An Anisotropic Landau-Lifschitz-Gilbert model of dissipation in qubits

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We derive a microscopic model for dissipative dynamics in a system of qubits interacting with an Ohmic bath that generalises the dissipative model of Landau-Lifschitz-Gilbert to the case of general anisotropic couplings to a thermal bath. Obtained from the Keldysh path integral formalism, it gives the classical (zero entanglement) limit of a system’s dynamics. The fluctuation dissipation relation enforces a close relationship between the thermal fluctuations and dissipative forcing terms of these dynamical equations. We use a representation that highlights the action of dissipation in confining the system to a reduced phase space. This model applies to a system of superconducting flux qubits whose coupling to the environment is necessarily anisotropic. We study the model in the context of the D-Wave computing device and show that the form of environmental coupling in this case produces dynamics that are closely related to several models proposed on phenomenological grounds.

I. INTRODUCTION

The Landau-Lifschitz-Gilbert equation provides a phenomenologically motivated description of the stochastic, dissipative evolution of a spin system. Conceived as a model for an open magnetic system, the dynamics consists of two terms corresponding to precessing, Hamiltonian evolution, and noisy, relaxing, dissipative evolution. This form of dissipation corresponds to the classical limit of a spin system coupled isotropically to a bosonic bath.

The behaviour of a Superconducting quantum device can be mapped to spin dynamics. In the case of a flux qubit, which has two low energy states, the low energy dynamics map on to those of a spin-$\frac{1}{2}$ magnetic moment, and an isolated qubit will exhibit only the Larmor precession described by a dissipation free Landau-Lifschitz-Gilbert equation; the qubit’s state evolves such that the absolute amplitudes of the two energy levels remain constant whilst the phase between them changes at a constant rate.

This relationship suggests that qubits coupled to the environment may display the same dissipative behaviour as magnetic moments, governed by the full dissipative Landau-Lifschitz-Gilbert equation. This model has been used to model an extended array of superconducting qubits in ref 4 whilst related vector models have been used in refs 5, 7.

However, the Landau-Lifschitz-Gilbert equation describes the dissipative dynamics of a two-level system that is exposed to isotropic environmental coupling, i.e. identical baths coupled to the $\hat{s}_x$, $\hat{s}_y$, and $\hat{s}_z$ operators. For qubits, these operators have different physical origins and so they will couple differently to noise. Hence, the dissipation will also be anisotropic, as the stochastic noise and dissipative terms are related by the fluctuation dissipation relation. As a result, systems with anisotropic couplings, such as a superconducting flux qubit, have a corresponding anisotropy in the dissipation and noise.

Due to the physical geometry of the superconducting flux qubit, stray flux and other environmental effects couple most strongly through the $\hat{s}_z$ operator. The anisotropy of this coupling to the environment introduces qualitatively new features into the system’s dynamics.

The dissipative dynamics of a two level system has been studied extensively in this work, we use a Landevi, description of the dynamics of a flux qubit accounting for the anisotropic coupling to the environment. This generalises previous dissipative spin models. We derive this Langvin equation from a Keldysh field theoretical description of the system.

An accurate model for the dissipative dynamics of a flux qubit can be used to assess the capabilities of putative quantum technologies. We apply our model to the D-Wave computing machine, which consists of a large array of controllable flux qubits. Extensive analysis has sought to correlate the behaviour of this machine with various quantum and classical models. Since classical dynamics correspond to a particular restriction upon fully quantum dynamics, the effectiveness of this approach is dependent upon identifying the appropriate restrictions that correspond to the classical limit. We show that our anisotropic Langevin equation may be reduced to obtain the heuristic models of refs 5 and 6 in an appropriate limit of dissipation.

II. THE ANISOTROPIC DISSIPATION OF FLUX QUBITS

A superconducting flux qubit will couple to its environment in various ways. The environmental degrees of freedom may consist of charge fluctuations, flux noise, coupling to nearby spins, or trapped vortices. When it is not possible to interrogate such environmental subsystems, the information contained in the states of these subsystems is lost from the system of interest—the qubit. This decoherence inhibits the ability of a qubit to remain in a given state indefinitely with good fidelity, and in turn limits the ability of an array of qubits to sustain entanglement.
The resulting loss of unitary evolution generates dissipative dynamics that, over sufficiently long times, drive the system towards certain equilibrium states, or dynamical fixed points. In this sense, the effects of decoherence are inherently inhomogeneous over the system’s Hilbert space, they affect different states in different ways, driving them towards different fixed points or along different paths to such fixed points.

