Renormalization Group Approach to Discretized Gravity

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Abstract

We summarize our renormalization group approach for the vector model as well as the matrix model which are the discretized quantum gravity in one- and two-dimensional spacetime. A difference equation is obtained which relates free energies for neighboring values of $N$. The reparametrization freedom in field space is formulated by means of the loop equation. The reparametrization identities reduce the flow in the infinite dimensional coupling constant space to that in finite dimensions. The matrix model gives a nonlinear differential equation as an effective renormalization group equation. The fixed point and the susceptibility exponents can be determined even for the matrix models in spite of the nonlinearity. They agree with the exact result.

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1 Introduction

The long-standing challenge for the quantum theory of gravity has met at least a partial success in low dimensions. The two-dimensional quantum gravity is particularly interesting from the viewpoint of string theories too. The discretized approach such as matrix models offers a possibility for a nonperturbative treatment [1]–[3]. Exact solutions of the matrix model have been obtained for two-dimensional quantum gravity coupled to minimal conformal matter with central charge $c \leq 1$. It has been very difficult to obtain results for two-dimensional quantum gravity coupled to conformal matter with central charge $c > 1$. Although one can easily write down matrix models for cases with $c > 1$, these models are not solvable up to now[4]. The numerical simulations suggest that it is not at all obvious if a matrix model candidate to describe a $c > 1$ model has the continuum description [3]. Therefore it is useful to obtain approximation schemes which enable us to calculate critical coupling constants and critical exponents for unsolved matrix models, especially for $c > 1$. In order to make use of such a scheme, we first need to make sure that the approximation method gives correct results for the exactly solved cases.

∥The speaker.
Brézin and Zinn-Justin have proposed a renormalization group approach to the matrix model. Consequences of their approach have been examined by several groups. A similar approach has been advocated previously for the $1/N$ expansion in somewhat different contexts. The vector model has been proposed for a discretized one-dimensional quantum gravity, in the same way as the matrix model for a discretized two-dimensional quantum gravity. Recently we have analyzed the vector model by means of the renormalization group approach and have clarified its validity and meaning. More recently, we have succeeded in extending our analysis of the renormalization group approach to matrix models.

The purpose of this paper is to present a new simplified derivation of our results on vector models and to summarize some of our results on matrix models in ref. We obtain infinitely many identities which express the freedom to reparametrize the field space. Thanks to these identities, we can rewrite the flow in the infinite dimensional coupling constant space as an effective flow in the space of finite number of coupling constants. In this note, we shall simplify the derivation of the reparametrization identities by using a new method similar to the loop equation for the matrix model. The resulting effective beta function determines the fixed points and the susceptibility exponents which agree with the exact results.

We also find that a similar procedure works for matrix models. Namely we can obtain a difference equation relating the free energy $-\log Z_{N-1}(g)$ to the free energy $-\log Z_N(g-\delta g)$ with shifts $\delta g$ of order $1/N$ in infinitely many coupling constants. We find infinitely many identities expressing the reparametrization freedom of matrix variables. The identities are sometimes called loop equations and enable us to rewrite the flow as an effective renormalization group flow in the space of finite number of coupling constants. A new characteristic feature of the matrix model is that the resulting effective renormalization group equations is nonlinear contrary to the vector model. In spite of the nonlinearity, we can obtain the fixed points and the critical exponents. As an explicit example, we analyzed the one matrix model with a single coupling constant and find a complete agreement with the exact results for the first critical point.

2 Renormalization Group Approach for Vector Models

It has been proposed that the free energy $F(N, g)$ of the matrix model should satisfy the following renormalization group equation:

$$\left[ N \frac{\partial}{\partial N} - \beta(g) \frac{\partial}{\partial g} + \gamma(g) \right] F(N, g) = r(g),$$

where $\beta(g)$ is called the beta function. The anomalous dimension and the inhomogeneous term are denoted as $\gamma(g)$ and $r(g)$ respectively. A fixed point $g_*$ is given by a zero of the beta function. In the following we shall show that this renormalization group equation is valid for vector models.

