The Information Geometry of Chaos

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In this Thesis, we propose a new theoretical information-geometric framework (IGAC, Information Geometrodynamical Approach to Chaos) suitable to characterize chaotic dynamical behavior of arbitrary complex systems. First, the problem being investigated is defined; its motivation and relevance are discussed. The basic tools of information physics and the relevant mathematical tools employed in this work are introduced. The basic aspects of Entropic Dynamics (ED) are reviewed. ED is an information-constrained dynamics developed by Ariel Caticha to investigate the possibility that laws of physics - either classical or quantum - may emerge as macroscopic manifestations of underlying microscopic statistical structures. ED is of primary importance in our IGAC. The notion of chaos in classical and quantum physics is introduced. Special focus is devoted to the conventional Riemannian geometrodynamical approach to chaos (Jacobi geometrodynamics) and to the Zurek-Paz quantum chaos criterion of linear entropy growth. After presenting this background material, we show that the ED formalism is not purely an abstract mathematical framework, but is indeed a general theoretical scheme from which conventional Newtonian dynamics is obtained as a special limiting case. The major elements of our IGAC and the novel notion of information geometrodynamical entropy (IGE) are introduced by studying two toy models. To illustrate the potential power of our IGAC, one application is presented. An information-geometric analogue of the Zurek-Paz quantum chaos criterion of linear entropy growth is suggested. Finally, concluding remarks emphasizing strengths and weak points of our approach are presented and possible further research directions are addressed. At this stage of its development, IGAC remains an ambitious unifying information-geometric theoretical construct for the study of chaotic dynamics with several unsolved problems. However, based on our recent findings, we believe it already provides an interesting, innovative and potentially powerful way to study and understand the very important and challenging problems of classical and quantum chaos.

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PREFACE

My research at the Joseph Henry Department of Physics, SUNY at Albany, has focused mainly on applications of geometric Clifford algebra to electrodynamics and 3+1 gravity and on applications of information physics to classical and quantum chaos. Thanks to my research, I became invited reviewer of several international scientific journals, among the others: AIP Conf. Proceedings in Mathematical and Statistical Physics, The Open Astronomy Journal, Journal of Electromagnetic Waves and Applications, Progress in Electromagnetics Research, Foundations of Physics and Mathematical Reviews (AMS).

During these years I worked under the precious supervision of Prof. Ariel Caticha. Moreover, I collaborated on several projects with my friends and colleagues Dr. Saleem Ali (SUNY at Albany), Prof. Salvatore Capozziello (University of Napoli, Italy), Dr. Christian Corda (University of Pisa, Italy), and Mr. Adom Giffin (SUNY at Albany). This thesis THE INFORMATION GEOMETRY OF CHAOS reports exclusively results of some of my research on applications of information physics to the study of chaos carried out between January 2004 and May 2008 at the Physics Department in Albany. Sections of my doctoral dissertation have been presented at International Conferences (Paris-France, Saratoga-USA, Erice-Italy) and they have appeared in a series of collaborative and/or single-authored conference proceedings and/or journal articles.
OVERVIEW

In this doctoral dissertation, I consider two important questions: First, are laws of physics practical rules to process information about the world using geometrical methods? Are laws of physics rules of inference? Second, since a unifying framework to describe chaotic dynamics in classical and quantum domains is missing, is it possible to construct a new information-geometric model, to develop new tools so that a unifying framework is provided or, at least, new insights and new understandings are given? After setting the scene of my thesis and after stating the problem and its motivations, I review the basic elements of the maximum relative entropy formalism (ME method) and recall the basics of Riemannian geometry with special focus to its application to probability theory (this is known as Information Geometry, IG). IG and ME are the fundamental tools that Prof. Ariel Caticha has used to build a form of information-constrained dynamics on statistical manifolds to investigate the possibility that Einstein’s general theory of gravity (or any classical or quantum theory of physics) may emerge as a macroscopic manifestation of an underlying microscopic statistical structure. This dynamics is known in the literature as Entropic Dynamics (ED). Therefore, since ED is an important element of this thesis, I review the key-points of such dynamics, emphasizing the most relevant points that I will use in my own information geometrodynamical approach to chaos (IGAC). Of course, before introducing my IGAC, I briefly review the basics of the conventional Riemannian geometrodynamical approach to chaos and discuss the notion of chaos in classical and quantum physics in general. After this background information, I start with my original contributions.

First, two chaotic entropic dynamical models are considered. The geometric structure of the statistical manifolds underlying these models is studied. It is found that in both cases, the resulting metric manifolds are negatively curved. Moreover, the geodesics on each manifold are described by hyperbolic trajectories. A detailed analysis based on the Jacobi-Levi-Civita equation for geodesic spread (JLC equation) is used to show that the hyperbolicity of the manifolds leads to chaotic exponential instability. A comparison between the two models leads to a relation among scalar curvature of the manifold ($R$), Jacobi field intensity ($J$) and information geometrodynamical entropy ($S$, $S$). I introduce the IGE entropy as a brand new measure of chaoticity. These three quantities, $R$, $J$, and $S$ are suggested as useful indicators of chaoticity. Indeed, in analogy to the Zurek-Paz quantum chaos criterion of linear entropy growth, a novel classical information-geometric criterion of linear IGE growth for chaotic dynamics on curved statistical manifolds is presented.

Second, in collaboration with Prof. Ariel Caticha, I show that the ED formalism is not purely an abstract mathematical framework; it is indeed a general theoretical scheme where conventional Newtonian dynamics is obtained
as a special limiting case. Newtonian dynamics is derived from prior information codified into an appropriate statistical model. The basic assumption is that there is an irreducible uncertainty in the location of particles so that the state of a particle is defined by a probability distribution. The corresponding configuration space is a statistical manifold the geometry of which is defined by the information metric. The trajectory follows from a principle of inference, the method of Maximum Entropy. No additional physical postulates such as an equation of motion, or an action principle, nor the concepts of momentum and of phase space, not even the notion of time, need to be postulated. The resulting entropic dynamics reproduces the Newtonian dynamics of any number of particles interacting among themselves and with external fields. Both the mass of the particles and their interactions are explained as a consequence of the underlying statistical manifold.

**Third**, I extend my study of chaotic systems (information geometrodynamical approach to chaos, IGAC) to an ED Gaussian model describing an arbitrary system of 3N degrees of freedom. It is shown that the hyperbolicity of a non-maximally symmetric 6N-dimensional statistical manifold $\mathcal{M}$ underlying the ED Gaussian model leads to linear information-geometrodynamical entropy growth and to exponential divergence of the Jacobi vector field intensity, quantum and classical features of chaos respectively. As a special physical application, the information geometrodynamical scheme is applied to investigate the chaotic properties of a set of $n$-uncoupled three-dimensional anisotropic inverted harmonic oscillators (IHOs) characterized by an Ohmic distributed frequency spectrum and I show that the asymptotic behavior of the information-geometrodynamical entropy is characterized by linear growth. Finally, the anisotropy of the statistical manifold underlying such physical system and its relationship with the spectrum of frequencies of the oscillators are studied.

Finally, I present concluding remarks emphasizing strengths and weak points of my approach and I address possible further research directions.
Chapter 1: Background and Motivation

I discuss the general setting of my thesis. I define the problem I am going to study, its motivation and its relevance. Moreover, I describe briefly the structure of the dissertation and, finally I state my original contributions

I. INTRODUCTION

In the orthodox scientific community, it is commonly accepted that laws of physics reflect laws of nature. In this Thesis we uphold a different line of thinking: laws of physics consist of rules designed for processing information about the world for the purpose of describing and, to a certain extent, understanding natural phenomena. The form of the laws of physics should reflect the form of rules for manipulating information.

This unorthodox point of view has important consequences for the laws of physics themselves: the identification of the relevant information-constraints of a particular phenomenon implies that the rules for processing information can take over, and thus, the laws of physics that govern the specific phenomenon should be determined. In principle, the field of applicability of the ideas explored in this Thesis is not limited to physics. This novel line of thinking may be useful to various scientific disciplines since all subject matters are investigated by applying the same universal methods and tools of reasoning.

One of our major line of research concerns the possibility that laws of physics can be derived from rules of inference. In the case of a positive answer, we expect that physics should use the same tools, the same kind of language that has been found useful for inference. We are well aware that most of our basic physics theories make use of the concepts of probability, of entropy, and of geometry.

We know that the evidence supporting the fact that laws of physics can be derived from rules of inference is already considerable. Indeed, most of the formal structure of statistical mechanics can already be derived from principles of inference [1]. Moreover, the derivation of quantum mechanics as information physics is already quite developed [2]. Finally, it seems plausible that even Einstein’s theory of gravity (GR, general relativity) might be derived from a more fundamental statistical geometrodynamics in analogy to the way in which thermodynamics can be derived from a microscopic statistical mechanics [3].

It is known that any attempt to build a unified theory of all forces is problematic: these problems arise from the difficulties of incorporating the classical theory of gravity with quantum theories of electromagnetic, weak and strong forces. Few situations, where gravitational and quantum phenomena coexist in a non-trivial way, can be
studied in some detail. It just happens that in these situations (for instance, Hawking’s black hole evaporation) thermodynamics plays an essential role. These considerations lead to believe that concepts such as entropy should play a key role in any successful theoretical construct attempting to unify all fundamental interactions in nature.

The relation between physics and nature is more complicated than has been usually assumed. An explicit admission of such a statement is represented by the recognition that statistical physics and quantum physics are theories of inference. This scientific awareness is leading to a gradual acceptance of the fact that the task of theoretical physics is that of putting some order in our understanding of natural phenomena, not to write the ultimate equations of the universe.

In this Thesis, we want to carry on this line of thought and explore the possibility that laws of physics become rules for the consistent manipulation of information: laws of physics work not because they are laws of nature but rather because they are laws of inference and they have been designed to work. Therefore, my first line of investigation will be the following:

First line of research I would like to push the possibility to extend the notion that laws of physics can be derived from rules of inference into new unexplored territories. I would like to construct new entropic dynamical models that reproduce recognizable laws of physics using merely the tools of inference (probability, information geometry and entropy).

However, I have to select the laws of physics that I want to reproduce.

It is known there is no unified characterization of chaos in classical and quantum dynamics. In the Riemannian and Finslerian (a Finsler metric is obtained from a Riemannian metric by relaxing the requirement that the metric be quadratic on each tangent space) geometrodynamical approach to chaos in classical Hamiltonian systems, an active field of research concerns the possibility of finding a rigorous relation among the sectional curvature, the Lyapunov exponents, and the Kolmogorov-Sinai dynamical entropy (i.e., the sum of positive Lyapunov exponents). The largest Lyapunov exponent characterizes the degree of chaoticity of a dynamical system and, if positive, it measures the mean instability rate of nearby trajectories averaged along a sufficiently long reference trajectory. Moreover, it is known that classical chaotic systems are distinguished by their exponential sensitivity to initial conditions and that the absence of this property in quantum systems has lead to a number of different criteria being proposed for quantum chaos. Exponential decay of fidelity, hypersensitivity to perturbation, and the Zurek-Paz quantum chaos criterion of linear von Neumann's entropy growth are some examples. These criteria accurately predict chaos in the classical limit, but it is not clear that they behave the same far from the classical realm.
I chose the second line of research below for the following three reasons: i) lack of a unifying understanding of chaotic phenomena in classical and quantum physics; ii) test the potential mathematical power of these information-geometric tools at my disposal; test the potential predicting power of entropic dynamical models.

**Second line of research** I would like to push the possibility to derive, explain and understand classical and quantum criteria of chaos from rules of inference. I would like to construct new entropic chaotic dynamical models that reproduce recognizable laws of mathematical-physics using merely the tools of inference (probability, information geometry and entropy).

II. PROBLEMS UNDER INVESTIGATION, THEIR RELEVANCE AND ORIGINAL CONTRIBUTIONS

First I review the basic elements of the maximum relative entropy formalism (ME method) and recall the basics of Riemannian geometry with special focus to its application to probability theory (this is known as Information Geometry, IG). IG and ME are the fundamental tools that Prof. Ariel Caticha has used to build a form of information-constrained dynamics on statistical manifolds to investigate the possibility that Einstein’s general theory of gravity (or any classical or quantum theory of physics) may emerge as a macroscopic manifestation of an underlying microscopic statistical structure. This dynamics is known in the literature as Entropic Dynamics (ED). Therefore, since ED is an important element of this thesis, I review the key-points of such dynamics, emphasizing the most relevant points that I use in my own information geometrodynamical approach to chaos (IGAC). Of course, before introducing my IGAC, I briefly review the basics of the conventional Riemannian geometrodynamics approach to chaos and discussed the notion of chaos in physics in general. After this long background information that is needed because of the originality and novelty of these topics, I begin with my original contributions.

First, two chaotic entropic dynamical models are considered. The geometric structure of the statistical manifolds underlying these models is studied. It is found that in both cases, the resulting metric manifolds are negatively curved. Moreover, the geodesics on each manifold are described by hyperbolic trajectories. A detailed analysis based on the Jacobi-Levi-Civita equation for geodesic spread (JLC equation) is used to show that the hyperbolicity of the manifolds leads to chaotic exponential instability. A comparison between the two models leads to a relation among scalar curvature of the manifold ($R$), Jacobi field intensity ($J$) and information geometrodynamical entropy ($S_M$). The IGE entropy is proposed as a brand new measure of chaoticity.

**First Contribution**[10–12]: I suggest that these three quantities, $R$, $J$, and $S_M$ are useful indicators of chaoticity.
for chaotic dynamical systems on curved statistical manifolds. Furthermore, I suggest a classical information-geometric criterion of linear information geometrodynamical entropy growth in analogy with the Zurek-Paz quantum chaos criterion.

Second, in collaboration with Prof. Ariel Caticha, I show that the ED formalism is not purely an abstract mathematical framework; it is indeed a general theoretical scheme where conventional Newtonian dynamics is obtained as a special limiting case.

Second Contribution 13: The reproduction of the Newtonian dynamics from first principles of probable inference and information geometric methods is another original contribution of my work.

Third, I extend my study of chaotic systems (information geometrodynamical approach to chaos, IGAC) to an ED Gaussian model describing an arbitrary system of $3N$ degrees of freedom. It is shown that the hyperbolicity of a non-maximally symmetric $6N$-dimensional statistical manifold $\mathcal{M}_S$ underlying the ED Gaussian model leads to linear information-geometrodynamical entropy growth and to exponential divergence of the Jacobi vector field intensity, quantum and classical features of chaos respectively. As a special physical application, the information geometrodynamical scheme is applied to investigate the chaotic properties of a set of $n$-uncoupled three-dimensional anisotropic inverted harmonic oscillators (IHOs) characterized by an Ohmic distributed frequency spectrum and I show that the asymptotic behavior of the information-geometrodynamical entropy is characterized by linear growth. Finally the anisotropy of the statistical manifold underlying such physical system and its relationship with the spectrum of frequencies of the oscillators are studied.

Third Contribution 14 I compute the asymptotic temporal behavior of the information geometrodynamical entropy of a set of $n$-uncoupled three-dimensional anisotropic inverted harmonic oscillators (IHOs) characterized by an Ohmic distributed frequency spectrum and I suggest the classical information-geometric analogue of the Zurek-Paz quantum chaos criterion in its classical reversible limit.

I am aware that several points in my IGAC need deeper understanding and analysis, however I hope that my work convincingly shows that:
**Point 1** Laws of physics are deeply geometrical because they are practical rules to process information about the world and geometry is the most natural tool to carry out that task. The notion that laws of physics are not laws of nature but rules of inference seems outrageous but cannot be simply dismissed. Indeed, it deserves serious attention and further research.

**Point 2** This is a novel and unorthodox research area and there are many risks and criticisms [15]. I believe the information geometrodynamical approach to chaos may be useful in providing a unifying criterion of chaos of both classical and quantum varieties, thus deserving further research and developments.
Chapter 2: Maximum entropy methods and information geometry

I review the basic information physics and mathematical tools employed in my work. First, I review the major aspects of the Maximum Entropy method (ME method), a unique general theory of inductive inference. Second, I recall some basic elements of conventional Riemannian differential geometry useful for the understanding of standard approaches to the geometrical study of chaos. Finally, I introduce the basics of Riemannian geometry applied to probability theory, namely Information Geometry (IG). IG and ME are the basic tools that I need to introduce in order to allow the reader to follow the description of the constrained information dynamics on curved statistical manifolds (entropic dynamics, ED).

III. INTRODUCTION

Inference is the process of drawing conclusions from available information. Information is whatever constraints rational beliefs. When the information available is sufficient to make unequivocal, unique assessments of truth we speak of making deductions: on the basis of this or that information we deduce that a certain proposition is true. The method of reasoning leading to deductive inferences is called logic. Situations where the available information is insufficient to reach such certainty lie outside the realm of logic. In these cases we speak of making a probable inference, and the method of reasoning is probability theory. An alternative name is "inductive inference". The word "induction" refers to the process of using limited information about a few special cases to draw conclusions about more general situations.

The main goal of inductive inference is to update from a prior probability distribution to a posterior distribution when new relevant information becomes available. Updating methods should be systematic and objective. The most important updating methods are the Bayesian updating method and the ME method. Jaynes' method of maximum entropy is a method to assign probabilities on the basis of partial testable information. Testable information is sufficient to make a prediction and predictions can be tested. MaxEnt arises as a rule to assign a probability distribution, however it can be extended to a full-fledged method for inductive inference. The extended method will henceforth be abbreviated as ME. In Jaynes's MaxEnt method, it is shown that statistical mechanics and thus thermodynamics are theories of inference. MaxEnt can be interpreted as a special case of ME when one updates from
a uniform prior using the Gibbs-Shannon entropy.

The nature of the information being processed dictates the choice between the Bayesian updating method and the ME method. Bayes’ theorem should be used to update our beliefs about the values of certain quantities \( \theta \) on the basis of information about the observed values of other quantities \( x \)- the data- and of the known relation between them- the conditional distribution \( p(x|\theta) \). If \( p(\theta) \) are the prior beliefs, the updated or posterior distribution is given by \( p(\theta|x) \propto p(\theta)p(x|\theta) \). The Bayesian method of updating is a consequence of the product rule for probabilities and therefore it is limited to situations where it makes sense to define the joint probability of \( x \) and \( \theta \), \( p(x, \theta) \). On the other hand, the ME method is designed for updating from a prior to a posterior probability distribution when the information to be processed takes the form of constraints on the family of acceptable posterior distributions. Although the terms ”prior” and ”posterior” are normally used only in the context of Bayes’ theorem, we will adopt the same terminology when using the ME method since we are concerned with the same goal of processing information to update from a prior to a posterior probability distribution. As a final remark, we point out that in general it is meaningless to use Bayes’ theorem to process information in the form of constraints, and conversely, it is meaningless to process data using ME. However, there are special cases where the same piece of information can be both interpreted as data and as constraint, In such cases, both methods can be used and it can be shown that they agree.

IV. WHAT IS THE MAXIMUM RELATIVE ENTROPY FORMALISM?

Consider a multidimensional discrete or continuous variable \( x \). Assume the prior probability distribution \( q(x) \) describes the uncertainty about \( x \). When new relevant information becomes available, our goal is to update from \( q(x) \) to a posterior probability distribution \( P(x) \). Information appears in the form of constraints and usually could be given in terms of expected values. The problem consists in selecting the proper \( p(x) \) among all those posterior probability distributions within the family defined by the available relevant constraints.

Skilling suggested that in order to select the posterior \( p(x) \) it seems reasonable to rank the candidate distributions in order of increasing preference. The ranking must be transitive: if distribution \( p_1 \) is preferred over distribution \( p_2 \), and \( p_2 \) is preferred over \( p_3 \), then \( p_1 \) is preferred over \( p_3 \). To each \( p(x) \) is assigned a real number \( S[p] \), which we will henceforth call entropy, in such a way that if \( p_1 \) is preferred over \( p_2 \), then \( S[p_1] > S[p_2] \). The probability distribution that maximizes the entropy \( S[p] \) will be the selected posterior distribution \( P(x) \). Therefore, it becomes evident to conclude that the Maximum Entropy method (ME) is a variational method involving entropies which are real numbers.
Moreover, to define the ranking scheme, a functional form of $S[p]$ must be chosen. Recall that the purpose of the ME method is to update from priors to posteriors and that the ranking scheme must depend on the particular prior $q$ and therefore the entropy $S$ must be a functional of both $p$ and $q$. Thus the entropy $S[p, q]$ produces a ranking of the distributions $p$ relative to the given prior $q$: $S[p, q]$ is the entropy of $p$ relative to $q$. Accordingly $S[p, q]$ is commonly called relative entropy. The modifier "relative" is redundant and will be dropped since all entropies are relative, even when relative to a uniform distribution. Moreover, since we deal with incomplete information the ME method cannot be deductive: the method must be inductive. Therefore, we may find useful to use those special cases where we know what the preferred distribution should be to eliminate those entropy functionals $S[p, q]$ that fail to provide the right update. In general, the known special cases are called the axioms of the theory. Since they define what makes one distribution preferable over another, they play a very crucial role in the ME updating method.

In what follows, we will briefly present the three axioms of the ME method. The axioms do not refer to merely three cases; any induction from such a weak foundation would hardly be reliable. The reason the axioms are convincing and so constraining is that they refer to three infinitely large classes of known special cases. Additional details and proofs are given in [21, 23].

**Axiom 1: Locality.** *Local information has local effects.*

Assume the information to be processed does not refer to a special subdomain $\mathcal{D}$ of the space $\mathcal{X}$ of $x$'s. The **PMU (Principle of Minimal Updating)** (PMU): *Beliefs should be updated only to the extent required by the new information* requires we do not change our minds about $\mathcal{D}$ in the absence of any new available information about $\mathcal{D}$. Therefore, we design the inference method so that the prior probability of $x$ conditional on $x \in \mathcal{D}$, $q(x|\mathcal{D})$, is not updated. The selected conditional posterior is $P(x|\mathcal{D}) = q(x|\mathcal{D})$. The consequence of axiom 1 is that non-overlapping domains of $x$ contribute additively to the entropy. Dropping multiplicative factors and additive terms that do not affect the overall ranking, the entropy functional becomes

$$S[p, q] = \int dx F(p(x), q(x), x),$$

where $F$ is some unknown function.

**Axiom 2: Coordinate invariance.** *The system of coordinates carries no information.*

Any of a variety of coordinate systems can be used to label the points $x$. The freedom to change coordinates should not affect the ranking of the distributions. The consequence of axiom 2 is that $S[p, q]$ can be written in terms of
coordinate invariants such as $dxm(x)$ and $p(x)/m(x)$, and $q(x)/m(x)$:

$$S[p, q] = \int dxm(x) \Phi \left( \frac{p(x)}{m(x)}, \frac{q(x)}{m(x)} \right).$$  \hfill (2)

(Multiplicative factors and additive terms that do not affect the overall ranking have been dropped.) Thus the unknown function $F(p(x), q(x), x)$ has been replaced by $m(x) \Phi \left( \frac{p(x)}{m(x)}, \frac{q(x)}{m(x)} \right)$. The unknown functions become now the density $m(x)$ and the function $\Phi$ with two arguments. Next we determine the density $m(x)$ by invoking the locality axiom 1 once again.

Axiom 1 (special case): When there is no new information there is no reason to change one’s mind.

The domain $\mathcal{D}$ in axiom 1 coincides with the whole space $\mathcal{X}$ when no new information is available. The conditional probabilities $q(x|\mathcal{D}) = q(x|\mathcal{X}) = q(x)$ should not be updated and the selected posterior distribution coincides with the prior, $P(x) = q(x)$. The consequence is that $S[p, q]$ becomes,

$$S[p, q] = \int dxq(x) \Phi \left( \frac{p(x)}{q(x)} \right).$$ \hfill (3)

Axiom 3: Consistency for independent subsystems. When a system is composed of subsystems that are known to be independent it should not matter whether the inference procedure treats them separately or jointly.

Assume the information on two independent subsystems 1 and 2 that are treated separately is such that the prior distributions $q_1(x_1)$ and $q_2(x_2)$ are respectively updated to $P_1(x_1)$ and $P_2(x_2)$. When treated as a single system the joint prior is $q_1(x_1)q_2(x_2)$ and the family of potential posteriors is $p(x_1, x_2) = p_1(x_1)p_2(x_2)$. The entropy functional must be such that the selected posterior is $P_1(x_1)P_2(x_2)$. The consequence of axiom 3 for this particular case of just two subsystems is that entropies are restricted to the one-parameter family given by

$$S_\eta[p, q] = \frac{1}{\eta(\eta + 1)} \left[ 1 - \int dxp(x) \left( \frac{p(x)}{q(x)} \right)^\eta \right].$$ \hfill (4)

Multiplicative factors and additive terms that do not affect the overall ranking scheme can be freely chosen. The $\eta = 0$ case reproduces the usual logarithmic relative entropy,

$$S[p, q] = -\int dxp(x) \log \left( \frac{p(x)}{q(x)} \right)$$ \hfill (5)

[Use $y^n = \exp(\eta \log y) \approx 1 + \eta \log y$ in (4) and let $\eta \to 0$ to get (5).]

In [23] it was argued that the index $\eta$ has to be the same for all systems. Consistency requires that $\eta$ must be a universal constant. From the success of statistical mechanics as a theory of inference it was inferred that the value of this constant must be $\eta = 0$ leading to the logarithmic entropy, eq. (5).
In conclusion, the ME updating method can be summarized as follows: 

**The ME method:** The objective is to update from a prior distribution \( q \) to a posterior distribution \( P(x) \) given the information that the posterior lies within a certain family of distributions \( p \). The selected posterior \( P(x) \) is that which maximizes the entropy \( S[p, q] \). Since prior information is valuable, the functional \( S[p, q] \) has been chosen so that beliefs are updated only to the extent required by the new information. No interpretation for \( S[p, q] \) is given and none is needed.

V. ELEMENTS OF RIEMANNIAN GEOMETRY

In this section I briefly recall some essential concepts and notations of Riemannian differential geometry which are used in this dissertation. The present section is only meant to facilitate the reader to follow the work presented in the next Chapters, so that my discussion will not be a rigorous treatment of the subject. For a more elaborate discussion, I refer the reader to references in [5], to textbooks of general relativity [24–26] or to a more mathematically oriented introductions to the subject given in [27]. Finally, a comprehensive and rigorous treatment, which goes far beyond what is needed to follow the exposition in this Thesis, can be found in Kobayashi and Nomizu [28]. As a side remark, I would like to emphasize that each work appearing in reference [5] has deeply shaped my own personal point of view concerning the relevance of geometry in the study of chaos. However, the author’s main concern in [5] was the investigation of classical chaos in the Riemannian and Finslerian geometric frameworks. My personal objective is to investigate both classical and quantum aspects of chaoticity in a hybrid information-geometric framework being aware that, thus far, no application of Finslerian geometry to probability theory is available in the literature [29].

A. Notes on Riemannian manifolds

A differentiable manifold \( \mathcal{M} \) is a set that can be covered with a collection, either finite or denumerable, of charts, such that each point of \( \mathcal{M} \) is represented at least on one chart, and the different charts are differentiably connected to each other. A chart is a set of coordinates on the manifold, i.e., it is a set of \( n \) real numbers \( (x_1, \ldots, x_n) \) which denote the ”position” of a point on the manifold. The number \( n \) of coordinates of a chart is the same for each connected part of the manifold (and for the whole manifold if the latter is connected, i.e., it cannot be split in two disjoint parts which are still manifolds); such a number is called the *dimension* of the manifold \( \mathcal{M} \). The union of the charts on \( \mathcal{M} \) is called an *atlas* of \( \mathcal{M} \).
1. Tangent vectors and tensors

A possible way to define a vector is using curves on the manifold \( \mathcal{M} \). Given a curve \( \gamma \) in \( \mathcal{M} \), represented in local coordinates by the parametric equations \( \theta = \phi(t) \), we define a tangent vector at \( P \in \mathcal{M} \) as the velocity vector of the curve in \( P \), i.e.,

\[
v = \dot{\gamma} = \lim_{t \to 0} \frac{\phi(t) - \phi(0)}{t}, \quad \phi(0) = P,
\]

so that the \( n \) components of the tangent vector \( v \) are given by

\[
v^i = \frac{d\phi^i}{dt}.
\]

The set of all the tangent vectors of \( \mathcal{M} \) in \( P \) is a linear space, referred to as the tangent space of \( \mathcal{M} \) in \( P \), and denoted by \( T_P \mathcal{M} \). Each tangent space is isomorphic to an \( n \)-dimensional Euclidean space. Given a chart \( (x_1, \ldots, x_n) \) in a neighborhood of \( P \), a basis \( (X_1, \ldots, X_n) \) of \( T_P \mathcal{M} \) can be defined, so that a generic vector \( v \) is expressed as a sum of the \( X_i \)'s weighted by its components,

\[
v = v^i X_i.
\]

The basis \( \{X_i\} \) is called a coordinate basis of \( T_P \mathcal{M} \), and its vectors \( X_i \) are often denoted by \( \partial/\partial x_i \) (the origin of this notation is in the fact that vectors can be defined as directional derivatives on \( \mathcal{M} \)). The basis depends on the chart: choosing another chart, \( (x'_1, \ldots, x'_n) \), we get another basis \( \{X'_i\} \). The components of \( v \) in the two different bases are connected by the following rule,

\[
v'^i = v^j \frac{\partial x'^i}{\partial x^j},
\]

referred to as the vector transformation rule. Indeed, one can define a vector as a quantity whose components transform according to (9). The union of all the tangent spaces \( T_P \mathcal{M} \) of the manifold \( \mathcal{M} \),

\[
T \mathcal{M} = \bigcup_{P \in \mathcal{M}} T_P \mathcal{M},
\]

is a \( 2n \)-dimensional manifold and is referred to as the tangent bundle of \( \mathcal{M} \).

A vector field \( V \) on \( \mathcal{M} \) is an assignment of a vector \( v_P \) at each point \( P \in \mathcal{M} \). If \( f \) is a smooth function,

\[
V(f)|P = v_P(f)
\]

is a real number for each \( P \in \mathcal{M} \), i.e., \( v(f) \) is a function on \( \mathcal{M} \). If such a function is smooth, \( V \) is called a smooth vector field on \( \mathcal{M} \). The curves \( \phi(t) \) which satisfy the differential equations

\[
\dot{\phi} = V(\phi(t))
\]
are called the *trajectories* of the field $V$, and the mapping $\phi_t : \mathcal{M} \to \mathcal{M}$ which maps any point $P$ of $\mathcal{M}$ along the trajectory of $V$ emanating from $P$ is called the *flow* of $V$. Given two vector fields $V, W$, one can define the *commutator* as the vector field $[V, W]$ such that

$$[V, W](f) = V(W(f)) - W(V(f)), \quad (13)$$

i.e., in terms of the local components,

$$[V, W]^j = V^i \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial V^j}{\partial x^i}. \quad (14)$$

We note that, if $\{X_i\}$ is a coordinate basis,

$$[X_i, X_j] = 0 \forall i, j, \quad (15)$$

and that, conversely, given $n$ nonvanishing and commuting vector fields that are linearly independent, there always exists a chart for which these vector fields are a coordinate basis.

Tangent vectors are not the only vector-like quantities that can be defined on a manifold $\mathcal{M}$: there are also *cotangent vectors*, which can be defined as follows. Let us recall that the *dual space* $V^*$ of a vector space $V$ is the space of *linear* maps from $V$ to the real numbers. Given a basis of $V$, $\{u_i\}$, a basis of $V^*$, $\{u^i\}$, called the *dual basis*, is defined by

$$u^i(u_j) = \delta^i_j, \quad (16)$$

The dual space of $T\mathcal{M}$, $T^*\mathcal{M}$, is called the *cotangent bundle* of $\mathcal{M}$. Its elements are called *cotangent vectors*, or sometimes *covariant vectors* (while the tangent vectors are sometimes denoted as *contravariant vectors*). The dual basis elements are usually denoted as $dx^1, ..., dx^n$, i.e., $dx^i$ is such that $dx^i(\partial/\partial x^j) = \delta^i_j$. The components $\omega_i$ of cotangent vectors transform according to the rule

$$\omega'_i = \omega_j \frac{\partial x^j}{\partial x'^i}, \quad (17)$$
to be compared with [9]. The common rule is to use subscripts to denote the components of dual vectors and superscripts for those of vectors.

A $(k, l)$*-tensor* $T$ over a vector space $V$ is a multilinear map

$$T : (V^* \times \ldots \times V^*) \times (V \times \ldots \times V) \to \mathbb{R} \quad (18)$$

i.e., acting on $k$ dual vectors and $l$ vectors, $T$ yields a number, and it does so in such a manner that if we fix all but one of the vectors or dual vectors, it is a linear map in the remaining variable. A $(0, 0)$ tensor is a scalar, a $(0,
1) tensor is a vector, and a (1, 0) tensor is a dual vector. The space \( T(k, l) \) of the tensors of type \((k, l)\) is a linear space; a \((k, l)\)-tensor is defined once its action on \(k\) vectors of the dual basis and on \(l\) vectors of the basis is known, and since there are \(n^k n^l\) independent ways of choosing these basis vectors, \(T(k, l)\) is a \(n^{k+l}\)-dimensional linear space. Two natural operations can be defined on tensors. The first one is called \textit{contraction} with respect to the \(i\)-th (dual vector) and the \(j\)-th (vector) arguments and is a map

\[
C : T(k, l) \ni T \mapsto CT \in T(k - 1, l - 1)
\]

defined by

\[
CT = \sum_{\sigma=1}^{n} T \left( v_{1}^{\sigma*} ; \ldots ; v_{j}^{\sigma} \ldots \right).
\]

The contracted tensor \(CT\) is independent of the choice of the basis, so that the contraction is a well-defined, invariant, operation. The second operation is the \textit{tensor product}, which maps an element \(T(k, l) \times T(k', l')\) into an element of \(T(k + k', l + l')\), i.e., two tensors \(T\) and \(T'\) into a new tensor, denoted by \(T \otimes T'\), defined as follows: given \(k + k'\) dual vectors \(v^1, \ldots, v^{k+k'}\) and \(l + l'\) vectors \(w_1, \ldots, w_{l+l'}\), then

\[
T \otimes T' (v^{1}, \ldots, v^{k+k'}; w_1, \ldots, w_{l+l'}) = T(v^{1}, \ldots, v^{k*}; w_1, \ldots, w_l) T'(v^{k+1}, \ldots, v^{k+k'}; w_{l+1}, \ldots, w_{l+l'}). \tag{20}
\]

The tensor product allows one to construct a basis for \(T(k, l)\) starting from a basis \(\{v_\mu\}\) of \(V\) and its dual basis \(\{v^\nu\}\): such a basis is given by the \(n^{k+l}\) tensors \(v_{\mu_1} \otimes \cdots \otimes v_{\mu_k} \otimes v^{\nu_1} \otimes \cdots \otimes v^{\nu_l}\). Thus, every tensor \(T \in T(k, l)\) allows a decomposition

\[
T = \sum_{\mu_1, \ldots, \nu_l=1}^{n} T_{\nu_1, \ldots, \nu_l}^{\mu_1, \ldots, \mu_k} v_{\mu_1} \otimes \cdots \otimes v_{\mu_k} \otimes v^{\nu_1} \otimes \cdots \otimes v^{\nu_l}; \tag{21}
\]

the numbers \(T_{\nu_1, \ldots, \nu_l}^{\mu_1, \ldots, \mu_k}\) are called the \textit{components} of \(T\) in the basis \(\{v_\mu\}\). The components of the contracted tensor \(CT\) are

\[
(CT)_{\nu_1, \ldots, \nu_l-1}^{\mu_1, \ldots, \mu_k-1} = T_{\nu_1, \ldots, \nu_l}^{\mu_1, \ldots, \mu_k} v_{\mu_{l+1}} \tag{22}
\]

and, the components of the tensor product \(T \otimes T'\) are

\[
(T \otimes T')_{\nu_1, \ldots, \nu_{l+l'}}^{\mu_1, \ldots, \mu_{k+k'}} = T_{\nu_1, \ldots, \nu_l}^{\mu_1, \ldots, \mu_k} T'_{\nu_{l+1}, \ldots, \nu_{l+l'}}^{\mu_{k+1}, \ldots, \mu_{k+k'}}. \tag{23}
\]

All these results are valid for a generic vector space, so that they hold in particular for the vector spaces of the tangent bundle \(TM\) of \(M\), over which tensors (and, analogously to vector fields, tensor fields) can be defined exactly as above.
2. Metrics on Riemannian manifolds

The length element $ds^2$ (the infinitesimal square distance, the metric) on $\mathcal{M}$ can be defined at each point $P \in \mathcal{M}$ by means of a $(0, 2)$-tensor $g$, provided it is symmetric, i.e., $g(v, w) = g(w, v)$, and nondegenerate, i.e., $g(v, w) = 0$ $\forall v \in T_PM$ if and only if $w = 0$. In fact, a $g$ with these properties induces on the tangent bundle $TM$ a nondegenerate quadratic form (called the scalar product),

$$g : (TM \times TM) \ni (v, w) \mapsto g(v, w) = \langle v, w \rangle \in \mathbb{R}.$$ \hspace{1cm} (24)

Then it is possible to measure lengths on the manifold. A manifold $\mathcal{M}$, equipped with a scalar product, is called a (pseudo)Riemannian manifold, and the scalar product is referred to as a (pseudo)Riemannian structure on $\mathcal{M}$. If the quadratic form (24) is positive-definite, then one speaks of a (proper) Riemannian metric. In the latter case the squared length element is always positive. For instance, one can define the length of a curve as

$$l(\gamma) = \int_{\gamma} \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle} dt.$$ \hspace{1cm} (25)

The curves $\gamma$ which are extremals of the length functional are called the geodesics of $\mathcal{M}$.

In a coordinate basis, we can expand the metric $g$ as

$$g = g_{ij} dx^i \otimes dx^j,$$ \hspace{1cm} (26)

so that one defines the invariant (squared) length element on the manifold, in local coordinates, as

$$ds^2 = g_{ij} dx^i dx^j.$$ \hspace{1cm} (27)

The scalar product of two vectors $v$ and $w$ is given, in terms of $g$, by

$$\langle v, w \rangle = g_{ij} v^i w^j = v_j w^j = v^i w_i.$$ \hspace{1cm} (28)

In the above equation we have made use of the fact that $g$ establishes a one-to-one correspondence between vectors and dual vectors, i.e., in components,

$$g_{ij} v^j = v_i.$$ \hspace{1cm} (29)

For this reason, the components of the inverse metric $g^{-1}$ are simply denoted by $g^{ij}$, instead of $(g^{-1})^{ij}$, and allow to pass from dual vector (covariant) components to vector (contravariant) components:

$$g^{ij} v_j = v^i.$$ \hspace{1cm} (30)
This operation of raising and lowering the indices can be applied not only to vector, but also to tensor components. This allows us to pass from \((k, l)\) tensor components to the corresponding \((k + 1, l - 1)\) tensor components and vice versa. What does not change in the operation is the sum \(k + l\) which is called the rank (or the order) of the tensor.

**B. Covariant differentiation on Riemannian manifolds**

Differential calculus on Riemannian manifolds is complicated by the fact that ordinary derivatives do not map vectors into vectors, i.e., the ordinary derivatives of the components of a vector \(w, dw^i/dt\), taken for instance at a point \(P\) along a given curve \(\gamma(t)\), are not the components of a vector in \(T_P\mathcal{M}\), because they do not transform according to the rule \([\mathbf{9}]\). The geometric origin of this fact is that the parallel transport of a vector from a point \(P\) to a point \(Q\) on a non-Euclidean manifold depends on the path chosen to join \(P\) and \(Q\). Since in order to define the derivative of a vector at \(P\), we have to move that vector from \(P\) to a neighboring point along a curve and then parallel-transport it back to the original point in order to measure the difference, we need a definition of parallel transport to define a derivative; conversely, given a (consistent) derivative, i.e., a derivative which maps vectors into vectors, one could define the parallel transport by imposing that a vector is parallel transported along a curve if its derivative along the curve is zero. The two ways are conceptually equivalent: we follow the first way, by introducing the notion of a connection and then using it to define the derivative operator. Such a derivative will be referred to as the covariant derivative.

A (linear) connection on the manifold \(\mathcal{M}\) is a map \(\nabla\) such that, given two vector fields (one could also consider tensor fields, but for the sake of simplicity we define connections using vectors) \(A\) and \(B\), it yields a third field \(\nabla_A B\) with the following properties:

1. \(\nabla_A B\) is bilinear in \(A\) and \(B\), i.e., \(\nabla_A (\alpha B + \beta C) = \alpha \nabla_A B + \beta \nabla_A C\) and \(\nabla_{\alpha A + \beta B} C = \alpha \nabla_A C + \beta \nabla_B C\);
2. \(\nabla f(A) B = f(\nabla_A B)\);
3. (Leibnitz rule) \(\nabla_A f(B) = (\partial_A f) B + f(\nabla_A B)\), where \(\partial_A\) is the ordinary directional derivative in the direction of \(A\).

The parallel transport of a vector \(V\) along a curve \(\gamma\), whose tangent vector field is \(\dot{\gamma}\), is then defined as the (unique) vector field \(W(t) = W(\gamma(t))\) along \(\gamma(t)\) such that

1. \(W(0) = V\);
2. \(\nabla_{\dot{\gamma}} W = 0\) along \(\gamma\).

The notion of covariant derivative now immediately follows: the covariant derivative \(DV/dt\) of \(V\) along \(\gamma\) is given
by the vector field
\[ \frac{DV}{dt} = \nabla_\gamma V. \] (31)

On the basis of equation (31), with a certain abuse of language, one often refers to \( \nabla_X Y \) as the covariant derivative of \( Y \) along \( X \), where \( X \) and \( Y \) are generic vector fields. Among all the possible linear connections, and given a metric \( g \), there is one and only one which (i) is symmetric, i.e.,
\[ \nabla_X Y - \nabla_Y X = [X, Y] \forall X, Y, \] (32)
and (ii) conserves the scalar product, i.e., the scalar product of two parallel vector fields \( P \) and \( P' \) is constant along \( \gamma \),
\[ \frac{d}{dt} \langle P, P' \rangle \equiv 0. \] (33)

Such a linear connection is obviously the natural one on a Riemannian manifold, and is referred to as the Levi-Civita (or Riemannian) connection. Whenever we refer to a covariant derivative without any specification, we mean the covariant derivative induced by the Riemannian connection. The components of the Riemannian connection \( \nabla \) with respect to a coordinate basis \( \{X_i\} \) are the Christoffel symbols, given by
\[ \Gamma^i_{jk} = \langle dx^i, \nabla_{X_j} X_k \rangle \] (34)
and are given, in terms of the derivatives of the components of the metric, by the following formula
\[ \Gamma^i_{jk} = \frac{1}{2} \left( \frac{\partial g^{im}}{\partial x^j} + \frac{\partial g^{jm}}{\partial x^k} - \frac{\partial g^{mk}}{\partial x^j} \right), \] (35)
where \( \partial_l = \partial/\partial x^l \). The expression in local coordinates of the covariant derivative \( (31) \) of a vector field \( V \) is then
\[ \frac{DV^i}{dt} = \frac{dV^i}{dt} + \Gamma^i_{jk} \frac{dx^j}{dt} V^k. \] (36)

1. Geodesic Equation

The geodesics are defined as the curves of extremal length on the manifold and can also be defined as self-parallel curves, i.e., curves such that the tangent vector \( \dot{\gamma} \) is always parallel transported. Thus geodesics are the curves \( \gamma(t) \) which satisfy the equation (referred to as the geodesic equation)
\[ \frac{D\dot{\gamma}}{dt} = 0 \] (37)
whose expression in local coordinates follows from (36), and is

\[ \frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0, \]  

(38)

provided \( t \) is an affine parameter. Since the norm of the tangent vector \( \dot{\gamma} \) of a geodesic is constant, \( |d\gamma/dt| = c \), the arc length of a geodesic is proportional to the parameter:

\[ s(t) = \int_{t_1}^{t_2} \left| \frac{d\gamma}{dt} \right| dt = c(t_2 - t_1). \]  

(39)

When the parameter is actually the arc length, i.e., \( c = 1 \), we say that the geodesic is *normalized*. Whenever we consider a geodesic, we assume it is normalized, if not explicitly stated otherwise. This means that equations (38) are nothing but the Euler-Lagrange equations for the length functional along a curve \( \gamma(s) \) parametrized by the arc length,

\[ l(\gamma) = \int ds. \]  

(40)

Given a congruence of geodesics \( \gamma(s) \) on \( M \), there exists a unique vector field \( G \) on \( T\!\! \! M \) such that its trajectories are \( (\gamma(s), \dot{\gamma}(s)) \). Such a vector field \( G \) is called the geodesic field and its flow \( (\gamma(s), \dot{\gamma}(s)) \) the geodesic flow on \( M \).

