ETA INVARIANTS AND REGULARIZED DETERMINANTS FOR ODD DIMENSIONAL HYPERBOLIC MANIFOLDS WITHCUSPS

JINSUNG PARK

ABSTRACT. We study eta invariants of Dirac operators and regularized determinants of Dirac Laplacians over hyperbolic manifolds with cusps and their relations with Selberg zeta functions. Using the Selberg trace formula and a detailed analysis of the unipotent orbital integral, we show that the eta and zeta functions defined by the relative traces are regular at the origin so that we can define the eta invariant and the regularized determinant. We also show that the Selberg zeta function of odd type has a meromorphic extension over $\mathbb{C}$, prove a relation of the eta invariant and a certain value of the Selberg zeta function of odd type, and derive a corresponding functional equation. These results generalize the earlier work of John Millson (see [12]) to hyperbolic manifolds with cusps. We also prove that the Selberg zeta function of even type has a meromorphic extension over $\mathbb{C}$, relate it to the regularized determinant, and obtain a corresponding functional equation.

1. INTRODUCTION

In the seminal paper [12], Millson derived a relation of the eta invariant of the odd signature operator and a certain value of Selberg zeta function of odd type for compact hyperbolic manifolds of dimension $(4n - 1)$. To prove this, Millson used the Selberg trace formula, which relates the spectral data to the geometric data, applied to a test function defined by the odd heat kernel of the odd signature operator. In [7], the corresponding work was done for the analytic torsion and the Ruelle zeta function for odd dimensional hyperbolic manifolds. These results have been generalized to the case of compact locally symmetric spaces of higher rank in [13], [14].

It would be an interesting problem to extend the aforementioned results to noncompact locally symmetric spaces with finite volumes. However, we encounter several serious difficulties when we discuss the extension of those results to noncompact locally symmetric spaces. First of all, the heat operator of the Laplacian is not of trace class. Hence we cannot use the trace of the heat operator as in the compact case. In [18], [19], M"{u}ller introduced the relative trace to overcome this kind of difficulty and defined corresponding relative eta invariants and relative regularized determinants. In this paper, we follow M"{u}ller’s approach to study the (relative) spectral invariants for noncompact hyperbolic manifolds with finite volumes. If the continuous spectrum of the operator has a gap near zero, the relative trace behaves as the usual trace in the compact case. This is the case of the Laplacians acting on

\[\text{2000 Mathematics Subject Classification. Primary:58J28, 58J52, Secondary:11M36.}\]
functions. The regularized determinants of these operators and their relations with Selberg zeta functions have been extensively studied for two and three dimensional manifolds with cusps in [1, 3, 10, 11, 17]. However, if the continuous spectrum of the operator has no gap near zero, we need to know the large time behavior of the relative trace, so its relation with the scattering theory. In this paper, we study the spectral invariants of Dirac operators and Dirac Laplacians for odd dimensional hyperbolic manifolds with cusps, whose continuous spectrums reach zero.

Our approach is to use the Selberg trace formula applied to test functions defined by the heat kernels. To do so, we analyze the corresponding geometric side of the Selberg trace formula, in particular, the unipotent orbital integral. A detailed analysis of these terms enables us to show that the Selberg zeta functions of odd/even type have meromorphic extensions over $\mathbb{C}$. These results can be considered as generalizations of an old result of Gangolli and Warner in [8] to the case of nontrivial locally homogeneous vector bundles over noncompact locally symmetric spaces.

We explain our result more precisely. Let $X = \Gamma \backslash \text{Spin}(2n+1)/\text{Spin}(2n+1)$ be a $(2n + 1)$-dimensional hyperbolic manifold with cusps. Here $\Gamma$ is a discrete subgroup of $G = \text{Spin}(2n+1,1)$ with finite co-volume. Throughout this paper, we also assume that the group generated by the eigenvalues of $\Gamma$ contains no root of unity. We now consider the Dirac operator $\mathcal{D}$ acting on $L^2(X, E)$. Here the spinor bundle $E$ over $X$ is a locally homogeneous vector bundle defined by the spin representation $\tau_n$ of the maximal compact subgroup $\text{Spin}(2n+1)$ of $\text{Spin}(2n+1,1)$. Let us observe that the restriction of $\tau_n$ to $\text{Spin}(2n) \subset \text{Spin}(2n+1)$ has the decomposition $\sigma_+ \oplus \sigma_-$ where $\sigma_\pm$ denotes the half spin representation of $\text{Spin}(2n)$. Let $\mathcal{K}_t$ be the family of functions over $G = \text{Spin}(2n+1,1)$ given by taking the local trace of the integral kernel of $e^{-t\widetilde{D}^2}$ or $\tilde{D}e^{-t\widetilde{D}^2}$ where $\tilde{D}$ is the lifting of $\mathcal{D}$ over the universal covering space of $X$. Now the Selberg trace formula applied to $\mathcal{K}_t$ has the following form,

\[
\sum_{\sigma = \sigma_\pm} \sum_{\lambda \in \sigma_\pm} \hat{\mathcal{K}}_t(\sigma, i\lambda_k) - \frac{1}{4\pi} \int_{-\infty}^{\infty} \text{Tr}(C_\Gamma(\sigma_+,-i\lambda)C'_\Gamma(\sigma_+,i\lambda)\pi_\Gamma(\sigma_+,i\lambda)(\mathcal{K}_t)) \, d\lambda
= I_\Gamma(\mathcal{K}_t) + H_\Gamma(\mathcal{K}_t) + U_\Gamma(\mathcal{K}_t)
\]

where $\sigma_p := \sigma_+ \cup \sigma_-$ gives the point spectrum of $\mathcal{D}$, $C_\Gamma(\sigma_+, i\lambda)$ is the intertwining operator (note that $C_\Gamma(\sigma_+, i\lambda) = C_\Gamma(\sigma_-, i\lambda)$ since $\sigma_+, \sigma_-$ are unramified) and $I_\Gamma(\cdot)$, $H_\Gamma(\cdot)$, $U_\Gamma(\cdot)$ are the identity, hyperbolic and unipotent orbital integrals, respectively. Following [15], [18], we define certain operators $\mathcal{D}_0(i)$ determined by $\mathcal{D}$ such that $e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} e^{-t\mathcal{D}_0(i)^2}$, $\mathcal{D}e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} \mathcal{D}_0(i)e^{-t\mathcal{D}_0(i)}$ are trace class operators on $L^2(X, E)$ where $\kappa$ is the number of the cusps of $X$. We also show that the spectral side of the above Selberg trace formula is equal to the relative trace $\text{Tr}(e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} e^{-t\mathcal{D}_0(i)^2})$ or $\text{Tr}(\mathcal{D}e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} \mathcal{D}_0(i)e^{-t\mathcal{D}_0(i)})$;
for example, if $\mathcal{K}_t$ is give by $e^{-t\tilde{D}^2}$,

$$
\text{Tr}(e^{-t\tilde{D}^2} - \sum_{i=1}^{\kappa} e^{-tD_0^2(i)})
= \sum_{\sigma=\sigma_{\pm}} \sum_{\lambda_k \in \sigma_k^+} \hat{\mathcal{K}}_t(\sigma, i\lambda_k) - \frac{1}{4\pi} \int_{-\infty}^{\infty} \text{Tr}(C^*(\sigma_+, -i\lambda)C_t^*(\sigma_+, i\lambda)\pi^*(\sigma_+, i\lambda)(\mathcal{K}_t)) \, d\lambda
= I_t(\mathcal{K}_t) + H_t(\mathcal{K}_t) + U_t(\mathcal{K}_t).
$$

A similar formula holds for $\text{Tr}(De^{-tD^2} - \sum_{i=1}^{\kappa} D_0(i)e^{-tD_0^2(i)})$ with the corresponding kernel function $\mathcal{K}_t$. The exact forms of $I_t(\mathcal{K}_t)$, $H_t(\mathcal{K}_t)$ are well-known, in particular, the hyperbolic orbital integrals $H_t(\mathcal{K}_t)$ provide us with the Selberg zeta function of odd/even type. Hence, a main task in this paper is the analysis of $U_t(\mathcal{K}_t)$. To do so, we use a result of Hoffmann in [1] and perform explicit computations for the weighted unipotent orbital integrals for our concerned cases. These explicit computations constitute some of the main ingredients of this paper. By these explicit computations, we can show that $U_t(\mathcal{K}_t) = 0$ if $\mathcal{K}_t$ is determined by $\tilde{D}e^{-t\tilde{D}^2}$.

Following Müller’s approach in [17], [18], [19], we define the eta function $\eta_D(s)$ and the zeta function $\zeta_{D^2}(s)$ using the relative traces $\text{Tr}(e^{-tD^2} - \sum_{i=1}^{\kappa} e^{-tD_0^2(i)})$, $\text{Tr}(De^{-tD^2} - \sum_{i=1}^{\kappa} D_0(i)e^{-tD_0^2(i)})$, respectively (see [1]), [30], [31] for the precise definitions of $\eta_D(s)$, $\zeta_{D^2}(s)$). The continuous spectrum of $\mathcal{D}$ is the whole real line, hence the large time contributions of the relative traces can also give rise to poles of $\eta_D(s)$, $\zeta_{D^2}(s)$. Therefore, we need to consider separately the small time contributions and the large time contributions for the meromorphic extensions of $\eta_D(s)$, $\zeta_{D^2}(s)$. We use the analytic expansion of the intertwining operator along the imaginary axis to get the large time contribution and we analyze all the terms in the geometric side of the Selberg trace formula for the small time contribution. We prove the following meromorphic structures of the eta and zeta functions.

**Theorem 1.1.** The eta function $\eta_D(z)$ and the zeta function $\zeta_{D^2}(z)$ have the meromorphic structures,

$$
\Gamma((z + 1)/2) \eta_D(z) = \sum_{k=0}^{\infty} \frac{-2\gamma_k}{z - 2k - 2} + K(z),
\Gamma(z) \zeta_{D^2}(z) = \sum_{k=-n}^{\infty} \frac{\beta_k}{z + k - \frac{1}{2}} + \frac{\beta_0}{(z - \frac{1}{2})^2} - \frac{h}{z} + \sum_{k=0}^{\infty} \frac{-\gamma_k'}{z - k - \frac{1}{2}} + H(z)
$$

where $\beta_k$, $\beta_0$ are locally computable constants, $\gamma_k$, $\gamma_k'$ are constants which are determined by the intertwining operator $C_t^*(\sigma_+, i\lambda)$, $h$ is the multiplicity of the zero eigenvalues of $\mathcal{D}$, and $K(z)$, $H(z)$ are holomorphic functions. In particular, $\eta_D(z)$ and $\zeta_{D^2}(z)$ are regular at $z = 0$. 


It follows from Theorem 1.1 that we can define the eta invariant by
\[ \eta(D) := \eta_D(0). \]
Following [12], we use the Selberg trace formula to derive the relation of the eta invariant \( \eta(D) \) and the value of the Selberg zeta function of odd type \( Z_H^o(s) \) at \( s = n \) (see [15] for the precise definition of \( Z_H^o(s) \)). Let us remark that \( Z_H^o(s) \) is defined a priori only for \( \text{Re}(s) \gg 0 \) and the meromorphic extension of \( Z_H^o(s) \) over \( \mathbb{C} \) is one of the main results of this paper. As we mentioned above, we show that all unipotent terms are vanishing in the Selberg trace formula applied to the odd heat kernel function. In conclusion, the equality of the relative trace and the orbital integrals for the odd kernel function is exactly the same as in the case of compact hyperbolic manifolds. Therefore we can expect the same formula of \( \eta(D) \) and \( Z_H^o(s) \) as in compact hyperbolic manifolds. However, in the corresponding functional equation, the term determined by the intertwining operator appears. The following theorem states our results for \( \eta(D) \) and \( Z_H^o(s) \).

**Theorem 1.2.** The Selberg zeta function of odd type \( Z_H^o(s) \) has a meromorphic extension over \( \mathbb{C} \) with \( s = n \) as a regular point and the following equalities hold:
\[
\eta(D) = \frac{1}{\pi i} \log Z_H^o(n),
\]
\[
Z_H^o(n + s)Z_H^o(n - s) = \exp(2\pi i \eta(D)) \left( \frac{\det C_+(s)C_-(0)}{\det C_-(s)C_+(0)} \right)^{-2^{n-1}} \quad \text{for} \quad s \in \mathbb{C}.
\]
Here \( C_\pm(s) \) are linear operators given by \( C_\Gamma(\sigma_+, s) = \begin{pmatrix} 0 & C_-(s) \\ C_+(s) & 0 \end{pmatrix} \).

We can define the zeta function \( \zeta_{D^2}(z, s) \) of the shifted Dirac Laplacian \( D^2 + s^2 \) where \( s \) is a positive real number. Now the continuous spectrum of \( D^2 + s^2 \) does not reach 0 and the large time contribution does not create any poles of \( \zeta_{D^2}(z, s) \). Therefore we can see that \( \zeta_{D^2}(z, s) \) is regular at \( z = 0 \) by Theorem 1.1. It follows that the regularized determinant
\[ \det(D^2, s) := \exp(-\zeta_{D^2}'(0, s)) \]
is well-defined. We show that \( \det(D^2, s) \) can be extended to a meromorphic function of \( s \) on \( \mathbb{C} \). Let us observe that \( \det(D^2, s) \neq \det(D^2, -s) \) as a meromorphic function over \( \mathbb{C} \) (see Remark 8.2). We use the Selberg trace formula to prove a relation between \( \det(D^2, s) \) and the geometric data, which consists of the Selberg zeta function of even type \( Z_H^e(s) \) (see [15] for the precise definition of \( Z_H^e(s) \)) and certain meromorphic functions over \( \mathbb{C} \) derived from the identity and unipotent orbital integrals. Let us also remark that \( Z_H^e(s) \) is defined a priori only for \( \text{Re}(s) \gg 0 \) and its meromorphic extension over \( \mathbb{C} \) is one of the main results of this paper. By a similar manner as in the previous case, we also derive a functional equation for \( \det(D^2, s) \) and \( Z_H^e(s) \) where the unipotent factor plays a non-trivial role. The following theorem states our results for \( \det(D^2, s) \) and \( Z_H^e(s) \).
Theorem 1.3. The Selberg zeta function of even type $Z_H^e(s)$ has a meromorphic extension over $\mathbb{C}$ and the following equalities hold for any $s \in \mathbb{C}$:

$$\text{Det}(D^2, s) = C Z_H^e(s + n) \Gamma(s + \frac{1}{2})^{-2n\kappa} \exp \left(4\pi \int_0^s p(\sigma_+, i\lambda) + P_U(i\lambda) \ d\lambda \right),$$

$$\text{Det}(D^2, s)^2 = C^2 (\det C_1(\sigma_+, s))^{-2^{n-1}} Z_H^e(n + s) Z_H^e(n - s) \left(\Gamma(s + \frac{1}{2}) \Gamma(-s + \frac{1}{2})\right)^{-2n\kappa},$$

where $C$ is a constant, $p(\sigma_+, s)$ is Plancherel measure for $\sigma_+$, $P_U(s)$ is an even polynomial of degree $(2n - 4)$, and $\kappa$ denotes the number of the cusps of $X$.

If we compare Theorem 1.2 with Theorem 1.3, we can see that there are no defect terms determined by the cusps in the relation of $\eta(D)$ and $Z_H^e(n)$. But the defect terms appear in the relation of $\text{Det}(D^2, s)$ and $Z_H^e(s)$.

This paper has the following structure. In Section 2, we define a Dirac operator $D$ for the locally homogeneous vector bundle $E$, which is defined by the spin representation $\tau_n$ of maximal compact subgroup Spin$(2n + 1)$. We introduce an operator $D_0(i)$ naturally determined by $D$, and using this we define the relative trace. In Section 3, we introduce the Selberg trace formula for nontrivial locally homogeneous vector bundles over noncompact locally symmetric spaces of rank 1. In Section 4, we use result from Section 3 to analyze the unipotent terms. In Section 5, we derive the relation between the relative trace and the spectral side of the Selberg trace formula. In Section 6, we define $\eta_D(s)$ and $\zeta_D^2(s)$ and prove Theorem 1.1. In Sections 7 and 8, we prove Theorem 1.2 and Theorem 1.3 using results of Sections 3, 4 and 5.

Acknowledgments. The author wants to express his gratitude to Werner Müller and Werner Hoffmann for their helpful comments on this paper. He also thanks Paul Loya, Morten Skarsholm Risager, Masato Wakayama and Krzysztof Wojciechowski for several help during the writing of this paper. Finally he thanks the anonymous referee for pointing out many mistakes and giving several comments, which improve this paper considerably. A part of this work was done during the author’s stay at ICTP and MPI. He wishes to express his thanks to ICTP and MPI for their financial support and hospitality.

