Complete Initial State QED Corrections to Off-Shell Gauge Boson Pair Production in $e^+e^-$ Annihilation

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Abstract

We study Standard Model four-fermion production in $e^+e^-$ annihilation at LEP2 energies and above using a semi-analytical approach. We derive the complete QED initial state corrections (ISR) to the reactions $e^+e^- \rightarrow (Z^0Z^0) \rightarrow f\bar{f}_1 f_2 \bar{f}_2$ and $e^+e^- \rightarrow (W^+W^-) \rightarrow f_1^u f_1^d f_2^u f_2^d$ with $f_1 \neq f_2$ and $f_i \neq e^\pm, \nu_e$. As compared to the well-known universal $s$-channel ISR, additional complexity arises due to non-universal, process-dependent ISR contributions from $t$- and $u$-channel fermion exchanges. The full set of formulae needed to perform numerical calculations is given together with samples of numerical results.

1 Introduction

The LEP2 $e^+e^-$ accelerator will finally operate at energies between 176 and 205 GeV [1] and thus pass the production thresholds for $W^\pm$ and $Z$ pairs. At LEP2, a typical process will be four-fermion production which is much more complex than fermion pair production as known from LEP1. This complexity shows up at tree level already, because of the many Feynman diagrams involved in four-fermion production. As tree-level amplitudes are not sufficient

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to describe experimental data, radiative corrections to four-fermion production are needed. It would be desirable to derive complete $O(\alpha)$ electroweak and QCD corrections to four-fermion production, but this has not yet been achieved, although many partial results were reported in [2–5] and in references quoted therein. In view of the anticipated experimental precision of LEP2 and future $e^+e^-$ colliders, it would be desirable to have theoretical predictions for cross-sections and distributions accurate at the level of half a percent.

In recent years, three major approaches to four-fermion production in $e^+e^-$ annihilation have been developed, namely Monte Carlo approaches [3], the semi-analytical approach [6–13], and the “deterministic approach” [14]. Monte Carlo and deterministic techniques use numerical integration for all phase space variables. Typically, the semi-analytical method performs analytical integrations over the five (seven, if ISR is included) angular degrees of freedom and uses high precision numerical integration for the remaining two (three, if ISR is included) squared invariant masses. It represents an approach to the high-dimensional, highly singular phase space integration problem inherent in this type of physical problem, which is elegant, fast, and numerically stable. Thus, it may serve as an ideal source of benchmarks for the other two approaches. From LEP1, we know that semi-analytical calculations are also relevant to experimentalists.

In this article we will present semi-analytical results for the gauge boson pair production reactions

\begin{align}
\text{CC3} & : \quad e^+e^- \rightarrow (W^+W^-) \quad \rightarrow \tilde{f}_1^\alpha f_1^\beta f_2^\gamma \bar{f}_2^\delta (\gamma) , \quad f_i \neq e^\pm, \nu_e \\
\text{NC2} & : \quad e^+e^- \rightarrow (Z^0Z^0) \quad \rightarrow f_1 \tilde{f}_1 f_2 \bar{f}_2 (\gamma) , \quad f_1 \neq f_2, \ f_i \neq e^\pm, \nu_e \\
\text{NC8} & : \quad e^+e^- \rightarrow (Z^0Z^0, Z^0\gamma, \gamma\gamma) \quad \rightarrow f_1 \tilde{f}_1 f_2 \bar{f}_2 (\gamma) , \quad f_1 \neq f_2, \ f_i \neq e^\pm, \nu_e \quad (1.1)
\end{align}

with complete initial state QED corrections. The three Feynman diagrams for the charged current CC3 process at tree level are given in figure 1. The two or eight diagrams for the neutral current NC2 and NC8 processes are depicted in figure 2\footnote{Strictly speaking, the NC2 process is well-defined (i.e. observable) only as on-shell reaction.}. Classifications of four-fermion processes may be found in references [8,15].

Initial state QED corrections (ISR) represent a dominant correction in $e^+e^-$ annihilation. Complete ISR to four-fermion production separates into a universal, factorizing, process-independent contribution and a non-universal, non-factorizing, process-dependent part. With the index $J$ labelling the CC3, NC2, and NC8 processes, the ISR corrected cross-sections can be generically written as
Fig. 1. The tree level Feynman diagrams for off-shell $W$ pair production (the C33 process). Left: $t$-channel diagram. Right: $s$-channel diagram. The particle momenta are given by the $k_i$ and $p_j$.

Fig. 2. The tree level Feynman diagrams for off-shell neutral gauge boson pair production ($\pi$C8 process). Left: $t$-channel. Right: $u$-channel. The $\pi$C2 process is obtained by neglecting diagrams with exchange photons.

$$\frac{d\sigma_{J,QED}(s)}{ds_1 ds_2} = \int \frac{ds'}{s} \left[ G(s'/s) \sigma_{J,0}(s', s_1, s_2) + \sigma_{J,QED}^{\text{non-univ}}(s, s', s_1, s_2) \right]$$

(1.2)

with invariant boson masses $s_1$ and $s_2$, reduced center of mass energy squared $s'$, and tree level four-fermion production cross-section $\sigma_{J,0}$. The factor $G(s'/s)$ contains all mass singularities $\ln(s/m^2_e)$ and incorporates the process-independent ISR radiators as known from $s$-channel $e^+e^-$ annihilation [16]. In addition, there is a non-universal contribution $\sigma_{J,QED}^{\text{non-univ}}$. It appears together with $t$-channel and $u$-channel amplitudes and is mass singularity free. For pure $s$-channel contributions it is absent.
Complete initial state QED corrections were shortly communicated for the $\text{CC}_3$ process in [7] and for the $\text{NC}_2$ and $\text{NC}_8$ processes in [11]. While a definition of initial state radiation is straightforward in the neutral current process, there is an arbitrariness for $W$ pair production in its definition. In [7], we restored the U(1)-invariance of the initial state photon emission by adding an auxiliary current. This arbitrariness is characteristic of charged current processes. Of course, it is the sum of the corrections what will be finally observable.

In this paper, the complete analytical formulae for the non-universal corrections, supplemented by a study of their numerical importance, will be presented for the first time.

For on-shell production,

\begin{align}
e^+e^- & \rightarrow W^+W^-(\gamma), \\
e^+e^- & \rightarrow Z^0Z^0(\gamma),
\end{align}

the generic cross-section may be obtained as follows:

\[ \bar{\sigma}_{J,\text{QED}}(s) = \int_4^{s} \frac{ds'}{s} \left[ G(s'/s) \bar{\sigma}_{J,0}(s', M^2_V, M^2_V) \right. \\
& \left. + \bar{\sigma}^{\text{non-univ}}_{J,\text{QED}}(s, s', M^2_V, M^2_V) \right]. \tag{1.4} \]

Here, $M_V$ represents the $W$ or $Z$ boson mass. The relation between $\bar{\sigma}_{J,0}$ and $\sigma_{J,0}$ on one hand and and $\bar{\sigma}^{\text{non-univ}}_{J,\text{QED}}$ and $\sigma^{\text{non-univ}}_{J,\text{QED}}$ on the other hand will be discussed in section 2.

For $Z$ pair production, there are no additional QED corrections for the on-shell case, while for $W$ pair production there are final state corrections and initial-final interferences. The non-universal initial state QED corrections were not known before as explicit analytical expressions. However, they have been determined as a part of the complete electroweak corrections to the processes (1.3) with numerical integrations in [17].

The outline of this report is as follows. Section 2 presents general features of our approach to the complete initial state QED corrections. In section 3 we give a detailed presentation of the non-universal cross-section contributions. Section 4 contains numerical results and section 5 concluding remarks. In a series of appendices, we give technical details of the performed computations. Some notations are introduced in appendix A. Our phase space parametriza-
tion and all relevant relations between particle four-momenta and phase space variables are presented in appendix B. In appendix C, the tree level, the real ISR, and the virtual ISR matrix elements are given. These matrix elements represent the starting point for the calculations in this paper. The analytical integrals needed for the integration of the angular phase space variables and, in the case of virtual corrections, loop momenta are collected in appendix D.

2 General Structure of Initial State QED Corrections

To include initial state QED corrections (ISR) to (1.1), the five-particle phase space is required to take into account four final state fermions and a bremsstrahlung photon with momentum $p$. We make use of the following parametrization:

$$d\Gamma_5 = \frac{1}{(2\pi)^{14}} \frac{\sqrt{\lambda(s, s', 0)}}{8s} \frac{\sqrt{\lambda(s', s_1, s_2)}}{8s'} \frac{\sqrt{\lambda(s_1, m_1^2, m_2^2)}}{8s_1} \frac{\sqrt{\lambda(s_2, m_3^2, m_4^2)}}{8s_2} \times ds' ds_1 ds_2 d\cos\theta d\Omega_R d\Omega_1 d\Omega_2.$$  (2.1)

In equation (2.1), the azimuth angle around the beam has already been integrated. We have adopted the usual definition of the $\lambda$ function,

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc,$$

$$\lambda \equiv \lambda(s, s_1, s_2).$$  (2.2)

We use $k_1$ and $k_2$ as the initial electron and positron four-momenta, while $p_1$, $p_2$, $p_3$, and $p_4$ label the final state fermion momenta as indicated in figures 1 and 2. The relevant invariant masses are\footnote{In our metric space-like four-vectors $k$ have positive $k^2$. Thus $k^2 = -m^2$ for on-shell particles of mass $m$.}

$$s = -(k_1 + k_2)^2 = -(p_1 + p_2 + p_3 + p_4 + p)^2,$$

$$s' = -(p_1 + p_2 + p_3 + p_4)^2,$$

$$s_1 = -(p_1 + p_2)^2,$$

$$s_2 = -(p_3 + p_4)^2.$$  (2.3)

The photon scattering angle $\theta$ is defined as the angle between $\vec{p}$ and $\vec{k}_2$ in the
center of mass system. The solid angle

\[ d\Omega_R = d\cos \theta_R d\phi_R \quad (2.4) \]

represents the production solid angle of the boson three vector \( \vec{v}_1 = \vec{p}_1 + \vec{p}_2 \) in the four-fermion rest frame \( \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4 = 0 \). \( \Omega_1 [\Omega_2] \) is the solid angle of \( \vec{p}_1 \) \( [\vec{p}_3] \) in the two-fermion rest frame \( \vec{p}_1 + \vec{p}_2 = 0 \) \( [\vec{p}_3 + \vec{p}_4 = 0] \). In this frame, the three-vectors \( \vec{p}_1 \) and \( \vec{p}_2 \) \( [\vec{p}_3 \text{ and } \vec{p}_4] \) are back to back. The polar and azimuthal decay angles in the above two-particle rest frames are defined via

\[ d\Omega_i = d\cos \theta_i \ d\phi_i, \quad i = 1, 2. \quad (2.5) \]

Further details of the kinematics and the five-particle phase space may be found in appendix B.

The processes (1.1) have the generic structure shown in figure 3. Their ISR-corrected cross-sections get two types of contributions, namely universal and non-universal ones. The first type is universal in the sense that it arises from photonic insertions to any of the basic diagrams and is independent of the details of the subsequent interactions. These universal corrections are, of course, related to the collinear divergences of the (radiating) initial state electrons and positrons and appear to be known, including higher order terms, from, e.g., the study of ISR corrections to the (single) Z line shape. They are exactly those of the s-channel ISR contributions. For diagrams with t- and u-channel exchanges and interferences of these diagrams among themselves and with s-channel diagrams, there are additional corrections which are not logarithmically enhanced and which depend on the details of the interfering amplitudes. These additional corrections are thus non-universal.

ISR-corrected cross-sections for the processes (1.1) with soft photon exponentiation may be described by the ansatz.
\( \sigma_{\text{ISR}}^J(s) = \int ds_1 \int ds_2 \int \frac{ds'}{s} \sum_k \frac{d^3\Sigma_j^{(k)}(s, s'; s_1, s_2)}{ds_1 ds_2 ds'} \) \hspace{1cm} (2.6)

with the threefold differential cross-section

\[
\frac{d^3\Sigma_j^{(k)}(s, s'; s_1, s_2)}{ds_1 ds_2 ds'} = \sigma_j^{(k)}(s', s_1, s_2) \left[ \beta_e v^{\beta_e - 1} S_j^{(k)} + H_j^{(k)} \right].
\] \hspace{1cm} (2.7)

where

\[
\beta_e = \frac{2\alpha}{\pi} \left[ \ln \left( \frac{s}{m_e^2} \right) - 1 \right], \\
v = 1 - \frac{s'}{s}.
\] \hspace{1cm} (2.8)

In equation (2.7) we have used several additional notations which will now be explained. The subscript \( J \) labels different processes, \( J \in \{ \text{CC3, NC2, NC8} \} \) and the superscript index \( k \) stands for cross-section contributions which stem from squared amplitudes or interferences with distinct Feynman topologies and coupling structures. Using

\[
\sigma_j^{(k,0)}(s; s_1, s_2) \equiv \sqrt{\lambda} \frac{\pi s^2}{s^2} \sigma_j^{(k)}(s; s_1, s_2),
\] \hspace{1cm} (2.9)

the soft+virtual contributions \( S_j^{(k)} \) and the hard contributions \( H_j^{(k)} \) take the form

\[
S_j^{(k)}(s, s'; s_1, s_2) = \left[ 1 + \bar{S}(s) \right] \sigma_j^{(k,0)}(s'; s_1, s_2) + \sigma_{\bar{S},J}^{(k)}(s'; s_1, s_2), \\
H_j^{(k)}(s, s'; s_1, s_2) = \bar{H}(s, s') \sigma_j^{(k,0)}(s'; s_1, s_2) + \sigma_{\bar{H},J}^{(k)}(s', s_1, s_2) \hspace{1cm} (2.10)
\]

with the well-known soft+virtual radiator \( \bar{S} \) and the hard radiator \( \bar{H} \),

\[
\bar{S}(s) = \frac{\alpha}{\pi} \left( \frac{\pi^2}{3} - \frac{1}{2} \right) + \frac{3}{4} \beta_e + \mathcal{O}(\alpha^2), \\
\bar{H}(s, s') = -\frac{1}{2} \left( 1 + \frac{s'}{s} \right) \beta_e + \mathcal{O}(\alpha^2). \] \hspace{1cm} (2.11)

Higher order terms [16] may be implemented exactly as described in [2,3,13].
We draw the attention to the definition of the (differential) effective tree level cross-section

\[
\frac{d\sigma_{J,0}(s'; s_1, s_2)}{ds_1 \, ds_2} = \frac{\sqrt{\lambda(s', s_1, s_2)}}{\pi s'^2} \sum_k C_J^{(k)}(s', s_1, s_2) G_J^{(k)}(s'; s_1, s_2)
\]

which is inherent in the universal ISR correction part. Coupling constants and boson propagators are collected in \(C_J^{(k)}\); while the \(G_J^{(k)}\) represent kinematical functions obtained from the analytical angular integration. Explicit expressions for \(C_J^{(k)}\) and \(G_J^{(k)}\) may be found in appendix A. If the index \(k\) is associated with \(s\)-channel \(e^+e^-\) annihilation only, non-universal ISR contributions are absent. The \(s\)-channel ISR diagrams are generically shown in figures 4 and 5. They contribute to \(W\) pair production. The non-universal cross-section contributions originate from the angular dependence of initial state \(t\) and \(u\)-channel propagators and contribute both to \(W\) - and \(Z\)-pair production.

To close the section, we comment on the on-shell cross-section (1.4). With the aid of equation (A.14) it is obtained by the following replacements:

\[
\bar{\sigma}_{J,0}(s, M_V^2, M_V^2) = \lim_{\Gamma_V \to 0} \int ds_1 \int ds_2 \sigma_{J,0}(s, s_1, s_2)
\]
\[
\int ds_1 \int ds_2 \lim_{\Gamma_V \to 0} \left[ \frac{\sqrt{\lambda}}{\pi s^2} \sum_k C^{(k)}_f(s, s_1, s_2) G^{(k)}_j(s, s_1, s_2) \right] \tag{2.13}
\]

and

\[
\bar{\sigma}^{\text{non-univ}}_{J, \text{QED}}(s, s', M_V^2, M_{V'}^2) = \lim_{\Gamma_{V'} \to 0} \int ds_1 \int ds_2 \sigma^{\text{non-univ}}_{J, \text{QED}}(s, s', s_1, s_2) \\
\equiv \int ds_1 \int ds_2 \sum_k \lim_{\Gamma_{V'} \to 0} C^{(k)}_f(s', s_1, s_2) \left[ \beta_e \beta_{e^-} \sigma^{(k)}_{S,J}(s'; s_1, s_2) + \sigma^{(k)}_{H,J}(s, s'; s_1, s_2) \right]. \tag{2.14}
\]

3 Non-universal Initial State Corrections

In this section, we present the final analytical results for the ISR corrected threefold differential cross-sections for the processes (1.1). The starting point of our calculations are matrix elements of CC3, NC2, and NC8 processes, which are given in Appendix C. With the help of the computer algebra packages SCHOONSCHIP, FORM, and Mathematica [18], matrix elements were squared, spin summation was performed, the scalar products were expressed in terms of the phase space variables, and algebraic manipulations of the resulting expressions were carried out. All analytical integrations were obtained from hand-made tables of canonical integrals. The kinematical relations needed for the treatment of real bremsstrahlung may be found in appendix B. For the virtual corrections, tree level kinematics may be used as it was explained in appendix B of [13]. In the course of performing the various steps of analytical integrations we proceeded as follows.

The virtual photonic corrections have been treated as a net sum of all contributing diagrams. The infrared singularity was isolated and, as a part of the universal corrections, subtracted from the net correction. Thus, the remaining, non-universal virtual corrections are by construction free of infrared problems. After tensor integration over the final state angular variables (see appendix D.1), the loop momentum integrations were performed with the aid of the integrals of appendix D.2 and the final integration over the vector boson production angle \(\vartheta\) in the center of mass system with the aid of appendix D.3.

Also the real photonic corrections were first integrated over the final state
angular variables (see again appendix D.1). Then we integrated over the production angles \( \phi_R \) and \( \theta_R \) of the vector boson in the two-boson rest frame (for the canonical integrals see appendix D.4) and finally over the photon production angle \( \theta \) in the center of mass system (see the table of canonical integrals in appendix D.5).

We have used the ultrarelativistic approximation for final state fermions and initial state electrons, i.e. the masses of these particles are neglected wherever possible. The various tables of canonical integrals were checked by Fortran programs with a precision of typically \( 10^{-10} \). At the end of our calculations, algebraic manipulations were carried out by hand to yield compactification of our final results given in equations (3.3), (3.5), (3.8), (3.10), (3.14), (3.16).

In the following two subsections, the non-universal corrections introduced in equation (2.10) will be explicitly given and commented.

### 3.1 The Neutral-Current Case

In the neutral current case, initial state QED corrections are represented by the Feynman diagrams of figures 6 and 7.

The non-universal NC\( 8 \) corrections are

\[
\sigma_{S,NC8}(s; s_1, s_2) = \frac{\alpha}{\pi} \frac{s_1 s_2}{8 \pi s^2} \sigma_{V,NC8}^{\text{non-univ}}(s; s_1, s_2),
\]

\[
\sigma_{H,NC8}(s, s'; s_1, s_2) = \frac{\alpha}{\pi} \frac{s_1 s_2}{\pi s} \sigma_{R,NC8}^{\text{non-univ}}(s, s'; s_1, s_2), \tag{3.1}
\]

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\( ^6 \) The latter was cross-checked against FORM outputs with the aid of auxiliary FORM codes.
where the subindex $V$ stands for virtual and the subindex $R$ for real photonic corrections.

