Certifying Separability in Symmetric Mixed States, and Superradiance

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Separability criteria are typically of the necessary-but-not-sufficient variety, in that satisfying some separability criterion, such as positivity of eigenvalues under partial transpose, does not strictly imply separability. Certifying separability amounts proving the existence of a decomposition of target mixed state into some convex combination of separable states; determining the existence of such a decomposition is “hard”. We show that it is effective to instead ask if the target mixed state “fits” some preconstructed separable form, in that one can generate a sufficient separability criterion relevant to all target states in some family by ensuring enough degrees of freedom in the preconstructed separable form. We demonstrate this technique by inducing a sufficient criterion for “diagonally symmetric” states of N qubits. A sufficient separability criterion opens the door to study precisely how entanglement is (not) formed; we use ours to prove that, counter-intuitively, entanglement is not generated in idealized Dicke Model superradiance despite its exemplification of many-body effects. We introduce a quantification of the extent to which a given preconstructed parametrization comprises the set of all separable states; for “diagonally symmetric” states our preconstruction is shown to be fully complete. This implies that our criterion is necessary in addition to sufficient, among other ramifications which we explore.

Despite extensive interest in many-body entanglement [1–4] the longstanding question of how, exactly, entanglement is generated at all remains open. To establish the minimal requisite common features of entanglement generation we must seek counter-intuitive instances to challenge our preconceptions. To that end, this research was motivated by initial indications which - inconclusively - suggested that entanglement may not be a feature of Dicke Model superradiance. Superradiance is a coherent radiative phenomenon resulting from collective and cooperative atomic effects [5–7],[8], and thus it possesses the typical hallmark of an entangling process; see, for example [9]. Various necessary criteria for separability [10–12] nevertheless failed to find signatures of entanglement. The extraordinary claim “superradiance occurs without entanglement”, demands the highest standard of evidence; to prove that superradiance need not be entangling we must certify its separability by employing some sufficient separability criterion.

For pure states, various methods can be employed to quantify entanglement [2–4], which are diagonal in the symmetric eigenbasis of N-partite 2-level Dicke states. Each Dicke-basis pure state is a superposition of equal-energy states; it is the normalized sum-over-all-permutations of a (separable) computational-basis state. Using bold font to indicate sets, such as $n = \{n_0, n_1\}$, we have

$$|D_n\rangle = w_n \sum_{\text{perms.}} \frac{|0...0, 1...1\rangle}{n_0! n_1!}$$

where $n_0 + n_1 = N$ and $w_n = \sqrt{n_0! n_1! / N!}$.

So for example

$$|D_{3,1}\rangle = \frac{|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle}{\sqrt{4}}.$$ 

The state $|D_n\rangle$ is entangled for all $0 < n_0 < N$; Dicke states are natural generalizations of the W state [15], and can also be described as the simultaneous eigenstates of total spin and spin-z operators with $J = N/2$ and $M = (n_1 - n_0) / 2$.
The most general mixed state which is diagonal in this basis can be parametrized as
\[
\rho_{\text{GDS}} = \sum_{n} \chi_{n} |D_{n}\rangle \langle D_{n}|
\] (3)
where the \(\chi_{n}\) represent the eigenvalues in the eigen-decomposition of \(\rho_{\text{GDS}}\), which, in the convention of quantum optics, we refer to as the populations of \(\rho_{\text{GDS}}\).

Next we preconstruct a set of separable states to serve as targets for our decomposition. We start with a completely generic single qubit pure state \(|\psi\rangle = \sqrt{\rho} |0\rangle + \sqrt{1-\rho} e^{i\phi} |1\rangle\), defined as \(\rho^{j} [y, \phi] \equiv |\psi\rangle \langle \psi|\) in operator form, where we take an \(N\)-fold tensor product of the single qubit state with itself, and mix uniformly over all phases but discretely over arbitrary amplitudes \(y_{j}\) with weights \(x_{j}\),
\[
\rho_{\text{SDS}} \equiv \int_{0}^{2\pi} (2\pi)^{-1} \sum_{j=1}^{j_{\text{max}}} x_{j} \rho^{j} [y_{j}, \phi]) \otimes^{N} d\phi.
\] (4)

We call such parametrized states separable diagonally symmetric (SDS) states., and the value of \(j_{\text{max}}\) depends on \(N\). Note that, by definition, all the variables \(x_{j}, y_{j}\) appearing in Eq. (4) must be real numbers between 0 and 1. Note also that our mixing protocol differs markedly from the Spherical Harmonics basis suggested in Ref. [13], and furthermore, the SDS states cannot be resolved by the partial-separability method of Ref. [16], as that protocol is incompatible with continuous mixtures.

