THE SPIN-BRAUER DIAGRAM ALGEBRA

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Abstract. We investigate the spin-Brauer diagram algebra, denoted $\text{SB}_n(\delta, \eta)$, that arises from studying an analogous form of Schur-Weyl duality for the action of the pin group on $V^\otimes n \otimes \Delta$. Here $V$ is the standard $N$-dimensional complex representation of $\text{Pin}(N)$ and $\Delta$ is the spin representation. When $\delta = \eta = N$ is a positive integer, we define a surjective map $\text{SB}_n(N, N) \to \text{End}_{\text{Pin}(N)}(V^\otimes n \otimes \Delta)$. We show $\text{SB}_n(\delta, \eta)$ is a cellular algebra and use cellularity to characterize its irreducible representations.

Contents

1. Introduction
2. The Spinor Representation
3. Spin-Brauer Diagram Algebra
4. Equivariant Projection and Injection Maps
5. Spin-Brauer Multiplication agrees with Composition
6. Cellularity of $\text{SB}_n(\delta, \eta)$
References

1. Introduction

Schur-Weyl duality is a seminal result in representation theory. It states that the actions of $\mathfrak{S}_n$ and $\text{GL}(V)$ on $V^\otimes n$ generate each others’ commutators. Here $\mathfrak{S}_n$ is the symmetric group on $n$ letters and $V$ is the standard representation.

In [Bra], Brauer pursued an analogous result to Schur-Weyl duality replacing the general linear group with the orthogonal group. Using invariant theory, he proved that the Brauer diagram algebra, denoted $B_n(\delta)$, surjects onto $\text{End}_{\text{O}(N)}(V^\otimes n)$. Here $\delta$ is the dimension of $V$. These diagram algebras, however, are well defined for any parameter $\delta$ and over any commutative ring $R$. Brauer proved this in his paper by giving a purely combinatorial description of multiplication in $B_n(\delta)$. He went on to show $B_n(\delta)$ possessed certain properties, but questions about its semi-simplicity and irreducible representations were not well understood until recently.

Following Brauer’s work, many variations of Schur-Weyl duality for matrix subgroups of $\text{GL}(V)$ and their corresponding centralizer algebras were investigated. We would like to recall two further examples. In 1989, Koike [Koi] and Turaev [Tur] independently discovered the walled-Brauer diagram algebra, $\text{Br}_{r,t}(N)$ as the centralizer of $\text{GL}(V)$ on $V^\otimes r \otimes (V^*)^\otimes t$. It has since been highly studied. For example, in [BCHLLS] the authors decompose $V^\otimes r \otimes (V^*)^\otimes t$ into irreducible $\text{GL}(V)$-modules. Then, in [CVDM] Cox, De Visscher, Doty and Martin discuss its blocks and semi-simplicity.

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Soon after in the 1990s, Martin [Mar1, Mar2] and Jones [Jon] independently discovered
the partition diagram algebra. It arose within the context of statistics as the centralizer
of the action of $\mathfrak{S}_n$ on $V^\otimes k$, the $k$-fold tensor product of the $n$-dimensional permutation
representation representation $V$. The partition algebra was then also extensively studied.
Notably in [Xi], Xi proved it is a cellular algebra in and used this to characterize its irreducible representations and discuss its semi-simplicity. Additionally, Halverson and Ram [HR] provided an explicit presentation by generators and relations and showed the existence of Murphy elements.

These examples suggest that diagram algebras arising from variations of Schur-Weyl duality are interesting objects of study in their own right. Additionally, they are easier to work with than the maps in the centralizer algebra. Results do not always immediately descend from the diagram algebras to their corresponding centralizers. However in [BEG], Bowman, Enyang and Goodman give a process for passing certain properties properties via quotient towers. As a result, defining and studying these diagram algebras as purely combinatorial objects also lies at the core of understanding the various forms of Schur-Weyl duality.

In [Koi2], Koike pursued another analogue of Schur-Weyl duality, replacing $O(N)$ with its double cover $\text{Pin}(N)$. We will use his results to prove that the spin-Brauer diagram algebra $SB_n(N,N)$ surjects onto $\text{End}_{\text{Pin}(N)}(V^\otimes n \otimes \Delta)$. Here $V$ is the standard representation of the orthogonal group and $\Delta$ is the spin representation of $\text{Pin}(N)$. For his purposes, Koike only considered this diagram algebra with a single variable $\delta$. We introduce a new variable $\eta$ with the future goal of defining a category of representations of the pin group to understand the kernel of the map $SB_n(N,N) \to \text{End}_{\text{Pin}(N)}(V^\otimes n \otimes \Delta)$, as Lehrer and Zhang did in [LZ]. As in the above examples, studying a centralizer algebra gave rise to a diagram algebra. Although Koike’s results imply that $SB_n(N,N)$ surjects onto the centralizer, he never gives an explicit combinatorial structure to $SB_n(\delta, \eta)$. Without this structure, this version of Schur-Weyl duality was largely incomplete.

Accordingly, the purpose of this paper is to complete this form of Schur-Weyl duality by defining and studying the associative algebra $SB_n(\delta, \eta)$. The existence of additional equivariant maps to and from $\Delta$ complicates the composition structure of $\text{End}_{\text{Pin}(N)}(V^\otimes n \otimes \Delta)$. Consequently, unlike the other diagram algebras mentioned above, defining a multiplication structure on $SB_n(\delta, \eta)$ is not a straight-forward extension of multiplication in $B_n(\delta)$. In Section 3, we provide this purely combinatorial description of $SB_n(\delta, \eta)$. In particular, we define a multiplication structure on basis elements and prove this structure is associative.

This allows us to define the spin-Brauer diagram algebra over any commutative ring, with $n \in \mathbb{Z}_{\geq 0}$ and for any parameters $\delta$ and $\eta$.

In Section 4, we construct the $\text{Pin}(N)$-equivariant projection $V \otimes \Delta \twoheadrightarrow \Delta$ (also discussed in [SS]) and injection $\Delta \hookrightarrow V \otimes \Delta$. We then prove equivariance. Koike mentioned these maps and defined them from an invariant theoretic standpoint but never formulated them in a basis. By defining these maps explicitly, we establish a more concrete connection between $SB_n(\delta, \eta)$ and $\text{End}_{\text{Pin}(N)}(V^\otimes n \otimes \Delta)$.

In Section 5, we use this explicit construction to prove that the combinatorial multiplication structure we define in Section 3 agrees with the composition of maps the diagrams represent, as described in Section 4. This shows our notion of multiplication is correct and completes this form of Schur-Weyl duality for the pin group.

Once $SB_n(\delta, \eta)$ is defined separately from the centralizer algebra, it becomes possible to ask many questions about its structure— as seen above for the other diagram algebras. In
particular, we may inquire about its semi-simplicity, the kernel of the map $SB_n(\delta, \eta) \to \text{End}_{\text{Pin}(N)}(V \otimes^n \Delta)$ and its cellularity. Following [BEG], we can also ask which of these properties descend to $\text{End}_{\text{Pin}(N)}(V \otimes^n \Delta)$. In this paper we only answer the question of cellularity, leaving the others for future work.

In [GL], Graham and Lehrer defined the notion of a cellular algebra and proved that $B_n(\delta)$ is cellular. Using properties of cellular algebras, they described the irreducible representations of the Brauer algebra completely. Many other diagram algebras were proved to be cellular. For example, both the partition algebra [Xi] and walled Brauer diagram algebra [CVDM, Theorem 2.7] are cellular.

In the Section 6, we provide a basis free proof that $SB_n(\delta, \eta)$ is cellular using a method developed in [Xi]. We then use Graham and Lehrer’s work to completely describe the irreducible representations of $SB_n(\delta, \eta)$.

1. Background and Notation. We always use $V$ to denote the complex $N$-dimensional standard representation of the orthogonal group $O(N)$ or equivalently the standard representation of $\text{Pin}(N)$. $\Delta$ will denote the spin representation of $\text{Pin}(N)$, which we define more explicitly below.

We assume a basic knowledge of the Clifford Algebra $C(Q)$ and the pin group $\text{Pin}(N, Q)$, where $Q$ is a bilinear form. We will define a bilinear form in Section 2 and use this same bilinear form throughout the paper. As a result, we suppress the bilinear form in $\text{Pin}(N, Q)$, just writing $\text{Pin}(N)$. We will use the fact that $\text{Pin}(N)$ is the simply-connected double cover of the orthogonal Lie group $O(N)$ with associated Lie algebra $\mathfrak{so}(N)$. Additionally, we recall $\text{Pin}(N)$ is not connected. Indeed, it has two connected components given by $\text{Pin}(N) \cap C^\text{even}(Q)$ and $\text{Pin}(N) \cap C^\text{odd}(Q)$.

Furthermore, the subgroup $\text{Spin}(N) \simeq \text{Pin}(N) \cap C^\text{even}(Q) \subset \text{Pin}(N)$ is a connected and simply-connected Lie group, studying $\text{Spin}(N)$-equivariant maps is equivalent to studying $\mathfrak{so}(N)$-equivariant maps. If we prove $\mathfrak{so}(N)$-equivariance, we can then deduce $\text{Pin}(N)$-equivariance by checking equivariance for one element in $\text{Pin}(N) \cap C^\text{odd}(Q)$. Indeed, this element will generate the odd degree subspace of $\text{Pin}(N) \cap C^\text{odd}(Q)$ as a $\text{Spin}(N)$-algebra.

We take this perspective because proving $\mathfrak{so}(N)$-equivariance is easier and more illuminating than working with the spin or pin groups. We identify $\mathfrak{so}(N)$ with $\bigwedge^2 V$, where $V$ is the $N$-dimensional standard representation. For further background the reader might consult [FH, § 20].

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2. The Spinor Representation

2.1. Basic Definitions. We begin by defining $\mathfrak{so}(N)$ for both even and odd-dimensional standard representations $V$ similar to [SS]. Let $W = C^m$ and $W_* = (C^m)^*$ its dual and put

\[ \nabla = W \oplus W_*, \quad V = \nabla \oplus C. \]

Let $e$ be a basis vector for the one dimensional space $C$ of $V$. We define an orthogonal form on $V$ so that $W$ and $W_*$ are both $m$-dimensional isotropic subspaces of $V$ and $C$ is a
one-dimensional space perpendicular to both of them. Define the orthogonal form $\omega'$ on $\nabla$ by

$$\omega'((v, f), (v', f')) = f'(v) + f(v').$$

Extend this to an orthogonal form $\omega$ on $V$ by setting $\omega(e, e) = 1$ and $\omega(e, v) = 0$ for all $v \in \nabla$. Notice that this construction can easily be modified to drop the extra copy of $C$. In this case, for even $N$, we use the vector space $\nabla$ with the orthogonal form $\omega'$. For odd $N = 2m + 1$, put

$$\mathfrak{so}(2m + 1) \cong \bigwedge^2 V = \bigwedge^2 W \oplus (W \otimes e) \oplus (W \otimes W_*) \oplus (W_* \otimes e) \oplus \bigwedge^2 W_*.$$

Here we use the standard decomposition of $\bigwedge^2 (W \oplus W_* \oplus e)$. For even $N = 2m$, we take $\nabla = W \oplus W_*$, so we have

$$\mathfrak{so}(2m) \cong \bigwedge^2 \nabla = \bigwedge^2 W \oplus (W \otimes W_*) \oplus \bigwedge^2 W_*.$$

Throughout this paper, we prove $\mathfrak{so}(N)$-equivariance by considering the actions of the above summands separately. We adopt the notation of [SS] and define elements of $\mathfrak{so}(N)$ as follows

- For $v, w \in W$ we let $x_{v, w} = v \wedge w$ and $x_v = v \otimes e$.
- For $v \in W$ and $\lambda \in W_*$ we let $h_{v, \lambda} = v \otimes \lambda$.
- For $\lambda, \mu \in W_*$ we let $y_{\lambda, \mu} = \lambda \wedge \mu$ and $y_{\lambda} = \lambda \otimes e$.

Now we define a map $\mathfrak{so}(N) \to \mathfrak{gl}(V)$. Suppose $u \in W \subset V$, then

$$x_{v, w}u = 0, \quad x_vu = 0, \quad h_{v, \lambda}u = \lambda(u)v, \quad y_{\lambda}u = \lambda(u)e, \quad y_{\lambda, \mu}u = \mu(u)\lambda - \lambda(u)\mu.$$

We define the action on $\eta \in W_*$ similarly:

$$x_{v, w}\eta = \eta(w)v - \eta(v)w, \quad x_v\eta = -\eta(v)e, \quad h_{v, \lambda}\eta = -\eta(v)\lambda, \quad y_{\lambda}\eta = 0, \quad y_{\lambda, \mu}\eta = 0.$$

Finally, put

$$x_{v, w}e = 0, \quad x_ve = v, \quad h_{v, \lambda}e = 0, \quad y_{\lambda}e = -\lambda, \quad y_{\lambda, \mu}e = 0.$$

As mentioned in [SS] this is a well defined representation of $\mathfrak{so}(N)$ that respects the orthogonal form on $V$ for odd dimension and $\nabla$ for even dimension. With this action, if $N$ is odd, $V$ is the standard representation of $\mathfrak{so}(N)$. If $N$ is even, $\nabla$ is the standard representation of $\mathfrak{so}(N)$.

