THE CORONA FACTORIZATION PROPERTY

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ABSTRACT. The corona factorization property is a property with connections to extension theory, $K$-theory and the structure of $C^*$-algebras. This paper is a short survey of the subject, together with some new results and open questions.

1. Introduction

Absorbing extensions have played an important role in various places in operator algebra theory. For example, they were used by Kasparov to give a nice characterization of $KK$-theory (see [1]). They are also important for the stable existence and stable uniqueness results of classification theory (see [2]). Still most recently, they have been used in new ways to classify some new interesting classes of nonsimple $C^*$-algebras (one class being purely infinite $C^*$-algebras with a unique ideal (see [3]); another class being Matsumoto algebras (these are “generalized Cuntz-Krieger algebras”) associated with primitive aperiodic substitutional subshifts (see [4]). (The definition of absorbing extensions is in section 3.)

The corona factorization property was originally motivated by extension theory problems related to classification theory - specifically, the theory of absorbing extensions! A major starting point was Elliott and Kucerovsky’s algebraic characterization of nuclearly absorbing extensions (see [5]). Basically, a separable nuclear stable $C^*$-algebra $B$ has the corona factorization property if $B$ has lots of absorbing extensions (see Theorem 3.1). Among other things, this includes automatically many of the extensions that have been important in classification theory, and leads to nice and useful characterizations in $KK$-theory. Most recently, the corona factorization property has been used to classify certain $C^*$-algebras associated with dynamical systems that were mentioned in the previous paragraph (see [6]).

It turns out that the corona factorization property is also connected with basic questions about the structure of $C^*$-algebras. Fundamental questions - like whether an extension of a stable $C^*$-algebra by a stable $C^*$-algebra has a stable extension algebra - seem to be related with the corona factorization property and the techniques used to study it. This is not completely surprising, as one of the main techniques of this subject (related with Elliott and Kucerovsky’s work on absorbing extensions) is the theory of stability for $C^*$-algebras, as developed by (Larry) Brown, Cuntz, Hjelmborg, Rordam and many others. Also, this subject often has the flavor of the theory of purely infinite simple $C^*$-algebras.

This paper is a survey of the theory, with proofs of several new results and many open questions. We would like to emphasize that the corona factorization property is a type of “regularity” property which demarcates between “nice” $C^*$-algebras (with “nice” $K$-theory, “nice” structure theory and possibly “nice” classification theory) and not so “nice” $C^*$-algebras. Indeed, our main examples of simple $C^*$-algebras with the corona factorization property are examples which (with the additional assumption of nuclearity etc.) have been amenable to the $K$-theoretic classification program. On the other hand, our main examples which don’t have corona factorization are exotic $C^*$-algebras (with very bad perforation etc.) constructed by Rordam (using, among other things, ideas of Villadsen).

In the next section, we introduce the corona factorization property and give basic examples and characterizations. In section 3, we state the equivalence of corona factorization with statements about extension theory and $KK$-theory. In section 4 (which is the largest section), we state connections with the structure theory of $C^*$-algebras. This section will also contain proofs of some new results as well as some open questions.

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Before ending this introduction, we fix one notation. All throughout this paper, the symbol “=_df” will roughly mean “is defined as” or “is defined to be”. For example, the statement “Let \( \mathcal{B} =_df \mathcal{A}_0 \otimes \mathcal{K} \)” reads as “Let \( \mathcal{B} \) be defined as the stabilization of \( \mathcal{A}_0 \).”

The survey component of this paper is joint work with Dan Kucerovsky.

2. Basic results and examples

The simplest statement of the corona factorization property (which, for the purposes of this paper, we take as the definition) is the following \( K \)-theoretic definition. Firstly, a positive element \( c \) of a \( C^* \)-algebra \( \mathcal{C} \) is said to be norm-full in \( \mathcal{C} \) if the \( C^* \)-algebra ideal that \( c \) generates is all of \( \mathcal{C} \); i.e., the \( C^* \)-subalgebra generated by \( CcC \) is all of \( \mathcal{C} \). (In much \( C^* \)-algebra literature, the terminology “full” is often used instead of “norm-full”, but here we are dealing with multiplier algebras which have the strict topology as well as the norm topology and we need to distinguish norm-full elements from strictly-full elements.)

Definition 2.1. Let \( \mathcal{B} \) be a separable stable \( C^* \)-algebra. Then \( \mathcal{B} \) is said to have the corona factorization property if every norm-full projection in \( \mathcal{M}(\mathcal{B}) \) is Murray-von Neumann equivalent to \( 1_{\mathcal{M}(\mathcal{B})} \).

With respect to this definition, one should think of the corona factorization property in the same terms as (though not too closely with!) “nice” properties like comparison of projections (which implies the corona factorization property!) We will see that this property is flexible enough to include many interesting \( C^* \)-algebras but also rigid enough to rule out many pathological examples.

Many simple unital separable nuclear \( C^* \)-algebras, which have been successfully classified using \( K \)-theory data, have (after stabilization) the corona factorization property. For example, we have the following proposition:

Proposition 2.1. Let \( \mathcal{A}_0 \) be a unital separable simple \( C^* \)-algebra.

