RESOLUTIONS AND HOMOLOGICAL DIMENSIONS OF DG-MODULES

BY

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Dedicated to L. Avramov on his 70th birthday

ABSTRACT

Recently, Yekutieli introduced projective dimension, injective dimension and flat dimension of DG-modules by generalizing the characterization of projective dimension, injective dimension and flat dimension of ordinary modules by vanishing of Ext or Tor-groups. In this paper, we introduce a DG-version of projective, injective and flat resolution for DG-modules over a connective DG-algebra which are different from the known DG-version of projective, injective and flat resolutions. An important feature of these resolutions is that, roughly speaking, the “length” of these resolutions gives projective, injective or flat dimensions. We show that these resolutions allow us to investigate basic properties of projective, injective and flat dimensions of DG-modules. As an application we introduce the global dimension of a connective DG-algebra and show that finiteness of the global dimension is derived invariant.

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1. Introduction

Differential graded (DG) algebra lies at the center of homological algebra and allows us to use techniques of homological algebra of ordinary algebras in a much wider context. The projective resolutions and the injective resolutions which are the fundamental tools of homological algebra already have their DG-versions, which are called a DG-projective resolution and a DG-injective resolution. The aim of this paper is to introduce a different DG-version for DG-modules over a connective DG-algebra. The motivation came from the projective dimensions and the injective dimensions for DG-modules introduced by Yekutieli.

We explain the details by focusing on the projective dimension and the projective resolution. Let $R$ be an ordinary algebra. One of the most fundamental and basic homological invariants to a (right) $R$-module $M$ is the projective dimension $\text{pd}_R M$. Avramov–Foxby [3] generalized the projective dimension to an object of the derived category $M \in \mathcal{D}(R)$. Recently, Yekutieli [15] introduced the projective dimension $\text{pd}_R M$ for an object $M \in \mathcal{D}(R)$ in the case where $R$ is a DG-algebra from the viewpoint that the number $\text{pd}_R M$ measures how the functor $\mathbb{R}\text{Hom}_R(M, -)$ changes the amplitude of the cohomology groups. (In the case where $R$ is an ordinary ring, for a DG-$R$-module $M \in \mathcal{C}(R)$ we have two projective dimensions, by Avramov–Foxby and by Yekutieli. It is explained in Remark 2.2 that they are essentially the same.)

Let $R$ be an ordinary ring and $M$ an $R$-module. Recall that the projective dimension $\text{pd}_R M$ is characterized as the smallest length of projective resolutions $P_*$:

$$0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to M \to 0.$$ 

There exists a notion of DG projective resolution, also called projectively cofibrant replacement and so on, which is a generalization of a projective resolution for a DG-module $M$ over a DG-algebra $R$. However, it is not suitable to measure the projective dimension. The aim of this paper is to introduce a notion of a sup-projective (sppj) resolution of an object of $M \in \mathcal{D}(R)$ which can measure the projective dimension, in the case where $R$ is a connective (=non-positive) DG-algebra.

Recall that a (cohomological) DG-algebra $R$ is called connective if the vanishing condition $H^{>0}(R) = 0$ of the cohomology groups is satisfied. There are rich sources of connective DG-algebras: the Koszul algebra $K_R(x_1, \ldots, x_d)$ in commutative ring theory, and an endomorphism DG-algebra $\mathbb{R}\text{Hom}(S, S)$ of a
silting object $S$ (for a silting object see [1]). We would like to point out that commutative connective DG-algebras are regarded as the coordinate algebras of derived affine schemes in derived algebraic geometry (see, e.g., [5]).

Let $R$ be a connective DG-algebra. We set $\mathcal{P} := \text{CoProd}^\oplus R \subset D(R)$ to be the full subcategory consisting of every $M \in D(R)$ which is a direct summand of a coproduct of copies of $R$. In sppj resolution, $\mathcal{P}$ plays the role of projective modules in the usual projective resolution.

A sppj resolution $P_\bullet$ of $M \in D^{<\infty}(R)$ is a sequence of exact triangles $\{E_i\}_{i \geq 0}$

$$E_i : M_{i+1} \xrightarrow{g_{i+1}} P_i \xrightarrow{f_i} M_i \to$$

such that $f_i$ is a sppj morphism (Definition 2.12), where we set $M_0 := M$. We often exhibit a sppj resolution $P_\bullet$ as below by splicing $\{E_i\}_{i \geq 0}$

$$P_\bullet : \cdots \to P_i \xrightarrow{\delta_i} P_{i-1} \xrightarrow{\delta_{i-1}} \cdots \to P_1 \xrightarrow{\delta_1} P_0 \to M$$

where we set $\delta_i := g_if_i$. It is analogous to that in the case where $R$ is an ordinary algebra; a projective resolution $P_\bullet$ of an $R$-module $M$ is constructed by splicing exact sequences

$$0 \to M_{i+1} \to P_i \to M_i \to 0$$

with $P_i$ projective.

We state the main result which gives equivalent conditions of $\text{pd} M = d$. For a DG-$R$-module $M$, we set $\text{sup} M := \text{sup}\{n \in \mathbb{Z} \mid H^n(M) \neq 0\}$. We denote by $D^{<\infty}(R)$ the derived category of DG-$R$-modules $M$ bounded from above, i.e., $\text{sup} M < \infty$. We note that if $M \in D(R)$ has finite projective dimension, then it belongs to $D^{<\infty}(R)$ (Lemma 2.3).

**Theorem 1.1 (Theorem 2.22):** Let $M \in D^{<\infty}(R)$ and $d \in \mathbb{N}$ be a natural number. Then the following conditions are equivalent:

1. $\text{pd} M = d$.
2. For any sppj resolution $P_\bullet$, there exists a natural number $e \in \mathbb{N}$ which satisfies the following properties:
   (a) $M_e$ belongs to $\mathcal{P}[-\text{sup}M_e]$.
   (b) $d = e + \text{sup}P_0 - \text{sup}M_e$.
   (c) The structure morphism $g_e : M_e \to P_{e-1}$ is not a split-monomorphism.
(3) $M$ has sppj resolution $P_\bullet$ of length $e$ which satisfies the following properties:

$$P_e \xrightarrow{\delta_e} P_{e-1} \xrightarrow{\delta_{e-1}} \cdots P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{f_0} M.$$ 

(a) $d = e + \sup P_0 - \sup P_e$.

(b) The $e$-th differential $\delta_e$ is not a split-monomorphism.

(4) The functor $F = \mathbb{R}\text{Hom}(M,-)$ sends the standard heart $\text{Mod}\, H^0(R)$ to $D[-\sup M, d-\sup M](R)$ and there exists $N \in \text{Mod} \, H^0$ such that

$$H^{d-\sup M}(F(N)) \neq 0.$$ 

(5) $d$ is the smallest number which satisfies

$$M \in \mathcal{P}[-\sup M] \ast \mathcal{P}[-\sup M + 1] \ast \cdots \ast \mathcal{P}[-\sup M + d].$$

The condition (4) tells that the projective dimension of $M$ can be measured by only looking at the standard heart $\text{Mod}\, H^0(R)$ of the derived category $D(R)$. The condition (5) says that the projective dimension $\text{pd} \, M$ is the smallest number of extensions by which we obtain $M$ from the “projective objects” $\mathcal{P}$ (see Definition 2.7).

Similar results for the injective dimension and the flat dimensions are given in Theorem 3.24 and Theorem 4.13.

In the final part of this paper, we introduce the global dimension $\text{gldim} \, R$ of a connective DG-algebra $R$. For an ordinary ring $R$, a key result to define the global dimension $\text{gldim} \, R$ is that the supremum of the projective dimensions $\text{pd} \, M$ of all $R$-modules $M$ and that of the injective dimensions $\text{id} \, M$ coincide. We provide a similar result for a connective DG-algebra $R$. It is well-known that the ordinary global dimensions are not preserved by derived equivalence, but their finiteness is preserved. We prove the DG-version of this result.

In the subsequent paper [11] we use resolutions developed in this paper to study connective commutative DG-algebras (CDGA), more precisely, piecewise Noetherian CDGA, which is a DG-counterpart of commutative Noetherian algebra. We develop basic notions (e.g., depth, localization, Bass numbers) and establish their properties (e.g., the Auslander–Buchsbaum formula). We show that results about minimal ifij resolutions are completely analogous to classical results about minimal injective resolution. Moreover, we observe that a DG-counter part $E_R(R/\mathfrak{p}')$ of the class of indecomposable injective modules are parametrized by prime ideals $\mathfrak{p} \in \text{Spec} \, H^0$ of the 0-th cohomology algebra

$$H^0 := H^0(R).$$
This fact is compatible with the viewpoint of derived algebraic geometry that the base affine scheme of the derived affine scheme \( \text{Spec} \, R \) associated to a CDGA \( R \) is the affine scheme \( \text{Spec} \, H^0 \).

These facts support that the \( \text{ifij} \) resolution is a proper generalization of the injective resolution. We can expect that it becomes an indispensable tool for studying DG-modules like an ordinary injective resolution for studying modules.

The paper is organized as follows. In Section 2, we introduce and study sppj resolutions. Section 3 deals with inf-injective(\( \text{ifij} \)) resolutions. Since the basic properties are proved in the same way as that of the similar statement of sppj resolutions, most of all proofs are omitted. However, we need to study the class \( I \) which plays the role of injective modules for an ordinary injective resolution. This class of DG-modules was already studied by Shaul in [14], in which he denoted \( I \) by \( \text{Inj} \, R \). But we take a different approach to the class \( I \).

Section 4 deals with a sup-flat (spft) resolution. A description of the class \( F \) of flat dimension 0 is still a conjecture. In Section 5 we introduce the global dimension gldim \( R \) of a connective DG-algebra \( R \) and prove that finiteness is preserved by derived equivalence.

1.1. Notation and convention. The basic setup and notations are the following.

Throughout the paper, we fix a base commutative ring \( k \) and a (DG, graded) algebra is a (DG, graded) algebra over \( k \). We denote by \( R = (R, \partial) \) a connective cohomological DG-algebra. Recall that “connective” means that \( H^>0(R) = 0 \). We note that every connective DG-algebra \( R \) is quasi-isomorphic to a DG-algebra \( S \) such that \( S^>0 = 0 \). Since quasi-isomorphic DG-algebras have equivalent derived categories, it is harmless to assume that \( R^>0 = 0 \) for our purpose.

For simplicity we denote by \( H := \text{H}(R) \) the cohomology algebra of \( R \), by \( H^0 := H^0(R) \) the 0-th cohomology algebra of \( R \). We denote by \( \text{Mod}^Z \, H \) the category of graded \( H \)-modules, by \( \text{Mod} \, H^0 \) the category of \( H^0 \)-modules.

We denote by \( C(R) \) the category of DG-\( R \)-modules and cochain morphisms, by \( K(R) \) the homotopy category of DG-\( R \)-modules and by \( D(R) \) the derived category of DG-\( R \)-modules. The symbol \( \text{Hom} \) denotes the \( \text{Hom} \)-space of \( D(R) \).

Let \( n \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\} \). The symbols \( D^{<n}(R), \, D^{>n}(R) \) denote the full subcategories of \( D(R) \) consisting of \( M \) such that \( H^{>n}(M) = 0, \, H^{\leq n}(M) = 0 \) respectively. We set

\[
D^{[a,b]}(R) = D^{>a}(R) \cap D^{\leq b}(R) \quad \text{for} \quad a, b \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}, \, a \leq b.
\]
We set
\[ D^b(R) := D^{<\infty}(R) \cap D^{>\infty}(R). \]

Since \( R \) is connective, the pair \((D^{\leq 0}(R), D^{\geq 0}(R))\) is a \( t \)-structure in \( D(R) \), which is called the standard \( t \)-structure. The truncation functors are denoted by \( \sigma^{<n}, \sigma^{>n} \). We identify the heart \( H = D^{\leq 0}(R) \cap D^{\geq 0}(R) \) of the standard \( t \)-structure with \( \text{Mod} H^0 \) via the functor \( \text{Hom}(H^0, -) \), which fits into the following commutative diagram:

\[
\begin{array}{ccc}
D(R) & \xrightarrow{\text{can}} & H^0 \\
\downarrow_{\text{Hom}(R, -)} & \swarrow_{\text{can}} & \searrow_{\text{Hom}(H^0, -)} \\
\text{Mod} H^0, & & D(R) \\
\end{array}
\]

where \text{can} is the canonical inclusion functor and \( f_* \) is the restriction functor along a canonical projection \( f : R \to H^0 \).

For a DG-\( R \)-module \( M \neq 0 \), we set \( \inf M := \inf \{ n \in \mathbb{Z} \mid H^n(M) \neq 0 \} \), \( \sup M := \sup \{ n \in \mathbb{Z} \mid H^n(M) \neq 0 \} \), \( \text{amp} M := \sup M - \inf M \). In the case \( \inf M > -\infty \), we use the abbreviation \( H^\inf(M) := H^{\inf M}(M) \). Similarly in the case \( \sup M < \infty \), we use the abbreviation \( H^{\sup}(M) := H^{\sup M}(M) \). We formally set \( \inf 0 := \infty \) and \( \sup 0 := -\infty \).

In the case where we need to indicate the DG-algebra \( R \), we denote \( \sup_R M \), \( \inf_R M \) and \( \text{amp}_R M \).

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2. Sup-projective resolutions of DG-modules

2.1. Projective dimension of $M \in \mathcal{D}(R)$ after Yekutieli. We recall the definition of the projective dimension of $M \in \mathcal{D}(R)$ introduced by Yekutieli.

**Definition 2.1** ([15, Definition 2.4]): Let $a \leq b \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$.

1. An object $M \in \mathcal{D}(R)$ is said to have **projective concentration** $[a, b]$ if the functor $F = R\text{Hom}_R(M, -)$ sends $\mathcal{D}^{[m,n]}(R)$ to $\mathcal{D}^{[m-b,n-a]}(k)$ for any $m \leq n \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$:

   $$F(\mathcal{D}^{[m,n]}(R)) \subset \mathcal{D}^{[m-b,n-a]}(k).$$

2. An object $M \in \mathcal{D}(R)$ is said to have **strict projective concentration** $[a, b]$ if it has projective concentration $[a, b]$ and doesn’t have projective concentration $[c, d]$ such that $[c, d] \subset [a, b]$.

3. An object $M \in \mathcal{D}(R)$ is said to have **projective dimension** $d \in \mathbb{N}$ if it has strict projective concentration $[a, b]$ for $a, b \in \mathbb{Z}$ such that $d = b - a$.

   In the case where $M$ does not have a finite interval as projective concentration, it is said to have **infinite projective dimension**.

   We denote the projective dimension by $\text{pd} M$.

**Remark 2.2:** Let $R$ be an ordinary algebra. Avramov–Foxby [3] introduced another projective dimension for a complex $M \in \mathcal{C}(R)$. If we denote by $\text{AF pd} M$ the projective dimension of Avramov–Foxby, then it is easy to see that

$$\text{AF pd} M = \text{pd} M - \sup M.$$

The following lemma is proved later after the proof of Theorem 3.10.

