Abstract

A standing conjecture in $L^2$-cohomology is that every finite CW-complex $X$ is of $L^2$-determinant class. In this paper, we prove this whenever the fundamental group belongs to a large class $G$ of groups containing e.g. all extensions of residually finite groups with amenable quotients, all residually amenable groups and free products of these. If, in addition, $X$ is $L^2$-acyclic, we also prove that the $L^2$-determinant is a homotopy invariant. Even in the known cases, our proof of homotopy invariance is much shorter and easier than the previous ones. Under suitable conditions we give new approximation formulas for $L^2$-Betti numbers.

Errata are added, rectifying some unproved statements about “amenable extension”: throughout, amenable extensions should be extensions with normal subgroups.

Keywords: $L^2$-determinant, $L^2$-Betti numbers, approximation, $L^2$-torsion, homotopy invariance

MSC: 58G26 (Primary), 55N25, 55P29 (Secondary)

1 Introduction

For a finite CW-complex $X$ with fundamental group $\pi$, $L^2$-invariants of the universal covering $\tilde{X}$ are defined in terms of the combinatorial Laplacians $\Delta_*$ on $C^*_c(\tilde{X}) = C^*_c(\tilde{X}) \otimes_{\mathbb{Z}\pi} l^2(\pi)$, which, after the choice of a cellular base, is a finite direct sum of copies of $l^2(\pi)$. $\Delta_p = (c_p \otimes \text{id})^*(c_p \otimes \text{id}) + \ldots$
(c_{p-1} \otimes \text{id})(c_{p-1} \otimes \text{id})^* \text{ becomes a matrix over } \mathbb{Z}\pi \subset \mathcal{N}\pi \text{ in this way, which acts on } l^2(\pi)^d \text{ via multiplication from the left. Here } \mathcal{N}\pi \text{ is the group von Neumann algebra with its natural trace } \text{tr}_\pi, \text{ defined as follows:}

1.1 \textbf{Definition.} For } \Delta = (a_{ij}) \in M(d \times d, \mathcal{N}\pi) \text{ set }
\text{tr}_\pi(\Delta) := \sum_i \text{tr}_\pi(a_{ii})

\text{where } \text{tr}_\pi(a) := a_1 = (a, 1)_{l^2(\pi)} \text{ is the coefficient of the trivial group element if } a = \sum_{g \in \pi} \lambda_g g \in \mathcal{N}\pi \subset l^2(\pi).

Particularly important are the spectral density functions

\[ F_p(\lambda) := F_{\Delta_p}(\lambda) := \text{tr}_\pi \chi_{[0, \lambda]}(\Delta_p). \tag{1.2} \]

The } L^2\text{-Betti numbers are defined as }

\[ b^{(2)}_p(X) := b^{(2)}_p(\Delta_p) := F_{\Delta_p}(0) = \dim_\pi(\ker(\Delta_p)). \]

These are invariants of the homotopy type of } X.

Another important invariant is the \textit{regularized determinant}:}

1.3 \textbf{Definition.} For a positive and self-adjoint operator } \Delta \in M(d \times d, \mathcal{N}) \text{ over a finite von Neumann algebra } \mathcal{N} \text{ with spectral density function } F_\Delta \text{ define }

\[ \ln \det_\mathcal{N}(\Delta) := \begin{cases} \int_{0^+}^{\infty} \ln(\lambda) dF_\Delta(\lambda); & \text{if the integral converges} \\ -\infty; & \text{otherwise} \end{cases} \]

Sometimes, this regularized determinant is called \textit{Fuglede-Kadison determinant}.

This gives rise to the definition:

1.4 \textbf{Definition.} A self-adjoint operator } \Delta \text{ as above is said to be of } \pi\text{-determinant class if and only if }

\[ \int_{0^+}^{1} \ln \lambda dF_\Delta(\lambda) > -\infty. \]

The space } X \text{ is said to be of } \pi\text{-determinant class if the Laplacian } \Delta_p \text{ is of determinant class for every } p.

1.5 \textbf{Conjecture.} \textit{Every finite CW-complex is of determinant class.}
If \( X \) is of \( \pi \)-determinant class and all \( L^2 \)-Betti numbers are zero, then we can define its (additive) \( L^2 \)-Reidemeister torsion

\[
T^{(2)}(X) := \sum_p (-1)^p p \ln \det_\pi \Delta_p.
\]

Burghelea et.al. \cite{1} show that \( L^2 \)-Reidemeister torsion is equal to \( L^2 \)-analytical torsion (for closed manifolds) and therefore is a generalization of the volume of a hyperbolic manifold.

Lück \cite{8, 1.4} shows that this torsion is an invariant of the simple homotopy type of \( X \). He conjectures

1.6 Conjecture. \( L^2 \)-Reidemeister torsion is a homotopy invariant,

and proves the following theorem \cite{8, 1.4}:

1.7 Theorem. The \( L^2 \)-Reidemeister torsion is a homotopy invariant of \( L^2 \)-acyclic finite CW-complexes with fundamental group \( \pi \) if and only if for every (invertible!) \( A \in \text{Gl}(d \times d, \mathbb{Z}\pi) \) the regularized determinant is zero: \( \ln \det_\pi (A^* A) = 0 \).

1.8 Remark. In fact, \( \ln \det_\pi (\cdot^* \cdot) \) factors through the Whitehead group of \( \pi \). The corresponding homomorphism is denoted \( \Phi \) by Lück, but we will write \( \ln \det_\pi \) for the map on \( W h(\pi) \) as well.

1.9 Remark. Mathai and Rothenberg \cite{10, 2.5} extend the study of the \( L^2 \)-determinants from the \( L^2 \)-acyclic to the general case, dealing with determinant lines instead of complex numbers. Without any difficulty, this could be done in our more general situation as well. Because this would only complicate the notation and seems not to be of particular importance, we will not carry this out.

Suppose that \( \pi \) is residual, i.e. it contains a nested sequence of normal subgroups \( \pi = \pi_1 \supset \pi_2 \supset \cdots \) such that \( \bigcap_i \pi_i = \{1\} \). Then we construct the corresponding coverings \( X_i \) of \( X \) with fundamental group \( \pi_i \). \( L^2 \)-invariants are defined for arbitrary normal coverings (the relevant von Neumann algebra is the one of the group of deck transformations). In the given situation we conjecture

1.10 Conjecture. In the situation just described, for every \( p \), the \( L^2 \)-Betti numbers \( b^{(2)}_p(X_i) \) converge as \( i \to \infty \) and

\[
\lim_{i \to \infty} b^{(2)}_p(X_i) = b^{(2)}_p(X).
\]
The projections $\pi \to \pi/\pi_i$ induce maps $p_i : M(d \times d, \mathbb{Z}\pi) \to M(d \times d, \mathbb{Z}\pi/\pi_i)$, and the Laplacian on $X_i$ (considered as such a matrix) is just the image of the Laplacian on $\tilde{X}$. Therefore, Conjecture 1.10 follows from the following conjecture about such matrices (and, in fact is equivalent as follows from the proof of [9, 2.2]).

**1.11 Conjecture.** For $A \in M(d \times d, \mathbb{Z}\pi)$ set $A_i := p_i(A)$. Then

$$\lim_{i \to \infty} \dim_{\pi/\pi_i}(\ker A_i) = \dim_{\pi}(\ker A).$$

The Conjectures 1.5, 1.6 and 1.10 are proven for residually finite groups by Lück [7, 0.5]. Using the ideas of Lück in a different context, Dodziuk and Mathai prove Conjecture 1.5 for amenable $\pi$ [3, 0.2]. They also establish an approximation theorem for $L^2$-Betti numbers of a slightly different type in this case.

Michael Farber [4] generalises Lück’s results to sequences of finite dimensional coefficients, which converge to $l^2(\pi)$ (one can interpret the covering cohomology as cohomology with coefficients in $l^2(\pi/\pi_i)$). In this sense, we are interested in special but infinite dimensional coefficient systems.

The aim of this paper is to extend the results of Lück and Dodziuk/Mathai to the following larger class of groups (Conjectures 1.10 and 1.11 have to be extended and modified suitably).

**1.12 Definition.** Let $\mathcal{G}$ be the smallest class of groups which contains the trivial group and is closed under the following processes:

- If $U < \pi$ is any subgroup such that $U \in \mathcal{G}$ and the discrete homogeneous space $\pi/U$ admits a $\pi$-invariant metric which makes it to an amenable discrete metric space, then $\pi \in \mathcal{G}$. (For our notion of amenability compare Section 4. The most important example is: $U$ normal and $\pi/U$ is an amenable group.)

