Variance Reduction Methods for Sublinear Reinforcement Learning

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Abstract

This paper is on hold until a technical problem is addressed. We will post a new arXiv version soon.

This work considers the problem of provably optimal reinforcement learning for episodic finite horizon MDPs, i.e., how an agent learns to maximize his/her long term reward in an uncertain environment. The main contribution is in providing a novel algorithm — Variance-reduced Upper Confidence Q-learning (vUCQ) — which enjoys a regret bound of \( \tilde{O}(\sqrt{H^5SA} + H^3SA) \), where the \( T \) is the number of time steps the agent acts in the MDP, \( S \) is the number of states, \( A \) is the number of actions, and \( H \) is the (episodic) horizon time. This is the first regret bound that is both sub-linear in the model size and asymptotically optimal. The algorithm is sub-linear in that the time to achieve \( \epsilon \)-average regret for any constant \( \epsilon \) is \( O(SA) \), which is a number of samples that is far less than that required to learn any non-trivial estimate of the transition model (the transition model is specified by \( O(S^2A) \) parameters).

The importance of sub-linear algorithms is largely the motivation for algorithms such as Q-learning and other “model free” approaches. vUCQ algorithm also enjoys minimax optimal regret in the long run, matching the \( \Omega(\sqrt{H^5SA}) \) lower bound.

Variance-reduced Upper Confidence Q-learning (vUCQ) is a successive refinement method in which the algorithm reduces the variance in Q-value estimates and couples this estimation scheme with an upper confidence based algorithm. Technically, the coupling of both of these techniques is what leads to the algorithm enjoying both the sub-linear regret property and the asymptotically optimal regret.

1 Introduction

This works considers the reinforcement learning problem where an agent seeks to (optimally) balance exploration with exploitation to maximize his/her long term reward ([SB98]). We study this problem in the context of an (unknown) episodic Markov decision processes (MDP), where we have an \( H \)-horizon MDP, with state set \( S \) and action set \( A \). Each run of the MDP is an episode of \( H \) time steps, where the agent starts at an arbitrary initial state, makes a sequence of \( H \) decisions and collects \( H \) rewards. The agent’s goal is to maximize his/her long term reward.

We measure the quality of an algorithm in terms of its regret, specified as follows: suppose that we run a reinforcement learning algorithm \( K \) for \( K \) episodes. Let \( T = KH \) be the total number of time steps elapsed, and let \( \{r_1, \ldots, r_T\} \) be the sequence of rewards generated in the learning process. The \( T \)-step expected regret for the algorithm \( K \) is

\[
\text{Regret}(T) := \mathbb{E}^{\pi^*} \left[ \sum_{t=1}^{T} r_t \right] - \mathbb{E}^{K} \left[ \sum_{t=1}^{T} r_t \right],
\]

where \( \pi^* \) is the optimal policy for the \( H \)-horizon MDP; \( \mathbb{E}^{\pi^*} \left[ \sum_{t=1}^{T} r_t \right] \) is the expectation of total rewards generated by playing \( \pi^* \) throughout; \( \mathbb{E}^{K} \left[ \sum_{t=1}^{T} r_t \right] \) is the expectation of total rewards generated by the learning algorithm \( K \).

The most basic question here is how quickly can we drive the average regret to 0. This question has been widely studied with various upper bounds on the regret having been established (see [JOA10, AJ17, AOM17]).
provides regret bounds for the state-of-art algorithms whose regret is $\tilde{O}(\sqrt{T})$. Let $T^*$ be a time when the average regret is $O(1)$. The “sublinear” column means $T^*$ is sublinear in the model size $|S|^2|A|$. See text for more details. The minimax lower bound on the regret is $\Omega(\sqrt{|S||A|T})$ ([JOA10, OVR16]). The results of [JOA10] and [AJ17] apply to a more general setting with infinite-horizon weakly communicating MDP with a diameter $D$ (which can be as small as 1 or scale superlinearly in $|S|$). In the special case of $H$-horizon MDP, note that the worst case diameter $D \leq H$ so these algorithms are comparable. We have not included results on the sample complexity of exploration from [SLW^+06, LH14a, LH14b, DB15, SS10, K^+03], as these bounds do not lead to optimal $O(\sqrt{T})$ regret bounds (though the results of [SLW^+06, SS10] are sub-linear).

Table 1: Regret bounds for reinforcement learning methods (we ignore polylog factors on $H|S||A|T$). We only list algorithms whose regret is $\tilde{O}(\sqrt{T})$. Let $T^*$ be a time when the average regret is $O(1)$. The “sublinear” column means $T^*$ is sublinear in the model size $|S|^2|A|$. See text for more details. The minimax lower bound on the regret is $\Omega(\sqrt{|S||A|T})$ ([JOA10, OVR16]). The results of [JOA10] and [AJ17] apply to a more general setting with infinite-horizon weakly communicating MDP with a diameter $D$ (which can be as small as 1 or scale superlinearly in $|S|$). In the special case of $H$-horizon MDP, note that the worst case diameter $D \leq H$ so these algorithms are comparable. We have not included results on the sample complexity of exploration from [SLW^+06, LH14a, LH14b, DB15, SS10, K^+03]; these latter results can be translated into regret bounds though they do not lead to regret bounds with the optimal dependence on $T$ (the optimal $T$ dependence is $O(\sqrt{T})$).

This work This work provides an algorithm which enjoys optimal (asymptotic) regret and in which this asymptotic regime kicks in at a “burn in” time which is sub-linear in the model size. We define the “burn in” time as the time $T$ at which the average regret is $O(1)$; crudely, the burn in time is the number of time steps until the agent has non-trivial average regret. Table 1 provides regret bounds for the state-of-art reinforcement learning methods which have $O(\sqrt{T})$ regret.

This work provides the variance-reduced Upper Confidence Q-learning (vUCQ) algorithm, which enjoys a regret bound of:

$$\tilde{O}\left(\sqrt{H|S||A|T + H^5|S||A|}\right).$$

This implies that, in order to obtain an average regret of less than $\epsilon$, it is sufficient for $T$ to be $O\left(\frac{|S||A|}{\epsilon^2}\right)$ (holding $H$ constant). Precisely, note that the burn-in time is the number of steps to get $\epsilon$-average regret (for constant $\epsilon$). vUCQ is sub-linear in that the time to achieve $\epsilon$-average regret (for any constant $\epsilon$) is $O(|S||A|)$, which is a number of samples that is far less than that required to learn any (non-trivial) estimate of the transition model (the transition model is specified by $O(|S|^2|A|)$ parameters). In contrast, the best prior result of [AOM17] requires a number of timesteps that is $O\left(\frac{|S||A|}{\epsilon^2} + \frac{|S|^2|A|}{\epsilon}\right)$ in order to obtain $\epsilon$-average regret.

Technically, we emphasize that this improvement is not simply based on a sharper analysis of the current upper confidence based algorithms; vUCQ achieves this sublinear property using a new algorithm which couples a method of variance reduction with upper confidence based algorithms. It is plausible to the authors that there are fundamental limitations to the current upper confidence based algorithms (such as those in [JOA10, AJ17, AOM17]) which prevent these algorithms from achieving sublinear linear regret; obtaining algorithmic lower bounds is an important open question. Finally, we should note that we do not know what the optimal dependence on the horizon time $H$ is in the lower order term is.

Overall, the results herein provide an algorithm that achieves near optimal regret within a sub-linear time, i.e. without having to observe an amount of samples comparable to the model size. In this sense, vUCQ is the first provable asymptotically optimal algorithm which is also “model free” in that the burn in time is sublinear.

\footnote{In fact their lower bound is for a more general setting, in which the underlying MDP is a weakly communicating infinite-horizon average reward MDP. The bound can be generalized to episodic MDP using the method described in [OVR16].}
2 Preliminaries

In this section we briefly review the basic model of episodic reinforcement learning and the notations to be used. We consider the undiscounted episodic reinforcement learning (RL) problem (see e.g. [BT95]). The same setting has been studied in [AOM17]. In this setting, the RL agent interacts with a stochastic environment modeled as a finite-horizon Markov decision process (MDP), which we denote as a tuple $\mathcal{M} = (\mathcal{S}, \mathcal{A}, P, r, H)$, where $\mathcal{S}$ is a finite set of states, $\mathcal{A}$ is a finite set of actions, $P \in \mathbb{R}^{\mathcal{S} \times \mathcal{A} \times \mathcal{S}}$ is a probability transition matrix, $r \in [0,1]^{\mathcal{S} \times \mathcal{A}}$ is the reward vector\(^2\) and $H \in \mathbb{Z}^+$ is the horizon parameter. We denote each row of $P$ as $P(s, a, \cdot) \in \Delta_S$, where $\Delta_S$ denotes the probability simplex on $\mathcal{S}$. The agent interacts with the environment episodically. For instance, consider each episode $k = 1, 2, \ldots, K$, the agent start from some state $s_1(k) \in \mathcal{S}$ chosen by the environment (which can be arbitrary and even history dependent) and then interacts with the environment for $H$ steps. The agent play a sequence of $H$ actions and observe a sequence of states and rewards from the environment based on an unknown $H$-horizon MDP. We denote the state and actions at episode $k$ as $(s_1(k), a_1(k)), (s_2(k), a_2(k)), \ldots, (s_H(k), a_H(k))$.

For now, let us omit the episode superscript $(k)$ and consider an $H$-horizon MDP. A policy is modeled as a map $\pi : \mathcal{S} \times [H] \rightarrow \mathcal{A}$. A value function with respect to a policy $\pi$ at stage $h = 1, 2, \ldots, H$ is a vector $V_h^\pi \in \mathbb{R}^S$ defined as,

$$\forall s \in \mathcal{S} : \quad V_h^\pi(s) := \mathbb{E}^\pi \left[ \sum_{k=h+1}^H r(s_k, a_k) \mid s_h = s \right],$$

namely, the expected total reward of playing $\pi$ in the remaining $H - h$ steps starting from state $s$. For a vector $V \in \mathbb{R}^S$, we denote $P V \in \mathbb{R}^{|S| \times |A|}$ as a $|S| \times |A|$-dimensional vector with $(P V)(s, a) = P(s, a)^T V$. For a given policy $\pi$, we denote $P_h^\pi \in \mathbb{R}^{|S| \times |S|}$ with $\forall s \in S : P_h^\pi(s, \cdot) := P(s, \pi(s, h))$. We denote a Q-function with respect to a vector $V$ as $Q(V) = r + PV$. Thus the Q-function with respect to a policy $\pi$ at stage $h$ is defined as $Q_h^\pi = r + PV_h^\pi$. The Bellman equation states that $\forall s \in \mathcal{S} : V_h^\pi(s) = Q_h^\pi(s, \pi(s, h))$. For an $H$-horizon MDP, there exists an optimal value functions $V_h^\ast$ for $h = 1, 2, \ldots, H - 1$ satisfying the Bellman equation $\forall s : V_h^\ast(s) = \max_{a \in \mathcal{A}} Q_h^\ast(s, a)$ and $V_H^\ast(s) = \max_{a \in \mathcal{A}} r(s, a)$. A policy $\pi^\ast$ achieving the optimal value function is called an optimal policy.

Let us now consider the reinforcement learning problem of an episodic MDP. Suppose the agent has played in total $K$ episodes, where each episode contains $H$ steps. We denote $T = KH$ as the total number of steps the agent has played. We can write the $T$-step regret as\(^3\)

$$\text{Regret}(T) = \mathbb{E} \left[ \sum_{k=1}^K V_{s_1(k)}^\ast - V_{s_1(k)}^\ast \right],$$

where $s_1(k)$ is the initial state of the $k$-th episode, determined by the environment, and the expectation is taken overall possible randomness in the learning process.

3 Main Results

In this section we give a technical summary of our algorithm and regret bounds. We also provide a roadmap of the regret analysis. Full developments of the algorithms and regret proofs are deferred to Sections 4-6.

3.1 A Short Description of the Algorithm vUCQ

The vUCQ algorithm is developed by carefully combining a number of algorithmic features. The algorithm can be viewed as an asynchronous and a bottom-up version of the randomized value iteration (see, e.g. [SWWY18]). We initialize our value function and Q-function as uniform upper bounds of the optimal functions, i.e., at the stage $h$, the value and Q-function is initialized as $H - h + 1$ at each state-action pair.

\(^2\)We use a deterministic reward function and require that each entry of $r$ is in the range $[0, 1]$. As it shown in [JOA10, AOM17], the difference of a regret bound from a randomized reward is only on the lower order terms.

\(^3\)Unlike the high probability measure used in [AOM17], the expected regret bound differs by a $\tilde{O}(H\sqrt{T})$ term less.
Our initial policy is an arbitrary policy. At the end of episode \(k - 1\), we partition the samples collected so far into at most \(\log(kH)\) many buckets for each state-action pair. For instance, for state-action pair \((s, a)\), the samples from \(P(\cdot|s, a)\) are partitioned into groups \(G_{s,a}^{(1)}, G_{s,a}^{(2)}, \ldots, G_{s,a}^{(\ell(s,a))}\) by their arrival time with \(|G_{s,a}^{(j)}| \asymp 2^j\). Thus the majority of the samples collected for each state-action pair is contained in the latest full bucket, i.e., \(G_{s,a}^{(\ell(s,a))}\). These geometrically increasing sized buckets allow us to successively refine the value and Q-functions. Since newer buckets are collected independently from older buckets, we are able to use the previous values and Q-functions as reference values for the next refinement. For instance, suppose we have obtained value function \(V_h^{(j)}\) and \(Q_h^{(j)}\) using the samples collected in \(\cup_{j'=1}^j G_{s,a}^{(j')}\), then \(V_h^{(j)}\) and \(Q_h^{(j)}\) are independent with the samples of \(\cup_{s,a} G_{s,a}^{(j+1)}\). By carefully designing confidence bounds, we demand the estimates of \(V_h^{(j+1)}\) and \(Q_h^{(j+1)}\) to be more accurate than \(V_h^{(j)}\) and \(Q_h^{(j)}\) (in a sense that is formally defined in Section 5). We use the values and Q-functions to obtain a greedy policy for the next episode.

### 3.2 Regret Analysis

Our main result is as follows.

**Theorem 3.1.** Let \(T = KH > 0\) be the total number of time steps. Then there exists an reinforcement learning algorithm, acting in the world of an \(H\)-episodic MDP, starting from any initial state, achieving an expected regret bound

\[
\text{Regret}(T) = \tilde{O}\left[\sqrt{|S||A|HT} + |S||A|H^3\right].
\]

Let us explain the result of Theorem 3.1. The leading-order term is \(\tilde{O}(\sqrt{|S||A|HT})\), which matches the regret lower bound suggested by [JOA10]. The low-order term is \(\tilde{O}(|S||A|H^5)\), which does not scale up as \(T\) increases. The low-order term will play a major role when \(T \leq |S||A|H^3\), which we refer to as the burn-in time before the algorithm starts to perform close-to-optimally. Our result appears to be the first one that achieves asymptotically tight regret bound as well as a sublinear burn-in time. See Table 1 for a comparison of existing results.

The burn-in time of a reinforcement learning algorithm plays a significant role in its practical performances. It is a transient period during which the algorithm has not achieved \(O(1)\) average regret. Our burn-in time is linear in the total number of state-action pairs \(|S||A|\), but is sublinear in the model size \(|S|^2|A|\). It suggests that the learning agent does not need to estimate the MDP precisely before starting to perform well.

When there is a generative model (introduced by [K+03]) that generates samples from any specified state-action pair, one can estimate the value of an MDP model up to \(\epsilon\) optimality using \(O(H^3|S||A|\epsilon^{-2})\) sample [AMK13]. Although this sample complexity result requires a stronger generative model for sampling, it essentially implies that a reasonable burn-in time should be linear in \(|S||A|\). In contrast, the previous work [AJ17] and [AOM17] obtain near-tight asymptotic regret bounds, but they suffer from superlinear burn-in times. A large burn-in time prevent their algorithms from warming up quickly. In this work, the proposed vUCQ algorithm appears to be the first regret-optimal method with sublinear burn-in time.

In what follows, we provide a roadmap for the regret analysis that leads to Theorem 3.1. In Section 4 we provide a full description of vUCQ algorithm. In Section 5 we give an analysis of the vUCQ using a Hoeffding-based bound. This analysis is not asymptotically tight but already shows sublinear burn-in time. The section provides an analytic framework for us to establish a more refined and asymptotically tight bounds. The proof of our main theorem is presented in Section 6. The near-tight asymptotic bound is obtained by seamlessly combining the Bernstein trick (of [AOM17]) and our variance reduction framework established in Section 5.

### 3.3 Related Works and Techniques

The work [AOM17] is the most related work to ours. The have developed the asymptotic near-tight regret bound on the same episodic MDP reinforcement learning model. Their contribution is the development of...
the upper confidence Q-learning algorithm (UCQ). In their setting, they learn a policy by estimating the optimal value function directly from the samples collected so far. They have carefully designed a confidence bound based on the number of samples collected per state-action pair. They then show that the sum of the confidence bounds relates to the regret bound of the algorithm. The fundamental obstacle prevents them from getting a sublinear burn-in time is the so-called “dependence” issue. For instance, in an $H$-horizon MDP, we have collected samples for each state-action pair, and use them to do value iteration. For a fixed $h$, the backup of the Bellman equation can be denoted as follows $Q_h \leftarrow r + \hat{P}V_{h+1}$, where $\hat{P}$ is the estimate of the probability transition matrix using the samples collected. Notice that $V_{h+1}$ is also obtained by the same set of samples. Thus $\hat{P}$ as a random vector is not independent with $V_{h+1}$. [AOM17] gets around this issue by showing that if we collect sufficiently many samples, i.e., $T \geq \text{poly}(H) |S|^2 |A|$, then the estimate $\hat{P}$ is already sufficiently accurate and dependence is no-longer a problem. This is the major reason for a large burn-in time. In fact, the same burn-in time source has been observed in [AJ17] and [JOA10] since they are essentially using similar tricks.

Getting around the “dependence” issue is a major step towards a sublinear burn-in time. In fact, if we separate the samples into $H$-different stages, i.e., we have $H$-independent estimates of the probability matrix, $\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_H$, then we can obtain a $\sqrt{H^3 |S||A|T}$ regret bound with sub-linear burn-in time (we include this analysis in the appendix (Section D)). However, since we are not using the full information (we discard samples since only $1/H$ fraction of the samples are used for computing backups), our regret bound is not tight in terms of $H$.

