Radiative spin polarization of twisted electrons in an ultrastrong magnetic field

Koen van Kruining, Felix Mackenroth, and Jörg B. Götte

1 Max Planck Institut für Physik komplexer Systeme, 01187 Dresden, Germany
2 University of Glasgow, Glasgow, G12 8QQ, UK

We present non-perturbative quantum electrodynamical calculations of the radiation rate for a twisted electron in a homogeneous magnetic field for low lying Landau levels \((N \leq 5)\). Our results apply for all field strengths including those beyond the critical field strength at which the spin contributes as much to the electron’s energy as its rest mass. As a consequence we are able to accurately calculate the spin flip rates for low lying Landau levels. This has hitherto only been achieved for higher lying Landau levels by Sokolov & Ternov, who found that the electron becomes partially spin polarized along the magnetic field. Our results suggest that the spin polarization becomes even stronger for low Landau levels, which are the relevant levels at ultra strong magnetic fields.

**Introduction**— A relativistic electron in a homogeneous magnetic fields is the oldest paradigm for the importance of spin. Mere months after Dirac presented his equation, Rabi found an incomplete solution allowing only for one spin orientation relative to the magnetic field \([1]\). Landau studied the problem extensively to explain diamagnetism and introduced a set of energy levels associated with the total angular momentum of the electron \([2, 3]\). Recently the similarity to free space electron vortex beams \([4–6]\) led to a renewed interest in Landau states \([7–11]\).

In 1963, Sokolov & Ternov (S&T) found that an electron in a magnetic field tends to orient its spin against the magnetic field (from now on spin down) with an equilibrium polarization, \((n_↓−n_↑)/(n_↓+n_↑) = 8\sqrt{3}/15 \approx 0.924\) \([3, 12]\) independent of the magnetic field strength. In their calculations they approximated the Laguerre-Gauß wave functions of the electrons by Bessel functions, an approximation which breaks down at low Landau levels where the finite widths of the electron wave functions become important.

For typical experiments in a Penning trap \([13]\) or a cyclotron the magnetic field strength does not exceed 10 T. At these field strengths the synchrotron radiation spectrum can be accurately computed by viewing the electron as a classical point particle without spin coupled to a classical \([14]\) or quantized \([15]\) radiation field. At high (compared to the Zeeman splitting) kinetic energies the S&T calculations \([12]\) yield excellent approximations to the electron’s actual spin polarization. New proposals for generating short-lived magnetic fields in plasma’s created by ultrashort laser pulses allow for field strengths of 0.1-1 MT \([16, 17]\), which is three to four orders of magnitude beyond the field strengths that are available from non-destructive magnets \([18]\). Even stronger magnetic fields, up to 10\(^{11}\) T can be found on the surface of neutron stars \([19, 22]\).

For such strong fields, classical electrodynamics breaks down, as an electron with energy \(E\) on a circular orbit in a magnetic field emits frequencies \(\omega \approx E^2|e|B/m_e^2\) where \(m_e\) is the electron mass, \(e\) its charge and \(B\) the magnetic field strength \([8]\), whence we conclude that for relativistic energies \(E \gtrapprox m_e^2/|e|B\) the emitted frequencies are larger than the electron’s energy, indicating that recoil will be important and the emission needs to be described in a quantum electrodynamics (QED) formalism. For strong magnetic fields \(B \gtrapprox B_{cr} = m_e^2/|e| \approx 4.4\) GT this condition is always satisfied. Physically, a magnetic field of this strength can be interpreted as imparting a momentum change of \(\Delta p \approx |e|B\beta\gamma c/\Delta t \approx m_e c\) onto a relativistic electron over a time interval \(\Delta t \sim h/m_e c^2\), forcing the virtual electron-positron pairs of QED onto circular orbits and hence effectively magnetizing the vacuum itself.

