On the topology of holomorphic bundles

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Abstract
In this work we study the topology of holomorphic rank two bundles over complex surfaces. We consider bundles that are constructed by glueing “local” holomorphic bundles and we show that under certain conditions the topology of the bundle does not depend on the glueing. As a consequence we present a simple and new classification of bundles on blown-up surfaces.

1 Introduction
Let $X$ be a complex manifold and let $A$ and $B$ be open sets that cover $X$. Given holomorphic bundles $E_A$ and $E_B$ defined over the subsets $A$ and $B$, we construct holomorphic bundles over $X$ by glueing the bundles $E_A$ and $E_B$ over $A \cap B$. We then compare the topology of bundles given by different glueings.

Our main motivation is the study of bundles over blown-up surfaces. Therefore we will focus our attention on the cases where the intersection $A \cap B$ is biholomorphic to $\mathbb{C}^m - \{0\}$. We then present simple holomorphic and topological classifications of bundles on blown-up surfaces.

2 Glueing bundles over a manifold
In this section we construct bundles over a complex manifold by glueing bundles defined on open subsets and then we compare their topology.
Consider the following data:

i) a complex manifold $X$ with $\dim \mathbb{C}X = m \geq 2$ and open subsets $A, B,$ and $C$ of $X$ satisfying $X = A \cup B$, $A \cap B = C$ with $C$ biholomorphic to $\mathbb{C}^m - \{0\}$

ii) holomorphic vector bundles $\pi_A : E_A \to A$ and $\pi_B : E_B \to B$ which are trivial when restricted to $C$

iii) a trivialization $F : E_A \to C \times \mathbb{C}^n$ of $E_A|_C$ and trivializations $G_i : E_B \to C \times \mathbb{C}^n$, $i = 0, 1$ of $E_B|_C$.

Given the above data, we define bundles $E_i$ over $X$ by the formula

$$E_i = E_A \bigcup_{F=G_i} E_B = (E_A \bigcup E_B)/\sim$$

where for $x \in E_A$ and $y \in E_B$ we define $x \sim y$ if $F(x) = G_i(y)$.

We have the following result about the bundles $E_i$.

**Proposition 2.1** The topology of the bundle $E_i$ is independent of the gluing, that is $E_0$ and $E_1$ are topologically equivalent vector bundles.

To prove this we present some preliminary lemmas.

**Lemma 2.2** If the trivializations $G_0$ and $G_1$ are homotopic, then $E_0$ and $E_1$ are topologically equivalent vector bundles.

**Proof:** By $G_0$ homotopic to $G_1$ we mean that there is a one parameter family of trivializations $G_t$ with $t \in [0, 1]$ taking $G_0$ to $G_1$. To prove the lemma we consider the product bundles $E_A \times I$ and $E_B \times I$ over $A \times I$ and $B \times I$ respectively, together with given trivializations $\tilde{F}$ and $\tilde{G}$ of $E_A \times I|_{C \times I}$ and $E_B \times I|_{C \times I}$ defined by $\tilde{F}(a, t) = (F(a), t)$, for $(a, t) \in A \times I$ and $\tilde{G}(b, t) = (G(b), t)$, for $(b, t) \in B \times I$. We have that $\tilde{G}(y, 0) = G_0(y)$ and $\tilde{G}(y, 1) = G_1(y)$. Defining the bundle $E$ over $X \times I$ by $E = (E_A \times I) \cup_{F=G} (E_B \times I)$ it immediately follows that $E|_{X \times \{0\}} \simeq E_0$ and $E|_{X \times \{1\}} \simeq E_1$ and consequently $E_0$ and $E_1$ are topologically equivalent.  

**Lemma 2.3** Consider the trivial vector bundle $D = C \times \mathbb{C}^n$ over a complex space $C$. Let $G_0$ and $G_1$ be two trivializations of $D$ over $C$ and let $\Phi : C \to GL(n, \mathbb{C})$ be the corresponding transition matrix, i.e. $\Phi(c)G_0(c) = G_1(c)$. Then $G_0$ and $G_1$ are homotopic if and only if $\Phi$ is nullhomotopic.

The proof is straightforward.
Lemma 2.4 Any holomorphic map \( f : \mathbb{C}^m - \{0\} \to GL(n, \mathbb{C}) \) for \( m \geq 2 \) is nullhomotopic.

