LETTER

Time scale separation in the low temperature East model: rigorous results

P Chleboun$^1$, A Faggionato$^2$ and F Martinelli$^1$

$^1$ Dipartimento di Matematica, Università Roma Tre, Largo S. L. Murialdo, I-00146 Roma, Italy
$^2$ Dipartimento di Matematica ‘G. Castelnuovo’, Università ‘La Sapienza’, Piazzale Aldo Moro 2, I-00185 Roma, Italy
E-mail: paul@chleboun.co.uk, faggiona@mat.uniroma1.it and martin@mat.uniroma3.it

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Abstract. We consider the non-equilibrium dynamics of the East model, a linear chain of 0–1 spins evolving under a simple Glauber dynamics in the presence of a kinetic constraint which forbids flips of those spins whose left neighbor is 1. We focus on the glassy effects caused by the kinetic constraint as $q \downarrow 0$, where $q$ is the equilibrium density of the 0 spins. Specifically we analyze time scale separation and dynamic heterogeneity, i.e. non-trivial spatio-temporal fluctuations of the local relaxation to equilibrium, one of the central aspects of glassy dynamics. For any mesoscopic length scale $L = O(q^{-\gamma})$, $\gamma < 1$, we show that the characteristic time scales associated with two length scales $d/q^\gamma$ and $d'/q^\gamma$ are indeed separated by a factor $q^{-a}$, $a = a(\gamma) > 0$, provided that $d'/d$ is large enough independently of $q$. In particular, the evolution of mesoscopic domains, i.e. maximal blocks of the form 111...10, occurs on a time scale which depends sharply on the size of the domain, a clear signature of dynamic heterogeneity. Finally we show that no form of time scale separation can occur for $\gamma = 1$, i.e. at the equilibrium scale $L = 1/q$, contrary to what was previously assumed in the physical literature based on numerical simulations.
Kinetically constrained spin models (KCMs) are stochastic particle models, usually defined in terms of a non-interacting Hamiltonian, whose dynamical evolution is determined by local rules encoding a kinetic constraint. A typical move of the KCM’s dynamics is the creation/destruction of a particle with rates determined by the given Hamiltonian through the usual detailed balance condition. However the move is inhibited unless the configuration of the neighboring particles satisfies a certain constraint. Examples include the \( f \)-facilitated models \[14\] which require \( f \) neighbors to be empty and anisotropic models in which certain preassigned neighbors must be empty.

The main interest for these models (see e.g. \[2, 4, 15\], \[18\]–\[20\]) stems from the fact that KCMs, in spite of their simplicity, display many key dynamical features of real glassy materials: an ergodicity breaking transition at some critical value \( q_c \) of the vacancy density \( q \), a huge relaxation time for \( q \) close to \( q_c \), super-Arrhenius behavior, dynamic heterogeneity (i.e. non-trivial spatio-temporal fluctuations of the local relaxation to equilibrium) and ageing, to mention but a few. Mathematically, KCMs pose very challenging and interesting problems because of the hardness of the constraint, with ramifications for bootstrap percolation problems \[21\], combinatorics \[9\], coalescence processes \[11, 13\] and random walks on triangular matrices \[17\]. Quite surprisingly, some of the tools developed for the analysis of the relaxation process of KCMs proved to be quite powerful also in other contexts such as card shuffling problems \[3\] and the random evolution of surfaces \[7\].

In this work we report on some rigorous findings \[8\] concerning the glassy dynamics of the East model, a popular KCM (see e.g. \[16, 19, 20\] and \[1, 5, 12\]) defined on a one-dimensional integer lattice of \( L \) sites, \( \Lambda = \{1, 2, \ldots, L\} \), for which each site can be in state 0 or 1, corresponding to empty or occupied respectively. We denote the state space by \( \Omega_\Lambda := \{0, 1\}^\Lambda \). The configuration evolves under Glauber type dynamics in the presence of the kinetic constraint which forbids flips of those spins whose left neighbor is in state 1. Each vertex, \( x \), waits an independent mean-1 exponential time and then, provided that the current configuration \( \sigma \) satisfies the constraint \( \sigma_{x-1} = 0 \), the value of \( \sigma \) at \( x \) is refreshed.
and set equal to 1 with probability 1 − q and to 0 with probability q. The leftmost vertex, \( x = 1 \), is unconstrained; equivalently we could imagine that at \( x = 0 \) there is a frozen 0 spin. Since the constraint at site \( x \) does not depend on the configuration at site \( x \), detailed balance is satisfied with respect to the product Bernoulli measure \( \pi := \prod_{x \in \Lambda} \pi_x \), where \( \pi_x \) is the Bernoulli measure with density \( p = 1 - q \).