As proven in the supplementary online materials, Eq. (4) can be equivalently expressed as
\[
\rho_{\text{SDS}} = N! \sum_{n} \sum_{j=1}^{j_{\text{max}}} x_{j} y_{j} \frac{n_{0}! n_{1}!}{n_{0, n_{1}!}} |D_{n}\rangle \langle D_{n}|
\] (5)
which more clearly parallels the form of Eq. (3). Orthogonality of the Dicke states allows us to match up terms inside the sums of Eq. (3) and Eq. (5), implying \(N+1\) polynomial equations [17] which define a decomposition the populations \(\chi\) of \(\rho_{\text{GDS}}\) into the parameters \(x, y\) of a \(\rho_{\text{SDS}}\). Explicitly, if we can successfully identify a mapping
\[
\forall n \quad \chi_{n} = N! \sum_{j=1}^{j_{\text{max}}} x_{j} y_{j} \frac{n_{0}! (1 - y_{j})^{n_{1}!}}{n_{0, n_{1}!}}
\] (6)
then we will have demonstrated that our particular \(\rho_{\text{GDS}}\) exists in the subspace defined by all possible \(\rho_{\text{SDS}}\), \(\rho_{\text{GDS}} \in \mathcal{Q}_{\text{GDS}}\), and thus that \(\rho_{\text{GDS}}\) is necessarily separable.

\(j_{\text{max}}\) is chosen in order for the system of equations (6) to be well behaved, i.e. that there should be exactly \(N+1\) variables \(x, y\) appearing in the \(N+1\) equations. Considering that \(x_{j}\) and \(y_{j}\) always come in pairs then plainly when \(N+1\) is even we should set \(j_{\text{max}} = (N+1)/2\). When \(N+1\) is odd the situation requires a manual adjustment.

we take \(j_{\text{max}} = [(N+1)/2]\) and fix the extraneous variable by forcing \(y_{(N+2j)/2} = 0\) [18]. To demonstrate, here is the system of polynomial equations for \(N = 4\) qubits,
\[
\begin{align*}
\chi_{4,0} &= x_{1}y_{1}^{4} + x_{2}y_{2}^{4} \\
\chi_{3,1} &= 4 \left( x_{1}y_{1}^{3}(1-y_{1}) + x_{2}y_{2}^{3}(1-y_{2}) \right) \\
\chi_{2,2} &= 6 \left( x_{1}y_{1}^{2}(1-y_{1})^{2} + x_{2}y_{2}^{2}(1-y_{2})^{2} \right) \\
\chi_{1,3} &= 4 \left( x_{1}y_{1}(1-y_{1})^{3} + x_{2}y_{2}(1-y_{2})^{3} \right) \\
\chi_{0,4} &= x_{1}(1-y_{1})^{4} + x_{2}(1-y_{2})^{4} + x_{3}
\end{align*}
\] (7)
Importantly, although the system of equations mapping \(\chi \leftrightarrow x, y\) can always be solved, the decomposition is valid only if it passes a “sanity check”. Explicitly, this decomposition certifies that \(\rho_{\text{GDS}}\) is separable if and only if convexity conditions on the coefficients parametrizing \(\rho_{\text{GDS}}\) are satisfied,
\[
\rho_{\text{GDS}} \in \mathcal{Q}_{\text{GDS}} \iff \exists x, y \text{ satisfying Eq. (6)}
\] such that \(\forall j : 0 \leq x_{j}, y_{j} \leq 1\) [20].

To be clear, conditions (8) are cumulatively a sufficient criterion for certifying separability, since
\[
\mathcal{Q}_{\text{SDS}} \subseteq \mathcal{Q}_{\text{SEP}} \cap \mathcal{Q}_{\text{GDS}} \subseteq \mathcal{Q}_{\text{GDS}}
\] (9)
where \(\mathcal{Q}_{\text{SEP}} \cap \mathcal{Q}_{\text{GDS}} \equiv \mathcal{Q}_{\text{SEP}} \cap \mathcal{Q}_{\text{GDS}}\) and where \(\subseteq \) and \(\subset\) are analogous to \(\leq \) and \(<\) respectively: \(\subset\) indicates a proper subset, categorically rejecting the possibility of equivalence. So, even though we have not yet ruled out the existence of a separable \(\rho_{\text{GDS}}\) incompatible with the SDS format, the criterion developed is already a sufficient one.

The ability to certify full separability is highly desired, as:

1. The necessary separability criterion of positivity under all partial transpositions [10, 11] does not imply biseparability along all bipartitions [21, 22].

2. A state can be partially separable, e.g. separable along all bipartitions, but still be entangled [23], even to the extent of serving as a resource for Bell inequality violations [24].

We emphasize that this method of generating sufficient (full) separability criteria is generic and adaptable: developing criteria for different states means parametrizing some separable states of similar form, so as to allow for parameter matching.

To demonstrate the utility of possessing a sufficient separability criterion we assess the candidacy of supersuperradiance for entanglement generation, per the original motivation for this research. A system initially in a pure Dicke state is said to evolve according to idealized pure Dicke Model superradiance [5] if it decays to the ground state according to the first-order differential equations
\[
\frac{\partial \chi_{n_{0}n_{1}}}{\partial \tau} = -(n_{0} + 1) n_{1} \chi_{n_{0}n_{1}} \frac{\partial \tau}{\partial \tau} + n_{0} (n_{1} + 1) \chi_{n_{0}-1, n_{1}+1} \frac{\partial \tau}{\partial \tau}
\] (10)
where \( \tau \) is a dimensionless time parameter, \( \tau = \Gamma t \) [25]. The idealization is that of perfect indistinguishability of the particles; experimentally it corresponds to the small-volume limit without dipole-dipole induced dephasing. Our question is whether such idealized superradiance can generate entanglement.