In the case of a system exchanging energy with an environmental bath, we anticipate the system will relax to a fixed point given by the Gibbs state. There remain, however, a plurality of dynamics that result in the system relaxing to this state, and we expect that different baths and couplings with different physical origins will result in different relaxation dynamics. This is relevant to controlled quantum systems in which the dynamics of interest occur before the system has relaxed to thermal equilibrium, but not necessarily on timescales where dissipation can be neglected.

The Landau-Lifshitz-Gilbert equation provides a description of the dissipative dynamics of a two-level system with isotropic coupling to the environment. It is often quoted in one of two equivalent forms:

\[
\dot{s} + s \times [(B + \eta) - \gamma \dot{s}] = 0, \quad (II.1a)
\]
\[
\dot{s} + \frac{1}{1 + s^2 \gamma^2} s \times [(B + \eta) + \gamma s \times (B + \eta)] = 0. \quad (II.1b)
\]

These are non-linear differential equations in a vector \(s\) of magnitude \(s\) that parametrises a qubit spin coherent state \(|s \rangle\). The equation is stated here in terms of an effective magnetic field, \(B\). More generally, the magnetic field may be replaced by an appropriate derivative of the spin Hamiltonian, \(B = -\nabla_s H(s)\). \(\eta\) describes a stochastic noise that satisfies the fluctuation dissipation relation: \(\langle \eta_i(\omega) \eta_j(\omega') \rangle = \gamma \omega \coth (\frac{\omega}{2T}) \delta_{ij} \delta(\omega + \omega')\).

This description closely resembles the phenomenological Bloch equations used to model nuclear magnetization, the only difference being that since the Landau-Lifshitz-Gilbert equation describes a single spin and not an ensemble decay, the magnetic field must preserve the spin \(|s\rangle = s\). Neglecting zero-point fluctuations, this Langevin equation describes the dissipative evolution of a single qubit in the presence of a magnetic field, \(B\), and coupled to its environment.

The dynamics of a system of multiple interacting qubits may be described by a set of coupled Landau-Lifshitz-Gilbert equations. These dynamics are less general and constitute a restriction to product states. Coherent dynamics are permitted for individual spins in these states, but there is no entanglement between spins. In this sense the equations correspond to the classical limit of the system.

Although originally developed to describe spins that are isotropically susceptible to noise, the Landau-Lifshitz-Gilbert equations have been used to model the classical dynamics of dissipative qubits, whilst other authors have also made use of similar vector models.

However, as argued above, the physics underlying superconducting flux qubits implies that noise and dissipation will be anisotropic. Whilst the microscopic origins of flux qubit decoherence are not fully understood, many environmental interactions are often modeled using linear \(s \cdot u\) couplings. These include flux noise and coupling to impurity spin or boson degrees of freedom, and have been shown to be significant contributions to decoherence. Whilst in the context of the D-Wave computing machine it has been argued that linear \(s\) coupling is the correct minimal model. Without loss of generality this linear coupling can be chosen to be \(s_z\) and can stand for longitudinal or transverse coupling. Due to the close relationship between the damping and noise terms enforced by the fluctuation dissipation relation, this cannot be remedied solely by an appropriate adjustment to the noise term: an adjustment to the noise term effects the damping to term in a manner that is difficult to guess.

### III. DERIVATION OF A LANGEVIN EQUATION WITH ANISOTROPIC DISSIPATION

Here we follow the approach of ref. to find the partition function \(Z = \int D|s|e^{iS|s|}\) for an open quantum system, and evaluate the dynamics of a qubit using a stationary phase approximation. In a closed system, the action of a single spin in a magnetic field, \(B\), is given by

\[
S_0 = \int_C dt L_0 = \int_C dt (\dot{s} \cdot A - s \cdot B), \quad (III.1)
\]

where \(A = \frac{1 - \cos \theta}{\sin \theta} \dot{\phi}\) is the single monopole vector potential. This action describes the precession of a spin around the axis of the magnetic field. When the spin is coupled to a bath of oscillators through the \(s_z\)-component, the action becomes

\[
S = S_0 + \int_C dt \left[ s_z \sum_\alpha g_\alpha x_\alpha + \sum_\alpha \frac{m}{2} \left( x_\alpha^2 - \omega_\alpha^2 x_\alpha^2 \right) \right]. \quad (III.2)
\]

We use Keldysh field theory to find the dynamics of this system. This provides a methodology for treating an open quantum system, enforcing the necessary fluctuation dissipation relation between the stochastic noise and deterministic damping terms induced by decoherence.