We will adopt the following perspective on the spin representation $\Delta$ of $\mathfrak{so}(N)$ found in [SS, FH]. Given the decomposition of the standard representation as $V = W \oplus W_* \oplus e$ or $\nabla = W \oplus W_*$ with $W$ $m$-dimensional, we put

$$\Delta = \bigwedge W = \bigwedge^0 W \oplus \cdots \oplus \bigwedge^m W$$

the exterior algebra on $W$.

As in [SS], we define the following operators on $\Delta$. For $v \in W$, let $X_v$ be the operator on $\Delta$ given by

$$X_v(w) = v \wedge w.$$

And for $\lambda \in W_*$, let $D_\lambda$ be the operator on $\Delta$ given by

$$D_\lambda(v_1 \wedge \cdots \wedge v_n) = \sum_{i=1}^n (-1)^{i-1} \lambda(v_i)v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_n.$$
With these definitions let
\[ H_{v,\lambda} = X_v D_\lambda. \]
This is the usual action of an element \( v \otimes \lambda \in \mathfrak{gl}(W) \) on \( \Delta \). Finally, define the operator \( D \) by
\[ D(v_1 \wedge \cdots \wedge v_n) = (-1)^n v_1 \wedge \cdots \wedge v_n. \]
Notice both the operators \( X_v \) and \( X_w \) supercommute for any \( v, w \in W \) as do the operators \( D_\lambda \) and \( D_\mu \) for any \( \lambda, \mu \in W_* \). That is, \( X_v X_w + X_w X_v = 0 \); so reversing the order of composition results in a sign change. We also have the following relationship between the operators \( X_v \) and \( D_\lambda \)
\[ (2.1.1) \quad X_v D_\lambda + D_\lambda X_v = \lambda(v). \]
Finally, we note that \( D \) supercommutes with both \( X_v \) and \( Y_\lambda \). Given these operators, we define a representation \( \rho \) of \( \mathfrak{so}(N) \) on \( \Delta \) as follows
\[
\begin{align*}
\rho(x_{v,w}) &= X_v X_w, & \rho(x_v) &= \frac{1}{\sqrt{2}} X_v D, \\
\rho(y_\lambda) &= \frac{1}{\sqrt{2}} D D_\lambda, & \rho(h_{v,\lambda}) &= H_{v,\lambda} - \frac{1}{2} \lambda(v), \\
\rho(y_{\lambda,\mu}) &= D_\lambda D_\mu.
\end{align*}
\]
This is a well-defined representation. In particular, the scalar \( \frac{1}{\sqrt{2}} \) ensures that the action respects the Lie bracket. We call \( \Delta \) the **Spinor Representation** of \( \mathfrak{so}(N) \).

### 3. Spin-Brauer Diagram Algebra

We follow the work of Brauer [Bra] and define the spin-Brauer diagram algebra \( \text{SB}_n(\delta, \eta) \) as a purely combinatorial object. In particular, we describe an associative multiplication structure.

**Definition 3.1.** For any parameters \( \delta, \eta \) and positive integer \( n \), a **spin-Brauer diagram** consists of five parts \( (U, U', \Gamma, \Gamma', f) \) where
- \( U \) and \( U' \) are subsets of \( T = \{1, \ldots, n\} \) and \( T' = \{1', \ldots, n'\} \) with a total order corresponding to the standard total order on \( T \) and \( T' \),
- \( \Gamma \) and \( \Gamma' \) are partial matchings on \( T \setminus U \) and \( T' \setminus U' \) respectively so that \( |T \setminus (U \cup V(\Gamma))| = |T' \setminus (U' \cup V(\Gamma'))| \),
- \( f \) is a bijection \( T \setminus (U \cup V(\Gamma)) \to T' \setminus (U' \cup V(\Gamma')) \).

We call \( (U, U', \Gamma, \Gamma', f) \) the **spin datum** for the spin-Brauer diagram.

We can think of this spin datum as a diagram by creating two rows of \( n \) vertices corresponding to \( T \) and \( T' \). The row for \( T \) will be on the top and \( T' \) on the bottom. We circle all the elements of \( U \) and \( U' \) and label them with their total order. We call these circled vertices **isolated vertices**. We then place an edge between \( x, y \in T \setminus U \) if \( (x, y) \in \Gamma \) and similarly draw an edge between the pairs in \( \Gamma' \). These edges are called **arcs**. Finally, connect \( x \in T \setminus (U \cup V(\Gamma)) \) to \( f(x) \in T' \setminus (U' \cup V(\Gamma')) \) with an edge. These edges are called **through strings**. Consider the following example converting a spin datum to a diagram.

**Example 3.2.** Let \( n = 5 \). The spin datum \( U = \{2, 5\}, U' = \{1', 4'\}, \Gamma = \{(1, 3)\}, \Gamma' = \{(2', 5')\} \) and \( f \) defined by \( f(4) = 3' \) corresponds to the following spin-Brauer diagram
Here $\Gamma$ and $\Gamma'$ describe the vertices connected by arcs. $U$ and $U'$ describe the isolated vertices in the first and second row respectively. Then $f$ describes how the through strings connect. It is also clear that we could reverse this process and easily read off the spin datum from the diagram. \hfill \Box

Fix a parameter $\delta$ and $n \in \mathbb{Z}_{\geq 0}$. Let $\Omega_1 = (U_1, U'_1, \Gamma_1, \Gamma'_1, f_1)$ and $\Omega_2 = (U_2, U'_2, \Gamma_2, \Gamma'_2, f_2)$ be spin-Brauer diagrams with $n$ vertices in each row.

We define a multiplication structure on spin-Brauer diagrams and extend this multiplication structure to linear combinations of spin-Brauer diagrams via the distributive property. Explicitly for any commutative ring $R$, to calculate $(\sum \alpha_i \Omega_i)(\sum \beta_j \Omega_j)$ with $\alpha_i, \beta_j \in R$, as a linear combination of spin-Brauer diagrams write
$$\left(\sum_i \alpha_i \Omega_i\right) \left(\sum_j \beta_j \Omega_j\right) = \sum_{i,j} \alpha_i \beta_j \Omega_i \Omega_j$$
and apply the following process to each of the $\Omega_i \Omega_j$. Let $\Omega = \Omega_2 \Omega_1$ be the diagram constructed as follows:

1) Place $\Omega_1$ on top of $\Omega_2$.
2) Define a new total ordering on all of the isolated vertices in $\Omega_1$ and $\Omega_2$ as $U_1 < U'_1 < U_2 < U'_2$ with the total orders on each of the $U_i$ preserved.
3) Any arcs and isolated vertices in the top row of $\Omega_1$ are also in the top row of $\Omega$ and similarly any arcs and isolated vertices in the bottom row of $\Omega_2$ are added to the bottom row of $\Omega$.
4) If we can follow any through strings from the top row of $\Omega_1$ in position $i$ to the bottom row of $\Omega_2$ in position $j$ draw a through string in $\Omega$ from vertex $i$ in the top row to vertex $j$ in the bottom row. Following a through string means walking along the path created when you place $\Omega_1$ on top of $\Omega_2$ that begins with vertex $i$.
5) If you follow a through string originating at a vertex $i$ in $\Omega_1$ to an isolated vertex with label $n_i$ in the total order, then vertex $i$ in the top row of $\Omega$ is isolated with label $n_i$. Similarly, if you follow a through string originating at a vertex $j$ in the bottom row of $\Omega_2$ to an isolated vertex with label $n_j$, the vertex $j$ in the bottom row of $\Omega$ is isolated with label $n_j$.

Notice the resulting diagram will not be a spin-Brauer diagram because the vertices will not be in the correct total order. We will discuss the combinatorial rule for placing these vertices back into their totally ordered state below called the spin-Clifford relation.

After this step, all the vertices in the top and bottom row of $\Omega$ will be fixed. We define a closed circuit in the product $\Omega_2 \Omega_1$ as a connected component in the graph created by identifying the vertices in the bottom row of $\Omega_1$ and top row of $\Omega_2$. As with the Brauer diagrams, closed circuits will scale the diagram $\Omega$ by a factor of $\delta$. We describe by example all the types of closed circuits that can occur in $\Omega$:

(I)
6) Scale the diagram $\Omega$ by $\delta$ for each closed circuit or type (I) and $\eta$ for each closed circuit of type (II) – (V).

7) Let $U \sqcup U'$ be the indices of the isolated vertices as they appear in $\Omega$. Reindex the remaining isolated vertices using the total order induced from step (2) so that the indexing is strictly increasing.

This accounts for any isolated vertices we may have removed in closed circuits. At this step the isolated vertices are not in the correct total order. To fix this, we must generalize the Clifford-relation discussed in [SS, §2.3]. Suppose $\Omega$ is a diagram resulting from this multiplication process with $U = \{n_1, \ldots, n_k\}$ and $U' = \{n'_1, \ldots, n'_\ell\}$ placed in the the total order from step (7). Furthermore, let $\Omega''$ be obtained by switching two consecutive elements $i, j \in U \sqcup U'$ and letting the new $U$ be the first $k$ elements and the new $U'$ the last $\ell$. Let $\Omega''$ be the diagram obtained by removing $i$ and $j$ from $U \sqcup U'$ and placing an edge between them.
Definition 3.3. The spin-Clifford relation is $\Omega + \Omega' = 2\Omega''$. □

Remark 3.4. Notice the spin-Clifford relation is a strict generalization of the Clifford relation $[SS]$. Indeed, we can swap isolated vertex indices within rows or across rows so long as they are consecutive. Furthermore, it applies to any diagram that arises from this multiplication process, not strictly spin-Brauer diagrams. We will see in Section 5 that this generalization is possible because diagrams arising from this multiplication process correspond to well defined compositions of maps in the centralizer algebra. Finally, notice that multiplying any spin-Brauer diagram by the identity restricts the spin-Clifford relation to Clifford relation on spin-Brauer diagrams. □

8) Use the spin-Clifford relation to place the isolated vertices back in increasing order, where the total order comes from step (2).

9) Reindex the isolated vertices in the bottom row so they begin at one in the total order.

Consider the following example of multiplying two spin-Brauer diagrams

Example 3.5.

This example will illustrate every piece of the multiplication process. Following step (2), we relabel all the isolated vertices to put them in a total order, increasing left to right and top to bottom. We preserve all of the arcs and isolated vertices as described in (3). Similarly, we draw through strings as described in (4). In this example, we have two through strings that terminate in isolated vertices. As described in step (5), the originating vertex of each of these through strings becomes isolated. Furthermore, according to step (6) each closed circuit scales by $\delta$. We have one closed circuit in this example of type (II) with 2 arcs. Applying all of these steps the first simplification is,

As mentioned, this is not a spin-Brauer diagram. Following step (7) we reindex the isolated vertices so they are consecutive integers while maintaining the order,

Again, this is not a spin-Brauer diagram. The isolated vertices of this diagram currently have total order $1 < 3 < 2 < 4$. To fix this, we need the isolated vertices to be strictly
increasing from left to right and top to bottom. Following the instructions in step (8), we swap the vertices labeled 2 and 3 in the total order via the spin-Clifford relation,

\[ -\eta \]
\[ \begin{array}{c}
\circ_3 \\
\circ_4 \\
\end{array} \]
\[ \begin{array}{c}
\circ_1 \\
\circ_2 \\
\end{array} \]
\[ \begin{array}{c}
\circ_4 \\
\circ_1 \\
\end{array} \]
\[ +2\eta \]
\[ \begin{array}{c}
\circ_3 \\
\circ_4 \\
\end{array} \]
\[ \begin{array}{c}
\circ_1 \\
\circ_2 \\
\end{array} \]
\[ \begin{array}{c}
\circ_4 \\
\circ_1 \\
\end{array} \].

Concluding, we proceed to (9), reindexing the isolated vertices to get,

\[ -\eta \]
\[ \begin{array}{c}
\circ_1 \\
\circ_4 \\
\end{array} \]
\[ \begin{array}{c}
\circ_2 \\
\circ_1 \\
\end{array} \]
\[ +2\eta \]
\[ \begin{array}{c}
\circ_3 \\
\circ_4 \\
\end{array} \]
\[ \begin{array}{c}
\circ_1 \\
\circ_2 \\
\end{array} \]
\[ \begin{array}{c}
\circ_4 \\
\circ_1 \\
\end{array} \].

**Definition 3.6.** For any ring \( R \), parameter \( \delta \) and \( n \in \mathbb{Z}_{\geq 0} \), define the algebra consisting of \( R \)-linear combinations of spin-Brauer diagrams on \( 2n \)-vertices with the above multiplication structure as the spin-Brauer Diagram Algebra denoted \( SB_n(\delta, \eta) \).

