Imaginary part of the zero angle light-by-light scattering tensor

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We calculate the bilinear combination of Dirac tensors describing the creation of a pair of charged particle (p-meson or leptons) by virtual photons integrated on the final particle phase volume. It can be interpreted as an s channel discontinuity of the zero angle light-by-light scattering tensor with both photons off mass shell. The expression of light-by-light scattering tensor is represented in the form where the gauge invariance and the Bose symmetry are explicitly shown. Some applications and the checks are discussed.

I. INTRODUCTION

A lot of attention was paid to the problem of describing the lowest order nonlinear effect of Quantum Electrodynamics, the scattering of photon on photon. One of earlier papers [1] reveals rather a complicated structure of tensor describing the light-by-light scattering (lbl) process

$$\gamma^\ast(k_1, \mu) + \gamma^\ast(k_2, \nu) \to \gamma^\ast(k_3, \rho) + \gamma^\ast(k_4, \sigma)$$

(1)

with all photons off mass shell have rather cumbersome form:

$$G_{\mu\nu\rho\sigma}(1234) = \sum_{24\text{perm}} \left\{ \frac{1}{8} A_{1234}^{2413} k_\mu^{(2)} k_\nu^{(1)} k_\rho^{(4)} k_\sigma^{(3)} + \frac{1}{4} A_{1234}^{2341} k_\mu^{(2)} k_\nu^{(3)} k_\rho^{(1)} k_\sigma^{(4)} + \frac{1}{2} A_{1234}^{2111} k_\mu^{(1)} k_\nu^{(1)} k_\rho^{(1)} k_\sigma^{(1)} + \frac{1}{2} A_{1234}^{2121} k_\mu^{(2)} k_\nu^{(1)} k_\rho^{(2)} k_\sigma^{(1)} + A_{1234}^{2311} k_\mu^{(2)} k_\nu^{(3)} k_\rho^{(1)} k_\sigma^{(1)} + A_{1234}^{2123} k_\mu^{(2)} k_\nu^{(1)} k_\rho^{(2)} k_\sigma^{(3)} + \frac{1}{2} B_{11}^{12} k_\mu^{(1)} k_\nu^{(2)} k_\rho^{(1)} k_\sigma^{(1)} + \frac{1}{2} B_{11}^{12} k_\mu^{(2)} k_\nu^{(1)} k_\rho^{(1)} k_\sigma^{(2)} + B_{11}^{13} k_\mu^{(1)} k_\rho^{(3)} k_\nu^{(3)} + \frac{1}{4} B_{11}^{14} k_\mu^{(3)} k_\rho^{(3)} k_\sigma^{(1)} + \frac{1}{8} C_{1234}^{1234} k_\mu^{(1)} k_\nu^{(1)} k_\rho^{(1)} k_\sigma^{(1)} \right\}$$

(2)

with the notation given in [1]. Detailed investigations of the light-by-light scattering tensor (lbl) were carried out in the 1960-1980s [2] (with references therein). Using the general formalism developed in these papers a series of applications was build for the cases when one, two, three and four photons were considered as real ones.

Among them are the elastic scattering of photon on photon, elastic scattering of real photon on Coulomb field of nucleon [3], splitting of photon to two real photons in the Coulomb field [4], annihilation of electron and positron to three gluon jets [5], neutral pion decay to four real photons [6], and the contribution to the anomalous magnetic moment of muon [7]. The intermediate states with hadrons including the Higgs boson were considered in a series of papers [8]. Recently, much attention was paid to the problem of calculation of the hadronic contribution to the anomalous magnetic moment of the muon [9], where the intermediate states with the light scalar and pseudoscalar mesons were shown to be relevant.

In these works definite improvements results [1, 2] was used. One of important applications concern the lbl tensor forward scattering kinematics off mass shell photons. The improvement of (2) becomes too complicate problem. Here we develop the independent approach using the light-cone basis.

The special form of the lbl tensor (forward scattering of the virtual photons) was used in the description of the peripheral collisions of hadrons [10], investigating, in particular, the unitarity problems in QED.

A similar form of the lbl tensor was used in the problem of calculation of the Gell-Mann $\beta$ function [11].

