Faddeev-Jackiw approach to gauge theories and ineffective constraints

J. ANTONIO GARCÍA* AND JOSEP M. PONS†

Departament d'Estructura i Constituents de la Matèria
Universitat de Barcelona i Institut de Física d'Altes Energies
Av. Diagonal 647
08028 Barcelona
Catalonia, Spain

September 1997

Abstract

The general conditions for the applicability of the Faddeev-Jackiw approach to gauge theories are studied. When the constraints are effective a new proof in the Lagrangian framework of the equivalence between this method and the Dirac approach is given. We find, however, that the two methods may give different descriptions for the reduced phase space when ineffective constraints are present. In some cases the Faddeev-Jackiw approach may lose some constraints or some equations of motion. We believe that this inequivalence can be related to the failure of the Dirac conjecture (that says that the Dirac Hamiltonian can be enlarged to an Extended Hamiltonian including all first class constraints, without changes in the dynamics) and we suggest that when the Dirac conjecture fails the Faddeev-Jackiw approach fails to give the correct dynamics. Finally we present some examples that illustrate this inequivalence.

UB-ECM-PF 97/20
hep-th/9803222
PACS: 04.20.Fy, 11.10.Ef, 11.15.-q

*electronic adress: garcia@ritae.ccm.ub.es
†electronic adress: pons@ecm.ub.es
1 Introduction

Some years ago, Faddeev and Jackiw [1] suggested a simple Lagrangian method for dealing with gauge theories. The distinctive feature of this method is the way to eliminate the gauge degrees of freedom: The reduction of the degrees of freedom to the physical ones is performed just by direct substitution of the holonomic constraints, derived from the variational principle, into the Lagrangian. This substitution takes place algorithmically, in several steps. Since the usual constraints that appear in the tangent space of a gauge theory are not holonomic (they usually involve velocities), the idea of Faddeev and Jackiw is to work with the canonical Lagrangian, which takes as a new configuration space the cotangent space (phase space) of the original theory. Then the canonical Lagrangian is at most linear in the velocities, and all the constraints become holonomic (they are the Hamiltonian constraints). Thus, the method of Faddeev and Jackiw avoids the sometimes cumbersome procedure pioneered by Dirac, and known as the Dirac method [2]. There is a price to pay nevertheless: the necessity to perform a non-trivial Darboux transformation at each stage of the new algorithm.

In a recent paper [3] we have proved, under some general assumptions, the equivalence of the Faddeev-Jackiw (F-J) method and the classical Dirac approach. Here we want to expand this result by considering the cases where some conditions required in [3] for the equivalence proof do not hold. The proof in [3] was produced under some conditions of regularity (summarized in the first section of that paper). In particular, we assumed that the constraints \( \phi_\mu(q,p) \) (primary, secondary...) that appear in the formalism allow for a canonical representation of the constraint surface, that is, there is a change of basis,

\[
\phi_\mu \rightarrow \xi_\mu = M_\mu^\nu(q,p)\phi_\nu, \quad \det M \neq 0, \tag{1.1}
\]

to a new set of functions \( \xi_\mu(q,p) \) that represent the same surface as the original constraints \( \phi_\mu(q,p) \), and where the functions \( \xi_\mu(q,p) \) are a subset of a new set of canonical variables. This assumption, crucial in [3], therefore takes for granted that all the constraints are effective, where by effective constraints we mean the following: A set of independent constraints is said to be effective—and ineffective otherwise— if the one-forms obtained by differentiating the constraints (that is, their gradients) are all independent on the constraint surface. Notice that the dimension of this space of one-forms is invariant under changes of the type (1.1).

The canonical transformation associated with (1.1) can not be realized for an ineffective representation of the constraint surface: As long as \( \det M \neq 0 \), an ineffective representation will remain so, and the \( \xi_\mu \) will never be a subset of a set of canonical variables. Since the assumption that the description of the constraint surface is effective was made at every stage in the Faddeev-Jackiw reduction algorithm, this means that only effective constraints were allowed in our proof in [3]. Of course, given an ineffective representation of the constraint surface, it is always possible to construct an effective representation for it. In more mathematical terms this effective representation is a basis of the ideal of functions that vanish on the constraint surface. This effective representation can be used to characterize this surface geometrically according to the
classification of its constraints as first class and second class. This characterization is also given by the rank of the symplectic structure projected from the phase space to the constraint surface.

The presence of ineffective constraints introduces some problems in the general theory of constrained systems. For instance, one can run into difficulties with the counting of the true –non gauge– degrees of freedom, or with the breakdown of the equivalence between the Dirac Hamiltonian formalism and the Extended Hamiltonian formalism –where all first class constraints are included in the Hamiltonian with independent Lagrange multipliers–, which is nothing but the failure of the Dirac conjecture [2]. Besides these theoretical aspects, it is interesting to analyze this type of constraint because they appear in some examples that exhibit a rich gauge algebraic structure, examples are the Siegel model [4] [5] and some models of $\mathcal{W}$-algebras in Euclidean space [6]. In this paper we will not discuss the problems that arise at the quantum level in case of ineffective constraints. This issue has been recently addressed for some simple examples from the point of view of the quantum projector method [7].

As regards the F-J method, we must remark that if for a given theory some ineffective constraints appear in Dirac’s stabilization algorithm, then the proof of equivalence of the F-J method with Dirac’s, given in [3], does not hold. In fact we will see that in some cases this equivalence is broken because the replacement of ineffective constraints in the variational principle for the canonical Lagrangian, as it is made in the F-J method for any constraint, leads to a loss of dynamical information. In this paper we will show that, in some cases, the presence of ineffective constraints makes the basic result obtained in [3] false, while in other cases the equivalence still holds. We will also show that in the cases when the equivalence does not hold, the correct method is that of Dirac, because in this case F-J method suffers a loss of dynamical information while Dirac’s method does not.

We will use throughout the paper a notation with finite number of degrees of freedom, though our results can be generalized to field theories. The class of constrained dynamical systems under our consideration will fulfill the following general properties:

(a) The Hessian matrix $\frac{\partial L}{\partial \dot{q}^i} \frac{\partial L}{\partial \dot{q}^j}$ is of constant rank on the constraint surface.

(b) Throughout the stabilization algorithm, second class constraints never become first class, or equivalently, the rank of the Poisson bracket among the constraints cannot decrease during the stabilization algorithm.

(c) We assume the existence of as many Noether gauge symmetries as primary first class constraints in phase space. This assumption goes beyond the results in [3].

(d) The primary constraints are always effective (this is required by Dirac theory [2]). In particular this last point ensures the equivalence between the Lagrangian approach and the Dirac method [9] with full generality.

