UNIFORM FAR-FIELD ASYMPTOTICS OF THE TWO-LAYERED GREEN FUNCTION IN 2D AND APPLICATION TO WAVE SCATTERING IN A TWO-LAYERED MEDIUM

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Abstract. In this paper, we establish new results for the uniform far-field asymptotics of the two-layered Green function (together with its derivatives) in 2D in the frequency domain. To the best of our knowledge, our results are the sharpest yet obtained. The steepest descent method plays an important role in the proofs of our results. Further, as an application of our new results, we derive the uniform far-field asymptotics of the scattered field to the acoustic scattering problem by buried obstacles in a two-layered medium with a locally rough interface. The results obtained in this paper provide a theoretical foundation for our recent work, where direct imaging methods have been developed to image the locally rough interface from phaseless total-field data or phased far-field data at a fixed frequency. It is believed that the results obtained in this paper will also be useful on its own right.

Key words. two-layered Green function, uniform far-field asymptotics, steepest descent method, acoustic scattering, two-layered medium.

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1. Introduction. This paper is concerned with the uniform far-field asymptotics of two-dimensional two-layered Green function (together with its derivatives) in the frequency domain and the application in the two-layered medium scattering problems. Two-layered Green functions have attracted much attention in many fields such as radar, remote sensing, ocean acoustics, exploration geophysics and outdoor sound propagation.

Two-layered Green functions play an important role in both direct and inverse scattering problems in two-layered media. Typically, the scattered waves for the scattering problems in two-layered media can be formulated by boundary integral equations associated with the two-layered Green functions (see, e.g., [2, 25, 30]). Therefore, efficient and accurate calculations of the two-layered Green functions have been extensively studied (see, e.g., [4, 6] for the method using high-frequency asymptotics, [8, 16, 31] for the method using contour deformations and [21] for the method using a new hybrid integral representation). Furthermore, with the benefit of high accuracy algorithms and special natures of the two-layered Green functions, there are many works concerning the efficient numerical methods for the direct scattering problems in two-layered media (see, e.g., [2, 25] for the approach combining the boundary integral equations and variational methods, [11] for the perfectly matched layer method and [13, 35] for the fast multipole method). Moreover, based on the properties of the two-layered Green functions, many fast and robust inversion algorithms have been developed for recovering the scatterers buried in two-layered media from a knowledge of the scattered-field data or the far-field data, such as the MUSIC method [1, 29],

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the linear sampling method [15], the sampling type method [22, 23] and the migration algorithm [26].

Let \( \mathbb{R}_+^2 := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\} \), \( \Gamma_0 := \{(x_1, 0) : x_1 \in \mathbb{R}\} \) and \( S_1^1 := \{x \in \mathbb{R}_+^2 : |x| = 1\} \). For any point \( x \in \mathbb{R}^2 \) with \(|x| \neq 0\), let \( \hat{x} := x/|x| \) denote the direction of \( x \). Then given any source point \( y \in \mathbb{R}_+^2 \cup \mathbb{R}_-^2 \), the two-dimensional two-layered Green function \( G(x, y) \) in the frequency domain is the solution to the following scattering problem

\[
\begin{align*}
\Delta_x G(x, y) + k_\pm^2 G(x, y) &= -\delta(x, y) \quad \text{in} \quad \mathbb{R}_+^2, \\
\left[G(x, y)\right] &= 0, \quad \left[\partial G(x, y)/\partial\nu(x)\right] = 0 \quad \text{on} \quad \Gamma_0, \\
\lim_{|x| \to +\infty} \sqrt{|x|} \left(\frac{\partial G(x, y)}{\partial|x|} - i k_\pm G(x, y)\right) &= 0 \quad \text{uniformly for all} \quad \hat{x} \in S_1^1,
\end{align*}
\]

where \( \delta(x, y) \) denotes the Dirac delta distribution, \( k_\pm = \omega/c_\pm > 0 \) are two different wave numbers in \( \mathbb{R}_\pm^2 \), respectively, \( \nu \) denotes the unit normal on \( \Gamma_0 \) pointing into \( \mathbb{R}_+^2 \) and \([\cdot]\) denotes the jump across the interface \( \Gamma_0 \). Here, (1.3) is called the Sommerfeld radiation condition, \( \omega \) is the wave frequency and \( c_\pm \) are the wave speeds in the half-spaces \( \mathbb{R}_\pm^2 \), respectively. For the derivation of the explicit form of \( G(x, y) \), we refer to [12, 1, 25, 30].

It is well-known from [14] that the solution to the scattering problem (1.1)–(1.3) in the limiting case \( k := k_+ = k_- \) (i.e. the fundamental solution of the homogeneous Helmholtz equation \( \Delta u + k^2 w = 0 \) in \( \mathbb{R}^2 \) with wave number \( k > 0 \)) is given by

\[
\Phi(x, y) := \frac{i}{4} H_0^{(1)}(k|x - y|) \quad \text{with} \quad H_0^{(1)}(x) \quad \text{denoting the Hankel function of the first kind of order 0 and} \quad \Phi(x, y) \quad \text{satisfies the far-field asymptotic behavior: for any} \quad y \in \mathbb{R}^2,
\]

\[
\Phi(x, y) = \frac{e^{ik|x|}}{\sqrt{|x|}} \sqrt{8\pi k} \left(e^{-ik\hat{x} \cdot y} + O(|x|^{-1})\right), \quad |x| \to +\infty, \quad x \in \mathbb{R}^2,
\]

uniformly for all angles \( \theta \in [0, 2\pi) \). Here, \( \theta \in [0, 2\pi) \) denotes the angle of the observation direction \( \hat{x} \in S_1^1 \) for any angle \( \hat{x} \in [0, \pi) \). For the derivation of the explicit form of \( G(x, y) \) is much more challenging since the explicit form of \( G(x, y) \) is more complicated due to the presence of the two-layered medium. In [1], it was proved that for any \( y \in \mathbb{R}^2 \), \( G(x, y) \) has the far-field asymptotic behavior:

\[
G(x, y) = \frac{e^{ik_+|x|}}{\sqrt{|x|}} G^\infty(\hat{x}, y) + G_{Res}(x, y), \quad |x| \to +\infty, \quad x \in \mathbb{R}_+^2,
\]

where \( G^\infty(\hat{x}, y) \) is the so-called far-field pattern of \( G(x, y) \) in \( x \in \mathbb{R}_+^2 \) (see [1, formula (9)]) and the remainder term \( G_{Res}(x, y) = O(|x|^{-1/2}) \) as \( |x| \to +\infty \) for any angle \( \theta \in (0, \pi) \); see [30, formula (2.59)] or (2.11) below for the complete definition of \( G^\infty(\hat{x}, y) \) with \( \hat{x} \in S_1^1 \) and \( y \in \mathbb{R}_+^2 \cup \mathbb{R}_-^2 \). Moreover, it was deduced in [30] (see also Appendix B of the supplementary materials in [7]) that for any \( y \in \mathbb{R}_+^2 \cup \mathbb{R}_-^2 \), \( G(x, y) \) has the far-field asymptotic behavior (1.5) with \( G_{Res}(x, y) \) satisfying that

\[
G_{Res}(x, y) = O(|x|^{-3/2}) \quad \text{as} \quad |x| \to +\infty
\]

for all angles \( \theta \in (0, \pi) \) in the case \( k_+ < k_- \) and for all angles \( \theta \in (0, \pi) \setminus \{\theta_\pm, \pi - \theta_\pm\} \in (0, \pi/2) \). Note that
θ and π − θ are called the critical angles of G(x, y) in the case k_+ > k_-; see Remarks 2.15 and 3.3 for discussions on the critical angles. Further, it can be deduced from [30, formula (2.27) and pages 31–32] that for any y ∈ R^2 \cup R^2, G(x, y) = O(|x|^{-3/2}) as |x| → +∞ for the angle θ ∈ \{0, π\}. Note that [30] only considered the pointwise (rather than uniform) estimates of the far-field asymptotics of G(x, y) with respect to the angle θ_k and these pointwise estimates can be dealt with in the same way for both cases k_+ < k_- and k_+ > k_-, as seen in [30]. Moreover, it should be pointed out that for the case k_+ > k_-, the far-field asymptotic estimates of G(x, y) obtained in [30] actually do not hold uniformly for the angles θ_k in the vicinity of the critical angles θ_k and π − θ_k; see Section 2 below for further discussions.

In this paper, we establish new results for the uniform far-field asymptotic properties of G(x, y) for all angles θ ∈ (0, π) \cup (π, 2π) and for both cases k_+ < k_- and k_+ > k_- The main contributions of our results are twofold. First, our results show that for any y ∈ R^2 \cup R^2, G(x, y) has the far-field asymptotic properties (1.5) and

\begin{equation}
G(x, y) = e^{ik_-|x|}G^\infty(\hat{x}, y) + G_{\text{Res}}(x, y), \quad |x| \rightarrow +\infty, \quad x \in R^2,
\end{equation}

where G^\infty(\hat{x}, y) in (1.7) is the far-field pattern of G(x, y) in x ∈ R^2 (see (3.4) below) and the remainder term G_{\text{Res}}(x, y) in (1.5) and (1.7) satisfies that

\begin{equation}
G_{\text{Res}}(x, y) = O(|x|^{-3/4}), \quad |x| \rightarrow +\infty, \quad x \in R^2 \cup R^2,
\end{equation}

uniformly for all angles θ ∈ (0, π) \cup (π, 2π) (including the critical angles) and all y ∈ B_{R_0}^+ \cup B_{R_0}^- with arbitrarily fixed R_0 > 0 (see Remark 3.4). Here, B_{R_0}^+ := \{y \in R^2 : |y| < R_0\} for R_0 > 0. We also prove that the uniform asymptotic property (1.8) is essentially sharp for the angles θ_k lying in the vicinity of any one of the critical angles and that the asymptotic property G_{\text{Res}}(x, y) = O(|x|^{-3/2}) does not hold for the critical angles (see Remarks 2.15 and 3.3). Secondly, for all angles θ ∈ (0, π) \cup (π, 2π) except for the critical angles, we obtain uniform upper bounds for the remainder term G_{\text{Res}}(x, y) in (1.5) and (1.7) with large enough |x|. The uniform upper bounds obtained show that for any critical angle θ, G_{\text{Res}}(x, y) in (1.5) and (1.7) satisfies that

\begin{equation}
G_{\text{Res}}(x, y) = O(|\theta − \theta_k|^{\frac{3}{2}}|x|^{-\frac{3}{2}}), \quad |x| \rightarrow +\infty,
\end{equation}

uniformly for all angles θ_k in a punctured neighborhood of θ and all y ∈ B_{R_0}^+ \cup B_{R_0}^- with arbitrarily fixed R_0 > 0 (see Theorems 2.14 and 3.1). Moreover, similarly to our analysis for G(x, y), we also derive uniform far-field asymptotic properties for \nabla_y G(x, y). To the best of our knowledge, our results on the uniform far-field asymptotics of G(x, y) and its derivatives are the sharpest yet obtained. Note that the steepest descent method plays an important role in the proofs of our results (see, e.g., [4, 5, 6, 9, 18, 27] for applications of the steepest descent method). The crucial part of our proofs is the singularity analysis of the relevant integrals for G(x, y) in the case when the angle θ_k is very close to any one of the critical angles. Further, as an application of our new results, we derive the uniform far-field asymptotics for the solution to the acoustic scattering problem by buried obstacles in a two-layered medium with a locally rough interface.

It is worth noting that this work is motivated by our recent work [24], where we have considered the inverse acoustic scattering in a two-layered medium with a locally rough interface and direct imaging methods have been proposed to numerically
reconstruct the locally rough interface from phaseless total-field data or phased far-field data. In fact, the results obtained in the present paper provide a theoretical foundation for the direct imaging methods given in [24].

It should be remarked that some related works have been done for the asymptotics of relevant Green functions. High-frequency asymptotic properties of the two-layered Green function in two dimensions have been studied in [4]. For the two-layered Green function in three dimensions, we refer to [3, 6] for the analysis of the high-frequency asymptotic properties, and [3, 6, 18] for the analysis of their far-field asymptotics. Moreover, [9] has derived the uniform far-field asymptotic expansion of the Green function for the Helmholtz equation in a half-plane with an impedance boundary condition.

The remaining part of the paper is organized as follows. Sections 2 and 3 are devoted to the theoretical analysis of the uniform far-field asymptotic property of the two-layered Green function $G(x, y)$ for the cases $x \in \mathbb{R}^2_+$ and $x \in \mathbb{R}^2_-$, respectively. In Section 4, as an application of the results obtained in Sections 2 and 3, we study the uniform far-field asymptotic property of the solution to the acoustic scattering problem by buried obstacles in a two-layered medium with a locally rough interface. Some concluding remarks are given in Section 5. We derive the formula (2.31) in Appendix A and prove Lemmas 2.9 and 2.10 in Appendix B.

2. Uniform far-field asymptotic analysis of $G(x, y)$ with $x \in \mathbb{R}^2_+$. This section is devoted to studying the uniform far-field asymptotic properties of $G(x, y)$ with $x \in \mathbb{R}^2_+$. To this end, we first introduce the following notations, which will be used throughout the paper. Define $n := k_- / k_+$. Denote the angle $\theta_c$ by

$$\theta_c := \begin{cases} \arccos(n) \in (0, \pi/2), & k_+ > k_-; \\ \arccos(1/n) \in (0, \pi/2), & k_- < k_+. \end{cases}$$

For any $x \in \mathbb{R}^2$ with $|x| \neq 0$, let $x = (x_1, x_2)$ and $\hat{x} = x / |x| = (\cos \theta_x, \sin \theta_x)$ with $\theta_x \in [0, 2\pi)$. For any $d \in \mathbb{S}^1 := \{d \in \mathbb{R}^2 : |d| = 1\}$, let $d = (d_1, d_2) = (\cos \theta_d, \sin \theta_d)$ with $\theta_d \in [0, 2\pi)$. Define $\mathbb{S}^1_\pm := \{x \in \mathbb{R}^2_+ : |x| = 1\}$. For any $y = (y_1, y_2) \in \mathbb{R}^2$, let $y' := (y_1, -y_2)$. For any $R_0 > 0$, define $\bar{B}_{R_0} := \{y \in \mathbb{R}^2_+ : |y| < R_0\}$. Let $\mathbb{C}_0, \mathbb{C}_1$ and $\mathbb{C}_2$ be given by

$$\mathbb{C}_0 := \mathbb{C} \setminus \{z \in \mathbb{C} : \text{Im}(z) = 0, \text{Re}(z) \leq 0\},$$
$$\mathbb{C}_1 := \mathbb{C} \setminus \{z \in \mathbb{C} : \text{Im}(z) \geq 0, \text{Re}(z) = 0\},$$
$$\mathbb{C}_2 := \mathbb{C} \setminus \{z \in \mathbb{C} : \text{Im}(z) \leq 0, \text{Re}(z) = 0\},$$

respectively. Then we introduce the functions $z^{\beta}, S_1(z)$ and $S_2(z)$ as follows.

1. For $\beta \in \mathbb{R}$ and $z \in \mathbb{C}_0$ with $z = |z|e^{i\theta_0}$ and $\theta_0 \in (-\pi, \pi)$, define $z^\beta := |z|^\beta e^{i\beta \theta_0}$. For $\beta > 0$ and $z = 0$, define $z^\beta := 0$. For simplicity, the function $z^\beta$ with $\beta = 1/2$ is also denoted by $\sqrt{z}$ as usual.

2. For $z \in \mathbb{C}_1$ with $z = |z|e^{i\theta_1}$ and $\theta_1 \in (-\pi, -\pi/2)$, let $S_1(z) := \sqrt{|z|}e^{i\theta_1/2}$. For $z = 0$ define $S_1(z) := 0$.

3. For $z \in \mathbb{C}_2$ with $z = |z|e^{i\theta_2}$ and $\theta_2 \in (-\pi, -\pi/2)$, let $S_2(z) := \sqrt{|z|}e^{i\theta_2/2}$. For $z = 0$ define $S_2(z) := 0$.

Moreover, for $z \in \mathbb{C}$ and $a > 0$ such that $z - a \in \mathbb{C}_1 \cup \{0\}$ and $z + a \in \mathbb{C}_2 \cup \{0\}$, define $S(z, a) := S_1(z - a)S_2(z + a)$ (note that $S(z, a) = 0$ for $z = \pm a$). For $z \in \mathbb{C}$ and $a > 0$ such that $z - a, z + a \in \mathbb{C}_2 \cup \{0\}$, define $S(z, a) := S_2(z - a)S_2(z + a)$. The branch cuts of $S(z, a)$ and $\tilde{S}(z, a)$ with $a > 0$ are depicted in Figure 1. For $z \in \{z \in \mathbb{C} :
Re(\(z = a\), Im(\(z \geq 0\)) and \(a > 0\), it can be seen that \(\lim_{h \in \mathbb{R}: h \to +0} S(z \pm h, a)\) exist, which are denoted as \(S_+(z, a)\), respectively. It is clear that

\[
S_+(z, a) = -S_-(z, a) = \tilde{S}(z, a) \quad \text{for } z \in \{z \in \mathbb{C} : \text{Re}(z) = a, \text{Im}(z) \geq 0\}.
\]

For \(c_1, c_2 \in \mathbb{R}\) with \(c_1 < c_2\), define the strip \(L_{c_1,c_2} := \{z \in \mathbb{C} : c_1 < \text{Im}(z) < c_2\}\). In particular, the strip \(L_{-c,c}\) with \(c > 0\) is denoted as \(L_c\). For \(\theta \in \mathbb{R}\), we define

\[
\mathcal{R}(\theta) := \frac{i \sin \theta + S(\cos \theta, n)}{i \sin \theta - S(\cos \theta, n)}, \quad \mathcal{T}(\theta) := \mathcal{R}(\theta) + 1.
\]

Throughout this paper, the constants may be different at different places. If not stated otherwise, for any \(a, b \in \mathbb{R} \cup \{\pm \infty\}\) and any function \(f\) defined in the complex plane, \(\int_a^b f(s)ds\) is always considered as an integral from \(a\) to \(b\) along the real axis.

Now we present the explicit formula for the two-layered Green function \(G(x, y)\) with \(x \in \mathbb{R}^2_+\). For \(x = (x_1, x_2) \in \mathbb{R}^2_+\) and \(y = (y_1, y_2) \in \mathbb{R}^2_+ \cup \mathbb{R}^2_+\), \(G(x, y)\) has the following form (see, e.g., [30, formula (2.27)] and [25, Appendix A])

\[
G(x, y) = \begin{cases} 
\frac{i}{4} H_0^{(1)}(|x - y|) + G_\mathcal{R}(x, y), & x \in \mathbb{R}^2_+, \ y \in \mathbb{R}^2_+, \\
G_\mathcal{T}(x, y), & x \in \mathbb{R}^2_+, \ y \in \mathbb{R}^2_+,
\end{cases}
\]

where \(G_\mathcal{R}(x, y)\) and \(G_\mathcal{T}(x, y)\) are given by the so-called Sommerfeld integrals, that is,

\[
G_\mathcal{R}(x, y) := \frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{S(\xi, k_+) - S(\xi, k_-)}{S(\xi, k_+) + S(\xi, k_-)} e^{i\xi(x_1 - y_1)} d\xi,
\]

\[
G_\mathcal{T}(x, y) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{S(\xi, k_-)y_2 - S(\xi, k_+)x_2}{S(\xi, k_+) + S(\xi, k_-)} e^{i\xi(x_1 - y_1)} d\xi,
\]

Moreover, by introducing the variable \(z = \xi/k_+\), (2.5) and (2.6) can be rewritten as

\[
G_\mathcal{R}(x, y) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{S(z, 1) - S(z, n)}{S(z, 1) + S(z, n)} e^{-ik_+|y|/(z \cos \theta_y - iS(z, 1) \sin \theta_y)} e^{ik_+|x|/p(z, \theta_x)} dz,
\]

\[
G_\mathcal{T}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-ik_+|y|/(z \cos \theta_y + iS(z, 1) \sin \theta_y)}}{S(z, 1) + S(z, n)} e^{ik_+|x|/p(z, \theta_x)} dz,
\]

where \(x = |x|/(\cos \theta_x, \sin \theta_x)\) with \(\theta_x \in (0, \pi)\), \(y = |y|/(\cos \theta_y, \sin \theta_y)\) with \(\theta_y \in (0, \pi) \cup (\pi, 2\pi)\) and \(p(z, \theta_x) := z \cos \theta_x + iS(z, 1) \sin \theta_x\).
As mentioned in Section 1, [30] derived the pointwise estimates of the far-field asymptotics of $G(x,y)$ for all angles $\theta_2 \in (0, \pi)$ in the case $k_+ < k_-$ and for all angles $\theta_2 \in (0, \pi) \setminus \{\theta_1, \pi - \theta_1\}$ in the case $k_+ > k_-$ (see Section 2.3.4 in [30]). These pointwise estimates were established in [30] by using the steepest descent method and were essentially based on the pointwise far-field asymptotic estimates of $G_R(x,y)$ and $G_T(x,y)$ given in [30, (2.57a) and (2.57b)].

In the following two subsections, we will study the uniform far-field asymptotic properties of the two-layered Green function $G(x,y)$ for the case $k_+ < k_-$ and for the case $k_+ > k_-$, respectively.

For the case $k_+ < k_-$, it is easy to prove that the far-field asymptotic estimates of $G_R(x,y)$ and $G_T(x,y)$ given in [30, (2.57a) and (2.57b)] hold uniformly for all angles $\theta_2 \in (0, \pi)$ by using the same approach as given in [30, Section 2.3.4]. This is essentially due to the fact that in the case $k_+ < k_-$, since $n = k_-/k_+ > 1$ then the distances between the saddle point $\cos \theta_2$ and the branch points $\pm n$ involved in relevant integrals given in [30, Section 2.3.4] have a uniform positive lower bound for all $\theta_2 \in (0, \pi)$. Therefore, one can easily obtain the uniform far-field asymptotic properties of $G(x,y)$ for all angles $\theta_2 \in (0, \pi)$ in the case $k_+ < k_-$; see Section 2.1 below.