To reduce the number of samples discarded and reduce the dependence issue, we were inspired by the recent development of fast sublinear time algorithms for solving MDPs developed in [SWWY18]4, in which they developed a variance reduction framework for obtaining the optimal policy through samples of an discounted-MDP with discount factor $\gamma$ (one can relates their model with ours by setting $H = (1 - \gamma)^{-1}$). Their computation model can be viewed as a simulator model, i.e., one can ask an oracle to provide as many samples as needed for each state-action pair. We summarize their framework in a “bottom-up” fashion as follows. They show that in a value iteration algorithm, one can reduce the sample complexity while preserving independence of estimators by successively refining the value function. More precisely, starting from some crude guesses of the optimal value function, one improves the value function by collecting a set of fresh samples. Each new refinement guarantees that the distance of the value function to the optimal one is decreased by a half. Let us denote the value function at the $j$-th refinement as $V^{(j)}$. Suppose that $\|V^{(j)} - V^*\|_\infty \leq u$. In the $(j+1)$-th refinement, the goal is to produce a $V^{(j+1)}$ yet demanding $\|V^{(j+1)} - V^*\|_\infty \leq u/2$. The idea is to first obtain a set of fresh samples $\mathcal{G}$ per state-action pair with $|\mathcal{G}| \propto 2^j$ and estimate $P V^{(j)}$ up to high precision. For each state-action pair, we collect another $H = \Theta((1 - \gamma)^{-1})$ sets of fresh samples $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_H$ each with $|\mathcal{H}_h| \propto H^2$. Using these $H$ sets of independent samples, we obtain independent estimates of $P^T (V^{(j+1)} - V^{(j)})$. By demanding that $\|V_h^{(j+1)} - V^{(j)}\|_\infty \leq \|V_h^{(j+1)} - V^*\|_\infty \leq u$ we can conclude that the independent samples, although of constant size, give good estimators for $P^T (V_h^{(j+1)} - V^{(j)})$, since the variance of estimators are at most $\|V_h^{(j+1)} - V^*\|_\infty^2 = u^2$ (this is the so-called “variance-reduction” trick). In this framework, we are not “discarding” too many samples since $\mathcal{G}^{(j)}$ for the largest $j$ contains the majority of the collected samples. The constant-sized sample $\mathcal{H}_h$ provide the sources for the burn-in, but there are only $H |S||A|$ many. Our algorithm shares with them a similar bucketing structure. However it becomes much more complicated once the samples collected for each state-action pair is non-synchronized, i.e., the agent is out of control for what next state he will observe – only the environment determines that. We will show in next section that we build non-synchronized hierarchical models for the underlying MDP, and use them to simulate a variance-reduction algorithm in the simulator setting. There are many difficulties relates to the non-synchronization. Fortunately, we manage to resolve these issues through a carefully design of reference vectors for each state-action pair and seamlessly combine variance reduction technique with the UCQ algorithm.

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4A recent work [SWW18], which is also co-authored by two of the authors of this paper, improves the running time and sample complexity of [SWWY18] to nearly optimal in solving discounted MDP with a generative model.
4 The Variance-Reduced Upper Confidence Q-Learning (vUCQ) Algorithm

In this section, we introduce our main variance-reduced algorithmic framework. We aim to show how variance reduction can be achieved in the reinforcement learning setting and how it helps us obtaining sublinear burn-in time. For ease of representation, we defer the full regret analysis to later sections.

**Known T v.s. Unknown T** To begin, suppose we have a known upper bound, denoted by $T$, on the number of steps an learning algorithm will run. The extension from known $T$ to unknown $T$ is standard with only an additional cost up to an $O(\log T)$ additive term on the regret bound. For instance, we first guess the true bound for the number of steps as $T'$, then we run the algorithm upto $T'$ steps. If the algorithm is still running, we then restart the algorithm and run for $2T'$ steps. Then we continue the process for $2^2T', 2^3T', \ldots, 2^iT'$ until we hit the true time upper bound $T$. If the algorithm gives a regret bound of the form $O(\sqrt{2^iT'} + C)$ in the $j$-th restart, we then obtain a final regret bound

$$\text{Regret}(T) = O\left(\sum_{j=1}^{T} \sqrt{2^iT'} + C\right) = O(\sqrt{T} + C\log T).$$

**Sample Collection and Dynamic Partition** Next, we describe our full algorithm, we run the vUCQ in the MDP, we will collect state transition samples at each state-action pair the agent has encountered. Suppose that the algorithm has run in $k$ episodes. For each state-action pair $(s, a) \in S \times A$ the agent has visited, we partition the collection of transition samples from $(s, a)$ into subsets with pre-specified sizes according to the arriving order of the samples:

$$G^{(1)}(s, a), H^{(1)}(s, a), G^{(2)}(s, a), H^{(2)}(s, a), \ldots, G^{(l(k,s,a))}(s, a), H^{(l(k,s,a))}(s, a).$$

We view each subset as a fixed-size bucket. As we collect more sample transitions from $(s, a)$, we fill the buckets one by one. In particular, we choose the bucket sizes such that for each $j < l(k, s, a)$, $|G^{(j)}(s, a)| = c_0 2^j$ and $|H^{(j)}(s, a)| = c_1 LH^3$ for some constant $c_0, c_1$ and $L = c_3 \log(|S|, |A|)$ for some constant $c_3$. Here we use $l(k, s, a)$ to denote the level number, i.e., the last bucket to be filled for $(s, a)$ at the $k$th episode, which is a random integer determined by the history and the learning algorithm. In addition, we partition each $H^{(j)}(s, a)$ into $H$ sub-buckets

$$H^{(0)}(s, a) \cup H^{(1)}(s, a) \cup \ldots \cup H^{(l(k,s,a))}(s, a),$$

where each $|H^{(j)}(s, a)| = c_1 LH^2$. As one may notice, samples from different buckets are independently drawn from the distribution $P(\cdot|s, a)$. Also note that the last buckets might not be filled, so it is possible that $|G^{(l(k,s,a))}| < c_0 2^{l(k,s,a)}$ or $|H^{(l(k,s,a))}| < c_1 LH^3$. We denote by $\ell^*(k, s, a)$ as the largest integer $j$ such that $|G^{(j)}| = c_0 2^j$ and $|H^{(j)}| = c_1 LH^3$ for samples collected before the $k$-th episode starts. We also call $\ell^*(k, s, a)$ as the latest full level up to episode $k$. Note that this number may vary as $(s, a)$ varies.

**Estimation of the MDP Model** The vUCQ algorithm constructs a sequence of successfully refined models for the unknown environment. Let us now consider the agent is at episode $k$. For each state action pair $(s, a)$ and another state $s'$, we denote

$$\forall j \leq \ell^*(k, s, a) : \quad \hat{p}^{(j)}(s'|s, a) = \frac{\sum_{s'' \in \mathcal{G}^{(j)}(s, a)} I(s'' = s')}{c_0 2^j} \quad \text{and} \quad \hat{p}^{(j)}(s'|s, a) = \frac{\sum_{s'' \in \mathcal{H}^{(j)}(s, a)} \hat{p}(s'' = s')}{c_1 LH^2}. \quad (1)$$

As one can observe, $\hat{p}^{(j)}(\cdot|s, a)$ is an estimate of $P(\cdot|s, a)$ using the data collected in $\mathcal{G}^{(j)}(s, a)$ and $\hat{p}^{(j)}(\cdot|s, a)$ is an estimate of $P(\cdot|s, a)$ using the data collected in $\mathcal{H}^{(j)}(s, a)$. Since for larger $j$, i.e. when $c_0 2^j \geq c_1 LH^2$, $\mathcal{H}^{(j)}(s, a)$ contains less number samples than that of $\mathcal{G}^{(j)}(s, a)$, $\hat{p}^{(j)}(\cdot|s, a)$ is noisier than $\hat{p}^{(j)}$. We also denote

$$\hat{p}^{(0)} = \hat{p}^{(0)} = 0 \in \mathbb{R}^{S \times A \times S}. \quad (2)$$

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For each \( j \leq \ell^*(k, s, a) \), we have a model of the environment defined by the estimated probability vector \( \tilde{P}^{(j)}(|s, a) \) and \( \tilde{P}^{(j)}_h(|s, a) \) for \( h = 1, 2, \ldots, H \). Notice that \( \ell^*(k, s, a) \) is not necessarily the same for different state-action pairs. Therefore, given a \( j \), if \( j > \ell^*(k, s, a) \), then we do not have estimates for \( \tilde{P}^{(j)}(|s, a) \) and \( \tilde{P}^{(j)}_h(|s, a) \). We call this phenomena the non-synchronization property. This is the major obstacle of achieving a similar variance reduction scheme as that used in a simulator model, e.g., [SWWW18]. In what follows, we will describe a method to get around this problem. We denote \( \ell^*(k) = \max_{s, a} \ell^*(k, s, a) \) as the largest full level. We can immediately bound \( \ell^*(k) \) by \( O(\log(kH)) \).

**Policy Improvement After Collecting Sufficient Samples** We now describe our policy updating procedure. For \( k = 1 \), we starting with an arbitrary policy \( \pi^{(1)} \) and acts in the environment to collect state samples and rewards. Once a state-action pair \((s, a)\) is full, i.e., at the end of the \((k-1)\)-th episode, it satisfies that \( |G^{(k, s, a)}| = c_0 2^{k} \) and \( |H^{(k, s, a)}| = c_1 L H^3 \), we update our policy by successively refining the value and policy of the \( H \)-horizon MDP for \( j = 1, 2, \ldots, \ell^*(k) \). For the \( j \)-th refinement of the value and policy, we use the estimates \( \tilde{P}^{(j)}(|s, a) \) and \( \tilde{P}^{(j)}_h(|s, a) \) if \( j \leq \ell^*(k, s, a) \) and \( \tilde{P}^{(\ell^*(k, s, a))}(|s, a) \) and \( \tilde{P}^{(\ell^*(k, s, a))}_h(|s, a) \) if \( j > \ell^*(k, s, a) \).

We now describe the refinement procedure. Let \( Q_{0,h}^{(j)}(s, a) = H - h + 1 \) for all \((s, a) \in S \times A\) and \( h \in [H] \). Notice that \( Q_{0,h}^{(j)} \) serves as the initial guess of the optimal Q-function. We will also define a reference vector \( V_{h,s,a}^{(j)} \) for each state-action pair \((s, a)\) and \( j \in [\ell^*(k)] \). This reference vector serves as a coarse estimation of \( V^*_h \), the optimal value function of the \( H \)-horizon MDP. As an initial guess, we set \( V_{h,s,a}^{(0)} := H - h + 1 \) for all \( h \in [H] \) and \((s, a) \in S \times A\). Notice that each \( V_{h,s,a}^{(0)} \) always upper bounds \( V^*_h \) for all \( h \in [H] \). To obtain \( Q_{h}^{(j)}(s, a) \), we solve the MDP using the dynamic programming defined as follows. We set \( Q_{h+1}^{(j)}(s, a) = 0 \) for all \((s, a) \in S \times A\) and \( j \in [\ell^*(k)] \). For \( h = H, H - 1, \ldots, 1 \), \((s, a) \in S \times A\) and \( j \in [\ell^*(k, s, a)] \), if \( 1 \leq \ell^*(k, s, a) < j \), we set,

\[
\tilde{P}^{(j)}(|s, a) := \tilde{P}^{(\ell^*(k, s, a))}(|s, a), \tilde{P}^{(j)}_h(|s, a) := \tilde{P}^{(\ell^*(k, s, a))}_h(|s, a) \quad \text{and} \quad V_{h,s,a}^{(j)} := V_{h,s,a}^{(\ell^*(k, s, a))},
\]

and if \( 1 \leq j \leq \ell^*(k, s, a) \), we set

\[
\tilde{P}^{(j)}(|s, a) = \tilde{P}^{(j)}(|s, a), \tilde{P}^{(j)}_h(|s, a) = \tilde{P}^{(j)}_h(|s, a) \quad \text{and} \quad V_{h,s,a}^{(j)} = \max_{a'} Q_{h+1}^{(j-1)}(s', a').
\]

Let \( V_{h+1}^{(j)}(s) = \max_{a} Q_{h+1}^{(j)}(s, a) \) for each \( s \). The above different assignment of \( \tilde{P}^{(j)}(|s, a) \) and \( \tilde{P}^{(j)}_h(|s, a) \) and \( V_{h+1}^{(j)}(s, a) \) for different \((s, a)\) is a key-step towards solving the non-synchronization problem. Next we successively refine \( Q_{h}^{(j)} \), for \( j = 1, \ldots, \ell^*(k) \) and \( h = H, H - 1, \ldots, 1 \), as

\[
Q_{h}^{(j)}(s, a) := \min_{a} \{ \hat{r}(s, a) + \tilde{P}^{(j)}(|s, a) V_{h+1}^{(j)}(s, a) + \tilde{P}^{(j)}_h(|s, a) V_{h+1}^{(j)}(s, a) + b^{(j)}_{h,s,a} Q_{h+1}^{(j-1)}(s, a) \},
\]

where \( \hat{r}(s, a) = r(s, a) \) if \((s, a)\) has been visited for at least once or 1, and \( b^{(j)}_{h,s,a} \) is the bonus function for exploration which we define by

\[
b^{(j)}_{h,s,a}(k) = \frac{1}{2} \cdot (H - h) \cdot \sqrt{\frac{L}{2^j}} \quad \text{if} \quad j \leq \ell^*(k, s, a) \quad \text{or} \quad \frac{1}{2} \cdot (H - h) \cdot \sqrt{\frac{L}{2^j}} \quad \text{otherwise}.
\]

Recall that \( L = c_3 \log(SAHT) \) for some large constant \( c_3 \).

**Intuition on Why the Successive Refinements Reduce Burn-in Time** In the above refinement formula \((4)\), we are guaranteed that \( Q_{h}^{(j)}(s, a) \leq Q_{h+1}^{(j-1)}(s, a) \) for any \( j = 1, 2, \ldots, \ell^*(k) \). Note that

\[
\hat{r}(s, a) + \tilde{P}^{(j)}(|s, a) V_{h+1}^{(j)}(s, a) + \tilde{P}^{(j)}_h(|s, a) V_{h+1}^{(j)}(s, a) + b^{(j)}_{h,s,a} Q_{h+1}^{(j-1)}(s, a) + b^{(j)}_{h,s,a}.
\]
is an approximate Bellman update. We will show that if \((s, a)\) has been observed, then
\[
\tilde{r}(s, a) + \tilde{P}^{(j)}(\cdot|s, a)\mathbf{v}_{h+1,s,a}^{(j)} + \tilde{P}^{(j)}_{h}(\cdot|s, a)\mathbf{v}_{h+1,s,a}^{(j)} - \mathbf{v}_{h+1,s,a}^{(j)}
\]
is “an unbiased estimator” of \(r(s, a) + P(\cdot|s, a)\mathbf{v}_{h+1}^{(j)}\) (note that the samples of different state-actions are not independent, we will deal with this issue in Appendix A). We add an additive term \(b_{h,s,a}^{(j)}(k)\). We will show in the next section that this term guarantees that \(Q_{h}^{(j)}(s, a)\) is always an over-estimator of \(Q_{h}^{*}(s, a)\) with high probability. Therefore, \(Q_{h}^{(j)}(s, a)\) become closer to \(Q_{h}^{*}(s, a)\) than \(Q_{h}^{(j-1)}(s, a)\). An important property we have used here is that, although \(\tilde{P}^{(j)}_{h}\) is noisier than \(\tilde{P}^{(j)}\), the infinity norm of \((\mathbf{v}_{h+1}^{(j)} - \mathbf{v}_{h+1,s,a}^{(j)})\) decreases as \(j\) increases. Thus the additive errors of the estimator \(\tilde{P}^{(j)}(\cdot|s, a)\mathbf{v}_{h+1,s,a}^{(j)} + \tilde{P}^{(j)}_{h}(\cdot|s, a)\mathbf{v}_{h+1,s,a}^{(j)} - \mathbf{v}_{h+1,s,a}^{(j)}\) become comparable. Since the estimator of \(\tilde{P}^{(j)}_{h}\) contains much less many samples, even if we have \(H\) of them, the overall number of samples used in estimating \(\tilde{P}^{(j)}_{h}\) is the lower order term compared to that of \(\tilde{P}^{(j)}\), which matches the leading order term of that in [AOM17]. Most importantly, \((\mathbf{v}_{h+1}^{(j)} - \mathbf{v}_{h+1,s,a}^{(j)})\) is an independent random vector with \(\tilde{P}^{(j)}_{h}\). This crucial property allows us to have an unbiased estimator with small error. In contrast, [AOM17] uses a biased estimator, which become accurate only when sufficient many samples for pair \((s, a)\) have been collected. This is the major advantage for us to get rid of the high burn-in time. Also note that the reference vectors \(\mathbf{v}_{h+1,s,a}^{(j)}\) are different for different \((s, a)\), which is a fundamental difference from the simulator case in [SWWY18].

After solving the above \(\ell^{*}(k)\) levels of dynamic programming, we play the policy
\[
\pi^{(k)}(s, h) = \arg\max_{a} Q_{h}^{(\ell^{*}(k))}(s, a)
\]
for further episodes until at least one state-action pair is full at the end of an episode. After that, we repeat the above mentioned procedure to update the policy. The complete algorithm is presented in Algorithm 1 and Algorithm 2. Algorithm 1 is a standard planner as is also used in [AOM17] except that it groups samples into buckets as described above. Algorithm 2 is the successive refinement sub-routine.