The partition function of the $O(N)$ symmetric vector model is given by

$$Z_N(g) = \int d^N \phi \exp \left[ -N \sum_{k=1}^{\infty} \frac{g_k}{2k} (\phi^2)^k \right],$$

(2)
where \( \phi \) is an \( N \) dimensional real vector \([4]\). Here we introduce infinitely many coupling constants \( g_k \), since we need all possible induced interactions after a renormalization group transformation even if we start with a few coupling constants only. The \( 1/N \) expansion of the logarithm of the partition function gives contributions from \( h \) loops as terms with \( N^{1-h} \). The vector model has the double scaling limit \( N \to \infty \) with \( N^{1/\gamma_1}(g - g_*) \) fixed, where the singular part of the free energy satisfies the scaling law \([9]\]

\[
- \log \left[ \frac{Z_N(g)}{Z_N(g_1 \neq 0, g_k = 0 (k \geq 2))} \right] \bigg|_{\text{sing}} = \sum_{h=0}^{\infty} N^{1-h}(g - g_*)^{2-\gamma_0-\gamma_1 h} a_h + \ldots. \tag{3}
\]

By inserting this form, one finds that the critical point corresponds to the fixed point \( \beta(g_*) = 0 \) and that susceptibility exponents \( \gamma_0, \gamma_1 \) are related to the derivative of the beta function

\[
\gamma_1 = 1/\beta'(g_*), \quad \gamma_0 = 2 - \gamma(g_*)/\beta'(g_*). \tag{4}
\]

In order to obtain a difference equation, we start with the partition function \( Z_{N-2}(g) \). After integrating over angular coordinates in \( \mathbb{R}^{N-2} \), we perform a partial integration in the radial coordinate \( x = \phi^2 \)

\[
Z_{N-2}(g) = \frac{\pi^{N-1}}{\Gamma(N/2 - 1)} \int_0^\infty dx \, x^{N/2-2} \exp \left[ -(N - 2) \sum_{k=1}^\infty g_k x^k \right] = \frac{\pi^{N-1}}{\Gamma(N/2 - 1)} \int_0^\infty dx \, x^{N/2-1} \left( \sum_{k=1}^\infty g_k x^{k-1} \right) \exp \left[ -(N - 2) \sum_{k=1}^\infty g_k x^k \right]. \tag{5}
\]

Identifying the right hand side with \((N - 2)g_1/2\pi)Z_N(g - 2\delta g)\), we obtain a difference equation for the logarithm of the partition function

\[
[(- \log Z_N(g)) - (- \log Z_N(g)) - \log \frac{(N - 2)g_1}{2\pi} = -[(- \log Z_N(g - 2\delta g)) - (- \log Z_N(g))]. \tag{6}
\]

where the shifts \( \delta g_k \) of the coupling constants are found to be

\[
\sum_{k=1}^\infty \frac{g_k}{k} x^k + \log \left( \sum_{k=1}^\infty \frac{g_k}{g_1} x^{k-1} \right) = N \sum_{k=1}^\infty \frac{\delta g_k}{k} x^k. \tag{7}
\]

We stress that no approximation is employed to obtain this difference eq. \([3]\) apart from neglecting the higher order terms in \( 1/N \) contrary to the perturbative method advocated in ref.\([6]\).

One can bring the quadratic term in the potential to the standard form \( \phi^2/2 \) since \( g_1 \) can be absorbed by a rescaling \( g_1 \phi^2 \to \phi^2 \). We have

\[
Z_N(g_1, g_2, g_3, \ldots) = g_1^{-N/2} Z_N(1, \tilde{g}_2, \tilde{g}_3, \ldots), \quad \tilde{g}_k = g_k/g_1^k. \tag{8}
\]