C. The curvature tensor

A way of measuring how much a Riemannian manifold \( (M, g) \) deviates from being Euclidean is by use of the curvature tensor. This quantity, also known as the Riemann-Christoffel tensor, is a tensor of order 4 defined as

\[ R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}, \]  

(41)

where \( \nabla \) is the Levi-Civita connection of \( M \). Observe that if \( M = \mathbb{R}^N \), then \( R(X, Y) = 0 \) for all the pairs of tangent vectors \( X, Y \), because of the commutativity of the ordinary derivatives. In addition, \( R \) measures the noncommutativity of the covariant derivative: in fact, if we choose a coordinate system \( \{x_1, \ldots, x_n\} \), we have, since \( \left[ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right] = 0 \),

\[ R \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \nabla_{\partial/\partial x_i} \nabla_{\partial/\partial x_j} - \nabla_{\partial/\partial x_j} \nabla_{\partial/\partial x_i}. \]  

(42)

In local coordinates, the components of the Riemann curvature tensor (considered here as a (1, 3)-tensor) are given by

\[ R^i_{jkl} = \frac{\partial}{\partial x^k} \Gamma^i_{j\ell} - \frac{\partial}{\partial x^j} \Gamma^i_{k\ell} + \Gamma^p_{j\ell} \Gamma^i_{pk} - \Gamma^p_{k\ell} \Gamma^i_{pj}. \]  

(43)
Thus, given a metric $g$, the curvature $R$ is uniquely defined. A manifold $(M, g)$ is called flat when the curvature tensor vanishes.

Given a positive function $f^2$, a *conformal transformation* is the transformation

$$(M, g) \rightarrow (M, \tilde{g}); \quad \tilde{g} = f^2 g,$$  \hspace{1cm} (44)

where $g$ is the metric tensor. Two Riemannian manifolds are said *conformally related* if they are linked by a conformal transformation. In particular, a manifold is $(M, g)$ *conformally flat* if it is possible to find a conformal transformation that sends $g$ into a flat metric. Conformally flat manifolds exhibit some remarkable simplifications for the calculation of the curvature tensor components (see [30]).

Closely related to the curvature tensor is the sectional — or Riemannian — curvature, which we define now. Let us consider two vectors $u, v \in T_PM$, and let us put

$$|u \wedge v| = \left( |u|^2 |v|^2 - \langle u, v \rangle \right)^{1/2},$$  \hspace{1cm} (45)

which is the area of the two-dimensional parallelogram determined by $u$ and $v$. If $|u \wedge v| \neq 0$ the vectors $u, v$ span a two-dimensional subspace $\pi \subset T_PM$. We define the *sectional curvature* at the point $P$ relative to $\pi$, as the quantity:

$$K(P; u, v) = K(P, \pi) = \frac{\langle R(u, v)u, v \rangle}{|u \wedge v|^2}$$  \hspace{1cm} (46)

which can be shown to be independent of the choice of the two vectors $u, v \in \pi$. In local coordinates, (46) becomes

$$K(P; u, v) = R_{ijkl} u^i v^j u^k v^l |u \wedge v|^2.$$  \hspace{1cm} (47)

The knowledge of $K$ for the $N(N-1)$ planes $\pi$ spanned by a maximal set of linearly independent vectors completely determines $R$ at $P$.

If dim$(M) = 2$ then $K$ coincides with the Gaussian curvature of the surface, i.e., with the product of the reciprocals of two curvature radii.

A manifold is called *isotropic* if $K(P, \pi)$ does not depend on the choice of the plane $\pi$. A remarkable result — Schur’s theorem [27] — is that in this case $K$ is also constant, i.e. it does not depend on the point $P$ either.

Some “averages” of the sectional curvatures are very important. The *Ricci curvature* $K_R$ at $P$ in the direction $v$ is defined as the sum of the sectional curvatures at $P$ relative to the planes determined by $v$ and the $N - 1$ directions orthogonal to $v$, i.e., if $\{e_1, \ldots, e_{N-1}, v = e_N\}$ is an orthonormal basis of $T_PM$ and $\pi_i$ is the plane spanned by $v$ and $e_i$,

$$K_R(P, v) = \sum_{i=1}^{N-1} K(P, \pi_i).$$  \hspace{1cm} (48)
The scalar curvature $\mathcal{R}$ at $P$ is the sum of the $N$ Ricci curvatures at $P$,

$$\mathcal{R} (P) = \sum_{i=1}^{N} K_R (P, e_i). \quad (49)$$

In terms of the components of the curvature tensor, such curvatures can be defined as follows (in the following formulae, we drop the dependence on $P$, because it is understood that the components are local quantities). We first define the Ricci tensor as the two-tensor whose components, $R_{ij}$, are obtained by contracting the first and the third indices of the Riemann tensor,

$$R_{ij} = R^k_{ikj}; \quad (50)$$

then,

$$K_R (v) = R_{ij} v^i v^j. \quad (51)$$

The right hand side of (51) is called "saturation" of $R_{ij}$ with $v$. The scalar curvature can be obtained as the trace of the Ricci tensor,

$$\mathcal{R} = R^i_i. \quad (52)$$

In the case of a constant curvature — or isotropic — manifold, the components of the Riemann curvature tensor have the remarkably simple form

$$R_{ijkl} = K (g_{ik} g_{jl} - g_{il} g_{jk}), \quad (53)$$

where $K$ is the constant sectional curvature, so that the components of the Ricci tensor are

$$R_{ij} = K g_{ij}, \quad (54)$$

and all the above defined curvatures are constants, and are related by

$$K = \frac{1}{N - 1} K_R = \frac{1}{N (N - 1)} \mathcal{R}. \quad (55)$$

D. The Jacobi-Levi-Civita equation

A brief derivation of the JLC (Jacobi-Levi-Civita) equation is presented in this subsection. We will proceed as follows: first, we will define the geodesic separation vector field $J$, then we will show that the field $J$ is actually a Jacobi field, i.e., obeys the Jacobi equation.
Let us define a geodesic congruence as a family of geodesics \( \{ \gamma(s) = \gamma(s, \tau) | \tau \in \mathbb{R} \} \) issuing from a neighborhood \( I \) of a manifold point, smoothly dependent on the parameter \( \tau \), and let us fix a reference geodesic \( \tilde{\gamma}(s, \tau_0) \). Denote then by \( \dot{\gamma}(s) \) the vector field tangent to \( \tilde{\gamma} \) in \( s \), i.e., the velocity vector field whose components are

\[
\dot{\gamma}^i = \frac{dx^i}{ds}, \tag{56}
\]

and by \( J(s) \) the vector field tangent in \( \tau_0 \) to the curves \( \gamma_s(\tau) \) for a fixed \( s \), i.e., the vector field of components

\[
J^i = \frac{dx^i}{d\tau}. \tag{57}
\]

The field \( J \) will be referred to as the geodesic separation field, and measures the distance between nearby geodesics. Let us now show that \( J \) is a Jacobi field. First of all, we notice that the field \( J \) commutes with \( \dot{\gamma} \), i.e., \([\dot{\gamma}, J] = 0\). In fact, from the definition of the commutator (14) and from the definitions of \( J \), (57), and of \( \dot{\gamma} \), (56), we have

\[
[\dot{\gamma}, J]^i = \dot{\gamma}^j \frac{\partial J^i}{\partial x^j} - J^j \frac{\partial \dot{\gamma}^i}{\partial x^j} = \frac{\partial x^i}{\partial s} \frac{\partial J^i}{\partial x^j} - \frac{\partial x^j}{\partial \tau} \frac{\partial \dot{\gamma}^i}{\partial x^j} = \frac{\partial J^i}{\partial s} - \frac{\partial \dot{\gamma}^i}{\partial \tau}, \tag{58}
\]

and using again (57) and (56), we find that

\[
\frac{\partial J^i}{\partial s} = \frac{\partial}{\partial s} \frac{\partial x^i}{\partial \tau} = \frac{\partial}{\partial \tau} \frac{\partial x^i}{\partial s} = \frac{\partial \dot{\gamma}^i}{\partial \tau}, \tag{59}
\]

so that \([\dot{\gamma}, J] = 0\). Now, let us compute the second covariant derivative of the field \( J \), \( \nabla^2 \dot{\gamma}J \). First of all, let us recall that our covariant derivative comes from a Levi-Civita connection, which is symmetric (see (32)), so that

\[
\nabla_{\dot{\gamma}} J - \nabla J \dot{\gamma} = [\dot{\gamma}, J], \tag{60}
\]

and having just shown that \([\dot{\gamma}, J] = 0\), we can write

\[
\nabla_{\dot{\gamma}} J = \nabla J \dot{\gamma}. \tag{61}
\]

Now, using this result, and the fact that \( \nabla_{\dot{\gamma}} \dot{\gamma} = 0 \) because \( \dot{\gamma} \) is a geodesic, we can write

\[
\nabla^2 \dot{\gamma} J = \nabla \dot{\gamma} \nabla_{\dot{\gamma}} J = \nabla \dot{\gamma} \nabla J \dot{\gamma} = [\nabla \dot{\gamma}, \nabla J] \dot{\gamma}, \tag{62}
\]

from which, using the definition of the curvature tensor (11) and, again, the vanishing of the commutator \([\dot{\gamma}, J] \), we get

\[
\nabla^2 \dot{\gamma} J = R(\dot{\gamma}, J) \dot{\gamma}, \tag{63}
\]

which is nothing but the Jacobi equation written in compact notation. In an explicit way the (63) can be written as,

\[
\frac{D^2 J^i}{ds^2} + R^{i}_{jkl} \frac{dq^j}{ds} \frac{dq^l}{ds} = 0 \tag{64}
\]
It is worth noticing that the normal component $J_\perp$ of $J$, i.e., the component of $J$ orthogonal to $\dot{\gamma}$ along the geodesic $\gamma$, is again a Jacobi field, since we can always write $J = J_\perp + \lambda \dot{\gamma}$: one immediately finds then that the velocity $\dot{\gamma}$ satisfies the Jacobi equation, so that $J_\perp$ must obey the same equation. This can allow us to restrict ourselves to the study of the normal Jacobi fields.

VI. WHAT IS INFORMATION GEOMETRY?

In the present section, I describe some of the basics of information geometry (IG). IG is the result of applying conventional Riemannian geometry to probability theory. Although interest in this subject can be traced back to the late 1960’s, it reached maturity only through the work of Amari in the 1980’s [31]. The development of the field of information geometry can only be said to have just begun.

A. Notes on Information Geometry

IG began as an investigation of the natural differential geometric structure possessed by families of probability distributions. As a rather simple example, consider the set of normal distributions with mean $\mu$ and variance $\sigma^2$:

$$p(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right).$$  \hspace{1cm} (65)

By specifying $(\mu, \sigma)$ we determine a particular normal distribution, and therefore the set may be viewed as a 2-dimensional space (manifold) which has $(\mu, \sigma)$ as a coordinate system. However, this is not an Euclidean space, but rather a Riemannian space with a metric which naturally follows from the underlying properties of the probability distributions. Probability distributions are the fundamental element over which fields such as statistics, stochastic processes, and information theory are developed. Therefore, the geometric structure of the space of probability distributions must play a fundamental role in these information sciences. In fact, considering statistical estimation from a differential geometric viewpoint has provided statistics with a new analytic tool which has allowed several previously open problems to be solved; information geometry has already established itself within the field of statistics. In the fields of information theory, stochastic processes, and systems, information geometry is being currently applied to allow the investigation of hitherto unexplored possibilities.

The utility of information geometry, however, is not limited to these fields. It has, for example, been applied productively to areas such as statistical physics and mathematical theory underlying neural networks.
From a mathematical point of view, quantum mechanics may be constructed as an extension of probability theory, and it is possible to generalize many concepts in probability theory to their quantum equivalents. The framework of information geometry for statistical models may also be extended to the quantum mechanical setting. A variety of important works related to differential geometrical aspects of quantum mechanics have so far been made by many researchers. However, the study of quantum information geometry has just started and we are far from getting its whole perspective at present.

One of the fundamental questions information geometry helps to answer is: "Given two probability distributions, is it possible to define a notion of "distance" between them?". This chapter, which introduces some of the basic concepts of information geometry, does not presuppose any knowledge of the theory of probability and distributions. Unfortunately however, it does require some knowledge of Riemannian geometry.

**B. Families of probability distributions as statistical manifolds**

For our purposes, a probability distribution over some field (or set) $\mathcal{X}$ is a distribution

$$p : \mathcal{X} \ni x \mapsto p(x) \in \mathbb{R}^+$$

such that

$$\int_{\mathcal{X}} dx p(x) = 1 \quad \text{and} \quad \int_{S} dx p(x) > 0 \quad \forall \text{ subset } S \subset \mathcal{X}. \quad (67)$$

In what follows, we will consider families of such distributions. In most cases these families will be parametrized by a set of continuous parameters $\theta = \{\theta^\mu\}_{\mu=1,\ldots,n}$ that take values in some open interval $I_\theta \subseteq \mathbb{R}^n$ and we write $p(x|\theta)$ to denote members of the family. For any fixed $\theta$, $p_\theta$ is the mapping from $\mathcal{X}$ to $\mathbb{R}$ with

$$p_\theta : \mathcal{X} \ni x \mapsto p_\theta(x) \equiv p(x|\theta) \in \mathbb{R}. \quad (68)$$

In information geometry, one extends a family of distributions $\mathcal{F}$,

$$\mathcal{F} = \{p_\theta | \theta \in I_\theta \subseteq \mathbb{R}^n\}, \quad (69)$$

to a manifold $\mathcal{M}$ such that the points in $\mathcal{M}$ are in a one to one relation with the distributions in $\mathcal{F}$. In doing so, one hopes to gain some insight into the structure of such a family. For example, one might hope to discover a reasonable measure of "nearness" of two distributions in the family. Having made the link between families of distributions and manifolds, one can try to identify which objects in the language of distributions naturally correspond to objects in the
language of manifolds and vice versa. The most important objects in the language of manifolds are tangent vectors. The tangent space \( T_{\theta} \) at the point in \( M \) with coordinates \( \{ \theta^\mu \}_{\mu=1,...,n} \) is seen to be isomorphic to the vector space spanned by the model parameters (function from \( X \) to \( \mathbb{R} \)) \( \frac{\partial \log p(x|\theta)}{\partial \theta^\mu} \). This space is called \( T_{\theta}^{(1)} \). Therefore, a vector field \( v(\theta) \in T(M) \),

\[
v : M \ni \theta \mapsto v(\theta) = v^\mu(\theta) \hat{e}_\mu \in T(M),
\]

is equivalent to a model parameter \( v_\theta(\cdot) \in T^{(1)}(M) \),

\[
v_\theta(x) = v^\mu(\theta) \frac{\partial \log p(x|\theta)}{\partial \theta^\mu}.
\]

Just as \( T(M) \) is the space of continuously differentiable mappings that assigns some vector \( v(\theta) \in T_\theta \) to each point \( \theta \in M \), \( T^{(1)}(M) \) assigns a model parameter \( v_\theta \in T^{(1)}_\theta \).

In view of the above equivalence, we will not find necessary to distinguish between the vector field \( v \) and the corresponding random variable \( v_\theta(\cdot) \). Equation (71) is called the 1-representation of the vector field \( v \). It is clearly possible to use some other basis of functionals of \( p(x|\theta) \) instead of \( \frac{\partial \log p(x|\theta)}{\partial \theta^\mu} \). Our present choice has the advantage that the 1-representation of a vector has zero expectation value,

\[
E \left[ \frac{\partial \log p(x|\theta)}{\partial \theta^\mu} \right] = \int_X d\mu(x|\theta) \frac{\partial \log p(x|\theta)}{\partial \theta^\mu} = \frac{\partial}{\partial \theta^\mu} \int_X d\mu(x|\theta) = 0.
\]

Using other functionals can be useful, and in fact the 1-representation turns out to be just one member of the family of \( \alpha \)-representations \( (\alpha \neq 1) \).

C. Distances between distributions: the Fisher-Rao information metric

In several applications we are interested in distances between distributions. For example, given a distribution \( p \in M \) and a submanifold \( S \subseteq M \) we may wish to find the distribution \( p' \in S \) that is "nearest" to \( p \) in some sense. In order to fulfill such a task we need a notion of distance on manifolds of distributions. In other words, we need a metric.

Consider a family of probability distributions \( p(x|\theta) \) defined in terms of some parameters \( \theta^\mu \) with \( \mu = 1,...,n \). The space of these distributions constitutes a manifold, the points of which are the distributions and, the parameters \( \theta^\mu \) are convenient set of coordinates. The structure of such manifolds is studied by introducing conventional differential geometrical notions. Ultimately, the problem is to quantify the extent to which we can distinguish between two neighboring probability distributions \( p(x|\theta) \) and \( p(x|\theta + d\theta) \). If \( d\theta \) is small enough the distributions overlap considerably, it is easy to confuse them and we are inclined to say that the distributions are near. More specifically we seek a
real positive number that provides a quantitative measure of the extent to which the two distributions can be distinguished. When we interpret this measure of distinguishability as a distance- the information metric- the manifold of distributions immediately acquires a geometric structure and we can proceed to study it using the mathematical techniques of differential geometry. It appears that the introduction of geometrical methods is the natural way to study spaces of probability distributions, to study how one changes one’s mind and effectively moves in such a space as a result of acquiring information. Perhaps, this is the reason why the models we develop to describe the world are so heavily geometrical.

We are looking for a quantitative measure of the extent that two probability distributions \( p(x|\theta) \) and \( p(x|\theta + d\theta) \) can be distinguished. An appealing and intuitive way to approach this problem is the following. Consider the relative difference,

\[
\frac{p(x|\theta + d\theta) - p(x|\theta)}{p(x|\theta)} = \frac{\partial \log p(x|\theta)}{\partial \theta^\mu} d\theta^\mu. \tag{73}
\]

It might seem that the expected value of the relative difference is a good candidate. However, it is not because it vanishes identically (see (72)),

\[
\int dx p(x|\theta) \frac{\partial \log p(x|\theta)}{\partial \theta^\mu} d\theta^\mu = d\theta^\mu \frac{\partial}{\partial \theta^\mu} \int dx p(x|\theta) = 0. \tag{74}
\]

Instead, the variance does not vanish and therefore it is a good choice,

\[
dl^2 = \int dx p(x|\theta) \frac{\partial \log p(x|\theta)}{\partial \theta^\mu} \frac{\partial \log p(x|\theta)}{\partial \theta^\nu} d\theta^\mu d\theta^\nu \defeq g_{\mu\nu} d\theta^\mu d\theta^\nu. \tag{75}
\]

This is the measure of distinguishability for which we are searching; a small value of \( \ndl^2 \) means the points \( \theta \) and \( \theta + d\theta \) are difficult to distinguish. The matrix \( g_{\mu\nu} \) is called the Fisher information matrix \[32\]. Thus far, no notion of distance has been introduced on the space of states. In general, it is said that the reason it is difficult to distinguish between two points is that they happen to be too close together. However, it is very tempting to invert the logic and assert that the two points \( \theta \) and \( \theta + d\theta \) must be very close together because they are difficult to distinguish. Moreover, notice that \( \ndl^2 \) is positive since it is a variance and it vanishes only when \( d\theta \) vanishes. Therefore it is natural to interpret \( g_{\mu\nu} \) as the metric tensor of a Riemannian space \[33\]. It is known as the Fisher-Rao information metric. This metric is a suitable metric for manifolds of distributions and it is given by,

\[
g_{\mu\nu} (\theta) = E [\partial_\mu l(\theta) \partial_\nu l(\theta)] \equiv \int dx p(x|\theta) \frac{\partial \log p(x|\theta)}{\partial \theta^\mu} \frac{\partial \log p(x|\theta)}{\partial \theta^\nu} \tag{76}
\]

with \( \partial_\mu \equiv \frac{\partial}{\partial \theta^\mu} \) and \( l(\theta) \equiv \log p(x|\theta) \). Notice that \( E [\partial_\mu l(\theta) \partial_\nu l(\theta)] = -E [\partial_\mu \partial_\nu l(\theta)] \). Obviously, the information
metric also defines an inner product: for two vector fields $v$ and $w$, we have

$$\langle v, w \rangle \overset{\text{def}}{=} g_{\mu\nu}(\theta) v^\mu w^\nu = E [v^\mu \partial_\mu l (\theta) w^\nu \partial_\nu l (\theta)] = E [v_\theta w_\theta]. \quad (77)$$

Rao recognized that $g_{\mu\nu}$ is a metric in the space of probability distributions and this recognition gave rise to the subject of information geometry \[31\]. This heuristic argument presents a disadvantage, namely it does not make explicit a crucial property of the Fisher-Rao metric: except for an overall multiplicative constant this Riemannian metric is unique \[34, 35\]. The coordinates $\theta$ are quite arbitrary and one can freely switch from one set to another. It is then easy to check that $g_{\mu\nu}$ are the components of a tensor, that is, the distance $dl^2$ is an invariant, a scalar.

Incidentally, $dl^2$ is also dimensionless. At first sight, the definition (76) may seem rather ad hoc. However, it has been proven \[31, 33, 36\] to be unique in having the following very appealing properties:

1. $g_{\mu\nu}(\theta)$ is invariant under reparametrizations of the sample space $\mathcal{X}$;

2. $g_{\mu\nu}(\theta)$ is covariant under reparametrizations of the manifold $\mathcal{M}$ (the parameter space).

The uniqueness of the Fisher-Rao information metric is the most remarkable aspect about this metric: except for a constant scale factor, it is the only Riemannian metric that adequately takes into account that points of the manifold are not structureless; that is, that they are probability distributions. As said before, a proof of such a result is given in \[34\]. Once the information metric is given, then connection coefficients, curvatures and other aspects of the geometry can be computed in the conventional way.

**D. Volume elements in curved statistical manifolds**

Once the distances among probability distributions have been assigned, a natural next step is to obtain measures for extended regions in the space of distributions.

Consider an $n$-dimensional volume of the statistical manifold $\mathcal{M}_x$ of distributions $p(x|\theta)$ labelled by parameters $\theta^\mu$ with $\mu = 1, \ldots, n$. The parameters $\theta^\mu$ are coordinates for the point $P$ and in these coordinates it may not be obvious how to write down an expression for a volume element $dV_{\mathcal{M}_x}$. However, within a sufficiently small region (volume element) any curved space looks flat. Curved spaces are "locally flat". The idea then is rather simple: within that very small region, we should use Cartesian coordinates and the metric takes a very simple form, it is the identity matrix $\delta_{\mu\nu}$. In locally Cartesian coordinates $\chi^\alpha$ the volume element is simply given by the product

$$dV_{\mathcal{M}_x} = d\chi^1 d\chi^2 \ldots d\chi^n, \quad (78)$$
which, in terms of the old coordinates, is

$$dV_{M_s} = \left| \frac{\partial \chi}{\partial \theta} \right| d\theta^1 d\theta^2 ... d\theta^n = \left| \frac{\partial \chi}{\partial \theta} \right| d^n \theta. \quad (79)$$

The problem consists in calculating the Jacobian $\left| \frac{\partial \chi}{\partial \theta} \right|$ of the transformation that takes the metric $g_{\mu\nu}$ into its Euclidean form $\delta_{\mu\nu}$.

Let the new coordinates be defined by $\chi^\mu = \Xi^\mu (\theta^1, ..., \theta^n)$. A small change in $d\theta$ corresponds to a small change in $d\chi$,

$$d\chi^\mu = X^\mu_m d\theta^m \quad \text{where} \quad X^\mu_m \overset{\text{def}}{=} \frac{\partial \chi^\mu}{\partial \theta^m}, \quad (80)$$

and the Jacobian is given by the determinant of the matrix $X^\mu_m$,

$$\left| \frac{\partial \chi}{\partial \theta} \right| = |\det (X^\mu_m)|. \quad (81)$$

The distance between two neighboring points is the same whether we compute it in terms of the old or the new coordinates,

$$dl^2 = g_{\mu\nu} d\theta^\mu d\theta^\nu = \delta_{\alpha\beta} d\chi^\alpha d\chi^\beta. \quad (82)$$

Therefore the relation between the old and the new metric is,

$$g_{\mu\nu} = \delta_{\alpha\beta} X^\alpha_\mu X^\beta_\nu. \quad (83)$$

Taking the determinant of (83), we obtain

$$g \overset{\text{def}}{=} \det (g_{\mu\nu}) = [\det (X^\alpha_\mu)]^2 \quad (84)$$

and, therefore

$$|\det (X^\alpha_\mu)| = \sqrt{g}. \quad (85)$$

Finally, we have succeeded in expressing the volume element totally in terms of the coordinates $\theta$ and the known metric $g_{\mu\nu} (\theta)$,

$$dV_{M_s} = \sqrt{g} d^n \theta. \quad (86)$$

The volume of any extended region on the manifold is given by,

$$V_{M_s} = \int dV_{M_s} = \int \sqrt{g} d^n \theta. \quad (87)$$
As a final remark, note that $\sqrt{|g|} d^n \theta$ is a scalar quantity and therefore is invariant under general coordinate transformations, $\theta \rightarrow \theta'$, preserving orientation. The square root of the metric tensor transforms as

$$\sqrt{g(\theta)} \rightarrow \sqrt{g(\theta')}$$

(88)

and the flat infinitesimal volume element $d^n \theta$ transforms as

$$d^n \theta \rightarrow \frac{\partial \theta'}{\partial \theta} d^n \theta'.$$

(89)

Thus, from (88) and (89), we obtain

$$\sqrt{g(\theta)} d^n \theta \rightarrow \sqrt{g(\theta')} d^n \theta'.$$

(90)

Equation (90) implies that the infinitesimal statistical volume element is invariant under general coordinate transformations that preserve orientation, that is with positive Jacobian.

VII. ME AND IG AT WORK: A SIMPLE EXAMPLE

In what follows, I will briefly illustrate a couple of examples where the ME method is employed. Further examples can be found in [18].

Let the microstates of a physical system be labelled by $x$, and let $m(x) \, dx$ be the number of microstates in the range $dx$. We assume that a state of the system, a macrostate, is defined by the known expected values $F^{\mu}$ of some $n_F$ variables $f^{\mu}(x)$ with $\mu = 1, \ldots, n_F$,

$$\langle f^{\mu}(x) \rangle = \int dx p(x) f^{\mu}(x) = F^{\mu}. $$

(91)

This limited information will certainly not be sufficient to answer all questions that one could conceivably ask about the system. Choosing the right set of variables $\{f^{\mu}\}$ is perhaps the most difficult problem in statistical mechanics. A crucial assumption is that (91) is not just any random information, instead it is the right information for our purposes.

It is convenient to think of each state as a point in an $n_F$-dimensional statistical manifold; the numerical values $F^{\mu}$ associated to each point form a convenient set of coordinates. The ME method allows us to associate a probability distribution to each point in the space of states. The probability distribution $p(x|F)$ that best reflects the prior information contained in $m(x)$, updated by the information $F^{\mu}$, is obtained by maximizing the relative logarithmic entropy

$$S[p|m] = - \int dx p(x) \log \left( \frac{p(x)}{m(x)} \right) $$

(92)
subject to the constraints (91) and to the normalization constraint \( \int dx p(x) = 1 \). Upon setting \( \delta S = 0 \) where \( S \) is given by

\[
S \equiv S[p|m] - \lambda \left( \int dx p(x) - 1 \right) - \lambda_\mu \left( \int dx p(x) f^\mu(x) - F^\mu \right)
\]

(93)

and using the normalization constraint, we obtain

\[
p(x|F) = \frac{m(x)}{Z(\lambda)} e^{-\lambda_\mu f^\mu(x)}.
\]

(94)

The partition function \( Z(\lambda) \) and the Lagrange multipliers \( \lambda_\mu \) are defined as,

\[
Z(\lambda) = \int dx m(x) e^{-\lambda_\mu f^\mu(x)} \quad \text{and} \quad -\frac{\partial \log Z(\lambda)}{\partial \lambda_\mu} = F^\mu.
\]

(95)

The maximized value of entropy is given by

\[
S(F) = S[p|m] \big|_{p=p(x|F)} = \log Z(\lambda) + \lambda_\mu F^\mu.
\]

(96)

For the sake of clarity, let us consider the following special case: assume the normalization and information constraints are given by

\[
\int dx p(x) = 1, \quad \int dx p(x)(x - \mu)^2 = \sigma^2
\]

(97)

where \( \mu = \int dx p(x) \). Upon maximizing \( S[p] = -\int dx p(x) \log p(x) \) (where we have assumed a uniform prior, \( m(x) = 1 \)) subject to the above constraints, we obtain

\[
p(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right).
\]

(98)

The probability distribution \( p(x|\mu, \sigma) \) in (98) is the well-known Gaussian probability distribution. As a last step, let us calculate the Fisher-Rao information metric associated with a statistical manifold of Gaussians. It is known that

\[
g_{ij} = \left( \frac{\partial^2 S(\theta'|\theta)}{\partial \theta'_i \partial \theta'_j} \right)_{\theta'=\theta}
\]

(99)

where,

\[
S(\theta'|\theta) = \int_{-\infty}^{+\infty} dx p(x|\theta') \log \left[ \frac{p(x|\theta')}{p(x|\theta)} \right] \quad \text{with} \quad \theta' \equiv (\mu, \sigma).
\]

(100)

Substituting (98) in (100), after some algebra, we obtain

\[
S(\theta'|\theta) = \log \left( \frac{\sigma}{\sigma'} \right) + \frac{\sigma^2}{2\sigma'^2} + \frac{(\mu' - \mu)^2}{2\sigma'^2}.
\]

(101)
Finally, substituting (101) in (99), we get

\[
g_{ij} = \begin{pmatrix}
\frac{1}{\sigma^2} & 0 \\
0 & \frac{2}{\sigma^2}
\end{pmatrix}
\]  

(102)

with a line element \( ds^2 \) given by

\[
ds^2 = g_{ij} d\theta^i d\theta^j = \frac{1}{\sigma^2} d\mu^2 + \frac{2}{\sigma^2} d\sigma^2.
\]  

(103)

The Gaussian distribution is quite remarkable, it applies to a wide variety of problems such as the distribution of errors affecting experimental data, the distribution of velocities of molecules in gases and liquids, the distribution of fluctuations of thermodynamical quantities, and so on. Basically, it arises whenever we deal with macroscopic variable that is the result of adding a large number of small independent contributions. Gaussian distributions can be derived as the distributions that codify information about the mean (first moment) and the variance (second moment) while remaining maximally ignorant about everything else. Gaussian distributions are successful when third and higher moments are irrelevant.
Chapter 3: Information-constrained dynamics, part I: Entropic Dynamics

In this Chapter, I describe (following reference [38]) the basic aspects of "Entropic Dynamics", a theoretical construct developed to investigate the possibility that Einstein’s general theory of gravity (or more generally, any classical or quantum theory of physics) may emerge as a macroscopic manifestation of an underlying microscopic statistical structure. ED is the starting point of the major contribution of this thesis (the IGAC, information geometrodynamical approach to chaos) and therefore I must revisit it. I emphasize the most relevant points that I will use in my own IGAC and suggest further improvements that, indeed, will be the object of Chapters 6 and 7.

VIII. INTRODUCTION

In Caticha’s Entropic Dynamics [38], the author explores the possibility that the laws of physics might be laws of inference rather than laws of nature. He explores what sort of dynamics can one derive from well-established rules of inference. Specifically, given relevant information codified in the initial and the final states, the problem is to study the trajectory that the system is expected to follow. It turns out the solution to this problem follows from a principle of inference, the principle of maximum entropy, and not from a principle of physics. The entropic dynamics derived this way exhibits some remarkable formal similarities with other generally covariant theories such as general relativity.

Dynamics is the study of changes occurring in nature. For instance, thermodynamics deals with changes between states of equilibrium and addresses the question of which final states can be reached from any given initial state. Mechanics is the study of changes known as motion, chemistry considers chemical reactions, quantum mechanics deals with transitions between quantum states, and so on. In all of these examples, the objective is predicting or explaining the observed changes on the basis of relevant available information that is codified in the "states" of the system. In some cases the final state can be predicted with certainty, in others the information available is incomplete and we can only assign probabilities. Thermodynamics holds a very special place among all these forms of dynamics. Thanks to the development of statistical mechanics by Maxwell, Boltzmann, Gibbs and others, and thanks to Jaynes’ work [1], thermodynamics became the first clear example of a fundamental physical theory that could be derived from general principles of probable inference. An appropriate choice of which states one is considering, plus well-known principles of inference [39], namely, consistency, objectivity, universality and honesty lead to the derivation of the entire theory of thermodynamics. These principles lead to a unique set of rules for processing information: these are the rules of probability theory [40] and the method of maximum entropy [1, 19]. There are strong indications...
that quantum mechanics can be deduced from principles of inference \[41\]. Many features of the theory follow from the correct identification of the subject matter plus general principles of inference. If the "fundamental" theory of quantum mechanics can be derived in this way, then it is possible that other forms of dynamics might ultimately reflect laws of inference rather than laws of nature. The fundamental equations of change, or motion, or evolution would follow from probabilistic and entropic arguments in the case that dynamics reflects laws of inference. The discovery of new dynamical laws would be reduced to the discovery of what is the necessary information for carrying out correct inferences. This is a very important point. Unfortunately, this search for the right variables has always been and remains to this day the major obstacle in the understanding of new phenomena.

In his work (ED), Caticha explores the possible connection between the fundamental laws of physics and the theory of probable inference. He explores the possibility that dynamics can be derived from inference and rather than starting with a known dynamical theory and attempting to derive it, he proceeds in the opposite direction. His ED arises simply from well-established rules of inference! In next section, the notation is introduced, the space of states is defined, and a brief review concerning the introduction of a natural quantitative measure of the change involved in going from one state to another turns the space of states into a metric space \[42\] is presented. (Such metric structures have been found useful in statistical inference, where the subject is known as Information Geometry \[31\], and in physics, to study both equilibrium \[43\] and nonequilibrium thermodynamics \[44\].) Typically, once the kinematics appropriate to a certain motion has been selected, one proceeds to define the dynamics by additional postulates. This is precisely the option Caticha wanted to avoid: in the dynamics developed here there are no such postulates. The equations of motion follow from an assumption about what information is relevant and sufficient to predict the motion.

In a previous work (irreversible entropic dynamics) \[42\], Caticha considered a similar problem. Assuming that the system evolves from a given initial state to other states, he studied the trajectory that the system is expected to follow. In this problem, the existence of a trajectory is assumed and, in addition, it is assumed that information about the initial state is sufficient to determine it. The application of a principle of inference, the method of maximum entropy (ME), to the only information available, the initial state and the recognition that motion occurred leads to the dynamical law. The resulting "entropic" dynamics is very simple: the system moves irreversibly and continuously along the entropy gradient. However, the question of whether the actual trajectory is the expected one remains unanswered and it depends on whether the information encoded in the initial state happened to be sufficient for prediction. For many systems more information is needed, even for those for which the dynamics is reversible. In
the reversible case, assuming that the system evolves from a given initial state to a given final state, the objective is to study what trajectory is the system expected to follow. Again, it is implicitly assumed that there is a trajectory, that in moving from one state to another the system will pass through a continuous set of intermediate states. Again, the equation of motion follows from a principle of inference, the principle of maximum entropy, and not from a principle of physics. (For a brief account of the ME method in a form that is convenient for our current purpose see previous Chapter). In the resulting "entropic" dynamics, the system moves along a geodesic in the space of states. The geometry of the space of states is curved and possibly quite complicated. Important features of this entropic dynamics are explored in section 4.

IX. THE FISHER-RAO INFORMATION METRIC

In this section, a quantitative description of the notion of change is briefly reviewed (for more details see [42]). First, change can be measured by distinguishability since the larger the change involved in going from one state to another, the easier it is to distinguish between them. Next, using the ME method one assigns a probability distribution to each state. This transforms the problem of distinguishing between two states into the problem of distinguishing between the corresponding probability distributions. The extent to which one distribution can be distinguished from another is given by the distance between them as measured by the Fisher-Rao information metric [31–33]. Thus, change is measured by distinguishability which is measured by distance. Let the microstates of a physical system be labelled by \( x \), and let \( m(x)dx \) be the number of microstates in the range \( dx \). We assume that a state of the system (i.e., a macrostate) is defined by the expected values \( \theta^\mu \) of some \( n_\Theta \) appropriately chosen variables \( \Theta^\mu(x) \) (\( \mu = 1, 2, \ldots, n_\Theta \)),

\[
\langle \Theta^\mu(x) \rangle = \int dx p(x) \Theta^\mu(x) = \theta^\mu \quad \text{with} \quad \mu = 1, 2, \ldots, n_\Theta.
\]  

A very important assumption is that the selected variables codify all the information relevant to answering the particular questions under investigation. Again, we emphasize there is no systematic procedure to choose the right variables. The selection of relevant variables is made on the basis of intuition guided by experiment. Essentially, it is a matter of trial and error. The variables should include those that can be controlled or observed experimentally, but there are cases where others must also be included. For instance, the success of equilibrium thermodynamics originates from the fact that a few variables are sufficient to describe a static situation, and being few, these variables are easy to identify. On the other hand, in fluid dynamics the selection is more difficult. One must include many
more variables, such as the local densities of particles, momentum, and energy, that are neither controlled nor usually observed. The states of the system form an $n_\theta$-dimensional manifold with coordinates given by the numerical values $\theta^\mu$. A probability distribution $p(x|\theta)$ is associated with each state. In order to obtain the distribution that best reflects the prior information contained in $m(x)$ updated by the information $\theta$, we maximize the logarithmic relative entropy

$$S[p|m] = -\int dx p(x) \log \left( \frac{p(x)}{m(x)} \right)$$

subject to the constraints \(104\). The distribution obtained this way is

$$p(x|\theta) = \frac{1}{Z} m(x) \exp \left[ \lambda_\mu \theta^\mu (x) \right],$$

where the partition function $Z$ and the Lagrange multipliers $\lambda_\mu$ are given by

$$Z(\lambda) = \int dx m(x) \exp \left[ \lambda_\mu \theta^\mu (x) \right] \text{ and } -\frac{\partial \log Z(\lambda)}{\partial \lambda_\mu} = \theta^\mu.$$

Furthermore, the change involved in going from state $\theta$ to the state $\theta + d\theta$ can be measured by the extent to which the two distributions can be distinguished. As discussed in \[31\], except for an overall multiplicative constant, the measure of distinguishability is given by the "distance" $d\ell$ between $p(x|\theta)$ and $p(x|\theta + d\theta)$,

$$d\ell^2 = g_{\mu\nu}(\theta) d\theta^\mu d\theta^\nu,$$

where

$$g_{\mu\nu}(\theta) = \int dx p(x|\theta) \frac{\partial \log p(x|\theta)}{\partial \theta^\mu} \frac{\partial \log p(x|\theta)}{\partial \theta^\nu}$$

is the Fisher-Rao metric \[32, 33\]. This metric is unique, it is the only Riemannian metric that adequately reflects the fact that the states $\theta$ are probability distributions, not "structureless points".