However, for the case $k_+ > k_-$, the far-field asymptotic estimates of $G_R(x,y)$ and $G_T(x,y)$ given in [30, (2.57a) and (2.57b)] do not hold uniformly for the angles $\theta_2$ in the vicinity of the critical angles $\theta_c$ and $\pi - \theta_c$. In fact, when the angle $\theta_2$ is in the vicinity of the critical angle $\theta_c$, this nonuniform property can be seen from the fact that the factor $|\sin \theta_c - \theta|^3/2$ involved in [30, formulas (2.54a) and (2.54b)] will blow up if $\theta$ tends to $\theta_c$, where the formulas (2.54a) and (2.54b) in [30] are related to $G_R(x,y)$ and $G_T(x,y)$, respectively, and where the notations $\theta$ and $\theta_c$ used in [30, formulas (2.54a) and (2.54b)] correspond to $\theta_2$ and $\theta_c$, respectively, used in the present paper. Moreover, when the angle $\theta_2$ is in the vicinity of the critical angle $\pi - \theta_c$, this nonuniform property can be easily seen from the above discussion on [30, formulas (2.54a) and (2.54b)] as well as a symmetry relation of $G(x,y)$ given in (2.90). Furthermore, it is worth noting that this nonuniform property is essentially due to the fact that in the case $k_+ > k_-$, since $n = k_-/k_+ < 1$ then $\cos \theta_2$ will be very close to $n$ (resp. $-n$) when $\theta_2$ approaches $\theta_c$ (resp. $\pi - \theta_c$), where $\cos \theta_2$ and $\pm n$ are the saddle point and the branch points, respectively, as mentioned above. Therefore, the method for deriving the far-field asymptotic estimates in [30, (2.57a) and (2.57b)] does not work for the derivation of the uniform far-field asymptotics of $G(x,y)$ for all angles $\theta_2 \in (0, \pi)$ in the case $k_+ > k_-$. In Remark 2.5 below, we give a detailed explanation on the main difficulties in deriving the uniform far-field asymptotic estimates of $G(x,y)$ for all angles $\theta_2 \in (0, \pi)$ in the case $k_+ > k_-$. We will also propose a method to address these difficulties after Remark 2.5 in Section 2.2.

### 2.1. The case $k_+ < k_-$. In this case, we have the following theorem on the uniform far-field asymptotic properties of $G(x,y)$, which is mainly based on the results in [30].

**Theorem 2.1.** Assume that $k_+ < k_-$ and let $R_0 > 0$ be an arbitrary fixed number. Suppose that $y = (y_1,y_2) \in B_{R_0}^+ \cup B_{R_0}^-$ and $x = |x| \hat{x} = |x|(\cos \theta_2, \sin \theta_2) \in \mathbb{R}_+^2$ with...
Let $\theta_\pm \in (0, \pi)$, then we have the asymptotic behavior

$$
G(x, y) = e^{ik_+ |x|} G^\infty(\hat{x}, y) + G_{Res}(x, y),
$$

(2.9)

$$
\nabla_y G(x, y) = e^{ik_+ |x|} H^\infty(\hat{x}, y) + H_{Res}(x, y),
$$

(2.10)

where $G^\infty$, $H^\infty$ are defined by

$$
G^\infty(\hat{x}, y) := e^{ik_+ \hat{x} \cdot y} \left\{ \begin{array}{rl}
e^{-ik_+ \hat{x} \cdot y} + R(\theta_\pm) e^{-ik_+ \hat{x} \cdot y}, & \hat{x} \in S^1_+, \ y \in \mathbb{R}^2, \\
T(\theta_\pm) e^{-ik_+ (y_1 \cos \theta_\pm + iy_2 \sin \theta_\pm, n)} & \hat{x} \in S^1_+, \ y \in \mathbb{R}^2,
\end{array} \right.
$$

(2.11)

$$
H^\infty(\hat{x}, y) := e^{-ik_+ \hat{x} \cdot y} \left\{ \begin{array}{rl}
e^{-ik_+ \hat{x} \cdot y} (\cos \theta_\pm) \cos(\theta_\pm) + R(\theta_\pm) e^{-ik_+ \hat{x} \cdot y}, & \hat{x} \in S^1_+, \ y \in \mathbb{R}^2, \\
T(\theta_\pm) e^{-ik_+ (y_1 \cos \theta_\pm + iy_2 \sin \theta_\pm, n)} & \hat{x} \in S^1_+, \ y \in \mathbb{R}^2,
\end{array} \right.
$$

(2.12)

and $G_{Res}$ and $H_{Res}$ satisfy the estimates

$$
|G_{Res}(x, y)|, |H_{Res}(x, y)| \leq C_{R_0} |x|^{-3/2}, \quad |x| \to +\infty,
$$

uniformly for all $\theta_\pm \in (0, \pi)$ and $y \in B^+_{R_0} \cup B^-_{R_0}$. Here, the constant $C_{R_0} > 0$ is independent of $x$ and $y$ but dependent of $R_0$.

Proof. By using the steepest decent method, it was proved in [30] that for fixed $\theta_\pm \in (0, \pi)$ and $y \in \mathbb{R}^2 \cup \mathbb{R}^2$, $G(x, y)$ and $\nabla_y G(x, y)$ satisfy (2.9) and (2.10), respectively, with $G_{Res}(x, y) = O(|x|^{-3/2})$ and $H_{Res}(x, y) = O(|x|^{-3/2})$ (see pages 39–41 in [30]). By using the same arguments as in [30], it can be proved that the statement of this theorem holds true.

2.2. The case $k_+ > k_-$. In view of (1.4) and (2.4), we need to investigate the uniform far-field asymptotic properties of $G_R(x, y)$ and $G_T(x, y)$ with $\theta_\pm \in (0, \pi)$. In what follows, we will give the detailed analysis of $G_R(x, y)$ with $\theta_\pm \in (0, \pi/2)$ (see Lemmas 2.4, 2.12 and 2.13 below). For $G_R(x, y)$ with $\theta_\pm \in (\pi/2, \pi)$ and $G_T(x, y)$ with $\theta_\pm \in (0, \pi)$, the theoretical analyses are similar. The main results of this subsection on the uniform far-field asymptotics of $G(x, y)$ for all angles $\theta_\pm \in (0, \pi)$ will be given in Theorem 2.14.

Motivated by the ideas in [4, 6, 9], we will use the steepest descent method to study $G_R(x, y)$. For this, we introduce some paths and curves (see Figure 2). Define the function $h_D(z) := -\text{sgn}(z) \arccos|\text{sech}(z)|$ for $z \in \mathbb{R}$, where $\text{sgn}$ denotes the sign function. It is clear that $h_D(z)$ is an odd and strictly monotonically decreasing function in $z \in \mathbb{R}$. Let the path $D$ denote the curve

$$
I_D := \{ \zeta \in \mathbb{C} : \text{Re}(\zeta) = \theta_\pm + h_D(\text{Im}(\zeta)) \}
$$

with the orientation from $-\pi/2 + \theta_\pm + i\infty$ to $\pi/2 + \theta_\pm - i\infty$ (see Figure 2(a) for the case $\theta_\pm \leq \theta_\pm < 2\pi$ and Figure 2(b) for the case $0 < \theta_\pm < \theta_\pm$). Note that at infinity the
Moreover, define the function $h_B(z) := \text{arccos}[\cos(\theta_c) \text{sech}(z)]$ for $z \in \mathbb{R}$. It is easy to see that $h_B(z)$ is an even function and is strictly monotonically increasing in $z \in [0, +\infty)$. Let $I_m := I_m^+ \cup I_m^-$ with $m \in \mathbb{Z}$, where

$$I_m^+ := \{ \zeta \in \mathbb{C} : \text{Re}(\zeta) = h_B(\text{Im}(\zeta)) + m\pi, \text{Im}(\zeta) \leq 0 \}$$

is a curve connecting $\theta_c + m\pi$ and $\pi/2 + m\pi - i\infty$, and

$$I_m^- := \{ \zeta \in \mathbb{C} : \text{Re}(\zeta) = -h_B(\text{Im}(\zeta)) + m\pi, \text{Im}(\zeta) \geq 0 \}$$

is a curve connecting $-\theta_c + m\pi$ and $-\pi/2 + m\pi + i\infty$. Note that for $m \in \mathbb{Z}$, both the curves $I_m^+$ and $I_{m+1}^-$ asymptotically approach the line $\text{Re}(\zeta) = \pi/2 + m\pi$ at infinity. Let $L_o$ denote the loop around the curve $I_0^+$ with the orientation indicated in Figure 2(b). For $l \in \mathbb{Z}$, let $I_{2l}^+ := \{ \text{Re}(z) - i\text{Im}(z) : z \in I_{2l}^+ \}$ denote the reflections of $I_{2l}^+$, respectively, about the real axis and define $\tilde{I}_{2l}^+ := \tilde{I}_{2l}^+ \cup \tilde{I}_{2l}^-$. Some properties of the curves $I_{2l}$, $I_{2l+1}$ and $\tilde{I}_{2l}$ with $l \in \mathbb{Z}$ are presented in Remark 2.3 below.

**Remark 2.2.** It is easily seen that the path $D$ will cross the real axis with only one intersection point $\theta_\zeta$. Moreover, by patient but straightforward calculations, we have

$$\sqrt{2k_c} e^{i\frac{\pi}{k_c}} \sin \left( \frac{\zeta - \theta_\zeta}{2} \right) = -\sqrt{\frac{k_c}{\cosh(\text{Im}(\zeta))}} \text{ for } \zeta \in D,$$

which is needed for later use. From this, it can be easily derived that $\text{Im}(ik_+ \cos(\zeta - \theta_\zeta)) = k_+$ for $\zeta \in D$ and $\text{Re}(ik_+ \cos(\zeta - \theta_\zeta)) = -k_+ \sinh^2(\text{Im}(\zeta))/\cosh(\text{Im}(\zeta)) < 0$ for $\zeta \in D \setminus \{ \theta_\zeta \}$. Actually, we note that $D$ is the steepest descent path of the function $ik_+ \cos(\zeta - \theta_\zeta)$ through the point $\theta_\zeta$. Here, for any analytic function $W(\zeta)$ of the complex variable $\zeta$, an integration path $Q$ in the complex plane is called a steepest descent path of $W(\zeta)$ if $\text{Im}(W(\zeta))$ is constant on $Q$ (see [27]). Furthermore, it is worth noting that $ik_+ \cos(\zeta - \theta_\zeta)$ has only one saddle point $\zeta = \theta_\zeta$ on the steepest descent path $D$. Thus, it can be seen from [4, Section 7.2] that $\text{Re}(ik_+ \cos(\zeta - \theta_\zeta))$ decreases most rapidly from $\theta_\zeta$ to $-\pi/2 + \theta_\zeta + i\infty$ and from $\theta_\zeta$ to $\pi/2 + \theta_\zeta - i\infty$ along the curve $I_D$. For more introductions and properties of steepest descent paths, we refer to [4, 5, 6, 27].

**Remark 2.3.** Let $A_1 := \bigcup_{l \in \mathbb{Z}} I_{2l}$, $A_2 := \bigcup_{l \in \mathbb{Z}} I_{2l+1}$ and $A_3 := \bigcup_{l \in \mathbb{Z}} \tilde{I}_{2l}$. By straightforward calculations, it is easily seen that

$$A_1 = \{ \zeta \in \mathbb{C} : \text{Re}(\cos(\zeta)) = n, \text{Im}(\cos(\zeta)) \geq 0 \},$$

$$A_2 = \{ \zeta \in \mathbb{C} : \text{Re}(\cos(\zeta)) = -n, \text{Im}(\cos(\zeta)) \leq 0 \},$$

$$A_3 = \{ \zeta \in \mathbb{C} : \text{Re}(\cos(\zeta)) = n, \text{Im}(\cos(\zeta)) \leq 0 \},$$

which imply that $S(\cos(\zeta), n)$ is analytic in $\zeta \in \mathbb{C} \setminus (A_1 \cup A_2)$ and $\tilde{S}(\cos(\zeta), n)$ is analytic in $\zeta \in \mathbb{C} \setminus (A_2 \cup A_3)$. Thus the curves in $A_1 \cup A_2$ are the branch cuts of $S(\cos(\zeta), n)$ and the curves in $A_2 \cup A_3$ are the branch cuts of $\tilde{S}(\cos(\zeta), n)$. Moreover, it can be deduced by direct calculations that the path $D$ lies in $\mathbb{C} \setminus (A_1 \cup A_2)$ for the case $\theta_\zeta \in (\theta_c, \pi/2)$, the path $D$ will cross $A_1 \cup A_2$ with only one intersection point $\theta_\zeta$ for the case $\theta_\zeta = \theta_c$ and the path $D$ lies in $\mathbb{C} \setminus (A_2 \cup A_3)$ for the case $\theta_\zeta \in (0, \theta_c)$ (see Figure 2). Here, we note that $\zeta = \theta_\zeta$ is a branch point of $S(\cos(\zeta), n)$ and $\tilde{S}(\cos(\zeta), n)$.
which shows that the function \( G_R(x,y) \) with \( \theta \in (0, \pi/2) \) can be rewritten as the sum of integrals over the steepest descent path \( D \) and (possibly) an integral over the loop \( L_o \).

Lemma 2.4. Assume that \( k_+ > k_- \). Let \( y = |y|(\cos \theta_y, \sin \theta_y) \in \mathbb{R}^2_+ \) with \( \theta_y \in (0, \pi) \) and \( x = |x|(\cos \theta_x, \sin \theta_x) \in \mathbb{R}^2_+ \) with \( \theta_x \in (0, \pi/2) \). Then the following statements hold.

1. For \( \theta_x \in [\theta_c, \pi/2) \) and \( |x| > |y| \), we have

\[
G_R(x,y) = G_R^{(1)}(x,y) + G_R^{(2)}(x,y),
\]

where \( G_R^{(1)} \) and \( G_R^{(2)} \) are given by

\[
G_R^{(1)}(x,y) := \frac{i}{4\pi} \int_D \frac{\cos(2\zeta) - n^2}{n^2 - 1} e^{ik_+(-|y| \cos(\zeta + \theta_y) + |x| \cos(\zeta - \theta_x))} d\zeta,
\]

\[
G_R^{(2)}(x,y) := \frac{i}{4\pi} \int_D \frac{2i \sin S(\cos \zeta, n)}{n^2 - 1} e^{ik_+(-|y| \cos(\zeta + \theta_y) + |x| \cos(\zeta - \theta_x))} d\zeta.
\]

2. For \( \theta_x \in (0, \theta_c) \) and \( |x| > |y|/\cos(\theta_x) \), we have

\[
G_R(x,y) = G_R^{(1)}(x,y) + G_R^{(3)}(x,y) + G_R^{(4)}(x,y),
\]

where \( G_R^{(1)} \) is given by (2.15), and \( G_R^{(3)} \) and \( G_R^{(4)} \) are given by

\[
G_R^{(3)}(x,y) := \frac{i}{4\pi} \int_D \frac{2i \sin S(\cos \zeta, n)}{n^2 - 1} e^{ik_+(-|y| \cos(\zeta + \theta_y) + |x| \cos(\zeta - \theta_x))} d\zeta,
\]

\[
G_R^{(4)}(x,y) := \frac{i}{4\pi} \int_{L_o} \frac{2i \sin S(\cos \zeta, n)}{n^2 - 1} e^{ik_+(-|y|/\cos(\zeta + \theta_y) + |x| \cos(\zeta - \theta_x))} d\zeta.
\]
Proof. Let $x$ and $y$ be fixed throughout the proof. Motivated by [1, 9], we introduce the change of variable
\begin{equation}
(2.20) \quad z = \begin{cases} 
\cos \eta, & 0 < \eta < \pi, \quad \text{for} -1 < z < 1, \\
\cos i \eta, & \eta \geq 0, \quad \text{for} \ z \geq 1, \\
\cos(\pi + i \eta), & \eta \leq 0, \quad \text{for} \ z \leq -1.
\end{cases}
\end{equation}

Then the formula (2.7) can be rewritten as
\begin{equation}
(2.21) \quad G_R(x, y) = \frac{i}{4\pi} \left\{ \int_0^\pi F(\eta, x, y) d\eta + \int_0^0 F(i\eta, x, y) id\eta + \int_0^\infty F(i\eta + \pi, x, y) id\eta \right\}
\end{equation}
where $F(z, x, y) := e^{ikz-|y| \cos(z+\theta_y)+|x| \cos(z-\theta_x)}(-i \sin z - S(\cos z, n))/(-i \sin z + S(\cos z, n))$ and $\mathcal{L}$ denotes the piecewise linear path with the orientation from $0+i \infty \to 0 \to \pi \to \pi - i \infty$ in the complex plane (see Figure 3). Here, we note that the term $S(z, 1)$ involved in (2.7) has two branch cuts and two zeros (see Figure 1). By the change of variable (2.20), such function $S(z, 1)$ is transformed to the entire function $-i \sin z$ and its reciprocal $(S(z, 1))^{-1}$ involved in (2.7) is eliminated due to the chain rule. This indeed leads to a simplified integral expression (2.21) of $G_R(x, y)$, which makes it more convenient to investigate the asymptotic properties of $G_R(x, y)$ later on in the present subsection.

Next, we rewrite $G_R(x, y)$ with the aid of Cauchy integral theorem. To this end, we distinguish between the following two cases.

Case 1: $\theta_x \in [\theta_c, \pi/2)$. By a straightforward calculation, we have that for $\zeta \in \mathbb{C}$,
\begin{equation}
(2.22) \quad \text{Im } (-|y| \cos(\zeta + \theta_y) + |x| \cos(\zeta - \theta_x)) = \sinh(\text{Im}(\zeta)) \left( |y| \sin(\text{Re}(\zeta) + \theta_y) - |x| \sin(\text{Re}(\zeta) - \theta_x) \right).
\end{equation}

Let $I_+ := \{ \zeta \in \mathbb{C} : \text{Im } (\zeta) > 0, -\frac{\pi}{2} + \theta_x \leq \text{Re}(\zeta) \leq 0 \}$. It is easy to verify that $\sin(\text{Re}(\zeta) - \theta_x) \leq \min(-\sin \theta_x, \sin(\text{Re}(\zeta) + \theta_y))$ for $\zeta \in I_+$. This, together with (2.22), implies that for $\zeta \in I_+$ and $|x| > |y|$,
\begin{equation}
\left| e^{ikz-|y| \cos(z+\theta_y)+|x| \cos(z-\theta_x)} \right| \leq e^{-k \sinh(\text{Im}(\zeta))(|x|+|y|) \sin \theta_x}.
\end{equation}

Further, choose $\varepsilon > 0$ small enough so that $\varepsilon < \min(\pi/2 - \theta_x, (\theta_x + \theta_y)/2)$ and let $I_{-\varepsilon} := \{ \zeta \in \mathbb{C} : \text{Im } (\zeta) < 0, -\frac{\pi}{2} + \theta_x - \varepsilon \leq \text{Re}(\zeta) \leq \pi \}$. Note that $\theta_x < \pi/2 - \varepsilon \leq \text{Re}(\zeta) - \theta_x \leq \pi - \theta_x$ and $\pi/2 + \varepsilon < \text{Re}(\zeta) + \theta_y < 2\pi$ for $\zeta \in I_{-\varepsilon}$. Then we can easily obtain that $\sin(\text{Re}(\zeta) - \theta_x) \geq \max(\sin \theta_x, \sin(\text{Re}(\zeta) + \theta_y))$ for $\zeta \in I_{-\varepsilon}$. Thus it follows from (2.22) that: for $\zeta \in I_{-\varepsilon}$ and $|x| > |y|$,
\begin{equation}
\left| e^{ikz-|y| \cos(z+\theta_y)+|x| \cos(z-\theta_x)} \right| \leq e^{-k \sinh(\text{Im}(\zeta))(|x|-|y|) \sin \theta_x}.
\end{equation}

It is easily seen that
\begin{equation}
(2.23) \quad |\sin(\zeta)| \leq |\sinh(\text{Im}(\zeta))| + \cosh(\text{Im}(\zeta)) \text{ for } \zeta \in \mathbb{C},
\end{equation}
thus we obtain
\begin{equation}
\frac{|-i \sin \zeta - S(\cos \zeta, n)|}{|-i \sin \zeta + S(\cos \zeta, n)|} \leq \frac{|\cos(2\zeta) - n^2|}{n^2 - 1} + \frac{2i \sin \zeta S(\cos \zeta, n)}{n^2 - 1} \leq C \left\{ 1 + |\sinh(\text{Im}(\zeta))| + \cosh(\text{Im}(\zeta))|^2 \right\} \text{ for } \zeta \in \mathbb{C},
\end{equation}
The curve \((2.24)\) Thus, using the similar arguments as in Case 1, we have that \(f\) or \(C > D = \pi \cdot \Re(\cos(\theta))\) with \(|\theta| \leq \theta_0 < \frac{\pi}{2}\). Therefore, using the above arguments and applying Cauchy integral theorem, we obtain (2.14).

**Case 2:** \(\theta_0 \in (0, \theta_c)\). In this case, it can be deduced that the path \(\mathcal{D}\) will cross the branch cut \(\mathcal{I}_0^+\) with only one intersection point, which is denoted as \(\zeta_o\) (see Figure 2(b)). Thus, using the similar arguments as in Case 1, we have that for \(|x| > |y|\),

\[
G_R(x, y) = G_R^{(1)}(x, y) + \frac{i}{4\pi} \int_{D_\theta} \frac{2i \sin \zeta S(\cos \zeta, n)}{n^2 - 1} e^{ik_{\zeta}(\cos(\zeta + \theta) + |x| \cos(\zeta - \theta_0))} d\zeta,
\]

where the path \(D_\theta\) is depicted in Figure 3(b). We note that, compared with the path \(\mathcal{D}\), the path \(D_\theta\) has an additional finite-length path around the branch cut \(\mathcal{I}_0^+\).

If we shrink the additional path for \(D_\theta\), then it follows from (2.2) and the property of \(\mathcal{I}_0^+\) given in Remark 2.3 that for \(|x| > |y|\),

\[
(2.24) \quad G_R(x, y) = G_R^{(1)}(x, y) + G_R^{(2)}(x, y) + G_{R, \zeta_0, 1}(x, y),
\]

where

\[
(2.25) \quad G_{R, \zeta_0, 1}(x, y) := \frac{i}{2\pi} \int_{\mathcal{I}_{\theta_c, \zeta_0}} \frac{2i \sin \zeta S(-\cos \zeta, n)}{n^2 - 1} e^{ik_{\zeta}(\cos(\zeta + \theta) + |x| \cos(\zeta - \theta_0))} d\zeta.
\]

Here, the path \(\mathcal{I}_{\theta_c, \zeta_0}\) denotes the curve \(\{\zeta \in \mathcal{I}_0^+: \Re(\zeta) \in [\theta_c, \Re(\zeta_0)]\}\) with its orientation from \(\theta_c\) to \(\zeta_o\).

