MIRKOVIC-VILONEN CYCLES AND POLYTOPES FOR A SYMMETRIC PAIR

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ABSTRACT. Let $G$ be a connected, simply-connected, and almost simple algebraic group, and let $\sigma$ be a Dynkin automorphism on $G$. Then $(G, G^\sigma)$ is a symmetric pair. In this paper, we get a bijection between the set of $\sigma$-invariant MV cycles (polytopes) for $G$ and the set of MV cycles (polytopes) for $G^\sigma$, which is the fixed point subgroup of $G$; moreover, this bijection can be restricted to the set of MV cycles (polytopes) in irreducible representations. As an application, we obtain a new proof of the twining character formula.

1. Introduction

Let $G$ be a connected semisimple algebraic group over $\mathbb{C}$, and let $G$ be the affine Grassmannian of $G$. Let $G_\lambda$ be the $G(\mathbb{C}[t])$-orbit on $G$ corresponding to a dominant coweight $\lambda$ on $G$. Let $IC_\lambda$ be the spherical perverse sheaf supported on $G_\lambda$. V. Ginzburg [G] and Mirković-Vilonen [MV] set up the geometric Satake correspondence, which says that the category of spherical perverse sheaves on $G$ is equivalent to the category of finite dimensional representations of the Langlands dual group $G^\vee$ of $G$; in particular, the irreducible representation $V(\lambda)$ of $G^\vee$ with highest weight $\lambda$ is identified with the cohomology group $H^*(G, IC_\lambda)$. Furthermore, Mirković and Vilonen [MV] discovered Mirković-Vilonen cycles which affords a natural basis of $V(\lambda)$.

In [A], Anderson studied the moment polytopes of Mirković-Vilonen cycles, which are called Mirković-Vilonen polytopes, and showed that these polytopes could be used to understand the combinatorics of representations of $G^\vee$. In [K1], Kamnitzer gave an explicit combinatorial description of the MV cycles and polytopes. He showed that canonical basis and MV cycles are governed by the same combinatorics, i.e MV cycles $\leftrightarrow$ MV polytopes $\leftrightarrow$ canonical basis, are bijections.

Let $\sigma$ be a nontrivial Dynkin automorphism of $G$. We have a Dynkin automorphism on $G^\vee$ induced from $\sigma$. Let $G^\sigma$ be the identity component of fixed point group of $\sigma$ on $G$. Let $\lambda$ be a $\sigma$-invariant dominant coweight of $G$, which can also be viewed as a dominant coweight of $G^\sigma$. Let $V(\lambda)$ be the irreducible representation of $G^\vee$ with highest weight $\lambda$. We have a natural action of $\sigma$ on $V(\lambda)$ induced from the action of the automorphism on $G^\vee$, which fixes the highest weight vector in $V(\lambda)$. For a $\sigma$-invariant coweight $\mu$ for $G$, $\sigma$ acts on the weight space $V_\mu(\lambda)$. The twining character is defined to be $\sum_{\sigma(\mu)=\mu} \text{trace}(\sigma|_{V_\mu(\lambda)}) e^\mu$. It is related to the character of the irreducible representation of $(G^\sigma)^\vee$ with highest weight $\lambda$ through the twining character formula, which is attributed to Jantzen [J] under the name of Jantzen theorem in [KLP]. Though there are many proofs in the literature (for example [J], [N], [KLP]), it seems that there is no satisfactory explanation for why Langlands dual appears in this formula.

In this paper, we consider the action of $\sigma$ on MV cycles and MV polytopes. The main result of the paper is to give an explicit bijection between $\sigma$-invariant MV cycles (polytopes) for $G$ to MV cycles (polytopes) for $G^\sigma$. In terms of polytopes, it sends $\sigma$-invariant MV polytopes $P$ for $G$, to $P^\sigma$, which is a MV polytope for $G^\sigma$. The bijection can be restricted to MV cycles (polytopes) in irreducible representation space.

In this paper, we also show that the automorphism on $G^\vee$ from Tannakian formalism is a Dynkin automorphism. On $V(\lambda)$, there are two actions of $\sigma$, where one is induced from $G^\vee$, and the other
one is induced from the action of \( \sigma \) on MV cycles. We show that both of them agree, then we get a new proof of twining character formula through geometric Satake correspondence.

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2. Dynkin automorphism

2.1. Notations. Let \( G \) be a connected, simply-connected and almost simple algebraic group of rank \( \ell \) over \( \mathbb{C} \). Let \( T \) be a maximal torus of \( G \) and let \( X^* = \text{Hom}(T, \mathbb{C}^\times) \), \( X_\ast = \text{Hom}(\mathbb{C}^\times, T) \) denote the weight and coweight lattices of \( T \). Then we have a natural perfect pairing \( \langle \cdot, \cdot \rangle : X_\ast \times X^* \to \mathbb{Z} \). Let \( W = N(T)/T \) denote the Weyl group.

Let \( I = \{1, \ldots, \ell\} \) denote vertices of the Dynkin diagram of \( G \). Let \( B \) be a Borel subgroup of \( G \) containing \( T \). Let \( \alpha_1, \alpha_2, \ldots, \alpha_\ell \) and \( \alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_\ell^\vee \) be simple roots and simple coroots of \( G \) with respect to \( B \), respectively. Then \( \alpha_{ij} = (\alpha_i^\vee, \alpha_j) \) is the entry of the Cartan matrix of \( G \). Note that \((X_\ast, X^*, \langle \cdot, \cdot \rangle, \alpha_i^\vee, \alpha_i; i \in I)\) is the root datum of \( G \). Let \( \lambda_1, \ldots, \lambda_\ell \in X_\ast \otimes \mathbb{R} \) be fundamental weights.

For \( i \in I \), let \( x_i: \mathbb{C} \to G \) and \( y_i: \mathbb{C} \to G \) be root homomorphisms (corresponding to \( \alpha_i \) and \( -\alpha_i \), respectively) which together with \( T \), \( B \) form a pinning of \( G \).

Let \( s_1, \ldots, s_\ell \in W \) be the set of simple reflections. Let \( w_0 \) be the longest element of \( W \), and let \( m \) be its length.

We use \( \geq \) for the usual partial order on \( X_\ast \), so that \( \mu \geq \nu \) if and only if \( \mu - \nu \) is a sum of positive coroots. More generally, for each \( w \in W \), we have the twisted partial order \( \geq_w \) on \( X_\ast \), where \( \mu \geq_w \nu \) if and only if \( w^{-1} \cdot \mu \geq w^{-1} \cdot \nu \).

A reduced word for an element \( w \in W \) is a sequence of indices \( i = (i_1, \ldots, i_k) \in I^k \) such that \( w = s_{i_1} s_{i_2} \cdots s_{i_k} \) is a reduced expression. In this paper, a reduced word will always mean a reduced word for \( w_0 \), where \( w_0 \) is the longest element in \( W \).

2.2. Group structure of \( G^\sigma \). Let \( \sigma: I \to I \) be a nontrivial bijection, satisfying \( a_{\sigma(i)\sigma(j)} = a_{ij} \) for all \( i, j \in I \). We assume that there are automorphisms \( \sigma: X^* \to X^* \) and \( \sigma: X_\ast \to X_\ast \) of \( \mathbb{Z} \)-modules satisfying \( \sigma(\alpha_i) = \alpha_{\sigma(i)} \) and \( \sigma(\alpha_i^\vee) = \alpha_{\sigma(i)}^\vee \) for any \( i \in I \). Then \( \sigma \) induces an automorphism \( \sigma: G \to G \) of algebraic groups, such that \( \sigma(x_i(a)) = x_{\sigma(i)}(a) \) and \( \sigma(y_i(a)) = y_{\sigma(i)}(a) \) (\( \forall i \in I \)). We call \( \sigma \) a Dynkin automorphism on \( G \). In particular, we have \( \sigma(B) = B \) and \( \sigma(T) = T \).