**Algorithm 1** vUCQ: variance reduced upper confidence reinforcement learning

1. **Input:** steps upper bond \(T > 0\), horizon \(H\) and initial state \(s^{*}\);
2. **Initialize:**
   3. \(G^{(1)}(s, a) = \emptyset\) for all \(j = 1, 2, \ldots\);
   4. \(H^{(j)}(s, a) = \emptyset\) for all \(j = 1, 2, \ldots\);
   5. Let \(t_{c} \leftarrow 0\); /*The last \(t\) used for solving \(\pi^{*}\);*/
   6. Let \(\ell(s, a) \leftarrow 0\) for all \((s, a)\);
   7. Let \(c_{0}, c_{1}\) be some large absolute constants;
8. **While** \(t \leq T;\)
9. Solve for \(\pi^{t} \leftarrow \text{vUCQVI}(T, G^{(1)}, G^{(2)}, \ldots, H^{(1)}, H^{(2)}, \ldots); \ell^{t} \leftarrow t;\)
10. **While** \(t_{c}\) unchanged:
11. For \(t = t^{t} + 1, t^{t} + 2, \ldots, t^{t} + H;\) /*enforcing that a policy is not modified in an \(H\)-long episode.*/
12. At state \(s^{t}\) play \(a^{t} = \pi^{t}(h, s^{t})\), obtain a reward \(r^{t}\) and the next state \(s^{t+1};\)
13. **If:** \(|\ell(s^{t}, a^{t})| < c_{0}2^{\ell(s^{t}, a^{t})}, \text{ Then: } G^{(\ell(s^{t}, a^{t}))} \leftarrow G^{(\ell(s^{t}, a^{t}))} \circ \{(s^{t}, a^{t}, s^{t+1}, r^{t})\} \quad \text{else: } H^{(\ell(s^{t}, a^{t}))} \leftarrow H^{(\ell(s^{t}, a^{t}))} \circ \{(s^{t}, a^{t}, s^{t+1}, r^{t})\};\)
14. **If:** \(|\ell(s^{t}, a^{t})| = c_{1}LH^{3}, \text{ Then: } \ell(s^{t}, a^{t}) \leftarrow \ell(s^{t}, \tilde{a}) + 1;\)
15. **If** some \(\ell(s, a)\) changes \(t_{c} \leftarrow t_{c} + 1; \)//*ready for re-resolving the MDP model*/
16. \(t \leftarrow t + H;\)

5 Analysis of vUCQVI

We prove in this section that our algorithm in the last section achieves an \(O(H \sqrt{|S||A|T} + H^{4}|S||A|)\) expected regret. This regret bound is not optimal yet. We will further tighten the analysis and prove a
Algorithm 2 vUCQVI: The Variance-reduced Successive Refinement Value Iteration Algorithm

\textbf{vUCQVI}(T, \mathcal{G}^{(1)}, \mathcal{G}^{(2)}, \ldots, \mathcal{H}^{(1)}, \mathcal{H}^{(2)}, \ldots):

1. For each \((s, a)\): let \(\ell^*(s, a)\) be the largest \(\ell\) such that \(|\mathcal{G}^{(\ell(s,a))}| = c_0 2^{\ell}\) and \(|\mathcal{H}^{(\ell(s,a))}| = c_1 H^3\);
2. Let \(\ell^* = \max_{s,a} \ell^*(s,a)\);
3. For each \(h \in [H + 1]\) and \((s,a) \in \mathcal{S} \times \mathcal{A}\): \(\mathcal{V}_{h,s,a}^{(0)} \leftarrow H - h + 1, Q_h^{(0)}(s,a) \leftarrow H - h + 1\);
4. For each \((h, j) \in [H] \times [\ell^*]\): Estimate \(\tilde{P}_h^{(j)}(\cdot|s,a), \tilde{P}_h^{(j)}(\cdot|s,a)\) for all \((s,a)\) using (1);
5. For \(j = 1, 2, \ldots, \ell^*\):
   \(\triangleright\) solve the \(\ell^*\) levels of models successively
6. \(V_h^{(j)} \leftarrow 0, \mathcal{V}_{h+1,s,a}^{(j)} \leftarrow 0\) for all \((s,a)\);
7. For \(h = H, H - 1, \ldots, 1\) \(\triangleright\) dynamic programming for \(H\) steps
   For \((s,a) \in \mathcal{S} \times \mathcal{A}\):
   \(\triangleright\)
9. If \(j \leq \ell^*\) Then
   10. \(\tilde{P}_h^{(j)}(\cdot|s,a) \leftarrow \tilde{P}_h^{(j)}(\cdot|s,a)\) and \(\tilde{P}_h^{(j)}(\cdot|s,a) \leftarrow \tilde{P}_h^{(j)}(\cdot|s,a)\), and \(\mathcal{V}_{h+1,s,a}^{(j)} \leftarrow V_h^{(j)}\);
   11. Else: \(\tilde{P}_h^{(j)}(\cdot|s,a) \leftarrow \tilde{P}_h^{(\ell^*(s,a))}(\cdot|s,a), \tilde{P}_h^{(j)}(\cdot|s,a) \leftarrow \tilde{P}_h^{(\ell^*(s,a))}(\cdot|s,a), \mathcal{V}_{h+1,s,a}^{(j)} = V_h^{(\ell^*(s,a)-1)}\);
12. Estimate \(Q_h^{(j)}(s,a)\) using (4);
13. \(V_h^{(j)}(s) \leftarrow \max_{a \in \mathcal{A}} Q_h^{(j)}(s,a)\) for all \(s \in \mathcal{S}\);
14. Let \(\pi(s,h) \leftarrow \arg\max Q_h^{(\ell^*)}(s,a)\) for all \((s,h) \in \mathcal{S} \times [H]\);
15. Return \(\pi\);

Sharper regret bound in Section 6. Since our underlying model is an \(H\)-horizon MDP, we denote \(\pi_k\) as the policy at the \(k\)-th episode.

Theorem 5.1. Let \(T > 0\) as a parameter. At any time \(t = KH \leq T\), Algorithm 1 achieves regret bound

\[ \text{Regret}(t) = O(H \sqrt{|S||A|L^3}) + O(|S||A|L^{5/2}H^4) \]

where \(L = O(\log |S||A|HT)\).

Before we present the formal proof Theorem 5.1, we present the core lemmas. We begin with some definitions that are crucial to our proof. At any given time \(t \leq T\), suppose we have collected samples \(\mathcal{G}^{(1)}, \mathcal{G}^{(2)}, \ldots\) and \(\mathcal{H}^{(1)}, \mathcal{H}^{(2)}, \ldots\.). Let \(\mathcal{L} = \{\ell^*(s,a) : (s,a) \in \mathcal{S} \times \mathcal{A}\}\) be the set of full levels for each \((s,a)\). Let \(\ell^* = \max_{\ell \in \mathcal{L}} \ell\). Suppose we run the algorithm vUCQVI (Algorithm 2) using these random samples. At at level \(j \in [\ell^*]\), we define the following random variables recursively. Denote \(\mathcal{V}^{(j)}_{H+1} = 0\) and for \(h = H, H - 1, \ldots, 1\), if \(0 \leq j \leq \ell^*(s,a)\), we set

\[ Q_h^{(j)}(s,a) := r(s,a) + P(\cdot|s,a)\mathcal{V}^{(j)}_{h+1} \]

if \(j > \ell^*(s,a)\), we set

\[ \tilde{Q}_h^{(j)}(s,a) := \min \left[r(s,a) + \tilde{P}_h^{(\ell^*(s,a))} V_{h+1}^{(\ell^*(s,a)-1)} + \tilde{P}_h^{(\ell^*(s,a))} \left(\mathcal{V}^{(j)}_{h+1} - V_{h+1}^{(\ell^*(s,a)-1)}\right) + \tilde{P}_h^{(\ell^*(s,a))} Q_{h+1}^{(j-1)}(s,a)\right] \]

and

\[ \tilde{V}^{(j)}_{H+1}(s) := \max_{a \in \mathcal{A}} \tilde{Q}_h^{(j)}(s,a) \]

Note that \(Q_h^{(0)}(s,a) = Q_h^*(s,a)\) for any \((s,a)\) and \(h \in [H]\). Next we show that, with high probability, our estimate of the value and \(Q\)-functions at any stage are overestimate of the optimal value and \(Q\) function. Moreover, in each stage, the estimation become closer to the random vectors described in (6), (7) and (8).

Lemma 1 (Bound on Improvement). At time \(t \leq T\), let \(\mathcal{L} = \{\ell^*(s,a) : (s,a) \in \mathcal{S} \times \mathcal{A}\}\) be the set of full levels for each \((s,a)\). Let \(\ell^* = \max_{\ell \in \mathcal{L}} \ell\). Let \(\tilde{Q}_h^{(j)}\) and \(\tilde{V}_h^{(j)}\) be defined in (6), (7) and (8) and let \(Q_h^{(j)}\) and \(V_h^{(j)}\) be the estimated \(Q\)-function and value function at level \(j\) and stage \(h\) in the algorithm vUCQVI. Then with probability at least \(1 - \delta/(|S|^3|A|^{3H^3}T^3)\), we have, for all \(j \in \{1, 2, 3, 4, \ldots, \ell^*\}\), for all \(h \in [H]\) and \((s,a) \in \mathcal{S} \times \mathcal{A}\):

1. \(Q_h^{(s,a)} \leq \tilde{Q}_h^{(j-1)}(s,a) \leq Q_h^{(j)}(s,a) \leq Q_h^*(s,a) + (H - h + 1)^2 \sqrt{L/2^{\ell^*-1}}\),
2. \(V_h^{(s)} \leq \tilde{V}_h^{(j-1)}(s) \leq V_h^{(j)}(s) \leq V_h^*(s) + (H - h + 1)^2 \sqrt{L/2^{\ell^*-1}}\),
3. \(V_h^{(s)} \leq V_h^{(j-1)}(s) \leq V_h^*(s) + (H - h + 1)^2 \sqrt{L/2^{\ell^*-1}}\),
4. $\tilde{P}^{(j)}(\cdot|s,a)\top \bar{V}_{h+1,s,a} + \bar{P}_{h}^{(j)}(\cdot|s,a)\top (V_{h+1}^{(j)} - \bar{V}_{h+1,s,a}) - \bar{P}(\cdot|s,a)\top V_{h+1}^{(j)} \leq (H - h)\sqrt{\frac{\log(|S||A|\sqrt{H}/\delta)}{2\min(n(k,s,a))}}/2,$

where $L = c_3 \log(|S||A|\sqrt{H}/\delta)$ for some sufficiently large constant $c_3$.

The proof of this lemma is by heavy machinery of induction analysis. The first two points of the lemma indicate that the random vectors $Q^*$ and $V^*$ are always upper bounds on the optimal value and lower bound of the outputs of the algorithm at each iteration. The third point indicate the monotonicity property, i.e., the value is always improving over each refinement. The last point indicates the variance reduction property, i.e., although $\bar{P}_h^{(j)}$ contains less samples than that of $\tilde{P}^{(j)}$, the estimation error is on the same order. This property allows us to use the majority of samples collected so far to bound the regret, while only paying a small constant term (sourced from these $H_h^{(j)}$ buckets) in the regret to resolve the independence issue. We postponped our proof to the appendix (Section C).

We are now ready to present the proof sketch of Theorem 5.1. The full proof is postponed to the appendix (Section C).

Proof Sketch of Theorem 5.1. The proof of this theorem depends on several parts. We denote $t_{c}(k)$ the number that policy $\pi^{c}(k)$ is used at episode $k$. We denote $t^{*}(k)$ as the largest $t^{*}(s,a)$ until episode $k$. As a first step, which is quite standard (see e.g. [AOM17]), we show that the regret at episode $k$ can be decomposed as

$$\text{Regret}_k \leq \mathbb{E}\left[\sum_{h=1}^{H} (\bar{Q}^{(h)}(s,h) - r(s,h) - \bar{P}(\cdot|s,h)\top \bar{V}_{h+1}^{(h)}) + e_h^{(k)}\right],$$

where $\bar{Q}^{(h)}(s,h) = Q(t^{*}(k))$, $s,a$ and $\bar{V}_{h+1}^{(h)} = V_{h+1}^{(h)}$ denote the corresponding $Q$-function and value function at the episode $k$. Here $e_h^{(k)}$ is a mean-0 random variable and thus only contributes 0 to the final regret bound. It remains to bound $|\bar{Q}^{(h)}(s,a) - r(s,a) - P(\cdot|s,a)\top \bar{V}_{h+1}^{(h)}|$. We will show that at each $(s,a)$, $|\bar{Q}^{(h)}(s,a) - r(s,a) - P(\cdot|s,a)\top \bar{V}_{h+1}^{(h)}| \leq b_k^{(h,s,a)}$ for all $j,h$ and $(s,a)$. We then show that each $b_k^{(h,s,a)} = O(H/\sqrt{2\min(n(k,s,a))}) \approx O(H/\sqrt{n(k,s,a)})$, where $n(k,s,a)$ is the number of visits to $(s,a)$ before episode $k$. Once we have shown $b_k^{(h,s,a)} \leq O(H/\sqrt{n(k,s,a)})$, the regret bound follows by simply taking a sum over all $(s,a)$ and making use of the pigeon-hole principle.

6 Achieving Nearly-Optimal Regret via Bernstein Technique

Finally we prove the near-optimal regret bound of Theorem 3.1 in this section. To improve the regret bound, we will modify Algorithms 1, 2, by changing the bucket sizes of $H^{(1)}, H^{(2)}, \ldots$, and the bonus function. We will use a smaller number of samples to precisely control the accumulated error, based on the observation that the error accumulation in the sequential process is actually much smaller than the previous upper bounds we used. In particular, we will use a Bernstein technique to augment the previous analysis, which involves using an iterative variance argument and a Bernstein inequality. This technique has also been used in [AMK13].

We first take $|H^{(j)}| = c_1 LH^3$, where $L = c_3 \log(|S||A|\sqrt{H}/\delta)$ for some large constant $c_3$ and error probability parameter $\delta \in (0,1)$ and integer $\beta \geq 4$. To set our bonus function, we first estimate the variance of our reference vector at each $(j,h,s,a)$ as

$$\tilde{\sigma}^{(j)}(s,a) = \tilde{P}^{(j)}(\cdot|s,a)\top \bar{V}_{h+1,s,a} - [\tilde{P}^{(j)}(\cdot|s,a)\top \bar{V}_{h+1,s,a}]^2.$$

Then we set our bonus function as

$$b^{(j)}_{h,s,a}(k) = \min\left[b^{(j)}_{1,h,s,a}(k), b^{(j)}_{2,h,s,a}(k)\right],$$

where $b^{(j)}_{h,s,a}(k)$ is the previous bonus function given in (5) and

$$b^{(j)}_{1,h,s,a}(k) = \frac{1}{4} \sqrt{\tilde{\sigma}^{(j)}(s,a) \cdot L} \cdot \frac{H - h}{4} \cdot \frac{L}{2\min(\ell^{*}(k,s,a))}$$

$$+ \frac{H - h}{4} \cdot \left(\frac{L}{2\min(\ell^{*}(k,s,a))}\right)^{3/4} + (H - h)^{(5 - \beta)/2} / 4 \cdot \sqrt{\frac{L}{2\min(\ell^{*}(k,s,a))}},$$

(11)

(10)
where the first two terms come from Bernstein inequality, the third term comes from the error of the empirical estimate of the variance and the last term comes from the estimate of $P(\cdot | s, a)^\top (V_h^{(j)} - V_{h,s,a}^{(j)})$. Next by an analog lemma of Lemma 1 with the newly chosen bonus function and $H$ size, Lemma 5 (presented in the Appendix C), we have the following theorem.

**Theorem 6.1.** Let $T$ be a parameter. Then there exists a reinforcement learning algorithm, acting in the world of an $H$-episodic MDP, starting from any initial state, at any time $t = KH \leq T$, achieving regret bound

$$\text{Regret}(t) = O\left( E\left[H^{(9-\beta)|S|(|A|t + |S|)|A|H^{3+1}L^3 + H^{3/2}L^{3/4}(|S||A|t)^{1/4}}\right] \right)$$

where $L = O(\log |S||A|HT)$, $\beta \geq 3$ is a constant, and $\sigma_h^k(s, a) = P(\cdot | s, a)^\top (V_h^k)^2 - [P(\cdot | s, a)^\top V_h^k]^2$.

The proof is similar to that of Theorem 6.1, but with more heavy machinery of bounding the variance estimations. We postpone the proof to the appendix (Section C). As a direct corollary of Theorem 6.1, we can prove the asymptotic tight regret bounds with a small burn-in time as given in Theorem 3.1.

**Proof of Theorem 3.1.** We will be applying the law of total variance (see e.g., [MM02, AOM17]) to bound

$$E\left[\sum_{h=1}^{H} \sigma_h^k(s_h^{(k)}, a_h^{(k)})\right] \leq H^2.$$

The rest of proof follows from choosing $\beta = 4$. ☐

Next we show that our lemma can also be applied to the communicating MDP with a worst case diameter $D$ (in this case the diameter induced by any policy $\pi$ is upper bounded by $D$). Note that we still require the underlying world to be an $H$-episodic MDP.

**Corollary 6.1.1.** Let $T$ as a parameter. Denote an $H$-horizon MDP $M$ with worst case diameter $D$. Then there exists an reinforcement learning algorithm, acting in the world of $M$, starting from any initial state, at any time $t = KH \leq T$, achieving regret bound

$$\text{Regret}(t) = O\left( LD\sqrt{|S||A||t + |S||A|H^5L^3}\right)$$

where $L = O(\log |S||A|HT)$.

**Proof.** We will be applying the same reasoning as used in [JOA10]. We argue that for any $\pi^k$, the a value function, $V_h^\pi$, satisfies, $\max_{s, s'} \in \mathcal{S} |V_h^\pi(s) - V_h^\pi(s')| \leq D$. Therefore, $\sigma_h^k(s_h^{(k)}, a_h^{(k)}) \leq D^2$ for all $h$ and $k$. Hence the lemma follows by a direct calculation. ☐

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A The Dependence Issue of RL Algorithms

As pointed out in [SL08], a sequence of samples obtained from a fixed state action pair \((s,a)\) cannot be treated as independent samples from the distribution \(P(\cdot|s,a)\). We can illustrate this case with the following three-state MDP. Suppose at each state, there is only one action. From state \(s_1\), there is 1/2 transition probability to \(s_2\) and 1/2 transition probability to \(s_3\). \(s_2\) always goes back to \(s_1\) and \(s_3\) self-loops forever. As such, one can predicts the sequence from \(s_1\) is always the form \(s_2,s_2,\ldots,s_2,s_3\). The longer the sequence, the worse the probability estimate of \(P(\cdot|s_1)\) is. However, as one might observe, long sequence from \(s_1\) can happens with very small probability, i.e., with probability at most \(2^{-t}\), where \(t\) is the length of the sequence. Thus a “bad” event happens with very small probability. [SL08] gives a formal analysis to justify that one can replace the samples at a fixed state-action pair with a sequence of independent samples to bound the probability of some trajectory. They do so by arguing that for any sequence of length \(m\), an algorithm can obtain such a sequence with probability at most that of obtaining this sequence by independent draws. Therefore, we can bound the probability of “bad” events happening by replacing samples with independent samples. Their formal lemma is presented as follows.

**Lemma 2** ([SL08], Claim C1). Consider an execution of a learning algorithm on an MDP. For a fixed state-action pair \((s,a)\), the probability that the sequence \(Q\) is observed by the learning agent (meaning that \(m\) experiences of \((s,a)\) do occur and each next-state and immediate reward observed after experiencing \((s,a)\) matches exactly the sequence in \(Q\)) is at most the probability that \(Q\) is obtained by a process of drawing \(m\) random next-states and rewards from distributions \(P(\cdot|s,a)\), respectively.

Based on the above lemma, we prove the following critical concentration lemma.

**Lemma 3.** Suppose we are running a deterministic reinforcement learning algorithm \(\mathcal{K}\). Suppose at some fixed time \(T\), we have collected a trajectory \(Q\) of length \(T\). For any state-action pair \((s,a)\), let \(Q_{t_1:t_2}^{s,a}\) be the set of samples collected from \((s,a)\) between arrival time \(t_1\) and \(t_2\) \((0 < t_1 < t_2)\). Let \(N(s,a)\) be the total number of samples collected at \((s,a)\). Let \(\hat{P}_{t_1:t_2}(\cdot|s,a)\) be the empirical estimator of the probability vector \(P(\cdot|s,a)\) using the samples from \(Q_{t_1:t_2}^{s,a}\). Let \(V := f(Q\setminus Q_{t_1:t_2}^{s,a})\) denote a random vector computed using samples other than these from \(Q_{t_1:t_2}^{s,a}\), where function \(f\) is a fixed function. Consider the event \(E_{t_1,t_2}\) defined as follows.

\[
E_{t_1,t_2} := \left\{ Q : N(s,a) \geq t_2 \text{ and } \|\hat{P}_{t_1:t_2}(\cdot|s,a)^TV - P(\cdot|s,a)^TV\|_{\infty} > H/\sqrt{\log(|S||A|T\delta^{-1})/(t_2 - t_1)} \right\},
\]

where we note that \(\|\hat{P}_{t_1:t_2}(\cdot|s,a)^TV - P(\cdot|s,a)^TV\|_{\infty}\) is a deterministic function on the samples \(Q\). Then

\[
\mathbb{P}[E_{t_1,t_2}] \leq \delta/\text{poly}(|S||A|T).
\]

**Proof.** content...

B Auxiliary Lemmas

**Theorem B.1** (Hoeffding Inequality). Let \(\delta \in (0, 1)\) be a parameter. Let \(p \in \Delta_d\) be a \(d\)-dimensional probability vector. Let \(s_1, s_2, \ldots, s_m\) be \(m\) i.i.d. samples from distribution \(p\). Let \(V \in \mathbb{R}^d\) be a \(d\)-dimensional vector. Then with probability at least \(1 - \delta\),

\[
\left| \frac{1}{m} \sum_{i=1}^{m} V(s_i) - p^TV \right| \leq \|V\|_\infty \cdot \sqrt{\frac{2 \log \frac{2}{\delta}}{m}}.
\]

**Theorem B.2** (Bernstein Inequality). Let \(\delta \in (0, 1)\) be a parameter. Let \(p \in \Delta_d\) be a \(d\)-dimensional probability vector. Let \(s_1, s_2, \ldots, s_m\) be \(m\) i.i.d. samples from distribution \(p\). Let \(V \in \mathbb{R}^d\) be a \(d\)-dimensional vector. Then with probability at least \(1 - \delta\)

\[
\left| \frac{1}{m} \sum_{i=1}^{m} V(s_i) - p^TV \right| \leq \sqrt{\frac{2 \text{Var}[V(s')] \log \frac{2}{\delta}}{m}} + \frac{2\|V\|_\infty \log \frac{2}{\delta}}{3m}.
\]
Induction on $L$ difference sequence with respect to $j$ proving (1). Since proving (2). Also $= 1$
these failure probabilities, we apply Lemma by bounding the probability of the complement of each event and the n applying a union bound. To bound

C. Missing Proofs from Section 5

Proof of Lemma 1. For simplicity, we denote $Q_1^{-1}(s, a) = H - h + 1$ and $Q_t^{-1}(s, a) = Q_t(s, a)$ for any $h, s, a$. We prove the lemma by induction on $j$.