We mention as well the problems with construction and checking the Monte - Carlo generators to describe inelastic processes in high energy hadrons (lepton-hadron) collisions. These reasons are the motivations of our paper.

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One of possible applications creation of lepton or pion pairs by two virtual photon of any possible polarization mechanism at collision of any (charged) particles. Creation of several pairs by the same mechanism. Constructing of the relevant Monte-Carlo generators [10]. Another one-an alternative approach to study of polarization of vacuum operator in gauge theory in two loop approximation $\Pi(q)$. In this case we have lbl tensor depending on two momenta - one of external boson, $q$ and other, $k$- the momentum of the internal vector boson. A further integration $k$ is implied [11]. We mention as well the problem of calculation of the photon (virtual) impact factor [10].

For these purposes, the explicit form of the lbl tensor for the forward scattering kinematics can be useful. It is the motivation of this paper. A rather compact expression for the $s$-channel discontinuity of the lbl tensor written in the explicit gauge invariant form and obeying the Bose-symmetry is presented below (see (15)). The $s$-channel discontinuity of the lbl tensor is associated with the scattering process

$$\gamma^*(k, \mu) + \gamma^*(q, \nu) \rightarrow \gamma^*(k, \mu_1) + \gamma^*(q, \nu_1),$$

namely,

$$\Delta_s L_{\mu\nu;\mu_1\nu_1}(k, q; k, q) = (4\pi\alpha)^2 \int T_{\mu\nu}(k, q; q_+, q_-)(T_{\mu_1\nu_1}(k, q; q_+, q_-))^* d\Gamma_2,$$

with the phase volume of the intermediate state with two on mass shell charged particles

$$\int d\Gamma_2 = \frac{1}{(2\pi)^2} \int d^4q_+ d^4q_- \delta(q_+^2 - m^2)\delta(q_-^2 - m^2)\delta^4(k + q - q_+ - q_-) =$$

$$= \frac{1}{2E_+ 2E_-} \delta^4(k + q - q_+ - q_-).$$

Write down the phase volume in the form (we will work in the center of mass reference frame $\vec{k} + \vec{q} = 0)$:

$$\int d\Gamma_2 = \frac{\beta}{8\pi} \frac{1}{2\pi} \int_0^{2\pi} \frac{d\phi}{2} \int_{-1}^1 dc,$$

with the 4-vector $q_-$ component defined as $q_- = (\sqrt{s}/2)(1, \beta \vec{n}), \vec{n} = (c, \sin \theta \cos \varphi, \sin \theta \sin \varphi)$ and

$$s = (k + q)^2 = (q_+ + q_-)^2; \quad \beta = \sqrt{1 - \frac{4m^2}{s}}, c = \cos \theta.$$ 

Here $\theta$ is the polar angle between the directions $\vec{k}, \vec{q}_-$ and $\phi$ is the azimuthal angle which define the direction of the 3-vector $\vec{q}_-$ in the frames with the $z$ axis directed along $\vec{k}$. The integration over the phase volume is essentially the angular averaging:

$$\Delta_s L_{\mu\nu;\mu_1\nu_1}(k, q; k, q) = (4\pi\alpha)^2 \frac{\beta}{8\pi} l_{\mu\nu;\mu_1\nu_1},$$

$$l_{\mu\nu;\mu_1\nu_1} = < T_{\mu\nu}(k; q_+, q_-)(T_{\mu_1\nu_1}(k; q_+, q_-))^* >,$$

$$< F > = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\phi}{2} \int_{-1}^1 dc F(c, \phi).$$

### II. LIGHT-LIKE BASIS. GENERAL FORM OF THE ZERO-ANGLE SCATTERING TENSOR

It is convenient to introduce two linear combinations of the photon 4-momenta $k = (k_1, k_2, k_3, k_4) = (k_0, |\vec{k}|, 0, 0)$; $q = (q_0, -|\vec{k}|, 0, 0)$

$$\chi = \frac{k + q}{\sqrt{s}} = (1, 0, 0, 0), \quad r = \frac{q_0k - k_0q}{|k|\sqrt{s}} = (0, 1, 0, 0)$$

$$\chi^2 = 1, \quad r^2 = -1, \quad (\chi r) = 0$$

and the "transversal" metric tensor

$$g^\perp_{\mu\nu} = \delta_{\mu,3}\delta_{\nu,3} + \delta_{\mu,4}\delta_{\nu,4}.$$
It has only one component transversal to 4-vectors $k,q$
\[ g_{\mu\nu}^k k_\mu = g_{\mu\nu}^q q_\mu = 0. \] (11)

