We study Koszul homology over Gorenstein rings. If an ideal is strongly Cohen-Macaulay, the Koszul homology algebra satisfies Poincaré duality. We prove a version of this duality which holds for all ideals and allows us to give two criteria for an ideal to be strongly Cohen-Macaulay. The first can be compared to a result of Hartshorne and Ogus; the second is a generalization of a result of Herzog, Simis, and Vasconcelos using sliding depth.

We study duality properties of Koszul homology and their implications. Let $I$ be an ideal of a commutative noetherian ring $R$ and $H_i(I)$ be its $i$th Koszul homology module. The differential graded algebra structure on the Koszul complex induces homomorphisms

$$H_{\ell - g - i}(I) \xrightarrow{\varphi_i} \text{Hom}_R(H_i(I), H_{\ell - g}(I)),$$

for all $i$, where $\ell$ is the minimal number of generators of $I$ and $g$ is its grade.

If every map $\varphi_i$ is an isomorphism then the Koszul homology algebra is said to be Poincaré, equivalently the ideal is said to satisfy Poincaré duality. In [9], Herzog proves that the Koszul homology algebra is Poincaré when the ring is Gorenstein and the ideal is strongly Cohen-Macaulay, that is when all of its non-vanishing Koszul homologies are Cohen-Macaulay modules. A proof of this fact can also be found in [5]. Herzog’s result extends the work of Avramov and Golod [1], where the authors prove that the maximal ideal of a local ring satisfies Poincaré duality if and only if the ring is Gorenstein.

We first give a version of this kind of duality which holds for all ideals of a Gorenstein ring; see Corollary 1.4.

**Theorem A.** Let $R$ be a Gorenstein ring and $I$ be an ideal of grade $g$ that is minimally generated by $\ell$ elements. Then there are isomorphisms

$$\text{Hom}_R(\text{Hom}_R(H_i(I), H_{\ell - g}(I)), H_{\ell - g}(I)) \cong \text{Hom}_R(H_{\ell - g - i}(I), H_{\ell - g}(I))$$

for all $i \geq 0$.

We also prove a generalized version of this theorem for Cohen-Macaulay rings with canonical module; see Theorem 1.6. In [4], Chardin proves these results in the graded setting using a spectral sequence. Our proof, which is the content of Section 1, employs a different spectral sequence.
In Sections 2 and 3, further analysis of this spectral sequence allows us to give several criteria for an ideal to be strongly Cohen-Macaulay in the presence of conditions ensuring that duality holds; see Proposition 2.3 and Remark 2.5.

Strongly Cohen-Macaulay ideals were formally introduced in [11] and since then have been the subject of intense study. The interest in these ideals is justified by the nice geometric properties of the schemes they define; under proper assumptions the Rees algebra and the symmetric algebra of these ideals are isomorphic and Cohen-Macaulay; see [13] and [14]. A large class of strongly Cohen-Macaulay ideals is given by ideals that are in the linkage class of a complete intersection, as shown by Huneke in [10]. Over a Gorenstein ring any ideal whose minimal number of generators $\mu(I)$ is at most $\text{grade}(I) + 2$ is strongly Cohen-Macaulay; see [12], [2].

In Section 2, we give the first criterion for an ideal to be strongly Cohen-Macaulay; see Theorem 2.7.

**Theorem B.** Let $R$ be a local Gorenstein ring and $I$ be an ideal of $R$. Let $h$ be an integer such that $h \geq \max\{2, \frac{1}{2}\dim(R/I)\}$. If, for all $i \geq 0$, the $R/I$-module $H_i(I)$ satisfies Serre’s condition $S_h$, then $I$ is strongly Cohen-Macaulay.

This theorem extends to all Koszul homology modules the following criterion of Hartshorne and Ogus [6]: Let $R$ be a Gorenstein local ring and $I$ be an ideal of $R$ such that $R/I$ is factorial. If $R/I$ satisfies $S_2$ and the inequality $\text{depth}(R/I)_p \geq \frac{1}{2}(\dim(R/I)_p) + 1$ holds for all prime ideals $p$ of $R/I$ of height at least 2, then $R/I$ is Cohen-Macaulay. Notice that by considering all the Koszul homologies, one can lower the depth required and remove the dependence on factoriality.

The proof of Theorem B comes from a careful analysis of the spectral sequence employed in the proof of Theorem 1.2, together with a generalization of a result of Huneke [11], namely Proposition 2.3, which is then used in the rest of the paper. Moreover, we use a result from [6]; see also [11, Lemma 5.8], which compares with Theorem B; see the discussion in Remark 2.8.

In Section 3, we give another criterion for an ideal to be strongly Cohen-Macaulay.

In [8], Herzog, Vasconcelos and Villareal introduce the notion of sliding depth which corresponds to the case $h = 0$ of the later generalization $h$-sliding depth $SD_h$ in [7]: an ideal $I$ satisfies $SD_h$ if

$$\text{depth}H_i(I) \geq \min\{\dim R - \text{grade}(I), \dim R - \mu(I) + i + h\}.$$ 

In [8], the authors prove that if a Cohen-Macaulay ideal $I$ satisfies the inequality $\mu(I_p) \leq \max\{\text{ht}(I), \text{ht}(p) - 1\}$ for every prime ideal $p$ containing $I$ and satisfies sliding depth, then $I$ is strongly Cohen-Macaulay. Using the spectral sequence from the proof of Theorem 1.2, we recover and extend this result in Theorem 3.4 by weakening the condition on $\mu(I)$ while strengthening sliding depth to $h$-sliding depth $SD_h$. Among the many corollaries we obtain, the most interesting is perhaps the following; see Corollaries 3.6 and 3.7.

**Theorem C.** Let $R$ be a local Gorenstein ring, and let $I$ be an ideal of $R$ such that $R/I$ satisfies Serre’s condition $S_2$. Suppose that $I$ satisfies $SD_1$ and one of the following two conditions holds

(a) $\mu(I_p) \leq \text{ht}(p)$ for all prime ideals $p \supseteq I$

(a') $\mu(I_p) \leq \text{ht}(p) + 1$ for all prime ideals $p \supseteq I$ and $H_1(I)$ satisfies $S_2$.

Then $I$ is strongly Cohen-Macaulay.
In Section 4 we present some results on the relationship between the Koszul homology modules for low dimensional ideals.