2. Dirac operators on odd dimensional hyperbolic manifolds with cusps

Let $G$ be a noncompact connected simple Lie group with finite center and let $\mathfrak{g}$ denote its Lie algebra. We assume that the real rank of $G$ is one. Let $\Gamma$ be a discrete subgroup of $G$ such that the group generated by the eigenvalues of $\Gamma$ contains no root of unity. Let $K$ be a maximal compact subgroup of $G$, $\theta$ the associated Cartan involution, $\mathfrak{k} \oplus \mathfrak{p}$ the associated Cartan decomposition, and $C(\ ,\ )$ the Killing form of $\mathfrak{g}$. Let $a$ be a maximal abelian subalgebra of $p$; $a$ is one dimensional since the rank of $G$ is 1. Let $\Phi$ be the set of roots of $(\mathfrak{g}, a)$. Choose an order for $\Phi$ and let $\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Phi^+} \mathfrak{g}_\lambda \oplus \sum_{\lambda \in \Phi^+} \mathfrak{g}_{-\lambda}$ be the root
space decomposition where $g_0$ is the centralizer of $a$. Let $\alpha_1$ be the unique simple positive root; then $\Phi^+ = \{\alpha_1, 2\alpha_1\}$. Let $n_{\alpha_1} := g_{\alpha_1}$ and $n_{2\alpha_1} := g_{2\alpha_1}$. This last space may be 0. If $N = \exp(n_{\alpha_1} \oplus n_{2\alpha_1})$ and $A = \exp(a)$, then $G = NAK$ is an Iwasawa decomposition. Let $P_0 = NAM$ (with $M$ the centralizer of $A$ in $K$) be the associated minimal parabolic. The $G$-conjugates of $P_0$ are the proper parabolic subgroups of $G$. A parabolic subgroup $P$ is called $\Gamma$-cuspidal if $\Gamma \cap N(P) \setminus N(P)$ is compact. Here $N(P)$ is the unipotent radical of $P$ which, we may assume, is $G$-conjugate to $N$. Let $P_\Gamma = \{P_1, \cdots, P_\kappa\}$ be a complete set of $\Gamma$-conjugacy classes of $\Gamma$-cuspidal subgroups of $G$.

From now on, we assume that $G = \text{Spin}(2n+1,1)$ and $K = \text{Spin}(2n+1)$. Let $\widetilde{X}$ denote the noncompact symmetric space given by $\text{Spin}(2n+1,1)/\text{Spin}(2n+1)$. The Killing form $C(\cdot, \cdot)$ provides us with an invariant metric on $G/K$ by $\langle Y, Z \rangle = \frac{1}{4n}C(Y, Z)$ for $Y, Z \in p$, which gives us constant curvature $(-1)$. We use the spin representation $(\sigma_w, \tau_w)$ of Spin(2n+1) to define a homogeneous vector bundle $\widetilde{E}$ over $\widetilde{X} = \text{Spin}(2n+1,1)/\text{Spin}(2n+1)$ by $\widetilde{E} = G \times V_{\tau_n}/\sim$ where

$$(g, v) \sim (g', v')$$ if $$(g', v') = (g_k, \tau_n(k^{-1})v)$$

for $g, g' \in G$, $k \in K$. We denote such equivalence classes by $[g, v]$. Note that $\widetilde{E}$ admits a left $G$ action defined by $g_0[g, v] = [g_0g, v]$. If we restrict the spin representation $\tau_n$ of $K = \text{Spin}(2n+1)$ to $M = \text{Spin}(2n)$, then $\tau_n$ decomposes into two half spin representations $\sigma_+, \sigma_-$ of Spin(2n). Two representations $(\sigma_+, H_{\sigma_+}), (\sigma_-, H_{\sigma_-})$ are unramified and $w\sigma_+ = \sigma_-, w\sigma_- = \sigma_+$ for the nontrivial element $w \in W(A) = M^*/M$ where $M^*$ is the normalizer of $A$ in $K$. We define the Dirac operator $\widetilde{D} : C^\infty(\widetilde{X}, \widetilde{E}) \rightarrow C^\infty(\widetilde{X}, \widetilde{E})$ by

$$\widetilde{D} = \sum_{i=1}^{2n+1} c(X_i)\nabla_{X_i}$$

where $\{X_i : 1 \leq i \leq 2n+1\}$ is a left invariant orthonormal frame such that $H := X_{2n+1}$ at $eK$ spans $a$, and $\nabla$ is the Levi-Civita connection on $\widetilde{E}$.

We consider a locally symmetric space given by $X = \Gamma \backslash \text{Spin}(2n+1,1)/\text{Spin}(2n+1)$ where $\Gamma$ is a discrete subgroup of Spin(2n+1,1) with unipotent elements satisfying the condition in the introduction. As a consequence of this assumption, $\Gamma$ is torsion-free and $\Gamma \cap P = \Gamma \cap N(P)$ so that $\Gamma \cap P \setminus N(P) = \Gamma \cap N(P) \setminus N(P)$. Then $X$ is a $(2n+1)$-dimensional hyperbolic manifold with cusps. We denote by $E$ the quotient space $\Gamma \backslash \widetilde{E}$. This is a locally homogeneous vector bundle over $X$. Moreover the Dirac operator $\widetilde{D}$ can be pushed down to $X$. We denote this operator by $D$ and its unique self-adjoint extension on $L^2(X, E)$ by $\mathcal{D}$. The hyperbolic manifold $X$ endowed with the metric $(\cdot, \cdot)$ has a following decomposition,

$$X = X_0 \cup W_1 \cup \cdots \cup W_\kappa$$

where $X_0$ is a compact manifold with boundaries and $W_i, i = 1, \cdots, \kappa$ are ends of $X$. (In general, $X_0$ and $W_i$'s may have the nonempty intersections each other.) For each end $W_i$
which we call as \textit{cusp}, \( W_i \cong [0, \infty) \times N_i \) and \( N_i \) can be identified with the flat torus \( T^{2n} \) with the metric \( dn^2 \), and the restriction of \( \langle \cdot, \cdot \rangle \) to \( W_i \) has the form \( dg^2(r, x) = dr^2 + e^{-2r}dn^2(x) \) for \( (r, x) \in [0, \infty) \times N_i \).

Now we have the following expression for \( D \) over the cusp \( W_i \),

\[
D = c(H)(\nabla_H + B - n \text{Id})
\]

where \( B = \sum_{i=1}^{2n} c(X_i)c(H)\nabla^N_{X_i} \) with Levi-Civita connection \( \nabla^N \) over \( E|_N \) (see (7) in \cite{2}). Note that \( B \) has the property \( c(H)B = -Bc(H) \) and \( E_0 := \ker(B) \) can be identified with \( V_n \). Let \( 0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots \to \infty \) be positive eigenvalues of \( B \), each eigenvalue repeated according to its multiplicity with corresponding eigensections \( \psi_j \). We denote by \( E_{\mu_j} \) the eigenspace corresponding to \( \mu_j \). We decompose

\[
L^2([0, \infty) \times N_i, E|_{[0, \infty) \times N_i}, dg^2(r, x))
\]

into

\[
L^2([0, \infty), E_0, e^{-2nr}dr) \oplus \bigoplus_{j=1}^{\infty} L^2([0, \infty), E_{\mu_j} \oplus c(H)E_{\mu_j}, e^{-2nr}dr)
\]

where \( [0, \infty) \) denotes \( [0, \infty) \). Note that \( E_0 \cong V_n \) is a symplectic vector space with a symplectic structure \( \langle c(H), \cdot \rangle \). We fix a Lagrangian subspace \( L \) of \( E_0 \cong V_n \) such that \( E_0 = L \oplus c(H)L \).

Then the map

\[
\alpha_j \psi_j + \alpha_{-j}c(H)\psi_j \to e^{-nr}\left(\begin{array}{c}
\alpha_j \\
\alpha_{-j}
\end{array}\right)
\]

gives a unitary equivalence

\[
L^2([0, \infty), E_0, e^{-2nr}dr) \oplus \bigoplus_{j=1}^{\infty} L^2([0, \infty), E_{\mu_j} \oplus c(H)E_{\mu_j}, e^{-2nr}dr)
\]

\[
\cong \bigoplus_{j=1}^{d(\sigma_+)} L^2([0, \infty), \mathbb{C}^2, dr) \oplus \bigoplus_{j=1}^{\infty} L^2([0, \infty), \mathbb{C}^2, dr)
\]

under which the Dirac operator \( D \) over the cusp \( W_i \) decomposes as follows

\[
D \to \bigoplus_{j=1}^{d(\sigma_+)} D_0 \oplus \bigoplus_{j=1}^{\infty} D_{\mu_j}
\]

where

\[
D_{\mu} = \begin{pmatrix}
0 & -d/dr + \mu \\
-d/dr & 0
\end{pmatrix}.
\]

Hence the Dirac Laplacian \( D^2 \) is transformed into \( \bigoplus_{\mu \in \text{spec}(B)} D_{\mu}^2 \) where \( D_{\mu}^2 = -\frac{d^2}{dr^2} + \mu^2 \) on \( L^2([0, \infty), \mathbb{C}, dr) \).

We now consider the operator

\[
c(H)\left(\frac{d}{dr} - n \text{Id}\right) : C_0^\infty([0, \infty), E_0) \to C_0^\infty([0, \infty), E_0)
\]

whose \( L^2 \)-extension ( with respect to \( e^{-2nr}dr \) ) is transformed to \( d(\sigma_+) \)-copies of \( D_0 \) in \( \mathbb{R} \).

Now, we put

\[
C_0^\infty([0, \infty), E_0, L) := \{ \phi \in C_0^\infty([0, \infty), E_0) \mid \phi(0) \in L \},
\]
then the following operator
\begin{equation}
(4) \quad c(H)(\frac{d}{dr} - n \text{Id}) : C_0^\infty(\mathbb{R}^+, E_0, L) \to L^2(\mathbb{R}^+, E_0, e^{-2nr}dr)
\end{equation}
is essentially self-adjoint. By the natural embedding of \( \mathbb{R}^+ \) into the geodesic rays in \( W_i \subset X \), we can regard \( L^2(\mathbb{R}^+, E_0, e^{-2nr}dr) \) as a subspace of \( L^2(X, E) \). The operator \( c(H)(\frac{d}{dr} - n \text{Id}) \) in \( \mathbb{R}^+ \) can be extended to the self-adjoint operator on \( L^2(X, E) \) by the zero map over the orthogonal complement of this subspace in \( L^2(X, E) \). For each \( W_i \), in this way we obtain the operator \( D_0(i) \), \( i = 1, \ldots, \kappa \). We can see that each \( D_0(i) \) has no point spectrum. Now we have

**Proposition 2.1.** The differences \( (e^{-tD^2} - \sum_{i=1}^{\kappa} e^{-tD_0^2(i)}) \), \( (\partial e^{-tD^2} - \sum_{i=1}^{\kappa} D_0(i)e^{-tD_0^2(i)}) \) are trace class operators on \( L^2(X, E) \) for \( t > 0 \).

**Proof.** First choose a smooth function \( g_s \) such that \( 0 < g_s \leq 1 \), \( g_s(x) = 1 \) for \( x \in X_0 \) and \( g_s((r, \cdot)) = 0 \) for \( (r, \cdot) \in W_i \) with \( r \geq s \) for \( 1 \leq i \leq \kappa \). Denote by \( U_{g_s}, U_{1-g_s} \) the operators in \( L^2(X, E) \) defined by multiplication by \( g_s, 1-g_s \) respectively. Then, for \( s \gg 0 \), \( U_{g_s}(\sum_{i=1}^{\kappa} e^{-tD_0^2(i)}) \) is of trace class and the support of \( U_{1-g_s}(\sum_{i=1}^{\kappa} e^{-tD_0^2(i)}) \) consists of disjoint \( \kappa \)-components lying in \( W_i \)'s. So, for our proof we may assume that the supports of \( e^{-tD_0^2(i)} \), \( W_i \)'s are disjoint each other. Now pick \( f \in C^\infty(X) \) such that \( 0 < f \leq 1 \), \( f(x) = 1 \) for \( x \in X_0 \) and \( f((r, \cdot)) = e^{-t} \) for \( (r, \cdot) \in W_i \) with \( r \gg 0 \) for \( 1 \leq i \leq \kappa \). Denote by \( U_f \) the operator in \( L^2(X, E) \) defined by multiplication by \( f \). Then we may write
\begin{align*}
e^{-tD^2} - \sum_{i=1}^{\kappa} e^{-tD_0^2(i)} &= (e^{-\frac{1}{2}D^2} - \sum_{i=1}^{\kappa} e^{-\frac{1}{2}D_0^2(i)}) \circ U_f^{-1} \circ U_f \circ e^{-\frac{1}{2}D^2} \\
&\quad + \sum_{i=1}^{\kappa} e^{-\frac{1}{2}D_0^2(i)} \circ U_f \circ U_f^{-1} \circ (e^{-\frac{1}{2}D^2} - \sum_{i=1}^{\kappa} e^{-\frac{1}{2}D_0^2(i)}).
\end{align*}
The heat kernel estimates for \( e^{-tD^2} \) and \( \sum_{i=1}^{\kappa} e^{-tD_0^2(i)} \) show that \( (e^{-\frac{1}{2}D^2} - \sum_{i=1}^{\kappa} e^{-\frac{1}{2}D_0^2(i)}) \circ U_f^{-1} \circ U_f \circ e^{-\frac{1}{2}D^2}, \sum_{i=1}^{\kappa} e^{-\frac{1}{2}D_0^2(i)} \circ U_f \circ U_f^{-1} \circ (e^{-\frac{1}{2}D^2} - \sum_{i=1}^{\kappa} e^{-\frac{1}{2}D_0^2(i)}) \) are Hilbert-Schmidt operators. The composition of Hilbert-Schmidt operators is of trace class, hence \( e^{-tD^2} - \sum_{i=1}^{\kappa} e^{-tD_0^2(i)} \) is a trace class operator. The remaining case is proved in the same way. \( \square \)

3. **The Selberg trace formula**

Let \( R_\Gamma \) be the right regular representation of \( G \) on \( L^2(\Gamma \backslash G) \) and \( f \) be a right \( K \)-finite function in the Harish-Chandra \( L^p \)-Schwartz space \( \mathcal{C}^p(G) \) where \( 0 < p < 1 \). The trace of the restriction \( R_\Gamma(f) \) to the discrete part \( L_2^0(\Gamma \backslash G) \) of \( L^2(\Gamma \backslash G) \) can be written as
\begin{equation*}
\text{Tr}(R_\Gamma(f)|_{L_2^0(\Gamma \backslash G)}) = I_\Gamma(f) + H_\Gamma(f) + U_\Gamma(f) + S_\Gamma(f) + T_\Gamma(f).
\end{equation*}
By (4.5) in [13], we have a function \( \tilde{\pi} \) unitary principal representation \( K \) from now on, we will denote by \( x, y \) given by \( K \) bundle, hence the kernel \( G \) consists of endomorphism valued functions on \( \rho \), respectively, relative to the Euclidean structure associated with the Killing form. Then the Haar measure \( d \gamma \) where \( \Gamma \) denotes the intertwining operator (see p.166–167 in [1], p.9–10 in [22] for the precise definitions of \( C_1(\Gamma), C_2(\Gamma), C'_1(\Gamma) \) and tempered distributions \( T_1, T_2, T'_1 \) will be discussed in the next section, and

\[
C_\Gamma(\sigma, \nu) : H_\Gamma(\sigma, \nu) \to H_\Gamma(\sigma, -\nu)
\]
denotes the intertwining operator (see p.166–167 in [1], p.9–10 in [22] for the precise definitions of \( C_1(\sigma, \nu), H_\Gamma(\sigma, \nu) \)). We refer to [11, 20, 22, 23] for detailed expositions of the Selberg trace formula. The various invariant measures that we use are normalized as in [20]. More precisely, let \( d_K \) be the Haar measure on \( K \) which assigns to \( K \) total volume one, let \( d_A, d_N \) be the exponentiation of the normalized Lebesgue measure on the Lie algebra \( \mathfrak{a}, \mathfrak{n} \) of \( A, N \), respectively, relative to the Euclidean structure associated with the Killing form. Then the Haar measure \( d_G \) is determined by

\[
d_G(x) = \alpha^{2\rho} d_N(n) d_A(a) d_K(k) \quad \text{for} \quad x = n a k
\]

where \( \rho \) is the sum of the positive roots of \((\mathfrak{g}, \mathfrak{a})\) divided by 2.

