SOLVING OPTIMAL CONTROL
PROBLEM USING HERMITE WAVELET

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Abstract. In this paper, we derive the operational matrices of integration, derivative and production of Hermite wavelets and use a direct numerical method based on Hermite wavelet, for solving optimal control problems. The properties of Hermite polynomials are used for finding these matrices. First, we approximate the state and control variables by Hermite wavelets basis; then, the operational matrices is used to transfer the given problem into a linear system of algebraic equations. In fact, operational matrices of Hermite wavelet are employed to achieve a linear algebraic equation, in place of the dynamical system in terms of the unknown coefficients. The solution of this system gives us the solution of the original problem. Numerical examples with time varying and time invariant coefficient are given to demonstrate the applicability of these matrices.

1. Introduction. The optimization theory is divided into four major parts, including mathematical programming, optimal control, game theory and differential game [4]. A branch of optimization theory is control theory that deals with extremizing a specified cost functional and at the same time satisfying some constraints. There are two general methods for solving optimal control problems. These methods are direct and indirect methods. Indirect methods (Pontryagins maximum principle, Bellmans dynamic programming) are based on converting the original optimal control problem into a two-point boundary value problem. In direct methods, the optimal solution is obtained by direct extremization of the performance index subject to constraints. Direct methods can be applied using discretization or parametrization methods. A review on control parametrization for constrained optimal control problems is given in [23]. In recent years, different direct methods based on orthogonal polynomials have been used to solve dynamic systems.

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Wavelets as a type of orthogonal polynomial has good property for approximating functions with discontinuities or sharp changes.

Many papers concerned with the application of orthogonal functions. Hwang et al. used Laguerre series for solving variational problems [14]. Special attention has been given to the applications of block pulse functions such as [15], Haddadi et al. used a hybrid of block pulse functions and Bernoulli polynomial for solving the optimal control of linear time-varying delay systems [11] and there are many similar papers which dedicated to solving variational problems via hybrid of block pulse functions and special polynomial as Legendre [21], Bernstein [5], Fourier series [8] and so on. Chen and Hsiao used Haar wavelet orthogonal functions and their integration matrices to optimize dynamic systems [6]. Several researchers are applying Legendre wavelet (see [22], [25], [26], [27]). Babolian and Fattahzadeh obtained numerical solution of differential equations using operational matrix of integration of Chebyshev wavelets basis [3]. Ghasemi and Tavassoli Kajani presented a solution of time-varying delay systems by Chebyshev wavelets [9].

The main characteristic of this technique is that it reduces the problem to solving a system of algebraic equations, thus to a great extent simplifying the problem. The approach is based on approximating various signals involved in equations by truncated orthogonal series and converting the underlying differential equations into an integral equations. Using operational matrix of integration and differentiation, eliminate the integral or derivative operation respectively, whenever needed.

This paper is organized into the following sections of which this introduction is the first. In Section 2, we introduce Hermite wavelet. Section 3 is about operational matrices of Hermite wavelet. In section 4 we present 5 numerical examples to illustrate the efficiency of introduced matrices in solving variational and optimal control problems. Finally, the paper is concluded with conclusion.

2. Hermite Wavelet. In this section we briefly describe Hermite wavelet that used in the next section. Wavelets constitute a family of functions constructed from dilation and translation of a single function \( \psi(t) \) called the mother wavelet. When the dilation parameter \( a \) and the translation parameter \( b \) vary continuously we have the following family of continuous wavelet as

\[
\psi(a, b) = |a|^{-\frac{1}{2}} \psi\left(\frac{t - b}{a}\right), \quad a, b \in \mathbb{R}, \quad a \neq 0, \quad (1)
\]

