DIRICHLET HEAT KERNEL ESTIMATES FOR A LARGE CLASS OF ANISOTROPIC MARKOV PROCESSES

KYUNG-YOUN KIM AND LIDAN WANG

Abstract. Let \( Z = (Z^1, \ldots, Z^d) \) be the \( d \)-dimensional Lévy process where \( Z^i \)'s are independent 1-dimensional Lévy processes with identical jumping kernel \( \nu^i(r) = r^{-1} \phi(r)^{-1} \). Here \( \phi \) is an increasing function with weakly scaling condition of order \( c, \pi \in (0, 2) \). We consider a symmetric function \( J(x, y) \) comparable to

\[
\begin{cases}
\nu^i(|x^i - y^i|) & \text{if } x^i \neq y^i \text{ for some } i \text{ and } x^j = y^j \text{ for all } j \neq i \\
0 & \text{if } x^i \neq y^i \text{ for more than one index } i.
\end{cases}
\]

Corresponding to the jumping kernel \( J \), there exists an anisotropic Markov process \( X \), see [KW22]. In this article, we establish sharp two-sided Dirichlet heat kernel estimates for \( X \) in \( C^{1,1} \) open set, under certain regularity conditions. As an application of the main results, we derive the Green function estimates.

1. Introduction

It is a well-known fact that the fundamental solution \( p(t, x, y) \) of the heat equation with a second order elliptic differential operator on \( \mathbb{R}^d \) is the transition density of the diffusion process related to the operator as an infinitesimal generator. The relation is also true for a large class of Markov processes corresponding to non-local operators with discontinuous sample paths. Correspondingly, there are lots of studies on the sharp two-sided heat kernel estimates for isotropic Markov processes (see, [CK03, CK08, CKK11] and other references therein).

For any open set \( D \subset \mathbb{R}^d \), one can define a killed processes upon leaving \( D \) and there exists a transition density \( p_D(t, x, y) \) if the distribution is absolutely continuous. The function \( p_D(t, x, y) \) is also called the Dirichlet heat kernel which describes an operator with zero exterior conditions. The Green function and solutions to Cauchy and Poisson problems with Dirichlet conditions can be expressed according to the Dirichlet heat kernel as well. However, it is difficult to study the Dirichlet heat kernel estimates for \( p_D(t, x, y) \), especially when the process is close to the boundary. Therefore, the Dirichlet heat kernel estimates were obtained relatively recently for the Laplacian operator (corresponding to the Brownian motion) in [Dav87, Hui92] for the upper bound, and in [Zha02] for the lower bound. In [CKS12], the authors estimated the Dirichlet heat kernel for the fractional Laplacian operator (corresponding to the isotropic stable Lévy process) in \( C^{1,1} \) open set over finite time intervals, and in bounded \( C^{1,1} \) open set for a large time. The elegant techniques developed in [CKS12] provide the framework on the Dirichlet heat kernel estimates for non-local operators. With successful application and extension of the framework, there are lots of results on discontinuous Markov processes: isotropic Lévy processes in [BGR14b, CKS14, CKK21], symmetric Lévy processes with Gaussian components in [CKS16] and symmetric Markov process (beyond the Lévy process) in [KK14, GKK20]. Also, such non-local operators can be used to describe models in diverse phenomena in the real world, see, [CT04, CV10, LS10, HL12].

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For the last two decades, there have been increasing interests in the anisotropic Markov processes. Consider the system of stochastic differential equations

$$dZ^i_t = \sum_{j=1}^d A_{ij}(Z_{t-})dL^j_t, \quad i = 1, \ldots, d.$$ 

where $L^i$ are identically distributed one-dimensional $\alpha$-stable process with $\alpha \in (0, 2)$. In [BC06], under mild conditions on the matrix $A = (A_{ij})$, the authors observed the unique weak solution to this system which forms a strong Markov process. In [KR18, KR20], they considered more specific conditions for $A$ and $\alpha$ to study the corresponding operators, and in [Cha19, Cha20, KRS21], they considered $L^i$ which are not necessarily same processes. It is worth to mention that there are fundamental differences between isotropic and anisotropic processes. For example, it is shown in [BC10] that Harnack inequality does not hold for cylindrical $\alpha$-stable processes. Intuitively, the significant difference lies in the fact that isotropic Markov processes can move (or jump) in any direction uniformly, while anisotropic Markov processes can only jump along one of the coordinate directions.

As for heat kernel estimates, they were firstly obtained (not in a sharp form) in [Xu13] for an anisotropic Markov process whose jumping measure is comparable to that of cylindrical $\alpha$-stable processes. With delicate analytic methods, sharp two-sided heat kernel estimates were established in [KKK22]. The results have been furtherly extended to more general anisotropic Markov processes in [KW22]. Estimates of Dirichlet heat kernels for anisotropic Markov processes is a completely new challenge. Very recently, in [CHZ], two-sided Dirichlet heat kernel estimates are studied when $L^i$ are $\alpha$-stable processes. In this article, we consider more general anisotropic Markov processes introduced in [KW22] and establish the two-sided Dirichlet heat kernel estimates in $C^{1,1}$ domains.

For any $0 < \underline{\alpha} \leq \alpha \leq \overline{\alpha} < 2$, let $\phi : (0, \infty) \to [0, \infty)$ be an increasing function with the following condition: there exist positive constants $\underline{\epsilon} \leq 1$ and $\overline{\epsilon} \geq 1$ such that

$$(\text{WS}) : \quad \frac{\epsilon}{\tau} \leq \frac{\phi(R)}{\phi(r)} \leq \frac{\epsilon}{\tau} \quad \text{for } 0 < r \leq R.$$ 

Using this $\phi$, define

$$\nu^1(r) := (r\phi(r))^{-1} \quad \text{for } r > 0. \quad (1.1)$$

Then (WS) implies

$$\int_{\mathbb{R}} \left(1 + |r|^2\right)\nu^1(|r|)dr \leq c \left(\int_0^1 r^{-\overline{\alpha}+1}dr + \int_1^\infty r^{-\underline{\alpha}-1}dr\right) < \infty$$

for some $c > 0$, so $\nu^1(dr) := \nu^1(|r|)dr$ is a Lévy measure. Consider an anisotropic Lévy process $Z$ in $\mathbb{R}^d$ defined by $Z = (Z^1, \ldots, Z^d)$, where each coordinate process $Z^i$ is an independent one-dimensional symmetric Lévy process with Lévy measure $\nu^1(dr)$. Then the corresponding Lévy measure $\nu$ of $Z$ is represented as

$$\nu(dh) = \sum_{i=1}^d \nu^1(|h^i|)dh^i \prod_{j \neq i} \delta_{\{0\}}(dh^j).$$

Intuitively, $\nu$ only measures the sets containing the line which is parallel to one of the coordinate axes. The corresponding Dirichlet form $(\mathcal{E}^\phi, \mathcal{F}^\phi)$ on $L^2(\mathbb{R}^d)$ is given by

$$\mathcal{E}^\phi(u, v) = \int_{\mathbb{R}^d} \left(\sum_{i=1}^d \int_{\mathbb{R}} (u(x + e_i\tau) - u(x))(v(x + e_i\tau) - v(x))J^\phi(x, x + e_i\tau)d\tau\right)dx,$$

$$\mathcal{F}^\phi = \{u \in L^2(\mathbb{R}^d) \mid \mathcal{E}^\phi(u, u) < \infty\},$$
where \(e_i\) is the unit vector in the positive \(x^i\)-direction and \(J^\phi(x, y)\) is the jumping kernel defined as follows:
\[
J^\phi(x, y) := \begin{cases} 
|x^i - y^i|^{-1} \phi(|x^i - y^i|)^{-1} & \text{if } x^i \neq y^i \text{ for some } i; \ x^j = y^j \text{ for all } j \neq i, \\
0 & \text{otherwise}.
\end{cases}
\]
We also consider a symmetric measurable function \(\kappa : \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty)\) satisfying that there exists \(\kappa_0 > 1\) such that
\[
\kappa_0^{-1} \leq \kappa(x, y) \leq \kappa_0 \quad \text{for } x, y \in \mathbb{R}^d.
\]
Let \(J : \mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag} \to (0, \infty)\) be a symmetric measurable function defined as
\[
J(x, y) := \kappa(x, y) J^\phi(x, y).
\]
By (WS), there exists \(c > 0\) such that for any constant \(a > 0\) and any \(i \in \{1, \ldots, d\}\), we have
\[
\int_{\{|\tau| > a\}} J(x, x + e_i \tau) d\tau < \frac{c}{\phi(a)} \quad \text{for } x \in \mathbb{R}^d \quad \text{and} \quad \int_{\{|\tau| > a\}} u^1(|\tau|) d\tau < \frac{c}{\phi(a)}.
\]
Consider the Dirichlet form \((\mathcal{E}, \mathcal{F})\) on \(L^2(\mathbb{R}^d)\) as follows:
\[
\mathcal{E}(u, v) := \int_{\mathbb{R}^d} \left( \sum_{i=1}^d \int_{\mathbb{R}} (u(x + e_i \tau) - u(x)) (v(x + e_i \tau) - v(x)) J(x, x + e_i \tau) d\tau \right) dx,
\]
\[
\mathcal{F} := \{ u \in L^2(\mathbb{R}^d) \mid \mathcal{E}(u, u) < \infty \}.
\]
Then we obtain in [KW22, Theorem 1.1] that the existence of a conservative Feller process \(X = (X^1, \ldots, X^d)\) associated with \((\mathcal{E}, \mathcal{F})\). Moreover, \(X\) has a jointly continuous transition density function \(p(t, x, y)\) on \(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d\), which enjoys the following estimates: there exists a constant \(C_1 > 1\) such that for any \(t > 0, x, y \in \mathbb{R}^d\),
\[
C_1^{-1} \phi^{-1}(t)^{-d} \prod_{i=1}^d \left( 1 \wedge \frac{t \phi^{-1}(t)}{|x^i - y^i| \phi(|x^i - y^i|)} \right) \leq p(t, x, y) \leq C_1 \phi^{-1}(t)^{-d} \prod_{i=1}^d \left( 1 \wedge \frac{t \phi^{-1}(t)}{|x^i - y^i| \phi(|x^i - y^i|)} \right),
\]
where \(a \wedge b := \min\{a, b\}\). \(X\) represents a large class of anisotropic Markov processes. More detailed examples can be found in [KW22, Remark 1.2].

