Diversification and limited information in the Kelly game

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The Kelly game

1. in one turn, a fraction $f$ of the current wealth can be invested
   - with the probability $p$, the invested amount is doubled
   - with the probability $1 - p$, the invested amount is lost

2. repeat (infinitely) many times

3. winning probability $p$ is constant and known
The Kelly game

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**Question**: How to find the optimal investment fraction?
The Kelly game

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**question:** how to find the optimal investment fraction?

**well-known answer:** maximise the exponential growth rate

\[
G(f) := \langle \ln (1 + fR_1) \rangle
\]

$R_1 =$ game return on one-turn basis
The Kelly game

- optimal investment fraction

\[ f^*(p) = \begin{cases} 
0 & p \in [0; \frac{1}{2}] \\
2p - 1 & p \in (\frac{1}{2}; 1] 
\end{cases} \]

- optimal growth rate

\[ G^*(p) = \ln 2 + p \ln p + (1 - p) \ln(1 - p) \]
The Kelly game

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- in real life:
  - simultaneous games
  - unknown game properties
  - \ldots

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Insider vs outsider: intro

- $M$ games simultaneously played
- **insider strategy**: know one game better
- **outsider strategy**: gain by diversification
Insider vs outsider: intro

- \( M \) games simultaneously played
- **insider strategy**: know one game better
- **outsider strategy**: gain by diversification
- when the outsider outperforms the insider?
  - 1. little insider’s information
  - 2. extensive outsider’s diversification
Insider vs outsider: framework

1. $M$ games

2. The winning probability of each game is either $p - \Delta$ or $p + \Delta$ (changing each turn randomly)

3. Insider knows the exact winning probability for one game (no diversification)

4. Outsider knows only the average winning probability $p$ (invests evenly in $M$ games)
Insider vs outsider: results

first approximation: $\Delta \approx (p - \frac{1}{2}) (\sqrt{2M} - 1)$
Limited information: framework

- even “noisy” information in the form $p \pm \Delta$ is artificial

- let’s assume that we use only $T$ past turns for learning

⋯L W W W L W W L L W
Limited information: framework

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\[
\begin{array}{cccccccc}
\ldots & \text{L W W W L W W L L W} \\
\end{array}
\]

↓

information about the game
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information about the game

our investment decision
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\[ \ldots L W W W L W W L L W \]

\[ T \text{ turns, } w \text{ wins} \]

↓

information about the game

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\[ \ldots \text{L W W W L W W L L W} \]
\[ \downarrow \]
information about the game
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our investment decision

$T$ turns, $w$ wins

$\varrho(p|w, T)$
even “noisy” information in the form $p \pm \Delta$ is artificial

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\[
\begin{array}{c}
\text{...L W W W L W W L L W} \\
\downarrow \\
\text{information about the game} \\
\downarrow \\
\text{our investment decision}
\end{array}
\]

$T$ turns, $w$ wins

$\varrho(p|w, T)$

$f^*(w, T)$
for any \( \varrho(p) \), \( G := \langle \ln(1 + f R_1) \rangle \) is maximised by

\[
f^*[\varrho] = 2\langle p \rangle_\varrho - 1
\]
Limited information: derivation

- For any $\varphi(p)$, $G := \langle \ln(1 + f R_1) \rangle$ is maximised by
  \[ f^*[\varphi] = 2\langle p \rangle_\varphi - 1 \]

- Observing $w$ wins in $T$ turns gives us the information
  \[ \varphi(p|w, T) \propto \pi(p) P(w|p, T) \]

- Here $P(w|p, T)$ is the binomial distribution
  \[ P(w|p, T) = \binom{T}{w} p^w (1 - p)^{T-w} \]

- $\pi(p)$ is the prior distribution of $p$
Limited information: results

- no prior information about the game: \( \pi(p) = 1 \) for \( p \in [0; 1] \)
Limited information: results

- no prior information about the game: $\pi(p) = 1$ for $p \in [0; 1]$
- the optimal investment fraction is

$$f^*(w, T) = \begin{cases} 
0 & w \leq T/2 \\
\frac{2w - T}{T + 2} & w > T/2 
\end{cases}$$

- two interesting cases:

$$\lim_{T \to \infty} f^*(w, T) = 2 \lim_{T \to \infty} \frac{w}{T} - 1 = 2p - 1$$

$$f^*(T, T) = \frac{T}{T + 2} < 1$$
Limited information: results

\[ G^* (p, T) \approx \ln 2 + p \ln p + (1 - p) \ln (1 - p) \]

Perfect information

\[ 1 - \frac{1}{2T} \]

Limited information

\[ p = 0.52, p = 0.60, p = 0.70 \]
Limited information: results

\[ G^*(p, T) \approx \ln 2 + p \ln p + (1 - p) \ln (1 - p) - \frac{1}{2T} \]

- Perfect information
- Limited information

![Graph showing the relationship between real return/ideal return and memory length for different values of p.](image)
The role of prior information

- what is $\pi(p)$?
  - a way how to quantify our prior lack of information
The role of prior information

- what is $\pi(p)$?
  1. a way how to quantify our prior lack of information
  2. aggregate information about $p$ evolving in time

Matúš Medo (University of Fribourg)  Diversification and limited information
Simple up and down economy

- simple pattern:
  - 80 good turns ($p = 0.8$)
  - 20 bad turns ($p = 0.2$)
  - repeated many times
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- perfect information: return 16.7%
“There cannot be a sure-win game!”
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- set $\pi(p) = 0$ for $p > p_{\text{max}}$
Additional sources of information

- “There cannot be a sure-win game!”
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- “Great, I have my posterior $P(p|w, T)$ but what if...”
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- set $P(\text{crisis comes}) = P_c$
- why necessary?
  because with enough data, prior beliefs are overruled!
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- our framework is too simple to allow for more realistic considerations…
we have seen:

- diversification and limited information in toy systems
- simple analytical results
Conclusion

- **we have seen:**
  - diversification and limited information in toy systems
  - simple analytical results

- **we haven’t seen:**
  - realistic risky games (e.g., log-normal returns)
  - all capabilities of the prior information $\pi(p)$
  - less frequent portfolio rebalancing
  - transaction costs
  - …
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Thank you for your attention