$B \to f_0(980)$ form factors and the width effect from light-cone sum rules

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Abstract

In this paper we calculate the $B \to f_0(980)$ form factors from light-cone sum rules with $B-$meson DAs, the quark-antiquark assignment of $f_0(980)$ is adopted and the three-particle correction from $B-$meson DAs are found to be smaller than 20% in the considered energy regions. We further explore the light-cone sum rules approach to study the $S-$wave $B \to \pi\pi$ form factors, the numerical fitting indicates that the $\sigma$ term should not be included in the resonance model and the $f_0+f_0'+f_0''$ model gives the reasonable prediction. As a by-product, we predict the strong coupling $|g_{f_0\pi\pi}| = 2.42 \pm 0.48$ GeV.
1 Introduction

The research of form factor from one hand promotes our knowledge of the hadron structure, such as the distribution amplitudes (DAs), and from the other hand provides a way to extract the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements in the semileptonic $B$ decays. In the literatures the heavy-to-light transition form factors are calculated by different approaches, in which the Lattice QCD (LQCD) gives the reliable simulations in the low recoil regions [1,2], while in the large and full recoil regions, the QCD-based analytical approaches like the light-cone sum rules (LCSRs) [3–13] and the perturbative QCD (PQCD) [14–20] are suggested.

There are many calculations for the heavy-to-light form factor with the final state being a pseudoscalar ($P$) or a vector ($V$) meson, while to our knowledge the calculation involved a scalar ($S$) meson still has many controversies, focusing mainly on the underlying structure and the width effect. It has been suggested that the scalar mesons with masses below or near 1 GeV (the isoscalar $\sigma/f_0(500)$ and $f_0(980)$, the isodoublet $\kappa$, and the isovector $a_0$) form a SU(3) flavor nonet, and the scalar mesons with masses around 1.5 GeV ($f_0(1370)$, $a_0(1450)$, $K_0^*(1430)$ and $f_0(1500)$) form another one. The combined analysis with the data [21,23] and the orbital angular momentum [24,25] imply that the heavier nonet states are more favourite the quark-antiquark assignment replenished with some possible gluon content, while the scalar mesons in the lighter nonet are more like to be the tetra-quark states. Moreover, the pictures of gluonball [26], hybrid state [27] and molecule state [28] are also discussed for the light scalar mesons. In a word, there is not a general agreement on the contents of $f_0(980)$.

Actually, there are some attempts to study the $B \to S$ form factors from light-cone sum rules with scalar meson DAs [29,33]. We would like to comment that these calculations consid-
ered only the scalar mesons in the heavier nonet, and the accuracy is debatable since the input DAs of scalar mesons still missing some important informations, like the standard conformal partial expansion and the width effect. In this paper, we propose to study the \( B \to f_0(980) \) form factor from the alternative LCSRs with \( B^- \) meson DAs, the conventional quark-antiquark assignment would be adopted for \( f_0(980) \) with the considerations: (i) the direct measurement has not ruled out this assignment yet for the light scalar mesons, (ii) this assignment indeed gives the accurate predictions for the mass of \( f_0(980) \) in the QCD sum rules \[34–37\], and (iii) in the \( B \) decay to the energetic and large recoiled \( f_0(980) \), this assignment is the natural choice because the possibility to form a tetra-quark state is power suppressed with the sea quark pair \[25\]. It is well known that the width of \( f_0(980) \) is suppressed by the phase space \[1\]. To access the width effect, we employ the approach proposed to calculate the \( B \to \pi\pi \) form factors in Ref. \[12\], with substituting the isovector dipion state by the isoscalar one.

The rest of this paper is organised as follows. In sec.2 we revisit and update the mass and the decay constant of \( f_0(980) \) in the two-point QCD sum rules. Section 3 and sec.4 are the main parts of this paper, where we present the LCSRs calculation for \( B \to f_0(980) \) form factors and generalizes it to study the \( S^- \) wave \( B \to \pi\pi \) form factors, respectively. We summary in sec.5. The coefficients in three-particle correction are complemented in Appendix A.

2 \( f_0(980) \) in the QCD sum rules revisited

QCD sum rules \[40\] is a powerful tool to study hadron spectrum. For the scalar meson with \( I^G(J^{PC}) = 0^+(0^{++}) \) in our interest, the two-point correlation function with the scalar quark current \( J_S(x) = \bar{q}_1(x)q_2(x) \) is

\[
\Pi_{2pSRs}(q) = i \int d^4xe^{iqx} \langle 0 | T \{ J_S(x), J_S(0) \} | 0 \rangle .
\] (1)

Generally speaking, the correlation function can be studied in twofold ways: the QCD calculation at quark-gluon level in the Euclidean momenta space, and the summing of intermediate states from the view of hadron. The QCD calculation in the negative half plane of \( q^2 \) is guaranteed by the operator-product-expansion (OPE) technology, and then the correlation function is written in terms of various quark and gluon condensates. On the other hand, the correlation function can be expressed as the sum of contributions from all possible intermediate states in the positive half-plane, possibly with some subtractions.

These two parallel calculations are equated by using the dispersion relation. After taking the quark-hadron duality to eliminate the contributions in the large energy regions, we obtain

\[1\] In fact the width of \( f_0 \) is smaller than it of \( \rho \) meson. We note that the width effect in \( B \to \rho \) transition is usually not be considered, because in the experimental analysis the \( \rho \) meson is identified with the P-wave \( \pi\pi \) signal when the dipion invariant mass locates in the \( \rho^- \) pole region \[38, 39\]. This makes the narrow-width treatment for \( B \to \rho \) form factor from LCSRs is consistent with the experimental measurement.
the result
\[
\frac{m_S^2 f_S^2}{m_S^2 - q^2} = \frac{1}{\pi} \int_0^{s_0} ds \frac{\text{Im} \Pi_{2pSRs}(s)}{s - q^2},
\]
where \(\Pi_{2pSRs}(s)\) is the OPE result for the correlation function, \(s_0\) is the threshold with taking into account only the ground state contribution. The mass and the modified decay constant of scalar meson are defined as
\[
\langle 0 | J_S | S(p) \rangle = m_S f_S(\mu), \quad f_S(\mu) = \frac{m_S}{m_{q_1(\mu)} - m_{q_2(\mu)}} f_S.
\]

The neutral scalar meson can not be produced via the vector current because \(f_S\) is vanished in the \(SU(3)/\)isospin limit with the charge conjugation invariance and the conservation of vector current. In order to improve the convergence of OPE calculation and to suppress the contributions from excited states and continuum spectrums, we apply the Borel transformation
\[
\text{on both sides of Eq}(2)\text{, and obtain the result with one-loop perturbative calculation and the vacuum condensate terms up to dimension six }^{35,41},
\]
\[
m_S^2 f_S^2 e^{-m_S^2/M^2} \left(\frac{\alpha_s(\mu)}{\alpha_s(M)}\right)^{8/3} \frac{3}{8\pi^2} M^4 \left[1 + \frac{\alpha_s(M)}{\pi} \left(\frac{17}{3} + 2 \frac{I(1)}{f(1)} - 2 \ln \frac{M^2}{\mu^2}\right) f(1)\right] \\
+ \frac{1}{8} \left(\frac{\alpha_s G^2}{\pi}\right) + \left(\frac{m_1}{2} + m_2\right) \langle \bar{q}_1 q_1 \rangle + \left(\frac{m_2}{2} + m_1\right) \langle \bar{q}_2 q_2 \rangle \\
- \frac{1}{M^2} \left(\frac{1}{2} m_2 \langle \bar{q}_1 g_s \sigma \cdot G q_1 \rangle + \frac{1}{2} m_1 \langle \bar{q}_2 g_s \sigma \cdot G q_2 \rangle\right) \\
- \frac{1}{3} \pi \alpha_s \langle \bar{q}_1 \gamma_{\mu} \lambda^a q_1 \bar{q}_1 \gamma_{\mu} \lambda^a q_1 \rangle - \frac{1}{3} \pi \alpha_s \langle \bar{q}_2 \gamma_{\mu} \lambda^a q_2 \bar{q}_2 \gamma_{\mu} \lambda^a q_2 \rangle \\
- \pi \alpha_s \langle \bar{q}_1 \sigma_{\mu \nu} \lambda^a q_2 \bar{q}_2 \sigma^{\mu \nu} \lambda^a q_1 \rangle.
\]

The functions defined in the perturbative terms are
\[
f(1) = 1 - e^{-\frac{s_0}{M^2}} \left(1 + \frac{s_0}{M^2}\right), \quad I(1) = \int_{e^{-\frac{s_0}{M^2}}}^1 dt \ln t \ln(-\ln t).
\]

We use the two-loop expression for the strong coupling \(^{42}\),
\[
\alpha_s(\mu) = \frac{\pi}{2\beta_1 \log(\mu/\Lambda)} \left(1 - \frac{\beta_2 \log(2 \log(\mu/\Lambda))}{\beta_1^2 \log(\mu/\Lambda)}\right),
\]
with the evolution kernel \(\beta_1 = (33 - 2n_f)/12\) and \(\beta_2 = (153 - 19n_f)/24\). The hadron scale \(\Lambda(n_f)\) = Which \(n_f = 3, 0.332, n_f = 4, 0.292, n_f = 5, 0.210\] is set to reproduce \(\alpha_s(m_Z) = 0.118\) and \(\alpha_s(1\text{ GeV}) = 0.474\). The input values for the nonperturbative vacuum condensates at default scale 1 GeV are taken as \(^{43,44}\):
\[
\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = (-0.25\text{ GeV})^2, \quad \langle \bar{s}s \rangle = 0.8 \langle \bar{q}q \rangle,
\]
\[
\langle g_s \bar{u} \sigma \cdot G u \rangle = \langle g_s \bar{d} \sigma \cdot G d \rangle = -0.8 \langle \bar{q} q \rangle, \quad \langle g_s \bar{s} \sigma \cdot G s \rangle = 0.8 \langle g_s \bar{u} \sigma \cdot G u \rangle,
\]
\[
\langle \alpha_s / \pi G^a_{\mu\nu} G^{a\mu\nu} \rangle = 0.012 \text{ GeV}^4,
\]
\[
\langle \bar{u} \gamma^\mu \lambda^a u \bar{d} \gamma^\mu \lambda^a d \rangle = -\frac{16}{3} \langle \bar{u} u \rangle \langle \bar{d} d \rangle.
\] 

