Notes on Lagrangean and Gamiltonian Symmetries

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Abstract

The Hamiltonization of local symmetries of the form $\delta q^A = \epsilon^a R_a^A (q, \dot{q})$ or $\delta q^A = \dot{\epsilon}^a R_a^A (q, \dot{q})$ for arbitrary Lagrangean model $L(q^A, \dot{q}^A)$ is considered. We show as the initial symmetries are transformed in the transition from $L$ to first order action, and then to the Hamiltonian action $S_H = \int d\tau (p_A \dot{q}^A - H_0 - v^a \Phi_a)$, where $\Phi_a$ are the all (first and second class) primary constraints. An exact formulae for local symmetries of $S_H$ in terms of the initial generators $R_a^A$ and all primary constraints $\Phi_a$ are obtained.
1 Introduction

In the majority of physically interesting models, the Lagrangians are symmetric with respect to some set of local transformations of the form

$$\delta q^A = R^A(\epsilon^a, q^A, \dot{q}^A).$$  \hspace{1cm} (1)

On application of the Dirac–Bergmann algorithm \cite{1,2} for the Hamiltonization of the theory under investigation, we obtain an equivalent description for the original classical dynamics in terms of canonical action

$$S_c = \int d\tau \left( -\frac{1}{2}C^{-1}_{AB}\Gamma^A\Gamma^B - H_0 - v^I\Phi_I \right),$$  \hspace{1cm} (2)

where $\Gamma^A \equiv (q^A, p_B)$ and $\Phi_I(q, p)$ are all first-class constraints of the theory. Constraints of the second class, if any, are taken into account by going to the Dirac bracket, such that $\{\Gamma^A, \Gamma^B\} = C^{AB}$. It is well known \cite{3,4} that local symmetries for Eq. (2) are the following transformations generated (in the sector $(q^A, p_B)$) by the first-class constraints $\Phi_I$:

$$\delta q^A = \epsilon^I \{q^A, \Phi_I\},$$
$$\delta p_A = \epsilon^I \{p_A, \Phi_I\},$$
$$\delta v^I = \epsilon^I + v^J \epsilon^K C_{JK}^I - \epsilon^J V_J^I,$$  \hspace{1cm} (3)

where the designations $\{\Phi_J, \Phi_K\} = C_{JK}^I\Phi_I$, $\{H_0, \Phi_I\} = V_I^J\Phi_J$.

For any specific model, numerous observations are available on how the Lagrangian symmetries, Eq. (1), and the Hamiltonian symmetries, Eq. (3), are bound \cite{5,6} but the question about the relation between them in a general theory still remains largely open. In particular, some researchers \cite{3,7–10} investigated the problem of reconstruction of Lagrangian symmetries by known Hamiltonian symmetries. However, provided that the Dirac–Bergmann algorithm is applied, the most natural statement of the problem seems to be as follows: how the Lagrangian transformations, Eq. (1), change when passing successively from the Lagrangian to the canonical action? In other words, concurrent with the procedure of Hamiltonization of the theory, we state the problem of Hamiltonization for the Lagrangian transformations with the aim to obtain an expression for the Hamiltonian action symmetries through the generators $R^A(\epsilon, q, \dot{q})$.

The solution of this problem may appear to be useful for a number of issues, in particular, in studying the corresponding algebras \cite{11}; in investigating the relation between the Lagrangian and the Hamiltonian BFV quantization \cite{12}; in consistent formulating the theory of the superparticle (superstring) on a curved background \cite{13}, and discussing Dirac’s conjecture \cite{8,14}.