The virtual contribution is a sum of contributions due to the different interferences of loop diagrams with tree level graphs:

$$\sigma_{V,\text{non-univ}}(s; s_1, s_2) = \sigma_{V,t}^{\text{non-univ}} + 2\sigma_{V,tu}^{\text{non-univ}} + \sigma_{V,u}^{\text{non-univ}}. \quad (3.2)$$

The squared $t$- and $u$-channel contributions are equal:

$$\sigma_{V,t}^{\text{non-univ}}(s; s_1, s_2) = \sigma_{V,u}^{\text{non-univ}}(s; s_1, s_2)$$

$$= \frac{3L_2}{2} \left[ l_+ + \delta l_- - s(s - \sigma) I_{12q} \right]$$

$$+ L_1 \left[ (s - 2\sigma)(s - \sigma) I_{12q} + \frac{3\sigma \delta}{2s} l_- - \frac{s - 4\sigma}{2} l_+ + s - \sigma \right]$$

$$+ L_0 \left[ - \left( \lambda + \sigma^2 + \frac{s^2}{2} \right) I_{12q} - \frac{5\delta}{2} l_- - 4\sigma l_+ + 8\sigma - 9s \right]$$

$$+ 2\sqrt{\lambda} \left[ 4 \left( 2 - l_+ \right) + \left( 3s - \sigma \right) I_{12q} \right]. \quad (3.3)$$

Here, the following notations have been used:

$$\sigma = s_1 + s_2,$$

$$\delta = s_1 - s_2,$$
\[ l_+ = \ln \frac{s_1}{s} + \ln \frac{s_2}{s}, \]
\[ l_- = \ln \frac{s_1}{s} - \ln \frac{s_2}{s} = \ln \frac{s_1}{s_2}, \]
\[ \mathcal{L}_0 = \ln \frac{s - \sigma + \sqrt{\lambda}}{s - \sigma - \sqrt{\lambda}}, \]
\[ \mathcal{L}_1 = \frac{s (s - \sigma)}{\lambda} \left[ \mathcal{L}_0 - \frac{2\sqrt{\lambda}}{s - \sigma} \right], \]
\[ \mathcal{L}_2 = \frac{s (s - \sigma)^3}{\lambda^2} \left[ \mathcal{L}_0 - \frac{2\sqrt{\lambda}}{s - \sigma} - \frac{2}{3} \left( \frac{\sqrt{\lambda}}{s - \sigma} \right)^3 \right], \]
\[ I_{12q} = \frac{1}{\sqrt{\lambda}} \left( \mathcal{D}_+ - \frac{1}{2} l_- \mathcal{L}_- \right), \]
\[ \mathcal{L}_- = \mathcal{L}_{12} - \mathcal{L}_{34}, \]
\[ \mathcal{L}_{12} = \ln \frac{s + \delta + \sqrt{\lambda}}{s + \delta - \sqrt{\lambda}}, \]
\[ \mathcal{L}_{34} = \ln \frac{s - \delta + \sqrt{\lambda}}{s - \delta - \sqrt{\lambda}}, \]
\[ \mathcal{D}_+ = \text{Li}_2 \left( -\frac{t_{\text{max}}}{s_1} \right) - \text{Li}_2 \left( -\frac{t_{\text{min}}}{s_1} \right) + \text{Li}_2 \left( -\frac{t_{\text{max}}}{s_2} \right) - \text{Li}_2 \left( -\frac{t_{\text{min}}}{s_2} \right), \]
\[ t_{\text{min}} = \frac{1}{2} \left( s - \sigma - \sqrt{\lambda} \right), \]
\[ t_{\text{max}} = \frac{1}{2} \left( s - \sigma + \sqrt{\lambda} \right). \]

The virtual corrections due to the \( tu \)-interferences are:

\[
\sigma^{\text{non-univ}}_{V,tu}(s; s_1, s_2) = \frac{3}{2} \mathcal{L}_2 \left[ s l_+ + \delta l_- - s(s - \sigma) I_{12q} \right]
+ \mathcal{L}_1 \left[ (s - \sigma) \left( 1 + 5 s I_{12q} + \frac{\delta}{2s} l_- \right) - \frac{9}{2} \left( s l_+ + \delta l_- \right) \right]
+ \mathcal{L}_0 \left\{ \frac{s^2}{s - \sigma} \left[ -\frac{7}{3} \mathcal{L}_0^2 - 16 \mathcal{D}_d + 4 \mathcal{D}_d + 4 \left( l_\sigma - 2 l_+ + \frac{3}{2} \right) l_\sigma
+ 4 l_- l_d - 4 \left( 1 - \frac{3 \pi^2}{2} \right) - 3 \left( l_+ - 3 \right) l_+ \right\}
\]
\begin{align*}
+ 2 s^2 & \left[ (3 h_1 + h_2) \left( l_\sigma - \frac{3}{4} l_+ \right) + \frac{3}{4} \left( 3 h_1 - h_2 \right) l_- + 2 h_{1+} \right] \\
+ s & \left[ 2 l_+ - 6 l_\sigma - \frac{15}{2} s I_{12q} - \frac{1}{d_1} - \frac{1}{d_2} - 3 \right] \\
+ & \sigma \left( l_+ - 2 l_\sigma \right) + \frac{5}{2} l_- \right) \right] \\
+ \frac{8 s^2}{s - \sigma} \left[ \mathcal{F}_{d+} - \mathcal{F}_{d-} - \mathcal{F}_{\sigma} + \left( \frac{3}{4} - \frac{1}{2} l_+ \right) D_{\sigma} + \frac{1}{2} l_+ D_{d-} \right] \\
+ & s^2 \left( 3 h_{1+} + h_{2+} \right) \left( \sqrt{\lambda} I_{12q} - 2 D_{\sigma} \right) \\
+ & s \sqrt{\lambda} \left( 2 I_{12q} + h_{1-} l_- \right) + 2 \left( 3 s + \sigma \right) D_{\sigma} .
\end{align*}

In equation (3.5), the following additional notations have been used:

\begin{align*}
  h_{1\pm} & = \frac{1}{s - s_1} \pm \frac{1}{s - s_2} , \\
  h_{2\pm} & = \frac{s_2}{(s - s_1)^2} \pm \frac{s_1}{(s - s_2)^2} , \\
  d_1 & = \frac{s - s_2}{s_1} , \\
  d_2 & = \frac{s - s_1}{s_2} , \\
  l_\sigma & = \ln \frac{s - \sigma}{s} , \\
  l_{d-} & = \ln d_1 - \ln d_2 , \\
  D_{\sigma} & = \text{Li}_2 \left( \frac{t_{\text{max}}}{s - \sigma} \right) - \text{Li}_2 \left( \frac{t_{\text{min}}}{s - \sigma} \right) , \\
  D_d & = \text{Re} \left[ \text{Li}_2 (d_1) + \text{Li}_2 (d_2) \right] , \\
  D_{d+} & = \text{Li}_2 \left( \frac{t_{\text{max}}}{s - s_2} \right) + \text{Li}_2 \left( \frac{t_{\text{min}}}{s - s_2} \right) + \text{Li}_2 \left( \frac{t_{\text{max}}}{s - s_1} \right) + \text{Li}_2 \left( \frac{t_{\text{min}}}{s - s_1} \right) , \\
  D_{d-} & = \text{Li}_2 \left( \frac{t_{\text{max}}}{s - s_2} \right) - \text{Li}_2 \left( \frac{t_{\text{min}}}{s - s_2} \right) - \text{Li}_2 \left( \frac{t_{\text{max}}}{s - s_1} \right) - \text{Li}_2 \left( \frac{t_{\text{min}}}{s - s_1} \right) ,
\end{align*}

13
\[ F_\sigma = \text{Li}_3 \left( \frac{t_{\text{max}}}{s - \sigma} \right) - \text{Li}_3 \left( \frac{t_{\text{min}}}{s - \sigma} \right), \]
\[ F_{d-} = \text{Li}_3 \left( \frac{t_{\text{max}}}{s - s_2} \right) - \text{Li}_3 \left( \frac{t_{\text{min}}}{s - s_2} \right) + \text{Li}_3 \left( \frac{t_{\text{max}}}{s - s_1} \right) - \text{Li}_3 \left( \frac{t_{\text{min}}}{s - s_1} \right), \]
\[ F_{t-} = \text{Li}_3 \left( -\frac{t_{\text{max}}}{d_1 t_{\text{min}}} \right) - \text{Li}_3 \left( -\frac{t_{\text{min}}}{d_1 t_{\text{max}}} \right) + \text{Li}_3 \left( -\frac{t_{\text{max}}}{d_2 t_{\text{min}}} \right) - \text{Li}_3 \left( -\frac{t_{\text{min}}}{d_2 t_{\text{max}}} \right). \]  

(3.6)

The non-universal real bremsstrahlung contribution to the NC8 process is given by:

\[ \sigma_{R,\text{NC8}}^{\text{non-univ}}(s, s'; s_1, s_2) = \sigma_{R,t}^{\text{non-univ}} + \sigma_{R,tu}^{\text{non-univ}} + \sigma_{R,u}^{\text{non-univ}}. \]  

(3.7)

Again, the squared t- and u-channel contributions are equal:

\[ \sigma_{R,t}^{\text{non-univ}}(s, s'; s_1, s_2) = \sigma_{R,u}^{\text{non-univ}}(s, s'; s_1, s_2) = \sqrt{\lambda} \left( \frac{1}{s'_-} - \frac{\sigma}{\lambda} \right) \]
\[ + \frac{\delta}{4s'_-} \left[ \frac{s'}{s} \left( 1 - \frac{ss'_+}{s'^2} \right) + \frac{\sigma(3s - s' - \sigma)}{\lambda} \right. \]
\[ + \frac{s}{s'_-} \left( 1 + \frac{\sigma^2}{\lambda} \right) + 12 \frac{ss_1s_2\sigma}{\lambda^2} \left( L_{c1} - L_{c2} \right) \]
\[ + \frac{1}{4} \left[ 1 + \frac{ss'_+}{ss'} - \frac{s - \sigma}{s'_-} - \frac{2s\sigma}{s'^2} + \frac{2\sigma}{\lambda} \left( s'_- - \sigma \right) \right. \]
\[ + \frac{2s_1s_2\sigma}{\lambda^2} \left( L_{c1} + L_{c2} \right) \]
\[ + \sqrt{\lambda} \left( \frac{s'_+ + \sigma}{4s'^2} + \frac{3\sigma s_1s_2}{2\lambda} \right) \left( L_{c3} + L_{c4} \right) + \frac{\sigma s'_+ s'_-}{2ss'^2} L_{c5} \]
\[ - \frac{1}{4\sqrt{\lambda}} \left[ 2\sigma + s + s'_+ + \frac{\sigma}{\lambda} \left( s(s'_- + \sigma) - 4s_1s_2 \right) \right. \]
\[ + \frac{12ss_1s_2\sigma}{\lambda^2} \left( L_{c12} \right) \]
+ \left(\frac{\sigma - s'}{4s'^2}\right) s'_\pm \left(D_1 + D_2\right) + \frac{\sigma s'_\pm - s's'_-}{4s'^2} \left(D_{z1r2} + D_{z2t1}\right). \quad (3.8)

In the above equation we have introduced the symbols:

\[ s'_\pm \equiv s \pm s', \quad \lambda' \equiv \lambda(s', s_1, s_2), \]

\[ \bar{\lambda} \equiv \lambda(s'_-, -s_1, -s_2) = s'_-^2 + 2s'_-\sigma + \delta^2. \quad (3.9) \]

The additional notations are introduced in appendix D.5. For easy reference, the number of each integral corresponding to the enumeration in appendix D.5 is listed in table 1.

The most cumbersome contribution is due to the tu-interference:

\[
\sigma_{R,tu}^{\text{non-univ}}(s, s'; s_1, s_2) = \frac{2 s'_\sigma}{s'(s' - \sigma)} L_{c5} + \frac{s'_+}{s'} \left(D_1 + D_2\right) \]

\[ - \frac{s^2 + s'^2}{2 s'_+(s' - \sigma)} \left(D_{z1r1} + D_{z2r2}\right) - \left(\frac{s}{s'} + \frac{s^2 + s'^2}{2 s'_-(s' - \sigma)}\right) \left(D_{z2t1} + D_{z1t2}\right) \]

\[ - \frac{s - 2s_1}{\sqrt{ss_1}} D_{t1u2} - \frac{s - 2s_2}{\sqrt{ss_2}} D_{t2u1} + \frac{s^2 + s'^2}{4 s'_-(s' - \sigma)} \left(D_{t1u2}^z + D_{t2u1}^z\right) \]

\[ + \frac{\sqrt{\lambda}}{2 (s'_- - 2\sigma)(s' - \sigma)} \left[ s(s' - \sigma) - s_1(s'_+ - 2\sigma) \right] D_{t1u2}^a \]

\[ + \frac{\sqrt{\lambda}}{2 (s'_+ - 2\sigma)(s' - \sigma)} \left[ s(s' - \sigma) - s_2(s'_+ - 2\sigma) \right] D_{t2u1}^a. \quad (3.10) \]

Again, additional notations are introduced and the number of each integral corresponding to the enumeration in appendix D.5 is listed in table 2.

3.2 The Charged-Current Case

It is in order to mention that, for the CC3 process, the separation of initial and final state radiation is not unique. Since, in the t-channel contribution, there is electric charge flow from the initial state to the final state, electric current
conservation is violated and one faces the problem to find a gauge-invariant definition of initial state radiation. As a solution we proposed in [7] what we call the current splitting technique (CST). In brief, the CST splits the electrically neutral $t$-channel neutrino flow into two oppositely flowing electric charges +1 and −1. Charge −1 is assigned to the initial state, charge +1 to the final state. This enables a gauge invariant definition of ISR with photon emission and absorption from the $t$-channel exchange particle. An auxiliary current is added to the naive charged current for this purpose in appendix C. Thus, the CC3 $t$-channel receives the same ISR corrections as the neutral current $t$-channel.

Of course, when performing a complete calculation, the final state corrections have to take into account opposite auxiliary terms so that the net auxiliary effect will vanish.

The non-universal corrections are due to the interferences of the $s$- and $t$-channel diagrams of figures 4 and 6 with the corresponding Born diagrams and of the diagrams of figures 5 and 7 among themselves. These interferences show less symmetry than the interferences of the $t$- and $u$-channel diagrams of the neutral current case. However, as mentioned earlier, the pure $s$-channel corrections have no non-universal parts:

$$\sigma_{s,CC3}^s = \sigma_{H,CC3}^s = 0. \quad (3.11)$$

Further, for the pure $t$-channel non-universal contributions to the CC3 process one finds, of course, expressions equal to those for the pure $t$-channel non-universal contributions to the NC8 process, namely
\[ \sigma_{S,CC3}^t(s; s_1, s_2) = \frac{\alpha}{\pi} \frac{s_1 s_2}{8 \pi s^2} \sigma_{V,t}^{\text{non-univ}}(s; s_1, s_2), \]

\[ \sigma_{H,CC3}^t(s, s'; s_1, s_2) = \frac{\alpha}{\pi} \frac{s_1 s_2}{\pi s} \sigma_{R,t}^{\text{non-univ}}(s, s'; s_1, s_2) \]

(3.12)

with \( \sigma_{V,t}^{\text{non-univ}} \) from equation (3.3) and \( \sigma_{R,t}^{\text{non-univ}} \) from equation (3.8).

What remains to be calculated in addition are the virtual and real \( st \)-interference contributions.

The virtual correction is given by

\[ \sigma_{S,CC3}^{st}(s; s_1, s_2) = \frac{\alpha}{\pi} \frac{s_1 s_2}{8 \pi s^2} \sigma_{V,at}^{\text{non-univ}}(s; s_1, s_2) \]

(3.13)

with

\[ \sigma_{V,at}^{\text{non-univ}}(s; s_1, s_2) = - \sqrt{\lambda} \left[ 7s + 3\sigma + \delta l_+ + \left( 4s\sigma + 2\lambda - \delta^2 \right) L_{12a} \right] 
+ L_0 \left[ 5s\sigma + 3\lambda - \frac{3\sigma^2}{2} - \frac{7\delta^2}{2} + \left( 2s\sigma + \lambda - \frac{\delta^2}{2} \right) l_+ + \delta \left( s + \frac{\sigma}{2} \right) l_- \right]. \]

(3.14)

The real bremsstrahlung contribution reads

\[ \sigma_{H,CC3}^{st}(s, s'; s_1, s_2) = \frac{\alpha}{\pi} \frac{s_1 s_2}{8 \pi} \sigma_{R,at}^{\text{non-univ}}(s, s'; s_1, s_2), \]

(3.15)

with

\[ \sigma_{R,at}^{\text{non-univ}}(s, s'; s_1, s_2) = - \frac{\delta s'}{s} \left( \frac{\sigma}{s} - 1 \right) \left( \frac{\sigma}{s} + 3 \right) + \frac{s_1 s_2}{s} \left( \frac{2}{s'} - \frac{1}{s} \right) \right) (L_{c1} - L_{c2}) 
+ \frac{s'}{s} \left[ \frac{1}{2} \left( \frac{\sigma}{s} - 1 \right) \left( \frac{\sigma}{s} + 3 \right) + \frac{s_1 s_2}{s} \left( \frac{2}{s'} - \frac{1}{s} \right) \right] (L_{c1} + L_{c2}) 
+ \sqrt{\lambda} \left[ \frac{s'}{s} \left( \frac{\sigma}{s} s' - 8 \right) + \frac{s'}{s} \left( \frac{\sigma}{s} + 3 \right) (L_{c3} + L_{c4}) \right] 
+ \frac{s'}{s} \left[ \frac{\sigma^2 s'^2}{s^2 s'^2} + 2s' \left( \frac{s_1 s_2 s'}{s^2 s'} - \frac{\sigma}{s} \right) + 4 \right] (L_{c5}). \]
\[-\frac{2\delta}{s} (D_1^t - D_2^t) + 2 \left( \frac{s_1 s_2 s_1'}{s s' s''} + \frac{\sigma}{s'} - \frac{s_1'}{s} \right) (D_1^t + D_2^t) \]

\[-2 \left( \frac{s_1 s_2 s_1'}{s s' s''} + \frac{\sigma}{s'} - 1 \right) (D_{zt1}^t + D_{zt2}^t) \]  

(3.16)

3.3 Discussion

To close the section we will now discuss some formal features of our results. Non-universal cross-section contributions are analytically rather involved and contain many di- and trilogarithms. Although expected, it is noteworthy that interferences are more involved than matrix element squared contributions, which is especially true for the tu-interference. Looking at the integrals collected in Appendix D, one sees that some of them are very complicated (in appendix D.5 e.g. the integral 31) for the real t-channel correction and integrals 36), 38), 40) for the real tu-interference). However, the final answers for t-, u-, and st-contributions, both virtual and real, are remarkably compact. The virtual contributions contain only one true dilogarithm, which arises from the three-point scalar integral I_{12q} (see equation (3.4) and integral 15) in appendix D.2). Both these virtual and real contributions do not contain true trilogarithms. In contrast, the tu-interferences are rather cumbersome. This may be traced back to the angular integrations involving products of different space-like fermionic propagators. Virtual tu-interferences contain several true trilogarithms and real bremsstrahlung tu-interference contributions exhibit complicated complex-valued dilogarithms. The latter, however, could indicate that some simplifications were overlooked.

We mention that the four non-universal cross-section contributions to the CC3 process became much more compact after inclusion of the auxiliary terms than without them.

The attentive reader will have noticed that the virtual+soft non-universal contributions are evaluated at \( s' \) rather than at \( s \). This is to avoid an unphysical, \( \delta \)-distribution-like concentration of non-universal virtual+soft ISR corrections at zero radiative energy loss.

In equations (3.1), (3.12), (3.13), and (3.15), both reasonable threshold and high-energy properties may be observed. For \( \sqrt{\lambda} \) or \( \sqrt{\lambda'} \) approaching zero, non-universal corrections vanish, which may be verified by inspection of the explicit expressions. Thus also the ISR corrected cross-section vanishes for the kinematical zero of the tree level or the universally corrected cross-section. At high energies, the screening property may be verified to hold. By screening
we mean the assurance that cross-sections fall sufficiently fast with rising $s$. For the $\text{CC3}$ case, this is ensured in Born approximation due to an interplay between the $s$- and $t$-channel matrix elements and relies a certain relation between the gauge couplings (see e.g. [19]). In the $\text{NC2}$ and $\text{NC8}$ cases, the same is achieved by an interplay between the $t$- and $u$-channel amplitudes leading to a cross-section proportional to the screening factor $s_1 s_2 / s^2$ (the couplings of the two channels are equal) [8]. From the universal QED corrections, no additional problems arise, because the interplay between the various contributing tree level matrix elements is not disturbed. This is completely different for the non-universal corrections. Here, the various interferences get different corrections and the interplay between them is destroyed. Fortunately, one may see from the explicit expressions that the screening factor observed in the net $\text{NC2}$ and $\text{NC8}$ tree level cross-sections is rediscovered not only for full cross-sections, but even for individual non-universal contributions. So, unitarity is not violated. We were not able to find the screening property for non-universal $\text{CC3}$ corrections without including the auxiliary corrections. Also in this respect, auxiliary corrections seem to be a quite natural, if not necessary ingredient of the calculation.

4 Numerical results

To illustrate the analytical results of sections 2 and 3, we present numerical results [20]. The effect of ISR on the boson pair production processes $\text{NC2}$ and $\text{CC3}$ is seen from figures 8 and 9 where inclusive cross-sections are presented. From both figures one realizes that, as discussed in section 2, the dominant part of the initial state QED corrections originates from universal ISR. Universal ISR is typically of $\mathcal{O}(10\%)$. Non-universal ISR on the contrary is suppressed and rises from a few parts per thousand in the LEP2 energy region to $\mathcal{O}(1\%)$ at 1 TeV. For the $\text{NC2}$ process at 1 TeV the non-universal contribution to the cross-section is 2.5%, whereas for $\text{CC3}$ at 1 TeV it is 1.4%. For reference we have given cross-section values for both $Z$ and $W$ pair production in table 3. The numerical precision is better than $10^{-4}$. As numerical input for table 3 we have used the standard LEP2 input (table 5 in [3]) which is reproduced in table 4. The weak mixing angle is determined by

$$\sin^2 \theta_W = \frac{\pi \alpha(2M_W)}{\sqrt{2} G_F M_W^2}$$

(4.1)

and we use the relation $G_\mu / \sqrt{2} = g^2 / (8M_W^2)$. 

Fig. 8. The inclusive total off-shell Z pair production cross-section $\sigma_{\text{tot}}^{ZZ}(s)$. In the inset, the relative deviations of the universally and completely ISR corrected cross-sections from the tree level cross-section are given.

In figure 10, we present, as an example for the NC8 process, the cross-section for

$$e^+e^- \rightarrow (Z^0Z^0, Z^0\gamma, \gamma\gamma) \rightarrow \mu^+\mu^- b\bar{b}(\gamma).$$  \hspace{1cm} (4.2)

Due to the negative slope of the cross-section curve ISR corrections are always positive. Universal ISR corrections vary between approximately 12% at $\sqrt{s}=130$ GeV and 21% at 600 GeV. Non-universal corrections rise from 0.9% at 130 GeV to 4.2% at 600 GeV. Thus, compared to the NC2 case, the NC8 non-universal corrections are considerably enhanced.

Finally it is worth mentioning that non-universal ISR tends to be harder than universal ISR, a fact which emerges from a numerical analysis of cross-sections with lower cuts on $s'$ [12]. This may be understood as follows: $O(\alpha)$ non-universal corrections are infrared finite, whereas the infrared divergent universal corrections are resummed by means of the soft photon exponentiation. Thus the universal corrections contain important soft resummed parts, and
Fig. 9. The inclusive total off-shell $W$ pair production cross-section $\sigma^{WW}_{\text{tot}}(s)$. In the inset, the relative deviations of the universally and completely ISR corrected cross-sections from the tree level cross-section are given.

the non-universal corrections do not.

