General Metrics of $G_2$ and Spin(7) Holonomy

Z.-W. Chong

*George P. & Cynthia W. Mitchell Institute for Fundamental Physics, Texas A&M University, College Station, TX 77843-4242, USA*

**ABSTRACT**

Using a method introduced by Hitchin we obtain the system of first order differential equations that determine the most general cohomogeneity one $G_2$ holonomy metric with $S^3 \times S^3$ principal orbits. The method is then applied to $G_2$ metric with $S^3 \times T^3$ principal orbits in which an analytic solution is obtained. The generalized metric has more free parameters than that previously constructed. After showing that the generalization is non-trivial a system of first order equations is obtained for new Spin(7) metric with principal orbits $S^7$. 
1 Introduction

Explicit metrics of special holonomy are of interest in both physics and mathematics. With the introduction of M-theory, seven and eight dimensional manifolds with $G_2$ and Spin(7) holonomy become particularly important since they provide natural candidates for minimally supersymmetric compactifications. (See for example, [1, 2].) Explicit complete metrics for such compact manifolds are unlikely, since they do not have continuous isometries. However, for non-compact manifolds explicit metrics do exist, and many $G_2$ and Spin(7) examples have been found [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

The $G_2$ metrics with principal orbits $S^3 \times S^3$ are of special interest because of their rich structure. The first non-singular example of this kind was obtained in [4, 5], in which the two $S^3$ are round. In [7] a generalization of the metric ansatz depending on nine unknown functions was given. An ansatz with four unknown functions was proposed in [8]. More general metrics of this kind were given in [12, 13]. For a review see [15].

The first example of a non-singular Spin(7) metric was given in [4, 5], along with $G_2$ metrics including the one described above. The principal orbits of this Spin(7) example are $S^7$, described as an $S^3$ bundle over $S^4$. A generalization of this metric was given in [6], by allowing the $S^3$ fibres to be “squashed”. This generalization was shown to be a special case in [10], in which a new family of Spin(7) metrics on a certain $\mathbb{R}^4$ bundle over $\mathbb{C}P^2$. For other constructions see [9, 11].

In all the above examples, the metrics are of cohomogeneity one. Hitchin [3] gave a general practical tool for calculating such special holonomy metrics of cohomogeneity one. The examples given there reproduce the Spin(7) metrics in [4, 5, 6] and the $G_2$ metrics in [4, 5, 8]. This construction was used in [14] to obtain more general metrics of $G_2$ holonomy. In this paper, we again employ this method to obtain larger classes of metrics with $G_2$ and Spin(7) holonomy.

The paper is organized as follows. In section 2, We consider the most general cohomogeneity one $G_2$ metric with $S^3 \times S^3$ principal orbits. We obtain first-order equations using the Hitchin approach, which guarantees the existence of $G_2$ holonomy. The metric is described by 18 functions satisfying 18 first-order differential equations, together with 7 consistent algebraic constraints. In section 3, we use the same approach to study $G_2$ metrics with principal orbits $S^3 \times T^3$ [16], which can be obtained by taking a contraction [13] of one
of the $S^3$ factors to $T^3$. We obtain an analytic solution that has more non-trivial parameters than those obtained in [16]. This demonstrates explicitly that our first-order system gives rise to a more general class of $G_2$ metrics than any known previously. In section 4, we apply this technique to constructing a more general class of Spin(7) metrics whose principal orbits are $S^7$. We obtain a system of first-order equations, which guarantees the existence of Spin(7) holonomy. We conclude our paper in section 5.

2 General $G_2$ holonomy metric with $S^3 \times S^3$ principal orbits

In this section we use the technique of [3, 14] to obtain the most general cohomogeneity one $G_2$ metric with $S^3 \times S^3$ principal orbits. A $G_2$ manifold is characterised by its associative 3-form $\Phi^3$, which has the structure

$$\Phi^3 = dt \wedge \omega + \rho, \quad (2.1)$$

where $\omega$ and $\rho$ are invariant 2-forms and 3-forms that do not involve $dt$, satisfying the necessary condition

$$\omega \wedge \rho = 0. \quad (2.2)$$

Since $S^3$ is an $SU(2)$ group manifold, we can write the vielbein for the $S^3 \times S^3$ in terms of two sets of left-invariant $SU(2)$ 1-forms $\sigma_i$ and $\Sigma_i$, satisfying

$$d\sigma_i = \frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k, \quad d\Sigma_i = \frac{1}{2} \epsilon_{ijk} \Sigma_j \wedge \Sigma_k. \quad (2.3)$$