The kernel \( \tilde{K}_t(x : y) \) of the integral operator \( e^{-t\tilde{D}^2} \) (or \( \tilde{D} e^{-t\tilde{D}^2} \)) over \( G/K \) is a section of \( \tilde{E} \otimes \tilde{E}^* \), the external tensor product of \( \tilde{E} \) and \( \tilde{E}^* \) over \( G \times G \). The bundle \( E \) is a trivial bundle, hence the kernel \( \tilde{K}_t(x : y) \) is an element of \( (C^\infty(G \times G) \otimes \text{End}(V_{\gamma_n}))^K \), which consists of endomorphism valued functions on \( G \times G \) invariant under \( K \). It follows that there exists a function \( \tilde{K}_t(x) : G \to \text{End}(V_{\gamma_n}) \) such that

\[
\tilde{K}_t(x : y) = \tilde{K}_t(x^{-1} y) \quad \text{and} \quad \tilde{K}_t(k^{-1}_1 x k_2) = \tau_n(k_1^{-1}) \tilde{K}_t(x) \tau_n(k_2)
\]

for \( x, y \in G, k_1, k_2 \in K \). We define the local trace of \( \tilde{K}_t(x) \) to be the scalar function on \( G \) given by \( K_t(x) := \text{tr}(\tilde{K}_t(x)) \). We denote by \( \hat{K}_t(\sigma, \nu) \) the Fourier transform of \( K_t \) for the unitary principal representation \( \pi_{\sigma, \nu} \) of \( G \), that is,

\[
\hat{K}_t(\sigma, \nu) := \text{Tr}(\pi_{\sigma, \nu}(K_t)).
\]

From now on, we will denote by \( K^*_t, K^*_\tau \) the scalar functions corresponding to \( e^{-t\tilde{D}^2}, \tilde{D} e^{-t\tilde{D}^2} \). By (4.5) in [13], we have
Proposition 3.1. For $\lambda \in \mathbb{R}$, we have

$$\hat{K}_t^e(\sigma, i\lambda) = e^{-t\lambda^2}, \quad \hat{K}_t^o(\sigma, i\lambda) = \lambda e^{-t\lambda^2}, \quad \hat{K}_t^o(\sigma, i\lambda) = -\lambda e^{-t\lambda^2}.$$ 

Since $\sigma_\pm$ is unramified, the intertwining operator $C_T(\sigma_+, \nu) = C_T(\sigma_-, \nu)$ acting on $\mathcal{H}_T(\sigma_+, \nu) = \mathcal{H}_T(\sigma_-, \nu)$ switches the subspaces induced by the representations $\sigma_+, \sigma_-$. Hence, $C_T(\sigma_+, \nu)$ takes the form

$$C_T(\sigma_+, \nu) = \begin{pmatrix} 0 & C_-(\nu) \\ C_+(\nu) & 0 \end{pmatrix}$$

with respect to the decomposition of $\mathcal{H}_T(\sigma_+, \nu)$. Therefore, we have

$$C_T(\sigma_+, \nu)^{-1}C_T'(\sigma_+, \nu) = \begin{pmatrix} C_-(\nu)C'_+(\nu) & 0 \\ 0 & C_+(\nu)C'_-(\nu) \end{pmatrix}$$

and

$$T_T(\cdot) = \frac{1}{4} \text{Tr}(C_T(\sigma_+, 0)\pi_T(\sigma_+, 0)(\cdot)) = 0.$$ 

Since $\tau_m|_M = \sigma_+ \oplus \sigma_-$, the Selberg trace formulas applied to $K_t^e$, $K_t^o$ are given by

$$\sum_{\sigma=\sigma_\pm} \sum_{\lambda_k \in \sigma_\pm} \hat{K}_t^e(\sigma, i\lambda_k) - \frac{1}{4\pi} \int_{-\infty}^{\infty} \text{Tr}(C_T(\sigma_+, -i\lambda)C_T'(\sigma_+, i\lambda)\pi_T(\sigma_+, i\lambda)(K_t^e)) \, d\lambda$$

$$= \sum_{\lambda_k \in \sigma_\pm} e^{-t\lambda_k^2} - \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} e^{-t\lambda^2} \text{tr}(C_T(\sigma_+, -i\lambda)C_T'(\sigma_+, i\lambda)) \, d\lambda$$

$$= I_T(K_t^e) + H_T(K_t^e) + U_T(K_t^e),$$

$$\sum_{\sigma=\sigma_\pm} \sum_{\lambda_k \in \sigma_\pm} \hat{K}_t^o(\sigma, i\lambda_k) - \frac{1}{4\pi} \int_{-\infty}^{\infty} \text{Tr}(C_T(\sigma_+, -i\lambda)C_T'(\sigma_+, i\lambda)\pi_T(\sigma_+, i\lambda)(K_t^o)) \, d\lambda$$

$$= \sum_{\lambda_k \in \sigma_\pm} \lambda_k e^{-t\lambda_k^2} - \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} \lambda e^{-t\lambda^2} \text{tr}(C_-(\sigma_+, i\lambda)) \, d\lambda$$

$$+ \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} \lambda e^{-t\lambda^2} \text{tr}(C_+(\sigma_+, -i\lambda)) \, d\lambda$$

$$= \sum_{\lambda_k \in \sigma_\pm} \lambda_k e^{-t\lambda_k^2} - \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} \lambda e^{-t\lambda^2} \text{tr}(C_-(\sigma_+, -i\lambda)C_T'(\sigma_+, i\lambda)) \, d\lambda$$

$$= I_T(K_t^o) + H_T(K_t^o) + U_T(K_t^o)$$

where $d(\sigma_\pm)$ is the degree of $\sigma_\pm$ (although we know $d(\sigma_+) = 2^{n-1}$, we will use the notation $d(\sigma_+)$ instead of $2^{n-1}$ since this indicates the origin of the constant factor) and

$$\text{tr}_s(C_T(\sigma_+, -\nu)C_T'(\sigma_+, \nu)) = \text{tr}(C_-(\sigma_+, -\nu)C_T'(\sigma_+, \nu)) - \text{tr}(C_+(\sigma_+, -\nu)C_T'(\sigma_+, \nu)).$$
For $I_\Gamma(K_t^\circ)$, we have
\[
I_\Gamma(K_t^\circ) = \text{Vol}(\Gamma\backslash G) \left( \int_{-\infty}^{\infty} \tilde{K}_t(\sigma_+, i\lambda) p(\sigma_+, i\lambda) \, d\lambda + \int_{-\infty}^{\infty} \tilde{K}_t(\sigma_-, i\lambda) p(\sigma_-, i\lambda) \, d\lambda \right)
\]
where $p(\sigma_\pm, i\lambda)$ is the Plancherel measure which is an even polynomial with respect to $\lambda$. The equalities $\tilde{K}_t(\sigma_\pm, i\lambda) = e^{-i\lambda^2}$ and $p(\sigma_+, i\lambda) = p(\sigma_-, i\lambda)$ give
\[
I_\Gamma(K_t^\circ) = 2\text{Vol}(\Gamma\backslash G) \int_{-\infty}^{\infty} e^{-i\lambda^2} p(\sigma_+, i\lambda) \, d\lambda
\]
and $\tilde{K}_t^\circ$ is odd with respect to $\lambda$ so that
\[
I_\Gamma(K_t^\circ) = 0.
\]
It is well-known that
\[
H_\Gamma(K_t^\circ) = \frac{1}{\sqrt{4\pi t}} \sum_{\gamma : \text{hyperbolic}} l(C_\gamma) j(\gamma)^{-1} D(\gamma)^{-1} \left( \frac{\chi_{\sigma_+}(m_\gamma) + \chi_{\sigma_-}(m_\gamma)}{a_\gamma m_\gamma} \right) e^{-\frac{l(C_\gamma)^2}{4t}},
\]
\[
H_\Gamma(K_t^\circ) = \frac{2\pi i}{(4\pi t)^{\frac{d}{2}}} \sum_{\gamma : \text{hyperbolic}} l(C_\gamma)^2 j(\gamma)^{-1} D(\gamma)^{-1} \left( \frac{\chi_{\sigma_+}(m_\gamma) - \chi_{\sigma_-}(m_\gamma)}{a_\gamma m_\gamma} \right) e^{-\frac{l(C_\gamma)^2}{4t}}
\]
where $l(C_\gamma)$ is the length of the closed geodesic $C_\gamma$, $j(\gamma)$ is the positive integer such that $\gamma = j(\gamma)$ for a primitive element $\gamma_0$, $D(\gamma) = e^{l(C_\gamma)} |\text{det}(\text{Ad}(a_\gamma m_\gamma)^{-1} - I_{|\gamma|})|$ for the element $a_\gamma m_\gamma \in A^+M$ which is conjugate to $\gamma$ and $\chi_\sigma$ is the character of $\sigma$.

4. UNIPOTENT TERMS

In this section we compute unipotent terms $U_\Gamma(K_t^\circ)$, $U_\Gamma(K_t^\circ)$. We employ the formula obtained by Hoffmann (see [9]) to compute these terms explicitly. By this explicit computation and Proposition 3.1, it follows that $U_\Gamma(K_t^\circ) = 0$. This simplifies many steps related to the application of the Selberg trace formula for $K_t^\circ$.

For a real rank 1 group $G$, the unipotent term for a right $K$-finite function $f$ in $C^p(G)$, $0 < p < 1$ is given by
\[
U_\Gamma(f) = \frac{1}{|\alpha_1|} (C_1(\Gamma) T_1(f) + C_2(\Gamma) T_2(f) + C_1'(\Gamma) T_1'(f)),
\]
(see theorem in p.299 of [20]). Here $\alpha_1$ is the unique simple positive root for $(g, a)$, the constants $C_1(\Gamma), C_2(\Gamma), C_1'(\Gamma)$ which depend on $\Gamma$ are computed in [3] and
\[
T_1(f) = \frac{1}{A(n_1)} \int_N \int_K f(k^{-1}nk) \, dk \, dn
\]
\[
T_2(f) = \frac{|\alpha_1|}{2} \left\{ \int_{G/G_0} f(xn_0x^{-1}) \, dx + \int_{G/G_{n_0}^{-1}} f(xn_0^{-1}x^{-1}) \, dx \right\}
\]
\[
T_1'(f) = \frac{m_1 + 2m_2}{A(n_1)} \int_{N_1} \int_{N_2} \int_K f(k^{-1}n_1n_2k) \log |\log(n_1)| \, dk \, dn_1 \, dn_2
\]
where $n_0$ is a representative of the nontrivial unipotent orbit in $n_2$, $m_i = \dim(n_i)$ and $A(n_1)$ is the volume of the unit sphere in $n_1$ and $G_n$ is the centralizer of an element $n$ in $G$.

In our case, $G = \text{Spin}(2n + 1, 1)$, the second term $T_2(f) = 0$ and $m_2 = 0$ in the third term $T_1'(f)$ since $N = N_1$. The first term $T_1$ for $K_t$ is given by

$$
T_1(K_t) = \frac{1}{2\pi A(n_1)} \left( \int_{-\infty}^{\infty} \hat{K}_t(\sigma_+, i\lambda) \ d\lambda + \int_{-\infty}^{\infty} \hat{K}_t(\sigma_-, i\lambda) \ d\lambda \right).
$$

We have $\hat{K}_t^e(\sigma_+, i\lambda) = e^{-t\lambda^2}$ and $\hat{K}_t^o(\sigma_+, i\lambda) = \pm \lambda e^{-t\lambda^2}$, which implies

$$
T_1(K_t^e) = \frac{1}{\pi A(n_1)} \int_{-\infty}^{\infty} e^{-t\lambda^2} \ d\lambda \quad \text{and} \quad T_1(K_t^o) = 0.
$$

The third term $T_1'(f)$ is more complicated and we need to introduce some notations. Let $T_M$ be a Cartan subgroup in $M$, so that $T = A \cdot T_M$ is a Cartan subgroup of $G$. Let $\Sigma_M$ denote the set of positive roots for $(m_C, t_m)$. Let $\rho_{\Sigma_M}$ be the half sum of elements in $\Sigma_M$. We denote by $\Sigma_A$ the set of positive roots of $(g_C, t_C)$ which do not vanish on $a_C$. The union of $\Sigma_M$ with $\Sigma_A$ gives the set of positive roots for $(g_C, t_C)$ denoted by $\Sigma_G$. Let $H_\alpha \in t_C$ be the coroot corresponding to $\alpha \in \pm \Sigma_G$, that is, $\alpha(H_\alpha) = 2, \alpha'(H_\alpha) \in \mathbb{Z}$ for all $\alpha, \alpha' \in \pm \Sigma_G$ and $\Pi := \prod_{\alpha \in \Sigma_M} H_\alpha$ is an element of the symmetric algebra $S(t_m)$. We denote the simple reflection corresponding to $\alpha \in \Sigma_G$ by $s_\alpha$. Following the corollary on p.96 of [9], we put

$$
I(f) := \frac{1}{2\pi} \sum_{\alpha \in \Sigma_A} \int_{-\infty}^{\infty} \Omega(\sigma, -i\lambda) \hat{f}(\sigma, i\lambda) \ d\lambda
$$

where

$$
\Omega(\sigma, i\lambda) = 2d(\sigma)\psi(1) - \frac{1}{2} \sum_{\alpha \in \Sigma_A} \lambda(H_\alpha) \frac{\Pi(s_\alpha \lambda_\alpha)}{\Pi(\rho_{\Sigma_M})} \times \left( \psi(1 + \lambda_\sigma(H_\alpha)) + \psi(1 - \lambda_\sigma(H_\alpha)) \right)
$$

where $d(\sigma)$ is the degree of $\sigma$, $\psi$ is the logarithmic derivative of the Gamma function and $\lambda_\alpha - \rho_{\Sigma_M}$ is the highest weight of $(\sigma, i\lambda) \in \hat{M} \times ia$.

The principal series representation $(\pi_{\sigma, \nu}, \mathcal{H}_{\sigma, \nu})$ depends on the parabolic subgroup $P$ and we denote its dependence on $P$ by $(\pi_{\sigma, \nu}(P), \mathcal{H}_{\sigma, \nu}(P))$. The intertwining operator

$$
J_{\pi|P}(\sigma, \nu) : \mathcal{H}_{\sigma, \nu}(P) \to \mathcal{H}_{\sigma, \nu}(\bar{P})
$$

is defined by

$$
(J_{\pi|P}(\sigma, \nu)\phi) = \int_{\bar{N}} \phi(x\bar{n}) \ d\bar{n}
$$

and satisfies

$$
J_{\pi|P}(\sigma, \nu)\pi_{\sigma, \nu}(P) = \pi_{\sigma, \nu}(\bar{P})J_{\pi|P}(\sigma, \nu).
$$

The restriction to $K$ defines an isomorphism from $\mathcal{H}_{\sigma, \nu}(P)$ to $\mathcal{H}_{\sigma}(P)$. Then $J_{\pi|P}(\sigma, \nu)$ can be considered as a family of operators from $\mathcal{H}_{\sigma}(P)$ to $\mathcal{H}_{\sigma}(\bar{P})$. Here $\mathcal{H}_{\sigma}(P)$ is the space of all measurable functions $v : K \to H_{\sigma}$ such that

$$
v(km) = \sigma(m)^{-1}v(k)
$$
for all \( m \in M, k \in K \). Let
\[
J_P(\sigma, \nu : f) := - \text{Tr}(\pi_{\sigma, \nu}(f) J_{P|P}(\sigma, \nu)^{-1} \partial_\nu J_{P|P}(\sigma, \nu))
\]
where the derivative \( \partial_\nu \) is taken with respect to \( \nu \) for the family of operators \( J_{P|P}(\sigma, \nu) \) acting on \( \mathcal{H}_\sigma(P) \). Then there exists the Harish Chandra C-function \( C_\tau(\sigma, \nu) \) such that
\[
T_\tau J_{P|P}(\sigma, \nu)^{-1} \partial_\nu J_{P|P}(\sigma, \nu) = C_\tau(\sigma, \nu)^{-1} \partial_\nu C_\tau(\sigma, \nu) T_\tau
\]
where \( T_\tau \) is the projection to the \( \tau \)-isotypic component of \( \mathcal{H}_{\sigma, \nu}(P) \). We refer to [6] for more detail. We have the following proposition for \( C_{\tau_n}(\sigma_\pm, \nu) \).

