GROWTH IN RIGHT-ANGLED GROUPS AND MONOIDS

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Abstract. We derive functional relationships between spherical generating functions of graph monoids, right-angled Artin groups and right-angled Coxeter groups. We use these relationships to express the spherical generating function of a right-angled Artin group in terms of the clique polynomial of its defining graph. We also describe algorithms for computing the geodesic generating functions of these structures.

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1. Overview

Let $\Gamma$ be a finite graph with nodes $m = \{1, \ldots, m\}$. Consider the set of equivalence classes of words in the alphabet $x_1, \ldots, x_m$, modulo the relation where two words are said to be equivalent if each can be obtained from the other by a sequence of substitutions called shuffles:

$$ux_ix_jv \rightarrow ux_jx_iv,$$

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for some words \( u \) and \( v \), and some and \( i, j \in m \) which are connected by an edge in \( \Gamma \). In other words, variables commute if and only if the corresponding nodes in the graph are connected by an edge.

Concatenation of words descends to a binary operation on equivalence classes of words, resulting in a monoid \( GM_\Gamma \) known as the graph monoid of \( \Gamma \). The identity element of this monoid is the empty word. Such monoids were studied by Cartier and Foata \cite{CF69} and therefore go by the name of Cartier-Foata monoids.

The right-angled Artin group \( RAAG_\Gamma \) associated to \( \Gamma \) is the group obtained by adding to \( GM_\Gamma \) the inverses of all its elements. Concretely, it may be regarded as the set of equivalence classes of words in the \( 2m \) symbols \( x_1, x_1^{-1}, \ldots, x_m, x_m^{-1} \), where, in addition to shuffles \( (1) \), substitutions of the form

\[
ux_i x_i^{-1} v \leftrightarrow uv \quad \text{and} \quad ux_i^{-1} x_i v \leftrightarrow uv,
\]

for \( i \in m \) are allowed. Right-angled Artin groups were introduced under the name of semifree groups by Baudisch \cite{Bau81} and were called graph groups by Servatius, Droms and Servatius \cite{SDS89}. Of late, they have attracted a lot of attention in geometric group theory (see, for example, the survey article by Charney \cite{Cha07}).

The right-angled Coxeter group \( RACG_\Gamma \) is the quotient of \( RAAG_\Gamma \) by its smallest normal subgroup containing \( x_i^2 \) for every \( i \in m \). In \( RACG_\Gamma \), each \( x_i \) is equal to its inverse, and therefore, just like in \( GM_\Gamma \), each element is represented by a word in the alphabet \( x_1, \ldots, x_m \). Now, in addition to shuffles \( (1) \), substitutions of the form

\[
ux_i x_i v \leftrightarrow uv,
\]

for each \( i \in m \) are allowed. The group \( RACG_\Gamma \) may be viewed as a graph product of cyclic groups of order two in the sense of Green \cite{Gre90} and Hermiller and Meier \cite{HM95}.

The length of an element \( x \) in any of the structures \( GM_\Gamma \), \( RAAG_\Gamma \) and \( RACG_\Gamma \) is the minimal length of a word that represents it, and is denoted \( l(x) \). Thus, the identity element is the only element of length 0 in all three structures.

\[\text{Example 1.}\] In \( GM_\Gamma \) and \( RACG_\Gamma \) there are \( m \) elements of length one, namely \( x_1, \ldots, x_m \), whereas in \( RAAG_\Gamma \) there are \( 2m \) elements of length one, namely \( x_1^{\pm 1}, \ldots, x_m^{\pm 1} \).
The spherical growth functions of these structures are defined as:

\[ GM_\Gamma(t) = \sum_{x \in GM_\Gamma} t^{l(x)}, \]

\[ RAAG_\Gamma(t) = \sum_{x \in RAAG_\Gamma} t^{l(x)}, \]

\[ RACG_\Gamma(t) = \sum_{x \in RACG_\Gamma} t^{l(x)}. \]

The theory of Tits systems gives the rationality of spherical generating functions of Coxeter groups in general (see, for example [Dav08, Cor. 17.1.6]). The rationality of spherical generating functions for right angled Artin groups follows from the results of Loeffler, Meier and Worthington [LMW02].