We consider the most general 3-form $\rho$ constructed from $\sigma_i$ and $\Sigma_i$, given by

$$\rho = n \Sigma_1 \Sigma_2 \Sigma_3 - m \sigma_1 \sigma_2 \sigma_3 + x_1 d(\sigma_1 \Sigma_1) + x_2 d(\sigma_2 \Sigma_2) + x_3 d(\sigma_3 \Sigma_3)$$
$$+ x_4 d(\sigma_1 \Sigma_2) + x_5 d(\sigma_2 \Sigma_1) + x_6 d(\sigma_2 \Sigma_3) + x_7 d(\sigma_3 \Sigma_2) + x_8 d(\sigma_3 \Sigma_1) + x_9 d(\sigma_1 \Sigma_3), \quad (2.4)$$

where $m$ and $n$ are constants, and $x_i$ are nine functions depending on $t$. In order to obtain $\omega$, we first consider the most general 4-form $\sigma$, involving nine $t$-dependent functions $y_i$:

$$\sigma = y_1 \sigma_2 \Sigma_3 \Sigma_3 + y_2 \sigma_3 \Sigma_3 \sigma_1 \Sigma_1 + y_3 \sigma_1 \sigma_2 \Sigma_2 + y_4 \sigma_2 \Sigma_3 \Sigma_1 + y_5 \sigma_3 \Sigma_2 \Sigma_1 + y_6 \sigma_3 \Sigma_1 \Sigma_2 + y_7 \sigma_1 \Sigma_2 \Sigma_1 + y_8 \sigma_1 \Sigma_2 \Sigma_3 + y_9 \sigma_2 \Sigma_1 \Sigma_2. \quad (2.5)$$

\footnote{Note that here we break the anti-invariance [3] under the $\mathbb{Z}/2$ that interchanges the two $S^3$ factors. This anti-invariance can be restored by setting $m = n$, $x_4 = x_5$, $x_6 = x_7$, and $x_8 = x_9$. It is similar for the following 4-form $\sigma$.}
we take the “square root” of the 4-form, writing it as $\sigma$
where 
where

The condition (2.2) now implies the algebraic constraints

$$-ax_4 + bx_5 - cx_2 + f x_1 - j x_7 + h x_8 = 0, \quad -bx_6 + cx_7 - f x_9 + gx_2 - h x_3 + k x_4 = 0,$$
$$ax_9 - cx_8 + ex_6 - gx_5 + j x_3 - k x_1 = 0, \quad -ax_5 + bx_4 + ex_1 - f x_2 + gx_9 - k x_6 = 0,$$
$$-bx_7 + cx_6 - ex_8 - gx_3 + h x_2 + j x_5 = 0, \quad ax_8 - cx_9 + f x_7 - h x_4 - j x_1 + k x_3 = 0. \quad (2.8)$$

Having obtained the ansatz for the associative 3-form $\Phi^{(a)}$, we can write down the metric for the $G_2$ manifold. We define the symmetric tensor density

$$B_{AB} = -\frac{1}{144} \Phi_{AC_1C_2} \Phi_{BC_3C_4} \Phi_{C_5C_6C_7} \varepsilon^{C_1...C_7}, \quad (2.9)$$

where $\varepsilon^{C_1...C_7}$ is the Levi-Civita tensor density in seven dimensions (with values $\pm 1$ and 0). The metric tensor is then given by

$$g_{AB} = \text{det}(B)^{-1/9} B_{AB}. \quad (2.10)$$

The Hamiltonian of the system can be written as $H = V(\rho) - 2W(\sigma)$, where $V(\rho)$ depends only on the tensor $\rho$, and $W(\sigma)$ depends only on $\sigma$. The function $V(\rho)$ is defined by

$$V(\rho) = \sqrt{-\frac{1}{6} K_a^b K_b^a}, \quad (2.11)$$

where

$$K_a^b \equiv \frac{1}{17} \rho_{c_1c_2c_4} \rho_{c_4c_5a} \varepsilon_{c_1c_2c_3c_4c_5}^b, \quad (2.12)$$

with $\varepsilon^{c_1...c_6}$ being the Levi-Civita tensor density in 6-dimensions. The function $W(\sigma)$ is calculated from

$$W(\sigma)^2 = \frac{1}{48} \varepsilon_{c_1...c_6} \tilde{\sigma}^{c_1c_2} \tilde{\sigma}^{c_3c_4} \tilde{\sigma}^{c_5c_6}, \quad (2.13)$$
where
\[ \tilde{\sigma}^{ab} = \frac{1}{24} \varepsilon^{abc_1c_2c_3c_4} \sigma_{c_1c_2c_3c_4}. \]

(2.14)