- If $\pi = \text{dirlim}_{i \in I} \pi_i$ is the direct limit of a directed system of groups $\pi_i \in \mathcal{G}$, then $\pi \in \mathcal{G}$, too.

- If $\pi = \text{invlim}_{i \in I} \pi_i$ is the inverse limit of a directed system of groups $\pi_i \in \mathcal{G}$, then $\pi \in \mathcal{G}$, too.

- The class $\mathcal{G}$ is closed under taking subgroups.

**1.13 Remark.** It follows immediately from the definition that $\mathcal{G}$ contains all amenable groups, is closed under directed unions and is residually closed. More details can be found in section 2.
The main theorem of the paper is the following:

1.14 Theorem. Suppose $\pi$ belongs to the class $\mathcal{G}$. Then for every CW-complex with fundamental group $\pi$ which has finitely many cells in each dimension, the Conjectures 1.5 and 1.6 are true. Approximation results which generalize 1.10 and 1.11 are valid under the condition that all the occurring groups belong to $\mathcal{G}$.

1.15 Remark. Clair proves in [2] the Conjectures 1.5, 1.6 and 1.10 for the large class residually amenable fundamental groups, using [10, 2.5].

Our general principle is to use the methods of Lück (and Dodziuk, Mathai, Rothenberg) and to check carefully which is the most general situation they apply to.

1.16 Remark. So far, no example of a countable group which does not belong to the class $\mathcal{G}$ has been constructed. Good candidates for such examples are finitely generated simple groups which are not amenable, e.g. groups containing a free group with two generators.

On the other hand, we can give no example of a non-residually amenable group which belongs to $\mathcal{G}$, either. In any case, our description of $\mathcal{G}$ leads easily to many properties like closedness under direct sums and free products, which (if true at all) are probably much harder to establish for the class of residually amenable groups.

In fact, we prove a little bit more than Theorem 1.14. Namely, we show that the relevant properties are stable under the operations characterizing the class $\mathcal{G}$. We use the following definitions:

1.17 Definition. Let $C$ be any property of discrete groups. It is said to be

- **stable under direct/inverse limits** if $C$ is true for $\pi$ whenever $\pi$ is a direct/inverse limit of a directed system of groups which have property $C$.

- **subgroup stable** if every subgroup $U < \pi$ of a group with property $C$ shares this property, too.

- **stable under amenable extensions** if $\pi$ has property $C$ whenever it contains a subgroup $U$ with property $C$ such that the homogeneous space $\pi/U$ is amenable.

The properties we have in mind are listed in the following definition:

1.18 Definition. Let $\pi$ be a discrete group. We say
• \( \pi \) is of determinant class, if \( \Delta \) is of \( \pi \)-determinant class \( \forall \Delta \in M(d \times d, \mathbb{Z}_\pi) \) which is positive and self-adjoint;

• \( \pi \) has semi-integral determinant if \( \ln \det_\pi(\Delta) \geq 0 \) for every \( \Delta \in M(d \times d, \mathbb{Z}_\pi) \) which is positive self-adjoint. In particular every such \( \Delta \) is of \( \pi \)-determinant class, i.e. \( \pi \) itself is of determinant class;

• \( \pi \) has Whitehead-trivial determinant if \( \ln \det_\pi(A^*A) = 0 \) \( \forall A \in Wh(\pi) \).

1.19 Theorem. Whitehead-trivial determinant is stable under direct and inverse limits and is subgroup stable.

1.20 Remark. In the light of this theorem, we can use the fact that the Fuglede-Kadison determinant must be trivial on trivial Whitehead groups, e.g. for every torsion free discrete and cocompact subgroup of a Lie group with finitely many components [5, 2.1]. Waldhausen shows that the Whitehead group is trivial for another class of groups, including torsion free one-relator groups and many fundamental groups of 3-manifolds [11, 17.5].

The validity of the isomorphism conjecture of Farrell and Jones would imply that the Whitehead group is trivial if \( \pi \) is torsion free.

We can define the class \( G' \) as the smallest class of groups which contains \( G \), in addition any class of groups whose Whitehead group is (known to be) trivial, and which is closed under taking subgroups, direct and inverse limits. Then every group in \( G' \) has Whitehead trivial determinant, i.e. \( L^2 \)-torsion is a homotopy invariant for \( L^2 \)-acyclic finite CW-complexes with such a fundamental group.

1.21 Theorem. The property “semi-integral determinant” is stable under direct and inverse limits, subgroup stable and stable under amenable extensions.

1.22 Theorem. If for a group \( \pi \) and \( \forall A \in Wh(\pi) \) we have \( \ln \det_\pi(A^*A) \geq 0 \), then the Fuglede-Kadison determinant is trivial on \( Wh(\pi) \).

In particular, semi-integral determinant implies Whitehead-trivial determinant.

Proof. \( A \in Wh(\pi) \) implies \( A \) has an inverse \( B \in Wh(\pi) \). Now by [8, 4.2]

\[
0 = \ln \det_\pi(\text{id}) = \ln \det_\pi((AB)^*AB) = \ln \det_\pi(A^*A) + \ln \det_\pi(B^*B) \geq 0 + \geq 0
\]

and the statement follows.  

It follows from the induction principle 2.2 and the fact that the trivial group has semi-integral determinant (Lemma 6.8) that 1.21 and 1.22 imply the first part of our main theorem 1.14.
2 Properties of the class $\mathcal{G}$

We proceed with a more precise definition of the class $\mathcal{G}$, similar to the description Linnel gives of his class $\mathcal{C}$ in [6, p. 570].

2.1 Definition. For each ordinal $\alpha$, we define a class of groups $\mathcal{G}_\alpha$ inductively.

- $\mathcal{G}_0$ consists of the trivial groups.
- If $\alpha$ is a limit ordinal, then $\mathcal{G}_\alpha$ is the union of $\mathcal{G}_\beta$ with $\beta < \alpha$.
- If $\alpha$ has a predecessor $\alpha - 1$, then $\mathcal{G}_\alpha$ consists of all groups which
  - are subgroups of groups of $\mathcal{G}_{\alpha-1}$
  - contain a subgroup $U$ which belongs to $\mathcal{G}_{\alpha-1}$ such that the quotient space is an amenable homogeneous space
  - are direct or inverse limits of directed systems of groups in $\mathcal{G}_{\alpha-1}$.

By definition, a group is in $\mathcal{G}$ if it belongs to $\mathcal{G}_\alpha$ for some ordinal $\alpha$.

The class $\mathcal{G}$ is defined by (transfinite) induction. Therefore, properties of the groups in $\mathcal{G}$ can be proven by induction, too. More precisely, the following induction principle is valid:

2.2 Proposition. Suppose a property $C$ of groups is shared by the trivial group, and the following is true:

- whenever $K$ has property $C$ and $K < \pi$ with $\pi/K$ an amenable homogeneous space, then $\pi$ has property $C$ as well;
- whenever $\pi$ is a direct or inverse limit of a directed system of groups with the property $C$, then $\pi$ has property $C$;
- property $C$ is inherited by subgroups.

Then property $C$ is shared by all groups in the class $\mathcal{G}$.

Proof. The proof of the induction principle is done by transfinite induction.

By assumption, $C$ holds for $\mathcal{G}_0$. We have to establish $C$ for every group in $\mathcal{G}_\alpha$, granted its validity for groups in $\mathcal{G}_\beta$ for all $\beta < \alpha$. If $\alpha$ is a limit ordinal, this is trivial. If $\alpha$ has a predecessor $\alpha - 1$, the assumptions just match the definition of $\mathcal{G}_\alpha$, so the statement follows.

Now, we study the properties of the class $\mathcal{G}$. 
2.3 Proposition. The class $\mathcal{G}$ is closed under directed unions.

Proof. A directed union is a special case of a directed direct limit. \hfill \square

2.4 Proposition. The class $\mathcal{G}$ is residually closed. This means that if $\pi$ contains a nested sequence of normal subgroups $\pi_1 \supset \pi_2 \supset \ldots$ with trivial intersection and if $\pi/\pi_i \in \mathcal{G}$ for all $i$ then also $\pi \in \mathcal{G}$.

Proof. The inverse system of groups $\pi/\pi_i$ has some inverse limit $G$. The system of maps $\pi \to \pi/\pi_i$ induces a homomorphism $\pi \to G$. If $g \in \pi$ is mapped to $1 \in G$, then $g$ has to be mapped to $1 \in \pi/\pi_i \forall i$, i.e. $g \in \bigcap_i \pi_i = \{1\}$. As a directed limit, $G \in \mathcal{G}$, and as a subgroup of $G$, also $\pi \in \mathcal{G}$, as well. \hfill \square

2.5 Theorem. If $U$ belongs to $\mathcal{G}$ and $i : U \to U$ is any group homomorphism, then the “mapping torus”-extension of $U$ with respect to $i$

$$\pi = \langle u \in U, t| t^{-1}ut = i(u), u \cdot v = (uv); \forall u, v \in U \rangle$$

also belongs to $\mathcal{G}$ (if $i$ is injective, this is a special example of an HNN-extension).