A state vector is sufficient to define the behaviour of a closed system. Non-equilibrium, open quantum systems require an ensemble of state vectors, or a density matrix, because of the loss of information to the bath. Evolving a density matrix \(\rho\) requires both pre- and post-multiplication by the time evolution operator \(\hat{U}(t_1, t_2) = \mathcal{T} \exp \left[ i \int_{t_1}^{t_2} dt H(t) \right]\). Thus \(\hat{\rho}_t = \hat{U}(t, 0) \hat{\rho}_0 \hat{U}(0, t)\) in contrast to the time evolution of a state vector which requires only one time evolution operator \(|\psi_t\rangle = \hat{U}(t, 0)|\psi_0\rangle\). This gives rise to a doubling of the degrees of freedom in the Keldysh theory of open systems compared to those required to describe the evolution of a closed system. These
degrees of freedom correspond to the forward and backward branches of the Keldysh contour, $C$, over which eqs. (III.1) and (III.2) are integrated.

It is useful in calculations to separate diagonal and off-diagonal contributions to the density matrix. The former are described by the symmetric combination of fields on the forwards and backwards Keldysh contour, and the latter by an antisymmetric combination. These fields are usually called the classical and quantum fields, respectively, and the rotation to this basis is called a Keldysh rotation. Performing a series expansion in the quantum fields constitutes a sequence of progressively higher order quantum corrections to the classical dynamics. Performing this rotation, and expanding to first order in the quantum fields results in the total action

$$S_0 + S_{\text{diss}} = \int_{-\infty}^{\infty} dt \left[ \sum_i s_i^q \left( \frac{\partial L_0}{\partial \dot{s}_i} - \frac{d}{dt} \frac{\partial L_0}{\partial s_i} \right) + 2 \sum_\alpha g_\alpha (s_z x_\alpha^q + s^q x_\alpha) + \sum_\alpha (x_\alpha x_\alpha^q) \left( \frac{0}{[D^R_\alpha]} - \frac{1}{[D^A_\alpha]} \right) \left( x_\alpha \right) \right].$$

(III.3)

The first term in eq. (III.3) encodes the Hamiltonian dynamics (closed system dynamics) of the spin. The remaining terms encode the dissipative dynamics induced by interaction with the bath.

### A. The Langevin Equation

The Langevin equation can be obtained from a limit of the Keldysh Theory as follows: Since for an open system the state of bath cannot be interrogated, it is integrated out to obtain the dissipative contribution to the action in terms of the spin alone:

$$S_{\text{diss}} = -\int_{-\infty}^{\infty} dt \left[ \sum_\alpha g_\alpha^2 (s_z s^q_z) \left( \frac{0}{[D^R_\alpha]} D^K_\alpha \left( s_z \right) \right) \right].$$

(III.4)

The Green’s functions $D^A_\alpha$, $D^R_\alpha$, and $D^K_\alpha$ can be evaluated in the case of an Ohmic bath, with spectral function $J(\omega) \propto \gamma \omega$. In the frequency basis, the advanced and retarded parts are given by $\frac{\gamma}{\pi} \frac{2}{\omega} \coth \left( \frac{\omega}{2T} \right) \approx 2i\gamma T$. The real constants give rise the an anisotropic potential $\sim s_z^2$, which may be absorbed into $S_0$ where it renormalises the capacitance. We consider the case where the only bias to the flux qubit is the magnetic field, $B$. Thus the renormalised value is assumed to be zero and these terms so do not appear henceforth. The fluctuation dissipation relation requires that the Keldysh component of the Green’s function is given by $\sum_\alpha g_\alpha^2 D^K_\alpha = \gamma (D^R - D^A)$, and $\gamma \omega \coth \left( \frac{\omega}{2T} \right) \approx 4i\gamma T$. The final approximation is valid if the temperature, $T$, is much greater than the characteristic frequency of the bath. The dissipative action,

$$S_{\text{diss}} = \gamma \int_{-\infty}^{\infty} dt \left( s_z^q s_z - s_z^q s_z + 4iT s_z^q s_z^q \right),$$

