A UNIFYING PERRON-FROBENIUS THEOREM FOR NONNEGATIVE TENSORS VIA MULTI-HOMOGENEOUS MAPS

ANTOINE GAUTIER∗, FRANCESCO TUDISCO†, AND MATTHIAS HEIN†

Abstract. Inspired by the definition of symmetric decomposition, we introduce the concept of shape partition of a tensor and formulate a general tensor spectral problem that includes all the relevant spectral problems as special cases. We formulate irreducibility and symmetry properties of a nonnegative tensor \( T \) in terms of the associated shape partition. We recast the spectral problem for \( T \) as a fixed point problem on a suitable product of projective spaces. This allows us to use the theory of multi-homogeneous order-preserving maps to derive a general and unifying Perron-Frobenius theorem for nonnegative tensors that either implies previous results of this kind or improves them by weakening the assumptions there considered. We introduce a general power method for the computation of the dominant tensor eigenpair, and provide a detailed convergence analysis.

Key words. Perron-Frobenius theorem, nonnegative tensor, tensor power method, tensor eigenvalue, tensor singular value, tensor norm

AMS subject classifications. 47H07, 47J10, 15B48, 47H09, 47H10

1. Introduction. Tensor eigenvalue problems have gained considerable attention in recent years as they arise in a number of relevant applications, such as best rank-one approximation in data analysis [6, 18], higher-order Markov chains [17], solid mechanics and the entanglement problem in quantum physics [5, 16], multi-layer network analysis [20], and many other. A number of contributions have addressed relevant issues both form the theoretical and numerical point of view. The multi-dimensional nature of tensors naturally gives rise to a variety of eigenvalue problems. In fact, the classical eigenvalue and singular value problems for a matrix can be generalized to the tensor setting following different constructions which lead to different notions of eigenvalues and singular values for tensors, all of them reducing to the standard matrix case when the tensor is assumed to be of order two. Moreover, the extension of the power method to the tensor setting, including certain shifted variants, is the best known method for the computation of tensor eigenpairs.

When the tensor has nonnegative entries, many authors have worked on tensor generalizations of the Perron-Frobenius theorem for matrices [2, 3, 8, 15, 17]. In this setting, existence, uniqueness and maximality of positive eigenpairs of the tensor are discussed, in terms of certain irreducibility assumptions. Moreover, as for the matrix case, Perron-Frobenius type results allow to address the global convergence of the power method for tensors with nonnegative entries [2, 5, 9, 17].

However, all the contributions that have appeared so far address particular cases of tensor spectral problems individually. In this work we formulate a general tensor spectral problem which includes the known formulations as special cases. Moreover, we prove a new Perron-Frobenius theorem for the general tensor eigenvalue problem which allows to retrieve the previous results as particular cases and, often, allows to significantly weaken the assumptions previously made. In addition, we prove the global convergence of a nonlinear version of the power method that allows to compute the dominant eigenpair for general tensor eigenvalues, under mild assumptions on the tensor and with an explicit upper-bound on the convergence rate.

∗Department of Mathematics and Computer Science, Saarland University, 66041 Saarbrücken, Germany (ag@cs.uni-saarland.de, hein@math.uni-sb.de).
†Department of Mathematics and Statistics, University of Strathclyde, G11XH Glasgow, UK (f.tudisco@strath.ac.uk).
For the sake of clearness, we first discuss the case of a square tensor of order three, \( T = (T_{i,j,k}) \in \mathbb{R}^{N \times N \times N} \). Let \( f_T: \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) denote the multilinear form induced by \( T \),

\[
f_T(x, y, z) = \sum_{i,j,k=1}^{N} T_{i,j,k} x_i y_j z_k \quad \forall x, y, z \in \mathbb{R}^n,
\]

and, for \( p, q, r \in (1, \infty) \), consider the following nonlinear Rayleigh quotients:

\[
\begin{align*}
\Phi^1(x) &= \frac{f_T(x, x, x)}{\|x\|_p^3}, \\
\Phi^2(x, y) &= \frac{f_T(x, y, y)}{\|x\|_p \|y\|_q^2}, \\
\Phi^3(x, y, z) &= \frac{f_T(x, y, z)}{\|x\|_p \|y\|_q \|z\|_r}.
\end{align*}
\]

Note that, since the tensor is nonnegative and has odd order, the maximum of \( \Phi^i \) over its domain provides a notion of norm of \( T \), for \( i = 1, 2, 3 \). Furthermore note that \( \Phi^1, \Phi^2 \) and \( \Phi^3 \) lead naturally to the definition of \( \ell^p \)-eigenvectors, \( \ell^{p,q} \)-singular vectors and \( \ell^{p,q,r} \)-singular vectors of the tensor \( T \) [15]. Indeed the latter are respectively defined as the solutions of the following spectral equations

\[
\begin{align*}
\mathcal{T}_1(x, x, x) &= \lambda \psi_p(x), \\
\mathcal{T}_2(x, y, y) &= \lambda \psi_q(y), \\
\mathcal{T}_3(x, y, z) &= \lambda \psi_r(z),
\end{align*}
\]

where \( \psi_p(x) = \frac{1}{p} \nabla \|x\|_p^p = (|x_1|^{p-2} x_1, \ldots, |x_N|^{p-2} x_N) \) and, for \( i = 1, 2, 3 \), the mapping \( \mathcal{T}_i(x, y, z) \) is the gradient of \( x_i \mapsto f_T(x_1, x_2, x_3) \).

It is well known that the singular values of a matrix always admit a variational characterization, whereas the same holds true for eigenvalues only if the matrix is symmetric. A similar situation occurs for tensors, where suitable symmetry assumptions on \( T \) are required in order to relate the critical points of the Rayleigh quotients in (1) with the solutions of the spectral equations in (2): If \( T \) is super symmetric, i.e. the entries of \( T \) are invariant under any permutation of its indices, then \( \nabla f_T(x, x, x) = 3 \mathcal{T}_1(x, x, x) \) and so the correspondence between the critical points of \( \Phi^1 \) and the solutions to \( \mathcal{T}_1(x, x, x) = \lambda \psi_p(x) \) is clear. If \( T \) is partially symmetric with respect to its second and third indices, i.e. \( T_{i,j,k} = T_{i,k,j} \) for every \( i,j,k \in [N] = \{1, \ldots, N\} \), then \( \nabla_x f_T(x, y, y) = \mathcal{T}_2(x, y, y) \) and \( \nabla_y f_T(x, y, y) = 2 \mathcal{T}_2(x, y, y) \) and, again, it can be verified that the critical points of \( \Phi^2 \) coincide with the solutions to the second system in (2). Finally, the third system in (2) always characterizes the critical points of \( \Phi^3 \) as \( \nabla f_T = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \). This latter case is the analogue of the singular value problem for matrices. In the case where \( T \) does not have such symmetries, then it can be shown that the critical points of \( \Phi^1 \) and \( \Phi^2 \) are solutions to spectral systems analogous to those in (2) but where the mapping \( \mathcal{T}_i \) is the gradient of \( x_i \mapsto f_S(x_1, x_2, x_3) \) and \( S \in \mathbb{R}^{N \times N \times N} \) is a symmetrized version of \( T \) whose construction depends on the considered problem. Note that this phenomenon is, again, aligned with the matrix case. In fact, the quadratic form associated to a matrix \( M \) always coincides with the form associated with the symmetric matrix \( (M^T + M)/2 \). We discuss this property in detail in Section 4.

Now, if \( T \) has nonnegative entries, i.e. \( T_{i,j,k} \geq 0 \) for all \( i,j,k \), then a simple argument shows that \( |\Phi^1(x)| \leq \Phi^1(|x|) \) for every \( x \in \mathbb{R}^N \setminus \{0\} \) and where the absolute value is taken component wise. In particular, this implies that the maximum of \( \Phi^1 \) is attained in the nonnegative orthant \( \mathbb{R}^N_+ = \{ x \in \mathbb{R}^N \mid x_i \geq 0, \forall i \in [N] \} \). Similar arguments show that the maxima of \( \Phi^2 \) and \( \Phi^3 \) are attained on nonnegative vectors.
as well. There is a vast literature on the study of the solutions to the systems in (2) in the particular setting where $T$ is nonnegative. We refer to it as the Perron-Frobenius theory for nonnegative tensors [4]. Let us briefly recall typical results of the latter theory. To this end, in this paragraph, we abuse the nomenclature and refer to a solution of one of the systems in (2) as an eigenpair of $T$. These eigenpairs are of the form $(\lambda, w)$ where $\lambda \in \mathbb{R}$ and $w$ belongs to $\mathbb{R}^N$, $\mathbb{R}^N \times \mathbb{R}^N$ or $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ depending on which problem is considered. First, it can be shown that when $p, q, r$ are large enough, then there is always a nonnegative maximal eigenpair $(\lambda^*, w^*)$, i.e. $\lambda^* \geq 0$ is an eigenvalue of largest magnitude and $w^*$ has nonnegative components. Note that this is consistent with the fact that the maximum of the Rayleigh quotient is attained at nonnegative vectors, as observed above. The maximal eigenvalue $\lambda^*$ has min-max and max-min characterizations which are usually referred to as Collatz-Wielandt formulas [3, 8, 9]. Furthermore, under additional irreducibility assumptions on $T$, known as weak irreducibility, it can be shown that $w^*$ has strictly positive components and that it is the unique eigenvector with this property. Moreover, assuming further irreducibility conditions, known as strong irreducibility, it can be shown that $T$ has a unique nonnegative eigenvector. Finally, it is possible to derive conditions under which ad-hoc versions of the power method converge to $w^*$. This computational aspect is particularly interesting as it allows to estimate accurately the maximum of the Rayleigh quotient with a simple and efficient iterative method. Convergence rates for such generalizations of the power method have been derived for instance in [8, 9, 13, 23] under quite restrictive assumptions on $p, q, r$ and the irreducibility of $T$.

In this paper, we address tensors of any order and propose a framework that allows us to unify the study of all spectral equations of the type shown in (2) and to prove a general Perron-Frobenius theorem which either improves the known results mentioned above or includes them as special cases. In particular, we give new conditions for the existence, uniqueness and maximality of positive eigenpairs for an ample class of tensor spectral equations, we prove new characterizations for the maximal eigenvalue and we discuss the convergence of the power method including explicit rates of convergence. This is done by introducing a parametrization, which we call shape partition, so that the three problems discussed in (2) can be recovered with a suitable choice of the partition. Moreover, shape partitions allow us to introduce general definitions of weak and strong irreducibility, which both reduce to existing counterparts for suitable choices of the partition. We discuss in detail the relationship between different types of irreducible nonnegative tensors and we show how they are related for different spectral equations.

A particular contribution of this paper is that we reformulate these tensor spectral problems in terms of suitable multi-homogeneous maps and the associated fixed points on a product of projective spaces. Thus, based on our results in [10], we show that most of the tensor spectral problems correspond to a multi-homogeneous mapping that is contractive with respect to a suitably defined projective metric. This relatively simple observation turns out to be very relevant as it allows to systematically weaken the assumptions made in the Perron-Frobenius literature for nonnegative tensors so far. The paper is written in a self-contained manner. However, for the proofs we rely heavily on our results from [10].

2. Preliminaries. In this section we fix the main notation and definitions that are required to formulate the Rayleigh quotients in (1) and the associated spectral problems in a unified fashion for the general case of a tensor of any order and with possibly different dimensions.
Let $T \in \mathbb{R}^{N_1 \times \cdots \times N_m}$ be a nonnegative tensor of order $m$, and define the induced multilinear form $f_T: \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_d} \to \mathbb{R}$ as

$$f_T(z_1, \ldots, z_m) = \sum_{j_i \in [N_1], \ldots, j_m \in [N_m]} T_{j_1, \ldots, j_m} z_1^{j_1} z_2^{j_2} \cdots z_m^{j_m},$$

where $[N_i] = \{1, \ldots, N_i\}$ for all $i$. Furthermore, let us consider the gradient of $f_T$, that is let $\nabla f_T = (\nabla T_1, \ldots, \nabla T_m)$ with $\nabla T_i = (\nabla T_{i,1}, \ldots, \nabla T_{i,N_i})$ and $\nabla T_{i,j_i}: \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_d} \to \mathbb{R}$ defined as

$$\nabla T_{i,j_i}(z_1, \ldots, z_m) = \sum_{j_1 \in [N_1], \ldots, j_{i-1} \in [N_{i-1}], j_{i+1} \in [N_{i+1}], \ldots, j_m \in [N_m]} T_{j_1, \ldots, j_m} z_1^{j_1} \cdots z_{i-1}^{j_{i-1}} z_{i+1}^{j_{i+1}} \cdots z_m^{j_m}.$$

As for the case of a square tensor of order three, described in the previous section, several Rayleigh quotients and spectral equations can be associated to $T$. For instance, we have now up to $m$ different choices of the norms in the denominator of (1). Moreover, various choices for the numerator are possible, depending on how one partitions the dimensions of $\mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_d}$. In order to formalize these properties for a general tensor $T$, we introduce here the concept of shape partition.

**Definition 2.1 (Shape partition).** We say that $\sigma$ is a shape partition of $T \in \mathbb{R}^{N_1 \times \cdots \times N_m}$ if $\sigma = \{\sigma_i\}_{i=1}^d$ is a partition of $[m]$, i.e. $\bigcup_{i=1}^d \sigma_i = [m]$ and $\sigma_i \cap \sigma_j = \emptyset$ for $i \neq j$, such that for every $i \in [d]$ and $j, j' \in \sigma_i$, it holds $N_j = N_{j'}$. Moreover, we always assume that:

(a) For every $i \in [d-1]$ and $j \in \sigma_i, k \in \sigma_{i+1}$ it holds $j \leq k$.

(b) If $d > 1$, then $|\sigma_i| \leq |\sigma_{i+1}|$ for every $i \in [d-1]$.

Observe that the conditions (a) and (b) in the above definition are not restrictive. Indeed, if $\sigma = \{\sigma_i\}_{i=1}^d$ is a partition of $[m]$ such that $N_j = N_{j'}$ for every $j, j' \in \sigma_i$ and $i \in [d]$, then there exists a permutation $\pi: [m] \to [m]$ such that $\tilde{\sigma} = \{\tilde{\sigma}_i\}_{i=1}^d$ defined as $\tilde{\sigma}_i = \{\pi(j): j \in \sigma_i\}, i \in [d]$ is a shape partition of the tensor $T$ defined as $\tilde{T}_{j_1, \ldots, j_m} = T_{\pi(j_1), \ldots, \pi(j_m)}$ for all $j_1, \ldots, j_m$. For instance if $T \in \mathbb{R}^{2 \times 3 \times 2}$ and $\sigma = \{\{1,3\}, \{2\}\}$, then one can define $\tilde{T}_{i,j,k} = T_{i,k,j}$ for all $i, j, k$ and $\tilde{\sigma} = \{\{1,2\}, \{3\}\}$.

**Remark 2.2.** The concept of shape partition of a tensor is strictly related with the integer partition of its order. More precisely, it is related with the Cartesian product of the integer partitions of the number of orders of $T$ having same dimension. Let us explain this with an example. Let $T \in \mathbb{R}^{N_1 \times N_1 \times N_2 \times N_2}$ be a tensor of order four. If $n_1 = n_2$ then the shape partitions of $T$ are $\sigma^1 = \{\{1,2,3,4\}\}, \sigma^2 = \{\{1\}, \{2,3,4\}\}$, $\sigma^3 = \{\{1,2\}, \{3,4\}\}$, $\sigma^4 = \{\{1\}, \{2\}, \{3,4\}\}$ and $\sigma^5 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$ which formally coincide with the integer partitions of the number 4, i.e. the order of $T$. Whereas, when $N_1 \neq N_2$, the shape partitions of $T$ formally coincide with the
Cartesian product of the integer partitions of the number 2. Precisely, when \(N_1 \neq N_2\), then \(\sigma^3, \sigma^4, \sigma^5\) are the shape partitions of \(T\). To help intuition, we show in Figure 1 the Ferrers diagrams of the shape partitions of the example tensor discussed here.

Shape partitions are useful and convenient for describing all spectral systems of the same form as (2) but for tensors of any order. In particular, throughout this paper we associate to each shape partition \(\sigma = \{\sigma_i\}_{i=1}^d\) of \(T \in \mathbb{R}^{N_1 \times \ldots \times N_m}\) the numbers \(s_1, \ldots, s_d, \nu_1, \ldots, \nu_d\), and \(n_1, \ldots, n_d\) defined as follows:

\[
\nu_i = |\sigma_i|, \quad s_i = \min\{a \mid a \in \sigma_i\}, \quad n_i = N_{s_i}, \quad \forall i \in [d],
\]

and \(s_{d+1} = m + 1\).

Given a shape partition \(\sigma\) we will always assume the definitions in (3), although the reference to the specific \(\sigma\) will be understood implicitly. Moreover, for convenience, we will very often use the \(n_i\) in place of the \(N_i\). The relation between these two numbers is made more clear by noting that the dimensions \(N_1 \times \ldots \times N_m\) of \(T\) can be rewritten as follows:

\[
\begin{align*}
\sigma_1 &\quad \frac{\text{\(N_1 \times \ldots \times N_{s_2-1}\)}}{n_1 \times \ldots \times n_{\nu_1}} &\quad \frac{\sigma_2}{n_2 \times \ldots \times n_{\nu_2}} &\quad \frac{\sigma_d}{n_d \times \ldots \times n_{\nu_d}} \\
\end{align*}
\]

Now, given \(p = (p_1, \ldots, p_d) \in (1, \infty)^d\) and the shape partition \(\sigma\) of \(T\), we define the Rayleigh quotient of \(T\) induced by \(\sigma\) and \(p\) as follows:

\[
\begin{align*}
\Phi(x_1, \ldots, x_d) &= \frac{f_T(x^{\sigma})}{\|x_1\|_{p_1}^p \|x_2\|_{p_2}^p \cdots \|x_d\|_{p_d}^p} \\
\end{align*}
\]

where \(x^{\sigma} = (x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_d, \ldots, x_d)\).