(7)

The light quark masses are estimated as the "current-quark" masses in the \(\overline{\text{MS}}~(\mu = 1 \text{ GeV})\)\(^{42}\):

\[ m_u = 0.00287 \text{ GeV}, \quad m_d = 0.00613 \text{ GeV}, \quad m_s = 0.125 \text{ GeV}. \] 

(8)

We consider the parameters running with one-loop accuracy \(^{45-47}\),

\[
m_q(\mu) = m_q(\mu_0) \left( \frac{\alpha_s(\mu_0)}{\alpha_s(\mu)} \right)^{-1/\beta_1}, \quad \langle \bar{q} q \rangle_\mu = \langle \bar{q} q \rangle_{\mu_0} \left( \frac{\alpha_s(\mu_0)}{\alpha_s(\mu)} \right)^{1/\beta_1},
\]
\[
\langle \alpha_s G^2 \rangle_\mu = \langle \alpha_s G^2 \rangle_{\mu_0}, \quad \langle g_s \bar{q} \sigma \cdot G q \rangle_\mu = \langle g_s \bar{q} \sigma \cdot G q \rangle_{\mu_0} \left( \frac{\alpha_s(\mu_0)}{\alpha_s(\mu)} \right)^{-1/(6\beta_1)}.
\] 

(9)

Table 1: The flavor mixing angle of \(f_0(980)\) in the literature.

| Channels | \(\theta\) (Degree) | Refs. |
|----------|-----------------|------|
| \(f_0(980), \sigma(500) \rightarrow \pi^+\pi^-\) | \(27 \pm 13\) or \(41 \pm 11\) | \(48\) |
| \(\phi(1020) \rightarrow f_0(980)\gamma \rightarrow 3\gamma\) | \(-48 \pm 6\) or \(86 \pm 3\) | \(49\) |
| \(D \rightarrow f_0(980)P \rightarrow \pi\pi(KK)P\) | \(34 \pm 6\) or \(46 \pm 6\) | \(50\) |
| \(\pi\pi \rightarrow \pi\pi, KK\) | strong indication for the gluonium nature of \(\sigma\) | \(51\) |
| \(K \rightarrow \pi\pi\nu\nu\) | 18.7 or 151.3 | \(52\) |
| \(B \rightarrow f_0(980)K^*\) | \([25, 40]\) or \([140, 165]\) | \(53\) |
| \(B_s \rightarrow J/\psi f_0(980)[\sigma]\) | \(34^{+5.1+5.1+4.8}_{-10.5-9.2-6.4}\) or \(146^{+10.5+9.1+6.4}_{-5.1-5.1-4.8}\) | \(54\) |
| \(B_{d,s}^0 \rightarrow D_0 f_0(980)\) | \([29, 46]\) or \([135, 158]\) | \(55\) |
| \(B_{d,s}^0 \rightarrow J/\psi f_0(980)[\sigma]\) | \(\sim 25\) | \(56\) |

The comparable branching ratios of the cascade decays \(B \rightarrow f_0(980) \rightarrow \pi\pi, KK\) \(^{42}\) indicates a quark flavor mixing in \(f_0(980)\),

\[
|f_0(980)\rangle = |s\bar{s}\rangle \cos \theta + |n\bar{n}\rangle \sin \theta,
\] 

(10)

where \(\bar{n}n = \frac{1}{\sqrt{2}}(\bar{u}u + \bar{d}d)\). In Tab.1 we enumerate some constraints for the mixing angle \(\theta\), and we preform to take the interval \(\theta \in [25, 45]\) (in unit of Degree) in our calculation.
Differentiating both sides of Eq. 4 by the Borel mass and taking the ratio of the two equations, one can obtain the mass without the quark flavor mixing dependence, which is actually put into the decay constant $\bar{f}_{f_0} \equiv \sin \theta \bar{f}_{f_0}$.

The interval of Borel mass is fixed by the rule of thumb that the contribution from high dimension condensate terms is no larger than ten percents in the truncated OPE, and simultaneously the contribution from excited and continuum states is smaller than thirty percents when summing up the hadrons. The threshold $s_0$ is usually close to the outset of the first excited state with the same quantum number and then a certain vicinity can be expected, we determine it with considering the maximal stability of physical quantities once the Borel mass has been set down. In Fig. 1, we displace the mass and the modified decay constant of $f_0(980)$ predicted from the 2pSRs, where the light gray band shows the uncertainty associated with the mixing angle $\theta$. Within the intervals $M^2 = 1.2 \pm 0.1$ GeV$^2$ and $s_0 = 1.8 \pm 0.2$ GeV$^2$, we obtain

$$m_{f_0(980)} = 0.98 \pm 0.04 \text{ GeV}, \quad \bar{f}_{f_0(980)} = 0.19 \pm 0.05 \text{ GeV}.$$ (11)

The mass agrees well with the PDG average value $m_{f_0(980)} = 0.99 \pm 0.02$ GeV, which indicates that the quark-antiquark assignment of $f_0(980)$ is feasible for phenomenology under the QCD sum rules.

3 \hspace{1em} B \to f_0(980) form factors from the LCSRs

The approach of LCSRs with B meson DAs was first proposed to calculate the $B \to P, V$ form factors \cite{9}, in this section we apply it to calculate the $B \to f_0(980)$ form factors.\footnote{Till now there is no evidence of $\sigma$ in the $B$ decays \cite{57}, so we take $f_0(980)$ as the lowest state in the scalar meson spectrums in $B$ decays.} We start with the correlation function

$$\Pi_\nu(p, q) = i \int d^4x e^{ip \cdot x} \langle 0 | T \{ J^u_S(x), J^T_\nu(0) \} | B^-(p + q) \rangle,$$ (12)
where the scalar current is \( J_s^B = \bar{u}u \), and the weak current is \( J_\nu^I = \bar{u}\Gamma_\nu b \). The indicator \( \mathcal{I} = A, T \) correspond to the lorentz structures \( \Gamma_\nu = \gamma_\nu\gamma_5 \) and \( \sigma_{\nu\mu}\gamma_5q^\mu \), respectively.\(^3\) After transiting to the heavy quark effective theory (HQET), the heavy-to-light current is reduced to the light-quark current and the correlation function is modified to

\[
\tilde{\Pi}_\nu(p, \tilde{q}) = i \int d^4xe^{ipx} \langle 0|T\{J_s^B(x), \tilde{J}_\nu^I(0)\}|B_v^-(p + \tilde{q})\rangle.
\]  

We use the notations \( p \) to denote the momentum carried by the scalar current \( J_s^B \), and \( \tilde{q} = q - m_bv \) for the effective current \( \tilde{J}_\nu^I = \bar{u}\Gamma_\nu^I b \). In the rest frame the effective \( b \)-quark field is defined as \( h_v(x) = e^{im_bvx}(x) \) with the unit vector \( v = (1, 0, 0, 0) \), and \( |B_v^-(p + \tilde{q})\rangle = |B_v^-(p + \tilde{q})\rangle \) is hold up to the \( O(1/m_b) \) accuracy. We discuss in the intervals \( |p^2|, |\tilde{q}^2| \gg \Lambda^2_{QCD}, (m_B - m_b)^2 \) where the correlation function does not fluctuate widely and the OPE calculation is applicable.

We contract the same flavor quark fields in the propagator

\[
s_u(x, 0, m_u) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \int_0^1 duG_{\mu\nu}(ux) \left[ \frac{ux\gamma^\mu}{p^2 - m_u^2} - \frac{\hat{p} + m_u}{2(p^2 - m_u^2)} \right],
\]

in which the first term is the freedom quark propagator in the QCD limit, and the second term respects the soft one-gluon correction. Two-particle and three-particle \( B \)-meson DAs are defined as\(^4\)

\[
\langle 0|\bar{u}_\alpha(x)h_{\nu\beta}(0)|B_v^-\rangle = -\frac{if_Bm_B}{4} \int_0^\infty d\omega e^{-i\omegaux} \left[ (1 + \hat{\nu}) \right]
\]

\[
\cdot \left\{ \frac{\phi_+^B(\omega)}{2v \cdot x} - \frac{\phi_-^B(\omega) - \phi_-^B(\omega)}{2v \cdot x} \right\} \gamma_5, \]

\[
\langle 0|\bar{u}_\alpha(x)G_{\rho\delta}(ux)h_{\nu\beta}(0)|B_v^-\rangle = \frac{f_Bm_B}{4} \int_0^\infty d\omega \int_0^\infty d\zeta e^{-1(\omega + u\zeta)x} \left[ (1 + \hat{\nu}) \right]
\]

\[
\cdot \left\{ (v_\rho\gamma_\delta - v_\delta\gamma_\rho)[\Psi_A(\omega, \zeta) - \Psi_V(\omega, \zeta)] - i\sigma_{\rho\delta}\Psi_V(\omega, \zeta)
\]

\[
- \left( \frac{x_\rho\gamma_\sigma - x_\sigma\gamma_\rho}{v \cdot x} \right) X_A(\omega, \zeta) + \left( \frac{x_\rho\gamma_\sigma - x_\sigma\gamma_\rho}{v \cdot x} \right) Y_A(\omega, \zeta) \right\} \gamma_5 \]  

Two variables \( \omega \) and \( \zeta \) are introduced to represent the plus components of light quark momentum and the gluon momentum, respectively. For the sake of simplicity we have omitted the path-ordered gauge factors on the left hand sides.