In this framework we analyze two questions: a) The reason for the inequivalence between the Dirac method and the F-J approach when some ineffective constraints are present. b) The possible relation between this inequivalence and the failure of the Dirac conjecture. The first question is addressed in full generality. As regards the second question, we think it is an interesting open problem that can help to understand under which conditions the Dirac conjecture fails. One of the essential assumptions in proofs
of the Dirac conjecture, as given for example in [10][11], is that all the constraints are effective. Indeed, to our knowledge, the counterexamples to this conjecture always present ineffective constraints at some stage of the Dirac algorithm. The relation of the Dirac conjecture and ineffective constraints can be established by an explicit construction of the gauge generator. In fact, we will see, by showing some examples, that there are cases where some secondary first class constraints do not need gauge fixing because they appear in the gauge generator as ineffective pieces. Because of a lack of a general theory of construction of Noether gauge generators for ineffective constrained theories, we cannot prove this relation in general.

In section 2 we recall the general properties of the F-J method when the constraints are effective, as described in [3]. In section 3 we analyze the way in which the presence of ineffective constraints affects the F-J method. In section 4 we suggest to relate the failure of F-J method when some ineffective constraints are present to the structure of the gauge generators and to the Dirac conjecture. In section 5 we give some examples that illustrate the incompleteness of the F-J method in such cases. Section 6 is devoted to conclusions. In Appendix A we show how the presence of ineffective constraints is dealt within Dirac’s approach.

## 2 The Faddeev-Jackiw method for effective constraints

Let us start by recalling some of the results obtained in our previous proof in [3] under the conditions (a)-(d) of the previous section. In this section we add the condition:

(e) All constraints in the stabilization algorithm are effective.

The F-J reduction procedure starts with a general Lagrangian of the form

$$L_c = p_i q^i - H_c(q, p) - \lambda^\mu \phi_\mu,$$

(2.1)

where $H_c$ is the canonical Hamiltonian, $\lambda^\mu$ are a set of Lagrange multipliers, and the primary constraints $\phi_\mu$ are taken as being effective and independent. By plugging these constraints into the Lagrangian we can eliminate as many variables as primary constraints, obtaining the reduced Lagrangian $L$, in terms of a reduced set of variables that we denote as $x^s$

$$L = a_s(x) \dot{x}^s - H(x)$$

(2.2)

where $a_s(x)$ are some specific functions that define the Liouville one-form, and give, after differentiation, the two-form symplectic structure in the reduced phase space. We can always perform a Darboux transformation

$$x^s \rightarrow Q^r, P_r, Z^a,$$

(2.3)

such that in these new coordinates $L$ takes the canonical form

$$L = P_r \dot{Q}^r - H(Q^r, P_r, Z^a).$$

(2.4)

\[^1\text{Notice that the sub or super-index carries information both on the type of variable –labeled by the letter– under consideration and the range of values that it can take.} \]
The $Z^a$ variables appear only when the two-form defined by $a_s(x)$ is degenerate, otherwise the functions $a_s(x)$ define the Dirac brackets in the reduced phase space. The equations of motion associated to $L$ for the $Z^a$ variables allow for the isolation of a subset of these variables together with some relations among the original variables $Q^r, P_r$:

$$\frac{\partial H}{\partial Z^a} = 0 \iff Z^{a_1} = f^{a_1}(Q^r, P_r, Z^{a_2}), \quad f_{a_2}(Q^r, P_r) = 0. \quad (2.5)$$

Substitution of $Z^{a_1} = f^{a_1}(Q^r, P_r, Z^{a_2})$ into $L$ yields (see equation (2.9) in [3])

$$L' = P_r \dot{Q}^r - H'(Q^r, P_r) - Z^{a_2} f_{a_2}(Q^r, P_r), \quad (2.6)$$

which again has the structure of the Lagrangian (2.1) with $Z^{a_2}$ as new Lagrange multipliers. The algorithmic procedure is thus established and the next step will be the elimination of another set of variables, by plugging the new constraints $f_{a_2} = 0$ into $L'$.

To summarize, at each stage of the algorithm, we plug the constraints into the action to get a reduced Lagrangian, and diagonalize (by means of a Darboux transformation) its associated symplectic form. If this form is degenerate we obtain as a byproduct new constraints that can be plugged again into the reduced Lagrangian. The procedure continues through a new diagonalization and ends up when no variables of the type $Z^a$ appear in the formalism. At this point we get a non-degenerate symplectic structure that actually represents the Dirac brackets in the reduced phase space.

Continuing with the assumption that all constraints are effective the following results were obtained in [3]:

(i) The F-J method is completely equivalent to the Dirac approach for gauge theories.

(ii) The Darboux transformation (2.3) is the projection on the constraint surface of a canonical transformation in the whole phase space that allow for a canonical representation of this surface.

(iii) The relations $Z^{a_1} - f^{a_1}(Q^r, P_r) = 0$ play the role of second class constraints with respect to a subset of the primary first class constraints that from now on become second class.

(iv) The $Z^{a_2}$ variables are canonical variables with respect to the remaining set of primary first class constraints and can be considered as Lagrange multipliers in (2.4). As it happens with the original Lagrange multipliers, these $Z^{a_2}$ variables are bound to become either pure gauge arbitrary variables or functions of the physical variables. In any case they will not play any role in the description of the reduced –physical– phase space.

### 3 Ineffective constraints and the Faddeev-Jackiw method

The results of the previous section rely on the condition that all constraints are effective at each step of the algorithm. Now let us drop this condition. At this point, the constraints in (2.3) may have appeared in $\partial H/\partial Z^a = 0$ as ineffective. Then, the substitution of $Z^{a_1}$ by $f^{a_1}$ in (2.4) may produce the disappearance of some of the
$Z^{a_2}$ variables. And more: If some of the constraints $f_{a_2} = 0$ are ineffective, it is not legitimate to interpret them as primary constraints associated to the new Lagrange multipliers $Z^{a_2}$. All these circumstances may lead to a loss of dynamical information—in the form of constraints or equations of motion—in the F-J method.

In order to deal with these problems we need to develop a new perspective of the reduction process that does not depend explicitly on the canonical representation of the constraint surface. We will use configuration-velocity space methods and discuss the validity of the F-J method by studying the Lagrangian equations of motion. As a byproduct we will obtain a new proof of the equivalence between the Dirac and F-J methods when no ineffective constraints appear.

The key idea is to analyze the two processes of reduction implied in the F-J method, namely the reduction from the Lagrangian (2.4) to (2.6) and the reductions of the new constraints $f_{a_2} = 0$ in (2.6). The basic difference between the analysis of the present section and the previous one is that we now allow the constraints $Z^{a_1} - f^{a_1} = 0$ and $f_{a_2} = 0$ to be ineffective.

