Motivic spheres and the image of Suslin’s Hurewicz map

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Abstract

We show that an old conjecture of A.A. Suslin characterizing the image of the “Hurewicz” map from Quillen K-theory in degree $n$ to Milnor K-theory in degree $n$ admits an interpretation in terms of unstable $\mathbb{A}^1$-homotopy sheaves of the general linear group. Using this identification, we establish Suslin’s conjecture in degree 5 for arbitrary fields having characteristic unequal to 2 or 3. We do this by linking the relevant unstable $\mathbb{A}^1$-homotopy sheaf of the general linear group to the stable $\mathbb{A}^1$-homotopy of motivic spheres.

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1 Introduction

The goal of this paper is to explore how concrete computations in $\mathbb{A}^1$-homotopy theory of ostensibly geometric origin have bearing on torsion phenomena in algebraic K-theory. More precisely, we investigate an old conjecture of Suslin, whose formulation we now recall. Suppose $F$ is an infinite field. By definition of the plus construction, the Hurewicz map induces a morphism

$$K^Q_n(F) = \pi_n (BGL(F)^+) \to H_n(BGL(F)^+) \to H_n(BGL(F));$$

here and henceforth, homology is taken with integer coefficients, which are suppressed from the notation. Suslin’s stabilization theorem [Sus84, Theorem 3.4] asserts:

(i) the stabilization maps $BGL_n(F) \to BGL_{n+1}(F)$ induce isomorphisms, functorially in $F$, of the form $H_i(BGL_n(F)) \to H_i(BGL_{n+1}(F))$ whenever $i \leq n$; in particular, there is an induced isomorphism $H_n(BGL_n(F)) \to H_n(BGL(F));$

∗Aravind Asok was partially supported by National Science Foundation Award DMS-1254892.
(ii) the cokernel of the stabilization map \(H_n(BGL_{n-1}(F)) \to H_n(BGL_n(F))\) coincides, functorially in \(F\) with \(K_n^M(F)\).

Putting all these facts together, one obtains a morphism

\[
K_n^Q(F) \longrightarrow H_n(BGL(F)) \longrightarrow H_n(BGL_n(F)) \longrightarrow K_n^M(F)
\]

which is functorial in \(F\). Abusing terminology slightly, we will refer to this map as Suslin’s Hurewicz map.

There is a natural graded ring homomorphism \(K_n^M(F) \to K_n^Q(F)\) induced by the identification \(K_1^M(F) = K_1^Q(F)\) and product maps in K-theory. Suslin showed that the composite of the natural homomorphism and the Hurewicz homomorphism

\[
K_n^M(F) \longrightarrow K_n^Q(F) \longrightarrow K_n^M(F)
\]

coincides with multiplication by \((-1)^n(n-1)!\). It follows that the image of Suslin’s Hurewicz map \(K_n^Q(F) \to K_n^M(F)\) contains \((n-1)!K_n^M(F)\). Suslin went on to make the following conjecture.

**Conjecture 1** (see [Sus84, p. 370, after Corollary 4.4]). For any infinite field \(F\), the image of Suslin’s Hurewicz map \(K_n^Q(F) \to K_n^M(F)\) coincides with \((n-1)!K_n^M(F)\).

Suslin refers to this conjecture as “very delicate”. As evidence for this assessment, he analyzed the first interesting case of this conjecture, i.e., \(n = 3\). In that case, he showed the conjecture was equivalent to the degree 3 case of the Milnor conjecture on quadratic forms, which was at that time still unproved, [Sus84, Proposition 4.5]. Since the degree 3 case of the Milnor conjecture on quadratic forms was established independently by Merkurjev–Suslin and Rost (see [JR89]), building on work of Arason [Ara75], we now know that Suslin’s conjecture holds in degree 3. (Alternatively, it follows from the results of Orlov–Vishik–Voevodsky [OVV07] which establish Milnor’s conjecture on quadratic forms in all degrees).

Suslin analyzed the \(n = 4\) case of the conjecture as well, but never published anything. In that case, the conjecture can be established affirmatively for various classes of fields, including number fields. Suslin then suggested an “absolute rigidity” conjecture for a certain motivic cohomology group (similar to [Dég08, Conjecture p. 242]) that would imply the conjecture for \(n = 4\) and arbitrary fields having characteristic 0. However, this absolute rigidity conjecture remains open. Furthermore, as far as we are aware, besides results for specific fields, nothing general is known about Suslin’s conjecture for larger values of \(n\).

The construction of Suslin’s Hurewicz map was generalized to local rings with infinite residue fields by Nesterenko–Suslin [NS89, §4] and Guin [Gui89, §4]. Moreover, the image of this more general Hurewicz map has the same properties as discussed above. While Suslin’s initial conjecture was proposed only for infinite fields, in light of these subsequent generalizations, it seems reasonable to replace the infinite field \(F\) by an arbitrary local ring with infinite residue field. In support of this generalization of Suslin’s conjecture, we offer the following result.

**Theorem 2** (See Theorem 3.3.12). If \(k\) is an infinite field having characteristic unequal to 2 or 3, then Suslin’s conjecture holds in degree 5 for any essentially smooth local \(k\)-algebra \(A\), i.e., Suslin’s Hurewicz map \(K_5^Q(A) \to K_5^M(A)\) has image precisely \(24K_5^M(A)\).
To establish this result, we connect Suslin’s Hurewicz homomorphism with another map $K_n^Q(F) \to K_n^M(F)$ that naturally appears in computations involving $\mathbb{A}^1$-homotopy sheaves of the general linear group defined in [AF14a]. The main goal of Section 2 is to establish this comparison. These results rely on the techniques of [AHW17, AHW18] and recent work of Schlichting [Sch17]. Granted the comparison result, we may then reformulate Suslin’s conjecture for general $n$ as a statement about the structure of $\mathbb{A}^1$-homotopy sheaves of $BGL_n$ (see Theorem 2.3.8 for a precise statement).

Once reformulated in terms of $\mathbb{A}^1$-homotopy sheaves, we establish Suslin’s conjecture in degree 5 by refining a key computation of [AWW17]. In particular, we establish the following result, which is one of the main results of Section 3.

**Theorem 3** (See Theorem 3.3.10). If $k$ is a field that has characteristic unequal to 2 or 3, then there is a short exact sequence of the form

$$0 \longrightarrow K^M_5/F \longrightarrow \pi^A_4(\mathbb{P}^3) \longrightarrow GW^3 \longrightarrow 0.$$ 

The key idea that permits this refinement is a comparison of unstable and stable computations of $\mathbb{A}^1$-homotopy sheaves and the proof of Theorem 3 relies on the recent computation of the first stable $\mathbb{A}^1$-homotopy sheaf of the motivic sphere spectrum by Röndigs–Spitzweck–Østvær [RSØ16]. In particular, we hope the technique of proof can be adapted to shed light on Suslin’s conjecture in general.

**Acknowledgements**

The first author would like to thank Sasha Merkurjev for explaining Suslin’s approach to his eponymous conjecture in degree 4; even though this makes no appearance here, it was still an essential input. The authors would also like to thank Marc Levine and the University of Duisberg-Essen where the initial idea of this paper was conceived and Kirsten Wickelgren for her collaboration in an early stage of this project. Finally, the authors would like to thank Oliver Röndigs for helpful discussions about [RSØ16], and for comments and corrections on a draft of this work.

**Preliminaries/Notation**

Throughout the paper, $k$ will denote a fixed base field. We write $\text{Sm}_k$ for the category of schemes that are separated, smooth and have finite type over $\text{Spec} k$. We write $\text{Spc}_k$ for the category of simplicial presheaves on $\text{Spec} k$; objects of this category will typically be written using a script font (e.g., $\mathcal{X}$). Our notation in Section 2 follows [AHW17, AHW18], and results from those papers will be used freely. For example, we will write $R_{\text{Zar}}$ for the Zariski fibrant replacement functor with respect to the injective Zariski local model structure on $\text{SpC}_k$, and $R_{\text{Nis}}$ for the corresponding construction in the Nisnevich local model structure. Our notation for $\mathbb{A}^1$-homotopy sheaves follows that of [AWW17] on which this paper builds.

## 2 Suslin’s Hurewicz homomorphism revisited

The goal of this section is to compare two homomorphisms from Quillen K-theory to Milnor K-theory: the first is Suslin’s Hurewicz homomorphism described in the introduction, and the second, constructed...
in [AF14a], arises naturally in motivic homotopy theory (it is related to a homomorphism defined by Suslin using Mennicke symbols). Using some ideas from $\mathbb{A}^1$-homotopy theory and some recent results of M. Schlichting, we will demonstrate that the two homomorphisms coincide.

In order to compare the two homomorphisms, we study homological stabilization results for spaces constructed out of general linear groups; Theorem 2.3.6 is related to the constructions of [HW15] who consider special linear groups, though our proof is somewhat different (in particular, it uses results of [AHW17, AHW18] in place of corresponding results from [Mor12]). The main result of this section is Theorem 2.3.8, which essentially shows that Suslin’s conjecture from the introduction, may be reformulated as providing a precise description of an $\mathbb{A}^1$-homotopy sheaf, building on the ideas of [AF14a].

### 2.1 The sheaf $S_n$ via homology of simplicial groups

In [AF14a], the first and second authors studied the $\mathbb{A}^1$-homotopy theory of $BGL_n$. Mirroring the situation in topology, there is a range in which these homotopy sheaves are “stable” and agree with those of the stable general linear group $BGL$, in which case they may be described in terms of algebraic K-theory. The first homotopy sheaf of $BGL_n$ lying outside of this stable range is that in degree $n$, and this sheaf is an extension of something stable by a sheaf $S_{n+1}$ which one might call the “non-stable” part of the computation. Our goal in this section is to reinterpret Suslin’s conjecture from the introduction as a statement about the structure of $S_{n+1}$. In order to do this, we recall the construction of $S_{n+1}$ in detail, and using the results of [AHW17, AHW18], we provide a “concrete” reinterpretation of the sections of this sheaf over local rings in terms of homology of simplicial groups.

#### Fiber sequences and homotopy sheaves

Suppose $k$ is a field, and $GL_n$ is the general linear $k$-group scheme. Consider the morphism of schemes $GL_{n-1} \to GL_n$ sending an invertible $n \times n$-matrix $M$ to the block matrix $\text{diag}(M,1)$. This morphism induces a map of simplicial classifying spaces $BGL_{n-1} \to BGL_n$ (thought of as simplicial presheaves on $\text{Sm}_k$) that we will refer to as the stabilization map.

For every integer $n \geq 1$ there is an $\mathbb{A}^1$-fiber sequence of the form

$$\mathbb{A}^n \setminus 0 \to BGL_{n-1} \to BGL_n$$

(we will refine and explain this fact in Proposition 2.1.3). Morel showed that $\mathbb{A}^n \setminus 0$ is $\mathbb{A}^1-(n-2)$-connected and that $\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) \cong K_{n}^{\text{MW}}$ where $K_{n}^{\text{MW}}$ is Morel’s unramified Milnor K-theory sheaf [Mor12, Corollary 6.39].