5 Concluding Remarks

In this paper we have presented the analytical results of the first complete gauge-invariant calculations of initial state QED corrections to off-shell vector boson pair production in $e^+e^-$ annihilation. We have decomposed the corrections into a dominant universal contribution and a numerically smaller non-universal contribution. Non-universal corrections were found to be smoothly behaving around the two boson production threshold and to be explicitly screened from unitarity violations at high energies. As a by-product, the angular distribution of radiated photons is available from intermediate steps of the calculation but has not been presented here.

By construction, we have not taken into account genuine weak corrections [22]
Table 3
Inclusive total off-shell cross-section values for the processes NC2 and CC3 with and without initial state QED corrections

| √s | σ\text{NC2} [pb] | σ\text{ISR} | σ\text{CC3} [pb] | σ\text{ISR} |
|-----|--------------------|-------------|------------------|-------------|
| 161 | 0.0154             | 0.0127      | 0.0127           | 4.8087      |
| 175 | 0.0648             | 0.0506      | 0.0508           | 15.9168     |
| 183 | 0.3811             | 0.2733      | 0.2743           | 17.6819     |
| 192 | 0.9783             | 0.7788      | 0.7817           | 18.4479     |
| 205 | 1.1833             | 1.0274      | 1.0313           | 18.5078     |
| 500 | 0.4097             | 0.4424      | 0.4479           | 7.3731      |
| 800 | 0.2124             | 0.2388      | 0.2435           | 3.9971      |

Table 4
Input parameters for table 3

| Quantity | Value           | Quantity | Value                      |
|----------|-----------------|----------|----------------------------|
| M\text{Z} | 91.1888 GeV    | α(0)    | 1/137.0359895            |
| Γ\text{Z} | 2.4974 GeV     | α(2M\text{W}) | 1/128.07               |
| M\text{W} | 80.23 GeV      | G\text{u} | 1.16639 · 10^{-5} GeV^{-2} |
| Γ\text{W} | 3G\text{u}M\text{W}^3/(√8π) | V\text{CKM} | 1                        |

or QED corrections related to the final state [17]. Compared to ISR, both are known to be smaller but may reach several percent nevertheless. Therefore they are comparable in size to the non-universal corrections calculated here. See also references [2,23]. Special suppressions of the order $O(Γ_{V}/M_{V})$ are estimated for interference effects between initial and final state radiation in resonant boson pair production and inter-bosonic final state interferences [24]. The complete photonic corrections to four-fermion production are not available although important steps towards their calculation have been achieved [25].

The presented calculation should be considered in connection with the program of semi-analytical treatments of complete processes at tree level [8]. It would be quite interesting to compute complete ISR not only for the off-shell production of vector boson pairs, but also for true four-fermion processes. Simplest are the so-called NC24 [9] and CC11 [13] processes, but also the combination of NC24 with Higgs production [10] should be straightforward. Because in these reactions all background Feynman diagrams, i.e. those diagrams that
must be added to CC3 [NC8] to get the sets for CC11 [NC24], are s-channel diagrams, there is nothing but universal ISR for the pure background. So the only missing pieces are the non-universal ISR contributions to the interferences between the boson pair signal and the background. The results may be of numerical relevance in the NC24 case below the ZZ production threshold. For CC11 and for the inclusion of Higgs production they will be of mere theoretical interest.

Acknowledgments

We would like to thank F. Jegerlehner, A. Leike, A. Olchevski, and U. Müller for helpful discussions. The contribution of M. Bilenky to the calculation of the virtual corrections to the CC3 process in 1992 is gratefully acknowledged. We are indebted to J. Blümlein for carefully reading the manuscript.
A Couplings and Factorizing Kinematical Functions

A.1 Couplings

The electroweak couplings and neutral boson propagators enter the Born and QED corrected cross-sections in certain combinations.

The NC8 process is described by one coupling function [9]:

\[
C_{\text{NC8}} = \frac{2}{(6\pi^2)^2} \sum_{V_i, V_j, V_k, V_l = \gamma, Z} \frac{\text{Re}}{D_{V_i}(s_1) D_{V_j}(s_2) D^*_{V_k}(s_1) D^*_{V_l}(s_2)} \frac{1}{}\times \left[ L(e, V_i) L(e, V_k) L(e, V_j) L(e, V_l) + R(e, V_i) R(e, V_k) R(e, V_j) R(e, V_l) \right] \\
\times \left[ L(f_1, V_i) L(f_1, V_k) + R(f_1, V_i) R(f_1, V_k) \right] N_c(f_1) N_p(V_i, V_k, m_1, s_1) \\
\times \left[ L(f_2, V_j) L(f_2, V_l) + R(f_2, V_j) R(f_2, V_l) \right] N_c(f_2) N_p(V_j, V_l, m_2, s_2). \tag{A.1}
\]

We have used the left- and right-handed fermion-boson couplings

\[
L(f, \gamma) = \frac{e Q_f}{2}, \quad L(f, Z) = \frac{g}{2 c_W} \left( I^f_3 - s_W^2 Q_f \right), \\
R(f, \gamma) = \frac{e Q_f}{2}, \quad R(f, Z) = -\frac{g}{2 c_W} s_W^2 Q_f \tag{A.2}
\]

where \(Q_f\) is the fermion charge in units of the positron charge \(e\), \(I^f_3\) is the fermion’s weak isospin third component, and \(s_W\) denotes the sine of the weak mixing angle. \(Q_e = -1\) and the fine structure constant is given by \(e = g s_W = \sqrt{4\pi\alpha}\). In addition, equation (A.1) contains boson propagators

\[
D_V(s) = s - M^2_V + i\Gamma_V/M_V. \tag{A.3}
\]

The photon has zero mass and width, \(M_\gamma = \Gamma_\gamma = 0\). The color factor \(N_c(f)\) is
unity for leptons and three for quarks. The phase space factors

\[ N_p(V_m, V_n, m, s) = \begin{cases} 
1 & \text{for } V_m \neq \gamma \text{ or } V_n \neq \gamma \\
\sqrt{1 - \frac{4m^2}{s} (1 + \frac{2m^2}{s})} & \text{for } V_m = V_n = \gamma
\end{cases} \]

in equation (A.1) correctly take into account that fermion masses may be neglected if the corresponding fermion pair couples to a Z resonance but may not if it only couples to a photon.

The cross-section for the \textit{NC2} process is obtained by omitting exchange photons in equation (A.1).

The \textit{CC3} process is described by three coupling functions \( C^{(k)}_{\text{CC3}} \), one for the \( t \)-channel, one for the \( s \)-channel, and one for the \( st \)-interference contribution [6,13]:

\[ C^t_{\text{CC3}} = \frac{2}{(6\pi^2)^2} \frac{1}{|D_W(s_1)|^2 |D_W(s_2)|^2} \times L^4(e, W) L^2(f_1, W) N_c(f_1) L^2(f_2, W) N_c(f_2), \quad (A.5) \]

\[ C^s_{\text{CC3}} = \frac{2}{(6\pi^2)^2} \sum_{V_i, V_j = \gamma, Z} \text{Re} \frac{1}{|D_W(s_1)|^2 |D_W(s_2)|^2 D_{V_i}(s) D_{V_j}(s)} \times \left[ L(e, V_i) L(e, V_j) + R(e, V_i) R(e, V_j) \right] \times g_3(V_i) g_3(V_j) L^2(f_1, W) N_c(f_1) L^2(f_2, W) N_c(f_2), \quad (A.6) \]

\[ C^{st}_{\text{CC3}} = \frac{2}{(6\pi^2)^2} \sum_{V_i = \gamma, Z} \text{Re} \frac{1}{|D_W(s_1)|^2 |D_W(s_2)|^2 D_{V_i}(s)} L^2(e, W) \times L(e, V_i) g_3(V_i) L^2(f_1, W) N_c(f_1) L^2(f_2, W) N_c(f_2). \quad (A.7) \]

The left-handed couplings \( L(f, W) \) of a weak isodoublet with the fermion \( f \) to a \( W \) boson are given by

\[ L(f, W) = \frac{g}{2\sqrt{2}}. \quad (A.8) \]

We neglect Cabibbo-Kobayashi-Maskawa mixing effects. The right-handed couplings \( R(f, W) \) vanish. The couplings \( g_3(V_i) \) originate from the three-boson vertices and are

\[ g_3(\gamma) = g s_W, \]
\[ g_3(Z) = g c_W. \] (A.9)

Using \( G_\mu/\sqrt{2} = g^2/(8M_W^2) \) in the \( C \) functions, one easily rewrites the cross-sections for heavy boson pair production, namely the processes \( \text{CC3} \) and \( \text{NC2} \), in terms of Breit-Wigner densities

\[ \rho_V(s_i) = \frac{1}{\pi |s_i - M_V^2 + is_i\Gamma_V/M_V|^2} \times \text{BR}(i), \quad V = Z, W^\pm \] (A.10)

with branching fractions \( \text{BR}(i) \) into the appropriate fermion pairs.

For the process \( \text{NC2} \) one gets

\[ \mathcal{C}_{\text{NC2}} = \frac{\rho_Z(s_1)\rho_Z(s_2)}{s_1s_2} \frac{(G_\mu M_Z^2)^2}{4} \left[ \left(1 - 2s_2^2W\right)^4 + \left(2s_2^2W\right)^4 \right]. \] (A.11)

For \( W^\pm \) pair production, the situation is slightly more involved. Defining

\[ \mathcal{C}_{\text{CC3}} = \frac{(G_\mu M_W^2)^2}{s_1s_2} \rho_W(s_1)\rho_W(s_2), \] (A.12)

one obtains by inspection of equations (A.5)–(A.7)

\[ \mathcal{C}_{\text{CC3}}^t = \mathcal{C}_{\text{CC3}} c_{\nu
u}, \]

\[ \mathcal{C}_{\text{CC3}}^s = \frac{4}{s^2} \mathcal{C}_{\text{CC3}} \left( c_{\gamma\gamma} + c_{\gamma Z} + c_{ZZ} \right), \]

\[ \mathcal{C}_{\text{CC3}}^{st} = \frac{1}{s} \mathcal{C}_{\text{CC3}} \left( c_{\nu\gamma} + c_{\nu Z} \right) \] (A.13)

with factors \( c_{ab} \) as defined in [8] (see also [6]).

For on-shell heavy boson pair production, \( \Gamma_V \to 0 \), Breit-Wigner densities are replaced by \( \delta \)-distributions,

\[ \rho_V(s) \stackrel{\Gamma_V \to 0}{\longrightarrow} \delta(s - M_V^2) \times \text{BR}(i). \] (A.14)
A.2 Kinematical Functions for the Factorizing QED Corrections

The kinematical function for the NC case is [26]:

\[ G_{NC2}(s; s_1, s_2) = G_{NC8}(s; s_1, s_2) = s_1s_2 \left[ \frac{s^2 + (s_1 + s_2)^2}{s - s_1 - s_2} \mathcal{L} - 2 \right]. \] (A.15)

For the CC case, one finds [6]:

\[ G_t^{CC3}(s; s_1, s_2) = \frac{1}{48} \left[ \lambda + 12s(s_1 + s_2) - 48s_1s_2 + 24(s - s_1 - s_2)s_1s_2 \mathcal{L} \right], \] (A.16)

\[ G_s^{CC3}(s, s_1, s_2) = \frac{\lambda}{192} \left[ \lambda + 12(ss_1 + ss_2 + s_1s_2) \right], \] (A.17)

\[ G^{st}_{CC3}(s; s_1, s_2) = \frac{1}{48} \left\{ (s - s_1 - s_2) \left[ \lambda + 12(ss_1 + ss_2 + s_1s_2) \right] - 24(ss_1 + ss_2 + s_1s_2)s_1s_2 \mathcal{L} \right\}, \] (A.18)

for the kinematical functions. The logarithm \( \mathcal{L} \) contained in integrated \( t \)- or \( u \)-channel contributions is defined by

\[ \mathcal{L} = \mathcal{L}(s; s_1, s_2) = \frac{1}{\sqrt{\lambda}} \ln \frac{s - s_1 - s_2 + \sqrt{\lambda}}{s - s_1 - s_2 - \sqrt{\lambda}}. \] (A.19)

B The 2 \( \rightarrow \) 5 Particle Phase Space

When considering initial state photon bremsstrahlung to a 2 \( \rightarrow \) 4 process, a photon with momentum \( p \) appears as fifth particle in the final state. The intermediate vector bosons (two-fermion systems) have momenta \( v_1 = p_1 + p_2 \), \( v_2 = p_3 + p_4 \). The five-particle phase space has eleven kinematical degrees of freedom. It is convenient to parametrize the five-particle Lorentz invariant phase space by a decomposition into subsequently decaying particles:

\[
\begin{align*}
\text{d}\Gamma_5 &= \frac{1}{(2\pi)^{14}} \sqrt{\lambda(s, s', 0)} \sqrt{\lambda(s', s_1, s_2)} \sqrt{\lambda(s_1, m_3^2, m_4^2)} \sqrt{\lambda(s_2, m_1^2, m_3^2)} \times \\
&\quad \times ds' \, ds_1 \, ds_2 \, d\cos\theta \, d\phi_R \, d\cos\theta_R \, d\phi_1 \, d\cos\theta_1 \, d\phi_2 \, d\cos\theta_2.
\end{align*}
\] (B.1)
where \( m_i \) is the mass of the final state fermion with momentum \( p_i \) and

\[ \phi : \text{Azimuth angle around the beam direction.} \]

\[ \theta : \text{Polar (scattering) angle of the photon with respect to the } e^+ \text{ direction } \vec{k}_2 \text{ in the center of mass frame.} \]

\[ \phi_R : \text{Azimuth angle around the photon direction } \vec{p}. \]

\[ \theta_R : \text{Polar angle of } \vec{v}_1 \text{ in the } (\vec{v}_1 + \vec{v}_2) \text{ rest frame with z axis along } \vec{p}. \]

\[ \phi_1 : \text{Azimuth angle around the } \vec{v}_1 \text{ direction.} \]

\[ \theta_1 : \text{Decay polar angle of } \vec{p}_1 \text{ in the } \vec{v}_1 \text{ rest frame with axis along } \vec{v}_1. \]

\[ \phi_2 : \text{Azimuth angle around the } \vec{v}_2 \text{ direction.} \]

\[ \theta_2 : \text{Decay polar angle of } \vec{p}_3 \text{ in the } \vec{v}_2 \text{ rest frame with axis along } \vec{v}_2. \]

\[ s_1 : \text{Invariant mass of the final state fermion pair } f_1\bar{f}_2 : \quad s_1 = -v_1^2. \]

\[ s_2 : \text{Invariant mass of the final state fermion pair } f_3\bar{f}_4 : \quad s_2 = -v_2^2. \]

\[ s' : \text{Reduced center of mass energy squared. This is the invariant mass of the final state four-fermion system:} \]

\[ s' = -(v_1 + v_2)^2 = -(p_1 + p_2 + p_3 + p_4)^2. \]

The ranges of the kinematical variables are

\[
\begin{align*}
(m_1 + m_2 + m_3 + m_4)^2 & \leq s' \leq s \\
(m_1 + m_2)^2 & \leq s_1 \leq (\sqrt{s'} - m_3 - m_4)^2 \\
(m_3 + m_4)^2 & \leq s_2 \leq (\sqrt{s'} - \sqrt{s_1})^2 \\
-1 & \leq \cos \theta \leq +1 \\
0 & \leq \phi_{(R,1,2)} \leq 2\pi \\
-1 & \leq \cos \theta_{(R,1,2)} \leq +1
\end{align*}
\]

(B.2)

or, alternatively,
Fig. B.1. Graphical representation of a $2 \rightarrow 5$ particle reaction. In general, the vectors $\vec{v}_1, \vec{v}_2, \vec{p}_1, \vec{p}_2, \vec{p}_3$, and $\vec{p}_4$ do not lie in the plane of the picture. In the figure, the angle $\theta$ is called $\theta_\gamma$ to emphasize that it belongs to the direction of the radiated photon.

\[
\begin{align*}
(m_1 + m_2)^2 &\leq s_1 \leq (\sqrt{s} - m_3 - m_4)^2 \\
(m_3 + m_4)^2 &\leq s_2 \leq (\sqrt{s} - \sqrt{s_1})^2 \\
(\sqrt{s_1} + \sqrt{s_2})^2 &\leq s' \leq s \\
-1 &\leq \cos \theta \leq +1 \\
0 &\leq \phi_{(R,1,2)} \leq 2\pi \\
-1 &\leq \cos \theta_{(R,1,2)} \leq +1
\end{align*}
\]  

(B.3)

An illustration of the $2 \rightarrow 5$ particle phase space is given in figure B.1.

For the processes under consideration, the integrations over the final state fermion decay angles $\phi_1$, $\theta_1$ and $\phi_2$, $\theta_2$ are separated from the other integrations and are carried out analytically with the help of invariant tensor integration (see appendix D.1). The remaining integrations are nontrivial. We will now explain, how the scalar products appearing in the squared matrix elements can be expressed in the center of mass frame with a Cartesian coordinate system as drawn in figure B.1 using the phase space variables introduced in equation (B.1).
The initial state vectors $k_1$ and $k_2$ are given by

$$k_1 = \left(k^0, k \sin \theta, 0, -k \cos \theta \right),$$
$$k_2 = \left(k^0, -k \sin \theta, 0, k \cos \theta \right)$$

(B.4)

with

$$k^0 = \frac{\sqrt{s}}{2}, \quad k = \frac{\sqrt{s}}{2} \beta, \quad \beta \equiv \sqrt{1 - \frac{4m_e^2}{s}}$$

(B.5)

where $m_e$ is the electron mass.

For the momentum of the photon radiated along the $z$ axis one finds in the center of mass system:

$$p = \left(p^0, 0, 0, p^0 \right) \quad \text{with} \quad p^0 = \frac{s - s'}{2\sqrt{s}}.$$ (B.6)

Thus, the invariant products $k_1k_2$ and $k_ip$ may be expressed easily.

The virtual boson pair recoils from the photon in the center of mass system:

$$\left(v_1 + v_2 \right) = \left(\frac{s + s'}{2\sqrt{s}}, 0, 0, -\frac{s - s'}{2\sqrt{s}} \right).$$ (B.7)

The vectors $v_1$ and $v_2$ in the two-boson rest frame (the $R$-system with $\vec{v}_{1,R} + \vec{v}_{2,R} = 0$) are easily derived:

$$v_{1,R} \equiv \left( p_{12}^{R,0}, 0, 0, \mathcal{P}_R \right) = \left( \frac{s' + s_1 - s_2}{2\sqrt{s'}}, 0, 0, \frac{\sqrt{\lambda(s', s_1, s_2)}}{2\sqrt{s'}} \right),$$

$$v_{2,R} \equiv \left( p_{34}^{R,0}, 0, 0, -\mathcal{P}_R \right) = \left( \frac{s' - s_1 + s_2}{2\sqrt{s'}}, 0, 0, -\frac{\sqrt{\lambda(s', s_1, s_2)}}{2\sqrt{s'}} \right).$$ (B.8)

Both $v_{1,R}$ and $v_{2,R}$ do not depend on angular variables.

We need $v_i$ in the center of mass system. The boost between the center of mass frame and the two-boson rest frame $R$ with zero component $\gamma^{(0)}$ and modulus of the spatial components $\gamma$ is
the center of mass system yields (vectors: are not fixed, equation (B.8) yields the following center of mass system four-vectors:

\[
\begin{align*}
\frac{1}{\gamma} v_1 &= (\gamma_1^{(0)} p_{i2}^{R,0} - \gamma \mathcal{P}_R \cos \theta_R, \mathcal{P}_R \sin \theta_R \cos \phi_R, - \mathcal{P}_R \sin \theta_R \sin \phi_R), \\
\frac{1}{\gamma} v_2 &= (\gamma_1^{(0)} p_{i3}^{R,0} + \gamma \mathcal{P}_R \cos \theta_R, - \mathcal{P}_R \sin \theta_R \cos \phi_R, \mathcal{P}_R \sin \theta_R \sin \phi_R), \\
&\quad -\gamma_1^{(0)} \mathcal{P}_R \cos \theta_R - \gamma p_{i1}^{R,0}).
\end{align*}
\]

Taking into account that the production angles of the virtual vector bosons are not fixed, equation (B.8) yields the following center of mass system four-vectors:

\[
\begin{align*}
\frac{1}{\gamma} v_1 &= (\gamma_1^{(0)} p_{i2}^{R,0} - \gamma \mathcal{P}_R \cos \theta_R, \mathcal{P}_R \sin \theta_R \cos \phi_R, - \mathcal{P}_R \sin \theta_R \sin \phi_R), \\
\frac{1}{\gamma} v_2 &= (\gamma_1^{(0)} p_{i3}^{R,0} + \gamma \mathcal{P}_R \cos \theta_R, - \mathcal{P}_R \sin \theta_R \cos \phi_R, \mathcal{P}_R \sin \theta_R \sin \phi_R), \\
&\quad -\gamma_1^{(0)} \mathcal{P}_R \cos \theta_R - \gamma p_{i1}^{R,0}).
\end{align*}
\]

The sum \((v_{1,R} + v_{2,R})\) in the \(R\)-system is simply \((\sqrt{s'}, 0, 0, 0)\). The boost into the center of mass system yields \((v_1 + v_2) = (\gamma_1^{(0)} \sqrt{s'}, 0, 0, -\gamma \sqrt{s'})\), which is exactly (B.7).