The present work is organized in the following way. In Sec. 2, we introduce designations and give the facts related to the Dirac–Bergmann algorithm that
are necessary for the subsequent discussion (see Ref. 2 for details). In Sec. 3, the symmetries for the action in the first-order formalism \( S_v \) and for the Hamiltonian action \( S_H \) are constructed by the local symmetries of the original action. Subsequently, a partial Hamiltonization of the transformations is performed for both cases, i.e., they are rewritten in terms of the Poisson bracket to the trivial (on-shell vanishes) symmetries of the first-order formalism. Section 4 deals with a special case of original Lagrangian symmetries where a “complete” Hamiltonization appears to be possible, namely, the symmetries \( S_H \) are expressed through all primary constraints of the theory (and through the generators of the original Lagrangian symmetries). It should be noted that this special class of symmetries is rather broad; in particular, the local fermion symmetries of the superparticle and the superstring theories in a covariant formulation satisfy the restrictions placed. The results are formulated and discussed in Conclusion.

2 The Dirac–Bergmann algorithm

Consider a mechanical system described by at most polynomial in velocities Lagrangian \( L(q^A, \dot{q}^A), A = 1, \ldots, N \), which will be assumed singular:

\[
\text{rank } M_{AB} \equiv \text{rank} \left( \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} \right) = K < N.
\]  

According to this equation, it is convenient to redefine the index \( A \equiv (i, \alpha) \), \( i = 1, \ldots, K, \alpha = K + 1, \ldots, N \), such that \( M_{ij} \) is non-singular and its inverse \( \tilde{M}_{ij} \) exists (note that for all known models this may be done without losing of manifestly Poincaré covariance). We also suppose that all variable \( q^A \) are even, the extension of all results to grassmanian case is formally straightforward. Further, let us consider an infinitesimal local transformations of the form

\[
\delta \dot{q}^A = \epsilon^a R_a^A(q^B, \dot{q}^B), \quad a = 1, \ldots, k',
\]  

and suppose the \( L \) is invariant up to an exact differential

\[
\delta \epsilon \dot{L} = [\epsilon^a D_a(q, \dot{q})].
\]  

If Eq. (5) essentially depends on the parameters \( \epsilon^a(\tau) \) (rank \( R_a^A = \text{max} \)), then \( K' \leq N - K \) as it will be seen from Eq. (27) below.

To go from the Lagrangian to the Hamiltonian formalism we first to pass into equivalent description of the initial dynamics in terms of first order action, defined on extended space \((q^A, p_A, v^A)\)

\[
S_v = \int d\tau [L(q^A, v^B) + p_A(\dot{q}^A - v^A)].
\]  

The equations of motion which follows from Eq. (7) may be identically rewritten in the Hamiltonian form by introducing of the Poisson bracket \( \{ , \} \) (defined
only for phase space sector \((q^A, p_A)\) of extended space), and of the Hamiltonian
\[
\hat{H}(q^A, p_A, v^A) \equiv p_A v^A - L(q, v).
\]
Then the dynamics is ruled by the following equations:
\[
\dot{q}^A = \{q^A, \hat{H}\},
\]
\[
\dot{p}^A = \{p_A, \hat{H}\},
\]
\[
\Phi_\alpha(q, p, v) \equiv p_\alpha - \frac{\partial L}{\partial v^\alpha} = 0, \quad \text{or} \quad \frac{\partial \hat{H}}{\partial v^\alpha} = 0,
\]
\[
\Phi_i(q, p, v) \equiv p_i - \frac{\partial L}{\partial v^i} = 0, \quad \text{or} \quad \frac{\partial \hat{H}}{\partial v^i} = 0.
\]
As a second step of Dirac–Bergmann algorithm, we solve Eq. (12)
\[
v^i = v^i(q^A, p_j, v^\alpha),
\]
and substitute these back into (9)–(12). Then we have the identities
\[
p_i - \frac{\partial L}{\partial v^i} \bigg|_v \equiv 0, \quad \text{or} \quad \frac{\partial \hat{H}}{\partial v^i} \bigg|_v \equiv 0,
\]
and the equations of motion in reduced space \((q^A, p_A, v^\alpha)\)
\[
\dot{q}_A = \{q^A, H\},
\]
\[
\dot{p}_A = \{p_A, H\},
\]
\[
\Phi_\alpha(q^A, p_A) \equiv p_\alpha - \frac{\partial L}{\partial v^\alpha} \bigg|_v = 0,
\]
where
\[
H(q^A, p_A, v^\alpha) \equiv \hat{H} \bigg|_v.
\]
Note that \(\{A(q, p), \hat{H}\} \bigg|_v \equiv \{A(q, p), \hat{H} \bigg|_v\} \) as a consequence of Eq. (14). Also, the left hand side of Eq. (17) do not depend on \(v^\alpha\), in accordance with the condition (4).

