Simulating symmetric time evolution with local operations

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New Journal of Physics \textbf{14} (2012) 123026 (28pp)
Received 23 July 2012
Published 14 December 2012
Online at http://www.njp.org/
doi:10.1088/1367-2630/14/12/123026

Abstract. In closed systems, dynamical symmetries lead to conservation laws. However, conservation laws are not applicable to open systems that undergo irreversible transformations. More general selection rules are needed to determine whether, given two states, the transition from one state to the other is possible. The usual approach to the problem of finding such rules relies heavily on group theory and involves a detailed study of the structure of the respective symmetry group. We approach the problem in a completely new way by using entanglement to investigate the asymmetry properties of quantum states. To this end, we embed the space state of the system in a tensor product Hilbert space, whereby symmetric transformations between two states are replaced with local operations on their bipartite images. The embedding enables us to use the well-studied theory of entanglement to investigate the consequences of dynamic symmetries. Moreover, under reversible transformations, the entanglement of the bipartite image states becomes a conserved quantity. These entanglement-based conserved quantities are new and different from the conserved quantities based on expectation values of the Hamiltonian symmetry generators. Our method is not group-specific and applies to general symmetries associated with any compact semi-simple Lie group.
1. Introduction

The evolution of most quantum systems is too complicated to be solved analytically or even simulated numerically in an efficient way, at least in the absence of powerful quantum computers [1, 2]. Many realistic situations involve either open dynamical systems or closed systems with Hamiltonians that contain numerous parameters, so that determining how they vary with time is at present not computationally tractable [3, 4]. In all such cases, symmetry-based approaches are powerful substitutes for actual detailed analysis of the complex dynamics involved. Noether’s theorem plays a central role in the study of dynamical symmetries of closed systems in classical mechanics [5]. The theorem, as well as its quantum mechanical counterparts, states that a Hamiltonian satisfying some symmetry is always accompanied by a corresponding conservation law [6].

On the other hand, open systems are more general and far more ubiquitous than closed systems. Their symmetric time evolutions are expressed by covariant completely positive (CP) maps that, in general, can be very different from unitary evolutions governed by symmetry-preserving Hamiltonians. Hence, no conserved quantities are associated with symmetric
Figure 1. Simulation of a covariant transformation $E_{\text{cov}}$ by an LOCC transformation $\tilde{E}_{\text{local}}$.

dynamics of open systems, and thus, the consequences of dynamical symmetries cannot always be reduced to selection rules based on conservation laws [7]. In fact, it was recently shown that, even for closed systems, conservation laws given by Noether’s theorem do not capture all the consequences of the symmetry in question [8]. It is therefore necessary to look beyond conservation laws in order to determine how states evolve under symmetric dynamics.

A system that has a certain symmetry cannot lose the symmetry as it evolves by a Hamiltonian or a master equation that preserves that type of symmetry. More generally, a state cannot become more asymmetric as it undergoes a symmetric time evolution. Comparing the asymmetry properties of two states can therefore be of great use in establishing whether one state can evolve to another under symmetric conditions. In other words, when a symmetry is imposed on the dynamics, the asymmetry of quantum states becomes a resource [7].

In this paper, we show that covariant CP-maps can be ‘simulated’ by a restricted subset of local operations and classical communications (LOCC). The key idea is to embed the system’s Hilbert space $\mathcal{H}$ within a larger tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$. The embedding is done with an isometry

$$\mathcal{H} \xrightarrow{\text{iso}} \mathcal{W} \subseteq \mathcal{H}_A \otimes \mathcal{H}_B$$

that has the following properties. First, the isometry maps symmetric states to separable states. Furthermore, consider two states $\rho$ and $\sigma$ that act on $\mathcal{H}$, and their corresponding bipartite image states $\tilde{\rho}_{AB}$ and $\tilde{\sigma}_{AB}$ that act on the image subspace $\mathcal{W}$. If there exists a covariant transformation $E_{\text{cov}}$ that maps $\rho$ to $\sigma$, i.e. $\sigma \equiv E_{\text{cov}}(\rho)$, then there must also exist a local transformation $\tilde{E}_{\text{local}}$ that maps $\tilde{\rho}_{AB}$ to $\tilde{\sigma}_{AB}$, i.e. $\tilde{\sigma}_{AB} = \tilde{E}_{\text{local}}(\tilde{\rho}_{AB})$ (figure 1). In this sense the local operator $\tilde{E}_{\text{local}}$ simulates the covariant map $E_{\text{cov}}$.

We show here that such isometries can be found for all covariant CP-maps that are associated with compact semi-simple symmetry Lie groups. Moreover, for any asymmetric state, we show that there exists an isometry that maps it to an entangled state. Hence, the entanglement in the image space captures all the asymmetry properties of the state. Our results follow from an application of the Wigner–Eckart theorem, generalized to all semi-simple groups, that determines the general form of the Kraus operators of covariant transformations [10].

The entanglement of the image state plays a somewhat similar role to biomarkers that are employed in biology in order to trace a biological process. Hence, the study of the evolution of entanglement governed by the LOCC map $\tilde{E}_{\text{local}}$ opens a new window to explore symmetric dynamics. In particular, it shows that the resource theory associated with the asymmetry of quantum states [7, 10] is equivalent to the resource theory of entanglement under a restricted subset of LOCC transformations.

A comprehensive collection of theorems and theoretical tools has been developed to study quantum entanglement for more than a decade [11–13]. The equivalence between

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3 http://pirsa.org/10120062/.
asymmetry and entanglement resources allows us to take advantage of the repertoire of tools of entanglement theory in order to study the asymmetry properties of quantum states. In particular, the established equivalence allows us to use any entanglement monotone and construct a corresponding ‘asymmetry monotone’. An asymmetry monotone, as the name suggests, is a real function defined on the set of quantum states such that its magnitude changes monotonically (i.e. non-increasing) during a symmetric evolution. In the case of reversible symmetric transformations, asymmetry monotones, of course, remain conserved. They can thus be regarded as generalizations of conserved quantities. Taking asymmetry monotones into account allows us to rule out classes of transformations that cannot be ruled out based on conservation laws alone.

A state that lacks a particular symmetry encodes information about the physical degrees of freedom that transform under that symmetry. In contrast, a symmetric state does not carry any such information. For example, the state of electrons with non-zero angular momentum along a particular direction in space is not symmetric under rotations and consequently encodes some information about that direction, whereas electrons in a rotationally invariant state of zero total angular momentum contain no information about any preferred direction.

So far, the study of asymmetry properties of quantum systems has mostly been focused on pure states. For example, interconversion of pure states under specific symmetry groups has been studied [10, 14, 15] and a general classification of pure-state asymmetry properties for arbitrary finite or compact Lie groups has been developed [7]. Prior to the present result, little was known about the general properties of mixed-state asymmetry and, with a few but important exceptions such as the $G$-asymmetry [16] (also known as the relative entropy of frameness [17]), asymmetry monotone functions of mixed states were not identified for symmetries associated with general groups.

Our work introduces a wide class of asymmetry monotones, defined for all states, pure or mixed. Some of the asymmetry monotones we construct can only be defined in terms of the entanglement of a bipartite system. A case in point is the negativity measure of entanglement [18]. Negativity is especially interesting as it provides us with an easily calculatable asymmetry monotone for all states and for all types of symmetry.

Although monotones are extremely useful tools in resource theories [12, 19], the conditions for the symmetric evolution of states need not always come in the form of asymmetry monotones. We derive a separate necessary condition for the existence of a covariant transformation from one state to another. However, the condition is such that it cannot be expressed in terms of asymmetry monotones, although for reversible symmetric transformations our necessary condition leads to new conserved quantities. We arrive at this condition by a new isometry embedding of the system’s Hilbert space into a different tensor product structure. This additional result shows that the isometry in equation (1) can be useful even if it does not simulate covariant transformations with LOCC.

The paper is organized as follows. In section 2, we go over the preliminaries of the asymmetry resource theory, as well as precise definitions of asymmetry monotones. We present our main result in section 3 starting with simply reducible compact Lie groups.

4 In [7] (http://pirsa.org/10120062/), it was called an ‘asymmetry measure’ and in [10] it was called a ‘frameness monotone’. Here we use the terminology of asymmetry monotones rather than asymmetry measures since these functions do not necessarily measure asymmetry, but can sometimes only detect it. To see it, consider, for example, the asymmetry monotone that is equal to 7.2 for asymmetric states and 0 for symmetric states. Clearly, this monotone does not measure asymmetry and only detects asymmetry.
In appendix A, we generalize the main result to general compact Lie groups. Section 4 focuses on specific examples of asymmetry monotones and how they compare with their entanglement counterparts. In section 5, we introduce a new isometry that, in general, does not simulate covariant maps with LOCC, but nonetheless leads to new results on time symmetric evolutions. Finally, we discuss our results and conclusions in section 6. Appendix B contains a special form of the general results for the case of Abelian groups.

2. Notations and preliminaries

In this section, we briefly discuss a few key elements of the resource theory of quantum asymmetry that we will be using in the rest of the paper. In particular, we go over $G$-covariant maps, irreducible tensor operators, the Wigner–Eckart theorem and asymmetry monotones. For a more detailed review on asymmetry, and its relation to reference frames and super-selection rules, see [7, 10, 14].

2.1. $G$-covariant transformations

Let $B(\mathcal{H})$ denote the set of bounded operators over $\mathcal{H}$. Consider a CP map $\mathcal{E} : B(\mathcal{H}) \to B(\mathcal{H})$ that takes density matrices to density matrices. Let $G$ be a group of transformations, and define the map $U(g)$ as

$$U(g) \rho U(g)^\dagger = \mathcal{E}(\rho).$$

where $U : G \to B(\mathcal{H}) : g \mapsto U(g)$ is a representation of the group $G$. In this paper, we only consider compact semi-simple Lie groups with fully reducible unitary representations. A semi-simple Lie group is a Lie group whose algebra is semi-simple. We will also assume that the representation of the group comes with a group-independent (Haar) measure. Compact Lie groups all have unitary representations with Haar measures. We say that the mapping $\mathcal{E}$ is symmetric with respect to $G$, or equivalently, that $\mathcal{E}$ is $G$-covariant, if for all $\rho$ and for all $g \in G$,

$$\mathcal{E} \circ U(g)(\rho) = U(g) \circ \mathcal{E}(\rho).$$

In particular, if the CP-map consists of a single unitary $\mathcal{E} = V(\bullet) V^\dagger$, then the condition of $G$-covariance in equation (3) becomes

$$[U(g), V] = 0, \quad \forall g \in G.$$ 

The unitary $V$ is called $G$-invariant in this case. Similarly, a symmetric state $\rho$ is any state that remains invariant under the application of the group representation, also known as a $G$-invariant state,

$$[U(g), \rho] = 0, \quad \forall g \in G,$$

which is equivalent to

$$\forall g \in G, \quad U(g) \rho U(g)^\dagger = \rho.$$

Our method relies heavily on the general form of the Wigner–Eckart theorem for semi-simple Lie groups. A much more complicated form of the Wigner–Eckart theorem exists for finite groups, sometimes known as the Koster–Wigner–Eckart theorem [23]. It might still be possible that this finite counterpart of the Wigner–Eckart theorem can lead to results analogous to ours, but we do not consider those cases in this paper.
Consider the uniform average of the group action:

\[ G[\rho] := \int d\mu_g \mathcal{U}(g)[\rho], \]

(7)

where \( d\mu_g \) denotes the group invariant (Haar) measure. In the case of discrete groups the integral is replaced with a sum, and the uniform measure with the inverse group size. The averaging superoperator \( G \) in equation (7) is known as the \( G \)-twirling operation. It follows from the uniformity of the measure that twirled states are invariant under the action of any element of the group, i.e. they are \( G \)-invariant. In fact, it can be shown that every \( G \)-invariant state can be expressed as the outcome of a twirling operation [14].

