Analytical results for the distribution of first return times of random walks on random regular graphs

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Abstract.
We present analytical results for the distribution of first return (FR) times of random walks (RWs) on random regular graphs (RRGs) consisting of $N$ nodes of degree $c \geq 3$. Starting from a random initial node $i$ at time $t = 0$, at each time step $t \geq 1$ an RW hops into a random neighbor of its previous node. We calculate the distribution $P(T_{FR} = t)$ of first return times to the initial node $i$. We distinguish between first return trajectories in which the RW retrocedes its own steps backwards all the way back to the initial node $i$ and those in which the RW returns to $i$ via a path that does not retrocede its own steps. In the retroceding scenario, each edge that belongs to the RW trajectory is crossed the same number of times in the forward and backward directions. In the non-retroceding scenario the subgraph that consists of the nodes visited by the RW and the edges it has crossed between these nodes includes at least one cycle. In the limit of $N \to \infty$ the RRG converges towards the Bethe lattice. The Bethe lattice exhibits a tree structure, in which all the first return trajectories belong to the retroceding scenario. Moreover, in the limit of $N \to \infty$ the trajectories of RWs on RRGs are transient in the sense that they return to the initial node with probability $< 1$. In this sense they resemble the trajectories of RWs on regular lattices of dimensions $d \geq 3$. The analytical results are found to be in excellent agreement with the results obtained from computer simulations.

Keywords: Random network, random regular graph, random walk, first passage time, first return time, recurrence, transience.
1. Introduction

Random walk (RW) models [1] provide useful tools for the analysis of dynamical processes on random networks [2,3] such as the spreading of rumours, opinions and infections [4–6]. Starting at time \( t = 0 \) from a random initial node \( i = x_0 \), at each time step \( t \geq 1 \) the RW hops randomly to one of the neighbors of its previous node. The resulting trajectory takes the form \( x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_t \rightarrow \ldots \), where \( x_t \) is the node visited at time \( t \). In some of the time steps the RW visits nodes that have not been visited before, while in other time steps it visits nodes that have already been visited at an earlier time. The mean number \( \langle S \rangle_t \) of distinct nodes visited by an RW on a random network up to time \( t \) was recently studied [7]. It was found that in the infinite network limit at sufficiently long times \( \langle S \rangle_t \simeq rt \), where the coefficient \( r < 1 \) depends on the network topology. These scaling properties resemble those obtained for RWs on high dimensional lattices and Cayley trees [8]. They imply that RWs on random networks revisit previously visited nodes less frequently than RWs on low dimensional lattices [9]. Therefore, RW models provide a highly effective framework for search and exploration processes on random networks.

For an RW starting from an initial node \( i \), the first return (FR) time \( T_{FR} \) is the first time at which the RW returns to \( i \) [10]. The first return time varies between different instances of the RW trajectory and its properties can be captured by a suitable distribution. The distribution of first return times may depend on the specific realization of the random network and on the choice of the initial node \( i \). The distribution of first return times for a given ensemble of random networks is denoted by \( P(T_{FR} = t) \). This distribution is calculated by averaging over many network instances drawn from the ensemble. For each network instance one needs to sample many RW trajectories starting from random initial nodes.

A more general problem involves the calculation of the first passage (FP) time \( T_{FP} \), which is the first time at which an RW starting from an initial node \( i \) visits a specified target node \( j \) [10–12]. The first return problem is a special case of the first passage problem, in which the initial node is also chosen as the target node. The distribution \( P(T_{FR} = t) \) of first return times was studied on the Bethe lattice, which exhibits a tree structure of an infinite size [13–15]. However, no closed-form analytical results are available for \( P(T_{FR} = t) \) on random networks of a finite size \( N \).

In this paper we present analytical results for the distribution of first return times of RWs on random regular graphs (RRGs) consisting of \( N \) nodes of degree \( c \geq 3 \). We consider separately the scenario in which the RW returns to the initial node \( i \) by retroceding (RETRO) its own steps and the scenario in which it does not retrocede its steps (¬RETRO) on the way back to \( i \). In the retroceding scenario an RW starting from the initial node \( i \) forms a random trajectory in the network and eventually returns to \( i \) by stepping backwards via the same edges that it crossed in the forward direction [Fig. 1(a)]. This implies that in the retroceding scenario each edge that belongs to the RW trajectory is crossed the same number of times in the forward and backward
Figure 1. Schematic illustrations of a retroceding first-return trajectory (a) and a non-retroceding first return trajectory (b) of an RW on an RRG. Note that in this illustration the RRG is of degree $c = 4$.

directions (which means that the total number of crossings of each edge must be an even number). As a result, in the retroceding scenario the first return time must be even, namely $T_{\text{FR}} = 2, 4, 6, \ldots$. In the infinite network limit, in which the RRG exhibits a tree structure, all the first return trajectories belong to the retroceding scenario. In this case the subgraph consisting of the nodes visited by the RW and of the edges it has crossed between these nodes exhibits a tree structure. Note that in an RRG of a finite size an RW may return at time $t'$ to a node that has been visited at an earlier time $t < t' - 2$, forming a cycle of length $t' - t$. There is a very rare possibility in which the RW may then retrocede its steps, crossing all the edges along the cycle in the backward direction to the node visited at time $t$ and then to the initial node $i$. In this case, the path is counted as a retroceding (RETRO) first return trajectory, in spite of the fact that it includes a cycle.

In the non-retroceding scenario an RW starting from $i$ forms a random trajectory in the network and eventually returns to $i$ without retroceding its own steps [Fig. 1(b)]. This means that in the non-retroceding scenario the trajectory includes at least three edges that are crossed an odd number of times. In the non-retroceding scenario the RW trajectory must include at least one cycle. Therefore, this scenario takes place only in RRGs of a finite size and is diminished in the infinite network limit. In the non-retroceding scenario the first return time may take any even or odd value that satisfies
First return times of random walks on random regular graphs

While the results for the retroceding trajectories are exact, in the analysis of the non-retroceding trajectories we use approximations that are accurate in the large \( N \) and long time limits. Therefore, the analytical results for the non-retroceding scenario are approximate results.

From the discussion above, it is clear that the retroceding and non-retroceding scenarios are mutually exclusive. More specifically, a first return RW trajectory in which every single edge is crossed the same number of times in the forward and backward directions belongs to the retroceding scenario, otherwise it belongs to the non-retroceding scenario. Using combinatorial and probabilistic methods we calculate the conditional distributions of first return times, \( P(T_{FR} = t | \text{RETRO}) \) and \( P(T_{FR} = t | \neg \text{RETRO}) \), in the retroceding and non-retroceding scenarios, respectively. We also calculate the mean and variance of each one of the two conditional distributions. We combine the results of the two scenarios with suitable weights and obtain the overall distribution of first return times \( P(T_{FR} = t) \) of RWs on RRGs of a finite size. In the limit of \( N \to \infty \) the RRG converges towards the Bethe lattice. The Bethe lattice exhibits a tree structure, in which all the first return trajectories belong to the retroceding scenario. It is also found that in the infinite network limit the trajectories of RWs on RRGs are transient in the sense that they return to the initial node with probability \(< 1 \). In this sense they resemble the trajectories of RWs on regular lattices of dimensions \( d \geq 3 \). The analytical results are found to be in excellent agreement with the results obtained from computer simulations.