In particular, we note that the functions \(\Phi^1, \Phi^2, \Phi^3\) of (1) can be recovered by setting \(\sigma^1 = \{(1, 2, 3)\}, \ p = p, \ \sigma^2 = \{(1), \{2, 3\}\}, \ p = (p, q)\) and \(\sigma^3 = \{(1), \{2\}, \{3\}\}, \ p = (p, q, r)\) respectively.

The Rayleigh quotient (4) is naturally related to a norm of the tensor which depends on both the shape partition \(\sigma\) and the choice of the norms \(\cdot \|_{p_i}\). We denote such norm as \(\|T\|_{(\sigma, p)} = \max_{x_1, \ldots, x_d \neq 0} \Phi(x_1, \ldots, x_d)\). Note that the absolute value in the definition of \(\|T\|_{(\sigma, p)}\) can be omitted when \(T\) is nonnegative. In fact, as discussed in the introduction, if \(T\) is nonnegative, then the maximum is always attained at nonnegative vectors. In the case \(d = m = 2\), \(p_1 = \ldots = p_m = 2\), \(\|T\|_{(\sigma, p)}\) is called the spectral norm of \(T\) and it is known that its computation is \(\text{NP}\)-hard in general (c.f. [12]). If \(d = m = 2\), then \(\|T\|_{(\sigma, p)}\) coincides with the \(\ell^p\)-norm of the matrix \(T\) and it is also known to be \(\text{NP}\)-hard for general matrices if, for instance, \(p_1 = p_2 \neq 1, 2\) is a rational number or \(1 \leq p_1 < p_2 \leq \infty\), see e.g. [11, 19].

A direct computation shows that the critical points of \(\Phi\) in (4) are solutions to the following spectral equation:

\[
\begin{align*}
\nabla_i f_T(x^{\sigma}) &= \lambda \psi_{p_i}(x_i), \quad \|x_i\|_{p_i} = 1 \quad \forall i \in [d],
\end{align*}
\]

where \(\nabla_i f_T(x^{\sigma}) \in \mathbb{R}^{n_i}\) denotes the gradient of the map \(x_i \mapsto f_T(x^{\sigma}), \ \psi_{p_i}(x_i) = (|x_{i,1}|^{p_i-1}\text{sign}(x_{i,1}), \ldots, |x_{i,n_i}|^{p_i-1}\text{sign}(x_{i,n_i}))\) for all \(x_i \in \mathbb{R}^{n_i}\) and \(\text{sign}(t) = t/|t|\) if \(t \neq 0\) and \(\text{sign}(0) = 0\).
It is important to note that $\nabla_i f_T(x^{[\sigma]})$ and $T_{si}(x^{[\sigma]})$ do not coincide in general, unless $\nu_i = 1$. Hence, we consider a more general class of spectral problems for tensors which is formulated as follows:

$$
T_{si}(x^{[\sigma]}) = \lambda \psi_{p_i}(x_i), \quad \|x_i\|_{p_i} = 1 \quad \forall i \in [d].
$$

Depending on the choice of $\sigma$, various known spectral problems related to nonnegative tensors can be recovered from (6). Indeed, if $d = m$, then $\sigma = \{\{1\},\ldots,\{m\}\}$ and we recover equation (1.2) in [8] which characterizes the $\ell^{p_1}\ldots\ell^{p_m}$-singular vectors of $T$. If $d = 2$, then $\sigma = \{\{1,\ldots,k\},\{k+1,\ldots,m\}\}$ for some $k \in [m-1]$ and we recover equation (2) in [16] which characterizes the $\ell^{p_1,p_2}$-singular vectors of the rectangular tensor $T$. Finally, if $d = 1$, then $\sigma = \{\{1,\ldots,m\}\}$ and we recover equation (7) in [15] which characterizes the $\ell^{p_1}$-eigenvectors of $T$. Perron-Frobenius type results have been established for each of the aforementioned spectral problems. In order to unify these results, we introduce here the following definition:

**Definition 2.3** $(\sigma,p)$-eigenvalues and eigenvectors. We say that $(\lambda,x)$ is a $(\sigma,p)$-eigenpair of $T$ if it satisfies (6). We call $\lambda$ a $(\sigma,p)$-eigenvalue of $T$ and $x$ a $(\sigma,p)$-eigenvector of $T$.

Key assumptions in the Perron-Frobenius theory of nonnegative tensor are strict nonnegativity, weak irreducibility and (strong) irreducibility. In order to address the general spectral problem of Definition 2.3, we recast such assumptions in terms of the chosen shape partition.

**Definition 2.4** $(\sigma$-nonnegativity and $\sigma$-irreducibility). For a nonnegative tensor $T \in \mathbb{R}_{+}^{N_1 \times \ldots \times N_m}$ and an associated shape partition $\sigma = \{\sigma_i\}_{i=1}^{d}$, consider the matrix $M \in \mathbb{R}_{+}^{(n_1+\ldots+n_d) \times (n_1+\ldots+n_d)}$ defined as

$$
M_{(i,t_i),(k,l_k)} = \frac{\partial}{\partial x_{k,l_k}} T_{i,t_i}(x^{[\sigma]}) \quad \forall (i,t_i),(k,l_k) \in I^\sigma = \bigcup_{i=1}^{d} \{i\} \times [n_i],
$$

where $1 = (1,\ldots,1)^T$ is the vector of all ones. We say that $T$ is:

- **$\sigma$-strictly nonnegative**, if $M$ has at least one nonzero entry per row.
- **$\sigma$-weakly irreducible**, if $M$ is irreducible.
- **$\sigma$-strongly irreducible**, if for every $x \in \mathbb{R}_{+}^{n_1} \times \ldots \times \mathbb{R}_{+}^{n_d}$ that is not entry-wise positive and is such that $x_i \neq 0$ for all $i \in [d]$, there exists $(k,l_k) \in I^\sigma$ such that $x_{k,l_k} = 0$ and $T_{k,l_k}(x^{[\sigma]}) > 0$.

These definitions coincide with most of the corresponding definitions introduced for special cases. Indeed, if $d = 1$, $\sigma$-strict nonnegativity reduces to the definition of strictly nonnegative tensor introduced in [13]. If $d = 1, 2, m$, $\sigma$-weak irreducibility reduces to the definition of weak irreducibility introduced in [8] and [16], respectively. If $d = 1, m$, $\sigma$-strong irreducibility reduces to the existing definitions of irreducibility introduced in [3] and [8]. However, in the case $d = 2$, $\sigma$-strong irreducibility is strictly less restrictive than the definition of irreducibility introduced in [5]. In Section 6.4 we give a detailed characterization of each of these classes of nonnegative tensors. In particular, we propose equivalent formulations of these class of tensors in terms of graphs and in terms of the entries of $T$. Furthermore, we show in Theorem 6.13 that $\sigma$-strong irreducibility implies $\sigma$-weak irreducibility which itself implies $\sigma$-strict nonnegativity. We also study how these classes are related, for a fixed tensor $T$ but different choices of $\sigma$. 
Using different shape partitions, one can associate several spectral problems to a
tensor $T$ via Definition 2.3 and sometimes one can transfer properties that hold true
for one formulation to another one. For instance, if a symmetric matrix $Q \in \mathbb{R}^{n \times n}$
is irreducible, i.e. $Q$ is $\{\{1,2\}\}$-irreducible, then its corresponding bipartite graph
is strongly connected, i.e. $Q$ is also $\{\{1\},\{2\}\}$-irreducible. In particular, this implies
that the classical Perron-Frobenius theorem holds not only for the eigenpairs of $Q$
but also for its singular pairs. A similar situation arises in the more general setting
of tensors. In order to formalize this property, we define the following partial order
on the set of shape partitions of $T$:

**Definition 2.5.** Let $\sigma = \{\sigma_i\}_{i=1}^d$, $\tilde{\sigma} = \{\tilde{\sigma}_i\}_{i=1}^d$
be two shape partitions of $T \in \mathbb{R}^{N_1 \times \ldots \times N_m}$, then we write $\sigma \subseteq \tilde{\sigma}$
if $d \geq \tilde{d}$ and there exists $\pi: [d] \rightarrow [\tilde{d}]$ such that
$\sigma_i \subseteq \tilde{\sigma}_{\pi(i)}$ for every $i \in [d]$.

Note that, for instance, in Remark 2.2 we have $\sigma^3 \subseteq \sigma^2 \subseteq \sigma^1$ whereas $\sigma^2 \not\subseteq \sigma^3$ and $\sigma^1 \not\subseteq \sigma^2$. Moreover, note that the shape partitions $\sigma = \{\{1\},\{2\}\}$ and
$\tilde{\sigma} = \{\{1,2\}\}$ of the symmetric matrix $Q$ above satisfy $\sigma \subseteq \tilde{\sigma}$ and irreducibility
with respect to $\sigma$ carries over to $\tilde{\sigma}$. More generally, we discuss in Sections 4 and 6 several
properties of the tensor $T$ preserved by the partial ordering $\subseteq$, that is properties that
automatically hold for $\tilde{\sigma}$ when holding for a shape partition $\sigma$ such that $\sigma \subseteq \tilde{\sigma}$. In
particular, this is the case of tensor symmetries that we define below in terms of $\sigma$.

We have already mentioned in the introduction that, as for the case of matrices,
symmetries in the entries of $T$ allow for different variational characterizations of the
associated spectrum. Therefore, given the shape partition $\sigma$ of $T$, we introduce the
definition of $\sigma$-symmetry. The latter is based on the concept of partially symmetric
tensors introduced in [7] which we recall for the sake of completeness:

**Definition 2.6** (Partially symmetric tensor, [7]). Let $T \in \mathbb{R}^{N_1 \times \ldots \times N_m}$
and let $\alpha \subset [m]$ be a subset of cardinality 2 at least. We say that $T$ is symmetric with respect
to $\alpha$ if $N_i = N_i^\alpha$ for each pair $\{i,i^\prime\} \subset \alpha$ and the value of $T_{j_1 \ldots j_m}$
does not change if we interchange any two indices $j_i,j_{i^\prime}$ for $i,i^\prime \in \alpha$ and any $j_k \in [N_k], k \in [m]$. We
agree that $T$ is symmetric with respect to each $\{i\}$ for $i \in [m]$.

**Definition 2.7** ($\sigma$-symmetry). Let $T \in \mathbb{R}^{N_1 \times \ldots \times N_m}$ and let $\sigma = \{\sigma_i\}_{i=1}^d$
be a shape partition of $T$. We say that $T$ is $\sigma$-symmetric if it is partially symmetric with
respect to $\sigma_i$ for all $i \in [d]$.

Observe that, in particular, every matrix is $\{\{1\},\{2\}\}$-symmetric and symmetric
matrices are $\{\{1,2\}\}$-symmetric. Moreover, if $T$ is $\sigma$-symmetric, then $T$ is $\tilde{\sigma}$-
symmetric for every shape partition $\tilde{\sigma}$ of $T$ such that $\tilde{\sigma} \subseteq \sigma$.

Similarly to the matrix case where only eigenpairs of symmetric matrices have
a variational characterization, we show in Lemma 4.1 that solving (5) is equivalent
to solve a problem of the form (6) where the tensor is $\sigma$-symmetric. Vice-versa, in
Lemma 4.2, we show that when the tensor is partially symmetric with respect to $\sigma$,
then the solutions of (6) are critical points of the Rayleigh quotient in (4).

3. Main results. In this section we describe the main results of this paper: A
complete characterization of the irreducibility properties of $T$ in terms of the shape
partition $\sigma$; a unifying Perron-Frobenius theorem for the general tensor spectral problem
of (6); and a generalized power method with a linear convergence rates that allows
to compute the dominant $(\sigma,p)$-eigenvalue and $(\sigma,p)$-eigenvector of $T$. These results
are based on a number of preliminary lemmas and results that we prove in the next
sections. Thus, for the sake of readability, we postpone the proofs of the main results.
to the end of the paper. We devote this section to describe the results and to relate them with previous work.

The first result is presented in the following:

**Theorem 3.1.** Let $T \in \mathbb{R}_+^{N_1 \times \ldots \times N_m}$ and let $\sigma = \{\sigma_i\}_{i=1}^d$ and $\tilde{\sigma} = \{\tilde{\sigma}_i\}_{i=1}^d$ be shape partitions of $T$ such that $\sigma \subseteq \tilde{\sigma}$. Then, the following holds:

(i) If $T$ is $\sigma$-weakly irreducible, then $T$ is $\sigma$-strictly nonnegative.

(ii) If $T$ is $\sigma$-strongly irreducible, then $T$ is $\sigma$-weakly irreducible.

(iii) If $T$ is $\sigma$-strictly nonnegative, then $T$ is $\tilde{\sigma}$-strictly nonnegative.

(iv) If $T$ is $\sigma$-weakly irreducible and $\tilde{\sigma}$-symmetric, then $T$ is $\tilde{\sigma}$-weakly irreducible.

(v) If $T$ is $\sigma$-strongly irreducible and $\tilde{\sigma}$-symmetric, then $T$ is $\tilde{\sigma}$-strongly irreducible.

**Proof.** See Section 6.4.

Few comments regarding the partial symmetry assumption in (iv) and (v) of the above theorem are in order: First, note that, as in the matrix case, the irreducibility of a tensor does not depend on the magnitude of its entries and so it is enough to assume that the nonzero pattern of $T$ is $\tilde{\sigma}$-symmetric. Second, by giving explicit examples, we note in Remarks 6.8 and 6.12 that assumption (v) can not be omitted in order to deduce $\tilde{\sigma}$-weak (resp. strong) irreducibility from $\sigma$-weak (resp. strong) irreducibility.

It is well known that in the case of nonnegative matrices, i.e. $m = 2$ and $d = 1$, $\sigma$-weak irreducibility and $\sigma$-strong irreducibility are equivalent. This equivalence is proved also for $m = 2$ and $d = 2$ in Lemma 3.1 [8]. Furthermore, (i), (ii) are known for the particular cases $d = 1, m$. Precisely, refer to Lemma 3.1 [8] for an equivalent of (ii) and to Proposition 8, (b) [9], Corollary 2.1. [13] for an equivalent of (i) in the cases $d = 1, m$ respectively. However, to our knowledge, the results of points (iii), (iv), (v) have not been proved before, in any setting.

Our second result is a new and unifying Perron-Frobenius theorem for $(\sigma, p)$-eigenpairs. First, let us consider the sets of nonnegative, nonnegative nonzero and positive tuples of vectors in $\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_d}$, that is: let $K^+_+ = \mathbb{R}^{n_1}_+ \times \ldots \times \mathbb{R}^{n_d}_+$, $K^+_{\sigma,0} = \{x \in K^+_{\sigma} \mid x_i \neq 0, i \in [d]\}$ and let $K^+_{\sigma,+}$ be the interior of $K^+_{\sigma}$. Furthermore,
let us define the \((\sigma, p)\)-spectral radius of \(T\):

\[
\rho^{(\sigma, p)}(T) = \sup \{ |\lambda| \mid \lambda \text{ is a } (\sigma, p)\text{-eigenvalue of } T \}.
\]

As mentioned before, the key of our Perron-Frobenius theorem is the relation with the theory of multi-homogeneous and order-preserving mappings [10]. In particular, let us consider \(F^{(\sigma, p)}: K^\sigma_+ \to K^\sigma_+\) defined as \(F^{(\sigma, p)} = (F_1^{(\sigma, p)}, \ldots, F_d^{(\sigma, p)})\) where \(F_i^{(\sigma, p)} = (F_{i,1}^{(\sigma, p)}, \ldots, F_{i,n_i}^{(\sigma, p)})\) and, for all \((i, j_i) \in I^\sigma\),

\[
F_{i,j_i}^{(\sigma, p)}(x) = \left( T_{i,j_i} (x[\sigma]) \right)^{p_i - 1}, \quad \text{where } p_i' = \frac{p_i}{p_i - 1}.
\]

We show in Lemma 5.1 that the nonnegative \((\sigma, p)\)-eigenpairs of \(T\) are in bijection with the multi-homogeneous eigenvectors of \(F^{(\sigma, p)}\), i.e., vectors \(x \in K^\sigma_+\) for which there exists \(\theta_1, \ldots, \theta_d \geq 0\) such that \(F_i^{(\sigma, p)}(x) = \theta_i x_i\) for all \(i \in [d]\). This key observation allows us to exploit the results proved in [10]. In particular, we consider the homogeneity matrix \(A(\sigma, p) \in \mathbb{R}^{d \times d}\) of \(F^{(\sigma, p)}\) given as

\[
A(\sigma, p) = \text{diag}(p_1' - 1, \ldots, p_d' - 1)(\nu 1^\top - I), \quad \nu = (|\sigma_1|, \ldots, |\sigma_d|)^\top,
\]

and let \(\rho(A(\sigma, p))\) be its spectral radius. In the following, \(A(\sigma, p)\) always refers to the homogeneity matrix of \(F^{(\sigma, p)}\), hence, when it is clear from the context, we omit the arguments \((\sigma, p)\) and write \(A\) instead of \(A(\sigma, p)\). Lemma 3.2 in [10] implies that \(\rho(A)\) is an upper bound on the Lipschitz constant of \(F^{(\sigma, p)}\) with respect to a suitable weighted Hilbert metric on \(K^\sigma_+\). Therefore, when \(\rho(A) \leq 1\), we can recast the \((\sigma, p)\)-eigenvalue problem for \(T\) in terms of a non-expansive map and derive the Perron-Frobenius theorem for \(T\) as a consequence.