Base case: $j = 0$. We first prove that the lemma holds for $j = 0$ deterministically, then we show higher levels also hold with high probability. In this case, $Q_0(s, a) = H - h + 1$ for any $h, s, a$ and $Q_H(s, a) = Q_H(s, a)$. Hence

$$Q_h(s, a) \leq Q_0(s, a) \leq Q_h(s, a) \leq Q_0(s, a) \leq Q_H(s, a) + (H - h + 1)^2 \sqrt{2L},$$

proving (1). Since $V_h(s, a) = \max_a Q_h(s, a) = V_h(s)$, we obtain

$$V_h(s) \leq V_h(s) \leq V_h(s) \leq V_h(s) \leq V_h(s) + (H - h + 1)^2 \sqrt{2L},$$

proving (2). Also $V_h(s) \leq V_1(s) \leq V_h(s) + (H - h + 1)^2 \sqrt{L}$ proves (3). Since $\tilde{P}(0) = 0$ and $\tilde{P}_h(0) = 0$
for all $h$, the last bullet of the lemma statement is also proved. Thus the base case is proved.

Induction Hypothesis: In the rest of the paragraph, we will define a sequence of events (for instance $E^{(1)}, E^{(2)}, \ldots$, which will become clear shortly). We will show that these events happen with high probability by bounding the probability of the complement of each event and then applying a union bound. To bound these failure probabilities, we apply Lemma ?? and assume the samples up to time $t$ are all independently sampled from each $P(\cdot|s, a)$.

For the rest of the proof, suppose there exist events $E^{(1)}, E^{(2)}, \ldots, E^{(j-1)}$, on which the lemma statement holds until $j - 1$. And that $P[E^{(1)} \cap E^{(2)} \ldots \cap E^{(j-1)}] \geq 1 - (j - 1)\delta/(|S|A^|H^3T^4|)$.

Induction on $j$: Next, we define an event $E^{(j)}$ as that, for all $h \in [H], (s, a) \in S \times A$,

$$|\tilde{P}(j)(s, a) - P(s, a)| |\nabla_{h,s,a}^{(j)} (V_{h+1}^{(j)} - V_{h+1,a}^{(j)}) | \leq (H - h + 1)^2 \sqrt{L}/2^{2\min(j,\ell^*(s, a))}/4,$$

where $L = c_3 \log(\max(|S|AHT))$ for some sufficiently large constant $c_3$. By Hoeffding bound (Theorem B.1), we have $P[E^{(j)}] \geq 1 - |S|A \cdot \delta/(4|S^4|A^4H^3T^4)$, provided sufficiently large constant $c_3$. Next, for each $h = 1, 2, \ldots, H$, we denote event $E_h^{(j)}$ as that for all $(s, a)$,

(i) $\left[A_1(j, s, a) - P(s, a)A_2(j, h+1, s, a) \right] \leq (H - h)\sqrt{L}/2^{2\min(j,\ell^*(s, a))}/4$;

(ii) $\left[A_1(j, s, a) - P(s, a)A_2(j, h+1, s, a) \right] \leq (H - h)\sqrt{L}/2^{2\min(j,\ell^*(s, a))}/4$;

(iii) $Q_h^*(s, a) \leq \tilde{Q}_h^{(j-1)}(s, a) \leq Q_h^*(s, a) \leq \tilde{Q}_h^{(j)}(s, a) \leq (H - h + 1)^2 \sqrt{L}/2^j$;

(iv) $V_h^*(s, a) \leq \tilde{V}_h^{(j-1)}(s, a) \leq V_h^*(s, a) \leq \tilde{V}_h^{(j)}(s, a) \leq (H - h + 1)^2 \sqrt{L}/2^j$.

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Note that if (i)-(iv) happen for all \( h = 1, 2, \ldots, H \), then for the statement of the lemma, (1) and (2) follow directly from (iii) and (iv); (3) follows from

\[
V_h^{(j-1)}(s) \leq \tilde{V}_h^{(j-1)}(s) + (H - h + 1)^2 L/2^{j-1} \leq V_h^{(j)}(s) + (H - h + 1)^2 L/2^{j-1};
\]

and (4) follows from (12) and (i).

As we have shown, \( \mathcal{E}^{(j)}_H \) happens with probability 1. We will show that if \( \mathcal{E}^{(j')}, \mathcal{E}^{(j)}_H, \mathcal{E}^{(j)}_E, \mathcal{E}^{(j)}_h, \mathcal{E}^{(j)}_{h-1}, \ldots, \mathcal{E}^{(j)}_{h+1} \) happen, then \( \mathcal{E}^{(j)}_H \) happens with probability at least \( 1 - \delta/(4|S|^4|A|^4H^4T^4) \).

We first note that the random vector \( \tilde{P}_h^{(j)}(\cdot|s,a) \) is independent with \( \mathcal{E}^{(j')}, \mathcal{E}^{(j)}_H, \mathcal{E}^{(j)}_E, \mathcal{E}^{(j)}_h, \mathcal{E}^{(j)}_{h-1}, \ldots, \mathcal{E}^{(j)}_{h+1} \).

Therefore, by Hoeffding bound (Theorem B.1), conditioning on \( \mathcal{E}^{(j)}_{h+1} \), with probability at least \( 1 - \delta/(4|S|^4|A|^4H^4T^4) \),

\[
\|\tilde{P}_h^{(j)}(\cdot|s,a) - P(\cdot|s,a)\|_1 = \|V_h^{(j)}(s,a) - V_h^{(j)}(s,a)\|_1 \leq \|V_h^{(j)} - V^{(j)}_h(s,a)\|_1 \leq \sqrt{L/(c_1LH^2)} \quad \text{and}
\]

\[
\|\tilde{P}_h^{(j)}(\cdot|s,a) - P(\cdot|s,a)\|_1 = \|V_h^{(j)} - V^{(j)}_h(s,a)\|_1 \leq \sqrt{L/(c_1LH^2)},
\]

for appropriately chosen constants \( c_1 \) and \( c_2 \). We condition on this event. Now we do case analysis on \( j \).

**Case \( \ell^*(s,a) < j \):** We then consider the case that \( \ell^*(s,a) < j \). We first have \( \tilde{V}_{h+1,s,a}^{(\ell^*(s,a)-1)} = V_h^{(\ell^*(s,a)-1)} \). Thus we have, by induction hypothesis, on \( \mathcal{E}^{(0)}, \ldots, \mathcal{E}^{(j-2)}, \mathcal{E}^{(j-1)} \) and \( \mathcal{E}^{(j')}, \mathcal{E}^{(j)}_H, \mathcal{E}^{(j)}_E, \mathcal{E}^{(j)}_h, \mathcal{E}^{(j)}_{h-1}, \ldots, \mathcal{E}^{(j)}_{h+1} \),

\[
\|V_h^{(\ell^*(s,a)-1)} - V_h^{(j)}\|_1 \leq \sum_{j' = \ell^*(s,a)}^j \|V_h^{(j'-1)} - V_h^{(j)}\|_1 \leq (H - h)^2 L/2^{j-1} \leq 5(H - h)^2 L/2^{\ell^*(s,a)}.\]

Since \( \|V_h^{(j)} - \tilde{V}_{h+1}^{(j)}\|_1 \leq (H - h)^2 \sqrt{L/2^j} \), we have,

\[
\|V_h^{(\ell^*(s,a)-1)} - \tilde{V}_{h+1}^{(j)}\| \leq 6(H - h)^2 \sqrt{L/2^{\ell^*(s,a)}}.
\]

Hence, we obtain

\[
\|\tilde{P}_h^{(j)}(\cdot|s,a) - P(\cdot|s,a)\|_1 = \|V_h^{(j)}(s,a) - V_h^{(j)}(s,a)\|_1 \leq (H - h) \sqrt{L/2^{\ell^*(s,a)}}/4 \quad \text{and}
\]

\[
\|\tilde{P}_h^{(j)}(\cdot|s,a) - P(\cdot|s,a)\|_1 = \|V_h^{(j)} - V_h^{(j)}(s,a)\|_1 \leq (H - h) \sqrt{L/2^{\ell^*(s,a)}}/4,
\]

for sufficiently large constant \( c_1 \), proving (i) and (ii).

Additionally, we have

\[
\bar{Q}_h^{(j)}(s,a) := \min \left[ r(s,a) + \bar{P}(\ell^*(s,a)) \mathbb{E} V_h^{(\ell^*(s,a)-1)} + \bar{P}_h^{(\ell^*(s,a))} \mathbb{E} \left[ \tilde{V}_{h+1}^{(j)}(s,a) \right] \right],
\]

and

\[
Q_h^{(j)}(s,a) := \min \left[ r(s,a) + \bar{P}(\ell^*(s,a)) \mathbb{E} V_h^{(\ell^*(s,a)-1)} + \bar{P}_h^{(\ell^*(s,a))} \mathbb{E} \left[ \tilde{V}_{h+1}^{(j)}(s,a) \right] \right],
\]

Since on \( \mathcal{E}^{(j)}_{h+1}, \tilde{V}_{h+1}^{(j)} \leq V_h^{(j)} \leq \tilde{V}_{h+1}^{(j)} + (H - h)^2 \sqrt{L/2^j} \), we thus have, \( \bar{Q}_h^{(j)}(s,a) \leq Q_h^{(j)}(s,a) \) and

\[
Q_h^{(j)}(s,a) - \bar{Q}_h^{(j)}(s,a) \leq \tilde{V}_{h+1}^{(j)} - \tilde{V}_{h}^{(j)} \leq (H - h)^2 \sqrt{L/2^j}.
\]

To show that \( \bar{Q}_h^{(j-1)}(s,a) \leq \bar{Q}_h^{(j)}(s,a) \), we additionally consider two cases: \( j - 1 > \ell^*(s,a) \) and \( j - 1 < \ell^*(s,a) \). If \( j - 1 > \ell^*(s,a) \), then we have

\[
\bar{Q}_h^{(j-1)}(s,a) := \min \left[ r(s,a) + \bar{P}(\ell^*(s,a)) \mathbb{E} V_h^{(\ell^*(s,a)-1)} + \bar{P}_h^{(\ell^*(s,a))} \mathbb{E} \left[ \tilde{V}_{h+1}^{(j-1)}(s,a) \right] \right],
\]
Since $\tilde{V}_{h+1}^{s,j-1} \leq \tilde{V}_{h+1}^{s,j}$, we have
\[ Q_h^*(s,a) \leq \tilde{Q}_h^{*(j-1)}(s,a) \leq \tilde{Q}_h^{*(j)}(s,a), \]  
proving (iii) and (iv).

Next we consider $j-1 \leq \ell^*(s,a)$. In this case,
\[ \tilde{Q}_h^{*(j-1)}(s,a) := r(s,a) + P(\cdot|s,a)^\top \tilde{V}_{h+1}^{*(j-1)}. \]

Thus, with (12), we have
\[
\tilde{Q}_h^{*(j)}(s,a) = \min \left[ r(s,a) + \tilde{P}^{(j)}(s,a)V_{h+1}^{(j)} - P(\cdot|s,a) \left[ V_{h+1}^{(j)} - V_{h+1}^{(j-1)} \right] + b_h^{(j)}(s,a), Q_{h+1}^{(j-1)}(s,a) \right]
\]
\[
\leq r(s,a) + P(\cdot|s,a)^\top \tilde{V}_{h+1}^{*(j-1)} + b_h^{(j)}(s,a)
\]
\[
\leq (H-h)\sqrt{L/2^{\ell^*(s,a)/2}}, \]
Since $b_h^{(j)}(s,a) = (H-h)\sqrt{L/2^{\ell^*(s,a)/2}}/2$, we have
\[ Q_h^*(s,a) \leq \tilde{Q}_h^{*(j-1)}(s,a) = r(s,a) + P(\cdot|s,a)^\top \tilde{V}_{h+1}^{*(j)} \leq \tilde{Q}_h^{*(j)}(s,a). \]  

In summary, with (14), (15) and (16), we have
\[ Q_h^*(s,a) \leq \tilde{Q}_h^{*(j-1)}(s,a) \leq \tilde{Q}_h^{*(j)}(s,a) \leq Q_h^{*(j)}(s,a) \leq (H-h)^2\sqrt{L/2^j}, \]  
proving (iii) and (iv).

**Case $\ell^*(s,a) > j$:** We then consider the case that $\ell^*(s,a) \geq j$. In this case, $V_{h+1}^{(j)} = V_{h+1}^{(j-1)}$. Additionally, by the algorithm definition, we have $V_{h+1}^{(j)} \leq V_{h+1}^{(j-1)}$. Applying the induction hypothesis, we have $\| V_{h+1}^{(j-1)} - \tilde{V}_{h+1}^{*(j-1)} \|_\infty \leq (H-h)^2\sqrt{L/2^{j-2}}$. And on $E_{h+1}^{(j)}$, we have
\[ \tilde{V}_{h+1}^{*(j-1)} \leq V_{h+1}^{(j)} \leq V_{h+1}^{(j)} + (H-h)^2\sqrt{L/2^j}. \]
Thus,
\[ \| V_{h+1}^{(j)} - \tilde{V}_{h+1}^{*(j)} \|_\infty \leq \| V_{h+1}^{(j-1)} - \tilde{V}_{h+1}^{*(j)} \|_\infty \leq (H-h)^2 \cdot \sqrt{L/2^{j-2}}, \]
We thus have
\[ \| V_{h+1}^{(j)} - \tilde{V}_{h+1}^{*(j)} \|_\infty \cdot \sqrt{1/(c_1 H^2)} \leq (H-h)\sqrt{L/2^j}/4 \]
provided sufficiently large $c_1$. With (12), we prove (i) and (ii). Together with the event $E_{h+1}^{(j)}$, we have
\[ |P(\cdot|s,a)^\top \tilde{V}_{h+1}^{(j)} + \tilde{P}(\cdot|s,a)^\top (V_{h+1}^{(j)} - \tilde{V}_{h+1}^{(j)}) - P(\cdot|s,a)^\top V_{h+1}^{(j)}| \leq (H-h)\sqrt{L/2^j}/2, \]
and
\[ Q_h^{(j)}(s,a) = r(s,a) + \tilde{P}(\cdot|s,a)^\top \tilde{V}_{h+1}^{(j)} + \tilde{P}(\cdot|s,a)^\top (V_{h+1}^{(j)} - \tilde{V}_{h+1}^{(j)}) + b_h^{(j)}(s,a)
\]
\[ = r(s,a) + P(\cdot|s,a)^\top V_{h+1}^{(j)} + b_h^{(j)}(s,a) \pm (H-h)\sqrt{L/2^j}/4 \pm (H-h)\sqrt{L/2^j}/4 \]
Denote $\tilde{Q}_h^{(j)}(s,a) := r(s,a) + P(\cdot|s,a)^\top V_{h+1}^{(j)}$. Since $b_h^{(j)}(s,a) = (H-h)\sqrt{L/2^j}/2$ we have
\[ \tilde{Q}_h^{(j)}(s,a) \leq Q_h^{(j)}(s,a) \leq \tilde{Q}_h^{(j)}(s,a) + (H-h)\sqrt{L/2^j}. \]
Moreover, since \( V_{h+1}^* \leq \tilde{V}_h^{(j)}(s, a) \leq V_{h+1} \leq (H-h)^2 \sqrt{L/2^j} \), we have
\[
\tilde{Q}_h^{(j)}(s, a) \leq r(s, a) + P(s, a) \tilde{V}_{h+1}^{(j)} + (H-h)^2 \sqrt{L/2^j} = \tilde{Q}_h^{(j)}(s, a) + (H-h)^2 \sqrt{L/2^j},
\]
and thus,
\[
Q_h^{(j)}(s, a) \leq \tilde{Q}_h^{(j)}(s, a) = r(s, a) + P(s, a) \tilde{V}_h^* \leq \tilde{Q}_h^{(j)}(s, a)
\]
\[
\leq Q_h^*(s, a) + (H-h)^2 \sqrt{L/2^j} + (H-h)^2 \sqrt{L/2^j} \leq \tilde{Q}_h^{(j)}(s, a) + (H-h+1)^2 \sqrt{L/2^j},
\]
proving (iii) and (iv).

