FINITENESS OF HITTING TIMES UNDER TABOO

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Abstract

We consider a continuous-time Markov chain with a finite or countable state space. For a site $y$ and subset $H$ of the state space, the hitting time of $y$ under taboo $H$ is defined to be infinite if the process trajectory hits $H$ before $y$, and the first hitting time of $y$ otherwise. We investigate the probability that such times are finite. In particular, if the taboo set is finite, an efficient iterative scheme reduces the study to the known case of a singleton taboo. A similar procedure applies in the case of finite complement of the taboo set. The study is motivated by classification of branching processes with finitely many catalysts.

Keywords and phrases: Markov chain, hitting time, taboo probabilities, catalytic branching process.

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1 Introduction

The concept of passage time under taboo for Markov chain has a long history and the first comprehensive exposition of the subject was given in the classical monograph [5]. Introduction of taboo probabilities and hitting times under taboo provided a powerful tool for study of functionals of Markov chains (see, e.g., [5], Ch. 2, Sec. 14), potential theory of Markov chains (see, e.g., [8], Ch. 4, Sec. 6), trajectory properties (see, e.g., [14]), matrix analytic methods in stochastic modeling (see, e.g., [9], Ch. 3, Sec. 5), etc. As far as we know the formula for probability of finiteness of a hitting time was derived only for the cases of empty taboo set (see [2], Ch. 2, Sec. 12) and the taboo set consisting of a single state (see [2]). Now we complete the general picture. The results are formulated as three theorems. The first one gives a representation for the probability of finiteness of a hitting time under taboo via taboo probabilities. The second theorem demonstrates the relations between such probabilities with different initial and target states when the taboo set changes by a single state. The latter result allows to construct a finite iterative scheme to evaluate the probability under consideration when either taboo set or its complement are finite. Theorem 3 covers an important particular case for a singleton taboo set. The proofs involve the Laplace-Stieltjes transform of functions appearing in a system of Chung’s integral equations of convolution type.

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Our interest in hitting times under taboo is motivated by their application to effective classification (see [4]) of branching processes with finitely many catalysts. For a single catalyst, the model was described in [3], although in a more restrictive framework called branching random walk on $\mathbb{Z}^d$, $d \in \mathbb{N}$, it was proposed in [13]. It turns out that in the branching random walk with a single catalyst the number of particles at the point of catalysis, coinciding with the process start point, can be investigated by means of a due Bellman-Harris process with two types of particles (see [11, 12]). However, the study of other local characteristics of the process with arbitrary start point can be performed in terms of a Bellman-Harris process with six types of particles (see [1, 3]). For the employment of such auxiliary processes, analysis of hitting times under taboo for a random walk on $\mathbb{Z}^d$ is indispensable as was shown in [2] and [11]. Note in passing that the results announced in [4] enable us to distinguish three types of asymptotic (as time tends to infinity) behavior of particles population in branching processes with finite set of catalysts producing taboo sets. The type is classified by the value of the Perron root (being less, equal or greater than 1) of a specified matrix with entries explicitly depending on the studied probabilities of finiteness of hitting times under taboo. So, intending to establish properties of branching processes with several catalysts (and the particles movement governed by a Markov chain), it is natural to prepare tools by treatment of hitting times under taboo for a Markov chain.

2 Main Results and Proofs

We consider an irreducible continuous-time Markov chain $\xi = \{\xi(t), t \geq 0\}$ generated by a conservative $Q$-matrix $A = (a(x, y))_{x,y \in S}$ having finite negative diagonal elements. Here $S$ is a finite or denumerable set. For $x \in S$, let $\tau_x := \{\xi(0) = x\} \cap \{t \geq 0 : \xi(t) \neq x\}$ where $\{B\}$ stands for the indicator of a set $B$. The stopping time $\tau_x$ (with respect to the natural filtration of the process $\xi$) is the first exit time from $x$ and $P_x(\tau_x \leq t) = 1 - e^{a(x,x)t}$, $t \geq 0$, (see, e.g., Theorem 5 in [3], Ch. 2, Sec. 5) where $P_x(\cdot) := P(\cdot | \xi(0) = x)$. Following Chung’s notation in [3], Ch. 2, Sec. 11, for an arbitrary, possibly empty set $H \subset S$ (“$\subseteq$” always means “$\subseteq$”) called henceforth the taboo set and for $t \geq 0$, denote by