Intuitively, this indistinguishable-particles idealization should yield the strongest entanglement possible, such that if less-idealized superradiance were to generate entanglement, then presumably entanglement would also be evident in this extremal model; see for example the discussion of volume-dependent many-body effects in Ref. [6][26]. To consider entanglement generation we utilize an unentangled, i.e. sufficient, criterion, a challenge where we numerically verified that for pure Dicke Model superradiance, conditions (8) are satisfied for all \( \tau \). This consistency-with-separability criterion. a necessary and sufficient indicator function which cuts off the integration whenever the populations violate the PPT conditions. Here the PPT conditions mean that all eigenvalues are nonnegative for all bipartitions of the qubits for partial transposition [35]. We find numerically that PPTGDSVol\(_{N=4}\) = \((3808 \pm 2) \times 10^{-6}\). In contrast, the volume of all GDS states, including entangled, follows from Eq. (13) absent the indicator function; GDSVol\(_{N=1}\) = \(1/N!\). For four qubits GDSVol\(_{N=4}\) = 41,666.6 \times 10^{-6}.

In principle one could calculate the volume of \( \varrho_{SDS} \) along the same lines as Eq. (13) but with a different indicator function based on conditions (8), but there is a much easier way to do it: perform the integration for SDSVol using \( \chi \) and \( \chi \) as the integration coordinates [33][34]. Thus

\[
PPTGDSVol_{N=4} = \iiint_{\chi \in GDS_{N=4}} 1_{PPT}(\chi) \delta(|\chi|_I) \, d\chi \tag{13}
\]

where \(|\chi|_I = \sum_n x_n \) and \( 1_{PPT}(\chi) = \begin{cases} 1 & \chi \in GDS_{PPT} \\ 0 & \chi \notin GDS_{PPT} \end{cases} \) is an indicator function which cuts off the integration whenever the populations violate the PPT conditions. Here the PPT conditions mean that all eigenvalues are nonnegative for all bipartitions of the qubits for partial transposition [35]. We find numerically that PPTGDSVol\(_{N=4}\) = \((3808 \pm 2) \times 10^{-6}\). In contrast, the volume of all GDS states, including entangled, follows from Eq. (13) absent the indicator function; GDSVol\(_{N=1}\) = \(1/N!\). For four qubits GDSVol\(_{N=4}\) = 41,666.6 \times 10^{-6}.

In principle one could calculate the volume of \( \varrho_{SDS} \) along the same lines as Eq. (13) but with a different indicator function based on conditions (8), but there is a much easier way to do it: perform the integration for SDSVol using \( \chi \) and \( \chi \) as the integration coordinates, thus eliminating the need for any indicator function whatsoever. To stay consistent to the originally established metric of the populations \( \chi \), we must insert a volume element in the integrand, namely the absolute value of the determinant of Jacobian matrix for the change-of-variable. For \( N = 4 \) there are five variable pairs \( x_1, x_2, x_3, y_1, y_2 \). The Jacobian’s determinant, happily a priori nonnegative, is \( \text{Jacc} = 96 x_1 x_2 (1-y_1)^2 (1-y_2)^2 (y_1-y_2)^2 \). Lastly we must ensure a one-to-one mapping between \( \chi \) and \( \chi \). To avoid the problematic interchangeability between the variable pairs \( x_1, x_2 \) and \( x_2, y_2 \) we impose the ordering \( x_1 \geq x_2 \).

Therefore

\[
SDSVol_{N=4} = \iiint_{x_1 \geq x_2} \text{Jacc} \times \delta(|\chi|_I) \, dx \, dy
\]

Lemma (12) may seem rather daunting; proving equivalence between separability criteria with formal logic is indeed an intimidating task. However, we can skip the logical proof and instead use integration to directly establish that volume of both \( \varrho_{SDS} \) and \( \varrho_{PPTGDS} \) are identical. To do so we establish a metric on the spaces of density matrices, the metric can be arbitrary but must be consistent; we choose the populations of \( \varrho_{GDS} \) as our integration coordinates [33][34].

\[
\varrho_{GDS} = \varrho_{SEPGDS} = \varrho_{PPTGDS} \tag{12}
\]

where we prove Lemma (12) for \( N = 4 \) and conjecture that it continues to hold for all \( N \) [32]. Demonstrating
we are forced to revise PPTGDSVol$_{N=4}$ to the upper limit of its uncertainty, which indicates convincingly that Lemma (12) is true for $N = 4$.

The authors suspect that Lemma (12) is true for all $N$ for reasons as follows: As previously mentioned, we found that Dicke Model superradiance time evolution, per Eq. (10), is PPT for any $\tau \geq 0$ for at least $N \leq 10$. Thus superradiance serves as a sort of representative sample of PPT\textsuperscript{\textsc{GDS}} states, or formally $\mathcal{G}_{\text{SUP-RAD}} \subset \mathcal{G}_{\text{PPT\textsuperscript{\textsc{GDS}}}}$. But also as mentioned earlier, we found that such systems apparently always fit the SDS form, in that they satisfy conditions (8) for any $\tau \geq 0$ for at least $N \leq 8$. If Lemma (12) were false, then the unflappable fitting of superradiant states into the SDS form would be surprising, as we would have expected $\mathcal{G}_{\text{SUP-RAD}} \not\subset \mathcal{G}_{\text{SDS}}$. Thus we have accumulated evidence-by-contraposition to support Lemma (12) for $N > 4$.