For our specific example, we find that
\[ V \equiv \sqrt{-U}, \]
\[ U = m^2 n^2 - 2mn \sum_{i=1}^{9} x_i^2 + \sum_{i=1}^{9} x_i^4 \]
\[ -4(m+n)(x_1x_2x_3 - x_3x_4x_5 - x_1x_6x_7 + x_4x_6x_8 + x_5x_7x_9 - x_2x_8x_9) \]
\[ -2x_1(x_2^2 + x_3^2 - x_1^2 - x_5^2 + x_6^2 + x_7^2 - x_8^2 - x_9^2) \]
\[ +2x_2(-x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 - x_8^2 - x_9^2) + 2x_3^2(-x_4^2 - x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2) \]
\[ +2x_4(-x_5^2 - x_6^2 + x_7^2 - x_8^2 + x_9^2) + 2x_5^2(x_6^2 - x_7^2 + x_8^2 - x_9^2) \]
\[ +2x_6^2(-x_7^2 - x_8^2 + x_9^2) + 2x_7^2(x_8^2 - x_9^2) + 2x_8^2(-x_9^2) \]
\[ +8x_1(x_2x_4x_5 + x_4x_7x_8 + x_5x_6x_9 + x_3x_8x_9) + 8x_3x_5x_6x_8 \]
\[ +8x_2x_4x_6x_9 + 8x_3x_4x_7x_9 + 8x_2x_3x_6x_7 + 8x_2x_5x_7x_8, \]
\[ W = (y_1 y_2 y_3 + y_4 y_6 y_8 + y_5 y_7 y_9 - y_3 y_4 y_5 - y_2 y_8 y_9 - y_1 y_6 y_7)^{-\frac{1}{2}}. \]

(2.15)

The manifold with $G_2$ holonomy is then governed by a set of first-order differential equations following from the Hamiltonian flow
\[ \dot{x}_i = -\frac{\partial H}{\partial y_i}, \quad \dot{y}_i = \frac{\partial H}{\partial x_i}, \]

(2.16)

where the dot denotes a derivative with respect to the “time” variable $t$, together with the Hamiltonian constraint $H = 0$. Thus we have
\[ \dot{x}_1 = \frac{y_2 y_3 - y_6 y_7}{W}, \quad \dot{x}_2 = \frac{y_1 y_3 - y_8 y_9}{W}, \quad \dot{x}_3 = \frac{y_1 y_2 - y_4 y_5}{W}, \]
\[ \dot{x}_4 = \frac{y_6 y_8 - y_3 y_5}{W}, \quad \dot{x}_5 = \frac{y_7 y_9 - y_5 y_4}{W}, \quad \dot{x}_6 = \frac{y_4 y_8 - y_1 y_7}{W}, \]
\[ \dot{x}_7 = \frac{y_5 y_9 - y_1 y_6}{W}, \quad \dot{x}_8 = \frac{y_4 y_9 - y_2 y_9}{W}, \quad \dot{x}_9 = \frac{y_5 y_7 - y_2 y_8}{W}. \]

(2.17)
\[\dot{y}_4 = [mn x_4 + (m + n)(x_6 x_8 - x_3 x_5) + x_4(-x_1^2 - x_2^2 + x_3^2 - x_4^2 + x_5^2 + x_6^2 - x_7^2 + x_8^2 - x_9^2) - 2(x_1 x_2 x_5 + x_1 x_7 x_8 + x_2 x_6 x_9 + x_3 x_7 x_9)]/W,\]
\[\dot{y}_5 = [mn x_5 + (m + n)(x_7 x_9 - x_3 x_4) + x_5(-x_1^2 - x_2^2 + x_3^2 - x_4^2 - x_5^2 - x_6^2 + x_7^2 + x_8^2 - x_9^2) - 2(x_1 x_2 x_4 + x_1 x_6 x_9 + x_3 x_6 x_8 + x_2 x_7 x_8)]/W,\]
\[\dot{y}_6 = [mn x_6 + (m + n)(x_4 x_8 - x_1 x_7) + x_6(-x_1^2 - x_2^2 - x_3^2 + x_4^2 + x_5^2 + x_6^2 - x_7^2 + x_8^2 - x_9^2) - 2(x_1 x_5 x_9 + x_3 x_5 x_8 + x_2 x_4 x_9 + x_2 x_3 x_7)]/W,\]
\[\dot{y}_7 = [mn x_7 + (m + n)(x_5 x_9 - x_1 x_6) + x_7(-x_1^2 - x_2^2 - x_3^2 - x_4^2 + x_5^2 + x_6^2 - x_7^2 - x_8^2 + x_9^2) - 2(x_1 x_4 x_8 + x_2 x_3 x_6 + x_3 x_4 x_9 + x_2 x_5 x_8)]/W,\]
\[\dot{y}_8 = [mn x_8 + (m + n)(x_4 x_6 - x_2 x_9) + x_8(-x_1^2 + x_2^2 - x_3^2 + x_4^2 - x_5^2 + x_6^2 - x_7^2 - x_8^2 + x_9^2) - 2(x_1 x_4 x_7 + x_1 x_3 x_9 + x_3 x_5 x_6 + x_2 x_5 x_7)]/W,\]
\[\dot{y}_9 = [mn x_9 + (m + n)(x_5 x_7 - x_2 x_8) + x_9(-x_1^2 + x_2^2 - x_3^2 - x_4^2 + x_5^2 - x_6^2 + x_7^2 + x_8^2 - x_9^2) - 2(x_1 x_5 x_6 + x_1 x_3 x_8 + x_2 x_4 x_6 + x_3 x_4 x_7)]/W. \quad (2.18)\]

