Dice: Compiling Discrete Probabilistic Programs for Scalable Inference

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Probabilistic programming languages (PPLs) are an expressive means of representing and reasoning about probabilistic models. The computational challenge of probabilistic inference remains the primary roadblock for applying PPLs in practice. Inference is fundamentally hard, so there is no one-size-fits all solution. In this work, we target scalable inference for an important class of probabilistic programs: those whose probability distributions are discrete. Discrete distributions are common in many fields, including text analysis, network verification, artificial intelligence, and graph analysis, but they prove to be challenging for existing PPLs.

We develop a domain-specific probabilistic programming language called Dice that features a new approach to exact discrete probabilistic program inference. Dice exploits program structure in order to factorize inference, enabling us to perform exact inference on probabilistic programs with hundreds of thousands of random variables. Our key technical contribution is a new reduction from discrete probabilistic programs to weighted model counting (WMC). This reduction separates the structure of the distribution from its parameters, enabling logical reasoning tools to exploit that structure for probabilistic inference. We (1) show how to compositionally reduce Dice inference to WMC, (2) prove this compilation correct with respect to a denotational semantics, (3) empirically demonstrate the performance benefits over prior approaches, and (4) analyze the types of structure that allow Dice to scale to large probabilistic programs.

1 INTRODUCTION

The primary analysis task in PPLs is probabilistic inference, computing the probability that an event occurs according to the distribution defined by the program. Probabilistic inference generalizes many well-known program analysis tasks, such as reachability, and hence inference for a sufficiently expressive language is an extremely hard program analysis task. The key to scaling inference is to strategically make assumptions about the structure of programs and place restrictions on which programs can be written, while retaining a useful and expressive language.

In this paper, we focus on scaling inference for an important class of probabilistic programs: those whose probability distributions are discrete. Most PPLs today focus on handling continuous random variables. In this setting one usually desires approximate inference techniques, such as forms of sampling [8, 14, 15, 29, 47, 52, 63, 80]. However, handling continuous variables typically requires making strong assumptions about the structure of the program: many of these inference techniques have strict differentiability requirements that preclude their application to programs with discrete random variables. For instance, momentum-based sampling algorithms like HMC and NUTS [41] and many variational approximations [53] are restricted to continuous latent random variables and almost-everywhere differentiability of the posterior distribution. Yet many application domains are naturally discrete: for example mixture models, networks and graphs, ranking and voting, and text. This key deficiency in some of the most popular PPLs has led to a recent rise in interest in handling discreteness in probabilistic programs [39, 64, 82].

In this work we focus entirely on the challenge of designing a fast and efficient discrete probabilistic program inference engine. We describe Dice, a domain-specific language for representing discrete probabilistic programs, along with a new algorithm for exact inference for such programs.
Dice extends a simple first-order, non-recursive functional language with support for making discrete probabilistic choices. It also provides first-class observations, which enables Dice to support Bayesian reasoning in the presence of evidence.

Discrete programs are not a new challenge, and there are existing PPLs that support exact inference for discrete probabilistic programs [2, 8, 20, 35, 36, 38, 62, 68, 74, 79]. However, we identify several compelling example programs from text analysis, network verification, and discrete graphical models that existing methods fail on. The reason that they fail is that the existing methods do not find and automatically exploit the necessary factorizations and structure. Dice’s inference algorithm is inspired by techniques for exact inference on discrete graphical models, which leverage the graphical structure to factorize the inference computation, decomposing it to avoid reasoning about all random variables jointly. For example, a common property is conditional independence: if a variable z is conditionally independent of x given y, then y acts as a kind of interface between x and z that allows inference to be split into two separate analyses. This kind of structure abounds in typical probabilistic programs. For example, a function call is conditionally independent of the calling context given the actual argument value. Dice’s inference algorithm automatically identifies and exploits these independences in order to factorize inference. This enables Dice to scale to extremely large discrete probabilistic programs: our experiments in Section 5 shows Dice performing exact inference on a real-world probabilistic program that is 1.9MB large.

Specifically, Dice builds on the knowledge compilation approach to inference [16, 18, 24, 31]. We show how to compile Dice programs to weighted Boolean formulas (WBF) and then perform exact inference via weighted model counting (WMC) on those formulas. We use binary decision diagrams (BDDs) to represent these formulas, They naturally identify and exploit conditional independence and other forms of structure, thereby avoiding exponential explosion for many classes of interesting programs. Further, BDDs support efficient WMC, linear in the size of the BDD.

Employing knowledge compilation for probabilistic inference in Dice requires us to generalize the prior approaches in several ways. In order to support logical compilation of traditional programming constructs such as conditionals, local variables, and arbitrarily nested tuples, we develop novel compilation rules that compositionally associate Dice programs with weighted Boolean formulae. A key challenge here is supporting arbitrary observations. To do this, a Dice program, as well as each Dice function, is compiled to two BDDs. Intuitively, one BDD represents all possible executions of the program, ignoring observations, and the other BDD represents all executions that satisfy the program’s observations. We show how to use WMC on these formulas to perform exact Bayesian inference, with arbitrary observations throughout the program. Second, Dice compiles functions modularly: each function is compiled to a BDD once, and we exploit efficient BDD composition operations to reuse this BDD at each call site. This technique produces the same final BDD that would otherwise be produced, but it allows us to amortize the costly BDD construction phase across all callers, which we demonstrate can provide orders-of-magnitude speedups.

In sum, this paper presents the following technical contributions:

- We describe the Dice language and illustrate its utility through three motivating examples (Section 2).
- We formalize Dice’s semantics (Section 3) and its compilation to weighted Boolean formulas (Section 4). We prove that the compilation rules are correct with respect to the denotational semantics: the probability distribution represented by a compiled Dice program is equivalent to that of the original program.
let x = flip₁ 0.1 in
let y = if x then flip₂ 0.2 else flip₃ 0.3 in
let z = if y then flip₄ 0.4 else flip₅ 0.5 in z

(a) Example Dice program. (b) Compiled BDD with weighted model counts.

Fig. 1. Illustration of compiling a Dice program that exploits factorization.

• We empirically compare Dice’s performance to that of prior PPLs with exact inference (Section 5). We describe new and challenging benchmark probabilistic programs from cryptography, network analysis, and discrete Bayesian networks, and show that Dice scales to orders-of-magnitude larger programs than existing approaches.

• We analyze some of the benefits of Dice’s compilation strategy in Section 6. First we note that Dice inference is PSPACE-hard. Then we characterize cases where Dice scales efficiently, and which types of structure it exploits in the distribution. We illustrate where to find that structure in the program code as well as the compiled BDD form. We use these results to provide a technical comparison with prior exact inference algorithms.

Finally, Section 7 discusses related work and Section 8 concludes.

2 AN OVERVIEW OF DICE

In this section we overview the Dice language and its inference algorithm. First we use a simple example program to show how Dice exploits program structure to perform inference in a factorized manner. Then we use an example from network verification to show how Dice exploits the modular structure of functions. Finally we use a cryptanalysis example to illustrate how inference in Dice is augmented to support Bayesian inference in the presence of evidence.

2.1 Factorizing Inference

Probabilistic programming languages (PPLs) endow traditional programming languages with probabilistic operations that enable the construction of probability distributions [2, 10, 20, 43, 51], and Dice is no exception. Specifically, Dice extends a first-order functional language that supports non-recursive functions and a form of bounded iteration. Despite its simplicity, this language can express a wide variety of statistical models, and exact probabilistic inference in Dice is fundamentally hard. In addition to its standalone usage, we anticipate Dice being used as a core language for discrete inference inside other probabilistic programming systems.

We begin with a simple motivating example that highlights the challenge of performing inference efficiently and how Dice meets this challenge. Consider the example Dice program in Figure 1a. The syntax is standard except for the introduction of a probabilistic expression flip θ, which flips a coin that returns true with probability θ and false with probability 1−θ. The subscript on each flip is not part of the syntax but rather used to refer to them uniquely in our discussion.

The goal of probabilistic inference is to produce a program’s output probability distribution, so in Figure 1a we desire the probability that z is true and the probability that z is false. Consider computing the probability that z is true, which we denote Pr(z = T). The most straightforward way to compute this quantity is via path enumeration: we can consider all possible assignments to all flips and sum the probability of all assignments under which z = T. A number of existing
PPLs directly implement path enumeration to perform inference \cite{2, 32, 36, 74}. Concretely this would involve computing the following sum of products:

\[
\begin{align*}
&0.1 \cdot 0.2 \cdot 0.4 + 0.1 \cdot 0.8 \cdot 0.5 + 0.9 \cdot 0.3 \cdot 0.4 + 0.9 \cdot 0.7 \cdot 0.5 \\
&\text{subject to } x=T, y=T, z=T, x=F, y=T, z=T, x=F, y=F, z=T
\end{align*}
\]  

\text{(1)}

In this work we focus on the problem of scaling inference, so we ask: how does exhaustive enumeration scale as this program grows in size? In this case we grow the program by adding one additional layer to the chain of flips that depends on the previous. With this growing pattern, the number of terms that a path enumeration must explore grows exponentially in the number of layers, so clearly exhaustive enumeration does not scale on this simple example. Despite its apparent simplicity, many existing inference algorithms cannot scale to large instances of this example; see Figure 10d in Section 5.

However, the sum in Equation 1 has redundant computation, and thus can be factorized as

\[
\begin{align*}
&0.1 \cdot \bigg( 0.2 \cdot 0.4 + 0.8 \cdot 0.5 \bigg) + 0.9 \cdot \bigg( 0.3 \cdot 0.4 + 0.7 \cdot 0.5 \bigg).
\end{align*}
\]

\text{(2)}

Such factorizations are abundant in this example, and in many others. Dice exploits these factorizations to scale, and in Section 5 we show that Dice scales to orders of magnitude larger programs than existing methods in part by exploiting these forms of factorization. Such factorizations are extremely common in probabilistic models, and finding and exploiting them is an essential strategy for scaling exact inference algorithms, for example for graphical models \cite{12, 18, 24, 49, 66}.

\textit{Factorized inference in Dice.} Inference in Dice is designed to find and exploit factorizations like the one shown above. The key insight is to separate the logical representation of the state space of the program from the probabilities, which allows Dice to identify factorizations implied by the structure of the program that are otherwise difficult to detect. This separation is achieved by compiling each program to a \textit{weighted Boolean formula}:

\textbf{Definition 2.1.} Let \( \varphi \) be a Boolean formula over variables \( X \), let \( L \) be the set of all literals (assignments to variables) over \( X \), and \( w : L \to \mathbb{R} \) be a function that associates a real-valued weight with each literal \( L \). The pair \((\varphi, w)\) is a \textit{weighted Boolean formula} (WBF).

To compile the program in Figure 1a into a WBF, we introduce one Boolean variable \( f_i \) for each expression \( \text{flip}_i \) \( \theta \) in the program. Our goal is for the resulting boolean formula over these variables to represent all possible flip valuations that cause \( z \) to be true, so one valid WBF is \( \varphi_{ex} = f_1 f_2 f_3 \lor f_1 \bar{f}_2 \bar{f}_3 \lor f_3 \bar{f}_5 \lor \bar{f}_1 \bar{f}_3 \bar{f}_5 \). Separately, the weight function represents the specific probabilities for each expression \( \text{flip}_i \) \( \theta \) from the program: the weight of \( f_i \) is \( \theta \) if \( f_i \) is true and \( 1 - \theta \) otherwise.