In the particular cases \(d = 1, 2, m\), typical assumptions on \(p_1, \ldots, p_m\) found in the literature on Perron-Frobenius theory of nonnegative tensors are \(p_i \geq m\) for every \(i \in [d]\), [8, 15, 16]. It is not difficult to see that if \(p_i \geq m\) for all \(i\), then \(\rho(A) \leq 1\), with equality if and only if \(p_1 = \ldots = p_m = m\). However, by the Collatz-Wielandt formula, we have \(\rho(A) = \min_{\nu \in \mathbb{R}^d_+} \max_{i \in [d]} (A^\top \nu)_i / \nu_i\), and thus it is clear that there are many choices of \(p_1, \ldots, p_d\) such that \(\rho(A) \leq 1\) but \(\min_{i \in [d]} p_i < m\). Moreover, note that the function \((p_1, \ldots, p_d) \mapsto \rho(A(\sigma, p))\) is strictly monotonically decreasing in the sense that for every \(p, \tilde{p} \in (1, \infty)^d\) with \(\tilde{p}_i \leq p_i\) for all \(i \in [d]\), it holds \(\rho(A(\sigma, \tilde{p})) \geq \rho(A(\sigma, p))\) with equality if and only if \(p = \tilde{p}\). An example comparing \(\rho(A) \leq 1\) with the conditions on \(p_1, \ldots, p_d\) given in [8, 9, 15, 16] is shown in Figure 2.

The following Perron-Frobenius theorem consists of five parts: The first one is a weak Perron-Frobenius theorem ensuring the existence of a maximal nonnegative \((\sigma, p)\)-eigenpair. The second characterizes \(\rho^{(\sigma, p)}(T)\) via a Collatz-Wielandt formula, a Gelfand type formula and a cone spectral radius formula. The third part, gives sufficient conditions for the existence of a positive \((\sigma, p)\)-eigenpair. The fourth part, gives conditions ensuring that \((\sigma, p)\)-eigenvectors which are nonnegative but not positive can not correspond to \(\rho^{(\sigma, p)}(T)\). The last part gives further conditions which guarantee that \(T\) has a unique nonnegative \((\sigma, p)\)-eigenvector.

Let us denote by \((F^{(\sigma, p)})^k\) the \(k\)-th composition of \(F^{(\sigma, p)}\) with itself, that is \((F^{(\sigma, p)})^k(x) = F^{(\sigma, p)}(F^{(\sigma, p)})^{k-1}(x)\). Moreover, let us define the following product of balls \(S^{(p, \sigma)}_+ = \{ x \in K^\sigma_+ \mid \|x\|_{p_i} = 1, \forall i \in [d] \}\) and its positive part \(S^{(p, \sigma)}_{+} = S^{(p, \sigma)}_+ \cap K^\sigma_+\).
Theorem 3.2. Let $\sigma = \{s_i\}_{i=1}^d$ be a shape partition of $T \in \mathbb{R}_+^{N_1 \times \cdots \times N_m}$. Furthermore let $r^{(\sigma,p)}(T)$, $F^{(\sigma,p)}$ and $A$ be as in (7), (8) and (9) respectively. Suppose that $T$ is $\sigma$-strictly nonnegative and $\rho(A) \leq 1$. Then, there exists a unique $b \in \mathbb{R}^d_+$ such that $A^T b = \rho(A) b$ and $\sum_{i=1}^d b_i = 1$. Furthermore, we have the following:

(i) There exists a $(\sigma, p)$-eigenpair $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{K}_+^{(\sigma)}$ of $T$ such that $\lambda = r^{(\sigma,p)}(T)$.

(ii) Let $\gamma = \sum_{i=1}^d \frac{b_i}{p_i}$, then $\gamma \in (1, \infty)$ and the following Collatz-Wielandt formula holds:

$$\inf_{x \in \mathbb{S}^{(p,\sigma)}_{++}} \mathcal{C}(F^{(\sigma,p)}, x) = r^{(\sigma,p)}(T) = \max_{y \in \mathbb{S}^{(p,\sigma)}_+} \mathcal{C}(F^{(\sigma,p)}, y)$$

where

$$\mathcal{C}(F^{(\sigma,p)}, x) = \prod_{i=1}^d \left( \max_{j_i \in [n_i]} \frac{F_{i,j_i}^{(\sigma,p)}(x)}{x_{i,j_i}} \right)^{(\gamma-1)b_i}$$

and

$$\mathcal{C}(F^{(\sigma,p)}, y) = \prod_{i=1}^d \left( \min_{j_i \in [n_i], y_{i,j_i} > 0} \frac{F_{i,j_i}^{(\sigma,p)}(y)}{y_{i,j_i}} \right)^{(\gamma-1)b_i}.$$ 

If additionally, $\rho(A) = 1$, then it holds

$$r^{(\sigma,p)}(T) = \sup_{x \in \mathcal{K}^{(\sigma)}_{++}, \lambda \rightarrow 1} \limsup_{k \rightarrow \infty} \left( \prod_{i=1}^d \|F^{(\sigma,p)}_{i,k}(z)\|_{p_i}^{b_i} \right)^{\frac{\gamma-1}{\lambda}}$$

$$= \lim_{k \rightarrow \infty} \left( \sup_{x \in \mathbb{S}^{(p,\sigma)}_{++}} \prod_{i=1}^d \|F^{(\sigma,p)}_{i,k}(z)\|_{p_i}^{b_i} \right)^{\frac{\gamma-1}{\lambda}}.$$ 

(iii) If either $\rho(A) < 1$ or $T$ is $\sigma$-weakly irreducible, then the $(\sigma, p)$-eigenvector $u$ of (i) can be chosen to be strictly positive, i.e. $u \in \mathcal{K}^{(\sigma)}_{++}$. Moreover, $u$ is then the unique positive $(\sigma, p)$-eigenvector of $T$.

(iv) If $T$ is $\sigma$-weakly irreducible, then for every $(\sigma, p)$-eigenpair $(\vartheta, x)$ of $T$ such that $x \in \mathcal{K}^{(\sigma)}_{++}$, it holds $\vartheta < r^{(\sigma,p)}(T)$.

(v) If $T$ is $\sigma$-strongly irreducible, then the $(\sigma, p)$-eigenvector $u$ of (i) is positive and it is the unique nonnegative $(\sigma, p)$-eigenvector of $T$.

Proof. See Section 7.

Note that Theorem 3.2, (i) is relatively obvious when $T$ is $\sigma$-symmetric. In fact, as shown in Lemma 4.2, in this case $r^{(\sigma,p)}(T) = \|T\|_{(\sigma,p)}$ and thus the existence of $u$ follows from the fact that a continuous function over a compact domain attains its maximum. In particular, this is always the case when $m = d$. However, when $d \neq m$ and $T$ is not $\sigma$-symmetric, proving the existence of $u$ is more delicate. The cases $d = 1, 2$ are proved in Theorem 2.3 [22] and Theorem 4.2 [16], but under the assumption that $p_i \geq m$, for all $i \in [d]$. Our Theorem 3.2, instead, addresses a more general case, but requires $\sigma$-strict nonnegativity of $T$. Although this is an additional requirement, we show for instance in Example 6.4 that this is a very mild assumption.

A particularly interesting consequence of the Collatz-Wielandt formula (10) is that every positive $(\sigma, p)$-eigenvector of $T$ must correspond to the maximal eigenvalue $r^{(\sigma,p)}(T)$. Such a formula is proved in Theorem 2.3 [22] and Theorem 1 [9] for the cases $d = 1$ and $d = m$ respectively. Both assume that $T$ has a positive $(\sigma, p)$-eigenvector and either $p_i \geq m$ if $d = 1$, or $(m-1)p_j \leq p_k(p_j - 1)$ for some $j \in [d]$ and all $k \in [m] \setminus \{j\}$, if $d = m$. In the case $d = 2$, a similar formula is proved in Theorem...
and 2.4 in [22], where it is proved that if all the entries of \( p \) of Theorem 3.2 is generally less restrictive than any known counterpart. In fact, the only result comparable with point (iv) we are aware of is Theorem 1.4 [3] and Theorem 14 [9]. Indeed, this result follows from the fact that (iv) of Theorem 3.2 and the characterizations of the spectral radius in (11) have not been proved before, besides the particular cases \( d = 1 \) and \( p_1 = m \). In fact, the only result comparable with point (iv) we are aware of is Theorem 1.4 [3] and Theorem 14 [9]. Indeed, this result follows from the fact that every nonnegative \((\sigma, p)\)-eigenvector of \( T \) has positive entries and its proof holds regardless of the choice of \( p_1, \ldots, p_d \in (1, \infty) \).

Our last main contribution concerns the computational aspects of the positive \((\sigma, p)\)-eigenvector \( u \) in Theorem 3.2. This vector can be computed using a nonlinear generalization of the power method. The usual power method for nonnegative tensors is formulated as follows: Let \( x^0 \in \mathcal{K}_{\sigma+} \), and, for \( k = 0, 1, 2, \ldots \), define

\[
x^{k+1} = \left( \frac{F^1_{1}(\sigma, p)(x^k)}{\| F^1_{1}(\sigma, p)(x^k) \|_{p_1}}, \ldots, \frac{F^d_{d}(\sigma, p)(x^k)}{\| F^d_{d}(\sigma, p)(x^k) \|_{p_d}} \right).
\]

This iterative process reduces to the one proposed in [17], [5], [8] for the cases \( d = 1, 2, m \) respectively. This sequence provides a natural generalization of the power method for computing eigenpairs of matrices. Usually, convergence towards \( u \) is only guaranteed when \( \rho(A) \leq 1 \) and the Jacobian matrix of \( F(\sigma, p) \) is primitive. However, we prove that when \( \rho(A) < 1 \) and \( T \) is \( \sigma \)-strictly nonnegative, or equivalently the matrix \( M \) of Definition 2.4 has at least one positive entry per row, this sequence converges towards \( u \) and we have a linear convergence rate for it.

If \( \rho(A) = 1 \), primitivity can be relaxed into irreducibility by considering a different sequence, which we define in the following. Let \( G(\sigma, p) : \mathcal{K}_+^\sigma \rightarrow \mathcal{K}_+^\sigma \) be defined as \( G(\sigma, p) = (G^1_{1}(\sigma, p), \ldots, G^d_{d}(\sigma, p)) \), \( G_i(\sigma, p) = (G^1_{i,1}(\sigma, p), \ldots, G^d_{i,n_i}(\sigma, p)) \) and

\[
G^i_{i,j}(\sigma, p)(x) = \sqrt{\lambda_{i,j} x_{i,j}} \end{aligned}
\]

and consider the sequence

\[
y^{k+1} = \left( \frac{G^1_{1}(\sigma, p)(y^k)}{\| G^1_{1}(\sigma, p)(y^k) \|_{p_1}}, \ldots, \frac{G^d_{d}(\sigma, p)(y^k)}{\| G^d_{d}(\sigma, p)(y^k) \|_{p_d}} \right),
\]

where \( k = 0, 1, 2, \ldots \) and \( z^0 \in \mathcal{K}^\sigma_{\sigma+} \). The convergence of the two sequences in (12) and (14) is proved in the next Theorem 3.3. In order to facilitate its statement, for \( k \geq 1 \), we let

\[
\tilde{c}_k = \tilde{c}_k F(\sigma, p, x^k), \quad \tilde{c}_k = \tilde{c}_k F(\sigma, p, x^k), \quad \tilde{c}_k = \tilde{c}_k (F(\sigma, p, y^k))^2, \quad \tilde{c}_k = \tilde{c}_k (F(\sigma, p, y^k))^2,
\]

where \( \tilde{c}_k, \tilde{c}_k \) are defined as in Theorem 3.2. By the continuity of \( \tilde{c}_k, \tilde{c}_k \) in \( \mathcal{K}_{\sigma+} \), if the sequence of \( x^k \), respectively \( y^k \), converges to a positive \((\sigma, p)\)-eigenvector \( u \) of \( T \), then \( \lim _{k \to \infty} \tilde{c}_k = \lim _{k \to \infty} \tilde{c}_k = r(\sigma, p)(T) \), respectively \( \lim _{k \to \infty} \tilde{c}_k = \lim _{k \to \infty} \tilde{c}_k = r(\sigma, p)(T) \).
Theorem 3.3. Assume that $T$ is $\sigma$-strictly nonnegative and has a positive $(\sigma, p)$-eigenvector $u$ and $\rho(A) \leq 1$. Furthermore, let $(x^k)_{k=0}^{\infty}$, $(y^k)_{k=0}^{\infty}$, $(\xi_k)_{k=1}^{\infty}$, $(\tilde{\xi}_k)_{k=1}^{\infty}$ be as above. Then, the following holds:

(i) If $\omega \in \{\xi, \xi\}$, then for all $k = 1, 2, \ldots$ it holds

$$\bar{\omega}_k \leq \bar{\omega}_{k+1} \leq r^{(\sigma, p)}(T) \leq \bar{\omega}_k,$$

and for every $\varepsilon > 0$, if $\bar{\omega}_k - \bar{\omega}_k \leq \varepsilon$, then

$$\left| \frac{\bar{\omega}_k + \bar{\omega}_k}{2} - r^{(\sigma, p)}(T) \right| \leq \frac{\varepsilon}{2}.$$

(ii) If $\rho(A) < 1$, then $\lim_{k \to \infty} x^k = u$ and, with $b$ as in Theorem 3.2,

$$\mu_b(x^k, u) \leq \left( \frac{\mu_b(x^1, x^0)}{1 - \rho(A)} \right) \rho(A)^k \quad \forall k = 1, 2, \ldots$$

where $\mu_b$ is the weighted Hilbert metric defined in (20).

(iii) If $T$ is $\sigma$-weakly irreducible, then $\lim_{k \to \infty} y^k = u$.

Proof. See Section 7.

To our knowledge, convergence of the power method for nonnegative tensors has been analyzed only for the cases $d = 1, 2, m$.

If $d = 1$, the known assumptions for the convergence of the power method towards $u$ are either $p_1 > m$ and $M$ primitive (Corollary 5.1, [8]), where $M$ is as in Definition 2.4, or $p_1 = m$ and $M$ irreducible (see Theorem 5.4 in [13]). Clearly, if $p_1 > m$, then the assumptions of Theorem 3.3, (ii) are considerably weaker as we only assume $T$ to be $\sigma$-strictly nonnegative. When $p_1 = m$, Theorem 3.3, (ii) is equivalent to Theorem 5.4 in [13] in terms of assumptions. However, note that the method in [13] uses an additive shift while we have a multiplicative shift. Observe, furthermore, that the convergence rate of [8] for the case $p_1 > m$ holds only asymptotically and assumes $T$ to be $\sigma$-weakly irreducible. Whereas, a linear convergence rate for the case $p_1 = m$ is proved under the assumption that $M$ is primitive in Theorem 4.1 [13].

For $d = 2$, results are known only in the case $p_1 = p_2 = m$. Precisely, in Theorem 7 [5] it is proved that $(x^k)_{k=1}^{\infty}$ converges towards $u$ if $p_1 = p_2 = m$ and $T$ is irreducible in the sense of Definition 1 in [5] which, as discussed above, is more restrictive than $T$ being $\sigma$-strongly irreducible. As $\sigma$-strong irreducibility implies $\sigma$-weak irreducibility, it is clear that Theorem 3.3, (iii) improves these results. A linear convergence rate is proved in Theorem 4 [23] for the case where $p_1 = p_2 = m$ but requires additional assumptions on $T$.

Finally, if $d = m$, then it is proved in Theorem 2, [9] that a variation of the power method converges to $u$ under the condition that $T$ is $\sigma$-weakly irreducible and $(m - 1)p_j \leq p_k(p_j - 1)$ for some $j \in [d]$ and all $k \in [m] \setminus \{j\}$, which, as discussed above, implies $\rho(A) < 1$ unless $p_1 = \ldots = p_d = m$, in which case $\rho(A) = 1$. Hence, in terms of convergence assumptions, Theorem 3.3 improves Theorem 2 in [9]. However, when $p_1 = \ldots = p_d = m$, the latter result provides an asymptotic convergence rate which is not implied by Theorem 3.3.

4. Tensor norms and spectral problems. In this section we study a number of relations between the critical points of the Rayleigh quotient $\Phi$ in (4) and the $(\sigma, p)$-eigenpairs of $T$. The goal of this discussion is twofold. First, it gives an optimization perspective on $(\sigma, p)$-eigenpairs and second it explains how to use our main results, in particular Theorem 3.3, for the computation of $\|T\|_{(\sigma, p)}$. 
In a first step, we prove in Lemma 4.1 how to construct a $\sigma$-symmetric tensor $S \in \mathbb{R}^{N_{1} \times \cdots \times N_{m}}$ so that $f_{T}(x[\sigma]) = f_{S}(x[\sigma])$ and $\nabla_{i}f_{T}(x[\sigma]) = \nu_{i}S_{a}(x[\sigma])$ for every $x \in \mathbb{R}^{n_{1} \times \cdots \times n_{m}}$, where $S_{a}(z) = \nabla f_{S}(z)$ for every $z \in \mathbb{R}^{N_{1} \times \cdots \times N_{m}}$. This construction has practical relevance, as it allows for a simple implementation of $\nabla_{i}f_{T}(x[\sigma])$ and it shows that partial symmetry is relevant when computing the critical points of $\Phi$. Furthermore, as $f_{T} = f_{S}$, we note that $S$ can be used in place of $T$ in the definition of $\Phi$, without changing the optimization problem. In particular, we have $\|T\|(\sigma, p) = \|S\|(\sigma, p)$.