From the definition of the path \(\mathcal{D}\), it is easily seen that

\[
\Re(\cos \zeta) = \cos(\Re(\zeta)) / \cos(\Re(\zeta) - \theta_0), \quad \Re(\zeta) - \theta_0 \in (-\pi/2, \pi/2) \quad \text{for} \quad \zeta \in \mathcal{D}.
\]
Then it easily follows that: for \( \zeta \in \mathcal{D} \), \( \text{Re}(\cos(\zeta)) \) is monotonously decreasing as \( \text{Re}(\zeta) \) increases. This, together with the fact that \( \text{Re}(\cos(\zeta_0)) = n \) (see Remark 2.3), implies that \( \text{Re}(\cos(\zeta)) < n \) for \( \zeta \in \mathcal{D}_c, \frac{x}{2} + \theta_x - i\infty \) and \( \text{Re}(\cos(\zeta)) > n \) for \( \zeta \in \mathcal{D}_c, \frac{x}{2} + \theta_x + i\infty, \zeta_0 \), where the path \( \mathcal{D}_c, \frac{x}{2} + \theta_x - i\infty \) denotes the part of the path \( \mathcal{D} \) starting from \( \zeta_0 \) and ending at \( \frac{x}{2} + \theta_x - i\infty \), and the path \( \mathcal{D}_c, \frac{x}{2} + \theta_x + i\infty, \zeta_0 \) denotes the part of the path \( \mathcal{D} \) starting from \( -\frac{x}{2} + \theta_x + i\infty \) and ending at \( \zeta_0 \). Thus, it follows from the definitions of the functions \( \tilde{S} \) and \( S \) that

\[
\begin{align*}
\tilde{S}(\cos(\zeta), n) &= S(\cos(\zeta), n), & \zeta \in \mathcal{D}_c, \frac{x}{2} + \theta_x + i\infty, \zeta_0, \\
\tilde{S}(\cos(\zeta), n) &= -S(\cos(\zeta), n), & \zeta \in \mathcal{D}_c, \frac{x}{2} + \theta_x - i\infty.
\end{align*}
\]

(2.26)

From this, it is deduced that

\[
(2.27) \quad G_{R, 2}^{(1)}(x, y) = G_{R, 3}^{(2)}(x, y) + G_{R, 2}(x, y),
\]

where

\[
(2.28) \quad G_{R, 2}(x, y) := \frac{i}{2\pi} \int_{\mathcal{D}_c, \frac{x}{2} + \theta_x - i\infty} 2i\sin\zeta S(\cos(\zeta), n) e^{ik_+(-|y| \cos(\zeta+\theta_x) + |x| \cos(\zeta-\theta_x))} d\zeta.
\]

Using (2.22) and (2.23), we have that for \( \zeta \in \{ \zeta \in \mathbb{C} : \text{Im}(\zeta) < 0, \frac{x}{2} - (\theta_x - \theta_y) \leq \text{Re}(\zeta) \leq \frac{x}{2} + \theta_x \}, \)

\[
\begin{align*}
\left| \frac{2i\sin\zeta S(\cos(\zeta), n)}{n^2 - 1} e^{ik_+(-|y| \cos(\zeta+\theta_x) + |x| \cos(\zeta-\theta_x))} \right| &
\leq C \left[ 1 + (|\sinh(\text{Im}(\zeta))| + \cosh(\text{Im}(\zeta)))^2 \right] e^{ik_+ \sinh(\text{Im}(\zeta))(|x| \cos(\theta_x) - |y|)},
\end{align*}
\]

where \( C > 0 \) is a constant independent of \( \zeta \). From this, (2.28) and the fact that \( S(\cos(\zeta), n) \) is analytic for \( \zeta \in \mathcal{I}_2 \) with \( \mathcal{I}_2 \) defined as in Case 1, we can apply Cauchy integral theorem to obtain that: for \( |x| > |y| / \cos(\theta_x) \),

\[
(2.29) \quad G_{R, 2}(x, y) = \frac{i}{2\pi} \int_{\mathcal{I}_c, \frac{x}{2} - i\infty} 2i\sin\zeta S(\cos(\zeta), n) e^{ik_+(-|y| \cos(\zeta+\theta_x) + |x| \cos(\zeta-\theta_x))} d\zeta,
\]

where the path \( \mathcal{I}_c, \frac{x}{2} - i\infty \) denotes the curve \( \{ \zeta \in \mathcal{I}_c^+ : \text{Re}(\zeta) \in (\text{Re}(\zeta_0), \frac{x}{2}) \} \) with the orientation from \( \zeta_0 \) to \( \frac{x}{2} - i\infty \). Thus it easily follows from (2.2), (2.25) and (2.29) that: for \( |x| > |y| / \cos(\theta_x) \),

\[
(2.30) \quad G_{R}^{(4)}(x, y) = G_{R, 1}(x, y) + G_{R, 2}(x, y).
\]

This, together with (2.24) and (2.27), implies that the formula (2.17) holds. The proof is thus complete.

Remark 2.5. As mentioned in Remark 2.2, \( ik_+ \cos(\zeta - \theta_x) \) has only one saddle point \( \zeta = \theta_x \) on the steepest descent path \( \mathcal{D} \). Note that \( ik_+ \cos(\zeta - \theta_x) \) appears in the exponential functions in both (2.16) and (2.18). Thus, by the steepest descent method, the asymptotic expansions of \( G_{R, 1}^{(2)}(x, y) \) and \( G_{R, 2}^{(2)}(x, y) \) for large \( |x| \) are expected to depend on the series expansions of \( S(\cos(\zeta), n) \) and \( \tilde{S}(\cos(\zeta), n) \), respectively, at the
saddle point $\zeta = \theta_+$ (see [4, Section 7.3]), where $S(\cos \zeta, n)$ and $\tilde{S}(\cos \zeta, n)$ appear in the amplitude functions in (2.16) and (2.18), respectively. However, since $\zeta = \theta_+$ is a branch point of $S(\cos \zeta, n)$ and $\tilde{S}(\cos \zeta, n)$ (see Remark 2.3), the modulus of every derivative of $S(\cos \zeta, n)$ and $\tilde{S}(\cos \zeta, n)$ at the saddle point $\zeta = \theta_+$ will blow up when the angle $\theta_+ \to \theta_+$. This leads to difficulties in the investigation of the uniform far-field asymptotics of $G_R^{(2)}(x, y)$ and $G_R^{(3)}(x, y)$ for the angles $\theta_+$ very close to the branch point $\theta_+$. On the other hand, because of the presence of the branch cuts, we need to analyze not only the asymptotics of the integrals $G_R^{(j)}(x, y)$ (j = 1, 2, 3) over the steepest descent path $D$ but also the asymptotics of the integral $G_R^{(4)}(x, y)$ over the loop $L_\circ$. This leads to another difficulty in investigating the uniform far-field asymptotics of $G_R(x, y)$, since the loop $L_\circ$ is around the branch cut $L^+_{\circ}$ of $S(\cos(-\cdot), n)$ appearing in the amplitude function in (2.19).

Remark 2.6. The steepest descent path $D$ was also used in [9, formula (9)] for deriving the uniform far-field asymptotic expansion of the Green function for the Helmholtz equation in a half-plane with impedance boundary condition. See [9, Section 2] for an equivalent definition of $D$. The integrand in [9, formula (9)] is analytic in the complex plane except for an infinite countable number of poles (see the discussion following [9, formula (7)]). In our case, however, due to the analyticity of $S(\cos(-\cdot), n)$ and $\tilde{S}(\cos(-\cdot), n)$ (see Remark 2.3), the integrands in (2.16) and (2.18) are analytic in the complex plane except for an infinite countable number of branch cuts. Therefore, the asymptotic analysis for [9, formula (9)] does not work for the study of the uniform far-field asymptotics of $G_R^{(2)}(x, y)$ and $G_R^{(3)}(x, y)$ for the angles $\theta_+$ very close to the branch point $\theta_+$, which is difficult as discussed in Remark 2.5.

To overcome the difficulties mentioned in Remark 2.5, in what follows we will introduce some useful integral identities (see Lemma 2.7 below) and give a rigorous analysis of the singularities of $S(\cos(-\cdot), n)$ and $\tilde{S}(\cos(-\cdot), n)$ (see Lemmas 2.9, 2.10 and 2.11 below).

For the subsequent use, we need some special functions. Let $D_\beta(z)$ denote the parabolic cylinder function and let $\Gamma(z)$ denote the Gamma function. We refer to [33] for the definitions and properties of $D_\beta(z)$ and $\Gamma(z)$. In particular, it is known that $\Gamma(z)$ is analytic except at the points $z = 0, -1, -2, \ldots$ (see, e.g., Section 12.1 in [33]) and $D_\beta(z)$ is analytic in $z \in \mathbb{C}$ for any $\beta \in \mathbb{R}$ (see, e.g., Section 16.5 in [33]). The following lemma presents some useful results relevant to $D_\beta(z)$ and $\Gamma(z)$, which are given in [6].

Lemma 2.7 (see formulas (A.3.25) and (A.3.26) in [6]). Assume $\beta \in \mathbb{R}$ with $\beta > -1$ and $\rho, b \in \mathbb{C}$ with $\text{Im}(\rho) > 0$, $\text{Im}(b) \neq 0$. Consider the integrals $F_2(\rho, b, \beta) := \int_\gamma^+ (s - b)^\beta e^{ips^2} ds$ and $F_3(\rho, b, \beta) := \int_{\gamma_2}^+ (s - b)^\beta e^{ips^2} ds$, where $\gamma_2$ is a loop around the branch cut $\{z \in \mathbb{C} : \text{Im}(z) = \text{Im}(b), \text{Re}(z) \leq \text{Re}(b)\}$ depicted in Figure 4. Then, we have

\begin{equation}
F_2(\rho, b, \beta) = \begin{cases}
  e^{i\rho b^2/2} \sqrt{2\pi} \left( \frac{1}{2\rho} \right)^{\beta+1} e^{i\pi(\beta+1)/4} D_\beta(\sqrt{2\rho b e^{i\pi/4}}), & \text{Im}(b) < 0, \\
  e^{i\rho b^2/2} \sqrt{2\pi} \left( \frac{1}{2\rho} \right)^{\beta+1} e^{i\pi(-\beta+1)/4} D_\beta(\sqrt{2\rho b e^{i\pi/4}}), & \text{Im}(b) > 0,
\end{cases}
\end{equation}

\begin{equation}
F_3(\rho, b, \beta) = \begin{cases}
  e^{i\rho b^2/2} \sqrt{2\pi} \left( \frac{1}{2\rho} \right)^{\beta+1} e^{i\pi(\beta+1)/4} D_\beta(\sqrt{2\rho b e^{i\pi/4}}), & \text{Im}(b) < 0, \\
  e^{i\rho b^2/2} \sqrt{2\pi} \left( \frac{1}{2\rho} \right)^{\beta+1} e^{i\pi(-\beta+1)/4} D_\beta(\sqrt{2\rho b e^{i\pi/4}}), & \text{Im}(b) > 0.
\end{cases}
\end{equation}
we get the following lemma.

\[ \theta \quad \text{hold.} \]

where

\[ (2.33) \]

and

\[ F_3(\rho, b, \beta) = \text{sgn}(\text{Im}(b))e^{i\rho} \left( \frac{1}{2\rho} \right) \frac{2\pi e^{-\pi i/2}}{\Gamma(-\beta)} D_{-\beta-1}(\sqrt{2\rho}b e^{i\theta}), \]

where \text{sgn} denotes the sign function.

Remark 2.8. There was a typo in [6, formula (A.3.25)] for the integral \( F_2(\rho, b, \beta) \) with \( \text{Im}(b) < 0 \) and [6, formula (A.3.25)] was given without a proof. In formula (2.31), we give the correct expression for this typo. For the proof of formula (2.31), see Appendix A.

To proceed further, we need the following Lemmas 2.9 and 2.10. The proofs of these two lemmas will be given in Appendices B.1 and B.2, respectively.

Lemma 2.9. Let \( f(s) := a_0 + a_1 P(s) + a_2 Q(s) \), \( s \in \mathbb{C} \), where the constants \( a_0 \geq 0, a_1 > 0, a_2 \in \mathbb{R} \), and the functions \( P(s) \) and \( Q(s) \) are given by

\[ P(s) := \sqrt{1 - \frac{s^2}{2k_+}}, \quad Q(s) := \frac{se^{-i\frac{\pi}{4}}}{\sqrt{2k_+}}. \]

Define \( w_0 := a_1 \sqrt{k_+}/\sqrt{a_1^2 + a_2^2} \). Then the following statements hold.

1. When \( a_2 \geq 0 \), we have \( \{f(s) : s \in L_{-w_0, \sqrt{k_+}} \} \cap \{s \in \mathbb{R} : s \leq 0\} = \emptyset \).
2. When \( a_2 < 0 \), we have \( \{f(s) : s \in L_{-\sqrt{k_+}, w_0} \} \cap \{s \in \mathbb{R} : s \leq 0\} = \emptyset \).

Lemma 2.10. Let \( A(z) \) and \( B(z) \) be two analytic functions in a strip \( L_w \) with \( w > 0 \). If \( A^2(z) = B^2(z) \) for \( z \in L_w \) and \( A(0) = B(0) \neq 0 \). Then we have \( A(z) = B(z) \) for \( z \in L_w \).

Let \( \arcsin(z) := \int_0^z \frac{1}{\sqrt{1-t^2}} dt \) denote the principal value of the inverse of the sine function in the complex plane, where the path of integration must not cross the branch cuts \( \{z \in \mathbb{R} : |z| \geq 1\} \) (see, e.g., Section 4.23 in [28]). It is known that \( \arcsin(z) \) is analytic in the domain \( \mathbb{C} \setminus \{z \in \mathbb{R} : |z| \geq 1\} \). Then, employing Lemmas 2.9 and 2.10 we get the following lemma.

Lemma 2.11. Let \( P(s) \) and \( Q(s) \) be the functions defined in Lemma 2.9. Assume \( \theta \in (0, \pi/2) \) and define \( \zeta(s) := 2\arcsin(Q(s)) + \theta \). Then the following statements hold.

1. \( \zeta(s) \) is analytic in the strip \( L_{\sqrt{k_+}} \) and satisfies

\[ (2.34) \quad P(s) = \cos \left( \frac{\zeta(s) - \theta}{2} \right), \quad Q(s) = \sin \left( \frac{\zeta(s) - \theta}{2} \right) \quad \text{for} \ s \in L_{\sqrt{k_+}}. \]
Moreover, \( \zeta(s) \) is a bijection from \( \mathbb{R} \) onto \( \mathcal{D} \) and \( \zeta(s) \) travels from \(-\pi/2 + \theta_z + i\infty\) to \(\pi/2 + \theta_z - i\infty\) on the path \( \mathcal{D} \) as \( s \) travels from \(-\infty\) to \(+\infty\) along the real axis.

2. In the case \( \theta_z \in (\theta_c, \pi/2) \), we have

\[
\sqrt{2/k_+e^{-\frac{i\pi}{k_+}}} H_{\theta_z}(s) H_{\pi-\theta_z}(s) \sqrt{s - s_b} \sqrt{s - s_b} = S(\cos(\zeta(s)), n) \quad \text{for } s \in \mathbb{R}
\]

and in the case \( \theta_z \in (0, \theta_c) \), we have

\[
-\sqrt{2/k_+e^{-\frac{i\pi}{k_+}}} H_{\theta_z}(s) H_{\pi-\theta_z}(s) \sqrt{s - s_b} \sqrt{s - s_b} = \tilde{S}(\cos(\zeta(s)), n) 
\]

where

\[
s_b := 2k_+ e^{\frac{i\pi}{k_+}} \sin \left( \frac{\theta_c - \theta_z}{2} \right), \quad s_b^* := 2k_+ e^{\frac{i\pi}{k_+}} \sin \left( \frac{\pi - \theta_c - \theta_z}{2} \right) \]

and \( H_{\theta}(s) := \sqrt{F^{(1)}_{\theta}(s)} \sqrt{F^{(2)}_{\theta}(s)} \sqrt{F^{(3)}_{\theta}(s)} \) for \( \theta \in (0, \pi) \) with \( F^{(j)}_{\theta}(s) \) \((j = 1, 2, 3)\) given by

\[
F^{(1)}_{\theta}(s) := 1 + P(s) \cos \left( \frac{\theta - \theta_z}{2} \right) + Q(s) \sin \left( \frac{\theta - \theta_z}{2} \right),
\]

\[
F^{(2)}_{\theta}(s) := P(s) \sin \left( \frac{\theta + \theta_z}{2} \right) + Q(s) \cos \left( \frac{\theta + \theta_z}{2} \right),
\]

\[
F^{(3)}_{\theta}(s) := \cos \left( \frac{\theta - \theta_z}{2} \right) + P(s).
\]

Moreover, \( H_{\theta_z}(s) \) and \( H_{\pi-\theta_z}(s) \) are analytic in the strip \( L_{\sigma_{\theta_z}} \) with \( \sigma_{\theta_z} := \sqrt{k_+ \min(\sin((\theta_c + \theta_z)/2), \cos((\theta_c - \theta_z)/2))} \).

3. For any \( \theta_z \in (0, \pi/2) \), the function \( H_{\theta_z}(s) H_{\pi-\theta_z}(s) \sqrt{s - s_b} \) is analytic in \( s \in L_{\sigma_{\theta_z}}^{(1)} \) with \( \sigma_{\theta_z}^{(1)} := \min(\sigma_{\theta_z}, \sqrt{k_+ \sin((\pi - \theta_c - \theta_z)/2))} \) and the function \( H_{\theta_z}(s) H_{\pi-\theta_z}(s) \sqrt{s - s_b} \sqrt{s - s_b} \) is analytic in \( s \in L_{\sigma_{\theta_z}}^{(2)} \) with \( \sigma_{\theta_z}^{(2)} := \sqrt{k_+ \sin((\theta_c - \theta_z)/2))} \). Moreover, we have

\[
0 < \sigma_{\min}^{(1)} \leq \sigma_{\theta_z}^{(1)} \leq \sigma_{\max}^{(1)} < \sqrt{k_+}
\]

with \( \sigma_{\min}^{(1)} := \min(\sqrt{k_+ \sin(\theta_c/2)}, \sqrt{k_+ \sin(\pi/2 - \theta_c/2)}) \) and \( \sigma_{\max}^{(1)} := \sqrt{k_+} \).

Proof. (1) From Section 4.15 in [28], it can be seen that \( \sin(z) \) is a conformal mapping from the domain \( \{z \in \mathbb{C} : -\pi/2 < \text{Re}(z) < \pi/2\} \) to the domain \( \mathbb{C}\{z \in \mathbb{R} : |z| \geq 1\} \) and its inverse is given by \( \text{arc}(\sin(z)) \). Further, it is clear that \( \text{Re}(\sqrt{z}) > 0 \) for \( z \in \mathbb{C}_0 \) and \( \text{Re}(\sin(z)) > 0 \) for \( z \in \{z \in \mathbb{C} : -\pi/2 < \text{Re}(z) < \pi/2\} \). These facts imply that \( \zeta(s) \) is analytic in \( L_{\sqrt{k_+}} \) and the formula (2.34) holds true. Moreover, from the formula (2.13), it follows that the mapping \( F(z) := \sqrt{2k_+e^{\frac{i\pi}{k_+}}} \sin((z - \theta_z)/2) \) is a bijection from \( \mathcal{D} \) to \( \mathbb{R} \) and \( F(z) \) travels from \(-\infty\) to \(+\infty\) along the real axis as \( z \) travels from \(-\pi/2 + \theta_z + i\infty\) to \(\pi/2 + \theta_z - i\infty\) on the path \( \mathcal{D} \). Hence, the proof of statement (1) is now completed from the fact that \( s = \sqrt{2k_+e^{\frac{i\pi}{k_+}}} \sin((\zeta(s) - \theta_z)/2) \) for \( s \in \mathbb{R} \) (see the second formula in (2.34)).
(2) From Lemma 2.9, we have \( F^{(3)}_{\theta_c}(s) \neq 0 \) for \( s \in L_{\sqrt{k_+}} \). Thus it follows from (2.34) that for \( s \in L_{\sqrt{k_+}} \).

\[
\left( \frac{2k_+ e^{i \pi/4}}{s} \right)^{-1} (\cos \zeta(s) - \cos \theta_c) = 4 \sin \left( \frac{\theta_c - \zeta(s)}{4} \right) \cos \left( \frac{\theta_c - 2\theta_c + \zeta(s)}{4} \right) \cos^2 \left( \frac{\theta_c - \zeta(s)}{4} \right) \sin \left( \frac{\zeta(s) + \theta_c}{2} \right) \\
= \left( \sin \left( \frac{\zeta(s) - \theta_c}{2} \right) - \sin \left( \frac{\theta_c - \theta_\epsilon}{2} \right) \right) \frac{\sqrt{2} e^{i \pi/4} \left( 1 + \cos \left( \frac{\theta_c - \zeta(s)}{2} \right) \right) \sin \left( \frac{\theta_c + \zeta(s)}{2} \right)}{\sqrt{k_+} \left( \cos \left( \frac{\theta_c - \theta_\epsilon}{2} \right) + \cos \left( \frac{\zeta(s) - \theta_c}{2} \right) \right)} \\
= i(s - s_0) \frac{F^{(1)}_{\theta_c}(s) F^{(2)}_{\theta_c}(s)}{F^{(3)}_{\theta_c}(s)} k_+^{-1}.
\]

Similarly as above, for \( s \in L_{\sqrt{k_+}} \), we have \( F^{(3)}_{\pi - \theta_c}(s) \neq 0 \) and

\[
\cos \zeta(s) + \cos \theta_c = (s - s_0^2) \frac{F^{(1)}_{\pi - \theta_c}(s) F^{(2)}_{\pi - \theta_c}(s)}{F^{(3)}_{\pi - \theta_c}(s)} \frac{\sqrt{2} e^{i \pi/4}}{\sqrt{k_+}}.
\]

Using Lemma 2.9 again, we obtain that \( \sqrt{F^{(3)}_{\theta_c}(s)} \) and \( \sqrt{F^{(3)}_{\pi - \theta_c}(s)} \) (\( j = 1, 2, 3 \)) are analytic functions in the strip \( L_{s_\theta} \). Thus, it follows that \( H_{\theta_c}(s) \) and \( H_{\pi - \theta_c}(s) \) are analytic in \( L_{s_\theta} \).

