First-order quantum correction to the Larmor radiation from a moving charge in spatially homogeneous time-dependent electric field

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First-order quantum correction to the Larmor radiation is investigated on the basis of the scalar QED on a homogeneous background of a time-dependent electric field, which is a generalization of a recent work by Higuchi and Walker so as to be extended for an accelerated charged particle in a relativistic motion. We obtain a simple approximate formula for the quantum correction in the limit of the relativistic motion when the direction of the particle motion is parallel to that of the electric field.

PACS numbers: 41.60.-m,12.20.-m,03.70.+k

I. INTRODUCTION

The Larmor radiation is the classical radiation from a charged particle in an accelerated motion [1]. In a recent paper by Higuchi and Walker [2], the quantum correction to the Larmor radiation is investigated on the basis of the scalar quantum electrodynamics (QED). In their approach, the mode function for the complex scalar field is constructed with the Wentzel-Kramers-Brillouin (WKB) approximation, in a form expanded with respect to $\hbar$. In a series of Higuchi and Martin’s work [3–5] (see also references therein), it has been well understood that the mode function reproduces the classical Larmor formula when the radiation energy is evaluated at the order of $\hbar^0$. The first-order quantum correction to the classical Larmor radiation is evaluated at the order of $\hbar$ in Ref. [2], though the investigation is limited to the non-relativistic motion of the charged particle.

In the present paper, we consider a simple generalization of Higuchi and Walker’s work [2], in order to investigate the case a relativistic motion of an accelerated charge. Assuming a homogeneous but time-varying background of electric field, we derive a formula for the radiation energy of the order of $\hbar$, the first-order correction due to the quantum effect. This generalized formula is applicable to the accelerated charge in a relativistic motion, and we focus our investigation on the first-order quantum correction to the Larmor radiation in the limit of the relativistic motion. This paper is organized as follows: In section 2, we present the general formula for the first-order quantum correction to the Larmor radiation. In section 3, we show that the formula reproduces the same result obtained in Ref. [2], in the limit of the non-relativistic motion of the accelerated charge. Then, an approximate formula in the limit of the relativistic motion is presented. Section 4 is devoted to summary and conclusions. In the appendix A, a brief derivation of the approximate formulas is summarized. In the appendix B, we consider the validity of the WKB approximation. Throughout this paper, we use units in which the velocity of light equals 1, unless stated otherwise.

II. FORMULATION

We consider the scalar QED with the action,

$$S = \int dt d^3 x \left[ (D_\mu \phi)^\dagger D^\mu \phi - \frac{m^2}{\hbar^2} \phi^\dagger \phi - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} \right],$$

where $D_\mu = (\partial/\partial x^\mu + ieA_\mu/\hbar)$, $e$ and $m$ are the charge and the mass of the massive scalar field, respectively, and $\mu_0$ is the magnetic permeability of vacuum. We work in the Minkowski spacetime, but consider the homogeneous electric background field $\mathbf{E}(t)$, which is related to the vector potential by $\mathbf{A}_\mu = (0, \mathbf{A}(t))$ and $\dot{\mathbf{A}}(t) = -\mathbf{E}(t)$, where the dot denotes the differentiation with respect to the time. The equation of motion of the free scalar field yields

$$\left( \frac{\partial^2}{\partial t^2} + \frac{(\mathbf{p} - e\mathbf{A}(t))^2 + m^2}{\hbar^2} \right) \varphi_p(t) = 0,$$

where $\varphi_p(t)$ is the coefficient of the Fourier expansion of the field, i.e., the mode function. Using the mode function, which is normalized so as to be $\varphi_p^* \varphi_p - \varphi_p^* \varphi_p = i$, the quantized field is constructed as

$$\phi(x) = \sqrt{\frac{\hbar}{L^3}} \sum_p \left( \varphi_p(t) b_p + \varphi^*_p(t) c^*_p \right) e^{i\mathbf{p} \cdot \mathbf{x}/\hbar},$$
where $L^3$ is the volume of the space, the creation and annihilation operators satisfy the commutation relations,

\[ [b_{\mathbf{p}}, b_{\mathbf{p}}^\dagger] = \delta_{\mathbf{p}, \mathbf{p}^\prime}, \quad [b_{\mathbf{p}}, b_{\mathbf{p}^\prime}] = [b_{\mathbf{p}}^\dagger, b_{\mathbf{p}^\prime}^\dagger] = 0, \]

and the same relations hold for $c_{\mathbf{p}}$ and $c_{\mathbf{p}}^\dagger$. We also quantize the free electromagnetic field as,

\[ A_\mu = \sqrt{\frac{\mu_0 c}{L^3}} \sum_{\lambda=1,2} \sum_{\mathbf{k}} e^\lambda_\mu \left( \frac{e^{-ikt}}{\sqrt{2k}} a_\mathbf{k}^\lambda + \text{h.c.} \right) e^{i\mathbf{k} \cdot \mathbf{x}}, \]

where $e^\lambda_\mu$ denotes the polarization vector, and $a_\mathbf{k}^\lambda$ and $a_\mathbf{k}^\lambda_\dagger$ are the creation and annihilation operators which satisfy the following commutation relation,

\[ [a_\mathbf{k}^\lambda, a_{\mathbf{k}^\prime}^\lambda_\dagger] = \delta^{\lambda \lambda^\prime} \delta_{\mathbf{k}, \mathbf{k}^\prime}. \]