The Hamiltonian constraint implies that

\[U = -4(y_1 y_2 y_3 + y_4 y_5 y_6 + y_7 y_8 y_9 - y_3 y_4 y_5 - y_2 y_3 y_9 - y_1 y_6 y_7) \quad (2.19)\]

Finally we present the explicit form of the metric, which is given by

\[ds^2 = dt^2 + g_{11} \sigma_1^2 + 2g_{12} \sigma_1 \sigma_2 + 2g_{13} \sigma_1 \sigma_3 + 2g_{14} \sigma_1 \Sigma_1 + 2g_{15} \sigma_1 \Sigma_2 + 2g_{16} \sigma_1 \Sigma_3 + 2g_{22} \sigma_2^2 + 2g_{23} \sigma_2 \sigma_3 + 2g_{24} \sigma_2 \Sigma_1 + 2g_{25} \sigma_2 \Sigma_2 + 2g_{26} \sigma_2 \Sigma_3 + 2g_{33} \sigma_3^2 + 2g_{34} \sigma_3 \Sigma_1 + 2g_{35} \sigma_3 \Sigma_2 + 2g_{36} \sigma_3 \Sigma_3 + 2g_{44} \Sigma_1^2 + 2g_{45} \Sigma_1 \Sigma_2 + 2g_{46} \Sigma_1 \Sigma_3 + 2g_{55} \Sigma_2^2 + 2g_{56} \Sigma_2 \Sigma_3 + 2g_{66} \Sigma_3^2, \quad (2.20)\]

where \(g_{ij}\) can be calculated in a straightforward way from (2.10). Owing to the complexity of the structures, we shall not present the explicit results here. We did verify that the system of first-order equations does imply the closure and co-closure of the associative 3-form, which demonstrates that the metric indeed has holonomy \(G_2\).

By the above construction we have obtained \(G_2\) metrics involving 18 functions \(x_i\) and \(y_i\), and two constants \(m\) and \(n\), governed by 7 algebraic equations (2.18 2.19), and \((18 - 7) = 11\) independent first-order equations.
3 \ G_2 \ holonomy \ metric \ with \ S^3 \times T^3 \ principal \ orbits

The SU(2) group associated with an S^3 can be contracted in three different ways, namely
the Euclidean, Heisenberg, and Abelian contractions (see, for example, [14]). Here we
consider the Abelian contraction for \( \sigma_i \). To do this, we define \( \sigma_i = \lambda \alpha_i \), and then send
\( \lambda \to 0 \). Thus we have \( d\alpha_i = 0 \), and correspondingly the S^3 becomes (locally) \( T^3 \).

We start with the 3-form \( \rho \) and 4-form \( \sigma \)

\[
\rho = n\Sigma_1\Sigma_2\Sigma_3 - m\alpha_1\alpha_2\alpha_3 + x_1d(\Sigma_1\alpha_1) + x_2d(\Sigma_2\alpha_2) + x_3d(\Sigma_3\alpha_3)
+ x_4d(\Sigma_1\alpha_2) + x_5d(\Sigma_2\alpha_1),
\]

\( (3.1) \)

\[
\sigma = y_1\Sigma_2\alpha_2\Sigma_3\alpha_3 + y_2\Sigma_3\alpha_3\Sigma_1\alpha_1 + y_3\Sigma_1\alpha_1\Sigma_2\alpha_2
+ y_4\Sigma_2\alpha_2\Sigma_3\alpha_1 + y_5\Sigma_3\alpha_2\Sigma_1\alpha_3,
\]

\( (3.2) \)

Note that a 3-form \( \rho \) without the \( x_4 \) and \( x_5 \) terms, and correspondingly a 4-form \( \sigma \) without
the \( y_4 \) and \( y_5 \) terms, were considered in [16]. We will see later that the more general 3-form \( \rho \) and 4-form \( \sigma \) considered here will give rise to an off-diagonal term in the metric.

The Hamiltonian is given by

\[
H = V - 2W = \sqrt{-U} - 2W
\]

\( (3.3) \)

where

\[
U = m^2n^2 + 4m(x_1x_2x_3 - x_3x_4x_5), \quad W = (y_1y_2y_3 - y_3y_4y_5)^\frac{1}{2}.
\]

\( (3.4) \)

The co-associative 3-form is \( \Phi(3) = dt \wedge \omega + \rho \), where

\[
\omega = \frac{y_2y_3}{W}\Sigma_1\alpha_1 + \frac{y_3y_1}{W}\Sigma_2\alpha_2 + \frac{W}{y_3}\Sigma_3\alpha_3
- \frac{y_3y_4}{W}\Sigma_2\alpha_1 - \frac{y_3y_5}{W}\Sigma_1\alpha_2.
\]