An important combinatorial invariant of a graph is its clique polynomial:

\[ p_\Gamma(t) = 1 + c_1 t + c_2 t^2 + \cdots, \]

where \( c_i \) is the number of cliques in \( \Gamma \) with \( i \) nodes. The spherical growth function of \( GM_\Gamma \) is has a nice expression in terms of \( p_\Gamma(t) \):

\[ (4) \quad GM_\Gamma(t) = \frac{1}{p_\Gamma(-t)}. \]

This formula can be obtained by substituting \( T_i = t \) for every \( i \) in the Cartier-Foata identity [CF69, Ch. 1, Eq. (1)]. For another elegant (and completely elementary) proof of (4) see Fisher [Fis89].

The theory of Coxeter groups gives a similar expression for right-angled Coxeter groups; see Davis [Dav08, Prop. 17.4.2]:

\[ (5) \quad RACG_\Gamma(t) = \frac{1}{p_\Gamma \left( \frac{-t}{1+t} \right)}. \]

The first main result of this article is a version of the formulae (4) and (5) for right-angled Artin groups:

\[ (6) \quad RAAG_\Gamma(t) = \frac{1}{p_\Gamma \left( \frac{-2t}{1+t} \right)}. \]

Example 2. Taking \( \Gamma \) to be the complete graph on \( m \) nodes gives a well-known identity:

The number of points in \( \mathbb{Z}^m \) which lie on the \( L^1 \)-sphere of radius \( n \) centered at the origin is the coefficient of \( t^n \) in \( \left( \frac{1}{1+t} \right)^m \).

Example 3. Taking \( \Gamma \) to be the graph with \( m \) vertices and no edges gives:
The number of words of length \( n \) in a free group on \( m \) generators is the coefficient of \( t^n \) in \( \frac{1+t}{1-(2m-1)t} \).

**Example 4.** Taking \( \Gamma \) to be the square graph (Fig. 1) yields the Cartesian square \( F_2 \times F_2 \) of the free group on two generators. The clique polynomial of this graph is \( p_\Gamma(t) = (1 + 2t)^2 \), and thus the spherical growth function of \( F_2 \times F_2 \) with respect to its standard generators is \( \left(\frac{1+t}{1-3t}\right)^2 \).

**Example 5.** The line graph \( A_m \) (Fig. 2) with \( m \) vertices has clique polynomial \( 1 + mt + (m-1)t^2 \). Therefore the spherical growth function of the corresponding right-angled Artin group is

\[
\frac{(1 + t)^2}{1 - 2(n - 1)t + (2n - 3)t^2}.
\]

The proof of Eq. (6) proceeds by deriving a functional relationship \( s \) between \( \text{RAAG}_\Gamma(t) \) and \( \text{RACG}_\Gamma(t) \), namely:

\[
(7) \quad \text{RAAG}_\Gamma(t) = \text{RACG}_\Gamma \left(\frac{2t}{1 - t}\right),
\]

which allows for the deduction of (7) from (5). This functional relationship is based upon the comparison of branching rules in the sense of Prasad [Pra14] for elements in \( \text{RAAG}_\Gamma \) and \( \text{RACG}_\Gamma \) (see Lemma 3 in the next section). A similar comparison of branching rules also gives a functional relationship between \( \text{GM}_\Gamma \) and \( \text{RACG}_\Gamma \):

\[
(8) \quad \text{GM}_\Gamma(t) = \text{RACG}_\Gamma \left(\frac{t}{1 - t}\right),
\]

which allows one to easily derive either of (4) and (5) from the other.
All the results in this article rely upon the classification of elements of our structures into types. The type $\tau(x)$ of an element $x$ is a subset of the vertex-set of the graph $\Gamma$ (see Definition 2). Let $M$ be any subset of the vertex-set of $\Gamma$. A generalization of the Cartier-Foata identity due to Krattenthaler [Kra06, Theorem 4.1] gives rise to a refinement of the identity (4):

$$\sum_{x \in GM_{\Gamma, \tau(x) \subseteq M}} t^{|l(x)|} = \frac{p_{\Gamma - M}(-t)}{p_{\Gamma}(-t)}.$$ 

