BOUNDEDNESS IN LOGISTIC KELLER–SEGEL MODELS WITH
NONLINEAR DIFFUSION AND SENSITIVITY FUNCTIONS

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Abstract. We consider the following fully parabolic Keller–Segel system
\begin{align}
\frac{du}{dt} &= \nabla \cdot (D(u)\nabla u - S(u)\nabla v) + u(1 - u^\gamma), \quad x \in \Omega, \ t > 0, \\
\frac{dv}{dt} &= \Delta v - v + u, \quad x \in \Omega, \ t > 0, \\
\frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0
\end{align}

over a multi-dimensional bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$. Here $D(u)$ and $S(u)$ are smooth functions satisfying: $D(0) > 0$, $D(u) \geq K_1 u^{m_1}$ and $S(u) \leq K_2 u^{m_2}$, $\forall u \geq 0$, for some constants $K_i \in \mathbb{R}_+$, $m_i \in \mathbb{R}$, $i = 1, 2$. It is proved that, when the parameter pair $(m_1, m_2)$ lies in some specific regions, the system admits global classical solutions and they are uniformly bounded in time. We cover and extend [22, 28], in particular when $N \geq 3$ and $\gamma \geq 1$, and [3, 29] when $m_1 > \gamma - \frac{2}{N}$ if $\gamma \in (0, 1)$ or $m_1 > \gamma - \frac{4}{N+2}$ if $\gamma \in [1, \infty)$. Moreover, according to our results, the index $\frac{2}{N}$ is, in contrast to the model without cellular growth, no longer critical to the global existence or collapse of this system.

1. Introduction. This paper investigates the following fully parabolic Keller–Segel system for $(u, v) = (u(x, t), v(x, t))$
\begin{align}
\frac{du}{dt} &= \nabla \cdot (D(u)\nabla u - S(u)\nabla v) + u(1 - u^\gamma), \quad x \in \Omega, \ t > 0, \\
\frac{dv}{dt} &= \Delta v - v + u, \quad x \in \Omega, \ t > 0, \\
\frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0, \\
u(x, 0) &= u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, \quad x \in \Omega, (1.1)
\end{align}

where $\Omega$ is a bounded domain in $\mathbb{R}^N, N \geq 2$, with smooth boundary $\partial \Omega$. (1.1) is a Keller–Segel model of chemotaxis, which describes the oriented movement of cellular organisms in response to heterogeneous spatial distribution of a chemical in the environment. Here $u(x, t)$ denotes cell population density at space–time location $(x, t)$, $v(x, t)$ the chemical concentration. $D(u)$ is the density–dependent motility of cells and it measures the ability of cells to move randomly in the environment, and $S(u)$ reflects the variation of cellular sensitivity with respect to the levels of cell.

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population density. The standard choice of logistic growth for the cell populations is made with intrinsic growth rate scaled to the unit; moreover, the attractive chemical is released by the cells and consumed by an enzyme in the environment meanwhile, both at a constant rate scaled to the unit. $γ$ is a positive constant that measures the cellular competition intensity when the population is crowded. $ν$ is the unit outer normal to the boundary $∂Ω$ and the non–flux boundary conditions interpret the assumption that the region is enclosed and inhibits both cellular and chemical flux across the boundary. The initial cell distribution $u_0 ∈ C^0(Ω)$ and chemical concentration $v_0 ∈ C^1(Ω)$ are assumed to be nonnegative but not identically zero.

The goal of this paper is to establish the global existence and uniform boundedness for (1.1). Throughout this paper, we shall assume that $D(u)$ and $S(u)$ are $C^2$–smooth functions of $u$ and there exist constants $K_i ∈ R^+$, $m_i ∈ R$, $i = 1, 2$, such that

$$0 < D(0), D(u) ≥ K_1 u^{m_1}, ∀ u ≥ 0,$$

and

$$0 ≤ S(u) ≤ K_2 u^{m_2}, ∀ u ≥ 0.$$

From the viewpoint of mathematical modeling, it is realistic to consider chemotaxis in 2D or 3D regions. Thus, we shall restrict our attention to the case of multi–dimensional domains with $N ≥ 2$, while the analysis carries over to 1D (See Remark 3 below). Our first main result is the following theorem.

**Theorem 1.1.** Let $Ω ∈ R^N, N ≥ 2$, be a bounded domain with smooth boundary $∂Ω$. Suppose that $γ ≥ 1$ and the smooth functions $D(u)$ and $S(u)$ satisfy (1.2) and (1.3) respectively with

$$m_2 - m_1 < \frac{3N + 2}{N(N + 2)}.$$

Then for any nonnegative initial data $(u_0, v_0) ∈ C^0(Ω) × C^1(Ω)$, there exists a unique couple $(u, v)$ of nonnegative bounded functions in $C^0(Ω × [0, ∞)) ∩ C^2(Ω × (0, ∞))$ which solves (1.1) classically. Moreover, there exists a positive constant $C$ such that the solution is uniformly bounded in time in the sense

$$∥u(·, t)∥_{L^∞(Ω)} + ∥v(·, t)∥_{L^∞(Ω)} < C, ∀ t ∈ (0, ∞).$$

**Remark 1.** According to [21, 22], (1.1) has global existence and boundedness if $0 < m_2 - m_1 < \frac{2}{N}$ and $γ ∈ (0, ∞)$, provided that the bounded domain $Ω$ is convex. Theorem 1.1 covers and extends these results when $N ≥ 3$ and $γ ≥ 1$. The convexity condition can be removed thanks to [9].

The logistic–type growth in first equation of (1.1) demonstrates the prevention of population overcrowding due to the competition between cellular organisms, in particular when the resource is limited. It is quite natural to expect that such damping effect tends to prevent blowup in (1.1) and a larger $γ$ has a stronger damping effect, however, whether or not it is sufficient remains unknown to this date, even for the classical Keller–Segel models with $m_1 = 0$, $m_2 = 1$. See [15, 27] for instance. Theorem 1.1 fails to reveal this fact due to technical reasons. To tackle this issue, we investigate the effect of $γ$ and present the second main result of this paper in the following theorem.

**Theorem 1.2.** Let $Ω ∈ R^N, N ≥ 2$, be a bounded domain with smooth boundary $∂Ω$. Suppose all the conditions in Theorem 1.1 hold except that the smooth functions
Lemma 3.2, we can easily show that Theorem 1.2 holds with (1.5) being replaced by 2. Then (1.7) always possesses blow-up solutions regardless of the size of total cell population; under further technical conditions on $D(u)$ and $S(u)$, blowup always occurs for any bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$. The fact that the blowup is finite in time is revealed in [4].

