A HYPERBOLIC SURFACE WITH A SQUARE GRID NET

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Abstract. We prove the existence of a hyperbolic surface spread over the sphere for which the projection map has all its singular values on the extended real line, and such that the preimage of the extended real line under the projection map is homeomorphic to the square grid in the plane. This answers a question raised by E. B. Vinberg.

1. Introduction

According to the uniformization theorem, an open simply connected Riemann surface is conformally equivalent to either the complex plane \( \mathbb{C} \), or the open unit disc \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). In the former case it is said to be parabolic, and in the latter it is called hyperbolic. We are interested in the application of the uniformization theorem to classically defined surfaces spread over the sphere.

A surface spread over the sphere is a pair \((X, \psi)\), where \( X \) is a topological surface and \( \psi : X \to \mathbb{C} \) a continuous, open and discrete map. Here \( \mathbb{C} \) denotes the Riemann sphere. The map \( \psi \) is called a projection. Two such surfaces \((X_1, \psi_1), (X_2, \psi_2)\) are equivalent, if there exists a homeomorphism \( \phi : X_1 \to X_2 \), such that \( \psi_1 = \psi_2 \circ \phi \).

According to a theorem of Stoilow [4], there exists a unique conformal structure on \( X \) such that \( \psi \) is a holomorphic map. We call this the pull-back structure on \( X \).

A point \( a \in \mathbb{C} \) is called a singular value of \( \psi \) if \( a \) is either a critical or an asymptotic value of \( \psi \). The set of critical points is a discrete subset of \( X \). A point \( a \in \mathbb{C} \) is called an asymptotic value of \( \psi \), if there exists a curve \( \gamma : [0, t_0) \to X \) such that

\[
\gamma(t) \to \infty \quad \text{and} \quad \psi(\gamma(t)) \to a \quad \text{as} \quad t \to t_0.
\]

A point \( a \in \mathbb{C} \) is called a singular value of \( \psi \) if \( a \) is either a critical or an asymptotic value of \( \psi \).

E. B. Vinberg [5] introduced the interesting class \( \mathcal{V} \) of surfaces spread over the sphere \((X, \psi)\), such that all singular values of \( \psi \) are contained in \( \mathbb{R} = \mathbb{R} \cup \{ \infty \} \). For such mappings \( \psi \), the components of the preimages of the upper and lower hemispheres are called cells. The preimage of the extended real line is a disjoint

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The first author was supported by the Alexander von Humboldt Foundation. The second author was supported by NSF grants DMS-0400636, DMS-0244421, and DMS-0244547.
union of the critical points, called *vertices*, and open arcs, called *edges*. Each vertex has an even number (which is greater than or equal to four) of edges emanating from it. The map $\psi$ in this case is called a *cellular map*. The set $\psi^{-1}(\mathbb{R})$, understood as an embedded graph $N$ in $X$, determines the topological properties of $\psi$. Such an embedded graph $N$ is called a *net*. Suppose that the edges of this graph are labeled by positive numbers such that the sum over the edges of each cell is $2\pi$. If $\overline{\mathbb{C}}$ is endowed with the spherical metric, we assume that $\psi$ maps an edge labeled by $\alpha$ onto an arc of length $\alpha$. Since the length of the real line in the spherical metric is $2\pi$, this is possible. Equivalently, one can label the vertices of a net and ends of edges going to infinity by points of $\overline{\mathbb{C}}$, so that the order of labels correspond to the natural order on $\overline{\mathbb{R}}$.

It is an interesting question in which cases the conformal type of $X$ is determined by the combinatorics of a net, independent of the labeling. G. MacLane [2] and Vinberg [3] considered the periodic net $\cos^{-1}(\mathbb{R})$. They showed that all labelings of this net by real numbers and $\infty$, with ends of edges going to infinity labeled by $\infty$, produce parabolic surfaces.

