AdS$_5$/CFT$_4$ Four-point Functions of
Chiral Primary Operators: Cubic Vertices

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ABSTRACT

We study the exchange diagrams in the computation of four-point functions of all chiral primary operators in $D = 4$, $\mathcal{N} = 4$ Super-Yang-Mills using AdS/CFT correspondence. We identify all supergravity fields that can be exchanged and compute the cubic couplings. As a byproduct, we also rederive the normalization of the quadratic action of the exchanged fields. The cubic couplings computed in this paper and the propagators studied extensively in the literature can be used to compute almost all the exchange diagrams explicitly. Some issues in computing the complete four-point function in the “massless sector” is discussed.
1 Introduction

Computing the four-point functions and understanding their structure is an important problem in the AdS/CFT correspondence [1]-[4]. There have been a lot of work on this subject over the past year [5]-[20], in which the most extensively studied example is the $D = 4, N = 4$ Super-Yang-Mills theory (SYM$_4$) and its string theory dual. Unlike many other CFTs in the correspondence, one can compute the correlation functions of SYM$_4$ in the perturbative regime and compare them with the strong coupling result obtained using the string theory on AdS. The detailed comparison of two- and three-point functions have shown that they are not renormalized [21, 22, 23]. Four-point functions in the two regimes are not expected to be related in a simple manner, and we hope to learn about non-trivial dynamics of SYM$_4$ in comparing them.

There is a technical difficulty in making such comparison. On the SYM$_4$ side, the easiest four-point functions to compute is those of the lowest dimensional chiral primary operators $\text{Tr} X^{(i} X^j)$, where $X^i$ are six real scalars of SYM$_4$[16]-[19]. On the other hand, in AdS, the easiest supergravity fields turn out to be the dilaton and the RR scalar, which correspond to (the supersymmetric completion of) $\text{Tr} F^2$ and $\text{Tr} F \wedge F$, respectively [6]-[15]. We wish to calculate the four-point functions of $\text{Tr} X^{(i} X^j)$ from the AdS side, since it will make a direct comparison between the two approaches possible. Note also that since $\text{Tr} F^2$ and $\text{Tr} F \wedge F$ are superconformal descendants of $\text{Tr} X^{(i} X^j)$ (one needs to act four supercharges), one can study the relation between the four-point functions of chiral primaries and those of descendants.\footnote{There are convincing arguments which indicate that the two- or three-point functions of chiral primaries completely determine those of descendants [23]. This argument is not valid for four-point functions.}

The main purpose of this paper is to compute the cubic couplings that are needed to evaluate the exchange diagrams for the four-point functions of arbitrary chiral primary operators of SYM$_4$, including, of course, $\text{Tr} X^{(i} X^j}$. Recall that there are two types of connected diagrams contributing to the four-point functions: exchange and contact. See Fig. 1. More specifically, we identify all supergravity fields that can be exchanged and compute their couplings to two of the external fields $s^i$ corresponding to the chiral primary operators. As a byproduct, we also determine the normalization of their quadratic action (thereby normalizing their propagator properly).

To evaluate the exchange diagrams, we also need the propagators and have to integrate over the location of the vertices in AdS. Fortunately, almost everything about the propagators and the integrals are known in the literature [9]-[14], with the propagator of a massive symmetric tensor particle being the only missing piece for our purpose.

At the end of this paper, we get back to our primary concern and restrict our attention to $\text{Tr} X^{(i} X^j)}$ and their superconformal descendants. Most physically interesting operators belong to this group, which form the massless multiplet of the $SU(2, 2|4)$ superconformal algebra.\footnote{The term “massless” comes from the fact that their supergravity dual all belong to the same supermultiplet as the massive ones.}
is known that the supergravity on $AdS_5$ can be truncated to contain the massless multiplet only \cite{30, 31, 32}. We discuss how this fact can be exploited in computing the complete four-point function of the (not necessarily chiral) operators in the massless multiplet.

As this work was being completed, we received reference \cite{33} which computes the same cubic couplings.

### 2 Identification of the Exchanged Fields

We wish to pick out the Kaluza-Klein modes of Type IIB supergravity on $AdS_5 \times S^5$ \cite{26, 27} which have non-zero cubic coupling with two $s^I$. Obviously, there cannot be any cubic coupling of a single fermion with two bosons, so we only need to examine the bosonic modes.