1. If \( \mathcal{A}_0 \) is exact, real rank zero, stable rank one, and has weakly unperforated ordered \( K_0 \)-group, then \( \mathcal{A}_0 \otimes \mathcal{K} \) has the corona factorization property.

2. If \( \mathcal{A}_0 \) is purely infinite, then \( \mathcal{A}_0 \otimes \mathcal{K} \) has the corona factorization property.

Proof. First, we prove (1). Let \( \mathcal{B} =_df \mathcal{A}_0 \otimes \mathcal{K} \). (For the notation “=_df”, see the remark in the second last paragraph of the introduction.) Let \( P \) be a norm-full projection in \( \mathcal{M}(\mathcal{B}) \). Since \( \mathcal{B} \) has real rank zero, let \( \{p_n\}_{n=1}^\infty \) be a sequence of pairwise orthogonal projections in \( \mathcal{B} \) (actually, in \( \mathcal{PB} \mathcal{P} \)) such that \( P = \sum_{n=1}^\infty p_n \), where the series converges in the strict topology on \( \mathcal{M}(\mathcal{B}) \). For each \( N \), let \( P_N =_df \sum_{n=1}^N p_n \). We view each \( P_N \) as a continuous function on \( T(\mathcal{A}_0) \) (the simplex of unital traces on \( \mathcal{A}_0 \)) in the natural way. Since \( \mathcal{B} \) has stable rank one and weak unperforation and since \( P \) is norm full in \( \mathcal{M}(\mathcal{B}) \), we must have that the sequence \( \{P_N\}_{N=1}^\infty \) increases to infinity pointwise on \( T(\mathcal{A}_0) \). Hence, by Dini’s Theorem, \( \{P_N\}_{N=1}^\infty \) must increase to infinity uniformly on \( T(\mathcal{A}_0) \). Hence, since \( \mathcal{B} \) has real rank zero, stable rank one and weak unperforation, \( 1_{\mathcal{M}(\mathcal{B})} \) must be Murray-von Neumann equivalent to a subprojection of \( P \). Hence, since \( \mathcal{B} \) is stable, \( P \) is Murray-von Neumann equivalent to the unit of \( \mathcal{M}(\mathcal{B}) \).

Now we prove (2). Again let \( \mathcal{B} =_df \mathcal{A}_0 \otimes \mathcal{K} \) (where \( \mathcal{A}_0 \) is now purely infinite and simple). Let \( \pi: \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{B})/\mathcal{B} \) be the natural quotient map. Suppose that \( P \) is a norm-full projection in \( \mathcal{B} \). Then \( \pi(P) \) is a nonzero projection in \( \mathcal{M}(\mathcal{B})/\mathcal{B} \). But by [27], \( \mathcal{M}(\mathcal{B})/\mathcal{B} \) is a simple, purely infinite (nonseparable) \( C^* \)-algebra. Hence, let \( x \in \mathcal{M}(\mathcal{B})/\mathcal{B} \) be such that \( \pi(x)\pi(P)\pi(x)^* = 1_{\mathcal{M}(\mathcal{B})/\mathcal{B}} \). Hence, let \( k \in \mathcal{B} \) be such that \( xPax^* = 1_{\mathcal{M}(\mathcal{B})/\mathcal{B}} + k \). Now since \( \mathcal{B} \) is stable, let \( S \) be an isometry in \( \mathcal{M}(\mathcal{B}) \) such that \( S^*kS \) is within \( \epsilon \) of zero. Then \( S^*Pax^*S \) is within \( \epsilon \) of \( 1_{\mathcal{M}(\mathcal{B})} \). Hence, since \( \epsilon > 0 \) is arbitrary, we can find \( y \in \mathcal{M}(\mathcal{B}) \) such that \( yPy^* = 1_{\mathcal{M}(\mathcal{B})} \). So the unit is Murray-von Neumann equivalent to a subprojection of \( P \). So since \( \mathcal{B} \) is stable, \( P \) is Murray-von Neumann equivalent to the unit of \( \mathcal{M}(\mathcal{B}) \).

In addition, type I \( C^* \)-algebras with a “finite-dimensionality” condition also have (after stabilization) the corona factorization property. The following result can be found in [16]:

Theorem 2.2. If \( \mathcal{B} \) is a separable stable type I \( C^* \)-algebra with finite decomposition rank then \( \mathcal{B} \) has the corona factorization property.

If \( \mathcal{B} \) has continuous trace then the decomposition rank of \( \mathcal{B} \) is the (ordinary topological) dimension of its spectrum. Hence, if \( X \) is a finite-dimensional compact separable metric space then \( C(X) \otimes \mathcal{K} \) has the corona factorization property. On the other hand, if \( X \) is not finite-dimensional, then this can fail. The following result can also be found in [16]:
Theorem 2.3. Let $X$ be the countably infinite Cartesian product of spheres. Then $C(X) \otimes K$ does not have the corona factorization property.

