INNER PRODUCT IN HIGHEST-WEIGHT REPRESENTATION

CHUANZHONG LI\textsuperscript{1}, ZHISHENG LIU\textsuperscript{2}, AND BAO SHOU\textsuperscript{3}

Abstract. In this paper, we study the inner product of states corresponding to weights of finite-dimensional highest-weight representations of classical groups. We prove that the action of the raising operators would reduce a state of highest-weight representation to a linear combination of states of highest-weight representation, with the level decreased by one. Then we propose an iterative algorithm for calculating the inner products of states efficiently, revealing the intricate structure of the representation.

As applications, we discuss the unitarity of the highest-weight representation and propose a conjecture. We determine the norm of a special class of states. And we completely determine the inner products of states of the minuscule representations. The algorithm proposed is applicable to the highest-weight representation of affine Lie algebra without modifications. These findings can be used to study the construction of solutions to Kapustin-Witten equations which are based on the fundamental solutions of Toda systems.


c

Contents

1. Introduction \hspace{1cm} 2
2. Preliminary on Highest-Weight Representations \hspace{1cm} 3
3. Inner Product \hspace{1cm} 5
3.1. Inner Product of Different States \hspace{1cm} 5
3.2. Inner Product of the Same State \hspace{1cm} 7
4. Applications \hspace{1cm} 13
4.1. Unitarity and Norm \hspace{1cm} 13
4.2. Minuscule Representations \hspace{1cm} 16
4.3. Affine Lie Algebras \hspace{1cm} 17
4.4. Kapustin-Witten Equations \hspace{1cm} 18
Acknowledgments \hspace{1cm} 19
Statements and Declarations \hspace{1cm} 20
Funding and/or Conflicts of interests/Competing interests \hspace{1cm} 20
Data Availability Statement \hspace{1cm} 20
Appendix A. Second Fundamental Representation of $G_2$ \hspace{1cm} 20
References \hspace{1cm} 24

2010 Mathematics Subject Classification. 16Z05.

Key words and phrases. Highest-weight representation, inner product, iterative algorithm, weight, Kapustin-Witten equations.
1. Introduction

Highest-weight representations are familiar to physicists. The representation of the Virasoro algebra can be used to characterize simple conformal field theories, such as minimal modes. The other states of representation can be obtained by successive applications of Virasoro operators on the highest-weight states. The inner product of states can be used to characterize the unitarity of representations, revealing the intricate structure of conformal theories.

In this paper, we study the inner product of states of highest-weight representations of simple Lie algebra $g$ and its applications. Any finite-dimensional irreducible representation has a unique highest-weight state $|\Lambda\rangle$. Starting from the highest-weight state, all the states in the representation space can be obtained by the action of the lowering operators of $g$ as follows

$$|\lambda\rangle = E^{-\beta}E^{-\gamma}\cdots E^{-\eta}|\Lambda\rangle \quad \text{for} \quad \beta, \gamma, \cdots, \eta \in \Delta_+.$$

The inner product between the state $|\lambda\rangle$ and the conjugate state $\langle \lambda |$ of another state $|\Lambda\rangle$ is

$$\langle \lambda | \lambda \rangle = \langle \Lambda | E^{+\beta'} \cdots E^{+\gamma'} E^{-\gamma} E^{-\beta} E^{+\eta} \cdots E^{-\eta}|\Lambda\rangle.$$  \hspace{1cm} (1.1)

For any state $|\lambda\rangle$, the norm of $|\lambda\rangle$ is positive definite

$$\langle \lambda | \lambda \rangle = \langle \Lambda | E^{+\eta} \cdots E^{+\gamma} E^{-\gamma} E^{+\beta} E^{-\beta} \cdots E^{-\eta}|\Lambda\rangle > 0.$$  \hspace{1cm} (1.1)

The inner product (1.1) appears in the solutions of the Kapustin-Witten (KW) equations. The following inner product is one of the factors conjectured in [1]

$$\langle v^i_w | v^j_w \rangle = \langle \Lambda_s | E^+_{j_1} \cdots E^+_{j_n} | E^-_{i_1} \cdots E^-_{i_m} \Lambda_s \rangle,$$  \hspace{1cm} (1.2)

where $E^-_{j_n} \cdots E^-_{j_1}$ and $E^+_{i_1} \cdots E^+_{i_m}$ are two sequences from the highest-weight $\Lambda_s$ to the weight $w$. The superscript $i$ and $j$ denote the $i$th and $j$th paths, respectively. The calculation of the commutation of operators in equation (1.2) is tedious and must be done manually for a large number of operators. In addition, to define the vector $|v_w(\omega)\rangle$, it need to consider all the paths from the highest-weight state to $w$. Each branch node in the diagram increases the number of paths, such as the node $(1, -1, 1)$ shown in Fig.(1). As the rank of $g$ increases, the number of paths, as well as the number of weights, grows rapidly. More details will be provided in Section 4.4.

We illustrate these problems through an example.

Example 1. Fundamental representation $\rho_2$ for $A_3$: Let $\omega$ be the weight vector ended. As shown in Fig.(1), there are four paths from the highest-weight $(0, 1, 0)$ to $(-1, -1, -1)$. To construct the norm of the state $|v_{(-1,-1,-1)}(\omega)\rangle$, we need to compute the following two norms

$$\langle v^1_{(-1,-1,-1)} | v^1_{(-1,-1,-1)} \rangle = \langle \Lambda_2 | E^+_2 E^+_3 E^+_3 E^-_1 E^-_2 | \Lambda_2 \rangle,$$
$$\langle v^2_{(-1,-1,-1)} | v^2_{(-1,-1,-1)} \rangle = \langle \Lambda_2 | E^+_2 E^+_3 E^+_1 E^-_1 E^-_2 | \Lambda_2 \rangle,$$
which correspond to two sequences from the highest-weight $(0, 1, 0)$ to $(-1, 1, -1)$. We also need to compute the following inner products,

$$
\langle v_1^{(-1,1,-1)} | v_2^{(-1,1,-1)} \rangle = \langle \Lambda_2 | E_2^+ E_1^+ E_3^+ E_1^- E_3^- E_2^- | \Lambda_2 \rangle,
$$

$$
\langle v_2^{(-1,1,-1)} | v_1^{(-1,1,-1)} \rangle = \langle \Lambda_2 | E_2^+ E_3^+ E_1^+ E_1^- E_3^- E_2^- | \Lambda_2 \rangle,
$$

where the states and their conjugate states are constructed by different paths.

In this paper, we focus on the calculation of the inner product (1.2). In Section 2, we introduce the highest-weight representations as a preparation. In Section 3, we propose an algorithm for calculating the inner product of states of the highest-weight representations, corresponding to a path in the weight diagram. The algorithm is an iterative process based on Theorem 4 which significantly reduces the computational workload. It can be used to calculate inner product of any irreducible finite-dimensional representations of classical groups in addition to the fundamental representations appearing in the KW equations.

In Section 4, we introduce applications of the algorithm. We discuss the unitarity of the highest-weight representations and propose one conjecture. We calculate the norms of a special kind of states. And we also determine the inner product of states in the minuscule representations. Finally, the application to the solution of KW equations is pointed out. In the appendix, we give an example to illustrate the proposed algorithm in detail. By using the proposed algorithm, the inner product of a state can be recorded in less than one page. In contrast, without the algorithm, it would take twenty-five pages to record the calculation process.

2. Preliminary on Highest-Weight Representations

In the Chevalley basis of a Lie algebra $\mathfrak{g}$, the corresponding raising and lowering operators are denoted by $E_i^{\pm}$ for a simple root $\alpha_i$, and the operators corresponding to the coroots are denoted by $H_i$. The commutation relations of these operators are given...
by
\[ [E^+_j, E^-_j] = \delta_{ji} H_j, \quad [H_i, E^\pm_j] = \pm A_{ji} E^\pm_j, \quad [H_i, H_j] = 0. \tag{2.1} \]

The finite-dimensional irreducible representation \( L_\Lambda \) has a unique highest-weight state \( |\Lambda\rangle \), satisfying
\[ E^\alpha |\Lambda\rangle = 0, \]
for any positive root \( \alpha \), which is a linear combination of simple roots \( \alpha_i \) with positive coefficients. All states \( |\lambda\rangle \) associated to the weight \( \lambda \) in the representation space \( L_\Lambda \) can be obtained by the action
\[ |\lambda\rangle = E^{-\beta} E^{-\gamma} \cdots E^{-\eta} |\Lambda\rangle \quad \text{for} \quad \beta, \gamma, \cdots, \eta \in \Delta_+, \]
where \( \Delta_+ \) is the set of positive roots. The states \( |\lambda\rangle \) satisfy the following identity
\[ H_i |\lambda\rangle = \lambda_i |\lambda\rangle. \tag{2.2} \]
The set of eigenvalues of all states in \( L_\Lambda \) is the weight system \( \Omega_\Lambda \). Any weight \( \lambda' \) in the set \( \Omega_\Lambda \) is such that \( \lambda - \lambda' \in \Delta_+ \). Consequently, \( \lambda \) is necessarily of the form \( \Lambda - \sum n_i \alpha_i \), with \( n_i \in \mathbb{Z}_+ \). We call \( \sum n_i \) the level of the weight \( \lambda \) in the representation \( \Lambda \). These weights can be obtained by the action of the lowering operators \( E^-_i \) of \( g \) as follows
\[ |\lambda\rangle = E^-_{j_{n(w)}} \cdots E^-_{j_1} |\Lambda\rangle, \]
whose conjugate state is defined by
\[ \langle \lambda | = \langle \Lambda | E^+_1 \cdots E^+_{j_{n(w)}} |\rangle. \]

