Critical Measures on Higher Genus Riemann Surfaces

Marco Bertola\textsuperscript{1,2}, Alan Groot\textsuperscript{3}, Arno B. J. Kuijlaars\textsuperscript{3}

\textsuperscript{1} Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve W., Montréal, Québec H3G 1M8, Canada. E-mail: marco.bertola@concordia.ca; Marco.Bertola@sissa.it
\textsuperscript{2} SISSA, International School for Advanced Studies, via Bonomea 265, Trieste, Italy
\textsuperscript{3} Department of Mathematics, Katholieke Universiteit Leuven, Leuven, Belgium.
E-mail: alan.groot@kuleuven.be, alangroot@gmail.com; arno.kuijlaars@kuleuven.be

Received: 15 July 2022 / Accepted: 11 August 2023
Published online: 14 September 2023 – © The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2023

Abstract: Critical measures in the complex plane are saddle points for the logarithmic energy with external field. Their local and global structure was described by Martínez-Finkelshtein and Rakhmanov. In this paper we start the development of a theory of critical measures on higher genus Riemann surfaces, where the logarithmic energy is replaced by the energy with respect to a bipolar Green’s kernel. We study a max-min problem for the bipolar Green’s energy with external fields $\text{Re} V$ where $dV$ is a meromorphic differential. Under reasonable assumptions the max-min problem has a solution and we show that the corresponding equilibrium measure is a critical measure in the external field. In a special genus one situation we are able to show that the critical measure is supported on maximal trajectories of a meromorphic quadratic differential. We are motivated by applications to random lozenge tilings of a hexagon with periodic weightings. Correlations in these models are expressible in terms of matrix valued orthogonal polynomials. The matrix orthogonality is interpreted as (partial) scalar orthogonality on a Riemann surface. The theory of critical measures will be useful for the asymptotic analysis of a corresponding Riemann–Hilbert problem as we outline in the paper.

1. Introduction

The notion of critical measures in the complex plane was developed by Martínez-Finkelshtein and Rakhmanov \cite{45,46,50} with the aim of studying asymptotic zero distributions of Heine-Stieltjes polynomials. It is related to the asymptotics of orthogonal polynomials with a non-hermitian orthogonality on contours in the complex plane as initiated by Stahl \cite{55,56} and Gonchar and Rakhmanov \cite{33}. The goal of this paper is to generalize the notion of critical measures to higher genus Riemann surfaces. Our motivation to do so comes from the analysis of certain tiling problems with periodic weightings as we will explain in Sect. 4 of this paper.
1.1. Critical measure in the complex plane. We start by recalling the notion of critical measure in the plane following [45]. It can be defined in a more general situation, but we restrict here to the case of an external field $\Re V$ where $V'$ is a rational function on $\mathbb{C}$. In fact, $V$ itself can be multi-valued, but $\Re V$ is assumed to be well-defined and single-valued on $\mathbb{C}$. The logarithmic energy of a probability measure $\mu$ in the external field $\varphi = \Re V$ is

$$E_\varphi [\mu] = \iint \log \frac{1}{|s-t|} d\mu(s)d\mu(t) + \int \varphi d\mu. \quad (1.1)$$

A critical measure is a probability measure $\mu$ such that (1.1) is stationary with respect to certain perturbations of $\mu$, known as Schiffer variations. For a continuous function $h : \mathbb{C} \to \mathbb{C}$, and a probability measure $\mu$, the one parameter family $(\mu_\varepsilon, h)_{\varepsilon \in \mathbb{R}}$ of probability measures is defined through their action on continuous functions $f$,

$$\int f d\mu_{\varepsilon, h} = \int f (s + \varepsilon h(s)) d\mu(s), \quad \varepsilon \in \mathbb{R}. \quad (1.2)$$

**Definition 1.1.** The probability measure $\mu$ is a critical measure in the external field $\varphi$ if

$$\lim_{\varepsilon \to 0} \frac{E_{\varphi} [\mu_{\varepsilon, h}] - E_{\varphi} [\mu]}{\varepsilon} = 0 \quad (1.3)$$

for every $C^1$ function $h$ with compact support.

The definition is a special case of [45, Definition 3.2]. By [45, Lemma 3.1], the limit (1.3) exists and is equal to

$$- \Re \left[ \iint \frac{h(s) - h(t)}{s-t} d\mu(s)d\mu(t) - \int h(s)V'(s)d\mu(s) \right].$$

By considering both $h$ and $ih$, it follows that $\mu$ is a critical measure if and only if

$$\iint \frac{h(s) - h(t)}{s-t} d\mu(s)d\mu(t) = \int h(s)V'(s)d\mu(s) \quad (1.4)$$

for every $C^1$ function $h$ with compact support. With a limiting argument (1.4) can be extended to $C^1$ functions $h$ that are bounded on the support of $\mu$.

The main result on critical measures is that they are supported on maximal trajectories of quadratic differentials. Recall that a trajectory of the quadratic differential $-Qdz^2$ is an (open or closed) contour $\Sigma$ such that $Q(z(s))(z'(s))^2 < 0$ where $s \mapsto z(s)$ is any smooth parametrization of $\Sigma$, see e.g. [57].

**Theorem 1.2** (Martínez-Finkelshtein and Rakhmanov [45]). Suppose $\varphi = \Re V$ where $V'$ is rational on $\mathbb{C}$. For a critical measure $\mu$ in the external field $\varphi$, we let

$$Q(z) = \left( \frac{V'(z)}{2} \right)^2 - \int \frac{V'(z) - V'(s)}{z-s} d\mu(s). \quad (1.5)$$

Then the following hold.
(a) $Q$ is a rational function with the property that

$$
\left[ \int \frac{d\mu(s)}{z-s} - \frac{V'(z)}{2} \right]^2 = Q(z), \quad m_2\text{-a.e.,}
$$

where $m_2$ denotes the planar Lebesgue measure.

(b) The support $\Sigma = \text{supp}(\mu)$ consists of a finite union of maximal trajectories of the quadratic differential $-Q(s)ds^2$, and on each trajectory we have (with $ds$ the complex line element and an appropriate branch of the square root)

$$
d\mu(s) = \frac{1}{\pi i} Q(s)^{1/2} ds, \quad s \in \Sigma.
$$

(c) The logarithmic potential $U^\mu(z) = \int \log \frac{1}{|z-s|} d\mu(s)$ of $\mu$ satisfies

$$
2U^\mu(z) + \text{Re } V(z) = c_j, \quad z \in \Sigma_j,
$$

with a constant $c_j$ that can be different for each connected component $\Sigma_j$ of $\Sigma$.

(d) Any point $z \in \Sigma$ that is not a zero of $Q$ has a neighborhood $D$ such that $D \cap \Sigma$ is an analytic arc, and

$$
\frac{\partial}{\partial n_+} (2U^\mu + \text{Re } V)(z) = \frac{\partial}{\partial n_-} (2U^\mu + \text{Re } V)(z), \quad z \in D \cap \Sigma,
$$

where $\frac{\partial}{\partial n_{\pm}}$ denote the two normal derivatives to $\Sigma$ at $z$.

**Proof.** See Theorem 5.1, Lemma 5.3 and Lemma 5.4 in [45].

The identity (1.8) is known as the $S$-property in the external field $\text{Re } V$ and $\Sigma$ is called an $S$-contour or $S$-curve in the external field $\text{Re } V$. Together with (1.7), it implies that the $g$-function

$$
g(z) = \int \log(z-s)d\mu(s)
$$

satisfies

$$
g_+(z) + g_-(z) - V(z) = \ell_j \quad \text{on } \Sigma_j
$$

for a complex constant $\ell_j$ that can be different for each connected component $\Sigma_j$ of $\Sigma$. Indeed, the real part of $g_+ + g_- - V$ is constant on each component by (1.7), while (1.8) implies that the imaginary part is constant on each component as well, as can be seen from an application of the Cauchy–Riemann equations.
1.2. Equilibrium measures. For a compact set $F \subset \mathbb{C}$ the equilibrium measure in the external field $\varphi = \text{Re } V$ is the probability measure on $F$ that minimizes the functional (1.1) among probability measures on $F$, see e.g. [23,51,52]. If $F$ is a continuum (i.e., a compact connected set with more than one point), or a union of continua, and $\text{Re } V$ is bounded from below on $F$, then there is a unique equilibrium measure $\mu^F$ in the external field $\text{Re } V$. It satisfies
\[ 2U^{\mu^F}(z) + \text{Re } V(z) = c, \quad \text{on } \text{supp}(\mu^F), \quad (1.10) \]
\[ 2U^{\mu^F}(z) + \text{Re } V(z) \geq c, \quad \text{on } F, \quad (1.11) \]
for some constant $c$. We emphasize that $c$ is the same on all connected components of $\text{supp}(\mu^F)$, which is in contrast to the situation in Theorem 1.2 (c). For a critical measure the constant can be different on each connected component of its support. We are interested in equilibrium measures that are also critical measures, and, given $V$, this will depend on the choice of a good compact set $F$.

To determine such $F$, Kuijlaars and Silva [44] introduced the notion of a critical set, based on the extremal compact considered by Rakhmanov in [50]. To describe it, we use
\[ E_\varphi(F) = E_\varphi[\mu^F]. \quad (1.12) \]

**Definition 1.3.** Let $F$ be a compact set such that $\varphi = \text{Re } V$ is bounded from below on $F$. Then $F$ is a critical set in the external field $\varphi$ if
\[ \lim_{\varepsilon \to 0} \frac{E_\varphi(F_{\varepsilon,h}) - E_\varphi(F)}{\varepsilon} = 0 \quad (1.13) \]
for every $C^1$ function $h$ with compact support, where
\[ F_{\varepsilon,h} = \{ x + \varepsilon h(x) \mid x \in F \}. \quad (1.14) \]

Note that (1.14) is the support of the deformed measure $\mu_{\varepsilon,h}$ if $\text{supp}(\mu) = F$.

Rakhmanov [50] essentially proved the following, see also [44].

**Proposition 1.4.** Let $F$ be a union of continua such that $\varphi = \text{Re } V$ is bounded from below on $F$. Let $\mu^F$ be its equilibrium measure in the external field $\varphi$.

(a) Then the limit in (1.13) exists and it is equal to the limit in (1.3), i.e.,
\[ \lim_{\varepsilon \to 0} \frac{E_\varphi(F_{\varepsilon,h}) - E_\varphi(F)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{E_\varphi[\mu^F_{\varepsilon,h}] - E_\varphi[\mu^F]}{\varepsilon}. \]

(b) $F$ is a critical set if and only if $\mu^F$ is a critical measure in the external field.

(c) Suppose $F$ belongs to a family $\mathcal{F}$ of union of continua such that for every $C^1$ function $h$ there is $\varepsilon_0 > 0$ such that $F_{\varepsilon,h} \in \mathcal{F}$ for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, where $F_{\varepsilon,h}$ is given by (1.14). Suppose also that
\[ E_\varphi(F) = \max_{F' \in \mathcal{F}} E_\varphi(F'). \quad (1.15) \]

Then $F$ is a critical set and $\mu^F$ is a critical measure in the external field $\varphi$. 

---
Proof. The proof of part (a) follows from the property that
\[ E_\varphi(F_{\varepsilon,h}) - E_\varphi[\mu^F_{\varepsilon,h}] = o(\varepsilon) \quad \text{as} \ \varepsilon \to 0, \]  
(1.16)
see [44, Section 4] or [50, Section 9.10] for details, and see also Proposition 3.8 below where the analogous statement is proved in the higher genus case. Part (b) is immediate from part (a) and the definitions of critical measure and critical set. If \( F \) satisfies the conditions of part (c), then \( E_\varphi(F_{\varepsilon,h}) \) has a local maximum at \( \varepsilon = 0 \) for every \( C^1 \) function \( h \). Hence (1.13) holds and \( F \) is a critical set. Then also \( \mu^F \) is a critical measure by part (b).

Part (c) shows that one may find critical sets from solving a max-min energy problem
\[ \max_{F \in \mathcal{F}} \min_{\text{supp}(\mu) \subset F} E_\varphi[\mu] \]
where the minimum is over probability measures on \( F \) and the maximum is over a suitable family \( \mathcal{F} \) satisfying the condition of part (c) of Proposition 1.4. If a maximizer \( F \) exists then clearly (1.15) holds and \( \mu^F \) is a critical measure in the external field \( \varphi \). Then Theorem 1.2 applies and it follows in particular that the support of \( \mu^F \) is a union of maximal trajectories of a quadratic differential.

It can be shown that under suitable conditions, a maximizer indeed exists. This program was first carried out by Kamvissis and Rakhmanov [39] in the case of weighted Green’s energy in the upper half plane, and later by Rakhmanov [50] in the case of weighted logarithmic energy in the complex plane; see also the work of Kuijlaars and Silva [44] for the case of a polynomial \( V \). Our extension to higher genus (see Theorem 3.11 below and its proof in Sect. 6) follows this approach.

There is a substantial literature on determining critical equilibrium measures in the external field \( \Re V \) where \( V \) is a polynomial. This is motivated in part by questions in random matrix theory, see the recent papers [2,11,12] and references cited therein. See also [47] for an extension to vector equilibrium measures.

2. Potential Theory on Riemann Surfaces

Before we can state the results of this paper we need to introduce certain notions from potential theory on a higher genus Riemann surface, see also Skinner [54] and a recent series of papers by Chirka [20–22]. Our exposition will focus on equilibrium measures in an external field.

Throughout, we use \( X \) to denote a compact Riemann surface with a distinguished point \( p_\infty \), which we refer to as the point at infinity.

2.1. Bipolar Green’s function. To extend Theorem 1.2 we first of all need the appropriate analogue of the logarithmic kernel that appears in (1.1) to define the logarithmic energy. This is provided by the bipolar Green’s function.

Proposition 2.1. Let \( X \) be a compact Riemann surface with a distinguished point \( p_\infty \in X \). There is a function \((p,q) \mapsto G(p,q)\) defined on \( X \times X \) such that
(a) for a fixed \( q \in X \setminus \{p_\infty\} \) the function \( p \mapsto G(p,q) \) is real-valued and harmonic on \( X \setminus \{p_\infty, q\} \),
(b) if \( z \) is a local coordinate at \( q \), then
\[
G(p, q) = -\log |z(p)| + O(1) \quad \text{as} \quad p \to q, \tag{2.1}
\]
(c) if \( z_{\infty} \) is a local coordinate at \( p_{\infty} \), then
\[
G(p, q) = \log |z_{\infty}(p)| + O(1) \quad \text{as} \quad p \to p_{\infty}, \tag{2.2}
\]
(d) \( G(p, q) = G(q, p) \).

As usual, a local coordinate at a point on a Riemann surface means a holomorphic chart in which the point corresponds to \( 0 \in \mathbb{C} \).

**Proof.** The existence of \( G \) satisfying parts (a), (b) and (c) of Proposition 2.1 can be found in Gamelin [32], Simon [53, Section 3.8]. We could not find an appropriate reference for part (d), although it may be known. The parts (a), (b) and (c) determine \( p \mapsto G(p, q) \) up to an additive constant that may depend on \( q \). It is noted by Skinner [54, p. 25] that it is not immediately obvious how to choose the constant such that \( G \) is symmetric. We give a proof in the appendix. \( \square \)

**Definition 2.2.** The function \( G \) satisfying the conditions of Proposition 2.1 is called the bipolar Green’s function with one pole at \( p_{\infty} \), or simply bipolar Green’s function.

The bipolar Green’s function is unique up to an additive constant, but the constant will not be important for us. Note that \( G(p, q) \to -\infty \) as \( p \to p_{\infty} \) and \( G(p, q) \to +\infty \) as \( p \to q \).

**Example 2.3.** (a) If \( X \) is the Riemann sphere with \( p_{\infty} \) the point at infinity, then \( G(p, q) = \log \frac{1}{|p-q|} \).

(b) On a complex torus \( X = \mathbb{C}/\Lambda \) with lattice \( \Lambda = \mathbb{Z} + \tau \mathbb{Z} \) and \( \Im \tau > 0 \), the bipolar Green’s function with pole at \( p_{\infty} = 0 \) (modulo \( \Lambda \)) is explicitly given by, see e.g. [20, 40, Section 2], [54],
\[
G(p, q) = \log \left| \frac{\theta_1(p)\theta_1(q)}{\theta_1(p-q)} \right| - \frac{2\pi}{\Im \tau} \left( \Im p \right) \left( \Im q \right) \tag{2.3}
\]
in terms of the Jacobi elliptic function \( \theta_1 \) that has a zero at 0 (see [48, Chapter 20] which however uses a different scaling of elliptic functions with periods \( \pi \) and \( \pi \tau \) instead of 1 and \( \tau \)), i.e.,
\[
\theta_1(z) = \theta_1(z; \tau) = -i \sum_{k=-\infty}^{\infty} (-1)^k e^{\frac{\pi i (k+\frac{1}{2})^2}{\tau} + (2k+1)\pi i z}. \tag{2.4}
\]
The following properties of \( \theta_1 \) can be found in [48, Chapter 20]. The Jacobi elliptic function is an odd entire function with simple zeros at every lattice point, and no other zeros. Moreover, it has the quasi-periodicity properties
\[
\theta_1(z + 1) = -\theta_1(z), \quad \theta_1(z + \tau) = -e^{-\pi i \tau - 2\pi i z} \theta_1(z). \tag{2.5}
\]

We can use (2.3) to construct the bipolar Green’s function on an arbitrary genus one Riemann surface \( X \), by composing it with the Abel map from \( X \) to \( \mathbb{C}/\Lambda \) that maps \( p_{\infty} \) to 0.
The local behavior of \(G(p, q)\) is determined by the logarithmic kernel \(\log \frac{1}{|p - q|}\) in the following sense.

**Lemma 2.4.** We have

(a) If \(z_\infty\) is a local coordinate at \(p_\infty\) (that maps \(p_\infty\) to 0) then we have

\[
G(p, q) - \log \frac{1}{|z_\infty(p) - 1 - z_\infty(q)|} = O(1)
\]

uniformly for \(p, q\) in a neighborhood of \(p_\infty\).

(b) If \(z_0\) is a local coordinate at \(p_0 \neq p_\infty\) (that maps \(p_0\) to 0) then we have

\[
G(p, q) - \log \frac{1}{|z_0(p) - z_0(q)|} = O(1)
\]

uniformly for \(p, q\) in a neighborhood of \(p_0\) that does not contain \(p_\infty\).