The paper is organized as follows. In Sec. 2 we briefly describe the random regular graph. In Sec. 3 we present the random walk model. In Sec. 4 we calculate the distribution of first return times of RWs on RRGs in the retroceding scenario. In Sec. 5 we calculate the distribution of first return times in the non-retroceding scenario. To this end, we derive a closed-form expression for \( \langle S \rangle_t \) in RRGs of a finite size, which is accurate at intermediate and long times. In Sec. 6 we combine the results obtained in sections 4 and 5 to obtain the overall distribution of first return times. In Sec. 7 we use the results obtained for the distribution of first return times to derive a more refined expression for \( \langle S \rangle_t \), which is accurate also at short times. In Sec. 8 we discuss the results and compare the return probability of an RW on an RRG to the return probability of an RW on a regular lattice with the same coordination number. The results are summarized in Sec. 9. In Appendix A we present an asymptotic expansion of Pólya’s constant, which provides the return probability of an RW on a \( d \)-dimensional hypercubic lattice.

2. The random regular graph

A random network (or graph) consists of a set of \( N \) nodes that are connected by edges in a way that is determined by some random process. For example, in a configuration model network the degree of each node is drawn independently from a given degree distribution \( P(k) \) and the connections are random and uncorrelated \cite{16,18}. The RRG
is a special case of a configuration model network, in which the degree distribution is a degenerate distribution of the form $P(k) = \delta_{k,c}$, namely all the nodes are of the same degree $c$. Here we focus on RRGs of a finite size $N$ and degree $c \geq 3$, which for a sufficiently large $N$ consist of a single connected component [19].

In the infinite network limit RRGs with a finite degree $c \geq 3$ exhibit a tree structure with no cycles. Thus, in this limit it coincides with a Bethe lattice whose coordination number is equal to $c$. In contrast, RRGs of a finite size exhibit a local tree-like structure, while at larger scales there is a broad spectrum of cycle lengths. In that sense RRGs differ from Cayley trees, which maintain their tree structure by reducing the most peripheral nodes to leaf nodes of degree 1.

A convenient way to construct an RRG of size $N$ and degree $c$ ($Nc$ must be an even number) is to prepare the $N$ nodes such that each node is connected to $c$ half edges or stubs [3]. At each step of the construction, one connects a random pair of stubs that belong to two different nodes $i$ and $j$ that are not already connected, forming an edge between them. This procedure is repeated until all the stubs are exhausted. The process may get stuck before completion in case that all the remaining stubs belong to the same node or to pairs of nodes that are already connected. In such case one needs to perform some random reconnections in order to complete the construction.

3. The random walk model

Consider an RW on an RRG, starting from a random initial node $i$ at time $t = 0$. At each time step $t \geq 1$ the RW hops randomly into one of the neighbors of its previous node. Since RRGs with $c \geq 3$ consist of a single connected component, an RW starting from any initial node can reach any other node in the network. At each time step $t \geq 2$ the RW may either step into a yet-unvisited node or into a node that has already been visited before. In Fig. 2 we present a schematic illustration of some of the events that may take place along the path of an RW on an RRG. In Fig. 2(a) we show a path segment in which at each time step the RW enters a node that has not been visited before. In Fig. 2(b) we show a path segment that includes a backtracking step, in which the RW moves back into the previous node (step no. 4). In Fig. 2(c) we show a path segment that includes a backtracking step (step no. 4) which is followed by a retroceding step (step no. 5). In Fig. 2(d) we show a path segment that includes a retracing step (step no. 5), in which the RW enters a node that was visited four time steps earlier. Retracing steps are not possible in the infinite network limit in which the RRG exhibits a tree structure.

The mean number of distinct nodes that are visited by an RW up to time $t$ is denoted by $\langle S \rangle_t$. The difference

$$\Delta_t = \langle S \rangle_t - \langle S \rangle_{t-1}$$

is the probability that at time $t$ the RW will visit a node that has not been visited before. For example, $\Delta_1 = 1$ and $\Delta_2 = (c-1)/c$. Using a generating function formulation based
on the cavity method, it was shown that the probability that an RW on an RRG of size $N \to \infty$ at time $t \gg 1$ will step into a yet-unvisited node is given by \[ \Delta_t = \frac{c - 2}{c - 1}. \] (2)

A similar result was obtained for RWs on the Bethe-Lattice \[ 14, 20]. \] The behavior of $\Delta_t$ in RRGs of a finite size has not been studied. However, Eq. (2) provides accurate results for $\Delta_t$ on finite RRGs in a range of intermediate times $1 \ll t \ll N$, namely at
times which are not too short but much shorter than the network size.

An RW that returns to the initial node with probability 1 is called a recurrent RW, while an RW that returns to the initial node with probability < 1 is called a transient RW [21]. Recurrent RWs return to the initial node infinitely many times while transient RWs return to the initial node only a finite number of times. In a seminal paper by G. Pólya, published 100 years ago, it was shown that an RW on a d-dimensional hypercubic lattice is recurrent in dimensions $d = 1, 2$ and transient in dimensions $d \geq 3$ [22]. Thus, in general, RWs on infinite systems may be either recurrent or transient, depending on the structure and dimension of the underlying network or lattice. In contrast, RWs on finite systems are always recurrent.

4. The distribution of first return times via retroceding trajectories

Consider an RW on an RRG, starting at $t = 0$ from a random initial node $i$. In the infinite network limit the RW may either return to $i$ via a retroceding trajectory or drift away without ever returning to $i$. In a finite network the RW returns to $i$ with probability 1 either via a retroceding trajectory or via a non-retroceding trajectory. We first consider the retroceding trajectories. The probability that an RW will first return to $i$ at time $t$, via a retroceding trajectory, is given by

$$P(T_{FR} = t, \text{RETRO}) = \frac{B(t)}{c^t},$$

where $B(t), t \geq 2$, is the number of retroceding RW trajectories that return to the initial node $i$ for the first time at time $t$. In the retroceding first return trajectories each edge is crossed the same number of times in the forward and in the backward directions. Therefore, retroceding trajectories exist only for even values of $t$. The first step is always an outward step (from $i$ to one of its neighbors), while the last step is always an inward step (from one of the neighbors back to $i$). The other $t - 2$ steps can be ordered in many different ways, as long as at each intermediate time the number of outward steps exceeds the number of inward steps by at least 1. The number of ways to order the inward and outward steps under this condition is given by the Catalan number $C_{\frac{t}{2} - 1}$, where

$$C_k = \frac{(2k)!}{k!(k + 1)!} = \frac{1}{2k + 1} \binom{2k + 1}{k + 1} = \frac{1}{k + 1} \binom{2k}{k}.$$  

The Catalan number appears in many combinatorial problems. For example, it counts the number of discrete mountain ranges of length $2k$ [23][24]. For an RRG of degree $c$ the number of possibility for the first (outward) step is $c$ while the number of possibilities for each one of the remaining $t/2 - 1$ outward steps is $c - 1$. The inward steps follow the path carved by the outer steps so they do not contribute additional factors of $c$. Multiplying the Catalan number by the factors of $c$ and $c - 1$, we obtain
$B(t) = \begin{cases} \frac{c(c-1)\frac{t}{2}-1}{2} & \text{if } t \text{ even} \\ 0 & \text{if } t \text{ odd.} \end{cases}$ (5)

The overall probability that an RW will first return to its initial node $i$ via a retroceding trajectory is given by

$$P(\text{RETRO}) = \sum_{t=2}^{\infty} P(T_{FR} = t, \text{RETRO}).$$ (6)

Inserting $P(T_{FR} = t, \text{RETRO})$ from Eq. (3) into Eq. (6) and carrying out the summation, we obtain

$$P(\text{RETRO}) = \frac{1}{c - 1}.$$ (7)

This result is in agreement with Eq. (30) in Ref. [13], which provides the probability of return in the Bethe lattice. Thus, in the infinite network limit, in which the RW is transient, the probability $P(R)$ that an RW will return to its initial node $i$ satisfies $P(R) = P(\text{RETRO})$. The complementary probability that an RW on a finite RRG will first return to $i$ via a non-retroceding path (or that an RW on an infinite RRG will never return to $i$), is given by

$$P(\neg\text{RETRO}) = \frac{c - 2}{c - 1}.$$ (8)

In Fig. 3 we present analytical results for the probability $P(\text{RETRO})$ that the first return process will take place via a retroceding trajectory and the probability $P(\neg\text{RETRO})$ that it will take place via a non-retroceding trajectory, as a function of the degree $c$ for random regular graphs of size $N = 1000$. The analytical results, obtained from Eqs. (7) and (8), respectively, are in excellent agreement with the results obtained from computer simulations (circles).