The transversal tensor can be written as
\[ g_{\mu\nu}^k = -g_{\mu\nu} + \chi_\mu \chi_\nu - r_\mu r_\nu. \] (12)

The photon momenta can be written in terms of the 4-vectors $\chi,r$ as
\[ k = k_0 \chi + pr, \quad q = q_0 \chi - pr, \quad p = \frac{1}{2} \sqrt{\frac{\Lambda}{s}}. \] (13)

The explicit form of $k_0, q_0, |\vec{k}| = |q|, \Lambda = \Lambda(s, k^2, q^2)$ is given in Appendix A.

Besides, we introduce two vectors orthogonal to $k, q$:
\[ \vec{k} = p \chi + k_0 r; \quad \vec{q} = p \chi - q_0 r; \quad (k \vec{k}) = 0; \quad (q \vec{q}) = 0. \] (14)

The $l_{\mu \nu_1 \nu_2}$ tensor can be written in terms of $g^\pm$ and $\vec{k}, \vec{q}$ which are explicit gauge-invariant quantities:
\[ l_{\mu \nu_1 \nu_2}^i = a_\nu^i F + a_\nu^i G + a_\nu^i H + a_\nu^i a + a_\nu^i b + a_\nu^i c + a_\nu^i d + a_\nu^i e + a_\nu^i f + a_\nu^i \vec{k}_\mu \vec{q}_\nu \nu_1, \quad i = \pi, \mu, \nu, \] (15)

with the c-number coefficients $a_j^i$ given below and the tensor structures
\[ F = g_{\nu \mu_1}^i g_{\nu \nu_1}^i; \quad G = g_{\nu \mu_1}^i g_{\nu \nu_1}^i; \quad H = g_{\nu \mu_1}^i g_{\nu \nu_1}^i; \] (16)

\[ a = g_{\mu_1 \nu_1} \vec{q}_\nu \nu_1; \quad b = g_{\mu_1 \nu_1} \vec{k}_\nu \nu_1; \quad c = g_{\nu \mu_1} \vec{q}_\nu \nu_1; \] (17)

\[ d = g_{\mu_1 \nu_1} \vec{k}_\nu \nu_1; \quad e = g_{\nu \mu_1} \vec{k}_\nu \nu_1; \quad f = g_{\nu \nu_1} \vec{k}_\nu \nu_1. \]

### III. RESULTS

In the case of pion pair production we obtain (details in Appendices A,B,C):
\[ a_H^\pi = < 4 + 16 \lambda \frac{1}{d} \beta_1 + 16 \frac{1}{d^2} \lambda^2 \beta_2 >; \] (18)
\[ a_F^\pi = a_F^e = 16 < \frac{1}{d} \lambda^2 \beta_2 >; \] (19)
\[ p^2 a_b^\pi = p^2 a_e^\pi = < 4 \frac{\lambda}{d} - 4 \frac{\lambda^2}{d^2} \beta_1 > - 4 \frac{1}{d} \lambda \beta_2 >; \] (20)
\[ p^2 a_d^\pi = p^2 a_e^\pi = < 4 \lambda \frac{1}{d} - 4 \frac{1}{d^2} \beta_1 >; \] (21)
\[ p^2 a_a^\pi = < 4 \lambda \beta_1 >; \] (22)
\[ p^2 a_f^\pi = < 4 \lambda \beta_2 >; \] (23)
\[ p^4 a_H^\pi = < 4 + 4(1 - \rho) \frac{\lambda}{d} + 4[16 + (1 - \rho)^2 \frac{\lambda^2}{d^2}] >. \] (24)