**Proof.** The two terms in the left hand sides of (2.6) and (2.7) have the same logarithmic singularities when \(p = q\). The two terms in (2.6) also have the same logarithmic singularity when \(p = p_\infty\) and by symmetry of \(G\) also when \(q = p_\infty\). Therefore the differences in both (2.6) and (2.7) are harmonic in both variables near \(p_\infty\) and \(p_0\), respectively, and therefore they remain bounded. \(\square\)

### 2.2. Equilibrium measure in external field.

The bipolar Green’s function allows us to extend concepts of logarithmic potential theory in the complex plane \([23,51,52]\) to compact Riemann surfaces.

**Definition 2.5.** Let \(\mu\) be a measure on \(X\) with compact support in \(X \setminus \{p_\infty\}\). Then we define its **bipolar Green’s energy**

\[
E[\mu] = \iint G(p, q)d\mu(p)d\mu(q).
\]  

(2.8)

For a lower semi-continuous \(\varphi: \text{supp}(\mu) \to \mathbb{R} \cup \{+\infty\}\) we define the bipolar Green’s energy of \(\mu\) in the external field \(\varphi\) by

\[
E_{\varphi}[\mu] = \iint G(p, q)d\mu(p)d\mu(q) + \int \varphi d\mu.
\]

(2.9)

We extend the definition (2.8) to signed measures \(\nu = \mu_1 - \mu_2\), provided \(E[\mu_1]\) and \(E[\mu_2]\) are finite. We need the crucial property

**Proposition 2.6.** If \(\mu_1, \mu_2\) are two compactly supported probability measures on \(X \setminus \{p_\infty\}\) with finite bipolar Green’s energy and \(\nu = \mu_1 - \mu_2\), then

\[
E[\nu] = \iint G(p, q)d\nu(p)d\nu(q) \geq 0,
\]

(2.10)

and \(E[\nu] = 0\) if and only if \(\mu_1 = \mu_2\).

**Proof.** A number of proofs are known in the literature. One proof involves the eigenfunctions of the Laplace operator as in [21] and [54]. Another proof relates (2.10) to the norm of certain functions in a Sobolev space; see [21] and [40]. \(\square\)
For a compact $K \subset X \setminus \{p_\infty\}$, let $\mathcal{E}^1(K)$ denote the set of all probability measures $\mu$ on $K$ with $E[\mu] < +\infty$. If $\mu \in \mathcal{E}^1(K)$ and $(\mu_n)_n$ is a sequence in $\mathcal{E}^1(K)$ such that
\[
\lim_{n \to \infty} E[\mu_n - \mu] = 0,
\]then we say that $(\mu_n)_n$ converges to $\mu$ in energy norm. Convergence in energy norm implies weak* convergence (see the second bullet point on [21, p. 306], or [37, Theorem 7.3.10] for the case of potential theory in $\mathbb{R}^n$), i.e.,
\[
\lim_{n \to \infty} \int f \, d\mu_n = \int f \, d\mu
\]
for every continuous $f$ on $K$.

For a compact set $F \subset X \setminus \{p_\infty\}$ and a lower semi-continuous $\varphi : F \to \mathbb{R} \cup \{+\infty\}$, we denote
\[
E_\varphi(F) = \inf_{\mu \in \mathcal{E}^1(F)} E_\varphi[\mu].
\]

**Definition 2.7.** Let $F \subset X \setminus \{p_\infty\}$ be compact and let $\varphi : F \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function. Suppose $E_\varphi(F) < +\infty$. Then a probability measure $\mu$ on $F$ that satisfies $E_\varphi(F) = E_\varphi[\mu]$ is called an equilibrium measure of $F$ in the external field $\varphi$, or simply an equilibrium measure. It will be denoted by $\mu^F$.

There is only one equilibrium measure, so that we speak of the equilibrium measure of $F$ in the external field $\varphi$, as is part of the following proposition.

**Proposition 2.8.** Suppose $F \subset X \setminus \{p_\infty\}$ is a compact set, and $\varphi : F \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous with $E_\varphi(F) < +\infty$. Then there is a unique equilibrium measure $\mu^F$ on $F$ in the external field $\varphi$. The equilibrium measure is the unique probability measure $\mu$ on $F$ such that for some constant $c$,
\[
2 \int G(p, q) d\mu(q) + \varphi(p) = c, \quad \text{q.e. on } \text{supp}(\mu),
\]
\[
2 \int G(p, q) d\mu(q) + \varphi(p) \geq c, \quad \text{q.e. on } F.
\]

Here q.e. means quasi-everywhere, i.e., except for a set of zero capacity.

**Proof.** With the aid of Proposition 2.6, the proof is similar to the proof for equilibrium measures in logarithmic potential theory given in [52, Theorem I.1.3], see also [21, Section 2.3] or [54, Lemma 3.4.5].

For the variational conditions (2.13) and (2.14) it is important that $G$ is symmetric, see Proposition 2.1 (d), since otherwise $2 \int G(p, q) d\mu(q)$ in (2.13) and (2.14) should be replaced by $\int \left( G(p, q) + G(q, p) \right) d\mu(q)$.

In the present work we will have that $F$ is a connected compact set with more than one point (i.e., a continuum), or a disjoint union of such sets, and in such a situation there are no exceptional sets of zero capacity, and (2.13) and (2.14) hold everywhere on their respective sets.
2.3. \textit{S-property}. For a compactly supported measure $\mu$ with $\text{supp}(\mu) \subset X \setminus \{p_\infty\}$, note that $p \mapsto -\int G(p, q) \mu(q)$ is a harmonic function on $X \setminus (\{p_\infty\} \cup \text{supp}(\mu))$. It is real-valued, but being harmonic it is locally the real part of a holomorphic function that we denote by $g$ and we call it the $g$-function. Depending on the situation, we may need to define certain branch cuts in order to make $g$ single-valued. Its real part

$$\text{Re} \ g(p) = -\int G(p, q) \mu(q)$$

is always single-valued, however.

In case $F = \gamma$ is a contour and $\varphi = \text{Re} \ V$ for a multi-valued, locally meromorphic function $V$ on $X$ with single-valued real part, the variational conditions (2.13)–(2.14) can also be formulated as

$$\text{Re} \ (g_+(p) + g_-(p) - V(p)) = c, \quad p \in \text{supp}(\mu),$$

$$\text{Re} \ (g_+(p) + g_-(p) - V(p)) \leq c, \quad p \in \gamma. \quad (2.15)$$

The contour will be oriented, which defines the + and − sides; the notation $g_+$ and $g_-$ refers to limiting values of $g$ on the + and − sides of $\gamma$.

\textbf{Definition 2.9.} A contour $\gamma$ has the \textbf{S-property in external field} $\text{Re} \ V$ if the $g$-function of its equilibrium measure $\mu$ in the external field $\text{Re} \ V$ has the following property: not only the real part of $g_+ + g_- - V$ is constant on $\text{supp}(\mu)$, see (2.15), but also the imaginary part of $g_+ + g_- - V$ is piecewise constant on each connected component of $\text{supp}(\mu)$.

In case $\gamma$ has more than one connected component, then the imaginary part of the constant can be different on the different components. The imaginary part can also be different due to a different choice of branches of $g$ in case a branch cut intersects $\gamma$.

3. \textbf{Statement of Results}

3.1. \textit{Cauchy kernel}. We would like to have the analogue of Theorem 1.2 (a) which in particular states that for a critical measure

$$\left[\int C(p, q)d\mu(q) - \frac{dV(p)}{2}\right]^2$$

is a meromorphic quadratic differential on $X$ where $C(p, q)$ is an appropriate Cauchy kernel defined below in terms of the bipolar Green’s function.

\textbf{Definition 3.1.} The \textbf{Cauchy kernel} $C(p, q)$ is given in terms of the bipolar Green’s function with pole at $p_\infty$, see Definition 2.2, by

$$C(p, q) = -2\partial_p G(p, q)dp. \quad (3.2)$$

To explain the notation, let us assume here that $p$ denotes at the same time the point and a complex local coordinate, with $p = x + iy$. Then the expression $\partial_p$ in (3.2) stands for the complex Cauchy–Riemann operator $\partial_p = \frac{1}{2}(\partial_x - i \partial_y)$, and $dp = dx + i \, dy$.

Let us further clarify by reminding the reader that if $f(x, y)$ is a harmonic real-valued function over some domain, $f_{xx} + f_{yy} \equiv 0$, then the expression $2\partial_p f := f_x - i \, f_y$ is immediately verified to be holomorphic over the same domain, i.e., it satisfies the
Cauchy–Riemann equations. Thus the expression \(2\partial_p f(p)\, dp\) defines a locally holomorphic differential and is independent of the choice of local coordinate used to compute it. Getting then back to our expression (3.2), since \(p \mapsto G(p, q)\) is a real-valued harmonic function over the domain \(p \in X \setminus \{q, p_\infty\}\), we deduce from this discussion that \(C(p, q)\) is a holomorphic differential with respect to the \(p\)-variable in \(X \setminus \{q, p_\infty\}\), while it remains harmonically dependent on the \(q\)-variable for any fixed \(q \in X \setminus \{p_\infty\}\). With respect to the variable \(p\) we can see that near \(q, p_\infty\) the differential has a simple pole (with residues \(-1, 1\), respectively): this follows from the fact that the singular part of \(G(p, q)\) described by eqs. (2.1) and (2.2) is the real part of the logarithm of a local coordinate. We summarize these properties as follows:

- \(C(\cdot, q)\) is a meromorphic differential on \(X\), with simple poles at \(q\) and at \(p_\infty\), and holomorphic otherwise, the pole at \(q\) has residue 1 and the pole at \(p_\infty\) has residue \(-1\) (these residues come from the behavior (2.1), (2.2) of the bipolar Green’s function)
- \(C(p, \cdot)\) is a harmonic function on \(X \setminus \{p, p_\infty\}\).

We also note that, since \(G(p, q)\) is harmonic and single-valued,

- \(C(\cdot, q)\) has purely imaginary periods, namely, for any closed path \(\gamma \in X \setminus \{q, p_\infty\}\) the integral \(\int_{\gamma} C(p, q)\) is purely imaginary.

The latter property follows from the fact that, by the Definition 3.1, \(G(p, q) = \text{Re} \int_{p}^{q} C(\cdot, q)\) (up to an additive constant), and the harmonic continuation of \(G(p, q)\) along any closed path \(\gamma\) in the variable \(p\) must yield the same value. The above three properties actually characterize uniquely the Cauchy kernel.

Getting now back to the expression (3.1), we conclude that for any measure \(\mu\) with compact support, the expression is a quadratic differential that is meromorphic on \(X \setminus \text{supp}(\mu)\) with poles at \(p_\infty\) and at the poles of \(dV\).

**Example 3.2.** (a) On the Riemann sphere we have

\[
C(p, q) = -2\partial_p \left( \log \frac{1}{|p-q|} \right) \, dp = \frac{dp}{p-q},
\]

which is the usual Cauchy kernel from complex analysis.

(b) On a complex torus \(X = \mathbb{C}/\Lambda\) as in Example 2.3 (b) we have

\[
C(p, q) = -2\partial_p \left( \log \left| \frac{\theta_1(p)\theta_1(q)}{\theta_1(p-q)} \right| - \frac{2\pi}{\text{Im}\, \tau} \left( \text{Im}\, p \right) \left( \text{Im}\, q \right) \right) \, dp
= - \left( \frac{\theta'_1(p)}{\theta_1(p)} - \frac{\theta'_1(p-q)}{\theta_1(p-q)} + \frac{2\pi i}{\text{Im}\, \tau} \text{Im}\, q \right) \, dp.
\]

The formula shows that \(q \mapsto C(p, q)\) is not meromorphic in the genus one case, but it is harmonic.

The notion of Cauchy kernel we use here could be further qualified as *imaginary normalized* and it is only harmonic with respect to \(q \neq p\). More commonly the term is used to refer to a kernel which is *meromorphic* in \(q\), with an additional pole at a prescribed collection of \(g\) points (where \(g\) is the genus of the Riemann surface). Possibly the first occurrence is in [3] but the notion appears ubiquitous in the literature on Riemann surfaces, notably in [30, 36, 58]. A related notion is used also later in this paper, see Sect. 3.5.
3.2. Critical measures. In order to define the notion of a critical measure, we need an analogue of the Schiffer variation formula (1.2). Instead of considering a function $h$, as we did in the genus zero case, we identify $h$ as a vector field on $X$, i.e., a section of the tangent bundle in the higher genus case. A vector field $h$ induces a flow $\Phi(t, p)$ on $X$ (for $t \in \mathbb{R}$ and $p \in X$, we have $\Phi(t, p) \in X$) satisfying $\frac{d\Phi}{dt} = h$ and $\Phi(0, p) = p$. Then, given a measure $\mu$ on $X$ we define $\mu_{\varepsilon, h}$ by its action on continuous functions $f$

$$\int f \, d\mu_{\varepsilon, h} = \int f(\Phi(\varepsilon, p)) \, d\mu(p), \quad \varepsilon \in \mathbb{R},$$

which is the analogue of (1.2). Thus $\mu_{\varepsilon, h}$ is the image of $\mu$ along the flow induced by $h$, and we may alternatively write

$$\mu_{\varepsilon, h} = \Phi(\varepsilon, \mu).$$

**Definition 3.3.** Let $\mu$ be a measure on $X$ with support in $X \setminus \{p_\infty\}$ and suppose that the external field $\varphi$ is a real-valued $C^1$ function on a neighborhood of $\text{supp}(\mu)$. Then $\mu$ is a critical measure in the external field $\varphi$ if

$$\lim_{\varepsilon \to 0} \frac{E_{\varphi}[\mu_{\varepsilon, h}] - E_{\varphi}[\mu]}{\varepsilon} = 0$$

(3.5)

for every $C^1$ vector field $h$.

Similar to [45, Lemma 3.1], there is a convenient identity for the limit in (3.5) in case $\varphi$ is given by the real part of $V$.

**Proposition 3.4.** Suppose that the external field $\varphi$ is given by $\varphi = \text{Re} \, V$ and suppose that $\mu$ is a measure with support in $X \setminus \{p_\infty\}$ such that $E[\mu] < \infty$ and $\varphi$ is bounded on $\text{supp}(\mu)$. Then the following holds.

(a) For every $C^1$ vector field $h$, the limit in (3.5) exists with

$$\lim_{\varepsilon \to 0} \frac{E_{\varphi}[\mu_{\varepsilon, h}] - E_{\varphi}[\mu]}{\varepsilon} = \text{Re} \, D_{V, h}(\mu),$$

(3.6)

where $D_{V, h}(\mu)$ is given by

$$D_{V, h}(\mu) = -\iint (h(p)C(p, q) + h(q)C(q, p)) \, d\mu(p) \, d\mu(q) + \int h dV \, d\mu$$

(3.7)

and $C(p, q)$ is the Cauchy kernel from Definition 3.1, see Remark 3.5.

(b) A critical measure $\mu$ satisfies $D_{V, h}(\mu) = 0$ for every $C^1$ vector field $h$.

The proof is given in Sect. 5.

**Remark 3.5.** Let us comment on the expression (3.7). The vector field $h$ acts on the meromorphic differential $dV$ in a natural way: the expression $hdV$ defines a $C^1$ function on $X$ away from the poles of $dV$. The function can be integrated against a measure and this is how the term $\int h dV \, d\mu$ should be interpreted in (3.7). It is the analogue of the right-hand side of (1.4). Also, the fact that the external field is harmonic means that we only need to consider vector fields in the holomorphic tangent bundle. See Remark 3.6 for more details.
The analogue of the left-hand side of (1.4) is the double integral with the Cauchy kernel \( C(p, q) \). Since \( C(p, q) \) is a differential in \( p \), the product \( h(p)C(p, q) \) is a function in both variables \( p \) and \( q \) that becomes infinite when \( p = q \). The same holds true for \( h(q)C(q, p) \), but the infinities for \( p = q \) disappear in the sum \( h(p)C(p, q) + h(q)C(q, p) \) if \( h \) is a \( C^1 \) vector field. The sum is a well-defined bounded and continuous function on \((X \setminus \{p_\infty\}) \times (X \setminus \{p_\infty\})\) that plays the role of the divided difference \( \frac{h(s) - h(t)}{s - t} \) in (1.4).

Remark 3.6. The energy functionals we are considering involve harmonic functions \( \varphi = \text{Re } V \) for their external field, and the bipolar Green’s function is also a harmonic function. This has the following effect: when we consider the Schiffer variations, it is sufficient for \( h(q)C(q, p) \), but the infinities for \( p = q \) disappear in the sum \( h(p)C(p, q) + h(q)C(q, p) \) if \( h \) is a \( C^1 \) vector field. The sum is a well-defined bounded and continuous function on \((X \setminus \{p_\infty\}) \times (X \setminus \{p_\infty\})\) that plays the role of the divided difference \( \frac{h(s) - h(t)}{s - t} \) in (1.4).

Definition 3.7. A real-valued harmonic function \( H(x, y) \) can be written locally as the real part of a (meromorphic) function \( f \) as \( H(x, y) = \text{Re } f(z) \). Acting with \( \nabla \) on such a function yields

\[
\nabla H(x, y) = \frac{1}{2} \left( (a + ib) \frac{\partial f(z)}{\partial \bar{z}} + (a - ib) \frac{\partial f(z)}{\partial z} \right) = \text{Re} \left( \nabla^{1,0} f \right).
\]

Having thus established that we need only to consider vectors in \( T^{1,0} \), we also now point out that given any (meromorphic) differential, written in local coordinate as \( \omega = f(z)dz \), the expression \( \frac{1}{\omega} \) can be interpreted as a (meromorphic) vector field in \( T^{1,0} \). This is easily verified by observing that it transforms as a vector in \( T^{1,0} \) does under holomorphic change of coordinates. In fact this also holds for smooth sections of the canonical bundle, namely, expressions \( \omega = f(z, \bar{z})dz \), with \( f \) a smooth (i.e. not necessarily satisfying the Cauchy–Riemann equations) complex-valued function. Even in this case, \( \frac{1}{\omega} \) again defines a smooth vector field wherever \( \omega \) is not zero.

3.3. Critical sets. For a vector field \( h \) with an associated flow \( \Phi \), we also consider the flow

\[
F_{\epsilon, h} = \Phi(\epsilon, F)
\]

of a compact set \( F \).