Let \( G^\sigma \) be the fixed point group of \( \sigma \) on \( G \), and let \( T^\sigma \) and \( B^\sigma \) be the fixed point groups of \( T \) and \( B \), respectively. Then \( G^\sigma \), \( B^\sigma \) and \( T^\sigma \) are connected, moreover \( G^\sigma \) is almost simple algebraic group, under our assumptions on \( G \), see \cite{SI}. We call \((G, G^\sigma)\) a symmetric pair.

We set \( X^*_\eta = \{ \lambda \in X_\ast | \sigma(\lambda) = \lambda \} \), and \( X^*_\eta \) is the character lattice of \( \lambda \). We have a perfect pairing \( X^*_\eta \times X^*_\eta \to \mathbb{Z} \) denoted again by \( \langle \cdot, \cdot \rangle \). Let \( \mathcal{I}_\eta \) be the set of \( \sigma \)-orbits on \( I \).

For any \( \eta \in \mathcal{I}_\eta \), let \( \alpha_{ij}^\eta = 2h \sum_{i \in I_\eta} \alpha_i^\vee \in X^*_\eta \), where \( h \) is the number of unordered pairs \((i, j)\) such that \( i, j \in \eta \), \( \alpha_i + \alpha_j \in \Phi \). Note that \( h = 1 \), if \( \eta = \{i, j\} \) and \( a_{ij} = -1 \); \( h = 0 \), otherwise. Let \( \theta: X^* \otimes \mathbb{R} \to X^*_\eta \otimes \mathbb{R} \) be the natural surjection induced from the perfect pairing \( \langle \cdot, \cdot \rangle : X_\ast \times X^* \to \mathbb{Z} \).
Set $\alpha_\eta = \theta(\alpha_i)$, and $\lambda_\eta = \frac{1}{k}\theta(\lambda_i)$, where $i$ is any element of $\eta$. We have the following proposition (see [KLP], [J]).

**Proposition 2.1.** $(X^*_n, X^*_n, \alpha'_n, \alpha_n)$ is a root datum of $G^\sigma$.

Define $x_\eta = \prod_{i \in \eta} x_i: \mathbb{C} \to G^\sigma$, by $x_\eta(a) = \prod_{i \in \eta} x_i(a)$, if $\eta$ has only one element, or $\forall \ i, j \in \eta$, with $i \neq j$, $a_{ij} = 0$; define $x_\eta: \mathbb{C} \to G^\sigma$, by $x_\eta(a) = x_i(a)x_j(2a)x_i(a)$, if $\eta = \{i, j\}$, $a_{ij} = -1$. We have the following lemma, see [LI].

**Lemma 2.2.** Let $x_1$, $x_2$ be two simple root subgroup homomorphisms of $G$ of type $A_2$ corresponding to $\alpha_1$ and $\alpha_2$. Then we have $x_1(a_1)x_2(a_2)x_1(a_3) = x_2(a_2a_3)x_1(a_1 + a_3)x_2(a_1a_3)$.

From this lemma, we see easily that $x_\eta$ is a group homomorphism. Similarly, we can define $y_\eta$, so that $x_\eta$ and $y_\eta$ are homomorphisms from $\mathbb{C}$ to $G^\sigma$. Since $tx_\eta(a)t^{-1} = x_\eta(\alpha_\eta(t)a)$, $x_\eta$ is a root subgroup homomorphism of $G^\sigma$ with root $\alpha_\eta$. We have

**Proposition 2.3.** $(\sigma^*, B^\sigma, x_\eta, y_\eta; \eta \in I_\eta)$ form a pinning of $G^\sigma$.

Clearly, $\sigma: G \to G$ induces an automorphism of $W$ denoted again by $\sigma$, satisfying $\sigma(s_i) = s_{\sigma(i)}$ for any $i \in I$. Let $W^\sigma = \{w \in W|\sigma(w) = w\}$. For any $\eta \in I_\eta$ we define $s_\eta \in W^\sigma$ to be the longest element in the subgroup of $W$ generated by $\{s_i; i \in \eta\}$. It is known that $W^\sigma$ is a Coxeter group on the generators $\{s_{\eta}; \eta \in I_\eta\}$. Any element $w \in W^\sigma$ can be restricted to $X^*_n$. Under this restriction, we can see that $W^\sigma$ is identified with the Weyl group of $G^\sigma$. For $w \in W^\sigma$, we denote by $\ell_\sigma(w)$ the length of $w$ in the Coxeter group $W^\sigma$.

3. **MV cycles and MV polytopes for the symmetric pair**

3.1. **Action of $\sigma$ on Affine Grassmannian.** Let $O = \mathbb{C}[t]$, and let $K$ be the quotient field of $O$. Let $G$ and $G_\sigma$ be affine Grassmannian of $G$ and $G^\sigma$ respectively. As the sets of rational points over $\mathbb{C}$, $G = G(K)/G(O)$, and $G_\sigma = G(K)^\sigma/G(O)^\sigma$. A coweight $\mu$ in $X_*$ gives a point in $G$, denoted by $\mathfrak{t}^\mu$. It is known that $\mathfrak{t}^\mu$ is a fixed points for the action of $T$ on $G$. In fact all the fixed points of $T$ are given in this way.

For a given dominant coweight $\lambda$, we set $G^\lambda = G(O) \cdot t^\lambda$. We have the decomposition $G = \bigsqcup_{\mu \in X^+_n} G^\lambda$, where $X^+_n$ is the set of dominant coweights.

Let $N$ be the unipotent radical of $B$. For $w \in W$, we set $N_w = wNw^{-1}$. For $w \in W$ and $\mu \in X_*$, define the semi-infinite cells by $S^\mu_w = N_w(K) \cdot \mathfrak{t}^\mu$. For simplicity, we set $S^\mu = S^\mu_w = N(K) \cdot \mathfrak{t}^\mu$. We have $G = \bigsqcup_{\mu \in X_\sigma} S^\mu$. The semi-infinite cells have the simple containment relation, $S^\mu_\sigma = \bigsqcup_{\nu \leq \mu} S^\nu_\sigma$. We see that if $S^\mu_\sigma \cap S^\nu_\sigma \neq \emptyset$, then $\nu \leq \mu$.

We have the closed embedding $i: G_\sigma \hookrightarrow G$. Since $\sigma(S^\lambda) = S^{\sigma(\lambda)}$, we have $G^\sigma = \bigsqcup_{\lambda \in X^2_\sigma} (S^\lambda)^\sigma$.

Set $U := \{g(t^{-1}) \in G(\mathbb{C}[t^{-1}])|g(0) = 1\}$. Then the fixed point set $U^\sigma = \{g(t^{-1}) \in G^\sigma(\mathbb{C}[t^{-1}])|g(0) = 1\}$. For a coweight $\lambda$, set $S(\lambda) := N(\mathbb{C}[t, t^{-1}]) \cap t^\lambda U t^{-\lambda}$ and $S_\sigma(\lambda) := N(\mathbb{C}[t, t^{-1}]) \cap t^\lambda U^\sigma t^{-\lambda}$. The following result should be well-known.

**Lemma 3.1.** Let $\lambda \in X_*$. Then the group $S(\lambda)$ acts simply-transitively on $S^\lambda$, i.e., $S(\lambda) \simeq S^\lambda$, with the map $g \mapsto g \cdot \mathfrak{t}^\lambda$.

**Proposition 3.2.** The fixed point subvariety of the action of $\sigma$ on $G$ is exactly identified with $G_\sigma$.

**Proof.** From Lemma 3.1, we are reduced to show $S(\lambda)^\sigma = S_\sigma(\lambda)$ for $\lambda \in X^*_\sigma$, and it is easy to see, since $S(\lambda)^\sigma = N(\mathbb{C}[t, t^{-1}])^\sigma \cap (t^\lambda U t^{-\lambda})^\sigma = N^\sigma(\mathbb{C}[t, t^{-1}]) \cap t^\lambda U^\sigma t^{-\lambda} = S_\sigma(\lambda)$.
Corollary 3.3. For $\lambda$ a $\sigma$-invariant, and $w$ a $\sigma$-invariant element in $W$, we have $(G^\lambda)^\sigma = G^\lambda$, $\overline{G^\lambda} = G^\lambda$, $(S^\mu_w)^\sigma = (S^\mu_w)^\lambda$, and $\overline{S^\mu_w} = (S^\mu_w)^\lambda$. 