Once the program is associated with a WBF, we can perform probabilistic inference via a \textit{weighted model count}. Formally, for a formula \( \varphi \) over variables \( X \), a sentence \( \omega \) is a \textit{model} of \( \varphi \) if it is a conjunction of literals, contains every variable in \( X \), and \( \omega \models \varphi \). We denote the set of all models of \( \varphi \) as \( \text{Mod}_\varphi(\omega) \). The \textit{weight of a model}, denoted \( w(\omega) \), is the product of the weights of each literal \( \omega_{\varphi} = \Pi_{\varphi \in \omega} w(l) \). Then, the following defines the WMC task:

\textbf{Definition 2.2.} Let \((\varphi, w)\) be a weighted Boolean formula. The \textit{weighted model count} (WMC) of \((\varphi, w)\) is the sum of the weights of each model, 

\[ \text{WMC}(\varphi, w) = \sum_{\omega \in \text{Mod}_\varphi} w(\omega). \]

What has been achieved? So far, not much! The WMC task is known to be \textit{#P}-hard for arbitrary Boolean formulas. Indeed, our formula \( \varphi_{ex} \) above is isomorphic to the structure of Equation 1, so the WMC calculation over it will be essentially equivalent. However, it has been observed in the
AI literature that certain representations of Boolean formulas – such as binary decision diagrams (BDDs) – both exploit the structure of a formula to minimize its representation and support linear time weighted model counting, and as such are useful compilation targets [13, 18, 26, 73]. The field of compiling Boolean formulas to representations that support tractable weighted model counting is broadly known as knowledge compilation, and inference via knowledge compilation is currently the state-of-the-art inference algorithm for certain kinds of discrete Bayesian networks [18] and probabilistic logic programs [30].

Dice utilizes the insights of knowledge compilation to perform factorized inference. First, the generated formula \( \varphi \) in a compiled WBF is represented as a BDD; Figure 1b shows the compiled BDD for the program in Figure 1a. A solid edge denotes the case where the parent variable is true and a dotted edge denotes the case where the parent variable is false. This BDD is logically equivalent to \( \varphi_{\text{ex}} \) but the BDD’s construction process exploits the program’s conditional independence to efficiently produce a compact canonical representation. Specifically, there is a single subtree for \( f_4 \), which is shared by both the path coming from \( f_2 \) and the path coming from \( f_3 \), and similarly for \( f_5 \). These shared sub-trees are induced by conditional independence: fixing \( y \) to a specific value – and hence guaranteeing that a path to \( f_4 \) is taken in the BDD – screens off the effect of \( x \) on \( z \), and hence reduces both the size of the final BDD and the cost of constructing it. The BDD automatically finds and exploits such factorization opportunities by caching and reusing repetitious logical sub-functions.

Dice performs inference on the original probabilistic program via WMC once the program is compiled to a BDD. Crucially, we can do so without exhaustively enumerating all paths or models. By virtue of the shared sub-functions, the BDD in Figure 1b directly describes how to compute the WMC in the factorized manner. Observe that each node is annotated with the weighted model count, which is computed in linear time in a single bottom-up pass of the BDD. For instance, the WMC at node \( f_2 \) is given by taking the weighted sum of the WMC of its children, \( 0.2 \times 0.4 + 0.8 \times 0.5 \). Finally, the sum taken at the root of the BDD (the node \( f_1 \)) is exactly the factorized sum in Equation 2.

### 2.2 Leveraging Functional Abstraction

The previous section highlights how Dice exploits factorization that comes from conditional independences in the program. One common source of such independences is functional abstraction: the behavior of a function call is independent of the calling context, given the actual argument. Dice inference as described above automatically exploits this structure as part of the BDD construction. In addition, Dice exploits functional abstraction in an orthogonal manner by modularly
compiling a BDD for each function once and then reusing this BDD at each call site, thereby amortizing the cost of the BDD construction across all callers.

To illustrate the benefits of functional abstraction, we adapt an example from recent work in probabilistic verification of computer networks via probabilistic programs [34]. Figure 2a shows a “diamond” network that contains four servers, labeled $S_i$. The network’s behavior is naturally probabilistic, to account for dynamics such as load balancing and congestion. In this case, server $S_1$ forwards an incoming packet to either $S_2$ or $S_3$, each with probability 50%. In turn, those servers forward packets received from $S_1$ to $S_4$, except that $S_3$ has a 0.1% chance of dropping such a packet. The diamond function in Figure 2b defines the behavior of this network as a probabilistic program in Dice. The argument boolean $s_1$ represents the existence of an incoming packet to $S_1$ from the left, and the function returns a boolean indicating whether a packet was delivered to $S_4$.

As mentioned above, Dice compiles functions modularly, so Dice first compiles the diamond function to a BDD, shown in Figure 2c. The variable $s_1$ represents the unknown input to the function, and the $f_i$ variables represent the flips in the function body, as in our previous example. Next Dice will create the BDD for the “main” expression in lines 7–9 of Figure 2b. During this process, the BDD for the diamond function is reused at each call site using standard BDD composition operations like conjunction (Section 4 describes this in more detail). The final BDD for the program is shown in Figure 2d, where each variable $f_{ij}$ represents the $i$th flip in the $j$th call to diamond.

The final BDD automatically identifies and exploits functional abstraction. For example, the structure of the BDD makes it clear that the third call to diamond depends only on the output of the second call to diamond, rather than the particular execution path taken to produce that output. As a result, even though there are three sub-networks, and therefore $2^6$ possible joint assignments to $f_{ij}$s, the BDD only has 8 nodes. More generally, this BDD will grow linearly in the number of composed diamond calls, though the number of possible executions grows exponentially. Hence functional abstraction both produces smaller BDDs, which leads to faster WMC computation, and reduces BDD compilation time by compiling each function once. We show in Section 5 that these capabilities provide orders of magnitude speedups in inference.

2.3 Bayesian Inference & Observations

Bayesian inference is a general and popular technique for reasoning about the probability of events in the presence of evidence. Dice, similar to other PPLs, supports Bayesian reasoning through an observe expression. Specifically, the expression “observe e” represents evidence (or an observation) that e has the value true; executions on which e does not have the value true are defined to have 0 probability.

Dice supports first-class observations, including inside of functions. An example is shown in Figure 3, which shows another rich class of discrete probabilistic inference problems that come from text analysis. For this problem the goal is to decrypt a given piece of ciphertext by inferring the most likely encryption key. We assume that the plaintext was encrypted using a Caesar cipher, which simply shifts
characters by a fixed but unknown constant, so the encryption key is an integer between 0 and 25 (e.g., with key 2, “abc” becomes “cde”).

The task of decrypting encrypted ciphertext can be cast as a probabilistic inference task by using frequency analysis [48]. In the English language each letter has a certain probability of being used: for instance, the frequency of letter “E” is 12.02%. In Figure 3, the function EncryptChar is a generative model for how each letter in the ciphertext was created. The function takes as an argument the encryption key as well as a received ciphertext character. First a plaintext character randomChar is chosen according to its empirical distribution (the ChooseChar function is not shown but straightforward). Then this character is encrypted with the given key. To make the inference problem more challenging and realistic, we assume that there is a chance that the encryptor mistakenly forgets to encrypt a character, so with some chance we observe that the ciphertext is the actual ciphertext character that we received, otherwise we observe nothing. Initially, the key (k) is assumed to be uniformly random (line 6). After invoking EncryptChar once for each received ciphertext character (lines 7–8), the posterior distribution on the key is returned.

The interaction of probabilistic inference with observations is subtle. Observations have a non-local and “backwards” effect on the probability distribution, which must be carefully preserved when performing inference. In our example, the observation inside of EncryptChar affects the posterior distribution of its argument key. These non-local effects are the bane of sampling-based inference algorithms: observations can impose complex constraints — such as the need in our example for ChooseChar to draw the right character — that make it challenging for sampling algorithms to find sufficiently many valid samples (we highlight this challenge in Section 5).

The WBF compilation strategy outlined in the previous section is inadequate for capturing the semantics of the EncryptChar function: this function always returns true, so its compiled BDD would be trivial. Clearly this is incorrect, since the EncryptChar function has an additional, implicit effect on the program, by making certain encryption keys more or less likely to be the correct one. To handle observations, we augment our compilation strategy to produce a second logical formula, which we call the accepting formula and denote \( \gamma \). The accepting formula represents all possible assignments to flips that cause all observes in the program to be satisfied. Together the formulas \( \phi \) and \( \gamma \) capture the meaning of the program: we can compute the posterior distribution on k by computing \( \frac{WMC(\phi \land \gamma, w)}{WMC(\gamma, w)} \). Note that \( \gamma \) serves two roles: it constrains \( \phi \) to only those executions that satisfy the observations, and its weighted model count computes the normalizing constant for the final probability distribution.

3 THE DICE LANGUAGE

Dice is a first-order functional language augmented with constructs for probabilistic programming. This section describes the language formally, providing its syntax and compositional semantics.

3.1 Syntax

We formalize a core subset of the Dice language, without support for bounded integers and iteration. These features are treated as syntactic sugar in our implementation, as described in Section 5.1. The core syntax of Dice is given in Figure 4. We enforce an A-normal form [33], which simplifies the semantics and compilation rules. A program is a sequence of functions followed by the ”main” expression. Each function is non-recursive and can only call functions that precede it. The language supports booleans, tuples, and typical operations over those types. In addition to this core syntax our Dice implementation supports convenient syntactic sugar for logical operations (\( \land \), \( \lor \), and \( \neg \)), statically bounded loops, bounded-size integers, and arbitrary function arity, as we describe in Section 5.1. We utilize these extensions in our examples freely.
Dice supports two probabilistic expressions. First, the expression `flip \ \theta`, where \( \theta \) is a real number between 0 and 1, denotes the distribution that has the value true with probability \( \theta \) and false with probability \( 1 - \theta \). Second, the expression `observe e` enables Bayesian reasoning by incorporating evidence. Specifically, `observe e` represents the observation that \( e \) has the value true. Semantically, executions on which \( e \) does not have the value true are defined to have 0 probability, which has the effect of implicitly increasing the probabilities of other executions. We define the expression `observe e` to always return the value true.

### 3.2 Semantics

The semantics of Dice programs is largely standard. We overview the semantics and highlight its key aspects and design choices. We begin with the semantics of Dice expressions, which are naturally represented as a probability distribution on values. Formally, let \( V \) be the set of all Dice values. Then, a discrete probability distribution on \( V \) is a function \( \text{Pr} : V \rightarrow [0,1] \) such that \( \sum_{\omega \in V} \text{Pr}(\omega) = 1 \).

Figure 5 provides the semantics for Dice expressions. The semantic function \( \llbracket \cdot \rrbracket \) maps expressions to their probability distributions. We defer discussion of function calls and observations to Sections 3.2.1 and 3.2.2 below. The semantics of values and tuple access are straightforward. For example, the semantics of the expression `fst (F,T)` is the probability distribution that assigns
To compute the probability that \( \text{flip} \ 0.8 \) denotes the distribution that assigns \( T \) the probability 0.8, \( F \) the probability 0.2, and all other values the probability 0.

The most interesting case in the semantics is for \texttt{let}, as it shows the path enumeration that is required when sequencing probabilistic expressions. Consider the example:

\[
\text{let } x = \text{flip} \ 0.1 \text{ in } \text{flip} \ 0.4 \lor x
\]

(ExLet)

To compute the probability that (ExLet) results in some value \( v \), we must consider all possible ways in which that value could result, based on all possible values \( v' \) for \( x \). Concretely, to evaluate \([\text{let } x = \text{flip} \ 0.1 \text{ in } \text{flip} \ 0.4 \lor x](T)\), the following sum is computed: 
\[
[\text{flip} \ 0.1](T) \times [\text{flip} \ 0.4 \lor x[x \mapsto T]](T) + [\text{flip} \ 0.1](F) \times [\text{flip} \ 0.4 \lor x[x \mapsto F]](T) = 0.1 \times 1.0 + 0.9 \times 0.4 = 0.46.
\]

3.2.1 Functions and Programs. Dice supports non-recursive functions. We generalize the semantics of expressions to functions in a natural way. Specifically, the semantics of a function \( f \) is a conditional probability distribution, which is a function from each value \( v \) to a probability distribution for \( f(v) \). Formally, the semantics of a function \([\text{func}] : V \rightarrow V \rightarrow [0, 1] \) is defined as follows:

\[
[\text{func } f(x : \tau) : \tau'\{e\}](v) \triangleq [e[x \mapsto v]]
\]

(3)

We can now give a semantics to function calls. To do so, we extend the semantics judgment to include a function table \( T \), which is a finite map from function names to their conditional probability distributions. Formally our semantics judgment for expressions now has the form \([e]_T : V \rightarrow [0, 1] \) and similarly for the semantics of function definitions above, but we leave \( T \) implicit when it is clear from the context. Figure 5 provides the semantics of a function call: the function’s conditional probability distribution is found in \( T \), and the probability distribution associated with the actual argument \( v \) is retrieved.