Here $p_{\Gamma - M}(t)$ is the clique polynomial of the induced subgraph of $\Gamma$ obtained after deleting the vertices in $M$. The functional relations (15) and (16) allow us to deduce similar identities for right-angled Artin groups and right-angled Coxeter groups:

$$\sum_{x \in RACG_{\Gamma, \tau(x) \subseteq M}} t^{|l(x)|} = \frac{p_{\Gamma - M}(-t)}{p_{\Gamma}(-t)},$$

$$\sum_{x \in RAAG_{\Gamma, \tau(x) \subseteq M}} t^{|l(x)|} = \frac{p_{\Gamma - M}(-2t)}{p_{\Gamma}(-2t)}.$$ 

A geodesic (or reduced) word in any of the structures $GM_{\Gamma}$, $RAAG_{\Gamma}$ and $RACG_{\Gamma}$ is any word which has minimal length among all the words that represent the same element. We let $GM_{\Gamma}^*$, $RAAG_{\Gamma}^*$ and $RACG_{\Gamma}^*$ denote the sets of geodesic words in $GM_{\Gamma}$, $RAAG_{\Gamma}$ and $RACG_{\Gamma}$ respectively. The geodesic generating functions for these structures are the functions:

$$GM_{\Gamma}^*(t) = \sum_{x \in GM_{\Gamma}^*} t^{|l(x)|},$$

$$RAAG_{\Gamma}^*(t) = \sum_{x \in RAAG_{\Gamma}^*} t^{|l(x)|},$$

$$RACG_{\Gamma}^*(t) = \sum_{x \in RACG_{\Gamma}^*} t^{|l(x)|}.$$

Our main result for geodesic generating functions is an algorithm for computing these geodesic generating functions. Again, the algorithm is based on branching rules for words based on type.

When $\Gamma$ is link-regular, all types of the same cardinality can be clubbed together for the purpose of computing geodesic generating functions. By doing so, we recover a result of Antolín and Ciobanu [AC13] on the geodesic growth functions of link-regular graphs.

The formulae and algorithms described in this paper have been implemented using the Sage Mathematical Software [S+14]. The code is
available from [http://www.imsc.res.in/~amri/growth/](http://www.imsc.res.in/~amri/growth/) This software allows the user to recover all the examples in [AC13, LMW02] with a few keystrokes.

### 2. Spherical Growth Functions

**Definition 1.** The type of an element $x$ in $\text{GM}_\Gamma$, $\text{RAAG}_\Gamma$ or $\text{RACG}_\Gamma$ is the subset of $\mathbf{m}$ consisting of those indices $i$ such that $x$ is represented by a reduced word ending in $x_i$ or in $x_i^{-1}$. We write $\tau(x)$ for the type of $x$.

In particular, the type of the identity element (which is represented by the empty word) is the empty set.

**Lemma 1.** For each element $x$ in $\text{GM}_\Gamma$, $\text{RAAG}_\Gamma$ or $\text{RACG}_\Gamma$, the elements of $\tau(x)$ form a clique in $\Gamma$.

*Proof.* This is clear for $\text{GM}_\Gamma$, because all words representing the same element can be obtained from one another by a sequence of shuffles as in (1). If a word ending in $x_i$ can be changed into a word ending in $x_j$ using a sequence of shuffles, then at some stage $x_i$ has to be shuffled past $x_j$, so $i$ and $j$ must be joined by an edge in $\Gamma$.

For $\text{RAAG}_\Gamma$ and $\text{RACG}_\Gamma$, it turns out that all *reduced* words representing the same element can be obtained from one another using a sequence of shuffles [Gre90, Theorem 3.9] and once again the lemma follows. □

**Lemma 2.** For each element $x$ in $\text{GM}_\Gamma$, $\text{RAAG}_\Gamma$ or $\text{RACG}_\Gamma$, if $l(xx_i^{\pm1}) > l(x)$, then $\tau(xx_i^{\pm1})$ is the unique maximal clique of $\tau(x) \cup \{i\}$ that contains $i$.