In his paper, Vinberg asked whether the conformal type is determined by combinatorics in the case of the square grid net. It is clear that if the labels on the vertices of every two adjacent square cells are symmetric with respect to their common edge, the resulting surface is parabolic, and the corresponding cellular map is a fractional linear transformation of some Weierstrass elliptic function. The main result of this paper is the following theorem, which answers the question of Vinberg.

**Theorem 1.** There exists a hyperbolic surface of class $\mathcal{V}$ whose net is homeomorphic to the square grid.

From the point of view of the type problem, the class $\mathcal{V}$ is a generalization of a classically studied class $\mathcal{S}$. A surface spread over the sphere $(X, \psi)$ is said to belong to the class $F(a_1, \ldots, a_q)$, if $\psi$ restricted to the complement of $\psi^{-1}(\{a_1, \ldots, a_q\})$ is a covering map onto its image $\overline{\mathbb{C}\backslash\{a_1, \ldots, a_q\}}$. One defines $F_q := \bigcup_{a_1, \ldots, a_q} F(a_1, \ldots, a_q)$ and $\mathcal{S} := \bigcup_q F_q$.

Surfaces of class $\mathcal{S}$ have a combinatorial representation in terms of so-called labeled Speiser graphs. Assume that $(X, \psi) \in F_q$ and $X$ is open. We fix a Jordan curve $L$, visiting the points $a_1, \ldots, a_q$ in cyclic order. The curve $L$ is usually called a base curve. It decomposes the sphere into two simply connected regions $H_1$, the region to the left of $L$, and $H_2$, the region to the right of $L$. Let $L_i$, $i = 1, 2, \ldots, q$, be the arc of $L$ from $a_i$ to $a_{i+1}$ (with indices taken modulo $q$). Let us fix points $p_1$ in $H_1$ and $p_2$ in $H_2$, and choose $q$ Jordan arcs $\gamma_1, \ldots, \gamma_q$ in $\overline{\mathbb{C}}$, such that each arc $\gamma_i$ has $p_1$ and $p_2$ as its endpoints, and has a unique point of intersection with $L$, which is in $L_i$. We take these arcs to be interiorwise disjoint, that is, $\gamma_i \cap \gamma_j = \{p_1, p_2\}$ when $i \neq j$. Let $\Gamma'$ denote the graph embedded in $\overline{\mathbb{C}}$, whose vertices are $p_1$, $p_2$, and whose edges are $\gamma_i$, $i = 1, \ldots, q$, and let $\Gamma = \psi^{-1}(\Gamma')$. We identify $\Gamma$ with its image in $\mathbb{R}^2$ under an orientation preserving homeomorphism of $X$ onto $\mathbb{R}^2$. The graph $\Gamma$ has the following properties: It is infinite, connected, homogeneous of degree $q$, and bipartite. A graph, properly embedded in the plane and having these properties is called a *Speiser graph*, also known as a *line complex*. The vertices of a Speiser graph
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Γ are traditionally marked by \( \times \) and \( \circ \), such that each edge of \( \Gamma \) connects a vertex marked \( \times \) with a vertex marked \( \circ \). Each face of \( \Gamma \), i.e. a connected component of \( \mathbb{R}^2 \setminus \Gamma \), has either a finite even number of edges along its boundary, in which case it is called an algebraic elementary region, or infinitely many edges, in which case it is called a logarithmic elementary region. Two Speiser graphs \( \Gamma_1, \Gamma_2 \) are said to be equivalent, if there is a sense-preserving homeomorphism of the plane, which takes \( \Gamma_1 \) to \( \Gamma_2 \).