Definition 3.7. \( F \) is a critical set in the external field \( \varphi = \text{Re } V \) if

\[
\lim_{\epsilon \to 0} \frac{E_\varphi(F_{\epsilon, h}) - E_\varphi(F)}{\epsilon} = 0
\]

for every \( C^1 \) vector field \( h \).
The following analogue of Proposition 1.4 holds.

**Proposition 3.8.** Let \( F \subset X \setminus \{p_\infty\} \) be a union of continua such that \( \text{Re } V \) is bounded from below on \( F \). Let \( \mu^F \) be the equilibrium measure of \( F \) in the external field \( \varphi = \text{Re } V \).

(a) Then for every \( C^1 \) vector field \( h \),

\[
\lim_{\varepsilon \to 0} \frac{E_\varphi(F_{\varepsilon,h}) - E_\varphi(F)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{E_\varphi(\mu^F_{\varepsilon,h}) - E_\varphi[\mu^F]}{\varepsilon}
\]

(3.12)

(b) \( F \) is a critical set in the external field \( \varphi \) if and only its equilibrium measure \( \mu^F \) is a critical measure in the external field \( \varphi \).

The proof is given in Sect. 5, and it is similar to the proof of Lemma 3.5 in [50] and the proof of Proposition 4.2 in [44]. Proposition 3.8 is the higher genus analogue of parts (a) and (b) of Proposition 1.4. The analogue of part (c) is now immediate as well.

**Corollary 3.9.** Suppose \( F \) belongs to a family \( \mathcal{F} \) of union of continua such that for every \( C^1 \) vector field \( h \) there is \( \varepsilon_0 > 0 \) such that \( F_{\varepsilon,h} \in \mathcal{F} \) for every \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \), where \( F_{\varepsilon,h} \) is given by (3.10). Suppose also that

\[
E_\varphi(F) = \max_{F' \in \mathcal{F}} E_\varphi(F').
\]

(3.13)

Then \( F \) is a critical set and \( \mu^F \) is a critical measure in the external field \( \varphi \).

### 3.4. Max–min energy problem

Next we describe a situation to which Corollary 3.9 applies and where we can prove the existence of a critical set \( F \) and a critical measure \( \mu^F \) in external field. We are not aiming at the most general set-up here, but instead we confine ourselves to a case that will be useful for the analysis of periodic tiling models, see Sect. 4.

We assume \( \varphi = \text{Re } V \), where \( V \) is locally meromorphic on \( X \) with logarithmic singularities at a finite number of points. The differential \( dV \) is meromorphic and single-valued on \( X \) with at most simple poles (i.e., an Abelian differential of the third kind). The logarithmic singularities of \( V \) correspond to simple poles of \( dV \). We assume the residues are real and negative, except for a pole \( p_0 \in X \setminus \{p_\infty\} \) with positive residue \( r_0 > 0 \) and a possible pole at \( p_\infty \). This implies that in a local coordinate \( z_0 \) at \( p_0 \) one has

\[
\text{Re } V(p) = r_0 \log |z_0(p)| + O(1), \quad \text{as } p \to p_0.
\]

(3.14)

We fix a local coordinate at \( p_0 \), that identifies a certain neighborhood of \( p_0 \) in \( X \) with a disk \( D(0, \delta_0) = \{|z_0| < \delta_0\} \) for some \( \delta_0 > 0 \). For \( \delta < \delta_0 \) we use \( U_\delta(p_0) \) to denote the neighborhood of \( p_0 \) that corresponds to \( D(0, \delta) \) in this local coordinate.

The differential \( dV \) could have simple pole at \( p_\infty \) as well, and then we use \( r_\infty = \text{Res}(dV, p_\infty) \) to denote the residue which could be positive or negative. If \( dV \) is holomorphic at \( p_\infty \), then we write \( r_\infty = 0 \). In either case we have in a fixed local coordinate \( z_\infty \) at \( p_\infty \)

\[
\text{Re } V(p) = r_\infty \log |z_\infty(p)| + O(1), \quad \text{as } p \to p_\infty.
\]

(3.15)

A neighborhood of \( p_\infty \) is identified with the disk \( D(0, \delta_\infty) \) for some \( \delta_\infty > 0 \). For \( \delta < \delta_\infty \) we use \( U_\delta(p_\infty) \) for the neighborhood of \( p_\infty \) that corresponds to the smaller disk \( D(0, \delta) \).
At all other poles (if any) there is a negative residue, which means that $\text{Re } V$ tends to $+\infty$ at those poles. Thus the external field $\text{Re } V$ is bounded from below on $X \setminus (U_\delta(p_0) \cup U_\delta(p_\infty))$ for any small enough $\delta > 0$. See Fig. 1 for a sketch of the neighborhoods $U_\delta(p_0)$ and $U_\delta(p_\infty)$.

**Definition 3.10.** Suppose $\delta_0$ and $\delta_\infty$ are as above with $\delta_\infty \geq \delta_0$ and such that the neighborhoods $U_\delta(p_0)$ and $U_\delta(p_\infty)$ are disjoint for every $\delta \in (0, \delta_0]$. Then for $\delta \in (0, \delta_0]$, we define the following

(a) We use $S_\delta$ to denote the set of simple closed oriented contours in $X \setminus (U_\delta(p_0) \cup U_\delta(p_\infty))$ that are not homotopic to a point in $X \setminus \{p_0, p_\infty\}$.

(b) We use $T_\delta$ to denote the set

$$T_\delta = \{ \gamma = \gamma_1 \cup \cdots \cup \gamma_n \mid n \in \mathbb{N}, \gamma_j \in S_\delta \text{ for every } j, \text{ and } \gamma \text{ is homologous in } X \setminus \{p_0, p_\infty\} \text{ to a circle around } p_0 \text{ in } U_\delta(p_0) \}.$$  (3.16)

(c) We define

$$\mathcal{F}_\delta = T_\delta, \quad \mathcal{F} = \bigcup_{0 < \delta \leq \delta_0} \mathcal{F}_\delta,$$  (3.17)

where the closure is with respect to the topology corresponding to Hausdorff distance on compact subsets of $X$.

An illustration of a possible multi-contour in $T_\delta$ is provided in Fig. 1.

The motivation for the definition (3.16) comes from the study of orthogonality on the Riemann surface, see the discussion in Sect. 4.5 below. The orthogonality is expressed through the vanishing of certain integrals of meromorphic differentials over a small contour around $p_0$. The meromorphic differentials have poles at $p_0$ and $p_\infty$, and in the absence of further poles, the integral can be deformed to a system of contours $\gamma$ as in (3.16) by Cauchy’s theorem. Within the class $T_\delta$ (with $\delta$ sufficiently small) we want to find an ideal $\gamma$. 

![Fig. 1. Neighborhoods $U_\delta(p_0)$ and $U_\delta(p_\infty)$, and an example of a multi-contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ in $T_\delta$ as defined in Definition 3.10. The three components belong to $S_\delta$](image-url)
The Hausdorff distance on compact subsets of $X$ that is used in (3.17) comes from a distance function $d$ on $X$ that is compatible with the complex structure; see (A.1). Then we write

$$d(p, F) = \min_{q \in F} d(p, q)$$

for the distance of $p \in X$ to a compact set $F \subset X$, and

$$d_H(F_1, F_2) = \max \left( \max_{p \in F_1} d(p, F_2), \max_{p \in F_2} d(p, F_1) \right)$$

for the Hausdorff distance between compact sets $F_1$ and $F_2$.

The Hausdorff distance restricted to compact subsets of a fixed compact defines a compact metric space. Thus $F_\delta$ is compact for every $\delta \in (0, \delta_0)$. The set $F$ from (3.17) does not depend on the precise choice of distance function on $X$. Recall that $E_\varphi(F)$ is given by (2.12).

**Theorem 3.11.** In the above setting, suppose that $r_0 = \text{Res}(dV, p_0) > 1$ and $r_\infty = \text{Res}(dV, p_\infty) > -1$. Then there exists $F \in \mathcal{F}$ (as defined in Definition 3.10) such that

$$E_\varphi(F) = \max_{F' \in \mathcal{F}} E_\varphi(F').$$

(3.18)

The set $F$ is critical and its equilibrium measure $\mu^F$ is a critical measure in the external field $\varphi$.

The condition $r_0 > 1$ in Theorem 3.11 implies that any $F \in \mathcal{F}$ that comes very close to $p_0$ has a small bipolar Green’s energy $E_\varphi(F)$ in a sense that will be made precise in Proposition 6.2. The condition $r_\infty > -1$ gives the same result for $F$ that come close to $p_\infty$.

The proof of Theorem 3.11 is in Sect. 6.

### 3.5. Final result.

The final result of the paper is a generalization of Theorem 1.2 to the case of a higher genus Riemann surface, albeit only for a special case of genus one that is given by a cubic equation

$$X : \quad w^2 = z(z - x_1)(z - x_2)$$

with $x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2 < 0$. We use $p = (w, z)$ to denote a generic point on $X$. We use $p_\infty$ for the point at infinity and we use $p_0 = (0, 0)$, $p_1 = (0, x_1)$, $p_2 = (0, x_2)$ to denote the branch points of (3.19). Then $X$ has two sheets that are copies of $\mathbb{C} \setminus ((-\infty, x_1] \cup [x_2, 0])$. The first sheet is such that $w > 0$ for $z > 0$ on the first sheet, while $w < 0$ for $z > 0$ on the second sheet. $X$ has an antiholomorphic involution

$$\sigma : X \to X : (w, z) \mapsto (\overline{w}, \overline{z}).$$

(3.20)

The real locus of $X$ is invariant under $\sigma$, and it consists of two parts. The bounded part is the real oval $C_1$ consisting of the points $p = (w, z)$ with $z \in [x_1, x_2]$, and the unbounded part $C_2$ consists of the points $p = (w, z)$ with $z \in [0, \infty)$ together with the point $p_\infty$ at infinity, see Fig. 2.
By means of the Abel map, we identify $X$ with a complex torus $\mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$ with $\tau \in i \mathbb{R}^+$, with $p_\infty$ identified with 0. The complex torus has the following $(2, -1)$-Cauchy kernel, which depends on a parameter $a$. Here $(2, -1)$ refers to the fact that this kernel is a quadratic differential with respect to one variable and a meromorphic vector field with respect to the other variable.

**Proposition 3.12.** For $u, v, a \in \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$, $a \neq 0, 1/2$, we have that

$$C^{(2,-1)}(u, v; a) = \frac{\theta_1'(0)\theta_1(u-a)^2\theta_1(v)^2\theta_1(u-v+2a)}{\theta_1(u-v)\theta_1(v-a)^2\theta_1(u)^2\theta_1(2a)} \, du^2 \, dv$$

(3.21)

satisfies the following.

(a) For fixed $u$ and $a$, (3.21) is a meromorphic vector field (i.e. a $-1$-differential by Remark 3.6) in $v$ with a simple pole at $v = u$, a double pole at $v = a$, a double zero at $v = 0$ and a simple zero at $v = u + 2a$.

(b) For fixed $v$ and $a$, (3.21) is a meromorphic quadratic differential in $u$, with a simple pole at $u = v$, a double pole at $u = 0$, a double zero at $u = a$, and a simple zero at $u = v - 2a$.

(c) The kernel (3.21) is normalized such that

$$C^{(2,-1)}(u, v; a) = \frac{1}{u-v} \, du^2 \, dv + O(1)$$

(3.22)

as $u \to v$ with $v \neq 0, v \neq a$ on the complex torus.

**Proof.** The periodicity properties (2.5) show that (3.21) is doubly periodic in both variables $u$ and $v$. The statements (a), (b) can be directly verified from the formula (3.21) with the fact that $\theta_1$ has a simple zero at 0 and no other zeros (modulo the lattice). Part (c) follows from the fact that in the limit $u \to v$ we have that $\theta_1(u-a) \to \theta_1(v-a)$, $\theta_1(u) \to \theta_1(v)$ and $\theta_1(u-v+2a) \to \theta_1(2a)$ while $\theta_1(u-v) = \theta_1'(0)(u-v) + O((u-v)^2)$.

Because of (3.22) we will say that $C^{(2,-1)}(u, v; a)$ has residue 1 at $u = v$. This notion of residue is conformally invariant if we transform $u$ and $v$ simultaneously by the same conformal map.
As a consistency check, we note that the poles in part (a) of Proposition 3.12 add up to \( u + 2a \), and this is where the simple zero is. The zeros and poles in part (b) both add up to \( v \), as it should be in view of Abel’s theorem. It is in accordance with the fact that \( C^{(2, -1)}(u, v; a) \frac{d\phi}{du} \) is doubly periodic with periods 1 and \( \tau \) in both \( u \) and \( v \).

With the Abel map we transform (3.21) to the Riemann surface (3.19) where \( \phi \) is given by the differential \( \omega \) where

\[
\omega = \left( \frac{dV(u)}{2} \right) - \int (C(u, q)dV(u) - C^{(2, -1)}(u, q; a)dV(q))d\mu(q).
\]

(b) The support of \( \mu \) is a union of analytic arcs or loops that are maximal trajectories of \( -\omega \). The measure \( \mu \) is absolutely continuous with respect to arclength, and it is given by the differential

\[
d\mu(u) = \frac{1}{\pi i} \omega(u)^{1/2}, \quad u \in \Sigma := \text{supp}(\mu),
\]

with an appropriate branch of the square root on each open arc or closed loop.
(c) The bipolar Green’s potential

\[ G^\mu(p) := \int G(p, q) d\mu(q) \]

of \( \mu \) satisfies

\[ 2G^\mu(p) + \text{Re } V(p) = c_j, \quad p \in \Sigma_j \]

with a constant \( c_j \) that can be different on each component \( \Sigma_j \) of \( \Sigma \).

(d) Any point \( p \in \Sigma \) that is not a zero of \( \omega \) has a neighborhood \( D \) such \( D \cap \Sigma \) is an analytic arc, and

\[ \frac{\partial}{\partial n_+} (2G^\mu + \text{Re } V) = \frac{\partial}{\partial n_+} (2G^\mu + \text{Re } V) \quad \text{on } D \cap \Sigma, \quad (3.27) \]

where \( \frac{\partial}{\partial n_\pm} \) denote the two normal derivatives to \( \Sigma \) inside \( D \cap \Sigma \).

The proof of Theorem 3.13 can be found in Sect. 7.2. The equality (3.27) is an analogue of the \( S \)-property (1.8) in the planar case, see also [45, Equation (5.30)] and [44, Definition 2.1]. Note that the equality of the normal derivatives is independent of the coordinate chart. The identity in (3.27) is equivalent to the \( S \)-property defined in Definition 2.9 by the Cauchy–Riemann equations.

3.6. Overview of the rest of the paper. In the next section we motivate the theory developed in the paper by relating it to certain periodic tiling models. We focus our discussion on lozenge tilings with periodic weightings.

In Sect. 5, we prove Propositions 3.4 and 3.8, which generalize the results [45, Lemma 3.1] on critical measures and [44, Proposition 4.2] (see also [50, Section 9.10]) on critical sets in the plane to compact Riemann surfaces of higher genus. In Sect. 6, we discuss the max-min energy problem and prove Theorem 3.11. We rely heavily on techniques developed in [39,44] and especially [50]. In Sect. 7, we prove Theorem 3.13; see in particular Sect. 7.2. The appendix contains a proof of Proposition 2.1 using the Green’s function for the Laplacian from Riemannian geometry.

4. Motivation: Periodic Tiling Models

4.1. Lozenge tilings of a hexagon. Our motivation comes from two-dimensional random tiling models with periodic weightings [43]. The main examples are domino tilings of the Aztec diamond [5,19,26] and lozenge tilings of a hexagon [15,17]. We focus in this discussion on the latter one. See Fig. 3 for the \( ABC \) hexagon and a possible tiling with lozenges of three types. The hexagon has vertices at \((0, 0), (B, 0), (B + C, C), (B + C, A + C), (C, A + C)\) and \((0, A)\), where \(A, B, C\) are positive integers. The vertices of each lozenge in a lozenge tiling have integer coordinates, see Fig. 3.

A weighting is an assignment of a weight to each possible tiling. We obtain such a weighting by first assigning a weight (i.e., a positive real number) to each lozenge depending on its type and on the position it has in the hexagon. We denote the weights by \( w_{\Box}(x, y) \), \( w_{\Diamond}(x, y) \), and \( w_{\Box}(x, y) \), where \((x, y) \in \mathbb{Z}^2\) is the position of the lower left vertex in case of \( w_{\Box}(x, y) \), \( w_{\Diamond}(x, y) \), and the position of the lower right vertex in case of \( w_{\Box}(x, y) \).
The weight \( w(T) \) of a tiling \( T \) is the product of the weights of the lozenges in the tiling, and the probability of \( T \) is proportional to its weight

\[
\mathbb{P}(T) = \frac{1}{Z} w(T), \quad Z = \sum_T w(T).
\] (4.1)

Two weightings are equivalent if they lead to the same probabilities (4.1). It is known that this model is determinantal, for any choice of positive weights. Different formulas for correlation kernels exist, coming either from the interpretation as a dimer model [41], or from the point of view of non-intersecting paths [27]. See [6, 18, 49] for connections between the different formulas.

4.2. Periodic weightings. The paper [26] gives a formula for the correlation kernel as a double contour integral in case of periodic weighting, where a weighting is periodic if there exist positive integers \( p \) and \( q \) such that

\[
w_{\square}(x + mp, y + nq) = w_{\square}(x, y), \quad (x, y, m, n \in \mathbb{Z}^2)
\] (4.2)

and similarly for \( w_{\rightarrow} \) and \( w_{\leftrightarrow} \). To describe the formula, we need that \( w_{\rightarrow}(x, y) = 1 \) for every \( (x, y) \in \mathbb{Z}^2 \), which is an assumption one can make without loss of generality, as for any weighting there is an equivalent weighting with this property, see e.g. [42] where this is called gauge equivalent. Then for each \( x \in \mathbb{Z} \) we consider a matrix \( T_x \) with entries

\[
T_x(y, y') = \begin{cases} 
  w_{\square}(x, y), & \text{if } y' = y, \\
  w_{\rightarrow}(x, y), & \text{if } y' = y + 1, \\
  0, & \text{otherwise},
\end{cases}
\] (4.3)

for \( (y, y') \in \mathbb{Z}^2 \), that describes a transition from the horizontal level \( x \) to \( x + 1 \). Each \( T_x \) is a two sided infinite matrix with two non-zero diagonals. The matrices \( T_x \) arise naturally as transition matrices in the non-intersecting path point of view on the hexagon tiling model [26].