\(^3\)We use the convention \( \sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \).

\(^4\)Recently, the renormalization group equations for three-particle distribution are resolved in the \( N_C \) limit and the models for higher-twist DAs of \( B \) meson are suggested \[^{58}\] \(^{59}\) \(^{62}\), following which the power suppressed correction are supplemented to \( B \) decays with energetic final states \[^{59}\] \(^{62}\). In this paper we concern more on the width effect which would be discussed in the next section, and postpone the completely correction from three-particle \( B \)-meson DAs for the future improvement.
The definition of $B \rightarrow S$ transition form factors is quoted as \cite{63,64}

$$\kappa(S(p)|J_{\nu}^A(0)|B^-(p + q)) = -i[F_+(q^2)p_\nu + F_-(q^2)q_\nu]$$

$$= -iF_1(q^2)[(2p + q)_\nu - \frac{m_B^2 - m_S^2}{q^2}q_\nu] - iF_0(q^2)\frac{m_B^2 - m_S^2}{q^2}q_\nu, \quad (17)$$

$$\kappa(S(p)|J_{\nu}^T(0)|B^-(p + q)) = -\frac{F_T(q^2)}{m_B + m_S}[q^2(2p + q)_\nu - (m_B^2 - m_S^2)q_\nu], \quad (18)$$

in which the normalizing factor is $\kappa = \sin \theta/\sqrt{2}$ for the neutral charged meson $f_0(980)$. Two equivalent definitions in Eq.\ref{17} indicate the following relation,

$$F_1(q^2) = \frac{F_+(q^2)}{2}(q^2), \quad F_0(q^2)\frac{(m_B^2 - m_S^2)}{q^2} = F_-(q^2) + \frac{F_+(q^2)(m_B^2 - m_S^2 - q^2)}{2}. \quad (19)$$

Following the Ref. \cite{9}, we calculate the correlation function in Eq.\ref{13} and obtain the $B \rightarrow f_0(980)$ form factors in the narrow width approximation,

$$F_{B \rightarrow f_0}^+(q^2) = \sin^2 \theta \frac{f_B m_B^2}{2} \int_0^{\sigma_0} d\sigma e^{-\frac{\sigma m_B^2}{M^2}} \left[ \frac{\bar{\sigma} m_B - m}{\bar{\sigma} m_B} \phi_+^B(\sigma m_B) \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r
\begin{align*}
+ m[\bar{\sigma}^2m_B^2 - q^2(1 - 2\sigma) - m^2] & \frac{[\phi_+^B(\sigma m_B) - \phi_-^B(\sigma m_B)]}{(\bar{\sigma}^2m_B^2 - q^2 + m^2)} \\
+ \left( m + 2mm_B[\bar{\sigma}^2m_B^2 - q^2(1 - 2\sigma) - m^2] \right) & \frac{[\phi_+^B(\sigma m_B)]}{(\bar{\sigma}^2m_B^2 - q^2 + m^2)} \\
+ \Delta F_{T,q}^{B \to S}(q^2, s_0, M^2) \right). (23) \end{align*}

The dimensionless variable \( \sigma \equiv \omega / m_B \) is the longitudinal momentum fraction of the light quark inside \( B \)-meson, and the virtuality of internal quark is \( s = m_B^2\sigma - (q^2\sigma - m^2)/\bar{\sigma} \). To obtain the above expressions, we have defined an auxiliary distribution

\[ \tilde{\phi}_\pm^B(\omega) \equiv \int_0^\omega d\tau [\phi_+^B(\tau) - \phi_-^B(\tau)] \] (24)

with the boundary \( \tilde{\phi}_\pm^B(0) = \tilde{\phi}_\pm^B(\infty) = 0 \). We give several comments timely following on the results: (i) our results shown in Eqs.(20-21) consist with the calculations presented for the \( B \to D_0^* \) form factors \cite{65} with considering the \( m_c \) effect, (ii) Eq.22 and Eq.23 are obtained by matching the coefficients associated to \( p_\nu \) and \( q_\nu \), respectively, and they should be equal to each other with the definition in Eq.18 and (iii) considering the Eqs.(20-23) at leading power, we get the relation \( F_T(q^2) = \frac{m_B - m_u}{m_B + m_s} F_-(q^2) \) in the heavy quark limit, and the relations \( F_+(0) = \frac{2m_B}{m_B + m_s} F_T(0) \), \( F_-(0) = 0 \) at the full recoil energy point.

The contributions from three-particle DAs of \( B \) meson are arranged in the universal form

\[
\Delta F_i^{B \to S}(q^2, s_0, M^2) = \int_0^{s_0} d\sigma e^{-s_0 \frac{m_B^2}{M^2}} \left( -I_{1,i}^{B \to S}(\sigma) + \frac{I_{2,i}^{B \to S}(\sigma)}{M^2} - \frac{I_{3,i}^{B \to S}(\sigma)}{2M^4} \right) \\
+ \left[ \frac{m_B^2}{m_B^2} \eta(\sigma) \left[ I_{2,i}^{B \to S}(\sigma) - \frac{1}{2} \left( \frac{1}{M^2} + \frac{d\eta}{d\sigma} \right) I_{3,i}^{B \to S}(\sigma) \right] \right] \bigg|_{\sigma = s_0}. (25) \]

The dimensionless variable \( \eta = \frac{\bar{\sigma}^2m_B^2}{\bar{\sigma}^2m_B^2 - q^2 + m^2} \) can be understood as the ratio between the minimal virtuality of the \( b \) quark field and the maximal virtuality carried by the internal light quark. The integral over the three-particle DAs is written as

\[
I_{N,i}^{B \to S}(\sigma) = \frac{1}{\bar{\sigma}N} \int_0^{s_0} \int_0^\infty d\omega \int_{\sigma m_B - \omega}^\infty \frac{d\zeta}{\zeta} \left[ C_{N,i}^{B \to S,\Phi_A}(\sigma, u, q^2) \Phi_A(\omega, \zeta) + C_{N,i}^{B \to S,\Phi_V}(\sigma, u, q^2) \Phi_V(\omega, \zeta) + C_{N,i}^{B \to S,X_A}(\sigma, u, q^2) \bar{X}_A(\omega, \zeta) + C_{N,i}^{B \to S,Y_A}(\sigma, u, q^2) \bar{Y}_A(\omega, \zeta) \right] \bigg|_{u=(\sigma m_B - \omega)/\zeta}, (26) \]

here another two auxiliary distributions are introduced as

\[
\bar{X}^B_A \equiv \int_0^\omega d\tau X^B_A(\tau, \zeta), \quad \bar{Y}^B_A \equiv \int_0^\omega d\tau Y^B_A(\tau, \zeta). (27) \]
In Eq.26, the lower indicator $N = 1, 2, 3$ stands for the power of Borel mass $M^{-2(N-1)}$ pre-multiplied with the integrals, the coefficients $C_{B \rightarrow S}^{B \rightarrow S}$ associated to each three-particle DA are presented in Appendix A.

We take the typical MS for $b$ quark mass $m_b(m_b) = 4.2$ GeV, and use $f_B = 0.207$ GeV obtained from the two-point QCD sum rules [66]. The inverse moment of $B-$meson DAs is set at $\lambda_B = 350$ MeV by considering $\mathcal{F}_{T,p}(0) = \mathcal{F}_{T,q}(0)$, this value is in agreement with the recent estimations [11,67], where $\lambda_B = 358^{+38}_{-30}(343^{+22}_{-20})$ MeV is obtained by comparing the $B \rightarrow \pi(\rho)$ form factor from LCSR with pion [68] (rho [8]) and $B-$meson DAs. In Fig.2 we plot the $B \rightarrow f_0(980)$ form factors derived under the narrow $f_0$ approximation, where the lightgray shadows reveal the total uncertainty came from the quark flavor mixing angle and the LCSR parameters. It is obvious to check the relations between the form factors in the full recoil energy point. The tensor form factors obtained with Eq.22 and Eq.23 consist very well with each other in the considered energy regions, so if we know these two results with a high accuracy, i.e., with including the NLO QCD corrections, we can determine/restrict the DAs of $B-$meson. The numerical results show that the three-particle contribution to the tensor form factor $\mathcal{F}_T(Q^2)$ is tiny, while the correction to the form factor $\mathcal{F}_+(Q^2)$ is about 10% in the considered energy regions, this prediction agrees with the result obtained recently in the covariant quark model.
For the form factor $F_{2-}(Q^2)$, the three-particle contribution is opposite in sign to the two-particle contribution, and is dominant in the large momentum transfer regions. If we want to go further accuracy prediction, the width effect of the intermediate scalar states should be considered.