Remark 2. Theorem 1.2 extends [28], in which max $\{m_1, \gamma\}$ is replaced by $\gamma$. On the other hand, suppose that $0 < \gamma < 1$, then according to Corollary 1 after Lemma 3.2 we can easily show that Theorem 1.2 holds with (1.5) being replaced by $2m_2 - m_1 < \max\{m_1, \gamma\} + \frac{4}{N+2}$. This also extends the results in [28].

Remark 3. When $N = 1$, according to our proofs of the main results, Theorem 1.1 and Theorem 1.2 also hold with (1.4) being replaced by $m_2 - m_1 < 2$ and (1.5) by $2m_2 - m_1 < \max\{m_1, \gamma\} + 2$, respectively. These results have been proved by [22, 28, 29].

1.1. Keller–Segel models with nonlinear diffusion. Theoretical and mathematical modeling of chemotaxis dates back to the works of Patlak [18] in the 1950s and Keller–Segel [12, 13, 14] in the 1970s. The chemotaxis PDE systems and their variants, now often called Keller–Segel models, have been extensively studied by many authors over the past few decades. We refer to the review papers [2, 6, 7, 8] for detailed descriptions of the models and their developments.

For example, in order to investigate the density–dependent chemotactic movement, considerable effort has been devoted to Keller–Segel models with nonlinear diffusion and sensitivity functions. For example, Painter and Hillen [17] proposed and studied the following volume–filling model

$$
\begin{aligned}
&u_t = \nabla \cdot ((Q(u) - uQ'(u))\nabla u - uQ(u)\nabla v), \quad x \in \Omega, \ t > 0, \\
v_t = \Delta v - v + u, \quad x \in \Omega, \ t > 0, \\
&u(x,0) = u_0(x) \geq 0, \ v(x,0) = v_0(x) \geq 0, \quad x \in \Omega,
\end{aligned}
$$

(1.6)

where $Q(u)$ denotes the density–dependent probability of the cell finding space at its neighboring location. It is assumed in [17] that $Q$ becomes identically zero for $u$ large enough. In particular, the authors considered a specific example with $Q(u) = \max\{u - u, 0\}$ to describe a situation where the intensity of cellular movement is lessened as the cell density increases, while cells become immobile where the density surpasses the critical level $\bar{u}$. This choice is made based on the so–called volume–filling effect: the movement of cells is inhibited near points where the cells are already packed. Global existence and boundedness of (1.6) are obtained in [5, 17].

In [25], Winkler relaxed the decay assumption on $Q(u)$ above and proposed the following model with general choices of smooth diffusion and sensitivity functions

$$
\begin{aligned}
&u_t = \nabla \cdot (D(u)\nabla u - S(u)\nabla v), \quad x \in \Omega, \ t > 0, \\
v_t = \Delta v - v + u, \quad x \in \Omega, \ t > 0, \\
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0, \\
u(x,0) = u_0(x) \geq 0, \ v(x,0) = v_0(x) \geq 0, \quad x \in \Omega.
\end{aligned}
$$

(1.7)

It is proved in [25] that, when $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a ball and $\frac{S(u)}{|\Omega|} \geq C u^{2+\epsilon}$, $\forall u \geq 1$, for some $C > 0$ and $\epsilon > 0$, [17] always possesses blow–up solutions regardless of the size of total cell population; under further technical conditions on $D(u)$ and $S(u)$, blowup always occurs for any bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$. The fact that the blowup is finite in time is revealed in [4]. On the other hand, Tao and Winkler [21] proved that, when the bounded domain $\Omega$ is convex, the solutions to (1.7) are global.
and bounded in time if $\frac{S(u)}{D(u)} \leq Cu^{\frac{\alpha}{N} - \epsilon}$, while the convexity condition was recently removed by Ishida et al. in \[9\]. Moreover, Ishida and Yokota \[10\] showed that (1.7) has global existence when $\frac{S(u)}{D(u)} \geq Cu^{\frac{\alpha}{N}}$ if the initial data are small; however the solutions may blow up in finite or infinite time if the initial data are large. Denote $\frac{S(u)}{D(u)} \simeq u^\alpha$ for $u$ large, then generally speaking, the index $\frac{\alpha}{N}$ is critical to (1.7) in the sense that its solutions exist globally and remain bounded in time if $\alpha < \frac{2}{N}$, while there may exist solutions which blow up in finite time if $\alpha \geq \frac{2}{N}$. We would like to mention that the same criticality of $\frac{S(u)}{D(u)} \simeq u^{\frac{2}{N}}$ holds in $\mathbb{R}^N$. See \[11, 19\] and the references therein.

1.2. Literature review and illustration of main results. Concerning the question to what extent the logistic growth of $u$ allows or prevents chemotactic collapse, (1.1) has been investigated by various authors. First of all, according to our discussions above, it is easy to see that the arguments for the global well–posedness of (1.7) in \[21\] naturally carry over to (1.1) thanks to the presence of logistic damping. This observation was confirmed by Wang et al. in \[22\], and they proved that, if $0 < m_2 - m_1 < \frac{2}{N}$ and $\gamma > 0$, (1.1) possesses a unique global classical solution which is nonnegative and uniformly bounded in time. Note that the index $\frac{3N+2}{N(N+2)}$
is larger than \( \frac{2}{N} \) for \( N \geq 3 \) (both equal 1 when \( N = 2 \)) and Theorem 1.1 covers and extends \cite{22} when \( N \geq 3 \) and \( \gamma \geq 1 \). Recently, Zhang and Li \cite{28} proved that, if \( m_2 - \frac{1}{N} m_1 < \frac{\gamma}{N} + \frac{1}{2} \) when \( \gamma \in (0, 1) \) or \( m_2 - \frac{1}{N} m_1 < \frac{\gamma}{N} + \frac{2}{N+2} \) when \( \gamma \geq 1 \), (1.1) possesses a unique global classical solution which is uniformly bounded. Moreover, we note that Theorem 1.2 covers and extends \cite{28}. In Figure 1, we present a schematic illustration of our results in Theorem 1.1 and Theorem 1.2. See Remarks 1 and 2. For each pair of \((m_1, m_2)\) in the shaded region of each plot as \( \gamma \) varies, (1.1) admits global bounded classical solutions. In particular, when \( \gamma \geq 1 \), Theorem 1.1 extends one borderline \( L_1 \) to \( L_1^* \). Here the borderlines have equations \( L_1^*: m_2 - m_1 = \frac{1}{N} + \frac{2}{N+2} \), \( L_2^*: m_2 - m_1 = \frac{2}{N^2} \) and \( L_3^*: m_2 - \frac{1}{N} m_1 = \frac{\gamma}{N} + \frac{2}{N+2} \).