If we restrict the parameters \( a \) and \( b \) to discrete values as \( a = a_0^{-k}, \ b = nb_0a_0^{-k}, \) \( a_0 > 1, \ b_0 > 0 \) we have the following family of discrete wavelets

\[
\psi_{k,n}(t) = |a_0|^{-\frac{k}{2}} \psi(a_0^k t - nb_0), \quad k, n \in \mathbb{R}, \quad (2)
\]

where form a wavelet basis for \( L^2(\mathbb{R}) \). In particular, when \( a_0 = 2 \) and \( b_0 = 1 \) then \( \psi_{k,n}(t) \) form an orthonormal basis [10]. The Hermite wavelets \( \psi_{n,m}(t) = \psi(k, n, m, t) \) involve four arguments \( n = 1, 2, \cdots, 2^{k-1}, k \) is any positive integer, \( m \) is the degree of Hermite polynomials and \( t \) is the normalized time. They are defined on the interval \([0, 1]\) by [2]:

\[
\psi_{n,m}(t) = \begin{cases} 2^{\frac{m}{2}} \sqrt{\frac{1}{2}} H_m(2^k t - 2n + 1) & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}, \\ 0 & \text{otherwise}, \end{cases} \quad (3)
\]

where \( m = 0, 1, 2, \cdots, M-1 \). Here \( H_m \) is the second Hermite polynomials of degree \( m \) with respect to the weight function \( w(t) = \sqrt{1 - t^2} \) in the interval \((-\infty, \infty)\)
and satisfy the following recursive formula
\[ H_0(t) = 1, \quad H_1(t) = 2t, \]
\[ H_{m+2}(t) = 2tH_{m+1}(t) - 2(m+1)H_m(t), \quad m = 0, 1, 2, 3, \ldots \]  
(4)

Function \( f(t) \in L^2(\mathbb{R}) \) can be expanded as
\[ f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} f_{n,m} \psi_{n,m}(t), \]
(5)

where \( f_{n,m} = \langle f, \psi_{n,m} \rangle \) where \( \langle \ldots \rangle \) denotes the inner product. If the infinite series in (5) are truncated, then it can be written as
\[ f(t) \approx \sum_{n=1}^{M-1} \sum_{m=0}^{M-1} f_{n,m} \psi_{n,m}(t) = F^T \Psi(t), \]
(6)

where \( F \) and \( \Psi(t) \) are \( M2^{k-1} \times 1 \) matrices as below:
\[
F = \begin{bmatrix}
    f_{1,0}, f_{1,1}, \cdots, f_{1,M-1}, f_{2,0}, f_{2,1}, \cdots, f_{2,M-1}, \cdots, \\
    f_{2k-1,0}, f_{2k-1,1}, \cdots, f_{2k-1,M-1}
\end{bmatrix}^T,
\]
\[
\Psi = \begin{bmatrix}
    \psi_{1,0}, \psi_{1,1}, \cdots, \psi_{1,M-1}, \psi_{2,0}, \psi_{2,1}, \cdots, \psi_{2,M-1}, \cdots, \\
    \psi_{2k-1,0}, \psi_{2k-1,1}, \cdots, \psi_{2k-1,M-1}
\end{bmatrix}^T.
\]
(7)

3. Operational Matrices of Hermite Wavelets. In this section we introduce three new Hermite wavelets operational matrices which will be used to solve the examples of this paper.

3.1. Hermite Wavelets Operational Matrix of Derivative. The following property of the Hermite wavelet will also be used. Let
\[
\frac{d\Psi(t)}{dt} \simeq D\Psi(t),
\]
(8)

where \( D \) is the \( M2^{k-1} \times M2^{k-1} \) operational matrix of derivative which is extracted as follows. For finding the mentioned matrix we use the undergoing relation
\[
\dot{H}_{m+1}(t) = 2(m+1)H_m(t) \Rightarrow \dot{\psi}_{n,m+1}(t) = 2^{k+1}(m+1)\psi_{n,m}(t),
\]
(9)

thus
\[
D = \begin{bmatrix}
    W & 0 & \cdots & 0 \\
    0 & W & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & W
\end{bmatrix}_{2^{k-1} \times 2^{k-1}},
\]
(10)

