PONTRYAGIN AND EULER FORMS AND CHERN-SIMONS TERMS IN WEYL-CARTAN SPACE

O. V. Babourova and B. N. Frolov

Department of Mathematics, Moscow State Pedagogical University,
Krasnoprudnaya 14, Moscow 107140, Russia;
E-mail: baburova.physics@mpgu.msk.su, frolovbn.physics@mpgu.msk.su

Abstract

The existence of the Pontryagin and Euler forms in a Weyl-Cartan space on the basis of the variational method with Lagrange multipliers are established. It is proved that these forms can be expressed via the exterior derivatives of the corresponding Chern-Simons terms in a Weyl-Cartan space with torsion and nonmetricity.

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I. Introduction

The dilatonic gravity is one of the attractive approaches to the modern gravitational theory. This is based on the fact that a low-energy effective string theory is reduced to the theory of interacting metric and scalar dilaton field.\textsuperscript{1} From the geometric point of view the dilatonic gravity is connected with the Weyl geometry of space-time.

The attractive feature of any quantum field theory is its renormalizability. It is well known that the Einstein gravity coupled to a scalar field is nonrenormalizable at the one-loop level.\textsuperscript{2} In order to get renormalizability one can add to the gravitational Lagrangian curvature-squared terms.\textsuperscript{3} In this connection the Gauss-Bonnet type identity for quadratic Lagrangians becomes the object of a considerable amount of attention. The generalization of the Gauss-Bonnet formula to four-dimensional pseudo-riemannian space $V_4$ was performed by Bach\textsuperscript{4} and Lanczos\textsuperscript{5} and on the basis of the variational method with Lagrange multipliers by Ray.\textsuperscript{6} It is well known that Bach-Lanczos identity in a Riemann space implies the one-loop renormalizability of pure gravitation.\textsuperscript{2} The generalization of the Bach-Lanczos identity to a Riemann-Cartan space $U_4$ was performed in Refs. 7-9.

The Gauss-Bonnet-Chern Theorem\textsuperscript{10} states that an integral of some form (called Euler form) over an oriented compact manifold without boundary does not depend on the choice of a metric and a connection of the manifold and therefore is the topological invariant of the manifold (its Euler characteristic). There exists another class of topological invariants of a manifold represented as integrals of Chern-Pontryagin forms over the manifold.\textsuperscript{11} These topological invariants are closely connected with the so-called topological charges of the space-time manifolds.
We shall obtain the Pontryagin type and the Euler type topological invariants in a Weyl-Cartan space \( Y_4 \) (the generalized Weyl space with torsion), which will be essential for the dilatonic gravitational theory with quadratic Lagrangians in the spaces with torsion and nonmetricity.

II. Topological invariants in general metric-affine space

In this section we consider a metric-affine space \((L_4,g)\) that is a connected 4-dimensional oriented differentiable manifold \( \mathcal{M} \) equipped with a linear connection \( \Gamma \) and a metric \( g \) of index 1.\(^{12}\) We shall use an anholonomic local vector frame \( e_a \) \((a = 1, 2, 3, 4)\) and a 1-form coframe \( \theta^a \) with \( e_a \mid \theta^b = \delta^b_a \) (\( \mid \) means the interior product). The vector basis \( e_a \) can be chosen to be pseudo-orthonormal with respect to a metric

\[
g = g_{ab} \theta^a \otimes \theta^b .
\] (2.1)

In this case one gets,

\[
g_{ab} := g(e_a, e_b) = \text{diag}(+1, +1, +1, -1) .
\] (2.2)

In \((L_4,g)\) a metric \( g \) and a connection \( \Gamma \) are not compatible in the sense that the \( GL(4, R) \)-covariant exterior differential \((\mathcal{D} := d + \Gamma \wedge \ldots)\) of the metric does not vanish,

\[
\mathcal{D} g_{ab} = dg_{ab} - \Gamma^c_{a} g_{cb} - \Gamma^c_{b} g_{ac} =: -Q_{ab} ,
\] (2.3)

where \( \Gamma^a_b \) is a connection 1-form and \( Q_{ab} \) is a nonmetricity 1-form.