**Lemma 2.3:** If $M \in \mathcal{D}(R)$ has finite projective dimension, then it belongs to $\mathcal{D}^{<\infty}(R)$.

The following lemma is deduced from the property of the standard $t$-structure of $\mathcal{D}(R)$.

**Lemma 2.4:** Let $M \in \mathcal{D}^{<\infty}(R)$ and $F := R\text{Hom}(M, -)$. Then for all $m \leq n$,

$$F(\mathcal{D}^{[m,n]}(R)) \subset \mathcal{D}^{[m-\sup M, \infty]}(R)$$

and there is $N \in \mathcal{D}^{[m,n]}(R)$ such that

$$H^{m-\sup M}(F(N)) \neq 0.$$
Proof. Let $\ell < m - \sup M$. Then $H^\ell(F(N)) = \text{Hom}(M, N[\ell]) = 0$ for $N \in D^{[m,n]}(R)$. This proves the first statement.

The standard $t$-structure induces a nonzero morphism $M \to H^{\sup}M[\sup M]$. Thus, $N = H^{\sup}(M)[-m]$ has the desired property. □

We deduce the following useful corollaries.

**Corollary 2.5:** An object $M \in D^{<\infty}(R)$ has projective dimension $d$ if and only if it has strict projective concentration $[\sup M - d, \sup M]$.

**Corollary 2.6:** Let $L \to M \to N \to$ be an exact triangle in $D(R)$. Then, we have

$$\text{pd } M - \sup M \leq \sup \{\text{pd } L - \sup L, \text{pd } N - \sup N\}.$$  

2.2. **The class $\mathcal{P}$ and sup-projective (sppj) resolution.** The class $\mathcal{P}$ plays the role of projective modules in the usual projective resolutions for sup-projective resolutions.

**Definition 2.7:** We denote by $\mathcal{P} = \text{CoProd}^\otimes \subset D(R)$ the full subcategory of direct summands of a coproduct of copies of $R$.

The basic properties of $\mathcal{P}$ are summarized in the lemma below. By $\text{Proj } H^0$ we denote the full subcategory of projective $H^0$-modules.

**Lemma 2.8:**

1. For $N \in D(R)$ and $P \in \mathcal{P}$, the map below associated to the $0$-th cohomology functor $H^0$ is an isomorphism:

$$\text{Hom}(P, N) \xrightarrow{\sim} \text{Hom}(H^0(P), H^0(N)).$$

2. For $N \in \text{Mod } H^0$, we have

$$\text{Hom}(P, N[n]) = \begin{cases} 
\text{Hom}_{\text{Mod } H^0}(H^0(P), N), & n = 0, \\
0, & n \neq 0.
\end{cases}$$

3. The functor $H^0$ induces an equivalence $\mathcal{P} \cong \text{Proj } H^0$.

**Remark 2.9:** Lurie [10, Section 7.2] studied the class $\mathcal{P}$ for $\mathbb{E}_1$-algebras and obtained the parallel result to (3) of the above lemma in [10, Corollary 7.2.2.19].

**Proof.** (1) is a consequence of the isomorphism $\text{Hom}(R, N) \cong H^0(N)$ for $N \in D(R)$. (2) is an immediate consequence of (1).
By (1), the functor $H^0 : \mathcal{P} \to \text{Proj} H^0$ is fully faithful. We prove it is essentially surjective. Let $Q \in \text{Proj} H^0$. We need to show that there exists $\mathcal{Q} \in \mathcal{P}$ such that $H(\mathcal{Q}) \cong Q$. The case where $Q$ is a free $H^0$-module is clear. We deal with the general cases. Then, there exists a free $H^0$-module $F$ and an idempotent element $e \in \text{End}_{H^0}(F)$ whose kernel is $Q$. Let $\mathcal{F} \in \mathcal{P}$ be such that $H(\mathcal{F}) \cong F$. By (1), there exists an idempotent element $e \in \text{End} \mathcal{F}$ which is sent to $e$ by the map associated to $H^0$. It is easy to check that the direct summand of $\mathcal{F}$ corresponding to $e$ has the desired property.

Remark 2.10: Almost all the results of the rest of this section are deduced from the properties of $\mathcal{P}$ given in Lemma 2.8. We can state and prove these results in an abstract setting of a triangulated category with $t$-structure whose heart has enough projectives.

By (1), it is clear that any object $P \in \mathcal{P}$ has projective dimension 0. Using this fact and standard arguments in triangulated categories, we obtain the following corollary.

Recall that for full subcategories $\mathcal{X}, \mathcal{Y} \subset \text{D}(R)$, we define a full subcategory $\mathcal{X} \ast \mathcal{Y}$ to be the full subcategory consisting of $Z \in \text{D}(R)$ which fits into an exact triangle $X \to Z \to Y \to$ with $X \in \mathcal{X}, Y \in \mathcal{Y}$. Note that this operation is associative, i.e., $(\mathcal{X} \ast \mathcal{Y}) \ast Z = \mathcal{X} \ast (\mathcal{Y} \ast Z)$. We remark that if $Z \in \text{D}(R)$ satisfies $\text{Hom}(Z, \mathcal{X}) = 0$, $\text{Hom}(Z, \mathcal{Y}) = 0$, then $\text{Hom}(Z, \mathcal{X} \ast \mathcal{Y}) = 0$.

Corollary 2.11: Let $a, b \in \mathbb{Z}$ such that $a \leq b$. Then for any object $M \in \mathcal{P}[a] \ast \mathcal{P}[a+1] \ast \cdots \ast \mathcal{P}[b]$, we have $\text{pd} M \leq b - a$.

When we construct a usual projective resolution for an ordinary module, we use surjective homomorphisms from a projective module. Next we introduce the notion which plays that role for our sup-projective resolution.

Definition 2.12 (sppj morphisms): Let $M \in \text{D}^{<\infty}(R), M \neq 0$.

1. A sppj morphism $f : P \to M$ is a morphism in $\text{D}(R)$ such that $P \in \mathcal{P}[-\text{sup} M]$ and the morphism $H^{\text{sup} M}(f)$ is surjective.

2. A sppj morphism $f : P \to M$ is called minimal if the morphism $H^{\text{sup} M}(f)$ is a projective cover.

By Lemma 2.8, for any $M \in \text{D}^{<\infty}(R)$, there exists a sppj morphism $f : P \to M$. The following two lemmas give a motivation to introduce sppj resolutions.
Lemma 2.13: Let $M \in D^{<\infty}(R)$ and $f : P \to M$ a sppj morphism and $N := \text{cn}(f)[-1]$ the cocone of $f$. Assume that $1 \leq \text{pd} M$. Then,

$$\text{pd} N = \text{pd} M - 1 - \sup M + \sup N.$$ 

Proof. Set $d = \text{pd} M, F = \mathbb{R}\text{Hom}(M, -)$ and let $g : N \to P$ be the canonical morphism. We may assume that $\sup M = 0$ by shifting the degree.

We claim that $N$ has projective concentration $[1 - d, 0]$. Let

$$m \leq n \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\} \quad \text{and} \quad L \in D^{[m,n]}(R).$$

We need to prove $\text{Hom}(N, L[i]) = 0$ for $i \in \mathbb{Z} \setminus [m, n + d - 1]$.

By the assumption, we have $\text{Hom}(M, L[i]) = 0$ for $i \in \mathbb{Z} \setminus [m, n + d]$. We also have $\text{Hom}(P, L[i]) = 0$ for $i \in \mathbb{Z} \setminus [m, n]$. We consider the exact sequence

$$(2-1) \quad \text{Hom}(P, L[i]) \to \text{Hom}(N, L[i]) \to \text{Hom}(M, L[i + 1]) \to \text{Hom}(P, L[i + 1]).$$

Using this, we can check that $\text{Hom}(N, L[i]) = 0$ for $i \in \mathbb{Z} \setminus [m - 1, n + d - 1]$. It remains to prove the case $i = m - 1$. In that case the exact sequence (2-1) becomes

$$(2-2) \quad 0 \to \text{Hom}(N, L[m - 1]) \to \text{Hom}(M, L[m]) \overset{f_*}{\to} \text{Hom}(P, L[m]).$$

Under the isomorphisms

$$\text{Hom}(M, L[m]) \cong \text{Hom}(H^0(M), H^m(L)), \quad \text{Hom}(P, L[m]) \cong \text{Hom}(H^0(P), H^m(L))$$

the map $f_*$ corresponds to the map $\text{Hom}(H^0(f), H^m(L))$, which is injective. Thus, we conclude that $\text{Hom}(N, L[m - 1]) = 0$. This completes the proof of the claim.

By Lemma 2.5 and the claim, $N$ has projective concentration $[1 - d, \sup N]$. Let $[a, b]$ be a strict projective concentration of $N$. Then, by Lemma 2.5,

$$1 - d \leq a = \sup N - \text{pd} N, b = \sup N.$$ 

Therefore to prove the desired formula $\text{pd} N = d - 1 + \sup N$. It is enough to show that there exists $m \leq n \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$ and $L \in D^{[m,n]}(R)$ such that $\text{Hom}(N, L[n + d - 1]) \neq 0$.

Since $d = \text{pd} M$, there exists $m \leq n \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$ and $L \in D^{[m,n]}(R)$ such that $\text{Hom}(M, L[n + d]) \neq 0$. Since we assume that $d \leq 1$, we have $\text{Hom}(P, L[n + d]) = 0$. Therefore from the $i = n + d - 1$ case of the exact sequence (2-1), we deduce that there exists a surjection

$$\text{Hom}(N, L[n + d - 1]) \twoheadrightarrow \text{Hom}(M, L[n + d]) \neq 0.$$ 

Thus, we conclude that $\text{Hom}(N, L[n + d - 1]) \neq 0$ as desired. ■
Lemma 2.14: Let $M \in \mathcal{D}(R) \setminus \{0\}$. Then $\text{pd } M = 0$ if and only if $M \in \mathcal{P}[\sup M]$. 

Proof. The “if” part is clear. We prove the “only if” part. Let $M \in \mathcal{D}(R)$ be such that $\text{pd } M = 0$. By Lemma 2.4, $\sup M < \infty$. Shifting the degree, we may assume $\sup M = 0$. We take an exact triangle below with $f$ sppj

\[(2-3) \quad N \xrightarrow{g} P \xrightarrow{f} M \rightarrow \]

Since $\mathcal{P}$ is closed under taking a direct summand, it is enough to show that $g$ is a split monomorphism.

Observe that $N \in \mathcal{D}^{\leq 0}(R)$. The above exact triangle induces the exact sequence

\[0 \rightarrow \text{Hom}(M, H^0(N)) \rightarrow \text{Hom}(P, H^0(N)) \xrightarrow{g_\ast} \text{Hom}(N, H^0(N)) \rightarrow 0\]

where we use $\text{Hom}(M, H^0(N)[1]) = 0$. Under the isomorphisms

\[
\begin{align*}
\text{Hom}(P, H^0(N)) &\cong \text{Hom}(H^0(P), H^0(N)), \\
\text{Hom}(N, H^0(N)) &\cong \text{Hom}(H^0(N), H^0(N))
\end{align*}
\]

the morphism $g_\ast$ corresponds to the morphism $\text{Hom}(H^0(g), H^0(N))$. Let $h : N \rightarrow H^0(N)$ be a canonical morphism. Then, there exists a morphism $k : P \rightarrow H^0(N)$ such that $kg = h$. Note that $h \in \text{Hom}(N, H^0(N))$ corresponds to $\text{id}_{H^0(N)} \in \text{Hom}(H^0(N), H^0(N))$ under the above isomorphism and hence

\[H^0(k)H^0(g) = \text{id}_{H^0(N)}.
\]

Since $h$ induces an isomorphism $\text{Hom}(P, N) \xrightarrow{\cong} \text{Hom}(P, H^0(N))$, there exists a morphism $\ell : P \rightarrow N$ such that $h\ell = k$.

Check that $H^0(\ell)H^0(g) = \text{id}_{H^0(N)}$. Therefore if we set $e = g\ell$, then $H^0(e)$ is an idempotent element of $\text{End}(H^0(P))$. Since the 0-th cohomology group functor $H^0$ induces an isomorphism $\text{End}(P) \cong \text{End}(H^0(P))$, we conclude that $e$ is an idempotent element. Let $Q$ be the corresponding direct summand of $P$. More precisely, $Q$ is an object of $\mathcal{D}(R)$ equipped with morphisms $i : Q \rightarrow P$.
and \( j : P \to Q \) such that \( ji = \text{id}_Q \) and \( ij = e \). Then, we have \( g\ell i = ei = iji = i \) and \( jg\ell i = \text{id}_Q \). This shows that there exists an isomorphism \( N \cong Q \oplus N' \) for some \( N' \in \text{D}(R) \) under which the morphism \( g \) corresponds to \( (i\ g') \) for some \( g' : N' \to P \).

\[
g : N \cong Q \oplus N' \xrightarrow{(i\ g')} P.
\]

If we show \( N' = 0 \), then \( Q \cong N \) and we finish the proof.

Assume on the contrary that \( N' \neq 0 \). First observe that \( \text{H}^0(Q) = \text{H}^0(N) \) by construction. Therefore, \( s := \sup N' < 0 \). We have \( \text{Hom}(Q \oplus N', \text{H}^s(N)[-s]) \neq 0 \).

On the other hand, applying \( \text{Hom}(-, \text{H}^s(N)[-s]) \) to the exact triangle (2-3), we obtain an exact sequence

\[
\text{Hom}(P, \text{H}^s(N)[-s]) \to \text{Hom}(N, \text{H}^s(N)[-s]) \to \text{Hom}(M, \text{H}^s(N)[-s + 1]).
\]

Since \( s < 0 \), both sides of this sequence are zero. Hence we conclude that \( \text{Hom}(Q \oplus N', \text{H}^s(N)[-s]) = 0 \). A contradiction. \( \blacksquare \)

In the proof we showed that

**Corollary 2.15:** Let \( M \in \text{D}(R) \) be such that \( \text{pd} M = 0 \) and \( f : P \to M \) a \text{sppj} morphism. Then, the cocone \( N = \text{cn} f[-1] \) has \( \text{pd} N = 0 \).

It is worth noting the following corollary:

**Corollary 2.16:** The full subcategory \( \mathcal{P} \subset \text{D}(R) \) consists of objects \( P \) such that either \( P = 0 \) or \( \text{pd} P = 0 \) and \( \sup P = 0 \).

Now from Lemma 2.13 and Lemma 2.14 it is clear how to define a resolution of a DG-module which computes its projective dimension.

**Definition 2.17** (sppj resolutions): (1) A **sppj resolution** \( P_* \) of \( M \) is a sequence of exact triangles for \( i \geq 0 \) with \( M_0 := M \),

\[
M_{i+1} \xrightarrow{g_{i+1}} P_i \xrightarrow{f_i} M_i,
\]

such that \( f_i \) is sppj.