For the simulations we generated a large number (typically 100) of random instances of the RRG consisting of $N$ nodes of degree $c$, using the procedure presented in Sec. 2. For each network instance, we generated a large number (typically 100) of RW trajectories, where each trajectory starts from a random initial node $i$ at time $t = 0$. The simulation results are obtained by averaging over all these trajectories. In the simulations, at each time step $t$ the RW selects randomly one of the $c$ neighbors of the node $x_{t-1}$, where the probability of each neighbor to be selected is $1/c$. It then hops to the selected node, denoted by $x_t$. Each RW trajectory is terminated upon its first return to $i$. The first return time $t$ is thus equal to the length of the trajectory. The trajectory $x_0, x_1, \ldots, x_t$ is recorded for further analysis. It is then determined whether the first return occurred via the retroceding or the non-retroceding scenario. More specifically, if all the edges along the RW trajectory are crossed an equal number of times in the forward and the backward directions (where the forward direction is the direction of the...
Figure 3. Analytical results for the probability $P(\text{RETRO})$ that the first return process will take place via a retroceding trajectory and the probability $P(\neg\text{RETRO})$ that it will take place via a non-retroceding trajectory, as a function of the degree $c$ for random regular graphs of size $N = 1000$. The analytical results, obtained from Eqs. (7) and (8), respectively, are in excellent agreement with the results obtained from computer simulations (circles).

first crossing), we conclude that the first return occurred via the retroceding scenario. Otherwise, we conclude that it occurred via the non-retroceding trajectory.

The distribution of first return times under the condition of retroceding paths is given by

$$P(T_{\text{FR}} = t | \text{RETRO}) = \frac{P(T_{\text{FR}} = t, \text{RETRO})}{P(\text{RETRO})}. \quad (9)$$

Inserting $P(T_{\text{FR}} = t, \text{RETRO})$ from Eq. (3) and $P(\text{RETRO})$ from Eq. (7) into Eq. (9), we obtain

$$P(T_{\text{FR}} = t | \text{RETRO}) = \begin{cases} C_{\frac{t}{2}-1} \left( 1 - \frac{1}{c} \right)^{\frac{t}{2}} \left( \frac{1}{c} \right) \left( \frac{1}{c} \right)^{t-1} & \text{t even} \\ 0 & \text{t odd} \end{cases}, \quad (10)$$

where $C_{\frac{t}{2}-1}$ is given by Eq. (4). Eq. (10) is in agreement with Eq. (24) in Ref. [8], which provides the distribution of first return times on Cayley trees.

The generating function of the distribution of first return times in the retroceding scenario is denoted by

$$V_{\text{RETRO}}(x) = \sum_{k=1}^{\infty} x^{2k} P(T_{\text{FR}} = 2k | \text{RETRO}). \quad (11)$$

Inserting $P(T_{\text{FR}} = 2k | \text{RETRO})$ from Eq. (10) into Eq. (11) and carrying out the summation, we obtain

$$V_{\text{RETRO}}(x) = \frac{c - \sqrt{c^2 - 4(c-1)x^2}}{2}. \quad (12)$$
The probability $P(T_{FR} = t|\text{RETRO})$ can be obtained from the generating function by differentiation via

$$P(T_{FR} = t|\text{RETRO}) = \left. \frac{1}{t!} \frac{d^t}{dx^t} \mathcal{V}_{\text{RETRO}}(x) \right|_{x=0}.$$  \hspace{1cm} (13)

Inserting $x = 1$ in Eq. (12) we obtain $\mathcal{V}(1) = 1$, which confirms the normalization of the distribution $P(T_{FR} = t|\text{RETRO})$.

The tail distribution of first return times is given by

$$P(T_{FR} > t|\text{RETRO}) = \sum_{t'=t+1}^{\infty} P(T_{FR} = t'|\text{RETRO}).$$  \hspace{1cm} (14)

Inserting $P(T_{FR} = t|\text{RETRO})$ from Eq. (10) into Eq. (14) and carrying out the summation, we obtain

$$P(T_{FR} > t|\text{RETRO}) = C \left\lfloor \frac{t}{2} \right\rfloor \left( \frac{c}{c-2} \right)^{\left\lfloor \frac{t}{2} \right\rfloor} \left( \frac{c-1}{c^2} \right)^{\left\lfloor \frac{t}{2} \right\rfloor} \mathcal{F}_1 \left[ \left\lfloor \frac{t}{2} \right\rfloor + 2, \frac{3}{2} - \frac{4(c-1)}{(c-2)^2} \right],$$  \hspace{1cm} (15)

where $\lfloor x \rfloor$ is the integer part of $x$. The function $\mathcal{F}_1[\ ]$ on the right hand side of Eq. (15) is the hypergeometric function, which is given by

$$\mathcal{F}_1 \left[ a, b, c \mid z \right] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!},$$  \hspace{1cm} (16)

where $(q)_n$ is the (rising) Pochhammer symbol [25].

In order to calculate the moments of $P(T_{FR} = t|\text{RETRO})$ we introduce the moment generating function $M_{\text{RETRO}}(x) = \mathcal{V}_{\text{RETRO}}(e^x)$. It is given by

$$M_{\text{RETRO}}(x) = \frac{c - \sqrt{c^2 - 4(c-1)e^{2x}}}{2}.$$  \hspace{1cm} (17)

Expanding $M_{\text{RETRO}}(x)$ to second order in powers of $x$, we obtain the mean first return time

$$E[T_{FR}|\text{RETRO}] = \frac{2(c-1)}{c-2},$$  \hspace{1cm} (18)

and the second moment

$$E[T_{FR}^2|\text{RETRO}] = \frac{4(c-1)(c^2 - 2c + 2)}{(c-2)^3}.$$  \hspace{1cm} (19)

Combining the results presented above for the first and second moments, we obtain the variance

$$\text{Var}[T_{FR}|\text{RETRO}] = \frac{4c(c-1)}{(c-2)^3}.$$  \hspace{1cm} (20)
From Eq. (18) we conclude that the mean first return time is of order $1$, and from Eq. (20) we conclude that the distribution of first return times is very narrow. This means that the retroceding scenario takes place only at very short times.

