Approximation of exponential-type functions on a uniform grid by shifts of a basis function

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Abstract

In this paper, we study the problem of interpolating a continuous function at \((n + 1)\) equally-spaced points in the interval \([0, 1]\), using shifts of a kernel on the \((1/n)\)-spaced infinite grid. The archetypal example here is approximation using shifts of a Gaussian kernel. We present new results concerning interpolation of functions of exponential type, in particular, polynomials on the integer grid as a step en route to solve the general interpolation problem. For the Gaussian kernel we introduce a new class of polynomials, closely related to the probabilistic Hermite polynomials and show that evaluations of the polynomials at the integer points provide the coefficients of the interpolants. Taking cue from the classical Newton polynomial interpolation, we derive a closed formula for the Gaussian interpolant of a continuous function on a uniform grid in the unit interval.

1 Introduction

In the mathematical literature pertaining to radial basis functions, there have been mainly two approaches to constructing interpolants with Gaussian kernels. The first involves interpolating a function \(f \in C(\mathbb{R})\) on the \(h\)-spaced

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grid \( h\mathbb{Z} \) by an interpolant of the form
\[
\sum_{z \in h\mathbb{Z}} \alpha_z \psi(x/h - z),
\]
where
\[
\psi(x) = \frac{\exp(-\|x\|^2/2)}{\sqrt{2\pi}}.
\]
Analysis of this so-called cardinal approximation has been done in a series of papers of Baxter, Riemenschneider and Sivakumar \[3, 9, 20, 21\].

The second concentrates on interpolating a continuous function on a finite subset \( Y \) of a compact interval (e.g \([0, 1]\)). Under this circumstance, the interpolant one seeks is of the form
\[
\sum_{y \in Y} \alpha_y \psi(x - y).
\]

There are multidimensional set-ups for both approaches. Modern mathematical literature abounds in developing error estimates for approximation schemes in this context. We refer readers to \[15, 13, 14, 12\] and the references therein.

Approximation methods involving sparse-grid algorithms have been recently proven effective and efficient; see \[11\]. Some sophisticated multi-level sparse grid kernel interpolation schemes have been constructed by authors of \[4, 10\]. We are currently motivated to develop sparse-grid algorithms for high-dimensional approximation with the Gaussian kernel and derive error estimates for \( C^k \) functions with polynomial growth. However, there are several obstacles en route to achieving these goals. The main purpose of the current paper is to clear a few obstacles out of the way. First and foremost, we face the problem of interpolating a function at the \((n + 1)\) equally-spaced points \( ih, i = 0, 1, \cdots, n \) with \( h = 1/n \), where \( n \in \mathbb{N} \). The approach we take here differs from those discussed in the above references. We first interpolate the given \((n + 1)\) data by a degree \( n \) polynomial, and then interpolate the polynomial by a radial basis function interpolant on \( h\mathbb{Z} \).

Functions of exponential type\(^1\) are often utilized as a half-way house in deriving error estimates for Sobolev space functions; see \[16, 17\]. As such, it is worthwhile to study the effect of the interpolation scheme when the

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\(^1\)They are also referred to as “band limited functions” in the literature.
target functions are of exponential types, and in particular, polynomials, which we anticipate to play a significant role in our future effort to obtain more nuanced error estimates for $C^k$–functions with polynomial growth. Interestingly enough, we observe in the analytic number theory literature that interpolation goes the opposite way in the sense that functions of exponential type are employed to approximate the Gaussian and other useful radial basis functions; see [5, 6, 7]. We hope that interactions of the two seemingly inverse research tracks will create synergetic results.

The layout of the paper is as follows. In Section 2 we will consider a general kernel $\psi$ and study the operator $T$ induced by the Toeplitz matrix $\psi(j - k), j, k \in \mathbb{Z}$

\[ We show that the operator is surjective on the class of functions of exponential type, and is injective on a restricted class of functions of exponential type. In particular, the operator is one-to-one on the linear space of polynomials. Furthermore, we demonstrate that the coefficients of an interpolant can be solved for explicitly. In Section 3 we investigate the special case in which the Gaussian kernel is employed and the target functions are polynomials. We introduce new classes of polynomials resembling the classical probabilistic Hermite polynomials, and derive closed formulas for the coefficients of a Gaussian interpolant in terms of these polynomials. In Section 4, we use a Newton type interpolation to derive a closed formula for the Gaussian interpolant of a continuous function on a uniform grid in the unit interval. The construction of the interpolants is stationary in the sense that the variance of the Gaussian used for approximation scales with the grid spacing $h$. This is not a convergent approximation scheme. However, it can be used as part of a residual correction scheme as is described in [10].