The following inequality holds:

\[
\sup M_{i+1} = \sup P_{i+1} \leq \sup P_i = \sup M_i.
\]

For a sppj resolution \( P_* \) with the above notations, we set \( \delta_i := g_{i-1} \circ f_i \),

\[
\delta_i : P_i \to P_{i-1}.
\]

Moreover we write

\[
\cdots \to P_i \xrightarrow{\delta_i} P_{i-1} \to \cdots \to P_1 \xrightarrow{\delta_1} P_0 \to M.
\]
(2) A sppj resolution $P_\bullet$ is said to have **length** $e$ if $P_i = 0$ for $i > e$ and $P_e \neq 0$.

(3) A sppj resolution $P_\bullet$ is called **minimal** if $f_i$ is minimal for $i \geq 0$.

Using sppj resolution, we can compute $\text{Hom}(M, N[n])$ for $N \in \text{mod} H^0$.

**Lemma 2.18:** Let $N \in \text{mod} H^0$, $M \in D^{<\infty}(R)$ and $P_\bullet$ be a sppj resolution of $M$. We denote the complexes below by $X_i, X_i'$:

$$
X_i : \text{Hom}(P_{i-1}, N[-\sup P_i]) \rightarrow \text{Hom}(P_i, N[-\sup P_i]) \rightarrow \text{Hom}(P_{i+1}, N[-\sup P_i]),
$$

$$
X_i' : \text{Hom}(H^{\sup}(P_{i-1}), N) \rightarrow \text{Hom}(H^{\sup}(P_i), N) \rightarrow \text{Hom}(H^{\sup}(P_{i+1}), N).
$$

Then,

$$
\text{Hom}(M, N[n]) = \begin{cases}
0, & n \neq i - \sup P_i \text{ for any } i \geq 0, \\
\text{H}(X_i), & n = i - \sup P_i \text{ for some } i \geq 0.
\end{cases}
$$

Moreover, in the case $n = i - \sup P_i$, we have

$$
\text{H}(X_i) = \begin{cases}
\text{Hom}(H^{\sup}(P_i), N), & \sup P_{i-1} \neq \sup P_i \neq \sup P_{i+1}, \\
\text{Ker[Hom}(H^{\sup}(P_i), N) \rightarrow \text{Hom}(H^{\sup}(P_{i+1}), N)], & \sup P_{i-1} \neq \sup P_i = \sup P_{i+1}, \\
\text{Cok[Hom}(H^{\sup}(P_{i-1}), N) \rightarrow \text{Hom}(H^{\sup}(P_i), N)], & \sup P_{i-1} = \sup P_i \neq \sup P_{i+1}, \\
\text{H}(X_i'), & \sup P_{i-1} = \sup P_i = \sup P_{i+1}.
\end{cases}
$$

We note that for $i \neq j$, we have

$$
i - \sup P_i \neq j - \sup P_j,
i - \sup P_i + 1 \neq j - \sup P_j + 1.
$$

We also note that a pair $i, j \geq 0$ satisfies $i - \sup P_i + 1 = j - \sup P_j$ if and only if $j = i + 1$ and $\sup P_i = \sup P_j$. 
Proof. For simplicity we set $t_i := -\sup P_i$. We have

$$\text{Hom}(P_i, N[n]) \cong \begin{cases} \text{Hom}_{\text{Mod } H^0}(H^{\sup}(P_i), N), & n = t_i, \\ 0, & n \neq t_i. \end{cases}$$

From the induced exact sequence

$$\text{Hom}(P_i, N[n-1]) \rightarrow \text{Hom}(M_{i+1}, N[n-1]) \rightarrow \text{Hom}(M_i, N[n]) \rightarrow \text{Hom}(P_i, N[n]),$$

we deduce the following isomorphism and exact sequences.

For $n \neq t_i, t_i + 1$, we have an isomorphism

$$(2-4) \quad \text{Hom}(M_{i+1}, N[n - 1]) \cong \text{Hom}(M_i, N[n]).$$

We have the exact sequence

$$0 = \text{Hom}(M_{i+1}, N[t_i - 1]) \rightarrow \text{Hom}(M_i, N[t_i]) \rightarrow \text{Hom}(P_i, N[t_i]) \rightarrow \text{Hom}(M_{i+1}, N[t_i]) \rightarrow 0.$$

The first term is zero since $\sup M_{i+1} \leq \sup P_i$. We note that in a similar way we see that the induced map $\text{Hom}(M_{i+1}, N[t_i]) \rightarrow \text{Hom}(P_{i+1}, N[t_i])$ is injective.

We have the exact sequence

$$(2-5) \quad \text{Hom}(P_{i-1}, N[t_i]) \rightarrow \text{Hom}(M_i, N[t_i]) \rightarrow \text{Hom}(M_{i-1}, N[t_i + 1]) \rightarrow \text{Hom}(P_{i-1}, N[t_i + 1]) = 0.$$ 

The last term is zero, since $t_i + 1 > t_{i-1}$.

Combining above observations, we obtain the following diagram, whose row and columns are exact:

\[
\begin{array}{cccc}
\text{Hom}(P_{i-1}, N[t_i]) & & & 0 \\
& \downarrow & \text{Hom}(M_i, N[t_i]) & \text{Hom}(P_i, N[t_i]) & \text{Hom}(M_{i+1}, N[t_i]) \\
0 & \rightarrow & \downarrow & \rightarrow & \downarrow \\
& \text{Hom}(M_{i-1}, N[t_i + 1]) & \rightarrow & \text{Hom}(P_{i+1}, N[t_i]) & \\
& & \downarrow & & \\
& 0 & & & \\
\end{array}
\]

Observe that the slant line is the complex $X_i$. Thus we conclude that

$$H(X_i) \cong \text{Hom}(M_{i-1}, N[t_i + 1]).$$
We deal with the case \( n = i + t_i \) for some \( i \geq 0 \). Then, \( n \neq j + t_j + 1, j + t_j \) for \( j = 0, \ldots, i - 2 \). Therefore we have the following isomorphisms:

\[
\text{Hom}(M, N[n]) \cong \text{Hom}(M_1, N[n - 1]) \cong \cdots \cong \text{Hom}(M_{i-1}, N[n - i + 1]).
\]

Since \( n - i + 1 = t_i + 1 \), we obtain the desired isomorphism

\[
\text{Hom}(M, N[n]) \cong \text{Hom}(M_{i-1}, N[t_i + 1]) \cong \text{H}(X_i).
\]

Next we deal with the case \( n \neq i + t_i \) for any \( i \geq 0 \). This case is divided into the following two cases:

(I) \( n = i + t_i + 1 \) for some \( i \geq 0 \).
(II) \( n \neq i + t_i + 1 \) for any \( i \geq 0 \).

We deal with the case (I). Then, \( n \neq j + t_j + 1, j + t_j \) for \( j = 0, \ldots, i - 1 \). Therefore we have the following isomorphisms:

\[
\text{Hom}(M, N[n]) \cong \text{Hom}(M_1, N[n - 1]) \cong \cdots \cong \text{Hom}(M_i, N[n - i]).
\]

Since \( n - i = t_i + 1 \),

\[
\text{Hom}(M, N[n]) = \text{Hom}(M_i, N[t_i + 1]).
\]

We only have to show that \( \text{Hom}(M_i, N[t_i + 1]) = 0 \). We have \( t_i \neq t_{i+1} \), since otherwise \( n = (i+1)+t_{i+1} \). Thus \( \text{Hom}(P_{i+1}, N[t_i]) = 0 \). As we mentioned before the induced map \( \text{Hom}(M_{i+1}, N[t_i]) \rightarrow \text{Hom}(P_{i+1}, N[t_i]) \) is injective. Thus, \( \text{Hom}(M_{i+1}, N[t_i]) = 0 \). Finally, from the exact sequence (2-5), we obtain a surjection \( \text{Hom}(M_{i+1}, N[t_i]) \rightarrow \text{Hom}(M_i, N[t_i + 1]) \). Thus, we conclude that \( \text{Hom}(M_i, N[t_i + 1]) = 0 \) as desired.

We deal with the case (II). Namely we assume \( n \neq i + t_i, i + t_i + 1 \) for any \( i \geq 0 \). This case is divided into the following two cases:

(II-i) \( n < t_0 \); (II-ii) \( n > t_0 \).

The case (II-i). Since \( M \in D^{\leq -t_0}(R), N[n] \in D^{> -t_0}(R) \), we have

\[
\text{Hom}(M, N[n]) = 0.
\]

The case (II-ii). Let \( i = n - t_0 + 1 \). Then we have isomorphisms

\[
\text{Hom}(M, N[n]) \cong \text{Hom}(M_1, N[n - 1]) \cong \cdots \cong \text{Hom}(M_i, N[n - i]).
\]

Since \( \text{sup} M_i = \text{sup} P_i = -t_i \leq -t_0 < -(n - i) \), using the standard \( t \)-structure as above, we deduce that \( \text{Hom}(M_i, N[n - i]) = 0 \).
The following theorem provides criteria for a natural number to be an upper bound of the projective dimension \( \text{pd} M \) of a DG-\( R \)-module \( M \).

**Theorem 2.19:** Let \( M \in \mathcal{D}^{<\infty}(R) \) and \( d \in \mathbb{N} \) be a natural number. Then, the following conditions are equivalent:

1. \( \text{pd} M \leq d \).
2. For any sppj resolution \( P_* \), there exists a natural number \( e \in \mathbb{N} \) such that \( M \in \mathcal{P}[-\sup P_e] \) and \( e + \sup P_0 - \sup M \leq d \). In particular, we have a sppj resolution of length \( e \):
   \[
   M \to P_{e-1} \to P_{e-2} \to \cdots \to P_1 \to P_0 \to M.
   \]
3. \( M \) has sppj resolution \( P_* \) of length \( e \) such that \( e + \sup P_0 - \sup P_e \leq d \).
4. The functor \( F = \mathbb{R}\text{Hom}(M, -) \) sends the standard heart \( \text{Mod} \, H^0 \) to \( \mathcal{D}[-\sup M, d-\sup M](R) \).
5. \( M \) belongs to \( \mathcal{P}[-\sup M] \ast \mathcal{P}[-\sup M + 1] \ast \cdots \ast \mathcal{P}[-\sup M + d] \).

We need preparation.

**Lemma 2.20:** Let \( Q_1, Q_2, Q_3 \) be objects of \( \mathcal{P} \), \( N_3 \xrightarrow{g_3} Q_2 \xrightarrow{f_2} N_2 \to \) be an exact triangle in \( \mathcal{D}(R) \) and \( f_3 : Q_3 \to N_3, g_2 : N_2 \to Q_1 \) morphisms in \( \mathcal{D}(R) \). We set

\[
\delta_3 := g_3 f_3, \quad \delta_2 := g_2 f_2.
\]

We consider the complex \( Q_3 \xrightarrow{\delta_3} Q_2 \xrightarrow{\delta_2} Q_1 \) inside \( \mathcal{D}(R) \). Assume, for any \( N \in \text{Mod} \, H^0 \), the induced complex below is exact:

\[
\text{Hom}_{\text{Mod} \, H^0}(H^0(Q_1), N) \to \text{Hom}_{\text{Mod} \, H^0}(H^0(Q_2), N) \to \text{Hom}_{\text{Mod} \, H^0}(H^0(Q_3), N).
\]

Then the complex \( Q_3 \xrightarrow{\delta_3} Q_2 \xrightarrow{\delta_2} Q_1 \) inside \( \mathcal{D}(R) \) splits as below:

\[
\begin{array}{ccccccc}
Q_3 & \xrightarrow{\delta_3} & Q_2 & \xrightarrow{\delta_2} & Q_1 \\
Q_2' \oplus Q_3' & \xrightarrow{(\text{id} 0 \, 0)} & Q_2' \oplus Q_2' & \xrightarrow{(0 \, 0 \, \text{id})} & Q_1' \oplus Q_2'
\end{array}
\]
Moreover, $Q_2''$ is a direct summand of $N_3$ and $Q_2'$ is a direct summand of $N_2$, and there are the following diagrams:

(i) $\begin{array}{ccc}
Q_3 & \xrightarrow{f_3} & N_3 \\
Q_2'' \oplus Q_3' & \xrightarrow{\text{id}_{Q_2''} 0} & Q_2'' \oplus N_3',
\end{array}$

(ii) $\begin{array}{ccc}
N_3 & \xrightarrow{g_3} & Q_2 \\
Q_2'' \oplus N_3' & \xrightarrow{\text{id}_{Q_2''} 0} & Q_2'' \oplus Q_2',
\end{array}$

(iii) $\begin{array}{ccc}
Q_2 & \xrightarrow{f_2} & N_2 \\
Q_2'' \oplus Q_2' & \xrightarrow{0 \text{id}_{Q_2'}} & N_2' \oplus Q_2',
\end{array}$

(iv) $\begin{array}{ccc}
N_2 & \xrightarrow{g_2} & Q_1 \\
N_2' \oplus Q_2' & \xrightarrow{0 \text{id}_{Q_2'}} & Q_1' \oplus Q_2'.
\end{array}$

Thus, the exact triangle $N_3 \xrightarrow{g_3} Q_2 \xrightarrow{f_2} N_2 \rightarrow$ is a direct sum of the three exact triangles

$0 \rightarrow Q_2'' \xrightarrow{\text{id}} Q_2' \rightarrow,$

$Q_2'' \xrightarrow{\text{id}} Q_2' \rightarrow 0 \rightarrow,$

$N_3' \rightarrow 0 \rightarrow N_2' \rightarrow .$

In particular $N_3' \cong N_2'[-1]$.

**Proof.** First we prove the splitting (2-8). It follows from the exactness of (2-7) for $N = \text{Cok} \, H^0(\delta_3)$ that the complex $H^0(Q_3) \xrightarrow{H^0(\delta_3)} H^0(Q_2) \xrightarrow{H^0(\delta_2)} H^0(Q_1)$ is exact and hence it is a truncated projective resolution of $\text{Cok} \, H^0(\delta_2)$. The cohomology of the complex (2-7) computes $\text{Ext}^1(\text{Cok} \, H^0(\delta_2), N)$. Thus, we conclude that the $H^0$-module $\text{Cok} \, H^0(\delta_2)$ is projective and the complex $H^0(Q_3) \xrightarrow{H^0(\delta_3)} H^0(Q_2) \xrightarrow{H^0(\delta_2)} H^0(Q_1)$ splits.

By Lemma 2.8.(3), we have an equivalence $\mathcal{P} \cong \text{Proj} \, H^0$. Therefore we have the splitting (2-8) as desired.