The above formulae allow to express the products \(k_i v_j\) and \(p v_j\). The product \(p_1 p_2\) [and analogously \(p_3 p_4\)] is most easily calculated in the center of mass system \(R_1 \ [R_2]\) defined by \(p_1 + p_2 = 0 \ [p_3 + p_4 = 0]\). This is explained in equations (B.5)–(B.11) of [13] for the 2→4 process. The discussion of reference [13] remains valid also with an additional photon in the initial state. For the remaining scalar products \(p p_i, k_j p_i, v_j p_i, \) and \(p_i p_j\) we used explicit expressions for the vectors \(p_i\) in the center of mass system. The vectors \(p_i\) are obtained from the vectors \(p_{i,R}\) in the \(R\)-system by rotating and applying the Lorentz boost (B.9):

\[
p_i = \begin{pmatrix}
\gamma_1^{(0)} p_{i,R}^{0}(s') - \gamma \left[ p_{i,R}^{x}(s') \cos \theta_R - p_{i,R}^{y}(s') \sin \theta_R \right] \\
\left[ p_{i,R}^{x}(s') \sin \theta_R + p_{i,R}^{y}(s') \cos \theta_R \right] \cos \phi_R + p_{i,R}^{y}(s') \sin \phi_R \\
\left[ -p_{i,R}^{x}(s') \sin \theta_R + p_{i,R}^{y}(s') \cos \theta_R \right] \sin \phi_R + p_{i,R}^{y}(s') \cos \phi_R \\
\gamma_1^{(0)} \left[ p_{i,R}^{x}(s') \cos \theta_R - p_{i,R}^{y}(s') \sin \theta_R \right] - \gamma p_{i,R}^{0}(s') 
\end{pmatrix}.
\]

The four-momenta \(p_{i,R}(s')\) are exactly the final state momenta \(p_i\) shown in equation (B.21) of [13], when these are evaluated at \(s'\) instead of \(s\). The frame \(R\) there has to be understood as frame \(R_1\) or \(R_2\) introduced here.

Now, the derivation of all momenta in the center of mass system in terms of the
integration variables is completed for the 2→5 process and all scalar products appearing in the initial state bremsstrahlung calculation can be expressed in terms of phase space variables.

For the calculation of virtual and soft photon corrections, the 2→4 phase space is needed together with the corresponding momentum representations. It may be taken over completely from appendix B of [13].

C Matrix Elements

In this appendix, we will present the matrix elements for the CC3 and NC2 processes. The matrix elements for the NC8 process are easily obtained from the NC2 matrix elements by adding amplitudes where $Z$ couplings (propagators) have been replaced by photon couplings (propagators).

C.1 Tree Level Amplitudes

For the calculation of soft and virtual corrections, the tree level matrix elements for the CC3 and NC2 processes are given in terms of $s$-channel, $t$-channel, and $u$-channel amplitudes corresponding to the Feynman diagrams of figures 1 and 2:

\[
\mathcal{M}_{\text{CC3}}^B = \mathcal{M}_{s,Z}^B + \mathcal{M}_{s,\gamma}^B + \mathcal{M}_{t,W}^B,
\]
\[
\mathcal{M}_{\text{NC2}}^B = \mathcal{M}_{t,Z}^B + \mathcal{M}_{u,Z}^B
\]

with

\[
\mathcal{M}_{s,Z}^B = \frac{g_{\gamma\gamma'}}{D_{Z}(s)} \frac{g_{\beta\beta'}}{D_{W}(s_1)} \frac{g_{\alpha\alpha'}}{D_{W}(s_2)} B_{s,Z}^{\gamma'} M_{12,W}^{\beta'} M_{34,W}^{\alpha'} g_3(Z) T^{\alpha\beta\gamma},
\]
\[
\mathcal{M}_{s,\gamma}^B = \frac{g_{\gamma\gamma'}}{D_{\gamma}(s)} \frac{g_{\beta\beta'}}{D_{W}(s_1)} \frac{g_{\alpha\alpha'}}{D_{W}(s_2)} B_{s,\gamma}^{\gamma'} M_{12,W}^{\beta'} M_{34,W}^{\alpha'} g_3(\gamma) T^{\alpha\beta\gamma},
\]
\[
\mathcal{M}_{t,W}^B = \frac{g_{\beta\beta'}}{D_{W}(s_1)} \frac{g_{\alpha\alpha'}}{D_{W}(s_2)} \frac{1}{q_t^2} B_{t,W}^{\alpha\beta} M_{12,W}^{\beta'} M_{34,W}^{\alpha'},
\]
\[
\mathcal{M}_{t,Z}^B = \frac{g_{\beta\beta'}}{D_{Z}(s_1)} \frac{1}{q_t^2} B_{t,Z}^{\alpha\beta} M_{12,Z}^{\beta'} M_{34,Z}^{\alpha'},
\]
\[
\mathcal{M}_{u,Z}^B = \frac{g_{\alpha\alpha'}}{D_{Z}(s_1)} \frac{1}{q_u^2} B_{u,Z}^{\alpha\beta} M_{12,Z}^{\alpha'} M_{34,Z}^{\beta'}.
\]
We have used the vector boson propagator denominators $D_V$ defined in equation (A.3), the trilinear coupling $g_3(V^0)$ given in equation (A.9), and

$$T^{\alpha\beta\gamma} = (v_1 - v_2) \gamma g^{\alpha\beta} - 2 v_1^{\alpha} g^{\beta\gamma} + 2 v_2^{\beta} g^{\alpha\gamma}.$$  (C.8)

In the massless approximation, the annihilation matrix elements $B_{s,V^0}^{\gamma'}$, $B_{t,V}^{\alpha\beta}$, $B_{u,V}^{\alpha\beta}$, and the decay matrix elements $M_{ij,V}^{\mu}$ are given by:

$$B_{s,V^0}^{\gamma'} = \bar{u}(-k_2)\gamma^\mu \left[ L(e, V^0) (1 + \gamma_5) + R(e, V^0) (1 - \gamma_5) \right] u(k_1),$$  (C.9)

$$B_{t,V}^{\alpha\beta} = 2 \bar{u}(-k_2)\gamma^\alpha \left[ L^2(e, V) (1 + \gamma_5) + R^2(e, V) (1 - \gamma_5) \right] \gamma^{\beta} u(k_1),$$  (C.10)

$$B_{u,Z}^{\alpha\beta} = 2 \bar{u}(-k_2)\gamma^\alpha \left[ L^2(e, Z) (1 + \gamma_5) + R^2(e, Z) (1 - \gamma_5) \right] \gamma^{\beta} u(k_1),$$  (C.11)

$$M_{ij,V}^{\mu} = \bar{u}(p_l)\gamma^\mu \left[ L(f_i, V) (1 + \gamma_5) + R(f_i, V) (1 - \gamma_5) \right] u(-p_j)$$  (C.12)

with left- and right-handed couplings as defined in equations (A.2) and (A.8). We have further used

$$q_t = k_1 - v_1 = -k_2 + v_2 = \frac{1}{2} (k_1 - k_2 - v_1 + v_2),$$  (C.13)

$$q_u = k_1 - v_2 = -k_2 + v_1 = \frac{1}{2} (k_1 - k_2 + v_1 - v_2).$$  (C.14)

### C.2 Virtual Initial State Corrections

The amplitudes for virtual initial state corrections are easily obtained from the matrix elements in equations (C.3) to (C.7) by substituting the initial state currents.

The $t$-channel NC2 virtual QED matrix element $\mathcal{M}^{V}_{i,Z}$ is represented by the
Feynman diagrams of figure 6 and given by

\[ M_{t,Z}^V = \frac{g_{\beta'\gamma'}}{D_Z(s_1)} \frac{g_{\alpha'\alpha}}{D_Z(s_2)} V_{\alpha\beta} M_{12,Z}^\beta M_{34,Z}^\alpha \] (C.15)

with

\[ V_{\alpha\beta} = e^2 \left\{ \frac{C_{\alpha\beta}}{16\pi^2} - i \mu^{(4-n)} \int \frac{d^n p}{(2\pi)^n} \left( V_{t,\text{vert}1} + V_{t,\text{vert}2} + V_{t,\text{self}} + V_{t,\text{box}} \right) \bigg|_{\mu=m_e} \right\} \] (C.16)

where dimensional regularization is applied with \( p \) being the loop photon momentum, and where

\[ V_{t,\text{vert}1} = \bar{u}(-k_2) \frac{(-2k_2^\mu - \gamma^\mu \gamma^\alpha (\gamma^\mu (\gamma^\nu - \gamma^\nu \gamma^\mu) (2k_{1,\mu} - \gamma^\alpha \gamma^\beta) \gamma^\beta)}{\left( p^2 - i\varepsilon \right) \left[ (p - q_i)^2 - i\varepsilon \right] \left( p^2 + 2k_{2,\mu} - i\varepsilon \right) q_i^2} \times 2 \left[ L^2(e, Z) (1 + \gamma_5) + R^2(e, Z) (1 - \gamma_5) \right] u(k_1), \]

\[ V_{t,\text{vert}2} = \bar{u}(-k_2) \frac{\gamma^\alpha \gamma^\mu (\gamma^\nu - \gamma^\nu \gamma^\mu) (2k_{1,\mu} - \gamma^\alpha \gamma^\beta) \gamma^\beta}{\left( p^2 - i\varepsilon \right) \left[ (p - q_i)^2 - i\varepsilon \right] \left( p^2 + 2k_{2,\mu} - i\varepsilon \right) q_i^2} \times 2 \left[ L^2(e, Z) (1 + \gamma_5) + R^2(e, Z) (1 - \gamma_5) \right] u(k_1), \]

\[ V_{t,\text{self}} = \bar{u}(-k_2) \frac{\gamma^\alpha \gamma^\mu (\gamma^\nu - \gamma^\nu \gamma^\mu) (2k_{1,\mu} - \gamma^\alpha \gamma^\beta) \gamma^\beta}{\left( p^2 - i\varepsilon \right) \left[ (p - q_i)^2 - i\varepsilon \right] \left( p^2 + 2k_{2,\mu} - i\varepsilon \right) q_i^2} \times 2 \left[ L^2(e, Z) (1 + \gamma_5) + R^2(e, Z) (1 - \gamma_5) \right] u(k_1), \]

\[ V_{t,\text{box}} = \bar{u}(-k_2) \frac{(-2k_2^\mu - \gamma^\mu \gamma^\alpha (\gamma^\nu - \gamma^\nu \gamma^\mu) (2k_{1,\mu} - \gamma^\alpha \gamma^\beta) \gamma^\beta)}{\left( p^2 - i\varepsilon \right) \left[ p^2 + 2k_1 p - i\varepsilon \right] \left( p^2 + 2k_2 p - i\varepsilon \right) \left[ (p - q_i)^2 - i\varepsilon \right]} \times 2 \left[ L^2(e, Z) (1 + \gamma_5) + R^2(e, Z) (1 - \gamma_5) \right] u(k_1). \] (C.17)

In the on-shell renormalization scheme, the \( t \)-channel counterterm part in equation (C.16) originates from the counterterms of two vertex and one electron self energy loops. It reads

\[ C_{t,Z}^{\alpha\beta} = \frac{P_{t,Z}^{\alpha\beta}}{q_i^2} (2P + 4P_{\text{IR}} - 4) \] (C.18)
with $B^\alpha_\beta_{t,V}$ from equation (C.10) and the ultraviolet and infrared poles

$$P \equiv \frac{1}{n-4} + \frac{1}{2} \gamma_E + \ln \left( \frac{m_e}{\mu} \right) - \ln \left( 2\sqrt{\pi} \right) \bigg|_{\mu=m_e} n = 4 - \varepsilon, \quad (C.19)$$

$$p^{IR} \equiv \frac{1}{n-4} + \frac{1}{2} \gamma_E + \ln \left( \frac{m_e}{\mu} \right) - \ln \left( 2\sqrt{\pi} \right) \bigg|_{\mu=m_e} n = 4 + \varepsilon. \quad (C.20)$$

Here, $\gamma_E$ is Euler’s constant.

The $u$-channel virtual QED matrix element $\mathcal{M}^V_{u,Z}$ is obtained from $\mathcal{M}^V_{t,Z}$ by the interchanges

$$v_1 \leftrightarrow v_2, \quad \mathcal{M}^\mu_{12,Z} \leftrightarrow \mathcal{M}^\mu_{34,Z}. \quad (C.21)$$

The symmetry (C.21) implies that, after the angular integrations, $\mathcal{M}^V_{u,Z}$ is not explicitly needed any more, because the terms corresponding to the interference of $\mathcal{M}^V_{u,Z}$ with the tree level matrix element is obtained from the interference of $\mathcal{M}^V_{t,Z}$ with the tree level matrix element by interchanging $s_1$ and $s_2$.

The virtual initial state QED matrix element for the $\text{CC3}$ $t$-channel is:

$$\mathcal{M}^V_{t,W} = \frac{g_{\beta\beta'}}{D_W(s_1)} \frac{g_{\alpha\alpha'}}{D_W(s_2)} V^\alpha_\beta_{t,W} M^\beta_\beta_{12,W} M^\alpha_\alpha_{34,W} \quad (C.22)$$

with

$$V^{\alpha_\beta}_{t,W} = e^2 \left\{ V^{\alpha_\beta}_{t,\text{aux}} - i \mu^{(4-n)} \int \frac{d^n p}{(2\pi)^n} V^{\alpha_\beta}_{t,\text{box}} \bigg|_{\mu=m_e} \right\} \quad (C.23)$$

where

$$V^{\alpha_\beta}_{t,\text{aux}} = \frac{C^{\alpha_\beta}_{t,W}}{16\pi^2} - i \mu^{(4-n)} \int \frac{d^n p}{(2\pi)^n} \left( V^{\alpha_\beta}_{t,\text{vert1}} + V^{\alpha_\beta}_{t,\text{vert2}} + V^{\alpha_\beta}_{t,\text{self}} \right) \bigg|_{\mu=m_e}. \quad (C.24)$$

The virtual and counter term contributions are those of (C.17) to (C.20) with corresponding $Z$ couplings replaced by $W$ couplings. The electron mass which, in principle, occurs in the $\text{NC2}$ $t$-channel propagators is absent in the charged current case where a neutrino is exchanged in the $t$-channel instead. Further, the $u$-channel diagrams are absent.
The virtual initial state QED matrix element for the $s$-channel annihilation represented by the amputated Feynman diagram in figure 4 is given by

$$ \mathcal{M}_{s,V^0}^V = \frac{g_{\gamma'\gamma}}{D_{V^0}(s)} \frac{g_{\beta\beta'}}{D_{W}(s_1)} \frac{g_{\alpha\alpha'}}{D_{W}(s_2)} V_{s,V^0}^{\gamma'} M_{12,W}^\beta M_{34,W}^\alpha g_3(V^0) T^{\alpha\beta\gamma} (C.25) $$

with

$$ V_{s,V^0}^{\gamma'} = e^2 \left\{ \frac{C_{s,V^0}^{\gamma'}}{16\pi^2} - i \mu^{(4-n)} \int \frac{d^n p}{(2\pi)^n} V_{s,\text{vert}}^{\gamma'} \bigg|_{\mu=m_e} \right\} (C.26) $$

and

$$ V_{s,\text{vert}}^{\gamma'} = \bar{u}(-k_2) \left\{ \frac{-2k_2^\mu - \gamma^{\mu} \not{\psi}}{p^2 + 2k_2p - i\varepsilon} \right\} \left[ \frac{p^2 - 2k_1p - i\varepsilon}{p^2 - i\varepsilon} \right] \left[ L(e, V^0) (1+\gamma_5) + R(e, V^0) (1-\gamma_5) \right] u(k_1). \quad (C.27) $$

The $s$-channel counterterm part is a mere vertex counterterm given by

$$ C_{s,V^0}^{\gamma'} = B_{s,V^0}^{\gamma'} \left( 2P + 4P^{\text{IR}} - 4 \right) \quad (C.28) $$

with $B_{s,V^0}^{\gamma'}$ from equation (C.9).

### C.3 Initial State Bremsstrahlung Amplitudes

As for the case of virtual initial state QED corrections, initial state bremsstrahlung matrix elements are obtained by altering the amplitudes in equations (C.3) to (C.7). From the Feynman diagrams of figure 7, one deduces the $t$-channel Bremsstrahlung matrix element for the NC2 case:

$$ \mathcal{M}_{t,Z}^{\alpha,\lambda} = \frac{g_{\beta\beta'}}{D_{Z}(s_1)} \frac{g_{\alpha\alpha'}}{D_{Z}(s_2)} M_{12,Z}^\beta M_{34,Z}^\alpha e R_{t,Z}^{\alpha\beta\mu} \varepsilon_\mu(p) \quad (C.29) $$

with

$$ R_{t,Z}^{\alpha\beta\mu} = \bar{u}(-k_2) \left\{ \gamma^\alpha \frac{\not{\psi} - \not{k}_2}{t_2} \gamma^\beta \frac{2k_1^\mu - \not{\psi} \gamma^\mu}{z_1} \right. \\ + \gamma^\alpha \frac{\not{\psi} - \not{k}_2}{t_2} \gamma^\mu \frac{\not{k}_1 - \not{\psi}}{t_1} \gamma^\beta \\ + \left. \gamma^\mu \not{\psi} - 2k_2^\mu \right\} \frac{\gamma^\alpha \not{k}_1 - \not{\psi}_1}{t_1} \gamma^\beta \right\} $$
\[ \times 2 \left[ L^2(e, Z) (1 + \gamma_5) + R^2(e, Z) (1 - \gamma_5) \right] u(k_1). \quad (C.30) \]

An analogous result is obtained for the NC2 \( u \)-channel initial state bremsstrahlung matrix element,

\[
M_{u, Z}^{\alpha, \alpha'} = \frac{g_{\gamma\gamma'}}{D_{Z}(s_1)} \frac{g_{\beta\beta'}}{D_{Z}(s_2)} \frac{M_{12, Z}^\alpha M_{34, Z}^{\alpha'}}{g_{\gamma\gamma'}} e R_{u, Z}^{\alpha\beta \mu} \varepsilon_\mu(p) \\
R_{u, Z}^{\gamma\beta \mu} = \bar{u}(-k_2) \left\{ \gamma^\alpha \gamma^\gamma 2k_1^\mu - \not{p} \gamma^\mu \frac{z_1}{u_2} \gamma^\beta \gamma_1^\beta \not{\gamma} \gamma_2^\beta \frac{z_1}{u_2} + \gamma^\mu \gamma_1^\beta \not{\gamma} \gamma_2^\beta \frac{z_1}{u_2} \right\} \\
\times 2 \left[ L^2(e, Z) (1 + \gamma_5) + R^2(e, Z) (1 - \gamma_5) \right] u(k_1). \quad (C.31) \]

For the \( s \)-channel bremsstrahlung matrix element of the CC3 process one has the Feynman diagrams of figure 5 and obtains

\[
M_{s, V^0}^{R, \lambda} = \frac{g_{\gamma\gamma'}}{D_{V^0}(s)} \frac{g_{\beta\beta'}}{D_{W}(s_1)} \frac{g_{\alpha\alpha'}}{D_{W}(s_2)} M_{12, W}^{\beta'} M_{34, W}^{\alpha'} g_{3}(V^0) T^{\alpha\beta \gamma} e R_{s, V^0}^{\gamma\mu \mu} \varepsilon_\mu(p) \\
(C.32) \]

with

\[
R_{s, V^0}^{\gamma\mu \mu} = \bar{u}(-k_2) \left\{ \gamma^\gamma \gamma_1^\mu - \not{\gamma} \gamma_1^\mu \frac{z_1}{u_2} + \gamma^\mu \gamma_1^\mu \not{\gamma} \gamma_2^\mu \frac{z_1}{u_2} \right\} \\
\times \left[ L(e, V^0) (1 + \gamma_5) + R(e, V^0) (1 - \gamma_5) \right] u(k_1). \quad (C.33) \]

All the initial state currents introduced so far fulfill the condition of U(1) current conservation,

\[
p_\mu R_{s, V^0}^{\gamma\mu \mu} = p_\mu R_{t}^{\alpha\beta \mu} = p_\mu R_{u}^{\alpha\beta \mu} = 0. \quad (C.34) \]

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There is no CC3 \( u \)-channel diagram. For the CC3 \( t \)-channel diagram, the situation is slightly more complicated. Naively, only the two left diagrams in figure 7 contribute to the CC3 \( t \)-channel ISR corrections. For these two, current conservation is violated and may be established by adding as a auxiliary current the additional Feynman diagram with radiation from the \( t \)-channel propagator [7] (cf. the discussion in section 3.2). Then, the charged current \( t \)-channel matrix element is just given by (C.29) with \( Z \) couplings replaced by \( W \) couplings.

In the above matrix elements the radiatively changed Mandelstam variables were introduced. In terms of the five-particle phase space parametrization from appendix B they are given by

\[
\begin{align*}
t_1 &= (k_1 - v_1)^2 + m_v^2 = a_1^t + b_1 \cos \theta_R + c \sin \theta_R \cos \phi_R, \\
t_2 &= (v_2 - k_2)^2 + m_v^2 = a_2^t + b_2 \cos \theta_R + c \sin \theta_R \cos \phi_R, \\
u_1 &= (k_1 - v_2)^2 + m_v^2 = a_1^u - b_1 \cos \theta_R - c \sin \theta_R \cos \phi_R, \\
u_2 &= (v_1 - k_2)^2 + m_v^2 = a_2^u - b_2 \cos \theta_R - c \sin \theta_R \cos \phi_R.
\end{align*}
\] (C.35)

Here \( m_v = m_e \) for NC2 and \( m_v = m_\nu = 0 \) for CC3. In (C.35), the kinematical parameters

\[
\begin{align*}
a_1^t &= m_v^2 - m_e^2 - s_1 + \frac{s' + \delta}{4 s'} (s' + s'_{\perp} \beta \cos \theta), \\
a_2^t &= m_v^2 - m_e^2 - s_2 + \frac{s' - \delta}{4 s'} (s'_{\perp} + s'_{\perp} \beta \cos \theta), \\
a_1^u &= m_v^2 - m_e^2 - s_2 + \frac{s' - \delta}{4 s'} (s'_{\perp} - s'_{\perp} \beta \cos \theta), \\
a_2^u &= m_v^2 - m_e^2 - s_1 + \frac{s' + \delta}{4 s'} (s'_{\perp} + s'_{\perp} \beta \cos \theta), \\
b_1 &= \frac{\sqrt{\lambda}}{4 s'} (s'_{\perp} \beta \cos \theta - s'_t), \\
b_2 &= \frac{\sqrt{\lambda}}{4 s'} (s'_{\perp} \beta \cos \theta + s'_t), \\
c &= -\frac{\sqrt{\lambda}}{2} \sqrt{\frac{s}{s'}} \beta \sin \theta
\end{align*}
\] (C.36)

are derived from the representations (B.4) and (B.10) in appendix B. The variables \( \delta, s'_{\perp}, s'_t, \sqrt{\lambda} \) are introduced in sections 2 and 3.
The two invariants related to radiation from external legs are

\[
\begin{align*}
  z_1 &= -2 k_1 p = \frac{s'}{2} \left( 1 + \beta \cos \theta \right) \equiv \frac{s'}{2} \tilde{z}_1, \\
  z_2 &= -2 k_2 p = \frac{s'}{2} \left( 1 - \beta \cos \theta \right) \equiv \frac{s'}{2} \tilde{z}_2.
\end{align*}
\]

(C.37)

The equations (C.36) and (C.37) are exact. In (C.37), one has to keep the electron mass, since it gives rise to mass singularities. The electron mass in (C.36) may be neglected, but it was retained in some integrals for the sake of numerical stability.