By representability of algebraic K-theory in the $\mathbb{A}^1$-homotopy category, one may show that $\pi_i^{\mathbb{A}^1}(BGL_n) \cong K_i^Q$ for $1 \leq i \leq n-1$, where $K_i^Q$ is the sheafification of the Quillen K-theory presheaf on $\text{Sm}_k$ for the Nisnevich topology. Stringing the associated long exact sequences in $\mathbb{A}^1$-homotopy sheaves together for different values of $n$, one obtains a composite morphism

$$K_{n+1}^{\text{MW}} = \pi_{n}^{\mathbb{A}^1}(\mathbb{A}^{n+1} \setminus 0) \to \pi_{n}^{\mathbb{A}^1}(BGL_n) \to \pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) = K_{n}^{\text{MW}}.$$ 

This composite map $K_{n+1}^{\text{MW}} \to K_{n}^{\text{MW}}$ is multiplication by $\eta$ if $n$ is even and 0 if $n$ is odd by [AF14a, Lemma 3.5]. Using the quotient map $K_{n}^{\text{MW}} \to K_n^M$ (the latter is an unramified Milnor K-theory sheaf),
which corresponds to forming the quotient by the subsheaf of $\eta$-divisible elements, one then obtains a commutative diagram of the form

\[
\begin{array}{ccc}
K_{n+1}^{MW} & \longrightarrow & K_n^{MW} \\
\downarrow & & \downarrow \\
\pi_{n-1}^{A^1}(BGL_n) & \longrightarrow & \pi_{n-1}^{A^1}(BGL_{n-1}) \\
\downarrow & & \downarrow \\
\pi_{n}^{A^1}(BGL_{n+1}) & \longrightarrow & K_n^M;
\end{array}
\]

where the dotted morphism exists for any $n \geq 2$. Since $\pi_{n}^{A^1}(BGL_{n+1}) = K_n^Q$, we conclude that

$$\psi_n : K_n^Q \longrightarrow K_n^M,$$

and we repeat [AF14a, Definition 3.6].

**Definition 2.1.1.** For any $n \geq 2$, define $S_n := \text{coker}(\psi_n)$.

The next result summarizes key properties of the sheaf $S_n$.

**Proposition 2.1.2.** Suppose $n \geq 2$ is an integer.

1. There is a canonical morphism $\mu_n : K_n^M \rightarrow K_n^Q$ extending the map induced by the isomorphism $K_1^M = K_1^Q$ and the product maps in K-theory.
2. The composite map $\psi_n \circ \mu_n$ is multiplication by $(n - 1)!$.
3. The canonical epimorphism $K_n^M \rightarrow K_n^M / 2$ factors through an epimorphism $S_n \rightarrow K_n^M / 2$.
4. The epimorphism $K_n^M \rightarrow S_n$ factors through an epimorphism $K_n^M / (n - 1)! \rightarrow S_n$.

**Proof.** The first statement is [AF14a, Lemma 3.7], and the latter two statements follow from [AF14a, Corollary 3.11] and its proof.

**Homotopy of the singular construction**

We now appeal to the results of [AHW17, AHW18] to recast the above results in terms of homotopy of classifying spaces of simplicial groups. Suppose now that $\mathcal{X}$ is a simplicial presheaf on $\text{Sm}_k$. Write $\Delta_k^n$ for the cosimplicial affine $k$-simplex, i.e., the cosimplicial object defined by $n \mapsto \text{Spec} k[x_0, \ldots, x_n]/(\sum_i x_i - 1)$ equipped with the usual coface and codegeneracy maps (see, e.g., [MV99, p. 88]). In that case, one defines the singular construction on $\mathcal{X}$ as the diagonal of a bisimplicial object:

$$\text{Sing}_{A^1} \mathcal{X} := \text{diag}(\text{Hom}(\Delta_{\bullet}, \mathcal{X})),$$

where $\text{Hom}$ is the internal hom in the category of simplicial presheaves.

A list of properties of the singular construction is provided on [MV99, p. 87]. The map $\mathcal{X} \rightarrow \text{Sing}_{A^1} \mathcal{X}$ is a monomorphism and $A^1$-weak equivalence, and $\text{Sing}_{A^1}$ commutes with the formation of finite limits (in particular, finite products). In the situations of interest to us, the simplicial presheaf $\text{Sing}_{A^1} \mathcal{X}$ is already $A^1$-local; the next result summarizes the facts we will need.
Proposition 2.1.3. Suppose $n \geq 1$ is an integer and $k$ is a field.

1. There is a Nisnevich local fiber sequence of the form

$$\text{Sing}^A_1(\mathbb{A}^n \setminus 0) \longrightarrow \text{Sing}^A_1BGL_{n-1} \longrightarrow \text{Sing}^A_1BGL_n.$$ 

2. For any smooth affine $k$-scheme $U$, the maps $\text{Sing}^A_1(\mathbb{A}^n \setminus 0)(U) \longrightarrow \text{R}_\text{Zar} \text{Sing}^A_1(\mathbb{A}^n \setminus 0)(U)$ and $\text{Sing}^A_1BGL_n(U) \longrightarrow \text{R}_\text{Zar} \text{Sing}^A_1BGL_n(U)$ are weak equivalences.

3. The spaces $\text{R}_\text{Zar} \text{Sing}^A_1(\mathbb{A}^n \setminus 0)$ and $\text{R}_\text{Zar} \text{Sing}^A_1BGL_n$ are Nisnevich local and $\mathbb{A}^1$-invariant.

Proof. The assumption $k$ is a field is only made for simplicity and can be weakened. Observe that there is a simplicial fiber sequence

$$GL_n/GL_{n-1} \longrightarrow BGL_{n-1} \longrightarrow BGL_n$$

essentially by definition (see [AHW18, §2.3] and Lemma 2.3.1). Applying $\text{Sing}^A_1$ to each term here, in light of [AHW17, Theorem 5.2.1], we conclude that there is a simplicial fiber sequence of the form

$$\text{Sing}^A_1GL_n/GL_{n-1} \longrightarrow \text{Sing}^A_1BGL_{n-1} \longrightarrow \text{Sing}^A_1BGL_n$$

by appeal to [AHW18, Proposition 2.1.1]. The “projection onto the first column” map $GL_n/GL_{n-1} \rightarrow \mathbb{A}^n \setminus 0$ is a Zariski locally trivial morphism with affine space fibers and thus by [AHW18, Lemma 4.2.4] the induced map $\text{Sing}^A_1GL_n/GL_{n-1} \rightarrow \text{Sing}^A_1(\mathbb{A}^n \setminus 0)$ is a weak equivalence after evaluation on affine schemes and the first result follows. The second and third statements are then contained in [AHW17, Theorem 5.1.3] and [AHW18, Theorem 2.3.2].

Corollary 2.1.4. Suppose $k$ is a field, and $A$ is the local ring of a smooth $k$-scheme $X$ at a point. The following statements hold:

1. $\pi_i(\text{Sing}^A_1(\mathbb{A}^n \setminus 0)(A)) = \begin{cases} 0 & \text{if } 1 \leq i \leq n-2, \\ \mathbb{K}^\text{MW}_n(A) & \text{if } i = n-1. \end{cases}$

2. $\pi_i(\text{Sing}^A_1BGL_n(A)) = \mathbb{K}^Q_i(A)$ if $1 \leq i \leq n-1$.

Proof. We know $\text{R}_\text{Zar} \text{Sing}^A_1(\mathbb{A}^n \setminus 0)$ is Nisnevich local and $\mathbb{A}^1$-invariant. In particular, $\pi^A_i(\mathbb{A}^n \setminus 0) = a_{\text{Nis}}\pi_i(\text{R}_\text{Zar} \text{Sing}^A_1(\mathbb{A}^n \setminus 0))$. By [Mor12, Chapter 6], we know that for any $i > 0$, the map $a_{\text{Zar}}\pi_i(\text{R}_\text{Zar} \text{Sing}^A_1(\mathbb{A}^n \setminus 0)) \rightarrow a_{\text{Nis}}\pi_i(\text{R}_\text{Zar} \text{Sing}^A_1(\mathbb{A}^n \setminus 0))$ is an isomorphism, i.e., the Zariski sheafification of the presheaf of homotopy groups is already a Nisnevich sheaf. Thus, for $A$ as in the statement, we see that

$$\pi^A_i(\mathbb{A}^n \setminus 0)(A) = \pi_i(\text{R}_\text{Zar} \text{Sing}^A_1(\mathbb{A}^n \setminus 0))(A) = \pi_i(\text{R}_\text{Zar} \text{Sing}^A_1(\mathbb{A}^n \setminus 0)(A)),$$

where the last equality follows essentially from the definition of $\text{R}_\text{Zar}$, i.e., from the fact that $\text{R}_\text{Zar} \text{Sing}^A_1(\mathbb{A}^n \setminus 0)$ has Zariski stalks that are fibrant simplicial sets.

On the other hand, for any smooth affine $k$-scheme $U$, the map $\text{Sing}^A_1(\mathbb{A}^n \setminus 0)(U) \rightarrow \text{R}_\text{Zar} \text{Sing}^A_1(\mathbb{A}^n \setminus 0)(U)$ is a weak equivalence, it follows that the same holds for $U = \text{Spec } A$. The result then follows from [Mor12, Corollary 6.39] as this computes the sheaf $\pi^A_i(\mathbb{A}^n \setminus 0)$ in the relevant cases.
2.2 Homological stability and Milnor K-theory

The second statement is deduced in a similar fashion. Using the connectivity statement for $\text{Sing}^A(A^{n+1} \backslash 0)/(S)$ mentioned above, the stabilization map $R_{\text{Zar}} \text{Sing}^A BGL_n \to R_{\text{Zar}} \text{Sing}^A BGL$ is an $(n - 1)$-equivalence upon evaluation at sections for $A$ as in the statement. The latter space represents algebraic K-theory by [Mor12, §4 Theorem 3.13] (though see [ST15, Theorem 4 and Remark 2 p. 1162] for some mild corrections and to establish the statement in the generality we need).

**Remark 2.1.5.** It follows immediately from Proposition 2.1.3(ii) and the argument in the beginning of Corollary 2.1.4 that the Zariski sheafification of $U \mapsto \pi_i(\text{Sing}^A BGL_n(U))$ is already a Nisnevich sheaf. For example, the Zariski sheaves $K^Q_n$ and $K^M_n$ described in the previous section are already Nisnevich sheaves. In addition, it follows that if $A$ is the local ring of a smooth $k$-scheme at a point, then $\pi_n(A)$ coincides with $\pi_n(\text{Sing}^A BGL_n(A))$; we will use this identification freely in the sequel; this provides the first link between $S_{n+1}$ and homotopy of $\text{Sing}^A BGL_n(A)$.

2.2 Homological stability and Milnor K-theory

Suslin’s conjecture is formulated in terms of homology of the classifying spaces of the discrete groups $GL_n(F)$ and results about homological stabilization for these groups. In the previous section, we saw that the homotopy of certain simplicial groups appeared naturally. Building on the homotopical results of the previous section, we proceed to analyze relative homology of the map $\text{Sing}^A BGL_n \to \text{Sing}^A BGL_{n+1}$. These results are natural simplicial counterparts of the results of Nesterenko–Suslin [NS89], which we quickly review. In particular, we establish Lemma 2.2.2, which is a preliminarily homological stabilization result, and Lemma 2.2.3 which yields an analog of Suslin’s morphism $H_n(BGL_n(F)) \to K^M_n(F)$ in the context of homology of $\text{Sing}^A BGL_n$.