2 General Kernels

For $f \in L(\mathbb{R})$, we use the following Fourier transform pair:

\[ \hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi x} f(x) dx \quad \hat{f}^{-1}(x) = \int_{\mathbb{R}} e^{2\pi i \xi x} f(\xi) d\xi. \]

\[ \text{The action of } T \text{ on an } f \in C(\mathbb{R}) \text{ takes the form:} \]

\[ \sum_{x \in \mathbb{Z}} \psi(j - k) f(j), \quad k \in \mathbb{Z}. \]

Appropriate growth conditions on $\psi$ and $f$ are assumed such that the series above converges absolutely.

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We assume that both Fourier transform and inverse Fourier transform have been properly extended to the Schwartz class of tempered distributions. Let $\mathbb{W}$ denote the collection of all functions $\psi \in C(\mathbb{R})$ for which there exist an $\epsilon > 0$ and a $C > 0$, such that the following inequality holds true:

$$|\psi(x)| \leq Ce^{-|x|^{1+\epsilon}}, \quad x \in \mathbb{R}.$$ 

Each $\psi \in \mathbb{W}$ induces a periodic function $\tilde{\psi}$ on $\mathbb{R}$:

$$\tilde{\psi}(x) := \sum_{z \in \mathbb{Z}} \psi(z)e^{2\pi i zx}, \quad x \in \mathbb{R}.$$ 

The period of the above function is 1. We will use $[0, 1]$ as the fundamental interval. We are particularly interested in the subset $\mathbb{W}^*$ of $\mathbb{W}$ defined by

$$\mathbb{W}^* := \{ \psi \in \mathbb{W} : \tilde{\psi}(x) \neq 0, \quad x \in [0, 1] \}. \quad (1)$$

For $\psi \in \mathbb{W}^*$, Wiener’s lemma [8, p. 228] asserts that there exists a sequence of complex numbers $a_z$, $z \in \mathbb{Z}$, such that $\sum_{z \in \mathbb{Z}} |a_z| < \infty$, and

$$\frac{1}{\psi(x)} = \sum_{z \in \mathbb{Z}} a_z e^{2\pi i zx}, \quad x \in \mathbb{R}.$$ 

For this reason, we call the inequality as displayed in (1) Wiener’s condition.

Let $\mathcal{S}$ and $\mathcal{S}'$ denote, respectively, the Schwartz classes of functions and tempered distributions. For each given $0 \leq \sigma < \infty$, let $\mathbb{E}_\sigma$ denote the class of analytic functions of exponential type $\sigma$.

**Proposition 1.** Let $\psi \in \mathbb{W}^*$. For each $\sigma \geq 0$, and every $f \in \mathbb{E}_\sigma$, there exists a $g \in \mathbb{E}_\sigma$, such that

$$f(j) = \sum_{z \in \mathbb{Z}} g(z)\psi(j - z), \quad j \in \mathbb{Z}.$$ 

**Proof:** By the Paley-Wiener theorem [23, p. 162], we may write

$$f = \hat{T}, \quad T \in \mathcal{S}', \quad \text{supp}(T) \subset [-\sigma, \sigma].$$

Since $\psi \in \mathbb{W}^*$, we have

$$\frac{1}{\psi(x)} = \sum_{z \in \mathbb{Z}} a_z e^{2\pi i zx}, \quad x \in \mathbb{R}.$$
Here $\sum_{z \in \mathbb{Z}} |a_z| < \infty$. The following equation defines $\frac{T}{\psi}$ as a Schwartz class distribution:

$$\langle \frac{T}{\psi}, \phi \rangle := \sum_{z \in \mathbb{Z}} a_z \langle T, e_z \cdot \phi \rangle, \quad \phi \in S.$$ 

Here $e_z$ denotes the function $x \mapsto e^{2\pi i z x}$. We also have

$$\text{supp} \left( \frac{T}{\psi} \right) \subset [-\sigma, \sigma].$$

It follows that $g := \left( \frac{T}{\psi} \right)^\wedge$ is a function of exponential type $\sigma$. Now consider the function

$$f^*(x) := \sum_{z \in \mathbb{Z}} g(x - z) \psi(z).$$

Fix each fixed $M > 0$ and every $x \in [-M, M]$, we have

$$\left| \sum_{z \in \mathbb{Z}} g(x - z) \psi(z) \right|$$

$$\leq C \sum_{z \in \mathbb{Z}} \exp \left( \sigma |x - z| - |z|^{1+\epsilon} \right)$$

$$\leq C e^{\sigma M} \sum_{z \in \mathbb{Z}} \exp \left( \sigma |z| - |z|^{1+\epsilon} \right) < \infty.$$ 

Thus the series converges uniformly on every compact subset of $\mathbb{R}$. Therefore the function $f^*$ is continuous on $\mathbb{R}$. We calculate its inverse Fourier transform:

$$(f^*)^\vee = (g)^\vee \cdot \tilde{\psi} = \frac{T}{\psi} \cdot \tilde{\psi} = T.$$ 