On the other hand, it seems necessary to point out that, under conditions (1.2) and (1.3), Cao \cite{3} proved that if \( m_2 < 1 \) and \( \gamma = 1 \), (1.1) admits a solution which is classical and uniformly bounded for any \( m_1 \in \mathbb{R} \). Therefore, the damping effect of logistic growth alone is sufficient to guarantee the global existence and boundedness for (1.1) as long as the sensitivity function is a sublinear function of cell density, regardless of the size of \( m_1 \). Combining the results from \cite{22}, Zheng \cite{29} generalized \cite{3} to \( 0 < m_2 - m_1 < \max\{\gamma - m_1, \frac{2}{N}\}, \gamma > 0 \). In Figure 2, we give a schematic illustration on these results and provide a relatively complete summary on the global existence for (1.1) with nonlinear diffusion and sensitivity functions. In this paper,

we shall prove Theorem 1.1 and Theorem 1.2 which establish the global existence and boundedness for (1.1) with a logistic growth of cell populations. Our results

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{An illustration of up-to-date summary results on global existence of the nonlinear diffusion system (1.1). For each pair of \((m_1, m_2)\) in the shaded region, (1.1) over bounded domain \( \Omega \subset \mathbb{R}^N, N \geq 2 \), admits global bounded classical solutions. The slope of \( L_1(L_1^*) \) is 1 and of \( L_2^* \) is \( \frac{1}{2} \); \( L_4 \) is the horizontal line \( m_2 = \gamma \).}
\end{figure}
indicate that, in contrast to (1.7) without cellular growth, the index $\frac{2}{N}$ is no longer critical to (1.1) in the presence of logistic cellular growth. Our results overlap with the results in [3, 22, 28, 29] when $N = 2$ and extends theirs when $N \geq 3$, in particular when $m_1$ is large.

2. Local existence and preliminary results. We first study the local existence of classical solutions of (1.1) following the fundamental theory developed by Amann [1].

**Proposition 1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial \Omega$. Suppose that the smooth functions $D(u)$ and $S(u)$ satisfy (1.2) and (1.3), respectively. Assume that $(u_0, v_0)$ belongs to $C^0(\Omega) \times C^1(\Omega)$ and $u_0, v_0 \geq, \neq 0$ in $\Omega$. Then there exist $T_{\text{max}} \in (0, \infty]$ and a unique couple $(u, v)$ of nonnegative functions in $C^0(\Omega \times [0, T_{\text{max}}]) \cap C^{2,1}(\Omega \times (0, T_{\text{max}}))$ which solve (1.1) classically in $\Omega \times (0, T_{\text{max}})$. Moreover $u(x, t) \geq 0$ and $v(x, t) \geq 0$ in $\Omega \times (0, T_{\text{max}})$ and the following dichotomy holds:

$$\text{either } T_{\text{max}} = \infty \text{ or } \limsup_{t \to T_{\text{max}}} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = \infty. \quad (2.1)$$

We now collect some basic properties of the local solutions.

**Lemma 2.1.** [Lemma 2.1 of [21], Lemma 2.2 of [23]] Let $(u, v)$ be a nonnegative classical solution of (1.1) in $\Omega \times (0, T_{\text{max}})$. Then there exists a positive constant $C$ such that

$$\|u(\cdot, t)\|_{L^1(\Omega)} + \|\nabla v(\cdot, t)\|_{L^2(\Omega)} \leq C, \forall t \in (0, T_{\text{max}}). \quad (2.2)$$

**Lemma 2.2.** Let $(u, v)$ be a nonnegative classical solution of (1.1) in $\Omega \times (0, T_{\text{max}})$. Suppose that $u \in L^\infty(0, t; L^p(\Omega))$ for some $p \in [1, \infty)$, then there exists a positive constant $C$ dependent on $q$, $\|u_0\|_{L^q(\Omega)}$ and $|\Omega|$ such that

$$\|v(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C \left(1 + \sup_{s \in (0, t)} \|u(\cdot, s)\|_{L^p(\Omega)}\right), \forall t \in (0, T_{\text{max}}), \quad (2.3)$$

where $q \in [1, \frac{Np}{N-p}]$ if $p \in [1, N)$, $q \in [1, \infty)$ if $p = N$ and $q = \infty$ if $p > N$.

**Proof.** The proof is standard by applying the $L^p - L^q$ estimates between semigroups $\{e^{t\Delta}\}_{t \geq 0}$ and we present it for the sake of completeness. Write the $v$–equation into the following abstract formula

$$v(\cdot, t) = e^{(\Delta-1)t}v_0 + \int_0^t e^{(\Delta-1)(t-s)}u(\cdot, s)ds. \quad (2.4)$$

By Lemma 1.3 of [20], there exists a positive constant $C_{21}$ such that

$$\|v(\cdot, t)\|_{W^{1,q}(\Omega)} = \left\|e^{(\Delta-1)t}v_0 + \int_0^t e^{(\Delta-1)(t-s)}u(\cdot, s)ds\right\|_{W^{1,q}(\Omega)}$$

$$\leq C_{21}\|v_0\|_{L^p(\Omega)} + C_{21} \int_0^t e^{-\sigma(t-s)} \left(1 + (t-s)^{-\frac{1}{2}} - \frac{2}{2} \left(\frac{1}{2} - \frac{1}{q}\right)\right)\|u(\cdot, s)\|_{L^p(\Omega)}ds, \quad (2.5)$$

where $\sigma$ is the first Neumann eigenvalue of $-\Delta$. On the other hand, under the conditions on $q$ after (2.4), we find

$$\sup_{t \in (0, \infty)} \int_0^t e^{-\sigma(t-s)} \left(1 + (t-s)^{-\frac{1}{2}} - \frac{2}{2} \left(\frac{1}{2} - \frac{1}{q}\right)\right)ds < \infty,$$

therefore (2.3) follows from (2.5).
Our proofs of Theorem 1.1 and Theorem 1.2 make frequent use of the following Gagliardo–Nirenberg interpolation inequality.

**Lemma 2.3.** (Lemma 2.2 in [23]) Let \( \Omega \subset \mathbb{R}^N \), \( N \geq 1 \), be a bounded domain with smooth boundary. Suppose that \( p, q \geq 1 \), \( (N - q)p < Nq \), and \( r \in (0, p) \) satisfy

\[
\frac{1}{p} = \alpha \left( \frac{1}{q} - \frac{1}{N} \right) + (1 - \alpha) \frac{1}{r} \quad \text{and} \quad \alpha \in (0, 1).
\]

Then there exists a positive constant \( C_0 \) such that for any \( f \in W^{1,q}(\Omega) \cap L^r(\Omega) \),

\[
\|f\|_{L^r(\Omega)} \leq C_0 \|\nabla f\|_{L^q(\Omega)}^{\alpha} \|f\|_{L^r(\Omega)}^{1 - \alpha} + C_0 \|f\|_{L^r(\Omega)}.
\]

We will also use the following fractional Gagliardo–Nirenberg inequality.