Consider a periodic weighting with periods \( p \) and \( q \). Then \( T_x \) is a block Toeplitz matrix with blocks of size \( q \times q \) because of the periodicity in the vertical direction. The symbol of \( T_x \) is the matrix valued function \( z \mapsto A_x(z) \) with

\[
A_x(z) = \left( T_x(y, y') \right)_{y, y' = 0}^{q-1} + z \left( T_x(y, y' + q) \right)_{y, y' = 0}^{q-1}, \quad z \in \mathbb{C}.
\] (4.4)
Due to the periodicity in the horizontal direction we have $T_{x+p} = T_x$, and $A_{x+p} = A_x$ for every $x$. Write

$$A(z) = A_0(z) A_1(z) \cdots A_{p-1}(z). \tag{4.5}$$

The positions of the $\square$ and $\blacktriangle$ lozenges in a random tiling give a random point process inside the hexagon. This random point process is determinantal and a formula for the correlation kernel $K$ is given (in a more general setting) in [26, Theorem 4.7]. Let us assume for simplicity that $A = qN$, $C = qM$, $B + C = pL$, for certain integers $N$, $M$, $L$ where $A$, $B$, $C$ are the dimensions of the hexagon. Then $K((px, qy), (px', qy'))$ is equal to the $(0, 0)$ entry of the matrix

$$= -\frac{X_{x>x'}}{2\pi i} \oint_{\gamma} A_{x-x'}(z) z^{y'-y} \frac{dz}{z}$$

$$+ \frac{1}{(2\pi i)^2} \oint_{\gamma} \oint_{\gamma} A^{L-x'}(w) R_N(w, z) A^x(z) \frac{w^{y'}}{z^{y'+1}} w^M dwdz \tag{4.6}$$

where the integrals are taken entrywise, and $R_N$ is a bivariate polynomial of degrees $\leq N - 1$ in both variables that has the reproducing kernel property

$$\frac{1}{2\pi i} \oint_{\gamma} P(w) A^L(w) \frac{R_N(w, z)}{w^M} dwdw = P(z) \tag{4.7}$$

for any matrix valued polynomial $P$ of degree $\leq N - 1$. The contour $\gamma$ in (4.6) and (4.7) is a simple closed contour going around the origin once in the positive direction. Other entries of the matrix (4.6) give the correlation kernel at positions $(px, qy+j)$, $(px', qy'+j')$ when $0 \leq j, j' \leq q - 1$ and there is a similar, but somewhat more complicated formula for the correlation kernel at arbitrary points when the first coordinates of the points are not multiples of the period $p$.

The reproducing kernel $R_N$ is related to matrix valued orthogonal polynomials (MVOPs). The monic MVOP $P_n$ of degree $n$ (if it exists) satisfies

$$\frac{1}{2\pi i} \oint_{\gamma} P_n(z) \frac{A(z)^L}{z^M} z^k dz = H_n \delta_{n,k} \quad \text{for } 0 \leq k \leq n, \tag{4.8}$$

with $\det H_n \neq 0$. If all MVOP up to degree $N - 1$ exist then

$$R_N(w, z) = \sum_{n=0}^{N-1} P_n^T(w) H_n^{-1} P_n(z). \tag{4.9}$$

In our setting it can be shown that the MVOP of degree $N$ exists, as well as certain other degrees, see also [34, Proposition 1.1], but not for all degrees up to $N - 1$, and the sum formula (4.9) is not valid. There is another formula for $R_N$ in terms of the solution of a Riemann–Hilbert problem which we describe next.
4.3. Riemann–Hilbert problem. Consider

$$
Y(z) = \begin{pmatrix}
P_N(z) & \frac{1}{2\pi i} \oint_{\gamma} \frac{P_N(s)A(s)^L}{s^{M+N}} \frac{ds}{s-z} \\
-H^{-1}_{N-1}P_{N-1}(z) & \frac{1}{2\pi i} \oint_{\gamma} \frac{P_{N-1}(s)A(s)^L}{s^{M+N}} \frac{ds}{s-z}
\end{pmatrix}, \quad z \in \mathbb{C} \setminus \gamma,
$$

(4.10)

where $P_N$ and $P_{N-1}$ are the monic MVOP of degrees $N$ and $N-1$. This matrix valued function of size $2q \times 2q$ is defined and analytic for $z \in \mathbb{C} \setminus \gamma$ and satisfies

$$
Y_+(z) = Y_-(z) \begin{pmatrix}
I_q & A(z)^L \\
0 & z^{-M-N}I_q
\end{pmatrix}, \quad z \in \gamma,
$$

(4.11)

where $Y_+(z)$ and $Y_-(z)$ denote the limiting values of $Y(z')$ as $z' \to z \in \gamma$ from the interior and exterior region, respectively. In addition, $Y$ has the asymptotic behavior

$$
Y(z) = (I_{2q} + O(z^{-1})) \begin{pmatrix}
z^NI_q & 0 \\
0 & z^{-N}I_q
\end{pmatrix} \text{ as } z \to \infty.
$$

(4.12)

The Riemann–Hilbert (RH) problem (4.11)–(4.12) for MVOP was formulated in [14,35] as a generalization of the well-known RH problem for orthogonal polynomials due to Fokas, Its and Kitaev [31]. The reproducing kernel is expressed in terms of $Y$ via the formula [25,26]

$$
R_N(w, z) = \frac{1}{w-z} \begin{pmatrix}
0 & I_q \\
I_q & 0
\end{pmatrix} Y^{-1}(w)Y(z) \begin{pmatrix}
I_q \\
0
\end{pmatrix},
$$

(4.13)

which can be seen as a matrix valued Christoffel-Darboux formula for the sum (4.9).
4.4. Transformation $Y \mapsto Z$. To proceed we follow [26, Section 5.3] to perform a transformation of the RH problem by using the spectral decomposition $A(z) = E(z) \Lambda(z) E(z)^{-1}$ of (4.5) where $\Lambda(z)$ is a diagonal matrix with the eigenvalues $\lambda_1(z), \ldots, \lambda_q(z)$ of $A(z)$ on the diagonal and the columns of $E(z)$ are eigenvectors of $A(z)$. The transformation

$$Z = Y \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$$

(4.14)

then gives a matrix valued function $Z$ with the jump condition

$$Z_+(z) = Z_-(z) \begin{pmatrix} I_q & \Lambda(z)^L \\ 0 & z^{M+N} I_q \end{pmatrix} \quad z \in \gamma.$$

(4.15)

The eigenvalues $\lambda_j(z)$ and eigenvectors in $E(z)$ will not be entire functions of the $z$ variable and will be single-valued only if we apply certain branch cuts. The definition (4.14) will also create a jump of $Z$ on these cuts that takes the form (with an appropriate choice of the eigenvectors in $E(z)$)

$$Z_+ = Z_- \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

(4.16)

for a permutation matrix $\sigma \in S_q$ that could depend on the branch cut.

4.5. Scalar orthogonality on the Riemann surface. The characteristic equation

$$X : \quad \det(\lambda I - A(z)) = 0$$

(4.17)

is a polynomial equation in $z$ and $\lambda$. It is the spectral curve in the sense of [43] (up to a sign change in $\lambda$ in case $n$ is odd), see in particular [43, p. 1048]. The algebraic curve $X$ is a Harnack curve which implies that it has the maximal number of real ovals [43, Theorem 5.1] in case it is nonsingular. If $X$ is singular then some of the real ovals are contracted to a point. In particular it has only real branch points when viewed as a $q$-fold cover of the $z$-plane, and the branch cuts in (4.16) are on the real line only. The case where $X$ has genus $\geq 1$ is of special interest as it allows for the appearance of a smooth (or gas) phase in the large scale limit, in addition to the more familiar solid and rough (or frozen and liquid) phases [43]. We show that the matrix valued orthogonality leads to scalar orthogonality conditions on $X$. See [8–10,16,29] for other recent contributions to orthogonality on a Riemann surface.

Returning to $Z$, we choose a row number $j \in \{1, \ldots, 2q\}$. Then for $k = 1, \ldots, q$, we consider $Z_{j,k}$ as a function on the $k$th sheet of $X$. The jump conditions (4.15) and (4.16) show that this defines a meromorphic function $f_j$ on $X$ with a pole at the point(s) at infinity. The entry $Z_{j,q+k}$ for $k = 1, \ldots, q$ is also considered on the $k$th sheet, and the jump properties (4.15) and (4.16) imply that they define a holomorphic function $\psi_j$ on $X \setminus \gamma_X$ (we use $\gamma_X$ to denote the contour on $X$ consisting of a copy of $\gamma$ on each of the sheets) with the jump

$$(\psi_j)_+ = (\psi_j)_- + f_j \frac{\lambda^L}{z^{M+N}} \quad \text{on} \ \gamma_X.$$
The asymptotic condition (4.12) and the definition (4.14) imply that \( f_j \) has a pole and \( \psi_j \) has a zero at each point at infinity. The precise order of the poles and zeros depends on the possible branching of \( X \) at infinity, and the behavior of \( E(z) \) at infinity. In any case the conditions (4.18) imply certain orthogonality conditions for \( f_j \) of the form

\[
\int_{\gamma(X)} f_j \, \frac{\lambda^L}{z^{M+N}} \omega = 0,
\]

for a large class of meromorphic differentials \( \omega \) on \( X \) with poles at infinity.

The pole conditions at infinity imply that \( f_j \) belongs to a vector space of meromorphic functions of dimension \( \approx qN \) and the orthogonality conditions (4.19) are for \( \omega \) in a vector space of meromorphic differentials of dimension \( \geq qN - c \) with a constant \( c \) that is independent of \( N, M, L \). Typically (4.19) will not be enough to characterize \( f_j \) as an orthogonal meromorphic function on \( X \), as the conditions (4.19) will have to be supplemented by \( \approx c \) additional conditions.

Since \( c \) is independent of \( N, M, L \) we expect that in generic cases the additional conditions do not influence the behavior of the bulk of the zeros of \( f_j \) in the large \( N, M, L \) limit. We expect that the zeros will accumulate on an \( S \)-curve in an external field on the Riemann surface \( X \), as it is known in the genus zero case. The equilibrium measure in external field on an \( S \)-curve has a \( g \)-function that will be useful in the next transformation in the steepest descent analysis of the Riemann–Hilbert problem.

To illustrate, let us take \( q = 2 \) and assume that \( X \) has branching at infinity. Let \( g_1 \) be the restriction of the \( g \) function to the first sheet, and \( g_2 \) to the second sheet. Then we expect that the following transformation \( Z \rightarrow U \),

\[
U = Z \operatorname{diag} \left( e^{-2N g_1}, e^{-2N g_2}, e^{2N g_1}, e^{2N g_2} \right)
\]

will be the appropriate next step in the steepest descent analysis.

4.6. The two-periodic Aztec diamond. The analogous model of the domino tilings of the Aztec diamond with two-periodic weightings was successfully analyzed in [26] but without the use of \( g \)-functions. The transformation \( Z \rightarrow U \) in [26, formula (5.20)] (note that \( X \) is used in [26] instead of \( Z \)) is explicit in terms of \( \lambda \) and \( z \). Let us show here that the transformation is actually of the form (4.20).

The Riemann surface \( X \) in [26] is associated with the equation

\[
X : \quad w^2 = z(z - x_1)(z - x_2)
\]

with real \( x_1, x_2 \) given by \( x_1 = -\alpha^2, x_2 = -\alpha^{-2} \) for some \( \alpha > 1 \). A point on \( X \) is denoted by \( p = (w, z) \) and we write \( w = w(p), z = z(p) \), for the projections onto the \( w \) and \( z \) coordinate. Then \( X \) is a two-sheeted compact Riemann surface with two sheets \( \mathbb{C} \setminus \{(-\infty, x_1] \cup [x_2, 0)\} \). The first sheet is such that \( w(p) > 0 \) for \( p = (w, z) \) with \( z(p) > 0 \) on the first sheet.

The meromorphic function \( \lambda \) on \( X \) has a double pole at \( p_1 \) where \( p_1 \) is the point on the first sheet with \( z(p_1) = 1 \), and a double zero at \( p_2 \), the point on the second sheet with \( z(p_2) = 1 \). There are no other zeros or poles [26, Lemma 5.2]. The bipolar Green’s function \( G(p, p_1) \) with pole at infinity and \( p_1 \) is explicit in terms of \( \lambda \), namely for some constant \( c \),

\[
G(p, p_1) = \frac{1}{4} \log |\lambda(p)| - \frac{1}{2} \log |z(p) - 1| + c
\]

(4.21)
since indeed the right-hand side satisfies the requirements of Proposition 2.1 (a), (b) and (c). Note that (4.21) is finite at \( p = p_2 \) since \( \lambda \) has a double zero at \( p_2 \). Without loss of generality we may assume that the bipolar Green’s function is taken such that \( c = 0 \) in the above formula.

Let \( 0 < r \leq 1 \) and let \( \mu_r \) be the uniform normalized Lebesgue measure on the circle \( F_r : |z - 1| = r \) on the first sheet. By conformal invariance it is the balayage of the point mass \( \delta_{p_1} \) onto \( F_r \), which means that \( \int G(p, q) d\mu_r(q) = G(p, p_1) \) for \( p \in X \) in the exterior \( \Omega_{\text{ext}} \) (which includes the entire second sheet). The equality extends to \( F_r \), and thus \( 2 \int G(p, q) d\mu_r(q) = \frac{1}{2} \log |\lambda(p)| - \log r \) on the support of \( \mu_r \), which means that \( \mu_r \) is the equilibrium measure of \( F_r \) in the external field

\[
\varphi = \Re V \quad \text{where} \quad V = -\frac{1}{2} \log \lambda
\]

cf. Proposition 2.8.

We verify that \( F_r \) has the \( S \)-property in the external field \( \Re V \). Since \( \frac{1}{4} \log |\lambda(p)| + \frac{1}{2} \log |z(p) - 1| \) is harmonic in the interior domain \( \Omega_{\text{int}} \) of \( F_r \) (including at \( p_1 \)) and agrees with \( G(p, p_1) + \log r \) on the circle. We conclude

\[
\int G(p, q) d\mu_r(q) = \begin{cases} 
\frac{1}{4} \log |\lambda(p)| - \frac{1}{2} \log |z(p) - 1|, & \text{in } \Omega_{\text{ext}}, \\
\frac{1}{4} \log |\lambda(p)| + \frac{1}{2} \log |z(p) - 1| - \log r, & \text{in } \Omega_{\text{int}}.
\end{cases}
\]

Then, with appropriate branches of the logarithm

\[
g(p) = \begin{cases} 
-\frac{1}{4} \log \lambda(p) + \frac{1}{2} \log(z(p) - 1), & \text{in } \Omega_{\text{ext}}, \\
-\frac{1}{4} \log \lambda(p) - \frac{1}{2} \log(z(p) - 1) - \log r, & \text{in } \Omega_{\text{int}}.
\end{cases}
\]

and indeed \( g_+ + g_- + \frac{1}{4} \log \lambda \) is constant on \( \text{supp}(\mu) \), which is the \( S \)-property in external field by Definition 2.9. It follows that the transformation in [26, Section 5.4] is indeed of the form (4.20).

The \( 2 \times k \) periodic Aztec diamond was studied by Berggren [5] using a Wiener-Hopf factorization technique developed in [7]. The formula (4.21) applies in the \( 2 \times k \) periodic case as well and it gives rise to an equilibrium measure in the same way as described above. Very recently, Berggren and Borodin [6] made a detailed study of the \( k \times l \) doubly periodic Aztec diamond.

4.7. Hexagon tilings. In the Aztec diamond example, one is fortunate to be able to find a contour with the \( S \)-property via an explicit construction. In the hexagon tiling models this does not seem to work and that is why we developed the methods of this paper in order to prove the existence of contours with the \( S \)-property in an external field.

The final result of this paper, Theorem 3.13, is restricted to a special situation of genus one. It will apply to lozenge tilings with periodic weightings of periods \( p = 3 \) and \( q = 2 \). In that case the spectral curve (4.17) has genus one and it can be put in the form

\[
x^2 = (z - x_1)(z - x_2)(z - x_3) \quad \text{with} \quad x_1 < x_2 < x_3 \leq 0.
\]

If \( x_3 = 0 \) then we are in the situation of (3.19). The external field is

\[
\varphi(z) = \Re V(z) = -\frac{b}{2} \log |\lambda| + \frac{1 + c}{2} \log |z| \quad (4.22)
\]
with \( b = \lim \frac{L}{N}, c = \lim \frac{M}{N}, 3b > 2c \), which is invariant under the involution (3.20).

The condition \( 3b > 2c \) comes from the fact that \( 3L = B + C > C = 2M \), hence \( \frac{3L}{N} > \frac{2M}{N} \), and we take the large \( N \) limit such that the strict inequality remains valid. Recall that \( \lambda \) is related to \( z \) via the characteristic equation (4.17). For \( q = 2 \) and \( p = 3 \) we will have that \( \lambda \) is meromorphic on \( X \) with a pole at \( \infty \) of order three, and three zeros that are on the positive real \( z \)-axis.

Then from (4.22) we get that \( dV \) has simple poles at 0, \( \infty \) and at the zeros of \( \lambda \). The simple pole at 0 has residue \( r_0 = 1 + c \) (due to the branching at 0). Since \( \lambda \) has a third order pole and \( z \) a second order pole at \( \infty \), one may calculate from (4.22) that the simple pole at \( \infty \) has residue \( r_\infty = \frac{3b}{2} - 2\frac{1+c}{2} = \frac{3b-2c}{2} - 1 \). Thus the residue conditions of Theorem 3.11 are satisfied, and thus there is a critical set \( F \) with equilibrium measure \( \mu^F \) that is a critical measure in the external field \( \varphi \). By symmetry we may assume that \( F \) and \( \mu^F \) are invariant under the involution \( \sigma \).

If the support of \( \mu^F \) does not intersect the bounded real oval \( C_1 \), then by Theorem 3.13, the support is a union of maximal trajectories of a quadratic differential \(-\omega\) given by (3.26). Note that by (3.25), \( \omega \) has double poles at the poles of \( dV \), but also at \( p_\infty \) due to the simple pole of \( \int C(u, q)d\mu(q) \) at \( u = p_\infty \). Counting multiplicities, \( \omega \) thus has a total of 8 poles. Because we are in genus one, we find that \( \omega \) then also has 8 zeros (counted with multiplicity).