### X. ENTROPIC DYNAMICS

The main objective of ED is deriving the expected trajectory of the system, assuming it evolves from a given initial state to a given final state. The entropic dynamical framework implicitly assumes that there exists a trajectory or, stated otherwise, that large changes are the result of a continuous succession of very many small changes. Therefore, the problem of studying large changes becomes the much simpler problem of studying small changes. Focusing on small changes and assuming that the change in going from the initial state $\theta_i$ to the final state $\theta_f = \theta_i + \Delta\theta$ is small...
enough, the distance $\Delta l$ between such states becomes

$\Delta l^2 = g_{\mu \nu} (\theta) \Delta \theta^\mu \Delta \theta^\nu. \quad (110)$

In what follows, we explain how to find which states are expected to lie on the trajectory between $\theta_i$ and $\theta_f$. First, in going from the initial to the final state the system must pass through a halfway point, that is, a state $\theta$ that is equidistant from $\theta_i$ and $\theta_f$. Chosen the halfway state, the expected trajectory of the system is determined. Indeed, there is nothing special about halfway states. Similarly, we could have argued that in going from the initial to the final state the system must first traverse a third of the way, that is, it must pass through a state that is twice as distant from $\theta_f$ as it is from $\theta_i$. In general, the system must pass through an intermediate states $\theta_\xi$ such that, having already moved a distance $dl$ away from the initial $\theta_i$, there remains a distance $\xi dl$ to be covered to reach the final $\theta_f$. Halfway states have $\xi = 1$, "third of the way" states have $\xi = 2$, and so on. Each different value of $\xi$ provides a different criterion to select the trajectory. If there are several ways to determine an (assumed) existing trajectory, consistency requires that all these ways should agree. The selected trajectory must be independent of $\xi$. Stated otherwise, the main ED problem becomes the following: "Initially, the system is in state $p(x|\theta_i)$ and new information is given to us. The system has moved to one of the neighboring states in the family $p(x|\theta_\xi)$. The problem becomes selecting the proper $p(x|\theta_\xi)$". This new formulation of the ED problem is precisely the kind of problem to be tackled using the ME method. Following [45] and what was reported in Chapter 2, we recall that the ME method is a method for processing information. It allows us to go from an old set of beliefs, described by the prior probability distribution, to a new set of beliefs, described by the posterior distribution, when the available information is just a specification of the family of distributions from which the posterior must be selected [46]. Usually, this family of posteriors is defined by the known expected values of some relevant variables. This is not necessary and the information-constraints need not be linear functionals. In ED, constraints are defined geometrically. Whenever one contemplates using the ME method, it is important to specify which entropy should be maximized. The selection of a distribution $p(x|\theta)$ requires that the entropies to be considered must be of the form

$S[p|q] = - \int dx p(x|\theta) \log \left( \frac{p(x|\theta)}{q(x)} \right). \quad (111)$

Equation (111) defines the entropy of $p(x|\theta)$ relative to the prior $q(x)$. The interpretation of $q(x)$ as the prior follows from the logic behind the ME method itself. As a side remark, following reference [45], I would like to recall that in the absence of new information there is no reason to change one’s mind. The selected posterior distribution should coincide with the prior distribution when there are no constraints. Since the distribution that maximizes $S[p|q]$
subject to no constraints is \( p \propto q \), we must set \( q(x) \) equal to the prior. That said, let us return to our ED problem. Assuming we know that the system is initially in state \( p(x|\theta_i) \) and we are not given the information that the system moved. Therefore, we have no reason to believe that any change has occurred. The prior \( q(x) \) should be chosen so that the maximization of \( S[p|q] \) subject to no constraints leads to the posterior \( p = p(x|\theta_i) \). The correct choice is \( q(x) = p(x|\theta_i) \). Instead, assuming we know that the system is initially in state \( p(x|\theta_i) \) and we are given the information that the system moved to one of the neighboring states in the family \( p(x|\theta_\xi) \), then the correct selection of the posterior probability distribution is obtained by maximizing the entropy

\[
S[\theta|\theta_i] = -\int dx p(x) \log \left( \frac{p(x|\theta)}{p(x|\theta_i)} \right),
\]

subject to the constraint \( \theta = \theta_\xi \). For the sake of simplicity, let us write \( \theta_\xi = \theta_i + d\theta \) and \( \theta_f = \theta_i + \Delta \theta \) so that \( S[\theta_\xi|\theta_i] \) becomes

\[
S[\theta_i + d\theta|\theta_i] = -\frac{1}{2} g_{\mu\nu}(\theta) d\theta^\mu d\theta^\nu,
\]

and the distances \( dl_i \) and \( dl_f \) from \( \theta_\xi \) to \( \theta_i \) and \( \theta_f \) are defined as

\[
dl_i^2 = g_{\mu\nu}(\theta) d\theta^\mu d\theta^\nu \quad \text{and} \quad \ndl_f^2 = g_{\mu\nu}(\theta)(\Delta \theta^\mu - d\theta^\mu)(\Delta \theta^\nu - d\theta^\nu).
\]

In order to maximize \( S[\theta_i + d\theta|\theta_i] \) under variations of \( d\theta \) subject to the constraint

\[
\xi \ndl_i = \ndl_f,
\]

we introduce a Lagrange multiplier \( \lambda \),

\[
\delta \left( -\frac{1}{2} g_{\mu\nu}(\theta) d\theta^\mu d\theta^\nu + \lambda (\xi^2 \ndl_i^2 - \ndl_f^2) \right) = 0.
\]

After some algebra, it can shown that

\[
d\theta^\mu = \eta \Delta \theta^\mu \quad \text{with} \quad \eta \equiv \left( 1 - \xi^2 - \frac{1}{2\lambda} \right)^{-1}.
\]

The multiplier \( \lambda \) and the quantity \( \eta \) are determined substituting back into the constraint \( \xi \ndl_i = \ndl_f \). From \( \ndl_i = \eta \Delta l \) and \( \ndl_f = (1 - \eta) \Delta l \), and therefore

\[
\eta = \frac{1}{1 + \xi} \quad \text{and} \quad \lambda = \frac{1}{2\xi (1 + \xi)}.
\]

Therefore, the intermediate state \( \theta_\xi \) selected by the maximum entropy method must satisfy the following relation

\[
dl_i + \ndl_f = \Delta l.
\]
The geometrical interpretation of (119) is straightforward: the triangle defined by the points \( \theta_i, \theta_\xi, \) and \( \theta_f \) degenerates into a straight line. This is sufficient to determine a short segment of the trajectory: all intermediate states lie on the straight line between \( \theta_i \) and \( \theta_f \). The generalization beyond short trajectories is immediate: if any three nearby points along a curve lie on a straight line the curve is a geodesic. This result is independent of the arbitrarily chosen value \( \xi \) so the potential consistency problem we mentioned before does not arise. Summarizing, the answer to the ED problem is the following [38]: "The expected trajectory is the geodesic that passes through the given initial and final states". As a final remark, we would like to point out that in ED the motion is predicted on the basis of a "principle of inference", the principle of maximum entropy, and not from a "principle of physics". ED is derived in an unusual way and one should expect some unusual features. Indeed, unusual features arise as soon as one asks any question concerning time. For example, ED determines the vector tangent to the trajectory \( \frac{\text{d} \theta^\mu}{\text{d}t} \), but not the actual "velocity" \( \frac{\text{d} \theta}{\text{d}t} \). This becomes clear since there is no reference to an external time \( t \) nowhere in the ED problem nor in any implicit background information. In order to find a relation between the distance \( l \) along the trajectory and the external time \( t \), additional information is required. In conventional forms of dynamics (ED is not a conventional form of dynamics) this information is implicitly encoded in a "principle of physics", in the Hamiltonian which fixes the evolution of a system relative to external clocks. However, the ED problem does not mention any external universe. The only clock available is the system itself, and the problem becomes that of deciding how this clock should be read. For instance, one of the variables \( \theta^\mu \) could be chosen as a clock variable and it could be arbitrarily called intrinsic time. Intrinsic time should be defined so that motion looks simple. Intrinsic time \( \tau \) may be considered as quantified change. A natural definition for the intrinsic time \( \tau \) consists in stipulating that the system moves with unit velocity, then \( \tau \) is given by the distance \( l \) itself, \( d\tau = dl \). A very special consequence of this definition of intrinsic time is that intervals between events along the trajectory are not known a priori. Intervals are determined only after the equations of motion are solved and the actual trajectory is determined. ED shares this peculiar feature with the theory of General Relativity (GR). In GR, as in ED, there is no reference to an external time. For instance, it is known that in GR the proper time interval along any curve between the initial and final three-dimensional geometries of space representing the given initial and final states is only determined after solving the Einstein equations of motion [48]. A serious impediment in understanding the classical theory of gravity is caused by the absence of an external time [49], since there is no clear understanding about which variables represent the true gravitational degrees of freedom. This absence of an external time gives rise to problems also in formulating a quantum theory of gravity [50], because of difficulties in defining equal-time commutators. In the following section, following reference [38], we point out some
 XI. CANONICAL FORMALISM FOR ENTROPIC DYNAMICS

ED can be derived from an "action" principle. The trajectory of the system is a geodesic and therefore the "action" is the length itself

\[ J[\theta] = \int_{\chi_i}^{\chi_f} d\chi \mathcal{L}(\theta, \dot{\theta}), \]  

(120)

where \( \chi \) is an arbitrary parameter along the trajectory. The Lagrangian \( \mathcal{L}(\theta, \dot{\theta}) \) is given by

\[ \mathcal{L}(\theta, \dot{\theta}) = \left( g_{\mu\nu} \dot{\theta}^\mu \dot{\theta}^\nu \right)^{\frac{1}{2}} \text{ and } \dot{\theta}^\mu = \frac{d\theta}{d\chi}. \]  

(121)

The action \( J[\theta] \) is invariant under reparametrizations \( \theta(\chi) \rightarrow \theta(f(\chi)) \) provided the end points are left unchanged, \( f(\chi_i) = \chi_i \) and \( f(\chi_f) = \chi_f \). Indeed, when the transformation is infinitesimal, \( f(\chi) = \chi + \varepsilon(\chi) \), the corresponding change in the action \( \delta J \) becomes,

\[ \delta J = \left( g_{\mu\nu} \dot{\theta}^\mu \dot{\theta}^\nu \right)^{\frac{1}{2}} \varepsilon(\chi)|_{\chi_i}^{\chi_f}. \]  

(122)

This change in equation (122) vanishes provided \( \varepsilon(\chi_i) = \varepsilon(\chi_f) = 0. \) As a side remark, we would like to point out that ED shares with GR the fact that both are generally covariant theories. Furthermore, as emphasized in [51], there is an important distinction between the symmetries of a generally covariant theory such as GR and the internal symmetries of a proper gauge theory. The action of a generally covariant theory (such as ED) is invariant under those reparametrizations that are restricted to map the boundary onto itself. For proper internal gauge transformations there are no such restrictions.

Recall that the standard Hamilton’s principle of least action for a nonrelativistic particle demands extremizing the action

\[ \int_{t_i}^{t_f} dt \left( \frac{m}{2} \delta_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} - U(x) \right) \]  

(123)

where \( t \) is "physical" time, and the interval between initial and final states \( t_f - t_i \) is given. Furthermore, recall that Jacobi’s principle of least action for a particle with energy \( E \) moving in a potential \( U(x) \) determines the trajectory by extremizing the action

\[ J[x] = \int_{\chi_i}^{\chi_f} d\chi \left( 2m \delta_{ij} \frac{dx^i}{d\chi} \frac{dx^j}{d\chi} \right)^{\frac{1}{2}} (E - U(x))^{\frac{1}{2}}. \]  

(124)
In Jacobi’s principle there is no reference to any time $t$. The time interval between initial and final states is not given, and the parameter $\chi$ is unphysical and arbitrary. The determination of the temporal evolution along the trajectory requires an additional constraint,

$$\frac{m}{2} \delta_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} + U(x) = E. \quad (125)$$

Therefore, we emphasize that the ED action [120] is an action of the Jacobi type, not of the Hamiltonian type. The natural choice for a supplementary constraint that defines $\tau$ and determines the evolution along the trajectory is given by

$$g_{\mu\nu} \frac{d\theta^\mu}{d\tau} \frac{d\theta^\nu}{d\tau} = 1. \quad (126)$$

It is known that GR is also described by a Jacobi-type action [52] and this leads to an additional formal similarity between GR and ED. In order to explore this similarity further it is convenient to construct the canonical Hamiltonian version of Jacobi’s action. The canonical momenta $\pi_\mu$ are defined as

$$\pi_\mu = \frac{\partial L}{\partial \dot{\theta}^\mu} = \frac{g_{\mu\nu} \dot{\theta}^\nu}{\left(g_{\alpha\beta} \dot{\theta}^\alpha \dot{\theta}^\beta\right)^{\frac{1}{2}}} \quad (127)$$

and have unit magnitude,

$$g^{\mu\nu} \pi_\mu \pi_\nu = 1. \quad (128)$$

The canonical Hamiltonian $H_{can}(\theta, \pi)$ vanishes identically,

$$H_{can}(\theta, \pi) = \dot{\theta}^\mu \pi_\mu - L(\theta, \dot{\theta}) \equiv 0, \quad (129)$$

because the Lagrangian is homogeneous of first degree in the $\dot{\theta}$’s. From a physics point of view, this is evident since the generator of time evolution (Hamiltonian) can be expected to vanish whenever there is no external time with respect to which the system could possibly evolve. We are led to consider the canonical action

$$\int_{\chi_i}^{\chi_f} d\chi \left( \dot{\theta}^\mu \pi_\mu - H_{can} \right) = \int_{\chi_i}^{\chi_f} d\chi \dot{\theta}^\mu \pi_\mu, \quad (130)$$

subject to the constrained variations of the momenta $\pi_\mu$ [128]. Therefore, the correct variational principle requires to extremize the action $I[\theta, \pi, N]$ given by

$$I[\theta, \pi, N] = \int_{\chi_i}^{\chi_f} d\chi \left[ \dot{\theta}^\mu \pi_\mu - Nh(\theta, \pi) \right] \quad (131)$$
where

$$h(\theta, \pi) = \frac{1}{2} g^{\mu\nu} \pi_\mu \pi_\nu - \frac{1}{2}$$

(132)

and $N(\chi)$ are Lagrange multipliers that enforce the constraint

$$h(\theta, \pi) = 0$$

(133)

for each value of $\chi$. For later convenience, the overall factor of $1/2$ in (132) is introduced (it amounts to rescaling $N$). Variation of $I[\theta, \pi, N]$ with respect to $\theta, \pi,$ and $N$ leads to the equations of motion,

$$\dot{\pi}_\mu = -N \frac{\partial h}{\partial \theta^\mu}, \dot{\theta}^\mu = N \frac{\partial h}{\partial \pi^\mu},$$

(134)

and (133). Of course, there is no equation of motion for $N$ and it must be determined from the constraint. It follows that,

$$N = \left(g_{\mu\nu} \dot{\theta}^\mu \dot{\theta}^\nu\right)^{\frac{1}{2}},$$

(135)

which, using the supplementary condition (126), implies

$$d\tau = N d\chi.$$  

(136)

The lapse function would be the analogue of $N$ in GR. Accordingly, the analogue of (133) in GR is called the Hamiltonian constraint. The quantity $N$ describes the increase of "intrinsic" time per unit increase of the unphysical parameter $\chi$. In terms of $\tau$ the equations of motion are given by

$$\frac{d\pi_\mu}{d\tau} = -\frac{\partial h}{\partial \theta^\mu} \text{ and } \frac{d\theta^\mu}{d\tau} = \frac{\partial h}{\partial \pi^\mu}.$$ 

(137)

In generally covariant theories (such as GR and ED) there is no canonical Hamiltonian (it vanishes identically) but there are constraints. Information-constraints play the role of generators of evolution and change in the unorthodox entropic dynamics.

**XII. CONCLUSIONS**

In this Chapter, following reference [38], we tried to describe the philosophy underlying the ED theoretical construct. We emphasized that entropic dynamics is formally similar to other generally covariant theories: the dynamics is reversible, the trajectories are geodesics, the system supplies its own notion of an intrinsic time, the motion can be
derived from a variational principle that turns out to be of the form of Jacobi’s action principle rather than the more familiar principle of Hamilton. Furthermore, we pointed out that “the canonical Hamiltonian formulation of ED is an example of a constrained information-dynamics where the information-constraints play the role of generators of evolution”. In conclusion, as a general remark, it would be worthwhile emphasizing that a reasonable physical theory must satisfy two key requirements: the first is that it must provide us with a set of mathematical models, the second is that the theory must identify real physical systems to which the models might possibly apply. The ED proposed in [38] satisfies the first requirement, but it fails with respect to the second. There are formal similarities with GR and whether Einstein’s theory of gravity will in the end turn out to be an example of ED is at this point no more than a speculation. A more definite answer may be achieved once the still unsettled problem of identifying those variables that describe the true degrees of freedom of the gravitational field is resolved [49, 50].

In what follows in this Thesis, I briefly explain the main idea that allowed Caticha and I to make progress in such ED theoretical construct. It is known that Caticha’s ultimate goal is to develop statistical geometrodynamics. The problem of GR is twofold: one is how geometry evolves, and the other is how particles move in a given geometry. My work on chaos focuses on how particles move in a given geometry and neglects the other problem (the evolution of the geometry). The realization that there exist two separate and distinct problems was a turning point in my research and lead to an unexpected result that I present in the next Chapters. Especially, in Chapter 6, I will argue that ED may lead to conventional physical theories such as Newtonian dynamics.
Chapter 4: The notion of chaos in physics

The notion of chaos in classical and quantum physics is introduced. The Zurek-Paz quantum chaos criterion of linear entropy growth (von Neumann entropy) is described. Moreover, I briefly review the basics of the conventional Riemannian geometric approach to chaos. Finally, the notion of Kolmogorov-Sinai dynamical entropy and that of Lyapunov exponents is introduced.

XIII. CLASSICAL CHAOTIC DYNAMICS

Since the time of Laplace and until rather recently most physicists believed that, given dynamic equations and initial conditions, the behavior of any macroscopic system can be reliably predicted. This confidence in the deterministic nature of classical physics amazingly coexisted with the experience of a large number of phenomena indicating the opposite: fluid turbulence, various kinds of plasma instabilities, games of chances (roulette, for example), and so on. The lack of predictability in this cases was attributed to "inessential" features, such as uncertainty of the initial conditions, the influence of uncontrolled external disturbances, or the participation of a very large number of degrees of freedom, which made predictions practically impossible.

Although the discovery of quantum mechanics destroyed the belief in the deterministic nature of physical systems at the microscopic scale, this was thought to be of no consequence for macroscopic systems that are well described by classical physics. Meanwhile, the understanding of the limited predictive capabilities of classical physics gathered force, especially due to the advent of electronic computers that permitted the systematic study of nonlinear dynamical systems. The importance of the remarkable work of H. Poincaré 53, who recognized the non-integrability of even simple dynamical systems and their chaotic properties, was not immediately recognized. Whereas, M. Born 54, for example, believed that the lack in long-term predictability of classical motions could be attributed to errors in initial data, the famous study of Fermi, Pasta and Ulam 55 surprisingly showed that not all nonlinear dynamical systems are characterized by stochastic behavior.

Eventually, mainly through the efforts of mathematicians, a qualitatively new concept of the nature of nonintegrable dynamical systems, namely local instability of the trajectories of the majority of nonlinear systems, was established. Numerous simple, apparently deterministic dynamical systems are characterized by extremely irregular and ultimately unpredictable motion, exclusively governed by the internal dynamics of the system. This chaotic behavior, which is in no way associated with the influence of external noise or uncertainty in the initial conditions, can be truly
called "dynamical stochasticity". Dynamical chaos is characteristic of many nonlinear dynamical systems in different branches of physics and other sciences: astronomy, chemistry, biology, meteorology, and even econometrics. In fact, chaotic behavior in mesoscopic systems is often the rule rather than the exception. This statement has a precise mathematical foundation in the famous Siegel theorem \[56\]: the nonintegrable Hamiltonians are dense among all analytic Hamiltonians (Hamiltonians that are infinitely differentiable and are locally described by a convergent power series \[57\]), but the integrable Hamiltonians are not. This allows us to say that nonintegrable systems are more abundant than the integrable ones.

Nonlinear dynamics was developed to its present form mainly by the efforts of mathematicians, such as Poincaré, Birkhoff, Siegel, Kolmogorov, Arnold, Moser, and Sinai.

As physicists we must be grateful for their far-sighted contributions, and we still have to explore the full implications of their deep insight into the physical world.

At the beginning of the last century, H. Poincaré \[58\] observed that a fully deterministic dynamics does not necessarily imply explicit predictions on the evolution of a dynamical system. This can be considered a milestone in the approach to the study of dynamical chaos. The content of the work by H. Poincaré and J. Hadamard is much more conceptually deep and subtle than I have resumed here. Anyway, chaos as an effect of instability of orbits in dynamical systems has remained for a long time a sort of pure mathematical subject.

Only in the fifties, the Kolmogorov-Arnold-Moser theorem (KAM) \[59\] and the numerical experiment on a chain of nonlinearly coupled oscillators by Fermi-Pasta-Ulam \[60\] have stressed again the fundamental relevance of dynamical chaos not only on a mathematical, but also on a physical ground. Later on, the works by E. Lorenz \[61\], M. Henon and C. Heiles \[62\] and B. V. Chirikov \[63\] have provided new insights on the origin of chaotic behaviors in dissipative as well as in conservative systems. The main conceptual improvement is the observation that dynamical chaos is not necessarily a consequence of the many degrees of freedom present in a system; on the other hand such a system, at the same time, can display, under certain conditions, ordered and very complex behaviors.

It is known that nonabelian gauge theories show a chaotic behavior in the classical limit. This manifests itself mainly in the rapid divergence of gauge fields configurations initially adjacent in configuration space which leads to a kind of saturation; a resemblance of a thermal state. Chaotic phenomena play an important role in the world of fundamental interactions, contributing to properties such as quark confinement, chiral symmetry breaking, and particle reactions at very high energy \[64\].
A. Integrable Dynamical Systems

The use of the $2n$-dimensional Hamiltonian phase space $\{(q_i, p_i); i = 1,\ldots, n\}$, where $q_i, p_i$ are the generalized coordinates and momenta, respectively, provides the ideal framework for the discussion of the concepts of integrability and local instability of trajectories. Assume that the Hamiltonian $H(q_i, p_i)$ is given in terms of analytic functions of the $q_i$ and $p_i$. From Hamiltonian equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (138)$$

it then follows Liouville’s theorem:

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \sum_i \left( \frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right) = 0, \quad (139)$$

where $\rho(p, q)$ is the constant phase space distribution along any trajectory in phase space. It is worthwhile mentioning that this particular property of Hamiltonian dynamics does not prevent the occurrence of dynamical stochastic motion (chaotic motion).

The integration of (138) depends on the existence, and identification, of so-called *integrals of motion*. In the Hamiltonian formalism, the time-dependence of dynamical quantities is conveniently expressed in terms of Poisson brackets. For some function $f(q_i, p_i, t)$ the total time derivative upon using (138) is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\} \equiv \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \left( \frac{\partial H}{\partial p_i} \frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial f}{\partial p_i} \right), \quad (140)$$

where $\{H, f\}$ is called the Poisson bracket of $H$ and $f$. If the function $f$ is explicitly time-independent, it is an integral (or constant) of motion if its Poisson bracket with the Hamiltonian $H$ vanishes. Evidently, for explicitly time-independent Hamiltonians, $H$ itself is a constant of motion, i.e. the total energy of the system is conserved.

A Hamiltonian system is said to be integrable, if there exist precisely $n$ independent isolating integrals of motion $I_i$:

$$I_i(p_1,\ldots, p_n; q_1,\ldots, q_n) = C_i, \quad i = 1,\ldots, n \quad (141)$$

where $C_i$ denotes the constant value of $I_i$. The independence of the quantities $I_i$ can be defined in terms of their mutual Poisson brackets:

$$\{I_i, I_j\} = 0 \, \forall i, j. \quad (142)$$

Usually, $H$ is taken as one of the constants of motion; this then implies the time-independence of the $I_i$: $\{H, I_i\} = 0$. 
Examples of integrable dynamical systems are all systems with a single degree of freedom described by an analytic Hamiltonian $\mathcal{H}(q, p)$ and all systems with $n$ degrees of freedom that are described by linear equations of motion. Such systems can be reduced to $n$ decoupled normal modes by linear transformation. There are examples of nonlinear dynamical systems that are integrable. A well-known example of a nonlinear integrable system is the so-called Toda chain corresponding to a chain of particles coupled by exponential two-body potentials. For a closed chain of three particles, the Toda Hamiltonian is:

$$\mathcal{H} = \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) + e^{-(q_1 - q_3)} + e^{-(q_2 - q_3)} + e^{-(q_3 - q_2)} - 3.$$ \hspace{1cm} (143)

Another familiar example of a completely integrable system is the two-body Kepler problem. Besides the constants of motion determined by space-time symmetry (energy and angular momentum), there exists a third integral of motion, the Runge-Lenz vector (as a result of a hidden dynamical $O(4)$ symmetry). The existence of this symmetry is the reason why the (elliptical) Kepler orbits are closed and the perihelion does not precess.

**B. Non-integrable (chaotic) dynamical system**

A dynamical system that is not integrable is called non-integrable. For instance, turbulent dynamical systems are non-integrable. However, turbulence and chaos are not synonyms! Turbulence is an example of a spatio-temporal complexity. Spatio-temporal complex dynamics is not yet completely understood, both from an experimental and a theoretical point of view. Turbulent motions are indeed chaotic, but chaotic motions need not be turbulent. Chaos may involve only a small number of degrees of freedom, that is it can be narrow band in space and/or time. There are numerous examples of chaotic systems characterized by temporal complexity but spatial simplicity, like the Lorenz’s system. Turbulence is different because it is always complex both in space and time. Turbulent motions are not time reversible even though it is governed by classical mechanics, i.e. Newton’s dynamics, which is time reversible.

1. *Chaos in the weather: the Lorenz model*

Sensitivity to initial conditions is what causes the seemingly unpredictable, long-term evolution of chaotic motion, because even a tiny error in the measurement of the initial conditions of a real dynamical system leads rapidly to a lack of predictability of its long-term behavior. As we cannot measure any real dynamical system with infinite precision, the long term prediction of chaotic motion in such systems is impossible, even if we know their equations of motion exactly.
The sensitive dependence on initial conditions of chaotic systems is more popularly known as the butterfly effect. This phenomenon was first discovered by Edward Lorenz during his investigation into a system of coupled ordinary differential equations (ODEs) used as a simplified model of 2D thermal convection, known as Rayleigh-Benard convection. These equations are now called the Lorenz equations, or Lorenz model.

In the Rayleigh-Benard convection there are two confining plates, a bottom plate at temperature \( T_b \) and a top plate at temperature \( T_t < T_b \). For small temperature differences between the two plates, heat is conducted through the stationary fluid between the plates. However, when \( T_b - T_t \) becomes large enough, buoyancy forces within the heated fluid overcome internal fluid viscosity and a pattern of counter-rotating, steady recirculating vortices is set up between the plates. Lorenz noticed that, in his simplified mathematical model of Rayleigh-Benard convection, very small differences in the initial conditions blew up and quickly led to enormous differences in the final behavior. Lorenz reasoned that if this type of behavior could occur in such a simple dynamical system, then it may also be possible in a much more complex physical system involving convection: the weather system. Thus, a very small perturbation, caused for instance by a butterfly flapping its wings, would lead rapidly to a complete change in future weather patterns. The Lorenz equations are

\[
\dot{x} = -\sigma (x - y), \quad \dot{y} = -xz + rx - y, \quad \dot{z} = xy - bz. 
\]  

This system has two nonlinearities, the \( xz \) term and the \( xy \) term, and exhibits both periodic and chaotic motion depending upon the values of the control parameters \( \sigma \), \( r \) and \( b \). The parameter \( \sigma \) is the Prandtl number which relates the energy losses within the fluid due to viscosity to those due to thermal conduction; \( r \) corresponds to the dimensionless measure of the temperature difference between the plates known as the Rayleigh number; \( b \) is related to the ratio of the vertical height of the fluid layer to the horizontal extent of the convective rolls within it. Note also that the variables \( x, y, z \) are not spatial coordinates but rather represent the convective overturning, horizontal temperature variation, and vertical temperature variation respectively.

XIV. WHAT IS QUANTUM CHAOS?

The problem of quantum chaos arose from the attempts to understand the very peculiar phenomenon of classical dynamical chaos in terms of quantum mechanics. Preliminary investigations immediately unveiled a very deep difficulty related to the fact that the two crucial properties of classical mechanics necessary for dynamical chaos to occur (continuous spectrum of motion and continuous phase space) are violated in quantum mechanics. Indeed, the
energy and frequency spectra of any quantum motion, bounded in phase space, are always discrete. According to the existing theory of dynamical systems such motion corresponds to the limiting case of regular motion. The ultimate origin of this fundamental quantum property is discreteness of the phase space itself or, in modern mathematical language, a non-commutative geometry of the latter. This is the very basis of all quantum physics directly related to the fundamental uncertainty principle which implies a finite size of an elementary phase-space cell,

\[ \Delta p \cdot \Delta x \simeq \hbar \text{ (per freedom).} \] (145)

The naive resolution of this difficulty would be the absence of any quantum chaos. For this reason it was even proposed to use the term "quantum chaology" which essentially means the study of the absence of chaos in quantum mechanics. If the above conclusions were true, a sharp contradiction would arise with the correspondence principle which requires the transition from quantum to classical mechanics for all phenomena including the new one: dynamical chaos. Does this really mean a failure of the correspondence principle as some authors insist? If it were so quantum chaos would, indeed, be a great discovery since it would mean that classical mechanics is not the limiting case of quantum mechanics but a distinct theory. Unfortunately, there exists a less radical (but also interesting and important) resolution of this difficulty.

A recent breakthrough in the understanding of quantum chaos has been achieved, particularly, due to a new philosophy which, either explicitly or implicitly, is generally accepted; namely the whole physical problem of quantum dynamics is considered as divided into two qualitatively different parts:

1. proper quantum dynamics as described by specific dynamical variables, the wavefunction \( \psi(t) \); and

2. quantum measurement including the recording of the result and hence the collapse of the wavefunction \( \psi(t) \).

The first part is described by some deterministic equation, for example, the Schrödinger equation and naturally belongs to the general theory of dynamical systems. The problem is well posed and this allows for extensive studies.

The second part still remains very vague to the extent that there is no common agreement even on the question whether this is a real physical problem or an ill-posed one so that the Copenhagen interpretation of quantum mechanics gives satisfactory answers to all the admissible questions. In any event, there exists as yet no dynamical description of quantum measurement including the \( \psi \)-collapse.

The absence of a classical-like chaos is true for the above mentioned first part of quantum dynamics only. Quantum measurement as far as the result is concerned, is a random process: all quantum measurements to date are thermodynamically irreversible. They are accompanied by an increase in entropy of the measured system together
with the measuring apparatus. This irreversibility can be used to locate the boundary between the classical and quantum domains. However, there are good reasons to believe that this randomness can be interpreted as a particular manifestation of dynamical chaos [71].

The separation of the first part of quantum dynamics, which is very natural from a mathematical point of view, was introduced by Schrodinger who, however, certainly underestimated the importance of the second part in physics.

A. The Zurek-Paz Quantum Chaos Criterion

In what follows, I will briefly describe the Zurek-Paz quantum chaos criterion of von Neumann’s linear entropy growth.

1. Introduction

It is now well-known that nonlinearity in classical systems generically leads to chaotic behavior. A necessary, but not sufficient, characteristic of this is a sensitive dependence of the orbits on initial conditions. Within this context the quantum analogue does not exist. Classical and quantum mechanics, however, are not very different when the dynamics of classical systems is rephrased in terms of the linear Liouville equation for phase space distributions. The sensitive dependence on initial conditions is mirrored, in this formulation, by the linear increase with time of the coarse-grained Gibbs entropy which is also known as the Shannon entropy. The quantum analogue of the Shannon entropy is the von Neumann entropy $\mathcal{S} \equiv S_{\text{von Neumann}}$ defined by

$$\mathcal{S} = - \text{tr}(\rho \log \rho),$$

(146)

$\rho$ being the density matrix of the system. For a Hamiltonian system unitary evolution implies

$$\frac{d\mathcal{S}}{dt} = 0.$$  

An ingredient which is crucial in quantum mechanics is measurement. Classically measurement on a system can be made such that there is an arbitrarily small disturbance on the system. In quantum mechanics this is not the case as can be appreciated from the Heisenberg uncertainty relations. The role of the environment in the quantum evolution of chaotic systems is well-known [72]. This line of thought suggests that a more natural way to restore the quantum classical correspondence is to consider physical systems not as being isolated from the rest of the universe but as undergoing constant and varied interactions with it. This is an inescapable fact of life and must be accounted for, at
least approximately, in any attempt to describe the real quantum behavior of microscopic and macroscopic objects. The destruction of phase coherence in a quantum system because of the continuous monitoring of its state by internal and external degrees of freedom is a process known as **decoherence**. In other words, decoherence is the loss of phase coherence between the set of preferred quantum states in the Hilbert space of the system due to the interaction with the environment.

The coupling of a quantum system to an environment turns on the decoherence process which leads to the emergence of classicality. In conclusion, classicality is an emergent property of an open quantum system (during a measurement process, information gets transformed from quantum to classical).

### 2. A toy model: the inverted harmonic oscillator

In a rigorous examination of the entropy approach to the classical-quantum correspondence problem Zurek and Paz have considered the completely tractable model of an inverted harmonic oscillator coupled to a high temperature (harmonic) bath. The Hamiltonian of the combined system is

\[
\mathcal{H}_{\text{tot}} = \mathcal{H}_{\text{system}} + \mathcal{H}_{\text{environment}} + \mathcal{H}_{\text{interaction}}
\]  

(147)

where \( \mathcal{H}_{\text{system}} \) is the inverted oscillator Hamiltonian

\[
\mathcal{H}_{\text{system}} (p, q) = \frac{p^2}{2} - \frac{\lambda^2 q^2}{2},
\]

(148)

\( \mathcal{H}_{\text{environment}} \) is the Hamiltonian of the chosen bath with canonical commutation relations \([q_n, p_m] = i\hbar\delta_{nm} \)

\[
\mathcal{H}_{\text{environment}} (p_n, q_n) = \sum_n \left( \frac{p_n^2}{2} + \frac{\omega_n^2 q_n^2}{2} \right),
\]

(149)

and \( \mathcal{H}_{\text{interaction}} \) is the Hamiltonian of interaction describing the (possibly time-dependent) coupling of the inverted harmonic oscillator, through its position variable of each of the environmental oscillators,

\[
\mathcal{H}_{\text{interaction}} (q, q_n) = -q c(t) \sum_n q_n.
\]

(150)

The potential energy function in is an inverted parabola with its apex at the origin. It is a model of instability in classical mechanics and the phase space dynamics governed by this Hamiltonian is an excellent model of a hyperbolic fixed point. It is certainly not a chaotic system — it lacks the folding requirement — but the parameter \( \lambda \) is analogous to a Lyapunov exponent in a genuinely chaotic system. This is because it induces the exponential rate of divergence (convergence) of nearby points on the unstable (stable) manifold in its 2-dimensional phase space. These
linear manifolds intersect at the origin, i.e. at the only fixed point of the dynamics. Let us consider the time evolution of a particle moving in the inverted oscillator potential \(V(q) = -\frac{\lambda q^2}{2}\), but now let us also consider it to be coupled through its position \(q\) to the position variables of each oscillator in the infinite set which we use as a model of a thermal bath at a high temperature. Further choosing the distribution of frequencies of this set to be of an Ohmic type \[75\] it is possible to derive a master equation for the reduced density matrix, \(\rho_r\), which describes the state of the particle at any time. The quantum Liouville equation reads \[8, 76\]

\[\frac{\partial \rho_r}{\partial t} = \frac{1}{i\hbar} [H, \rho_r] - \gamma (x - y) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \rho_r - \frac{D}{\hbar^2} (x - y)^2 \rho_r, \tag{151}\]

where \(\rho_r(x, y) = \langle x | \rho_r | y \rangle\) is the reduced density matrix in the position representation, \(D = 2m\gamma k_B T\) and \(\gamma\) describes the strength of coupling to the environment and serves as a dissipation parameter. Upon making the weak coupling assumption of \(\gamma \ll 1\), Zurek and Paz have solved the equation corresponding to the above for the Wigner function. This task is made considerably easier by the fact that the form of the potential for the inverted oscillator implies that all the quantum correction terms vanish identically and the quantum Liouville equation in Wigner’s representation becomes

\[\frac{\partial W}{\partial t} = -\lambda q \frac{\partial W}{\partial p} - p \frac{\partial W}{\partial q} + D \frac{\partial^2 W}{\partial p^2}. \tag{152}\]

For more general potentials, 3rd or higher order derivatives of \(V(q)\) would appear in \[152\]. On calculating the rate of change of von Neumann entropy it can be shown that \[8\]

\[\frac{dS(t)}{dt} \approx \lambda (1 + ae^{-2\lambda t})^{-1} t \rightarrow \infty \lambda \tag{153}\]

that is,

\[S(t) \uparrow \approx \lambda t \tag{154}\]

where \(S(t) = -tr(\rho_r(t) \log \rho_r(t))\) is the von Neumann entropy of the system, \(\rho_r(t)\) is the reduced density matrix of the system at time \(t\) and \(\lambda\) is the same as in \[148\]. Here \(a\) is a constant dependent on the initial choice of density matrix. The quantum entropy production rate is determined by the classical instability parameter \(\lambda\). Given that the classical Lyapunov exponent to which \(\lambda\) is analogous is equal to the Kolmogorov-Sinai (KS) entropy of the system, this is indeed a remarkable characterization. It suggests that after a time, a quantum, classically chaotic system loses information to the environment at a rate determined entirely by the rate at which the classical system loses information as a result of its dynamics, namely, the KS entropy. Notice that the KS entropy is not really an entropy.
It is an entropy per unit time, or an "entropy rate". Further details on this last remark will appear at the end of this Chapter.

The inverted harmonic oscillator is intended as a model of instability and, in fact, the dynamical behavior in phase space is dominated by a hyperbolic point at the origin. The unstable and stable directions and the rate at which initial phase space distributions expand and contract in these directions, respectively, are determined by $\lambda$. In this sense we call $\lambda$ an instability parameter analogous to a Lyapunov exponent in a classical chaotic system. Indeed, at any point on a trajectory the sum of the Lyapunov exponents is zero. For a chaotic trajectory there must be at least two nonzero Lyapunov exponents.

The Zurek-Paz conjecture is not free from criticisms [77]. Equation (154) has been derived with the help of various simplifying assumptions. Their conjecture is supported by the fact that hyperbolic points, such as that exhibited at the origin in the inverted oscillator model, are ubiquitous in the phase space of a chaotic system. The inverted harmonic oscillator model is, therefore, a possible representation of the local behavior in chaotic classical evolution.

However, there are a number of reasons why we should question any conclusions drawn as to the implications for a real chaotic system based on so simple a model. First, there are no quantum corrections to the Wigner function evolution for this quadratic potential (derivatives of third and higher order vanish, and with them the quantum corrections). The model does not allow for these influences on the dynamics, which, though small in the presence of an environment in comparison to the classical terms, nonetheless are generally always present. Moreover, Zurek and Paz let the dissipation parameter $\gamma \to 0$ while keeping $D = 2m\gamma k_B T$ constant. This means, essentially, increasing the environmental temperature and/or the mass of the particle. However, the dissipation term can never be dropped completely since this would entail setting $\gamma = 0$ identically, implying decoupling from the environment and hence $D = 0$ too. A third and related model-dependent assumption concerns the choice of a thermal bath as environment. Assumed are such features as a very large or even infinite number of degrees of freedom in the bath, the special choice of the density of frequencies of the oscillators which comprise it and the independent, non-interacting nature of these oscillators.

Moreover, phase-space considerations may be addressed as well. The stable and unstable manifolds associated with all hyperbolic points in Hamiltonian chaotic systems intersect one another and those intersection points are associated with other hyperbolic points (hyperbolic points are sometimes called saddle points; they are fixed points in phase space with the property that the trajectories are hyperbolae around them [78]). In this way homoclinic and heteroclinic points are formed (homoclinic points are intersections of the stable and unstable manifolds of the same
cycle point while heteroclinic points are intersections of the stable and unstable manifolds of different cycle points \(^{78, 79}\). The stable and unstable manifolds of the inverted oscillator intersect only at the hyperbolic origin in phase space. Clearly, therefore, the effect that the complicated distribution of homoclinic points might have on the open dynamics is not taken into account. Neither, of course, is the effect of heteroclinic points.

The inverted oscillator model is not chaotic. It has fixed stable and unstable directions which intersect at the origin of phase space, the only fixed point of the dynamics. Thus, it does not take into account the effect of elliptic points, homoclinic points, heteroclinic points, stable islands or cantori \(^{78}\) (cantori are invariant Cantor sets in the irregular or stochastic region of phase space remaining after destruction of KAM surfaces and create partial barriers to transport in chaotic regions) on the open quantum dynamics \(^{80}\). In short, it is not a good model of the extremely complex mixed phase spaces in which trajectories of generic Hamiltonian systems evolve. Importantly, too, the inverted oscillator model does not take into account the folding mechanism which, along with stretching, characterizes classical chaos. Indeed, the fixed direction of the stable and unstable manifolds are quite inadequate to represent the rapid change in the direction of the local stable and unstable manifolds typically seen in genuinely chaotic systems. In ignoring this essential ingredient of chaos one is, in effect, ignoring the fact that the directions of squeezing and contraction change rapidly along a typical trajectory. Farini et al. \(^{81}\) have illustrated the dangers of ignoring the folding effect by studying a driven particle in a quartic double well potential in the absence of an environment (see \(^{82}\) for a study with an environment).

Notwithstanding these objections, however, the inverted oscillator remains a tractable model of instability both for a closed system and for an open system in the presence of an environment. As such, it deserves attention for the insights it might give regarding the qualitative and maybe quantitative behavior of genuine, open quantum analogs of classically chaotic systems.

The purpose of studying such an elementary system is to build up some degree of intuition as to the behavior of quantum chaotic systems coupled to an environment. A priori the claims of applicability of an inverted oscillator to modeling a chaotic system should be treated with caution. To a certain degree, the results for the oscillator can serve as a guide to actual quantum behavior in chaotic systems. There is indeed some value in using the inverted harmonic oscillator as a toy model of instability in open quantum systems. The conjectures that follow from it should, however, be tested in more systems that are classically chaotic.
XV. GEOMETRY AND CHAOS

The investigations on the occurrence of regular and chaotic behavior in $N$-dimensional dynamical systems are performed with a variety of methods and mathematical tools. Recently, this ensemble widened with the inclusion of the Riemannian and Finslerian geometric approaches [5, 6].

As other new approaches, this tool has been suggested and applied to the study of stability properties of general dynamical systems, in the hope to bring the phenomenological analysis of their possibly chaotic behavior back to an inquiry directed towards an explanation, at least qualitative, of the mechanisms responsible for the onset of chaos. Within the framework of the geometrical picture, this explanation was sought through a possible link between a change in the curvature properties of the underlying manifold and a modification of the qualitative dynamical behavior of the system. Within the Hamiltonian approach, the ingredients needed to make chaos lie basically on the presence of stretching and folding of dynamical trajectories; i.e., in the existence of a strong dependence on initial conditions, which, together with a bound on the extension of the phase space, yield to a substantial unpredictability on the long time evolution of a system. Usually, the strong dependence is detected looking at the occurrence of an exponentially fast increase of the separation between initially arbitrarily close trajectories. To have true chaos, this last property must be however supplemented by the compactness of the ambient space where dynamical trajectories live, this simply in order to discard trivial exponential growths due to the unboundedness of the volume at disposal of the dynamical system. Stated otherwise, the folding is necessary in order to have a dynamics actually able to mix the trajectories, making practically impossible, after a finite interval of time, to discriminate between trajectories which were very nearby each other at the initial time. When the space isn’t compact, even in presence of strong dependence on initial conditions, it could be possible, in some instances (though not always), to distinguish among different trajectories originating within a small distance and then evolved subject to exponential instability. In the geometric description of dynamics, the recipe to find chaos is essentially the same, with some minor differences, which nevertheless prove sometimes to be very relevant for the understanding of the qualitative behavior of the system. When the geometrization procedure is accomplished, the study of dynamical trajectories is brought back to the analysis of a geodesic flow on a suitable manifold $\mathcal{M}$. In order to define chaos, $\mathcal{M}$ should be compact, and geodesics on it have to deviate exponentially fast.
XVI. RIEMANNIAN GEOMETRIZATION OF HAMILTONIAN DYNAMICS

Consider a classical Hamiltonian dynamical systems with $N$ degrees of freedom, confined in a finite volume (usually systems defined on a lattice are considered), whose Hamiltonian is of the form

$$H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + U(q_1, ..., q_N), \quad (155)$$

where the $q$'s and the $p$'s are, respectively, the coordinates and the conjugate momenta of the system. Our emphasis is on systems with a large number of degrees of freedom. The dynamics of the system (155) is defined in the $2N$-dimensional phase space spanned by the $q$'s and the $p$'s. It is possible to relate the dynamical and the statistical properties of the system (155) with the geometrical and topological properties of the phase space where the dynamical trajectories of the system live [5]. It turns out that as long as we consider Hamiltonians of the form (155) we can restrict ourselves to the study of the geometry and the topology of the $N$-dimensional configuration space (actually, an enlarged configuration space with two extra dimensions may also be considered) without losing information. In fact, the dynamical trajectories can be seen as geodesics of the configuration space, provided the latter has been endowed with a suitable metric.

A Hamiltonian system whose kinetic energy is a quadratic form in the velocities is referred to as a natural Hamiltonian system. Every Newtonian system, that is a system of particles interacting through forces derived from a potential, i.e. of the form (155), belongs to this class. The trajectories of a natural system can be seen as geodesics of a suitable Riemannian manifold. This classical result is based on a variational formulation of dynamics. In fact Hamilton’s principle states that the motions of a Hamiltonian system are the extrema of the functional (Hamiltonian action $\mathcal{I}$)

$$\mathcal{I} = \int L dt \quad (156)$$

where $L$ is the Lagrangian function of the system, and the geodesics of a Riemannian manifold are the extrema of the length functional

$$l = \int ds \quad (157)$$

where $s$ is the arc-length parameter. Once a connection between length and action is established, by means of a suitable choice of the metric, it will be possible to identify the geodesics with the physical trajectories.
A. Geometry and dynamics

Even if we restrict ourselves to the case of natural systems, the Riemannian formulation of classical dynamics is far from unique. There are many possible choices for the ambient space and its metric. The most commonly known choice — dating back to the nineteenth century — is the so-called Jacobi metric on the configuration space of the system. Actually this was the geometric framework of Krylov’s work [83]. There are other possibilities, for instance a metric originally introduced by Eisenhart on an enlarged configuration space-time [84], but this will not be discussed here. The choice of the metric to be used will be dictated mainly by convenience. These choices certainly do not contain all the possibilities of geometrizing conservative dynamics. For instance, with regard to systems whose kinetic energy is not quadratic in the velocities — the classical example is a particle subject to conservative as well as velocity-dependent forces, such as the Lorentz force — it is impossible to give a Riemannian geometrization, but becomes possible in the more general framework of a Finsler geometry [6]. However, we will not consider this here, and restrict ourselves to standard Hamiltonian systems.

1. The Jacobi metric tensor

Consider an autonomous dynamical system, i.e., a system with interactions which do not explicitly depend on time, whose Lagrangian can be written as

\[ \mathcal{L} = T - U = \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j - U(q_1, \ldots, q_N), \]  

(158)

where the dot stands for a derivative with respect to the parameter on which the \( q \)'s depend (such a parameter is the time \( t \) here, but could also be the arc-length \( s \)).

