SEMI-INARIANT $\xi^\perp$-SUBMANIFOLDS OF GENERALIZED QUASI-SASAKIAN MANIFOLDS

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Dedicated to the memory of Prof. Stere Ianuș (1939 – 2010)

Abstract. A structure on an almost contact metric manifold is defined as a generalization of well-known cases: Sasakian, quasi-Sasakian, Kenmotsu and cosymplectic. Then we consider a semi-invariant $\xi^\perp$-submanifold of a manifold endowed with such a structure and two topics are studied: the integrability of distributions defined by this submanifold and characterizations for the totally umbilical case. In particular we recover results of Kenmotsu [8], Eum [9] and Papaghiuc [12].

1. Preliminaries and basic formulae

An interesting topic in the differential geometry is the theory of submanifolds in spaces endowed with additional structures. In 1978, A. Bejancu (in [2]) studied CR-submanifolds in Kähler manifolds. Starting from it, several papers have been appeared in this field. Let us mention only few of them: a series of papers of B.Y. Chen (e.g. [5]), of A. Bejancu and N. Papaghiuc (e.g. [3] in which the authors studied semi-invariant submanifolds in Sasakian manifolds). See also [10]. The study was extended also to other ambient spaces, for example A. Bejancu in [4] also studied QR-submanifolds in quaternionic manifolds and M. Barros in [11] investigated CR-submanifolds in quaternionic manifolds. Several important results above CR-submanifolds are being brought together in [4], [5], [9], [10], [11] and the corresponding references. The purpose of the present paper is to investigate the semi-invariant $\xi^\perp$-submanifolds in a generalized Quasi-Sasakian manifold.

Let $\tilde{M}$ be a real $(2n + 1)$-dimensional smooth manifold endowed with an almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$:

\[
\begin{align*}
\phi^2 &= -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi \xi = 0 \\
\eta(X) &= \tilde{g}(X, \xi), \quad \tilde{g}(\phi X, Y) + \tilde{g}(X, \phi Y) = 0
\end{align*}
\]

for any vector fields $X, Y$ tangent to $\tilde{M}$ where $I$ is the identity on sections of the tangent bundle $TM$, $\phi$ is a tensor field of type $(1,1)$, $\eta$ is a 1-form, $\xi$ is a vector field and $\tilde{g}$ is a Riemannian metric on $\tilde{M}$. Throughout the paper all manifolds and vectors fields are smooth.
maps are smooth. We denote by $\mathcal{F}(\tilde{M})$ the algebra of the smooth functions on $\tilde{M}$ and by $\Gamma(E)$ the $\mathcal{F}(\tilde{M})$-module of the sections of a vector bundle $E$ over $\tilde{M}$.

The almost contact manifold $\tilde{M}(\phi, \xi, \eta)$ is said to be normal if

$$N_\phi(X, Y) + 2d\eta(X, Y)\xi = 0$$

where

$$N_\phi(X, Y) = [\phi X, \phi Y] + \phi^2 [X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y], \quad X, Y \in \Gamma(T\tilde{M})$$

is the Nijenhuis tensor field corresponding of the tensor field $\phi$.

The fundamental 2-form $\Phi$ on $\tilde{M}$ is defined by $\Phi(X, Y) = \tilde{g}(X, \phi Y)$.

In [8], the author studied hypersurfaces of an almost contact metric manifold $\tilde{M}$ whose structure tensor fields satisfy the following relation

$$(\tilde{\nabla}_X \phi) Y = \tilde{g}(\tilde{\nabla}_X \phi Y, Y)\xi - \eta(Y)\tilde{\nabla}_X \xi$$

(1)

where $\tilde{\nabla}$ is the Levi-Civita connection of the metric tensor $\tilde{g}$. See also [6, 7]. For the sake of simplicity we say that a manifold $\tilde{M}$ endowed with an almost contact metric structure satisfying (1) is a generalized Quasi-Sasakian manifold, in short G.Q.S. Define a $(1, 1)$ type tensor field $F$ by

$$FX = -\tilde{\nabla}_X \xi.$$ (2)

**Proposition 1.** If $\tilde{M}$ is a G.Q.S manifold then any integral curve of the structure vector field $\xi$ is a geodesic i.e. $\tilde{\nabla}_\xi \xi = 0$. Moreover $d\Phi = 0$ if and only if $\xi$ is a Killing vector field.