The above construction is reversible. Suppose that the faces of a Speiser graph \( \Gamma \) are labeled by \( a_1, \ldots, a_q \), so that when going counterclockwise around a vertex \( \times \), the indices are encountered in their cyclic order, and around \( \circ \) in the reversed cyclic order. We fix a simple closed curve \( L \subset \mathbb{C} \) passing through \( a_1, \ldots, a_q \). Let \( H_1, H_2, L_1, \ldots, L_q \) be as before. Let \( \Gamma^* \) be the planar dual of \( \Gamma \). If \( e \) is an edge of \( \Gamma^* \) from a face of \( \Gamma \) marked \( a_j \) to a face of \( \Gamma \) marked \( a_{j+1} \), let \( \psi \) map \( e \) homeomorphically onto the corresponding arc \( L_j \) of \( L \). This defines \( \psi \) on the edges and vertices of \( \Gamma^* \). We then extend \( \psi \) to map the faces of \( \Gamma^* \) homeomorphically to \( H_1 \) and \( H_2 \), respectively. This defines a surface spread over the sphere \( (\mathbb{R}^2, \psi) \in F(a_1, \ldots, a_q) \).

The conformal structure is induced by the conformal structure of \( \mathbb{C} \) if one considers a copy of \( H_1 \) or \( H_2 \) for each face of \( \Gamma^* \), and glues \( H_1 \) to \( H_2 \) using the identity map whenever the corresponding faces are adjacent. See [3] for further details.

The proof of Theorem 1 is in two parts. In Section 2 we construct a hyperbolic surface of class \( \mathcal{S} \) whose net is a certain degeneration of the square grid, namely some of the sides of the grid are collapsed to a point. In Section 3 we use quasiconformal deformations to obtain a surface of class \( \mathcal{V} \) with the desired net. For a background on the theory of quasiconformal maps see [1].

We would like to mention two interesting open questions connected to this result. First, is it possible to find a hyperbolic surface spread over the sphere with combinatorics of the square grid, which has a labeling that is symmetric with respect to a line? This would correspond to a real meromorphic map in the unit disc such that the preimage of the extended real line is homeomorphic to the square grid. The second question is whether MacLane-Vinberg’s result still holds if one allows non-symmetric labeling of the cosine net, i.e. whether the cosine net always gives a parabolic surface in the case when we do not assume that labels on the ends of edges going to infinity in the upper half-plane are the same as in the lower half-plane.

ACKNOWLEDGMENTS. We would like to thank Alexandre Eremenko for suggesting this problem and Mario Bonk for helpful discussions. We would also like to thank the referee for valuable comments and for suggesting a simplification of the proof in Section 2.

2. A HYPERBOLIC SURFACE OF CLASS \( \mathcal{S} \)

Consider a planar graph \( G \) whose vertices form the set \( \{(m, n) : m, n \in \mathbb{Z}\} \), and whose edge set is

\[
\{(m, n), (m, n + 1)\} : m, n \in \mathbb{Z}\} \cup \{(m, n), (m + 1, n)\} : m, n \in \mathbb{Z}, n \geq 0\}
\cup \{(m, n), (m + 1, n)\} : m, n \in \mathbb{Z}, n < 0, m < 0\}.
\]
We denote by $\circ$ every vertex $(m,n)$ with $|m+n|$ even, and by $\times$ every vertex $(m,n)$ with $|m+n|$ odd. By replacing edges of the set 
\[
\{(m,-2n), (m,-2n+1)\}: \ m,n \in \mathbb{Z}, m \geq 0, n \geq 1
\]
in the graph $G$ by multiple edges of the same form, we obtain a homogeneous graph of degree four. This is a Speiser graph which we denote by $\Gamma$ (see Figure 1). We label its faces by $0, 1/e, e, \infty$, so that this order is preserved when going counterclockwise around $\times$, and the face adjacent to the vertex $(0,0)$ which is in the first quadrant is labeled by $e$.

The Speiser graph $\Gamma$ with so prescribed labeling defines a unique surface $(X, \psi)$ of class $S$, or, more precisely, of class $F_{4}(0, 1/e, e, \infty)$. Here we assume that the base curve is the extended real line. The rest of this section is devoted to the proof that the surface $(X, \psi)$ is of hyperbolic type.

We will show hyperbolicity by first cutting the surface into two parts $A$ and $B$, and finding explicit quasiconformal maps to horizontal strips. We then calculate the asymptotics of the gluing maps and use a Theorem of L. I. Volkovyskii to show hyperbolicity of the resulting surface.