We consider the process, in which one photon is emitted from a charged particle, as shown in Fig. 1. Note that this process is prohibited without the background electric field because of the Lorentz invariance of the Minkowski spacetime, which ensures existence of the frame that the charged particle is at rest. However, on the electric field background, we have the radiation energy from the process, which can be evaluated, as follows. Using the in-in formalism [6, 7], we may compute the radiation energy at the lowest order of the coupling constant,

\[ E = \sum_{\lambda} \int d^3k \hbar \langle a_{\mathbf{k}^\lambda}^\dagger a_{\mathbf{k}^\lambda} \rangle = \hbar^{-2} \sum_{\lambda} \int d^3k \hbar \text{Re} \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} dt_1 \langle \text{in} | H_1(t_1) a_{\mathbf{k}^\lambda}^\dagger a_{\mathbf{k}^\lambda} H_1(t_2) | \text{in} \rangle, \]

where we adopted the range of the integration from the infinite past to the infinite future, and $| \text{in} \rangle$ denotes the initial state, which we choose as one charged particle state with the momentum $\mathbf{p}_i$, i.e., $| \text{in} \rangle = b_{\mathbf{p}_i}^\dagger | 0 \rangle$ and $H_1(t) = -\frac{ie}{\hbar} \int d^3x A^{\mu} \left\{ \left( \partial_\mu - \frac{ie}{\hbar} A_\mu \right) \phi^\dagger \phi - \phi^\dagger \left( \partial_\mu + \frac{ie}{\hbar} A_\mu \right) \phi \right\}$. Expression (7) leads to the lowest contribution corresponding to Fig. 1.

\[ E = -\frac{e^2}{\epsilon_0} \int \frac{d^3k}{(2\pi)^3} \left\{ \left[ \int dt \frac{e^{ikt}}{\sqrt{2k}} \left( \frac{\partial}{\partial t} \varphi_{\mathbf{p}_f}(t)^\ast \varphi_{\mathbf{p}_f}(t) - \varphi_{\mathbf{p}_f}(t)^\ast \left( \frac{\partial}{\partial t} \varphi_{\mathbf{p}_f}(t) \right) \right) \right]^2 \right. \]

\[ - \left. \int dt \frac{e^{ikt}}{\sqrt{2k}} \left( \frac{i(\mathbf{p}_f - e\mathbf{A})}{\hbar} \varphi_{\mathbf{p}_f}(t)^\ast \varphi_{\mathbf{p}_f}(t) + \varphi_{\mathbf{p}_f}(t)^\ast \left( \frac{i(\mathbf{p}_f - e\mathbf{A})}{\hbar} \varphi_{\mathbf{p}_f}(t) \right) \right) \right\}, \]

where $\mathbf{p}_f = \mathbf{p}_i - \hbar \mathbf{k}$, and $\epsilon_0$ is the permittivity of vacuum, which is related to $\mu_0$ by $\epsilon_0 \mu_0 = 1/c^2 = 1$. Performing the partial integral and using Eq. (2), we have

\[ E = -\frac{e^2}{\epsilon_0} \int \frac{d^3k}{(2\pi)^3} \left\{ \left[ \int dt \frac{e^{ikt}}{\sqrt{2k}} \left( \mathbf{k} \cdot (\mathbf{p}_i + \mathbf{p}_f - 2e\mathbf{A}) \right) \varphi_{\mathbf{p}_f}(t)^\ast \varphi_{\mathbf{p}_f}(t) \right]^2 \right. \]

\[ - \left. \int dt \frac{e^{ikt}}{\sqrt{2k}} \frac{\mathbf{p}_i + \mathbf{p}_f - 2e\mathbf{A}}{\hbar} \varphi_{\mathbf{p}_f}(t)^\ast \varphi_{\mathbf{p}_f}(t) \right\} \right]. \]
where \( \mathbf{k} \) is the unit vector of \( \mathbf{k} \), i.e., \( \mathbf{k} = \mathbf{k}/|\mathbf{k}| \). We consider the following WKB solution for the mode function

\[
\varphi_p(t) = \frac{1}{\sqrt{2 \Omega_p(t)}} \exp \left[ -i \int_{\Omega_p(t')}^{t} dt' \right]
\]

(11)

with

\[
\Omega_p(t) = \sqrt{(p - eA(t))^2 + m^2}/\hbar,
\]

(12)

then, Eq. (10) gives

\[
E = -\frac{e^2}{2\epsilon_0} \int \frac{d^3k}{(2\pi)^3} \int d\xi \int d\xi' \left\{ \left( \frac{\mathbf{k} \cdot (\mathbf{p}_i - e\mathbf{A})}{\sqrt{\mathbf{p}_i - e\mathbf{A}}^2 + m^2} \right)^2 + \frac{\hbar^2}{2} \left( \frac{\mathbf{k} \cdot (\mathbf{p}_i - e\mathbf{A})}{\sqrt{\mathbf{p}_i - e\mathbf{A}}^2 + m^2} \right)^2 \right\} e^{i\mathbf{k} \cdot \mathbf{x} - ik\xi}
\]

(13)

we used the notations \( \mathbf{x} = \mathbf{x}(t), \mathbf{x}' = \mathbf{x}(t') \), \( \mathbf{A} = \mathbf{A}(t), \mathbf{A}' = \mathbf{A}(t'), \mathbf{A}'' = \mathbf{A}(t'') \), and we introduced the new variable \( \xi \) instead of \( t \)

\[
\xi = t - \int_{t'}^{t} \frac{\mathbf{k} \cdot (\mathbf{p}_i - e\mathbf{A}(t''))}{\sqrt{\mathbf{p}_i - e\mathbf{A}(t'')}} dt''
\]

(16)

and \( \xi' \) is defined in the same way as \( \xi \) but with replacing \( t \) by \( t' \). Furthermore, we introduce the quantities parametrized by \( t \) (or \( \xi \)),

\[
\frac{dx}{d\tau} = \mathbf{p}_i - e\mathbf{A}, \quad \frac{dt}{d\tau} = \sqrt{(\mathbf{p}_i - e\mathbf{A})^2 + m^2},
\]

(17)

(18)

then Eq. (15) is rephrased as

\[
E = -\frac{e^2}{2\epsilon_0} \int \frac{d^3k}{(2\pi)^3} \int d\xi \int d\xi' \left\{ \left( \frac{\mathbf{k} \cdot (\mathbf{p}_i - e\mathbf{A}(t''))}{\sqrt{\mathbf{p}_i - e\mathbf{A}(t'')}} \right)^2 + \frac{\hbar^2}{2} \left( \frac{\mathbf{k} \cdot (\mathbf{p}_i - e\mathbf{A}(t''))}{\sqrt{\mathbf{p}_i - e\mathbf{A}(t'')}} \right)^2 \right\} e^{i\mathbf{k} \cdot \mathbf{x} - ik\xi}
\]