The equilibrium measure in external field and its associated \( g \)-function will be useful in the steepest descent analysis of the RH problem as we indicated. The details of the steepest descent analysis, and their consequences for the random tiling model are under current investigation.

For a very specific class of periodic weightings, we are indeed able to determine the critical measure with the \( S \)-property. In these examples the support of the critical measure intersects the bounded oval in a point. Hence the assumptions of Theorem 3.13 are not satisfied and we cannot use the theorem as stated. However, certain symmetry arguments can be used to identify the appropriate quadratic differential and the conclusions of the theorem are valid.

We finally remark that the Riemann surface associated with hexagon tilings may have multiple infinities. The potential theory developed in this paper has to be modified in such a situation. Suppose \( p_{\infty(1)}, \ldots, p_{\infty(d)} \) are the distinct points at infinity. Each point at infinity comes with its own bipolar Green’s function. If we use \( G_{p_{\infty(j)}} \) to denote the bipolar Green’s function with one pole at \( p_{\infty(j)} \), then instead of \( G \) one has to use a linear combination of the kernels \( G_{p_{\infty(j)}} \) with coefficients that depend on the branching numbers at the various points at infinity. The results on equilibrium measures and critical measures remain valid, with appropriately modified residue conditions in Theorem 3.11.

5. Proofs of Propositions 3.4 and 3.8

5.1. Proof of Proposition 3.4.

Proof. Let \( \Phi \) denote the flow associated with the \( C^1 \) vector field \( h \). We have

\[
\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} V(\Phi(\varepsilon, p)) = h(p)dV(p) \quad (5.1)
\]
and thus by (3.4)

\[
\int V d\mu_{\varepsilon,h} = \int V(\Phi(\varepsilon, p))d\mu(p) \\
= \int Vd\mu + \varepsilon \int hVd\mu + o(\varepsilon) \quad \text{as } \varepsilon \to 0.
\]

(5.2)

Since \( p \mapsto G(p,q) \) is harmonic, it is locally the real part of a holomorphic function \( \mathcal{G}(p,q) \) and

\[
C(p,q) = -2\partial_p G(p,q) dp = -\partial_p \mathcal{G}(p,q) dp.
\]

Then for \( p \neq q \),

\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \mathcal{G}(\Phi(\varepsilon, p), q) dp = -h(p)C(p,q)
\]

and by taking the real part

\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} G(\Phi(\varepsilon, p), q) dp = -\operatorname{Re} (h(p)C(p,q)).
\]

Since \( G \) is symmetric in the two variables, we then also have

\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} G(p, \Phi(\varepsilon, q)) dp = -\operatorname{Re} (h(q)C(q,p)),
\]

and by the chain rule

\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} G(\Phi(\varepsilon, p), \Phi(\varepsilon, q)) dp = -\operatorname{Re} (h(p)C(p,q) + h(q)C(q,p)) \quad (5.3)
\]

Then because of (2.8), (3.4) and (5.3).

\[
E[\mu_{\varepsilon,h}] = \int \int G(\Phi(\varepsilon, p), \Phi(\varepsilon, q))d\mu(p)d\mu(q) \\
= E[\mu] - \varepsilon \operatorname{Re} \int \int (h(p)C(p,q) + h(q)C(q,p))d\mu(p)d\mu(q) + o(\varepsilon)
\]

(5.4)

as \( \varepsilon \to 0 \). Combining (5.2) and (5.4) we obtain the limit in part (a).

If \( \mu \) is critical then the expression in part (a) is zero for every \( h \), and this implies that the real part of \( D_{V,h}(\mu) \) vanishes. Changing \( h \) to \( ih \), we find that the imaginary part vanishes as well, and part (b) follows. \( \square \)
5.2. Convergence of perturbed measures in energy norm. The next lemma concerns the convergence of the measures in energy norm; see (2.11).

Lemma 5.1. Suppose $\mu$ is compactly supported in $X \setminus \{p_\infty\}$ with $E[\mu] < +\infty$. Let $h$ be a $C^1$ vector field. Then the measures $\mu_{\varepsilon,h}$ tend to $\mu$ as $\varepsilon \to 0$ in energy norm.

Proof. If $f$ is a continuous function, then

$$\int f \, d\mu_{\varepsilon,h} = \int f(\Phi, p) \, d\mu(p)$$

and $f(\Phi, p) \to f(p)$ for every $p$. Since $f$ is bounded on a compact neighborhood of $\text{supp}(\mu)$ in $X \setminus \{p_\infty\}$, we can apply dominated convergence to conclude that $\int f \, d\mu_{\varepsilon,h} \to \int f \, d\mu$ as $\varepsilon \to 0$. This proves the weak$^*$ convergence of $(\mu_{\varepsilon,h})_\varepsilon$ to $\mu$.

Since $q \mapsto \int G(p, q) \, d\mu(p)$ is lower semi-continuous, we have by weak$^*$ convergence

$$E[\mu] = \int \int G(p, q) \, d\mu(p) \, d\mu(q) \leq \liminf_{\varepsilon \to 0} \int \int G(p, q) \, d\mu(p) \, d\mu_{\varepsilon,h}(q).$$

From (5.4) we conclude $E[\mu_{\varepsilon,h}] \to E[\mu]$, and therefore

$$\limsup_{\varepsilon \to 0} E[\mu_{\varepsilon,h}] - E[\mu] = \limsup_{\varepsilon \to 0} \left( E[\mu_{\varepsilon,h}] + E[\mu] - 2 \int \int G(p, q) \, d\mu(p) \, d\mu_{\varepsilon,h}(q) \right)$$

$$\leq E[\mu] + E[\mu] - 2E[\mu] = 0,$$

which is the convergence in energy norm. \qed

For later use, we need a uniformity in the limit (3.6) for measures on a fixed compact.

Lemma 5.2. Let $K \subset X \setminus \{p_\infty\}$ be a compact set such that $\varphi = \text{Re} \, V$ is bounded on $K$. Let $h$ be a $C^1$ vector field. Then for every probability measure $\mu$ on $K$ with $E[\mu] < +\infty$, we have

$$E[\varphi][\mu_{\varepsilon,h}] - E[\varphi][\mu] - \varepsilon \, \text{Re} \, D_{V,h}(\mu) = o(\varepsilon) \quad \text{as} \quad \varepsilon \to 0,$$

(5.5)

where the $o(\varepsilon)$ only depends on $h$ and $K$, but not on $\mu$.

Proof. Let $\Phi$ be the flow associated with the vector field $h$. Since $\varphi$ is bounded on $K$, the set $K$ does not contain any of the poles of $dV$. Then there is $\varepsilon > 0$ small enough such that

$$\bigcup_{t \in [-\varepsilon, \varepsilon]} \bigcup_{p \in K} \Phi(t, p)$$

(5.6)

is a compact subset of $X \setminus \{p_\infty\}$ that does not contain any of the poles of $dV$. Thus $\varphi$ is bounded on the set (5.6).

We already observed in Remark 3.5 that

$$(p, q) \mapsto H(p, q) = h(p)C(p, q) + h(q)C(q, p)$$

(5.7)

is a well defined function on $(X \setminus \{p_\infty\})^2$. It is clearly $C^1$ function for $p \neq q$, since we assumed that $h$ is $C^1$. 

Suppose $p, q$ are in a local chart, where $p$ and $q$ correspond to $z$ and $w$ respectively in the local coordinate. Then

$$C(p, q) = \frac{1}{z - w} dz + \tilde{C}(z, w) dz$$

where $\tilde{C}(z, w)$ is $C^\infty$ smooth. Write $h = h_1 \frac{\partial}{\partial z} + h_2 \frac{\partial}{\partial \bar{z}}$ with $C^1$ functions $h_1$ and $h_2$. Then by (5.7)

$$H(p, q) = h_1(z) \left( \frac{1}{z - w} + \tilde{C}(z, w) \right) + h_1(w) \left( \frac{1}{w - z} + \tilde{C}(w, z) \right)$$

which shows that $H$ is a continuous function in a local chart. Therefore $H$ is uniformly continuous on the compact set (5.6). It follows from the uniform continuity that

$$\sup_{|t| \leq \varepsilon} \sup_{p, q \in K} |H(\Phi(t, p), \Phi(t, q)) - H(p, q)| = o(1) \quad (5.8)$$

as $\varepsilon \to 0$, where the $o$-term only depends on $h$ and the compact set $K$.

Next, by integrating (5.3) and recalling (5.7), we find

$$G(\Phi(\varepsilon, p), \Phi(\varepsilon, q)) - G(p, q) + \varepsilon \text{Re } H(p, q)$$

$$= - \text{Re } \int_0^\varepsilon [H(\Phi(s, p), \Phi(s, p)) - H(p, q)] ds.$$

Combining this with (5.8), we arrive at the estimate

$$|G(\Phi(\varepsilon, p), \Phi(\varepsilon, q)) - G(p, q) + \varepsilon \text{Re } H(p, q)| = o(\varepsilon), \quad p, q \in K,$$

as $\varepsilon \to 0$, where the $o$-term only depends on $h$ and $K$. We integrate with respect to $d\mu(p)d\mu(q)$ where $\mu$ is a probability measure on $K$ with $E[\mu] < \infty$, to find that as $\varepsilon \to 0$,

$$\left| \int \int G(p, q)d\mu_{\varepsilon, h}(p)d\mu_{\varepsilon, h}(q) - \int \int G(p, q)d\mu(p)d\mu(q) \right. + \varepsilon \text{Re } \int \int H(p, q)d\mu(p)d\mu(q) \left| = o(\varepsilon), \quad (5.9) \right.$$

with the $o$-term independent of $\mu$.

Using (5.1) and the fact that $\text{Re } V$ is bounded on (5.6) we find in an analogous way that for any probability measure $\mu$ on $K$, as $\varepsilon \to 0$,

$$\left| \int \text{Re } V d\mu_{\varepsilon, h} - \int \text{Re } V d\mu - \varepsilon \int h dV d\mu \right| = o(\varepsilon). \quad (5.10)$$

Combining (5.9) and (5.10), and recalling the definitions, we obtain (5.5). \qed
5.3. Proof of Proposition 3.8.

Proof. Let $F \subset X \setminus \{p_\infty\}$ be a union of continua such that $\operatorname{Re} V$ is bounded from below on $F$ and let $\mu^F$ be the equilibrium measure of $F$ in the external field $\varphi$.

Part (b) follows directly from the definition once part (a) has been established, so we only need to prove part (a).

To establish (3.12) it suffices to show that

$$E_\varphi(F_{\varepsilon,h}) - E_\varphi(F) = E_\varphi[\mu^F_{\varepsilon,h}] - E_\varphi[\mu^F] + o(\varepsilon)$$

(5.11)
as $\varepsilon \to 0$, since we already know from Proposition 3.4 that the limit in the right-hand side of (3.12) exists, and having (5.11) we conclude that the limit in the left-hand side also exists and the equality (3.12) holds.

For a $C^1$ vector field $h$, and $\varepsilon \in \mathbb{R}$, we are going to use the four probability measures $\mu^F$, $\mu^F_{\varepsilon,h}$, $\mu^{F_{\varepsilon,h}}$ and $(\mu^{F_{\varepsilon,h}})_{-\varepsilon,h}$. To simplify notation, we write

$$\mu = \mu^F, \quad \mu_\varepsilon = \mu^F_{\varepsilon,h}, \quad \mu^\varepsilon = \mu^{F_{\varepsilon,h}}, \quad \mu_{-\varepsilon} = (\mu^{F_{\varepsilon,h}})_{-\varepsilon,h}.$$ 

Thus $\mu^\varepsilon$ is the equilibrium measure of $F_{\varepsilon,h}$ (see (3.10)) in the external field $\varphi$, and therefore

$$E_\varphi(F_{\varepsilon,h}) = E_\varphi[\mu^\varepsilon] \leq E_\varphi[\mu_\varepsilon],$$

(5.12)
since $\mu_\varepsilon$ is supported on $F_{\varepsilon,h}$. Since $\mu_{-\varepsilon}$ is the image of $\mu^\varepsilon$ under the reverse flow associated with $h$, it is supported on $F$, and hence

$$E_\varphi(F) = E_\varphi[\mu] \leq E_\varphi[\mu_{-\varepsilon}],$$

(5.13)
Applying Lemma 5.2 to $\mu$ and $\mu^\varepsilon$ (with $-\varepsilon$ instead of $\varepsilon$ for the latter measure), we obtain

$$E_\varphi[\mu_{-\varepsilon}] = E_\varphi[\mu] + \varepsilon \operatorname{Re} D_{V,h}(\mu) + o(\varepsilon),$$

$$E_\varphi[\mu^\varepsilon] = E_\varphi[\mu^\varepsilon_{-\varepsilon}] + \varepsilon \operatorname{Re} D_{V,h}(\mu^\varepsilon) + o(\varepsilon).$$

(5.14)
Subtracting the two identities in (5.14) and using the inequalities (5.12) and (5.13), we get

$$0 \leq E_\varphi[\mu_{-\varepsilon}] - E_\varphi[\mu^\varepsilon] \leq \varepsilon \operatorname{Re} D_{V,h}(\mu) - \varepsilon \operatorname{Re} D_{V,h}(\mu^\varepsilon) + o(\varepsilon)$$

(5.15)
The expression (3.7) shows that $D_{V,h}(\mu^\varepsilon)$ remains bounded as $\varepsilon \to 0$ and therefore by (5.15)

$$\lim_{\varepsilon \to 0} (E_\varphi[\mu_{-\varepsilon}] - E_\varphi[\mu^\varepsilon]) = 0.$$ 

(5.16)
Next, we use the identity

$$E[\mu_{-\varepsilon} - \mu^\varepsilon] = 2E_\varphi[\mu_{-\varepsilon}] + 2E_\varphi[\mu^\varepsilon] - 4E_\varphi \left[\frac{\mu_{-\varepsilon} + \mu^\varepsilon}{2}\right]$$

and the inequality

$$E_\varphi(F_{\varepsilon,h}) \leq E_\varphi \left[\frac{\mu_{-\varepsilon} + \mu^\varepsilon}{2}\right],$$
which holds since $\frac{\mu_\varepsilon + \mu^\varepsilon}{2}$ is a probability measure on $F_{\varepsilon, h}$, to find that

$$0 \leq E[\mu_\varepsilon - \mu^\varepsilon] \leq 2E_\varphi[\mu_\varepsilon] - 2E_\varphi[\mu^\varepsilon].$$

Hence in view of (5.16)

$$\lim_{\varepsilon \to 0} E[\mu_\varepsilon - \mu^\varepsilon] = 0. \tag{5.17}$$

Since $\mu_\varepsilon$ tends to $\mu$ in energy norm, see Lemma 5.1, we find from (5.17) that $\mu^\varepsilon$ also tends to $\mu$ in energy norm, and in particular in weak* sense. Then

$$\lim_{\varepsilon \to 0} DV_{\varphi, h}(\mu_\varepsilon) = DV_{\varphi, h}(\mu) \tag{5.18}$$

by weak* convergence, since the functions that appear in the integrals (3.7) are bounded and continuous. Using (5.18) in (5.15), we find

$$E_\varphi[\mu_\varepsilon] - E_\varphi[\mu^\varepsilon] = o(\varepsilon)$$

as $\varepsilon \to 0$. We obtain (5.11), and the proof of Proposition 3.8 is complete. $\Box$

6. Proof of Theorem 3.11

6.1. A preliminary lemma.

Lemma 6.1. Suppose $F \in F_\delta$, and let $\tilde{F}$ be a connected component of $F$.

(a) If $\tilde{F} \subset U_{\delta_0}(p_0)$ then $\tilde{F}$ separates $p_0$ from $\partial U_{\delta_0}(p_0)$.

(b) If $\tilde{F} \subset U_{\delta_\infty}(p_\infty)$ then $\tilde{F}$ separates $p_\infty$ from $\partial U_{\delta_\infty}(p_\infty)$.

(c) There is $\eta > 0$, depending only on $\delta$, such that $\text{diam}(\tilde{F}) \geq \eta$.

Proof. Any contour $\gamma \in S_\delta$ has the properties (a) and (b), since otherwise $\gamma$ would be homotopic to a point in $X \setminus \{p_0, p_\infty\}$, which would contract the definition of $S_\delta$ in Definition 3.10 (a).

We next prove that

$$\inf_{\gamma \in S_\delta} \text{diam}(\gamma) > 0. \tag{6.1}$$

By compactness of $X$ there is an $\eta > 0$ such that every $\eta$ neighborhood of a point is contractible. Thus any $\gamma \in S_\delta$ that is not contractible in $X$ must have $\text{diam}(\gamma) \geq \eta$. Now suppose $\gamma \in S_\delta$ is contractible in $X$. Then $X \setminus \gamma$ has a simply connected component that contains at least one of $p_0, p_\infty$ since otherwise $\gamma$ would be contractible in $X \setminus \{p_0, p_\infty\}$ which again would contradict Definition 3.10 (a). If the simply connected component contains both $p_0$ and $p_\infty$ then $\gamma$ will have a minimal diameter. If the simply connected component contains only one of $p_0$ and $p_\infty$, then $\gamma$ separates $p_0$ from $p_\infty$ and it will have a minimum diameter since $\gamma$ is outside of $U_{\delta}(p_0)$ and $U_{\delta}(p_\infty)$. Thus (6.1) holds.

Thus properties (a), (b), and (c) hold if $F = \gamma \in S_\delta$. Then by (3.16) it also holds if $F = \gamma \in T_\delta$. The properties are preserved under taking closure in the Hausdorff metric and the lemma follows. $\Box$
6.2. Estimates of $E_\varphi(F)$. We start with a proposition, whose proof should be compared to that of [44, Proposition 5.1].

**Proposition 6.2.** Suppose $\varphi$ has the behavior (3.14) near $p_0$ and the behavior (3.15) near $p_\infty$. Let $F$, and $\delta_0, \delta_\infty$ be as in Definition 3.10. Then there is a constant $C$ such that the following hold.