That is, both $f$ and $f^*$ are the Fourier transform of the distribution $T$, meaning that they are the same function. In particular, we have

$$\sum_{z \in \mathbb{Z}} g(j - z) \psi(z) = \sum_{z \in \mathbb{Z}} g(z) \psi(j - z) = f(j), \quad j \in \mathbb{Z}.$$ 

This completes the proof. $\square$

For the uniqueness of the coefficients, we have the following result.
Proposition 2. Assume that $0 < \epsilon < \pi$. Let $g \in E_{\pi-\epsilon}$, and let $\psi \in C(\mathbb{R})$ be such that $\sum_{z \in \mathbb{Z}} |\psi(z)| < \infty$, and the series $\sum_{z \in \mathbb{Z}} g(z)\psi(x-z)$ converges uniformly on every compact subset of $\mathbb{R}$. Thus the function $f$ defined by the following “discrete convolution”:

$$f(x) := \sum_{z \in \mathbb{Z}} g(z)\psi(x-z),$$

is continuous on $\mathbb{R}$. Assume furthermore that the periodic function $\tilde{\psi}$ satisfies Wiener’s condition. Then, in order that $f(j) = 0$, $j \in \mathbb{Z}$, it is necessary and sufficient that $g(\zeta) = 0$, $\zeta \in \mathbb{C}$.

Proof: Of course, only the necessity part needs a proof. Assume that $f(j) = 0$, $j \in \mathbb{Z}$. Write

$$f(j) = \sum_{z \in \mathbb{Z}} g(z)\psi(j-z) = \sum_{z \in \mathbb{Z}} g(j-z)\psi(z), j \in \mathbb{Z},$$

and consider the function $f^*$ defined by,

$$f^*(x) = \sum_{z \in \mathbb{Z}} g(x-z)\psi(z).$$

The (distributional) Fourier transform of $f^*$ can be easily calculated to be

$$\hat{f}^*(\xi) := \hat{g}(\xi) \sum_{z \in \mathbb{Z}} \psi(z)e^{2\pi i \xi z} = \hat{g}(\xi) \tilde{\psi}(\xi).$$

Since $g \in E_{\pi-\epsilon}$, $\hat{g}$ is supported in $[-\pi+\epsilon, \pi-\epsilon]$, as then is $\hat{f}$. Hence $f^* \in E_{\pi-\epsilon}$ also. Resorting to Carlson’s theorem (an analytic function in $E_{\pi-\epsilon}$ which is zero on the positive integers is identically zero [19]), we reach the conclusion that $f^*(\zeta) = 0$, $\zeta \in \mathbb{C}$. The Fourier transform of $f^*$ is therefore also zero. Since $\hat{f}^*(\zeta) = \hat{g}(\zeta)\tilde{\psi}(\zeta)$, and $\tilde{\psi}(\zeta) \neq 0$, we have $\hat{g}(\zeta) = 0$, $\zeta \in \mathbb{R}$. That is, $\hat{g}$ is the zero distribution. Hence $g$ is identically zero.

Propositions 1 and 2 imply the following result.

Corollary 1. Let $0 < \epsilon < \pi$ be given, and let $\psi \in \mathbb{W}^*$. For each $g \in E_{\pi-\epsilon}$, there exists a unique $f \in E_{\pi-\epsilon}$, such that

$$\sum_{z \in \mathbb{Z}} \psi(j-z)f(z) = g(j), \quad j \in \mathbb{Z}. $$
Suppose that \( f \) is radial (even), and that for some \( \delta > 0 \) we have
\[
|f(x)| \leq A(1 + x^2)^{-(1/2+\delta)}, \quad |\hat{f}(\xi)| \leq A(1 + \xi^2)^{-(1/2+\delta)},
\]
where \( A > 0 \) is a constant. Then the following Poisson summation formula holds true; see [22, p.252].
\[
\sum_{z \in \mathbb{Z}} f(z) e^{2 \pi i z x} = \sum_{z \in \mathbb{Z}} \hat{f}(x + z).
\]
Thus, Wiener’s condition is satisfied if both \( \psi \) and \( \hat{\psi} \) have the above decay rate, and \( \hat{\psi} \) is positive. Specifically, the Gaussian kernel satisfies this condition.

In the rest of this section, we concentrate on interpolating polynomials by integer translates of a fixed function \( \psi \in C(\mathbb{R}) \) with appropriate decay condition. We remind readers that for any \( 0 < \epsilon < \pi \), and any polynomial \( p \), we have \( p \in \mathbb{E}_{\pi - \epsilon} \). To interpolate a polynomial, we do not need the extra decay rate as imposed on functions from \( \mathbb{W} \). Any function \( \psi \in S \) satisfying Wiener’s condition will suffice. For such a \( \psi \), all the moments \( M_k \):
\[
M_k = \sum_{z \in \mathbb{Z}} z^k \psi(z),
\]
are finite.