We focus on case \( q \ll 1 \) because this is the regime where the glassy features of the time evolution are more pronounced. It is also the situation which requires more theoretical work, rigorous and non-rigorous, because of the huge relaxation times which make numerical simulation infeasible. The equilibrium density of zeros, \( q \), is often written as \( q = e^{-\beta/(1 + e^{-\beta})} \), where \( \beta \) is the inverse temperature, so small \( q \) corresponds to the low temperature regime.

Before describing our results we first briefly review some previous work. It is known rigorously that the stationary dynamics relax exponentially fast on a time scale \( T_{\text{rel}} \) which is uniformly bounded in the length \( L \) of the chain \([1, 5]\) and diverges very rapidly as \( q \downarrow 0 \) (cf (3) below). It is also known \([6]\) that, after time \( T_{\text{rel}} \) and starting from a large class of initial laws, the local vacancy density converges exponentially fast to \( q \).

However the most interesting and challenging dynamical behavior occurs for \( q \ll 1 \) on time scales shorter than the relaxation time, \( T_{\text{rel}} \). It was first argued in \([19, 20]\) and then later proved in \([11]\) that the dynamics of the infinite East chain in a space window \([1, 2^N]\) and up to time scales \(^3\) \( O(1/q^N) \) is well approximated, as \( q \downarrow 0 \), by a certain hierarchical coalescence process (HCP) \([13]\). In this HCP, vacancies are isolated and domains, i.e. maximal intervals of the form 111...10, with cardinality between \( 2^{n-1} + 1 \) and \( 2^n \), \( n \leq N \), are able to coalesce with the domain on their right only at time scales \( \sim (1/q)^n \). As a result, ageing and dynamic heterogeneity in the above regime emerge in a natural way, with a scaling limit\(^4\) for the relevant quantities in the same universality class as several other mean field coalescence models of statistical physics \([10]\).

The above result says nothing about the dynamics and its characteristic time scales at intermediate (mesoscopic) length scales \( L = 1/q^\gamma \), \( 0 < \gamma < 1 \), or at the typical inter-vacancy equilibrium scale \( 1/q \). At these length scales the low temperature dynamics of the East model is no longer predominantly driven by an effective energy landscape, but entropic effects become crucial and a subtle entropy/energy competition comes into play. As an example of these effects, we observe that the natural extrapolation to the equilibrium scale \( L_c = 1/q \) of the characteristic time scale \( T_{\text{rel}}^{(n)} \approx 1/q^n \), appropriate for domains of length \( L_n \approx 2^n \) with \( n = O(1) \), led (for example see \([20]\)) to the wrong prediction of a relaxation time \( \sim 2^{\theta^2} \), where \( \theta = \log_2(1/q) \), to be compared with the correct scaling \( 2^{\theta^2/2(1+o(1))} \) (see (4) with \( \gamma = 1 \)). Notice that for small temperatures \( \theta \simeq \beta/\ln(2) \).

2. Results

Our first result is to show that, in the low temperature limit, for a given system of size \( L = O(1/q) \), the three natural characteristic time scales of the East model are equivalent (up to universal constants). This equivalence is important because of various notions of

\(^3\) Recall that \( f = O(g) \) means that there exists \( c > 0 \) independent of \( q \) such that \( |f| \leq cg \).

\(^4\) We emphasize that the physical literature (see e.g. \([10, 20]\)) assumed the existence of a scaling limit. One of the main contributions of \([13]\) was to actually prove this, together with giving a complete classification of the universality classes.
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‘equilibration time’ which appear in the physics literature. The three characteristic times considered, for a system of size \( L \), are as follows:

(i) The relaxation time \( T_{\text{rel}}(L) \), defined as the inverse of the spectral gap of the infinitesimal generator of the process, i.e. the poorest relaxation time found by fully diagonalizing the master equation.

(ii) The mixing time, \( T_{\text{mix}}(L) \), which is the time required for the chain to be close (in the sense of total variation distance) to the equilibrium distribution \( \pi \), starting from the worst case initial distribution.

(iii) The mean first-passage time \( T_{\text{hit}}(L) \); starting from the state \( \sigma = (1, 1, \ldots, 1, 0) \) with a single vacancy at site \( L \), we define \( T_{\text{hit}}(L) \) as the mean time for the spin at site \( L \) to flip to 1.

As observed in [20], the time \( T_{\text{hit}}(L) \) gives some insight into the low temperature dynamics, because of the following argument. For small \( q \) the process quickly reaches a state in which there are few isolated 0 spins separated by domains of 1 spins. If two vacancies are separated by a domain of cardinality \( \ell \) then, conditioned on the vacancy on the left surviving, the mean time for removing the vacancy on the right is exactly \( T_{\text{hit}}(\ell) \).