In a second step, we prove in Lemma 4.2 that the $(\sigma, p)$-eigenvector and $(\sigma, p)$-eigenvalues of the $\sigma$-symmetric tensor $S$ are precisely the critical points, resp. values, of $\Phi$. In particular, this means that $\|S\|(\sigma, p) = r(\sigma, p)(S)$ and thus, if $S$ satisfies the assumptions of Theorem 3.3, the power method converges to a global maximizer $u$ of $\Phi$ and $f_{T}(u[\sigma]) = \|T\|(\sigma, p)$.

Finally, we discuss in Lemma 4.3 cases where $\|T\|(\sigma, p) = \|T\|(\sigma, \tilde{p})$ for different shape partitions $\sigma, \tilde{\sigma}$.

**Lemma 4.1.** Let $\sigma = \{\sigma_{i}\}_{i=1}^{d}$ be a shape partition of $T \in \mathbb{R}^{N_{1} \times \cdots \times N_{m}}$. For $i \in [d]$, let $\mathcal{S}_{i}$ be the permutation group of $\sigma_{i}$, and define $S \in \mathbb{R}^{N_{1} \times \cdots \times N_{m}}$ as

$$S_{j_{1}, \ldots, j_{m}} = \sum_{i=1}^{d} \frac{1}{\nu_{i}} \sum_{\pi_{i} \in \mathcal{S}_{i}} T_{\pi_{i}(1) \cdots \pi_{i}(s_{i}+1-1) \cdots \pi_{i}(s_{i})}$$

for all $j_{k} \in [N_{k}]$, $k \in [m]$. Then, we have $f_{T}(x[\sigma]) = f_{S}(x[\sigma])$ for all $x$. Furthermore, $S$ is $\sigma$-symmetric and, if $S = \nabla f_{S}$, then $\nabla_{i}f_{T}(x[\sigma]) = \nu_{i}S_{a}(x[\sigma])$ for all $x$ and $i \in [d]$.

**Proof.** For $x \in \mathbb{R}^{n_{1} \times \cdots \times n_{m}}$, let $Z \in \mathbb{R}^{N_{1} \times \cdots \times N_{m}}$ be the tensor defined as $Z_{j_{1}, \ldots, j_{m}} = T_{j_{1}, \ldots, j_{m}} \prod_{i \in [d]} \prod_{t \in [\sigma_{i}]} x_{i,j_{t}}$ for all $j_{1}, \ldots, j_{m}$. We have

$$f_{S}(x[\sigma]) = \sum_{i \in [d]} \sum_{t \in [\sigma_{i}]} \sum_{j_{t} \in [n_{i}]} Z_{j_{1}, \ldots, j_{m}}$$

$$= \sum_{i \in [d]} \sum_{t \in [\sigma_{i}]} \sum_{j_{t} \in [n_{i}]} \sum_{a \in [d]} \sum_{\pi_{a} \in \mathcal{S}_{a}} Z_{j_{1}, \ldots, j_{a} \pi_{a}, \ldots, j_{a+1}, \ldots, j_{m}}$$

$$= \sum_{i \in [d]} \sum_{t \in [\sigma_{i}]} \sum_{j_{t} \in [n_{i}]} S_{j_{1}, \ldots, j_{m}, x_{1,j_{1}}, \ldots, x_{d,j_{m}}} = f_{S}(x[\sigma]).$$

To conclude, note that, as $S$ is partially symmetric with respect to $\sigma_{i}$, Equation (4) in [15] implies $\nu_{i}S_{a}(x[\sigma]) = \nabla_{i}f_{S}(x[\sigma]) = \nabla_{i}f_{T}(x[\sigma])$. \qed

Now, we show that the converse of Lemma 4.1 is also true.

**Lemma 4.2.** Let $\sigma = \{\sigma_{i}\}_{i=1}^{d}$ be a shape partition of $T \in \mathbb{R}^{N_{1} \times \cdots \times N_{m}}$ and $p \in (1, \infty)^{d}$. If $T$ is $\sigma$-symmetric, then the $(\sigma, p)$-eigenvectors of $T$ are critical points of the Rayleigh quotient $\Phi$ defined in (4). Furthermore, it holds $\|T\|(\sigma, p) = r(\sigma, p)(T)$.

**Proof.** As $T$ is symmetric with respect to $\sigma_{i}$ for $i \in [d]$, we have $S = T$ where $S$ is as in (18). Thus Lemma 4.1 implies that $\nabla_{i}f_{T}(x[\sigma]) = \nu_{i}T_{a}(x[\sigma])$ for every $i$. Hence, if $x \in K_{\sigma}^{\pi}$ satisfies $T_{a}(x[\sigma]) = \lambda_{\psi_{p_{i}}}(x_{i})$ for all $i$, then we have $\nabla_{i}f_{T}(x[\sigma]) = \nu_{i}\lambda_{\psi_{p_{i}}}(x_{i})$ for every $i$, i.e. $x$ is a critical point of $\Phi$. Finally, note that $\lambda$ is the critical value associated to $x$ since $f_{T}(x[\sigma]) = (T_{a}(x[\sigma]), x_{i}) = \lambda\|x_{i}\|_{p_{i}}$, as $f_{T}(z)$ is linear in $z_{s_{i}}$. Therefore, we have $\|T\|(\sigma, p) = r(\sigma, p)(T).$ \qed
Finally, we show below that if $\sigma = \{\sigma_i\}_{i=1}^d$, $\tilde{\sigma} = \{\tilde{\sigma}_i\}_{i=1}^d$ are shape partitions of $T$, $\sigma \subseteq \tilde{\sigma}$ and $T$ is partially symmetric with respect to $\tilde{\sigma}$, then the corresponding tensor norms are equivalent for suitable choices of the $p_i, p_j$. This result is essentially a corollary of Theorem 1 in [1].

**Lemma 4.3.** Let $\sigma = \{\sigma_i\}_{i=1}^d$, $\tilde{\sigma} = \{\tilde{\sigma}_i\}_{i=1}^d$ be two shape partitions of $T \in \mathbb{R}^{N_1 \times \ldots \times N_m}$. If $\sigma \subseteq \tilde{\sigma}$ and $F \in (1, \infty)^d$, $\tilde{\sigma} \in (1, \infty)^d$ are such that $p_i = p_j$ whenever $\sigma_i \subset \tilde{\sigma}_j$. Then we have $\|T\|_{(\sigma, F)} \leq \|T\|_{(\tilde{\sigma}, F)}$.

**Proof.** If $\sigma = \tilde{\sigma}$, there is nothing to prove, so let us assume $\sigma \neq \tilde{\sigma}$. Clearly, we have $\|T\|_{(\sigma, F)} \geq \|T\|_{(\tilde{\sigma}, F)}$. We prove the reverse inequality. First, note that by Lemma 4.1, by substituting $T$ with $S$ if necessary, we may assume without loss of generality that $T$ is $\tilde{\sigma}$-symmetric. Now, let $(x_i^*, \ldots, x_j^*)$ be such that $\|x_i^*\|_{F_i} = 1$ for all $i \in [d]$ and $\|T\|_{(\sigma, F)} = f_t((x^*)^{(\sigma)})$. As $\sigma \subseteq \tilde{\sigma}$ and $\sigma \neq \tilde{\sigma}$, there exists $i, j \in [d], k \in [d]$ such that $i < j$, $\sigma_i \subset \tilde{\sigma}_k$ and $\sigma_j \subset \tilde{\sigma}_k$. Then $p_i = p_j$ by assumption and we have

$$\|T\|_{(\sigma, F)} = \max_{x, y, z, k \neq 0} \frac{f_t((x_i^*, x_k^*)^* \|x_i\|_{F_i}^\nu_i \|x_j\|_{F_j}^\nu_j)}$$

where $\nu_i = |\sigma_i|$ and $\nu_j = |\sigma_j|$. Now, as $T$ is partially symmetric with respect to $\tilde{\sigma}_k$, Theorem 1 in [1] implies that we there exists $(y_i^*, \ldots, y_d^*)$ with $\|y_i^*\|_{F_i} = 1, i \in [d]$ such that $\|T\|_{(\sigma, F)} = f_t((y^*)^{(\sigma)})$ and $y_j^* = y_j^*$. Continuing this argument for every $i, j \in [d], k \in [d]$ as above, we deduce that there exists $(z_i^*, \ldots, z_d^*)$ with $\|z_i^*\|_{F_i} = 1, i \in [d]$. $\|T\|_{(\sigma, F)} = f_t((z^*)^{(\sigma)})$ and the following property: For every $i, j \in [d]$ such that there exists $k \in [d]$ with $\sigma_i \subset \tilde{\sigma}_k$ and $\sigma_j \subset \tilde{\sigma}_k$, it holds $z_i^* = z_j^*$. It follows that there exists $\sigma$ and $z \in K_{\sigma, 0}$ such that $\sigma[z^*] = z^*$. Hence, we have

$$\|T\|_{(\sigma, F)} = f_t((z^*)^{(\sigma)}) = f_t((\sigma[z^*])^{(\sigma)}) = f_t(\sigma[z^*]) \leq \|T\|_{(\sigma, F)}$$

which concludes the proof.

5. The multi-homogeneous setting. In order to gain intuition on the reasoning for introducing the mapping $F^{(\sigma, p)}$ in (8), let us first consider the matrix case. Let $Q \in \mathbb{R}^{n \times n}$ be a square matrix. We know that the eigenvectors of $Q$ are fixed points of the homogeneous map $x \mapsto Qx$ in the projective space of $\mathbb{R}^n$, that is, if $Qx = \lambda x$ for some $x$ with $\|x\| = 1$, then $g(x) = x$ where $g(z) = Qz/\|Qz\|$.

We extend this observation to the tensor setting by means of the map $F^{(p, \sigma)}$. Precisely, we prove in Lemma 5.1 that the $(\sigma, p)$-eigenvectors of $T$ are exactly the fixed points of $F^{(\sigma, p)}$ in the product of projective spaces corresponding to $\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_d}$. This observation is useful as, for nonnegative tensors $T$, the mapping $F^{(\sigma, p)}$ is order-preserving and multi-homogeneous and we can apply the nonlinear Perron-Frobenius theorem discussed in [10] to derive conditions on the dominant $(\sigma, p)$-eigenvector of nonnegative tensors. In particular, we discuss how the spectral radius of $F^{(\sigma, p)}$ relates to the $(\sigma, p)$-spectral radius of $T$.

In the end of this section we briefly recall a number of relevant irreducibility assumptions on $F^{(\sigma, p)}$ that we will then transfer to $T$ afterwards in Section 6.4.

Let us first recall some useful concepts and notation from [10]. For $x, y \in K_{\sigma, 0}$, we write $x \leq_K y$ if $y - x \in K_{\sigma, 0}$. $x \leq_K y$ and $x <_K y$ if $y - x \in K_{\sigma, 0}$, $y - x \in K_{\sigma, 0} \setminus \{0\}$ and $y - x \in K_{\sigma, 0}$ respectively. A mapping $F: K_{\sigma, 0} \to K_{\sigma, 0}$ is said to be order-preserving if

$$x \leq_K y \quad \implies \quad F(x) \leq_K F(y) \quad \forall x, y \in K_{\sigma, 0}.$$
Furthermore, $F$ is said to be multi-homogeneous if there exists $B \in \mathbb{R}^{d \times d}$ such that for all $x \in K^\sigma_+$, $\alpha > 0$, it holds
\[
F_i(x_1, \ldots, x_{j-1}, \alpha x_j, x_{j+1}, \ldots, x_d) = \alpha^{B_{ij}} F_i(x) \quad \forall i, j \in [d].
\]
The matrix $B$ is called homogeneity matrix of $F$.

For a matrix $B \in \mathbb{R}^{d \times d}$ and a vector $\alpha \in \mathbb{R}^d$ we write $\alpha^B \in \mathbb{R}^d$ to denote the vector whose coordinates are $(\alpha_i^B)_i = \prod_{j=1}^d \alpha_{ij}^{B_{ij}}$, $i \in [d]$ and for every vector $x = (x_1, \ldots, x_d) \in K_+$ we write $\alpha \otimes x = (\alpha_1 x_1, \ldots, \alpha_d x_d) \in K_+$. With this notation we can now compactly write $F(\alpha \otimes x) = \alpha^B \otimes F(x)$ where $B$ is the homogeneity matrix of $F$. Furthermore, the set of equations $F_i(x) = \theta_i x_i$ for all $i \in [d]$, can be rewritten as $F(x) = \theta \otimes x$, where $\theta = (\theta_1, \ldots, \theta_d)$.

A natural (semi-)metric for the study of eigenvectors in cones is the so-called Hilbert projective metric $\mu$: $\mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ defined as
\[
\mu(x, y) = \ln \left( \frac{\mathcal{M}(x/y) \mathcal{M}(y/x)}{\mathcal{m}(x/y)} \right) = \ln \left( \frac{\mathcal{M}(x/y)}{\mathcal{m}(x/y)} \right),
\]
where $\mathcal{M}(\cdot/\cdot), \mathcal{m}(\cdot/\cdot): \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ are defined as
\[
\mathcal{M}(x/y) = \max_{i \in [n]} \frac{x_i}{y_i} \quad \text{and} \quad \mathcal{m}(x/y) = \min_{i \in [n]} \frac{x_i}{y_i}.
\]
Indeed, if $G: \mathbb{R}^n_+ \to \mathbb{R}^n_+$ and $(x, \theta) \in \mathbb{R}^n_+ \times \mathbb{R}_+$, then it holds $G(x) = \theta x$ if and only if $\mu(G(x), x) = 0$.

While this metric is useful for the study of positive $(\sigma, \rho)$-eigenvectors when $d = 1$, the case $d > 1$ is more delicate. In fact, it has been shown in [10] that, for $d > 1$, the following weighted product $\mu_b$ of Hilbert metrics on $\mathbb{K}_+^\sigma = \mathbb{R}^n_+ \times \ldots \times \mathbb{R}^n_+$ is a better choice: For $b \in \mathbb{R}^d_+$, let us define $\mu_b: \mathbb{K}_+^\sigma \times \mathbb{K}_+^\sigma \to \mathbb{R}_+$ as
\[
\mu_b(x, y) = \sum_{i=1}^d b_i \mu_i(x_i, y_i) = \ln \left( \prod_{i=1}^d \frac{\mathcal{M}_i(x_i/y_i)^{b_i}}{\mathcal{m}_i(x_i/y_i)^{b_i}} \right),
\]
where $\mu_i$ is the Hilbert projective metric on $\mathbb{R}^n_+$ defined as in (19). While $\mu_b$ is just a rescaling of $\mu$ when $d = 1$, it is shown in [10] that, for any $d \geq 1$, $(x, \theta) \in \mathbb{K}_+^\sigma \times \mathbb{R}_+$ satisfies $F(x) = \theta \otimes x$ if and only if $\mu_b(F(x), x) = 0$. Note that this property holds for any choice of the weights $b \in \mathbb{R}^d_+$. However, there are natural choices of $b$ based on the homogeneity matrix $B$ of $F$. In fact, for any multi-homogeneous order-preserving map $F$ with homogeneity matrix $B$, the spectral radius $\rho(B)$ is an optimal upper-bound on the Lipschitz constant of $F$ with respect to $\mu_b$, obtained by choosing $b$ to be an eigenvector of $B^\top$, c.f. [10].

When $T$ is a nonnegative tensor and $\sigma$ is any shape partition of $T$, the map $F(\sigma, p)$ defined in (8) is order-preserving and multi-homogeneous for any choice of $p$, with homogeneity matrix given by
\[
A = \text{diag}(p_1', \ldots, p_d') = (\nu 1^T - I),
\]
where $\nu = (|\sigma_1|, \ldots, |\sigma_d|)^T$ and $p_i' = p_i/(p_i - 1)$ for $i = 1, \ldots, d$. Note that, for every shape partition $\sigma$ of $T$ and any $p$, the matrix $A$ is nonnegative and irreducible. Therefore there exists a unique positive eigenvector $b$ of $A^\top$ and thus in the following we shall always assume that $b$ is chosen in the following way:
\[
b_i > 0, \quad \forall i \in [d], \quad A^\top b = \rho(A)b, \quad \sum_{i=1}^d b_i = 1.
\]
These observations allow us to use the nonlinear Perron-Frobenius theory for \( F(\sigma, p) \) to analyze the \((\sigma, p)\)-spectrum of the nonnegative tensor \( T \).

The following lemma establishes the correspondence between the fixed points of \( F(\sigma, p) \) in the product of projective spaces and the \((\sigma, p)\)-eigenvectors of \( T \). Further, it also explains how to reconstruct the corresponding eigenvalues.

**Lemma 5.1.** Let \( x \in K_{+}^{d} \), then the following are equivalent:

\( a \) \((x_{1}, \ldots, x_{d}) = (\|x_{1}\|_{p_{1}}, \ldots, \|x_{d}\|_{p_{d}}) \) is a \((\sigma, p)\)-eigenvector of \( T \).

\( b \) There exists \( \theta \in \mathbb{R}_{+}^{d} \) such that \( F_{i}(\sigma, p)(x) = \theta_{i}x_{i} \) for all \( i \in [d] \).