(19)
The mathematical technique adopted in Ref. [2] is equivalent to replacing $k$ in Eq. (19) by the partial differentiation with respect to $\xi$ or $\xi'$ which operates to $e^{ik\xi-i\xi'}$. The partial integrations lead to

$$E = E^{(0)} + E^{(1)} + O(h^2),$$

where we defined

$$E^{(0)} = -\frac{e^2}{2\varepsilon_0(2\pi)^3} \int d\Omega_k \int_0^\infty dk \int d\xi \int d\xi' e^{ik(\xi-\xi')} \left( \left( \hat{k} \cdot \frac{d^2x}{d\xi'^2} \right) \left( \hat{k} \cdot \frac{d^2x'}{d\xi'^2} \right) - \left( \frac{d^2x}{d\xi'^2} \cdot \frac{d^2x'}{d\xi'^2} \right) \right),$$

$$E^{(1)} = -\frac{e^2}{2\varepsilon_0(2\pi)^3} \int d\Omega_k \int_0^\infty dk \int d\xi \int d\xi' e^{ik(\xi-\xi')} \times \left\{ \frac{i\hbar}{4} \left( \frac{d}{d\xi} - \frac{d}{d\xi'} \right) \left( \frac{d}{d\xi} \frac{d}{d\xi'} \right) \left( \hat{k} \cdot \frac{d^2x}{d\xi'^2} \right) \left( \hat{k} \cdot \frac{d^2x'}{d\xi'^2} \right) - \frac{d}{d\xi} \cdot \frac{d}{d\xi'} \left( \frac{d}{d\xi} \cdot \frac{d}{d\xi'} \right) \left( \hat{k} \cdot \frac{dx}{dt} \frac{dx}{dt'} + \hat{k} \cdot \frac{dx'}{dt'} \frac{dx'}{dt} \right) \right\} + \frac{i\hbar}{2} \frac{d^2}{d\xi'^2} \int d\xi' \left[ \left( \hat{k} \cdot \frac{dx}{d\xi} \right) \left( \hat{k} \cdot \frac{dx'}{d\xi} \right) - \frac{d}{d\xi} \cdot \frac{d}{d\xi'} \left( \frac{d}{d\xi} \cdot \frac{d}{d\xi'} \right) \right] \int_{\xi(t')}^{\xi(t)} d\xi'' \frac{d}{d\xi''} \left( 1 - \left( \hat{k} \cdot \frac{dx''}{dt''} \right)^2 \right).$$

where $E^{(0)}$ and $E^{(1)}$ are the terms of the order of $\hbar^0$ and $\hbar^1$, respectively. Here, we assumed the boundary terms can be neglected, as is the case in Ref. [2]. The integration with respect to $k$ yields

$$E^{(0)} = \frac{e^2}{(4\pi)^2\varepsilon_0} \int d\Omega_k \int d\xi \left( \frac{d^2x}{d\xi'^2} \right)^2 \left( \hat{k} \cdot \frac{d^2x}{d\xi'^2} \right)^2.$$  (23)

The expression (23) yields the classical formula of the Larmor radiation from a charged particle. The first-order quantum correction of the order of $\hbar$ is described by Eq. (22), which yields

$$E^{(1)} = \frac{e^2\hbar}{(4\pi)^3\varepsilon_0} \int d\Omega_k \int d\xi \int d\xi' \int_0^\infty d\xi'' \frac{1}{\xi-\xi'} \times \left\{ \left( \frac{d}{d\xi} - \frac{d}{d\xi'} \right) \left( \frac{d}{d\xi} \frac{d}{d\xi'} \right) \left( \hat{k} \cdot \frac{d^2x}{d\xi'^2} \right) \left( \hat{k} \cdot \frac{d^2x'}{d\xi'^2} \right) - \frac{d}{d\xi} \cdot \frac{d}{d\xi'} \left( \frac{d}{d\xi} \cdot \frac{d}{d\xi'} \right) \left( \hat{k} \cdot \frac{dx}{dt} \frac{dx}{dt'} + \hat{k} \cdot \frac{dx'}{dt'} \frac{dx'}{dt} \right) \right\} + \frac{e^2\hbar}{(4\pi)^3\varepsilon_0} \int d\xi' \left( \hat{k} \cdot \frac{dx}{d\xi} \right) \left( \hat{k} \cdot \frac{dx'}{d\xi} \right) \int_{\xi(t')}^{\xi(t)} d\xi'' \frac{d}{d\xi''} \left( 1 - \left( \hat{k} \cdot \frac{dx''}{dt''} \right)^2 \right).$$  (24)

Equation (23) transforms into Eq. (13). Other useful formulas are summarized in the appendix A.

### III. APPROXIMATE FORMULAS

In the non-relativistic limit, where the velocity $v = dx/dt$ is small enough compared with the velocity of light, $|v| \ll 1$, Eqs. (23) and (24) reduce to

$$E^{(0)} = \frac{e^2}{6\pi\varepsilon_0} \int dt \dot{v}(t) \cdot \dot{v}(t),$$

$$E^{(1)} = \frac{e^2\hbar}{6\pi^2\varepsilon_0 m} \int dt \int_{t'}^{t} dt'' \dot{v}(t) \cdot \dot{v}(t') \frac{\dot{v}(t) \cdot \dot{v}(t')}{t-t''},$$

respectively. A brief derivation is summarized in the appendix A. Equation (26) was found for the first time by Higuchi and Walker in Ref. [2]. In the case of the periodic electric field, $|E| = E_0 \sin \omega t$, where $E_0$ is a constant, we have the periodic acceleration, $|\dot{v}| = (eE_0/m) \sin \omega t$. Then

$$\frac{dE^{(0)}}{dt} = \frac{e^4E_0^2\sin^2 \omega t}{m^2 6\pi\varepsilon_0},$$

$$\frac{dE^{(1)}}{dt} = \frac{\hbar e^4E_0^2}{m^2 12\pi\varepsilon_0 m} \omega.$$

(27)

(28)
After taking an average over a long time-duration, we have
\[
\frac{E^{(1)}}{E^{(0)}} = -\frac{\hbar \omega}{mc^2},
\]  
(29)
where \( c \) is the light velocity, which is restored here. The quantum effect becomes important when the time scale of the acceleration multiplied by \( c \) is comparable to the Compton wavelength, namely, when the wavelike feature of the particle appears.