**Putting it together** By a union bound over all \( s, a \), \( \mathcal{E}_h^{(j)} \) happens with probability at least \( 1 - \delta/(|S|^3|A|^3H^3T^4) \). By a union bound over all \( h = H-1, H-2, \ldots, 1, \) we obtain, with probability at least, \( 1 - \delta/(4|S|^3|A|^3H^3T^4) \), all \( \mathcal{E}_1^{(j)}, \mathcal{E}_2^{(j)}, \ldots, \mathcal{E}_H^{(j)} \) happen. Denote \( \mathcal{E}^{(j)} = \mathcal{E}^{(j)} \cap \mathcal{E}^{(j)}_1 \cap \mathcal{E}^{(j)}_2 \cap \ldots \cap \mathcal{E}^{(j)}_H \), thus \( \mathbb{P}[\mathcal{E}^{(j)}] \geq 1 - \delta/(|S|^3|A|^3H^3T^4) \). Thus by a union bound, \( \mathcal{E}^{(1)}, \mathcal{E}^{(2)}, \ldots, \mathcal{E}^{(j)} \) all happen with probability at least \( 1 - j \cdot \delta/(|S|^3|A|^3H^3T^4) \). Thus the lemma statement holds for all \( j \in [\ell^*] \) with probability at least \( 1 - \delta/(|S|^3|A|^3H^3T^4) \). This completes the proof. \( \square \)

**Proof of Theorem 5.1.** For the rest of the proof, we denote \( \delta \in (0, 1) \) as a small constant. Denote \( t = KH \leq T \). At an episode \( k \), we denote \( t_c(k) \) as the number of calls of vUCQVI up to episode \( k \). Let \( \ell^*(k) \) be the highest full level over all state-action pairs before episode \( k \) (also see the definition in the statement of Lemma 1). Denote \( \pi(k) = \pi^{t_c(k)} \) and
\[
V_h^{(k)} = V_h^{(\ell^*(k))} \quad \text{and} \quad Q_h^{(k)} = Q_h^{(\ell^*(k))}
\]
in vUCQVI. Thus \( V_h^{(k)} \) and \( Q_h^{(k)} \) are the latest estimation of the value function and Q-function at episode \( k \). For the rest of the proof, we denote the following regret definition.
\[
\overline{\text{Regret}}(KH) = \sum_{k=1}^{K} [V_1^{(\ell^*(k))} - V_1^{(s_1^{(k)})}]. \tag{18}
\]
Thus we have
\[
\overline{\text{Regret}}(KH) = \mathbb{E}[\overline{\text{Regret}}(KH)], \tag{19}
\]
To show the regret at episode \( k \), we look at the following vector,
\[
\Delta_h^{(k)} = V_h^* - V_h^{(s_1^{(k)})} \quad \text{and} \quad \overline{\Delta}_h^{(k)} = V_h^{(\ell^*(k))} - V_h^{(s_1^{(k)})}.
\]
Denote \( \delta_h^{(k)} = \Delta_h^{(k)}(s_h^{(k)}) \), we then have
\[
\overline{\text{Regret}}(KH) = \sum_{k=1}^{K} \delta_h^{(k)}.
\]
Let \( \mathcal{E}_k \) be the event that Lemma 1 holds at episode \( k \). By Lemma 1, we have \( \mathbb{P}[\mathcal{E}_k] \geq 1 - \delta/(|S|^3|A|^3H^3T^3) \). On \( \mathcal{E}_k \), we have
\[
\forall h \in [H], s \in S, a \in A : \quad Q_h^*(s, a) \leq Q_h^{(k)}(s, a).
\]
Thus, on \( \mathcal{E}_k \), for all \( h \in [H] \),
\[
\Delta_h^{(k)} \leq \overline{\Delta}_h^{(k)}.
\]
Moreover, since \( \|\Delta_h^{(k)}\|_\infty \leq H \) and \( \|\overline{\Delta}_h^{(k)}\|_\infty \leq H \), we have
\[
\mathbb{E}[\Delta_h^{(k)}] \leq \mathbb{E}[\overline{\Delta}_h^{(k)}] + O(H^{-2}T^{-2}).
\]

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Next, we consider $\Delta_h^{(k)}$. Let $Q_h^{(k)}(s, a) = r(s, a) + P(\cdot | s, a)\top V_{h+1}^{(k)}$. By Lemma 1, on $\mathcal{E}_k$, we have for all $h \in [H], (s, a) \in \mathcal{S} \times \mathcal{A}$,

$$
\Delta_h^{(k)} = P\pi^k V_{h+1}^{(k)} - P\pi^k \hat{V}_{h+1}^{(k)} + b_h^{(k)}
$$

Thus $\Delta_h^{(k)} = \Delta_h^{(k)} + b_h^{(k)}$ for a random vector $b_h^{(k)} \in \mathbb{R}^{|\mathcal{S}|}$. Using the same notation as in Lemma 1, we have,

$$
Q_h^{(k)}(s, a) := Q_h^{(\ell^*(k))}(s, a) = \hat{r}(s, a) + \hat{P}(\ell^*(k))(\cdot | s, a)\hat{V}_{h+1,s,a}^{(k)} + \hat{P}(\ell^*(k))(\cdot | s, a)\hat{V}_{h+1,s,a}^{(k)} - \nabla_s \hat{V}_{h+1,s,a}^{(k)}.
$$

Then conditioning on $\mathcal{E}_k$, we have

$$
\forall s \in \mathcal{S} : \quad b_h^{(k)}(s) \leq (H - h)\sqrt{L/2\ell^*(k,\pi^k(s))}.
$$

Thus

$$
\forall s \in \mathcal{S} : \quad \mathbb{E}[b_h^{(k)}(s)] \leq \mathbb{E}[(H - h)\sqrt{L/2\ell^*(k,\pi^k(s))}] + O(H^{-2}\bar{T}^{-2}).
$$

Next, we define

$$
\delta_h^{(k)} = \hat{\Delta}_h^{(k)}(s_h^{(k)}).
$$

Then we have

$$
\delta_h^{(k)} := P(\cdot | s_h^{(k)}, \pi^k(s_h^{(k)}))\top \hat{\Delta}_h^{(k)} + b_h^{(k)}(s_h^{(k)})
$$

Let

$$
e_h^{(k)} = P(\cdot | s_h^{(k)}, \pi^k(s_h^{(k)}))\top \hat{\Delta}_h^{(k)} - \delta_h^{(k)}.
$$

We obtain,

$$
\delta_1^{(k)} = \sum_{j=1}^{H} e_h^{(k)}(s_h^{(k)}) + \sum_{j=1}^{H} b_j^{(k)}(s_h^{(k)}),
$$

where we denote $e_H^{(k)} = 0$. Conditioning on $\mathcal{E}_k$, we have

$$
\sum_{h=1}^{H} b_h^{(k)}(s_h^{(k)}) \leq \sum_{h=1}^{H} (H - h)\sqrt{L/2\ell^*(k,s_h^{(k)},\pi^k(s_h^{(k)}))} \leq H \sum_{h=1}^{H} \sqrt{L/2\ell^*(k,s_h^{(k)},\pi^k(s_h^{(k))})}.
$$

Let $N(k, s, a)$ be the number of observations of state action pair $(s, a)$ up to stage $k$. Notice that

$$
N(k, s, a) = \ell^*(k,s,a) = \sum_{\ell=0}^{\ell^*(k,s,a)} (c_02^\ell + c_1LH^3)
$$

thus we have $\ell^*(k,s,a) \leq \log N(k, s, a)$ and

$$
[N(k, s, a) - c_1 \log N(k, s, a) \cdot LH^3] / 2 \leq [N(k, s, a) - c_1 \ell^*(k, s, a) \cdot LH^3] / 2 \leq c_02^{\ell^*(k,s,a)} \leq N(k, s, a).
$$
For $N(k, s, a) \geq 2c_1L^2H^3$, in which case $N(k, s, a) \geq 2c_1 \log N(k, s, a) \cdot LH^3$, for sufficiently large constant $c_3$ (recall that $L = c_3 \log(|S||A|HT)$). We have

$$N(k, s, a)/4 \leq c_02^{e_h(k, s, a)} \leq N(k, s, a).$$

Therefore, we can bound

$$\sum_{h=1}^{H} \sqrt{L/2^{e_h(k, s, a)}} \leq \sum_{h=1}^{H} \left[ \sqrt{L} \cdot \mathbb{I}(N(k, s, a) < 2c_1L^2H^3) + \sqrt{4L/N(k, s, a)} \right],$$

where we slightly abuse the notation by assuming that $N(k, s, a) \geq 1$ (the $N(k, s, a) = 0$ part has been absorbed to the first term). Thus, conditioning on $E_1, E_2, \ldots, E_K$, we have

$$\sum_{k=1}^{K} \sum_{h=1}^{H} b_h(k)(s_h(k)) \leq H \sum_{k=1}^{K} \sum_{h=1}^{H} \left[ \sqrt{L} \cdot \mathbb{I}(N(k, s_h(k), a_h(k)) < 2c_1L^2H^3) + \sqrt{4L/N(k, s_h(k), a_h(k))} \right]$$

$$\leq H \sum_{k=1}^{K} \sum_{h=1}^{H} \left[ \sqrt{L} \cdot \mathbb{I}(n < 2c_1L^2H^3) + \sqrt{4L/n} \right]$$

$$\leq 2c_1|\mathcal{S}||\mathcal{A}|L^{5/2}H^4 + H \sum_{k,s,a} \sqrt{4LN(k,s,a)}$$

$$\leq 2c_1|\mathcal{S}||\mathcal{A}|L^{5/2}H^4 + H \sqrt{4|\mathcal{S}||\mathcal{A}|Lt}.$$

Therefore,

$$\mathbb{E} \left[ \sum_{k=1}^{K} \sum_{h=1}^{H} b_h(k)(s_h(k)) \right] \leq 2c_1|\mathcal{S}||\mathcal{A}|L^{5/2}H^4 + H \sqrt{4|\mathcal{S}||\mathcal{A}|Lt} + O(H^{-2}T^{-1}).$$

Next, for each $e_h^{(k)}$, we have

$$\mathbb{E}[e_h^{(k)}] = \mathbb{E}[P(\cdot|s_h^{(k)}, \pi^{k}(s_h^{(k)}))^{\top} \Delta_h^{(k)} - \delta_h^{(k)}]$$

$$= \mathbb{E}_{\Delta_h^{(k)}} \left\{ \mathbb{E}[P(\cdot|s_h^{(k)}, \pi^{k}(s_h^{(k)}))^{\top} \Delta_h^{(k)} - \delta_h^{(k)}|\Delta_h^{(k)}] \right\} = 0$$

where we use the fact $\mathbb{E}[\delta_h^{(k)}|\Delta_h^{(k)}] = P(\cdot|s_h^{(k)}, \pi^{k}(s_h^{(k)}))^{\top} \Delta_h^{(k)}$. In summary, we obtain

$$\mathbb{E} \left[ \sum_{k,h} \delta_{k,h} \right] \leq \mathbb{E} \left[ \sum_{k,h} \tilde{\delta}_{k,h} \right] + O(H^{-2}T^{-1}) \leq 3H \sqrt{|\mathcal{S}||\mathcal{A}|Lt} + 2c_1|\mathcal{S}||\mathcal{A}|L^{5/2}H^4.$$

Thus

$$\text{Regret}(T) \leq \mathbb{E} \left[ \sum_{k,h} \delta_{k,h} \right] \leq 3H \sqrt{|\mathcal{S}||\mathcal{A}|Lt} + 2c_1|\mathcal{S}||\mathcal{A}|L^{5/2}H^4.$$

\[\square\]

### C.2 Proofs Missing from Section 6

Before we prove this lemma, we first show that the bonus function gives a sufficiently good upper bound on the error of the estimation of $P(\cdot|s, a)^{\top} \nabla_{h,s,a}^{(j)}$.

**Lemma 4 (Variance Estimation).** Let $P$ be a distribution on $\mathcal{S}$. Let $s_1, s_2, \ldots, s_m$ be $m$ independent samples from $P$. For an arbitrary vector $V$, denote

$$\bar{\sigma} = \frac{1}{m} \sum_i V^2(s_i) - \left[ \frac{1}{m} \sum_i V(s_i) \right]^2.$$
Let $\sigma = \text{Var}_{s \sim P}[V(s)]$. Then with probability at least $1 - \delta/(16|S|^4|A|^4H^4T^4)$,
\[
|\bar{\sigma} - \sigma| \leq \|V\|_\infty^2 \cdot \sqrt{L/m/4}
\]
provided $L = c_3 \log(|S||A|HT)$ for sufficiently large constant $c_3$.

**Proof.** Firstly, by Hoeffding’s inequality (Theorem B.1), with probability at least $1 - \delta/(16|S|^4|A|^4H^4L^4)$ we have
\[
\left| \frac{1}{m} \sum_i V^2(s_i) - P^TV^2 \right| \leq \|V\|_\infty^2 \cdot \sqrt{L/m/8} \quad \text{and} \quad \left| \frac{1}{m} \sum_i V(s_i) - P^TV \right| \leq \|V\|_\infty \cdot \sqrt{L/m/16}
\]
provided sufficiently large constant $c_3$. Conditioning on this event, we have
\[
|\bar{\sigma} - \sigma| \leq \left| \frac{1}{m} \sum_i V^2(s_i) - P^TV^2 \right| + \left| \left( \frac{1}{m} \sum_i V(s_i) \right)^2 - (P^TV)^2 \right|
\leq \frac{\|V\|_\infty^2}{8} \cdot \sqrt{\frac{L}{m}} + \left| \frac{1}{m} \sum_i V(s_i) - P^TV \right| \left( \frac{1}{m} \sum_i V(s_i) + P^TV \right)
\leq \|V\|_\infty^2 \cdot \sqrt{L/m}/4.
\]

**Lemma 5** (Bound on Improvement). At time $t \leq T$, let $L = \{\ell^*(s,a) : (s,a) \in S \times A\}$ be the set of full levels for each $(s,a)$. Let $\ell^* = \max_{\ell \in L} \ell$. Let $\tilde{Q}^{(j)}_h$ and $\tilde{V}^{(j)}_h$ be defined in (6), (7) and (8) and let $Q^{(j)}_h$ and $V^{(j)}_h$ be the estimated $Q$-function and value function at level $j$ and stage $h$ in the algorithm vUCQVI. Then with probability at least $1 - \delta/(|S|^3|A|^3H^3T^3)$, for all $j \in \{1, 2, 3, 4, \ldots, \ell^*\}$, we have, for all $h \in [H]$ and $(s,a) \in S \times A$:

1. $Q^*_h(s,a) \leq \tilde{Q}^{(j-1)}_h(s,a) \leq \tilde{Q}^{(j)}_h(s,a) \leq Q^{(j)}_h(s,a) \leq \tilde{Q}^{(j)}_h(s,a) + (H - h + 1)^2 \sqrt{L/2^{j-1}},$
2. $V^*_h(s) \leq \tilde{V}^{(j-1)}_h(s) \leq \tilde{V}^{(j)}_h(s) \leq V^{(j)}_h(s) \leq \tilde{V}^{(j)}_h(s) + (H - h + 1)^2 \sqrt{L/2^{j-1}},$
3. $V^{(j)}_h(s) \leq V^{(j-1)}_h(s) \leq V^{(j)}_h(s) + (H - h + 1)^2 \sqrt{L/2^{j-1}},$
4. $|\tilde{P}^{(j)}_h(|s,a|) - \tilde{V}^{(j-1)}_h(s,a) + \tilde{P}^{(j)}_h(|s,a|) \tilde{V}^{(j)}_h(s,a) - P(|s,a|) \tilde{V}^{(j)}_h(s,a) + \tilde{P}^{(j-1)}_h(s,a)| \leq \delta_{h,s,a}(k),$

where $L = c_3 \log(|S||A|HT/\delta)$ for some sufficiently large constant $c_3$.

**Proof of Lemma 5.** The proof of this lemma is very similar to that of Lemma 1 with only slight modifications to some bounds. For completeness, we present the full proof here. For simplicity, we denote $Q^{(j-1)}_h(s,a) = H - h + 1$ and $\tilde{Q}^{(j-1)}_h(s,a) = Q^*_h(s,a)$ for any $h, s, a$. We prove the lemma by induction on $j$.

**Base case:** $j = 0$. We first prove that the lemma holds for $j = 0$ deterministically, then we show higher levels also hold with high probability. In this case, $Q^{(0)}_h(s,a) = H - h + 1$ for any $h, s, a$ and $\tilde{Q}^{(0)}_h(s,a) = Q^*_h(s,a)$. Hence
\[
Q^*_h(s,a) \leq \tilde{Q}^{(0)}_h(s,a) \leq \tilde{Q}^{(0)}_h(s,a) \leq Q^{(0)}_h(s,a) \leq (H - h + 1)^2 \sqrt{2L},
\]
proving (1). Since $\tilde{V}^{(0)}_h(s) = \max_a \tilde{Q}^{(0)}_h(s,a) = V^*_h(s)$, we obtain
\[
V^*_h(s) \leq \tilde{V}^{(0)}_h(s) \leq \tilde{V}^{(0)}_h(s) \leq \tilde{V}^{(0)}_h(s) \leq \tilde{V}^{(0)}_h(s) + (H - h + 1)^2 \sqrt{2L},
\]
proving (2). Also $V^{(0)}_h(s) \leq V^{(0)}_h(s) \leq (H - h + 1)^2 \sqrt{L}$ proves (3). Since $\tilde{P}^{(0)}_h = 0$ and $\tilde{P}^{(0)}_h = 0$ for all $h$, the last bullet of the lemma statement is also proved. Thus the base case is proved.
**Induction Hypothesis:** For the rest of the proof, suppose there exist events $E^{(1)}, E^{(2)}, \ldots, E^{(j-1)}$, on which the lemma statement holds until $j - 1$. And that $\mathbb{P}[E^{(1)} \cap E^{(2)} \cap \ldots \cap E^{(j-1)}] \geq 1 - (j - 1)\delta/(|S|^3 |A|^4 H^3 T^4)$.

**Induction on $j$:** Recall that
\[
\tilde{\sigma}_h^{(j)}(s,a) = \tilde{P}^{(j)}(s,a) T_{V_{h+1},s,a} - \tilde{P}^{(j)}(s,a) T_{V_{h+1},s,a}^2.
\] (21)

Denote $\sigma_h^{(j)}(s,a) = \text{Var}_{s' \sim P(|s,a)} V_{h+1,s,a}(s')$. Thus by Lemma 4, with probability at least $1 - \delta/(16|S|^4 |A|^4 H^3 T^4)$,
\[
|\tilde{\sigma}_h^{(j)}(s,a) - \sigma_h^{(j)}(s,a)| \leq \frac{(H - h)^2}{4} \cdot \sqrt{\frac{L}{2\min(j, \ell^*(s,a))}}.
\] (22)

Denote
\[
b_{2,h,s,a}^{(j)} = 4 \cdot \sqrt{\frac{\sigma_h^{(j)}(s,a) \cdot L}{2\min(j, \ell^*(s,a))}} + \frac{H - h}{8} \cdot \left(\frac{L}{2\min(j, \ell^*(s,a))}\right)^{3/4} + \frac{1}{4} \cdot \frac{(H - h)L}{2\min(j, \ell^*(s,a))}.
\]

Next, we define an event $E^{(j)}$ as that, for all $(s,a)$,
\[
|\tilde{P}^{(j)}(s,a) T_{V_{h+1},s,a} - P(s,a) T_{V_{h+1},s,a}| \leq \min \left\{ b_{2,h,s,a}^{(j)}, \frac{H - h}{4} \cdot \sqrt{\frac{L}{2\min(j, \ell^*(s,a))}} \right\}
\] (23)

where $L = c_3 \log(|S||A|HT/\delta)$ for some sufficiently large constant $c_3$. By Hoeffding’s inequality (Theorem B.1), for all $(s,a) \in S \times A$, with probability at least $1 - |S||A| \cdot \delta/(8|S|^4 |A|^4 H^3 T^4)$,
\[
|\tilde{P}^{(j)}(s,a) T_{V_{h+1},s,a} - P(s,a) T_{V_{h+1},s,a}| \leq \frac{H - h}{4} \cdot \sqrt{\frac{L}{2\min(j, \ell^*(s,a))}}.
\]

By Bernstein’s inequality (Theorem B.2), with probability at least $1 - |S||A| \cdot \delta/(16|S|^4 |A|^4 H^3 T^4)$, for all $(s,a) \in S \times A$,
\[
|\tilde{P}^{(j)}(s,a) T_{V_{h+1},s,a} - P(s,a) T_{V_{h+1},s,a}| \leq \frac{1}{4} \cdot \sqrt{\frac{\sigma_h^{(j)}(s,a) \cdot L}{2\min(j, \ell^*(s,a))}} + \frac{1}{4} \cdot \frac{(H - h)L}{2\min(j, \ell^*(s,a))}.
\]

Combining with (22), with a union bound, we have
\[
\mathbb{P}[E^{(j)}] \geq 1 - |S||A| \cdot \delta/(4|S|^4 |A|^4 H^3 T^4),
\]

provided sufficiently large constant $c_3$. Next, for each $h = 1, 2, \ldots, H$, we denote event $E_h^{(j)}$ as that for all $(s,a)$,

(i) \( |[\tilde{P}_h^{(j)}(s,a) - P(s,a)]^T (V_{h+1} - T_{V_{h+1},s,a})| \leq (H - h)^{(5 - \beta)/2} \cdot \sqrt{L/2\min(j, \ell^*(s,a))} / 4; \)

(ii) \( |[\tilde{P}_h^{(j)}(s,a) - P(s,a)]^T (V_{h+1} - T_{V_{h+1},s,a})| \leq (H - h)^{(5 - \beta)/2} \cdot \sqrt{L/2\min(j, \ell^*(s,a))} / 4; \)

(iii) \( Q_h(s,a) \leq \tilde{Q}_h^{(j-1)}(s,a) \leq \tilde{Q}_h^{(j)}(s,a) \leq Q_h^{(j)}(s,a) \leq \tilde{Q}_h^{(j)}(s,a) + (H - h + 1)^2 \sqrt{L/2}; \)

(iv) \( V_h(s) \leq \tilde{V}_h^{(j-1)}(s) \leq \tilde{V}_h^{(j)}(s) \leq V_h^{(j)}(s) \leq \tilde{V}_h^{(j)}(s) + (H - h + 1)^2 \sqrt{L/2}. \)

\textsuperscript{5}Recall that to bound the failure probability, we have replaced the samples with independent samples by applying Lemma ??.
Note that if (i)-(iv) happen for all $h = 1, 2, \ldots, H$, then for the statement of the lemma, (1) and (2) follow directly from (iii) and (iv); (3) follows from
\[
V_h^{(j-1)}(s) \leq \tilde{V}_h^{(j-1)}(s) + (H-h+1)^2 \sqrt{L/2^{j-1}} \leq V_h^{(j)}(s) + (H-h+1)^2 \sqrt{L/2^{j-1}};
\]
and (4) follows from (23) and (i).