**Proposition 4.1.** For the half spin representation \( \sigma_\pm \) of \( \text{Spin}(2n) \),
\[
C_{\tau_n}(\sigma_+, i\lambda) = C_{\tau_n}(\sigma_-, i\lambda) = \frac{(2n - 1)!}{(n - 1)!} \frac{\Gamma(i\lambda + \frac{3}{2})}{\Gamma(i\lambda + n + \frac{3}{2})} \text{Id.}
\]

**Proof.** This follows from theorem 8.2 in [6]. \( \square \)

By the equality (8) (see also (49)) in [9], the weighted orbital integral \( T_1^\prime \) is given by
\[
T_1^\prime(f) = \frac{m_1}{A(n_1)} \left( I(f) + \frac{1}{2\pi} \text{p.v.} \sum_{\sigma \in M} d(\sigma) \int_{-\infty}^{\infty} J_P(\sigma, i\lambda : f) d\lambda + \sum_{\sigma \in M} d(\sigma) \frac{n(\sigma)}{2} \hat{f}(\sigma, 0) \right)
\]
where p.v. means the Cauchy principal value and \( 2n(\sigma) \) is the order of the zero of \( p(\sigma, \nu) \) at \( \nu = 0 \). Using that \( n(\sigma_\pm) = 0 \) and Proposition 4.1, we obtain
\[
T_1^\prime(K_1^\tau) = \frac{m_1}{2\pi A(n_1)} \int_{-\infty}^{\infty} e^{-t\lambda^2} \left( \Omega(\sigma_+, -i\lambda) + \Omega(\sigma_-, -i\lambda) - 2d(\sigma_+) \partial_{i\lambda} \log C_{\tau_n}(\sigma_+, i\lambda) \right) d\lambda,
\]
\[
T_1^\prime(K_1^\tau) = \frac{m_1}{2\pi A(n_1)} \int_{-\infty}^{\infty} \lambda e^{-t\lambda^2} \left( \Omega(\sigma_+, -i\lambda) - \Omega(\sigma_-, -i\lambda) \right) d\lambda.
\]
Now we consider \( \Omega(\sigma_\pm, i\lambda) \) in the following proposition.

**Proposition 4.2.** For the half spin representation \( \sigma_\pm \) of \( \text{Spin}(2n) \), we have
\[
\Omega(\sigma_+, i\lambda) = \Omega(\sigma_-, i\lambda)
\]
\[
= -\frac{d(\sigma_+)}{2} \left( \psi(i\lambda - n + \frac{1}{2}) + \psi(-i\lambda - n + \frac{1}{2}) + \psi(i\lambda + \frac{1}{2}) + \psi(-i\lambda + \frac{1}{2}) \right) + P^n(\lambda)
\]
where \( P^n(\lambda) \) is an even polynomial of degree \( (2n - 4) \) for \( n \geq 2 \) and \( P^1(\lambda) \) is a constant.

**Proof.** The \( n = 1 \) case can be computed in the same way as for the \( n \geq 2 \) cases, hence we may assume that \( n \geq 2 \) in the following proof. The highest weight of the half spin representation \( \sigma_\pm \) of \( \text{Spin}(2n) \subset \text{Spin}(2n + 1) \) is given by
\[
\frac{1}{2} (e_2 + e_3 + \cdots + e_n \pm e_{n+1})
\]
with respect to the standard basis \( \{ e_i \} \). This implies that
\[
i\lambda e_1 + \sigma_\pm + p_{\Sigma M} = i\lambda e_1 + (n - \frac{1}{2}) e_2 + (n - \frac{3}{2}) e_3 + \cdots + (n + \frac{3}{2}) e_n \pm \frac{1}{2} e_{n+1}.
\]
The positive roots $\alpha \in \Sigma_A$ are given by $e_1 - e_j, e_1 + e_j$ for $2 \leq j \leq n + 1$. Then we can see that $s_{(e_1 - e_j)}(i\lambda e_1 + \sigma_\pm + \rho_{\Sigma_M})$ is

$$i\lambda e_j + (n - \frac{1}{2})e_2 + \cdots + (n - j + \frac{3}{2})e_1 + \cdots + \frac{3}{2}e_n \pm \frac{1}{2}e_{n+1} \quad \text{if} \quad 2 \leq j \leq n,$$

$$i\lambda e_{n+1} + (n - \frac{1}{2})e_2 + \cdots + \frac{3}{2}e_n \pm \frac{1}{2} e_1 \quad \text{if} \quad j = n + 1$$

and $s_{(e_1 + e_j)}(i\lambda e_1 + \sigma_\pm + \rho_{\Sigma_M})$ is

$$-i\lambda e_j + (n - \frac{1}{2})e_2 + \cdots + (n - j + \frac{3}{2})e_1 + \cdots + \frac{3}{2}e_n \pm \frac{1}{2}e_{n+1} \quad \text{if} \quad 2 \leq j \leq n,$$

$$-i\lambda e_{n+1} + (n - \frac{1}{2})e_2 + \cdots + \frac{3}{2}e_n \mp \frac{1}{2}e_1 \quad \text{if} \quad j = n + 1.$$

These give us the formula

$$\Pi(s_{(e_1 \pm e_j)}(i\lambda e_1 + \sigma_\pm + \rho_{\Sigma_M}))$$

$$= C_j(\lambda^2 + (n - \frac{1}{2})^2) \cdot (\lambda^2 + (n - \frac{3}{2})^2) \cdots (\lambda^2 + (n - j + \frac{5}{2})^2)$$

$$\cdot (-\lambda^2 - (n - j + \frac{1}{2})^2) \cdots (-\lambda^2 - (\frac{3}{2})^2) \cdot (-\lambda^2 - (\frac{1}{2})^2)$$

where

$$C_j = \prod_{1 \leq k < l \leq n, k \neq j, l \neq j}((n - k + \frac{1}{2})^2 - (n - l + \frac{1}{2})^2)$$

$$= \prod_{1 \leq k < l \leq n, k \neq j, l \neq j}(l - k)(2n - k - l + 1)$$

for $2 \leq j \leq n + 1$. In particular, we obtain

$$\Pi(s_{e_1 \pm e_j}(i\lambda e_1 + \sigma_+ + \rho_{\Sigma_M})) = \Pi(s_{e_1 \pm e_j}(i\lambda e_1 + \sigma_- + \rho_{\Sigma_M}))$$

$$\Pi(s_{e_1 - e_j}(i\lambda e_1 + \sigma_+ + \rho_{\Sigma_M})) = \Pi(s_{e_1 + e_j}(i\lambda e_1 + \sigma_- + \rho_{\Sigma_M}))$$

for $2 \leq j \leq n + 1$. From now on, we denote by $P_j(\lambda)$ the polynomial of $\lambda$ in (13) for $2 \leq j \leq n + 1$. Note that $P_j(\lambda)$ is an even polynomial of degree $2(n - 1)$. On the other hand, $(i\lambda e_1 + \sigma_\pm + \rho_{\Sigma_M})(H_\alpha)$ is given by

$$i\lambda - (n - j + \frac{3}{2}) \quad \text{if} \quad \alpha = e_1 - e_j, \quad 2 \leq j \leq n$$

$$i\lambda \mp \frac{1}{2} \quad \text{if} \quad \alpha = e_1 - e_{n+1}$$

$$i\lambda + (n - j + \frac{3}{2}) \quad \text{if} \quad \alpha = e_1 + e_j, \quad 2 \leq j \leq n$$

$$i\lambda \pm \frac{1}{2} \quad \text{if} \quad \alpha = e_1 + e_{n+1}.$$
Then the pair \( \left( \psi(1 + (i\lambda e_1 + \sigma_\pm + \rho_{\Sigma_M})(H_\alpha)), \psi(1 - (i\lambda e_1 + \sigma_\pm + \rho_{\Sigma_M})(H_\alpha)) \right) \) is given by

\[
\begin{align*}
\psi(i\lambda - n + j - \frac{1}{2}), & \quad \psi(-i\lambda + n - j + \frac{5}{2}) \quad \text{for } e_1 - e_j, \quad 2 \leq j \leq n \\
\psi(i\lambda + \frac{1}{2}), & \quad \psi(-i\lambda + \frac{3}{2}) \quad \text{for } e_1 - e_{n+1}, \quad \sigma = \sigma_+ \\
\psi(i\lambda + \frac{3}{2}), & \quad \psi(-i\lambda + \frac{1}{2}) \quad \text{for } e_1 - e_{n+1}, \quad \sigma = \sigma_- \\
\psi(i\lambda + n - j + \frac{5}{2}), & \quad \psi(-i\lambda - n + j - \frac{1}{2}) \quad \text{for } e_1 + e_j, \quad 2 \leq j \leq n \\
\psi(i\lambda + \frac{3}{2}), & \quad \psi(-i\lambda + \frac{1}{2}) \quad \text{for } e_1 + e_{n+1}, \quad \sigma = \sigma_+ \\
\psi(i\lambda + \frac{1}{2}), & \quad \psi(-i\lambda + \frac{3}{2}) \quad \text{for } e_1 + e_{n+1}, \quad \sigma = \sigma_-.
\end{align*}
\]

Comparing the two sets

\[
\begin{align*}
\left\{ \Pi(s_\alpha(i\lambda e_1 + \sigma_+ + \rho_{\Sigma_M})), 1 + (i\lambda e_1 + \sigma_+ + \rho_{\Sigma_M})(H_\alpha), 1 - (i\lambda e_1 + \sigma_+ + \rho_{\Sigma_M})(H_\alpha) \right\},
\left\{ \Pi(s_\alpha(i\lambda e_1 + \sigma_- + \rho_{\Sigma_M})), 1 + (i\lambda e_1 + \sigma_- + \rho_{\Sigma_M})(H_\alpha), 1 - (i\lambda e_1 + \sigma_- + \rho_{\Sigma_M})(H_\alpha) \right\}
\end{align*}
\]

we see that they are equal to each other, so that \( \Omega(\sigma_+, i\lambda) = \Omega(\sigma_-, i\lambda) \). We now compute the exact form of \( \Omega(\sigma_+, i\lambda) = \Omega(\sigma_-, i\lambda) \). Using the relations \( \psi(z + 1) = \frac{1}{z} + \psi(z) \), we have

\[
\psi(i\lambda - n + j - \frac{1}{2}) + \psi(-i\lambda - n + j - \frac{1}{2}) + \psi(i\lambda + n - j + \frac{5}{2}) + \psi(-i\lambda + n - j + \frac{5}{2})
= \frac{1}{\lambda^2 + (\frac{1}{2})^2} + \cdots + \frac{2(n-j+\frac{1}{2})}{\lambda^2 + (n-j+\frac{1}{2})^2} + \frac{-2(n-j+\frac{5}{2})}{\lambda^2 + (n-j+\frac{5}{2})^2} + \cdots + \frac{-2(n-\frac{1}{2})}{\lambda^2 + (n-\frac{1}{2})^2}
+ \psi(i\lambda - n + \frac{1}{2}) + \psi(-i\lambda - n + \frac{1}{2}) + \psi(i\lambda + \frac{1}{2}) + \psi(-i\lambda + \frac{1}{2}).
\]

Now using the formula (see the last line of p.95 in [9]),

\[
\sum_{\alpha \in \Sigma} \Pi(s_\alpha \lambda_\sigma) = 2\Pi(\lambda_\sigma),
\]

we decompose

\[
\frac{1}{2} \sum_{\alpha \in \Sigma} \frac{\Pi(s_\alpha \lambda_\sigma)}{\Pi(\rho_{\Sigma_M})} \times \left( \psi(1 + \lambda_\sigma(H_\alpha)) + \psi(1 - \lambda_\sigma(H_\alpha)) \right)
\]

into

\[
\frac{d(\sigma_+)}{2} \left( \psi(i\lambda - n + \frac{1}{2}) + \psi(-i\lambda - n + \frac{1}{2}) + \psi(i\lambda + \frac{1}{2}) + \psi(-i\lambda + \frac{1}{2}) \right)
\]

and

\[
\frac{1}{4\Pi(\rho_{\Sigma_M})} \sum_{j=2}^{n+1} P_j(\lambda) R_j(\lambda)
\]
where

\[ R_j(\lambda) = \frac{1}{\lambda^2 + \left(\frac{1}{2}\right)^2} + \cdots + \frac{2(n-j+\frac{1}{2})}{\lambda^2 + (n-j+\frac{1}{2})^2} + \frac{-2(n-j+\frac{5}{2})}{\lambda^2 + (n-j+\frac{3}{2})^2} + \cdots + \frac{-2(n-\frac{1}{2})}{\lambda^2 + (n-\frac{1}{2})^2}. \]

We can see, from the definitions of \( P_j(\lambda) \) and \( R_j(\lambda) \), that \( P_j(\lambda)R_j(\lambda) \) is an even polynomial of degree \( 2n - 4 \). This ends the proof. □

For the constants in (11), comparing proposition 6.2 in [1] with theorem 2 in [3], we obtain

(14) \[ C'(\Gamma) \frac{m_1}{|\alpha_1|} \frac{m_1}{A(n_1)} = \frac{\kappa}{2}. \]

Now we have the following corollary.

Corollary 4.3.

\[ U_\Gamma(K_\sigma) = 0, \quad U_\Gamma(K_\sigma^e) = \frac{2}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha^2}(P_U(\lambda) + Q(\lambda)) \, d\lambda \]

where \( P_U(\lambda) \) is an even polynomial of degree \( (2n - 4) \) and

\[ Q(\lambda) = -\kappa \frac{d(\sigma_+)}{2} \left( \psi(i\lambda + \frac{1}{2}) + \psi(-i\lambda + \frac{1}{2}) \right). \]

Proof. The first claim follows easily from (12), (14), Proposition 1.1 and 1.2. For the second claim, \( Q(\lambda) \) is a priori given by

\[ -\kappa \frac{d(\sigma_+)}{2} \left( \frac{1}{2} \left( \psi(i\lambda - n + \frac{1}{2}) + \psi(-i\lambda - n + \frac{1}{2}) \right) \right) \]

\[ -\frac{1}{2} \left( \psi(i\lambda + n + \frac{1}{2}) + \psi(-i\lambda + n + \frac{1}{2}) \right) \right) + \left( \psi(i\lambda + \frac{1}{2}) + \psi(-i\lambda + \frac{1}{2}) \right). \]

If we use the relation \( \psi(z + 1) = \frac{1}{z} + \psi(z) \), then we can reduce the above formula to the claimed one for \( Q(\lambda) \). □

5. Relative Traces and Spectral Sides

In this section we study relations of the relative traces with the spectral sides of the Selberg trace formulas applied to the test functions \( K_\sigma^e, K_\sigma^o \). A formula of this type was proved by Müller for the similar cases in [15], [18]. Following [15], [18], we prove the corresponding formula for Dirac operators acting on spinor bundles over hyperbolic manifolds with cusps.

First, let us observe that \( L^2(X,E) \) can be identified with the space

\[ (L^2(\Gamma \setminus G) \otimes V_{n\sigma})^K = \{ f \in L^2(\Gamma \setminus G) \otimes V_{n\sigma} \mid f(xk) = \tau_n(k)^{-1}f(x) \quad \text{for} \quad k \in K, x \in G \}. \]

The decomposition of \( L^2(\Gamma \setminus G) = L^2_d(\Gamma \setminus G) \oplus L^2_c(\Gamma \setminus G) \) allows us to decompose

(15) \[ (L^2(\Gamma \setminus G) \otimes V_{n\sigma})^K = (L^2_d(\Gamma \setminus G) \otimes V_{n\sigma})^K \oplus (L^2_c(\Gamma \setminus G) \otimes V_{n\sigma})^K. \]
The continuous part \( (L^2_0(\Gamma \backslash G) \otimes V_{\tau_n})^K \) is spanned by the wave packets with the Eisenstein series, so we need to know how \( D \) acts on the Eisenstein series. For \( \Phi \in (\mathcal{H}_\Gamma(\sigma_+, \nu) \otimes V_{\tau_n})^K \), the Eisenstein series attached to \( \Phi \) is defined by

\[
E(\Phi : \nu : x) := \sum_{\gamma \in \Gamma \cap P \backslash \Gamma} \Phi(\gamma x),
\]

which is defined a priori for \( \text{Re}(\nu) \gg 0 \) and has the meromorphic extension over \( \mathbb{C} \). Assume that \( \Phi_{j, \pm} \) is in the \( \pm i \)-eigenspace of \( c(H) \) on \( (\mathcal{H}_\Gamma(\sigma_+, \nu) \otimes V_{\tau_n})^K \) for \( j = 1, \cdots, d(\sigma_+) \). The Eisenstein series \( E(\Phi_{j, \pm} : i\lambda : x) \) for \( \nu = i\lambda \) satisfies

\[
(16) \quad DE(\Phi_{j, \pm} : i\lambda : x) = \pm \lambda E(\Phi_{j, \pm} : i\lambda : x).
\]

For \( \phi \in \mathcal{C}^\infty_0(X, E) \), the decomposition \((15)\) provides us with the formula

\[
(17) \quad \phi(x) = \sum_k (\phi, \phi_k) \phi_k(x)
\]

\[
+ \frac{1}{4\pi} \sum_{j=1}^{d(\sigma_+)} \int_{-\infty}^\infty E(\Phi_{j, +} : i\lambda : x) \int_X E(\Phi_{j, +} : -i\lambda : y) \phi(y) \ dy \ d\lambda
\]

\[
+ \frac{1}{4\pi} \sum_{j=1}^{d(\sigma_+)} \int_{-\infty}^\infty E(\Phi_{j, -} : i\lambda : x) \int_X E(\Phi_{j, -} : -i\lambda : y) \phi(y) \ dy \ d\lambda
\]

where \( \{\phi_k\} \) is an orthonormal basis of \( (L^2_0(\Gamma \backslash G) \otimes V_{\tau_n})^K \). Since \( D \) preserves the decomposition of \((15)\), we may assume that each \( \phi_k \) is an eigensection of \( D \). Therefore \((16)\) and \((17)\) imply

\[
D\phi(x) = \sum_{\lambda_k} \lambda_k (\phi, \phi_k) \phi_k(x)
\]

\[
+ \frac{1}{4\pi} \sum_{j=1}^{d(\sigma_+)} \int_{-\infty}^\infty \lambda E(\Phi_{j, +} : i\lambda : x) \int_X E(\Phi_{j, +} : -i\lambda : y) \phi(y) \ dy \ d\lambda
\]

\[
- \frac{1}{4\pi} \sum_{j=1}^{d(\sigma_+)} \int_{-\infty}^\infty \lambda E(\Phi_{j, -} : i\lambda : x) \int_X E(\Phi_{j, -} : -i\lambda : y) \phi(y) \ dy \ d\lambda.
\]