(III.5)

is then made linear in $s_z^q$ by a Hubbard-Stratonovich transformation. This introduces a variable $\eta$ that later appears as the stochastic term in the Langevin equation:

$$S_{\text{diss}} = \int_{-\infty}^{\infty} dt \left( -2\gamma s_z s_z + 2s^q \eta + i\frac{\eta^2}{4\gamma T} \right).$$

(III.6)

The statistics of the stochastic term $\eta = \eta \hat{z}$, $\langle \eta(t) \rangle = 0$, $\langle \eta(t)\eta(t') \rangle = 2\gamma T \delta(t - t')$ are defined by the Gaussian ensemble introduced in eq. (III.6).

The Langevin equation, eq. (III.8a), is a stochastic differential equation with a solution given by an ensemble of pure state trajectories. While it is possible to obtain the dynamics within any manifold of quantum states, in order to obtain the classical dynamics of a many qubit system, we restrict the pure state trajectories to product states. This sub-manifold of the full Hilbert corresponds to separable states in which there is zero entanglement between the different qubits. Thus, mutatis mutandis, the same calculation is valid for systems of many qubits.

The Langevin equations, eqs. (III.8a) and (III.8b), obtained by this method include, by construction, the dissipative and stochastic effects induced by an anisotropic bath. These dynamics are different from those obtained for an isotropic bath, eqs. (II.1a) and (II.1b), with important consequences.

### IV. DYNAMICS OF THE MODEL

The Langevin equation of eqs. (III.8a) and (III.8b) describes the dissipative dynamics of a system of interacting, non-entangled flux qubits, with environmental coupling solely through the $\hat{s}_z$ operator. We now compare the dynamics of qubits with isotropic couplings, eqs. (II.1a) and (II.1b), and anisotropic couplings,
eqs. (III.8a) and (III.8b), to their environments. Both models relax to the same distribution, minimising the free energy, but exhibit different sets of dynamics at intermediate times. Moreover, the energy conserving dynamics of the two models are equivalent, and they consist of the moment \( s \) precessing about the external field, \( B \). This is expected as for zero coupling to the bath, the Hamiltonians \( H_0 \) for the two systems are the same. However, the non-Hamiltonian dissipative terms have qualitatively different effects.

The dissipation term in the Landau-Lifshitz-Gilbert equation has several special properties: firstly the dissipation drives relaxation towards \( B \), and drives motion that is at all times perpendicular to that induced by the precessional dynamics. Secondly, the rate of this relaxation towards \( B \), and hence the rate of energy dissipation, depends only upon the angle between \( s \) and \( B \), i.e. the energy remaining. Otherwise stated, the Hamiltonian term drives motion only along energy equipotentials, and the dissipative dynamics drives motion only perpendicular to them. Thus, to find the future state of the system it is not necessary to consider the interplay between these two dynamics, as only dissipative motion drives changes in the azimuthal angle between \( s \) and \( B \), whereas only the precession term drives changes in the polar angle.

In contrast, in more general dissipative models there is no such separation of the effects of precession and dissipation, and their interplay remains important. In the Langevin equation for \( \dot{s} \) coupling, eq. (II.8), it is clear that the dissipative term drives rotation about \( \hat{z} \), regardless of the orientation of \( B \). The system relaxes indirectly, through the interplay of dynamics and the state dependent modulation of the rate of dissipation. The effect of this interplay is highlighted by the appearance of regions of novel behaviour such as dissipation free precession, retrograde motion, and effective dimensional reduction.

To discuss the dynamics of the \( \dot{s} \) coupling model, it is useful to introduced the timescales \( t_p^{-1} = B \) and \( t_d^{-1} = \gamma_s B \sin^2 \theta^* \) where \( \theta^* \) is the polar angle of the field \( B \) and \( B = |B| \). \( t_p \) and \( t_d \) which are characteristic of the precessional motion and dissipative motion, respectively. In the limits where these scales are widely separated, the system’s behaviour is dominated by the faster dynamics on short timescales, whilst some effective dynamics emerge on longer timescales.