We finish the introduction by settling some notation. Given a sequence \( y \) of \( \ell \) elements in \( R \), we denote by \( K(y) \) the Koszul complex on the elements \( y \) and by \( H_i(y) \) its \( i \)th homology module. The \( i \)th cohomology module of the complex \( \text{Hom}_R(K(y), R) \) is the Koszul cohomology, and it is denoted by \( H^i(y) \). For an \( R \)-module \( M \), its Koszul complex and its Koszul homology and cohomology modules are denoted by \( K(y; M) \), \( H_i(y; M) \) and \( H^i(y; M) \), respectively. The Koszul complex of an ideal \( I \) is computed on a minimal set of generators, and it is denote by \( K(I) \). Its homology and cohomology modules are denoted by \( H_i(I) \) and by \( H^i(I) \). Moreover, we will often use the isomorphism \( H^i(y; M) \cong H_{\ell-i}(y; M) \). If the sequence \( y \) is of length \( \ell \) then the only non-zero homology modules are \( H_i(y) \) for \( i = 0, \ldots, \ell - g \), where \( g \) is the grade of the ideal generated by \( y \), and each of these has dimension equal to the dimension of \( R/I \) since they have the same support over \( R \).

In the first half of the paper, we give our results in terms of Koszul homology on a sequence of elements rather than the Koszul homology of an ideal. We do this because in the proofs of our applications we often localize and lose minimality. However, in Sections 3 and 4 the results are given for ideals since, historically, that is the main case of interest.

Throughout the paper we use Serre’s conditions \( S_n \): An \( R \)-module \( M \) is said to satisfy \( S_n \) if the inequality \( \text{depth} M_p \geq \min\{n, \dim R_p\} \) holds for every prime ideal \( p \) in \( R \). We also use the following fact:

**Fact.** Let \( M \) and \( N \) be \( R \)-modules which satisfy \( S_2 \) and \( S_1 \), respectively. Assume that the \( R \)-homomorphism \( \varphi : M \to N \) gives an isomorphism when localized at any prime ideal \( p \) such that \( \text{ht}(p) \leq 1 \). Then \( \varphi \) is an isomorphism.

1. Spectral sequence and generalized duality result

The aim of this section is to give a duality modeled on Poincaré duality that holds for all ideals of a Gorenstein ring. We also prove a more general version for Cohen-Macaulay rings with a canonical module \( \omega_R \). In [4, Lemma 5.7], Chardin proves this in the graded setting; his proof invokes a different spectral sequence from the one that we use, which we describe below.

1.1 Construction. Let \( y \) be a sequence of \( \ell \) elements generating an ideal \( \langle y \rangle \) of grade \( g \). Let \( J \) be a minimal injective resolution of \( R \), and let \( K \) denote the Koszul complex \( K(y) \). Consider the double complex \( C = \text{Hom}_R(K, J) \) with \( C^{p,q} = \text{Hom}_R(K_p, J^q) \) and its two associated spectral sequences. Taking homology first in the vertical \( q \) direction, one obtains a collapsing spectral sequence, yielding the information that

\[
H^n(\text{Tot}C) = H_n(\text{Hom}_R(K, R)) \cong H^n(y) \cong H_{\ell-n}(y).
\]

The second convergent spectral sequence, obtained by taking homology first in the horizontal \( p \) direction, is therefore of the form

\[
E_2^{p,q} = \text{Ext}_R^q(H_p(y), R) \Rightarrow H_{\ell-(p+q)}(y).
\]

Note that \( E_2^{p,q} = 0 \) for \( q < g \), since the ideal generated by \( y \) has grade \( g \) and annihilates \( H_i(y) \). Furthermore, when \( R \) is Gorenstein of dimension \( d \), one has that \( E_2^{p,q} = 0 \) for \( q > d \).
1.2 Theorem. Let $R$ be a Gorenstein ring and $y$ be a sequence of $\ell$ elements that generate an ideal of grade $g$. Then there is an isomorphism
\[
\text{Ext}_R^g(\text{Ext}_R^g(H_i(y), R), R) \cong \text{Ext}_R^g(H_{\ell-g-i}(y), R)
\]
for every $i \geq 0$.

The idea is that one more application of the functor $\text{Ext}_R^g(-, \omega_R)$, itself a duality on Cohen-Macaulay modules of dimension $d - g$, removes the lower-dimensional obstructions to it being a duality on Koszul homology.

Before we prove this theorem, we restate it in a way that enables one to compare it to classical Poincaré duality. To do so, we employ some well-known isomorphisms.

1.3 Remark. Let $R$ be a Cohen-Macaulay ring and $y$ a sequence of $\ell$ elements that generate an ideal $I$ of grade $g$. It is well-known that the top non-vanishing Koszul homology satisfies $H_{\ell-g}(y) \cong \text{Ext}_R^g(R/I, R)$. Let $M$ be an $R/I$-module. Choosing any $R$-regular sequence $x$ of length $g$ in the ideal $I$, one therefore obtains natural isomorphisms (the third one being adjunction) for all $i$,
\[
\text{Hom}_R(M, H_{\ell-g}(y)) \cong \text{Hom}_R(M, \text{Ext}_R^g(R/I, R)) \\
\cong \text{Hom}_R(M, \text{Hom}_R(R/I, R/(x))) \\
\cong \text{Hom}_R(M, R/(x)) \\
\cong \text{Ext}_R^g(M, R).
\]

The identifications in the remark now yield the following version of Theorem 1.2.

1.4 Corollary. Let $R$ be a Gorenstein ring and $y$ a set of $\ell$ elements that generate an ideal of grade $g$. Then there is an isomorphism
\[
\text{Hom}_R(\text{Hom}_R(H_i(y), H_{\ell-g}(y)), H_{\ell-g}(y)) \cong \text{Hom}_R(H_{\ell-g-i}(y), H_{\ell-g}(y))
\]
for all $i \geq 0$.

Now we give the proof of the theorem.

Proof of Theorem 1.2. Set $d$ to be the dimension of $R$ and $I$ the ideal generated by the sequence $y$. Consider the spectral sequence from Construction 1.1. The edge homomorphisms provide maps
\[
\psi_i : H_{\ell-g-i}(y) \to \text{Ext}_R^g(H_i(y), R)
\]
and hence maps
\[
\text{Ext}_R^g(\text{Ext}_R^g(H_i(y), R), R) \xrightarrow{\text{Ext}_R^g(\psi_i, R)} \text{Ext}_R^g(H_{\ell-g-i}(y), R).
\]
Both modules above satisfy Serre’s condition $S_2$ as $R/I$-modules, for they can be rewritten in the form $\text{Hom}_{R/I}(-, R/(x))$, where $x$ is any $R$-regular sequence of length $g$ in $I$. Thus, by the fact from the introduction, to show that the map $\text{Ext}_R^g(\psi_i, R)$ is an isomorphism for each $i$, it suffices to prove that it is an isomorphism in codimension 1 over the ring $R/I$. Localizing at a prime ideal of $R$ whose image in $R/I$ has height at most 1, one sees that it remains to prove that $\text{Ext}_R^g(\psi_i, R)$ is an isomorphism for the case that $\text{dim} R/I \leq 1$, that is, $d - g \leq 1$.