3.2. MV cycles and MV polytopes. Let $\mu_1, \mu_2$ be coweights with $\mu_1 \geq \mu_2$. Following Anderson [A], an irreducible component of $S^{\mu_1}_w \cap S^{\mu_2}_w$ is called an MV cycle with coweight $(\mu_1, \mu_2)$. This definition of an MV cycle is a generalization of the original one in [MV]. $X_\sigma$ acts on $G$ by $\nu \cdot L := t^\nu \cdot L$. Since $T$ normalizes $N_w$, we see that $\nu \cdot S^\mu_w = S^{\mu+\nu}_w$. If $A$ is a component of $S^{\mu_1}_w \cap S^{\mu_2}_w$, then $\nu \cdot A$ is a component of $S^{\mu_1+\nu}_w \cap S^{\mu_2+\nu}_w$. Hence $X_\sigma$ acts on the set of all MV cycles. The orbit of an MV cycle with coweight $(\mu_1, \mu_2)$ is called a stable MV cycle with coweight $\mu_2 - \mu_1$. Note that a stable MV cycle with coweight $\mu$ has a unique representative with coweight $(\nu, \nu + \mu)$ for a fixed coweight $\nu$.

Let $\text{MVC}_G$ denote the set of stable MV cycles for $G$, and let $\text{MVC}_G^w$ denote the set of those with coweight $\mu$. For a $T$-invariant closed subvariety $A$ of the affine Grassmannian, let $\Phi(A) \subset t_\mathbb{R} := X_\sigma \otimes \mathbb{R}$ be the moment polytope of $A$, which is exactly the convex hull of $\{\mu \in X_\sigma | t^\mu \in A\}$.

If $A$ is an MV cycle with coweight $(\mu_1, \mu_2)$, then we say that $\Phi(A)$ is an MV polytope with coweight $(\mu_1, \mu_2)$. The action of $X_\sigma$ on the set of MV cycles gives an action of $X_\sigma$ on the set of MV polytopes. It is easy to see that $\nu \cdot P = P + \nu$. The orbit of $X_\sigma$ on an MV polytope with coweight $(\mu_1, \mu_2)$ is called a stable MV polytope with coweight $\mu_2 - \mu_1$.

Let $\text{MVP}_G$ be the set of stable MV polytopes for $G$, and let $\text{MVP}_G^w$ be the set of stable MV polytopes for $G$ with coweight $\mu$. As mentioned in [A], there is a natural bijection between $\text{MVC}_G$ and $\text{MVP}_G$. Let $C$ be an MV cycle, and $[C]$ be its stable MV cycle. Let $\text{P}_C$ be the corresponding MV polytope of $C$, and $\text{P}_C$ be its stable MV polytope. If there is no confusion, we write $C$ (resp. $P$) for both MV cycle (or polytope) and stable MV cycle (resp. polytope).

Suppose we are given a collection of coweights $\mu_\bullet = (\mu_w)_{w \in W}$ such that $\mu_w \leq_w \mu_w$ for all $v, w \in W$. Then we define the corresponding pseudo-Weyl polytope by:

$$P(\mu_\bullet) := \cap_w C^w = \{\alpha | \alpha \cdot w \cdot \lambda_i \leq \langle \mu_w, w \cdot \lambda_i \rangle, \forall w \in W, \text{ and } i \in I\}.$$ 

For a collection $(\mu_w)_{w \in W}$ with coweights such that $\mu_y \leq_w \mu_w$, for any $y, w \in W$, set $A(\mu_\bullet) = \cap_w S^\mu_w$, and let Conv$(\mu_\bullet)$ be the convex hull of $(\mu_w)_{w \in W}$ in $t_\mathbb{R}$. $A(\mu_\bullet)$ is called a GGMS stratum, and it is a candidate of MV cycles. If it is not empty, then the moment polytope of $A(\mu_\bullet)$ is exactly Conv$(\mu_\bullet)$ (see Lemma 2.3, [K1]), which also coincides with $P(\mu_\bullet)$. That is, Conv$(\mu_\bullet) = P(\mu_\bullet)$.

The following theorem gives a criterion for the closure of a GGMS stratum to be an MV cycle.

Theorem 1 (Kamnitzer [K1]). Let $(\mu_w)_{w \in W}$ be the set with coweights, such that $\mu_y \leq_w \mu_w$, for any $y, w \in W$. Then $A(\mu_\bullet) = \cap_w S^\mu_w$ is an MV cycle if and only if Conv$(\mu_\bullet)$ is an MV polytope.

Let $P$ be an MV polytope with vertices $(\mu_w)_{w \in W}$. Then $P$ is the moment polytope of an MV cycle $\cap_w S^\mu_w$. In this case, $\sigma(\cap_w S^\mu_w) = \cap_w S^\sigma(\mu_w)$ is also an MV cycle, and its moment polytope is exactly Conv$(\sigma(\mu_{\sigma^{-1}(w)}))$. Hence it is an MV polytope with vertices $(\sigma(\mu_{\sigma^{-1}(w)}))_{w \in W}$, which coincides with $P$.

Lemma 3.4. Let $(\mu_w)_{w \in W}$ be the vertices of an MV polytope $P$, and let $A(\mu_\bullet)$ be the corresponding GGMS stratum, such that $A(\mu_\bullet)$ is an MV cycle. Then the following statements are equivalent:

1. $P$ is $\sigma$-invariant.
2. $A(\mu_\bullet)$ is $\sigma$-invariant.
3. $A(\mu_\bullet)$ is $\sigma$-invariant.
4. $\sigma(\mu_w) = \mu_{\sigma(w)}$, $\forall w \in W$.

Proof. Since MV cycles are parametrized by MV polytopes bijectively, it is easy to see that the moment polytope of $\sigma(\cap_w S^\mu_w)$ is $\sigma(P)$. So $P$ is $\sigma$-invariant if and only if $A(\mu_\bullet)$ is $\sigma$-invariant, i.e.,

(1) $\Leftrightarrow$ (2).
Assume \( A(\mu) \) is \( \sigma \)-invariant. Then \( \overline{NS_w} = \overline{S^\sigma(w^{-1}(w))} \). Since \( \overline{S^\sigma(w)} \) and \( \overline{S^\sigma(w^{-1}(w))} \) are locally closed, we have \( (\overline{S^\sigma(w)}) \cap (\overline{S^\sigma(w^{-1}(w))}) \neq \emptyset \). It implies that, \( \forall w \in W, S^\sigma_w \cap S^\sigma(w^{-1}(w)) \neq \emptyset \). Hence \( \mu_w = \sigma(\mu_{w^{-1}}) \), \( \forall w \in W \). So \((2) \Rightarrow (4)\).

It is easy to see \((3) \Leftrightarrow (4)\), and \((4)\) implies \((1)\) immediately.

3.3. Lusztig datum. Let \( i \) be a reduced word, and \( n_\bullet \in \mathbb{N}^m \). Recall some results in [K1]. We define \( \{\mu_w\}_{0 \leq k \leq m} \) inductively by \( \mu_0 = 0 \) and \( \mu_{w_k} = \mu_{w_{k-1}} - n_k w_{k-1}^{-1}(\alpha_i \mathbf{x}) \), for any \( 1 \leq k \leq m \). Then \( A^I(n_\bullet) = \overline{S^\mu(w)} \). If \( n_\bullet \) is an \( \sigma \)-invariant, then \( \overline{A^I(n_\bullet)} \) is dense in \( \overline{A^I(n_\bullet)} \).

We give a necessary and sufficient condition on the \( i \)-Lusztig datum \( n_\bullet \), so that \( P \) is \( \sigma \)-invariant.