### 3.1 General reduction for holonomic constraints

Consider a configuration space locally described by the coordinates $x_a, q_j$. Suppose a general Lagrangian of the form

$$L(x_a, q_j; \dot{x}_a, \dot{q}_j),$$

and, for some regular functions $g_a(q)$, let $x_a - g_a(q) = 0$ define a surface in this configuration space.

The pull-back of $L$ to this surface will define the reduced Lagrangian $L_{\text{red}}$ as

$$L_{\text{red}}(q; \dot{q}) = L(g_a(q), q_j; \partial g_a \partial q_i \dot{q}_i, \dot{q}_j),$$

and there is the following relationship between the Euler-Lagrange derivatives for these two Lagrangians:

$$[L_{\text{red}}]_{q_j} = [L]_{q_j} |_{x=g(q)} + \frac{\partial g_a}{\partial q_j} [L]_{x_a} |_{x=g(q)}. \tag{3.1}$$

In the next subsections we will make use repeatedly of this result.

### 3.2 Type 1 problems: elimination of $Z^{a_1}$

Now let us substitute $Z^{a_1} = f^{a_1}(Q^r, P_r, Z^{a_2})$ from (2.5), into (2.4). This defines the partially reduced Lagrangian $L'$,

$$L'(Q^r, P_r, \dot{Q}^r, \dot{P}_r; Z^{a_2}) = L(Q^r, P_r, \dot{Q}^r, \dot{P}_r; f^{a_1}(Q^r, P_r, Z^{a_2}), Z^{a_2}). \tag{3.2}$$

whose equations of motion satisfy, according to (3.1),

$$[L']_{Z^{a_2}} = [L]_{Z^{a_2}} |_{Z^{a_1}=f^{a_1}} + \frac{\partial f^{a_1}}{\partial Z^{a_2}} [L]_{Z^{a_1}} |_{Z^{a_1}=f^{a_1}}.$$
Now, according to (2.5), and noticing that the equations of motion for the \( Z \) variables are just extremity conditions, we have

\[
\tag{3.3}
 f_{a_2} = 0 \implies [L']Z^{a_2} = 0.
\]

The implication is not a two way implication because some of the variables \( Z^{a_2} \) may disappear from \( L' \). This may happen if some of the relations \( Z^{a_1} - f^{a_1} = 0 \), which are effective constraints, have originally appeared in (2.5) in an ineffective form. Let us produce an example. Consider \( L = PQ + Z_2(Z_1 - f_1(Q,P))^2 + (Z_1 - f_1(Q,P))f_2(Q,P) \).

Application of (2.5) gives \( Z_1 - f_1(Q,P) = 0 \) and \( f_2(Q,P) = 0 \), but when we substitute \( f_1(Q,P) \) for \( Z_1 \) in \( L' \) we get \( L' = PQ \). The variable \( Z_2 \) disappears and the constraint \( f_2(Q,P) = 0 \) is not retrievable from \( L' \).

This analysis implies that we can only guarantee that

\[
[L]Z^a = 0 \implies [L']Z^{a_2} = 0, \quad Z^{a_1} - f^{a_1}(Q,P;Z^{a_2}) = 0.
\]

Denoting the rest of the variables, \( Q \) and \( P \), as \( X \), we have, using again (3.1),

\[
\tag{3.4}
[L']_X = [L]_X \bigg|_{Z^{a_1} = f^{a_1}} + \frac{\partial f^{a_1}}{\partial X}[L]_{Z^{a_1} = f^{a_1}}.
\]

On the surface defined by \( f_{a_2} = 0 \) the last term vanishes, and the rest is

\[
[L']_X \bigg|_{f_{a_2} = 0} = [L]_X \bigg|_{[L]Z^a = 0} = 0.
\]

Therefore we arrive at

\[
\tag{3.5}
\begin{cases}
[L]_X = 0 \\
[L]Z^a = 0
\end{cases} \quad \implies \quad \begin{cases}
[L']_X = 0 \\
[L']Z^{a_2} = 0 \\
Z^{a_1} - f^{a_1}(Q,P;Z^{a_2}) = 0
\end{cases}.
\]

This one-way-only implication is the type 1 problem with the F-J method. The equivalence only holds when (3.3) is indeed an equivalence, that is, when

\[
\tag{3.6}
f_{a_2} = 0 \iff [L']Z^{a_2} = 0.
\]

This equivalence is guaranteed if the \( Z^{a_1} \) type variables are auxiliary variables \(^2\) \( i.e., [L]Z^{a_1} = 0 \iff Z^{a_1} - f^{a_1} = 0 \).

We conclude that there is a possible loss of dynamical information when the original Lagrangian \( L \) is partially reduced to \( L' \) by plugging into it the relations \( Z^{a_1} = f^{a_1}(P,Q;Z^{a_2}) \). This loss of information originates in the one-way implication displayed in equation (3.3). This non-equivalence has its roots in the fact that some of these relations may appear within ineffective constraints.

\(^2\)Auxiliary variables are a set of variables that can be obtained (as a set) by using their own equations of motion in terms of the rest of the variables that describe the system. For details see [10].
3.3 Type 2 problems: reduction to the surface \( f_{a_2}(Q, P) = 0 \)

There is a second source of problems, also related to ineffectiveness, that haunts the F-J method. Suppose we still have an equivalence in (3.6) and let us complete the reduction of the Lagrangian (2.4) by plugging the constraints \( f_{a_2}(Q, P) = 0 \) into (2.6). Here we consider that all these constraints are independent. In case they are not, their number will be reduced accordingly and so will be the number of \( Z^{a_2} \) variables appearing in \( L' \).

As we already note in (2.6), \( L' \) takes the form

\[
L'(Q^r, P_r, \dot{Q}^r, \dot{P}_r; Z^{a_2}) = P_r\dot{Q}^r - H'(Q^r, P_r) - Z^{a_2} f_{a_2}(Q^r, P_r)
\]

that is, \( L' \) is at most linear in the variables \( Z^{a_2} \). Let us change the variables \( Q, P \) to variables \( y_m, x_{a_2} \) such that \( f_{a_2}(Q, P) = 0 \) \( \Leftrightarrow \) \( x_{a_2} = 0 \).