If $\text{dim} R/I = 0$, that is, $d = g$, the spectral sequence collapses to a single row at $q = g$, and so the edge homomorphism $\psi_i$ is already an isomorphism for all $i \geq 0$. 

We now give a duality theorem for all ideals of a Gorenstein ring.
If \( \dim R/I = 1 \), that is, \( d = g + 1 \), the spectral sequence has at most two nonzero rows and thus satisfies \( E_2 = E_\infty \) and yields particularly short filtrations of \( H^n(\text{Tot}C) \), namely given by the exact sequences involving the edge homomorphisms

\[
0 \to \Ext^d_R(H_i(y), R) \to H_{\ell-g-i}(\psi) \to \Ext^g_R(H_i(y), R) \to 0.
\]

Set \( L = \Ext^d_R(H_i(y), R) \), and apply \( \Ext^g_R(-, R) \) to the sequence above. Since \( R \) is Gorenstein, the module \( L \) has finite length, and so \( \Ext^j_R(L, R) \) vanishes for all \( j < d \), yielding the desired statement.

**1.5 Remark.** The isomorphisms in Theorem 1.2 are induced by the edge homomorphism from the spectral sequence of Construction 1.1.

We now extend Theorem 1.2 to the setting of Cohen-Macaulay rings. This generalization is also given by Chardin in [4] for \( M = R \) in the graded case.

**1.6 Theorem.** Let \( R \) be a Cohen-Macaulay noetherian ring with canonical module \( \omega_R \). Let \( M \) be a maximal Cohen-Macaulay module, and let \( y \) be a sequence of elements that generate an ideal of grade \( g \). Then there is an isomorphism

\[
\Ext^g_R(\Ext^g_R(H_i(y; M), \omega_R), \omega_R) \cong \Ext^g_R(H_{\ell-g-i}(y; \Hom_R(M, \omega_R)), \omega_R)
\]

for every \( i \geq 0 \).

**Proof.** The proof is the same as that of Theorem 1.2 with the following minor modifications. Consider instead the double complex \( C = \Hom_R(K \otimes_R M, J) \), where \( K \) is the Koszul complex \( K(y) \) and \( J \) is an injective resolution of \( \omega_R \), and the two spectral sequences associated to it. Taking homology first in the vertical \( q \) direction, one again obtains a collapsing spectral sequence as each \( K_p \otimes_R M \) is maximal Cohen-Macaulay and hence \( IE_{p,q}^1 = \Ext^q_R(K_p \otimes_R M, \omega_R) \) vanishes for all \( q > 0 \). This implies that \( IE_{p,0}^2 \) is the \( p \)th homology of the total complex \( C \). So, one can see that

\[
H^n(\text{Tot}C) \cong H_n(\Hom_R(K \otimes_R M, \omega_R))
\]

\[
\cong H_n(\Hom_R(K, \Hom_R(M, \omega_R)))
\]

\[
\cong H_{\ell-n}(y, \Hom_R(M, \omega_R)).
\]

In the other direction one obtains a convergent spectral sequence of the form

\[
II_E_{p,q}^2 = \Ext^q_R(H_p(y; M), \omega_R) \Rightarrow H_{\ell-(p+q)}(y; \Hom_R(M, \omega_R)).
\]

Now the same argument as in Theorem 1.2 applied to this spectral sequence yields the desired result since \( \omega_R \) has injective dimension \( d \) and is a maximal Cohen-Macaulay module, hence the longest \( \omega_R \)-sequence in the ideal \( (y) \) has length \( g \).

2. Consequences and related results

In this section we first give extensions of two classical results, the first of Herzog from [9] and the second of Huneke from [11], to the case of sequences of elements. We need these more general versions for the theorems in this section and the next.

Then, in the main theorem of the section, by a careful analysis of the spectral sequence from Construction 1.1, we show that the Koszul homologies are Cohen-Macaulay if they satisfy Serre’s conditions \( S_h \) for \( h \) at least half their dimension and at least 2. The result extends a theorem of Hartshorne and Ogus in [6] from \( R/I \) to all the Koszul homology modules.
We begin by proving a version of the result of Herzog from [9] for arbitrary sequences of elements.

**2.1 Proposition.** Let $R$ be a Gorenstein ring and $h$ be a nonnegative integer. Let $y$ be a sequence of $\ell$ elements that generate an ideal of grade $g$. If the module $H_i(y)$ is Cohen-Macaulay for all $i \leq h$, then there is an isomorphism

$$\text{Ext}_R^g(H_i(y), R) \cong H_{\ell - g - i}(y)$$

for each $i \leq h + 1$.

In particular, $H_{\ell - g - i}(y)$ is Cohen-Macaulay for every $i \leq h$. Moreover, the following hold

$$\text{ann}(H_i(y)) = \text{ann}(H_{\ell - g - i}(y)) \text{ for } 0 \leq i \leq h,$$

$$\text{ann}(H_{h+1}(y)) \subseteq \text{ann}(H_{\ell - g - (h+1)}(y))$$

**Proof.** Set $H_i = H_i(y)$ for all $i \geq 0$. Consider the spectral sequence

$$E_2^{p,q} = \text{Ext}_R^q(H_p, R) \Rightarrow H_{\ell - (p+q)}$$

from Construction 1.1. As before, one has that $E_2^{p,q} = 0$ for all $q < g$, but now the Cohen-Macaulay hypothesis implies that $E_2^{p,q} = 0$ when $p \leq h$ and $q > g$ as well. So one has that $E_2^{p,g} = E_3^{p,g} = \cdots = E_\infty^{p,g}$ for all $p \leq h + 1$. Therefore, by renaming the index $p$ by $i$ one gets

$$E_\infty^{p,g} = \text{Ext}_R^g(H_i, R) \cong H_{g+i}(\text{TotC}) = H_{\ell - g + i} \cong H_{\ell - g - i},$$

for $i = 0, \ldots, h + 1$, and so the first part of thesis holds. Since $R$ is Gorenstein and $H_i$ is Cohen-Macaulay for $i \leq h$, the module $H_i(R)$ is Cohen-Macaulay for each $i \leq h$. For the remaining assertions, notice that for $i \leq h + 1$

$$\text{ann}(H_i) \subseteq \text{ann}(\text{Ext}_R^g(H_i, R)) = \text{ann}(H_{\ell - g - i}).$$

For $i \leq h$ the other inclusion follows from the following isomorphisms

$$H_i \cong \text{Ext}_R^g(\text{Ext}_R^g(H_i, R), R) \cong \text{Ext}_R^g(H_{\ell - g - i}, R),$$

where the first one is [3, Theorem 3.3.10].