Proposition 3.5. Let \( w_0 = s_{\eta_1} s_{\eta_2} \cdots s_{\eta_m} \) be a reduced expression of \( w_0 \) relative to the Coxeter group \( W^\sigma, \) where \( \eta_1, \eta_2, \cdots, \eta_m \), are orbits of \( \sigma \) in \( I \). For each \( \eta \), we fix a reduced expression of \( s_\eta \) as an element of \( W \), and denote by \( \overline{\mu}_{w_\eta} \), \( \mu_{w_\eta} \) the corresponding MV polytope of type \( R_\eta \). Then \( P \) is \( \sigma \)-invariant if and only if \( n_1 = n_2 = \cdots = n_r, \) \( n_{r+1} = n_{r+2} = \cdots = n_{r+r_2}, \cdots \), where \( r_\eta \) is the length of \( s_\eta \) as an element of \( W \).

Proof. For any orbit \( \eta \) of \( \sigma \), let \( R_\eta \) be the root system generated by \( \{\alpha_i; i \in \eta\} \). Let \( W_\eta \) be the Coxeter group generated by \( \{s_i; \text{ for } i \in \eta\} \). Then \( s_\eta \) is the longest element in \( W_\eta \).

Recall that \( n_\eta \) is the length of the edge connecting \( \mu_{w_{k-1}} \) and \( \mu_{w_k} \), i.e. \( \mu_{w_k} - \mu_{w_{k-1}} = -n_{k} w_{k-1}^{-1}(\alpha_i \mathbf{x}) \). The convex hull of \( \{\mu_w; w \in W_\eta\} \) forms an MV polytope for an algebraic group of type \( R_\eta \). We denote it by \( P_{\eta 1} \). From \( \mu_{w_{1}} \cdots, \mu_{w_{r_\eta}} \), we get a Lusztig datum \( (n_1, n_2, \cdots, n_{r_\eta}) \) along the chosen reduced word of \( s_\eta \). The convex hull of \( \{\mu_w; w = s_\eta y, \text{ for } y \in W_{\eta_0}\} \) forms an MV polytope of type \( R_\eta \). We denote it by \( P_{\eta 2} \). From \( \mu_{w_{1}} \cdots, \mu_{w_{r_\eta}} \), we get a Lusztig datum \( (n_{r_\eta+1}, n_{r_\eta+2}, \cdots, n_{r_\eta+r_2}) \) along the chosen reduced word of \( s_{\eta_2} \). Similarly, we get subsequently MV polytopes \( P_{\eta 3}, \cdots, P_{\eta m} \), with type \( R_{\eta_3}, \cdots, R_{\eta_m} \). We also get their corresponding Lusztig data along the chosen reduced words of \( s_{\eta_i} \).

Now let us return to the proof. If \( P \) is \( \sigma \)-invariant, we have \( \sigma(\mu_w) = \mu_{\sigma(w)} \), for all \( w \in W \), by Lemma 3.4. Applying Lemma 3.4, we see that \( P_{\eta k} \) for all \( k \), are \( \sigma \)-invariant.

Let us that there are two possibilities: \( A_2 \) and \( A_1 \times A_1 \times \cdots \times A_1 \) (with \( l \) copies of \( A_1 \), where \( l = 2 \) or 3) for \( R_\eta \). Hence the sufficient part is reduced to the following two cases which are easy to check.

1. \( A_2 \), if \( P \) is \( \sigma \)-invariant, then \( n_1 = n_2 = n_3 \).
2. \( A_1 \times A_1 \times \cdots \times A_1 \), if \( P \) is \( \sigma \)-invariant, then \( n_1 = n_2 = \cdots = n_l \).

Conversely, from \( A^I(n_\bullet) = \cap_k S^\mu_{w_k} \), we have \( \sigma(A^I(n_\bullet)) = A^I(n_\bullet) \), where \( j = (\sigma(i_1), \sigma(i_2), \cdots, \sigma(i_m)) \).

From the condition of \( n_\bullet \), it is easy to see \( R^I(n_\bullet) = n_\bullet \). Hence their closures coincide, i.e. the corresponding MV cycle of this \( i \)-Lusztig datum is \( \sigma \)-invariant. By Lemma 3.4, \( P \) is \( \sigma \)-invariant. 
 \( \square \)
3.4. The bijection between MV cycles (polytopes) for a symmetric pair. Let $P$ be a $σ$-invariant MV polytope for $G$. In this subsection, we will show that $P^σ$ is an MV polytope for $G^σ$, and then we get the bijection between $MV$ polytopes for a symmetric pair.

Consider the symmetric pair $(A_4, B_2)$. For the longest element in the Weyl group $W$, we have reduced expressions $w_0 = s_1s_4 \cdot s_2s_3s_2 \cdot s_1s_4 = s_2s_3s_2 \cdot s_1s_4 \cdot s_2s_3s_2 \cdot s_1s_4$. We get two reduced words $ι_σ$ and $ι'_σ$ for $G^σ$ from these two expressions of $w_0$. From $ι_σ$ and $ι'_σ$, we naturally get 2 reduced words for $G, i = (1, 4, 2, 3, 2, 1, 4, 2, 3, 2)$, $i' = (2, 3, 2, 1, 4, 2, 3, 2, 1, 4)$, respectively. Let $n_*$, $n'_*$ be Lusztig data along $i$ and $i'$ for $P$, respectively. According to Proposition 3.5, we may write $n_*$ and $n'_*$ as follows

\[ n_* = (n_1, n_2, n_3, n_4, n_5, n_6) \in \mathbb{N}^{10}, \]

\[ n'_* = (n'_1, n'_2, n'_3, n'_4, n'_5, n'_6) \in \mathbb{N}^{10}, \]

where $n_k, n'_k$ are non-negative integers.

Set $n^σ_*(i) = (n_1, n_2, n_3, n_4)$. By sending $n_*$ to $n^σ_*$, we get a bijection between $ι$-Lusztig data of $σ$-invariant $MV$ polytopes for $G$ and $ι'$-Lusztig data of $MV$ polytopes for $G^σ$. We shall show this bijection is intrinsic, and independent of the choice of reduced words. Note that the above procedure works for general case.

For any subvariety $Y \subset G$, we set $Y^σ := \{ y \in Y | σ(y) = y \}$.

Let $B(n_*^σ) = \{(b_*) \in K^{ℓ(w_0)} | val(b_*) = n_k, \forall k \}$ and $B_σ(n^σ_*) = \{(b_*) \in K^{ℓσ(w_0)} | val(b_*) = n^σ_k, \forall k \}$, where $val$ is the valuation function on $K$. Define a map $j_σ$ from $B_σ(n^σ_*)$ to $B(n_*^σ)$ by $j_σ(b_1, b_2, b_3, b_4) = (b_1, b_2, 2b_2, b_3, b_3, b_4, 2b_4, b_4)$. We get two $ι$ and $ι'$ are reduced words of $G$ resulting from the reduced words of $G^σ$, $ι_σ$ and $ι'_σ$ respectively, in the sense of Proposition 3.5.

Lemma 3.6. Let $n_*$ be a $σ$-invariant $ι$-Lusztig datum. Then $A^ι(n_*^σ) = A^ι'(n^σ_*)$.

Proof. We only show this lemma for the pair $(A_4, B_2)$, and the following argument works in general.

Let $ι : A^ι(n^σ_*) \hookrightarrow G$ be the natural imbedding, which is the restriction of $ι : G_σ \hookrightarrow G$. We have surjections $ι_ι : B_σ(n^σ_*) \twoheadrightarrow A^ι(n^σ_*)$, and $ι_ι : B(n_*^σ) \rightarrow A^ι(n_*^σ)$, which are given by

\[ π_ι(b_1, b_2, b_3, b_4) = [η_ιw^{-1}(x_{ι_1}(b_1)x_{ι_2}(b_2)x_{ι_3}(b_3)x_{ι_4}(b_4))], \]

\[ π_ι(b_1, b_2, b_3, b_4, 2b_4, b_4) = \]

\[ [η_ιw^{-1}(x_{ι_1}(b_1)x_{ι_2}(b_2)x_{ι_3}(2b_2)x_{ι_2}(b_2) \cdot x_1(b_3)x_3(2b_3)x_2(b_3) \cdot x_2(b_4)x_3(2b_4)x_2(b_4))]. \]

where $x_{ι_1}$ and $x_{ι_2}$ are root subgroup homomorphisms for $G^σ$, and we denote by $\lceil \rceil$ the projection from $G(Κ)$ to $G$. For the definition of $η_ιw$, see (section 4.4, [11]). Since $x_1(b_1)x_3(b_3) = x_{ι_1}(b_1), x_2(b_2)x_2(b_2) = x_{ι_2}(b_2)$, for $ι = 1$ or 3, and $x_2(b_2)x_2(b_2) = x_{ι_2}(b_2)$, for $ι = 2$ or 4, we can see that $ι \circ ι_ι = π_ι \circ j_σ$, i.e., we have the following commutative diagram

\[ \begin{array}{ccc}
B_σ(n^σ_*) & \xrightarrow{j_σ} & B(n_*^σ) \\
\downarrow π_ι & & \downarrow π_ι \\
A^ι(n^σ_*) & \xrightarrow{ι} & A^ι(n_*^σ). 
\end{array} \]

Since $π_ι(B_σ(n^σ_*)) = A^ι(n^σ_*)$, we have $A^ι(n^σ_*) \subset A^ι(n_*^σ)^σ$.