\( (3.5) \)

A \( G_2 \) holonomy metric is obtained if \( x_i \) and \( y_i \) satisfy the Hamiltonian flow equation

\[
\dot{x}_i = -\frac{\partial H}{\partial y_i}, \quad \dot{y}_i = \frac{\partial H}{\partial x_i},
\]

\( (3.6) \)

which results in

\[
\dot{x}_1 = \frac{y_2y_3}{W}, \quad \dot{x}_2 = \frac{y_3y_1}{W}, \quad \dot{x}_3 = \frac{y_1y_2 - y_4y_5}{W},
\]

\[
\dot{x}_4 = -\frac{y_3y_5}{W}, \quad \dot{x}_5 = -\frac{y_3y_4}{W},
\]

\[
\dot{y}_1 = \frac{2mx_2x_3}{\sqrt{-U}}, \quad \dot{y}_2 = \frac{2mx_3x_1}{\sqrt{-U}}, \quad \dot{y}_3 = \frac{2m(x_1x_2 - x_4x_5)}{\sqrt{-U}},
\]

\[
\dot{y}_4 = -\frac{2mx_3x_5}{\sqrt{-U}}, \quad \dot{y}_5 = -\frac{2mx_3x_4}{\sqrt{-U}}.
\]

\( (3.7) \)
A simpler system of equations can be obtained by a change of variable from $t$ to $\tilde{t}$ [16],

$$\frac{dt}{d\tilde{t}} = 4\sqrt{y_1y_2y_3 - y_3y_4y_5},$$

which is equivalent to considering the Hamiltonian flow

$$\tilde{H} = -m^2n^2 + 4m(x_1x_2x_3 - x_3x_4x_5) - 4(y_1y_2y_3 - y_3y_4y_5) = -m^2n^2 - 4mX - 4Y \quad (3.9)$$

with

$$X = x_1x_2x_3 - x_3x_4x_5, Y = y_1y_2y_3 - y_3y_4y_5.$$ \hspace{1cm} (3.10)

The flow equation becomes

$$
\begin{align*}
x_1' &= 4y_2y_3, & x_2' &= 4y_3y_1, & x_3' &= 4(y_1y_2 - y_4y_5), \\
x_4' &= -4y_3y_5, & x_5' &= -4y_4y_3, \\
y_1' &= 4mx_2x_3, & y_2' &= 4mx_3x_1, & y_3' &= 4m(x_1x_2 - x_4x_5), \\
y_4' &= -4mx_3x_5, & y_5' &= -4mx_3x_4.
\end{align*}
$$

(3.11)

where the prime denotes a derivative with respect to $\tilde{t}$. In addition $x_i$ and $y_i$ satisfy the constraint from the requirement $\omega \wedge \rho = 0$, namely

$$y_2x_4 - y_1x_5 - y_4x_2 + y_5x_1 = 0.$$ \hspace{1cm} (3.12)

Suggested by [16] we find the following conserved quantities

$$
\begin{align*}
x_2y_2 - x_1y_1 &= k_1, & x_5y_5 - x_4y_4 &= k_2, & x_1y_1 + x_4y_4 - x_3y_3 &= k_3 \\
x_5y_1 + x_2y_4 &= x_1y_5 + x_4y_2 = \lambda,
\end{align*}
$$

(3.13)

where $k_1$, $k_2$, $k_3$ and $\lambda$ are constants. Defining $z_3 = x_3y_3$, we find that

$$
\begin{align*}
\frac{dz_3}{dt} &= 4Y - 4mX = m^2n^2 + 8Y \\
\frac{d^2z_3}{dt^2} &= 8\frac{dY}{dt} = -32m[3z_3^2 + 2(k_1 + k_2 + 2k_3)z_3 + k_3(k_1 + k_2 + k_3) - \lambda^2].
\end{align*}
$$

(3.14, 3.15)

This can be integrated explicitly, and $z_3$ can be written in terms of Weierstrass function, which has a second order pole. Near a pole $x_i(\tilde{t})$ and $y_i(\tilde{t})$ takes the approximate form
\[ x_i \sim y_i \sim \frac{1}{t} \] Written in terms of \( t \) through the relation (3.8), we have

\[
\begin{align*}
x_1 &= A_1 t^2, & y_1 &= 4A_2 A_3 t^2, \\
x_2 &= A_2 t^2, & y_2 &= 4A_3 A_1 t^2, \\
x_3 &= A_3 t^2, & y_3 &= 4(A_1 A_2 - A_4 A_5) t^2, \\
x_4 &= A_4 t^2, & y_4 &= -4A_3 A_5 t^2, \\
x_5 &= A_5 t^2, & y_5 &= -4A_3 A_4 t^2.
\end{align*}
\] (3.16)