Proof. There is a canonical projection $\pi \to \mathbb{Z}$ sending $u \in U$ to $0$ and $t$ to $1$. Denote its kernel by $K$. We will show that $K$ belongs to $\mathcal{G}$, then so does $\pi$ because it is an extension of $K$ with amenable quotient $\mathbb{Z}$.

Now $K$ is the direct limit of the sequence

$$U \xrightarrow{i} U \xrightarrow{i} U \xrightarrow{i} U \ldots$$

and belongs to $\mathcal{G}$, which is closed under taking direct limits. \hfill \square

2.6 Remark. Although such a mapping-torus extension of a finitely presented group is finitely presented again, the kernel $K$ we used in the proof may very well not even admit a finite set of generators. This is one instance where it is useful to allow arbitrary groups, even if one is only interested in fundamental groups of finite CW-complexes.

For the next property, we use the induction principle.

2.7 Proposition. $\mathcal{G}$ is closed under forming

(1) direct sums and direct products;

(2) free products.
Proof. We have to check the conditions for the induction principle. Fix \( \pi \in \mathcal{G} \).

(1) If \( U < G \) then \( U \times \pi < G \times \pi \). If \( G \times \pi \in \mathcal{G} \), then the same is true for \( U \times \pi \).

If \( G \) is the (direct or inverse) limit of the directed system of groups \( G_i \), then \( G \times \pi \) is the limit of the system \( G_i \times \pi \) (compare Lemma 2.8 or 2.9). If (by assumption) \( G_i \times \pi \in \mathcal{G} \), then \( G \times \pi \in \mathcal{G} \). Finally, if \( U < G \), and \( G/U \) is amenable, then \( U \times \pi < G \times \pi \) with the same amenable quotient. Therefore, \( U \times \pi \in \mathcal{G} \Rightarrow G \times \pi \in \mathcal{G} \).

(2) For the free products, the first step is to prove: \( \ast_{i \in I} \mathbb{Z}/4 \in \mathcal{G} \) for every index set \( I \). This is the direct limit of finite free products of copies of \( \mathbb{Z}/4 \), therefore we have to prove the statement for finite \( I \). Now \( \ast_{i=1}^n \mathbb{Z}/4 \) is a subgroup of \( \mathbb{Z}/4 \ast \mathbb{Z}/4 \) (contained in the kernel of the projection onto one factor), and \( \mathbb{Z}/4 \ast \mathbb{Z}/4 \) is virtually free, i.e. an extension of a residually finite (the free group) with an amenable (finite) group. Therefore \( \mathbb{Z}/4 \ast \mathbb{Z}/4 \) belongs to \( \mathcal{G} \).

Next we show that \( \pi \ast (\ast_{j \in J} \mathbb{Z}/4) \in \mathcal{G} \) for every set \( J \). We prove this using the induction principle. For \( \pi = 1 \) this is the conclusion of the first step. If \( \pi \) is a limit of \( (\pi_i)_{i \in I} \), or a subgroup of \( G \), then \( \pi \ast (\ast_{j \in J} \mathbb{Z}/4) \) is a subgroup of the limit of \( \pi_i \ast (\ast_{j \in J} \mathbb{Z}/4) \) (compare Lemma 2.8 and Lemma 2.9) or a subgroup of \( G \ast (\ast_{j \in J} \mathbb{Z}/4) \), and we can apply that \( \mathcal{G} \) is subgroup closed. If \( U < \pi \) and \( \pi/U \) is amenable, \( \pi \ast (\ast_{j \in J} \mathbb{Z}/4) \) acts on \( \pi/U \). We get a new point stabilizer, which is isomorphic to the free product of \( U \) with \( \ast_{G/U} (\ast_{i \in I} \mathbb{Z}/4) \). Fortunately, the induction hypothesis applies with the free product of an arbitrary number of copies of \( \mathbb{Z}/4 \).

As the next step we show that \( \ast_{i \in I} \pi \in \mathcal{G} \). This follows (as \( \ast_{i \in I} \pi \) is a direct limit) from the corresponding statement for \( I \) finite, and these are subgroups of \( \pi \ast \mathbb{Z}/4 \), contained in the kernel of the projection onto \( \mathbb{Z}/4 \).

\( \pi_1 \ast \pi_2 \) is contained in \( (\pi_1 \times \pi_2) \ast (\pi_1 \times \pi_2) \), and the general statement follows by induction and taking limits.

In the proof of Proposition 2.7 we have used the following two lemmas.

2.8 Lemma. If \( \pi \) is the direct limit of a system of groups \( \pi_i \) and \( G \) is any group, then \( \pi \ast G \) is the direct limit of \( \pi_i \ast G \), and \( \pi \times G \) is the direct limit of \( \pi_i \times G \).
Proof. There are obvious maps from $\pi_* G$ to $\pi * G$ and from $\pi \times G$ to $\pi \times G$.

Suppose one has consistent maps from $\pi_* G$ (or $\pi_* G$) to some group $X$. Since $\pi$ and $G$ both are subgroups of $\pi_* G$ (or of $\pi \times G$), this means that we have a consistent family of maps on $\pi_i$ multiplied with a fixed map on $G$. Therefore (from the properties of products) there exists exactly one map from $\pi_* G$ (or $\pi \times G$) to some group $X$ making all the diagrams commutative (for the commutative product note that the union of the images of $\pi_i$ in $X$ commutes with the image of $G$).

2.9 Lemma. If $\pi$ is an inverse limit of a system of groups $\pi_i$ and $G$ is any group, then $\pi_* G$ is contained in the inverse limit $X$ of $\pi_* G$.

The inverse limite of $\pi \times G$ is $\pi \times G$.

Proof. First, we look at the free products:
We have a consistent family of homomorphisms from $\pi_* G$ to $\pi_* G$, therefore a homomorphism from $\pi * G$ to $X$. An element $x = p_1 g_1 \ldots p_n g_n \in \pi * G$ is in the kernel of this homomorphism iff it is mapped to $1 \in \pi_* G$ for every $i \in I$. This can not happen if $1 \neq x \in \pi$. It remains to check the case $1 \neq g_1 \in G$. We may assume that $g_2 \neq 1$ iff $p_2 \neq 1$. If $\phi_i : \pi \rightarrow \pi_i$ is the natural homomorphism, then $x$ is mapped to $\phi_i(p_1)\phi_i(p_2)g_2 \ldots g_n \in \pi_i * G$. If this is trivial, but $g_1 \neq 1$, necessarily $\phi_i(p_2) = 1 \forall i$. This implies $p_2 = 1$, i.e. $x = p_1 g_1$, since we wrote $x$ in normal form. But then $\phi_i * \text{id}(x) \neq 1 \forall i$, and the kernel of the map to $X$ is trivial, as required.

For the commutative product, let $X$ be a group together with a consistent family of morphisms to $\pi_i \times G$. These have the form $x \mapsto (\phi_i(x), f_i(x))$. Composition with the projections to $\pi_i$ or to $G$ shows that $\pi_i$ is a consistent family of morphisms to $\pi_i$, and $f = f_i : X \rightarrow G$ all coincide. Let $\phi : X \rightarrow \pi$ be the limit. Then $\phi \times f : X \rightarrow \pi \times G$ is a unique homomorphism which makes all relevant diagrams commutative.

3 Passage to subgroups

Suppose $U \subset \pi$ is a subgroup of a discrete group. A positive self-adjoint matrix $A \in M(d \times d, \mathbb{Z}U)$ can also be considered as a matrix over $\mathbb{Z}\pi$. Denote the operators with $A_U$ and $A_\pi$, respectively. Recall the following well known fact:

3.1 Proposition. The spectral density functions of $A_U$ and $A_\pi$ coincide.
**Proof.** Choose a set of representatives \( \{g_i\}_{i \in I} \) with \( 0 \in I \) and \( g_0 = 1 \), to write 
\[ \pi = \bigoplus_{i \in I} U g_i. \]
Then 
\[ \ell^2(\pi)^d = \bigoplus_{i \in I} \ell^2(U)^d g_i. \]
With respect to this splitting, the action of \( A_\pi \) on \( \ell^2(\pi) \) is diagonal and, restricted to each of the summands \( \ell^2(U)^d g_i \), is multiplication by \( A_u \) from the left. It follows that every spectral projection \( \chi_{[0,\lambda]}(A_\pi) \) is diagonal with \( \chi_{[0,\lambda]}(A_U) \) on the diagonal. Then
\[ F_{A_\pi}(\lambda) = \sum_{k=1}^d \langle \chi_{[0,\lambda]}(A_\pi) e^U_k, e^U_k \rangle = \sum_{k=1}^d \langle \chi_{[0,\lambda]}(A_\pi) e^U_k \cdot 1, e^U_k \cdot q \rangle = \sum_{k=1}^d \langle \chi_{[0,\lambda]}(A_U) e^U_k, e^U_k \rangle = F_{A_U}(\lambda). \]

**3.2 Corollary.** The properties of Definition 1.18 are inherited by subgroups.

In particular, we have proven the subgroup part of Theorems 1.19 and 1.21.