Finally, we define the semantics of programs \([\text{p}]_T : V \rightarrow [0, 1] \). Intuitively, each function is given a semantics in the context of the prior functions, and then the semantics of the program is defined as the semantics of the “main” expression. We formalize this semantics inductively via the following two rules, where \( \bullet \) denotes the empty sequence and \( \eta(\text{func}) \) denotes the name of the function \text{func}:

\[
[\bullet e]_T \triangleq [e]_T \quad [\text{func } p]_T \triangleq [p]_T \cup \{\eta(\text{func}) \rightarrow [\text{func}]_T\}.
\]

(4)

3.2.2 Observations & Bayesian Conditioning. Observations complicate the goal of associating a probability distribution with each program expression. Our semantics of \texttt{observe} in Figure 5 follows prior work by assigning probability 0 to a failed observation \([10, 20, 43, 51, 63]\). Now consider the following example program:

\[
\text{let } x = \text{flip} \ 0.6 \text{ in } \text{let } y = \text{flip} \ 0.3 \text{ in } \text{let } = \text{observe } x \lor y \text{ in } x
\]

(ObsProg)

Because the \texttt{observe} expression is falsified when both \( x \) and \( y \) are false, that scenario has probability 0. Hence according to our semantics \([\text{ObsProg}]_T = 0.6 \) and \([\text{ObsProg}]_F = 0.12 \). As a result the meaning of this program is not a valid probability distribution.

The standard approach to handling this issue is to treat the semantics as producing an **unnormalized** distribution, which need not sum to 1 and which is normalized at the very end to produce a valid probability distribution for the entire program. Here we explore the subtle properties of this unnormalized distribution, which will serve a crucial purpose later during our compilation
strategy. We explicitly label the normalizing constant as $[\cdot]_A$ and let $[\cdot]_D$ denote the normalized distribution for a program. These two quantities can be straightforwardly computed from the unnormalized semantics in Figure 5:

$$
[e]_A \triangleq \sum_v [e](v), \quad [e]_D(v) \triangleq \frac{1}{[e]_A}[e](v).
$$

(5)

For instance, in the above example $[\text{ObsProg}]_A = 0.12 + 0.6 = 0.72$, $[\text{ObsProg}]_D(T) = 0.6/0.72 \approx 0.83$, and $[\text{ObsProg}]_D(F) = 0.12/0.72 \approx 0.17$.

It is clear that, by construction, $[\cdot]_D$ always yields a probability distribution, so we call it the \textit{distributional semantics}. This is the quantity that we need in order to answer inference queries on the program. What does $[\cdot]_A$ represent? Typically it is not given a meaning but rather simply considered to be an arbitrary normalizing constant computed for the entire program. And indeed, the normalizing constant is irrelevant for the purposes of performing global inference: the probabilities in the unnormalized semantics can be scaled arbitrarily without changing $[\cdot]_D$. This “normalize at the end” mode of operation is standard for many PPLs that use an unnormalized semantics, and so $[\cdot]_A$ is not typically given a semantic interpretation in these contexts, other than as an unknown normalizing constant [20, 30].

However, when reasoning about partial programs, the distributional semantics alone is not sufficient. For example, consider these two functions:

\begin{align*}
\text{fun } f(x:\text{Bool}):\text{Bool} \{ \text{let } y = x \lor \text{flip}(0.5) \text{ in let } z = \text{observe } y \text{ in } y \} & \quad (6) \\
\text{fun } g(x:\text{Bool}):\text{Bool} \{ \text{true } \} & \quad (7)
\end{align*}

Because the observation in $f$ requires $y$ to be true, the two functions have the identical distributional semantics: they both return true with probability 1, regardless of the argument $x$. However, these two functions are not equivalent! Specifically, the observation in $f$ has the effect of changing the probability distribution of the argument $x$ when the function is called. Concretely,

\begin{align*}
[\text{let } x = \text{flip } 0.1 \text{ in let } \text{obs } = f(x) \text{ in } x]_D(T) &= 0.1/0.55 \\
[\text{let } x = \text{flip } 0.1 \text{ in let } \text{obs } = g(x) \text{ in } x]_D(T) &= 0.1
\end{align*}

The quantity $[\cdot]_A$ carries exactly the information needed to distinguish these functions. Specifically, $[\cdot]_A$ represents the probability that $e$ has an \textit{accepting} execution, which satisfies all observations, so we call it the \textit{accepting semantics}. In the above example, $[g(F)]_A = 1$ but $[f(F)]_A = 0.5$: the function call $f(F)$ will succeed only half of the time. This quantity allows us to precisely compute the effect of the observation on any caller.

In summary, the semantics in Figure 5 computes an unnormalized distribution. However, since the normalizing constant is exactly the accepting probability, the semantics has the effect of computing two key quantities on each program fragment, both of which are necessary to characterize its meaning: its normalized probability distribution and its probability of accepting. Later this accepting semantics will be explicitly represented during compilation as the accepting formula.

4 PROBABILISTIC INFERENCE FOR DICE

This section formalizes our approach to probabilistic inference in Dice via reduction to \textit{weighted model counting} (WMC). In this style, a probabilistic model is compiled to a \textit{weighted Boolean formula} (WBF) such that WMC queries on the WBF exactly correspond to inference queries on the original model. This approach has been successfully used to perform exact inference in discrete Bayesian networks as well as probabilistic databases [18, 30]. However, to our knowledge it has
We describe this compilation in stages: first on the Boolean sub-language, then with the addition of all observations, and similarly not been previously applied to a probabilistic programming language with traditional programming language constructs, functions, and first-class observations.

The bulk of this section formalizes our novel algorithm for compiling Dice programs to WBF. We describe this compilation in stages: first on the Boolean sub-language, then with the addition of tuples, and finally with the addition of functions. We also state and prove a correctness theorem, which formally relates WMC queries over a program’s compiled WBF to the semantics from the previous section. Finally we illustrate how we use BDDs to represent WBFs, which enables the approach to automatically perform factorized inference.

4.1 Compiling Boolean Dice Expressions

The formal compilation judgment for Boolean Dice expressions has the form \( e \leadsto (\varphi, \gamma, w) \), where \( \varphi \) and \( \gamma \) are logical formulas and \( w \) is a weight function (this judgment form will be extended later to accommodate other language features). We call \( \varphi \) the unnormalized formula: it represents all possible assignments to variables and flips for which \( e \) evaluates to true, ignoring observations. We call \( \gamma \) the accepting formula: it represents all possible assignments to variables and flips that cause all observations in \( e \) to succeed. Before showing the formal rules, we present two examples to build intuition on the compilation to WBF and how it is used to perform inference.

**Example 4.1.** The expression (ExLet) from the previous section compiles to the unnormalized formula \( \varphi = f_1 \land f_2 \), where \( f_1 \) and \( f_2 \) are Boolean variables associated with flip 0.1 and flip 0.4 respectively. Since there are no observations, \( \gamma = T \) for this example. The weight function \( w \) assigns weights to the literals of \( f_1 \) and \( f_2 \) that correspond with their probabilities in (ExLet). Then we have that \( [\text{ExLet}] (T) = \text{WMC}(\varphi, w) = 0.46 \) and \( [\text{ExLet}] (F) = \text{WMC}(\overline{\varphi}, w) = 0.54 \).

**Example 4.2.** The program (ObsProg) from the previous section compiles to the unnormalized formula \( \varphi = f_1 \) and the accepting formula \( \gamma = f_1 \land f_2 \), where \( f_1 \) corresponds with flip 0.6 and \( f_2 \) with flip 0.3. Hence the formula \( \varphi \land \gamma \) is true if and only if the program evaluates to \( T \) and satisfies all observations, and similarly \( \overline{\varphi} \land \gamma \) is true if and only if the program evaluates to \( F \) and satisfies all observations. Then, with the appropriate weight function \( w \), we can perform Bayesian inference on (ObsProg) via two weighted model counts: \( [(\text{ObsProg})]_D (T) = \text{WMC}(\varphi \land \gamma, w) / \text{WMC}(\gamma, w) \approx 0.83 \) and \( [(\text{ObsProg})]_D (F) = \text{WMC}(\overline{\varphi} \land \gamma, w) / \text{WMC}(\gamma, w) \approx 0.17 \).

---

### Formal Rules

| Rule | Syntax |
|------|--------|
| C-TRUE | \( T \leadsto (T, T, \emptyset) \) |
| C-FALSE | \( F \leadsto (F, T, \emptyset) \) |
| C-IDENT | \( x \leadsto (x, T, \emptyset) \) |
| C-FLIP | \( \text{flip} \theta \leadsto (f, T, (f \mapsto \theta, T, \overline{T} \mapsto 1 - \theta)) \) |
| C-OBS | \( \text{observe} \ aexp \leadsto (\varphi, T, \emptyset) \) |
| C-ITE | if \( aexp \) then \( e_1 \) else \( e_2 \) \( \leadsto ((\varphi_g \land \varphi_F) \lor ((\overline{\varphi_g} \land \varphi_E), ((\varphi_g \land \gamma_T) \lor ((\overline{\varphi_g} \land \gamma_E), w_T \cup w_E) \) |
| C-LET | let \( x = e_1 \) in \( e_2 \) \( \leadsto (\varphi_1, \gamma_1, w_1) \) \( \varphi_2[x \mapsto \varphi_1], \gamma_1 \land \gamma_2[x \mapsto \varphi_1], w_1 \cup w_2 \) |

Fig. 6. Compiling Boolean expressions to WBFs.
The formal compilation rules are shown in Figure 6. The above examples show how closed programs are compiled, but expressions can also have free variables in them. The rule C-IDENT handles a free variable x simply by introducing a corresponding Boolean variable. To illustrate the rule C-FLIP, flip 0.4 \sim (f, T, w) where w maps f to 0.4 and \bar{f} to 0.6, and f is a fresh Boolean variable. Hence \text{WMC}(f \land T, w) = 0.4 = [\text{flip} 0.4] (T) and \text{WMC}(\bar{f}, w) = 0.6 = [\text{flip} 0.4] (\bar{f}).

The rule C-OBS handles observables. Since an expression’s unnormalized formula ignores observations, the unnormalized formula for observe aexp is simply T. The metavariable aexp ranges over values and identifiers and hence compiles to an accepting formula of T and an empty weight function. Finally, the unnormalized formula of aexp becomes the accepting formula of observe aexp, in order to capture all ways that the observation is satisfied.

The rule C-ITE encodes the usual logical semantics of conditionals. Finally, the C-LET rule shows how to represent expression sequencing. The logical substitution \[\varphi_1[x \mapsto \varphi_2]\] replaces all occurrences of x in \varphi_1 with the formula \varphi_2. For the accepting formula, the expression let x = e_1 in e_2 only accepts if both expressions accept, so we simply conjoin the accepting formulas. To illustrate the rule, we show the derivation through the rules for our example (EXLET), assuming the obvious rule for compiling logical disjunction (which is syntactic sugar for a conditional expression):

\[
\begin{array}{c|c|c}
\text{fresh } f_1 & x \sim (x, T, \emptyset) & \text{fresh } f_2 \\
\hline
\text{flip } 0.1 \sim (f_1, T, w_1) & \text{flip } 0.4 \sim (f_2, T, w_2) & \text{flip } 0.4 \lor x \sim (f_2 \lor x, T, w_2) \\
1 \text{let } x = \text{flip } 0.1 \text{ in flip } 0.4 \lor x \sim (f_2 \lor x[x \mapsto f_1], T, w_1 \cup w_2) & \text{(EXLETCompilation)}
\end{array}
\]

This compilation matches Example 4.1 above and shows how logical substitution captures expression sequencing. We safely combine weights as w_1 \cup w_2 because no two subexpressions can share flips, so there can be no conflicts.

The statement of correctness for Boolean Dice expressions connects our compilation rules to the formal semantics from the previous section:

**Lemma 4.3 (Boolean Expression Correctness).** Let e be a Boolean Dice expression with free variables x_1, \ldots, x_n and suppose e \sim (\varphi, \gamma, w). Then for any Boolean values v_1, \ldots, v_n:

- \[[e[x_i \mapsto v_i]]_{A} = \text{WMC}(y[x_i \mapsto v_i], w, \varphi, \gamma, w)
- \text{for any Boolean value } v, \ [[e[x_i \mapsto v_i]]_{D}(v) = \text{WMC}((\varphi \iff v) \land \gamma)[x_i \mapsto v_i], w) / \text{WMC}(y[x_i \mapsto v_i], w)].