The coefficients of the tensor for the lepton pair in the intermediate state are (index $\mu$ corresponds to $\mu$ - meson)
\[ p^2 a_b^\mu = \frac{8 \lambda^2}{d^2} (1 - d) \epsilon \delta + [\frac{1}{4} d - 2 \lambda^2 \beta_1 - \lambda^2 \beta_2]; \] (25)
\[ p^2 a_a^\mu = \frac{8 \lambda^2}{d^2} (1 - d) \epsilon \delta + [\frac{1}{4} d - 2 \lambda^2 \beta_1 - \lambda^2 \beta_2]; \] (26)
\[ p^2 a_c^\mu = \frac{8 \lambda^2}{d^2} (1 - d) \epsilon \delta + [\frac{1}{4} d - 2 \lambda^2 \beta_1 - \lambda^2 \beta_2]; \] (27)
\[ p^2 a_f^\mu = \frac{8 \lambda^2}{d^2} (1 - d) \epsilon \delta + [\frac{1}{4} d - 2 \lambda^2 \beta_1 - \lambda^2 \beta_2]; \] (28)
\[ p^4 a_H^\mu = \frac{8(1 - d)^2}{d^2} (1 - d) \epsilon \delta + [\frac{1}{4} d - 2 \lambda^2 \beta_1 - \lambda^2 \beta_2]. \] (29)
Moreover,
\[ a_\mu^\alpha = a_\mu^\alpha (\epsilon \leftrightarrow \delta); \quad p^2 a_\mu^\alpha = a_\mu^\alpha; \quad a_\mu^\alpha = a_\mu^\nu. \]  
(19)

Here we use the notation for scalar coefficients
\[ \epsilon = \frac{k^2}{s}; \quad \delta = \frac{q^2}{s}; \quad \sigma = \frac{q_0}{\sqrt{s}}; \quad \eta = \frac{k_0}{\sqrt{s}}; \]
\[ \lambda = \frac{1}{\epsilon + \delta - 1}; \quad \rho = \frac{\Lambda}{s}, \quad \beta_1 = \frac{1}{2} \beta^2 (1 - c^2); \quad \beta_2 = \frac{1}{8} \beta^4 (1 - c^2)^2. \]  
(20)

The relevant integrals needed to perform the integration over the polar angle are given in Appendix C.

IV. CONCLUSION. THE CASE OF REAL PHOTONS

In the case of both photons on the mass shell the 4-vectors \( \hat{k}, \hat{q} \) become the light-like ones. Using
\[ p = \sqrt{s}; \quad \rho = 1, \quad \lambda = -1; \quad \eta = \sigma = \frac{1}{2}, \quad \epsilon = \delta = 0, \]  
(21)

we obtain
\[ a_\mu^\pi = \langle -4 - 16 \frac{d}{d^2} \beta_1 + 16 \frac{1}{d^2} \beta_2 \rangle; \]
\[ a_\mu^\alpha = a_\mu^\nu = 16 \langle \frac{1}{d^2} \beta_2 \rangle; \]
\[ \frac{s}{4} a_\mu^\alpha = \frac{s}{4} a_\mu^\nu = \langle -4 \beta_1 \frac{1}{d} + 4 \frac{1}{d^2} \rangle - 4 + 8 \frac{1}{d^2} \rangle; \]
\[ \frac{s}{4} a_\mu^\nu = p^2 a_\mu^\nu = 0; \]
\[ \frac{s}{4} a_\mu^\nu = \frac{s}{4} a_\mu^\nu = \langle \beta_1 \frac{1}{d} - \frac{1}{d^2} \rangle \rangle; \]
\[ \frac{s^2}{16} a_\mu^\alpha = 4 < 1 + 16 \frac{1}{d^2} \rangle. \]  
(22)

and
\[ a_\mu^\rho = \langle \frac{8}{d^2} \frac{1}{4} d - \beta_2 \rangle; \]
\[ a_\mu^\nu = \langle -8 \frac{1}{d^2} \frac{1}{4} d - \beta_1 + \beta_2 \rangle; \]
\[ a_\mu^\nu = \langle \frac{8}{d^2} (1 - d)(\frac{1}{4} d - \beta_1) - \beta_2 \rangle; \]
\[ a_\mu^\mu = a_\mu^\nu = -a_\mu^\nu = -a_\mu^\nu = \frac{2}{d} \beta_1; \]
\[ a_\mu^\nu = a_\mu^\nu = \frac{4}{d^2} (1 - d)(\frac{1}{2} d - \beta_1), \]
\[ d = 1 - \beta^2 c^2. \]  
(23)