Let us consider a more general case, when the electric field \( |E| = E_0 f(t/t_0) \), where \( f \) is a function of \( t/t_0 \) with a constant \( t_0(>0) \). In this case, the acceleration is \( \dot{\mathbf{v}} = (eE_0/m)f(t/t_0) \), and Eqs. (23) and (24) give
\[
E^{(0)} = \frac{e^4 E^2_{0t_0}}{6\pi \epsilon_0 m^2} \int d\tau f^2(\tau),
\]
(30)
\[
E^{(1)} = -\frac{\hbar e^4 E^2_0}{6\pi \epsilon_0 m^3} \int d\tau d\tau' \frac{f(\tau)f(\tau') - f(\tau')f(\tau)}{\tau - \tau'},
\]
(31)
then, we have
\[
\frac{E^{(1)}}{E^{(0)}} = -\frac{\hbar}{\pi mc^2 t_0} D,
\]
(32)
where we defined
\[
D = \left( \int d\tau'' f^2(\tau'') \right)^{-1} \int d\tau d\tau' \frac{f(\tau)f(\tau') - f(\tau')f(\tau)}{\tau - \tau'},
\]
(33)
where \( f, \tau(\tau') \) means the differentiation of \( f(\tau) \) with respect to \( \tau \). Here, let us consider the following three cases: (1) \( f(\tau) = 1 - (\tau^2)^n \) for \(|\tau| \leq 1 \) and \( f(\tau) = 0 \) for \(|\tau| > 1 \), (2) \( f(\tau) = 1/(1 + \tau^2)^n \) for \(-\infty < \tau < \infty \), (3) \( f(\tau) = 1/(\cosh \tau)^n \) for \(-\infty < \tau < \infty \), \( f(\tau) = 1 - |\tau|^m \) for \( 0 < \tau < 1 \), and \( f(\tau) = 0 \) for \(|\tau| > 1 \). Figure 2 show \( D \) as a function of \( n \) for cases (1) – (3), for which one can see that \( D \) is positive. Figure 3 show \( D \) as a function of \( n \) and \( m \) for case (4).

Thus, in all of the above cases, the first-order quantum correction \( E^{(1)} \) is negative. Also, the quantum effect is very small as long as the motion of the particle is non-relativistic. This result is consistent with that found in Refs. [2, 8]. The quantum effect might become important when the emitted photon energy becomes of order of \( mc^2 \) [2]. Note that this speculation is based on the result with the non-relativistic approximation.

Next, let us consider the relativistic limit, \( |p_i| \gg |eA|, m \). For simplicity, we consider the case when the direction of the particle motion is always parallel to that of the background electric field, i.e., \( \mathbf{v} \propto \mathbf{A} \). Namely, we consider the case when the directions of the particle’s motion and the background electric field are parallel at any moment, and adopt this direction as the z axis. Then, we may write \( \mathbf{A} = (0, 0, A(t)) \), \( \mathbf{A} = (0, 0, -E(t)) \), \( \mathbf{v} = (0, 0, v) \), and \( p_i = (0, 0, p_i) \). In this case, we have
\[
E^{(0)} = \frac{e^4}{(4\pi)^2 \epsilon_0} \int d\Omega_k (1 - \cos^2 \theta) \int dt \frac{m^4 e^2 \dot{A}^2(t)}{p_i^6 (1 - v \cos \theta)^5}.
\]
(34)
The integration with respect to \( \hat{k} \) yields
\[
E^{(0)} = \frac{1}{6\pi \epsilon_0} \frac{m^4 e^4}{p_i^6} \int dt \frac{\dot{A}^2(t)}{(1 - v^2)^3}.
\]
(35)
We consider the case \( p_i \gg |eA|, m \). We also assume \( |A| \sim |A/\omega| \sim |\dot{A}/\omega^2| \), where \( 1/\omega \) is a timescale of a time-varying background electric field. In this relativistic limit, we have the leading order expression for the quantum correction (see also appendix A),
\[
E^{(1)} \sim -\frac{e^2 \hbar}{(4\pi)^2 \epsilon_0} \int d\Omega_k (1 - \cos^2 \theta) \int d\xi \int d\xi' \frac{m^2}{\xi - \xi'} \frac{1}{p_i^6 (1 - v \cos \theta)^2 (1 - v' \cos \theta)^2}
\times e^2 \left\{ \dot{A}(t)\dot{A}(t') \left( \frac{-v^2 \cos \theta}{(1 - v \cos \theta)(1 - v' \cos \theta)} + \frac{(2 + v' \cos \theta)v'}{(1 - v' \cos \theta)^2} \right) + \dot{A}(t')\dot{A}(t) \left( \frac{-v'^2 \cos \theta}{(1 - v \cos \theta)(1 - v' \cos \theta)} + \frac{(2 + v \cos \theta)v}{(1 - v' \cos \theta)^2} \right) \right\}.
\]
(36)
Adopting the approximation, $v = v' = \bar{v} \simeq 1$, and $\xi - \xi' \simeq (t - t')(1 - \bar{v} \cos \theta)$, we have

$$E^{(1)} \simeq - \frac{e^2 \hbar}{4(2\pi)^3 \epsilon_0} \int d\Omega_k (1 - \cos^2 \theta) \int dt \int dt' \frac{1}{(1 - \bar{v} \cos \theta)^5} \frac{m^2}{p_i^5} \frac{e^2 (\dot{A}(t)\dot{A}(t') - \dot{A}(t)\dot{A}(t'))}{t - t'}.$$  \hfill (37)

The integration with respect to $\hat{k}$ yields

$$E^{(1)} \simeq - \frac{e^4 \hbar}{3(2\pi)^2 \epsilon_0} \frac{m^2}{p_i^5} \int dt \int dt' \frac{1}{(1 - \bar{v}^2)^3} \frac{\dot{A}(t)\dot{A}(t') - \dot{A}(t)\dot{A}(t')}{t - t'}. \hfill (38)$$