The proof of the previous result uses ideas of Rordam (and Villadsen!). In particular, it is motivated by Rordam’s example of an extension of a stable $C^*$-algebra by a stable $C^*$-algebra such that the extension algebra is not stable (see [31]). Using another of Rordam’s “exotic” constructions, we also have a simple example. The following can be found in [17] (this is related to Rordam’s discovery that “stability is not a stable property”. See also [30]):

Theorem 2.4. There is a simple stable AH-algebra $\mathcal{B}$ with stable rank one such that $\mathcal{B}$ does not have the corona factorization property.

We note that the corona factorization property does not rule out every possible type of “exotic” behaviour. For example, in [19], it is shown that there exists a simple AH-algebra with perforation as well as the corona factorization property.

Finally, we end this section with some basic alternate characterizations. The full proof of the following can be found in [17]:

Theorem 2.5. Let $\mathcal{B}$ be a separable stable $C^*$-algebra. Then the following are equivalent:

1. $\mathcal{B}$ has the corona factorization property.
2. If $P$ is a norm-full projection in $\mathcal{M}(\mathcal{B})/\mathcal{B}$ then there exists $z \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ such that $zPz^* = 1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}$.
3. If $P$ is a norm-full projection in $\mathcal{M}(\mathcal{B})/\mathcal{B}$ then $P$ is properly infinite.
4. If $c$ is a positive element of $\mathcal{M}(\mathcal{B})/\mathcal{B}$ such that $C^*(c)$ does not nontrivially intersect any proper ideal of $\mathcal{M}(\mathcal{B})/\mathcal{B}$, then $c$ is properly infinite.

Sketch of proof. We prove (2) implies (1) and (3) implies (1).

First for (2) implies (1): Let $\pi : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ be the natural quotient map. Since $P$ is norm-full in $\mathcal{M}(\mathcal{B})$, $\pi(P)$ is norm-full in $\mathcal{M}(\mathcal{B})/\mathcal{B}$. Hence, by (2), let $x \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ be such that $\pi(x)\pi(P)\pi(x)^* = 1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}$. Hence, let $k \in \mathcal{B}$ be such that $xPx^* = 1_{\mathcal{M}(\mathcal{B})} + k$. Since $\mathcal{B}$ is stable, let $S$ be an isometry in $\mathcal{M}(\mathcal{B})$ such that $S^*kS$ is within $\epsilon$ of zero. Hence, $S^*xPx^*S$ is within $\epsilon$ of the unit of $\mathcal{M}(\mathcal{B})$. Since $\epsilon > 0$ is arbitrary and since $\mathcal{B}$ is stable, $P$ is Murray-von Neumann equivalent to the unit of $\mathcal{M}(\mathcal{B})$.

Next for (3) implies (1): Suppose that $P$ is a norm-full projection in $\mathcal{M}(\mathcal{B})$. Since $P$ is norm-full in $\mathcal{M}(\mathcal{B})$, there is a positive integer $N$ such that the $N$-times direct sum of $P$ with itself is Murray-von Neumann equivalent to the unit of $\mathcal{M}(\mathcal{B})$. But by (3), $P$ is properly infinite. Hence, $P$ is Murray-von Neumann equivalent to the unit of $\mathcal{M}(\mathcal{B})$. \hfill $\square$

Finally, we note that property (2) in Theorem 2.5 is formally similar to one of the characterizations of purely infiniteness for simple unital separable $C^*$-algebras. Hence, it is not surprising that the theory of corona factorization is related to the structure and stability theory of $C^*$-algebras and often has the flavour of purely infinite simple $C^*$-algebras.

3. THE CORONA FACTORIZATION PROPERTY AND EXTENSION THEORY

The corona factorization property first arose in our study of $C^*$-algebras which are “nice” from the point of view of absorbing extensions. This is, for instance, one of the first places where the corona factorization property enters into the classification program for simple unital separable nuclear $C^*$-algebras. For basic extension theory (Busby invariant, BDF-sums etc...) we refer the reader to [11] and [22].

We say that an extension $\tau$ is absorbing if $\tau +_{\text{BDF}} \rho$ is BDF-equivalent to $\rho$ for every trivial extension $\rho$, where $+_{\text{BDF}}$ is the BDF-sum. Note that the definition implies that $\tau$ is not unital. For unital $\tau$, the definition is exactly the same except that the trivial extension $\rho$ is required to lift to a unital $*$-homomorphism (into the multiplier algebra $\mathcal{M}(\mathcal{B})$). Finally, if in the above definition we require that $\rho$ be weakly nuclear then we say that $\tau$ is nuclearly absorbing. (Note that if either the ideal algebra or quotient algebra is a nuclear $C^*$-algebra, then every extension is weakly nuclear.) Absorbing extensions play an important role in a number of places - among other things, Kasparov used them to give a clean characterization of $KK$-theory (see [11]). They are also important in the stable existence and stable uniqueness theorems of classification theory (see, for example, [3]), as well as recent classification results for nonsimple $C^*$-algebras (see [7] and [6]).
To see the connection between the corona factorization property and absorbing extensions, we need to first discuss norm-full extensions.