Hence we can answer inference queries on the original expression via two WMC queries on the compiled WBF. The following key lemma directly implies the above theorem; a stronger version of it is proven in Appendix B.2:

**Lemma 4.4.** Let e be a Boolean Dice expression with free variables x_1, \ldots, x_n and suppose e \sim (\varphi, \gamma, w). Then for any Boolean values v_1, \ldots, v_n and Boolean value v,

\[[e[x_i \mapsto v_i]](v) = \text{WMC}((\varphi \iff v) \land \gamma)[x_i \mapsto v_i], w).

**4.2 Tuples & Typed Compilation**

Next we extend our compilation rules to support arbitrarily nested tuples. This extension requires that we generalize the compilation judgment, which now has the following form:

\[\Gamma \vdash e : \tau \sim (\varphi, \gamma, w)\]

First, our compilation is now **typed**: \(\Gamma\) is the usual type environment for free variables and \(\tau\) is the type of \(e\). The types are necessary to determine how to properly encode program variables in the
We define typed substitution inductively as follows:

\[ \Gamma(x) = \tau \]
\[ \Gamma \vdash x : \tau \rightsquigarrow (F_\tau(x), T, \emptyset) \] (C-IDENT) \[ \Gamma(x_1) = \tau_1, \Gamma(x_2) = \tau_2 \]
\[ \Gamma \vdash (x_1, x_2) : \tau_1 \times \tau_2 \rightsquigarrow ((F_{\tau_1}(x_1), F_{\tau_2}(x_2)), T, \emptyset) \] (C-TUP)

\[ \Gamma(x) = \tau_1 \times \tau_2 \]
\[ \Gamma \vdash \text{fst } x : \tau_1 \rightsquigarrow (F_{\tau_1}(x_1), T, \emptyset) \] (C-FST) \[ \Gamma(x) = \tau_1 \times \tau_2 \]
\[ \Gamma \vdash \text{snd } x : \tau_2 \rightsquigarrow (F_{\tau_2}(x_2), T, \emptyset) \] (C-SND)

\[ \Gamma \vdash \text{aexp : } \text{Bool} \rightsquigarrow (\varphi_g, T, \emptyset) \]
\[ \Gamma \vdash \text{e}_T : \tau \rightsquigarrow (\varphi_T, y_T, w_T) \]
\[ \Gamma \vdash \text{e}_E : \tau \rightsquigarrow (\varphi_E, y_E, w_E) \]

\[ \Gamma \vdash \text{if } \text{aexp then } \text{e}_T \text{ else } \text{e}_E : \tau \rightsquigarrow \left( ((\varphi_g \land \varphi_T) \lor ((\varphi_g \land \gamma_T) \lor (\varphi_g \land y_E) \lor (w_T \cup w_E)) \right) \] (C-ITE)

\[ \Gamma \vdash \text{let } x : \tau_1 = \text{e}_1 \text{ in } \text{e}_2 : \tau_2 \rightsquigarrow (\varphi_2[\varphi_1 \mapsto \varphi_1], y_1 \land y_2[\varphi_1 \mapsto \varphi_1], w_1 \cup w_2) \] (C-LET)

Fig. 7. Typed compilation for tuples. These assume, without loss of generality but for simplicity, that \text{fst}, \text{snd}, and tuple construction are only ever performed with identifiers as arguments.

Compiled logical formulas. Second, compilation produces a collection of Boolean formulas, one per occurrence of the type \text{Bool} in \tau. The new metavariable \varphi is defined inductively as either a Boolean formula \varphi or a pair of the form (\varphi_1, \varphi_2).

Figure 7 shows the new rules for compiling tuples and also presents updated versions of the rules from Figure 6, other than the Boolean-specific rules. The extended compilation for tuples is structurally very similar to Boolean compilation, but requires generalizing the Boolean operations in a natural way to accommodate tuples (Appendix A summarizes this new notation). The new version of C-IDENT uses the form function \( F_\tau(x) \), which constructs the logical representation of a variable \( x \) based on its type \( \tau \). It is defined inductively as \( F_{\text{Bool}}(x) \triangleq x \) and \( F_{\tau_1 \times \tau_2}(x) \triangleq (F_{\tau_1}(x_1), F_{\tau_2}(x_2)) \). Note the subscripts \( x_1 \) and \( x_2 \) that lexically distinguish the left and right elements. This function also allows us to naturally define the compilation for tuple creation as well as \text{fst} and \text{snd} in Figure 7.

The C-ITE rule shows how we generalize the compilation of conditionals to accommodate tuples. The rule requires that we conjoin a Boolean expression \( \varphi_g \) (the compiled guard) with a potential tuple of formulas (the compiled then and else branches). To do this, we define \text{broadcasted conjunction}, denoted \( \varphi_g \land \hat{\varphi} \), as conjoining \( \varphi_g \) with all the Boolean expressions in the tuple \( \hat{\varphi} \). Formally, we define it as \( \varphi_a \land \varphi_b \triangleq \varphi_a \land \varphi_b \land (\varphi_{b_1}, \varphi_{b_2}) \triangleq (\varphi_a \land \hat{\varphi}_{b_1}, \varphi_a \land \hat{\varphi}_{b_2}) \). In addition to broadcasted conjunction, C-ITE also requires \text{point-wise disjunction}, denoted \( \hat{\varphi}_r \lor \hat{\varphi}_f \). Point-wise disjunction is defined inductively as \( \varphi_1 \lor \varphi_2 \triangleq \varphi_1 \lor \varphi_2 \land (\varphi_{11}, \varphi_{12}) \lor (\varphi_{21}, \varphi_{22}) \triangleq (\varphi_{11} \lor \hat{\varphi}_{21}, \varphi_{12} \lor \hat{\varphi}_{22}) \).

Finally, to generalize the compilation of \text{let} expressions, in the C-LET rule we employ \text{typed substitution} \( \varphi_2[x \mapsto \varphi_1] \) to substitute the compiled version of \( e_1 \) into the compiled version of \( e_2 \). We define typed substitution inductively as follows:

\[ \varphi_2[x \mapsto \varphi_1] \triangleq \varphi_2[x \mapsto \varphi_1], \quad \varphi_2[x \mapsto a \times b] \triangleq (\varphi_a \land \hat{\varphi}_b)[x \mapsto \varphi_2]\]

\[ (\varphi_1, \varphi_2)[x \mapsto \varphi] \triangleq (\varphi_1[x \mapsto \varphi], \varphi_2[x \mapsto \varphi]). \]

We can state and prove a natural generalization of our key lemma from the previous subsection, Lemma 4.4. The lemma depends on \text{pointwise iff}, denoted \( \varphi_1 \iff \varphi_2 \) and defined inductively as
We conclude our development of Fig. 8. Compiling functions and programs. These assume without loss of generality but for simplicity that \( \phi \) follows:

- \( \text{p simply compile the function’s body in an appropriate type environment.} \)
- \( \text{The judgment} \)
- \( \Phi \) substitutes the actual argument for the formal argument. One subtlety is that we must ensure that the rule for compiling function calls. \( \text{The rule simply looks up the function’s compiled WBF and sub-} \)

\[
\begin{align*}
\Gamma \cup \{ x_1 : \tau_1 \}, \Phi \vdash e : \tau_2 \rightsquigarrow (\phi, \gamma, w) & \quad \text{(C-FUNC)} \\
\Gamma, \Phi \vdash \text{fun } f(x_1 : \tau_1) : \tau_2 \{ e \} \rightsquigarrow (\phi, \gamma, w) & \quad \text{(C-PROG1)} \\
\Gamma, \Phi \vdash \text{fun } f(x_1 : \tau_1) : \tau_2 \{ e \} \rightsquigarrow (\phi_f, \gamma_f, w_f) & \quad \text{(C-PROG2)} \\
\Gamma \cup \{ f : \tau_1 \rightarrow \tau_2 \}, \Phi \cup \{ f : \rightarrow (x_1, \phi_f, \gamma_f, w_f) \} \vdash p : \tau \rightsquigarrow (\phi, \gamma, w) & \quad \text{(C-PROG2)} \\
\Gamma, \Phi \vdash f(x_1 : \tau_1) : \tau_2 \{ e \} p : \tau \rightsquigarrow (\phi, \gamma, w) & \quad \text{(C-PROG2)} \\
\end{align*}
\]

Fig. 8. Compiling functions and programs. These assume without loss of generality but for simplicity that function calls are only ever given identifiers as arguments.

follows: \( \phi_1 \leftrightarrow \phi_2 \Delta \phi_3 \leftrightarrow \phi_2 \) and \( (\phi_1, \phi_2) \leftrightarrow (\phi_1', \phi_2') \Delta (\phi_1 \leftrightarrow \phi_1') \lor (\phi_2 \leftrightarrow \phi_2') \). Then, the following correctness lemma is proved in Appendix B.2:

**Lemma 4.5 (Typed Correctness Without Functions).** Let \( e \) be a Dice expression without function calls, and suppose \( \{ x_1 : \tau_1 \} \vdash e : \tau \rightsquigarrow (\phi, \gamma, w) \). Then for any values \( \{ v_i : \tau_i \} \) and \( v : \tau \), we have that \( \langle \llbracket e[x_1 \mapsto v_1] \rrbracket \rangle (v) = \text{WMC} \left( (\phi \leftrightarrow v) \land \gamma \right) [x_1 \mapsto v_1], w \).

### 4.3 Functions & Programs

We conclude our development of Dice compilation by introducing functions and programs in Figure 8. We introduce a new piece of context \( \Phi \) into our judgment, which maps function names to their compiled function bodies. Function names are mapped to a 4-tuple \((x_{arg}, \phi, \gamma, w)\) where \( x_{arg} \) is the logical variable for the function’s formal argument and the other items are respectively the function body’s compiled unnormalized formula, accepting formula, and weight function.

The judgment \( \Gamma, \Phi \vdash \text{func } \rightsquigarrow (\phi, \gamma, w) \) compiles function definitions. As shown in C-FUNC, we simply compile the function’s body in an appropriate type environment. The judgment \( \Gamma, \Phi \vdash p : \tau \rightsquigarrow (\phi, \gamma, w) \) compiles programs by compiling each function in order, followed by the “main” expression. The rules C-PROG1 and C-PROG2 perform this compilation. After each function is compiled, its compiled WBF is added to \( \Phi \) and its type is added to \( \Gamma \), for use subsequent compilation.

The final judgment form for expressions is \( \Gamma, \Phi \vdash e : \tau \rightsquigarrow (\phi, \gamma, w) \), and C-FUNCALL shows the rule for compiling function calls. The rule simply looks up the function’s compiled WBF and substitutes the actual argument for the formal argument. One subtlety is that we must ensure that the flips in each call to a function are independent of one another. Our compilation approach makes it straightforward to do so: simply replace all of the variables in \( \phi \) and \( \gamma \), aside from the formal argument \( x_{arg} \), with fresh variables. We use an auxiliary function \( \text{RefreshFlips}(x_{arg}, \phi, \gamma, w) \) for this purpose. We now state the full correctness theorem for Dice compilation:

**Theorem 4.6 (Compilation Correctness).** Let \( p \) be a Dice program and \( \emptyset, \emptyset \vdash p : \tau \rightsquigarrow (\phi, \gamma, w) \). Then: (1) \( \llbracket p \rrbracket_\Delta = \text{WMC}(\gamma, w) \), and (2) for any value \( v : \tau \), \( \llbracket p \rrbracket_D (v) = \text{WMC}((\phi \leftrightarrow v) \land \gamma, w) / \text{WMC}(\gamma, w) \).