For any open set \(D \subset \mathbb{R}^d\), we define the first exit time \(\tau_D := \inf\{t > 0 : X_t \notin D\}\) of \(D\) for the process \(X\), and consider the subprocess \(X^D_t\) of \(X\) killed upon leaving \(D\) as follows:
\(X^D_t(\omega) = X_t(\omega)\) if \(t < \tau_D(\omega)\) and \(X^D_t(\omega) = \partial\) if \(t \geq \tau_D(\omega)\) where \(\partial\) is a cemetery point. In this article, we investigate that \(X^D\) has a continuous transition density \(p_D(t, x, y)\) with respect to the Lebesgue measure, then obtain the two-sided bounds for \(p_D(t, x, y)\) in \(C^{1,1}\) open set \(D \subset \mathbb{R}^d\) for different ranges of time \(t\).

We call \(D \subset \mathbb{R}^d\) a \(C^{1,1}\) open set if there exists a localization radius \(R > 0\) and a constant \(\Lambda > 0\) such that for every \(z \in \partial D\) there exists a \(C^{1,1}\) function \(\varphi = \varphi_z : \mathbb{R}^{d-1} \to \mathbb{R}\) satisfying
\[
\varphi(0) = 0, \ \nabla \varphi(0) = (0, \ldots, 0), \ ||\nabla \varphi||_\infty \leq \Lambda \quad \text{and} \quad |\nabla \varphi(x) - \nabla \varphi(w)| \leq \Lambda |x-w|
\]
and an orthonormal coordinate system \(CS_z\) of \(z = (z^1, \ldots, z^{d-1}, z^d) := (\bar{z}, z^d)\) with an origin at \(z\) such that \(D \cap B(z, R) = \{y = (\tilde{y}, y^d) \in B(0, R) \in CS_z : y^d > \varphi(\tilde{y})\}\). The pair \((R, \Lambda)\) is called the \(C^{1,1}\) characteristics of the open set \(D\), and we may assume that \(R < 1\) and \(\Lambda > 2\).

Note that a \(C^{1,1}\) open set \(D\) with characteristics \((R, \Lambda)\) can be unbounded and disconnected, and the distance between two distinct components of \(D\) is at least \(R\). \(C^{1,1}\) open sets in \(\mathbb{R}\) with
a characteristic $R > 0$ can be written as the union of disjoint intervals so that the infimum of the lengths of all these intervals is at least $R$. It is well known that if $D$ is $C^{1,1}$ open set with the characteristics $(R, \Lambda)$, then $D$ satisfies the interior and exterior ball conditions with the characteristic $R_1 \leq R$, that is, there exist balls $B_1, B_2 \subset \mathbb{R}^d$ with radius $R_1$ such that $B_1 \subset D \subset B_2$ satisfying $\delta_{B_1}(x) \leq \delta_D(x) \leq \delta_{B_2}(x)$ for any $x \in B_1$. Here we use the definition $\delta_A(x) := \text{dist}(x, \partial A)$ for any $A \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$. Throughout this article, without loss of generality, we always assume that $R = R_1$.

Throughout this article, we assume additional conditions for the regularities of $\kappa$ and $\phi$ appeared in (1.3).

(\textbf{K}_\eta): There are $\kappa_1 > 0$ and $\pi/2 < \eta \leq 1$ such that $|\kappa(x, x + h) - \kappa(x, x)| \leq \kappa_1|h|^{\eta}$ for every $x, h \in \mathbb{R}^d$, $|h| \leq 1$.

(\textbf{SD}): $\phi \in C^1(0, \infty)$ and $r \rightarrow - (\nu^1)'(r)/r$ is decreasing.

We now state our main results of this article. Let

$$
\Psi(t, x) := \left(1 \wedge \frac{\phi(\delta_D(x))}{t}\right) \quad \text{for } t > 0, x \in \mathbb{R}^d.
$$

\textbf{Theorem 1.1}. Suppose that $X$ is a symmetric pure jump Hunt process whose jumping intensity kernel $J$ satisfies the conditions (1.3), (\textbf{SD}) and (\textbf{K}_\eta). Suppose that $D$ is a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics $(R, \Lambda)$. Then, for each $T > 0$, there exists $C_2 = C_2(\phi, \kappa_0, \kappa_1, \eta, R, \Lambda, T, d) > 0$ such that the transition density $p_D(t, x, y)$ of $X^D$ has the following upper estimate:

$$
p_D(t, x, y) \leq C_2 \Psi(t, x) \Psi(t, y)p(t, x, y), \quad \text{for any } (t, x, y) \in (0, T] \times D \times D.
$$

For the lower bound estimate, we need extra conditions: for any $u \in \mathbb{R}^d, a \in \mathbb{R}$ and $1 \leq i \leq d$, define a new point $[u]_a := (u^1, \ldots, u^{i-1}, a, u^{i+1}, \ldots, u^d) \in \mathbb{R}^d$.

For any $\gamma \in (0, 1]$, we consider the condition

(\textbf{D}_\gamma): For any $x, y \in U$ and for any permutation $\{i_1, i_2, \ldots, i_d\}$ of $\{1, 2, \ldots, d\}$, let $\xi_{(0)} = \xi_{(1)} = \ldots = \xi_{(d)}$ be points that trace from $x$ to $y$ with

$$
\xi_{(0)} := x, \quad \xi_{(1)} := [x]_{y^1}, \quad \xi_{(2)} := [\xi_{(1)}]_{y^2}, \ldots, \quad \xi_{(d-1)} := [\xi_{(d-2)}]_{y^{d-1}} \quad \text{and} \quad \xi_{(d)} := [\xi_{(d-1)}]_{y^d} = y.
$$

We say that $U$ satisfies condition (\textbf{D}_\gamma) if for any $r > 0$ and $x, y \in U$ with $\delta_U(x) \wedge \delta_U(y) \geq r$, there exists a permutation $\{i_1, i_2, \ldots, i_d\}$ of $\{1, 2, \ldots, d\}$ such that $B(\xi_{(k)}, r) \subset U$, for all $k = 1, 2, \ldots, d$.

By the definition of (\textbf{D}_\gamma), one can easily see that

$$
\xi_{(l)}^{i_k} = \begin{cases} 
y^{i_k} & \text{if } l \geq k \\
x^{i_k} & \text{if } l < k.
\end{cases}
$$

\textbf{Theorem 1.2}. Under the setting of \textbf{Theorem 1.1}, assume in addition that $D$ satisfies (\textbf{D}_\gamma) for some $\gamma \in (0, 1]$.

(1) Then for any $T > 0$, there exists $C_3 := C_3(\phi, \kappa_0, \kappa_1, \eta, R, \Lambda, T, d, \gamma) > 0$ such that for any $t \in (0, T]$ and $x, y \in D$,

$$
p_D(t, x, y) \geq C_3 \Psi(t, x) \Psi(t, y)p(t, x, y).
$$

(2) Assume that $D$ is bounded, then there exists $C_4 := C_4(\phi, \kappa_0, \kappa_1, \eta, R, \Lambda, d, \gamma, \text{diam}(D)) > 1$ such that

$$
C_4^{-1} \leq \lambda^D \leq C_4.
$$
where \(-\lambda^D < 0\) is the first eigenvalue of \(-\mathcal{L}\) on \(D\).