Notice that the equivalence (3.8), does not guarantee the effectiveness of \( f_{a_2}(Q, P) \); for instance it could be that \( f_{a_2}(Q, P) = (x_{a_2})^2 \). Now consider the further reduction of \( L' \), to the surface \( x_{a_2} = 0 \):

\[
L_R(y) = L'(x_{a_2} = 0, y_m, \dot{x}_{a_2} = 0, \dot{y}_m).
\]

Notice that \( Z^{a_2} \) disappears from \( L_R \). Applying (3.1), we have

\[
[L_R]_{y_m} = [L']_{y_m} \bigg|_{x_{a_2} = 0}.
\]

Let us now discuss separately the two cases we can find according to the effectiveness or ineffectiveness of the constraints.

A) Consider the case when the constraints \( f_{a_2}(P, Q) = 0 \) are truly effective: Without loss of generality we can take the variables \( x, y \) such that \( f_{a_2}(P, Q) = x_{a_2} \). In this case,

\[
[L_R]_{y_m} = [L']_{y_m} \big|_{x_{a_2} = 0} = [A]_{y_m} \big|_{x_{a_2} = 0}.
\]

On the other hand,

\[
[L']_{x_{a_2}} \big|_{x_{a_2} = 0} = [A]_{x_{a_2}} \big|_{x_{a_2} = 0} - Z^{a_2}.
\]

So we have,

\[
\begin{align*}
[L']_{Z^{a_2}} &= 0 \\
[L']_{y_m} &= 0 \\
[L']_{x_{a_2}} &= 0
\end{align*} \iff \begin{align*}
x_{a_2} &= 0 \\
[L_R]_{y_m} &= 0 \\
Z^{a_2} &= [A]_{x_{a_2}} \big|_{x=0}
\end{align*}.
\]

The number of \( x \)-type variables may be larger than the number of functions \( f_{a_2} \) because one of these functions being ineffective may kill more than one degree of freedom. This is the case for instance of the square of the norm of a vector in Euclidean space. For the sake of simplicity we use the same indices for the \( x \) variables and the \( f_{a_2} \) functions.
As it is argued in [3], the $Z^{a_2}$ variables are irrelevant because either they are gauge variables or they become determined through constraints as functions of the physical (gauge invariant) variables. In any case we can get rid of them and hence the equations

$$Z^{a_2} = [A]_{x_{a_2}} \bigg|_{x_{a_2}=0} = 0$$

(3.11)
can be ignored. The equations for the relevant set of $y$ variables (until subsequent reductions further cut down this set) are therefore $[L_R]_{y_m} = 0$.

This proves the correctness of this stage of the F-J reduction procedure as long as ineffective constraints do not appear in the formalism. If at each stage, no ineffective constraints appear, we have produced a new proof of the correctness of the F-J method, that is, its equivalent to the canonical Lagrangian analysis which in turn is equivalent to the Dirac’s method.

B) Consider, for the sake of simplicity, that all the constraints $f_{a_2}(Q, P) = 0$ are ineffective. In such case (3.10) is modified to

$$\begin{cases}
[\mathcal{L}]_{Z^{a_2}} = 0 \\
[\mathcal{L}]_{y_m} = 0 \\
[\mathcal{L}]_{x_{a_2}} = 0
\end{cases} \iff \begin{cases}
x_{a_2} = 0 \\
[L_R]_{y_m} = 0 \\
[A]_{x_{a_2}} \big|_{x_{a_2}=0} = 0
\end{cases}.$$

(3.12)

Notice that the reduced equations of motion, $[L_R]_{y_m} = 0$, are potentially incomplete because of the presence of new equations for the $y$ variables: those given by

$$[A]_{x_{a_2}} \big|_{x_{a_2}=0} = 0.$$

(3.13)

This is the type 2 problem with F-J reduction method. Only when these new equations are empty or do not add new information to the reduced equations derived from $L_R$ (as in the case where at some stage of the F-J algorithm the set $f_{a_2}$ is empty), can we say that the Faddeev–Jackiw method still works. Otherwise, there is a loss of dynamical information, for the equations of motion derived from the reduced Lagrangian $L_R$ are not the whole set of equations of motion for the reduced variables. Whether this loss of information consists in the loss of some constraints or of true equations of motion (that is, equations with velocities in the lhs) will be explored in the next subsection.

In a general case, when some of the constraints $f_{a_2} = 0$ are effective and some are ineffective, both types of equations, (3.11) and (3.13), will appear, the first associated with the effective constraints and the second with the ineffective ones. The potential incompleteness of the F-J method comes in this case from this last type of equations.

### 3.4 Losing constraints and equations of motion in the type 2 problems

The previous results can be reformulated in a more transparent way by using a canonical representation for the variables describing the constraint surface. In order to simplify the notations let us suppose that all the constraints $f_{a_2}$ are ineffective and recall (3.8),

$$f_{a_2}(Q, P) = 0 \iff x_{a_2} = 0.$$
In order to cover the most general case, take for the coordinates $x_{a_2}, y_m$ the canonical form $x_{a_2} = \{Q^a, P_s, P_u\}$ and $y_m = \{Q^u, Q^t, P_t\}$ where coordinates and momenta with the same label are canonical pairs. In these new coordinates the Lagrangian (3.7) can be written as

$$L' = P_u \dot{Q}^u + P_s \dot{Q}^s + P_t \dot{Q}^t - H'(Q^u, P_u, Q^s, P_s, Q^t, P_t) - Z^a_2 f_{a_2}(Q^a, P_s, P_u, Q^u, Q^t, P_t),$$

where we have kept, for simplicity, the notations of (2.6) for all functions involved. Now

$$f_{a_2} = 0 \iff Q^s = P_s = P_u = 0.$$  \hfill (3.14)

The reduced Lagrangian (3.9) becomes

$$L_R = P_t \dot{Q}^t - H_R(Q^u, Q^t, P_t),$$

where $H_R = H'(Q^u, P_u = 0, Q^s = 0, P_s = 0, Q^t, P_t)$. As we know from the previous analysis this Lagrangian may not contain all the dynamical information of the reduced dynamics. Its equations of motion $[L_R]_{y_m} = 0$ are

$$\dot{Q}^t - \frac{\partial H_R}{\partial P_t} = 0, \quad -\dot{P}_t + \frac{\partial H_R}{\partial Q^t} = 0, \quad \frac{\partial H_R}{\partial Q^u} = 0,$$

where the last set of equations are constraints. The dynamical information loss is contained in the equations of motion (3.13) $[A]_{x_{a_2}} = 0$. Thanks to the canonical representation of the constraint surface, these equations take the simple form

$$\frac{\partial H'}{\partial P_s} \bigg|_{x_{a_2}=0} = 0, \quad -\frac{\partial H'}{\partial Q^s} \bigg|_{x_{a_2}=0} = 0, \quad \dot{Q}^u - \frac{\partial H'}{\partial P_u} \bigg|_{x_{a_2}=0} = 0.$$  \hfill (3.17)

Notice that the first two sets of equations in (3.17) are just constraints, maybe new ones, maybe not, whereas the last one contains only true equations of motion, which are all new at this level. This difference –possible new constraints versus true equations of motion– has its roots in the first and second class character of the surface $f_{a_2} = 0$, which is revealed after the effectivization of its defining constraints. Part of these effectivized constraints, $P_u = 0$, are first class and the rest, $Q^s = 0, P_s = 0$, second class. Then, as we see in (3.17), the possible loss of constraints comes from the sector of the second class effectivized constraints, whereas the loss of equations of motion comes from the sector of the first class effectivized constraints.