$$H_{\tau_{xy}}(t) := P_x(\tau_x = y, \xi(u) \notin H, \min\{\tau_x, t\} < u < t), \quad x, y \in S,$$

the transition probability from $x$ to $y$ in time $t$ under taboo $H$. In the case $H = \emptyset$ the function $p_{xy}(\cdot) = \varphi p_{xy}(\cdot)$ is an ordinary transition probability. Note that $H_{\tau_{xy}}(\cdot) \equiv 0$ for $x \in S$, $y \in H$ and $x \neq y$ whereas $H_{\tau_{xy}}(\cdot) = e^{a(x,x)t}$ for $x \in H$ and $H_{\tau_{xy}}(\cdot) \geq e^{a(x,x)t}$ for $x \notin H, t \geq 0$.

Set

$$H_{\tau_{xy}} := \{\xi(0) = x\} \cap \{t \geq \tau_x : \xi(t) = y, \xi(u) \notin H, \tau_x < u < t\}, \quad x, y \in S,$$

where, as usual, we assume that $\inf\{t \in \emptyset\} = \infty$. The stopping time $H_{\tau_{xy}}$ is the first entrance time from $x$ to $y$ under taboo $H$ whenever $x \neq y$ and is the first return time to $x$ under taboo $H$ when $x = y$. Let $H_{F_{xy}}(t) := P_x(H_{\tau_{xy}} \leq t)$ and $F_{xy}(t) := P_x(\varphi_{\tau_{xy}} \leq t), t \geq 0$, be (improper) distribution functions of $H_{\tau_{xy}}$ and $\varphi_{\tau_{xy}}$, respectively. Clearly, $H_{\tau_{xy}}(\cdot) \equiv H_{\{y\}F_{xy}(\cdot)}$ almost surely (a.s.) for $y \in H$ and, consequently, $H_{F_{xy}}(\cdot) \equiv H_{\{y\}F_{xy}(\cdot)}$ for $y \in H$. So, it is sufficient to consider $H_{F_{xy}}(\cdot)$ for $x, y \in S$ and $y \notin H$. 

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According to Theorem 8 in [5], Ch. 2, Sec. 11, the following first entrance formulae are true for \(x, y \in S, z \notin H\) and \(t \geq 0\)

\[
H P_{xy}(t) = z, H P_{xy}(t) + \int_0^t H P_{zy}(t-u) d H F_{xz}(u), \quad (1)
\]

\[
H F_{xy}(t) = z, H F_{xy}(t) + \int_0^t H F_{zy}(t-u) d y, H F_{xz}(u), \quad z \neq y, \quad (2)
\]

where we write \(z, H P_{xy}(t)\) instead of \(z \cup H P_{xy}(t)\) and similarly for other symbols. Prior to applying the Laplace transform to functions from (1) and (2), set

\[
\hat{p}(\lambda) := \int_0^\infty e^{-\lambda t} p(t) dt, \quad \hat{F}(\lambda) := \int_0^\infty e^{-\lambda t} d F(t), \quad P(t) := \int_0^t p(u) du, \quad \lambda, t > 0,
\]

for any taboo probability \(p\) and distribution function \(F\).

Recall (see, e.g., [5], Ch. 2, Sec. 10) that the irreducible Markov chain \(\xi\) is recurrent (i.e. \(P_x(\{t \geq \tau_x : \xi(t) = y\}\) is unbounded) = 1 for any \(x, y \in S\) iff \(P_{xy}(\infty) = \infty\) where \(P_{xy}(\infty) := \lim_{t \to \infty} P_{xy}(t)\). In a similar way, \(\xi\) is transient (i.e. for each \(x, y \in S\) one has \(\mathbb{P}(\{t \geq \tau_x : \xi(t) = y\}\) is unbounded) = 0) iff \(P_{xy}(\infty) < \infty\). We stress that \(\xi\) is either recurrent or transient (see, e.g., Theorem 4 and Corollary 2 in [5], Ch. 2, Sec. 10). For the properties of the so-called Green function \(G(x,y) := P_{xy}(\infty), x, y \in S\), see, e.g., [10], Ch. 4, Sec. 1 and 2. In accordance with Theorems 1, 3 and relation (5) in [5], Ch. 2, Sec. 12, identity (1) implies that