**Review of some results of Nesterenko–Suslin**

We now recall some results of Suslin as extended by Nesterenko–Suslin/Guin. The maps $GL_m \times GL_n \to GL_{m+n}$ given by block sum:

$$(X_1, X_2) \mapsto \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$$

induce maps of classifying spaces $BGL_m \times BGL_n \cong B(GL_m \times GL_n) \to BGL_{m+n}$. For any commutative unital ring $A$, these maps induce external product maps $H_m(BGL_m(A)) \otimes H_n(BGL_n(A)) \to H_{m+n}(BGL_{m+n}(A))$ (if coefficients are not indicated, homology will always be taken with coefficients in $\mathbb{Z}$), which are studied in [Sus84, §] or [NS89, §3]. These external products map equip $\bigoplus_{n \geq 0} H_n(BGL_n(A))$ with the structure of a ring.

A result of Suslin [Sus84, Corollary 2.7.2], generalized by Nesterenko–Suslin/Guin, shows that that for any local ring $A$ with infinite residue field, the exterior product map $H_1(BGL_1(A))^{\times n} \to H_n(BGL_n(A))$

factors through a map

$$\theta : K^M_n(A) \to H_n(BGL_n(A))/H_n(BGL_{n-1}(A)).$$
Suslin constructed an explicit splitting of this map \cite[Corollaries 2.4.1 and 2.7.4]{Suslin} and concluded in \cite[Theorem 3.4]{Suslin} that $\theta$ is an isomorphism. This result was generalized in \cite[Theorem 3.25]{NS89}.

**Definition 2.2.1.** The map $s_n$, defined as the composite

$$H_n(BGL_n(A)) \to H_n(BGL_n(A))/H_n(BGL_{n-1}(A)) \xrightarrow{\theta^{-1}} K^M_n(A),$$

will be called **Suslin’s morphism**.

The maps $s_n$ just described can be put together to yield:

$$\bigoplus_{n \geq 0} s_n : \bigoplus_{n \geq 0} H_n(BGL_n(A)) \to \bigoplus_{n \geq 0} K^M_n(A).$$

By appeal to \cite[Lemma 2.6.1]{Suslin} and \cite[Lemma 3.23]{NS89}, the direct sum on the left hand side has naturally the structure of a graded ring. In fact, this ring is graded commutative (essentially because the direct sum operation on vector spaces is a symmetric monoidal structure). It is well-known that the Milnor K-theory ring is graded commutative as well. By \cite[Corollary 3.28]{NS89}, $\bigoplus_{n \geq 0} s_n$ is in fact a homomorphism of graded rings.

**Weak homological stability**

We now analyze an analog of Suslin’s homomorphism after applying the singular construction.

**Lemma 2.2.2.** Suppose $n \geq 1$ is an integer, $k$ is a field and $A$ is an essentially smooth local $k$-algebra. The morphisms:

$$H_i(\text{Sing}^A_1 BGL_{n}(A)) \to H_i(\text{Sing}^A_1 BGL_{n+1}(A))$$

are isomorphisms for $i \leq n - 1$ and split surjective for $i = n$.

**Proof.** For $n$ as in the statement, the fact that the map in question is an isomorphism for $i \leq n - 1$ and surjective for $i = n$ follows by combining the relative Hurewicz theorem and the connectivity estimate for $\text{Sing}^A_1(\mathbb{A}^{n+1} \setminus 0)(A)$ in Corollary 2.1.4.

To construct the splitting, we proceed as follows. Consider the following diagram

$$\begin{array}{ccccccc}
H_n(BGL_n(A)) & \to & H_n(BGL_{n+1}(A)) & \to & \cdots & \to & H_n(BGL(A)) \\
\downarrow & & \downarrow & & & & \downarrow \\
H_n(\text{Sing}^A_1 BGL_n(A)) & \to & H_n(\text{Sing}^A_1 BGL_{n+1}(A)) & \to & \cdots & \to & H_n(\text{Sing}^A_1 BGL(A))
\end{array}$$

where the horizontal maps are the stabilization maps and the vertical maps are induced by the map from a simplicial pre-sheaf to its singular construction.

The maps in the first row may be analyzed by appeal to \cite{Suslin} and \cite{NS89} or \cite{Gui89}. In particular, they prove (see \cite[Theorem 3.4(c)]{Suslin} and \cite[Theorem 3.25]{NS89} \cite[Théorème 1]{Gui89}) that for any local ring $A$ with infinite residue field, the map $H_n(BGL_n(A)) \to H_n(BGL_{n+j}(A))$ is an isomorphism for any integer $j \geq 1$. Thus, all the maps in the top row are isomorphisms.
We claim that the map $BGL(A) \to \text{Sing}^k BGL(A)$ is actually a homology isomorphism. To see this, recall that $\text{Sing}^k BGL(A)$ represents Karoubi-Villamayor K-theory of $A$. The homotopy groups of the space $\text{Sing}^k BGL(A)$ are precisely the Karoubi-Villamayor K-theory groups [Wei13, Definition 11.4]. Since Karoubi-Villamayor K-theory coincides with Quillen K-theory for regular rings [Wei13, Corollary 12.3.2], by appeal to the $+ = Q$-theorem [Wei13, Corollary 7.2], we conclude that $\text{Sing}^k BGL(A)$ can be taken as a model for the $+$ construction of $BGL(A)$ and the claim follows.

Combining these observations, we conclude that the composite map

$$H_n(BGL_n(A)) \to H_n(BGL(A)) \to H_n(\text{Sing}^k BGL(A))$$

is an isomorphism. On the other hand, using the homology stabilization results for the singular construction we mentioned at the beginning of this proof, we conclude that $H_n(\text{Sing}^k BGL_{n+1}(A)) \to H_n(\text{Sing}^k BGL(A))$ is an isomorphism. Therefore, the composite map

$$H_n(BGL_n(A)) \to H_n(BGL_{n+1}(A)) \to H_n(\text{Sing}^k BGL_{n+1}(A))$$

is also an isomorphism. By commutativity of the left-most square, we conclude that the inclusion map $H_n(BGL_n) \to H_n(\text{Sing}^k BGL_n)$ is injective and provides a splitting of the stabilization map as claimed.

**Milnor K-theory and the relative Hurewicz theorem**

There is another way to link Milnor K-theory and the stabilization map $\text{Sing}^k BGL_{n-1} \to \text{Sing}^k BGL_n$ via the relative Hurewicz theorem, which we now describe. Again, assume $k$ is a field, and begin by observing that, if $A$ is an essentially smooth local $k$-algebra, then $\pi_1(\text{Sing}^k BGL_n(A)) = G_n(A)$ for any integer $n \geq 1$; this follows, e.g., from the second point of Corollary 2.1.4. There is an action of $\pi_1(\text{Sing}^k BGL_{n-1}(A))$ on the homotopy groups of the homotopy fiber of the map $\text{Sing}^k BGL_{n-1}(A) \to \text{Sing}^k BGL_n(A)$, i.e., on $\pi_i(\text{Sing}^k(\mathbb{A}^n \setminus 0)(A))$. The relative Hurewicz theorem describes the relative homology of the above map in the first non-vanishing degree as a quotient of the relative homotopy group by this action.

**Lemma 2.2.3.** For any field $k$ and any essentially smooth local $k$-algebra $A$, there is a short exact sequence of the form

$$H_n(\text{Sing}^k BGL_{n-1}(A)) \to H_n(\text{Sing}^k BGL_n(A)) \xrightarrow{\delta_n} K_n^M(A).$$

**Proof.** It follows from [AF14b, p. 2590] and the proof of [AF15, Proposition 3.5.1] that the action of $G_n(A)$ on $\pi_{n-1}(\text{Sing}^k(\mathbb{A}^n \setminus 0)(A))$ is the standard action of $G_n(A)$ on $K_n^{\text{MW}}(A)$. The quotient of $K_n^M(A)$ by the standard action is precisely the quotient by the subgroup of $\eta$-divisible elements and therefore is precisely $K_n^M(A)$.}

**Remark 2.2.4.** Recall that one may extend the definition of Milnor K-theory to local rings (e.g., [NS89, §3]). The affirmation of the Gersten conjecture for Milnor K-theory for regular local rings containing a field [Ker09, Theorem 7.1] allows us to conclude that the evident map $K_n^M(A) \to K_n^M(A)$ is actually an isomorphism, if $A$ is a regular local ring. We use this identification without further mention in what follows.
2.3 Suslin’s Hurewicz morphism and the stabilization theorem

Suppose now \( k \) is a field, and \( A \) is an essentially smooth local \( k \)-algebra with infinite residue field. We now compare Suslin’s morphism \( s_n \) to the boundary map \( \delta_n \) in the relative Hurewicz theorem of Lemma 2.2.3. In light of Suslin’s stabilization theorem we can identify the target of \( s_n \) with the relative homology group \( H_n(BGL_n(A), BGL_{n-1}(A)) \). Thus, functoriality of the singular construction and the relative homology exact sequence in conjunction with Lemma 2.2.3 and [NS89, Theorem 3.25] yield the following commutative diagram of exact sequences:

\[
\begin{array}{ccccccc}
H_n(BGL_{n-1}(A)) & \to & H_n(BGL_n(A)) & \to & K_n^M(A) & \to & H_{n-1}(BGL_{n-1}(A)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_n(\text{Sing}^k BGL_{n-1}(A)) & \to & H_n(\text{Sing}^k BGL_n(A)) & \to & K_n^M(A) & \to & H_{n-1}(\text{Sing}^k BGL_{n-1}(A)) & \cdots
\end{array}
\]

and our goal is to study the maps \( K_n^M(A) \to K_n^M(A) \). The above diagram can be very explicitly analyzed for small values of \( n \).

**Lemma 2.3.1.** The map \( BGL_1 \to Sing^A BGL_1 \) is the identity map of simplicial presheaves.

**Proof.** By definition, the usual bar model of \( BGL_1 \) is a simplicial object with \( i \)-simplices given by \( G_n^i \). Since \( \text{Hom}(\Delta^n, G_m^\times) = G_m^\times \), it follows directly from the definition of \( Sing^A \) that the map \( BGL_1 \to Sing^A BGL_1 \) is the identity map. \( \square \)

**Lemma 2.3.2.** If \( k \) is a field, and if \( A \) is an essentially smooth local \( k \)-algebra with infinite residue field, then the map \( BGL_2(A) \to Sing^A BGL_2(A) \) induces isomorphism on homology in degrees \( \leq 2 \).