We shall define the free energy for the vector model

\[
F(N, g_1, g_2, g_3, \ldots) = -\frac{1}{N} \log \frac{Z_N(g_1, g_2, \ldots)}{Z_N(g_1, 0, \ldots)} = F(N, \tilde{g}_2, \tilde{g}_3, \ldots). \tag{9}
\]
We sometimes write the free energy as a function of $\tilde{g}_2, \tilde{g}_3, \ldots$, since it is independent of $g_1$ because of the rescaling identity (8) if one uses $g_1, g_2, \cdots$ as independent variables. We denote the partial derivatives with respect to $g_1, g_2, g_3, \cdots$ by $|_g$, and those with $g_1, g_2, g_3, \cdots$ by $|_{\tilde{g}}$.

$$\frac{\partial}{\partial g_1}|_g = \frac{\partial}{\partial g_1}|_{\tilde{g}} - \sum_{k=2}^{\infty} \frac{\tilde{g}_k}{g_1} \frac{\partial}{\partial \tilde{g}_k}|_{\tilde{g}}, \quad \frac{\partial}{\partial g_k}|_g = \frac{1}{g_1^{k-1}} \frac{\partial}{\partial \tilde{g}_k}|_{\tilde{g}}. \quad (10)$$

In the $N \to \infty$ limit, we can obtain a differential equation from the difference equation (8) as a renormalization group equation for the free energy $F$

$$\left[ N \frac{\partial}{\partial N} + 1 \right] F(N, g) = -\frac{1}{2} + \sum_{k=1}^{\infty} N \delta g_k \left. \frac{\partial F(N, g)}{\partial g_k} \right|_g \equiv G,$n

$$G = -\frac{1}{2} + N \delta g_1 \frac{1}{2g_1} + \sum_{k=2}^{\infty} N \left( \frac{\delta g_k}{g_1} - \frac{\delta g_1}{g_1} k \tilde{g}_k \right) \left. \frac{\partial F(N, \tilde{g})}{\partial \tilde{g}_k} \right|_{\tilde{g}}. \quad (11)$$

This renormalization group equation clearly shows a flow in the infinite dimensional coupling constant space as is usual in the Wilson’s renormalization group approach. Eq. (11) shows that the anomalous dimension is given by $\gamma(\tilde{g}) = 1$. Therefore two susceptibility exponents are related by $\gamma_0 + \gamma_1 = 2$.

### 3 Reparametrization Identities

Our key observation in using the renormalization group equation is the ambiguity to identify the renormalization group flow in the coupling constant space [14]. Though the above equation (11) seems to describe a renormalization group flow in the infinite dimensional coupling constant space, the direction of the flow is in fact ambiguous because all the differential operators $(\partial/\partial \tilde{g}_k)$ are not linearly independent. Since the model is $O(N)$ invariant, we can obtain new informations only from reparametrizations of the radial coordinate $x = \phi^2$ which become reparametrizations of a half real line. In order to facilitate the derivation of the reparametrization identities, it is useful to make the following reparametrization

$$\phi^2 \to \phi'^2 = \phi^2 \left( 1 + \varepsilon \frac{1}{\zeta - \phi^2} \right), \quad (12)$$

where $\varepsilon$ is an infinitesimal parameter and $\zeta$ is a parameter to generate all possible reparametrizations as a power series. Substituting (12) in (8), we obtain an identity

$$\left( \frac{N}{2} - 1 \right) \left( \frac{1}{\zeta - \phi^2} \right) + \frac{\zeta}{(\zeta - \phi^2)^2} \right) - N \left( \frac{\phi'^2}{\zeta - \phi'^2} V'(\phi'^2) \right) = 0, \quad (13)$$

$$\langle O \rangle \equiv \frac{1}{Z_N(\gamma)} \int d^N \phi \ O \ e^{-NV(\phi^2)} \quad (14)$$

we shall call this identity a loop equation since it resembles that in the matrix model [3]. It is convenient to define the expectation value $W(\zeta)$ of a resolvent

$$W(\zeta) = \left( \frac{1}{\zeta - \phi^2} \right) = \sum_{j=0}^{\infty} \frac{\langle (\phi^2)^j \rangle}{\zeta^{j+1}} = \frac{1}{\zeta} + \sum_{j=1}^{\infty} \frac{2j}{\zeta^{j+1}} \left. \frac{\partial F(N, g)}{\partial q_j} \right|_{q_j}, \quad (15)$$
By expanding the identity (13), we can obtain various reparametrization identities which form a representation of the Virasoro algebra [10]. We see that derivatives of the free energy in terms of infinitely many coupling constants $\tilde{g}_k$ are related by infinitely many reparametrization identities. Thus one can expect that only a finite number of derivatives are linearly independent. The loop equation for the vector model (13) can be solved as