Also for any \(T > 0\), there exists \(C_5 = C_5(\phi, \kappa_0, \kappa_1, \eta, R, \Lambda, T, d, \gamma, \text{diam}(D)) > 1\) such that for all \(t \in [T, \infty)\) and \(x, y \in D\),

\[
C_5^{-1} e^{-t\lambda^D} \sqrt{\phi(\delta_D(x))} \sqrt{\phi(\delta_D(y))} \leq p_D(t, x, y) \leq C_5 e^{-t\lambda^D} \sqrt{\phi(\delta_D(x))} \sqrt{\phi(\delta_D(y))}.
\]

We now consider \(x, y \in D\) with \(x^i \neq y^i\) for all \(1 \leq i \leq d\). By integrating the heat kernel estimates in Theorem 1.1 and Theorem 1.2 with respect to \(t \in (0, \infty)\), one gets the following Green function estimates.

**Theorem 1.3.** Suppose that \(X\) is a symmetric pure jump Hunt process whose jumping intensity kernel \(J\) satisfies the conditions (1.3), (SD) and (\(K_\eta\)). Suppose that \(D\) is a bounded \(C^{1,1}\) open set in \(\mathbb{R}^d\) with characteristics \((R, \Lambda)\) under the condition \((D_\eta)\). For any \((x, y)\) such that \(x^i \neq y^i\) for all \(i \in \{1, \ldots, d\}\). Then when \(d \geq 2\), there exists \(C_6 > 1\) such that

\[
C_6^{-1} \left(1 + \frac{\sqrt{\phi(\delta_D(x))}}{\sqrt{\phi(r_1(x, y))}}\right) \left(1 + \frac{\sqrt{\phi(\delta_D(y))}}{\sqrt{\phi(r_1(x, y))}}\right) \phi(r_1(x, y))^{d+1} \prod_{i=1}^{d} \frac{1}{|x^i - y^i| \phi(|x^i - y^i|)} \leq G_D(x, y) \\
\leq G_D(x, y) \leq C_6 \left(1 + \frac{\sqrt{\phi(\delta_D(x))}}{\sqrt{\phi(r_2(x, y))}}\right) \left(1 + \frac{\sqrt{\phi(\delta_D(y))}}{\sqrt{\phi(r_2(x, y))}}\right) \phi(r_2(x, y))^{d+1} \prod_{i=1}^{d} \frac{1}{|x^i - y^i| \phi(|x^i - y^i|)},
\]

where \(r_1(x, y) := \min_{i \in \{1, \ldots, d\}} |x^i - y^i|, r_2(x, y) := \max_{i \in \{1, \ldots, d\}} |x^i - y^i|\). When \(d = 1\), letting \(a(x, y) := \sqrt{\phi(\delta_D(x))} \phi(\delta_D(y))\) and \(x^+ := \max(x, 0)\), we have that

\[
G_D(x, y) \geq \frac{a(x, y)}{|x - y|} \wedge \left(\frac{a(x, y)}{\phi^{-1}(a(x, y))} + \left(\int_{|x-y|}^{\phi^{-1}(a(x,y))} \frac{\phi(s)}{s^2} ds\right)^+\right).
\]

The rest of this article is organized as follows. In Section 2, we compute some key upper bounds of the generator of \(Z\) with testing functions defined on \(C^{1,1}\) open sets, and discuss the key estimates on exit distributions for \(X\), see Theorem 2.8. In Section 3, we first discuss the regularity of the transition density \(p_D(t, x, y)\) of \(X^D\), that is, \(p_D(t, x, y)\) can be refined to be Hölder continuous for \((t, x, y) \in \mathbb{R}_+ \times D \times D\). Then we adopt the method in [GKK20, Section 5] to obtain the upper bound of \(p_D(t, x, y)\) stated in Theorem 1.1. However, a variation of the key lemma [GKK20, Lemma 5.1] is needed since the process here is anisotropic (see, Lemma 3.5). In Section 4, we prove the lower bound of \(p_D(t, x, y)\) in Theorem 1.2(1), by utilizing a preliminary lower bound result in Proposition 4.5 as well as the estimates of exit distributions presented in Theorem 2.8. The large time Dirichlet heat kernel estimates in Theorem 1.2(2) are proved in Section 5. In Section 6, as an application of Theorem 1.1 and Theorem 1.2, we derive the Green function estimates. We would like to mention that this is the very first attempt to observe the Green function estimates of anisotropic Markov processes, though only for the points with different coordinates in all directions.

**Notations.** For a function space \(\mathbb{F}(U)\) on an open set \(U\) in \(\mathbb{R}^d\), we let

\[
\mathbb{F}_c(U) := \{f \in \mathbb{F}(U) : f\ \text{has compact support}\}.
\]

We use \((\cdot, \cdot)\) to denote the inner product in \(L^2(\mathbb{R}^d)\) and \(\|\cdot\|_k := \|\cdot\|_{L^k(\mathbb{R}^d)}\). For \(x \in \mathbb{R}^d\) and \(r > 0\), we use \(B(x, r)\) to denote a ball centered at \(x\) with radius \(r\), and \(Q(x, r)\) to denote a cube centered at \(x\) with side length \(r\). The letter \(c = c(a, b, \ldots)\) will denote a positive constant depending on \(a, b, \ldots\), and it may change at each appearance. The labeling of the constants \(c_1, c_2, \ldots\) begins anew in the proof of each statement. The notation \(\approx\) is to be read as “is
defined to be $f \asymp g$ if the quotient $f/g$ is comparable to some positive constants.

2. Analysis on $Z$ and Exit Distribution for $X$

Recall that $Z = (Z^1, \ldots, Z^d)$ is the $d$-dimensional Lévy process where $Z^i$'s are independent 1-dimensional Lévy processes with the jumping kernel $J^d(x, y) = |x^i - y^i|^{-1} \phi(|x^i - y^i|)^{-1}$, where $\phi$ is an increasing function satisfying (WS) and (SD). $X = (X^1, \ldots, X^d)$ is the anisotropic Markov process with the jumping kernel $J$ satisfying (SD) and $(K_\eta)$.

In Subsection 2.1, we introduce the renewal function $V$ corresponding to the coordinate process $Z^d$, and obtain the estimates for the generator of $Z$ applied to the testing functions on $C^{1,1}$ open sets, see Proposition 2.4. In Subsection 2.2, we give the key estimates on exit distributions for $X$, see Theorem 2.8.

2.1. Analysis on $Z$. Recall that $Z^d$ is a pure jump Lévy process with Lévy measure $\nu^1(dr)$. The Lévy-Khintchine (characteristic) exponent of $Z^d$ has the form

$$\psi(\xi) = \int_\mathbb{R} (1 - \cos(\xi r)) \nu^1(dr) \quad \text{for } \xi \in \mathbb{R}.$$ 

Let $M_t := \sup_{s \leq t} Z^d_s$ and $L$ be the local time at 0 for $M - Z^d$, the reflected process of $Z^d$ from the supremum. For the right-continuous inverse process $L^{-1}$, define the ascending ladder-height process $H_s := Z^d_{L_s-1} = M_{L_s-1}$. Then the Laplace exponent of $H$ is

$$\chi(\xi) = \exp\left(\frac{1}{\pi} \int_0^\infty \frac{\log(\psi(\theta \xi))}{1 + \theta^2} \, d\theta\right) \quad \text{for } \xi \geq 0$$

(see, [Fri74, Corollary 9.7]). Define the renewal function $V$ of $H$ as follows:

$$V(x) := \int_0^\infty \mathbb{P}(H_s \leq x) \, ds \quad \text{for } x \in \mathbb{R}.$$ 

Then $V(x) = 0$ for $x < 0$, $V(\infty) = \infty$ and $V$ is non-decreasing. Also $V$ is sub-additive (see [Ber96, p.74]), that is,

$$V(x + y) \leq V(x) + V(y) \quad \text{for } x, y \in \mathbb{R}.$$

(2.1)

Since the distribution of $Z^d_t$ is absolutely continuous for every $t > 0$, the resolvent measure of $Z^d_t$ is absolutely continuous as well, see, [Fuk72, Theorem 6]. From [Sil80, Theorem 2] and [Sil80, (1.8) and Theorem 1], we see that $V(x)$ is absolutely continuous and harmonic on $(0, \infty)$ for $Z^d_t$, and $V'$ is a positive harmonic function on $(0, \infty)$ for $Z^d_t$. Therefore, $V$ is actually (strictly) increasing.