An important test is the correspondence with the cross sections of pair production in real photon collisions. Really,
\[ \frac{d\sigma}{dc} = \frac{\alpha^2}{2(s - 4m^2)} g_{\mu\nu} g_{\mu\nu} g_{\mu\nu} g_{\mu\nu}. \]  
(24)

We obtain for the cross section of a pair of charged scalar particle production \[ d\sigma = \frac{\alpha^2}{2s} \beta[1 - \frac{8m^2}{s} \frac{1}{d} + \frac{32m^4}{s^2} \frac{1}{d^2}], \]  
(25)

and for a fermion pair production
\[ d\sigma = \frac{2\alpha^2 \beta^2}{s} \frac{1 + \beta^2 c^2}{d^4} + \frac{8m^2}{s} \frac{1}{ds} - \frac{32m^4}{s^2 d^2}. \]  
(26)
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Appendix A: The charged pi-meson pair intermediate state. Kinematics

The matrix element of the subprocess has the form

\[ M_{\gamma'^{\rightarrow}q^-q^+}^{\text{scal}} = 4\pi\alpha T_{\mu\nu}, \] (A1)

with

\[ T_{\mu\nu} = \frac{(2q_+ - k)\mu(2q_+ - q)\nu}{d_-} + \frac{(k - 2q_+)\mu(2q_+ - q)\nu}{d_+} - 2g_{\mu\nu}, \]
\[ d_- = (q_+ - k)^2 - m^2 = k_2 - 2q_- k; \quad d_+ = (-q_+ + k)^2 - m^2 = k_2 - 2q_- q. \] (A2)

One can check the fulfillment of the gauge-invariance condition \( T_{\mu\nu}k_\mu = T_{\mu\nu}q_\nu = 0 \).

It seems to be convenient to rewrite the Dirac tensor \( T_{\mu\nu} \) as

\[ T_{\mu\nu} = Aq_+q_- - 2q_-C_\nu - 2q_-D_\mu + r_{\mu\nu} + 2g^{\mu\nu}; \]
\[ A = \frac{1}{d_-} + \frac{1}{d_+}; \quad C_\nu = \frac{(2k + q)}{d_-}, \quad D_\mu = \frac{(k + 2q_+)\mu}{d_-}; \]
\[ r_{\mu\nu} = \frac{1}{d_-}k_\mu(2k + q)\nu + \frac{1}{d_+}q_\nu(2q_+ + k)\mu - 2\chi_\mu\chi_\nu + 2r_{\mu\nu} ; \]
\[ g^{\mu\nu} = -g_{\mu\nu} + \chi_\mu\chi_\nu - r_{\mu\nu}. \] (A3)

Using the energy-momentum conservation law

\[ k + q = q_+ + q_- \]

we find

\[ k_0 = \frac{s + k^2 - q^2}{2\sqrt{s}}, \quad q_0 = \frac{s + q^2 - k^2}{2\sqrt{s}}; \]
\[ \vec{k}^2 = \frac{\Lambda}{4s}, \quad \Lambda = R(s, k^2, q^2) = s^2 + (k^2)^2 + (q^2)^2 - 2sk^2 - 2sq^2 - 2k^2q^2. \] (A4)

Below we will express the lbl tensor in terms of \( g^\perp \) and \( \vec{k}, \vec{q} \) which are explicit gauge-invariant quantities. So we have (see (9)):

\[ \Delta_\perp L_{\mu\nu;\mu_1\nu_1}(k, q; k, q) = \frac{\beta}{8\pi}(4\pi\alpha)^2 < T_{\mu\nu}(T_{\mu_1\nu_1})^\ast >. \] (A5)

