$ for $t \in (0, t_{\text{max}})$. Our arguments also involve the inversion time $t_*$ of Eqs. (4.3.34), (4.3.37); we will treat separately the cases $t_{\text{max}} \leq t_*$ and $t_{\text{max}} > t_*$. 

i) $t_{\text{max}} \leq t_*$. Let us consider any time $t \in (0, t_{\text{max}})$; then we have $Y < x(t), Y < y(t) < (F/V)^{1/(2t)}$ and Eqs. (4.3.29), (4.3.30) give

$$\int_Y^{x(t)} \frac{dx}{\sqrt{E + V_1 z^{2t}}} = \frac{\sqrt{2}}{t+1} \int_{Y}^{y(t)} \frac{dy}{\sqrt{F - V_2 z^{2t}}}.$$  \hspace{1cm} (D.1)

Clearly, the above chain of identities is verified if and only if

$$\int_Y^{x(t)} \frac{dz}{\sqrt{E + V_1 z^{2t}}} = \int_Y^{y(t)} \frac{dz}{\sqrt{F - V_2 z^{2t}}} \quad \text{for all } t \in (0, t_{\text{max}}).$$ \hspace{1cm} (D.2)

On the other hand, since $1/\sqrt{E + V_1 z^{2t}} < 1/\sqrt{F - V_2 z^{2t}}$ for all $z \in [Y, (F/V)^{1/(2t)})$, the above identity (D.2) is certainly violated if $x(t) \leq y(t)$. This suffices to infer that

$$x(t) > y(t) > Y \quad \text{for all } t \in (0, t_{\text{max}}).$$ \hspace{1cm} (D.3)

This also imply $y(t) > Y > 0 > -x(t)$. Therefore, summing up we have $x(t) > y(t) > -x(t)$, as required.

81
ii) \( t_{\max} > t_\star \). In this case, the relations written in Eqs. (4.3.29), (4.3.36), (D.2) hold true for \( t \in (0, t_\star) \) and, by continuity, even for \( t = t_\star \). Then, repeating the considerations which led to Eq. (D.3), we infer
\[
x(t) > y(t) > Y \quad \text{for all} \quad t \in (0, t_\star] .
\] (D.4)
The above inequalities also imply \( y(t) > Y > 0 > -x(t) \), which allows us to infer
\[
-x(t) < y(t) < x(t) \quad \text{for all} \quad t \in (0, t_\star] .
\] (D.5)
Finally, let \( t \in (t_\star, t_{\max}) \). Recalling that \( x(t) \) is a strictly increasing function, and using Eq. (D.5) at time \( t_\star \) we infer \( x(t) > x(t_\star) > y(t_\star) = (3/V_2)^{1/(2\ell)} \); on the other hand, from Eq. (4.3.15) we know that in any case \( -(3/V_2)^{1/(2\ell)} \leq y(t) \leq (3/V_2)^{1/(2\ell)} \). Summing up, we have
\[
-x(t) < y(t) < x(t) \quad \text{for all} \quad t \in (t_\star, t_{\max})
\] (D.6)
and this fact, along with Eq. (D.5), ensures \( -x(t) < y(t) < x(t) \) for all \( t \in (0, t_{\max}) \).

D.2 The \( t \to t_{\max}^- \) asymptotics of \( \tau(t) \): proof of Eq. (4.3.52)

Let us consider the integral representation of \( \tau(t) \) given in Eq. (4.3.7). Due to Eqs. (4.3.32), (4.3.38), the integrand function therein has the asymptotic expansion
\[
(x^2(t') - y^2(t'))^{\frac{\ell+1}{2}} = \frac{\ell+1}{(\ell-1)\sqrt{2V_1}} (t_{\max} - t')^{-1} \left(1 + O((t_{\max} - t')^{\frac{2}{\ell-1}})\right) \quad \text{for} \ t' \to t_{\max}^- .
\] (D.1)
We now re-write Eq. (4.3.7) isolating the dominant contribution for \( t \to t_{\max}^- \), which gives
\[
\tau(t)/\theta = \int_0^t dt' \left( x^2(t') - y^2(t') \right)^{\frac{\ell+1}{2}} = \int_0^t dt' \left( x^2(t') - y^2(t') \right)^{\frac{\ell+1}{2}} - \frac{\ell+1}{(\ell-1)\sqrt{2V_1}} (t_{\max} - t')^{-1} .
\] (D.2)
Computing the first integral above and breaking the second one in two parts, we obtain
\[
\tau(t)/\theta = \frac{\ell+1}{(\ell-1)\sqrt{2V_1}} \log \left( \frac{t_{\max}}{t_{\max} - t} \right) + \left(\int_0^{t_{\max}} - \int_t^{t_{\max}}\right) dt' \left( x^2(t') - y^2(t') \right)^{\frac{\ell+1}{2}} - \frac{\ell+1}{(\ell-1)\sqrt{2V_1}} (t_{\max} - t')^{-1} .
\] (D.3)
Let us write \(...\) for the expression appearing above between square brackets; according to Eq. (D.1), we have \(...\) = \( O((t_{\max} - t')^{-1+\frac{2}{\ell-1}}) \) for \( t' \to t_{\max}^- \); hence \( \int_0^{t_{\max}} dt' [... \) is convergent and \( \int_t^{t_{\max}} dt' [... \) is convergent. Thus, \( \int_0^{t_{\max}} dt' O((t_{\max} - t')^{-1+\frac{2}{\ell-1}}) = O((t_{\max} - t)^{\frac{2}{\ell-1}}) \) for \( t \to t_{\max}^- \). In conclusion,
\[
\tau(t)/\theta = \frac{\ell+1}{(\ell-1)\sqrt{2V_1}} \log \left( \frac{t_{\max}}{t_{\max} - t} \right) + \left(\int_0^{t_{\max}} - \int_t^{t_{\max}}\right) dt' \left( x^2(t') - y^2(t') \right)^{\frac{\ell+1}{2}} - \frac{\ell+1}{(\ell-1)\sqrt{2V_1}} (t_{\max} - t')^{-1} + O((t_{\max} - t)^{\frac{2}{\ell-1}}) ,
\] (D.4)
as stated in Eq. (4.3.52).
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