Corresponding to the Lévy measure $\nu^1(dz)$, the Pruitt function (see [Pru81]) is defined as follows:

$$h(r) := \int_\mathbb{R} (1 \wedge |z|^2 r^{-2}) \nu^1(dz) \quad \text{for } r > 0,$$

and by [BGR14a, Corollary 3] and [BGR15, Proposition 2.4],

$$h(r) \asymp [V(r)]^{-2} \psi(r^{-1}) \quad \text{for } r > 0.$$ 

Since $s \rightarrow \phi(s^{-1})^{-1}$ satisfies (WS), (1.1) and [BGR14a, Proposition 28] yield that

$$\psi(r) \asymp \phi(r^{-1})^{-1} \quad \text{for } r > 0.$$

Combining the above observations, we conclude that

$$\phi(r) \asymp [V(r)]^2 \quad \text{and} \quad \nu^1(r) \asymp [V(r)]^{-2} r^{-1} \quad \text{for } r > 0.$$

(2.2)
Also (WS) condition of $\phi$ yields the weakly scaling properties of $V$ as follows: there exists $C_V > 1$ such that
\[ C_V^{-1} \left( \frac{R}{r} \right)^{d/2} \leq \frac{V(R)}{V(r)} \leq C_V \left( \frac{R}{r} \right)^{\frac{d}{2}} \quad \text{for any } 0 < r \leq R. \tag{2.3} \]
By (2.2) and (2.3), it is clear that for any $r > 0$, there is $c = c(\mathcal{A}, \mathcal{C}) > 0$ such that
\[ \int_{\{|t| > r\}} V(|t|)\nu^1(|t|)dt \leq \frac{c}{V(r)} \quad \text{and} \quad \int_{\{|t| > r\}} \nu^1(|t|)dt \leq \frac{c}{[V(r)]^2}. \tag{2.4} \]

The following proposition is directly obtained by [KR16, Theorem 1.1] since $V$ and $V'$ are harmonic on $(0, \infty)$ for $Z^d$, and the assumption [KR16, (A)] is satisfied by (SD).

**Proposition 2.1.** The function $x \mapsto V(x)$ is twice-differentiable for any $x > 0$, and there exists a positive constant $C_{2.1}$ such that
\[ V'(x) \leq C_{2.1} \frac{V(x)}{x \wedge 1} \quad \text{and} \quad |V''(x)| \leq C_{2.1} \frac{V'(x)}{x \wedge 1} \quad \text{for any } x > 0. \]

For any $x = (\bar{x}, x^d) \in \mathbb{R}^d$, define $w(x) = V(x^d \vee 0)$. Then we have the proposition which follows from [GKK20, Proposition 3.2]. For the reader’s convenience, we give the proof.

**Proposition 2.2.** For $\lambda > 0$, there exists $C_{2.2} = C_{2.2}(\lambda, \phi) > 0$ such that for any $r > 0$ and $i \in \{1, \ldots, d\}$, we have
\[ \sup_{\{x \in \mathbb{R}^d : 0 < x^d \leq \lambda r\}} \int_{\{|t| > r\}} w(x + e_i t)\nu^1(|t|)dt < \frac{C_{2.2}}{V(r)}. \]

**Proof.** Since $w(x + z) \leq V(x^d + V(|z|)$ for $x^d > 0$ and $z \in \mathbb{R}^d$, it follows that
\[ \int_{\{|t| > r\}} w(x + e_i t)\nu^1(|t|)dt \leq V(x^d) \int_{\{|t| > r\}} \nu^1(|t|)dt + \int_{\{|t| > r\}} V(|t|)\nu^1(|t|)dt. \]

By (2.4) and (2.1), we conclude that
\[ \sup_{\{x \in \mathbb{R}^d : 0 < x^d \leq \lambda r\}} \int_{\{|t| > r\}} w(x + e_i t)\nu^1(|t|)dt \leq c_1 \left( \frac{V(\lambda r)}{[V(r)]^2} + \frac{1}{V(r)} \right) \leq \frac{c_2}{V(r)}. \]

For each $i \in \{1, \ldots, d\}$, we define an operator $\mathcal{L}_Z^{(i)}$ by
\[ \mathcal{L}_Z^{(i)} f(x) := \lim_{\epsilon \downarrow 0} \int_{\{|t| > \epsilon\}} (f(x + e_i t) - f(x))\nu^1(|t|)dt =: \lim_{\epsilon \downarrow 0} \mathcal{L}_Z^{(i), \epsilon} f(x) \quad \text{for } x \in \mathbb{R}^d. \tag{2.5} \]

Applying [GKK20, Theorem 3.3] with Proposition 2.1 and Proposition 2.2, we have the following Theorem.

**Theorem 2.3.** For any $x = (\bar{x}, x^d) \in \mathbb{R}^d$ with $x^d > 0$, $\mathcal{L}_Z^{(i)} w(x)$ is well-defined and $\mathcal{L}_Z^{(i)} w(x) = 0$ for any $i \in \{1, \ldots, d\}$.

For the rest of this paper, let $D$ be a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics $(R, \Lambda)$. We may assume that $R < 1$ and $\Lambda > 2$. For $Q \in \partial D$, consider the coordinate system $CS_Q$ such that
\[ B(Q, R) \cap D = \{\bar{y}, y^d\} \in B(Q, R) \text{ in } CS_Q: y^d > \varphi_Q(\bar{y}). \]

Define a function $\rho_Q(y) := y^d - \varphi_Q(\bar{y})$ in $CS_Q$. Then
\[ \frac{\rho_Q(y)}{\sqrt{1 + \Lambda^2}} \leq \delta_D(y) \leq \rho_Q(y) \quad \text{for any } y \in B(Q, R) \cap D. \tag{2.6} \]

For $r_1, r_2 > 0$, let
\[ D_Q(r_1, r_2) := \{y \in D : |\bar{y}| < r_1, 0 < \rho_Q(y) < r_2\}. \]
Proposition 2.4. For any $Q \in \partial D$ and $R_0 \leq R/2$, we define

$$h_{R_0}(y) = h_{Q,R_0}(y) := V(\delta_D(y)) \mathbf{1}_{D \cap B(Q,R_0)}(y) \quad \text{for } y \in \mathbb{R}^d.$$ 

Let $r_0 := R_0/4$. Then there exists a constant $c = c(\phi, R, \Lambda) > 0$ independent of $Q$ such that for any $i \in \{1, \ldots, d\}$, $L_{Z}^{(i)}h_{R_0}$ is well-defined in $D_Q(r_0, r_0)$ and

$$|L_{Z}^{(i)}h_{R_0}(x)| \leq \frac{c}{V(R_0)} \quad \text{for any } x \in D_Q(r_0, r_0).$$

Proof. For any $x \in D_Q(r_0, r_0)$, let $x_0 \in \partial D$ satisfy $\delta_D(x) = |x - x_0|$. Let $\varphi := \varphi_{x_0}$ be the $C^{1,1}$ function and $CS := CS_{x_0}$ be an orthogonal coordinate system with $x_0$ so that

$$\varphi(\tilde{0}) = 0, \quad \nabla \varphi(\tilde{0}) = (0, \ldots, 0), \quad \|\nabla \varphi\|_{\infty} \leq \Lambda, \quad |\nabla \varphi(\tilde{y}) - \nabla \varphi(\tilde{z})| \leq \Lambda |\tilde{y} - \tilde{z}|$$

for any $\tilde{y}, \tilde{z} \in \mathbb{R}^d$ and $D \cap B(x_0, R) = \{y = (\tilde{y}, y^d) \in B(0, R) \text{ in } CS : y^d > \varphi(\tilde{y})\}$. We fix the function $\varphi$ and the coordinate system $CS$, and we define a function

$$g_x(y) = V(\delta_H(y)) \mathbf{1}_{D \cap B(Q,R_0)}(y) = V(y^d) \mathbf{1}_{D \cap B(Q,R_0)}(y) \quad \text{for } y \in H$$

where $H = \{y = (\tilde{y}, y^d) \in CS : y^d > 0\}$ is the half space in $CS$. Note that $h_{R_0}(x) = g_x(x) = V(x^d)$ and that $L_{Z}^{(i)}h_{R_0} - g_x = L_{Z}^{(i)}\tilde{h}_{R_0}$ by Theorem 2.3. Therefore it suffices to show that $L_{Z}^{(i)}\tilde{h}_{R_0}$ is well-defined, and that there exists a constant $c_1 = c_1(\phi, R, \Lambda) > 0$ such that

$$\int_{\{t \in \mathbb{R} : x + e_i t \in \partial D \cap H\}} |h_{R_0}(x + e_i t) - g_x(x + e_i t)| \nu^1(|t|) dt \leq c_1 V(R_0)^{-1}. \quad (2.7)$$

Define $\tilde{\varphi} : B(\tilde{0}, R_0) \to \mathbb{R}$ by $\tilde{\varphi}(\tilde{z}) := 2\Lambda \tilde{z}^2$. Since $\varphi(\tilde{0}) = 0$, the mean value theorem implies that $|\tilde{\varphi}(\tilde{z})| \leq \Lambda \tilde{z}^2 \leq \tilde{\varphi}(\tilde{z})$ for any $\tilde{z} = (\tilde{z}, z^d) \in D \cup H$. Let

$$A := \{z \in D \cap B(x_0, r_0) : |z^d| \leq \tilde{\varphi}(\tilde{z})\} \quad \text{and} \quad E := \{z \in D : |\tilde{z}| < r_0, \tilde{\varphi}(\tilde{z}) < z^d < 3r_0 \Lambda\}.$$