Let $B$ be the subsurface of $X$ with boundary which is obtained by gluing together hemispheres corresponding to the vertices of the Speiser graph $\Gamma$ from the set \{(m,n): m > 0, n \leq 0\}, together with quarterspheres from the set \{(0,n): n \leq 0\}. Quarterspheres here are the parts of the hemispheres outside the unit disc. Let $A$ be the complement of $B$ in $X$. Then the interiors of $A$ and $B$ are simply connected and the common boundary is a simple curve (see Figure 1).

The part $A$ is uniformized by the strip $S_{1} = \{z: 0 < \text{Im } z < \pi\}$, using the map 
\[
\varphi(\alpha(\exp(z))),
\]
where $\alpha$ is the quasiconformal map given in polar coordinates by $(r, \theta) \mapsto (r, 3\theta/2)$, $0 < \theta < \pi$, and $\varphi$ is the conformal map of a rectangle \{x+iy: 0 < x < \pi/2, 0 < y < \tau\} to the domain \{|z| > 1, \text{Im } z > 0\}, with $\varphi(0) = 1, \varphi(\pi/2) = e, \varphi(\pi/2 + i\tau) = \infty$ and $\varphi(i\tau) = -1$. By reflection $\varphi$ extends to an elliptic function in the plane with periods $2\pi$ and $2i\tau$. The map $\alpha \circ \exp$ maps $S_{1}$ to quadrants I–III, and $\varphi$ maps these quadrants to $A$ (see Figure 2). The map $\varphi$ sends the real line to the real line, and
the imaginary axis to the unit circle. The critical points are at \( \pm \pi/2, \pm \pi/2 + i\tau \) and all equivalent points. The critical values are \( \wp(\pi/2) = e, \wp(-\pi/2) = 1/e, \wp(\pi/2 + i\tau) = \infty \) and \( \wp(-\pi/2 + i\tau) = 0. \)

The part \( B \) is uniformized by the strip \( S_2 = \{ z : \pi < \text{Im} z < 2\pi \} \), using the map
\[
\exp(\sin(\beta(\exp(z))))
\]
where \( \beta \) is the quasiconformal map given in polar coordinates by \((r,\theta) \mapsto (r,\theta/2)\), \(-\pi < \theta < 0\). The map \( \beta \circ \exp \) maps \( S_2 \) to quadrant IV, and \( \exp \circ \sin \) maps this quadrant to \( B \) (see Figure 2).

The gluing map \( f \) from the boundary \( \{ \text{Im} z = 0 \} \) of \( S_1 \) to the boundary \( \{ \text{Im} z = 2\pi \} \) of \( S_2 \) and the gluing map \( g \) from the boundary \( \{ \text{Im} z = \pi \} \) of \( S_1 \) to the boundary \( \{ \text{Im} z = \pi \} \) of \( S_2 \) are given by
\[
f(t) = \log(e^t + p(e^t)), \quad g(t) = \log \left( -\text{arsinh} \left( \frac{\pi e^t + q(e^t)}{t} \right) \right),
\]
where \( p(x) \) and \( q(x) \) are real-analytic periodic functions with derivatives bounded below by \( c > -1 \). Differentiating, we find that \( f'(t) \geq (c+1)/2 \) and \( g'(t) \leq C/t \) for \( C > 0 \) and \( t \) large enough. By [6, Theorem 24, p. 89] (see also the remark following it), the resulting surface is hyperbolic.

3. Unraveling logarithmic staircases

In this section we will use quasiconformal deformations of \( (X, \psi) \) to obtain a surface of the same type with the net homeomorphic to the square grid.

