Convergence to equilibrium distribution.

The Klein-Gordon equation coupled to a particle

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Abstract

We consider the Hamiltonian system consisting of a Klein-Gordon vector field and a particle in \( \mathbb{R}^3 \). The initial data of the system is a random function with a finite mean density of energy which also satisfies a Rosenblatt- or Ibragimov-type mixing condition. Moreover, initial correlation functions are translation-invariant. We study the distribution \( \mu_t \) of the solution at time \( t \in \mathbb{R} \). The main result is the convergence of \( \mu_t \) to a Gaussian measure as \( t \to \infty \), where \( \mu_\infty \) is translation-invariant.

Key words and phrases: a Klein-Gordon vector field coupled to a particle; random initial data; mixing condition; correlation matrices; characteristic functional; convergence to statistical equilibrium.

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1 Introduction

The paper concerns problems of long-time convergence to an equilibrium distribution in a coupled system. We have proved the convergence for partial differential equations of hyperbolic type in $\mathbb{R}^n$, $n \geq 2$, in [1, 2, 3, 6]; for harmonic crystals in [4, 5], and for a scalar field coupled to a harmonic crystal in [7]. Here we treat a particle coupled to a Klein-Gordon or wave vector equation.

Let us outline our main result and the strategy of the proof. (For the formal definitions and statements, see Section 2.) Consider the Hamiltonian system consisting of a real-valued vector field $\varphi(x)$, $x \in \mathbb{R}^3$, and a particle with position $q \in \mathbb{R}^3$. The Hamiltonian functional is

$$H(\varphi, q, \pi, p) = \sum_{n=1}^{d} \int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla \varphi_n(x)|^2 + \frac{m^2_n |\varphi_n(x)|^2}{2} + \frac{|\pi_n(x)|^2}{2} + \varphi_n(x) q \cdot \nabla \rho_n(x) \right) dx + \frac{1}{2} (|p|^2 + \omega^2 |q|^2).$$

(1.1)

Here $m_n \geq 0$, $\omega > 0$, $\varphi(x) = (\varphi_1(x), \ldots, \varphi_d(x)) \in \mathbb{R}^d$ and correspondingly for $\pi(x)$, $\rho(x)$. The function $Y$ stands for the scalar product in the Euclidean space $\mathbb{R}^3$. Taking formally variational derivatives in (1.1), the coupled dynamics becomes

$$\dot{\varphi}_n(x, t) = \pi_n(x, t) = (\Delta - m_n^2) \varphi_n(x, t) - q(t) \cdot \nabla \rho_n(x), \quad n = 1, \ldots, d, \quad (1.2)$$

$$\dot{q}(t) = p(t), \quad \dot{p}(t) = -\omega^2 q(t) - \sum_{n=1}^{d} \int \varphi_n(x, t) \nabla \rho_n(x) dx, \quad t \in \mathbb{R}. \quad (1.3)$$

To state our main results we formulate some assumptions on a constant $\omega > 0$ and a coupled function $\rho(x) = (\rho_1(x), \ldots, \rho_d(x))$.

A1. The matrix $(\omega^2 - m_n^2)I - K$ is positive definite, where $m_* = 0$ if $m = (m_1, \ldots, m_d) = 0$ and $m_* = \min\{m_n : m_n \neq 0, n = 1, \ldots, d\}$ if $m \neq 0$, $K = (K_{ij})_{i,j=1}^{3}$ is the $3 \times 3$ matrix with the matrix elements $K_{ij}$,

$$K_{ij} = \sum_{n=1}^{d} \frac{1}{(2\pi)^3} \int \frac{k_i k_j |\hat{\rho}_n(k)|^2}{k^2 + m_n^2 - m_*^2} dk.$$

A2. The function $\rho(x)$ is a vector real-valued smooth function, $\rho(-x) = \rho(x)$, $\rho(x) = 0$ for $|x| \geq R_\rho$.

A3. For all $n = 1, \ldots, d$ and $k \in \mathbb{R}^3 \setminus \{0\}$,

$$\hat{\rho}_n(k) = \int e^{ikx} \rho_n(x) dx \neq 0.$$

This assumption can be weakened (see Remark 7.6).

We study the Cauchy problem for the system (1.2)–(1.3) with initial conditions

$$\varphi(x, 0) = \varphi^0(x), \quad \pi(x, 0) = \pi^0(x), \quad q(0) = q^0, \quad p(0) = p^0. \quad (1.4)$$

Let us write $Y_0 \equiv (\varphi^0(x), q^0, \pi^0(x), p^0)$, $Y(t) \equiv (\varphi(x, t), q(t), \pi(x, t), p(t))$. Then the system (1.2)–(1.4) writes as

$$\dot{Y}(t) = F(Y(t)), \quad t \in \mathbb{R}; \quad Y(0) = Y_0. \quad (1.5)$$
We assume that the initial date $Y_0$ is a random element of a real functional space $\mathcal{E}$ consisting of states with a finite local energy, see Definition 2.2 below. The distribution of $Y_0$ is a probability measure $\mu_0$ of mean zero satisfying conditions $S1–S3$. In particular, the correlation functions of the measure $\mu_0$ is translation-invariant. For a given $t \in \mathbb{R}$, we denote by $\mu_t$ the probability measure on $\mathcal{E}$ defining the distribution of the solution $Y(t)$ to the problem (1.5).

Our main result gives the weak convergence of the measures $\mu_t$ to a limit measure $\mu_\infty$,

$$\mu_t \rightharpoonup \mu_\infty, \quad t \to \infty. \quad (1.6)$$

The measure $\mu_\infty$ is a translation-invariant Gaussian measure on $\mathcal{E}$. Similar results hold as $t \to -\infty$ because the dynamics is time-reversible. Moreover, in Section 6.3 we prove the convergence of the correlation functions of the measures $\mu_t$ to a limit as $t \to \infty$.

We prove the convergence (1.6) by using the strategy of [1, 2] in two steps:

I. The family of measures $\mu_t$, $t \geq 0$, is weakly compact in an appropriate Fréchet space.

II. The characteristic functionals converge to a Gaussian functional,

$$\hat{\mu}_t(Z) = \int \exp(i\langle Y, Z \rangle) \mu_t(dY) \to \exp\{-\frac{1}{2}Q_\infty(Z, Z)\}, \quad t \to \infty, \quad (1.7)$$

where $Z$ is an arbitrary element of the dual space and $Q_\infty$ is the quadratic form defined by (2.17), $\langle \cdot, \cdot \rangle$ stands for the inner product in a real Hilbert space $L^2(\mathbb{R}^3) \otimes \mathbb{R}^N$ with different $N = 1, 2, \ldots$.

Let us explain the main idea of the proof. At first we derive the decay of the order $(1 + |t|)^{-3/2}$ (and the exponential decay in the case when $m = 0$) for the local energy of the solution $Y(t)$ to (1.5) assuming that the initial date $Y_0$ has a compact support (see Proposition 4.1). Then we apply the integral representation (5.4) of $Y(t)$ and prove a uniform bound (5.1) for the mean local energy with respect to the measure $\mu_t$, $t \geq 0$. Finally property I follows from the Prokhorov Theorem.

To prove of II we derive the asymptotic behavior of the solution $Y(t)$ (see Proposition 6.1), namely,

$$\langle Y(t), Z \rangle \sim \sum_{n=1}^d \langle W_n(t)(\varphi_n^0, \pi_n^0), \psi_n^Z \rangle, \quad t \to \infty, \quad (1.8)$$

where $W_n(t)$ is a solving operator to the Cauchy problem (2.9) for the free wave or Klein-Gordon equation, the functions $\psi_n^Z$ are expressed by $Z$ (see formula (2.13)). Then we apply the results of [1, 2], where the weak convergence of the statistical solutions is proved for free wave and Klein-Gordon equations.

2 Main results

2.1 Notation

We assume that the initial data $Y_0$ are given by an element of the real phase space $\mathcal{E}$ defined below.
Definition 2.1 Let $\mathcal{H} \equiv H_{1,\text{loc}}^1(\mathbb{R}^3) \oplus H_{0,\text{loc}}^0(\mathbb{R}^3)$ be the Fréchet space of pairs $\phi \equiv (\varphi(x), \pi(x))$ with $\mathbb{R}^d$-valued functions $\varphi(x)$ and $\pi(x)$, which is endowed with the local energy seminorms

$$
\|\phi\|_{E,R}^2 = \int_{|x|<R} (|\varphi(x)|^2 + |\nabla \varphi(x)|^2 + |\pi(x)|^2) dx < \infty, \ \forall R > 0. \tag{2.1}
$$

Definition 2.2 Let $\mathcal{E} \equiv \mathcal{H} \oplus \mathbb{R}^d \oplus \mathbb{R}^3$ be the Fréchet space of vectors $Y \equiv (\varphi(x), q, p)$, with the local energy seminorms

$$
\|Y\|^2_{\mathcal{E},R} = \|\phi\|^2_{E,R} + |q|^2 + |p|^2, \ \forall R > 0. \tag{2.2}
$$

Now we formulate the following condition on $\rho$ (cf. condition A1).

A1’. The matrix $\omega^2 I - K_0$ is positive definite, where $K_0 = (K_{0,ij})_{i,j=1}^3$ is the $3 \times 3$ matrix with the matrix elements $K_{0,ij}$,

$$
K_{0,ij} = \sum_{n=1}^d \frac{1}{(2\pi)^3} \int \frac{k_i k_j \rho_n(k)}{k^2 + m_n^2} dk.
$$

Note that if $m = 0$ the conditions A1 and A1’ coincide. If $m \neq 0$, condition A1’ is weaker than A1.

Proposition 2.3 Let conditions A1’ and A2 hold. Then (i) for every $Y_0 \in \mathcal{E}$, the Cauchy problem (1.5) has a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$.

(ii) For every $t \in \mathbb{R}$, the operator $U(t) : Y_0 \mapsto Y(t)$ is continuous on $\mathcal{E}$. Moreover, for every $R > R_0$, $T > 0$,

$$
sup_{|t| \leq T} \|U(t)Y_0\|^2_{\mathcal{E},R} \leq C(T) \|Y_0\|^2_{\mathcal{E},R+T}. \tag{2.3}
$$

Proposition 2.3 is proved in Section 3.