**Lemma 2.4.** (Lemma 2.5 in [9]) Let \( \Omega \subset \mathbb{R}^N \), \( N \geq 1 \), be a bounded domain with smooth boundary. Suppose that \( q, s \geq 1 \), \( k > 0 \) and \( \alpha \in (0, 1) \) satisfy

\[
\frac{1}{2} - \frac{k}{N} = \alpha \left( \frac{1}{2} - \frac{1}{N} \right) + (1 - \alpha) \frac{q}{s} \quad \text{and} \quad k \leq \alpha.
\]

Then there exists a positive constant \( C_1 \) such that for any \( f \in W^{1,2}(\Omega) \cap L^r(\Omega) \),

\[
\|f\|_{W^{1,2}(\Omega)} \leq C_1 \|\nabla f\|_{L^q(\Omega)} \|f\|_{L^r(\Omega)}^{1 - \alpha} + C_1 \|f\|_{L^r(\Omega)}.
\]

### 3. A priori estimates.

According to Proposition 1.1 and Theorem 1.2, it is sufficient to show that \( \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \) is bounded for all \( t \in (0, T_{\text{max}}) \), then \( T_{\text{max}} = \infty \) and the solution is global in time. Indeed, we shall show that the solution is uniformly bounded for \( t \in (0, \infty) \). To this end, it suffices to prove that \( \|u(\cdot, t)\|_{L^p(\Omega)} \) is bounded for some \( p \) large thanks to Lemma 2.2 as we shall see later on. The main vehicle of our approach is the combined estimate on \( \int_0^1 u^p + \int_{\Omega} |\nabla v|^2 q \) for both \( p \) and \( q \) large, based on an idea recently developed in [21, 24] etc.

### 3.1. Combined a priori estimates.

For any \( p \geq 2 \), we multiply the \( u \)-equation in (1.1) by \( u^{p-1} \) and then integrate it over \( \Omega \) by parts

\[
\frac{1}{p} \frac{d}{dt} \int_\Omega u^p = \int_\Omega u^{p-1} \nabla \cdot (D(u) \nabla u) - \int_\Omega u^{p-1} \nabla \cdot (S(u) \nabla v) + \int_\Omega u^p (1 - u^\gamma) = -(p-1) \int_\Omega D(u) u^{p-2} |\nabla u|^2 + (p-1) \int_\Omega S(u) u^{p-2} \nabla u \cdot \nabla v + \int_\Omega u^p (1 - u^\gamma).
\]

Here and in the sequel we skip \( dx \) in the integrals for the succinctness.