2.2. Irreducible tensor operators and the Wigner–Eckart theorem

Let the irreps of the group \( G \) be labeled by the letter \( j \). In general, \( j \) can be a shorthand notation for a multiple of independent labels that together fully label the irreps. As the irreps are unitary, \( j \) denotes the highest weight of each irrep and is a vector of dimension \( \ell \), where \( \ell \) is the rank of the associated algebra. Also, let \( m \) label the basis vectors of the irrep, i.e. the basis vectors spanning the invariant subspace of the \( j \)th irrep. In fact, each \( m \) denotes a weight of the irrep labeled by \( j \) and is thus also an \( \ell \)-dimensional vector in the weight space of the irreps.

Also, let us decompose the Hilbert space as

\[ \mathcal{H} = \bigoplus_{j, \lambda} \mathcal{H}_{j,\lambda}, \]

(8)

where \( \mathcal{H}_{j,\lambda} \) carries an irrep labeled by \( j \), where the range of \( j \) is assumed to be unbounded. The index \( \lambda \) labels the multiplicity of the irrep. With this decomposition of \( \mathcal{H} \), the \( G \)-twirling of a state \( \rho \) has the form

\[ G[\rho] = \sum_{j, \lambda} p_{j, \lambda} \Pi_{j, \lambda}, \]

(9)

where \( \Pi_{j, \lambda} \) is the projection onto subspace \( \mathcal{H}_{j,\lambda} \) that carries the \( j \)th irrep.

The definition of \( G \)-covariance in equation (3) is equivalent to

\[ \mathcal{E} = \mathcal{U}(g) \circ \mathcal{E} \circ \mathcal{U}(g^{-1}), \quad \forall g \in G. \]

(10)

Clearly, if \( \{K_i\} \) is a set of Kraus operators of a \( G \)-covariant CP-map \( \mathcal{E} \), then from equation (10), it follows that \( \{U(g)K_iU(g)^\dagger\} \) is also a set of Kraus operators for \( \mathcal{E} \). Now, two operator sum representations of the same channel \( \mathcal{E} \) are related by a unitary matrix. Therefore, it follows that

\[ U(g)K_iU(g)^\dagger = \sum_{i'} u_{i'i}(g) K_{i'}, \]

(11)

where \( u_{i'i}(g) \) are the elements of a unitary matrix \( u(g) \). It was shown in [10] that if the \( \{K_i\} \) are linearly independent, then \( u(g) \) is also a representation of the group \( G \). Furthermore, bringing the matrix \( u(g) \) to the block diagonal form,

\[ u(g) = \bigoplus_{j, \lambda} u_{j,\lambda}(g), \]

(12)

of the group’s irreps, simply amounts to a different unitary remixing of the Kraus operators, and is thus allowed. This, in turn, means that the Kraus operators of a \( G \)-covariant CP-map can
be grouped into subsets that mix only among themselves, each labeled by the irrep labels of the group.

Thus, every $G$-covariant CP-map admits a Kraus decomposition labeled $K_{j,m,\alpha}$, with $\alpha$ being a multiplicity index, such that

$$K_{j,m,\alpha} = \sum_{m'} u_{m,m'}^{(j)}(g) K_{j,m',\alpha}, \quad \forall g \in G.$$ (13)

For each irrep label $j$, Kraus operators of the set $\{K_{j,m,\alpha}\}$ are called irreducible tensor operators of rank $j$.

A CP-map with a Kraus decomposition composed of a set of irreducible tensor operators,

$$\mathcal{E}_{j,\alpha}(\bullet) = \sum_{m} K_{j,m,\alpha}(\bullet) K_{j,m,\alpha}^\dagger,$$ (14)

is an irreducible $G$-covariant operation. Every $G$-covariant CP-map can be expressed as a sum of irreducible $G$-covariant operations.

2.2.1. The Wigner–Eckart theorem. The Wigner–Eckart theorem determines the matrix elements of the irreducible tensor operators with respect to the $SU(2)$ algebra, also known as spherical tensor operators (see, e.g., pp 193–5 of [22]). In fact, the Wigner–Eckart theorem can be generalized and applied to any compact, semi-simple group and its associated Lie algebra. For simplicity of the exposition, we will first assume that the Kronecker product of the algebra associated with the group is simply reducible. That is, the coupling of two irreps has no outer multiplicities (i.e. multiplicities that arise due to coupling). We leave the generalization to all semi-simple compact groups to appendix A. The Wigner–Eckart theorem then specifies the form of the matrix elements of $K_{j,M,\alpha}$ as we now discuss.

Let $\{|j, \lambda; m\rangle\}$ be the set of basis vectors spanning the Hilbert space $\mathcal{H}$. Here $m$ labels the weights of the $j$th irrep, as before, and $\lambda$ labels the multiplicity of the irrep. The Wigner–Eckart theorem states that the matrix elements of $K_{j,m,\alpha}$ are given by

$$\langle j', \lambda'; m'|K_{j,M,\alpha}|j, \lambda; m\rangle = \begin{pmatrix} j & m \\ J & M \end{pmatrix} \langle j', \lambda' || K_{j,\alpha}|| j, \lambda \rangle,$$ (15)

where $\langle j', \lambda' || K_{j,\alpha}|| j, \lambda \rangle$ is the reduced matrix element independent of $m$ and $m'$, and $\begin{pmatrix} j & J \\ m & M \end{pmatrix}$ are the (general) Clebsch–Gordan (CG) coefficients.

2.2.2. The Clebsch–Gordan coefficients. The generalized CG coupling coefficients

$$\begin{pmatrix} j_1 & j_2 \\ m_1 & m_2 \end{pmatrix} \begin{pmatrix} j_3 \\ m_3 \end{pmatrix}$$

relate the basis $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$ to the basis $|j_3, m_3; (j_1, j_2)\rangle$ that reduces the Kronecker product of the two irreps,

$$|j_3, m_3; (j_1, j_2)\rangle = \sum_{m_1,m_2} \begin{pmatrix} j_1 & j_2 \\ m_1 & m_2 \end{pmatrix} |j_1, m_1\rangle \otimes |j_2, m_2\rangle.$$ (16)

Here, we have dropped the multiplicity index $\lambda$, as the CG-coefficients do not depend on the multiplicity. In the rest of the paper, we use $|j, m\rangle$, or $|j, \lambda; m\rangle$ instead of $|j, m; (j_1, j_2)\rangle$ or $|j, \lambda, m; (j_1, j_2)\rangle$ for brevity whenever the context is clear.
2.3. Monotones

Every restriction on quantum operations defines a resource theory, determining how quantum states that cannot be prepared under the restriction may be manipulated and used to circumvent the restriction. Here we discuss briefly how the resourcefulness of these quantum states is quantified. We will focus on entanglement theory and the theory of asymmetry that is associated with a group $G$ of transformations. In entanglement theory, the quantum operations or CP-maps are confined to LOCC, and only separable states can be prepared by LOCC (assuming no access to previously existing entanglement). In the resource theory of quantum asymmetry, the only allowed operations are $G$-covariant CP maps, and the only states that can be prepared without any resources are $G$-invariant states.

A quantum state cannot turn into a stronger resource by the set of restricted (or allowed) operations. Therefore, the strength of the resource must be quantified by functions that do not increase under the set of allowed operations. Such functions are called monotones. We now give a precise definition for monotones that we will use in the rest of the paper and that apply to both entanglement and asymmetry.

The most general quantum transformation converts an initial state $\rho$ into one of a set of possible final states, say $\sigma_x$, that occurs with probability $p_x$. Such a general quantum transformation is described by a CP map $E: B(\mathcal{H}) \to B(\mathcal{H})$ that is itself composed of a number of CP (in general, trace decreasing) maps $\{E_x\}$, so that $E = \sum_x E_x$, and

$$\sigma_x := E_x[\rho]/p_x,$$

where the probability $p_x = \text{Tr}(E_x[\rho])$. We say that $E$ is $G$-symmetric if all $\{E_x\}$ are $G$-covariant.

The ensemble of outcomes is written as $\{\sigma_x, p_x\}$. This ensemble can be equivalently expressed as a density operator

$$\tilde{\sigma} := \sum_x p_x \sigma_x \otimes |x\rangle \langle x|,$$

where $\{|x\rangle\}$ are a set of mutually orthogonal unit states. The state $\tilde{\sigma}$ can be prepared out of the ensemble $\{\sigma_x, p_x\}$ by annexing an ancilla in the state labeled by the index $x$ and then forgetting the value of $x$. Reversely, the ensemble can always be reproduced from the density operator $\tilde{\sigma}$ by making the measurement $M = \{|x\rangle \langle x|\}_x$.

**Definition 1.** Using the above notations, a function $A: B(\mathcal{H}) \to \mathbb{R}^+$ is called an asymmetry (entanglement) monotone if it satisfies

$$A(\rho) \geq A(\tilde{\sigma})$$

for all CP maps $E$ that are $G$-covariant (LOCC).

We can further classify the asymmetry (entanglement) monotones into another category:

**Definition 2.** The asymmetry (entanglement) monotone $A: B(\mathcal{H}) \to \mathbb{R}^+$ is called an ensemble monotone if it satisfies

$$A(\rho) \geq \sum_x p_x A(\sigma_x)$$

for all CP maps $E$ that are $G$-symmetric (LOCC).

*New Journal of Physics* 14 (2012) 123026 (http://www.njp.org/)
Note that the set of ensemble monotones is a strict subset of the monotones defined in equation (19). The most restrictive monotones are monotones that do not increase under any non-deterministic (trace-non-increasing) CP-map $\mathcal{E}_\pi$.