### 4 Amenable extensions

**4.1 Definition.** A discrete homogeneous space \( \pi/U \) is called amenable, if on \( \pi/U \) we find a \( \pi \)-invariant integer-valued metric \( d: \pi/U \times \pi/U \to \mathbb{N} \) such that

- sets of finite diameter are finite
- for every \( K > 0 \), \( \epsilon > 0 \) there is a finite subset \( X \subset \pi/U \) with
  \[ |N_K(X)| \leq \epsilon |X| \]
  where \( N_K(X) := \{ x \in \pi/U; d(x, X) \leq K \text{ and } d(x, \pi/U - X) \leq K \} \) is the \( K \)-neighborhood of the boundary of \( X \).

A nested sequence of finite subsets \( K_1 \subset K_2 \subset \ldots \) is called an amenable exhaustion of \( \pi/U \) if \( \bigcup K_\alpha = \pi/U \) and if \( \forall K > 0 \) and \( \epsilon > 0 \) we find \( N \in \mathbb{N} \) so that \( |N_K(K_i)| \leq \epsilon |K_i| \forall i \geq N \).

**4.2 Lemma.** Every amenable homogeneous space \( \pi/U \) admits an amenable exhaustion.
Proof. By assumption for \(n, K \in \mathbb{N}\) we find \(X_{n,K}\) with \(|N_K(X_{n,K})| \leq \frac{1}{n}|X_{n,K}|\). Fix some base point in \(\pi/U\). Since \(\pi\) acts transitively on \(\pi/U\) and the metric is \(\pi\)-invariant, we may assume after translation that the base point is contained in each of the \(X_{n,K}\). Now we construct the exhaustion \(E_i\). Set \(E_1 := X_{1,1}\) inductively. For the induction suppose \(E_1, \ldots, E_n\) are constructed with \(|N_k(E_k)| \leq \frac{1}{k}|E_k|\) for \(k = 1, \ldots, n\). Suppose \(E_n\) has diameter \(d \in \mathbb{N}\) with \(2d \geq n + 1\). Set \(E_{n+1} := E_n \cup X_{n+1,2d}\). Then \(N_{n+1}(E_{n+1}) \subset N_{2d}(E_{n+1})\). On the other hand, by the triangle inequality \(N_d(E_n) \subset X_{n+1,2d}\) and therefore \(N_{2d}(E_{n+1}) = N_{2d}(X_{n+1,2d})\), and it follows
\[
|N_{n+1}(E_{n+1})| \leq |N_{2d}(X_{n+1,2d})| \leq \frac{1}{n+1} |X_{n+1,2d}| \leq \frac{1}{n+1} |E_{n+1}|.
\]
The claim follows.

4.3 Example. If \(U\) is a normal subgroup and \(\pi/U\) is an amenable group, it is an amenable homogeneous space, too.

4.4 Definition. Suppose \(\pi\) is a group with subgroup \(U\) and amenable quotient \(\pi/U\). Choose an amenable cover \(X_1 \subset X_2 \subset \cdots \subset \pi/U\). For \(B \in M(d \times d, NU)\) set
\[
\text{tr}_m(B) := \frac{1}{|X_m|} \text{tr}_U(B).
\]
For \(\Delta \in M(d \times d, NU)\) positive and self-adjoint set \(\Delta_m := P_m \Delta P_m\) where \(P_m = \text{diag}(p_m)\) with \(p_m \in B(l^2(\pi))\) is given by projection onto the closed subspace generated by the inverse image of \(X_m\). Then \(\Delta_m\) no longer belongs to \(NU\) but still to \(NU\) and we define (by slight abuse of notation)
\[
F_{\Delta_m}(\lambda) := \text{tr}_m(\chi_{[0,\lambda]}(\Delta_m)),
\]
\[
\ln \text{det}_U(\Delta_m) := \int_{0^+} \ln(\lambda) \, dF_{\Delta_m}(\lambda)
\]
using the new \(F_{\Delta_m}\).

Here \(\Delta_m\) is considered as operator on the image of \(P_m\). This subspace is \(NU\)-isomorphic to \(l^2(U)^d|X_m|\).

Note that there are two meanings of \(F_{\Delta_m}(\lambda)\) and \(\ln \text{det}_U(\Delta_m)\) (using either \(\text{tr}_U\) or \(\text{tr}_m\)), but in the amenable case we will always use the variant where we divide by the volume of the sets \(X_m\).

The following is one of the key lemmas which make our (respectively Lück’s) method work:
4.5 Lemma. In the situation above, we find $K \in \mathbb{R}$ independent of $m$, so that
\[ \|\Delta\| \leq K \quad \text{and} \quad \|\Delta_m\| \leq K \quad \forall m \in \mathbb{N}. \]

Proof. This is an immediate consequence of the fact that $\|P\| \leq 1$ for every projection $P$ and $\Delta_m = P_m \Delta P_m$ with projections $P_m$. \hfill \square

We now establish the second key lemma. It generalizes a corresponding result of Dodziuk/Mathai [3, 2.3] where $U$ is trivial. We need the result only for matrices over $\mathbb{C}_\pi$, but for possible other applications we prove a more general statement here.

4.6 Lemma. Let $p(x) \in \mathbb{C}[x]$ be a polynomial. Suppose $\Delta \in M(d \times d, N\pi)$. Then
\[ \text{tr}_\pi p(\Delta) = \lim_{m \to \infty} \text{tr}_m p(\Delta_m). \]

Proof. By linearity it suffices to prove the statement for the monomials $x^N$, $N \in \mathbb{N}$.

Pull the metric on $\pi/U$ back to $\pi$ to get some semimetric on $\pi$. Denote the inverse image of $X_k$ in $\pi$ with $X'_k$. We have to compare $(\Delta^N g \epsilon_k, g \epsilon_k)$ and $(\Delta^N_m g \epsilon_k, g \epsilon_k)$ for $g = g_i \in X'_m$, in particular for those $g_i$ with $B_a(g_i) \subset X'_m$. Of course, we don’t find $a \in \mathbb{R}$ such that the difference is zero. However, we will show that (for fixed $N$) we can find $a$ such that the difference is sufficiently small.

First observe that $P_m g \epsilon_k = g \epsilon_k$ if $g \in X'_m$, and (since $P_m$ is self-adjoint)
\[ ((P_m \Delta P_m)^N g \epsilon_k, g \epsilon_k) = (\Delta P_m \Delta \ldots P_m \Delta g \epsilon_k, g \epsilon_k). \]

Now the following sum is a telescope and therefore
\[
\begin{align*}
\Delta P_m \Delta \ldots P_m \Delta &= \\
\Delta^N - \Delta(1 - P_m)\Delta^{N-1} - \Delta P_m \Delta(1 - P_m)\Delta^{N-2} - \ldots - \Delta P_m \ldots \Delta(1 - P_m)\Delta.
\end{align*}
\]

(4.7)

It follows for $g \in X'_m$
\[
\begin{align*}
| (\Delta^N g \epsilon_k, g \epsilon_k) - (\Delta^N_m g \epsilon_k, g \epsilon_k) | &\leq \sum_{i=1}^{N-1} \left| (1 - P_m)\Delta^i g \epsilon_k, (\Delta^* P_m)^{N-i} g \epsilon_k \right| \\
&\leq \sum_{i=1}^{N-1} |(1 - P_m)\Delta^i g \epsilon_k| \cdot \|\Delta^*\|^{N-i}.
\end{align*}
\]
Here we used the fact that the norm of a nontrivial projector is 1 and \(|ge_k| = 1\).

Fix \(\epsilon > 0\). For \(i = 1, \ldots, N - 1\) and \(k = 1, \ldots, d\) we have \(\Delta^i g e_k \in l^2(\pi)^d\).