(a) If $F \in F$ intersects $\partial U_\delta(p_0)$ for some $\delta < \delta_0/2$ then

$$E_\varphi(F) \leq (r_0 - 1) \log \delta + C. \quad (6.2)$$

(b) If $F \in F$ intersects $\partial U_\delta(p_\infty)$ for some $\delta < \delta_\infty/2$, then

$$E_\varphi(F) \leq (1 + r_\infty) \log \delta + C. \quad (6.3)$$

**Proof.** (a) Take $0 < \delta < \delta_0/2$ and assume $F \in F$ intersects $\partial U_\delta(p_0)$. Let $\tilde{F}$ be a connected component of $F$ that intersects $\partial U_\delta(p_0)$. Then either $\tilde{F}$ is fully contained in $U_{2\delta}(p_0)$ and we put $F_0 = \tilde{F}$, or it contains a sub-continuum $F_0 \subset \tilde{F}$ that intersects both $\partial U_\delta(p_0)$ and $\partial U_{2\delta}(p_0)$ and that is contained in $U_{2\delta}(p_0)$.

In both cases $z_0(F_0)$ is a continuum in $\mathbb{C}$ and we claim that

$$\text{diam}(z_0(F_0)) \geq \delta, \quad (6.4)$$

where $z_0$ is the local coordinate at $p_0$. The claim (6.4) follows immediately if $F_0$ intersects both $\partial U_\delta(p_0)$ and $\partial U_{2\delta}(p_0)$, since in that case the $z_0$-images of the two intersection points are in $z_0(F_0)$ with distance $\geq \delta$. The claim (6.4) also follows if $F_0 = \tilde{F} \subset U_{2\delta}(p_0)$, since then $z_0(F_0)$ is a continuum in $\mathbb{C}$ that intersects $\partial D(0, \delta)$ and separates $p_0$ from $p_\infty$ by Lemma 6.1.

Let $\omega$ be the equilibrium measure from usual logarithmic potential theory for the continuum $z_0(F_0) \subset \overline{D(0, 2\delta)}$. From (6.4) it follows that the capacity of $z_0(F_0)$ is at least the capacity of a straight line segment of length $\delta$, which is $\delta/4$, see e.g. [51, Theorem 5.3.2]. Therefore

$$\iint \log \frac{1}{|z - w|} d\omega(z) d\omega(w) \leq -\log \frac{\delta}{4}. \quad (6.5)$$

Because of (2.7) we have for some constant $C_1$

$$G(p, q) \leq \log \frac{1}{|z_0(p) - z_0(q)|} + C_1, \quad p, q \in U_{\delta_0}(p_0). \quad (6.6)$$

Let $\mu$ be the pullback of $\omega$ by the local coordinate $z_0$. Then $\mu$ is a probability measure on $F \cap \overline{U_{2\delta}(p_0)} \subset U_{\delta_0}(p_0)$ so that by (6.6), the properties of the pullback measure, and (6.5)

$$\iint G(p, q) d\mu(p) d\mu(q) \leq \iint \log \frac{1}{|z_0(p) - z_0(q)|} d\mu(p) d\mu(q) + C_1$$

$$= \iint \log \frac{1}{|z - w|} d\omega(z) d\omega(w) + C_1$$

$$\leq -\log \delta + \log 4 + C_1. \quad (6.7)$$

Because of (3.14) there is $C_2$ such that

$$\varphi(p) \leq r_0 \log |z_0(p)| + C_2, \quad \text{for } p \in U_{\delta_0}(p_0).$$
Since $|z_0(p)| \leq 2\delta$ for $z \in \text{supp}(\mu) \subset \overline{U_{2\delta}(p_0)}$, we thus find

$$\int \varphi(p) d\mu(p) \leq r_0 \log(2\delta) + C_2. \quad (6.8)$$

Combining (6.7) and (6.8) we obtain

$$E_{\varphi}[\mu] \leq (r_0 - 1) \log \delta + (r_0 + 2) \log 2 + C_1 + C_2.$$ 

Since $\mu$ is a probability measure on $F_0 \subset F$, we have $E_{\varphi}(F) \leq E_{\varphi}[\mu]$ and (6.2) follows.

(b) The proof of part (b) is similar. We take $0 < \delta < \delta/2$ and we assume that $F \in \mathcal{F}$ intersects $\partial U_\delta(p_\infty)$. Let $\tilde{F}$ be a connected component of $F$ that intersects $\partial U_\delta(p_\infty)$.

Similar to (6.4) we now find a sub-continuum $F_0 \subset \tilde{F} \cap \overline{U_{2\delta}(p_\infty)}$ with

$$\text{diam}(z_\infty(F_0)) \geq \delta.$$ 

Let $\omega$ be the equilibrium measure of $z_\infty(F_0)$, and $\mu$ its pullback under the mapping $p \mapsto z_\infty(p)$. Then (6.5) again holds, and from (2.6) we find for some constant $C_1$,

$$\int \int G(p, q) d\mu(p) d\mu(q) \leq \int \int \log \frac{1}{|z_\infty(p)^{-1} - z_\infty(q)^{-1}|} d\mu(p) d\mu(q) + C_1$$

$$= \int \int \log \frac{1}{|z^{-1} - w^{-1}|} d\omega(z) d\omega(w) + C_1$$

$$= \int \int \log \frac{|z|}{|z - w|} d\omega(z) d\omega(w) + C_1$$

$$\leq -\log \frac{\delta}{4} + 2 \int \log |z| d\omega(z) + C_1. \quad (6.9)$$

Since $|z| \leq 2\delta$ for $z \in \text{supp}(\omega)$ we obtain

$$\int \int G(p, q) d\mu(p) d\mu(q) \leq -\log \delta + \log 4 + 2 \log(2\delta) + C_1$$

$$= \log \delta + 4 \log 2 + C_1, \quad (6.9)$$

which is the analogue of (6.7), but note the different sign with $\log \delta$.

Because of (3.15) there is $C_2$ such that $\varphi(p) \leq r_\infty \log |z_\infty(p)| + C_2$ for $p \in U_{\delta_0}(p_\infty)$, which gives us the analogue of (6.8)

$$\int \varphi(p) d\mu(p) \leq r_\infty \log(2\delta) + C_2 \quad (6.10)$$

since $\mu$ is supported on $U_{2\delta}(p_\infty) \subset U_{\delta_0}(p_\infty)$ and $z_\infty(p) \leq 2\delta$ for $p \in \text{supp}(\mu)$.

Adding (6.9) and (6.10) we find

$$E_{\varphi}[\mu] \leq (1 + r_\infty) \log \delta + (r_\infty + 4) \log 2 + C_1 + C_2.$$ 

Since $\mu$ is a probability measure on $F$ we have $E_{\varphi}(F) \leq E_{\varphi}[\mu]$ and (6.3) follows. □
6.3. The max–min problem is well-posed. If \( r_0 > 1 \) then it follows from (6.2) that \( E_\varphi(F) \) is small (i.e., very negative) in case \( F \) comes very close to \( p_0 \). If \( r_\infty > -1 \) then similarly \( E_\varphi(F) \) is small if \( F \) comes close to \( p_\infty \). This allows us to prove that the max-min energy problem is well-posed.

**Lemma 6.3.** Suppose \( r_0 > 1 \) and \( r_\infty > -1 \). Then the max-min energy problem is well-posed: that is, \( \sup_{F \in \mathcal{F}} E_\varphi(F) \) is finite.

**Proof.** Let \( m = \max(r_0 - 1, r_\infty + 1) > 0 \) and let \( C \) be as in Proposition 6.2. We fix \( \delta = \delta_0/4 \), so that by Proposition 6.2 (and the standing assumption \( \delta_0 \leq \delta_\infty \)),

\[
E_\varphi(F) \leq m \log \delta + C < \infty \tag{6.11}
\]

for all \( F \in \mathcal{F} \) that intersect \( \partial U_\delta(p_0) \) or \( \partial U_\delta(p_\infty) \).

Now if \( F \in \mathcal{F} \) does not intersect \( \partial U_\delta(p_0) \) or \( \partial U_\delta(p_\infty) \), any connected component \( \tilde{F} \) of \( F \) satisfies \( \tilde{F} \subset U_\delta(p_0) \), \( \tilde{F} \subset U_\delta(p_\infty) \) or \( \tilde{F} \subset X \setminus (U_\delta(p_0) \cup \partial U_\delta(p_\infty)) \). In the first case, we can take a \( \delta' < \delta \) for which \( \tilde{F} \cap \partial U_\delta(p_0) \neq \emptyset \) and apply Proposition 6.2 to conclude that

\[
E_\varphi(F) \leq (r_0 - 1) \log \delta' + C \leq m \log \delta + C.
\]

That is, the same upper bound as in (6.11) can be used. Similarly,

\[
E_\varphi(F) \leq (r_\infty + 1) \log \delta' + C \leq m \log \delta + C
\]

in the second case and the upper bound from (6.11) again works. It remains to find an upper bound for all \( F \in \mathcal{F} \) with \( F \subset X \setminus (U_\delta(p_0) \cup \partial U_\delta(p_\infty)) \).

Let \( \Gamma \) be a simple curve connecting \( p_0 \) and \( p_\infty \) without containing any further poles of \( dV \). Then for all \( \varepsilon > 0 \) small enough, the fattened set \( \Gamma_\varepsilon = \{ p \in X : d(p, \Gamma) \leq \varepsilon \} \) does not contain any other poles of \( dV \) either. We fix such a value of \( \varepsilon \) so that \( \varepsilon < \delta \) as well and define \( K = \Gamma_\varepsilon \setminus (U_\delta(p_0) \cup U_\delta(p_\infty)) \). The external field \( \varphi \) is continuous and hence bounded on the compact set \( K \), say \( |\varphi| \leq M \) on \( K \) for some \( M \in \mathbb{R} \).

As \( K \) is compact, we can cover it with a finite number of coordinate charts \((U_j, z_j)\), \( j = 1, \ldots, n \). By Lemma A.1, there are \( \eta > 0 \) and \( C > 0 \) such that for every subset \( K' \) of \( K \) with diameter at most \( \eta \), we have \( K' \subset U_j \) for some \( j \) and \( \text{diam}(K') \leq C \text{diam}(z_j(K')) \). Without loss of generality, we may assume \( \eta \leq \varepsilon \).

Now take \( F \subset \mathcal{F} \) with \( F \subset X \setminus (U_\delta(p_0) \cup \partial U_\delta(p_\infty)) \). Then \( K \cap F \) contains a continuum \( F_0 \) with diameter \( \eta \), namely a continuum that connects \( \partial \Gamma_\eta \) with \( \Gamma \). Hence there is a \( j = 1, \ldots, n \) such that \( F_0 \subset \tilde{U}_j \). Since \( \text{diam}(F_0) \leq C \text{diam}(z_j(F_0)) \), it follows that \( z_j(F_0) \) is a continuum in \( \mathbb{C} \) with (Euclidean) diameter at least \( \frac{\eta}{C} \).

Let \( \omega \) be the equilibrium measure of \( z_j(F_0) \) (in the usual logarithmic potential theory in the plane) without external field, and let \( \mu \) be the pullback of \( \omega \) by the local coordinate \( z_j \). By similar arguments as given in the proof of Proposition 6.2, we obtain

\[
\iint G(p, q) \mu(p) \mu(q) \leq -\log \left( \frac{\eta}{4C} \right) + C_j = -\log \eta + \log 4 + \log C + C_j
\]

for some constant \( C_j \) that only depends on \( U_j \). Taking \( \tilde{C} = \max_j C_j \) and using that \( |\varphi| \leq M \) on \( \text{supp}(\mu) \subset K \), we thus find

\[
E_\varphi[\mu] = \iint G(p, q) \mu(p) \mu(q) + \int \varphi(p) \mu(p)
\]

\[
\leq -\log \eta + \log 4 + \log C + C_j + M \leq -\log \eta + 2 \log 2 + \log C + \tilde{C} + M.
\]
Consequently,

\[ E_\varphi(F) \leq E_\varphi(F_0) \leq E_\varphi[\mu] \leq - \log \eta + 2 \log 2 + \log C + \tilde{C} + M. \]

As the upper bound is independent of \( F \), this concludes the proof. \( \Box \)

6.4. Continuity of the energy functional. Rakhmanov [50, Theorem 9.8] established the following fundamental result for the complex plane. His arguments extend to the higher genus case and this was already done by Chirka [21, Section 2.9] in the unweighted case. We formulate the continuity of the weighted energy functional for the case of external field \( \varphi = \text{Re } V \) that is of interest in the paper, but it holds for more general \( \varphi \).

**Proposition 6.4.** The weighted energy functional \( F \mapsto E_\varphi(F) \) is continuous on \( F \).

To prepare for the proof of Proposition 6.4 we need three lemmas. The first lemma provides an estimate for the usual Green's function \( G_\Omega(p, q) \) of \( \Omega \subset X \setminus F \) with \( F \in \mathcal{F}_\delta \). Recall that the Green's function \( (p, q) \mapsto G_\Omega(p, q) \) is non-negative for \( p, q \in X \times X \), zero whenever \( p \in F \) or \( q \in F \) and for a fixed \( q \in \Omega \), \( p \mapsto G_\Omega(p, q) \) is continuous on \( X \), harmonic on \( \Omega \), with

\[ G_\Omega(p, q) = - \log |z(p)| + O(1) \text{ as } p \to q \]

where \( p \mapsto z(p) \) is a local coordinate around \( q \). The following estimate for the Green's function is contained in [50, Lemma 9.9] for compacts in the complex plane. The extension to the higher genus case is mentioned in [21, p. 331].

**Lemma 6.5.** For every \( 0 < \delta < \delta_0 \) there is a constant \( C = C(\delta) > 0 \) such that for every \( F \in \mathcal{F}_\delta \) and every \( p \in X \setminus U_{\delta/2}(p_\infty) \), one has

\[ G_\Omega(p, p_\infty) \leq C \sqrt{d_H(p, F)}. \]

where \( \Omega = X \setminus F \).

**Proof.** The argument of [21, p. 331] applies since the diameter of each component of \( F \) is at least \( \eta > 0 \) by Lemma 6.1 (c). \( \Box \)

Of course, it is possible that \( F \in \mathcal{F}_\delta \) contains a pole of \( dV \) where \( \varphi \to +\infty \). The next lemma shows that for our purposes, we may replace \( \varphi \) by a continuous external field on \( \mathcal{F}_\delta \). For every \( m \in \mathbb{R} \), we denote \( \min(\varphi, m) \) by \( \varphi_m \), which is a continuous function on \( F \).

The proof is inspired by the proof of Theorem I.1.3(b) in [52]. However, in [52], the underlying set of the equilibrium problem is fixed, while here we have a family of sets. Therefore, we decided to give a full proof.

**Lemma 6.6.** For every \( \delta < \delta_0 \), there exists an \( m \in \mathbb{R} \) such that for every \( F \in \mathcal{F}_\delta \), the equilibrium measures of \( F \) in the external field \( \varphi \) and \( \varphi_m = \min(\varphi, m) \) are the same. Moreover, for the shared equilibrium measure \( \mu \), we have \( \text{supp } \mu \subset \{ \varphi \leq m \} \) and \( \varphi = \varphi_m \) on \( \text{supp } \mu \). Finally, \( E_\varphi \) and \( E_{\varphi_m} \) agree as functions on \( \mathcal{F}_\delta \).
Let \( \delta < \delta_0 \). Since the max-min energy problem is well-posed (see Lemma 6.3), the number \( M = \sup_{F \in \mathcal{F}_\delta} E_F \) is finite.

We write \( K = X \setminus (U_\delta(p_0) \cup U_\delta(p_\infty)) \) and note that every \( F \in \mathcal{F}_\delta \) is contained in \( K \). Because \( G \) is bounded away from \(-\infty\) away from \( p_\infty \) and the external field \( \varphi \) is only \(-\infty\) at \( p_0 \) and possibly at \( p_\infty \) by assumption, we have

\[
m_G = \min_{p,q \in K} G(p,q) > -\infty \quad \text{and} \quad m_\varphi = \min_{p \in K} \varphi(p) > -\infty.
\]

We take \( m \geq m_\varphi \) such that \( m_G + \frac{1}{2}m + \frac{1}{2}m_\varphi \geq M + 1 \) and write \( K_m = \{ p \in K : \varphi(p) \leq m \} \). Note that \( m \) is independent of \( F \in \mathcal{F}_\delta \).

Moreover, if \( (p,q) \notin K_m \times K_m \), then \( \frac{1}{2}(\varphi_m(p) + \varphi_m(q)) \geq \frac{1}{2}(m + m_\varphi) \). Hence

\[
G(p,q) + \frac{1}{2}(\varphi_m(p) + \varphi_m(q)) \geq m_G + \frac{1}{2}m + \frac{1}{2}m_\varphi \geq M + 1, \quad (p,q) \notin K_m \times K_m.
\]

We now turn to the proof of the main statement. Let \( F \in \mathcal{F}_\delta \) and write \( F_m = F \cap K_m = \{ p \in F : \varphi(p) \leq m \} \). Take \( \mu \) as the equilibrium measure of \( F \) in the external field \( \varphi_m \) and let \( \mu^F \) be the equilibrium measure of \( F \) in the external field \( \varphi \). Then \( E_{\varphi_m}[\mu] \leq E_{\varphi_m}[\mu^F] \leq E_{\varphi}[\mu^F] = E_{\varphi}(F) < M + 1 \), so that \( \mu(F_m) > 0 \) by (6.12).

Hence \( \tilde{\mu} = \mu|_{F_m}/\mu(F_m) \) is well-defined. Moreover, by (6.12),

\[
E_{\varphi_m}[\mu] = \left( \iint_{F_m \times F_m} + \iint_{(X \times X) \setminus (F_m \times F_m)} \right) [G(p,q) + (1/2)(\varphi_m(p) + \varphi_m(q))]d\mu(p)d\mu(q) \\
\geq \mu(F_m)^2 E_{\varphi_m}[\tilde{\mu}] + (M + 1)(1 - \mu(F_m)^2).
\]

Since \( E_{\varphi_m}[\mu] < M + 1 \), the above leads to the contradiction \( E_{\varphi_m}[\tilde{\mu}] < E_{\varphi_m}[\mu] \) unless \( \mu(F_m) = 1 \), in which case \( E_{\varphi_m}[\mu] \geq E_{\varphi_m}[\tilde{\mu}] \). But then \( \mu = \tilde{\mu} \) by the uniqueness of the equilibrium measure, so that \( \text{supp}(\mu) \subset F_m \).

