Finite-sample concentration of the empirical relative entropy around its mean

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Abstract

In this note, we show that the relative entropy of an empirical distribution of \( n \) samples drawn from a set of size \( k \) with respect to the true underlying distribution is exponentially concentrated around its expectation, with central moment generating function bounded by that of a gamma distribution with shape \( 2k \) and rate \( n/2 \). This improves on recent work of Bhatt and Pensia [BP21] on the same problem, who showed such a similar bound with an additional polylogarithmic factor of \( k \) in the shape, and also confirms a recent conjecture of Mardia et al. [MJTNW20]. The proof proceeds by reducing the case \( k > 3 \) of the multinomial distribution to the simpler case \( k = 2 \) of the binomial, for which the desired bound follows from standard results on the concentration of the binomial.

1 Introduction

Given \( n \) samples from some distribution \( P = (p_1, \ldots, p_k) \) on a finite set of size \( k \), the realized fraction of samples corresponding to each element \((X_1, \ldots, X_k)\) is a natural estimator (and in fact the maximum likelihood estimator) of the underlying distribution \( P \). Since the Neyman–Pearson lemma [NP33] reduces optimal hypothesis testing to understanding the distribution of the likelihood ratio statistic, in this case we are led to study the empirical relative entropy with respect to the true distribution:

**Definition 1.1.** Given a distribution \( P = (p_1, \ldots, p_k) \) on a finite set of size \( k \) and multinomially distributed random variables \((X_1, \ldots, X_k) \sim \text{Multi}(n; (p_1, \ldots, p_k))\) for a positive integer \( n \), the empirical relative entropy is

\[
V_{n,k,P} = D\left(\left(\frac{X_1}{n}, \ldots, \frac{X_k}{n}\right) \middle\| (p_1, \ldots, p_k)\right) = \sum_{i=1}^{k} \frac{X_i}{n} \log \frac{X_i}{np_i},
\]

and is such that \( 2nV_{n,k,P} \) is the likelihood-ratio statistic of the hypothesis that the probabilities of \( X \) are \( P \), where

\[
D\left((q_1, \ldots, q_k) \middle\| (p_1, \ldots, p_k)\right) = \sum_{i=1}^{k} q_i \log \frac{q_i}{p_i}
\]

denotes the relative entropy or Kullback–Leibler (KL) divergence of \( Q \) with respect to \( P \).\(^1\)

\(^1\)All logarithms and exponentials are in the natural base.
For fixed $k$, it is well known (from e.g. Wilks’ theorem [Wil38]) that the likelihood ratio statistic $2nV_{n,k,P}$ converges in distribution to $\chi^2_{k-1}$ a chi-squared distribution with $k-1$ degrees of freedom as $n$ goes to infinity (assuming $P$ is not supported on a set of size smaller than $k$). For the specific case we are interested in of the multinomial distribution, there are also known finite-sample bounds, most notably the now-standard bound obtained via the method of types [Csi98] that

$$\Pr[V_{n,k,P} \geq \varepsilon] \leq \binom{n+k-1}{k-1} \cdot \exp(-n\varepsilon)$$

for all real $\varepsilon \geq 0$, which has optimal decay in $\varepsilon$ as $n$ and $\varepsilon$ go to infinity, but is trivial for $\varepsilon$ close to $\mathbf{E}[V_{n,k,P}] \leq \log\left(1 + \frac{k-1}{n}\right) \leq \frac{k-1}{n}$ [Pan03]. Mardia et al. [MJTNW20] recently substantially improved this bound, giving a roughly quadratic improvement in the $\binom{n+k-1}{k-1}$ factor while maintaining the decay in $\varepsilon$, and also posed several conjectures about improved bounds. Subsequently, the author gave an incomparable exponential bound [Agr20b] (further improved by Guo and Richardson [GR21]) which becomes non-trivial for $\varepsilon > \frac{k-1}{n}$ but which has decay like $n\varepsilon \cdot (1 - o(1))$ for large $\varepsilon$, by bounding the moment generating function of $V_{n,k,P}$.

Most of the above bounds focused on the question of bounding the probability that $V_{n,k,P}$ exceeds 0 by some $\varepsilon$, but it is also natural to ask about concentration around $\mathbf{E}[V_{n,k,P}]$. In particular, Mardia et al. [MJTNW20] posed the following conjecture:

**Conjecture 1.2 ([MJTNW20, Conjecture 2]).** There are positive constants $c_1$ and $c_2$ such that for every $n$, $k$, $P$, and $\varepsilon \geq 0$ it holds that

$$\Pr\left[|V_{n,k,P} - \mathbf{E}[V_{n,k,P}]| \geq \varepsilon\right] \leq c_1 \exp\left(-c_2 \min\left\{\frac{n^2\varepsilon^2}{k-1}, n\varepsilon\right\}\right).$$

By standard results on subgamma random variables (e.g. [BLM13, §2.4, Theorem 2.3]), Conjecture 1.2 is equivalent to upper bounding the central moment generating function of $V_{n,k,P}$ by that of a gamma distribution with shape $C_1(k-1)$ and rate $C_2n$ on a ball around the origin of radius $C_3n$ for positive constants $C_1$, $C_2$, and $C_3$.