Assume $n_*$ is of coweight $µ$. It is known that $X(µ) = S^σ_0 \cap S^w_µ = \bigcup A^ι(n_*)$, where the union is taken over $n_*$, such that $n_*$ is an $ι$-Lusztig datum with coweight $µ$. Hence we have

\[ X(µ)^σ = \bigcup A^ι(n_*)^σ, \]

where $A^ι(n_*)$ appear in the right hand side.
Corollary 3.7. Let $P = \text{Conv}(\mu) = \bigcup A^1(\mu)$, where the union is taken over $\mu$ such that $\mu$ is an $i_\sigma$-Lusztig datum with coweight $\mu$.

Let $m_\bullet = (m_1, m_2, m_3, m_4)$ be an $i_\sigma$-Lusztig datum, such that $\overline{A^1(m_\bullet)}$ is an MV cycle for $G^\sigma$ with coweight $\mu$. Let $n''_\bullet = (n_1, n_2, n_3, n_4)$. Then $n''_\bullet$ is $\sigma$-invariant, and hence $A^1(n_\bullet) \subset A^1(n''_\bullet)^\sigma$. By comparing decompositions of $X(\mu)^\sigma$ in [1] and [2], we obtain $A^1(n_\bullet)^\sigma = A^1(n''_\bullet)^\sigma$.

Remark 3.1. From this lemma, we see that the closure of the fixed point set of $\sigma$ on some open subset of a $\sigma$-invariant MV cycle $C$ is an MV cycle for $G^\sigma$. We believe that the fixed point set of $\sigma$ on $\sigma$-invariant MV cycle for $G$ is an MV cycle for $G^\sigma$.

Corollary 3.7. If $A^1(n_\bullet)^\sigma$ is not $\sigma$-invariant, then $A^1(n''_\bullet)^\sigma$ is empty.

Lemma 3.8. If $n_\bullet$ is a $\sigma$-invariant $i$-Lusztig datum, and $R_i^1(n_\bullet) = n'_\bullet$, then $(A^1(n_\bullet) \cap A^1(n''_\bullet))^\sigma$ contains an open dense subset.

Proof. We can change $i$ to $i'$ by combining several braid $d$-moves.

If $(\cdots, i_k, i_{k+1}, i_{k+2}, i_{k+3}, \cdots) \mapsto (\cdots, i_k, i_{k+2}, i_{k+1}, i_{k+3}, \cdots, i_{k+4}, \cdots)$, $(d = 2)$, define a rational map from $B(n_\bullet)$ to $B(n''_\bullet)$, by

$(\cdots, b_k, b_{k+1}, b_{k+2}, b_{k+3}, \cdots) \mapsto (\cdots, b_k, b_{k+2}, b_{k+1}, b_{k+3}, \cdots)$.

If $(\cdots, i_k, i_{k+1}, i_{k+2}, i_{k+3}, i_{k+4}, \cdots) \mapsto (\cdots, i_k, i_{k+2}, i_{k+1}, i_{k+4}, i_{k+3}, \cdots)$, $(d = 3)$, then we define a rational map from $B(n_\bullet) \cap B(n''_\bullet)$ by

$(\cdots, b_k, b_{k+1}, b_{k+2}, b_{k+3}, b_{k+4}, \cdots) \mapsto (\cdots, b_k, b_{k+2}b_{k+3}, b_{k+4}b_{k+1} + b_{k+3}b_{k+2}, b_{k+4}, b_{k+3}, \cdots)$.

It is well-known that, by several braid $d$-moves, we can arrive at $Y$ from $i$. Let $i \mapsto i_1 \mapsto i_2 \mapsto \cdots \mapsto i'$ be one such path, where $\mapsto$ represents a braid $d$-move. For a path from $i$ to $i'$, we denote the rational map $f$ by combining those in each step defined above. Assume $f(b_1, \cdots, b_m) = (b'_1, \cdots, b'_m)$. It is easy to see that $b'_k$ is a rational function with numerator and denominator as nonzero polynomials with nonnegative integral coefficients. Consider the diagram

\[
\begin{array}{ccc}
B(n_\bullet) & \xrightarrow{f} & B(n''_\bullet) \\
\downarrow \pi_1 & & \downarrow \pi_1 \\
A^1(n_\bullet) & \xrightarrow{f} & A^1(n''_\bullet)
\end{array}
\]

where $\pi_1$ is as in the proof of Lemma 3.6 and dashed arrows denote rational maps. We have $\pi_1 = \pi_1 \circ f$.

Let $F$ be the product of all denominators appearing in every step of $d$-moves, so it is a nonzero polynomial with nonnegative integral coefficients. Let $U = \{(b_1, \cdots, b_m) \in B(n_\bullet) | F(b_1, \cdots, b_m) \neq 0\}$. Then $f$ is well-defined on $U$, and so $\pi_1(U) \subset A^1(n_\bullet) \cap A^1(n''_\bullet)$.

There exists $y \in U$, such that $\pi_1(y) \in \pi_1(U) \subset A^1(n_\bullet) \cap A^1(n''_\bullet)$, and if $\pi_1(y)$ is $\sigma$-invariant. Hence $(A^1(n_\bullet) \cap A^1(n''_\bullet))^\sigma$ is nonempty. Since $\pi_1$ is an open map, $\pi_1(U)$ is open in $A^1(n_\bullet)$. We only show it in the case of $(A_{4}, B_{2})$. Since $A^1(n_\bullet)$ is $\sigma$-invariant, we have $n_\bullet = (\bar{n}_1, \bar{n}_2, \bar{n}_3, \bar{n}_4, \bar{n}_5, \bar{n}_6)$. Now take $y = (t^{\bar{n}_1}, t^{\bar{n}_2}, t^{\bar{n}_3}, t^{\bar{n}_4}, t^{\bar{n}_5}, t^{\bar{n}_6}) \in B(n_\bullet)$, then $F(y) \neq 0$. In the general case, we have the similar argument.

Since $A^1(n_\bullet)$ is irreducible by Lemma 3.6, we have $(A^1(n_\bullet) \cap A^1(n''_\bullet))^\sigma$ is dense in $A^1(n''_\bullet)^\sigma$.

\[
\text{Lemma 3.9. Let } \text{Conv}(\mu_w) = \text{Conv}(\mu_{w'}) \subset W^\sigma. \text{ If } \text{the MV polytope } P = \text{Conv}(\mu_w) \text{ is } \sigma-\text{invariant, then } P^\sigma = \text{Conv}(\mu_{w'}) \subset W^\sigma.
\]
Proof. Since $P$ is $\sigma$-invariant, we have $\sigma(\mu_w) = \mu_w$, for $w \in W^\sigma$. We can easily see that $\sigma$ acts trivially on Conv$(\mu_w)_{w \in W^\sigma}$, so Conv$(\mu_w)_{w \in W^\sigma} \subset P^\sigma$.