$$W(\zeta) = \frac{2}{1 - 2\zeta V'(\zeta)} \left[ \sum_{k=1}^{\infty} \frac{d^k (\zeta V'(\zeta))}{d\zeta^k} \right] \langle (\zeta - \phi^2)^{k-1} \rangle - \frac{1}{N} \zeta W'(\zeta).$$

(16)

The resolvent to the leading order in $1/N$ expansion is given by omitting the last term.

To illustrate the use of the reparametrization identities, we shall first take the case of a single coupling constant. Let us consider a point in the coupling constant space $(g_1, \tilde{g}_2, \tilde{g}_3, \tilde{g}_4, \ldots) = (g_1, \tilde{g}_2, 0, 0, \ldots)$. The coupling constant shifts $\delta g_k$ given by eq.(7) are then found to be

$$N\delta g_k = (-1)^{k+1}(\tilde{g}_2g_1)^k + g_1\delta_{k,1} + \tilde{g}_2g_1^2\delta_{k,2}.$$  

(17)

Inserting the shifts $\delta g_k$ to the right hand side $G$ of the renormalization group equation (11), we find

$$G = -\frac{1}{2} + \frac{1}{2} \int_{-\infty}^{-(\tilde{g}_2g_1)} d\zeta \left( W(\zeta) - \frac{1}{\zeta} \right) + g_1 \frac{\partial F}{\partial g_1} \bigg|_{\tilde{g}} + \tilde{g}_2g_1^2 \frac{\partial F}{\partial g_2} \bigg|_{\tilde{g}}$$

$$= \frac{1}{2} \int_{-\infty}^{-(\tilde{g}_2g_1)} d\zeta \left( W(\zeta) - \frac{1}{\zeta} \right) - \tilde{g}_2 \frac{\partial F}{\partial \tilde{g}_2} \bigg|_{\tilde{g}}$$

(18)

By using the solution for the resolvent (16) to the leading order in $1/N$, we find that the renormalization group flow in infinitely many coupling constant space is now reduced to an effective flow in single coupling constant $\tilde{g}_2$. Thus we obtain a renormalization group equation with the effective beta function $\beta_{\text{eff}}(\tilde{g}_2)$ and the inhomogeneous term $r(\tilde{g}_2)$

$$\left[ N \frac{\partial}{\partial N} + 1 \right] F(N, \tilde{g}_2) = G = \beta_{\text{eff}}(\tilde{g}_2) \frac{\partial F(N, \tilde{g}_2)}{\partial \tilde{g}_2} + r(\tilde{g}_2),$$

(19)

where the effective beta function and the inhomogeneous term are given in terms of $\Delta = 4\tilde{g}_2 + 1$ as

$$\beta_{\text{eff}}(\tilde{g}_2) = \frac{1}{4} \left[ 1 - \Delta - \frac{(1 - \Delta)^2}{2\sqrt{\Delta}} \log \left( \frac{1 + \sqrt{\Delta}}{1 - \sqrt{\Delta}} \right) \right]$$

(20)

$$r(\tilde{g}_2) = \frac{1}{2\sqrt{\Delta}} \left[ \left( \frac{1 + \sqrt{\Delta}}{2} \right)^2 \log \left( \frac{1 + \sqrt{\Delta}}{2} \right) - \left( \frac{1 - \sqrt{\Delta}}{2} \right)^2 \log \left( \frac{1 - \sqrt{\Delta}}{2} \right) \right].$$

(21)