*Proof.* Since $l(xx_i^{\pm1}) > l(x)$, a reduced word for $xx_i^{\pm1}$ can be obtained by appending $x_i^{\pm1}$ to a reduced word for $x$. Now $x_i^{\pm1}$ can be shuffled to the end of $xx_i^{\pm1}$ if and only if either $j = i$, or $x_j \in \tau(x)$ and $x_j$ can be shuffled past $x_i$. In other words, $\tau(xx_i^{\pm1})$ consists of those elements of $\tau(x) \cup \{i\}$ which share an edge with the node $i$ of $\Gamma$, which is nothing but the unique maximal clique in $\tau(x) \cup \{i\}$ containing $i$. □

**Definition 2.** Let $x$ be an element of $\text{GM}_\Gamma$, $\text{RAAG}_\Gamma$ or $\text{RACG}_\Gamma$ of positive length. Let $i = \max \tau(x)$. Then $x$ has a reduced word ending in $x_i^\epsilon$ where $\epsilon = \pm 1$. The element $x' = xx_i^{-\epsilon}$ is called the parent of $x$ and $x$ is called a child of $x'$.

\[1\] For Coxeter groups, this is a well-known theorem of Tits [Tit69, Théorème 3].
While each element has a unique parent, it can have several children. The parent-child relationship for elements can be understood in terms of the following notion of branching for cliques (see Lemma 3):

**Definition 3.** Let \( C \) and \( C' \) be cliques in \( \Gamma \). Say that \( C' \) branches to \( C \) (denoted \( C' \rightarrow C \)) if the following conditions hold:

1. \( C - C' \) is a singleton set (call its unique element \( i \)),
2. \( C \) is the maximal clique in \( C' \cup \{i\} \) containing \( i \), and
3. \( i > j \) for every \( j \in C \).

We write \( C' \rightarrow C \).

Note that if \( i > j \) for every \( j \in C' \) then the maximal clique containing \( i \) in \( C' \cup \{i\} \) is the unique branch of \( C' \) with \( C - C' = \{i\} \). Otherwise, \( C' \) has no branches with \( C' - C = \{i\} \).

**Lemma 3.** Let \( C' \) be a clique of \( \Gamma \). The children of an element of type \( C' \) are described by the following rules:

1. If \( x' \in \text{GM}_\Gamma \) has type \( C' \) then \( x' \) has a unique child of type \( C \) for every branch \( C \) of \( C' \), one child of type \( C' \) and no other children.
2. If \( x' \in \text{RAAG}_\Gamma \) has type \( C' \) then \( x' \) has two children of type \( C \) for every branch \( C \) of \( C' \), one child of type \( C' \) and no other children.
3. If \( x' \in \text{RACG}_\Gamma \) has type \( C' \) then \( x' \) has one child of type \( C \) for every branch \( C \) of \( C' \) and no other children.

**Proof.** The proofs of the three assertions are slight variations on the same theme:

Suppose that \( x' \in \text{GM}_\Gamma \) has type \( C' \). Let \( j = \max C' \). Then \( x'x_j \) is a child of \( x' \) of type \( C' \). If \( i > j \), then by Lemma 2, \( x'x_i \) is a child of \( x' \) of type \( C \) where \( C \) is the branch of \( C' \) with \( C - C' = \{i\} \). For all other values of \( i \), \( x'x_i \) is not a child of \( x' \). This proves the assertion for \( \text{GM}_\Gamma \).

Suppose \( x' \in \text{RAAG}_\Gamma \) and \( \tau(x') = C' \). Let \( i = \max C' \). If \( x' \) has a reduced word ending in \( x_i \), then \( xx_i \) is a child of \( x \) of type \( C' \). If \( x' \) has a reduced word ending in \( x_i^{-1} \), then \( xx_i^{-1} \) is again child of \( x \) of type \( C' \). In either case, \( x' \) has a unique child of type \( C' \). On the other hand, if \( i > j \) for every \( j \in C' \), then \( xx_i \) and \( xx_i^{-1} \) are both children of type \( C \) where \( C \) is the branch of \( C' \) with \( C - C' = \{i\} \). For \( i < j \), \( x'x_i^{-1} \) is not a child of \( x' \). This proves the assertion for \( \text{RAAG}_\Gamma \).