For a given \( L = \text{O}(1/q) \), we establish that the ratio of any two of the above time scales is bounded by a universal constant.

Having established the above equivalence, our next result concerns the dependence of the characteristic time scales on the length \( L \). In particular, we address the issue of time scale separation up to the equilibrium scale \( 1/q \). For two system sizes \( L', L \) which do not depend on \( q \), \( T_{\text{rel}}(L') \gg T_{\text{rel}}(L) \) if and only if\(^5\) \([\log_2 L'] \gg [\log_2 L]\) (where \( f \gg g \) means that \( f \) dominates \( g \) by a factor of \( 1/q^\alpha \), \( \alpha > 0 \), as \( q \downarrow 0 \)). For mesoscopic length scales, that is for \( L = d/q^\gamma \) for some \( d > 0 \) and \( \gamma \in (0, 1) \), we show that there exists some \( \lambda > 1 \) depending only on \( \gamma \) such that \( T_{\text{rel}}(\lambda L) \gg T_{\text{rel}}(L) \). Moreover, if \( \gamma < 1/2 \) then \( \lambda = 2 \).

While at finite lengths (independent of \( q \)) the question of time scale separation is completely characterized, for mesoscopic lengths of order \( \text{O}(1/q^\gamma) \), for \( \gamma \in (0, 1) \), our knowledge is less detailed. Although we show that time scale separation occurs between length scales whose ratio is above a certain threshold \( \lambda \), we do not know, for example, whether it occurs in a ‘continuous’ fashion. Given \( \gamma \in (0, 1] \) we say that continuous time scale separation occurs at length scale \( 1/q^\gamma \) if \( T_{\text{rel}}(d'/q^\gamma) \gg T_{\text{rel}}(d/q^\gamma) \) whenever \( d' > d \).

This still remains an interesting open problem on mesoscopic length scales.

In [19, 20] a continuous time scale separation was conjectured for \( \gamma = 1 \), i.e. for length scales of the order of the equilibrium inter-vacancy distance. The following hypothesis was put forward on the basis of numerical simulations:

\[
\left( \frac{T_{\text{hit}}(d/q)}{T_{\text{rel}}(\infty)} \right)^{1/\log(1/q)} \rightarrow f(d) \quad \text{as} \quad q \downarrow 0, \tag{1}
\]

for some positive, strictly increasing function \( f(d) \). This hypothesis was fundamental to the so-called super-domain dynamics proposed by Evans and Sollich for describing the time evolution of the stationary East model [19, 20]. Notice that, because of the equivalence of the time scales \( T_{\text{hit}}(L), T_{\text{rel}}(L) \) for \( L = \text{O}(1/q) \), we could have replaced \( T_{\text{hit}}(d/q) \) in (1) by \( T_{\text{rel}}(d/q) \) without changing the scaling function \( f \).

\(^5\) Recall that \([x]\) is the smallest integer \( n \) such that \( n \geq x \).
Our next result shows that there is no form of time scale separation at the equilibrium length scale, that is for $\gamma = 1$; in particular, the above conjecture does not hold. In fact we are able to show that, as $q \downarrow 0$,

$$T_{\text{rel}}(d/q)/T_{\text{rel}}(d'/q) \leq c(d,d'),$$

with $c(d',d)$ independent of $q$. So the ratio of the characteristic time scales for system sizes which differ on the equilibrium length does not diverge as $q \downarrow 0$; in particular, this implies that $f$ in (1) is constant. This is an important example of a case in which numerical simulations can be misleading for systems where the characteristic times are so large as to not be practically accessible through simulations, and emphasizes the need for rigorous results.

Finally, building on these results, we are able to prove [8] that the East process exhibits dynamic heterogeneity on mesoscopic length scales, $1/q^\gamma$ for $\gamma \in (0,1)$, in the following sense. For a fixed $\gamma < 1$, starting from any initial configuration, after a time $T_{\text{rel}}(1/q^\gamma)$ domains much shorter than $1/q^\gamma$ are very unlikely to be present, whereas any domain that was much larger than $1/q^\gamma$ initially is very likely to have survived.