We note that $s^I$ are singlets under the $SL(2, \mathbb{R})$ symmetry group of the Type IIB supergravity. This means that two $s^I$ can have a cubic coupling with a third field only if the latter is again a singlet. In fact, it is sufficient to consider the $U(1)_Y$ subgroup of $SL(2, \mathbb{R})$. \footnote{It was argued in \cite{25} that the $U(1)_Y$ symmetry goes beyond the supergravity approximation and becomes an exact symmetry of the operator product expansions for which at least two of the three operators are short. This is the case for our discussion.} In $D = 10$, the RR-scalar, the dilaton and the two two-form fields have non-zero charge under the $U(1)_Y$ and the same is true of all their Kaluza-Klein modes. Hence they do not couple to $s^I$ at cubic order.

The graviton and the RR four-form potential in $D = 10$ are $U(1)_Y$ singlets, and their Kaluza-Klein modes may couple to $s^I$. It turns out that all but two Kaluza-Klein towers have non-zero cubic couplings. Before explaining the reason for the exceptions, we need to recall the definition massless graviton. Since mass is not a Casimir of the AdS supersymmetry, most of the fields in this multiplet have non-zero AdS mass.

Figure 1: The two types of supergravity diagram contributing to the AdS computation of four-point functions of chiral primary operators.
of the modes following references [26]. After fixing a gauge and solving for the constraints, the
spherical harmonics decomposition of the physical modes read
\[
\begin{align*}
h^\alpha_\alpha &= h^I_2 Y^I, & a_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} &= b^I_4 \epsilon_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \nabla^\alpha Y^I \\
h_{\mu \alpha} &= h^I_{\mu \alpha} Y^I, & a_{\mu \alpha_1 \alpha_2} &= \phi^I_\mu \epsilon_{\alpha_1 \alpha_2 \alpha_3} \nabla_{\beta} Y^I_\alpha \\
h_{(\alpha \beta)} &= \phi^I (Y_{(\alpha \beta)}), & a_{\mu \nu \alpha \beta} &= b^I_{\mu \nu} Y^I_{(\alpha \beta)} \\
h^I_{(\mu \nu)} &= H^I_{(\mu \nu)} Y^I, & h^I_4 &= H^I_4 Y^I.
\end{align*}
\]
Following [21], we define two mass eigenstate scalars
\[
s^I = \frac{1}{20(k + 2)} \{h^I_2 - 10(k + 4)b^I\}, \quad t^I = \frac{1}{20(k + 2)} \{h^I_2 + 10kb^I\}. \tag{2.2}
\]
As in [27], we also define two mass eigenstate vectors
\[
V_\mu = h_\mu - 4(k + 3)\phi_\mu, \quad W_\mu = h_\mu + 4(k + 1)\phi_\mu. \tag{2.3}
\]
The scalar \(\phi^I\), symmetric traceless tensor \(H^I_{(\mu \nu)}\) and the two-forms \(b^I_{\mu \nu} \pm \) are mass eigenstates on their own.

Among these Kaluza-Klein towers, \(b^I_{\mu \nu} \pm \) are the only two that decouple from \(s^I\) completely. The reason is that the Clebsch-Gordan coefficient of two \(s^I\) and a \(b^I_{\mu \nu} \pm \) for the \(SU(4) = \text{Spin}(6)\) \(R\)-symmetry group is zero at all levels. The \(R\)-symmetry quantum number of the antisymmetric tensor spherical harmonics \(Y^I_{[\alpha \beta]} \) and \(Y^I_{(\alpha \beta)} \) are \((k + 1, k, 0)\) and \((k + 1, k + 1, 2)\), respectively, where the \(i\)-th number in each triplet is the number of boxes on the \(i\)-th row of an \(SU(4)\) Young tableaux. On the other hand, the \(R\) symmetry quantum number of \(s^I\) is \((k, k, 0)\). Using the standard tensor-product rule in terms of Young tableaux, one can show that the tensor product of \((k_1, k_1, 0)\) and \((k_2, k_2, 0)\) do not contain \((k_3 + 1, k_3 - 1, 0)\) or \((k_3 + 1, k_3 + 1, 2)\) for any \(k_3\).