Proof: By Hartog’s Theorem, \( f \) extends to a holomorphic function \( \tilde{f} : \mathbb{C}^m \to M(n, \mathbb{C}) \), where \( M(n, \mathbb{C}) \) denotes the space of all \( n \times n \) matrices with complex coefficients. We claim that \( \tilde{f}(0) \in GL(n, \mathbb{C}) \). In fact, if \( \tilde{f}(0) \notin GL(n, \mathbb{C}) \) then \( \det(\tilde{f}(0)) = 0 \). But then \( (\det \circ \tilde{f})^{-1}(0) = \{0\} \subset \mathbb{C}^m \) which is a contradiction, because \( (\det \circ \tilde{f}) : \mathbb{C}^m \to \mathbb{C} \) is holomorphic and the pre-image of a point by a holomorphic function in \( \mathbb{C}^m \) is either empty or has codimension 1. Which proves the claim.

Thus, we can write \( \tilde{f} : \mathbb{C}^m \to GL(n, \mathbb{C}) \). Hence \( f \) factors through \( \mathbb{C}^m \), i.e. \( f = \tilde{f} \circ i \) where \( i \) is the inclusion \( i : \mathbb{C}^m - \{0\} \to \mathbb{C}^m \). As \( \mathbb{C}^m \) is contractible it follows that \( f \) is nullhomotopic.

Alternative proof of Lemma 2.4: Consider the holomorphic function \( g = \det \circ f : \mathbb{C}^m - \{0\} \to \mathbb{C} \). By Hartog’s theorem we have that both \( g \) and \( 1/g \) extend to holomorphic functions defined on \( \mathbb{C}^m \). Let \( \tilde{g} \) be the extension of \( g \) to \( \mathbb{C}^m \). Then, because \( 1/g \) also extends as a holomorphic function to \( \mathbb{C}^m \), it follows that \( \tilde{g}(0) \neq 0 \). We have \( g = i \circ \tilde{g} \) where \( i \) is the inclusion \( i : \mathbb{C}^m - \{0\} \to \mathbb{C}^m \) and since \( \mathbb{C}^m \) is contractible it follows that \( g \) is nullhomotopic.

Proof of Proposition 2.1: Lemmas 2.3 and 2.4 imply that any two holomorphic trivializations of the trivial bundle \( (\mathbb{C}^m - \{0\}) \times \mathbb{C}^n \) over \( \mathbb{C}^m - \{0\} \) are homotopic and then Lemma 2.2 implies that the bundles obtained using these trivializations are topologically equivalent.

3 Bundles on Blown-up Surfaces

In this section we apply Proposition 2.1 to give a simple topological description of bundles on some blown-up surfaces. Let us consider the following case:

i) \( X = \tilde{S} \) is the blow-up of a complex surface \( S \) at a point \( P \) and \( \ell \) is the exceptional divisor

ii) the open subsets \( A = N_{\ell} \) and \( B = \tilde{S} - \ell \simeq S - \{P\} \) are respectively a neighborhood of the exceptional divisor \( \ell \) and the complement of the exceptional divisor.

iii) \( A \cap B \simeq \mathbb{C}^2 - \{0\} \).

Some elementary examples of such surfaces are the blow up of the projective plane \( \mathbb{P}^2 \) at a point or the blow-up of a Hirzebruch surface.
$S_n = P(\mathcal{O}(n) \oplus \mathcal{O})$ at a point. Studying successive blow-ups on these basic surfaces leads to a similar topological classification of bundles on any rational surface. This just follows from the classification of rational surfaces, see Griffiths and Harris [3].

To state our topological classification we first quote some results from previous papers.

We write $\tilde{C}^2 = U \cup V$, where $U = C^2 = \{(z, u)\}$, $V = C^2 = \{((\xi, v))\}$, $U \cap V = (C - \{0\}) \times C$ with the change of coordinates $(\xi, v) = (z^{-1}, z u)$.