Because \( \text{supp}(\mu) \subset F_m \), we also have \( \varphi = \varphi_m \) on \( \text{supp}(\mu) \). It directly follows that \( E_{\varphi_m}[\mu] = E_{\varphi}[\mu] \). Since \( \varphi_m \leq \varphi \) on \( F \), we have

\[
E_{\varphi}(F) \leq E_{\varphi}[\mu] = E_{\varphi_m}[\mu] = E_{\varphi_m}(F) \leq E_{\varphi}(F).
\]

Consequently, we have equality throughout the equation and the equilibrium measures of \( F \) in the external fields \( \varphi \) and \( \varphi_m \) are the same. The final statement follows directly, which concludes the proof. \( \square \)

The third lemma gives an estimate on the harmonic extension. Suppose \( \varphi \) is a continuous function on \( F \). Then there is a harmonic function \( \tilde{\varphi} \) on \( X \setminus F \) with \( \tilde{\varphi} = \varphi \) on \( F \). It is simply the solution of the Dirichlet problem. In our setting \( \varphi \) itself is harmonic, wherever it is finite.

Since \( \varphi \) is not necessarily continuous on \( F \), we replace \( \varphi \) by \( \varphi_m = \min(\varphi, m) \) from the lemma above. The following is an estimate on \( |\tilde{\varphi}_m - \varphi_m| \) near \( F \) that is due to Rakhmanov [50, Lemma 9.7] in the genus zero case (in a much more precise form, actually). In the higher genus case we can follow the same proof as all the arguments are local in nature.

**Lemma 6.7.** For every \( \delta < \delta_0, \varepsilon > 0 \) and \( m \in \mathbb{R} \), there is \( \eta > 0 \) such that for every \( F \in \mathcal{F}_\delta \) and every \( p \in X \) with \( d_H(p,F) < \eta \) one has

\[
|\tilde{\varphi}_m(p) - \varphi_m(p)| \leq \varepsilon,
\]

where \( \tilde{\varphi}_m \) is the harmonic extension of \( \varphi_m = \min(\varphi, m) \) to \( X \setminus F \).
Proof of Proposition 6.4. It is enough to show that $F \mapsto E_\psi(F)$ is continuous on $\mathcal{F}_\delta$ for every $0 < \delta < \delta_0$. We fix $\delta$ and use Lemma 6.6 to take an $m \in \mathbb{R}$ so that $E_\psi(F) = E_{\psi_m}(F)$ for all $F \in \mathcal{F}_\delta$.

Let $\varepsilon > 0$. Then by Lemmas 6.5 and 6.7 there is $\eta > 0$ such that for any $F \in \mathcal{F}_\delta$ and $p \in X$ with $d_H(p, F) < \eta$ we have $G_\Omega(p, p_\infty) \leq \frac{\varepsilon}{3}$ and $|\tilde{\phi}_m(p) - \phi_m(p)| \leq \frac{\varepsilon}{3}$ where $\Omega = X \setminus F$ and $\tilde{\phi}_m$ is the harmonic extension of $\phi_m$ from $F$ to $X \setminus F$.

Now take $F_1, F_2 \in \mathcal{F}_\delta$ such that $d_H(F_1, F_2) < \eta$. Let $\mu_1 = \mu_{F_1}$ be the equilibrium measure of $F_1$ in the external field $\phi$ and let $\mu_2$ be the balayage of $\mu_1$ to $F_2$. Then $\mu_2$ is a probability measure on $F_2$ such that

$$
\int \phi_m d\mu_2 = \int \tilde{\phi}_m d\mu_1,
$$

where $\tilde{\phi}_m$ is the harmonic extension of $\phi_m$ relative to $F_2$. Thus

$$
\left| \int \phi_m d\mu_2 - \int \phi_m d\mu_1 \right| = \left| \int (\tilde{\phi}_m - \phi_m) d\mu_1 \right| \leq \frac{\varepsilon}{3} \quad (6.13)
$$

since $|\tilde{\phi}_m - \phi_m| \leq \frac{\varepsilon}{3}$ on $F_1$.

Using the Green’s function $G_\Omega$ for the open set $\Omega = X \setminus F_2$, it follows that

$$
\int \int G(p, q) d\mu_2(p) d\mu_2(q)
$$

$$
= \int \int G(p, q) d\mu_1(p) d\mu_1(q) - \int \int G_\Omega(p, q) d\mu_1(p) d\mu_1(q)
$$

$$
+ 2 \int G_\Omega(p, p_\infty) d\mu_1(p)
$$

$$
\leq \int \int G(p, q) d\mu_1(p) d\mu_1(q) + 2 \int G_\Omega(p, p_\infty) d\mu_1(p),
$$

where the first identity is the analogue of [50, Lemma 9.6]. Hence

$$
\int \int G(p, q) d\mu_2(p) d\mu_2(q) \leq \int \int G(p, q) d\mu_1(p) d\mu_1(q) + \frac{2\varepsilon}{3} \quad (6.14)
$$

since $G_\Omega(p, p_\infty) \leq \frac{\varepsilon}{3}$ for $p \in F_1$. Consequently, by (6.14), (6.13) and Lemma 6.6,

$$
E_\psi(F_2) = E_{\psi_m}(F_2) \leq \int \int G(p, q) d\mu_2(p) d\mu_2(q) + \int \phi_m d\mu_2
$$

$$
\leq \int \int G(p, q) d\mu_1(p) d\mu_1(q) + \int \phi_m d\mu_1 + \varepsilon
$$

$$
= E_{\psi_m}(F_1) + \varepsilon = E_\psi(F_1) + \varepsilon.
$$

By symmetry, we also have $E_\psi(F_1) \leq E_\psi(F_2) + \varepsilon$ and hence $|E_\psi(F_1) - E_\psi(F_2)| \leq \varepsilon$. This shows that $F \mapsto E_\psi(F)$ is uniformly continuous on $\mathcal{F}_\delta$ and concludes the proof. \qed
6.5. Proof of Theorem 3.11. Since an extremal set stays away from \( p_0 \) and \( p_\infty \) in case both \( r_0 > 1 \) and \( r_\infty > -1 \), the continuity of the weighted energy functional allows us to prove Theorem 3.11.

**Proof of Theorem 3.11.** Let \( m = \max(1 - r_0, -1 + r_\infty) > 0 \) and let \( \delta_0, \delta_\infty \) and \( C \) be as in Proposition 6.2. Pick an arbitrary \( F_0 \in \mathcal{F} \), and take \( \delta \in (0, 1) \) with \( 2 \delta < \min(\delta_0, \delta_\infty) \) small enough such that \( F_0 \in \mathcal{F}_\delta \) and \( m \log \delta + C < E_\varphi(F_0) \).

Since \( \mathcal{F}_\delta \) is compact in the Hausdorff distance, and \( E_\varphi \) is continuous on \( \mathcal{F}_\delta \), there is \( F \in \mathcal{F}_\delta \) where \( E_\varphi \) takes its maximum on \( \mathcal{F}_\delta \). If \( F' \in \mathcal{F} \setminus \mathcal{F}_\delta \), then either \( F' \) intersects \( \partial U_{\delta'}(p_0) \) for some \( \delta' \leq \delta \), or \( F' \) intersects \( \partial U_{\delta'}(p_\infty) \) for some \( \delta' \leq \delta \). In the first case, we find by part (a) of Proposition 6.2

\[
E_\varphi(F') \leq (1 - r_0) \log \delta' + C \leq (1 - r_0) \log \delta + C \leq m \log \delta + C \leq E_\varphi(F).
\]

Similarly, in the second case we use part (b) and we also find \( E_\varphi(F') \leq E_\varphi(F) \).

The continuum \( F \) satisfies the conditions of Corollary 3.9 and the theorem follows.

\( \square \)

7. Proof of Theorem 3.13

7.1. Preparation for the proof. By Proposition 3.4 a critical measure \( \mu \) satisfies (3.24) for every \( C^1 \) vector field \( h \). As indicated in Sect. 3, we use the \((2, -1)\)-Cauchy kernel defined in Proposition 3.12 to extract relevant information from (3.24).

In the genus zero case one has (1.4) and \( h \) is a function (not a vector field). For the proof of Theorem 1.2 one takes

\[
h(s) = \frac{1}{z - s}
\]

with a fixed \( z \in \mathbb{C} \setminus \text{supp} \mu \). Then

\[
\frac{h(s) - h(t)}{s - t} = \frac{1}{(z - s)(z - t)}
\]

and

\[
\iint \frac{h(s) - h(t)}{s - t} d\mu(s)d\mu(t) = \left[ \int \frac{d\mu(s)}{z - s} \right]^2
\]

which is a crucial step in the proof of Theorem 1.2, see e.g. [45, Proof of Lemma 5.1] or [44, Proof of Proposition 3.7].

We assume that \( X \) has the form (3.19) as discussed in Sect. 3.5 with bounded real oval \( C_1 \). We have an analogue of (7.3) in case \( \mu \) is invariant under the involution (3.20) and its support does not intersect \( C_1 \).

**Proposition 7.1.** Suppose \( \mu \) is a compactly supported measure on \( X \setminus ((p_\infty) \cup C_1) \) that is invariant under the involution \( \sigma \) from (3.20). Then there exists \( a \in C_1 \) such that

\[
\iint \left( C(p, q)C^{(2, -1)}(u, p; a) + C(q, p)C^{(2, -1)}(u, q; a) - C(u, p)C(u, q) \right) d\mu(p)d\mu(q) = 0, \quad u \in X.
\]

(7.4)
Proof. We compare the expressions

\[ C(p, q)C^{(2,-1)}(u, p; a) + C(q, p)C^{(2,-1)}(u, q; a) \quad (7.5) \]

and

\[ C(u, p)C(u, q). \quad (7.6) \]

Both (7.5) and (7.6) are, for fixed \( p, q, \) and \( a, \) meromorphic quadratic differentials in \( u, \) with simple poles in \( u = p, u = q \) and a double pole at \( u = p_\infty. \) The residue at \( u = p \) is \( C(p, q) \) for both expressions, and the residue at \( u = q \) is \( C(q, p), \) again the same for both of them. These two poles disappear if we take the difference, and the only pole is the double pole at \( u = p_\infty. \) Then the left-hand side of (7.4) is also a meromorphic quadratic differential in \( u \) with a possible double pole at \( u = p_\infty \) only.

Suppose \( a \in X \setminus \{ p_\infty \cup \text{supp}(\mu) \} \) is a zero of the meromorphic differential \( \int C(u, q)d\mu(q). \) Then it is a double zero of \( \int \int C(u, p)C(u, q)d\mu(p)d\mu(q) = \int C(u, p)d\mu(p) \) and since both \( C^{(2, -1)}(u, p; a) \) and \( C^{(2, -1)}(u, q; a) \) have a double zero at \( u = a \) as well, see Proposition 3.12, we find that (7.4) has a double zero at \( u = a. \) It has at most a double pole at \( u = p_\infty, \) and there are no other poles and zeros (since \( X \) has genus one), unless it vanishes identically. The only non-zero meromorphic quadratic differentials on \( X \) that have double zeros and a double pole at \( p_\infty \) are \( z \frac{dz}{w^2}, (z - z_1) \frac{dz}{w^2}, \)

\( (z - z_2) \frac{dz}{w^2}, \) and their scalar multiples. Thus if \( a \not\in \{ p_0, p_1, p_2 \} \) where \( p_0 = (0, 0), \)

\( p_1 = (0, z_1), p_2 = (0, z_2) \) then (7.4) holds.

We now show that the differential \( \int C(u, q)d\mu(q) \) has exactly two zeros on \( C_1: \) if at least one of them is not at the branch point, then the proof follows by the above discussion; we then will consider the special case where both zeros are at a branch point. Now, let \( \mu \) be as in the statement of the proposition. Then

\[ G^\mu(u) := \int G(u, q)d\mu(q) \]

is real analytic on the cycle \( C_1, \) since the support of \( \mu \) is disjoint from \( C_1. \) Since \( \mu \) is invariant under \( \sigma, \) we also have \( G^\mu(\sigma(u)) = G^\mu(u) \) which implies that \( \frac{\partial}{\partial x} G^\mu(u) = 0 \) for \( u \in C_1, \) if we use \( z = x + iy \) as the local coordinate at a point \( u = (w, z) \) on \( C_1. \) By compactness, \( G^\mu \) attains a maximum and a minimum on \( C_1. \) If an extremum is attained at \( a \in C_1, \) then we also have \( \frac{\partial}{\partial x} G^\mu(u) = 0 \) at \( u = a. \) Hence by (3.2) we have that \( a \) is a zero of

\[ \int C(u, q)d\mu(q) = - (\partial_x - i\partial_y) \int G(u, q)d\mu(q). \]

Thus, by what we already proved, if \( G^\mu \) attains an extremum on \( C_1 \) at a point \( a \not\in \{ p_1, p_2 \} \) then (7.4) holds.

Next, we consider the special case \( \mu = \delta_q \) with \( q \in C_2 \setminus \{ p_0, p_\infty \} \) on the unbounded cycle. On the complex torus the extrema of \( G(u, q) \) are attained at the zeros of

\[ u \mapsto \frac{\theta_1'(u - v)}{\theta_1(u - v)} - \frac{\theta_1'(u)}{\theta_1(u)} \quad (7.7) \]

where \( v \) is the image of \( q \) under the Abel map, see (3.3) with \( \text{Im} \ v = 0. \) The elliptic function (7.7) has two simple poles at \( u = 0 \) and \( u = v. \) By Abel’s theorem the two
zeros add up to \( v \) (modulo \( \Lambda = \mathbb{Z} + \tau \mathbb{Z} \)). Since \( v \neq \frac{1}{2} \), (since \( v = \frac{1}{2} \) corresponds to \( q = p_0 \) in \( X \)), the two zeros cannot be \( \frac{1}{2} \tau \) and \( \frac{1}{2} + \frac{1}{2} \tau \). On \( X \) this means that \( C(u, q) \) does not vanish at both \( p_1 \) and \( p_2 \).

Finally, we consider the case that \( G^\mu \) attains its extrema on \( C_1 \) only at the branch points \( p_1 \) and \( p_2 \), so that \( \int C(u, p)d\mu(p) \) vanishes at both \( p_1 \) and \( p_2 \). Pick \( q \) on the unbounded cycle, \( q \neq p_0 \), \( q \neq p_\infty \) and consider \( \mu_t = \mu + t\delta_q \). Since \( C(u, q) \) does not vanish at both \( p_1 \) and \( p_2 \), we find that for \( t > 0 \),

\[
\int C(u, p)d\mu_t(p) = \int C(u, p)d\mu(p) + tC(u, q)
\]

is not zero at both \( p_1 \), \( p_2 \), and therefore it will have a zero somewhere else on the cycle. The zero depends on \( t > 0 \), say \( a_t \in C_1 \), and the identity (7.4) holds for \( \mu_t \) and \( a_t \).

Letting \( t \to 0^+ \) and by using a continuity and compactness argument, we find that (7.4) holds for \( t = 0 \) as well, where \( a \in C_1 \) is any limit point of \( (a_t)_{t > 0} \) as \( t \to 0^+ \).

If \( \int |C(u, q)|d\mu(q) < \infty \), which is the case for \( u \text{-a.e. on } X \), then (7.4) can be rewritten to

\[
\int \left( C(p, q)C^{(2, -1)}(u, p; a) + C(q, p)C^{(2, -1)}(u, q; a) \right) d\mu(p)d\mu(q) = \left[ \int C(u, q)d\mu(q) \right]^2.
\]

(7.8)

The identity (7.8) can be viewed as a genus one analogue of (7.3). Note however that we do not have an analogue of the divided difference identity (7.2).

7.2. Proof of Theorem 3.13.

Proof. Suppose \( \mu \) is a critical measure in the external field \( \varphi = \Re V \) that is invariant under the involution \( \sigma \). Suppose \( \text{supp}(\mu) \subset X \setminus \{ p_\infty \} \cup C_1 \). Let \( a \in C_1 \) be as in Proposition 7.1.

We take \( h(p) = C^{(2, -1)}(u, p; a) \) which is a \( C^1 \) vector field with a double pole at \( p = a \) and a simple pole at \( p = u \). Thus \( h \) is a \( C^1 \) vector field on \( \text{supp}(\mu) \) if \( u \notin \text{supp}(\mu) \), as it is already assumed that \( a \notin \text{supp}(\mu) \). Thus by Proposition 3.4 (b) we have \( D_{V, h}(\mu) = 0 \) provided that \( u \notin \text{supp}(\mu) \).

With an approximation argument as in [44, Lemma 3.5] we can extend the equality \( D_{V, h}(\mu) = 0 \) from \( u \notin \text{supp}(\mu) \) to any \( u \in X \setminus \{ p_\infty \} \) for which

\[
\int \frac{d\mu(q)}{d(u, q)} < +\infty.
\]

which holds for a.e. \( u \) on \( X \).

From (3.7) we thus have

\[
\int \left( C(p, q)C^{(2, -1)}(u, p; a) + C(q, p)C^{(2, -1)}(u, q; a) \right) d\mu(p)d\mu(q) = \int C^{(2, -1)}(u, q; a)dV(q)d\mu(q), \quad u \text{ - a.e. on } X.
\]

(7.9)
Using (7.4) we get from (7.9) that
\[ \left[ \int C(u, q)d\mu(q) \right]^2 = \int C^{(2,-1)}(u, q; a)dV(q)d\mu(q), \quad u - \text{a.e. on } X, \]
which is easily seen to be (3.25) with \( \omega \) given by (3.26).

Note that \( \omega \) is indeed a meromorphic quadratic differential on \( X \), since the pole at \( u = q \) cancels out in the difference
\[ C(u, q)dV(u) - C^{(2,-1)}(u, q; a)dV(q) \]
and therefore the integral transform in (3.25) indeed extends analytically across the support \( \Sigma_1 \) of the measure \( \mu \).

For the proof of (b) we can follow the proof of Proposition 3.8 in [44, Proposition 3.8], since all arguments are local in nature.