It follows that we find \(R > 0\) so that

\[
|(1 - P_{B_R(g)}) \Delta^i g e_k| \leq \epsilon
\]  

(4.8)

where \(P_{B_R(g)}\) is the projector onto the closed subspace spanned by the elements in \(\bigcup_{k=1}^d B_R(g) e_k\). Since \(\Delta\) and the semimetric are \(\pi\)-invariant, this holds for every \(g \in \pi\) with \(R\) independent of \(g\). If the range of \(P_{B_R(g)}\) is contained in the range of \(P_m\), i.e. if \(B_R(g) \subseteq X'_m\) then (4.8) implies

\[
|(1 - P_m) \Delta^i g e_k| \leq \epsilon
\]

(since we have even more trivial Fourier coefficients in the standard orthonormal base coming from \(\pi\) of \(l^2(\pi)^d\)). Taken together, we get

\[
|\text{tr}_\pi \Delta^N - \text{tr}_m \Delta^N_m| \leq \frac{1}{|X_m|} \sum_{k=1}^d \sum_{i \in X_m} |(\Delta^N g_i e_k, g_i e_k) - ((\Delta^N_m) g_i e_k, g_i e_k)|
\]

\[
\leq \frac{1}{|X_m|} \sum_{k=1}^d \sum_{i \in X_m} \sum_{j=1}^{N-1} |(1 - P_m) \Delta^i g_i e_k| \cdot \|\Delta^i\|^{N-j}
\]

\[
\leq \frac{1}{|X_m|} \sum_{k=1}^d \sum_{i \in X_m - N_R(X_m)} \sum_{j=1}^{N-1} |(1 - P_m) \Delta^i g_i e_k| \cdot \|\Delta^i\|^{N-j}
\]

\[
\leq \frac{1}{|X_m|} \sum_{i \in N_R(X_m)} \sum_{k=1}^d \sum_{j=1}^{N-1} |(1 - P_m) \Delta^i g_i e_k| \cdot \|\Delta^i\|^{N-j}
\]

\[
\leq \epsilon d N \max_{j=1, \ldots, N-1} \{\|\Delta^i\|^{j}\} + \frac{N R(X_m)}{|X_m|} d N \max_{j=1, \ldots, N} \|1 - P_m\| \max_{j=1, \ldots, N} \{\|\Delta^i\|^{j} \cdot \|\Delta^i\|^{N-j}\}
\]

\[
= C_N
\]

Note that \(C_N\) and \(C'_N\) are independent of \(m\) and \(\epsilon\). Since \(X_m\) is an amenable extension, for every \(R\) we find \(m_R\) so that \(\frac{N_R(X_m)}{|X_m|}\) is smaller than \(\epsilon\) for every \(m \geq m_R\). Since \(\epsilon\) was arbitrary, the assertion of the lemma follows. \(\square\)

5 Direct and inverse limits

5.1 Remark. In this section, we study the properties of Definition 1.18 for direct and inverse limits. However, we will only deal with the apparently
weaker statements that each of the conditions holds for every \( \Delta \) of the form \( \Delta = A^*A \). The general case is a consequence of this since we can easily compare the self-adjoint \( \Delta \) with \( \Delta^2 = \Delta^* \Delta \), because \( F_\Delta(\lambda) = F_{\Delta^2}(\lambda^2) \).

We describe now the situation we are dealing with in this section:

**5.2 Definition.** Suppose the group \( \pi \) is the direct or inverse limit of a directed system of groups \( \pi_i, i \in I \). The latter means that we have a partial ordering \( < \) on \( I \), and \( \forall i, j \in I \) we find \( k \in I \) with \( i < k \) and \( j < k \). In the case of a direct limit, let \( p_i : \pi_i \rightarrow \pi \) the natural maps, in the case of an inverse limit, \( p_i : \pi \rightarrow \pi_i \).

Suppose \( A \in M(d \times d, \mathbb{C} \pi) \) is given. If \( \pi \) is an inverse limit, let \( A_i = p_i(A) \) be the image of \( A \) under the projection \( M(d, \pi) \rightarrow M(d, \pi_i) \). Set \( \Delta_i := A_i^*A_i \) (this follows from the algebraic description of the adjoint [7, p. 465]). In particular, all of the operators \( \Delta_i \) are positive. Define

\[
\text{tr}_i(\Delta_i) := \text{tr}_{\pi_i}(\Delta_i).
\]

\( F_{\Delta_i}(\lambda) \) is defined using the trace on the von Neumann algebra \( \pi_i (i \in I) \).

If we want to give a similar definition in the case where \( \pi \) is a direct limit, we have to make additional choices. Namely, let \( A = (a_{kl}) \) with \( a_{kl} = \sum_{g \in \pi} \lambda^g_{kl}g \). Then, only finitely many of the \( \lambda^g_{kl} \) are nonzero. Let \( V \) be the corresponding finite collection of \( g \in \pi \). Since \( \pi \) is the direct limit of \( \pi_i \) we find \( j_0 \in I \) such that \( V \subset p_{j_0}(\pi_{j_0}) \). Choose an inverse image for each \( g \) in \( \pi_{j_0} \). This gives a matrix \( A_{j_0} \in M(d \times d, \mathbb{C} \pi_{j_0}) \), which is mapped to \( A_i \in M(d \times d, \pi_i) \) for \( i > j_0 \). Now we apply the above constructions to this net \( (A_i)_{i>j_0} \). Note that this definitely depends on the choices.

For notational convenience, we choose some \( j_0 \in I \) also when we deal with an inverse limit.

Now, we will establish in this situation the two key lemmas corresponding to Lemma 4.5 and Lemma 4.6.

For the first lemma, instead of working with the norm of operators, we will use another invariant which gives an upper bound for the norm but is much easier to read off:

**5.3 Definition.** Let \( \pi \) be a discrete group, \( \Delta \in M(d \times d, \mathbb{Z} \pi) \). Set

\[
K(\Delta) := d^2 \max_{i,j} |a_{i,j}|_1 \text{ where } |\cdot|_1 \text{ is the } L^1-\text{norm on } \mathbb{C} \pi \subset l^1(\pi).
\]
5.4 Lemma. Adopt the situation of Definition 5.2. One can find $K \in \mathbb{R}$, independent of $i$, such that
\[ \|A_i\| \leq K \quad \forall i > j_0 \quad \text{and} \quad \|A\| \leq K. \]
Proof. Lück [7, 2.5] shows that $\|A_i\| \leq K(A_i)$. It follows from the construction of $A_i$ that $K(A_i) \leq K(A)$ in the case of an inverse limit, and $K(A_i) \leq K(A_{j_0})$ in the case of a direct limit, with $j_0$ as above. In both cases, we obtain a uniform bound for $\|A_i\|$.

5.5 Lemma. Adopt the situation of Definition 5.2. Let $p(x) \in \mathbb{C}[x]$ be a polynomial. There exists $i_0 \in I$ depending on the matrix $A$ and on $p$ such that
\[ \text{tr}_{\pi}(p(A)) = \text{tr}_{i}(p(A_i)) \quad \forall i > i_0. \]
Proof. Suppose $\pi$ is the inverse limit of the $\pi_i$. We follow [7, 2.6]. Let $p(A) = \left( \sum_{g \in \pi} \lambda_g^{kl} g \right)_{k,l=1,\ldots,d}$. Then
\[ \text{tr}_{\pi}(p(A)) = \sum_k \lambda_{1k}^{kk} \quad \text{and} \quad \text{tr}_{\pi_i}(p(A_i)) = \sum_k \sum_{g \in \ker p_i} \lambda_{g}^{kk}. \]
Since only finitely many of $\lambda_{ij}^{ij} \neq 0$ and $\pi$ is the inverse limit of the $\pi_i$, we find $i_0 \in I$ such that $\lambda_{ij}^{kk} \neq 0$ and $g \in \ker p_{i_0}$ implies $g = 1$. For $i > i_0$ the assertion is true.

If $\pi$ is the direct limit we have chosen $A_{j_0}$ with $p_{j_0}(A_{j_0}) = A$. Then $p_{j_0}(p(A_{j_0})) = p(A)$. However, there may be a $g \in \pi_{j_0}$ with nontrivial coefficient in $p(A_{j_0})$ with $p_{j_0}(g) = 1$, and this means that the relevant traces may differ. But still there are only finitely many $g$ with nontrivial coefficient in $p(A_{j_0})$, and since $\pi$ is the direct limit of $(\pi_i)_{j_0}^i$ we find $i_0$ such that $p_{i_0}(g) = 1$ for $g \in \pi_{i_0}$ with nontrivial coefficient in $p(A_{i_0})$ implies $g = 1$. Then the above reasoning shows $\text{tr}_{\pi_i}(p(A_i)) = \text{tr}_{\pi}(p(A)) \forall i > i_0$. Here for $i > j_0$ $A_i$ is the image of $A_{j_0}$ induced by the map $\pi_{j_0} \rightarrow \pi_i$. \qed

6 Approximation properties and proofs of stability statements

In this section, we use the information gathered so far to proof the statements of the introduction, in particular Theorem 1.21. This is done by studying limits of operators, therefore the same treatment yields approximation results for $L^2$-Betti numbers. The precise statements and conditions are given below.
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We have already dealt with the passage to subgroups.