### 4 The width effect and the $B \to \pi \pi$ form factors

To discuss the width effect of the intermediate states, let’s look back to the dispersion relation of the correlation function in Eq.12

$$\Pi_\nu(p,q) = \frac{1}{\pi} \int_0^\infty ds \frac{\text{Im} \Pi_\nu(s,q^2)}{s - p^2 - i\epsilon}, \quad (28)$$

where the imaginary part is obtained rigorously by interpolating a complete set of intermediate states to retain the unitarity,

$$2 \text{Im} \Pi_\nu(s,q^2) = \sum_n \int d\tau_n \langle 0|J^\nu_S|S_n(p_n)\rangle \langle S_n(p_n)|J^i_\nu(q^2)|B^-(q + p_n)\rangle (2\pi)^4\delta(s - p_n^2). \quad (29)$$

In the above equation, $p_n$ is the momentum of each interpolated state $|S_n\rangle$, $d\tau_n$ denotes the integration over the phase space volume of these state. An immediate way to investigate the width effect is to substitute the interpolation of scalar mesons by the stable multi-meson states, such as the $\pi\pi$, $4\pi$, $KK$ and their continuum states \[12\].

With the interpolation of the single meson states, the correlation function is rewritten as

$$\Pi_\nu(p^2,q^2) = \frac{2\sqrt{\alpha g f}_\pi f_0 \langle f_0(p)|J^\nu_S(p)|B^-(p + q)\rangle}{m_S^2 - p^2 - i\epsilon} + \frac{1}{\pi} \int_{s_0}^\infty ds \frac{\text{Im} \Pi_\nu(s,q^2)}{s - p^2 - i\epsilon}, \quad (30)$$

where the contribution from the ground state is singled out while the rest contributions are retained in the integral, $s_0$ is again the threshold truncated the excited states. Eq.30 has been used to derive the LCSRs result in sec.3.

For the case with the multi-meson states interpolating, Eq.28 is modified to

$$\Pi_\nu(k^2,q^2) = \frac{1}{\pi} \int_{4m_\pi^2}^\infty ds \frac{\int d\tau_{2\pi} \frac{J^\nu_{2\pi}(k^2)}{\frac{2m_\pi^2}{2m_\pi} \delta_{\rho\sigma} \langle \pi^\rho(k_1)\pi^\sigma(k_2)|J^i_\nu(q)|B^-(k + q)\rangle}{s - k^2 - i\epsilon}} + \cdots. \quad (31)$$

The ellipsis denotes the contributions from other interpolating states with higher thresholds, like $4\pi$, $K\bar{K}$, and etc. As we will see later, the threshold of scalar pion form factor is $s_0^{2\pi} \sim 2.4$ GeV$^2$, then the phase space opens for the $4\pi$ and $K\bar{K}$ states. But in fact, the contribution from the $K\bar{K}$ state is expected to be suppressed since it is not the main channel coupled to $B^-$.

\[6\] In the case with vector pion form factor \[12\], the contributions from $4\pi$, $K\bar{K}$ states are suppressed by the threshold space $s_0^{2\pi} \lesssim 1.0 - 1.5$ GeV$^2$. \[70\]
can be extrapolated to a high energy calculation, we would adopt the phase updated from the amplitude analysis \[83, 84\], which forced to employ a model \[81\] or to adopt the perturbative QCD approximation \[82\]. In our theory loses the unitarity and the analyticity when going to more higher energies, and one is well as the low energy Roy equation \[85\].

The remaining matrix element in Eq.31 is expressed in terms of the Borel mass \(M^2\) with the normalization \(s_0^\pi = m^\pi_\pi^2\) under the chiral symmetry, the averaged light quark mass is \(m = (m_u + m_d)/2\). The once-subtracted omenés representation of the form factor is \[72\]

\[
\Gamma_\pi(t) = P(t) \exp \left[ \frac{t}{4m^2_\pi} \int_{4m^2_\pi}^{\infty} ds \frac{\delta_{\Gamma^\pi}(s)}{s(s-t)} \right],
\]

with the normalization \(\Gamma_\pi(0) = m^\pi_\pi^2\) under the chiral symmetry, the averaged light quark mass is \(\bar{m} = (m_u + m_d)/2\). The once-subtracted omenés representation of the form factor is \[72\]

\[
\Gamma_\pi(t) = P(t) \exp \left[ \frac{t}{4m^2_\pi} \int_{4m^2_\pi}^{\infty} ds \frac{\delta_{\Gamma^\pi}(s)}{s(s-t)} \right],
\]

where \(\delta_{\Gamma^\pi}\) is defined as the phase of \(\Gamma_\pi(t)/P(t)\). The function \(P(t)\) is the polynomial to consider the zero values \[72\] in the case of free zeros \(P(t) = \Gamma_\pi(0)\). The Watson theorem makes sure that the identity \(\delta_{\Gamma^\pi} = \delta^0\) holds in the elastic region \(s \leq 4m^2_K\). Beyond the \(K\bar{K}\) threshold, the generalized Watson theorem with considering the coupled-channel \((\pi\pi - K\bar{K})\) is still available to predict the phase up to the energy \(s = 1.5^2\) GeV² \[76–80\]. Unfortunately, the dispersion theory loses the unitarity and the analyticity when going to more higher energies, and one is forced to employ a model \[81\] or to adopt the perturbative QCD approximation \[82\]. In our calculation, we would adopt the phase updated from the amplitude analysis \[83, 84\], which can be extrapolated to a high energy \(\sim 10\) GeV² by considering all the measured data, the \(\pi\pi - K\bar{K}\) final state interaction, the mass difference between the charged and neutral kaon, as well as the low energy Roy equation \[85\].

The remaining matrix element in Eq.31 is expressed in terms of the \(B \to \pi\pi\) form factors. For the axial-vector current sandwiched between the \(B\)-meson and the dipion state, we have \[86\]

\[
-i\langle \pi^+(k_1)\pi^-(k_2)|\bar{u}\gamma_\nu\gamma_5 b|B^-(q + k)\rangle
\]

\[
= F_t(q^2, k^2, q \cdot k) \frac{q_\nu}{\sqrt{q^2}} + F_0(q^2, k^2, q \cdot k) \frac{2\sqrt{q^2}}{\sqrt{\lambda_B}} \left( k_\nu - \frac{k \cdot q}{q^2} q_\nu \right)
\]

\[1\]Hereafter we take \(f_0, f_0', f_0''\) and \(f_0''\) to denote the scalar state \(f_0(980), f_0(1370)\) and \(f_0(1700)\), respectively.

\[2\]The experiment data \[73\] have made known that in the \(S\)-wave isoscalar \(\pi\pi\) elastic scattering, the phase \(\delta^0\) is very close to \(\pi\) and the moduli of amplitude has a sharp dip around the \(K\bar{K}\) threshold, which indicates the corresponding form factor may have a zero in this interval and the phase at there can not be understood without taking into account the \(K\bar{K}\) channel \[74, 75\]. When \(\delta^0 = \pi\) happens below the threshold \(s_1 < 4m^2_K\), the form factor has a zero at this energy. On the contrast, the zero value does not appear if \(\delta^0 = \pi\) occurs after opening the \(K\bar{K}\) channel, but a dip close arbitrarily to zero is allowed around the threshold.
\[ F\parallel(q^2, k^2, q \cdot \bar{k}) \frac{1}{\sqrt{k^2}^2} (\bar{k}_\nu - \frac{4(q \cdot k)(q \cdot \bar{k})}{\lambda_B} k_\nu + \frac{4k^2(q \cdot \bar{k})}{\lambda_B} q_\nu). \]  

(34)

The dot products appeared above are written down by three independent variables,

\[ q \cdot k = \frac{1}{2}(m_B^2 - k^2 - q^2), \quad q \cdot \bar{k} = \frac{\sqrt{\lambda_B}}{2} \beta_\pi(k^2) \cos \theta_\pi, \]  

(35)

where \( k^2 \) is the invariant mass of dipion system, \( q^2 \) denotes the squared momentum transfer in the weak decay, and \( \theta_\pi \) represents the angle between the 3-momentum of \( \pi^-k_2 \) and \( B \) meson in the dipion rest frame. What’s more, \( \beta_\pi(k^2) = \sqrt{1 - 4m^2/\lambda_B} \) is the phase factor of dipion system, and \( \lambda_B \equiv \lambda(m_B^2, q^2, k^2) = m_B^4 + k^4 + q^4 - 2(m_B^2k^2 + m_B^2q^2 + k^2q^2) \) is the kinematic Källén function. By the way, the transition matrix element in Eq.31 can also be expressed as,

\[ H_\lambda = \langle \pi^+(k_1)\pi^-k_2|\bar{u}\gamma_\nu\gamma_5b|B^-(q + k)\rangle, \]  

(36)

with different helicities of the dipion state (\( \lambda = t, 0, +, - \)). An advantage of the expression in Eq.36 is that the helicity amplitudes can be expanded in terms of the associated Legendre polynomials, then the contributions from different partial waves can be studied. For our calculation the expression in Eq.34 is more convenient with the orthogonal momentum vectors, so we translate the partial wave expansion from the helicity amplitudes \( H_\lambda \) to the form factors \( F_i \),

\[ F_{0,l}(q^2, k^2, q \cdot \bar{k}) = \sum_{l=0}^\infty \sqrt{2l + 1} \frac{F(l)_{0,l}(q^2, k^2)}{l} P_l(0)(\cos \theta_\pi), \]

\[ F\parallel(q^2, k^2, q \cdot \bar{k}) = \sum_{l=1}^\infty \sqrt{2l + 1} \frac{F(l)\parallel(q^2, k^2)}{l} P_l(1)(\cos \theta_\pi) \frac{\sin \theta_\pi}{\sin \theta_\pi}. \]  

(37)