Furthermore, suppose that \( \|x_{i}\|_{p_{i}} = 1 \), then we have the following:

\( c \) If \( F_{i}(\sigma, p)(x) = \theta_{i}x_{i} \) for all \( i \in [d] \), then there exists \( \lambda \in \mathbb{R}_{+} \) such that \( \theta_{i} = \lambda^{p_{i}-1} \) and \((\sigma, x)\) is a \((\sigma, p)\)-eigenpair of \( T \).

\( d \) If \((\lambda, x)\) is a \((\sigma, p)\)-eigenpair of \( T \), then \( F_{i}(\sigma, p)(x) = \lambda^{p_{i}-1}x_{i} \) for all \( i \in [d] \).

**Proof.** Let \( \nu_{i} = |\sigma_{i}| \) for all \( i \in [d] \). First assume that \( b \). Then, we have \( \lambda \geq 0 \) such that for every \( i \in [d] \), it holds

\[
\lambda\|x_{i}\|_{p_{i}}^{-\nu_{i}}\psi_{p_{i}}(x_{i}) = \lambda\psi_{p_{i}}(x_{i}) = T_{i}(x_{i}) = \left(\|x_{i}\|_{p_{i}}^{-\nu_{i}}\prod_{j=1}^{d}\|x_{j}\|_{p_{j}}^{-\nu_{j}}\right)T_{i}(x_{i}).
\]

By rearranging the above equation and composing it by \( \psi_{p_{i}} \), we get

\[
F_{i}(\sigma, p)(x) = \psi_{p_{i}}(T_{i}(x_{i})) = \left(\lambda\|x_{i}\|_{p_{i}}^{-\nu_{i}}\prod_{j=1}^{d}\|x_{j}\|_{p_{j}}^{-\nu_{j}}\right)^{p_{i}-1}x_{i}
\]

and thus (a) implies (b). In particular, note that if \( \|x_{i}\|_{p_{i}} = 1 \) for all \( i \in [d] \), then (d) follows from the above equation.

Now suppose that there exists \( \theta \in \mathbb{R}_{+}^{d} \) such that \( F_{i}(\sigma, p)(x) = \theta_{i}x_{i} \) for all \( i \) and set \( \tilde{x} = \left(\|x_{1}\|_{p_{1}}^{-\nu_{1}}, \ldots, \|x_{d}\|_{p_{d}}^{-\nu_{d}}\right) \). Then, we have \( F_{i}(\sigma, p)(\tilde{x}) = \tilde{\theta}_{i}\tilde{x}_{i} \) with \( \tilde{\theta}_{i} = \theta_{i}(|\tilde{x}_{i}|_{p_{i}}^{-\nu_{i}}\prod_{j=1}^{d}\|\tilde{x}_{j}\|_{p_{j}}^{-\nu_{j}})^{p_{i}-1} \) for all \( i \in [d] \). Hence, we get

\[
T_{i}(\tilde{x}_{i}) = \psi_{p_{i}}(F_{i}(\sigma, p)(\tilde{x})) = \tilde{\theta}_{i}^{-1}\psi_{p_{i}}(\tilde{x}_{i}), \quad \forall i \in [d].
\]

So the last thing we need to prove is that there exists \( \lambda \geq 0 \) such that \( \tilde{\theta}_{i}^{-1} = \lambda \) for all \( i \in [d] \). This follows form the fact that \( T_{i}(\tilde{x}_{i}) = \tilde{\theta}_{i}^{-1}\psi_{p_{i}}(\tilde{x}_{i}) \) as we have

\[
f_{T}(x_{i}) = \left(\tilde{x}_{i}, T_{i}(x_{i})\right) = \tilde{\theta}_{i}^{-1}(\tilde{x}_{i}, \psi_{p_{i}}(\tilde{x}_{i})) = \tilde{\theta}_{i}^{-1}\|\tilde{x}_{i}\|_{p_{i}}^{-\nu_{i}}\psi_{p_{i}}(\tilde{x}_{i}) = \tilde{\theta}_{i}^{-1}\|\tilde{x}_{i}\|_{p_{i}}^{-\nu_{i}}\psi_{p_{i}}(\tilde{x}_{i})
\]

Finally, if \( \|x_{i}\|_{p_{i}} = 1 \) for all \( i \), then \( \tilde{\theta}_{i} = \theta_{i} = f_{T}(\tilde{x}_{i})^{p_{i}-1} \) for all \( i \) which proves (c).}

We explain the connection between the spectral radius of the order-preserving multi-homogeneous mapping \( F(\sigma, p) \) and the \((\sigma, p)\)-spectral radius of the tensor \( T \). To this end, let us denote by \( S_{+}^{\sigma, p} \) the product of \( p_{i} \)-spheres in \( K_{+}^{d} \), i.e., \( S_{+}^{\sigma, p} = \{ x \in K_{+}^{d} \mid \|x\|_{p_{i}} = 1, \ i \in [d] \} \). Following [10], Section 4, the spectral radius of an order-preserving multi-homogeneous mapping \( F : K_{+}^{d} \to K_{+}^{d} \) is defined as

\[
r_{b}(F) = \sup \left\{ \prod_{i=1}^{d} \theta_{i}^{p_{i}} \mid F(x) = \theta \otimes x \text{ for some } x \in S_{+}^{\sigma, p} \right\}.
\]
We relate \( r_b(F(\sigma,p))(T) \) and \( r(\sigma,p)(T) \) in the following:

**Lemma 5.2.** Suppose that \( b \in \mathbb{R}_{++}^d \) satisfies \( \sum_{i=1}^d b_i = 1 \) and there exists a \((\sigma,p)\)-eigenpair \((\lambda,x) \in \mathbb{R}_+ \times \mathbb{S}_+^{(\sigma,p)}\) of \( T \) such that \( \lambda = r(\sigma,p)(T) \). Furthermore, suppose that there exists \((\theta,y) \in \mathbb{R}_+^d \times \mathbb{S}_+^{(\sigma,p)}\) such that \( F(y) = \theta \otimes y \) and \( \prod_{i=1}^d \theta_i = r_b(F(\sigma,p)) \), then

\[
(23) \quad r(\sigma,p)(T) = r_b(F(\sigma,p))^{\gamma-1}, \quad \text{where} \quad \gamma = \frac{\sum_{i=1}^d b_i \lambda_i}{\sum_{i=1}^d b_i \lambda_i - 1} \in (1, \infty).
\]

**Proof.** Let \( \gamma' = \sum_{i=1}^d b_i \lambda_i \), then \( \gamma' \geq \min_{j \in [d]} p'_j \sum_{i=1}^d b_i = \min_{j \in [d]} \sigma_j > 1 \), thus \( \gamma = \frac{\gamma'}{\sum_{i=1}^d b_i \lambda_i} \in (1, \infty) \) and \((\gamma - 1)(\gamma' - 1) = 1\). Now, Lemma 5.1, (d) implies that \( F(\sigma,p)(x) = \lambda \otimes x \) with \( \lambda_i = \lambda \sigma_i^{-1} \), hence we have

\[
(24) \quad r(\sigma,p)(T)^{\gamma-1} = \lambda^{\gamma-1} = \prod_{i=1}^d \lambda_i^{\sigma_i^{-1}} = \prod_{i=1}^d \lambda_i^{b_i} \leq r_b(F(\sigma,p)).
\]

On the other hand, by Lemma 5.1, (c) we know that there exists \( \theta \in \mathbb{R}_+ \) such that \( \theta_i = \theta \sigma_i^{-1} \) for all \( i \in [d] \) and \( \theta \) is a \((\sigma,p)\)-eigenvalue of \( T \). Hence, we have

\[
(25) \quad r_b(F(\sigma,p)) = \prod_{i=1}^d \theta_i^{b_i} = \prod_{i=1}^d \theta_i^{b_i(\sigma_i^{-1})} = \theta^{\gamma - 1} \leq r(\sigma,p)(T)^{\gamma-1}.
\]

Let \( F: \mathcal{K}_+^{\sigma} \rightarrow \mathcal{K}_+^{\sigma} \) be a multi-homogeneous mapping. The Perron-Frobenius theorem discussed in [10] has mainly three types of irreducibility assumptions for \( F \). Before concluding this section, let us briefly comment on each of them. Then, by considering the particular map \( F = F(\sigma,p) \), in the next Section 6 we will recast these assumptions in terms of the entries of the tensor \( T \).

The first one is that \( F: \mathcal{K}_+^{\sigma} \rightarrow \mathcal{K}_+^{\sigma} \) to have to satisfy \( F(\mathcal{K}_+^{\sigma}) \subset \mathcal{K}_+^{\sigma} \). This assumption guarantees that the distance \( \mu_b(F(x),F(y)) \) is always well defined. We will see in the next section that for \( F = F(\sigma,p) \), this assumption is equivalent to requiring \( T \) to be \( \sigma \)-nonnegative.

The second type of assumptions are on the Jacobian matrix \( DF(u) \) of \( F \), evaluated at its positive eigenvector \( u \). This is important because, if \( F(u) = \theta \otimes u \) for some \( u \in \mathcal{K}_+^{\sigma} \), then the irreducibility of the matrix \( DF(u) \) ensures that \( u \) is unique and, if in addition \( DF(u) \) is primitive, then the normalized iterates of \( F \) will converge to \( u \) (see e.g. Theorems 5.2 and 6.2 [10]).

Finally, if \( G(F) \), the graph of \( F \), is strongly connected, then Theorem 4.3 [10] ensures the existence of a positive eigenvector. The definition of \( G(F) \) can be found in Definition 4.2 [10] and is recalled here for the sake of completeness:

**Definition 5.3** (Graph of a multi-homogeneous mapping). The graph \( G(F) \) of an order-preserving multi-homogeneous mapping \( F: \mathcal{K}_+^{\sigma} \rightarrow \mathcal{K}_+^{\sigma} \) is the pair \( G(F) = (\mathcal{I}^{\sigma}, \mathcal{E}(F)) \), where \( \mathcal{I}^{\sigma} \) is the set of nodes and an edge \((i,j),(k,l)\) \( \in \mathcal{E}(F) \) exists if and only if \( \lim_{n \to \infty} F_{k,l}(e^{(k,l)}(z)) = \infty \), where \( e^{(k,l)}: \mathbb{R}_+ \rightarrow \mathcal{K}_+^{\sigma} \) is defined as \( e^{(k,l)}(z))(k,l) = z \) and \( (e^{(k,l)}(z))(\eta,j) = 1 \) for all \( (\eta,j) \in \mathcal{I}^{\sigma} \setminus \{(k,l)\} \).

6. **Classes of nonnegative tensors.** We discuss here the different classes of nonnegative tensors given in Definition 2.4. We propose characterizations in terms of graphs for each of them and explain how they relate to the irreducibility assumptions.
on $F^{(\sigma,p)}$ discussed in the previous section. To this end, we first introduce the $\sigma$-graph of a nonnegative tensor $T$ and discuss some of its properties. Then, we consider each class separately, characterize them and discuss how they relate for different shape partitions of a fixed tensor. Finally, we discuss a hierarchy between these classes and give a proof of Theorem 3.1.

6.1. $\sigma$-graphs of nonnegative tensors. We propose the a definition of graph associated to a nonnegative tensor and with respect to one of its shape partition. We call this graph the $\sigma$-graph of $T$ and denote it $G^\sigma(T)$. Simply put, the set of nodes of $G^\sigma(T)$ is $I^\sigma$ and there is an edge from $(k,l_k)$ to $(i,t_i)$ if the variable $x_{i,j}$ effectively appear in the expression of $T_{k,l_k}(x^{(\sigma)})$. Formally, we have the following:

Definition 6.1 ($\sigma$-graph of a nonnegative tensor). Let $\sigma = \{\sigma_i\}_{i=1}^d$ be a shape partition of $T \in \mathbb{R}_{+}^{n_1 \times \ldots \times n_m}$. The $\sigma$-graph of $T$ is the directed graph $G^\sigma(T) = (I^\sigma,E^\sigma(T))$ defined as follows: The set of nodes is $I^\sigma = \cup_{i=1}^d \{i\} \times [n_i]$ and there is an edge $((k,l_k),(i,t_i)) \in E^\sigma(T) \subset I^\sigma \times I^\sigma$ if one of the following condition holds:

- $(k,l_k) \neq (i,t_i)$ and there exists $j_1,\ldots,j_m$ such that $T_{j_1,\ldots,j_m} > 0$, $j_{s_k} = l_k$ and $t_i \in \{j_a : a \in \sigma_i\}$.
- $(k,l_k) = (i,t_i)$ and there exists $j_1,\ldots,j_m$ such that $T_{j_1,\ldots,j_m} > 0$, $j_{s_k} = l_k$ and $t_i \in \{j_a : a \in \sigma_i \setminus \{s_i\}\}$.

Note that in the cases $d = 1,m$, $G^\sigma(T)$ coincide with the graphs associated to $T$ introduced in Sections 4 and 1 of [8] respectively. Furthermore, when $d = 2$, $G^\sigma(T)$ coincide with the graph associated to $T$ introduced in Section 4 of [16]. In particular, if $T \in \mathbb{R}^{n \times n}$ is a square matrix, then the shape partitions of $T$ are $\sigma = \{1,2\}$ and $\bar{\sigma} = \{1\}$, $\{2\}$. Furthermore, $G^\sigma(T)$ is the graph with $n$ nodes and adjacency matrix $T$, whereas $G^\sigma(T)$ is the bipartite graph with $n + n$ nodes and biadjacency matrix $T$. Next we illustrate the three graphs induced by a square tensor of order 3.

Example 6.2. Let $T \in \mathbb{R}^{3 \times 3 \times 3}$ be defined as

$$T_{2,2,1} = T_{3,2,1} = T_{1,3,1} = T_{2,2,2} = T_{2,1,3} = 1 \quad \text{and} \quad T_{i,j,k} = 0 \quad \text{otherwise.}$$

Furthermore, let $\sigma^1, \sigma^2, \sigma^3$ be the shape partitions of $T$, namely:

$$\sigma^1 = \{1,2,3\}, \quad \sigma^2 = \{1\}, \{2,3\} \quad \text{and} \quad \sigma^3 = \{1\}, \{2\}, \{3\}.$$  

Then, $T$ induces the following graphs:

The following lemma shows that $G^\sigma(T) = G(F^{(\sigma,p)})$ and that for every $x \in K^\sigma_{+}$, $DF^{(\sigma,p)}(x)$ is an adjacency matrix for these graphs.

Lemma 6.3. Let $G^\sigma(T) = (I^\sigma,E^\sigma(T))$ be the $\sigma$-graph of $T$ and $G(F^{(\sigma,p)}) = (I^\sigma,E(F^{\sigma,p}))$ be the graph of $F^{(\sigma,p)}$ as multi-homogeneous mapping. Then, for every $(k,l_k),(i,t_i) \in I^\sigma$, the following are equivalent:
(i) \((k,l_k),(i,t_i)\) \(\in \mathcal{E}^\sigma(T)\).

(ii) For all \(x \in K_{++}\), it holds \(\frac{\partial}{\partial x_{i,t_i}} F_{k,l_k}^{(\sigma,p)}(x) > 0\).

(iii) \((k,l_k),(i,t_i)\) \(\in \mathcal{E}(F^{(\sigma,p)})\).

Proof. (i) \(\Rightarrow\) (ii): If \((k,l_k),(i,t_i)\) \(\in \mathcal{E}^\sigma(T)\), there exist indexes \(j_1, \ldots, j_m\) such that \(T_{j_1,\ldots,j_m} > 0\), \(j_{s_k} = l_k\) and \(t_i \in \{j_{a} | a \in \sigma_i\}\). Hence, for \(x \in K_{++}\), we have

\[
\frac{\partial}{\partial x_{i,t_i}} F_{k,l_k}^{(\sigma,p)}(x) \geq \left( T_{j_1,\ldots,j_m} \left( \prod_{t \in \sigma_i, t \neq s_i} x_{i,j_t} \right) \prod_{t \neq i} x_{i,j_t} \right)^{p_k-1} > 0.
\]

(ii) \(\Rightarrow\) (iii): First of all, note that as \(F^{(\sigma,p)}\) is order-preserving, by Theorem 1.3.1 in [14], we know that the Jacobian matrix of \(F^{(\sigma,p)}\) at \(x \in K_{++}\) is nonnegative. Now, for \(z > 1\), we have

\[
\frac{\partial}{\partial x_{i,t_i}} F_{k,l_k}^{(\sigma,p)}(e^{(i,t_i)}(z)) \geq \frac{\partial}{\partial x_{i,t_i}} F_{k,l_k}^{(\sigma,p)}(e^{(i,t_i)}(1)) > 0.
\]

Lemma 2.5 in [10] implies that \(A_{k,i} > 0\) as \(\frac{\partial}{\partial x_{i,t_i}} F_{k,l_k}^{(\sigma,p)}(x) > 0\) for \(x \in \mathcal{K}_{++}\) and

\[
A_{k,i} F_{k,l_k}^{(\sigma,p)}(e^{(i,t_i)}(z)) = \sum_{j_1=1}^{n_1} \left( e^{(i,t_i)}(z) \right)_{i,j_1} \frac{\partial}{\partial x_{i,t_i}} F_{k,l_k}^{(\sigma,p)}(e^{(i,t_i)}(z)) \geq \left( e^{(i,t_i)}(z) \right)_{i,t_i} \frac{\partial}{\partial x_{i,t_i}} F_{k,l_k}^{(\sigma,p)}(e^{(i,t_i)}(z)) \geq z \frac{\partial}{\partial x_{i,t_i}} F_{k,l_k}^{(\sigma,p)}(e^{(i,t_i)}(1)).
\]

By taking the limit as \(z \to \infty\) we obtain the desired result.