If Lemma (12) is true for all $N$, as evidence suggests, then the ramifications are numerous. First, it implies that conditions (8) amount to a necessary and sufficient criterion for separability. Second, it implies that the basic PPT criterion is a sufficient separability test for diagonally symmetric states. Third, we can generate novel practical necessary (but not sufficient) separability criteria by simply considering weaker extensions of conditions (8). For example, presuming that all separable diagonally symmetric states fit the form of Eq. (6) allows us to identify "separable maxima" for the populations such that if even a single population exceeds its "maximum separable value" then entanglement is incontrovertible. We find that for $\rho_{\text{GDS}}$ to be separable it is necessary (but not sufficient) to satisfy this weaker form of Eq. (6) expressed as

$$\forall n \left\{ \chi_{n_0,n_1} \leq \left( \frac{n_0!n_1!}{N!} \right) \frac{1}{\max_{0 \leq y < 1} \left[ y^{n_0} (1-y)^{n_1} \right]} \right\}$$

$$\therefore \left\{ \chi_{n_0,n_1} \leq \frac{n_0!n_1!}{n_0!} \left( \frac{n_1!}{n_1!} \right) \left( \frac{N!}{N^N} \right) \right\}$$

(14)

which is computationally optimal as a first-pass test to detect entanglement.

The symmetric basis of Dicke states can be extended to general qudits. We desire a generalization of Eq. (6) for qudits, and we wonder if said generalization would also be necessary in addition to sufficient, à la Lemma (12). We hope to consider this in a future work.

In conclusion, what was originally an analysis of superradiance has led to broad approach for studying multipartite entanglement. We found that a Guess & Check technique can be surprisingly efficient, as evidenced by the derivation of conditions (8) which apply for all states diagonal in the symmetric basis. Moreover, the derived criterion is a completely tight characterization of separability properties, since we found that it maps out a volume of states no smaller than that defined by the PPT criterion. Additionally, our motivating question has been firmly answered in the negative; pure Dicke Model superradiance cannot generate entanglement, begging the question "What is, then, the essential prerequisite of entanglement?"? We hope that our techniques for generating sufficient separability criteria, and for certifying the sufficiency of known necessary separability criteria, may prove useful in furthering the understanding of entanglement.

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[17] There are $N+1$ ways to choose $n$, since $n_0, n_1 \in \mathbb{Z}^+$ and $n_0 + n_1 = N$.
[18] Setting $y(N+2)/2 = 0$ when $N$ is an even number actually doesn’t induce loss of generality, evidenced in that PPTGDSVol = SDSVol still holds when $N$ is an even number.
[19] Note that $x_3$ does not appear until the final equation, this is a consequence of having set $y_3 = 0$ to ensure that only free variables exist in the five equations (7).
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[25] Eq. (10) corresponds to Eq. (4.7) in Ref. [5], but with \( t \times \Gamma \to \tau, J - M \to n_0, J + M \to n_1 \), and using \( \rho_M \to \chi_{n_0,n_1} \) for the populations.

[26] Note that this presumption does not constitute proof; we cannot confidently infer an absence of entanglement in the realistic cases of dephasing and lower symmetry from our null finding of entanglement in the pure Dick Model. Proof of inference is desirable for future research.

[27] Alternative separable initial states include the SDS states (which are not pure), pure superpositions of Dicke states, and even mixed states outside of the GDS manifold. The authors consider such variants of initial conditions, along with other generalizations of superradiance, in another paper now in preparation.

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[31] PPT of superradiance may be demonstrated, for example, by graphing the eigenvalues of the partial transpositions of superradiant \( \rho \) as functions of time, analogous to the graphs of the decomposition parameters in the supplementary online materials.

[32] States diagonal in the symmetric basis are a subset of general permutation-symmetric states, \( \mathcal{Q}_{\text{GDS}} \subset \mathcal{Q}_{\text{SYM}} \), thus Lemma (12) is both trivially true for \( N = 2,3 \) [28, 29] and consistent with the existence of permutation-symmetric PPT-entangled states [29, 30].

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[35] The permutation symmetry of \( \rho_{\text{GDS}} \) means we need only consider two bipartitions: partial transposition of the first qubit \( \rho^{\text{PT}_{1/3}} \) or of the first two qubits \( \rho^{\text{PT}_{2/2}} \), akin to the considerations in Ref. [29]. Both bipartitions are rejected by the PPT indicator function (and both are necessary). The PPT indicator function may be thought of as a multidimensional Heaviside step function of the eigenvalues of \( \rho^{\text{PT}_{1/3}} \) and \( \rho^{\text{PT}_{2/2}} \). The authors used a different but equivalent formulation when evaluating Eq. (13).