**2.2 Remark.** The isomorphisms in the previous proposition are given by the edge homomorphism of the spectral sequence from Construction 1.1.

A surprising fact is that the Koszul homology algebra is Poincaré under the much weaker assumption of reflexivity on the Koszul homology modules, rather than the full hypothesis of Cohen-Macaulayness. This fact was noted by Huneke in [11, Prop 2.7] for ideals. Below we use Theorem 1.2 to obtain a version, modulo the identifications from Remark 1.3, that works for a single value of $i$ and any sequence of elements.

**2.3 Proposition.** Let $R$ be a Gorenstein ring and $y$ be a sequence of $\ell$ elements that generate an ideal of grade $g$. Assume that for some integer $i$ the Koszul homology $H_i(y)$ satisfies Serre’s condition $S_2$ as an $R/(y)$-module. Then the edge homomorphism from the spectral sequence of Construction 1.1 gives an isomorphism $H_i(y) \cong \text{Ext}_R^g(H_{\ell - g - i}(y), R)$. 

\[\square\]
Proof. Denote the dual $\text{Ext}^g_R(\cdot, R)$ by $(\cdot)^\vee$. By Theorem 1.2 and Remark 1.5, the dual $\psi^\vee$ of the edge homomorphism $\psi: H_i(y) \to H_{\ell-g-i}(y)^\vee$ is an isomorphism. Dualizing once more one obtains a commutative diagram

\[
\begin{array}{ccc}
H_i(y) & \xrightarrow{\psi} & H_{\ell-g-i}(y)^\vee \\
\downarrow & & \downarrow \\
H_i(y)^{\vee\vee} & \xrightarrow{\psi^{\vee\vee}} & H_{\ell-g-i}(y)^{\vee\vee}
\end{array}
\]

whose bottom row is an isomorphism and vertical maps are the natural maps to the double dual after one makes the following identifications for $R/\langle y \rangle$-modules

\[
(\cdot)^\vee = \text{Ext}^g_R(\cdot, R) \cong \text{Hom}_R(\cdot, R/\langle x \rangle),
\]

where $x$ is any $R$-regular sequence of length $g$ in $(y)$. The desired conclusion follows from the fact that the vertical maps are isomorphisms. Indeed, since the modules in the top row satisfy $S^2$ also as $R/\langle x \rangle$-modules, the first by hypothesis and the second by identifying it with a Hom as above, they are reflexive as $R$ is Gorenstein. □

2.4 Remark. An even simpler proof of the proposition above can be given if one knows that, for all $j$, the Koszul homology $H_j(y)$ satisfies $S^2$ as an $R/\langle y \rangle$-module (or even just for all $j \leq \ell - g - i$) along the lines of Huneke’s original proof. By Proposition 2.1 and Remark 2.2, the edge homomorphism $\psi$ is an isomorphism when localized at primes of height at most 2 in $R/\langle x \rangle$, where $x$ is any $R$-regular sequence of length $g$ in $(y)$. As in the previous proof one sees that both the domain and target modules satisfy $S^2$ over $R/\langle x \rangle$, and hence $\psi$ is an isomorphism by the fact from the introduction.

2.5 Remark. The converse of Proposition 2.3 is true as well: If the isomorphism holds for some $i$, then $H_i(y)$ satisfies $S_2$ as an $R/\langle y \rangle$-module. Indeed, the identification of the target as $\text{Ext}^g_R(\cdot, R) \cong \text{Hom}_R(\cdot, R/\langle x \rangle)$ implies that this module satisfies $S_2$ as an $R/\langle x \rangle$-module and hence as an $R/\langle y \rangle$ module.

Notice that this, together with Huneke’s result [11, Prop 2.7], implies that the following conditions are equivalent:

1. The ideal $I$ satisfies Poincaré duality, i.e., the maps from the introduction induced by the algebra structure on the Koszul complex are isomorphisms.
2. The Koszul homologies of $I$ satisfy $S_2$.
3. The edge homomorphisms from the spectral sequence are isomorphisms.

We do not know if the duality maps from (1) and (3) are the same, but clearly they are isomorphisms at the same time.

The results in the rest of the paper come from further analysis of the spectral sequence from Construction 1.1 in situations where the edge homomorphism is an isomorphism, for example as in Proposition 2.3 above. Indeed, in such circumstances certain strong conditions are forced on the spectral sequence, as the following key lemma shows.

2.6 Lemma. Let $R$ be a Gorenstein ring and $y$ be a sequence of $\ell$ elements that generate an ideal of grade $g$. Consider the spectral sequence

\[
E_2^{p,q} = \text{Ext}^p_R(H_q(y), R) \Rightarrow H_{\ell-(p+q)}(y),
\]
from Construction 1.1. Fix an $i \geq 0$. If the edge homomorphism $H_i(y) \to Ext^g_R(H_{\ell-g-i}(y), R)$ is an isomorphism, then

1. the differential $d^{\ell-g-i,g}_{r} : E^{\ell-g-i,g}_r \to E^{\ell-g-i-r+1,g+r}_r$ is zero for all $r \geq 2$, and
2. the modules $E^{\ell,q}_{g_r}$ are zero for any $q > g$ with $p + q = \ell - i$.

Proof. Set $H_i = H_i(y)$ for all $i$. The composition of the homomorphisms

$$H_i \to E^{\ell-g-i,g}_\infty \to E^{\ell-g-i,g}_2 = Ext^g_R(H_{\ell-g-i}, R)$$

is the edge homomorphism. Since it is an isomorphism, we obtain that $E^{\ell-g-i,g}_\infty = Ext^g_R(H_{\ell-g-i}, R)$, which implies that $d^{\ell-g-i,g}_r = 0$ for all $r \geq 2$.

Since the spectral sequence converges there is a filtration $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{d-g}$ such that $F_{d-g} \cong H_i$ and $F_{j+1}/F_j \cong E^{\ell,q}_{g_r}$ with $p + q = \ell - i$ and $q = d - j$. As the map $H_i \to E^{\ell-g-i,g}_\infty$ is an isomorphism, the filtration reduces to $F_j = 0$ for all $j = 0, \ldots, d - g - 1$ and therefore $E^{\ell,q}_{g_r} = 0$ for $q > g$ with $p + q = \ell - i$. □

Now we are ready to state and prove an extension of a theorem of Hartshorne and Ogus from [6]. Notice that by requiring the proper Serre’s condition on all the Koszul homology modules, one can relax them by one and also remove the factoriality hypotheses on $R/I$.