(iii) \(\Rightarrow\) (i): We prove that if (i) does not hold, then (iii) does not hold either. Indeed, if \((k,l_k),(i,t_i)\) \(\in \mathcal{E}(F^{(\sigma,p)})\), then by construction of \(e^{(i,t_i)}(z)\), with \(j_{s_k} = l_k\) we have

\[
F_{k,l_k}^{(\sigma,p)}(e^{(i,t_i)}(z)) = \left( \sum_{j=1}^{m} \sum_{j \neq s_k} T_{j_1,\ldots,j_m} \right)^{p_k-1}.
\]

As this expression is bounded and constant in \(z\), (iii) cannot hold. 

\(\square\)

6.2. \(\sigma\)-strict nonnegativity. Note that the Hilbert projective metric in (19) is defined for strictly positive vectors and thus we have to assume \(F^{(\sigma,p)}(x) \in \mathcal{K}_{++}\) for every \(x \in \mathcal{K}_{++}\) in order to exploit its properties. This assumption motivates the definition of \(\sigma\)-strict nonnegativity which is discussed here. Before showing that these conditions are equivalent, we wish to stress that \(\sigma\)-strict nonnegativity is a very mild condition as it still allows \(T\) to be very sparse. This is illustrated in the following:

Example 6.4. Let \(T \in \mathbb{R}^{n \times n \times \cdots \times n}\) be an \(m\)-th order tensor so that \(T_{j_1,\ldots,j_m} > 0\) if and only if \(j_1 = \ldots = j_m\). Then, for any shape partition \(\sigma = \{\sigma_i\}_{i=1}^d\) of \(T\), \(F^{(\sigma,p)}\) satisfies \(F^{(\sigma,p)}(x) \in \mathcal{K}_{++}\) for every \(x \in \mathcal{K}_{++}\). Note that this tensor has \(n\) positive entries and \(n^m - n\) zero entries.

The following lemma characterizes the \(\sigma\)-strict nonnegativity assumption.

Lemma 6.5. The followings are equivalent:

(i) \(T\) is \(\sigma\)-strictly nonnegative.

(ii) \(\mathcal{I}^{(\sigma,p)}(x) \in \mathcal{K}_{++}\) for every \(x \in \mathcal{K}_{++}\).

(iii) \(F^{(\sigma,p)}(1) \in \mathcal{K}_{++}\).

(iv) For every \((i,l_i)\) \(\in \mathcal{I}\), there exists \(j_1,\ldots,j_m\) with \(T_{j_1,\ldots,j_m} > 0\) and \(j_{s_k} = i_l\).

Proof. (i) \(\Rightarrow\) (ii): Let \((i,l_i) \in \mathcal{I}\) and \(x \in \mathcal{K}_{++}\), we show that \(F_{i,l_i}^{(\sigma,p)} > 0\). As \(T\) is \(\sigma\)-strictly nonnegative, there exists \((k,j_k) \in \mathcal{I}\) such that the matrix \(M\) of Definition 2.4 satisfies \(M_{(i,l_i),k,j_k} > 0\). Lemma 6.3 then implies \(\frac{\partial}{\partial x_{k,j_k}} T_{i,l_i}(x^{(\sigma)}) > 0\) and so

\[
F_{i,l_i}^{(\sigma,p)}(x) = \left( T_{i,l_i}(x^{(\sigma)}) \right)^{p_i-1} = \left( \frac{1}{p_k} \sum_{k=1}^{n_k} \frac{\partial}{\partial x_{k,l_i}} T_{i,l_i}(x^{(\sigma)}) x_{k,j_k} \right)^{p_i-1} > 0,
\]
where we have used Euler’s theorem for homogeneous functions in the second equality. 

(ii) ⇒ (iii) is obvious. (iii) ⇒ (iv): Let \((i, t_i) \in I^\sigma\), then \(0 < F^{(\sigma, p)}(1)\). The claim follows from \(F^{(\sigma, p)}(1) = (\sum_{i=1}^{\#I_{t_i}} \sum_{j_t \neq j_{t_i}} T_{j_t \ldots j_{t_i}})^{\eta_i - 1}\), where \(j_{s_i} = t_i\). (iv) ⇒ (i): Let \((i, t_i) \in I^\sigma\). There exists \(j_1, \ldots, j_m\) such that \(T_{j_1 \ldots j_m} > 0\) and \(j_{s_i} = t_i\). Then Lemma 6.3 implies that for any \(k\), it holds \((i, t_i, (k, j_{s_i})) \in E^\sigma(T)\) where \(t_k = s_k\) if \(d > 1\) and \(t_k = s_k + 1\) otherwise. Hence, \(M_{(i,t_i),(k,t_k)} > 0\) and the proof is done. 

As a direct consequence of Lemma 6.5 (iv), we have that the condition \(F^{(\sigma, p)}(1) \in K_+^\sigma\) holds independently of the choice of \(p\) and this condition is inherited by larger partitions in the partial ordering of shape partitions.

**Lemma 6.6.** Let \(\sigma = \{\sigma_i\}_{i=1}^d\), \(\tilde{\sigma} = \{\tilde{\sigma}_i\}_{i=1}^d\) be shape partitions of \(T\) such that \(\sigma \subseteq \tilde{\sigma}\). If \(T\) is \(\sigma\)-strictly nonnegative, then it is \(\tilde{\sigma}\)-strictly nonnegative.

However, if \(\sigma \not\subseteq \tilde{\sigma}\), it can be that \(F^{(\sigma, p)}(1) \in K_+^\sigma\) and \(F^{(\sigma', p)}(1) \not\in K_+^\sigma\).

**6.3. \(\sigma\)-weak irreducibility.** Lemmas 6.3 and 6.5 imply that if \(T\) is \(\sigma\)-weakly irreducible, then \(T\) is \(\sigma\)-strictly nonnegative. Furthermore, Lemma 6.3 implies that \(T\) is \(\sigma\)-weakly irreducible if and only if \(G^\sigma(T)\) is strongly connected. Now, we show that when \(T\) is partially symmetric with respect to \(\sigma\), \(G^\sigma(T)\) is undirected and a result analogous to Lemma 6.6 can be derived for \(\sigma\)-weak irreducibility.

**Lemma 6.7.** Suppose that \(T\) is \(\sigma\)-symmetric. Then \(G^\sigma(T)\) is undirected. Furthermore, if \(\tilde{\sigma}\) is a shape partition of \(T\) such that \(\sigma \subseteq \tilde{\sigma}\) and \(T\) is \(\sigma\)-weakly irreducible, then \(T\) is \(\tilde{\sigma}\)-weakly irreducible.

**Proof.** Let \(\sigma = \{\sigma_i\}_{i=1}^d\) and let \((k, l_k), (i, t_i) \in I^\sigma\) be such that \(((k, l_k), (i, t_i)) \in E^\sigma(T)\). If \((k, l_k) = (i, t_i)\) then clearly \(((i, t_i), (k, l_k)) \in E^\sigma(T)\). If \((k, l_k) \neq (i, t_i)\), there exists \(j_1, \ldots, j_m\) with \(T_{j_1 \ldots j_m} > 0\) and \(t_i \in \{j_a \mid a \in \sigma_i\}\). Let \(s_i' \in \sigma_i\) be such that \(t_i = j_{s_i'}\). As \(T\) is partially symmetric with respect to \(\sigma_i\), we have \(T_{j_k'} \ldots j_m' = T_{j_1 \ldots j_m} > 0\) where \(j_k' = j_{s_i'}, j_k' = j_{k'}, j_k' = j_{k'}\) and \(j_k' = j_{k'}\) otherwise. In particular, this implies that \(((i, t_i), (k, l_k)) \in E^\sigma(T)\) and thus \(G^\sigma(T)\) is undirected.

Now, assume that \(T\) is \(\sigma\)-weakly irreducible and let \(\tilde{\sigma} = \{\tilde{\sigma}_i\}_{i=1}^d\) be a shape partition of \(T\) such that \(\sigma \subseteq \tilde{\sigma}\). Let \(\emptyset \neq V^1, V^2 \subset I^\sigma\) be such that \(V^1 \cup V^2 = \emptyset\) and \(V^1 \cap V^2 = I^\sigma\).

We show that there is an edge between \(V^1\) and \(V^2\) in order to conclude that \(T\) is \(\tilde{\sigma}\)-weakly irreducible. As \(\sigma \subseteq \tilde{\sigma}\), there exists \(\pi: [d] \to [d]\) such that \(\sigma_i \subset \tilde{\sigma}_i\) for all \(i \in [d]\). For \(k = 1, 2, i \in [d]\) and \(j \in [d]\) let \(V_{k}^i = \{t_i \mid (i, t_i) \in V^k\}\) and \(V_{j}^k = V_{\pi(j)}^k\). Furthermore, set \(V_{k}^i = \bigcup_{j=1}^{d} \{j\} \times V_{j}^k\) for \(k = 1, 2\). Then \(V^1, V^2\) forms a partitioning of \(I^\sigma\) into nonempty disjoints subsets. As \(G^\sigma(T)\) is strongly connected, there exists \((k, l_k) \in V^1\) and \((i, t_i) \in V^2\) such that \(((k, l_k), (i, t_i)) \in E^\sigma(T)\). We claim that \(((\pi(k), l_k), (\pi(i), t_i)) \in E^\sigma(T)\), as \((\pi(k), l_k) \in V^1\) and \((\pi(i), t_i) \in V^2\), this will conclude the proof. Let \(s_i = \min\{a \mid a \in \tilde{\sigma}_i\}\) for \(i \in [d]\). There exists \(j_1, \ldots, j_m\) such that \(T_{j_1 \ldots j_m} > 0\), \(j_{s_i} = k\) and either \((k, l_k) = (i, t_i)\) and \(t_i \in \{j_a \mid a \in \sigma_i \setminus \{s_i\}\}\) or \((k, l_k) \neq (i, t_i)\) and \(t_i \in \{j_a \mid a \in \sigma_i\}\). In either cases, one can use the partial symmetry of \(T\) and rearrange the \(j_1, \ldots, j_m\) into \(j_1', \ldots, j_m'\) so that \(T_{j_1' \ldots j_m'} > 0\), \(j_{s_{s(i)}} = j_{s(i)}\), \(j_{a'} = j_{s_{s(i)}}\) and \(j_{a'} = j_{a}\) for all \(a \in [m]\) \(\setminus \{s_{s(i)}\}\). In particular this implies our claim and the proof is done.

Note that the partial symmetry assumption in Lemma 6.7 cannot be omitted. For instance, the tensor of Example 6.2 is \(\sigma^1\)-weakly irreducible for \(i = 1, 3\) but not \(\sigma^2\)-weakly irreducible where \(\sigma^1, \sigma^2, \sigma^3\) are defined as in (26). In facts, already in the subset \(\{0, 1\}^{3 \times 3 \times 3}\) of third order tensors, any combination can happen, that is for any \(\Omega \subset [3]\), there is a tensor which is \(\sigma^1\)-weakly irreducible for \(i \in \Omega\) and not \(\sigma^2\)-weakly irreducible for \(i \in \Omega\).
irreducible for \( i \in [3] \setminus \Omega \). We prove the latter statement in the following remark where we exhibit such tensors for all \( \Omega \subset [3] \). As \([3]\) has 8 different subsets \( \Omega \), for the sake of brevity, we simply list all entries of these tensors in the reverse lexicographic order as a binary string of length 27. So, for instance, the tensor \( T \) of Example 6.2 can be compactly described as

\[
\begin{array}{cccccccccccccccccccccc}
\tau_{1,1,1} & \tau_{2,1,1} & \tau_{3,1,1} & \tau_{1,2,1} & \tau_{2,2,1} & \tau_{3,2,1} & \tau_{1,3,1} & \ldots & \tau_{1,1,3} & \ldots & \tau_{3,3,3} \\
\hline
0 & 0 & 0 & 0 & 1 & 1 & & & & & 100000001000000000000000000
\end{array}
\]  

(27)

Remark 6.8. Let \( \sigma^1, \sigma^2, \sigma^3 \) be as in (26). For every \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{0, 1\} \) there exists a tensor \( T(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{0, 1\}^{3 \times 3 \times 3} \) such that for \( i = 1, 2, 3, \) \( T(\varepsilon_1, \varepsilon_2, \varepsilon_3) \) is \( \sigma^i \)-weakly irreducible if \( \varepsilon_i = 1 \) and \( T(\varepsilon_1, \varepsilon_2, \varepsilon_3) \) is not \( \sigma^i \)-weakly irreducible if \( \varepsilon_i = 0 \). Precisely, with the notation introduced in (27) we have

\[
\begin{aligned}
T(0,0,0) &= 000000000000000000000000000000000000000000, \\
T(0,1,0) &= 011100100000000000000000000000000000000000, \\
T(1,0,0) &= 011101000000000000000000000000000000000000, \\
T(1,1,0) &= 011101000000000000000000000000000000000000, \\
T(0,0,1) &= 001001000000000000000000000000000000000000, \\
T(0,1,1) &= 001001000000000000000000000000000000000000, \\
T(1,0,1) &= 001001000000000000000000000000000000000000, \\
T(1,1,1) &= 011101001000000000000000000000000000000000.
\end{aligned}
\]

Note that all the tensors given above are not \( \sigma^i \)-strongly irreducible, for \( i = 1, 2, 3 \).

6.4. \( \sigma \)-strong irreducibility. Now, we characterize \( \sigma \)-strong irreducibility and discuss its implications on \( F(\sigma, p) \). In particular, we prove that \( T \) is \( \sigma \)-strongly irreducible if and only if for every \( x^0 \in K_{\sigma^0}^\sigma \), there exists an integer \( N = N(x) \) such that \( x^N \in K_{\sigma^+}^\sigma \) where \( x^{k+1} = x^k + F(\sigma, p)(x^k) \) for \( k = 0, 1, \ldots \). This property implies that every nonnegative \( (\sigma, p) \)-eigenvector of \( T \) is strictly positive. Indeed, we show that if \( T \) is \( \sigma \)-strongly irreducible, then \( F(\sigma, p) \) satisfies the assumptions of the following lemma:

**Lemma 6.9.** Suppose that for every \( x^0 \in K_{\sigma^0}^\sigma \setminus K_{\sigma^+}^\sigma \), there exists \( N \) such that \( x^N \in K_{\sigma^+}^\sigma \), where \( x^{k+1} = x^k + F(\sigma, p)(x^k) \) for \( k = 0, 1, \ldots \). Then, for every \( (\theta, x) \in \mathbb{R}_+^d \times K_{\sigma^0}^\sigma \), such that \( F(\sigma, p)(x) = \theta \otimes x \), we have \( x \in K_{\sigma^+}^\sigma \).

**Proof.** If \( x \in K_{\sigma^+}^\sigma \), there is nothing to prove so let us assume that \( x \notin K_{\sigma^+}^\sigma \). Set \( x^0 = x \) and let \( N \) be such that \( x^N \in K_{\sigma^+}^\sigma \). Note that for every \( k \) we have \( x^k = \delta^{(k)} \otimes x \) where \( \delta^{(k)} \in \mathbb{R}_+^d \) is given by \( \delta^{(0)} = 1^T \) and \( \delta^{(j+1)} = \delta^{(j)} + (\delta^{(j)})^A \otimes \theta \) for all \( j = 0, 1, \ldots \). In particular \( \delta^{(N)} \in \mathbb{R}_+^d \) and so \( \delta^{(N)} \otimes x = x^N \in K_{\sigma^+}^\sigma \) implies that \( x \notin K_{\sigma^+}^\sigma \), which concludes the proof.

We give equivalent characterizations of \( \sigma \)-strong irreducibility:

**Lemma 6.10.** The following statements are equivalent:

(i) \( T \) is \( \sigma \)-strongly irreducible.

(ii) For every \( \theta \neq V \subset I^\sigma \) such that \( V = \{ l_i \in [n_i] \mid (i, l_i) \in V \} \neq [n_i] \) for all \( i \in [d] \), the following holds: There exists \( k \in [d] \) and \( j_1, \ldots, j_m \) such that \( T_{j_1, \ldots, j_m} > 0 \), \( j_s \in V_k \), \( j_t \in [n_k] \setminus V_k \), \( t \in \sigma_k \setminus \{ s \} \) and \( j_s \in [n_k] \setminus V_k \), \( t \in [n_k] \setminus V_k \), \( s \in I_k \), \( i \in [d] \setminus \{ k \} \).

(iii) There exists \( N \leq n_1 + \ldots + n_d - d \) such that for all \( j = (j_1, \ldots, j_d) \in [n_1] \times \ldots \times [n_d] \), it holds \( e_j^N \in K_{\sigma^+}^\sigma \), where \( e_j^{k+1} = e_j^k + T(e_j^k, [\sigma^1]) \) and \( e_j^0 \) \( k, l_k = 1 \) if \( l_k = j_k, e_j^0 \) \( k, l_k = 0 \) else.