**Definition 3.** An asymmetry (entanglement) monotone $A : \mathcal{B}(\mathcal{H}) \to \mathbb{R}^+$ is faithful if

\[
A(\rho) = 0 \quad \text{iff} \quad \rho \text{ is } G\text{-invariant (separable)}.
\]  

(21)

We are now ready to present our main result that connects $G$-covariant transformations to LOCC transformations and entanglement monotones to asymmetry monotones.

### 3. Simulating $G$-covariant transformations

As discussed in section 1, the central idea of this paper is to embed the system’s Hilbert space within a larger Hilbert space in such a way that the covariant transformations between original states map to LOCC transformations in the larger Hilbert space. We now proceed to make precise the concepts and procedures involved. We use the notations introduced in sections 1 and 2.

**Definition 4.** An LOCC-simulating isometry is an isometry $\mathcal{C} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{W})$, with a bipartite image space $\mathcal{W} \subseteq \mathcal{H}_A \otimes \mathcal{H}_B$ (see equation (1)), that satisfies the following three conditions:

1. For any $G$-covariant map, $\mathcal{E}_{\text{cov}}$, the map $\mathcal{C} \circ \mathcal{E}_{\text{cov}} \circ \mathcal{C}^{-1} \equiv \mathcal{E}_{\text{local}} : \mathcal{B}(\mathcal{W}) \to \mathcal{B}(\mathcal{W})$ is local; that is, $\mathcal{E}_{\text{local}}$ can be implemented by LOCC.
2. If $\rho$ is $G$-invariant, then $\mathcal{C}(\rho)$ is separable.
3. There exists an asymmetric state (i.e. non-$G$-invariant state) $\sigma$ for which $\mathcal{C}(\sigma)$ is entangled.

The third point excludes trivial isometries that map every state, whether $G$-invariant or not, to a separable state. One example of such a trivial isometry is simply adding an ancilla state to every state $\rho$, i.e.

\[
\rho \mapsto \mathcal{C}(\rho) := \rho \otimes |0\rangle\langle 0|.
\]  

(22)

Trivial isometries of this sort are of course always possible, but they differentiate neither between $G$-invariant and non-invariant states nor between $G$-covariant and non-covariant transformations. Thus, they tell us nothing about the states’ asymmetry properties or about the conditions under which covariant transformations are possible. The other extreme, that of mapping every asymmetric state to an entangled state, although ideal, is not likely to always be possible. The isometries that we consider here do not fall under either extreme. Nevertheless, we are able to find a set of isometries that is complete in the sense that for any asymmetric state there exists at least one isometry in the set that takes it to an entangled state. In this sense, entanglement captures all aspects of asymmetry.

#### 3.1. The main isometry

The Wigner–Eckart theorem determines the matrix elements of an irreducible tensor operator, such as the Kraus operators of $G$-covariant transformations, in the basis $|j, \lambda; m\rangle$ introduced in section 2.1 (see equation (15)). An important consequence of the Wigner–Eckart theorem
is the existence of the so-called selection rules. The generalized CG coupling coefficients 
\( (j, m; J, M|j', m') \) are zero unless the weights \( m, M \) and \( m' \) satisfy the relation
\[
m + M = m'.
\] (23)
The matrix elements that do not satisfy equation (23) must vanish. It thus follows from the
Wigner–Eckart theorem that the only thing a \( G \)-covariant Kraus operator \( K_{J,M,\alpha} \) does on the
weight \( m \) of a basis state is to translate it by \( M \), independently of the other relevant parameters
\( j, J, \lambda \) and \( \alpha \). We exploit this fact in the following definition and theorem when we introduce
an isometry that satisfies the three conditions of definition 4.

**Definition 5.** Let \( \mathcal{H}_B \) denote the Hilbert space spanned by kets \( |m\rangle \) where \( m \) ranges over the
representation weights of the associated algebra of the group, and let
\[
W := \text{span} \{ |j, \lambda; m\rangle \otimes |m\rangle \} \subset \mathcal{H} \otimes \mathcal{H}_B.
\] (24)
The isometry \( \mathcal{C} \) is defined by its action on the basis kets as
\[
|j, \lambda; m\rangle \xrightarrow{\mathcal{C}} |j, \lambda; m\rangle \otimes |m\rangle.
\] (25)

We now show that \( \mathcal{C} \) satisfies all the conditions of definition 4.

**Proposition 1.** \( \mathcal{C} \) is an LOCC-simulating isometry.

**Proof.** To see that the first condition in definition (4) is satisfied, consider a \( G \)-covariant CP-map
\( \mathcal{E}_{\text{cov}} \) whose operator sum representation is given in terms of Kraus operators \( \{K_{J,M,\alpha}\} \). We define
\[
\tilde{K}_{J,M,\alpha} := K_{J,M,\alpha} \otimes T_M,
\] (26)
where
\[
T_M := \sum_m |m + M\rangle \langle m|
\] (27)
is a translation operator. Let \( \mathcal{E}_{\text{local}} \) denote the CP-map whose operator sum representation
corresponds to the Kraus operators \( \tilde{K}_{J,M,\alpha} \) given in equation (26). Note that from equation (23)
it follows that \( \mathcal{E}_{\text{local}} = \mathcal{C} \circ \mathcal{E}_{\text{cov}} \circ \mathcal{C}^{-1} \). We need to show that \( \mathcal{E}_{\text{local}} \) can be implemented by LOCC.
Indeed, note that \( T_M \), being merely a translation operator, is unitary (assuming that the range
of the weights in the decomposition (8) is unbounded). Therefore, the map \( \mathcal{E}_{\text{local}} \) can be
implemented as follows: Alice makes a ‘local’ measurement described by the Kraus operators
\( \{K_{J,M,\alpha}\} \) and sends the part \( M \) of her measurement outcome to Bob, who then performs the
unitary transformation \( T_M \). Hence, the first criterion of definition 4 is satisfied.

Secondly, recall that any \( G \)-invariant state \( \rho \) is equal to its own \( G \)-twirling (see equation (9)),
\[
\rho = \sum_{j,\lambda} p_{j,\lambda} \Pi_{j,\lambda},
\] (28)
where the projection \( \Pi_{j,\lambda} \) is equal to
\[
\Pi_{j,\lambda} = \sum_m |j, \lambda; m\rangle \langle j, \lambda; m|.
\] (29)
The state $C(\rho)$ is thus equal to
\[
C(\rho) = \sum_{j, \lambda} p_{j, \lambda} \sum_m |j, \lambda; m\rangle \langle j, \lambda; m| \otimes |m\rangle \langle m|,
\]
which is clearly a separable state.

Finally, a state of the form
\[
|\psi\rangle = c_1 |j_1, \lambda_1; m_1\rangle + c_2 |j_2, \lambda_2; m_2\rangle
\]
(31)
is mapped to the entangled state
\[
|\tilde{\psi}\rangle = c_1 |j_1, \lambda_1; m_1\rangle \otimes |m_1\rangle + c_2 |j_2, \lambda_2; m_2\rangle \otimes |m_2\rangle.
\]
(32)
This completes the proof. \(\square\)

The example in equation (31) suggests that if a state has coherence in $m$ it is mapped to an entangled state. In the next proposition, we make this claim rigorous and give the necessary and sufficient conditions for a general mixed state $\rho$ to be mapped to an entangled state by the isometry $C$.

**Proposition 2.** Let $\Pi_m$ be the projection
\[
\Pi_m := \sum_{j, \lambda} |j, \lambda; m\rangle \langle j, \lambda; m|.
\]
Then, the isometry $C$ maps a state $\rho$ to an entangled state if and only if there exists $m$ such that $[\rho, \Pi_m] \neq 0$; i.e. $\rho$ has coherence in $m$.

**Proof.** Every state $\tilde{\rho}$ acting on $\mathcal{W}$ is the image of some state acting on $\mathcal{H}$, i.e. $\tilde{\rho} = C(\rho)$. If $\tilde{\rho}$ is a separable state, it must have a pure state decomposition composed of product states
\[
\tilde{\rho} = \sum_i q_i |\tilde{\phi}_i\rangle \langle \tilde{\phi}_i|,
\]
where each $|\tilde{\phi}_i\rangle$ is both a product state and the image of some state $|\phi_i\rangle$ under the isometry $C$. This is because $C$ is a linear invertible map, and any pure state decomposition of $\rho$ corresponds to a pure state decomposition of $\tilde{\rho}$ and vice versa. Thus, since $|\tilde{\phi}_i\rangle = C(|\phi_i\rangle)$ is a product state, $|\phi_i\rangle$ must have the form
\[
|\phi_i\rangle = \sum_{j, \lambda} c_{i; j, \lambda} |j, \lambda; m_i\rangle,
\]
(33)
where $c_{i; j, \lambda}$ are some complex coefficients and the superposition above consists of a single value for $m = m_i$. Otherwise, containing two different values for $m$ in the above expansion necessarily renders the state $|\tilde{\phi}_i\rangle$ entangled. Consequently, the form of the initial state $\rho$ corresponding to $\tilde{\rho} = C(\rho)$ must be
\[
\rho = \sum_i q_i |\phi_i\rangle \langle \phi_i|,
\]
with $|\phi_i\rangle$ as in equation (33). According to equation (33) each $|\phi_i\rangle \langle \phi_i|$ commutes with $\Pi_m$ for all $m$ and therefore $[\rho, \Pi_m] = 0$. The argument works in the other direction as well. In other words, if every pure state decomposition of $\rho$ contains at least one pure state that is in a
coherent superposition of two or more eigenstates with different values of $m$, then $\mathcal{C}(\rho)$ will be an entangled state. This completes the proof.

The isometry $\mathcal{C}$ does not map all asymmetric states to entangled states. For example, the state $|\phi\rangle = |j, \lambda; m\rangle$ is not $G$-invariant (assuming that $G$ is non-Abelian and $j$ does not label the identity irrep), and yet it is mapped to the product state

$$|	ilde{\phi}\rangle = |j, \lambda; m\rangle \otimes |m\rangle.$$ 

However, as we now show, we can define another LOCC-simulating isometry, similar to $\mathcal{C}$, that maps $|j, \lambda; m\rangle$ to an entangled state.

### 3.2. A complete set of local operations and classical communications-simulating isometries

In definition 5 we have used the basis $\{|j, \lambda; m\rangle\}$ to define the isometry $\mathcal{C}$. However, there is nothing special about the choice of the irrep weights $m$. In fact, the set of states

$$|j, \lambda; m\rangle_g := U(g)|j, \lambda; m\rangle$$

forms an equally valid basis for the irreps, labeled by the new weights $m_g$ (the multiplicity index $\lambda$ can always be relabeled if it is needed). On the other hand, by definition, the irrep basis states mix among themselves under the action of the group

$$U(g)|j, \lambda; m\rangle = \sum_{m'} D_{m,m'}^{(j)}(g) |j, \lambda; m'\rangle,$$

where $D_{m,m'}^{(j)}(g)$ is the matrix representation of the $j$th irrep. Reversing equation (34), we obtain

$$|j, \lambda; m\rangle = \sum_{m'} D_{m,m'}^{(j)}(g^{-1}) |j, \lambda; m'\rangle_g.$$ 

Hence, if we had defined the isometry relative to the new weights, the state $|j, \lambda; m\rangle$ would be mapped to an entangled state. In fact, the isometry $\mathcal{C}$ is only one of a class of isometries that can be defined for different choices of $g \in G$ relative to the weights $\{m_g\}$. The mapping $\mathcal{C}$ is merely the isometry corresponding to the identity element of the group.