Let $\phi : Q_3 \rightarrow Q_2''$ and $\phi' : Q_2'' \rightarrow Q_3$ be the retraction and the section of the left direct sum decomposition of (2-8). Let $\psi : Q_2 \rightarrow Q_2''$ and $\psi' : Q_2'' \rightarrow Q_2$ be the retraction and the section of the middle direct sum decomposition (2-8).
Note that we have $\delta_3 = \psi' \phi.$

\[
\begin{array}{ccc}
Q_3 & \xrightarrow{\delta_3} & Q_2 \\
\downarrow \psi'' & & \downarrow \psi' \ast \\
Q''_2 \oplus Q'_3 & \xrightarrow{\text{id} \ 0 \ 0} & Q''_2 \oplus Q'_2
\end{array}
\]

Then $\psi g_3 : N_3 \to Q''_2$ is a split epimorphism with a section $f_3 \phi' : Q''_2 \to N_3$. Moreover, if we set $N'_3 = \ker \psi g_3$, then the morphisms $f_3, g_3$ are of the following forms

\[
\begin{array}{cccc}
Q_3 & \xrightarrow{f_3} & N_3 & \xrightarrow{g_3} & Q_2 \\
Q''_2 \oplus Q'_3 & \xrightarrow{\text{id}_{Q''_2} \ 0 \ \text{id}_{Q'_3}} & N''_2 \oplus N'_3, & \xrightarrow{\text{id}_{Q''_2} \ 0 \ \text{id}_{N'_3}} & Q''_2 \oplus Q'_2
\end{array}
\]

where $f_3, g_3$ denote the restriction morphisms. The first diagram gives (i). The second one gives (ii) except the vanishing of the $(2, 2)$-component, i.e., $g_3 = 0$. We denote $\alpha := g_3$.

In a similar way, we obtain (iv) and (iii) except the vanishing of the $(1, 1)$-component, which we denote by $\beta$.

Observe that the exact triangle $N_3 \xrightarrow{g_3} Q_2 \xrightarrow{f_2} N_2 \to$ is a direct sum of two exact sequences:

\[
\begin{array}{c}
Q''_2 \xrightarrow{\text{id}} N''_2 \xrightarrow{\beta} N'_2, \quad N'_3 \xrightarrow{\alpha} Q'_2 \xrightarrow{\text{id}} Q'_2
\end{array}
\]

Therefore we conclude that $\alpha = 0, \beta = 0$.

**Proof of Theorem 2.19.** The implication $(2) \Rightarrow (3)$ is clear.

We prove the implication $(3) \Rightarrow (1)$. From the assumption, we deduce that $M$ belongs to $\mathcal{P}[-\sup P_0] \ast \mathcal{P}[-\sup P_1 + 1] \ast \cdots \ast \mathcal{P}[-\sup P_e + e]$. Thus by Corollary 2.11, we conclude that

\[
\text{pd } M \leq e - \sup P_e + \sup P_0 \leq d.
\]

We prove the implication $(1) \Rightarrow (2)$. If for $i \geq 1$ we have $\text{pd } M_{i-1} > 0$, then by Corollary 2.15, we must have $\text{pd } M_{j-1} > 0$ for $j \leq i$. Therefore by Lemma 2.13, $\text{pd } M_j = \text{pd } M_{j-1} - 1 - \sup M_{j-1} + \sup M_j$ for $j \leq i$. Thus, we have

\[
\text{pd } M_i + i + \sup M - \sup M_i = \text{pd } M
\]
for $i$ such that $\text{pd} M_{i-1} > 0$. Since $i + \sup M - \sup M_i \geq i$, the set
$$\{i \geq 1 \mid \text{pd} M_{i-1} > 0\}$$
is finite. If we set $e := \max\{i \geq 1 \mid \text{pd} M_{i-1} > 0\}$, then $\text{pd} M_e = 0$ and hence $M \in \mathcal{P}[-\sup M_e]$ by Lemma 2.14. From the equation (2-9) and $\text{pd} M \leq d$, we deduce the desired inequality.

The implication (5) $\Rightarrow$ (1) is proved in Corollary 2.11. The implications (2) $\Rightarrow$ (5), (1) $\Rightarrow$ (4) are clear.

It remains to prove the implication (4) $\Rightarrow$ (1).

Claim 2.21: $M$ has a sppj resolution of finite length.

We postpone the proof of the claim. Assume that $M$ has a sppj resolution of length $\ell < \infty$. We prove that if $N \in \mathcal{D}^{[a,b]}(R)$, then $F(N) \in \mathcal{D}^{[a+t_\ell,b+t_\ell+d]}(k)$. First, the case $a, b \in \mathbb{Z}$ can be proved by induction on $b - a$. Next, we deal with the case $a = -\infty$ and $b \in \mathbb{Z}$. Since $\text{Hom}(\mathcal{P}, \mathcal{D}^{<0}(R)) = 0$, we have $F(\sigma^{<c} N) = 0$ for $c = -t_\ell - \ell$. Thus, $F(N) = F(\sigma^{\geq c} N)$ and the problem is reduced to the first case. Finally we deal with the case $a \in \mathbb{Z}$ and $b = \infty$. Since $\text{Hom}(\mathcal{P}, \mathcal{D}^{>0}(R)) = 0$, we have $F(\sigma^{>-t_\ell} N) = 0$. Thus $F(N) \cong F(\sigma^{<-t_\ell} N)$ and the problem is reduced to the first case. To finish the proof we only have to show the claim.

Proof of Claim 2.21. By shifting the degree, we may assume that $\sup M = 0$. Let $P_\bullet$ be a sppj resolution of $M$. We set $t_i := -\sup P_i$. We take $\ell$ to be a natural number such that $\ell + t_\ell > d - \sup M$. Then for $N \in \text{Mod} H^0$ and $j \geq \ell$, we have $\text{Hom}(M, N[j + t_j]) = 0$.

The situation is divided into the following two cases:

(A) $t_j = t_\ell$ for $j \geq \ell$,

(B) there exists $k > \ell$ such that $t_\ell = t_{\ell-1} = \cdots = t_{k-1} \neq t_k$.

First we deal with the case (A). We prove that $M_{\ell+1}$ belongs to $\mathcal{P}[-\sup M_{\ell+1}]$ and hence that $M$ has a sppj resolution of length $\ell + 1$. Thanks to Lemma 2.18, we can apply (the shifted version of) Lemma 2.20 to $P_{j+1} \to P_j \to P_{j-1}$ for $j > \ell$.

The sppj morphism $f_j : P_j \to M_j$ is of the form
$$\begin{pmatrix} 0 & 0 \\ 0 & \text{id} \end{pmatrix} : P''_j \oplus P'_j \to M'_j \oplus P'_j$$

and $M'_{j+1} \cong M'_j[-1]$. We note that since $H^{-t_\ell}(f_j)$ is surjective, we have $H^{-t_\ell}(M'_j) = 0$. 

Therefore for $i \geq 0$,
\[
H^{-t_\ell-i}(M'_{\ell+1}) = H^{-t_\ell}(M'_{\ell+1}[-i]) = H^{-t_\ell}(M'_{\ell+1+i}) = 0.
\]
Hence $H^{\leq -t_\ell}(M'_{\ell+1}) = 0$. On the other hand, since $\text{sup} M_{\ell+1} = -t_\ell$, we have $H^{\geq -t_\ell}(M_{\ell+1}) = 0$. Thus, $H(M'_{\ell+1}) = 0$. We conclude that $M'_{\ell+1} = 0$ and $M_{\ell+1} \cong P'_{\ell+1}$ as desired.

Next we deal with the case (B). We prove that the morphism
\[
f_{k-1}: P_{k-1} \rightarrow M_{k-1}
\]
is an isomorphism and hence that $M$ has a sppj resolution of length $k-1$.

First we assume that $t_{k-1} \neq t_k \neq t_{k+1}$. Then $P_k = 0$ by Lemma 2.18. Thus in this case, $M_k = 0$. Hence $f_{k-1}: P_{k-1} \rightarrow M_{k-1}$ is an isomorphism.

Next we assume that $t_{k-1} \neq t_k = t_{k+1}$. Then by Lemma 2.18 the morphism $\delta_{k+1} : P_{k+1} \rightarrow P_k$ is a split epimorphism. Consequently, the morphism $g_{k+1} : M_{k+1} \rightarrow P_k$ is a split epimorphism. Therefore, there exists $L_{k+1}$ and an isomorphism $M_{k+1} \cong L_{k+1} \oplus P_k$ under which $g_{k+1}$ corresponds to the canonical projection. Thus we conclude that $f_k = 0$ and by the definition of a sppj morphism $M_k = 0$. Hence $f_{k-1}: P_{k-1} \rightarrow M_{k-1}$ is an isomorphism.

This completes the proof of Theorem 2.19.

The following is the main theorem of this section.

**Theorem 2.22:** Let $M \in D^{<\infty}(R)$ and $d \in \mathbb{N}$ be a natural number. Then the following conditions are equivalent:

1. $\text{pd} M = d$.
2. For any sppj resolution $P_\bullet$, there exists a natural number $e \in \mathbb{N}$ which satisfies the following properties:
   (a) $M_e \in \mathcal{P}[-\text{sup} M_e]$.
   (b) $d = e + \text{sup} P_0 - \text{sup} M_e$.
   (c) $g_e$ is not a split-monomorphism.
3. $M$ has sppj resolution $P_\bullet$ of length $e$ which satisfies the following properties:
   (a) $d = e + \text{sup} P_0 - \text{sup} P_e$.
   (b) $\delta_e$ is not a split-monomorphism.
(4) The functor \( F = \mathbb{R}\text{Hom}(M, -) \) sends the standard heart \( \text{Mod} H^0 \) to \( \mathcal{D}[\sup_{-} M, d - \sup M](R) \) and there exists \( N \in \text{Mod} H^0 \) such that
\[
H^{d - \sup M}(F(N)) \neq 0.
\]

(5) \( d \) is the smallest number which satisfies
\[
M \in \mathcal{P}[- \sup M] * \mathcal{P}[- \sup M + 1] * \cdots * \mathcal{P}[- \sup M + d].
\]

Proof. The implication (2) \( \Rightarrow \) (3) is clear.

We prove the implication (1) \( \Rightarrow \) (2). In the proof of Theorem 2.19, we showed that if we set \( e = \max\{i \mid \text{pd } M_{i-1} > 0\} \), then
\[
M_e \in \mathcal{P}[- \sup M_e] \quad \text{and} \quad \text{pd } M = e + \sup M - \sup M_e.
\]
If the morphism \( g_e : M_e \to P_{e-1} \) is a split-monomorphism, then \( M_{e-1} \) is a direct summand of \( P_e \) and \( \text{pd } M_{e-1} = 0 \). This contradicts to the definition of \( e \).

We prove the implication (3) \( \Rightarrow \) (1). We remark that the condition that \( \delta_e \) is not a split-monomorphism implies that \( P_e \neq 0 \). By Theorem 2.19, it is enough to prove that there exists \( N \in \text{Mod} H^0 \) such that \( \text{Hom}(M, N[d - \sup M]) \neq 0 \).

Assume that \( \sup P_{e-1} \neq \sup P_e \). Then by Lemma 2.18 for \( N \in \text{Mod} H^0 \),
\[
\text{Hom}(M, N[d - \sup M]) \cong \text{Hom}(H^{\sup}(P_e), N).
\]
Thus, \( N = H^{\sup}(P_e) \) has the desired property.

Assume that \( \sup P_{e-1} = \sup P_e \). Then by Lemma 2.18 for \( N \in \text{Mod} H^0 \),
\[
\text{Hom}(M, N[d - \sup M]) \cong \text{Cok}[\text{Hom}(H^{\sup}(P_{e-1}), N) \to \text{Hom}(H^{\sup}(P_e), N)].
\]
Since \( \delta_e \) is not a split-monomorphism, \( N = H^{\sup}(P_e) \) has the desired property.

The equivalence (4) (resp. (5)) to the other conditions follows from Theorem 2.19.

The length \( e \) of a sppj resolution \( P_\bullet \) of a DG-module \( M \) possibly has several values.

Example 2.23: Let \( n \in \mathbb{N} \) and \( M(n) := R \oplus R[n] \). Then, it can be directly checked that \( \text{pd } M(n) = n \). The DG-module \( M(n) \) has a sppj resolution of length 1,
\[
P_\bullet : R[n-1] \to R \xrightarrow{(1)} M(n).
\]
On the other hand, we have the exact triangles
\[
E_m : M_{(m-1)} \to R \xrightarrow{(1,0)} M_{(m)}.
\]
Since \( M(0) = R^{\oplus 2} \), splicing \( \{ E_m \}_{m=1}^n \) we obtain a sppj resolution of length \( n \),

\[
M(0) \to R^{\oplus 2} \to \cdots \to R^{\oplus 2} \to M(n).
\]

2.3. The subcategory of DG-modules of finite projective dimension.

Theorem 2.22 has the following consequences.

We denote by \( D(R)_{\text{fpd}} \) the full subcategory consisting of \( M \) having finite projective dimension.

**Proposition 2.24:**

\[
D(R)_{\text{fpd}} = \text{thick } \mathcal{P} = \bigcup \mathcal{P}[a] \ast \mathcal{P}[a+1] \ast \cdots \ast \mathcal{P}[b]
\]

where \( a, b \) run over all the pairs of integers such that \( a \leq b \).

We denote by \( D_{\text{mod } H^0}(R) \) the full subcategory consisting of \( M \) such that \( H^i(M) \) is a finitely generated graded \( H^0 \)-module for \( i \in \mathbb{Z} \). We set

\[
D_{\text{mod } H^0}(R)_{\text{fpd}} = D_{\text{mod } H^0}(R) \cap D(R)_{\text{fpd}}, \quad \mathcal{P}_{\text{mod } H^0} = D_{\text{mod } H^0}(R) \cap \mathcal{P}.
\]

Note that \( \mathcal{P}_{\text{mod } H^0} = \text{add } R \), that is, objects of \( \mathcal{P}_{\text{mod } H^0} \) are precisely direct summands of finite direct sums of \( R \). Therefore \( \text{thick } \mathcal{P}_{\text{mod } H^0} = \text{thick } R \) is nothing but a perfect derived category \( \text{perf } R \) of \( R \).

We recall the notion of a piecewise Noetherian DG-algebra, which plays a similar role to that of Noetherian algebras in the theory of ordinary rings.

**Definition 2.25:** We call a DG-algebra \( R \) **right piecewise Noetherian** if \( H^0 \) is right Noetherian and \( H^{-i} \) is finitely generated as a right \( H^0 \)-module for \( i \geq 0 \).

The name is taken from [4]. The same notion is called right cohomological pseudo-Noetherian in [15] and right Noetherian in [13, 14].

**Proposition 2.26:** If \( R \) is piecewise Noetherian, then

\[
D_{\text{mod } H^0}(R)_{\text{fpd}} = \text{perf } R = \bigcup \mathcal{P}_{\text{mod } H^0}[a] \ast \mathcal{P}_{\text{mod } H^0}[a+1] \ast \cdots \ast \mathcal{P}_{\text{mod } H^0}[b]
\]

where \( a, b \) run through all the pairs of integers such that \( a \leq b \).

**Proof.** It is clear that the first one contains the second one and that the second one contains the third one.

For an object \( M \in D_{\text{mod } H^0}(R) \), we can construct a sppj resolution \( P_\bullet \) of \( M \) such that \( P_i \) belongs to \( \mathcal{P}_{\text{mod } H^0} \). Thus if moreover \( \text{pd } M < \infty \), it belongs to the third one.
2.4. Tensor product. For $P \in \mathcal{P}$ and $L \in \text{Mod}(H^0)^{\text{op}}$,

$$H^n(P \otimes_R^L L) = \begin{cases} H^0(P) \otimes_{H^0} L, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

In a similar way to Lemma 2.18, we can prove the following lemma.