On the other hand, ordinary perturbative QED is not applicable, as in a magnetic field of strength \(B \gtrapprox m_e/eB\), where \(d\) is the field’s spatial extent, an electron’s cyclotron radius becomes \(r = \beta\gamma c/B \lesssim \beta\gamma d\), indicating that in the non-relativistic approximation \(\gamma \equiv 1\) the electron’s motion is bound and the magnetic field must be accounted for by non-perturbative QED techniques \([23, 24]\). The S&T approximation breaks down in this regime too, because even low lying states \(N \lesssim 10\) already have kinetic energies in excess of the electron’s rest mass and it becomes energetically prohibitive to occupy the much higher Landau levels which the S&T approximation presupposes. Thus the magnetic field must be accounted for by a non-perturbative technique. Such a non-perturbative QED scattering theory can be described in the Furry picture of quantum dynamics \([25]\) through replacing the vacuum electron states by solutions of the Dirac equation in a magnetic field \([10]\). Analogous studies of radiation emission in this nonlinear QED regimes were previously conducted in the interaction of relativistic electrons with ultra-intense laser fields \([20]\), a regime which is complementary to the pure magnetic field case studied here. In the absence of strong fields, the scattering theory of non-plane wave states has been explored \([27, 28]\) using techniques which are useful in strong-field situations too.

In this Letter, we analytically investigate and numerically compute the decay rates between low lying Landau levels and find that the relative difference between up-
to-down and down-to-up spin flip channels is even larger than the factor found by S&T.

**Analytical model**— We will take the magnetic field to be constant, homogeneous with field strength $B$ and pointing in the positive $z$-direction, which is the quantization axis for angular momenta too. We use $\eta_{\mu\nu} = \text{diag}(+,−,−,−)$ as our metric, express energies in eV and use $c = h = 1$ and $\epsilon_0 = \mu_0 = 1/4\pi$, which implies $|e| = \sqrt{\alpha} \approx 1/\sqrt{137}$.

For this geometry there exists a complete exact basis of nondiffracting Laguerre-Gauss beam solutions with their beam axes along the magnetic field [10]. They are specified by a momentum along the beam axis, $p$, radial quantum number, $n$, orbital quantum number, $l$, and spin $\sigma = \pm \frac{1}{2}$. Although spin-orbit mixing makes it impossible to attribute an integer orbital angular momentum and a half-integer spin to a given state, one can still assign an integer $l$ and half integer $\sigma$ by assuming the limit $B \to 0$. The total angular momentum $j = l + \sigma$ is always half-integer. The energy of the electron is $\mathcal{E} = \sqrt{n^2 \gamma^2 + p^2 + 2B|e|N}$ with $N = n + \frac{1}{2}(l+|l|) + \sigma + \frac{1}{2}$. The electron states are normalizable in the transverse plane with normalization $\mathcal{N} = 4\pi \mathcal{E}(m_e + \mathcal{E})(n+|l|)!/B|e|n!$.

The most effective way to keep track of angular momentum changes of the electron when it radiates is to expand the photon field too in a basis of eigenstates of angular momentum along the beam axis. These are the photon Bessel modes with total angular momentum $j$, momentum along the beam axis $k$, transverse momentum $\kappa$ and energy $\omega = \sqrt{\kappa^2 + \kappa^2}$. For the photon polarization we use a basis of left- (−) and right (+) handed helicity. For a photon emitted in the positive $z$-direction positive helicity corresponds to a predominantly positive spin whereas for a photon emitted backward it corresponds to a predominantly negative spin, with the expectation value for the photon spin along the beam axis continuously decreasing with decreasing $k$. Using the Coulomb gauge these photon modes are described by the vector potential

$$A_{k\kappa \pm} = \frac{e^{i(kz-\omega t)}}{2} \left[ \left( \frac{1}{2} \begin{pmatrix} 1 & ± \frac{\kappa}{|\kappa|} \end{pmatrix} J_{j,-1}(\kappa r)e^{i(j,-1)\phi} + (1 ± \frac{\kappa}{|\kappa|}) J_{j,+1}(\kappa r)e^{i(j,+1)\phi} \right) - \left( \frac{1}{2} \begin{pmatrix} 1 & ± \frac{\kappa}{|\kappa|} \end{pmatrix} J_{j,-1}(\kappa r)e^{i(j,-1)\phi} + (1 ± \frac{\kappa}{|\kappa|}) J_{j,+1}(\kappa r)e^{i(j,+1)\phi} \right) \right].$$

(1)