To estimate (3.1), we have from \( D(u) \geq K_1 u^{m_1} \) in (1.2) that

\[
-(p-1) \int_\Omega D(u) u^{p-2} |\nabla u|^2 \leq - K_1 (p-1) \int_\Omega u^{p+m_1-2} |\nabla u|^2 \leq - \frac{4K_1(p-1)}{(p+m_1)^2} \int_\Omega |\nabla u|^{p+m_1} |u|^{p+m_1}.
\]
where the identity holds due to the fact \(u^{p+m_1-2} |\nabla u|^2 = \frac{4}{(p+m_1)^2} |\nabla u^{p+m_1}|^2\). Moreover, we can apply Young’s inequality to obtain
\[
(p-1) \int_{\Omega} S(u) u^{p-2} \nabla u \cdot \nabla v \\
\leq \frac{K_1(p-1)}{2} \int_{\Omega} u^{p+m_1-2} |\nabla u|^2 + \frac{(p-1)}{2K_1} \int_{\Omega} S^2(u) u^{p-m_1-2} |\nabla v|^2 \\
\leq \frac{2K_1(p-1)}{(p+m_1)^2} \int_{\Omega} |\nabla u^{p+m_1}|^2 + \frac{K_2^2(p-1)}{2K_1} \int_{\Omega} u^{p-m_1+2m_2-2} |\nabla v|^2
\]
and
\[
(1 + \frac{1}{p}) \int_{\Omega} u^p \leq \frac{1}{2} \int_{\Omega} u^{p+\gamma} + C_{31}, \tag{3.4}
\]
where \(C_{31}\) is a positive constant dependent on \(p\). Thanks to (3.2)–(3.4), (3.1) implies
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + \frac{1}{p} \int_{\Omega} u^p + \frac{2K_1(p-1)}{(p+m_1)^2} \int_{\Omega} |\nabla u^{p+m_1}|^2 + \frac{1}{2} \int_{\Omega} u^{p+\gamma} \\
\leq \frac{K_2^2(p-1)}{2K_1} \int_{\Omega} u^{p-m_1+2m_2-2} |\nabla v|^2 + C_{31}. \tag{3.5}
\]
On the other hand, for any \(q \geq 2\), we have from the \(v\)-equation in (1.1)
\[
\frac{1}{2q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} = \int_{\Omega} |\nabla v|^{2q-2} \nabla v \cdot \nabla \Delta v \\
= \int_{\Omega} |\nabla v|^{2q-2} \nabla v \cdot \Delta v + \int_{\Omega} |\nabla v|^{2q-2} \nabla v \cdot \nabla (-v + u). \tag{3.6}
\]
In light of the pointwise identity \(\nabla v \cdot \Delta v = \frac{1}{2} \Delta |\nabla v|^2 - |D^2 v|^2\), we first estimate \(I_1\) in (3.6) through the integration by parts
\[
I_1 = \int_{\Omega} |\nabla v|^{2q-2} \Delta |\nabla v|^2 - \int_{\Omega} |\nabla v|^{2q-2} |D^2 v|^2 \\
= \int_{\Omega} |\nabla v|^{2q-2} \Delta |\nabla v|^2 - \int_{\Omega} |\nabla v|^{2q-2} |\nabla |\nabla v|^2|^2 \\
= \int_{\Omega} |\nabla v|^{2q-2} \frac{\partial |\nabla v|^2}{\partial v} - \frac{(q-1)}{2} \int_{\Omega} |\nabla v|^{2q-4} |\nabla |\nabla v|^2|^2 \\
\leq \frac{C_{\Omega}}{2} \int_{\partial \Omega} |\nabla v|^{2q-2} \frac{\partial |\nabla v|^2}{\partial v} \leq \frac{C_{\Omega}}{2} \int_{\partial \Omega} |\nabla v|^{2q} = \tilde{C}_{\Omega} \|\nabla v\|^{q} \tag{3.8}
\]
moreover, for \(r \in (0, \frac{1}{2})\) being an arbitrary number, we apply the Sobolev trace embedding (e.g. (1.9), Lemma 2.3 and Lemma 2.4 in [9]) that there exists a positive constant \(C_{32}\)
\[
\|\nabla v\|^{q} \leq C_{32} \|\nabla v\|_{W^{r+\frac{1}{2}, 2}(\Omega)}^2, \tag{3.9}
\]
Choose \( k = r + \frac{1}{2} \) and \( s = 2 \) in Lemma 2.4 and then we have that \( \alpha_1 \in (0, 1) \) satisfies
\[
\frac{1}{2} - \frac{r + \frac{1}{2}}{N} = \left( 1 - \alpha_1 \right) \frac{q}{2} + \alpha_1 \left( \frac{1}{2} - \frac{1}{N} \right),
\]
or equivalently
\[
\alpha_1 = \frac{\frac{q}{2} - \left( \frac{1}{2} - \frac{1}{N} \right)}{\frac{q}{2} - (r + 1)} \in (r + \frac{1}{2}, 1),
\]
hence we can apply (2.6) to \( f = |\nabla v|^q \) to obtain
\[
\left\| |\nabla v|^q \right\|_{W^{r+\frac{1}{2},2}(\Omega)} \leq C_{33} \left\| |\nabla v|^q \right\|_{L^2(\Omega)}^{\frac{\alpha_1}{2}} \left\| |\nabla v|^q \right\|_{L^2(\Omega)}^{1-\frac{\alpha_1}{2}} + C_{33} \left\| |\nabla v|^q \right\|_{L^2(\Omega)}^{\frac{\alpha_1}{2}} + C_{34}, \tag{3.10}
\]
where the second inequality is due to the boundedness of \( \left\| |\nabla v|^q \right\|_{L^2(\Omega)} \) in (2.2). In view of (3.9) and (3.10), we apply Young's inequality to (3.8) and have
\[
I_{11} \leq 2C_{31}C_{32}^2 \left( C_{34}^2 \right) \left\| |\nabla v|^q \right\|_{L^2(\Omega)}^{2\alpha_1} + C_{34},
\]
\[
\leq \frac{(q - 1)}{q^2} \int_{\Omega} |\nabla |\nabla v|^q |^2 + C_{35}, \tag{3.11}
\]
where \( C_{35} \) is a positive constant dependent on \( q \). On the other hand, through the pointwise identity \( |\nabla v|^{2q-4} |\nabla |\nabla v|^q |^2 = \frac{4}{q} |\nabla |\nabla v|^q |^2 \), we can rewrite \( I_{12} \) as
\[
I_{12} = \frac{2(q - 1)}{q^2} \int_{\Omega} |\nabla |\nabla v|^q |^2. \tag{3.12}
\]
Substituting (3.11) and (3.12) into (3.7) gives us
\[
I_1 \leq -\frac{(q - 1)}{q^2} \int_{\Omega} |\nabla |\nabla v|^q |^2 \int_{\Omega} |\nabla v|^{2q-2} |D^2 v|^2 + C_{35}. \tag{3.13}
\]
To estimate \( I_2 \) in (3.6), we first have from the integration by parts
\[
I_2 = \int_{\Omega} |\nabla v|^{2q-2} |\Delta v| \cdot \nabla (-v + u) \cdot \nabla v \cdot \nabla v - \int_{\Omega} |\nabla v|^{2q}
\]
\[
= -\int_{\Omega} u \nabla \cdot (|\nabla v|^{2q-2} \nabla v) - \int_{\Omega} |\nabla v|^{2q}
\]
\[
= -\int_{\Omega} u |\nabla v|^{2q-2} \Delta v - (q - 1) \int_{\Omega} u |\nabla v|^{2q-4} |\nabla v|^2 \cdot \nabla v - \int_{\Omega} |\nabla v|^{2q}. \tag{3.14}
\]
Then we can apply Young's inequality to derive
\[
-I_{21} \leq \frac{N}{4} \int_{\Omega} u^2 |\nabla v|^{2q-2} + \frac{1}{N} \int_{\Omega} |\nabla v|^{2q-2} |\Delta v|^2
\]
\[
\leq \frac{N}{4} \int_{\Omega} u^2 |\nabla v|^{2q-2} + \frac{1}{2} \int_{\Omega} |\nabla v|^{2q-2} |D^2 v|^2, \tag{3.15}
\]
where the second inequality follows from the pointwise inequality $|\Delta v|^2 \leq N |D^2 v|^2$, and

$$-I_{22} \leq 2(q-1) \int_{\Omega} u^2 |\nabla v|^{2q-2} + \frac{(q-1)}{8} \int_{\Omega} |\nabla v|^{2q-4} |\nabla |\nabla v|^2|^2$$

$$= 2(q-1) \int_{\Omega} u^2 |\nabla v|^{2q-2} + \frac{(q-1)}{2q^2} \int_{\Omega} |\nabla |\nabla v|^2|^2. \hspace{1cm} (3.16)$$

Collecting (3.15) and (3.16), we infer from (3.14) and Hölder’s inequality to estimate Lemma 3.1.

Let $u,v$ be a positive classical solution of (1.1) in $\Omega \times (0,T_{\text{max}})$. Suppose that $\gamma \geq 1$ and $m_1,m_2 \in \mathbb{R}$ satisfy $m_2 - m_1 < \frac{3N+2}{N(N+2)}$. Then for any $p \geq 2$ and $q \geq 2$, there exists a positive constant $C(p,q)$ such that

$$\int_{\Omega} u^p + \int_{\Omega} |\nabla v|^{2q} \leq C(p,q), \forall t \in (0,\infty). \hspace{1cm} (3.20)$$