Then we have
\[
H_i |\lambda\rangle = H_i E^-_{j_{n(w)}} \cdots E^-_{j_1} |\Lambda\rangle \\
= (E^-_{j_{n(w)}} H_i - E^-_{j_{n(w)}} A_{j_{n(w)},i}) \cdots E^-_{j_1} |\Lambda\rangle \\
= - \sum_{b=1}^{n(w)} E^-_{j_{n(w)}} \cdots A_{j_{b,i}} E^-_{j_{b,i}} \cdots E^-_{j_1} |\Lambda\rangle + E^-_{j_{n(w)}} \cdots E^-_{j_1} H_i |\Lambda\rangle \\
= (\Lambda_i - \sum_{b=1}^{n(w)} A_{j_{b,i}}) E^-_{j_{n(w)}} \cdots E^-_{j_1} |\Lambda\rangle. \tag{2.3}
\]

According to formula (2.2), we have
\[ \lambda_i = \Lambda_i - \sum_{b=1}^{n(w)} A_{j_{b,i}}. \tag{2.4} \]

There are two positive integers \( p_i \) and \( q_i \), such that
\[
(E^+_i)^{p_i+1} |\lambda\rangle \sim E^+_i |\lambda + p_i \alpha_i\rangle = 0, \tag{2.5}
(E^-_i)^{q_i+1} |\lambda\rangle \sim E^-_i |\lambda - q_i \alpha_i\rangle = 0, \tag{2.6}
\]
for any simple root \( \alpha_i \). The integers \( p_i \) and \( q_i \) satisfy the following identity
\[ 2 (\alpha_i, \lambda) |\alpha_i|^2 = (\alpha_i^\vee, \lambda) = -(p_i - q_i). \]
This identity is crucial for determining all the weights in the weight system $\Omega_\Lambda$. By progressing level by level, we determine the value of $p_i$ at each step. Clearly, $\lambda - \alpha_i$ is also a weight if $q_i$ is nonzero, that is, if $\lambda_i + p_i > 0$. We emphasize the following string of weights corresponding to a weight $\lambda$

$$\lambda + p_\alpha\lambda_i, \cdots, \lambda, \cdots, \lambda - q_\alpha q_i.$$  \hfill (2.7)

Note that the weight $\lambda + p_\alpha\lambda_i$ satisfies formula (2.5), which will play a significant role in the algorithm proposed in Section 3.

The procedure for constructing all the weights in the representation is as follows. We start with the highest-weight $\Lambda = (\Lambda_1, \ldots, \Lambda_r)$:

1. Construct the sequence of weights $\Lambda - \alpha_i, \Lambda - 2\alpha_i, \cdots, \Lambda - r\alpha_i$.
2. Repeat this process with $\Lambda$ replaced by each of the weights just obtained.
3. Continue this process until no more weights with positive Dynkin labels are produced.

Simple examples will clarify the method.

**Example 2.** Consider the fundamental representation $\rho_2$ of $A_3$, characterized by the highest weight $(0, 1, 0)$, as illustrated in Fig. (1). The weights at each step can be derived using the aforementioned procedure.

Another example is the fundamental representation $\rho_2$ of $G_2$, whose weights can be read from Fig. (3) in Section 3.

### 3. Inner Product

In this section, we first prove that the inner product of the states defined by different weights is zero. And then, we examine the inner product of states defined by the same weight but with different paths from the highest weight to the terminating weight in the weight diagram.

In Theorem 4, we prove that the action of the raising operators would reduce a state of highest-weight representation to a linear combination of states of highest-weight representation, with the level decreased by one. Based on this crucial result, we propose an iterative algorithm for calculating the inner product efficiently, which is the main result of this paper.

#### 3.1. Inner Product of Different States.

Let $|\nu_w^m\rangle$ denote a state of level $m$ along a path in the weight diagram ended with the weight $w$. And $|\nu_w^m,n\rangle$ denote a state of level $m$ along the $n$th path from the highest-weight vector to the weight $w$.

First, we derive a useful identity. According to the commutation relations (2.1), for the highest-weight $\Lambda = \sum \lambda_a \omega_a$, we have the following identity,

$$E_a^+ E_{j_n}^- E_{j_{n-1}}^- \cdots E_{j_1}^- |\Lambda\rangle = (\delta_{a,j_n} H_a + E_{j_n}^- E_a^+) E_{j_{n-1}}^- \cdots E_{j_1}^- |\Lambda\rangle$$

$$= \sum_{i=1}^n E_{j_n}^- E_{j_{n-1}}^- \cdots E_{j_i}^- \delta_{a,j_i} H_a E_{j_i}^- E_{j_{i-1}}^- \cdots E_{j_1}^- |\Lambda\rangle$$  \hfill (3.1)

$$= \sum_{i=1}^n \delta_{a,j_i} (\lambda_a - (\sum_{i=1}^{i-1} A_{j_i,a})) E_{j_n}^- E_{j_{n-1}}^- \cdots E_{j_i}^- E_{j_{i-1}}^- \cdots E_{j_1}^- |\Lambda\rangle,$$
Proof. Assume the states

Theorem 1. The inner product of the states with different level is zero.

Proof. Assume the states \(|v^m\rangle\) and \(|v^n\rangle\) with levels \(m\) and \(n\) are given by

\[|v^m\rangle = |E_{i_1}^- \cdots E_{i_m}^- |Λ\rangle, \quad |v^n\rangle = |E_{j_1}^- \cdots E_{j_n}^- |Λ\rangle.\]

Without loss of generality, assume the level \(m > n\). Then the inner product of \(|v^m\rangle\) and \(|v^n\rangle\) is

\[
\langle v^m | v^n \rangle = \langle Λ | E_{i_1}^+ \cdots E_{i_m}^+ |E_{j_1}^- \cdots E_{j_n}^- |Λ\rangle
\]
\[
= \langle Λ | E_{i_1}^+ \cdots E_{i_m}^+ (δ_{i_m,j_n} H_{i_m} + E_{j_n}^- E_{i_m}^+) E_{j_{n-1}}^- \cdots E_{j_1}^- |Λ\rangle
\]
\[
= \langle Λ | E_{i_1}^+ \cdots E_{i_{m-1}}^+ \left( \sum_{k=1}^{n} E_{j_k}^- \cdots E_{j_{k+1}}^- δ_{i_{m-j_n},j_k} H_{i_{m-j_n}} E_{j_{k-1}}^- \cdots E_{j_1}^- \right) |Λ\rangle.
\]

According to formula (3.1), the operator \(δ_{i_m,j_n} H_{i_m}\) can be seen as an undetermined constant \(h_{i_{m-j_n},j_k}\) because it is the eigenvalue of the operator \(δ_{i_m,j_n} H_{i_m}\) acting on the state \(E_{j_{n-1}}^- \cdots E_{j_1}^- |Λ\rangle\). The action of the operator \(E_{s}^+\) decrease the number of the operators \(E_{s}^-\) of \(|v_n\rangle\) by one. Then there is no \(E_{s}^-\) left after \(n\) times of the actions of the operators \(E_{s}^+\). And the remaining operators \(E_{i_1}^+ \cdots E_{i_{m-n}}^+\) annihilate \(|Λ\rangle\). Thus, we have

\[
\langle v^m | v^n \rangle = \langle Λ | E_{i_1}^+ \cdots E_{i_{m-n}}^+ (\sum_{k=1}^{n} E_{j_k}^- \cdots h_{i_{m-j_n},j_k} E_{j_{k-1}}^- \cdots E_{j_1}^- ) |Λ\rangle
\]
\[
= \langle Λ | E_{i_1}^+ \cdots E_{i_{m-n}}^+ \left( \sum h_{i_j} δ_{i_j} \right) |Λ\rangle.
\]

We draw the conclusion.

Before examining the inner product of different states at the same level, we present the following lemma.

Lemma 2. If the states are obtained by the same contents of operators \(E_{s}^-\) acting on the highest-weight with different orders, the weights corresponding to the states are the same.

Proof. Let two states be generated by the operators \(E_{1}^- , \cdots , E_{n}^-\) acting on the highest weight \(Λ\) in different sequences. The weights of these two states are the same due to the equality

\[w_1^3 = w_2^2 = Λ - α_1 - \cdots - α_n.\]

We then state the following theorem.

Theorem 3. The inner product of different states with the same level is zero.
Proof. $|v^{n,1}_{u_1}|$ and $|v^{n,2}_{u_2}|$ are two different states with the same level $n$ and $w^1 \neq w^2$. According to Lemma 2, they contain the same number of operators $E^-_s$ but have different contents. The inner product is,

$$\langle \Lambda | E^+_i \cdots E^+_i | E^-_j \cdots E^-_j | \Lambda \rangle$$

$$= \langle \Lambda | E^+_i \cdots E^+_i (H_{i_n,j_n} \delta_{i_n,j_n} + E^-_{i_n} E^+_i) E^-_{j_{n-1}} \cdots E^-_j | \Lambda \rangle$$

$$= \langle \Lambda | E^+_i \cdots E^+_i \sum_{k=1}^{n} E^-_{j_n} \cdots E^-_{j_{k+1}} H_{i_k,j_k} \delta_{i_k,j_k} E^-_{j_{k-1}} \cdots E^-_j | \Lambda \rangle$$

$$= \langle \Lambda | E^+_i \cdots E^+_i \sum_{k=1}^{n} (E^-_{j_n} \cdots E^-_{j_{k+1}} h_{i_k,j_k} \delta_{i_k,j_k} E^-_{j_{k-1}} \cdots E^-_j) | \Lambda \rangle$$

$$= \sum h_{i,j} \delta_{i,j}$$

$$= 0.$$  

The last equality is zero because each term of the fourth equality vanish.

\[ \square \]

3.2. Inner Product of the Same State. In this section, we examine the inner product of states defined by different paths, which terminate with the same weight in the weight diagram. Considering a special case of identity (3.1),

$$E^+_i (E^-_i)^n | \Lambda \rangle = (H_i + E^-_i E^+_i) (E^-_i)^{n-1} | \Lambda \rangle$$

$$= \sum_{l=1}^{n-1} (E^-_i)^l H_i (E^-_i)^{n-1-l} | \Lambda \rangle$$

$$= \sum_{l=1}^{n-1} (E^-_i)^l (\lambda_i - (n - 1) - l) A_{ii} (E^-_i)^{n-1-l} | \Lambda \rangle$$

$$= n(\lambda_i - (n - 1)) (E^-_i)^{n-1} | \Lambda \rangle$$

$$= R.H.S$$  \[ (3.2) \]

where we have used formula (2.3) in the third step.