Substituting Eq.34 and Eq.37 into the numerator in Eq.31, we obtain the \( S \)–wave contribution to the imaginary part as

\[ \int d\tau_{2\pi} \frac{\Gamma^*_\pi(k^2)}{2m} \langle \pi^+(k_1)\pi^-k_2|J^A_\nu(q)|B^-(p + k)\rangle \]

\[ = \frac{i}{16\pi} \frac{\beta_\pi(s)}{m} \left[ F(l=0)_{0,0}(q^2, s) \frac{2\sqrt{q^2}}{\sqrt{\lambda_B}} k_\nu + \left( \frac{F(l=0)_{l,0}(q^2, k^2)}{\sqrt{q^2}} - \frac{F(l=0)_{l,0}(q^2, s)}{\sqrt{\lambda_B}} \frac{2\sqrt{q^2}}{k \cdot q} q_\nu \right) \right]. \]  

(38)

In fact, the phase space \( d\tau_{2\pi} \) plays as a \( S \)–wave projector for the timelike-helicity form factors \( F_{l,0}(q^2, k^2, q \cdot \bar{k}) \), which means that only the \( S \)–wave component \( F_{l,0}(q^2, k^2) \) survives after integrating over the angle \( \theta_\pi \). For the form factor \( F\parallel(q^2, k^2, q \cdot \bar{k}) \), the \( S \)–wave projector vanishes and the contribution starts from the \( D \)–wave component (\( l = 2n, n = 1, 2, 3 \cdots \)), which part is expected tiny in the \( B \rightarrow \pi\pi \) transition and would not be discussed in this paper. In order to eliminate the contribution from higher power condensate terms and the excited states, we take the semi-local duality with the effective threshold \( s_0^{2\pi} \)

\[ \int_{s_0^{2\pi}}^\infty ds e^{-s/M^2} \text{Im}\Pi_\nu(s, q^2) = \int_{s_0^{2\pi}}^\infty ds e^{-s/M^2} \text{Im}\Pi_\nu^{\text{OPE}}(s, q^2). \]  

(39)
The LCSRs results for the $S$–wave $B \to \pi \pi$ transition are obtained as,

\[
\int_{4m_π^2}^{s_π^*} ds \, e^{-s/M^2} \frac{\beta_π(s)}{\hat{m}} \frac{Γ_π(s)}{m} \Gamma^{(t=0)}_π(q^2, s) \sqrt{q^2/\sqrt{s_B}} = \frac{f_B m_B^2}{2} \left\{ \int_0^{s_π^*} ds \frac{e^{-s(M_π^2/\hat{m})}}{32π^2} \frac{Γ_π(s)}{m} \left( \frac{F^{(t=0)}_π(q^2, k^2)}{\sqrt{q^2}} - \frac{F^{(t=0)}_0(q^2, s)}{2\sqrt{q^2} k \cdot q} \right) \right\}, \tag{40}
\]

\[
\int_{4m_π^2}^{s_π^*} ds \, e^{-s/M^2} \frac{β_π(s)}{\hat{m}} \frac{Γ_π(s)}{m} \frac{Γ^{(t=0)}_π(q^2, k^2)}{\sqrt{q^2}} \left[ \frac{1 - σ_m^B(σm_B)}{2} + \Delta f^{B→S}_π(q^2, s^2_π^*, M^2) \right]. \tag{41}
\]

Multiplying both sides of the Eq. 17 and Eq. 34 by $q^ν$, we obtain the matrix elements deduced by the pseudo-scalar current $J^P = i\bar{m}_b \bar{u} γ_5 b$,

\[
\langle S(p)|J^P(0)|B^−(p + q)⟩ = (m_B^2 - m_S^2) F_0(q^2), \tag{42}
\]

\[
\langle π^+(k_1)π^−(k_2)|J^P|B^−(q + k)⟩ = √q^2 F_π(q^2, k^2, q \cdot k). \tag{43}
\]

Eq. 42 suggests an auxiliary LCSRs for the $B \to f_0(980)$ form factor $F_0$,

\[
F^{B→f_0}_0(q^2) = \frac{sin^2 θ}{2} \frac{f_B m_B^3 m_b}{m_f_0_f_S(m_B^2 - m_f^2)} \left\{ \int_0^{σ_0} dσ e^{-s(M_π^2/\hat{m})} \left[ \frac{2m_B^2 - q^2 + m^2}{2σ^2 m_B^2} - \frac{m}{σ_m B} \right] \phi_+(σm_B) \right. \right.
\]

\[
\left. + \left[ \frac{m^2}{σ_m^2 m_B^2 - q^2 + m^2} - \frac{m}{2σ_m B} \right] \phi_+^B(σm_B) \right] \right. \left. + \left[ \frac{2σ_m^2 m_B^2}{σ^2 m_B^2 - q^2 + m^2} + \frac{6m}{2σ^2 m_B^2} - \frac{3}{σ_m B} \right] \phi_+(σm_B) \right] + ΔF^{B→S}_0(q^2, s^2_0, M^2). \tag{44}
\]

Eq. 43 hints another LCSRs for the timelike-helicity $B \to \pi \pi$ form factor $F_π$, where the phase space integral of the dipion state is modified to

\[
\int dτ_π \frac{Γ_π^∗(k^2)}{2m} \langle π^+(k_1)π^−(k_2)|J^P|B^−(p + k)⟩ = \frac{Γ_π^∗(s)}{m} √q^2 F_π^{(t=0)}(q^2, s), \tag{45}
\]

and the corresponding LCSRs is

\[
\int_{4m_π^2}^{s_π^*} ds \, e^{-s/M^2} \frac{β_π(s)}{\hat{m}} \frac{Γ_π(s)}{m} \frac{Γ^{(t=0)}_π(q^2, s)}{\sqrt{q^2}} = \frac{f_B m_B^3 m_b}{2} \left\{ \int_0^{s_π^*} dσ e^{-s(M_π^2/\hat{m})} \left[ \frac{2m_B^2 - q^2}{2σ^2 m_B^2} - \frac{3}{σ_m B} \phi_+(σm_B) \right] \right\}. \tag{46}
\]
4.2 Models

Eqs. (40-46) are the main results in this section. A trouble we encountered immediately is that we can not solve out the $B \to \pi\pi$ form factors in terms of the $B$ meson DAs, because they are convoluted with the scalar pion form factor in the integrand. We have to introduce models to parameterise the $S-$wave $B \to \pi\pi$ form factors, and the first candidate coming into mind is the single resonance model written as

\[ F_0^{(l=0)}(s, q^2) \frac{\sqrt{q^2}}{\sqrt{\lambda_B}} = \frac{1}{\sqrt{\lambda_B}} \frac{g_{f_0\pi^+\pi^-} - \mathcal{F}_{B \to f_0}^B(q^2)}{m_{f_0} - s + i\sqrt{s}\Gamma_{f_0}(s)} e^{i\phi_{f_0}(s, q^2)}, \]

\[ \frac{1}{2} \left( \frac{F_t^{(l=0)}(s, q^2)\sqrt{q^2}}{\sqrt{\lambda_B}} \right)^2 - F_0^{(l=0)}(s, q^2)\frac{\sqrt{q^2}}{\sqrt{\lambda_B}} \frac{m_B^2 - s - q^2}{q^2} = \frac{1}{\sqrt{\lambda_B}} \frac{g_{f_0\pi^+\pi^-} - \mathcal{F}_{B \to f_0}^B(q^2)}{m_{f_0} - s + i\sqrt{s}\Gamma_{f_0}(s)} e^{i\phi_{f_0}(s, q^2)}. \]

\[ \frac{1}{2} F_t^{(l=0)}(s, q^2)\sqrt{q^2} = \frac{1}{\sqrt{\lambda_B}} \frac{g_{f_0\pi^+\pi^-} - \mathcal{F}_{B \to f_0}^B(q^2)(m_B^2 - m_{f_0}^2)}{m_{f_0}^2 - s + i\sqrt{s}\Gamma_{f_0}(s)} e^{i\phi_{f_0}(s, q^2)}. \]
A condition underlying in Eqs.(40,41,46) is the reality\(^9\) of the imaginary part,
\[
\text{Im} \left[ \Gamma^*_\pi(s) F_i^{(l=0)}(q^2, s) \right] = 0. \tag{50}
\]

We introduce a strong phase \(\phi_{f_0}\) to compensate the phase difference between the pion form factor and the \(f_0\) model for \(B \to \pi\pi\) form factors. Generally speaking, \(\phi_{f_0}\) should depend on both the two variables \(s\) and \(q^2\), but the \(q^2\) dependence does not appear in the single \(f_0\) resonance model,
\[
\delta_{f_\pi}(s) - \phi_{f_0}(s) = \text{Arg} \left[ \frac{g_{f_0\pi\pi}}{m_{f_0}^2 - s + i\sqrt{s}\Gamma_{f_0}(s)} \right]. \tag{51}
\]