A curvature 2-form \( \Omega^a_b \) and a torsion 2-form \( \mathcal{T}^a \),

\[
\Omega^a_b = \frac{1}{2} R^a_{bcd} \theta^c \wedge \theta^d , \quad \mathcal{T}^a = \frac{1}{2} T^a_{bc} \theta^b \wedge \theta^c ,
\] (2.4)
are defined by virtue of the Cartan’s structure equations,

\[ \Omega^a_b = d\Gamma^a_b + \Gamma^a_c \wedge \Gamma^c_b, \]
\[ T^a = \mathcal{D}\theta^a = d\theta^a + \Gamma^a_b \wedge \theta^b. \]

Let us consider the 4-form

\[ \Pi = B^b_{a,p} \Omega^a_b \wedge \Omega^p_q, \]

where \( B^b_{a,p} \) is an unknown \( GL(4,R)\)-invariant tensor. From the form of (2.7) it is easy to get the following symmetry property of this tensor,

\[ B^b_{a,p} = B^q_{p,a}. \]

The 4-form (2.7) is proportional to the volume 4-form \( \eta \) of the 4-dimensional manifold \( M \), where

\[ \eta = \frac{1}{4!} \eta_{abcd} \theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d, \quad \eta_{abcd} = \sqrt{-\det g_{kl}} \varepsilon_{abcd}. \]

Here \( \varepsilon_{abcd} \) is the components of the totally antisymmetric \( GL(4,R)\)-invariant Levi-Civita 4-form density \( 1^2 \) \( (\varepsilon_{1234} = -1) \).

Since \( D\eta = d\eta = 0 \) as a 5-form on the 4-dimensional manifold \( M \), one has the identity\( ^{13} \)

\[ D\eta_{abcd} = -\frac{1}{2} Q \eta_{abcd}, \quad Q := g^{pq} Q_{pq}. \]

The explicit form of the tensor (2.8) should be determined on the basis of the condition that the integral

\[ \int_M \Pi \]

over the oriented 4-dimensional manifold \( M \) without boundary does not depend on the choice of a metric and a connection and therefore the variation of the integrand of (2.11) with respect to a metric and a connection should be
equal to an exact form. Here we consider the manifold \( \mathcal{M} \) without boundary for the simplicity. For the manifolds with boundary some additional surface terms should be taken into account.\(^{11}\)

As a consequence of (2.1) the variation with respect to a metric \( g \) is determined only by variations of 1-forms \( \theta_a \) because of the fact that the variation \( \delta g_{ab} = 0 \) when one chooses the pseudo-orthonormal basis \( e_a \) and gets the condition (2.2). The tensor \( B^{bq}_{ap} \) in (2.7) also should not to be varied when the local vector basis \( e_a \) is chosen to be anholonomic and pseudo-orthonormal because it can be constructed (as an \( GL(4, R) \)-invariant tensor) only from the metric tensor, Kronecker delta \( \delta_a^b \) and the \( GL(4, R) \)-invariant totally antisymmetric Levi-Civita density \( \epsilon_{abpq} \).

The variation of (2.7) yields the expression,

\[
\delta \Pi = 2\delta \Gamma^a_b \wedge (\mathcal{D}B^{bq}_{ap}) \wedge \Omega^p_q + d(2\delta \Gamma^a_b \wedge B^{bq}_{ap} \Omega^p_q) .
\]

(2.12)

Here the following relation has been used,

\[
\delta \Omega^a_b \wedge \Phi^b_a = d(\delta \Gamma^a_b \wedge \Phi^b_a) + \delta \Gamma^a_b \wedge \mathcal{D} \Phi^b_a ,
\]

(2.13)

that valids for an arbitrary 2-form \( \Phi^a_b \).