The Hamiltonian \( \mathcal{H} = T + U \) is an integral of motion, whose value, the energy \( E \), is a conserved quantity. Hence Hamilton’s principle can be cast in Maupertuis’ form [85]: the natural motions of the system are the stationary paths in the configuration space \( \Gamma \) for the functional

\[ \mathcal{F} = \int_{\gamma(t)} p_i dq^i = \int_{\gamma(t)} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \dot{q}^i dt \]  

(159)

among all the isoenergetic curves, i.e. the curves \( \gamma(t) \) connecting the initial and final points parametrized so that the Hamiltonian \( \mathcal{H}(p, q) \) is a constant equal to the energy \( E \). The fact that the curves must be isoenergetic with energy \( E \) implies that the accessible part of the configuration space is not the whole \( \Gamma \), but only the subspace \( \Gamma_E \subset \Gamma \) defined
by

$$\Gamma_E = \{ q \in \Gamma : U(q) \leq E \}. \quad (160)$$

In fact a curve $\gamma'$ that lies outside $\Gamma_E$ will never be parametrizable in such a way that the energy is $E$, because $\gamma'$ will then pass through points where $U > E$ and the kinetic energy is positive.

The kinetic energy $T$ is a homogeneous function of degree two in the velocities, hence Euler’s theorem implies that

$$2T = q^i \frac{\partial L}{\partial \dot{q}^i}, \quad (161)$$

and Maupertuis’ principle reads as

$$\delta F = \delta \int 2T dt = 0. \quad (162)$$

The configuration space $\Gamma$ of a dynamical system with $N$ degrees of freedom has a differentiable manifold structure, and the Lagrangian coordinates $(q_1, \ldots, q_N)$ can be regarded as local coordinates on $\Gamma$. The latter becomes a Riemannian manifold once a proper metric is defined. Consider systems of the form \[\text{[155]}\] where the kinetic energy matrix is given by $m_{ij}$. If we write

$$g_{ij} = 2 [E - U(q)] m_{ij}, \quad (163)$$

then, recalling that $T = \frac{1}{2} m_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt}$, equation \[\text{[162]}\] becomes

$$0 = \delta \int 2T dt = \delta \int (g_{ij} \dot{q}^i \dot{q}^j) \frac{1}{2} dt = \delta \int ds, \quad (164)$$

so that the motions are the geodesics of $\Gamma$ provided $ds$ is the arc-length element, i.e., the metric on $\Gamma$ is given by the tensor whose components are just the $g_{ij}$ defined in \[\text{[163]}\]. This metric is referred to as the \textit{Jacobi metric}, and its arc-length element is

$$ds^2 \equiv g_{ij} dq^i dq^j = 2 [E - U(q)] \frac{dq^i}{dt} \frac{dq^j}{dt} dt^2 = 4 [E - U(q)]^2 dt^2. \quad (165)$$

The geodesic equations written in the local coordinate frame $(q^1, \ldots, q^N)$ are

$$\frac{D \dot{q}^i}{ds} = \frac{d^2 q^i}{ds^2} + \Gamma^i_{jk} \frac{dq^j}{ds} \frac{dq^k}{ds} = 0, \quad (166)$$

where $D/ds$ is the covariant derivative along the curve $\gamma(s)$, $\dot{\gamma} = dq/ds$ is the velocity vector of the geodesic and the $\Gamma^i_{jk}$ are the Christoffel symbols. Using the definition of the Christoffel symbols, it is straightforward to show that \[\text{[166]}\] becomes

$$\frac{d^2 q^i}{ds^2} + \frac{1}{2(E-U)} \left[ 2 \frac{\partial (E-U)}{\partial q_j} \frac{dq^i}{ds} \frac{dq^j}{ds} - g^{ij} \frac{\partial (E-U)}{\partial q_j} g_{km} \frac{dq^k}{ds} \frac{dq^m}{ds} \right] = 0, \quad (167)$$
whence, using (165), Newton’s equations are recovered,

\[
\frac{d^2 q^i}{dt^2} = - \frac{\partial U}{\partial q_i}. \tag{168}
\]

Note that the Jacobi metric is obtained by a conformal change of the kinetic energy metric \(m_{ij}\). In fact the general result for the Riemannian geometrization of natural Hamiltonian dynamics \([86]\) states that given a dynamical system on a Riemannian manifold \((\Gamma, m)\), i.e., a dynamical system whose Lagrangian is

\[
\mathcal{L} = \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j - U(q), \tag{169}
\]

then it is always possible to find a conformal transformation of the metric,

\[
g_{ij} = \Phi(q) m_{ij} \tag{170}
\]

such that the geodesics of \((\Gamma, g)\) are the trajectories of the original dynamical system; this transformation is defined by

\[
\Phi(q) = E - U(q). \tag{171}
\]

### B. Stability and curvature

The study of the stability of the trajectories of a dynamical system finds a natural framework in the geometrization of the dynamics since it links the latter with the stability of the geodesics; this stability is completely determined by the curvature of the manifold, as shown below.

Studying the stability of the dynamics means determining the evolution of perturbations of a given trajectory. This implies that one should follow the evolution of the linearized (tangent) flow along the reference trajectory. For a Newtonian system, writing the perturbed trajectory as

\[
\tilde{q}^k(t) = q^k(t) + \eta^k(t), \tag{172}
\]

substituting this expression in the equations of motion

\[
\ddot{q}^k + \frac{\partial U(q)}{\partial q^k} = 0, \tag{173}
\]

and retaining terms up to first order in the \(\eta\)'s, one finds that the perturbation obeys the so-called tangent dynamics equation which reads as

\[
\ddot{\eta}^k + \left( \frac{\partial^2 U(q)}{\partial q^k \partial q^l} \right)_{q^k=q^k(t)} \eta^l = 0. \tag{174}
\]
This equation should be solved together with the dynamics in order to determine the stability or instability of the trajectory: when the norm of the perturbations grows exponentially, the trajectory is unstable, otherwise it is stable. Let us now translate the stability problem into geometric language. By writing, in close analogy to what has been done above in the case of dynamical systems, a perturbed geodesic as

\[ \tilde{q}^k(s) = q^k(s) + J^k(s), \] (175)

and then inserting this expression in the equation for the geodesics, one finds that the evolution of the perturbation vector \( J \) is given by the following equation:

\[
\frac{D^2 J^k}{ds^2} + R^k_{lmn} \frac{dq^l}{ds} \frac{dq^m}{ds} = 0, \tag{176}
\]

where \( R^k_{lmn} \) are the components of the Riemann curvature tensor. Equation (176) is referred to as the Jacobi equation, and the tangent vector field \( J \) as the Jacobi field. This equation was first studied by Levi-Civita and is also often referred to as the equation of Jacobi and Levi-Civita.

The remarkable fact is that the evolution of \( J \) — and then the stability or instability of the geodesic — is completely determined by the curvature of the manifold. Therefore, if the metric is induced by a physical system, as in the case of Jacobi or Eisenhart metrics, such an equation links the stability or instability of the trajectories to the curvature of the ambient manifold.

C. Mechanical manifolds and curvature

In this subsection, we have to give explicit expressions for the curvature of the mechanical manifolds, i.e., of those manifolds whose Riemannian structure is induced by the dynamics via the Jacobi or the Eisenhart metric.

We already observed that the Jacobi metric is a conformal deformation of the kinetic-energy metric, whose components are given by the kinetic energy matrix \( m_{lm} \). In the case of systems whose kinetic energy matrix is diagonal, this means that the Jacobi metric is conformally flat. This greatly simplifies the computation of curvatures. It is convenient to define then a symmetric tensor \( A_{lm} \) whose components are

\[
A_{lm} = \frac{N - 2}{4(E - U)^2} \left[ 2(E - U) \partial_l \partial_m U + 3 \partial_l U \partial_m U - \frac{\delta_{lm}}{2} \left| \nabla U \right|^2 \right], \tag{177}
\]

where \( U \) is the potential, \( E \) is the energy, and \( \nabla \) and \( || \) stand for the Euclidean gradient and norm, respectively. The curvature of \( (\Gamma, g_J) \) can be expressed through \( A_{lm} \). In fact, the components of the Riemann tensor are

\[
R_{ijkm} = \frac{1}{N - 2} \left[ A_{jk} \delta_{im} - A_{jm} \delta_{ik} + A_{im} \delta_{jk} - A_{ik} \delta_{jm} \right]. \tag{178}
\]
By contraction of the first and third indices, we obtain the Ricci tensor, whose components are

$$ R_{lm} = \frac{N - 2}{4(E - U)^2} \left[ 2(E - U) \partial_l \partial_m U + 3 \partial_l U \partial_m U \right] + \frac{\delta_{lm}}{4(E - U)} \left\{ 2(E - U) \Delta U - (N - 4) |\nabla U|^2 \right\}, \quad (179) $$

and by a further contraction we obtain the scalar curvature

$$ \mathcal{R} = \frac{N - 1}{4(E - U)} \left[ 2(E - U) \Delta U - (N - 6) |\nabla U|^2 \right], \quad (180) $$

where $\Delta = \nabla^2$. To summarize, the dynamical trajectories of a Hamiltonian system of the form \[155\] can be seen as geodesics of the configuration space once a suitable metric is defined. The general relationship which holds between dynamical and geometrical quantities regardless of the precise choice of the metric can be sketched as follows:

| Dynamics          | Geometry          |
|-------------------|-------------------|
| $t$-time          | $s$-arclength     |
| $U$-potential energy | $g$-metric       |
| $\partial U$-force | $\Gamma$-Christoffel symbols |
| $\partial^2 U, (\partial U)^2 \text{"curvature"}$ of the potential | $\mathcal{R}$-curvature of the manifold |

Furthermore, the stability of the dynamical trajectories can be mapped onto the stability of the geodesics, which is completely determined by the curvature of the manifold.

### D. Integrability and Killing Vectors

In the Riemannian geometrodynamical approach to chaos (Jacobi geometrodynamics), the strategy consists in making use of the Hamiltonian formulation of the dynamical system and then in reducing the dynamics to a geodesic flow. This reduction is performed at the level of the least action principle. The most fascinating feature of this approach is that the problem (often very complicated) of dynamics is reduced to geometrical properties of a single object — the manifold on which geodesic flow is induced. In the Jacobi reformulation, all of the dynamical information is collected into a single geometric object in which all the available manifest symmetries are retained.

For example, the sensitive dependence of trajectories on initial conditions, which is a key ingredient of chaos, can be investigated starting from the equation of geodesic equation. The integrability of the system, instead, is connected with the existence of Killing vectors and tensors on this manifold \[87, 88\]. Consider an $n$-dimensional manifold $\mathcal{M}$ with metric tensor $g_{\mu\nu}(x)$. Any 4-vector $\xi_\alpha(x)$ that satisfies the Killing equation

$$ \mathcal{L}_\xi g = \xi_\alpha ; \beta + \xi_\beta ; \alpha = 0 $$

\[182\]
is said to form a Killing vector of the metric \( g_{\mu\nu}(x) \). The quantity \( \mathcal{L}_\xi g \) is the Lie derivative of the metric tensor \( g \) along \( \xi \) while the semi-colon in \( \mathcal{L}_\xi g \) represents the standard covariant derivative on curved manifolds. For the Lie derivative \( \mathcal{L}_\xi g \) to vanish, requires that the geometry to be unchanged as one moves in the \( \xi \)-direction, that is, \( \xi \) represents a direction of symmetry of \( \mathcal{M} \). Killing vectors provide the first integrals of the geodesic equation. Killing equations in general are enormously difficult to solve. Construction of Killing vectors is a relatively simple task in the case of conformally flat spaces (which is the case in the majority of mechanical problems). For instance, a compact manifold \( \mathcal{M} \) with negative Ricci curvatures has no nontrivial Killing vector field (Theorem of Bochner, \[89\]). As a simple example, consider the Poincaré upper half plane \( \mathcal{M}_{\text{Poincaré}} = \{(x, y) : y > 0\} \) with the Poincaré line element

\[
ds^2 = \frac{1}{y} (dx^2 + dy^2).
\]

Since the metric coefficients are independent of \( x \), \( \xi \equiv \frac{\partial}{\partial x} \) is a Killing vector field. The vector \( \xi \) has a length \( \|\xi\| \) that tends to infinity as we approach the \( x \)-axis (\( y \to 0 \)).

An \( n \)-dimensional manifold \( \mathcal{M} \) is said to be maximally symmetric if it has \( \frac{n(n+1)}{2} \) Killing vectors. The most familiar examples of maximally symmetric spaces of Euclidean signatures are the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) and the \( n \)-dimensional spheres \( S^n \). For Euclidean signatures, the flat maximally symmetric spaces are planes or appropriate higher-dimensional generalizations, while the positively curved ones are spheres. Maximally symmetric Euclidean spaces of negative curvature are hyperboloids. There are, of course, maximally symmetric manifolds with Lorentzian signatures. The maximally symmetric spacetime with \( \mathcal{R} = 0 \) is simply Minkowski space. The positively curved maximally symmetric spacetime is called de Sitter space, while that with negative curvature is labeled anti-de Sitter space. In particular, any space with vanishing curvature tensor is maximally symmetric; the converse, however, is not true. If a manifold is maximally symmetric, the curvature is the same everywhere and the same in every direction. Hence, if we know the curvature of a maximally symmetric space at one point, we know it everywhere. Indeed, there are only a small number of possible maximally symmetric spaces; they are classified by scalar curvature \( \mathcal{R} \) (which will be constant everywhere), the dimensionality \( n \), the metric signature, and perhaps some discrete pieces of information relating to the global topology.

In any maximally symmetric space \( \mathcal{M} \), at any point, in any coordinate system,

\[
R_{\mu\nu\rho\sigma} = \frac{\mathcal{R}}{n \, (n-1)} \left( g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right).
\]

It may be useful to introduce the Weyl projective curvature tensor \( W_{\mu\nu\rho\sigma} \) defined as,

\[
W_{\mu\nu\rho\sigma} \overset{\text{def}}{=} R_{\mu\nu\rho\sigma} - \frac{\mathcal{R}}{n \, (n-1)} \left( g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right).
\]

The Weyl projective curvature tensor should not to be confused with Weyl’s conformal curvature tensor \[26\]. Weyl’s
projective tensor $W_{\mu\nu\rho\sigma}$ measures the deviation from isotropy of a given manifold. For an isotropic manifold $W_{\mu\nu\rho\sigma} = 0$. As a final remark, we emphasize that for a maximally symmetric manifold (isotropic manifold), the following relations among the scalar curvature $R$, Ricci curvature tensor $R_{\mu\nu}$ and Gaussian scalar curvature $K$ follow \[26\],

$$R \overset{\text{def}}{=} R_{\mu\nu\rho\sigma} g^{\mu\rho} g^{\nu\sigma} = \sum_{\rho \neq \sigma} K(e_\rho, e_\sigma) = n(n - 1) K,$$

and,

$$R_{\mu\nu} = (n - 1) K g_{\mu\nu}. \quad (186)$$

The quantities $K(e_\rho, e_\sigma)$ in \[185\] are the sectional curvatures of planes spanned by pairs of orthonormal basis elements. For two arbitrary vectors $a = a^\mu e_\mu$ and $b = b^\nu e_\nu$, the sectional curvature $K(a, b)$ is defined as \[90\],

$$K(a, b) \overset{\text{def}}{=} \frac{R_{\mu\nu\rho\sigma} a^\mu b^\nu a^\rho b^\sigma}{(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) a^\mu b^\nu a^\rho b^\sigma}. \quad (187)$$

**XVII. LYAPUNOV EXPONENTS AND THE KOLMOGOROV-SINAI ENTROPY**

In strict mathematical terms, chaotic motion is defined in terms of the long-term exponential divergence of neighboring trajectories in phase space. Neighboring orbits of integrable systems, on the other hand, are either performing stable oscillations around each other or diverge at most as a finite power of time.

The rate of exponential divergence is quantitatively measured by the positive Lyapunov exponents $\lambda_i > 0$. These exponents can be introduced in the context of general dynamical systems governed by the first-order differential equations,

$$\dot{x}_i = F_i(x_1, ..., x_N), \quad i = 1, ..., N. \quad (188)$$

Given a solution $\tilde{x}_i(t)$ of \[188\], we can linearize the equations of motion around this reference orbit and obtain a set of linear differential equations for the deviations $\delta x_i(t) = x_i(t) - \tilde{x}_i(t)$:

$$\frac{d}{dt} (\delta x_i(t)) = \sum_{j=1}^{N} \delta x_j(t) \left( \frac{\partial F_i}{\partial x_j} \right)_{x_j=\tilde{x}_j(t)}. \quad (189)$$

The length of the vector $\delta \tilde{x}(t)$,

$$d(t) = \left( \sum_{i, j=1}^{N} [\delta_{ij} \delta x_i(t) \delta x_j(t)] \right)^{\frac{1}{2}} = \left( \sum_{k=1}^{N} [\delta x_k(t)]^2 \right)^{\frac{1}{2}}, \quad (190)$$

provides a measure of the divergence of the two neighboring trajectories $\tilde{x}_i(t)$ and $x_i(t)$. The maximal Lyapunov exponent $\lambda_1$ is defined as the long-time average of its logarithmic growth rate:

$$\lambda_1 = \lim_{t \to \infty} \lim_{d(0) \to 0} \frac{1}{t} \log \left( \frac{d(t)}{d(0)} \right). \quad (191)$$
The double limit is required because the accessible phase space is usually bounded, and hence $d(t)$ cannot continue to grow forever, given a fixed initial distance $d(0)$. Regions of phase space for which $\lambda_1 > 0$ exhibit sensitive dependence on the initial conditions. An infinitesimal change in the initial data results in macroscopic deviations after a sufficiently long time:

$$d(t) \overset{t \to \infty}{\approx} d(0) \exp(\lambda_1 t).$$

(192)

The exponential instability of motion means positive maximal Lyapunov exponent $\lambda_1 > 0$.

An attractor is a subset of the manifold $\mathcal{M}$ toward which almost all sufficiently close trajectories converge asymptotically, covering it densely as the time goes on [78]. Strange attractors are called chaotic attractors. Chaotic attractors have at least one finite positive Lyapunov exponent. On the other hand, random (noisy) attractors have an infinite positive Lyapunov exponent, as no correlation exists between one point on the trajectory and the next (no matter how close they are).

If we calculate the Lyapunov exponent for orthogonal directions of maximum divergence in phase space, we obtain a set of Lyapunov exponents ($\lambda_1, \ldots, \lambda_n$) where $n$ is the dimension of the phase space. This set of Lyapunov exponents is known as the Lyapunov spectrum and is usually ordered from the largest positive Lyapunov exponent $\lambda_1$, down to the largest negative exponent, $\lambda_n$, i.e. maximum divergence to maximum convergence.

The reason why the exponentially unstable motion is called chaotic is that almost all trajectories are unpredictable in the following sense: according to the Alekseev-Brudno theorem [91] in the algorithmic theory of dynamical systems, the information $I(t)$ associated with a segment of a trajectory of length $|t|$ is equal asymptotically to [92]

$$h_{KS} = \lim_{|t| \to \infty} \frac{I(t)}{|t|} = \sum_{j} \lambda_j^{(+)}$$

(193)

where $\sum_{j} \lambda_j^{(+)}$ is the sum of all positive Lyapunov exponents and $h_{KS}$ is the so-called Kolmogorov-Sinai dynamical entropy (KS entropy; indeed, $h_{KS}$ is not really an entropy but an entropy per unit time, or an "entropy rate"), a statistical indicator of chaos. The quantity $I(t)$ in (193) is formally called the Kolmogorov algorithmic complexity [93] and, in other words, we may say that the Alekseev-Brudno theorem states that "the KS entropy measures the algorithmic complexity of classical trajectories" [94]. In computer science, the Kolmogorov complexity $I_U(x)$ of a string $x$ with respect to a universal computer $U$ (Turing machine) is defined as [94],

$$I_U(x) \overset{\text{def}}{=} \min_p \{ I(x) : U(p) = x \},$$

(194)

and it represents the minimum length over all binary programs $p$ that print $x$ and halt. Thus, $I_U(x)$ is the shortest description length of $x$ over all descriptions interpreted by the computer $U$. Equation (193) shows that in order to
predict each new segment of a chaotic trajectory, one needs an additional information proportional to the length of this segment and independent of the full previous length of trajectory (trajectories correspond to infinitely long strings $x$). This means that this information cannot be extracted from observation of the previous motion, even an infinitely long one! If the instability is not exponential but, for example, only a power law, then the required information per unit time is inversely proportional to the full previous length of the trajectory and, asymptotically, the prediction becomes possible. The important condition $h_{KS} > 0$, which characterizes chaotic motion, is not invariant with respect to the change of time variable. All Lyapunov exponents of an integrable system are zero. Conversely, the existence of a single positive Lyapunov exponent demonstrates the nonintegrability of a dynamical system.

Chaos is characterized by the positivity of at least one Lyapunov exponent of the system and therefore the significance of the concept of chaos depends essentially on the invariance of the Lyapunov exponents. It is known that under Lorentz transformations, $\lambda$ and $h_{KS}$ change, but their positivity is preserved for chaotic systems [95]. Under Rindler transformations [96], $\lambda$ and $h_{KS}$ change in such a way that systems, which are chaotic for an accelerated Rindler observer, can be nonchaotic for an inertial Minkowski observer. Therefore, the concept of chaos is observer-dependent [95]. Furthermore, a Lyapunov exponent (which is a "per time" measure of having exponential separation of nearby trajectories in "time") is not invariant under transformations of the "time" coordinate! The fact that the Lyapunov exponents are strongly gauge dependent quantities leads to additional problems connected with the characterization of chaos, especially in general relativity (where a gauge invariant measure of chaoticity is still missing). Therefore, the risk of inventing chaotic solutions in an artificial way is present and, because of that, extra care is needed in order to characterize "true chaos" [97].
Chapter 5: Curvature, entropy and Jacobi fields

In this Chapter, we study chaos in the context of two models the dynamics of which is entropic dynamics on curved statistical manifolds. Two chaotic entropic dynamical models are considered. The geometric structure of the statistical manifolds underlying these models is studied. It is found that in both cases, the resulting metric manifolds are negatively curved. Moreover, the geodesics on each manifold are described by hyperbolic trajectories. A detailed analysis based on the Jacobi-Levi-Civita equation for geodesic spread (JLC equation) is used to show that the hyperbolicity of the manifolds leads to chaotic exponential instability. A comparison between the two models leads to a relation among scalar curvature of the manifold ($R$), Jacobi field intensity ($J$) and information geometrodynamical entropy ($IGE, S_M$). The IGE is a convenient new tool we introduce to study chaotic entropic dynamics. We propose the IGE entropy as a new measure of chaoticity. These three quantities, $R$, $J$, and $S_M$ are suggested as useful indicators of chaoticity on curved statistical manifolds. Indeed, in analogy to the Zurek-Paz quantum chaos criterion (in its classical reversible limit), a classical information-geometric chaos criterion of linear IGE growth is suggested.

XVIII. INTRODUCTION

Entropic Dynamics (ED) [98] is a theoretical framework constructed on statistical manifolds to explore the possibility that the laws of physics, either classical or quantum, might be laws of inference rather than laws of nature. It is known that thermodynamics can be obtained by means of statistical mechanics which can be considered a form of statistical inference [1] rather than a pure "physical" theory. Indeed, even some features of quantum physics can be derived from principles of inference [99]. Recent research considers the possibility that Einstein’s theory of gravity is derivable from general principles of inductive inference [100]. Unfortunately, the search for the correct variables that encode relevant information about a system is a major obstacle in the description and understanding of its evolution. The manner in which relevant variables are selected is not straightforward. This selection is made, in most cases, on the basis of intuition guided by experiment. The Maximum relative Entropy (ME) method [101] is used to construct ED models. The ME method is designed to be a tool of inductive inference. It is used for updating from a prior to a posterior probability distribution when new information in the form of constraints becomes available. We use known techniques [98] to show that this principle leads to equations that are analogous to equations of motion. Information is processed using ME methods in the framework of Information Geometry (IG) [31] that is, Riemannian geometry applied to probability theory. In our approach, probability theory is a form of generalized logic of plausible inference.
It should apply in principle, to any situation where we lack sufficient information to permit deductive reasoning.

In this Chapter, we focus on two special entropic dynamical models. In the first model (ED1), we consider an hypothetical system whose microstates span a 2D space labelled by the variables $x_1 \in \mathbb{R}^+$ and $x_2 \in \mathbb{R}$. We assume that the only testable information pertaining to the quantities $x_1$ and $x_2$ consists of the expectation values $\langle x_1 \rangle$, $\langle x_2 \rangle$ and the variance $\Delta x_2$. In the second model (ED2), we consider a 2D space of microstates labelled by the variables $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$. In this case, we assume that the only testable information pertaining to the quantities $x_1$ and $x_2$ consists of the expectation values $\langle x_1 \rangle$ and $\langle x_2 \rangle$ and of the variances $\Delta x_1$ and $\Delta x_2$. Our models may be extended to more elaborate systems (highly constrained dynamics) where higher dimensions are considered. However, for the sake of clarity, we restrict our considerations to the above relatively simple cases. Given two known boundary macrostates, we investigate the possible trajectories of systems on the manifolds. The geometric structure of the manifolds underlying the models is studied. The metric tensor, Christoffel connections coefficients, Ricci and Riemann curvature tensors are calculated in both cases and it is shown that in both cases the dynamics takes place on negatively curved manifolds. The geodesics of the dynamical models are hyperbolic trajectories on the manifolds. A detailed study of the stability of such geodesics is presented using the equation of geodesic deviation (Jacobi equation). The notion of statistical volume elements is introduced to investigate the asymptotic behavior of a one-parameter family of neighboring geodesics. It is shown that the behavior of geodesics on such manifolds is characterized by exponential instability that leads to chaotic scenarios on the manifolds. These conclusions are supported by the asymptotic behavior of the Jacobi vector field intensity. Finally, a relation among entropy-like quantities, instability and curvature in the two models is presented.

**XIX. CURVED STATISTICAL MANIFOLDS**

In the case of ED1, a measure of distinguishability among the states of the system is achieved by assigning a probability distribution $p\left(\vec{x} | \vec{\theta}\right)$ to each state defined by expected values $\theta_1^{(1)}$, $\theta_1^{(2)}$, $\theta_2^{(2)}$ of the variables $x_1$, $x_2$ and $(x_2 - \langle x_2 \rangle)^2$. In the case of ED2, one assigns a probability distribution $p\left(\vec{x} | \vec{\theta}\right)$ to each state defined by expected values $\theta_1^{(1)}$, $\theta_2^{(1)}$, $\theta_1^{(2)}$, $\theta_2^{(2)}$ of the variables $x_1$, $(x_1 - \langle x_1 \rangle)^2$, $x_2$ and $(x_2 - \langle x_2 \rangle)^2$. The process of assigning a probability distribution to each state provides the statistical manifolds of the ED models with a metric structure. Specifically, the Fisher-Rao information metric [32] defined in (201) is used to quantify the distinguishability of probability distributions $p\left(\vec{x} | \vec{\theta}\right)$ that live on the manifold (the family of distributions $\left\{ p^{(\text{tot})} \left(\vec{x} | \vec{\theta}\right) \right\}$ is as a manifold, each distribution $p^{(\text{tot})} \left(\vec{x} | \vec{\theta}\right)$ is a point with coordinates $\theta^i$ where $i$ labels the macrovariables). As such, the Fisher-Rao metric assigns an IG to the space of states.
A. The Statistical Manifold $\mathcal{M}_{S_1}$

Consider a hypothetical physical system evolving over a two-dimensional space. The variables $x_1 \in \mathbb{R}^+$ and $x_2 \in \mathbb{R}$ label the 2D space of microstates of the system. We assume that all information relevant to the dynamical evolution of the system is contained in the probability distributions. For this reason, no other information (such as external fields) is required. We assume that the only testable information pertaining to the quantities $x_1$ and $x_2$ consists of the expectation values $\langle x_1 \rangle$, $\langle x_2 \rangle$ and the variance $\Delta x_2$. Therefore, these three expected values define the 3D space of macrostates $\mathcal{M}_{S_1}$ of the ED1 model. Each macrostate may be thought as a point of a three-dimensional statistical manifold with coordinates given by the numerical values of the expectations $\theta_1^{(1)}$, $\theta_1^{(2)}$, $\theta_2^{(2)}$. The available information can be written in the form of the following constraint equations,

$$\langle x_1 \rangle = \int_0^\infty dx_1 x_1 p_1 \left(x_1|\theta_1^{(1)}\right), \quad \langle x_2 \rangle = \int_{-\infty}^\infty dx_2 x_2 p_2 \left(x_2|\theta_1^{(2)}, \theta_2^{(2)}\right),$$

$$\Delta x_2 = \sqrt{\left(\langle x_2 \rangle - \langle x_2 \rangle^2\right)} = \left[\int_{-\infty}^\infty dx_2 \left(x_2 - \langle x_2 \rangle\right)^2 p_2 \left(x_2|\theta_1^{(2)}, \theta_2^{(2)}\right)\right]^{\frac{1}{2}},$$

where $\theta_1^{(1)} = \langle x_1 \rangle$, $\theta_1^{(2)} = \langle x_2 \rangle$ and $\theta_2^{(2)} = \Delta x_2$. The probability distributions $p_1$ and $p_2$ are constrained by the conditions of normalization,

$$\int_0^\infty dx_1 p_1 \left(x_1|\theta_1^{(1)}\right) = 1, \quad \int_{-\infty}^\infty dx_2 p_2 \left(x_2|\theta_1^{(2)}, \theta_2^{(2)}\right) = 1.$$  

Information theory identifies the exponential distribution as the maximum entropy distribution if only the expectation value is known. The Gaussian distribution is identified as the maximum entropy distribution if only the expectation value and the variance are known (see the simple example presented in Chapter 2). ME methods allow us to associate a probability distribution $p^{(\text{tot})}(\vec{x}|\theta)$ to each point in the space of states. The distribution that best reflects the information contained in the prior distribution $m(\vec{x})$ updated by the constraints ($\langle x_1 \rangle$, $\langle x_2 \rangle$, $\Delta x_2$) is obtained by maximizing the relative entropy

$$\left[S\left(\vec{\theta}\right)\right]_{\text{ED1}} = -\int_0^\infty \int_{-\infty}^\infty dx_1 dx_2 p^{(\text{tot})}(\vec{x}|\theta) \log \left[\frac{p^{(\text{tot})}(\vec{x}|\theta)}{m(\vec{x})}\right],$$

where $m(\vec{x}) \equiv m$ is the uniform prior probability distribution. The prior $m(\vec{x})$ is set to be uniform since we assume the lack of initial available information about the system (postulate of equal a priori probabilities). Upon maximizing (197), given the constraints (195) and (196), we obtain

$$p^{(\text{tot})}(\vec{x}|\theta) = p_1 \left(x_1|\theta_1^{(1)}\right) p_2 \left(x_2|\theta_1^{(2)}, \theta_2^{(2)}\right) = \frac{1}{\mu_1} e^{-\frac{x_1^2}{\mu_1}} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}},$$

(198)
where \( \theta^{(1)}_1 = \mu_1, \ \theta^{(2)}_1 = \mu_2 \) and \( \theta^{(2)}_2 = \sigma_2 \). The probability distribution (198) encodes the available information concerning the system and \( M_{s_1} \) becomes,

\[
M_{s_1} = \left\{ p^{(\text{tot})}(\vec{x}|\vec{\theta}) = \frac{1}{\mu_1} e^{-\frac{\mu_1^2}{\mu_1}} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(\sigma_2^2-\mu_2)^2}{2\sigma_2^2}} \right\}, \quad (199)
\]

where \( \vec{x} \in \mathbb{R}^+ \times \mathbb{R} \) and \( \vec{\theta} \equiv (\mu_1, \mu_2, \sigma_2) \). Note that we have assumed uncoupled constraints between the microvariables \( x_1 \) and \( x_2 \). In other words, we assumed that information about correlations between the microvariables did not need to be tracked. This assumption leads to the simplified product rule in (198). Coupled constraints however, would lead to a generalized product rule in (198) and to a metric tensor (201) with non-trivial off-diagonal elements (covariance terms). Correlation terms may be fictitious. They may arise for instance from coordinate transformations. On the other hand, correlations may arise from interaction of the system with external fields. Such scenarios would require more delicate analysis.

1. The Metric Tensor on \( M_{s_1} \)

We cannot determine the evolution of microstates of the system since the available information is insufficient. Instead we can study the distance between two total distributions with parameters \( (\mu_1, \mu_2, \sigma_2) \) and \( (\mu_1 + d\mu_1, \mu_2 + d\mu_2, \sigma_2 + d\sigma_2) \). Once the states of the system have been defined, the next step concerns the problem of quantifying the notion of change in going from the state \( \vec{\theta} \) to the state \( \vec{\theta} + d\vec{\theta} \). For our purpose a convenient measure of change is distance. The measure we seek is given by the dimensionless ”distance” \( ds \) between \( p(\vec{x}|\vec{\theta}) \) and \( p(\vec{x}|\vec{\theta} + d\vec{\theta}) \):

\[
ds^2 = g_{ij} d\theta^i d\theta^j, \quad (200)
\]

where

\[
g_{ij} = \int d\vec{x} p(\vec{x}|\vec{\theta}) \frac{\partial \log p(\vec{x}|\vec{\theta})}{\partial \theta^i} \frac{\partial \log p(\vec{x}|\vec{\theta})}{\partial \theta^j}
\]

is the Fisher-Rao information metric. Substituting (198) into (201), the metric \( g_{ij} \) on \( M_{s_1} \) becomes,

\[
(g_{ij})_{M_{s_1}} = \left( \begin{array}{ccc} \frac{1}{\mu_1} & 0 & 0 \\ 0 & \frac{1}{\sigma_2} & 0 \\ 0 & 0 & \frac{2}{\sigma_2^2} \end{array} \right).
\]

(202)

Substituting (202) into (200), the ”length” element reads,

\[
(ds^2)_{M_{s_1}} = \frac{1}{\mu_1^2} d\mu_1^2 + \frac{1}{\sigma_2^2} d\mu_2^2 + \frac{2}{\sigma_2^2} d\sigma_2^2.
\]

(203)
Notice that the metric structure of $\mathcal{M}_{s_1}$ is an emergent structure and is not itself fundamental. It arises only after assigning a probability distribution $p\left(\vec{x}|\vec{\theta}\right)$ to each state $\vec{\theta}$.

2. The Curvature of $\mathcal{M}_{s_1}$

In this paragraph we calculate the statistical curvature $R_{\mathcal{M}_{s_1}}$. This is achieved via application of standard differential geometric methods to the space of probability distributions $\mathcal{M}_{s_1}$. Recall the definitions of the Ricci tensor $R_{ij}$ and Riemann curvature tensor $R_{\alpha\mu\nu\rho}$,

$$R_{ij} = g^{ab} R_{aibj} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{ij} \Gamma^m_{kn} - \Gamma^m_{ik} \Gamma^k_{jm}, \quad (204)$$

and

$$R_{\alpha\mu\nu\rho} = \partial_\nu \Gamma^\alpha_{\mu\rho} - \partial_\rho \Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\beta\nu} \Gamma^\beta_{\mu\rho} - \Gamma^\alpha_{\beta\rho} \Gamma^\beta_{\mu\nu}, \quad (205)$$

The Ricci scalar $R_{\mathcal{M}_{s_1}}$ is obtained from (204) or (205) via appropriate contraction with the metric tensor $g_{ij}$ in (202), namely

$$\mathcal{R} = R_{ij} g^{ij} = R_{\alpha\beta\gamma\delta} g^{\alpha\gamma} g^{\beta\delta}, \quad (206)$$

where $g^{ik} g_{kj} = \delta^i_j$ so that $g^{ij} = (g_{ij})^{-1} = \text{diag}(\mu^2, \sigma^2, \sigma^2)$. The Christoffel symbols $\Gamma^k_{ij}$ appearing in (204) and (205) are defined by,

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} (\partial_i g_{mj} + \partial_j g_{im} - \partial_m g_{ij}). \quad (207)$$

Substituting (202) into (207), we calculate the non-vanishing components of the connection coefficients,

$$\Gamma^1_{11} = -\frac{1}{\mu_1}, \quad \Gamma^3_{22} = \frac{1}{2\sigma_2}, \quad \Gamma^3_{33} = -\frac{1}{\sigma_2}, \quad \Gamma^2_{23} = \Gamma^2_{32} = -\frac{1}{\sigma_2}. \quad (208)$$

By substituting (208) in (204) we determine the Ricci tensor components,

$$R_{11} = 0, \quad R_{22} = -\frac{1}{2\sigma_2}, \quad R_{33} = -\frac{1}{\sigma_2}. \quad (209)$$

The non-vanishing Riemann tensor component is,

$$R_{2323} = -\frac{1}{\sigma_2}. \quad (210)$$

Finally, by substituting (209) or (210) into (206) and using $(g_{ij})^{-1}$ we obtain the Ricci scalar,

$$R_{\mathcal{M}_{s_1}} = -1 < 0. \quad (211)$$
From (211) we conclude that $\mathcal{M}_{s_1}$ is a manifold of constant negative $(-1)$ curvature. We remark that the scalar curvature of $\mathcal{M}_{s_1}$ arises from the presence of the Gaussian distribution. Instead, the exponential distribution does not contribute to $\mathcal{R}_{\mathcal{M}_{s_1}}$.

**B. The Statistical Manifold $\mathcal{M}_{S_2}$**

In this case we assume that the 2D space of microstates of the system is labelled by the variables $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$. We assume, as in subsection (2.1), that all information relevant to the dynamical evolution of the system is contained in the probability distributions. Moreover, we assume that the only testable information pertaining to the quantities $x_1$ and $x_2$ consist of the expectation values $\langle x_1 \rangle$ and $\langle x_2 \rangle$ and of the variances $\Delta x_1$ and $\Delta x_2$. Therefore, these four expected values define the 4D space of macrostates $\mathcal{M}_{S_2}$ of the ED2 model. Each macrostate may be thought as a point of a four-dimensional statistical manifold with coordinates given by the numerical values of the expectations $\theta^{(1)}_1$, $\theta^{(1)}_2$, $\theta^{(2)}_1$, $\theta^{(2)}_2$. We emphasize the fact that entropic dynamic is not defined on the space of microstates but on the space of macrostates. The available information can be written in the form of the following constraint equations,

$$\langle x_1 \rangle = \int_{-\infty}^{+\infty} dx_1 x_1 p_1 \left( x_1 | \theta^{(1)}_1, \theta^{(1)}_2 \right), \quad \langle x_2 \rangle = \int_{-\infty}^{+\infty} dx_2 x_2 p_2 \left( x_2 | \theta^{(2)}_1, \theta^{(2)}_2 \right),$$

$$\Delta x_1 = \sqrt{\left( \langle x_1 \rangle - \langle x_1 \rangle \right)^2} = \left[ \int_{-\infty}^{+\infty} dx_1 \left( x_1 - \langle x_1 \rangle \right)^2 p_1 \left( x_1 | \theta^{(1)}_1, \theta^{(1)}_2 \right) \right]^{1/2},$$

$$\Delta x_2 = \sqrt{\left( \langle x_2 \rangle - \langle x_2 \rangle \right)^2} = \left[ \int_{-\infty}^{+\infty} dx_2 \left( x_2 - \langle x_2 \rangle \right)^2 p_2 \left( x_2 | \theta^{(2)}_1, \theta^{(2)}_2 \right) \right]^{1/2},$$

where $\theta^{(1)}_1 = \langle x_1 \rangle$, $\theta^{(1)}_2 = \Delta x_1$, $\theta^{(2)}_1 = \langle x_2 \rangle$ and $\theta^{(2)}_2 = \Delta x_2$. The probability distributions $p_1$ and $p_2$ are constrained by the conditions of normalization,

$$\int_{-\infty}^{+\infty} dx_1 p_1 \left( x_1 | \theta^{(1)}_1, \theta^{(1)}_2 \right) = 1, \quad \int_{-\infty}^{+\infty} dx_2 p_2 \left( x_2 | \theta^{(2)}_1, \theta^{(2)}_2 \right) = 1. \quad (213)$$

The distribution that best reflects the information contained in the uniform prior distribution $m(\vec{x}) \equiv m$ updated by the constraints ($\langle x_1 \rangle$, $\Delta x_1$, $\langle x_2 \rangle$, $\Delta x_2$) is obtained by maximizing the relative entropy

$$\left[ S \left( \vec{\theta} \right) \right]_{\text{ED2}} = -\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx_1 dx_2 p(\text{tot}) \left( \vec{x} | \vec{\theta} \right) \log \left[ \frac{p(\text{tot}) \left( \vec{x} | \vec{\theta} \right)}{m(\vec{x})} \right]. \quad (214)$$

Upon maximizing (214), given the constraints (212) and (213) we obtain

$$p^{(\text{tot})} \left( \vec{x} | \vec{\theta} \right) = \frac{1}{\sqrt{2\pi} \sigma_1} e^{-\frac{\left( x_1 - \mu_1 \right)^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi} \sigma_2} e^{-\frac{\left( x_2 - \mu_2 \right)^2}{2\sigma_2^2}}. \quad (215)$$
The probability distribution (215) encodes the available information concerning the system and $\mathcal{M}_{s_2}$ becomes,

$$
\mathcal{M}_{s_2} = \left\{ p^{(\text{tot})}(\vec{x}, \vec{\theta}) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(x_1 - \mu_1)^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(x_2 - \mu_2)^2}{2\sigma_2^2}} \right\},
$$

(216)

where $\vec{x} \in \mathbb{R} \times \mathbb{R}$ and $\vec{\theta} \equiv (\mu_1, \sigma_1, \mu_2, \sigma_2)$.

1. The Metric Tensor on $\mathcal{M}_{s_2}$

Proceeding as in (2.1.1), we determine the metric on $\mathcal{M}_{s_2}$. Substituting (215) into (201), the metric $g_{ij}$ on $\mathcal{M}_{s_2}$ becomes,

$$
(g_{ij})_{\mathcal{M}_{s_2}} = \begin{pmatrix}
\frac{1}{\sigma_1^2} & 0 & 0 & 0 \\
0 & \frac{1}{\sigma_1^2} & 0 & 0 \\
0 & 0 & \frac{1}{\sigma_2^2} & 0 \\
0 & 0 & 0 & \frac{1}{\sigma_2^2}
\end{pmatrix}.
$$

(217)

Finally, substituting (217) into (200), the "length" element reads,

$$
(ds^2)_{\mathcal{M}_{s_2}} = \frac{1}{\sigma_1^2} d\mu_1^2 + \frac{2}{\sigma_1^2} d\sigma_1^2 + \frac{1}{\sigma_2^2} d\mu_2^2 + \frac{2}{\sigma_2^2} d\sigma_2^2.
$$

(218)

2. The Curvature of $\mathcal{M}_{s_2}$

Proceeding as in (2.1.2), we calculate the statistical curvature $\mathcal{R}_{\mathcal{M}_{s_2}}$ of $\mathcal{M}_{s_2}$. Notice that $g^{ij} = (g_{ij})^{-1} = \text{diag}(\sigma_1^2, \sigma_1^2, \sigma_2^2, \sigma_2^2)$. Substituting (217) into (207), the non-vanishing components of the connection coefficients become,

$$
\Gamma^1_{12} = \Gamma^1_{21} = -\frac{1}{\sigma_1^2}, \quad \Gamma^2_{22} = -\frac{1}{\sigma_1^2}, \quad \Gamma^2_{11} = \frac{1}{2\sigma_1^2}, \quad \Gamma^3_{34} = \Gamma^3_{43} = -\frac{1}{\sigma_2^2}, \quad \Gamma^4_{33} = \frac{1}{2\sigma_2^2}, \quad \Gamma^4_{44} = -\frac{1}{\sigma_2^2}.
$$

(219)

By substituting (219) in (204) we determine the Ricci tensor components,

$$
R_{11} = -\frac{1}{2\sigma_1^2}, \quad R_{22} = -\frac{1}{\sigma_1^2}, \quad R_{33} = -\frac{1}{2\sigma_2^2}, \quad R_{41} = -\frac{1}{\sigma_2^2}.
$$

(220)

The non-vanishing Riemann tensor components are,

$$
R_{1212} = -\frac{1}{\sigma_1^2}, \quad R_{3434} = -\frac{1}{\sigma_2^2}.
$$

(221)

Finally, by substituting (220) or (221) into (206) and using $(g_{ij})^{-1}$, we obtain the Ricci scalar,

$$
\mathcal{R}_{\mathcal{M}_{s_2}} = -2 < 0.
$$

(222)

From (222) we conclude that $\mathcal{M}_{s_2}$ is a manifold of uncoupled Gaussian probability distributions of constant negative $(-2)$ curvature.
XX. THE ED MODELS

The dynamics can be derived from a standard principle of least action (Maupertuis-Euler-Lagrange-Jacobi-type) [85, 98]. The main differences are that the dynamics being considered here are defined on a space of probability distributions $\mathcal{M}_s$, not on an ordinary linear space $V$. Also, the standard coordinates $q_j$ of the system are replaced by statistical macrovariables $\theta^j$.