**Definition 3.1.** Suppose that $A$ and $B$ are separable $C^*$-algebras such that $B$ is stable. An extension $\tau : A \to M(B)/B$ is said to be norm-full if for every nonzero, positive element $a \in A$, $\tau(a)$ is a norm-full element of $M(B)/B$.

We note that the Lin and Kasparov extensions as well as many other useful extensions (e.g. useful to classification theory; see [3]) are norm full extensions.

Note that an absorbing extension is necessarily norm-full (in the definition of absorbing, we can always choose the trivial extension to be norm-full). On the other hand, it is not always the case that the converse is true (see below). The converse, however, is always true (for nuclearly absorbing) exactly in the case where the ideal algebra has the corona factorization property. In [18], we prove the following:

**Theorem 3.1.** Suppose that $B$ is a separable, stable $C^*$-algebra. Then the following conditions are equivalent:

1. $B$ has the corona factorization property.
2. Every norm-full extension of $B$ is nuclearly absorbing.
3. Every norm-full extension of $B$ is nuclearly absorbing in the sense of approximate unitary equivalence (with unitaries in the corona).
4. Every norm-full trivial extension of $B$ is nuclearly absorbing.

Note that from Theorems 2.3 and 3.1, it follows that if $X$ is the countably infinite Cartesian product of spheres then $C(X) \otimes K$ has a nonabsorbing full extension. For more details, see [16].

In connection to the above, the corona factorization property also gives a clean characterization of $KK^1$ as well as a nice uniqueness theorem.

When the context is clear, we use the terminologies “full extension” and “full $*$-homomorphism” instead of “norm-full extension” and “norm-full $*$-homomorphism”. (When confusion with the strict topology on the multiplier algebra is not possible...)

**Theorem 3.2.** Let $B$ be a separable stable nuclear $C^*$-algebra. Then the following conditions are equivalent:

1. $B$ has the corona factorization property.
2. For every separable nuclear $C^*$-algebra $A$, $KK^1(A,B)$ is the group of full extensions under unitary equivalence by multiplier unitaries.
3. Let $A$ be a unital separable nuclear $C^*$-algebra. Let $\phi, \psi : A \to M(B)/B$ be two unital full $*$-monomorphisms Then $[\phi] = [\psi]$ in $KL(A,M(B)/B)$ if and only if $\phi$ and $\psi$ are approximately unitarily equivalent, with unitaries coming from the corona algebra.

We finally note that Theorems 3.1 and 3.2 have played a role in recent classification theory for certain nonsimple $C^*$-algebras with a distinguished ideal (see, for example, [7] and [6]).

4. THE CORONA FACTORIZATION PROPERTY AND THE STRUCTURE OF $C^*$-ALGEBRAS.

It turns out that the corona factorization property is also of interest from the point of view of the structure of $C^*$-algebras. This (long) section is an exposition of results so far obtained. There will also be some proofs of new results as well as some open questions.

The corona factorization property demarcates among “nice” and not so “nice” $C^*$-algebras. Firstly, Rordam has constructed an example of a simple AH-algebra $B_0$, with stable rank one, such that there is a positive integer $n \geq 2$ with $M_n(B_0)$ stable but $B_0$ itself is not stable (see [30]). In the context of the corona factorization property, this phenomenon is ruled out. In [17], we prove the following:

**Theorem 4.1.** Suppose that $B$ is a separable, stable $C^*$-algebra. Then the following are equivalent:

1. $B$ has the corona factorization property.
2. Suppose that $D$ is a full, hereditary subalgebra of $B$. Suppose that there is an integer $n \geq 1$ such that $M_n(D)$ is stable. Then $D$ itself is stable.

Hence, in the context of the corona factorization property, stability is a stable property for full hereditary subalgebras.
Next, Rordam has constructed an example of an extension of a separable stable C*-algebra, by a separable stable C*-algebra, such that the extension algebra is not stable (see [31]). When the ideal algebra has the corona factorization property, this behaviour is also ruled out. In [17], we prove the following:

**Theorem 4.2.** Suppose that \( J, E \) and \( A \) are separable C*-algebras, such that \( J \otimes K \) has the corona factorization property. Suppose that we have an exact sequence of the form
\[
0 \to J \to E \to A \to 0.
\]
Then \( E \) is stable if-and-only-if \( J \) and \( A \) are stable.

Moreover, under appropriate hypotheses, we actually have a converse. In [17], we also prove the following:

**Theorem 4.3.** Suppose that \( B \) is a stable, separable, simple, real rank zero C*-algebra with cancellation of projections. Then the following are equivalent:

1. \( B \) has the corona factorization property.
2. Every extension of \( B \), by a separable stable C*-algebra, gives a stable extension algebra.

By the same argument as the above, we also have the following:

**Theorem 4.4.** Suppose that \( B \) is a stable, separable, simple, real rank zero C*-algebra with cancellation of projections. Then the following are equivalent:

1. \( B \) has the corona factorization property.
2. Every extension of \( B \), by the compact operators \( K \), gives a stable extension algebra.
3. Suppose that
\[
0 \to B \to C \to K \to 0
\]
is an essential quasidiagonal extension such that \( C \) has real rank zero. Then \( C \) is stable.