Using the above formula, we can further generalize identity (3.1).

Proposition 1. For the highest-weight $\Lambda = \Sigma_\alpha \lambda_\alpha \omega_\alpha$, we have

$$E^+_i (E^-_i)^n \prod_{b=1}^{m} E^-_{j_b} | \Lambda \rangle = n(\lambda_i - (n - 1) - \sum_{b=1}^{m} A_{j_b,i}) (E^-_i)^{n-1} \prod_{b=1}^{m} E^-_{j_b} | \Lambda \rangle + (E^-_i)^n E^+_i \prod_{b=1}^{m} E^-_{j_b} | \Lambda \rangle.$$
Proof. According to Eq.(3.2), we have
\[
L.H.S = \sum_{a=0}^{n-1} (E^-_i)^a H_i (E^-_i)^{n-1-a} \prod_{b=1}^{m} E^-_{j_b} |\Lambda\rangle + (E^-_i)^n E^+_i \prod_{b=1}^{m} E^-_{j_b} |\Lambda\rangle
\]
\[
= \sum_{a=0}^{n-1} (E^-_i)^a (\lambda_i - (n - 1 - a)A_{ii} - \sum_{b=1}^{m} A_{j_b,i}) (E^-_i)^{n-1-a} \prod_{b=1}^{m} E^-_{j_b} |\Lambda\rangle
\]
\[
+ (E^-_i)^n E^+_i \prod_{b=1}^{m} E^-_{j_b} |\Lambda\rangle
\]
\[
= n(\lambda_i - (n - 1) - \sum_{b=1}^{m} A_{j_b,i}) (E^-_i)^{n-1} \prod_{b=1}^{m} E^-_{j_b} |\Lambda\rangle + (E^-_i)^n E^+_i \prod_{b=1}^{m} E^-_{j_b} |\Lambda\rangle.
\]
\(\square\)

For \(n = 0\), this formula reduce to Eq.(3.1). For \(m = 0\), we recover Eq.(3.2). Using this formula, we can calculate any inner product directly; however, the computational efficiency is not acceptable. With the notation \(|\Lambda^1\rangle = \prod_{b=1}^{m} E^-_{j_b} |\Lambda\rangle\), the identity in Proposition \(\square\) becomes
\[
E^+_i (E^-_i)^n \prod_{b=1}^{m} E^-_{j_b} |\Lambda\rangle = n(\Lambda^1 - (n - 1)) (E^-_i)^{n-1} |\Lambda^1\rangle + (E^-_i)^n E^+_i |\Lambda^1\rangle \prod_{b=1}^{m} E^-_{j_b} |\Lambda\rangle. \quad (3.3)
\]
The subscript \(|\Lambda^1\rangle\) denotes the state defined by the operators on the right acting on \(|\Lambda\rangle\)
\[
|\Lambda^1\rangle = \prod_{b=1}^{m} E^-_{j_b} |\Lambda\rangle.
\]

The following formula is the basis of our main result, which is a generalization of the identities (3.3).

**Proposition 2.** Let
\[
|\lambda\rangle = \prod_{b=1}^{m_1} E^-_{j_b} (E^-_i)^{n_1} |\Lambda^1\rangle \prod_{b=1}^{m_2} E^-_{j_b} (E^-_i)^{n_2} |\Lambda^2\rangle \cdots \prod_{b=1}^{m_l} E^-_{j_b} (E^-_i)^{n_l} |\Lambda^l\rangle \prod_{b=1}^{m_{l+1}} E^-_{j_b} |\Lambda\rangle \quad (3.4)
\]
with the highest-weight is given by \(\Lambda = \Sigma_a \Lambda_a \omega_a\). And \((E^-_i)^{n_1}, \ldots, (E^-_i)^{n_l}\) are the product factors related to the operator \(E^-_i\). Then we have
\[
E^+_i \prod_{b=1}^{m_1} E^-_{j_b} (E^-_i)^{n_1} |\Lambda^1\rangle \prod_{b=1}^{m_2} E^-_{j_b} (E^-_i)^{n_2} |\Lambda^2\rangle \cdots \prod_{b=1}^{m_l} E^-_{j_b} (E^-_i)^{n_l} |\Lambda^l\rangle \prod_{b=1}^{m_{l+1}} E^-_{j_b} |\Lambda\rangle \quad (3.5)
\]
\[
= \sum_{k=1}^{l} \prod_{b=1}^{m_k} E^-_{j_b} (E^-_i)^{n_k} \cdots n_k (\Lambda^k_i - (n_k - 1)) \prod_{b=1}^{m_k} E^-_{j_b} (E^-_i)^{n_k-1} \cdots \prod_{b=1}^{m_l} E^-_{j_b} (E^-_i)^{n_l} \prod_{b=1}^{m_{l+1}} E^-_{j_b} |\Lambda\rangle,
\]

where \(i \neq j_b\).
Proof. According to Eq.(2.1), we have \([E_i^+, E_j^-] = 0\) for \(i \neq j\). According to Eq.(3.2), we have

\[
L.H.S = E_i^+ \prod_{b=1}^{m_1} E_{j_b}^- (E_i^-)^{n_1} |\Lambda^1\rangle
\]

\[
= n_1 (\Lambda_i^1 - (n_1 - 1)) \prod_{b=1}^{m_1} E_{j_b}^- (E_i^-)^{n_1-1} |\Lambda^1\rangle + \prod_{b=1}^{m_1} E_{j_b}^- (E_i^-)^{n_1} E_i^+ |\Lambda^1\rangle
\]

\[
= n_1 (\Lambda_i^1 - (n_1 - 1)) \prod_{b=1}^{m_1} E_{j_b}^- (E_i^-)^{n_1-1} |\Lambda^1\rangle + \cdot \cdot \cdot
\]

\[
+ \prod_{b=1}^{m_1} E_{j_b}^- (E_i^-)^{n_1} \cdot \cdot \cdot n_k (\Lambda_i^k - (n_k - 1)) \prod_{b=1}^{m_k} E_{j_b}^- (E_i^-)^{n_k-1} |\Lambda^k\rangle
\]

\[
+ \prod_{b=1}^{m_1} E_{j_b}^- (E_i^-)^{n_1} \cdot \cdot \cdot \prod_{b=1}^{m_k} E_{j_b}^- (E_i^-)^{n_k} E_i^+ |\Lambda^k\rangle
\]

\[
= \cdot \cdot \cdot = R.H.S.
\]

\[\square\]

Fortunately, there are not always numerous terms on the right hand side of the identity (3.5). Let \(|\Lambda^t\rangle\) be the first state satisfying \(E_i^+ |\Lambda^t\rangle = 0\) in \(\lambda\) (3.4). Consequently, the sequences will terminate at the \(t\)th factor containing \(E_i^-\). And then we have

\[
E_i^+ \prod_{b=1}^{m_1} E_{j_b}^- (E_i^-)^{n_1} \prod_{b=1}^{m_2} E_{j_b}^- (E_i^-)^{n_2} |\Lambda^2\rangle \cdot \cdot \cdot \prod_{b=1}^{m_t} E_{j_b}^- (E_i^-)^{n_t} |\Lambda^t\rangle \prod_{b=1}^{m_{t+1}} E_{j_b}^- |\Lambda^t\rangle
\]

\[= \sum_{k=1}^{m_{t+1}} \prod_{b=1}^{m_1} E_{j_b}^- (E_i^-)^{n_1} \cdot \cdot \cdot n_k (\Lambda_i^k - (n_k - 1)) \prod_{b=1}^{m_k} E_{j_b}^- (E_i^-)^{n_k-1} \cdot \cdot \cdot \prod_{b=1}^{m_t} E_{j_b}^- (E_i^-)^{n_t} |\Lambda^t\rangle.
\]

Moreover, we find that the terms on the right hand side of the identity (3.6) correspond to paths in the weight diagram.

**Theorem 4.** The action of the \(E_i^+\) would reduce a state corresponding to a path in the weight diagram to states corresponding to paths in the same weight diagram.

**Proof.** We prove the theorem by induction on the level of the weight. Assuming the proposition is true for weights of level \(k\), we prove the theorem for weights of level \(k+1\). The weight at the level \(k+1\) is denoted as \(w^{k+1}\). For a weight of level \(k+1\), the case where only one arrow pointing it are shown in Fig.(2)(b),(c),and (d). The case where at least two arrows pointing it is shown in Fig.(2)(a).