As before, we prove this via the following stronger property, which is proven in Appendix B.3:

**Theorem 4.7 (Typed Program Correctness).** Let \( p \) be a Dice program \( \emptyset, \emptyset \vdash p : \tau \rightsquigarrow (\phi, \gamma, w) \). Then for any \( v : \tau \), we have that \( \llbracket p \rrbracket (v) = \text{WMC}((\phi \leftrightarrow v) \land \gamma, w) \).
4.4 Binary Decision Diagrams as WBF

Weighted model counting on WBFs is still #P-hard, so our compilation above is not necessarily advantageous. Now we reap the benefits of this translation by representing WBF with binary decision diagrams (BDDs), a data structure that facilitates efficient inference by exploiting the program structure to minimize the size of the WBF. A BDD is a popular data-structure for representing Boolean formulas, and there is a rich literature of using BDDs to represent the state-space of non-probabilistic programs during model checking [21, 46].

The compilation rules in the previous subsections were deliberately designed to facilitate BDD compilation. Consider the example compilation (ExLETCompilation) from Section 4.1. Each step in this derivation can be translated into a corresponding BDD operation, as illustrated by the BDD derivation tree in Figure 9. The final BDD is compiled compositionally, at each step exploiting program structure to produce a minimal, canonical representation (for the given variable ordering). The operations necessary for constructing this derivation tree – BDD conjunction, disjunction, and substitution – are all standard operations that are available in BDD packages such as CUDD [76]. The cost of Dice inference is then exactly the cost of constructing the corresponding BDD derivation tree: once the final BDD is constructed, inference is linear time in the size of the BDD.

Note that, even though inference is linear time in the size of the BDD, constructing the BDD can still be computationally hard. The key benefit is that the BDD can exploit program structure in order to scale efficiently on many useful examples. The remainder of this paper is devoted to showing that this derivation tree can be efficient to construct for useful programs. In Section 5, we show this experimentally, and Section 6 characterizes the hardness of Dice inference.

5 DICE IMPLEMENTATION & EMPIRICAL EVALUATION

Now we describe our implementation and empirical evaluation of Dice. Dice is implemented in OCaml and uses CUDD as its backend for compiling BDDs [76]. First we describe extensions to the core Dice syntax that make programming in Dice more ergonomic and enable us to more easily implement some of the benchmark programs. Then we describe our empirical evaluation of Dice’s performance in comparison with prior PPLs on a suite of benchmarks. In Section 6 we give context to these experiments and discuss why Dice succeeds on many benchmarks where others fail.

5.1 Dice Extensions & Ergonomics

Our implementation extends the core Dice syntax from Figure 4 in several ways. We relax the constraint on A-normal form, allowing more arbitrary placement of expressions. We also include

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1Dice is available at https://github.com/SHoltzen/dice.
syntactic sugar for the usual Boolean operators $\land$, $\lor$ and $\neg$. Finally, we include support for bounded integers and bounded iteration, both of which are described in more detail next. Further details of our implementation can be found in Appendix C.

5.1.1 Bounded Integers. Dice supports probability distributions over integers with the `discrete` keyword: for instance, the expression `discrete(0.1, 0.4, 0.5)` defines a discrete distribution over $\{0, 1, 2\}$ where 0 has probability 0.1, 1 has probability 0.4, and 2 has probability 0.5. There are a number of possible strategies for encoding integers into a WBF. The simplest — and the one we implemented — is a one-hot encoding. Specifically, a distribution over $n$ integers is represented as tuple of $n$ Boolean variables, each representing one integer value, and flips are used to ensure that each variable is true with the specified probability. For example, here is the encoding of our example distribution above:

```
let v0 = flip(0.1) in
let v1 = ¬v0 ∧ flip(0.4/(0.4 + 0.5)) in
let v2 = ¬v0 ∧ ¬v1 in (v0, (v1, v2))
```

Formally, for a discrete distribution `discrete(\theta_1, \theta_2, \ldots, \theta_n)`, the encoded value $v_i$ is true only if (1) $\land_{k<i} \neg v_k$ holds and (2) a coin flipped with probability $\theta_i/\sum_{j\geq i} \theta_j$ is true. Dice also supports the standard modular arithmetic operations like $(\times)$ on integers.

5.1.2 Statically Bounded Iteration. Iteration and loops are challenging program constructs to support in PPLs. Dice, like many other PPLs, supports bounded iteration: loops that always terminate after a finite number of iterations [20, 23, 35, 38, 68]. It does so via the syntax `iterate(f, init, k)`, where $f$ is a function name, `init` is an initialization expression, and $k$ is an integer indicating the number of times to call $f$:

```
iterate(f, init, k) ⇝ f(f(\ldots f(init)))^k
```

Many useful examples — such as the network reachability example from Section 2 — can be expressed as bounded iteration.

5.2 Empirical Performance Evaluation

We have faithfully implemented the compilation strategy and use of BDDs as described in Section 4. Section 2 highlights some program structure that BDD compilation exploits, and Section 6 explores this structure further, but the question remains: does this structure exist in practice, and can Dice effectively exploit it? We investigate these questions from two angles:

Q1: Raw Performance How quickly can Dice perform exact inference on benchmark probabilistic programs from the literature? We evaluate this question in Section 5.2.1.

Q2: Exploiting Functions What are the performance benefits of modular compilation for functions? We evaluate this question in Section 5.2.2 by comparing Dice’s performance with and without inlining function calls.

To evaluate these research questions we compare Dice against state-of-the-art PPLs that employ two different classes of exact inference algorithms:

Algebraic Methods The first class are algebraic inference methods that represent the probability distribution as a symbolic expression or algebraic decision diagrams (ADDs) [20, 28, 35, 62].
Table 1. Comparison of inference algorithms (times are milliseconds). A "✗" denotes a timeout at 2 hours of running. The total time for Dice is reported under the "Dice" column, and the total size of the final compiled BDD is reported in the "BDD Size" column.

We discuss this class of inference algorithms more thoroughly in Section 6.3. In this class, we compare experimentally against Psi [35].

**Enumerative Methods** The second class of inference methods work by exhaustively enumerating all paths through the probabilistic program, possibly using dynamic programming to reduce the search space [2, 19, 32, 36, 38, 74, 80]. Both Psi and WebPPL [38] have a mode that supports dynamic-programming exact inference, and we compare against them experimentally.

Comparing the performance of probabilistic program inference is challenging because performance is closely tied to the intricacies of how the program is structured: semantically equivalent programs may have vastly differing performance. Throughout this section we made a best-effort attempt at representing the programs in a way that was maximally performant in each language.

### 5.2.1 Performance Comparison

Table 1 summarizes the results of our performance experiments. Each row is a different benchmark. The benchmarks are separated into two categories: common baselines that are well-known and a new challenging set of discrete Bayesian networks, separated by a horizontal divider. The "Psi", "DP", and "Dice" columns give the amount of time (in milliseconds) for respectively (1) Psi’s default inference algorithm [35], (2) Psi’s built-in dynamic programming inference algorithm that is specialized for finite discrete programs, and (3) the total time for Dice to compile a BDD and perform weighted model counting.

We include two other columns, "# Paths" and "BDD Size", that give a proxy for how hard each inference problem is. The "# Paths" column gives how many paths would be explored by a path enumeration algorithm. The "BDD Size" gives the final compiled BDD generated by Dice, which

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We used Psi version 2d21f9fe04cf3aac533e08ccc2df18179947baad
in conjunction with the “# Paths” gives a metric for how much structure Dice is exploiting. Now we describe the two categories of experiments in more detail.\(^3\)

**Baselines.** Benchmarks 1—8 are well-known and commonly used discrete examples from the literature, which includes all of the discrete programs that Psi and R2 were evaluated on [9, 35, 63]. These examples are small and thus relatively easy for exact inference, but they serve as an important sanity check. All of the PPLs can solve these benchmarks, but Dice is generally faster by an order-of-magnitude or more.

**Discrete Bayesian Networks.** There is currently a lack of challenging discrete probabilistic program benchmarks in the literature. To more rigorously establish the relative performance of Dice and existing algorithms, we supplement the existing baselines with challenging discrete Bayesian networks that we translated into probabilistic programs; these are experiments 9—17. These networks were selected from the Bayesian Network Repository, an online repository of well-known Bayesian networks.\(^4\) These programs are realistic examples of probabilistic inference: each of these examples has been used to answer scientific research questions in various domains, and have on the order of thousands or tens of thousands of random variables. The repository has examples with a variety of sizes, and we chose representative examples from a range of sizes and complexity. Table 3 in the appendix summarizes the sizes of these Bayesian network benchmarks and files. These examples are extremely challenging: to give a sense of scale, the “Munin” example is a 1.9 megabyte program.

The goal for each benchmark is to compute the marginal probability of a specific leaf variable in the network, which is a standard Bayesian network query. The prior PPLs succeed on only two of these benchmarks, while Dice succeeds on all nine of them, further demonstrating that Dice can effectively exploit program structure to scale.

5.2.2 **Modular Compilation.** We return to the motivating examples from Section 2 to see how Dice compares with existing methods, and against a version of itself where all function calls are inlined. Figure 10 shows how different algorithms scale as the size of the problem grows (note that all plots are in log-log scale).

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\(^3\)See the supplementary materials for the source code for all the benchmarks used in this section.

\(^4\)https://www.bnlearn.com/bnrepository/
Encryption. Figure 3 introduced the Caesar cipher motivating example, and Figure 10a shows how exact inference on this example scales as the number of characters being encrypted increases. Dice is about an order of magnitude faster than the case when function calls are inlined, and multiple orders of magnitude faster than WebPPL and Psi. In particular, Psi’s default algebraic inference fails on to handle the encryption of a single character; we explore why in Section 6.3.

Approximate inference approaches generally struggle with these kinds of programs, due to the low probability of finding samples that satisfy the observations. To illustrate this, we also report the time it took for rejection sampling to draw 10 accepted samples. Rejection sampling scales exponentially in this case and thus is not a feasible route around the state-space explosion problem.

Network Reachability. Next we examine how separate compilation helps in the path reachability task described in Figure 2. Figure 10b shows how exact inference scales in the number of diamond subnetworks. We see a modest benefit over inlining: compiling the diamond function multiple times is not very expensive since it is so small. Both versions of Dice are multiple orders of magnitude faster than Psi and WebPPL due to the exponential number of paths.

We expect to see overall linear scaling of Dice for many network topologies due to conditional independence. To evaluate this, Figure 10c shows a version where instead of diamonds we use a ladder network of the following structure: \[ \cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \]. The goal is to determine the probability of a packet reaching the end of a network that consists of a chain of ladder subnetworks where each has a similar probabilistic routing policy to the diamond network. Dice continues to scale well, while this example is challenging for the other methods, in part since the number of paths is exponential in the length of the network.

5.2.3 Query Amortization. One of the benefits of our compilation is that a single (potentially expensive) compilation, once completed, can be efficiently reused to perform many queries. Instead of computing the probability of a single marginal in the program, we can have our function return the entire joint distribution of all variables as a tuple, and compute marginals of this joint distribution efficiently by calling that function repeatedly. We highlight this capability in Table 2, which shows the cost of compiling the full joint distribution of the example discrete Bayesian networks.\(^5\) These compilations take on the order of several seconds; however, once compiled, computing each marginal probability – or any other query with a small BDD, such as disjoining together several variables – takes milliseconds. For comparison, Psi cannot compute a single marginal within two hours on most of these instances.

| Benchmark   | Dice (ms) | BDD Size |
|-------------|-----------|----------|
| Cancer      | 5         | \(4.6 \times 10^1\) |
| Alarm       | \(1.6 \times 10^3\) | \(4.3 \times 10^5\) |
| Hailfinder  | \(1.8 \times 10^5\) | \(2.1 \times 10^5\) |
| Survey      | 1         | \(1.2 \times 10^2\) |
| Insurance   | \(4.0 \times 10^2\) | \(2.3 \times 10^5\) |
| Hepar2      | \(3.7 \times 10^3\) | \(5.4 \times 10^5\) |
| Pigs        | \(1.0 \times 10^4\) | \(2.6 \times 10^5\) |
| Water       | \(3.2 \times 10^4\) | \(6.8 \times 10^4\) |
| Munin       | \(4.7 \times 10^5\) | \(2.2 \times 10^7\) |

Table 2. Compiling the joint distribution

6 DISCUSSION & ANALYSIS

The previous section demonstrates empirically that Dice can perform exact inference orders of magnitude faster than existing inference algorithms on a range of benchmarks. In this section we provide discussion and analysis that provide context for these results. First we ask in Section 6.1: how hard is exact inference in Dice? We show that inference is PSPACE-hard, which means that it is likely harder than inference on discrete Bayesian networks. This begs the question: why do

\(^5\)See the programs in the "full-joint" directory in the supplement.
the experiments in Section 5 succeed at all? We explore this question in Section 6.2 by identifying different forms of program structure that Dice exploits in order to scale. Finally, Section 6.3 considers algebraic representations as an alternative compilation target for probabilistic programs and discusses the forms of structure that they are and are not capable of exploiting.