Summing up: There are two sources, both related to ineffective constraints, for incompleteness in the F-J method: the reduction of the $Z^{a_1}$ type variables and the reduction of the ineffective constraints among the set of functions $f_{a_2}$. Sometimes this incompleteness amounts to a loss of constraints, whereas in some other cases there is a loss of equations of motion. If the basis of the ideal of functions vanishing on the surface defined by the ineffective functions $f_{a_2} = 0$ contains first class constraints, then we conclude that there is a true loss of dynamical information in the form of equations of motion, i.e., involving velocities. Whether there is a real loss of information for the physical variables or not, cannot be decided until the F-J algorithm is completed. Our examples in section 5 will show cases where this loss is real.
4 Relation with Dirac’s conjecture

In this section we want to examine the failure of F-J reduction method in presence of ineffective constraints from another perspective. Here we will work in Dirac’s formalism in order to consider the gauge generators of the theory and how the process of gauge fixing is modified by the presence of ineffective constraints.

Constraint analysis in Dirac formalism has at least two conceptually different applications: the stabilization algorithm, on one side, and the construction of the gauge generators, on the other.

The gauge generators are made up of first class constraints in a chain that involves some arbitrary functions and their derivatives. Since we are working with the canonical Lagrangian, there is a role, too, for the Lagrange multipliers as new variables. In a given theory, the complete gauge generator contains as many arbitrary functions as there are independent gauge transformations in the theory. This number coincides with the number of primary first class constraints.

Notice that these two aspects of the constraint analysis become complementary with regard to the determination of the correct number of (physical) degrees of freedom of our theory. The determination of the constraint surface is a first step in this direction. Next we need to know the gauge generator of the theory in order to eliminate further the spurious degrees of freedom associated with the gauge symmetries.

All the constraints appearing in $G$ are first class (this fact was first proved by Dirac [2]), but nothing prevents that some of them be ineffective. Even more, there are examples [5] where the effectivization of some constraints involved in $G$ is second class!

The presence of ineffective constraints at the secondary, or tertiary, etc., level of the constraint algorithm is intrinsic to the dynamical system under consideration. As far as the stabilization algorithm is concerned we can, at each step, make these ineffective constraints effective by a wise choice of a new set of functions that generate the ideal of functions vanishing at the constraint surface that has been determined so far. In this sense the stabilization algorithm is essentially unaffected by the presence of ineffective constraints. Certainly, this “effectivization” of constraints is the standard way to proceed in Dirac’s method [9] [12] but it is not mandatory. In fact, in appendix A we sketch how the stabilization algorithm works in the presence of ineffective constraints.

With regard to the construction of the gauge generator, this “effectivization” of constraints is not allowed\(^4\). Let us elaborate on this important point. As we show in a generator $G$ of a Noether symmetry depending on $q, p, t, \lambda, \dot{\lambda}, ...$, where $\lambda$ denotes the set of Lagrange multipliers associated with the primary constraints, is characterized by the property

$$\frac{D G}{D t} + \{ G, H_D \} = pc, \quad \frac{D}{D t} = \frac{\partial}{\partial t} + \dot{\lambda} \frac{\partial}{\partial \lambda} + \ddot{\lambda} \frac{\partial}{\partial \dot{\lambda}} + ... ,$$  \(4.1\)

where $H_D = H_c + \lambda^{m_1} \phi_{m_1}$, $H_c$ is the canonical Hamiltonian and $\phi_{m_1}$ are the primary constraints, and $pc$ stands for a linear combination of primary constraints. The Noether

\(^4\)Notice that every ineffective constraint is first class, regardless of whether its “effectivization” is first or second class.
transformations for the canonical Lagrangian $L_c = P_i \dot{Q}^i - H_D$ are defined by

$$\delta Q = \{Q, G\}, \quad \delta P = \{P, G\}$$

and $\delta \lambda^\mu$ is defined so that $\delta L_c$ becomes a total time derivative. In some cases a pure gauge generator depending only on $q, p, t$ may be constructed and the condition (4.1) splits into (see also [14])

$$\frac{\partial G}{\partial t} + \{G, H_c\} = pc, \quad \{G, pc\} = pc. \quad (4.2)$$

These last conditions are, in general, more restrictive, and as a consequence a solution $G$ of (4.2) may not exist while a solution $G$ to (4.1) can still be constructed. With the notation $\phi_{m_k}$ for the $k$-ary (primary, secondary...) first class constraints we can write the gauge generator in the form

$$G = \sum \mu^{m_k}(q, p, t, \lambda, \dot{\lambda}, ...) \phi_{m_k}(q, p) \quad (4.3)$$

and recover the general results of [15] under weaker assumptions. For details see [13].

As is well established, [15], [10], the standard counting of the degrees of freedom for systems that do not exhibit ineffective constraints is as follows: If the dimension of the original phase space is $2N$, the number of first class constraints is $m$, and the corresponding (even) number of second class constraints is $2s$, the total number of degrees of freedom is $2F = 2N - 2m - 2s$.

If there are ineffective constraints in the gauge generators, then we must not introduce any gauge fixing constraints for them. This is so because these constraints do not generate any motion in the constraint surface and therefore they do not transform the dynamical trajectories. This means that the gauge fixing constraints must only be included for the secondary, tertiary,... first class effective constraints in $G$. For details on the gauge fixing procedure, see [16]. The counting of degrees of freedom is obviously affected by this circumstance, and for instance, the final number of physical degrees of freedom may be odd, as it happens in some of the examples in the next section.