Similarly, in [Agr20a], the author conjectured that the bound on the non-centered moment generating function of $V_{n,k,P}$ by that of the gamma distribution with shape $k-1$ and rate $n$ on the positive reals [Agr20b, Theorem 1.3] also holds for the centered version (i.e. with constants $C_1 = C_2 = 1$, but for the positive reals), which from the above would suffice to prove a one-sided version of Conjecture 1.2:

**Conjecture 1.3 ([Agr20a, Conjecture 4.4.5]).** For every $n$, $k$, and $P$, we have for all $0 \leq t < n$ that

$$\log \mathbf{E}\left[\exp\left(t\left(V_{n,k,P} - \mathbf{E}[V_{n,k,P}]\right)\right)\right] \leq (k-1) \log\left(\frac{\exp(-t/n)}{1-t/n}\right),$$

where the right-hand side is the centered log moment generating function of a gamma distribution of shape $k-1$ and rate $n$.

Significant progress towards these conjectures was made in recent work of Bhatt and Pensia [BP21], who established near-optimal bounds in the case that the probabilities of the multinomial distribution are bounded away from 0.
Theorem 1.4 (Equivalent form of [BP21, Theorem 1]). There are positive constants $C_1$ and $C_2$ such that for all $n, k,$ and $P$, it holds that

$$\log \mathbb{E} \left[ \exp \left( t \left( V_{n,k,P} - \mathbb{E}[V_{n,k,P}] \right) \right) \right] \leq C_1 \cdot k \log \left( \frac{k}{\min_i p_i} \right) \cdot \log \left( \frac{\exp(-C_2 t/n)}{1 - C_2 t/n} \right),$$

for all $|t| < C_2 n$.

However, Theorem 1.4 does not suffice to prove Conjecture 1.2 due to the additional polylogarithmic factors in $k$ and $\min_i p_i$.

In this work, we close this gap, proving Conjecture 1.2 by giving an upper bound on the centered moment generating function of $V_{n,k,P}$ with explicit constants, though falling short of those conjectured by Conjecture 1.3.

Theorem 1.5 (This work, main result). For every $n, k,$ and $P$, we have for all $t < n/2$ that

$$\log \mathbb{E} \left[ \exp \left( t \left( V_{n,k,P} - \mathbb{E}[V_{n,k,P}] \right) \right) \right] \leq \min \left\{ \frac{4kt^2/n^2}{1 - 2t/n}, 2k \log \left( \frac{\exp(-2t/n)}{1 - 2t/n} \right) \right\},$$

and so in particular the centered cumulant generating function of $V_{n,k,P}$ is bounded by that of a gamma distribution with shape $2k^2$ and rate $n/2$.

Remark 1.6. At first glance the validity of the above bound for the entire negative real line rather than an interval of length $Cn$ appears qualitatively stronger than what is necessary for Conjecture 1.2, but in fact such bounds are equivalent because $V_{n,k,P} \geq 0$ implies the trivial upper bound for $t \leq 0$ that

$$\log \mathbb{E} \left[ \exp \left( t \left( V_{n,k,P} - \mathbb{E}[V_{n,k,P}] \right) \right) \right] \leq |t| \cdot \mathbb{E}[V_{n,k,P}] \leq |t| \cdot \frac{k - 1}{n},$$

which already establishes the claim for $t \leq -Cn$ for any positive constant $C$, and in fact is stronger than Theorem 1.5 for sufficiently negative $t$.

Corollary 1.7. For all $\varepsilon \geq 0$ we have that

$$\Pr \left[ V_{n,k,P} \geq \mathbb{E}[V_{n,k,P}] + \varepsilon \right] \leq \left( 1 + \frac{n\varepsilon}{4k} \right)^{2k} \cdot \exp \left( -\frac{n\varepsilon}{2} \right) \leq \exp \left( -\frac{3n^2\varepsilon^2}{48k + 8n\varepsilon} \right) \leq \exp \left( -\min \left\{ \frac{n^2\varepsilon^2}{24k}, \frac{n\varepsilon}{8} \right\} \right)$$

and for all $0 \leq \varepsilon \leq 2k/n$ we have that

$$\Pr \left[ V_{n,k,P} \leq \mathbb{E}[V_{n,k,P}] - \varepsilon \right] \leq \exp \left( -k \left( 1 - \sqrt{1 - \frac{n\varepsilon}{2k}} \right)^2 \right) \leq \exp \left( -\frac{n^2\varepsilon^2}{16k} \right).$$

In particular, Conjecture 1.2 holds with $c_1 = 2$ and $c_2 = 1/48$.  