Let us choose a function $\zeta(x) \in C_0^\infty(\mathbb{R}^3)$ with $\zeta(0) \neq 0$. Denote by $H_{s,\text{loc}}^1(\mathbb{R}^3)$, $s \in \mathbb{R}$, the local Sobolev spaces of $\mathbb{R}^d$-valued functions $\varphi$, i.e., the Fréchet spaces of distributions $\varphi \in D'(\mathbb{R}^3)$ with the finite seminorms $\|\varphi\|_{s,R} := \|\Lambda^s(\zeta(x/R)\varphi)\|_{L^2(\mathbb{R}^3)}$, where $\Lambda^s \psi := F_{k=1}^{-1} \langle |k|^s \hat{\psi}(k) \rangle$, $\hat{\psi} := \sqrt{|k|^2 + 1}$, and $\hat{\psi}$ is the Fourier transform of a tempered distribution $\hat{\psi}$.

Definition 2.4 For $s \in \mathbb{R}$ denote $\mathcal{H}^s \equiv H_{\text{loc}}^{1+s}(\mathbb{R}^3) \oplus H_{0,\text{loc}}^s(\mathbb{R}^3)$, $\mathcal{E}^s \equiv \mathcal{H}^s \oplus \mathbb{R}^d \oplus \mathbb{R}^3$.

Using standard techniques of pseudodifferential operators and Sobolev’s Theorem (see, e.g., [8]), it is possible to prove that $\mathcal{E}^0 = \mathcal{E} \subset \mathcal{E}^{-\varepsilon}$ for every $\varepsilon > 0$, and the embedding is compact.

2.2 Random solution. Convergence to equilibrium

Let $(\Omega, \Sigma, P)$ be a probability space with expectation $E$ and $\mathcal{B}(\mathcal{E})$ denote the Borel $\sigma$-algebra in $\mathcal{E}$. We assume that $Y_0 = Y_0(\omega, x) \in (1.5)$ is a measurable random function with values in $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$. In other words, $(\omega, x) \mapsto Y_0(\omega, x)$ is a measurable map $\Omega \times \mathbb{R}^3 \to \mathbb{R}^{2d+6}$ with respect to the (completed) $\sigma$-algebra $\Sigma \times \mathcal{B}(\mathbb{R}^3)$ and $\mathcal{B}(\mathbb{R}^{2d+6})$. Then $Y(t) = U(t)Y_0$ is also a measurable random function with values in $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$ owing to Proposition 2.3. We denote by $\mu_0(dY_0)$ a Borel probability measure in $\mathcal{E}$ giving the distribution of $Y_0$. Without loss of generality, we assume $(\Omega, \Sigma, P) = (\mathcal{E}, \mathcal{B}(\mathcal{E}), \mu_0)$ and $Y_0(\omega, x) = \omega(x)$ for $\mu_0(d\omega) \times dx$-almost all $(\omega, x) \in \mathcal{E} \times \mathbb{R}^3$. 


Definition 2.5 \( \mu_t \) is a Borel probability measure in \( E \) which gives the distribution of \( Y(t) \):

\[
\mu_t(B) = \mu_0(U(-t)B), \quad \forall B \in B(E), \quad t \in \mathbb{R}.
\]

Our main objective is to prove the weak convergence of the measures \( \mu_t \) in the Fréchet spaces \( \mathcal{E}^{-\varepsilon} \) for each \( \varepsilon > 0 \),

\[
\mu_t \xrightarrow{\varepsilon-\varepsilon} \mu_\infty \quad \text{as} \quad t \to \infty, \tag{2.4}
\]

where \( \mu_\infty \) is a limit measure on \( E \). This means the convergence

\[
\int f(Y) \mu_t(dY) \to \int f(Y) \mu_\infty(dY) \quad \text{as} \quad t \to \infty
\]

for any bounded continuous functional \( f(Y) \) on \( \mathcal{E}^{-\varepsilon} \).

Definition 2.6 The correlation functions of measure \( \mu_t \) are defined by

\[
Q_t(x,y) \equiv \mathbb{E}(Y(x,t) \otimes Y(y,t)), \quad \text{for almost all} \quad x,y \in \mathbb{R}^3,
\]

if the expectations in the right hand side are finite.

We set \( D = D_0 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \), \( D_0 := [C_0^\infty(\mathbb{R}^3) \otimes \mathbb{R}^d]^2 \), and

\[
\langle Y, Z \rangle := \langle \phi, \psi \rangle + q \cdot u + p \cdot v
\]

for \( Y = (\phi, q, p) \in \mathcal{E} \), and \( Z = (\psi, u, v) \in D \). For a probability measure \( \mu \) on \( E \) denote by \( \hat{\mu} \) the characteristic functional (Fourier transform)

\[
\hat{\mu}(Z) \equiv \int \exp(i \langle Y, Z \rangle) \mu(dY), \quad Z \in D.
\]

A measure \( \mu \) is called Gaussian (with zero expectation) if its characteristic functional has the form

\[
\hat{\mu}(Z) = \exp\left\{ - \frac{1}{2} Q(Z, Z) \right\}, \quad Z \in D,
\]

where \( Q \) is a real nonnegative quadratic form in \( D \). A measure \( \mu \) is called translation-invariant if

\[
\mu(T_hB) = \mu(B), \quad \forall B \in \mathcal{B}(E), \quad h \in \mathbb{R}^3,
\]

where \( T_hY(x) = Y(x - h) \).

2.3 Main theorem

We assume that the initial measure \( \mu_0 \) has the following properties \textbf{S0}–\textbf{S3}.

\textbf{S0} \( \mu_0 \) has zero expectation value, \( \mathbb{E}Y_0(x) \equiv \int Y_0(x) \mu_0(dY_0) = 0 \), \( x \in \mathbb{R}^3 \).

\textbf{S1} \( \mu_0 \) has finite mean energy density, i.e.,

\[
E\left( |\varphi^0(x)|^2 + |\nabla \varphi^0(x)|^2 + |\pi^0(x)|^2 + |q^0|^2 + |p^0|^2 \right) < \infty. \tag{2.5}
\]
Denote by $\nu_0 := P\mu_0$, where $P : (\phi^0, q^0, p^0) \in \mathcal{E} \rightarrow \phi^0 \in \mathcal{H}$.

**S2** The correlation functions of the measure $\nu_0$ are translation invariant, i.e., for $n, n' \in \mathbb{Z}$,

$$E\left(\varphi_n^0(x) \otimes \varphi_{n'}^0(y)\right) := q_{0,nn'}^0(x - y), \quad E\left(\pi_n^0(x) \otimes \pi_{n'}^0(y)\right) := q_{0,nn'}^1(x - y),$$

$$E\left(\tau_n^0(x) \otimes \varphi_{n'}^0(y)\right) := q_{0,nn'}^{01}(x - y), \quad E\left(\tau_n^0(x) \otimes \tau_{n'}^0(y)\right) := q_{0,nn'}^{11}(x - y). \quad (2.6)$$

Now we formulate the mixing condition for the measure $\nu_0$.

Let $O(r)$ be the set of all pairs of open convex subsets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^3$ at the distance not less than $r$, $d(\mathcal{A}, \mathcal{B}) \geq r$, and let $\sigma(\mathcal{A})$ be the $\sigma$-algebra in $\mathcal{H}$ generated by the linear functionals $\phi \mapsto \langle \phi, \psi \rangle$, where $\psi \in \mathcal{D}_0$ with $\text{supp} \psi \subset \mathcal{A}$. Define the Ibragimov mixing coefficient of a probability measure $\nu_0$ on $\mathcal{H}$ by the rule (cf [9, Def. 17.2.2])

$$\varphi(r) \equiv \sup_{(\mathcal{A}, \mathcal{B}) \in O(r)} \sup_{A \in \sigma(\mathcal{A}), B \in \sigma(\mathcal{B})} \frac{|\nu_0(A \cap B) - \nu_0(A)\nu_0(B)|}{\nu_0(B)} > 0 \quad (2.7)$$

**Definition 2.7** We say that the measure $\nu_0$ satisfies the strong uniform Ibragimov mixing condition if $\varphi(r) \rightarrow 0$ as $r \rightarrow \infty$.

**S3** The measure $\nu_0$ satisfies the strong uniform Ibragimov mixing condition, and

$$\int_0^{+\infty} r^2 \varphi^{1/2}(r)dr < \infty. \quad (2.8)$$

Consider the following Cauchy problem

$$\begin{cases} 
\varphi(x, t) = (\Delta - m_n^2)\varphi(x, t), & t \in \mathbb{R}, \\
\varphi(x, t)|_{t=0} = \phi^0(x), \quad \varphi(x, t)|_{t=0} = \pi^0(x), & x \in \mathbb{R}^3,
\end{cases} \quad (2.9)$$

where $m_n \geq 0$, $\varphi(x, t) \in \mathbb{R}^3$. The following lemma is proved in [1, p.7], [2, p.1225].

**Lemma 2.8** (i) For any $\phi^0 = (\phi^0, \pi^0) \in \mathcal{H}_1 \equiv H^1_{loc}(\mathbb{R}^3) \oplus L^2_{loc}(\mathbb{R}^3)$, there exists a unique solution $\phi(t) = (\varphi(x, t), \varphi(x, t)) \in C(\mathbb{R}, \mathcal{H}_1)$ to the Cauchy problem (2.9).

(ii) For any $t \in \mathbb{R}$, the operator $W_n(t) : \phi^0 \mapsto \phi(t)$ is continuous in $\mathcal{H}_1$.

We define the operator $W(t)$ on the space $\mathcal{H} = [\mathcal{H}_1]^d$ by the rule

$$W(t)(\phi_1^0, \ldots, \phi_d^0) = (W_1(t)\phi_1^0, \ldots, W_d(t)\phi_d^0). \quad (2.10)$$

Let $\mathcal{E}_n(x)$ be the fundamental solution of the operator $-\Delta + m_n^2$, $n \in \mathbb{Z}$. For almost all $x, y \in \mathbb{R}^3$, introduce the matrix-valued function $Q_n^\tau(x, y) = (q_{n,n'}^\tau(x, y))_{n,n'=1}^d$, where

$$q_{n,n'}^\tau = \left(q_{n,n'}^{ij}\right)_{i,j=0}^1 = \chi_{nn'} \frac{1}{2} \begin{pmatrix} q_{0,nn'}^{00} + \mathcal{E}_n * q_{0,nn'}^{11} & q_{0,nn'}^{01} & -q_{0,nn'}^{10} \\
q_{0,nn'}^{10} & q_{0,nn'}^{11} & q_{0,nn'}^{00} \end{pmatrix}, \quad (2.11)$$

where $\chi_{nn'} = 1$ if $m_n = m_{n'}$, and $\chi_{nn'} = 0$ otherwise, the functions $q_{0,nn'}^{ij}, i, j = 0, 1$, are defined in (2.6); and $*$ stands for the convolution of distributions.
Remark 2.9 According to [9, Lemma 17.2.3], the derivatives $\partial^{i+j}_{0,nn'}$ are bounded by mixing coefficient: $\forall \alpha \in \mathbb{Z}^3_+$ with $|\alpha| \leq 2 - i - j$ (including $\alpha = 0$), $i,j = 0,1$,

$$|\partial^{i+j}_{0,nn'}(z)| \leq C\varphi^{1/2}(|z|), \quad \forall z \in \mathbb{R}^3, \quad n,n' \in \bar{d}.$$ Therefore, $\partial^{i+j}_{0,nn'} \in L^p(\mathbb{R}^3)$, $p \geq 1$ (see [1, p.16]). Hence, $q^i_{\infty,nn'} \in L^1(\mathbb{R}^3)$ if $m_n \neq 0$ by the bound (2.11). If $m_n = 0$, (2.8) implies the existence of the convolution $\mathcal{E}_n \ast q^1_{0,nn'}$.