Let us now make contact with Dirac’s conjecture. In a modern interpretation, this conjecture says that it is always possible to enlarge the Dirac Hamiltonian (which already contains the primary first class constraints with their Lagrange multipliers) with the addition of all the remaining, secondary, tertiary, etc. first class constraints, and their new associated Lagrange multipliers, without any change of the dynamics of the theory and its physical interpretation. This enlarged Hamiltonian is known as the Extended Hamiltonian. It can be proved that if all constraints are effective, the Extended Hamiltonian and the Dirac Hamiltonian give equivalent results, and the reason is that we can always introduce a gauge fixing for both Hamiltonians that yield the same dynamics in the same reduced phase space [10]. The differences arise when there exist ineffective constraints. In such case, if the first class constraints that are added to the Dirac Hamiltonian to define the Extended one, are given in an effective representation, then there will be gauge fixings for all these first class constraints, even for the ones that come from the effectivization of ineffective ones. So we see that in this
case there is an “excess of gauge fixing” that makes the Extended theory inequivalent to the one described by the original Dirac Hamiltonian. We think it plausible, and all examples that we have studied support it, that there is a link between the failure of the Dirac conjecture and the failure of the F-J method to describe the correct reduced dynamics in such a way that

\[
\text{Dirac conjecture fails } \implies \text{F-J method fails,} \tag{4.4}
\]

but we are not able to prove this relation because of a lack of a general theory of the construction of gauge generators for ineffective constrained theories.

As a final comment let us mention that, using the antibracket cohomology [17] our results can be reformulated as follows: In the case of effective constraints it is proved in [10] that the BRST cohomology at ghost number zero consist of all the observables of a given physical theory. On the other hand, the BRST cohomology is invariant with respect to the elimination of the auxiliary variables (in our notations \(Z^{a_1}\) in the first step of reduction and \(Z^{a_2}, x^{a_2}\) in the second step) [17] and therefore we can conclude that the reduction process produces the same results, that is, the reduced (by eliminating the auxiliary variables) theory is completely equivalent with the original theory. But in the case of ineffective constraints these theorems no longer apply because some of the \(Z^{a_1}, Z^{a_2}\) type variables are not auxiliary variables. The loss of dynamical information produced in the reduction process by this fact was analyzed in section 3. It will be of interest, specially for field theories, to analyze this loss of dynamical information from a cohomological perspective. In this respect, it may be helpful to analyze how the symmetries of the theory are altered in the reduction process.

5 Examples

In order to exhibit in a transparent way the inequivalence results obtained in section 3 and their relation with the failure of the Dirac conjecture, we choose some simple examples. First, we choose a model that by some ineffective \(Z^{a_1}\)-type variables lose a first class constraint upon reduction. By constructing the gauge generator we show that the Dirac conjecture is violated. Then we analyze another model that presents two secondary ineffective constraints (effectivized second class). The model is such that the equations (3.13) give a new constraint. Only when this new constraint is not considered in the model does the F-J approach give the correct reduced dynamics, because then equations (3.13) are empty. By an explicit construction of the gauge generator we show that the F-J approach lose an equation of motion. By an explicit construction of the gauge generator we show that the system violates the Dirac conjecture [17].

\footnote{In this section we use only subindex notation for clarity in exposition.}
5.1 Type 1 problems

In order to exhibit a simple case where the substitution of the $Z^{a_1}$ variables (the first step in the F-J reduction) produces a loss of information, let us consider the canonical Lagrangian

$$L_c = \sum_{i=1}^{4} \dot{q}_i p_i - \sum_{i=1}^{4} \frac{1}{2} p_i^2 - p_3 q_2 - q_1 q_2^2 - \lambda_1 p_1 - \lambda_2 p_2.$$  \hfill (5.1)

Here $\lambda_1, \lambda_2$ are Lagrange multipliers.

In Dirac’s constraint analysis, the Dirac Hamiltonian is

$$H_D = \sum_{i=1}^{4} \frac{1}{2} p_i^2 + p_3 q_2 + q_1 q_2^2 + \lambda_1 p_1 + \lambda_2 p_2.$$  \hfill (5.2)

The stabilization of the primary constraints $p_1 = 0$ and $p_2 = 0$ gives $q_2^2 = 0$ and $p_3 = 0$ as secondary constraints. A subsequent stabilization determines $\lambda_2 = 0$. The only Lagrange multiplier that remains arbitrary is $\lambda_1$. Reducing the second class constraints and introducing a gauge fixing for the constraint $p_1 = 0$ (for instance, $q_1 = 0$), we find that the final reduced dynamics for the physical degrees of freedom is given by the equations of motion

$$\dot{p}_4 = 0, \quad \dot{q}_4 = p_4, \quad \dot{q}_3 = 0.$$  \hfill (5.3)

Notice that the total number of degrees of freedom is odd, i.e., $q_3(0), q_4(0), p_4(0)$.

The gauge generator for this theory is

$$G = \epsilon (q_2 p_3 + q_1 q_2^2 + \lambda_1 p_1 + \lambda_2 p_2) + \dot{\epsilon} (q_2 p_2 - q_1 p_1).$$  \hfill (5.4)

From the structure of this generator we observe that:

(a) The number of arbitrary parameters in $G$ is equal to the number of primary first class constraints.

(b) Only the piece containing $p_1$ generates true gauge transformations on the constraint surface, all other terms are ineffective (in fact, under the Lagrangian equations of motion, $\lambda_2$ is the time derivative $\dot{q}_2$ of the constraint $q_2$).

(c) The secondary first class constraint $p_3 = 0$ does not generate a gauge transformation on the constraint surface, that is, the Dirac conjecture is violated.

Because the Dirac conjecture fails the Extended Hamiltonian formalism also fails. To see this let us construct the Extended Hamiltonian

$$H_E = \sum_{i=1}^{4} \frac{1}{2} p_i^2 + p_3 q_2 + q_1 q_2^2 + \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 q_2 + \lambda_4 p_3.$$  \hfill (5.5)

Upon eliminating the second class constraints $q_2, p_2$ and using the corresponding (trivial) Dirac bracket we obtain

$$H'_E = \frac{1}{2} p_4^2 + \frac{1}{2} p_3^2 + \lambda_1 p_1 + \lambda_4 p_3.$$  \hfill (5.6)
Now to fix the dynamics we need two gauge fixing conditions. Take for example \( q_1 = 0, q_3 = 0 \); the result is free particle motion in the space \((q_4, p_4)\). This result is different from the correct Dirac analysis and also different from the one given by the F-J method as we will see.