**Definition 3.7.** We define \( B(SB_n(\delta, \eta)) \) to be the basis for \( SB_n(\delta, \eta) \) consisting of all spin-Brauer diagrams on \( n \) vertices.

**Theorem 3.8.** For any ring \( R \), \( n \in \mathbb{Z}_{\geq 0} \) and parameters \( \delta \) and \( \eta \), \( SB_n(\delta, \eta) \) is an associative algebra. Furthermore, the multiplication coefficients for the basis \( B(SB_n(\delta, \eta)) \) are in \( \mathbb{Z}[\delta] \).

**Proof.** It is clear from the multiplication construction that all the multiplication coefficients will be polynomials in \( \delta \) with integer coefficients.

As for associativity, this follows from not completing step (8) until after combining all three diagrams. In this case, you have the same closed circuits of the same type appear regardless of the order of composition so you will scale by the same constant \( \delta \) or \( \eta \). You can also follow through strings from the top row of the first diagram to the bottom row of the last diagram, i.e. following through strings is associative. This is also a consequence of the fact that Brauer diagrams have an associative multiplication structure.

Finally, if a through string terminates in an isolated vertex with index \( j \), this would occur regardless of the order of multiplication. Hence all the simplifications described in the multiplication would occur, changing the order of multiplication merely changes which simplifications occur first, but importantly does not change the simplifications themselves. See an example of this below. \[\square\]
Example 3.9. Consider the following multiplication of diagrams:

Regardless of the order of multiplication, we have a through string from the first vertex in the top row to the second vertex in the bottom row. Similarly, we get a closed circuit given by the first and last vertex of the bottom row of the first diagram and the second and last vertex of the top row of the third diagram. Whence, we scale by $\delta$. As noted in the proof, we merely change the order of the simplifications necessary to get this closed circuit. If we compose the top two diagrams first we have

If we compose the last two diagrams first we have

In either case, the same number of closed circuits appear. All other simplifications occur, just in opposite order. Consequently, regardless of the order of multiplication we get the same diagram with the same unordered isolated vertex numbers. Then, when we apply the spin-Clifford relation the same expansion as a sum of basis elements appears. In this case,
the resulting diagram is

\[ \eta \bullet \quad \bullet \quad \bullet \quad \⊙_1 \quad \bullet \quad \⊙_2. \]

4. Equivariant Projection and Injection Maps

We now define the equivariant projection \( V \otimes \Delta \to \Delta \) and injection \( \Delta \hookrightarrow V \otimes \Delta \) mentioned in [Koi2] but never described in a basis. This explicit construction is the key to linking multiplication in \( \text{SB}_n(\delta, \eta) \) with the composition of the corresponding maps.

**Definition 4.1.** Define the map \( \pi: V \otimes \Delta \to \Delta \) on basis elements by

\[
\pi(v \otimes x) := \sqrt{2}X_v(x) \quad \pi(\lambda \otimes x) := \sqrt{2}D_\lambda(x) \quad \pi(e \otimes x) := D(x).
\]

Extend by linearity to all of \( V \otimes \Delta \). We call \( \pi \) the spin projection. We also let \( \pi_i \) denote the projection of the \( i \)th tensor position of \( V \otimes \Delta \).

**Lemma 4.2.** The spin projection, \( \pi \), is a \( \text{Pin}(N) \)-equivariant projection map.

**Proof.** It is clear that this is a surjective map. By bilinearity of the tensor product, linearity of \( \pi \) and linearity of the action of \( \mathfrak{so}(N) \), it suffices to prove equivariance on basis elements of \( \Delta \) under the action of our basis for \( \mathfrak{so}(N) \).

In §2 we decomposed \( \mathfrak{so}(N) \) as

\[
\bigwedge^2 W \oplus (W \otimes e) \oplus (W \otimes W^*) \oplus (W^* \otimes e) \oplus \bigwedge^2 W^*.
\]

We prove equivariance under the action of each of the summands. First assume \( \dim(V) = 2m + 1 \) is odd. We will see the even case is naturally contained in the odd case.

It suffices to prove equivariance with respect to \( w_i \wedge w_j \in \bigwedge^2 W \) when we map \( w_k \otimes a, w^*_k \otimes a \) and \( e \otimes a \) where \( a \in \Delta \) is arbitrary. Consider the action on \( w_k \otimes a \),

\[
\pi((w_i \wedge w_j) \cdot w_k \otimes a) = \pi(w_k \otimes X_{w_i}X_{w_j}(a)) \\
= (\sqrt{2})X_{w_k}X_{w_i}X_{w_j}(a) \\
= (\sqrt{2})X_{w_i}X_{w_j}X_{w_k}(a) \\
= (w_i \wedge w_j) \cdot \pi(w_k \otimes a).
\]

The third equality holds because our operators super-commute. Now, when we consider the action on \( w^*_k \otimes a \) we have

\[
\pi((w_i \wedge w_j) \cdot w^*_k \otimes a) = \pi \left( (w^*_k(w_j)w_i - w^*_k(w_i)w_j) \otimes a + w^*_k \otimes X_{w_i}X_{w_j}(a) \right) \\
= \sqrt{2} \cdot w^*_k(w_j)X_{w_i}(a) - \sqrt{2} \cdot w^*_k(w_i)X_{w_j}(a) + \sqrt{2} \cdot D_{w^*_k}X_{w_i}X_{w_j}(a) \\
= \sqrt{2} \cdot X_{w_i}X_{w_j}D_{w^*_k}(a) \\
= (w_i \wedge w_j) \cdot \pi(w^*_k \otimes a).
\]
In the third equality, we use the relationship (2.1.1) between the $X_\nu$ and $D_\lambda$ operators. Finally, if we consider the action on $e \otimes a$, we see
\[
\pi((w_i \wedge w_j) \cdot e \otimes a) = \pi(e \otimes X_{w_i}X_{w_j}(a)) = DX_{w_i}X_{w_j}(a) = X_{w_i}X_{w_j}D(a) = (w_i \wedge w_j) \cdot \pi(e \otimes a).
\]

Here, we recall that the linear operator $D$ super-commutes with all other linear operators. This establishes equivariance for the $\bigwedge^2 W$ summand. The verification process for $\bigwedge^2 W_*$ is similar so we leave it to the reader.

Similarly, we work through the case of $W_* \otimes e$ and leave $W \otimes e$-equivariance to the reader. Consider $w_j^* \otimes e$, a basis element of $W_* \otimes e$. As before, we first consider the action of $w_j^* \otimes e$ on $w_i \otimes a \in V \otimes \Delta$. We have
\[
\pi((w_j^* \otimes e) \cdot w_i \otimes a) = \pi(w_j^*(w_i)e \otimes a + w_i \otimes \frac{1}{\sqrt{2}} DD_{w_j^*}(a)) = w_j^*(w_i)D(a) + X_{w_i}DD_{w_j^*}(a) = DD_{w_j^*}X_{w_i}(a) = (w_j^* \otimes e) \cdot \pi(w_i \otimes a).
\]

Once again the third equality follows from (2.1.1). Now consider the action on the element $w_i^* \otimes a$,
\[
\pi((w_j^* \otimes e) \cdot w_i^* \otimes a) = \pi(w_j^* \otimes \frac{1}{\sqrt{2}} DD_{w_j^*}(a)) = D_{w_i^*}DD_{w_j^*}(a) = DD_{w_j^*}D_{w_i^*}(a) = (w_j^* \otimes e) \cdot \pi(w_i^* \otimes a).
\]

This follows from the skew commutativity of the operators. Finally consider the action on the element $e \otimes a$,
\[
\pi((w_j^* \otimes e) \cdot e \otimes a) = \pi(-w_j^* \otimes a + e \otimes \frac{1}{\sqrt{2}} DD_{w_j^*}(a)) = -\sqrt{2}D_{w_j^*}(a) + \frac{1}{\sqrt{2}} D_{w_j^*}(a) = \frac{1}{\sqrt{2}} DD_{w_j^*}(a) = (w_j^* \otimes e) \cdot \pi(e \otimes a).
\]

As before, we use skew commutativity of the operators to obtain the third equality. This proves $W_* \otimes e$-equivariance. $W \otimes e$-equivariance is similar and so we leave it to the reader. It remains to prove $gl(W)$ equivariance, but this is clear from the definition of our maps.

If $N$ is even, we check equivariance for $\bigwedge^2 W$, $gl(W)$ and $\bigwedge^2 W^*$ exactly as above but now we do not have to consider the action of $e \otimes a$. Hence, the even case is naturally contained in the above work. As $\pi$ is equivariant with respect to each of the summands, it is $so(N)$-equivariant for any positive integer $N$.

This establishes $so(N)$-equivariance and hence $Spin(N)$-equivariance. As discussed in §1, to prove $Pin(N)$-equivariance, it suffices to prove $\pi$ is equivariant under the action of \(\frac{1}{\sqrt{2}}(w_1 - w_1^*) \in Pin(N) \cap e^{odd}(\omega)\).
Remark 4.3. We note that \( \frac{1}{\sqrt{2}}(w_1 - w_1^*) \in \Pin(N) \cap \mathcal{C}^{\text{odd}}(\omega) \). Clearly it is in the odd part of the Clifford algebra. To see it is in \( \Pin(N) \) notice

\[
\omega \left( \frac{1}{\sqrt{2}}(w_1 - w_1^*), \frac{1}{\sqrt{2}}(w_1 - w_1^*) \right) = -1.
\]

\[\Box\]

We check equivariance when acting on \( w_i \otimes a \). For ease of notation, let \( \gamma = \frac{1}{\sqrt{2}}(w_1 - w_1^*) \).

For explicit descriptions of the representations \( V \) and \( \Delta \) of \( \Pin(N) \), we refer the reader to Fulton-Harris [FH, §20]. We do note that in the definition of our bilinear form, we do not scale by 2 like Fulton-Harris. Accordingly, the element \( w_1^* \in \Pin(N) \) acts by \( D_{w_1^*} \), not \( 2D_{w_1^*} \).

When we consider the action of \( \gamma \) on \( w_i \otimes a \), we see

\[
\pi(\gamma \cdot (w_i \otimes a)) = \pi(\gamma \cdot w_i \otimes \gamma \cdot a)
= \pi((-w_i + w_1^*(w_i)w_1 - w_1^*(w_i)w_1^*)) \otimes \left( \frac{1}{\sqrt{2}}X_{w_1}(a) - \frac{1}{\sqrt{2}}D_{w_1}(a) \right)
= -X_{w_i}X_{w_1}(a) - w_1^*(w_i)D_{w_1}X_{w_1}(a) + X_{w_i}D_{w_1}^*(a) - w_1^*(w_i)X_{w_1}D_{w_1^*}(a)
= -X_{w_i}X_{w_1}(a) + X_{w_i}D_{w_1}^*(a) - w_1^*(w_i)a
= \gamma \cdot (\frac{1}{\sqrt{2}}X_{w_1}(a))
= \gamma \cdot \pi(w_i \otimes a).
\]

Checking equivariance for \( w_i^* \otimes a \) and \( e \otimes a \) are extremely similar and so are left to the reader. This proves \( \Pin(N) \)-equivariance. \[\Box\]

Definition 4.4. When \( V \) is even dimensional, with \( \dim(V) = 2m \) define the map \( \iota: \Delta \to V \otimes \Delta \) by

\[
\iota(a) := \sqrt{2} \left( \sum_{i=1}^{m} w_i \otimes D_{w_i^*}(a) + w_i^* \otimes X_{w_i}(a) \right).
\]

When \( V \) is odd dimensional, with \( \dim(V) = 2m + 1 \) define

\[
\iota(a) := \sqrt{2} \left( \sum_{i=1}^{m} w_i \otimes D_{w_i^*}(a) + w_i^* \otimes X_{w_i}(a) \right) + e \otimes D(a).
\]

We call \( \iota \) the spin injection. We also let \( \iota_j \) denote the injection of \( \Delta \) into the \( j \)-th tensor position of \( V^{\otimes n} \otimes \Delta \).

Lemma 4.5. The spin injection, \( \iota \), is a \( \Pin(N) \)-equivariant injection.