If we set \( A_1 A_2 A_3 - A_3 A_1 A_5 = \frac{m}{64} \), a \( G_2 \) metric of topology \( \mathbb{R}^4 \times T^3 \) can be obtained.

\[
ds^2 = dt^2 + \frac{1}{4} t^2 (\Sigma_0^2 + \Sigma_1^2 + \Sigma_2^2) + 16[(A_1^2 - A_3^2)\alpha_1^2 + 2(A_1 A_4 + A_2 A_5)\alpha_1 \alpha_2 + (A_2^2 - A_4^2)\alpha_2^2 + A_3^2 \alpha_3^2]
\] (3.17)

In the construction of [16], the \( G_2 \) metric had two free parameters. Here, however, the generalized metric has four independent parameters. Also we note that there is an off-diagonal term in the metric which results from the extra \( x_4 \) and \( x_5 \) terms in the 3-form (3.1). This shows that the metric considered here is more general, though it has the same topology as that in [16].

### 4 Spin(7) metric with principle orbit \( S^7 \)

In this section we will consider new Spin(7) metric with principal orbits \( S^7 \) described as an \( SU(2) \) bundle over \( S^4 \). The three invariant \( SU(2) \) connection 1-forms are \( \alpha_i \ (i=1,2,3) \), and \( \omega_i \ (i=1,2,3) \) are the curvature 2-forms. The structure equations for the principal orbits are

\[
\begin{align*}
d\Sigma_1 &= \omega_1 - 2\Sigma_2 \Sigma_3, & d\Sigma_2 &= \omega_2 - 2\Sigma_3 \Sigma_1, & d\Sigma_3 &= \omega_3 - 2\Sigma_1 \Sigma_2, \\
&d\omega_1 = 2(\omega_2 \Sigma_3 - \omega_3 \Sigma_2), & d\omega_2 = 2(\omega_3 \Sigma_1 - \omega_1 \Sigma_3), & d\omega_3 = 2(\omega_1 \Sigma_2 - \omega_2 \Sigma_1)
\end{align*}
\] (4.1)

with

\[
\begin{align*}
\omega_1 &= -(\Sigma_0 \Sigma_1 + \Sigma_2 \Sigma_3), & \omega_2 &= -(\Sigma_0 \Sigma_2 + \Sigma_3 \Sigma_1), & \omega_3 &= -(\Sigma_0 \Sigma_3 + \Sigma_1 \Sigma_2),
\end{align*}
\] (4.2)

where the basis \( \Sigma_\mu \ (\mu=0,1,2,3) \) give the standard metric \( ds^2 = \Sigma_0^2 + \Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2 \) on \( S^4 \).
We consider the following exact 4-form constructed from $\alpha_i$ and $\omega_i$
\[
\rho = x_1 d(\alpha_1 \omega_1) + x_2 d(\alpha_2 \omega_2) + x_3 d(\alpha_3 \omega_3) + x_4 d(2\alpha_1 \alpha_2 \alpha_3) + x_5 d(\alpha_1 \omega_2)
\]
(4.3)
\[
= 2(x_1 + x_2 + x_3)\Sigma_0 \Sigma_1 \Sigma_2 \Sigma_3 - 2(-x_1 + x_2 + x_3 + x_4)\alpha_2 \alpha_3 (\Sigma_0 \Sigma_1 + \Sigma_2 \Sigma_3) \\
-2(x_1 - x_2 + x_3 + x_4)\alpha_3 \alpha_1 (\Sigma_0 \Sigma_2 + \Sigma_3 \Sigma_1) \\
-2(x_1 + x_2 - x_3 + x_4)\alpha_1 \alpha_2 (\Sigma_0 \Sigma_3 + \Sigma_1 \Sigma_2) \\
+2x_5[\alpha_2 \alpha_3 (\Sigma_0 \Sigma_2 + \Sigma_3 \Sigma_1) + \alpha_3 \alpha_1 (\Sigma_0 \Sigma_1 + \Sigma_2 \Sigma_3)].
\]
(4.4)

Note that the above 4-form $\rho$ but without the $x_5$ term was considered in [3], reproducing the Spin(7) metrics obtained in [6]. Also note that the most general 4-form constructed from $\alpha_i$ and $\Sigma_\mu$ was written down in [17]. In order to follow Hitchin’s procedure, one should then calculate the metric that is implied by the choice of 4-form, and solve the equations following from the Hamiltonian flow. In [17] the form of the metric was instead imposed as an additional constraint, and in fact this was more restrictive than was implied by the choice of 4-form, leading to a highly constrained solution set. In this paper, by contrast, we consider a more modest generalization of previous choices for the 4-form ansatz (by including the $x_5$ term in (4.4)), but we do follow the Hitchin procedure and derive the form of the metric, rather than imposing it as an additional ansatz. We shall see that, as in Section 3, the extra $x_5$ term will give rise to an off-diagonal term in the Spin(7) metric. (Such terms were not included in the metric ansatz considered in [17].) In consequence, we obtain new Spin(7) metrics that were not found in the analysis in [17]. The extension to most general 4-form ansatz considered in [17], and with the metric derived from this ansatz, gives a more complicated system of first-order equations, analogous to those obtained for $G_2$ metrics in section 2. We shall not present these here, since they are rather involved but straightforward to derive.