### 3 Quadratic Action and Cubic Couplings

#### 3.1 Quadratic Action

The part of the \(D = 5\) action that is relevant to our discussion can be written as
\[
S = \frac{4N^2 \omega_5}{(2\pi)^5} \int d^5 x \sqrt{-g} \{L_2 + L_3\} \tag{3.1},
\]
where the constant in front the integral is (the inverse of) the five dimensional Newton’s constant including the volume of a unit 5-sphere \(\omega_5 = \pi^3\). The integrands \(L_2\) and \(L_3\) are the quadratic and
cubic terms of the Lagrangian. The quadratic Lagrangian takes the following form.

\[ 2L_2 = -A_1^4 \left\{ (\nabla s^I)^2 + (m_1^I)^2 (s^I)^2 \right\} - A_2^4 \left\{ (\nabla t^I)^2 + (m_2^I)^2 (t^I)^2 \right\} - A_3^4 \left\{ (\nabla \phi)^2 + (m_3^I)^2 (\phi^I)^2 \right\} \\
\quad - A_4^4 \left\{ \frac{1}{2} F_{\mu\nu}^2 (V^I) + (m_4^I)^2 (V^I)^2 \right\} - A_5^4 \left\{ \frac{1}{2} F_{\mu\nu}^2 (W^I) + (m_5^I)^2 (W^I)^2 \right\} \\
\quad - A_6^4 \left\{ (\nabla_{\lambda} H_{(\mu\nu)})^2 + (m_6^I)^2 (H^I_{\mu\nu})^2 \right\}, \]

(3.2)

where \( F_{\mu\nu} (V^I) \equiv \nabla_{\mu} V_{\nu}^I - \nabla_{\nu} V_{\mu}^I \) and similarly for \( F_{\mu\nu} (W^I) \). The normalization constants and the masses are given by

\[ A_1^4 = 32 \frac{k(k-1)(k+2)}{k+1} z(k), \quad (m_1^I)^2 = k(k-4), \]
\[ A_2^4 = 32 \frac{(k+4)(k+5)(k+2)}{k+3} z(k), \quad (m_2^I)^2 = (k+4)(k+8), \]
\[ A_3^4 = \frac{1}{2} z(k), \quad (m_3^I)^2 = k+4, \]
\[ A_4^4 = \frac{1}{2} \frac{k+1}{k+3} z(k), \quad (m_4^I)^2 = (k-1)(k+1), \]
\[ A_5^4 = \frac{1}{2} \frac{k+3}{2(k+2)} z(k), \quad (m_5^I)^2 = (k+3)(k+5), \]
\[ A_6^4 = \frac{1}{2} z(k), \quad (m_6^I)^2 = k(k+4) - 2. \]

(3.3)

The normalization of the quadratic action for \( s^I \) was first computed in [21]. Subsequent papers [28, 29] pointed out some flaws in the original derivation, but confirmed that the answer was correct. The normalization constants for other fields were first computed in [28]. As will be explained in subsection 3.4, our computation of the cubic couplings provide another way to fix the normalizations.

### 3.2 Cubic Couplings: Results

As we will show in the next section, the cubic Lagrangian in (3.1) is given by

\[ 2L_3 = -\frac{1}{3} G_{1JK}^1 s^I s^J s^K - G_{1JK}^2 t^I t^J s^K - G_{1JK}^3 \phi^I \phi^J s^K \\
\quad - G_{1JK}^4 V_{\mu}^I s^J \nabla_{\nu} s^K - G_{1JK}^5 W_{\mu}^I s^J \nabla_{\nu} s^K - G_{1JK}^6 H_{(\mu\nu)}^I s^J \nabla(\mu \nabla_{\nu}) s^K, \]

(3.4)

where the coupling constants are defined by

\[ G_{1,12,3}^1 = \frac{2^2 \sigma (\sigma^2 - 1)(\sigma^2 - 4) \alpha_1 \alpha_2 \alpha_3}{(k_1 + 1)(k_2 + 1)(k_3 + 1)} a(k_1, k_2, k_3) \langle C^{1_t} C^{1 t} C^{1 s} \rangle, \]
\[ G_{1,12,3}^2 = \frac{2^2 (\sigma + 2) \alpha_1 ! (\alpha_2 + 2) (\alpha_3 + 2)}{(\alpha_1 - 5) ! (k_1 + 3)(k_2 + 1)(k_3 + 1)} a(k_1, k_2, k_3) \langle C^{1 t} C^{1 t} C^{1 s} \rangle, \]
\[ G_{1,12,3}^3 = \frac{2^2 \sigma (\sigma + 1)(\alpha_1 - 1)(\alpha_1 - 2)}{(k_2 + 1)(k_3 + 1)} h(k_1, k_2, k_3) \langle T^{1 t} C^{1 t} C^{1 s} \rangle, \]
\[
G_{1_1 1_2 I_3}^4 = 2^5 \frac{(k_1 + 1)(\sigma^2 - \frac{1}{4})(\sigma + \frac{3}{2})(\alpha_1 - \frac{1}{2})}{(k_1 + 2)(k_2 + 1)(k_3 + 1)} e(k_1, k_2, k_3) \langle V^{I_1 C^{I_2} C^{I_3}} \rangle,
\]
\[
G_{1_1 1_2 I_3}^5 = 2^5 \frac{(k_1 + 3)(\sigma + \frac{3}{2})(\alpha_1 - \frac{1}{2})(\alpha_1 - \frac{5}{2})}{(k_1 + 2)(k_2 + 1)(k_3 + 1)} e(k_1, k_2, k_3) \langle V^{I_1 C^{I_2} C^{I_3}} \rangle,
\]
\[
G_{1_1 1_2 I_3}^6 = 2^5 \frac{(\sigma + 1)(\sigma + 2)\alpha_1(\alpha_1 - 1)}{(k_1 + 2)(k_3 + 1)} a(k_1, k_2, k_3) \langle C^{I_1 C^{I_2} C^{I_3}} \rangle.
\]