**Proof.** By appeal to Lemma 2.3.1 we conclude that the map \( H_1(Sing^A BGL_1(A)) \to H_1(Sing^A BGL_2(A)) \) is an isomorphism. It follows from Lemma 2.2.3 that the map \( \delta_2 \) is surjective and that there is a commutative diagram of exact sequences of the form

\[
\begin{array}{ccccccc}
H_2(BGL_1(A)) & \to & H_2(BGL_2(A)) & \to & K_2^M(A) & \to & 0 \\
\sim & & \downarrow & & \downarrow & & \downarrow \\
H_2(\text{Sing}^A BGL_1(A)) & \to & H_2(\text{Sing}^A BGL_2(A)) & \to & K_2^M(A) & \to & 0
\end{array}
\]

Now the map \( H_2(BGL_2(A)) \to H_2(\text{Sing}^A BGL_2(A)) \) is split injective by Lemma 2.2.2. Therefore, the composite of this splitting and the map \( H_2(BGL_2(A)) \to K_2^M(A) \) is surjective. A diagram chase thus implies that the vertical map \( K_2^M(A) \to K_2^M(A) \) is also surjective. Then, the five lemma implies that \( H_2(BGL_2(A)) \to H_2(\text{Sing}^A BGL_2(A)) \) must be surjective as well and thus it is an isomorphism. \( \square \)

**Comparison of stabilization maps**

To analyze the comparison maps in general, we will use pass through an auxiliary space. Following Schlichting [Sch17, §6], we define \( \tilde{E}_n(U) \) to be the maximal perfect subgroup of the kernel of the map \( GL_n(\Gamma(U, \mathcal{O}_U)) \to \pi_0(Sing^A GL_n(\Gamma(U, \mathcal{O}_U))) \). By construction, \( \tilde{E}_n(U) \) is a subgroup of \( SL_n(U) \).
since it maps to zero in the commutative group $GL_n(U)/SL_n(U) = \mathcal{O}_U(U)^\times$. If either $n \geq 3$ or $n = 2$ and the residue fields of smooth $k$-schemes have $> 3$ elements, then the inclusion of presheaves $\tilde{E}_n \to SL_n$ becomes an isomorphism after Zariski sheafification [Sch17, Lemma 6.5] and $\tilde{E}_n(U)$ is a presheaf of perfect groups. We write $BGL_+^n$ for the simplicial presheaf obtained by appeal to the functorial version of the plus construction [BK72, Chapter VII §6] applied to $BGL_n$ and the presheaf of perfect groups $\tilde{E}_n(R)$.

**Proposition 2.3.3.** Suppose $k$ is an infinite field. If $n \geq 1$ is an integer, then the following statements hold:

1. the map $BGL_n \to \text{Sing}^\Lambda_1 BGL_n$ factors through $BGL_+^n$ and these factorizations are functorial in $n$, and
2. the map $L_{\mathbf{A}_1} BGL_n \to L_{\mathbf{A}_1} BGL_+^n$ is split in the simplicial homotopy category.

**Proof.** The first statement is immediate from the definition of $\tilde{E}_n(R)$ and the definition of the plus construction. For the second statement, observe that $BGL_n \to \text{Sing}^\Lambda_1 BGL_n$ is an $\mathbf{A}_1$-weak equivalence, and the target is $\mathbf{A}_1$-local by [AHW18]. Therefore, this map becomes a simplicial weak equivalence after $\mathbf{A}_1$-localization. The second point then follows immediately from the first. \qed

**Proposition 2.3.4.** Suppose $k$ is an infinite field. For any $n \geq 2$, the map on simplicial homotopy fibers induced by the square

$$
\begin{array}{ccc}
BGL_+^{n-1} & \to & BGL_+^n \\
\downarrow & & \downarrow \\
\text{Sing}^\Lambda_1 BGL_{n-1} & \to & \text{Sing}^\Lambda_1 BGL_n
\end{array}
$$

induces an isomorphism on homotopy sheaves in degrees $\leq n - 1$.

**Proof.** Suppose $k$ is an infinite field and $A$ is an essentially smooth local $k$-algebra. Write $F_n$ for the homotopy fiber of the map in the top row. By [Sch17, Theorem 5.38], one knows that the presheaves $\pi_i(F_n)(A)$ vanish for $i \leq n - 2$ and have Zariski sheafification $K_{MW}^n$ for $i = n - 1$. Likewise, Proposition 2.1.3 states that the homotopy fiber of the bottom horizontal map is $\text{Sing}^\Lambda_1 \mathbf{A}_1^n \setminus 0$. Corollary 2.1.4 shows that this homotopy fiber is also $(n - 2)$-connected and has $(n - 1)$-st Zariski homotopy sheaf isomorphic to $K_{MW}^n$ in degree $i - 1$.

To establish the result, it suffices to show that the induced map $K_{MW}^n \to K_{MW}^n$ is an isomorphism. However, by Proposition 2.3.3 it follows that this map is actually split injective. The endomorphism ring of $K_{MW}^n$ coincides with $(K_{MW}^n)_{-n}(k) \cong GW(k)$ for any $n \geq 1$ by Lemma 3.3.6. Since the ring $GW(k)$ contains no idempotents besides 0 or 1 by [KRW72, Theorem 3.9] (see Example 3.11 for details), it follows that our map $K_{MW}^n \to K_{MW}^n$ must be an isomorphism, as required. \qed

**Lemma 2.3.5.** Suppose $k$ is an infinite field. For any integer $n \geq 1$, the map of presheaves $\pi_1(BGL_+^n) \to \pi_1(\text{Sing}^\Lambda_1 BGL_n)$ is an isomorphism after Zariski sheafification.

**Proof.** We know that $\pi_1(BGL_+^n(R)) = GL_n(R)/\tilde{E}_n(R)$ by definition, and [Sch17, Lemma 6.5] implies that the Zariski sheafification of $\tilde{E}_n(R)$ coincides with that of $SL_n(R)$. In particular, the map $GL_n(R)/\tilde{E}_n(R) \to R^\times$ given by the determinant is an isomorphism after Zariski sheafification. Likewise, the Zariski sheafification of $\pi_1(\text{Sing}^\Lambda_1 BGL_n)$ coincides with $\mathbb{G}_m$. Now the map in question
factors through the natural map \( \pi_1(BGL_n) \rightarrow \pi_1(\text{Sing}^A_1BGL_n) \) by Proposition 2.3.3. However, the map \( \pi_1(BGL_n) \rightarrow \pi_1(\text{Sing}^A_1BGL_n) \) is the map sending \( GL_n(R) \) the quotient by the subgroup of matrices homotopic to the identity, and the result follows. \( \square \)

**Refined weak homotopy invariance results**

**Theorem 2.3.6.** If \( k \) is an infinite field, and \( A \) is an essentially smooth local \( k \)-algebra then the following statements hold.

1. There is a commutative diagram of the form

\[
\begin{array}{cccccc}
H_n(BGL_{n-1}(A)) & \longrightarrow & H_n(BGL_n(A)) & \longrightarrow & K^M_n(A) & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & \\
H_n(\text{Sing}^A_1BGL_{n-1}(A)) & \longrightarrow & H_n(\text{Sing}^A_1BGL_n(A)) & \delta_n & \longrightarrow & K^M_n(A) & \longrightarrow 0,
\end{array}
\]

i.e., \( \delta_n \) is surjective.

2. The maps \( H_i(\text{Sing}^A_1BGL_n(A)) \rightarrow H_i(\text{Sing}^A_1BGL_{n+1}(A)) \) are isomorphisms for \( i \leq n \).

3. The maps \( H_i(BGL_n(A)) \rightarrow H_i(\text{Sing}^A_1BGL_n(A)) \) are isomorphisms when \( i \leq n \).

**Proof.** The diagram in question arises from the long exact sequence in relative homology. Now, the maps \( BGL_n \rightarrow \text{Sing}^A_1BGL_n \) factor through \( BGL_n^+ \) compatibly with \( n \) by Proposition 2.3.3. We analyze this situation first. Observe that \( H_n(BGL_n^+(A), BGL_{n-1}^+(A)) \cong K^M_n(A) \) by [Sch17, Theorem 5.38] and Lemma 2.3.5 and the same relative Hurewicz argument as in Lemma 2.2.3. Moreover, it follows that the induced map

\[
H_n(BGL_n^+(A), BGL_{n-1}^+(A)) \rightarrow H_n(\text{Sing}^A_1BGL_n(A), \text{Sing}^A_1BGL_{n-1}(A))
\]

is an isomorphism. In other words, there is a commutative diagram of exact sequences of the form:

\[
\begin{array}{cccccc}
H_n(BGL_{n-1}^+(A)) & \longrightarrow & H_n(BGL_n^+(A)) & \longrightarrow & K^M_n(A) & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & \\
H_n(\text{Sing}^A_1BGL_{n-1}(A)) & \longrightarrow & H_n(\text{Sing}^A_1BGL_n(A)) & \delta_n & \longrightarrow & K^M_n(A) & \longrightarrow 0.
\end{array}
\]

where the map \( K^M_n(A) \rightarrow K^M_n(A) \) is an isomorphism. Since the maps \( BGL_n(A) \rightarrow BGL_n^+(A) \) are homology isomorphisms, the result follows. It follows immediately that \( \delta_n \) is surjective.

The fact that \( \delta_n \) is surjective implies the map \( H_n(\text{Sing}^A_1BGL_n(A)) \rightarrow H_n(\text{Sing}^A_1BGL_{n+1}(A)) \) is an isomorphism, i.e., point (2). Thus, we conclude that the stabilization map \( H_n(\text{Sing}^A_1BGL_n(A)) \rightarrow H_n(\text{Sing}^A_1BGL(A)) \) is an isomorphism as well. In the proof of Lemma 2.2.2, we established that the map \( H_n(BGL_n(A)) \rightarrow H_n(\text{Sing}^A_1BGL(A)) \) is an isomorphism, and this map factors through the map \( H_n(BGL_n(A)) \rightarrow H_n(\text{Sing}^A_1BGL_n(A)) \), which we therefore also conclude is an isomorphism; thus Point (3) is established. \( \square \)

**Remark 2.3.7.** If the map \( H_i(BGL_n(A)) \rightarrow H_i(\text{Sing}^A_1BGL_n(A)) \) is an isomorphism in degrees \( \leq n \), we say weak homotopy invariance holds for the (integral) homology of \( GL_n \) in degrees \( \leq n \) [HW15,
Definition 2.3. The notion of weak homotopy invariance was introduced by F. Morel in his approach to the Friedlander–Milnor conjecture (cf. [Mor11], though this notion does not explicitly appear there). In particular, Theorem 2.3.6 implies that weak homotopy invariance holds in degrees \( \leq n \) for \( GL_n \) over local rings. In contrast, if one replaces \( GL_n \) by \( SL_2 \), one knows that weak homotopy invariance fails in degree 3, e.g., by [HW15, Theorem 1].

The comparison theorem

**Theorem 2.3.8.** Assume \( k \) is an infinite field, \( n \geq 1 \) is an integer, and \( A \) is an essentially smooth local \( k \)-algebra.

1. There is an exact sequence of the form
   \[
   K_n^{Q}(A) \to K_n^{M}(A) \to S_n(A) \to 0,
   \]
   where the left horizontal map is Suslin’s Hurewicz homomorphism.

2. Suslin’s conjecture holds in degree \( n \) if and only if the canonical surjection \( K_n^{M}/(n-1)!(A) \to S_n(A) \) is an isomorphism.