Given the initial macrostate and that the system evolves to a fixed final macrostate, we investigate the expected trajectories of the ED models on $\mathcal{M}_{s_1}$ and $\mathcal{M}_{s_2}$. The classical dynamics of a particle can be derived from the principle of least action in the Maupertuis-Euler-Lagrange-Jacobi form [85],

$$\delta J_{\text{Jacobi}}[q] = \delta \int_{s_i}^{s_f} ds \mathcal{F} \left( q_j, \frac{dq_j}{ds}, s, E \right) = 0,$$  \hspace{1cm} (223)

where $q_j$ are the coordinates of the system, $s$ is an affine parameter along the trajectory and $\mathcal{F}$ is a functional defined as

$$\mathcal{F} \left( q_j, \frac{dq_j}{ds}, s, E \right) = \left( 2 (E - U) \right)^{\frac{1}{2}} \left( \sum_{j,k} a_{jk} \frac{dq_j}{ds} \frac{dq_k}{ds} \right)^{\frac{1}{2}}.$$  \hspace{1cm} (224)

For a non-relativistic system, the energy $E$ is,

$$E = T + U (q) = \frac{1}{2} \sum_{j,k} a_{jk} (q) \dot{q}_j \dot{q}_k + U (q)$$  \hspace{1cm} (225)

where the coefficients $a_{jk}$ are the reduced mass matrix coefficients and $\dot{q} = \frac{dq}{ds}$ is the time derivative of the canonical coordinate $q$. We now seek the expected trajectory of the system assuming it evolves from $\theta^{\mu}_{\text{old}} = \theta^\mu \equiv (\mu_1 (s_i), \mu_2 (s_i), \sigma_2 (s_i))$ to $\theta^{\mu}_{\text{new}} = \theta^\mu + d\theta^\mu \equiv (\mu_1 (s_f), \mu_2 (s_f), \sigma_2 (s_f))$. Such a system moves along a geodesic in the space of states, which is a curved manifold with the appropriately chosen metric [98]. Since the trajectory of the system is a geodesic, the ED-action is itself the length; that is,

$$J_{\text{ED}} [\theta] = \int (ds)^2 \left( g_{ij} d\theta^i d\theta^j \right)^{\frac{1}{2}} = \int_{s_i}^{s_f} ds \left( g_{ij} \frac{d\theta^i}{ds} \frac{d\theta^j}{ds} \right)^{\frac{1}{2}} \equiv \int_{s_i}^{s_f} ds \mathcal{L}(\theta, \dot{\theta})$$ \hspace{1cm} (226)

where $\dot{\theta} = \frac{d\theta}{ds}$ and $\mathcal{L}(\theta, \dot{\theta})$ is the Lagrangian of the system,

$$\mathcal{L}(\theta, \dot{\theta}) = (g_{ij} \dot{\theta}^i \dot{\theta}^j)^{\frac{1}{2}}.$$  \hspace{1cm} (227)

A useful choice for $s$ is one satisfying the condition $g_{ij} \frac{d\theta^i}{ds} \frac{d\theta^j}{ds} = 1$. Therefore, we formally identify the affine parameter $s$ with the temporal evolution parameter $\tau$, $s \equiv \tau$. Performing a standard calculus of variations with $s \equiv \tau$, we obtain

$$\delta J_{\text{ED}} [\theta] = \int d\tau \left( \frac{1}{2} \frac{\partial g_{ij}}{\partial \theta^k} \dot{\theta}^i \dot{\theta}^j - \frac{d\dot{\theta}^k}{d\tau} \right) \delta \theta^k = 0, \forall \delta \theta^k.$$  \hspace{1cm} (228)
Note that from (228), \( \frac{d\theta^k}{d\tau} = \frac{1}{2} \frac{\partial g_{ij}}{\partial \theta^k} \dot{\theta}^i \dot{\theta}^j \). This differential equation shows that if \( \frac{\partial g_{ij}}{\partial \theta^k} = 0 \) for a particular \( k \) then the corresponding \( \dot{\theta}^k \) is conserved. This suggests to interpret \( \dot{\theta}^k \) as momenta. Equations (228) and (207) lead to the geodesic equation,

\[
\frac{d^2 \theta^k(\tau)}{d\tau^2} + \Gamma^k_{ij} \frac{d\theta^i(\tau)}{d\tau} \frac{d\theta^j(\tau)}{d\tau} = 0. \tag{229}
\]

Observe that (229) are nonlinear, second order coupled ordinary differential equations. These equations describe a dynamics that is reversible and their solution is the trajectory between an initial and a final macrostate. The trajectory can be equally well traversed in both directions.

A. Model ED1: Geodesics on \( M_{s_1} \)

We seek the explicit form of (229) for ED1. Substituting (208) in (229), we obtain,

\[
\begin{align*}
\frac{d^2 \mu_1}{d\tau^2} - & \frac{1}{\mu_1} \left( \frac{d\mu_1}{d\tau} \right)^2 = 0, \\
\frac{d^2 \mu_2}{d\tau^2} - & \frac{2}{\sigma_2} \frac{d\mu_2}{d\tau} \frac{d\sigma_2}{d\tau} = 0, \\
\frac{d^2 \sigma_2}{d\tau^2} - & \frac{1}{\sigma_2} \left[ \left( \frac{d\sigma_2}{d\tau} \right)^2 - \frac{1}{2} \left( \frac{d\mu_2}{d\tau} \right)^2 \right] = 0. \tag{230}
\end{align*}
\]

Integrating this set of differential equations, we obtain

\[
\begin{align*}
\mu_1(\tau) &= A_1 \left[ \cosh(\alpha_1 \tau) - \sinh(\alpha_1 \tau) \right], \\
\mu_2(\tau) &= B_1 \frac{1}{2\beta_1} \cosh(2\beta_1 \tau) - \sinh(2\beta_1 \tau) + \frac{B_2}{8\beta_1} + C_1, \\
\sigma_2(\tau) &= B_3 \frac{[\cosh(\beta_1 \tau) - \sinh(\beta_1 \tau)]}{\cosh(2\beta_1 \tau) - \sinh(2\beta_1 \tau) + \frac{B_1}{8\beta_1}} + C_2. \tag{231}
\end{align*}
\]

The integration constants arising from the exponential contribution to the geodesic equations are \( A_1 = A_1(\mu_1(0), \dot{\mu}_1(0)) = \mu_1(0) \) and \( \alpha_1 = (\mu_1(0), \dot{\mu}_1(0)) = -\frac{\dot{\mu}_1(0)}{\mu_1(0)} \in \mathbb{R}^+ \) with \( \dot{\mu}_1(0) = \left( \frac{d\mu_1(\tau)}{d\tau} \right)_{\tau=0} \).

The integration constants arising from the Gaussian contribution to the geodesic equations are \( B_1 = B_1(\mu_2(0), \sigma_2(0), \dot{\mu}_2(0), \dot{\sigma}_2(0)) \), \( C_1 = C_1(\mu_2(0), \sigma_2(0), \dot{\mu}_2(0), \dot{\sigma}_2(0)) \), \( C_2 = C_2(\mu_2(0), \sigma_2(0), \dot{\mu}_2(0), \dot{\sigma}_2(0)) \) and \( \beta_1 = (\mu_2(0), \sigma_2(0), \dot{\mu}_2(0), \dot{\sigma}_2(0)) \). In total, there are the six real integration constants since there are six initial
conditions: three for the initial values of the macrovariables labelling points on $M_{s_1}$, and three for the initial values of the first derivative of the macrovariables. However, the normalization constraint 
\[
g_{ij} \left( \frac{d \theta^i}{d s} \frac{d \theta^j}{d s} \right)^{\frac{1}{2}} = \left( \dot{\theta}_j \dot{\theta}_i \right)^{\frac{1}{2}} = 1
\]
leads to five independent initial conditions.

Note that the set of equations (231) parametrizes the evolution surface of the statistical submanifold $m_{s_1}$ of $M_{s_1}$,
\[
m_{s_1} = \left\{ p^{(tot)}(\vec{x}|\vec{\theta}) \in M_{s_1}: \vec{\theta} \text{ satisfy (230)} \right\}.
\]
(232)
The set of points in $m_{s_1}$ are the expected intermediate macrostates of the system in its evolution from a given initial macrostate $\vec{\theta}_i \equiv (\mu_1(0), \mu_2(0), \sigma_2(0))$ to a given final macrostate $\vec{\theta}_f \equiv (\mu_1(\tau), \mu_2(\tau), \sigma_2(\tau))$. In other words, the measure of $m_{s_1}$ describes the portion of statistical volume in configuration space accessed by the system in its information-constrained evolution between two given points of the manifold $M_{s_1} \supset m_{s_1}$. A different set of initial conditions would lead to consider a different submanifold $m'_{s_1}$, $M_{s_1} \supset m'_{s_1} \neq m_{s_1}$.

B. Model ED2: Geodesics on $M_{s_2}$

We seek the explicit form of (229) for ED2. Substituting (219) in (229), we obtain,
\[
\frac{d^2 \mu_1}{d \tau^2} - \frac{2}{\sigma_1} \frac{d \mu_1}{d \tau} \frac{d \sigma_1}{d \tau} = 0,
\]
\[
\frac{d^2 \sigma_1}{d \tau^2} - \frac{1}{\sigma_1} \left[ \left( \frac{d \sigma_1}{d \tau} \right)^2 - \frac{1}{2} \left( \frac{d \mu_1}{d \tau} \right)^2 \right] = 0,
\]
\[
\frac{d^2 \mu_2}{d \tau^2} - \frac{2}{\sigma_2} \frac{d \mu_2}{d \tau} \frac{d \sigma_2}{d \tau} = 0,
\]
\[
\frac{d^2 \sigma_2}{d \tau^2} - \frac{1}{\sigma_2} \left[ \left( \frac{d \sigma_2}{d \tau} \right)^2 - \frac{1}{2} \left( \frac{d \mu_2}{d \tau} \right)^2 \right] = 0.
\]
(233)
Integrating this set of differential equations, we obtain
\[
\mu_1(\tau) = \frac{A_2^2}{2 \alpha_2} \frac{1}{\cosh (2 \alpha_2 \tau) - \sinh (2 \alpha_2 \tau)} \frac{1}{\cosh (2 \alpha_2 \tau) - \sinh (2 \alpha_2 \tau)} + C_1,
\]
\[
\sigma_1(\tau) = \frac{A_2^2}{2 \alpha_2} \frac{\cosh (\alpha_2 \tau) - \sinh (\alpha_2 \tau)}{\cosh (2 \alpha_2 \tau) - \sinh (2 \alpha_2 \tau)} + C_2,
\]
\[
\mu_2(\tau) = \frac{B_2^2}{2 \beta_2} \frac{1}{\cosh (2 \beta_2 \tau) - \sinh (2 \beta_2 \tau)} + C_3,
\]

\[
\sigma_2(\tau) = B_2 \frac{[\cosh(\beta_2 \tau) - \sinh(\beta_2 \tau)]}{\cosh(2\beta_2 \tau) - \sinh(2\beta_2 \tau) + \frac{B_2^2}{8\beta_2^2}} + C_4. \tag{234}
\]

The eight integration constants \(A_2, B_2, C_1, C_2, C_3, C_4, \alpha_2\) and \(\beta_2\) assume real values and they are not independent from each other. They satisfy the normalization condition

\[
\left( g_{ij} \frac{\partial \theta^i}{\partial x^s} \frac{\partial \theta^j}{\partial x^d} \right)^{1/2} = \left( \dot{\theta}_j \dot{\theta}_j \right)^{1/2} = 1.
\]

Furthermore, they are functions of the initial values of the macrovariable \(\vec{\theta} \equiv (\mu_1, \mu_2, \sigma_1, \sigma_2)\) and of its first derivative \(\frac{d\vec{\theta}}{d\tau} \equiv (\dot{\mu}_1, \dot{\mu}_2, \dot{\sigma}_1, \dot{\sigma}_2)\).

Again, note that the set of equations (234) parametrizes the evolution surface of the statistical submanifold \(m_{s_2}\) of \(M_{s_2}\),

\[
m_{s_2} = \left\{ p^{(tot)} \left( \vec{x} | \vec{\theta} \right) \in M_{s_1} : \vec{\theta} \text{ satisfy (233)} \right\}. \tag{235}
\]

The set of points in \(m_{s_1}\) are the expected intermediate macrostates of the system in its evolution from a given initial macrostate \(\vec{\theta}_i \equiv (\mu_1(0), \mu_2(0), \sigma_1(0), \sigma_2(0))\) to a given final macrostate \(\vec{\theta}_f \equiv (\mu_1(\tau), \mu_2(\tau), \sigma_1(\tau), \sigma_2(\tau))\). In other words, the measure of \(m_{s_2}\) describes the portion of statistical volume in configuration space accessed by the system in its information-constrained evolution between two given points of the manifold \(M_{s_2} \supset m_{s_2}\). A different set of initial conditions would lead to consider a different submanifold \(m'_{s_2}, M_{s_2} \supset m'_{s_2} \neq m_{s_2}\).

XXI. CHAOTIC INSTABILITY IN THE ED MODELS

The Riemannian curvature of a manifold is closely connected with the behavior of the geodesics on it, i.e., with the motion of the corresponding dynamical system [85]. If the Riemannian curvature of a manifold is positive (as on a sphere or ellipsoid), then the nearby geodesics oscillate about one another in most cases; whereas if the curvature is negative (as on the surface of a hyperboloid of one sheet), geodesics rapidly diverge from one another.

A. Instability in ED1

In this subsection, the stability of ED1 is considered. It is shown that neighboring trajectories are exponentially unstable under small perturbations of initial conditions. In the rest of the Chapter, we only assume that in ED1 we have \(\beta_1 = \alpha_1 = -\frac{\mu_1(0)}{\mu_1(0)} \equiv \alpha \in \mathbb{R}^+\); in ED2, we assume we have \(\alpha_2 = \beta_2 \equiv \alpha \in \mathbb{R}^+\). We make this choice because we are selecting the quantity "\(\alpha\)" as the common parameter (the one-parameter characterizing families of geodesics on configuration spaces) labelling different geodesics on manifolds \(M_{s_1}\) and \(M_{s_2}\) and our objective is to compare the degree of chaoticity of both dynamics (ED1 and ED2) on their respective underlying curved statistical manifolds.

At this stage of our discussion, the parameter "\(\alpha\)" is a quantity extracted from the continuous entropic dynamical
evolution equations (231) and (234). Of course, a different set of initial conditions would lead to a different parameter \( \alpha \). Once the choice is made, we assume that \( \alpha \) is roughly constant over accessible region of configuration space. This assumption would not be new in the literature, see reference \[103\] for example. In the following sections, we will discover that \( \alpha \) plays the role of a standard Lyapunov exponent.

We could show that \( \alpha \) does play the role of a Lyapunov exponent already at this stage of our discussion. Lyapunov exponents are asymptotic quantities: they are defined in the limit as time approaches infinity. The finite Lyapunov exponent in the direction \( e \in \mathbb{R}^n \) of a trajectory \( x(\tau, x_0) \) satisfying the differential equation \( \dot{x} = A(\tau) x \) with \( x \in \mathbb{R}^n \) and with initial condition \( x(0, x_0) = x_0 \) is defined as \[104\],

\[
\lambda(e) \equiv \lim_{\tau \to \infty} \log \left[ \frac{\sqrt{\langle Xe, Xe \rangle}}{\sqrt{\langle e, e \rangle}} \right].
\] (236)

The brackets \( \langle \cdot, \cdot \rangle \) in (236) denote the standard scalar product in \( \mathbb{R}^n \). \( X = X(\tau; x(\tau, x_0)) \) is the asymptotically regular fundamental matrix of the differential equation \( \dot{x} = A(\tau) x \) \[103\]. In the ED1 model, the differential equation to consider is

\[
\frac{d\vec{\theta}_{ED1}(\tau)}{d\tau} = A_{ED1}(\tau) \vec{\theta}_{ED1}(\tau),
\] (237)

where \( \vec{\theta}_{ED1}(\tau) \equiv (\mu_1(\tau), \mu_2(\tau), \sigma_2(\tau)) \) are given in (231). In the asymptotic limit, the \( 3 \times 3 \) matrix \( A_{ED1}(\tau) \) can be approximated by a diagonal matrix with constant coefficients, \( A_{ED1}(\tau) \underset{\tau \to \infty}{\approx} \text{diag}(\alpha_1, 0, \beta_1) \). A straightforward calculation would lead to an asymptotically regular \( 3 \times 3 \) fundamental matrix

\[
X_{ED1}(\tau) \underset{\tau \to \infty}{\approx} \text{diag}(c_1 \exp(\alpha_1 \tau), c_2 \tau, c_3 \exp(\beta_1 \tau))
\] (238)

with \( c_i \in \mathbb{R}, \forall i = 1, 2, 3 \). Therefore, equation (236) would lead to the following interesting result

\[
\lambda_{\text{max}}(e) = \max_{\mathbb{R}^+} \{\alpha_1, \beta_1\}, \forall e \in \mathbb{R}^3.
\] (239)

We have shown that, under our assumptions, the leading Lyapunov exponent \( \lambda_{\text{max}} \) is given by \( \alpha \).

1. The Geodesic Length \( \Theta_{M_{s1}} \)

Consider the one-parameter family of geodesics \( F_{G_{M_{s1}}} \equiv \{\theta^\mu_{M_{s1}}(\tau; \alpha)\}_{\alpha \in \mathbb{R}^+} \) where \( \theta^\mu_{M_{s1}} \) are solutions of (230), the "selector parameter" \( \alpha \) tells which geodesic is being considered and the affine parameter \( \tau \) tells where is the point being considered on a given geodesic. The length of geodesics in \( F_{G_{M_{s1}}} \) is defined as,

\[
\Theta_{M_{s1}}(\tau; \alpha) \overset{\text{def}}{=} \int_0^\tau \left[ \frac{1}{\mu_1^2} \left( \frac{d\mu_1}{d\tau'} \right)^2 + \frac{1}{\sigma_2^2} \left( \frac{d\sigma_2}{d\tau'} \right)^2 + \frac{2}{\sigma_2^2} \left( \frac{d\mu_2}{d\tau'} \right)^2 \right] d\tau',
\] (240)
where \( g_{ij} = (g_{ij})_{M_{s_1}} \) is given in (202). Substituting (231) in (240) and considering the asymptotic expression of \( \Theta_{M_{s_1}} (\tau; \alpha) \), we obtain
\[
\Theta_{M_{s_1}} (\tau \to \infty; \alpha) \equiv \Theta_1 (\tau; \alpha) \tau \to \infty \approx \alpha \tau.
\] (241)

In evaluating (240), we have not imposed the conventional normalization condition \( (\dot{\theta} \dot{\theta}^i)_{\tau} = 1 \). Indeed, in our case \( (\dot{\theta} \dot{\theta}^i)_{\tau} = \text{constant} \); what matters is that we will use this very same normalization constant in evaluating the length of geodesics in \( F_{G_{M_{s_2}}} (\alpha) \) so that we may compare both lengths using the same ”meter” (statistical affine parameter \( \tau \)).

In order to roughly investigate the asymptotic behavior of two neighboring geodesics labelled by the parameters \( \alpha \) and \( \alpha + \delta \alpha \), we consider the following difference,
\[
\Delta \Theta_1 \equiv |\Theta_1 (\tau; \alpha + \delta \alpha) - \Theta_1 (\tau; \alpha)| = \left| \frac{\partial \Theta_1}{\partial \alpha} \right|_{\tau} \delta \alpha \tau \to \infty \approx |\delta \alpha| \tau.
\] (242)

It is clear that \( \Delta \Theta_1 \) diverges, that is, the lengths of two neighboring geodesics with slightly different parameters \( \alpha \) and \( \alpha + \delta \alpha \) differ in a remarkable way as the evolution parameter \( \tau \to \infty \). This hints at the onset of instability of the hyperbolic trajectories on \( M_{s_1} \).

2. Evolution of Volumes \( V_{M_{s_1}} \) on the Statistical Manifold \( M_{s_1} \)

The instability of ED1 can be further explored by studying the behavior of the one-parameter family of statistical volume elements \( F_{V_{M_{s_1}}} (\alpha) \equiv \{ V_{M_{s_1}} (\tau; \alpha) \}_\alpha \). Recall that \( M_{s_1} \) is the space of probability distributions \( p^{(\text{tot})} (\vec{x} | \vec{\theta}) \) labeled by parameters \( \theta_1^{(1)}, \theta_1^{(2)}, \theta_2^{(2)} \). These parameters are the coordinates of the point \( p^{(\text{tot})} \), and in these coordinates a 3D volume element \( dV_{M_{s_1}} \) reads
\[
dV_{M_{s_1}} = \sqrt{g} d\theta_1^{(1)} d\theta_1^{(2)} d\theta_2^{(2)} \equiv \sqrt{g} d\mu_1 d\mu_2 d\sigma_2,
\] (243)

where in the ED1 model here presented, \( g = |\det (g_{ij})_{M_{s_1}}| = \frac{2}{\mu_1^2 \sigma_2^2} \). Hence, the volume element \( dV_{M_{s_1}} \) is given by,
\[
dV_{M_{s_1}} = \frac{\sqrt{2}}{\mu_1 \sigma_2} d\mu_1 d\mu_2 d\sigma_2.
\] (244)

The volume increase of an extended region of \( M_{s_1} \) is given by,
\[
V_{M_{s_1}} (\tau; \alpha) \defeq \int_{\tilde{\theta}_{M_{s_1}} (0; \alpha)} dV_{M_{s_1}} = \int_{\mu_1 (0)}^{\mu_1 (\tau)} \int_{\mu_2 (0)}^{\mu_2 (\tau)} \int_{\sigma_2 (0)}^{\sigma_2 (\tau)} \frac{\sqrt{2}}{\mu_1 \sigma_2} d\mu_1 d\mu_2 d\sigma_2.
\] (245)
Integrating (245) using (231), we obtain

\[ V_{M_{s,1}}(\tau; \alpha) = \frac{\tau}{\sqrt{2}} e^{\alpha \tau} - \frac{\ln A}{\sqrt{2\alpha}} e^{\alpha \tau} + \frac{\ln A}{\sqrt{2\alpha}}. \] (246)

The quantity that actually encodes relevant information about the stability of neighboring volume elements is the average volume \( \langle V_{M_{s,1}}(\tau; \alpha) \rangle \),

\[ \langle V_{M_{s,1}}(\tau; \alpha) \rangle = \frac{1}{\tau} \int_0^\tau V_{M_{s,1}}(\tau'; \alpha) d\tau' = \frac{1}{\tau} \left\{ \frac{1}{\sqrt{2\alpha^2}} (\alpha \tau - 1) e^{\alpha \tau} - \frac{\ln A}{\sqrt{2\alpha^2}} e^{\alpha \tau} + \frac{\ln A}{\sqrt{2\alpha}} \right\}. \] (247)

For convenience, let us rename \( \langle V_{M_{s,1}}(\tau; \alpha) \rangle \equiv V_{M_{s,1}}(\tau; \alpha) \). Therefore, the asymptotic expansion of \( V_{M_{s,1}}(\tau; \alpha) \) for \( \tau \to \infty \) reads,

\[ V_{M_{s,1}}(\tau; \alpha) \sim \approx \frac{1}{\sqrt{2\alpha}} e^{\alpha \tau}. \] (248)

This asymptotic evolution in (248) describes the exponential increase of average volume elements on \( M_{s,1} \). The exponential instability characteristic of chaos forces the system to rapidly explore large areas (volumes) of the statistical manifolds. It is interesting to note that this asymptotic behavior appears also in the conventional description of quantum chaos [102] where the entropy increases linearly at a rate determined by the Lyapunov exponents [106]. The linear entropy increase as a quantum chaos criterion was introduced by Zurek and Paz. In our information-geometric approach a relevant variable that will be useful for comparison of the two different degrees of instability characterizing the two ED models is the relative entropy-like quantity defined as,

\[ S_{M_{s,1}}(\tau; \alpha) \overset{\text{def}}{=} \log V_{M_{s,1}}(\tau; \alpha). \] (249)

Substituting (248) in (249) and considering the asymptotic limit \( \tau \to \infty \), we obtain

\[ S_{M_{s,1}}(\tau; \alpha) \sim \approx \alpha \tau. \] (250)

Notice that the late-time limit \( (\tau \to \infty) \) is necessary to describe chaos [107]. Furthermore, the study of the asymptotic behavior of average quantities is very common in chaotic dynamics [103, 107]. The entropy-like quantity \( S_{M_{s,1}} \) in (250) may be interpreted as the asymptotic limit of the natural logarithm of a statistical weight \( V_{M_{s,1}}(\tau; \alpha) \overset{\text{def}}{=} \langle V_{M_{s,1}}(\tau; \alpha) \rangle \), defined on \( M_{s,1} \). Equation (250) is the information-geometric analog of the Zurek-Paz chaos criterion.

As a final remark, we would like to emphasize that the connection between \( \alpha \) and \( S_{M_{s,1}}(\tau; \alpha) \) may be compared to the connection between the KS entropy and the "physical" entropy (the entropy of the second law of thermodynamics) [108]; the steps we used to construct the IGE resemble the ones that Toda and Ikeda used to propose a quantal Lyapunov exponent [109].
3. The Jacobi Vector Field $\tilde{J}_{M_{1}}$

We study the behavior of the one-parameter family of neighboring geodesics $\mathcal{F}_{G_{M_{1}}} (\alpha) \equiv \{ \theta^\mu_{M_{1}} (\tau; \alpha) \}_{\alpha \in \mathbb{R}^+}$ where,

$$\theta^1 (\tau; \alpha) = \mu_1 (\tau; \alpha) = Ae^{\alpha \tau}, \quad \theta^2 (\tau; \alpha) = \mu_2 (\tau; \alpha) = \frac{A^2}{2\alpha e^{-2\alpha \tau} + \frac{A^2}{8\alpha^2}},$$

$$\theta^3 (\tau; \alpha) = \sigma_2 (\tau; \alpha) = \frac{A e^{-\alpha \tau}}{e^{-2\alpha \tau} + \frac{A^2}{8\alpha^2}}.$$ (251)

The relative geodesic spread is characterized by the Jacobi equation [26, 110],

$$D^2 (\delta \theta^i) + R_{lmkj} \frac{\partial \theta^k}{\partial \tau} \frac{\partial \theta^l}{\partial \tau} \delta \theta^m = 0$$ (252)

where $i = 1, 2, 3$ and,

$$\delta \theta^i \equiv \delta_\alpha \theta^i \equiv \left( \frac{\partial \theta^i (\tau; \alpha)}{\partial \alpha} \right) \tau \delta \alpha.$$ (253)

Equation (252) forms a system of three coupled ordinary differential equations linear in the components of the deviation vector field (253) but nonlinear in derivatives of the metric (201). It describes the linearized geodesic flow: the linearization ignores the relative velocity of the geodesics. When the geodesics are neighboring but their relative velocity is arbitrary, the corresponding geodesic deviation equation is the so-called generalized Jacobi equation [111]. The nonlinearity is due to the existence of velocity-dependent terms in the system.

Neighboring geodesics accelerate relative to each other with a rate directly measured by the curvature tensor $R_{\alpha \beta \gamma \delta}$. The Riemann tensor is, within this information-geometric framework, an indicator of the strength of "statistical tidal forces". Multiplying both sides of (252) by $g_{ij}$ and using the standard symmetry properties of the Riemann curvature tensor, the geodesic deviation equation becomes,

$$g_{ji} \frac{D^2 (\delta \theta^i)}{D \tau^2} + R_{lmkj} \frac{\partial \theta^k}{\partial \tau} \frac{\partial \theta^l}{\partial \tau} \delta \theta^m = 0.$$ (254)

Recall that the covariant derivative $\frac{D^2 (\delta \theta^\mu)}{D \tau^2}$ in (252) is defined as,

$$\frac{D^2 \delta \theta^\mu}{D \tau^2} = \frac{d^2 \delta \theta^\mu}{d \tau^2} + 2 \Gamma^\mu_{\alpha \beta} \frac{d \delta \theta^\alpha}{d \tau} \frac{d \theta^\beta}{d \tau} + \Gamma^\mu_{\alpha \beta \gamma} \delta \theta^\alpha \frac{d^2 \theta^\beta}{d \tau^2} + \Gamma^\mu_{\alpha \beta \nu} \frac{d \theta^\nu}{d \tau} \frac{d \theta^\beta}{d \tau} \delta \theta^\alpha +$$

$$+ \Gamma^\mu_{\alpha \beta} \frac{d \theta^\sigma}{d \tau} \frac{d \theta^\beta}{d \tau} \delta \theta^\rho\frac{d \theta^\rho}{d \tau}.$$ (255)

and that the only non-vanishing Riemann tensor component is $R_{2323} = -\frac{1}{\sigma_1^4}$. Therefore, the three differential equations
for the geodesic deviation are,

\[
\frac{d^2 (\delta \theta^1)}{d\tau^2} + 2 \Gamma_{11}^1 \frac{d \delta \theta^1}{d\tau} + \partial_1 \Gamma_{11}^1 \left( \frac{d \delta \theta^1}{d\tau} \right)^2 \delta \theta^1 = 0,
\]

(256)

\[
\frac{d^2 (\delta \theta^2)}{d\tau^2} + 2 \left[ \Gamma_{23}^2 \frac{d \delta \theta^3}{d\tau} + \Gamma_{32}^2 \frac{d \delta \theta^1}{d\tau} \right] + \partial_3 \Gamma_{23}^2 \left( \frac{d \delta \theta^3}{d\tau} \right)^2 \delta \theta^2 + \Gamma_{32}^2 \Gamma_{33}^3 \left( \frac{d \delta \theta^3}{d\tau} \right)^2 \delta \theta^2 = 0,
\]

(257)

\[
= \frac{1}{g_{22}} R_{2323} \frac{d \delta \theta^2}{d\tau} \frac{d \delta \theta^3}{d\tau} - \frac{1}{g_{22}} R_{2323} \left( \frac{d \delta \theta^3}{d\tau} \right)^2 \delta \theta^2,
\]

\[
= \frac{1}{g_{33}} R_{2323} \frac{d \delta \theta^2}{d\tau} \frac{d \delta \theta^3}{d\tau} - \frac{1}{g_{33}} R_{2323} \left( \frac{d \delta \theta^2}{d\tau} \right)^2 \delta \theta^2.
\]

(258)

Substituting (208), (210) and (251) in equations (256), (257) and (258) and considering the asymptotic limit \( \tau \to \infty \), the geodesic deviation equations become,

\[
\frac{d^2 (\delta \theta^1)}{d\tau^2} + 2 \alpha \frac{d (\delta \theta^1)}{d\tau} + \alpha^2 \delta \theta^1 = 0,
\]

(259)

\[
\frac{d^2 (\delta \theta^2)}{d\tau^2} + 2 \alpha \frac{d (\delta \theta^2)}{d\tau} + \frac{16 \alpha^2}{A} e^{-\alpha \tau} \frac{d (\delta \theta^3)}{d\tau} + \left( \alpha^2 - \frac{8 \alpha^3}{A} e^{-\alpha \tau} \right) \delta \theta^3 = 0,
\]

(260)

\[
\frac{d^2 (\delta \theta^3)}{d\tau^2} + 2 \alpha \frac{d (\delta \theta^3)}{d\tau} + \left( \alpha^2 - \frac{32 \alpha^4}{A^2} e^{-2\alpha \tau} \right) \delta \theta^3 - \frac{8 \alpha^2}{A} e^{-\alpha \tau} \frac{d (\delta \theta^2)}{d\tau} - \frac{8 \alpha^3}{A} e^{-\alpha \tau} \delta \theta^2 = 0.
\]

(261)

Neglecting the exponentially decaying terms in \( \delta \theta^3 \) in (260) and (261) and assuming that,

\[
\lim_{\tau \to \infty} \left( \frac{16 \alpha^2}{A} e^{-\alpha \tau} \frac{d (\delta \theta^3)}{d\tau} \right) = 0, \quad \lim_{\tau \to \infty} \left( \frac{8 \alpha^2}{A} e^{-\alpha \tau} \frac{d (\delta \theta^2)}{d\tau} \right) = 0, \quad \lim_{\tau \to \infty} \left( \frac{8 \alpha^3}{A} e^{-\alpha \tau} \delta \theta^2 \right) = 0,
\]

(262)

the geodesic deviation equations finally become,

\[
\frac{d^2 (\delta \theta^1)}{d\tau^2} + 2 \alpha \frac{d (\delta \theta^1)}{d\tau} + \alpha^2 \delta \theta^1 = 0,
\]

(263)

\[
\frac{d^2 (\delta \theta^2)}{d\tau^2} + 2 \alpha \frac{d (\delta \theta^2)}{d\tau} + \alpha^2 \delta \theta^2 = 0,
\]

\[
\frac{d^2 (\delta \theta^3)}{d\tau^2} + 2 \alpha \frac{d (\delta \theta^3)}{d\tau} + \alpha^2 \delta \theta^3 = 0.
\]

Note that in order to prove that our assumptions in (202) are correct, we will check a posteriori its consistency. Integrating the system of differential equations (263), we obtain

\[
\delta \mu_1 (\tau) = (a_1 + a_2 \tau) e^{-\alpha \tau},
\]
\[ \delta \mu_2 (\tau) = (a_3 + a_4 \tau) e^{-\alpha \tau} - \frac{1}{2\alpha} \sigma_1^2 e^{-2\alpha \tau} + a_6, \]

\[ \delta \sigma_2 (\tau) = (a_3 + a_4 \tau) e^{-\alpha \tau}, \]  \hspace{1cm} (264)

where \( a_i, i = 1, ..., 6 \) are integration constants. Note that conditions (262) are satisfied and therefore our assumption are compatible with the solutions obtained. Finally, consider the vector field components \( J^k \equiv \delta \theta^k \) defined in and its magnitude \( J \),

\[ J^2 = J^i J_i = g_{ij} J^i J^j. \]  \hspace{1cm} (265)

The magnitude \( J \) is called the Jacobi field intensity. In our case (265) becomes,

\[ J_{M_{s_1}}^2 \approx \frac{1}{\mu_1^2} \left( \delta \mu_1 \right)^2 + \frac{1}{\sigma_2^2} \left( \delta \mu_2 \right)^2 + \frac{2}{\sigma_2^2} \left( \delta \sigma_2 \right)^2. \]  \hspace{1cm} (266)

Substituting (251) and (264) in (266), and keeping the leading term in the asymptotic expansion in \( J_{M_{s_1}}^2 \), we obtain

\[ J_{M_{s_1}} \approx C_{M_{s_1}} e^{\alpha \tau}, \]  \hspace{1cm} (267)

where the constant coefficient \( C_{M_{s_1}} = \frac{A a_3^2}{2 \sqrt{2\alpha}} \) encodes information about initial conditions and depends on the model parameter \( \alpha \). We conclude that the geodesic spread on \( M_{s_1} \) is described by means of an exponentially divergent Jacobi vector field intensity \( J_{M_{s_1}} \). It is known that classical chaotic systems exhibit exponential sensitivity to initial conditions. This characterization, quantified in terms of Lyapunov exponents, is an important ingredient in any conventional definition of classical chaos. In our approach, the quantity \( \lambda_J \approx \lim_{\tau \to \infty} \frac{1}{\tau} \log \left( \frac{\| J(\tau) \|}{\| J(0) \|} \right) \) with \( J \) given in (267) would play the role of the conventional Lyapunov exponents.

### B. Instability in ED2

In this subsection, the instability of the geodesics on \( M_{s_2} \) is studied. We proceed as in section (5.4.1).

#### 1. The Geodesic Length \( \Theta_{M_{s_2}} \)

Consider the one-parameter family of geodesics \( \mathcal{F}_{G_{M_{s_2}}} (\alpha) \equiv \left\{ \theta_{\alpha}^{\mu} (\tau; \alpha) \right\}_{\alpha \in \mathbb{R}^+}^{\mu = 1, 2, 3, 4} \) where \( \theta_{\alpha}^{\mu} \) are solutions of (253). The length of geodesics in \( \mathcal{F}_{G_{M_{s_2}}} (\alpha) \) is defined as,

\[ \Theta_{M_{s_2}} (\tau; \alpha) \equiv \int_0^\tau \left( g_{ij} d\theta^i d\theta^j \right)^{\frac{1}{2}} = \int_0^\tau \left[ \frac{1}{\sigma_1^2} \left( \frac{d\mu_1}{d\tau'} \right)^2 + \frac{1}{\sigma_1^2} \left( \frac{d\sigma_1}{d\tau'} \right)^2 + \frac{2}{\sigma_2^2} \left( \frac{d\mu_2}{d\tau'} \right)^2 + \frac{2}{\sigma_2^2} \left( \frac{d\sigma_2}{d\tau'} \right)^2 \right]^\frac{1}{2} d\tau', \]  \hspace{1cm} (268)
where \( g_{ij} = (g_{ij})_{M_{s_2}} \) defined in (217). Substituting (234) in (268) and considering the asymptotic limit of \( \Theta_{M_{s_2}} (\tau; \alpha) \) when \( \tau \to \infty \), we obtain

\[
\Theta_{M_{s_2}} (\tau \to \infty; \alpha) \equiv \Theta_2 (\tau; \alpha) \overset{\tau \to \infty}{\approx} 2\alpha \tau. \tag{269}
\]

In evaluating (269), we have not imposed the conventional normalization condition \( \left( \dot{\theta}_i \dot{\theta}_j \right)^{\frac{1}{2}} = 1 \). Indeed, in our case \( \left( \dot{\theta}_i \dot{\theta}_j \right)^{\frac{1}{2}} \) = constant; what matters is that we have used the very same normalization constant used in evaluating the length of geodesics in \( F_{G_{M_{s_1}}} (\alpha) \) so that we may compare both lengths using the same "meter" (statistical affine parameter \( \tau \)). In order to roughly investigate the asymptotic behavior of two neighboring geodesics labelled by the parameters \( \alpha \) and \( \alpha + \delta \alpha \), we consider the following difference,

\[
\Delta \Theta_2 \equiv |\Theta_2 (\tau; \alpha + \delta \alpha) - \Theta_2 (\tau; \alpha)| \approx \left( \frac{\partial \Theta_2}{\partial \alpha} \right)_{\tau} \delta \alpha \bigg|_{\tau \to \infty} \approx 2 |\delta \alpha| \tau. \tag{270}
\]

It is clear that \( \Delta \Theta_2 \) diverges, that is the lengths of two neighboring geodesics with slightly different parameters \( \alpha \) and \( \alpha + \delta \alpha \) differ in a significant way as the evolution parameter \( \tau \to \infty \). This hints at the onset of instability of the hyperbolic trajectories on \( M_{s_2} \).