Motivated by the above results, we look for connections between the corona factorization property and interesting properties about the structure of simple separable unital C*-algebras. One such property is Rordam’s notion of regularity:

**Definition 4.1.** A C*-algebra \( B \) is said to be regular if every full hereditary subalgebra of \( B \), with no nonzero unital quotients and no nonzero bounded traces, is stable.

Firstly, regularity implies the corona factorization property:

**Lemma 4.5.** Suppose that \( B \) is a separable, stable C*-algebra. If \( B \) is regular, then \( B \) has the corona factorization property.

*Proof.* Suppose that \( D \) is a full, hereditary subalgebra of \( B \) such that there is a positive integer \( n \geq 1 \) with \( M_n(D) \) being a stable C*-algebra. Then \( D \) has no nonzero unital quotient and no nonzero bounded trace (for otherwise, \( M_n(D) \) would have such - which is impossible since \( M_n(D) \) is stable). Hence, by regularity \( D \) must be stable. But \( D \) was arbitrary. Hence, by Theorem 4.1, \( B \) has the corona factorization property. \( \square \)

The converse of the above result is not known (even with the additional assumptions of simplicity and real rank zero).

Regularity is an interesting property from the point of view of C*-algebra structure. The next lemma is due to Hjelmborg and Rordam. We present the short argument for the convenience of the reader.

**Lemma 4.6.** Suppose that \( C \) is a simple, stable, exact, separable C*-algebra which is regular. Then \( C \) is either purely infinite or stably finite.

*Proof.* Suppose that \( C \) is not stably finite. Then no nonzero hereditary subalgebra of \( C \) is stably finite. Let \( D \) be a nonunital hereditary subalgebra of \( C \). Since \( D \) is not stably finite, and since \( D \) is exact, \( D \) has no nonzero bounded traces. Hence, by hypothesis, \( D \) is stable. But \( D \) was arbitrary. Hence, every nonzero hereditary subalgebra of \( C \) is either unital or stable. Hence, by a result of Zhang’s [13] (see also [12]), the algebra \( C \) is either the compact operators or purely infinite. \( \square \)

Henceforth, we will say that a C*-algebra \( C \) has dichotomy if \( C \) is either stably finite or purely infinite (in the sense of Kirchberg and Rordam). Note that this property passes on to full, hereditary subalgebras.

Next, we weaken the notion of regularity by introducing the notion of asymptotic regularity.
Definition 4.2. Suppose that $B$ is a separable, stable $C^*$-algebra. Then $B$ is said to be asymptotically regular, if whenever $D$ is a full hereditary subalgebra of $B$ with no nonzero unital quotients and no nonzero bounded traces, there is a positive integer $n \geq 1$ such that $M_n(D)$ is stable.

It is interesting to try and understand the relationship between regularity and asymptotic regularity. In the case of exact, simple, stable, real rank zero algebras, the two notions are the same. To prove this, we require the following proposition, which can be found in [2] Theorem 4.23:

Proposition 4.7. Let $B$ be a separable, stable $C^*$-algebra. Let $P$ be a projection in $\mathcal{M}(B)$. Then $PBP$ is a stable, full, hereditary subalgebra of $B$ if-and-only-if $P$ is Murray-von Neumann equivalent to the unit of $\mathcal{M}(B)$.

We also need the following lemma which is in [14] Lemma 10:

Lemma 4.8. Let $B$ be a separable stable $C^*$-algebra. Then for every hereditary subalgebra $B_0$ of $B$, there exists a multiplier projection $P \in \mathcal{M}(B)$ such that $PBP \cong B_0$.

Proposition 4.9. Let $B$ be a separable, exact, simple, stable, real rank zero $C^*$-algebra. Then $B$ is regular if-and-only-if $B$ is asymptotically regular.

Proof. The “only-if” direction is clear.

We proceed to prove the “if” direction. If $B$ is type I, then $B$ is the compact operators over a separable, infinite dimensional Hilbert space. Hence, $B$ is automatically regular. Hence, we may assume that $B$ is not type I. First, suppose that there is a projection $p$ of $B$ such that the hereditary subalgebra $pBp$ has a unital trace (hence, $B$ is stably finite).

Suppose that $B$ is asymptotically regular. Suppose, to the contrary, that $B$ is not regular. Hence, let $D$ be a nonzero hereditary subalgebra of $B$, with no unit, and no nonzero bounded traces, such that $D$ is not stable. By Lemma 4.8, let $P$ be a projection in $\mathcal{M}(B)$ such that $PBP$ is isomorphic to $D$. By asymptotic regularity of $B$, let $n$ be the least integer such that $M_n(D)$ is stable (stably finite). For each positive integer $m$, let $Q_m = \bigoplus_{i=1}^m P$. Since $B$ is stable, $\mathcal{M}(B)$ contains a copy of $O_m$ for each $m$; and hence for each positive integer $n$, $Q_n$ is defined, up to Murray-von Neumann equivalence, as a projection in $\mathcal{M}(B)$. Hence, by Proposition 4.7, $n$ is the least integer such that $Q_n$ is Murray-von Neumann equivalent to $1_{\mathcal{M}(B)}$, the unit of $\mathcal{M}(B)$. In particular, $Q_1 = P$ is not Murray-von Neumann equivalent to $1_{\mathcal{M}(B)}$.