For the first case with one arrow, we have

\[|w^{k+1}\rangle = E_i^- |w^k\rangle,
\]

where \(|w^k\rangle\) is a state corresponding to a path from the highest-weight state to the weight of level \(k\). Then the inner product is

\[\langle w^{k+1} |w^{k+1}\rangle = \langle w^k |E_i^+ |w^{k+1}\rangle.
\]
We need to prove that the expansion of $E_i^+ |w^{k+1}_5\rangle$ are states corresponding to paths from the highest-weight state to the weight of level $k$.

- For $E_i^+ |w^k_3\rangle = 0$ as shown in Fig. (2)(b), according to the assumption made, we have
  $$E_i^+ |w^{k+1}_2\rangle = E_i^+ E_i^- |w^k_3\rangle = H_i |w^k_3\rangle = h_i |w^k_3\rangle,$$
  which leads to the conclusion.

- For $E_i^+ |w^k_4\rangle \neq 0$ and $|w^k_4\rangle = E_i^- |w^{k-1}_3\rangle$ as shown in Fig. (2)(c), according to the assumption made, we have
  $$E_i^+ |w^k_4\rangle = a_1 |w^{k-1-j_1}_1\rangle + a_2 |w^{k-1,j_2}_2\rangle + \cdots + a_n |w^{k-1,j_n}_n\rangle. \tag{3.7}$$

The states on the right hand side are states corresponding to paths from the highest-weight state to the weights of level $k - 1$. Then

$$E_i^+ |w^{k+1}_3\rangle = E_i^+ E_i^- |w^k_4\rangle = (H_i + E_i^- E_i^+)|w^k_4\rangle = h_i |w^k_4\rangle + a_1 E_i^- |w^{k-1,j_1}_1\rangle + a_2 E_i^- |w^{k-1,j_2}_2\rangle + \cdots + a_n E_i^- |w^{k-1,j_n}_n\rangle,$$

where we have used the expansion of $E_i^+ |w^k_4\rangle \ (3.7)$. The states on the last equality are states corresponding to paths from the highest-weight state to the weights of level $k$. Thus we draw the conclusion.

- For $E_i^+ |w^k_4\rangle \neq 0$ and $|w^k_5\rangle = E_j^- |w^{k-1}_5\rangle$, $(i \neq j)$, then $|w^k_5\rangle = E_j^- E_i^- |w^{k-2}_1\rangle$ as shown in Fig. (2)(d). According to the assumption made, we have
  $$\langle w^k_5| w^{k-2}_5 \rangle = \langle w^{k-1,1}_4| E_i^+ E_j^- |w^{k-1,2}_5\rangle = \langle w^{k-2,1}_1| E_j^+ E_i^+ E_j^- E_i^- |w^{k-2,2}_1\rangle = \langle w^{k-2,1}_1| E_j^+ E_j^- E_i^+ E_i^- |w^{k-2,2}_1\rangle.$$
According to the assumption made, we have
\[
E_i^+ E_i^- |w_1^{k-2,2}\rangle = a'_i |w_1^{k-2,j_1}\rangle + a'_2 |w_1^{k-2,j_2}\rangle + \cdots + a'_n |w_1^{k-2,j_n}\rangle.
\]

The states on the right hand side correspond to paths from the highest-weight state to the weights of level \(k - 2\). As shown in Fig. (2)(d), we have
\[
\begin{align*}
\langle w_5^{k,1} | w_5^{k,2} \rangle &= \langle w_1^{k-2,1} | E_j^+ E_j^- (a'_1 |w_1^{k-2,j_1}\rangle + a'_2 |w_1^{k-2,j_2}\rangle + \cdots + a'_n |w_1^{k-2,j_n}\rangle) \\
&= \langle w_1^{k-2,1} | E_j^+ (a_1 |w_1^{k-1,j_1}\rangle + a_2 |w_1^{k-1,j_2}\rangle + \cdots + a_n |w_1^{k-1,j_n}\rangle) \tag{3.8}
\end{align*}
\]

where \(|w_1^{k-1,j_i}\rangle\) on the right hand side are states corresponding to paths from the highest-weight state to the weights of level \(k - 1\). Then
\[
E_i^+ |w_4^{k+1}\rangle = E_i^+ E_i^- E_i^- |w_1^{k-2}\rangle = (H_i + E_i^- E_i^+) E_i^- |w_1^{k-2}\rangle = h_i E_i^- |w_1^{k-2}\rangle + a_1 E_i^- |w_1^{k-1,j_1}\rangle + a_2 E_i^- |w_1^{k-1,j_2}\rangle + \cdots + a_n E_i^- |w_1^{k-1,j_n}\rangle,
\]
where we have used the expansion of \(E_i^+ |w_5^k\rangle\) (3.8). The states on the last equality are states corresponding to paths from the highest-weight state to the weights of level \(k\). We draw the conclusion.

Next we consider case where two arrows pointing to the state \(w_4^{k+1}\) as shown in Fig. (2)(a). \(|w_1^{k,1}\rangle\) and \(|w_1^{k,2}\rangle\) are states of level \(k\) satisfying
\[
|w_1^{k,1}\rangle = E_j^- |w_1^{k-1}\rangle, \quad |w_1^{k,2}\rangle = E_i^- |w_1^{k-1}\rangle.
\]

According to the assumption made, we have
\[
E_j^+ |w_1^{k,1}\rangle = E_j^+ E_j^- |w_1^{k-1,1}\rangle = a_1 |w_1^{k-1,j_1}\rangle + a_2 |w_1^{k-1,j_2}\rangle + \cdots + a_n |w_1^{k-1,j_n}\rangle, \tag{3.9}
\]
\[
E_i^+ |w_1^{k,2}\rangle = E_i^+ E_i^- |w_1^{k-1,2}\rangle = b_1 |w_1^{k-1,i_1}\rangle + b_2 |w_1^{k-1,i_2}\rangle + \cdots + b_m |w_1^{k-1,i_m}\rangle. \tag{3.10}
\]

For the inner product \(\langle w_1^{k+1,2} | w_1^{k+1,1}\rangle = \langle w_1^{k,2} | E_i^+ E_j^- | w_1^{k,1}\rangle\), we have to prove \(E_i^+ E_j^- |w_1^{k,1}\rangle\) are states corresponding to paths from the highest-weight state to the weight of level \(k\). We have
\[
\begin{align*}
\langle w_1^{k+1,2} | w_1^{k+1,1}\rangle &= \langle w_1^{k,2} | E_i^+ E_j^- | w_1^{k,1}\rangle \\
&= \langle w_1^{k-1,2} | E_j^+ E_i^+ E_j^- E_i^- | w_1^{k,1}\rangle \\
&= \langle w_1^{k-1,2} | E_j^+ E_j^- E_i^+ E_i^- | w_1^{k,1}\rangle \\
&= \langle w_1^{k-1,2} | E_j^+ E_j^- (b_1 |w_1^{k-1,i_1}\rangle + b_2 |w_1^{k-1,i_2}\rangle + \cdots + b_m |w_1^{k-1,i_m}\rangle) \\
&= \langle w_1^{k-1,2} | E_j^+ (b_1 E_j^- |w_1^{k-1,i_1}\rangle + b_2 E_j^- |w_1^{k-1,i_2}\rangle + \cdots + b_m E_j^- |w_1^{k-1,i_m}\rangle),
\end{align*}
\]
where we have used the expansion (3.10). The states of the last equality in brackets correspond to paths from the highest-weight state to the weights of level \(k\). Thus we draw the conclusion.

For the inner product \(\langle w_1^{k+1,1} | w_1^{k+1,2}\rangle = \langle w_1^{k,1} | E_j^+ E_i^- | w_1^{k,2}\rangle\), we have to prove \(E_j^+ E_i^- |w_1^{k,2}\rangle\) are states corresponding to paths from the highest-weight state to the weight of level \(k\).
Then we get
\[
\langle w^{k+1,1}|w^{k+1,2}\rangle = \langle w_1^{k,1}|E_i^+ E_j^- |w_1^{k,2}\rangle \\
= \langle w_1^{k-1,1}|E_i^+ E_j^- E_i^- E_j^+ |w_1^{k-1,2}\rangle \\
= \langle w_1^{k-1,1}|E_i^+ E_j^- |w_1^{k-1,2}\rangle \\
= \langle w_1^{k-1,1}|E_i^+ (a_1 |w^{k-1,j_1}\rangle + a_2 |w^{k-1,j_2}\rangle + \cdots + a_n |w^{k-1,j_n}\rangle) \rangle \\
= \langle w_1^{k-1,1}|E_i^+ (a_1 E_i^- |w^{k-1,j_1}\rangle + a_2 E_i^- |w^{k-1,j_2}\rangle + \cdots + a_n E_i^- |w^{k-1,j_n}\rangle),
\]
where we have used the expansion (3.9). The states of the last equality in brackets correspond to path from the highest-weight state to the weights of level \(k\). Thus we draw the conclusion.

For the inner product of \(\langle w^{k+1,1}|w^{k+1,1}\rangle\) and \(\langle w^{k+1,2}|w^{k+1,2}\rangle\), the proofs are reduced to the first case. \(\square\)

**Remark 1.** The actions of the operators reveal the global information of the weights of the highest-weight representations.

By using formula (3.9) again and again, the algorithm for calculating the inner product present itself now.