From this we can see that the following equalities hold in the distributional sense:

\[
(18) \quad e^{-tD^2}(x : y) = \sum_{\lambda_k \in \sigma_p} e^{-t\lambda_k^2} \phi_k(x) \otimes \phi_k^*(y)
\]

\[
+ \frac{1}{4\pi} \sum_{j=1}^{d(\sigma_+)} \int_{-\infty}^\infty e^{-t\lambda^2} E(\Phi_{j, +} : i\lambda : x) \otimes E(\Phi_{j, +} : -i\lambda : y)^* \ d\lambda
\]

\[
+ \frac{1}{4\pi} \sum_{j=1}^{d(\sigma_+)} \int_{-\infty}^\infty e^{-t\lambda^2} E(\Phi_{j, -} : i\lambda : x) \otimes E(\Phi_{j, -} : -i\lambda : y)^* \ d\lambda,
\]
Proposition 5.1. (Maass-Selberg) Here $\Phi$ is the Maass-Selberg relation. We choose $R_0$ such that $W_{i,R}$’s are disjoint each other for $R \geq R_0$. We put $W_R := \bigcup_{i=1}^{\kappa} W_{i,R}$ and $X_R := X - W_R$ for $R \geq R_0$. The constant term of the Eisenstein series $E(\Phi : i\lambda : x)$ over $W_i$ has the form

$$e^{(-i\lambda+n)r_\Phi} + e^{(i\lambda+n)r} \left( (C_T(\sigma_+,i\lambda) \otimes \text{Id})\Phi \right).$$

Here $\Phi_i$ is the component of $\Phi$ over $W_i$ and note that the operator $C_T(\sigma_+,i\lambda) \otimes \text{Id}$ acts on $(\mathcal{H}(\sigma_+ , i\lambda) \otimes V_{\tau_0})^\kappa$. From now on, we assume that $||\Phi_i|| = 1$ for $i = 1, \cdots , \kappa$. We now discuss the Maass-Selberg relation in our context.

**Proposition 5.1.** (Maass-Selberg) We have

$$(19) \quad D e^{-t\nu^2}(x : y) = \sum_{\lambda_k \in \Sigma_0} \lambda_k e^{-\lambda_k^2} \phi_k(x) \otimes \phi_k^*(y)$$

$$+ \frac{1}{4\pi} \sum_{j=1}^{d(\sigma_+)} \int_{-\infty}^{\infty} \lambda e^{-\lambda^2} E(\Phi_{j,+} : i\lambda : x) \otimes E(\Phi_{j,+} : -i\lambda : y)^* \, d\lambda$$

$$- \frac{1}{4\pi} \sum_{j=1}^{d(\sigma_-)} \int_{-\infty}^{\infty} \lambda e^{-\lambda^2} E(\Phi_{j,-} : i\lambda : x) \otimes E(\Phi_{j,-} : -i\lambda : y)^* \, d\lambda.$$

Recalling the decomposition of $X$, we consider the subset $W_{i,R} := [R, \infty) \times N_i$ in $W_i$. We have

$$\int_{X_R} |E(\Phi_{i,:} : i\lambda : x)|^2 \, dx = 2\kappa R - \text{tr} \left( C_{\pm}(i\lambda)C_{\pm}'(-i\lambda) \right) + O(e^{-cR})$$

$$= 2\kappa R - \text{tr} \left( C_{\pm}'(-i\lambda)C_{\pm}(i\lambda) \right) + O(e^{-cR})$$

where $c$ is a positive constant.

**Proof.** We will consider only the case of $\Phi_+$ so we use the notation $\Phi$ instead of $\Phi_+$ in the following proof. The case of $\Phi_-$ can be done in the same way. It follows from Green’s formula that

$$(\lambda - \lambda')(E(\Phi : i\lambda : x), E(\Phi : i\lambda' : x))_{X_R}$$

$$= \langle DE(\Phi : i\lambda : x), E(\Phi : i\lambda' : x) \rangle_{X_R} - \langle E(\Phi : i\lambda : x), DE(\Phi : i\lambda' : x) \rangle_{X_R}$$

$$= \langle \delta(H)E(\Phi : i\lambda : x), E(\Phi : i\lambda' : x) \rangle_{\partial X_R}.$$
where $C_{\pm}(i\lambda)_{ik}$ is a component of $C_{\pm}(i\lambda)$. Now we use the functional equation

$$C_{\Gamma}(\sigma_+, i\lambda)C_{\Gamma}(\sigma_+, -i\lambda) = \begin{pmatrix} 0 & C_-(i\lambda) \\ C_+(i\lambda) & 0 \end{pmatrix} \begin{pmatrix} 0 & C_-(i\lambda) \\ C_+(i\lambda) & 0 \end{pmatrix} = \text{Id},$$

which implies the equality

$$\langle c(H)E(\Phi : i\lambda : x), E(\Phi : i\lambda' : x) \rangle_{\partial W_i, R} = i e^{-i(\lambda-\lambda')R} - i e^{i(\lambda-\lambda')R}$$

$$- i e^{i(\lambda-\lambda')R} \sum_{k=1}^{\kappa} C_+(i\lambda)_{ik}C_-(-i\lambda')_{ki}$$

$$+ i e^{i(\lambda-\lambda')R} \sum_{k=1}^{\kappa} C_+(i\lambda)_{ik}C_-(-i\lambda)_{ki} + O(e^{-cR}).$$

Combining (22) and (23), we have

$$(\lambda - \lambda')(E(\Phi : i\lambda : x), E(\Phi : i\lambda' : x))_{\lambda R} = i \kappa e^{-i(\lambda-\lambda')R} - i \kappa e^{i(\lambda-\lambda')R}$$

$$+ i e^{i(\lambda-\lambda')R} \sum_{i,k=1}^{\kappa} C_+(i\lambda)_{ik} \left( C_-(-i\lambda)_{ki} - C_-(-i\lambda')_{ki} \right) + O(e^{-cR}).$$

If we pass to the limit $\lambda \to \lambda'$ after dividing each side of (24) by $\lambda - \lambda'$, we get the equality (21).

The next task is to consider the corresponding formula for $D_0(i)$. We fix a Lagrangian subspace $L$ in $V_{\tau_n}$ as before. For an orthonormal basis $\{\psi_j\} \subset L$, we define

$$\phi_{j,+} := \frac{1}{\sqrt{2}} \left( \psi_j - i c(H)\psi_j \right), \quad \phi_{j,-} := \frac{1}{\sqrt{2}} \left( \psi_j + i c(H)\psi_j \right)$$

and

$$e(\phi_{j,+} : i\lambda : x) := e^{(n-i\lambda)r} \phi_{j,+} + e^{(n+i\lambda)r} \phi_{j,-},$$

$$e(\phi_{j,-} : i\lambda : x) := e^{(n-i\lambda)r} \phi_{j,-} + e^{(n+i\lambda)r} \phi_{j,+}.$$
with $\phi_c \in C^\infty_0(W_i, E)$ and $\phi_c|_{\partial W_i} \in L$.

\[ D_0(i)\phi = \frac{1}{4\pi} \sum_{j=1}^{d(\sigma_+)} \int_{-\infty}^{\infty} \lambda E^i(\phi_{j,+} : i\lambda : x) \int_X E^i(\phi_{j,+} : -i\lambda : y) \phi(y) \, dy \, d\lambda \]

\[ -\frac{1}{4\pi} \sum_{j=1}^{d(\sigma_+)} \int_{-\infty}^{\infty} \lambda E^i(\phi_{j,-} : i\lambda : x) \int_X E^i(\phi_{j,-} : -i\lambda : y) \phi(y) \, dy \, d\lambda. \]

The following equality can be proved in the same way as in the proof of Proposition 5.1.

\[ \sum_{j=1}^{\kappa} \int_{X_R} |E^i(\phi_{j,\pm} : i\lambda : x)|^2 \, dx = 2\kappa R + O(e^{-cR}) \]

for some positive constant $c$. (Let us remark that there is no contribution over $\partial W_i$ by the choice of $\phi_{j,\pm}$ when we apply the Green formula as in (22).) It follows from (21), (26) that

\[ \int_X |E(\Phi_{j,\pm} : i\lambda : x)|^2 dx = \lim_{R \to \infty} \int_{X_R} |E(\Phi_{j,\pm} : i\lambda : x)|^2 dx \]

\[ = -\text{tr}(C_{\mp}(-i\lambda)C_\pm^r(i\lambda)). \]

Finally, the following proposition is the result of (5), (6), (8), (18), (19), (25), (27) and Corollary 4.3.

**Proposition 5.2.** We have

\[ \text{Tr}(e^{-tD^2} - \sum_{i=1}^{\kappa} e^{-tD_0^2(i)}) \]

\[ = \sum_{\lambda_k} e^{-t\lambda^2_k} - \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} e^{-t\lambda^2} \text{tr}(C_T(\sigma_+, -i\lambda)C_T^r(\sigma_+, i\lambda)) \, d\lambda \]

\[ = H_T(K_t^\gamma) + U_T(K_t^\gamma), \]

\[ \text{Tr}(DE^{-tD^2} - \sum_{i=1}^{\kappa} D_0(i)e^{-tD_0^2(i)}) \]

\[ = \sum_{\lambda_k} \lambda_k e^{-t\lambda^2_k} - \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} \lambda e^{-t\lambda^2} \text{tr}(C_T(\sigma_+, -i\lambda)C_T^r(\sigma_+, i\lambda)) \, d\lambda \]

\[ = \sum_{\lambda_k} \lambda_k e^{-t\lambda^2_k} - \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} \lambda e^{-t\lambda^2} \text{tr}(C_-(\sigma_+)C_0^r(i\lambda)) \, d\lambda \]

\[ + \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} \lambda e^{-t\lambda^2} \text{tr}(C_+(\sigma_+)C_0^r(i\lambda)) \, d\lambda \]

\[ = H_T(K_t^\gamma). \]
Remark 5.3. Although the definition of $D_0(i)$ depends on the choice of the Lagrangian subspace $L$, the relative traces on the left sides of (28), (29) do not depend on this choice. This is because the right side of (26) does not depend on the choice of $L$.

6. Meromorphic continuations of the eta and zeta functions

In this section, we prove Theorem 11 which provides us with the pole structures of the eta function and the zeta function over $\mathbb{C}$. We follow [18] and [19] and define

$$
\eta_{D,1}(z) := \frac{1}{\Gamma \left( \frac{z+1}{2} \right)} \int_0^1 t^{\frac{z-1}{2}} \text{Tr}(D e^{-tD^2} - \sum_{i=1}^K D_0(i)e^{-tD_0^2(i)}) \, dt,
$$

(30)

$$
\zeta_{D,1}(z) := \frac{1}{\Gamma(z)} \int_0^1 t^{z-1} \left[ \text{Tr}(e^{-tD^2} - \sum_{i=1}^K e^{-tD_0^2(i)}) - h \right] \, dt
$$

for $\text{Re}(z) \gg 0$ and

$$
\eta_{D,2}(z) := \frac{1}{\Gamma \left( \frac{z+1}{2} \right)} \int_1^\infty t^{\frac{z-1}{2}} \text{Tr}(D e^{-tD^2} - \sum_{i=1}^K D_0(i)e^{-tD_0^2(i)}) \, dt,
$$

(31)

$$
\zeta_{D,2}(z) := \frac{1}{\Gamma(z)} \int_1^\infty t^{z-1} \left[ \text{Tr}(e^{-tD^2} - \sum_{i=1}^K e^{-tD_0^2(i)}) - h \right] \, dt
$$

for $\text{Re}(z) \ll 0$ where $h$ is the multiplicity of zero eigenvalues of $D^2$. To define the eta invariant and the regularized determinant, we need to study the meromorphic extensions of $\eta_{D,i}(z)$ and $\zeta_{D^2,i}(z)$ near $z = 0$ for $i = 1, 2$. The difficulty, which is not present for closed manifolds, is the presence of continuous spectrum. Moreover, the continuous spectrum of $D$ is equal to the whole real line in our case. Hence the meromorphic extensions of $\eta_{D,2}(z)$ and $\zeta_{D^2,2}(z)$ have nontrivial poles.

We start with $\eta_{D,1}(z)$. It follows from (10) and (29) that

$$
\text{Tr}(D e^{-tD^2} - \sum_{i=1}^K D_0(i)e^{-tD_0^2(i)}) = H_{\Gamma}(K_i^0)
$$

(32)

$$
= \frac{2\pi i}{(4\pi t)^{\frac{3}{2}}} \sum_{\gamma: \text{hyperbolic}} l(C_{\gamma})^2 j(\gamma)^{-1} D(\gamma)^{-1} (\chi_{\sigma_+}(m_{\gamma}) - \chi_{\sigma_-}(m_{\gamma})) e^{-\frac{i\ell(C_{\gamma})^2}{4t}}.
$$

The number $c := \min_{(\gamma: \text{hyperbolic})} l(C_{\gamma})$ is a positive real number, hence as $t \to 0$,

$$
\frac{2\pi i}{(4\pi t)^{\frac{3}{2}}} \sum_{\gamma: \text{hyperbolic}} l(C_{\gamma})^2 j(\gamma)^{-1} D(\gamma)^{-1} (\chi_{\sigma_+}(m_{\gamma}) - \chi_{\sigma_-}(m_{\gamma})) e^{-\frac{i\ell(C_{\gamma})^2}{4t}} \sim a e^{-\frac{\pi^2}{4t}}
$$

(33)

for a constant $a$. Now (32) and (33) give

$$
\text{Tr}(D e^{-tD^2} - \sum_{i=1}^K D_0(i)e^{-tD_0^2(i)}) \sim a e^{-\frac{\pi^2}{4t}} \quad \text{as} \quad t \to 0.
$$

This means that $\eta_{D,1}(z)$ can be extended to the whole complex plane without poles.
For $\zeta_{D^2,1}(z)$, we use (28) to get
\[
\text{Tr}(e^{-tD^2} - \sum_{i=1}^{k} e^{-tD^2(i)}) = I_\Gamma(K^e_1) + H_\Gamma(K^e_1) + U_\Gamma(K^e_1).
\]
Recall that
\[
H_\Gamma(K^e_1) = \frac{1}{\sqrt{4\pi t}} \sum_{\gamma: \text{hyperbolic}} l(C_\gamma) j(\gamma)^{-1} D(\gamma)^{-1} \left( \chi_{\sigma_+}(m_\gamma) + \chi_{\sigma_-}(m_\gamma) \right) e^{-\frac{t(C_\gamma)^2}{4}}.
\]
As $H_\Gamma(K^e_1)$, we have
\[
(35) \quad H_\Gamma(K^e_1) \sim a' e^{-\frac{t}{4}} \quad \text{as} \quad t \to 0
\]
for a constant $a'$. We can see that $H_\Gamma(K^e_1)$ does not give any poles in the meromorphic continuation of $\zeta_{D^2,1}(z)$. Next we consider $I_\Gamma(K^e_1)$ and $U_\Gamma(K^e_1)$. An elementary computation shows
\[
(36) \quad I_\Gamma(K^e_1) = \sum_{k=0}^{n} a_k t^{-k+\frac{1}{2}}
\]
for some constants $a_k$. By Corollary 4.3, we have
\[
U_\Gamma(K^e_1) = \frac{2}{2\pi} \int_{-\infty}^{\infty} e^{-t\lambda^2} (P_U(\lambda) + Q(\lambda)) \, d\lambda
\]
where $P_U(\lambda)$ is an even polynomial of degree $(2n - 4)$ and $Q(\lambda)$ is given by
\[
-\kappa \frac{d(\gamma_+)}{2} \left( \psi(i\lambda + \frac{1}{2}) + \psi(-i\lambda + \frac{1}{2}) \right).
\]
An elementary computation leads to
\[
(37) \quad \int_{-\infty}^{\infty} e^{-t\lambda^2} P_U(\lambda) \, d\lambda = \sum_{k=0}^{n-2} b_k t^{-k+\frac{1}{2}}
\]
for some constants $b_k$. Using the relations
\[
\psi(x + 1) = \frac{1}{x} + \psi(x), \quad \psi(x) + \psi(x + \frac{1}{2}) = 2(\psi(2x) - \log 2),
\]
we have
\[
Q(\lambda) = -\kappa \frac{d(\gamma_+)}{2} \left( 2(\psi(2i\lambda + 1) + \psi(-2i\lambda + 1) - 2 \log 2) - (\psi(i\lambda + 1) + \psi(-i\lambda + 1)) \right).
\]
To deal with the digamma function $\psi$, we use the following asymptotic expansion
\[
\psi(z + 1) \sim \log z + \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)z^{2k}} \quad \text{as} \quad z \to \infty
\]
where $B_{2k}$ are Bernoulli numbers. This implies the expansion
\[
(38) \quad \int_{-\infty}^{\infty} e^{-t\lambda^2} Q(\lambda) \, d\lambda \sim \sum_{k=0}^{\infty} c_k t^{-k+\frac{1}{2}} + d_0 t^{-\frac{1}{2}} \log t \quad \text{as} \quad t \to 0
\]
for some constants $c_k$ and $d_0$. By \((35), (36), (37), (38)\), we have
\[
I_\Gamma(K^e_t) + H_\Gamma(K^e_t) + U_\Gamma(K^e_t) \sim \sum_{k=-\infty}^{\infty} \beta_k t^{k-\frac{1}{2}} + \beta_0 t^{\frac{1}{2}} \log t \quad \text{as} \quad t \to 0
\]
for constants $\beta_k, \beta_0'$. Therefore $\zeta_{D^2,1}(z)$ is well defined for $\Re(z) > n + \frac{1}{2}$ and we can extend $\zeta_{D^2,1}(z)$ to a meromorphic function on $\mathbb{C}$, with poles determined by
\[
\Gamma(z)\zeta_{D^2,1}(z) = \sum_{k=-n}^{\infty} \frac{\beta_k}{z + k - \frac{1}{2}} + \frac{\beta_0'}{(z - \frac{1}{2})^2} - \frac{h}{z} + H_1(z)
\]
where $\beta_k, \beta_0'$ are constants and $H_1(z)$ is a holomorphic function.