2.7 Theorem. Let $R$ be a local Gorenstein ring and $y$ be a sequence of $\ell$ elements which generate an ideal of grade $g$. Let $h$ be an integer such that $h \geq \max\{2, \frac{1}{2} \dim(R/(y))\}$. If, for all $i \geq 0$, the $R/(y)$-module $H_i(y)$ satisfies Serre’s condition $S_h$, then $H_i(y)$ is Cohen-Macaulay for all $i = 0, \ldots, \ell - g$.

Proof. Set $H_i = H_i(y)$ for all $i$. Note that if $\dim R/(y) \leq 2$ or $h \geq d - g$ there is nothing to prove, so one may assume that $\dim R/(y) \geq 3$ and $h < d - g$. Since the conditions in the hypotheses localize, we may thus assume by induction that the ideal $(y)$ is strongly Cohen-Macaulay on the punctured spectrum.

We use the spectral sequence

$$E^{p,q}_2 = Ext^q_R(H_p, R) \Rightarrow H_{\ell-(p+q)}$$

from Construction 1.1. As before, one has that $E^{p,q}_2 = 0$ for $q < g$, but now the hypothesis that $H_i$ satisfies $S_h$ implies that $E^{p,q}_2 = 0$ holds for all $q > d - h$ as well. We use induction on $i$ to show that $H_i$ and $H_{\ell-g-i}$ are Cohen-Macaulay.

Taking as a base case for the induction $i = -1$, one sees that it holds trivially as $H_{-1} = 0 = H_{\ell-g+1}$. For the inductive step, we suppose that $H_p$ and $H_{\ell-g-p}$ are Cohen-Macaulay for all $p \leq i$, and we show that $H_{i+1}$ and $H_{\ell-g-(i+1)}$ are Cohen-Macaulay. The inductive hypothesis implies that $E^{p,q}_2 = 0$ whenever $q > g$ and either $p \leq i$ or $p \geq \ell - g - i$.

Serre’s condition $S_h$ on the homology modules and Proposition 2.3 imply that the modules $H_k$ and $Ext^q_R(H_{\ell-g-k}, R)$ are isomorphic via the edge homomorphism for any $k \geq 0$. It follows from Lemma 2.6 that the differentials emanating from row $g$ on any page of the spectral sequence are zero maps. Hence, the top two possibly nonzero modules $E^{p,q}_2$ in column $\ell - g - (i+1)$, namely those with $(p, q) = (\ell - g - i - 1, d - h)$ and $(\ell - g - i - 1, d - h - 1)$ (or just the one module $E^{p,q}_2$ with $(p, q) = (\ell - g - i - 1, d - h)$ if $d - h - 1 = g$), have all differentials coming into or out of them equal to zero on any page of the spectral sequence. Therefore,
for these values of \((p, q)\) one has \(E^p_q = E^{p+1}_q = \cdots = E^\infty_q\). On the other hand, Lemma 2.6 also yields that \(E^p_{\infty} = 0\) for any \(q > g\). So one gets that
\[
\text{Ext}^{d-h}_R(H_{\ell-g-i-1}, R) = 0 = \text{Ext}^{d-h-1}_R(H_{\ell-g-i-1}, R)
\]
(or if \(d-h-1 = g\) just that \(\text{Ext}^{d-h-1}_R(H_{\ell-g-i-1}, R) = 0\) and so \(H_{\ell-g-i-1}\) is clearly Cohen-Macaulay). This yields the improved inequality depth \(H_{\ell-g-i-1} \geq h + 2\) and therefore
\[
\text{depth} H_{i+1} + \text{depth} H_{\ell-g-i-1} \geq 2h + 2 \geq \text{dim}(R/(y)) + 2.
\]
holds. Together with the earlier assumption that \((y)\) is strongly Cohen-Macaulay on the punctured spectrum, this inequality now implies that \(H_{i+1}\) and \(H_{\ell-g-i-1}\) are Cohen-Macaulay as desired by the remark below applied to the ring \(S = R/(x)\) where \(x\) is an \(R\)-regular sequence of length \(g\) in the ideal \((y)\), using the fact that the supports of \(R/(y)\) and \(R/(x)\) are the same.

2.8 Remark. Let \(S\) be a Cohen-Macaulay ring with canonical module \(\omega_S\). If an \(S\)-module \(M\) and its dual \(M^\vee = \text{Hom}_S(M, \omega_S)\) satisfy
\[
\text{depth} M_p + \text{depth}(M^\vee)_p \geq \text{dim}(R/(y))_p + 2
\]
for all prime ideals \(p\) in \(S\) of height at least 3, and if \(M_p\) is Cohen-Macaulay for all prime ideals \(p\) in \(S\) of height at most 2, then \(M\) is Cohen-Macaulay. This can essentially be found in the paper \([6]\) of Hartshorne and Ogus, but we give the version as stated and proved in \([11, \text{Lemma 5.8}]\).

Theorem 2.7 can also be compared to this result for \(M = H_i(y)\) and \(S = R/(x)\) where \(x\) is an \(R\)-regular sequence of length \(g\) in the ideal \((y)\). In fact, by Proposition 2.3 and Remark 1.3
\[
H_{\ell-g-i}(y) \cong \text{Hom}_{R/(x)}(H_i(y), R/(x)) \cong H_i(y)^\vee.
\]
The hypotheses of the theorem give the inequalities
\[
\text{depth} H_i(y)_p + \text{depth}(H_i(y)^\vee)_p \geq \text{dim}(R/(x))_p
\]
for all \(i\) and still yield that all \(H_i(y)\) are Cohen-Macaulay. Notice that by requiring high enough depth for all the Koszul homologies, we can weaken hypotheses on the depth required to get a conclusion of Cohen-Macaulay.

3. Sliding depth

The main result of the section is a generalization of \([8, \text{Theorem 1.4}]\), in which the authors show that ideals satisfying sliding depth are strongly Cohen-Macaulay in the presence of conditions limiting the number of local generators of the ideal.

3.1 Definition. An ideal \(I\) of \(R\) is strongly Cohen-Macaulay if all the non-vanishing Koszul homology modules are Cohen-Macaulay.