Denote by $Q^\nu_\infty(\psi,\psi)$ a real quadratic form on $S_0 \equiv [S(\mathbb{R}^3) \otimes \mathbb{R}^d]^2$ defined by

$$Q^\nu_\infty(\psi,\psi) = \langle Q^\nu_\infty(x,y),\psi(x) \otimes \psi(y) \rangle = \sum_{n,n'=1}^d \langle q^\nu_{\infty,nn'}(x-y),\psi_n(x) \otimes \psi_{n'}(y) \rangle.$$ (2.12)

The following result can be proved by an easy adaptation of the proof of Theorem B of [1, 2], where the result is proved in the case $d = 1$.

**Theorem 2.10** Let conditions $S_0 - S_3$ hold. Then

(i) the measures $\nu_t \equiv W(t)^*\nu_0$ weakly converge as $t \to \infty$ on the space $\mathcal{H}^{-\varepsilon}$ for each $\varepsilon > 0$.

(ii) The limit measure $\nu_\infty$ is a translation-invariant Gaussian measure on $\mathcal{H}$.

(iii) The characteristic functional of $\nu_\infty$ is of the form

$$\hat{\nu}_\infty(\psi) = \exp \left\{ -\frac{1}{2} Q^\nu_\infty(\psi,\psi) \right\}, \quad \psi \in S_0.$$ Let $Z = (\psi,u,v) \in D$, i.e., $\psi = (\psi_1,\ldots,\psi_d) \in D_0$, $(u,v) \in \mathbb{R}^3 \times \mathbb{R}^3$. Denote

$$\psi^Z = (\psi^Z_1,\ldots,\psi^Z_d), \quad \psi^Z_k := \psi_k(x) - \sum_{n=1}^d \theta_k(x) + \alpha_k(x) \cdot u + \beta_k(x) \cdot v, \quad k \in \bar{d}.$$ (2.13)

Here $\alpha_k = (\alpha^1_k,\alpha^2_k,\alpha^3_k), \beta_k = (\beta^1_k,\beta^2_k,\beta^3_k),$

$$\alpha^i_k \equiv \alpha^i_k(x) = -\sum_{r=1}^d \int_0^{+\infty} \mathcal{N}_n(s)W'_k(-s) \left( \begin{array}{c} \nabla_r \rho_k(x) \\ 0 \end{array} \right) ds,$$ (2.14)

$$\beta^i_k \equiv \beta^i_k(x) = -\sum_{r=1}^d \int_0^{+\infty} \mathcal{N}_n(s)W'_k(-s) \left( \begin{array}{c} 0 \\ \nabla_r \rho_k(x) \end{array} \right) ds,$$ (2.15)

$$\theta_k(x) := \sum_{i=1}^3 \int_0^{+\infty} W'_k(-s) \alpha^i_k(x) \langle W_n(s)\mathcal{R}_m,\psi_n \rangle ds, \quad \mathcal{R}_m := \left( \begin{array}{c} 0 \\ \nabla_i \rho_n \end{array} \right),$$ (2.16)

the matrix $\mathcal{N}(s) = (\mathcal{N}_n(s))_{i,j=1}^3$ is defined in (4.17), the operator $W'_n(t)$ is adjoint to the operator $W_n(t)$:

$$\langle \phi, W'_n(t)\psi \rangle = \langle W_n(t)\phi, \psi \rangle, \quad \psi \in [S(\mathbb{R}^3)]^2, \quad \phi \in \mathcal{H}_1, \quad t \in \mathbb{R}.$$ Denote by $Q_\infty(Z,Z)$ a real quadratic form in $D$ of the form

$$Q_\infty(Z,Z) = Q^\nu_\infty(\psi^Z,\psi^Z),$$ (2.17)

where $\psi^Z$ is defined in (2.13). Our main result is the following theorem.
Theorem 2.11 Let conditions A1–A3 and S0–S3 hold. Then
(i) the convergence in (2.4) holds for any $\varepsilon > 0$.
(ii) The limit measure $\mu_{\infty}$ is a Gaussian equilibrium measure on $E$.
(iii) The limit characteristic functional has the form
\[ \hat{\mu}_{\infty}(Z) = \exp\{-\frac{1}{2}Q_{\infty}(Z,Z)\}, \quad Z \in D. \]
(iv) The measure $\mu_{\infty}$ is invariant, i.e., $U(t)^*\mu_{\infty} = \mu_{\infty}$, $t \in \mathbb{R}$.

Remark 2.12 Instead of the strong uniform Ibragimov mixing condition, it suffices to assume the uniform Rosenblatt mixing condition [13] together with a higher degree ($>2$) in the bound (2.5), i.e., to assume that there exists a $\delta$, $\delta > 0$, such that
\[ E\left(\sum_{n=1}^{d} \left(\|\nabla \varphi_n\|^2 + m_n^2\|\varphi_n\|^2 + \|\pi_n\|^2\right) + |q|^2 + |p|^2\right) < \infty. \]

In this case, the condition (2.8) needs the following modification:
\[ \int_{0}^{+\infty} r^2 \alpha^p(r)dr < \infty, \]
where $p = \min(\delta/(2 + \delta), 1/2)$, $\alpha(r)$ is the Rosenblatt mixing coefficient defined as in (2.7) but without $\nu_0(B)$ in the denominator.

3 Existence of solutions, a priori estimates

In this section we prove Proposition 2.3 by the similar arguments as in [10, Lemma 6.3].

Let us represent the solution $Y(t)$ as the pair of the functions $(Y^0(t), Y^1(t))$, where $Y^0(t) = (\varphi(t), q(t))$, $Y^1(t) = (\pi(t), p(t))$.

Denote by $H^s(\mathbb{R}^3)$ the Sobolev space of $\mathbb{R}^d$-valued functions. Let $E$ be the Hilbert space of pairs $Y = (Y^0, Y^1)$, where $Y^0 = (\varphi, q) \in H^1(\mathbb{R}^3) \oplus \mathbb{R}^3$, $Y^1 = (\pi, p) \in H^0(\mathbb{R}^3) \oplus \mathbb{R}^3$, with the finite norm $\|Y\|_E^2 := \sum_{n=1}^{d} \left(\|\nabla \varphi_n\|^2 + m_n^2\|\varphi_n\|^2 + \|\pi_n\|^2\right) + |q|^2 + |p|^2$. Here $\|\cdot\|$ stands for the norm in $H^0(\mathbb{R}^3)$. Now we prove the auxiliary lemma.

Lemma 3.1 Let conditions A1’ and A2 hold. Then
(i) for every $Y_0 \in E$, the Cauchy problem (1.5) has a unique solution $Y(t) \in C(\mathbb{R}, E)$.
(ii) For every $t \in \mathbb{R}$, the operator $U(t) : Y_0 \mapsto Y(t)$ is continuous on $E$.
(iii) The energy is conserved and finite,
\[ H(Y(t)) = H(Y_0) \quad \text{for} \quad t \in \mathbb{R}. \quad (3.1) \]

Proof. Step (i) In the case when $\rho = 0$ the existence and uniqueness of solution $Y(t) \in C(\mathbb{R}, E)$ to the problem (1.5) is proved by Fourier transform. Therefore the problem (1.5) for $Y(t) \in C(\mathbb{R}, E)$ is equivalent to
\[ Y(t) = e^{A_0 t} Y_0 + \int_{0}^{t} e^{A_0(t-s)} BY(s) ds, \quad (3.2) \]
where

\[
A_0 = \begin{pmatrix} 0 & I \\ -H_0 & 0 \end{pmatrix}, \quad H_0 Y^0 = ((-\Delta + m_1^2)\varphi_1, \ldots, (-\Delta + m_d^2)\varphi_d, \omega^2 q),
\]

\[
B(Y^0, Y^1) = (0, RY^0), \quad RY^0 := -\left(q \cdot \nabla \rho_1, \ldots, q \cdot \nabla \rho_d, \sum_{n=1}^d \int \varphi_n(x) \nabla \rho_n(x) \, dx \right)
\]

for \( Y^0 = (\varphi_1, \ldots, \varphi_d, q) \). Note that \( \|e^{A_0 t} Y_0\|_E \leq C\|Y_0\|_E \); and the second term in (3.2) is estimated by

\[
\sup_{|t| \leq T} \| e^{A_0 (t-s)} B Y(s) \|_E \leq C T \sup_{|s| \leq T} \| Y(s) \|_E.
\]

This bound and the contraction mapping principle imply the existence and uniqueness of the local solution \( Y(t) \in C([-\varepsilon, \varepsilon], E) \) with an \( \varepsilon > 0 \).

Step (ii) To prove (3.1) let us assume that \( \phi^0_n = (\varphi^0_n, \pi^0_n) \in C^3(\mathbb{R}^3) \times C^2(\mathbb{R}^3) \) and \( \phi^0_n(x) = 0 \) for \( |x| \geq R_0 \), \( n \in \mathbb{N} \). Then \( \varphi_n(x, t) \in C^2(\mathbb{R}^3 \times \mathbb{R}) \) and

\[
\varphi_n(x, t) = 0 \quad \text{for} \quad |x| \geq |t| + \max\{R_0, R_\rho\}
\]

by the integral representation (3.2) and condition \( A_2 \). Therefore, for such initial data, the equality (3.1) can be proved by integration by parts. Hence the energy conservation (3.1) follows from the continuity of \( U(t) \) and the fact that \( [C_0^3(\mathbb{R}^3)]^d \oplus \mathbb{R}^3 \oplus [C_0^2(\mathbb{R}^3)]^d \oplus \mathbb{R}^3 \) is dense in \( E \).