With these photon states and the electron states from [10], one can compute the transition matrix element $\mathcal{M} = \int \Psi_f \mathcal{A}^* \Psi_i dV$ with $\Psi_f$ the adjoint final electron state, $\Psi_i \gamma_0$, and slash denoting contraction with the Dirac matrices $\mathcal{A}^* = \gamma^\mu A^\mu_i$. Performing the integrations over the coordinate along the beam axis, time, and the azimuthal angle will yield three delta functions which we will take out of $\mathcal{M}$. Then, only the radial integration remains. Unlike for plane waves, there is no fourth delta function as there is no fourth conserved quantity whose operator commutes with the angular momentum operator and therefore the electron and photon states cannot be simultaneous eigenstates of four conserved quantities. To compute the decay rate from the state $N, j, \sigma$ to $N', j', \sigma'$ with $N > N'$, we have to compute the squared transition matrix element and integrate it over all outgoing coaxial electron momenta, all coaxial and transverse photon momenta and sum over the photon’s angular momentum and both photon polarizations. We take the initial coaxial momentum of the electron $p = 0$ for definiteness. The decay rates for an electron moving along the beam axis can be found by a Lorentz transformation. Taking our wave functions confined to a disc of thickness $L$ and radius $R$ (which we will take to infinity), the outgoing electron’s density of states is $L/2\pi$. The photon’s density of states (for a single angular momentum and polarization) is $L/2\pi \times R/\pi$. Putting all these ingredients together, the decay rate is

$$\Gamma_{Nj\sigma \to N'j'\sigma'} = (2\pi)^2 \epsilon^2 \sum_{\pm} \sum_{j,v} \iint \delta(\mathcal{E} - \mathcal{E}' - \omega) \delta(p - p' - k) \delta_{j,v} \delta_{j',v} \frac{|\mathcal{M}|^2}{LN LN' LN'_\gamma} \frac{L dp'}{2\pi} \frac{LR dkd\kappa}{2\pi^2}.$$  

(2)

Primes refer to the properties of the final electron state. Using the condition $\int \mathcal{A} \cdot \mathcal{A}^* dV = 2\pi/\omega$ yields, for $R \gg \kappa^{-1}$, $N_\gamma \approx LR\omega/2\kappa$. Substituting the transverse normalization factors in eq. (2), $L$ and $R$ disappear and the size of the disc can be taken to infinity.

Together with the delta functions the condition $\omega = \sqrt{\kappa^2 + \kappa^2}$ will remove two integrals and the sum over photon angular momenta. The remaining integral has to be integrated numerically.

**Numerical results**— We numerically computed all
transition rates for electron states in a magnetic field with \( N \leq 5 \) and \( j \geq -9/2 \) to final states with \( N' \leq 4 \) and \( j' \geq -17/2 \). The highest allowable \( j \) for a given \( N \) is always \( j = N - 1/2 \). The reason to choose a lower \( j \) cutoff for the final state is that the electron tends to lose angular momentum when it decays more often than it gains it. To compute the spin flip rate we take the sum \( \sum_{Nj'\sigma'} \Gamma_{Nj\sigma \rightarrow N'j'\sigma'} \) over all energetically allowed final states in this data set.

To compare our results for the relative spin flip rates of twisted electrons to conventional results, we use the plane wave approximation from \([3]\). Here the electron is considered to be a plane wave with the time evolution \( e^{-iHt} \), with \( H \) the Hamiltonian of the problem under consideration. This approximation takes the recoil of the electron when it emits a photon into account properly and is thus an improvement over classical electrodynamics if \( E \geq m^2/e|e|B \). It reproduces the S&T spin flip rates exactly \([3]\). We choose a plane wave momentum equal to the expected tangential momentum of the highest \( f \) state for a given \( N \). This momentum can be written using Euler gamma functions.

\[
P_L = \sqrt{2B|e|} \frac{\Gamma(N + \frac{1}{2})}{\Gamma(N)}. \tag{3}
\]

Interestingly, taking this expression for the tangential momentum of the electron implies that in the weak field limit the change of momentum the electron gets upon decaying far exceeds the emitted photon’s momentum, which is \( \sim B|e|/2m_e \). Thus upon emitting a photon, the electron always gets a ‘superkick’ \([29]\), caused by the spatial extent of the (twisted) photon mode being much larger than the spatial extent of the electron’s wave function. When angular momentum is transferred from one to the other, the electron’s tangential momentum must change by much more than the emitted photon’s tangential momentum. Because the electron’s tangential momentum averaged over one orbit is zero, momentum is nonetheless conserved.