Applying the F-J method we eliminate the primary constraints \( p_1, p_2 \) to obtain
\[
L'_c = \dot{q}_3 p_3 + \dot{q}_4 p_4 - \frac{1}{2} p_4^2 - \frac{1}{2} p_3^2 - p_3 q_2 - q_1 q_2^2, \tag{5.7}
\]
and the variables \( q_1, q_2 \) are now \( Z \)-type variables. The equations of motion associated with these variables are \( q_2^2 = 0 \) and \( p_3 + 2 q_1 q_2 = 0 \) respectively. The first relation allows for the isolation of the variable \( q_2 \) as \( q_2 = 0 \), which is a \( Z_{a_1} \)-type variable. The second one implies \( p_3 = 0 \), which is a constraint of the type \( f_{a_2} \). The F-J reduction procedure first dictates to plug \( q_2 = 0 \) into the canonical Lagrangian. We get
\[
L_R = \dot{q}_3 p_3 + \dot{q}_4 p_4 - \frac{1}{2} p_4^2 - \frac{1}{2} p_3^2. \tag{5.8}
\]
This Lagrangian has no dependence whatsoever on \( q_1 \), and as a consequence the constraint \( p_3 = 0 \) can not be obtained from \( L_R \). As expected, the reason for this information loss is the ineffective character of the constraint \( q_2^2 = 0 \). The reduced Lagrangian \( L_R \) is regular and no new constraints arise from its dynamics. We can conclude that the violation of the Dirac conjecture is related to the failure of the F-J reduction process to give the correct reduced dynamics.

### 5.2 Type 2 problems: some constraints are missing

In this example we analyze a system with two primary first class constraints that have a tertiary constraint in the Dirac method that is missing in the F-J approach. Here we want to illustrate the failure of the F-J method when the constraints \( \{3.8\} \) contain a set of second class effective constraints among themselves at some level of the reduction process. Consider the following Lagrangian:
\[
L_c = \sum_{i=1}^{n} \dot{q}_i p_i - H_c(q, p) - \lambda_1 p_1 - \lambda_2 p_2, \tag{5.9}
\]
where
\[
H_c(q, p) := H_R(q_r, p_r) + q_1 p_3^2 + q_2 q_3^2 + F(q_r, p_r) p_3, \quad i = 1,...,n, \quad r \geq 4. \tag{5.10}
\]
\( H_R \) and \( F \) are functions that we do not need to specify. The variables \( q_1, q_2 \) are considered, by construction, as \( Z_{a_2} \)-type variables.

The application of the Dirac method is straightforward: The two primary first class constraints \( p_1 = 0, p_2 = 0 \) produce the ineffective constraints \( p_3^2 = 0, q_3^2 = 0 \) which define a pair of second class constraints upon effectivization, namely \( p_3 = 0, q_3 = 0 \). A new stabilization of these constraints yields \( F(q_r, p_r) = 0 \). This new constraint can
give rise to other new constraints upon the application of the stabilization algorithm. Suppose for definiteness that there are no new constraints, that is,

\[ \{ F, H_R \} = 0. \] (5.11)

In that case the gauge generator is

\[
G = \left( \frac{q_1}{q_2} \epsilon_2 - \frac{1}{4q_2} \epsilon_1 + \frac{3\lambda_1}{2q_2} \epsilon_2 + \frac{\lambda_1}{2q_2} \epsilon_2 \right) + q_1 \epsilon_1 + \lambda_1 \epsilon_1 + 2q_1(2q_1 \epsilon_2 - \dot{\epsilon}_1) + \frac{\lambda_2}{4q_2^2}(-4q_1 \epsilon_2 + \dot{\epsilon}_1 - 2\lambda_1 \epsilon_2) )p_1 \\
+ \left( \frac{1}{2} \epsilon_2 + 3q_2 \epsilon_1 + \lambda_2 \epsilon_1 - 4q_2 q_1 \epsilon_2 \right)p_2 \\
+ \left( \frac{q_1}{q_2} \epsilon_2 - \frac{1}{4q_2} \epsilon_1 + \frac{\lambda_1}{2q_2} \epsilon_2 + q_1 \epsilon_1 \right)p_3^2 \\
+ \left( \frac{1}{2} \epsilon_2 + q_2 \epsilon_1 \right)q_3^2 + (2q_1 \epsilon_2 - \dot{\epsilon}_1)q_3 p_3 \\
- \dot{\epsilon}_2 F q_3 + \epsilon_2 F^2 + \epsilon_1 F p_3.
\]

This gauge generator has an effective action only on the primary first class constraints. The rest of the action is ineffective and as a consequence does not produce gauge transformations on the constraint surface. This means that the only gauge fixings needed are for the primary first class constraints \( p_1 = 0, p_2 = 0 \). Then we conclude that the first class constraint \( F = 0 \) does not need any gauge fixing. The system violates the Dirac conjecture and the extended formalism fails.

On the other hand, by noting that, after elimination of the constraints \( p_1 = 0 \) and \( p_2 = 0 \), the variables \( q_1 \) and \( q_2 \) are \( Z \)-type (see (2.6)), the F-J method produces the reduced Lagrangian

\[
L_R = p_r \dot{q}_r - H_R(q_r, p_r).
\] (5.12)

There are no new constraints and the algorithm stops here. It is clear that the constraint that appear in the Dirac formalism, namely

\[ F(q_r, p_r) = 0, \] (5.13)

is not present in the F-J approach. Note that this constraint come from the equations of motion (3.13) that the F-J approach is not able to produce. As we expected from the general analysis given in the previous sections, the two procedures yield very different outcomes for the description of reduced phase space. The two approaches coincide only if \( F = 0 \) because the equations (3.13) are in this case empty.

### 5.3 Type 2 problems: some equations of motion are missing

This example is designed to illustrate the failure of the F-J method in the case when the constraints (3.8) contain, upon effectivization, only first class constraints. It illustrates
also the case when the reduced phase space may have an odd number of degrees of freedom. Consider the Lagrangian

\[ L_c = \sum_{i=1}^{n} p_i \dot{q}_i - H_c(q, p) - \lambda p_2, \]  

(5.14)

where

\[ H_c(q, p) := H_R(q_r, p_r) + q_2 p_1^2 + F(q_1, q_r, p_r) p_1, \quad i = 1 \ldots n, \quad r \geq 3, \]  

(5.15)

where \( F \) and \( H_R \) are functions that we do not need to specify.

In the Dirac analysis, the theory contains a primary first class constraint \( p_2 = 0 \), which upon stabilization gives rise to a new ineffective constraint \( p_1^2 = 0 \). The gauge generator

\[ G = (\epsilon - 2 \epsilon \frac{\partial F}{\partial q_1}) p_2 + \dot{\epsilon} p_1^2 \]  

(5.16)

contains an ineffective piece that does not need a gauge fixing. The model violates the Dirac conjecture and presents a weakly (that is, only on shell) gauge invariant observable \( q_1 \). As we expected from the general analysis, the F-J procedure is unable to reproduce this result because it considers only the strong (on and off shell) gauge invariant observables. The equations of motion that will be lost in F-J method, (3.13), are

\[ \dot{q}_1 - F(q_1, q_r, p_r) = 0. \]  

(5.17)

The system presents a phase space with an odd number of degrees of freedom.