**Proof.** The first assertion follows immediately from (1) with $X = Y = \xi$, and taking into account that $\eta(\tilde{\nabla}_\xi \xi) = 0$. Next, we deduce

$$3d\Phi(X, Y, Z) = \tilde{g}(\tilde{\nabla}_X \phi Z, Y) + \tilde{g}(\tilde{\nabla}_Z \phi Y, X) + \tilde{g}(\tilde{\nabla}_Y \phi X, Z) + \eta(X)\left(\tilde{g}(Y, \tilde{\nabla}_{\phi Z} \xi) + \tilde{g}(\phi Z, \tilde{\nabla}_Y \xi)\right) + \eta(Y)\left(\tilde{g}(Z, \tilde{\nabla}_{\phi X} \xi) + \tilde{g}(\phi X, \tilde{\nabla}_Z \xi)\right) + \eta(Z)\left(\tilde{g}(X, \tilde{\nabla}_{\phi Y} \xi) + \tilde{g}(\phi Y, \tilde{\nabla}_X \xi)\right).$$

If we suppose that $\xi$ is Killing then, from the last equation, we obtain $d\Phi = 0$.

Conversely, suppose that $d\Phi = 0$. Taking into account the first part of the statement, for $X = \xi, \eta(Y) = \eta(Z) = 0$, the last relation implies

$$\tilde{g}(Y, \tilde{\nabla}_{\phi Z} \xi) + \tilde{g}(\phi Z, \tilde{\nabla}_Y \xi) = 0.$$

Finally, by replacing $Z$ with $\phi Z$ and $Y$ by $Y - \eta(Y)\xi$ we deduce that $\xi$ is a Killing vector field. \qed

The next result can be obtained by direct calculation:

**Proposition 2.** A G.Q.S manifold $\tilde{M}$ is normal and

$$\phi \circ F = F \circ \phi, \quad F\xi = 0, \quad \eta \circ F = 0, \quad \tilde{\nabla}_\xi \phi = 0.$$ (3)

**Remark 1.**

a) It is easy to see that on such manifold $\tilde{M}$ the structure vector field $\xi$ is not necessarily a Killing vector field i.e. $\tilde{M}$ is not necessarily a K-contact manifold.

b) It is also interesting to pointed out that the following particular situations hold

1) $FX = -\phi X$ then $\tilde{M}$ is Sasakian.
2) \( FX = -X + \eta(X)\xi \) then \( \tilde{M} \) is Kenmotsu
3) \( FX = 0 \) then \( \tilde{M} \) is cosymplectic
4) if \( \xi \) is a Killing vector field then \( \tilde{M} \) is a quasi-Sasakian manifold.

Now, let \( \tilde{M} \) be a G.Q.S manifold and consider an \( m \)-dimensional submanifold \( M \), isometrically immersed in \( \tilde{M} \). Denote by \( g \) the induced metric on \( M \) and by \( \nabla \) its Levi-Civita connection. Let \( \nabla^{\perp} \) and \( h \) be the normal connection induced by \( \tilde{\nabla} \) on the normal bundle \( TM^{\perp} \) and the second fundamental form of \( M \), respectively. Then one has the direct sum decomposition \( T\tilde{M} = TM \oplus TM^{\perp} \). Recall the Gauss and Weingarten formulae

\[
\begin{align*}
(G) & \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \\
(W) & \quad \tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \quad X, Y \in \Gamma(TM)
\end{align*}
\]

where \( A_N \) is the shape operator with respect to the normal section \( N \) and satisfies

\[
\tilde{g}(h(X, Y), N) = g(A_N X, Y), \quad X, Y \in \Gamma(TM), \quad N \in \Gamma(TM^{\perp}).
\]

The purpose of the present paper is to investigate the semi-invariant \( \xi^{\perp}\)-submanifolds in a G.Q.S manifold. More precisely, we suppose that the structure vector field \( \xi \) is orthogonal to the submanifold \( M \). According to Bejancu [4] we say that \( M \) is a semi-invariant \( \xi^{\perp}\)-submanifold if there exist two orthogonal distributions, \( D \) and \( D^{\perp} \), in \( TM \) such that:

\[
TM = D \oplus D^{\perp}, \quad \phi D = D, \quad \phi D^{\perp} \subseteq TM^{\perp}
\]

(4)

where \( \oplus \) denotes the orthogonal sum. If \( D^{\perp} = \{0\} \) then \( M \) is an invariant \( \xi^{\perp}\)-submanifold. The normal bundle can also be decomposed as \( TM^{\perp} = \phi D^{\perp} \oplus \mu \), where \( \phi \mu \subseteq \mu \). Hence \( \mu \) contains \( \xi \).