Now we prove that the formula (2.35) holds for the case \( \theta_\epsilon \in (\theta_c, \pi/2) \). From Lemma 2.9 and (2.34), it follows that \( \zeta'(s) \) is analytic in \( s \in L_{\sqrt{k_+}} \) and is given by

\[
(2.38) \quad \zeta'(s) = \sqrt{2/k_+} e^{-i \pi/4} / P(s), \ s \in L_{\sqrt{k_+}}.
\]

This, together with the definition of \( P(s) \), implies that \( |\zeta'(s)| \) is uniformly bounded for \( s \in L_{\sqrt{k_+}} / 2 \). Further, for any fixed \( \theta_\epsilon \in (\theta_c, \pi/2) \), it can be seen that \( \mathcal{D} = \{ \zeta(s) : s \in \mathbb{R} \} \subset II \) and the distance between \( \mathcal{D} \) and the boundary of \( II \) has a positive lower bound, where \( II \) is defined as in the proof of Lemma 2.4. Thus, it easily follows from Taylor formula and Cauchy inequality (see, e.g., [32, Corollary 4.3]) that, for any fixed \( \theta_\epsilon \in (\theta_c, \pi/2) \), there exists small enough \( \varepsilon_{\theta_\epsilon} > 0 \) such that \( \{ \zeta(s) : s \in L_{s_{\theta_\epsilon}} \} \subset II \). By this and Remark 2.3, we obtain that \( \mathcal{S}(\cos(\zeta(s)), n) \) with \( \theta_\epsilon \in (\theta_c, \pi/2) \) is analytic in \( s \in L_{s_{\theta_\epsilon}} \). Clearly, \( \sqrt{s - s_\theta_0} \) and \( s - s_0^2 \) are analytic in \( s \in L_{\arg(s_\theta)} \) and in \( s \in L_{\arg(s_0^2)} \), respectively. Moreover, by a straightforward calculation, we have

\[
(2.39) \quad \frac{\sqrt{2}}{\sqrt{k_+} e^{i \pi/4}} \sqrt{-s_0} - s_0^2 H_{\theta_c}(0) H_{\pi - \theta_c}(0) = \mathcal{S}(\cos(\zeta(0)), n) = -i \sqrt{n^2 - \cos^2 \theta_\epsilon} \neq 0 \quad \text{for } \theta_\epsilon \in (\theta_c, \pi/2).
\]

Hence, using the above arguments and applying Lemma 2.10, we obtain that the formula (2.35) holds for the case \( \theta_\epsilon \in (\theta_c, \pi/2) \).
Next, we prove that the formula (2.36) holds for the case \( \theta_x \in (0, \theta_c) \). Let \( \theta_x \in (0, \theta_c) \) in the rest proof of statement (2). Denote by \( q_1(z) := \sin((z - \theta_x)/2) \) and \( q_2(z) := \cos((z - \theta_x)/2) \). A straightforward calculation gives that

\[
\text{Im} \left( \sqrt{2k_+ e^{iz} q_1(z)} \right) = \sqrt{k_+} \left( q_1(\text{Re}(z)) \cosh \left( \frac{\text{Im}(z)}{2} \right) + q_2(\text{Re}(z)) \sinh \left( \frac{\text{Im}(z)}{2} \right) \right).
\]

(2.40)

It is easily seen that \( q_1(\text{Re}(z)) \cosh \left( \frac{\text{Im}(z)}{2} \right) \) and \( q_2(\text{Re}(z)) \sinh \left( \frac{\text{Im}(z)}{2} \right) \) are both nonnegative or both nonpositive in \( z \in A_2 \cup A_3 \), where \( A_2 \) and \( A_3 \) are defined in Remark 2.3. This, together with (2.40), implies that

\[
\left| \text{Im} \left( \sqrt{2k_+ e^{iz} q_1(z)} \right) \right| \geq \sqrt{k_+} \left| q_1(\text{Re}(z)) \cosh \left( \frac{\text{Im}(z)}{2} \right) \right| \geq |\text{Im}(s_b)| \quad \text{for } z \in A_2 \cup A_3.
\]

(2.41)

Thus, combining (2.41) and the second formula in (2.34), we deduce that

\[
\{ \zeta(s) : s \in L_{|\text{Im}(s_b)|} \} \cap (A_2 \cup A_3) = \emptyset.
\]

Hence, it follows from Remark 2.3 that \( \tilde{S}(\cos(\zeta(s)), n) \) is analytic in \( s \in L_{|\text{Im}(s_b)|} \).

Further, it is easy to verify that

\[
-\sqrt{2/k_+ e^{-iz} \sqrt{-s_0} \sqrt{-s_b} H_{\theta_x}(0) H_{\pi - \theta_x}(0)} = \tilde{S}(\cos(\zeta(0)), n) = \sqrt{\cos^2 \theta_x - n^2} \neq 0
\]

for \( \theta_x \in (0, \theta_c) \). Therefore, employing Lemma 2.10 again, we obtain that the formula (2.36) holds for the case \( \theta_x \in (0, \theta_c) \).

(3) For \( \theta_x \in (0, \pi/2) \), it is clear that \( \sigma^{(2)}_{\theta_x} \leq \sigma^{(1)}_{\theta_x} \) and \( \sigma^{(1)}_{\theta_x} = \sqrt{k_+} \min(\sin((\theta_x + \theta_x)/2), \cos((\theta_x + \theta_x)/2)) \). Thus by statement (2), the analyticity of \( \sqrt{s - s_b} \) and \( \sqrt{s - s_b} \) and a direct calculation, we can obtain that statement (3) holds.

With the aid of Lemmas 2.7 and 2.11, we are now ready to study the uniform farfield asymptotic estimates of the functions \( G_R^{(j)}(x, y) \) \((j = 1, 2, 3, 4)\) defined in Lemma 2.4. We mention that in the proofs of Lemmas 2.12 and 2.13 below, an important role is played by the technically involved singularity analysis of the relevant integrals for \( G_R(x, y) \) when the angle \( \theta_x \) is very close to the branch point \( \theta_c \).

**Lemma 2.12.** Assume that \( k_+ > k_- \) and let \( R_0 > 0 \) be an arbitrary fixed number. Suppose that \( y = |y| (\cos \theta_y, \sin \theta_y) \in B^+_R \) with \( \theta_y \in (0, \pi) \) and \( x = |x| (\cos \theta_x, \sin \theta_x) \in \mathbb{R}^2_+ \) with \( \theta_x \in [\theta_c, \pi/2) \), then we have the asymptotic behavior

\[
G_R^{(1)}(x, y) = e^{ik_+ |x|} e^{iz} \sqrt{8\pi k_+} \left( \frac{2 \cos^2 \theta_x - 1 - n^2}{n^2 - 1} \right) e^{-ik_+ |y| \cos(\theta_x + \theta_y)} + G_R^{(1), \text{Res}}(x, y)
\]

(2.42)

and

\[
G_R^{(2)}(x, y) = e^{ik_+ |x|} e^{iz} \sqrt{8\pi k_+} \left( \frac{2 \sin \theta_x \tilde{S}(\cos \theta_x, n)}{n^2 - 1} \right) e^{-ik_+ |y| \cos(\theta_x + \theta_y)} + G_R^{(2), \text{Res}}(x, y),
\]

(2.43)
where $G^{(1)}_{R, \text{Res}}(x, y)$ and $G^{(2)}_{R, \text{Res}}(x, y)$ satisfy

\begin{align}
|G^{(1)}_{R, \text{Res}}(x, y)| \leq C_{R_0} |x|^{-3/2}, & \quad |x| \to +\infty, \\
|G^{(2)}_{R, \text{Res}}(x, y)| \leq C_{R_0} |x|^{-3/4}, & \quad |x| \to +\infty,
\end{align}

uniformly for all $\theta_0 \in [\theta_c, \pi/2)$ and $y \in B^+_R$, and that

\begin{align}
|G^{(2)}_{R, \text{Res}}(x, y)| \leq C_{R_0} |\theta_c - \theta_0| |x|^{-3/4}, & \quad |x| \to +\infty,
\end{align}

uniformly for all $\theta_0 \in (\theta_c, \pi/2)$ and $y \in B^+_R$. Here, the constant $C_{R_0} > 0$ is independent of $x$ and $y$ but dependent of $R_0$.

Proof. Let $|x|$ be sufficiently large throughout the proof. As mentioned in Remark 2.3, the steepest descent path $\mathcal{D}$ will not cross the branch cuts of $\mathcal{S}(\cos \zeta, n)$ for the case $\theta_0 \in (\theta_c, \pi/2)$, and will cross the branch cuts of $\mathcal{S}(\cos \zeta, n)$ with only one intersection point $\theta_0$ for the case $\theta_0 = \theta_c$. Consequently, we will apply the steepest descent method (see, e.g., [4, 6, 9]) to rewrite $G^{(j)}_R(x, y) (j = 1, 2)$ as integrals over the real axis. By statement (1) of Lemma 2.11, we introduce the change of variable $\zeta = \zeta(s)$ to rewrite $G^{(j)}_R(x, y) (j = 1, 2)$ as

\begin{equation}
G^{(j)}_R(x, y) = \frac{ie^{ik_+|x|}}{4\pi} \int_{-\infty}^{+\infty} v_j(s) F(s) \frac{d\zeta(s)}{ds} e^{-i|x|s^2} ds,
\end{equation}

where $\zeta(s)$ is defined as in Lemma 2.11 and

\begin{align}
v_1(s) & := \cos(2\zeta(s)) - \frac{n^2}{n^2 - 1}, \quad v_2(s) := \frac{2i \sin \zeta(s) \mathcal{S}(\cos(\zeta(s)), n)}{n^2 - 1}, \\
F(s) & := e^{-ik_+|y| \cos(\zeta(s) + \theta_0)}.
\end{align}

Next, the proof is divided into two steps. The first step is to estimate $G^{(2)}_R(x, y)$ and the second step is to estimate $G^{(1)}_R(x, y)$.

**Step 1:** Estimates of $G^{(2)}_R(x, y)$. From Lemma 2.11 and formula (2.47), it is easily seen that $G^{(2)}_R(x, y)$ can be rewritten as

\begin{equation}
G^{(2)}_R(x, y) = \frac{ie^{ik_+|x|}}{4\pi} \int_{-\infty}^{+\infty} \sqrt{s - s_0} g(s) e^{-i|s|s^2} ds,
\end{equation}

where $g(s)$ is given by

\begin{equation}
g(s) := \sqrt{2/k_+} e^{-i\pi/4} H_{\theta_0}(s) H_{\theta_0}(s) \sqrt{s - s_0^2} F(s) \frac{d\zeta(s)}{ds} \frac{2i \sin \zeta(s)}{n^2 - 1}
\end{equation}

and is an analytic function in the strip $L_{\sigma_{\theta_0}^{(1)}}$ with $\sigma_{\theta_0}^{(1)}$ defined in Lemma 2.11. For $s \in L_{\sqrt{k_+}}$, with the aid of (2.34), we have

\begin{align}
\sin(\zeta(s) - \theta_0) = 2P(s)Q(s), \quad \cos(\zeta(s) - \theta_0) = 2P^2(s) - 1,
\end{align}

which implies that

\begin{align}
\sin(\zeta(s)) = 2P(s)Q(s) \cos(\theta_\zeta + (2P^2(s) - 1) \sin(\theta_\zeta), \\
\cos(\zeta(s) + \theta_\zeta) = (2P^2(s) - 1) \cos(\theta_\zeta + \theta_0) - 2P(s)Q(s) \sin(\theta_\zeta + \theta_0).
\end{align}
Note that \( \text{Re}(P(s)) > 0 \) in \( s \in \mathbb{L}_{k_{+}}^{-} \), \( \min[\cos((\theta_{c} - \theta_{\pm})/2), \cos((\pi - \theta_{c} - \theta_{\pm})/2)] \geq \sin(\theta_{c}/2) \) and \( \sigma_{\theta_{\pm}}^{(1)} \leq \sigma_{\max}^{(1)} < \sqrt{k_{+}} \) (see statement (3) of Lemma 2.11). Thus we have \( |F_{\theta_{\pm}}^{(3)}(s)| \geq C(1 + |s|), |F_{\pi - \theta_{\pm}}^{(3)}(s)| \geq C(1 + |s|) \) and \( C(1 + |s|) \leq |P(s)| \leq 2(1 + |s|)/\sqrt{2k_{+}} \) in \( s \in \mathbb{L}_{\sigma_{\theta_{\pm}}^{(1)}}^{-} \) for some constant \( C > 0 \). Hence, by using the formula (2.38), we have:

\[
(2.51) \quad |g(s)| \leq C_{R_{0}}(1 + |s|)^{5/2} e^{\tilde{C}_{R_{0}}|s|^{2}}, \quad s \in \mathbb{L}_{\sigma_{\theta_{\pm}}^{(1)}},
\]

where the positive constants \( C_{R_{0}} \) and \( \tilde{C}_{R_{0}} \) are independent of \( \theta_{\pm} \). Then it follows from Cauchy inequality (see, e.g., [32, Corollary 4.3]), (2.37) and the estimate (2.51) that for \( n = 0, 1, 2, \ldots \),

\[
\frac{d^{5} g}{ds^{5}}(s) \leq \frac{C_{R_{0}n!(1 + \sigma_{\theta_{\pm}}^{(1)} + |s|)^{5/2} e^{\tilde{C}_{R_{0}}n!(1 + |s|)^{2}}}}{\sigma_{\theta_{\pm}}^{(1)}}, \quad s \in \mathbb{R}.
\]

The rest proof of this step is divided into the following two parts.

**Part I:** We prove that \( G_{R}^{(2)}(x, y) \) has the form (2.43) with \( G_{R, \text{Rez}}^{(2)}(x, y) \) satisfying (2.46) uniformly for all \( \theta_{\pm} \in (\theta_{c}, \pi/2) \) and \( y \in B_{R_{0}}^{+} \).

Assume \( \theta_{\pm} \in (\theta_{c}, \pi/2) \). Let \( G(s) = \sqrt{s - s_{b}} g(s) \) and choose \( \sigma = \sigma_{\theta_{\pm}}^{(2)} \) with \( \sigma_{\theta_{\pm}}^{(2)} \) defined in Lemma 2.11. By statement (3) of Lemma 2.11, \( G(s) \) is an analytic function in the strip \( \mathbb{L}_{\sigma} \). Define the function \( J(s) := G(s) - G(0) - G'(0)s \). Then it easily follows from (2.39) and (2.49) that

\[
G_{R}^{(2)}(x, y) = \frac{e^{ik_{+}|x|}}{4\pi} \left( \int_{-\infty}^{+\infty} G(0)e^{-|x|s^{2}} ds + \int_{-\infty}^{+\infty} J(s)e^{-|x|s^{2}} ds \right)
\]

\[
(2.53) = \frac{e^{ik_{+}|x|}}{\sqrt{|x|}} \frac{e^{i\pi/2} 2iS(\cos(\theta_{\pm}, \sigma)n)}{\sqrt{2\pi k_{+}}^{n+1}} \sin\theta_{\pm} e^{-ik_{+}|y|} e^{-|y|n(\cos(\theta_{\pm}, \sigma)n)} + \frac{i e^{ik_{+}|x|}}{4\pi} (I_{1} + I_{2}),
\]

where \( I_{1} \) and \( I_{2} \) are given by

\[
I_{1} := \int_{-\sqrt{s}}^{\sqrt{s}} J(s)e^{-|x|s^{2}} ds, \quad I_{2} := \int_{\sqrt{s}}^{+\infty} (J(s) + J(-s))e^{-|x|s^{2}} ds.
\]

Next, we estimate \( I_{1} \). By mean-value theorem, we have: for any \( s \in \mathbb{R} \), there exist \( \alpha_{1}, \alpha_{2} \in [0, 1] \) such that

\[
\text{Re}(J(s)) = \text{Re} \left( \sqrt{\alpha_{1}s - s_{b}} g''(\alpha_{1}s) + \frac{g'(\alpha_{1}s)}{\sqrt{\alpha_{1}s - s_{b}}} - \frac{g(\alpha_{1}s)}{4(\alpha_{1}s - s_{b})^{2}} \right) s^{2}
\]

and

\[
\text{Im}(J(s)) = \text{Im} \left( \sqrt{\alpha_{2}s - s_{b}} g''(\alpha_{2}s) + \frac{g'(\alpha_{2}s)}{\sqrt{\alpha_{2}s - s_{b}}} - \frac{g(\alpha_{2}s)}{4(\alpha_{2}s - s_{b})^{2}} \right) s^{2}.
\]

Here, we note that \( \sqrt{\alpha_{j}s - s_{b}} \neq 0 \) \((j = 1, 2)\) due to \( \text{Im}(s_{b}) < 0 \). Further, it is easy to verify that \( |s - s_{b}| \geq \sigma \) for \( s \in \mathbb{R} \) and that \( \sigma < \sqrt{k_{+}} \). These, together with (2.37),
(2.51) and (2.52), yield that
\[
|J(s)| \leq C_{R_0} \left(1 + \sigma^{-1/2} + \sigma^{-3/2}\right) s^2 \leq C_{R_0} \sigma^{-3/2} s^2, \quad s \in (-\sqrt{\sigma}, \sqrt{\sigma}).
\]

Hence, we get
\[
(2.54) \quad |I_1| \leq \frac{C_{R_0}}{\sigma^{3/2}} \int_{-\infty}^{+\infty} s^2 e^{-|s|^2} ds = \frac{C_{R_0} \Gamma\left(\frac{3}{2}\right)}{2\sigma^{3/2} |x|^2}.
\]

For the estimate of \(I_2\), we introduce the following function
\[
\Psi(t) := \int_{\sqrt{\sigma}}^{t} (J(s) + J(-s)) e^{-r_0 s^2} ds,
\]
where \(r_0\) is a fixed positive constant satisfying \(r_0 > 2\tilde{C}_{R_0}\) with \(\tilde{C}_{R_0}\) given in (2.51).

By (2.51), it is deduced that
\[
\sup_{t > \sqrt{\sigma}} |\Psi(t)| \leq 2 \int_{-\infty}^{+\infty} \left(|G(s)| + |G(0)|\right) e^{-r_0 s^2} ds
\]
\[
\leq \int_{-\infty}^{+\infty} C_{R_0} (1 + |s|)^3 e^{-(r_0 - \tilde{C}_{R_0})s^2} ds \leq C_{R_0}.
\]

From this it follows that for \(|x|\) large enough,
\[
(2.55) \quad |I_2| = \left|\int_{\sqrt{\sigma}}^{+\infty} \Psi(s) e^{-|s|-r_0)^2} ds\right| = \left|2 \int_{\sqrt{\sigma}}^{+\infty} (|s| - r_0) s \Psi(s) e^{-(|s|-r_0)^2} ds\right|
\]
\[
\leq C_{R_0} \int_{\sqrt{\sigma}}^{+\infty} 2(|s| - r_0) s e^{-|s|-r_0)^2} ds \leq C_{R_0} e^{-|s|-r_0)\sigma \leq C_{R_0} e^{-|s|\sigma}.
\]

Thus, in terms of (2.43) and (2.53), we can apply (2.54) and (2.55) to obtain that
\[
|G_{R, R_{\text{res}}}(x, y)| = |(I_1 + I_2) / (4\pi)| \leq C_{R_0} \left((\sigma |x|)^{-3/2} + e^{-|x|\sigma}\right)
\]
\[
(2.56) \quad \leq \frac{C_{R_0}}{|\sin((\theta_\epsilon - \theta_\hat{x})/2)|^{\frac{3}{2}} |x|^{\frac{3}{2}}}
\]
for large enough \(|x|\). By using this and (2.53) it is easily obtained that \(G_{R, R_{\text{res}}}(x, y)\) has the form (2.43) with \(G_{R, R_{\text{res}}}(x, y)\) satisfying (2.46) uniformly for all \(\theta_\hat{x} \in (\theta_\epsilon, \pi/2)\) and \(y \in B_{R_0}^+\).

**Part II:** We prove that \(G_{R}(x, y)\) has the form (2.43) with \(G_{R, R_{\text{res}}}(x, y)\) satisfying (2.45) uniformly for all \(\theta_\hat{x} \in [\theta_\epsilon, \pi/2)\) and \(y \in B_{R_0}^+\). To do this, we consider the following three cases.

**Case 1:** \(\theta_\hat{x} \in (\theta_\epsilon, \pi/2)\) with \(|\sin((\theta_\epsilon - \theta_\hat{x})/2)| \geq (2\sqrt{\kappa_+ |x|})^{-1}\). The proof of this case can be easily obtained by using (2.53) and (2.56).

**Case 2:** \(\theta_\hat{x} \in (\theta_\epsilon, \pi/2)\) with \(|\sin((\theta_\epsilon - \theta_\hat{x})/2)| < (2\sqrt{\kappa_+ |x|})^{-1}\). Define \(g_1(s) := [g(s) - g(s_0)] / [s(s - s_0)]\). Due to the analyticity of \(g(s)\), it is easily seen that \(g_1(s)\) is analytic in \(s \in L_{\sigma_\hat{x}}^{(1)}\). Then following the idea in [3, Section
2] and [6, Section A.3.3], we can employ (2.52) and integration by parts to rewrite (2.49) as

\begin{equation}
G^{(2)}_R(x, y) = \frac{ie^{ik_x|x|}}{4\pi} \left\{ \int_{-\infty}^{\infty} \left[ g(s_b)\sqrt{s-s_b} + \frac{g(s_b) - g(0)}{s_b} (s-s_b)^2 \right] e^{-|x|^2 s} ds + G^{(2)}_{R,1}(x, y) \right\},
\end{equation}

where

\begin{equation}
G^{(2)}_{R,1}(x, y) := \frac{1}{2|x|} \int_{-\infty}^{\infty} \sqrt{s-s_b} \left( (s-s_b)g_1'(s) + 3 \frac{s}{2}g_1(s) \right) e^{-|x|^2 s} ds.
\end{equation}

It can be seen that \( g(s_b) = h_c(\theta_x) \) and \( \text{Im}(s_b) < 0 \), where the function \( h_c(\theta) \) is defined by

\begin{equation}
h_c(\theta) := \frac{2^2 e^{5\pi i/8} }{k^+_+ \left[ \cos \left( \frac{\theta_x - \theta}{2} \right) \right]^{3/2} (\tan \theta_c)^{1/2}} e^{-ik_y|\theta_c \cos(\theta_c + \theta_y)} , \quad \theta \in (0, \pi/2).
\end{equation}

Thus it follows from (2.31) that

\begin{equation}
G^{(2)}_R(x, y) = \frac{ie^{ik_x|x|}}{4\pi} \left\{ g(s_b)F_2(i|x|, s_b, 1/2) + \frac{g(s_b) - g(0)}{s_b} F_2(i|x|, s_b, 3/2) + G^{(2)}_{R,1}(x, y) \right\}
\end{equation}

\begin{equation}
= \frac{e^{11\pi i/8 + k_x|x|}}{\sqrt{\pi}|x|} \left[ \cos \left( \frac{\theta_x - \theta}{2} \right) \right]^{3/2} (\tan \theta_c)^{1/2} \left( D^\perp(u) + \frac{A-1}{u} D^\perp(u) \right)
\end{equation}

\begin{equation}
+ \frac{ie^{ik_x|x|}}{4\pi} \left[ G^{(2)}_{R,1}(x, y) \right],
\end{equation}

where \( A := g(0)/g(s_b) \) and \( u := 2\sqrt{k_x|x|} e^{3\pi i/4} \sin((\theta_x - \theta_x)/2) \).