From (3.25) and working in a local coordinate around \( u \in \Sigma \), we have
\[ \left( \int C(u, q)d\mu(q) - \frac{dV(u)}{2} \right) = \pm Q(u)^{1/2}du \]
where \( \pm \) denote the boundary limits on \( \Sigma \). Thus
\[ \left( \int C(u, q)d\mu(q) - \frac{dV(u)}{2} \right)_+ = -\left( \int C(u, q)d\mu(q) - \frac{dV(u)}{2} \right)_- \]
or put otherwise
\[ \left( \int C(u, q)d\mu(q) \right)_+ + \left( \int C(u, q)d\mu(q) \right)_- - dV(u) = 0 \quad \text{on } \Sigma \]
as \( dV \) does not have a jump on \( \Sigma \). If \( g(u) = \int G(u, p)d\mu(p) \), then
\[ g_+(u) + g_-(u) + V(u) = \text{const} \quad \text{on } D \cap \Sigma. \]
Taking the real part we obtain part (c). Taking the imaginary part and using the Cauchy–Riemann equations, we obtain part (d). \( \square \)

Funding M.B. is supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC) grant RGPIN-2016-06660. A.G. is supported by long term structural funding-Methusalem grant of the Flemish Government. A.B.J.K. is supported by long term structural funding-Methusalem grant of the Flemish Government, and by FWO Flanders projects EOS 30889451 and G.0910.20.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
A Existence and Symmetry of the Bipolar Green’s Function

Let \( X \) be a Riemann surface and let \( p_\infty \) be a distinguished point at infinity. Our goal is to prove Proposition 2.1 and in particular part (d) that gives us the symmetry \( G(p, q) = G(q, p) \) of the bipolar Green’s function. For this proof we rely on Riemannian geometry where the existence of a symmetric Green’s function for the Laplacian is known.

A.1 Riemann surfaces as Riemannian manifolds. We first recall some relevant notions from Riemannian geometry. The main reference in this section is [38]. As shown in Lemma 2.3.3 in [38], \( X \) admits a conformal Riemannian metric \( \rho \), which is given in the local coordinate \((U, z)\) by

\[
\rho_U(z)^2 dz d\bar{z} , \quad \rho_U(z) > 0,
\]

where \( \rho_U \) is smooth and which transforms correctly under a holomorphic change of local coordinates. This turns \( X \) into a Riemannian manifold. The Riemannian metric allows for the definition of the length of a curve \( \gamma \subset U \) and area of a measurable set \( B \subset U \), namely

\[
\ell(\gamma) := \int_\gamma \rho_U(z)|dz| , \quad \text{area}(B) := \frac{i}{2} \int_B \rho_U(z)^2 dz d\bar{z};
\]

see also [38, p. 21]. The length of general curves is computed by splitting the curve into a finite number of pieces, each of which is contained in a single coordinate chart; a similar method works for the area. Note that area \( (X) \) is finite as \( X \) is compact.

The Riemannian metric \( \rho \) globally defines an area form (i.e., a nowhere-vanishing 2 form) \( dA \) (the dependence on \( \rho \) is suppressed in the notation) in the local coordinate \((U, z)\) by

\[
\frac{i}{2} \rho_U(z)^2 dz d\bar{z}.
\]

The area form \( dA \) will be used to integrate functions.

The distance \( d : X \times X \to [0, +\infty) \) between two points \( p \) and \( q \) can then be defined as

\[
d(p, q) := \inf \{ \ell(\gamma) \mid \gamma : [0, 1] \to X \text{ is a curve with } \gamma(0) = p \text{ and } \gamma(1) = q \}.
\]

(A.1)

The metric topology on \( X \) defined by \( d \) coincides with the original topology on \( X \). This is because in a coordinate chart \((U, z)\), the induced distance \( d(z(p), z(q)) \) is equivalent to the Euclidean distance in \( z(U) \) in the sense that for a fixed compact set \( K \subset U \),

\[
c|z(p) - z(q)| \leq d(z(p), z(q)) \leq C|z(p) - z(q)|, \quad p, q \in K,
\]

(A.2)

where \( c > 0 \) and \( C > 0 \) are the minimum and maximum of \( \rho_U \) on \( K \) respectively. See also [1, Theorem 1.18].

Lemma A.1. Let \( K \subset X \) be compact and let \( \{U_j\}_{j=1}^n \) be a finite open cover of \( K \) with coordinate charts \((U_j, z_j)\). Then there exists an \( \eta > 0 \) and a \( C > 0 \) such that for every subset \( K' \) of \( K \) with diameter at most \( \eta \), \( K' \subset U_j \) for some \( j \) and

\[
diam(K') \leq C \, diam(z_j(K')).
\]
Note that we have the diameter with respect to $d$ (see (A.1)) on the left and the standard Euclidean diameter on the right.

**Proof of Lemma A.1.** Let $K \subset X$ be compact and let $\{U_j\}_{j=1}^n$ be a finite open cover of $K$ with coordinate charts $(U_j, z_j)$ be given. By the properties of the metric topology, we can find open sets $\tilde{U}_j$ that are compactly contained in $U_j$ (that is, the closure of $\tilde{U}_j$ is compact and contained in $U_j$) such that $K \subset \bigcup_{j=1}^n \tilde{U}_j$. Hence by (A.2), for every $j = 1, \ldots, n$, there are numbers $C_j > 0$ such that

$$d(p, q) \leq C_j |z_j(p) - z_j(q)|, \quad p, q \in \tilde{U}_j.$$  

Let $\eta > 0$ be a Lebesgue number for the open cover $\{\tilde{U}_j\}_{j=1}^n$ of $K$ (so every subset of $K$ with diameter at most $\eta$ is contained in some $\tilde{U}_j$) and take $C = \max_j C_j > 0$. Suppose that $K' \subset K$ has diameter $\text{diam}(K') \leq \eta$. Then $K'$ lies in some $\tilde{U}_j$. Moreover,

$$\text{diam}(K') = \sup_{p, q \in K'} d(p, q) \leq C_j \sup_{p, q \in K'} |z_j(p) - z_j(q)| \leq C \sup_{p, q \in K'} |z_j(p) - z_j(q)| = C \text{diam}(z_j(K')),$$

which concludes the proof. \(\Box\)

The Riemannian metric $\rho$ moreover defines the Laplace-Beltrami operator $\Delta$ on functions $f \in C^2(X)$ by locally setting

$$\Delta f = \frac{4}{\rho_U(z)^2} \frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}} f,$$

see [38, Definition 2.3.3] and also [20, Section 1.1]. Note that $\Delta f$ is a function on $X$. Moreover,

$$\Delta f = -\star d \star df$$

where $\star$ denotes the Hodge star operator on $k$-forms (see [38, Section 5.2]). The 2-form $d \star df$ is independent of $\rho$ (and is sometimes taken as a definition for the Laplacian, see e.g. [28]) and hence $\Delta$ only depends on $\rho$ through the application of the Hodge star operator on $d \star df$.

**A.2 Proof of Proposition 2.1.** Because $X$ can be turned into a compact Riemannian manifold, it carries a Green’s function of the Laplacian [1, Theorem 4.13] (see also [4, Theorem 2.1]). This is a real-valued function $\tilde{G}$ defined on $X \times X$ minus the diagonal that is smooth, symmetric

$$\tilde{G}(p, q) = \tilde{G}(q, p) \quad \text{for } p \neq q, \quad \text{(A.3)}$$

and satisfies the distributional identity

$$\Delta_p \tilde{G}(p, q) = \delta_q(p) - \text{area}(X)^{-1}, \quad \text{(A.4)}$$

that is,

$$\int \tilde{G}(p, q) \Delta f(p) dA(p) = f(q) - \text{area}(X)^{-1} \int f dA$$
for all $C^2$ functions $f$. Moreover, (A.3) and (A.4) define $\tilde{G}$ uniquely up to an additive constant.

Furthermore, it has the following local behavior: if $z$ is a local coordinate around a point $p_0 \in X$, then

$$\tilde{G}(p, q) = -\frac{1}{2\pi} \log |z(p) - z(q)| + O(1)$$

(A.5)

uniformly for $p$ and $q$ in a neighborhood of $p_0$, as follows e.g. from the proof of Theorem 4.13(c) in [1] combined with (A.2).

It should be noted that $\tilde{G}$ does not have any special behavior at $p_\infty$. From $\tilde{G}$ we obtain the bipolar Green’s function with pole at $p_\infty$ as follows.

**Proposition A.2.** The function defined by

$$G(p, q) = 2\pi[\tilde{G}(p, q) - \tilde{G}(p, p_\infty) - \tilde{G}(q, p_\infty)]$$

(A.6)

satisfies the properties stated in Proposition 2.1.

**Proof.** Parts (b) and (c) of Proposition 2.1 follow directly from (A.5) and (A.6) and part (d) follows from (A.6) and the symmetry (A.3) of $\tilde{G}$. Hence it remains to check part (a).

Fix a $q \in X \setminus \{p_\infty\}$. The function $p \mapsto G(p, q)$ is clearly real-valued on $X \setminus \{p_\infty, q\}$. Moreover, it follows from (A.4) and (A.6) that

$$\Delta_p G(p, q) = 2\pi(\delta_q(p) - \delta_{p_\infty}(p))$$

in a distributional sense. The right-hand side is zero for $p \notin \{p_\infty, q\}$, hence $p \mapsto G(p, q)$ is weakly harmonic on $X \setminus \{p_\infty, q\}$. By Weyl’s lemma (see e.g. [38, Theorem 3.4.2]), the function $p \mapsto G(p, q)$ is then harmonic on $X \setminus \{p_\infty, q\}$. This concludes the proof. \(\square\)

**References**

1. Aubin, T.: Some Nonlinear Problems in Riemannian Geometry. Springer-Verlag, Berlin (1998)
2. Barhoumi, A., Bleher, P., Deaño, A., Yattselev, M.: Investigation of the two-cut phase region in the complex cubic ensemble of random matrices, arXiv:2201.12871
3. Behnke, H., Stein, K.: Entwicklung analytischer Funktionen auf Riemannschen Flächen. Math. Ann. **120**, 430–461 (1949)
4. Beltrán, C., Corral, N., Criado del Rey, J.G.: Discrete and continuous green energy on compact manifolds. J. Approx. Theory **237**, 160–185 (2019)
5. Berggren, T.: Domino tilings of the Aztec diamond with doubly periodic weightings. Ann. Probab. **49**, 1965–2011 (2021)
6. Berggren, T., Borodin, A.: Geometry of the doubly periodic Aztec dimer model, arXiv:2306.07482
7. Berggren, T., Duits, M.: Correlation functions for determinantal processes defined by infinite block Toeplitz minors. Adv. Math. **356**, 106766 (2019)
8. Bertola, M.: Padé approximation on Riemann surfaces and KP tau functions. Anal. Math. Phys. **11**, 149 (2021)
9. Bertola, M.: Abelianization of matrix orthogonal polynomials, arXiv:2107.12998
10. Bertola, M.: Nonlinear steepest descent approach to orthogonality on elliptic curves. J. Approx. Theory **276**, 105717 (2022)
11. Bertola, M., Bleher, P., Gharakhloo, R., McLaughlin, K.T-R., Tovbis, A.: Openness of regular regimes of complex random matrix models, arXiv:2203.11348
12. Bleher, P., Gharakhloo, R., McLaughlin, K.T-R.: Phase diagram and topological expansion in the complex random matrix model, arXiv:2112.09412
13. Borodin, A., Duits, M.: Biased $2 \times 2$ periodic Aztec diamond and an elliptic curve, arXiv:2203.11885
14. Cassatella-Contra, G.A., Mañas, M.: Riemann-Hilbert problems, matrix orthogonal polynomials and
discrete matrix equations with singularity confinement. Stud. Appl. Math. 128, 252–274 (2012)
15. Charlier, C.: Doubly periodic lozenge tilings of a hexagon and matrix valued orthogonal polynomials.
Stud. Appl. Math. 146(1), 3–80 (2021)
16. Charlier, C.: Matrix orthogonality in the plane versus scalar orthogonality on a Riemann surface, Trans.
Math. Appl. 5 (2021)
17. Charlier, C., Duits, M., Kuijlaars, A.B.J., Lenells, J.: A periodic hexagon tiling model and non-Hermitian
orthogonal polynomials. Comm. Math. Phys. 378, 401–466 (2020)
18. Chhita, S., Duits, M.: On the domino shuffle and matrix refactorizations, preprint arXiv:2208.01344
19. Chhita, S., Johansson, K.: Domino statistics of the two-periodic Aztec diamond. Adv. Math. 294, 37–149
(2016)
20. Chirka, E.M.: Potentials on a compact Riemann surface. Proc. Steklov Inst. Math. 301, 272–303 (2018)
21. Chirka, E.M.: Equilibrium measures on a compact Riemann surface. Proc. Steklov Inst. Math. 306,
296–334 (2019)
22. Chirka, E.M.: Capacities on a compact Riemann surface. Proc. Steklov Inst. Math. 311, 1075–1131 (2021)
23. Deift, P.: Orthogonal Polynomials and Random Matrices: a Riemann–Hilbert Approach, Courant Lecture
Notes in Mathematics 3. Amer. Math. Soc, Providence, RI (1999)
24. Deift, P., Kriecherbauer, T., McLaughlin, K.T.-R., Venakides, S., Zhou, X.: Uniform asymptotics for
polynomials orthogonal with respect to varying exponential weights and applications to universality
questions in random matrix theory. Comm. Pure Appl. Math. 52, 1335–1425 (1999)
25. Delvaux, S.: Average characteristic polynomials for multiple orthogonal polynomial ensembles. J. Ap-
prox. Theory 162, 1033–1067 (2010)
26. Duits, M., Kuijlaars, A.B.J.: The two-periodic Aztec diamond and matrix valued orthogonal polyno-
mials. J. Eur. Math. Soc. 23, 1075–1131 (2021)
27. Eynard, B., Mehta, M.L.: Matrices coupled in a chain I. Eigenvalue correlations. J. Phys. A 31, 4449–4456
(1998)
28. Farkas, H.M., Kra, I.: Riemann Surfaces. Springer-Verlag, New York-Berlin (1980)
29. Fasondini, M., Olver, S., Xu, Y.: Orthogonal polynomials on planar cubic curves, arXiv:2011.10884, to
appear in Found. Comp. Math
30. Fay, J.D.: Theta Functions on Riemann Surfaces. Lecture Notes in Mathematics, vol. 352. Springer-Verlag,
Berlin-New York (1973)
31. Fokas, A., Its, A., Kitaev, A.: The isomonodromy approach to matrix models in 2D quantum gravity.
Comm. Math. Phys. 147, 395–430 (1992)
32. Gamelin, T.W.: Complex Analysis. Springer-Verlag, New York (2001)
33. Gonchar, A.A., Rakhmanov, E.A.: Equilibrium distributions and the rate of rational approximation of
analytic functions, Mat. Sb. (N.S.) 134(176) (1987), no. 3, 306–352. English translation in Math. USSR-
Sb. 62 (1989), no. 2, 305–348
34. Groots, A., Kuijlaars, A.B.J.: Matrix-valued orthogonal polynomials related to hexagon tilings. J. Approx.
Theory 270, 105619 (2021)
35. Grünbaum, F.A., de la Iglesia, M.D., Martínez-Finkelshtein, A.: Properties of matrix orthogonal poly-
nomials via their Riemann-Hilbert characterization. SIGMA 7, 098 (2011)
36. Gusman, S.J., Rodin, J.L.: The kernel of an integral of Cauchy type on closed Riemann surfaces (Russian).
Sibirsk. Mat. Z. 3, 527–531 (1962)
37. Helms, L.L.: Potential Theory. Universitext, 2nd edn. Springer Verlag, London (2014)
38. Jost, J.: Compact Riemann Surfaces. Springer-Verlag, Berlin (2006)
39. Kamvissis, S., Rakhmanov, E.A.: Existence and regularity for an energy maximization problem in two
dimensions. J. Math. Phys. 46, 083505 (2005)
40. Kang, N-G., Makarov, N.G.: Calculus of conformal fields on a compact Riemann surface, arXiv:1708.07361
41. Kenyon, R.: Local statistics of lattice dimers. Ann. Inst. Henri Poincaré Probab. Stat. 33, 591–618 (1997)
42. Kenyon, R.: Lectures on dimers. In: Statistical Mechanics, IAS/Park City Math. Ser. 16, Amer. Math.
Soc., Providence, RI, pp. 191–230 (2009)
43. Kenyon, R., Okounkov, A., Sheffield, S.: Dimers and amoebae. Ann. Math. 163, 1019–1056 (2006)
44. Kuijlaars, A.B.J., Silva, G.L.F.: S-curves in polynomial external fields. J. Approx. Theory 191, 1–37
(2015)
45. Martínez-Finkelshtein, A., Rakhmanov, E.A.: Critical measures, quadratic differentials and weak limits
of zeros of Stieljes polynomials. Comm. Math. Phys. 302, 53–111 (2011)
46. Martínez-Finkelshtein, A., Rakhmanov, E.A.: Do orthogonal polynomials dream of symmetric curves?
Found. Comput. Math. 16, 1697–1736 (2016)
47. Martínez-Finkelshtein, A., Silva, G.L.F.: Critical measures for vector energy: global structure of traject-
ories of quadratic differentials. Adv. Math. 302, 1137–1232 (2016)
48. Olver, F.W., Lozier, D.W., Boisvert, R.F., Clark, C.W. (eds.): NIST Handbook of Mathematical Functions. Cambridge University Press, Cambridge (2010)
49. Petrov, L.: Asymptotics of random lozenge tilings via Gelfand-Tsetlin schemes. Probab. Theory Related Fields 160, 429–487 (2014)
50. Rakhmanov, E.A.: Orthogonal polynomials and S-curves, in: Recent advances in orthogonal polynomials, special functions, and their applications, Contemp. Math. 578, Amer. Math. Soc., Providence RI, pp. 195–239 (2012)
51. Ransford, T.: Potential Theory in the Complex Plane. Cambridge Univ. Press, Cambridge (1995)
52. Saff, E.B., Totik, V.: Logarithmic Potentials with External Fields. Springer-Verlag, Berlin (1997)
53. Simon, B.: Harmonic Analysis, A Comprehensive Course in Analysis, Part 3. Amer. Math. Soc, Providence RI (2015)
54. Skinner, B.: Logarithmic Potential Theory on Riemann Surfaces, Dissertation (Ph.D.), California Institute of Technology, https://thesis.library.caltech.edu/8915
55. Stahl, H.: Orthogonal polynomials with complex-valued weight function. I, II. Constr. Approx. 2(3), 225–240 (1986)
56. Stahl, H.: Orthogonal polynomials with respect to complex-valued measures, In: Orthogonal Polynomials and their Applications (Erice, 1990), volume 9 of IMACS Ann. Comput. Appl. Math., Baltzer, Basel, pp. 139–154 (1991)
57. Strebel, K.: Quadratic Differentials. Springer Verlag, Berlin (1984)
58. Zverovich, E.I.: Boundary value problems in the theory of analytic functions in Hölder classes on Riemann surfaces. Russ. Math. Surv. 26, 117–192 (1971)

Communicated by K. Johansson