SUPPLEMENTARY ONLINE MATERIALS

Explicit Separability Certification for N=4

In the main text we consider Dicke Model superradiance to be governed by the differential equations of Eq. (10) subject to initial conditions given by Eq. (11), namely

\[
\forall n \frac{\partial \chi_{n_0,n_1}}{\partial \tau} = n_0 (n_1 + 1) \chi_{n_0-1,n_1+1} [\tau] - (n_0 + 1) n_1 \chi_{n_0,n_1+1} [\tau]
\]

such that \( \chi_{n_0,n_1} [\tau \to 0] = \begin{cases} 1 & n_1 = N, n_0 = 0 \\ 0 & n_1 < N, n_0 > 0 \end{cases} \) .

which for \( N = 4 \) yields the solutions

\[
\begin{align*}
\chi_{0,4} &= e^{-4\tau} \\
\chi_{1,3} &= 2e^{-4\tau} - 2e^{-6\tau} \\
\chi_{2,2} &= 6e^{-6\tau} (-2\tau - 1) + 6e^{-4\tau} \\
\chi_{3,1} &= 36e^{-4\tau} (\tau - 1) + 36e^{-6\tau} (\tau + 1) \\
\chi_{4,0} &= e^{-6\tau} (-24\tau^2 - 28) + e^{-4\tau} (27 - 36\tau) + 1
\end{align*}
\]

which are plotted in Fig. 1.

Per Eq. (6) in the main text, the decomposition parameters are solved from the simultaneous polynomial equations defined by

\[
\forall n \chi_{n_0,n_1} = N! \sum_{j=1}^{J_{\text{max}}} x_j y_j^{n_0} (1 - y_j)^{n_1} \frac{(n_0)! n_1!}{n_0! n_1!} .
\]
Enumerated explicitly for $N = 4$ the decompositions equations are

\[
\begin{align*}
\chi_{4,0} &= x_1(y_1)^4 + x_2(y_2)^4 \\
\chi_{3,1} &= 4 \left( x_1(y_1)^3 (1 - y_1) + x_2(y_2)^3 (1 - y_2) \right) \\
\chi_{2,2} &= 6 \left( x_1(y_1)^2 (1 - y_1)^2 + x_2(y_2)^2 (1 - y_2)^2 \right) \\
\chi_{1,3} &= 4 \left( x_1(y_1)(1 - y_1)^3 + x_2(y_2)(1 - y_2)^3 \right) \\
\chi_{0,4} &= x_1(1 - y_1)^4 + x_2(1 - y_2)^4 + x_3
\end{align*}
\]

which also appear as Eq. (7) in the main text. One can readily solve Eqs. (A.4) analytically. To express the solutions it is convenient to relabel $y_1 = y_+, x_1 = x_+, y_2 = y_-$, and $x_2 = x_-$ so that we may compactly state

\[
y_\pm = \frac{9[y_+^3]^2 - 18[y_+^2][y_-] + 3[2][y_-] (\frac{1}{3} - 8[y_-]^2)}{4[y_-^2] + 6 \big( \frac{1}{3} - 4[y_-^2] \big)^2 + 9[y_-^2] - 9[y_-^2] \big( \frac{1}{3} + 4[y_-^2] \big)} \\
\]

\[
x_\pm = \frac{y_\pm^2 \chi_{2,2} - 6(y_\pm - 1)^2 \chi_{4,0}}{6y_\pm^2 (y_\pm - y_+) (y_\pm (2y_\pm - 1) - y_+)} \\
x_3 = 1 - x_+ - x_-
\]

where in Eq. (A.5) we used $\chi_{n_0,n_1}$ as merely a horizontally compact form of $\chi_{n_0,n_1}$. Taking the populations to be as per Eqs. (A.2) and then plotting the decomposition parameters as functions $\tau$ we obtain Fig. 2 where it is plainly evident that the extrama of $\{\bar{x}(\tau), \bar{y}(\tau)\}$ lie between zero and one. We know that the superradiating systems starts off in a separable state (the maximally excited state) and that it tends to a separable state (the ground state) and so if there were entanglement generated then it would have to build and then dissipate. As such, we confidently establish permanent separability when we are able to bound the extrama of $\{\bar{x}(\tau), \bar{y}(\tau)\}$ as between zero and one. This visually certifies the perpetual separability of the system, and hence the inability of pure Dicke Model superradiance to generate entanglement.

**FIG. 1.** The system is initially entirely in the maximally excited state, so the population $\chi_{0,4}$ initially equals 1. The system then cascades through the lower levels, such that the lower populations achieve their peak filling in chronological sequence, with the system asymptotically tending towards the ground state, defined by $\chi_{0,4} = 1$. Observe that the sum of the five populations is equal to 1 at all times by virtue of normalization.