Similar to the base case $j = 0$, $E_h^{(j)}$ happens with probability 1. We will show that if $E_h^{(j)}, E_{h-1}^{(j)}, E_{h-2}^{(j)}, \ldots, E_{h-1}^{(j)}$ happen, then $E_h^{(j)}$ happens with probability at least $1 - \delta/(|S|^3|A|^3H^3T^4)$. We note that the random probability vector $\tilde{P}_h^{(j)}(\cdot|s,a)$ is independent with $E_h^{(j)}, E_{h-1}^{(j)}, E_{h-2}^{(j)}, \ldots, E_{h-1}^{(j)}$. Therefore, by Hoeffding bound (Theorem B.1), conditioning on $E_{h+1}^{(j)}$, with probability at least $1 - \delta/(|S|^3|A|^3H^3T^4)$,
\[
\begin{align*}
&\left\|\left[\tilde{P}_h^{(j)}(\cdot|s,a) - P(\cdot|s,a)\right]\top (V_h^{(j)} - \nabla_{a,s,a}^{(j)})\right\| \leq \left\|V_h^{(j)} - \nabla_{a,s,a}^{(j)}\right\|_{\infty} \cdot \sqrt{L/(c_1LH^\beta - 1)} \quad \text{and} \\
&\left\|\left[\tilde{P}_h^{(j)}(\cdot|s,a) - P(\cdot|s,a)\right]\top (\tilde{V}_h^{(j)} - \nabla_{a,s,a}^{(j)})\right\| \leq \left\|\tilde{V}_h^{(j)} - \nabla_{a,s,a}^{(j)}\right\|_{\infty} \cdot \sqrt{L/(c_1LH^\beta - 1)},
\end{align*}
\]
for appropriately chosen constants $c_1$ and $c_3$. We condition on this event. Now we do case analysis on $j$.

Case $\ell^*(s,a) < j$: We then consider the case that $\ell^*(s,a) < j$. We first have $V_{h+1}^{(\ell^*(s,a)-1)} = V_{h+1}^{(j-1)}$. Thus we have, by induction hypothesis, on $E_h^{(0)}, E_h^{(j-2)}, E_h^{(j-1)}$ and $E_h^{(j)}, E_{h-1}^{(j)}, E_{h-2}^{(j)}, \ldots, E_{h-1}^{(j)}$,
\[
\begin{align*}
&\|V_{h+1}^{(\ell^*(s,a)-1)} - V_{h+1}^{(j)}\|_{\infty} \leq \sum_{j'=\ell^*(s,a)}^{j} \|V_{h+1}^{(j-1)} - V_{h+1}^{(j)}\|_{\infty} \leq \sum_{j'=\ell^*(s,a)}^{j} (H-h)^2 \sqrt{L/2^{j'-1}} \leq 5(H-h)^2 \sqrt{L/2^{\ell^*(s,a)}}.
\end{align*}
\]
Since $\|V_{h+1}^{(j)} - \tilde{V}_{h+1}^{(j)}\|_{\infty} \leq (H-h)^2 \sqrt{L/2^j}$, we have,
\[
\|V_{h+1}^{(\ell^*(s,a)-1)} - \tilde{V}_{h+1}^{(j)}\|_{\infty} \leq 6(H-h)^2 \sqrt{L/2^{\ell^*(s,a)}}.
\]
Hence, we obtain
\[
\begin{align*}
&\left\|\left[\tilde{P}_h^{(j)}(\cdot|s,a) - P(\cdot|s,a)\right]\top (V_{h+1}^{(j)} - \nabla_{a,s,a}^{(j)})\right\| \leq (H-h)^{(5-\beta)/2} \cdot \sqrt{L/2^{\ell^*(s,a)}}/4 \quad \text{and} \\
&\left\|\left[\tilde{P}_h^{(j)}(\cdot|s,a) - P(\cdot|s,a)\right]\top (\tilde{V}_{h+1}^{(j)} - \nabla_{a,s,a}^{(j)})\right\| \leq (H-h)^{(5-\beta)/2} \cdot \sqrt{L/2^{\ell^*(s,a)}}/4,
\end{align*}
\]
for sufficiently large constant $c_1$, proving (i) and (ii).

Next, we have
\[
\tilde{Q}_h^{(j)}(s,a) := \min \left[ (r(s,a) + \hat{P}(\ell^*(s,a))\top V_{h+1}^{(\ell^*(s,a)-1)}) + \hat{P}(\ell^*(s,a))\top [\tilde{V}_{h+1}^{(j)} - V_{h+1}^{(\ell^*(s,a)-1)}] + b_{h,s,a}^{(\ell^*(s,a))}, Q_{h+1}^{(j-1)}(s,a) \right],
\]
and
\[
Q_h^{(j)}(s,a) := \min \left[ (r(s,a) + \hat{P}(\ell^*(s,a))\top V_{h+1}^{(\ell^*(s,a)-1)}) + \hat{P}(\ell^*(s,a))\top [V_{h+1}^{(j)} - V_{h+1}^{(\ell^*(s,a)-1)}] + b_{h,s,a}^{(\ell^*(s,a))}, Q_{h+1}^{(j-1)}(s,a) \right],
\]
Since on $E_{h+1}^{(j)}$, $\tilde{V}_{h+1}^{(j)} \leq V_{h+1}^{(j)} \leq \tilde{V}_{h+1}^{(j)} + (H-h)^2 \sqrt{L/2^j}$, we thus have, $\tilde{Q}_h^{(j)}(s,a) \leq Q_h^{(j)}(s,a)$ and
\[
Q_h^{(j)}(s,a) - \tilde{Q}_h^{(j)}(s,a) \leq \|\tilde{V}_{h+1}^{(j)} - V_{h+1}^{(j)}\|_{\infty} \leq (H-h)^2 \sqrt{L/2^j}.
\]
To show that $\tilde{Q}_{h+1}^{(j-1)}(s,a) \leq \tilde{Q}_h^{(j)}(s,a)$, we additionally consider two cases: $j - 1 > \ell^*(s,a)$ and $j - 1 \leq \ell^*(s,a)$. 

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If \( j - 1 > \ell^*(s, a) \), then we have
\[
\tilde{Q}_h^{(j-1)}(s, a) := \min \left[ r(s, a) + \hat{P}(\ell^*(s, a))\top V^{(\ell^*(s, a)-1)}_{h+1} + \hat{P}(\ell^*(s, a))\top \left[ V^{(\ell^*(s, a)-1)}_{h+1} - V^{(\ell^*(s, a)-1)}_{h+1} \right] + b_h^{(\ell^*(s, a))} + Q^{(j-2)}_{h+1}(s, a) \right].
\]
Since \( \tilde{V}^{(j)}_{h+1} \leq V^{(j)}_{h+1} \), we have
\[
Q_h^*(s, a) \leq \tilde{Q}_h^{(j-1)}(s, a) \leq \tilde{Q}_h^{(j)}(s, a),
\]
proving (iii) and (iv).

Next we consider \( j - 1 \leq \ell^*(s, a) \). By induction hypothesis, on \( \mathcal{E}^{(j-1)} \) with \( (26) \) and event \( \mathcal{E}^{(j)}' \), we have
\[
\tilde{Q}_h^{(j)}(s, a) = \min \left[ r(s, a) + \hat{P}(\ell^*(s, a))V^{(\ell^*(s, a)-1)}_{h+1} + \hat{P}(\ell^*(s, a))\left[ V^{(\ell^*(s, a)-1)}_{h+1} - V^{(\ell^*(s, a)-1)}_{h+1} \right] + b_h^{(\ell^*(s, a))} + Q^{(j-1)}_{h+1}(s, a) \right]
\]
\[
= \min \left[ r(s, a) + P(\cdot | s, a)\top \tilde{V}^{(j)}_{h+1} + b_h^{(\ell^*(s, a))} + \pm \min \left\{ b_{2, h, s, a}^{(j)}, \frac{H - h}{4} \right\} \right] + \left( H - h \right)^{(5-\beta)/2} \cdot \sqrt{L/2}\tilde{\epsilon}(s, a)/4, \quad (28)
\]
Since \( b_h^{(\ell^*(s, a))} = \min \left\{ b_{2, h, s, a}^{(j)}, \frac{H - h}{4} \right\} \), we have
\[
Q_h^*(s, a) \leq \tilde{Q}_h^{(j-1)}(s, a) \leq \tilde{Q}_h^{(j)}(s, a) \leq Q_h^{(j)}(s, a) \leq \tilde{Q}_h^{(j)}(s, a) + \left( H - h \right)^2 \sqrt{L/2}/2,
\]
proving (iii) and (iv).

**Case** \( \ell^*(s, a) \geq j \): We then consider the case that \( \ell^*(s, a) \geq j \). In this case, \( \tilde{V}^{(j)}_{h+1, s, a} = V^{(j-1)}_{h+1} \). Additionally, by the algorithm definition, we have \( V^{(j)}_{h+1} \leq V^{(j-1)}_{h+1} \). Applying the induction hypothesis, we have
\[
\| V^{(j-1)}_{h+1} - \tilde{V}^{(j-1)}_{h+1} \| \leq \left( H - h \right)^2 \sqrt{L/2}\tilde{\epsilon}(s, a)/2.
\]
And on \( \mathcal{E}^{(h+1)} \), we have
\[
\tilde{V}^{(j-1)}_{h+1} \leq \tilde{V}^{(j)}_{h+1} \leq V^{(j)}_{h+1} \leq \tilde{V}^{(j)}_{h+1} + \left( H - h \right)^2 \sqrt{L/2}/2.
\]
Thus,
\[
\| V^{(j)}_{h+1} - \tilde{V}^{(j)}_{h+1, s, a} \| \leq \| V^{(j-1)}_{h+1} - \tilde{V}^{(j)}_{h+1} \| \leq \left( H - h \right)^2 \sqrt{L/2}\tilde{\epsilon}(s, a)/2.
\]
We thus have
\[
\| V^{(j)}_{h+1} - \tilde{V}^{(j)}_{h+1, s, a} \| \cdot \sqrt{1/(c_1 H^2 - 1)} \leq \left( H - h \right)^{(5-\beta)/2} \cdot \sqrt{L/2}/4
\]
provided sufficiently large \( c_1 \), proving (i) and (ii).

Together with the event \( \mathcal{E}^{(j)}' \), by \( (12) \), we have
\[
\tilde{P}(\cdot | s, a)\top \tilde{V}^{(j)}_{h+1, s, a} + \tilde{P}_h^{(j)}(\cdot | s, a)\top \left( V^{(j)}_{h+1} - \tilde{V}^{(j)}_{h+1, s, a} \right) = P(\cdot | s, a)\top V^{(j)}_{h+1} + b_h^{(j)}(s, a) \pm \min \left\{ b_{2, h, s, a}^{(j)}, (H - h) \cdot \sqrt{L/2}/4 \right\}
\]
\[
\pm \left( H - h \right)^{(5-\beta)/2} \cdot \sqrt{L/2}/4
\]
Denote $\widetilde{Q}^{(j)}(s, a) := r(s, a) + P^{(j)}(s, a) V^{(j)}_{h+1}$. Since 

$$h^{(j)}_{h, s, a} = \min \{ h^{(j)}_{2, h, s, a}, \sqrt{(H - h) \cdot \frac{L}{2^j}} + (H - h)^{(5 - \beta)/2} \cdot \sqrt{L/2^j} \},$$

we have 

$$\widetilde{Q}^{(j)}(s, a) \leq \hat{V}^{(j)}_{h+1} \leq \widetilde{Q}^{(j)}(s, a) \leq \hat{V}^{(j)}_{h+1} + (H - h)^2 \sqrt{L/2^j}.$$

Moreover, since $V^{(j)}_{h+1} \leq \widetilde{V}^{(j)}_{h+1} \leq V^{(j)}_{h+1} + (H - h)^2 \sqrt{L/2^j}$, we have 

$$\widetilde{Q}^{(j)}(s, a) \leq r(s, a) + P(\cdot | s, a)^T V^{(j)}_{h+1} + (H - h)^2 \sqrt{L/2^j} = \check{Q}^{(j)}(s, a) + (H - h)^2 \sqrt{L/2^j},$$

and thus, 

$$\check{Q}^{(j)}(s, a) = r(s, a) + P(\cdot | s, a)^T V^{(j)}_{h+1} \leq \check{Q}^{(j)}(s, a) + (H - h)^2 \sqrt{L/2^j} \leq \check{Q}^{(j)}(s, a) + (H - h)^2 \sqrt{L/2^j},$$

proving (iii) and (iv).

Therefore, by a union bound over all $s, a$, with probability at least, $1 - \delta/(|S|^3|A|^3 H^4 T^4)$, $\check{Q}^{(j)}$ happens. By a union bound over all $h = H - 1, H - 2, \ldots, 1$, we obtain, with probability at least, $1 - \delta/(|S|^3|A|^3 H^4 T^4)$, all $\check{Q}^{(j)}_1, \check{Q}^{(j)}_2, \ldots, \check{Q}^{(j)}_H$ happen. Denote $\check{Q}^{(j)} = \check{Q}^{(j)}_1 \cap \check{Q}^{(j)}_2 \cap \ldots \check{Q}^{(j)}_H$, thus $\mathbb{P}[(\check{Q}^{(j)})^c] \geq 1 - \delta/(|S|^3|A|^3 H^4 T^4)$. Thus by a union bound, $\check{Q}^{(j)}_1, \check{Q}^{(j)}_2, \ldots, \check{Q}^{(j)}_H$ all happen with probability at least $1 - j \delta/(|S|^3|A|^3 H^4 T^4)$. Thus the lemma statement holds for all $j \in [\ell^c]$ with probability at least $1 - \delta/(|S|^3|A|^3 H^4 T^3)$. This completes the proof.

Before we show our main result, we note that a similar bound for regret still holds as in Theorem 5.1. This is because the bonus defined in (5) is an upper bound of (10). And more samples of $\check{H}^{(j)}$ give more accurate estimation of $P(\cdot | s, a)^T (V^{(j)}_{h+1} - \widetilde{V}^{(j)}_{h, s, a})$. This will serve as a coarse estimation of the regret bound.

We formally denote this as the following lemma, whose proof can be inferred from that of Theorem 5.1.

**Lemma 6.** Let $t = KH \leq T$ be the current time. Consider Algorithm 1 with bonus function setting to be (10) and each $|H^{(j)}(s, a)| = c_2 H \beta^3$ for $\beta \geq 3$. For each episode $k$ of the algorithm, let $\ell^c(k)$ be the largest level among all state-action pairs up to episode $k$. Let $V^{\pi^k}_h := V^{(j)}_{h, s, a}$ denote the estimated value function up to episode $k$. Let $V^{\pi^k}_h$ be the true value function of the policy $\pi^k$. Then with probability at least $1 - \delta/(|S|^3|A|^3 H^3 T^3)$, for all $h \in [H]$

$$\left| \sum_{k=1}^{K} V^{(j)}_{h}(s^{(k)}_h) - V^{\pi^k}_h(s^{(k)}_h) \right| \leq O\left(H \sqrt{|S||A|Lt} + |S||A| L^{5/2} H^{\beta + 1} \right).$$

We also show an theorem that bounds the summation of the variance.