Let \( p_k(x) = x^k \), and define
\[
I^\psi[p_k](x) := \sum_{z \in \mathbb{Z}} a_z \psi(x - z).
\]
We will show that for interpolation of polynomials the coefficients \( a_z \) are polynomial in \( z \). Furthermore, the polynomial coefficients are constructible in a recursive fashion.

**Theorem 3.** For \( k = 0, 1, \cdots, n \),
\[
\sum_{z \in \mathbb{Z}} z^k \psi(z - j) = \sum_{m=0}^{k} \binom{k}{m} M_{k-m} j^m.
\]

**Proof:** We first observe that
\[
M_k = \sum_{z \in \mathbb{Z}} (z - j)^k \psi(z - j)
\]
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by simply renumbering the sum. Thus the case \( k = 0 \) is trivial. Let us now assume the result is true for all \( 0 \leq m \leq k < n \). Then, for \( j \in \mathbb{Z} \), since \( k + 1 \leq n \),

\[
M_{k+1} = \sum_{z \in \mathbb{Z}} (z - j)^{k+1} \psi(z - j)
\]

\[
= \sum_{z \in \mathbb{Z}} \left\{ \sum_{m=1}^{k+1} (-1)^m \binom{k+1}{m} z^{k+1-m} j^m \right\} \psi(z - j)
\]

\[
+ \sum_{z \in \mathbb{Z}} z^{k+1} \psi(z - j).
\]

We can rearrange the first sum above

\[
\sum_{z \in \mathbb{Z}} \left\{ \sum_{m=1}^{k+1} (-1)^m \binom{k+1}{m} z^{k+1-m} j^m \right\} \psi(z - j)
\]

\[
= \sum_{m=0}^{k} (-1)^m \binom{k+1}{m} j^m \left\{ \sum_{z \in \mathbb{Z}} z^{k+1-m} \psi(z - j) \right\}
\]

\[
= \sum_{m=0}^{k} (-1)^m \binom{k+1}{m} j^m \left\{ \sum_{l=0}^{k+1-m} \binom{k+1-m}{l} M_{k+1-m-l} j^l \right\}
\]

\[
= \sum_{s=1}^{n+1} (-1)^s M_{k+1-s} j^s \sum_{l=0}^{s-1} (-1)^l \binom{k+1}{s-l} \binom{k+1-s+l}{l},
\]

where we have put \( m + l = s \). The sum on the right hand side can be simplified by using the following formula:

\[
\binom{k+1}{s-l} \binom{k+1-s+l}{l} = \frac{(k+1)!(k+1-s+l)!}{(k+1-s+l)!(s-l)!(k-1+s)l!}
\]

\[
= \frac{(k+1)!}{s!(k+1-s)!(s-l)l!}
\]

\[
= \binom{k+1}{s} \binom{s}{l}.
\]
Therefore, we have

\[
\sum_{l=0}^{s-1} (-1)^l \binom{k+1}{s-l} \binom{k+1-s+l}{l} = \binom{k+1}{s} \sum_{l=0}^{s-1} (-1)^l \binom{s}{l} = (-1)^{s+1} \binom{k+1}{s}.
\]

Here we have used the simple fact that

\[0 = (1 - 1)^s = \sum_{l=0}^{s-1} (-1)^l \binom{s}{l} + (-1)^s.\]

Substituting this into (3), we have

\[
\sum_{z \in \mathbb{Z}} \left\{ \sum_{m=1}^{k+1} (-1)^m \binom{k+1}{m} z^{k+1-m} j^m \right\} \psi(z - j) = \sum_{s=1}^{k+1} (-1)^s (-1)^{s+1} M_{k+1-s} \binom{k+1}{s} j^s,
\]

which, upon being combined with Equation (2), gives us that

\[
M_{k+1} = -\sum_{s=1}^{k+1} M_{k+1-s} \binom{k+1}{s} j^s + \sum_{z \in \mathbb{Z}} z^{k+1} \psi(j - z).
\]

Rearranging the last equation we have

\[
\sum_{z \in \mathbb{Z}} z^{k+1} \psi(j - z) = \sum_{s=0}^{k+1} M_{k+1-s} \binom{k+1}{s} j^s,
\]

completing the induction. \(\square\)