Proof. For the consistency of notations, we denote

$$\lambda_1 = p - m_1 + 2m_2 - 2, \lambda_2 = 2, \hspace{1cm} (3.21)$$

and

$$\kappa_1 = 2, \kappa_2 = 2(q-1). \hspace{1cm} (3.22)$$

Let $\mu_i > 1$ be an arbitrary real number and $\mu_i^* := \frac{\mu_i}{\mu_i - 1}$ be its conjugate. We apply Hölder’s inequality to estimate $I_{31}$ in (3.19) as

$$I_{31} \leq \left( \int_{\Omega} u^{p(m_1 - 2m_2 - 2)} \right)^{\frac{1}{p}} \cdot \left( \int_{\Omega} |\nabla v|^{2\mu_1^*} \right)^{\frac{1}{m_1}} = \left( \int_{\Omega} u^{\lambda_1 m_1} \right)^{\frac{1}{m_1}} \cdot \left( \int_{\Omega} |\nabla v|^{\kappa_1 \mu_1^*} \right)^{\frac{1}{\kappa_1}}$$

and

$$I_{32} \leq \left( \int_{\Omega} u^{2\mu_2} \right)^{\frac{1}{\mu_2}} \cdot \left( \int_{\Omega} |\nabla v|^{2(q-1)\mu_2^*} \right)^{\frac{1}{m_2}} = \left( \int_{\Omega} u^{\lambda_2 m_2} \right)^{\frac{1}{m_2}} \cdot \left( \int_{\Omega} |\nabla v|^{\kappa_2 \mu_2^*} \right)^{\frac{1}{\kappa_2}}.$$
which, in light of (3.21) and (3.22), can be written in the general form

\[ I_{3i} \leq \left( \int_{\Omega} u^{i 2 -} \right)^{\frac{1}{i 2 -}} \left( \int_{\Omega} |\nabla v|^{r \cdot \mu_i} \right)^{\frac{1}{r \cdot \mu_i}}, \quad i = 1, 2. \tag{3.23} \]

By Gagliardo–Nirenberg interpolation inequality in Lemma 2.3 and the fact that \((a + b)^\rho \leq \rho^a (a^\rho + b^\rho), \forall a, b \geq 0 \) and \( \rho > 0 \), we can find positive constants \( C_{37}, \ldots, C_{310} \) such that in (3.23)

\[
\left( \int_{\Omega} u^{i \cdot \mu_i} \right)^{\frac{1}{i 2 -}} \leq C_{37} \left\| \nabla u^{\frac{p + m_1}{2}} \right\|_{L^\frac{p + m_1}{p + m_1} (\Omega)}^{2 \lambda_i \cdot \mu_i} \cdot \left\| \nabla u^{\frac{p + m_1}{2} (1 - \alpha_{2i})} \right\|_{L^\frac{p + m_1}{p + m_1} (\Omega)}^{2 \lambda_i \cdot \mu_i} + C_{37} \left\| u^{\frac{p - m_1}{2}} \right\|_{L^\frac{p - m_1}{p - m_1} (\Omega)}^{2 \lambda_i \cdot \mu_i},
\]

\[
\leq C_{38} \left\| \nabla u^{\frac{p + m_1}{2}} \right\|_{L^2 (\Omega)}^{2 \lambda_i \cdot \mu_i} + C_{38}
\]

with

\[
\alpha_{2i} = \frac{p + m_1}{2} - \frac{p + m_1}{2 \cdot \mu_i}, \quad (3.25)
\]

and

\[
\left( \int_{\Omega} |\nabla v|^{r \cdot \mu_i} \right)^{\frac{1}{r \cdot \mu_i}} \leq \left\| |\nabla v|^q \right\|_{L^\frac{q}{q-1} (\Omega)}^{\frac{q}{q-1} \cdot \alpha_{3i}},
\]

\[
\leq C_{39} \left\| \nabla |\nabla v|^q \right\|_{L^\frac{q}{q-1} (\Omega)}^{\frac{q}{q-1} \cdot \alpha_{3i}} \cdot \left\| |\nabla v|^q \right\|_{L^\frac{q}{q-1} (\Omega)}^{\frac{q}{q-1} \cdot \alpha_{3i}} + C_{39} \left\| |\nabla v|^q \right\|_{L^\frac{q}{q-1} (\Omega)}^{\frac{q}{q-1} \cdot \alpha_{3i}}
\]

\[
\leq C_{310} \left\| |\nabla v|^q \right\|_{L^2 (\Omega)}^{\frac{q}{q-1} \cdot \alpha_{3i}} + C_{310}
\]

with

\[
\alpha_{3i} = \frac{q}{2} - \frac{\kappa_i \mu_i}{q-i}, \quad (3.27)
\]

where in (3.24) and (3.26) we have applied the boundedness of \( \|u\|_{L^2} \) and \( \|\nabla u\|_{L^2} \) due to (2.2).

For large \( p \) and \( q \), choosing \( \mu_1 = \frac{q}{q-i} \) and \( \mu_2 = \frac{q}{2}, \mu_i \) being their conjugates, we claim that

\[
\frac{2 \lambda_i \cdot \mu_i}{p + m_1} \geq 1, \frac{\kappa_i \mu_i}{q} \geq 1, 0 < \alpha_{2i}, \alpha_{3i} < 1
\]

and under condition (1.4)

\[
\xi_i(p, q) := \frac{2 \lambda_i}{p + m_1}, \cdot \alpha_{2i} + \frac{\kappa_i}{q}, \cdot \alpha_{3i} = \frac{\lambda_i - \frac{1}{m_i}}{p + m_1} - \frac{\frac{q}{2} - \frac{1}{m_i}}{q} < 2.
\]

We assume that (3.28) and (3.29) hold for now. On the other hand, it is easy to show that if \( \delta_1 + \delta_2 < 2, \delta_1 > 0 \), then for any \( \epsilon > 0 \), there exists \( C_{\epsilon} > 0 \) such that

\[
(a^{\delta_1} + 1)(b^{\delta_2} + 1) \leq \epsilon (a^{2} + b^{2}) + C_{\epsilon}, \forall a, b \geq 0, \text{ therefore, we derive from (3.24) and (3.26) that}
\]

\[
I_{3i} \leq C_{38} C_{310} \left( \left\| \nabla u^{\frac{p + m_1}{2}} \right\|_{L^2 (\Omega)}^{\frac{2 \lambda_i \cdot \mu_i}{p + m_1} \cdot \alpha_{2i}} + 1 \right) \left( \left\| |\nabla v|^q \right\|_{L^2 (\Omega)}^{\frac{\kappa_i \mu_i}{q} \cdot \alpha_{3i}} + 1 \right)
\]

\[
\leq \epsilon \int_{\Omega} |\nabla u^{\frac{p + m_1}{2}}|^2 + \epsilon \int_{\Omega} |\nabla |\nabla v|^q|^2 + C_{311}.
\]

(3.30)
Choosing $\epsilon > 0$ small, we plug (3.30) into (3.19) and obtain for all $t \in (0, \infty)$
\[
\frac{d}{dt} \left( \frac{1}{p} \int_{\Omega} u^p + \frac{1}{2q} \int_{\Omega} |\nabla v|^{2q} \right) + \frac{1}{p} \int_{\Omega} u^p + \frac{1}{2q} \int_{\Omega} |\nabla v|^{2q} \leq C(p, q),
\]
and solving this ODI gives rise to (3.28). Therefore, in order to prove Lemma 3.1, we are left to verify (3.28) and (3.29).