In the remaining part of this case, we further assume that \( |x| \) is large enough so that \( |x| \geq 8 \left( \sigma_{\min}^{(1)} \right)^{-2} \) with \( \sigma_{\min}^{(1)} \) given in Lemma 2.11. Then, due to the assumption \( |\sin((\theta_x - \theta_x)/2)| < (2\sqrt{k_x|x|})^{-1} \) and Lemma 2.11, we have \( |s_b| < \sigma^{(1)}_{\theta_x}/4 \). Hence, we can use the analyticity of \( g(s) \) in \( L_{\sigma^{(1)}_{\theta_x}} \) and the Taylor formula to obtain that

\begin{equation}
g_1(s) = \frac{g(s) - g(0) - (g(s_b) - g(0))s/s_b}{s(s-s_b)}
\end{equation}

\begin{equation}
= \sum_{n=1}^{+\infty} \frac{d^n g(0)}{s^n} (s_b)^{n-1} / n! - \sum_{n=1}^{+\infty} \frac{d^n g(0)}{s^n} (s_b)^{n-1} / n!
\end{equation}

for \( |s| < \sigma^{(1)}_{\theta_x} \). These, together with (2.37) and (2.52), imply that

\begin{equation}
|g_1(s)| \leq \sum_{n=2}^{+\infty} (n-1) \left| \frac{d^n g(0)}{n!} \left( \frac{\sigma^{(1)}_{\theta_x}}{n!} \right)^{n-2} \right|
\end{equation}

\begin{equation}
\leq C_R_0 \left( \sigma^{(1)}_{\theta_x} \right)^{-2} \sum_{n=2}^{+\infty} \frac{n-1}{2^{n-2}} \leq C_R_0 \text{ for } |s| \leq \sigma^{(1)}_{\theta_x}/2.
\end{equation}
Further, similarly as above, we can apply the Taylor formula and (2.52) to obtain that

$$\tag{2.61} \left| g(0) - g(s_b) \right| / |s_b| \leq \sum_{n=1}^{+\infty} \frac{d^n g}{ds^n}(0)|s_b|^{n-1}/n! \leq C_{R_0} \sum_{n=1}^{+\infty} \left( \sigma^{(1)}_{\theta} \right)^{n-1} |s_b|^n \leq C_{R_0}. $$

From (2.37), (2.51), (2.61) and \( |s_b| < \sigma^{(1)}_{\theta} / 4 \), we have

$$\tag{2.62} \left| g_1(s) \right| \leq \left| g(s) - g(0) \right| / (|s| |s - s_b|) + C_{R_0} / |s - s_b| \leq C_{R_0} (1 + |s|)^{\frac{\bar{C}_{R_0}}{2}} |s|^2 $$

for \( s \in L_{\sigma_{\theta}^{(1)}} \) with \( |s| > \sigma^{(1)}_{\theta} / 2 \). Thus it follows from (2.60) and (2.62) that

$$\tag{2.63} \left| g_1(s) \right| \leq C_{R_0} (1 + |s|)^{\frac{\bar{C}_{R_0}}{2}} |s|^2 \quad \text{for} \ s \in L_{\sigma_{\theta}^{(1)}}. $$

Then similarly as in deriving the estimate (2.52), we can use (2.37), (2.63) and the Cauchy inequality to obtain

$$\left| g_1(s) \right| \leq C_{R_0} (1 + |s|)^{\frac{\bar{C}_{R_0}}{2}} |s|^2, \quad s \in \mathbb{R}. $$

Hence, by this and (2.63) we arrive at the result

$$\tag{2.64} \left| G_{R,1}^{(2)}(x, y) \right| \leq C_{R_0} |x|^{-1} \int_{-\infty}^{+\infty} (1 + |s|)^{4} e^{(\bar{C}_{R_0} - |x|)|s|^2} ds \leq C_{R_0} |x|^{-3/2}. $$

From the explicit expression of \( g(s_b) \) and (2.61), we have

$$\tag{2.65} \left| (A - 1)/u \right| = \left| (g(0) - g(s_b)) / \left( i\sqrt{2|x|s_b g(s_b)} \right) \right| \leq C_{R_0} |x|^{-1/2}. $$

Moreover, since \( |u| < 1 \) due to the assumption \( |\sin ((\theta - \theta_z)/2)| < (2 \sqrt{k_+ |x|})^{-1} \), we can apply the fact that \( D_{1/2}(z) \) and \( D_{3/2}(z) \) are analytic for any \( z \in \mathbb{C} \) to obtain that \( |D_{1/2}(u)| \leq C \) and \( |D_{3/2}(u)| \leq C \) for some constant \( C > 0 \). This, together with (2.59), (2.64), (2.65) and the fact that \( |\cos ((\theta - \theta_z)/2)| \geq \cos (\pi/4 - \theta_z/2) \) for any \( \theta \in (\theta_z, \pi/2) \), implies that

$$\tag{2.66} \left| G_{\mathcal{R}}^{(2)}(x, y) \right| \leq C_{R_0} |x|^{-3/4} (1 + |x|^{-1/2}) + C_{R_0} |x|^{-3/2} \leq C_{R_0} |x|^{-3/4}. $$

Further, using the assumption \( |\sin ((\theta - \theta_z)/2)| < (2 \sqrt{k_+ |x|})^{-1} \) again, we have

$$\left| \cos \theta_z - \cos \theta_c \right| = \left| 2 \sin ((\theta - \theta_z)/2) \sin ((\theta + \theta_z)/2) \right| \leq k_+^{-1/2} |x|^{-1/2}, $$

which yields that

$$\tag{2.67} \left| \frac{e^{ik_+ |x|}}{\sqrt{|x|}} \frac{e^{i\bar{z}}}{\sqrt{8\pi k_+}} \frac{2iS(\cos \theta_z, n)}{n^2 - 1} \sin \theta_z e^{-ik_+ |y|} \cos \theta_z \right| \leq C |x|^{-3/4} $$

for some constant \( C > 0 \). Therefore, it follows from (2.66) and (2.67) that \( G_{\mathcal{R}}^{(2)}(x, y) \) has the form (2.43) with \( G_{\mathcal{R}, \text{Res}}^{(2)}(x, y) \) satisfying (2.45) uniformly for all \( \theta_z \in (\theta_c, \pi/2) \) with \( |\sin ((\theta - \theta_z)/2)| < (2 \sqrt{k_+ |x|})^{-1} \) and \( y \in B_{R_0}^{k_+} \).
**Case 3:** $\theta_\tilde{x} = \theta_c$. Taking the limit $\theta_\tilde{x} \to +\theta_c$ along the real axis in (2.35), we obtain that for $\theta_\tilde{x} = \theta_c$,

$$S(\cos(\zeta(s)), n) = \left\{ \begin{array}{ll}
\sqrt{2/k_+} e^{-i\pi/2} \hat{H}_{\theta_c}(s) \sqrt{s - s_0^+} \sqrt{s_0^+}, & s \geq 0,
\sqrt{2/k_+} e^{-i\pi/2} \hat{H}_{\theta_c}(s) \sqrt{s - s_0^-} \sqrt{-s}, & s < 0.
\end{array} \right.$$  

Then, by (2.47), $G_{R}^{(2)}(x, y)$ can be written as

$$G_{R}^{(2)}(x, y) = i e^{ik x |x|/4\pi} \int_0^{+\infty} (g(s) + ig(-s)) \sqrt{s} e^{-|x| s^2} ds$$

$$= \frac{i e^{ik x |x|/4\pi}}{4\pi} \int_0^{+\infty} \left\{ (g(0) + ig(0)) \sqrt{s} e^{-|x| s^2} + [(g(s) - g(0)) + i(g(-s) - g(0))] \sqrt{s} e^{-|x| s^2} \right\} ds$$

$$= \frac{i e^{ik x |x|/4\pi}}{4\pi} \left\{ 2^{-1/2} e^{\pi i/4} g(0) \Gamma(3/4) |x|^{-3/4} \right\}$$

$$+ \int_0^{+\infty} [(g(s) - g(0)) + i(g(-s) - g(0))] \sqrt{s} e^{-|x| s^2} ds \right\},$$  

(2.68)

where $g(s)$ is given by (2.50). Similarly as in the discussion in Part I of this step, we can apply (2.37), (2.52) and the mean-value theorem for $Re(g(s) - g(0))$ and $Im(g(s) - g(0))$ to obtain that

$$|(g(s) - g(0))/s| \leq C_{R_0} (1 + |s|)^{5/2} e^{-\tilde{C}_{R_0} |x|^2}$$

for $s \in \mathbb{R}$. Thus it follows that

$$\left| \int_0^{+\infty} [(g(s) - g(0)) + i(g(-s) - g(0))] \sqrt{s} e^{-|x| s^2} ds \right|$$

$$\leq C_{R_0} \int_0^{+\infty} (s^{3/2} + s^4) e^{-|x| \tilde{C}_{R_0} s^2} ds$$

$$\leq C_{R_0} \left[ \left( |x| - \tilde{C}_{R_0} \right)^{-5/4} + \left( |x| - \tilde{C}_{R_0} \right)^{-5/2} \right] \leq C_{R_0} |x|^{-5/4}$$  

(2.69)

for large enough $|x|$. Hence, using (2.68) and (2.69), we have $|G_{R}^{(2)}(x, y)| \leq C_{R_0} |x|^{-3/4}$. This, together with the fact that $S(\cos \theta_c, n) = 0$, implies that $G_{R}^{(2)}(x, y)$ has the form (2.43) with $G_{R,R_{\text{res}}}(x, y)$ satisfying (2.45) uniformly for $\theta_\tilde{x} = \theta_c$ and all $y \in B_{R_0}$.

Based on the discussions for the above three cases, we obtain that $G_{R}^{(2)}(x, y)$ has the form (2.43) with $G_{R,R_{\text{res}}}(x, y)$ satisfying (2.45) uniformly for all $\theta_\tilde{x} \in [\theta_c, \pi/2)$ and $y \in B_{R_0}$.

**Step 2:** Estimate of $G_{R}^{(1)}(x, y)$. From (2.47) and statement (1) of Lemma 2.11, it is easily seen that

$$G_{R}^{(1)}(x, y) = \frac{i e^{ik x |x|/4\pi}}{4\pi} \int_{-\infty}^{+\infty} g_2(s) e^{-|x| s^2} ds,$$

where $g_2(s) := v_1(s) F(s) \zeta(s)/ds$ is analytic in $L_{\sqrt{k_+}}$. A straightforward calculation
where
\[ G^{(1)}_R(x, y) = \frac{\text{Re}(\hat{\mathcal{G}}_{\mathcal{R}, \text{Res}}(x, y))}{4\pi} \left( \int_{-\infty}^{+\infty} g_2(0)e^{-|x|s^2} ds + \int_{-\infty}^{+\infty} g_2'(0)sec^{-|x|s^2} ds + G^{(1)}_{\mathcal{R}, \text{Res}}(x, y) \right) \]
\[ = \frac{e^{ik_+|x|}}{\sqrt{|x|}} e^{i\frac{\pi}{8}} \left( \frac{2 \cos^2 \theta_\pm - 1 - n^2}{n^2 - 1} \right) e^{-ik_+|y|cos(\theta_\pm + \theta_0)} + \frac{ie^{ik_+|x|}}{4\pi} G^{(1)}_{\mathcal{R}, \text{Res}}(x, y), \]
where \( G^{(1)}_{\mathcal{R}, \text{Res}}(x, y) := \int_{-\infty}^{+\infty} g_2_{\text{Res}}(s)e^{-|x|s^2} ds \) with \( g_2_{\text{Res}}(s) := g_2(s) - g_2(0) - g_2'(0)s \).

Let \( \tilde{\sigma} \) be a fixed number in \((0, \sqrt{k_+})\). Note that \(|P(s)| \neq 0\) in \( s \in L_{\tilde{\sigma}} \). Then similarly to the derivation of (2.51), we can get that \(|g_2(s)| \leq C_{R_0, \tilde{\sigma}}(1 + |s|)^3 e^{C_{\tilde{\sigma}}|s|^2}\)
for any \( s \in L_{\tilde{\sigma}} \), where the constant \( C_{R_0, \tilde{\sigma}} \) depends on \( R_0 \) and \( \tilde{\sigma} \). Thus it follows from Cauchy inequality that
\[
\left| \frac{d^2 g_2_{\text{Res}}(s)}{ds^2} \right| \leq 2C_{R_0, \tilde{\sigma}}(1 + |s| + \tilde{\sigma})^3 e^{C_{\tilde{\sigma}}|s|^2}/\tilde{\sigma}^2 \leq C_{R_0, \tilde{\sigma}}(1 + |s|)^3 e^{C_{\tilde{\sigma}}|s|^2}
\]
for \( s \in \mathbb{R} \). Hence, similarly as in the discussion in Part I of Step 1, we can apply the mean-value theorem for \( \text{Re}(g_2_{\text{Res}}) \) and \( \text{Im}(g_2_{\text{Res}}) \) to obtain that
\[
\left| \frac{g_2_{\text{Res}}(s)}{s^2} \right| \leq C_{R_0, \tilde{\sigma}}(1 + |s|)^3 e^{C_{\tilde{\sigma}}|s|^2} \text{ for } s \in \mathbb{R}.
\]
For simplicity, we choose \( \tilde{\sigma} = \sqrt{k_+}/2 \) and so it follows that
\[
|G^{(1)}_{\mathcal{R}, \text{Res}}(x, y)| \leq C_{R_0, \sqrt{k_+}/2} \int_0^{+\infty} (s^2 + s^5) e^{-\left(|x|-C_{R_0}\right)s^2} ds \]
\[
\leq C_{R_0, \sqrt{k_+}/2} \left[ \left(|x|-C_{R_0}\right)^{-3/2} + \left(|x|-C_{R_0}\right)^{-3} \right] \leq C_{R_0}|x|^{-3/2}
\]
for \( |x| \) large enough. This, together with (2.70), implies that \( G^{(1)}_{\mathcal{R}}(x, y) \) has the form (2.42) with \( G^{(1)}_{\mathcal{R}, \text{Res}}(x, y) \) satisfying (2.44) uniformly for all \( \theta_\pm \in [\theta_c, \pi/2] \) and \( y \in B^+_{R_0} \).

**Lemma 2.13.** Assume that \( k_+ > k_- \) and let \( R_0 > 0 \) be an arbitrary fixed number. Suppose that \( y = |y| (\cos \theta_\pm, \sin \theta_\pm) \in B^+_{R_0} \) with \( \theta_\pm \in (0, \pi) \) and \( x = |x| (\cos \theta_\pm, \sin \theta_\pm) \in \mathbb{R}^2_+ \) with \( \theta_\pm \in (0, \pi) \), then the following statements hold.

1. \( G^{(1)}_{\mathcal{R}}(x, y) \) has the asymptotic behavior (2.42) with \( G^{(1)}_{\mathcal{R}, \text{Res}}(x, y) \) satisfying (2.44) uniformly for all \( \theta_\pm \in (0, \theta_c) \) and \( y \in B^+_{R_0} \).

2. \( G^{(3)}_{\mathcal{R}}(x, y) \) has the asymptotic behavior
\[
G^{(3)}_{\mathcal{R}}(x, y) = \frac{e^{ik_+|x|}}{\sqrt{|x|}} \frac{e^{i\frac{\pi}{8}}}{\sqrt{8\pi k_+}} \left( \frac{2i \sin \theta_\pm \hat{S}(\cos \theta_\pm, n)}{n^2 - 1} \right) e^{-ik_+|y|cos(\theta_\pm + \theta_0)} + G^{(3)}_{\mathcal{R}, \text{Res}}(x, y)
\]
with \( G^{(3)}_{\mathcal{R}, \text{Res}}(x, y) \) satisfying that
\[
|G^{(3)}_{\mathcal{R}, \text{Res}}(x, y)| \leq C_{R_0} |x|^{-3/4}, \quad |x| \to +\infty,
\]
\[
|G^{(3)}_{\mathcal{R}, \text{Res}}(x, y)| \leq C_{R_0} |\theta_\pm - \theta_\pm|^\frac{3}{2} |x|^{-\frac{3}{2}}, \quad |x| \to +\infty,
\]
uniformly for all \( \theta_\pm \in (0, \theta_c) \) and \( y \in B^+_{R_0} \).
3. $G^{(4)}_{R}(x, y)$ satisfies that

$$
|G^{(4)}_{R}(x, y)| \leq C_{R_0}|x|^{-3/4}, \quad |x| \to +\infty,
$$

$$
|G^{(4)}_{R}(x, y)| \leq C_{R_0} |\theta_c - \theta_\xi|^2 |x|^2, \quad |x| \to +\infty,
$$

uniformly for all $\theta_\xi \in (0, \theta_c)$ and $y \in B^+_{R_0}$.

Here, the constant $C_{R_0} > 0$ is independent of $x$ and $y$ but dependent of $R_0$.

**Proof.** Let $|x|$ be sufficiently large throughout the proof. Statement (1) can be obtained in a same way as in Step 2 of the proof of Lemma 2.12. Moreover, as mentioned in Remark 2.3, the steepest descent path $D$ will not cross the branch cuts of $\tilde{S}(\cos \zeta, n)$ for the case $\theta_\xi \in (0, \theta_c)$. Thus statement (2) can be proved in a similar way as in Step 1 of the proof of Lemma 2.12. Note that formula (2.32) is needed for deducing statement (2) since $\text{Im}(s_b) > 0$ for $\theta_\xi \in (0, \theta_c)$ (compare the derivation of (2.59)).

Now we only need to prove statement (3). From (2.26), (2.28), (2.30) and the fact that $S_-(\cos \zeta, n) = -\tilde{S}(\cos \zeta, n)$ for $\zeta \in I_{\theta_c, \xi o}$ (see (2.2)), it can be seen that

$$
G^{(4)}_{R}(x, y) = -\frac{i}{2\pi} \left[ \int_{\theta_c, \xi o} + \int_{D_{\theta_c, \xi o} + \theta_\xi \to -\infty} \right] \frac{2i \sin \zeta \tilde{S}(\cos \zeta, n)}{n^2 - 1} e^{ik_y (-|y| \cos(\zeta + \theta_y) + |x| \cos(\zeta - \theta_x))} d\zeta.
$$

Here, the paths $I_{\theta_c, \xi o}$ and $D_{\theta_c, \xi o} + \theta_\xi \to -\infty$ are defined in the proof of Lemma 2.4. Then it follows from Remark 2.3 and Cauchy integral theorem that

$$
G^{(4)}_{R}(x, y) = -\frac{i}{2\pi} \left[ \int_{\theta_c, \xi o} + \int_{D_{\theta_c, \xi o} + \theta_\xi \to -\infty} \right] \frac{2i \sin \zeta \tilde{S}(\cos \zeta, n)}{n^2 - 1} e^{ik_y (-|y| \cos(\zeta + \theta_y) + |x| \cos(\zeta - \theta_x))} d\zeta,
$$

where $D_{\theta_c, \pi/2 + \theta_\xi \to -\infty}$ denotes the part of the path $D$ starting from $\theta_\xi$ and ending at $\pi/2 + \theta_\xi \to -\infty$. Let $\zeta(s)$ be defined in Lemma 2.11 and $I_{s_b, 0}$ be the path $\{ \sqrt{2k_y} e^{i\pi/4 s} : t \in \mathbb{R} \ s.t. 0 < t < \sin((\theta_c - \theta_\xi)/2) \}$ with the orientation from $s_b$ to 0. From the arguments in the proof of statement (1) of Lemma 2.11, it can be seen that $\zeta(s)$ is a conformal mapping from $L_{\sqrt{k_y}}$ to $\{ \zeta(s) : s \in L_{\sqrt{k_y}} \}$. Since $I_{s_b, 0}$ lies in $L_{\sqrt{k_y}}$, it follows from the change of variable $\zeta = \zeta(s)$ and statement (2) of Lemma 2.11 that

$$
G^{(4)}_{R}(x, y) = -\frac{i e^{ik_y |x|}}{2\pi} \left[ \int_{I_{s_b, 0}^+} + \int_{0}^{+\infty} \right] \frac{2i \sin \zeta(s) \tilde{S}(\cos(\zeta(s)), n)}{n^2 - 1} F(s) \frac{d\zeta(s)}{ds} e^{-|x|s^2} ds
$$

$$
(2.73) \quad = -\frac{i e^{ik_y |x|}}{2\pi} \left[ \int_{I_{s_b, 0}^+} + \int_{0}^{+\infty} \right] \sqrt{s - s_b f(s)} e^{-|x|s^2} ds
$$

with $F(s)$ defined in (2.48) and $f(s)$ given by

$$
f(s) := -\sqrt{2/k_y} e^{-i\pi/4} H_{\theta_c}(s) H_{\pi - \theta_\xi}(s) \sqrt{s - s_b^2} F(s) \frac{d\zeta(s)}{ds} \frac{2i \sin \zeta(s)}{n^2 - 1}.
$$

Further, it can be seen from statement (3) of Lemma 2.11 that $f(s)$ is an analytic function in the strip $L_{\sigma(s)}^{(1)}$ with $\sigma(s)$ given as in Lemma 2.11. Moreover, by same
Due to (2.37), it follows from Cauchy inequality and (2.74) that for
where the positive constants \( C_{R_0} \) and \( C_{\bar{R}_0} \) are independent of \( \theta \) but dependent of \( R_0 \). Due to (2.37), it follows from Cauchy inequality and (2.74) that for \( n = 0, 1, 2, \ldots, \)

\[
\left| \frac{d^n f}{ds^n}(s) \right| \leq \frac{C_{R_0} n! (1 + |s|)^{n/2} e^{C_{\bar{R}_0} |s|^{2/3}}}{(\sigma_{\min}^{(1)})^{n}} \leq \frac{C_{R_0} n! (1 + |s|)^{n/2} e^{C_{\bar{R}_0} |s|^{2/3}}}{(\sigma_{\min}^{(1)})^{n}},
\]

(2.75)

The rest of the proof is divided into the following two parts.

**Part I:** We prove that \( G_R^{(4)}(x, y) \) satisfies (2.72) uniformly for all \( \theta \in (0, \theta_c) \) and \( y \in \mathbb{B}_R. \)

Due to \( \text{Re}(s_b) > 0 \) and \( \text{Im}(s_b^2) > 0 \), we let the path \( \mathcal{D}_{s_b} \) denote the curve \( \{s \in \mathbb{C} : \text{Re}(s) \in (\text{Re}(s_b), +\infty), \text{Im}(s) = (\text{Re}(s))^{-1}\text{Im}(s_b^2)/2 \} \), with the orientation from \( s_b \) to \( +\infty \). Clearly, \( \text{Im}(s^2) = \text{Im}(s_b^2) \) on \( s \in \mathcal{D}_{s_b} \) and \( \mathcal{D}_{s_b} \subset L_{\sigma_{\min}^{(1)}} \). It is worth noting that \( \mathcal{D}_{s_b} \) is a part of the steepest descent path of the function \( h(s) = -s^{2} \) crossing \( s_b \). Thus, using (2.73), (2.74), the analyticity of \( f(s) \) and Cauchy integral theorem, we can rewrite \( G_R^{(4)}(x, y) \) as the integral along the path \( \mathcal{D}_{s_b} \), that is,

\[ G_R^{(4)}(x, y) = -ie^{iky + |x|}(2\pi)^{-1} \int_{\mathcal{D}_{s_b}} \sqrt{s-s_b} f(s) e^{-|x|^2 ds}. \]

Define the function \( \phi(t) := (s_b^2 + t)^{1/2} \). Since \( \text{Re}(\phi(t)) > \text{Re}(s_b) \) and \( \text{Im}(\phi^2(t)) = \text{Im}(s_b^2) \) for \( t \in (0, +\infty) \), it is easily seen that \( \phi(t) \) travels from \( s_b \) to \( +\infty \) on the path \( \mathcal{D}_{s_b} \) as \( t \) travels from 0 to \( +\infty \) along the real axis. Thus introducing the change of variable \( s = \phi(t) \) and applying the fact that \( \text{Im}(s^2) = \text{Im}(s_b^2) \) for \( s \in \mathcal{D}_{s_b} \), we have

\[ G_R^{(4)}(x, y) = -ie^{iky + |x|}(2\pi)^{-1} \int_{0}^{+\infty} \sqrt{t} f_2(t) e^{-|x|^2 dt}, \]

(2.76)

where \( f_2(t) := [2(\phi(t) + s_b)^{1/2} \phi(t)]^{-1} f(\phi(t)) \). Note that \( \left| (\phi(t) + s_b)^{1/2} \right| \geq |s_b|^{1/2} \) and \( |\phi(t)| \geq |s_b| \) for \( t \in [0, +\infty) \). Hence, combining (2.74) and (2.76), we have that for \( |x| \) large enough,

\[
\left| G_R^{(4)}(x, y) \right| \leq C_{R_0} |s_b|^{-3/2} \int_{0}^{+\infty} \sqrt{t} (1 + |s_b| + \sqrt{t}) e^{C_{\bar{R}_0} |s_b|^{2/3}} e^{-|x|^2 dt}
\leq C_{R_0} |s_b|^{-3/2} \int_{0}^{+\infty} (\sqrt{t} + t^{7/4}) e^{-|x|^{2/3} t} dt
\leq C_{R_0} |s_b|^{-3/2} \left[ (|x| - C_{R_0})^{-3/2} + (|x| - C_{R_0})^{-11/4} \right]
\leq C_{R_0} |s_b|^{-3/2} \left[ |x| - (\theta_c - \theta) \right]^{-3/2} |x|^{-3/2},
\]

(2.77)

From this, we easily obtain that \( G_R^{(4)}(x, y) \) satisfies (2.72) uniformly for all \( \theta \in (0, \theta_c) \) and \( y \in \mathbb{B}_R \).
Part II: We prove that $G^{(4)}_R(x, y)$ satisfies (2.71) uniformly for all $\theta_\xi \in (0, \theta_c)$ and $y \in B_{R_0}^\circ$. For this aim, we distinguish between the following two cases.