2. Integrability of Distributions on a Semi-invariant \( \xi^{\perp}\)-Submanifold

Let \( M \) be a semi-invariant \( \xi^{\perp}\)-submanifold of a G.Q.S manifold \( \tilde{M} \). Denote by \( P \) and \( Q \) the projections of \( TM \) on \( D \) and \( D^{\perp} \) respectively, namely for any \( X \in \Gamma(TM) \)

\[
X = PX + QX.
\]

(5)

Moreover, for any \( X \in \Gamma(TM) \) and \( N \in \Gamma(TM^{\perp}) \) we put

\[
\phi X = tX + \omega X
\]

(6)

\[
\phi N = BN + CN
\]

(7)

with \( tX \in \Gamma(D) \), \( BN \in \Gamma(TM) \) and \( \omega X, CN \in \Gamma(TM^{\perp}) \). We also consider, for \( X \in \Gamma(TM) \), the decomposition

\[
FX = \alpha X + \beta X, \quad \alpha X \in \Gamma(D), \quad \beta X \in \Gamma(TM^{\perp}).
\]

(8)

The purpose of this section is to study the integrability of both distributions \( D \) and \( D^{\perp} \). With this scope in mind, we state first the following result.

**Proposition 3.** Let \( M \) be a semi-invariant \( \xi^{\perp}\)-submanifold of a G.Q.S manifold \( \tilde{M} \). Then we have

a) \( (\nabla_X t)Y = A_{\omega Y} X + Bh(X, Y) \),

b) \( (\nabla_X \omega)Y = Ch(X, Y) - h(X, tY) + g(FX, \phi Y)\xi, \quad X, Y \in \Gamma(TM) \).
Proof. The statement follows immediately from (6)–(8).

Taking into consideration the decomposition of $TM^\perp$, it can be easily proved:

**Proposition 4.** Let $M$ be a semi-invariant $\xi^\perp$-submanifold of a G.Q.S manifold $\tilde{M}$. Then for any $N \in \Gamma(TM^\perp)$ one has:

a) $BN \in D^\perp$,

b) $CN \in \mu$.

**Proposition 5.** If $M$ is a semi-invariant $\xi^\perp$-submanifold of a G.Q.S manifold $\tilde{M}$ then

$$A_\omega Z W = A_\omega W Z$$

for any $Z, W \in \Gamma(D^\perp)$.

The following two results give necessary and sufficient conditions for the integrability of the two distributions.

**Theorem 1.** Let $M$ be a semi-invariant $\xi^\perp$-submanifold of a G.Q.S manifold $\tilde{M}$. Then the distribution $D^\perp$ is integrable.

**Proof.** Let $Z, W \in \Gamma(D^\perp)$. Then from (6), (9) and (10) we deduce that

$$t[Z, W] = A_\omega Z W - A_\omega W Z = 0.$$ 

Hence the conclusion.

**Theorem 2.** If $M$ is a semi-invariant $\xi^\perp$-submanifold of a G.Q.S manifold $\tilde{M}$ then the distribution $D^\perp$ is integrable if and only if

$$h(tX, Y) - h(X, tY) = (L_\xi\tilde{g})(X, \phi Y) \xi, \quad X, Y \in \Gamma(D).$$

**Proof.** The statement yields directly from (6) and (9)

$$\omega([X, Y]) = h(X, tY) - h(tX, Y) + (L_\xi\tilde{g})(X, \phi Y) \xi.$$

Notice that the two results above are analogue those obtained in the Kenmotsu case in [12] and for the cosymplectic case in [14]. See also [10] when the submanifold is tangent to the structure vector field of the Sasakian manifold.

Moreover, from (8) we deduce

**Proposition 6.** Let $M$ be a $\xi^\perp$-semi-invariant submanifold of a G.Q.S manifold $\tilde{M}$. Then

$$A_\xi X = \alpha X, \quad \nabla_\xi^\perp \xi = -\beta X, \quad X \in \Gamma(TM).$$

Let now $\{e_i, \phi e_i, e_{2p+j}\}$, $i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, q\}$ be an adapted orthonormal local frame on $M$, where $q = \dim D^\perp$ and $2p = \dim D$. One can state the following

**Theorem 3.** If $M$ is a $\xi^\perp$-semi-invariant submanifold of a G.Q.S manifold $\tilde{M}$ one has

$$\eta(H) = \frac{1}{m} \text{trace}(A_\xi), \quad m = 2p + q.$$
Proof. Using a general formula for the mean curvature, e.g. \( H = \frac{1}{m} \sum_{a=1}^{q} \text{trace}(A_{\xi_a}) \xi_a \), where \( \{\xi_1, \ldots, \xi_q\} \) is an orthonormal basis in \( TM^\perp \), the conclusion holds by straightforward computations. □

In the case when the ambient space is a Kenmotsu manifold we retrieve the known result from [12, p. 614].