First of all, we see that (3.28) is equivalent as
\[
\frac{1}{2} - \frac{1}{N} < \frac{p+m_1}{2\lambda_i\mu_i} \leq 1 \quad \text{and} \quad \frac{1}{2} - \frac{1}{N} < \frac{q}{\kappa_i\mu_i} \leq 1,
\]
which, in terms of (3.21), (3.22) and $\mu_1 = \frac{q}{q-1}$, $\mu_2 = \frac{q}{2}$, become
\[
\frac{1}{2} - \frac{1}{N} < \frac{2(p+m_1)(q-1)}{2(p-m_1+2m_2-2)} \leq 1, \quad \frac{1}{2} - \frac{1}{N} < \frac{1}{2} \leq 1
\]
and
\[
\frac{1}{2} - \frac{1}{N} < \frac{p+m_1}{2p} \leq 1, \quad \frac{1}{2} - \frac{1}{N} < \frac{(p-2)q}{2p(q-1)} \leq 1
\]
respectively. Then it is easy to check that all the inequalities hold if $p$ and $q$ are sufficiently large hence (3.28) is verified.

We now proceed to prove $\xi_i(p, q) < 2$ in (3.29). Again, in light of (3.21), (3.22) and $\mu_i$, they become
\[
\xi_1(p, q) = \frac{p-m_1+2m_2-2-\frac{1}{\mu_1}}{p+m_1} - \left( \frac{1}{2} - \frac{1}{N} \right) + \frac{1}{2} - \frac{1}{N} < 2
\]
and
\[
\xi_2(p, q) = \frac{2-\frac{1}{\mu_2}}{p+m_1} - \left( \frac{1}{2} - \frac{1}{N} \right) + \frac{(q-1)-\frac{1}{\mu_2}}{2} - \left( \frac{1}{2} - \frac{1}{N} \right) < 2
\]
or equivalently
\[
\frac{q}{2} > \zeta_1 \left( \frac{p+m_1}{2} - \left( \frac{1}{2} - \frac{1}{N} \right) \right) + \left( \frac{1}{2} - \frac{1}{N} \right)
\]
and
\[
\frac{q}{2} < \zeta_2 \left( \frac{p+m_1}{2} - \left( \frac{1}{2} - \frac{1}{N} \right) \right) + \left( \frac{1}{2} - \frac{1}{N} \right),
\]
where
\[
\zeta_1 = \zeta_1(p, q) := \frac{\frac{1}{2} - \frac{1}{\mu_1}}{m_1-m_2+\frac{1}{2}+\frac{1}{N}+\frac{1}{2(\frac{1}{2}-\frac{1}{N})}} > 0
\]
and
\[
\zeta_2 = \zeta_2(p, q) := \frac{\frac{1}{2} + \frac{p-2}{2p} - \left( \frac{1}{2} - \frac{1}{N} \right)}{1 - \frac{1}{p}} > 0,
\]
where the denominator in $\zeta_1$ is positive under the condition $m_2-m_1 < \frac{3N+2}{N(N+2)}$ in (1.4); moreover, passing the limit $(p, q) \to (\infty, \infty)$ gives us
\[
\zeta_2(\infty, \infty) - \zeta_1(\infty, \infty) = \left( \frac{1}{2} + \frac{1}{N} \right) - \frac{\frac{1}{2}}{m_1-m_2+\frac{1}{2}+\frac{1}{N}}
\]
$$
= \frac{m_2-m_1 - \frac{3N+2}{N(N+2)}}{(m_2-m_1 - (1+\frac{1}{N}))(\frac{1}{2}+\frac{1}{N})} > 0,
$$
then we can always find $p$ and $q$ sufficiently large such that $\zeta_2(p, q) > \zeta_1(p, q)$ which implies that (3.34) and (3.35) hold under the same conditions, therefore $\xi_i(p, q) < 2$ in (3.29) for both $i = 1, 2$, and this completes the proof of Lemma 3.1. \qed
3.2. Alternative estimates. In this subsection, we shall provide estimates for (3.19) by a different approach in the following lemma. Our proof is based on and slightly extends [28].

Lemma 3.2. Let \((u, v)\) be a positive classical solution of (1.1) in \(\Omega \times (0, T_{\text{max}})\). Suppose that \(\gamma \geq 1\) and \(m_1, m_2 \in \mathbb{R}\) satisfy \(2m_2 - m_1 < \max\{\gamma, m_1\} + \frac{4}{N+2}\). Then for any \(p \geq 2\) and \(q \geq 2\), there exists a positive constant \(C(p, q)\) such that

\[
\int_\Omega u^p + \int _\Omega |\nabla v|^{2q} \leq C(p, q), \forall t \in (0, \infty).
\]

Proof. First of all, for \(p\) large, choosing
\[
\alpha_4 = \frac{p+m_1}{2} - \left(\frac{1}{2} - \frac{1}{N}\right) \in (0, 1),
\]
we apply Gagliardo–Nirenberg interpolation inequality in Lemma 2.3 to obtain

\[
\int_\Omega u^{p+m_1} = \|u^{p+m_1}\|_{L^2(\Omega)}^2 \leq C_{312} \|\nabla u^{p+m_1}\|_{L^2(\Omega)}^{2\alpha_4} \|u^{p+m_1}\|_{L^{p/(p-m_1-2m_2)}(\Omega)}^{2(1-\alpha_4)} + C_{312} \|u^{p+m_1}\|_{L^{p/(p-m_1-2m_2)}(\Omega)}^2
\]

\[
\leq C_{313} \|\nabla u^{p+m_1}\|_{L^2(\Omega)}^{2\alpha_4} + C_{313}, \tag{3.37}
\]

where the second inequality is due to (2.2).

To estimate (3.19), we apply Young’s inequality to have that for some positive constants \(C_{314}\) and \(C_{315}\)

\[
\frac{K^2(p-1)}{2K_1} I_{31} \leq \frac{1}{4} \int_\Omega (u^{p-m_1-2m_2-2})^{p + \max\{\gamma, m_1\}} + C_{314} \int_\Omega |\nabla v|^{2\max\{\gamma, m_1\}}^{p + \max\{\gamma, m_1\}} + \max\{\gamma, m_1\} - 2m_2 - 2
\]

\[
= \frac{1}{4} \int_\Omega u^{p + \max\{\gamma, m_1\}} + C_{314} \int_\Omega |\nabla v|^{\theta_1}, \tag{3.38}
\]

and

\[
\frac{N + 8(q-1)}{4} I_{32} \leq \frac{1}{4} \int_\Omega (u^2)^{\max\{\gamma, m_1\}} + C_{315} \int_\Omega |\nabla v|^{2(q-1)} \max\{\gamma, m_1\} - 2m_2 - 2
\]

\[
= \frac{1}{4} \int_\Omega u^{p + \max\{\gamma, m_1\}} + C_{315} \int_\Omega |\nabla v|^{\theta_2}. \tag{3.39}
\]

where we denote

\[
\theta_1 := \theta_1(p, q) = \frac{2(p + \max\{\gamma, m_1\})}{\max\{\gamma, m_1\} + m_1 - 2m_2 + 2} \tag{3.40}
\]

and

\[
\theta_2 := \theta_2(p, q) = \frac{2(q-1)(p + \max\{\gamma, m_1\})}{p + \max\{\gamma, m_1\} - 2}. \tag{3.41}
\]

We note that \(\theta_i\) are well defined since \(\max\{\gamma, m_1\} > 2m_2 - m_1 - 2\) in light of (1.5).