At all field strengths and all initial \( N \) and \( j \) quantum numbers we computed, we find that the spin flip rate for spin down-to-up exceeds the spin up-to-down flip rate by more than the ratio found by S&T (see FIG. 1). For magnetic field strength far below the critical field, the difference is many orders of magnitude and the spin-down-to-up transition rate scales as \( B^2 \) relative to the up-to-down transition rate (see FIG. 1). Assuming a simple model of electrons undergoing spin flips at their respective rates, the spin polarization of the electrons is expected to reach an equilibrium. Our results suggest that for low lying Landau levels at low magnetic field strengths, the equilibrium spin polarization is so good as to be considered perfect for all practical purposes (see FIG. 2).

The large discrepancy in spin flip rates between our computations and the established results stem from the commonly made assumption to use a plane wave approximation \([3]\) or replace the Laguerre-Gauss states by Bessel states \([12, 30]\). Especially for the low \( n \) states we considered, replacing a function having a finite transverse extent with a function that has an infinite transverse extent opens decay channels that for the actual Laguerre-Gauss states are suppressed. This causes two kinds of overestimation of the spin flip rates.

First, at low field strength and low quantum numbers the wavelength of the emitted photon is approximately \( m_e/B|e| \) whereas the size of the electron beam is roughly \( \sqrt{2(2n + |l| + 1)/B|e|} \). This means that for \( 2n + |l| + 1 \ll 2m_e^2/B|e| \), the electron only ‘probes’ the center of the photon mode, which for a vortex mode is
FIG. 3. Time scale at which spin equilibrium is reached. Interestingly, beyond the critical field (gray area), lower lying states equilibrate faster. Note the diverging S&F rate beyond the critical field strength, indicating that the S&F approximation is not applicable in that regime.

a dark spot. Thus the local vortex structure of the photon modes creates a set of selection rules at weak fields, dictating at what order in $B|e|/m_e^2$ a particular decay process becomes relevant. The derivation of these selection rules is beyond the scope of this article, but one can see their striking effects in FIG. 1. The up-to-down spin flip rate agrees well with the S&T result for weak magnetic fields whereas the down-to-up spin flip rate is suppressed by many orders of magnitude. A hint as to why this happens is that an electron can decay from spin up to spin down whilst keeping its $n$ and $l$ quantum numbers the same whereas this is energetically impossible for an down-to-up transition.

The second effect occurs because Laguerre-Gauß states with vastly different $l$ quantum numbers and low $n$ quantum numbers do not overlap well, as one lies in the dark spot of the other. Transitions between these states should be suppressed at all field strengths. Again, assuming the electron wave function has an infinite spatial extent overlooks this effect.

The time scale over which an electron reaches its equilibrium spin can be inferred from a simple two-state model. Assuming there is only one spin up and one spin down state with electrons transitioning back and forth between them, the equilibrium spin is approached as $e^{-t/\tau_{eq}}$, with $\tau_{eq} = 1/(\Gamma_{\uparrow\downarrow} + \Gamma_{\downarrow\uparrow})$, the reciprocal of the sum of the spin flip rates. In FIG. 2 we plot the spin equilibration times for several Landau levels. The equilibration time depends little on the Landau level, but decreases quadratically with $B$ up to the critical field. At one Tesla the spin equilibration time is on the order of a millennium. The actual dynamics is more complicated, with the electron being able to decay to different Landau levels, both via spin-preserving and spin flip decays. At low magnetic fields the time scales of spin preserving and spin flip decays separate, as can be seen in FIG. 4. This separation of time scales stems from the spin preserving decays having higher decay rates than the spin flip decays, as one can see in FIG. 4. For low magnetic field strengths the electron decays first to an $N = 1$ spin up state before undergoing spin flip to an $N = 0$ spin down state much later. At high field strengths both processes occur at similar rates and the electron can undergo spin flip before decaying to an $N = 1$ spin up state, as is signified by the lower occupation of these states at all times.

For higher $n$ states we expect that the overestimation of decay rates due to these effects becomes smaller and the established results become more reliable.

Discussion—Our calculations show that for low-lying states, finite-extent effects influence the emission of synchrotron radiation. The most striking consequence is that, even in excited states, an electron in a magnetic field can become highly spin polarized.