5. The distribution of first return times via non-retroceding trajectories

Consider an RW starting from a random initial node $i$ at $t = 0$. The mean number of distinct nodes visited by the RW up to time $t$ is denoted by $\langle S \rangle_t$. The probability $P(T_{FR} \leq t | \neg \text{RETRO})$ that the RW will return to the initial node up to time $t$ under the condition that it will occur via the non-retroceding scenario is essentially the probability that one of the $\langle S \rangle_{t-1}$ nodes visited during $t' = 1, 2, \ldots, t$ is the initial node $i$. This probability is given by

$$P(T_{FR} \leq t | \neg \text{RETRO}) = \begin{cases} 0 & t = 0, 1, 2 \\ \frac{\langle S \rangle_{t-1} - 2}{N-2} & t \geq 3. \end{cases}$$

(21)

The numerator on the right hand side of Eq. (21), for $t \geq 3$, represents the expected number of distinct nodes visited by an RW in the time interval $1 \leq t' \leq t$, apart from the nodes visited at $t' = 1$ and $t' = 2$. The point is that the nodes visited at times $t' = 1$ and $t' = 2$ in the non-retroceding scenario are clearly distinct from the initial node $i$. In contrast, at any later time, $t' \geq 3$ the RW may return to $i$ via the non-retroceding scenario. The denominator on the right hand side of Eq. (21) represents the number of nodes in the network, apart from the nodes visited at $t' = 1$ and $t'$ = 2, namely the same two nodes that are offset in the numerator. While the RW may visit these two nodes at times $t' \geq 3$, such visits would not make any contribution to $\langle S \rangle_{t-1}$ because these would be return visits. In the RRG all the nodes are of the same degree while the connectivity is random. Therefore, all the nodes in the network (apart from the nodes visited at $t' = 1$ and $t' = 2$) have the same probability to be visited by the RW in the time interval $3 \leq t' \leq t$. Therefore, the right hand side of Eq. (21) represents the probability that an RW will return to the initial node $i$ up to time $t$. The proper normalization of this expression is apparent from the fact that $(\langle S \rangle_{t-1} - 2)/(N-2) \to 1$ in the limit of $t \to \infty$.

The complementary probability, namely the probability that the RW will not return to $i$ within the first $t$ time steps, under the condition that the first return process will take place via the non-retroceding scenario, is given by

$$P(T_{FR} > t | \neg \text{RETRO}) = \begin{cases} 1 & t = 0, 1, 2 \\ 1 - \frac{\langle S \rangle_{t-1} - 2}{N-2} & t \geq 3. \end{cases}$$

(22)

In order to utilize Eq. (22) we derive below a closed-form expression for $\langle S \rangle_t$. To this end, we first consider the probability $\Delta_t$, which at intermediate times $1 \ll t \ll N$ it is given by Eq. (2). At longer times, $\Delta_t$ is proportional to the fraction of yet-unvisited nodes among all the nodes in the network, apart from the nodes visited at times $t - 1$ and $t - 2$. Therefore, a saturation term emerges and
Eq. (23) indicates that the probability of an RW at time $t$ to enter a yet-unvisited node depends on $t$ via the expected number of distinct nodes $\langle S \rangle_{t-1}$ that have been visited up to time $t-1$. Thus, the probability of an RW that has already visited $s$ distinct nodes to enter a yet-unvisited node in the next time step is given by

$$\Delta(t) = \begin{cases} 
\frac{1}{c-1} \left(1 - \frac{\langle S \rangle_{t-1} - 2}{N-2}\right) & t = 1 \\
\frac{c-2}{c-1} \left(1 - \frac{s-2}{N-2}\right) & s \geq 2.
\end{cases}$$

(23)

In Fig. 4 we present analytical results for the probability $\Delta(s)$ of an RW that has already visited $s$ distinct nodes to hop in the next time step into a node that has not been visited before, on RRGs of size $N = 1000$ and degrees $c = 3$ (solid line), $c = 4$ (dashed line) and $c = 10$ (dotted line). The analytical results, obtained from Eq. (24), are in excellent agreement with the results obtained from computer simulations (circles).

Inserting $\Delta_t$ from Eq. (23) into Eq. (1), we obtain a recursion equation for $\langle S \rangle_t$ of the form

$$\langle S \rangle_t = \frac{c-2}{c-1} \left(1 + \frac{2}{N-2}\right) + \left[1 - \frac{c-2}{c-1}\right] \frac{1}{N-2} \langle S \rangle_{t-1},$$

(25)

that applies to $t \geq 2$. Solving Eq. (25) with the initial condition $\langle S \rangle_1 = 2$, we obtain

$$\langle S \rangle_t = 2 + (N - 2) \left\{1 - \left[1 - \frac{c-2}{c-1}\right] \frac{1}{N-2} \right\}^{t-1}.$$

(26)
Approximating the square brackets on the right hand side of Eq. (26) by an exponential function, we obtain

$$\langle S \rangle_t \simeq 2 + (N - 2) \left\{ 1 - \exp \left[ - \left( \frac{c - 2}{c - 1} \right) \frac{t - 1}{N - 2} \right] \right\}. \quad (27)$$

Eq. (27) depends on time only via \((t - 1)/(N - 2)\), which highlights the fact that \(\langle S \rangle_t\) is a slowly varying quantity. It can be characterized by the time \(\tau\) at which the RW is expected to complete visiting half of the nodes in the network, or formally \(\langle S \rangle_\tau = N/2\). The time \(\tau\), given by

$$\tau = \left( \frac{c - 1}{c - 2} \right) N \ln(2), \quad (28)$$

is analogous to the half-life time of radioactive materials, which is the time it takes until half of the nuclei undergo radioactive decay.

In Fig. 5 we present analytical results for the mean number \(\langle S \rangle_t\) of nodes visited by an RW up to time \(t\) on an RRG of size \(N = 1000\) and degrees \(c = 3\) (solid line), \(c = 4\) (dashed line) and \(c = 10\) (dotted line). The analytical results, obtained from Eq. (27), are in excellent agreement with the results obtained from computer simulations (circles). The results demonstrate the linear behavior of \(\langle S \rangle_t\) vs. \(t\) at early times. As \(c\) is increased the slope \(d\langle S \rangle_t/dt\) at early times becomes steeper. This implies that as \(c\) is increased the number of distinct nodes visited by the RW up to a given time \(t\) increases.

In Fig. 6 we present analytical results for the probability \(\Delta_t\) of an RW at time \(t\) to step into a node that has not been visited before, on an RRG of size \(N = 1000\) and degrees \(c = 3\) (solid line), \(c = 4\) (dashed line) and \(c = 10\) (dotted line). The analytical results, obtained from Eq. (23), in which \(\langle S \rangle_{t-1}\) is given by Eq. (27) are in excellent agreement with the results obtained from computer simulations (circles).
Figure 6. Analytical results for the probability $\Delta_t$ of an RW at time $t$ to hop into a new node that has not been visited before, on a random regular graph of size $N = 1000$ and degrees $c = 3$ (solid line), $c = 4$ (dashed line) and $c = 10$ (dotted line). The analytical results, obtained from Eq. (23), in which $\langle S \rangle_t - 1$ is given by Eq. (27), are in excellent agreement with the results obtained from computer simulations (circles).

agreement with the results obtained from computer simulations (circles).