6.1 Computational Hardness of Exact Dice Inference

The experiments in Section 5 raise a natural question: how hard is the exact inference challenge for Dice programs? The complexity of exact inference has been well-studied in the context of discrete Bayesian networks. In particular, the decision problem of determining whether or not the probability of an event in a Bayesian network exceeds a certain threshold is \(\text{PP}\)-complete [55, 56]. The canonical \(\text{PP}\)-complete problem is \(\text{MAJ}\text{SAT}\), the problem of deciding whether or not the majority of truth assignments satisfy a logical formula. It is clear that exact Dice is \(\text{PP}\)-hard: indeed, some of our experiments in Section 5 utilize a polynomial-time reduction from discrete Bayesian networks to Dice programs. However, in fact exact inference for Dice is \(\text{PSPACE}\)-hard, and therefore likely harder than discrete Bayesian network inference as \(\text{PP} \subseteq \text{PSPACE}\):

**Theorem 6.1.** Exact inference in Dice is \(\text{PSPACE}\)-hard.

A proof sketch is in Appendix C.4. This result depends on the expressiveness of functions, which Bayesian networks lack. We leave for future work the investigation of tighter complexity bounds for Dice inference.

6.2 When is Dice Inference Fast?

Dice inference, in the worst case, is extremely hard. Why, then, do the experiments in Section 5 succeed? Put another way: when can we guarantee that the BDD derivation tree is efficient to construct (i.e., polynomial in the size of the program)? In this section we explore two sources of tractability in Dice inference, both of which are structural properties that a programmer can consciously exploit while designing Dice programs. The first source of structure is independence, which implies the existence of factorizations. The second is a more subtle property called local structure that implies that, even in some cases without independence, it can still be efficient to construct the BDD derivation tree [12, 17]. These forms of structure were first introduced in the context of graphical models for capturing conditional probability tables with various forms of structure. We show that these insights can be generalized to Dice programs.

6.2.1 Independence. The independence property implies that two program parts communicate only over a limited interface. It is the key reason why Dice performs so well in many of the benchmarks (Section 5.2.1). Programs naturally have conditional independence, implied by their control flow, function boundaries, etc. In the motivating example in Figure 1b, variable \(z\) does not depend on \(x\) given an assignment to \(y\). This is commonly called conditional independence of \(x\) and \(z\) given \(y\).

Dice naturally exploits conditional independence. We can formalize this by giving bounds on the cost of composing BDDs that are conditionally independent. In general, the operation \(B_1 \land B_2\) on two BDDs \(B_1\) and \(B_2\) has time and space complexity \(O(|B_1| \times |B_2|)\), and similarly for \(B_1 \lor B_2\) [60]. This implies a worst-case exponential blowup as BDDs are composed. However, Dice can exploit conditional independence – among other properties – to avoid this exponential blowup in practice:

**Proposition 6.2.** Let \(B_1\) and \(B_2\) be BDDs that share no variables other than some variable \(z\), and let \(|B|\) be the size of the BDD \(B\). Then we say \(B_1\) and \(B_2\) are conditionally independent given \(z\), and computing \(B_1 \land B_2\) and \(B_1 \lor B_2\) has time and space complexity \(O(|B_1| + |B_2|)\) for a variable order that orders the variables in \(B_1\) before \(z\) and \(z\) before the variables in \(B_2\).
let z = flip 0.5 in
let x = if z then flip 0.6 else flip 0.7 in
let y = if z then flip 0.7 else x in (x, y)

(a) Context-specific independence.

fun foo(a:Bool, b:Bool, c:Bool):Bool {
  a v b v c
}

(c) Structure without independence.

(b) Compiled BDD.

(d) Compiled BDD.

Fig. 11. Dice programs and their compiled BDDs illustrating different degrees of structure.

A proof sketch can be found in Appendix C.3. Proposition 6.2 implies that compositional rules that utilize conjunction and disjunction to compose Dice programs – like C-LET – can be efficient in the presence of conditional independence. One useful source of conditional independence is function calls: they are conditionally independent from all other expressions given their arguments and return value. The motivating example in Figure 2 illustrates an example of this form of conditional independence. Each call to the diamond procedure is independent of all prior calls given only the immediately previous call. It follows that the size of the BDD for the example in Figure 2d grows as \(O(|\text{diamond}| \times c)\), where \(c\) is the number of calls to the diamond procedure and \(|\text{diamond}|\) is the size of the compiled BDD for the procedure.

Dice exploits another, more fine-grained form of independence called context-specific independence. Historically, context-specific independence has led to significant speedups in graphical model inference [12]. We briefly sketch its benefits here. Two BDDs \(B_1\) and \(B_2\) are contextually independent given \(z = v\), for some variable \(z\) and value \(v\), if \(B_1[z \leftrightarrow v]\) and \(B_2[z \leftrightarrow v]\) share no variables [12]. As for conditional independence, composing contextually independent BDDs can often be efficient.

An example program that exhibits context-specific independence is show in Figure 11a. The variables \(x\) and \(y\) are correlated if \(z = F\) or if \(z\) is unknown, but they are independent if \(z = T\). Thus, \(x\) is independent of \(y\) given \(z = T\). Figure 11b shows how our compilation strategy exploits this independence. Since the program evaluates to a tuple, it is compiled to a tuple of two BDDs. However, in our implementation these BDDs share nodes wherever possible, so they can be equivalently viewed as a single, multi-rooted BDD. The left and right element of the tuple are represented by the \(l\) and \(r\) roots respectively. The program’s context-specific independence implies that there will be no shared sub-BDD between \(l\) and \(r\) if \(f_1\) is false. We refer to Boutilier et al. [12] for more on the performance benefits of exploiting context-specific independence in probabilistic graphical models.

6.2.2 Local Structure. Finally, it is possible for the BDD compilation process to be efficient even in the absence of independence if the program has structure that is amenable to efficient BDD compilation. Chavira and Darwiche [17] showed that exploiting local structure led to significant speedups in Bayesian network inference. Figure 11c gives an example Dice program that computes the disjunction of three flips. Figure 11d shows the compiled BDD for this function. It is compact and hence exploiting the program structure. Note that, if the number of variables disjoined
together were to increase, the size of the BDD – and the cost of compiling it – would increase only linearly with the number of variables. Dice implicitly exploits this structure during inference.

6.3 Algebraic Representations

Previous sections have shown that BDDs naturally capture and exploit factorization and procedure reuse. While these are common and useful program properties, they are not the only possible ones, and different compilation targets will naturally exploit others. In this section we consider algebraic compilation targets as a foil to our approach, to highlight their relative strengths and weaknesses.

In contrast to our WMC approach that explicitly separates the logical representation from probabilities, algebraic approaches integrate probabilities directly into the compilation target. A common algebraic target are algebraic decision diagrams (ADDs) [4], which are similar to binary decision diagrams except that they have numeric values as leaves. This makes them a natural choice for compactly encoding probability distributions in the probabilistic programming and probabilistic model checking communities, with different encoding strategies from Dice [20, 28, 54]. As an example, Figure 12, shows an ADD for the program in Figure 1a if it returned a tuple of \(x, y\), and \(z\). ADDs encode probabilities of total assignments of variables: in this example, a probability of 0.008 is given to the assignment \(x = y = z = T\).

ADDs have several similarities with BDDs. First, they support composition operations and so can offer a compositional compilation target [20], albeit very different from the one described by our compilation rules. Second, they support efficient inference once the ADD is constructed. Despite these similarities, ADDs have strikingly different scaling properties from BDDs because they exploit different underlying structure of the program. The key difference is that BDDs are agnostic to the flip parameters: they naturally exploit logical program structure such as independence and local structure in order to scale without needing to know what any probabilities are. As the previous subsections have argued, BDDs excel at this task. In contrast, ADDs naturally exploit global repetitious probabilities: repeated probabilities of possible worlds in the entire distribution. This is shown in Figure 12, which collapses states with the same probability – for example, if \(x = y = F\), then the ADD terminates with a node that does not depend on \(z\)’s value.

Global repetitious probabilities are an orthogonal property to independence. ADDs do not exploit independence in the same way as Dice. ADDs must explicitly represent the probability of each total instantiation of the variables of interest, corresponding to each possible value of the returned tuple. In our example, this means that the ADD cannot exploit the conditional independence of \(z\) and \(x\) given \(y\), and needs to enumerate their joint probabilities.

Hence, unlike Dice’s BDD representation, the size of a compiled ADD is sensitive to the precise parameters chosen for flips in the program. If these parameters are chosen such that the probability of each total assignment is distinct, and we are interested in a tuple of all the random variables, then the number of leaves in the ADD will equal the number of paths in the probabilistic program. As shown in Table 1, this can be prohibitively large for many examples; the BDD size is typically many orders of magnitude smaller than the number of paths on these real-world programs.

7 RELATED WORK

There is a large literature on probabilistic programming languages and inference algorithms. At a high level, Dice is distinguished from existing PPLs by being the first to use weighted model
counting to perform exact inference for a PPL that includes traditional programming language constructs, functions, and first-class observations. In this section we survey the existing literature on probabilistic program inference and provide context for how each relates to Dice.

Path-based inference algorithms. The most common class of probabilistic program inference algorithms today are operational, meaning that they work by executing the probabilistic program on concrete values. Common examples include sampling algorithms [14, 15, 37, 44, 57, 58, 67, 72, 77, 81] and variational approximations [8, 29, 52, 61, 80]. Other approaches use symbolic techniques to perform inference but are similar in spirit, in the sense that they separately enumerate paths through the program [2, 32, 36, 74]. These approaches do not factorize the program: they consider entire execution paths as a whole. Additionally, sampling and variational algorithms are distinguished from our approach by being approximate rather than exact inference algorithms. In general, these techniques can be applied to both discrete and continuous distributions, though they often rely on program continuity or differentiation to be effective [14, 40, 41, 52, 61, 80]. In contrast, Dice performs factorized, exact inference on non-smooth, non-differentiable, discrete programs.

Algebraic inference algorithms. A number of PPL inference algorithms work by translating the probabilistic program into an algebraic expression that encodes its probability distribution, and then using symbolic algebra tools in order to manipulate that expression and perform probabilistic inference. Examples include Psi [35], Hakaru [62], and approaches that employ algebraic decision diagrams [20, 28]. Algebraic representations exploit fundamentally different program structure from our approach based on weighted model counting; see Section 6.3 for a discussion.

Graphical model compilation. There exists a large number of PPLs that perform inference by converting the program into a probabilistic graphical model [11, 59, 61, 69]. These compilation strategies are limited by the semantics of graphical models: key program structure — such as functions, conditional branching, etc. — is usually lost during compilation and so cannot be exploited during inference. Further, graphical models can express conditional independence via the graphical structure, but typical inference algorithms such as variable elimination cannot exploit more subtle, context-specific forms of independence that our approach exploits, as shown in Section 6.2.1 [24].

Probabilistic Logic Programs. Closest to our approach are techniques for exact inference in probabilistic logic programs [27, 30, 71, 78]. Similar to our work, these techniques reduce probabilistic inference to weighted model counting and employ representations that support efficient WMC, such as BDDs [13] or sentential decision diagrams [25]. Unlike that work, Dice supports traditional programming language constructs, including functions, and it supports first-class observations rather than only observations at the very end of the program. We show how to exploit functional abstraction for modular compilation, and first-class observations require us to explicitly account for an accepting probability in both the semantics and the compilation strategy.

Programmer-Guided Inference Decomposition. Several PPLs provide a sublanguage that allows the programmer to provide information that can be used to decompose program inference into multiple separate parts [42, 58, 70]. Hence the goal is similar in spirit to our goal of automated program factorization. These approaches are complementary to ours: Dice automatically finds and exploits program factorizations and local structure, while these approaches can perform sophisticated decompositions through explicit programmer guidance.