To deal with the meromorphic extensions of $\eta_{D,2}(z)$ and $\zeta_{D^2,2}(z)$, we consider the right sides of the following equalities,
\[
\begin{align*}
\text{Tr}(De^{-tD^2} - \sum_{i=1}^{\infty} D_0(i)e^{-tD_0^2(i)}) &= \sum_{\lambda_k} \lambda_k e^{-t\lambda_k} \\
&- \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} \lambda e^{-t\lambda^2} \text{tr}_s(C_\Gamma(\sigma_+, -i\lambda)C'_\Gamma(\sigma_+, i\lambda)) \, d\lambda, \\
\text{Tr}(e^{-tD^2} - \sum_{i=1}^{\infty} e^{-tD_0^2(i)}) - h &= \sum_{\lambda_k \neq 0} e^{-t\lambda_k^2} \\
&- \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} e^{-t\lambda^2} \text{tr}(C_\Gamma(\sigma_+, -i\lambda)C'_\Gamma(\sigma_+, i\lambda)) \, d\lambda.
\end{align*}
\]
The discrete eigenvalues give
\[
\sum_{\lambda_k} \lambda_k e^{-t\lambda_k} \sim e^{-ct}, \quad \sum_{\lambda_k \neq 0} e^{-t\lambda_k^2} \sim e^{-ct} \quad \text{as} \quad t \to \infty
\]
for a positive constant $c$. The operator $C_\Gamma(\sigma_+, i\lambda)$ is analytic along the imaginary axis and
\[
\begin{align*}
\text{tr}_s(C_\Gamma(\sigma_+, -i\lambda)C'_\Gamma(\sigma_+, i\lambda)) &= -\text{tr}_s(C_\Gamma(\sigma_+, i\lambda)C'_\Gamma(\sigma_+, -i\lambda)), \\
\text{tr}(C_\Gamma(\sigma_+, -i\lambda)C'_\Gamma(\sigma_+, i\lambda)) &= \text{tr}(C_\Gamma(\sigma_+, i\lambda)C'_\Gamma(\sigma_+, -i\lambda)),
\end{align*}
\]
hence we have the following analytic expansion at $\lambda = 0$:
\[
\begin{align*}
\text{tr}_s(C_\Gamma(\sigma_+, -i\lambda)C'_\Gamma(\sigma_+, i\lambda)) &= \sum_{k=0}^{\infty} f_{2k+1}\lambda^{2k+1}, \\
\text{tr}(C_\Gamma(\sigma_+, -i\lambda)C'_\Gamma(\sigma_+, i\lambda)) &= \sum_{k=0}^{\infty} g_{2k}\lambda^{2k}
\end{align*}
\]
for some constants $f_{2k+1}, g_{2k}$. Therefore we have
\[
\int_{-\infty}^{\infty} \lambda e^{-t\lambda^2} \text{tr}_s(C_\Gamma(\sigma_+, -i\lambda)C'_\Gamma(\sigma_+, i\lambda)) \, d\lambda \sim \sum_{k=0}^{\infty} \gamma_k t^{-(k+\frac{1}{2})} \quad \text{as} \quad t \to \infty,
\]
\[ (41) \quad \int_{-1}^{1} e^{-t\lambda^2} \text{tr}(C_T(\sigma_+, -i\lambda)C'_T(\sigma_+, i\lambda)) \, d\lambda \sim \sum_{k=0}^{\infty} \gamma_k' t^{-(k+\frac{1}{2})} \text{ as } t \to \infty. \]

The corresponding integrals over \((-\infty, 1]\lambda \cup [1, \infty)\lambda\) converge to 0 exponentially as \(t \to \infty\). Hence, the expansion \((40)\) shows that \(\eta_{D,2}(z)\) is well defined for \(\text{Re}(z) < 1\) and extends to the whole complex plane with the following pole structure:

\[ (42) \quad \Gamma((z + 1)/2)\eta_{D,2}(z) = \sum_{k=0}^{\infty} \frac{-2\gamma_k}{z - 2k - 2} + K_2(z) \]

for constants \(\gamma_k\) and a holomorphic function \(K_2(z)\). In the same way, the expansion \((41)\) implies that \(\zeta_{D,2}(z)\) is well defined for \(\text{Re}(z) < \frac{1}{2}\) and extends to the whole complex plane with poles determined by the equality

\[ (43) \quad \Gamma(z)\zeta_{D,2}(z) = \sum_{k=0}^{\infty} \frac{-\gamma_k'}{z - k - \frac{1}{2}} + H_2(z) \]

for the constants \(\gamma_k'\) and a holomorphic function \(H_2(z)\).

We define the eta and zeta functions by

\[ (44) \quad \eta_D(z) := \eta_{D,1}(z) + \eta_{D,2}(z), \quad \zeta_{D}(z) := \zeta_{D,1}(z) + \zeta_{D,2}(z). \]

Here now the right sides of these equalities are meromorphic functions over \(\mathbb{C}\) with the poles described in the above. The equalities \((39), (42), (43)\) give the following theorem

\textbf{Theorem 6.1.} \textit{The poles of the eta function \(\eta_D(z)\) and the zeta function \(\zeta_{D}(z)\) are determined by the equations}

\[ \Gamma((z + 1)/2)\eta_D(z) = \sum_{k=0}^{\infty} \frac{-2\gamma_k}{z - 2k - 2} + K(z), \]

\[ \Gamma(z)\zeta_D(z) = \sum_{k=-n}^{\infty} \frac{\beta_k}{z + k - \frac{1}{2}} + \frac{\beta_0'}{z - \frac{1}{2}} + \frac{h}{z^2} + \sum_{k=0}^{\infty} \frac{-\gamma_k'}{z - k - \frac{1}{2}} + H(z) \]

\textit{where} \(K(z)\) \textbf{and} \(H(z)\) \textit{are holomorphic. In particular,} \(\eta_D(z)\) \textbf{and} \(\zeta_{D}(z)\) \textit{are regular at} \(z = 0\).

\textbf{7. Eta invariants, Zeta functions of odd type and functional equations}

In this section, we study the eta invariant and its relation with the Selberg zeta function of odd type. We use the Selberg trace formula to prove a generalization of Millson’s theorem in \([12]\) for hyperbolic manifolds with cusps. Since the unipotent term in our situation vanishes (see Corollary \([13]\)), we obtain the same formula as in the case of Millson in \([12]\). We also derive the functional equation for the eta invariant and the Selberg zeta function of odd type.
By the analysis in Section 6, the eta function $\eta_{\mathcal{D}}(z)$ is regular at $z = 0$ and we can put $z = 0$ in the equality

$$\eta_{\mathcal{D}}(z) = \frac{1}{\Gamma\left(\frac{z+1}{2}\right)} \int_{0}^{\infty} t^{\frac{z+1}{2}} \text{Tr}(\mathcal{D}e^{-tD^2}) - \sum_{i=1}^{\kappa} \mathcal{D}_0(i)e^{-tD_0^2(i)} \, dt.$$ 

We define the eta invariant of $\mathcal{D}$ by

$$\eta(\mathcal{D}) := \eta_{\mathcal{D}}(0) = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} t^{-\frac{1}{2}} \text{Tr}(\mathcal{D}e^{-tD^2}) - \sum_{i=1}^{\kappa} \mathcal{D}_0(i)e^{-tD_0^2(i)} \, dt.$$ 

Let us recall that

$$\text{Tr}(\mathcal{D}e^{-tD^2}) = \frac{2\pi i}{(4\pi t)^{\frac{1}{2}}} \sum_{\gamma: \text{hyperbolic}} l(C_{\gamma}) j(\gamma)^{-1} D(\gamma)^{-1}(\chi_{\sigma_+}(m_{\gamma}) - \chi_{\sigma_-}(m_{\gamma})) e^{-\frac{l(C_{\gamma})^2}{4t}}.$$ 

Using the elementary equality $\int_{0}^{\infty} e^{-s^2t} e^{-r^2/4t} dt = \frac{e^{-sr}}{4\pi r}$, we have

$$\int_{0}^{\infty} e^{-s^2t} \text{Tr}(\mathcal{D}e^{-tD^2}) - \sum_{i=1}^{\kappa} \mathcal{D}_0(i)e^{-tD_0^2(i)} \, dt = \frac{i}{2} \sum_{\gamma: \text{hyperbolic}} l(C_{\gamma}) j(\gamma)^{-1} D(\gamma)^{-1}(\chi_{\sigma_+}(m_{\gamma}) - \chi_{\sigma_-}(m_{\gamma})) e^{-sl(C_{\gamma})}.$$ 

We define the Selberg zeta function of odd type by

$$(45) \quad Z^o_H(s) := \exp \left( - \sum_{\gamma: \text{hyperbolic}} j(\gamma)^{-1} |\text{det}(\text{Ad}(a_{\gamma}m_{\gamma})^{-1} - I|_n)|^{-1}(\chi_{\sigma_+}(m_{\gamma}) - \chi_{\sigma_-}(m_{\gamma})) e^{-sl(C_{\gamma})} \right)$$

for $\text{Re}(s) \gg 0$. In Proposition 7.2 we will show that $Z^o_H(s)$ has a meromorphic extension over $\mathbb{C}$ and $Z^o_H(s)$ is regular at $s = n$. Now we have

$$\int_{0}^{\infty} e^{-s^2t} \text{Tr}(\mathcal{D}e^{-tD^2}) - \sum_{i=1}^{\kappa} \mathcal{D}_0(i)e^{-tD_0^2(i)} \, dt = \frac{i}{2} \frac{d}{ds} \log Z^o_H(s + n).$$

Following the argument on p. 27 of [12], we use the equality

$$t^{\frac{z+1}{2}} = \frac{2}{\Gamma\left(\frac{z+1}{2}\right)} \int_{0}^{\infty} s^{-\frac{z+1}{2}} e^{-s^2t} \, ds.$$
for \( \text{Re}(z) < 1 \) to get

\[
\eta_D(z) = \frac{2}{\Gamma\left(\frac{z+1}{2}\right)\Gamma\left(\frac{1-z}{2}\right)} \int_0^\infty \int_0^\infty s^{-z} e^{-s^2t} \, ds \, \text{Tr}(De^{-tD^2}) - \sum_{i=1}^\kappa D_0(i)e^{-tD_0^2(i)} \, dt
\]

\[
= \frac{2}{\Gamma\left(\frac{z+1}{2}\right)\Gamma\left(\frac{1-z}{2}\right)} \int_0^\infty \int_0^\infty e^{-s^2t} \, \text{Tr}(De^{-tD^2}) - \sum_{i=1}^\kappa D_0(i)e^{-tD_0^2(i)} \, dt \, ds
\]

\[
= \frac{i}{\Gamma\left(\frac{z+1}{2}\right)\Gamma\left(\frac{1-z}{2}\right)} \int_0^\infty s^{-z} \frac{d}{ds} \log Z_H^0(s+n) \, ds.
\]

If we evaluate the above equality at \( z = 0 \), we get the following theorem.

**Theorem 7.1.** For the Dirac operator \( D \) over a \((2n+1)\)-dimensional hyperbolic manifold with cusps, the eta invariant \( \eta(D) \) and the Selberg zeta function of odd type \( Z_H^0(s) \) satisfy

\[
(46) \quad \eta(D) = \frac{1}{\pi i} \log Z_H^0(n).
\]

Now let us show that \( Z_H^0(s) \) has the meromorphic extension over \( \mathbb{C} \). We select a smooth odd function \( g(u) \) such that \( |g(u)| = 1 \) if \( |u| > c \), \( g(u) = 0 \) near 0 and \( \int_0^\infty g'(u)du = 1 \) where \( c = \min_{\{\gamma: \text{hyperbolic}\}} l(C_\gamma) \). We define

\[
\begin{cases}
H_s(\sigma_+, \lambda) := \int_{-\infty}^\infty g(u)e^{-su}e^{i\lambda u}du, \\
H_s(\sigma_-, \lambda) := -H_s(\sigma_+, \lambda)
\end{cases}
\]

for a complex parameter \( s \). By the Paley-Wiener theorem, (see theorem 2.2 in [12]), there exists \( f_s \) over \( G \) with \( \hat{f}_s(\sigma_+, i\lambda) = H_s(\sigma_+, \lambda), \hat{f}_s(\sigma, i\lambda) = 0 \) if \( \sigma \neq \sigma_\pm \). Applying the Selberg trace formula to the one parameter family of functions \( f_s \) on \( G \) for \( \text{Re}(s) \gg 0 \), we get

\[
\sum_{\lambda_j \in \sigma^+_\nu} H_s(\sigma_+, \lambda_j) + \sum_{\lambda_j \in \sigma^-_\nu} H_s(\sigma_-, \lambda_j)
\]

\[
- \frac{d(s\pm)}{4\pi} \int_{\sigma^+_\nu} H_s(\sigma_+, \lambda) \text{tr}_s(C_\Gamma(\sigma_+, -i\lambda)C_\Gamma'(\sigma_+, i\lambda)) \, d\lambda
\]

\[
= \sum_{\gamma: \text{hyperbolic}} l(C_\gamma) j(\gamma)^{-1} D(\gamma)^{-1} \left( \chi_{\sigma_+}(m_\gamma) - \chi_{\sigma_-}(m_\gamma) \right) e^{-sl(C_\gamma)}
\]

\[
= \frac{d}{ds} \log Z_H^0(s+n).
\]

Here we used the fact that the identity, unipotent orbital integrals vanish by the definition of \( H_s(\sigma_\pm, \lambda) \) and results in the previous sections. We shall use this equality to get the meromorphic extension of

\[
Z(s) := \frac{d}{ds} \log Z_H^0(s+n)
\]

over \( \mathbb{C} \) and to investigate its poles.
Discrete eigenvalue term: Integration by part gives
\[H_s(\sigma, \lambda) = \pm \frac{1}{s - i\lambda} \int_0^\infty g'(u)e^{-(s-i\lambda)u} du \mp \frac{1}{s + i\lambda} \int_0^\infty g'(u)e^{-(s+i\lambda)u} du.\]
This equality provides the meromorphic extension of \(H_s(\sigma, \lambda_j)\) over \(\mathbb{C}\) and we see that \(H_s(\sigma, \lambda_j)\) has simple poles at \(i\lambda_j, -i\lambda_j\) with residues \(\pm m_j\) for \(\lambda_j \in \sigma^+_R\) where \(m_j\) is the multiplicity of \(\lambda_j\).