Plugging in $\langle S \rangle_t$ from Eq. (27) into Eq. (22) and taking the long-time limit, we obtain the tail distribution of first return times via the non-retroceding scenario. It is given by

$$P(T_{FR} > t | \neg \text{RETRO}) = \begin{cases} 1 & t = 0, 1, 2 \\ \exp \left[ - \left( \frac{c-2}{c-1} \right) \frac{t-2}{N-2} \right] & t \geq 3. \end{cases} \quad (29)$$

The probability mass function of the distribution of first return times is given by

$$P(T_{FR} = t | \neg \text{RETRO}) = P(T_{FR} > t - 1 | \neg \text{RETRO}) - P(T_{FR} > t | \neg \text{RETRO}), \quad (30)$$

or

$$P(T_{FR} = t | \neg \text{RETRO}) = \begin{cases} 0 & t = 0, 1, 2 \\ \exp \left[ - \left( \frac{c-2}{c-1} \right) \frac{t-3}{N-2} \right] - \exp \left[ - \left( \frac{c-2}{c-1} \right) \frac{t-2}{N-2} \right] & t \geq 3. \end{cases} \quad (31)$$

The generating function of the distribution of first return times in the non-retroceding scenario is given by

$$V_{\neg \text{RETRO}}(x) = \sum_{t=3}^{\infty} x^t P(T_{FR} = t | \neg \text{RETRO}). \quad (32)$$

Inserting $P(T_{FR} = t | \neg \text{RETRO})$ from Eq. (31) into Eq. (32), we obtain
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\[ V_{\text{RETRO}}(x) = x^2 \left\{ 1 + \frac{x - 1}{1 - x \exp \left[ \left( \frac{c-2}{c-1} \right) \frac{1}{N-2} \right] } \right\}. \quad (33) \]

In order to calculate the moments of \( P(T_{FR} = t | \neg \text{RETRO}) \) we introduce the moment generating function \( M_{\text{RETRO}}(x) = V_{\text{RETRO}}(e^x) \). It is given by

\[ M_{\text{RETRO}}(x) = e^{2x} \left\{ 1 + \frac{e^x - 1}{1 - e^x \exp \left[ \left( \frac{c-2}{c-1} \right) \frac{1}{N-2} \right] } \right\}. \quad (34) \]

We also introduce the cumulant generating function \( K_{\text{RETRO}}(x) = \ln M_{\text{RETRO}}(x) \). It is given by

\[ K_{\text{RETRO}}(x) = 2x + \ln \left\{ 1 + \frac{e^x - 1}{1 - e^x \exp \left[ \left( \frac{c-2}{c-1} \right) \frac{1}{N-2} \right] } \right\}. \quad (35) \]

Expanding \( K_{\text{RETRO}}(x) \) in powers of \( x \), we obtain

\[ K_{\text{RETRO}}(x) = \left\{ 2 + \frac{1}{1 - \exp \left[ \left( \frac{c-2}{c-1} \right) \frac{1}{N-2} \right] } \right\} x \]
\[ + \frac{\exp \left[ \left( \frac{c-2}{c-1} \right) \frac{1}{N-2} \right] }{2 \left\{ 1 - \exp \left[ \left( \frac{c-2}{c-1} \right) \frac{1}{N-2} \right] \right\}^2} x^2 + O(x^3). \quad (36) \]

Thus, the mean first return time in the non-retroceding scenario is given by

\[ \mathbb{E}[T_{FR} | \neg \text{RETRO}] = 2 + \frac{1}{1 - \exp \left[ \left( \frac{c-2}{c-1} \right) \frac{1}{N-2} \right]}, \quad (37) \]

and the variance of the distribution of first return times is given by

\[ \text{Var}[T_{FR} | \neg \text{RETRO}] = \frac{\exp \left[ \left( \frac{c-2}{c-1} \right) \frac{1}{N-2} \right] }{\left\{ 1 - \exp \left[ \left( \frac{c-2}{c-1} \right) \frac{1}{N-2} \right] \right\}^2}. \quad (38) \]

Expanding these expressions for large \( N \), we obtain

\[ \mathbb{E}[T_{FR} | \neg \text{RETRO}] \simeq \left( \frac{c-1}{c-2} \right) N + \frac{1}{2} - \frac{2}{c-2}, \quad (39) \]

and

\[ \text{Var}[T_{FR} | \neg \text{RETRO}] \simeq \left( \frac{c-1}{c-2} \right)^2 (N-2)^2. \quad (40) \]

Therefore, in the large \( N \) limit both the mean and the standard deviation of the distribution of first return times scale like \( N \). This implies that the distribution of first return times via the non-retroceding scenario is a broad distribution.
6. The overall distribution of first return times

The overall distribution of first return times can be expressed in the form

\[
P(T_{FR} = t) = P(T_{FR} = t | RETRO) P(RETRO) \\
+ P(T_{FR} = t | \neg RETRO) P(\neg RETRO). \tag{41}
\]

Since \( \mathbb{E}[T_{FR} | RETRO] \) is of order 1, while \( \mathbb{E}[T_{FR} | \neg RETRO] \) is of order \( N \), there is a clear separation of time scales between the retroceding and the non-retroceding scenarios. Inserting \( P(T_{FR} = t | RETRO) \) from Eq. (10) and \( P(T_{FR} = t | \neg RETRO) \) from Eq. (31) into Eq. (41), we obtain

\[
P(T_{FR} = t) = \begin{cases} \frac{B(t)}{c^t} + \left( \frac{c-2}{c} \right) \exp \left[ - \left( \frac{c-2}{c} \right) \frac{t}{N-2} \right] - \exp \left[ - \left( \frac{c-2}{c} \right) \frac{t}{N-2} \right] & t = 0, 1, 2 \\
\frac{B(t)}{c^t} + \left( \frac{c-2}{c} \right) \exp \left[ - \left( \frac{c-2}{c} \right) \frac{t}{N-2} \right] \left[ 1 - \left( \frac{c-2}{c} \right) \frac{t}{N-2} \right] & t \geq 3. \end{cases} \tag{42}
\]

where \( B(t) \) is given by Eq. (5). In the limit of \( N \to \infty \), Eq. (42) is reduced to

\[
P(T_{FR} = t) = \frac{B(t)}{c^t}. \tag{43}
\]

The generating function of \( P(T_{FR} = t) \) is given by

\[
V(x) = \frac{c - \sqrt{c^2 - 4(c-1)x^2}}{2(c-1)} + x^2 \left\{ 1 + \left( \frac{c-2}{c-1} \right) \frac{x-1}{1-x} \exp \left[ - \left( \frac{c-2}{c-1} \right) \frac{1}{N-2} \right] \right\}. \tag{44}
\]

The tail distribution of first return times can be expressed in the form

\[
P(T_{FR} > t) = P(T_{FR} > t | RETRO) P(RETRO) \\
+ P(T_{FR} > t | \neg RETRO) P(\neg RETRO). \tag{45}
\]

Inserting \( P(T_{FR} > t | RETRO) \) from Eq. (15) and \( P(T_{FR} > t | \neg RETRO) \) from Eq. (29) into Eq. (45), we obtain

\[
P(T_{FR} > t) = C_{\frac{1}{2}} \frac{c}{(c-2)^2} \left( \frac{c-1}{c^2} \right)^{\frac{1}{2}} _2F_1 \left[ \begin{array}{c} 1, \frac{3}{2} \\ \frac{1}{2} + 2 \end{array} \right] - \frac{4(c-1)}{(c-2)^2} \\
+ \left( \frac{c-2}{c-1} \right) \exp \left[ - \left( \frac{c-2}{c-1} \right) \frac{t-2}{N-2} \right]. \tag{46}
\]

In the limit of \( N \to \infty \) Eq. (46) is reduced to

\[
P(T_{FR} > t) = C_{\frac{1}{2}} \frac{c}{(c-2)^2} \left( \frac{c-1}{c^2} \right)^{\frac{1}{2}} _2F_1 \left[ \begin{array}{c} 1, \frac{3}{2} \\ \frac{1}{2} + 2 \end{array} \right] - \frac{4(c-1)}{(c-2)^2} \\
+ \left( \frac{c-2}{c-1} \right). \tag{47}
\]
The mean first return time $\langle T_{FR} \rangle$ of RWs on RRGs can be obtained exactly using the Kac lemma, which employs general properties of discrete stochastic processes [26]. Applying the Kac lemma to an RW on an undirected random graph consisting of a single connected component of size $N$, it implies that the mean first return time to a given initial node $i$ is $\langle T_{FR} \rangle_i = 1/P(i)$, where $P(i)$ is the fraction of time at which the RW resides at node $i$ in the limit of an infinitely long trajectory. In general, $P(i)$ is given by $P(i) = k_i/[N\langle K \rangle]$, where $k_i$ is the degree of $i$ and $\langle K \rangle$ is the mean degree of the network [6]. In the case of an RRG consisting of $N$ nodes, $P(i) = 1/N$ for all the nodes. As a result, the mean first return time for all nodes is $\langle T_{FR} \rangle = N$.