A simply connected subsurface \( (Y, \psi) \) with \( Y \subset X \) is a logarithmic staircase over \( a \in \mathbb{R} \) in \( (X, \psi) \) if there exists \( c \in \mathbb{R} \) such that \( \psi|_Y : Y \to D_c^\infty(a) \) is a regular covering map, i.e. there exists a conformal map \( \phi : Y \to \{ \text{Re} z < c \} \) such that \( \psi(z) = a + e^{\phi(z)} \) for \( z \in Y \). The preimage of the real line consists of infinitely many half-lines \( l_k = (-\infty, c) + i\pi k, k \in \mathbb{Z} \), in the \( \phi \)-coordinate. A simply connected subsurface \( (Y, \psi) \) is a logarithmic staircase over \( \infty \) if \( (Y, 1/\psi) \) is a logarithmic staircase over 0. The surface \( (X, \psi) \) contains infinitely many logarithmic staircases over 0 and \( \infty \).
Figure 3. Topological picture of the net of the cos-spine.

In order to obtain a surface whose net is homeomorphic to the square grid, we will replace all of those logarithmic staircases by cosine spines. We say that \((Y, \psi)\) is a cos-spine in \((X, \psi)\) if \(Y \subset X\) is simply connected, \(\psi\) is a cellular map, and \((\psi|_Y)^{-1}(\mathbb{R})\) is a union of infinitely many lines \(l_0, l_1, \ldots\), where \(l_0\) has one endpoint on \(\partial Y\), all other lines have two endpoints on \(\partial Y\) and \(l_0\) intersects every \(l_k\) once, and none of the other \(l_k\) intersect (see Figure 3). The line \(l_0\) is the axis of symmetry of the cos-spine.

Theorem 2. Let \((X, \psi)\) be a Riemann surface spread over the sphere and \((Y, \psi)\) a logarithmic staircase over some \(a \in \mathbb{R}\) in \((X, \psi)\). Let \(z_0\) be a point on \(\partial Y\) with \(\psi(z_0) \in \mathbb{R}\). Endow \(X\) with the pull-back complex structure. Then there exists a universal constant \(K > 1\) and a \(K\)-quasiregular map \(\tilde{\psi} : X \to \mathbb{C}\) with \(\tilde{\psi} = \psi\) outside \(Y\) and \((Y, \tilde{\psi})\) being a cos-spine whose axis of symmetry passes through \(z_0\).

An immediate corollary is the following.

Corollary 1. Let \((X, \psi)\) be a Riemann surface spread over the sphere and \((Y_1, \psi), (Y_2, \psi), \ldots\) be mutually disjoint logarithmic staircases in \((X, \psi)\). Furthermore, let \(z_j\) be a point on \(\partial Y_j\) with \(\psi(z_j) \in \mathbb{R}\). Then there exists a Riemann surface \((X, \tilde{\psi})\) of the same type as \((X, \psi)\), where \(\tilde{\psi} = \psi\) outside \(\bigcup_j Y_j\), and \(\tilde{\psi}|_{Y_j}\) is a cos-spine for each \(j\) with its axis of symmetry passing through \(z_j\).

Proof. If we apply Theorem 2 to each logarithmic staircase separately, we obtain a quasiregular map \(\tilde{\psi} : X \to \mathbb{C}\) with the desired net. This implies that the conformal structures induced by \(\psi\) and \(\tilde{\psi}\) are quasiconformally equivalent. The type of a simply connected Riemann surface is preserved under quasiconformal homeomorphisms.

Before proving Theorem 2, we need some preliminary results.

Let \(S\) denote the strip \(\{0 < \text{Im} z < 1\}\).

Lemma 1. Let \(f_0, f_1 : \mathbb{R} \to \mathbb{R}\) be diffeomorphisms and \(M \geq 1\) with \(1/M \leq f_1(x) \leq M\) and \(|f_0(x) - f_1(x)| \leq M - 1\) for \(k = 0, 1\) and all \(x \in \mathbb{R}\). Then there exists a \(K(M)\)-quasiconformal map \(f : S \to S\) with \(f(x + ik) = f_k(x) + ik\) for \(k = 0, 1\). Furthermore, \(\lim_{M \to 1} K(M) = 1\).
Figure 4. Topological picture of the net of the function in Lemma 2.