**Proof.** For Point (1), recall from the proof of Lemma 2.2.2 that the map \( BGL^+(A) \to Sing^A BGL(A) \) is a weak equivalence. On the other hand, Theorem 2.3.6, and functoriality of the Hurewicz map yield a commutative diagram of the form

\[
\begin{array}{ccccccccc}
\pi_n(BGL^+(A)) & \to & H_n(BGL^+(A)) & \to & H_n(BGL_n(A)) & \to & K_n^M(A) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\pi_n(Sing^A BGL(A)) & \to & H_n(Sing^A BGL(A)) & \to & H_n(Sing^A BGL_n(A)) & \to & \delta_n K_n^M(A) & \to & 0
\end{array}
\]

where all the vertical morphisms are isomorphisms. Thus, Suslin’s Hurewicz morphism coincides with the composite in the bottom row under these isomorphisms; we call this composite \( \psi_n \).

Next, we claim that the composite in the bottom row coincides with the morphism \( \psi_n \). Indeed, this is essentially a consequence of functoriality of the relative Hurewicz morphism. To this end, observe that the stabilization map \( \pi_n(Sing^A BGL_{n+1}(A)) \equiv \pi_n(Sing^A BGL(A)) \) is an isomorphism by Lemma 2.2.2. The relative Hurewicz map \( \pi_{n-1}(Sing^A BGL_{n}\backslash 0(A)) \to H_{n-1}(Sing^A BGL_n(A), Sing^A BGL_{n-1}(A)) \) coincides with the standard quotient map \( K_n^{MW}(S) \to K_n^M(S) \) by the proof of Lemma 2.2.3. By functoriality of (relative) Hurewicz maps, we therefore have a commutative square of the form

\[
\begin{array}{ccccccccc}
\pi_n(Sing^A BGL_n(A)) & \to & K_n^{MW}(A) \\
\downarrow & & \downarrow \\
H_n(Sing^A BGL_n(A)) & \to & K_n^M(A) 
\end{array}
\]

Now, the map \( \psi_n \) is defined by factorization through the stabilization map \( \pi_n(Sing^A BGL_n(A)) \to K_n^M(A) \). Using the splitting of Lemma 2.2.2, it follows immediately that the map \( H_n(Sing^A BGL_n(A)) \to K_n^M(A) \) also factors through the stabilization map \( H_n(Sing^A BGL_n(A)) \to H_n(Sing^A BGL_{n+1}(A)) \).
Combining these observations and appealing again to functoriality of Hurewicz maps, we conclude that \( \psi_n \) coincides with \( \psi'_n \).

Point (2) follows immediately from Point (1). \( \square \)

## 3 The sheaf \( \pi^{A_1}_{4}(S^{3+3\alpha}) \) revisited

In the previous section, we showed that Suslin’s conjecture from the introduction was equivalent to showing that the natural epimorphism \( K^M_n \to S_n \) induced an isomorphism \( K^M_n/(n-1)! \to S_n \). In this section, we verify this assertion in the case \( n = 5 \). To do this, we proceed in two steps.

We refine [AWW17, Theorem 5.2.5], which describes the \( A_1 \)-homotopy sheaf \( \pi^{A_1}_{4}(\mathbb{P}^1 \wedge 3) \) under certain restrictions on the base field in two ways. By analyzing real and \( \ell \)-adic realizations, we show that most of the restrictions on the base field in the statement of [AWW17, Theorem 5.2.5] are superfluous. The outcome of this analysis is a description of the sheaf \( \pi^{A_1}_{4}(\mathbb{P}^1 \wedge 3) \) that parallels that of \( \pi^{A_1}_{n}(BGL_n) \) in §2.1: it may be described as an extension of a sheaf defined in terms of higher Grothendieck–Witt theory by a “non-stable” part, which is related to the sheaf \( S_5 \) analyzed in the previous section (though we caution the reader that “non-stable” is in quotes here to distinguish it from analyzing what happens after \( \mathbb{P}^1 \)-stabilization, which will also be of interest to us).

Second, by comparing our refined description to the stable results of [RSØ16], we are able to conclude that \( S_5 \) coincides with \( K^M_5/24 \). The main computational result is achieved in Theorem 3.3.10. We conclude with Theorem 3.3.12, which shows Suslin’s conjecture holds in degree 5 under appropriate hypotheses.

### 3.1 The degree map and realization

In this section, we establish some preliminary results about real realization of homotopy groups and, in particular, the interaction between real realization and module structures on \( A_1 \)-homotopy sheaves.

#### Compositions in \( A_1 \)-homotopy groups

Fix an integer \( p \geq 2 \). For any integer \( q \geq 0 \) we may consider the motivic sphere \( S^{p+q\alpha} \). The abelian group \( [S^{p+q\alpha}, S^{p+q\alpha}]_{A_1} \) of \( A_1 \)-homotopy endomorphisms of \( S^{p+q\alpha} \) admits a natural ring structure via composition. If \( q = 0 \), this ring is isomorphic to \( \mathbb{Z} \), while if \( q > 0 \), this ring is isomorphic to \( K^0_{\text{MW}}(k) \) by Morel’s computations. More generally, we recall the following result of F. Morel.

**Lemma 3.1.1** ([Mor12, Corollary 6.43]). If \( p \geq 2 \) is an integer and \( r, q \geq 0 \) are integers, then

\[
[S^{p+q\alpha}, S^{p+r\alpha}] = \begin{cases} 
K^0_{\text{MW}}(k) & \text{if } r > 0 \\
\mathbb{Z} & \text{if } r = q = 0 \\
0 & \text{otherwise}. 
\end{cases}
\]

Note that \( G_m \)-suspension induces group homomorphisms

\[
[S^{p+q\alpha}, S^{p+r\alpha}]_{A_1} \to [S^{p+(q+1)\alpha}, S^{p+(r+1)\alpha}]_{A_1}.
\]

If \( q, r > 0 \), then these maps are isomorphisms. If \( r = q = 0 \), then the map is the unit map \( \mathbb{Z} \to K^0_{\text{MW}}(k) \), while if \( r = 0 \) and \( q \neq 0 \), the map is the zero map.
Given a third integer \( s \geq 0 \), composition yields homomorphisms
\[
[S^{p+q\alpha}, S^{p+r\alpha}]_{\mathbb{A}^1} \times [S^{p+q\alpha}, S^{p+r\alpha}]_{\mathbb{A}^1} \rightarrow [S^{p+q\alpha}, S^{p+r\alpha}]_{\mathbb{A}^1}
\]
\[(f, g) \mapsto f \circ g.
\]
These maps are bilinear and associative and compatible with \( \mathbb{G}_m \)-suspension as described above. In particular, we obtain an \( \mathbb{N} \times \mathbb{N} \)-graded ring structure on \( \bigoplus_{q,r \geq 0} [S^{p+q\alpha}, S^{p+r\alpha}]_{\mathbb{A}^1} \).

Real realization of endomorphism rings

If \( k \) is a field and \( k \hookrightarrow \mathbb{R} \) is an embedding, then sending a smooth \( k \)-scheme \( X \) to the topological space \( X(\mathbb{R}) \) equipped with its usual structure of a real manifold extends to a functor
\[
\mathcal{R} : \mathcal{H}_\ast(k) \rightarrow \mathcal{H}_\ast.
\]
At the level of homotopy categories, this functor was exposed in [MV99, §3 pp. 121-122] and was later described in terms of a Quillen adjunction in [DI04, §5.3]. It follows from the construction of [DI04] that real realization is a simplicial functor of simplicial model categories.

Since the real realization of \( \mathbb{G}_m \) is homotopy equivalent to \( S^0 \), it follows that the real realization of \( S^{p+q\alpha} \) is \( S^p \) for any \( q \geq 0 \). Therefore, by functoriality, real realization induces a ring homomorphism
\[
\mathcal{R} : \bigoplus_{q,r \geq 0} [S^{p+q\alpha}, S^{p+r\alpha}]_{\mathbb{A}^1} \rightarrow [S^p, S^p] = \mathbb{Z}.
\]
We now analyze this ring homomorphism; we begin with the degree \((0,0)\)-part.

Following [Mor12, p. 53], set \( \epsilon = -(-1) \in \mathbb{K}^{\text{MW}}(k) \). The element \( \epsilon \) represents the \( \mathbb{A}^1 \)-homotopy class of the endomorphism of \( S^{2+2\alpha} \) obtained by switching the two factors of \( \mathbb{G}_m \) under the identification \( [S^{2+2\alpha}, S^{2+2\alpha}]_{\mathbb{A}^1} \cong \mathbb{K}^{\text{MW}}(k) \) [Mor04, Lemma 6.1.1(2)].

The next result is a straightforward consequence of the the description of the Grothendieck–Witt ring of the real numbers combined with the fact that the real realization of \( \epsilon \) is the identity map \( S^2 \rightarrow S^2 \).

**Lemma 3.1.2.** For any integers \( p, q, p \geq 2, q \geq 0 \), the ring map \( [S^{p+q\alpha}, S^{p+r\alpha}] \rightarrow [S^p, S^p] \) induced by real realization is an isomorphism if \( q = 0 \) and the surjection \( \mathbb{Z}[\epsilon]/(\epsilon^2 - 1) \rightarrow \mathbb{Z} \) given by evaluation at 1 if \( q \neq 0 \).

The groups \( \mathbb{K}_q^{\text{MW}}(\mathbb{R}) \) are generated by the expressions \([a]\) for \( a \in \mathbb{R}^\times \) and an element \( \eta \). Therefore, to understand real realization more generally, we need to understand the real realizations of these elements.

**Proposition 3.1.3.** The following statements hold about real realization:

1. for any \( a \in \mathbb{R}^\times \),
\[
\mathcal{R}([a]) = \begin{cases} 
0 & \text{if } a > 0, \text{ and} \\
1 & \text{if } a < 0;
\end{cases}
\]

2. for any \( a \in \mathbb{R}^\times \),
\[
\mathcal{R}((a)) = \begin{cases} 
1 & \text{if } a > 0, \text{ and} \\
-1 & \text{if } a < 0; \text{ and}
\end{cases}
\]
3. if \( \eta : S^{1+2\alpha} \to S^{1+\alpha} \) is the usual Hopf map, then \( \mathfrak{R}(\eta) = -2 \)

**Proof.** For the first point, by definition \([a] \) is the stabilization of the map \( S^0_k \to \mathbb{G}_m \) sending the non-base-point of \( S^0_k \) to the element \( a \in \mathbb{G}_m(k) \). It follows that if \( a > 0 \), then the real realization of this map is the constant map, which has degree 0. Likewise, if \( a < 0 \), then the real realization of this map is the identity map \( S^0 \to S^0 \), whose suspensions all have degree 1.

For the second point, consider the element \( \langle a \rangle \in \mathbb{K}_{GW}^n(\mathbb{R}) \). If \( a > 0 \), then \( \langle a \rangle \cong \langle 1 \rangle = 1 \) and is sent to 1 under real realization. If \( a < 0 \), then \( \langle a \rangle \cong \langle -1 \rangle \). Since \( \mathfrak{R}(\epsilon) = 1 \) by Lemma 3.1.2, it follows that \( \mathfrak{R}(-1) = -1 \).

Finally, since \( (-1)^{1+n} = 1 + \eta[-1] \) and real realization is a ring homomorphism, we conclude that \( \mathfrak{R}(\eta) = -2 \) from the discussion above. \( \square \)

Given these preliminary results, we may state the main result about realization we will use.