2In fact, the techniques in this work are capable of establishing shape $Ck$ for a constant $1 < C < 2$, see Remark 2.12. More generally, here and in the corollaries we give explicit values of constants, but have not tried to optimize them.
Remark 1.8. For $\varepsilon > 2k/n$, Theorem 1.5 implies that $\Pr[V_{n,k,p} \leq E[V_{n,k,p}] - \varepsilon] = 0$, but since $E[V_{n,k,p}] \leq \log(1 + \frac{k-1}{n}) \leq \frac{k-1}{n}$ [Pan03], as in Remark 1.6 this (and the corresponding part of Corollary 1.7) is subsumed by the fact that $V_{n,k,p} \geq 0$ implies $\Pr[V_{n,k,p} \leq E[V_{n,k,p}] - \varepsilon] = 0$ for all $\varepsilon > E[V_{n,k,p}]$.

Similarly, Theorem 1.5 also implies moment bounds, recovering a weaker version of a variance upper bound of [MJTNW20] and strengthening the moment bounds from [Agr20b; BP21].

Corollary 1.9. We have that $\text{Var}(V_{n,k,p}) \leq 8k/n^2$, and more generally, for all integers $q \geq 1$ we have

$$E\left[\left(V_{n,k,p} - E[V_{n,k,p}]\right)^{2q}\right] \leq \frac{2^6q(k^q q! + (2q)!)}{n^{2q}},$$

so that in particular for all real $q \geq 1$ we have that

$$\sqrt[q]{E\left[\left|V_{n,k,p} - E[V_{n,k,p}]\right|^q\right]} \leq \frac{24}{n} \left(\sqrt{kq} + q\right).$$

The proof of Theorem 1.5 follows a similar outline as that of the earlier work [Agr20b] for the non-centered moment generating function, later extended to the centered version by [BP21], namely it reduces the multinomial case to the simpler case $k = 2$ of the binomial and then bounding the binomial. Our point of departure is in the reduction used: the aforementioned works used a reduction that takes advantage of the dependence between the variables $X_i$ for $i \in \{1, ..., k\}$ and as a result has bounds in terms of $k - 1$, but does not adapt as easily to the centered case (though it can be done, as in [BP21]); by contrast we use a reduction that shows we can consider independent $X_i$ by incurring a quadratic loss, resulting in a simpler proof and stronger bound in the centered case (though weaker in the non-centered case, both via the quadratic loss and by depending on $k$ rather than $k - 1$). It would be interesting to find an approach that worked cleanly for both cases without incurring these losses.

2 Proof

2.1 Reduction from the multinomial to binomial

In this section, we reduce the case of an alphabet of size $k$ to that of an alphabet of size 2. To state the result, it is convenient to work with a slightly modified formulation of the relative entropy, commonly used when considering it as a $f$-divergence [Csi63; Mor63; AS66].

Definition 2.1. Let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be the function on the non-negative reals given by $\phi(x) = x \log x - x + 1$ for $x > 0$ and $\phi(0) = 1$, and let $\phi_+(x) = \phi(x) \cdot 1_{x \geq 1}$ and $\phi_-(x) = \phi(x) \cdot 1_{x \leq 1}$.

Lemma 2.2. $\phi$ is continuous, convex, non-negative, decreasing on $[0, 1]$, increasing on $[1, \infty)$, and has $\phi(1) = 0$. In particular, $\phi_+$ and $\phi_-$ are also continuous, convex, non-negative, are respectively non-decreasing and non-increasing, and satisfy $\phi = \phi_+ + \phi_-.$

Lemma 2.3. The relative entropy satisfies

$$D((q_1, ..., q_k) \| (p_1, ..., p_k)) = \sum_{i=1}^{k} p_i \cdot \phi\left(\frac{q_i}{p_i}\right)$$
where \( 0 \cdot \phi(q_i/0) = \infty \) if \( q_i > 0 \).

With this definition, we can state the main result of this section:

**Proposition 2.4.** For all \( n, k, P = (p_1, \ldots, p_k) \), and \( t \in \mathbb{R} \), it holds that

\[
\mathbb{E} \left[ \exp \left( t \cdot \left( V_{n,k,P} - \mathbb{E}[V_{n,k,P}] \right) \right) \right]
\leq \prod_{i=1}^{k} \mathbb{E} \left[ \exp \left( 2t \cdot \left( p_i \phi_+(X_i/np_i) - \mathbb{E}[p_i\phi_+(X_i/np_i)] \right) \right) \right]
\leq \prod_{i=1}^{k} \mathbb{E} \left[ \exp \left( 2t \cdot \left( p_i \phi_-(X_i/np_i) - \mathbb{E}[p_i\phi_-(X_i/np_i)] \right) \right) \right]
\]

where \( (X_1, \ldots, X_k) \) is multinomially distributed with \( n \) samples and probabilities \( P \).