**Algorithm 1.** The states \(|v^1_w\rangle\) and \(|v^2_w\rangle\) are defined by paths from the highest-weight vector \(\Lambda\) to the weight \(w\). To calculate the inner product,
\[
\langle v^1_w|v^2_w\rangle = \langle \Lambda_s|E_{i_1}^+ \cdots E_{i_n}^+ E_{j_1}^- \cdots E_{j_n}^- |\Lambda_s\rangle,
\]
the operators defining the state \(|v^1_w\rangle\) act on the state \(|v^2_w\rangle\) in sequence from right to left using formula (3.3). The actions of the raising operators would reduce the states on the right to the states one to one correspondence with paths in the weight diagram. These processes continue until no operators are left.

The algorithm is an iterative process because of Theorem 4. Thus the efficiency of calculations of inner product is improved greatly.

Finally, we would like to point out a fact.

**Theorem 5.** The inner product of states depends on paths defining the states.

![Figure 3. Weights in the fundamental representation (0, 1) of G2.](image-url)

We illustrate this fact through examples. As shown in Fig.(3), there are two paths from the highest-weight state \((0, 1)\) to the weight \((0, 0)\). These two paths correspond to the following states
\[
|v_{(0,0)}^{5,1}\rangle = E_2^- E_1^- E_1^- E_2^- |\Lambda_2\rangle, \\
|v_{(0,0)}^{5,2}\rangle = E_2^- E_2^- E_1^- E_1^- |\Lambda_2\rangle.
\]
The inner products can be calculated by commutating the operators directly as follows:

\[
\langle \psi^{5,1}_{(0,0)} | \psi^{5,1}_{(0,0)} \rangle = \langle \Lambda_2 | E_2^+ E_1^+ E_1^- E_2^- | E_2^- E_1^- E_1^- E_2^+ | \Lambda_2 \rangle = 72,
\]

\[
\langle \psi^{5,2}_{(0,0)} | \psi^{5,1}_{(0,0)} \rangle = \langle \Lambda_2 | E_2^+ E_1^+ E_1^- E_2^- | E_2^- E_1^- E_1^- E_2^+ | \Lambda_2 \rangle = 24,
\]

\[
\langle \psi^{5,2}_{(0,0)} | \psi^{5,2}_{(0,0)} \rangle = \langle \Lambda_2 | E_2^+ E_1^+ E_1^- E_2^- | E_2^- E_1^- E_1^- E_2^+ | \Lambda_2 \rangle = 36,
\]

which are different from each other. In the appendix, we will repeatedly calculate these inner products by using Algorithm [1].

4. Applications

In previous sections, we proposed an iterative algorithm for calculating the inner product of states of highest-weight representation. In this section, we discuss the applications of the algorithm, such as the norm of states and the unitarity of representations. And we completely determine the inner products of states of the minuscule representations. Algorithm [1] also work for the highest-weight representation of affine Lie algebra, which is infinite dimension. Finally, we note that it can be used to study the solution of the KW equations, where the inner products are related to the fundamental representations only.

4.1. Unitarity and Norm. As an application of Algorithm [1] we discuss the calculations of the norms of states. For the state \( |\lambda\rangle = E_{j_1}^- \cdots E_{j_n}^- |\Lambda_s\rangle \), the norm is

\[
\langle \lambda | \lambda \rangle = \langle \Lambda_s | E_{j_1}^- \cdots E_{j_n}^- | E_{j_1}^- \cdots E_{j_n}^- | \Lambda_s \rangle.
\]

The state \( |\lambda\rangle \) in the height weight representation is given by

\[
|\lambda\rangle = (E_1^-)^{n_1} |\Lambda_1\rangle \prod_{b=1}^{m_1} E_{j_b}^- (E_1^-)^{n_2} |\Lambda_2\rangle \cdots \prod_{b=1}^{m_l} E_{j_b}^- (E_1^-)^{n_l} |\Lambda_l\rangle \prod_{b=1}^{m_{l+1}} E_{j_b}^- |\Lambda|.
\]  

(4.1)

The notations have the same means with that in equality (3.6), with the subscript \( i \) of \( m_i \) begin from two.

To calculate the norm of \( |\lambda\rangle \), we need to take the action \( E_{i}^+ |\lambda\rangle \) firstly. Thus, we rewrite the identity (3.6) as follows

\[
E_{i}^+ (E_{i}^-)^{n_1} |\Lambda_1\rangle \prod_{b=1}^{m_1} E_{j_b}^- (E_{i}^-)^{n_2} |\Lambda_2\rangle \cdots \prod_{b=1}^{m_l} E_{j_b}^- (E_{i}^-)^{n_l} |\Lambda_l\rangle \prod_{b=1}^{m_{l+1}} E_{j_b}^- |\Lambda| = \sum_{k=1}^{t} (E_{i}^-)^{n_1} \cdots n_k (\Lambda_{i}^k - (n_k - 1)) \prod_{b=1}^{m_k} E_{j_b}^- (E_{i}^-)^{n_k-1} \cdots \prod_{b=1}^{m_l} E_{j_b}^- (E_{i}^-)^{n_l} |\Lambda^l|.
\]  

(4.2)

We have the following conjecture for the coefficients on the right hand side of the identity.

**Conjecture 1.** \( n_k (\Lambda_{i}^k - (n_k - 1)) \geq 0 \).

Unfortunately, we are not able to prove it presently. We have verified it through numerous examples and some of which are given in the appendix. The validity of the conjecture is based on the paths in the weight diagram of the highest-weight representation defining the state \( |\lambda\rangle \). This conjecture guarantees that coefficients are positive for
each step of Algorithm[1] Thus, the norm is positive for any state in the representation, which implies the unitary of the space. This also holds for linear combinations of such states. To calculate the norm of $|\lambda\rangle$, we need to take the following action.

\[ E_i^+ |\lambda\rangle = (E_i^+)^{n_1} |\lambda\rangle = \prod_{b=1}^{m_2} E_{j_b}^- (E_i^+)^{n_1} |\Lambda^{n_1}_2\rangle \prod_{b=1}^{m_1} E_{j_b}^- (E_i^+)^{n_1} |\Lambda_i^{n_1}_1\rangle \prod_{b=1}^{m_{l+1}} E_{j_b}^- |\Lambda\rangle \]

By equality is

\[ (E_i^+)^{n_1-1} \sum_{k=1}^{t} \prod_{b=1}^{m_2} E_{j_b}^- (E_i^+)^{n_1} |\Lambda^{n_1+\alpha k}_2\rangle \prod_{b=1}^{m_1} E_{j_b}^- (E_i^+)^{n_1} |\Lambda_i^{n_1+\alpha k}_1\rangle \prod_{b=1}^{m_{l+1}} E_{j_b}^- |\Lambda\rangle = \frac{1}{n_k (\Lambda_i^k - (n_k - 1))} \prod_{b=1}^{m_k} E_{j_b}^- |\Lambda^k\rangle \prod_{b=1}^{m_{l+1}} E_{j_b}^- |\Lambda\rangle \]

The operator $E_i^+$ decrease one $E_i^-$ in each product factor of $|\lambda\rangle$ containing $E_i^-$. After the action of operators $(E_i^+)^{n_1}$, the number of terms on the right hand side of the last equality is

\[ C_{m_1}^{k_1} + C_{m_1-k_1}^{k_2} + \cdots + C_{m_1-k_1-k_2-k_3-k_4-k_5-k_6-k_7-k_8-k_9-k_{10}}^{k_{11}} = 1 \]

with $k_1 + k_2 + \cdots + k_l = n_1$. For the last term, we have

\[ C_{m_1-k_1-k_2-k_3-k_4-k_5-k_6-k_7-k_8-k_9-k_{10}}^{k_{11}} = C_{k_1}^{k_{11}} = 1. \]

There are many difficulties to get a closed formula of norm of $\lambda$. According to formula [3.1], the operators in the conjunction state $|\lambda\rangle$ would decrease the same operators in the state $|\lambda\rangle$.

1. The combination formula [4.4] imply there is no simple closed formula to describe the norm of $|\lambda\rangle$.

2. The two adjacent product factors of $E_i^-$ become one when the product factor $\prod_{b=1}^{m_l} E_{j_b}^-$ are decreased by the operators in $|\lambda\rangle$. The actions of operators lead to more complicated combination formula than formula [1.4].

3. Note that $|\Lambda^k\rangle$ would not be annihilated by operators except $E_i^+$.

Special states: There are states corresponding to a particular type of paths in the weight diagram, whose norms can be fully determined.

These states and the norms are given by the following proposition.