Case 1: $\theta_\xi \in (0, \theta_c)$ with $\left| \sin((\theta_c - \theta_\xi)/2) \right| \geq (2\sqrt{k_x|x|})^{-1}$. The proof of this case can be easily obtained by applying (2.77).

Case 2: $\theta_\xi \in (0, \theta_c)$ with $\left| \sin((\theta_c - \theta_\xi)/2) \right| < (2\sqrt{k_x|x|})^{-1}$. In this case, assume that $|x|$ is large enough s.t. $|x| \geq 8 \left( \sigma_{\min}^{(1)} \right)^{-2}$ and thus $|s_b| < \sigma_{\min}^{(1)}/4 \leq \sigma_{\min}^{(1)}/4$. Then from (2.73), (2.74), the analyticity of $f(s)$, the fact that $\Im(s_b) = |s_b|/\sqrt{2} < \sigma_{\min}^{(1)}/(4\sqrt{2})$ and Cauchy integral theorem, it follows that $G^{(4)}_R(x, y)$ can be rewritten as

$$G^{(4)}_R(x, y) = -ie^{ik_\perp|x|}(2\pi)^{-1} \int_{I_{s_b, s_b + \infty}} \sqrt{s - s_b} f(s) e^{-|x|^2 s} ds,$$

where $I_{s_b, s_b + \infty}$ denotes the path $\{s_b + t : 0 < t < +\infty\}$ with the orientation from $s_b$ to $s_b + \infty$. Define $f_1(s) := \left[ f(s) - f(s_b) \right] / (s - s_b)/s_b$. Due to the analyticity of $f(s)$, it is clear that $f_1(s)$ is analytic in $s \in L_{\theta_\xi}$. Thus similarly to the derivation of (2.57), we can apply (2.75) and (2.78) to get that

$$G^{(4)}_R(x, y) = -\frac{ie^{ik_\perp|x|}}{2\pi} \int_{I_{s_b, s_b + \infty}} \left[ f(s_b) \sqrt{s - s_b} + \frac{f(s_b) - f(0)}{s_b} (s - s_b)^{\frac{3}{2}} \right] e^{-|x|^2 s} ds,$$

(2.79)

$$-\frac{ie^{ik_\perp|x|}}{2\pi} G^{(4)}_{R,1}(x, y),$$

where

$$G^{(4)}_{R,1}(x, y) := \frac{1}{2|x|} \int_{I_{s_b, s_b + \infty}} \sqrt{s - s_b} \left( (s - s_b)f_1'(s) + \frac{3}{2} f_1(s) \right) e^{-|x|^2 s} ds.$$

Since $\Im(-s_b) < 0$, it follows from a change of variable $t = -s$ that $\int_{I_{s_b, s_b + \infty}} (s - s_b)^{\beta} e^{-|x|^2 s} ds = 2^{-1} e^{-i\pi \beta} F_3(i|x|, -s_b, \beta)$ for $\beta = 1/2$ and $\beta = 3/2$, where $F_3$ is defined in Lemma 2.7. This, together with (2.33) and (2.79), implies that

$$G^{(4)}_R(x, y) + \frac{ie^{ik_\perp|x|}}{2\pi} G^{(4)}_{R,1}(x, y)$$

$$= -\frac{e^{ik_\perp|x|}}{4\pi} \left\{ f(s_b) F_3(i|x|, -s_b, 1/2) - \frac{f(s_b) - f(0)}{s_b} F_3(i|x|, -s_b, 3/2) \right\}$$

(2.81)

$$\frac{f(s_b) e^{ik_\perp|x|} \cos^2((\theta_c - \theta_\xi)/2) + \frac{3}{2}}{|x|^2 2^3 \Gamma(-1/2)} \left( D - \frac{\bar{A}}{\bar{u}} + \frac{\Gamma(-1/2)}{\Gamma(-3/2)} \left( \frac{\bar{A} - 1}{\bar{u}} \right) D - \frac{\bar{A}}{\bar{u}} \right),$$

where $\bar{u} := 2\sqrt{k_x|x|} e^{\mp i \frac{\pi}{2}} \sin((\theta_c - \theta_\xi)/2)$ and $\bar{A} := f(0)/f(s_b)$. It is easily seen that $f(s_b) = -h_c(\theta_\xi)$ with the function $h_c$ defined in (2.58). Similarly to the derivation of (2.61), we can apply Taylor formula and (2.75) to deduce that

$$|f(0) - f(s_b)|/|s_b| \leq C_{R_0},$$

(2.82)

which implies that

$$|(\bar{A} - 1)/\bar{u}| = \left| (f(0) - f(s_b)) / \left( 2|x|s_b f(s_b) \right) \right| \leq C_{R_0} |x|^{-1/2},$$

(2.83)
Similarly to the derivation of (2.63), we can employ the analyticity of \( f(s) \), Taylor formula, the fact that \(|s_b| < \sigma_{\theta_z}^{(1)}/4\) and formulas (2.37), (2.74), (2.75) and (2.82) to get that

\[
|f_1(s)| \leq C_{R_0}(1 + |s|)^{\frac{5}{2}} e^{\tilde{C}_{R_0}|s|^2}, \quad s \in L_{\sigma_{\theta_z}^{(1)}}.
\]

Then similarly to the derivation of (2.75), we can use the above formula, Cauchy inequality and (2.37) to obtain that: for \( n = 0, 1, \)

\[
\left| \frac{d^n f_1}{ds^n}(s) \right| \leq C_{R_0} n! (1 + |s|)^{n+5/2} e^{\tilde{C}_{R_0}|s|^2} \left( \sigma_{\theta_z}^{(1)}/2 \right)^n, \quad s \in L_{\sigma_{\theta_z}^{(1)}}^{n+5/2}.
\]

This, together with the facts that \( s - s_b > 0 \) for \( s \in I_{s_b, s_0 + \infty} \), \( \text{Re}(s_b) > 0 \) and \( \text{Re}(s_b^2) = 0 \), implies that for \( n = 0, 1, \)

\[
\frac{d^n f_1}{ds^n}(s)e^{-|x|^2} \leq C_{R_0} (1 + |s - s_b|)^5 e^{-(|x| - \tilde{C}_{R_0})(s - s_b)^2} \quad \text{for} \ s \in I_{s_b, s_0 + \infty}.
\]

Thus, combining this and (2.80) and using a change of variable \( t = s - s_b \), we obtain

\[
|G_{R, 1}^{(4)}(x, y)| \leq C_{R_0} |x|^{-1} \int_0^{+\infty} \left( t^{1/2} + t^4 \right) e^{-(|x| - \tilde{C}_{R_0})t^2} dt
\]

\[
\leq C_{R_0} |x|^{-1} \left( (|x| - \tilde{C}_{R_0})^{-3/4} + (|x| - \tilde{C}_{R_0})^{-5/2} \right) \leq C_{R_0} |x|^{-7/4}
\]

for \( |x| \) large enough. Further, since \( |\tilde{\upsilon}| < 1 \) in this case, it follows from the analyticity of the function \( D_{\beta} \) with \( \beta \in \mathbb{R} \) that \( |D_{-3/2}(\tilde{\upsilon})| \leq C \) and \( |D_{-5/2}(\tilde{\upsilon})| \leq C \) for some constant \( C > 0 \). This, together with (2.81), (2.83), (2.84) and the fact that \( |\cos((\theta - \theta_c)/2)| \geq \cos(\theta_c/2) \) for any \( \theta \in (0, \theta_c) \), implies that

\[
|G_{R}^{(4)}(x, y)| \leq C_{R_0} |x|^{-3/4} (1 + |x|^{-1/2}) + C_{R_0} |x|^{-7/4} \leq C_{R_0} |x|^{-3/4}.
\]

From the discussions in the above two cases, we obtain that \( G_{R}^{(4)}(x, y) \) satisfies (2.71) uniformly for all \( \theta_z \in (0, \theta_c) \) and \( y \in B_{R_0}^+ \). Therefore, the proof is complete. \( \square \)

Based on Lemmas 2.4, 2.12 and 2.13, we get the following uniform far-field asymptotic estimates of the two-layered Green function and its derivatives.

**Theorem 2.14.** Assume that \( k_+ > k_- \) and let \( R_0 > 0 \) be an arbitrary fixed number. Suppose that \( y = (y_1, y_2) = |y| (\cos \theta_y, \sin \theta_y) \in B_{R_0}^+ \cup B_{R_0}^-, \) \( \text{with} \ \theta_y \in (0, \pi) \cup (\pi, 2\pi) \) and \( x = (x_1, x_2) = |x| \tilde{x} = |x| (\cos \theta_x, \sin \theta_x) \in \mathbb{R}_+^2 \) with \( \theta_x \in (0, \pi) \), then we have the asymptotic behaviors

\[
G(x, y) = \frac{e^{ik_+ |x|}}{\sqrt{|x|}} G^\infty(\tilde{x}, y) + G_{Res}(x, y),
\]

\[
\nabla_y G(x, y) = \frac{e^{ik_+ |x|}}{\sqrt{|x|}} H^\infty(\tilde{x}, y) + H_{Res}(x, y),
\]

where \( G^\infty \) and \( H^\infty \) are given by (2.11) and (2.12), respectively, and \( G_{Res} \) and \( H_{Res} \) satisfy

\[
|G_{Res}(x, y)|, |H_{Res}(x, y)| \leq C_{R_0} |x|^{-3/4}, \quad |x| \to +\infty,
\]
uniformly for all \( \theta \in (0, \pi) \) and \( y \in B_{R_0}^+ \cup B_{R_0}^- \).

(2.88) \( |G_{Res}(x, y)|, |H_{Res}(x, y)| \leq C_{R_0} |\theta - \theta_0|^{-\frac{3}{2}} |x|^{-\frac{7}{2}}, \ |x| \to +\infty, \)

uniformly for all \( \theta \in (0, \theta_c) \cup (\theta_c, \pi/2) \) and \( y \in B_{R_0}^+ \cup B_{R_0}^- \), and

(2.89) \( |G_{Res}(x, y)|, |H_{Res}(x, y)| \leq C_{R_0} |\pi - \theta - \theta_0|^{-\frac{3}{2}} |x|^{-\frac{7}{2}}, \ |x| \to +\infty, \)

uniformly for all \( \theta \in [\pi/2, \pi - \theta_0) \cup (\pi - \theta_0, \pi) \) and \( y \in B_{R_0}^+ \cup B_{R_0}^- \). Here, the constant \( C_{R_0} > 0 \) is independent of \( x \) and \( y \) but dependent of \( R_0 \).

**Proof.** First, we consider the asymptotic behavior of \( G(x, y) \). To this end, we distinguish between the following four cases.

**Case 1:** \( \theta \in (0, \pi/2) \) and \( \theta_0 \in (0, \pi) \). Note that \( S(\cos \theta, n) = \tilde{S}(\cos \theta, n) \) for \( \theta_0 \in (0, \theta_c) \). Thus, it follows from (1.4), (2.4) and Lemmas 2.4, 2.12 and 2.13 that \( G(x, y) \) has the asymptotic behavior (2.85) with \( G_{Res}(x, y) \) satisfying (2.87) uniformly for all \( \theta \in (0, \pi/2) \) and \( y \in B_{R_0}^+ \) and satisfying (2.88) uniformly for all \( \theta \in (0, \theta_0) \cup (\theta_0, \pi/2) \) and \( y \in B_{R_0}^- \).

**Case 2:** \( \theta = \pi/2 \) and \( \theta_0 \in (0, \pi) \). By using similar arguments as in the proof of Theorem 2.1, we can obtain that \( G(x, y) \) has the asymptotic behavior (2.85) with \( G_{Res}(x, y) \) satisfying (2.87) uniformly for all \( \theta \in (0, \pi/2) \) and \( y \in B_{R_0}^+ \). This directly implies that \( G_{Res}(x, y) \) in (2.85) satisfies (2.87) and (2.89) uniformly for \( \theta = \pi/2 \) and all \( y \in B_{R_0}^- \).

**Case 3:** \( \theta \in (\pi/2, \pi) \) and \( \theta_0 \in (0, \pi) \). Since \( S(\xi, a) = S(-\xi, a) \) for any \( \xi \in \mathbb{R} \) and \( a > 0 \), it easily follows from (2.4) and (2.5) that

(2.90) \( G(v, w) = G(x, y) \)

with \( v := (-x_1, x_2) \) and \( w := (-y_1, y_2) \). Note that \( \theta_0 = \pi - \theta_0' \). Thus, with the aid of the results in Case 1, we deduce that \( G(x, y) \) has the asymptotic behavior (2.85) with \( G_{Res}(x, y) \) satisfying (2.87) uniformly for all \( \theta \in (\pi/2, \pi) \) and \( y \in B_{R_0}^+ \) and satisfying (2.89) uniformly for all \( \theta \in (\pi/2, \pi - \theta_0) \cup (\pi - \theta_0, \pi) \) and \( y \in B_{R_0}^- \).

**Case 4:** \( \theta \in (0, \pi) \) and \( \theta_0 \in (\pi, 2\pi) \). Analogous to the above three cases, we can also study the asymptotic behavior of \( G_T(x, y) \) by using similar arguments as in the proofs of Theorem 2.1 and Lemmas 2.4, 2.12 and 2.13. Consequently, for \( y \in B_{R_0}^- \), we can obtain from (2.4) that \( G(x, y) \) has the asymptotic behavior (2.85) with \( G_{Res}(x, y) \) satisfying all its properties presented in this theorem.

Secondly, we consider the asymptotic behavior of \( \nabla_y G(x, y) \). By (2.5) and (2.6) we have

\[
\nabla_y G_R(x, y) = \frac{-1}{4\pi} \int_{-\infty}^{+\infty} S(\xi, k_+) - S(\xi, k_-) - e^{-S(\xi, k_+)|x_2 + y_2|} \, S(\xi, k_-) d\xi, \ y_2 > 0,
\]

\[
\nabla_y G_T(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{S(\xi, k_-)y_2} - e^{S(\xi, k_+)x_2} \, S(\xi, k_-) d\xi, \ y_2 < 0.
\]

From [14], it is easy to see that the Hankel function \( H_0^{(1)} \) satisfies

\[
\nabla_y \left( H_0^{(1)}(k_+|x - y|) \right) = \frac{e^{ik_+|x|}}{\sqrt{|x|}} e^{-i\frac{k_+}{8\pi} \sqrt{\frac{k_+}{|x|} (\hat{x} e^{-ik_+y} + O(|x|^{-1}))}}
\]
as \( |x| \to +\infty \) uniformly for all \( \theta_x \in (0, \pi) \) and \( y \in \mathbb{R}^2 \) with \( |y| < R_0 \). Note that \( H^\infty(\hat{x}, y) = \nabla_y G^\infty(\hat{x}, y) \). Thus, using (2.4) and similar arguments as in the derivations of the asymptotic behavior of \( G(x, y) \), we can obtain that \( \nabla_y G(x, y) \) has the asymptotic behavior (2.86) with \( H_{Res}(x, y) \) satisfying all its properties presented in this theorem.

Remark 2.15. Let \( y \) be an arbitrary fixed point in \( \mathbb{R}_2^+ \cup \mathbb{R}_2^- \) and \( G_{Res}(x, y) \) be given as in Theorem 2.14. It has been proved in [30, Section 2.3.4] that

\[
G_{Res}(x, y) = O(|x|^{-3/2}), \quad |x| \to +\infty,
\]

for all \( \theta_x \in (0, \pi) \backslash \{\theta_e, \pi - \theta_e\} \). The above result is also a direct consequence of Theorem 2.14. Moreover, we can further deduce that

\[
\lim_{|x| \to +\infty} |G_{Res}(x, y)||x|^{3/4} \neq 0 \quad \text{for} \quad \theta_x \in \{\theta_e, \pi - \theta_e\},
\]

which directly implies that (2.91) does not hold for \( \theta_x \in \{\theta_e, \pi - \theta_e\} \). In fact, for the case \( y \in \mathbb{R}_2^+ \), (2.92) can be easily proved by Lemmas 2.4 and 2.12 and formulas (1.4), (2.4), (2.68), (2.69) and (2.90) (note that \( g(0) \in (2.68) \) is equal to non-zero value \( h_e(\theta_x) \) with \( \theta_e \) defined in (2.58)).

3. Uniform far-field asymptotic analysis of \( G(x, y) \) with \( x \in \mathbb{R}_2^+ \). In this section, we study the uniform far-field asymptotics of \( G(x, y) \) with \( x \in \mathbb{R}_2^+ \). Let \( \theta_e \) be defined as in (2.1). Note that in the case \( k_+ < k_- \), there are two critical angles \( \pi + \theta_e \) and \( 2\pi - \theta_e \) for \( G(x, y) \). However, the difficulties in the investigation of the uniform far-field asymptotics of \( G(x, y) \) for the angles \( \theta_x \) in the vicinity of these two critical angles can be resolved by using a property of \( G(x, y) \) presented below.

Let the functions \( \bar{\mathcal{R}}(\theta) \) and \( \bar{T}(\theta) \) be defined by

\[
\bar{\mathcal{R}}(\theta) := \frac{i \sin \theta - S(\cos \theta, 1/n)}{i \sin \theta + S(\cos \theta, 1/n)}, \quad \bar{T}(\theta) := \bar{\mathcal{R}}(\theta) + 1 \quad \text{for} \quad \theta \in \mathbb{R}.
\]

By [30, formula (2.27)], \( G(x, y) \) with \( x = (x_1, x_2) \in \mathbb{R}_2^+ \) and \( y = (y_1, y_2) \in \mathbb{R}_2^+ \cup \mathbb{R}_2^- \) has the following explicit formula (see also [25, Appendix A])

\[
G(x, y) = \begin{cases} 
G_\bar{\mathcal{R}}(x, y), & x \in \mathbb{R}_2^+, \quad y \in \mathbb{R}_2^+, \\
\frac{i}{4} \mathcal{H}_0^{(1)}(k_- |x - y|) + G_\bar{\mathcal{R}}(x, y), & x \in \mathbb{R}_2^+, \quad y \in \mathbb{R}_2^-,
\end{cases}
\]

where \( G_\bar{\mathcal{R}} \) and \( G_\mathcal{R} \) are given by the Sommerfeld integrals as follows

\[
G_\bar{\mathcal{R}}(x, y) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-S(\xi, k_+)y_2 + S(\xi, k_-)x_2} \frac{e^{i\xi(x_1 - y_1)}}{S(\xi, k_+) + S(\xi, k_-)} d\xi,
\]

\[
G_\mathcal{R}(x, y) := \frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{S(\xi, k_-) - S(\xi, k_+)}{S(\xi, k_+) + S(\xi, k_-)} e^{-S(\xi, k_-)|x_2 + y_2|} e^{i\xi(x_1 - y_1)} d\xi.
\]

Let \( G^*(x, y) := G(x', y') \) for \( x' \in \mathbb{R}_2^+ \) and \( y' \in \mathbb{R}_2^+ \cup \mathbb{R}_2^- \) with \( x' \) and \( y' \) defined in Section 2. By formulas (2.4) and (3.1), it is easy to verify that \( G^*(x, y) \) is the two-layered Green function satisfying (1.1)–(1.3) with the wave numbers \( k_+ \) and \( k_- \) replaced by
Suppose that $y = (y_1, y_2) \in B_{R_0}^+ \cup B_{R_0}^-$ and $x = (x_1, x_2) = |x| \hat{x} = |x|(\cos \theta_x, \sin \theta_x) \in \mathbb{R}_+^2$ with $\theta_x \in (\pi, 2\pi)$, then we have the asymptotic behaviors

$$G(x, y) = \frac{e^{ik_- |x|}}{\sqrt{|x|}} G^\infty(\hat{x}, y) + G_{Res}(x, y),$$

$$\nabla_y G(x, y) = \frac{e^{ik_- |x|}}{\sqrt{|x|}} H^\infty(\hat{x}, y) + H_{Res}(x, y),$$

where $G^\infty$, $H^\infty$ are defined by

$$(3.4) \quad G^\infty(\hat{x}, y) := \frac{e^{\frac{i\pi}{2}}}{\sqrt{8\pi k_-}} \begin{cases} T(\theta_x) e^{-ik_- (y_1 \cos \theta_x - iy_2 S(\cos \theta_x, 1/n))}, & \hat{x} \in S_1^+, \quad y \in \mathbb{R}_+^2, \\ e^{-ik_- \hat{x} \cdot y} + R(\theta_x) e^{-ik_- \hat{x} \cdot y'}, & \hat{x} \in S_-^1, \quad y \in \mathbb{R}_-^2, \end{cases}$$

$$(3.5) \quad H^\infty(\hat{x}, y) := e^{-i\frac{\pi}{4}} \sqrt{\frac{k_-}{8\pi}} \begin{cases} T(\theta_x) e^{-ik_- (y_1 \cos \theta_x - iy_2 S(\cos \theta_x, 1/n))} \begin{pmatrix} \cos \theta_x \\ -i S(\cos \theta_x, 1/n) \end{pmatrix}^T, & \hat{x} \in S_1^+, \quad y \in \mathbb{R}_+^2, \\ e^{-ik_- \hat{x} \cdot y} \begin{pmatrix} \cos \theta_x \\ \sin \theta_x \end{pmatrix}^T + R(\theta_x) e^{-ik_- \hat{x} \cdot y'} \begin{pmatrix} \cos \theta_x \\ -\sin \theta_x \end{pmatrix}^T, & \hat{x} \in S_-^1, \quad y \in \mathbb{R}_-^2, \end{cases}$$

and $G_{Res}$ and $H_{Res}$ satisfy the estimates

$$(3.6) \quad |G_{Res}(x, y)|, \quad |H_{Res}(x, y)| \leq C_{R_0} |x|^{-3/4}, \quad |x| \to +\infty,$$

uniformly for all $\theta_x \in (\pi, 2\pi)$ and $y \in B_{R_0}^+ \cup B_{R_0}^-$. 

$$|G_{Res}(x, y)|, \quad |H_{Res}(x, y)| \leq C_{R_0} |\pi + \theta_c - \theta_x|^{-\frac{3}{4}} |x|^{-\frac{3}{4}}, \quad |x| \to +\infty,$$

uniformly for all $\theta_x \in (\pi, \pi + \theta_c) \cup (\pi + \theta_c, 3\pi/2)$ and $y \in B_{R_0}^+ \cup B_{R_0}^-$, and

$$|G_{Res}(x, y)|, \quad |H_{Res}(x, y)| \leq C_{R_0} |2\pi - \theta_c - \theta_x|^{-\frac{3}{4}} |x|^{-\frac{3}{4}}, \quad |x| \to +\infty,$$

uniformly for all $\theta_x \in (3\pi/2, 2\pi - \theta_c) \cup (2\pi - \theta_c, 2\pi)$ and $y \in B_{R_0}^+ \cup B_{R_0}^-$. Here, the constant $C_{R_0} > 0$ is independent of $x$ and $y$ but dependent of $R_0$.