Scattering term: We consider the scattering term
\[\Phi(z) := \text{tr}_s(C_T(\sigma, -z)) \frac{d}{dz} C_T(\sigma, z) = \frac{d}{dz} \left( \log \det C_+(z) - \log \det C_-(z) \right).\]
Using the equalities \(C_T(\sigma, z)C_T(\sigma, -z) = \text{Id}\), \(C_T(\sigma, z)^* = C_T(\sigma, \bar{z})\) and the first displayed formula on p. 518 and (6.8) of [10], we have
\[\det C_+(z) = \det C_+(0) p_+^z \prod_{\text{Re}(q) < 0} \frac{z + q_+}{z - q_+},\]
\[\det C_-(z) = \det C_-(0) p_-^z \prod_{\text{Re}(q) < 0} \frac{z + q_-}{z - q_-}\]
for some constants \(p_+, p_-\). Here the infinite products are taken over the poles \(\{q_\pm\}\) of \(\det C_+\) respectively. Note that \(\det C_\pm\) is holomorphic over the half plane with \(\text{Re}(z) \geq 0\). Hence \(\Phi(z)\) has the following form over \(\mathbb{C}\):
\[(49) \quad \Phi(z) = -\sum_{\text{Re}(q) > 0} \frac{2\text{Re}(q_+)}{(z - q_+)(z + q_+)} + \sum_{\text{Re}(q) < 0} \frac{2\text{Re}(q_-)}{(z - q_-)(z + q_-)} + \log p_+ - \log p_-.
\]
Now we consider the contour integral
\[\mathcal{L}_R := \frac{1}{4\pi i} \int_{L_R} H_s(\sigma, z) \text{tr}_s(C_T(\sigma, -iz)) C_T'(\sigma, iz) dz\]
where \(L_R = [-R, R] \cup \{Re^\theta \mid 0 \leq \theta \leq \pi\}\). As in proposition 3.10 of [8], we can apply the Cauchy integral formula and obtain
\[(50) \quad \lim_{R \to \infty} \mathcal{L}_R = i \left( \sum_{\text{Re}(q) < 0} \frac{1}{s - q_+} \int_0^\infty g'(u)e^{-(s-q_+)u} du - \sum_{\text{Re}(q) < 0} \frac{1}{s - q_-} \int_0^\infty g'(u)e^{-(s-q_-)u} du \right)\]
for \(\text{Re}(s) > 0\). Now the right side of (50) gives us the meromorphic extension over \(\mathbb{C}\). On the other hand, we can also show that
\[\lim_{R \to \infty} \mathcal{L}_R = \frac{1}{4\pi i} \int_{-\infty}^\infty H_s(\sigma, \lambda) \text{tr}_s(C_T(\sigma, -i\lambda)) C_T'(\sigma, i\lambda) d\lambda.\]
Proposition 7.2. The Selberg zeta function of odd type $Z_H^o(s + n)$ has a meromorphic extension over $\mathbb{C}$, and is regular at $s = 0$.

Remark 7.3. The zeros of $Z_H^o(s + n)$ are located at $\pm i\lambda_j$ for $\lambda_j \in \sigma_p^\pm$, at $q_+$ for $\text{Re}(q_+) < 0$ and their orders are $m_j$, $d(\sigma_+)b_+$. The poles of $Z_H^o(s + n)$ are located at $\mp i\lambda_j$ for $\lambda_j \in \sigma_p^\pm$, at $q_-$ for $\text{Re}(q_-) < 0$ and their orders are $m_j$, $d(\sigma_+)b_-$. 

Let us study the functional equation of $\eta(D)$ and $Z_H^o(s)$. We set 

$$R(s) := Z(s) - Z(-s) + d(\sigma_+)\Phi(s).$$

Then $R(s)$ is an odd entire function of $s$. Let $h(s)$ be an odd function which decreases sufficiently rapidly as $\text{Im}(s) \rightarrow \infty$ in the strip \(\{ s \in \mathbb{C} \mid |\text{Re}(s)| < n + \epsilon, \epsilon > 0 \}\) and consider the contour integral 

$$L_T := \frac{1}{2\pi i} \int_{L_T} h(s)Z(s) \, ds$$

where $L_T$ is the rectangle with the corners $a+iT, a-iT, -a+iT, -a-iT$ with $n < a < n+\epsilon$. Then we have 

$$\lim_{T \to \infty} L_T = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} h(s)(Z(s) - Z(-s)) \, ds + 2 \frac{1}{2\pi i} \int_{-a+i\infty}^{-a-i\infty} h(s)Z(s) \, ds.$$ 

We apply the Cauchy integral theorem to get the equality 

$$\lim_{T \to \infty} L_T = 2 \sum_{\lambda_j} m_j h(i\lambda_j) + \sum_{-a<\text{Re}(q_k)<0} d(\sigma_+)b_k h(q_k)$$

where we use the notations $q_k, b_k$ instead of $q_\pm, \pm b_\pm$. Because the simple poles of $Z(s)$ are located at the $q_k$'s with residues $d(\sigma_+)b_k$, between $\text{Re}(s) = -a$ and $\text{Re}(s) = 0$, we have 

$$\frac{1}{2\pi i} \int_{-a+i\infty}^{-a-i\infty} h(s)Z(s) \, ds = \frac{1}{2\pi i} \int_{i\infty}^{-i\infty} h(\lambda)Z(\lambda) \, d\lambda + \sum_{-a<\text{Re}(q_k)<0} d(\sigma_+)b_k h(q_k).$$

Here the integral over the semicircle of radius $R$ vanishes as $R \to \infty$ by the definition of $H_s(\sigma_+, z)$ and (39). Therefore the meromorphic extension of the scattering term has simple poles at $q_\pm$ for $\text{Re}(q_\pm) < 0$ with residues $\pm d(\sigma_+)b_\pm$ where $b_\pm$ denotes the order of the pole of $\det C_T(\sigma_+, z)$ at $q_\pm$.

Combining the contributions of the $H_s(\sigma_+, \lambda_j)$'s and the scattering term, we see that $Z(s)$ has a meromorphic extension over $\mathbb{C}$ and its simple poles are located at $\pm i\lambda_j$, $\mp i\lambda_j$ for $\lambda_j \in \sigma_p^\pm$, and at $q_\pm$ for $\text{Re}(q_\pm) < 0$ with residues $m_j$, $-m_j$ and $\pm d(\sigma_+)b_\pm$ respectively. In particular, we can see that $Z(s)$ is regular at $s = 0$. This implies the following proposition.
The simple poles at $-q_k$ of $Z(-s)$ between $\text{Re}(s) = a$ and $\text{Re}(s) = 0$ give rise to the equality

\[ \int_{a-i\infty}^{a+i\infty} h(s)(Z(s) - Z(-s))\,ds = \int_{-i\infty}^{i\infty} h(\lambda)(Z(\lambda) - Z(-\lambda))\,d\lambda + \sum_{0<\text{Re}(-q_k)<a} d(\sigma_+)b_k h(-q_k). \]

By (51), (52) and (53), we have

\[ 2 \sum_{\lambda_j} m_j h(i\lambda_j) = \frac{1}{2\pi i} \int_{i\infty}^{-i\infty} h(\lambda)(Z(\lambda) + \frac{d(\sigma_+)}{2}\Phi(\lambda))\,d\lambda \]

\[ + \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} h(\lambda)(Z(\lambda) - Z(-\lambda) + d(\sigma_+)(\Phi(\lambda))\,d\lambda. \]

If we change variables $\lambda \rightarrow i\lambda$, we see that the first term of the right side in (54) is equal to two times of

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} h(i\lambda) \sum_{C:\text{hyperbolic}} l(C') j(\gamma)^{-1} D(\gamma)^{-1} \left( \chi_{\sigma_+}(m_{\gamma}) - \chi_{\sigma_-}(m_{\gamma}) \right) e^{-i\lambda l(C')}\,d\lambda \]

\[ + \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} h(i\lambda)\Phi(i\lambda)\,d\lambda. \]

By (53), (55) and the Selberg trace formula applied to $h(i\lambda)$, we have

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} h(\lambda) R(\lambda)\,d\lambda = 0 \]

for any odd holomorphic $h(s)$ satisfying suitable growth conditions. Since $R(s)$ is an entire function, this implies that $R(s) = 0$ over $\mathbb{C}$. Hence we have the equality

\[ \frac{d}{ds} \log(Z_H^0(n+s)Z_H^0(n-s)) = -d(\sigma_+)\Phi(s) \]

for any $s \in \mathbb{C}$. Finally, recalling the equalities

\[ \Phi(s) = \frac{d}{ds} \left( \log \det C_+(s) - \log \det C_-(s) \right), \]

\[ \eta(D) = \frac{1}{\pi i} \log Z_H^o(n), \]

we get the following theorem.

**Theorem 7.4.** We have

\[ Z_H^o(n+s)Z_H^o(n-s) = \exp(2\pi i\eta(D)) \left( \frac{\det C_+(s)C_-(0)}{\det C_-(s)C_+(0)} \right)^{-d(\sigma_+)} \]

for $s \in \mathbb{C}$. 
Remark 7.5. In the above equality, $Z_H^H(n+s)Z_H^H(n-s)$ has zeros at $q_+, -q_+$ of orders $d(\sigma_+)b_+$ for $\text{Re}(q_+) < 0$ and poles at $q_-, -q_-$ of orders $d(\sigma_+)b_-$ for $\text{Re}(q_-) < 0$. These zeros and poles coincide with the zeros and poles of $(\frac{\det C_+(s)C_+(0)}{\det C_-(s)C_+(0)})^{-d(\sigma_+)}$.

8. Cusp contributions for regularized determinants and functional equations

In this section, we compute the unipotent factor in the relation between the regularized determinant and the Selberg zeta function of even type. We also derive the functional equation for the regularized determinant and the Selberg zeta function of even type where the unipotent factor plays a nontrivial role. This equation is the even type counterpart of the functional equation for the eta invariant and the Selberg zeta function of odd type proved in Section 7.

We define the zeta function with the factor $e^{-ts^2}$ for a positive real number $s$ by

$$\zeta_{D^2}(z,s) := \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \text{Tr}(e^{-tD^2} - \sum_{i=1}^\kappa e^{-tD_0^2(i)}) e^{-ts^2} dt.$$  

Note that $\zeta_{D^2}(0,0) = \zeta_{D^2}(z)$ if the kernel of $D^2$ is trivial. We proved that $\zeta_{D^2}(z)$ is regular at $z = 0$ in Section 6. Due to the factor $s^2$, the continuous spectrum of $D^2 + s^2$ has minimum $s^2$ and the large time part of the zeta function $\zeta_{D^2}(z,s)$ does not produce any poles. Therefore $\zeta_{D^2}(z,s)$ is regular at $z = 0$. We define the regularized determinant of $D^2 + s^2$ by

$$\text{Det}(D^2, s) := \exp(-\zeta'_{D^2}(0, s))$$

for a positive real number $s$. Now we observe that $\zeta_{D^2}(0, s) = 0$ by the results of Section 6. Then we have

$$\left. \frac{d}{dz} \right|_{z=0} \zeta_{D^2}(z,s) = \int_0^\infty t^{z-1} \text{Tr}(e^{-tD^2} - \sum_{i=1}^\kappa e^{-tD_0^2(i)}) e^{-ts^2} dt \bigg|_{z=0}.$$  

Recall that

$$\text{Tr}(e^{-tD^2} - \sum_{i=1}^\kappa e^{-tD_0^2(i)})$$

\begin{align*}
= \sum_{\lambda_k} e^{-t\lambda_k^2} & - \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} e^{-t\lambda^2} \text{tr}(C_+(\sigma_+, -i\lambda)C_+(\sigma_+, i\lambda)) d\lambda \\
= I_\Gamma(K_+^e) + H_\Gamma(K_+^e) + U_\Gamma(K_+^e),
\end{align*}
and put

\[ I(z, s) := \int_0^\infty t^{z-1} I_\Gamma(K_t^e) e^{-ts^2} \, dt, \]

\[ H(z, s) := \int_0^\infty t^{z-1} H_\Gamma(K_t^e) e^{-ts^2} \, dt, \]

\[ U(z, s) := \int_0^\infty t^{z-1} U_\Gamma(K_t^e) e^{-ts^2} \, dt. \]

Then we have

\[ \frac{1}{2s} \frac{d}{ds} \log \Det(D^2, s) = I(1, s) + H(1, s) + U(1, s). \]

Now we want to find \( Z_I(s), Z^e_H(s), Z_U(s) \) such that

\[ I(1, s) = \frac{1}{2s} \frac{d}{ds} \log Z_I(s), \quad H(1, s) = \frac{1}{2s} \frac{d}{ds} \log Z^e_H(s), \quad U(1, s) = \frac{1}{2s} \frac{d}{ds} \log Z_U(s). \]

First, we recall that the Selberg zeta function of even type,

\[ Z^e_H(s) := \exp(-\sum_{\sigma_{\pm}} \sum_{\gamma:\text{hyperbolic}} j(\gamma)^{-1} |\det(\Ad(a_{\gamma} m_{\gamma})^{-1} - I_n)|^{-1} \chi_{\sigma} m_{\gamma} e^{-sl(C_{\gamma})}) \]

is defined for \( \Re(s) \gg 0 \) and we will show that this has a meromorphic extension over \( \mathbb{C} \).

We use (9) and elementary equality

\[ \int_0^\infty \frac{1}{\sqrt{\pi t}} \exp\left(-\left(\frac{r^2}{4t} + ts^2\right)\right) dt = \frac{1}{2s} e^{-rs} \]

to see

\[ H(1, s) = \frac{1}{2s} \frac{d}{ds} \log Z^e_H(s + n). \]

It follows from Corollary 4.3 that \( U(z, s) \) is the Mellin transform with factor \( e^{-ts^2} \) of the following terms

\[ \frac{2}{2\pi} \int_{-\infty}^\infty e^{-t\lambda^2} \left(P_U(\lambda) - \kappa \frac{d(\sigma_{\pm})}{2} \left(\psi(i\lambda + \frac{1}{2}) + \psi(-i\lambda + \frac{1}{2})\right)\right) d\lambda \]

where \( P_U(\lambda) \) is an even polynomial of degree \((2n - 4)\). We deal with the term \( \psi(\pm i\lambda + \frac{1}{2}) \)

using the Cauchy integral formula

\[ \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty e^{-t(\lambda^2 + s^2)} \psi(\pm i\lambda + \frac{1}{2}) \, d\lambda \, dt \]

\[ = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{1}{\lambda^2 + s^2} \psi(\pm i\lambda + \frac{1}{2}) \, d\lambda = \frac{1}{2s} \psi(s + \frac{1}{2}). \]

For the part \( P_U(\lambda) \) in (61), we have

\[ \frac{1}{2s} \frac{d}{ds} \log \exp\left(\int_0^s P_U(i\lambda) \, d\lambda\right) = \frac{1}{2\pi} \int_0^\infty e^{-ts^2} \int_{-\infty}^\infty e^{-t\lambda^2} P_U(\lambda) \, d\lambda \, dt \]

by Lemma 3 in [7]. We define

\[ Z_U(s) := \Gamma(s + \frac{1}{2})^{-2\kappa d(\sigma_{\pm})} \exp\left(2 \int_0^s P_U(i\lambda) \, d\lambda\right). \]
Then this satisfies

\begin{equation}
U(1, s) = \frac{1}{2s} \frac{d}{ds} \log Z_U(s).
\end{equation}

For \( I(z, s) \), we can treat this as for \( P_U(\lambda) \) since \( p(\sigma_+, \lambda) = p(\sigma_-, \lambda) \) is an even polynomial of \( \lambda \). Hence, we can see

\begin{equation}
Z_I(s) := \exp(2 \int_0^s P_I(i\lambda) \, d\lambda)
\end{equation}

where \( P_I(\lambda) = 2\pi \text{Vol}(\Gamma \setminus G) p(\sigma_+, \lambda) \). By (57), (59), (61) and (62), we have the following theorem

**Theorem 8.1.** The following equality holds for any \( s \in \mathbb{C} \),

\begin{equation}
\text{Det}(\mathcal{D}^2, s) = C Z_H^e(s+n) \Gamma(s + \frac{1}{2})^{-2\kappa d(\sigma_+)} \exp(2 \int_0^s P_I(i\lambda) + P_U(i\lambda) \, d\lambda)
\end{equation}

where \( C \) is a constant and \( P_I(\lambda) = 2\pi \text{Vol}(\Gamma \setminus G) p(\sigma_+, \lambda) \).

**Proof.** A priori, the equality \((63)\) holds for a real number \( s \gg 0 \) since \( Z_H^e(s+n) \) is defined only for \( \text{Re}(s) \gg 0 \). But the right side of \((63)\) gives the meromorphic extension over \( \mathbb{C} \) by Proposition 8.3 and this also gives the meromorphic extension of the left side of \((63)\) over \( \mathbb{C} \).

**Remark 8.2.** Let us remark that \( \text{Det}(\mathcal{D}^2, s) \neq \text{Det}(\mathcal{D}^2, -s) \) as a meromorphic function over \( \mathbb{C} \). This is because the equality \((63)\) holds for a real number \( s \gg 0 \) a priori and the meromorphic extension of \( \text{Det}(\mathcal{D}^2, s) \) is given by the right side of \((63)\), which does not satisfy this property.