It is well known that the Hamiltonian \(H\) may be identically rewritten in the form
\[
H = H_0(q^A, p_i) + v^\alpha \Phi_\alpha,
\]
where
\[
H_0 \equiv \left( p_i v^i - L(q, v) + v^\alpha \frac{\partial L}{\partial v^\alpha} \right) \bigg|_v,
\]
the last may depend only on \(q^A\) and \(p_i\) variables.
As the result, the Hamiltonian dynamics may be described in terms of Hamiltonian action
\[
S_H \equiv S_v = \int d\tau (p_A \dot{q}^A - H_0 - v^\alpha \Phi_\alpha),
\]
(21)
where \(\Phi_\alpha\) are the all (first and second class) primary constraints. Equations (15)–(17) follows from variation of Eq. (21) with respect to \(q, p, v\) variables.

In conclusion of this section let us write some identities will be used below. By differentiating of Eq. (14) one get
\[
\frac{\partial v^i}{\partial p_A} = \tilde{M}^{ij} |_v \delta_j^A,
\]
(22)
\[
\frac{\partial v^i}{\partial v^\alpha} = -\tilde{M}^{ij} M_{j\alpha} |_v,
\]
(23)
\[
\frac{\partial v^i}{\partial q^A} = -\tilde{M}^{ij} \frac{\partial^2 L}{\partial v^j \partial q^A} |_v,
\]
(24)
Where, from now \(M_{AB} \equiv \frac{\partial^2 L(q, v)}{\partial v^A \partial v^B}\). Then, from the identity \(\partial (\Phi_\alpha |_v) / \partial v^\beta \equiv 0\) and from Eq. (23) one finds
\[
(M_\alpha^\beta - M_{\alpha i} \tilde{M}^{ij} M_{j\alpha} |_v \equiv 0.
\]
(25)

3 Hamiltonization of local symmetries

To rewrite the local symmetries of \(L\) in terms of \(S_v\) and \(S_H\), let us first to arrive at a consequences, followed from the condition (6). By standard algebraic manipulations, one finds an expression for an exact differential
\[
\frac{\partial L}{\partial q^A} R_a^A = D_a,
\]
(26)
and the following Lagrangian identities
\[
\frac{\partial^2 L}{\partial q^A \partial \dot{q}^B} R_a^A(q, \dot{q}) = 0,
\]
\[
\frac{\partial L}{\partial q^A \partial \dot{q}^B} R_a^A - \frac{\partial L}{\partial q^B \partial \dot{q}^A} \dot{q}^B R_a^A = 0.
\]
(27)
As it fulfilled for arbitrary \(q^A(\tau)\) these equations remain valid after the substitution \(\dot{q}^A \rightarrow v^A\) and identically fulfilled for arbitrary functions \(q^A(\tau)\) and \(v^A(\tau)\).