In the case of the periodic background of the electric field, $\dot{A}(t) = -E_0 \sin \omega t$, where $E_0$ is a constant, we have

$$\frac{dE^{(0)}}{dt} = \frac{e^4 m^4}{6\pi \epsilon_0 p_i^8} \frac{E_0^2 \cos^2 \omega t}{(1 - \bar{v}^2)^3}, \hfill (39)$$

$$\frac{dE^{(1)}}{dt} = \frac{\hbar c^4 m^2}{12\pi \epsilon_0 p_i^5} \frac{E_0^3 \omega}{(1 - \bar{v}^2)^3}. \hfill (40)$$

After averaging over sufficiently long time-duration, we have

$$\frac{E^{(1)}}{E^{(0)}} = \frac{p_i}{mc} \frac{\hbar \omega}{mc^2}. \hfill (41)$$

Note that the quantum correction $E^{(1)}$ is positive, which is a contrast to the non-relativistic case.
Similar to the non-relativistic limit, we next consider the case, \( \dot{A}(t) = -E_0 f(t/t_0) \), with a general function \( f(\tau) \). In the case, we have

\[
E^{(0)} = \frac{\epsilon^4 m^4 E_0^2 t_0}{6\pi\epsilon_0 p_i^2(1 - \epsilon^2)^3} \int d\tau f^2(\tau),
\]

\[
E^{(1)} = \frac{\hbar \epsilon^4 m^2 E_0^2}{12\pi^2 \epsilon_0 p_i^2(1 - \epsilon^2)^3} \int \int d\tau d\tau' \frac{f(\tau)f(\tau') - f(\tau)f(\tau')}{\tau - \tau'},
\]

and

\[
\frac{E^{(1)}}{E^{(0)}} = \frac{p_i}{\pi mc^2 t_0} D,
\]

where \( D \) is defined by Eq. [33]. When we adopt the three function of \( f(\tau) \) of (1) - (3) as in the case of the non-relativistic limit, \( D \) is positive. Thus, in contrast to the non-relativistic case, the quantum correction \( E^{(1)} \) is positive again, for all the cases in the present paper.

For the radiation from an electron in a periodic electric field, e.g., by a laser field, Eq. (11) is estimated as

\[
\frac{E^{(1)}}{E^{(0)}} \sim 2.6 \times 10^{-3} \left( \frac{p_i c}{\text{GeV}} \right) \left( \frac{mc^2}{0.5\text{MeV}} \right)^{-2} \left( \frac{\omega}{10^{15}\text{s}^{-1}} \right),
\]

where \( \omega \sim 10^{15}\text{s}^{-1} \) corresponds to an x-ray laser. The quantum effect becomes significant when the electron kinetic energy reaches the TeV scale. The above formula is derived under the condition, \( p_i \gg |eA| \). For a periodic electric field of large amplitude, \( p_i \sim |eA| \), the condition of the relativistic motion cannot be always guaranteed, because the physical momentum might become \( |p_i - eA| \sim m \). In this case, it is difficult to express the quantum correction in a simple analytic form. We need a more general treatment including fully numerical calculation, because the non-locality plays an important role. Potentially, there is a lot of room for discussion about how to detect the quantum effect of the Larmor radiation experimentally, but this is outside of the scope of the present paper.

### IV. SUMMARY

In the present paper, we obtained the general formula, Eq. (24) or Eq. (33), for the first-order quantum correction to the Larmor radiation from a charged particle moving in spatially homogeneous time-dependent electric field. This formula reproduces the same result as that in Ref. [2], in the limit of a non-relativistic motion of the charged particle. Our result is useful to investigate the case of a relativistic motion. When the direction of a particle’s motion is parallel to that of the background electric field, a simple formula was derived. In the limit of the relativistic motion of the charged particle, we obtained the formula [36]. Similar to the case of the non-relativistic motion [2], the leading quantum effect is described by a non-local difference between \( \dot{A}(t)\dot{A}(t') \) and \( \dot{A}(t)\dot{A}(t') \), as is demonstrated in Eq. [38]. This quantum effect disappears when \( \dot{A} \) is constant. Note that Eq. [38] is the leading term in the limit of the ultra-relativistic motion, assuming \( p_i \gg eA \), and \( |A| \sim |A/\omega| \sim |A/\omega|^2 \). We discarded the other sub-leading terms. For example, the term in proportion to \( \dot{A}(t)\dot{A}(t')(\dot{A}(t) - \dot{A}(t')) \) appears in the sub-leading terms, but also disappears when \( \dot{A} \) is constant. Thus, the essence of the quantum effect of the Larmor radiation should be the nonlocality, which reflects the fact that the exact solution of motion cannot be represented with simple classical trajectories in quantum theory [2].

We also note that the expression in the non-relativistic limit [26] is not simply connected to that in the relativistic limit [38]. The leading contributions for these opposite limits come from different sources. In the non-relativistic limit, the leading contribution comes only from terms in [A2], i.e., the phase of the mode function. On the other hand, in the relativistic limit, the leading contribution comes from both [A1] and [A2], i.e., the amplitude and the phase of the mode function. An interesting question might be how these facts are related to the difference of our final results in the opposite limits.

By applying the formula to the cases of a periodic acceleration and possible function of acceleration, it was demonstrated that the leading quantum effect enhances the radiation in the relativistic limit and that it decreases in the non-relativistic limit. This quantum effect will become important when the incident kinetic electron energy approaches TeV scale for a periodic electric field background with an x-ray laser. However, this result is obtained assuming that the charged particle is moving in the direction parallel to that of the background electric field. In a practical situation, this assumption is somewhat ideal. Here too there is a lot of room for further investigations of more general cases (cf. Ref.[2]), but this is outside of the scope of the present paper.
Our work, which is based on the QED theoretical framework, will be useful to investigate the feature of the radiation from an electron under a strong electric field. Investigation of the quantum effect in the Larmor radiation could be related to the subject of testing the QED process in the strong field background. For example, Chen and Tajima claimed the possibility of detecting the Unruh effect in the radiation from an electron under an ultraintense laser background. Possible signature of the Unruh effect in the radiation from an electron accelerated by an electric field of strong lasers is under debate (cf. [11, 12]). According to Ref. [9], the radiation from the Unruh effect could be of order of $h$. The characteristic signature of the Unruh effect claimed in [9] is in proportion to $E_0^3$ at order of $h$. As mentioned in the above, in our approach, the term in proportion to $\dot{A}(t)\dot{A}(t')\dot{A}(t)\dot{A}(t')$ appears in the sub-leading terms in evaluating Eq. (23). This might give a contribution in proportion to $E_0^3$. However, the angular dependence is different. In our approach, the quantum radiation of the order $h$ emitted in the direction of the motion, $\theta = 0$, is exactly zero from Eq. (19). This is a difference between our result and the prediction in Ref. [9], which might be tested experimentally. However, in our approach, it is difficult to separate the signature of the Unruh effect from other effects, even if they existed. This is a disadvantage of our approach.