**Lemma 7.** Consider the same setting of Lemma 6, at any episode $k$, let 

$$\sigma^{(k)}_h(s, a) = \mathbb{P}(\cdot | s, a)^T (V^{(j)}_{h, s, a})^2 - [P(\cdot | s, a)^T V^{(j)}_{h, s, a})^2].$$

Let 

$$\sigma^k_h(s) = \mathbb{P}(\cdot | s, \pi^k(s))^T (V^{\pi^k}_{h+1})^2 - [P(\cdot | s, \pi^k(s))^T V^{\pi^k}_{h+1})^2].$$

Then there exists an event $\mathcal{E}_k$, which happens with probability at least $1 - 2\delta/(|S|^3|A|^3 H^3 T^3)$, such that 

$$\sum_{h=1}^{H} \sigma^{(k)}_h(s^{(k)}_h, a^{(k)}_h) \leq 2H \sum_{h=1}^{H} \mathbb{P}(\cdot | s^{(k)}_h, \pi^k(s^{(k)}_h))^T (V^{\pi^k}_{h+1} - V^{\pi^k}_{h+1}) + 4H^2 \sum_{h=1}^{H} \sqrt{L/2^j(s^{(k)}_h, a^{(k)}_h)}. $$
Proof. For notation simplicity, we ignore the superscript \((k)\). In particular, we denote \(s_{h} := s_{h}^{(k)}\), \(a_{h} = \pi_{h}^{(k)}(s_{h}^{(k)})\), \(\bar{\sigma}_{h} := \tilde{\sigma}_{h}^{(k)}(s_{h}^{(k)}, a_{h}^{(k)})\), \(\sigma_{h} := \sigma_{h}^{(k)}(s_{h}^{(k)}, a_{h}^{(k)})\), \(\nabla_{h+1} := \nabla_{h+1}^{(k)}(s_{h}^{(k)}, a_{h}^{(k)})\), \(\bar{P}_{h} := \bar{P}^{(k)}(|s_{h}, a_{h})\), and \(P_{h} := P(\cdot|s_{h}, a_{h})\). By straightforward calculation, we have
\[
\bar{\sigma}_{h} - \sigma_{h} = \bar{P}_{h}^{\top}(\nabla_{h+1})^{2} - (\bar{P}_{h}^{\top}\nabla_{h+1})^{2} - P_{h}^{\top}(V_{h+1}^{\pi_{h}^{(k)}})^{2} + (P_{h}^{\top}V_{h+1}^{\pi_{h}^{(k)}})^{2} \\
= \bar{P}_{h}^{\top}(\nabla_{h+1})^{2} - P_{h}^{\top}(\nabla_{h+1})^{2} + P_{h}^{\top}(\nabla_{h+1})^{2} - P_{h}^{\top}(V_{h+1}^{\pi_{h}^{(k)}})^{2} \\
+ (P_{h}^{\top}V_{h+1}^{\pi_{h}^{(k)}})^{2} - (P_{h}^{\top}\nabla_{h+1})^{2} + (P_{h}^{\top}V_{h+1}^{\pi_{h}^{(k)}})^{2} - (P_{h}^{\top}\nabla_{h+1})^{2}.
\]
Let \(\mathcal{E}_{k}\) denote the event that Lemma 5 holds up to episode \(k\). Note that on \(\mathcal{E}_{k}\), we have \(V_{h}^{\pi_{h}^{(k)}} \leq V_{h}^{\pi} \leq \bar{V}_{h} \leq H - h + 1\). Denote another event \(\hat{\mathcal{E}}_{k}\) that for all \((h, s, a)\),
\[
|\bar{P}(\cdot|s, a)^{\top}(\nabla_{h,s,a}^{(k)})^{2} - P(\cdot|s, a)^{\top}(\nabla_{h,s,a}^{(k)})^{2}| \leq (H - h + 1)^{2}\sqrt{L/2\ell^{(k,s,a)}}.
\]
Then by Hoeffding bound and with a union bound on all state-action pairs, we obtain
\[
P(\hat{\mathcal{E}}_{k}) \geq 1 - \delta/(|S|^{3}A^{3}H^{3}T^{3}),
\]
provided sufficiently large \(c_{3}\) in \(L\). Let \(\mathcal{E}'_{k} = \mathcal{E}_{k} \cap \hat{\mathcal{E}}_{k}\). Then \(P(\mathcal{E}'_{k}) \geq 1 - 2\delta/(|S|^{3}A^{3}H^{3}T^{3})\). On \(\mathcal{E}'_{k}\), we have \((P_{h}^{\top}V_{h+1}^{\pi_{h}^{(k)}})^{2} - (P_{h}^{\top}\nabla_{h+1})^{2} \leq 0\) and
\[
|\bar{\sigma}_{h} - \sigma_{h}| \leq 2H^{2}\sqrt{L/2\ell^{(k,s,a)}} + P_{h}^{\top}(\nabla_{h} - V_{h}^{\pi_{h}^{(k)}})(\nabla_{h} + V_{h}^{\pi_{h}^{(k)}}) \\
\leq 2H^{2}\sqrt{L/2\ell^{(k,s,a)}} + 2(H - h + 1)P_{h}^{\top}(\nabla_{h} - V_{h}^{\pi_{h}^{(k)}}).
\]

Proof of Theorem 6.1. We will run Algorithm 1 but with modifications on the size of \(\mathcal{H}^{(1)}, \mathcal{H}^{(2)}, \ldots\) and the bonus function. We will take \(|\mathcal{H}^{(j)}| = c_{1}LH^{\beta}\) for \(\beta \geq 3\) and \(L = c_{3}\log(|S||A|HT/\delta)\) for some large constant \(c_{3}\) and error probability parameter \(\delta \in (0, 1)\). The bonus function will be set to \((10)\).

The formal proof of the regret bound is similar to that of Theorem 5.1. Denote \(t = KH \leq T\). At an episode \(k\), we denote \(t_{c}(k)\) the time of the last call of \(vUCQVI\). Let \(\ell^{*}(k)\) be the highest level over all state-action pairs during episode \(k\) (also see the definition in the statement of Lemma 1). Denote \(\pi^{(k)} = \pi^{\ell^{*}(k)}\) and \(V_{h}^{(k)} = V_{h}^{\ell^{*}(k)}\) and \(Q_{h}^{(k)} = Q_{h}^{\ell^{*}(k)}\) in \(vUCQVI\). Thus \(V_{h}^{(k)}\) and \(Q_{h}^{(k)}\) are the latest estimation of the value function and Q-function at state \(k\).

We rewrite the regret definition as
\[
\widetilde{\text{Regret}}(KH) = \sum_{k=1}^{K} V_{1}^{(k)}(s^{(k,1)}) - V_{1}^{(k)}(s^{(k,1)}),
\]
(31)

To show the regret at episode \(k\), we look at the following vector,
\[
\Delta_{h}^{(k)} = V_{h}^{\pi_{h}^{(k)}} - V_{h}^{\pi_{h}^{(k)}} \quad \text{and} \quad \Delta_{h}^{(k)} = V_{h}^{(k)} - V_{h}^{\pi_{h}^{(k)}}.
\]
Denote \(\delta_{h}^{(k)} = \Delta_{h}^{(k)}(s^{(k)})\), we then have
\[
\widetilde{\text{Regret}}(KH) = \sum_{k=1}^{K} \delta_{1}^{(k)}.
\]
Let $\mathcal{E}_k$ be the event that Lemma 5 holds at episode $k$. By Lemma 5, we have $\mathbb{P}[\mathcal{E}_k] \geq 1 - \delta/|S|^3 |A|^3 H^3 T^3$. On $\mathcal{E}_k$, we have
\[
\forall h \in [H], s \in \mathcal{S}, a \in \mathcal{A} : \quad Q_h^*(s, a) \leq Q_h^k(s, a).
\]
Thus, on $\mathcal{E}_k$, for all $h \in [H]$,
\[
\Delta_h^k \leq \tilde{\Delta}_h^k.
\]
Next, we consider $\tilde{\Delta}_h^k$. Let $\tilde{Q}_{k,h}(s, a) = r(s, a) + P(\cdot|s, a)^\top V_{k,h+1}$. By Lemma 5, on $\mathcal{E}_k$, we have for all $h \in [H], (s, a) \in \mathcal{S} \times \mathcal{A},$
\[
\tilde{Q}_h^k(s, a) \leq Q_h^k(s, a) \leq \tilde{Q}_h^k(s, a) + 2b_h^k(k),
\]
where $\ell^*(k, s, a)$ is the maximum full level of $(s, a)$ at episode $k$. Since $V_h^k(s) = \max_a Q_h^k(s)$, we have, on $\mathcal{E}_k$,
\[
V_h^k(s) = Q_h^k(s, \pi_h^k(s)) \leq \tilde{Q}_h^k(s, \pi_h^k(s)) + 2b_h^k(k).
\]
We denote,
\[
\tilde{\Delta}_h^k = P^{\pi_h^k} V_{h+1}^k - P^{\pi_h^k} V_{h+1}^\pi + b_h^k = P^{\pi_h^k} \tilde{\Delta}_h^k + b_h^k
\]
for a random vector $b_h^k \in \mathbb{R}^{|\mathcal{S}|}$. Using the same notation as in Lemma 5, we have,
\[
Q_h^k(s, a) := Q_h^\ell^*(k)(s, a) = r(s, a) + \tilde{P}^{\ell^*(k)}(\cdot|s, a) V_{h+1, s, a}^\ell^*(k) + \hat{P}_h^{\ell^*(k)}(\cdot|s, a) [V_{h+1}^\ell^*(k) - V_{h+1, s, a}^\ell^*(k)].
\]
Then conditioning on $\mathcal{E}_k$, we have
\[
\forall s \in \mathcal{S} : \quad |b_h^k(s)| \leq 2b_h^k(s, \pi_h^k(s)).
\]
Next, we define
\[
\tilde{\delta}_h^k = \tilde{\Delta}_h^k(s_h^k).
\]
Then we have
\[
\tilde{\delta}_h^k := P(|s_h^k, \pi_k(s_h^k)) \top \tilde{\Delta}_h^k + b_h^k(s_h^k)
\]
\[
= P(|s_h^k, \pi_k(s_h^k)) \top \tilde{\Delta}_h^k - \tilde{\delta}_h^k + b_h^k(s_h^k).
\]
Let
\[
e_h^k = P(|s_h^k, \pi_k(s_h^k)) \top \tilde{\Delta}_h^k - \tilde{\delta}_h^k.
\]
We obtain,
\[
\tilde{\delta}_h^k = \sum_{j=1}^{H} e_h^k + \sum_{j=1}^{H} b_h^k(s_h^k),
\]
where we denote $e_H^k = 0$. Conditioning on $\mathcal{E}_k$, we have
\[
\left| \sum_{h=1}^{H} b_h^k(s_h^k) \right| \leq 2 \sum_{h=1}^{H} b_h^k(s_h^k, \pi_h^k(s_h^k)) \leq \frac{1}{2} \left( \sum_{h=1}^{H} \frac{\alpha_h^j(s_h^k, \pi_h^k(s_h^k)) \cdot L}{2^{\ell^*(k, s_h^k, \pi_h^k(s_h^k))}} + \frac{H - h}{2} \sum_{h=1}^{H} \frac{L}{2^{\ell^*(k, s_h^k, \pi_h^k(s_h^k))}} \right)^{3/4}\left( \sum_{h=1}^{H} \frac{L}{2^{\ell^*(k, s_h^k, \pi_h^k(s_h^k))}} \right)^{3/4} + \frac{1}{2} \cdot \sum_{h=1}^{H} \frac{(H - h)^{\beta - \beta} \cdot L}{2^{\ell^*(k, s_h^k, \pi_h^k(s_h^k))}}.
\]
Let $N(k, s, a)$ be the number of observations of state action pair $(s, a)$ up to stage $k$. Notice that
\[
N(k, s, a) = \sum_{\ell=0}^{\ell^*(k, s, a)} (c_0 2^\ell + c_1 LH^\beta)
\]
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thus we have $\ell^*(k, s, a) \leq \log N(k, s, a)$ and

$$[N(k, s, a) - c_1 \log N(k, s, a) \cdot LH^2]/2 \leq [N(k, s, a) - c_1 \ell^*(k, s, a) \cdot LH^2]/2 \leq c_0 2^{\ell^*(k, s, a)} \leq N(k, s, a).$$

For $N(k, s, a) \geq 2c_1 L^2 H^2$, in which case $N(k, s, a) \geq 2c_1 \log N(k, s, a) \cdot LH^2$, for sufficiently large constant $c_3$ (recall that $L = c_3 \log(|S||A|HT)$). We have

$$N(k, s, a)/4 \leq c_0 2^{\ell^*(k, s, a)} \leq N(k, s, a).$$

Therefore, we can bound

$$\sum_{h=1}^{H} \frac{1}{2^{\ell^*(k, s, a)}} \leq \sum_{h=1}^{H} \left[ \mathbb{I}(N(k, s, a, a_h(k)) < 2c_1 L^2 H^2) + 4/N(k, s, a, a_h(k)) \right],$$

where $a_h(k) = \pi(k,s)_h(a_h(k))$,

$$\sum_{h=1}^{H} \left( \frac{1}{2^{\ell^*(k, s, a)}} \right)^{3/4} \leq \sum_{h=1}^{H} \left[ \mathbb{I}(N(k, s, a, a_h(k)) < 2c_1 L^2 H^2) + (4/N(k, s, a, a_h(k)))^{3/4} \right],$$

and

$$\sum_{h=1}^{H} \left( \frac{1}{2^{\ell^*(k, s, a)}} \right)^{1/2} \leq \sum_{h=1}^{H} \left[ \mathbb{I}(N(k, s, a, a_h(k)) < 2c_1 L^2 H^2) + (4/N(k, s, a, a_h(k)))^{1/2} \right].$$

Note that here we slightly abuse the notation by assuming that $N(k, s, a, a_h(k)) \geq 1$. Thus, conditioning on $E_1, E_2, \ldots, E_K$, we have

$$\sum_{h=1}^{H} \sum_{k=1}^{K} H_b^{(k)}(s_k^{(k)}, a_h^{(k)}) \leq H \cdot (\sqrt{L} + L/2 + L^{3/4}/2) \cdot \sum_{k=1}^{K} \sum_{h=1}^{H} \left[ \mathbb{I}(N(k, s_h^{(k)}, a_h^{(k)}) < 2c_1 L^2 H^2) \right]$$

$$+ \sum_{k=1}^{K} \sum_{h=1}^{H} \frac{\sigma_h^{(k)}(s_h^{(k)}, a_h^{(k)}) \cdot L}{N(k, s_h^{(k)}, a_h^{(k)})} + \sum_{k=1}^{K} \sum_{h=1}^{H} \frac{H H^5/\beta}{N(k, s_h^{(k)}, a_h^{(k)})}$$

$$+ 2 \sum_{k=1}^{K} \sum_{h=1}^{H} \frac{H L}{N(k, s_h^{(k)}, a_h^{(k)})} + 2 H L^{3/4} \sum_{k=1}^{K} \sum_{h=1}^{H} \left( \frac{1}{N(k, s_h^{(k)}, a_h^{(k)})} \right)^{3/4}$$

$$\leq 3|S||A|^{H^{5/4} + 1} L^3 + \sqrt{H^{5/4} |S||A|} + 2 H L^{3/4} (|S||A|)^{1/4} + 2 H L |S||A| \log t$$

$$+ \sum_{k=1}^{K} \sum_{h=1}^{H} \frac{\sigma_h^{(k)}(s_h^{(k)}, a_h^{(k)}) \cdot L}{N(k, s_h^{(k)}, a_h^{(k)})},$$

where we apply the following inequalities,

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \frac{1}{N(k, s_h^{(k)}, a_h^{(k)})} = \frac{\sum_{k=1}^{K} \sum_{s,a} N(k, s,a)}{n} \leq 2 \sum_{s,a} \sqrt{N(k, s,a)} \leq 2(|S||A| \sum_{s,a} N(k, s,a))^{1/2} = 2 |S||A|^{1/2};$$

and

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \frac{1}{N(k, s_h^{(k)}, a_h^{(k)})}^{3/4} \leq 4 (|S||A| \sum_{s,a} N(k, s,a))^{1/4} \leq 4 (|S||A|)^{1/4}$$

and

$$\sum_{k=1}^{K} \sum_{s,a} N(k, s,a) \leq |S||A| \log t.$$
We apply the Azuma-Hoeffding's inequality to bound the sum of $e_h^{(k)}$: with probability at least $1 - \delta/(2|S|^3|A|^3T^3H^3)$,

\[
\left| \sum_{k=1}^{K} \sum_{h=1}^{H} e_h^{(k)} \right| \leq \sqrt{H^2Lt} \quad \text{and} \quad \left| \sum_{k=1}^{K} \sum_{h=1}^{H-1} e_h^{(k)} \right| \leq \sqrt{H^2Lt},
\]

provided sufficiently large $L$. We denote this event as $E'$. Then, by union bound,

\[
\mathbb{P}[E' \cap \bigcap_{k=0}^{K} E_k \cap E'_K] \geq 1 - \sum_{k=1}^{K} \frac{3\delta}{|S|^3|A|^3T^3H^3} - \frac{\delta}{2|S|^3|A|^3T^3H^3} \geq 1 - \frac{7\delta}{2|S|^3|A|^3H^2T^2}.
\]

Conditioning on $E_k$, we have

\[
\delta_h^{(k)} \leq \tilde{\delta}_h^{(k)}.
\]

Therefore, conditioning on $E_1, E_2, \ldots, E_K$, we have

\[
\sum_{k,h} \delta_h^{(k)} \leq \sum_{k,h} \tilde{\delta}_h^{(k)}
\]

for some constant $C'$. Thus the event $E'' := E' \cap E_1 \cap E_2 \cap \ldots \cap E_K$ happens with probability at least $1 - 3\delta/(2|S|^3|A|^3H^2T^2)$. We obtain

\[
\mathbb{E}\left[ \sum_{k,h} \delta_h^{(k)} \right] \leq \mathbb{E}\left[ \sum_{k,h} \tilde{\delta}_h^{(k)} \right] + O(H^{-2}T^{-1}) \leq \mathbb{E}\left[ \sum_{k=1}^{K} \sum_{h=1}^{H} \delta_h^{(k)}(s_h^{(k)})|E''| \right] + O(H^{-2}T^{-1}).
\]

Therefore,

\[
\text{Regret}(t) = \mathbb{E}[\bar{\text{Regret}}(t)] = O\left[ |S||A|H^{\beta+1}L^3 + \sqrt{H^5-\beta|S||A|t} + HL^{3/4}(|S||A|t)^{1/4} + HL|S||A| \log t \right] + \mathbb{E}\left( \sum_{k=1}^{K} \sum_{h=1}^{H} \sqrt{\frac{\tilde{\sigma}_h^{(k)}(s_h^{(k)}, a_h^{(k)}) \cdot L}{N(k, s_h^{(k)}, a_h^{(k)})}} \right),
\]

We now consider the last term, by Cauchy-Schwarz inequality,

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \sqrt{\tilde{\sigma}_h^{(k)}(s_h^{(k)}, a_h^{(k)}) \cdot L} \leq \sqrt{L} \cdot \left( \sum_{k=1}^{K} \sum_{h=1}^{H} \tilde{\sigma}_h^{(k)}(s_h^{(k)}, a_h^{(k)}) \cdot \sum_{s,a} \frac{1}{N(k, s, a)} \right)^{1/2}
\]

\[
\leq L \left( |S||A| \sum_{k=1}^{K} H \sum_{h=1}^{H} \tilde{\sigma}_h^{(k)}(s_h^{(k)}, a_h^{(k)}) \right)^{1/2}.
\]

Now, we apply Lemma 4, recall that $E'_K$ denotes the event that

\[
\sum_{h=1}^{H} \tilde{\sigma}_h^{(k)}(s_h^{(k)}, a_h^{(k)}) \leq \sum_{h=1}^{H} \sigma_h^{(k)}(s_h^{(k)}) + 2H \sum_{h=1}^{H} \mathbb{P}(\mathbb{1}_{s_h^{(k)} = \pi_h^{(k)}}) + (V_{h+1}^{(k)} - V_{h}^{(k)})
\]

\[
+ 4H^2 \sum_{h=1}^{H} \sqrt{L/\tilde{\sigma}_h^{(k)}(s_h^{(k)}, a_h^{(k)})}
\]

\[
\leq \sum_{h=1}^{H} \sigma_h^{(k)}(s_h^{(k)}) + 2H \sum_{h=1}^{H} e_h^{(k)} + \tilde{\delta}_h^{(k)} + 4H^2 \sum_{h=1}^{H} \sqrt{L/\tilde{\sigma}_h^{(k)}(s_h^{(k)}, a_h^{(k)})}.
\]