Step (iii) Note that

\[
\frac{1}{2} \|\nabla \varphi_n(x)\|^2 + \frac{1}{2}m_n^2 \|\varphi_n(x)\|^2 + \langle \varphi_n(x), q \cdot \nabla \rho_n(x) \rangle
\]

\[
= \frac{1}{2(2\pi)^3} \left( \|k^2 + m_n^2 \hat{\varphi}_n(k)\|^2 + \frac{1}{\sqrt{k^2 + m_n^2}} i q \cdot k \hat{\rho}_n(k) \right) + \frac{|q|^2}{\sqrt{k^2 + m_n^2}} \hat{\rho}_n(k)\|^2 - \left( \frac{1}{\sqrt{k^2 + m_n^2}} \right)^2.
\]

Hence the Hamiltonian functional is nonnegative, since

\[
H(Y) = \frac{1}{2} \sum_{n=1}^d \left\{ \|\pi_n\|^2 + \frac{1}{(2\pi)^3} \|k^2 + m_n^2 \hat{\varphi}_n(k)\|^2 + \frac{1}{\sqrt{k^2 + m_n^2}} i q \cdot k \hat{\rho}_n(k) \right\} + \frac{1}{2} \left( \omega^2 |q|^2 + q \cdot K_0 q \right) \geq 0
\]

by condition \( A_1' \). Moreover, by (3.1) and (3.4), for \( |t| < \varepsilon \), we obtain that

\[
\|Y(t)\|_E \leq C H(Y(t)) = C H(Y_0).
\]

On the other hand, for \( Y_0 = (\varphi^0, \pi^0, q^0, p^0) \),

\[
H(Y_0) \leq \sum_{n=1}^d \left( \|\nabla \varphi_n^0\|^2 + \frac{1}{2}m_n^2 \|\varphi_n^0\|^2 + \frac{1}{2} \|\pi_n^0\|^2 \right) + \frac{1}{2} \left( \omega^2 + \|\rho\|^2 \right) |q^0|^2 + \frac{1}{2} |p^0|^2.
\]

Hence, we obtain the \( a \ priori \) estimate

\[
\|Y(t)\|_E \leq C_1 \|Y_0\|_E \quad \text{for} \quad t \in \mathbb{R}.
\]

(3.6)
Therefore, properties (i)–(iii) of Lemma 3.1 for arbitrary \( t \in \mathbb{R} \) follow from the bound (3.6).

We return to the proof of Proposition 2.3. Let us choose \( R > R_\rho \) with \( R_\rho \) from condition A2. Then by the integral representation (3.2) the solution \( Y(t) \) for \(|x| < R\) depends only on the initial data \( Y_0(x) \) with \(|x| < R + |t|\). Thus the continuity of \( U(t) \) in \( E \) follows from the continuity in \( E \).

For every \( R > 0 \) we define the local energy seminorms by

\[
\|Y\|_{E(R)}^2 := \sum_{n=1}^{d} \int_{|x|<R} \left( |\nabla \varphi_n(x)|^2 + m_n^2 |\varphi_n(x)|^2 + |\pi_n(x)|^2 \right) dx + |q|^2 + |p|^2 \tag{3.7}
\]

for \( Y = (\varphi, \pi, q, p) \). By the estimate (3.6), we obtain the following local energy estimates:

\[
\|U(t)Y_0\|_{E(R)}^2 \leq C \|Y_0\|_{E(R+|t|)}, \tag{3.8}
\]

for \( R > R_\rho \) and \( t \in \mathbb{R} \). Hence, the bound (2.3) follows from (3.8).

4 Decay of local energy

**Proposition 4.1** Let conditions A1–A3 hold and let \( Y_0 \in E \) be such that

\[
\varphi^0(x) = \pi^0(x) = 0 \text{ for } |x| > R_1, \tag{4.1}
\]

with some \( R_1 > 0 \). Then there exists a constant \( C = C(R, R_1) > 0 \) such that the following bound holds for every \( R > 0 \),

\[
\|Y(t)\|_{E,R} \leq C \varepsilon_m(t) \|Y_0\|_{E,R_1}, \quad t \geq 0, \tag{4.2}
\]

where

\[
\varepsilon_m(t) = \begin{cases} 
  e^{-\delta |t|} \text{ with } \delta > 0, & \text{if } m = 0 \\
  (1 + |t|)^{-3/2}, & \text{if } m \neq 0.
\end{cases} \tag{4.3}
\]

In the case when \( m = (m_1, \ldots, m_d) = 0 \) Proposition 4.1 is extension of Proposition 7.1 from [10], where a similar result is established in the case when \( d = 1 \) and \( \rho(x) = \rho_r(|x|) \).

In the case when \( m \neq 0 \) we apply the methods of [11].

To prove Proposition 4.1 we solve the Cauchy problem (1.5) applying the Fourier-Laplace transform,

\[
\tilde{Y}(\lambda) = \int_0^{+\infty} e^{-\lambda t} Y(t) \, dt, \quad \text{Re } \lambda > 0.
\]

Then the system (1.2)–(1.4) becomes

\[
\begin{align*}
-\varphi^0_n(x) + \lambda \tilde{\varphi}_n(x, \lambda) & = \tilde{\pi}_n(x, \lambda), \quad n = 1, \ldots, d, \tag{4.4} \\
-\pi^0_n(x) + \lambda \tilde{\pi}_n(x, \lambda) & = (\Delta - m_n^2) \tilde{\varphi}_n(x, \lambda) - \tilde{q}(\lambda) \cdot \nabla \rho_n(x), \tag{4.5} \\
-q^0 + \lambda \tilde{q}(\lambda) & = \tilde{\rho}(\lambda), \tag{4.6} \\
-p^0 + \lambda \tilde{p}(\lambda) & = -\omega^2 \tilde{q}(\lambda) - \sum_{n=1}^{d} \int \tilde{\varphi}_n(y, \lambda) \nabla \rho_n(y) \, dy. \tag{4.7}
\end{align*}
\]
From (4.4)–(4.7) we obtain
\[
\tilde{\varphi}_n(x, \lambda) = (-\Delta + m_n^2 + \lambda^2)^{-1} \left( \lambda \varphi_n^0 + \pi_n^0(x) \right) - (-\Delta + m_n^2 + \lambda^2)^{-1} \nabla \rho_n(x) \cdot \tilde{q}(\lambda),
\]
\[
(\lambda^2 + \omega^2) \tilde{q}(\lambda) = H(\lambda) \tilde{q}(\lambda) + \mathcal{R}(\lambda, Y_0),
\]
where
\[
\mathcal{R}(\lambda, Y_0) := -\sum_{n=1}^d \int (-\Delta + m_n^2 + \lambda^2)^{-1} \left( \pi_n^0(y) + \lambda \varphi_n^0(y) \right) \nabla \rho_n(y) dy + (p^0 + \lambda q^0),
\]
and \( H(\lambda) \) is the \( 3 \times 3 \) matrix with the matrix elements \( H_{ij}(\lambda) \),
\[
H_{ij}(\lambda) = \sum_{n=1}^d \int \nabla_i \rho_n(y) (-\Delta + m_n^2 + \lambda^2)^{-1} \nabla_j \rho_n(y) \, dy.
\]
Therefore, equation (4.8) rewrites as
\[
\tilde{q}(\lambda) = \left[ (\lambda^2 + \omega^2)I - H(\lambda) \right]^{-1} \mathcal{R}(\lambda, Y_0) \equiv \tilde{N}(\lambda) \mathcal{R}(\lambda, Y_0),
\]
where by \( \tilde{N}(\lambda) \) we denote a \( 3 \times 3 \) matrix of the form
\[
\tilde{N}(\lambda) = D^{-1}(\lambda), \quad D(\lambda) := (\lambda^2 + \omega^2)I - H(\lambda) \quad \text{for} \quad \Re \lambda > 0.
\]

### 4.1 Time decay for \( q(t) \) and \( p(t) \)

In this subsection we prove the exponential decay for \( q(t) \) and \( p(t) \) in the case when \( m = 0 \). The case \( m \neq 0 \) is considered in Appendix.

**Theorem 4.2** Let \( m = 0 \), conditions A1–A3 and (4.1) hold. Then there exists a \( \delta > 0 \) such that the following bound holds
\[
|q(t)| + |p(t)| \leq C e^{-\delta t} \| Y_0 \|_{E,R_1}.
\]

To prove Theorem 4.2 we first investigate the properties of the matrix \( D(\lambda) \).

Denote \( \mathcal{C}_\beta := \{ \lambda \in \mathcal{C} : \Re \lambda > \beta \} \) for \( \beta \in \mathbb{R} \).

**Lemma 4.3** Let \( m = 0 \) and conditions A1–A3 hold. Then (i) \( D(\lambda) \) admits an analytic continuation to \( \mathcal{C} \); (ii) for every \( \beta > 0 \) \( \exists N_\beta > 0 \) such that \( v \cdot D(\lambda)v \geq C|v|^2|\lambda|^2 \) for \( \lambda \in \mathcal{C}_{-\beta} \) with \( |\lambda| \geq N_\beta \) and every \( v \in \mathbb{R}^3 \). (iii) There exists a \( \delta > 0 \) such that \( v \cdot D(\lambda)v \neq 0 \) for \( \lambda \in \mathcal{T}_{-\delta} \) and for every \( v \neq 0 \).

Lemma 4.3 is a modification of Lemma 7.2 of [10], where the result is proved in the case when \( d = 1 \) and \( \rho(x) = \rho_r(|x|) \).

**Proof.** (i) We rewrite the entries of the matrix \( H(\lambda) \) as
\[
H_{ij}(\lambda) = \sum_{n=1}^d \int \int \nabla_i \rho_n(y) \frac{e^{-\lambda |y-z|}}{4\pi |y-z|} \nabla_j \rho_n(z) \, dydz.
\]
Hence, $H_{ij}(\lambda)$ is defined and has an analytic continuation to $\mathbb{C}$. Therefore property (i) follows.

(ii) From (4.14) it follows that

$$H_{ij}(\lambda) \to 0 \text{ as } |\lambda| \to \infty \text{ with } \lambda \in \mathbb{C}_{-\beta}$$

(4.15)

what implies (ii).