One can see that the variation (2.12) is equal to an exact form, if the tensor \( B^{bq}_{ap} \) satisfies the condition,

\[
\mathcal{D}B^{bq}_{ap} = 0 .
\]

(2.14)

In a general metric-affine space \( (L_4, g) \) there are only two possibilities to satisfy (up to constant factors) the condition (2.14),

\[
(a) \quad B^{bq}_{ap} = \delta^b_a \delta^q_p , \quad (b) \quad B^{bq}_{ap} = \delta^b_p \delta^q_a .
\]

(2.15)

In the case (a) the 4-form \( \Pi \) (2.7) reads,

\[
\Pi_\Omega = \Omega^a_b \wedge \Omega^b_a = \text{Tr}(\Omega \wedge \Omega) ,
\]

(2.16)
and in the case (b) one has,

$$
\Pi_{\tr\Omega} = \Omega_a^a \wedge \Omega_b^b = \Tr\Omega \wedge \Tr\Omega .
$$

(2.17)

We see that the 4-forms (2.16) and (2.17) are equal up to constant factors to the well-known Pontryagin forms.\textsuperscript{11,12}

### III. The topological invariants in a Weyl-Cartan space

A Weyl-Cartan space $Y_4$ is a space with a metric, curvature, torsion and non-metricity which obeys the constraint,

$$
Q_{ab} = \frac{1}{4} g_{ab} Q .
$$

(3.1)

This constraint can be introduced into the variational approach with the help of the method of Lagrange multipliers. In this case the integral (2.11) has to be modified,

$$
\int_M \left( \Pi + \Lambda^{ab} \wedge (Q_{ab} - \frac{1}{4} g_{ab} Q) \right) ,
$$

(3.2)

where the Lagrange multiplier $\Lambda^{ab}$ is a tensor-valued 3-form with the properties,

$$
\Lambda^{ab} = \Lambda^{ba} , \quad \Lambda^a_a = 0 .
$$

(3.3)

The variation of (3.2) with respect to $\theta^a$, $\Gamma^a_b$ and the Lagrange mutiplier yields that the following variational derivatives have to vanish identically,

$$
\delta \Gamma^a_b : \mathcal{D}(B^{ab}_{\theta^a}) \wedge \Omega^a_q - \Lambda^a_b = 0 ,
$$

(3.4)

$$
\delta \Lambda^{ab} : \quad Q_{ab} - \frac{1}{4} Q g_{ab} = 0 .
$$

(3.5)
As in the previous section the variational derivative with respect to $\theta^a$ is absent because of the fact that there is no an explicit dependence on $\theta^a$ of the integrand expression in (3.2).

The identity (3.4) in $Y_4$ is equivalent to the following identities,

$$\Lambda^{ba} = (\mathcal{D} - \frac{1}{4}Q) B^{[ba]} Q_p \wedge \Omega^p_q , \quad (3.6)$$

$$\mathcal{D} B^{[ba]} Q_p \wedge \Omega^p_q = 0 , \quad (3.7)$$

$$\mathcal{D} B^{[a]} Q_p \wedge \Omega^p_q = 0 . \quad (3.8)$$

The identities (3.7), (3.8) are satisfied in the following four cases:

$$\begin{align*}
(a) & \quad B^{baq} Q_p = g^{ba} \delta^q_p , \\
(b) & \quad B^{baq} Q_p = \delta^b_p g^{qa} , \\
(c) & \quad B^{baq} Q_p = g^{bq} \delta^a_p , \\
(d) & \quad B^{baq} Q_p = \eta^{baq} .
\end{align*} \quad (3.9)$$

In the case (d) one has to use (2.3), (3.1) and (2.10).

The equality (3.6) determines the Lagrange multiplier. In all cases (a)-(d) one has $\Lambda^{ab} = 0$. This means that the Weyl-Cartan constraint (3.7) can be imposed both before and after the variational procedure.

The cases (a) and (b) coincide with (2.15) and yield for a Weyl-Cartan space $Y_4$ the Pontryagin forms (2.16) and (2.17) of the previous section. The cases (c) and (d) appear in $Y_4$ but not in $(L_4, g)$.

In the case (c) one has the Pontryagin form,

$$\Pi_{CW} = \Omega^{ab} \wedge \Omega_{ab} = \text{Tr}(\Omega \wedge \Omega^T) , \quad (3.11)$$

where $\Omega^T$ means the transpose of $\Omega$. In $Y_4$ with the help of the relation,

$$\Omega_{ab} = \Omega_{[ab]} + \frac{1}{4} g_{ab} \text{Tr} \Omega , \quad (3.12)$$