**FIG. 2.** Observe that all five decomposition parameters remain bounded between zero and one, which is to say that conditions (8) of the main text are satisfied, and the system is perpetually fully separable. That the state is initially fully excited can be seen in that $x_2 + x_3 = 1$ at $\tau = 0$ and, although $y_3 \equiv 0$, we see that $y_2$ also equals zero when $\tau = 0$. Normalization of the state imposes $x_1 + x_2 + x_3 = 1$ at all times.
Explicit Separability Certification for N=8

Again we consider Dicke Model superradiance per Eq. (A.1). For N = 8 the superradiant populations are given by

\[ \chi_{0,8} = e^{-8\tau} \]
\[ \chi_{1,7} = \frac{4}{3}e^{-14\tau} (e^{6\tau} - 1) \]
\[ \chi_{2,6} = \frac{1}{15}e^{-18\tau} (-70e^{4\tau} + 28e^{10\tau} + 42) \]
\[ \chi_{3,5} = \frac{14}{5}e^{-20\tau} (9e^{2\tau} - 5e^{6\tau} + e^{12\tau} - 5) \]
\[ \chi_{4,4} = \frac{14}{3}e^{-20\tau} (-60\tau + 54e^{2\tau} - 10e^{6\tau} + e^{12\tau} - 45) \]
\[ \chi_{5,3} = \frac{28}{3}e^{-20\tau} (75(4\tau + 5) - 25e^{6\tau} + e^{12\tau} + 27e^{2\tau}(20\tau - 13)) \]
\[ \chi_{6,2} = 28e^{-20\tau} (-50e^{6\tau}(3\tau - 2) + e^{12\tau} - 162e^{2\tau}(5\tau - 2) - 25(12\tau + 17)) \]
\[ \chi_{7,1} = \frac{196}{5}e^{-20\tau} (125e^{6\tau}(2\tau - 1) + 125(2\tau + 3) + e^{12\tau}(10\tau - 7) + 81e^{2\tau}(10\tau - 3)) \]
\[ \chi_{8,0} = -800e^{-14\tau}(7\tau - 3) - 196e^{-20\tau}(20\tau + 31) + \frac{49}{5}e^{-8\tau}(23 - 40\tau) + \frac{1568}{5}e^{-18\tau}(11 - 45\tau) + 1 \]

which are plotted in Fig. 3. Recall again that the decomposition parameters are solved from Eq. (A.3). Enumerated explicitly for N = 8 the decomposition equations are

\[ \chi_{0,8} = x_1 (1 - y_1)^8 + x_2 (1 - y_2)^8 + x_3 (1 - y_3)^8 + x_4 (1 - y_4)^8 + x_5 \]
\[ \chi_{1,7} = \frac{1}{8}x_1 y_1 (1 - y_1)^7 + x_2 (1 - y_2)^7 y_2 + x_3 (1 - y_3)^7 y_3 + x_4 (1 - y_4)^7 y_4 \]
\[ \chi_{2,6} = \frac{1}{28}x_2 y_2^2 (1 - y_1)^6 + x_2 (1 - y_2)^6 y_2^2 + x_3 (1 - y_3)^6 y_3^2 + x_4 (1 - y_4)^6 y_4^2 \]
\[ \chi_{3,5} = \frac{3}{56}x_3 y_3^3 (1 - y_1)^5 + x_2 (1 - y_2)^5 y_3^3 + x_3 (1 - y_3)^5 y_3^3 + x_4 (1 - y_4)^5 y_4^3 \]
\[ \chi_{4,4} = \frac{4}{70}x_1 (1 - y_1)^4 y_4^4 + x_2 (1 - y_2)^4 y_4^4 + x_3 (1 - y_3)^4 y_4^4 + x_4 (1 - y_4)^4 y_4^4 \]
\[ \chi_{5,3} = \frac{5}{56}x_1 (1 - y_1)^3 y_5^5 + x_2 (1 - y_2)^3 y_5^5 + x_3 (1 - y_3)^3 y_5^5 + x_4 (1 - y_4)^3 y_4^5 \]
\[ \chi_{6,2} = \frac{6}{28}x_2 (1 - y_1)^2 y_6^6 + x_2 (1 - y_2)^2 y_6^6 + x_3 (1 - y_3)^2 y_6^6 + x_4 (1 - y_4)^2 y_4^6 \]
\[ \chi_{7,1} = \frac{7}{8}x_1 (1 - y_1) y_7^7 + x_2 (1 - y_2) y_7^7 + x_3 (1 - y_3) y_7^7 + x_4 (1 - y_4) y_7^7 \]
\[ \chi_{8,0} = x_1 y_1^8 + x_2 y_2^8 + x_3 y_3^8 + x_4 y_4^8 \]

which we do not attempt to give an analytic solution to. We stress that the system of equations defined by Eq. (A.3) is trivially enumerated for arbitrary N. Furthermore, most any program can solve the system of equations for numeric values of \( \chi \).

Since the system of equations is readily solvable numerically, just as with N = 4 we take the populations as governed by superradiance, now per Eqs. (A.6), and for N = 8 we restrict our consideration to numerical values of \( \tau \). This restriction is entirely irrelevant, however, as our end-goal is to plot the decomposition parameters as (numeric) functions \( \tau \). Doing so, we obtain Fig. 4 where again it is plainly evident that the extrama of \( \{ \bar{x}(\tau), \bar{y}(\tau) \} \) lie between zero and one. This visually certifies the perpetual separability of the system, and hence the inability of pure Dicke Model superradiance to generate entanglement.
FIG. 3. The state is initially entirely in the maximally-excited state, so the population $\chi_{0,8}$ initially equals 1. The system then cascades through the lower levels, such that the lower populations achieve their peak filling in chronological sequence, with the system asymptotically tending towards the ground state, defined by $\chi_{8,0} = 1$.