\section*{D Phase Space Integrals}

In this appendix, we collect the analytical solutions of the integrals needed for the semi-analytical cross-section calculation of processes (1.1). These integrals are necessary to obtain the twofold differential cross-section \( d^2 \sigma / ds_1 ds_2 \) in the case of virtual ISR and the threefold differential cross-section \( d^3 \sigma / ds' ds_1 ds_2 \) for initial state bremsstrahlung. Integrals are presented in the ultrarelativistic approximation, i.e. the electron mass is neglected wherever possible. The integrals have been numerically checked. The azimuth around the beam yields a factor \( 2\pi \). We do not elaborate on the pure s-channel in this appendix, because it is well-known that its initial state radiation is fully covered by the universal ISR corrections.

\subsection*{D.1 Fermion Decay Angle Tensor Integrals}

Integrations over final state fermion decay angles are carried out with the help of invariant tensor integration. Using the decay matrix element \( M_{12,V}^{\mu} \) from equation (C.12) one obtains

\[
\begin{align*}
  \sum_{Spins} \int d\Gamma_{12} M_{12,V}^{\mu} M_{12,V}^{\nu*} & \equiv \sqrt{\frac{\lambda(s_1,m_1^2,m_2^2)}{8 s_1}} \sum_{Spins} \int_{-1}^{1} d\cos \theta_1 \int_{0}^{2\pi} d\phi_1 M_{12,V}^{\mu} M_{12,V}^{\nu*} \\
  & = \sqrt{\frac{\lambda(s_1,m_1^2,m_2^2)}{8 s_1}} \frac{4\pi}{3} \left[ L^2(f_1,V) + R^2(f_1,V) \right] \\
  & \quad \times \left[ s_1 g^{\mu\nu} + (p_1^\mu + p_2^\mu) (p_1^\nu + p_2^\nu) \right].
\end{align*}
\]

(D.1)
with \( L(f_1, V) \) and \( R(f_1, V) \) from (A.2) and (A.8). An analogous result is valid for the integration of \( M_{34,V}^\mu M_{34,V}^{\mu*} \) over \( d\Gamma_{34} \). The result of the above decay phase space integral factorizes in the cross-section. Therefore one can use formula (D.1) to integrate the final state fermion decay angles for both the \( 2 \to 4 \) and the initial state bremsstrahlung \( 2 \to 5 \) particle phase space.

### D.2 Loop Integrals

For the computation of the virtual initial state QED corrections to the studied processes, apart from the integration over final state fermion decay angles, two sets of integrals are needed. The first set is presented in this subsection and consists of the integration over the loop momentum \( p \):

\[
[A]_n = \mu^{(4-n)} \int \frac{d^n p}{(2\pi)^n} A \bigg|_{\mu=m_e} \quad \text{with} \quad |n-4| \ll 1. \quad (D.2)
\]

The second set, the integration over \( \cos \theta \) (in \( 2 \to 4 \) kinematics), is given in section D.3. It should be noticed that we present only the loop integrals for the virtual \( s \)- and \( t \)-channel graphs, because, in the twofold differential cross-section, the interferences of the virtual \( u \)-channel graphs with the tree level \( t \)- and \( u \)-channel graphs equal the interferences of the virtual \( t \)-channel graphs with the tree level \( u \)-channel and \( t \)-channel graphs. This is easily verified by symmetry arguments. We use the definitions

\[
\begin{align*}
\Pi_0 &= \frac{1}{p^2 - i\varepsilon} \\
\Pi_1 &= \frac{1}{(p-k_1)^2 + m_e^2 - i\varepsilon} = \frac{1}{p^2 - 2pk_1 - i\varepsilon} \\
\Pi_2 &= \frac{1}{(p+k_2)^2 + m_e^2 - i\varepsilon} = \frac{1}{p^2 + 2pk_2 - i\varepsilon} \\
\Pi_q &= \frac{1}{(p-q_1)^2 - i\varepsilon}
\end{align*}
\]

and \( q_2^2 = t \). For later use we define \( q_2^2 = u \). Dimensional regularization is used to consistently treat divergent loop integrals. The ultraviolet and infrared poles are taken from equations (C.19) and (C.20). For the sake of compact presentation of the subsequent integrals we introduce two symmetry operations, namely
\[ \mathcal{T} \left[ f(s_1, s_2) \right] \equiv f(s_2, s_1) \quad \text{(D.4)} \\
\mathcal{U} \left[ g(k_1, k_2) \right] \equiv g(-k_2, -k_1). \quad \text{(D.5)} \]

Further we introduce the notations

\[ l_0 \equiv \ln \left( -\frac{s}{m^2_e - i\varepsilon} \right) = \ln \frac{s}{m^2_e} - i\pi \equiv l_\beta - i\pi, \]
\[ l_1 \equiv \ln \left( -\frac{s_1}{m^2_e - i\varepsilon} \right) = \ln \frac{s_1}{m^2_e} - i\pi, \]
\[ l_2 \equiv \ln \left( -\frac{s_2}{m^2_e - i\varepsilon} \right) = \ln \frac{s_2}{m^2_e} - i\pi, \]
\[ l_t \equiv \ln \frac{q^2}{m^2_e}. \quad \text{(D.6)} \]

One obtains the following table of canonical integrals:

1) \[ \left[ \frac{1}{\Pi_0 \Pi_1} \right]_n = \left[ \frac{1}{\Pi_0 \Pi_2} \right]_n = \frac{i}{16 \pi^2} \left( -2P + 2 \right) \]
2) \[ \left[ \frac{p^\mu}{\Pi_0 \Pi_1} \right]_n = \mathcal{U} \left( \left[ \frac{p^\mu}{\Pi_0 \Pi_2} \right]_n \right) = \frac{i k^\mu_1}{16 \pi^2} \left( -P + \frac{1}{2} \right) \]
3) \[ \left[ \frac{1}{\Pi_0 \Pi_q} \right]_n = \frac{i}{16 \pi^2} \left( -2P - l_t + 2 \right) \]
4) \[ \left[ \frac{p^\mu}{\Pi_0 \Pi_q} \right]_n = \frac{q^\mu}{2} \left[ \frac{1}{\Pi_0 \Pi_q} \right]_n = \frac{i q^\mu}{16 \pi^2} \left( -P - \frac{l_t}{2} + 1 \right) \]
\[ \frac{1}{\Pi_1 \Pi_2} = \frac{i}{16 \pi^2} (\pi^2 - 2P - l_0 + 2) \]

\[ \left[ \frac{p^\mu}{\Pi_1 \Pi_2} \right]_n = \frac{(k_1 - k_2)^\mu}{2} \left[ \frac{1}{\Pi_1 \Pi_2} \right]_n = \frac{i}{16 \pi^2} (k_1 - k_2)^\mu \left( -P - \frac{l_0}{2} + 1 \right) \]

\[ \frac{1}{\Pi_1 \Pi_2} = \mathcal{T} \left( \left[ \frac{1}{\Pi_2 \Pi_q} \right]_n \right) = \frac{i}{16 \pi^2} (2P - l_1 + 2) \]

\[ \left[ \frac{p^\mu}{\Pi_1 \Pi_q} \right]_n = \mathcal{U} \left( \left[ \frac{p^\mu}{\Pi_2 \Pi_q} \right]_n \right) = \frac{i}{16 \pi^2} (q^\mu + k_1^\mu) \left( -P - \frac{l_1}{2} + 1 \right) \]

\[ \frac{1}{\Pi_0 \Pi_1 \Pi_2} = \frac{i}{16 \pi^2} \left[ -\frac{2l_0}{s} p^{\mu \nu} + \frac{1}{s} \left( \frac{\pi^2}{6} - \frac{l_0^2}{2} \right) \right] \]

\[ \left[ \frac{p^\mu}{\Pi_0 \Pi_1 \Pi_2} \right]_n = -\frac{i}{16 \pi^2} (k_1 - k_2)^\mu \frac{l_0}{s} \]

\[ \left[ \frac{p^\mu p^\nu}{\Pi_0 \Pi_1 \Pi_2} \right]_n = \frac{i}{16 \pi^2} \left[ g^{\mu \nu} \left( -\frac{1}{2} P - \frac{l_0}{4} + \frac{3}{4} \right) \right. \]

\[ \left. + \left( k_1^\mu k_1^\nu + k_2^\mu k_2^\nu \right) \frac{1 - l_0}{2s} + \left( k_1^\mu k_2^\nu + k_2^\mu k_1^\nu \right) \frac{1}{2s} \right] \]

\[ \frac{1}{\Pi_0 \Pi_1 \Pi_q} = \mathcal{T} \left( \left[ \frac{1}{\Pi_0 \Pi_2 \Pi_q} \right]_n \right) \]

\[ = \frac{i}{16 \pi^2} \frac{1}{t + s_1} \left[ \frac{1}{2} \left( l_t - l_1 \right) \left( 3l_1 - l_t \right) - 2 \text{Li}_2 \left( \frac{s_1 + t - i\varepsilon}{s_1} \right) \right] \]

\[ \left[ \frac{p^\mu}{\Pi_0 \Pi_1 \Pi_q} \right]_n = \mathcal{U} \left( \left[ \frac{p^\mu}{\Pi_0 \Pi_2 \Pi_q} \right]_n \right) \]

\[ = \frac{i}{16 \pi^2} \frac{1}{t + s_1} \left\{ \begin{array}{l} q^\mu (l_t - l_1) \\ + k_1^\mu \left( -l_1 + t \left[ \frac{1}{\Pi_0 \Pi_1 \Pi_q} \right]_n - \frac{2t}{t + s_1} (l_t - l_1) \right) \end{array} \right\} \]
\[ 14) \left[ \frac{p^\mu p^\nu}{\Pi_0 \Pi_1 \Pi_2 q} \right]_n = \mathcal{U} \left( \left[ \frac{p^\mu p^\nu}{\Pi_0 \Pi_1 \Pi_2 q} \right]_n \right) \]

\[
= \frac{i}{16 \pi^2} \left\{ g^{\mu\nu} \left( -\frac{1}{2} P + \frac{3}{4} - \frac{1}{4} \left( t + s_1 \right) \left( t l_t + s_1 l_l \right) \right) 
+ \frac{k^\mu_1 k^\nu_1}{t + s_1} \left\{ \left( \frac{1}{2} + \frac{t}{t + s_1} \right) \left( 1 - l_1 \right) - \frac{3 t^2}{(t + s_1)^2} \left( l_t - l_1 \right) 
+ \frac{t^2}{t + s_1} \left[ \frac{1}{\Pi_0 \Pi_1 \Pi_2 q} \right]_n \right\} 
+ \frac{k^\mu_1 q^\nu + k^\nu_1 q^\mu}{2 (t + s_1)} \left\{ \frac{t}{t + s_1} \left( l_t - l_1 \right) - 1 \right\} 
+ \frac{q^\mu q^\nu}{2 (t + s_1)} \left( l_t - l_1 \right) \right\} 
\]

\[ 15) \left[ \frac{1}{\Pi_1 \Pi_2 q} \right]_n = \frac{i}{16 \pi^2} I_{12q} \]

\[ 16) \left[ \frac{p^\mu}{\Pi_1 \Pi_2 q} \right]_n = \frac{i}{16 \pi^2} \lambda \times \left\{ q^\mu \left[ s (s - \sigma) I_{12q} - (s + \delta) \ln \frac{s_1}{s} - (s - \delta) \ln \frac{s_2}{s} \right] 
+ k^\mu_1 \left[ - s_2 (s + \delta) I_{12q} + (s - \sigma) \ln \frac{s_1}{s} + 2 s_2 \ln \frac{s_2}{s} \right] 
+ k^\mu_2 \left[ s_1 (s - \delta) I_{12q} - 2 s_1 \ln \frac{s_1}{s} - (s - \sigma) \ln \frac{s_2}{s} \right] \right\} \]

\[ 17) \left[ \frac{p^\mu p^\nu}{\Pi_1 \Pi_2 q} \right]_n = \frac{i}{16 \pi^2} \left\{ g^{\mu\nu} F_{21} + q^\mu q^\nu F_{22} + k^\mu_1 k^\nu_1 F_{23} + k^\mu_1 k^\nu_2 F_{24} 
+ \left( k^\mu_2 k^\nu_2 + k^\nu_2 k^\mu_2 \right) F_{25} + \left( k^\mu_1 q^\nu + k^\nu_1 q^\mu \right) F_{26} 
+ \left( k^\mu_2 q^\nu + k^\nu_2 q^\mu \right) F_{27} \right\} \]

\[ F_{21} = \frac{1}{4} \left\{ \frac{1}{\lambda} \left[ s_1 (s - \delta) \ln \frac{s_1}{s} + s_2 (s + \delta) \ln \frac{s_2}{s} - 2 s s_1 s_2 I_{12q} \right] 
- 2 P + 3 - l_0 \right\} \]
\[ F_{22} = \frac{1}{\lambda} \left\{ -s - \frac{1}{2} \left[ 3s + \delta + \frac{6ss_1}{\lambda} (s-\delta) \right] \ln \frac{s_1}{s} \right. \]
\[- \frac{1}{2} \left[ 3s - \delta + \frac{6ss_2}{\lambda} (s+\delta) \right] \ln \frac{s_2}{s} \]
\left. + s^2 \left[ 1 + \frac{6s_1s_2}{\lambda} \right] I_{12q} \right\} \]

\[ F_{23} = \frac{1}{\lambda} \left\{ -s_2 + \frac{1}{2} \left[ s - s_1 - 3s_2 - \frac{6s_1s_2}{\lambda} (s-\delta) \right] \ln \frac{s_1}{s} \right. \]
\[- \frac{3}{2} s^2 \left( s + \delta \right) \ln \frac{s_2}{s} + s_2^2 \left[ 1 + \frac{6s_1s_2}{\lambda} \right] I_{12q} \left\} \right. \]

\[ F_{24} = \frac{1}{\lambda} \left\{ -s_1 + \frac{1}{2} \left[ s - s_2 - 3s_1 - \frac{6s_2s_1}{\lambda} (s+\delta) \right] \ln \frac{s_2}{s} \right. \]
\[- \frac{3}{2} s^2 \left( s - \delta \right) \ln \frac{s_1}{s} + s_1^2 \left[ 1 + \frac{6s_2s_1}{\lambda} \right] I_{12q} \left\} \right. \]

\[ F_{25} = \frac{1}{2\lambda} \left\{ s - \sigma + s_1 \left[ 1 + \frac{6s_2}{\lambda} (s+\delta) \right] \ln \frac{s_1}{s} \right. \]
\left. + s_2 \left[ 1 + \frac{6s_1}{\lambda} (s-\delta) \right] \ln \frac{s_2}{s} \right. \]
\left. + 2s_1s_2 \left[ 1 - \frac{3s}{\lambda} (s-\sigma) \right] I_{12q} \left\} \right. \]

\[ F_{26} = \frac{1}{\lambda} \left\{ \frac{s - \delta}{2} + \left[ \frac{s + s_2}{2} + \frac{6s_1s_2}{\lambda} \right] \ln \frac{s_1}{s} \right. \]
\[- \left[ \frac{s_2}{2} - \frac{3ss_2}{\lambda} (s-\sigma) \right] \ln \frac{s_2}{s} \]
\left. - ss_2 \left[ 1 + \frac{3s_1}{\lambda} (s-\delta) \right] I_{12q} \right\} \]
\[ F_{27} = -\frac{1}{\lambda} \left\{ \frac{s + \delta}{2} + \left[ \frac{s + s_1}{2} + \frac{6 ss_1 s_2}{\lambda} \right] \ln \frac{s_2}{s} \right. \]
\[ \left. - \left[ \frac{s_1}{2} - \frac{3 ss_1}{\lambda} (s - \sigma) \right] \ln \frac{s_1}{s} \right. \]
\[ \left. - ss_1 \left[ 1 + \frac{3 s_2}{\lambda} (s + \delta) \right] I_{12q} \right\} \]

\[ 18) \left[ \frac{2 p q}{\Pi_0 \Pi_1 \Pi_2 \Pi_3} \right]_n = \frac{i}{16 \pi^2} \left\{ I_{12q} + \frac{1}{s} \left[ l_0 (l_1 + l_2 - 2 l_t) + \frac{1}{2} l_t^2 \right] \right\} \]

D.3 Virtual Corrections’ Phase Space Integrals

After the integration over the loop momentum, there is one non-trivial integration left when the virtual initial state corrections to the NC2 or CC3 processes are evaluated. This is the integration over the boson scattering angle \( \vartheta \) in the center of mass system. We recall that the below integrals do not cover the interferences of virtual \( u \)-channel graphs with tree level \( t \)- and \( u \)-channel graphs, because these interferences are deduced from the interferences of the virtual \( t \)-channel graphs with the tree level \( u \)-channel and \( t \)-channel graphs by symmetry arguments. We introduce the notation

\[ [A]_V \equiv \frac{\sqrt{\lambda}}{2} \int_{-1}^{+1} d\cos \vartheta A = \int_{t_{\min}}^{t_{\max}} dt A \]

with \( t_{\min} \) and \( t_{\max} \) as defined in equation (3.4).

Below, we use

\[ t_{12} \equiv s_1 + t \]
\[ t_{34} \equiv s_2 + t \]

and adopt the notations of equations (3.4) and (3.6). The following list of integrals is obtained:
1) \([1_t]_V = \sqrt{\lambda}\)

2) \([t]_V = \frac{\sqrt{\lambda}}{2} (s - \sigma)\)

3) \([t^2]_V = \frac{\sqrt{\lambda}}{3} \left\{ (s - \sigma)^2 - s_1 s_2 \right\}\)

4) \(\left[ \frac{1}{t} \right]_V = \left[ \frac{1}{u} \right]_V = \ln \frac{s - \sigma + \sqrt{\lambda}}{s - \sigma - \sqrt{\lambda}} = \mathcal{L}_0\)

5) \(\left[ \frac{1}{t^2} \right]_V = \left[ \frac{1}{u^2} \right]_V = \frac{\sqrt{\lambda}}{s_1 s_2}\)

6) \(\left[ \frac{1}{t u} \right]_V = \frac{1}{s - \sigma} \left( \left[ \frac{1}{t} \right]_V + \left[ \frac{1}{u} \right]_V \right) = \frac{2}{s - \sigma} \mathcal{L}_0\)

The above integrals 1) to 6) are familiar from the calculation of the tree level cross-section, while the integrals 7) to 29) genuinely originate from the virtual corrections.