Appendix B: The charged lepton pair intermediate state

The matrix element of the subprocess has the form

\[ M_{q^+_\mu q^-\mu q_+}^{\gamma'^{\rightarrow}q^-q^+} = 4\pi\alpha T_{\mu\nu}(q_+, q_-), \] (B1)
with
\[
T_{\mu\nu(q_+q_-)} = \bar{u}(q_-)[\gamma_\mu - \hat{k}_- + m_d \gamma_\nu + \gamma_\nu - \hat{k}_+ + m_u \gamma_\mu]v(q_+), \quad \hat{a} = a_\mu \gamma_\mu.
\]

Using the on mass shell conditions of leptons it can be written as
\[
T_{\mu\nu(q_+q_-)} = \bar{u}(q_-) [Q_\mu \gamma_\nu - \frac{1}{d_-} \gamma_\mu \hat{k}_\nu + \frac{1}{d_+} \gamma_\nu \hat{k}_\mu] v(q_+),
\]
\[
Q_\mu = 2\left( \frac{q_\mu}{d_-} - \frac{q_\mu}{d_+} \right).
\]

Rather tedious calculations lead to the result given above.

**Appendix C: details of calculations**

The expression for the denominators are
\[
d_- = \frac{k^2 + q^2 - s}{2} (1 + bc), \quad d_+ = \frac{k^2 + q^2 - s}{2} (1 - bc), \quad b = \beta \frac{\sqrt{k^2 + q^2 - s}}{k^2 + q^2 - s}, \quad c = \cos \theta, \quad \theta = \frac{\hat{q}}{k}. \quad (C1)
\]

The product of two Dirac tensors for a pion pair has the form (notation in (A3)):
\[
T_{\mu\nu}T_{\mu_1\nu_1} = 16A^2(q_+q_-q_{\mu_1}q_{\nu_1} - 8AC_\nu(\nu_{\mu_1}q_{\nu_1} - q_{\mu_1}q_{\nu_1}) - 8AD_{\mu_1}(q_{\nu_1}q_{\mu_1} - q_{\mu_1}q_{\nu_1}) - 
\]
\[
8AD_{\mu_1}(q_{\nu_1}q_{\mu_1} - q_{\mu_1}q_{\nu_1}) + 4A_{\mu_1\nu_1}(q_{\mu_1}q_{\nu_1}) + 4A_{\mu_1\nu_1}(q_{\mu_1}q_{\nu_1}) + 4C_\nu C_\nu(q_{\mu_1}q_{\nu_1} - q_{\mu_1}q_{\nu_1}) - 
\]
\[
2C_\nu r_{\mu_1\nu_1} q_{\mu_1} - 2C_\nu r_{\mu_1\nu_1} q_{\nu_1} - 2D_{\mu_1} r_{\mu_1\nu_1} q_{\mu_1} + r_{\mu_1\nu_1} q_{\mu_1} + r_{\mu_1\nu_1} q_{\nu_1} + 4q_{\mu_1} q_{\nu_1} + 4q_{\nu_1} q_{\mu_1} + 
\]
\[
2g_{\mu\nu}[4A(q_{\mu_1}q_{\nu_1} - q_{\mu_1}q_{\nu_1}) - 2D_{\mu_1} q_{\nu_1} + 2D_{\nu_1} q_{\mu_1} + g_{\mu\nu} q_{\mu_1}].
\]

The averaging of the relevant 4-vector product gives
\[
< q_{-\mu} > = \frac{\sqrt{s}}{2} < (\chi + \beta c)_\mu >;
\]
\[
< q_{-\mu} q_{-\nu} > = \frac{s}{4} [\chi^2 + \beta \epsilon(\epsilon r) + \beta^2 (c^2 r^2 + \frac{1}{2}(1 - c^2) g_{\perp})_{\mu\nu} >;
\]
\[
< q_{-\mu} q_{-\nu} > = \frac{s}{4} [\chi^3 + (\chi^2 r) \beta c + \beta^2 (\epsilon r^2 + \frac{1}{2}(1 - c^2) g_{\perp})_{\mu\nu} >;
\]
\[
< q_{-\mu} q_{-\nu} > = \frac{s}{4} [\chi^4 + \beta \epsilon (\chi^3 r) + 
\]
\[
\beta^2 (c^2 (\chi^2 r^2) + \frac{1}{2}(1 - c^2) (\chi^2 g_{\perp})_{\mu\nu} >;
\]
\[
\beta^2 (c^2 (\chi^2 r^2) + \frac{1}{2}(1 - c^2) (\chi^2 g_{\perp})_{\mu\nu} >.
\]