To calculate the metric we first construct the dual tensor density
\[
\tilde{\rho}^{abc} = \frac{1}{4!} \varepsilon^{abcd_1d_2d_3d_4} \rho_{d_1d_2d_3d_4}.
\]
(4.5)

Then, by defining the symmetric tensor density
\[
H^{ab} = -\frac{1}{144} \rho^{ae_1c_2} \rho^{be_3c_4} \varepsilon^{e_5e_6e_7} \varepsilon_{c_1c_2...c_7},
\]
(4.6)
we can calculate the volume from $V = |\det H|^{1/12}$, finding
\[
V = a^2 (a_1^4 b_1^2 b_2^2 b_3^2 - b_1 b_2 \nu_1^2)^{1/2},
\]
(4.7)
where
\[
a^4 = 2(x_1 + x_2 + x_3), \quad a^2 b_2 b_3 = 2(-x_1 + x_2 + x_3 + x_4),
\]
\[
a^2 b_3 b_1 = 2(x_1 - x_2 + x_3 + x_4), a^2 b_1 b_2 = 2(x_1 + x_2 - x_3 + x_4), \quad v_1 = -2x_5. \tag{4.8}
\]

The gradient flow equation is given by
\[
\begin{align*}
\frac{\partial V}{\partial x_1} &= 2(-\dot{x}_1 + \dot{x}_2 + \dot{x}_3 + \dot{x}_4), \\
\frac{\partial V}{\partial x_2} &= 2(\dot{x}_1 - \dot{x}_2 + \dot{x}_3 + \dot{x}_4), \\
\frac{\partial V}{\partial x_3} &= 2(\dot{x}_3 + \dot{x}_2 - \dot{x}_3 + \dot{x}_4), \\
\frac{\partial V}{\partial x_4} &= 2(\dot{x}_1 + \dot{x}_2 + \dot{x}_3), \\
\frac{\partial V}{\partial x_5} &= -2\dot{x}_5.
\end{align*} \tag{4.9}
\]

It can be rewritten as
\[
\begin{align*}
da^4 &= \frac{a^6}{V} b_1 b_2 b_3 (b_1 + b_2 + b_3) - \frac{a^2 v_1^2}{V}, \\
da^2 b_2 b_3 &= \frac{V}{2a^4} + \frac{a^6}{V} b_1 b_2 b_3 (-b_1 + b_2 + b_3) - \frac{a^2 v_2^2}{V}, \\
da^2 b_3 b_1 &= \frac{V}{2a^4} + \frac{a^6}{V} b_1 b_2 b_3 (b_1 - b_2 + b_3) - \frac{a^2 v_3^2}{V}, \\
da^2 b_1 b_2 &= \frac{V}{2a^4} + \frac{a^6}{V} b_1 b_2 b_3 (b_1 + b_2 - b_3) - \frac{a^2 v_4^2}{V}, \\
dv_1 &= 2\frac{a^4 b_1 b_2}{V} v_1. \tag{4.10}
\end{align*}
\]

The metric is obtained from
\[
h_{ab} = |\det H|^{1/6} H_{ab} \tag{4.11}
\]

and it can be written as
\[
ds^2 = dt^2 + h_{11} \sigma_1^2 + h_{22} \sigma_2^2 + h_{33} \sigma_3^2 + 2h_{12} \sigma_1 \sigma_2 + h_{23} \sigma_2 \sigma_3 + h_{31} \sigma_3 \sigma_1 + h_{12}^2 + h_{23}^2 + h_{31}^2 + \Sigma^2, \tag{4.12}
\]
where
\[
\begin{align*}
h_{11} &= \frac{b_1 b_2 (a^4 b_1^2 b_3^2 + v_1^2)}{a^4 b_1 b_2 b_3^2 - v_1^2}, & h_{22} &= \frac{b_1 b_2 (a^4 b_2^2 b_3^2 + v_1^2)}{a^4 b_1 b_2 b_3^2 - v_1^2}, & h_{33} &= b_3^2 - \frac{v_1^2}{a^4 b_1 b_2}, \\
h_{12} &= \frac{a^2 b_1 b_2 (b_1 + b_2) b_3 v_1}{a^4 b_1 b_2 b_3^2 - v_1^2}, & h_{23} &= a^2.
\end{align*} \tag{4.13}
\]

This metric together with the system of first-order equations (4.10) guarantees
\[
\frac{\partial \rho}{\partial t} = d(*) \rho. \tag{4.14}
\]

which shows that the metric determined by (4.10) has Spin(7) holonomy. We see that
\[h_{12}\] gives rise to an off-disgonal term in (4.12), which is proportional to \(x_5\). Through the
relation in (4.8), this off-diagonal term results from the extra \(x_5\) term in (4.4).
5 Discussion