A key result required for proving the findings outlined above was the establishing of extremely precise bounds on the relaxation time as a function of $L$ and $q$. We prove that for any $d > 0$ and $L \leq d/q$, there exist positive exponents $\alpha, \alpha'$ depending only on $d$ such that

$$\frac{n!}{q^{\nu_2(\frac{d}{2})}} q^\alpha \leq T_{\text{rel}}(L) \leq \frac{n!}{q^{\nu_2(\frac{d}{2})}} q^{-\alpha'},$$

where $n = \lceil \log_2 L \rceil$. (3)

In particular, on mesoscopic length scales $L = O(1/q^\gamma)$, $\gamma < 1$, the constants $\alpha$ and $\alpha'$ depend only on $\gamma$. For $L = \lceil d/q^\gamma \rceil$, with $\gamma \in (0,1]$ and some $d > 0$, (3) gives the leading order dependence of the relaxation time on $\theta = \log_2(1/q)$ and $\gamma$ as follows:

$$2^{F(\theta,\gamma) - c \theta} \leq T_{\text{rel}}(\lceil d/q^\gamma \rceil) \leq 2^{F(\theta,\gamma) + c' \theta},$$

where $F(\theta,\gamma) = \gamma (1 - \gamma/2) \theta^2 + \gamma \theta \log_2 \theta$. (4)

for two positive constants $c, c'$ which are independent of $\theta$. The bounds in (3) are also sufficient to show that time scale separation does occur on mesoscopic length scales ($\gamma < 1$).

**Remark.** We emphasize that (3) alone does not exclude some weak form of time scale separation at the equilibrium scale ($\gamma = 1$) because of the presence of the unknown exponents $\alpha, \alpha'$. Thus (2) does not follow from (3); the two results required different mathematical analysis.

Heuristically the above behavior of the relaxation time can be justified as follows. It was shown in [9, 19, 20] that, starting from the state $\sigma = (1,1,\ldots,1,0)$ with a single vacancy at $L = 2^n$, in order to flip the rightmost spin the system has to create at least $n$ extra vacancies. Since the creation of a vacancy requires the overcoming of a local energy barrier, in a first approximation the non-equilibrium dynamics of the East model for $q \ll 1$ is driven by a non-trivial energy landscape whose characteristic activation time $t_n$ is of order $1/q^n$. The other main contribution in (3), $n!/2^{\nu_2(n)}$, is much more subtle and more difficult to justify heuristically. It is an entropic term related, in some sense, to the number of configurations inside $\{1,2,\ldots,2^n - 1\}$ which can be reached from the state $\sigma$ using at most $n$ vacancies. For such a quantity, call it $V(n)$, the following bounds were established.
in [9]:

\[ c_1 n! \frac{\binom{n}{2}}{2} \leq V(n) \leq c_2 n! \frac{\binom{n}{2}}{2} \]

with \( c_1, c_2 \) positive constants in \((0, 1)\).

A first naive guess would be that the actual relaxation time is the activation time \( t_n \) reduced by a factor proportional to \( V(n)^{-1} \) (see [6] for a rigorous lower bound on \( T_{rel}(L) \) based on this idea). Notice however that the true reduction factor in (3) is much smaller, and equal to \( \frac{2^{\binom{n}{2}}}{n!} \). Thus only a tiny fraction \( \left( \frac{1}{n!} \right)^2 \) of the configurations reachable with \( n \) vacancies really matter. What happens is that most configurations with \( n \) vacancies will return quickly to the initial state \( \sigma \) before removing the vacancy at \( 2^n \) and are therefore not visited during the last excursion which overcomes the energy barrier and removes the vacancy.

3. Summary

In this note we reported on some rigorous results concerning time scale separation and dynamic heterogeneities for the low temperature East model. We prove a strong equivalence of three characteristic time scales which appeared in the physical literature and we established a strong form of time scale separation up to, but excluding, the equilibrium length scale \( 1/q \). A basic ingredient for these results is a novel rigorous approach to estimating precisely the relaxation time, in which, besides the activation time across energy barriers, key entropic contributions are accounted for. We prove the absence of any form of time scale separation at the equilibrium length scale, including the one conjectured by Evans and Sollich which was used to formulate their super-domain dynamics. According to our results, the scaling limit of the stationary East model on scales \( L \geq 1/q \) remains an open problem. We observe that our findings are consistent with the following conjecture [1]:

**Conjecture.** As \( q \downarrow 0 \), the stationary East process in \([0, +\infty)\), after rescaling space by \( q \) and speeding up time by the relaxation time, converges to a limit point process \( X_t \) on \([0, +\infty)\) which can be described as follows:

(i) At fixed time \( t \), \( X_t \) is a Poisson (rate-1) process of particles \((\equiv 0 \text{ spins})\).

(ii) For each \( \ell > 0 \) and with a positive rate depending on \( \ell \), each particle deletes all particles to its right up to a distance \( \ell \) and replaces them by a new Poisson (rate-1) process of particles.

At this stage we do not have a detailed understanding of the dynamics on the equilibrium length scale, and thus there is no proposed form for the dependence of the rate on the distance \( \ell \), beyond the fact that it is decreasing in \( \ell \).

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