where \( W \) is as follows
\[
W = 2^{k+1} \begin{bmatrix}
    0 & 0 & 0 & \cdots & 0 & 0 \\
    1 & 0 & 0 & \cdots & 0 & 0 \\
    0 & 2 & 0 & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & M-1 & 0
\end{bmatrix}_{M \times M},
\]
(11)
3.2. Hermite Wavelets Operational Matrix of Integration. Now, we extract the operational matrix of integration $P$ for Hermite wavelet by the aim of the derivative of Hermite wavelet. The operational matrix of integration is defined as follows

$$\int_0^t \Psi(\tau)d\tau \simeq P\Psi(t),$$  \hspace{1cm} (12)$$

where $P$ is a $M2^{k-1} \times M2^{k-1}$ matrix given by

$$P =\begin{bmatrix} S & F & F & \cdots & F \\ 0 & S & F & \cdots & F \\ 0 & 0 & S & \cdots & F \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & S \end{bmatrix},$$  \hspace{1cm} (13)$$

where $S$ and $F$ are $M \times M$ matrices given as below

$$S = \begin{bmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 \\ a_2 & 0 & b_2 & 0 & \cdots & 0 \\ a_3 & 0 & 0 & b_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{M-1} & 0 & 0 & 0 & \cdots & b_{M-1} \\ a_M & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad F = \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ c_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{M-1} & \vdots & \vdots & 0 \\ c_M & 0 & \cdots & 0 \end{bmatrix},$$  \hspace{1cm} (14)$$

where

$$a_m = -\frac{H_m(-1)}{2^{k+1}m}, \quad b_m = \frac{1}{2^{k+1}m}, \quad c_m = \frac{H_m(1) - H_m(-1)}{2^{k+1}m}. \hspace{1cm} (15)$$

The procedure of finding $S$ and $F$ is as follows

$$\dot{\psi}_{n,m+1}(t) = 2^{k+1}(m+1)\psi_{n,m}(t)$$  

$$\Rightarrow \int_0^t \dot{\psi}_{n,m+1}(\tau)d\tau = 2^{k+1}(m+1) \int_0^t \psi_{n,m}(\tau)d\tau$$  

$$\Rightarrow \int_0^t \dot{\psi}_{n,m}(\tau)d\tau = \frac{1}{2^{k+1}(m+1)} \int_0^t \dot{\psi}_{n,m+1}(\tau)d\tau. \hspace{1cm} (16)$$

According to the definition of $\dot{\psi}_{n,m}$ we consider three cases for $t$
1. $t < \frac{n-1}{2^k-1} \Rightarrow \int_0^t \psi_{n,m}(\tau)d\tau = 0$

2. $\frac{n-1}{2^k-1} \leq t < \frac{n}{2^k-1} \Rightarrow \int_0^t \psi_{n,m}(\tau)d\tau$
   \[= \frac{1}{2^{k+1}(m+1)} \int_{\frac{n-1}{2^k-1}}^{\frac{n}{2^k-1}} \frac{\psi_{n,m+1}(\tau)}{\pi^2} d\tau \]
   \[= \frac{1}{2^{k+1}(m+1)} \psi_{n,m+1}(t) - \frac{H_{m+1}(-1)}{2^{k+1}(m+1)} \psi_{n,0}(t) \]
   \[= b_{m+1}\psi_{n,m+1}(t) + a_{m+1}\psi_{n,0}(t), \]

3. $t \geq \frac{n}{2^k-1} \Rightarrow \int_0^t \psi_{n,m}(\tau)d\tau$
   \[= \frac{1}{2^{k+1}(m+1)} \int_{\frac{n}{2^k-1}}^{\frac{n}{2^k-1}} \frac{\psi_{n,m+1}(\tau)}{\pi^2} d\tau \]
   \[= (H_{m+1}(1) - H_{m+1}(-1))\psi_{n,0}(t) = c_{m+1}\psi_{n,0}(t), \]

$n'$ depend on $t$.