Now, by using Eqs. (26) and (27), one may easily check that following local transformations
\[
\delta q^A = \epsilon^a R_a^A(q, v),
\]
\[
\delta_v p_A = \frac{\partial^2 L}{\partial q^A \partial p^B} \delta_v q^B, \\
\delta_v v^A = (\delta_v q^A).
\]  

leaves the action \( S_v \) (7) invariant up to boundary terms. Further, these formulae may be identically expressed in terms of Poisson brackets as follows:

\[
\delta_v q^A = \epsilon^a \{ q^A, \Phi_B R_a^B \}, \\
\delta_v p_A = \epsilon^a \{ p_A, \Phi_B R_a^B \} + \Phi_B^a \frac{\partial R_a^B}{\partial q^A}, \\
\delta_v v^A = \dot{v} R_a^A + \epsilon^a \left( \frac{\partial R_a^A}{\partial q^B} v^B + \frac{\partial R_a^A}{\partial v^B} \dot{v}^B \right) + \epsilon^a \frac{\partial R_a^A}{\partial q^B} (\dot{q}^B - v^B),
\]

with \( \Phi_B(q, p, v) \) from Eqs. (11) and (12). The last terms in Eqs. (30) and (31) (on-shell vanishes in the first order formalism) can be neglected. Indeed, from an expression for an arbitrary variation of \( S_v \)

\[
\delta S_v = - \int d\tau \left\{ \Phi_A \delta v^A + \left( \dot{p}_A - \frac{\partial L}{\partial q^A} \right) \delta q^A + \delta p_A (q^A - v^A) + \text{boundary terms} \right\},
\]

it follows that the transformations with parameters \( w^A_B \) and \( f^{AB} \)

\[
\delta p_A = \Phi_B w^A_B, \\
\delta v^A = w^A_B (\dot{q}^B - v^B); \\
\delta q^A = -\Phi_B f^{BA}, \\
\delta v^A = f^{AB} \left( \dot{p}_A - \frac{\partial L}{\partial q^A} \right);
\]

are the symmetries of \( S_v \) by itself. The last terms of Eqs. (30) and (31) are precisely of this kind. Neglecting these terms, we observe that the quantities \( \Phi_B R_a^B \) acts as the generators of local symmetries for \( S_v \) (7).

In a similar manner, one may separating out the generators \( R_a^A \) from the Poisson brackets in Eqs. (29), (30) and also neglect all irrelevant terms. Then one finds symmetries \( \delta_v q^A = \epsilon^a R_a^B \{ q^A, \Phi_B \}, \delta_v p_A = \epsilon^a R_a^B \{ p_A, \Phi_B \} \), accompanied by some complicated expression for transformations of \( v^A \).

The transition to the case of \( S_H \) is straightforward. Indeed, note that if \( \delta S_v = 0 \) for arbitrary \( v^i \), than, in particular, \( \delta S_v |_{v^i} = 0 \). Further, from Eqs. (21) and (7) we have (for arbitrary variation of \( q^A, p_A, v^\alpha \))

\[
\delta S_H = \delta(S_v |_{v^i}) = \delta S_v |_{v^i} + \Phi_1(q, p, v) |_{v} \delta v^i \equiv \delta S_v |_{v^i}.
\]

Therefore, the local symmetries for \( S_H \) (21) derives from Eq. (28) (or Eqs. (29)–(31)) by dropping \( \delta v^i \) and by direct substitution of \( v^i(q^A, p_j, v^\alpha) \) in the remains. The most transparent and symmetrical expressions for these formulae will be obtained in the next section for the case of Lagrangian transformations of a special form.
4 Local symmetries of a special form

Let us consider a special case when $\delta L = 0$ under the transformations of Eq. (5). Then, in comparison with previous section, we have an identity

$$\frac{\partial L}{\partial v^i} R_a^i(q, v) + \frac{\partial L}{\partial v^\alpha} R_a^\alpha(q, v) = 0,$$

(37)
in addition to Eq. (27). It allows us to write the following relations for some linear combinations of all primary constraints

$$\Phi_\alpha(q, p) R_a^\alpha|_v = p_A R_a^\alpha|_v,$$

(38)

$$\Phi_\alpha(q, p) \frac{\partial R_a^\alpha}{\partial v^j} |_v = p_A \frac{\partial R_a^\alpha}{\partial v^j} |_v.$$