Combining the results obtained above for the retroceding and the non-retroceding scenarios, the mean first return time $\langle T_{FR} \rangle$ can be expressed in the form

$$
\langle T_{FR} \rangle = E[T_{FR}|RETRO]P(RETRO) \\
+ E[T_{FR}|\neg RETRO]P(\neg RETRO). \tag{48}
$$

Inserting $E[T_{FR}|RETRO]$ from Eq. (18) and $E[T_{FR}|\neg RETRO]$ from Eq. (37) into Eq. (48), we obtain

$$
\langle T_{FR} \rangle = \frac{2}{c-2} + \left(\frac{c-2}{c-1}\right) \left\{2 + \frac{1}{1 - \exp\left[-\frac{1}{N-2}\right]}\right\}. \tag{49}
$$

In the large network limit of $N \gg 1$, we obtain

$$
\langle T_{FR} \rangle = N + \mathcal{O}(1). \tag{50}
$$

This result essentially agrees with the Kac lemma, apart from a discrepancy of order 1, which does not scale with $N$. The discrepancy is due to the approximations used in
Figure 8. Analytical results for the mean first return time $\langle T_{FR} \rangle$ (solid line), the mean first return time $E[T_{FR}|RETRO]$ conditioned on retroceding trajectories (dotted line), and the mean first return time $E[T_{FR}|\neg RETRO]$ conditioned on non-retroceding trajectories (dashed line), as a function of the degree $c$ for random regular graphs of size $N = 1000$. The inset shows a magnified view of $E[T_{FR}|RETRO]$, revealing the $c$-dependence of this curve. The analytical results, obtained from Eqs. (49), (18) and (37), are in excellent agreement with the results obtained from computer simulations (circles).

the derivation of the distribution of first return times in the non-retroceding scenario, in which we took the limits of large $N$ and long times.

In Fig. 7 we present analytical results for the tail distribution of first return times of RWs on an RRG of size $N = 1000$ and $c = 3$, $c = 4$ and $c = 10$. The analytical results, obtained from Eq. (46), are in excellent agreement with the results obtained from computer simulations (circles).

In Fig. 8 we present analytical results for the mean first return time $\langle T_{FR} \rangle$ (solid line), the mean first return time conditioned on retroceding trajectories $E[T_{FR}|RETRO]$ (dotted line) and on non-retroceding trajectories $E[T_{FR}|\neg RETRO]$ (dashed line), as a function of the degree $c$ for random regular graphs of size $N = 1000$. The analytical results, obtained from Eqs. (49), (18) and (37), respectively, are in excellent agreement with the results obtained from computer simulations (circles).

7. A more accurate analytical expression for $\langle S \rangle_t$

Consider an RW path segment of the form $x_0 \rightarrow x_1 \rightarrow x_2 \cdots \rightarrow x_t$ on an RRG consisting of $N$ nodes of degree $c$. For any initial node $x_0$, the probability of a given path of length $t$ to occur is $1/c^t$. Thus, in an infinitely long RW trajectory, the path segment $x_0 \rightarrow x_1 \rightarrow x_2 \cdots \rightarrow x_t$ (referred to as the forward path) would occur with the same frequency as the time reversal (or backward) path $x_t \rightarrow x_{t-1} \rightarrow x_{t-2} \rightarrow \cdots \rightarrow x_1 \rightarrow x_0$ in which the RW visits the same sequence of nodes but in the opposite direction.

Below we use an ensemble of such path segments to explore the relation between
the distribution of first return times $P(T_{FR} > t)$ and the probability $\Delta_t$. The probability $P(T_{FR} > t)$ that the FR time of an RW starting from a random node $x_0$ will be larger than $t$ is equal to the fraction of forward path segments that satisfy $x_{t'} \neq x_0$ for $t' = 1, 2, \ldots, t$, drawn from an ensemble of RW path segments. Similarly, the probability $\Delta_t$ that an RW will step into a previously unvisited node at time $t$ is equal to the fraction of forward path segments that satisfy $x_t \neq x_{t'}$ for $t' = 0, 1, 2, \ldots, t$. Since the statistical weights of the backward path segments are equal to the weights of the corresponding forward path segments, the probability $P(T_{FR} > t)$ is also equal to the fraction of backward path segments that satisfy $x_t \neq x_{t'}$ for $t' = t - 1, t - 2, \ldots, 0$.

In general, closed form expressions for $\Delta_t$ can be obtained using the recursion equations

$$
\Delta_t = P(T_{FR} > t).
$$

In the limit of $N \to \infty$ the probability $P(T_{FR} > t)$ is given by Eq. (47). Using Eqs. (47) and (51) we evaluate $\Delta_t$ for the first few steps of the RW:

$$
\Delta_t = \begin{cases}
1 & t = 0, 1 \\
n - 1 \times (c - 1)^2 (c + 1) & t = 2, 3 \\
(c - 1)^3 (c^2 + 2c + 2) & t = 4, 5 \\
(c - 1)^3 (c^2 + 2c + 2) & t = 6, 7.
\end{cases}
$$

(52)

In general, closed form expressions for $\Delta_t$ can be obtained using the recursion equations

$$
\Delta_t = 2 \left( \frac{c - 1}{c^2} \right) \left\{ \left[ \frac{4 \lfloor t/2 \rfloor - 3}{2(c - 1)} \right] \Delta_{t-2} - \frac{2 \lfloor t/2 \rfloor - 3}{2(c - 1)} \Delta_{t-4} \right\},
$$

(53)

where the initial conditions are given by Eq. (52). This equation is obtained from Eq. (47). More specifically, the initial conditions are $\Delta_0$ and $\Delta_2$ for even times and $\Delta_1$ and $\Delta_3$ for odd times. For $t \geq 2$ the probability $\Delta_t$ can also be expressed in the form

$$
\Delta_t = c \left( \frac{c - 1}{c^2} \right) Q_{\lfloor t/2 \rfloor-1}(c),
$$

(54)

where $Q_n(c)$ is a polynomial of order $n$ in $c$. These polynomials can be generated by the recursion equation

$$
Q_{k-1}(c) = \left[ \frac{4k - 6}{k} + \frac{c^2}{(c - 1)} \right] Q_{k-2}(c) - \left( \frac{4k - 6}{k} \right) \frac{c^2}{c - 1} Q_{k-3}(c),
$$

(55)

where $Q_0(c) = 1$ and $Q_1(c) = c + 1$.

The generating function associated with the probability $\Delta_t$ is given by

$$
W(x) = \sum_{k=0}^{\infty} x^{2k} \Delta_{2k}.
$$

(56)
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Note that the sum includes only the even terms due to the fact that $\Delta_{2k+1} = \Delta_{2k}$. Carrying out the summation, we obtain

$$W(x) = \frac{c - 2 + \sqrt{c^2 - 4(c-1)x^2}}{2(c-1)(1-x^2)}. \quad (57)$$

The probability $\Delta_{2k}$ can be obtained from the generating function according to

$$\Delta_{2k} = \frac{1}{(2k)!}\frac{d^{2k}}{dx^{2k}}[x=0] W(x). \quad (58)$$

In the long time limit of $t \gg 1$ the hypergeometric function in Eq. (47) converges towards 1. Therefore, in this limit

$$\Delta_t \simeq \frac{c}{\sqrt{\pi(c-1)(c-2)^2} \left[ \frac{c}{2} \right]^\frac{c}{2}} \left[ \frac{4(c-1)}{c^2} \right]^{\frac{c-2}{2}} \frac{c - 2}{c - 1}. \quad (59)$$

The first term on the right hand side of Eq. (59) decays exponentially with a characteristic time scale of

$$\tau = \frac{2}{\ln \left[ \frac{c^2}{4(c-1)} \right]}. \quad (60)$$

As a result

$$\Delta_t \to \frac{c - 2}{c - 1}, \quad (61)$$

in agreement with Eq. (2).