**Definition 6.** For every $g \in G$, we define the isometry $\mathcal{C}_g$ as

$$\mathcal{C}_g := (U(g) \otimes \mathcal{I}_B) \circ \mathcal{C} \circ U^\dagger(g),$$

where $U(g) := U(g)(\bullet)U^\dagger(g)$, and $\mathcal{I}_B$ is the identity superoperator acting on $\mathcal{H}_B$.

The isometry $\mathcal{C}_g$ acts on basis states, $|j, \lambda; m\rangle_g$, and maps them to

$$|j, \lambda; m\rangle_g \mapsto |j, \lambda; m\rangle_g \otimes |m\rangle.$$

Note that the image space of all the isometries $\{\mathcal{C}_g\}$ is the space $\mathcal{W}$ in (24). Clearly, the proof of proposition 1 can be easily modified to apply to all the set of isometries $\{\mathcal{C}_g\}$. Moreover, note that the state $|\phi\rangle \equiv |j, \lambda; m\rangle$ is mapped to

$$|\phi\rangle \xrightarrow{\mathcal{C}_g} |\tilde{\phi}\rangle = \sum_{m'} D_{m,m'}^{(j,\lambda)}(g^{-1}) |j, \lambda; m'\rangle_g \otimes |m'\rangle,$$

which is, in general, an entangled state.
It is instructive at this stage to look at the specific group of rotations \(SO(3)\), or similarly, the group \(SU(2)\) to gain some intuition. The weights \(m\) of the associated algebra \(su(2)\) are one dimensional and correspond to the eigenvalues of the angular momentum operator, \(J_z\), along the \(z\)-direction. Each irrep is labeled by the single number \(j\) corresponding to the maximum \(z\)-eigenvalue of angular momentum, and the total angular momentum, \(J^2\), is equal to \(j(j + 1)\). There is obviously nothing special about the choice of the \(z\)-axis. The \(z\)-axis can be rotated to a new axis \(\hat{n}\), which corresponds to applying the respective group representation on the quantum states. One way to specify an element of the group is to determine the axis \(\hat{n}\) to which it takes the \(z\)-axis. In other words, each isometry in the class of definition 6 is identified by the choice of a new \(z\)-direction and can be denoted as \(C_{\hat{n}}\).

Thus, to take full advantage of the entanglement features of the embedding, one has to take more than one isometry into consideration. As we shall now see, if \(\rho \in B(\mathcal{H})\) is an asymmetric state, then there exists \(g \in G\) such that \(C_g(\rho)\) is an entangled state. In fact, for the \(SU(2)\) group we will see that only two directions are needed to characterize all the asymmetry properties of a state. That is, if \(C_{\hat{n}}(\rho)\) is separable for two independent choices of \(\hat{n}\), then \(\rho\) is necessarily \(G\)-invariant.

Also for more general connected groups, there exists a finite number of isometries \(\{C_{g_i}\}\) (associated with a finite number of group elements \(\{g_i\}\)) that capture all the asymmetry properties of a state. That is, if a state is mapped to a separable state by all the isometries in the finite set \(\{C_{g_i}\}\), then the state must be symmetric. This allows, in principle, to check whether a state is \(G\)-invariant or not, by considering its bipartite image states only for a finite number of isometry elements. Otherwise, all the infinite isometries, each associated with a member of the group, must have been considered before such an assessment could be made.

Before proving the above claim rigorously, let us illustrate the idea of the proof with the simple and more familiar example of the group \(SU(2)\). Suppose that \(C(\rho)\) is separable for some state \(\rho\). Then, according to proposition 2 the state \(\rho\) has no coherence in \(m\), the eigenvalue of the \(J_z\) operator. It means, in turn, that the state \(\rho\) commutes with \(J_z\). By the same argument, if \(C_{\hat{n}}(\rho)\) is separable, then the state \(\rho\) commutes with \(J_z\). Therefore, if both \(C(\rho)\) and \(C_{\hat{n}}(\rho)\) are separable, then \(\rho\) commutes with both \(J_z^\dagger\) and \(J_z\). But since \([J_z, J_z^\dagger] = iJ_y\), \(\rho\) also commutes with \(J_y\) and so it must commute with all the elements of the group which means that \(\rho\) is an \(SU(2)\)-invariant state. This line of argument can be generalized to other groups, as we now demonstrate.

Suppose \(G\) is a simply connected group parametrized by \(r\) parameters. Let \(\mathfrak{g}\) be the associated algebra of \(G\) of rank \(\ell\), and let \(\mathfrak{h}\) be its \(\ell\)-dimensional Cartan subalgebra. Denote the operator representation of the infinitesimal generators of the group as \(X_{\lambda} : \mathcal{H} \to \mathcal{H}\), for \(a = 1, \ldots, r\). Similarly, denote the representation of the Cartan operators spanning \(\mathfrak{h}\) as \(H_i : \mathcal{H} \to \mathcal{H}\), where \(i = 1, \ldots, \ell\).

Now, let \(S \subset G\) be the subgroup of \(G\) whose members permute the infinitesimal generators of the group among themselves. By this we mean that, for every \(s \in S\),

\[ U(s) X_a U(s)^\dagger = X_{a'(s)}, \quad a, a' \in \{1, \ldots, r\}. \]  

(39)

As both \(\mathfrak{g}\) and \(\mathfrak{h}\) are finite, the subgroup \(S\) contains only a finite number of elements. We are now ready to prove the general case.

**Proposition 3.** Let \(\rho \in B(\mathcal{H})\). Then, \(\rho\) is \(G\)-invariant if and only if for all \(s\) belonging to the finite subgroup \(S \subset G\), the state \(C_s(\rho)\) is separable.
Proof. If $\rho$ is $G$-invariant, then $C_s(\rho)$ is separable for all $g \in G$, and thus for all $s \in S$, since $\{C_s\}$ is a set of LOCC-simulating maps.

We therefore assume that $C_s(\rho)$ is separable for all $s \in S$. The requirement that $C_s(\rho)$ is separable implies that $\rho$, when expressed in the basis $|j, \lambda; m\rangle_s$, has no coherence in $m$.

Consider the projection

$$\Pi_m^{(s)} := \sum_{j, \lambda} |j, \lambda; m\rangle_s \langle j, \lambda; m|.$$

The condition for separability is equivalent to the requirement that $[\rho, \Pi_m^{(s)}] = 0$ for all $m$ (see proposition 2).

The set of operators, $H_i^{(s)} := U(s) H_i U(s)^\dagger$, are all diagonal in the new basis,

$$H_i^{(s)} |j, \lambda; m\rangle_s = m_i |j, \lambda; m\rangle_s,$$

and form a representation for new Cartan operators. It follows that $H_i^{(s)} = \sum_m m_i \Pi_m^{(s)}$. Thus, if $C_s(\rho)$ is separable, $\rho$ must satisfy

$$[\rho, H_i^{(s)}] = 0, \quad i = 1, \ldots, \ell.$$

But this is true for all $s \in S$ (including the identity $e$, where $H_i \equiv H_i^{(e)}$). Every $X_a$ can be constructed from the commutators of $H_i^{(s)}$, once all the $H_i^{(s)}$ for all $s \in S$ are included. It follows that the state $\rho$ commutes with all the generators $X_a$, and consequently, with all the elements of the group as well. In other words, the state is $G$-invariant.

In the next section, we see how entanglement of the embedded state changes under $G$-covariant transformations of the original state. This, in turn, enables us to relate the asymmetry features of the original state to the ensuing entanglement.

### 3.3. Constructing asymmetry monotones from entanglement monotones

Roughly speaking, propositions 1 and 3 imply that the evolution of asymmetry can be simulated by the evolution of entanglement. In particular, we can define asymmetry monotones for the states acting on $\mathcal{H}$ in terms of the entanglement monotones of the states acting on $\mathcal{W}$ to which they are mapped.

**Definition 7.** For every bipartite entanglement monotone $E$, we define the corresponding asymmetry monotone as

$$A_g^E : B(\mathcal{H}) \rightarrow \mathbb{R}^+ : \rho \mapsto E \left( C_g(\rho) \right).$$

The following proposition ensures that $A_g^E$ is indeed an asymmetry monotone, assuming that $E$ is an entanglement monotone.

**Proposition 4.** Consider an entanglement monotone $E$. If $\rho \xrightarrow{\text{cov}} \sigma$ is possible, then for every $g \in G$,

$$E \left( C_s(\rho) \right) \geq E \left( C_s(\sigma) \right).$$

**Remark.** A similar inequality holds in the case of non-deterministic $G$-covariant CP-maps for the average of $E$, assuming that $E$ is an ensemble monotone (see section 2.3).
Proof. The result follows directly from definition 6 and the extension of proposition 1 to all isometries $C_g$. 

As not all asymmetric states are taken to entangled states, the asymmetry monotone $A^g_E$ is not faithful even if $E$ itself a faithful entanglement monotone. However, proposition 3 allows us to define a faithful asymmetry monotone from the monotones $A^g_E$:

**Proposition 5.** The function

\[ A^\text{sup}_E := \sup_{g \in G} A^g_E, \]  

where $\sup_{g \in G}$ stands for the supremum taken over all $g$ in $G$, is a faithful asymmetry monotone, provided that $E$ is a faithful entanglement monotone.

Replacing the supremum above by a maximum over the finite number of elements in $S \subset G$ (see proposition 3) will also lead to a faithful asymmetry monotone. For example, if $G = SU(2)$, then the function

\[ \max_{\hat{n} \in \{\hat{z}, \hat{x}\}} A^{\hat{n}}_E \]

is also an asymmetry monotone.

### 3.4. Unitary transformation

If the CP-map is reversible, i.e. a unitary operation, then the condition of monotonicity for the monotones (40) must be true in both directions, which in turn implies that the monotone functions must remain constant.

**Proposition 6.** Consider an entanglement monotone $E$. If $\rho \xleftrightarrow{G} \sigma$ is a reversible $G$-covariant transformation, then for every $g \in G$, $A^g_E$ is a conserved quantity; i.e.

\[ E(C_g(\rho)) = E(C_g(\sigma)). \]  

Thus, for closed systems governed by a symmetric Hamiltonian, every entanglement monotone $E$ leads to new conserved quantities, $\{A^g_E\}_{g \in G}$. For a Hamiltonian that is symmetric with respect to the group $G$, the expectation values of the generators of $G$ are also conserved quantities. However, unlike $A^g_E$, for open systems these expectation values are not behaving monotonically.