Then for any $z \in D \cap B(x_0, r_0) \setminus A$, since $x \in D_Q(r_0, r_0)$,

$$\tilde{\varphi}(\tilde{z}) < z^d \leq |z^d - x^d| + |x^d| \leq r_0 + \delta_D(x) \leq r_0 + \sqrt{2}r_0 \leq 3r_0,$$

so that $z \in E$. Also for any $z \in E$, since $Q \in \partial D$ and $x_0 \in D_Q(r_0, r_0)$,

$$|z - Q| \leq |z - x_0| + |x_0 - Q| \leq \sqrt{|z^d|^2 + |\tilde{z}|^2 + 2r_0} \leq \sqrt{(3r_0 \Lambda)^2 + r_0^2} + \sqrt{2}r_0 \leq 4 \Lambda r_0 = R_0,$$

so that $z \in D \cap B(Q, R_0)$. Therefore,

$$D \cap B(x_0, r_0) \setminus A \subset E \subset D \cap B(Q, R_0).$$

Now we decompose (2.7) as follows:

$$\int_{\{|t| \geq r_0\}} |h_{R_0}(x + e_i t) - g_x(x + e_i t)| \nu^1(|t|) dt + \int_{\{|t| < r_0\}} |h_{R_0}(x + e_i t) - g_x(x + e_i t)| \nu^1(|t|) dt$$

$$\leq \int_{\{|t| \geq r_0\}} (h_{R_0}(x + e_i t) + g_x(x + e_i t)) \nu^1(|t|) dt + \int_{\{|t| < r_0\}} (h_{R_0}(x + e_i t) + g_x(x + e_i t)) \nu^1(|t|) dt$$

$$+ \int_{\{x + e_i t \in A\}} |h_{R_0}(x + e_i t) - g_x(x + e_i t)| \nu^1(|t|) dt =: \text{I} + \text{II} + \text{III}.$$

For any fixed $x \in D_Q(r_0, r_0)$, we denote $y_t := x + e_i t, t > 0$. In $CS$, let $P$ be the hyperplane tangent to $\partial D$ at $x_0$. Consider a function $\Gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ defined by $\Gamma(\tilde{y}) := \nabla \varphi(\tilde{x}) \cdot \tilde{y}$ which describes the plane $P$. Let $\nabla \varphi(\tilde{x}) := (\alpha^1, \ldots, \alpha^d)$ and $\theta_1$ be the angle between the normal vector $\nabla \varphi(\tilde{x})$ of the hyperplane $P$ and $y_t = x + e_i t$. Then since

$$\alpha^i t = (\nabla \varphi(\tilde{x}), e_i t) = |\nabla \varphi(\tilde{x})||t| \cos \theta_1, \quad |\cos \theta_1| = \frac{|\alpha^i|}{|\nabla \varphi(\tilde{x})|}$$

and

$$|\tilde{y}_t| = |t \sin \theta_1| \quad \text{and} \quad |y^d - x^d| = |t \cos \theta_1|$$

(2.8)
Now we estimate I, II and III as follows:

I: Let |t| ≥ r_0. Since R_0 = 4Λ r_0 ≤ 4A|t|, by the definition of h_{R_0} with (2.1), (2.4) and Proposition 2.2, we have

\[ I \leq c_2 \int_{|t| \geq r_0} V(|t|) \nu^1(|t|) dt + \sup_{x \in \mathbb{R}^d : 0 < x^d < r_0} \int_{|t| \geq r_0} g_x(y) \nu^1(|t|) dt \]
\[ \leq c_3 V(r_0)^{-1} + \sup_{x \in \mathbb{R}^d : 0 < x^d < r_0} \int_{|t| \geq r_0} w(x + \epsilon t) \nu^1(|t|) dt \leq (c_3 + C_{2.2}) V(r_0)^{-1}. \tag{2.9} \]

II: For \( y_t = x + \epsilon_t t \in A \), that is, \( \theta_1 \in (0, \pi) \), hence \( |\tilde{y}_t| \neq 0 \). Since \( |y_t^{|t|} \cup \delta_D(y_t) \leq 2 |\tilde{\varphi}(\tilde{y}_t)| \leq 4A|\tilde{y}_t| \leq 4A|t| \) by (2.8) and \( V \) is increasing,
\[ h_{R_0}(y_t) + g_x(y_t) \leq c_4 V(|\tilde{y}_t|) \leq c_4 V(|t|). \]

For the surface measure \( m_{d-1}(dy) \) on \( \mathbb{R}^{d-1} \), \( m_{d-1}(\{ \tilde{y}_t \in \mathbb{R}^d : |\tilde{y}_t| = s \sin \theta_1, |y_t^{|t|} \leq 2A|\tilde{y}_t|^2 \}) \leq c_5(s \sin \theta_1)^d \leq c_5 s^d \). Then by (2.2) and (2.3)
\[ II \leq c_4 \int_{\{y_t \in A\}} V(|\tilde{y}_t|) \nu^1(|t|) dt \leq c_6 \int_0^{r_0} \frac{V(s)}{|V(s)|^2} s^d ds = c_6 \int_0^{r_0} \frac{s^{d-1}}{V(s)} ds \]
\[ \leq \frac{c_7}{(d - \frac{2}{2})} \frac{1}{V(r_0)}. \tag{2.10} \]

III: For \( y_t \in E \subset D \), we first note that
\[ |y_t^{|t|} - \delta_D(y_t)| \leq c_8 |\tilde{y}_t|^2. \tag{2.11} \]

To obtain this, we consider the cases \( y_t^{|t|} = \delta_E(y_t) \leq \delta_D(y_t) \) and \( y_t^{|t|} = y_t^{|t|} = \delta_E(y_t) \) separately. If \( 0 < y_t^{|t|} = \delta_E(y_t) \leq \delta_D(y_t) \), by the outer ball condition
\[ \delta_D(y_t) - y_t^{|t|} \leq (\sqrt{|\tilde{y}_t|^2 + (R + y_t^{|t|})^2} - R) - y_t^{|t|} = \frac{|\tilde{y}_t|^2}{\sqrt{|\tilde{y}_t|^2 + (R + y_t^{|t|})^2} + (R + y_t^{|t|})} \leq \frac{|\tilde{y}_t|^2}{R}. \]

Similarly, if \( y_t^{|t|} = \delta_E(y_t) \geq \delta_D(y_t) \), the interior ball condition implies that
\[ y_t^{|t|} - \delta_D(y_t) \leq y_t^{|t|} - (R - \sqrt{|\tilde{y}_t|^2 + (R - y_t^{|t|})^2}) = \frac{|\tilde{y}_t|^2}{\sqrt{|\tilde{y}_t|^2 + (R - y_t^{|t|})^2} + (R - y_t^{|t|})} \leq \frac{|\tilde{y}_t|^2}{R/2}. \]

For \( |t| \leq r_0 \), by (2.8), since \( |\tilde{y}_t| \leq r_0 \) and \( |y_t^{|t|} \leq |x^d| + |t| \leq 2r_0 \),
\[ \delta_D(y_t)^2 \leq |y_t - x_0|^2 = |\tilde{y}_t|^2 + |y_t^{|t|}|^2 \leq 5r_0^2. \]

Using the above observation with the scale invariant Harnack inequality for \( Z^d \) (see, [CKK09, Theorem 1.4]) applied to \( V' \), Proposition 2.1 and (2.3), we have that for \( |t| \leq r_0 \),
\[ \sup_{u \in [y_t^{|t|} \delta_D(y_t), y_t^{|t|} \delta_D(y_t)]} V'(u) \leq \sup_{u \in [y_t^{|t|} \delta_D(y_t), \sqrt{5} r_0]} V'(u) \leq c_9 \inf_{u \in [y_t^{|t|} \delta_D(y_t), \sqrt{5} r_0]} V'(u) \leq c_9 V'(\sqrt{5} r_0) \leq c_{10} V(r_0)/r_0. \tag{2.12} \]

Combining (2.11)–(2.12), the mean value theorem yields that for \( y_t \in E \),
\[ |h_{R_0}(y_t) - g_x(y_t)| \leq \sup_{u \in [y_t^{|t|} \delta_D(y_t), y_t^{|t|} \delta_D(y_t)]} V'(u) |y_t^{|t|} - \delta_D(y_t)| \leq c_{11} V(r_0) |t|^2/r_0. \]