**Proof.** Define \( f(x + iy) := (1 - y)f_0(x) + yf_1(x) + iy \) in \( S \). The fact that \( f_0 \) and \( f_1 \) are increasing homeomorphisms of the real line immediately implies that \( f \) is a homeomorphism of \( S \) onto itself, preserving boundary components. The partial derivatives are

\[
\begin{align*}
    f_x(x + iy) &= (1 - y)f'_0(x) + yf'_1(x), \\
    f_y(x + iy) &= f_1(x) - f_0(x) + i.
\end{align*}
\]

Viewing the complex plane as \( \mathbb{R}^2 \), the matrix of the differential is

\[
Df(x, y) = \begin{pmatrix}
    (1 - y)f'_0(x) + yf'_1(x) & f_1(x) - f_0(x) \\
    0 & 1
\end{pmatrix}
\]

Thus \( Jf(x, y) = (1 - y)f'_0(x) + yf'_1(x) \geq 1/M \) and \( ||Df(x, y)|| \leq \max(1, (1 - y)f'_0(x) + yf'_1(x) + |f_1(x) - f_0(x)|) \leq 2M - 1 \). For the dilatation these estimates give

\[
Kf(x, y) = \frac{||Df(x, y)||^2}{Jf(x, y)} \leq M(2M - 1)^2,
\]

which is finite for all \( M \) and goes to 1 for \( M \to 1 \), as claimed.

Let \( D \subset \mathbb{C} \) be a domain which is symmetric w.r.t. the real line. A function \( G : D \to \mathbb{C} \) is called real if \( G(\overline{w}) = \overline{G(w)} \) for all \( w \in D \).

**Lemma 2.** There exists a real quasiregular function \( G : \{\Re w \leq 1\} \to \mathbb{C} \) and a constant \( M > 0 \) such that

1. \( G(w) = e^w \) for \( \Re w = 1 \) and for \( \Re w \leq -M - 1 \)
2. \( G \) has one real critical point and the preimage of the real line consists of infinitely many mutually disjoint rays to \( \infty \) that start on the line \( \{\Re w = 1\} \), and an arc which has two endpoints on that line, intersects the real axis once, and is disjoint from all other rays (see Figure 4).

**Proof.** Define

\[
H(w) := \frac{w + 4}{w - 4} e^w.
\]

We want to interpolate between \( H \) and \( \exp \) (locally) quasiconformally in the vertical strips \( S' = \{0 < \Re w < 1\} \) and \( S'' = \{-M - 1 < \Re w < -M\} \), where \( M \) is to be
chosen. For interpolation in \(S'\) we will use Lemma 1 and thus we have to check the conditions for the lifts (logarithms) of the boundary mappings of \(H\) on \(\{\text{Re } w = 0\}\) and \(\text{exp}\) on \(\{\text{Re } w = 1\}\). In our case \(f_1\) is the identity mapping as the imaginary part of the logarithm of \(\text{exp}(1 + ix)\). For \(f_0\) we get

\[
 f_0(x) = \text{Im } \log H(ix)
 = \text{Im } \left( \log \frac{4 + ix}{4 + ix} + ix \right)
 = x + \arctan \frac{8x}{16 + x^2}.
\]

From the boundedness of the arctangent we immediately have \(|f_1(x) - f_0(x)| \leq \frac{\pi}{2}\). Furthermore,

\[
 f'_0(x) = 1 + \frac{8(16 - x^2)}{(16 + x^2)^2 + 64x^2},
\]

from which we get

\[
 |f'_0(x) - 1| = \frac{|8(16 - x^2)|}{(16 + x^2)^2 + 64x^2} \leq \frac{1}{2}
\]

for all \(x \in \mathbb{R}\). An application of Lemma 1 shows that there exists a quasiconformal map \(\tilde{G}\) in \(S'\) with boundary values \(\log H(w)\) on \(\{\text{Re } w = 0\}\) and \(w\) on \(\{\text{Re } w = 1\}\). Then \(G = \text{exp} \circ \tilde{G}\) is quasiregular in \(S'\) with boundary values \(H\) and \(\text{exp}\), respectively.