(3.11) can be decomposed as follows,

$$\Omega^{ab} \wedge \Omega_{ab} = \Omega^{[ab]} \wedge \Omega_{[ab]} + \frac{1}{4} \text{Tr} \Omega \wedge \text{Tr} \Omega . \quad (3.13)$$
On the other hand the Pontryagin form (2.16) in $Y_4$ has the decomposition,

$$\Omega^a_b \wedge \Omega^b_a = -\Omega^{[ab]} \wedge \Omega_{[ab]} + \frac{1}{4} \text{Tr} \Omega \wedge \text{Tr} \Omega . \quad (3.14)$$

Therefore in a Weyl-Cartan space $Y_4$ one has two fundamental Pontryagin forms, which are equal up to constant factors to

$$\Pi_C = \Omega^{[ab]} \wedge \Omega_{[ab]} , \quad \Pi_W = \text{Tr} \Omega \wedge \text{Tr} \Omega . \quad (3.15)$$

The former form is the volume preserving Pontryagin form and the latter one is the dilatonic Pontryagin form.

In the case (d) we get the Euler form in a Weyl-Cartan space $Y_4$,

$$\mathcal{E} = \eta^{bq}_{ap} \Omega^a_b \wedge \Omega^p_q . \quad (3.16)$$

One can use the holonomic coordinate basis $e_\alpha = \partial_\alpha$ and express the topological invariant corresponding to (3.16) in the component form,

$$\int_{\mathcal{M}} \mathcal{E} = \int_{\mathcal{M}} E \sqrt{-g} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 , \quad (3.17)$$

$$E = R^2 - (R_{\alpha\beta} + \tilde{R}_{\alpha\beta})(R^{\beta\alpha} + \tilde{R}^{\beta\alpha}) + R_{\alpha\beta\mu\nu} R^{\mu\nu\alpha\beta} , \quad (3.18)$$

where $R^\alpha_{\beta\mu\nu}$ are the components of the curvature 2-form in a holonomic basis, the following notations being used, $R_{\alpha\beta} = R^\sigma_{\alpha\sigma\beta} , \tilde{R}_{\alpha\beta} = R_{\alpha\sigma\beta} \tilde{\sigma} , R = R^\sigma_{\sigma}$. The Gauss-Bonnet-Chern Theorem\textsuperscript{10} states the relation of the integral (3.17) over the oriented compact manifold $\mathcal{M}$ without boundary with the Euler characteristic of this manifold. The explicit proof using a holonomic basis of the independence of (3.17) on the choice of a metric and a connection of a Weyl-Cartan space $Y_4$ is explained in Ref. 14.
IV. Chern-Simons terms in a Weyl-Cartan space

It is well known that in \((L_4,g)\) the Pontryagin forms can be represented as the exterior derivatives of the \(GL(4,R)\) Chern-Simons terms,

\[
\Pi_\Omega = dC_\Omega , \quad C_\Omega = \Gamma_a^b \wedge \Omega^a_b - \frac{1}{3} \Gamma_a^b \wedge \Gamma_c^a \wedge \Gamma_b^c , \quad (4.1)
\]

\[
\Pi_W = dC_W , \quad C_W = \frac{1}{2} Q \wedge \Omega^a_a . \quad (4.2)
\]

It is easy to see that Pontryagin form \(\Pi_C\) (3.15) in a Weyl-Cartan space \(Y_4\) can be represented in an analogous manner,

\[
\Pi_C = dC_C , \quad C_C = \Gamma[ab] \wedge \Omega^{ab} - \frac{1}{3} \Gamma[ab] \wedge \Gamma^{[ac]} \wedge \Gamma^{cb]} . \quad (4.3)
\]

As it was pointed out in Ref. 12, the Euler form (3.16) in the framework of a Riemann-Cartan space can be expressed in terms of the corresponding Chern-Simons type construction,

\[
C_E = dC_E , \quad C_E = \eta^{bp}_{aq} \left( \Omega^a_b \wedge \Gamma^p_q - \frac{1}{3} \Gamma^a_b \wedge \Gamma^p_f \wedge \Gamma^f_q \right) . \quad (4.4)
\]

Let us prove that formula (4.4) is also valid in a Weyl-Cartan space \(Y_4\). The proof is based on the two Lemmas.

Lemma 1. If the equality

\[
D\eta^b_q_{ap} = 0 , \quad (4.5)
\]

is valid, then the identity (4.4) is fulfilled.