The effective beta function $\beta_{\text{eff}}(\tilde{g}_2)$ exhibits a zero at $\tilde{g}_2 = -1/4 \equiv g_2^{\text{ct}}$. Furthermore, we get the susceptibility exponents $\gamma_1 = 3/2$, $\gamma_0 = 1/2$ from the derivative of the effective beta function by using (4). The fixed point and the susceptibility exponents are in complete agreement with the exact results for the $m = 2$ critical point of the vector model corresponding to pure gravity [4]. The beta function $\beta_{\text{eff}}(\tilde{g}_2)$ has also a trivial fixed point at $\tilde{g}_2 = 0$, which is ultraviolet unstable since $\partial \beta_{\text{eff}} / \partial \tilde{g}_2 (\tilde{g}_2 = 0) = -1 < 0$. 
By a systematic expansion of the free energy in powers of $1/N$, we can extract the complete information from the renormalization group flow namely from the difference equation and the reparametrization identities

$$F(N, \tilde{g}_2) = \sum_{h=0}^{\infty} N^{-h} f_h(\tilde{g}_2). \quad (22)$$

We should evaluate the difference equation (6) and solve the loop equation (16) both up to the order $N^{-h}$. Then we obtain an ordinary differential equation for $f_h$

$$(1 - h)f_h(\tilde{g}_2) - \beta^{\text{eff}}(\tilde{g}_2) \frac{\partial f_0}{\partial \tilde{g}_2}(\tilde{g}_2) = r_h(\tilde{g}_2), \quad (23)$$

where the effective beta function is common to all $h$, but the inhomogeneous terms $r_h$ depend on $h$. We see immediately that the general solution is given by a sum of an arbitrary multiple of the solution of the homogeneous equation and a particular solution of the inhomogeneous equation. Since both the effective beta function $\beta^{\text{eff}}(\tilde{g}_2)$ and the inhomogeneous term $r_h(\tilde{g}_2)$ are analytic in $\tilde{g}_2$ around the fixed point $\tilde{g}_{2*}$, the singular behavior of $f_h$ comes from the solution of the homogeneous equation. It is important to notice that the singular term corresponding to the continuum physics (the so-called universal term) is specified by the beta function alone. The normalization $a_h$ of singular terms, however, cannot be obtained from the renormalization group equation

$$f_h(\tilde{g}_2) = f_h(\tilde{g}_2)_{\text{sing}} + f_h(\tilde{g}_2)_{\text{analytic}}$$

$$f_h(\tilde{g}_2)_{\text{sing}} = (\tilde{g}_2 - \tilde{g}_{2*})^{(1 - \gamma)h} a_h + \cdots. \quad (24)$$

If we sum over the contributions from various $h$, we find that the renormalization group equation determines the combinations of variables appropriate to define the double scaling limit. On the other hand, the functional form of the singular part of the free energy $F_{\text{sing}}$ in the scaling variable $N^{1/\gamma_1}(\tilde{g}_2 - \tilde{g}_{2*})$ is undetermined corresponding to the undetermined normalization factor $a_h$ for the singular terms of each $h$

$$F(N, \tilde{g})_{\text{sing}} = \sum_{h=0}^{\infty} N^{-h} f_h(\tilde{g}_2)_{\text{sing}} = \sum_{h=0}^{\infty} N^{-h}(\tilde{g}_2 - \tilde{g}_{2*})^{2 - \gamma_0 - \gamma h} a_h$$

$$= (\tilde{g}_2 - \tilde{g}_{2*})^{2 - \gamma_0} f(N^{1/\gamma_1}(\tilde{g}_2 - \tilde{g}_{2*}))_{\text{sing}}. \quad (25)$$

Let us note that the free energy is completely determined if we impose the condition $F(N, \tilde{g}_2 = 0) = 0$, which follows from the definition (9).