Static Analysis & Model Checking. Forms of symbolic model checking often represent the reachable state space of a program as a BDD [6, 46]. Our work can be thought of as enriching this representation with probabilities: we track the possible assignments to each flip and the accepting formula in order to do exact Bayesian inference via WMC. Static analysis techniques have also been generalized to analyze probabilistic programs. For example, probabilistic abstract interpretation [22] provides a general framework for static analysis of probabilistic programs. However,
these techniques seek to acquire lower or upper bounds on probabilities, while we target exact inference. Probabilistic model checking (PMC) is a mature generalization of traditional model checking with multiple high-quality implementations [28, 54]. The goal of PMC is typically to verify that a system meets a given probabilistic temporal logic formula. They can also be used to perform probabilistic inference, but they have not used weighted model counting for inference and instead typically rely on ADDs, which gives them different scaling properties than Dice as we discussed earlier.

8 CONCLUSION

We presented a new approach to exact inference for discrete probabilistic programs and implement it in Dice. We (1) showed how to reduce exact inference for Dice to weighted model counting, (2) proved this translation correct, (3) demonstrated the performance of this inference strategy over existing methods, and (4) characterized the efficiency of compiling Dice in key scenarios.

In the future we hope to extend Dice in several ways. First, we believe that the insights of Dice can be cleanly integrated into many existing probabilistic programming systems, even those with approximate inference that can handle continuous random variables. We see this as an exciting avenue for extending the reach of approximate inference algorithms, which currently struggle with discreteness. Second, we believe that Dice can be extended to handle more powerful data structures and programming constructs, notably forms of loops and recursion. And finally, we hope to further explore the landscape of weighted model counting approaches.

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Table 3. Sizes of each discrete Bayesian network benchmark. Each network is annotated with a size in bytes (B) that is the size of the generated Dice program.

| j20 Nodes | 20–50 Nodes | 50–100 Nodes | 100–1000 Nodes | >1000 Nodes |
|-----------|-------------|--------------|---------------|------------|
| Survey (920B) | Alarm (18KB) | Hailfinder (91KB) | Pigs (213KB) | Munin (1.9MB) |
| Cancer (725B) | Insurance (33KB) | Hepar2 (54KB) | Water (302KB) | Dice program. |

A NOTATION

Form Denoted $F_{\tau}(x)$, converts a syntactic variable $x$ into a tuple of type $\tau$:
- $F_{\text{Bool}}(x) = x$.
- $F_{\tau_1 \times \tau_2}(x) = (F_{\tau_1}(x_1), F_{\tau_2}(x_r))$.

Typed substitution $\varphi_1[x \mapsto \hat{\varphi}_2]$, where $\Gamma(x) = \tau$ and $\Gamma \vdash \varphi_2 : \tau$. Intuitively, remaps each identifier in $x$ to its corresponding Boolean formula in $\hat{\varphi}_2$.

Broadcasted conjunction $\varphi_a \wedge \hat{\varphi}_b$, where $\Gamma(\varphi_a) = \text{Bool}$ and $\Gamma \vdash \hat{\varphi}_b : \tau$: conjoins a Boolean expression $\varphi_a$ with a tuple $\hat{\varphi}_b$. Intuitively, conjoins each element in $\hat{\varphi}_b$ with the Boolean expression $\varphi_a$:
- $\varphi_a \wedge \hat{\varphi}_b \triangleq \varphi_a \land \hat{\varphi}_b$.
- $\varphi_a \wedge_{\tau_1 \times \tau_2} (\hat{\varphi}_b^l, \hat{\varphi}_b^r) \triangleq \left( \varphi_a \wedge \hat{\varphi}_b^l, \varphi_a \wedge \hat{\varphi}_b^r \right)$

Point-wise disjunction $\hat{\varphi}_1 \lor \hat{\varphi}_2$, where $\Gamma \vdash \hat{\varphi}_1 : \tau, \Gamma \vdash \hat{\varphi}_2 : \tau$:
- $\hat{\varphi}_1 \lor \hat{\varphi}_2 \triangleq \hat{\varphi}_1 \lor \hat{\varphi}_2$.
- $(\hat{\varphi}_1^l, \hat{\varphi}_2^l) \lor (\hat{\varphi}_1^r, \hat{\varphi}_2^r) \triangleq (\hat{\varphi}_1^l \lor \hat{\varphi}_1^r, \hat{\varphi}_2^l \lor \hat{\varphi}_2^r)$

Pointwise iff $\hat{\varphi}_1 \leftrightarrow \hat{\varphi}_2$, where $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are of type $\tau$.
- $\text{Bool}(x) \leftrightarrow \varphi_2 \triangleq x \leftrightarrow \varphi_2$.
- $(\hat{\varphi}_1, \hat{\varphi}_2) \leftrightarrow (\hat{\varphi}_1^l, \hat{\varphi}_2^l) \triangleq (\hat{\varphi}_1^l \leftrightarrow \hat{\varphi}_1^r) \land (\hat{\varphi}_2^l \leftrightarrow \hat{\varphi}_2^r)$

B PROOFS

B.1 Key Lemmas

Lemma B.1 (Independent Conjunction). Let $\alpha$ and $\beta$ be Boolean sentences which share no variables; we call such sentences independent. Then, for any weight function $w$, WMC($\alpha \land \beta, w$) = WMC($\alpha, w$) $\times$ WMC($\beta, w$).

Proof. The proof relies on the fact that, if two sentences $\alpha$ and $\beta$ share no variables, then any model $\omega$ of $\alpha \land \beta$ can be split into two components, $\omega_{\alpha}$ and $\omega_{\beta}$, such that $\omega = \omega_{\alpha} \land \omega_{\beta}$, $\omega_{\alpha} \Rightarrow \alpha$, and $\omega_{\beta} \Rightarrow \beta$, and $\omega_{\alpha}$ and $\omega_{\beta}$ share no variables. Then: WMC($\alpha \land \beta, w$) = $\sum_{\omega \in \text{Models}(\alpha \land \beta)} \prod_{l \in \omega} w(l) = \left[ \sum_{\omega_{\alpha} \in \text{Models}(\alpha)} \prod_{a \in \omega_{\alpha}} w(a) \right] \times \left[ \sum_{\omega_{\beta} \in \text{Models}(\beta)} \prod_{b \in \omega_{\beta}} w(b) \right] = \text{WMC}(\alpha, w) \times \text{WMC}(\beta, w)$. □
PROPOSITION B.2 (INCLUSION-EXCLUSION). For any two formulae \( \varphi_1 \) and \( \varphi_2 \) and weight function \( w \), \( \text{WMC}(\varphi_1 \lor \varphi_2, w) = \text{WMC}(\varphi_1, w) + \text{WMC}(\varphi_2, w) - \text{WMC}(\varphi_1 \land \varphi_2, w) \). Note the important mutual exclusion case when \( \varphi_1 \land \varphi_2 = F \).

B.2 Correctness of Expression Compilation

LEMMA B.3 (VALUE CORRECTNESS). For any values \( v \) and \( v' \) of type \( \tau \), \( [v] (v') = \text{WMC}(v \leftrightarrow v', \emptyset) \).

PROOF. By induction on \( \tau \):

- \( \tau = \text{Bool} \). Then case analysis:
  - \([T]\) \( (T) = 1 = \text{WMC}(T \leftrightarrow T, \emptyset) \)
  - \([T]\) \( (F) = 0 = \text{WMC}(T \leftrightarrow F, \emptyset) \)
  - \([F]\) \( (F) = 1 = \text{WMC}(F \leftrightarrow F, \emptyset) \)
  - \([F]\) \( (T) = 0 = \text{WMC}(F \leftrightarrow T, \emptyset) \)
- Inductive step: \( \tau = \tau_1 \times \tau_2 \). Then,
  \[
  [(v_1, v_2)]((v'_1, v'_2)) = [v_1] (v'_1) \times [v_2] (v'_2)
  = \text{WMC}(v_1 \leftrightarrow v'_1, \emptyset) \times \text{WMC}(v_2 \leftrightarrow v'_2, \emptyset) \quad \text{Induction Hyp.}
  = \text{WMC}(v_1 \leftrightarrow v'_1, \emptyset) \times \text{WMC}(v_2 \leftrightarrow v'_2, \emptyset) \quad \text{Independent Conj.}
  = \text{WMC}(v_1 \leftrightarrow v'_1, \emptyset) \times \text{WMC}(v_2 \leftrightarrow v'_2, \emptyset).
  \]

\( \square \)

LEMMA B.4 (TYPED SUBSTITUTION). For any values \( v, v_x : \tau \), it holds that \( (v \leftrightarrow v_x) = (F_x(x) \leftrightarrow v)[x \mapsto v_x] \).

PROOF. By induction on \( \tau \):

- \( \tau = \text{Bool} \). Then, \( (v \leftrightarrow v_x) = (v \leftrightarrow x)[x \mapsto v_x] = (v \leftrightarrow F_x(x))[x \mapsto v_x] \).
- \( \tau = \tau_1 \times \tau_2 \). Then, let \( v = (v', v') \) and \( v_x = (v'_x, v'_x) \). Then,
  \[
  (v', v') \leftrightarrow (v'_x, v'_x) = (v' \leftrightarrow v'_x) \land (v' \leftrightarrow v'_x)
  = (v' \leftrightarrow F_x(x))[x \mapsto v'_x] \land (v' \leftrightarrow F_x(x))[x \mapsto v'_x] \quad \text{Ind. Hyp.}
  = (v' \leftrightarrow F_x(x) \land (v' \leftrightarrow F_x(x))[x \mapsto v'_x])[x \mapsto v'_x]
  = ((v', v') \leftrightarrow F_x \times F_x(x))[x \mapsto v'_x](v', v').
  \]

\( \square \)

LEMMA B.5 (TYPED CORRECTNESS WITHOUT PROCEDURES). Let \( e \) be a Dice expression without procedure calls. Let \( \{x_1 : \tau_i\} \vdash e : \tau \sim (\varphi, y, w) \). Then for any values \( \{v_1 : \tau_i\} \) and \( v : \tau \), we have that \([e[x_1 \mapsto v_1]](v) = \text{WMC}\left( ((v \leftrightarrow \varphi) \land \gamma)[x_1 \mapsto v_1], w\right) \).

PROOF. The proof is by structural induction on the syntax of Boolean Dice programs. First, we prove that the theorem holds for the non-inductive terms:

- \( e = T \) and \( e = F \) follow directly from Lemma B.3.
- \( e = \text{flip} \theta \). Then, \( \Gamma \vdash \text{flip} \theta : \text{Bool} \sim (f, T, w) \) for a fresh \( f \). Then, \( \text{WMC}(f \land T, w) = \theta = [\text{flip} \theta] (T) \) and \( \text{WMC}(f, w) = 1 - \theta = [\text{flip} \theta] (F) \).
• e = x. Then, \( \Gamma \vdash x : \tau \rightsquigarrow (\phi, \top, \emptyset) \), and let \( v_x : \tau \) be the value substituted for \( x \).

\[
\left[ x \mapsto x \right] (v) = [v_x] (v) \\
= \text{WMC}(\left( v_x \mapsto v \right) \land \top, \emptyset) \\
= \text{WMC}(\left( (F_{\tau}(x) \mapsto v) \land \top \right) [x \mapsto v_x], \emptyset) \quad \text{Lemma B.3}
\]

• e = \text{fst } x. Assume \( \Gamma(x) = \tau_1 \times \tau_2 \). Then, \( \Gamma \vdash \text{fst } x : \tau_1 \rightsquigarrow (F_{\tau_1}(x), \top, \emptyset) \). Let \( v_x = (v_{x_1}^l, v_{x_2}^r) : \tau_1 \times \tau_2 \) be the value substituted for \( x \). Then,

\[
\left[ \text{fst } x \mapsto x \mapsto v_x \right] (v) = [v_x] (v) \\
= \text{WMC}(\left( v_x^l \mapsto v \right) \land \top, \emptyset) \\
= \text{WMC}(\left( (F_{\tau_1}(x) \mapsto v) \land \top \right) [x \mapsto v_x], \emptyset) \quad \text{Lemma B.4}
\]

An analogous argument holds for \( \text{snd } x \).