For the rest of this section, assume the following situation:

6.1 Situation. The group $\pi$ is the direct or inverse limit of a directed system of groups $\pi_i$, or an amenable extension $U \to \pi \to \pi/U$ (write $\pi_i = U$ also in this case).

As described in [4.4 or 5.2] any matrix $\Delta$ over $\mathbb{C}\pi$ then gives rise to matrices $\Delta_i$ over $\pi_i$ (after the choice of an inverse image in the case of a direct limit, and after the choice of an amenable exhaustion for amenable extensions). Without loss of generality we assume that $\Delta = A^*A$ for another matrix $A$ over $\mathbb{C}\pi$ (this is explained in Remark 5.1). We also get spectral density functions $\Delta A_i(\lambda)$ defined using the group $\pi_i$ (remember that in the amenable case there is an additional normalisation).

The problem now is to obtain information about $\Delta A(\lambda)$ from the family $\Delta A_i(\lambda)$.

In particular, we want to show that $\Delta A_i(0)$ converges to $\Delta A(0)$. (Translated to geometry this means that certain $L^2$-Betti numbers converge.)

In short: we have

- A group $\pi$ and a matrix $\Delta \in M(d \times d, \mathbb{C}\pi)$
- A family $(\Delta_i)i \in I$ of matrices over $\mathbb{C}\pi_i$ which approximate $\Delta$ ($I$ is a directed system)
- Positive and normal trace functionals $tr_i$ (on a von Neumann algebra which contains $\Delta_i$) which are normalized in the following sense: If $\Delta = \text{id} \in M(d \times d, \mathbb{Z}\pi)$ then $tr_i(\Delta_i) = d \forall i$.
- If $\Delta$ lives over $\mathbb{Z}\pi$, then $\Delta_i$ is a matrix over $\mathbb{Z}\pi_i$.

6.2 Definition. Define

$$\Delta F(\lambda) := \lim \sup_i F_{\Delta_i}(\lambda),$$

$$\Delta F(\lambda) := \lim \inf_i F_{\Delta_i}(\lambda).$$

Remember $\lim \sup_{i \in I} \{x_i\} = \inf_{i \in I} \{\sup_{j \leq i} \{x_j\}\}$.

6.3 Definition. Suppose $F : [0, \infty) \to \mathbb{R}$ is monotone increasing (e.g. a spectral density function). Then set

$$F^+(\lambda) := \lim_{\epsilon \to 0^+} F(\lambda + \epsilon)$$

i.e. $F^+$ is the right continuous approximation of $F$. In particular, we have defined $\Delta F^+$ and $\Delta F^+$. 

6.4 Remark. Note that by our definition a spectral density function is right continuous, i.e. unchanged if we perform this construction.

To establish the first step in our program we have to establish the following functional analytical lemma (compare [7] or [2]):

6.5 Lemma. Let $\mathcal{N}$ be a finite von Neumann algebra with positive normal and normalized trace $\text{tr}_\mathcal{N}$. Choose $\Delta \in M(d \times d, \mathcal{N})$ positive and self-adjoint. If for a function $p_n : \mathbb{R} \to \mathbb{R}$
\[ \chi_{[0,\lambda]}(x) \leq p_n(x) \leq \frac{1}{n} \chi_{[0,K]}(x) + \chi_{[0,\lambda+1/n]}(x) \quad \forall 0 \leq x \leq K \quad (6.6) \]
and if $\|\Delta\| \leq K$ then
\[ F_{\Delta}(\lambda) \leq \text{tr}_\mathcal{N} p_n(\Delta) \leq \frac{1}{n} d + F_{\Delta}(\lambda + 1/n). \]
Here $\chi_S(x)$ is the characteristic function of the subset $S \subset \mathbb{R}$.

Proof. This is a direct consequence of positivity of the trace, of the definition of spectral density functions and of the fact that $\text{tr}_\mathcal{N}(1 \in M(d \times d, \mathcal{N})) = d$ by the definition of a normalized trace. □

6.7 Proposition. For every $\lambda \in \mathbb{R}$ we have
\[ \overline{F_{\Delta}}(\lambda) \leq F_{\Delta}(\lambda) = F_{\Delta}^+(\lambda) \leq F_{\Delta}^+(\lambda), \]
\[ F_{\Delta}(\lambda) = F_{\Delta}^+(\lambda) = F_{\Delta}^+(\lambda). \]

Proof. The proof only depends on our key lemmas [4.5] [4.6] [5.4] [5.5] These say
- $\exists K \in \mathbb{R}$ such that $\|\Delta_i\| \leq K \forall i \in I$
- For every polynomial $p \in \mathbb{C}[x]$ we have $\text{tr}_\pi(p(\Delta)) = \lim_i \text{tr}_i(p(\Delta_i))$.

For each $\lambda \in \mathbb{R}$ choose polynomials $p_n \in \mathbb{R}[x]$ such that inequality (6.6) is fulfilled. Note that by the first key lemma we find a uniform upper bound $K$ for the spectrum of all of the $\Delta_i$. Then by Lemma [6.6]
\[ F_{\Delta_i}(\lambda) \leq \text{tr}_i(p_n(\Delta_i)) \leq F_{\Delta_i}(\lambda + 1/n) + \frac{d}{n} \]
We can take the limes inferior and superior and use the second key lemma to get
\[ \overline{F_{\Delta}}(\lambda) \leq \text{tr}_\pi(p_n(\Delta)) \leq \overline{F_{\Delta}}(\lambda + 1/n) + \frac{d}{n}. \]
Now we take the limes for $n \to \infty$. We use the fact that $\text{tr}_\pi$ is normal and $p_n(\Delta)$ converges strongly inside a norm bounded set to $\chi_{[0,\lambda]}(\Delta)$. Therefore the convergence even is in the ultrastrong topology.

This implies

$$F_\Delta(\lambda) \leq F_\Delta^+(\lambda).$$

For $\epsilon > 0$ we can now conclude since $F_\Delta$ and $\overline{F_\Delta}$ are monoton

$$F_\Delta(\lambda) \leq F_\Delta(\lambda + \epsilon) \leq \overline{F_\Delta}(\lambda + \epsilon) \leq F_\Delta(\lambda + \epsilon).$$

Taking the limit $\epsilon \to 0^+$ gives (since $F_\Delta$ is right continuous)

$$F_\Delta(\lambda) = \overline{F_\Delta}^+(\lambda) = \overline{F_\Delta}^+(\lambda).$$

Therefore both of the inequalities are established.

The next step is to proof convergence results without taking right continuous approximations (at least for $\lambda = 0$). We are able to do this only under additional assumptions:

- From now on, $\Delta$ and therefore also $\Delta_i \forall i \in I$ are matrices over the integral group ring.

The following statement is used as start for the induction.

6.8 Lemma. The trivial group has semi-integral determinant.

Proof. Take $\Delta \in M(d \times d, \mathbb{Z})$ positive and self-adjoint. Then $\det_1(\Delta)$ is the product of all nonzero eigenvalues and therefore the lowest nonzero coefficient in the characteristic polynomial. Therefore, it is an integer $\neq 0$ and $\ln \det_1(\Delta) \geq 0$.

Now we give a proof of Theorem 6.21 and prove the corresponding approximation result.

6.9 Theorem. Suppose $\pi_i$ has “semitotal determinant” $\forall i \in I$, then the same is true for $\pi$, and $\dim_\pi(\ker \Delta) = F_\Delta(0) = \lim_i F_{\Delta_i}(0)$.