**Lemma 2.27:** Let $L \in \text{Mod}(H^0)^{\text{op}}$, $M \in \mathcal{D}^{<\infty}(R)$ and $P^\bullet$ be a sppj resolution of $M$. We denote the complexes below by $Y_i, Y_i'$:

- $Y_i : H^{\text{sup} P_i}(P_{i+1} \otimes_R^L L) \to H^{\text{sup} P_i}(P_i \otimes_R^L L) \to H^{\text{sup} P_i}(P_{i-1} \otimes_R^L L)$,
- $Y_i' : H^{\text{sup} P_i}(P_{i+1}) \otimes_{H^0} L \to H^{\text{sup} P_i}(P_i) \otimes_{H^0} L \to H^{\text{sup} P_i}(P_{i-1}) \otimes_{H^0} L$.

Then,

$$H^n(M \otimes_R^L L) = \begin{cases} 0, & n \neq -i + \text{sup} P_i \text{ for any } i \geq 0, \\ H(Y_i), & n = -i + \text{sup} P_i \text{ for some } i \geq 0. \end{cases}$$

Moreover, in the case $n = -i + \text{sup} P_i$, we have

$$H(Y_i') = \begin{cases} H^{\text{sup} P_i}(P_i) \otimes_{H^0} L, & \text{sup} P_{i-1} \neq \text{sup} P_i \neq \text{sup} P_{i+1}, \\ \text{Cok}[H^{\text{sup} P_i}(P_{i+1}) \otimes_{H^0} L \to H^{\text{sup} P_i}(P_i) \otimes_{H^0} L], & \text{sup} P_{i-1} \neq \text{sup} P_i = \text{sup} P_{i+1}, \\ \text{Ker}[H^{\text{sup} P_i}(P_i) \otimes_{H^0} L \to H^{\text{sup} P_i}(P_{i-1}) \otimes_{H^0} L], & \text{sup} P_{i-1} = \text{sup} P_i \neq \text{sup} P_{i+1}, \\ H(Y_i'), & \text{sup} P_{i-1} = \text{sup} P_i = \text{sup} P_{i+1}. \end{cases}$$

2.5. Minimal sppj resolution. From Lemma 2.18 we deduce the following corollary.

**Corollary 2.28:** Assume that $M \in \mathcal{D}^{<\infty}(R)$ admits a minimal sppj resolution $P^\bullet$. Then for a simple $H^0$-module $S$ we have

$$\text{Hom}(M, S[n]) = \begin{cases} 0, & n \neq i - \text{sup} P_i \text{ for any } i \geq 0, \\ \text{Hom}(H^{\text{sup} P_i}(P_i), S), & n = i - \text{sup} P_i \text{ for some } i \geq 0. \end{cases}$$

From Lemma 2.27 we deduce the following corollary.
Corollary 2.29: Assume that $M \in \mathcal{D}^{< \infty}(R)$ admits a minimal sppj resolution $P_\bullet$. Then for a simple $(H^0)^{\text{op}}$-module $T$ we have

$$H^n(M \otimes_R T) = \begin{cases} 0, & n \neq -i + \sup P_i \text{ for any } i \geq 0, \\ H^{\sup}(P_i) \otimes_{H^0} T, & n = -i + \sup P_i \text{ for some } i \geq 0. \end{cases}$$

3. Inf-injective resolutions of DG-modules

3.1. Injective dimension of $M \in \mathcal{D}(R)$ after Yekutieli. We recall the definition of the injective dimension of $M \in \mathcal{D}(R)$ introduced by Yekutieli.

Definition 3.1 ([15, Definition 2.4]): Let $a \leq b \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$.

1. An object $M \in \mathcal{D}(R)$ is said to have injective concentration $[a, b]$ if $a, b$ are such that the functor $F = \mathbb{R}\text{Hom}_R(-, M)$ sends $\mathcal{D}^{[m,n]}(R)$ to $\mathcal{D}^{[a-n, b-m]}(k)$ for any $m \leq n \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$:

$$F(\mathcal{D}^{[m,n]}(R)) \subset \mathcal{D}^{[a-n, b-m]}(k).$$

2. An object $M \in \mathcal{D}(R)$ is said to have strict injective concentration $[a, b]$ if it has injective concentration $[a, b]$ and doesn’t have injective concentration $[c, d]$ such that $[c, d] \subset [a, b]$.

3. An object $M \in \mathcal{D}(R)$ is said to have injective dimension $d \in \mathbb{N}$ if it has strict injective concentration $[a, b]$ for $a, b \in \mathbb{Z}$ such that $d = b - a$.

In the case where $M$ does not have a finite interval as injective concentration, it is said to have infinite injective dimension.

We denote the injective dimension by $\text{id} M$.

Remark 3.2: Let $R$ be an ordinary algebra. Avramov–Foxby [3] introduced another injective dimension for a complex $M \in \mathcal{C}(R)$. If we denote by $\text{AF id} M$ the injective dimension of Avramov–Foxby, then it is easy to see that

$$\text{AF id} M = \text{id} M + \inf M.$$

The following lemma is the injective version of Lemma 2.3.

Lemma 3.3: If $M \in \mathcal{D}(R)$ has finite injective dimension, then it belongs to $\mathcal{D}^{>-\infty}(R)$.

Proof. Since $R \in \mathcal{D}^{<0}(R)$, the complex $M = \mathbb{R}\text{Hom}(R, M)$ belong $\mathcal{D}^{>-\infty}(k)$. □
The following lemma is the injective version of Lemma 2.4. We omit the proof, since it can be proved in a similar way.

**Lemma 3.4:** Let $M \in \mathbb{D}^{> -\infty}(R)$ and $F := \mathbb{R}\text{Hom}(\cdot, M)$. Then for any $m \leq n$, 
$$F(\mathbb{D}^{[m,n]}(R)) \subset \mathbb{D}^{[-n + \inf M, \infty]}(k).$$
Moreover, there exists $N \in \mathbb{D}^{[m,n]}(R)$ such that $H^{-n + \inf M}(F(N)) \neq 0$.

We deduce the following useful corollaries.

**Corollary 3.5:** Let $M \in \mathbb{D}^{> -\infty}(R)$ and $d \in \mathbb{N}$. Then $\text{id} M = d$ if and only if $M$ has a strict injective concentration $[\inf M, \inf M + d]$.

**Corollary 3.6:** Let $L \to M \to N \to$ be an exact triangle in $\mathbb{D}(R)$. Then, we have
$$\text{id} M + \inf M \leq \sup \{\text{id} L + \inf L, \text{id} N + \inf N\}.$$ 

3.2. **The class $\mathcal{I}$**. The aim of Section 3.2 is to introduce the full subcategory $\mathcal{I} \subset \mathbb{D}(R)$ which plays the role of $\mathcal{P}$ for inf-injective resolutions.

**Definition 3.7:** Let $A = \bigoplus_{i \leq 0} A^i$ be a graded algebra. By $\text{Inj}^0 A \subset \text{Mod}^\mathbb{Z} A$, we denote the full subcategory consisting of injective graded $A$-modules $J$ cogenerated by degree 0-part. More precisely, $J$ is assumed to satisfy the following conditions:

(a) $J$ is a graded injective $A$-module.
(b) $J^0$ is an essential submodule of $J$.

It might be worth noting that $J^{< 0} = 0$ follows from the second condition.

A point is that $\text{Inj}^0 A$ is equivalent to the category $\text{Inj} A^0$ of injective $A^0$-modules as shown in the lemma below.

**Lemma 3.8:** Let $A = \bigoplus_{i \leq 0} A^i$ be a graded algebra. Then the following assertion is true:

1. We have an adjoint pair
   $$\phi : \text{Mod}^\mathbb{Z} A \leftrightarrow \text{Mod} A^0 : \psi$$
   defined by $\phi(M) := M^0$ and $\psi(L) := \text{Hom}_{A^0}^\bullet(A, L)$.
2. The adjoint pair $\phi \dashv \psi$ is restricted to give an equivalence $\text{Inj}^0 A \cong \text{Inj} A^0$. 
Proof. (1) is standard and so is left to the readers.

(2) We can check that $\phi$ and $\psi$ restrict to functors between $\text{Inj}^0 A$ and $\text{Inj} A^0$. It is also easy to check that $\phi \circ \psi = \text{id}_{\text{Inj} A^0}$.

For $J \in \text{Inj}^0 A$, we have a canonical map $f : J \to \text{Hom}^\bullet_{A^0}(A, J^0) = \psi \circ \phi(J)$ which is the identity map $f^0 = \text{id}_{J^0}$ at degree 0. This implies that $\text{Ker} \ f \cap J^0 = 0$, hence by the definition on $\text{Inj}^0 A$, $\text{Ker} \ f = 0$. Thus $\psi \circ \phi(J) \cong J \oplus J'$ for some $J'$. However, $J' \cap \psi \circ \phi(J)^0 = 0$; we conclude that $J' = 0$. This proves that $\psi \circ \phi \cong \text{id}_{\text{Inj}^0 A}$.

Since $R$ is a DG-algebra, we may equip $\psi_R(K) := \text{Hom}^\bullet_{R^0}(R, K)$ with the differential induced from that of $R$ and regard it as a DG-$R$-module.

We denote by $\text{Inj}^0 R$ the full subcategory of $\mathcal{C}(R)$ consisting of the DG-$R$-modules of the form $\psi_R(K) = \text{Hom}^\bullet_{R^0}(R, K)$ for some $K \in \text{Inj} R^0$.

Definition 3.9: By $\mathcal{I} \subset \mathcal{D}(R)$, we denote the full subcategory consisting of (the quasi-isomorphism class of) $\psi_R(K)$ for $K \in \text{Inj} R^0$. In case we need to emphasize the DG-algebra $R$, we denote $\mathcal{I}$ by $\mathcal{I}(R)$.

The properties of $\mathcal{I}$ which are used to construct the inf-injective resolution are summarized in the following theorem, which is an injective version of Lemma 2.8.

Theorem 3.10: The following hold:

1. For $N \in \mathcal{D}(R)$ and $I \in \mathcal{I}$,
   $$\text{Hom}(N, I[n]) \cong \text{Hom}_{\text{Mod} H^0}(H^{-n}(N), H^0(I)).$$

2. For $N \in \text{Mod} H^0$ and $I \in \mathcal{I}$,
   $$\text{Hom}(N, I[n]) \cong \begin{cases} \text{Hom}_{\text{Mod} H^0}(H^0(N), H^0(I)), & n = 0, \\ 0, & n \neq 0. \end{cases}$$

3. The cohomology functor $H : \mathcal{D}(R) \to \text{Mod}^\mathbb{Z} H$ induces an equivalence
   $$H : \mathcal{I} \xrightarrow{\cong} \text{Inj}^0 H.$$

4. Therefore, the 0-th cohomology functor $H^0 : \mathcal{D}(R) \to \text{Mod}^\mathbb{Z} H$ induces an equivalence
   $$H^0 : \mathcal{I} \xrightarrow{\cong} \text{Inj} H^0.$$
We need preparations. Let $\pi : R^0 \to H^0$ be the canonical projection. We denote by $\pi^!$ the functor $\pi^!(M) := \text{Hom}_{R^0}(H^0, M)$: 

$$\pi^! : \text{Mod} R^0 \to \text{Mod} H^0.$$ 

It is easy to see that the functor $\pi^!$ can be restricted to the functor $\pi^! : \text{Inj} R^0 \to \text{Inj} H^0$.

**Lemma 3.11:** Let $K \in \text{Inj} R^0$. Then for a DG-$R$-module $M$, we have isomorphisms

1. $\text{Hom}^\bullet_R(M, \psi_R(K)) \cong \text{Hom}^\bullet_{R^0}(M, K)$,
2. $\text{Hom}_{C(R)}(M, \psi_R(K)) \cong Z^0(\text{Hom}^\bullet_R(M, \psi_R(K))) \cong \text{Hom}^0_{R^0}(M/B(M), K)$,
3. $\text{Hom}_{K(R)}(M, \psi_R(K)) \cong H^0(\text{Hom}^\bullet_R(M, \psi_R(K)))$

$$\cong \text{Hom}^0_H(H(M), \psi_H \pi^1(K))$$

$$\cong \text{Hom}_{\text{Mod} H^0}(H^0(M), \pi^1 K).$$

**Proof.**
1. We have the adjunction isomorphism $\gamma$ below:

$$\gamma : \text{Hom}^\bullet_R(M, \psi_R(K)) \cong \text{Hom}^\bullet_{R^0}(M, K), \quad \gamma(\phi)(m) := \phi(m)(1).$$

It is an isomorphism of graded $k$-modules. We can check that $\gamma$ is compatible with the differentials of both sides and hence it gives an isomorphism in $C(k)$.

2. Since $K$ is regarded as a DG-$R^0$-module with zero differential, for a homogeneous element $f \in \text{Hom}^\bullet_{R^0}(M, K)$, we have $\partial(f) = (-1)^{|f|+1} f \circ \partial_M$. Therefore we have 

$$Z^0 \text{Hom}^\bullet_{R^0}(M, K) \cong \text{Hom}^0_{R^0}(M/BM, K).$$

3. This follows from the string of isomorphisms

$$\text{H}^0 \text{Hom}^\bullet_{R^0}(M, K) \cong \text{Hom}^0_{R^0}(H(M), K) \cong \text{Hom}_{\text{Mod} R^0}(H^0(M), K)$$

$$\cong \text{Hom}_{\text{Mod} H^0}(H^0(M), \pi^1 K)$$

$$\cong \text{Hom}^\bullet_H(H(M), \psi_H \pi^1(K)) \quad \text{where} \ \cong \ \text{follows from the fact that} \ K \ \text{is injective} \ R^0 \text{-module,} \ \cong \ \text{is a consequence of the fact that} \ K \ \text{is concentrated at the 0-th degree,} \ \cong \ \text{is deduced from the equation} \ B^0H(M) = 0 \ \text{and} \ \cong \ \text{is nothing but the adjoint isomorphism.}$$

The above lemma has two corollaries.
Corollary 3.12: For $K \in \text{Inj} \, R^0$, we have

1. $H \psi_R(K) \cong \psi_H \pi^1(K)$,
2. $H^0 \psi_R(K) \cong \pi^1(K)$.

Proof. (1) The first isomorphism is clear. Substituting $M = R$ in Lemma 3.11.(3), we obtain the second isomorphism.

(2) This is obtained from (1) by applying $\phi_H$.

Recall that a DG-$R$-module $I$ is called **DG-injective** if its underlying graded $R$-module is injective and the Hom-complex $\text{Hom}^\bullet_R(A,I)$ is acyclic for any acyclic DG-$R$-module $A$. An important property of DG-injective module $M$ is that for any DG-$R$-module $M$, the Hom-space $\text{Hom}_{K(R)}(M,I)$ in the homotopy category $K(R)$ is isomorphic to Hom-space $\text{Hom}(M,I)$ in the derived category $D(R)$ via the canonical map

$$\text{Hom}_{K(R)}(M,I) \cong \text{Hom}(M,I)$$

(see, for example, [8, 12]).