Theorem 3.2. Assume that $k_+ > k_-$ and let $R_0 > 0$ be an arbitrary fixed number. Suppose that $y = (y_1, y_2) \in B_{R_0}^+ \cup B_{R_0}^-$ and $x = |x| \hat{x} = |x|(\cos \theta_x, \sin \theta_x) \in \mathbb{R}_+^2$ with $\theta_x \in (\pi, 2\pi)$, then we have the asymptotic behaviors

$$G(x, y) = \frac{e^{ik_- |x|}}{\sqrt{|x|}} G^\infty(\hat{x}, y) + G_{Res}(x, y),$$

$$\nabla_y G(x, y) = \frac{e^{ik_- |x|}}{\sqrt{|x|}} H^\infty(\hat{x}, y) + H_{Res}(x, y),$$

$k_-$ and $k_+$, respectively. Note further that $x' \in \mathbb{R}_+^2$ and $\theta_x = 2\pi - \theta_0$ for $x \in \mathbb{R}_+^2$. Therefore, applying Theorems 2.1 and 2.14 to $G^*(x, y)$, we can directly obtain the following two theorems for the uniform far-field asymptotics of $G(x, y)$ with $x \in \mathbb{R}_+^2$.
where $G^\infty$ and $H^\infty$ are given by (3.4) and (3.5), respectively, and $G_{Res}$ and $H_{Res}$ satisfy
\[
|G_{Res}(x, y)|, \ |H_{Res}(x, y)| \leq C_{R_0}|x|^{-3/2}, \quad |x| \to +\infty,
\]
uniformly for all $\theta z \in (\pi, 2\pi)$ and $y \in B^{+}_{R_0} \cup B^{-}_{R_0}$. Here, the constant $C_{R_0} > 0$ is independent of $x$ and $y$ but dependent of $R_0$.

**Remark 3.3.** In this remark, we restrict our attention to the case $k_+ < k_-$. Let $y$ be an arbitrary fixed point in $\mathbb{R}^2$ and $G_{Res}(x, y)$ be given as in Theorem 3.1. It easily follows from Theorem 3.1 that $G_{Res}(x, y)$ satisfies (2.91) for all $\theta z \in (\pi, 2\pi) \setminus \{\pi + \theta_c, 2\pi - \theta_c\}$. Moreover, combining Remark 2.15 and the arguments above Theorem 3.1, we can obtain that
\[
(\text{3.7}) \quad \lim_{|x| \to +\infty} |G_{Res}(x, y)||x|^{3/4} \neq 0 \quad \text{for} \ \theta z \in \{\pi + \theta_c, 2\pi - \theta_c\},
\]
which directly implies that $G_{Res}(x, y)$ does not satisfy (2.91) for $\theta z \in \{\pi + \theta_c, 2\pi - \theta_c\}$. Further, it is easily seen from (3.7) that the uniform asymptotic estimate of $G_{Res}(x, y)$ in (3.6) is essentially sharp. Therefore, we call $\pi + \theta_c$ and $2\pi - \theta_c$ the critical angles for the case $k_+ < k_-.

**Remark 3.4.** Let $G_{Res}(x, y)$ and $H_{Res}(x, y)$ be given as in Theorems 2.1 and 3.1 for the case $k_+ < k_-$. We be given as in Theorems 2.1 and 3.2 for the case $k_+ > k_-$. Then it easily follows from the results in Section 2 and this section that, for the cases $k_+ < k_- \quad \text{and} \quad k_+ > k_-$, $G_{Res}$ and $H_{Res}$ satisfy the estimates
\[
|G_{Res}(x, y)|, \ |H_{Res}(x, y)| \leq C_{R_0}|x|^{-3/4}, \quad |x| \to +\infty,
\]
uniformly for all $\theta z \in (0, \pi) \cup (\pi, 2\pi)$ and $y \in B^{+}_{R_0} \cup B^{-}_{R_0}$ with arbitrarily fixed $R_0 > 0$.

**Remark 3.5.** For any source point $y$ lying on the interface $\Gamma_0$, due to the well-posedness of the scattering problem in a two-layered medium (see [2, 34]), we can define the two-layered Green function $G(x, y)$ such that $G(x, y)$ is the unique solution satisfying $G(\cdot, y) - \Phi_0(\cdot, y) \in H^1_{loc}(\mathbb{R}^2)$, $\Delta_x G(x, y) + \kappa^2 G(x, y) = -\delta(x, y)$ in $\mathbb{R}^2$ (in the distributional sense) and the Sommerfeld radiation condition (1.3), where $\Phi_0(\cdot, y)$ denotes the fundamental solution of the Laplace equation $\Delta w = 0$ in $\mathbb{R}^2$ and where $\kappa$ is the wave number defined by $\kappa := k_+ \quad \text{in} \quad \mathbb{R}^+_+ \quad \text{and} \quad \kappa := k_- \quad \text{in} \quad \mathbb{R}^-_+$. Here, $H^1_{loc}(\mathbb{R}^2)$ denotes the space of all functions $\phi : \mathbb{R}^2 \to \mathbb{C}$ such that $\phi \in H^1(B)$ for all open balls $B \subset \mathbb{R}^2$. From the expression of the Hankel function $H^{(1)}_0(\cdot)$ given in [14, Section 3.5] as well as the expression of $G(x, y)$ given in (2.4) and (3.1), it can be seen that for any $y \in \mathbb{R}^2 \cup \mathbb{R}^2$, $G(x, y)$ also satisfies $G(\cdot, y) - \Phi_0(\cdot, y) \in H^1_{loc}(\mathbb{R}^2)$.

Then by using the well-posedness of the scattering problem in a two-layered medium again, we can apply the elliptic interior $H^2$-regularity (see, e.g., [19, Section 6.3]) and the Sobolev inequality given in [19, formula (32) in Section 5.6.3] to deduce that $G(\cdot, y) \in C(\mathbb{R}^2 \setminus \{y\})$ for any $y \in \mathbb{R}^2$ and that $G(x, \cdot) \in C(\mathbb{R}^2 \setminus \{x\})$ for any $x \in \mathbb{R}^2$. Thus, employing the local regularity estimate in [10, Theorem 2.7], we can obtain that $G(\cdot, y) \in C^1(\mathbb{R}^2 \setminus \{y\})$ for any $y \in \mathbb{R}^2$. Furthermore, it is easy to see from (2.4) and (3.1) that $G(x, y)$ satisfies the reciprocity relation $G(x, y) = G(y, x)$ for any $x, y \in \mathbb{R}^2 \cup \mathbb{R}^2$ with $x \neq y$ (see also [30, (2.28)]). Hence, by the above discussions, we have $G(x, \cdot) \in C^1(\mathbb{R}^2 \setminus \{x\})$ for any $x \in \mathbb{R}^2$. On the other hand, it easily follows from a direct calculation that for any $\hat{x} \in S^1_+ \cup S^1_-$, $G^\infty(\hat{x}, \cdot)$ (see (2.11) and (3.4)) and $H^\infty(\hat{x}, \cdot)$ (see (2.12) and (3.5)) can be extended as continuous functions in $\mathbb{R}^2$, which we denote by $G^\infty(\hat{x}, \cdot)$ and $H^\infty(\hat{x}, \cdot)$, respectively, again. Therefore, it can be seen that Theorems 2.1, 2.14, 3.1 and 3.2 still hold with $y \in B^{+}_{R_0} \cup B^{-}_{R_0}$ replaced by $y \in \{y \in \Gamma_0 : |y| < R_0\}$. 
Remark 3.6. By Lebesgue’s theorem, it follows that for any $x \in \mathbb{R}^d_+$, $G_R(x, \cdot)$ given in (2.5) (resp. $G_T(x, \cdot)$ given in (2.6)) can be extended as a function in $C^\infty(\overline{\mathbb{R}^d_+})$ (resp. $C^\infty(\overline{\mathbb{R}^d_+})$). Similarly, for any $x \in \mathbb{R}^d_+$, $G_R(x, \cdot)$ given in (3.2) (resp. $G_T(x, \cdot)$ given in (3.3)) can be extended as a function in $C^\infty(\overline{\mathbb{R}^d_+})$ (resp. $C^\infty(\overline{\mathbb{R}^d_+})$). By such extensions, we can employ the continuity of $G(x, y)$ presented in Remark 3.5 as well as (2.4) and (3.1) to obtain that for any $x \in \mathbb{R}^d_+$, $G(x, y) = \frac{1}{2}H_0^1(k_+|x-y|) + G_R(x, y) = G_T(x, y)$ on $y \in \Gamma_0$ and that for any $x \in \mathbb{R}^d_+$, $G(x, y) = G_T(x, y) = \frac{1}{2}H_0^1(k_-|x-y|) + G_R(x, y)$ on $y \in \Gamma_0$.

4. Uniform far-field asymptotics of the solution to the scattering problem in a two-layered medium. In this section, as an application of the results in Sections 2 and 3, we study the uniform far-field asymptotics of the solution to the acoustic scattering problem by buried obstacles in a two-layered medium with a locally rough interface. To this end, we introduce some notations. Let $\Gamma := \{(x_1, x_2) : x_2 = h_1(x_1), x_1 \in \mathbb{R}\}$ represent a locally rough surface, where $h_1$ is a Lipschitz continuous function with compact support in $\mathbb{R}$. Let $\Gamma_p := \{(x_1, x_2) : x_2 = h_1(x_1), x_1 \in \text{Supp}(h_1)\}$ denote the local perturbation of $\Gamma$. Let $\Omega_{\pm} := \{(x_1, x_2) : x_2 \geq h_1(x_1), x_1 \in \mathbb{R}\}$ denote the homogenous media above and below $\Gamma$, respectively. We assume that the scattering obstacle $D$, described by a bounded domain with $C^2$-boundary $\partial D$ and a connected complement, is fully embedded in the lower medium $\Omega_-$. Let $k_\pm = \omega/c_\pm$ be the wave numbers in $\Omega_{\pm}$, respectively, with $\omega$ being the wave frequency and $c_\pm$ being the wave speeds in the homogenous media $\Omega_{\pm}$, respectively. Let $B_R := \{x \in \mathbb{R}^d : |x| < R\}$ be a disk with radius $R > 0$. Let $H^1_{\text{loc}}(\mathbb{R}^d)$ be defined as in Remark 3.5 and let $H^1_{\text{loc}}(\mathbb{R}^d \setminus \overline{D})$ be the space of all functions $\phi : \mathbb{R}^d \setminus \overline{D} \rightarrow \mathbb{C}$ such that $\phi \in H^1((\mathbb{R}^d \setminus \overline{D}) \cap B)$ for all open balls $B$ containing $\overline{D}$.

Now we describe the considered scattering problem. Consider the time-harmonic ($e^{-i\omega t}$ time dependence) incident plane wave $u^0(x, d) := e^{i k x \cdot d}$ propagating in the direction $d = (\cos \theta_d, \sin \theta_d) \in \mathbb{S}^d_\pm$ with $\theta_d \in (\pi, 2\pi)$. Then the acoustic scattering problem by buried obstacles in a two-layered medium is to find the total field $u^{\text{tot}}(x, d) = u^0(x, d) + u^*(x, d)$, which is the sum of the reference field $u^0(x, d)$ and the scattered field $u^*(x, d)$. The reference wave $u^0(x, d)$ is generated by the incident field $u^0(x, d)$ and the two-layered medium, and is given by (see, e.g., (2.13a) and (2.13b) in [30])

$$u^0(x, d) := \begin{cases} u^0(x, d) + u^*(x, d), & x \in \mathbb{R}^d_+, \\ u^0(x, d), & x \in \mathbb{R}^d_- \end{cases},$$

where the reflected wave $u^r(x, d)$ and transmitted wave $u^t(x, d)$ are given by

$$u^r(x, d) := \mathcal{R}(\pi + \theta_d)e^{i k^+ x \cdot d^*}, \quad u^t(x, d) := \mathcal{T}(\pi + \theta_d)e^{i k^- x \cdot d^*},$$

respectively. Here, $d^* := (\cos \theta_d, -\sin \theta_d)$ denotes the reflection direction in $\mathbb{S}^d_\pm$, $d^* := n^{-1}(\cos \theta_d, n, iS(\cos \theta_d, n))$ with $n$ given as in Section 2, and $\mathcal{R}(\pi + \theta_d)$ and $\mathcal{T}(\pi + \theta_d)$ are the reflection and transmission coefficients, respectively, with $\mathcal{R}$ and $\mathcal{T}$ given by (2.3). The definition of $S(\cdot, \cdot)$ gives that

$$d^* = \begin{cases} n^{-1} \cos \theta_d, -\sqrt{1 - (n^{-1} \cos \theta_d)^2} & \text{if } n^{-1} |\cos \theta_d| \leq 1, \\ n^{-1} \cos \theta_d, -i \sqrt{(n^{-1} \cos \theta_d)^2 - 1} & \text{if } n^{-1} |\cos \theta_d| > 1. \end{cases}$$
In particular, if \( n^{-1}|\cos \theta_d| \leq 1 \), then \( d^0 = (\cos \theta_d^0, \sin \theta_d^0) \) is the transmission direction in \( S^1 \) with \( \theta_d^0 \in [\pi, 2\pi] \) satisfying \( \cos \theta_d = n^{-1} \cos \theta_d \). It is easily seen that for any \( d \in S^1 \), the reference wave \( u^0(x, d) \in H^1_{\text{loc}}(\mathbb{R}^2) \) and \( u^0(x, d) \) satisfies the Helmholtz equations by the unperturbed two-layered medium together with the transmission condition on \( \Gamma_0 \), that is,

\[
\Delta u^0 + k^2 u^0 = 0 \quad \text{in} \quad \mathbb{R}^2_+,
\]

\[
[u^0] = 0, \quad [\partial u^0 / \partial \nu] = 0 \quad \text{on} \quad \Gamma_0,
\]

where \( \nu \) denotes the unit normal on \( \Gamma_0 \) pointing into \( \mathbb{R}^2_+ \) and \([\cdot]\) denotes the jump across the interface \( \Gamma_0 \).

When \( D \) is a penetrable obstacle, the total field \( u^{\text{tot}}(x, d) \) and the scattered field \( u^s(x, d) \) satisfy the following scattering problem:

\[
\begin{align*}
\Delta u^{\text{tot}} + k_+^2 u^{\text{tot}} &= 0 \quad \text{in} \quad \Omega_+, \\
\Delta u^{\text{tot}} + k_-^2 u^{\text{tot}} &= 0 \quad \text{in} \quad \Omega_-,
\end{align*}
\]

\[
[u^{\text{tot}}] = 0, \quad [\partial u^{\text{tot}} / \partial \nu] = 0 \quad \text{on} \quad \Gamma,
\]

\[
\lim_{|x| \to +\infty} \sqrt{|x|} \left( \frac{\partial u^s}{\partial |x|} - i k_\pm u^s \right) = 0 \quad \text{uniformly for all} \quad \hat{x} \in S^1_+,
\]

where \( n_D(x) \in L^\infty(\Omega_-) \) denotes the refractive index with \( \text{Re}(n_D) > 0, \text{Im}(n_D) \geq 0 \) and \( \text{Supp}(n_D - 1) = \overline{D} \). \( \nu \) denotes the unit normal on \( \Gamma \) pointing into \( \Omega_+ \), \([\cdot]\) denotes the jump across the interface \( \Gamma \), and (4.4) is the Sommerfeld radiation condition.

When \( D \) is an impenetrable obstacle, the total field \( u^{\text{tot}}(x, d) \) and the scattered field \( u^s(x, d) \) satisfy the following scattering problem:

\[
\begin{align*}
\Delta u^{\text{tot}} + k_+^2 u^{\text{tot}} &= 0 \quad \text{in} \quad \Omega_+,
\Delta u^{\text{tot}} + k_-^2 u^{\text{tot}} &= 0 \quad \text{in} \quad \Omega_- \\
[u^{\text{tot}}] = 0, \quad [\partial u^{\text{tot}} / \partial \nu] = 0 \quad \text{on} \quad \Gamma \setminus \overline{D},
\end{align*}
\]

\[
\mathcal{B}(u^{\text{tot}}) = 0 \quad \text{on} \quad \partial D,
\]

\[
\lim_{|x| \to +\infty} \sqrt{|x|} \left( \frac{\partial u^s}{\partial |x|} - i k_\pm u^s \right) = 0 \quad \text{uniformly for all} \quad \hat{x} \in S^1_+.
\]

Here, \( \mathcal{B} \) denotes one of the following three boundary conditions

\[
\begin{align*}
\mathcal{B}(u^{\text{tot}}) &:= u^{\text{tot}} \quad \text{on} \quad \partial D, \quad \text{if} \ D \text{ is a sound-soft obstacle}, \\
\mathcal{B}(u^{\text{tot}}) &:= \partial u^{\text{tot}} / \partial \nu \quad \text{on} \quad \partial D, \quad \text{if} \ D \text{ is a sound-hard obstacle}, \\
\mathcal{B}(u^{\text{tot}}) &:= \partial u^{\text{tot}} / \partial \nu + i \lambda u^{\text{tot}} \quad \text{on} \quad \partial D, \quad \text{if} \ D \text{ is an impedance obstacle},
\end{align*}
\]

where \( \nu \) is the unit outward normal to \( \partial D \), and the impedance function \( \lambda \) is a real-valued, continuous and nonnegative function. See Figure 5 for the problem geometry.

In the following theorem, we present some useful results for the well-posedness of the scattering problem (4.1)–(4.4) and the scattering problem (4.5)–(4.9), which are mainly based on [2, 34]. We refer to [17] for the well-posedness of the electromagnetic scattering problem in a two-layered medium. Throughout the paper, we assume that the total field \( u^{\text{tot}}(x, d) \) and the scattered field \( u^s(x, d) \) are given in the sense of Theorem 4.1.

**Theorem 4.1.** Let \( R > 0 \) be an arbitrary fixed number such that \( \Gamma_p \cup \overline{D} \subset B_R \). Then given the reference field \( u^0(x, d) \) generated by the incident field \( u^i(x, d) \) with \( d \in S^1 \) and the two-layered medium, the following statements hold.
1. For any \( d \in S^1_+ \), there exists a unique solution \( u^s(x, d) \in H^1_{loc}(\mathbb{R}^2) \) such that
\[
u^{tot}(x, d) := u^0(x, d) + u^s(x, d) \in H^1_{loc}(\mathbb{R}^2)
\]
and the total field \( u^{tot}(x, d) \) and the scattered field \( u^s(x, d) \) solve the scattering problem (4.1)–(4.4). Furthermore, \( \|u^s(\cdot, d)\|_{H^1(B_R)} \) is uniformly bounded for all \( d \in S^1_+ \).

2. For any \( d \in S^1_- \), there exists a unique solution \( u^s(x, d) \in H^1_{loc}(\mathbb{R}^2 \setminus \overline{D}) \) such that
\[
u^{tot}(x, d) := u^0(x, d) + u^s(x, d) \in H^1_{loc}(\mathbb{R}^2 \setminus \overline{D})
\]
and the total field \( u^{tot}(x, d) \) and the scattered field \( u^s(x, d) \) solve the scattering problem (4.5)–(4.9). Furthermore, \( \|u^s(\cdot, d)\|_{H^1(B_R \setminus \overline{D})} \) is uniformly bounded for all \( d \in S^1_- \).

\begin{proof}
(1) This statement is a direct consequence of Theorem 2.5 in [2] and the facts that \( u^0(\cdot, d) \in H^1_{loc}(\mathbb{R}^2) \) and \( \|u^0(\cdot, d)\|_{H^1(B_R)} \) is uniformly bounded for all \( d \in S^1_+ \).

(2) The uniqueness of the scattering problem (4.5)–(4.9) has been proved in [34].

We can argue similarly as in the proof of Theorem 2.5 in [2] to show that this statement holds.
\end{proof}

Define the function spaces
\[
C(S^1_+) := \{ \varphi \in C(S^1_+) : \varphi \text{ is uniformly continuous on } S^1_+ \}
\]
with the norms \( \|\varphi\|_{C(S^1_+)} := \sup_{x \in S^1_+} |\varphi(x)| \), respectively, and define the function spaces
\[
C^1(S^1_+) := \{ \varphi \in C^1(S^1_+) : \varphi \text{ and } \text{Grad} \varphi \text{ are uniformly continuous on } S^1_+ \}
\]
with the norms \( \|\varphi\|_{C^1(S^1_+)} := \sup_{x \in S^1_+} |\varphi(x)| + \sup_{x \in S^1_+} |\text{Grad} \varphi(x)| \), respectively, where \( \text{Grad} \) denotes the surface gradient on \( S^1 \). With the help of Theorem 4.1 and the uniform far-field asymptotic estimates of \( G(x, y) \) presented in Theorems 2.1 and 3.1, we have the following theorem for the uniform far-field asymptotics of the scattered field \( u^s(x, d) \) in the case \( k_+ < k_- \).