For the converse, the perfect pairing $(X_+ \otimes \mathbb{R}) \times (X_- \otimes \mathbb{R}) \to \mathbb{R}$ descends to $(X_+ \otimes \mathbb{R}) \times (X_- \otimes \mathbb{R}) \to \mathbb{R}$ (see Section 2.2). Note that $t^\sigma_\mathbb{R}$ can be identified with $X_+ \otimes \mathbb{R}$.

For any $\beta \in P^\sigma \subset P$, and $w \in W^\sigma$, we have $\langle \beta, w \cdot \lambda_i \rangle \leq \langle \mu_w, w \cdot \lambda_i \rangle$. By descent, we have $\langle \beta, w \cdot \lambda_\eta \rangle \leq \langle \mu_w, w \cdot \lambda_\eta \rangle$, for all orbit $\eta$ of $\sigma$ in $I$, where $\lambda_\eta$ is the fundamental weight for $G^\sigma$ corresponding to $\lambda_\eta$, for $i \in I$. Since $P^\sigma \subset t^\sigma_\mathbb{R}$, we see that

$$P^\sigma \subset \{ \beta \in t^\sigma_\mathbb{R} | \langle \beta, w \cdot \lambda_\eta \rangle \leq \langle \mu_w, w \cdot \lambda_\eta \rangle, \forall \eta, \forall w \in W^\sigma \}.$$ 

The right hand side is exactly Conv$(\mu_w)_{w \in W^\sigma}$.

\[ \square \]

Theorem 3.10. If $P$ is a $\sigma$-invariant MV polytope for $G$, then $P^\sigma$ is an MV polytope for $G^\sigma$.

Proof. Let $\mu_i$ be the reduced word $i_\sigma$ for $G^\sigma$, and let $n^\sigma_i$ be the corresponding $i_\sigma$-Lusztig datum of $P$.

Let $\mathcal{I}$ be the fixed reduced word for $G$ from $i_\sigma$, in the sense of Proposition 3.9. Let $J = \{ (i', n'_{i'}) | i' \}$ be a reduced word for $G$ from some reduced word $i'_\sigma$ for $G^\sigma$, and $R^{i'}_{\mathcal{I}}(n^\sigma_i) = n^\sigma_i$. We have $\cap_{(i', n'_{i'})} A^{i'}(n^\sigma_i)$ contains an open and dense subset of $A^{i'}(n^\sigma_i)$ from Lemma 3.8 since the intersection of finite open dense subsets is still open and dense.

Recall $A^{i'}(n^\sigma_i) = \cap S_{w_k}$, and $A^{i'}_{\mathcal{I}}(n^\sigma_i) = \cap (S_{w_k})_{\mathcal{I}}$. By Lemma 3.6 we have $(\cap_{(i', n'_{i'})} A^{i'}(n^\sigma_i))^{\mathcal{I}} = \cap (A^{i'}(n^\sigma_i))^{\mathcal{I}} = A^0(\mu_w)_{w \in W^\sigma}$, where $A(\mu_w)_{w \in W^\sigma} = \cap_{w \in W^\sigma} (S_{w_k})_{\mathcal{I}}$. The last equality holds, since for any $w \in W^\sigma$, there exists some reduced word $i'_\sigma$ of $G^\sigma$ and some integer $k$, such that $w = w_k$. Therefore, we have $A^{i'}_{\mathcal{I}}(n^\sigma_i) = A^0(\mu_w)_{w \in W^\sigma} = (\cap_{(i', n'_{i'})} A^{i'}(n^\sigma_i))^{\mathcal{I}} = A^0(\mu_w)_{w \in W^\sigma}$. That means, the moment polytope of the MV cycle $A^{i'}_{\mathcal{I}}(n^\sigma_i)$ is Conv$(\mu_w)_{w \in W^\sigma}$, which is exactly $P^\sigma$, by Lemma 3.6. Hence $P^\sigma$ is really an MV polytope for $G^\sigma$.

\[ \square \]

Corollary 3.11. Let $(i, n_i)$ and $(i', n_{i'})$ be two $\sigma$-invariant Lusztig data. If $R^{i'}_{\mathcal{I}}(n^\sigma_i) = n^\sigma_i$, then $R^{i'}_{\mathcal{I}}(n^\sigma_{i'}) = n^\sigma_i$

Theorem 3.12. We have a bijection $\theta_P : MVP_G \to MVP_{G^\sigma}$, given by $P \to P^\sigma$, which preserves coweights. Induced from $\theta_P$, we have a bijection $\theta_G : MVC_G \to MVC_{G^\sigma}$

Proof. Let $P$ be a $\sigma$-invariant MV polytope for $G$. By Theorem 3.10 we have a well-defined map $\theta_P : MVP_G \to MVP_{G^\sigma}$ by $\theta_P(P) = P^\sigma$.

Fix a reduced word $i_\sigma$ for $G^\sigma$. Let $i$ be a reduced word coming from $i_\sigma$. For any MV polytope for $G$ (resp. $G^\sigma$), we have the corresponding $i$ (resp. $i_\sigma$) Lusztig datum. According to Proposition 3.9, $\theta_P$ is injective. Let $Q$ be any MV polytope for $G^\sigma$, and let $m_i$ be the $i_\sigma$-Lusztig datum of $Q$. By Lemma 3.6 and its proof, there exists a unique $i$-Lusztig datum $n_i$ such that $A^i(n_i)$ is contained in $A^{i'}(n^\sigma_i)$, and $n_i$ is $\sigma$-invariant. Let $P_Q$ be the MV polytope of $A^i(n_i)$. We have $P_Q = Q$, since $P_Q$ has the same $i$-Lusztig datum as $Q$. So $\theta_P$ is surjective.

Hence $\theta_P$ is a bijection, and it is easy to see that it preserves the coweights of MV polytopes.

\[ \square \]

3.5. The bijection in highest weight case. Let $\lambda, \mu$ be $\sigma$-invariant coweights, we set $X(\lambda, \mu) := S^\lambda \cap S^\mu$, and $X(\mu - \lambda) = S^\mu \cap S^\mu - \lambda$. In this subsection, we have the same assumptions on the $i$ and $i_\sigma$ as in Subsection 3.4.

The following lemma is given by Anderson [A].
Lemma 3.13. An irreducible component of \( X(\lambda, \mu) \) is contained in \( \overline{G^\lambda} \) if and only if it appears as basis in \( V_p(\lambda) \).

First of all, we have a decomposition:

\[
X(\lambda, \mu) = \lambda \cdot X(\mu - \lambda) = \bigsqcup_{n} \lambda \cdot A^i(n),
\]

where the union is taken over \( n \) which are i-Lusztig data with coweight \( \mu - \lambda \). Then

\[
S^A_{\lambda} \cap S_{\mu}^{w_0} \cap \overline{G^\lambda} = \bigsqcup_{1} \lambda \cdot A^i(n) \cup \bigsqcup_{2} (\lambda \cdot A^i(n) \cap \overline{G^\lambda}),
\]

where the first union 1 is taken over those \( n \) in \( \{3\} \) such that \( \lambda \cdot A^i(n) \subset \overline{G^\lambda} \); the second union 2 is taken over those \( n \) in \( \{3\} \) such that \( \lambda \cdot A^i(n) \not\subset \overline{G^\lambda} \).

If \( \lambda \cdot A^i(n) \not\subset \overline{G^\lambda} \), then \( \lambda \cdot A^i(n) \cap \overline{G^\lambda} \) is of lower dimension than \( A^i(n) \).