2. Evolution of Volumes \( V_{M_{s_2}} \) on the Statistical Manifold \( M_{s_2} \)

The instability of ED2 can be explored by studying the behavior of the one-parameter family of statistical volume elements \( F_{V_{M_{s_2}}} (\alpha) \equiv \{ V_{M_{s_2}} (\tau; \alpha) \} \). Recall that \( M_{s_2} \) is the space of probability distributions \( p^{(tot)} (\vec{x}; \vec{\theta}) \) labeled by parameters \( \theta_1^{(1)}, \theta_2^{(1)}, \theta_1^{(2)}, \theta_2^{(2)} \). These parameters are the coordinates of the point \( p^{(tot)} \), and in these coordinates a 4D infinitesimal volume element \( dV_{M_{s_2}} \) reads,

\[
dV_{M_{s_2}} = \sqrt{g} d\theta_1^{(1)} d\theta_2^{(1)} d\theta_1^{(2)} d\theta_2^{(2)} \equiv \sqrt{g} d\mu_1 d\sigma_1 d\mu_2 d\sigma_2, \tag{271}
\]

where in the ED2 model here presented, \( g = |\det (g_{ij})_{M_{s_2}}| = \frac{1}{\sigma_1 \sigma_2} \). Hence, the infinitesimal volume element \( dV_{M_{s_2}} \) is given by,

\[
dV_{M_{s_2}} = \frac{2}{\sigma_1 \sigma_2} d\mu_1 d\sigma_1 d\mu_2 d\sigma_2. \tag{272}
\]

The volume of an extended region of \( M_{s_2} \) is defined by,

\[
V_{M_{s_2}} (\tau; \alpha) \overset{\text{def}}{=} \int_{\tilde{M}_{s_2} (0; \alpha)} \hat{d}_{M_{s_2}} (\tau; \alpha) \overset{\text{def}}{=} \int_{\mu_1 (0)}^{\mu_1 (\tau)} \int_{\mu_2 (0)}^{\mu_2 (\tau)} \int_{\sigma_1 (0)}^{\sigma_1 (\tau)} \int_{\sigma_2 (0)}^{\sigma_2 (\tau)} \frac{2}{\sigma_1 \sigma_2} d\mu_1 d\sigma_1 d\mu_2 d\sigma_2, \tag{273}
\]
Integrating (273) and using (234), we obtain
\[ V_{M_{s_2}}(\tau; \alpha) = \frac{A^2}{2\alpha^2} e^{2\alpha \tau} - \frac{A^2}{2\alpha^2}. \] (274)

The accessible volume, on average, on the configuration space \( M_{s_2} \) is
\[ \langle V_{M_{s_2}}(\tau; \alpha) \rangle_\tau \equiv \frac{1}{\tau} \int V_{M_{s_2}}(\tau'; \alpha) \, d\tau' = \frac{A^2}{4\alpha^3} e^{2\alpha \tau} - \frac{A^2}{2\alpha^2}. \] (275)

For convenience, let us rename \( \langle V_{M_{s_2}}(\tau; \alpha) \rangle_\tau \equiv V_{M_{s_2}}(\tau; \alpha) \). Therefore, the asymptotic expansion of \( V_{M_{s_2}}(\tau; \alpha) \) for \( \tau \to \infty \) reads,
\[ V_{M_{s_2}}(\tau; \alpha) \xrightarrow{\tau \to \infty} \frac{A^2}{4\alpha^3} e^{2\alpha \tau}. \] (276)

In analogy to (249) we introduce,
\[ S_{M_{s_2}}(\tau; \alpha) \equiv \log V_{M_{s_2}}(\tau; \alpha). \] (277)

Substituting (276) in (277) and considering its asymptotic limit, we obtain
\[ S_{M_{s_2}}(\tau; \alpha) \xrightarrow{\tau \to \infty} 2\alpha \tau. \] (278)

3. The Jacobi Vector Field \( \vec{J}_{M_{s_2}} \)

We proceed as in (5.4.1.3). Study the behavior of the one-parameter \((\alpha)\) family of neighboring geodesics on \( M_{s_2} \), \( \{ \theta^i(\tau; \alpha) \}_{i=1, 2, 3, 4} \) with
\[ \theta^3(\tau; \alpha) \equiv \theta^1(\tau; \alpha) = \mu_1(\tau; \alpha) = \frac{A^2}{2\alpha} e^{-2\alpha \tau} + \frac{A^2}{8\alpha^2}, \] (279)
\[ \theta^4(\tau; \alpha) \equiv \theta^2(\tau; \alpha) = A e^{-\alpha \tau} e^{-2\alpha \tau} + \frac{A^2}{8\alpha^2}. \] (280)

Note that because we will compare the two Jacobi fields \( J_{M_{s_1}} \) on \( M_{s_1} \) and \( J_{M_{s_2}} \) on \( M_{s_2} \), we assume the same initial conditions as considered in (4.1.3). Recall that the non-vanishing Riemann tensor components are \( R_{1212} = -\frac{1}{\sigma_1} \) and \( R_{3434} = -\frac{1}{\sigma_2} \) given in (221). Therefore two of the four differential equations describing the geodesic spread are,
\[
\frac{d^2 (\delta \theta^1)}{d\tau^2} + 2 \left[ \Gamma^1_{12} \frac{d\theta^1}{d\tau} \frac{d(\delta \theta_1)}{d\tau} + \Gamma^1_{21} \frac{d\theta^1}{d\tau} \frac{d(\delta \theta^2)}{d\tau} \right] + \delta_2 \Gamma^1_{12} \left( \frac{d\theta^2}{d\tau} \right)^2 \delta \theta^1 + \Gamma^1_{21} \Gamma^2_{22} \left( \frac{d\theta^2}{d\tau} \right)^2 \delta \theta^1
= \frac{1}{g_{11}} R_{1212} \frac{d\theta^1}{d\tau} \frac{d\theta^2}{d\tau} \delta \theta^2 - \frac{1}{g_{11}} R_{1212} \left( \frac{d\theta^2}{d\tau} \right)^2 \delta \theta^1,
\] (281)
\[
\frac{d^2 (\delta \theta^2)}{d\tau^2} + 2 \left[ \Gamma_{11}^1 \frac{d}{d\tau} \left( \frac{d \delta \theta^1}{d\tau} \right) + \Gamma_{22}^2 \frac{d}{d\tau} \left( \frac{d \delta \theta^2}{d\tau} \right) \right] + \partial_2 \Gamma_{22}^2 \left( \frac{d \delta \theta^2}{d\tau} \right)^2 \delta \theta^2 + \Gamma_{11}^1 \frac{d^2}{d\tau^2} \frac{d \delta \theta^1}{d\tau} \delta \theta^1 \\
= \frac{1}{g_{22}} R_{1212} \frac{d}{d\tau} \delta \theta^1 - \frac{1}{g_{22}} R_{1212} \left( \frac{d \delta \theta^1}{d\tau} \right)^2 \delta \theta^2. \tag{282}
\]

The other two equations can be obtained from \[[251]\] and \[[252]\] substituting the index 1 with 3 and 2 with 4. Thus, we will limit our considerations just to the above two equations. Using equations \[[219]\], \[[221]\], \[[279]\] and \[[280]\] in \[[251]\] and \[[252]\] and considering the asymptotic limit \(\tau \to \infty\), the two equations of geodesic deviation become,

\[
\frac{d^2 (\delta \theta^1)}{d\tau^2} + 2\alpha \frac{d}{d\tau} \left( \frac{d \delta \theta^1}{d\tau} \right) + \frac{16\alpha^2}{A} e^{-\alpha \tau} \frac{d}{d\tau} \left( \frac{d \delta \theta^2}{d\tau} \right) + \left( \alpha^2 - \frac{8\alpha^3}{A} e^{-\alpha \tau} \right) \delta \theta^2 = 0, \tag{283}
\]

\[
\frac{d^2 (\delta \theta^2)}{d\tau^2} + 2\alpha \frac{d}{d\tau} \left( \frac{d \delta \theta^2}{d\tau} \right) + \left( \alpha^2 - \frac{32\alpha^4}{A^2} e^{-2\alpha \tau} \right) \delta \theta^2 - \frac{8\alpha^2}{A} e^{-\alpha \tau} \frac{d}{d\tau} \left( \frac{d \delta \theta^1}{d\tau} \right) - \frac{8\alpha^3}{A} e^{-\alpha \tau} \delta \theta^1 = 0. \tag{284}
\]

Neglecting the exponentially decaying terms in \(\delta \theta^2\) in \[[283]\] and \[[284]\] and assuming

\[
\lim_{\tau \to \infty} \left( \frac{16\alpha^2}{A} e^{-\alpha \tau} \frac{d}{d\tau} \left( \frac{d \delta \theta^2}{d\tau} \right) \right) = 0, \lim_{\tau \to \infty} \left( \frac{8\alpha^2}{A} e^{-\alpha \tau} \frac{d}{d\tau} \left( \frac{d \delta \theta^1}{d\tau} \right) \right) = 0, \lim_{\tau \to \infty} \left( \frac{8\alpha^3}{A} e^{-\alpha \tau} \delta \theta^1 \right) = 0 \tag{285}
\]

the geodesic deviation equations in \[[283]\] and \[[284]\] become,

\[
\frac{d^2 (\delta \theta^1)}{d\tau^2} + 2\alpha \frac{d}{d\tau} \left( \frac{d \delta \theta^1}{d\tau} \right) + \alpha^2 \delta \theta^2 = 0, \frac{d^2 (\delta \theta^2)}{d\tau^2} + 2\alpha \frac{d}{d\tau} \left( \frac{d \delta \theta^2}{d\tau} \right) + \alpha^2 \delta \theta^2 = 0. \tag{286}
\]

The consistency of the assumptions in \[[285]\] will be checked \textit{a posteriori} after integrating equations in \[[280]\]. It follows that the geodesics spread on \(M_{s_2}\) is described by the temporal evolution of the following deviation vector components,

\[
\delta \mu_1 (\tau) = (a_1 + a_2 \tau) e^{-\alpha \tau} - \frac{1}{2\alpha} a_3 e^{-2\alpha \tau} + a_4, \delta \sigma_1 (\tau) = (a_1 + a_2 \tau) e^{-\alpha \tau} \tag{287}
\]

\[
\delta \mu_2 (\tau) = (a_5 + a_6 \tau) e^{-\alpha \tau} - \frac{1}{2\alpha} a_7 e^{-2\alpha \tau} + a_8, \delta \sigma_2 (\tau) = (a_5 + a_6 \tau) e^{-\alpha \tau}
\]

where \(a_i, i = 1, \ldots, 8\) are integration constants. Note that \(a_4\) and \(a_8\) in \[[287]\] equal \(a_6\) in \[[264]\]. Furthermore, note that these solutions above are compatible with the assumptions in \[[285]\]. Finally, consider the Jacobi vector field intensity \(J_{M_{s_2}}\) on \(M_{s_2}\),

\[
J_{M_{s_2}}^2 = \frac{1}{\sigma_1^2} (\delta \mu_1)^2 + \frac{2}{\sigma_1^2} (\delta \sigma_1)^2 + \frac{1}{\sigma_2^2} (\delta \mu_2)^2 + \frac{2}{\sigma_2^2} (\delta \sigma_2)^2. \tag{288}
\]

Substituting \[[279]\], \[[280]\] and \[[287]\] in \[[288]\] and keeping the leading term in the asymptotic expansion in \(J_{M_{s_2}}^2\), we obtain

\[
J_{M_{s_2}} \overset{\tau \to \infty}{\approx} C_{M_{s_2}} e^{\alpha \tau}. \tag{289}
\]

where the constant coefficient \(C_{M_{s_2}} = \frac{A a_3^3}{\sqrt{2} \alpha e} = 2C_{M_{s_1}}\) encodes information about initial conditions and it depends on the model parameter \(\alpha\). We conclude that the geodesic spread on \(M_{s_2}\) is described by means of an \textit{exponentially} divergent Jacobi vector field intensity \(J_{M_{s_2}}\).
Many problems in mathematical statistics, information theory and in stochastic processes can be tackled using differential geometric methods on curved statistical manifolds. For instance, an important class of statistical manifolds is that arising from the exponential family and one particular family is that of gamma probability distributions. These distributions have been shown to have important uniqueness properties for stochastic processes. In this Chapter, two statistical manifolds of negative curvature $\mathcal{M}_{s_1}$ and $\mathcal{M}_{s_2}$ have been considered. They are representations of smooth families of probability distributions (exponentials and Gaussians for $\mathcal{M}_{s_1}$, Gaussians for $\mathcal{M}_{s_2}$). They represent the "arena" where the entropic dynamics takes place. The instability of the trajectories of the ED1 and ED2 on $\mathcal{M}_{s_1}$ and $\mathcal{M}_{s_2}$ respectively, has been studied using the statistical weight $\langle V_{\mathcal{M}_s} \rangle_\tau$ defined on the curved manifold $\mathcal{M}_s$ and the Jacobi vector field intensity $J_{\mathcal{M}_s}$. Does our analysis lead to any possible further understanding of the role of statistical curvature in physics and statistics? We argue that it does.

The role of curvature in physics is fairly well understood. It encodes information about the field strengths for all the four fundamental interactions in nature. The curvature plays a key role in the Riemannian geometric approach to chaos. In this approach, the study of the Hamiltonian dynamics is reduced to the investigation of geometrical properties of the manifold on which geodesic flow is induced. For instance, the stability or local instability of geodesic flows depends on the sectional curvature properties of the suitable defined metric manifold. The sectional curvature brings the same qualitative and quantitative information that is provided by the Lyapunov exponents in the conventional approach. Furthermore, the integrability of the system is connected with existence of Killing vectors on the manifold. However, a rigorous relation among curvature, Lyapunov exponents and Kolmogorov-Sinai entropy is still under investigation. In addition, there does not exist a well defined unifying characterization of chaos in classical and quantum physics due to fundamental differences between the two theories. Even in other fields of research (for instance, statistical inference) the role of curvature is not well understood. The meaning of statistical curvature for a one-parameter model in inference theory was introduced in . Curvature served as an important tool in the asymptotic theory of statistical estimation. The higher the scalar curvature at a given point on the manifold, the more difficult it is to do estimation at that point. Our analysis may be useful to shed light in statistical inference theory as well.

Recall that the entropy-like quantity $S$ is the asymptotic limit of the natural logarithm of the average of the statistical volume $\langle V_{\mathcal{M}_s} \rangle_\tau$ associated to the evolution of the geodesics on $\mathcal{M}_s$. Considering equations and
we obtain,
\[ S_{M_{s_2}} \approx 2S_{M_{s_1}}. \] (290)

Furthermore, the relationship between the statistical curvatures on the curved manifolds \( M_{s_1} \) and \( M_{s_2} \) is,
\[ R_{M_{s_2}} = 2R_{M_{s_1}}. \] (291)

In view of (290) and (291), it follows that there is a direct proportionality between the curvature \( R_{M_i} \) and the asymptotic expression for the entropy-like quantity \( S \) characterizing the ED on manifolds \( M_i \) with \( i = 1, 2 \), namely
\[ \frac{R_{M_{s_2}}}{R_{M_{s_1}}} = \frac{S_{M_{s_2}}}{S_{M_{s_1}}}. \] (292)

Moreover, from (267) and (289), we obtain the following relation,
\[ J_{M_{s_2}} \approx 2J_{M_{s_1}}. \] (293)

The two manifolds \( M_{s_1} \) and \( M_{s_2} \) are exponentially unstable and the intensity of Jacobi vector field \( J_{M_{s_2}} \) of manifold \( M_{s_2} \) with curvature \( R_{M_{s_2}} = -2 \) is asymptotically twice the intensity of the Jacobi vector field \( J_{M_{s_1}} \) of manifold \( M_{s_1} \) with curvature \( R_{M_{s_1}} = -1 \). Considering (291) and (293), we obtain
\[ \frac{R_{M_{s_2}}}{R_{M_{s_1}}} = \frac{J_{M_{s_2}}}{J_{M_{s_1}}}. \] (294)

It seems there exists a direct proportionality between the curvature \( R_{M_i} \) and the intensity of the Jacobi field \( J_{M_i} \) characterizing the degree of chaoticity of a statistical manifold of negative curvature \( M_i \). Finally, comparison of (292) and (294) leads to the formal link between curvature, entropy and chaoticity:
\[ R \sim S \sim J. \] (295)

Though several points need deeper understanding and analysis, we hope that our work shows that this information-geometric approach may be useful in providing a unifying framework to study chaos on statistical manifolds underlying entropic dynamical models.

**XXIII. CONCLUSIONS**

Two chaotic entropic dynamical models have been considered: a 3D and 4D statistical manifold \( M_{s_1} \) and \( M_{s_2} \) respectively. These manifolds serve as the stage on which the entropic dynamics takes place. In the former case, macro-coordinates on the manifold are represented by the expectation values of microvariables associated with Gaussian and
exponential probability distributions. In the latter case, macro-coordinates are expectation values of microvariables associated with two Gaussians distributions. The geometric structure of $\mathcal{M}_{s_1}$ and $\mathcal{M}_{s_2}$ was studied in detail. It was shown that $\mathcal{M}_{s_1}$ is a curved manifold of constant negative curvature $-1$ while $\mathcal{M}_{s_2}$ has constant negative curvature $-2$. The geodesics of the ED models are hyperbolic curves on $\mathcal{M}_{s_i}$ ($i = 1, 2$). A study of the stability of geodesics on $\mathcal{M}_{s_1}$ and $\mathcal{M}_{s_2}$ was presented. The notion of statistical volume elements was introduced to investigate the asymptotic behavior of a one-parameter family of neighboring volumes $F_{V_{\mathcal{M}_s}}(\alpha) \equiv \{V_{\mathcal{M}_s}(\tau; \alpha)\}_{\alpha \in \mathbb{R}^+}$. An information-geometric analog of the Zurek-Paz chaos criterion was presented. It was shown that the behavior of geodesics is characterized by exponential instability that leads to chaotic scenarios on the curved statistical manifolds. These conclusions are supported by a study based on the geodesic deviation equations and on the asymptotic behavior of the Jacobi vector field intensity $J_{\mathcal{M}_s}$ on $\mathcal{M}_{s_1}$ and $\mathcal{M}_{s_2}$. A Lyapunov exponent analog similar to that appearing in the Riemannian geometric approach was suggested as an indicator of chaos. On the basis of our analysis a relationship among an entropy-like quantity, chaoticity and curvature in the two models ED1 and ED2 is proposed, suggesting to interpret the statistical curvature as a measure of the entropic dynamical chaoticity.

The implications of this work are twofold. Firstly, it helps to understand the possible future use of the statistical curvature in modelling real processes by relating it to conventionally accepted quantities such as entropy (be it the KS entropy, the Shannon entropy, the IGE entropy (that we have introduced in this Chapter), the Kolmogorov complexity, the von Neumann entropy, the quantum dynamical entropy, etc. etc.) and chaos. On the other hand, it serves to cast what is already known in physics regarding curvature in a new light as a consequence of its proposed link with inference. Finally we remark that based on the results obtained from the chosen ED models, it is not unreasonable to think that should the correct variables describing the true degrees of freedom of a physical system be identified, perhaps deeper insights into the foundations of models of physics and reasoning (and their relationship to each other) may be uncovered.
Chapter 6: Information-constrained dynamics, Part II: Newtonian entropic dynamics

In collaboration with Prof. Ariel Caticha, I show that the ED formalism is not purely a mathematical framework; it is indeed a general theoretical scheme where conventional Newtonian dynamics is obtained as a special limiting case. Newtonian dynamics is derived from prior information codified into an appropriate statistical model. The basic assumption is that there is an irreducible uncertainty in the location of particles so that the state of a particle is defined by a probability distribution. The corresponding configuration space is a statistical manifold the geometry of which is defined by the information metric. The trajectory follows from a principle of inference, the method of Maximum Entropy. No additional "physical" postulates such as an equation of motion, or an action principle, nor the concepts of momentum and of phase space, not even the notion of time, need to be postulated. The resulting entropic dynamics reproduces the Newtonian dynamics of any number of particles interacting among themselves and with external fields. Both the mass of the particles and their interactions are explained in terms of the underlying statistical manifold.

XXIV. INTRODUCTION

In this Chapter, we use well established principles of inference to derive Newtonian dynamics from relevant prior information codified into a statistical model [13]. We do not assume equations of motion or principles of least action. Moreover, neither the concept of momentum nor that of the associated phase space is assumed. Indeed, not even the notion of an absolute Newtonian time is postulated. Firstly, we construct a suitable statistical model of the space of states of a system of particles. The statistical configuration space is automatically endowed with a geometry and this information geometry turns out to be unique [31, 34]. Secondly, we tackle the dynamics: given the initial and the final states, we investigate what trajectory the system is expected to follow. In the conventional approach one postulates an equation of motion or an action principle that presumably reflects a "law of nature". However, in our theoretical framework, the dynamics follows from a principle of inference, the method of Maximum (relative) Entropy, ME [115]. We show that with a suitable choice of the statistical manifold the resulting "entropic dynamics" reproduces Newtonian dynamics, or more properly, Newtonian entropic dynamics (NED).
XXV. CONFIGURATION SPACE AS A STATISTICAL MANIFOLD

Let us consider a single particle moving in space $\mathbb{R}^3$: the configuration space is a three dimensional manifold $\mathcal{M}_s$ with some unknown metric tensor $g_{ij}(\theta)$. Our main assumption is that there is a certain fuzziness to space $\mathbb{R}^3$; there is an irreducible uncertainty in the location of the particle. Thus, the assertion “the particle is at the point $\theta^\nu$ means that its "true" position $x$ is somewhere in the vicinity of $\theta$. This leads us to associate a probability distribution $p(x|\theta)$ to each point $\theta$ and the configuration space is thus transformed into a statistical manifold $\mathcal{M}_s$ : a point $\theta$ is no longer a structureless dot but a probability distribution. Remarkably there is a unique measure of the extent to which the distribution at $\theta$ can be distinguished from the neighboring distribution at $\theta + d\theta$. It is the information metric of Fisher and Rao [31]. Thus, physical space, when viewed as a statistical manifold, inherits a metric structure from the distributions $p(x|\theta)$. We will assume that the originally unspecified metric $g_{ij}(\theta)$ is precisely the information metric induced by the distributions $p(x|\theta)$.

A. The Gaussian Model and the Covariance Problem

Given a manifold $\mathcal{M}_S$ of probability distributions $\{p(x|\theta)\}$, the problem is to find the corresponding information metric $g_{\mu\nu}(\theta)$. This is commonly called the direct problem. Its solution might be laborious but it is quite mechanical. In this Chapter, we are tackling what is called the inverse problem: constructing a statistical manifold $\mathcal{M}_S$ with a given metric tensor $g_{\mu\nu}(\theta)$. In [117], it was proposed that a Gaussian model,

$$p(x|\theta) = \frac{\gamma^{1/2}(\theta)}{(2\pi)^{3/2}} \exp \left[-\frac{\gamma_{ij}(\theta)(x^i - \theta^i)(x^j - \theta^j)}{2}\right],$$

(296)

where $\gamma = \det \gamma_{ij}$, encodes the physically relevant information, which consists of an estimate of the particle position,

$$\langle x^i \rangle = \int dx \, p(x|\theta) \, x^i = \theta^i,$$

(297)

and of its uncertainty, given by the covariance matrix $\tilde{\gamma}^{ij}(\theta)$,

$$\langle (x^i - \theta^i)(x^j - \theta^j) \rangle = \tilde{\gamma}^{ij}(\theta),$$

(298)

where $\tilde{\gamma}^{ij} = \int dx \, p(x|\theta)(x^i - \theta^i)(x^j - \theta^j)$ is the inverse of $\gamma_{ij}$, $\tilde{\gamma}^{ik}\gamma_{kj} = \delta^i_j$. It is worthwhile noticing that the expected values in eqs. (297) and (298) are not covariant under coordinate transformations. Indeed, the transformation $x'^i = f^i(x)$ does not lead to $\theta'^i = f^i(\theta)$ because in general $\langle f(x) \rangle \neq f(\langle x \rangle)$ except when uncertainties are small. Our Gaussian model can at best be an approximation valid when $p(x|\theta)$ is sharply localized in a very small region within
which curvature effects are negligible. Fortunately, this is all we need for our present purpose. The information
distance between \( p(x|\theta) \) and \( p(x|\theta + d\theta) \) is calculated from (see e.g., [31])

\[
d\ell^2 = G_{\mu\nu} \, d\theta^\mu \, d\theta^\nu. \tag{299}
\]

with \( G_{\mu\nu} = \int dx \, p(x|\theta) \partial_\mu \log p(x|\theta) \partial_\nu \log p(x|\theta) \) and \( \partial_\mu = \frac{\partial}{\partial \theta^\mu} \). Consider the nine-dimensional space of Gaussians

\[
p(x|\theta, \gamma) = \frac{\gamma^{3/2}}{(2\pi)^{3/2}} \exp \left[ -\frac{\gamma_{ij}(x^i - \theta^i)(x^j - \theta^j)}{2} \right]. \tag{300}
\]

Here the parameters \( \theta^\mu \) include the three \( x^i \) plus six independent elements of the symmetric matrix \( \gamma_{ij} \). Eq. (299) defines the information distance between \( p(x|\theta, \gamma) \) and \( p(x|\theta + d\theta, \gamma + d\gamma) \) as

\[
d\ell^2 = G_{ij} \, d\theta^i \, d\theta^j + \frac{1}{2} \tilde{\gamma}^{ik} \tilde{\gamma}^{jl} \, d\gamma_{ij} \, d\gamma_{kl}, \tag{301}
\]

where \( G_{ij} = \gamma_{ij}, \, G_{k}^{ij} = 0 \) and, \( G^{ij \, kl} = \frac{1}{4} (\tilde{\gamma}^{ik}\tilde{\gamma}^{jl} + \tilde{\gamma}^{il}\tilde{\gamma}^{jk}) \) with \( \tilde{\gamma}^{ik} \gamma_{kj} = \delta^i_j \). Therefore,

\[
d\ell^2 = \gamma_{ij} \, d\theta^i \, d\theta^j + \frac{1}{2} \gamma_{ij} \gamma_{kl} \, d\gamma_{ij} \, d\gamma_{kl}. \tag{302}
\]

This is the metric of the full nine-dimensional manifold, but it is not what we need. What we want is the metric of the embedded three-dimensional submanifold where \( \gamma_{ij} = \gamma_{ij}(\theta) \) is some function of \( \theta \). The tensorial behavior of \( \gamma_{ij} \) in equation (296) is our major concern. As we said, the notion of expected value is not covariant. Our approach is an approximation that is valid when the position uncertainty is much smaller than the local radius of curvature of the manifold; or, alternatively, the effects of space curvature are negligible within the region where \( p(x|\theta) \) is appreciable.

Moreover, \( g_{ij} \) is a tensor while \( \tilde{\gamma}^{\mu\nu} \) does not behave as a tensor (in general). However, it can be shown that special cases exist where the tensorial behavior of \( \gamma_{ij} \) can be restored. Namely, if we consider exclusively linear change of coordinates \( \theta^k \to \theta'^r = \theta'^r (\theta^k) \) such that,

\[
\frac{\partial^2 \theta'^k}{\partial \theta^\mu \partial \theta^\nu} = 0 \tag{303}
\]

that is, if we assume that the new set of coordinates \( \{ \theta'^r \} \) are not allowed to depend on the old set \( \{ \theta^k \} \) in a nonlinear way, than we may conclude that \( \gamma_{ij} \) transforms as a tensor. In these very restrictive conditions, we may conclude that \( \gamma_{ij} \) has tensorial behavior provided that the new coordinates are not allowed to depend on the old coordinates (source coordinates) in a nonlinear way.

### B. The Gaussian-like Model and the Solution to the Covariance Problem

One of the major points in the indirect problem is choosing the correct variance-covariance field tensor \( \gamma_{ij} (\theta) \) that leads to a given metric tensor \( g_{ij} (\theta) \) on \( M_S \). We want to devise fully covariant models. This can be achieved by
constructing Gaussian-like probability distributions that, in the limit of small uncertainties, approximate the Gaussian distributions in (296). Such a distribution may be defined as,

\[ p(x|\theta) = \frac{1}{\zeta} \gamma^{1/2}(x) \exp \left[ -\frac{\ell^2(x, \theta)}{2\sigma^2(\theta)} \right]. \]  

(304)

The quantity \( \gamma \) defined in (304) is the determinant of the positive definite metric tensor field \( \gamma_{ij} \) satisfying the relation,

\[ d\ell^2 = \gamma_{ij} d\theta^i d\theta^j. \]  

(305)

The quantity \( \sigma(\theta) \) is a scalar field, \( \ell(x, \theta) \) is the \( \gamma \)-length along the \( \gamma \)-geodesic from the point \( \theta \) to the point \( x \) and \( \zeta \) is a normalization constant. The proposed probability distribution in (304) is a manifestly covariant quantity: the normalization constant \( \zeta \), the \( \gamma \)-length \( \ell(x, \theta) \), the scalar field \( \sigma(\theta) \) and, \( dx_{\gamma}^{1/2}(x) \) are all invariant quantities. We remark that the space of spherically symmetric Gaussians with prescribed \( \sigma(\theta) \),

\[ p(x|\theta) = \frac{1}{(2\pi\sigma^2)^{3/2}} \exp \left[ -\frac{1}{2\sigma^2} \delta_{ij}(x^i - \theta^i)(x^j - \theta^j) \right], \]  

(306)

is a special case of the distributions defined in (304). In this case, the variance-covariance matrix in (296) is diagonal and proportional to the unit matrix,

\[ \gamma_{ij} = \frac{1}{\sigma^2(\theta)} \delta_{ij}, \quad \tilde{\gamma}_{ij} = \sigma^2(\theta) \delta_{ij} \quad \text{and}, \quad \gamma = \frac{1}{\sigma^6(\theta)}. \]  

(307)

Differentiating \( \gamma_{ij} \) and using the fact that \( d\sigma = \partial_k \sigma(\theta) d\theta^k \) with \( \partial_k \equiv \frac{\partial}{\partial \theta^k} \), we get

\[ d\gamma_{ij} = -\frac{2\delta_{ij}}{\sigma^3(\theta)} \partial_k \sigma(\theta) d\theta^k. \]  

(308)

Substituting (308) in (302), we obtain

\[ d\ell^2 = \frac{1}{\sigma^2(\theta)} \delta_{ij} d\theta^i d\theta^j + \frac{1}{2} \sigma^4(\theta) \delta^{ik}\delta^{jl} \frac{2\delta_{ij}}{\sigma^3(\theta)} \sigma^2(\theta) \partial_k \sigma(\theta) \partial_l \sigma(\theta) d\theta^n d\theta^n \]  

(309)

which, using \( \delta^{ik}\delta^{jl}\delta_{kl} = \delta^k \delta^j = \delta^k \), simplifies to

\[ d\ell^2 = \frac{1}{\sigma^2(\theta)} [\delta_{ij} + 6\partial_i \sigma(\theta) \partial_j \sigma(\theta)] d\theta^i d\theta^j. \]  

(310)

In the case of probability distributions (306), the metric tensor \( g_{ij}(\theta) \) on \( \mathcal{M}_S \) becomes,

\[ g_{ij}(\theta) = \frac{1}{\sigma^2(\theta)} [\delta_{ij} + 6\partial_i \sigma(\theta) \partial_j \sigma(\theta)]. \]  

(311)

If the \( \sigma(\theta) \) field varies very slowly

\[ \partial_i \sigma << 1, \]  

(312)
then to first order the metric $d\ell^2$ is independent of the derivatives $\partial_i \sigma(\theta)$,

$$d\ell^2 \approx \frac{1}{\sigma^2(\theta)} \delta_{ij} d\theta^i d\theta^j,$$

(313)

and the metric tensor $g_{ij}(\theta)$ is just a conformal transformation of flat metric $\delta_{ij}$,

$$g_{ij}(\theta) \approx \frac{1}{\sigma^2(\theta)} \delta_{ij}.$$

(314)

We point out that condition (312) is compatible with the definition given in (323), $\sigma^2(\theta) = \frac{\sigma^2}{\Phi(\theta)}$. Rewriting $\sigma(\theta)$ in terms of $\Phi(\theta)$, equation (312) reads,

$$\partial_i \sigma \equiv -\frac{1}{2} \sigma_0 \Phi^{-\frac{1}{2}}(\theta) \partial_i \Phi(\theta) << 1.$$  

(315)

Therefore, we are allowed to consider very small $\partial_i \sigma$ by simply making the parameter $\sigma_0$ small and keeping $\partial_i \Phi(\theta) \sim F_i$ ($F_i$ would be the information geometric analog of a conservative force appearing in Newton’s second law) arbitrarily big. For small $\sigma(\theta)$, the distribution in (304) can be written as,

$$p(x|\theta) = \frac{1}{\zeta(\theta)^{1/2}(x)} \exp \left[ -\frac{\gamma_{ij}(\theta)}{2\sigma^2(\theta)} (x^i - \theta^i)(x^j - \theta^j) \right].$$

(316)

In local Cartesian coordinates (LCC), $\theta(\text{old}) \xrightarrow{\text{LCC}} \theta(\text{new}) \equiv \theta'$, we obtain

$$\gamma_{ij}(\theta) \xrightarrow{\text{LCC}} \gamma_{ij}^{(\text{new})}(\theta') = \delta_{ij} \text{ with } \partial_i \gamma_{ij}^{(\text{new})}(\theta') = 0$$

(317)

and (316) becomes,

$$p^{(\text{old})}(x|\theta) \xrightarrow{\text{LCC}} p^{(\text{new})}(x'|\theta') \approx \frac{1}{\zeta(\theta')^{1/2}(x')} \exp \left[ -\frac{\delta_{ij}}{2\sigma^2(\theta')} (x'^i - \theta'^i)(x'^j - \theta'^j) \right].$$

(318)

In the case of probability distributions (318), the metric tensor $g_{ij}(\theta)$ on $\mathcal{M}_S$ becomes,

$$g_{ij}(\theta) \approx \frac{1}{\sigma^2(\theta)} \delta_{ij} = \frac{1}{\sigma^2(\theta)} \gamma_{ij}(\theta).$$

(319)

Since (319) is a tensorial equation, its validity is preserved in arbitrary coordinates system and,

$$g_{ij}(\theta) \approx \frac{1}{\sigma^2(\theta)} \gamma_{ij}(\theta),$$

(320)

the tensorial relation between $g_{ij}(\theta)$ and $\gamma_{ij}(\theta)$ is covariantly preserved. In conclusion, in this subsection we provided a solution to the inverse problem suggesting a fully covariant Gaussian-like model of probability distributions \{p(x|\theta)\} forming a curved statistical manifold $\mathcal{M}_S$ with a given metric tensor given, in the limit of small uncertainties, by $g_{ij}(\theta)$ in (320). The possibility to extend this result in regions of high curvature, such as near singularities, remains to be ascertained.
Our objective is to construct ED models on statistical manifolds of Gaussian-like probability distributions presented in the previous section. We follow the work presented in Chapter 3. Assume there exists a continuous path, the key question is to find the trajectory the system is expected to follow, given an initial and a final state. A large change is the result of a succession of very many small changes and therefore we only need to determine the properties of a short segment of the trajectory. The idea behind entropic dynamics is that as the system moves from a point $\theta$ to a neighboring point $\theta + \Delta \theta$ it must pass through a halfway point. The basic dynamical question can now be rephrased as follows: the system is initially described by the probability distribution $p(x|\theta)$ and we are given the information that it has moved to one of the neighboring states in the family $p(x|\theta')$ where the $\theta'$ lie on the plane halfway between the initial $\theta$ and the final $\theta + \Delta \theta$. Which $p(x|\theta')$ do we select? The answer is given by the method of maximum (relative) entropy, ME. The selected distribution is that which maximizes the entropy of $p(x|\theta')$ relative to the prior $p(x|\theta)$ subject to the constraint that $\theta'$ is equidistant from $\theta$ and $\theta + \Delta \theta$. The result is that the selected $\theta'$ minimizes the distance to $\theta$ and therefore the three points $\theta$, $\theta'$ and $\theta + \Delta \theta$ lie on a straight line. Since any three neighboring points along the trajectory must line up, the trajectory predicted by entropic dynamics is the geodesic that minimizes the length

$$J = \int_{\xi_i} d\xi \left[ g_{ij}(\theta) \dot{\theta}^i \dot{\theta}^j \right]^{1/2}$$

with $\dot{\theta}^i = \frac{d\theta^i}{d\xi}$, (321)

where $\xi$ is any parameter that labels points along the curve, $\theta^i = \theta^i(\xi)$. In entropic dynamics, the minimal-length geodesics represent the only family of curves that is singled out as special. The construction of useful physics models does not require any additional structure and therefore none will be introduced. The simplest statistical model is a three-dimensional manifold of spherically symmetric Gaussians with constant variance $\sigma^2_0$. From (319) follows that the corresponding information metric is

$$g^{(0)}_{ij}(\theta) = \gamma^{(0)}_{ij}(\theta) = \frac{1}{\sigma^2_0} \delta_{ij},$$

(322)

where $\delta_{ij}$ is the familiar metric of flat Euclidean space. We point out that already in such a simple model entropic dynamics reproduces the familiar straight line trajectories that are commonly associated with Galilean inertial motion. However, non-trivial entropic dynamical models require some curvature. For instance, consider the model of spherically symmetric Gaussians where the variance is a non-uniform scalar field $\sigma^2(\theta)$. It is convenient to write the corresponding
information metric $g_{ij}(\theta)$ as the Euclidean metric eq. (322) modulated by a positive conformal factor $\Phi(\theta)$,
\[
g_{ij}(\theta) = \gamma_{ij}(\theta) = \frac{\Phi(\theta)}{\sigma_0^2} \delta_{ij}, \quad \sigma^2(\theta) = \frac{\sigma_0^2}{\Phi(\theta)}. \quad (323)
\]
The conformal factor $\Phi(\theta)$ defines an angle-preserving transformation (conformal transformation) and its effect is a local dilation. It is useful to rewrite the length eq. (321) with the metric (323) in the form
\[
\mathcal{J} = 2^{1/2} \int \xi \mathcal{L} (\theta, \dot{\theta}),
\]
with a "Lagrangian" function $\mathcal{L}(\theta, \dot{\theta})$ given by,
\[
\mathcal{L}(\theta, \dot{\theta}) = \left[ \Phi(\theta) T_\xi \right]^{1/2}, \quad T_\xi \overset{\text{def}}{=} \frac{\sigma_0^2}{\frac{\Phi(\theta)}{T_\xi}} \frac{\delta_{ij}}{d_\xi^i d_\xi^j}. \quad (325)
\]
The geodesics follow from the Lagrange equations,
\[
\frac{d}{d\xi} \frac{\partial \mathcal{L}(\theta, \dot{\theta})}{\partial \dot{\theta}^i} - \frac{\partial \mathcal{L}(\theta, \dot{\theta})}{\partial \theta^i} = 0 \quad (326)
\]
that is, substituting (325) in (326),
\[
\frac{1}{\sigma_0^2} \left( \frac{\Phi}{T_\xi} \right)^{1/2} \frac{d}{d\xi} \left[ \left( \frac{\Phi}{T_\xi} \right)^{1/2} \frac{d\theta^i}{d\xi} \right] - \frac{\partial \Phi}{\partial \theta^i} = 0. \quad (327)
\]
These equations can be simplified considerably once we notice that the parameter $\xi$ is arbitrary. Let us replace the original $\xi$ with a new parameter $\tau$ defined as
\[
d\tau = \left( \frac{T_\xi}{\Phi} \right)^{1/2} d\xi \quad \text{or} \quad \frac{d}{d\tau} = \left( \frac{\Phi}{T_\xi} \right)^{1/2} \frac{d}{d\xi}. \quad (328)
\]
In terms of the new $\tau$ the equation of motion (327) becomes,
\[
\frac{1}{\sigma_0^2} \frac{d^2\theta^i}{d\tau^2} - \frac{\partial \Phi}{\partial \theta^i} = 0. \quad (329)
\]
In order to allow us the possibility of considering the limit of small uncertainties, $\sigma_0 \ll 1$, we redefine the relation between $\xi$ and $\tau$ introducing a new parameter $\kappa$ ($\kappa$ has the dimensions of a time) such that,
\[
d\tau = \kappa \left( \frac{T_\xi}{\Phi} \right)^{1/2} d\xi \quad \text{or} \quad \frac{d}{d\tau} = \left( \frac{\Phi}{T_\xi} \right)^{1/2} \frac{d}{d\xi}. \quad (330)
\]
From eq. (330) the new $\tau$ is such that
\[
\Phi = \kappa^2 T_\xi \left( \frac{d\xi}{d\tau} \right)^2 = T_\tau, \quad T_\tau \overset{\text{def}}{=} \frac{1}{2} \frac{\kappa^2}{\sigma_0^2} \delta_{ij} \frac{d\theta^i}{d\tau} \frac{d\theta^j}{d\tau}. \quad (331)
\]
Eqs. (329) and (331) are equivalent to Newtonian dynamics. To make it explicit we introduce a "mass" $m$ and a "potential" $\phi(\theta)$ through a mere change of notation,
\[
m \overset{\text{def}}{=} \frac{\varepsilon \kappa^2}{\sigma_0^2} \quad \text{and} \quad \varepsilon \Phi(\theta) \overset{\text{def}}{=} E - \phi(\theta). \quad (332)
\]
The parameter $\varepsilon$ has been introduced because of dimensional analysis convenience ($\varepsilon$ has the dimensions of an energy; $\sigma_0$ has the dimensions of a length) and the constant $E$ reflects the freedom to add a constant to the potential. The result is the Newtonian information-dynamical equation,

$$m \frac{d^2 \theta^i}{d\tau^2} + \frac{\partial \phi}{\partial \theta^i} = 0 ,$$

and energy conservation,

$$\frac{1}{2} m \delta_{ij} \frac{d\theta^j}{d\tau} \frac{d\theta^i}{d\tau} + \phi(\theta) = E .$$

Thus, the constant $E$ is interpreted as energy. We have just derived $F = ma$ purely from principles of inference applied to the relevant information codified into a statistical model! From eq.(321) onwards our inference approach is formally identical to the Jacobi action principle of classical mechanics but we did not need to know this.

Within our theoretical construct, both the mass $m$ of the particles and their interactions are explained in terms of an irreducible uncertainty of their positions. Masses and interactions are features of the curved statistical manifold underlying the information-dynamics. Even though our formalism describes a non-relativistic model, there already appears a "unification" between mass $m$ and potential energy $\phi(\theta)$: they are different aspects of the same thing, the particle’s "intrinsic" position uncertainty $\sigma_0$ modulated throughout space by the field $\Phi(\theta)$. The derivation presented in this section illustrates the main idea but has two important limitations. First, it applies to a single particle with a fixed constant energy $E$ and this means that we consider only isolated systems. Second, even though we have identified a convenient parameter $\tau$, we do not know that it actually represents "true" time. It could be that $\tau$ is the universal Newtonian time. It could be that $\tau$ is just a parameter that applies only to one particular isolated particle.

The original formulation in terms of the "Jacobi" action, eq.(324), is completely timeless. Therefore the appearance of time is obscure. The solution to both these problems emerges as we apply the formalism to the motion of the only system known to be completely isolated: the whole universe. In this case, the fact that the energy $E$ is a fixed constant does not represent a restriction. Moreover, since the preferred time parameter would be associated to the whole universe, it would not be at all inappropriate to call it the universal time.

**XXVII. NEWTONIAN ENTROPIC DYNAMICS OF THE WHOLE UNIVERSE**

Our theoretical scheme may be generalized to arbitrary $N$-particles interacting with external potentials and also with each other. However, to simplify our notation we will consider a universe that consists of $N = 2$ particles. For the 2-particle system the position $\theta = (\theta_1, \theta_2)$ is denoted by 6 coordinates $\theta^A$ with $A = 1, 2, \ldots, 6$. Let $\theta^A = (\theta^1,$
\( \theta^i \) with \( i_1 = 1, 2, 3 \) for particle 1 and \( i_2 = 4, 5, 6 \) for particle 2. A point in the \( N = 2 \) configuration space is a Gaussian distribution,

\[
p(x|\theta) = \frac{1}{(2\pi)^{3/2}} \exp \left[ -\frac{\gamma_{AB}(\theta)(x^A - \theta^A)(x^B - \theta^B)}{2} \right].
\]  
(335)

The simplest model for two (possibly non-identical) particles assigns uniform variances \( \sigma_1^2 \) and \( \sigma_2^2 \) to each particle. The corresponding metric, analogous to eq. (322), is

\[
g^{(0)}_{AB} = \gamma^{(0)}_{AB} = \tilde{m}_{AB},
\]  
(336)

where \( m_{AB} \) is a constant \( 6 \times 6 \) diagonal matrix,

\[
\tilde{m}_{AB} \overset{\text{def}}{=} \begin{bmatrix}
\delta_{i_1,j_1}/\sigma^2_1 & 0 \\
0 & \delta_{i_2,j_2}/\sigma^2_2
\end{bmatrix},
\]  
(337)

where each entry represents a \( 3 \times 3 \) matrix. The metric \( m_{AB} \) describes a flat space; the trajectories are familiar "straight" lines and the particles move independently of each other; they do not interact. However, non-trivial dynamics requires the introduction of curvature and the simplest way to do this is through an overall conformal field \( \Phi(\theta) \) with \( \theta = (\theta_1, \theta_2) \). We propose

\[
g_{AB}(\theta) = \gamma_{AB}(\theta) = \Phi(\theta)\tilde{m}_{AB} .
\]  
(338)

The equation of motion for the \( N = 2 \) universe is the geodesic that minimizes

\[
\mathcal{J} = 2^{1/2} \xi_f \int_{\xi_i}^{\xi_f} d\xi \mathcal{L}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) ,
\]  
(339)

where \( \mathcal{L}(\theta, \dot{\theta}) = \{\Phi(\theta)T_\xi(\dot{\theta})\}^{1/2} \) and \( T_\xi(\dot{\theta}) = \frac{1}{2} \tilde{m}_{AB}\dot{\theta}^A\dot{\theta}^B \). The Lagrange equations yield,

\[
\tilde{m}_{AB} \left( \frac{\Phi}{T_\xi} \right)^{1/2} \frac{d}{d\xi} \left[ \left( \frac{\Phi}{T_\xi} \right)^{1/2} \frac{d\theta^B}{d\xi} \right] - \frac{\partial \Phi}{\partial \theta^A} = 0 ,
\]  
(340)

which suggests introducing a new parameter \( \tau \) defined by

\[
d\tau = \kappa \left( \frac{T_\xi}{\Phi} \right)^{1/2} d\xi \quad \text{or} \quad \kappa \frac{d}{d\tau} = \left( \frac{\Phi}{T_\xi} \right)^{1/2} \frac{d}{d\xi} .
\]  
(341)

In terms of the new parameter the equations of motion are

\[
m_{AB} \frac{d^2\theta^A}{d\tau^2} - \varepsilon \frac{\partial \Phi}{\partial \theta^A} = 0 .
\]  
(342)

where \( m_{AB} \overset{\text{def}}{=} \varepsilon \kappa^2 \tilde{m}_{AB} \) is a diagonal matrix. Equation (342) becomes

\[
\varepsilon \kappa^2 \frac{d^2\theta^i_n}{d\tau^2} - \varepsilon \frac{\partial}{\partial \theta^i_n} \Phi(\theta_1, \theta_2) = 0 ,
\]  
(343)
for each of the particles, \( n = 1, 2 \). The motion of particle 1 depends on the location of particle 2: \textit{these are interacting particles!} The new time parameter \( \tau \), eq.(341), is such that

\[
\Phi = \kappa^2 T_\xi \left( \frac{d\xi}{d\tau} \right)^2 = T_\tau, \quad T_\tau = \frac{1}{2} \kappa^2 \hat{m}_{AB} \frac{d\theta^A}{d\tau} \frac{d\theta^B}{d\tau}.
\]

As before, the equivalence to Newtonian dynamics is made explicit by a change of notation,

\[
\varepsilon \frac{\kappa^2}{\sigma_n^2} \overset{\text{def}}{=} m_n \quad \text{and} \quad \varepsilon \Phi(\theta) = E - \phi(\theta).
\]  

The result is

\[
m_n \frac{d^2 \theta^i}{d\tau^2} + \frac{\partial}{\partial \theta^i} \phi(\theta_1, \theta_2) = 0,
\]

with

\[
\varepsilon \Phi(\theta_1, \theta_2) = \frac{1}{2} m_{AB} \frac{d\theta^A}{d\tau} \frac{d\theta^B}{d\tau} = E - \phi(\theta_1, \theta_2).
\]

The constant \( E \) in (347) is the total energy of the universe and there are no restrictions on the energy of individual subsystems. The choice for the conformal factor \( \Phi(\theta_1, \theta_2) \) is quite general. For instance, we may consider \( \Phi(\theta_1, \theta_2) \),

\[
\Phi(\theta_1, \theta_2) = -V_1(\theta_1) - V_2(\theta_2) - U(\theta_1, \theta_2) + E,
\]

so the particles can interact with external potentials \( V_1 \) and \( V_2 \) and also with each other through \( U(\theta_1, \theta_2) \). The definition of the parameter \( \tau \) requires taking into account all the particles in the universe. We started with a completely timeless theory, eq.(339), and in fact, no \textit{external} time has been introduced. What we have is a convenient \( \tau \) parameter associated to the change of the total system, which in this case is the whole universe. The universe is its own clock and it measures universal time. It is worthwhile noticing that the reparametrization that allowed us to introduce a Newtonian time was possible only because the same conformal factor \( \Phi(\theta) \) applies equally to all particles.