The mean number of distinct nodes visited by an RW on an RRG up to time $t$ can be expressed in the form

$$\langle S \rangle_t = \sum_{t' = 0}^{t} \Delta_{t'} \quad (62)$$

Replacing $\Delta_t$ in Eq. (62) by $P(T_{FR} > t)$ and performing summation by parts [25], we obtain

$$\langle S \rangle_t = (t + 1)P(T_{FR} > t) + \sum_{t' = 1}^{t} t'P(T_{FR} = t'). \quad (63)$$

Inserting $P(T_{FR} > t)$ from Eq. (17) $P(T_{FR} = t')$ from Eq. (13) into Eq. (63), we obtain

$$\langle S \rangle_t = 2 + C_{\frac{1}{2}} \frac{c(t+1)}{(c-2)^2} \left( \frac{c - 1}{c^2} \right)^{\frac{c}{2}} \left[ \frac{1}{2} \right]_{c}^{2F1} \left[ \frac{3}{2} \left[ \frac{1}{2} \right] + 2 - \frac{4(c-1)}{(c-2)^2} \right]$$

$$+ \frac{2}{c-1} \left[ -1 + \sum_{k=1}^{\lfloor t/2 \rfloor} kC_{k-1} \left( \frac{1}{c} \right)^k \left( \frac{1}{c} \right)^{k-1} \right] + \frac{c - 2}{c - 1}(t - 1). \quad (64)$$
Carrying out the summation, we obtain

\[
\langle S \rangle_t = 2 + C [\tfrac{1}{2}] \frac{c(t+1)}{(c-2)^2} \left( \frac{c-1}{c^2} \right)^{\lfloor \frac{t}{2} \rfloor} \binom{1, \frac{3}{2}}{\lfloor \frac{t}{2} \rfloor + 2} \frac{4(c-1)}{(c-2)^2} \\
- 2C [\tfrac{1}{2}] \frac{c(\lfloor \frac{t}{2} \rfloor + 1)}{(c-2)^2} \left( \frac{c-1}{c^2} \right)^{\lfloor \frac{t}{2} \rfloor} \binom{1, \frac{1}{2}}{\lfloor \frac{t}{2} \rfloor + 1} \frac{4(c-1)}{(c-2)^2} \\
+ \frac{2}{(c-1)(c-2)} + \frac{c-2}{c-1}(t-1). 
\]

In the long time limit, the terms in Eq. (65) that include the hypergeometric functions decay to zero. As a result, Eq. (65) is reduced to

\[
\langle S \rangle_t \simeq 2 + \frac{2}{(c-1)(c-2)} + \frac{c-2}{c-1}(t-1). 
\]

In the case of finite networks, in the long-time limit, the last equation can be extended to the form

\[
\langle S \rangle_t \simeq 2 + (N-2) \left\{ 1 - \exp \left[ -\frac{2}{(c-1)(c-2)(N-2)} - \frac{(c-2)}{c-1} \frac{t-2}{N-2} \right] \right\}. 
\]

This equation is more accurate than Eq. (27), because it takes into account the contribution of the first few steps to \( \langle S \rangle_t \) in an exact way. The essence of this is that at early times the discovery rate \( \Delta t \) of new, yet-unvisited nodes, given by Eq. (52), is slightly larger than its asymptotic value, given by Eq. (61).

8. Discussion

First passage processes are an important landmark in the life-cycle of RWs on networks. The characteristic time scale of these processes is of order \( t \sim N \). Another landmark is the first hitting process, which is the first time in which the RW enters a previously visited node \[27, 28\]. The characteristic time scale of the first hitting process is \( t \sim \min\{c, \sqrt{N}\} \), namely \( t \sim c \) in dilute networks and \( t \sim \sqrt{N} \) in dense networks. In both cases the first hitting time is much shorter than the first passage time \[27, 28\]. Yet another important event which occurs at much longer time scales is the step at which the RW completes visiting all the nodes in the network. The time at which this happens is called the cover-time, which scales like \( t \sim N \ln N \) \[29\]. This means that on average an RW visits each node \( \ln N \) times before it completes visiting all the nodes in the network at least once.

It is interesting to compare the results obtained in this paper for RRGs in the infinite network limit with the corresponding results for regular lattices with the same coordination numbers. For example, the coordination number of a hypercubic lattice in \( d \) dimensions is \( 2d \). Thus, in terms of the connectivity it is analogous to an RRG of
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degree \( c = 2d \). An RW on an infinite one dimensional lattice returns to the initial site with probability \( P(R) = 1 \). In fact, this result can be obtained by inserting \( c = 2 \) in Eq. (7) above. The distribution of first return times of an RW on a one dimensional lattice is given by

\[
P(T_{FR} = t) = \begin{cases} \frac{(t-2)!}{2^{t-1}(2-1)(\frac{t}{2})!} & \text{if } t \text{ even} \\ 0 & \text{if } t \text{ odd.} \end{cases}
\]

(68)

This result is obtained by inserting \( c = 2 \) in Eq. (10) above. The mean first return time of an RW on a one dimensional lattice diverges. This is consistent with Eq. (18) above, whose right hand side diverges for \( c = 2 \).

An RW on a two dimensional square lattice also returns to the initial site with probability \( P(R) = 1 \). In contrast, for RWs on hypercubic lattices of dimensions \( d \geq 3 \) the probability to return to the initial site is \( P(R) < 1 \) [22]. As shown above, an RW on an RRG of degree \( c \geq 3 \) and infinite size returns to the initial node with probability \( P(R) = 1/(c-1) < 1 \). This means that RWs on RRGs behave qualitatively like RWs on regular lattices of dimension \( d \geq 3 \). For an RW on a hypercubic lattice of dimension \( d \geq 3 \), the probability of return to the initial site is given by [30–32]

\[
P(R) = 1 - \left\{ \int_0^\infty e^{-t} \left[ I_0 \left( \frac{t}{d} \right) \right]^d \ dt \right\}^{-1},
\]

(69)

where \( I_0(x) \) is the modified Bessel function [25]. In Appendix A we evaluate the right hand side of Eq. (69) as a Taylor expansion in powers of \( 1/d \). We obtain

\[
P(R) = \frac{1}{2d} + \frac{1}{2d^2} + O \left( \frac{1}{d^3} \right).
\]

(70)

The return probability of an RW on an RRG with the same coordination number can be obtained by inserting \( c = 2d \) in Eq. (7). It yields

\[
P(R) = \frac{1}{2d-1}.
\]

(71)

Writing Eq. (71) as a power series in \( 1/d \), we obtain

\[
P(R) = \frac{1}{2d} + \frac{1}{4d^2} + O \left( \frac{1}{d^3} \right).
\]

(72)