**Theorem 3.1** [1, Thm. 2.1] Let $E$ be a holomorphic rank two vector bundle on $\tilde{C}^2$ with vanishing first Chern class and let $j$ be the non-negative integer that satisfies $E_\ell \simeq \mathcal{O}(j) \oplus \mathcal{O}(-j)$. Then $E$ has a transition matrix of the form

$$
\begin{pmatrix}
  z^j & p \\
  0 & z^{-j}
\end{pmatrix}
$$

from $U$ to $V$, where $p$ is a polynomial given by

$$
p = \sum_{i=1}^{2j-2} \sum_{l=i-j+1}^{j-1} p_{il} z^l u^i.
$$

**Theorem 3.2** [2, Thm. 3.3] Let $E$ be a holomorphic rank two vector bundle on $\mathcal{O}(-k)$ whose restriction to the zero section is $E_\ell \simeq \mathcal{O}(j_1) \oplus \mathcal{O}(j_2)$, with $j_1 \geq j_2$. Then $E$ has a transition matrix of the form

$$
\begin{pmatrix}
  z^{j_1} & p \\
  0 & z^{j_2}
\end{pmatrix}
$$

from $U$ to $V$, where the polynomial $p$ is given by

$$
p = \sum_{i=1}^{[j_1-j_2-2]/k} \sum_{l=ki+j_2+1}^{j_1-1} p_{il} z^l u^i
$$

and $p = 0$ if $j_1 < j_2 + 2$.

**Theorem 3.3** [2, Cor. 4.2] Holomorphic bundles on the blow up of a surface are trivial on a neighborhood of the exceptional divisor minus the exceptional divisor.

As a consequence of Theorems 3.1 and 3.3 we have the following holomorphic classification of bundles on $\tilde{S}$. 

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Corollary 3.4  Every holomorphic rank two vector bundle over $\tilde{S}$ with vanishing first Chern class is completely determined (up to isomorphism) by a 4-tuple $(E, j, p, \Phi)$ where $E$ is a holomorphic rank two bundle on $S$ with vanishing first Chern class, $j$ is a non-negative integer, $p$ is a polynomial, and $\Phi : \mathbb{C}^2 \setminus \{0\} \to GL(2, \mathbb{C})$ is a holomorphic map.

Proof: The essential ingredient here is that by theorem 3.3 every holomorphic bundle on $\tilde{S}$ is trivial on $N_\ell - \ell$ for some neighborhood $N_\ell$ of the exceptional divisor. It follows that outside $\ell$ we may take a pull-back bundle $\pi^*(E|_{S-\ell})$ of a holomorphic rank two bundle $E$ on $S$ with vanishing first Chern class and glue it to a bundle on $N_\ell$ using the function $\Phi$. Now we use Theorem 3.1 to see that a bundle on $N_\ell$ is determined by a non-negative integer $j$ and a polynomial $p$ whose form is explicitly known.

The corresponding classification for nonvanishing first Chern class is the following.

Corollary 3.5  Every holomorphic rank two vector bundle over $\tilde{S}$ is completely determined (up to isomorphism) by a 5-tuple $(E, j_1, j_2, p, \Phi)$ where $E$ is a holomorphic rank two bundle on $S$, $j_1$ and $j_2$ are integers, $p$ is a polynomial, and $\Phi : \mathbb{C}^2 \setminus \{0\} \to GL(2, \mathbb{C})$ is a holomorphic map.

The proof is analogous to the one for Corollary 3.4.

4  Topology of bundles on $\tilde{S}$

We now deduce the topological counterparts of Corollaries 3.4 and 3.5.

Corollary 4.1  Every holomorphic rank two vector bundle over $\tilde{S}$ with vanishing first Chern class is topologically determined by a triple $(E, j, p)$ where $E$ is a holomorphic rank two bundle on $S$, $j$ is a non-negative integer, and $p$ is a polynomial.

Proof: By Corollary 3.4 we know that such a bundle is holomorphically determined by a 4-tuple $(E, j, p, \Phi)$ and Proposition 2.1 shows that topologically the choice of the map $\Phi$ is irrelevant.

A straightforward generalization of Corollary 4.1 for the case of nonvanishing first Chern class is the following result.
Corollary 4.2 Every holomorphic rank two vector bundle over $\tilde{S}$ is topologically determined by a 4-tuple $(E, j_1, j_2, p)$ where $E$ is a holomorphic rank two bundle on $S$, $j_1$ and $j_2$ are integers, and $p$ is a polynomial.

The proof is analogous to the proof of 4.1.