(iii) At first note that the matrix $D(\lambda)$ is positive definite if $\text{Im } \lambda = 0$. Indeed, from condition $A1$ it follows that, for any $v \in \mathbb{R}^3 \setminus \{0\}$,

$$v \cdot D(\lambda)v = (\lambda^2 + \omega^2)|v|^2 - \sum_{n=1}^{d} \frac{1}{(2\pi)^3} \int \frac{(v \cdot k)^2}{k^2 + \lambda^2} |\hat{\rho}_n(k)|^2 \, dk$$

$$\geq \omega^2|v|^2 - \sum_{n=1}^{d} \frac{1}{(2\pi)^3} \int \frac{(v \cdot k)^2}{k^2} |\hat{\rho}_n(k)|^2 \, dk > 0.$$ 

Secondly, for $y \in \mathbb{R}^1 \setminus \{0\}$, we find

$$H_{ij}(iy + 0) = \sum_{n=1}^{d} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{k_i k_j |\hat{\rho}_n(k)|^2}{k^2 - (y - i0)^2} \, dk = \int_{0}^{+\infty} \frac{r^4 g_{ij}(r)}{(r - y + i0)(r + y - i0)} \, dr,$$

where

$$g_{ij}(r) := \sum_{n=1}^{d} \frac{1}{(2\pi)^3} \int |\theta \theta | |\hat{\rho}_n(r\theta)|^2 \, dS_\theta.$$ 

Note that $g_{ij} \in C^\infty([0, +\infty)), \text{ max } r \in [0, \infty) (1 + r)^N |g_{ij}(r)| < \infty$ for every $N > 0$ by condition $A2$. Applying the Plemelj formula (see [12]) and condition $A3$ we obtain

$$v \cdot \text{Im} \, H(iy + 0)v = -\pi \frac{y^3}{2} \sum_{n=1}^{d} \frac{1}{(2\pi)^3} \int_{|\theta| = 1} (v \cdot \theta)^2 |\hat{\rho}_n(|y|\theta)|^2 \, dS_\theta \neq 0$$

(4.16)

for any vector $v \in \mathbb{R}^3 \setminus \{0\}$. Hence $v \cdot \text{Im} \, D(iy + 0)v = -v \cdot \text{Im} \, H(iy + 0)v \neq 0$. Lemma 4.3 is proved.

Denote by $\mathcal{N}(t)$ an inverse Laplace transformation of $\tilde{\mathcal{N}}(\lambda)$,

$$\mathcal{N}(t) = \frac{1}{2\pi i} \int_{-\infty - \delta}^{i\infty - \delta} e^{\lambda t} \tilde{\mathcal{N}}(\lambda) \, d\lambda \text{ for } t > 0.$$

(4.17)

**Lemma 4.4** Let $m = 0$ and conditions $A1$–$A3$ hold. Then for $j = 0, 1, \ldots,$

$$|\mathcal{N}^{(j)}(t)| \leq C e^{-\delta t}, \text{ t > 1.}$$

(4.18)

**Proof.** By Lemma 4.3 the bound on $\mathcal{N}(t)$ follows. To prove the bound for $\mathcal{N}(t)$ we consider $\lambda \tilde{\mathcal{N}}(\lambda)$ and prove the following bound:

$$\left| v \cdot (\lambda \tilde{\mathcal{N}}(\lambda))'v \right| \leq \frac{C |v|^2}{1 + |\lambda|^2}, \text{ for } \lambda \in \mathbb{T}_{-\delta}. $$

(4.19)
This implies that \( \max_{t \in [0, +\infty)} |te^{\delta t} \mathcal{N}(t)| < \infty \), and the bound (4.18) for \( \mathcal{N}(t) \) follows. To prove (4.19) it suffices to establish that \( |\tilde{\mathcal{N}}_i'(\lambda)| \leq C(1 + |\lambda|)^{-3} \). From (4.14) it follows that

\[
H'_{ij}(\lambda) \rightarrow 0 \quad \text{as} \quad |\lambda| \rightarrow \infty \quad \text{with} \quad \lambda \in \mathbb{C}_{-\delta}.
\]

Therefore, Lemma 4.3 and (4.12) imply that, for \( i, j = 1, 2, 3 \),

\[
|\tilde{\mathcal{N}}'_i(t)| \leq C(1 + |\lambda|)^{-3},
\]

and the bound (4.18) for \( \dot{\mathcal{N}}(t) \) follows. To prove (4.19) it suffices to establish that

\[
|\tilde{\mathcal{N}}'_i(t)| \leq C(1 + |\lambda|)^{-3}.
\]

From (4.14) it follows that \( H'_{ij}(\lambda) \rightarrow 0 \) as \( |\lambda| \rightarrow \infty \) with \( \lambda \in \mathbb{C}_{-\delta} \).

Therefore, Lemma 4.3 and (4.12) imply that, for \( i, j = 1, 2, 3 \),

\[
|\tilde{\mathcal{N}}'_i(t)| \leq C(1 + |\lambda|)^{-3}.
\]

This implies (4.19). The bound (4.18) with \( j \geq 2 \) is proved similarly.

**Proof of Theorem 4.2.** Denote by \( \varphi_n^1(t) \equiv \varphi_n^1(x, t) \) the solution of the Cauchy problem (2.9) with the initial data \( \varphi_0^0 = (\varphi_0^0, \pi_0^0) \). Then (4.9) and (4.11) imply that for \( t > 0 \)

\[
q(t) = I(t, \phi^0) + R(t, \dot{\phi^0}, \dot{p^0}), \quad (4.20)
\]

where \( \phi^0 = (\phi_1^0, \ldots, \phi_d^0) \) and

\[
I(t, \phi^0) := -\sum_{n=1}^d \int_0^t \langle \varphi_n^1(t - s), \mathcal{N}(s) \nabla \rho_n \rangle ds
\]

\[
= \sum_{n=1}^d \int_0^t \langle \mathcal{N}(s) \nabla \varphi_n^1(t - s), \rho_n \rangle ds, \quad (4.21)
\]

\[
R(t, \dot{\phi^0}, \dot{p^0}) := \mathcal{N}(t)\dot{\phi^0} + \dot{\mathcal{N}}(t)\dot{p^0}. \quad (4.22)
\]

First, by the bound (4.18) we have

\[
|R(t, \dot{\phi^0}, \dot{p^0})| \leq C e^{-\delta t}(|\dot{\phi^0}| + |\dot{p^0}|). \quad (4.23)
\]

Secondly, since \( m = 0 \),

\[
W_n(t)\phi_n^0 = 0 \quad \text{for} \quad x \in B_R \quad \text{and} \quad t > R + R_1. \quad (4.24)
\]

due to the condition (4.1) and the strong Huygens principle. Hence, from (4.18) and (4.21) we obtain (see (3.6)) that, for \( t > R_1 + R_\rho \),

\[
|I(t, \phi^0)| \leq \sum_{n=1}^d C(p_n) \int_0^t |\mathcal{N}(s)||\nabla \varphi_n^1(t - s)||_{L^2(B_{R_\rho})} ds
\]

\[
\leq \sum_{n=1}^d C_1 \int_{t - R_1 - R_\rho}^t e^{-\delta s} \|\phi_n^0\|_{K_1} ds \leq C e^{-\delta t} \|\phi^0\|_{K_1}. \quad (4.25)
\]

Relations (4.20), (4.23) and (4.25) imply the bound (4.13) for \( q(t) \). The bound for \( p(t) \) is proved similarly.
4.2 Time decay of field components

Now we finish the proof of Proposition 4.1. From equations (1.2) we have

\[ \phi_n(x, t) = W_n(t)\phi_n^0 - \int_0^t W_n(t - s) \left( 0 \nabla \rho_n(x) \right) \cdot q(s) \, ds, \quad n \in \bar{d}. \]  

(4.26)

Let \( m_n = 0 \). Then the bound (4.24), the condition (4.1) and Theorem 4.2 imply

\[ \|\phi_n(t)\|_R \leq C \int_{t - R + R}^t e^{-\delta s} \|Y_0\|_{\mathcal{E}, R_1} \, ds \leq Ce^{-\delta t} \|Y_0\|_{\mathcal{E}, R_1} \]

for \( t > R + \max\{R_p, R_1\} \). Let \( m_n \neq 0 \). Then instead of (4.24) we apply the following well-known bound:

\[ \|W_n(t)\phi_0^0\|_R \leq C(1 + t)^{-3/2} \|\phi_0^0\|_{R_1}, \quad t \geq 0. \]

Hence Theorem 7.1 and the representation (4.26) yield

\[ \|\phi_n(t)\|_R \leq C(1 + t)^{-3/2} \|Y_0\|_{\mathcal{E}, R_1}. \]

The bound (4.2) is proved.

5 Compactness of measures \( \mu_t \)

**Lemma 5.1** Let conditions A1–A3 and S0–S2 hold. Then

\[ \sup_{t \geq 0} E\|U(t)Y_0\|^2_{\mathcal{E}, R} \leq C(R) < \infty, \quad \forall R > 0. \]  

(5.1)

**Proof** Let us write \( U_0(t) = e^{A_0 t} \) (see (3.3)). At first, note that

\[ \sup_{t \geq 0} E\|U_0(t)Y_0\|^2_{\mathcal{E}, R} \leq C(R), \quad \forall R > 0. \]  

(5.2)

Indeed (see (2.1) and (2.2)),

\[ \|U_0(t)Y_0\|^2_{\mathcal{E}, R} = \|W(t)\phi_0^0\|^2_R + |q_0(t)|^2 + |\dot{q}_0(t)|^2, \]

where the operator \( W(t) \) is defined in (2.10) and \( q_0(t) \) is a solution to the Cauchy problem

\[ \dot{q}_0(t) + \omega^2 q_0(t) = 0, \quad t \in \mathbb{R}, \quad (q_0(t), \dot{q}_0(t))|_{t=0} = (q_0^0, p_0^0). \]

Hence, \( |q_0(t)| + |\dot{q}_0(t)| \leq C(|q_0^0| + |p_0^0|) \). From [1, Proposition 3.2] and [2, Proposition 3.1] it follows that

\[ \sup_{t \geq 0} E\|W(t)\phi_0^0\|^2_R = \sup_{t \geq 0} \sum_{n=1}^d E\|W_n(t)\phi_n^0\|^2_R \leq C(R), \quad \forall R > 0. \]  

(5.3)
It implies the bound (5.2). Further, we represent the solution to (1.5) as follows

$$U(t)Y_0 = U_0(t)Y_0 + \int_0^t U(t-s)BU_0(s)Y_0\,ds,$$  

(5.4)

where the operator $B$ is defined in (3.3). Hence, (4.2) and (5.2) yield

$$E\|U(t)Y_0\|^2_{L,R} \leq E\|U_0(t)Y_0\|^2_{L,R} + E\int_0^t \|U(t-s)BU_0(s)Y_0\|^2_{L,R}\,ds$$

$$\leq C(R) + \int_0^t \varepsilon_m(t-s)E\|U_0(s)Y_0\|^2_{L,R}\,ds \leq C_1(R) < \infty. \quad \blacksquare$$

6 Convergence of characteristic functionals

6.1 Asymptotic behavior of $Y(t)$

Proposition 6.1 Let conditions A1–A3 and S0–S3 hold. Then

(i) the following bounds hold,

$$E|q_i(t) - \sum_{k=1}^d \langle W_k(t)\phi^0_k, \alpha^i_k \rangle|^2 \leq C\varepsilon_m(t),$$

(6.1)

$$E|p_i(t) - \sum_{k=1}^d \langle W_k(t)\phi^0_k, \beta^i_k \rangle|^2 \leq C\varepsilon_m(t), \quad t > 1,$$

(6.2)

where $\varepsilon_m(t) = e^{-2\delta t}$, if $m = 0$, and $\varepsilon_m(t) = (1 + t)^{-1}$ otherwise, the functions $\alpha^i_k, \beta^i_k$ are defined in (2.14) and (2.15), $i = 1, 2, 3$.