Proof. This is also clearly an injection. We will proceed as in the proof of Lemma 4.2. First assume that \( \dim(V) = 2m + 1 \) is odd. We will prove equivariance in this case and deduce
equivariance in the even dimensional. Let \( w_j \wedge w_k \in \bigwedge^2 W \) and \( a \in \Delta \), we have

\[
\iota((w_j \wedge w_k) \cdot a) = \iota(X_{w_j}X_{w_k}(a))
\]

\[
= \sqrt{2} \left[ \sum_{i=1}^{m} w_i \otimes D_{w_i}^* X_{w_j}X_{w_k}(a) + w_i^* \otimes X_{w_i}X_{w_j}X_{w_k}(a) \right] + e \otimes DX_{w_j}X_{w_k}(a)
\]

\[
= \sqrt{2} \left[ \sum_{i=1}^{m} w_i \otimes X_{w_j}X_{w_k}(a) \right] + \sqrt{2} \cdot w_j \otimes X_{w_k}(a) - \sqrt{2} \cdot w_k \otimes X_{w_j}(a) + e \otimes X_{w_j}X_{w_k}D(a)
\]

\[
= (w_j \wedge w_k) \cdot \left[ \sqrt{2} \sum_{i=1}^{m} w_i \otimes D_{w_i}^*(a) + w_i^* \otimes X_{w_i}(a) + e \otimes D(a) \right]
\]

\[
= (w_j \wedge w_k) \cdot \iota(a).
\]

The case for \( w_j^* \wedge w_k^* \) is similar and left to the reader. Now consider the action of the element \( w_j^* \otimes e \),

\[
\iota((w_j^* \otimes e) \cdot a) = \iota\left(\frac{1}{\sqrt{2}} DD_{w_j}^*(a)\right)
\]

\[
= \sum_{i=1}^{m} w_i \otimes D_{w_i}^* DD_{w_j}^*(a) + w_i^* \otimes X_{w_i} DD_{w_j}^*(a) + \frac{1}{\sqrt{2}} e \otimes D_{w_j}^*(a)
\]

\[
= \sum_{i=1}^{m} w_i \otimes DD_{w_j}^* D_{w_i}^*(a) + w_i^* \otimes DD_{w_j}^* X_{w_i}(a) - w_j^* \otimes D(a) + \frac{1}{\sqrt{2}} e \otimes D_{w_j}^*(a)
\]

\[
= \sum_{i=1}^{m} w_i \otimes DD_{w_j}^* D_{w_i}^*(a) + w_i^* \otimes DD_{w_j}^* X_{w_i}(a) - w_j^* \otimes D(a) + \frac{2}{\sqrt{2}} e \otimes DD_{w_j}^* D(a)
\]

The case for \( w_j \otimes e \) is similar and left to the reader. It remains to verify \( gl(W) \)-equivariance, which is straightforward from the construction of our maps.

As in the proof of Lemma 4.2, the even dimensional case is contained in the above. Thus \( \iota \) is \( so(N) \)-equivariant and so \( Spin(N) \)-equivariant. We now check equivariance under the
action of $\gamma = \frac{1}{\sqrt{2}}(w_1 - w_1^*)$,

$$\iota(\gamma \cdot a) = \iota(\frac{1}{\sqrt{2}}X_{w_1}(a) - \frac{1}{\sqrt{2}}D_{w_1^*}(a))$$

$$= \sum_{i=1}^{m} w_i \otimes D_{w_i^*}(X_{w_1} - D_{w_1^*})(a) + w_i^* \otimes X_{w_i}(X_{w_1} - D_{w_1^*})(a) + \frac{1}{\sqrt{2}} e \otimes D(X_{w_1} - D_{w_1^*})(a)$$

$$= \sum_{i=1}^{m} -w_i \otimes (X_{w_1} - D_{w_1^*})D_{w_1^*}(a) - w_i^* \otimes (X_{w_1} - D_{w_1^*})X_{w_i}(a) + w_1 \otimes a - w_i^* \otimes a - \frac{1}{\sqrt{2}} e \otimes (X_{w_1} - D_{w_1^*})D(a)$$

$$= \sum_{i=1}^{m} -w_i \otimes (X_{w_1} - D_{w_1^*})D_{w_1^*}(a) - w_i^* \otimes (X_{w_1} - D_{w_1^*})X_{w_i}(a) + w_1 \otimes a - w_1 \otimes (X_{w_1} - D_{w_1^*})X_{w_1}(a) - w_1^* \otimes (X_{w_1} - D_{w_1^*})X_{w_1}(a) - \frac{1}{\sqrt{2}} e \otimes (X_{w_1} - D_{w_1^*})D(a)$$

$$= \gamma \cdot \iota(a).$$

This proves equivariance with respect to $\gamma$ and so $\text{Pin}(N)$-equivariance. \hfill \Box

For $\sigma \in \mathfrak{S}_n$, let $\tau_\sigma$ be the equivariant map $V^\otimes n \otimes \Delta \to V^\otimes n \otimes \Delta$ sending the vector in tensor position $i$ to tensor position $\sigma(i)$. We call $\tau_\sigma$ the swap operator.

We also have an equivariant map $\psi_{i,j}$ from [Koi2, §8]. Let $\psi_{i,j}$ be the linear immersion of the invariant element

$$\sum_{i=1}^{m} w_i \otimes w_i^* + w_i^* \otimes w_i + (e \otimes e)$$

into the $i,j$ tensor positions of $V^\otimes n \otimes \Delta$ where $\dim(V) = 2m + 1$. If $\dim(V) = 2m$, the $\mathfrak{so}(N)$-invariant element is

$$\sum_{i=1}^{m} w_i \otimes w_i^* + w_i^* \otimes w_i.$$

To realize the spin-Brauer diagrams as elements of the centralizer algebra, we need an additional equivariant map $\kappa_{i,j}$ called the contraction. This map is given by contracting the elements in the $i^{th}$ and $j^{th}$ tensor positions of $V^\otimes n \otimes \Delta$ using the bilinear form on $V$. By construction, $\mathfrak{so}(N)$ respects this bilinear form so this is an equivariant map.

**Remark 4.6.** Any operators that act on different tensor positions commute. This is clear and we leave verification to the reader, but it will be important. \hfill \Box

### 5. Spin-Brauer Multiplication agrees with Composition

In this section we discuss the correspondence between $\text{SB}_n(\delta, \eta)$ and maps in $\text{End}_{\text{Pin}(N)}(V^\otimes n \otimes \Delta)$. We conclude this section by proving our combinatorial description of multiplication agrees with the corresponding composition of maps.

Let $\Omega \in \mathcal{B}(\text{SB}_n(N,N))$ be a spin-Brauer diagram with spin datum $\Omega = (U,U',\Gamma,\Gamma',f)$ and let $T = \{1, \ldots, n\}$. Furthermore, let $\sigma \in \mathfrak{S}_n$ be the permutation induced by $f$. That
is, $\sigma(i) = f(i)$ if $i \in T \setminus (U \cup V(\Gamma))$ and $\sigma(i) = i$ otherwise. Then $\Omega$ corresponds to the following equivariant map

$$\Omega \mapsto \prod_{(i,j) \in \Gamma'} \psi_{i,j} \circ \prod_{j \in U'} t_j \circ \tau_{\sigma} \circ \prod_{i \in U} \pi_i \circ \prod_{(i,j) \in \Gamma} \kappa_{i,j} =: f_\Omega.$$ (5.1)

Extend the correspondence in (5.1) by linearity to all of $SB_n(\delta, \eta)$.

**Theorem 5.2.** Under this correspondence, for $n, N \in \mathbb{Z}^+$, $SB_n(N, N)$ surjects onto the centralizer algebra $\text{End}_{\text{Pin}(N)}(V^\otimes n \otimes \Delta)$ and for $N \geq 2n$ the map in (5.1) is an isomorphism.

**Proof.** This follows from [Koi2, §5, §7]. Indeed, Koike gives formulas for decomposing his maps into a composition of projections, injections, immersions and contractions as in (5.1) [Koi2, §6, Theorem 8.1]. Accordingly, it suffices to prove our maps on one tensor component agree up to a scalar.

From Lemmas 4.2 and 4.5 the spin projection and injection are $\text{Pin}(N)$-equivariant and thus $\mathfrak{so}(N)$-equivariant. Due to the semi-simplicity of $\mathfrak{so}(N)$, every finite dimensional representation can be decomposed as a direct sum of irreducible representations. In particular, we can view $\iota$ and $\pi$ as equivariant maps between the irreducible components of $\Delta$ and $V \otimes \Delta$.

These maps are unique on each of the components up to a scalar by Schur’s lemma for semi-simple Lie algebras. Accordingly, to show uniqueness of $\iota$ and $\pi$ it suffices to show $\Delta$ and $V \otimes \Delta$ decompose into a direct sum of irreducible representations with no multiplicities.

**Lemma 5.3.** $\Delta$ and $V \otimes \Delta$ decompose into a direct sum of irreducible representations with no multiplicities.

**Proof.** The spin representation $\Delta$ is irreducible when $\dim(V)$ is odd and is the direct sum of the two distinct irreducible half spin representations when $\dim(V)$ is even [FH, §20].

The weight diagram of $V \otimes \Delta$ is generated by $\alpha + \beta$ where $\alpha$ is a weight of $V$ and $\beta$ is a weight of $\Delta$. $\Delta$ has a rational highest weight that varies with the dimension of $V$. While $V$ has weights generated by $L_1, \ldots, L_n$ defined in [FH, §12]. Whence, the weight diagrams for $V$ and $\Delta$ are disjoint. As a result, when we generate the weight lattice for $V \otimes \Delta$, no two weights can sum to the same weight. Equivalently, all weights will have multiplicity 1. This implies every irreducible representation in $V \otimes \Delta$ will occur without multiplicity. \qed

We conclude from Lemma 5.3 that $\pi$ and $\iota$ are uniquely determined up to a scalar. This implies Koike’s equivariant maps $\Delta \hookrightarrow V \otimes \Delta$ and $V \otimes \Delta \twoheadrightarrow \Delta$ must agree with $\iota$ and $\pi$ up to a scalar.

Koike used invariant theory to prove that the image of his Generalized-Brauer diagrams span the centralizer algebra [Koi2, Lemma 5.6]. By the above, the images of our spin-Brauer diagrams must span the centralizer algebra as well. \qed

**Theorem 5.4.** If $n, N \in \mathbb{Z}_{\geq 0}$, for any $\Omega_1, \Omega_2 \in SB_n(N, N)$ we have $f_{\Omega_2 \Omega_1} = f_{\Omega_2} \circ f_{\Omega_1}$.

**Proof.** By linearity it suffices to verify $f_{\Omega_2 \Omega_1} = f_{\Omega_2} \circ f_{\Omega_1}$ for $\Omega_1, \Omega_2 \in B(SB_n(N, N))$ and for an arbitrary basis element of $V^\otimes n \otimes \Delta$.

Let $\Omega_1, \Omega_2 \in B(SB_n(N, N))$. Notice, our multiplication construction agrees with Brauer’s [Bra]. That is, $B_n(N)$ is a subalgebra of $SB_n(N, N)$. As a result, we know our theorem holds for any components of the diagrams that appear in Brauer diagrams. We may therefore assume $\Omega_1$ and $\Omega_2$ do not have any through strings creating a path from the top row of $\Omega_1$ to the bottom row of $\Omega_2$. Furthermore, we can assume there are no arcs that form closed circuit (I).
Now, if we consider the composition $f_{\Omega_2} \circ f_{\Omega_1}$ we need to simplify the maps so that the composition is of the form (5.1). This will correspond to a sum of maps which we will show is $f_{\Omega_2}f_{\Omega_1}$.

Notice the spin projections and contractions in the first row of $\Omega_1$ must remain as do the spin injections and immersions in the bottom row of $\Omega_2$. Indeed, it suffices to compute what Koike calls the inside homomorphism. That is, the compositions of maps between the spin contractions and projections of $\Omega_1$ and the spin injections and immersions of $\Omega_2$. For further discussion we refer the reader to [Koi2, §9].

By assumption, we can follow every through string originating from a vertex in the top row of $\Omega_1$ or bottom row of $\Omega_2$ to an isolated vertex. To prove the theorem, it remains to prove:

(1) All the closed circuits correspond to scaling by $\dim(V)$.
(2) Through strings leading to isolated vertices become isolated.

To prove part (2), we must also show the Spin-Clifford relation agrees with the corresponding composition of maps. In summary, the theorem breaks down into a series of lemmas.

**Lemma 5.5.** Closed circuit (II) corresponds to scaling by $N = \dim(V)$.

**Proof.** We are considering a closed circuit in our diagram, so no through strings will begin or end in any of our vertices. Equivalently, after applying the first projections maps we project away all of the entries in these tensor positions. We will use this fact in all of the following lemmas. We keep track of entries we project away with a dash. For example, if we contract the first and third tensor position of $v_{1} \otimes v_{2} \otimes v_{3} \otimes a$, we write $\omega(v_{1}, v_{3})(- \otimes v_{2} - \otimes a)$.

As noted, it suffices by linearity of our maps to prove this lemma for simple tensors. Furthermore, because we sum over all basis vectors, we can permute the indices in the sum corresponding to every closed circuit of the form (II) so that the sum resembles the example given in (II). Accordingly, it suffices to prove the result for this example. Suppose our circuit has length $k < n$. Then, circuit (II) corresponds to the maps

$$K_{k,k-1} \circ \cdots \circ K_{1,2} \circ \psi_{k-2,k-1} \circ \cdots \circ \psi_{2,3} \circ \iota_{k} \circ \iota_{1}.$$  

Let $- \otimes \cdots \otimes - \otimes a \in V^{\otimes k} \otimes \Delta$ with $a = w_{i_{1}} \wedge \cdots \wedge w_{i_{k}}$ a simple tensor in the basis for $\Delta$. Here we only consider the $k$ tensor positions in $V^{\otimes m} \otimes \Delta$ involved in our closed circuit. First, suppose $\dim(V) = 2m$.