FIG. 4. Observe that all nine decomposition parameters remain bounded between zero and one, which is to say that conditions (8) of the main text are satisfied, and the system is perpetually fully separable. Normalization is evident in that the sum of the $x$ totals one at all times $\tau$.

**Complete Derivation of the SDS Form**

In the main text it is claimed that the definitions of the SDS form given in Eq. (4) and Eq. (5) are equivalent, meaning that

$$
\int_0^{2\pi} (2\pi)^{-1} \sum_{j=1}^{j_{\text{max}}} x_j (\rho^1 [y_j, \phi]) \otimes \rho^N \, d\phi = N! \sum_{n=1}^{n_{\text{max}}} \sum_{j=1}^j x_j y_j \frac{(1 - y_j)^{n_1}}{n_0! n_1!} |D_n\rangle \langle D_n|
$$

(A.8)

which we formally prove below.

1. As in the main text preceding Eq. (4) we take a completely generic normalized single-qubit pure state $|\psi\rangle \equiv \sqrt{y} |0\rangle + \sqrt{1 - y} e^{i\phi} |1\rangle$ and use it to form a pure single-qubit product state, $\rho^1 [y, \phi] \equiv |\psi\rangle \langle \psi|$. Explicit expansion tells us that

$$
\rho^1 [y, \phi] = y |0\rangle \langle 0| + (1 - y) |1\rangle \langle 1| + \sqrt{y(1 - y)} (e^{-i\phi} |0\rangle \langle 1| + e^{i\phi} |1\rangle \langle 0|)
$$

(A.9)

2. Next take the tensor product of the single qubit product state with itself $N$ times, $\rho^N [y, \phi] \equiv \rho^1 [y, \phi] \otimes N$. Raising a sum to a power $N$ results in a sum of products. Here the exponents $\gamma_{00}, \gamma_{10}, \gamma_{01}, \gamma_{11}$ appearing in the products below are to be understood as ranging over nonnegative integers $\gamma \in \mathbb{Z}^+$ in such a manner that the sum of the exponents total $N$, $\gamma_{00} + \gamma_{10} + \gamma_{01} + \gamma_{11} = N$.

$$
\rho^N [y, \phi] \equiv \rho^1 [y, \phi] \otimes N = \sum_{\{\text{all } \gamma\}}^N y^{(\gamma_{00} + \gamma_{10}/2)} (1 - y)^{(\gamma_{11} + \gamma_{10}/2)} e^{i\phi (\gamma_{10} - \gamma_{01})} \sum_{\text{operator permutations}}^{} \rho \left[ \begin{array}{cc} \gamma_{00} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} \end{array} \right]
$$

(A.10)

where we have introduced a convenient generalization of computational basis states for product states,

$$
\rho \left[ \begin{array}{cc} \gamma_{00} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} \end{array} \right] \equiv (|0\rangle \langle 0|)^{\otimes \gamma_{00}} (|1\rangle \langle 1|)^{\otimes \gamma_{11}} (|0\rangle \langle 0|)^{\otimes \gamma_{01}} (|1\rangle \langle 1|)^{\otimes \gamma_{10}}.
$$

(A.11)

Note that the sum over operator permutations is intentionally not normalized as each permutation of each has equal weight in the expansion of $\rho^1 [y, \phi] \otimes N$. 
3. The next step is to mix uniformly over all $\phi$, namely $\rho^N[y] \equiv (2\pi)^{-1} \int_0^{2\pi} \rho^N[y, \phi] \, d\phi$. The trick in this step is that

$$
\int_0^{2\pi} e^{i\phi(y_{10} - y_{01})} \, d\phi = \begin{cases}
0 & y_{10} \neq y_{01} \\
1 & y_{10} = y_{01}
\end{cases}
$$

which allows us to perform a change-of-variable such that $y_{10} = y_{01} \rightarrow \kappa$, $y_{00} \rightarrow n_0 - \kappa$, $y_{11} \rightarrow n_1 - \kappa$, yielding simply

$$
\rho^N[y] = \sum_n \sum_\kappa y_n^\kappa (1 - y)^{n_1} \rho \begin{pmatrix} n_0 - \kappa \\ \kappa \end{pmatrix} \begin{pmatrix} \kappa \\ n_1 - \kappa \end{pmatrix}
$$

where instead of summing over the four $\gamma$'s we are summing over $n_0, n_1$, and $\kappa$. In these variables new the condition $y_{00} + y_{10} + y_{01} + y_{11} = N$ is automatically satisfied, but to preserve the positivity of both $y_{00}$ and $y_{11}$ we must be careful to upper bound $\kappa \leq \min[n_0, n_1]$.