In Ref. [8], the quantum radiation from a charged particle moving in an expanding or contracting universe was investigated. It was shown that the radiation can be regarded as the Larmor radiation from a charged particle in an accelerated (accelerated) motion, because the physical momentum of the particle decreases (increases) as the background universe expands (contracts) [8, 13]. The approach developed in the present paper is useful to investigate the quantum effect of this process [14].

Acknowledgment K.Y. thanks K. Homma, T. Takahashi, H. Nomura, M. Sasaki, H. Okamoto, K. Yokoya and A. Higuchi for useful communication when the topic of the present paper was initiated. We thank R. Kimura for useful discussions and comments. This work was supported by the Japan Society for Promotion of Science (JSPS) Grants-in-Aid for Scientific Research (No. 21540270, No. 21244033). This work was also supported by JSPS Core-to-Core Program “International Research Network for Dark Energy.”

Appendix A: Brief Summary of Derivation of Approximate Formulas

It is straightforward to derive the following formulas,

\[
\frac{d}{d\xi} \frac{d}{d\xi'} \frac{d}{d\xi''} \left[ \left( \frac{\mathbf{k} \cdot d\mathbf{x}}{d\xi} \right) \left( \frac{d\mathbf{x}'}{d\xi'} \right) - \frac{d\mathbf{x}}{d\xi} \frac{d\mathbf{x}'}{d\xi'} \right] \left( \frac{\mathbf{k} \cdot d\mathbf{x}}{dt} \right) + \left( \frac{\mathbf{k} \cdot d\mathbf{x}'}{dt} \right) \frac{d\mathbf{x}}{d\xi} \frac{d\mathbf{x}'}{d\xi'} \right] \right] 
\]

\[
\times \left( \frac{\mathbf{k} \cdot d\mathbf{x}}{dt} \frac{d\mathbf{x}'}{d\xi'} \right) - \left( \frac{\mathbf{k} \cdot d\mathbf{x}'}{dt} \frac{d\mathbf{x}}{d\xi'} \right) \left( \frac{\mathbf{k} \cdot d\mathbf{x}'}{dt} \frac{d\mathbf{x}}{d\xi} \right) \right]
\]

\[
+ 2 \left( \left( \frac{\mathbf{k} \cdot d\mathbf{x}}{d\xi} \right) \left( \frac{\mathbf{k} \cdot d\mathbf{x}'}{d\xi'} \right) - \frac{d\mathbf{x}}{d\xi} \frac{d\mathbf{x}'}{d\xi'} \right) \left( \frac{d\mathbf{x}}{d\xi} \right) \right]
\]

\[
+ \left( \left( \frac{\mathbf{k} \cdot d\mathbf{x}}{d\xi} \right) \left( \frac{\mathbf{k} \cdot d\mathbf{x}'}{d\xi'} \right) - \frac{d\mathbf{x}}{d\xi} \frac{d\mathbf{x}'}{d\xi'} \right) \left( \frac{d\mathbf{x}}{d\xi} \right) \right]
\]

\[
- \left( \left( \frac{\mathbf{k} \cdot d\mathbf{x}}{d\xi} \right) \left( \frac{\mathbf{k} \cdot d\mathbf{x}'}{d\xi'} \right) - \frac{d\mathbf{x}}{d\xi} \frac{d\mathbf{x}'}{d\xi'} \right) \left( \frac{d\mathbf{x}}{d\xi} \right) \right]
\]

\[
- \left( \left( \frac{\mathbf{k} \cdot d\mathbf{x}}{d\xi} \right) \left( \frac{\mathbf{k} \cdot d\mathbf{x}'}{d\xi'} \right) - \frac{d\mathbf{x}}{d\xi} \frac{d\mathbf{x}'}{d\xi'} \right) \left( \frac{d\mathbf{x}}{d\xi} \right) \right] (A1)
\]
Then, we find

\[
E^{(1)} = \frac{e^2 h}{(4\pi)^3 \epsilon_0} \int d\Omega_k \int d\xi \int d\xi' \frac{1}{\xi - \xi'} \times \left\{ \left( \hat{k} \cdot \frac{d^3 x}{d\xi^2} \right) \left( \hat{k} \cdot \frac{d^3 x'}{d\xi'^2} \right) - \frac{d^3 x}{d\xi^2} \cdot \frac{d^3 x'}{d\xi'^2} \right\} \int_{\xi'} d\xi'' \frac{d\tau''}{d\xi''} \left( 1 - \left( \hat{k} \cdot \frac{d\xi''}{d\tau''} \right)^2 \right) \\
+ 2 \left( \hat{k} \cdot \frac{d^2 x}{d\xi^2} \right) \left( \hat{k} \cdot \frac{d^2 x'}{d\xi'^2} \right) - \frac{d^2 x}{d\xi^2} \cdot \frac{d^2 x'}{d\xi'^2} \left( \frac{d}{d\xi} \left( \hat{k} \cdot \frac{d\xi'}{d\tau} \right) \frac{d}{d\xi'} \left( \hat{k} \cdot \frac{d\xi'}{d\tau'} \right) \right) - \left( \hat{k} \cdot \frac{d^3 x}{d\xi^3} \right) \left( \hat{k} \cdot \frac{d^3 x'}{d\xi'^3} \right) - \frac{d^3 x}{d\xi^3} \cdot \frac{d^3 x'}{d\xi'^3} \left( \frac{d}{d\xi} \left( \hat{k} \cdot \frac{d\xi'}{d\tau} \right) \frac{d}{d\xi'} \left( \hat{k} \cdot \frac{d\xi'}{d\tau'} \right) \right) - \left( \hat{k} \cdot \frac{d^4 x}{d\xi^4} \right) \left( \hat{k} \cdot \frac{d^4 x'}{d\xi'^4} \right) - \frac{d^4 x}{d\xi^4} \cdot \frac{d^4 x'}{d\xi'^4} \left( \frac{d}{d\xi} \left( \hat{k} \cdot \frac{d\xi'}{d\tau} \right) \frac{d}{d\xi'} \left( \hat{k} \cdot \frac{d\xi'}{d\tau'} \right) \right) \\
+ 2 \left( \hat{k} \cdot \frac{d^3 x}{d\xi^3} \right) \left( \hat{k} \cdot \frac{d^3 x'}{d\xi'^3} \right) - \frac{d^3 x}{d\xi^3} \cdot \frac{d^3 x'}{d\xi'^3} \left( \frac{d}{d\xi} \left( \hat{k} \cdot \frac{d\xi'}{d\tau} \right) \frac{d}{d\xi'} \left( \hat{k} \cdot \frac{d\xi'}{d\tau'} \right) \right) \right\}. \tag{A3}
\]