**Corollary 1.** There does not exist a minimal semi-invariant \( \xi^\perp \)-submanifold of a Kenmotsu manifold.

Also it is not difficult to prove:

**Theorem 4.** Let \( M \) be a semi-invariant \( \xi^\perp \)-submanifold of a G.Q.S manifold \( \widetilde{M} \). Then

1. the distribution \( D \) is integrable and its leaves are totally geodesic in \( M \) if and only if \( h(X, Y) \in \Gamma(\mu) \) where \( X, Y \) belong to \( D \);
2. any leaf of the integrable distribution \( D^\perp \) is totally geodesic in \( M \) if and only if \( h(X, Z) \in \Gamma(\mu) \) if \( X \in \Gamma(D) \) and \( Z \in \Gamma(D^\perp) \).

**Proof.** Let us prove only the first statement. For any \( Z \in D^\perp \) we have
\[
g(h(X, Y), \phi Z) = \tilde{g}(\tilde{\nabla}X Y, \phi Z) = -\tilde{g}(Y, \tilde{\nabla}X(\phi Z)) =
\]
\[
= -\tilde{g}(Y, (\tilde{\nabla}X \phi) Z) - \tilde{g}(\phi Y, \tilde{\nabla}X Z) = g(\tilde{\nabla}X(\phi Y), Z).
\]
Let \( M^* \) be a leaf of the integrable distribution \( D \) and \( h^* \) the second fundamental form of \( M^* \) in \( M \).

For any \( Z \in \Gamma(D^\perp) \) we have:
\[
g(h^*(X, Y), Z) = \tilde{g}(\tilde{\nabla}X tY, Z) = \tilde{g}((\tilde{\nabla}X \varphi) Y + \varphi(\tilde{\nabla}X Y), Z) = -\tilde{g}(h(X, Y), \varphi Z)
\]
which proves that the leaf \( M^* \) of the integrable \( D \) is totally geodesic in \( M \) if and only if \( h(X, Y) \in \Gamma(\mu) \).

Notice that the part (2) of the previous Theorem was obtained in the Kenmotsu case by Papaghiuc in [13, p. 115]. □

We end this section with the following

**Corollary 2.** If the leaves of the integrable distribution \( D \) are totally geodesic in \( M \) then the structure vector field \( \xi \) is \( D \)-Killing, that is \((L_\xi g)(X, Y) = 0, X, Y \in \Gamma(D)\).

### 3. Totally umbilical semi-invariant \( \xi^\perp \)-submanifolds

The main purpose of this section is to obtain a complete characterization of a totally umbilical semi-invariant \( \xi^\perp \)-submanifold of a G.Q.S manifold \( \widetilde{M} \). Recall that for a totally umbilical submanifold we have
\[
h(X, Y) = g(X, Y)H, \quad X, Y \in \Gamma(TM).
\]

First we state:
Theorem 5. An invariant $\xi^\perp$-submanifold $M$ of a G.Q.S manifold is totally umbilical if and only if
$$h(X, Y) = \frac{1}{m}g(X, Y)\text{trace}(A_\xi).$$

Proof. If $M$ is an invariant $\xi^\perp$-submanifold then for any $X, Y \in \Gamma(TM)$ we have $h(X, \phi Y) = \phi h(X, Y) - g(A_\xi \phi X, Y)$. Let us consider an orthonormal frame $\{e_i, e_{p+i}\}, i = 1, \ldots, p$ on $M$; from the above relation one obtains that $\phi H = 0$. Again, since $M$ is an invariant submanifold:
$$H = g(H, \xi)\xi = \frac{1}{m} \sum_{i=1}^{m} g(h(e_i, e_i), \xi)\xi = \frac{1}{m}\text{trace}(A_\xi)\xi$$
and the proof is complete. $\Box$

Corollary 3. A semi-invariant $\xi^\perp$-submanifold of a quasi-Sasakian manifold is minimal.

The case of a semi-invariant $\xi^\perp$-submanifold in a G.Q.S manifold $\tilde{M}$ is solved in the next Theorem.