Note that the expectations in Proposition 2.4 involve only a single \( X_i \) at a time, i.e. we have broken their dependence. To do so, we use the fact that the variables are negatively associated in the sense of Joag-Dev and Proschan [JP83].

**Definition 2.5** ([JP83, Definition 2.1]). A collection of real-valued random variables \( (Z_1, \ldots, Z_m) \) is said to be negatively associated if for all disjoint subsets \( A_1, A_2 \subseteq \{1, \ldots, m\} \) and (pointwise) non-decreasing functions \( f_i : \mathbb{R}^{|A_i|} \to \mathbb{R} \), it holds that \( \text{Cov}(f_1(Z_i), i \in A_1), f_2(Z_j, j \in A_2)) \leq 0 \).

**Lemma 2.6** ([JP83, Properties P2 and P6]). If \( (Z_1, \ldots, Z_m) \) are negatively associated random variables, then for all functions \( f_1, \ldots, f_m : \mathbb{R} \to \mathbb{R} \) which are either all non-increasing or non-decreasing, the random variables \( (f_1(Z_1), \ldots, f_m(Z_m)) \) are negatively associated. In particular, if each \( f_i(Z_i) \geq 0 \) almost surely, then \( \mathbb{E}[f_1(Z_1) \cdots f_m(Z_m)] \leq \mathbb{E}[f_1(Z_1)] \cdots \mathbb{E}[f_m(Z_m)] \).

**Lemma 2.7** ([JP83, 3.1(a)]). For all positive integers \( n, k \) and probabilities \( P = (p_1, \ldots, p_k) \), the random variables \( (X_1, \ldots, X_k) \) distributed multinomially with \( n \) samples and probabilities \( P \) are negatively associated.

We would like to apply Lemma 2.6 to the KL divergence, but cannot do so directly since the function \( \phi \) is not monotone; however, since \( \phi \) can be written as the sum of the monotone functions \( \phi_+ \) and \( \phi_- \), we can apply it after first separating the two parts by Cauchy–Schwarz, incurring a quadratic penalty.

**Proof of Proposition 2.4.** Fix \( n, k, P = (p_1, \ldots, p_k) \), and \( t \in \mathbb{R} \). Then by Lemma 2.3 and linearity of expectation we have that

\[
V_{n,k,P} - \mathbb{E}[V_{n,k,P}] = \sum_{i=1}^{k} p_i \phi_+(X_i/np_i) - \mathbb{E}[p_i\phi_+(X_i/np_i)] + \sum_{i=1}^{k} p_i \phi_-(X_i/np_i) - \mathbb{E}[p_i\phi_-(X_i/np_i)],
\]
and so by Cauchy–Schwarz we have
\[
\mathbb{E}\left[ \exp\left( t \cdot (V_{n,k,P} - \mathbb{E}[V_{n,k,P}]) \right) \right] \\
\leq \sqrt{\mathbb{E}\left[ \prod_{i=1}^{k} \exp\left( 2t \cdot \left( p_i \phi_+ (X_i/nP_i) - \mathbb{E}[p_i \phi_+ (X_i/nP_i)] \right) \right) \right]} \\
\cdot \sqrt{\mathbb{E}\left[ \prod_{i=1}^{k} \exp\left( 2t \cdot \left( p_i \phi_- (X_i/nP_i) - \mathbb{E}[p_i \phi_- (X_i/nP_i)] \right) \right) \right]}.
\]

Now, since \( \phi_+ \) and \( \phi_- \) are monotone (Lemma 2.2), we have that the functions
\[
f_i(x) = \exp\left( 2t \left( p_i \phi_+ (x/nP_i) - \mathbb{E}[\phi_+ (X_i/nP_i)] \right) \right) \\
g_i(x) = \exp\left( 2t \left( p_i \phi_- (x/nP_i) - \mathbb{E}[\phi_- (X_i/nP_i)] \right) \right)
\]
are for each \( i \) respectively non-decreasing and non-increasing if \( t \geq 0 \) and respectively non-increasing and non-decreasing if \( t \leq 0 \). Thus, since the exponential function is non-negative, the result follows Lemmas 2.6 and 2.7.

### 2.2 Bounding the binomial

It remains to bound the centered moment generating function of the random variables \( np \cdot \phi_+ (X/nP) \) for \( X \) binomially distributed with \( n \) trials and success probability \( p \), where \( \phi_+ \in \{ \phi_+, \phi_- \} \). Such bounds can be derived using standard results on subgamma random variables (as done in [BP21] following [BLM13, §2.4]), but we do so explicitly here both for completeness and to derive (less-standard) bounds in terms of the moment generating function of the gamma distribution itself for comparison to Conjecture 1.3.