(ii) Let $\psi \in [C_0^\infty(\mathbb{R}^3)]^2$ with supp $\psi \subset B_R$. Then, for $n \in \tilde{d}$ and $t \geq 1$,

$$E\left|\langle \phi_n(t), \psi \rangle - \langle W_n(t)\phi^0_n, \psi \rangle + \sum_{k=1}^d \langle W_k(t)\phi^0_k, \theta_{kn} \rangle \right|^2 \leq C\varepsilon_m(t),$$

(6.3)

where the functions $\theta_{kn}(x), \, k, n \in \tilde{d}$, are defined in (2.16) with $\psi_n = \psi$.

Proof. (i) At first, relations (4.20)–(4.22), the bounds (4.23) and (7.11) yield

$$E|q_i(t) + \sum_{k=1}^d \sum_{r=1}^3 \int_0^t \langle \varphi^1_k(t-s), \mathcal{N}_{ir}(s)\nabla_r p_k \rangle \,ds|^2 \leq C\varepsilon_m^2(t)$$

(6.4)

with $\varepsilon_m(t)$ from (4.3). Secondly,

$$E \left| \int_t^{+\infty} \langle \varphi^1_k(t-s), \mathcal{N}_{ir}(s)\nabla_r p_k \rangle \,ds \right|^2$$

$$= \int_t^{+\infty} \mathcal{N}_{ir}(s_1) \,ds_1 \int_t^{+\infty} \mathcal{N}_{ir}(s_2) E \left( \langle \varphi^1_k(t-s_1), \nabla_r p_k \rangle \langle \varphi^1_k(t-s_2), \nabla_r p_k \rangle \right) \,ds_2.$$
For any \( t, s_1, s_2 \in \mathbb{R} \), we have
\[
|E\left( \langle \varphi_k(t - s_1), \nabla_r \rho_k \rangle \langle \varphi_k(t - s_2), \nabla_r \rho_k \rangle \right)| \leq C \sup_{\tau \in \mathbb{R}} E|\langle \varphi_k^1(\tau), \nabla_r \rho_k \rangle|^2 \\
\leq C_1 \sup_{\tau \in \mathbb{R}} E\|\varphi_k^1(\tau)\|_{L^2(B_{R_\rho})}^2 \leq C_2 < \infty
\]
by the bound (5.3). Hence, applying the bounds (4.18) and (7.11) we obtain that
\[
E \left| \int_{t}^{\infty} \langle \varphi_k^1(t - s), \mathcal{N}_{ir}(s) \nabla_r \rho_k \rangle \, ds \right|^2 \leq C \tilde{\varepsilon}_m(t). \tag{6.5}
\]
Therefore, (6.1) follows from (6.4), (6.5) and (2.14), since
\[
\langle \varphi_k^1(t - s), \mathcal{N}_{ir}(s) \nabla_r \rho_k \rangle = \left\langle W_k(t) \phi_k^0, W_k(-s) \left( \begin{array}{c} \mathcal{N}_{ir}(s) \nabla_r \rho_k \\ 0 \end{array} \right) \right\rangle.
\]
The bound (6.2) can be proved by similar way.

(ii) Let \( \psi \in [C_0^\infty(\mathbb{R}^3)]^2 \) with \( \text{supp} \, \psi \subset B_{R}. \) From (4.26) it follows that
\[
\langle \phi_n(t), \psi \rangle = \langle W_n(t) \phi_n^0, \psi \rangle - \frac{3}{2} \int_{0}^{t} q_i(t - s) \langle W_n(s) \mathcal{R}_{in}, \psi \rangle \, ds, \quad \mathcal{R}_{in} \equiv \left( \begin{array}{c} 0 \\ \nabla_i \rho_n \end{array} \right). \tag{6.6}
\]
Note that
\[
\langle W_n(s) \mathcal{R}_{in}, \psi \rangle = \begin{cases} 0 & \text{for } s > R_\rho + R \quad \text{if } m_n = 0, \\
\mathcal{O}\left((1 + s)^{-3/2}\right) & \text{if } m_n \neq 0. \tag{6.7}
\end{cases}
\]
Then from (6.1) and (6.7) it follows that
\[
E \left| \int_{0}^{t} \left( q_i(t - s) - \sum_{k=1}^{d} \langle W_k(t - s) \phi_k^0, \alpha_k^0 \rangle \right) \langle W_n(s) \mathcal{R}_{in}, \psi \rangle \, ds \right|^2 \leq C \tilde{\varepsilon}_m(t). \tag{6.8}
\]
Denote
\[
I_n(t) := E \left| \int_{0}^{t} \sum_{k=1}^{d} \langle W_k(t - s) \phi_k^0, \alpha_k^0 \rangle \langle W_n(s) \mathcal{R}_{in}, \psi \rangle \, ds \right|^2.
\]
If \( m_n = 0 \), \( I_n(t) = 0 \) for \( t > R_\rho + R \) by (6.7). If \( m_n \neq 0 \),
\[
|I_n(t)| \leq C \tilde{\varepsilon}_m(t). \tag{6.9}
\]
It follows from the following estimate: for \( \tau \in \mathbb{R} \),
\[
E|\langle W_k(\tau) \phi_k^0, \alpha_k^0 \rangle|^2 = E \left| \int_{0}^{\infty} \sum_{r=1}^{d} \langle \mathcal{N}_{ir}(s) \left( W_k(\tau - s) \phi_k^0, \left( \begin{array}{c} \nabla_r \rho_k \\ 0 \end{array} \right) \right) \rangle \, ds \right|^2 \leq C < \infty,
\]
by (5.3) and (4.18) and (7.11). The relation (6.6) and the bounds (6.8) and (6.9) imply the bound (6.3).

**Corollary 6.2** Let \( Z = (\psi, u, v) \in \mathcal{D} = \mathcal{D}_0 \times \mathbb{R}^3 \times \mathbb{R}^3 \). Then
\[
\langle Y(t), Z \rangle = \langle W(t) \phi^0, \psi \rangle + r(t),
\]
where \( \langle Y(t), Z \rangle = \langle \phi(t), \psi \rangle + q(t) \cdot u + p(t) \cdot v \), \( Y(t) = (\phi(t), q(t), p(t)) \) is a solution to the Cauchy problem (1.5), the function \( \psi^Z \) is defined in (2.13) and \( E|r(t)|^2 \leq C \tilde{\varepsilon}_m(t) \).
6.2 The end of the proof of Theorem 2.11

Proposition 6.3 Let all assumptions of Theorem 2.11 be fulfilled. Then, for \( Z \in \mathcal{D} \),
\[
E \exp\{i(Y(t), Z)\} \to \exp \left\{ -\frac{1}{2} Q_\infty(Z, Z) \right\}, \quad t \to \infty.
\]

Proof. By triangle inequality we have
\[
\left| E e^{i(Y(t), Z)} - \exp\left\{ -\frac{1}{2} Q_\infty(Z, Z) \right\} \right| \leq \left| E \left( e^{i(Y(t), Z)} - e^{i(W(t)\phi^0, \psi^Z)} \right) \right| + \left| E e^{i(W(t)\phi^0, \psi^Z)} - \exp\left\{ -\frac{1}{2} Q_\infty(Z, Z) \right\} \right|.
\]

The first term in the RHS of (6.10) is estimated by
\[
\left| E \left( e^{i(Y(t), Z)} - e^{i(W(t)\phi^0, \psi^Z)} \right) \right| \leq E|Y(t), Z) - (W(t)\phi^0, \psi^Z)| \leq E|t| \leq \left( E|t|^2 \right)^{1/2} \leq C_\varepsilon^{1/2}(t) \to 0 \quad \text{as} \quad t \to \infty,
\]
by Corollary 6.2. It remains to prove the convergence of \( E \exp\{i(W(t)\phi^0, \psi^Z)\} \equiv \hat{v}_t(\psi^Z) \) to a limit as \( t \to \infty \). In [1, 2] we proved the convergence of \( \hat{v}_t(\psi) \) to a limit for \( \psi \in \mathcal{D}_0 \). However, generally, \( \psi^Z \notin \mathcal{D}_0 \). Now we introduce a space \( H_m \) such that \( \psi^Z \in H_m \) and the characteristic functionals \( \hat{v}_t(\psi), t \in \mathbb{R} \), are equicontinuous in \( H_m \).

Definition 6.4 \( H_m \) is the space of the pairs \( \psi = (\psi^0, \psi^1) \) of \( \mathbb{R}^d \)-valued functions \( \psi^0 = (\psi^0_1, \ldots, \psi^0_d) \) and \( \psi^1 = (\psi^1_1, \ldots, \psi^1_d) \), such that \( \psi^0_n, \psi^1_n \in L^2(\mathbb{R}^3), \psi^1_n \in H^1(\mathbb{R}^3), \quad n \in \tilde{d} \), with the finite norm
\[
\|\psi\|_m^2 := \sum_{n=1}^d \left( \|\psi^0_n\|^2 + \|\psi^1_0\|^2 \right) + \|\psi^1_1\|^2 + \|\nabla \psi^1_1\|^2).
\]

The formulas (2.13)–(2.16) and condition A2 imply that the functions \( \psi^Z \) satisfying the bound \( \sup_{k \in \mathbb{R}^3} (1 + |k|)^N |\hat{\psi}^Z(k)| < \infty \) for every \( N > 0 \), and \( \psi^Z \in H_m \). Note that if all \( m_n \neq 0 \), \( H_m = L^2(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3) \).