\section{A. Remarks on Newtonian Entropic Dynamics}

Newtonian entropic dynamics offers a new perspective on the concept of mass and interactions. In order to see this, notice that since \( \gamma_{AB} \) in (338) is diagonal the distribution (335) is a product,

\[
p(x|\theta) = p(x_1|\theta_1, \theta_2)p(x_2|\theta_1, \theta_2).
\]

Although the model represents interacting particles, the distribution is a product: the uncertain variables \( x_1 \) and \( x_2 \) are statistically independent. The coupling arises through the conditioning on \( \theta = (\theta_1, \theta_2) \). Focusing our attention on
particle 1 (similar remarks also apply to particle 2), we note the distribution $p(x_1|\theta_1, \theta_2)$ is a spherically symmetric Gaussian,

$$p(x_1|\theta_1, \theta_2) \propto \exp \left[ -\frac{\delta_{ij}(x^i - \theta^i)(x^j - \theta^j)}{2\sigma_1^2(\theta_1, \theta_2)} \right].$$

(350)

The uncertainty in the position of particle 1 is given by $\sigma_1(\theta_1, \theta_2) = \left[ \Phi(\theta_1, \theta_2)m_1 \right]^{-1/2}$. The mass $m_1$ is interpreted in terms of a uniform background contribution to the uncertainty. Mass is a manifestation of an uncertainty in location; higher mass reflects a lower uncertainty. On the other hand, interactions arise from the non-uniformity of $\sigma_1(\theta_1, \theta_2)$ that depends on the location of other particles through the modulating field $\Phi(\theta_1, \theta_2)$. It is amusing to note that even though this is a non-relativistic model there already appears a "unification" between mass and (potential) energy: they are different aspects of the same thing, the position uncertainty.

XXVIII. CONCLUSIONS

In this Chapter, we have shown that the tools of inference- probability, information geometry and entropy- are sufficiently rich that one can construct entropic dynamics models that reproduce recognizable laws of physics. Indeed, preliminary steps towards an entropic dynamics approach to general relativity appeared in [117]. Moreover, an entropic dynamics approach to the study of chaos has already lead to interesting results [119–121]. An extended version of these results will be presented in Chapters 7 and 8. We emphasize that NED has limited applicability being a nonrelativistic model. Our theoretical construct invokes two metrics: there is the metric of flat three-dimensional Euclidean space, $\delta_{ij}$, that appears in the kinetic energies and there is the information metric $g_{ij}$ that accounts for mass $m$ and interactions $\phi$ and applies to the curved configuration space $\mathcal{M}_s$. This is a reflection of the fact that a system of $N$ particles is described as a point in a $3N$-dimensional configuration space $\mathcal{M}_s$ instead of $N$ points living within the same evolving three-dimensional space $\mathbb{R}^3$. Furthermore, we the choice of the modulating field $\Phi(\theta)$ does not arise from first principles, it is merely an educated guess. This is no different from Newton’s dictum *hypothesis non fingo*: a choice of $\Phi(\theta)$ is justified by its explanatory success. Finally, at this very premature point in the development of our Newtonian entropic dynamics, we do not offer any physical insight about the underlying fuzziness of space. This will be one of our major concerns in future works.
Chapter 7: Information geometrodynamical approach to chaos: An Application

I extend my study of chaotic systems (information geometrodynamical approach to chaos, IGAC) to an ED Gaussian model describing an arbitrary system of $3N$ degrees of freedom. It is shown that the hyperbolicity of a non-maximally symmetric $6N$-dimensional statistical manifold $\mathcal{M}_S$ underlying the ED Gaussian model leads to linear information-geometrodynamical entropy growth and to exponential growth of the Jacobi vector field intensity, quantum and classical features of chaos respectively. As a special physical application, the information geometrodynamical scheme is applied to investigate the chaotic properties of a set of $n$-uncoupled three-dimensional anisotropic inverted harmonic oscillators (IHOs) characterized by an Ohmic distributed frequency spectrum. I show that the asymptotic temporal behavior of the information geometrodynamical entropy of such a system exhibits linear growth and I suggest the system studied may be considered to be the classical information-geometric analogue of the Zurek-Paz quantum chaos criterion in its classical reversible limit.

XXIX. INTRODUCTION

The lack of a unified characterization of chaos in classical and quantum dynamics is well-known. In the Riemannian [5] and Finslerian [6] (a Finsler metric is obtained from a Riemannian metric by relaxing the requirement that the metric be quadratic on each tangent space) geometrodynamical approach to chaos in classical Hamiltonian systems, an active field of research concerns the possibility of finding a rigorous relation among the sectional curvature, the Lyapunov exponents, and the Kolmogorov-Sinai dynamical entropy (i.e. the sum of positive Lyapunov exponents) [7]. The largest Lyapunov exponent characterizes the degree of chaoticity of a dynamical system and, if positive, it measures the mean instability rate of nearby trajectories averaged along a sufficiently long reference trajectory. Moreover, it is known that classical chaotic systems are distinguished by their exponential sensitivity to initial conditions and that the absence of this property in quantum systems has lead to a number of different criteria being proposed for quantum chaos. Exponential decay of fidelity, hypersensitivity to perturbation, and the Zurek-Paz quantum chaos criterion of linear von Neumann’s entropy growth [8] are some examples [9]. These criteria accurately predict chaos in the classical limit, but it is not clear that they behave the same far from the classical realm. The present work makes use of Entropic Dynamics (ED) [122]. ED is a theoretical framework that arises from the combination of inductive inference (Maximum relative Entropy Methods, [123]) and Information Geometry (Riemannian geometry applied to
probability theory) (IG) \[31\]. As such, ED is constructed on statistical manifolds. It is developed to investigate the possibility that laws of physics - either classical or quantum - might reflect laws of inference rather than laws of nature. This Chapter contains works that follow up a series of my works \[10–12\]. In this Chapter, the ED theoretical framework is used to explore the possibility of constructing a unified characterization of classical and quantum chaos. We investigate a system whose microstates \( \{ \vec{X} \} \) are characterized by \( 3N \) degrees of freedom \( \{ x^{(\alpha)}_a \}_{a=1, 2, 3} \). Each degree of freedom is Gaussian-distributed and it is described by two pieces of relevant information, its mean expected value and its variance. This leads to consider an ED model on a non-maximally symmetric \( 6N \)-dimensional statistical manifold \( M_s \). It is shown that \( M_s \) possesses a constant negative Ricci curvature that is proportional to the number of degrees of freedom of the system, \( R_{M_s} = -3N \). It is shown that the system explores statistical volume elements on \( M_s \) at an exponential rate. We define a dynamical information-geometric entropy \( S_{M_s} \) of the system and we show it increases linearly in time (statistical evolution parameter) and is moreover, proportional to the number of degrees of freedom of the system. The geodesics on \( M_s \) are hyperbolic trajectories. Using the Jacobi-Levi-Civita (JLC) equation for geodesic spread, it is shown that the Jacobi vector field intensity \( J_{M_s} \) diverges exponentially and is proportional to the number of degrees of freedom of the system. Thus, \( R_{M_s}, S_{M_s} \) and \( J_{M_s} \) are proportional to the number of Gaussian-distributed microstates of the system. This proportionality leads to conclude there is a substantial link among these information-geometric indicators of chaoticity. Finally, as a special physical application, the information geometrodynamical scheme is applied to investigate the chaotic properties of a set of \( n \)-uncoupled three-dimensional anisotropic inverted harmonic oscillators (IHOs) characterized by an Ohmic distributed frequency spectrum. We study the three-dimensional IHOs for the sake of generality. However we also illustrate the main idea in a simpler example studying the IGAC (Information Geometrodynamical Approach to Chaos) of two uncoupled inverted one-dimensional harmonic oscillators. I show that the asymptotic temporal behavior of the information geometrodynamical entropy of such a system presents linear growth and I suggest the system studied may be considered the classical information-geometric analogue of the Zurek-Paz quantum chaos criterion in its classical reversible limit.

XXX. SPECIFICATION OF THE GAUSSIAN ED-MODEL

Maximum relative Entropy (ME) methods are used to construct an ED model that follows from an assumption about what information is relevant to predict the evolution of the system. Given a known initial macrostate (probability distribution) and that the system evolves to a final known macrostate, the possible trajectories of the system are examined. A notion of distance between two probability distributions is provided by IG. As shown in \[32, 33\] this
distance is quantified by the Fisher-Rao information metric tensor.

We consider an ED model whose microstates span a 3N-dimensional space labelled by the variables \( \{ \vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(N)} \} \) with \( \vec{x}^{(\alpha)} \equiv (x_1^{(\alpha)}, x_2^{(\alpha)}, x_3^{(\alpha)}) \), \( \alpha = 1, \ldots, N \) and \( x_a^{(\alpha)} \in \mathbb{R} \) with \( a = 1, 2, 3 \). We assume the only testable information pertaining to the quantities \( x_a^{(\alpha)} \) consists of the expectation values \( \langle x_a^{(\alpha)} \rangle \) and variance \( \Delta x_a^{(\alpha)} = \sqrt{\left\langle \left( x_a^{(\alpha)} - \langle x_a^{(\alpha)} \rangle \right)^2 \right\rangle} \). The set of these expectation values define the 6N-dimensional space of macrostates. A measure of distinguishability among macrostates is obtained by assigning a probability distribution \( P(\vec{X} \mid \vec{\Theta}) \) to each macrostate \( \vec{\Theta} \) where \( \{ \vec{\Theta} \} = \{ (1) \theta_a^{(\alpha)}, (2) \theta_a^{(\alpha)} \} \) with \( \alpha = 1, 2, \ldots, N \) and \( a = 1, 2, 3 \). The process of assigning a probability distribution to each state endows \( \mathcal{M}_S \) with a metric structure. Specifically, the Fisher-Rao information metric defined in (362) is a measure of distinguishability among macrostates. It assigns an IG to the space of states.

A. The Gaussian statistical manifold \( \mathcal{M}_S \)

We consider an arbitrary system evolving over a 3N-dimensional space. The variables \( \{ \vec{X} \} = \{ \vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(N)} \} \) label the 3N-dimensional space of microstates of the system. All information relevant to the dynamical evolution of the system is assumed to be contained in the probability distributions. For this reason, no other information is required. Each macrostate may be viewed as a point of a 6N-dimensional statistical manifold with coordinates given by the numerical values of the expectations \( (1) \theta_a^{(\alpha)} = \langle x_a^{(\alpha)} \rangle \) and \( (2) \theta_a^{(\alpha)} = \Delta x_a^{(\alpha)} \equiv \sqrt{\left\langle \left( x_a^{(\alpha)} - \langle x_a^{(\alpha)} \rangle \right)^2 \right\rangle} \). The available information is contained in the following 6N information constraint equations,

\[
\langle x_a^{(\alpha)} \rangle = \int_{-\infty}^{+\infty} dx_a^{(\alpha)} P_a^{(\alpha)} \left( x_a^{(\alpha)} \left| (1) \theta_a^{(\alpha)}, (2) \theta_a^{(\alpha)} \right. \right),
\]

\[
\Delta x_a^{(\alpha)} = \left[ \int_{-\infty}^{+\infty} dx_a^{(\alpha)} \left( x_a^{(\alpha)} - \langle x_a^{(\alpha)} \rangle \right)^2 P_a^{(\alpha)} \left( x_a^{(\alpha)} \left| (1) \theta_a^{(\alpha)}, (2) \theta_a^{(\alpha)} \right. \right) \right]^{\frac{1}{2}},
\]

where \( (1) \theta_a^{(\alpha)} = \langle x_a^{(\alpha)} \rangle \) and \( (2) \theta_a^{(\alpha)} = \Delta x_a^{(\alpha)} \) with \( \alpha = 1, 2, \ldots, N \) and \( a = 1, 2, 3 \). The probability distributions \( P_a^{(\alpha)} \) are constrained by the conditions of normalization,

\[
\int_{-\infty}^{+\infty} dx_a^{(\alpha)} P_a^{(\alpha)} \left( x_a^{(\alpha)} \left| (1) \theta_a^{(\alpha)}, (2) \theta_a^{(\alpha)} \right. \right) = 1.
\]

The Gaussian distribution is identified by information theory as the maximum entropy distribution if only the expectation value and the variance are known. ME methods allows to associate a probability distribution \( P(\vec{X} \mid \vec{\Theta}) \) to each
point in the space of states $\vec{\Theta}$. The distribution that best reflects the information contained in the prior distribution $m \left( \vec{X} \right)$ updated by the information $\left( \left\langle x_a^{(\alpha)} \right\rangle, \Delta x_a^{(\alpha)} \right)$ is obtained by maximizing the relative entropy

$$S \left( \vec{\Theta} \right) = - \int_{\{ \vec{X} \}} d^{3N} \vec{X} P \left( \vec{X} | \vec{\Theta} \right) \log \left( \frac{P \left( \vec{X} | \vec{\Theta} \right)}{m \left( \vec{X} \right)} \right).$$

(353)

As a working hypothesis, the prior $m \left( \vec{X} \right)$ is set to be uniform since we assume the lack of prior available information about the system (postulate of equal a priori probabilities). Upon maximizing (353), given the constraints (351) and (352), we obtain

$$P \left( \vec{X} | \vec{\Theta} \right) = \prod_{a=1}^{3N} \prod_{a=1}^{\alpha} P_a^{(\alpha)} \left( x_a^{(\alpha)} | \mu_a^{(\alpha)}, \sigma_a^{(\alpha)} \right)$$

(354)

where

$$P_a^{(\alpha)} \left( x_a^{(\alpha)} | \mu_a^{(\alpha)}, \sigma_a^{(\alpha)} \right) = \left( \frac{1}{2\pi \sigma_a^{(\alpha)}} \right)^{\frac{1}{2}} \exp \left[ -\frac{\left( x_a^{(\alpha)} - \mu_a^{(\alpha)} \right)^2}{2 \sigma_a^{(\alpha)}^2} \right]$$

(355)

and (1) $\theta_a^{(\alpha)} = \mu_a^{(\alpha)}$, (2) $\theta_a^{(\alpha)} = \sigma_a^{(\alpha)}$. For the rest of the Chapter, unless stated otherwise, the statistical manifold $\mathcal{M}_S$ will be defined by the following expression,

$$\mathcal{M}_S = \left\{ P \left( \vec{X} | \vec{\Theta} \right) \text{ in (354)} : \vec{X} \in \mathbb{R}^{3N}, \vec{\Theta} \in D_\Theta = \left( -\infty, +\infty \right)_\mu \times (0, +\infty)_\sigma \right\}^{3N}.$$

(356)

The probability distribution (354) encodes the available information concerning the system. Note we assumed uncoupled constraints among microvariables $x_a^{(\alpha)}$. In other words, we assumed that information about correlations between the microvariables need not to be tracked. This assumption leads to the simplified product rule (354). However, coupled constraints would lead to a generalized product rule in (354) and to a metric tensor (362) with non-trivial off-diagonal elements (covariance terms). For instance, the total probability distribution $P (x, y | \mu_x, \sigma_x, \mu_y, \sigma_y)$ of two dependent Gaussian distributed microvariables $x$ and $y$ reads

$$P (x, y | \mu_x, \sigma_x, \mu_y, \sigma_y) = \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-r^2}} x$$

$$\times \exp \left\{ -\frac{1}{2 (1-r^2)} \left[ \frac{(x - \mu_x)^2}{\sigma_x^2} - 2r \frac{(x - \mu_x)(y - \mu_y)}{\sigma_x \sigma_y} + \frac{(y - \mu_y)^2}{\sigma_y^2} \right] \right\},$$

(357)

where $r \in (-1, +1)$ is the correlation coefficient given by

$$r = \frac{\langle (x - \langle x \rangle)(y - \langle y \rangle) \rangle}{\sqrt{\langle x - \langle x \rangle \rangle^2} \sqrt{\langle y - \langle y \rangle \rangle^2}} = \frac{\langle xy \rangle - \langle x \rangle \langle y \rangle}{\sigma_x \sigma_y}.$$
The metric induced by (357) is obtained by use of (362), the result being

\[
g_{ij} = \begin{bmatrix}
-\frac{1}{\sigma^2_x(r^2-1)} & 0 & \frac{r}{\sigma_x\sigma_y(r^2-1)} & 0 \\
0 & -\frac{2r^2}{\sigma^2_x(r^2-1)} & 0 & \frac{r^2}{\sigma_x\sigma_y(r^2-1)} \\
\frac{r}{\sigma_x\sigma_y(r^2-1)} & 0 & -\frac{1}{\sigma_y^2(r^2-1)} & 0 \\
0 & \frac{r^2}{\sigma_x\sigma_y(r^2-1)} & 0 & -\frac{2r^2}{\sigma_y^2(r^2-1)} \\
\end{bmatrix},
\]

where \( i, j = 1, 2, 3, 4 \). The Ricci curvature scalar associated with manifold characterized by (359) is given by

\[
R = g^{ij}R_{ij} = -8 \frac{(r^2 - 2) + 2r^2 (3r^2 - 2)}{8(r^2 - 1)},
\]

(360)

It is clear that in the limit \( r \to 0 \), the off-diagonal elements of \( g_{ij} \) vanish and the scalar \( R \) reduces to the result obtained in [11], namely \( R = -2 < 0 \). Correlation terms may be fictitious. They may arise for instance from coordinate transformations. On the other hand, correlations may arise from external fields in which the system is immersed. In such situations, correlations among \( x_a^{(\alpha)} \) effectively describe interaction between the microvariables and the external fields. Such generalizations would require more delicate analysis. Before proceeding, a comment is in order. Most probability distributions arise from the maximum entropy formalism as a result of simple statements concerning averages (Gaussians, exponential, binomial, etc.). Not all distribution are generated in this manner however. Some distributions are generated by combining the results of simple cases (multinomial from a binomial) while others are found as a result of a change of variables (Cauchy distribution). For instance, the Weibull and Wigner-Dyson distributions can be obtained from an exponential distribution as a result of a power law transformation [124].

1. Metric structure of \( M_S \)

We cannot determine the evolution of microstates of the system since the available information is insufficient. Not only is the information available insufficient but we also do not know the equation of motion. In fact there is no standard "equation of motion". Instead we can ask: how close are the two total distributions with parameters \( (\mu_a^{(\alpha)}, \sigma_a^{(\alpha)}) \) and \( (\mu_a^{(\alpha)} + d\mu_a^{(\alpha)}, \sigma_a^{(\alpha)} + d\sigma_a^{(\alpha)}) \)? Once the states of the system have been defined, the next step concerns the problem of quantifying the notion of change from the state \( \Theta \) to the state \( \Theta + d\Theta \). A convenient measure of change is distance. The measure we seek is given by the dimensionless distance \( ds \) between \( P(\vec{X} | \Theta) \) and \( P(\vec{X} | \Theta + d\Theta) \),

\[
ds^2 = g_{\mu\nu} d\Theta^\mu d\Theta^\nu \quad \text{with} \quad \mu, \nu = 1, 2, \ldots, 6N,
\]

(361)

where

\[
g_{\mu\nu} = \int d\vec{X} P(\vec{X} | \Theta) \frac{\partial \log P(\vec{X} | \Theta)}{\partial \Theta^\mu} \frac{\partial \log P(\vec{X} | \Theta)}{\partial \Theta^\nu}
\]

(362)
is the Fisher-Rao information metric. Substituting (354) into (362), the metric \( g_{\mu\nu} \) on \( M_s \) becomes a \( 6N \times 6N \) matrix \( M \) made up of \( 3N \) blocks \( M_{2 \times 2} \) with dimension \( 2 \times 2 \) given by,

\[
M_{2 \times 2} = \begin{pmatrix}
\left( \sigma_{a}^{(n)} \right)^{-2} & 0 \\
0 & 2 \times \left( \sigma_{a}^{(n)} \right)^{-2}
\end{pmatrix}
\]  

(363)

with \( \alpha = 1, 2, \ldots, N \) and \( a = 1, 2, 3 \). From (362), the "length" element (361) reads,

\[
ds^2 = \sum_{\alpha=1}^{3N} \sum_{a=1}^{3N} \left[ \frac{1}{\left( \sigma_{a}^{(n)} \right)^2} d\mu_{\alpha}^{(n)/2} + \frac{2}{\left( \sigma_{a}^{(n)} \right)^2} d\sigma_{a}^{(n)/2} \right].
\]  

(364)

We bring attention to the fact that the metric structure of \( M_s \) is an emergent (not fundamental) structure. It arises only after assigning a probability distribution \( P(\vec{X} | \vec{\Theta}) \) to each state \( \vec{\Theta} \).

2. Curvature of \( M_s \)

Given the Fisher-Rao information metric, we use standard differential geometry methods applied to the space of probability distributions to characterize the geometric properties of \( M_s \). Recall that the Ricci scalar curvature \( R \) is given by,

\[
R = g^{\mu\nu} R_{\mu\nu},
\]  

(365)

where \( g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho \) so that \( g^{\mu\nu} = (g_{\mu\nu})^{-1} \). The Ricci tensor \( R_{\mu\nu} \) is given by,

\[
R_{\mu\nu} = \partial_\nu \Gamma^\gamma_{\mu\rho} - \partial_\rho \Gamma^\gamma_{\mu\nu} + \Gamma^\gamma_{\nu\lambda} \Gamma^\rho_{\gamma\lambda} - \Gamma^\rho_{\nu\gamma} \Gamma^\gamma_{\rho\lambda}.
\]  

(366)

The Christoffel symbols \( \Gamma^\rho_{\mu\nu} \) appearing in the Ricci tensor are defined in the standard manner as,

\[
\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left( \partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right).
\]  

(367)

Using (363) and the definitions given above, we can show that the Ricci scalar curvature becomes

\[
R_{M_s} = R^\alpha_{\alpha} = \sum_{\rho \neq \sigma} K(e_\rho, e_\sigma) = -3N < 0.
\]  

(368)

The scalar curvature is the sum of all sectional curvatures of planes spanned by pairs of orthonormal basis elements \( \{ e_\rho = \partial_{\Theta_\rho(p)} \} \) of the tangent space \( T_p M_s \) with \( p \in M_s \),

\[
K(a, b) = \frac{R_{\mu\rho\sigma\lambda} a^\mu b^\rho a^\sigma b^\lambda}{(g_{\mu\sigma} g_{\nu\lambda} - g_{\mu\rho} g_{\nu\sigma}) a^\mu b^\rho a^\sigma b^\lambda}, a = \sum_\rho (a, h^\rho) e_\rho.
\]  

(369)
where \( \langle e_\rho, h^{\sigma} \rangle = \delta^\sigma_\rho \). Notice that the sectional curvatures completely determine the curvature tensor. From (368) we conclude that \( \mathcal{M}_s \) is a \( 6N \)-dimensional statistical manifold of constant negative Ricci scalar curvature. A detailed analysis on the calculation of Christoffel connection coefficients using the ED formalism for a four-dimensional manifold of Gaussians can be found in [11].

3. Anisotropy and Compactness

It can be shown that \( \mathcal{M}_s \) is not a maximally symmetric multidimensional manifold. The first way this can be understood is from the fact that the Weyl Projective curvature tensor \( W_{\mu\nu\rho\sigma} \) (or the anisotropy tensor) defined by

\[
W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{\mathcal{R}_{\mathcal{M}_s}}{n(n-1)}(g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma}), \tag{370}
\]

with \( n = 6N \) in the present case, is non-vanishing. In (370), the quantity \( R_{\mu\nu\rho\sigma} \) is the Riemann curvature tensor defined in the usual manner by

\[
R^\alpha_{\beta\rho\sigma} = \partial_\sigma \Gamma^\alpha_{\beta\rho} - \partial_\rho \Gamma^\alpha_{\beta\sigma} + \Gamma^\alpha_{\lambda\sigma} \Gamma^\lambda_{\beta\rho} - \Gamma^\alpha_{\lambda\rho} \Gamma^\lambda_{\beta\sigma}. \tag{371}
\]

Considerations regarding the negativity of the Ricci curvature as a strong criterion of dynamical instability and the necessity of compactness of \( \mathcal{M}_s \) in “true” chaotic dynamical systems is under investigation [14].

The issue of symmetry of \( \mathcal{M}_s \) can alternatively be understood from consideration of the sectional curvature. In view of (369), the negativity of the Ricci scalar implies the existence of expanding directions in the configuration space manifold \( \mathcal{M}_s \). Indeed, from (368) one may conclude that negative principal curvatures (extrema of sectional curvatures) dominate over positive ones. Thus, the negativity of the Ricci scalar is only a sufficient (not necessary) condition for local instability of geodesic flow. For this reason, the negativity of the scalar provides a strong criterion of local instability. Scenarios may arise where negative sectional curvatures are present, but the positive ones could prevail in the sum so that the Ricci scalar is non-negative despite the instability in the flow in those directions. Consequently, the signs of the sectional curvatures are of primary significance for the proper characterization of chaos.

Yet another useful way to understand the anisotropy of the \( \mathcal{M}_s \) is the following. It is known that in \( n \) dimensions, there are at most \( \frac{n(n+1)}{2} \) independent Killing vectors (directions of symmetry of the manifold). Since \( \mathcal{M}_s \) is not a pseudosphere, the information metric tensor does not admit the maximum number of Killing vectors \( K_\nu \) defined as

\[
\mathcal{L}_K g_{\mu\nu} = D_\mu K_\nu + D_\nu K_\mu = 0, \tag{372}
\]
where \( D_\mu \), given by

\[
D_\mu K_\nu = \partial_\mu K_\nu - \Gamma^\rho_{\nu\mu} K_\rho, \tag{373}
\]

is the covariant derivative operator with respect to the connection \( \Gamma \) defined in (367). The Lie derivative \( \mathcal{L}_K g_{\mu\nu} \) of the tensor field \( g_{\mu\nu} \) along a given direction \( K \) measures the intrinsic variation of the field along that direction (that is, the metric tensor is Lie transported along the Killing vector) \[125\]. Locally, a maximally symmetric space of Euclidean signature is either a plane, a sphere, or a hyperboloid, depending on the sign of \( \mathcal{R} \). In our case, none of these scenarios occur. As will be seen in what follows, this fact has a significant impact on the integration of the geodesic deviation equation on \( \mathcal{M}_s \). At this juncture, we emphasize it is known that the anisotropy of the manifold underlying system dynamics plays a crucial role in the mechanism of instability. In particular, fluctuating sectional curvatures require also that the manifold be anisotropic. However, the connection between curvature variations along geodesics and anisotropy is far from clear and is currently under investigation.

Krylov was the first to emphasize \[126\] the use of \( \mathcal{R} < 0 \) as an instability criterion in the context of an \( N \)-body system (a gas) interacting via Van der Waals forces, with the ultimate hope to understand the relaxation process in a gas. However, Krylov neglected the problem of compactness of the configuration space manifold which is important for making inferences about exponential mixing of geodesic flows \[127\]. Mixing provides statistical independence of different parts of a trajectory. This is the condition for application of probability theory that allows to calculate statistical properties such as diffusion, relaxation etc. Why is compactness so significant in the characterization of chaos? True chaos should be identified by the occurrence of two crucial features: 1) strong dependence on initial conditions and exponential divergence of the Jacobi vector field intensity, i.e., stretching of dynamical trajectories; 2) compactness of the configuration space manifold, i.e., folding of dynamical trajectories. Compactness \[6, 128\] is required in order to discard trivial exponential growths due to the unboundedness of the "volume" available to the dynamical system. In other words, the folding is necessary to have a dynamics actually able to mix the trajectories, making practically impossible, after a finite interval of time, to discriminate between trajectories which were very nearby each other at the initial time. When the space is not compact, even in presence of strong dependence on initial conditions, it could be possible in some instances (though not always), to distinguish among different trajectories originating within a small distance and then evolved subject to exponential instability.

The statistical manifold \( \mathcal{M}_s \) defined in \[356\] is compact provided that the parameter space \( D_\Theta \) is compact. This can be seen as follows. It is known from IG that there is a one-to-one relation between elements of the statistical manifold and the parameter space. More precisely, the statistical manifold \( \mathcal{M}_s \) is homeomorphic to the parameter
space $\mathcal{D}_\Theta$. This implies the existence of a continuous, bijective map $h_{\mathcal{M}_S, \mathcal{D}_\Theta}$,

$$h_{\mathcal{M}_S, \mathcal{D}_\Theta} : \mathcal{M}_S \ni P\left(\vec{X} \mid \Theta\right) \rightarrow \Theta \in \mathcal{D}_\Theta$$  \hspace{1cm} (374)

where $h_{\mathcal{M}_S, \mathcal{D}_\Theta}^{-1}(\Theta) = P\left(\vec{X} \mid \Theta\right)$. The inverse image $h_{\mathcal{M}_S, \mathcal{D}_\Theta}^{-1}$ is the so-called homeomorphism map. In addition, since homeomorphisms preserve compactness, it is sufficient to restrict ourselves to a compact subspace of the parameter space $\mathcal{D}_\Theta$ in order to ensure that $\mathcal{M}_S$ is itself compact. In our specific case, the accessible region of the statistical manifold $\mathcal{M}_s$ is given by

$$\mathcal{M}_S = \left\{ P\left(\vec{X} \mid \Theta\right) \text{ in (354)} : \vec{X} \in \mathbb{R}^{3N}, \Theta \in \mathcal{D}_\Theta \right\},$$  \hspace{1cm} (375)

where the $6N$ dimensional $\mathcal{D}_\Theta$ is defined as

$$\mathcal{D}_\Theta \overset{\text{def}}{=} I_\mu \times I_\sigma = [\mu_{\min}, \mu_{\max}]^{3N} \times [\sigma_{\min}, \sigma_{\max}]^{3N}.$$  \hspace{1cm} (376)

Assuming $\tau \in \mathbb{R}^+$, from equations (380), we obtain

$$\mu_{\min} = \frac{4\beta B^2}{8\beta^2 + B^2} + C, \quad \mu_{\max} = 4\beta + C, \quad \sigma_{\min} = D, \quad \sigma_{\max} = \sqrt{2\beta} + D.$$  \hspace{1cm} (377)

In conclusion, $\mathcal{D}_\Theta$ is compact provided we restrict ourselves to consider only the accessible macrostates on $\mathcal{M}_s$.

XXXI. CANONICAL FORMALISM FOR THE GAUSSIAN ED-MODEL

Jacobi was the first one to develop a technique of classical mechanics where a Hamiltonian system is geometrized by transforming it into a geodesic flow on a suitable manifold with a convenient Riemannian metric \cite{129}. The two key steps in obtaining the geometrization of a Hamiltonian system are the introduction of a conformal transformation of the metric and the rescaling of the time parameter \cite{130} (and, for more details, Chapter 6). The reformulation of dynamics in terms of a geodesic problem allows the application of a wide range of well-known geometrical techniques in the investigation of the solution space and properties of equations of motions. The power of the Jacobi reformulation is that all of the dynamical information is collected into a single geometric object - the manifold on which geodesic flow is induced - in which all the available manifest symmetries are retained. For instance, integrability of the system is connected with the existence of Killing vectors and tensors on this manifold \cite{88, 131, 129}. In this section we study the trajectories of the system on $\mathcal{M}_S$. We emphasize ED can be derived from a standard principle of least action (of Maupertuis-Euler-Lagrange-Jacobi type) \cite{85, 122}. The main differences are that the dynamics being considered here, namely ED, is defined on a space of probability distributions $\mathcal{M}_s$, not on an ordinary
vectorial space $V$ and the standard coordinates $q_\mu$ of the system are replaced by statistical macrovariables $\Theta^\mu$. The geodesic equations for the macrovariables of the Gaussian ED model are given by,

$$\frac{d^2 \Theta^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{d\Theta^\nu}{d\tau} \frac{d\Theta^\rho}{d\tau} = 0$$

with $\mu = 1, 2, \ldots, 6N$. The geodesic equations are nonlinear second order coupled ordinary differential equations. They describe a reversible dynamics whose solution is the trajectory between an initial and a final macrostate. The trajectory can be equally well traversed in both directions.

### A. Geodesics on $M_s$

We determine the explicit form of (378) for the pairs of statistical coordinates $(\mu_\alpha^a, \sigma_\alpha^a)$. Substituting the expression of the Christoffel connection coefficients into (378), the geodesic equations for the macrovariables $\mu_\alpha^a$ and $\sigma_\alpha^a$ associated to the microstate $x_\alpha^a$ become,

$$\frac{d^2 \mu_\alpha^a}{d\tau^2} - \frac{2}{\sigma_\alpha^a} \frac{d\mu_\alpha^a}{d\tau} \frac{d\sigma_\alpha^a}{d\tau} = 0, \quad \frac{d^2 \sigma_\alpha^a}{d\tau^2} - \frac{1}{\sigma_\alpha^a} \left( \frac{d\sigma_\alpha^a}{d\tau} \right)^2 + \frac{1}{2\sigma_\alpha^a} \left( \frac{d\mu_\alpha^a}{d\tau} \right)^2 = 0,$$

with $\alpha = 1, 2, \ldots, N$ and $a = 1, 2, 3$. The $3N$ Gaussians are not coupled to each other, however each Gaussian is characterized by coupled macrovariables $\mu_\alpha^a$ and $\sigma_\alpha^a$. Equation (379) describes a set of coupled ordinary differential equations, whose solutions are

$$\mu_\alpha^a(\tau) = \frac{\left( B_\alpha^a \right)^2}{2\beta_\alpha^a} \frac{1}{\cosh \left( 2\beta_\alpha^a \tau \right) - \sinh \left( 2\beta_\alpha^a \tau \right) + \frac{\left( B_\alpha^a \right)^2}{8\beta_\alpha^a}} + C_\alpha^a,$$

$$\sigma_\alpha^a(\tau) = B_\alpha^a \frac{\cosh \left( \beta_\alpha^a \tau \right) - \sinh \left( \beta_\alpha^a \tau \right)}{\cosh \left( 2\beta_\alpha^a \tau \right) - \sinh \left( 2\beta_\alpha^a \tau \right) + \frac{\left( B_\alpha^a \right)^2}{8\beta_\alpha^a}} + D_\alpha^a.$$

The quantities $B_\alpha^a, C_\alpha^a, D_\alpha^a, \beta_\alpha^a$ are real integration constants that can be evaluated upon specification of boundary conditions. We are interested in the stability of the trajectories on $M_s$. It is known [85] that the Riemannian curvature of a manifold is intimately related to the behavior of geodesics on it. If the Riemannian curvature of a manifold is negative, geodesics (initially parallel) rapidly diverge from one another. We observe that since every maximal geodesic (one that cannot be extended to any larger interval) is well-defined for all temporal parameters $\tau$, $M_s$ constitute a geodesically complete manifold [132]. It is therefore a natural setting within which one may consider global questions and search for a weak criterion of chaos [6].
XXXII. EXPONENTIAL DIVERGENCE OF THE JACOBI VECTOR FIELD INTENSITY

The actual interest of the Riemannian formulation of the dynamics stems from the possibility of studying the instability of natural motions through the instability of geodesics of a suitable manifold, a circumstance that has several advantages. First of all a powerful mathematical tool exists to investigate the stability or instability of a geodesic flow: the Jacobi-Levi-Civita equation for geodesic spread [133]. The JLC-equation describes covariantly how nearby geodesics locally scatter. It is a familiar object both in Riemannian geometry and theoretical physics (it is of fundamental interest in experimental General Relativity). Moreover the JLC-equation relates the stability or instability of a geodesic flow with curvature properties of the ambient manifold, thus opening a wide and largely unexplored field of investigation of the connections among geometry, topology and geodesic instability, hence chaos.

Consider the behavior of the one-parameter (at fixed \(a\) and \(\alpha\), \(\lambda_\alpha \equiv \lambda\)) family of neighboring geodesics \(F_{G_{M_s}} (\lambda) \equiv \{\Theta^\mu_{\lambda_s} (\tau; \lambda)\}_{\lambda \in \mathbb{R}^+}^{\mu=1,\ldots,6N}\) where

\[
\mu_\alpha (\tau; \lambda) = \frac{\Lambda_\alpha^2}{2\Lambda_\alpha^{(\alpha)} \cosh \left(2\Lambda_\alpha^{(\alpha)} \tau\right) - \sinh \left(2\Lambda_\alpha^{(\alpha)} \tau\right)} + \frac{\Lambda_\alpha^{(\alpha)2} - \cosh \left(2\Lambda_\alpha^{(\alpha)} \tau\right) + C_\alpha}{8\Lambda_\alpha^{(\alpha)2}}
\]

with \(\alpha = 1, 2,\ldots, N\) and \(a = 1, 2, 3\). The relative geodesic spread on a (non-maximally symmetric) curved manifold as \(M_s\) is characterized by the Jacobi-Levi-Civita equation, the natural tool to tackle dynamical chaos [125, 133],

\[
\frac{D^2 \delta \Theta^\mu}{D\tau^2} + R^\mu_{\nu\rho\sigma} \frac{\partial \Theta^\nu}{\partial \tau} \frac{\partial \Theta^\rho}{\partial \tau} = 0
\]

where the Jacobi vector field \(J^\mu\) is defined as,

\[
J^\mu \equiv \delta \Theta^\mu \equiv \delta_\alpha^{(\alpha)} \Theta^\mu = \left(\frac{\partial \Theta^\mu (\tau; \lambda)}{\partial \lambda_\alpha^{(\alpha)}}\right)_{\tau=\text{const}} \delta \lambda_\alpha^{(\alpha)}.
\]

Notice that the JLC-equation appears intractable already at rather small \(N\). For isotropic manifolds, the JLC-equation can be reduced to the simple form,

\[
\frac{D^2 J^\mu}{D\tau^2} + K J^\mu = 0, \mu = 1,\ldots, 6N
\]

where \(K\) is the constant value assumed throughout the manifold by the sectional curvature. The sectional curvature of manifold \(M_s\) is the 6\(N\)-dimensional generalization of the Gaussian curvature of two-dimensional surfaces of \(\mathbb{R}^3\). If
$K < 0$, unstable solutions of equation \((384)\) assumes the form
\[
J(\tau) = \frac{1}{\sqrt{-K}} \omega(0) \sinh(\sqrt{-K} \tau)
\]
(385)
once the initial conditions are assigned as $J(0) = 0$, $\frac{dJ(0)}{d\tau} = \omega(0)$ and $K < 0$. Equation \((382)\) forms a system of $6N$ coupled ordinary differential equations linear in the components of the deviation vector field \((383)\) but nonlinear in derivatives of the metric \((362)\). It describes the linearized geodesic flow: the linearization ignores the relative velocity of the geodesics. When the geodesics are neighboring but their relative velocity is arbitrary, the corresponding geodesic deviation equation is the so-called generalized Jacobi equation \([111, 134]\). The nonlinearity is due to the existence of velocity-dependent terms in the system. Neighboring geodesics accelerate relative to each other with a rate directly measured by the curvature tensor $R_{\alpha\beta\gamma\delta}$. Substituting \((381)\) in \((382)\) and neglecting the exponentially decaying terms in $\delta \Theta^\mu$ and its derivatives, integration of \((382)\) leads to the following asymptotic expression of the Jacobi vector field intensity,
\[
J_{M_S} = ||J|| = \left(\sum_{\alpha=1}^{3} \sum_{a=1}^{3} \exp(\lambda^{(\alpha)} \tau) \right)^{\frac{1}{2}}
\]
(386)
As a side remark, we point out that if we consider the special case where different Gaussians are characterized by the same initial conditions leading to the same $\lambda \equiv \lambda^{(\alpha)}_a = \lambda^{(\alpha')}_{a'} \forall a, a' = 1, 2, 3$ and $\forall \alpha, \alpha' = 1, ..., N$, equation \((386)\) becomes,
\[
J_{M_S} \xrightarrow{\tau \to \infty} 3N \exp(\lambda \tau).
\]
(387)
We conclude that the geodesic spread on $M_s$ is described by means of an exponentially divergent Jacobi vector field intensity $J_{M_s}$, a classical feature of chaos. In our approach the quantity $\lambda_J$,
\[
\lambda_J \overset{\text{def}}{=} \lim_{\tau \to \infty} \frac{1}{\tau} \ln \left( \frac{||J_{M_S}(\tau)||}{||J_{M_S}(0)||} \right)
\]
(388)
would play the role of the conventional Lyapunov exponents.

XXXIII. LINEARITY OF THE INFORMATION GEOMETRODYNAMICAL ENTROPY

The statistical manifold $M_s$ is the space of probability distributions $\{P(\vec{X} | \Theta)\}$ labeled by $6N$ statistical parameters $\Theta$. These parameters are the coordinates for the point $P$, and in these coordinates a volume element $dV_{M_s}$ reads,
\[
dV_{M_S} = \sqrt{g} d^{6N} \Theta = \prod_{\alpha=1}^{N} \prod_{a=1}^{3} \frac{\sqrt{2}}{\sigma^{(\alpha)}_a} d\mu^{(\alpha)}_a d\sigma^{(\alpha)}_a.
\]
(389)
The volume of an extended region $\Delta V_M(\tau; \lambda)$ of $M_s$ is defined by,

$$\Delta V_M(\tau; \lambda) \equiv \prod_{a=1}^{N} \prod_{\alpha=1}^{3} \int_{\mu_{\alpha}^{(a)}(\tau)}^{\mu_{\alpha}^{(a)}(0)} \int_{\sigma_{\alpha}^{(a)}(\tau)}^{\sigma_{\alpha}^{(a)}(0)} \frac{\sqrt{2}}{\sqrt{\sigma_{\alpha}^{(a)}}} \ d\mu_\alpha^{(a)} d\sigma_\alpha^{(a)} \quad (390)$$

where $\mu_{\alpha}^{(a)}(\tau)$ and $\sigma_{\alpha}^{(a)}(\tau)$ are given in (381). The quantity that encodes relevant information about the stability of neighboring volume elements is the average volume $V_M(\tau; \lambda) \equiv \langle \Delta V_M(\tau; \lambda) \rangle_T$.

$$V_M(\tau; \lambda) \equiv \frac{1}{\tau} \int_0^\tau \Delta V_M(\tau'; \lambda) \ d\tau' \ \tau \to \infty \approx \exp \left[ \sum_{\alpha=1}^{N} \sum_{a=1}^{3} \lambda_{\alpha}^{(a)} \tau \right] \quad (391)$$

Again, as a side remark, we point out that if we consider the special case where different Gaussians are characterized by the same initial conditions, equation (386) becomes,

$$V_M(\tau; \lambda) \ \tau \to \infty \approx \exp (3N\lambda \tau) \quad (392)$$

This asymptotic regime of evolution in (391) describes the exponential increase of average volume elements on $M_s$.