To begin, we use the standard fact that these (non-centered) random variables satisfy strong tail bounds, via e.g. the classical Hoeffding inequality:

**Lemma 2.8.** If \( X \) is binomially distributed with \( n \) trials of success probability \( p \), then the random variables
\[
Z_+ = n D\left( \left( \frac{X}{n}, 1 - \frac{X}{n} \right) \left| (p, 1 - p) \right) \cdot 1_{X \geq np} \\
Z_- = n D\left( \left( \frac{X}{n}, 1 - \frac{X}{n} \right) \left| (p, 1 - p) \right) \cdot 1_{X \leq np}
\]
are both stochastically dominated by the exponential distribution, that is, \( \Pr[Z_i \geq x] \leq \exp(-x) \) for all \( x \geq 0 \) and \( i \in \{ +, - \} \).

**Proof.** By Hoeffding’s inequality [Hoe63] we have that for any real \( k \geq np \) (resp. \( k \leq np \)) it holds that \( \Pr[X \geq k] \) (resp. \( \Pr[X \leq k] \)) is at most \( \exp(-n \cdot f(k)) \) for \( f(k) = D\left( \left( \frac{k}{n}, 1 - \frac{k}{n} \right) \left| (p, 1 - p) \right) \), so since \( f(k) \) is increasing in \( k \) for \( k \geq np \) and decreasing in \( k \) for \( k \leq np \), we get by inverting \( f \) that \( \Pr[n \cdot f(X) \cdot 1_{X \geq np} \leq \varepsilon] \leq \exp(-\varepsilon/n) = \exp(-\varepsilon) \) as desired, and analogously for the other tail.
Corollary 2.9. If $X$ is binomially distributed with $n$ trials of success probability $p$, then $n p \cdot \phi_+(X / np)$ and $n p \cdot \phi_-(X / np)$ are both stochastically dominated by an exponential random variable.

Proof. We have that
\[
\begin{align*}
n D\left(\left(\frac{X}{n}, 1 - \frac{X}{n}\right) \ \bigg| \ \cdot (p, 1 - p) \right) \cdot 1_{X \geq np} &= np \cdot \phi_+\left(\frac{X}{np}\right) + n(1 - p) \cdot \phi_-\left(\frac{n - X}{n(1 - p)}\right) \\
&\geq np \cdot \phi_+\left(\frac{X}{np}\right) \geq 0
\end{align*}
\]
\[
\begin{align*}
n D\left(\left(\frac{X}{n}, 1 - \frac{X}{n}\right) \ \bigg| \ \cdot (p, 1 - p) \right) \cdot 1_{X \leq np} &= np \cdot \phi_-\left(\frac{X}{np}\right) + n(1 - p) \cdot \phi_+\left(\frac{n - X}{n(1 - p)}\right) \\
&\geq np \cdot \phi_-\left(\frac{X}{np}\right) \geq 0
\end{align*}
\]
so that the result follows from Lemma 2.8.

Finally, we show that random variables satisfying such tail bounds have their centered moment generating function bounded by that of a gamma distribution.

Lemma 2.10. Let $Z$ be a non-negative random variable. Then for all $t \in \mathbb{R}$, we have that
\[
\log \mathbb{E}\left[\exp\left(t \left(Z - \mathbb{E}[Z]\right)\right)\right] = \log \left(1 + t \mathbb{E}[Z] + \int_0^\infty t(\exp(tx) - 1) \Pr[Z \geq x] \, dx\right) - t \mathbb{E}[Z].
\]

Proof. Since $Z$ is non-negative, we have that $\mathbb{E}[Z] = \int_0^\infty \Pr[Z \geq x] \, dx$, and by integration by parts (or non-negativity of the exponential) also that
\[
\mathbb{E}[\exp(tZ)] = 1 + \int_0^\infty t \exp(tx) \Pr[Z \geq x] \, dx
\]
for all $t \in \mathbb{R}$.

Proposition 2.11. Let $Z$ be a non-negative random variable stochastically dominated by the exponential distribution, i.e. such that $\Pr[Z \geq x] \leq \exp(-x)$ for all $x \geq 0$. Then for all $t \in (-\infty, 1)$, it holds that
\[
\log \mathbb{E}\left[\exp\left(t \left(Z - \mathbb{E}[Z]\right)\right)\right] \leq B(t)
\]
where
\[
B(t) = \begin{cases} 
\frac{t^2}{1 - t} & t \leq 0 \\
\max\left\{\log\left(1 + \frac{t^2}{1 - t} - \frac{t^2}{5}\right), \log\left(1 + \frac{t^2}{5} + \frac{t^2}{1 - t} - \frac{t^2}{5}\right)\right\} & t \geq 0
\end{cases}
\]
satisfies the upper bounds
\[
B(t) \leq \begin{cases} 
\frac{t^2}{1 - t} & t \leq 0 \\
\log\left(1 + \frac{t^2}{1 - t}\right) & t \geq 0
\end{cases}
\]
for all $t < 1$. 7
Remark 2.12. By optimizing over the set of random variables stochastically dominated by the exponential, one can (with more work) establish an upper bound of the form \( C \log \left( \frac{\exp(-t)}{1-t} \right) \) for an explicit constant \( C < 2 \), but since the result does not hold under the stated assumptions for \( C = 1 \), we do not attempt to optimize this constant beyond the minimal work we do here to give \( C = 2 \).