From decomposition \( \{4\} \) and Corollary \( \{5\} \) we have

\[
S^A_{\lambda} \cap S_{\mu}^{w_0} \cap \overline{G^\lambda} = (S^A_{\lambda})^\sigma \cap (S_{\mu}^{w_0})^\sigma \cap \overline{G^\lambda})^\sigma = \bigsqcup_{3} \lambda \cdot A^i(n)^\sigma \cup \bigsqcup_{4} (\lambda \cdot A^i(n) \cap \overline{G^\lambda})^\sigma,
\]

where the first union 3 is taken over those \( n \) in \( \{3\} \), such that \( \lambda \cdot A^i(n) \subset \overline{G^\lambda} \) and \( n \) is \( \sigma \)-invariant; the second union 4 is taken over those \( n \) in \( \{3\} \), such that \( \lambda \cdot A^i(n) \not\subset \overline{G^\lambda} \) and \( n \) is \( \sigma \)-invariant. From the point view of \( G^\sigma \), we also have a decomposition

\[
(S^A_{\lambda})^\sigma \cap (S_{\mu}^{w_0})^\sigma \cap \overline{G^\lambda})^\sigma = \bigsqcup_{5} \lambda \cdot A^i(m) \cup \bigsqcup_{6} (\lambda \cdot A^i(m) \cap \overline{G^\lambda})^\sigma,
\]

where the first union 5 is taken over \( m \) which are \( i^\sigma \)-Lusztig data with coweight \( \mu - \lambda \), satisfying \( \lambda \cdot A^i(m) \subset \overline{G^\lambda} \); the second union 6 is taken over \( m \) which are \( i^\sigma \)-Lusztig data with coweight \( \mu - \lambda \), satisfying \( \lambda \cdot A^i(m) \not\subset \overline{G^\lambda} \).

If \( \lambda \cdot A^i(m) \not\subset \overline{G^\lambda} \), then \( \lambda \cdot A^i(m) \cap \overline{G^\lambda} \) is of lower dimension than \( A^i(m) \).

Lemma 3.14. \( \overline{G^\lambda} = \cap S_{w}^{w^{-\lambda}} \).

Proof. We know \( \cap S_{w}^{w^{-\lambda}} \) is an MV cycle with coweight \( (\lambda, w_0 \cdot \lambda) \), and it is contained in \( \overline{G^\lambda} \). Since both of them are of the same dimension \( 2(\lambda, \rho) \), and both of them are irreducible, we have \( \overline{G^\lambda} = \cap S_{w}^{w^{-\lambda}} \).

Lemma 3.15. If \( \lambda \cdot A^i(n) \not\subset \overline{G^\lambda} \), and \( n \) is \( \sigma \)-invariant, then \( (\lambda \cdot A^i(n) \cap \overline{G^\lambda})^\sigma \) is of lower dimension than \( A^i(n)^\sigma \).

Proof. With the same reason as in the proof of lemma \( \{3\} \) we can find an open subset \( U \subset B(n) \), such that \( \pi_1(U) \subset \cap (\cap_{i}A^i(n)) = \cap_{w} S_{w}^{w^{-\lambda}} \) is open in \( A^i(n) \).

Note that \( (\cap \lambda \cdot S_{w}^{w^{-\lambda}}) \cap \overline{G^\lambda} \) is empty. Otherwise, if there exists a point \( p \in (\cap \lambda \cdot S_{w}^{w^{-\lambda}}) \cap \overline{G^\lambda} \), then

\[
p \in (\cap \lambda \cdot S_{w}^{w^{-\lambda}}) \cap \overline{G^\lambda} = (\cap \lambda \cdot S_{w}^{w^{-\lambda}}) \cap \cap S_{w}^{w^{-\lambda}} \subset (\cap \lambda \cdot S_{w}^{w^{-\lambda}}) \cap S_{w}^{w^{-\lambda}}.
\]

That is, \( \forall w \in W \), \( p \) must be contained in \( \lambda \cdot S_{w}^{w^{-\lambda}} \cap S_{w}^{w^{-\lambda}} \). From \( S_{w}^{w^{-\lambda}} = \bigsqcup_{\mu \leq w \cdot \lambda} S_{\mu}^{w} \), we have \( \mu_\lambda + \lambda \leq w \cdot \lambda \). We get that \( \text{Conv}(\mu_\lambda) + \lambda \subset \text{Conv}(W \cdot \lambda) \). According to Anderson’s theorem on multiplicity of weight space \( \overline{G^\lambda} \), we have \( \lambda \cdot A^i(n) \) is an MV cycle in \( V_p(\lambda) \). By Lemma 3.13 it is a contradiction to the condition that \( \lambda \cdot A^i(n) \not\subset \overline{G^\lambda} \). As in Lemma 3.8 there exists a point \( p \in \lambda \cdot A^i(n) \). So \( \lambda \cdot A^i(n)^\sigma \cap \overline{G^\lambda} \) has lower dimension than \( A^i(n)^\sigma \).
By Lemma 3.15, and by comparing the two decompositions (5) and (9), we have that the set \( \{ A^\tau(\nu) \} \) is \( \tau \)-invariant and of coweight \( \mu - \lambda \), and \( \lambda \cdot A^\tau(\nu) \subseteq G^\lambda \) is in bijection with the set \( \{ A^\tau(\nu) \} \) is of coweight \( \mu - \lambda \), and \( \lambda \cdot A^\tau(\nu) \subseteq G^\lambda \), by sending \( A^\tau(n) \to A^\tau(n)^\sigma \). We thus obtain the following theorem.

**Theorem 3.16.** We have a bijection \( \theta_G^\lambda : MVC_G(\lambda)^\sigma \to MVC_G(\lambda) \), which is the restriction of \( \theta_C \) in Theorem 3.12.

4. Twining character formula

Recall that \( \text{Perv}_{\mathcal{G}(G)}(\mathcal{G}) \) is a tensor category \([MV]\), and it is easy to see the tensor functor \( \sigma^* \) induced from the action of \( \sigma \) on affine Grassmannian is a tensor equivalence. From the functoriality of Tannakian formalism \([DX]\), we have a natural automorphism \( \tilde{\sigma} \) on \( G^\vee \).

Fix a \( \sigma \)-invariant coweight \( \lambda \), and choose an isomorphism \( \phi : IC_\lambda \simeq \sigma^*(IC_\lambda) \), which is compatible with the action of \( \sigma \) on \( MV \) cycles (as the basis of \( V(\lambda) \)).

**Lemma 4.1.** The action of \( \tilde{\sigma} \) on \( G^\vee \) is compatible with the natural action of \( \sigma \) on \( V(\lambda) \) induced from \( \phi \).

**Proof.** Let \( T \) be the functor from \( \text{Perv}_{\mathcal{G}(G)}(\mathcal{G}) \) to \( \text{Rep}(G^\vee) \), such that \( T(IC_\lambda) = (\rho_\lambda, V(\lambda)) \), where \( \rho_\lambda : G^\vee \to GL(V(\lambda)) \) is the corresponding representation.

From \( \sigma^* : \text{Perv}_{\mathcal{G}(G)}(\mathcal{G}) \to \text{Perv}_{\mathcal{G}(G)}(\mathcal{G}) \), we get \( T(\sigma^*(IC_\lambda)) = (\rho_\lambda \circ \tilde{\sigma}, V(\lambda)) \). Let \( \tilde{\sigma} \) be the functor from \( \text{Rep}(G^\vee) \) to \( \text{Rep}(G^\vee) \), by sending \( (\rho_\lambda, V(\lambda)) \) to \( (\rho_\lambda \circ \tilde{\sigma}, V(\lambda)) \). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Perv}_{\mathcal{G}(G)}(\mathcal{G}) & \xrightarrow{T} & \text{Rep}(G^\vee) \\
\downarrow{\sigma^*} & & \downarrow{\tilde{\sigma}} \\
\text{Perv}_{\mathcal{G}(G)}(\mathcal{G}) & \xrightarrow{T} & \text{Rep}(G^\vee)
\end{array}
\]

By applying \( T \) to \( \phi : IC_\lambda \simeq \sigma^*(IC_\lambda) \), we obtain an isomorphism \( \sigma = T(\phi) : (\rho_\lambda, V(\lambda)) \to (\rho_\lambda \circ \tilde{\sigma}, V(\lambda)) \) in \( \text{Rep}(G^\vee) \). In other words, there exists a linear isomorphism \( \sigma : V(\lambda) \to V(\lambda) \) satisfying

\[
\sigma(\rho_\lambda(g) \cdot v) = (\rho_\lambda \circ \tilde{\sigma})(g) \cdot \sigma(v) = \rho_\lambda(\tilde{\sigma}(g)) \cdot \sigma(v), (g \in G^\vee, v \in V(\lambda)).
\]

\[\square\]

**Theorem 4.2.** \( \tilde{\sigma} \) is a Dynkin automorphism on \( G^\vee \).

**Proof.** Let \( \text{Vect}_X \) be the tensor category of \( X_\sigma \)-graded vector spaces. The action of \( \sigma \) on \( X_\sigma \) induces an tensor functor \( \sigma^* \) on \( \text{Vect}_X \). From Mirkovic-Vilonen’s paper \([MV]\), we know that there is a tensor functor \( F \) from \( \text{Perv}_{\mathcal{G}(G)}(\mathcal{G}) \) to \( \text{Vect}_X \), and it’s easy to see \( \sigma^* \) and \( \sigma^* \) are compatible with \( F \).