Therefore, by (2.2)–(2.3) with the fact that \( r_0 < 1 \),
\[ III \leq c_{11} \frac{V(r_0)}{r_0} \int_0^{r_0} \frac{s}{|V(s)|^2} ds \leq \frac{c_{12} r_0^{d-1}}{V(r_0)} \int_0^{r_0} s^{d-1} ds \leq \frac{c_{13} r_0^d}{V(r_0)} \leq \frac{c_{13}}{V(r_0)}. \tag{2.13} \]

Hence, we conclude our claim by (2.9), (2.10) and (2.13) using the definition of \( r_0 \) and (2.3). ■
2.2. Estimates on exit distribution for $X$. Recall that $X$ is a conservative Feller process with the jumping kernel $J(x, y)$ defined in (1.3) satisfying the conditions (SD) and (K$g$). For each $i \in \{1, \ldots, d\}$, define operators $\mathcal{L}^{(i)}$ and $\mathcal{L}$ on $C_c^2(\mathbb{R}^d)$ as

$$
\mathcal{L}^{(i)} f(x) := \lim_{\epsilon \downarrow 0} \int_{|t| > \epsilon} (f(x + e_i t) - f(x)) J(x, x + e_i t) dt =: \lim_{\epsilon \downarrow 0} \mathcal{L}^{(i)} f(x)
$$

and $\mathcal{L} f(x) := \sum_{i=1}^{d} \mathcal{L}^{(i)} f(x)$. For any $i \in \{1, \ldots, d\}$, $g \in C_c^2(\mathbb{R}^d)$ and $0 < \epsilon < r < 1$, we have

$$
\mathcal{L}_\epsilon^{(i)} g(x) = \kappa(x, x) \int_{|t| < \epsilon} (g(x + e_i t) - g(x) - (e_i t) \cdot \nabla g(x)) \nu^1(|t|) dt + \int_{|t| < \epsilon} (g(x + e_i t) - g(x)) (\kappa(x, x + e_i t) - \kappa(x, x)) \nu^1(|t|) dt + \int_{|t| \leq \epsilon} (g(x + e_i t) - g(x)) J(x, x + e_i t) dt.
$$

For $x \in \mathbb{R}^d$ and $|t| \leq 1$, by (K$g$), we first obtain that

$$
|(g(x + e_i t) - g(x))(\kappa(x, x + e_i t) - \kappa(x, x))| \leq \|g\|_{\infty} \kappa_1 |t|^{\eta + 1}.
$$

Also (WS) with the fact that $1 \geq \eta > \varpi/2 > \varpi - 1$ implies that for $0 < r < 1$,

$$
\int_{|t| < r} |t|^2 \nu^1(|t|) dt \leq \int_{|t| < r} |t|^\eta \nu^1(|t|) dt < \infty. \quad (2.14)
$$

Therefore, $\mathcal{L}^{(i)} g$ is well defined and $\mathcal{L}_\epsilon^{(i)} g$ converges to $\mathcal{L}^{(i)} g$ uniformly on $\mathbb{R}^d$. Furthermore, for every $0 < r < 1$,

$$
\mathcal{L}^{(i)} g(x) = \kappa(x, x) \int_{|t| < r} (g(x + e_i t) - g(x) - (e_i t) \cdot \nabla g(x)) \nu^1(|t|) dt + \int_{|t| < r} (g(x + e_i t) - g(x)) (\kappa(x, x + e_i t) - \kappa(x, x)) \nu^1(|t|) dt + \int_{|t| \leq r} (g(x + e_i t) - g(x)) J(x, x + e_i t) dt. \quad (2.15)
$$

The following proof is almost the same as that of [GKK20, Lemma 2.2].

**Lemma 2.5.** For any $g \in C_c^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, there exists a $\mathbb{P}_x$-martingale $M^g_t$ with respect to the filtration of $X$ such that

$$
M^g_t = g(X_t) - g(X_0) - \int_0^t \mathcal{L} g(X_s) ds \quad \mathbb{P}_x\text{-a.s.}
$$

In particular, for any stopping time $S$ with $\mathbb{E}_x [S] < \infty$, we have

$$
\mathbb{E}_x [g(X_S)] - g(x) = \mathbb{E}_x \left[ \int_0^S \mathcal{L} g(X_s) ds \right]. \quad (2.16)
$$

**Proof.** Let $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ be the $L^2$-generator of the semigroup $T_t$ with respect to $X$. By the definition of $\mathcal{L}$, we could apply the similar proof as that of [SU07, Proposition 2.5] to obtain that $C_c^2(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{A})$ and $\mathcal{A}|_{C_c^2(\mathbb{R}^d)} = \mathcal{L}|_{C_c^2(\mathbb{R}^d)}$. Since $T_t$ is strongly continuous (see e.g., [Fuk80, Theorem 1.3.1 and Lemma 1.3.2]), for any $f \in \mathcal{D}(\mathcal{A})$ and $t \geq 0$,

$$
\left\| (T_t f - f) - \int_0^t T_s Af ds \right\|_{L^2} = 0
$$

(see e.g. [EK86, Proposition 1.5]). Hence for $g \in C_c^2(\mathbb{R}^d)$,

$$
T_t g(x) - g(x) = \int_0^t T_s \mathcal{L} g(x) ds, \quad a.e. \ x \in \mathbb{R}^d. \quad (2.17)
$$
Note that by (1.1), (1.2), (K_\eta) and (1.4), the absolute value of (2.15) is bounded as follows:
\[ |L^{(i)} g(x)| \leq c_1 \left( \kappa_0 \| \phi \|^2 \| g \|_\infty \int_0^r \frac{s}{\phi(s)} ds + \kappa_1 \| \nabla g \|_\infty \int_0^r \frac{s \eta}{\phi(s)} ds \right) + 2 \| g \|_\infty \frac{c_2}{\phi(r)}. \]  
(2.18)
Similar as (2.14), by (WS) with the fact that \( \eta > \alpha/2 > \alpha - 1 \), we have that
\[ \int_0^r \frac{s}{\phi(s)} ds \leq c_3 \frac{r^2}{\phi(r)} \quad \text{and} \quad \int_0^r \frac{s \eta}{\phi(s)} ds \leq c_4 \frac{r^{\eta+1}}{\phi(r)}. \]
Applying these inequalities to (2.18), we conclude that there exists a constant \( c = c(\phi, \eta, \kappa_0, \kappa_1) > 0 \) such that for any function \( g \in C^2_c(\mathbb{R}^d) \) and \( 0 < r < 1 \),
\[ |L^{(i)} g| \leq \frac{c}{\phi(r)} (r^2 \| \phi \|^2 g \|_\infty + r^{\eta+1} \| \nabla g \|_\infty + \| g \|_\infty). \]  
(2.19)
Thus, \( L = \sum_{i=1}^d L^{(i)} g(x) \) is bounded.
Let \( H_t(x) := \int_0^T T_s L g(x) ds \), then \( H_t(x) \leq t \| L g \|_\infty \) which is bounded for fixed \( t > 0 \). Since \( T_\varepsilon \), \( \varepsilon > 0 \), is strong Feller, \( T_\varepsilon H_{t-\varepsilon} \in C_b(\mathbb{R}^d), \varepsilon \in (0, t) \). Also since
\[ |H_t(x) - T_\varepsilon H_{t-\varepsilon}(x)| = |H_t(x)| \leq \varepsilon \| L u \|_\infty, \]
\( H_t \) is continuous and therefore (2.17) holds for any \( x \in \mathbb{R}^d \). Therefore, Markov property implies that
\[ M_t^g = g(X_t) - g(X_0) - \int_0^t L g(X_s) ds \]
is \( \mathbb{P} \)-martingale for any \( x \in \mathbb{R}^d \). Since \( |M_t^g| \leq 2 \| g \|_\infty + t \| L g \|_\infty \), (2.16) follows by the optional stopping theorem. \( \square \)

The following proof is almost the same as that of [GKK20, Lemma 2.3].