The simple model in Eqs.(47,49) is inspired by the physics that the sum rules obtained for the \(B \to \pi\pi\) form factor, in the narrow width approximation, should recover the sum rules for the \(B \to f_0(980)\) form factor. To check this point, let's consider the energy-dependent \(f_0 \to \pi\pi\) width\(^1\)\(^\text{10}\)\(^\text{11}\):
\[
\Gamma_{f_0}(s) = \frac{g_{f_0\pi\pi}^2\beta_\pi(s)}{16\pi \sqrt{s}} \Theta(s - 4m^2_\pi) = \Gamma_{f_0}^{\text{tot}} \frac{\beta(s)}{\beta(m_{f_0})} \frac{m_{f_0}}{\sqrt{s}} \Theta(s - 4m^2_\pi), \tag{52}
\]

where \(\Gamma_{f_0}^{\text{tot}}\) is the total width of \(f_0(980)\), and the strong coupling is normalized as
\[
\langle \pi^+(k_1)\pi^-(k_2)|f_0(980)(k_1 + k_2)\rangle = g_{f_0\pi^+\pi^-} \tag{53}
\]
with the relation \(g_{f_0\pi^0\pi^0} = g_{f_0\pi^+\pi^-}/\sqrt{2}\). The energy-dependence of the width represents the loop effects of two pions coupling to the \(f_0\) state, from this point we take intentionally the \(f_0\)–dominance approximation for the scalar pion form factor\(^1\)\(^\text{11}\):
\[
\left. \frac{\Gamma^*_\pi(s)}{m_{f_0}} \right|_{f_0} = \frac{g_{f_0\pi\pi} m_{f_0} \tilde{f}_{f_0}}{m_{f_0}^2 - s - i\sqrt{s}\Gamma_{f_0}(s)} e^{-i\phi_{f_0}(s,q^2)}. \tag{54}
\]

Substituting Eqs.(47,54) into Eq.40 and taking into account Eq.52, the l.h.s of Eq.40 becomes
\[
\frac{g_{f_0\pi\pi}^2 m_{f_0} \tilde{f}_{f_0}}{m_{f_0}^2 - s - i\sqrt{s}\Gamma_{f_0}(s)} e^{-i\phi_{f_0}(s,q^2)} \int_{4m^2_\pi}^{s_\pi^0} dse^{-s/M^2} \frac{\Gamma_{f_0}(s)\sqrt{s}}{\pi (m_{f_0}^2 - s)^2 + s\Gamma_{f_0}(s)} \Gamma_{f_0}^{\text{tot} \to 0} \to m_{f_0} \tilde{f}_{f_0} \mathcal{F}^{B \to f_0}_+ (q^2) e^{-m_{f_0}^2/M^2}, \tag{55}
\]
and we recover the result for the form factor \(\mathcal{F}^{B \to f_0}_+\) in the zero-width limit \(\Gamma_{f_0}^{\text{tot} \to 0}\). Similarly, we can recover the LCSRs for the form factors \(\mathcal{F}_-^{B \to f_0}\) and \(\mathcal{F}_0^{B \to f_0}\) with taking into account the resonance models in Eqs.48 and 49, respectively. What’s more, the relation defined in Eq.19 also holds well in the resonance models in Eqs.(47,48,49).

\(^9\)Here we consider the more strict local reality at each point of invariant mass.

\(^\text{10}\)We discuss in the general formula without specifying detail expression of \(\Gamma_{f_0}(s)\), i.e., the Flatté model [87].

\(^\text{11}\)In fact, the Breit-Wigner formula cannot be used directly to describe the scalar resonance. The case is different in our previous work about the \(P\)–wave \(B \to \pi\pi\) form factor [12], where the magnitude of timelike pion form factor is well measured [88] and parameterized in the Gounaris-Sakurai model [89].
We employ the $z$–series expansion \cite{90} to parameterise the \( B \to f_0 \) form factors,

\[
\mathcal{F}_j^{B\to f_0}(q^2) = \frac{\mathcal{F}_j^{B\to f_0}(0)}{1 - q^2/m_B^2} \left\{ 1 + b_{\mathcal{F}_j} \zeta(q^2) + c_{\mathcal{F}_j} \zeta^2(q^2) \right\}, \tag{56}
\]

where \( j = +, - , 0 \), \( \mathcal{F}_j^{B\to f_0}(0) \) is the values at the full recoil energy point, and the parameters \( b_{\mathcal{F}_j}, c_{\mathcal{F}_j} \) indicate the coefficients associated with the \( \zeta \)–functions,

\[
\zeta(q^2) = z(q^2) - z(0), \tag{57}
\]

\[
z(q^2) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}} \tag{58}
\]

with the definitions \( t_+ \equiv (m_B \pm m_{f_0})^2 \) and \( t_0 \equiv t_+ (1 - \sqrt{1 - t_- / t_0}) \).

\subsection*{4.3 Numerics}

With the single \( f_0 \) model in Eqs.\((47,48,49)\), we rewrite the LCSRs predictions for the \( S \)–wave \( B \to \pi\pi \) form factors in a more general formula,

\[
\frac{\kappa_{\mathcal{F}_j} + \eta_{\mathcal{F}_j} \zeta(q^2) + \rho_{\mathcal{F}_j} \zeta^2(q^2)}{1 - q^2/m_B^2} \times \mathcal{F}_j I(s_0^{2\pi}, M^2, \Gamma_{f_0}^{\text{tot}}) = I_j^{\text{OPE}}(s_0^{2\pi}, M^2, q^2), \tag{59}
\]

where for the sake of brevity we introduce the following notations:

\[
\kappa_{\mathcal{F}_j} \equiv |g_{f_0\pi^+\pi^-}| \mathcal{F}_j^{B\to f_0}(0)/\mathcal{X}, \quad \eta_{\mathcal{F}_j} \equiv b_{\mathcal{F}_j} |g_{f_0\pi^+\pi^-}| \mathcal{F}_j^{B\to f_0}(0)/\mathcal{X}, \quad \rho_{\mathcal{F}_j} \equiv c_{\mathcal{F}_j} |g_{f_0\pi^+\pi^-}| \mathcal{F}_j^{B\to f_0}(0)/\mathcal{X}, \quad X_{\mathcal{F}_+} = X_{\mathcal{F}_-} = 1, \quad X_{\mathcal{F}_0} = (m_B^2 - m_{f_0}^2). \tag{60}
\]

The integral coefficient on the l.h.s reads as

\[
I(s_0^{2\pi}, M^2, \Gamma_{f_0}^{\text{tot}}) = \frac{1}{16\pi^2} \int_{4m_{\pi}^2}^{s_0^{2\pi}} ds e^{-s/M^2} \frac{\beta_{\pi}(s) \left| \Gamma_{\pi}(s)/\tilde{m} \right|^2}{\sqrt{\left( m_{f_0}^2 - s \right)^2 + s \Gamma_{f_0}^2(s)}}. \tag{61}
\]

\( I_j^{\text{OPE}} \) represent the OPE calculations on the r.h.s of Eqs.\((40,41,46)\).

There is no physics requirement that the threshold \( s_0^{2\pi} \) should be equal to \( s_0 \). We fixed it in an independent way by considering again the correlation function in Eq.\(1 \) but with interpolating the multi-meson states with the same quantum number. The 2pSRs is then written in terms of the \( S \)–wave isoscalar pion form factor as

\[
\int_{4m_{\pi}^2}^{s_0^{2\pi}} ds e^{-s/M^2} \frac{\beta_{\pi}(s) \left| \Gamma_{\pi}(s) \right|^2}{16\pi^2 \sqrt{2\tilde{m}}} = \Pi_{2\text{pSRs}}^{\text{OPE}}(s_0^{2\pi}, M^2). \tag{62}
\]
Figure 4: The moduli and phase of the Omenè function (Eq.33) in the $S$–wave isoscalar $\pi\pi$ scattering [83].

With the form factor expressed in Eq.33 and displaced in Fig.4, we obtain the threshold $s_{0}^{2\pi} = 2.4$ GeV$^{2}$, which is larger than the threshold $s_{0} = 1.8$ GeV$^{2}$ in the case of single meson interpolating. This is easy to be understood by the broad structure of the $S$–wave isoscalar $\pi\pi$ state. It is apparent from Fig.4 that there are two smooth peaks appear in the $S$–wave isoscalar pion form factor, so it is nature to question what’s the roles of $\sigma$ and the excited states, like $f_{0}', f_{0}''$, in the $B \to \pi\pi$ transition. To convince ourselves, we suggest additionally the $f_{0} + \sigma$ model and the $f_{0} + f_{0}' + f_{0}''$ model.

We consider the first one by appending $\sigma$ to the single $f_{0}$ model, and modify the Eq.(49) to

$$
\frac{1}{2}F^{(l=0)}_{j}(s, q^{2})\sqrt{q^{2}} = \sum_{S=f_{0},\sigma} \frac{g_{S\pi}\pi}{{\kappa}_{S} \sum} \frac{\mathcal{F}_{j}^{B\to S}(q^{2})(m_{B}^{2} - m_{S}^{2})}{m_{S}^{2} - s + i\sqrt{s}\Gamma_{S}(s)} e^{i\phi_{S}(s)}. \tag{63}
$$

We again tacitly assume that the strong phases $\phi_{S}$ associated to each resonance are only dependent on the invariant mass of dipion state, the simplest way to achieve it is to take the same $q^{2}$–evolution for both the two resonances: $\mathcal{F}_{j}^{B\to \sigma}(q^{2}) = \gamma\mathcal{F}_{j}^{B\to f_{0}}(q^{2})$. The dimensionless parameter $\gamma\mathcal{F}_{j}^{B\to \sigma}$ indicates the relative contribution from $\sigma$ comparing to the contribution from $f_{0}$ which is normalized as unit. In this way, the general formula in Eq.(59) is rewritten in the $f_{0} + \sigma$ model,

$$
\sum_{S=f_{0},\sigma} \left[ \gamma\mathcal{F}_{j}^{S} X_{j}^{S} I^{S}(s_{0}^{2\pi}, M^{2}, \Gamma_{S}^{\text{tot}}) \right] \frac{\kappa_{\mathcal{F}_{j}} + \eta_{\mathcal{F}_{j}}(q^{2}) + \rho_{\mathcal{F}_{j}}(q^{2})}{1 - q^{2}/m_{B}^{2}} = I_{j}^{\text{OPE}}(s_{0}^{2\pi}, M^{2}, q^{2}). \tag{64}
$$