### 4. Examples

We now review in more detail some examples of asymmetry monotones that are constructed from entanglement monotones through the class of LOCC simulating isometries.

#### 4.1. The negativity of entanglement as a measure of asymmetry

Many totally new asymmetry monotones can be constructed from entanglement monotones using the isometry $C$. Here we introduce two such monotones for the first time. One such monotone uses the negativity of entanglement, and the other uses the logarithmic negativity [18, 24].
Definition 8. The negativity of asymmetry is defined as

\[ A_N(\rho) := \frac{\| C(\rho)^\Gamma \|_1 - 1}{2}, \]  

(44)

and the logarithmic negativity of asymmetry is

\[ A_{LN}(\rho) := \log \| C(\rho)^\Gamma \|_1, \]  

(45)

where \( \Gamma \) denotes partial transpose and \( \| \cdot \|_1 \) is the one-norm

\[ \| \rho \|_1 = \text{Tr} \sqrt{\rho^\dagger \rho}. \]  

(46)

Both negativity and logarithmic negativity are particularly useful monotones as they are very easily computable for all states, pure or mixed. Note, however, that the negativity and the logarithmic negativity do not reduce to entropy functions for pure states. For pure states, the negativity of asymmetry can be expressed in a very simple closed form. As discussed in [10], every pure state can be brought to a standard form with no explicit multiplicity index by \( G \)-covariant transformations. Consider the pure state in the standard form \(|\psi\rangle = \sum_{j, m} \sqrt{p_{j, m}} |j; m\rangle\). The norm of the partial transpose is

\[ \| C(|\psi\rangle\langle\psi|)^\Gamma \|_1 = \left( \sum_{j, m} \sqrt{p_{j, m}} \right)^2. \]  

(47)

It follows that the logarithmic negativity of asymmetry is equal to

\[ A_{LN}(|\psi\rangle\langle\psi|) = 2 \log \left( \sum_{j, m} \sqrt{p_{j, m}} \right). \]  

(48)

After simplifying the equations, the negativity of asymmetry can be expressed as

\[ A_N(|\psi\rangle\langle\psi|) = \sum_{j \neq j', m \neq m'} \sqrt{p_{j, m} p_{j', m'}}. \]  

(49)

4.2. Asymmetry monotones based on the squashed entanglement

Squashed asymmetry is another new monotone constructed from the squashed entanglement monotone [25].

Definition 9. The squashed asymmetry is defined as

\[ A_{sq}(\rho) := E_{sq}(C(\rho)), \]  

(50)

where

\[ E_{sq}(C(\rho)) = \frac{1}{2} \inf_c S(A : B \mid C) \]  

(51)

is the squashed entanglement. \( A \) and \( B \) denote the systems associated with the Hilbert spaces \( \mathcal{H} \) and \( \mathcal{H}_B \), respectively. \( C \) denotes an auxiliary system with the Hilbert space \( \mathcal{H}_C \). The minimum is taken over all extensions of \( C(\rho) \) to a tripartite state \( \sigma^{ABC} \) acting on \( \mathcal{H} \otimes \mathcal{H}_B \otimes \mathcal{H}_C \), and the function \( S(A : B | C) \) is the conditional mutual entropy.

Squashed entanglement is known to be an additive monotone over the tensor product of states [25]. It is also a lower bound on entanglement of formation and an upper bound on the distillable entanglement. Its asymmetry counterpart introduced here could shed light on the properties of multiple copy \( G \)-covariant transformations.
4.3. Measures based on distance

Monotones based on how far states are from the set of non-resources are known as distance measures [11]. The geometric intuition at play here can apply to various resources, not just entanglement. If the resource is entanglement, then the more entangled a state is, the further away it is from the set of separable states. The ‘distance’ between any two states \( \rho \) and \( \sigma \) is measured by a function \( D(\rho, \sigma) \) with distance-like properties (e.g. \( D(\rho, \sigma) \geq 0 \) with equality if and only if \( \sigma = \rho \)). The function \( D \), however, need not be literally a metric. All that is needed is that \( D \) preserve the partial order, and that \( D(\rho, \rho) = 0 \) for all \( \rho \). \( D \) need not satisfy the triangle inequality, for instance, and it need not even be symmetric. The distance-based monotone is defined as the minimum distance to the target set \( Q \):

\[
E_D(\rho) := \inf_{\sigma \in Q} D(\rho, \sigma).
\]

In the case of entanglement, the target set is the set \( \text{SEP} \) of separable states. If the function \( D(\rho, \sigma) = \text{Tr} \left[ \rho \log \rho - \rho \log \sigma \right] \) is the relative entropy, then \( E_D \) above is called the relative entropy of entanglement (REE). The REE has many nice properties and plays a crucial role in the theory of entanglement [12, 13].

Just as in the previous subsection, we can use equation (40) to define an asymmetry monotone that is based on the REE. We call this monotone the relative entropy of asymmetry (REA). However, unlike the monotones in the previous subsection, distance-based monotones of asymmetry can also be defined directly by choosing the target set \( Q \) to be the set of \( G \)-invariant states. In this case, if \( D \) is taken to be the relative entropy, then the resulting monotone is known to be the \( G \)-asymmetry [16, 17]. How the \( G \)-asymmetry is related to the REA is an important question which we discuss here only partially. A more detailed study of the comparison is left to future work.

4.3.1. The relative entropy of asymmetry. As discussed above, an important and widely studied entanglement distance monotone is the REE,

\[
E_R(\rho) = \min_{\sigma \in \text{SEP}} S(\rho || \sigma),
\]

where the relative entropy

\[
S(\rho || \sigma) = -S(\rho) - \text{Tr}(\rho \log \sigma)
\]

is the distance function and where the infimum can be replaced with a minimum [21]. The relative entropy is not symmetric and does not preserve the triangle inequality. Following section 3.3, we can define a class of asymmetry monotones

\[
A^g_R(\rho) := E_R(C_g(\rho)), \quad \forall g \in G.
\]

Finally, we define the REA to be the maximum monotone.

**Definition 10.** The REA is the monotone,

\[
A^\text{max}_R(\rho) := \max_{s \in S} A^s_R(\rho),
\]

where the finite subgroup \( S \subset G \) was defined by the property in equation (39).

From the discussion in section 3.3 it follows that \( A^\text{max}_R \) is faithful, i.e. \( A^\text{max}_R(\rho) = 0 \) if and only if \( \rho \) is \( G \)-invariant.
4.3.2. Comparison with G-asymmetry. Choosing the set INV of G-invariant as the target set \( Q \) for the states acting on \( \mathcal{H} \) leads to a measure known as the G-asymmetry \([16]\) or, alternatively, the relative entropy of frameness \([17]\),

\[
A_g := \min_{\sigma \in \text{INV}} S(\rho \parallel \sigma) = S(G(\rho)) - S(\rho). \tag{56}
\]

We refer to \( A_g \) as G-asymmetry in the rest of the paper. Here, \( G(\rho) \) is the twirling operation discussed in equation (7) of section 2.1.

In order to compare G-asymmetry with REA, let us first consider a slightly different function, also based on the REE but with a different target set relative to which the distance is minimized.

Each isometry \( C_s \), for \( s \in S \), leads, in general, to a strict distinct subset of SEP that acts on \( \mathcal{H} \otimes \mathcal{H}_B \). We denote the set by \( \text{SEP}_s \). That is, \( \text{SEP}_s \) is the intersection of \( \text{SEP} \) with the image of \( C_s \) (see figure 2). We also denote the image of the set of \( G \)-invariant states under \( C_s \) as \( C_s[\text{INV}] \). Note that if \( G \) is not Abelian, then \( C_s[\text{INV}] \) is a strict subset of \( \text{SEP}_s \). For example, as we saw earlier, \( \text{SEP}_s \) also contains product states \( |\tilde{\phi}\rangle = |j, \lambda; m\rangle_s \otimes |m\rangle \) that are the images of the states \( |j, \lambda; m\rangle_s \). However, the eigenstates \( |j, \lambda; m\rangle_s \) are not \( G \)-invariant when \( j \neq 0 \). We now define the function \( A_{R_s}^* \) as

\[
A_{R_s}^*(\rho) := \min_{\sigma \in C_s[\text{INV}]} S(C_s(\rho) \parallel \sigma). \tag{57}
\]

The function \( A_{R_s}^* \) can, in general, be greater than \( A_{R}^{\max} \) but can never be smaller.

**Proposition 7.** For every \( s \in S \), \( A_{R_s}^* \) are greater than or equal to the REA.

\[
A_{R_s}^{\max}(\rho) \leq A_{R_s}^*(\rho), \quad \forall \rho \in \mathcal{B}(\mathcal{H}). \tag{58}
\]

---

6 If \( G \) is Abelian, then all separable states in \( \text{SEP}_s \) are images of invariant states and thus \( C_s[\text{INV}] = \text{SEP}_s \) (see appendix B).
Proof. For any given \( s \in S \), \( C_s[\text{INV}] \subseteq \text{SEP} \subseteq \text{SEP} \). It follows that \( A_R^s \leq A_R^{s^*} \), since \( A_R^s \) is obtained by minimizing the relative entropy over the larger set \( \text{SEP} \) that includes \( C_s[\text{INV}] \). As this is true for all \( s \in S \), \( A_R^{s^*} \) is greater than or equal to the maximum \( A_R^{\text{max}} \) too. \( \square \)

The isomorphism between the two sets \( \text{INV} \) and \( C_s[\text{INV}] \) implies that the minimum taken over \( C_s[\text{INV}] \) in the definition of \( A_R^{s^*}(\rho) \) coincides with the minimum of \( G\text{-asymmetry} \) \( A_G \) in equation (56). By this we mean that the separable state that minimizes the relative entropy in equation (57) is the image, under the isometry \( C_s \), of the invariant state that minimizes the relative entropy in equation (56).

To see this, consider the spectral decomposition of states \( \rho \) and \( \sigma \) acting on \( \mathcal{H} \), namely, 
\[
\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \quad \text{and} \quad \sigma = \sum_j q_j |\phi_j\rangle\langle\phi_j|.
\]
Recall that \( C_s \), being an isometry, preserves the inner product between pure states. It follows that the spectral decomposition of the image states are 
\[
C_s(\rho) = \sum_i p_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| \quad \text{and} \quad C_s(\sigma) = \sum_j q_j |\tilde{\phi}_j\rangle\langle\tilde{\phi}_j|,
\]
where \( |\tilde{\psi}_i\rangle \) and \( |\tilde{\phi}_j\rangle \) are the images of \( |\psi_i\rangle \) and \( |\phi_j\rangle \), i.e. \( |\psi_i\rangle \mapsto |\tilde{\psi}_i\rangle \) and \( |\phi_j\rangle \mapsto |\tilde{\phi}_j\rangle \). Hence, for every two states \( \rho \) and \( \sigma \), the two relative entropies \( S(\rho \parallel \sigma) \) and \( S(C_s(\rho) \parallel C_s(\sigma)) \) must be equal. Two corollaries follow:

Corollary 8. For every \( s \in S \), the functions \( A_R^{s^*} \) and \( A_G \) are identical, \( A_R^{s^*} \equiv A_G \).