Now let us analyze the reduced dynamics that result from the F-J approach. A direct application of the F-J ideas gives rise to the reduced Lagrangian

\[ L_R = \dot{q}_r p_r - H_R(q_r, p_r). \]  

(5.18)

As expected the F-J analysis lose the equations (5.17). We know from our general analysis that the equation for \( q_2 \) can be eliminated via a gauge fixing procedure, i.e., the coordinate \( q_2 \) is a pure gauge variable.

6 Conclusions

In this paper we have studied in detail the F-J method for gauge theories in the presence of ineffective constraints. We have singled out the two different sources (type 1 and type 2 problems) of incompleteness that the F-J reduction method may suffer in this case. As a byproduct of this analysis we obtained a new proof of the equivalence between the F-J reduction algorithm and that of Dirac when ineffective constraints are not present. This new proof is based on Lagrangian methods.

The type 1 problems produce the loss of some constraints. The type 2 problems may be split into two cases: the possible loss of constraints and the loss of equations of motion. Our analysis allows one to identify when the type 2 problems will lead to
the first or the second case. The loss of constraints is associated with sets of ineffective constraints of the type $f_{a2}$ [2.3], that become second class upon effectivization, whereas the loss of equations of motion is associated with first class constraints.

We give examples of every situation and we show in a specific case (Section 5.1) that the Dirac method, the F-J approach, and the Extended Dirac method may give three different dynamics. The correct one is, of course, the Dirac dynamics, which is always equivalent to the Lagrangian formulation.

The structure of the gauge generators and the gauge fixing procedure turns out to be one of the keys for an understanding of the source of problems originated by the presence of ineffective constraints. The discussion of these issues is neatly illustrated in the examples. The example in Section 5.3 has an odd number of degrees of freedom, completely compatible with a canonical formulation. The presence of ineffective constraints in the gauge generator makes the gauge fixing procedure different from the standard case, for ineffective constraints do not require any gauge fixing. This is exactly the cause of failure of the Dirac conjecture when the Extended Hamiltonian is built by using the effectivized form of the constraints. We then suggest that the failure of Dirac’s conjecture always implies the failure, in the sense of incompleteness, of the F-J method.

Finally, a word is in order concerning the treatment of ineffective constraints in the framework of the stabilization algorithm for the standard Dirac procedure. We have devoted appendix A to showing that the stabilization algorithm works perfectly well in the presence of ineffective constraints. The discussion of these issues is neatly illustrated in the examples. The example in Section 5.3 has an odd number of degrees of freedom, completely compatible with a canonical formulation. The presence of ineffective constraints in the gauge generator makes the gauge fixing procedure different from the standard case, for ineffective constraints do not require any gauge fixing. This is exactly the cause of failure of the Dirac conjecture when the Extended Hamiltonian is built by using the effectivized form of the constraints. We then suggest that the failure of Dirac’s conjecture always implies the failure, in the sense of incompleteness, of the F-J method.

Appendix A: Ineffective constraints and Dirac’s method

In this appendix we show that the Dirac method can be applied when some constraints are ineffective. We can choose either to stabilize the ineffective constraints or the effectivized ones. Consider the simplest case we can think of an ineffective constraint. Let $f = \phi^2$ be such a constraint, which represents the same surface –the points where $f$ vanishes– as $\phi$. We represent this surface by the notation

$$f \simeq 0,$$

or, equivalently

$$\phi \simeq 0.$$

Here we can be working either in the tangent space or in the cotangent space (phase space) of a constrained dynamical system. Let us assume also that $\phi$ is indeed effective. Let us suppose that we are performing a stabilization algorithm for some dynamics.
defined by the vector field $X$. Notice that

$$X(f) = 2\phi X(\phi) \simeq 0,$$

so one may be tempted to claim that the stabilization algorithm has finished, for the action of $X$ on $f$ vanishes in the constraint surface. But this is incorrect, as one can check by going to the next order. In fact, the requirement

$$X^2(f) = 2(X(\phi))^2 + 2\phi X^2(\phi) \simeq 0,$$

enforces the new constraint

$$(X(\phi))^2 \simeq 0,$$

which is again ineffective but defines the same surface as

$$X(\phi) \simeq 0.$$ 

Thus we notice that, in this case, the stabilization of an ineffective constraint does not stops at first order, even though the first order stabilization does not introduce new constraints. Moreover, we eventually get the same new restriction, that is, $X(\phi) \simeq 0$, as the stabilization of an effective constraint, $\phi$, gives at first order.

To understand why it is so, we should consider the meaning of the stabilization algorithm. The dynamics generated by the vector field $X$ defines trajectories

$$x(t) = e^{tX} x,$$

where $x(0) = x$. Tangency of these trajectories to the surface defined by $f = 0$ means that we must require $f(x(t)) = 0$ for any $x = x(0)$ such that $f(x) = 0$. But

$$f(x(t)) = (e^{tX} f)(x) = f(x) + t(Xf)(x) + \frac{1}{2} t^2(X^2 f)(x) + \ldots,$$

and we get an infinite set of requirements,

$$(Xf)(x) = 0, \quad (X^2 f)(x) = 0, \ldots.$$ 

In general only a few of these terms will introduce new restrictions. Notice that when $f$ is ineffective, then the fact that the first order requirement, $(Xf)(x) = 0$, is automatically satisfied does not imply that the second order, $(X^2 f)(x) = 0$, is satisfied.

In conclusion, when dealing with ineffective constraints the stabilization algorithm does not finish at the level where we find no new restrictions. We must proceed further until we are sure that all the tangency conditions have emerged.

According to these remarks, there are two ways to deal with ineffective constraints within the framework of Dirac’s method. Either we proceed through the lines sketched above or we can take effective constraints at any stage of the stabilization algorithm to represent the constraint surface. This second method is the one applied in the geometrization of Dirac algorithm [12]. Geometrically, the relevant information is the constraint surface and not the specific determination of the functions one uses to describe it. Both ways to realize the stabilization procedure are equivalent. The second is advantageous from the algorithmic point of view, while the first is more suitable if one wants to construct, for instance, the generators of the gauge transformations [5].
Acknowledgements

We would like to thank Larry Shepley for a careful reading of the manuscript. J.M.P. acknowledges support by the CICIT (contract numbers AEN95-0590 and GRQ93-1047). J.A.G. is supported by CONACyT postgraduate fellowship and also thanks the Departament d’Estructura i Constituents de la Matèria at the Universitat de Barcelona for its hospitality.

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