Theorem 6. Let $M$ be a semi-invariant $\xi^\perp$-submanifold of a G.Q.S manifold $\tilde{M}$ with $\text{dim}D^\perp > 1$. Then $M$ is totally umbilical if and only if (13) holds.

Proof. Let $X \in \Gamma(D)$ be a unit vector field and $N \in \Gamma(\mu) \setminus \text{span}\{\xi\}$. By direct calculation it results that:
$$g(H, N) = g(h(X, X), N) = g(\nabla_X \phi X - (\nabla_X \phi)X, \phi N) = g(h(X, \phi X), \phi N) = 0$$
which proves that $H \in \phi D^\perp \oplus \text{span}\{\xi\}$.

For $Z, W \in \Gamma(D^\perp)$, from (9) we derive $QA_{\phi Z}W = -g(Z, W)\phi H$ i.e.
$$g(Z, \phi H)g(W, \phi H) = g(Z, W)g(\phi H, \phi H).$$
(15)

If we take $Z = W$ orthogonal to $\phi H$, since $\text{dim}D^\perp > 1$, from the above relation we infer $\phi H = 0 \Rightarrow H \in \text{span}\{\xi\}$. At this point the conclusion is straightforward.

Conversely, if (13) is supposed to be true, then we get (14) which together with (13) we deduce that $M$ is totally umbilical. $\Box$

Let us remark that when $\tilde{M}$ is a Kenmotsu manifold the result of the Theorem 6 was proved in [12].

Corollary 4. Every $\xi^\perp$-hypersurface of a G.Q.S manifold $\tilde{M}$ is totally umbilical.

Proof. If $M$ is a hypersurface then $TM^\perp = \text{span}\{\xi\}$ that is $h(X, Y) \in \text{span}\{\xi\}$. Next, from (14) it follows (13). $\Box$

In the particular case of a Kenmotsu manifold this result was obtained by Papaghiuc in [12] p. 617.

As a consequence of Theorem 6 we obtain

Theorem 7. If $M$ is a totally umbilical semi-invariant $\xi^\perp$-submanifold of a G.Q.S manifold $\tilde{M}$ with $\text{dim}D^\perp > 1$, then $M$ is a semi-invariant product.
Here, by a semi-invariant product we mean a semi-invariant $\xi^\perp$-submanifold of $\tilde{M}$ which can be locally written as a Riemannian product of a $\phi$-invariant submanifold and a $\phi$-anti-invariant submanifold of $\tilde{M}$, both of them orthogonal to $\xi$.

**Proof.** From the definition of totally umbilical submanifold we have $h(X, Z) = 0$ for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$, so that, by b) of Theorem 4, the leaves of $D^\perp$ are totally geodesic submanifolds of $M$. By Theorem 6 we have $h(X, Y) \in \text{span}\{\xi\} \subset \mu$ for any $X, Y \in D$. By virtue of a) of Theorem 1, this implies that the invariant distribution $D$ is integrable and its integral manifolds are totally geodesic submanifolds of $M$. Therefore, we conclude that $M$ is a semi-invariant product. □

Without any restriction on the dimension of $D^\perp$, we have the following

**Theorem 8.** Let $M$ be a totally umbilical semi-invariant $\xi^\perp$-submanifold of a G.Q.S manifold $\tilde{M}$. If $D$ is integrable, then each leaf of $D$ is a totally geodesic submanifold of $M$.

**Proof.** By using b) of Proposition 3 for any $X \in \Gamma(D)$, we have

$$\omega(\nabla_X X) = -g(X, X)CH - g(FX, \phi Y)\xi.$$

Since $CH \in \mu$ by b) of Lemma 3 and $\omega U \in \phi D^\perp$ for any $U \in \Gamma(TM)$, from the above equation we deduce that $\omega(\nabla_X X) = 0$, or equivalently

$$\nabla_X X \in D, \quad \forall X \in \Gamma(D).$$

Replacing $X$ by $X + Y$, we get $\nabla_X Y + \nabla_Y X \in \Gamma(D)$ for all $X, Y \in \Gamma(D)$. This condition, together with the integrability of $D$, implies

$$\nabla_X Y \in D, \quad \forall X, Y \in \Gamma(D). \quad (16)$$

As $D$ is integrable, Frobenius theorem ensures that $M$ is foliated by leaves of $D$. Combining this fact with (16), we conclude that the leaves of $D$ are totally geodesic submanifolds of $M$. □

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