7) \(\left[ \frac{1}{t_{12}} \right]_V = \mathcal{T} \left( \left[ \frac{1}{t_{34}} \right]_V \right) = \frac{\sqrt{\lambda}}{ss_1}\)

8) \(\left[ \frac{1}{t_{12}} \right]_V = \mathcal{T} \left( \left[ \frac{1}{t_{34}} \right]_V \right) = \ln \frac{s + \delta + \sqrt{\lambda}}{s + \delta - \sqrt{\lambda}} = \mathcal{L}_{12}\)

9) \([l_t]_V = \frac{s - \sigma}{2} \mathcal{L}_0 + \frac{\sqrt{\lambda}}{2} \mathcal{L}_S - \sqrt{\lambda}\)

\[\mathcal{L}_S = \left( 2 l_\beta + \ln \frac{s_1}{s} + \ln \frac{s_2}{s} \right)\]

10) \([t l_t]_V = \frac{(s - \sigma)^2}{4} - 2 s_1 s_2 \mathcal{L}_0 + \frac{(s - \sigma) \sqrt{\lambda}}{4} (\mathcal{L}_S - 1)\)

11) \(\left[ \frac{l_t}{t} \right]_V = \frac{1}{2} \mathcal{L}_0 \mathcal{L}_S\)

12) \(\left[ \frac{l_t}{u} \right]_V = \ln \frac{s - \sigma}{m_e^2} \mathcal{L}_0 - \text{Li}_2 \left( \frac{t_{\text{max}}}{s - \sigma} \right) + \text{Li}_2 \left( \frac{t_{\text{min}}}{s - \sigma} \right)\)
13) \[ \left[ \frac{t_1}{t_{12}} \right]_V = \mathcal{T} \left( \left[ \frac{t_1}{t_{12}} \right]_V \right) = \frac{\mathcal{L}_{12} \mathcal{L}_S}{2} - \frac{\mathcal{L}_0}{2} \ln \frac{s_1}{s} + \text{Li}_2 \left( -\frac{t_{\text{max}}}{s_1} \right) - \text{Li}_2 \left( -\frac{t_{\text{min}}}{s_1} \right) \]

14) \[ \left[ \frac{l_t}{t^2} \right]_V = \frac{1}{2} \frac{\sqrt{\lambda}}{s_1 s_2} \left( \mathcal{L}_S + 2 \right) - \left( s - \sigma \right) \mathcal{L}_0 \]

15) \[ \left[ \frac{l_t}{t_{12}^2} \right]_V = \mathcal{T} \left( \left[ \frac{l_t}{t_{12}^2} \right]_V \right) = \frac{1}{s_1} \left( \frac{s - \delta}{2s} \mathcal{L}_0 - \mathcal{L}_{12} + \frac{\sqrt{\lambda}}{2s} \mathcal{L}_S \right) \]

16) \[ \left[ l_t^2 \right]_V = \frac{s - \sigma}{2} \left( \mathcal{L}_S - 2 \right) \mathcal{L}_0 + \sqrt{\lambda} \left( \frac{\mathcal{L}_0^2}{4} + \frac{\mathcal{L}_S^2}{4} - \mathcal{L}_S + 2 \right) \]

17) \[ \left[ t_1 t_2^2 \right]_V = \frac{(s - \sigma)^2 - 2s_1 s_2}{4} \mathcal{L}_0 \left( \mathcal{L}_S - 1 \right) + \left( s - \sigma \right) \frac{\sqrt{\lambda}}{4} \left( 1 - \mathcal{L}_S + \frac{\mathcal{L}_0^2 + \mathcal{L}_S^2}{2} \right) \]

18) \[ \left[ \frac{l_t^2}{t} \right]_V = \mathcal{L}_0 \left( \frac{\mathcal{L}_0^2}{12} + \frac{\mathcal{L}_S^2}{4} \right) \]

19) \[ \left[ \frac{l_t^2}{u} \right]_V = \left( \ln^2 \frac{s - \sigma}{m_0^2} - \frac{\pi^2}{6} \right) \mathcal{L}_0 - \mathcal{L}_S \left[ \text{Li}_2 \left( \frac{t_{\text{max}}}{s - \sigma} \right) - \text{Li}_2 \left( \frac{t_{\text{min}}}{s - \sigma} \right) \right] + 2 \left[ \text{Li}_3 \left( \frac{t_{\text{max}}}{s - \sigma} \right) - \text{Li}_3 \left( \frac{t_{\text{min}}}{s - \sigma} \right) \right] \]

20) \[ \left[ \frac{l_t^2}{t_{12}} \right]_V = \mathcal{T} \left( \left[ \frac{l_t^2}{t_{12}} \right]_V \right) = \frac{\mathcal{L}_0^2 + \mathcal{L}_S^2}{4} \mathcal{L}_{12} - \frac{\mathcal{L}_0 \mathcal{L}_S}{2} \ln \frac{s_1}{s} \]

\[
+ \left( \mathcal{L}_0 + \mathcal{L}_S \right) \text{Li}_2 \left( -\frac{t_{\text{max}}}{s_1} \right) + \left( \mathcal{L}_0 - \mathcal{L}_S \right) \text{Li}_2 \left( -\frac{t_{\text{min}}}{s_1} \right) - 2 \text{Li}_3 \left( -\frac{t_{\text{max}}}{s_1} \right) + 2 \text{Li}_3 \left( -\frac{t_{\text{min}}}{s_1} \right) \]

21) \[ \left[ \frac{l_t^2}{t^2} \right]_V = \frac{1}{s_1 s_2} \left[ \sqrt{\lambda} \left( \frac{\mathcal{L}_0^2}{4} + \frac{\mathcal{L}_S^2}{4} + \mathcal{L}_S + 2 \right) - \frac{s - \sigma}{2} \left( \mathcal{L}_S + 2 \right) \mathcal{L}_0 \right] \]

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\[ \left[ \frac{l_{12}^2}{t_{12}^2} \right]_V = \mathcal{T} \left( \left[ \frac{l_{12}^2}{t_{34}^2} \right]_V \right) = \frac{1}{s_1} \left[ \frac{s-\delta}{2} \mathcal{L}_0 \mathcal{L}_S + \frac{\ln s_1}{s} \mathcal{L}_0 - \mathcal{L}_S \mathcal{L}_{12} \right. \\
+ \frac{\sqrt{\lambda}}{4s} (\mathcal{L}_0^2 + \mathcal{L}_S^2) - 2 \text{Li}_2 \left( - \frac{t_{\text{max}}}{s_1} \right) + 2 \text{Li}_2 \left( - \frac{t_{\text{min}}}{s_1} \right) \right] \]

\[ \left[ \text{Li}_2 \left( \frac{t_{12} - i \varepsilon}{s_1} \right) \right]_V = \mathcal{T} \left( \left[ \text{Li}_2 \left( \frac{t_{34} - i \varepsilon}{s_2} \right) \right]_V \right) \\
= \frac{s-\sigma}{2} \mathcal{L}_0 + \frac{\sqrt{\lambda}}{2} (\mathcal{L}_S - 2 l_1 - 2) \\
+ \left( s_1 + t_{\text{max}} \right) \text{Li}_2 \left( 1 + \frac{t_{\text{max}} - i \varepsilon}{s_1} \right) \\
- \left( s_1 + t_{\text{min}} \right) \text{Li}_2 \left( 1 + \frac{t_{\text{min}} - i \varepsilon}{s_1} \right) \]

\[ \left[ t \text{Li}_2 \left( \frac{t_{12} - i \varepsilon}{s_1} \right) \right]_V = \mathcal{T} \left( \left[ t \text{Li}_2 \left( \frac{t_{34} - i \varepsilon}{s_2} \right) \right]_V \right) = -\frac{\sqrt{\lambda}}{8} (s-5s_1-s_2) \\
+ \frac{(s-\sigma)(s-3s_1-s_2) - 2s_1s_2}{8} \mathcal{L}_0 - \frac{\sqrt{\lambda}}{8} (s-3s_1-s_2) (l_- - 2\pi i) \\
+ \frac{t_{\text{max}}^2 - s_1^2}{2} \text{Li}_2 \left( 1 + \frac{t_{\text{max}} - i \varepsilon}{s_1} \right) - \frac{t_{\text{min}}^2 - s_1^2}{2} \text{Li}_2 \left( 1 + \frac{t_{\text{min}} - i \varepsilon}{s_1} \right) \]

\[ \left[ \frac{1}{t} \text{Li}_2 \left( \frac{t_{12} - i \varepsilon}{s_1} \right) \right]_V = \mathcal{T} \left( \left[ \frac{1}{t} \text{Li}_2 \left( \frac{t_{34} - i \varepsilon}{s_2} \right) \right]_V \right) \\
= \frac{\pi^2}{6} \mathcal{L}_0 - 2 \text{Li}_3 \left( - \frac{t_{\text{max}}}{s_1} \right) + 2 \text{Li}_3 \left( - \frac{t_{\text{min}}}{s_1} \right) \\
+ \left( \ln \frac{t_{\text{max}}}{m_e^2} - l_1 \right) \text{Li}_2 \left( - \frac{t_{\text{max}}}{s_1} \right) \\
- \left( \ln \frac{t_{\text{min}}}{m_e^2} - l_1 \right) \text{Li}_2 \left( - \frac{t_{\text{min}}}{s_1} \right) \]
26) \( \text{Re} \left \{ \frac{1}{t} \text{Li}_2 \left( \frac{t_{12} - i \epsilon}{s_1} \right) \right \}_V = \mathcal{T} \left \{ \text{Re} \left \{ \frac{1}{t} \text{Li}_2 \left( \frac{t_{34} - i \epsilon}{s_2} \right) \right \}_V \right \} \)

\[
= \mathcal{L}_0 \left[ -\frac{\mathcal{L}_0^2}{6} - \frac{1}{2} \ln(d_1) \ln \frac{s_1}{s_2} - \ln^2(d_1) - 2 \text{Li}_2 \left( \frac{1}{d_1} \right) + \frac{2\pi^2}{3} \right ]
- \ln \frac{t_{\text{max}}}{s_1} \text{Li}_2 \left( \frac{t_{\text{min}}}{s - s_2} \right ) + \ln \frac{t_{\text{min}}}{s_1} \text{Li}_2 \left( \frac{t_{\text{max}}}{s - s_2} \right )
+ \text{Li}_3 \left( \frac{t_{\text{max}}}{s_1} \right ) - \text{Li}_3 \left( \frac{t_{\text{min}}}{s_2} \right ) - \text{Li}_3 \left( \frac{t_{\text{max}}}{s - s_2} \right ) + \text{Li}_3 \left( \frac{t_{\text{min}}}{s - s_2} \right )
+ \text{Li}_3 \left( \frac{t_{\text{max}}}{d_1 t_{\text{min}}} \right ) - \text{Li}_3 \left( \frac{t_{\text{min}}}{d_1 t_{\text{max}}} \right )
\]

27) \( \frac{1}{t_{12}} \text{Li}_2 \left( \frac{t_{12} - i \epsilon}{s_1} \right ) \right \}_V = \mathcal{T} \left \{ \frac{1}{t_{34}} \text{Li}_2 \left( \frac{t_{34} - i \epsilon}{s_2} \right ) \right \}_V \)

\[
= \text{Li}_3 \left( 1 + \frac{t_{\text{max}} - i \epsilon}{s_1} \right ) - \text{Li}_3 \left( 1 + \frac{t_{\text{min}} - i \epsilon}{s_1} \right )
\]

28) \( \frac{1}{t^2} \text{Li}_2 \left( \frac{t_{12} - i \epsilon}{s_1} \right ) \right \}_V = \mathcal{T} \left \{ \frac{1}{t^2} \text{Li}_2 \left( \frac{t_{34} - i \epsilon}{s_2} \right ) \right \}_V \)

\[
= -\frac{1}{t_{\text{max}}} \text{Li}_2 \left( 1 + \frac{t_{\text{max}} - i \epsilon}{s_1} \right ) + \frac{1}{t_{\text{min}}} \text{Li}_2 \left( 1 + \frac{t_{\text{min}} - i \epsilon}{s_1} \right )
+ \frac{1}{s_1} \left[ \text{Li}_2 \left( 1 + \frac{s_1}{t_{\text{max}} - i \epsilon} \right ) - \text{Li}_2 \left( 1 + \frac{s_1}{t_{\text{min}} - i \epsilon} \right ) \right ]
\]

29) \( \frac{1}{t_{12}^2} \text{Li}_2 \left( \frac{t_{12} - i \epsilon}{s_1} \right ) \right \}_V = \mathcal{T} \left \{ \frac{1}{t_{34}^2} \text{Li}_2 \left( \frac{t_{34} - i \epsilon}{s_2} \right ) \right \}_V \)

\[
= \frac{1}{s_1} \left[ \mathcal{L}_{12} - \frac{1 - s_1 + s_2}{2s} \mathcal{L}_0 + \frac{\sqrt{\lambda}}{2s} (l_- - 2\pi i)
- \frac{t_{\text{min}} + s_1}{s} \text{Li}_2 \left( 1 + \frac{t_{\text{max}} - i \epsilon}{s_1} \right ) + \frac{t_{\text{max}} + s_1}{s} \text{Li}_2 \left( 1 + \frac{t_{\text{min}} - i \epsilon}{s_1} \right ) \right ]
\]
D.4 Bremsstrahlung Integrals – First Series

For the calculation of bremsstrahlung to processes (1.1), the phase space was parametrized as exposed in equation (B.3) and appendix B. After integration over the final state fermion decay angles, the first series of bremsstrahlung integrals is over the scattering azimuth and polar angles $\phi_R$ and $\theta_R$ of the boson three-vector $v_1$ in the two-boson rest frame:

$$[A]_R \equiv \frac{1}{4\pi} \int_0^{2\pi} d\phi_R \int_{-1}^{1} d\cos\theta_R A.$$ (D.9)

We introduce a third symmetry operation,

$$S[f(\cos\theta)] \equiv f(-\cos\theta)$$ (D.10)

Taking into account that the contribution of the bremsstrahlung $u$-channel matrix element squared to the threefold differential cross-section equals the contribution of the bremsstrahlung $t$-channel matrix element, the following bremsstrahlung integrals are needed for the first series:

1) $[1]_R = 1 \quad 2) [\cos\phi_R f(\cos\theta_R)]_R = 0$

3) $[\cos\theta_R]_R = 0 \quad 4) [\cos^2\phi_R]_R = 1/2$
5) \([\cos^2 \theta_R]_R = 1/3\)

7) \[\begin{array}{c}
\left[ \frac{1}{t_1} \right]_R = \frac{2s'}{\sqrt{X} (s'_+ - s'_- \cos \theta)} \ln \left( \frac{s'_+ (s'_+ + \sqrt{X}) - 2s' \sigma + s'_+ \delta - (s'_+ + \delta + \sqrt{X}) s'_+ \cos \theta}{s'_- (s'_- - \sqrt{X}) - 2s' \sigma + s'_- \delta - (s'_- + \delta - \sqrt{X}) s'_- \cos \theta} \right) \\
≡ \frac{2s'}{\sqrt{X} S_{d1} t_{t1}}
\end{array}\]

8) \[\left[ \frac{1}{t_2} \right]_R = ST \left( \left[ \frac{1}{t_1} \right]_R \right) \equiv \frac{2s'}{\sqrt{X} S_{d2} t_{t2}}\]

9) \[\left[ \frac{1}{u_1} \right]_R = T \left( \left[ \frac{1}{t_1} \right]_R \right) \equiv \frac{2s'}{\sqrt{X} S_{d1} u_{u1}}\]

10) \[\left[ \frac{1}{u_2} \right]_R = S \left( \left[ \frac{1}{t_1} \right]_R \right) \equiv \frac{2s'}{\sqrt{X} S_{d2} u_{u2}}\]

11) \[\left[ \frac{\cos \theta_R}{t_1} \right]_R = \frac{16 s'^2 b_1}{X' S_{d1}^2} \left( 1 - a_1^t \left[ \frac{1}{t_1} \right]_R \right)\]

12) \[\left[ \frac{\cos \theta_R}{t_2} \right]_R = \frac{16 s'^2 b_2}{X' S_{d2}^2} \left( 1 - a_2^t \left[ \frac{1}{t_2} \right]_R \right)\]

13) \[\left[ \frac{1}{t_1} \right]_R = \frac{4s'}{s_1 (s - \delta + \sqrt{X} - s'_- \cos \theta) (s - \delta - \sqrt{X} - s'_- \cos \theta)} \equiv \frac{4s'}{s_1 d_1^t d_1}\]

14) \[\left[ \frac{1}{t_2} \right]_R = ST \left( \left[ \frac{1}{t_1} \right]_R \right) \equiv \frac{4s'}{s_2 d_2^t d_2}\]
$$15) \left[ \frac{1}{t_1 t_2} \right]_R = \frac{1}{2S_{t_1 t_2}} \ln \left( \frac{A_{1-}^t A_{2+}^t}{A_{1+}^t A_{2-}^t} \right) \equiv \frac{1}{2S_{t_1 t_2}} l_{t12}$$

$$S_{t_1 t_2} = \sqrt{\lambda'} \left( \sqrt{\lambda + \delta + s'_- \cos \theta} \right) \left( \sqrt{\lambda - \delta - s'_- \cos \theta} \right)$$

$$\equiv \frac{\sqrt{\lambda'}}{8s'} S_{12+}^t S_{12-}^t$$

$$A_{1-}^t = a_{1-} + b_{1-} \cos \theta \quad A_{1+}^t = a_{1+} + b_{1+} \cos \theta$$

$$A_{2-}^t = a_{2-} + b_{2-} \cos \theta \quad A_{2+}^t = a_{2+} + b_{2+} \cos \theta$$

$$a_{1-} = s^2 \sigma - s \delta^2 + \frac{s}{2} s'_- (s' + \delta - 2\sigma) - \sqrt{\lambda'} \left( s \delta - \frac{s}{2} s'_- \right)$$

$$b_{1-} = - \left( \sqrt{\lambda'} + s' + \delta \right) \frac{s}{2} s'_-$$

$$a_{1+} = s^2 \sigma - s \delta^2 + \frac{s}{2} s'_- (s' + \delta) - \sqrt{\lambda'} \left( s \delta - \frac{s}{2} s'_- \right)$$

$$b_{1+} = + \left( \sqrt{\lambda'} - s' - \delta \right) \frac{s}{2} s'_-$$

$$a_{2-} = s^2 \sigma - s \delta^2 + \frac{s}{2} s'_- (s' - \delta - 2\sigma) - \sqrt{\lambda'} \left( s \delta + \frac{s}{2} s'_- \right)$$

$$b_{2-} = - \left( \sqrt{\lambda'} - s' + \delta \right) \frac{s}{2} s'_-$$

$$a_{2+} = s^2 \sigma - s \delta^2 + \frac{s}{2} s'_- (s' - \delta) + \sqrt{\lambda'} \left( s \delta + \frac{s}{2} s'_- \right)$$

$$b_{2+} = + \left( \sqrt{\lambda'} + s' - \delta \right) \frac{s}{2} s'_-$$
16) \[
\frac{1}{[u_1 u_2]_{R}} = S\left(\frac{1}{[t_1 t_2]_{R}}\right) \equiv \frac{1}{2S_{u_1 u_2}} l_{u_12}
\]

17) \[
\frac{1}{[t_1 u_1]_{R}} = \frac{1}{a_{1}^{ut}}\left(\frac{1}{[t_1]_{R}} + \frac{1}{[u_1]_{R}}\right)
\]

\[
a_{1}^{ut} = \frac{1}{2}(s_{+}' - s_{-}' \cos \theta) - \sigma \equiv \frac{a}{2}(1 - b \cos \theta)
\]

\[
a = s_{+}' - 2\sigma, \quad b = \frac{s_{+}'}{s_{+}' - 2\sigma}, \quad 0 \leq b < 1
\]

18) \[
\frac{1}{[t_2 u_2]_{R}} = \frac{1}{a_{2}^{ut}}\left(\frac{1}{[t_2]_{R}} + \frac{1}{[u_2]_{R}}\right)
\]

\[
a_{2}^{ut} = \frac{1}{2}(s_{+}' + s_{-}' \cos \theta) - \sigma = \frac{a}{2}(1 + b \cos \theta)
\]

19) \[
\frac{1}{[t_1 u_2]_{R}} = \frac{1}{2\sqrt{C_{12}}} \left[\ln \left(\frac{A_{1}^{-} B_{2}^{+}}{A_{1}^{+} B_{2}^{-}}\right) + 2\ln \left(\frac{a_{s1} + b_{d}}{a_{s1} - b_{d}}\right)\right] \equiv \frac{1}{2\sqrt{C_{12}}} l_{t_1 u_2}
\]

\[
A_{1}^{-} = \sqrt{C_{12}} b_{d} (a_{1}^{t} + b_{1}) + C_{12} + B_{1}^{t} (a_{s1} - b_{d}) / 2
\]

\[
A_{1}^{+} = \sqrt{C_{12}} b_{d} (a_{1}^{t} - b_{1}) + C_{12} + B_{1}^{t} (a_{s1} + b_{d}) / 2
\]

\[
B_{2}^{-} = \sqrt{C_{12}} b_{d} (a_{2}^{u} - b_{2}) + C_{12} + B_{2}^{u} (a_{s1} - b_{d}) / 2
\]

\[
B_{2}^{+} = \sqrt{C_{12}} b_{d} (a_{2}^{u} + b_{2}) + C_{12} + B_{2}^{u} (a_{s1} + b_{d}) / 2
\]

\[
C_{12} = \left(\frac{\sqrt{X}}{2s'}\right)^{2} (A_{12} \cos^{2} \theta + B_{12})
\]

\[
A_{12} = -s_{+}^{2} s_{1} (s - s_{1})
\]

\[
B_{12} = s_{+}^{2} s (s - s_{1}) - 2s s_{+} (s - \sigma) (s - s_{1}) + s^{2} (s - \sigma)^{2}
\]

\[
B_{1}^{t} = -2 \left[b_{1} (a_{1}^{t} b_{2} + a_{2}^{u} b_{1}) + a_{s1} c^{2}\right]
\]

\[
B_{2}^{u} = -2 \left[b_{2} (a_{1}^{t} b_{2} + a_{2}^{u} b_{1}) + a_{s1} c^{2}\right]
\]

\[
a_{s1} = a_{1}^{t} + a_{2}^{u} = \frac{s_{+}'}{2s'} (s' + \delta) - 2s_{1}
\]

\[
b_{d} = b_{2} - b_{1} = \sqrt{\frac{X}{2s'} s_{+}'}
\]
20) \[ \frac{1}{t_2 u_1} \bigg|_R = \mathcal{T} \left( \left[ \frac{1}{t_1 u_2} \right]_R \right) \equiv \frac{1}{2 \sqrt{C_2}} l_{2u_1} \]

21) \[ \frac{\cos \theta_R}{t_1^2} \bigg|_R = \frac{16 s^2 b_1}{X S_{d1}^2} \left( \left[ \frac{1}{t_1} \right]_R - a_t \left[ \frac{1}{t_1^2} \right]_R \right) \]

22) \[ \frac{\cos \theta_R}{t_2^2} \bigg|_R = \frac{16 s^2 b_2}{X S_{d2}^2} \left( \left[ \frac{1}{t_2} \right]_R - a_t \left[ \frac{1}{t_2^2} \right]_R \right) \]

23) \[ \frac{\cos^2 \theta_R}{t_1} \bigg|_R = \frac{16 s^2}{X S_{d1}^2} \left\{ -a_t^2 + \frac{1}{2} \left( 2a_t^2 + c^2 \right) \left[ \frac{1}{t_1} \right]_R \right\} \]
\[ \quad + \frac{384 s^4}{X^2 S_{d1}^4} a_t^2 \left\{ 1 - a_t \left[ \frac{1}{t_1} \right]_R \right\} \]

24) \[ \frac{\cos^2 \theta_R}{t_2} \bigg|_R = \mathcal{T} \left( \left[ \frac{\cos^2 \theta_R}{t_2} \right] \bigg|_R \right) \]