**Proposition 3.1.4.** Suppose \( n \geq 2 \) is an integer. For any integer \( j > 0 \), the image of the homomorphism given by real realization

\[
\text{im}(\pi^A_{n-1+j\alpha}(S^{n-1+n\alpha})(\mathbb{R})) \to \pi_{n-1}(S^{n-1})) = \begin{cases} (1) & \text{if } j < n; \\ (2^{n-j}) & \text{if } j \geq n. \end{cases}
\]

More precisely, for \( j = n \) the map factors as \( \mathbb{GW}^n(\mathbb{R}) \to \mathbb{W}(\mathbb{R})^{sgn} \mathbb{Z} \).

**Proof.** In every case, the source group is \( \mathbb{K}_{n-j}^{GW}(\mathbb{R}) \). If \( j < n \), this group contains the \( (n-j) \)-fold product \([-1] \cdots [-1]\) and the result follows immediately from Proposition 3.1.3(1) and compatibility of real realization with the product structures described above.

If \( j = n \), the result follows from Lemma 3.1.2. If \( j > n \), then the source of the map is the free abelian group generated by \( \eta^{j-n} \) and the result follows from Proposition 3.1.3(3). \( \square \)

**Étale realization**

We refer the reader to [Isa04] for a detailed discussion of étale realization. Let us quickly summarize the main points. If \( \ell \) is prime, one begins by defining \( \ell \)-complete étale realization functor on the category of schemes: given a scheme \( X \), its étale realization is an \( \ell \)-complete pro-simplicial set that we will denote by \( \tilde{\mathcal{E}}(X) \). The construction has the property that a morphism of schemes \( f : X \to Y \) induces a weak equivalence \( \tilde{\mathcal{E}}(X) \to \tilde{\mathcal{E}}(Y) \) if and only if \( f^* : H^*_{\tilde{\mathcal{E}}}(Y; \mathbb{Z}/(\ell)) \to H^*_{\tilde{\mathcal{E}}}(X; \mathbb{Z}/(\ell)) \) is an isomorphism. By [Isa04], if \( k \) is a field and \( \ell \) is different from the characteristic of \( k \), the assignment \( X \mapsto \tilde{\mathcal{E}}(X) \) on smooth \( k \)-schemes extends to a functor on the pointed \( \mathbb{A}^1 \)-homotopy category \( \mathcal{H}(k) \); abusing notation slightly, we will also denote this functor by \( \tilde{\mathcal{E}} \). If \( k \) is furthermore separably closed, it follows from the Künneth isomorphism in étale cohomology with \( \mathbb{Z}/(\ell) \)-coefficients that the functor \( \tilde{\mathcal{E}} \) preserves finite products and smash products of pointed spaces. Moreover, by construction, the functor \( \tilde{\mathcal{E}} \) commutes with the formation of homotopy colimits.

Assume now that \( k \) is separably closed and write \( R \) for the ring of Witt vectors in \( k \). Choose an algebraically closed field \( K \) and fix embeddings \( R \hookrightarrow K \) and \( C \hookrightarrow K \). For any split reductive group \( G \), these morphism yield maps of the form:

\[
G_k \longrightarrow G_R \leftarrow G_K \longrightarrow G_C
\]

which we will use to compare the étale realization over \( k \) with complex realization.
Lemma 3.1.5. If $k$ is a separably closed field having characteristic $p$ and $\ell$ is a prime different from $p$, then for any integers $i, j \geq 0$, 
\[ \mathcal{E}(S^{i+j}) \cong (S^{i+j})^\wedge. \]

Proof. The comparison maps described before the statement applied to $G = \mathbb{G}_m$ yield identifications 
\[ \mathcal{E}(\mathbb{G}_m) \cong ((\mathbb{G}_m(\mathbb{C}))^\wedge \cong (S^1)^\wedge. \] The étale realization of the simplicial circle is also 
\[ (S^1)^\wedge \] since étale realization preserves homotopy colimits and the simplicial circle can be realized as a homotopy pushout of the diagram $* \leftarrow S^0_k \rightarrow *$. To conclude, we use the fact that étale realization preserves smash products of pointed spaces. \[ \square \]

Remark 3.1.6. In contrast to the case of real realization, over a separably closed field, the étale realization of the endomorphisms of the motivic sphere are determined wholly by the realization of the identity map and the realization of $\eta$.

3.2 Realization of some homotopy sheaves of spheres

The purpose of this section is to describe the behavior of various homotopy sheaves of spheres under real and étale realization. In particular, we analyze the $\mathbb{R}$-realization homomorphism 
\[ \pi^\mathbb{A}^1(X^{2+3\alpha}) \to \pi_3(S^2) \] and correct [AF15, Corollary 5.4.1]. Along the way, we remind the reader of the computation of this $\mathbb{A}^1$-homotopy sheaf, which will appear in subsequent sections.

The computation of $\pi^\mathbb{A}^1(X^{2+3\alpha})$ revisited

We begin by recalling some results about $\mathbb{A}^1$-fiber sequences from [AF15, §4.2]. First, for any integer $n \geq 1$ there is a pull-back diagram of linear algebraic groups

\[
\begin{array}{ccc}
Sp_{2n} & \longrightarrow & SL_{2n+1} \\
\downarrow & & \downarrow \\
Sp_{2n+2} & \longrightarrow & SL_{2n+2}.
\end{array}
\]

This diagram yields isomorphisms of $k$-schemes $SL_{2n+2}/Sp_{2n+2} \cong SL_{2n+1}/Sp_{2n}$. If we define $X_n = SL_{2n}/Sp_{2n}$, then by [AF15, Proposition 4.2.2] we obtain $\mathbb{A}^1$-fiber sequences of the form

\[ (3.2.1) \quad X_n \longrightarrow X_{n+1} \longrightarrow \mathbb{A}^{2n+1} \setminus 0. \]

In the case $n = 2$, there is an exceptional isomorphism $X_2 = SL_4/Sp_4 \cong SL_3/SL_2$ and thus $X_2$ is $\mathbb{A}^1$-weakly equivalent to $\mathbb{A}^3 \setminus 0$, yielding an $\mathbb{A}^1$-fiber sequence

\[ S^{2+3\alpha} \cong \mathbb{A}^3 \setminus 0 \longrightarrow X_3 \longrightarrow \mathbb{A}^5 \setminus 0 \cong S^{4+5\alpha}. \]

The schemes $X_n$ are symmetric varieties, and $X_\infty := \text{colim}_{n \geq 0} X_n$ is a model for $\mathbb{A}^1$-connected component of the base-point in the space $GL/Sp$ arising in higher Grothendieck–Witt theory [ST15]. In particular, as observed in [AF15, Proposition 4.2.2], $\pi^\mathbb{A}^1(X_n) \cong GW^3_{i+1}$ for $i \leq 2n - 2$, at least if we work over a field having characteristic unequal to 2. Using the long exact sequence in $\mathbb{A}^1$-homotopy sheaves associated with the above fiber sequence, there is an associated exact sequence of the form

\[ GW^3_5 \longrightarrow K^{MW}_5 \longrightarrow \pi^\mathbb{A}^1_3(S^{2+3\alpha}) \longrightarrow GW^3_4 \longrightarrow 0. \]
The cokernel of the morphism $\mathbf{GW}^3_5 \to \mathbf{K}_{5}^{MW}$ is called $\mathbf{F}_5$ in [AF15, Theorem 4.3.1], and it is observed that $\mathbf{F}_5$ is a quotient of the fiber product $S_5 \times \mathbf{K}_{5}^{MW}/\mathbf{I}^5 =: T_5$ from [AF14a, Theorem 3.14].

Behavior of $\pi_{3+j\alpha}^{3+5\alpha}(S^{2+3\alpha})$ under realization

There are natural homeomorphisms $X_n(\mathbb{R}) \approx SL_{2n}(\mathbb{R})/Sp_{2n}(\mathbb{R})$. Replacing the groups by their homotopy equivalent maximal compact subgroups, we obtain weak equivalences of the form

$$X_n(\mathbb{R}) \simeq SO(2n)/U(n).$$

Furthermore, the fiber sequence (3.2.1) corresponds under $\mathbb{R}$-realization to the fiber sequence

$$X_n(\mathbb{R}) \to X_{n+1}(\mathbb{R}) \to S^{2n}.$$

While real realization need not necessarily preserve fiber sequences, it follows from this observation that the real realization of $X_n \to X_{n+1} \to R_{2n+1} \setminus 0$ is sent to a fiber sequence by real realization.

In a range depending on $n$, the homotopy groups of $X_{n+1}(\mathbb{R})$ may be computed by means of real Bott periodicity. For instance, the stabilization map

$$\pi_3(X_3(\mathbb{R})) = \pi_3(SO(6)/U(3)) \to \pi_3(SO/U) = \pi_3(O/U) = 0$$

is an isomorphism. We therefore obtain, from a portion of the long exact homotopy sequence of (3.2.2), a presentation

$$\pi_4(X_3(\mathbb{R})) \to \pi_4(S^4) \to \pi_3(S^2) \to 0.$$

Since $\pi_4(S^4) \cong \pi_3(S^2) \cong \mathbb{Z}$, we conclude that the map $\pi_4(S^4) \to \pi_3(S^2)$ is an isomorphism.

Let $j \geq 0$. One obtains a ladder diagram from the long exact homotopy sequences of (3.2.1) and (3.2.2) and the $\mathbb{R}$ realization map $\rho$:

$$\begin{array}{cccccc}
\pi_4^{3+5\alpha}(S^{2+3\alpha}) & \to & \pi_3^{3+5\alpha}(S^2) & \to & 0 \\
\rho' & & \rho & & \\
\pi_4(S^4) & \cong & \pi_3(S^2) & \to & 0 & \to 0.
\end{array}$$

According to [AF15, Proposition 5.2.1], $f$ is surjective when $j \geq 5$, and is an isomorphism when $j \geq 6$. The next result describes the image of the left hand map and corrects [AF15, Corollary 5.4.1].

**Proposition 3.2.1.** For any integer $j \geq 0$, the image of the homomorphism given by $\mathbb{R}$-realization satisfies

$$\text{im}(\pi_3^{3+5\alpha}(S^{2+3\alpha})(\mathbb{R}) \to \pi_3(S^2)) = \begin{cases} 
(1) & \text{if } j < 5 \\
(2j-5) & \text{if } j \geq 5.
\end{cases}$$

**Proof.** In view of the commutativity of Diagram 3.2.3, this result is an immediate consequence of the corresponding fact for $\rho'$, which is contained in Proposition 3.1.4. \qed

**Remark 3.2.2.** It is known that $\pi_{3+5\alpha}(A^3 \setminus 0) \cong \mathbb{Z}/(24) \times \mathbb{Z}/(2)$ $\mathbf{W}$ as a $\mathbf{GW}$-module [AF15, Proposition 5.2.1], but this will not be needed in the sequel.
3.3 The EHP sequence and homotopy sheaves of $S^{3+3\alpha}$

The sheaf $\pi_{3+6\alpha}^1(S^{5+6\alpha})$ is computed from $\pi_{3+3\alpha+j\alpha}^1(S^{2+3\alpha})$ by appeal to the simplicial EHP sequence of [AWW17, Theorem 3.3.13]. In particular, for any base field $k$, there is a short exact sequence of the form

\[
\pi_{3+3\alpha+j\alpha}^1(S^{2+3\alpha}) \longrightarrow \pi_{4+3\alpha+j\alpha}^1(S^{3+3\alpha}) \longrightarrow 0,
\]

where the morphism $P$ is induced by composition with the Whitehead square of the identity. By construction, this sequence is natural in $k$. In [AWW17, Theorem 5.2.5], after some preliminary results, the image of $P_k$ in Diagram 3.3.1 is identified; our goal is to perform a similar analysis of the EHP sequence here.