We can immediately deduce the following corollary from Lemma 3.11.(3).

Corollary 3.13: $\psi_R(K)$ is a DG-injective DG-$R$-module. In particular, for $M \in C(R)$ we have

$$\text{Hom}(M,\psi_R(K)) \cong \text{Hom}_{K(R)}(M,\psi_R(K)).$$

The next lemma is the last preparation.

Lemma 3.14: $\pi^1 : \text{Inj} \, R^0 \to \text{Inj} \, H^0$ is essentially surjective.

Proof. Let $J \in \text{Inj} \, H^0$ and let $E = E_{R^0}(J)$ denote the injective hull of $J$ as an $R^0$-module. Then, we prove that $J = \pi^1 E$.

It is clear that $J \subset \pi^1 E$. Let $x \in \pi^1 E$. In other words, $x$ is an element of $E$ such that $B^0 x = 0$. Then the submodule $J + Rx$ is regarded as an $H^0$-module. Since the extension $J \subset E$ is essential, so is the extension $J \subset J + Rx$. Thus, $J \subset J + Rx$ is an essential extension of $H^0$-modules. As injective $H^0$-modules have a maximality of essential extensions (see e.g. [2, 18.11]), we have

$$J = J + Rx.$$

In particular we have $x \in J$. This shows that $\pi^1 E \subset J$. ■
Proof of Theorem 3.10. Let $I := \psi_R(K)$ for some $K \in \text{Inj}^0 R$. Then, combining Lemma 3.11.(3), Corollary 3.12.(2) and Corollary 3.13 we obtain an isomorphism

$$\text{Hom}(N, I) \cong \text{Hom}_{K(R)}(N, \phi_R(K))$$

$$\cong \text{Hom}_{\text{Mod} H^0}(H^0(N), \pi^1 K)$$

$$\cong \text{Hom}_{\text{Mod} H^0}(H^0(N), H^0(I))$$

for $N \in D(R)$. Considering shifts, we obtain (1).

(2) immediately follows from (1).

We prove (4). By (1) the functor $H^0 : \mathcal{I} \to \text{Inj} H^0$ is fully faithful.

By Lemma 3.12, the upper square of the following diagram is commutative:

The commutativity of the lower triangle is obvious. Thus the functor

$$H^0 := \phi_H : \mathcal{I} \to \text{Inj} H^0$$

is essentially surjective by Lemma 3.14. This completes the proof of (4).

(3) follows from (4) by Lemma 3.8. ■

Proof of Lemma 2.3. Let $F := \mathbb{R}\text{Hom}(M, -)$. By the assumption, there exists an integer $a$ such that $F(D_{\geq 0}(R)) \subset D_{\geq a}(k)$. Let $J \in \text{Inj} H^0$ and $I \in \mathcal{I}$ be a unique object such that $H^0(I) \cong J$. By Theorem 3.10.(1), we have

$$H^n(F(I)) \cong \text{Hom}(M, I[n]) \cong \text{Hom}(H^{-n}(M), J).$$

Therefore we have $\text{Hom}(H^{-n}(M), J) = 0$ for any $J \in \text{Inj} H^0$ provided that $n < a$. Since $\text{Mod} H^0$ has an injective cogenerator, it follows that

$$H^{> -a}(M) = 0.$$  ■
3.2.1. Inf-inj (ifij) morphism.

**Definition 3.15** (ifij morphism and ifij resolution): Let $M \in D^{> -\infty}(R)$, $M \neq 0$.

1. An **ifij morphism** $f : M \to I$ is a morphism in $D(R)$ such that $I \in \mathcal{I}[-\inf M]$ and the morphism $H^{\inf M}(f)$ is injective.

2. An ifij morphism $f : M \to I$ is called **minimal** if the morphism $H^{\inf M}(f)$ is an injective envelope.

As a consequence of Theorem 3.10, we deduce the following corollary.

**Corollary 3.16:** For any $M \in D^{> -\infty}(R)$, there exists an ifij morphism $f : M \to I$.

Observe that if we replace $D(R)$ with the opposite triangulated category $D(R)^{\text{op}}$ and $\mathcal{P}$ with $\mathcal{I}^{\text{op}}$, then the argument of the proof of Lemma 2.14 works thanks to Theorem 3.10. Thus, we can deduce the following lemma, which is a dual version of an injective version of Lemma 2.14.

**Lemma 3.17:** We have

\[ \mathcal{I} = \{ I \in D(R) \mid I = 0 \text{ or } \inf I = 0, \text{id } I = 0 \} . \]

**Remark 3.18:** Lemma 3.17 tells us that the class $\mathcal{I} \subset D(R)$ coincides with $\text{Inj } R$ introduced by Shaul in [14].

In this remark we explain a relationship between this section and [14]. In that paper Shaul defined the full subcategory $\text{Inj } R \subset D(R)$ as a full subcategory consisting of objects $I$ such that either $I = 0$ or $\inf I = 0, \text{id } I = 0$. He showed that an object $I \in \text{Inj } R$ satisfies the properties of Theorem 3.10.(1) and (2) and proved that the statement of Theorem 3.10.(3) holds for $\text{Inj } R$.

In this paper, we took a converse way. We define the full subcategory $\mathcal{I}$ as a full subcategory of $D(R)$ consisting of DG-modules defined concretely. Then we prove Theorem 3.10 for $\mathcal{I}$ and, finally, identify $\mathcal{I}$ with $\text{Inj } R$.

We will make use of the concrete construction of objects of $\mathcal{I}$ in Section 3.6 and the subsequent work [11].

3.3. Inf-injective (ifij) resolutions. We introduce the notion of an inf-injective (ifij)-resolution of $M \in D^{> -\infty}(R)$ and show its basic properties. Since for almost all their proofs we can apply arguments of similar statements of sppj resolutions by replacing $D(R)$ with the opposite triangulated category $D(R)^{\text{op}}$ and $\mathcal{P}$ with $\mathcal{I}^{\text{op}}$, we omit them.
Definition 3.19 (ifij morphism and ifij resolution): Let $M \in \mathbb{D}^{>\infty}(R), M \neq 0$.

1. A **ifij morphism** $f : M \rightarrow I$ is a morphism in $\mathbb{D}(R)$ such that $I \in \mathbb{I}[-\inf M]$ and the morphism $H^{\inf M}(f)$ is injective.

2. A ifij morphism $f : M \rightarrow I$ is called **minimal** if the morphism $H^{\inf M}(f)$ is an injective envelope.

3. A **ifij resolution** $I_* \mathbb{R}$ of $M$ is a sequence of exact triangles for $i \leq 0$ with $M_0 := M$,

$$M_i \xrightarrow{f_i} I_i \xrightarrow{g_i} M_{i-1},$$

such that $f_i$ is ifij.

The following inequality holds:

$$\inf M_i = \inf I_i \leq \inf I_{i-1} = \inf M_{i-1}.$$

For an ifij resolution $I_*$ with the above notations, we set $\delta_i := f_{i-1} \circ g_i$:

$$\delta_i : I_i \rightarrow I_{i-1}.$$

Moreover, we write

$$M \rightarrow I_0 \xrightarrow{\delta_0} I_{-1} \xrightarrow{\delta_{-1}} \cdots \rightarrow I_{-i} \xrightarrow{\delta_{-i}} I_{-i-1} \rightarrow \cdots.$$

4. An ifij resolution $I_*$ is said to have **length** $e$ if $I_{-i} = 0$ for $i > e$ and $I_{-e} \neq 0$.

5. An ifij resolution $I_*$ is called **minimal** if $f_i$ is minimal for $i \leq 0$.

We give basic properties of ifij resolutions.

**Lemma 3.20:** Let $M \in \mathbb{D}(R)$. Then, $\id M = 0$ if and only if $\inf M > -\infty$ and $M \in \mathbb{I}[-\inf M]$.

**Lemma 3.21:** Let $a, b \in \mathbb{Z}$ such that $a \leq b$. Then for any object

$$M \in \mathbb{I}[a] \ast \mathbb{I}[a+1] \ast \cdots \ast \mathbb{I}[b],$$

we have $\id M \leq b - a$.

**Lemma 3.22:** Let $M \in \mathbb{D}^{>\infty}(R), M \neq 0$ and $M \xrightarrow{f} I \xrightarrow{g} N$ be an exact triangle with $f$ ifij. Assume that $\id M \geq 1$. Then

$$\id N = \id M - 1 + \inf M - \inf N.$$
Lemma 3.23: Let $N \in \text{mod } H^0$, $M \in D^{> -\infty}(R)$ and $I_\bullet$ be an ifij resolution of $M$. We denote the complexes below by $Z_i, Z'_i$:

$$Z_{-i} : \text{Hom}(N, I_{-i+1}[\inf I_{-i}]) \to \text{Hom}(N, I_{-i}[\inf I_{-i}]) \to \text{Hom}(N, I_{-i-1}[\inf I_{i}]),$$

$$Z'_i : \text{Hom}(N, H^{\inf}(I_{-i+1})) \to \text{Hom}(N, H^{\inf}(P_{-i})) \to \text{Hom}(N, H^{\inf}(P_{-i-1})).$$

Then,

$$\text{Hom}(N, M[n]) = \begin{cases} 
0, & n \neq i + \inf I_{-i} \text{ for any } i \geq 0, \\
H(Z_{-i}), & n = i + \inf I_{-i} \text{ for some } i \geq 0.
\end{cases}$$

Moreover, in the case $n = i - \inf I_{-i}$, we have

$$H(Z_{-i}) = \begin{cases} 
\text{Hom}(N, H^{\inf}(I_{-i})), & \inf I_{-i+1} \neq \inf I_{-i} \neq \inf I_{-i+1}, \\
\text{Ker}[\text{Hom}(N, H^{\inf}(I_{-i})) \to \text{Hom}(N, H^{\inf}(I_{-i-1}))], & \inf I_{-i+1} \neq \inf I_{-i} = \inf I_{-i-1}, \\
\text{Cok}[\text{Hom}(N, H^{\inf}(I_{-i+1})) \to \text{Hom}(N, H^{\inf}(I_{-i}))], & \inf I_{-i+1} = \inf I_{-i} \neq \inf I_{-i-1}, \\
H(Z'_i), & \inf I_{-i+1} = \inf I_{-i} = \inf I_{-i-1}.
\end{cases}$$

We give an ifij version of Theorem 2.22

Theorem 3.24: Let $M \in D^{> -\infty}(R)$ and $d$ be a natural number. Set

$$F := \mathbb{R} \text{Hom}(-, M).$$

Then the following conditions are equivalent:

1. $\text{id } M = d$.
2. For any ifij resolution $I_\bullet$, there exists a natural number $e \in \mathbb{N}$ which satisfies the following properties:
   - (a) $M_e \in I[-\inf M_e]$.
   - (b) $d = e + \inf M_e - \inf I_0$.
   - (c) $g_{-e}$ is not a split-epimorphism.
3. $M$ has ifij resolution $I_\bullet$ of length $e$ which satisfies the following properties:
   - (a) $d = e + \inf I_{-e} - \inf I_0$.
   - (b) $\delta_e$ is not a split-epimorphism.
(4) The functor $F := \mathbb{R}\text{Hom}(-, M)$ sends the standard heart $\text{Mod}H^0$ to $D^{[\inf M, \inf M + d]}(k)$ and there exists $N \in \text{Mod}H^0$ such that

$$H^{\inf M + d}(F(N)) \neq 0.$$ 

(5) The following conditions hold:

(a) The functor $F = \mathbb{R}\text{Hom}(-, M)$ sends finitely generated $H^0$-modules to $D^{[\inf M, \inf M + d]}(k)$.

(b) There exists a finitely generated $H^0$-module $N \in \text{Mod}H^0$ such that $H^{\inf M + d}(F(N)) \neq 0$.

(6) $d$ is the smallest number which satisfies

$$M \in \mathcal{I}[-\inf M] \ast \mathcal{I}[-\inf M + 1] \ast \cdots \ast \mathcal{I}[-\inf M + d].$$

To deduce the other conditions from condition (5) we need to use the following lemma.

**Lemma 3.25:** A complex $J_\bullet : J_3 \xrightarrow{d_3} J_2 \xrightarrow{d_2} J_1$ of injective $H^0$-modules is exact and splits if and only if a complex below is exact for every finitely generated $H^0$-module $N$:

$$\text{Hom}_{H^0}(N, J_3) \rightarrow \text{Hom}_{H^0}(N, J_2) \rightarrow \text{Hom}_{H^0}(N, J_1).$$

**Proof.** The “only if” part is clear. We prove the “if” part. Substituting $N = H^0$ we see that the complex $J_\bullet$ is exact.

We prove that $J_\bullet$ splits. Let $K := \text{Ker}d_3, L := \text{Im}d_3 = \text{Ker}d_2$. From the assumption, we can show that the induced map $\text{Hom}_{H^0}(N, J_3) \rightarrow \text{Hom}_{H^0}(N, L)$ is surjective for any finitely generated $H^0$-module $N$. This shows that

$$\text{Ext}^1_{H^0}(N, K) = 0$$

for any finitely generated $H^0$-module $N$. By the Baer criterion (see [2, 18.3]), $K$ is an injective $H^0$-module. Thus the inclusion $K \hookrightarrow J_3$ splits and consequently $L$ is an injective $H^0$-module. Therefore the complex $J_\bullet$ splits. 

3.4. The subcategory of DG-modules of finite injective dimension.

As with the sppj resolution, Theorem 3.24 has the following consequences. We denote by $\text{D}(R)_{\text{fid}}$ the full subcategory consisting of $M$ having finite injective dimension.
Proposition 3.26:

\[ D(R)_{\text{fid}} = \text{thick} \mathcal{I} = \bigcup \mathcal{I}[a] * \mathcal{I}[a + 1] * \cdots * \mathcal{I}[b], \]

where \( a, b \) runs through all the pairs of integers such that \( a \leq b \).

We set \( D_{\text{mod}} H^0(R)_{\text{fid}} := D_{\text{mod}} H^0(R) \cap D(R)_{\text{fid}} \). Assume that the base ring \( k \) is a field and \( R \) is locally finite-dimensional, i.e., \( \dim_k H^i < \infty \) for \( i \in \mathbb{Z} \). Then the \( k \)-dual complex \( R^* := \text{Hom}_k(R, k) \) has a canonical DG-\( R \)-module structure such that \( H(R^*) = H^* \) as graded \( H \)-modules. Therefore \( R^* \) belongs to \( I_{\text{mod}} H^0 := \mathcal{I} \cap D_{\text{mod}} H^0(R) \). Since the \( H^0 \)-module \( (H^0)^* \) is an injective cogenerator of \( \text{mod} H^0 \), we have \( I_{\text{mod}} H^0 = \text{add} R^* \).

By the same argument of Proposition 2.26 we obtain the following proposition.