25) \[ \frac{1}{t_1 t_2} \bigg|_R = \frac{1}{S_{t2}} \left\{ -a_t^2 + a_t^2 \left( a_t a_d - b_1 b_d \right) \left[ \frac{1}{t_1^2} \right]_R \right\} \]
\[ + \left( b_2 \left( a_t b_2 - a_t b_1 \right) - a_t c^2 \right) \left[ \frac{1}{t_1 t_2} \right]_R \right\} \]
\[ a_d = a_2 - a_1 = -\frac{s'}{2s'} (\delta - s' \cos \theta) \]

26) \[ \frac{1}{t_1 t_2^2} \bigg|_R = \mathcal{T} \left( \left[ \frac{1}{t_1 t_2} \right] \bigg|_R \right) \]

27) \[ \frac{1}{t_1 t_2 u_1} \bigg|_R = \frac{1}{a_t} \left( \left[ \frac{1}{t_1 t_2} \right]_R + \left[ \frac{1}{t_2 u_1} \right]_R \right) \]

28) \[ \frac{1}{t_1 t_2 u_2} \bigg|_R = \frac{1}{a_2} \left( \left[ \frac{1}{t_1 t_2} \right]_R + \left[ \frac{1}{t_1 u_2} \right]_R \right) \]
\begin{align*}
29) & \left[ \frac{1}{t_1 u_1 u_2} \right]_{R} = \frac{1}{a_1^{ut}} \left( \left[ \frac{1}{u_1 u_2} \right]_{R} + \left[ \frac{1}{t_1 u_2} \right]_{R} \right) \\
30) & \left[ \frac{1}{t_2 u_1 u_2} \right]_{R} = \frac{1}{a_2^{ut}} \left( \left[ \frac{1}{u_1 u_2} \right]_{R} + \left[ \frac{1}{t_2 u_1} \right]_{R} \right) \\
31) & \left[ \frac{1}{t_1 t_2 u_1 u_2} \right]_{R} = \frac{1}{a_1^{ut}} \frac{1}{a_2^{ut}} \left( \left[ \frac{1}{t_1 t_2} \right]_{R} + \left[ \frac{1}{u_1 u_2} \right]_{R} + \left[ \frac{1}{t_1 u_2} \right]_{R} + \left[ \frac{1}{t_2 u_1} \right]_{R} \right)
\end{align*}

D.5 Bremsstrahlung Integrals – Second Series

Below, the integrals needed for the last analytical integration in the bremsstrahlung case are given. This integration is over the photon scattering angle $\theta$. Using the ultrarelativistic approximation and the notation

$$[A]_{\theta} \equiv \frac{1}{2} \int_{-1}^{+1} d\cos \theta \ A,$$

(D.11)

the following integrals are required for the “second series” of bremsstrahlung integrals:

1) \([1]_{\theta} = 1\)

$$2) \left[ \frac{1}{z_1} \right]_{\theta} = \left[ \frac{1}{z_2} \right]_{\theta} = \frac{1}{2} l_{\beta}$$

$$3) \left[ \frac{m_{e}^{2}}{z_{1}^{2}} \right]_{\theta} = \left[ \frac{m_{e}^{2}}{z_{2}^{2}} \right]_{\theta} = \frac{s}{4}$$

$$4) \left[ \frac{1}{S_{d1}} \right]_{\theta} = \left[ \frac{1}{S_{d2}} \right]_{\theta} = \frac{1}{2s'_{s}} \ln \left( \frac{s}{s'} \right)$$

$$5) \left[ \frac{1}{S_{d1}^{2}} \right]_{\theta} = \left[ \frac{1}{S_{d2}^{2}} \right]_{\theta} = \frac{1}{4ss'}$$

$$6) \left[ \frac{1}{S_{d1}^{3}} \right]_{\theta} = \left[ \frac{1}{S_{d2}^{3}} \right]_{\theta} = \frac{s'_{s}}{16 s^{2} s'^{2}}$$
7) \([l_{t1}]_\theta = - \left(1 + \frac{s' - \delta}{2s_{-}}\right) L_{c1} + \frac{\sqrt{\lambda}}{2s_{-}} L_{c3} + L_{c5}\)

\[L_{c1} = \ln \left(\frac{s'_l(s' + \sqrt{\lambda}) + s's\sigma - s\delta}{s'_l(s' - \sqrt{\lambda}) + s's\sigma - s\delta}\right)\]

\[L_{c3} = \ln \left(1 + \frac{s'_l(s - \delta)}{s's_{2}}\right)\]

\[L_{c5} = \ln \left(\frac{s' - \sigma + \sqrt{\lambda}}{s' - \sigma - \sqrt{\lambda}}\right)\]

8) \([l_{t2}]_\theta = \mathcal{T}( [l_{t1}]_\theta ) = - \left(1 + \frac{s' + \delta}{2s_{-}}\right) L_{c2} + \frac{\sqrt{\lambda}}{2s_{-}} L_{c4} + L_{c5}\)

\[L_{c2} = \ln \left(\frac{s'_l(s' + \sqrt{\lambda}) + s's\sigma + s\delta}{s'_l(s' - \sqrt{\lambda}) + s's\sigma + s\delta}\right)\]

\[L_{c4} = \ln \left(1 + \frac{s'_l(s + \delta)}{s's_{1}}\right)\]

9) \([l_{t12}]_\theta = \left(1 + \frac{s' - \delta}{2s_{-}}\right) L_{c1} + \left(1 + \frac{s' + \delta}{2s_{-}}\right) L_{c2} - \frac{\sqrt{\lambda}}{2s_{-}} (L_{c3} + L_{c4})\]

10) \(\left[\frac{1}{z_{1}} l_{t1}\right]_\theta = \left[\frac{1}{z_{2}} l_{t2}\right]_\theta = \frac{1}{2} \left( l_{\beta} L_{t1} - D_{z_{1}t1} \right)\)

\[L_{t1} = L_{c5} - L_{c1}\]

\[D_{z_{1}t1} = \text{Li}_2\left(\frac{s'_l(s + s'_l - \delta - \sqrt{\lambda})}{2s(s_{-}' - \delta) + s'(\sigma + \delta)}\right) - \text{Li}_2\left(\frac{s'_l(s + s'_l - \delta + \sqrt{\lambda})}{2s(s_{-}' - \delta) + s'(\sigma + \delta)}\right)\]

11) \(\left[\frac{1}{z_{1}} l_{t2}\right]_\theta = \left[\frac{1}{z_{2}} l_{a1}\right]_\theta = \frac{1}{2} \left( l_{\beta} L_{c5} - D_{z_{1}t2} \right)\)

\[D_{z_{1}t2} = \text{Li}_2\left(\frac{-s'_l(s' + \delta - \sqrt{\lambda})}{s'(\sigma + \delta)}\right) - \text{Li}_2\left(\frac{-s'_l(s' + \delta + \sqrt{\lambda})}{s'(\sigma + \delta)}\right)\]
12) \[ \left[ \frac{1}{z_2} l_{t1} \right]_\theta = \left[ \frac{1}{z_1} l_{u2} \right]_\theta = \frac{1}{2} \left( l_\beta L_{c5} - D_{z2t1} \right) \]

\[ D_{z2t1} = \text{Li}_2 \left( \frac{-s_l'(s' - \delta - \sqrt{\lambda})}{s'(\sigma - \delta)} \right) - \text{Li}_2 \left( \frac{-s_l'(s' - \delta + \sqrt{\lambda})}{s'(\sigma - \delta)} \right) \]

13) \[ \left[ \frac{1}{z_2} l_{t2} \right]_\theta = \left[ \frac{1}{z_1} l_{u1} \right]_\theta = \frac{1}{2} \left( l_\beta L_{r2} - D_{z2t2} \right) \]

\[ L_{r2} = L_{c5} - c_{e2} \]

\[ D_{z2t2} = \text{Li}_2 \left( \frac{s'_l(s + s'_l + \delta - \sqrt{\lambda})}{2s(s'_l + \delta) + s'(\sigma - \delta)} \right) - \text{Li}_2 \left( \frac{s'_l(s + s'_l + \delta + \sqrt{\lambda})}{2s(s'_l + \delta) + s'(\sigma - \delta)} \right) \]

14) \[ \left[ \frac{1}{S_{d1}} l_{t1} \right]_\theta = \left[ \frac{1}{S_{d2}} l_{u2} \right]_\theta \]

\[ = \frac{1}{2s'_l} \text{Re} \left[ - \text{Li}_2 \left( \frac{s(s' + \delta + \sqrt{\lambda})}{s'(\sigma + \delta)} \right) + \text{Li}_2 \left( \frac{s' + \delta + \sqrt{\lambda}}{\sigma + \delta} \right) \right] + \text{Li}_2 \left( \frac{s(s' + \delta - \sqrt{\lambda})}{s'(\sigma + \delta)} \right) - \text{Li}_2 \left( \frac{s' + \delta - \sqrt{\lambda}}{\sigma + \delta} \right) \]

\[ \equiv \frac{1}{2s_l} D_1 \]

15) \[ \left[ \frac{1}{S_{d2}} l_{t2} \right]_\theta = \left[ \frac{1}{S_{d1}} l_{u1} \right]_\theta = \mathcal{T} \left( \left[ \frac{1}{S_{d1}} l_{t1} \right]_\theta \right) \equiv \frac{1}{2s_l} D_2 \]

\[ ^7 \text{A comment on the treatment of arguments of logarithms and dilogarithms is in order here. The logarithm is undefined for negative real values, the dilogarithm has a cut for real numbers larger than 1. This means that, in principle, infinitesimal imaginary parts from the propagator denominators have to be carried along in the whole calculation. This, however, is an unnecessary nuisance, because all integrands are real and regular. If infinitesimal imaginary parts were needed, because logarithm or dilogarithm arguments lay on cuts in final results, they were attributed to the invariant masses } s_1 \text{ and } s_2. \text{ As integrands are real and regular, this is a correct treatment, because it is then irrelevant for the integrand how an infinitesimal imaginary part is entered. This technique will be used in subsequent integrals without further notice.} \]
\[ \left[ \frac{1}{z_1} l_{112} \right]_\theta = \left[ \frac{1}{z_2} l_{u12} \right]_\theta = \frac{1}{2} \left( l_\beta L_{c1} + D_{z11} + D_{z12} \right) \]

\[ \left[ \frac{1}{z_2} l_{112} \right]_\theta = \left[ \frac{1}{z_1} l_{u12} \right]_\theta = \frac{1}{2} \left( l_\beta L_{c2} + D_{z21} + D_{z22} \right) \]

\[ \left[ \frac{1}{a_{1u}^u} l_{112} \right]_\theta = \left[ \frac{1}{a_{2u}^u} l_{u12} \right]_\theta = \frac{1}{s} \text{ Re} \left[ L_{c8} L_{c6} + D_{a1}^{lu} \right] \]

\[ L_{c8} = \ln \frac{s - \sigma}{s' - \sigma} \]

\[ L_{c6} = \ln \frac{(a_{1-} b + b_{1-})(a_{2+} b + b_{2+})}{(a_{1+} b + b_{1+})(a_{2-} b + b_{2-})} \]

\[ D_{a1}^{lu} = - \text{Li}_2 \left( \frac{b_{1-}(1 + b)}{a_{1-} b + b_{1-}} \right) + \text{Li}_2 \left( \frac{b_{1-}(1 - b)}{a_{1-} b + b_{1-}} \right) \]

\[ - \text{Li}_2 \left( \frac{b_{2+}(1 + b)}{a_{2+} b + b_{2+}} \right) + \text{Li}_2 \left( \frac{b_{2+}(1 - b)}{a_{2+} b + b_{2+}} \right) \]

\[ + \text{Li}_2 \left( \frac{b_{1+}(1 + b)}{a_{1+} b + b_{1+}} \right) - \text{Li}_2 \left( \frac{b_{1+}(1 - b)}{a_{1+} b + b_{1+}} \right) \]

\[ + \text{Li}_2 \left( \frac{b_{2-}(1 + b)}{a_{2-} b + b_{2-}} \right) - \text{Li}_2 \left( \frac{b_{2-}(1 - b)}{a_{2-} b + b_{2-}} \right) \]

\[ \left[ \frac{1}{a_{2u}^u} l_{112} \right]_\theta = \left[ \frac{1}{a_{1u}^u} l_{u12} \right]_\theta = \frac{1}{s} \text{ Re} \left[ L_{c8} L_{c7} + D_{a2}^{lu} \right] \]

\[ L_{c7} = \mathcal{T} (L_{c6}) \]

\[ D_{a2}^{lu} = \mathcal{T} (D_{a1}^{lu}) \]

\[ \left[ \frac{1}{S_{a1}^2} l_{11} \right]_\theta = \mathcal{T} \left( \left[ \frac{1}{S_{a2}^2} l_{12} \right]_\theta \right) \]

\[ = \frac{1}{4 s' s_1} \left( \sqrt{\lambda} \ln \frac{s'}{s} + \frac{\sqrt{\lambda}}{2} L_{c3} - \frac{s' - \sigma}{2} L_{c5} \right) \]

\[ + s(s' + \delta) - s'(\sigma + \delta) \frac{L_{t1}}{2s} \]
21) \[ \left[ \frac{1}{S^2_{d1}} l_{t1} \right]_\theta = T \left( \left[ \frac{1}{S^2_{d2}} l_{t2} \right]_\theta \right) \]

\[ = \frac{1}{16 s' L_{c3}} \left[ \frac{s' \sqrt{X}}{ss'^2 s_1} + \frac{(s' + \delta) \sqrt{X}}{2 s'^2 s_1^2} L_{c3} \right. \]

\[ + \frac{\sigma^2 - \delta^2 + 2\sigma\delta + 2s' (\sigma - \delta) - 2s'^2}{4 s'^2 s_1^3} L_{c5} \]

\[ + \frac{(s' + \delta) \sqrt{X}}{s'^2 s_1^2} \ln \frac{s'}{s} + \left( \frac{s'^2 - s' (\sigma - \delta) + \delta^2}{2 s'^2 s_1^2} - \frac{1}{s^2} \right) L_{t1} \]

22) \[ \left[ \frac{1}{S^4_{d1}} l_{t1} \right]_\theta = T \left( \left[ \frac{1}{S^4_{d2}} l_{t2} \right]_\theta \right) \]

\[ = \frac{1}{48 s' L_{c5}} \left[ - \frac{s' \sqrt{X}}{ss'^3 s_1} \left( \frac{s'}{2s} + \frac{s' + \delta}{s_1} \right) - \frac{1}{s^3} L_{t1} + \frac{1}{s s^3} L_{c5} \right. \]

\[ + \frac{2s'^2 + 2\delta^2 + 3s'\delta - s'\sigma}{4s'^3 s_1^3} \sqrt{X} \left( 2 \ln \frac{s'}{s} + L_{c3} \right) \]

\[ - \frac{(s' + \delta)(2s'^2 + 2\delta^2 + s'\delta - 3s'\sigma)}{4s'^3 s_1^3} L_{c1} \]

23) \[ \left[ \frac{m^2_{e}}{z_1^2} l_{t2} \right]_\theta = \left[ \frac{m^2_{e}}{z_2^2} l_{t1} \right]_\theta = \left[ \frac{m^2_{e}}{z_1^2} l_{u2} \right]_\theta = \left[ \frac{m^2_{e}}{z_2^2} l_{u1} \right]_\theta = \frac{s}{4} L_{c5} \]

24) \[ \left[ \frac{1}{d^+_1 d^-_1} \right]_\theta = T \left( \left[ \frac{1}{d^+_2 d^-_2} \right]_\theta \right) = \frac{1}{4 s' \sqrt{X}} L_{c1} \]

25) \[ \left[ \frac{\cos \theta}{d^+_1 d^-_1} \right]_\theta = \frac{1}{4 s' \sqrt{X}} \left( \frac{s - \delta}{\sqrt{X}} L_{c1} - L_{c3} \right) \]

26) \[ \left[ \frac{\cos \theta}{d^+_2 d^-_2} \right]_\theta = - \frac{1}{4 s' \sqrt{X}} \left( \frac{s + \delta}{\sqrt{X}} L_{c2} - L_{c4} \right) \]

27) \[ \left[ \frac{1}{S^4_{12} + S^4_{12}} \right]_\theta = \frac{1}{4 s' \sqrt{X}} L_{\lambda} \]

\[ L_{\lambda} = \ln \left( \frac{s' + \sigma + \sqrt{X}}{s' + \sigma - \sqrt{X}} \right) \]
28) \[
\cos \theta \left[ \frac{S_{12}^t}{\bar{S}_{12}^t} \right]_{\theta} = \frac{1}{4 s_-^2} \left( \ln \frac{s_1}{s_2} - \frac{\delta}{\sqrt{\lambda}} L_\lambda \right)
\]

29) \[
\cos \theta \left[ \frac{1}{(S_{12}^t)^2 (\bar{S}_{12}^t)^2} \right]_{\theta} = \frac{1}{8 s_-^2 \lambda} \left( \frac{1}{\sqrt{\lambda}} L_\lambda + \frac{s_1^t \sigma + \delta^2}{2 s_-^2 s_1 s_2} \right)
\]

30) \[
\cos \theta \left[ \frac{l_{12}}{\bar{S}_{12}^t} \right]_{\theta} = \frac{\delta}{8 s_-^2 \lambda} \left( -\frac{1}{\sqrt{\lambda}} L_\lambda + \frac{s_1^t + \sigma}{2 s_1 s_2} \right)
\]

31) \[
\cos \theta \left[ \frac{l_{12}}{\bar{S}_{12}^t} \right]_{\theta} = \frac{1}{4 s_-^2 \lambda} \text{Re} \left[ -\text{Li}_2 \left( \frac{c_{++} a_{--} e_+}{d} \right) + \text{Li}_2 \left( \frac{c_{--} a_{--} e_+}{d} \right) + \text{Li}_2 \left( \frac{c_{++} a_{--} e_-}{d} \right) - \text{Li}_2 \left( \frac{c_{--} a_{--} e_-}{d} \right) + \text{Li}_2 \left( \frac{c_{++} a_{++} e_+}{d} \right) - \text{Li}_2 \left( \frac{c_{--} a_{++} e_+}{d} \right) + \text{Li}_2 \left( \frac{c_{++} a_{++} e_-}{d} \right) - \text{Li}_2 \left( \frac{c_{--} a_{++} e_-}{d} \right) + \text{Li}_2 \left( \frac{c_{++} a_{+++} e_+}{d} \right) - \text{Li}_2 \left( \frac{c_{--} a_{+++} e_+}{d} \right) + \text{Li}_2 \left( \frac{c_{++} a_{+++} e_-}{d} \right) - \text{Li}_2 \left( \frac{c_{--} a_{+++} e_-}{d} \right) \right]
\]

\[ = \frac{1}{4 s_-^2 \sqrt{\lambda}} D_{12}^t \]

\[ a_{\pm \pm} = s \pm \sqrt{\lambda} \pm \sqrt{\lambda} \]

\[ c_{\pm \pm} = \delta \pm s_- \pm \sqrt{\lambda} \]

\[ e_{\pm} = s' - \sigma \pm \sqrt{\lambda} \]

\[ d = 8 s s_1 s_2 \]
32) \[
\frac{l_{112}}{(S_{12+}^t)^2 (S_{12-}^t)^2} = \frac{1}{2 \lambda} \left[ \frac{l_{112}}{S_{12+}^t S_{12-}^t} \right]_{\theta} \\
+ \frac{1}{8 s_t} \left[ \frac{\sqrt{\lambda} \sqrt{\lambda}}{2 s s_1 s_2} L_{c_1} + \frac{s_- - \delta}{2 s_-} L_{c_2} + \frac{s_- + \delta}{2 s_-} L_{c_2} \\
+ \frac{\lambda - s(s' + \sigma)}{4 s s_1 s_2} (L_{c_1} + L_{c_2}) - \frac{\sqrt{\lambda} (s' + \sigma)}{4 s s_1 s_2} (L_{c_3} + L_{c_4}) \right]
\]

33) \[
\frac{\cos \theta \ l_{112}}{(S_{12+}^t)^2 (S_{12-}^t)^2} = \frac{\delta}{s_t} \left[ \frac{l_{112}}{(S_{12+}^t)^2 (S_{12-}^t)^2} \right]_{\theta} \\
+ \frac{1}{8 s_t} \left[ \frac{\sqrt{\lambda} \ln s_1}{2 s s_1 s_2} - \frac{1}{2 s_-} L_{c_1} + \frac{1}{2 s_-} L_{c_2} - \frac{s' - \sigma}{4 s s_1 s_2} (L_{c_1} - L_{c_2}) - \frac{\sqrt{\lambda}}{4 s s_1 s_2} (L_{c_3} - L_{c_4}) \right]
\]

34) \[
\frac{l_{112}}{(S_{12+}^t)^3 (S_{12-}^t)^3} = \frac{3}{4 \lambda} \left[ \frac{l_{112}}{(S_{12+}^t)^2 (S_{12-}^t)^2} \right]_{\theta} + \frac{1}{32 s_t \sqrt{\lambda}} \left[ \frac{\sqrt{\lambda} \sqrt{\lambda} (s' \sigma + 2\delta^2)}{4 s_1 s_2^2} + \frac{\sqrt{\lambda} \delta}{4 s_1 s_2^2} L_{c_1} + \frac{\sqrt{\lambda}}{4 s_1 s_2^2} L_{c_1} \\
+ \frac{\sqrt{\lambda}}{4 s_1 s_2^2} L_{c_2} - \frac{\sqrt{\lambda} (2 s s_1 s_2 + \lambda (s' - \sigma))}{4 s_1 s_2^2} L_{c_1} \\
- \frac{\sqrt{\lambda} (6 s s_1 s_2 - s^2 (s' + \sigma) + \lambda (s' - \sigma))}{8 s_1 s_2^2} (L_{c_1} + L_{c_2}) + \frac{\sqrt{\lambda} \sqrt{\lambda} (s(s' + \sigma) - 2 s s_1 s_2 - \lambda)}{8 s_1 s_2^2} (L_{c_3} + L_{c_4}) \right]
\]

\[ L_{c_12} = \frac{1}{s_-} \left( \frac{s_1^2}{s_-} L_{c_2} - \frac{s_2^2}{s_-} L_{c_1} - \frac{\delta \sqrt{\lambda}}{s} \right) \]
Integrals 36) to 41) were not directly calculated from the result of the \( \Omega^8 \) integration. Instead, a Feynman parametrization was used to “linearize” the \( \beta \) parameter.