Proof. Note that \( t(\exp(tx) - 1) \geq 0 \) for all \( x \geq 0 \) and \( t \in \mathbb{R} \), so that \( t(\exp(tx) - 1) \Pr[Z \geq x] \leq t(\exp(tx) - 1) \exp(-x) \), and thus by Lemma 2.10 we have for \( t < 1 \) that

\[
\log \mathbb{E} \left[ \exp \left( t \left( Z - \mathbb{E}[Z] \right) \right) \right] \leq \log \left( 1 + t \mathbb{E}[Z] + \int_0^\infty t(\exp(tx) - 1) \exp(-x) \, dx \right) - t \mathbb{E}[Z]
\]

\[
= \log \left( 1 + t \mathbb{E}[Z] + \frac{t^2}{1-t} \right) - t \mathbb{E}[Z]
\]

(1)

The upper bound for \( t \leq 0 \) follows from the fact that \( \log(1 + x) \leq x \) for all \( x \in \mathbb{R} \). It remains to show the upper bound for \( t \geq 0 \), which we do in two cases based on \( \mathbb{E}[Z] \).

If \( \mathbb{E}[Z] \geq 1/5 \), then since Eq. (1) is decreasing in \( \mathbb{E}[Z] \) (e.g. by elementary calculus), we have that

\[
\log \mathbb{E} \left[ \exp \left( t \left( Z - \mathbb{E}[Z] \right) \right) \right] \leq \log \left( 1 + \frac{t}{5} + \frac{t^2}{1-t} \right) - \frac{t}{5}
\]

as desired. On the other hand, if \( \mathbb{E}[Z] \leq 1/5 \), then by Markov’s inequality we have for all \( x \geq 0 \) that \( \Pr[Z \geq x] \leq (\mathbb{E}[Z])/x \leq 1/(5x) \), which is smaller than \( \exp(-x) \) on an interval containing [3/10, 5/2]. In particular, we can bound

\[
\int_0^\infty t(\exp(tx) - 1) \Pr[Z \geq x] \, dx
\]

\[
\leq \int_0^\infty t(\exp(tx) - 1) \exp(-x) \, dx - \int_{3/10}^{5/2} t(\exp(tx) - 1)(\exp(-x) - 1/(5x)) \, dx,
\]

where since \( \exp(tx) - 1 \geq tx \) we have

\[
\int_{3/10}^{5/2} t(\exp(tx) - 1)(\exp(-x) - 1/(5x)) \, dx \geq \int_{3/10}^{5/2} tx(\exp(-x) - 1/(5x)) \, dx \geq \frac{t^2}{5}.
\]

In particular, we get that

\[
\log \mathbb{E} \left[ \exp \left( t \left( Z - \mathbb{E}[Z] \right) \right) \right] \leq \log \left( 1 + t \mathbb{E}[Z] + \frac{t^2}{1-t} - \frac{t^2}{5} \right) - t \mathbb{E}[Z] \leq \log \left( 1 + \frac{t^2}{1-t} - \frac{t^2}{5} \right)
\]

where the second inequality is because the function is decreasing in \( \mathbb{E}[Z] \).

Finally, we prove the upper bounds on \( B \). The first bound follows from the fact that \( \log \) is an increasing function, \( \log(1 + x) \leq x \) for all \( x \), and that \( \log(C + x) - x \) is a decreasing function of \( x \geq 0 \) for \( C \geq 1 \). For the second bound, elementary calculus shows that

\[
2 \log \left( \frac{\exp(-t)}{1-t} \right) - B(t)
\]

is non-increasing on the non-positive reals and non-decreasing on the non-negative reals, so that since it is 0 at 0 the bound follows. □
2.3 Putting it together

We can now prove the main results as stated in the introduction.