**Proposition 3.27:** Assume that the base ring \( k \) is a field and \( R \) is locally finite-dimensional. Then we have

\[ D_{\text{mod}} H^0(R) = \text{thick} R^* = \bigcup I_{\text{mod}} H^0[a] * I_{\text{mod}} H^0[a + 1] * \cdots * I_{\text{mod}} H^0[b], \]

where \( a, b \) runs through all the pairs of integers such that \( a \leq b \).

Jin [6] introduced the notion of Gorenstein DG-algebra and studied their representation theory. In [6] a **Gorenstein** DG-algebra is defined to be a connective DG-algebra which is proper, i.e., \( \dim_k \sum_{i \in \mathbb{Z}} H^i < \infty \) and satisfies the condition (J) \( \text{perf} R = \text{thick} R^* \). We note that for a locally finite-dimensional connective DG-algebra \( R \) over a field \( k \), properness follows from the condition (J).

On the other hand, in ordinary ring theory, Iwanaga–Gorenstein algebra is defined to be an (ordinary) algebra \( A \) such that \( \id A A < \infty, \id_{A^{op}} A < \infty \). Thus, there are obvious DG-generalizations of the notion of an Iwanaga–Gorenstein algebra. In the next proposition, we show these two generalizations coincide with each other.

**Proposition 3.28:** Assume that the base ring \( k \) is a field. Then for a locally finite-dimensional DG-algebra \( R \), the following conditions are equivalent:

1. \( \text{perf} R = \text{thick} R^* \).
2. \( \id R R < \infty, \id_{R^{op}} R < \infty \).
Proof. (1)⇒(2). Since $R \in \text{perf} R = \text{thick} \, R^*$, we see that $\text{id}_R \, R < \infty$. Since the $k$-duality gives a contravariant equivalence

$$D_{\text{mod}} H^0(R)^{\text{op}} \simeq D_{\text{mod}} H^0(R^{\text{op}}),$$

we have $\text{perf} R^{\text{op}} = \text{thick} \, R^*$ inside $D(R^{\text{op}})$. Thus in the same way as above we see that $\text{id}_{R \text{op}} \, R < \infty$.

(2)⇒(1). It follows from $\text{id}_R \, R < \infty$ that $R \in \text{thick} \, R^*$ and that $\text{perf} R \subset \text{thick} \, R^*$. Similarly we have $\text{perf} R^{\text{op}} \subset \text{thick} \, R^*$ inside $D(R^{\text{op}})$. The $k$-duality sends the latter inclusion to $\text{thick} R^* \subset \text{perf} R$ inside $D(R)$. Thus we see that

$$\text{perf} R = \text{thick} \, R^*$$

as desired. ■

3.5. Minimal ifij resolution. From Lemma 3.23 we deduce the following corollary. We note that since every module over an ordinary algebra has an injective-hull, every object $M \in D^{-\infty} R$ admits a minimal ifij resolution.

**Corollary 3.29:** Let $M \in D^{-\infty} R$ and $I_\bullet$ be a minimal ifij resolution of $M$. Then for a simple $H^0$-module $S$ we have

$$\text{Hom}(S, M[n]) = \begin{cases} 0, & n \neq i + \inf I_{-i} \text{ for any } i \geq 0, \\ \text{Hom}(S, H^{\inf}(I_{-i})), & n = i + \inf I_{-i} \text{ for some } i \geq 0. \end{cases}$$

3.6. The Bass–Papp theorem. Recall that the Bass–Papp theorem states that an ordinary algebra $A$ is right Noetherian if and only if any direct sum of injective (right) $A$-modules is injective (see [9, Theorem 3.46]). Shaul [14] gave a DG-version of it. We provide another proof by using the method developed here.

**Theorem 3.30** (Shaul [14]): A connective DG-algebra $R$ is right piecewise Noetherian if and only if the class $\mathcal{I}$ is closed under direct sums in $D(R)$.

**Proof.** For $J \in \text{Inj} \, H^0$, we set

$$G(J) := \psi_R(E_{R^0}(J)) = \text{Hom}_{R^0}(R, E_{R^0}(J))$$

where $E_{R^0}(J)$ denotes the injective-hull of $J$ as $R^0$-module. Then, $G(J)$ belongs to $\mathcal{I}$. Moreover, by Corollary 3.12.(2) and the proof of Lemma 3.14, we have

$$H^0 G(J) \cong J \quad \text{for } J \in \text{Inj} \, H^0.$$
We prove the “only if” part by showing that if a small family \( \{ J_\lambda \}_{\lambda \in \Lambda} \) in \( \text{Inj} H^0 \) is given, then the DG-\( R \)-module \( \bigoplus_{\lambda \in \Lambda} G(J_\lambda) \) is quasi-isomorphic to \( G(J) \) for some \( J \in \text{Inj} H^0 \). What we actually prove is that if we set \( J := \bigoplus_{\lambda \in \Lambda} J_\lambda \), then there exists a quasi-isomorphism

\[
 f : \bigoplus_{\lambda \in \Lambda} G(J_\lambda) \xrightarrow{\sim} G(J).
\]

Let \( \iota_\lambda : J_\lambda \to J \) be a canonical inclusion. It can be extended to a homomorphism \( \underline{\iota}_\lambda : E_{R^0}(J_\lambda) \to E_{R^0}(J) \) between injective-hulls. Then, the morphism \( f \) induced from the collection \( \{ G(\underline{\iota}_\lambda) \}_{\lambda \in \Lambda} \) satisfies the desired property:

\[
 f := (G(\underline{\iota}_\lambda))_{\lambda \in \Lambda} : \bigoplus_{\lambda \in \Lambda} G(J_\lambda) \to G(J).
\]

Indeed, taking the cohomology group of \( G(\underline{\iota}_\lambda) \), we obtain the morphism below:

\[
 \text{Hom}(H, \iota_\lambda) : \text{Hom}_{H^0}(H, J_\lambda) \to \text{Hom}_{H^0}(H, J).
\]

Therefore, we have the equality of morphisms

\[
 H(f) = (\text{Hom}(H, \iota_\lambda))_{\lambda \in \Lambda} : \bigoplus_{\lambda \in \Lambda} \text{Hom}_{H^0}(H, J_\lambda) \to \text{Hom}_{H^0}(H, J).
\]

Finally, we observe that the right hand side is ensured to be an isomorphism by the assumption that \( R \) is right piecewise Noetherian.

We prove the “if” part. Let \( \{ J_\lambda \}_{\lambda \in \Lambda} \) be a small family in \( \text{Inj} H^0 \). We set \( J := \bigoplus_{\lambda \in \Lambda} J_\lambda \). By the assumption, the direct sum \( I := \bigoplus_{\lambda \in \Lambda} G(J_\lambda) \) belongs to \( \mathcal{I} \). Thus \( J \cong H^0(I) \) is an injective \( H^0 \)-module by Theorem 3.10. We have shown that the class of injective \( H^0 \)-modules is closed under a direct sum. Thus, by the Bass–Papp theorem for ordinary rings, we conclude that \( H^0 \) is right Noetherian.

Since \( J \) belongs to \( \text{Inj} H^0 \), the canonical morphism \( f : I \to G(J) \) defined as above becomes an isomorphism after taking the 0-th cohomology morphism \( H^0(f) \). Therefore, \( f \) is an isomorphism by Theorem 3.10.

Observe that the \( i \)-th cohomology morphism of \( f \) is a canonical morphism induced from the universal property of the direct product

\[
 \bigoplus_{\lambda \in \Lambda} \text{Hom}_{H^0}(H^{-i}, J_\lambda) \cong H^i(I) \xrightarrow{H^i(f)} H^i(G(J)) \cong \text{Hom}_{H^0}(H^{-i}, \bigoplus_{\lambda \in \Lambda} J_\lambda).
\]

It follows from Lemma 3.31 below that \( H^{-i} \) is finitely generated. □
Lemma 3.31: Let $A$ be a right Noetherian algebra. Then an $A$-module $M$ is finitely generated if and only if for any family $\{J_\lambda\}_{\lambda \in \Lambda}$ of injective $A$-modules, the canonical morphism below is an isomorphism:

$$\gamma : \bigoplus_{\lambda \in \Lambda} \text{Hom}_A(M, J_\lambda) \to \text{Hom}_A \left( M, \bigoplus_{\lambda \in \Lambda} J_\lambda \right).$$

Proof. The “only if” part is clear. We prove the “if” part by showing that if $M$ is infinitely generated, then the canonical morphism $\gamma$ happens to become not surjective.

An infinitely generated $A$-module $M$ has a strictly increasing sequence of submodules of $M$,

$$0 =: M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M.$$

Let

$$J_i := E_A(M_i/M_{i-1})$$

be the injective-hull of $M_i/M_{i-1}$. We set

$$N := \bigcup_{i \geq 1} M_i, \quad J = \bigoplus_{i \geq 1} J_i.$$

Then the composite morphism of the canonical morphisms

$$f_i : M_i \to M_i/M_{i-1} \to J_i$$

extends to a morphism $f'_i : N \to J_i$ of $A$-modules. Observe that the collection $\{f'_i\}$ induces a morphism $f : N \to J$. Since $A$ is right Noetherian, $J$ is injective. Therefore $f$ extends to $\tilde{f} : M \to J$.

Let $\pi_i : J \to J_i$ be a canonical projection. Then, we have $\pi_i(\tilde{f}|_{M_i}) = f_i$:

$$f_i : M_i \hookrightarrow M \xrightarrow{\tilde{f}} J \xrightarrow{\pi_i} J_i.$$

Thus, in particular, $\pi_i \tilde{f} \neq 0$ for any $i \geq 1$. This shows that $\tilde{f}$ does not belong to the image of $\gamma$. \qed

4. Sup-flat resolutions of DG-modules

4.1. Flat dimension of $M \in \text{D}(R)$ after Yekutieli. We denote the opposite DG-algebra of $R$ by $R^{\text{op}}$ and identify left DG-$R$-modules with (right) DG-$R^{\text{op}}$-modules.
Definition 4.1 ([15, Definition 2.4]): Let \( a \leq b \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\} \).

1. An object \( M \in \mathcal{D}(R) \) is said to have **flat concentration** \([a, b]\) if the functor \( F = M \otimes^L_R \) sends \( \mathcal{D}^{[m,n]}(R^{\text{op}}) \) to \( \mathcal{D}^{[a+m,b+n]}(k) \) for any \( m \leq n \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\} \),

\[
F(\mathcal{D}^{[m,n]}(R^{\text{op}})) \subset \mathcal{D}^{[a+m,b+n]}(k).
\]

2. An object \( M \in \mathcal{D}(R) \) is said to have **strict flat concentration** \([a, b]\) if it has flat concentration \([a, b]\) and does not have flat concentration \([c, d]\) such that \([c, d] \subsetneq [a, b]\).

3. An object \( M \in \mathcal{D}(R) \) is said to have **flat dimension** \( d \in \mathbb{N} \) if it has strict flat concentration \([a, b]\) for \( a, b \in \mathbb{Z} \) such that \( d = b - a \). In the case where \( M \) does not have a finite interval as flat concentration, it is said to have **infinite flat dimension**.

We denote the flat dimension of \( M \) by \( \text{fd} M \).

Remark 4.2: Let \( R \) be an ordinary algebra. Avramov–Foxby [3] introduced another flat dimension for a complex \( M \in \mathcal{C}(R) \). If we denote by \( AF \text{fd} M \) the projective dimension of Avramov–Foxby, then it is easy to see that

\[
AF \text{fd} M = \text{fd} M - \text{sup} M.
\]

The following flat version of Lemma 2.3 follows from the isomorphism

\[
M \otimes^L_R R \cong M.
\]

Lemma 4.3: If \( M \in \mathcal{D}(R) \) has finite flat dimension, then it belongs to \( \mathcal{D}^{<\infty}(R) \).

The following lemma is deduced from the property of the derived tensor products.

Lemma 4.4: Let \( M \in \mathcal{D}^{<\infty}(R) \) and \( F := M \otimes^L_R - \). Then for all \( m \leq n \),

\[
F(\mathcal{D}^{[m,n]}(R^{\text{op}})) \subset \mathcal{D}^{[-\infty,n+\text{sup} M]}(k)
\]

and there is \( N \in \mathcal{D}^{[m,n]}(R^{\text{op}}) \) such that \( H^{n+\text{sup} M}(F(N)) \neq 0 \).

Proof. The first statement follows from computation of \( M \otimes^L_R N \) by using a DG-projective resolution of \( M \) (see e.g. [12]).

We set \( b = \text{sup} M \). Then \( H^b(M \otimes^L_R H^0) \cong H^b(M) \otimes_{H^0} H^0 \cong H^b(M) \). Thus, \( N := H^0[-n] \) has the desired property. \( \blacksquare \)
4.2. The class $\mathcal{F}$ and sup-flat (spft) resolution. The class $\mathcal{F}$ plays the role of flat modules in the usual flat resolutions for sup-flat resolutions. However, an explicit description like $\mathcal{P}, \mathcal{I}$ has not yet been obtained.

**Definition 4.5:** We denote by $\mathcal{F} \subset \text{D}(R)$ the full subcategory of those objects $F \in \text{D}(R)$ such that $\text{fd} F = 0$.

The basic properties of $\mathcal{F}$ are summarized in the lemma below.

**Lemma 4.6:** Let $F \in \mathcal{F}$. Then the following statements hold:

1. Let $N \in \text{D}(R^{\text{op}})$. The canonical morphism $N \to \sigma^{>n} N$ induces an isomorphism below for $n \in \mathbb{Z}$,
   
   $$ H^n(F \otimes_R^L N) \to H^n(F \otimes_R^L \sigma^{>n} N). $$

2. Let $N \in \text{D}(R^{\text{op}})$. The canonical morphism $\sigma^{\leq n} N \to N$ induces an isomorphism below for $n \in \mathbb{Z}$,
   
   $$ H^n(F \otimes_R^L \sigma^{\leq n} N) \to H^n(F \otimes_R^L N). $$

3. For $N \in \text{Mod} H^0$, we have
   
   $$ H^n(F \otimes_R^L N) = \begin{cases} 
   H^0(F) \otimes_{H^0} N, & n = 0, \\
   0, & n \neq 0/
   \end{cases} $$

4. For $N \in \text{D}(R^{\text{op}})$, we have $H^n(F \otimes_R^L N) \cong H^0(F) \otimes_{H^0} H^n(N)$.

5. $H^0(F)$ is a flat $H^0$-module and $H(F) \cong H^0(F) \otimes_{H^0} H$.

**Remark 4.7:** Lurie [10, Section 7.2] studied the class $\mathcal{F}$ for connective $\mathbb{E}_1$-algebras and characterized flat $\mathbb{E}_1$-modules by the condition (5) of above lemma in [10, Theorem 7.2.2.15].

**Proof.** (1) Since $F \otimes_R^L \sigma^{<n} N$ belongs to $\text{D}^{<n}(R)$, we have $H^i(F \otimes_R^L \sigma^{<n} N) = 0$ for $i = n, n + 1$. Now the desired isomorphism is derived from the canonical exact triangle

$$ \sigma^{<n} N \to N \to \sigma^{\geq n} N \to. $$

(2) This is proved in a similar way to (1).