The definitions of the symbols \(\sigma, \alpha_{1,2,3}\), the functions \(a(k_1, k_2, k_3)\), \(e(k_1, k_2, k_3)\), \(h(k_1, k_2, k_3)\) and the brackets \(\langle C^{I_1 C^{I_2} C^{I_3}} \rangle, \langle V^{I_1 C^{I_2} C^{I_3}} \rangle, \langle T^{I_1 C^{I_2} C^{I_3}} \rangle\) are given in Appendix A.

### 3.3 Cubic Couplings: Derivation

We use the method developed in [21] to calculate the cubic couplings. The starting point is the quadratic corrections to the field equation for \(s^I\),

\[
\{\nabla^2 - (m_1^I)^2\} s^I = \frac{1}{2(k + 2)} \left\{ (k + 4)(k + 5)Q_1 + Q_2 + (k + 4)(\nabla_\mu Q^I_3 + Q_4) \right\}^I,
\]

(3.6)

where \(Q_{1,2,3,4}\) are defined in equations (3.24) and (3.25) of [21]. To calculate the coupling of two \(s^I\) with another field, say \(\Psi\), one looks for terms proportional to \(s\Psi\) on the right-hand-side of (3.6).

The collection of such terms usually contain higher derivative terms of \(s^I\) which can be removed by a field redefinition. The final product of the calculation takes the form

\[
\{\nabla^2 - (m_1^I)^2\} s^I = \frac{1}{2} \lambda_{IJK}^1 s^I s^K + \lambda_{IJK}^2 \phi^I s^K + \lambda_{IJK}^3 \phi^I s^K
\]

\[
+ \lambda_{IJK}^4 V^I_\mu \nabla_\mu s^K + \lambda_{IJK}^5 W^I_\mu \nabla_\mu s^K + \lambda_{IJK}^6 H^I_{(\mu)} \nabla(\mu \nabla^\nu) s^K.
\]

(3.7)

If we multiply it by the normalization constant \(A_{1_I}^I\), we obtain

\[
A_{1_I}^I \{\nabla^2 - (m_1^I)^2\} s^I = \frac{1}{2} G_{IJK}^1 s^I s^K + G_{IJK}^2 s^I s^K + G_{IJK}^3 s^K
\]

\[
+ G_{IJK}^4 V^I_\mu \nabla_\mu s^K + G_{IJK}^5 W^I_\mu \nabla_\mu s^K + G_{IJK}^6 H^I_{(\mu)} \nabla(\mu \nabla^\nu) s^K.
\]

(3.8)

Remarkably, all the coupling constants \(G_{IJK}^n\) satisfy non-trivial conditions which ensures that the equation of motion can be derived from an action of the form (3.1), (3.2), (3.4). In the following, we briefly sketch the intermediate steps of the calculation that lead to (3.7) for each coupling.

### 3.3.1 Coupling to Scalars: \(sss, sst, s\phi\)

We recall the derivation of \(sss\) vertices from [21]. Collecting all the terms that are quadratic in \(s\) on the right-hand-side of (3.6), one finds

\[
\{\nabla^2 - (m_1^I)^2\} s^I = D_{IJK} s^I s^K + E_{IJK} \nabla_\mu s^I \nabla^\mu s^K + F_{IJK} \nabla(\mu \nabla^\nu) s^I \nabla(\mu \nabla^\nu) s^K.
\]