This means that to leading order in \( 1/d \) the return probability of an RW on a high dimensional hypercubic lattice coincides with the return probability of an RW on an RRG with the same coordination number. However, the subleading terms are not the same. For a \( d \)-dimensional hypercubic lattice it is \( 1/(2d^2) \) while for the corresponding RRG it is \( 1/(4d^2) \). The difference is due to the contribution of the shortest cycles in the hypercubic lattice, whose length is \( \ell = 4 \). This is due to the fact that the probability of an RW on a hypercubic lattice to return to the initial site at time \( t = 4 \) via such cycle is
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Figure 9. The return probability $P(R)$ of an RW on a $d$-dimensional hypercubic lattice (+ symbols) as a function of $d$, obtained from Eq. (69). These probabilities are compared to the return probabilities of RWs on RRGs with the same coordination numbers $c = 2d$ (circles), given by Eq. (71). It is found that the two curves coincide for $d = 1$. For $d \geq 2$ the return probability on a $d$-dimensional hypercubic lattice is larger than on the corresponding RRG. This is due to the existence of short cycles in the hypercubic lattice, which provide additional channels for the RW to return to the initial node. As $d$ is increased the contribution of these short cycles diminishes and the two curves converge towards each other.

\[ P(T_{FR} = 4) = \frac{1}{4d^2} + O \left( \frac{1}{d^3} \right). \] (73)

In Fig. 9 we present the return probability $P(R)$ of an RW on a $d$-dimensional hypercubic lattice (+ symbols) as a function of $d$, obtained from Eq. (69). The results are compared to the return probability of an RW on an RRG with the same coordination number (circles), given by Eq. (71). It is found that as $d$ is increased the two curves converge towards each other. This means that in the limit of high dimensions the return probabilities of RWs on hypercubic lattices coincide with those of the corresponding RRGs. Thus, Eq. (71) provides a simple asymptotic expression for Eq. (69) in the large $d$ limit. Also note that the return probabilities coincide for $d = 1$. This is typical of the Bethe-Peierls approximation that recovers the exact one dimensional result as well as the results in high dimensions [33, 34].

Another important quantity that is interesting to compare between RRGs and regular lattices is $\langle S \rangle_t$. In the case of $d$-dimensional hypercubic lattices, it was shown that $\langle S \rangle_t \sim \sqrt{t}$ for $d = 1$, $\langle S \rangle_t \sim t/\ln t$ for $d = 2$, while for $d \geq 3 \langle S \rangle_t = \left[ 1 - P(R) \right] t$, where $P(R)$ is given by Eq. (69) [9]. Thus, the results obtained for $\langle S \rangle_t$ in RRGs resemble the corresponding results on regular lattices of dimensions $d \geq 3$.

After returning to the initial node $i$ for the first time, the RW continues its trajectory and may return to $i$ again at a later time. In the limit of $N \to \infty$ the RW is transient and it thus returns to $i$ only a finite number of times. The distribution of the number
of times an RW on an infinite RRG returns to $i$ is given by

$$P(N_{\text{visits}} = n) = \frac{c - 2}{c - 1} \left( \frac{1}{c - 1} \right)^n,$$

where $n = 0, 1, 2, \ldots$. This is a geometric distribution whose moment generating function is given by

$$M(x) = \frac{c - 2}{c - 1 - e^x}.$$

The cumulant distribution function of $P(N_{\text{visits}} = n)$ is given by

$$K(x) = \ln M(x) = \ln \left( \frac{c - 2}{c - 1 - e^x} \right).$$

Expanding $K(x)$ to second order in $x$, we obtain the mean and variance of $P(N_{\text{visits}} = n)$, which are given by

$$\mathbb{E}[N_{\text{visits}}] = \frac{1}{c - 2},$$

and

$$\text{Var}(N_{\text{visits}}) = \frac{c - 1}{(c - 2)^2},$$

respectively. Thus, the expected number of visits decreases as $c$ is increased.

9. Summary

We presented analytical results for the distribution of first return times of RWs on RRGs. In the analysis we distinguished between the scenario in which the RW returns to the initial node $i$ by retroceding its own steps and the scenario in which it does not retrocede its steps on the way back to $i$. We calculated the conditional distributions of first return times, $P(T_{\text{FR}} = t|\text{RETRO})$ and $P(T_{\text{FR}} = t|\neg\text{RETRO})$, in the retroceding and non-retroceding scenarios, respectively. We also calculated the mean and the variance of each one of the two conditional distributions. It was found that the distribution of first return times in the retroceding scenario is narrow and its mean is of order 1. In contrast, the distribution of first return times in the non-retroceding scenario is broad and its mean is $\simeq N$. Thus, there is a clear separation of time scales between the two scenarios. We combined the results of the two scenarios and obtained the overall distribution of first return times $P(T_{\text{FR}} = t)$ of RWs on RRGs of a finite size. The retroceding scenario is retained in the infinite network limit, while the non-retroceding scenario exists only in finite networks. It was also found that in the infinite network limit the trajectories of RWs on RRGs are transient in the sense that they return to the initial node with probability $< 1$. In this sense they resemble the trajectories of RWs on regular lattices of dimensions $d \geq 3$.

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Appendix A. Asymptotic expansion of Pólya’s constant

The probability that an RW on a $d$-dimensional hypercubic lattice will return to the initial site is given by

$$P(R) = 1 - \frac{1}{U(d)}, \quad \text{(A.1)}$$

where

$$U(d) = \int_0^\infty e^{-t} \left[ I_0 \left( \frac{t}{d} \right) \right]^d dt, \quad \text{(A.2)}$$

is Pólya’s constant and $I_0(x)$ is the modified Bessel function of the first kind [25]. Changing variables from $t$ to $s = t/d$, one can express $U(d)$ in the form

$$U(d) = d \int_0^\infty e^{-ds} [I_0(s)]^d ds, \quad \text{(A.3)}$$

which is $d$ times the Laplace transform of $[I_0(s)]^d$. In order to carry out the integration, we use the Taylor expansion of $[I_0(s)]^d$ around $s = 0$ and integrate it term by term. The expansion is given by [35–37]

$$[I_0(s)]^d = 1 + d \sum_{k=0}^\infty \frac{P_k(d)}{[(k+1)!]^2} \left( \frac{s}{2} \right)^{2(k+1)}, \quad \text{(A.4)}$$

where $P_k(d)$ are the Bender-Brody-Meister polynomials, which are expressed in powers of $d$ and have integer coefficients. The lowest order polynomials and the recursion equation for generating higher order polynomials appear in Ref. [35]. Inserting $[I_0(s)]^d$ from Eq. (A.4) into Eq. (A.3) and carrying out the integration term by term, we obtain

$$U(d) = 1 + d \sum_{k=1}^\infty \left( \frac{2k}{k} \right) \frac{P_{k-1}(d)}{(2d)^{2k}}. \quad \text{(A.5)}$$

Using the lowest order polynomials $P_0(d), \ldots, P_6(d)$, which are given explicitly in Ref. [35], we obtain

$$U(d) = 1 + \frac{1}{2d^2} + \frac{3}{4d^3} + \frac{3}{2d^3} + \frac{15}{4d^4} + \frac{355}{32d^5} + \frac{595}{16d^6} + \frac{8715}{64d^7} + \mathcal{O}\left( \frac{1}{d^8} \right). \quad \text{(A.6)}$$

Inserting $U(d)$ from Eq. (A.6) into Eq. (A.1), we obtain

$$P(R) = 1 - \frac{1}{2d^2} + \frac{1}{2d^2} + \frac{7}{8d^3} + \frac{35}{16d^4} + \frac{215}{32d^5} + \frac{1501}{16d^6} + \frac{5677}{64d^7} + \mathcal{O}\left( \frac{1}{d^8} \right). \quad \text{(A.7)}$$

This 7th order expansion in powers of $1/d$ provides an approximation for Eq. (A.1) whose error is less than 1% for $d \geq 4$ and less than 0.01% for $d \geq 10$. 
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