For interpolation in \(S''\), fix an increasing \(C^\infty\)-function \(\eta : \mathbb{R} \to [0, 1]\) with \(\eta(t) = 0\) for \(t < -1\) and \(\eta(t) = 1\) for \(t > 0\). Define

\[
 G(w) = \eta(M + \text{Re } w)H(w) + (1 - \eta(M + \text{Re } w))e^w
 = e^w \left[ \eta(M + \text{Re } w) \frac{w + 4}{w - 4} + 1 - \eta(M + \text{Re } w) \right]
 = e^w \left[ \eta(M + \text{Re } w) \frac{8}{w - 4} + 1 \right]
\]

for \(\text{Re } w \leq 0\). This agrees with the earlier definition of \(G\) on the imaginary axis. The function \(G\) is smooth in the left half-plane, and it is holomorphic outside \(S''\).

We claim that \(G\) is locally quasiconformal there when \(M\) is sufficiently large. The partial derivatives of \(G\) in \(S''\) are (we omit the argument \(M + \text{Re } w\) for \(\eta\) and \(\eta'\) here)

\[
 \frac{\partial G}{\partial w} = e^w \eta' \frac{4}{w - 4}
\]

and

\[
 \frac{\partial G}{\partial w} = e^w \left[ \eta \frac{8}{w - 4} + 1 + \eta' \frac{4}{w - 4} - \eta' \frac{8}{(w - 4)^2} \right],
\]

thus we have the estimates (assuming \(M > 4\))

\[
 \left| e^{-w} \frac{\partial G}{\partial w} \right| \leq \frac{4\|\eta'\|}{M - 4},
\]

and

\[
 \left| e^{-w} \frac{\partial G}{\partial w} \right| \geq 1 - \frac{8}{M - 4} - \frac{4\|\eta'\|}{M - 4} - \frac{8}{(M - 4)^2}.
\]
where \( \| \cdot \| \) stands for the supremum norm. Choosing \( M \) large enough, we can ensure \( |e^{-w} \frac{\partial G}{\partial w}| \leq 1/3 \) and \( e^{-w} \frac{\partial G}{\partial w} \geq 2/3 \), which gives \( \frac{\partial G}{\partial w} / \frac{\partial G}{\partial w} \leq 1/2 \) for the complex dilatation of \( G \). This shows that \( G \) is locally quasiconformal in \( S'' \), thus it is a quasiregular map. \( \square \) (Lemma 2)

**Proof of Theorem 2.** Assuming that \( a \in \mathbb{R} \), let \( \phi : Y \to \{ \Re w < c \} \) be a uniformization of \( Y \) with \( \phi(z_0) \in \mathbb{R} \) and \( \psi(z) = a + e^{\phi(z)} \) for \( z \in Y \). To obtain the desired quasiregular map we modify the exponential map in the half-plane \( \{ \Re w < c \} \). Define
\[
g(w) := e^{-k(M+2)}G(w-c+k(M+2))
\]
for \( -M-1 < \Re w - c + k(M+2) \leq 1 \), where \( k = 0, 1, 2, \ldots \) and \( G \) is the function from Lemma 2. The map \( \tilde{\psi} \) is given by
\[
\tilde{\psi}(z) = \begin{cases} 
\psi(z), & z \notin Y, \\
a + g(\phi(z)), & z \in Y.
\end{cases}
\]
If \( a = \infty \), the function \( \tilde{\psi} \) is defined in \( Y \) by the formula \( \tilde{\psi}(z) = 1/g(\phi(z)) \). \( \square \) (Theorem 2)

**Proof of Theorem 1 continued.** Theorem 1 now follows from Corollary 1 applied to the surface \( (X, \psi) \) constructed in Section 2 and logarithmic staircases over \( 0 \) and \( \infty \). The points \( z_j \) should be chosen so that \( \psi(z_j) \) belongs to a regularly covered interval of the real line between a critical and an asymptotic values. \( \square \) (Theorem 1)

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