Corollary 9. The \( G\text{-asymmetry} \) is greater than or equal to the REA.
\[
A_R^{\text{max}}(\rho) \leq A_G(\rho), \quad \forall \rho \in B(\mathcal{H}). \tag{59}
\]

The relationship between \( G\text{-asymmetry} \) and the REA goes deeper than what we have discussed so far, and our discussion here must be viewed only as an introductory treatment of the subject. We leave a more complete discussion to future work.

5. Other entanglement-based selection rules and conservation laws

In this section, we consider a different isometry that has been used implicitly in the literature concerning symmetry and quantum reference frames [10, 14, 17]. The isometry is quite natural to consider, but as we will see, in general, it is not an LOCC-simulating isometry. Nevertheless, we will show that it still leads to new and independent necessary conditions for the manipulation of asymmetric states.

We start by considering the Hilbert space decomposition of equation (8). Irreps carrying subspaces \( \mathcal{H}_{j,\lambda} \) for a fixed \( j \) are equivalent. Their direct sum
\[
\mathcal{H}_j := \bigoplus\limits_\lambda \mathcal{H}_{j,\lambda} \tag{60}
\]
is isomorphic to \( \mathcal{H}_j \cong \mathcal{M}_j \otimes \mathcal{N}_j \), where \( \mathcal{M}_j \) carries the \( j \)th irrep, and \( \mathcal{N}_j \) is the so-called multiplicity space carrying the trivial representation of the group [14]. It follows that \( \mathcal{H} \cong \mathcal{W}_\mathcal{L} \), where \( \mathcal{W}_\mathcal{L} := \bigoplus\limits_j \mathcal{M}_j \otimes \mathcal{N}_j. \tag{61} \]

In [14] the isomorphism of \( \mathcal{H} \) and \( \mathcal{W}_\mathcal{L} \) was assumed implicitly, but now we explicitly introduce the isometry connecting them.

\[ \text{In fact, as is apparent from definition 5, the isometry } C_S \text{ merely ‘repeats’ the weight label } m \text{ for each eigenket } |j, \lambda; m\rangle, \text{ by attaching to it the ket } |m\rangle, \text{ i.e. } |j, \lambda; m\rangle \mapsto |j, \lambda; m\rangle \otimes |m\rangle. \]

New Journal of Physics 14 (2012) 123026 (http://www.njp.org/)
Definition 11. Let \( \{ |j, m \rangle \}_m \) and \( \{ |j, \lambda \rangle \}_\lambda \) be the basis states spanning the spaces \( \mathcal{M}_j \) and \( \mathcal{N}_j \), respectively. Then \( \mathcal{L} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{W}_L) \) is the isometry that maps
\[
|j, \lambda; m \rangle \xrightarrow{\mathcal{L}} |j, m \rangle \otimes |j, \lambda \rangle .
\] (62)

Note that \( \mathcal{W}_L \subset \mathcal{M} \otimes \mathcal{N} \), where \( \mathcal{M} := \bigoplus_j \mathcal{M}_j \) and \( \mathcal{N} := \bigoplus_j \mathcal{N}_j \). Therefore, states in the image of \( \mathcal{L} \) (i.e. states in \( \mathcal{W}_L \)) can be viewed as bipartite states. Moreover, if \( \rho \) is a \( G \)-invariant state, then from equation (9) it follows that
\[
\mathcal{L}(\rho) = \sum_{j, \lambda} p_{j, \lambda} \left( \sum_m |j, m \rangle \langle j, m| \right) \otimes |j, \lambda \rangle \langle j, \lambda|,
\] (63)
which is a separable state (see also [14]). Similarly, any coherent superposition of states with different values of \( j \),
\[
|\phi \rangle = \sum_{j, m, \lambda} c_{j, \lambda, m} |j, \lambda; m \rangle ,
\] (64)
is mapped to an entangled state,
\[
|\tilde{\phi} \rangle = \sum_{j, m, \lambda} c_{j, \lambda, m} |j, m \rangle \otimes |j, \lambda \rangle .
\] (65)
Thus, \( \mathcal{L} \) satisfies conditions (2) and (3) in definition 4 of an LOCC-simulating isometry. However, \( \mathcal{L} \) is not an LOCC-simulating isometry since it does not, in general, satisfy condition (1) of definition 4, as we show now for the group \( G = SU(2) \).

5.1. \( \mathcal{L} \) is not a local operations and classical computations-simulating isometry

We now show that the entanglement of the bipartite states in the image of the isometry \( \mathcal{L} \) can, in fact, be increased by covariant transformations. Consider the 1/2-spin state \( \Psi = |\psi \rangle \langle \psi| \), where \( |\psi \rangle = |1/2; 1/2 \rangle \), where we ignore the multiplicity index \( \lambda \), as it plays no role in what follows. Note that \( \mathcal{L}(\Psi) \) is a product state. Using equation (15), we see that the map \( \mathcal{E}_{1/2} \) takes \( \Psi \) to a state whose image is entangled. We only deal with fixed \( \alpha \) in (15), so we can remove it from our notation as well. Consider the operator sum representation of the irreducible \( SU(2) \)-covariant map \( \mathcal{E}_{1/2} \) consisting of two Kraus operators \( K_{1/2, 1/2} \) and \( K_{1/2, -1/2} \). Because of the freedom in the choice of \( SU(2) \)-covariant Kraus operators, we can choose them so that they act on \( |\psi \rangle \) up to a normalization factor as
\[
K_{1/2, 1/2}|\psi \rangle \propto |1; 1 \rangle \xrightarrow{\mathcal{E}} |1; 1 \rangle \otimes |1 \rangle ,
\] (66)
\[
K_{1/2, -1/2}|\psi \rangle \propto |1; 0 \rangle + |0; 0 \rangle \xrightarrow{\mathcal{E}} |1; 0 \rangle \otimes |1 \rangle + |0; 0 \rangle \otimes |0 \rangle .
\]
The state \( \mathcal{L}(\mathcal{E}_{1/2}(\Psi)) \) is an equal mixture of the two states in the rhs of equation (66) and is thus an entangled state. It follows that the transformation
\[
\mathcal{L}(\Psi) \mapsto \mathcal{L}(\mathcal{E}_{1/2}(\Psi))
\] (67)
cannot be accomplished by LOCC.
5.2. Necessary conditions for the manipulation of asymmetric states

Our motivation for introducing the isometries between the original and the Kronecker product Hilbert spaces is to learn about $G$-covariant transformations. In particular, we study how the entanglement of the image states changes. In order to better understand how the entanglement changes under the isometry $\mathcal{L}$, we now focus on the form of the maps that act on the image states and mimic $G$-covariant transformations. The Wigner–Eckart theorem implies that, up to a projection to the subspace $\mathcal{W}_L$ of equation (61), those are separable maps, i.e. of the form

$$\tilde{E}_{\text{sep}}(\bullet) = \sum_x \tilde{V}_x \otimes \tilde{K}_x(\bullet) \tilde{V}_x^\dagger \otimes \tilde{K}_x^\dagger.$$  

(61)

To see this, let $\Pi_{\mathcal{W}_L}$ denote the projection to the $\mathcal{W}_L$-space. As we saw in section 2.2, every $G$-covariant transformation can be constructed from a set of irreducible tensor operators $K_{J,M,\alpha}$. So we need only consider how $K_{J,M,\alpha}$ are mimicked in the $\mathcal{W}_L$-space. If $\rho$ is mapped to $\sigma$ by $K_{J,M,\alpha}(\sigma$ is in general subnormalized), then $\mathcal{L}(\rho)$ is mapped to $\mathcal{L}(\sigma)$ by the operator

$$\tilde{K}_{J,M,\alpha} := \tilde{V}_{J,M} \otimes \tilde{K}_{J,\alpha},$$  

(68)

followed by $\Pi_{\mathcal{W}_L}$. The matrix elements of $\tilde{V}_{J,M}$ and $\tilde{K}_{J,\alpha}$ are, following the Wigner–Eckart theorem, equal to the CG coefficient and the reduced matrix, respectively,

$$\langle j_2, m_2 | \tilde{V}_{J,M} | j_1, m_1 \rangle = \begin{pmatrix} j_1 & J \\ m_1 & m_2 \end{pmatrix},$$  

(69)

$$\langle j_2, \lambda_2 | \tilde{K}_{J,\alpha} | j_1, \lambda_1 \rangle = \langle j_2, \lambda_2 | K_{J,\alpha} | j_1, \lambda_1 \rangle.$$  

(70)

Again, here we consider only simply reducible groups. For the generalization of the results of this section to all semi-simple Lie groups see appendix A.

The entanglement of the image states can be increased only because of the projection $\Pi_{\mathcal{W}_L}$ in equation (68). We can express the projection as $\Pi_{\mathcal{W}_L} = \sum_j \Pi_j$, where

$$\Pi_j = \Pi_{\mathcal{H}_j} \otimes \Pi_{\mathcal{K}_j} := \sum_m |j, m\rangle |j, m\rangle \otimes \sum_\lambda |j, \lambda\rangle |j, \lambda\rangle.$$  

(71)

(72)

Responsible for creating or increasing the entanglement are the cross terms $\Pi_j$ and $\Pi_{j'}$ acting on both sides of $\mathcal{L}(\rho)$ as

$$\mathcal{L}(\rho) \mapsto \Pi_{\mathcal{W}_L} \tilde{K}_{J,M,\alpha} \mathcal{L}(\rho) \tilde{K}_{J,M,\alpha}^\dagger \Pi_{\mathcal{W}_L}.$$  

(71)

In order to get rid of the cross terms, we proceed as follows: assume a given $G$-covariant CP-map $\mathcal{E}$ acting on $\rho$, and the corresponding map on the bipartite state,

$$\tilde{E}[\mathcal{L}(\rho)] = \Pi_{\mathcal{W}_L} \left( \tilde{E}_{\text{sep}}[\mathcal{L}(\rho)] \right) \Pi_{\mathcal{W}_L},$$  

(72)

where $\tilde{E}_{\text{sep}}$ has an operator sum representation in terms of Kraus operators defined in equation (68). If, instead, we consider the transformation

$$\mathcal{L}(\rho) \mapsto \tilde{\sigma} = \sum_j \Pi_j \tilde{E}[\mathcal{L}(\rho)] \Pi_j$$  

$$= \sum_j \Pi_j \left( \tilde{E}_{\text{sep}}[\mathcal{L}(\rho)] \right) \Pi_j,$$

(73)
then the overall map remains a separable one. Note that the $\Pi_j$ are themselves separable. In fact, the transformation in (73) can be implemented by LOCC. The reason is this: the superoperator $\tilde{E}_{\text{sep}}$ is composed of operators $\tilde{V}_{J,M} \otimes \tilde{K}_{J,\alpha}$. The projections $\Pi_{J', \gamma} \tilde{V}_{J,M}$ are unitary operators acting on the irrep-subspace $\mathcal{M}_j$, as their matrix elements are simply the CG-coefficients corresponding to a change of basis in $\mathcal{M}_j$. Thus, the whole transformation can be implemented by a series of local measurements by Alice, corresponding to operators $\Pi_{J', \gamma} \tilde{K}_{J,\alpha}$, followed by the unitaries $\Pi_{J, \gamma} \tilde{V}_{J,M}$ performed by Bob.