(39)
The first can be tested by using of Eqs. (37) and (14). To derive the second, we differentiate the Eq. (37) with respect to $v^j$ and use the first from Eq. (27) yield

$$\frac{\partial L}{\partial v^i} \frac{\partial R_a^i}{\partial v^j} + \frac{\partial L}{\partial v^\alpha} \frac{\partial R_a^\alpha}{\partial v^j} = 0.$$

(40)

Then, substituting $v^i(q^A, p_j, v^\alpha)$ and by virtue of Eqs. (14) and (17), we get the desired result.

Using these identities, it is not difficult to substitute $v^i(q^A, p_j, v^\alpha)$ from Eq. (13) to the Eq. (28) or Eqs. (29)–(31) to derive the symmetries of $S_H$ (21). We get after some algebra

$$\delta \epsilon R_a^\alpha|_v \{q^A, \Phi_\alpha(q, p)\},$$

(41)

$$\delta \epsilon p_A = \epsilon R_a^\alpha|_v \{p_A, \Phi_\alpha(q, p)\},$$

(42)

$$\delta \epsilon v^\alpha = (\delta \epsilon q^\alpha),$$

(43)

where $R_a^\alpha$ are the generators of initial Lagrangian symmetries and $\Phi_\alpha(q, p) \approx 0$ are the all (first and second class) primary constraints. Note that it is an exact formulae, namely, all terms of the form (33), (34) identically cancel out during the calculations. The equations (41)–(43) present the Hamiltonian form of initial local Lagrangian symmetries for the system under consideration. As in previous section, Eqs. (41) and (42) may be identically rewritten so that the $R_a^\alpha$ generators are incorporated in the Poisson bracket, neglecting trivial symmetries of the form (33), (34).

In conclusion, note that all of the preceding remain valid for transformations of the form $\delta q^a = \epsilon^a R_a A$ instead of Eq. (5).
5 Conclusion

We have discussed how the local Lagrangian symmetries of an arbitrary con- strained system are transformed in passing to the first order action $S_v$, Eq. (7), and further to the Hamiltonian action $S_H$, Eq. (21), which includes all primary constraints.

The symmetry transformations for $S_v$ are written in the explicit form in Eqs. (28). The formulas obtained are rewritten in terms of the Poisson bracket to the on-shell vanishes of the first-order formalism, Eqs. (33) and (34), and have the final form of Eqs. (29)–(31).

The explicit form of the symmetry transformations for $S_H$ are given by Eqs. (41)–(43). Let us give some comments concerning the structure of these equations.

(a) Equations (41) and (42) have the form similar to that of Eq. (3) but involve a “composite” parameter, $\epsilon^\alpha \equiv \epsilon^\alpha R_\alpha^\alpha \big|_{v}$. Essential parameters in the transformations, Eqs. (41) and (42), like in the Lagrangian transformations, Eqs. (5), are the arbitrary functions $\epsilon^a(\tau)$, $a = 1, \ldots, K'$.

(b) The local symmetry generators for $S_H$ are all primary constraints $\Phi_\alpha(q,p)$ (and not only first-order constraints, contrary to Eq. (3)).

(c) Involved in Eqs. (41)–(43) are only the $R_\alpha^\alpha$ generators of the complete set of generators, $R_\alpha^A$. It is not surprising since in the Lagrangian formalism the generators $R_\alpha^i$ can also be expressed through the rest ones, using identity (27), as $R_\alpha^i \equiv -M^ijM_{ij}R_\alpha^\alpha$.

It would be of interest to generalize the statements of the present work to the case of a complete Hamiltonian action $S_c$, Eq. (2), i.e., to take into account all secondary constraints of the theory, and to discuss the problem of the deformation of the algebra of original Lagrangian transformations in the transition $S \to S_v \to S_H \to S_c$. The work in the direction is in progress.

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