Substituting (3.38)–(3.41) into (3.19), we derive

\[
\frac{d}{dt} \left(\frac{1}{p} \int_\Omega u^p + \frac{1}{2q} \int_\Omega |\nabla v|^{2q}\right) + \left(\frac{1}{p} \int_\Omega u^p + \frac{1}{2q} \int_\Omega |\nabla v|^{2q}\right) + \frac{(q-1)}{2q^2} \int_\Omega |\nabla v|^2 \leq C_{314} \int_\Omega |\nabla v|^{\theta_1} + C_{315} \int_\Omega |\nabla v|^{\theta_2} + C_{316}. \tag{3.42}
\]
Again, we apply the Gagliardo–Nirenberg interpolation inequality to have
\[
\int_\Omega |\nabla v|^{\theta_i} = \left\| |\nabla v|^q \right\|_{L^\frac{2}{q}(\Omega)}^{\frac{\theta_i}{q}} \leq C_{317} \left\| |\nabla v|^q \right\|_{L^\frac{2}{q}(\Omega)}^{\frac{\theta_i}{q}} \frac{1}{\frac{\theta_i}{q}(\Omega)} + C_{317} \left\| |\nabla v|^q \right\|_{L^\frac{2}{q}(\Omega)}^{\frac{\theta_i}{q}} \frac{1}{\frac{\theta_i}{q}(\Omega)}
\]
\[
\leq C_{317} \left\| |\nabla v|^q \right\|_{L^\frac{2}{q}(\Omega)}^{\frac{\theta_i}{q}} + C_{318},
\]
with
\[
\alpha_{5i} = \frac{\theta_i}{q - \frac{\theta_i}{q}},
\]
where we have applied the boundedness of $|\nabla v|_{L^2(\Omega)}$ in the last inequality.

For further estimates, we want to claim that under condition (1.5) there exist $p$ and $q$ large such that the followings hold
\[
\frac{\theta_i}{q} \geq 1, 0 < \alpha_{5i} < 1, \text{ and } 0 < \frac{\theta_i}{q} \alpha_{5i} < 2.
\]
Assume (3.43) for now, then we have from Young’s inequality that for any $\epsilon > 0$, there is a positive constant $C_\epsilon$ such that
\[
\int_\Omega |\nabla v|^{\theta_i} \leq \epsilon \left\| |\nabla v|^q \right\|_{L^2(\Omega)}^2 + C_\epsilon.
\]
Substituting (3.44) into (3.42), we can easily derive
\[
\frac{d}{dt} \left( \frac{1}{p} \int_\Omega u^p + \frac{1}{2q} \int_\Omega |\nabla v|^{2q} \right) + \left( \frac{1}{p} \int_\Omega u^p + \frac{1}{2q} \int_\Omega |\nabla v|^{2q} \right) \leq C_{319}
\]
and solving this inequality gives rise to (3.36).

Finally, we need to verify the inequalities in (3.43) hold, or equivalently
\[
\theta_i \geq q, \forall i > 2 \text{ and } q > \frac{\theta_i}{2} - \frac{2}{N}.
\]
Accordingly $\theta_i > 2$ and $\theta_2 \geq q$ hold because both $p$ and $q$ are large, therefore we shall only need to verify that $\theta_i \geq q$ and $q > \frac{\theta_i}{2} - \frac{2}{N}, i = 1, 2$. To this end, we have from straightforward calculations that these inequalities become
\[
q_1(p) = \frac{1}{\max\{\gamma, m_1\} + m_1 - 2m_2 + 2 \left( p + \max\{\gamma, m_1\} \right)} - \frac{2}{N},
\]
\[
q_2(p) = \frac{N + 2}{2N} \left( p + \max\{\gamma, m_1\} \right) - \frac{2}{N},
\]
and
\[
q_3(p) = \frac{2}{\max\{\gamma, m_1\} + m_1 - 2m_2 + 2 \left( p + \max\{\gamma, m_1\} \right)}.
\]
Obviously $q_3(p) > q_1(p)$. On the other hand, we can have from straightforward calculations that $\max\{\gamma, m_1\} + m_1 - 2m_2 + 2 \leq \frac{N + 2}{2N}$ if $2m_2 - m_1 < \max\{\gamma, m_1\} + \frac{4}{N + 2}$. Under this condition, we can always find $p$ and $q$ large such that $q_2(p) > q_1(p)$ and this verifies (3.43). Therefore, (3.36) holds for $p$ and $q$ being sufficiently large, and we can apply Hölder’s inequality to verify it for all $p, q \in [2, \infty)$. The proof of Lemma 3.2 completes. □
Corollary 1. If $\gamma \in (0, 1)$ in (1.1), we choose $s = N - 1$ in Lemma 2.3, then by the same arguments as above, we can show that Lemma 3.2 also holds with condition (1.5) being replaced by $2m_2 - m_1 < \max\{m_1, \gamma\} + \frac{2}{N}$. Therefore, (1.1) has global existence and boundedness as claimed in Remark 2 after Theorem 1.2. This slightly extends the results in [28] where $2m_2 - m_1 < \gamma + \frac{2}{N}$ is assumed.

4. Global existence and boundedness. Finally, we present the proofs of our main results.

Proof of Theorem 1.1. The proof is rather standard and we shall only sketch the main steps. First of all, by taking $p > N$, we have from Lemma 2.2 and Lemma 3.1 that $\|v(\cdot, t)\|_{W^{1, \infty}}$ is bounded for all $t > 0$. Then one can apply the classical Moser–Alikakos $L^p$ iteration arguments of Lemma A.1 in [21] to show that $\|u(\cdot, t)\|_{L^\infty}$ is uniformly bounded for $t \in (0, T_{\max})$. Therefore $T_{\max} = \infty$ and the solution is global in time according to Proposition 1.

Proof of Theorem 1.2. The proof is the same as that of Theorem 1.1 by using Lemma 3.2.

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