Inspired by the Eq.(19) we also apply the relation $2\mathcal{F}_{j}^{B\to S}(0) = \mathcal{F}_{j}^{B\to S}(0)$ in the fitting. The fitting result is shown in Tab.2 where the first errors come from the total widths of the intermediate resonances ($\Gamma_{f_{0}}^{\text{tot}} = 0.055 \pm 0.045$ GeV and $\Gamma_{\sigma_{0}}^{\text{tot}} = 0.55 \pm 0.15$ GeV [42]), and the second errors are obtained by varying the sum rules parameters ($M^{2} = 1.2 \pm 0.1$ GeV$^{2}$ and $s_{0}^{2\pi} = 2.4 \pm 0.2$ GeV$^{2}$). We quote the values of the integral coefficients to show the relative contributions from the two resonances,

$$
I_{f_{0}} = (7.29^{+1.93}_{-0.85} \pm 0.45) \times 10^{-2}, \quad I_{\sigma_{0}} = (5.45^{+1.20}_{-0.81} \pm 0.21) \times 10^{-2}. \tag{65}
$$
In Fig.5, we plot the $B \rightarrow \pi \pi$ form factor $\sqrt{q^2} F_t^{(\ell=0)}(s, q^2)$ obtained within the $f_0 + \sigma$ model. Because the widths of intermediate resonances are still varying in a broad region (i.e., $\Gamma_{f_0}^{\text{tot}} \in [0.01, 0.1]$ GeV), the huge uncertainties (extending in two orders of magnitude) appear around the resonance poles, this result can be read straightly from the denominators in Eq.63. Let us focus on the full recoil energy point $q^2 = 0$, where the form factors has been calculated from the LCSRs with asymptotic dipion DAs [91]. In that work, the phase-shift of $\pi\pi$ scattering and the once-subtracted Omnès solution are adopted to express the asymptotic coefficient $B_{10}^\parallel(s)$. As mentioned above for the isoscalar pion form factor, with the aim to take into account the contributions from the $\pi\pi - KK$ scattering, it is more reasonable to take the phase of $\pi\pi$ scattering and the once-subtracted Omnès solution. Moreover, the normalizations are different in the isoscalar dipion DAs and the isoscalar pion form factor: $B_{10}^\parallel(0) = -5/9$ and $\Gamma_{\pi}(0)/\hat{m} = 2m_{\pi}^0$. We revisit the work in Ref. [91] with the same inputs (formulas) adopted here and update the asymptotic result from the dipion LCSRs: $\sqrt{q^2} F_t^{(\ell=0)}(4m_{\pi}^2, 0)/m_B = 5.40 \pm 1.00$. As depicted in Fig.5, the result obtained here under the $f_0 + \sigma_0$ model is $\sqrt{q^2} F_t^{(\ell=0)}(4m_{\pi}^2, 0)/m_B = 36.0 \pm 10.0$, which is impossible to reconcile with the asymptotic prediction. We conclude temporarily that the $\sigma-$term should not been included in the resonance model to describe the $S-$wave $B \rightarrow \pi\pi$ transition with the small

Table 2: The fitting result for form factor $F_i (i = +, 0)$ within the $f_0 + \sigma_0$ model.

| $|g_{S \pi^+ \pi^-}|F_j/\pi$ | $\kappa_{F_j}^{f_0}$ | $\eta_{F_j}^{f_0} (\text{GeV})$ | $\rho_{F_j}^{f_0} (\text{GeV})$ | $\gamma_{F_j}^{\sigma_0}$ |
|-----------------|----------------|----------------|----------------|----------------|
| $|g_{S \pi^+ \pi^-}|F_+/\pi$ | 1.01$^{+0.08+0.01}_{-0.10-0.04}$ | $-2.71^{+0.28+0.17}_{-0.24-0.08}$ | $-2.25^{+0.23+1.09}_{-0.11-0.99}$ | 1.00$^{+0.13+0.04}_{-0.25-0.00}$ |
| $|g_{S \pi^+ \pi^-}|F_0/\pi$ | 0.51$^{+0.04+0.01}_{-0.05-0.02}$ | $2.06^{+0.20+0.31}_{-0.25-0.43}$ | $2.91^{+0.42+3.01}_{-0.77-3.79}$ | 1.00$^{+0.13+0.04}_{-0.25-0.00}$ |

\[^{12}\text{In the case of the } P-\text{wave } B \rightarrow \pi\pi \text{ form factor, the expansion coefficient of the asymptotic isovector dipion DAs and the timelike pion form factor have the same normalization: } B_{01}^V(0) = F_{\pi}(0) = 1, \text{ then the results obtained in these two ways are consistent }^{12}\text{[92,93].}\]

Figure 5: $\sqrt{q^2} F_t^{(\ell=0)}(s, q^2)$ obtained within the $f_0 + \sigma_0$ model.
dipion invariant mass.

We then do the similar fit within the $f_0 + f'_0 + f''_0$ resonance model, again the assumptions $\mathcal{F}^f_j(q^2) = \gamma^{f'}_j \mathcal{F}^f_j(q^2)$ and $\mathcal{F}^{f''}_j(q^2) = \gamma^{f''}_j \mathcal{F}^{f''}_j(q^2)$ are adopted to simplify the phase elimination, and the particular relation $\gamma^{S}_{f_0} = \gamma^{S}_{f_+}$ is complemented to each resonance\footnote{In fact, the Eq.\ref{eq:5.5} generalized with multi-resonances implies a strict relation at the full recoil energy point

\begin{equation}
\sum_s 2(m_B^2 - m_S^2)|g_{S\pi\pi}|\mathcal{F}_s^{B\to S}(0) = \sum_s (m_B^2 - m_S^2)|g_{S\pi\pi}|\mathcal{F}_s^{B\to S}(0),
\end{equation}

which is further reduced in terms of the strength parameters $\gamma^S_{f_i}$:

\begin{equation}
\sum_s (m_B^2 - m_S^2)\gamma^S_{f_0} = \sum_s (m_B^2 - m_S^2)\gamma^S_{f_+}.
\end{equation}

In the $f_0 + \sigma$ model, $\gamma^{S}_{f_0} = \gamma^{S}_{f_+}$ is the exactly solution. And in the $f_0 + f'_0 + f''_0$ model discussed following, $\gamma^{S}_{f_0} = \gamma^{S}_{f_+}$ is one set of the particular solution.}. The fitting result within the $f_0 + f'_0 + f''_0$ model are presented in Tab.3. At the bottom of Tab.3 for the comparison we supplement the $B \to f_0$ form factors calculated directly from the LCSR parameters. From the result list in the second column, we can extract the strong coupling $|g_{f_0\pi\pi}| = 1.98 \pm 0.38 \text{ GeV}$ and obtain subsequently $|g_{f_0\pi\pi}| = 2.42 \pm 0.48 \text{ GeV}$, which is consistent with the result $|g_{f_0\pi\pi}| = 1.60 \pm 0.80 \text{ GeV}$ calculated directly from Eq.\ref{eq:5.52} within the uncertainty. Because the dependences on the quark flavor mixing angle cancel between $\kappa^f_{f_0}$ and $\mathcal{F}^{B\to f_0}(0)$, the coupling extracted in this way has the smaller uncertainty.

The contribution from each resonance to the OPE result is list plotted in Fig.\ref{fig:6} from which we can see that $f_0$ plays as the leading role as expected. We also find that the contribution from $f''_0$ is at least one times larger than it from $f'_0$, the underlying physics is that $f'_0$ couples only to the $\rho\rho$ state while $f''_0$ can couples directly to the $\pi\pi$ state. We plot in Fig.\ref{fig:7} for the $S$–wave

| $g_{S\pi\pi}$ | $\mathcal{F}_j$ | $\kappa^f_{f_0}$ (GeV) | $\eta^f_{f_0}$ (GeV) | $\rho^f_{f_0}$ (GeV) | $\gamma^f_{f_0}$ | $\gamma^f_{f_0}$ |
|---|---|---|---|---|---|---|
| $g_{S\pi\pi}$ | $\mathcal{F}_j$ | $1.12^{+0.03+0.00}_{-0.06-0.17}$ | $-2.99^{+0.16+0.39}_{-0.08-0.39}$ | $-2.49^{+0.14+1.36}_{-0.11-0.83}$ | $0.55^{+0.16+0.92}_{-0.00-0.00}$ | $1.06^{+0.15+0.10}_{-0.65-0.12}$ |
| $g_{S\pi\pi}$ | $\mathcal{F}_j$ | $0.50^{+0.01+0.00}_{-0.03-0.09}$ | $1.57^{+0.07+0.17}_{-0.03-0.66}$ | $-4.67^{+2.02+3.06}_{-0.93-3.67}$ | $0.55^{+0.16+0.92}_{-0.00-0.00}$ | $1.06^{+0.15+0.10}_{-0.62-0.15}$ |

\begin{table}[h!]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$\mathcal{F}_+=f_0(0)/\mathcal{X}$ & $\mathcal{F}_0\to f_0(0)/\mathcal{X}$ & \hline
$0.57 \pm 0.06$ & $0.27 \pm 0.03$ & \hline
\end{tabular}
\end{table}
$B \to \pi\pi$ form factors obtained within the $f_0 + f'_0 + f''_0$ model, where for convenience we also show the result obtained from the single $f_0$ fit with the blue curves. In addition to the sharp peak in the $f_0$ region, we find there is another significant enhancement around $f''_0$, while the contribution from $f'_0$ is invisible. The result at the full recoil energy point $\sqrt{q^2} F_i^{(l=0)}(4m_{\pi}^2, 0) = 11.0 \pm 5.0$ is close to the asymptotic result obtained from the LCSRs with dipion DAs, on the basis of this we suggest that the $f_0 + f'_0 + f''_0$ model is the more reasonable one to describe the $S-$wave $B \to \pi\pi$ from factors.