Suppose \( x' \in \text{RACG}_\Gamma \) and \( \tau(x') = C' \). Let \( j = \max C' \). Since \( x' \) has a reduced word ending in \( x_j \), \( l(xx_i) < l(x) \), as \( x_j^2 = 1 \). On the other hand, if \( i > j \) for every \( j \in C' \), then \( xx_i \) is a child of \( x' \) type \( C \) where
C is the branch of \( C' \) with \( C - C' = \{ i \} \). For \( i < j \), \( x'x_i \) is not a child of \( x' \). This proves the assertion for \( \mathrm{RACG}_\Gamma \). □

For any clique \( C \) in \( \Gamma \), let \( m_n(C) \), \( a_n(C) \) and \( c_n(C) \) denote the number of elements of length \( n \) and type \( C \) in \( \mathrm{GM}_\Gamma \), \( \mathrm{RAAG}_\Gamma \) and \( \mathrm{RACG}_\Gamma \) respectively. Lemma 3 implies that

\[
  m_n(C) = m_{n-1}(C) + \sum_{C' \rightarrow C} m_{n-1}(C'),
\]

\[
  a_n(C) = a_{n-1}(C) + 2 \sum_{C' \rightarrow C} a_{n-1}(C'),
\]

\[
  c_n(C) = \sum_{C' \rightarrow C} c_{n-1}(C').
\]

Enumerate all the non-empty cliques of \( \Gamma \) in some order: \( C_1, C_2, \ldots \)

Let \( \mathbf{m}_n, \mathbf{a}_n \) and \( \mathbf{c}_n \) be the column vectors whose \( i \)th entries are \( m_n(C_i) \), \( a_n(C_i) \) and \( c_n(C_i) \) respectively. By Example 1, \( a_1 = 2m_1 = 2c_1 \).

Let \( B_0 \) be the matrix whose \((i, j)\)th entry is given by

\[
  B_0(i, j) = \begin{cases} 
    1 & \text{if } C_j \rightarrow C_i, \\
    0 & \text{otherwise.}
  \end{cases}
\]

Lemma 3 can be expressed in matrix form as follows:

\[
  \mathbf{m}_n = (I + B_0)\mathbf{m}_{n-1},
\]

\[
  \mathbf{a}_n = (I + 2B_0)\mathbf{a}_{n-1},
\]

\[
  \mathbf{c}_n = B_0\mathbf{c}_{n-1}.
\]

Iterating these identities gives

\[
  \mathbf{m}_n = (I + B_0)^{n-1}\mathbf{m}_1,
\]

\[
  \mathbf{m}_n = (I + 2B_0)^{n-1}\mathbf{a}_1,
\]

\[
  \mathbf{c}_n = B_0^{n-1}\mathbf{c}_1,
\]

giving rational expressions for vector-valued generating functions:

\begin{align*}
  \sum_{n=1}^\infty \mathbf{m}_nt^n & = t[I - (I + B_0)t]^{-1}\mathbf{m}_1, \quad (12) \\
  \sum_{n=1}^\infty \mathbf{m}_nt^n & = t[I - (I + 2B_0)t]^{-1}\mathbf{a}_1, \quad (13) \\
  \sum_{n=1}^\infty \mathbf{c}_nt^n & = t[I - B_0t]^{-1}\mathbf{c}_1. \quad (14)
\end{align*}
Furthermore, functional relationships between such generating functions are obtained:

\[
\sum_{n=1}^{\infty} m_n t^n = \sum_{n=1}^{\infty} (I + B_0)^{n-1} t^n m_1 \\
= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \binom{n-1}{k} B_0^k t^n m_1 \\
= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \binom{n-1}{k-1} B_0^{k-1} t^n m_1 \\
= \sum_{k=1}^{\infty} B_0^{k-1} \sum_{n=1}^{\infty} \binom{n-1}{k-1} t^n m_1 \\
= \sum_{k=1}^{\infty} B_0^{k-1} \left( \frac{t}{1-t} \right)^k m_1 \\
= \sum_{k=1}^{\infty} c_k \left( \frac{t}{1-t} \right)^k.
\]