3.2 Definition. An ideal \(I\) is said to satisfy \(h\)-sliding depth or \(SD_h\) if it satisfies
\[
\text{depth} H_i(I) \geq \min\{d - g, d - \ell + i + h\}.
\]
The condition \(SD_h\) localizes, as was shown in \([7]\), where it was introduced.
3.3 Remark. Let $y$ a sequence of $\ell$ elements, and set $K_i = K_i(y)$ and $H_i = H_i(y)$. Let $Z_i$ and $B_i$ be the submodules of $K_i$ of cycles and boundaries, respectively. From the exact sequences

$$0 \rightarrow Z_{i+1} \rightarrow K_{i+1} \rightarrow B_i \rightarrow 0 \quad \text{and}$$

$$0 \rightarrow B_i \rightarrow Z_i \rightarrow H_i \rightarrow 0$$

one can easily verify that the following inequalities hold for all $i \geq 0$

$$\text{depth } Z_i \geq \min\{d, d - \ell + i + h + 1\}$$

when $I$ satisfies $SD_h$.

Given an ideal $I$, denote by $\mu(I)$ its minimal number of generators. The following theorem is a generalization of the main result in [8, Theorem 1.4]. Notice that for $j \geq 0$ condition (a) below is equivalent to $\mu(I_p) \leq \max\{ht(I), ht(p) + j\}$; with this formulation the theorem in [8] corresponds to the case $j = -1$ in the theorem below (their assumption of Cohen-Macaulayness for $R/I$ is not in fact necessary; see Remark 3.5). In fact, the proof that we give works for the case $j = -1$ as well, and hence provides a shorter proof of their result.

3.4 Theorem. Let $R$ be a local Gorenstein ring, and let $I$ be an ideal of $R$. Fix an integer $j \geq 0$. Suppose that the following conditions hold

(a) $\mu(I_p) \leq \text{ht}(p) + j$ for all prime ideals $p \supseteq I$.

(b) $I$ satisfies $SD_h$ where $h = \lceil \frac{j + 1}{2} \rceil$.

(c) $H_i(I)$ satisfies Serre’s condition $S_2$ as an $R/I$-module for $i = 0, \ldots, \ell - g$.

Then $I$ is strongly Cohen-Macaulay.

3.5 Remark. Notice that conditions (a) and (b) give that the homology modules $H_i(I)$ automatically satisfy Serre’s condition $S_2$ for all $i \geq \lceil \frac{j + 1}{2} \rceil + 1$. Indeed, by [8, §6(ii)] the condition $SD_h$ localizes. This implies that

$$\text{depth } H_i(I_p) \geq \dim R_p - \mu(I_p) + i + h \geq -j + i + h = h' + i + 1$$

for $h' = \lceil \frac{j + 1}{2} \rceil$ since $h' + h = j + 1$. For example, for $j = 0$ (and $j = -1$), one gets that $H_i(I)$ satisfies $S_2$ for all $i > 0$, and hence Theorem 3.4 immediately yields Corollary 3.6 below. Similarly, for $j = 1$, only $H_0(I)$ and $H_1(I)$ do not automatically satisfy $S_2$ from the sliding depth condition, and hence one obtains Corollary 3.7 below.

3.6 Corollary. Let $R$ be a local Gorenstein ring, and let $I$ be an ideal of $R$ such that $R/I$ satisfies Serre’s condition $S_2$. Suppose the following conditions hold

(a) $\mu(I_p) \leq \text{ht}(p)$ for all prime ideals $p \supseteq I$.

(b) $I$ satisfies $SD_1$.

Then $I$ is strongly Cohen-Macaulay.

3.7 Corollary. Let $R$ be a local Gorenstein ring, and let $I$ be an ideal of $R$ such that $R/I$ satisfies Serre’s condition $S_2$. Suppose the following conditions hold

(a) $\mu(I_p) \leq \text{ht}(p) + 1$ for all prime ideals $p \supseteq I$.

(b) $I$ satisfies $SD_1$.

(c) $H_i(I)$ satisfies Serre’s condition $S_2$ as an $R/I$-module.

Then $I$ is strongly Cohen-Macaulay.
We first give a quick proof of Theorem 3.4 using the spectral sequence from Construction 1.1. Since there is an elementary proof modeled on that of [8, Theorem 1.4] that avoids spectral sequences, we also include it afterwards.

**Proof.** Denote by $H_i$ the homology modules $H_i(I)$. First note that if $\dim R/I \leq 2$ then the assertion follows trivially from condition (c). Assume that $\dim R/I \geq 3$. Let $\ell$ be the minimal number of generators of $I$ and $g$ be the grade of $I$; by adding a set of indeterminates to $R$ and to $I$ one may assume that $g > \ell - g + h + 1$. Furthermore, since the conditions on $I$ localize, one may assume by induction that $I$ is strongly Cohen-Macaulay on the punctured spectrum of $R$. Consider the spectral sequence

$$E_2^{p,q} = \Ext_R^q(H_p, R) \Rightarrow H_{\ell-(p+q)}$$

from Construction 1.1. As before, one has that $E_2^{p,q} = 0$ for $q < g$.

We use induction on $i$ to prove that $H_i$ and $H_{\ell-g-i}$ are Cohen-Macaulay. Note that for all $i \leq h$, by assumption (b), the module $H_{\ell-g-i}$ is Cohen-Macaulay and hence $H_i$ is also Cohen-Macaulay as one has $H_i(I) \cong \Ext_R^q(H_{\ell-g-i}(I), R)$ by Proposition 2.3. We use induction to prove that $H_i$ is Cohen-Macaulay for the remaining range $h < i < \ell - g - h$, considering the base cases ($i \leq h$) done. For the inductive step, we suppose that $H_p$ and $H_{\ell-g-p}$ are Cohen-Macaulay for all $p < i$ and we show that $H_i$ and $H_{\ell-g-i}$ are Cohen-Macaulay. The sliding depth hypothesis $SD_h$ yields inequalities

$$\text{depth } H_i \geq d - \ell + i + h, \quad \text{and } \quad \text{depth } H_{\ell-g-i} \geq d - \ell + (\ell - g - i) + h.$$

If they are not Cohen-Macaulay, we improve the inequality for depth $H_{\ell-g-i}$ by one in order to be able to apply Remark 2.8.

The inductive hypothesis implies that $E_2^{p,q} = 0$ whenever $q > g$ and either $p < i$ or $p > \ell - g - i$. In addition, by Proposition 2.3 and by Lemma 2.6 the differentials emanating from row $g$ on any page of the spectral sequence are zero maps. Therefore, all differentials landing in the term $E_2^{p,q}$ with $p_0 = \ell - g - i$ and $q_0 = d - (d - g - i + h) = g + i - h$ are zero.