We can extend the analysis to more general situations of finitely many coupling constants, such as the case of the two coupling constants $g_1 = 1, \quad \tilde{g}_2 \neq 0, \quad \tilde{g}_3 \neq 0, \quad \tilde{g}_k = 0 \ (k \geq 4)$. We can write down the shifts of coupling constants $\delta g_k$ in eq.(15) explicitly, and we can obtain the resolvent as a solution of the loop equation (13). By inserting these to the right hand side $G$ of the renormalization group equation, we obtain effective beta functions and inhomogeneous terms. As reported in ref.[10], we found a perfect agreement with the exact result for $m = 2, 3$ cases. Moreover, we were able to explicitly draw the renormalization group flow diagram in the space of two coupling constant space [10]. Higher multicritical points can be analyzed similarly.
4 Nonlinear Renormalization Group Equation for Matrix Models

In this last section we report briefly on our new results on matrix models \[11\]. The partition function \(Z_N(g)\) of the matrix model is defined by an integral over an \(N \times N\) hermitian matrix \(\Phi\) with a potential \(V(\Phi)\)

\[
Z_N(g) = \int dN^2 \Phi \exp \left[ -N \text{tr} V(\Phi) \right], \quad V(\Phi) = \sum_{k=1}^{\infty} \frac{g_k}{k} \Phi^k
\]  

The cubic interaction with a single coupling constant \(g\) corresponds to \(V(\Phi) = 1/2 \Phi^2 + g/3 \Phi^3\). We can integrate over the angular variables to obtain an integral over the eigenvalues \(\lambda\)

\[
Z_N(g) = c_N \int \prod_{i=1}^{N} d\lambda_i \Delta_N^2(\lambda) \exp \left[ -N \sum_{i=1}^{N} V(\lambda_i) \right]
\]  

where \(\Delta_N\) denotes the Van der Monde determinant \(\Delta_N(\lambda) = \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)\), and \(c_N = \pi^{N(N-1)/2} / \prod_{p=1}^{N} p!\).

In order to relate \(Z_{N+1}\) to \(Z_N\), we shall integrate one of the eigenvalues \(\lambda_{N+1}\) in \(Z_{N+1}\)

\[
Z_{N+1}(g) = c_{N+1} \int \prod_{i=1}^{N} d\lambda_i \Delta_{N+1}^2(\lambda) \exp \left[ -(N+1) \sum_{i=1}^{N+1} V(\lambda_i) \right]
\]  

\[
Z_{N+1}(g) = c_{N+1} \int \prod_{i=1}^{N} d\lambda_i \Delta_{N}^2(\lambda) e^{-(N+1)^2 \sum_{i=1}^{N} V(\lambda_i)} \int d\lambda_{N+1} \prod_{i=1}^{N} |\lambda_{N+1} - \lambda_i|^2 e^{-(N+1) V(\lambda_{N+1})}
\]  

The \(\lambda_{N+1}\) integral can be evaluated by a saddle point method to the leading order in \(1/N\). The saddle point \(\lambda_{N+1}^s\) is determined as a function of \(\text{tr} \Phi^n\) by the saddle point equation

\[
V'(\lambda_{N+1}^s) = \frac{2}{N} \text{tr} \left( \frac{1}{\lambda_{N+1}^s - \Phi} \right) = \frac{2}{N} \sum_{n=0}^{\infty} \frac{\text{tr} \Phi^n}{(\lambda_{N+1}^s)^{n+1}}
\]  

By inserting the saddle point \(\lambda_{N+1}^s\) into the partition function, we obtain

\[
\frac{Z_{N+1}(g)}{Z_N(g)} = \frac{c_{N+1}}{c_N} \int dN^2 \Phi \exp \left[ -(N+1) \text{tr} V(\Phi) - NV(\lambda_{N+1}^s) + 2 \text{tr} \log |\lambda_{N+1}^s - \Phi| + O(N^0) \right] \int dN^2 \Phi \exp \left[ -N \text{tr} V(\Phi) \right] = \left\langle \exp \left[ -\text{tr} V(\Phi) - NV(\lambda_{N+1}^s) + 2 \text{tr} \log |\lambda_{N+1}^s - \Phi| + \log \frac{c_{N+1}}{c_N} + O(N^0) \right] \right\rangle
\]  