**Theorem 2.13** (Theorem 1.5 restated). For every $n, k,$ and $P,$ we have for all $t < n/2$ that

$$\log E\left[\exp\left(t(V_{n,k,P} - E[V_{n,k,P}])\right)\right] \leq \min\left\{\frac{4kt^2/n^2}{1 - 2t/n}, 2k \log \left(\frac{\exp(-2t/n)}{1 - 2t/n}\right)\right\}$$

**Proof.** Proposition 2.4 and Corollary 2.9 show that the centered moment generating function of $V_{n,k,P}$ at $t \in \mathbb{R}$ is dominated by

$$\sqrt{\prod_{i=1}^{k} \exp\left(\frac{2t}{n} \cdot (Z_i - E[Z_i])\right)} \cdot \sqrt{\prod_{i=1}^{k} \exp\left(\frac{2t}{n} \cdot (Z'_i - E[Z'_i])\right)}$$

where $Z_i$ and $Z'_i$ are non-negative random variables stochastically dominated by the exponential distribution, so the result follows from Proposition 2.11.

**Corollary 2.14** (Corollary 1.7 restated). For all $\varepsilon \geq 0$ we have that

$$\Pr\left[V_{n,k,P} \geq E[V_{n,k,P}] + \varepsilon\right] \leq \left(1 + \frac{2k \varepsilon}{4}\right)^{2k} \cdot \exp\left(-\frac{n\varepsilon}{2}\right) \leq \exp\left(-\frac{3n^2\varepsilon^2}{48k + 8n\varepsilon}\right) \leq \exp\left(-\min\left\{\frac{n^2\varepsilon^2}{24k}, \frac{n\varepsilon}{8}\right\}\right)$$

and for all $0 \leq \varepsilon \leq 2k/n$ we have that

$$\Pr\left[V_{n,k,P} \leq E[V_{n,k,P}] - \varepsilon\right] \leq \exp\left(-k\left[1 - \sqrt{1 - \frac{n\varepsilon}{2k}}\right]^2\right) \leq \exp\left(-\frac{n^2\varepsilon^2}{16k}\right).$$

In particular, Conjecture 1.2 holds with $c_1 = 2$ and $c_2 = 1/48.$

**Proof.** The first inequality in each chain is immediate from Theorem 1.5 by computing the optimal Chernoff bound from (i.e. the convex conjugate of) $2k \log \left(\frac{\exp(-2t/n)}{1 - 2t/n}\right)$ for the upper tail and $\frac{4kt^2/n^2}{1 - 2t/n}$ for the lower tail. The relaxed bounds follow from the elementary inequalities $\log(1 + x) \leq \frac{x + 6}{2x + 3}$ (e.g. [Top07]) for $x \geq 0$ and $1 - \sqrt{1 - x} \geq x/2$ for $x \leq 1.$ The implication for Conjecture 1.2 is because $k \geq 2$ implies $k \leq 2(k - 1).$

**Corollary 2.15** (Corollary 1.9 restated). We have that $\text{Var}(V_{n,k,P}) \leq 8k/n^2,$ and more generally, for all integers $m \geq 1$ we have

$$E\left[\left(V_{n,k,P} - E[V_{n,k,P}]\right)^{2m}\right] \leq \frac{2^{6m}(k^m m! + (2m)!)n^2}{n^2 m},$$

so that in particular for all real $q \geq 1$ we have that

$$\sqrt{E\left[\left|V_{n,k,P} - E[V_{n,k,P}]\right|^q\right]} \leq \frac{24}{n}(\sqrt{kq} + q).$$
Proof. The variance bound follows from the fact that $\text{Var}(Z) = \lim_{t \to 0} \frac{\log \mathbb{E}[\exp(t(Z - \mathbb{E}[Z]))]}{t^2/2}$ for a random variable $Z$ with moment generating function finite around 0, and the general claim for integer $m$ follows from standard results on sub-gamma random variables, e.g. [BLM13, Theorem 2.3] applied to the bound from Theorem 1.5.

The in particular claim follows because $\sqrt[n]{\mathbb{E} \left[ \left| V_{n,k,p} - \mathbb{E} [V_{n,k,p}] \right|^q \right]}$ is a non-decreasing function of $q$ by Jensen’s inequality, so we have that if $2m \leq q + 2$ is the smallest even integer at least $q$, then

\[
\sqrt[n]{\mathbb{E} \left[ \left| V_{n,k,p} - \mathbb{E} [V_{n,k,p}] \right|^q \right]} \leq \frac{8}{n} \sqrt[n]{k^m m! + (2m)!} \\
\leq \frac{8}{n} \left( \sqrt[n]{k^m m^m} + \sqrt[n]{(2m)^{2m}} \right) \leq \frac{24}{n} (\sqrt{kq} + q)
\]

where the last line is because $q \geq 1$ and $2m \leq q + 2$ implies $2m \leq 3q$. \qed

References

[Agr20a] R. Agrawal. “Deriving Indistinguishability from Unpredictability: Tools and Applications in Pseudorandomness.” PhD thesis. Cambridge, Massachusetts, USA: Harvard University, Sept. 10, 2020. 212 pp. (cit. on p. 2).