By [Mor12, Corollary 6.43], $\pi_{3+3\alpha+j\alpha}^1(S^{2+3\alpha}) \cong K_{0-j}^{MW}$. On the other hand, the description of $\pi_{3+3\alpha+j\alpha}^1(S^{2+3\alpha})$ for large $j$ follows from [AF15, Lemma 5.1.1]: $\pi_{3+3\alpha+j\alpha}^1(S^{2+3\alpha}) \cong 0$ for $j \geq 6$ and the latter sheaf coincides with the sheaf $W$. Granted these two facts, the EHP exact sequence above takes the form:

\[
\pi_{3+3\alpha+j\alpha}^1(S^{2+3\alpha}) \longrightarrow \pi_{4+3\alpha+j\alpha}^1(S^{3+3\alpha}) \longrightarrow 0
\]

On the map $P_k$

There are identifications $\text{Hom}_{K_{0}^{MW}}(K_{0}^{MW}, W) \cong W(k) \cong \text{Hom}_{GW(k)}(GW(k), W(k))$. The map $P_k$ of (3.3.2) therefore determines an element $[P_k]$ of $W(k)$. The next result, which follows by unwinding definitions, gives some conditions equivalent to the surjectivity of $P_k$.

**Lemma 3.3.1.** The following are equivalent:

1. The map $P_k$ is surjective;
2. The element $[P_k]$ generates $W(k)$ as a $GW(k)$ module;
3. The element $[P_k]$ is a unit of $W(k)$;
4. The map $P_k(k)$ is surjective.

We now prove some descent and ascent results relating the maps $P_k$ for different classes of fields.

**Lemma 3.3.2.** The following statements hold:

1. If $K/k$ is a field extension, and if $P_k$ is surjective, then $P_K$ is surjective as well.
2. If $k$ is not formally real, $k^*$ is a separable closure of $k$ and $P_{k^*}$ is surjective, then $P_k$ is surjective.
3. If $P_{\mathbb{R}}$ is surjective, then $P_{\mathbb{Q}}$ is surjective.

**Proof.** In each case, we appeal to the equivalent conditions of Lemma 3.3.1: the map $P_k$ is surjective if and only if the element $[P_k]$ is a unit of $W(k)$. The latter statement may be checked by appeal to a classical result of Pfister [Lam05, Theorem 8.7].

For the first point, the definition of $P_k$ is natural in $k$, which is to say that $[P_K]$ is the image of $[P_k]$ under the functorial ring map $W(k) \rightarrow W(K)$. Since the element $[P_k]$ is a unit, so too is $[P_K]$. 

For the second point we know $W(k^s) = \mathbb{Z}/(2)$, and $W(k) \to W(k^s)$ is the dimension map (modulo 2). By [Lam05, Theorem VIII.8.7], the class $[P_k] \in W(k)$ is a unit if and only if it remains a unit in $W(k^s)$ under this map.

For the third point, begin by observing that the field $\mathbb{Q}$ is formally real and has a unique real closure, the field $\mathbb{R}^{alg}$ of real algebraic numbers. By [Lam05, Theorem VIII.8.7], the class $[P_\mathbb{Q}] \in W(\mathbb{Q})$ is a unit if and only if $[P_{\mathbb{R}^{alg}}] \in W(\mathbb{R}^{alg})$ is a unit. Moreover, a form $q$ is a unit if and only if its signature with respect to the unique ordering of $\mathbb{R}^{alg}$ is $\pm 1$. Since this last statement can be checked by passing to $\mathbb{R}$, we conclude. \hfill \Box

A vanishing result

We are now in a position to strengthen [AWW17, Proposition 5.2.3].

**Proposition 3.3.3.** If $k$ is a field having characteristic unequal to 2, then $\pi_{4+6\alpha}(S^{3+3\alpha}) = 0$.

**Proof.** Contemplating the exact sequence in Diagram (3.3.2), by appeal to Lemmas 3.3.1 and 3.3.2(1), it suffices to show that $[P_k]$ is a unit for $k$ any prime field. We treat the characteristic zero case in Lemma 3.3.4 and the positive characteristic case in Lemma 3.3.5.

**Lemma 3.3.4.** The element $[P_\mathbb{Q}] \in W(\mathbb{Q})$ is a unit.

**Proof.** By Lemma 3.3.2(3), it is sufficient to show that the map $P_\mathbb{R}(\mathbb{R})$ in the following diagram, obtained by evaluating (3.3.1), is surjective:

\[
\begin{array}{ccc}
\pi_{5+6\alpha}^1(S^{5+6\alpha})(\mathbb{R}) & \xrightarrow{P_\mathbb{R}(\mathbb{R})} & \pi_{3}^1(S^{2+3\alpha})(\mathbb{R}) \\
\mathbb{Z}[\epsilon]/(\epsilon^2 - 1) & \xrightarrow{\pi_{4+6\alpha}^1(S^{3+3\alpha})(\mathbb{R})} & \mathbb{Z};
\end{array}
\]

the equality on the left is contained in Lemma 3.1.2.

Since $\mathbb{R}$-realization is functorial and the real realization of the relevant portion of the simplicial EHP sequence coincides with the corresponding portion of the EHP sequence of the real points, we obtain a commuting diagram of exact sequences:

\[
\begin{array}{ccccccccc}
\pi_{5+6\alpha}^1(S^{5+6\alpha})(\mathbb{R}) & \xrightarrow{P} & \pi_{3+6\alpha}^1(S^{2+3\alpha})(\mathbb{R}) & \xrightarrow{E} & \pi_{4+6\alpha}^1(S^{3+3\alpha})(\mathbb{R}) & \xrightarrow{H} & 0 \\
\pi_5(S^5) & \xrightarrow{|P|} & \pi_3(S^2) & \xrightarrow{\rho} & \pi_4(S^3) & \xrightarrow{\lambda} & 0.
\end{array}
\]

It is well known that the lower sequence takes the form $\mathbb{Z}_{\ell_5} \to \mathbb{Z}_{\eta_{hop}} \to \mathbb{Z}/(2) \to 0$, and in particular that $|P|_{\ell_5} = 2\eta_{hop}$.

The rest of the argument is a diagram chase. Write $P_\mathbb{R}(1) = n$. We wish to show that $n = \pm 1$. The left hand vertical map is the surjection $\mathbb{Z}[\epsilon]/(\epsilon^2 - 1) \to \mathbb{Z}$ given by evaluation at 1, again by Lemma 3.1.2; it sends the class of the identity map to the class of the identity. Therefore, $2\mathbb{Z}_{\eta_{hop}}$ is the image of the composite map $\pi_{5+6\alpha}^1(S^{5+6\alpha})(\mathbb{R}) \to \pi_5(S^5)$.
3.3 The EHP sequence and homotopy sheaves of $S^{3+3\alpha}$

We showed in Proposition 3.2.1 that the image of the map $\rho$ is $2\mathbb{Z}_{\text{top}}$. Following the left hand square in the clockwise direction, we see that the image of the composite map $\pi_{5+6\alpha}^1(S^{5+6\alpha})(\mathbb{R}) \to \pi_3(S^2)$ is $2n\mathbb{Z}_{\text{top}}$. Consequently, $n = \pm 1$ as required.

**Lemma 3.3.5.** If $k$ is a finite field having characteristic unequal to 2, then $[P_k]$ is a unit.

**Proof.** By appeal to Lemma 3.3.2(2), it is sufficient to show that the map $P_{k^s}(k^s)$ is surjective.

\[
\pi_{5+6\alpha}^1(S^{5+6\alpha})(k^s) \xrightarrow{P_{k^s}(k^s)} \pi_{3+6\alpha}^1(S^{2+3\alpha})(k^s) \xrightarrow{E_{k^s}(k^s)} \pi_{4+6\alpha}^1(S^{3+3\alpha})(k^s) \to 0
\]

Take $\ell = 2$. By appeal to Lemma 3.1.5 and functoriality of étale realization, we obtain a commutative diagram of the form:

\[
\begin{array}{c}
\pi_{5+6\alpha}^1(S^{5+6\alpha})(k^s) \\
\downarrow \\
\pi_{11}(S^{11})_2
\end{array}
\begin{array}{c}
\xrightarrow{P} \\
\downarrow |P| \\
\pi_{9}(S^5)_2
\end{array}
\begin{array}{c}
\xrightarrow{E} \\
\downarrow \rho \\
\pi_{10}(S^6)_2
\end{array}
\to 0.
\]

The sequence $\pi_{11}(S^{11}) \to \pi_9(S^5) \to \pi_{10}(S^6) \to 0$ is a portion of the classical EHP exact sequence, and one knows that $\pi_{11}(S^{11}) = \mathbb{Z}$, $\pi_9(S^5) = \mathbb{Z}/(2)$ and $\pi_{10}(S^6) = 0$ and thus this sequence remains exact after 2-completion, i.e., the diagram above is a commutative diagram of exact sequences. Therefore, we conclude that $|P|$ is necessarily surjective.

The identity map lies in the image of the leftmost vertical arrow, and so the composite map $\pi_{5+6\alpha}^1(S^{5+6\alpha})(k^s) \to \pi_9(S^5)_2 \cong \mathbb{Z}/(2)$ is surjective. The result then follows from a diagram chase. 

**Some results on strictly $\mathbb{A}^1$-invariant sheaves**

The vanishing result of Proposition 3.3.3 can be used to deduce information about the sheaves $\pi_{4+3\alpha}^j(S^{3+3\alpha})$ for $j < 6$ by appeal to some technical results on strictly $\mathbb{A}^1$-invariant sheaves that were established in [AWW17, §5.1]. We restate the necessary results here for the convenience of the reader.

**Lemma 3.3.6 ([AWW17, Lemma 5.1.3]).** Suppose $M$ is a strictly $\mathbb{A}^1$-invariant sheaf.

1. For any integer $n \geq 1$, there are isomorphisms

$$\text{Hom}(K_{n}^{\text{MW}}, M) \cong M_{-n}(k).$$

2. If $n \geq 2$, the evident map $\text{Hom}(K_{n}^{\text{MW}}, M) \to \text{Hom}(K_{n-1}^{\text{MW}}, M_{-1})$ induced by contraction is an isomorphism compatible with the identification of Point (1).

**Lemma 3.3.7 ([AWW17, Lemma 5.1.5]).** Fix a base field $k$. If $\phi : K_{n}^{\text{MW}} \to M$ is a morphism of sheaves such that $\phi_{-j} = 0$, then

1. assuming $n \geq j \geq 0$, the morphism $\phi$ is trivial; and
2. assuming $0 \leq n < j$, the morphism $\phi$ factors through a morphism $K_{n}^{\text{MW}}/I^j \to M$. 