A. Weak coupling limit, \( t_p \ll t_d \)

For weak coupling, the system’s behaviour remains dominated by the precession found in the fully closed system dynamics. As shown in fig. 1a the spin will generally perform many rotations about the magnetic field, \( B \), before reaching the neighbourhood of the ground state. This allows the state dependence of the dissipation to be averaged out. In this regime the Langevin equation may be approximated by a Landau-Lifshitz-Gilbert equation

\[
\dot{s} + s \times [(B + \eta) + \gamma_{\text{eff}} s \times (B + \eta)] = 0 \quad \text{(IV.1)}
\]

where \( \gamma_{\text{eff}} = \frac{1}{2} \gamma \sin \theta^* \) is the effective dissipation at polar angle, \( \theta^* \). This results in dynamics similar to the isotropic case (eqs. (II.1a) and (II.1b)), thus the limit \( t_p \ll t_d \) offers no novel dynamics.

B. Strong coupling limit, \( t_p \gg t_d \)

In the limit \( t_p \gg t_d \), the system’s dynamics are initially dominated by the dissipative term, which drives rotation about the \( z \) axis. However, this dissipation goes to zero when \( B, s \) and \( z \) are coplanar. Once the state of the system approaches this one-dimensional manifold, the dynamics, shown in fig. 1b changes to one in which evolution is governed by the interplay of the dissipative and Hamiltonian dynamics. The state is unlikely to leave the neighbourhood of this manifold and so, on timescales longer than the time, \( t_d \), required to relax to the reduced manifold, this confinement constitutes a dissipative reduction of the phase space from O(3) dynamics to O(2).

Since the system quickly relaxes to a state in which \( B, s \) and \( z \) are approximately coplanar, effective dynamics can be obtained by expanding about this point. One obtains an ordinary differential equation in \( \theta \), the polar angle of \( s \), coupled to a mean reverting stochastic process.
in $\phi$, the azimuthal angle;

$$\dot{\theta} = -B \sin \theta^* \phi$$

$$\dot{\phi} = B \frac{\sin(\theta - \theta^*)}{\sin \theta} - Bs\gamma \sin \theta^* \sin \theta \phi + \eta. \quad \text{(IV.2)}$$

These dynamical equations govern the dynamics on timescales longer than the initial relaxation time $t_d$. Eliminating $\phi$ yields effective dynamics for the strong coupling limit, which consists of an O(2) spin confined to the $B\hat{z}$ plane, at an angle $\theta$ to $\hat{z}$

$$\dot{\theta} + B \sin \theta^* \left( \gamma s \sin \theta \dot{\theta} + B \frac{\sin(\theta - \theta^*)}{\sin \theta} + \eta \right) = 0. \quad \text{(IV.3)}$$

The qualitative structure of the dynamics can be investigated by linearising about the fixed point $\theta^*$, giving

$$\ddot{\theta} + \frac{\dot{\theta}}{t_d} + \frac{\theta - \theta^*}{t_p^2} + \frac{\eta}{\sqrt{\gamma^2 s^2 t_d^2 t_p^2}} = 0. \quad \text{(IV.4)}$$

These dynamics show that at low temperature the system, confined to a one dimensional manifold, relaxes as a noisy damped oscillator.

V. CONSEQUENCES OF ANISOTROPIC DISSIPATION

The Langevin equation derived in the previous section exhibits markedly distinct behaviours in different regimes. The effects of this have implications for the usefulness of the system for computation.

A. Different behaviours for the same system

For a qubit to be useful for quantum computation it must be sufficiently manipulable. The DiVincenzo criteria for universal quantum computation require that a universal set of quantum gates can be implemented.31

For a trivial system of one qubit, this in effect requires the implementation of at least two non-parallel magnetic fields. Thus, for studying qubits, the field, $B$, is assumed to be a tunable parameter of the system, whereas the environmental coupling, $\gamma$, is fixed at some finite value.