**Proposition 3.** For the highest-weight $\Lambda = \Sigma_{a} \lambda_a \omega_a$, we have

\[ |\lambda\rangle = \prod_{b=1}^{m_1} E_{j_b}^- (E_i^+)^{n_1} |\Lambda^{n_1}_2\rangle \prod_{b=1}^{m_2} E_{j_b}^- (E_i^+)^{n_1} |\Lambda_i^{n_1}_1\rangle \prod_{b=1}^{m_{l+1}} E_{j_b}^- |\Lambda\rangle, \]

\[ |\lambda^k\rangle = \prod_{b=m_0}^{m_k} E_{j_{m_0-k}}^- (E_i^+)^{n_k} |\Lambda^k\rangle \prod_{b=1}^{m_l} E_{j_b}^- (E_i^+)^{n_l} |\Lambda^l_i\rangle \prod_{b=1}^{m_{l+1}} E_{j_b}^- |\Lambda\rangle, \]

where the states satisfy the following constraints

\[ E_{i_k}^+ |\Lambda^k\rangle = 0, \quad E_{j_{m_0-k}}^- |\lambda^k\rangle = 0, \quad m_0 = 1, \ldots, m_k, \quad k = 1, \ldots, l. \]
Then the norm of $|\lambda\rangle$ is given by

$$\langle \lambda | \lambda \rangle = \prod_{k=1}^{l} (\Lambda_{i_k}^{k})^2.$$ 

And the norm of $|\lambda'\rangle$ is given by

$$\langle \lambda' | \lambda' \rangle = \prod_{b=k}^{l} (\Lambda_{i_b}^{b})^2.$$ 

**Proof.** According to the construction of the path, we have $n_k = \Lambda_{i_k}^{k}$, $(k = 1, \cdots, l)$ and

$$\langle \lambda | \lambda \rangle$$

$$= \langle \cdots E_{j_1}^+ \prod_{b=1}^{m_1} E_{j_b}^- (E_{i_1}^-)^{n_1}_{|\lambda^{1\prime}} \prod_{b=1}^{m_2} E_{j_b}^- (E_{i_2}^-)^{n_2}_{|\lambda^{2\prime}} \cdots \prod_{b=1}^{m_i} E_{j_b}^- (E_{i_i}^-)^{n_i}_{|\lambda^i} \prod_{b=1}^{m_{i+1}} E_{j_b}^- |\lambda \rangle.$$ 

$$= \langle \cdots E_{j_2}^+ \prod_{b=2}^{m_1} E_{j_b}^- (E_{i_1}^-)^{n_1}_{|\lambda^{1\prime}} \prod_{b=1}^{m_2} E_{j_b}^- (E_{i_2}^-)^{n_2}_{|\lambda^{2\prime}} \cdots \prod_{b=1}^{m_i} E_{j_b}^- (E_{i_i}^-)^{n_i}_{|\lambda^i} \prod_{b=1}^{m_{i+1}} E_{j_b}^- |\lambda \rangle.$$ 

$$= \langle \cdots (E_{n_1}^+)^{n_1-1} E_{n_1}^+ (E_{i_1}^-)^{n_1}_{|\lambda^{1\prime}} \prod_{b=1}^{m_2} E_{j_b}^- (E_{i_2}^-)^{n_2}_{|\lambda^{2\prime}} \cdots \prod_{b=1}^{m_i} E_{j_b}^- (E_{i_i}^-)^{n_i}_{|\lambda^i} \prod_{b=1}^{m_{i+1}} E_{j_b}^- |\lambda \rangle.$$ 

$$= \langle \cdots (E_{n_1}^+)^{n_1-1} (E_{i_1}^-)^{n_1}_{|\lambda^{1\prime}} \prod_{b=1}^{m_2} E_{j_b}^- (E_{i_2}^-)^{n_2}_{|\lambda^{2\prime}} \cdots \prod_{b=1}^{m_i} E_{j_b}^- (E_{i_i}^-)^{n_i}_{|\lambda^i} \prod_{b=1}^{m_{i+1}} E_{j_b}^- |\lambda \rangle.$$ 

$$= \langle \cdots (E_{n_2}^+)^{n_2} \prod_{b=1}^{m_2} E_{j_b}^+ (E_{i_1}^-)^{n_2}_{|\lambda^2} \prod_{b=1}^{m_2} E_{j_b}^- (E_{i_2}^-)^{n_2}_{|\lambda^{2\prime}} \cdots \prod_{b=1}^{m_i} E_{j_b}^- (E_{i_i}^-)^{n_i}_{|\lambda^i} \prod_{b=1}^{m_{i+1}} E_{j_b}^- |\lambda \rangle.$$ 

$$= \cdots$$

$$= \prod_{k=1}^{l} (\Lambda_{i_k}^{k})^2.$$ 

Similarly, we can get the norm of $|\lambda'\rangle$. 

**Example 3.** The states corresponding to the weights in Fig.12 belong to states in this proposition.

We have a generalization of formula (4.5). The state is

$$|\lambda\rangle = (E_{i_0}^-)^{n_0}_{|\lambda^0} \prod_{b=1}^{m_1} E_{j_b}^- (E_{i_1}^-)^{n_1}_{|\lambda^{1\prime}} \prod_{b=1}^{m_2} E_{j_b}^- (E_{i_2}^-)^{n_2}_{|\lambda^{2\prime}} \cdots \prod_{b=1}^{m_i} E_{j_b}^- (E_{i_i}^-)^{n_i}_{|\lambda^i} \prod_{b=1}^{m_{i+1}} E_{j_b}^- |\lambda \rangle,$$
where \( n_0 \neq \Lambda_{i_0}^0 \). The norm of this state is

\[
\langle \lambda'' | \lambda'' \rangle = \langle \cdots (E_{i_0}^+)^{n_0} (E_{i_0}^-)^{n_0} | \Lambda^0 \rangle \prod_{b=1}^{m_1} E_{j_b}^- (E_{i_1}^-)^{n_1} | \Lambda^1 \rangle \prod_{b=1}^{m_2} E_{j_b}^- (E_{i_2}^-)^{n_2} | \Lambda^2 \rangle \cdots \prod_{b=1}^{m_l} E_{j_b}^- (E_{i_l}^-)^{n_l} | \Lambda^l \rangle \prod_{b=1}^{m_{l+1}} E_{j_b}^- | \Lambda \rangle
\]

\[
= \langle \cdots (E_{i_0}^+)^{n_0-1} (E_{i_0}^-)^{n_0-1} | \Lambda^0 \rangle \prod_{b=1}^{m_1} E_{j_b}^- (E_{i_1}^-)^{n_1} | \Lambda^1 \rangle \prod_{b=1}^{m_2} E_{j_b}^- (E_{i_2}^-)^{n_2} | \Lambda^2 \rangle \cdots \prod_{b=1}^{m_l} E_{j_b}^- (E_{i_l}^-)^{n_l} | \Lambda^l \rangle \prod_{b=1}^{m_{l+1}} E_{j_b}^- | \Lambda \rangle
\]

\[
= (\Lambda_{i_0}^0 - (n_0 - 1)) n_0
\]

\[
= \prod_{k=1}^{n_0} (\Lambda_{i_0}^0 - k + 1) k \langle \lambda | \lambda \rangle
\]

\[
= \prod_{k=1}^{n_0} (\Lambda_{i_0}^0 - k + 1) k \prod_{k=1}^{l} (\Lambda_{i_k}^k) \sqrt{2}.
\]

4.2. Minuscule Representations. For minuscule representations, the fundamental weight is the highest weight. Table II presents a complete list of minuscule fundamental weights for simple Lie algebras \([18]\), which are the highest-weight vectors. For the minuscule representations, all the strings in the weight spaces are two terms long,

\[
\langle \lambda, \alpha^\vee \rangle = 2 \frac{\langle \lambda, \alpha \rangle}{(\alpha, \alpha)} \leq 1.
\]

In \([19]\), we conjecture a formula of the factors in the braces of formula (4.11) for minuscule representations. Calculating the inner product of states in these representations would help prove this conjecture. The weights in the fundamental representation \( \rho_2 \) of \( A_4 \) are displayed as follows.

\[
(0, 1, 0, 0) \quad \downarrow \alpha_2
\]

\[
(1, -1, 1, 0) \xmapsto{\alpha_1} (0, 1, 0, 0) \quad \downarrow \alpha_3
\]

\[
(0, -1, 1, 1) \xmapsto{\alpha_1} (0, -1, 0, 1) \quad \downarrow \alpha_2
\]

\[
(1, 0, 0, -1) \xmapsto{\alpha_3} (0, 1, 0, -1) \quad \downarrow \alpha_4
\]

Since there are only two elements in the string of weights \((2.7)\) for minuscule representations, we have \( n_k = 1, (k = 1, \cdots, l) \) in formula \((1.5)\). Thus, the norms of the states \(| \lambda'' \rangle \) is

\[
\langle \lambda'' | \lambda'' \rangle = 1.
\]

Furthermore, the inner product of weights in these representations can be completely determined. The proof of the following theorem show the subtleties of Algorithm II for calculating inner products.

**Theorem 6.** For minuscule representations, the inner product of states defined by paths ending with the same weight in the weight diagram is one.
Proof. The states \(|v^{n,i}_w\rangle\) and \(|v^{n,j}_w\rangle\) corresponding to weight \(w\) with level \(n\) are given by
\[
|v^{n,i}_w\rangle = |E_{i_1}^- \cdots E_{i_l}^-|\Lambda\rangle, \quad |v^{n,j}_w\rangle = |E_{j_1}^- \cdots E_{j_l}^-|\Lambda\rangle.
\]
To prove the theorem, we need to consider two cases as shown in Fig. (2)(a) and (b). For the first case, the inner product of \(|v^{n,i}_w\rangle\) and \(|v^{n,j}_w\rangle\) is
\[
\langle v^{n,j}_w|v^{n,i}_w\rangle = \langle \Lambda|E_{i_1}^+ \cdots E_{i_l}^+|E_{j_1}^- \cdots E_{j_l}^-|\Lambda\rangle
= \langle \Lambda|E_{i_1}^+ \cdots E_{i_{l-2}}^+ (E_{j_1}^- E_{i_{l-1}}^- E_{j_2}^- E_{i_{l-2}}^- E_{j_3}^- \cdots E_{j_1}^- |\Lambda\rangle
= \langle \Lambda|E_{i_1}^+ \cdots E_{i_2}^+ E_{j_2}^+ E_{j_1}^- H_{i_3} E_{i_2}^- E_{j_3}^- E_{j_1}^- \cdots E_{j_{l-2}}^- E_{j_{l-1}}^- |\Lambda\rangle
= \langle \Lambda|E_{i_1}^+ \cdots E_{i_2}^+ E_{j_2}^- E_{j_1}^- \cdots E_{j_{l-2}}^- E_{j_{l-1}}^- |\Lambda\rangle
= 1. \tag{4.7}
\]
For the second case, the inner product is
\[
\langle v^{n,j}_w|v^{n,i}_w\rangle = \langle \Lambda|E_{i_1}^+ \cdots E_{i_{l-1}}^+ E_{i_{l-1}}^- E_{j_1}^- \cdots E_{j_{l-1}}^- |\Lambda\rangle
= \langle \Lambda|E_{i_1}^+ \cdots E_{i_{l-1}}^+ (E_{i_{l-1}}^- E_{i_{l-1}}^-) E_{j_1}^- \cdots E_{j_{l-1}}^- |\Lambda\rangle
= \langle \Lambda|E_{i_1}^+ \cdots E_{i_{l-1}}^- (H_{i_{l-1}} E_{i_{l-1}}^-) E_{j_1}^- \cdots E_{j_{l-1}}^- |\Lambda\rangle
= \langle \Lambda|E_{i_1}^+ \cdots E_{i_{l-1}}^- E_{j_{l-1}}^- \cdots E_{j_{l-1}}^- |\Lambda\rangle
= 1. \tag{4.8}
\]
We draw the conclusion. \(\square\)

In [19], another proof of Theorem 6 is provided by induction on the level of weights.