When we apply all the injections we have

$$2 \sum_{i_{1},i_{2}=1}^{m} w_{i_{1}}^{*} \otimes w_{j_{1}} \otimes w_{j_{2}}^{*} \otimes \cdots \otimes w_{j_{(k-2)/2}}^{*} \otimes w_{j_{(k-2)/2}} \otimes w_{i_{2}} \otimes X_{w_{i_{2}}} \circ D_{w_{i_{1}}}^{*}(a) +$$

$$w_{i_{1}} \otimes w_{j_{1}}^{*} \otimes w_{j_{1}} \otimes \cdots \otimes w_{j_{(k-2)/2}}^{*} \otimes w_{j_{(k-2)/2}} \otimes w_{i_{2}}^{*} \otimes D_{w_{i_{2}}}^{*} \circ X_{w_{i_{1}}}^{*}(a) +$$

$$\cdots.$$  

There are far more terms in the sum corresponding to all the possible permutations of the spin immersions. However, it suffices to consider the terms that alternate between elements of $W$ and $W^{*}$. Indeed, $W$ and $W^{*}$ are isotropic, so when we apply the spin contraction the only pairs of basis elements that survive are $w_{i_{1}} \otimes w_{j_{k}}^{*}$ where $i_{1} = j_{k}$.

When we apply the contraction map to tensor positions 1 and 2, it equates the indices in the sum. We record this by reindexing the sum from $i_{1} = j_{1} = 1$ to $m$. We continue applying the contractions. Each time we apply a contraction, we equate two more indices. When we
apply the final contraction map, we have equated all of the indices and projected every tensor position. In the end, we have forced the string of equalities \( i_1 = j_1 = j_2 = \cdots = j_{(k-2)/2} = i_2 \). Our sum is now

\[
2 \sum_{i=1}^{m} \mathbf{a} = - \otimes \cdots \otimes - \otimes X_{w_i} \circ D_{w_i}^*(a) +
- \otimes \cdots \otimes - \otimes D_{w_i}^* \circ X_{w_i}(a).
\]

The map \( X_{w_i} \circ D_{w_i}^*(a) \) will be the identity if \( w_i \) is one of the components of \( a \in \Delta \) and zero otherwise. On the other hand, \( D_{w_i}^* \circ X_{w_i}(a) \) is the identity when \( w_i \) is not a component of \( a \) and zero otherwise. As we sum over all basis vectors, we get one copy of \( - \otimes \cdots \otimes - \otimes a \) for each index \( i = 1, \ldots, m \). This leaves us with

\[
(5.6) \quad (2m)(- \otimes \cdots \otimes - \otimes a).
\]

This is precisely the identity map scaled by \( 2m = \dim(V) \).

If \( \dim(V) = 2m + 1 \), we are in a similar situation. However, now we have additional terms corresponding to the spanning element \( e \). In particular, the immersions now contain an additional \( e \otimes e \). When we contract a term containing \( e \), if any other tensor position is not \( e \) the tensor vanishes. As a result, one additional term in the sum will be nonzero after the spin contractions. This is the term where every tensor position contains \( e \),

\[
e \otimes e \otimes \cdots \otimes e \otimes e \otimes e \otimes D \circ D(a).
\]

Notice that \( D \circ D \) is the identity. So when we apply the spin contraction to each of these positions what remains is

\[- \otimes \cdots \otimes - \otimes a,
\]

one additional copy of our original tensor. Adding this to \((5.6)\), we see the closed circuit corresponds to the map

\[- \otimes \cdots - \otimes a \mapsto (2m + 1)(- \otimes \cdots - \otimes a).
\]

Once again, this is precisely the identity map scaled by \( 2m + 1 = \dim(V) \).

\[ \square \]

**Lemma 5.7.** Closed circuit (III) corresponds to scaling by \( N = \dim(V) \).

**Proof.** We proceed as in Lemma 5.5. Suppose our circuit has length \( k < n \). We can permute the indices in the sum corresponding to every closed circuit of the form (III) so that the sum equals the example given in (III). Accordingly, it suffices to prove this result for this example. Circuit (III) corresponds to the map

\[
\pi_k \circ \pi_1 \circ \kappa_{2,3} \circ \cdots \circ \kappa_{k-2,k-1} \circ \psi_{k,k-1} \circ \cdots \circ \psi_{1,2}.
\]

After applying the immersions we have

\[
- \otimes \cdots - \otimes a \mapsto \sum_{j_1, \ldots, j_{k/2} = 1}^{m} w_{j_1} \otimes w_{j_1}^* \otimes w_{j_2} \otimes w_{j_2}^* \otimes \cdots \otimes w_{j_{k/2}} \otimes w_{j_{k/2}}^* \otimes a +
- w_{j_1}^* \otimes w_{j_1} \otimes w_{j_2}^* \otimes w_{j_2} \otimes \cdots \otimes w_{j_{k/2}}^* \otimes w_{j_{k/2}} \otimes a +
\cdots.
\]
The remaining terms will all vanish when we apply the spin contractions because some pair will contain two elements from \( W \) or \( W^* \). Applying the contractions forces equality among the indices, i.e. \( j_1 = j_2 = \cdots = j_{k/2} \). This gives us

\[
\sum_{i=1}^{m} w_i \otimes - \otimes - \otimes \cdots \otimes - \otimes w_i^* \otimes a + w_i^* \otimes - \otimes - \otimes \cdots \otimes - \otimes w_i \otimes a.
\]

Now if we apply our spin projections we recognize the same sum from Lemma 5.5,

\[
2 \sum_{i=1}^{m} - \otimes - \otimes - \otimes - \otimes \cdots - \otimes - \otimes D w_i^* X w_i(a) +
\]

\[
- \otimes - \otimes - \otimes - \otimes \cdots - \otimes - \otimes X w_i D w_i^*(a).
\]

This sum is precisely \((2m)(- \otimes \cdots \otimes - a)\). So we scale by \( \dim(V) \). When \( \dim(V) = 2m + 1 \) there will be one more term that does not vanish when we apply the spin contractions,

\[
e \otimes e \otimes \cdots \otimes e \otimes a.
\]

After the contractions and spin projections this term is sent to

\[
- \otimes - \otimes \cdots \otimes - \otimes DD(a) = - \otimes - \otimes \cdots - \otimes - a.
\]

Thus if \( \dim(V) = 2m + 1 \) this closed circuit corresponds to the map sending \(- \otimes \cdots - \otimes - a\) to \((2m + 1)(- \otimes \cdots \otimes - a)\). \qed

**Lemma 5.8.** Closed circuit (IV) corresponds to scaling by \( N = \dim(V) \).

**Proof.** We proceed as in Lemmas 5.5 and 5.7. Suppose our circuit has length \( k < n \). Then circuit (IV) corresponds to the map

\[
\pi_k \circ \kappa_{1,2} \circ \cdots \circ \kappa_{k-2,k-1} \circ \psi_{2,3} \circ \cdots \circ \psi_{k-1,k} \circ \iota_1.
\]

Applying all the immersions,

\[- \otimes \cdots - \otimes - a \mapsto \sqrt{2} \sum_{i,j_1,\ldots,j_{k-2}=1}^{m} w_i \otimes w_i^* \otimes w_{j_1} \otimes \cdots \otimes w_{j_{k-2}/2} \otimes w_{j_{k-2}/2} \otimes D w_i^*(a) +
\]

\[
w_i^* \otimes w_{j_1} \otimes w_{j_1}^* \otimes \cdots \otimes w_{j_{k-2}/2} \otimes w_{j_{k-2}/2}^* \otimes X w_i(a) + e \otimes e \otimes e \otimes \cdots \otimes e \otimes D(a).
\]

Applying the contractions equates all of the indices. Then when we apply the spin projection to the \( k^{th} \) tensor position we have

\[
2 \sum_{i=1}^{m} - \otimes - \otimes \cdots - \otimes - X w_i D w_i^*(a) +
\]

\[
- \otimes - \otimes \cdots - \otimes - D w_i^* X w_i(a) +
\]

\[
- \otimes - \otimes \cdots - \otimes - DD(a).
\]

This is the sum we saw in the previous two lemmas. Using the same reasoning, we can conclude this closed circuit results in scaling by \( \dim(V) \). \qed

**Lemma 5.9.** Closed circuit (V) corresponds to scaling by \( N = \dim(V) \).
Proof. Apply the same proof as in Lemma 5.8, but we now inject into the last tensor position and project from the first.

Lemma 5.10. If we follow a through string to an isolated vertex this corresponds to replacing the originating vertex of the through string with the corresponding isolated vertex.

Proof. It suffices to consider two cases:

(1) When the through string terminates directly in an isolated vertex.
(2) When the through string is connected to an isolated vertex via one spin contraction.

Indeed, if we travel along multiple immersions and contractions to reach an isolated vertex as in

\[
\begin{array}{c}
\bullet & \bullet & \circ \\
\circ & \circ
\end{array}
\]

we equate all of the indices we introduced as in the proofs of Lemmas 5.5-5.9. The corresponding sum is the same as the sum associated to the maps in the diagram

\[
\begin{array}{c}
\bullet & \circ \\
\circ & \circ
\end{array}
\]

The same reasoning applies for any diagram with the terminal isolated vertex in the second row. In this case, the sum corresponding to these diagrams reduces to the sum associated to

\[
\begin{array}{c}
\circ
\end{array}
\]

Consider a diagram of this type. The isolated vertex corresponds to \( n_i \) in the total order. Suppose the through string originates in tensor position \( k \) and terminates in tensor position \( \ell \). Clearly, the following two operations are equivalent

- Send a vector \( v \) in tensor position \( k \) to tensor position \( \ell \) then apply a spin projection to position \( \ell \) in the appropriate order corresponding to \( n_i \).
- Project the vector \( v \) from tensor position \( k \) as the \( n_i^{th} \) projection.
This corresponds to replacing the origin of the through string with the isolated vertex of index $n_i$. Now consider the second case. That is,

$$
\bullet \quad \bigcirc^{n_i} \quad \bullet
$$

This is a slightly more interesting case because a spin immersion becomes a spin projection. To see how this occurs, suppose the through string originates in tensor position $k$ and terminates in position $\ell$. When we consider the corresponding maps, we have

$$
v \otimes - \otimes - \otimes a \mapsto - \otimes v \otimes - \otimes a.
$$

Without loss of generality, we assume the first tensor position is position $k$, the second is $\ell$ and the last is $\ell + 1$. We can do this because the maps we consider only affect these tensor positions. Furthermore, any application of linear operators in the spin representation tensor position occur consecutively, so we may isolate these maps. When we apply the spin injection we have

$$
\sqrt{2} \left[ \sum_{i=1}^{m} - \otimes v \otimes w_i \otimes D_{w_i^*}^*(a) + - \otimes v \otimes w_i^* \otimes X_{w_i}(a) \right] + - \otimes v \otimes e \otimes D(a).
$$

Suppose $v = \sum_{i=1}^{m} \alpha_i w_i + \beta_i w_i^* + \gamma e$. The contraction projects tensor positions $\ell$ and $\ell + 1$ and scales by the bilinear form applied to these positions. The only nonzero terms are of the form $\alpha_i \omega(v, w_i^*)$, $\beta_i \omega(v, w_i)$ and $\gamma \omega(v, e)$. Here $\omega$ is the bilinear form defined in Section 2. The sum in (5.11) becomes

$$
\sqrt{2} \left[ \sum_{i=1}^{m} - \otimes - \otimes - \otimes \beta_i D_{w_i^*}(a) + - \otimes - \otimes - \otimes \alpha_i X_{w_i^*}(a) \right] + - \otimes - \otimes - \otimes \gamma D(a).
$$

This is precisely the spin projection applied to the vector $v$ in position $k$. \qed

**Lemma 5.12.** The spin-Clifford relation (Definition 3.3) when applied to a diagram resulting from step (7) of the multiplication process corresponds to interchanging the two corresponding spin injection(s) and/or projection(s).

**Proof.** Assume we have completed step (7) of the multiplication process for some spin-Brauer diagrams. This produces a diagram, not necessarily spin-Brauer. Lemmas 5.5-5.10 imply the simplifications in steps (2)-(7) agree with the corresponding simplification of maps. Hence, these diagrams correspond to some map in the centralizer with the spin projections and injections out of order. We will show that swapping the order of these maps corresponds to the spin-Clifford relation.

We first address swapping the order of isolated vertices across rows. If we want to swap two isolated vertices that occur in the same vertex position of the top and bottom row, i.e.
a diagram of the form
\[(5.13)\]
\[\delta \cdot \bullet \quad \odot^2 \quad \bullet \quad \odot^1.\]

This diagram must come from a composition where the isolated vertices follow through strings, for example a composition of the form
\[(5.14)\]
\[\bullet \quad \odot^1 \quad \odot^2 \quad \bullet \quad \odot^1 \quad \odot^2 \quad \bullet \quad \odot^1.\]

In this case, we notice that the entries in the tensor positions will remain the same. What we mean is that we will spin project the original entry in position three and then spin inject. What will change is the order of composition of the maps in the spin representation tensor position.