Proof. In anholonomic orthonormal frames one has \(d\eta^b_q_{ap} = 0\), and therefore (4.5) yields,

\[
\Gamma^b_f \eta^f_q_{ap} - \Gamma^f_a \eta^b_q_{fp} + \Gamma^q_f \eta^b_q_{af} - \Gamma^f_p \eta^b_q_{af} = 0 . \quad (4.6)
\]
After multiplying (4.6) externally by the 3-form $\Gamma^a_s \wedge \Gamma^b_q \wedge \Gamma^f_q$, one gets the $Y_4$-identity,

$$\eta^{bq}_{ap} \Gamma^a_s \wedge \Gamma^b_q \wedge \Gamma^f_q = 0 .$$  \hspace{1cm} (4.7)

After multiplying (4.6) externally by the 3-form $\Omega^a_b \wedge \Gamma^p_q$, one gets the second $Y_4$-identity,

$$\eta^{bq}_{ap} (2\Omega^a_b \wedge \Gamma^p_q + \Omega^a_f \wedge \Gamma^f_q + \Gamma^a_b \wedge \Gamma^p_q) = 0 .$$  \hspace{1cm} (4.8)

Now using the identities (4.7) and (4.8), the Cartan’s structure equation (2.5) and the Bianchi identity,

$$\mathcal{D}\Omega^a_b = d\Omega^a_b + \Gamma^a_f \wedge \Omega^f_b - \Omega^a_f \wedge \Gamma^f_b = 0 ,$$  \hspace{1cm} (4.9)

let us perform the exterior differentiation of the Chern-Simons term $C_E$ (1.4) and get,

$$dC_E - E = \frac{1}{3} \eta^{bq}_{ap} \Gamma^a_s \wedge \Gamma^b_q \wedge \Gamma^f_q$$

$$- \frac{2}{3} \eta^{bq}_{ap} (2\Omega^a_b \wedge \Gamma^p_q + \Omega^a_f \wedge \Gamma^f_q + \Gamma^a_b \wedge \Gamma^p_q) = 0 ,$$  \hspace{1cm} (4.10)

as was to be proved.

**Lemma 2.** The equality (4.5) is valid if and only if the space under consideration is a Weyl-Cartan space $Y_4$.

**Proof.** In a general $(L_4,g)$ space one has,

$$\mathcal{D}\eta^{bq}_{ap} = \eta^{bq}_{ap} \mathcal{D}\tilde{Q}^{bn} - \eta^{bq}_{ap} \tilde{Q}^{bn} ,$$  \hspace{1cm} (4.11)

where $\mathcal{Q}^{bn} := Q^{bn} - \frac{1}{4} g^{bn} Q$ is the tracefree part of the nonmetricity 1-form, $\mathcal{Q}^b = 0$. For a Weyl-Cartan space $Y_4$ one has $\tilde{Q}^{bn} = 0$ and the sufficient condition of the Lemma is evident. The necessary condition of the Lemma is the consequence of the fact that the vanishing of (4.11) leads to the equality,

$$g^{ab} \tilde{Q}^{pq} - g^{bp} \tilde{Q}^{aq} - g^{aq} \tilde{Q}^{bp} + g^{pq} \tilde{Q}^{ab} = 0 ,$$  \hspace{1cm} (4.12)

which yields $\tilde{Q}^{pq} = 0$, as was to be proved.
V. Conclusions

We have proved the existence of the Pontryagin and Euler forms in a Weyl-Cartan space on the basis of the variational method with Lagrange multipliers. It has been discovered that the Pontryagin form, $\Pi_C = \Omega^{(ab)} \wedge \Omega_{[ab]}$, and Euler form, $\mathcal{E} = \eta_{a[p}^b \Omega^p_{b]} \wedge \Omega^q_{q}$, which are specific for a Riemann-Cartan space, also exist in a Weyl-Cartan space. With the help of these forms the topological invariants of a Weyl-Cartan space which do not depend on the choice of a metric and a connection are constructed. It has been proved that these forms can be expressed via the exterior derivatives of the corresponding Chern-Simons terms in a Weyl-Cartan space (see (4.3) and (4.4), respectively). From the Lemma 2 proved it follows that the relation (4.4) is not valid in the more general geometry than the Weyl-Cartan one.
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