Now since $B$ has real rank zero, let $\{p_k\}_{k=1}^\infty$ be a sequence of pairwise orthogonal projections in $B$ such that $P = \sum_{k=1}^\infty p_k$, where the sum on the right hand side converges in the strict topology in $\mathcal{M}(B)$. Note that for each unital trace $\tau$ in $pBp$, $\tau$ extends naturally to a semifinite trace on $B$. This trace, in turn, gives a trace on the positive cone of $\mathcal{M}(B)$. We use “$\tau^*$” to denote any one of these traces. Now since $Q_n$ is Murray-von Neumann equivalent to $1_{\mathcal{M}(B)}$, we must have that for each unital trace $\tau$ of $pBp$, $\tau(P) = \infty$. Hence, for each unital trace $\tau$ of $pBp$, $\sum_{k=1}^\infty \tau(p_k) = \infty$.

Now by [23] Theorem 5.8, and since $B$ is not type I, $B$ is weakly divisible. Hence, for each $k$, for each strictly positive integer $l \geq 2$, let $m_{k,l}$ be an integer greater than $l$, and let $q_{k,l,1}, q_{k,l,2}, ..., q_{k,l,m_{k,l}}$ be pairwise orthogonal projections in $B$ such that

1. for each $i$, $q_{k,l,i}$ is a subprojection of $p_k$;
2. for each $i,j$, $q_{k,l,i}$ is Murray-von Neumann equivalent to $q_{k,l,j}$;
3. $q_{k,l,1} + q_{k,l,2} + ... + q_{k,l,m_{k,l}}$ is a subprojection of $p_k$; and
4. $p_k$ is a subprojection of $q_{k,l,1} + q_{k,l,2} + ... + q_{k,l,m_{k,l}}$.

Now let $T(pBp)$ be the simplex of unital traces of $pBp$. $T(pBp)$ is a compact, convex set (compact in the pointwise topology). As we have observed in previous paragraphs, for each $\tau \in T(pBp)$, $\tau(p) = \sum_{k=1}^\infty \tau(p_k) = \infty$. Moreover, by the compactness of $T(pBp)$, $\sum_{k=1}^\infty p_k$ is a sequence of positive, (affine) continuous functions on $T(pBp)$ which blows up to infinity uniformly on $T(pBp)$. We recursively define a sequence $\{r_i\}_{i=2}^{\infty}$ of projections in $B$, and an increasing sequence $\{n_i\}_{i=2}^{\infty}$ of positive integers as follows:

Let $n_2$ be an integer such that $\sum_{k=1}^{n_2} \tau(q_{k,1,1}) \geq 1$, for all $\tau \in T(pBp)$. Let $r_1 = \sum_{k=1}^{n_2} q_{k,1,1}$.

Suppose that both $n_i$ and $r_i$ have been chosen. Let $n_{i+1}$ be an integer such that $\sum_{k=n_{i+1}}^{n_{i+1} + \tau(q_{k,i+1,1}))} \geq 1$, for all $\tau \in T(pBp)$. Let $r_{i+1} = \sum_{k=n_{i+1}}^{n_{i+1} + \tau(q_{k,i+1,1})} q_{k,i+1,1}$.

The recursion is complete. Now let $R = \sum_{i=1}^{\infty} r_i$. The sum converges in the strict topology in $\mathcal{M}(B)$. Hence, $R$ is a projection in $\mathcal{M}(B)$. By our choice of the $r_i$, $\tau(R) = \infty$ for all $\tau \in T(pBp)$ (where, as usual,
Definition 4.4. Let $B$ be a unital, separable $C^*$-algebra, and let $B = \mathcal{M}(B)$ be the stabilization of $B$. Then $B$ is regular if-and-only-if $B$ is asymptotically regular and has the corona factorization property.

Proof. The “only-if” direction follows by definition.

Now for the “if” direction. Suppose that $D$ is a full, hereditary subalgebra of $B$, with no nonzero unital quotients and no nonzero bounded traces. By asymptotic regularity, let $n$ be a positive integer such that $\mathcal{M}_n(D)$ is stable. Then since $B$ has the corona factorization property, and by Theorem 4.1, $D$ must be stable. So since $D$ is arbitrary, $B$ is regular.

Now let $B$ be a stable $C^*$-algebra, and let $\pi$ be a nonzero $*$-representation of $B$. Then $\pi$ extends to a unique strictly continuous, surjective $*$-homomorphism $\pi'' : \mathcal{M}(B) \to \mathcal{M}(\pi(B))$ (between multiplier algebras). Let $\mathcal{I}_\pi$ be the proper, norm-closed ideal of $\mathcal{M}(B)$ given by $\mathcal{I}_\pi = \{c \in \mathcal{M}(B) : \pi''(c) \in \pi(B)\}$.