From the definition of \(\xi\) by Eq. (16), we have

\[
\frac{d\xi}{dt} = 1 - \hat{k} \cdot \frac{dx}{dt} = 1 - \psi, \tag{A4}
\]
where we defined $\dot{v} = \hat{k} \cdot \mathbf{v}$. Then, we also have
\begin{align}
\frac{d\mathbf{x}}{d\xi} &= \frac{\mathbf{v}}{1 - \dot{v}}, \\
\frac{d^2\mathbf{x}}{d\xi^2} &= \frac{\dot{v}}{(1 - \dot{v})^2} + \frac{\dot{v}_\perp}{(1 - \dot{v})^3}, \\
\frac{d^3\mathbf{x}}{d\xi^3} &= \frac{\ddot{v}}{(1 - \dot{v})^3} + \frac{\ddot{v}_\perp}{(1 - \dot{v})^4} + \frac{3\dot{v}_\perp^2}{(1 - \dot{v})^4} + \frac{3\dot{v}_\perp^2}{(1 - \dot{v})^5},
\end{align}
and
\begin{align}
\frac{d\tau}{dt} &= \frac{1}{\sqrt{(\mathbf{p}_i - e\mathbf{A})^2 + m^2}}, \\
\frac{d}{d\xi} \left( \frac{d\tau}{dt} \right) &= \frac{1}{1 - \dot{v}} \left( \frac{\dot{v}}{\sqrt{(\mathbf{p}_i - e\mathbf{A})^2 + m^2}} - \frac{\mathbf{v}(\mathbf{p}_i - e\mathbf{A}) \cdot (-e\mathbf{A})}{\sqrt{(\mathbf{p}_i - e\mathbf{A})^2 + m^2}} \right), \\
\frac{d^2}{d\xi^2} \left( \frac{d\tau}{dt} \right) &= \frac{1}{(1 - \dot{v})^3} \left( \frac{\dot{v}}{\sqrt{(\mathbf{p}_i - e\mathbf{A})^2 + m^2}} - \frac{\mathbf{v}(\mathbf{p}_i - e\mathbf{A}) \cdot (-e\mathbf{A})}{\sqrt{(\mathbf{p}_i - e\mathbf{A})^2 + m^2}} \right) + \frac{1}{(1 - \dot{v})^2} \left( \frac{\dot{v}}{\sqrt{(\mathbf{p}_i - e\mathbf{A})^2 + m^2}} + \frac{3\mathbf{v}(\mathbf{p}_i - e\mathbf{A}) \cdot (-e\mathbf{A})}{\sqrt{(\mathbf{p}_i - e\mathbf{A})^2 + m^2}} \right) \\
&\quad - \frac{2\mathbf{v}(\mathbf{p}_i - e\mathbf{A}) \cdot (-e\mathbf{A}) + \mathbf{v}(\mathbf{p}_i - e\mathbf{A}) \cdot (-e\mathbf{A}) + \mathbf{v}(\mathbf{p}_i - e\mathbf{A}) \cdot (-e\mathbf{A})}{\sqrt{(\mathbf{p}_i - e\mathbf{A})^2 + m^2}}.
\end{align}

In the limit of a non-relativistic motion of a charged particle, we use the following approximate formulas,
\begin{align}
\frac{dx}{d\xi} &\sim \mathbf{v}, \quad \frac{d^2x}{d\xi^2} \sim \dot{v}, \quad \frac{d^3x}{d\xi^3} \sim \ddot{v}, \\
\frac{d\tau}{dt} &\sim \frac{1}{m}, \quad \frac{d}{d\xi} \left( \frac{d\tau}{dt} \right) \sim \frac{\dot{v}}{m}, \quad \frac{d^2}{d\xi^2} \left( \frac{d\tau}{dt} \right) \sim \frac{\ddot{v}}{m},
\end{align}
then, we have the following expression by neglecting sub-leading terms,
\begin{align}
E^{(1)} &= \frac{e^2\hbar}{(4\pi)^{3/2}} \int d\Omega_k \int dt \int dt' \frac{1}{t - t'} \\
&\quad \times \frac{2}{m} \left\{ (\dot{v} \cdot \dot{v}' - \ddot{v} \cdot \ddot{v}') (\dot{v} - \dot{v}') (t - t') - 2(\dot{v} \cdot \ddot{v}' - \ddot{v} \cdot \dot{v}') + 2(\dot{v} \cdot \ddot{v}' - \ddot{v} \cdot \dot{v}) \right\}. \quad (A11)
\end{align}

We also neglect the first term of the right-hand side of (A11), which is of order of $\dot{v}^3$. After the integration with respect to $k$, we have
\begin{align}
E^{(1)} &= \frac{e^2\hbar}{(4\pi)^{3/2}} \int dt \int dt' \frac{A}{3m} \left\{ -\ddot{v} \cdot \ddot{v}' + \frac{2(\dot{v} \cdot \ddot{v}' - \ddot{v} \cdot \dot{v}')}{t - t'} \right\}. \quad (A12)
\end{align}

The first term of the right-hand side of (A12) gives no contribution by assuming that the acceleration is zero at the boundaries of the time. Then, Eq. (20) is obtained.