[Agr20b] R. Agrawal. “Finite-Sample Concentration of the Multinomial in Relative Entropy.” In: IEEE Transactions on Information Theory 66.10 (Oct. 2020), pp. 6297–6302. ISSN: 1557-9654. DOI: 10.1109/TIT.2020.2996134. arXiv: 1904.02291 (cit. on pp. 2, 4).

[AS66] S. M. Ali and S. D. Silvey. “A General Class of Coefficients of Divergence of One Distribution from Another.” In: Journal of the Royal Statistical Society. Series B (Methodological) 28.1 (1966), pp. 131–142. ISSN: 0035-9246. JSTOR: 2984279 (cit. on p. 4).

[BLM13] S. Boucheron, G. Lugosi, and P. Massart. Concentration Inequalities: A Nonasymptotic Theory of Independence. 1st ed. Oxford: Oxford University Press, Feb. 7, 2013. 481 pp. ISBN: 978-0-19-953525-5. DOI: 10.1093/acprof:oso/9780199535255.001.0001 (cit. on pp. 2, 6, 10).

[BP21] A. Bhatt and A. Pensia. Sharp Concentration Inequalities for the Centered Relative Entropy. Sept. 18, 2021. arXiv: 2109.09028 (cit. on pp. 1–4, 6).

[Csi63] I. Csiszár. “Eine informationstheoretische Ungleichung und ihre Anwendung auf den Beweis der Ergodizität von Markoffschen Ketten.” In: A Magyar Tudományos Akadémia. Matematikai Kutató Intézetének Közléményei 8 (1963), pp. 85–108 (cit. on p. 4).

[Csi98] I. Csiszár. “The Method of Types.” In: IEEE Transactions on Information Theory 44.6 (Oct. 1998), pp. 2505–2523. ISSN: 0018-9448. DOI: 10.1109/18.720546 (cit. on p. 2).

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[GR21] F. R. Guo and T. S. Richardson. “Chernoff-Type Concentration of Empirical Probabilities in Relative Entropy.” In: *IEEE Transactions on Information Theory* 67.1 (Jan. 2021), pp. 549–558. ISSN: 1557-9654. DOI: 10.1109/TIT.2020.3034539. arXiv: 2003.08614 (cit. on p. 2).

[Hoe63] W. Hoeffding. “Probability Inequalities for Sums of Bounded Random Variables.” In: *Journal of the American Statistical Association* 58.301 (1963), pp. 13–30. ISSN: 0162-1459. DOI: 10.2307/2282952. JSTOR: 2282952 (cit. on p. 6).

[JP83] K. Joag-Dev and F. Proschan. “Negative Association of Random Variables with Applications.” In: *The Annals of Statistics* 11.1 (Mar. 1983), pp. 286–295. ISSN: 0090-5364, 2168-8966. DOI: 10.1214/aos/1176346079 (cit. on p. 5).

[MJTNW20] J. Mardia, J. Jiao, E. Tánczos, R. D. Nowak, and T. Weissman. “Concentration Inequalities for the Empirical Distribution of Discrete Distributions: Beyond the Method of Types.” In: *Information and Inference: A Journal of the IMA* 9.4 (Dec. 16, 2020), pp. 813–850. ISSN: 2049-8764, 2049-8772. DOI: 10.1093/imaiai/iaz025. arXiv: 1809.06522 (cit. on pp. 1, 2, 4).

[Mor63] T. Morimoto. “Markov Processes and the H-Theorem.” In: *Journal of the Physical Society of Japan* 18.3 (Mar. 15, 1963), pp. 328–331. ISSN: 0031-9015. DOI: 10.1143/JPSJ.18.328 (cit. on p. 4).

[NP33] J. Neyman and E. S. Pearson. “On the Problem of the Most Efficient Tests of Statistical Hypotheses.” In: *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* 231.694-706 (Jan. 1, 1933), pp. 289–337. ISSN: 1364-503X, 1471-2962. DOI: 10.1098/rsta.1933.0009 (cit. on p. 1).

[Pan03] L. Paninski. “Estimation of Entropy and Mutual Information.” In: *Neural Computation* 15.6 (June 1, 2003), pp. 1191–1253. ISSN: 0899-7667. DOI: 10.1162/089976603321780272 (cit. on pp. 2, 4).

[Top07] F. Topsøe. “Some Bounds for the Logarithmic Function.” In: *Inequality Theory and Applications*. Vol. 4. Nova Sci. Publ., New York, 2007, pp. 137–151. ISBN: 978-1-59454-874-1 (cit. on p. 9).

[Wil38] S. S. Wilks. “The Large-Sample Distribution of the Likelihood Ratio for Testing Composite Hypotheses.” In: *The Annals of Mathematical Statistics* 9.1 (Mar. 1938), pp. 60–62. ISSN: 0003-4851, 2168-8990. DOI: 10.1214/aoms/1177732360 (cit. on p. 2).