Here we imply
\[
(ab)_{\mu_1}\cdots \cdots \mu_2 = a_\mu b_\nu + a_\nu b_\mu; \quad (a^\alpha)_{\mu_1}\cdots \cdots \mu_2 = a_\mu \cdots \cdots a_\mu; \quad (a^2 b)_{\mu_1\mu_2\mu_3} = a_\mu a_\mu b_\mu + \cdots + b_\mu a_\mu a_\mu; \quad (a^2 g)_{\mu_1\mu_2\mu_3} = a_\mu g_{\mu_2\mu_3} + a_\mu g_{\mu_3\mu_2} + a_\mu g_{\mu_2\mu_3} + a_\mu g_{\mu_3\mu_2} + \cdots + b_\mu a_\mu a_\mu a_\mu; \quad (a^2 g)_{\mu_1\mu_2\mu_3} = a_\mu g_{\mu_2\mu_3} + a_\mu g_{\mu_3\mu_2} + a_\mu g_{\mu_2\mu_3} + a_\mu g_{\mu_3\mu_2} + \cdots + b_\mu a_\mu a_\mu a_\mu;
\]
\[
(g_{\perp})_{\mu_1}\cdots \cdots \mu_2 = g_{\mu_1\mu_2\mu_3} + g_{\mu_2\mu_3\mu_1} + g_{\mu_3\mu_1\mu_2} + g_{\mu_1\mu_2\mu_3} + g_{\mu_2\mu_3\mu_1} + g_{\mu_3\mu_1\mu_2} + g_{\mu_1\mu_2\mu_3} + g_{\mu_2\mu_3\mu_1} + g_{\mu_3\mu_1\mu_2}.
\]

The relevant integrals needed to perform the integration over the polar angle are (d = 1 - b^2 e^2):
\[
\varphi_1 = \frac{1 - c^2}{d} = \frac{1 - b^2}{2b} \left( 1 - \frac{1 - b^2}{2b} L \right); \quad \varphi_2 = \frac{(1 - c^2)^2}{d^2} = \frac{1}{2b^4} \left[ 3 - b^2 - \frac{1 - b^2}{2b} (3 + b^2) L \right];
\]
\[
\varphi_3 = \frac{1}{d} = \frac{1}{b} L; \quad \varphi_4 = \frac{c^2 (1 - c^2)}{d^2} = \frac{1}{2b^4} \left[ \frac{3}{2} + \frac{1}{4} (7 - 3b^2) \frac{1}{b} L \right];
\]
\[
\varphi_5 = \frac{1}{d^2} = \frac{1}{1 - b^2} + \frac{1}{2b} L; \quad \varphi_6 = \frac{c^2}{d^2} = \frac{1}{b^2} \left[ \frac{1}{1 - b^2} - \frac{1}{2b} L \right]; \quad L = \ln \frac{1 + b}{1 - b}. \quad (C5)
\]
For the charged $\pi$-meson case we have

\begin{align*}
a_H^\pi &= 4 + 4\beta^2 \lambda \varphi_1 + \beta^4 \lambda^2 \frac{1}{2} \varphi_2; \\
a_G^\pi &= a_F^\pi = \frac{1}{2} \beta^4 \lambda^2 \varphi_2; \\
p^2 a_h^\pi &= p^2 a_c^\pi = -4 + \beta^2 \lambda (\varphi_1 - 2 \varphi_2 \lambda) + \varphi_3 \lambda (-5 + \rho); \\
p^2 a_c^\pi &= p^2 a_d^\pi = \frac{1}{2} \beta^2 \lambda^2 (\rho - 1) b^2 \varphi_4; \\
p^2 a_e^\pi &= \frac{1}{2} \beta^2 \eta^2 \lambda^2 b^2 \varphi_4; \\
p^4 a_h^\pi &= 4 + 2 \lambda (1 - \rho) \varphi_3 + (16 + (1 - \rho)^2) \lambda^2 \varphi_5. \tag{C6}
\end{align*}

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