**Lemma 2.6.** There exists a constant \( c = c(\phi, \eta, \kappa_0, \kappa_1, d) > 0 \) such that, for any \( r \in (0, 1) \), \( Q \in \partial D \), and any stopping time \( S \) (with respect to the filtration of \( X \)), we have
\[ \mathbb{P}_x (|X_S - Q| \geq r) \leq \frac{c \mathbb{E}_x[S]}{\phi(r)} \quad \text{for } x \in B(Q, r/2). \]  
(2.20)
In particular, there exists a constant \( C_{2.6} = C_{2.6}(\phi, \eta, \kappa_0, \kappa_1, d) > 0 \) such that, for any \( r \in (0, 1) \), \( Q \in \partial D \), and any open sets \( U \) with \( D_Q(r, r) \subset U \subset D \),
\[ \mathbb{P}_x (X_{\tau_U} \in D) \leq C_{2.6} \frac{\mathbb{E}_x[\tau_U]}{\phi(r)} \quad \text{for } x \in D \cap B(Q, r/2 \sqrt{1 + \Lambda^2}). \]

**Proof**. For any \( x \in B(Q, r/2) \), this lemma is clear when \( \mathbb{E}_x[S] = \infty \) so we may assume that \( \mathbb{E}_x[S] < \infty \). Let \( g \in C^\infty_c(\mathbb{R}^d) \) be a function \( -1 \leq g \leq 0 \) satisfying
\[ g(y) = \begin{cases} -1, & \text{if } |y| < 1/2 \\ 0, & \text{if } |y| \geq 1. \end{cases} \]
Then
\[ \sum_{i=1}^d \left\| \frac{\partial g}{\partial y_i} \right\|_\infty + \sum_{i,j=1}^d \left\| \frac{\partial^2 g}{\partial y_i \partial y_j} \right\|_\infty = c_1 < \infty. \]
For any \( r \in (0, 1) \), define \( g_r(y) = \frac{g(y - Q)}{r} \). Then \( -1 \leq g_r \leq 0 \),
\[ g_r(y) = \begin{cases} -1, & \text{if } |y - Q| < r/2 \\ 0, & \text{if } |y - Q| \geq r, \end{cases} \]  
(2.21)
and
\[ \sum_{i=1}^d \left\| \frac{\partial g_r}{\partial y_i} \right\|_\infty < c_1 r^{-1} \quad \text{and} \quad \sum_{i,j=1}^d \left\| \frac{\partial^2 g_r}{\partial y_i \partial y_j} \right\|_\infty < c_1 r^{-2}. \]
According to (2.19), there exists $c_2 = c_2(\phi, \eta, \kappa_0, \kappa_1, d) > 0$ such that for $0 < r < 1$,

$$
\|L g_r\|_\infty \leq \sum_{i=1}^{d} \|L^{(i)} g_r\|_\infty \leq \frac{c_2}{\phi(r)}.
$$

(2.22)

Therefore, by Lemma 2.5 with (2.21) and (2.22), we obtain that for any $x \in B(Q, r/2)$ with $E_x[S] < \infty$,

$$
P_x(|X_S - Q| \geq r) = \mathbb{E}_x [1 + g_r(X_S) : |X_S - Q| \geq r] 
\leq \mathbb{E}_x [1 + g_r(X_S)] = -g_r(x) + \mathbb{E}_x [g_r(X_S)]
= \mathbb{E}_x \left[ \int_0^S L g_r(X_t) dt \right] \leq \|L g_r\|_\infty \mathbb{E}_x[S] \leq \frac{c_2}{\phi(r)} \mathbb{E}_x[S],
$$

which gives the proof of (2.20). For the second assertion, for any $r \in (0, 1)$ and $Q = (\tilde{Q}, Q^d) \in \partial D$, let $U$ be an open set satisfying that $D_Q(r, r) \subset U \subset D$. Since $\rho_Q(z) \geq r$ or $|\tilde{z} - Q| \geq r$ for $z = (\tilde{z}, z^d) \in D \setminus U$, using (2.6), we obtain that

$$
|z - Q| \geq \delta_D(z) \vee |\tilde{z} - Q| \geq \frac{r}{\sqrt{1 + \Lambda^2}}.
$$

Therefore, (2.20) and (WS) imply that for $x \in D \cap B(Q, r/2\sqrt{1 + \Lambda^2})$,

$$
P_x(X_{\tau_U} \in D) \leq P_x \left( |X_{\tau_U} - Q| \geq \frac{r}{\sqrt{1 + \Lambda^2}} \right) \leq c \frac{E_x[\tau_U]}{\phi(r)}.
$$

[Lemma 2.7]

For any $R_0 \leq R/2$, let $r_0 = R_0/4\Lambda$. For any $Q \in \partial D$ and $k \in \mathbb{N}$ satisfying that $2^{-k} < r_0/2$, let

$$
D_k := \left\{ y \in D \cap B(Q, R) : |\overline{y} - \overline{Q}| \leq r_0 - \frac{1}{2^k}, \frac{1}{2^k} < \rho_Q(y) < r_0 - \frac{1}{2^k} \right\}.
$$

Then for every $u \in \mathbb{R}^d$ with $|u| < 2^{-k}$, recall that the function $h_{R_0} = h_{Q, R_0}$ is defined in Proposition 2.4, then

$$
L_u^{(i)} h_{Q, R_0}(w) := \lim_{\epsilon \downarrow 0} \int_{\{|t|>\epsilon\}} (h_{Q, R_0}(w - u + e_i t) - h_{Q, R_0}(w - u)) J(w, w + e_i t) dt
$$

(2.23)
is well defined in $D_k$ and there exists $C_{2.7} = C_{2.7}(\phi, \kappa_0, \kappa_1, R, \Lambda, \eta, d) > 0$ independent of $Q \in \partial D$ and $k \in \mathbb{N}$ such that

$$
|L_u^{(i)} h_{Q, R_0}(w)| \leq \frac{C_{2.7}}{V(R_0)},
$$

for all $w \in D_k$ and $|u| < 2^{-k}$.

Proof. For any $Q \in \partial D$ and $R_0 < R/2$, let $h(y) := h_{Q, R_0}(y)$ and $r_0 = R_0/4\Lambda$. For any $w \in D_k$ and $|u| < 2^{-k}$, let $x := w - u$ and $\kappa_u(x, y) := \kappa(u + x, u + y)$. Then $x \in D_Q(r_0, r_0)$, and the integral of the right hand side of (2.23) is decomposed by

$$
G_{\epsilon}(x) + \kappa_u(x, x) \cdot L_u^{(i)} \overline{h}(x) + H(x)
$$

where

$$
G_{\epsilon}(x) := \int_{\{t < |t| \leq \epsilon\}} (h(x + e_i t) - h(x)) (\kappa_u(x, x + e_i t) - \kappa_u(x, x)) \nu^1(|t|) dt,
$$

$$
H(x) := \int_{\{1 < |t|\}} (h(x + e_i t) - h(x)) (\kappa_u(x, x + e_i t) - \kappa_u(x, x)) \nu^1(|t|) dt,
$$

and $L_u^{(i)} \overline{h}$ is defined in (2.5).
Since $\delta_D(y) \leq |y - Q| \leq R_0$ for any $y \in B(Q, R_0)$, $|h(x + e_1 t) - h(x)| \leq c_1$ by the definition of $h$. Therefore by (1.2) and (1.4) with the fact that $R_0 < 1$, we first obtain

$$|H(x)| \leq c_1 \int_{|t| < 1} \nu^1(|t|)dt \leq c_2 V(R_0)^{-1}. \quad (2.24)$$

For any $x, y \in D$, let $z_x, z_y \in \partial D$ be the points satisfying that $|z_x - x| = \delta_D(x)$ and $|z_y - y| = \delta_D(y)$. Let $x_0 \in D$ be a point constructing a parallelogram with $z_x, z_y, x$ so that $|x - x_0| = |z_x - z_y|$. Since

$$|z_x - z_y|^2 = |x - y|^2 + |\varphi_Q(x) - \varphi_Q(y)|^2 \leq (1 + \Lambda^2)|x - y|^2,$$

we have that

$$|\delta_D(y) - \delta_D(x)| \leq |x_0 - y| \leq |x - y| + |x - x_0| \leq 3\Lambda|x - y|.$$

Therefore, by (2.1)

$$|h(x + e_1 t) - h(x)| \leq c_3 V(|t|).$$

Also $|\kappa_u(x, x + e_1 t) - \kappa_u(x, x)| \leq (\kappa_1|t|^\eta \wedge 2\kappa_0)$ by (1.2) and (K). Therefore,

$$|G_\varepsilon(x)| \leq c_3 \left( \kappa_1 \int_{\{|t| \leq R_0\}} V(|t|)|t|^\eta \nu^1(|t|)dt + 2\kappa_0 \int_{\{|t| > R_0\}} V(|t|)\nu^1(|t|)dt \right) =: c_4 (I + II).$$

By (2.2) and (2.3) with $|t| \leq R_0$, for $\eta > \sigma/2$,

$$|I| \leq c_5 \int_{\{|t| \leq R_0\}} V(|t|)^{1/2} dt \leq \frac{c_6 R_0^{\sigma/2}}{V(R_0)} \int_{\{|t| \leq R_0\}} |t|^{-\sigma/2 + \eta - 1} dt \leq \frac{c_7}{V(R_0)}$$

and by (2.4), $|II| \leq \frac{c_8}{V(R_0)}$. Therefore

$$\lim_{\varepsilon \downarrow 0} G_\varepsilon(x) \text{ exists and } \lim_{\varepsilon \downarrow 0} |G_\varepsilon(x)| \leq \frac{c_7 + c_8}{V(R_0)}. \quad (2.25)$$

Finally from Proposition 2.4, $\lim_{\varepsilon \downarrow 0} L_Z^{(i), \varepsilon} h(x)$ exists and

$$\lim_{\varepsilon \downarrow 0} L_Z^{(i), \varepsilon} h(x) \leq \frac{C_{2.4}}{V(R_0)}. \quad (2.26)$$

Hence combining (2.24), (2.25) and (2.26), we have the conclusion.