It follows that the average entanglement of the state $\tilde{\sigma}$ cannot exceed the entanglement of the initial state $\mathcal{L}(\rho)$. We state the result in the following proposition.

**Proposition 10.** Let $E$ be an ensemble entanglement monotone. We further assume that $E$ is faithful and convex. The $G$-covariant transformation $\rho \mapsto \sigma$ is possible only if the following condition holds:

$$E(\mathcal{L}(\rho)) \geq E(\tilde{\sigma}).$$

(74)

**Proof.** The proposition is an immediate consequence of the fact that the transformation in (73) can be implemented by LOCC.

Can we restate the condition of equation (74) in terms of new asymmetry monotones? Let us define the average initial state as

$$\bar{\rho} := \sum_j \Pi_j \mathcal{L}(\rho) \Pi_j.$$  

(75)

Clearly, the entanglement $E(\mathcal{L}(\rho)) \geq E(\bar{\rho})$. But does $E(\bar{\rho})$ exceed $E(\tilde{\sigma})$ as well? If this were true, then we could define an ensemble asymmetry monotone as $A^\text{new}_E(\rho) := E(\bar{\rho})$ after all. However, this is not the case. Consider the group $G = SU(2)$, and let $\rho = |\phi\rangle \langle \phi|$, where

$$|\phi\rangle := \frac{1}{\sqrt{2}} |3/2; 1/2\rangle + \frac{1}{\sqrt{2}} |1/2; 1/2\rangle.$$

The image state $\mathcal{L}(\rho) = |\tilde{\phi}\rangle \langle \tilde{\phi}|$, where

$$|\tilde{\phi}\rangle := \frac{1}{\sqrt{2}} |3/2; 1/2\rangle \otimes |3/2\rangle + \frac{1}{\sqrt{2}} |1/2; 1/2\rangle \otimes |1/2\rangle$$

is an entangled state. Also consider the irreducible $SU(2)$-covariant CP-map $\mathcal{E}_{1/2}$ (see section 2.2). The state $\tilde{\rho}$ is a separable state, whereas the ensuing state $\tilde{\sigma}$ of equation (73) is entangled. In other words,

$$E(\tilde{\sigma}) \not\leq E(\tilde{\rho}) = 0.$$

Note that, in accordance with proposition 10, it is still true that $0 < E(\tilde{\sigma}) \leq E(\mathcal{L}(\rho))$.

In summary, proposition 10 provides a necessary condition that all $G$-covariant transformations must satisfy. Let us call such a necessary condition a **general selection rule**.

We have shown that the general selection rule in proposition 10 is not expressible in terms of asymmetry monotones of the initial and final states, but it is expressible in terms of the entanglement of their image states. This is an example of how asymmetry monotones are not the only relevant quantities in the study of symmetries of open systems.
5.3. Conserved quantities

If we further restrict ourselves to reversible $G$-covariant transformations, still more interesting results can be deduced from the $\mathcal{L}$-isometry. Unitary operations have only one Kraus operator. If $G$ is non-Abelian, $G$-covariant unitaries exist only among $G$-covariant transformations labeled by the identity representation, $J = 0$, denoted by $\mathcal{E}_{0,\alpha} = \mathcal{K}_{0,0,\alpha}$. (We consider the case of Abelian groups in appendix B.)

The unitary $\mathcal{K}_{0,0,\alpha}$ maps each subspace $\mathcal{H}_j$ in (60) to itself, and the corresponding bipartite operator $\tilde{\mathcal{K}}_{0,0,\alpha}$ has the form

$$\tilde{\mathcal{K}}_{0,0,\alpha} = \left( \sum_j \Pi_{\mathcal{H}_j} \right) \otimes \tilde{\mathcal{K}}_{0,\alpha}.$$  \hspace{1cm} (76)

The above form is a direct consequence of the CG-coefficients in equation (A.4) for the case $J = M = 0$.

Substituting for $\mathcal{E}$ in equation (72) shows that in this case the overall projection $\Pi_{\mathcal{W}_\mathcal{L}}$ into the subspace $\mathcal{W}_\mathcal{L}$ can be dropped, because $\tilde{\mathcal{K}}_{0,0,\alpha}$ maps $\mathcal{W}_\mathcal{L}$ to itself. Equivalently, $\Pi_{\mathcal{W}_\mathcal{L}} \Pi_j = \Pi_j$, so that

$$\Pi_{\mathcal{W}_\mathcal{L}} \tilde{\mathcal{K}}_{0,0,\alpha} = \tilde{\mathcal{K}}_{0,0,\alpha}.$$

The operator $\tilde{\mathcal{K}}_{0,0,\alpha}$ is, of course, a local unitary. It thus follows that for every reversible $G$-covariant transformation $\mathcal{E}$, the entanglement of the image state in equation (72) remains constant. In other words, we have identified a conserved quantity.

**Proposition 11.** For reversible $G$-covariant transformations, $\mathcal{E}_{0,\alpha}$, the function

$$L(\rho) := E (\mathcal{L}(\rho))$$  \hspace{1cm} (77)

is a conserved quantity.

Hence, we have obtained new conservation laws for closed systems. The new conservation laws are not of the form of the expectation value of a generator of a Hamiltonian symmetry, but are instead in terms of entanglement monotones. In the case of open systems and irreversible transformations, the conservation law is replaced with a general selection rule, again in terms of entanglement monotones.

6. Conclusion

The present paper contains two major innovations: firstly, the notion of using local operations to simulate symmetric dynamics, and secondly, the idea of applying the well-known and well-studied resource theory of entanglement to a totally different resource theory. Symmetric time evolutions described by covariant transformations are based on group structures, invariant subspaces and representation theory. It is not evident, at first, that such structures have any connections to local operations and tensor products of two or more systems. However, the link exists, and once found, is actually very simple. We found that the effect of an irreducible covariant operator on a ket $|m\rangle$, labeled by the weight $m$ of the algebra (ignoring the other labels), is a simple translation by some fixed amount $M$, $|m\rangle \rightarrow |m+M\rangle$. Thus, the local
operators that simulate the $G$-covariant transformations exploit a common feature of all Lie groups, i.e. how the weights are transformed.

In this lies the strength of our method, as it applies equally to all compact semi-simple Lie groups and links them all to a sub-class of local operations. In turn, this enables entanglement theory, as the resource theory of the restriction to LOCC, to be applied to the study of covariant transformations, irrespective of the symmetry group involved. Entanglement has been the focus of intense study and plays a central role in quantum information theory. This fact is reflected in the abundance of well-investigated entanglement measures and monotones, each of which can now be used to extract information about the asymmetry of quantum states. One important consequence has been the realization that, for closed systems, entanglement serves as a conserved quantity, or a constant of motion.

There are various directions one can go from here. First, we can ask: what do entanglement considerations tell us about the specifics of $G$-covariant transformations? For example, majorization of the Schmidt coefficients of the final pure state by the coefficients of the initial state is the necessary and sufficient condition for pure state to pure state transitions under LOCC. If we apply the majorization condition to the images of the initial and final states for different isometries $C_g$, would we retrieve the exact form of the corresponding $G$-covariant transformations?

A second line of study concerns the case of finite groups. The isometries we introduced derive from the form of the Wigner–Eckart theorem for Lie groups. For finite groups, the form of the Wigner–Eckart theorem is different and more complicated [23]. If the finite-group version of the Wigner–Eckart theorem lends itself to the construction of LOCC-simulating isometries, then entanglement theory can be directly applied to finite groups as well.

On a different note, we have not considered the case of many-copy transformations and asymptotic limits in this paper. Many questions of interest can be asked in this respect, including additivity of the measure and possible applications to the problem of distillation of asymmetry resources.

Finally, a fourth direction for future research suggested by our result is to look for similar conditions in other resource theories. For example, the restriction to Gaussian operations results in a new resource theory where non-Gaussian states are resources [26]. Another example is thermodynamics. Thermodynamics has been recognized as an energy preserving resource theory where transformations are restricted to operations that do not increase the total energy [29], and already, connections between thermodynamics, viewed as a resource theory, and entanglement have been demonstrated [27–31]. If the restricted set of operations in any of those resource theories are simulated by local operations, then it would be possible to employ entanglement theory for the study of those resource theories as well.

Acknowledgments

We acknowledge valuable discussions with Varun Narasimhachar and Iman Marvian. BT also acknowledges valuable discussions with P S Turner. This research has been supported by Alberta Innovates, the Natural Sciences and Engineering Research Council, General Dynamics Canada and the Canadian Centre of Excellence for Mathematics of Information Technology and Complex Systems (MITACS).
Appendix A. Generalized Wigner–Eckart theorem in the presence of outer multiplicities

The main results of the paper can be extended to the general case where the Kronecker product of the algebra associated with the group is not simply reducible. An algebra $H$ is not simply reducible when the algebra has outer multiplicities, i.e. multiplicities arising due to the coupling of the irreps. We now consider the general form of the Wigner–Eckart theorem

$$\langle j', \lambda'; m' | K_{J,M,\alpha} | j, \lambda; m \rangle = \sum_{\mu} \left( j \atop m \right| J \atop M \left( j', \mu \atop m' \right) \langle j', \lambda' || K_{J}\parallel j, \lambda \rangle_{\mu},$$

(A.1)

where $\mu$ is the outer multiplicity index for the irrep $[j']$ due to the coupling $[j] \otimes [J] \mapsto [j']$. Here, we have used the symbol $[j]$ to denote the representation labeled by $j$ and similarly for other representations. The terms $\left( j \atop m \right| J \atop M \left( j', \mu \atop m' \right)$ are the general CG coefficients, depending, in the general case, on the outer multiplicity $\mu$ in addition to the irrep and weight labels.

If the transformation $K_{J,M,\alpha}$ is unitary, $J$ and $M$ remain the labels of the identity representation, $J = M = 0$. Coupling to the identity representation never results in outer multiplicities. Thus, the results for $G$-covariant unitaries in the paper are valid for the general case.