The exponential instability characteristic of chaos forces the system to rapidly explore large areas (volumes) of the statistical manifold. It is interesting to note that this asymptotic behavior appears also in the conventional description of quantum chaos where the entropy increases linearly at a rate determined by the Lyapunov exponents [106]. The linear increase of entropy as a quantum chaos criterion was introduced by Zurek and Paz [8]. In our information-geometric approach a relevant quantity that can be useful to study the degree of instability characterizing the ED model is the information-geometric entropy defined as,

$$S_{M_s} \equiv \lim_{\tau \to \infty} \log V_M(\tau; \lambda) \quad (393)$$

Substituting (391) in (393), we obtain

$$S_{M_s} = \lim_{\tau \to \infty} \log \left\{ \frac{1}{\tau} \int_0^\tau \prod_{\alpha=1}^{N} \prod_{a=1}^{3} \int_{\mu_{\alpha}^{(a)}(\tau)}^{\mu_{\alpha}^{(a)}(0)} \int_{\sigma_{\alpha}^{(a)}(\tau)}^{\sigma_{\alpha}^{(a)}(0)} \frac{\sqrt{2}}{\sqrt{\sigma_{\alpha}^{(a)}}} \ d\mu_\alpha^{(a)} d\sigma_\alpha^{(a)} \right\} \ \tau \to \infty \approx \sum_{\alpha=1}^{N} \sum_{a=1}^{3} \lambda_{\alpha}^{(a)} \tau \quad (394)$$

In the special case of same initial conditions for different Gaussians, (394) becomes,

$$S_{M_s}(\tau; \lambda) \ \tau \to \infty \approx 3N\lambda \tau \quad (395)$$

The entropy $S_{M_s}$ in (394) is the asymptotic limit of the natural logarithm of the statistical weight $\langle \Delta V_M_{\alpha} \rangle_T$ defined on $M_s$. Its linear growth in time is reminiscent of the aforementioned quantum chaos criterion. Indeed, equation (391) may be considered the information-geometric analog of the Zurek-Paz chaos criterion.

In conclusion, we have shown that

$$R_{M_s} = -3N \ J_{M_s} (\tau; \lambda) \ \tau \to \infty \approx \sum_{\alpha=1}^{N} \sum_{a=1}^{3} \exp (\lambda_{\alpha}^{(a)} \tau) \ \ , \ S_{M_s}(\tau; \lambda) \ \tau \to \infty \approx \sum_{\alpha=1}^{N} \sum_{a=1}^{3} \lambda_{\alpha}^{(a)} \tau \quad (396)$$
Each indicator of chaos behaves as expected: $R_{M_s}$ is negative (this is a sufficient but not necessary condition for chaos), $J_{M_S}$ grows exponentially in $\tau$ and, $S_{M_s}$ grows linearly in $\tau$ and is proportional to the sum of positive Lyapunov exponents of the system. Furthermore, it is reasonable to state that the temporal complexity (chaoticity) of a system ought to grow linearly as a function of the number of its variables and it seems reasonable to assume that this complexity should not depend on the special choice of the initial conditions of the system but only on its dynamical evolution. The selection of a special set of initial conditions should not affect the degree of chaoticity of a dynamical system. Because of these considerations, we are allowed to choose any special set of convenient initial conditions and evaluate the behavior of the indicators of chaos in such special case. As we have showed, assuming the same initial conditions for different Gaussians, we obtain

$$R_{M_s} = -3N, S_{M_s} \sim 3N\lambda \tau, J_{M_S} \sim 3Ne^{\lambda \tau}. \quad (397)$$

The Ricci scalar curvature $R_{M_s}$ grows as a function of the number of the microvariables of the system, the information-geometric entropy $S_{M_s}$ grows linearly as a function of the microvariables of the system and the Jacobi vector field intensity $J_{M_S}$ grows exponentially as a function of the microvariables of the system: $R_{M_s}$, $S_{M_s}$ and $J_{M_S}$ behave as proper indicators of chaoticity and are proportional to the number of Gaussian-distributed microstates of the system. This proportionality leads to the conclusion that there exists a formal link among these information-geometric measures of chaoticity. Formally,

$$R_{M_s} \sim S_{M_s} \sim J_{M_S}. \quad (398)$$

Equation (398), together with the information-geometric analog of the Zurek-Paz quantum chaos criterion, equation (394), represent the fundamental results of this work. We are aware that equation (398) is reliable in the restrictive assumption of Gaussianity and for very special initial conditions. However, we believe that with some additional technical machinery, more general conclusions can be achieved and this connection among indicators of chaoticity may be strengthened. Furthermore, we believe our theoretical modelling scheme may be used to describe actual systems where transitions from quantum to classical chaos scenario occur, but this requires additional analysis. In the following section, we briefly consider some similarities among the von Neumann, Kolmogorov-Sinai and Information-Geometro dynamical entropies.
XXXIV. ON THE VON NEUMANN, KOLMOGOROV-SINAI AND INFORMATION GEOMETRODYNAMICAL ENTROPIES

In conventional approaches to chaos, the notion of entropy is introduced, in both classical and quantum physics, as the missing information about the systems fine-grained state \[1, 9\]. Following the first work in reference \[9\], we consider a classical system and suppose that the phase space is partitioned into very fine-grained cells of uniform volume \(\Delta v\), labelled by an index \(j\). If one does not know which cell the system occupies, one assigns probabilities \(p_j\) to the various cells; equivalently, in the limit of infinitesimal cells, one can use a phase-space density \(\rho(X_j) = \frac{p_j}{\Delta v}\).

Then, in a classical chaotic evolution, the asymptotic expression of the information needed to characterize a particular coarse-grained trajectory out to time \(\tau\) is given by the Shannon information entropy (measured in bits) \[9\],

\[
S^{\text{(chaotic\ classical)}} = - \int dX \rho(X) \log_2 (\rho(X) \Delta v) = - \sum_j p_j \log_2 p_j \sim K \tau.
\]  (399)

where \(\rho(X)\) is the phase-space density and \(p_j = \frac{\rho_j}{\Delta v}\) is the probability for the corresponding coarse-grained trajectory. \(S^{\text{(chaotic\ classical)}}\) is the missing information about which fine-grained cell the system occupies. Equation (399) can be explained with the following approximate reasoning \[9\]: the number of pieces in the partition of the evolved pattern grows as \(2^{K \tau}\) (i.e., \(#\{j\} \approx 2^{K \tau}\)), each piece having approximately the same phase-space volume and, therefore, the same probability \(p_j = \frac{\rho_j}{\Delta v} \approx 2^{-K \tau}\). That said, equation (399) follows in a straightforward way. However, we think this picture is not the most clear one. We believe that within our IGAC, a better and clearer understanding of what actually is happening may be achieved. The quantity \(K\) represents the linear rate of information increase and it is called the Kolmogorov-Sinai entropy (or metric entropy). \(K\) quantifies the degree of classical chaos (for a more detailed discussion about \(K\), see Chapter 4). It is worthwhile emphasizing that the quantity that grows asymptotically as \(K \tau\) is really the average of the information on the left side of equation (399). This distinction can be ignored however, if we assume that the chaotic system has roughly constant Lyapunov exponents over the accessible region of phase space.

In quantum mechanics the fine-grained alternatives are normalized state vectors in Hilbert space. From a set of probabilities for various state vectors, one can construct a density operator

\[
\hat{\rho} = \sum_j \lambda_j |\psi_j\rangle \langle \psi_j|, \quad \hat{\rho} |\psi_j\rangle = \lambda_j |\psi_j\rangle.
\]  (400)

The normalization of the density operator, \(\text{tr} (\hat{\rho}) = 1\), implies that the eigenvalues make up a normalized probability distribution. The von Neumann entropy (natural generalization of both Boltzmann’s and Shannon’s entropy) of the
density operator $\hat{\rho}$ (measured in bits) $^{136, 137}$,

$$S_{\text{quantum}}^{(\text{chaotic})} = -\text{tr} (\hat{\rho} \log_2 \hat{\rho}) = -\sum_j \lambda_j \log_2 \lambda_j \sim K_q \tau \quad (401)$$

can be thought of as the missing information about which eigenvector the system is in. Entropy quantifies the degree of unpredictability about the system’s fine-grained state. In quantum mechanics, the von Neumann entropy plays a role analogous to that played by the Shannon entropy in classical probability theory. They are both monotone functionals of the state. The von Neumann entropy reduces to the Shannon entropy for diagonal density matrices. However, in general the von Neumann entropy is a subtler object than its classical counterpart. The quantity $K_q$ in (401) can be interpreted as the non-commutative (quantum theory is a non-commutative probability theory) quantum analog of the Kolmogorov-Sinai dynamical entropy, the so-called quantum dynamical entropy $^{138}$. Examples of quantum dynamical entropies applied to quantum chaos and quantum information theory are the Alicki-Fannes (AF) $^{139}$ entropy and the Connes-Narnhofer-Thirring (CNT) $^{140}$ entropy. Both the AF and CNT entropy coincide with the KS entropy on classical dynamical systems. They also coincide on finite-dimensional quantum systems. However, they differ when moving from finite to infinite quantum systems. Furthermore, recall that decoherence is the loss of phase coherence between the set of preferred quantum states in the Hilbert space of the system due to the interaction with the environment. Moreover, decoherence induces transitions from quantum to classical systems. Therefore, classicality is an emergent property of an open quantum system. Motivated by such considerations, Zurek and Paz investigated implications of the process of decoherence for quantum chaos.

They considered a chaotic system, a single unstable harmonic oscillator characterized by a potential $V(x) = -\frac{\omega^2 x^2}{2}$ ($\lambda$ is the Lyapunov exponent), coupled to an external environment. In the reversible classical limit $^{141}$, the von Neumann entropy of such a system increases linearly at a rate determined by the Lyapunov exponent,

$$S_{\text{quantum}}^{(\text{chaotic}) \text{ (Zurek-Paz)}} \tau \sim \infty \omega \tau \quad (402)$$

Notice that the consideration of $3N$ uncoupled identical unstable harmonic oscillators characterized by potentials $V_i(x) = -\frac{\omega^2 x^2}{2}$ ($\omega_i = \omega_j \equiv \omega$; $i, j = 1, 2, ..., 3N$) would simply lead to

$$S_{\text{quantum}}^{(\text{chaotic}) \text{ (Zurek-Paz)}} \tau \sim \infty 3N \omega \tau \quad (403)$$

The resemblance of equations (394) and (403) is remarkable. In what follows, we apply our information geometrical method to a set of two ($n = 2$) uncoupled inverted anisotropic harmonic oscillators and show we obtain asymptotic linear IGE growth. The case for an arbitrary $n$-set in three dimensions is presented in the Appendix.
In this section, our objective is to characterize chaotic properties of a set of two one-dimensional inverted harmonic oscillators, each with different frequency $\omega_1 \neq \omega_2$. We will study the asymptotic behavior of the geometrodynamical entropy and the functional dependence of the Ricci scalar curvature of the 2-dimensional manifold $M^{(2)}_{IHO}$ underlying the ED model of the IHOs on the frequencies $\omega_i, i = 1, 2$. In Chapter 6, we explored the possibility of using well established principles of inference to derive Newtonian dynamics from relevant prior information codified into an appropriate statistical manifold [15]. In what follows, we introduce the basics of the general formalism for a set of n IHOs. This approach is similar (mathematically but not conceptually) to the geometrization of Newtonian dynamics used in the Riemannian geometrodynamical to chaos [5, 142].

A. Informational geometrization of Newtonian dynamics

In what follows, we apply the general formalism developed in Chapter 6 to our specific problem under consideration. The system under investigation has $n$ degrees of freedom and a point on the $n$ dimensional configuration space manifold $M^{(n)}_{IHO}$ is parametrized by the $n$ Lagrangian coordinates $(\theta_1, \ldots, \theta_n)$. Moreover, the system under investigation is described by the Lagrangian $L$,

$$L = T(\dot{\theta}_1, \ldots, \dot{\theta}_n) - \Phi(\theta_1, \ldots, \theta_n) = \frac{1}{2} \sum_{i=1}^{n} \dot{\theta}_i \dot{\theta}_i + \frac{1}{2} \sum_{j=1}^{n} \omega_j^2 \theta_j^2$$

(404)

so that the Hamiltonian function $H = T + \Phi \equiv E$ is a constant of motion. For the sake of simplicity, let us set $E = 1$. According to the principle of stationary action - in the form of Maupertuis - among all the possible isoenergetic paths $\gamma(\tau)$ with fixed end points, the paths that make vanish the first variation of the action functional

$$I = \int_{\gamma(\tau)} \frac{\partial L}{\partial \dot{\theta}_i} \dot{\theta}_i d\tau$$

(405)

are natural motions. As the kinetic energy $T$ is a homogeneous function of degree two, we have $2T = \dot{\theta}_i \frac{\partial L}{\partial \dot{\theta}_i}$, and Maupertuis’ principle reads

$$\delta I = \delta \int_{\gamma(\tau)} 2T d\tau = 0.$$  

(406)

The manifold $M^{(n)}_{IHO}$ is naturally given a proper Riemannian structure. In fact, let us consider the matrix

$$g_{ij}(\theta_1, \ldots, \theta_n) = [1 - \Phi(\theta_1, \ldots, \theta_n)] \delta_{ij}$$

(407)
so that Maupertuis’ principle becomes

\[
\delta \int_{\gamma(\tau)} T d\tau = \delta \int_{\gamma(\tau)} \left( T^2 \right)^{\frac{1}{2}} d\tau = \delta \int_{\gamma(\tau)} \left\{ [1 - \Phi (\theta_1, \ldots, \theta_n)] \delta_{ij} \dot{\theta}_i \dot{\theta}_j \right\}^{\frac{1}{2}}
\]

\[
= \delta \int_{\gamma(\tau)} \left( g_{ij} \dot{\theta}_i \dot{\theta}_j \right)^{\frac{1}{2}} d\tau = \delta \int_{\gamma(s)} ds = 0, \quad ds^2 = g_{ij} d\theta^i d\theta^j
\]  

(408)

thus motions are geodesics of \( M^{(n)}_{1H} \), provided we define \( ds \) as its arclength. The metric tensor \( g(\cdot, \cdot) \) of \( M^{(n)}_{1H} \) is then defined by

\[
g = g_{ij} d\theta^i \otimes d\theta^j
\]

(409)

where \((d\theta^1, \ldots, d\theta^n)\) is a natural base of \( T^*_\theta M^{(n)}_{1H} \) - the cotangent space at the point \( \theta \) - in the local chart \((\theta^1, \ldots, \theta^n)\).

This is known as the Jacobi metric (or kinetic energy metric). Denoting by \( \nabla \) the canonical Levi-Civita connection, the geodesic equation is defined as \[143\],

\[
\nabla_{\dot{\gamma}} \dot{\gamma} = 0, 
\]

(410)

where \( \dot{\gamma} \) is the tangent vector of the allowed paths \( \gamma \) at constant energy \( E \). In the local chart \((\theta^1, \ldots, \theta^n)\), equation \[110\] becomes,

\[
\frac{d^2 \theta^i}{ds^2} + \Gamma^i_{jk} \frac{d\theta^j}{ds} \frac{d\theta^k}{ds} = 0 
\]

(411)

where the Christoffel coefficients are the components of \( \nabla \) defined by

\[
\Gamma^i_{jk} = \left( d\theta^i, \nabla_j e_k \right) = \frac{1}{2} \theta^m \left( \partial_j g_{km} + \partial_k g_{mj} - \partial_m g_{jk} \right), 
\]

(412)

with \( \partial_i = \frac{\partial}{\partial \theta^i} \). Since \( g_{ij} (\theta_1, \ldots, \theta_n) = [1 - \Phi (\theta_1, \ldots, \theta_n)] \delta_{ij} \), from the geodesic equation we obtain

\[
\frac{d^2 \theta^i}{ds^2} + \frac{1}{2 (1 - \Phi)} \left[ 2 \frac{\partial (1 - \Phi)}{\partial \theta_j} \frac{d\theta^j}{ds} \frac{d\theta^i}{ds} - g^{ij} \frac{\partial (1 - \Phi)}{\partial \theta_j} g_{km} \frac{d\theta^k}{ds} \frac{d\theta^m}{ds} \right] = 0,
\]

(413)

whereupon using \( ds^2 = (1 - \Phi)^2 d\tau^2 \), we verify that \[113\] reduces to

\[
\frac{d^2 \theta^i}{d\tau^2} + \frac{\partial \Phi (\theta_1, \ldots, \theta_n)}{\partial \theta^i} = 0, \quad i = 1, \ldots, n.
\]

(414)

Equation \[114\] are Newton’s equations. It is worthwhile emphasizing that the transformation to geodesic motion on a curved statistical manifold is obtained in two key steps: the conformal transformation of the metric, \( \delta_{ij} \rightarrow g_{ij} = (1 - \Phi) \delta_{ij} \) and, the rescaling of the temporal evolution parameter, \( d\tau^2 \rightarrow ds^2 = 2 (1 - \Phi)^2 d\tau^2 \).
B. Two uncoupled inverted one-dimensional harmonic oscillators

As a simple physical example, we examine the IG associated with a set of two one-dimensional IHOS. In this case, the metric tensor $g_{ij}$ appearing in (407) takes the form

$$g_{ij} (\theta_1, \theta_2) = [1 - \Phi (\theta_1, \theta_2)] \delta_{ij} \text{ with } i, j = 1, 2.$$  \hfill (415)

where the function $\Phi (\theta_1, \theta_2)$ is given by,

$$\Phi (\theta_1, \theta_2) = \sum_{j=1}^{2} \Phi_j (\theta_j), \Phi_j (\theta_j) = -\frac{1}{2} \omega_j^2 \theta_j^2.$$  \hfill (416)

Hence the metric tensor $g_{ij}$ on $M_{/(2)} \text{} \text{IHOS}$ becomes,

$$g_{ij} = \begin{pmatrix} 1 + \frac{1}{2} (\omega_1^2 \theta_1^2 + \omega_2^2 \theta_2^2) & 0 \\ 0 & 1 + \frac{1}{2} (\omega_1^2 \theta_1^2 + \omega_2^2 \theta_2^2) \end{pmatrix}.$$  \hfill (417)

Using the standard definition of the Ricci scalar (365), we obtain

$$R_{M_{/(2)} \text{} \text{IHOS}} (\omega_1, \omega_2) = \frac{4 (\theta_1^2 \omega_1^4 + \theta_2^2 \omega_2^4) - 4 (\theta_1^2 + \theta_2^2) \omega_1^2 \omega_2^2 - 8 (\omega_1^2 + \omega_2^2)}{(\theta_1^2 \omega_1^4 + \theta_2^2 \omega_2^4 + 2)^3},$$  \hfill (418)

For $\omega_1 = \omega_2 = \omega$, the scalar curvature (418) is always negative,

$$R_{M_{/(2)} \text{} \text{IHOS}} (\omega) = \frac{-16 \omega^2}{[2 + (\theta_1^2 + \theta_2^2) \omega^2]^3} < 0, \forall \omega \geq 0.$$  \hfill (419)

However, in presence of distinct frequency values, $\omega_1 \neq \omega_2$, it is possible to properly choose the $\omega$’s so that $R_{M_{/(2)} \text{} \text{IHOS}} (\omega_1, \omega_2)$ becomes either negative or positive. In addition, we notice that the manifold underlying the IHOS model is anisotropic since its associated Weyl projective curvature tensor components are non-vanishing. For the special case, $\omega_1 = \omega_2$, we obtain

$$W_{1212} (\omega) = \frac{8 \omega^4 (\theta_1^2 + \theta_2^2) + 2 \omega^6 (\theta_1^2 + \theta_2^2) + 4 \omega^6 \theta_1^2 \theta_2^2}{(\theta_1^2 \omega^2 + \theta_2^2 \omega^2 + 2)^3},$$  \hfill (420)

Clearly, the frequency parameter $\omega$ drives the degree of anisotropy of the statistical manifold $M_{/(2)} \text{} \text{IHOS}$ and, as expected, in the limit of vanishing $\omega$, we recover the flat ($R = 0$), isotropic ($W = 0$) Euclidean manifold characterized by metric $\delta_{ij}$. This result is a concrete example of the fact that conformal transformations change the degree of anisotropy of the ambient statistical manifold underlying the Newtonian dynamics. Our only remaining task is to compute the information geometrodynamical entropy $S_{M_{/(2)} \text{} \text{IHOS}} (\tau; \omega_1, \omega_2)$, defined as

$$S_{M_{/(2)} \text{} \text{IHOS}} (\tau; \omega_1, \omega_2) \defeq \lim_{\tau \to \infty} \log \left[ \langle \Delta V_{M_{/(2)} \text{} \text{IHOS}} (\tau; \omega_1, \omega_2) \rangle_{\tau} \right].$$  \hfill (421)
The quantity \( \langle \Delta V_{\mathcal{M}_{IHO}}(\tau; \omega_1, \omega_2) \rangle_\tau \) appearing in (421) is the average volume element, defined by

\[
\langle \Delta V_{\mathcal{M}_{IHO}}(\tau; \omega_1, \omega_2) \rangle_\tau = \frac{1}{\tau} \int_0^\tau \Delta V_{\mathcal{M}_{IHO}}(\tau; \omega_1, \omega_2) \, d\tau',
\]

with the statistical volume element \( \Delta V_{\mathcal{M}_{IHO}}(\tau; \omega_1, \omega_2) \) given by

\[
\Delta V_{\mathcal{M}_{IHO}}(\tau; \omega_1, \omega_2) = \int_{\{\vec{0}\}} 1 + \frac{1}{2} (\omega^2 \theta_1^2 + \omega^2 \theta_2^2) \, d\theta_1' d\theta_2'
\]

\[
\tau \to \infty \quad \approx \frac{1}{6} \theta_1' \theta_2' (\omega^2 \theta_1^2 + \omega^2 \theta_2^2).
\]

Recall that the two Newtonian equations of motion for each inverted harmonic oscillator are given by,

\[
\frac{d^2 \theta_j}{d\tau^2} - \omega^2 \theta_j = 0, \forall j = 1, 2.
\]

Hence, the asymptotic behavior of such macrovariables on manifold \( \mathcal{M}_{IHO}^{(2)} \) is given by,

\[
\theta_j(\tau) \quad \tau \to \infty \quad \approx \Xi_j e^{\omega_j \tau}, \Xi_j \in \mathbb{R}, \forall j = 1, 2.
\]

Substituting \( \theta_1(\tau') = \Xi_1 e^{\omega_1 \tau'} \) and \( \theta_2(\tau') = \Xi_2 e^{\omega_2 \tau'} \) into (423), we obtain

\[
\Delta V_{\mathcal{M}_{IHO}}(\tau; \omega_1, \omega_2) \quad \tau \to \infty \quad \approx \frac{\Xi_1 \Xi_2}{6} e^{(\omega_1 + \omega_2)\tau} (\Xi_1^2 e^{2 \omega_1 \tau} \omega_1 + \Xi_2^2 e^{2 \omega_2 \tau} \omega_2).
\]

By direct computation, we find the average of (426) is given by,

\[
\langle \Delta V_{\mathcal{M}_{IHO}}(\tau; \omega_1, \omega_2) \rangle_\tau \quad \tau \to \infty \quad \approx \frac{1}{\tau} \int_0^\tau \left[ \frac{\Xi_1 \Xi_2}{6} e^{(\omega_1 + \omega_2)\tau'} \left( \Xi_1^2 e^{2 \omega_1 \tau'} \omega_1 + \Xi_2^2 e^{2 \omega_2 \tau'} \omega_2\right) \right] d\tau'.
\]

Assuming as a working hypothesis that \( \Xi_1 = \Xi_2 = \Xi \), we obtain

\[
\frac{1}{\tau} \int_0^\tau \left[ \frac{\Xi_1 \Xi_2}{6} e^{(\omega_1 + \omega_2)\tau'} \left( \Xi_1^2 e^{2 \omega_1 \tau'} \omega_1 + \Xi_2^2 e^{2 \omega_2 \tau'} \omega_2\right) \right] d\tau' = \begin{cases} 
\frac{1}{12} \Xi^6 \omega \exp(4\omega \tau), & \text{if } \omega_1 = \omega_2, \\
\frac{1}{18} \Xi^6 \omega_1 \exp(3\omega_1 \tau), & \text{if } \omega_1 > \omega_2, \\
\frac{1}{18} \Xi^6 \omega_2 \exp(3\omega_2 \tau), & \text{if } \omega_2 > \omega_1.
\end{cases}
\]

Finally, substituting (428) in (421), we obtain

\[
S_{\mathcal{M}_{IHO}}^{(2)}(\tau; \omega_1, \omega_2) \quad \tau \to \infty \quad \approx \begin{cases} 
2\omega \tau, & \text{if } \omega_1 = \omega_2, \\
\omega_1 \tau, & \text{if } \omega_1 > \omega_2, \\
\omega_2 \tau, & \text{if } \omega_2 > \omega_1.
\end{cases}
\]
It is clear that the information-geometrodynamical entropy $S_{M_{IHO}}^{(2)}(\tau; \omega_1, \omega_2)$ exhibits classical linear behavior in the asymptotic limit, with proportionality coefficient $\Omega = \omega_1 + \omega_2$.

$$S_{M_{IHO}}^{(2)}(\tau; \omega_1, \omega_2)^{\tau \to \infty} \propto \Omega \tau. \quad (430)$$

Equation (430) expresses the asymptotic linear growth of our information geometrodynamical entropy for the IHO system considered. This result (for $n = 2$) extends the result of Zurek-Paz in a classical information-geometric setting. This result, together with my previous works \[11, 12\] lend substantial support for the IGAC approach advocated in the present Chapter.

XXXVI. CONCLUSIONS

A Gaussian ED statistical model has been constructed on a $6N$-dimensional statistical manifold $M_s$. The macro-coordinates on the manifold are represented by the expectation values of microvariables associated with Gaussian distributions. The geometric structure of $M_s$ was studied in detail. It was shown that $M_s$ is a curved manifold of constant negative Ricci curvature $-3N$. The geodesics of the ED model are hyperbolic curves on $M_s$. A study of the stability of geodesics on $M_s$ was presented. The notion of statistical volume elements was introduced to investigate the asymptotic behavior of a one-parameter family of neighboring volumes $F_{V_{M_s}}(\lambda) \equiv \{V_{M_s}(\tau; \lambda)\}_{\lambda \in \mathbb{R}^+}$. An information-geometric analog of the Zurek-Paz chaos criterion was suggested. It was shown that the behavior of geodesics is characterized by exponential instability that leads to chaotic scenarios on the curved statistical manifold. These conclusions are supported by a study based on the geodesic deviation equations and on the asymptotic behavior of the Jacobi vector field intensity $J_{M_s}$ on $M_s$. A Lyapunov exponent analog similar to that appearing in the Riemannian geometric approach to chaos was suggested as an indicator of chaoticity. On the basis of our analysis a relationship among an entropy-like quantity, chaoticity and curvature is proposed, suggesting to interpret the statistical curvature as a measure of the entropic dynamical chaoticity.

The results obtained in this work are significant, in our opinion, since a rigorous relation among curvature, Lyapunov exponents and Kolmogorov-Sinai entropy is still under investigation \[7\]. In addition, there does not exist a well defined unifying characterization of chaos in classical and quantum physics \[9\] due to fundamental differences between the two theories. In addition, the role of curvature in statistical inference is even less understood. The meaning of statistical curvature for a one-parameter model in inference theory was introduced in \[113\]. Curvature served as an important tool in the asymptotic theory of statistical estimation. Therefore the implications of this work is twofold. Firstly,
it helps understanding possible future use of the statistical curvature in modelling real processes by relating it to conventionally accepted quantities such as entropy and chaos. On the other hand, it serves to cast what is already known in physics regarding curvature in a new light as a consequence of its proposed link with inference.

As a simple physical example, we considered the information-geometry $\mathcal{M}_{IHO}^{(2)}$ associated with a set of two inverted harmonic oscillators. It was determined that in the limit of a flat frequency spectrum ($\omega_1 = \omega_2 = \omega$), the scalar curvature $\mathcal{R}_{\mathcal{M}_{IHO}^{(2)}}(\omega_1, \omega_2)$ is constantly negative. In the case of distinct frequencies, i.e., $\omega_1 \neq \omega_2$, it is possible - for appropriate choices of $\omega_1$ and $\omega_2$ - to obtain either negative or positive values of $\mathcal{R}_{\mathcal{M}_{IHO}^{(2)}}(\omega_1, \omega_2)$. Moreover, it was shown that $\mathcal{M}_{IHO}^{(2)}$ is an anisotropic manifold since the Weyl projective curvature tensor has a non-vanishing component $W_{1212}$. It was found that the information geometrodynamical entropy of the IHO system exhibits asymptotic linear growth. This IHO example is generalized to arbitrary values of $n$ in the Appendix.

The descriptions of a classical chaotic system of arbitrary interacting degrees of freedom, deviations from Gaussianity and chaoticity arising from fluctuations of positively curved statistical manifolds are being investigated [14].

XXXVII. APPENDIX

A. The set of $n$ uncoupled inverted anisotropic three-dimensional harmonic oscillators

1. Ohmic frequency spectrum

We now generalize the results obtained in this Chapter for a set of $n$ IHOs. The information metric on the $3n$-dimensional statistical manifold $\mathcal{M}_{IHO}^{(3n)}$ is given by

$$g_{ij}(\theta_1, ..., \theta_{3n}) = [1 - \Phi(\theta_1, ..., \theta_{3n})] \delta_{ij},$$

where

$$\Phi(\theta_1, ..., \theta_{3n}) = \sum_{j=1}^{3n} \Phi_j(\theta_j), \quad \Phi_j(\theta_j) = -\frac{1}{2} \omega_j^2 \theta_j^2.$$ (432)

The information geometrodynamical entropy $S_{\mathcal{M}_{IHO}^{(3n)}}(\tau; \omega_1, ..., \omega_{3n})$ is defined as

$$S_{\mathcal{M}_{IHO}^{(3n)}}(\tau; \omega_1, ..., \omega_{3n}) \equiv \lim_{\tau \to \infty} \log \left[ \left\langle \Delta V_{\mathcal{M}_{IHO}^{(3n)}}(\tau; \omega_1, ..., \omega_{3n}) \right\rangle_\tau \right],$$

where the average volume element $\Delta V_{\mathcal{M}_{IHO}^{(3n)}}$ is given by

$$\left\langle \Delta V_{\mathcal{M}_{IHO}^{(3n)}}(\tau; \omega_1, ..., \omega_{3n}) \right\rangle_\tau = \frac{1}{\tau} \int_0^\tau \Delta V_{\mathcal{M}_{IHO}^{(3n)}}(\tau'; \omega_1, ..., \omega_{3n}) d\tau',$$ (434)
and the statistical volume element $\Delta V_{M_{IHO}}^{(3n)}$ is defined as

$$
\Delta V_{M_{IHO}}^{(3n)}(\tau'; \omega_1, ..., \omega_n) = \int \frac{d^{3n} \theta'}{\theta'} \left( 1 + \frac{1}{2} \sum_{j=1}^{3n} \omega_j^2 \theta_j'^2 \right)^{\frac{3n}{2}}.
$$

Substituting (434) and (435) in (433) we obtain the general expression for $S_{M_{IHO}}^{(3n)}(\tau; \omega_1, ..., \omega_{3n})$,

$$
S_{M_{IHO}}^{(3n)}(\tau; \omega_1, ..., \omega_{3n}) \overset{\text{def}}{=} \lim_{\tau \to \infty} \log \left\{ \frac{1}{\tau} \int_0^\tau \int \frac{d^{3n} \theta'}{\theta'} \left( 1 + \frac{1}{2} \sum_{j=1}^{3n} \omega_j^2 \theta_j'^2 \right)^{\frac{3n}{2}} d\tau' \right\}.
$$

To evaluate (436) we observe $\Delta V_{M_{IHO}}^{(3n)}$ can be written as

$$
\Delta V_{M_{IHO}}^{(3n)}(\tau'; \omega_1, ..., \omega_{3n}) = \int \frac{d^{3n} \theta'}{\theta'} \left( 1 + \frac{1}{2} \sum_{j=1}^{3n} \omega_j^2 \theta_j'^2 \right)^{\frac{3n}{2}},
$$

$$
= \int d\theta'_1 \int d\theta'_2 ... \int d\theta'_{3n-1} \left[ \int \left( 1 + \frac{1}{2} \sum_{j=1}^{3n} \omega_j^2 \theta_j'^2 \right)^{\frac{3n}{2}} d\theta'_{3n} \right],
$$

$$
\approx \frac{1}{3n} \frac{1}{2^{\frac{3n}{2}}} \left( \prod_{i=1}^{3n} \theta_i' \right) \left[ \sum_{j=1}^{3n} \omega_j^2 \theta_j'^2 \right]^{\frac{3n}{2}}.
$$

Since the $n$-Newtonian equations of motions for each IHO are given by

$$
\frac{d^2 \theta_j}{d\tau^2} - \omega_j^2 \theta_j = 0, \forall j = 1, ..., 3n,
$$

the asymptotic behavior of such macrovariables on manifold $M_{IHO}^{(3n)}$ is given by

$$
\theta_j(\tau) \overset{\tau \to \infty}{\approx} \sum_j \Theta_j e^{\omega_j \tau}, \sum \Theta_j \in \mathbb{R}, \forall j = 1, ..., 3n.
$$

We therefore obtain

$$
\Delta V_{M_{IHO}}^{(3n)}(\tau; \omega_1, ..., \omega_{3n}) \overset{\tau \to \infty}{\approx} \frac{1}{3n} \frac{1}{2^{\frac{3n}{2}}} \left( \prod_{i=1}^{3n} \Theta_i \right) \cdot \exp \left( \sum_{i=1}^{3n} \omega_i \tau \right) \left[ \sum_{j=1}^{3n} \sum \Theta_j^2 e^{2\omega_j \tau} \right]^{\frac{3n}{2}}.
$$

Upon averaging (440) we find

$$
\left\langle \Delta V_{M_{IHO}}^{(3n)}(\tau; \omega_1, ..., \omega_{3n}) \right\rangle \overset{\tau \to \infty}{\approx} \frac{1}{\tau} \int_0^\tau \left\{ \frac{1}{3n} \frac{1}{2^{\frac{3n}{2}}} \left( \prod_{i=1}^{3n} \Theta_i \right) \cdot \exp (\Omega \tau') \left[ \sum_{j=1}^{3n} \sum \Theta_j^2 e^{2\omega_j \tau'} \right]^{\frac{3n}{2}} \right\} d\tau'.
$$
where $\Omega = \sum_{i=1}^{3n} \omega_i$. As a working hypothesis, we assume $\Xi_i = \Xi_j \equiv \Xi \forall i, j = 1,.., 3n$. Furthermore, assume that $n \to \infty$ so that the spectrum of frequencies becomes continuum and, as an additional working hypothesis, assume this spectrum is linearly distributed (Ohmic frequency spectrum),

$$\rho_{\text{Ohmic}}(\omega) = \frac{2}{\Omega_{\text{cut-off}}} \omega \text{ with } \int_0^{\Omega_{\text{cut-off}}} \rho_{\text{Ohmic}}(\omega) \, d\omega = 1, \Omega_{\text{cut-off}} = \xi \Omega, \xi \in \mathbb{R}. \quad (442)$$

Therefore, we obtain

$$\langle \Delta V_{M^{(3n)}_{\text{IHO}}} (\tau; \omega_1,.., \omega_{3n}) \rangle_{\tau \to \infty} \approx \frac{1}{3n} \left( \frac{\xi^2 \Omega^2}{2} \right)^{\frac{6n}{2}} \exp \left( \frac{\xi \Omega}{\tau} \right). \quad (443)$$

Finally, substituting (443) into (433), we obtain the remarkable result

$$S_{M^{(3n)}_{\text{IHO}}} (\tau; \omega_1,.., \omega_{3n}) \to \infty \Omega \tau, \Omega = \sum_{i=1}^{3n} \omega_i. \quad (444)$$

Equation (444) displays the asymptotic, linear information geometrodynamical entropy growth of the generalized $n$-set of inverted harmonic oscillators and extends the result of Zurek-Paz to an arbitrary set of anisotropic inverted harmonic oscillators in a classical information-geometric setting.

The Ohmic frequency spectrum case leads to asymptotic IGE growth. However, in this case, we are not able to compactify the parameter space of our statistical model. The compactification of the parameter space (and therefore of the statistical manifold) is required for true chaos where the folding mechanism must be present (indeed, the lack of such feature has been one of the most important criticisms to the Zurek-Paz model). The folding mechanism required for true chaos is not even restored in a statistical sense (averaging over $\omega$ and $\tau$),

$$\tilde{\theta}_{\text{Ohmic}} (\tau; \omega) \equiv \lim_{\tau, \omega \to \infty} \left\{ \frac{1}{\tau} \int_0^\tau \int_0^\omega \rho_{\text{Ohmic}} (\omega') \, d\omega' \, d\tau' \right\} \approx \lim_{\tau, \omega \to \infty} \frac{\exp (\omega \tau)}{\tau^2} \to \infty. \quad (445)$$

This missing feature may lead us to consider in the near future the possibility of considering other frequency spectra.
Chapter 8: Concluding Remarks and Future Research Directions

I present concluding remarks emphasizing strengths and weaknesses of my approach and I address possible further research directions.

XXXVIII. CONCLUDING REMARKS

In this doctoral dissertation, I considered two important questions:

First, are laws of physics practical rules to process information about the world using geometrical methods? Are laws of physics rules of inference?

Second, since a unifying framework to describe chaotic dynamics in classical and quantum domains is missing, is it possible to construct a new information-geometric model, to develop new tools so that a unifying framework is provided or, at least, new insights and new understandings are given?

After setting the scene of my thesis and after stating the problem and its motivations, I reviewed the basic elements of the maximum relative entropy formalism (ME method) and recall the basics of Riemannian geometry with special focus to its application to probability theory (this is known as Information Geometry, IG). IG and ME are the fundamental tools that Prof. Ariel Caticha has used to build a form of information-constrained dynamics on statistical manifolds to investigate the possibility that Einstein’s general theory of gravity (or any classical or quantum theory of physics) may emerge as a macroscopic manifestation of an underlying microscopic statistical structure. This dynamics is known in the literature as Entropic Dynamics (ED). Therefore, since ED was an important element of this thesis, I reviewed the key-points of such dynamics, emphasizing the most relevant points that I used in my own information geometrodynamical approach to chaos (IGAC). Of course, before introducing my IGAC, I briefly reviewed the basics of the conventional Riemannian geometrodynamics approach to chaos and discussed the notion of chaos in physics in general. After this long background information that was needed because of the originality and novelty of these topics, I started with my original contributions. Two entropic dynamical models are considered. The geometric structure of the statistical manifolds underlying these models is studied. It is found that in both cases, the resulting metric manifolds are negatively curved. Moreover, the geodesics on each manifold are described by hyperbolic trajectories. A detailed analysis based on the Jacobi-Levi-Civita equation for geodesic spread (JLC equation) is used to show that the hyperbolicity of the manifolds leads to chaotic exponential instability. A comparison between the two models leads to a relation among scalar curvature of the manifold ($\mathcal{R}$), Jacobi field intensity ($J$) and information geometrodynamical
entropy (IGE, $S_M$). The IGE entropy is proposed as a brand new measure of chaoticity.

**First Contribution** [10–12]: I suggest that these three quantities, $R$, $J$, and $S_M$ are useful indicators of chaoticity for chaotic dynamical systems on curved statistical manifolds. Furthermore, I suggest a classical information-geometric criterion of linear information geometrodynamical entropy growth in analogy with the Zurek-Paz quantum chaos criterion.

In collaboration with Prof. Ariel Caticha, I show that the ED formalism is not purely a mathematical framework; it is indeed a general theoretical scheme where conventional Newtonian dynamics can be obtained as a special limiting case. Newtonian dynamics is derived from prior information codified into an appropriate statistical model. The basic assumption is that there is an irreducible uncertainty in the location of particles so that the state of a particle is defined by a probability distribution. The corresponding configuration space is a statistical manifold the geometry of which is defined by the information metric. The trajectory follows from a principle of inference, the method of Maximum Entropy. No additional ”physical” postulates such as an equation of motion, or an action principle, nor the concepts of momentum and of phase space, not even the notion of time, need to be postulated. The resulting entropic dynamics reproduces the Newtonian dynamics of any number of particles interacting among themselves and with external fields. Both the mass of the particles and their interactions are explained as a consequence of the underlying statistical manifold.

**Second Contribution** [13]: The derivation of the Newtonian dynamics from first principles of probable inference and information geometric methods is another original contribution of my work in collaboration with Prof. Ariel Caticha.

Third, I extend my study of chaotic systems (information geometrodynamical approach to chaos, IGAC) to an ED Gaussian model describing an arbitrary system of $3N$ degrees of freedom. It is shown that the hyperbolicity of a non-maximally symmetric $6N$-dimensional statistical manifold $\mathcal{M}_S$ underlying the ED Gaussian model leads to linear information-geometrodynamical entropy growth and to exponential divergence of the Jacobi vector field intensity. As a special physical application, the information geometrodynamical scheme is applied to investigate the chaotic properties of a set of $n$-uncoupled three-dimensional anisotropic inverted harmonic oscillators coupled to an internal environment and I show that the asymptotic behavior of the information-geometrodynamical entropy is characterized
by linear growth. Finally, considerations concerning the anisotropy of the statistical manifold underlying such physical system and its relationship with the spectrum of frequencies of the oscillators are carried out.

**Third Contribution** [14]: *I compute the asymptotic temporal behavior of the information geometrodynamical entropy of a set of $n$-uncoupled three-dimensional anisotropic inverted harmonic oscillators (IHOs) characterized by an Ohmic distributed frequency spectrum and I suggest the classical information-geometric analogue of the Zurek-Paz quantum chaos criterion in its classical reversible limit.*

I am aware that several points in my IGAC need deeper understanding and analysis, however I hope that my work convincingly shows that this information-geometric approach may be useful in providing a unifying criterion of chaos, thus deserving further research and developments.

**XXXIX. FUTURE RESEARCH DIRECTIONS**

It is evident that my work requires additional improvements and deeper understanding of some of its results in view of its comparison to more recent results obtained in more orthodox approaches.

I am working on the possibility of extending the IGAC to chaotic systems with an arbitrary number of interacting microscopic degrees of freedom. This would lead to the problem of considering an information-geometric line element with non trivial off-diagonal matrix elements. This is relevant because it allow us to study not only macroscopic chaos but also microscopic dynamics characterizing several known chaotic Newtonian dynamical systems. In addition, this would increase the understanding of the relationship between the microscopic and macroscopic behaviors of a physical system.

I am considering the IGAC arising from arbitrary (non-uniform) prior probability distributions together with possible deviations from Gaussianity and chaoticity arising from fluctuations of positively curved statistical manifolds. I would like to take into consideration the study of more realistic physical, biological, complex systems in general (brain dynamics, finance, etc.). I am searching for a better understanding of the possible relevance of my formalism to key concepts in modern quantum information theory: chaos, decoherence and entanglement. I am thinking about the possibility of employing my IGAC to describe situations of transitions from quantum to classical chaotic physical systems.

These last two points require a generalization of the IGAC in an appropriate manner to facilitate the study of quantum mechanical systems. In collaboration with Dr. Saleem Ali, I already started some additional works in this
direction. We are trying to construct a quantum Hilbert space from a classical curved statistical manifold. This extension is implemented via the introduction of complex and symplectic structure tensors that are compatible with the underlying Riemannian geometry induced by the Fisher-Rao information metric. Vectors on the resulting manifold \( \mathcal{H} \) are interpreted as state vectors associated with points (i.e., probability density functions) on \( \mathcal{M} \).

As a final remark, I would like to point out that my primary concern in the near future is refining some of my latest controversial work on regular and chaotic quantum energy level statistics and, possibly, extending the application of the IGAC to soft chaos regimes [15].
Acknowledgments

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