5 Conclusion

In this paper we calculate the $B \to f_0(980)$ form factor as the first time from the light-cone sum rules with $B-$meson DAs, and explore the calculation to study the $S-$wave $B \to \pi\pi$ form factors. To ensure the usefulness of the quark-antiquark assignment for $f_0(980)$, we revisit the 2pSRs with scalar current and predict the mass of $f_0(980)$ with considering the quark flavor mixing. For the $B \to f_0(980)$ form factor, we find that the three-particle $B-$meson DAs gives no more than 20\% correction to the leading twist contribution. In order to investigate the width effect, we suggest two resonance models to parameterize the $S-$wave $B \to \pi\pi$ from factors, and the fitting results show that (i) the $f_0 + \sigma$ model is not apposite because the $\sigma$ resonance gives one times enhancement to the form factor at the full recoil energy point, which pulls the result far away from the asymptotic calculation processed in the LCSRs with dipion DAs, (ii) the $f_0 + f'_0 + f''_0$ model gives the compatible result to the asymptotic calculation, and as a by-product, provides a new way to determine the strong coupling $|g_{f_0\pi\pi}| = 2.42 \pm 0.48$ GeV.

Our predictions still have large uncertainty, mainly comes from the freedom to choose the widths of $f_0$ and $f'_0$ states. Further improvements on this project include mainly the follows: (i) completing the rest corrections from the three-particle DAs of $B-$meson, which part is indispensable to provide the final result at subleading power, (ii) pushing the OPE calculation
Figure 7: $\sqrt{q^2}F_0^{(l=0)}(s,q^2)$ obtained within the $f_0 + f'_0 + f''_0$ model, we remark $\sqrt{q^2}F_0^{(l=0)}(s,0) = \sqrt{q^2}F_1^{(l=0)}(s,0)$ at the full recoil energy point.

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A Coefficients in the three-particle correction

We present here the coefficients appeared in the three-particle correction for $B \rightarrow S$ transition.\footnote{We omit the upper indicator $B \rightarrow S$ for the simplicity, and use the character $m$ to denote the light quark mass of the internal propagator.}

\begin{align*}
C_{1,f_+}^{\Phi_+ \Phi_-} &= \frac{2u - 2}{\sigma m_B^2}, & C_{2,f_+}^{\Phi_+ \Phi_-} &= -\frac{(2u - 2)(m_B^2 - q^2 + m^2) + (2u + 1)\sigma^2 m_B^2 - 3m_B\bar{\sigma}}{\bar{\sigma} m_B^2}, \\
C_{2,f_+}^{\Phi_-} &= -(6u\bar{\sigma} - \frac{6m}{m_B}), & C_{2,f_+}^{\bar{\chi}_A} &= \frac{(2u - 1)\sigma m_B - 2(2u + 1)m}{\bar{\sigma} m_B^2}, \\
C_{3,f_+}^{\bar{\chi}_A} &= \frac{4(2u - 1)m^2\sigma m_B - 4m\bar{\sigma}^2 m_B^2 + 2[(2u + 1)m - (2u - 1)\sigma m_B](\sigma^2 m_B^2 - q^2 + m^2)}{\bar{\sigma} m_B^2}, \\
C_{2,f_+}^{\bar{y}_A} &= \frac{18}{m_B}, & C_{3,f_+}^{\bar{y}_A} &= -\left(\frac{24u m^2}{m_B} + \frac{6m^2}{m_B} - 6m\bar{\sigma} + 8u m\bar{\sigma}\right); \quad (69)
\end{align*}

\begin{align*}
C_{1,f_-}^{\Phi_+ \Phi_-} &= \frac{2u - 2}{\sigma m_B^2}, & C_{2,f_-}^{\Phi_+ \Phi_-} &= -\frac{(2u - 2)(m_B^2 - q^2 + m^2) - (2u + 1)\sigma\bar{\sigma} m_B^2 - 3m_B\sigma}{\bar{\sigma} m_B^2}, \\
C_{2,f_-}^{\Phi_-} &= 6u\sigma + \frac{6m}{m_B}, & C_{2,f_-}^{\bar{\chi}_A} &= -\frac{3(2u - 1)\sigma m_B + 2(2u + 1)m}{\bar{\sigma} m_B^2}, \\
C_{3,f_-}^{\bar{\chi}_A} &= \frac{4(2u - 1)m^2\sigma m_B + 4m\sigma m_B^2 + 2[(2u + 1)m + (2u - 1)\sigma m_B](\sigma^2 m_B^2 - q^2 + m^2)}{\bar{\sigma} m_B^2}, \\
C_{2,f_-}^{\bar{y}_A} &= \frac{18}{m_B}, & C_{3,f_-}^{\bar{y}_A} &= -\left(\frac{24u m^2}{m_B} + \frac{6m^2}{m_B} + 6m\sigma - 8u m\sigma\right), \quad (70)
\end{align*}

\begin{align*}
C_{2,f_{T,p}}^{\Phi_-} &= -\frac{2u - 1}{m_B}, & C_{2,f_{T,p}}^{\Phi_-} &= -\frac{3u}{m_B}, & C_{2,f_{T,p}}^{\bar{\chi}_A} &= \frac{2(2u - 1)}{\bar{\sigma} m_B^2}, \\
C_{3,f_{T,p}}^{\bar{\chi}_A} &= \frac{2(2u - 1)(\sigma^2 m_B^2 - q^2 + m^2) + 4m\sigma m_B}{\bar{\sigma} m_B^2}, & C_{3,f_{T,p}}^{\bar{y}_A} &= -\frac{4(2u - 3)m}{m_B}, \quad (71)
\end{align*}

\begin{align*}
C_{1,f_{T,q}}^{\Phi_-} &= -\frac{2u - 1}{2\bar{\sigma} m_B}, & C_{2,f_{T,q}}^{\Phi_-} &= -\frac{(2u - 1)[\sigma^2 m_B^2 - q^2(1 - 2\sigma) + m^2]}{2\bar{\sigma} m_B}, \\
C_{1,f_{T,q}}^{\Phi_-} &= -\frac{3u}{2\bar{\sigma} m_B}, & C_{2,f_{T,q}}^{\Phi_-} &= -\frac{3u[\sigma^2 m_B^2 - q^2(1 - 2\sigma) + m^2]}{2\bar{\sigma} m_B}, & C_{1,f_{T,q}}^{\bar{\chi}_A} &= \frac{2u - 1}{2\bar{\sigma} m_B^2}, \\
C_{2,f_{T,q}}^{\bar{\chi}_A} &= \frac{(2u - 1)(\sigma^2 m_B^2 - q^2 + m^2) + 2\sigma m m_B - (2u - 1)[\sigma^2 m_B^2 - q^2(1 - 2\sigma) - m^2]}{\bar{\sigma}^2 m_B^2}.
\end{align*}
\[ C_{3, f_T, q} = -\frac{[2u - 1)(\bar{\sigma}^2 m_B^2 - q^2 + m^2) + 2\bar{\sigma} mm_B][\bar{\sigma}^2 m_B^2 - q^2(1 - 2\bar{\sigma}) - m^2]}{\bar{\sigma}^2 m_B^2}, \]
\[ C_{2, f_T, q} = -\frac{2(2u - 3)m}{\bar{\sigma} m_B}, \quad C_{3, f_T, q} = -\frac{2(2u - 3)m[\bar{\sigma}^2 m_B^2 - q^2(1 - 2\bar{\sigma}) - m^2]}{\bar{\sigma} m_B}, \]
\[ (72) \]
\[ C_{1, f_0} = \frac{3(2u - 1)}{2\bar{\sigma} m_B}, \quad C_{2, f_0} = \frac{3(2u - 1)}{2\bar{\sigma} m_B}(\bar{\sigma}^2 m_B^2 - q^2 + m^2) - 3m, \]
\[ C_{1, f_0} = \frac{3u}{\bar{\sigma} m_B}, \quad C_{2, f_0} = -\frac{3u}{\bar{\sigma} m_B}(\bar{\sigma}^2 m_B^2 - q^2 + m^2) - 6m, \]
\[ C_{1, f_0} = \frac{2u + 1}{\bar{\sigma}^2 m_B^2}, \quad C_{2, f_0} = (2u + 7) - \frac{4um}{\bar{\sigma} m_B} - \frac{2(2u + 1)(\bar{\sigma}^2 m_B^2 - q^2 + m^2)}{\bar{\sigma}^2 m_B^2}, \]
\[ C_{3, f_0} = -4(2u - 1)m^2 - \frac{4um}{\bar{\sigma} m_B}(\bar{\sigma}^2 m_B^2 - q^2 + m^2) + \frac{2u + 1}{\bar{\sigma}^2 m_B^2}(\bar{\sigma}^2 m_B^2 - q^2 + m^2)^2, \]
\[ C_{1, f_0} = \frac{2m(3 - 2u)}{\bar{\sigma} m_B}, \quad C_{2, f_0} = 12(1 - 2u)m^2 - \frac{2(3 - 2u)m}{\bar{\sigma} m_B}(\bar{\sigma}^2 m_B^2 - q^2 + m^2). \]
\[ (73) \]

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