(3.9)
where $D$, $E$ and $F$ are some functions of $I$, $J$ and $K$. One can remove the derivative terms on the right-hand-side of (3.9) by a field redefinition

$$s^I = s'^I + J_{IJK} s'^J s'^K + L_{IJK} \nabla^\mu s'^J \nabla_\mu s'^K,$$

(3.10)

where

$$L_{IJK} = \frac{1}{2} F_{IJK}, \quad J_{IJK} = \frac{1}{4} F_{IJK} \{(m_1^I)^2 - (m_1^J)^2 - (m_1^K)^2 + 8\} \tag{3.11}$$

such that (3.9) takes the form of (3.7) with

$$\lambda^I_{IJK} = D_{IJK} - \{(m_1^J)^2 + (m_1^K)^2 - (m_1^I)^2\} J_{IJK} - \frac{2}{5} L_{IJK} (m_1^J)^2 (m_1^K)^2 \tag{3.12}$$

In order to compute the $ss^I$ and $ss^\phi$ vertices, one collect the terms proportional to $s^I$ or $s^\phi$ from (3.6). The $s^I$ and $s^\phi$ terms appear on the right-hand-side of (3.9) with the same number of derivatives. The derivative terms can be removed by a field redefinition similar to (3.10), (3.11) with appropriate values of the masses for different fields.

### 3.3.2 Coupling to Vectors: $ss^V$, $ss^W$

The terms proportional to $s^V$ and $s^W$ on the right-hand-side of (3.6) add up to yield

$$\{\nabla^2 - (m_1^I)^2\} s^I = D^4_{IJK} V^J_\mu \nabla^\mu s^k + D^5_{IJK} W^J_\mu \nabla^\mu s^k + E^4_{IJK} \nabla_\mu V^J_\nu \nabla^\nu s^K + E^5_{IJK} \nabla_\mu W^J_\nu \nabla^\nu s^K \tag{3.13}$$

Consider the field redefinition

$$s^I = s'^I + J^4_{IJK} V^J_\mu \nabla^\mu s'^K + J^5_{IJK} W^J_\mu \nabla^\mu s'^K \tag{3.14}$$

To remove the $E_{IJK}$ terms, we set $J_{IJK} = \frac{1}{2} E_{IJK}$. Then, in terms of the redefined field, the equation of motion reads

$$\{\nabla^2 - (m_1^I)^2\} s^I = \lambda^4_{IJK} V^J_\mu \nabla^\mu s^k + \lambda^5_{IJK} W^J_\mu \nabla^\mu s^k, \tag{3.15}$$

where

$$\lambda^{4,5}_{IJK} = D^{4,5}_{IJK} - \frac{1}{2} \{(m_1^J)^2 + (m_1^K)^2 - (m_1^I)^2 - 8\}. \tag{3.16}$$

### 3.3.3 Coupling to Symmetric Tensors: $ss^H_{(\mu\nu)}$

The same method is applicable here, except that there is no need to make a field redefinition.
3.4 Quadratic Action Revisited

In the previous subsection, we obtained the coupling constants using the equation of motion of $s^I$ only. One could use the field equations for other fields to derive the same coupling. For example, the quadratic correction to the equation for $t^I$ will contain a term like

$$\{ \nabla^2 - (m_I^2)^2 \} t^I = \frac{1}{2} \tilde{\lambda}_{IJK} s^J s^K$$

after the same kind of field redefinition we made before. In order for an action of the type (3.1) to exist, the normalization of the $t^I$ kinetic term must satisfy

$$A^2_{tI} \tilde{\lambda}_{IJK} = G^2_{IJK}.$$  \tag{3.18}

where the $G^2_{IJK}$ was obtained in (3.8). This means that one can determine $A^2_{tI}$ by comparing the two equations of motion if $A^1_{tI}$ is known.

One can use this method to determine the normalization constants for all fields from that of $s^I$. One can also reverse the logic and determine the normalization constant of $s^I$ from that of some other field. For example, the Kaluza-Klein tower $H^{I}_{(\mu\nu)}$ contains the $D = 5$ graviton and its massive counterparts. On a general ground, even though a simple form of the covariant action in $D = 10$ is not available, one expect that the action in $D = 5$ will take precisely the form

$$S = \frac{V}{2\kappa^{10}} \int d^5 x \sqrt{-g} \{ R + \cdots \},$$

where $V$ is the volume of the internal space and $R$ is the curvature of the 5-dimensional metric. This fixes the normalization for $H^{I}_{(\mu\nu)}$ completely, and one can use it to determine those for other fields. We have rederived all the normalization constants this way, compared them with the previous results in [28] and found perfect agreement. The details of the computation is not particularly illuminating and we omit it here.