\[
F_{xy}(\infty) = 1 \quad \text{if} \quad \xi\text{ is recurrent}, \quad (3)
\]

\[
F_{xy}(\infty) = \frac{G(x,y)}{G(y,y)} \in (0,1) \quad \text{if} \quad \xi\text{ is transient and } x \neq y, \quad (4)
\]

\[
F_{xx}(\infty) = 1 + \frac{1}{a(x,x)G(x,x)} \in (0,1) \quad \text{if} \quad \xi\text{ is transient}. \quad (5)
\]

Our aim is to find the value \(H F_{xy}(\infty)\) being the probability of finiteness of \(H \tau_{xy}\) conditioned on \(\{\xi(0) = x\}\). As was already noted it is sufficient to consider \(x, y \in S\) and \(y \notin H\). The following statement shows that \(H F_{xy}(\infty)\) can be expressed in terms of \(H P_{xy}(\infty)\) and \(H P_{yy}(\infty)\) similarly to (4) and (5).

**Theorem 1** For any nonempty taboo set \(H\) and \(x, y \in S, y \notin H\), one has

\[
H F_{xy}(\infty) = \frac{H P_{xy}(\infty)}{H P_{yy}(\infty)}, \quad x \neq y, \quad (6)
\]

\[
H F_{xx}(\infty) = 1 + \frac{1}{a(x,x)H P_{xx}(\infty)} \in (0,1), \quad x \notin H, \quad (7)
\]

where \(0 \leq H P_{xy}(\infty) < \infty\) and \(0 < H P_{yy}(\infty) < \infty\).
Due to (2) we have the following system of two linear integral equations in functions $\hat{F}_{xy}(\lambda)$ and $\hat{F}_{xz}(\lambda)$

\begin{align*}
\hat{F}_{xy}(\lambda) &= \frac{\hat{P}_{xy}(\lambda) - y, H \hat{P}_{xy}(\lambda)}{H \hat{P}_{yy}(\lambda)}, \quad \lambda > 0.
\end{align*}

Since $y, H p_{xy}(\cdot) \equiv 0$ for $x \neq y$, relation (8) implies (6) due to identity $F(\infty) = \lim_{\lambda \to 0^+} \hat{F}(\lambda)$ for a distribution function $F$ having support in $[0, \infty)$. The identity holds by the monotone convergence theorem applied to the Lebesgue integral representing $\hat{F}(\lambda)$. Equality (8) also entails (7), since $x, H p_{xx}(t) = e^{a(x,x)t}$, $t \geq 0$, and $x, H \hat{P}_{xx}(\lambda) = (\lambda - a(x,x))^{-1}$, $\lambda \geq 0$. Theorem 1 is proved. □

In the next theorem the value $H F_{xy}(\infty)$ is expressed in terms of $H' F_{x'y'}(\infty)$ with appropriate choice of a collection of states $x'$, $y' \in S$ and a certain set $H'$ such that $H' \subset H$ or $H \subset H'$. Thus, for a finite nonempty set $H$, the evaluation of $H F_{xy}(\infty)$ can be reduced to the case when $H$ consists of a single state. The same procedure can be performed when $S \setminus H$ is a finite set. Below we use the Kronecker delta $\delta_{xy}$ for $x, y \in S$.