A.1. The set of isometries $\{C_g\}$

All the CG coefficients are identically zero unless, as before, the weights labeling the bra and the ket and the tensor operator satisfy the relation

$$m + M = m'.$$

It follows that, as far as the weights are concerned, the same translation operator as in equation (27) applies to all the terms in the rhs of (A.1), and thus the same set of isometries $\mathcal{C}$ and $\mathcal{C}_g$ in definitions 5 and 6 of section 3 still satisfy all the conditions of an LOCC-simulating isometry of definition 3.1.

A.2. The isometry $\mathcal{L}$

The situation is more complicated for the isometry $\mathcal{L}$. The existence of outer multiplicities implies that we must define new Hilbert spaces to embed the original Hilbert space, i.e. Hilbert spaces that include the outer multiplicities in the label of their basis states. Let

$$\mathcal{M} = \text{span} \{ |j, \mu; m\rangle \}_{j,\mu,m}$$

be the space spanned by the basis states $|j, \mu; m\rangle$. Here, $j$ and $m$ are, as before, the irrep label and the weight label, respectively. We have included an additional label $\mu$, ranging over $\mu = 0, \ldots, \infty$, that we will shortly relate to the outer multiplicities. Similarly, let

$$\mathcal{N} = \text{span} \{ |j, \mu; \lambda\rangle \}_{j,\mu,\lambda},$$

where $\lambda$ is the label for the (initial) irrep multiplicities. Also, define $\mathcal{M}_j = \text{span} \{ |j, 0; m\rangle \}_m$ and $\mathcal{N}_j = \text{span} \{ |j, 0; \lambda\rangle \}_\lambda$, and

$$\mathcal{W}_L := \bigoplus_j \mathcal{M}_j \otimes \mathcal{N}_j.$$

As before, we can define the isometry $\mathcal{L}$ by specifying how it acts on the basis states.
Definition 12. \( \mathcal{L} : B(\mathcal{H}) \rightarrow B(\mathcal{W}_\ell) \) is the isometry that maps
\[
|j, \lambda; m\rangle \mapsto |j, 0; m\rangle \otimes |j, 0; \lambda\rangle.
\] (A.2)

Clearly, \( \mathcal{W}_\ell \subset \mathcal{M} \otimes \mathcal{N} \), and thus the states in the image of \( \mathcal{L} \) (i.e. states in \( \mathcal{W}_\ell \)) are bipartite states. Let \( K_{j,M,\alpha} \) be an irreducible \( G \)-covariant operator (see equation (13) in section 2.2). The operator acting on \( \mathcal{M} \otimes \mathcal{N} \) that mimics \( K_{j,M,\alpha} \) can again be expressed as a separable state followed by a projection to the image subspace \( \mathcal{W}_\ell \). Assume that \( \rho \) is mapped to a (in general subnormalized) \( \sigma \) by \( K_{j,M,\alpha} \), \( \mathcal{L}(\rho) \) is then mapped to \( \mathcal{L}(\sigma) \) by the operator
\[
\tilde{K}_{j,M,\alpha} := \tilde{V}_{j,M} \otimes \tilde{K}_{j,\alpha},
\] (A.3)
followed by \( \Pi_{\mathcal{W}_\ell} \). The general form of the Wigner–Eckart theorem (A.1) implies that
\[
\langle j_2, \mu_2; m_2 | \tilde{V}_{j,M} | j_1, \mu_1; m_1 \rangle = \begin{pmatrix} j_1 & J \\ m_1 & M \end{pmatrix} \begin{pmatrix} j_2, \mu_2 \\ m_2 \end{pmatrix},
\]
\[
\langle j_2, \mu_2; \lambda_2 | \tilde{K}_{j,\alpha} | j_1, \mu_1; \lambda_1 \rangle = \langle j_2, \lambda_2 || K_{j,\alpha} || j_1, \lambda_1 \rangle_{\mu_2}.
\] (A.4)

Note that the rhs does not depend on the value of \( \mu_1 \) in either equation. The map \( \Pi_{\mathcal{W}_\ell} \) is
\[
\Pi_{\mathcal{W}_\ell} = \sum_{j,\mu} \Pi_{j,\mu} \Pi_{\mathcal{W}_\ell},
\]
where
\[
\Pi_{j,\mu} := \Pi_{\mathcal{H}_j} \otimes \Pi_{\mathcal{W}_\ell},
\]
for a given \( G \)-covariant CP-map \( \mathcal{E} \) acting on \( \rho \), the corresponding map on the bipartite state is
\[
\tilde{\mathcal{E}}[\mathcal{L}(\rho)] = \Pi_{\mathcal{W}_\ell}(\tilde{\mathcal{E}}_{\text{sep}}[\mathcal{L}(\rho)]) \Pi_{\mathcal{W}_\ell},
\] (A.6)
where \( \tilde{\mathcal{E}}_{\text{sep}} \) has an operator sum representation in terms of Kraus operators defined in equation (68). \( \tilde{\mathcal{E}}[\mathcal{L}(\rho)] \) is still not an LOCC-simulating CP-map. The cross terms \( \Pi_{j,\mu} \) and \( \Pi_{j',\mu'} \) acting on both sides of \( \mathcal{L}(\rho) \) render the overall CP-map a non-separable one. However, we can destroy the cross terms here too, by applying the set of projections \( \Pi_{j,\mu} \) separately on both sides and then taking the average of the maps, as follows:
\[
\mathcal{L}(\rho) \mapsto \bar{\sigma} = \sum_{j,\mu} \Pi_{j,\mu} \tilde{\mathcal{E}}[\mathcal{L}(\rho)] \Pi_{j,\mu}
\]
\[
= \sum_{j,\mu} \Pi_{j,\mu} \left( \tilde{\mathcal{E}}_{\text{sep}}[\mathcal{L}(\rho)] \right) \Pi_{j,\mu}.
\] (A.7)
The overall map is separable now, and the condition \( E(\rho) \geq E(\bar{\sigma}) \) must hold if the transition from \( \rho \) to \( \sigma \) is possible.

**Appendix B. Abelian Lie groups**

The irreducible representations of Abelian groups are one dimensional. The irrep label is always the highest weight and one-dimensional irreps have only one weight. Thus, the irrep label and the weight label are the same. We use the label \( n \) for the irreps of an Abelian group, and to conform to the notation of the rest of the paper, we label the basis states as \(|n, \lambda; n\rangle\). We now show that the results of the paper are greatly simplified in the case of Abelian groups. In particular, we show that the isometries \( C_g \) are all equivalent to each other, and are furthermore equivalent to the isometry \( \mathcal{L} \).
Definition 13. \( \mathcal{C} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}_G) \) is the isometry that maps
\[
|n, \lambda; n\rangle \rightarrow |n, \lambda; n\rangle \otimes |n\rangle.
\] (B.1)

Definition 14. \( \mathcal{L} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}_G) \) is the isometry that maps
\[
|n, \lambda; n\rangle \rightarrow |n; n\rangle \otimes |n; \lambda\rangle.
\] (B.2)

First, note that the action of a group element on the basis kets is to merely add a phase,
\[
U(g)|n, n, \lambda\rangle = e^{i\theta(g)}|n, n, \lambda\rangle.
\]

Thus, the definition 6 of \( \mathcal{C}_g \) implies that
\[
\mathcal{C}_g \equiv \mathcal{C}, \quad \forall g \in G.
\] (B.3)

The form of the irreducible \( G \)-covariant transformation is also simplified to
\[
\langle n', \lambda'; n|K_{N,N,\alpha}|n, \lambda; n\rangle = \delta_{n',n+N}\langle n', \lambda' || K_{N,\alpha} || n, \lambda\rangle,
\] (B.4)
or equivalently
\[
K_{N,\alpha} = \sum_n c^{(N,\alpha)}_{n,\lambda,\lambda'} |n + N, \lambda'; n + N\rangle \langle n, \lambda; n|,
\] (B.5)
where \( c^{(N,\alpha)}_{n,\lambda,\lambda'} = \langle n', \lambda' || K_{N,\alpha} || n, \lambda\rangle \).

Assume that \( \rho \) is mapped to \( \text{a (in general subnormalized state)} \sigma \) by \( K_{f,M,\alpha}. \mathcal{C}(\rho) \) is then mapped to \( \mathcal{C}(\sigma) \) by the operator
\[
\tilde{K}_{N,\alpha}^\mathcal{C} = K_{N,\alpha} \otimes \sum_n |n + N\rangle \langle n|.
\] (B.6)

Equivalently, \( \mathcal{L}(\rho) \) is mapped to \( \mathcal{L}(\sigma) \) by the operator
\[
\tilde{K}_{N,\alpha}^\mathcal{L} = \sum_n |n + N; n + N\rangle \langle n; n| \otimes \sum_n c^{(N,\alpha)}_{n,\lambda,\lambda'} |n + N, \lambda'; n\rangle \langle n, \lambda|.
\] (B.7)

\( \tilde{K}_{N,\alpha}^\mathcal{C} \) can be implemented by an LOCC-transformation, as \( \mathcal{C} \) is an LOCC-simulating isometry.

Now, interestingly, the simulating operator of the second isometry, \( \tilde{K}_{N,\alpha}^\mathcal{L} \), is implementable by LOCC transformations as well. So, in the case of the Abelian groups, the isometry \( \mathcal{L} \) is also an LOCC-simulating isometry. In fact, the forms of \( \tilde{K}_{N,\alpha}^\mathcal{C} \) and \( \tilde{K}_{N,\alpha}^\mathcal{L} \) are similar, both composed of the tensor product of a copy of the original \( G \)-covariant operator \( K_{N,\alpha} \) and a translation operator, and the isometry
\[
|n, \lambda; n\rangle \otimes |n\rangle \rightarrow |n; n\rangle \otimes |n; \lambda\rangle
\]
maps one set of LOCC-transformations to an equivalent set of LOCC transformations. In this sense, the two isometries \( \mathcal{C} \) and \( \mathcal{L} \) are equivalent.

The image state under either isometry is an entangled state if and only if the initial state has no coherence in \( n \), i.e. if the state is a coherent superposition of states with different values of \( n \). States acting on the original Hilbert space \( \mathcal{H} \) with no coherence in \( n \) are the \( G \)-invariant states, as the twirling operations destroy the coherence in \( n \).

Proposition 12. If \( G \) is an Abelian group, then the image state \( \mathcal{C}(\rho) \) (or equivalently \( \mathcal{L}(\rho) \)) is a separable state if and only if the initial state \( \rho \in \mathcal{B}(\mathcal{H}) \) is \( G \)-invariant.

Finally, as a corollary we note that the ‘average state’, \( \bar{\sigma} \) of equation (73), is always a separable state and has no entanglement.
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New Journal of Physics **14** (2012) 123026 (http://www.njp.org/)