4 On the Massless Multiplet

In view of the fact that most of work on the four-point functions deal with the operators in the massless multiplet, it is of particular interest to examine the couplings of $s^{(ij)}$ corresponding to the operator $\text{Tr} X^{(i}X^{j)}$. Consider the cubic couplings in (3.5) for $k_2 = k_3 = 2$. From the way the $SO(6)$ tensors are contracted, one finds that the couplings can be non-zero only for $k_1 \leq 4$. It is easy to compute the couplings for all such $k_1$ to find that the only non-vanishing couplings are $G^4_{I_1I_2I_3}$ for $k_1 = 1$ and $G^6_{I_1I_2I_3}$ for $k_1 = 0$. From the mass formula in (3.3), one notices that the supergravity fields that couple to two $s^{(ij)}$ are a massless vector and the graviton. They correspond to the R-symmetry current and the energy-momentum tensor operators of SYM$_4$, respectively.
This should not come as a surprise. It is known that the Type IIB supergravity on $AdS_5 \times S^5$ can be consistently truncated to contain the massless multiplet. The resulting theory is identified with the $D=5$, $N=8$, $SO(6)$ gauged supergravity [30, 31, 32]. The fields $t^I$, $\phi^I$ and $W^I_{\mu}$ do not have a component in the massless sector, so in order for the truncation to be possible, they should not couple to two $s^{ij}$. Our computation explicitly shows that it is indeed the case.

One can make a step further. In the gauged supergravity, the coupling of $s^{ij}$ with the vector and the graviton at cubic order is completely determined by gauge invariance and general covariance. That is, the action must take the form (we suppress the parenthesis in $s^{ij}$ below)

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left[ R + 12 - \frac{1}{8g^2} (F_{\mu\nu}^{ij})^2 - \frac{1}{4} \left\{ (Ds^{ij})^2 - 4(s^{ij})^2 \right\} + \cdots \right], \quad (4.1)$$

where the gauge-covariant derivative is defined by

$$D_\mu s^{ij} = \nabla_\mu + V^i_{\mu} s^{kj} + V^j_{\mu} s^{ki}, \quad (4.2)$$

and $V^i_{\mu}$ are real and antisymmetric tensors of $SO(6)$. $F^{ij}_{\mu\nu}$ are the non-abelian field strength.

We want to compare (4.1) with the massless sector of (3.1), (3.2), (3.3), (3.4) and (3.5). First, we can absorb $A^I_1$ into the normalization of $s^{ij}$ to bring the kinetic term into the form (4.1). General covariance of (4.1) then uniquely fixes the $ssH_{(\mu\nu)}$ coupling, since it comes from the expansion of the metric multiplying the kinetic term. Not surprisingly, the result agree with (3.5). Next, we change the normalization of $V_\mu$ such that the cubic coupling in (3.5) is identified with that derived by expanding the covariant derivative (4.2). This fixes the value of Yang-Mills coupling constant $g^2$ in (4.1). Keeping track of numerical factors carefully, one finds that $g^2 = 4$. This is in agreement with an independent analysis of [32].

Note that, contrary to the complicated derivation of the cubic couplings from the Kaluza-Klein reduction of Type IIB supergravity, the action of the gauged supergravity (4.1) is rather simple. Moreover, the complete action and supersymmetry transformation rules are known [30, 31]. Therefore, in order to compute the correlation functions of operators in the massless multiplet only, it may be easier to start from the gauged supergravity rather than the full $D=10$ supergravity. This is perhaps the right way to tackle the formidable task of computing the quartic vertices if one is interested in the four-point functions of the operators in the massless multiplet only.

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Appendix

A Spherical Harmonics Integrals

A.1 Scalar, Vector and Tensor Spherical Harmonics

A self-contained introduction to spherical harmonics is given in [21]. Here we summarize the defining properties of spherical harmonics on $S^5$.

- Scalar spherical harmonics $Y^I$ are defined to be the functions in $\mathbb{R}^6$ of the form
  \[Y^I = C^I_{i_1 \cdots i_k} x^{i_1} \cdots x^{i_k},\]  
  where $x^i$ are the Cartesian coordinates of $\mathbb{R}^6$ and $C^I$ are totally symmetric and traceless tensors of $SO(6)$.

- Vector spherical harmonics $Y^I_\alpha$ are defined to be the tangent components of the vector field in $\mathbb{R}^6$ of the form
  \[Y^I_\alpha = (V^I)^a_{i_1 \cdots i_k} x^{i_1} \cdots x^{i_k},\]  
  where the tensor $(V^I)^a_{i_1 \cdots i_k}$ are symmetric and traceless in $i_1, \cdots, i_k$ and vanishes when symmetrized over $a$ and all $i$s.