If the isolated vertices are ordered consecutively, this means the corresponding linear operators will be composed consecutively. As a result, swapping the isolated vertex indexing corresponds to swapping the order of the composition of the linear operators in the spin representation tensor position. Thus, the spin-Clifford relation reduces to the super commutativity of the operators.

These linear operators, however, are not strictly super-commutative. $X_v$ and $D_\lambda$ satisfy the identity \[(2.1.1)\]. As a result, when we swap we must account for the other terms that appear in the sum.

Suppose we want to apply a spin injection into position $\ell$ and projection from position $k$. It suffices to restrict our attention to these tensor positions. So we consider $v \otimes - \otimes a$ where $v$ is in tensor position $k$ and we spin inject into tensor position $\ell$. We may assume this position has been projected away. Indeed, if this is not the case it implies the diagram did not result from step \((7)\) in the multiplication of two diagrams.

Let $v = \sum_{i=1}^{m} \alpha_i w_i + \beta_i w_i^* + \gamma e$. When we spin project $v$, it corresponds to applying the linear operators $\sqrt{2} \sum_{i=1}^{m} \alpha_i X_{w_i} + \beta_i D_{w_i}^* + \gamma D$ in the spin representation tensor position. If we spin inject into position $\ell$ then spin project position $k$ we have,
\[
2 \sum_{j=1,i=1}^{m} - \otimes \alpha_i w_j^* \otimes X_{w_i} X_{w_j}(a) + - \otimes \alpha_i w_j \otimes X_{w_i} D_{w_j}(a) + - \otimes \alpha_i e \otimes X_{w_i} D(a) + \\
- \otimes \beta_i w_j^* \otimes D_{w_i}^* X_{w_j}(a) + - \otimes \beta_i w_j \otimes D_{w_i}^* D_{w_j}(a) + - \otimes \beta_i e \otimes D_{w_i}^* D(a) + \\
(\sqrt{2}) \left[ - \otimes \gamma w_j^* \otimes DX_{w_j}(a) + - \otimes \gamma w_j \otimes DD_{w_j}(a) \right] + - \otimes \gamma e \otimes DD(a).
\]
\[(5.15)\]

Once again, we only consider the $k$ and $\ell$ tensor positions as well as the spin representation tensor position. When we swap all of these operators, we have the same sum but scaled by $(-1)$. However, in any positions containing the linear operators $X_{w_i}$ and $D_{w_i}^*$, we get an
extra term from (2.1.1) when \( i = j \). Explicitly, let \( \Omega' \) be the sum in (5.15). When we swap all the operators we have

\[
-\Omega' + 2 \sum_{i=1}^{m} - \otimes \alpha_i w_i \otimes a + - \otimes \beta_i w_i^* \otimes a + - \otimes \gamma e \otimes a.
\]

Notice this simplifies to \(-\Omega' + 2(- \otimes v \otimes a)\) where \( v \) is now in the \( \ell \)-tensor position and all of the other components of the diagram and corresponding maps are the same. This shows that the following operations correspond:

- Swap the order of composition of a consecutively ordered spin projection and injection.
- In the corresponding diagram, swap the indices of the isolated vertices and scale \((-1)\). Furthermore, add the diagram where we draw a through string between the isolated vertices we swapped.

This is the spin-Clifford relation.

If we wish to swap two spin injections in the bottom row the same proof applies. Indeed, when we swap the indexing of consecutive isolated vertices, it corresponds to swapping the order of composition of the corresponding linear operators. Once again, we scale by \((-1)\) and account for the additional terms that arise. However, in this case the additional terms result in the following sum,

\[
-\Omega' + 2 \sum_{i=1}^{m} w_i \otimes w_i^* \otimes a + w_i^* \otimes w_i \otimes a + e \otimes e \otimes a.
\]

This second summand is exactly the spin immersion into the corresponding tensor positions. When \( \dim(V) \) is even the same proof applies removing all of the terms that contain the vector \( e \). This establishes the spin-Clifford relation.

\[ \square \]

Lemmas 5.5-5.10 show that simplifying the inside homomorphism agrees with our multiplication structure. We then have a composition of the form (5.1), but the projections and injections are not in the correct order. Lemma 5.12 proves that placing the spin projection and injection maps into the correct order agrees with the operation for swapping the indices in the diagrams. As a result, each step of the simplification agrees. We conclude that \( f_{\Omega_2} \circ f_{\Omega_1} = f_{\Omega_2 \Omega_1} \).

\[ \square \]

6. Cellularity of \( SB_n(\delta, \eta) \)

For certain parameters \( \delta, \eta \) and \( n \) we just showed the multiplication structure of \( SB_n(\delta, \eta) \) agrees with the composition of maps in the centralizer algebra. As a result, we may study the simpler \( SB_n(\delta, \eta) \) to discover properties of \( \text{End}_{\text{Pin}(N)}(V^{\otimes n} \otimes \Delta) \) in this form of Schur-Weyl duality. In particular, we will prove \( SB_n(\delta, \eta) \) is a cellular algebra over any field \( k \). This will allow us to describe all of its irreducible representations.

Throughout this section we fix a field \( k \). Furthermore, \( k\Sigma_\ell \) will denote the symmetric group algebra over \( k \) on \( \ell \) letters.

We refer the reader to Graham-Lehrer [GL] for the classical definition of a cellular algebra. We will use Xi’s basis-free characterization to prove cellularity of \( SB_n(\delta, \eta) \). Before we state this definition we need the following terminology.

**Definition 6.1.** [Xi, Definition 3.2] Let \( A \) be an \( k \)-algebra. Assume there is an involution \( i \) on \( A \). A two-sided ideal \( J \) in \( A \) is called a **cell ideal** if and only if \( i(J) = J \) and there
exists a left ideal $\nabla \subset J$ such that $\nabla$ is finitely generated and free over $R$ and that there is an isomorphism of $A$-bimodules $\alpha : J \simeq \nabla \otimes_R i(\nabla)$ making the following diagram commute

$$
\begin{array}{ccc}
J & \xrightarrow{\alpha} & \nabla \otimes_R i(\nabla) \\
i & \downarrow & x \otimes y \mapsto i(y) \otimes i(x) \\
J & \xrightarrow{\alpha} & \nabla \otimes_R i(\nabla).
\end{array}
$$

The algebra $A$ (with involution $i$) is called cellular if and only if there is an $k$-module decomposition $A = J_1' \oplus J_2' \oplus \cdots \oplus J_n'$ (for some $n$) with $i(J_j') = J_j'$ for each $j$ and such that setting $J_j = \bigoplus_{i=1}^j J_i'$ gives a chain of two-sided ideals of $A$: $0 = J_0 \subset J_1 \subset \cdots \subset J_n = A$ (each fixed by $i$) and for each $j$ the quotient $J_j'/J_{j-1}$ is a cell ideal with respect to the involution induced by $i$ on the quotient $A/J_{j-1}$.

The $\nabla$ associated to each $J_j/J_{j-1}$ are called Weyl Modules or in [GL] Cell Representations. With this terminology we recall an important lemma.

**Lemma 6.2 (Lemma 3.3, [Xi]).** Let $A$ be an algebra with an involution $i$. Suppose there is a decomposition

$$A = \bigoplus_{j=1}^m V_j \otimes_k V_j \otimes_k B_j$$

where $V_j$ is a vector space and $B_j$ is a cellular algebra with respect to an involution $\alpha_j$ and a cell chain $J_1^{(j)} \subset \cdots \subset J_{s_j}^{(j)} = B_j$ for each $j$. Define $J_t = \bigoplus_{j=1}^t V_j \otimes_k V_j \otimes_k B_j$. Assume that the restriction of $i$ on $V_j \otimes_k V_j \otimes_k B_j$ is given by $w \otimes v \otimes b \mapsto v \otimes w \otimes \sigma_j(b)$. If for each $j$ there is a bilinear form $\varphi_j : V_j \otimes_k V_j \to B_j$ such that $\sigma_j(\varphi_j(w, v)) = \varphi_j(v, w)$ for all $w, v \in V_j$ and that the multiplication of two elements in $V_j \otimes V_j \otimes B_j$ is governed by $\varphi_j$ (mod $J_{j-1}$), that is, for $x, y, u, v \in V_j$ and $b, v \in B_j$, we have

$$(x \otimes y \otimes b)(u \otimes v \otimes v) = x \otimes v \otimes b \varphi_j(y, u)c$$

(mod $J_{j-1}$) and if $V_j \otimes V_j \otimes J_{\ell}^{(j)} + J_{j-1}$ is an ideal in $A$ for all $\ell$ and $j$, then $A$ is a cellular algebra.

We will use this lemma to prove the following theorem.

**Theorem 6.3.** For any field $k$, $n \in \mathbb{Z}_{\geq 0}$ and $\delta, \eta$ arbitrary parameters $SB_n(\delta, \eta)$ is a cellular algebra.

**Proof.** The proof will consist of many parts broken into lemmas and propositions showing $SB_n(\delta, \eta)$ satisfies all the hypotheses of Lemma 6.2.

We follow a general framework used in [Xi]. We begin by introducing some notation. For $n$ a positive integer, we denote by $E_n$:

$$E_n := \left\{ \rho = ((\rho_1), (\rho_2), \ldots, (\rho_k)) \mid \emptyset \neq (\rho_i) \subset \{1, \ldots, n\}, \bigcup_{i=1}^k (\rho_i) = \{1, \ldots, n\}, \right.$$\n
$$\left. (\rho_i) \cap (\rho_j) = \emptyset \ (i \neq j), |(\rho_i)| \leq 2 \right\}.$$ 

Consider the following example,
Example 6.4.

\[ E_3 = \{(12)(3), (13)(2), (23)(1), (1)(2)(3)\}. \]

Notice, we force each partition \( \rho_i \) to contain at most two elements, so the partition \((123)\) is excluded. \(\square\)

Define a counting function \( m_1: E_n \rightarrow \mathbb{N} \) that counts the number of parts of a partition \( \rho \) that have size 1. So if \( \rho = ((\rho_1)\ldots(\rho_k)) \), \( m_1(\rho) \) is the number of \( \rho_i \) such that \( |\rho_i| = 1 \).

We now construct a vector space that will encode spin-Brauer diagrams. Define a vector space \( V_\ell \) with basis given by the set

\[ (6.5) \quad S_\ell = \{ (\rho, S) \mid \rho \in E_\ell, m_1(\rho) \geq \ell, S \subset \rho, |S| = m_1(S) = \ell \}. \]

Basis elements are pairs \((\rho, S)\) with \( \rho \) a partition of \( n \) into pieces of size 1 or 2, such that there are at least \( \ell \) partition elements of size one. Given such a partition \( \rho \), we pair it with a subpartition \( S \) consisting only of \( (\rho_i) \) such that \( |\rho_i| = 1 \).

For \( \Omega \in \mathcal{B}(\text{SB}_n(\delta, \eta)) \), let \( \ell \) be the number of through strings in \( \Omega \). We will associate to \( \Omega \) a basis element \((x, S) \otimes (y, T) \otimes \sigma \in V_\ell \otimes V_\ell \otimes k\Sigma_\ell \) and show this association has an inverse and hence is a bijection.

Label the vertices in the top and bottom rows of \( \Omega \) as \{1, \ldots, n\}. The top and bottom rows of \( \Omega \) partition \{1, \ldots, n\} into subsets of size 1 or 2 in a natural way. Every isolated vertex and originating vertex of a through string corresponds to a subset of size 1. Arcs correspond to subsets of size 2.

Let \( x \) be the partition of the top row of \( \Omega \) and \( y \) the partition of the bottom row. Next, put \( S \) and \( T \) as the subsets of vertex numbers in the top and bottom row respectively where through strings originate. Hence, \( |S| = |T| = \ell \), i.e. \((x, S) \otimes (y, T) \in V_\ell \otimes V_\ell \).

Now \( \sigma \) encodes how the through strings connect. Suppose

\[ S = (S_1, S_2, \ldots, S_\ell), \quad T = (T_1, T_2, \ldots, T_\ell). \]

Let \( f \) be the bijection in the spin datum of \( \Omega \). Put \( \sigma \) as the permutation induced by \( f \). This gives a well defined element in \( k\Sigma_\ell \). We then encode \( \Omega \) as \((x, S) \otimes (y, T) \otimes \sigma \in V_\ell \otimes V_\ell \otimes k\Sigma_\ell \).