Proof. Choose $K \in \mathbb{R}$ such that $K > \|\Delta\|$ and $K > \|\Delta_i\| \forall i$. This is possible because of the key Lemma 4.5 or 5.4. Then

$$\ln \det_{\pi_i}(\Delta_i) = \ln(K)(F_{\Delta_i}(K) - F_{\Delta_i}(0)) - \int_{0^+}^K \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda} d\lambda.$$
If this is (by assumption) $\geq 0$, then since $F_{\Delta_i}(K) = \text{tr}_i(1_d) = d$ by our normalisation

$$
\int_{0^+}^K \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda} \, d\lambda \leq \ln(K)(d - F_{\Delta_i}(0)) \leq \ln(K)d.
$$

We want to establish the same estimate for $\Delta$. If $\epsilon > 0$ then

$$
\int_{\epsilon}^K \frac{F_{\Delta}(\lambda) - F_{\Delta}(0)}{\lambda} \, d\lambda = \int_{\epsilon}^K \frac{F_{\Delta}^+(\lambda) - F_{\Delta}(0)}{\lambda} \, d\lambda = \int_{\epsilon}^K \frac{F_{\Delta}(\lambda) - F_{\Delta}(0)}{\lambda}
$$

(since the integrand is bounded, the integral over the left continuous approximation is equal to the integral over the original function)

$$
\leq \int_{\epsilon}^K \frac{F_{\Delta}(\lambda) - \overline{F_{\Delta}}(0)}{\lambda} \leq \int_{\epsilon}^K \frac{\liminf_i F_{\Delta_i}(\lambda) - \limsup_i F_{\Delta_i}(0)}{\lambda}
\leq \int_{\epsilon}^K \frac{\liminf_i (F_{\Delta_i}(\lambda) - F_{\Delta_i}(0))}{\lambda}
\leq \liminf_i \int_{\epsilon}^K \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda} \leq d\ln(K).
$$

Since this holds for every $\epsilon > 0$, we even have

$$
\int_{0^+}^K \frac{F_{\Delta}(\lambda) - F_{\Delta}(0)}{\lambda} \, d\lambda \leq \sup \liminf_{\epsilon > 0} \int_{\epsilon}^K \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda} \, d\lambda \leq d\ln(K).
$$

The second integral would be infinite if $\lim_{\delta \to 0} \overline{F_{\Delta}(\delta)} \neq \overline{F_{\Delta}(0)}$. It follows that $\limsup_i F_{\Delta_i}(0) = F_{\Delta}(0)$. Since we can play the same game for every subnet of $I$, also $\liminf_i F_{\Delta_i}(0) = F_{\Delta}(0)$ i.e. the approximation property is true.
For the estimate of the determinant note that in the above inequality
\[
\sup_{\epsilon > 0} \liminf_{m \to \infty} \int_{\epsilon}^{K} \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda} \, d\lambda \\
\leq \liminf_{i} \sup_{\epsilon > 0} \int_{\epsilon}^{K} \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda} \, d\lambda = \liminf_{i} \int_{0^+}^{K} \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda} \, d\lambda \\
\leq \ln(K)(d - F_{\Delta_i}(0)).
\]
Therefore
\[
\ln \det_{\pi}(\Delta) = \ln(K)(d - F_{\Delta}(0)) - \int_{0^+}^{K} \frac{F_{\Delta}(\lambda) - F_{\Delta}(0)}{\lambda} \, d\lambda \\
\geq \ln(K)(d - \lim_{i} F_{\Delta_i}(0)) - \liminf_{i} \int_{0^+}^{K} \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda} \, d\lambda \\
= \limsup_{i} \left( \ln(K)(d - F_{\Delta_i}(0)) - \int_{0^+}^{K} \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda} \, d\lambda \right) \\
= \limsup_{i} \ln \det_{\pi_i}(\Delta_i) \geq 0. \quad \square
\]

6.10 Remark. The above reasoning does not show that the determinants converge. An unpublished example of Lück shows that this in general is wrong for matrices over the complex group ring of \(\mathbb{Z}\). But he shows that the statement is true for matrices over \(\mathbb{Z}[\mathbb{Z}]\). For other groups, the question is completely open.

6.11 Remark. The assumption “semi-integral determinant” is very strong. Originally, Lück as well as Dodziuk/Mathai and Clair use the following in some sense weaker property: a discrete group \(\pi\) is spectrally sublogarithmic if for \(\Delta \in M(d \times d, \mathbb{Z}\pi)\) positive and self-adjoint
\[
F_{\Delta}(\lambda) - F_{\Delta}(0) \leq d \frac{\ln R(\Delta)}{-\ln \lambda} \quad \text{for} \quad 0 < \lambda < 1.
\]
However, it is not clear how to find \(R(\Delta)\) such that the property is stable under limits as well as under amenable extensions.

As observed by Clair [2], instead of \(1/\ln(x)\) any other function which is right continuous at zero would work equally well.

6.12 Remark. We can establish the approximation results only under the assumption that the groups \(\pi_i\) have good properties, e.g. belong to the class \(\mathcal{G}\). It is interesting to note that for amenable groups every quotient group is amenable and belongs to \(\mathcal{G}\). It follows that Conjecture \[\ref{conjecture:1.10}\] holds if \(\pi_1(X)\) is amenable without additional assumptions.
If $\pi \in G$ and $U < \pi$ then also $U \in G$. Therefore, in case of an amenable exhaustion (i.e. $\pi/U$ is amenable and we approximate using this fact) the conditions which imply convergence are automatically fulfilled.

6.13 Remark. It would be possible to give geometrical interpretations of the more general approximation results. However, this seems to be very artificial and therefore is omitted here.

Proof of Theorem 1.19

It remains to show that Whitehead trivial determinant is stable under direct and inverse limits.

Proof of Theorem 1.19. We are still in the situation described at the beginning of this section, and, in addition, we assume that $\Delta$ is invertible in $M(d \times d, \mathbb{Z}_\pi)$ with inverse $B \in M(d \times d, \mathbb{Z}_\pi)$. In case $\pi$ is an inverse limit $\Delta_i$ and $B_i$ are images of projections of $\Delta$ and $B$, and therefore remain inverse to each other.

In case $\pi$ is a direct limit, we first lifted $\Delta$ to some $\Delta_{j_0}$. We may assume that we also can lift $B$ to $B_{j_0}$. Then $\Delta_{j_0}B_{j_0}$ is mapped to the identity over $\pi$. Since it has only finitely many nonzero coefficients, there is $j_1$ such that the image of $\Delta_{j_0}B_{j_0}$ over $\pi_{j_1}$ already is the identity, and similar for $B_{j_0}\Delta_{j_0}$. Therefore, we may assume that the lifts $\Delta_{j_0}$ and $B_{j_0}$ are inverse to each other. The same is then true for $\Delta_i$ and $B_i$ for $i > j_0$, i.e. $\Delta_i$ represents an element in $Wh(\pi_i)$.

By assumption $\ln \det_{\pi_i}(\Delta_i^*\Delta_i) = 0$. Note that the proof of Theorem 6.9 applies to our situation and we conclude $\ln \det_{\pi}(\Delta^*\Delta) \geq 0$. Since $\Delta \in Wh(\pi)$ was arbitrary, Theorem 1.22 implies the result.

6.14 Remark. It is not possible to proceed along similar lines in the case that $U$ has an amenable quotient (even if $\pi$ is amenable itself). The problem is that we approximate the matrix $\Delta$ (over $\mathbb{Z}_\pi$) by matrices over $\mathbb{Z}_U$ of larger and larger dimension. One can show that these matrices are invertible over $\mathbb{N}U$, if $\Delta$ itself was invertible. However, even if the inverse of $\Delta$ is a matrix over $\mathbb{Z}_\pi$, in general this is not true for the approximating matrices over $\mathbb{Z}_U$.

This finishes the proof of Theorems 1.19, 1.21 and 1.14. We proceed with some side remarks.
7 Complex approximation

In this section, we will address the question whether the approximation results we have obtained in section 6 are valid not only for matrices over the integral group ring, but also over the complex group ring. In particular, we adopt Situation 6.1: A group $\pi$ is approximated by groups $\pi_i$, and a matrix $\Delta = A^*A$ by matrices $\Delta_i$.

Also, we try to relate the approximation problem to the Atiyah conjecture, which says

7.1 Conjecture. Suppose $\pi$ is torsion free. Then $\dim_\pi(\ker A) \in \mathbb{Z}$ whenever $A \in M(d \times d, \mathbb{C}\pi)$.

This conjecture is true for abelian groups, free groups, and for extensions with amenable quotient (compare Linnell [6]).

Essentially, we will give a positive answer to our question only for free abelian groups. We start with a general observation.

7.2 Lemma. Suppose in the situation 6.1 that $\ker \Delta = 0$. Then the approximation result holds without integrality assumptions: $\lim_i F(\Delta_i)(0) = 0 = \dim_\pi(\ker \Delta)$.

More generally, if $\lambda$ is not an eigenvalue of $\Delta$ then $\lim_i F(\Delta_i)(\lambda) = F(\Delta)(\lambda)$.