- Tensor spherical harmonics $Y^I_{(\alpha \beta)}$ are defined to be the tangents component of the tensor field in $\mathbb{R}^6$ of the form
  \[Y^I_{ab} = (T^I)^{ab}_{i_1 \cdots i_k} x^{i_1} \cdots x^{i_k},\]  
  where the tensor $(T^I)^{ab}_{i_1 \cdots i_k}$ are symmetric and traceless in $i_1, \cdots, i_k$ and $a, b$ and vanishes when symmetrized over $b$ and all $i$s with fixed $a$.

A.2 Quadratic Integrals

The following integrals of the spherical harmonics are needed in the calculation of the main text.

\[
\frac{1}{\omega_5} \int Y^I_1 Y^I_2 = z(k) \langle C^I_1 C^I_2 \rangle, \\
\frac{1}{\omega_5} \int \nabla_\alpha Y^I_1 \nabla^\alpha Y^I_2 = f(k) z(k) \langle C^I_1 C^I_2 \rangle, \\
\frac{1}{\omega_5} \int \nabla_{(\alpha} Y^I_1 \nabla^{\alpha \beta)} Y^I_2 = q(k) z(k) \langle C^I_1 C^I_2 \rangle, \\
\frac{1}{\omega_5} \int Y^I_1 Y^I_2 g^{\alpha \beta} = z(k) \langle V^I_1 V^I_2 \rangle, \\
\frac{1}{\omega_5} \int Y^I_{(\alpha \beta)} Y^I_{(\gamma \delta)} g^{\alpha \gamma} g^{\beta \delta} = z(k) \langle T^I_1 T^I_2 \rangle,
\]  

(A.4)
where we normalized the integral by the volume of a unit 5-sphere $\omega_5 = \pi^3$. The brackets in the formulas are defined by

\[
\langle C^{I_1} C^{I_2} \rangle \equiv C^{I_1}_{i_1 \cdots i_k} C^{I_2}_{i_1 \cdots i_k},
\]

\[
\langle V^{I_1} V^{I_2} \rangle \equiv \langle V^{I_1} \rangle_{i_1 \cdots i_k} (V^{I_2})_{i_1 \cdots i_k},
\]

\[
\langle T^{I_1} T^{I_2} \rangle \equiv \langle T^{I_1} \rangle_{i_1 \cdots i_k} (T^{I_2})_{i_1 \cdots i_k},
\]

(A.5)

The functions $z$, $f$ and $q$ are defined by

\[
z(k) = \frac{1}{(k + 1)(k + 2)2^k},
\]

\[
f(k) = k(k + 4),
\]

\[
q(k) = \frac{4}{3}k(k - 1)(k + 4)(k + 5),
\]

(A.6)

The method of deriving (A.4) is explained in [21].

### A.3 Cubic Integrals

We also need the following cubic integrals of spherical harmonics,

\[
\frac{1}{\omega_5} \int Y^{I_1} Y^{I_2} Y^{I_3} = a(k_1, k_2, k_3) \langle C^{I_1} C^{I_2} C^{I_3} \rangle,
\]

\[
\frac{1}{\omega_5} \int Y^{I_1} \nabla_\alpha Y^{I_2} \nabla_\alpha Y^{I_3} = b(k_1, k_2, k_3) \langle C^{I_1} C^{I_2} C^{I_3} \rangle,
\]

\[
\frac{1}{\omega_5} \int \nabla^{(\alpha} \nabla^{\beta)} Y^{I_1} \nabla_\alpha Y^{I_2} \nabla_\beta Y^{I_3} = c(k_1, k_2, k_3) \langle C^{I_1} C^{I_2} C^{I_3} \rangle,
\]

\[
\frac{1}{\omega_5} \int Y^{I_1} \nabla^{(\alpha} \nabla^{\beta)} Y^{I_2} \nabla^\alpha \nabla^\beta Y^{I_3} = d(k_1, k_2, k_3) \langle C^{I_1} C^{I_2} C^{I_3} \rangle,
\]

\[
\frac{1}{\omega_5} \int Y^{I_1} Y^{I_2} \nabla_\alpha Y^{I_3} = e(k_1, k_2, k_3) \langle V^{I_1} C^{I_2} C^{I_3} \rangle,
\]

\[
\frac{1}{\omega_5} \int Y^{I_1}_\alpha \nabla_\beta Y^{I_2} \nabla^{(\alpha} \nabla^{\beta)} Y^{I_3} = f(k_1, k_2, k_3) \langle V^{I_1} C^{I_2} C^{I_3} \rangle,
\]