Conversely, if \((x, S) \otimes (y, T) \otimes \sigma \in V_\ell \otimes V_\ell \otimes k\Sigma_\ell \), we construct a spin-Brauer diagram \( \Omega' \). Let \( U \) consist of all \((x, i) \in x \setminus S\) of size one. Similarly, let \( U' \) be all \((y, i) \in y \setminus T\) of size one. Put \( \Gamma \) as all \((x, i) \in x \) with \(|x_i| = 2\). Define \( \Gamma' \) similarly. The bijection \( f \) is then induced by the permutation \( \sigma \) on \( S = x \setminus (U \cup V(\Gamma)) \rightarrow y \setminus (U' \cup V(\Gamma')) = T \). Here \( f \) sends the element \( S_i \in S \) to \( T_{(\sigma(i))} \in T \). This gives us a well defined spin datum and hence spin-Brauer diagram \( \Omega' \in \mathcal{B}(\text{SB}_n(\delta, \eta)) \).

**Lemma 6.6.** This construction gives a bijective correspondence between \( \Omega \in \mathcal{B}(\text{SB}_n(\delta, \eta)) \) and the basis for \( V_\ell \otimes V_\ell \otimes k\Sigma_\ell \) described in (6.5), where \( \ell \in \{0,1,\ldots,n\} \).

**Proof.** This construction is clearly invertible. \(\square\)

Another piece of the cell-datum necessary to prove cellularity is an involution \( i: \text{SB}_n(\delta, \eta) \rightarrow \text{SB}_n(\delta, \eta) \). Given a spin-Brauer diagram \( \Omega \in \mathcal{B}(\text{SB}_n(\delta, \eta)) \) with spin-datum \((U, U', \Gamma, \Gamma', f)\) define the **spin involution** \( i \) as

\[ i(\Omega) = (U', U, \Gamma', \Gamma, f^{-1}). \]

With the total order on \( U' \) and \( U \) reversed. Extend \( i \) by linearity to all of \( \text{SB}_n(\delta, \eta) \). We leave it to the reader to verify that this is a well-defined element of \( \text{SB}_n(\delta, \eta) \).

**Lemma 6.7.** The linear map \( i \) is an anti-automorphism of \( \text{SB}_n(\delta, \eta) \) with \( i^2 = id \).
Proof. Linearity of the map is clear from the definition. Furthermore, by construction \( i^2 = id \). It remains to check that \( i(\Omega_1 \Omega_2) = i(\Omega_2) i(\Omega_1) \) on basis elements. Linearity will then imply the result in general. This follows immediately from the realization of \( \Omega_1 \) and \( \Omega_2 \) as diagrams. The map \( i \) exchanges the rows. When realized as diagrams, it is clear the following operations are equivalent:

- Exchange the rows of both diagrams, reverse the total order and take the product.
- Take the product and exchange the rows. \( \square \)

We now define a bilinear form \( \varphi_\ell : V_\ell \otimes V_\ell \rightarrow k\Sigma_\ell \). Let \( (\rho, S) \in \mathcal{S}_\ell \) with \( S = \{S_1, S_2, \cdots, S_\ell\} \) and \( S_1 < S_2 < \cdots < S_\ell \). Given \( \mu \in E_n \) and \( \nu \in E_m \) we define \( \mu \cdot \nu \) to be the smallest partition created by merging all parts of \( \mu \) and \( \nu \) with common elements. For example,

Example 6.8. \( \mu = \{(13), (2), (45)\} \) and \( \nu = \{(12), (3), (4), (5)\} \), then \( \mu \cdot \nu = \{(123), (45)\} \). \( \square \)

Given a partition \( \mu \in E_n \) let \( \text{sing}(\mu) \) be all the components of the partition with size 1. We call these the \textbf{singletons} in the partition.

Given \( (x, S), (y, T) \in V_\ell \), consider

\[
\text{sing}(x) \setminus S = \{\gamma_1, \ldots, \gamma_k\} \quad \text{and} \quad \text{sing}(y) \setminus T = \{\gamma_{k+1}, \ldots, \gamma_{k+m}\}.
\]

Here we place \( \gamma_i \) in the natural order corresponding to the elements they represent in the partition. These are the lists of isolated vertices. Define \( \Gamma_S \) to be the ordered set of \( \gamma_j \in \{\gamma_1, \ldots, \gamma_{k+m}\} \) such that there is some \( 1 \leq i, j \leq \ell \) where \( S_i \) and \( \gamma_j \) are in a component of \( x \cdot y \). We define \( \Gamma_T \) similarly.

Define \( \beta(\Gamma_S, \Gamma_T) \) as the minimal number of pairs \( (i + 1, i) \) such that after inductively removing \( \gamma_{i+1} \) from \( \Gamma_S \) and \( \gamma_i \) from \( \Gamma_T \) and reindexing the \( \gamma_j \), the remaining elements in \( \Gamma_S \) are all less than the remaining elements of \( \Gamma_T \). For example,

Example 6.9. If \( \Gamma_S = \{\gamma_1, \gamma_2, \gamma_5, \gamma_6\} \) and \( \Gamma_T = \{\gamma_3, \gamma_4, \gamma_7\} \), then \( \beta(\Gamma_S, \Gamma_T) = 2 \). Indeed, we first remove \( (5, 4) \) then after reindexing we have the sets \( \{\gamma_1, \gamma_2, \gamma_4\} \) and \( \{\gamma_3, \gamma_5\} \). We remove \( (4, 3) \) and the order is correct. \( \square \)

The choice of pairs we remove is always uniquely determined. This gives the number of isolated vertices we need to swap across rows using the Spin-Clifford relation in our diagram multiplication.

Finally, let \( \text{Cr}((x, S), (y, T)) \) be the number of pairs \( (i, j) \) so that \( S_i \) and \( T_j \) are contained in a component of \( x \cdot y \). This component would have to be unique. This counts the number of through strings that connect in our two diagrams.

Now define \( \varphi_\ell((x, S) \otimes (y, T)) \) to be zero if any of the following occur,

1. There exists some \( i, j \) with \( 1 \leq i, j \leq \ell \) and \( i \neq j \) such that there is a part of \( x \cdot y \) containing both \( S_i \) and \( T_j \). Or dually, if there is a part of \( x \cdot y \) containing both \( T_i \) and \( S_j \).
2. \( |\Gamma_S| \neq |\Gamma_T| \).
3. \( \text{Cr}((x, S), (y, T)) + |\Gamma_S| \neq \ell \) or equivalently \( \text{Cr}((x, S), (y, T)) + |\Gamma_T| \neq \ell \).
4. \( \beta(\Gamma_S, \Gamma_T) \neq |\Gamma_S| \) or \( \beta(\Gamma_S, \Gamma_T) \neq |\Gamma_T| \). Equivalently, \( \text{Cr}((x, S), (y, T)) + \beta(\Gamma_S, \Gamma_T) \neq \ell \).

Otherwise, let \( \varphi_\ell \) be the following element of \( k\Sigma_\ell \). First, scale by \( \delta \) for each component of \( x \cdot y \) that does not contain any element from \( S \) or \( T \) but only consist of unions of pairs from \( x \) and \( y \). That is, unions of elements of size 2. These correspond to closed circuits of type (I).
Then, scale by \( \eta \) for each component of \( x \cdot y \) that does not contain any element from \( S \) or \( T \) and contains some element from either \( \Gamma_S \) or \( \Gamma_T \). These corresponds to closed circuits of types (II)-(V). Next, scale by \( 2^{|\Gamma_S|} = 2^{|\Gamma_T|} \). This accounts for the factor of two in the spin-Clifford relation.

Then, if there exists a component of \( x \cdot y \) containing \( S_i \) and \( T_j \) let our permutation in \( \Sigma_\ell \) send \( i \) to \( j \). Remove \( S_i \) from \( S \) and \( T_j \) from \( T \).

Since, we are assuming \( \varphi \) is nonzero, (2) and (3) imply the following

- Every remaining element of \( S \) corresponds to some \( \gamma_i \in \Gamma_S \).
- Every remaining element of \( T \) corresponds to some \( \gamma_j \in \Gamma_T \).

Indeed, if one of the remaining elements of \( S \) does not correspond to an element of \( \Gamma_S \) this means \( \text{Cr}((x, S), (y, T)) + |\Gamma_S| < \ell \). Similarly for \( T \).

As in the definition of \( \beta \), inductively remove pairs \((i + 1, i)\) such that after removing \( \gamma_{i+1} \) from \( \Gamma_S \) and \( \gamma_i \) from \( \Gamma_T \) and reindexing, all the elements of \( \Gamma_S \) are less than the elements of \( \Gamma_T \). During this process, suppose we remove the pair \((i + 1, i)\). If \( S_k \) corresponds to \( \gamma_{i+1} \) and \( T_m \) corresponds to \( \gamma_i \), then our permutation sends \( k \) to \( m \).

By (4) every element of \( \Gamma_S \) will be less than the elements in \( \Gamma_T \). Indeed, we need \( \beta(\Gamma_S, \Gamma_T) = |\Gamma_S| = |\Gamma_T| \). Whence, this process gives a pairing of all the remaining elements of \( S \) and \( T \), i.e. a permutation in \( k\Sigma_\ell \). Extend this map by linearity to \( V_\ell \otimes V_\ell \). As an example consider

**Example 6.10.** Let \( \ell = 3 \). If \( x = \{(1)(2)(3)(4)(5)(67)(8)\} \), \( S = \{(1)(4)(6)\} \) and \( y = \{(13)(2)(4)(5)(6)(78)\} \). Then we have

\[
x \cdot y = \{(13)(2)(4)(5)(678)\}.
\]

We notice \( \text{Cr}((x, S), (y, T)) = 1 \) because \( S_3 \) and \( T_3 \) are both in a component of \( x \cdot y \).

Furthermore, \( \text{sing}(x) \setminus S = \{(2)(3)(5)\} \) and \( \text{sing}(y) \setminus T = \{(4)\} \). With the notation we have been using, we say \( \{(2)(3)(5)\} = \{\gamma_1, \gamma_2, \gamma_3\} \) and \( \{(4)\} = \{\gamma_4\} \). Now proceeding by definition we construct \( \Gamma_S \) and \( \Gamma_T \). We see \( (4) = S_2 = \gamma_4 \in x \cdot y \). Also, \( (1) = S_1 \) and \( \gamma_2 \) are in the element \( 13 \in x \cdot y \). Hence \( \Gamma_S = \{\gamma_2, \gamma_4\} \).

Next, we see \( (2) = T_1 = \gamma_1 \) and \( (5) = T_2 = \gamma_3 \) are also both in \( x \cdot y \). Hence \( \Gamma_T = \{\gamma_1, \gamma_3\} \). This implies \( \beta(\Gamma_S, \Gamma_T) = 2 \) as we must remove \( \gamma_2 \) and \( \gamma_1 \) as well as \( \gamma_3 \) and \( \gamma_4 \) for \( \Gamma_S \) to be less than \( \Gamma_T \). It is important to remember we can only remove consecutively indexed elements.

We check that the bilinear form is nonzero. The first condition does not occur. \( |\Gamma_S| = |\Gamma_T| = \beta(\Gamma_S, \Gamma_T) = 2 \) and \( \text{Cr}((x, S), (y, T)) + \beta(\Gamma_S, \Gamma_T) = 3 = \ell \).

There are no closed circuits because every element of \( x \cdot y \) contains some \( S_i \) or \( T_j \). So we do not scale by a power of \( \delta \) or \( \eta \). Now we must check what the permutation should be. When computing the crossing number we saw that both \( S_3 \) and \( T_3 \) were in the component (678) of \( x \cdot y \). Hence our permutation will fix 3.

To discover the final part of the permutation when finding \( \beta(\Gamma_S, \Gamma_T) \) we had to remove both the pairs \( (\gamma_2, \gamma_1), (\gamma_4, \gamma_3) \in \Gamma_S \times \Gamma_T \). We saw that \( S_1 \) corresponds to \( \gamma_2 \) and \( T_1 \) corresponds to \( \gamma_1 \). Accordingly, our permutation fixes 1. Similarly, \( S_2 \) corresponds to \( \gamma_4 \) and \( T_2 \) corresponds to \( \gamma_3 \) so our permutation fixes 2. In conclusion, the resulting permutation is the identity. So in this example \( \varphi((x, S), (y, T)) = 2^2 \cdot id \in k\Sigma_3 \).

\[\square\]

**Remark 6.11.** The map \( \varphi_\ell : V_\ell \otimes_k V_\ell \to k\Sigma_\ell \) is a bilinear form.

\[\square\]

We wish to show that multiplication of two diagrams with \( \ell \) through strings is encoded by \( \varphi_\ell \pmod{J_{\ell-1}} \). Before we can prove this, we need the following result.
Lemma 6.12. \( J_t := \sum_{j=0}^{t} V_j \otimes V_j \otimes k \Sigma_j \) is an ideal of \( \text{SB}_n(\delta, \eta) \).

*Proof.* This is stated in Kojke [Koi2, p. 69]. Concretely, this is the ideal of all diagrams with at most \( t \) through strings. The number of through strings only decreases upon multiplication. 