When coupling to the environment is anisotropic, whether the coupling is weak or strong depends not only upon fixed parameters intrinsic to the system, but also upon the orientation of the magnetic field, $B$. This is important when characterising such a device. In particular a system that is analysed under conditions when $B$ and $z$ are nearly aligned will appear weakly coupled, despite that for other orientations of $B$ the dynamics may be entirely dominated by environment-induced dissipative dynamics. It is a general feature of qubit systems that inhomogeneous environmental couplings will result in dynamics that are correspondingly inhomogeneous. Certain states may evolve as if near dissipation free, whilst others may be dominated by dissipative or noisy dynamics.

B. A model for lossy Qubit Arrays

We now show that the anisotropic Langevin equation eq. (IV.3) is equivalent to models for the dynamics of the D-Wave machine that were previously proposed on phenomenological grounds. The D-Wave machine consists of a controllable array of flux qubits. Aspects of its dynamics have been heralded as evidence of quantum mechanics.17,19 However, refs 5 and 7 developed heuristic classical models of lossy qubit arrays that reproduced these behaviours and generated further comments and investigations.4,6,18–20,33–35

The model of refs 5 and 7 was chosen for its classical structure to show that the reproduced dynamics were not inherently quantum. The authors did not attempt to justify it as microscopically authentic model of flux qubit dynamics. The model consisted of a set of classical O(2) spins undergoing an exploration of phase space, that is deterministic in ref 5 and stochastic in ref 7 with a bias for relaxing towards the local magnetic field.

In section IV B we showed formally how the dynamics of a semiclassical $\hat{O}(3)$ spin are driven to a reduced $\hat{O}(2)$ manifold when strongly anisotropically coupled to the environment through the $\hat{s}_z$ operator. The models introduced by refs 5, 7, and 18 corresponds to different limits of eq. (IV.3). The deterministic $\hat{O}(2)$ dynamics of ref 5 correspond to eq. (IV.4) in the absence of noise. Similarly the Metropolis-Hastings stochastic model of refs 7 and 18 corresponds to a Monte-Carlo Boltzmann sampling of the underlying partition function. This was noted in ref 35 where it was shown that the $\hat{O}(2)$ spin model can be derived from a physical system, although the authors had to artificially introduce a restriction to the correct $\hat{O}(2)$ manifold—we show it can be found as a natural consequence of environmental coupling.

The equivalence of the phase space exploration encoded by the stochastic dynamics in eqs. (IV.3) and (IV.4) with that of the sampling of the partition function can be seen most explicitly in the structure of the partition function given in eq. (III.7). The acceptance probability of a Monte Carlo update step can be related to the transition probability between the two states, which is encoded by integrating the dynamical equation. This extends the relationship between $\hat{O}(2)$ spins and the classical limit of Quantum Monte Carlo noted in ref 35 by providing a physical justification for the restriction to $\hat{O}(2)$ dynamics.

Thus the Langevin equation, in the strongly coupled limit, resolved over unentangled states, recovers the dynamics introduced on phenomenological grounds as the classical limit of an array of coupled qubits. This corresponds physically to the case when environmentally induced decoherence prevents entanglement from persisting on dynamical timescales.
VI. CONCLUSION

We have developed a Langevin description of the dynamics of qubits that allows for anisotropic coupling to the environment. This is a natural generalisation of the Landau-Lifshitz-Gilbert equations developed to describe the dissipative dynamics of spins with isotropic coupling to the environment. The fluctuation-dissipation relation applied to the bath degrees of freedom has the important consequence that anisotropic noise inevitably leads to anisotropic dissipation.

This model applies explicitly to qubits experiencing dissipation due to fluctuations in the gap energy (environmental coupling to the $\delta_z$ operator). Anisotropy results in unexpected structure in the dynamics unseen in the isotropic case. The dissipative dynamics drive phase space reduction to lower dimensional manifolds, and transitions between fast and slow dynamics. The rapid relaxation to a reduced $O(2)$ manifold of constrained dynamics constitutes a microscopic derivation of the heuristic model of refs 1, 17, and 18, which was capable of reproducing several observed behaviours of the D-Wave machine.

Understanding of the effects of state-dependent noise and anisotropic coupling to the environment is crucial for the proper control of quantum devices. As we have shown in the case of the D-Wave machine, these effects can bias the system dynamics in unexpected ways. Used constructively, this may be harnessed to useful ends. If ignored, the dynamics may completely differ from that intended.

VII. ACKNOWLEDGEMENTS

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