3.3. Hermite wavelets operational matrix of production. One of the operators that plays an important role in modelling of equations is operational matrix of production ($\tilde{F}$) which is defined as follows

$$\Psi(t)\Psi^T(t)F \simeq \tilde{F}\Psi(t), \quad (17)$$

Here $F$ is a given $M2^{k-1} \times 1$ vector and $\tilde{F}$ is a $M2^{k-1} \times M2^{k-1}$ matrix. For finding $\tilde{F}$ we use the following property of Hermite polynomials.

$$H_m(t)H_q(t) = \sum_{r=0}^{\min(m,q)} r!2^r \left( \begin{array}{c} m \\ r \end{array} \right) \left( \begin{array}{c} q \\ r \end{array} \right) H_{m+q-2r}(t),$$

$$\psi_{n,m} \cdot \psi_{k,q}(t) = \begin{cases} 0, & n \neq k, \\ 2^k \sqrt{\frac{2}{\pi}} \sum_{r=0}^{\min(m,q)} r!2^r \left( \begin{array}{c} m \\ r \end{array} \right) \left( \begin{array}{c} q \\ r \end{array} \right) \psi_{m+q-2r}(t), & n = k. \end{cases} \quad (18)$$

To illustrate the calculation procedure of $\tilde{F}$ we choose $M = 3$ and $k = 1$, thus we have

$$F^T = [f_{10} f_{11} f_{12}],$$

$$\Psi\Psi^T = \begin{pmatrix} \psi_{10}\psi_{10} & \psi_{10}\psi_{11} & \psi_{10}\psi_{12} \\ \psi_{11}\psi_{10} & \psi_{11}\psi_{11} & \psi_{11}\psi_{12} \\ \psi_{12}\psi_{10} & \psi_{12}\psi_{11} & \psi_{12}\psi_{12} \end{pmatrix} = 2^k \sqrt{\frac{2}{\pi}} \begin{pmatrix} \psi_{10} & \psi_{11} & \psi_{12} \\ \psi_{11} & 2\psi_{10} + \psi_{12} & 4\psi_{11} \\ \psi_{12} & 4\psi_{11} & 8\psi_{10} + 8\psi_{12} \end{pmatrix},$$

$$\Rightarrow \tilde{F} = 2^k \sqrt{\frac{2}{\pi}} \begin{pmatrix} f_{10} & f_{11} & f_{12} \\ 2f_{11} & f_{10} + 4f_{12} & f_{11} \\ 8f_{12} & 4f_{11} & f_{10} + 8f_{12} \end{pmatrix}. \quad (19)$$
In general case, $\tilde{F}$ is a $M 2^{k-1} \times M 2^{k-1}$ block diagonal matrix and has the following form

$$\tilde{F} = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{2^{k-1}} \end{pmatrix},$$ (20)

where the method of obtaining each $B_i$, $i = 1, 2, \cdots, 2^{k-1}$, similar to the one described above.

4. **Illustrative Examples.** In this section, we solve some linear and nonlinear optimal control problem via operational matrices which introduced in the previous section. The method which is used in each example is described briefly in it.

**Example 4.1.** Let us consider the undergoing optimal control problem [24]

$$J = \frac{1}{2} \int_0^1 (x^2(t) + u^2(t)) dt,$$ (21)

subject to the system

$$\dot{x}(t) = -x(t) + u(t),$$
$$x(0) = 1.$$ (22)

The exact solution is given as

$$x(t) = \cosh(\sqrt{2}t) + \beta \sinh(\sqrt{2}t),$$
$$u(t) = (1 + \beta \sqrt{2}) \cosh(\sqrt{2}t) + (\sqrt{2} + \beta) \sinh(\sqrt{2}t),$$
$$\beta = -\frac{\cosh \sqrt{2} + \frac{1}{\sqrt{2}} \sinh \sqrt{2}}{\sqrt{2} \cosh \sqrt{2} + \sinh \sqrt{2}}.$$