Let \(q^\psi_k(x) = \sum_{i=0}^{k} M_{k-i} \binom{k}{i} x^i\), and

\[
I^\psi[p_k](x) = \sum_{z \in \mathbb{Z}} b_{k,z} \psi(z - x),
\]

be the interpolant to the monomial \(p_k(x) = x^k\), where the coefficients \(b_{k,z}\) are to be determined in the sequel.
Suppose we fix $k$ and let $A_{l,s} = M_{l-s}(l^{-1})$, $l, \ldots, k$, $s = 1, \ldots, l$. Then, since
\[
\sum_{z \in \mathbb{Z}} z^{l-1} \psi(j - z) = \sum_{s=1}^{l} A_{l,s} j^{s-1}, \quad l = 1, \ldots, k, \tag{4}
\]
we can use backward elimination to see that
\[
j^{l-1} = \sum_{z \in \mathbb{Z}} \left\{ \sum_{s=1}^{l} B_{l,s} z^{s-1} \right\} \psi(j - z), l = 1, \ldots, k,
\]
for some numbers $B_{l,s}$, $s = 1, \ldots, l$, $l = 1, \ldots, k$. Let $A$ and $B$ be the matrices with entries $A_{l,s}, B_{l,s}$, $s = 1, \ldots, l$, $l = 1, \ldots, k$, and zero otherwise, respectively. Then, substituting (4) into the last equation we have
\[
j^{l-1} = \sum_{s=1}^{l} B_{l,s} \sum_{m=1}^{s} A_{s,m} j^{m-1}
= \sum_{s,m=1}^{n} B_{l,s} A_{s,m} j^{m-1}, \quad l = 1, \ldots, k,
\]

since the additional matrix entries in the sum are zero. Equating coefficients of $j^{l-1}$, $l = 1, \ldots, k$ we have $B = A^{-1}$. We also have the following important relationship:
\[
A_{i,j} = \frac{i}{j} A_{i+k,j+k}, \quad 1 \leq j \leq i \leq n+1, \quad k \leq n - i. \tag{5}
\]

**Theorem 4.** For $k \in \mathbb{N}$,
\[
\sum_{z \in \mathbb{Z}} (z - x)^k \psi(z - x) = \sum_{i=0}^{k} \binom{k}{i} q_i^\psi(-x) I^\psi[p_{k-i}](x). \tag{6}
\]

**Proof:** Let $j \in \mathbb{Z}$. Denote $A_i$ the $i$th row of $A$ and $B_i^T$, the $i$th row of
B. We have

\[ \sum_{i=0}^{k} \binom{k}{i} q_i \psi(-x) I_\psi[p_{k-i}](x) \]

\[ = \sum_{i=0}^{k} \binom{k}{i} \left\{ \sum_{j=1}^{i+1} A_{i,j} (-x)^{i-j} \right\} \sum_{z \in \mathbb{Z}} \left\{ \sum_{l=1}^{k-i+1} B_{k-i+1,l} z^l \right\} \psi(z-x) \]

\[ = \sum_{z \in \mathbb{Z}} \left\{ \sum_{i=0}^{k} \sum_{l=1}^{k-i-j} (-1)^{i-l} x^i z^j \left( \frac{k}{i+1} \right) \sum_{l=1}^{k-i-l} A_{k+1-i-l} B_{k+2-i-l,j+1} \right\} \psi(z-x) \]

using (5). Therefore,

\[ \sum_{i=0}^{k} \binom{k}{i} q_i \psi(-x) I_\psi[p_{k-i}](x) \]

\[ = \sum_{z \in \mathbb{Z}} \left\{ \sum_{i=0}^{k} \sum_{j=0}^{i} (-1)^{i-j} x^i z^j \left( \frac{k}{i} \right) \sum_{l=1}^{k-i-j} A_{k+1-i-l} B_{k+2-l-j+1} \right\} \psi(z-x) \]

where the second equation follows from the fact that \( B = A^{-1} \).

Let us now define the error in interpolation \( E^\psi_k(x) = I_\psi[p_k](x) - p_k(x) \), and

\[ \chi_k(x) = \sum_{z \in \mathbb{Z}} (z-x)^k \psi(z-x) - M_k, \quad k = 0, 1, \cdots. \]

Then we have the following result for the errors in interpolation:

**Corollary 2.** For \( k = 0, 1, \cdots \),

\[ \chi_k(x) = \sum_{i=0}^{k} \binom{k}{i} q_i \psi(-x) E^\psi_{k-i}(x). \]
Proof: Since \( I^\psi[p_{k-i}](j) = p_{k-i}(j) \), \( i = 0, 1, \cdots, k, j \in \mathbb{Z} \), by Theorem 4, we have for \( k \in \mathbb{Z} \),

\[
\sum_{i=0}^{k} \binom{k}{i} q_i^\psi(-j)j^k = \sum_{z \in \mathbb{Z}} (j - z)^k \psi(j - z) = M_k.
\]

Since a polynomial is uniquely determined by its values on the integers we have

\[
\sum_{i=0}^{k} \binom{k}{i} q_i^\psi(-x)x^k = M_k.
\]