Moreover, the hypothesis of $SD_h$ implies that $E_2^{p,q} = 0$ for $q > d - (d - g - p + h)$, that is, they vanish above the line though $E_2^{p_0,q_0}$ with a slope of $-1$. Therefore, all differentials emanating from $E_2^{p_0,q_0}$ are zero as well, and so one sees that $E_2^{p_0,q_0} = E_3^{p_0,q_0} = \cdots = E_\infty^{p_0,q_0}$. But Proposition 2.3 and Lemma 2.6 yield that $E_\infty^{p_0,q_0} = 0$; therefore one gets that $E_2^{p_0,q_0} = 0$, which yields the improved depth inequality

$$\text{depth } H_{\ell-g-i-1} \geq d - \ell + (\ell - g - i) + h + 1.$$

In conclusion, one has

$$\text{depth } H_{\ell-g-i} + \text{depth } H_i \geq d - \ell + (\ell - g - i) + h + 1 + d - \ell + i + h \geq (d - g) + (d + j - \ell) + 2 \geq \dim R/I + 2,$$

where the last inequality follows from the fact that $d + j - \ell \geq 0$: Indeed condition (a) at $p = m$ yields the inequality $\ell \leq d + j$; note that even if $j = -1$ this inequality would hold as $\dim R/I \geq 3$.

Together with the earlier assumption that $I$ is strongly Cohen-Macaulay on the punctured spectrum, this inequality now implies that $H_i$ and $H_{\ell-g-i}$ are Cohen-Macaulay as desired by the result [11, Lemma 5.8] or [6], as described in Remark 2.8, applied to the ring $S = R/(\mathbf{x})$ where $\mathbf{x}$ is an $R$-regular sequence of length $g$ in the ideal $(y)$, using the fact that the supports of $R/(y)$ and $R/(\mathbf{x})$ are the same. □
The following is a proof of Theorem 3.4 that avoids spectral sequences.

Alternate Proof of Theorem 3.4. Let \( \ell \) be the minimal number of generators of \( I \) and \( g \) be the grade of \( I \). As in the other proof, we may assume that one has \( \dim R/I \geq 3 \) and \( g > \ell - g + h + 1 \) and that \( I \) is strongly Cohen-Macaulay on the punctured spectrum of \( R \).

Set \( H_i = H_i(I) \) for all \( i \). Since \( H_i \) satisfies Serre’s condition \( S_2 \) as an \( R/I \)-module for all \( i \), there is an isomorphism \( H_i \cong \text{Ext}^g_R(H_{\ell - g - i}, R) \) by Proposition 2.3. Next note by condition (b) that \( H_{\ell - g - i} \), and hence \( H_i \) by Proposition 2.3, are Cohen-Macaulay for \( i = 0, \ldots h \). In particular, one sees that \( R/I \) is Cohen-Macaulay. In order to prove that \( H_i \) and \( H_{\ell - g - i} \) are Cohen-Macaulay for the remaining range \( h < i < \ell - g - h \) we prove the inequality

\[
\text{depth} \ H_{\ell - g - i} + \text{depth} \ H_i \geq \dim R/I + 2
\]

and the desired conclusion follows from Remark 2.8 as in the other proof.

Since \( I \) satisfies \( SD_h \) one has inequalities

\[
\text{depth} \ H_{\ell - g - i} + \text{depth} \ H_i \geq d - \ell + (\ell - g - i) + h + d - \ell + i + h
\]

\[
\geq (d - g) + (d + j - \ell) + 1.
\]

Notice, as in the previous proof, that condition (a) at \( p = m \) yields the inequality \( \ell \leq d + j \) and so gives that \( d + j - \ell \geq 0 \).

If, in fact, \( d + j - \ell \geq 1 \), then for all \( i \) one has that

\[
\text{depth} \ H_{\ell - g - i} + \text{depth} \ H_i \geq d - \ell + 2 \geq d - g + 2 = \dim R/I + 2
\]

and hence \( H_i \) is Cohen-Macaulay for all \( i \).

If \( d + j - \ell = 0 \), we use induction on \( i \) to prove that \( H_i \) (equivalently, \( H_{\ell - g - i} \)) is Cohen-Macaulay for all \( i \), or, equivalently, that the inequality 1 holds. For \( i = 0 \) the conclusion is obvious as \( R/I \) is Cohen-Macaulay. Note further that depth \( Z_{\ell - g + 1} \geq d - g + 2 \) since \( Z_{\ell - g + 1} \cong B_{\ell - g + 1} \) and \( B_{\ell - g + 1} \) has projective dimension at most \( g - 2 \) as \( H_i = 0 \) for \( i \geq \ell - g + 1 \).

For the inductive step, we prove a seemingly stronger statement. We assume that \( H_i \) is Cohen-Macaulay and that depth \( Z_{\ell - g - (i - 1)} \geq d - g + 2 \) and prove that \( H_{i+1} \) is Cohen-Macaulay and that depth \( Z_{\ell - g - i} \geq d - g + 2 \).

We use the following exact sequences from Remark 3.3

\[
0 \to Z_{t+1} \to K_{t+1} \to B_t \to 0 \tag{3}
\]

\[
0 \to B_t \to Z_t \to H_t \to 0 \tag{4}
\]

From the exact sequence (3) with \( t = \ell - g - i \), depth-counting yields that depth \( B_{\ell - g - i} \geq d - g + 1 \) from the inductive hypothesis. Thus the exact sequence (4) with \( t = \ell - g - i \) gives that

\[
\text{depth} \ Z_{\ell - g - i} \geq \min\{\text{depth} \ B_{\ell - g - i}, \text{depth} \ H_{\ell - g - i}\} = d - g,
\]

where equality holds since \( H_{\ell - g - i} \) is Cohen-Macaulay.