Proof. We know that $F(\Delta)(x) = F(\Delta)(x)$ for every $x \in \mathbb{R}$. If $F(\Delta)(\lambda) \leq F(\Delta)(\lambda) - \epsilon$ then $F(\Delta)(x) \leq F(\Delta)(\lambda) - \epsilon$ for every $x < \lambda$, i.e. $F(\Delta)(x) \leq F(\Delta)(\lambda) - \epsilon \forall x < \lambda$. By assumption, the eigenspace of $\Delta$ to $\lambda$ is trivial, therefore $F(\Delta)$ is continuous at $\lambda$ and $\epsilon$ can only be zero. \qed

7.3 Proposition. If $\pi$ is torsion free and fulfills the Atiyah conjecture for $1 \times 1$-matrices, then the above approximation result $\lim_i F(\Delta_i)(0) = F(\Delta)(0)$ holds $\forall \Delta = A^*A \in \mathbb{C}\pi$.

Proof. Take $0 \neq \Delta \in \mathbb{C}\pi$. By assumption, since $\ker \Delta \neq l^2(\pi)$, $\ker \Delta = 0$. Conclude with lemma 7.2. \qed

7.4 Proposition. If $\pi$ is free abelian and a (subgroup of an) inverse limit —e.g. if we approximate $\pi$ residually— then the approximation result holds $\forall \Delta \in \mathbb{C}\pi$.

Proof. Since all properties are stable under directed unions, it suffices to consider a finitely generated free abelian group $\pi \cong \mathbb{Z}^n$. Proposition 7.3 shows that the statement holds for $1 \times 1$-matrices. We will use commutativity to reduce the general case to the one-dimension case. Embed $\mathbb{C}\pi$ into its
ring of fractions. Let \( A \in M(d \times d, \mathbb{C}[\mathbb{Z}^n]) \). Linear algebra tells us that we find \( X, Y \in \text{Gl}(d \times d, \mathbb{C}([\mathbb{Z}^n])) \) such that \( A = X \text{diag}(1, \ldots, 1, 0, \ldots, 0)Y \).

Collecting the denominators in \( X \) and \( Y \) we find \( 0 \neq c \in \mathbb{C}[[\mathbb{Z}^n]] \) and \( X', Y' \in M(d \times d, \mathbb{C}[[\mathbb{Z}^n]]) \cap \text{Gl}(d \times d, \mathbb{C}_\pi) \) such that

\[
cA = X' \text{diag}(1, \ldots, 1, 0, \ldots, 0)Y'
\]

The corresponding equation holds after passage to matrices over \( \pi_i \). Since \( c, X' \) and \( Y' \) have trivial kernel, lemma 7.2 implies the desired convergence result.

7.5 Remark. With a little bit more effort, one can get similar approximation results also in the other two contexts we are studying, in particular for amenable exhaustions of \( \mathbb{Z}^n \).

7.6 Remark. Approximation with complex coefficients implies that not only the dimensions or the kernels but of the eigenspaces to every complex number converge, since we simply replace \( A \) by \( A - \lambda \). Together with the fact that the right continuous \( \limsup \) of the \( F_{\Delta_i} \) is the spectral density function of \( \Delta \), we have convergence of the spectral density function at every point.

8 Quotients

To enlarge the class \( \mathcal{G} \), it is important to find other operations under which our main properties: determinant class and semi-integrality are inherited.

We indicate just one partial result:

**8.1 Proposition.** Suppose \( 1 \to F \to \pi \to Q \overset{p}{\to} 1 \) is an extension of groups and \( |F| < \infty \), and \( \pi \) is of determinant class. Then also \( Q \) is of determinant class.

**Proof.** We only indicate the proof, which was discussed with M. Farber during a conference in Oberwolfach, and which uses the theory of virtual characters of Farber [4]. \( l^2(\pi) \) corresponds to the Dirac character \( \delta_1 \). The representations \( V_k \) of the finite group \( F \) give rise to characters \( \chi_k \) of \( \pi \) with support contained in \( F \). Since \( l^2(F) = \bigoplus \mu_k V_k \), \( \delta_1 = \frac{1}{|F|} \sum \mu_k \chi_k \) with \( \mu_k > 0 \). Since the operator \( \Delta \) we are interested in arises from \( c^* \otimes \text{id} \) on \( C^* \otimes l^2(\pi) \), \( F_\Delta(\lambda) = \sum \frac{\mu_k}{|F|} F^{\chi_k}_\Delta(\lambda) \) (compare [4, 7.2]). Now the trivial representation \( V_1 \) of \( F \) corresponds to the quotient representation \( l^2(Q) \), and \( F_\Delta^{l^2(Q)} \) is just \( F^{l^2(Q)}_{p(\Delta)} \). By assumption \( \int_{\lambda_0^+} \ln(\lambda) \ dF_\Delta(\lambda) > -\infty \). Since
\[ \int \ln(\lambda) \, dF^{\chi_k}_\Delta(\lambda) < \infty \quad \forall k, \] 

it follows in particular

\[ \int_{0^+}^{\infty} \ln(\lambda) \, dF^Q_{p(\Delta)}(\lambda) > -\infty. \]

Since \( p \) is surjective, this is true for every matrix over \( \mathbb{Z} \pi \) we have to consider. This concludes the proof. \( \square \)

8.2 Remark. If we have an extension \( 1 \to \mathbb{Z}^n \to \pi \xrightarrow{p} Q \to 1 \), we can not write the character of \( l^2(\pi) \) as a direct sum, but as a direct integral (over the dual space \( \hat{T}^n = T^n \)). Then \( F_{\Delta}(\lambda) = \int_{T^n} F_{\Delta}^{\chi_q}(\lambda) \, d\eta \), and \( F_{\Delta}^{\chi}(\lambda) = F^Q_{p(\Delta)}(\lambda) \).

After establishing an appropriate continuity property, we can conclude as above that if \( \pi \) is of determinant class then

\[ \int_{0^+}^{\infty} \ln(\lambda) \, dF^Q_{p(\Delta)}(\lambda) > -\infty, \]

i.e. also \( Q \) is of determinant class.

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Errata to “Integrality of $L^2$-Betti numbers”,
“$L^2$-determinant class and approximation of
$L^2$-Betti numbers”, and work based on these

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Last compiled March 30, 2022

In [4, Section 4], a general notion of “amenable extension” $U \leq \pi$ is defined,
and in [4, Definition 1.12] a class of group $\mathcal{G}$ is defined which is in particular
closed under such (generalized) amenable extensions.

A particular example of a generalized amenable extension is a normal amenable
extension, i.e. $U$ is a normal subgroup of $\pi$ and $\pi/U$ is amenable.

The main result, [4, Theorem 1.14] is then proved for groups in the class $\mathcal{G}$,
based on claimed stability of the relevant properties of groups under “amenable
extension”.

Unfortunately, as pointed out by Christian Wegner, the proof of the relevant
stability and approximation property for the general notion of amenable
extension is not valid, it is based on a commutation relation which can’t be
established.

Nonetheless, the proof works perfectly well for usual normal amenable
extensions. Therefore, the assertions of the paper must be restricted to normal
amenable extensions; in particular, the definition of $\mathcal{G}$ must be modified such
that “amenable extension” has to be replaced by the (a priori more restrictive)
notion of “normal amenable extension”.

The definition of $\mathcal{G}$ and the approximation result for amenable extensions
has been taken up in [5, Definition 3 and Proposition 1]. Consequently, also
in this paper each use of “amenable extension” has to be restricted to “normal
amenable extension”, the class $\mathcal{G}$ has to be redefined accordingly.

The definition of $\mathcal{G}$ has also been taken up in [1, Definition 1.8], and, based on
the methods of [5, generalized approximation theorems and further properties
are claimed to be established for the groups in $\mathcal{G}$. Again, these statements are
established only if, throughout, “amenable extension” is changed to “normal
amenable extension” and the class $\mathcal{G}$ has to be replaced by the (a priori smaller)
class which is closed only under “normal amenable extension”.

Similarly, the definition of $\mathcal{G}$ is taken up in [2, Definition 4.6], and based on
it a class $\hat{\mathcal{G}}$ is defined; and the results of [5] are used. Therefore, as before, the
definitions have to be modified to allow only “normal amenable extensions” to
have valid proofs for the statements made about groups in $\mathcal{G}$ in [2].
Similarly, the definition of $G$ has been taken up in [3, Situation 3.1] and the notion of (generalized) “amenable extension” in [3, Definition 5]. The results stated in [3] for generalized “amenable extensions” and for groups in $G$, e.g. are generalizations of and based on the methods of [3]; consequently they again have to be modified by replacing the (generalized) “amenable extensions” by “normal amenable extensions” throughout, and by using the (a priori) smaller class $G$ based on this.

0.1 Remark. To the authors knowledge, no example of an (generalized) amenable extension which is not a normal amenable extension is know, in particular no such example and has been used explicitly in the literature.

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