In our calculations we ignore level shifts due to higher order perturbations, this includes the lifting of the spin-degeneracy due to the electron’s anomalous magnetic moment. For the emitted photons we ignore the effects of vacuum birefringence at strong background magnetic fields. Even at the critical field strength the magnitude of these effects is $\sim \alpha/2\pi$. The main effect of the shifting of the electron’s energy levels is that the available phase space for the various decay channels change. The decay rates should change proportionally to the available phase space. Therefore we expect the relative changes in the decay rates to be on the order of $\alpha/2\pi$ too. The only qualitatively new feature that can occur due to level shifts is that it allows decays between previously degen-
erate states, most importantly spin up and spin down with the same $N$. Our method is unable to make predictions about decay rates between such near-degenerate states, but from the small available phase space we expect them to be small even at the critical field strength $O((\alpha B/2n B_c)^2)$. The vacuum birefringence primarily affects the propagation of the emitted photons. The changes it causes in the coupling of the photons to the electron are of second order in $\alpha$. Having considered these effects we believe our calculations are still fairly reliable at the critical field strength.

Outlook—Our numerical integration over all outgoing photon states assumed a homogeneous density of states, but it can be adapted to inhomogeneous densities as well, making it well suited for cavity QED problems in a strong magnetic field.

At low magnetic field strengths ($\sim 1T$), the spin equilibration times are exceedingly long for the Landau levels we investigated. One can imagine investigating higher lying Landau levels, which we expect to trade off some equilibrium spin polarization purity (as $\sqrt{2(2n+|l|+1)/B|e|}$, is larger and transition selectivity is less strict) for a faster equilibration time, finding states which combine reasonably good equilibrium spin polarization purity with a reasonable equilibration time.

Our results were obtained using transition matrix elements obtained from first principles without making any approximation apart from ignoring higher order perturbative effects. This approach can potentially be used for many more problems including quantum corrections and scattering in a strong magnetic field. To complement our exact approach, for weak magnetic fields, the small spatial extent of the electron wave function compared to the wavelength of the radiated photons allows for constructing a simpler approximate model.