The exact sequence (4) yields the long exact sequence

\[
0 = \text{Ext}^{g-1}_R(H_{\ell - g - i}, R) \to \text{Ext}^{g-1}_R(Z_{\ell - g - i}, R) \to \text{Ext}^{g-1}_R(B_{\ell - g - i}, R) \to \text{Ext}^{g}_R(H_{\ell - g - i}, R) \to \text{Ext}^{g}_R(Z_{\ell - g - i}, R) \to \text{Ext}^{g}_R(B_{\ell - g - i}, R) = 0, \tag{5}
\]
where the last equality comes from the fact that depth $B_{t-g-i} \geq d - g + 1$. The exact sequences (3) and (4), together with the fact that $\text{Ext}^j_R(H_i, R) = 0$ for all $j < g$, yield a series of isomorphisms (note $g - 1 - i > 0$ for $i = 0, \ldots, \ell - g$)

$$\text{Ext}^{g-1}_R(B_{t-g-i}, R) \cong \text{Ext}^{g-2}_R(Z_{t-g-i+1}, R) \cong \cdots \cong \text{Ext}^{g-1-i}_R(B_{t-g}, R).$$

One can see that the last Ext module is isomorphic to $H_i$ by noting that the tail of the Koszul complex is a free resolution of $B_{t-g}$. Furthermore, the term $\text{Ext}^g_R(H_{t-g-i}, R)$ is also isomorphic to $H_i$. The exact sequence (5) then becomes

$$0 \to \text{Ext}^{g-1}_R(Z_{t-g-i}, R) \to H_i \xrightarrow{\psi} H_i \to \text{Ext}^g_R(Z_{t-g-i}, R) \to 0.$$

As the module $Z_{t-g-i}$ is a second syzygy, say of $W$, one has that $\text{Ext}^g_R(Z_{t-g-i}, R) \cong \text{Ext}^{g+2}_R(W, R)$. But for primes $p$ of height $g$ or $g + 1$ the module $\text{Ext}^{g+2}_R(W, R)_p$ vanishes as $R_p$ has injective dimension at most $g + 1$. So $\psi$ is surjective, hence an isomorphism for such primes; in particular, $\psi$ is an isomorphism in codimension 1 over $R/I$. Since $H_i$ satisfies $S_2$ as an $R/I$-module, this implies that $\psi$ is an isomorphism. Therefore, $\text{Ext}^{g-1}_R(Z_{t-g-i}, R)$ and $\text{Ext}^g_R(Z_{t-g-i}, R)$ vanish, yielding depth $Z_{t-g-i} \geq d - g + 2$.

By the exact sequence (3) for $t = \ell - g - i - 1$ a depth count yields that depth $B_{t-g-i-1} \geq d - g + 1$. Similarly, the exact sequence (4) for $t = \ell - g - i$ yields

$$\text{depth } H_{t-g-i-1} \geq \min\{\text{depth } B_{t-g-i-1} - 1, \text{depth } Z_{t-g-i-1}\} \geq d - g - i + h$$

where the last inequality is due to Remark (3.3) and the fact that $i > h$. Thus, the inequality (1) holds with $i$ replaced by $i + 1$, completing the inductive step. \qed

4. Low Dimension

For low dimensional ideals, some surprising relationships between the Koszul homologies fall out of the spectral sequence, as was also noted by Chardin in [4].

4.1 Proposition. Let $R$ be a local Gorenstein ring and let $I$ be an ideal such that $\dim R/I = 2$. Assume that the Koszul homology module $H_{t-g-i-1}(I)$ is Cohen-Macaulay for some $i$. If either $H_i(I)$ or $H_{t-g-i-2}(I)$ has positive depth, then $H_i(I)$ is Cohen-Macaulay.

Proof. Consider the spectral sequence from Construction 1.1. Set $H_p = H_p(I)$ for all $p$. Since $\dim R/I = 2 = d - g$, the second page $E_2$ has at most 3 nonzero rows, namely rows $g, g + 1, g + 2$. Thus, one has $E_2^{g, g+1} = E_2^{g+1, g+1}$. Since $H_{t-g-i-1}$ is Cohen-Macaulay, then the edge map $H_{t-g-i-1} \to \text{Ext}^g_R(H_{i+1}, R)$ is an isomorphism by Proposition 2.3. One then has that $\text{Ext}^{d-1}_R(H_i, R) = E_2^{i, g+1} = E_2^{g, g+1} = 0$, where the last equality is by Lemma 2.6. This implies that depth $H_i \neq 1$. Therefore, if $H_i(I)$ has positive depth, then $H_i$ is Cohen-Macaulay.

Moreover, Lemma 2.6 also yields that the differential $d^{g+1}_2$ is zero. Thus, the equality $E_2^{g, d} = E_2^{g, d}$ holds. Therefore, $\text{Ext}^{g+2}_R(H_i, R) = E_2^{g, d}$ is isomorphic to a submodule of $H_{t-g-i-2}$. Since $R$ is Gorenstein and $g + 2 = d$, the module $\text{Ext}^{g+2}_R(H_i, R)$ has finite length. If $H_{t-g-i-2}$ has positive depth then $\text{Ext}^{g+2}_R(H_i, R)$ vanishes and so $H_i(I)$ is Cohen-Macaulay. \qed
Similar results can be deduced in other low dimensions. As an example, consider the following result.

4.2 Proposition. Let $R$ be a local Gorenstein ring and let $I$ be an ideal such that $\dim R/I = 3$. Assume that the Koszul homology modules $H_{r-g-i-1}(I)$ and $H_{r-g-i-2}(I)$ satisfy Serre’s $S_2$ condition for some $i$. If both $H_i(I)$ and $H_{i-1}(I)$ have positive depth, then $H_i(I)$ is Cohen-Macaulay.

Proof. Consider the spectral sequence from Construction 1.1. Set $H_p = H_p(I)$ for all $p$. Since $\dim R/I = 3 = d - g$, the second page $E_2$ has at most 4 nonzero rows, namely rows $g$, $g + 1$, $g + 2$, and $g + 3$. Therefore, the only possible nonzero differential coming in or out of $E_2^{i,g+1}$ on any page is the one to $E_2^{i-1,g+3}$ on page 2, but the fact that $H_{i-1}(I)$ has positive depth implies that $E_2^{i-1,g+3} = 0$. Therefore, one has that $E_2^{i,g+1} = E_2^{i,g+3}$. But since the edge map $H_{r-g-i-1} \to \Ext^g_R(H_{i+1}, R)$ is an isomorphism by Proposition 2.3, Lemma 2.6 implies that $E_2^{i,g+1}$ vanishes. Therefore, depth $H_i \neq 2$. Lemma 2.6 also yields that the differential $d_2^{i+1,g}$ is zero. This being the only possible nonzero differential coming in or out of $E_2^{i,g+2}$, one gets that $E_2^{i,g+2} = E_2^{i+1,g+2}$. However, since the edge map $H_{r-g-i-2} \to \Ext^g_R(H_{i+2}, R)$ is an isomorphism by Proposition 2.3, Lemma 2.6 implies that $E_2^{i,g+2}$ vanishes. Therefore, depth $H_i \neq 1$. Since depth $H_i \neq 0$ by hypothesis, $H_i(I)$ must be Cohen-Macaulay. \qed

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