Now let $B_0$ be a separable, unital $C^*$-algebra, and let $B := B_0 \otimes \mathcal{K}$ be the stabilization of $B_0$. Suppose that $\tau$ is a unital trace on $B_0$. Then $\tau$ extends canonically to a trace on the positive cone of $\mathcal{M}(B)$, which we also denote by “$\tau$”. We let $\mathcal{J}_\tau$ be the proper ideal of $\mathcal{M}(B)$ given by the norm-closure of $\{b \in \mathcal{M}(B) : \tau(b^*b) < \infty\}$.

Definition 4.3. Let $B_0$ be a separable, unital $C^*$-algebra, and let $B = \mathcal{M}(B_0)$ be the stabilization of $B_0$. Let $\mathcal{I}$ be a proper ideal of $\mathcal{M}(B)$, which contains $B$. We say that the ideal $\mathcal{I}$ of $\mathcal{M}(B)$ is regular if

1. $\mathcal{I}$ is contained in an ideal of the form $\mathcal{I}_\pi$, for some nonzero $*$-representation $\pi$ of $B$; or
2. $\mathcal{I}$ is contained in an ideal of the form $\mathcal{J}_\tau$, for some unital trace $\tau$ on $B_0$.

Otherwise, we say that $\mathcal{I}$ is nonregular.

If $B_0$ is the Cuntz algebra, then both $B_0 \otimes \mathcal{K}$ and $\mathcal{K}$ have simple corona algebras (see [20]). Hence, it follows, by definition, that both $\mathcal{M}(B_0 \otimes \mathcal{K})$ and $\mathcal{M}(\mathcal{K})$ have no nonregular ideals. If $B_0$ is a simple, unital AF-algebra, such that the tracial simplex of $B_0$ has only $n < \infty$ extreme points, then $\mathcal{M}(B_0 \otimes \mathcal{K})$ has $n$ maximal, proper ideals. These ideals are all regular, since they come from traces on $B_0$ (see [20]).

Definition 4.4. Let $B_0$ be a unital $C^*$-algebra, and let $B = \mathcal{M}(B_0)$ be the stabilization of $B_0$. We say that $B$ has property $R$, if whenever $p$ is a projection contained inside a proper ideal of $\mathcal{M}(B)$, $p$ is also contained inside a regular ideal.

Under the conditions of exactness, simplicity and real rank zero, property $R$ is rather strong.

Proposition 4.11. Let $B_0$ be a simple, exact, unital, separable, real rank zero $C^*$-algebra, and let $B = \mathcal{M}(B_0)$ be the stabilization of $B_0$. Suppose that $B$ has property $R$. Then

1. $B$ has dichotomy; and
2. $B$ has the corona factorization property.
Proof. We first prove statement i). Suppose, to the contrary, that $B$ is neither purely infinite, nor stably finite. By [27] Theorem 3.2, $\mathcal{M}(B)$ has a proper ideal $J$, which properly contains $B$. By [44], since $B$ has real rank zero, $J$ is the norm linear span of its projections. Hence, let $P$ be a projection in $J$ such that $P$ is not contained in $B$. Since $B_0$ is exact and is not stably finite, $B_0$ has no unital traces. From this, and the simplicity of $B$, $\mathcal{M}(B)$ has no regular ideals other than $B$. Hence, $P$ is a projection, contained in a proper ideal of $\mathcal{M}(B)$, such that $P$ is not contained in a regular ideal. This contradicts property $R$. Hence, $B$ must be either purely infinite or stably finite.

We now proceed to prove the second statement, ii). Suppose, to the contrary, that $B$ does not have the corona factorization property. By i), we know that $B$ must be stably finite (since, by Proposition 2.1, a stable, separable, simple, purely infinite $C^*$-algebra always has the corona factorization property). Hence, by exactness, $B_0$ has a unital trace. So let $T(B_0)$ be the simplex of unital traces on $B_0$, $T(B_0)$ is a compact, convex set (compact with respect to the pointwise topology).

Since $B$ does not have the corona factorization property, let $P$ be a norm-full projection in $\mathcal{M}(B)$ such that $P$ is not Murray-von Neumann equivalent to $1_{\mathcal{M}(B)}$, the unit of $\mathcal{M}(B)$. By norm-fullness, there is an integer $n \geq 2$ such that $\bigoplus_{k=1}^n P$ is Murray-von Neumann equivalent to $1_{\mathcal{M}(B)}$. Hence, for every $\tau \in T(B_0)$, $\tau(P) = \infty$ (again, we are using “$\tau$” to denote also the natural extension to the positive cone of $\mathcal{M}(B)$).

Since $B$ has real rank zero, let $\{p_k\}_{k=1}^{\infty}$ be a sequence of pairwise orthogonal projections in $B$ such that $P = \sum_{k=1}^{\infty} p_k$, where the sum converges in the strict topology in $\mathcal{M}(B)$. Hence, for each $\tau \in T(B_0)$, $\sum_{k=1}^{\infty} \tau(p_k) = \infty$.