Subtracting this equation from (6) we see that

\[
\chi_k(x) = \sum_{i=0}^{k} \binom{k}{i} q_i^\psi(-x)I^\psi[p_{k-i}](x) - M_k
\]

\[
= \sum_{i=0}^{k} \binom{k}{i} q_i^\psi(-x)I^\psi[p_{k-i}](x) - \sum_{i=0}^{k} \binom{k}{i} q_i^\psi(-x)p_{k-i}(x)
\]

\[
= \sum_{i=0}^{k} \binom{k}{i} q_i^\psi(-x)(I^\psi[p_{k-i}](x) - p_{k-i}(x))
\]

\[
= \sum_{i=0}^{k} \binom{k}{i} q_i^\psi(-x)E^\psi_{k-i}(x). \quad \square
\]

Remark 1. Theorem 4 gives a recursive formula for computing the interpolant of any polynomial, as long as one knows the interpolants for polynomials of lower degree. Likewise, the result of the corollary expresses the error between a polynomial and its interpolant in the same fashion. If the Gaussian kernel is employed to do interpolation, then we will have more interesting information to offer. We will pick up this topic in the next section.

3 The Gaussian Kernel

In this section we devote our study more specifically to the case of \( \psi(x) = \frac{\exp(x^2/2)}{\sqrt{2\pi}} \). Pertinent to the contents of this paper will be the probabilistic
Hermite polynomials $H_e_k, k = 0, 1, \cdots$. These may be defined in a number of ways, but for us perhaps the most appropriate is via Rodrigues formula:

$$H_e_k(x) := \exp(x^2/2) \frac{d^k}{dx^k} \exp(-x^2/2), \quad k = 0, 1, \cdots.$$ 

We have the following explicit representation of these polynomials (see e.g. [2]):

$$H_e_k(x) = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^{k-i} \frac{k!}{i!(k-2i)!} \frac{x^{k-2i}}{2^i}, \quad k = 0, 1, \cdots,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$. The polynomial $H_e_k$ has a close cousin that is often referred to as the probabilistic polynomial of negative variance:

$$N_e_k(x) = \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{k!}{i!(k-2i)!} \frac{x^{k-2i}}{2^i}, \quad k = 0, 1, \cdots,$$

which have the same coefficients in absolute value, but the coefficients are all positive.

The probabilistic polynomials of negative variance arise very naturally in this study as they are the result of the continuous convolution of the Gaussian with the polynomials of appropriate degree:

**Lemma 1.** Let $\psi(x) = \exp(-x^2/2)$. Then

$$N_e_k(x) = \int_{-\infty}^{\infty} y^k \psi(y-x)dy$$

$$= \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i} C_{2i} x^{k-2i},$$

where

$$C_k = \int_{-\infty}^{\infty} y^k \psi(y)dy = \begin{cases} (k-1)!!, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

**Proof:** It is well-known (see e.g. [18]) that

$$\int_{-\infty}^{\infty} y^k \psi(y)dy = \begin{cases} (k-1)!!, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$
To prove the first equation, we make a simple change of variable \( w = y - x \):

\[
\int_{-\infty}^{\infty} y^k \psi(x - y) dy = \int_{-\infty}^{\infty} (w + x)^k \psi(w) dw
\]

\[
= \sum_{i=0}^{k} \binom{k}{i} x^i \int_{-\infty}^{\infty} w^{k-i} \psi(w) dw
\]

\[
= \sum_{i=0}^{k} \binom{k}{i} C_{k-i} x^i
\]

\[
= \sum_{i=0}^{k} \binom{k}{i} C_i x^{k-i}
\]

\[
= \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{i} C_{2i} x^{k-2i}.
\]

We remind readers that all the odd degree terms have zero coefficient. If we substitute the value for \( C_{2i} \) we see that \( Ne_k \) is the probabilistic Hermite polynomial of negative variance:

\[
Ne_k(x) = \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{k!}{i!(k-2i)!} x^{k-2i} \quad \square
\]

A fascinating relationship between \( He_k \) and \( Ne_k \) is the so-called umbral composition (see [2]):

\[
\sum_{i=0}^{\lfloor k/2 \rfloor} \frac{k!}{i!(k-2i)!} \frac{He_{k-2i}(x)}{2^i} = \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{k!}{i!(k-2i)!} \frac{Ne_{k-2i}(x)}{2^i} = x^k.
\]

Using Lemma \( \Box \) and the second equation above we have

\[
x^k = \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{(-1)^i}{2^i i!(k-2i)!} \int_{-\infty}^{\infty} y^{k-2i} \psi(x - y) dy
\]

\[
= \int_{-\infty}^{\infty} He_k(y) \psi(y - x) dy,
\]