Lemma 6.5 (i) The quadratic form \( Q_\ell^r(\psi, \psi) = \int |\langle \phi^0, \psi \rangle|^2 \nu_\ell(d\phi^0), \ t \in \mathbb{R} \), are equicon- tinuous in \( H_m \).
(ii) The characteristic functionals \( \hat{v}_t(\psi), t \in \mathbb{R} \), are equiconcuous in \( H_m \).

Proof. (i) It suffices to prove the uniform bounds
\[
\sup_{t \in \mathbb{R}} |Q_\ell^r(\psi, \psi)| \leq C \|\psi\|_m^2, \quad \psi \in H_m.
\]

Note that \( Q_\ell^r(\psi, \psi) = \sum_{n,n'=1}^d \langle g_{0,n'}(x-y), W_n(t)\psi_n(x) \otimes W_{n'}(t)\psi_{n'}(y) \rangle \). Then by Remark 2.9 we have
\[
\sup_{t \in \mathbb{R}} |Q_\ell^r(\psi, \psi)| \leq C \sup_{t \in \mathbb{R}} \sum_{n=1}^d \|W_n(t)\psi_n\|_{L^2}^2 \leq C \|\psi\|_m^2.
\]

(6.12)
By the Cauchy-Schwartz inequality and (6.12) we obtain that
\[
\lim_{t \to \infty} |\hat{\nu}_t(\psi_1) - \hat{\nu}_t(\psi_2)| \leq \sqrt{Q^\nu_t(\psi_1 - \psi_2, \psi_1 - \psi_2)} \leq C\|\psi_1 - \psi_2\|_m.
\]

Since \( \psi^Z \in H_m \), Proposition 3.3 of [1] (or Proposition 3.2 of [2]) and Lemma 6.5, (ii) yield
\[
\hat{\nu}_t(\psi^Z) \to \exp\{-\frac{1}{2}Q^\nu_\infty(\psi^Z, \psi^Z)\} \quad \text{as} \quad t \to \infty,
\]
where \( Q^\nu_\infty \) is defined by (2.12). This completes the proof of Theorem 2.11.

### 6.3 Convergence of correlation functions

**Proposition 6.6** Let all assumptions of Theorem 2.11 be fulfilled. Then, for \( Z_1, Z_2 \in \mathcal{D} \),
\[
E(\langle Y(t), Z_1 \rangle \langle Y(t), Z_2 \rangle) \to Q_\infty(Z_1, Z_2), \quad t \to \infty.
\]

**Proof.** It is enough to prove the convergence of \( E|\langle Y(t), Z \rangle|^2 \) to a limit as \( t \to \infty \). From Corollary 6.2 it follows that, for \( Z \in \mathcal{D} \),
\[
E|\langle Y(t), Z \rangle|^2 = E|\langle W(t)\phi^0, \psi^Z \rangle|^2 + o(1) = Q^\nu_t(\psi^Z, \psi^Z) + o(1), \quad t \to \infty,
\]
where \( \psi^Z \) is defined in (2.13) and \( \psi^Z \in H_m \). Therefore, Proposition 6.2 of [1] (or Lemma 4.4 of [2]) and Lemma 6.5, (i) imply that \( \lim_{t \to \infty} Q^\nu_t(\psi^Z, \psi^Z) = Q^\nu_\infty(\psi^Z, \psi^Z) \). Formula (2.17) implies (6.13).

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### 7 Appendix: Time decay of \( q(t) \), \( p(t) \): Case \( m \neq 0 \)

Here we prove the following result, which is a modification of Lemma 17.1 of [11].

**Theorem 7.1** Let \( m \neq 0 \), the conditions A1–A3 hold, and \( Y_0 \in \mathcal{E} \) with \( \phi^0(x) = 0 \) for \( |x| > R_1 \). Then \( q(t) \) and \( p(t) \) are continuous and the following bound holds
\[
|q(t)| + |p(t)| \leq C(1 + t)^{-3/2}, \quad t \geq 0.
\]
Ar first, we rewrite the system (4.4)–(4.7) in the form
\[
(A - \lambda)\tilde{Y}(\lambda) = -Y_0, \quad \text{Re} \lambda > 0,
\]
(7.1)
where \(A = A_0 + B\) and the operators \(A_0, B\) are defined in (3.3). Hence, the solution \(\tilde{Y}\) is given by
\[
\tilde{Y}(\lambda) = -(A - \lambda)^{-1}Y_0, \quad \text{Re} \lambda > 0,
\]
if the resolvent \(R(\lambda) = (A - \lambda)^{-1}\) exists for \(\text{Re} \lambda > 0\).

**Proposition 7.2** The operator-valued function \(R(\lambda) : E \to E\) is analytic for \(\text{Re} \lambda > 0\).

**Proof.** It suffices to prove that the operator \((A - \lambda) : E \to E\) has a bounded inverse operator for \(\text{Re} \lambda > 0\).

Let us prove that \(\text{Ker}(A - \lambda) = 0\) for \(\text{Re} \lambda > 0\). Indeed, let \(\tilde{Y}(\lambda) \in E\) is a solution (7.1) with \(Y_0 = 0\). The function \(Y(t) = \tilde{Y}(\lambda)e^{\lambda t} \in C(\mathbb{R}, E)\) is the solution to the equation \(\dot{Y} = AY\). Then the Hamiltonian \(H(Y(t)) = e^{2\lambda t}H(\tilde{Y}(\lambda))\) grows exponentially by (3.4). This grow contradicts (3.1). Hence, (3.4) implies that \(\tilde{\varphi}(x, \lambda) = 0\) and \(\tilde{\varphi}(\lambda) = 0\) because \(\lambda \neq 0\).

Remember that \(A = A_0 + B\) (see (3.3)), where the operator \(B\) is finite-dimensional and the operator \(A_0^{-1}\) is bounded in \(E\). We rewrite
\[
A - \lambda = (A_0 - \lambda)(I + (A_0 - \lambda)^{-1}B),
\]
where \((A_0 - \lambda)^{-1}B\) is a compact operator. Since \(\text{Ker}(I + (A_0 - \lambda)^{-1}B) = 0\), the operator \(I + (A_0 - \lambda)^{-1}B\) is invertible by the Fredholm theory.

Denote by \(g_{\lambda,n}\) the fundamental solution of the operator \(-\Delta + m_n^2 + \lambda^2\),
\[
g_{\lambda,n}(y) = \frac{e^{-\kappa_n|y|}}{4\pi|y|}, \quad n \in \tilde{d},
\]
(7.2)
where \(\kappa_n^2 = m_n^2 + \lambda^2\), \(\text{Re} \kappa_n > 0\) for \(\text{Re} \lambda > 0\). From (4.4)–(4.7) it follows that
\[
M(\lambda) \left( \begin{array}{c} \tilde{q}(\lambda) \\ \tilde{p}(\lambda) \end{array} \right) = \left( \begin{array}{c} q^0 \\ \tilde{p}^0(\lambda) \end{array} \right),
\]
where \(\tilde{p}^0(\lambda) = p^0 - \sum_{n=1}^{d} \langle g_{\lambda,n} * (\lambda \varphi_n^0(\lambda) + \tilde{\pi}_n^0(\lambda)), \nabla \rho_n \rangle\),
\[
M(\lambda) := \begin{pmatrix} \lambda I & -I \\ \omega^2 I - H(\lambda) & \lambda I \end{pmatrix},
\]
(7.3)
and the matrix \(H(\lambda)\) is defined in (4.10). The entries of \(H(\lambda)\) are of the form
\[
H_{ij}(\lambda) = \sum_{n=1}^{d} \int \nabla_i \rho_n(y)(g_{\lambda,n} * \nabla_j \rho_n)(y) \, dy = \sum_{n=1}^{d} \frac{1}{(2\pi)^3} \int \frac{k_i k_j |\hat{\rho}_n(k)|^2}{k^2 + m_n^2 + \lambda^2} \, dk.
\]
(7.4)
The following result is proved in [11, p.351].
Lemma 7.3  (i) The operator $-\Delta + m_n^2 + \lambda^2$ is invertible in $L^2(\mathbb{R}^3)$ for $\text{Re} \lambda > 0$ and its fundamental solution (7.2) decays exponentially as $|y| \to \infty$.

(ii) For every fixed $y \neq 0$, the Green function $g_{\lambda,n}(y)$ admits an analytic continuation (in variable $\lambda$) to the Riemann surface of the algebraic function $\sqrt{\lambda^2 + m_n^2}$ with the branching points $\pm im_n$ if $m_n \neq 0$.

Lemma 7.3 and notation (7.3) imply that $M(\lambda)$ admits an analytic continuation from the domain $\text{Re} \lambda > 0$ on the Riemann surface with the branching points $\pm im_n$ with $m_n \neq 0$, $n \in \hat{d}$. Moreover, the matrix $M^{-1}(\lambda)$ exists for large $\lambda$. It follows from (7.3) since $H(\lambda) \to 0$ as $\text{Re} \lambda \to \infty$ by (7.4).

Corollary 7.4  (i) The matrix $M(\lambda)$ is invertible for $\text{Re} \lambda > 0$, and

$$\left( \begin{array}{c} \tilde{q}(\lambda) \\ \tilde{p}(\lambda) \end{array} \right) = M^{-1}(\lambda) \left( \begin{array}{c} q^0 \\ p^0(\lambda) \end{array} \right), \quad \text{Re} \lambda > 0. \quad (7.5)$$

(ii) The matrix $M^{-1}(\lambda)$ admits meromorphic continuation from the domain $\text{Re} \lambda > 0$ to the Riemann surface with the branching points $\pm im_n$ with $m_n \neq 0$, $n \in \hat{d}$.

Now we investigate the limit values of $M^{-1}(\lambda)$ at the imaginary axis $\lambda = ix$, $x \in \mathbb{R}$. Without loss of generality, we assume that $d = 2$ and $0 < m_1 < m_2$. The other cases can be considered similarly. The limit matrix

$$M(ix + 0) = \left( \begin{array}{cc} ixI & -I \\ \omega^2 I - H(ix + 0) & ixI \end{array} \right), \quad x \in \mathbb{R}, \quad (7.6)$$

exists, and its entries are continuous functions of $x \in \mathbb{R}$, smooth for $|x| < m_1$, $m_1 < |x| < m_2$, and $|x| > m_2$.