Let \( \# : \text{SB}_n(\delta, \eta) \rightarrow \mathbb{Z}_{\geq 0} \) be the function that maps a sum of spin-Brauer diagram to the maximal number of through strings in the sum. We call \( \#(\sum_i \Omega_i) \) the **maximal crossing number**.

Lemma 6.13. Let \( \Omega_1, \Omega_2 \in \mathcal{B}(\text{SB}_n(\delta, \eta)) \). If \( \Omega_1 = (u, R) \otimes (x, S) \otimes \sigma_1 \in V_\ell \otimes V_\ell \otimes k \Sigma_\ell \) and \( \Omega_2 = (y, T) \otimes (v, Q) \otimes \sigma_2 \in V_\ell \otimes V_\ell \otimes k \Sigma_\ell \), then

\[
\Omega_2 \Omega_1 = (u, R) \otimes (v, Q) \otimes \sigma_1 \varphi_\ell((x, S), (y, T)) \sigma_2,
\]

modulo \( J_{\ell-1} \).

*Proof.* If \( \psi_{\ell}((x, S), (y, T)) = 0 \), then by definition of \( \varphi_\ell \) we see \( \#(\Omega_2 \Omega_1) < \ell \). As in each situation (1)-(4) we lose a through string. Furthermore, these are all the possible ways we could decrease the crossing number. This implies every element in the sum corresponding to \( \Omega_2 \Omega_1 \) is contained in \( J_{\ell-1} \).

Now assume \( \varphi_\ell((x, S), (y, T)) = 2^j \delta^a \eta^b \in k \Sigma_\ell \) as defined above. It remains to show \( 2^j \delta^a \eta^b (u, R) \otimes (v, Q) \otimes \sigma_1 \otimes \sigma_2 \) corresponds to the element \( \Omega_2 \Omega_1 \pmod{J_{\ell-1}} \).

If \( \#(\Omega_2 \Omega_1) = \ell \) there will only be one diagram in the sum decomposition with \( \ell \) through strings. Specifically, the diagram \( \Omega \) in the sum resulting from repeatedly applying the spin-Clifford relation to swap isolated vertex indices across rows. All other diagrams will have less than \( \ell \) through strings. Indeed, after we apply the first spin-Clifford relation, the resulting diagram will have \(|\Gamma_S| = |\Gamma_T| = \ell - C((x, S), (y, T)) - 1\). Hence, we can create at most \( \ell - 1 \) through strings by applying the spin-Clifford relation.

As all other diagrams in the sum decomposition of \( \Omega_2 \Omega_1 \) will have less than \( \ell \) through strings it suffices to show

\[
\Omega = 2^{|\Gamma_S|} \delta^k \eta^a (u, R) \otimes (v, Q) \otimes \sigma_1 \sigma_2.
\]

The scalars are correct because we must apply the spin-Clifford relation exactly \(|\Gamma_S|\) times by condition (4). This results in scaling by \( 2^{|\Gamma_S|} \). Furthermore, \( \Omega \) will be scaled by \( \delta^a \eta^b \) where \( j \) is the number of closed circuits or type (I) and \( a \) is the number of closed circuits of types (II)-(V). This follows from our construction of \( \varphi_\ell \).

Furthermore, \( \Omega \) will have the same \((u, S)\) determining its top row and \((v, Q)\) its bottom row. Indeed, \( \Omega \) has \( \ell \) through strings, so every through string has to be preserved. Additionally, in the multiplication of \( \Omega_2 \Omega_1 \) we cannot change the originating vertex of a through string. This forces complete preservation of isolated vertices, arcs and through string origins.

It remains to prove \( \sigma_1 \sigma_2 \) is the correct permutation of the through strings where we recall that we compose permutations left to right. First, if we consider a through string such that \( S_i, T_j \in x \cdot y \) then by definition \( \sigma(i) = j \). Suppose \( S_i \) is connected to \( R_m \) and \( T_j \) is connected to \( Q_t \), i.e. \( \sigma_1(m) = i \) and \( \sigma_1(j) = t \). In the final diagram, we need to send \( R_m \) to \( Q_t \). This clearly occurs as we compose from left to right \( m \to i \to j \to t \).

Now consider the part of the diagram consisting of through strings that terminate in an isolated vertex. Suppose the through string originating in \( R_m \) and connecting to \( S_i \) ultimately terminates in an isolated vertex \( \gamma_{k+1} \in \Gamma_S \). Then by condition (4) after reindexing, we can assume without loss of generality that \( \gamma_k \in \Gamma_T \). This corresponds to some \( T_j \) which is
connected to $Q_t$. After applying the spin-Clifford relation, we create a through string between $R_m$ and $Q_t$. Hence our permutation must send $m$ to $t$. This is the case as $\sigma_1(m) = i$, then $\sigma(i) = j$ by definition and $\sigma_2(j) = t$. We continue applying the spin-Clifford relation to generate all the other through strings in $\Omega$. By the same reasoning, every through string is correctly encoded by $\sigma_1 \sigma_2$.

This proves $\Omega = (u, R) \otimes (v, Q) \otimes \sigma_1 \varphi_\ell((x, S), (y, T)) \sigma_2 \pmod{J_{\ell-1}}$. \hfill \Box

**Lemma 6.14.** If $\Omega_1 = (x, S) \otimes (y, T) \otimes \sigma \in V_\ell \otimes V_\ell \otimes k \Sigma_\ell$ then $i(\Omega_1) = (y, T) \otimes (x, S) \otimes \sigma^{-1}$

**Proof.** This is a consequence of definitions and Lemma 6.13. \hfill \Box

**Lemma 6.15.** Let $\tau: k \Sigma_\ell \rightarrow k \Sigma_\ell$ be the involution on $k \Sigma_\ell$ defined by $\sigma \mapsto \sigma^{-1}$ for all $\sigma \in \Sigma_\ell$. Then $i(\varphi_\ell(v_1, v_2)) = \varphi_\ell(v_2, v_1)$ for $v_i \in V_\ell$.

**Proof.** Assume $v_1 = (x, S)$ and $v_2 = (y, T)$. If $\varphi_\ell(v_1, v_2) = 0$ then by construction $\varphi_\ell(v_2, v_1) = 0$ as well. So assume $\varphi_\ell(v_1, v_2) \neq 0$.

If this is the case, we notice that the scalar $\delta \eta^2 |T| \ell$ does not change when we interchange $S$ and $T$. Indeed, the size of $\Gamma_S$ and $\Gamma_T$ are preserved. Furthermore, the number of closed circuits of each type remains the same because $x \cdot y = y \cdot x$. It remains to check that the permutation is inverted.

Let $\sigma$ be the permutation described in the construction of $\varphi_\ell$. If $S_i$ and $T_{\sigma(i)}$ are contained in the same part of $x \cdot y$ then $T_i$ and $S_{\sigma^{-1}(i)}$ are contained in the same part of $y \cdot x = x \cdot y$. Hence the permutation associated to $\varphi_\ell(v_2, v_1)$ is $\sigma^{-1}$ for through strings that connect.

Next we consider through strings that terminate in isolated vertices. Suppose $S_i$ and $T_{\sigma(i)}$ are associated to $\gamma_{k+1}$ and $\gamma_k$. Without loss of generality, assume $\gamma_k$ is the largest element of $\Gamma_S$. When we apply $i$, the orders of $\Gamma_S$ and $\Gamma_T$ are reversed. That is,

$$\gamma_j \rightarrow \gamma_{2k-j+1}.$$ 

$S_i$ is now in a component of $x \cdot y$ with the isolated vertex corresponding to $\gamma_{2k-(k+1)+1} = \gamma_k$. Similarly, $T_{\sigma(i)}$ is associated to $\gamma_{k+1}$. We remove these elements and inductively apply the above reasoning to see that in general, $T_{\sigma(i)}$ and $S_i$ are mapped to each other. That is, the permutation associated to $\varphi_\ell(v_2, v_1)$ is described by $\sigma(i) \mapsto i$. This permutation is $\sigma^{-1}$. \hfill \Box

We are now ready to prove the theorem. Put $J_{-1} = 0, \Sigma_0 = \{1\}$ and $B_\ell = k \Sigma_\ell$. Then $SB_n(\delta, \eta)$ has a description

$$SB_n(\delta, \eta) = V_0 \otimes_k V_0 \otimes_k B_0 \oplus \cdots \oplus V_\ell \otimes_k V_\ell \otimes_k B_\ell \cdots \oplus V_n \otimes_k V_n \otimes_k B_n.$$ 

This follows from Lemma 6.6. Note that $B_\ell$ is a cellular algebra with respect to the involution $\sigma \mapsto \sigma^{-1}$ for $\sigma \in \Sigma_\ell$ (see [Xi, Proof of Theorem, pg. 107]). By all the Lemmas in this section this description of the spin-Brauer algebra satisfies all the necessary conditions in Lemma 6.2. Hence $SB_n(\delta, \eta)$ is a cellular algebra. \hfill \Box

We extract some immediate consequences of cellularity.

**Corollary 6.16.** The Weyl Modules of $SB_n(\delta, \eta)$ are $\nabla_\ell(\lambda) := V_\ell \otimes v_\ell \otimes \nabla(\lambda)$ where $\ell \in \{0, 1, \ldots, n\}$ and $\lambda$ is a partition of $\ell$, $v_\ell$ is a fixed nonzero element of $V_\ell$ and $\nabla(\lambda)$ is a Weyl module of $k \Sigma_\ell$. For $\ell = 0$, we take $\lambda = (0)$ and $\nabla(0) = k$.

**Proof.** This is an immediate consequence of the proof of Theorem 6.3. These are precisely the $\nabla$ associated to each $J_\ell/J_{\ell-1}$. We also refer the reader to [Xi] for further discussion. \hfill \Box
The existence of Weyl-Modules is an immediate consequence of the cellular structure. For a specific definition we refer the reader to [GL, Section 2]. These modules play a significant role in understanding the representations of a cellular algebra. In particular, we can define a symmetric bilinear form $\Phi_\ell$ on them. This bilinear form is described using the bilinear form in the cellular datum (6.11). It turns out that over a field, non-degeneracy of this bilinear form for each $\ell$ is equivalent to semi-simplicity [GL, Theorem 3.8]. Furthermore, the $\ell$ for which $\Phi_\ell$ is non-degenerate parametrize all the absolutely irreducible representations of a cellular algebra [GL, Theorem 3.4].

This means we can parametrize all the irreducible representations of $\text{SB}_n(\delta, \eta)$. First we must make a definition.

**Definition 6.17.** Let $a \in \mathbb{Z}_{\geq 0}$. A partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of $m$ is $a$-regular if there is no $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+a-1}$ for any $i = 1, \ldots, \ell - a$. This just means that every part of $\lambda$ appears at most $a - 1$ times when $a > 0$. Every partition is $0$-regular.

**Corollary 6.18.** Let $\text{SB}_n(\delta, \eta)$ be the spin-Brauer diagram algebra for some $n > 1$ over a field $k$ of characteristic $a$ where $a$ can be zero. If $\delta, \eta \neq 0$, then the nonisomorphic irreducible representations of $\text{SB}_n(\delta, \eta)$ are parametrized by

$$\{(m, \lambda) \mid 0 \leq m \leq n, \lambda \text{ an } a\text{-regular partition of } m\}.$$ 

If $\delta = \eta = 0$ then $m = 0$.

**Proof.** We follow [Xi, GL]. From Corollary 6.16 and [GL] the irreducible representations of $\text{SB}_n(\delta, \eta)$ are parametrized by $\{(\ell, \lambda) \mid \Phi_{\ell, \lambda} \neq 0\}$. Here $\Phi_{\ell, \lambda}$ is a bilinear form on Weyl Modules defined in [GL, §2]. If $\ell \neq 0$ then $\Phi_{\ell, \lambda} \neq 0$ if and only if the corresponding linear form $\Phi_\lambda$ for the cellular algebra $k\Sigma_\ell$ is not zero. Here we use the fact that $\varphi_\ell((x, S), (x, S)) = \delta^c \eta^d \text{id} \in k\Sigma_\ell$, where $c + d = |x| - \ell$. This implies from the definition of $\Phi_{\ell, \lambda}$ that for any $\ell \neq 0$ and $(x, S) \otimes v_\ell \otimes \nabla(\lambda)$ the bilinear form $\Phi_{\ell, \lambda}((x, S) \otimes v_\ell \otimes \nabla_1(\lambda), (x, S) \otimes v_\ell \otimes \nabla_2(\lambda)) = \delta^c \eta^d \cdot \Phi_\lambda(\nabla_1(\lambda), \nabla_2(\lambda))$ which will be nonzero if and only if $\Phi_\lambda$ is nonzero because $\nabla_1(\lambda)$ and $\nabla_2(\lambda)$ are arbitrary standard modules of $k\Sigma_\ell$.

It follows from [DJ, (7.6)] that $\Phi_\lambda$ is nonzero if and only if $\lambda$ is an $a$-regular partition of $\ell$. If $m = 0$, then $\Phi_{\ell, \lambda} \neq 0$ if and only if $\delta = \eta \neq 0$. 

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