Examples: Let us write down some examples to clarify the statement of Corollary 4.1. First we fix a holomorphic rank two bundle $E$ with vanishing first Chern class over the surface $S$. Then we would like to see which are the possible bundles $\tilde{E}$ over $\tilde{S}$ that are a pull-back of $E$ outside the exceptional divisor. According to Corollary 4.1 such bundles are topologically given by a choice of an integer $j$ and a polynomial $p$ whose form is given in Theorem 3.1 as $p = \sum_{i=1}^{j-1} \sum_{l=i-j+1}^{\frac{j-1}{2}} \sum_{k=0}^{l} u^i$.

If $j = 0$, then $p = 0$ and it follows that $\tilde{E} = \pi^*E$ is globally a pull-back bundle. That is, applying Corollary 3.4, we verify the well known fact that bundles over a blown-up surface that are trivial when restricted to the exceptional divisor are pull-backs.

If $j = 1$, then also $p = 0$. In this case we see that on a neighborhood $N_\ell$ of $\ell$ the bundle is $\mathcal{O}(1) \oplus \mathcal{O}(-1)$. If follows that all holomorphic bundles $\tilde{E}$ over $\tilde{S}$ whose restriction to the exceptional is $\mathcal{O}(1) \oplus \mathcal{O}(-1)$, are topologically equivalent. However, clearly these bundles are not pull-backs and are not topologically equivalent to any of the bundles we obtained in the previous case.

If $j = 2$, then $p = (p_{10} + p_{11}z)u + p_{21}zu^2$ depends on three complex parameters. However these are not effective parameters in the sense that some different choices of the polynomial $p$ will give isomorphic bundles over $N_\ell$ (holomorphically and hence also topologically) and therefore will also lead to globally isomorphic bundles over $\tilde{S}$. This leads us immediately to the question of determining the “local moduli space” structure. That is, to see for a fixed value of $j$ what polynomials define isomorphic bundles over $N_\ell$. We call $\mathcal{M}_j$ the moduli space of isomorphism classes of bundles on $N_\ell$ whose restriction to $\ell$ equals $\mathcal{O}(j) \oplus \mathcal{O}(-j)$. The answer to the local moduli question is given by the following results.

Theorem 4.3 [1, Thm. 3.4] The moduli space $\mathcal{M}_2$ is homeomorphic to the union $\mathbb{P}^1 \cup \{q_1, q_2\}$, of a complex projective plane $\mathbb{P}^1$ and two points, with a basis of open sets given by

$$U \cup \{q_1 \cup \{q_1, q_2, U \in U - \phi\} \cup \{q_1, q_2, U \in U - \phi\},$$

where $U$ is a basis for the standard topology of $\mathbb{P}^1$. 

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Theorem 4.4 [1, Thm.3.5] The generic set of the moduli space $\mathcal{M}_j$ is a complex projective space of dimension $2j - 3$.

**Examples:** Let us continue the analysis of the case $j = 2$ started in the preceding example. We have seen that for $j = 2$ the polynomial $p$ is given by three complex parameters. However, by Theorem 4.3 we see that nonequivalent choices of $p$ are parametrized by a non-Hausdorff space formed by a projective line $\mathbb{P}^1$ with two extra points. Therefore, it follows from Corollary 3.4 that for each fixed choice of glueing $\Phi$, isomorphism classes of bundles are parametrized by $\mathbb{P}^1$ (with the standard topology) plus two extra points. However it is simple to see that any such choices will produce topologically equivalent bundles.

For each value of the integer $j$ we can reproduce an analysis similar to the ones in the previous examples. For a chosen bundle $E$ over $S$ and a particular choice of glueing $\Phi$ we have generically a projective space $\mathbb{P}^{2j-3}$ parametrizing nonisomorphic bundles $\tilde{E}$. Details for the topology as well as explicit calculations of Chern classes for these bundles will appear in a subsequent paper.

Let us represent holomorphic bundles on the blown-up surface $\tilde{S}$ by $(E, j, p, \Phi)$, according to Corollary 3.4. Then we have just proved the following.

**Theorem 4.5** Let $E$ be a holomorphic rank two bundle with vanishing first Chern class. For fixed $\Phi$ and $j$ the family of isomorphism classes of holomorphic bundles on $\tilde{S}$ which are a pull-back of $E$ outside the exceptional divisor is generically parametrized by $\mathbb{P}^{2j-3}$.

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