(3) The case $n = 0$ is well-known. The case $n \neq 0$ follows from (1) and (2).
(4) Combining (1), (2) and (3), we obtain the desired isomorphism as below:
\[ H^n(F \otimes_R^L N) \cong H^n(F \otimes_R^L \sigma_{\leq n} \sigma_{\geq n} N) \]
\[ \cong H^n(F \otimes_R^L H^n(N)[-n]) \]
\[ \cong H^0(F \otimes_R^L H^n(N)) \cong H^0(F) \otimes_{H^0} H^n(N). \]

(5) The first statement follows from (3). The second statement follows from (4). □

Using standard arguments of triangulated categories, we obtain the following lemma.

Lemma 4.8: Let \( a, b \in \mathbb{Z} \) such that \( a \leq b \). Then for any object
\[ M \in \mathcal{F}[a] \ast \mathcal{F}[a+1] \ast \cdots \ast \mathcal{F}[b], \]
we have \( \text{fd} M \leq b - a. \)

We give the definition of a sup-flat (spft) resolution of \( M \in \text{D}^{<\infty}(R) \).

Definition 4.9 (spft morphism and spft resolution): Let \( M \in \text{D}^{<\infty}(R), M \neq 0. \)

1. A spft morphism \( f : F \to M \) is a morphism in \( \text{D}(R) \) such that
   \( F \in \mathcal{F}[-\sup M] \) and the morphism \( H^{\sup M}(f) \) is surjective.

2. A spft morphism \( f : F \to M \) is called minimal if the morphism
   \( H^{\sup M}(f) \) is a flat cover.

3. A spft resolution \( F_* \) of \( M \) is a sequence of exact triangles for \( i \geq 0 \)
   with \( M_0 := M, \)
   \[ M_{i+1} \xrightarrow{g_{i+1}} F_i \xrightarrow{f_i} M_i, \]
such that \( f_i \) is spft.
   The following inequality holds:
   \[ \sup M_{i+1} = \sup F_{i+1} \leq \sup F_i = \sup M_i. \]
   For a spft resolution \( F_* \) with the above notations, we set \( \delta_i := g_{i-1} \circ f_i, \)
   \[ \delta_i : F_i \to F_{i-1}. \]
   Moreover, we write
   \[ \cdots \to F_i \xrightarrow{\delta_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{\delta_1} F_0 \to M. \]

4. A spft resolution \( F_* \) is said to have length \( e \) if \( F_i = 0 \) for \( i > e \) and \( F_e \neq 0. \)

5. A spft resolution \( F_* \) is called minimal if \( f_i \) is minimal for \( i \geq 0. \)
It is clear that $P \subset F$. Therefore every $M \in D^{< \infty}(R)$ admits a spft morphism $F \to M$.

We collect basic properties of spft resolutions.

**Lemma 4.10:** Let $M \in D^{< \infty}(R)$ and $f : F \to M$ be a spft morphism and $N := \operatorname{cn}(f)[-1]$ the cocone of $f$. Assume that $1 < \text{fd } M$. Then,

$$\text{fd } N = \text{fd } M - 1 - \sup M + \sup N.$$

**Lemma 4.11:** Let $M \in D^{< \infty}(R)$ and $d \in \mathbb{N}$ be a natural number. Then the following conditions are equivalent:

1. $\text{fd } M \leq d$.
2. For any spft resolution $F_\bullet$, there exists a natural number $e \in \mathbb{N}$ such that $M_e \in \mathcal{F}[-\sup F_e]$ and $e + \sup F_0 - \sup M_e \leq d$. In particular, we have a spft resolution of length $e$,

$$M_e \to F_{e-1} \to F_{e-2} \to \cdots \to F_1 \to F_0 \to M.$$ 

3. $M$ has spft resolution $F_\bullet$ of length $e$ such that $e + \sup F_0 - \sup F_e \leq d$.

**Lemma 4.12:** Let $L \in \operatorname{Mod}(H^0)^{\text{op}}$, $M \in D^{< \infty}(R)$ and let $F^\bullet$ be a spft resolution of $M$. We denote the complexes below by $Y_i, Y'_i$:

$$Y_i : H^{\sup F_i}(F_{i+1} \otimes_R L) \to H^{\sup F_i}(F_i \otimes_R L) \to H^{\sup F_i}(F_{i-1} \otimes_R L),$$

$$Y'_i : H^{\sup F_i}(F_{i+1}) \otimes_{H^0} L \to H^{\sup F_i}(F_i) \otimes_{H^0} L \to H^{\sup F_i}(F_{i-1}) \otimes_{H^0} L.$$

Then,

$$H^n(M \otimes_R L) = \begin{cases} 0, & n \neq -i + \sup F_i \text{ for any } i \geq 0, \\ H(Y_i), & n = -i + \sup F_i \text{ for some } i \geq 0. \end{cases}$$

Moreover, in the case $n = -i + \sup F_i$, we have

$$H(Y_i) = \begin{cases} H^{\sup F_i}(F_i) \otimes_{H^0} L, & \text{sup} F_{i-1} \neq \sup F_i = \sup F_{i+1}, \\ \operatorname{Cok}[H^{\sup F_i}(F_{i+1}) \otimes_{H^0} L \to H^{\sup F_i}(F_i) \otimes_{H^0} L], & \text{sup} F_{i-1} \neq \sup F_i = \sup F_{i+1}, \\ \operatorname{Ker}[H^{\sup F_i}(F_i) \otimes_{H^0} L \to H^{\sup F_i}(F_{i-1}) \otimes_{H^0} L], & \text{sup} F_{i-1} = \sup F_i \neq \sup F_{i+1}, \\ H(Y'_i), & \text{sup} F_{i-1} = \sup F_i = \sup F_{i+1}. \end{cases}$$
We recall the notion of pure-injection. Let $A$ be an algebra. A morphism $f : M \to N$ between $A$-modules said to be a **pure-injection** if

$$f \otimes_A N : M \otimes_A N \to N \otimes_A N$$

is injective for all $A^{\text{op}}$-modules $N$.

**Theorem 4.13:** Let $M \in \mathcal{D}^{<\infty}(R)$ and $d \in \mathbb{N}$ be a natural number. Then the following conditions are equivalent:

1. $\text{fd} M = d$.
2. For any spft resolution $F_\bullet$, there exists a natural number $e \in \mathbb{N}$ which satisfies the following properties:
   a. $M_e \in \mathcal{F}[-\text{sup} M_e]$.
   b. $d = e + \text{sup} F_0 - \text{sup} M_e$.
   c. $H^{\text{sup} M_e}(g_e)$ is not a pure-injection.
3. $M$ has spft resolution $F_\bullet$ of length $e$ which satisfies the following properties:
   a. $d = e + \text{sup} F_0 - \text{sup} F_e$.
   b. $H^{\text{sup} F_e}(\delta_e)$ is not a pure-injection.

As with the sppj resolution, Theorem 4.13 has the following consequences. We denote by $\mathcal{D}(R)_{\text{ffd}}$ the full subcategory consisting of $M$ having finite flat dimension.

**Proposition 4.14:**

$$\mathcal{D}(R)_{\text{ffd}} = \text{thick } \mathcal{F} = \bigcup \mathcal{F}[a] * \mathcal{F}[a + 1] * \cdots * \mathcal{F}[b],$$

where $a, b$ runs through all the pairs of integers such that $a \leq b$.

### 4.3. Conjecture about an explicit description of $\mathcal{F}$

The following is a spft version of Lemma 2.14 and is still a conjecture.

We denote by $\mathcal{F}' \subset \mathcal{D}(R)$ the full subcategory of those objects $F' \in \mathcal{D}(R)$ which form a quasi-isomorphism class of DG-$R$-modules of the forms $F^0 \otimes_{R^0} R$ for some flat $R^0$-module $F^0$. It is easy to check that $\mathcal{F}' \subset \mathcal{F}$

**Conjecture 4.15:** $\mathcal{F}' = \mathcal{F}$.
5. The global dimension

We introduce the notion of the global dimension of a connective DG-algebra $R$.

**Theorem 5.1:** Let $R$ be a connective DG-algebra. Then the following numbers are the same:

$$\sup\{ \text{pd } M - \text{amp } M \mid M \in \mathcal{D}^{<\infty}(R) \},$$

$$\sup\{ \text{pd } M \mid M \in \text{Mod } H^0 \},$$

$$\sup\{ \text{id } M - \text{amp } M \mid M \in \mathcal{D}^{>\infty}(R) \},$$

$$\sup\{ \text{id } M \mid M \in \text{Mod } H^0 \}.$$

This common number is called the (right) global dimension of $R$ and is denoted by $\text{gldim } R$.

**Proof.** For simplicity, we set the $i$-th value in question to be $v_i$ for $i = 1, 2, 3, 4$. For example,

$$v_2 := \inf\{ n \in \mathbb{N} \mid \text{pd } M \leq n \text{ for all } M \in \text{Mod } H^0 \}.$$

We claim $v_1 = v_2$. It is clear that $v_2 \leq v_1$. Therefore it is enough to prove that $\text{pd } M \leq v_2 + \text{amp } M$ for any $M \in \mathcal{D}^{<\infty}$. In the case where $\text{amp } M = \infty$, there is nothing to prove. We deal with the case $\text{amp } M < \infty$ by the induction on $a = \text{amp } M$. The case $\text{amp } M = 0$ is obvious. Assume that the case $< a$ is already proved. Let $\sigma^{<\sup} M \rightarrow M \rightarrow H^{\sup}(M)[-\sup] \rightarrow$ be the exact triangle induced from the standard $t$-structure. Note that

$$\text{amp } \sigma^{<\sup} M < \text{amp } M.$$

Using Lemma 2.6 and the induction hypothesis, we deduce the desired inequality.

We can prove $v_3 = v_4$ in the same way.

We prove $v_2 \leq v_4$. If $v_4 = \infty$, then there is nothing to prove. Assume that $v_4 < \infty$. Let $M \in \text{Mod } H^0$. Then for any $N \in \text{Mod } H^0$, we have $\mathbb{R}\text{Hom}(M, N) \in \mathcal{D}^{[0,v_4]}$. Thus, by Theorem 2.19, we deduce that $\text{pd } M \leq v_4$. This proves the desired inequality.

We can show $v_4 \leq v_2$ in the same way and prove the equality $v_2 = v_4$. 

**Remark 5.2:** For any connective DG-algebra $R$, we have

$$\sup\{ \text{pd } M \mid M \in \mathcal{D}^{<\infty}(R) \} = \infty,$$

since $\text{pd}(R \oplus R[1]) = n$ for $n \in \mathbb{N}$. 


Observe that if $R$ is an ordinary algebra, then the global dimension defined in Theorem 5.1 coincides with the ordinary global dimension. The ordinary global dimensions are not preserved by derived equivalence, but their finiteness is preserved. We prove the DG-version.

Let $R$ and $S$ be connective DG-algebra. Assume that they are derived equivalent to each other. Namely, there exists an equivalence $\mathcal{D}(R) \simeq \mathcal{D}(S)$ of triangulated categories, by which we identify $\mathcal{D}(R)$ with $\mathcal{D}(S)$.

**Proposition 5.3:** In the above situation the following assertions hold:

1. $\mathrm{pd}_S R < \infty$.
2. $\mathrm{gldim} S \leq \mathrm{gldim} R + \mathrm{pd}_S R$.
3. $\mathrm{gldim} R < \infty$ if and only if $\mathrm{gldim} S < \infty$.

**Proof.** (1) It is well-known that $R \in \mathbf{thick} S$. Thus, in particular, $\mathrm{pd}_S R < \infty$.

(2) If $\mathrm{gldim} R = \infty$, then there is nothing to prove. We assume $\mathrm{gldim} R < \infty$.

Let $M \in \mathbf{Mod} H^0(S)$. Then, $R \mathrm{Hom}(R, M) \in \mathcal{D}[-\sup_S R, \mathrm{pd}_S R - \sup_S R](k)$. Therefore $M$ belongs to $\mathcal{D}[-\sup_S R, \mathrm{pd}_S R - \sup_S R](R)$. Thus we have

$$\mathrm{amp}_R M \leq \mathrm{pd}_S R < \infty$$

and

$$\mathrm{amp}_R M - \sup_R M - \sup_S R = -\inf_R M - \sup_S R \leq 0.$$  

Note that $\mathrm{pd}_R M \leq \mathrm{gldim} R + \mathrm{amp}_R M < \infty$. Let $P_\bullet$ be a sppj resolution of $M$ as an object of $\mathcal{D}(R)$ of length $e$ such that $\mathrm{pd}_R M = e + \sup_R M - \sup_R e$. Then, $M \in \mathcal{P}[-\sup_R P_0] \cdots \mathcal{P}[e - \sup_R P_e]$. Since for $P \in \mathcal{P}$ we have $\mathrm{pd}_S P = \mathrm{pd}_S R$, using Corollary 2.6, we deduce the inequality below:

$$\mathrm{pd}_S M \leq \mathrm{pd}_S R + e - \sup_S P_e = \mathrm{pd}_S R + \mathrm{pd}_R M - \sup_R M + \sup_R P_e - \sup_S P_e.$$  

Observe that $\sup_R P_e - \sup_S P_e = -\sup_S R$. Thus, combining the inequalities (5-11), (5-10) and $\mathrm{pd}_R M \leq \mathrm{gldim} R + \mathrm{amp}_R M$, we obtain the desired inequality:

$$\mathrm{pd}_S M \leq \mathrm{pd}_S R + \mathrm{pd}_R M - \sup_R M + \sup_R P_e - \sup_S P_e \leq \mathrm{pd}_S R + \mathrm{gldim} R + \mathrm{amp}_R M - \sup_R M - \sup_S R.$$  

(3) follows from (1) and (2).

We give a characterization of connective DG-algebras of global dimension 0.
Proposition 5.4: For a connective DG-algebra $R$, the following conditions are equivalent:

1. $\text{gldim } R = 0$.
2. $R$ is an ordinary algebra (i.e., $H^{<0} = 0$) which is semi-simple.

Proof. The implication (2) $\Rightarrow$ (1) is clear. We prove the implication (1) $\Rightarrow$ (2).

We only have to show that $H^{<0} = 0$. Since $\text{pd } H^0 = 0$, the canonical morphism $R \to H^0$ which is sppj splits in $D(R)$. Thus, the canonical exact sequence $0 \to H^{<0} \to H \to H^0 \to 0$ of graded $H$-modules splits. It follows that $H^{<0} = 0$.

To finish this section we point out the following theorem, suggested by the referee.

Theorem 5.5: Assume that the base commutative ring $k$ is noetherian. Let $R$ be a commutative DG-algebra of bounded cohomology such that the cohomology algebra $H = H(R)$ is essentially of finite type. If $\text{gldim } R < \infty$, then $R$ is concentrated in the 0-th degree, i.e., $H^i = 0$ for $i \neq 0$.

This theorem is nothing but a reformulation, by using Proposition 2.26, of Yekutieli’s result [15, Theorem], which is based on Jorgensen’s result [7, Theorem 0.2].

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