Now let us show that \( Z_H^e(s) \) has the meromorphic extension over \( \mathbb{C} \). We consider an even smooth function \( g(|u|) \) where \( g(u) \) is the given function in Section 7. We set

\[ H_s(\sigma_\pm, \lambda) = \int_{-\infty}^{\infty} g(|u|) e^{-s|u|} e^{i\lambda u} \, du \]

for a complex parameter \( s \). Then integration by parts gives

\begin{equation}
H_s(\sigma_\pm, \lambda) = \frac{1}{s - i\lambda} \int_0^\infty g'(u) e^{-(s-i\lambda)u} \, du + \frac{1}{s + i\lambda} \int_0^\infty g'(u) e^{-(s+i\lambda)u} \, du.
\end{equation}
Let us apply the Selberg trace formula to the one parameter family of functions \( f_s \) on \( G \) with \( \text{Re}(s) > 0 \), such that \( \hat{f}_s(\sigma, i\lambda) = H_s(\sigma, \lambda), \hat{f}_s(\sigma, i\lambda) = 0 \) if \( \sigma \neq \sigma \pm \). Then we get

\[
\sum_{\lambda_j \in \sigma_p^+} H_s(\sigma_+, \lambda_j) + \sum_{\lambda_j \in \sigma_p^-} H_s(\sigma_-, \lambda_j)
- \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} H_s(\sigma_+, \lambda) \tr(C_\Gamma(\sigma_+, -i\lambda)C_\Gamma^*(\sigma_+, i\lambda)) \, d\lambda
= \sum_{\gamma: \text{hyperbolic}} l(C_\gamma)j(\gamma)^{-1}D(\gamma)^{-1}\left(\chi_{\sigma_+}(m_\gamma) + \chi_{\sigma_-}(m_\gamma)\right)e^{-s\text{Im}(C_\gamma)}
- \frac{\kappa d(\sigma_+)}{2\pi} \int_{-\infty}^{\infty} H_s(\sigma_+, \lambda)\psi(i\lambda + \frac{1}{2}) + \psi(-i\lambda + \frac{1}{2}) \, d\lambda
+ \frac{2}{2\pi} \int_{-\infty}^{\infty} H_s(\sigma_+, \lambda)(P_I(\lambda) + P_U(\lambda)) \, d\lambda.
\]

Let us recall the equality

\[
\frac{d}{ds} \log Z_H^c(s + n) = \sum_{\gamma: \text{hyperbolic}} l(C_\gamma)j(\gamma)^{-1}D(\gamma)^{-1}\left(\chi_{\sigma_+}(m_\gamma) + \chi_{\sigma_-}(m_\gamma)\right)e^{-s\text{Im}(C_\gamma)},
\]

and now we investigate the other terms in (65) to get the meromorphic extension of \( \frac{d}{ds} \log Z_H^c(s + n) \) over \( \mathbb{C} \) and to determine its poles.

**Discrete eigenvalue term:** Using (64) as before, we can see that \( H_s(\sigma, \lambda_j) \) has a meromorphic extension over \( \mathbb{C} \) and has the simple poles at \( i\lambda_j \) and \(-i\lambda_j \) for \( \lambda_j \in \sigma_p^\pm \) with the residue \( m_j \) where \( m_j \) is the multiplicity of \( \lambda_j \).

**Scattering term:** Now we consider the scattering term

\[
- \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} H_s(\sigma_+, \lambda) \tr(C_\Gamma(\sigma_+, -i\lambda)C_\Gamma^*(\sigma_+, i\lambda)) \, d\lambda.
\]

As in the previous case, we can show that the function

\[
\Psi(z) := \tr(C_\Gamma(\sigma_+, -z)\frac{d}{dz}C_\Gamma(\sigma_+, z))
\]

has the following form over \( \mathbb{C} \):

\[
\Psi(z) = \sum_{\text{Re}(q_k) < 0} -\frac{2\text{Re}(q_k)}{(z - q_k)(z + \overline{q_k})} + \log p
\]

for some constant \( p \). Here the sum is taken over the set of poles of \( \text{det} C_\Gamma(\sigma_+, z) \). We repeat the method in Section 7 to prove that the scattering term has a meromorphic extension over \( \mathbb{C} \) and has poles at \( q_k \) for \( \text{Re}(q_k) < 0 \) with residues \( d(\sigma_+)b_k \). As before \( b_k \) denotes the order of the pole of \( \text{det} C_\Gamma(\sigma_+, z) \) at \( q_k \).

**Identity and Unipotent term:** It follows from proposition 3.9 in [8] that

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} H_s(\sigma_+, \lambda)(P_I(\lambda) + P_U(\lambda)) \, d\lambda = 0.
\]
We now turn our attention to the other unipotent terms. As in proposition 3.7 of [8], we use the Cauchy integral formula to get
\[
\frac{\kappa d(\sigma_+)}{2\pi} \int_{-\infty}^{\infty} H_s(\sigma_+, \lambda)(\psi(i\lambda + \frac{1}{2}) + \psi(-i\lambda + \frac{1}{2})) \, d\lambda = 2\kappa d(\sigma_+)\psi(s + \frac{1}{2})
\]
for Re(s) ≫ 0 and the right side gives us the meromorphic extension over \( \mathbb{C} \) of the left side.

Considering the equality (65) and the above analysis of other terms, we conclude

**Proposition 8.3.** The Selberg zeta function of even type \( Z^e_H(s) \) has a meromorphic extension over \( \mathbb{C} \).

**Remark 8.4.** The zeros of \( Z^e_H(s + n)\Gamma(s + \frac{1}{2})^{-2\kappa d(\sigma_+)} \) are located at \( i\lambda_j, -i\lambda_j \) for \( \lambda_j \in \sigma_p^\pm \), and at \( q_k \) for Re(\( q_k \)) < 0 and their orders are \( m_j, d(\sigma_+)b_k \).

We now prove the functional equation for Det\( (D^2, s) \) and \( Z^e_H(s) \). First let us define
\[
Z(s) := \frac{d}{ds} \log Z^e_H(s + n) - 2\kappa d(\sigma_+)\psi(s + \frac{1}{2})
\]
and we can see that
\[
R(s) := Z(s) + Z(-s) + d(\sigma_+)\Psi(s)
\]
is an even entire function of \( s \). Now let \( h(s) \) be an even function which decreases sufficiently rapidly as Im(\( s \)) → ∞ in the strip \( \{ s \in \mathbb{C} \mid |\text{Re}(s)| < n + \epsilon, \epsilon > 0 \} \). We follow Section 7 and consider the contour integral
\[
L_T := \frac{1}{2\pi i} \int_{L_T} h(s)Z(s) \, ds
\]
where \( L_T \) is the rectangle with the corners \( a+iT, a-iT, -a+iT, -a-iT \) with \( n < a < n+\epsilon \). Then we have
\[
\lim_{T \to \infty} L_T = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} h(s)(Z(s) + Z(-s)) \, ds + \frac{2}{2\pi i} \int_{-a+i\infty}^{-a-i\infty} h(s)Z(s) \, ds.
\]
The Cauchy integral theorem gives
\[
\lim_{T \to \infty} L_T = 2 \sum_{\lambda_j} m_j h(i\lambda_j) + \sum_{-a<\text{Re}(q_k)<0} d(\sigma_+)b_k h(q_k).
\]
The simple poles at \( q_k \) of \( Z(s) \) in the strip between Re(\( s \)) = \(-a \) and Re(\( s \)) = 0 have residues \( d(\sigma_+)b_k \) and give the equality
\[
\frac{1}{2\pi i} \int_{-a+i\infty}^{-a-i\infty} h(s)Z(s) \, ds = \frac{1}{2\pi i} \int_{i\infty}^{-i\infty} h(\lambda)Z(\lambda) \, d\lambda + \sum_{-a<\text{Re}(q_k)<0} d(\sigma_+)b_k h(q_k).
\]
Similarly, the simple poles at \(-q_k\) of \(Z(-s)\) in a strip between \(\text{Re}(s) = a\) and \(\text{Re}(s) = 0\) have residues \(-d(\sigma_+)b_k\) and provide us with

\[
\begin{align*}
(68) \quad & \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} h(s) (Z(s) + Z(-s)) \, ds \\
= & \frac{1}{2\pi i} \int_{-\infty}^{\infty} h(\lambda) (Z(\lambda) + Z(-\lambda)) \, d\lambda + \sum_{0<\text{Re}(q_k)<a} -d(\sigma_+)b_k h(-q_k).
\end{align*}
\]

We use (66), (67) and (68) to obtain

\[
(69) \quad 2 \sum_{\lambda_j} m_j h(i\lambda_j) = 2 \frac{1}{2\pi i} \int_{-\infty}^{\infty} h(\lambda) (Z(\lambda) + \frac{d(\sigma_+)}{2} \Psi(\lambda)) \, d\lambda
\]

\[
+ \frac{1}{2\pi i} \int_{-\infty}^{i\infty} h(\lambda) (Z(\lambda) + Z(-\lambda) + d(\sigma_+) \Psi(\lambda)) \, d\lambda.
\]

Replacing \(\lambda\) by \(i\lambda\) in the first term of the right side of the equality (69), we see that this term is equal to

\[
(70) \quad 2 \sum_{\gamma \text{ hyperbolic}} l(C_\gamma) j(\gamma)^{-1} D(\gamma)^{-1} \left( \frac{m_\gamma}{\chi_{\sigma_+}(m_\gamma)} + \frac{m_\gamma}{\chi_{\sigma_-}(m_\gamma)} \right) \frac{1}{2\pi} \int_{-\infty}^{\infty} h(i\lambda) e^{-i\lambda(C_\gamma)} \, d\lambda
\]

\[
- 2 \frac{1}{2\pi} \int_{-\infty}^{\infty} h(i\lambda) 2kd(\sigma_+) \psi(i\lambda + \frac{1}{2}) \, d\lambda + \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{i\infty} h(i\lambda) \Psi(i\lambda) \, d\lambda.
\]

By (69), (70) and the Selberg trace formula applied to \(h(i\lambda)\), we have

\[
\frac{1}{2\pi i} \int_{-\infty}^{i\infty} h(\lambda) (R(\lambda) + 4P_I(-i\lambda) + 4P_U(-i\lambda)) \, d\lambda = 0
\]

for any even holomorphic \(h(s)\) satisfying suitable growth conditions. Since

\[
R(s) + 4P_I(-is) + 4P_U(-is) = R(s) + 4P_I(is) + 4P_U(is)
\]

is an entire function, we have

\[
Z(s) + Z(-s) + 4P_I(is) + 4P_U(is) = -d(\sigma_+) \Psi(s)
\]

over \(\mathbb{C}\). Now we get the equality

\[
\frac{d}{ds} \log \left( Z_H(n + s) \Gamma(s + \frac{1}{2})^{-2kd(\sigma_+)} \right) + 4P_I(is) + 4P_U(is)
\]

\[
= \frac{d}{ds} \log \left( Z_H(n - s) \Gamma(-s + \frac{1}{2})^{-2kd(\sigma_+)} \right) - d(\sigma_+) \frac{d}{ds} \log(\det C_\Gamma(\sigma_+, s)),
\]

which leads to the formula

\[
(71) \quad Z_H(n + s) \Gamma(s + \frac{1}{2})^{-2kd(\sigma_+)} \exp(4 \int_0^s P_I(i\lambda) + P_U(i\lambda) \, d\lambda)
\]

\[
= (\det C_\Gamma(\sigma_+, s))^{-d(\sigma_+)} Z_H^n(n - s) \Gamma(-s + \frac{1}{2})^{-2kd(\sigma_+)}. \]
This equality holds a priori up to constant $c$. As in the proof of lemma 4.3 in [8], we multiply $s^{-2m_0}$ to both sides of (71) where $m_0$ is the multiplicity of the zero eigenvalue of $D$. Doing this removes the spectral zero of $Z_H^e(n \pm s)$ at $s = 0$. If we compare the remaining parts at $s = 0$, we can see that the constant $c$ equals 1. From (63), we have

$$\det(D^2, s)^2 = C^2 Z_H^e(n + s)^2 \Gamma(s + \frac{1}{2})^{-4\kappa d(\sigma_+)} \exp(4 \int_0^s P_I(i\lambda) + P_U(i\lambda) \ d\lambda).$$

Finally, we combine (71) and this equality to get

**Theorem 8.5.** For any $s \in \mathbb{C}$, we have

(72) $\det(D^2, s)^2$

$$= C^2 (\det C_\Gamma(\sigma_+, s))^{-d(\sigma_+)} Z_H^e(n + s)Z_H^e(n - s) \left( \Gamma(s + \frac{1}{2}) \Gamma(-s + \frac{1}{2}) \right)^{-2\kappa d(\sigma_+)}.$$

**Remark 8.6.** The right side of (72) has the zeros at $i\lambda_j, -i\lambda_j$ of order $2m_j$ for $\lambda_j \in \sigma^\pm_+$, and at $q_k$ of order $2d(\sigma_+)b_k$ for $\text{Re}(q_k) < 0$. These are the zeros of $\det(D^2, s)^2$ as we expected.

**References**

[1] Dan Barbasch and Henri Moscovici, *L²-index and the Selberg trace formula*, J. Funct. Anal., **53**, (1983) 151-201.
[2] Christian Bär, *The Dirac operator on Hyperbolic manifolds of Finite volume*, J. Diff. Geom., **54**, (2000) 439-488.
[3] David Lee DeGeorge, *On a Theorem of Osborne and Warner. Multiplicities in the Cuspidal Spectrum*, J. Funct. Anal., **48**, (1982) 81-94.
[4] Isaac Efrat, *Determinants of Laplacians on surfaces of finite volume*, Comm. Math. Phys., **119**, (1988) 443-451.
[5] Isaac Efrat, *Erratum: “Determinants of Laplacians on surfaces of finite volume”*, Comm. Math. Phys., **138**, (1991) 607.
[6] Masaki Koyama, Shin Koizumi and Masaoichi Mamiuda, *The expression of the Harish-Chandra C-functions of semisimple Lie groups Spin(n,1), SU(n,1)*, J. Math. Soc. Japan, **51**, (1999) 955-985.
[7] David Fried, *Analytic torsion and closed geodesics on hyperbolic manifolds*, Invent. Math., **84**, (1986) 523-540.
[8] Ramesh Gangolli and Garth Warner, *Zeta functions of Selberg’s type for some noncompact quotients of symmetric spaces of rank one*, Nagoya Math. J., **78**, (1980) 1-44.
[9] Werner Hoffmann, *The Fourier transforms of weighted orbital integrals on semisimple groups of real rank one*, J. Reiner Angew. Math., **489**, (1997) 53-97.
[10] Shin-ya Koyama, *Determinant expression of Selberg zeta functions.I*, Trans. Amer. math. Soc., **324**, (1991) 149-168.
[11] Shin-ya Koyama, *Determinant expression of Selberg zeta functions.II*, Trans. Amer. math. Soc., **329**, (1992) 755-772.
[12] John Millson, *Closed geodesic and the η-invariant*, Ann. of Math., **108**, (1978) 1-39.
[13] Henri Moscovici and Robert Stanton, *Eta invariants of Dirac operators on locally symmetric manifolds*, Invent. Math., **95**, (1989) 629-666.
[14] Henri Moscovici and Robert Stanton, *R-torsion and zeta functions for locally symmetric manifolds*, Invent. Math., **105**, (1991) 185-216.
[15] Werner Müller, *Spectral theory for Riemannian manifolds with cusps and a related trace formula*, Math. Nachr., **111**, (1983) 197-288.
[16] Werner Müller, The trace class conjecture in the theory of automorphic forms, Ann. of Math., 130, (1989) 473-529.

[17] Werner Müller, Spectral geometry and scattering theory for certain complete surfaces of finite volume, Invent. Math., 109, (1992) 265-305.

[18] Werner Müller, Eta invariants and Manifolds with boundary, J. Diff. Geom., 40, (1994) 311-377.

[19] Werner Müller, Relative zeta functions, relative determinants and scattering theory, Comm. Math. Phys., 192, (1998) 309-347.

[20] M. Scott Osborne and Garth Warner, Multiplicities of the Integrable Discrete Series: The case of a Nonuniform Lattice in an R-rank One Semisimple Group, J. Funct. Anal., 30, (1978) 287-320.

[21] Peter Sarnak, Determinants of Laplacians, Comm. Math. Phys., 110, (1987) 113-120.

[22] Peter Sarnak and Masato Wakayama, Equidistribution of Holonomy about Closed Geodesics, Duke Math. J., 100, (1999) 1-57.

[23] Garth Warner, Selberg’s trace formula for nonuniform lattices: The R-rank one case, Adv. Math. Suppl. Stud., 6, (1979) 1-142.

Mathematisches Institut, Universität Bonn, Beringstrasse 1, D-53115 Bonn, Germany
E-mail address: jpark@math.uni-bonn.de