4.3. Affine Lie Algebras. The algorithm described in [11] for calculating the inner product is applicable for affine Lie algebra \(\hat{g}\). The Dynkin diagram of \(\hat{g}\) is obtained from that of \(g\) by adding an extra node representing the extra simple root \(\alpha_0\). Given a set of affine simple roots and a scalar product [20], the extended Cartan matrix of affine Lie algebras is defined by
\[
\hat{A}_{ji} = (\alpha_i, \alpha_j^\vee) \quad 0 \leq i, j \leq r.
\]
The matrix \( \hat{A} \) encodes the whole structure of \( \hat{\mathfrak{g}} \). In the Chevally basis, the communication relations for the generators associated with the simple roots of \( \hat{\mathfrak{g}} \) can be written as

\[
[E_i^+, E_j^-] = \delta_{ji} H_j, \quad [H_i, E_j^\pm] = \pm \hat{A}_{ji} E_j^\pm, \quad [H_i, H_j] = 0, \quad (4.9)
\]

where now \( i, j = 0, 1, \cdots, r \). The communication relations are the same as those in formula (2.1), with the exception of the operators corresponding to \( \alpha_0 \). However, this formulation does not explicitly reveal the infinite-dimensional nature of \( \hat{\mathfrak{g}} \).

The procedure that lists the weights in irreducible highest-weight representations of \( \mathfrak{g} \) also works for \( \hat{\mathfrak{g}} \). We simply have to keep track of an additional Dynkin label. However, this algorithm does not terminate in the affine case.

Since the communication relation for the generators and the construction of states are the same as those for the semisimple Lie algebra \( \mathfrak{g} \), Algorithm 1 for calculating the inner product of states in the highest-weight representations is applicable to the affine Lie algebra \( \hat{\mathfrak{g}} \).

4.4. Kapustin-Witten Equations. We begin by reviewing the Kapustin-Witten equations. An extensive introduction to this topic can be found in [1].

The localization equations of the twisted \( N = 4 \) super Yang-Mills theory can be applied to the description of the Khovanov homology of knots [3, 4, 5]. On a half space \( V = \mathbb{R}^3 \times \mathbb{R}_+ \), the supersymmetry conditions lead to the Kapustin-Witten equations. As described in [4], the \( \text{KW} \) equations are

\[
F - \phi \wedge \phi + *d_A \phi = 0 = d_A * \phi, \quad (4.10)
\]

where \( d_A \) is the covariant exterior derivative associated with a connection \( A \), and \( \phi \) is one-form valued in the adjoint of the gauge group \( G \). There is a Lie product understood in the \( \phi \wedge \phi \) terms and * denotes the Hodge duality. Different reductions of the equations lead to other well known equations *e.g.*, Nahm’s equations, Bogomolny equations or Hitchin equations.

The solutions of the model were studied in [4] [6] [7]. As we all know, all kinds of Toda systems were studied a lot recently in [8] and the Toda systems have very nice structures from the point of solutions. For any simple compact gauge group, after reducing to a Toda system, in [1], V. Mikhaylov conjectured a formula of the solutions of the model for the boundary ‘t Hooft operator. The ‘t Hooft operator corresponds to a cocharacter \( \omega \in \Gamma_{ch}^\vee \). Let \( \Delta \) be the set of simple roots \( \alpha_i \), and then \( \alpha_i(\hat{\omega}) = m_i \) with \( \hat{\omega} = \omega + \delta^\vee \), \( \delta^\vee \in \mathfrak{b} \) is the dual of the Weyl vector in the sense that \( \alpha_i(\delta^\vee) = 1 \). \( E_\alpha \) are the raising generators corresponding to the simple roots, and then the explicit fields on the solution are

\[
\phi_0 = -\frac{i}{2\rho} \partial_\sigma \chi(\sigma),
\]

\[
\varphi = \frac{1}{r} \sum_{\alpha \in \Delta} \exp \left[ \alpha(i\omega + \frac{1}{2}\chi(\sigma)) \right] E_\alpha,
\]

\[
A = -i \left( \hat{\omega} + \frac{y}{2\sqrt{y^2 + r^2}} \partial_\sigma \chi(\sigma) \right) d\theta,
\]
where $\chi(\sigma) = \sum \chi_i(\sigma) H_i$ with coroots $H_i$. The functions $\chi_i(\sigma)$ are conjectured as follows,

$$e^{-\chi_s(\sigma)} = 2^{-B_s} \sum_{w \in \Delta_s} \left[ \exp \left( 2\sigma w(\hat{\omega}) \right) \left< v_w(\hat{\omega})\mid v_w(\hat{\omega}) \right> (-1)^{n(w)} \prod_{\beta_a \in \Delta_+} \left( \beta_a(\hat{\omega}) \right)^{-2\left< w, \beta_a \right>/\left< \beta_a, \beta_a \right>} \right],$$

where $B_s = 2 \sum_j A_{s,j}^{-1}$ with Cartan matrix $A_{i,j}$. For a weight $w = \Lambda_s - \sum_{i=1}^{n(w)} \alpha_{ji}$, $\alpha_{ji} \in \Delta$ in the fundamental representation $\rho_s$, the vector $\mid v_w(\hat{\omega}) \rangle$ is defined as follows

$$\mid v_w(\hat{\omega}) \rangle = \sum_{s} \prod_{a=1}^{n(w)} \frac{1}{w(\hat{\omega}) - w_a(\hat{\omega})} E_{j_{n(w)}}^- \cdots E_{j_1}^- |\Lambda_s \rangle,$$

where $s$ enumerate ways in which the weight $w$ can be reached from the highest-weight, i.e. each $s$ corresponds to the following sequences

$$E_{j_{n(w)}}^- \cdots E_{j_1}^- |\Lambda_s \rangle.$$

In order to prove this conjecture, we need to check the following boundary condition

$$e^{-\chi_s(\sigma)} \mid_{\sigma \to 0} = 0.$$

For other related work on these equations, see [9][10][11][12][13].

To check the boundary condition (4.12), we have to compute the following inner product

$$W_w = \langle v_w(\hat{\omega})\mid v_w(\hat{\omega}) \rangle,$$

which involves the inner products

$$\langle v^i_w \mid v^j_w \rangle = \langle \Lambda_s \mid E_{i_{1}}^+ \cdots E_{i_{n}}^+ |E_{j_{n}}^- \cdots E_{j_1}^- \mid \Lambda_s \rangle,$$

where $E_{j_{n}}^- \cdots E_{j_1}^-$ and $E_{i_{n}}^+ \cdots E_{i_1}^+$ are two sequences from the highest-weight $\Lambda_s$ to weight $w$.

Algorithm [1] and the results in previous subsection would be helpful for the calculation of the inner product

$$\langle v^i_w \mid v^j_w \rangle = \langle \Lambda_s \mid E_{i_{1}}^+ \cdots E_{i_{n}}^+ |E_{j_{n}}^- \cdots E_{j_1}^- \mid \Lambda_s \rangle,$$

In [19], we give another form of the factors in the square bracket of formula (4.11) for certain weights of highest-weight representations of simply laced Lie algebras, which bypass the above computation difficulties.

Acknowledgments

Chuanzhong Li is supported by the National Natural Science Foundation of China under Grant No.12071237. This work was supported by a grant from the Postdoctoral Foundation of Zhejiang Province.
Funding and/or Conflicts of interests/Competing interests.

The authors do not have any possible conflicts of interest.

Data Availability Statement.

The data that support the findings of this study are available from the corresponding author, [BaoShou], upon reasonable request.

APPENDIX A. SECOND FUNDAMENTAL REPRESENTATION OF $G_2$

In this Appendix, using Algorithm 1, we calculate several inner products of states corresponding to the weights of the fundamental representation $(0, 1)$ of $G_2$ as shown in Fig. (3). There are four paths from the highest-weight vector to the lowest weight vector as shown in Figs. (4) and (5), corresponding to the four states $|v_{(0,1)}^a\rangle$, $|v_{(0,1)}^b\rangle$, $|v_{(0,1)}^c\rangle$, and $|v_{(0,1)}^d\rangle$.