Applying Tannakian formalism, from \( F \) we get the forgetful functor from \( \text{Rep}(G^\vee) \) to \( \text{Rep}(T^\vee) \), where \( T^\vee \) is a torus of \( G^\vee \), and \( \sigma^*, \sigma^* \) induce automorphisms on \( G^\vee \) and \( T^\vee \), respectively. Since \( \sigma^* \) and \( \sigma^* \) are compatible with \( F \), we have \( \tilde{\sigma} \) preserve the torus \( T^\vee \), i.e, \( \tilde{\sigma}(T^\vee) = T^\vee \). It induces the action of \( \sigma \) on \( X^\vee(T^\vee) \).

Let \( B^\vee \) be the maximal subgroup of \( G^\vee \), which stabilizes the highest weight line \( V_\lambda(\lambda) \) in \( V(\lambda) \), for every \( \sigma \)-invariant dominant weight \( \lambda \). It’s easy to see \( B^\vee \) is a Borel subgroup of \( G \), and contains \( T^\vee \); furthermore, \( \sigma(B^\vee) = B^\vee \).

The coroots of \( G_{\alpha_i}^\vee, i \in I \), can be viewed as the roots of \( G^\vee \), and send the root \( \alpha_i^\vee \) to \( \alpha_{\sigma(i)}^\vee \) automatically, since under the identification of \( X^\vee(T^\vee) \) and \( X_\sigma \), the actions of \( \sigma \) are compatible.

Let \( \mathfrak{g}^\vee \) be the Lie algebra of \( G^\vee \). Let \( \tau \) be the automorphism on \( \mathfrak{g}^\vee \) induced from \( \tilde{\sigma} \). From the following Lemma 4.3, we know \( \tau \) acts trivially on the simple root space \( \mathfrak{g}_{i}^\vee \), for \( i \) fixed by \( \sigma \). Lift \( \tau \) to \( \tilde{\sigma} \) on \( G^\vee \), then \( \tilde{\sigma} \) act trivially on the root subgroup \( U_{\alpha_i^\vee} \) and \( U_{-\alpha_i^\vee} \), for \( i, \sigma(i) = i \). Hence we are
able to find root subgroup homomorphisms $x_i^{\gamma} : \mathbb{C} \to G$ and $y_i^{\gamma} : \mathbb{C} \to G$, corresponding to $\alpha_i^{\gamma}$ and $-\alpha_i^{\gamma}$, such that $\sigma(x_i^{\gamma}(a)) = x_i^{\gamma}(\sigma(a))$ and $\sigma(y_i^{\gamma}(a)) = y_i^{\gamma}(\sigma(a))$ for all $i \in I$.

Hence $\sigma$ is a Dynkin automorphism with respect to a pinning of $G^{\vee}$, where the actions of $\tau$ on $G^{\vee}$ and $\sigma$ on $\mathbb{C}_G$ are compatible.

From Lemma 4.1 we have $\sigma([a,b]) = [\tau(a), \sigma(b)]$, for two arbitrary elements $a$ and $b$ in $G^{\vee}$. By Schur's lemma, we have $\tau = c \cdot \sigma$, for some constant $c$. Let $\gamma$ be the corresponding coroot of highest root $\gamma^{\vee}$, so it is $\sigma$-invariant. Since $[e_{\gamma^{\vee}}, e_{-\gamma^{\vee}}] \in \mathbb{C} : \gamma$, we have $[e_{\gamma^{\vee}}, e_{-\gamma^{\vee}}] = \tau([e_{\gamma^{\vee}}, e_{-\gamma^{\vee}}]) = [\tau(e_{\gamma^{\vee}}), \tau(e_{-\gamma^{\vee}})] = c^2 \cdot [e_{\gamma^{\vee}}, e_{-\gamma^{\vee}}]$. Hence $c^2 = 1$.

If $G^{\vee}$ is of type $A_2n$, there exist two adjacent simple roots $\alpha_i^{\vee}$ and $\alpha_j^{\vee}$, such that $\sigma(i) = j$, for $i$ and $j \in I$. Then we have $\tau([e_{\alpha_i^{\vee}}, e_{\alpha_j^{\vee}}]) = [e_{\alpha_i^{\vee}}, e_{\alpha_j^{\vee}}] = -[e_{\alpha_i^{\vee}}, e_{\alpha_j^{\vee}}]$. Since $\alpha_i^{\vee} + \alpha_j^{\vee}$ is also $\sigma$-invariant, it forces $c = -1$.

If $G^{\vee}$ is of other type. Let $h_i = [e_{\alpha_i^{\vee}}, e_{-\alpha_i^{\vee}}]$. Since $\sigma([e_{\alpha_i^{\vee}}, e_{-\alpha_i^{\vee}}]) = [\tau(e_{\alpha_i^{\vee}}), \tau(e_{-\alpha_i^{\vee}})] = c \cdot [e_{\alpha_i^{\vee}}, e_{-\alpha_i^{\vee}}]$, we have $\sigma(h_i) = c \cdot h_{\sigma(i)}$. Then $\{h_i\}_{i \in I}$ is a basis of $\mathbb{H}^{\vee}$. Since there exists $i \in I$, such that $\sigma(i) = i$, when $G^{\vee}$ is not of type $A_2n$, it’s easy to see $\text{trace}(\sigma|_{\mathbb{H}^{\vee}}) > 0$. Moreover, $\sigma$ interchanges MV cycles on $\mathbb{H}^{\vee}$, so $\text{trace}(\tau|_{\mathbb{H}^{\vee}}) > 0$. We thus have $c = 1$.

Remark 4.1. We can give another construction of Dynkin automorphism on $G^{\vee}$ which is compatible with the action of $\sigma$ on $\mathbb{C}_G$ cycles, by using Vasserot’s explicit construction of the action of dual group on cohomology of perverse sheaves [\math]. Moreover, this automorphism coincides with the one from Tannakian formalism.

Recall that twining character is defined to be $\chi^\sigma(V(\lambda)) := \sum_{\mu \in P(\lambda)^\sigma} \text{trace}(\sigma|_{V_\mu(\lambda)}) e^\mu$ for a Dynkin automorphism $\sigma$, where $\lambda$ is $\sigma$-invariant.

Proposition 4.4.

$$\chi^\sigma(V(\lambda)) = \frac{\sum_{w \in W^{\sigma}} (-1)^{\ell(w)} e^{w(\lambda + \rho)}}{\sum_{w \in W^{\sigma}} (-1)^{\ell(w)} e^{w(\rho)}}.$$  

Proof. Let $V^\sigma(\lambda)$ be the irreducible representation of $(G^\sigma)^{\vee}$ with highest weight $\lambda$. By Weyl character formula for $G^\sigma$, we have $\sum_{\mu \in P(\lambda)^\sigma} \dim V^\sigma_\mu(\lambda) e^\mu = \sum_{w \in W^{\sigma}} (-1)^{\ell(w)} e^{w(\lambda + \rho)}$.

Comparing with our definition of twining character for $G$, we see that it is equivalent to show $\text{trace}(\sigma|_{V_\mu(\lambda)}) = \dim V^\sigma_\mu(\lambda)$, for any $\mu \in P(\lambda)^\sigma$. By Lemma 4.1 $\text{trace}(\sigma|_{V_\mu(\lambda)}) = z(\text{MVC}_G^\mu(\lambda)^\sigma)$. Hence our proposition follows from Theorem 4.10.

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