(7)

so that we can recover the monomials by integrating against the probabilistic Hermite polynomials. Of course, this gives us an idea of what will happen in the discrete case.
To do this, we need an analogue of the probabilistic Hermite polynomial for the discrete case. We define
\[
\tilde{H}_k(x) = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{k}{2i} M_{2i} x^{k-2i}, \quad k = 0, 1, \ldots.
\] (8)

We also let
\[
\tilde{N}_k = q_k^\psi, \quad k = 0, 1, \ldots,
\]
when \(\psi\) is the Gaussian. In other words
\[
\tilde{N}_k(x) = \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i} M_{2i} x^{k-2i}.
\] (9)

Equation (7) suggests that a good prototype for a closed formula for the interpolant \(I_\psi[p_k](x)\) is
\[
\sum_{z \in \mathbb{Z}} \tilde{H}_e(z) \psi(z-x), \quad x \in \mathbb{R}.
\]

In the next result, we will show that this is indeed the case (up to a constant very close to 1) for \(k = 0, 1, 2, 3\).

**Lemma 2.** For \(k = 0, 1, \ldots, \) and \(j \in \mathbb{Z}\),
\[
\sum_{z \in \mathbb{Z}} \tilde{H}_e(z) \psi(j-z) = \sum_{i=0}^{\lfloor k/4 \rfloor} \left\{ \sum_{j=0}^{2i} (-1)^j \binom{k}{2j} \binom{k-2j}{4i-2j} M_{2j} M_{4i-2j} \right\} j^{k-4i}.
\]

**Proof:** By Theorem 3, Equations (9) and (5), we have
\[
\sum_{z \in \mathbb{Z}} \tilde{H}_e(z) \psi(j-z) = \sum_{z \in \mathbb{Z}} \left\{ \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{k}{2i} M_{2i} z^{k-2i} \right\} \psi(j-z)
\]
\[
= \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{k}{2i} M_{2i} \left\{ \sum_{z \in \mathbb{Z}} z^{k-2i} \psi(j-z) \right\}
\]
\[
= \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{k}{2i} M_{2i} \left\{ \sum_{j=0}^{\lfloor (k-2i)/2 \rfloor} (-1)^j \binom{k-2i}{2j} M_{2j} x^{k-2i-2j} \right\}.
\]
Rearranging we obtain that
\[
\sum_{z \in \mathbb{Z}} \tilde{e}(z)\psi(j - z) = \sum_{i=0}^{\lfloor k/2 \rfloor} \left\{ \sum_{j=0}^{i} (-1)^{j} \binom{k}{2j} \binom{k - 2j}{2i - 2j} M_{2j} M_{2i - 2j} \right\} j^{k - 2i}
\]
\[
= \sum_{i=0}^{\lfloor k/4 \rfloor} \left\{ \sum_{j=0}^{2i} (-1)^{j} \binom{k}{2j} \binom{k - 2j}{4i - 2j} M_{2j} M_{4i - 2j} \right\} j^{k - 4i},
\]
where we have used the fact that, if \(i\) is odd, then
\[
\sum_{j=0}^{i} (-1)^{j} \binom{k}{2j} \binom{k - 2j}{4i - 2j} M_{2j} M_{4i - 2j} = 0,
\]
which is true because
\[
\binom{k}{2j} \binom{k - 2j}{4i - 2j} M_{2j} M_{4i - 2j} = k! \left( \frac{(k - 2j)! (2i)! (2j)!}{(k - 2i + 2j)!} \right) \]
\[
= \left( \frac{k!}{(k - 2i + 2j)! (2i)! (2j)!} \right) \left( \binom{k}{2i} \binom{k - 2(i - j)}{2j} \right).
\]

As we can see, for \(k = 0, 1, 2, 3\), the above lemma gives an exact formula. Interestingly, if \(M_k = C_k\), \(k = 0, 1, \cdots\), then the correction terms above are all zero. This is why we get the umbral composition formula for the probabilistic Hermite polynomials. For higher degrees we need to modify the polynomial in the summation for interpolation. To this end we introduce the polynomials \(Q_k(x)\), which we define by
\[
Q_k = \frac{1}{M_0^2} N e_k, \quad k = 0, 1, 2, 3,
\]
and for \(k = 4, 5, \cdots\),
\[
Q_k = \frac{1}{M_0^2} N e_k - \sum_{i=1}^{\lfloor k/4 \rfloor} \left\{ \sum_{j=0}^{2i} (-1)^{j} \binom{k}{2j} \binom{k - 2j}{4i - 2j} M_{2j} M_{4i - 2j} \right\} Q_{k - 4i}.
\]

Using Lemma 2 we immediately get the main result of this section

**Theorem 5.** For \(k = 0, 1, \cdots\), and \(\psi\) the Gaussian, we have
\[
I_\psi[p_k](x) = \sum_{z \in \mathbb{Z}} Q_k(z)\psi(z - j).
\]
4 A Newton Type Gaussian Interpolant

In this section, we give a closed formula of a new Gaussian kernel interpolant to a function defined at equally spaced points \(0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\). The construction of this interpolant utilizes the full \((1/n)\) spaced infinite grid. As such, it is different from most of Gaussian kernel interpolants constructed with conventional procedures. However, for all the practical computational purposes, only a small number of centres outside of the interval of interpolation are required. The rapid decay of the Gaussian kernel offsets the error incurred by dropping terms (shifts) of the interpolant far from the interpolation interval.