• e = \text{let } e_1 \text{ in } e_2. Assume \( \Gamma \vdash e_1 : \tau_1 \rightsquigarrow (\hat{\phi}_1, \gamma_1, w_1) \) and \( \Gamma \cup \{ x : \tau_1 \} \vdash e_2 : \tau_2 \rightsquigarrow (\hat{\phi}_2, \gamma_2, w_2) \). For notational simplicity, assume that the substitution \( [x \mapsto v_1] \) has been applied to \( \hat{\phi}_1, \gamma_1, \hat{\phi}_2, \gamma_2 \), and that all weighted model counts are performed with the weight \( w_1 \cup w_2 \). Then,

\[
\left[ \text{let } x = e_1 \text{ in } e_2 \right] [x \mapsto v_1] (T) \\
= \sum_{v} [e_1[x_i \mapsto v_i]] (T) \times [e_2[x_i \mapsto v_i, x \mapsto v]] (T) \\
= \sum_{v_x \in \tau_1} \text{WMC}(\left( \phi_1 \mapsto v_x \right) \land \gamma_1 \times \text{WMC}(\left( \phi_2 \mapsto v \right) \land \gamma_2) [x \mapsto v_x]) \quad \text{Ind. Hyp.}
\]

\[
= \sum_{v_x \in \tau_1} \text{WMC}(\left( \phi_1 \mapsto v_x \right) \land \gamma_1 \land \left( \phi_2 \mapsto v \right) \land \gamma_2) [x \mapsto v_x]) \quad \text{Indep. Conj.}
\]

\[
= \sum_{v_x \in \tau_1} \text{WMC}(\left( \phi_1 \mapsto v_x \right) \land \gamma_1 \land \left( \phi_2 \mapsto v \right) \land \gamma_2) [x \mapsto v_x]) \quad \text{Mut. Excl.}
\]

\[
= \text{WMC}(\left( \phi_2 \mapsto v_2 \right) \land \gamma_1 \land \gamma_2) [x \mapsto \hat{\phi}_1])
\]
\begin{itemize}
\item $e = \text{observe } g$. Assume $\Gamma \vdash g : \text{Bool} \rightsquigarrow (\varphi, T, w)$. This case relies on interpreting the semantics of $\llbracket \text{observe } g[x_i \mapsto v_i] \rrbracket (v)$ as $\llbracket g[x_i \mapsto v_i] \rrbracket (T) \times [T] (v)$. Then,
\[
\llbracket \text{observe } g[x_i \mapsto v_i] \rrbracket (v) = \llbracket g[x_i \mapsto v_i] \rrbracket (T) \times [T] (v) = \text{WMC}(\varphi \land T, w) \times \text{WMC}(v \land T).
\]
\end{itemize}

\begin{itemize}
\item $e = \text{if } g \text{ then } e_T \text{ else } e_E$. Assume $\Gamma \vdash g : \text{Bool} \rightsquigarrow (\varphi_g, T, w_g), \Gamma \vdash e_T : \tau \rightsquigarrow (\psi_T, \gamma_T, w_T), \Gamma \vdash e_E : \tau \rightsquigarrow (\psi_E, \gamma_E, w_E)$. Again assume for notational simplicity that all weighted model counts are performed with the weight function $w_g \cup w_2 \cup w_g$ and that the substitutions $[x_i \mapsto v_i]$ have been performed on the compiled formulae. Then,
\[
\llbracket \text{if } g \text{ then } e_T \text{ else } e_E \rrbracket (v) = \llbracket g \rrbracket (T) \times \llbracket e_T \rrbracket (v) + \llbracket g \rrbracket (F) \times \llbracket e_E \rrbracket (v) = \text{WMC}(\varphi_g \land T) \times \text{WMC}((\psi_T \mapsto v) \land \gamma_T) + \text{WMC}(\varphi_g \land T) \times \text{WMC}(\varphi_E \mapsto v) \land \gamma_E) \quad \text{Ind. Hyp.}
\]
\[
\llbracket e_T \rrbracket (v) = \text{WMC}(\varphi_g \land (\psi_T \mapsto v) \land \gamma_T) + \text{WMC}(\varphi_g \land (\psi_T \mapsto v) \land \gamma_E) \quad \text{Indep. Conj.}
\]
\[
\llbracket e_E \rrbracket (v) = \text{WMC}(\varphi_g \land (\psi_T \mapsto v) \land \gamma_T) \lor (\varphi_g \land (\psi_E \mapsto v) \land \gamma_E)) \quad \text{Mut. Excl.}
\]
\[
\llbracket e \rrbracket (v) \supseteq \varphi_g \land \gamma_T \lor (\varphi_g \land \gamma_E))
\]
\end{itemize}

\section*{B.3 Theorem 4.7}

First we extend Lemma 4.5 to show that Boolean function call compilation is correct. First we need some preliminaries. The semantics and compilation of an expression can only be compared if the function context they are compiled in is \emph{compatible}:

\begin{definition}[Table Compatibility]
Let $\Phi$ be a compiled function table, $T$ be a function table, and $\Gamma$ be a type environment. Then we say $T$ and $\Phi$ are \emph{compatible} if for any function identifier $x$, where $\Gamma(x) = \tau_1 \rightarrow \tau_2$ and $\Phi(x) = (x, \varphi, \gamma, w)$, it holds for any argument value $\nu^x : \tau_1$ and value $\nu : \tau_2$. $T(x)(\nu^x)(\nu) = \text{WMC}(((\varphi \mapsto v) \land \gamma)[x \mapsto \nu^x], w)$.
\end{definition}

Then, we can extend Lemma 4.5 to assume compatible tables:

\begin{theorem}[Boolean Correctness with Procedure Calls]
Let $e$ be a dice expression with function calls, $T$ and $\Phi$ be compatible tables, let $\{x_i : \tau_i\}, \Phi \vdash e : \tau \rightsquigarrow (\varphi, \gamma, w)$. Then, for any values $\{v_i : \tau_i\}$ and $\nu : \tau$, we have that $\llbracket e[x_i \mapsto v_i] \rrbracket (v) = \text{WMC}(((\varphi \mapsto v) \land \gamma)[x_i \mapsto v_i], w)$.
\end{theorem}

\begin{proof}
The proof is identical to the proof of Lemma 4.5 except for the addition of the function call syntax, which we prove here.

Assume $e = x_1(x_2)$ and assume $\Phi(x_1) = (x_{arg}, \varphi, \gamma, w)$. Assume $(\varphi', \varphi', w) = \text{RefreshFlips}(\varphi, \gamma, w)$. Then, $x_1(x_2) \rightsquigarrow (\varphi[x_{arg} \mapsto x_2], \gamma[x_{arg} \mapsto x_2], w)$. Then the result follows directly from table compatibility:
\[
\llbracket x(\nu^x) \rrbracket (T) = T(x)(\nu^x)(T)
\]
\[
= \text{WMC}(((\varphi \mapsto v) \land \gamma)[x \mapsto \nu^x], w) \quad \text{Table Compatibility}
\]
\[
= \text{WMC}(((\varphi' \mapsto v) \land \gamma')[x \mapsto \nu^x], w) \quad \text{Defn. of RefreshFlips}
\]
\end{proof}
Now we are ready for the main theorem:

**Theorem B.8 (Typed Program Correctness).** Let \( \varphi \) be a Dice program \( \Gamma \vdash \varphi : \tau \rightsquigarrow (\phi, \gamma, w) \). Then for any \( v : \tau \), we have that \( [\varphi]_A (v) = \text{WMC}(\phi \rightleftarrows v) \land \gamma, w \).

**Proof.**
- **Base case:** \( \varphi = e \). Assume \( \Gamma, \Phi \models e : \tau \rightsquigarrow (\phi, \gamma, w) \). Then, \( [\varphi]_D (v) = [\phi]_D (v) = \text{WMC}(\phi \rightleftarrows v) \land \gamma, w \), by Theorem B.7.
- **Inductive step:** The program is of the form \( \varphi = \text{fun } x_1(x_2) \{ e \} \).

Assume that \( \Gamma, \Phi \models \text{fun } x_1(x_2) \{ e \} : \tau_1 \rightarrow \tau_2 \rightsquigarrow (\phi_f, \gamma_f, w_f) \). Let \( T' = T \cup \{ x_1 \rightarrow [\text{func}] \} \) and \( \Phi' = \Phi \cup \{ x_1 \mapsto (x_2, \phi_f, \gamma_f, w_f) \} \). Then, Theorem B.7 guarantees that \( T' \) and \( \Phi' \) are compatible tables. Let \( \Gamma \cup \{ x_1 \mapsto \tau_1 \rightarrow \tau_2 \}, \Phi' \vdash p_2 : \tau \rightsquigarrow (\phi, \gamma, w) \). Then,

\[
[\text{fun } x_1(x_2) \{ e \} p_2]_{T'} (v) = [p_2]_{T'} (v) = \text{WMC}(\phi \rightleftarrows v) \land \gamma, w)
\]

By Ind. Hyp.

Finally we prove Theorem 4.6, restated here for convenience:

**Theorem B.9 (Compilation Correctness).** Let \( \varphi \) be a Dice program and \( \emptyset, \emptyset \vdash \varphi : \tau \rightsquigarrow (\phi, \gamma, w) \). Then:

- \([\varphi]_A = \text{WMC}(\gamma, w)\)
- **for any value** \( v : \tau \), \([\varphi]_D (v) = \text{WMC}(\phi \rightleftarrows v) \land \gamma, w) \text{WMC}(\gamma, w)\).

**Proof.** Let \( \{ \} \vdash \varphi : \tau \rightsquigarrow (\phi, \gamma, w) \). Then,

\[
[\varphi]_A = \sum_v \text{WMC}(\phi \rightleftarrows v) \land \gamma, w) \quad \text{Theorem 4.7}
\]

\[
= \text{WMC} \left( \bigvee_v ((\phi \rightleftarrows v) \land \gamma, w) \right) \quad \text{Mut. Excl.}
\]

Then, \([\varphi]_D (v) = \sum_v [\varphi] (v) \land \gamma, w) \text{WMC}(\gamma, w) \) by Theorem 4.7 and the above argument.

**C IMPLEMENTATION DETAILS**

**C.1 Variable Ordering**

The variable ordering – the order in which variables are branched on in a BDD – is a critical parameter that determines how compactly a BDD can represent a particular logical formula [13, 60]. Finding the optimal order – the one that minimizes the size of the BDD – is NP-hard, so one must typically resort to heuristics for choosing orderings that work well in practice. Dice orders variables according to the syntactic order in which they occur in the program, mirroring the topological variable ordering heuristic from Bayesian networks [24]. We anticipate future work in deriving more sophisticated variable ordering heuristics from static program analyses.

**C.2 Multi-rooted BDDs**
Dice typically needs to represent many BDDs at the same time that share structure. The accepting and unnormalized formulae may share sub-formulas, or tuples may compile to formulae that share some substructure. *Multi-rooted BDDs* naturally exploit this repeated substructure to compactly represent multiple Boolean formulae in a single data-structure. For instance, the following example program that returns a tuple is compiled into the multi-rooted BDD in Figure 13:

```
let x = flip₁ 0.6 in let y = x ∧ flip₂ 0.4 in (x, y)
```

**C.3 Proposition 6.2**

**Proof Sketch.** The proof is by construction. For instance, for conjunction, BDDs for \(B₁, B₂\), and \(B₁ ∧ B₂\) are of the form:

\[
B₁ = \quad B₂ = \quad B₁ ∧ B₂ =
\]

where \(B₁'\) is the BDD for \(B₁\) with \(z\) separated out and \(B₂ | z\) is the BDD for \(B₂\) with \(z = \top\). The BDD for \(B₁ ∧ B₂\) can be constructed in linear time by traversing \(B₁'\) and replacing the node \(z\) with \(B₂ | z\), and similarly for the false case.

**C.4 Theorem 6.1**

**Proof Sketch.** The PSPACE-hardness of Dice inference follows directly from the expressiveness of non-recursive Boolean programs. In particular, there is a polynomial-time reduction from the quantified Boolean formula (QBF) problem, which is PSPACE-complete, to such a program. This reduction can also be used to reduce QBF to the problem of determining the probability that a Dice program outputs true.