**Theorem 4.2.** Assume that \( k_+ < k_- \). Let \( R > 0 \) be large enough such that \( \Gamma_p \cup \overline{D} \subset B_R \) and suppose that \( x = |x|\hat{x} = |x|(\cos \theta_2, \sin \theta_2) \in \Omega_+ \cup \Omega_- \) with \( \theta_2 \in (0, \pi) \cup (\pi, 2\pi) \) and \( |x| > R \). For \( d \in S^1_- \), let \( u^s(x, d) \) be the scattered field for either the scattering problem (4.1)–(4.4) or the scattering problem (4.5)–(4.9). Then \( u^s(x, d) \)
has the asymptotic behavior

\begin{align}
\tag{4.10}
\text{for } x \in \Omega_+ \setminus B_R,
\end{align}

\begin{align}
\frac{e^{ik|x|}}{\sqrt{|x|}} u^\infty(x, d) + u^s_{\text{Res}}(x, d) & \quad \text{for } x \in \Omega_+ \setminus B_R,
\end{align}

with the far-field pattern \( u^\infty(\hat{x}, d) \) of the scattered field given by

\begin{align}
\tag{4.11}
\int_{\partial B_R} \left[ \frac{\partial G^\infty(\hat{x}, y)}{\partial \nu(y)} u^s(y, d) - \frac{\partial u^s(y, d)}{\partial \nu(y)} G^\infty(\hat{x}, y) \right] ds(y), \quad \hat{x} \in S_1^+ \cup S_1^-,
\end{align}

where \( u^\infty(\hat{x}, d) \) satisfies \( u^\infty(\cdot, d) \in C^1(S_1^+) \), \( u^\infty(\cdot, d) \in C(S_1^-) \) and \( \text{Grad}_\hat{x} u^\infty(\cdot, d) \in L^1(S_1^-) \) with

\[ \|u^\infty(\cdot, d)\|_{C^1(S_1^+)} \leq C, \quad \|u^\infty(\cdot, d)\|_{C(S_1^-)} \leq C \quad \text{for all } d \in S_1^- , \]

and \( u^s_{\text{Res}}(x, d) \) satisfies

\[ |u^s_{\text{Res}}(x, d)| \leq C|x|^{-3/2}, \quad |x| \to +\infty, \]

uniformly for all \( \theta \leq (0, \pi) \) and \( d \in S_1^- \),

\[ |u^s_{\text{Res}}(x, d)| \leq C|x|^{-3/4}, \quad |x| \to +\infty, \]

uniformly for all \( \theta \leq (\pi, 2\pi) \) and \( d \in S_1^- \),

\[ |u^s_{\text{Res}}(x, d)| \leq C|\pi + \theta_c - \theta|^{-3} |x|^{-2}, \quad |x| \to +\infty, \]

uniformly for all \( \theta \leq (\pi, \pi + \theta_c) \cup (\pi + \theta_c, \pi/2) \) and \( d \in S_1^- \), and

\[ |u^s_{\text{Res}}(x, d)| \leq C|2\pi - \theta_c - \theta|^{-3} |x|^{-2}, \quad |x| \to +\infty, \]

uniformly for all \( \theta \leq (3\pi/2, 2\pi - \theta_c) \cup (2\pi - \theta_c, 2\pi) \) and \( d \in S_1^- \). Here, \( G^\infty(\hat{x}, y) \) is defined by (2.11) and (3.4), and \( C > 0 \) is a constant independent of \( x \) and \( d \).

**Proof.** Since \( G(x, y) \) and \( u^s(x, d) \) satisfy the Sommerfeld radiation condition (see formulas (1.3), (4.4) and (4.9)), we can use a similar argument as in [14, Theorem 2.5] to obtain that for any \( x \in (\Omega_+ \cup \Omega_-) \setminus B_R \),

\begin{align}
\tag{4.12}
u^s(x, d) = \int_{\partial B_R} \left[ \frac{\partial G(x, y)}{\partial \nu(y)} u^s(y, d) - \frac{\partial u^s(y, d)}{\partial \nu(y)} G(x, y) \right] ds(y),
\end{align}

where \( \nu \) denotes the outward unit normal to the boundary \( \partial B_R \) (see also \([7, (3.11)]\)).

With the aid of Theorem 4.1 and elliptic regularity estimates (see [20]), it follows that

\[ \|u^s(\cdot, d)\|_{L^2(\partial B_R)} \|\partial u^s(\cdot, d)/\partial \nu\|_{L^2(\partial B_R)} \leq C_R \quad \text{for all } d \in S_1^-, \]

where \( C_R > 0 \) is a constant independent of \( d \) but dependent of \( R \). These, together with Theorems 2.1 and 3.1, imply that \( u^s(x, d) \) has the form (4.10) with \( u^\infty(\hat{x}, d) \) given
by (4.11) and with \(u^*_{\text{Res}}(x,d)\) satisfying all its properties presented in this theorem. Further, since \(n > 1\), it is easily deduced that
\[
\|S(\cos(\cdot), n)\|_{C^1[0, \pi]}, \|S(\cos(\cdot), 1/n)\|_{C[\pi, 2\pi]}, \int_{\pi}^{2\pi} \left| \frac{\partial S(\cos(\theta), 1/n)}{\partial \theta} \right| d\theta \leq C.
\]
Thus, from (2.11), (2.12), (3.4) and (3.5), it is easy to verify that \(G^\infty(\cdot, y), H^\infty(\cdot, y) \in C^1(S^1_+)\), \(G^\infty(\cdot, y), H^\infty(\cdot, y) \in C(S^1_+)\) and \(\text{Grad}_x G^\infty(\cdot, y), \text{Grad}_x [H^\infty(\cdot, y) \cdot \nu(y)] \in L^1(S^1_+)\) with
\[
\|G^\infty(\cdot, y)\|_{C^1(S^1_+)}, \|G^\infty(\cdot, y)\|_{C(S^1_+)}, \|\text{Grad}_x G^\infty(\cdot, y)\|_{L^1(S^1_+)} \leq C,
\]
\[
\|H^\infty(\cdot, y)\|_{C^1(S^1_+)}, \|H^\infty(\cdot, y)\|_{C(S^1_+)}, \|\text{Grad}_x [H^\infty(\cdot, y) \cdot \nu(y)]\|_{L^1(S^1_+)} \leq C
\]
for all \(y \in \partial B_R \setminus \Gamma_0\). Using this, (4.11) and (4.13) and noting that \(\nabla_y G^\infty(\hat{x}, y) = H^\infty(\hat{x}, y)\) for \(\hat{x} \in S^1_+ \cup S^1_+ \) and \(y \in \mathbb{R}^2_+ \cup \mathbb{R}^2_+\), we obtain that \(u^\infty(\hat{x}, d)\) satisfies all its properties presented in this theorem. The proof is thus complete.

For the case \(k_+ > k_-\), we note that (4.12) also holds and thus we can employ Theorems 2.14, 3.2 and 4.1 to obtain the following uniform far-field asymptotic estimates of \(u^*(x,d)\). The proof is similar to that of Theorem 4.2 and is thus omitted.

**Theorem 4.3.** Assume that \(k_+ > k_-\). Let \(R > 0\) be large enough such that \(\Gamma_R \cup \Gamma_0 \subset B_R\) and suppose that \(x = |x| \hat{x} = |x|(\cos \theta_x, \sin \theta_x) \in \Omega_+ \cup \Omega_-\) with \(\theta_x \in (0, \pi) \cup (\pi, 2\pi)\) and \(|x| > R\). For \(d \in S^1_+\), let \(u^*(x,d)\) be the scattered field given as in Theorem 4.2. Then \(u^*(x,d)\) has the asymptotic behavior (4.10) with the far-field pattern \(u^\infty(\hat{x}, d)\) of the scattered field given by (4.11), where \(u^\infty(\hat{x}, d)\) satisfies \(u^\infty(\cdot, d) \in C(S^1_+)\), \(\text{Grad}_x u^\infty(\cdot, d) \in L^1(S^1_+)\) and \(u^\infty(\cdot, d) \in C^1(S^1_+)\) with
\[
\|u^\infty(\cdot, d)\|_{C(S^1_+)}, \|\text{Grad}_x u^\infty(\cdot, d)\|_{L^1(S^1_+)} \leq C \quad \text{for all } d \in S^1_+,
\]
and \(u^*_{\text{Res}}(x,d)\) satisfies
\[
|u^*_{\text{Res}}(x,d)| \leq C|x|^{-3/4}, \quad |x| \to +\infty,
\]
uniformly for all \(\theta_x \in (0, \pi)\) and \(d \in S^1_+\),
\[
|u^*_{\text{Res}}(x,d)| \leq C|\theta_c - \theta_x|^{-3/2}|x|^{-3/4}, \quad |x| \to +\infty,
\]
uniformly for all \(\theta_x \in (0, \theta_c) \cup (\theta_c, \pi/2)\) and \(d \in S^1_+\),
\[
|u^*_{\text{Res}}(x,d)| \leq C|\pi - \theta_c - \theta_x|^{-3/2}|x|^{-3/4}, \quad |x| \to +\infty,
\]
uniformly for all \(\theta_x \in [\pi/2, \pi - \theta_c) \cup (\pi - \theta_c, \pi)\) and \(d \in S^1_-\), and
\[
|u^*_{\text{Res}}(x,d)| \leq C|x|^{-3/2}, \quad |x| \to +\infty,
\]
uniformly for all \(\theta_x \in (\pi, 2\pi)\) and \(d \in S^1_-\). Here, \(C > 0\) is a constant independent of \(x\) and \(d\).

**Remark 4.4.** Let \(u^*_{\text{Res}}(x,d)\) be given as in Theorem 4.2 for the case \(k_+ < k_-\) and be given as in Theorem 4.3 for the case \(k_+ > k_-\). Then it easily follows from Theorems 4.2 and 4.3 that, for both the case \(k_+ < k_-\) and the case \(k_+ > k_-\), \(u^*_{\text{Res}}(x,d)\) satisfies the estimate
\[
|u^*_{\text{Res}}(x,d)| \leq C|x|^{-3/4}, \quad |x| \to +\infty,
\]
uniformly for all \(\theta_x \in (0, \pi) \cup (\pi, 2\pi)\) and \(d \in S^1_-\).
5. Conclusion. In this paper, we have established new results for the uniform far-field asymptotics of the two-dimensional two-layered Green function \( G(x, y) \) (together with its derivatives) in the frequency domain. We note that, to the best of our knowledge, our results are the sharpest yet obtained. The proofs of our results are based on the steepest descent method. As an application of our results for \( G(x, y) \) and its derivatives, we have derived the uniform far-field asymptotic behaviors of the scattered field to the acoustic scattering problem by buried obstacles in a two-layered medium with a locally rough interface. Further, the results obtained in this paper provide a theoretical foundation for our recent work [24], where direct imaging methods have been proposed to numerically recover the locally rough interface from phaseless total-field data or phased far-field data at a fixed frequency. It is believed that the uniform asymptotic results obtained in this paper will also be useful on its own right. Moreover, it is interesting to study the uniform far-field asymptotics of the Green function with the background medium consisting of more than two layers. This is more challenging and will be considered as a future work.

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Appendix A. Proof of formula (2.31) in Lemma 2.7.

In order to prove formula (2.31) in Lemma 2.7, we need the following lemma given in [5].

Lemma A.1 (see formula (9.4.28) on page 384 in [5]). Assume \( \beta > -1 \). Define the function \( W^\pm_\beta(z) := \int_{C_+} t^\beta e^{-zt} dt \), where the path \( C_+ \) is depicted in Figure 6. Then we have

\[
W_\beta^+(z) = \sqrt{2\pi e^{i\pi/4}} D_\beta(i z).
\]

We now prove formula (2.31).

Proof of formula (2.31). Assume \( \text{Im}(b) < 0 \) and let \( \alpha = -b \). Let \( \text{Im}(\rho) > 0 \) and then we can write \( \rho = \rho e^{i\theta_\rho} \) with \( \theta_\rho \in (0, \pi) \). By a change of variable \( t = s - b \), we have

\[
F_2(\rho, b, \beta) = \int_{l_{\mathbb{R}+\alpha}} t^\beta e^{i\rho t(1-\alpha)^2} dt,
\]

where \( l_{\mathbb{R}+\alpha} \) denotes the path \( \{ s + \alpha : s \in \mathbb{R} \} \) with the orientation from \(-\infty + \alpha\) to \(+\infty + \alpha\).
Let the path \( l_\alpha \) denote the curve \( \{ (\sqrt{2}\rho)^{-1} e^{i\pi/4}(s + \alpha) : s \in \mathbb{R} \} \) with the orientation from \( (\sqrt{2}\rho)^{-1} e^{i\pi/4}(-\infty + \alpha) \) to \( (\sqrt{2}\rho)^{-1} e^{i\pi/4}(\infty + \alpha) \) (see Figure 7). Here, we note that \( (\sqrt{2}\rho)^{-1} e^{i\pi/4} = (2|\rho|^{-1/2} e^{i(\pi/4 - \theta/2)}, \pi/4 - \theta/2 \in (0, \pi/4) \) for the case \( \text{Re}(\rho) > 0 \), \( \pi/4 - \theta/2 \in (-\pi/4, 0) \) for the case \( \text{Re}(\rho) < 0 \) and \( \pi/4 - \theta/2 = 0 \) for the case \( \text{Re}(\rho) = 0 \). For the case \( \text{Re}(\rho) > 0 \), let the paths \( l_\alpha \) denote the curves \( \{ z \in l_\alpha : \pm \text{Im}(z) > 0 \} \), respectively, with the same orientations as \( l_\alpha \) (see Figure 7(a)). We now claim that

\[
F_2(\rho, b, \beta) = \begin{cases} 
\int_{l_\alpha} t^\beta e^{i\rho(t-\alpha)^2} dt, & \text{if Re}(\rho) \leq 0, \\
\int_{l_\alpha} t^\beta e^{i\rho(t-\alpha)^2} dt + e^{i2\pi\beta} \int_{l_\alpha} t^\beta e^{i\rho(t-\alpha)^2} dt, & \text{if Re}(\rho) > 0,
\end{cases}
\]

(2.1) We only prove (2.2) since the proof of (2.1) is similar and easier. In what follows, we assume that \( \text{Re}(\rho) > 0 \). Let \( p_\alpha \in \mathbb{R} \) denote the intersection point of \( l_\alpha \) and real axis, and let the path \( \mathbb{R}_{p_\alpha} \) denote the curve \( \{ z \in \mathbb{R} : z < p_\alpha \} \) with the orientation from \( -\infty \) to \( p_\alpha \). Moreover, define

\[
I_1 := \{ z \in \mathbb{C} : \text{Im}(z) > \text{Im}(\alpha), \text{Im}(\sqrt{2}pe^{-i\pi/4}z) < \text{Im}(\alpha) \}, \\
I_2 := \{ z \in \mathbb{C} : 0 < \text{Im}(z) < \text{Im}(\alpha), \text{Im}(\sqrt{2}pe^{-i\pi/4}z) > \text{Im}(\alpha) \},
\]

and let \( l_3 \) denote the domain enclosed by \( l_\alpha^- \) and \( \mathbb{R}_{p_\alpha} \) (see Figure 7(a)). For \( t \in \mathbb{R}_{p_\alpha} \), define \( t_{+}^\beta := \lim_{\epsilon \in \mathbb{R}, t \to +\alpha}(t \pm i\epsilon)^\beta \). It is clear that \( t_{+}^\beta \) exist and

\[
t_{+}^\beta = e^{i2\pi\beta} t_{-}^\beta \quad \text{for } t \in \mathbb{R}_{p_\alpha},
\]

(3.3) since \( p_\alpha < 0 \) due to \( \text{Re}(\rho) > 0 \) and \( \text{Im}(\alpha) > 0 \). Further, for any \( r \in \mathbb{R} \) it is easy to verify that \( |t|^r \cdot |e^{i(\alpha - t)}| \to 0 \) as \( |t| \to +\infty \) uniformly for all \( t \in I_1 \cup I_2 \). This, together with (3.3) and Cauchy integral theorem, gives that

\[
F_2(\rho, b, \beta) = \int_{l_\alpha} t^\beta e^{i\rho(t-\alpha)^2} dt + \int_{\mathbb{R}_{p_\alpha}} t_{+}^\beta e^{i\rho(t-\alpha)^2} dt \\
= \int_{l_\alpha} t^\beta e^{i\rho(t-\alpha)^2} dt + e^{i2\pi\beta} \int_{\mathbb{R}_{p_\alpha}} t_{+}^\beta e^{i\rho(t-\alpha)^2} dt \\
= \int_{l_\alpha} t^\beta e^{i\rho(t-\alpha)^2} dt + e^{i2\pi\beta} \int_{l_\alpha} t_{+}^\beta e^{i\rho(t-\alpha)^2} dt.
\]

Thus we obtain (2.2).

On the other hand, for any \( \eta \in \mathbb{C}_0 \), we write \( \eta = |\eta|e^{i\theta_\eta} \) with \( \theta_\eta \in (-\pi, \pi) \). It is easily seen that \( \theta_\rho/2 - \pi/4 \in (-\pi/4, \pi/4) \) and

\[
\begin{align*}
\theta_\rho &\in \left[ \frac{\pi}{2}, \pi \right), \quad \theta_\eta \in (0, \pi), \quad \frac{\pi}{2} + \theta_\eta - \frac{\theta_\rho}{2} \in (-\frac{\pi}{2}, \pi), & \text{if Re}(\rho) \leq 0, \eta \in l_{\mathbb{R}+\alpha}, \\
\theta_\rho &\in (0, \frac{\pi}{2}), \quad \theta_\eta \in (0, \pi), \quad \frac{\pi}{2} + \theta_\eta - \frac{\theta_\rho}{2} \in (0, \pi), & \text{if Re}(\rho) > 0, e^{i\pi/4}/\sqrt{2}\rho \in l_{\alpha}^-, \\
\theta_\rho &\in (0, \frac{\pi}{2}), \quad \theta_\eta \in (0, \pi), \quad \frac{\pi}{2} + \theta_\eta - \frac{\theta_\rho}{2} \in (\pi, 2\pi), & \text{if Re}(\rho) > 0, e^{i\pi/4}/\sqrt{2}\rho \in l_{\alpha}^+.
\end{align*}
\]

Hence, it follows from the definition of \((\cdot)^\beta\) that \( (\sqrt{2}pe^{-i\pi/4})^\beta = (2\rho)^{\beta/2}e^{-i\beta\pi/4} \) and

\[
\begin{align*}
\left( \frac{1}{\sqrt{2}pe^{-i\pi/4}} \right)^\beta &= \left( \frac{1}{\sqrt{2}pe^{-i\pi/4}} \right)^\beta \eta^{\beta}, & \text{if Re}(\rho) \leq 0, \eta \in l_{\mathbb{R}+\alpha}, \\
\left( \frac{1}{\sqrt{2}pe^{-i\pi/4}} \right)^\beta &= \left( \frac{1}{\sqrt{2}pe^{-i\pi/4}} \right)^\beta \eta^{\beta}, & \text{if Re}(\rho) > 0, e^{i\pi/4}/\sqrt{2}\rho \in l_{\alpha}^+, \\
\left( \frac{1}{\sqrt{2}pe^{-i\pi/4}} \right)^\beta &= e^{-i2\pi\beta} \left( \frac{1}{\sqrt{2}pe^{-i\pi/4}} \right)^\beta \eta^{\beta}, & \text{if Re}(\rho) > 0, e^{i\pi/4}/\sqrt{2}\rho \in l_{\alpha}^-.
\end{align*}
\]
Thus, inserting $t = t(z) := (\sqrt{2p})^{-1} e^{i\frac{z}{\sqrt{2}}} z$ into (A.1) and (A.2), we have

$$F_2(\rho, b, \beta) = e^{i\beta k^2} \int_{\mathcal{C}_0} \left( \frac{1}{2p} \right)^{\frac{3}{2}+\frac{1}{2}} e^{\frac{i\pi(\beta+1)}{4}} z^{\gamma} e^{-\frac{1}{2}z^2-\gamma z} dz$$

for both the case $\text{Re}(\rho) \leq 0$ and the case $\text{Re}(\rho) > 0$, where $\gamma := -\sqrt{2p}e^{-\frac{i\pi}{4}}\alpha$ and we use the fact that $i\rho(t - \alpha)^2 - i\rho z^2 = -\frac{1}{2}z^2 - \gamma z$. Note that $\mathcal{C}_0$ lies in the upper-half complex plane due to $\text{Im}(b) < 0$, and $|\rho e^{-\frac{1}{2}z^2-\gamma z}| \to 0$ as $|z| \to +\infty$ uniformly for all $z \in \{z \in \mathbb{C} : 0 \leq \text{Im}(z) \leq \text{Im}(\alpha)\}$. Thus using Cauchy integral theorem, we deduce that

$$F_2(\rho, b, \beta) = e^{i\beta k^2} \int_{\mathcal{C}_0} \left( \frac{1}{2p} \right)^{\frac{3}{2}+\frac{1}{2}} e^{\frac{i\pi(\beta+1)}{4}} z^{\gamma} e^{-\frac{1}{2}z^2-\gamma z} dz.$$
for $s \in IV$, which implies that $\text{Im}(P(s)) > 0$ for $s \in III$ and $\text{Im}(P(s)) < 0$ for $s \in IV$. This, together with the fact that $\text{Im}(s) - \text{Re}(s) > 0$ for $s \in III$ and $\text{Im}(s) - \text{Re}(s) < 0$ for $s \in IV$, gives $\text{Im}(f(s)) = a_1 \text{Im}(P(s)) + a_2 (\text{Im}(s) - \text{Re}(s))/(2\sqrt{k_+}) \neq 0$ for $s \in III \cup IV$. Moreover, for $s \in Q_1 \cap L_{-w_0, \sqrt{k_+}}$, we have

$$f(s) = a_0 + a_1 P(s) + a_2 \frac{se^{-i\phi}}{\sqrt{2k_+}} = a_0 + a_1 \sqrt{1 - \frac{(\text{Im}(s))^2}{k_+}} + \frac{a_2}{\sqrt{k_+}} \text{Im}(s) > 0.$$  

For $s \in (L_{-w_0, \sqrt{k_+}} \cap Q_2) \backslash \{0\}$, we obtain from (B.1) that $\text{Re}(f(s)) = a_0 + a_1 \text{Re}(P(s)) > 0$. Therefore, from the above arguments, we obtain that statement (1) holds.

Secondly, we consider the statement (2). Let $a_2 < 0$. By similar arguments as in the proof of statement (1), we can deduce that $\text{Im}(f(s)) > 0$ for $s \in I$, $\text{Im}(f(s)) < 0$ for $s \in II$, $\text{Re}(f(s)) > 0$ for $s \in III \cup IV$, $f(s) > 0$ for $s \in L_{-\sqrt{k_+}, w_0} \cap Q_1$ and $\text{Re}(f(s)) > 0$ for $s \in (L_{-\sqrt{k_+}, w_0} \cap Q_2) \backslash \{0\}$. Thus it easily follows that statement (2) holds.

**B.2. Proof of Lemma 2.10.**

Proof. From the analyticity of $A(z)$ and $B(z)$, there exists a disk $B_\varepsilon := \{z \in \mathbb{C} : |z| < \varepsilon\}$ such that: for $z \in B_\varepsilon$,

$$A(z) = \sum_{j=0}^{+\infty} A_j z^j, \quad B(z) = \sum_{j=0}^{+\infty} B_j z^j, \quad A^2(z) = \sum_{j=0}^{+\infty} A^{(2)}_j z^j, \quad B^2(z) = \sum_{j=0}^{+\infty} B^{(2)}_j z^j.$$  

Since $A^2(z) = B^2(z)$ in $L_w$, it is easily seen that

$$A^{(2)}_m = \sum_{j=0}^{m} A_j A_{m-j} = B^{(2)}_m = \sum_{j=0}^{m} B_j B_{m-j}, \quad m \geq 0.$$  

Due to the fact that $A_0 = A(0) = B_0 = B(0) \neq 0$, we can apply (B.2) with $m = 1$ to obtain that $A_1 = B_1$. Then, by repeating the same argument, we can apply (B.2) with $m = 2, 3, \ldots$ to obtain that $A_j = B_j$ for $j = 0, 1, 2, \ldots$. This implies that $A(z) = B(z)$ in $B_\varepsilon$. Therefore, the statement of this lemma follows from the analyticity of $A(z)$ and $B(z)$ in $L_w$. 

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