\[
\frac{1}{\omega_5} \int \nabla_\alpha Y^{I_1} Y^{I_2} \nabla^{(\alpha} \nabla^{\beta)} Y^{I_3} = g(k_1, k_2, k_3) \langle V^{I_1} C^{I_2} C^{I_3} \rangle,
\]

\[
\frac{1}{\omega_5} \int Y^{I_1}_{(\alpha \beta)} \nabla^{\alpha} Y^{I_2} \nabla^{\beta} Y^{I_3} = h(k_1, k_2, k_3) \langle T^{I_1} C^{I_2} C^{I_3} \rangle,
\]

\[
\frac{1}{\omega_5} \int \nabla_\gamma Y^{I_1}_{(\alpha \beta)} \nabla^{(\alpha} \nabla^{\beta)} Y^{I_2} \nabla_\gamma Y^{I_3} = i(k_1, k_2, k_3) \langle T^{I_1} C^{I_2} C^{I_3} \rangle.
\]

The brackets are defined by

\[
\langle C^{I_1} C^{I_2} C^{I_3} \rangle \equiv C^{I_1}_{\{\alpha_1\}} C^{I_2}_{\{\alpha_2\}} C^{I_3}_{\{\alpha_3\}} C^{I_3}_{\{\alpha_1\} \{\alpha_2\}},
\]

\[
\langle V^{I_1} C^{I_2} C^{I_3} \rangle \equiv \frac{1}{\omega_5} \langle V^{I_1} \rangle_{\{\alpha_2\} \{\alpha_3\}} C^{I_2}_{\{\alpha_3+1/2\} \{\alpha_3+1/2\}} C^{I_3}_{\{\alpha_1-1/2\} \{\alpha_1-1/2\}} C^{I_3}_{\{\alpha_2-1/2\} \{\alpha_2-1/2\}}
\]

\[
= -\langle V^{I_1} C^{I_3} C^{I_2} \rangle,
\]

\[
\langle T^{I_1} C^{I_2} C^{I_3} \rangle \equiv \langle T^{I_1} \rangle_{\{\alpha_2\} \{\alpha_3\}} C^{I_2}_{\{\alpha_3\} \{\alpha_3-1\}} C^{I_3}_{\{\alpha_1\} \{\alpha_1\}},
\]

(A.8)

where we defined

\[
\sigma = \frac{1}{2}(k_1 + k_2 + k_3), \quad \alpha_i = \sigma - k_i \quad (i = 1, 2, 3).
\]

(A.9)
The functions are defined by

\[
a(k_1, k_2, k_3) = \frac{1}{(\sigma+2)!2^{\sigma-1}k_1!k_2!k_3!},
\]
\[
b(k_1, k_2, k_3) = \frac{1}{2}(f_2 + f_3 - f_1)a(k_1, k_2, k_3),
\]
\[
c(k_1, k_2, k_3) = \frac{1}{4}(f_2^2 - f_1^2 - f_3^2 + 2f_2f_3)a(k_1, k_2, k_3) + \frac{3}{8}f_1b(k_1, k_2, k_3),
\]
\[
d(k_1, k_2, k_3) = \frac{1}{2}(f_2 + f_3 - f_1 - 8)b(k_1, k_2, k_3) - \frac{3}{8}f_2f_3a(k_1, k_2, k_3).
\]
\[
e(k_1, k_2, k_3) = \frac{1}{(\sigma+2)!2^{\sigma-3/2}(\alpha_1 - \frac{3}{2})(\alpha_2 - \frac{3}{2})(\alpha_3 - \frac{3}{2})!}k_1!k_2!k_3!,
\]
\[
f(k_1, k_2, k_3) = \left\{\frac{1}{2}(f_2 + f_3 - f_1 - 3) - \frac{1}{6}f_3\right\}e(k_1, k_2, k_3),
\]
\[
g(k_1, k_2, k_3) = \left\{\frac{1}{2}(f_1 + f_3 - f_2 - 5) - \frac{1}{6}f_3\right\}e(k_1, k_2, k_3).
\]
\[
h(k_1, k_2, k_3) = 2(\sigma + 2)\alpha_1a(k_1, k_2, k_3),
\]
\[
i(k_1, k_2, k_3) = \frac{1}{2}(f_2 - f_1 - f_3 - 8)h(k_1, k_2, k_3).
\]

Again, these formulas can be derived using the method of [21].
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