Here \(\langle \quad \rangle\) represents the normalized average with respect to the measure \(dN^2 \Phi \exp \left[ -N \text{tr} V(\Phi) \right]\). We observe from the above equation and eq. (31) that infinitely many numbers of operators of the form \(\text{tr} \Phi^n \text{tr} \Phi^n \cdots\) are induced.
The situation, however, simplifies if we appeal to the large-$N$ limit. In this limit a multi-point function of $U(N)$-invariant operators factorizes into a product of one-point functions

$$\langle O O' \cdots \rangle = \langle O \rangle \langle O' \rangle \cdots + O \left( \frac{1}{N^2} \right). \quad (32)$$

In favor of this factorization property, eq. (31) reads

$$Z_{N+1}(g) = \exp \left[ -\langle \text{tr} V(\Phi) \rangle - NV(\langle \lambda_{N+1}^s \rangle) + 2 \langle \text{tr} \log |\lambda_{N+1}^s - \Phi| \rangle + \log \frac{c_{N+1}}{c_N} + O(N^0) \right], \quad (33)$$

where $\langle \lambda_{N+1}^s \rangle$ is determined in terms of $\langle \text{tr} \Phi^m \rangle$ by averaging eq. (30),

$$V'(\langle \lambda_{N+1}^s \rangle) - 2 \langle \frac{1}{N} \text{tr} \frac{1}{\langle \lambda_{N+1}^s \rangle - \Phi} \rangle = 0. \quad (34)$$

Let us introduce the free energy which is normalized to vanish for vanishing coupling constants

$$F(N, g) \equiv -\frac{1}{N^2} \log \left[ \frac{Z_N(g)}{Z_N(g_1 = 0, g_2 = 1, g_k = 0 (k \geq 2))} \right]. \quad (35)$$

where $Z_N(g_1 = 0, g_2 = 1, g_k = 0 (k \geq 2)) = 2^{N/2}(\pi/N)^{N^2/2}$. By taking the $N \to \infty$ limit, we find the following differential equation as a renormalization group equation for the free energy

$$\left( N \frac{\partial}{\partial N} + 2 \right) F(N, g) = \left\langle \frac{1}{N} \text{tr} V(\Phi) \right\rangle + V'(|\lambda_{N+1}^s|) - 2 \left\langle \frac{1}{N} \text{tr} \log |\lambda_{N+1}^s - \Phi| \right\rangle - \frac{3}{2} + O \left( \frac{1}{N} \right). \quad (36)$$

We observe that this renormalization group equation describes a flow in infinite dimensional coupling constant space analogous to the vector model. We stress that this RG equation does not involve a perturbation with respect to coupling constants, unlike the approximation schemes proposed in ref. [3].

Similarly to the vector model, we can find infinitely many identities expressing the reparametrization freedom of matrix variables. In order to derive these identities, it is most convenient to formulate in terms of the loop equations [3], [12]. Let us define the expectation value of the resolvent

$$W(\zeta) = \left\langle \text{tr} \left( \frac{1}{\zeta - \Phi} \right) \right\rangle \quad (37)$$

By a reparametrization of matrix variables, we can obtain the loop equation to the leading order in $1/N$ explicitly

$$W(\zeta)^2 - V'(\zeta)W(\zeta) + Q(\zeta; V) = 0, \quad (38)$$

$$Q(\zeta; V) \equiv \sum_{k \geq 1} \frac{1}{k!} V^{(k+1)}(\zeta) \left\langle \frac{1}{N} \text{tr} (\Phi - \zeta)^{k-1} \right\rangle.$$

If we consider the space of single coupling constant for the cubic interaction $(g_1, g_2, g_3, \cdots) = (0, 1, g_3, 0, \cdots)$, we can relate derivatives with respect to all the other coupling constants to that with respect to the single coupling constant $g_3$ by means of the loop equation.
Therefore we obtain to the leading order in $1/N$

$$
\left(N \frac{\partial}{\partial N} + 2\right) F(N, g) = -1 - \frac{g_3}{2} \frac{\partial F}{\partial g_3} + V(\lambda_{N+1}^*) - 2 \log(\lambda_{N+1}^*) - 2 \int_{-\infty}^{(\lambda_{N+1}^*)} d\zeta \left( W(\zeta) - \frac{1}{\zeta} \right) \\
\equiv G\left(g_3, \frac{\partial F_N}{\partial g_3}\right). \tag{39}
$$

Contrary to the vector model, this effective renormalization group equation contains nonlinear terms in $\partial F_N/\partial g_3$.

$$
G\left(g_3, \frac{\partial F}{\partial g_3}\right) = \sum_{n=0}^\infty \beta_n(g_3) \left( \frac{\partial F}{\partial g_3} \right)^n. \tag{40}
$$

This nonlinearity is the most characteristic feature of the matrix model.