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We have \( \mathbb{P}[^E_k] \geq 1 - 2\delta/(|S|^3|A|^3H^3T^3) \). Additionally, by Lemma 6, with probability at least \( 1 - \delta/(|S|^3|A|^3HT^2) \), for all \( h \in [H] \),

\[
K \sum_{k=1}^{H} \sum_{h=1}^{K} \delta_{h+1}^{(k)} \leq C(H^2\sqrt{|S||A|L} + |S||A|L^{5/2}H^{\beta+2}),
\]

for some absolute constant \( C \), which we denote as \( E''_k \). Thus, conditioning on \( E'_1, E'_2, \ldots, E'_K, E''_1, E''_2, \ldots, E''_K \), we obtain,

\[
K \sum_{k=1}^{H} \sum_{h=1}^{K} \sigma_{h}^{(k)}(s_h^{(k)}, a_h^{(k)}) \leq \sum_{h=1}^{H} \sum_{k=1}^{H} \sigma_{h}^{(k)}(s_h^{(k)}) + C'(H^2\sqrt{L}t + H^3\sqrt{|S||A|L} + |S||A|L^{5/2}H^{\beta+3})
\]

for some constant \( C' \). Since \( E'_1, E'_2, \ldots, E'_K, E''_1, E''_2, \ldots, E''_K \) happens with probability at least \( 1 - O(\delta/(|S|^3|A|^3H^2T)) \), we have

\[
\text{Regret}(t) = O \left[ \mathbb{E} \left( L \left( |S||A| \sum_{k=1}^{H} \sigma_{h}^{(k)}(s_h^{(k)}, a_h^{(k)}) \right)^{1/2} \right) + |S||A|H^{\beta+1}L^3 + \sqrt{H(5-\beta)|S||A|t} \right] + H^{3/2}L^{3/4}(|S||A|t)^{1/4}
\]

\[
\leq O \left[ \mathbb{E} \left( L \left( \sum_{k=1}^{H} \sum_{h=1}^{K} \sigma_{h}^{(k)}(s_h^{(k)}, a_h^{(k)}) \right)^{1/2} \right) + |S||A|H^{\beta+1}L^3 + \sqrt{H(5-\beta)|S||A|t} \right] + H^{3/2}L^{3/4}(|S||A|t)^{1/4}
\]

where the last inequality follows from Jensen’s inequality.

\[\square\]

D A Sublinear Burn-in Algorithm With \( \sqrt{H^3|S||A|T} \) Regret

In this section, we propose a simpler algorithm for obtaining nearly optimal regret. Firstly, instead of collecting all the samples into one stage, we collect samples into \( H \) stages. To bound the error probability, we will instead assume independence over samples (i.e., Lemma 7). In the independent case, the probability estimations are independent across stages. Now we describe the UCB-Q-VI algorithm as follows. For simplicity, we chose

\[
\text{bonus}(N, V) = \|V\|_{\infty} \cdot \sqrt{L/\max(N, 1)}
\]

(33)

where \( L = \Theta[\log(|S||A|T/\delta)] \) for some \( \delta \in (0, 1) \). Note that in the algorithm, \( r(s, a) \leq \bar{r}(s, a) \) for any \( (s, a) \in S \times A \) and any time step \( t \). In particular if \( \sum_{h \in [H]} N_{k,h}(s,a) \geq 1 \), then \( \bar{r}(s, a) = r(s, a) \). In particular we have the following lemma.
Theorem D.1. The bounds of the two definitions essentially take the same form. We further notice that the Algorithm 1 satisfies, Lemma 9. of the optimal value function. First, we show that the value function estimated through the UCB-Q-VI algorithm is always an over estimator of the Q-function, and \( \hat{r}(s, a) \) value function.

Proof. Here we sketch the proof and then we fill in the details. We first show that with high probability for each \( h \in [H] \),

\[
Q^*_h \leq Q_{K,h} \quad \text{and} \quad V^*_h \leq V_{K,h}
\]

where \( Q_{K,h} \in \mathbb{R}^{[S] \times [A]} \) and \( V_{K,h} \in \mathbb{R}^{[S]} \) are defined in the algorithm as the estimated Q-function and value function, and “\( \leq \)” denotes the coordinate-wise comparison.

Proof. Here we sketch the proof and then we fill in the details. We first show that with high probability for each \( h \in [H] \),

\[
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\]

Lemma 8. For any \((k, h) \in [K] \times [H]\) and any \((s, a) \in S \times A\),

\[
r(s, a) \leq \tilde{r}(s, a) \leq r(s, a) + \sqrt{\frac{1}{N_{k,h}(s, a)}}.
\]

Thus, we denote

\[
\forall k, h, s, a : \quad \tilde{b}_{k,h}(s, a) = b_{k,h}(s, a) + \sqrt{\frac{1}{N_{k,h}(s, a)}}.
\] (34)

We further notice that the Q-function estimated by the algorithm never exceeds \( H \).

Our first theorem about the regret is denoted as follows.

Theorem D.1. For any given time \( T = KH \) for some \( K \geq 1 \), with probability at least \( 1 - \delta \), the regret of Algorithm 1 satisfies,

\[
\hat{\text{Regret}}(T) \leq 7\sqrt{H^3 |S||A|LT},
\]

where \( L = \Theta(\log(|S||A|T)) \) is defined in the bonus function (33).

Before proving this theorem, we first present several lemmas that indicate the properties of the algorithm. First, we show that the value function estimated through the UCB-Q-VI algorithm is always an over estimator of the optimal value function.

Lemma 9. Denote \( Q^*_h \in \mathbb{R}^{[S] \times [A]}, V^*_h \in \mathbb{R}^{[S]} \) as the optimal Q-function and value function at stage \( h \). Let \( T = KH \) and \( L = c \cdot \log(|S||A|T) \) for some sufficiently large constant \( c > 0 \) for the bonus function (33). Let \( N_K = \{ N_{K,h}(s, a) : s \in S, a \in A, h \in [H] \} \) be the set of counts for each state-action pairs. Then with probability least \( 1 - \delta / (10 |S|^2 |A|^2 T^2) \), we have

\[
Q^*_h \leq Q_{K,h} \quad \text{and} \quad V^*_h \leq V_{K,h}
\]

where \( Q_{K,h} \in \mathbb{R}^{[S] \times [A]} \) and \( V_{K,h} \in \mathbb{R}^{[S]} \) are defined in the algorithm as the estimated Q-function and value function, and “\( \leq \)” denotes the coordinate-wise comparison.
Then we show that $V_{K,H}^* \leq V_{K,H}$. The lemma follows by applying the monotonicity of the probability matrix: if $V_{h+1}^* \leq V_{K,h+1}$ then $P_{K,h} \cdot V_{h+1}^* \leq \tilde{P}_{K,h} \cdot V_{K,h+1}$.

We now show the first step. For each $(s, a) \in S \times A$, by an Hoeffding bound (Theorem B.1) and applying Lemma 9, with probability at least $1 - \delta/(10|S|^3 |A|^3 T^3)$,

$$Q_{K,h}^*(s, a) = r(s, a) + P(\cdot|s, a)^\top V_{K,h}^* \leq r(s, a) + \tilde{P}_{K,h}(\cdot|s, a)^\top V_{h+1}^* + b_{K,h}(s, a)$$

provided appropriate constant factor $c$ in $L$. Therefore, by a union bound, with probability at least $1 - \delta/(10|S|^2 |A|^2 T^2)$, for each $h \in [H]$,

$$Q_{K,h}^* \leq r + \tilde{P}_{K,h} V_{h+1}^* + b_{K,h},$$

which we condition on for the rest of the proof.

Next, for a given $s \in S$, since if $\exists a \in A$ s.t. $N_{K,H}(s, a) = 0$, then $V_{K,H}(s) = 1 \geq V_{h}(s)$; or if $\min_{a \in A} N_{K,H}(s, a) > 0$ then $V_{K,H}(s) = \max_a r(s, a) = V^*(s)$. Therefore, $V_{h}^* \leq V_{K,H}^*$ with probability 1. Thus, for $h = H - 1, H - 2, \ldots, 1$, we have, for each $(s, a)$

$$Q_{K,h}^*(s, a) \leq \min(H - h + 1, r(s, a) + \tilde{P}_{K,h}(\cdot|s, a)^\top V_{h+1}^* + b_{K,h})$$

$$\leq \min(H - h + 1, \tilde{r}(s, a) + \tilde{P}_{K,h}(\cdot|s, a)^\top V_{K,h}^* + b_{K,h}) = Q_{K,h}(s, a)$$

$$V_{h}^*(s) = \max_a Q_{K,h}^*(s, a) \leq \max_a Q_{K,h}(s, a) = V_{K,h}(s).$$

This completes the proof. \qed

Next we show a lemma that each expectation is approximated well.

**Lemma 10.** Let $Q_{h}^*, V_{h}^*$, $Q_{K,h}^*, V_{K,h}^*$, $N_{K}, L$ be defined the same as in Lemma 9. Let $T = KH$. Then with probability at least $1 - \delta/(10|S|^2 |A|^2 T^2)$, we have

$$\forall 1 \leq h \leq H - 1 : r + PV_{K,h+1} \leq Q_{K,h} \leq r + PV_{K,h+1} + 2\bar{b}_{K,h},$$

where $\bar{b}_{K,h}$ is defined in (34).

**Proof.** Recall that

$$Q_{K,h} = \tilde{r} + \tilde{P}_{K,h} V_{K,h+1} + b_{K,h}.$$

To compute the failure probability, $\tilde{P}_{K,h}$ can be viewed as independent with $V_{K,h+1}$, we have, by a Hoeffding bound (again, with Lemma 9), with probability at least $1 - \delta/(10|S|^3 |A|^3 T^3)$

$$\forall (s, a) \in S \times A : |\tilde{P}_{K,h}(\cdot|s, a)^\top V_{K,h+1} - P(\cdot|s, a)^\top V_{K,h+1}| \leq b_{K,h}(s, a),$$

provided appropriately chosen constants in $L$ for the bonus function (33). Additionally, by definition of $\tilde{r}$, we have

$$r(s, a) \leq \tilde{r}(s, a) \leq r(s, a) + \sqrt{1/\max(N_{k,h}(s, a), 1)}.$$

Thus

$$r(s, a) + P(\cdot|s, a)^\top V_{K,h+1} \leq \min[H - h + 1, r(s, a) + \tilde{P}_{K,h}(\cdot|s, a)^\top V_{K,h+1} + b_{K,h}(s, a)] \leq Q_{K,h}(s, a)$$

and

$$Q_{K,h}(s, a) = \min[H - h + 1, \tilde{r}(s, a) + \tilde{P}_{K,h}(\cdot|s, a)^\top V_{K,h+1} + b_{K,h}(s, a)]$$

$$\leq r(s, a) + b_{K,h}(s, a) + P(\cdot|s, a)^\top V_{K,h+1} + b_{K,h}(s, a)$$

$$\leq r(s, a) + P(\cdot|s, a)^\top V_{K,h+1} + 2\bar{b}_{K,h}(s, a).$$

We then complete the proof by applying a union bound over all $h \in [H], s \in S, a \in A$. \qed
Proof of Theorem D.1. We define the regret as

$$\widetilde{\text{Regret}}(KH) = \sum_{k=1}^{K} V^*_{1}(s_{k,1}) - V^*_{1}^{k}(s_{k,1}),$$

(35)

where we use $\pi^k$ for the $k$-th non-stationary policy, $V_h$ for the value function in the $h$-th stage and $s_{k,h}$ for the state encountered by the algorithm at time $t = (k - 1)H + h$. To show the regret at stage $k$, we look at the following vector,

$$\Delta_{k,h} = V^*_h - V^*_{\pi^k}$$

and

$$\tilde{\Delta}_{k,h} = V_{k,h} - V^*_{\pi^k}.$$

Denote $\delta_{k,h} = \Delta_{k,h}(s_{k,h}, \pi^k(s_{k,h}))$, we then have

$$\widetilde{\text{Regret}}(KH) = \sum_{k=1}^{K} \delta_{k,1},$$

Let $E_{1,k}$ be the event that for all $h \in [H],

$$Q^*_k \leq Q_{k,h}.$$ By Lemma 9, we have

$$P[E_{1,k}] \geq 1 - \frac{\delta}{10|S|^2|A|^2(kH)^2}.$$ Thus, on $E_{1,k}$, for all $h \in [H],

\Delta_{k,h} \leq \tilde{\Delta}_{k,h}.$

Next, we consider $\tilde{\Delta}_{k,h}$. Let $Q_{k,h} = r + PV_{k,h+1}$. Let $E_{2,k}$ be the event that, for all $h \in [H],

$$Q_{k,h} \leq Q_{k,h} \leq Q_{k,h} + 2\tilde{\delta}_{k,h}.$$ By Lemma 10, we have,

$$P[E_{2,k}] \geq 1 - \frac{\delta}{10|S|^2|A|^2(kH)^2}.$$ Since $V_{k,h}(s) = \max_a Q_{k,h}$, we have, on $E_{2,k},$

$$V_{k,h}(s) = Q_{k,h}(s, \pi^k_h(s)) \leq Q_{k,h}(s, \pi^k_h(s)) + 2\tilde{\delta}_{k,h}(s, \pi^k_h(s)).$$

We denote,

$$\tilde{\Delta}_{k,h} = P\pi^k V_{k,h+1} - P\pi^k V^*_{k+1} + b'_{k,h} = P\pi^k \tilde{\Delta}_{k,h+1} + b'_{k,h}$$

for a random vector $b'_{k,h} \in \mathbb{R}^{[S]}$. Then conditioning on $E_{2,k}$, we have

$$\forall s \in S : \ b'_{k,h}(s) \leq 2\tilde{\delta}_{k,h}(s, \pi^k(s)).$$

Next, we define

$$\tilde{\delta}_{k,h} = \tilde{\Delta}_{k,h}(s_{k,h}).$$

Then we have

$$\tilde{\delta}_{k,h} := P(s_{k,h}, \pi^k(s_{k,h}))^\top \tilde{\Delta}_{k,h+1} + b'_{k,h}(s_{k,h})$$

$$= P(s_{k,h}, \pi^k(s_{k,h}))^\top \tilde{\Delta}_{k,h+1} - \tilde{\delta}_{k,h+1} + \tilde{\delta}_{k,h+1} + b'_{k,h}(s_{k,h}).$$

Let

$$e_{k,h} = P(s_{k,h}, \pi^k(s_{k,h}))^\top \tilde{\Delta}_{k,h+1} - \tilde{\delta}_{k,h+1}.$$
We obtain,
\[ \tilde{\delta}_{k,1} = \sum_{j=1}^{H} e_{k,h} + \sum_{j=1}^{H} b'_{k,h}(s_{k,h}), \]
where we denote \( e_{k,H+1} = 0 \). Conditioning on \( \mathcal{E}_{2,k} \), we have
\[ \sum_{j=1}^{H} b'_{k,h}(s_{k,h}) \leq 2 \sum_{j=1}^{H} \tilde{b}_{k,h}(s_{k,h}, a_{k,h}) = 2 \sum_{j=1}^{H} \tilde{b}_{k,h}(s_{k,h}, a_{k,h}) \leq 3 \sum_{j=1}^{H} \sqrt{\frac{H^2 L}{\max(N_{k,h}(s_{k,h}, a_{k,h}), 1)}} \]
where \( a_{k,h} = \pi^k(s_{k,h}) \), provided \( H \geq 1 \). We further denote the event \( \mathcal{E}_2 = \mathcal{E}_{2,1} \cap \mathcal{E}_{2,2} \cap \mathcal{E}_{2,3} \ldots \mathcal{E}_{2,K} \). Then, by union bound,
\[ \mathbb{P}[\mathcal{E}_2] \geq 1 - \sum_{k=1}^{K} \frac{\delta}{10 \cdot |S|^2 |A|^2 K^2 H^2} \geq 1 - \frac{\pi^2 \cdot \delta}{10 \cdot 6 \cdot |S|^2 |A|^2 H^2} \geq 1 - \frac{\delta}{3 \cdot |S|^2 |A|^2 H^2}. \]
Thus, conditioning on \( \mathcal{E}_2 \), we have
\[ \sum_{k=1}^{K} \sum_{h=1}^{H} b'_{k,h}(s_{k,h}) \leq 3 \sum_{k=1}^{K} \sum_{h=1}^{H} \sqrt{\frac{H^2 L}{\max(N_{k,h}(s_{k,h}, a_{k,h}), 1)}} \leq 3 \sum_{s,a,h} \sqrt{H^2 L} \cdot \sqrt{N_{k,h}(s,a)} \leq 6 \sqrt{H^2 L} \cdot \sqrt{|A| |A| L T}. \]
(36)
Next, for each \( e_{k,h} \), we have
\[ \mathbb{E}[e_{k,h}] = \mathbb{E}[P(\pi^k(s_{k,h}))^\top \tilde{\Delta}_{k,h+1} - \tilde{\delta}_{k,h+1}] = \mathbb{E}_{\tilde{\Delta}_{k,h+1}} \{ \mathbb{E}[P(\pi^k(s_{k,h}))^\top \tilde{\Delta}_{k,h+1} - \tilde{\delta}_{k,h+1} | \tilde{\Delta}_{k,h+1}] \} = 0 \]
where we use the fact \( \mathbb{E}[\tilde{\delta}_{k,h+1} | \tilde{\Delta}_{k,h+1}] = P(\pi^k(s_{k,h}))^\top \tilde{\Delta}_{k,h+1} \). Let \( \mathcal{F}_{k,h} \) be the \( \sigma \)-algebra of fixing samples from \( k' = 1, 2, \ldots, k - 1 \). Then \( X_{k,h} = P(\pi^k(s_{k,h}))^\top \tilde{\Delta}_{k,h+1} - \tilde{\delta}_{k,h+1} \) for \( k = 1, 2, \ldots \) and \( h = 1, 2, \ldots, H \) form a martingale difference sequence with respect to \( \mathcal{F}_{k,h} \). Furthermore, by the algorithm definition, the estimated Q-function is upper bounded by \( H \). Thus we have
\[ |e_{k,h}| \leq H \]
almost surely. Then, by Azuma-Hoeffding inequality, we have, with probability at least \( 1 - 2 \exp[-t^2/(2H^2 T)] \)
\[ \left| \sum_{k=1}^{K} \sum_{h=1}^{H} e_{k,h} \right| \leq t. \]
Taking \( t = \sqrt{H^2 T L} \), we obtain, with probability at least \( 1 - \delta/(3|S|^2 |A|^2 T^3) \),
\[ \left| \sum_{k=1}^{K} \sum_{h=1}^{H} e_{k,h} \right| \leq \sqrt{H^2 T L}, \]
which we denote as event \( \mathcal{E}_3 \). Next, we denote event \( \mathcal{E}_1 = \mathcal{E}_{1,1} \cap \mathcal{E}_{1,2} \cap \mathcal{E}_{1,3} \ldots \mathcal{E}_{1,K} \). Then, by union bound,
\[ \mathbb{P}[\mathcal{E}_1] \geq 1 - \sum_{k=1}^{K} \frac{\delta}{10|S|^2 |A|^2 K^2 H^2} \geq 1 - \frac{\delta}{3|S|^2 |A|^2 H^2}. \]
By definition of $\mathcal{E}_{1,k}$, on which, we have
\[ \delta_{k,h} \leq \tilde{\delta}_{k,h}. \]
Therefore, conditioning on $\mathcal{E}_1$,
\[ \sum_{k,h} \delta_{k,h} \leq \sum_{k,h} \tilde{\delta}_{k,h}. \]
In summary, conditioning on $\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3$, which happens with probability at least $1 - \delta / (|S|^2|A|^2H^2)$, we obtain
\[ \hat{\text{Regret}}(KH) = \sum_{k,h} \delta_{k,h} \leq \sum_{k,h} \tilde{\delta}_{k,h} \leq 7\sqrt{H^3|S||A|LT}. \]