If $B$ is type I, then it is the compact operators on a separable Hilbert space; and hence, $B$ automatically has the corona factorization property. Hence, let us assume that $B$ is not type I. Then, by [24] Theorem 5.8, $B$ is weakly divisible. Hence, for each $k$, for each strictly positive integer $l \geq 2$, let $m_{k,l}$ be a positive integer greater than $l$, and let $q_{k,l,1}, q_{k,l,2}, \ldots, q_{k,l,m_{k,l}}$ be pairwise orthogonal projections in $B$ such that

1. For each $i$, $q_{k,l,i}$ is a subprojection of $p_k$;
2. For each $i,j$, $q_{k,l,i}$ is Murray-von Neumann equivalent to $q_{k,l,j}$;
3. $q_{k,l,1} + q_{k,l,2} + \ldots + q_{k,l,m_{k,l} - 1}$ is a subprojection of $p_k$; and
4. $p_k$ is a subprojection of $q_{k,l,1} + q_{k,l,2} + \ldots + q_{k,l,m_{k,l}}$.

Now for each $i$, define $r_i$ as in the proof of Proposition 1.9. And, as in the proof of Proposition 1.9, let $R = \sum_{i=1}^{\infty} r_i$, where the sum converges in the strict topology in $\mathcal{M}(B)$. As in Proposition 1.9, for each integer $m$, $\bigoplus_{j=1}^m R$ is not Murray-von Neumann equivalent to $1_{\mathcal{M}(B)}$. Hence, $R$ is contained in a proper ideal of $\mathcal{M}(B)$. But also, as in Proposition 1.9, $\tau(R) = \infty$ for every $\tau \in T(B_0)$. Hence, $R$ is a projection which is contained in a proper ideal of $\mathcal{M}(B)$, but which is not contained in a regular ideal. This contradicts property $R$.

Next, we relate property $R$ to regularity, for the case of simple, exact, separable, stable $C^*$-algebras.

**Proposition 4.12.** Let $B_0$ be a simple, exact, separable $C^*$-algebra, and let $B := B_0 \otimes K$ be the stabilization of $B_0$. Then $B$ is regular if-and-only-if $B$ has property $R$ and the corona factorization property.

**Proof.** We first prove the “only-if” direction. Suppose that $B$ is regular. By Lemma 4.5, $B$ has the corona factorization property. Hence, it suffices to prove that $B$ has property $R$. So let $P$ be a projection in $\mathcal{M}(B)$ such that $P$ is contained in a proper ideal of $\mathcal{M}(B)$. If $P$ is an element of $B$, then automatically, $P$ is contained in a regular ideal of $\mathcal{M}(B)$. Hence, we may assume that $P$ is not contained in $B$.

Now suppose, to the contrary, that $P$ is not contained inside any regular ideal of $\mathcal{M}(B)$. Hence, $PBP$ must be a full hereditary subalgebra of $B$, with no unit and no nonzero bounded traces. Hence, since $B$ is regular, $PBP$ is stable. Hence, by Proposition 1.7, $P$ is Murray-von Neumann equivalent to the unit of $\mathcal{M}(B)$. This contradicts our assumption that $P$ is contained in a proper ideal of $\mathcal{M}(B)$. Hence, $P$ is contained in a regular ideal of $\mathcal{M}(B)$. By the arbitrariness of $P$, $B$ has property $R$.

Next, we prove the “if” direction. Suppose that $B$ has both property $R$ and the corona factorization property. Suppose that $D$ is a nonzero hereditary subalgebra of $B$, with no unit and no nonzero bounded traces. By Lemma 1.8, let $P$ be a projection in $\mathcal{M}(B)$ such that $PBP$ is isomorphic to $D$. Since $D$ has no nonzero bounded traces, $P$ cannot be contained in any regular ideal of $\mathcal{M}(B)$. Hence, since $B$ has property $R$, $P$ is a norm-full element of $\mathcal{M}(B)$. Hence, by the corona factorization property and by [32] Lemma 16.2, $P$ must be Murray-von Neumann equivalent to the unit of $\mathcal{M}(B)$. Hence, $PBP$ is stable. Hence, $D$ is stable. Since $D$ is arbitrary, $B$ must be regular.
Corollary 4.13. Suppose that $B$ is a simple, separable, stable, exact, real rank zero $C^*$-algebra. Then $B$ is regular if and only if $B$ has property $R$.

Collecting the above results and restricting to the simple, real rank zero case, we have the following:

Theorem 4.14. Suppose that $B$ is a simple, separable, stable, exact, real rank zero $C^*$-algebra. Then the following are equivalent:

1. $B$ is regular.
2. $B$ is asymptotically regular.
3. $B$ has property $R$.

Question 4.15. Suppose that $B$ is a separable, stable $C^*$-algebra. If $B$ has the corona factorization property, then is $B$ regular? What if, in addition, $B$ is simple and has real rank zero?

Question 4.16. With possibly additional assumptions, what is (are) the connection(s) of the above statements with the statement that every extension of $B$, by a separable stable $C^*$-algebra, gives a stable extension algebra? What about dichotomy?

Question 4.17. Does every simple separable stable nuclear real rank zero $C^*$-algebra have the corona factorization property? What if we also assumed that the $C^*$-algebra was $A^H$?

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