Similarly, one shows

\[
\sum_{n=1}^{\infty} a_n t^n = \sum_{k=1}^{\infty} c_k \left( \frac{2t}{1-t} \right)^k.
\]

For each clique \( C \) in \( \Gamma \) consider the generating functions

\[
\text{GM}_{\Gamma}^C(t) = \sum_{n=1}^{\infty} m_n(C) t^n, \\
\text{RAAG}_{\Gamma}^C(t) = \sum_{n=1}^{\infty} a_n(C) t^n \quad \text{and} \\
\text{RACG}_{\Gamma}^C(t) = \sum_{n=1}^{\infty} c_n(C) t^n.
\]

Then the calculations above show that

\[
(15) \quad \text{GM}_{\Gamma}^C(t) = \text{RACG}_{\Gamma}^C \left( \frac{t}{1-t} \right) \\
(16) \quad \text{RAAG}_{\Gamma}^C(t) = \text{RACG}_{\Gamma}^C \left( \frac{2t}{1-t} \right).
\]

Summing over all cliques \( C \) in \( \Gamma \) give the identities (7) and (8) from Section I. Inverting the identity (8) gives \( \text{RACG}_{\Gamma}^C(t) = \text{GM}_{\Gamma}^C(t/(1+t)) \). Substituting this into (7) gives \( \text{RAAG}_{\Gamma}^C(t) = \text{GM}_{\Gamma}^C(2t/(1+t)) \).
Thus the expression (6) for the spherical generating function of $\text{RAAG}_\Gamma$ follows from the corresponding expression (4) for $\text{GM}_\Gamma$, and similarly, the identities (10) and (11) follow from (9).

3. Geodesic Growth Functions

The parent-child relationship for words is simpler than for elements:

**Definition 4.** Let $w$ be a word in $\text{GM}_\Gamma$, $\text{RAAG}_\Gamma$ or $\text{RACG}_\Gamma$. Then the word $w'$ obtained by removing the last letter of $w$ is called the parent of $w$. The word $w$ is said to be a child of $w'$.

The analog of Lemma 3 for words uses a weaker notion of branching for cliques:

**Definition 5.** Let $C$ and $C'$ be cliques in $\Gamma$. Say that $C'$ branches weakly to $C$ (denoted $C' \rightarrow C$) if the following conditions hold:

1. $C - C'$ is a singleton set (call its unique element $i$),
2. $C$ is the maximal clique in $C' \cup \{i\}$ containing $i$, and

We write $C' \rightsquigarrow C$ if $C'$ branches weakly to $C$.

The only difference between weak branching defined here and the notion of branching in Definition 3 is that the third condition of Definition 3 has been dropped.

The notion of type can be extended from elements of $\text{GM}_\Gamma$, $\text{RAAG}_\Gamma$ and $\text{RACG}_\Gamma$ to reduced words in these structures: the type of a word is just the type of the element that it represents.

**Lemma 4.** Let $C'$ be a clique of $\Gamma$. The children of a word $w'$ of type $C'$ are described by the following rules:

1. If $w' \in \text{GM}_\Gamma$ has type $C'$ then $w'$ has a unique child of type $C$ for every weak branch $C'$ of $C'$, $|C'|$ children of type $C'$ and no other children.
2. If $w' \in \text{RAAG}_\Gamma$ has type $C'$ then $w'$ has two children of type $C$ for every weak branch $C'$ of $C'$, $|C'|$ children of type $C'$ and no other children.
3. If $w' \in \text{RACG}_\Gamma$ has type $C'$ then $w'$ has one child of type $C$ for every weak branch $C'$ of $C'$ and no other children.