In spite of the nonlinearity, we can find the fixed points and the critical exponents in the following way \cite{11}. If we expand the coefficients $\beta_n$ in powers of the coupling constant around the fixed point $g_{3*}$, they are analytic

$$
\beta_n(g_3) = \sum_{k=0}^{\infty} \beta_{nk}(g_3 - g_{3*})^k. \tag{41}
$$

The non-analyticity of the free energy should come from solving the differential equation (39). We assume that the free energy has singular and analytic terms as in the vector model (24)

$$
F(N, g_3) = F_{\text{analytic}} + F_{\text{sing}}, \quad F_{\text{analytic}} = \sum_{k=0}^{\infty} c_k(g_3 - g_{3*})^k, \quad F_{\text{sing}} = \sum_{k=0}^{\infty} d_k(g_3 - g_{3*})^{k+\gamma}. \tag{42}
$$

By comparing the power series expansion of the renormalization group equation, we find consistency conditions for the above expansion to be valid. The first four conditions read

$$
0 = d_0 \gamma \sum_{n=1}^{\infty} \beta_{n0}n c_1^{n-1}, \quad 2c_1 = \sum_{n=0}^{\infty} \beta_{n1}c_n^{n}, \\
2d_0 = d_0 \gamma \left[ \sum_{n=1}^{\infty} \beta_{n1}nc_1^{n-1} + 2c_2 \sum_{n=2}^{\infty} \beta_{n0}n(n-1)c_1^{n-2} \right], \tag{43} \\
2c_2 = \sum_{n=0}^{\infty} \beta_{n2}c_n^{n} + 2c_2 \sum_{n=1}^{\infty} \beta_{n1}nc_1^{n-1} + 2(c_2)^2 \sum_{n=2}^{\infty} \beta_{n0}n(n-1)c_1^{n-2}.
$$

By solving these four equations we can determine four quantities, namely the fixed point, the coefficient $c_1$, the critical exponent $\gamma$, and the coefficient $c_2$. Therefore these equations are enough to determine the fixed point and the critical exponents. We have found that the above set of equations in fact has a solution that agrees with the exact result \cite{11} $g_{3*} = \pm 2^{-1} \cdot 3^{-3/4}$, $\gamma_1 = \gamma = 5/2$.

Similarly we can determine all the coefficients $c_k$ ($k \geq 0$) and $d_k/d_0$ ($k \geq 1$) except for the overall normalization of the singular term $d_0$. Moreover we can show that higher genus free energies have singular terms precisely needed for the double scaling behavior

$$
F(N, g_3)_{\text{sing}} = \sum_{h=0}^{\infty} N^{-2h} f_h(g_3)_{\text{sing}}, \quad f_h(g_3)_{\text{sing}} = (g_3 - g_{3*})^{(1-h)\gamma_1} d_0^h + \cdots. \tag{44}
$$
The singular terms are determined up to the normalization \( d_0^h \) for each genus. These features are analogous to the case of the vector model. However, a distinctive feature of the matrix model is that the two coefficients \( c_1 \) and \( c_2 \) are needed to determine the fixed point \( g_* \) and the critical exponent \( \gamma_1 \). This novel new feature is a direct consequence of the nonlinear nature of the renormalization group equation for the matrix model (41). It is intriguing to observe that these two coefficients can be determined self-consistently without referring to all the other coefficients \( c_k \) \((k \neq 1, 2)\). We can perform a similar analysis for cases with more couplings corresponding to the higher multicritical points, and also multi-matrix models [11].

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