**Theorem 2** If $H$ is a nonempty subset of $S$ and $x, y, z \in S$, $y, z \notin H$, $z \neq y$, then

\begin{align*}
z, H F_{xy}(\infty) &= \frac{H F_{xy}(\infty) - H F_{zx}(\infty) H F_{zy}(\infty)}{1 - H F_{yz}(\infty) H F_{zy}(\infty)}, \quad H F_{xy}(\infty) < 1.
\end{align*}

where $H F_{yz}(\infty) H F_{zy}(\infty) < 1$. If $H$ is any subset of $S$ and $x, y \in S$, $x \notin H$, $x \neq y$, then

\begin{align*}
H F_{xy}(\infty) &= \frac{x, H F_{xy}(\infty)}{1 - y, H F_{xx}(\infty)}.
\end{align*}

Moreover, for any $H \subset S$ and $x, y \in S$ one has

\begin{align*}
H F_{xy}(\infty) &= (\delta_{xy} - 1) \frac{a(x,y)}{a(x,x)} - \sum_{z \in S, z \neq x, z \neq y, z \notin H} \frac{a(x,z)}{a(x,x)} H F_{zy}(\infty).
\end{align*}

**Proof.** Due to (2) we have the following system of two linear integral equations in functions $z, H F_{xy}(\cdot)$ and $y, H F_{xz}(\cdot)$

\begin{align*}
\begin{cases}
H F_{xy}(t) &= z, H F_{xy}(t) + \int_{0}^{t} H F_{zy}(t - u) \, dy, H F_{xz}(u),
H F_{xz}(t) &= y, H F_{xz}(t) + \int_{0}^{t} H F_{yz}(t - u) \, dz, H F_{zy}(u).
\end{cases}
\end{align*}

Applying the Laplace-Stieltjes transform and using its convolution property (see, e.g., [2], Ch. 13, Sec. 2, property (i)) we get a new system of equations in $z, H \hat{F}_{xy}(\lambda)$ and $y, H \hat{F}_{xz}(\lambda)$

\begin{align*}
\begin{cases}
\hat{F}_{xy}(\lambda) &= z, H \hat{F}_{xy}(\lambda) + y, H \hat{F}_{xz}(\lambda) \hat{F}_{zy}(\lambda),
\hat{F}_{xz}(\lambda) &= y, H \hat{F}_{xz}(\lambda) + z, H \hat{F}_{zy}(\lambda) \hat{F}_{yz}(\lambda).
\end{cases}
\end{align*}
Solving this system we obtain
\[ z, H \tilde{F}_{xy}(\lambda) = \frac{H \tilde{F}_{xy}(\lambda) - H \tilde{F}_{xz}(\lambda) H \tilde{F}_{zy}(\lambda)}{1 - H \tilde{F}_{yz}(\lambda) H \tilde{F}_{zy}(\lambda)}. \]
Letting \( \lambda \to 0+ \) in the latter relation we come to \( (9) \). The inequality \( H \tilde{F}_{yz}(\infty) H \tilde{F}_{zy}(\infty) < 1 \) holds true, since \( H \tilde{F}_{yz}(\infty) H \tilde{F}_{zy}(\infty) \leq H \tilde{F}_{yy}(\infty) \) in view of \( (2) \) and \( H \tilde{F}_{yy}(\infty) < 1 \) by virtue of \( (7) \). We come to relation \( (10) \) applying the Laplace-Stieltjes transform to \( (2) \) when \( z = x \) and letting \( \lambda \to 0+ \). The claim \( (11) \) ensues from the identity
\[
H \tilde{F}_{xy}(t) = (\delta_{xy} - 1) (1 - e^{a(x,x)t}) \frac{a(x,y)}{a(x,x)} - \sum_{z \in S, z \neq x, z \neq y, z \notin H} \frac{a(x,z)}{a(x,x)} \int_{0}^{t} (1 - e^{a(x,x)(t-u)}) dH \tilde{F}_{zy}(u)
\]
that is valid due to the strong Markov property of \( \xi \) (see, e.g., Theorem 3 in [5], Ch. 2, Sec. 9) involving the stopping time \( \tau_x \). The proof is complete. \( \square \)

The following result can be viewed as the complement to relation \( (9) \) when \( H \) is an empty set and \( \xi \) is a transient Markov chain. The last hypothesis permits us to obtain the formula involving the Green functions.