**Proof.** The proof of these assertions is quite similar to that of Lemma 3 and is omitted. □

For each clique $C$, let $m'_n(C)$, $a'_n(C)$ and $c'_n(C)$ denote the number of reduced words of length $n$ and type $C$ in $\text{GM}_\Gamma$, $\text{RAAG}_\Gamma$ and $\text{RACG}_\Gamma$.
respectively. Then Lemma 4 tells us that
\[ m_n^*(C) = |C|m_{n-1}^*(C) + \sum_{C' \sim C} m_{n-1}^*(C'), \]
\[ a_n^*(C) = |C|a_{n-1}^*(C) + 2 \sum_{C' \sim C} a_{n-1}^*(C'), \]
\[ c_n^*(C) = \sum_{C' \sim C} c_{n-1}^*(C'). \]

As in Section 2, enumerate the non-empty cliques of \( \Gamma \) as \( C_1, C_2, \ldots \).

Let \( B_1 \) be the matrix whose \((i, j)\)th entry is given by
\[ B_1(i, j) = \begin{cases} 
1 & \text{if } C_j \sim C_i, \\
0 & \text{otherwise}. 
\end{cases} \]

Also, let \( D \) denote the diagonal matrix whose \((i, i)\)th entry is \( |C_i| \). Let \( m_n^* \), \( a_n^* \) and \( c_n^* \) denote the column vectors whose \( i \)th entries are \( m_n^*(C_i) \), \( a_n^*(C_i) \) and \( c_n^*(C_i) \) respectively. Then \( m_1^* = m_1 \), \( a_1^* = a_1 \) and \( c_1^* = c_1 \).

Moreover, the above recurrences for these vectors can be written in matrix form as
\[ m_n^* = (D + B_1)m_{n-1}^*, \]
\[ a_n^* = (D + 2B_1)a_{n-1}^*, \]
\[ c_n^* = B_1c_{n-1}^*. \]

giving, as in Section 2, rational expressions for vector-valued generating functions:
\[ \sum_{n=1}^{\infty} m_n^* t^n = t[I - (D + B_1)t]^{-1} m_1, \quad (17) \]
\[ \sum_{n=1}^{\infty} a_n^* t^n = t[I - (D + 2B_1)t]^{-1} a_1, \quad (18) \]
\[ \sum_{n=1}^{\infty} c_n^* t^n = t[I - B_1t]^{-1} c_1. \quad (19) \]

Adding them up gives a rational expression for the geodesic generating functions:
\[ GM_\Gamma^*(t) = 1 + 1't[I - (D + B_1)t]^{-1} m_1, \]
\[ RAAG_\Gamma^*(t) = 1 + 1't[I - (D + 2B_1)t]^{-1} a_1, \]
\[ RACG_\Gamma^*(t) = 1 + 1't[I - B_1t]^{-1} c_1, \]

where \( 1' \) denotes the all-ones row vector of dimension equal to the number of non-empty cliques in \( \Gamma \).
Since $D$ and $B_1$ do not usually commute, it is not possible to take the binomial expansion of $(D+B_1)^n$ and obtain anything like the functional relations (15) and (16).

4. Link-Regular Graphs

Recall that the link of a clique $C$ in $\Gamma$ is the set of all nodes in $\Gamma$ which are not in $C$, but which are connected by an edge to each of the nodes in $C$. We write $Lk(C)$ for the link of $C$.

**Definition 6** (Link-regularity). The graph $\Gamma$ is said to be link regular if any two cliques of the same cardinality have links of the same cardinality.

The computation of geodesic generating functions for link-regular graphs can be simplified considerably by clubbing together all the types which are cliques of the same cardinality.

**Lemma 5.** Let $\Gamma$ be a link-regular graph. If $C'_1$ and $C'_2$ are cliques of the same cardinality in $\Gamma$, then for each non-negative integer $r$, the number of $r$-cliques that are weak branches of $C'_1$ is equal to the number of $r$-cliques that are weak branches of $C'_2$.

**Proof.** For each non-negative integer $r$, let $L_r$ denote the cardinality of the link of any $r$-clique in $\Gamma$. Fix a $k$-clique $C$. For each subset $S \subset C$, define

$$F_C(S) = \{x \in \Gamma - C \mid Lk(x) \cap C = S\}.$$  

It follows from Definition 5 that:

(20) $C$ branches to $S \cup \{x\}$ weakly if and only if $x \in F_C(S)$.