For solving the above problem, assume the state and control variables $x(t)$ and $u(t)$ can be expressed as a truncated sum of Hermite wavelet.

$$x(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}^1 \psi_{n,m}(t) = C_1^T \Psi(t),$$ (23)
$$u(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}^2 \psi_{n,m}(t) = C_2^T \Psi(t),$$ (24)

by integrating the two sides of Eq.(22) we will have

$$x(t) - x(0) = \int_0^t \dot{x}(t) dt,$$
$$C_1^T \Psi(t) - d_0^T \Psi(t) = -C_1^T P \Psi(t) + C_2^T P \Psi(t) \Rightarrow C_1^T + C_1^T P - C_2^T P = d_0^T,$$ (25)

where

$$d_0^T = \begin{bmatrix} \frac{1}{2^{2^{k-1}}} \sqrt{\frac{2}{\pi}}, 0, \cdots, 0 \mid \frac{1}{2^{2^{k-1}}} \sqrt{\frac{2}{\pi}}, 0, \cdots, 0 \mid \cdots \mid \frac{1}{2^{2^{k-1}}} \sqrt{\frac{2}{\pi}}, 0, \cdots, 0 \end{bmatrix}.$$ (26)
Substituting (23) and (24) into (21) gives
\[ J \simeq C_1^T \Lambda C_1 + C_2^T \Lambda C_2, \] (27)
where
\[ \Lambda = \int_0^1 \Psi(t)\Psi^T(t)dt. \] (28)

The optimal control problem becomes to minimization \( J \) given in Eq.(27), subject to the constraint of Eq.(25). The curves of exact and approximate solution of \( x(t) \) and \( u(t) \) are shown in Fig.1 and Fig.2, respectively. Also a comparison of the optimal costs in some papers is made in Table 1. In Table 2 we present the result obtained and compared it with the exact solution for \( x(t) \) and \( u(t) \).

### Table 1. Comparison of the optimal values of \( J \) (Example 4.1)

|                | Kafash et al. [17] | Saberi Nik et al. [24] | Approximated solution via HW |
|----------------|---------------------|------------------------|-----------------------------|
| Exact value of \( J \) | 0.1926992981       | 0.192914197            | 0.1929092981                 |

### Table 2. The exact and approximated values of \( x(t) \) and \( u(t) \) for Example 4.1

| Time | Approximated solution via HW | Exact solution | Approximated solution via HW | Exact solution |
|------|-------------------------------|----------------|-------------------------------|----------------|
| 0.0  | 1.0000                        | 1.0000         | -0.3859                       | -0.3858        |
| 0.2  | 0.7584                        | 0.7594         | -0.2769                       | -0.2769        |
| 0.4  | 0.5799                        | 0.5799         | -0.1902                       | -0.1902        |
| 0.6  | 0.4472                        | 0.4472         | -0.1189                       | -0.1189        |
| 0.8  | 0.3505                        | 0.3505         | -0.0571                       | -0.0571        |
| 1    | 0.2820                        | 0.2820         | 0.0000                        | 0.0000         |

### Figure 1. Approximate (linestyle is -) and exact (linestyle is :) solution for \( x(t) \)

**Example 4.2.** The second example concerns with the minimization of the following optimal control problem [19]

\[ J = \frac{1}{2} \int_0^1 (3x^2(t) + u^2(t))dt, \] (29)
\[ \dot{x}(t) = -x(t) + u(t), \] (30)
\[ x(0) = 0, \quad x(1) = 2. \] (31)

The exact solution is given as
\[ x(t) = \frac{2}{\sinh 2} \sinh(2t), \]
\[ u(t) = \frac{2}{\sinh 2}(2 \cosh 2t + \sinh 2t). \]