\[
35) \left[ \frac{\cos \theta \ l_{12}}{(S_{12}^t)^3 (S_{12}^{-})^3} \right]_\theta = \frac{1}{4 \lambda} \left[ \frac{\cos \theta \ l_{12}}{(S_{12}^t)^2 (S_{12}^{-})^2} \right]_\theta + \frac{\delta}{4 s'_\lambda} \left[ \frac{l_{12}}{(S_{12}^t)^2 (S_{12}^{-})^2} \right]_\theta
\]

\[
- \frac{\delta}{s'_\lambda} \left[ \frac{l_{12}}{(S_{12}^t)^3 (S_{12}^{-})^3} \right]_\theta + \frac{1}{32 s'_2 \lambda} \left[ \frac{\sqrt{\lambda} \lambda \delta}{2 s s_1^2 s_2^2} \right]
\]

\[
+ \frac{\bar{\lambda}^2}{4 s_1^2 s_2^2} L_{c12} + \frac{1}{2 s'_1 s_1} L_{c1} - \frac{1}{2 s'_2 s_2} L_{c2}
\]

\[
+ \frac{\sqrt{\lambda}}{8 s^2 s_1^2 s_2^2} \left( \frac{(s'-\sigma) \bar{\lambda} + 2 s_1 s_2}{2 \ln \frac{s_1}{s_2} - L_{c3} + L_{c4}} \right) (L_{c1} - L_{c2})
\]

\[
36) \left[ \frac{l_{1u2}}{2 \sqrt{C_{12}}} \right]_\theta = \frac{i c_{12}}{x_0} \left[ \operatorname{Li}_2 \left( \frac{\beta_{12} + i x_0}{\tau_1} \right) + \operatorname{Li}_2 \left( \frac{\beta_{12} + i x_0}{\tau_2} \right) \right.
\]

\[
+ \operatorname{Li}_2 \left( \frac{\beta_{12} + i x_0}{\tau_1^*} \right) + \operatorname{Li}_2 \left( \frac{\beta_{12} + i x_0}{\tau_2^*} \right)
\]

\[
- \operatorname{Li}_2 \left( \frac{-\beta_{12} + i x_0}{\tau_1} \right) - \operatorname{Li}_2 \left( \frac{-\beta_{12} + i x_0}{\tau_2} \right)
\]

\[
- \operatorname{Li}_2 \left( \frac{-\beta_{12} + i x_0}{\tau_1^*} \right) - \operatorname{Li}_2 \left( \frac{-\beta_{12} + i x_0}{\tau_2^*} \right) \right]
\]

\[
\equiv c_{12} D_{1u2}
\]

\[
c_{12} = \frac{s'}{s'_\lambda \sqrt{\lambda} s s_1}
\]

\[
x_0 = \sqrt{\frac{s - s_1}{s}}
\]

\[
\beta_{12} = \sqrt{\frac{s_1 - 4 m_e^2}{s}}
\]

\[
\tau_1 = \frac{+ q_{12} - i p_{12}}{b_{12} + i r_{12}} \quad q_{12} = (ss' - ss_2 - s' s_1 \sqrt{\frac{s_1}{s}})
\]

\[
p_{12} = s'_2 s_1 x_0
\]

\[
\tau_2 = \frac{- q_{12} - i p_{12}}{b_{12} + i r_{12}} \quad r_{12} = s' \sqrt{\lambda} s s_1 x_0
\]

\[
b_{12} = s_1 (s - \delta)
\]

\[\text{Integrals 36) to 41) were not directly calculated from the result of the } \Omega_R \text{ integration. Instead, a Feynman parametrization was used to “linearize” } 1/t_{1u2} \text{ and } 1/t_{2u1}: 1/t_{1u2} = \int_0^1 d\alpha/t_{1+}^2 (1 - \alpha)^2. \text{ Then, } \Omega_R, \theta, \text{ and finally the Feynman parameter } \alpha \text{ were integrated.} \]
37) \[ \left[ \frac{l_{t_{2u1}}}{2 \sqrt{C_{21}}} \right]_\theta = \mathcal{T} \left( \left[ \frac{l_{t_{1u2}}}{2 \sqrt{C_{12}}} \right]_\theta \right) \equiv c_{34} D_{t_{2u1}} \]

38) \[ \left[ \frac{l_{t_{1u2}}}{z_1 2 \sqrt{C_{12}}} \right]_\theta = \left[ \frac{l_{t_{1u2}}}{z_2 2 \sqrt{C_{12}}} \right]_\theta = \]

\[
\frac{s'}{4 \sqrt{\lambda'} (s's' - ss_2 - s's_1)} \left\{ \begin{array}{c}
2 l_\beta (2 L_{c5} - L_{c1}) \\
+ \ln^2 \left( \frac{2 x_1}{x_1 - 1} \right) - \ln^2 \left( \frac{2 x_1}{x_1 + 1} \right) + \text{Li}_2 \left( -\frac{x_1 + 1}{x_1 - 1} \right) - \text{Li}_2 \left( -\frac{x_1 - 1}{x_1 + 1} \right) \\
- \ln \left( \frac{x_1 - x_2}{x_1 + x_2} \right) \left( \ln \frac{x_2 - 1}{x_2 + 1} - \ln \frac{x_1 - 1}{x_1 + 1} \right) \\
+ \ln \frac{x_1 - 1}{x_1 - x_2} \ln \left| \frac{x_2 - 1}{x_1 - x_2} \right| - \ln \frac{x_1 + 1}{x_1 - x_2} \ln \left| \frac{x_2 + 1}{x_1 - x_2} \right| \\
+ \ln \frac{x_1 + 1}{x_1 + x_2} \ln \left| \frac{x_2 - 1}{x_1 + x_2} \right| - \ln \frac{x_1 - 1}{x_1 + x_2} \ln \left| \frac{x_2 + 1}{x_1 + x_2} \right| \\
+ 2 \text{Li}_2 \left( -\frac{x_2 - 1}{x_1 - x_2} \right) - 2 \text{Li}_2 \left( -\frac{x_2 + 1}{x_1 - x_2} \right) \\
+ \frac{s'}{\sqrt{s\lambda'}} \text{Re} \left[ \sum_{i=1}^{2} \frac{(-1)^{i+1}}{\sqrt{\beta^2 x_i^2 - x_0^2}} \times \sum_{k=1}^{8} (-1)^{k+1} \left( \frac{t_{k+}^{(i)}}{t_{k-}^{(i)}} \right) \left( \ln \left[ u_+^{(i)}(+\beta) \right] - \ln \left[ u_+^{(i)}(-\beta) \right] - 2 \pi i \Theta(x_i) \right) \right. \\
\left. \quad - \ln \frac{t_{k-}^{(i)}}{t_{k+}^{(i)}} \left( \ln \left[ u_-^{(i)}(+\beta) \right] - \ln \left[ u_-^{(i)}(-\beta) \right] - 2 \pi i \Theta(-x_i) \right) \right) \\
\quad - \text{Li}_2 \left[ -\frac{u_+^{(i)}(+\beta)}{t_{k+}^{(i)}} \right] + \text{Li}_2 \left[ -\frac{u_+^{(i)}(-\beta)}{t_{k+}^{(i)}} \right] \\
\quad + \text{Li}_2 \left[ -\frac{u_-^{(i)}(+\beta)}{t_{k-}^{(i)}} \right] - \text{Li}_2 \left[ -\frac{u_-^{(i)}(-\beta)}{t_{k-}^{(i)}} \right] \right) \right\} \]

\equiv \frac{s'}{4 \sqrt{\lambda'} (ss' - ss_2 - s's_1)} \left\{ 2 l_\beta (2 L_{c5} - L_{c1}) + D_{t_{1u2}}^z \right\}
\[
\begin{align*}
\alpha_{12} &= \frac{s - \delta}{s_\lambda} \\
t_1^0 &= \tau_1 \\
t_3^0 &= \tau_2 \\
t_5^0 &= \tau_2^* \\
t_7^0 &= \tau_1^* \\
t_0^{(i)} &= t_0^0 + t_\pm^{(i)} \\
u_\pm^{(i)}(x) &= t(x) - t_\pm^{(i)} \\
t_\pm^{(i)} &= \frac{ix_0 \pm \sqrt{\beta^2 x_i^2 - x_0^2}}{x_i} \\
t(x) &= \frac{ix_0 + \sqrt{x^2 - x_0^2}}{x} \\
x_{1/2} &= \frac{s' s_1 (s - \delta) \pm \sqrt{\lambda'}}{s' s_1 - s\lambda'} \\
\beta &= \sqrt{1 - \frac{4 m_e^2}{s}} \\
\end{align*}
\]

\[
\begin{align*}
39) \quad \left[ \frac{l_{t_{1u1}}}{\tilde{z}_1 2 \sqrt{C_{21}}} \right]_\theta &= \left[ \frac{l_{t_{2u1}}}{\tilde{z}_2 2 \sqrt{C_{21}}} \right]_\theta = \mathcal{T} \left( \left[ \frac{l_{t_{1u2}}}{\tilde{z}_1 2 \sqrt{C_{12}}} \right]_\theta \right) \\
&\equiv \frac{s'}{4 \sqrt{\lambda'}} \frac{\left( s_1 (s - \delta) \pm \lambda' \right)}{s' s_1 - s\lambda'} \left\{ 2 \beta \left( 2L_{c5} - L_{c2} \right) + D_{t2u1}^z \right\} \\
40) \quad \left[ \frac{l_{t_{1u2}}}{a_{1t}^u 2 \sqrt{C_{12}}} \right]_\theta &= \left[ \frac{l_{t_{1u2}}}{a_{2u}^u 2 \sqrt{C_{12}}} \right]_\theta = \frac{s'}{s_\lambda} \text{Re} \left( X_0 + X_1 + X_2 \right) \equiv \frac{s'}{s_\lambda} D_{t1u2}^a \\
\end{align*}
\]

In the calculation of the integrals \( X_i \) the following quantities appear:

\[
\begin{align*}
A_3 &= s_1 (s'_+ - 2\sigma)^2 - \beta^2 s\lambda' \\
B_3 &= s_1 (s'_+ - 2\sigma) (s - \delta) \\
\Delta_3 &= \lambda' \left\{ (s' s_2 - s'_+ s_1 - 2s_1 (s - \sigma))^2 \\
&\quad \quad - 4m_e^2 \left\{ s_1 (s - \delta)^2 + \lambda' (s - s_1) \right\} \right\} \\
x_{3/4} &= \beta \frac{-B_3 \pm \sqrt{\Delta_3}}{A_3} \\
\end{align*}
\]

In principle, depending on the values of \( x_3 \) and \( x_4 \), four cases must be
distinguished:

(i) \( x_{3/4} \in \mathbb{IR}; \ |x_4| \geq |x_3| > \beta \)

(ii) \( x_{3/4} \in \mathbb{IR}; \ |x_4| > \beta \geq |x_3| \)

(iii) \( x_{3/4} \in \mathbb{IR}; \ \beta \geq |x_4| \geq |x_3| \)

(iv) \( \Delta_3 \leq 0; \ x_3 = x_4^* \in \mathbb{C} \)

However, since the integrand is regular, it is sufficient to present the solution for case (i). The solutions for the other three cases are then obtained by analytical continuation. In case (iv), which is relevant for only a very small fraction of the phase space, the problem of a numerically correct treatment arises. This problem was solved by computing the integral’s value for a very nearby phase space point so that the expression of case (i) could be used. Having in mind that the integral is regular, it is clear that thus only a negligible error is introduced. The expressions \( X_0, X_1, \) and \( X_2 \) for case (i) are presented below.

\[
X_0 = \frac{2}{\sqrt{\Delta_3}} \, L_{\theta 8} \left[ \ln \left( \frac{\beta - x_3}{\beta + x_3} \right) - \ln \left( \frac{\beta - x_4}{\beta + x_4} \right) \right]
\]

\[
X_1 = \frac{\sqrt{s_1} \left( s_+ - \sigma \right)}{2 \sqrt{s \lambda \Delta_3}} \sum_{i=3}^{4} \frac{(-1)^{i+1} \left( x_i - a_{34} \right)}{\sqrt{x_i^2 - x_0^2}} \times
\]

\[
\sum_{k=1}^{8} (-1)^{k+1} \left( \ln \left( \frac{t_{k+}^{(i)}}{t_0^{(i)}} \right) \left( \ln \left[ u_+^{(i)}(+\beta) \right] - \ln \left[ u_+^{(i)}(-\beta) \right] - 2\pi i \Theta(x_i) \right) \right)
\]

\[
- \ln \left( \frac{t_{k+}^{(i)}}{t_0^{(i)}} \right) \left( \ln \left[ u_-^{(i)}(+\beta) \right] - \ln \left[ u_-^{(i)}(-\beta) \right] - 2\pi i \Theta(-x_i) \right)
\]

\[
- \text{Li}_2 \left[ \frac{-u_+^{(i)}(+\beta)}{t_{k+}^{(i)}} \right] + \text{Li}_2 \left[ \frac{-u_+^{(i)}(-\beta)}{t_{k+}^{(i)}} \right]
\]

\[
+ \text{Li}_2 \left[ \frac{-u_-^{(i)}(+\beta)}{t_{k-}^{(i)}} \right] - \text{Li}_2 \left[ \frac{-u_-^{(i)}(-\beta)}{t_{k-}^{(i)}} \right]
\]

\[65\]
\[a_{34} = -\frac{s - \delta}{s'_+ - 2\sigma}\]

\[t^{(i)}_{\pm k} = t_k^0 + t^{(i)}_{\pm}\]

\[t^{(i)}_{\pm} = \frac{i x_0 \pm \sqrt{x_i^2 - x_0^2}}{x_i}\]

\[u^{(i)}_{\pm}(x) = t(x) - t^{(i)}_{\pm}\]

\[t(x) = \frac{i x_0 + \sqrt{x^2 - x_0^2}}{x}\]

\[X_2 = -\frac{1}{2 \sqrt{\Delta_3}} \times\]

\[
\sum_{i=1}^{2} \sum_{j=1}^{2} (-1)^{j+1} \left( \ln \frac{x_i - x_j - i\varepsilon}{x_i + x_j - i\varepsilon} \ln \frac{x_j - \beta}{x_j + \beta} \right.
\]

\[- \text{Li}_2 \left[ - \frac{x_j - \beta}{x_i - x_j - i\varepsilon} \right] + \text{Li}_2 \left[ - \frac{x_j + \beta}{x_i - x_j - i\varepsilon} \right]
\]

\[+ \text{Li}_2 \left[ \frac{x_j - \beta}{x_i + x_j - i\varepsilon} \right] - \text{Li}_2 \left[ \frac{x_j + \beta}{x_i + x_j - i\varepsilon} \right] \left)\right.
\]

\[41) \left[ \frac{l_{t_{2u_3}}}{a_{1u}^2 2 \sqrt{C_{21}}} \right]_{\theta} = \left[ \frac{l_{t_{2u_3}}}{a_{2u}^2 2 \sqrt{C_{21}}} \right]_{\theta} = \mathcal{T} \left( \left[ \frac{l_{t_{1u_2}}}{a_{1u}^2 2 \sqrt{C_{12}}} \right]_{\theta} \right) \equiv \frac{s'_t}{s'_t} D^{a}_{12u_1}\]
References

[1] G. Altarelli, T. Sjöstrand and F.Zwirner, eds., Report of the Workshop on Physics at LEP2, CERN 96-01 (1996).

[2] W. Beenakker et al., Report of the Working Group on WW Cross-sections and Distributions, in [1].

[3] D. Bardin et al., Report of the Working Group on Event Generators for WW Physics, in [1].

[4] F. Boudjema et al., Report of the Working Group on Standard Model Processes, in [1].

[5] M.L. Mangano et al., Report of the Working Group on Event Generators for Discovery Physics, in [1].

[6] T. Muta, R. Najima and S. Wakaizumi, Mod. Phys. Lett. A1 (1986) 203.

[7] D. Bardin, M. Bilenky, A. Olchevski and T. Riemann, Phys. Lett. B308 (1993) 403; E: Phys. Lett. B357 (1995) 725.

[8] D. Bardin, M. Bilenky, D. Lehner, A. Olchevski and T. Riemann, Semi-analytical approach to four-fermion production in $e^+e^-$ annihilation, in: T. Riemann and J. Blümlein, eds., Proc. of the Zeuthen Workshop on Elementary Particle Theory “Physics at LEP200 and Beyond”, Nucl. Phys. B (Proc. Suppl.) 37B (1994), 148-157.

[9] D. Bardin, A. Leike and T. Riemann, Phys. Lett. B344 (1995) 383.

[10] D. Bardin, A. Leike and T. Riemann, Phys. Lett. B353 (1995) 513.

[11] D. Bardin, D. Lehner and T. Riemann, Complete initial state radiation to off-shell $Z$ pair production in $e^+e^-$ annihilation, in: B.B. Levchenko, ed., Proceedings of the IX International Workshop on High Energy Physics and Quantum Field Theory (Moscow Univ. Publ. House, 1995) 221-226.

[12] D. Lehner, Ph.D. thesis, Humboldt-Universität zu Berlin, Germany (1995), Internal Report DESY-Zeuthen 95-07, [hep-ph/9512301] and references therein.

[13] D. Bardin and T. Riemann, preprint DESY 95-167 (1995), [hep-ph/9509341], to appear in Nucl. Phys. B.

[14] G. Montagna, O. Nicrosini, G. Passarino and F. Piccinini, Phys. Lett. B348 (1995) 178; G. Passarino, Univ. Torino preprint, [hep-ph/9602302] (1996), submitted to Comp. Phys. Comm..

[15] F.A. Berends, R. Pittau and R. Kleiss, Nucl. Phys. B424 (1994) 308.

[16] F.A. Berends, G.J.H. Burgers and W.L. van Neerven, Nucl. Phys. B297 (1988) 429, E: B304 (1988) 921;
B.A. Kniehl, M. Krawczyk, J.H. Kühn and R. Stuart, Phys. Lett. B209 (1988) 337.
[17] W. Beenakker, K. Kołodziej and T. Sack, *Phys. Lett.* B258 (1991) 469;  
W. Beenakker, F.A. Berends and T. Sack, *Nucl. Phys.* B367 (1991) 287;  
J. Fleischer, K. Kołodziej and F. Jegerlehner, *Phys. Rev.* D47 (1993) 830;  
J. Fleischer, F. Jegerlehner, K. Kołodziej and G.J. van Oldenborgh, *Comp. Phys. Comm.* 85 (1995) 29, [hep-ph/9405380].

[18] M. Veltman, “SCHOONSCHIP – A Program for Symbol Handling”, 1989;  
H. Strubbe, *Comp. Phys. Comm.* 8 (1974) 1;  
J. Vermaseren, “Symbolic Manipulations with FORM” (Computer Algebra Nederland, Amsterdam, 1991);  
M.J. Abell and J.P. Brasilto, “The Mathematica Handbook” (Academic Press, San Diego, CA, 1992);  
N. Blachman, “Mathematica: A Practical Approach” (Prentice Hall, Englewood Cliffs, N. J., 1992).

[19] W. Alles, Ch. Boyer and A. Buras, *Nucl. Phys.* B119 (1977) 125.

[20] D.Bardin, M. Bilenky, D. Lehner, A. Leike, A. Olchevski and T. Riemann, Fortran programs gentle4fan.f and gentle_nc_qed.f. The codes are available from the authors upon E-Mail request or via WWW: http://www.ifh.de/bardin/gentle4fan.uu and http://www.ifh.de/lehner/gentle_nc_qed.uu. A short description may be found in [3], p. 68.

[21] Particle Data Group (L. Montanet et al.), “Review of Particle Properties”, *Phys. Rev.* D50 (1994).

[22] M. Böhm, A. Denner, T. Sack, W. Beenakker, F.A. Berends and H. Kuijf, *Nucl. Phys.* B304 (1988) 463;  
J. Fleischer, F. Jegerlehner and M. Zralek, *Z. Phys.* C42 (1989) 409.

[23] W. Beenakker, *Status of Standard-Model Corrections to On-Shell W Pair Production*, in: T. Riemann and J. Blümlein, eds., *Proc. of the Zeuthen Workshop on Elementary Particle Theory “Physics at LEP200 and Beyond”*, Nucl. Phys. B (Proc. Suppl.) 37B (1994), 59-74.

[24] V.S. Fadin, V.A. Khoze and A.D. Martin, *Phys. Lett.* B311 (1993) 311; *Phys. Lett.* B320 (1994) 141; *Phys. Rev.* D49 (1994) 2247.

[25] A. Aeppli and D. Wyler, *Phys. Lett.* B262 (1991) 125;  
G.J. van Oldenborgh, P.J. Franzini and A. Borrelli, *Comp. Phys. Comm.* 83 (1994) 14;  
J. Fujimoto et al., *Non-resonant diagrams in radiative four-fermion processes* in: T. Riemann and J. Blümlein, eds., *Proc. of the Zeuthen Workshop on Elementary Particle Theory “Physics at LEP200 and Beyond”*, Nucl. Phys. B (Proc. Suppl.) 37B (1994), 169-174;  
G.J. van Oldenborgh, Leiden Univ. preprint INLO-PUB-95-04 (1995), submitted to *Nucl. Phys.* B;  
K. Melnikov and O. Yakovlev, preprint MZ-TH/95-01 (1995), [hep-ph/9501358].

[26] V. Baier, V. Fadin and V. Khoze, *Sov. J. Phys. JETP* 23 (1966) 104.