**Theorem 3** Let \( \xi \) be a transient Markov chain and \( x, y, z \in S \). Then \( z \tilde{F}_{xy}(\infty) \in [0, 1) \) and
\[
\begin{align*}
    z \tilde{F}_{xy}(\infty) & = \frac{G(x,y)G(z,z) - G(x,z)G(z,y)}{G(z,z)G(y,y) - G(y,z)G(z,y)}, \quad x \neq y, \ x \neq z, \ y \neq z, \quad (12) \\
    z \tilde{F}_{yy}(\infty) & = 1 + \frac{G(z,z)}{a(y,y) (G(y,y)G(z,z) - G(y,z)G(z,y))}, \quad y \neq z, \quad (13) \\
    z \tilde{F}_{zy}(\infty) & = -\frac{G(z,y)}{a(z,z) (G(y,y)G(z,z) - G(y,z)G(z,y))}, \quad y \neq z. \quad (14)
\end{align*}
\]

**Proof.** Examining the proof of Theorem \( 2 \) we see that, for transient \( \xi \), formula \( (9) \) is true even for \( H = \emptyset \) as \( F_{yz}(\infty) F_{zy}(\infty) < 1 \) on account of \( (1) \). Thus, substituting \( (1) \) in \( (2) \) we derive \( (12) \). According to \( (2) \) and \( (1) \) one has \( z \tilde{F}_{xy}(\infty) \leq F_{xy}(\infty) < 1 \). In a similar way we obtain \( (13) \) and \( (14) \) by employing \( (1) \) and \( (5) \). Theorem \( 3 \) is established. \( \square \)

For recurrent \( \xi \), formula \( (9) \) fails for \( H = \emptyset \), since \( F_{yz}(\infty) F_{zy}(\infty) = 1 \) in view of \( (3) \).
So, in general, there is no counterpart of the previous theorem differing from assertion of Theorem \( 1 \) for a singleton taboo. However for symmetric, space-homogeneous random walk on \( \mathbb{Z} \) or \( \mathbb{Z}^2 \) having finite variance of jump sizes (such random walk is transient on \( \mathbb{Z}^d \) with \( d \geq 3 \)) it is possible to provide representation for \( z \tilde{F}_{xy}(\infty) \) alternative to those in Theorem \( 1 \). This follows from Theorems 1 and 2 in [2].

To conclude we return to the general case of Markov chains. In contrast to \( H \tau_{xy} \) define the hitting time of state \( y \) under taboo \( H \) after the first exit out of the starting state \( x \) as
\[
H \bar{\tau}_{xy} := \mathbb{I}\{\xi(0) = x\} \inf\{t \geq 0 : \xi(t + \tau_x) = y, \ \xi(u) \notin H, \ \tau_x < u < t + \tau_x\}.
\]
Such random variables arise naturally in study of catalytic branching processes. Evidently, 
\[ H\tau_{xy} = \tau_x + H\tau_{xy} \] and \( \mathbb{P}_x(H\tau_{xy} = 0) = (\delta_{xy} - 1)a(x, y)a(x, x)^{-1} \). Moreover, by virtue of the
strong Markov property of \( \xi \) random variables \( \tau_x \) and \( H\tau_{xy} \) are independent. Therefore, taking into account the convolution formula and the expression for \( \mathbb{P}_x(\tau_x \leq t) \) we get

\[
H_F^{xy}(t) = \int_{0^-}^{t} (1 - e^{a(x,x)(t-u)}) \, dH_F^{xy}(u) \quad (15)
\]

where \( H_F^{xy}(t) := \mathbb{P}_x(H\tau_{xy} \leq t), \ t \geq 0 \). Hence, \( H_F^{xy}(\infty) = H_F^{xy}(\infty) \) for any \( x, y \in S \), \( H \subset S \), and the assertions of Theorems 1 – 3 hold true if \( H_F^{xy}(\infty) \) is replaced by \( H_F^{xy}(\infty) \).

Note also that on account of (15) the distribution function \( H_F^{xy}() \) has a bounded density. Consequently, the function \( H_F^{xy}() \) has also a density (which is not bounded in general) in view of the following equality

\[
H_F^{xy}(\infty) - H_F^{xy}(t) = \sum_{z \in S, z \notin H, z \neq x, z \neq y} \frac{a(x, z)}{-a(x, x)}(H_F^{zy}(\infty) - H_F^{zy}(t))
\]

derived similarly to Lemma 3 in [2]. Thus the results established for \( H\tau_{xy} \) are valid for \( H\tau_{xy} \) as well.

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