For solving this problem, as example1 we substitute Eqs.(23) and (24) in the performance index (29), thus we have
\[ J \approx \frac{1}{2}(3C_1^T \Lambda C_1 + C_2^T \Lambda C_2), \] (32)
with the same method as what we do in example1 Eq.(30) converts to the following linear system of equations
\[ -C_1 - P^T C_1 + P^T C_2 = 0, \] (33)
for second boundary condition we have
\[ x(1) = 2 \Rightarrow C_1^T \Psi(1) = 2. \]

Now the optimal control problem is approximated by a quadratic programming problem. The optimal value of vectors \( C_1 \) and \( C_2 \) can be obtained from a standard quadratic programming method. The exact and approximate curves of \( x(t) \) and \( u(t) \) are shown in Fig.3 and Fig.4, respectively. Comparison of the optimal value of performance index is given in Table 3. In Table 4 we present the result obtained and compared it with the exact solution for \( x(t) \) and \( u(t) \).

**Table 3.** Comparison of the optimal values of \( J \) (Example 4.2)

|                   | Exact solution [19] | Hashemi Mehne and Hashemi Borzabadi[12] | Approximated solution via HW |
|-------------------|---------------------|----------------------------------------|----------------------------|
|                   | 6.1586              | 6.1748                                 | 6.1495                     |

**Table 4.** The exact and approximated values of \( x(t) \) and \( u(t) \) for Example 4.2

| Time t | Approximated solution | Exact solution | Approximated solution | Exact solution |
|--------|-----------------------|----------------|-----------------------|----------------|
|        | via HW                | via HW         | via HW                | via HW         |
| 0.0    | 0.0000                | 0.0000         | 1.0025                | 1.0034         |
| 0.2    | 0.2264                | 0.2265         | 1.4185                | 1.4188         |
| 0.4    | 0.4896                | 04897          | 1.9646                | 1.9648         |
| 0.6    | 0.8321                | 0.8324         | 2.8293                | 2.8293         |
| 0.8    | 1.3097                | 1.3100         | 4.1515                | 4.1526         |
| 1      | 2.0000                | 2.0000         | 6.1300                | 6.1493         |
Example 4.3. We want to find an optimal controller $u(t)$ that minimizes the following performance index ([1], [16], [18])

$$J = \int_0^1 [x_1^2(t) + x_2^2(t) + 0.005u^2(t)]dt,$$

subject to

$$\begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= -x_2(t) + u(t), \\
x_1(0) &= 0, \\
x_2(0) &= -1.
\end{align*}$$

For solving this example we use the operational matrix of derivative as follows

$$\begin{align*}
x_1(t) &= C_1^T \Psi(t) \Rightarrow \dot{x}_1(t) = C_1^T D \Psi(t), \\
x_2(t) &= C_2^T \Psi(t) \Rightarrow \dot{x}_2(t) = C_2^T D \Psi(t), \\
u(t) &= C_3^T \Psi(t).
\end{align*}$$

We substitute (35), (36) and (37) in the dynamic system of (34). Similar to the two preceding examples we have

$$J \simeq C_1^T AC_1 + C_2^T AC_2 + 0.005C_3^T AC_3,$$

In addition according to initial values we have

$$x_1(0) = 0 \Rightarrow C_1^T \Psi(0) = 0,$$
\[ x_2(0) = -1 \Rightarrow C_2^T \Psi(0) = -1, \]
\[ \Psi(0) = 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} [H_0(-1), H_1(-1), \ldots, H_{M-1}(-1) | 0, \ldots, 0 | \ldots | 0, \ldots, 0]^T. \]

Now we have a system of linear algebraic equation with quadratic performance index. Approximate solution of \( J \) is given in Table 5.