4. To proceed we must notice that

$$
\sum_{\text{operator permutations}} \sum_{\kappa} \rho \begin{pmatrix} (n_0 - \kappa) \\ \kappa \end{pmatrix} \begin{pmatrix} \kappa \\ (n_1 - \kappa) \end{pmatrix} = \left( \sum_{\text{perms.}} |0\rangle_0 |0\rangle_n |1\rangle_n \right) \left( \sum_{\text{perms.}} |0\rangle_n |0\rangle_n |1\rangle_1 \right)
$$

which makes use of a binomial theorem argument. The left hand side of Eq. (A.14) is a double sum, over permutations of the four operators as well as over all possible partition schemes indexed by $\kappa$. This is equivalent to the right hand side of Eq. (A.14), namely taking the product of unpaired permutation summations. This counting scheme follows from

$$
\sum_{\text{operator permutations}} \sum_{\kappa} \rho \begin{pmatrix} (n_0 - \kappa) \\ \kappa \end{pmatrix} \begin{pmatrix} \kappa \\ (n_1 - \kappa) \end{pmatrix} = \sum_{\text{perms.}} \rho \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \rho \begin{pmatrix} 2 \\ 1 \end{pmatrix}
$$

which is equivalent to

$$
\sum_{\text{operator permutations}} \rho \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \rho \begin{pmatrix} 2 \\ 1 \end{pmatrix}
$$

At this point it is constructive to review Eq. (1) in the main text, which defined the Dicke states as $|D_n\rangle = w_n \sum_{\text{perms.}} |0\rangle_0 |0\rangle_n |1\rangle_n$. Note that the special case of $n = \{3, 1\}$ considered in Eq. (A.15) is the identically the example of Eq. (2) in the main text. Making use of Eq. (A.14) with Eq. (1) allows for a direct substitution such that we have

$$
\rho^N[y] = \sum_n \frac{y_n^\kappa (1 - y)^{n_1}}{w_n^2} |D_n\rangle \langle D_n|.
$$

5. The last step in our construction is to take an arbitrary finite convex mixture over multiple possible $y_j$ so that each $\rho^N[y_j]$ gets weighted by some parameter $x_j$, $\rho_{\text{DS}} = \sum_{j=1}^{\text{max}} x_j \rho^N[y_j]$. We substitute in for the definition of $w_n = \sqrt{n_0!n_1!/N!}$ to finally match up with the quoted form of Eq. (5) in the main text,

$$
\rho_{\text{DS}} = N! \sum_n \frac{\sum_{j=1}^{\text{max}} x_j y_n^\kappa (1 - y)^{n_1}}{n_0!n_1!} |D_n\rangle \langle D_n|.
$$
thereby proving that

\[
\int_0^{2\pi} (2\pi)^{-1} \sum_{j=1}^{j_{\text{max}}} x_j \rho \{ y_j, \phi \} \otimes \sum_{j=1}^{j_{\text{max}}} x_j y_j \frac{n_0 (1 - y_j)^{n_1}}{n_0! n_1!} |D_n\rangle \langle D_n| d\phi = N! \sum_{n} \sum_{j=1}^{j_{\text{max}}} x_j y_j \frac{n_0 (1 - y_j)^{n_1}}{n_0! n_1!} |D_n\rangle \langle D_n|
\]

(A.18)

as claimed.

For completeness, recall that we define \( j_{\text{max}} = \lceil (N + 1)/2 \rceil \) with the special restriction such that \( y_{(N+2)/2} = 0 \) when \( N \) is even, per the discussion subsequent to Eq. (6) in the main text.

---

**Volume of the Separable States for arbitrary N**

The volume calculations to determine PPTGDSVol and SDSVol can be done (easily and analytically) for the trivial cases of \( N = 2, 3 \) and indeed we find perfect analytic agreement between the PPTGDSVol\(_N\) and the SDSVol\(_N\) for those cases. Such small-\( N \) considerations are useful only insofar as verifying the methodology, as the PPT criterion is known to be necessary and sufficient for separability in those regimes [29, 30].

It is interesting to consider larger \( N \) however, for which PPTGDSVol\(_N\), the generalization of Eq. (13) from the main text, becomes computationally intractable. On the other hand SDSVol\(_N\) can be readily calculated analytically up through \( N \sim \mathcal{O}(10) \), as its discontinuous indicator function appearing in the integrand is much simpler. For SDSVol\(_N\) the purpose of the indicator function is merely to ensure a one-to-one mapping between \( \chi \) and \( x, y \), and it can be substituted for nothing more than division by the multiplicity of solutions to the polynomial system of equations produced by Eq. (6) of the main text, leaving the integrand as just a lone volume element. We tabulated SDSVol\(_N\) for many \( N \) and found that it fits the formula

\[
\text{SDSVol}_N = \prod_{z=1}^{N} \frac{z^{(z-1)}}{(2n-1)!} (n-1)! \]

(A.19)

although we have not yet been able to derive this from first principles. Eq. (A.19) provably yields the volume of the separable GDS states for \( N \leq 4 \), as for those cases we thoroughly demonstrated that \( \varrho_{\text{SDS}} = \varrho_{\text{SEP} \cap \text{GDS}} \). If one also accepts that Lemma (12) of the main text holds true for all \( N \) then one has that Eq. (A.19) yields the volume of the separable GDS states for all \( N \).