Define

$$G_C(S) = \{x \in \Gamma - C \mid Lk(x) \cap C \supset S\}.$$  

Then $G_C(S)$ consists of all the points of $\Gamma - C$ which lie in $Lk(S)$. If $|S| = r$, then

(21) $|G_C(S)| = L_r - (k - r),$

which depends only on $k$ and $r$ (and not on the particular choices of cliques $C$ and $S$ of cardinality $k$ and $r$ respectively). Write $G_k(r)$ for $G_C(S)$. 
By the principle of inclusion and exclusion,

\[ F_C(S) = \sum_{T \supset S} (-1)^{|T-S|} G(T) \]

\[ = \sum_{i=r}^{k} (-1)^{i-r} \binom{k-r}{i-r} G_k(i), \]

which is again independent of \(C\) and \(S\), so long as their cardinalities are preserved. Write \(F_k(r)\) for \(F_C(S)\).

By (20), a \(k\)-clique weakly branches to \(\binom{k}{r} F_k(r)\) many \(r+1\)-cliques, and the Lemma follows. \(\square\)

Let \(\overline{m}_n(k)\), \(\overline{a}_n(k)\) and \(\overline{c}_n(k)\) denote the numbers of words of length \(n\) and having type of cardinality \(k\). Then combining Lemma 4 with Lemma 5 gives:

\[ \overline{m}_n(i) = [i + F_j(i-1)] m_{n-1}(j), \]

\[ \overline{a}_n(i) = [i + 2F_j(i-1)] a_{n-1}(j), \]

\[ \overline{c}_n(i) = F_j(i-1) c_{n-1}(j) \]

for \(n \geq 2\).

Let \(d\) be the cardinality of the largest clique of \(\Gamma\). Let \(\overline{B}_1\) be the \(d \times d\) matrix whose \((i, j)\)th entry is given by

\[ \overline{B}_1(i, j) = F_j(i-1). \]

Let \(\overline{D}\) be the diagonal matrix with diagonal entries 1, 2, \ldots, \(d\). Let \(\overline{m}_n\), \(\overline{a}_n\) and \(\overline{c}_n\) denote the column vectors of length \(d\) whose \(i\)th coordinates are \(m_n(i), a_n(i)\) and \(c_n(i)\) respectively, for \(i = 1, \ldots, d\). Then \(\overline{m}_1 = \overline{c}_1 = 2\overline{m}_1\).

Then as in Section 3, we obtain rational expressions for vector-valued generating functions:

\[ \sum_{n=1}^{\infty} \overline{m}_n t^n = t[I - (\overline{D} + \overline{B}_1) t]^{-1} \overline{m}_1, \]

\[ \sum_{n=1}^{\infty} \overline{a}_n t^n = t[I - (\overline{D} + 2\overline{B}_1) t]^{-1} \overline{a}_1, \]

\[ \sum_{n=1}^{\infty} \overline{c}_n t^n = t[I - \overline{B}_1 t]^{-1} \overline{c}_1. \]

In the above calculation of geodesic generating functions for link-regular graphs, the only information that we have used is the number \(m\) of nodes in \(\Gamma\) and the numbers \(L_1, L_2, L_3, \ldots\). The information contained
in this data is the same as the information contained in the clique polynomial of $\Gamma$ because of the identities

$$nc_n = c_{n-1}L_{n-1}$$ for $n = 2, 3, 4, \ldots$.

Thus we recover Theorem 5.1 of Antolín and Ciobanu [AC13]:

**Theorem 1.** If two link-regular graphs $\Gamma_1$ and $\Gamma_2$ have the same clique polynomial, then there is an equality of associated geodesic generating functions:

$$GM_{\Gamma_1}^*(t) = GM_{\Gamma_2}^*(t)$$

$$RAAG_{\Gamma_1}^*(t) = RAAG_{\Gamma_2}^*(t)$$

$$RACG_{\Gamma_1}^*(t) = RACG_{\Gamma_2}^*(t).$$

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