| Table 5. Comparison between different methods for optimal value of \( J \) (Example 4.3) |
|-----------------------------------------------|--------|--------|--------|--------|--------|
| Exact value | Hsieh[13] | Jaddu[16] | Majdalawi[18] | Our proposed method |
| 0.06936994 | 0.0702 | 0.0693689 | 0.069368896 | 0.0693688962 |

**Example 4.4.** Consider the problem of finding the minimum of

\[ J = \int_0^1 [\dot{x}^2(t) + t\dot{x}(t)]dt, \]  
(39)

with the boundary conditions

\[ x(0) = 0, \quad x(1) = \frac{1}{4}. \]  
(40)

Suppose that \( \dot{x}(t) \) can be expanded approximately as

\[ \dot{x}(t) \simeq \sum_{n=1}^{2^k-1} \sum_{m=0}^{M-1} c_{nm} \Psi_{nm}(t) = C^T \Psi(t), \]  
(41)

by substituting Eq.(41) in Eq.(39) we get

\[ J \simeq \int_0^1 [C^T \Psi(t) \Psi^T(t) C + d^T \Psi(t) \Psi^T(t) C]dt \]
\[ = C^T \Lambda C + d^T \Lambda C, \]  
(42)

where

\[ t = d^T \Psi(t), \]
\[ d^T = \left[\begin{array}{c}
\frac{b_1}{a}, \frac{1}{a}, 0, \ldots, 0 | \frac{b_2}{a}, \frac{1}{a}, 0, \ldots, 0 | \ldots | \frac{b_{2^k-1}}{a}, \frac{1}{a}, 0, \ldots, 0
\end{array}\right], \]
\[ a = \sqrt{\frac{2}{\pi}} 2^{\frac{k}{2}} 2^{k+1}, \quad b_i = -2(-2i + 1) i = 1, 2, \cdots, 2^{k-1}. \]

According to boundary condition we have

\[ x(1) = C^T \int_0^1 \Psi(t) dt = \frac{1}{4}. \]  
(43)

In Table 6, a comparison is made between the present method and the exact solution.

| Table 6. The approximate and exact values of \( J \) (Example 4.4) |
|--------|--------|--------|
| Exact value | Approximated value via HW | Error |
| 0.16666666666 | 0.16666666666 | 0.4 \( \times 10^{-14} \) |
Example 4.5. Find an optimal control which minimizes the following problem \[ J = \frac{1}{2} \int_0^1 (x^2(t) + u^2(t))dt, \] \[ \dot{x}(t) = tx(t) + u(t), \] \[ x(0) = 1. \] (44) (45) (46)

For solving the above problem we use the operational matrix of production as follows

\[ t = \Psi^T(t)d, \]
\[ x(t)t \simeq X^T\Psi(t)\Psi^T(t)d \simeq X^T\tilde{d}\Psi(t), \quad x(0) = X^T_0\Psi(t), \]

by integrating from Eq.(45) we will have

\[ \int_0^t \dot{x}(\tau)d\tau = \int_0^t \tau x(\tau)d\tau + \int_0^t u(\tau)d\tau, \]
\[ \Rightarrow X^T - X^T_0 = X^T\tilde{d}P + U^TP. \] (47)

Now the differential equation convert to a system of algebraic equations. For cost functional (44) we do as foregoing examples. Approximate solution is given in Table 7.

| Elnagar[7] | Jaddu [16] | Abu Haya[1] | Rafiei[20] | Our method via HW |
|------------|------------|-------------|------------|------------------|
| 0.48427022 | 0.4842676003 | 0.4842678105 | 0.4842677529 | 0.4842676962 |

5. Conclusion. In this paper, the Hermite wavelets were used as a new orthogonal polynomials to parameterize the state and control variables and we derived the operational matrices of integration and derivative, for this wavelet. The Hermite wavelet operational matrices, $P$ and $D$, together with the integration of two Hermite wavelet vectors production, are used to solve the examples. It is also shown that the Hermite wavelets provide an acceptable solution for problems. We can use these matrices for solving delay and multi delay optimal control problems, too.

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