Now let us consider the case when the particle is moving with a relativistic speed in the direction parallel to the electric field. We choose the $z$ axis parallel to this direction. Then, we may write $\mathbf{A} = (0, 0, A(t))$, $\mathbf{v} = (0, 0, v)$, and $\mathbf{p}_i = (0, 0, p_i)$. In this case, we have
\begin{align}
\dot{v} &\simeq -\frac{m^2 e \dot{A}}{p_i^3}, \quad \ddot{v} \simeq -\frac{m^2 e \ddot{A}}{p_i^3}.
\end{align}
We also have
\[
\frac{d^2 z}{d\xi^2} \approx \frac{-1}{(1 - v \cos \theta)^3} \frac{m^2 e \hat{A}}{p_i^3}, \quad \frac{d^2 z}{d\xi^3} \approx \frac{-1}{(1 - v \cos \theta)^4} \frac{m^2 e \hat{A}}{p_i^3},
\]
\[
\frac{d\tau}{dt} \approx \frac{1}{p_i}, \quad \frac{d}{d\xi} \left( \frac{d\tau}{dt} \right) \approx \frac{v \cos \theta}{1 - v \cos \theta} \frac{e \hat{A}}{p_i^3},
\]
and
\[
\frac{d^2}{d\xi^2} \left( \frac{d\tau}{dt} \right) \approx \frac{v \cos \theta}{1 - v \cos \theta} \frac{e \hat{A}}{p_i^3} - \frac{v \cos \theta}{(1 - v \cos \theta)^2} \frac{(e \hat{A})^2 m^2}{p_i^5},
\]
in the limit of the relativistic motion. These approximate formulas yield the expression Eq. \([11]\) at the leading order. In this derivation, we here note that the following term of the integration included in Eq. \([A3]\),
\[
\int_{\xi'}^\xi d\xi'' \rho_d'' \left( 1 - \left( \hat{k} \cdot \frac{d\mathbf{x}''}{dt''} \right)^2 \right) = \int_{\xi'}^\xi d\xi'' \rho_d'' \left( 1 - \chi^2(t'') \right), \tag{A13}
\]
gives no contribution at the leading order. Using Eqs. \([A4]\) and \([A5]\), the right-hand side of Eq. \([A13]\) is written as
\[
\int_{\xi'}^\xi d\xi'' \frac{\rho_d''}{(p_i - e \mathbf{A})^2 + m^2} \left( 1 - \chi^2(t'') \right) = \int_{\xi'}^\xi d\xi'' \frac{1 + \chi(t'')}{(p_i - e \mathbf{A})^2 + m^2}. \tag{A14}
\]
In the limit of the relativistic motion, the leading term is
\[
\int_{\xi'}^\xi d\xi'' \frac{1 + \chi(t'')}{(p_i - e \mathbf{A})^2 + m^2} \approx \frac{1}{p_i} \int_{\xi'}^\xi d\xi'' \left( 1 + \frac{e \mathbf{A} \cdot \mathbf{p}_i}{|\mathbf{p}_i|^2} \right) (1 + \chi(t''))
\]
\[
\approx \frac{1 + \cos \theta}{p_i} (\xi - \xi'). \tag{A15}
\]
The contribution of this term to \(E^{(1)}\) is zero by assuming that the acceleration is zero at the boundaries of the time.

### Appendix B: Validity of WKB Approximation

We consider the validity of using the WKB approximation, which breaks down when the background field varies rapidly. The following condition is necessary to use the WKB approximation (e.g., \([13]\)),
\[
\frac{1}{2\Omega_p^2} \left| \frac{\dot{\Omega}_p}{\Omega_p} - \frac{3}{2} \frac{\dot{\Omega}_p^2}{\Omega_p} \right| \ll 1. \tag{B1}
\]
Using the expression \([B2]\), this condition yields
\[
\frac{\hbar^2}{2(p_i^2 - e \mathbf{A})^2 + m^2}) \left| \frac{5}{2} (e \mathbf{A} \cdot (p_i - e \mathbf{A}))^2 + ((p_i - e \mathbf{A})^2 + m^2)(e \hat{A} \cdot (p_i - e \mathbf{A}) - (e \hat{A})^2) \right| \ll 1. \tag{B2}
\]
In the relativistic limit, \(|p_i| \gg |e \mathbf{A}|, m\), as is considered in section 3, \([B2]\) reduces to
\[
\frac{\hbar^2}{2(p_i^2)^3} \left| \frac{5}{2} (e \mathbf{A} \cdot p_i)^2 + p_i^2 (e \mathbf{A} \cdot p_i) \right| \ll 1. \tag{B3}
\]
In the case of the periodic electric field, \(\dot{A} = -E_0 \sin \omega t\), \([B3]\) requires
\[
\frac{\hbar^2 e^2 E_0^2}{p_i^4} \ll 1, \quad \text{and} \quad \frac{\hbar^2 e E_0 \omega}{p_i^3} \ll 1, \tag{B4}
\]
which can be rewritten as

\[ \left( \frac{h \omega}{p_i} \right)^2 \left( \frac{e E_0}{p_i \omega} \right)^2 \ll 1, \quad \text{and} \quad \left( \frac{h \omega}{p_i} \right)^2 \left( \frac{e E_0}{p_i \omega} \right)^2 \ll 1. \] (B5)

We impose \( e E_0 / \omega \ll p_i \), as a condition of the limit of the relativistic motion. Then, the first inequality of (B5) is satisfied when the second inequality of (B5) is satisfied. Then, the condition required for the WKB approximation is written as,

\[ 1.3 \times 10^{-10} \left( \frac{\omega}{10^{15} \text{s}^{-1}} \right) \left( \frac{e E_0}{1 \times 10^{15} \text{eV/m}} \right) \left( \frac{p_i c}{\text{MeV}} \right)^{-3} \ll 1. \] (B6)