Set the normalization of the highest-weight state to be one.

$$\langle v_{(0,1)}|v_{(0,1)}\rangle = \langle \Lambda_2|\Lambda_2 \rangle = 1.$$  

The inner product of state $|v_{(3,-1)}\rangle$ is 

$$\langle v_{(3,-1)}|v_{(3,-1)}\rangle = \langle \Lambda_2|E_2^+E_2^-|\Lambda_2 \rangle = 1.$$  

The inner product of state $|v_{(1,0)}\rangle$ is 

$$\langle v_{(1,0)}|v_{(1,0)}\rangle = \langle \Lambda_2|E_2^+E_1^+E_1^-E_2^-|\Lambda_2 \rangle = \langle v_{(3,-1)}|E_1^+E_1^-|v_{(3,-1)}\rangle = 3\langle v_{(3,-1)}|v_{(3,-1)}\rangle = 3.$$  

Figure 4. Paths $a$ and $b$ from the highest-weight state to the lowest weight state in the fundamental representation $(0, 1)$ of $G_2$. 

(a)  

(b)
The inner product of state \(|\psi_{(-1,1)}\rangle\) is
\[
\langle \psi_{(-1,1)} | \psi_{(-1,1)} \rangle = \langle \Lambda_2 | E_2^+ E_1^+ E_1^- E_1^- E_2^- | \Lambda_2 \rangle = \langle \psi_{(3,-1)} | E_1^+ 4E_1^- | \psi_{(3,-1)} \rangle = 4 \langle \psi_{(1,0)} | \psi_{(1,0)} \rangle = 12.
\]

The inner product of state \(|\psi_{(-3,2)}\rangle\) is
\[
\langle \psi_{(-3,2)} | \psi_{(-3,2)} \rangle = \langle \Lambda_2 | E_2^+ E_1^+ E_1^+ E_1^- E_1^- E_2^- | \Lambda_2 \rangle = \langle \psi_{(3,-1)} | E_1^+ 3E_1^- E_1^- | \psi_{(3,-1)} \rangle = 3 \langle \psi_{(-1,1)} | \psi_{(-1,1)} \rangle = 36.
\]

The inner product of state \(|\psi_{(2,-1)}\rangle\) is
\[
\langle \psi_{(2,-1)} | \psi_{(2,-1)} \rangle = \langle \Lambda_2 | E_2^+ E_1^+ E_1^+ E_2^- E_1^- E_2^- | \Lambda_2 \rangle = \langle \psi_{(-1,1)} | \psi_{(-1,1)} \rangle = 12.
\]

The inner product of state \(|\psi_{(0,0)}\rangle\) is
\[
\langle \psi_{(0,0)} \rangle | \psi_{(0,0)\rangle} \rangle = \langle \Lambda_2 | E_2^+ E_1^+ E_2^+ E_1^- E_2^- E_1^- E_2^- | \Lambda_2 \rangle = \langle \psi_{(2,-1)} | 2\psi_{(2,-1)} \rangle = 24.
\]
The inner product $\langle v^2_{(0,0)}|v^1_{(0,0)}\rangle$ is
\[
= \langle \Lambda_2|E^+_2 E^+_1 E^+_1 E^+_2 |E^-_2 E^-_1 E^-_1 E^-_2|\Lambda_2\rangle = \langle v_{(-3,2)}|v_{(-3,2)}\rangle = 36.
\]

The inner product of state $|v^2_{(0,0)}\rangle$ is
\[
\langle v^2_{(0,0)}|v^2_{(0,0)}\rangle = \langle \Lambda_2|E^+_2 E^+_1 E^+_1 E^+_2 |E^-_2 E^-_1 E^-_1 E^-_2|\Lambda_2\rangle = \langle v_{(-3,2)}|2v_{(-3,2)}\rangle = 72.
\]

The inner product $\langle v^1_{(0,0)}|v^2_{(0,0)}\rangle$ is
\[
= \langle \Lambda_2|E^+_2 E^+_1 E^+_1 E^+_2 |E^-_2 E^-_1 E^-_1 E^-_2|\Lambda_2\rangle = \langle \Lambda_2|E^+_2 E^+_1 E^+_1 E^+_2 |E^-_2 E^-_1 E^-_1 E^-_2|\Lambda_2\rangle = 3\langle v_{(2,-1)}|v_{(2,-1)}\rangle = 36.
\]

Next we compute the inner product $\langle v^a_{(0,-1)}|v^a_{(0,-1)}\rangle$ which is the most complicated one for the inner product in the representation
\[
\langle v^a_{(0,-1)}|v^a_{(0,-1)}\rangle = \langle \Lambda_2|E^+_2 E^+_1 E^+_1 E^+_2 |E^-_2 E^-_1 E^-_1 E^-_2|\Lambda_2\rangle = \langle \Lambda_2|E^+_2 E^+_1 E^+_1 E^+_2 |E^-_2 E^-_1 E^-_1 E^-_2|\Lambda_2\rangle = \langle \Lambda_2|E^+_2 E^+_1 E^+_1 E^+_2 |E^-_2 E^-_1 E^-_1 E^-_2|\Lambda_2\rangle = \langle \Lambda_2|E^+_2 E^+_1 E^+_1 E^+_2 |E^-_2 E^-_1 E^-_1 E^-_2|\Lambda_2\rangle = 0\langle v_{(-1,0)}|v_{(-1,0)}\rangle + 0\langle v_{(1,-1)}|v_{(1,-1)}\rangle + 3\langle v_{(0,0)}|v_{(0,0)}\rangle + 36\langle v_{(3,-1)}|v_{(3,-1)}\rangle.
\]

The first zero is the coefficient $(\Lambda^0 - (n_0 - 1))n_0 = (1 - (2 - 1))2 = 0$. And the second zero is the eigenvalue of the operators $H_1$. So the first two terms vanish and the last one is calculated as follows,
\[
\langle v^a_{(0,-1)}|v^a_{(0,-1)}\rangle = 3\langle \Lambda_2|E^+_2 E^+_1 E^+_1 E^+_2 |E^-_2 E^-_1 E^-_1 E^-_2|\Lambda_2\rangle = 3\langle \Lambda_2|E^+_2 E^+_1 E^+_1 E^+_2 |E^-_2 E^-_1 E^-_1 E^-_2|\Lambda_2\rangle = 12\langle \Lambda_2|E^+_2 E^+_1 E^+_1 E^+_2 |E^-_2 E^-_1 E^-_1 E^-_2|\Lambda_2\rangle = 30\langle \Lambda_2|E^+_2 E^+_1 E^+_1 E^+_2 |E^-_2 E^-_1 E^-_1 E^-_2|\Lambda_2\rangle = 12*36,
\]

where we have used formula (A.1).
Since the algorithm is an iterative process, we can use the results of the level $k$ when we do the calculations for the step for level $k + 1$. For each step, the coefficients are positive, which are consistent with Conjecture [1].
REFERENCES

[1] V. Mikhaylov, “On the Solutions of Generalized Bogomolny Equations,” [arXiv:1202.4848]
[2] A. Kapustin and E. Witten, “Electric-magnetic duality and the geometric Langlands program,” [arXiv:hep-th/0604151]
[3] E. Witten, “Analytic Continuation Of Chern-Simons Theory,” [arXiv:1001.2933]
[4] E. Witten, “Fivebranes and knots,” [arXiv:1101.3216]
[5] D. Gaiotto and E. Witten, “Knot Invariants from Four-Dimensional Gauge Theory,” [arXiv:1106.4789]
[6] M. Henningson, “Boundary conditions for GL-twisted N=4 SYM,” [arXiv:1106.3845]
[7] M. Henningson, “’t Hooft Operators in the Boundary,” [arXiv:1109.2393]
[8] C. Z. Li, J. S. He, “On the Extended Multi-component Toda Hierarchy”, Mathematical Physics Analysis and Geometry, 17(2014) 377-407.
[9] M. Gargiardo, K. Uhlenbeck, “Geometric aspects of the Kapustin-Witten equations”, J Fixed Point Theory Appl.11(2012) 185-198.
[10] R. Mazzeo, E. Witten, “The Nahm Pole Boundary Condition”, [arXiv:1311.3167]
[11] S. Q. He, “Rotationally Invariant Singular Solutions to the Kapustin-Witten Equations”, [arXiv:1510.07706]
[12] Y. Tanaka, “On the singular sets of solutions to the Kapustin-Witten equations on compact Kahler surfaces”, [arXiv:1510.07739]
[13] T. Huang, “A lower bound on the solutions of Kapustin-Witten equations,” [arXiv:1601.07986]
[14] R. N. Cahn, “Semisimple Lie algebras and their representations,” Benjamin-Cummings, 1984.
[15] D. Gaiotto and E. Witten, “Supersymmetric boundary conditions in N=4 super Yang-Mills theory,” J. Stat. Phys. 135 (2009) 789-855. [arXiv:0804.2902]
[16] B. Kostant, “The solution to a generalized Toda lattice and representation theory,” Adv. in Math. 34(1979), 3.
[17] P. Mansfield, “Solution Of Toda systems,” Nucl. Phys. B 208(1982), 277.
[18] P.M. Green, “Combinatorics of Minuscule Representations”, Cambridge Tracts in Mathematics, Cambridge University Press, 2013
[19] C.Z. Li, Z.S. Liu, B. Shou, ” Solutions to Kapustin-Witten equations for ADE type groups”, preprint, 30pp, [arXiv:1604.07172]
[20] P.D. Francesco, P. Mathieu, D. Senechal, “Conformal Field Theory,” Springer, 1996

1 College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, 266590, P.R. China

2 Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing, 100190, China

3 Center of Mathematical Sciences, Zhejiang University, Hangzhou, 310027, China

Email address: lichuanzhong@sdu.edu.cn, zsliu@itp.ac.cn, bsoul@zju.edu.cn