Lemma 7.5 The limit matrix $M(ix + 0)$ is invertible for $x \in \mathbb{R}$.

Proof. (i) Let $|x| \leq m_1$. Then the matrix $(\omega^2 - x^2)I - H(ix + 0)$ is positive definite. Indeed, for every $v \in \mathbb{R}^3 \setminus \{0\}$, by the condition A1 with $m_* = m_1$,

$$v \cdot ((\omega^2 - x^2)I - H(ix + 0))v = (\omega^2 - x^2)|v|^2 - \sum_{n=1}^{\hat{d}} (2\pi)^{-3} \int \frac{(k \cdot v)^2 |\hat{\rho}_n(k)|^2 dk}{k^2 + m_n^2 - x^2}$$

$$\geq (\omega^2 - m_1^2)|v|^2 - v \cdot K v > 0.$$

(ii) Let $m_1 < |x| \leq m_2$. Then $v \cdot \text{Im} H(ix + 0)v \neq 0$ for every $v \in \mathbb{R}^3 \setminus \{0\}$. Indeed,

$$\text{Im} H_{ij}(ix + 0) = \text{Im} (2\pi)^{-3} \int \frac{k_i k_j |\hat{\rho}_1(k)|^2 dk}{k^2 + m_1^2 - (x - i0)^2}.$$

For $\varepsilon > 0$, consider the function

$$h_{ij}(ix + \varepsilon) = \int \frac{k_i k_j |\hat{\rho}_1(k)|^2}{k^2 + m_1^2 - (x - i\varepsilon)^2} dk, \quad |x| > m_1.$$

Denote $D_{\varepsilon}(k) = k^2 + m_1^2 - (x - i\varepsilon)^2$. For $|x| > m_1$, $D_{0}(k) = 0$ if $|k| = \sqrt{x^2 - m_1^2}$. We fix a small $\delta > 0$ and introduce a cutoff function $\zeta \in C_0^\infty(\mathbb{R}^3)$ such that $\zeta(k) \geq 0$, if $|k| < \delta$.

19
\[ \zeta(k) = 1 \] when \(|D_0(k)| < \delta\) and \(\zeta(k) = 0\) when \(|D_0(k)| \geq 2\delta\). Note that \(\text{Im} \ h_{ij}(ix + 0) = \text{Im} \ h_{ij}^\delta(ix + 0)\), where
\[
h_{ij}^\delta(ix + 0) = \lim_{\varepsilon \to 0} \int \zeta(k) \frac{k_i k_j |\hat{\rho}_1(k)|^2}{D_\varepsilon(k)} \, dk.
\]
Denote \(a(k) = \sqrt{k^2 + m_i^2}\). Assume that \(x > 0\). Since
\[
\frac{1}{D_\varepsilon(k)} = \frac{1}{2a(k)(a(k) - x + i\varepsilon)} + \frac{1}{2a(k)(a(k) + x - i\varepsilon)}.
\]
\(\text{Im} \ h_{ij}^\delta(ix + 0) = \text{Im} \ h_{ij}^\delta(ix + 0)\), where
\[
h_{ij}^\delta(ix + \varepsilon) := \int \zeta(k) \frac{k_i k_j |\hat{\rho}_1(k)|^2}{2a(k)(a(k) - x + i\varepsilon)} \, dk.
\]
We rewrite \(h_{ij}^\delta(ix + \varepsilon)\) as
\[
h_{ij}^\delta(ix + \varepsilon) = \int \frac{g(\alpha)}{\alpha + i\varepsilon} \, d\alpha, \quad g(\alpha) = \int \zeta(k) \frac{k_i k_j |\hat{\rho}_1(k)|^2}{2a(k)|\nabla a(k)|} \, dS.
\]
Hence \(\text{Im} \ h_{ij}^\delta(ix + 0) = -\pi g(0)\) by the Plemelj formula. Finally, for \(x > m_1 > 0\),
\[
\text{Im} \ h_{ij}(ix + 0) = -\pi \int_{|k| = \sqrt{x^2 - m_i^2}} \frac{k_i k_j |\hat{\rho}_1(k)|^2}{2|k|} \, dS.
\]
Hence, applying condition \(A3\) we obtain that, for \(m_1 < |x| \leq m_2\),
\[
v \cdot \text{Im} \ H(ix + 0)v = -\text{sign}(x)\pi(2\pi)^{-3} \int_{|k| = \sqrt{x^2 - m_i^2}} \frac{(v \cdot k)^2 |\hat{\rho}_1(k)|^2}{2|k|} \, dS \neq 0.
\]
(iii) Let \(|x| > m_2\). In this case we find
\[
v \cdot \text{Im} \ H(ix + 0)v = -\text{sign}(x)\pi \sum_{n=1}^{2} (2\pi)^{-3} \int_{|k| = \sqrt{x^2 - m_i^2}} \frac{(v \cdot k)^2 |\hat{\rho}_n(k)|^2}{2|k|} \, dS \neq 0.
\]
In general case \(d = 1, 2, \ldots\), we enumerate \(m_1, \ldots, m_d\) in the increasing order, \(0 \leq m_1 \leq m_2 \leq \ldots \leq m_d\). If \(m_k \neq m_{k+1}\) and \(m_k < |x| \leq m_{k+1}\),
\[
v \cdot \text{Im} \ H(ix + 0)v = -\text{sign}(x)\pi \sum_{n=1}^{k} (2\pi)^{-3} \int_{|k| = \sqrt{x^2 - m_i^2}} \frac{(v \cdot k)^2 |\hat{\rho}_n(k)|^2}{2|k|} \, dS \neq 0 \tag{7.7}
\]
by condition \(A3\). For \(|x| > m_d\), the formula (7.7) holds with \(k = d\).

**Remark 7.6** We use condition \(A3\) only in the estimate (7.7). Hence, instead of condition \(A3\) it suffices to assume that for any \(v \in \mathbb{R}^3 \setminus \{0\}\) and \(x > 0\)
\[
\sum_{n: m_n < x} \left( |k|^3 \int_{|\theta| = 1} (v \cdot \theta)^2 |\hat{\rho}_n(|k|\theta)|^2 \, dS_{\theta} \right)_{|k| = \sqrt{x^2 - m_n^2}} \neq 0.
\]
Corollary 7.7 The matrix $M^{-1}(ix + 0)$ is smooth w.r.t. $x \in \mathbb{R}$ outside the points $x = \pm im_l$ with $m_l \neq 0$.

Lemma 7.8 (i) The matrix $M^{-1}(ix+0)$ admits the following Puiseux expansion in a neighborhood of $\pm im_l$ ($m_l \neq 0$): there exists an $\varepsilon > 0$ such that

$$M^{-1}(ix + 0) = \sum_{k=0}^{\infty} c_k^\pm (x \mp m_l)^{k/2}, \quad |x \mp m_l| < \varepsilon, \quad x \in \mathbb{R}. \quad (7.8)$$

(ii) There exists a matrix $R_0$ and a matrix-valued function $R_1(x)$ such that

$$M^{-1}(ix + 0) = \frac{1}{x} R_0 + R_1(x), \quad |x| > \max_n m_n + 1, \quad x \in \mathbb{R}, \quad (7.9)$$

where $|\partial_x^k R_1(x)| \leq C_k/|x|^2$ for $|x| > \max_n m_n + 1$, $x \in \mathbb{R}$, $k = 0, 1, \ldots$.

Proof. (i) Formula (7.8) follows from (7.2) and (7.4).

(ii) Let $f \in L^2(\mathbb{R}^3)$ with supp $f \subset B_R$. Then (see formula (16.7) of [11])

$$\|\partial_x^k[-\Delta + m_n^2 + (ix + 0)^2]^{-1}f\|_{L^2(B_R)} \leq \frac{C_k(R)}{|x|} \|f\|_{L^2(B_R)}, \quad |x| \geq m_n + 1,$$

for every $R > 0$. Therefore (see (4.10)) we obtain that $|\partial_x^k H_{ij}(ix + 0)| \leq C_k/|x|$ for $|x| > \max_n m_n + 1$. Then formula (7.6) implies (7.9).

Proof of Theorem 7.1. Applying (7.5) we obtain that

$$\left( \begin{array}{c} q(t) \\ p(t) \end{array} \right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ixt} M^{-1}(ix + 0) \left( \begin{array}{c} q^0 \\ p^0(i_x + 0) \end{array} \right) dx. \quad (7.10)$$

Without loss of generality, let us assume that $0 < m_1 < m_2 < \ldots < m_d$. We split the Fourier integral (7.10) into $d+1$ terms by using the partition of unity $\zeta_0(x) + \ldots + \zeta_d(x) = 1$, $x \in \mathbb{R}$:

$$\left( \begin{array}{c} q(t) \\ p(t) \end{array} \right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ixt} (\zeta_0(x) + \ldots + \zeta_d(x)) M^{-1}(ix + 0) \left( \begin{array}{c} q^0 \\ p^0 \end{array} \right) dx$$

$$= I_0(t) + \ldots + I_d(t),$$

where the functions $\zeta_k(x) \in C^\infty(\mathbb{R})$ are supported by

- $\text{supp} \zeta_0 \subset \{ x \in \mathbb{R} : |x| > m_d + 1 \text{ or } |x| < m_1/2 \}$,
- $\text{supp} \zeta_1 \subset \{ x \in \mathbb{R} : m_1/3 < |x| < (m_1 + m_2)/2 \}$,
- $\text{supp} \zeta_2 \subset \{ x \in \mathbb{R} : m_2 - 2(m_2 - m_1)/3 < |x| < (m_2 + m_3)/2, \ldots, \}$
- $\text{supp} \zeta_d \subset \{ x \in \mathbb{R} : m_d - 2(m_d - m_{d-1})/3 < |x| < m_d + 2 \}$.

Then (i) the function $I_0(t) \in C[0, +\infty)$ decays faster than any power of $t$ due to Lemma 7.8, and (ii) the functions $I_k(t) \in C^\infty(\mathbb{R})$, $k = 1, \ldots, d$, decay like $(1 + |t|)^{-3/2}$ by virtue to (7.8). Theorem 7.1 is proved.

Corollary 7.9 Let $m \neq 0$ and conditions A1–A3 hold. Then

$$|\mathcal{N}^{(j)}(t)| \leq C(1 + |t|)^{-3/2}, \quad j = 0, 1, \quad (7.11)$$

where $\mathcal{N}^{(j)}(t)$ is defined in (4.17) and (4.12). This bound can be proved by similar way as Theorem 7.1.
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