In this paper we have obtained a class of cohomogeneity one $G_2$ metrics with $S^3 \times S^3$ principal orbits, derived from the most general invariant 3-forms $\rho$ and 4-forms $\sigma$ which break the anti-invariance under the $\mathbb{Z}/2$ action interchanging the two $S^3$ factors. This should be the most general cohomogeneity one $G_2$ metric that one can obtain with principal orbits $S^3 \times S^3$. Although we cannot obtain the explicit solution to the highly coupled system of non-linear first-order differential equations, we luckily obtained an analytic solution to the metric with principal orbits $S^3 \times T^3$, which arises as a group contraction \cite{14} of one of the two $S^3$ factors. This type of contraction was considered in \cite{16}. By showing that our solution has more free parameters than the solutions obtained in \cite{16}, we can conclude that this generalization is a non-trivial one.

We also considered Spin(7) metrics with principal orbits $S^7$, with a similar generalization of previous results obtained by writing down a more general exact 4-form, constructed from the $SU(2)$-connection 1-form $\alpha_i$ and curvature 2-form $\omega_i$. It was shown there that the most general exact 4-forms one can consider has ten parameters. In this paper we only considered a five parameter exact 4-form, which is already sufficient to imply the new feature of off-diagonal elements in the Spin(7) metrics. We concentrated on how the generalization modified the system of differential equations which determined the Spin(7) metrics. We also showed how the metrics looks. The formalism presented here can also be applied in a straightforward way to Spin(7) metrics whose principal orbits are Aloff-Wallach spaces, i.e. $SU(3)/U(1)$ cosets.

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References

[1] M. Atiyah, J. Maldacena and C. Vafa, An M-theory flop as a large n duality, J. Math. Phys. 42, 3209 (2001), [hep-th/0011256]
[2] M. Atiyah and E. Witten, *M-theory dynamics on a manifold of G(2) holonomy*, Adv. Theor. Math. Phys. 6, 1 (2003), hep-th/0107177.

[3] N. Hitchin, *Stable forms and special metrics*, math.DG/0107101.

[4] R.L. Bryant and S. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. 58, 829 (1989).

[5] G.W. Gibbons, D.N. Page and C.N. Pope, *Einstein metrics on S^3, R^3 and R^4 bundles*, Commun. Math. Phys. 127, 529 (1990).

[6] M. Cvetič, G. W. Gibbons, H. Lü and C. N. Pope, *New complete non-compact Spin(7) manifolds*, Nucl. Phys. B 620, 29 (2002) hep-th/0103155. *New cohomogeneity one metrics with Spin(7) holonomy*, math.DG/0105119.

[7] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *Supersymmetry M3-branes and G_2 manifolds*, Nucl. Phys. B620, 3 (2002), hep-th/0106026.

[8] A. Brandhuber, J. Gomis, S.S. Gubser and S. Gukov, *Gauge theory at large N and new G(2) holonomy metrics*, Nucl. Phys. B611, 179 (2001), hep-th/0106034.

[9] H. Kanno and Y. Yasui, *On Spin(7) holonomy metric based on SU(3)/U(1)*, J. Geom. Phys. 43, 293 (2002) hep-th/0108226.

[10] S. Gukov and J. Sparks, *M-theory on Spin(7) manifolds. I*, Nucl. Phys. B625 (2002) 3, hep-th/0109025.

[11] H. Kanno and Y. Yasui, *On Spin(7) holonomy metric based on SU(3)/U(1). II*, J. Geom. Phys. 43, 310 (2002) hep-th/0111198.

[12] M. Cvetič, G. W. Gibbons, H. Lü and C. N. Pope, *A G(2) unification of the deformed and resolved conifolds*, Phys. Lett. B 534, 172 (2002) hep-th/0112138.

[13] A. Brandhuber, *G(2) holonomy spaces from invariant three-forms*, Nucl. Phys. B 629, 393 (2002) hep-th/0112113.

[14] Z. W. Chong, M. Cvetič, G. W. Gibbons, H. Lü, C. N. Pope and P. Wagner, *General metrics of G(2) holonomy and contraction limits*, Nucl. Phys. B 638, 459 (2002) hep-th/0204064.
[15] M. Cvetic, G. W. Gibbons, H. Lü and C. N. Pope, *Special holonomy spaces and M-theory*, hep-th/0206154.

[16] S. Gukov, S. T. Yau and E. Zaslow, *Duality and fibrations on G(2) manifolds*, hep-th/0203217.

[17] H. Kanno and Y. Yasui, *Harmonic forms and deformation of ALC metrics with Spin(7) holonomy*, Nucl. Phys. B 650, 449 (2003), hep-th/0208139.