(iv) For every \( x^0 \in K_{\sigma^0}^\sigma \), there exists a positive integer \( N_x \) such that \( x^{N_x} \in K_{\sigma^+}^\sigma \), where \( x^{k+1} = x^k + F(\sigma, p)(x^k) \) and \( k = 0, 1, 2 \ldots \)

(v) \( Q(F(x, \sigma, p)(x)) \ni Q(x) \) for every \( z \in K_{\sigma^0}^\sigma \setminus K_{\sigma^+}^\sigma \), where, for every \( x \in K_{\sigma^0}^\sigma \), \( Q(x) = \{ (i, j) \in I^\sigma \mid x_{ij} = 0 \} \).
Proof. Note that the equivalence (i) \(\Leftrightarrow\) (v) is direct. We show the other implications by a circular argument, i.e. (ii) \(\Rightarrow\) \(\Rightarrow\) (v) \(\Rightarrow\) (ii).

(ii) \(\Rightarrow\) (iii): Let \(z \in K_{+,0}^+ \setminus K_{+}^\sigma\) and let \(Q(z)\) be defined as in (v). Let \(V = Q(z)\), then \(V \neq \emptyset\) and \(V_i = \{j_i \mid z_{i,j_i} = 0\} \neq [n_i]\) for all \(i \in [d]\). Now, there exists \(k \in [d]\) and \(j_1, \ldots, j_m\) such that \(T_{j_1, \ldots, j_m} > 0, J_k \in V_k, j_i \in [n_k] \setminus V_k, t \in \sigma_k \setminus \{s_k\}\) and \(j_t \in [n_i] \setminus V_t, t \in \sigma, i \in [d] \setminus \{k\}\). It follows that

\[
T_{k,j_k}(z^{[\sigma]}) \geq T_{j_1, \ldots, j_m} \left( \prod_{t \in \sigma_k, t \neq s_k} z_{k,j_t} \right) \prod_{i=1, i \neq k}^{d} \prod_{t \in \sigma_i} z_{i,j_t} > 0,
\]

and so \((k, j_k) \in Q(z) \setminus Q(T(z^{[\sigma]}))\). This shows that \(|Q(z)| > |Q(z + T(z^{[\sigma]}))|\) for all \(z \in K_{+,0}^+ \setminus K_{+}^\sigma\). It follows that for all \(j \in [n_1] \times \cdots \times [n_d]\), we have \(|Q(e_j^k)| > |Q(e_j^{k+1})|\) for all \(k\) such that \(|Q(e_j^k)| > 0\). Finally, note that if \(e_j^k \in K_{+,0}^+\), then \(e_j^{k+1} \in K_{+,0}^+\) for all \(l \geq 0\) and \(|Q(e_j^l)| \leq n_1 + \cdots + n_d - d\), so that \(|Q(e_j^{n+1})| = 0\) for all \(j \in [n_1] \times \cdots \times [n_d]\) which concludes this part.

(iii) \(\Rightarrow\) (iv): Let \(x^0 \in K_{+,0}^+ \setminus K_{+}^\sigma\), then there exists \(j \in [n_1] \times \cdots \times [n_d]\) and \(\delta^{(0)} \in \mathbb{R}_{++}^d\) such that \(\delta^{(0)} \otimes e_j^0 \leq x^0\). We prove by induction that for every \(k\) there exists \(\delta^{(k)} \in \mathbb{R}_{++}^d\) such that \(\delta^{(k)} \otimes e_j^k \leq x^k\). The case \(k = 0\) is discussed above, so suppose it is true for \(k \geq 0\) and let \(\delta^{(k)} \in \mathbb{R}_{++}^d\) be such that \(\delta^{(k)} \otimes e_j^k \leq x^k\). Set

\[
\alpha_i = \min \left\{ \left( T_{i,j_i}, ((e_j^k)^{[\sigma]}) \right)^{p_i-2} \mid j_i \in [n_i] \text{ and } T_{i,j_i}, ((e_j^k)^{[\sigma]}) > 0 \right\} \quad \forall i \in [d],
\]

and let \(\delta^{(k+1)} = \min \{\delta^{(k)}, ((\delta^{(k)})^A) \cdot \alpha_i \} > 0\) for \(i \in [d]\). Then, as \(F^{(\sigma,p)}\) is order-preserving, we have

\[
\delta^{(k+1)} \otimes e_j^{k+1} \leq K \delta^{(k)} \otimes e_j^k + (\alpha \circ (\delta^{(k)})^A) \otimes T((e_j^k)^{[\sigma]})
\]

\[
\leq K \delta^{(k)} \otimes e_j^k + F^{(\sigma,p)}(\delta^{(k)} \otimes e_j^k) \leq K x^k + F^{(\sigma,p)}(x^k) = x^{k+1}.
\]

This concludes our induction proof. In particular, we have \(0 < K \delta^{(N)} \otimes e_j^N \leq x^N\) for all \(N \geq n_1 + \cdots + n_d\) which shows the claim.

(iv) \(\Rightarrow\) (v): We show that if (v) does not hold, then (iv) does not hold either. Note that for \(x, y \in K_{+}^\sigma\), if \(Q(x) = Q(y)\), then there exists \(\alpha, \beta \in \mathbb{R}_{++}^d\) such that \(\alpha \otimes y = K x \leq K \beta \otimes y\) which implies that \(Q(F^{(\sigma,p)}(x)) = Q(F^{(\sigma,p)}(y))\) as we then have \(\alpha^A \otimes F^{(\sigma,p)}(y) \leq K F^{(\sigma,p)}(x) \leq K \beta^A \otimes F^{(\sigma,p)}(y)\). Now, suppose that there exists \(x^0 \in K_{+,0}^+\) with \(\emptyset \neq Q(x^0) \subset Q(F^{(\sigma,p)}(x^0))\). Then, we have

\[
Q(x^1) = Q(x^0 + F^{(\sigma,p)}(x^0)) = Q(x^0) \cap Q(F^{(\sigma,p)}(x^0)) = Q(x^0).
\]

Using induction and the arguments above, if \(Q(x^k) = Q(x^0)\) for \(k > 0\), then

\[
Q(x^{k+1}) = Q(F^{(\sigma,p)}(x^k)) \cap Q(x^k) = Q(F^{(\sigma,p)}(x^0)) \cap Q(x^k) = Q(x^0).
\]

Hence, \(Q(x^k) \neq \emptyset\) for every \(k > 0\) and thus (iv) can not be satisfied.

(v) \(\Rightarrow\) (ii): Let \(\emptyset \neq V \subset I^\sigma\) be such that \(V_i = \{j_i \mid (i, j_i) \in V\} \neq [n_i]\) for all \(i\). Define \(z \in K_{+}^\sigma\) as \(z_{i,j_i} = 0\) if \((i, j_i) \in V\) and \(z_{i,j_i} = 1\) else. Then \(z \in K_{+,0}^+\) as \(V_i \neq [n_i]\) for all \(i\), and \(z \notin K_{+,+}^\sigma\) as \(V \neq \emptyset\). Now, we have \(Q(F^{(\sigma,p)}(z)) \not\subset Q(z)\) and so there exists \((k, l_k) \in I^\sigma\) such that \(F^{(\sigma,p)}(z) > 0\) and \(z_{k,l_k} = 0\). \(F^{(\sigma,p)}(z) > 0\) implies the
existence of \( j_1, \ldots, j_m \) such that

\[
T_{j_1, \ldots, j_m} \left( \prod_{t \in \sigma_k, t \neq s_k} z_{k, j_t} \right) \prod_{i=1}^d \prod_{t \in \sigma_k} z_{i, j_t} > 0.
\]

Hence, we have \( T_{j_1, \ldots, j_m} > 0 \) and \( z_{i, j_t} > 0 \) for all \( t \in \sigma_k, i \neq k \) and \( z_{k, j_t} > 0 \), \( t \in \sigma_k \setminus \{s_k\} \). As \( z_{i, j_t} > 0 \) implies that \((i, j_t) \notin V\), this concludes the proof. \( \square \)

Let us point out that the second characterization in the above lemma reduces to the definition of irreducibility introduced for the cases \( d = 1 \) and \( d = m \) in [3] and [8], respectively. Furthermore, the third characterization is particularly relevant as it allows to introduce a simple algorithm for checking \( \sigma \)-irreducibility. In particular, observe that, when \( d = 1 \), such characterization reduces to Theorem 5.2 of [21].

We prove the equivalent of Lemma 6.7 and Remark 6.8 for strong irreducibility.

**Lemma 6.11.** If \( T \) is \( \sigma \)-strongly irreducible and \( \sigma \)-symmetric, then for any shape partition \( \sigma \) of \( T \) such that \( \sigma \subseteq \tilde{\sigma} \), \( T \) is \( \tilde{\sigma} \)-strongly irreducible.

**Proof.** By Lemma 6.10, we may assume without loss of generality that \( p_i = \tilde{p}_j = 2 \) for all \( i \in [d], j \in [d] \). Now, there exists \( \tilde{\sigma} \) such that \( x_{\tilde{\sigma}} \in K^*_{\#} \) for all \( x \in K^*_\sigma \). Let \( x^0 \in K^*_{\tilde{\sigma}+} \setminus K^*_{\tilde{\sigma}+} \) and, for \( k \in \mathbb{N} \), define \( x^{k+1} = x^k + F^{(\sigma, \rho)}(x^k) \). We show that \( x^K \in K^*_{\tilde{\sigma}+} \) for some \( K > 0 \) so that the claim follows from Lemma 6.10. Define \( z^0 = (x^0)^{\tilde{\sigma}} \in K^*_{\tilde{\sigma}+} \) and \( z^{k+1} = z^k + F^{(\sigma, \rho)}(z^k) \) for \( k \in \mathbb{N} \). As \( T \) is \( \sigma \)-symmetric, we have \( z^K = (x^K)^{\tilde{\sigma}} \) for all \( k \). Lemma 6.10 implies the existence of \( K > 0 \) such that \( z^K \in K^*_{\tilde{\sigma}+} \) and thus \( x^K \in K^*_{\tilde{\sigma}+} \), which concludes the proof. \( \square \)

Again, the partial symmetry can not be omitted in the above lemma.

**Remark 6.12.** Let \( \sigma^1, \sigma^2, \sigma^3 \) be as in (26). For every \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{0, 1\} \), there exists a tensor \( T^{(\varepsilon_1, \varepsilon_2, \varepsilon_3)} \in \{0, 1\}^{3 \times 3 \times 3} \) such that for \( i = 1, 2, 3 \), \( T^{(\varepsilon_1, \varepsilon_2, \varepsilon_3)} \) is \( \sigma^i \)-strongly irreducible if \( \varepsilon_i = 1 \) and \( T^{(\varepsilon_1, \varepsilon_2, \varepsilon_3)} \) is not \( \sigma^i \)-strongly irreducible if \( \varepsilon_i = 0 \). Precisely, with the notation of (27) we have

\[
T^{(0, 0, 0)} = 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T^{(1, 0, 0)} = 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
T^{(0, 1, 0)} = 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T^{(1, 1, 0)} = 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
T^{(0, 0, 1)} = 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T^{(1, 0, 1)} = 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T^{(0, 1, 1)} = 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T^{(1, 1, 1)} = 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Note that all the tensors given above are \( \sigma^i \)-weakly irreducible, for \( i = 1, 2, 3 \).

Furthermore, in the case \( d = 2 \), it follows from Theorem 2.4 in [24] that if \( T \) is irreducible in the sense of Definition 1 in [5], then \( T \) is \( \sigma \)-strongly irreducible. However, the converse is not true. In fact, for example, the tensor \( T^{(0, 1, 0)} \) given above 6.12 is \( \sigma^2 \)-strongly irreducible, but, with \( x = ((1, 0, 0)\top, (0, 1, 0)\top) \), we have \( T(x^{\sigma^2}) = ((0, 0, 0)\top, (1, 0, 0)\top) \) and so, by Lemma 2 in [5], \( T \) can not be irreducible in the sense of Definition 1 in [5].

We are now almost ready to prove the first of our main theorems.

**6.5. Proof of Theorem 3.1.** Note that points (iv) and (v) of Theorem 3.1 follow immediately from Lemmas 6.7 and 6.11, respectively. Thus, we only need to prove that \( \sigma \)-strong irreducibility implies \( \sigma \)-weak irreducibility and that \( \sigma \)-weak irreducibility implies \( \sigma \)-strict nonnegativity. This is addressed by the following lemma.

For completeness, let us remark that in the particular cases \( d = 1, m \), it is known that (strong) irreducibility implies weak irreducibility (see Lemma 3.1 [8]). Furthermore, still for the particular cases \( d = 1, m \), it was proved in Proposition 8, (b) of [9]
and Corollary 2.1. [13] that weak irreducibility implies strict nonnegativity. All these results are particular cases of the following:

**Lemma 6.13.** If $T$ is $\sigma$-strongly irreducible, then it is $\sigma$-weakly irreducible. If $T$ is $\sigma$-weakly irreducible, then it is $\sigma$-strictly nonnegative.

**Proof.** The case $d = 1$ follows from Corollary 2.1. [13] and Lemma 3.1 [8]. Now, suppose $d > 1$. Clearly, if $G_\sigma(T)$ is strongly connected, then Lemmas 6.3 and 6.5 imply that $T$ is $\sigma$-strictly nonnegative. Now, suppose that $T$ is $\sigma$-strongly irreducible and let us show that $T$ is $\sigma$-weakly irreducible. To this end, we first show that $T$ is $\sigma$-strictly nonnegative. Suppose by contradiction that it is not the case. By Lemma 6.5, there exists $x \in K_{\sigma+}$ and $(k, l_k) \in \mathcal{I}_\sigma$ such that $F_{k,l_k}(x) = 0$. Let $z \in K_{\sigma+}$ be defined as $z_{i,j_i} = x_{i,j_i}$ for all $(i, j_i) \in \mathcal{I}_\sigma \setminus \{(k, l_k)\}$ and $z_{k,l_k} = 0$. Then, as $z \leq_K x$, we have $F_{k,l_k}(z) \leq F_{k,l_k}(x) = 0$ which contradicts Lemma 6.10, (v).

Now, to show that $T$ is $\sigma$-weakly irreducible, we show that for every nonempty subset $V^1, V^2 \subset \mathcal{I}_\sigma$ with $V^1 \cap V^2 = \emptyset$ and $V^1 \cup V^2 = \mathcal{I}_\sigma$ there exists $(k, l_k) \in V^1$ and $(i, t_i) \in V^2$ such that $((k, l_k), (i, t_i)) \in \mathcal{E}_\sigma(T)$. So let $V^1, V^2$ be such a partition of $\mathcal{I}_\sigma$ and set $V_i^j = \{t_i \in [n_i] \mid (i, t_i) \in V^j\}$ for $i \in [d], j = 1, 2$. First, assume that $V_i^1 \neq \emptyset$ for all $i \in [d]$. Then, as $T$ is $\sigma$-strongly irreducible, there exists $k \in [d]$ and $j_1, \ldots, j_m$ such that $T_{j_1, \ldots, j_m} > 0$, $j_k \in V_k^1$, $j_t \in V_t^2$, $t \in \sigma_k \setminus \{s_k\}$ and $j_t \in V_t^2$, $t \in \sigma_i, i \in [d] \setminus \{k\}$. It follows that $((k, j_k), (i, j_t)) \in \mathcal{E}_\sigma(T)$ for all $i \neq k$ and we are done. Now, suppose that there exists $k \in [d]$ such that $V_k^1 = \emptyset$. We claim that if there is no edge between $V^1$ and $V^2$ in $G_\sigma(T)$, then $T$ is not $\sigma$-strictly nonnegative which contradicts our previous argument. Indeed, suppose that $((k, l_k), (i, t_i)) \notin \mathcal{E}_\sigma(T)$ for all $(k, l_k) \in V^1$ and $(i, t_i) \in V^2$. Let $(i, t_i) \in V^2$. Note that $i \neq k$ as $V_k^1 = \emptyset$. Furthermore, we have $T_{j_1, \ldots, j_m} = 0$ for all $j_1, \ldots, j_m$ such that $j_{s_i} = t_i$ and $j_t \in [n_T^k]$. By Lemma 6.5, (iv), this implies that $T$ is not $\sigma$-strictly nonnegative, a contradiction. Thus, there exists $(k, l_k) \in V^1$ and $(i, t_i) \in V^2$ such that $((k, l_k), (i, t_i)) \in \mathcal{E}_\sigma(T)$ and as this is true for every partition of $\mathcal{I}_\sigma$, it follows that $G_\sigma(T)$ is connected.

We finally have all the tools for the proof of Theorem 3.1, which is now a simple consequence of what have been discussed so far.

**Proof of Theorem 3.1.** (i), (ii), (iii) follow from Lemma 6.13 and (iv), (v) follow from Lemmas 6.7 and 6.11.\hfill $\Box$

We conclude the paper by proving the other two main results of Section 3.

**7. Proof of Theorems 3.2 and 3.3.** First, let us recall that the homogeneity matrix $A$ of $F(\sigma, p)$ is given as

$$A = \text{diag}(p_1^1 - 1, \ldots, p_d^1 - 1)(\nu^\top - I), \quad \nu = (|\sigma_1|, \ldots, |\sigma_d|),$$

and $b \in \mathbb{R}^d_{++}$ is the unique positive vector such that $A^\top b = \rho(A)b$ and $\sum_{i=1}^d b_i = 1$.

Now, for the proof of Theorem 3.2, we first need the following additional lemma.

**Lemma 7.1.** Suppose that $\rho(A) \leq 1$ and $T$ is $\sigma$-strictly nonnegative. If $(\theta, u) \in \mathbb{R}^d_+ \times S_{\sigma, p}(\sigma)$ satisfies $F_i^{(\sigma, p)}(u) = \theta_i u$ for all $i \in [d]$ and $\prod_{i=1}^d \theta_i^{b_i} = r_b(F(\sigma, p))$, then $\theta_i^{p_i-1} = r(\sigma, p)(T)$ for all $i \in [d]$.