For \(f \in C(\mathbb{R})\), let \(S_tf(x) = f(tx)\) denote the dilation operator. Given \(n \in \mathbb{N}\) let \(h = 1/n\). Let \(\psi_h = nS_h\psi\), and \(I_{\psi}^{n}[p]\) be the interpolant to \(p \in \Pi_k\), the polynomials of degree \(k\), on \(h\mathbb{Z}\) using \(\psi_h\) as the basis function. Then,

\[
I_{\psi}^{n}[p] = S_hI_{\psi}[S_np], \quad p \in \Pi_k.
\]

First we construct the Newton form of the interpolant to \(S_nf \in C[0,1]\) defined on \(X_n = \{i, i = 0, 1, \cdots, n\}\). The Newton polynomials are

\[
R_{n,k}(x) = \prod_{i=0}^{k}(x - i), \quad k = 0, \cdots, n.
\]

For a given \(n \in \mathbb{N}\), let us define the \(k\)th order divided differences, \(k = 0, 1, \cdots, n\) of the function \(f\) by \([f]_{n,0}(i) = f(i), i = 0, 1, \cdots, n\) and

\[
[f]_{n,k}(i) = \frac{[f]_{n,k-1}(i + 1) - f_{n,k-1}(i)}{k}, \quad i = 0, 1, \cdots, n - k. \quad (10)
\]

Then the polynomial interpolant to \(S_nf\) on \(nX\) is

\[
T_n(x) = \sum_{k=0}^{n}[S_nf]_{n,k}(0)R_{n,k}(x).
\]

In order to make use of the results of the previous section, we need to write the Newton polynomials in monomial form

\[
R_{n,k}(x) = \prod_{i=0}^{k}(x - i) = \sum_{i=0}^{k}s_{k,i}x^i,
\]
where \(s_{k,i}, i = 0, 1, \ldots, k\) are the Stirling numbers of the first kind; see e.g. [1, Chapter 24]. Thus,

\[
T_n(f) = \sum_{k=0}^{n} [S_n f]_{n,k}(0) \sum_{i=0}^{k} s_{k,i} x^i
\]

\[
= \sum_{i=0}^{n} x^i \sum_{k=i}^{n} [S_n f]_{n,k}(0) s_{k,i}.
\]

Using the results of the previous section we immediately have the main result of this section.

**Theorem 6.** Let \(f \in C[0, 1]\), \(n \in \mathbb{N}\) and \(h = 1/n\). Then

\[
f(ih) = \sum_{z \in \mathbb{Z}} P_n[f](z) \psi(z - i), \quad i = 0, 1, \ldots, n,
\]

where

\[
P_n[f] = \sum_{i=0}^{n} C_{n,i}[f] Q_i,
\]

with

\[
C_{n,i}[f] = \sum_{k=i}^{n} [S_n f]_{n,k}(0) s_{k,i}.
\]

The numbers \([S_n f]_{n,k}(0)\) are defined by (10) and \(s_{k,i}\) are Stirling numbers of the first kind.

### 5 Main Conclusion

As main results of this paper, we have shown that the interpolant to a polynomial using a suitable kernel has polynomial coefficients. More importantly, a kernel interpolant to a polynomial is constructible recursively, as is the way in which we express the error between the polynomial and its kernel interpolant. For the Gaussian kernel, we provide closed formulas for the coefficients of the kernel interpolant to a polynomial. These are given in terms of a new class of polynomials that closely resemble the classical probabilistic Hermite polynomials. Via interpolating polynomials, we find a way to construct a kernel interpolant to a function defined on an equally-spaced grid of
a compact interval. In theory, this interpolant uses shifts of the kernel on a full infinite grid. In numerical implementation, however, only a small number of shifts of the kernel centered outside of the interpolation interval is needed thanks to the rapid decay of the kernel. These have cleared the way for our future work in which we will investigate numerical aspects of this process. Our goals are to obtain stable and efficient algorithms for the computation of the interpolant, and to develop error estimates for $C^k$—functions having polynomial growth. With the stationary interpolation scheme, the error will not go to zero as the grid spacing contracts, but the errors estimate will be useful for analysing the residual approximation algorithm that is detailed in [10].

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