\[1\] I. I. Rabi, Z. Phys. 49, 507 (1928)
\[2\] L. D. Landau and J. M. Lifshitz, Quantum mechanics (non-relativistic theory), 3rd ed., Course of theoretical physics, Vol. 3 (Butterworth-heinemann, 1991).
\[3\] J. M. Lifshitz, L. P. Pitajevski, and W. B. Berenstetstki, Quantenelektrodynamik, 7th ed., Lehrbuch der Theoretischen Physik, Vol. 4 (Harri Deutsch, 1991).
\[4\] K. Y. Bliokh, Y. P. Bliokh, S. Savel’ev, and F. Nori, Phys. Rev. Lett. 99, 190404 (2007); K. Y. Bliokh, M. R. Dennis, and F. Nori, ibid. 107, 174802 (2011).
\[5\] R. van Boxem, J. Verbeeck, and B. Partoens, Europhys. Lett. 102, 40010 (2013).
\[6\] S. M. Barnett, Phys. Rev. Lett. 118, 114802 (2017)
\[7\] K. Y. Bliokh, P. Schattschneider, J. Verbeeck, and F. Nori, Phys. Rev. X 2, 041011 (2012).
\[8\] C. Greenshields, R. L. Stamps, and S. Franke-Arnold, New J. Phys. 14, 103040 (2012) C. R. Greenshields, S. Franke-Arnold, and R. L. Stamps, New J. Phys. 17, 093015 (2015).
\[9\] G. Guzzinati, P. Schattschneider, K. Y. Bliokh, F. Nori, and J. Verbeeck, Phys. Rev. Lett. 110, 093601 (2013)
\[10\] P. Schattschneider, T. Schachinger, M. Stöger-Pollach, S. Löfler, A. Steiger-Thirsfeld, K. Y. Bliokh, and F. Nori, Nat. Comm. 5, 4586 (2014).
\[11\] K. van Kruining, A. G. Hayrapetyan, and J. B. Götte, Phys. Rev. Lett. 119, 030401 (2017)
\[12\] A. J. Silenko, P. Zhang, and L. Zou, Phys. Rev. Lett. 121, 043202 (2018)
\[13\] A. A. Sokolov and I. M. Ternov, Sov. Phys. Dokl. 8, 1203 (1964).
\[14\] D. Hanneke, S. Fogwell Hoogerheide, and G. Gabrielse, Phys. Rev. A 83, 052122 (2011)
\[15\] L. D. Landau, J. M. Lifshitz, and L. P. Pitajevski, The classical theory of fields, 4th ed., Course of theoretical physics, Vol. 2 (Butterworth Heinemann, 1994).
\[16\] O. V. Bogdanov, P. O. Kazinski, and G. Y. Lazarenko, Phys. Rev. A 97, 033837 (2018)
\[17\] Z. Lecz, I. V. Konopolev, A. Serji, and A. Andrejev, Sci. Repts. 6, 36139 (2016)
\[18\] D. J. Stark, T. Toncian, and A. V. Arefeev, Phys. Rev. Lett. 116, 185603 (2016)
\[19\] J. R. Sims, D. G. Rickel, C. A. Swenson, J. B. Schillig, G. W. Ellis, and C. N. Ammerman, IEEE Trans. Appl. Supercond. 18, 587 (2008)
\[20\] B. Paczyński, Act. Astr. 42, 145 (1992).
\[21\] C. Thompson and R. C. Duncan, Mon. Not. R. Astr. Soc. 275, 255 (1995) Astrophys. J. 473, 322 (1996)
\[22\] G. Vasisht and E. V. Gotthelf, Astrophys. J. 486, L129 (1997)
\[23\] C. Kouveliotou, S. Dieters, T. Strohmayer, J. van Paradijs, G. J. Fishman, C. A. Meegan, K. Hurley, J. Kommers, I. Smith, D. Frail, and T. Murakami, Nature 393, 295 (1998)
\[24\] V. Ritus, J. Sov. Laser Res. 6, 101007/BF01120220 (1985).
\[25\] V. N. Baier, V. M. Katkov, and V. M. Strakhovenko, Electromagnetic processes at high energies in oriented single crystals (World Scientific, 1998).
\[26\] W. H. Furry, Phys. Rev. 81, 115 (1951).
\[27\] F. Mackenroth and A. Di Piazza, Phys. Rev. A 83, 032106 (2011) A. Di Piazza, C. Mueller, K. Z. Hatagorsytan, and C. H. Keitel, Rev. Mod. Phys. 84, 1177 (2012) F. Mackenroth, N. Neitz, and A. D. Piazza, Plasma Phys. Contr. Fusion 55, 124018 (2013) A. Angioi, F. Mackenroth, and A. Di Piazza, Phys. Rev. A 93, 052102 (2016)
\[28\] I. P. Ivanov, Phys. Rev. D 83, 093001 (2011) I. P. Ivanov and V. G. Serbo, Phys. Rev. A 84, 033804 (2011) I. P. Ivanov, Phys. Rev. D 85, 076001 (2012) V. G. Serbo, I. P. Ivanov, S. Fritzsche, D. Seipt, and A. Surzhikov, Phys. Rev. A 92, 012705 (2015) O. Matula, A. G. Hayrapetyan, V. G. Serbo, A. Surzhikov, and S. Fritzsche, New. J. Phys. 16, 053024 (2014) I. P. Ivanov, D. Seipt, A. Surzhikov, and S. Fritzsche, Phys. Rev. D 94, 076001 (2016) V. A. Zaitsev, V. G. Serbo, and V. M. Shabakhet, Phys. Rev. A 95, 012702 (2017)
\[29\] P. Schattschneider, S. Löfler, M. Stöger-Pollach, and J. Verbeeck, Ultramicroscopy 136, 81 (2014) R. Juchtmans, A. Béché, A. Abakumov, M. Batuk, and J. Verbeeck, Phys. Rev. B 91, 094112 (2015) R. Juchtmans and J. Verbeeck, Phys. Rev. B 92, 134108 (2015)
[29] S. M. Barnett and M. V. Berry, J. Opt. 15, 125701 (2013).

[30] A. A. Sokolov and I. M. Ternov, JETP 4, 396 (1957).

[31] D. Constantinescu, Nucl. Phys. B 36, 121 (1972); Nucl. Phys. B 44, 288 (1972).