\section*{The computation}

Granted the results above, we are now in a position to establish our refinement of [AWW17, Theorem 5.2.5]. Recall from 3.3.1 that there is an exact sequence of the form

\[ \pi^A_{5+j\alpha}(S^{5+6\alpha}) \xrightarrow{P_k} \pi^A_{3+j\alpha}(S^{2+3\alpha}) \xrightarrow{E_k} \pi^A_{4+j\alpha}(S^{3+3\alpha}) \to 0. \]

The sheaf \( \pi^A_{3+j\alpha}(S^{2+3\alpha}) \) was described in greater detail at the beginning of Section 3.2: for \( j = 0 \) it is an extension of \( GW_4^3 \) by a sheaf called \( F_5 \). Since \( \pi^A_{5}(S^{5+6\alpha}) = K^M_6 \), \( \text{Hom}(K^M_6, GW^3_4) = (GW^3_4)_{-6} \) by Lemma 3.3.6, and \( (GW^3_4)_{-6} = 0 \) by [AF15, Proposition 3.4.3], we conclude that (i) the image of the map \( K^M_6 \to \pi^A_3(S^{2+3\alpha}) \) is contained in \( F_5 \) and (ii) \( \pi^A_3(S^{2+3\alpha}) \to GW^3_4 \) factors through a map \( \pi^A_4(S^{3+3\alpha}) \to GW^3_4 \).

**Proposition 3.3.8.** There is an exact sequence

\[ S_5 \to \pi^A_4(S^{3+3\alpha}) \to GW^3_4 \to 0 \]

which becomes short exact after 4-fold contraction.

**Proof.** At the beginning of Section 3.2 we recalled from [AF15, Theorem 4.3.1] the fact that there is a morphism \( T_5 \to F_5 \) that becomes an isomorphism after 4-fold contraction. In particular, we conclude that the map \( \text{Hom}(K^M_6, T_5) \to \text{Hom}(K^M_6, F_5) \) is an isomorphism by Lemma 3.3.6. Therefore, the map \( K^M_6 \to F_5 \) from the EHP sequence lifts uniquely to a morphism \( K^M_6 \to T_5 \). By definition, \( T_5 \) is a fiber product of \( S_5 \) and \( I^5 \) along \( K^M_6/2 \) and there is an exact sequence of the form

\[ 0 \to I^6 \to T_5 \to S_5 \to 0. \]

However, since \( \text{Hom}(K^M_6, S_5) = 0 \) by combining [AF14a, Lemma 2.7 and Corollary 3.11]. Therefore, the image of the map \( K^M_6 \to T_5 \) factors through a map \( K^M_6 \to I^5 \).

Now, we appeal to Proposition 3.3.3. Indeed, it follows from this vanishing result that the map

\[ \pi^A_{5+6\alpha}(S^{5+6\alpha}) \to \pi^A_{3+6\alpha}(S^{2+3\alpha}) \]

is necessarily surjective. Unwinding the definitions, we conclude that the map \( K^M_6 \to I^6 \) of the previous paragraph is surjective, and we obtain the required exact sequence. Exactness after 4-fold contraction is immediate from [AF15, Theorem 4.3.1]. \( \square \)

**Comparison with the stable result**

In order to obtain complete the calculation of \( \pi^A_4(S^{3+3\alpha}) \) stated in the introduction, we compare with calculations that have been carried out \( P^1 \)-stably in [RS016]. To this end, recall that there is a stabilization map \( S^{3+3\alpha} \to \Omega^\infty_{p^1} \Sigma^\infty_{p^1} S^{3+3\alpha} \). By [AF17, Theorem 4.4.5] there are a sequence of morphisms \( S^{(n-1)+\alpha} \to \Omega^\infty_{p^1} O \) that stabilize to the degree map from the motivic sphere spectrum. As a consequence, the following diagram commutes:

\[ \begin{array}{ccc} \pi^A_4(S^{3+3\alpha}) & \to & GW_4^3 \\ & \downarrow & \downarrow \\ \pi^A_4(\Omega^\infty_{p^1} \Sigma^\infty_{p^1} S^{3+3\alpha}) & \to & GW_4^3. \end{array} \]
In combination with Proposition 3.3.8 we observe that there is an exact sequence of strictly $A^1$-invariant sheaves of the form

$$K^M_5/24 \rightarrow \pi^{A^1}_4(S^{3+3\alpha}) \rightarrow GW^3_4 \rightarrow 0.$$  

Moreover, both the groups appearing in this extension are stably non-trivial.

Now, suppose $k$ is a field having characteristic exponent $p$ and consider $\mathbb{Z}[\frac{1}{p}]$. Work of Röndigs, Spitzweck and Østvær [RSØ16, Theorem 5.5] establishes a short exact sequence of the form:

$$(3.3.5)\quad 0 \rightarrow K^M_5/24 \otimes \mathbb{Z}[\frac{1}{p}] \rightarrow \pi^{A^1}_4(\Omega^{\infty}_{p_1} \Sigma^{\infty}_{p_1} S^{3+3\alpha}) \otimes \mathbb{Z}[\frac{1}{p}] \rightarrow GW^3_4 \otimes \mathbb{Z}[\frac{1}{p}] \rightarrow 0.$$  

We remark that their computation does not look exactly like this, but their statement implies this one because the map out of $\pi^{A^1}_4(\Omega^{\infty}_{p_1} \Sigma^{\infty}_{p_1} S^{3+3\alpha})$ in their description is the unit map from the sphere spectrum to the spectrum $KO$.

In light of the discussion above involving [AF17, Theorem 4.4.5], the stabilization map then induces a commutative diagram of the form:

$$(3.3.6)\quad \begin{array}{ccc}
K^M_5/24 & \rightarrow & \pi^{A^1}_4(S^{3+3\alpha})
\downarrow f
\rightarrow & GW^3_4 & \rightarrow 0
\end{array}$$

some comments are in order about this morphism of exact sequences. We have suppressed the tensoring with $\mathbb{Z}[1/p]$ on the two terms on the left in the bottom row since $p$ will, momentarily, be taken to be different from 2 or 3; our goal is to analyze the left-hand vertical map and, to this end, we use the following result about endomorphisms of unramified Milnor K-theory sheaves.

**Lemma 3.3.9.** Let $n \geq 0$ and let $m$ be a nonegative integer. Then

$$\text{Hom}(K^n_M/m, K^n_M/m) = \mathbb{Z}/(m)$$

generated by the identity map.

**Proof.** The case $n = 0$ is trivial. We observe that if $n \geq 1$, then by Lemma 3.3.6, $\text{Hom}(K^n_M/m, K^n_M/m)$ is a subgroup of $(K^n_M/m)_n = K^n_M/m = \mathbb{Z}/(m)$. But the $n$-fold contraction functor sends the identity map in $\text{Hom}(K^n_M/m, K^n_M/m)$ to a generator of $\text{Hom}(K^n_M/m, K^n_M/m) = \mathbb{Z}/(m)$. □

**Theorem 3.3.10.** Let $k$ be a field having characteristic different from 2 or 3. There is short exact sequence

$$0 \rightarrow K^M_5/24 \rightarrow \pi^{A^1}_4(S^{3+3\alpha}) \rightarrow GW^3_4 \rightarrow 0.$$  

**Proof.** We would like to deduce that this diagram of (3.3.6) is essentially an isomorphism of short exact sequences. First of all, we show $f$ is an isomorphism. By Lemma 3.3.9, it suffices to check this after 5-fold contraction. The diagram becomes

$$(3.3.7)\quad \begin{array}{ccc}
\mathbb{Z}/(24) & \rightarrow & \pi^{A^1}_{4+5\alpha}(S^{3+3\alpha})
\downarrow f_{5}
\rightarrow & 0 & \rightarrow 0
\end{array}$$

$$\begin{array}{ccc}
0 & \rightarrow & \mathbb{Z}/(24)
\rightarrow & \pi^{A^1}_{4+5\alpha}(\Omega^{\infty}_{p_1} \Sigma^{\infty}_{p_1} S^{3+3\alpha}) & \rightarrow & 0 & \rightarrow 0.
\end{array}$$
Since \( \mathbb{Z}/(24) \) is independent of the field, we may verify \( f \) is an isomorphism after passage to an algebraically closed field. In characteristic 0, it follows from [RSO16, Remark 5.7] or [AWW17] that the \( \mathbb{Z}/(24) \) in both the source and the target of \( f_5 \) is generated by the motivic Hopf map \( \nu \). Since 24 is only divisible by 2 and 3, it follows using étale realization and lifting from characteristic 0 that \( \nu \) generates the stable group. Thus, in either case, the map sends a generator to a generator and must be an isomorphism. Therefore, the induced map \( K_5^M/24 \to K_5^M/24 \) is an isomorphism. Since there is an epimorphism \( K_5^M/24 \to S_5 \) factoring this isomorphism, we also conclude that \( S_5 \cong K_5^M/24 \). □

Suslin’s conjecture in degrees 3 and 5

We now analyze Suslin’s conjecture in 5, before doing this we make a remark about the degree 3 case.

Remark 3.3.11. The techniques we have developed above also show for any infinite field \( k \) having characteristic unequal to 2 and any essentially smooth local \( k \)-algebra \( A \), Suslin’s Hurewicz map \( K^Q_3(A) \to K^M_3(A) \) has image precisely \( 2K^M_3(A) \), which recovers Suslin’s original result. Indeed, granted Theorem 2.3.8, this result essentially follows from Morel’s identification of \( \pi^A_3(BGL_2) = K^MW_2 \). In more detail, the long exact sequence in \( A^1 \)-homotopy sheaves associated with the \( A^1 \)-fiber sequence \( A^3 \setminus 0 \to BGL_2 \to BGL_3 \); this takes the form

\[
\pi^A_3(BGL_3) \longrightarrow K^MW_3 \longrightarrow K^MW_2 \longrightarrow K^MW_2 \longrightarrow 0.
\]

The epimorphism \( K^MW_2 \to K^M_2 \) is the canonical epimorphism with kernel \( I^3 \) and one may show that the induced map \( K^MW_3 \to I^3 \) is the canonical epimorphism with kernel \( 2K^M_3 \), e.g., by appealing to étale and real realization along the lines of the proof of Lemma 3.3.5. It follows that the image of \( \pi^A_3(BGL_3) \) in \( K^MW_3 \), i.e., the image of \( \psi_3 \), is precisely \( 2K^M_3 \).

Theorem 3.3.12. For any infinite field \( k \) having characteristic unequal to 2 or 3 and any essentially smooth local \( k \)-algebra \( A \), Suslin’s Hurewicz map \( K^Q_5(A) \to K^M_5(A) \) has image precisely \( 4!K^M_5(A) \).

Proof. In the proof of Theorem 3.3.10 we showed that the canonical epimorphism \( K^M_5/24 \to S_5 \) is an isomorphism of strictly \( A^1 \)-invariant sheaves. The result then follows immediately from Theorem 2.3.8 by evaluation at stalks (recall that \( A^1 \)-homotopy sheaves have the property that the Zariski sheafification is already a Nisnevich sheaf, and so one may evaluate at Zariski stalks). □

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