**Proof.** By Lemma 6.5, we have that the $\sigma$-strict nonnegativity of $T$ implies $F(\sigma, p)(x) \in K_{\sigma+}$ for all $x \in K_{\sigma+}$. Now, as $F_i^{(\sigma, p)}(u) = \theta_i u$ for all $i \in [d]$, Lemma 5.1 implies the existence of $\lambda \in \mathbb{R}_+$ such that $\theta_i^{p_i-1} = \lambda$ for all $i \in [d]$.\hfill $\Box$
and \((\lambda, \mathbf{u})\) is a \((\sigma, p)\)-eigenpair of \(T\). We prove that \(\lambda = r^{(\sigma, p)}(T)\). Clearly, we have \(\lambda \leq r^{(\sigma, p)}(T)\). Now, let \((\varrho, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_d}\) be any \((\sigma, p)\)-eigenpair of \(T\). Then, by definition, we have \(T_i(\mathbf{v}^{(\sigma)}) = \varrho \psi_{\varrho_i}^{(\sigma)}(\mathbf{v}_i)\) for every \(i \in [d]\). It follows that \(\psi_{\varrho_i}^{(\sigma)}(T_{i,j_i}(\mathbf{v}^{(\sigma)})) = |\varrho|^{p_i'}|\text{sign}(\varrho)|\mathbf{v}_i\) for all \(i\). In particular, by the triangle inequality, with \(\mathbf{w} = |\varrho|\), i.e. \(\mathbf{w}\) is the component-wise absolute value of \(\mathbf{v}\), we have \(|\psi_{\varrho_i}^{(\sigma)}(T_{i,j_i}(\mathbf{v}^{(\sigma)}))| \leq F_{i,j_i}^{(\sigma, p)}(\mathbf{w})\). Hence, for \((i, j_i) \in I^\sigma\) such that \(w_{i,j_i} > 0\), it holds
\[
|\varrho|^{p_i'} = \frac{|\psi_{\varrho_i}^{(\sigma)}(T_{i,j_i}(\mathbf{v}^{(\sigma)}))|}{w_{i,j_i}} \leq F_{i,j_i}^{(\sigma, p)}(\mathbf{w}).
\]

Now, as \(\|\mathbf{v}_i\|_{p_i} = 1\) for all \(i \in [d]\), we have \(\mathbf{w} \in S_+^{(\sigma, p)}\) and thus Theorem 5.1 in [10] implies that, with \(\gamma' = \sum_{i=1}^d b_i^{p_i'}\), it holds \(|\varrho|^{\gamma' - 1} = \prod_{i=1}^d |\varrho|^{b_i(p_i'-1)} \leq r_b(F^{(\sigma, p)}) = \lambda^{\gamma' - 1}\). Finally, as \(\gamma' = \frac{\gamma}{\gamma - 1}\), where \(\gamma\) is defined as in Lemma 5.2, we have \(\gamma' > 1\) and thus it follows that \(|\varrho| \leq \lambda\) implying that \(\lambda \geq r^{(\sigma, p)}(T)\) which concludes the proof. 

**Proof of Theorem 3.2.** The existence of \(\mathbf{b}\) is discussed below Equation (22). By Lemma 6.5 we have \(F^{(\sigma, p)}(\mathbf{x}) \in K_{++}^\sigma\) for all \(\mathbf{x} \in K_{++}^\sigma\) as \(T\) is \(\sigma\)-strictly nonnegative.

(i) First note that \(\gamma \in (1, \infty)\) by Lemma 5.2. To show the existence of a \((\sigma, p)\)-eigenpair \((\lambda, \mathbf{u}) \in \mathbb{R}_+ \times K_{++}^\sigma\) of \(T\) such that \(\lambda = r^{(\sigma, p)}(T)\), it is enough, by Lemma 7.1, to show that there exists \((\theta, \mathbf{u}) \in \mathbb{R}_+^d \times S_+^{(\sigma, p)}\) such that \(F^{(\sigma, p)}(\mathbf{u}) = \theta \otimes \mathbf{u}\) and \(\prod_{i=1}^d \theta_i^{b_i} = r_b(F^{(\sigma, p)})\). If \(\rho(A) = 1\), the existence of \((\theta, \mathbf{u})\) follows from Theorem 4.1 in [10]. If \(\rho(A) < 1\), then Theorem 3.1 in [10] implies the existence of \((\theta, \mathbf{u}) \in \mathbb{R}_+^d \times S_+^{(\sigma, p)}\) such that \(F^{(\sigma, p)}(\mathbf{u}) = \theta \otimes \mathbf{u}\). As \(\mathbf{u}\) is positive, Theorem 5.1 in [10] implies that \(\prod_{i=1}^d \theta_i^{b_i} = r_b(F^{(\sigma, p)})\) and thus we can choose \((\theta, \mathbf{u}) = (\tilde{\theta}, \mathbf{u})\). In any cases, we have proved the existence of \((\theta, \mathbf{u})\) with the desired property and it follows from Lemma 7.1 that \(r^{(\sigma, p)}(T)\) is a \((\sigma, p)\)-eigenpair of \(T\).

(ii) Lemma 5.2 implies that \(r^{(\sigma, p)}(T) = r_b(F^{(\sigma, p)})\gamma^{-1}\) and \(\gamma \in (1, \infty)\). Thus, (10) and (11) follow respectively from Theorems 5.1 and 4.1 in [10].

(iii) First note that as \(\rho(A) \leq 1\) and \(F^{(\sigma, p)}(\mathbf{x}) \in K_{++}^\sigma\) for all \(\mathbf{x} \in K_{++}^\sigma\), from Theorem 5.1 in [10], that for any \((\tilde{\theta}, \mathbf{u}) \in \mathbb{R}_+^d \times S_+^{(\sigma, p)}\) such that \(F^{(\sigma, p)}(\mathbf{u}) = \tilde{\theta} \otimes \mathbf{u}\), we have \(\prod_{i=1}^d \tilde{\theta}_i^{b_i} = r_b(F^{(\sigma, p)})\). Now, if \(\rho(A) < 1\), then Theorem 3.1 in [10] implies that there exists a unique \(\mathbf{u} \in S_+^{(\sigma, p)}\) such that \(F^{(\sigma, p)}(\mathbf{u}) = \tilde{\theta} \otimes \mathbf{u}\) for some \(\tilde{\theta} \in \mathbb{R}_+^d\). If \(\rho(A) = 1\), then by Lemma 6.3, we know that the \(\sigma\)-weak irreducibility of \(T\) implies that the graph of the multi-homogeneous mapping \(F^{(\sigma, p)}\) is strongly connected. Hence, Theorem 4.3 in [10] implies the existence of \((\tilde{\theta}, \mathbf{u}) \in \mathbb{R}_+^d \times S_+^{(\sigma, p)}\) such that \(F^{(\sigma, p)}(\mathbf{u}) = \tilde{\theta} \otimes \mathbf{u}\). Furthermore, as \(T\) is \(\sigma\)-weakly irreducible, by Lemma 6.3 we know that \(DF^{(\sigma, p)}(\mathbf{x})\) is irreducible for every \(\mathbf{x} \in K_{++}^\sigma\). Hence, Theorem 5.2 in [10] implies that \(\mathbf{u}\) is the unique vector in \(S_+^{(\sigma, p)}\) such that \(F^{(\sigma, p)}(\mathbf{u}) = \tilde{\theta} \otimes \mathbf{u}\). In any cases, we have that there exists a unique \((\tilde{\theta}, \mathbf{u}) \in \mathbb{R}_+^d \times S_+^{(\sigma, p)}\) with \(F^{(\sigma, p)}(\mathbf{u}) = \tilde{\theta} \otimes \mathbf{u}\). Hence, Lemma 7.1 implies that the \((\sigma, p)\)-eigenvector \(\mathbf{u}\) of (ii) can be chosen strictly positive. Finally, if \(\mathbf{v} \in S_+^{(\sigma, p)}\) is a \((\sigma, p)\)-eigenvector of \(T\) such that \(\mathbf{v} \neq \mathbf{u}\), then, by Lemma 5.1, there exists \(\lambda \in \mathbb{R}_+^d\) such that \(F^{(\sigma, p)}(\mathbf{v}) = \lambda \otimes \mathbf{v}\) which is a contradiction as we have shown that \(\mathbf{u}\) is the unique vector in \(S_+^{(\sigma, p)}\) having this property.

(iv) If \((\varrho, \mathbf{x})\) is a \((\sigma, p)\)-eigenpair of \(T\) and \(\mathbf{x} \in K_{++}^\sigma \setminus K_{++}^\sigma\), then by Lemma 5.1,
we have $F_i^{(σ, p)}(x) = \varphi_i^{p_i - 1} x_i$ for all $i \in [d]$. Now, Theorem 5.2 in [10] implies that, with $\gamma' = \frac{γ}{γ - 1} = \sum_{i=1}^d b_i p_i$, we have $θ = (θ')^{γ - 1} < r_b(F^{(σ, p)})^{γ - 1} = r_b^T(T)$, where we have used Lemma 7.1 for the last equality.

(v) Let $(θ, x)$ be a $(σ, p)$-eigenpair of $T$ such that $x ∈ K^σ_{++}$. As $T$ is $σ$-strongly irreducible, Lemma 6.10 implies that $F^{(σ, p)}$ satisfies the assumption of Lemma 6.9. In particular, as $F_i^{(σ, p)}(x) = \varphi_i^{p_i - 1} x_i$ for all $i \in [d]$, Lemma 6.9 implies that $x ∈ S_{++}$. As $σ$-strong irreducibility implies $σ$-weak irreducibility by Theorem 3.1, we know by (iii) that $u$ is the unique positive $(σ, p)$-eigenpair of $T$ in $S_{++}$ and thus $x = u$.

To prove Theorem 3.3, we first introduce the following preliminary lemma:

**Lemma 7.2.** Let $G^{(σ, p)}$ be defined as in (13). Then, the following hold:

(a) $G^{(σ, p)}$ is an order-preserving multi-homogeneous mapping. Furthermore, the homogeneity matrix of $G^{(σ, p)}$ is given by $B = (A + 1)/2$ and $B^T b = ρ(B)b$, where $A$ is the homogeneity matrix of $F^{(σ, p)}$.

(b) If $T$ is $σ$-strictly nonnegative, then $G^{(σ, p)}(x) ∈ K^σ_{++}$ for all $x ∈ K^σ_{++}$.

(c) For every $u ∈ K^σ_{++}$, we have $F^{(σ, p)}(u) = θ ⊗ u$ if and only if $G^{(σ, p)}(u) = \tilde{θ} ⊗ u$ with $\tilde{θ} = θ_i$ for all $i ∈ [d]$.

(d) It holds $r_b(G^{(σ, p)})^2 = r_b(F^{(σ, p)})$.

(e) If $T$ is $σ$-weakly irreducible, then the Jacobian matrix $DG^{(σ, p)}(x)$ is primitive for every $x ∈ K^σ_{++}$.

**Proof.** (a)-(d) follow by a straightforward calculation. For (e), note that

$$DG^{(σ, p)}(x) = \frac{1}{2} \text{diag}(G^{(σ, p)}(x))^{-1} \left( \text{diag}(F^{(σ, p)}(x)) + DF^{(σ, p)}(x) \right).$$

As $T$ is $σ$-weakly irreducible, $DF^{(σ, p)}(x)$ is irreducible by Lemma 6.3. It follows that $DG^{(σ, p)}(x)$ is primitive.

**Proof of Theorem 3.3.** First we note some general observations: As $u ∈ S_{++}^{(σ, p)}$ is a positive $(σ, p)$-eigenvector of $T$, we know by (10) that its corresponding $(σ, p)$-eigenvalue is $λ = r_b^{(σ, p)}(T)$. Furthermore, Lemmas 5.1 and 7.2 imply that $F_i^{(σ, p)}(u) = λ^{p_i - 1} u_i$ and $G_i^{(σ, p)}(u) = λ^{p_i - 1} u_i$ for all $i ∈ [d]$. Lemmas 5.2 and 7.2 imply that $λ = r_b(F^{(σ, p)})^{γ - 1} = r_b(G^{(σ, p)})^2(γ - 1)$. To show (i), let $ω ∈ \{ξ, ζ\}$. Then (15) follow from Lemma 6.4 in [10]. Now, suppose that $ε > 0$ and $\tilde{ω}_k - \tilde{ω}_k < ε$. Then, (16) is obtained by subtracting $(\tilde{ω}_k + \tilde{ω}_k)/2$ from $\tilde{ω}_k ≤ λ \le \tilde{ω}_k$. Finally, with Lemma 7.2, (e), we have that (ii) and (iii) both follow from Theorem 6.1 in [10].

**Acknowledgments.** The authors are grateful to Shmuel Friedland and Lek-Heng Lim for a number of insightful discussions and for pointing out relevant references. This work has been funded by the ERC starting grant NOLEPRO 307793. The work of F.T. has been also partially funded by the MSC individual fellowship MAGNET 744014.

**References**

[1] S. Banach, Über homogene Polynome in $(L^2)$, Studia Math., 7 (1938), pp. 36–44.
[2] D. W. Boyd, The power method for $θ^\ast$ norms, Linear Algebra Appl., 9 (1974), pp. 95–101.
[3] K. C. Chang, K. Pearson, and T. Zhang, Perron-Frobenius theorem for nonnegative tensors, Commun. Math. Sci., 6 (2008), pp. 507–520.
[4] K. C. Chang, L. Qi, and T. Zhang, A survey on the spectral theory of nonnegative tensors, Numer. Linear Algebra Appl., 20 (2013), pp. 891–912.
[5] K. C. Chang, L. Qi, and G. Zhou, Singular values of a real rectangular tensor, J. Math. Anal. Appl., 370 (2010), pp. 284–294.
[6] L. De Lathauwer, B. De Moor, and J. Vandewalle, On the best rank-1 and rank-(r₁, r₂, . . . , rₙ) approximation of higher-order tensors, SIAM J. Matrix Anal. Appl., 21 (2000), pp. 1324–1342.
[7] S. Friedland, Best rank one approximation of real symmetric tensors can be chosen symmetric, Front. Math. China, 8 (2013), pp. 19–40.
[8] S. Friedland, S. Gaubert, and L. Han, Perron-Frobenius theorem for nonnegative multilinear forms and extensions, Linear Algebra Appl., 438 (2013), pp. 738–749.
[9] A. Gautier and M. Hein, Tensor norm and maximal singular vectors of nonnegative tensors – A Perron-Frobenius theorem, a Collatz-Wielandt characterization and a generalized power method, Linear Algebra Appl., 505 (2016), pp. 313–343.
[10] S. Friedland, S. Gaubert, and L. Han, Perron-Frobenius theorem for nonnegative multilinear forms and extensions, Linear Algebra Appl., 438 (2013), pp. 738–749.
[11] L. De Lathauwer, B. De Moor, and J. Vandewalle, On the best rank-1 and rank-(r₁, r₂, . . . , rₙ) approximation of higher-order tensors, SIAM J. Matrix Anal. Appl., 21 (2000), pp. 1324–1342.
[12] S. Friedland, Best rank one approximation of real symmetric tensors can be chosen symmetric, Front. Math. China, 8 (2013), pp. 19–40.
[13] S. Friedland, S. Gaubert, and L. Han, Perron-Frobenius theorem for nonnegative multilinear forms and extensions, Linear Algebra Appl., 438 (2013), pp. 738–749.
[14] A. Gautier and M. Hein, Tensor norm and maximal singular vectors of nonnegative tensors – A Perron-Frobenius theorem, a Collatz-Wielandt characterization and a generalized power method, Linear Algebra Appl., 505 (2016), pp. 313–343.
[15] J. M. Hendrickx and A. Olshevsky, Matrix p-norms are NP-hard to approximate if p ≠ 1, 2, ∞, SIAM Journal on Matrix Analysis and Applications, 31 (2010), pp. 2802–2812.
[16] C. J. Hillar and L.-H. Lim, Most tensor problems are NP-hard, J. ACM, 60 (2013), pp. 1–38.
[17] S. Hu, Z. Huang, and L. Qi, Strictly nonnegative tensors and nonnegative tensor partition, Sci. China Math., 57 (2014), pp. 181–195.
[18] B. Lemmens and R. D. Nussbaum, Nonlinear Perron-Frobenius theory, Cambridge University Press, general ed., 2012.
[19] L.-H. Lim, Singular values and eigenvalues of tensors: a variational approach, in IEEE CAMSAP’05, no. 1, 2005, pp. 129–132.
[20] C. Ling and L. Qi, fth-Singular values and spectral radius of rectangular tensors, Front. Math. China, 8 (2013), pp. 63–83.
[21] M. Ng, L. Qi, and G. Zhou, Finding the largest eigenvalue of a nonnegative tensor, SIAM J. Matrix Anal. Appl., 31 (2009), pp. 1090–1099.
[22] Q. Yang and Y. Yang, Further results for the Perron-Frobenius theorem for nonnegative tensors II, SIAM J. Matrix Anal. Appl., 32 (2011), pp. 1236–1250.
[23] Y. Yang and Q. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors, SIAM J. Matrix Anal. Appl., 31 (2010), pp. 2517–2530.
[24] L. Zhang, Linear convergence of an algorithm for largest singular value of a nonnegative rectangular tensor, Front. Math. China, 8 (2013), pp. 141–153.
[25] G. Zhou, L. Caccetta, and L. Qi, Convergence of an algorithm for the largest singular value of a nonnegative rectangular tensor, Linear Algebra Appl., 438 (2013), pp. 909–968.