We note the following Lévy system for $X$: for any $x \in \mathbb{R}^d$, stopping time $S$ (with respect to the filtration of $X$), and non-negative measurable function $f$ on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s, y, y) = 0$ for all $y \in \mathbb{R}^d$ and $s \geq 0$, we have that

$$\mathbb{E}_x \left[ \sum_{s \leq S} f(s, X_{s-}, X_s) \right] = \mathbb{E}_x \left[ \int_0^S \left( \sum_{i=1}^d \int_\mathbb{R} f(s, X_s, X_s + e_i h)J(X_s, X_s + e_i h)dh \right) ds \right], \quad (2.27)$$

(see, e.g., [CK03, Lemma 4.7] and [CK08, Appendix A] for the proof). We now arrive at the estimates on exit bounds for $X$.

**Theorem 2.8.** (1) There exist constants $\lambda_0 = \lambda_0(\phi, \kappa_0, \kappa_1, R, \Lambda, \eta, d) \geq 1$ and $C_{2.8.1} = C_{2.8.1}(\phi, \kappa_0, \Lambda, d) > 0$ such that for any $Q \in \partial D$, $s \leq R/(2\lambda_0)$ and $x \in D_Q(s, s)$,

$$\mathbb{E}_x[\tau_{D_Q(s, s)}] \leq C_{2.8.1} V(s) V(\delta_D(x)).$$

(2) There exists a constant $C_{2.8.2} = C_{2.8.2}(\phi, \kappa_0, \Lambda, d) > 0$ such that for any $Q \in \partial D$, $s \leq R/2$, $\lambda \geq 4\Lambda$ and $x \in D_Q(\Lambda^{-1}s, \Lambda^{-1}s)$,

$$\mathbb{P}_x \left( X_{\tau_{D_Q(\Lambda^{-1}s, \Lambda^{-1}s)}} \in D_Q(s, s) \setminus D_Q(s, \Lambda^{-1}s) \right) \geq C_{2.8.2} V(\delta_D(x))/V(s).$$

**Proof.** For any $Q \in \partial D$ and $R_0 \leq R/2$, define $h_{R_0}(y) := h_{Q,R_0}(y) = V(\delta_D(y))1_{D_D \cap B(Q, R_0)}(y)$. Consider a non-negative function $f \in C^\infty(\mathbb{R}^d)$ such that $f(y) = 0$ for $|y| > 1$ and $\int_{\mathbb{R}^d} f(y)dy = 0$. Theorem 2.8 then follows from the above estimates. □
1. For any \( k \geq 1 \), define \( f_k(y) := 2^{kd} f(2^k y) \) and \( h^{(k)} := f_k \ast h_{R_0} \in C^2_c(\mathbb{R}^d) \). Let
\[
D^\lambda_k := \left\{ y \in D \cap B(Q,R) : |\tilde{y} - \tilde{Q}| \leq \frac{R_0}{\lambda^k} - \frac{1}{2^k}, \frac{1}{2^k} < \rho_Q(y) < \frac{R_0}{\lambda^k} - \frac{1}{2^k} \right\} \text{ for } \lambda \geq 4A.
\]
Since \( h^{(k)} \in C^2_c(\mathbb{R}^d) \), \( \mathcal{L}^{(i)} h^{(k)} \) is well defined (see, the discussion before (2.15)) and for any \( w \in D^\lambda_k \),
\[
\begin{align*}
&\int_{|t| > \varepsilon} (h^{(k)}(w + e_t) - h^{(k)}(w)) J(w, w + e_t) dt \\
= &\int_{|t| > \varepsilon} \int_{\mathbb{R}^d} f_k(u) (h_{R_0}(w + e_t u) - h_{R_0}(w - u)) du J(w, w + e_t) dt \\
= &\int_{|u| < 2^{-k}} f_k(u) \left( \int_{|t| > \varepsilon} (h_{R_0}(w + e_t u) - h_{R_0}(w - u)) J(w, w + e_t) dt \right) du.
\end{align*}
\]
Lemma 2.7 implies that for \( w \in D^\lambda_k \) and \( u \in B(0, 2^{-k}) \),
\[
\lim_{\varepsilon \downarrow 0} \int_{|t| > \varepsilon} (h_{R_0}(w + e_t u) - h_{R_0}(w - u)) J(w, w + e_t) dt = \mathcal{L}^{(i)}_u h_{R_0}(w),
\]
and \(-C_{2.7} V(R_0)^{-1} \leq \mathcal{L}^{(i)}_u h_{R_0}(w) \leq C_{2.7} V(R_0)^{-1}\). So by letting \( \varepsilon \downarrow 0 \) in (2.28) with the dominated convergence theorem, it follows that for \( w \in D^\lambda_k \),
\[
|\mathcal{L}^{(i)} h^{(k)}(w)| = \int_{|u| < 2^{-k}} f_k(u) \mathcal{L}^{(i)}_u h_{R_0}(w) du \leq \frac{C_{2.7}}{V(R_0)} \int_{|u| < 2^{-k}} f_k(u) du = \frac{C_{2.7}}{V(R_0)}.
\]
Therefore,
\[
|\mathcal{L} h^{(k)}(w)| \leq \sum_{i=1}^d |\mathcal{L}^{(i)} h^{(k)}(w)| \leq d \frac{C_{2.7}}{V(R_0)}. \tag{2.29}
\]
Applying Lemma 2.5 to \( \tau_{D^\lambda_k} \) and \( h^{(k)} \) with (2.29), for any \( x \in D^\lambda_k \), we have
\[
\mathbb{E}_x \left[ h^{(k)}(X_{\tau_{D^\lambda_k}}) \right] - \frac{dC_{2.7}}{V(R_0)} \mathbb{E}_x \left[ \tau_{D^\lambda_k} \right] \leq h^{(k)}(x) \leq \mathbb{E}_x \left[ h^{(k)}(X_{\tau_{D^\lambda_k}}) \right] + \frac{dC_{2.7}}{V(R_0)} \mathbb{E}_x \left[ \tau_{D^\lambda_k} \right].
\]
Let \( D_0 := D_Q(\lambda^{-1} R_0, \lambda^{-1} R_0) \). For any \( x \in D_0 \subset B(Q,R_0) \), letting \( k \to \infty \) we have that
\[
\begin{align*}
V(\delta_D(x)) &\geq \mathbb{E}_x \left[ h_{R_0} \left( X_{\tau_{D_0}} \right) \right] - \frac{dC_{2.7}}{V(R_0)} \mathbb{E}_x \left[ \tau_{D_0} \right], \tag{2.30} \\
V(\delta_D(x)) &\leq \mathbb{E}_x \left[ h_{R_0} \left( X_{\tau_{D_0}} \right) \right] + \frac{dC_{2.7}}{V(R_0)} \mathbb{E}_x \left[ \tau_{D_0} \right]. \tag{2.31}
\end{align*}
\]
We may assume that \( R < 1/4 \). Let \( r_1 := R_0/\lambda < 1/(32A) \). For \( z \in D_0 \), since
\[
|z - Q|^2 \leq r_1^2 + (\varphi_Q(r_1) + r_1)^2 \leq r_1^2 + (\lambda r_1^2 + r_1)^2 < 3r_1^2, \quad D_0 \subset B(Q, \sqrt{3}r_1). \tag{2.32}
\]
In \( CS_Q \), let \( \Pi \) be the hyperplane tangent to \( \partial D \) at \( Q \). Let \( \Gamma(\tilde{y}) := \nabla \varphi_Q(\tilde{Q}) \cdot \tilde{y} \) be a function defined on \( \mathbb{R}^{d-1} \) which describes the hyperplane \( \Pi \) and let \( v := v_Q := \nabla \varphi(\tilde{x})/|\nabla \varphi(\tilde{x})| \) be a unit normal vector of the hyperplane \( \Pi \). Then there exists an index \( i_0 \in \{1, \ldots, d\} \) such that the angle between \( v \) and \( e_{i_0} \) denoted by \( \theta_1 := \arg(v,e_{i_0}) \in [0, \frac{\pi}{2}] \), in other words, \( (v,e_{i_0}) = \cos \theta_1 \in (\frac{1}{\sqrt{2}}, 1] \) and \( |\nabla \varphi_Q(\tilde{Q})| = |\tan \theta_1| < 1 \). For any \( y \in \mathbb{R}^d \), we use the new notation \( \tilde{y} := (y^1, \ldots, y^{i_0}, y^{i_0+1}, \ldots, y^d) \in \mathbb{R}^{d-1} \). Define two sets \( D_1, D_2 \subset \mathbb{D} \) by
\[
\begin{align*}
P_1 &:= Q + 5r_1 e_{i_0} \quad \text{and} \quad D_1 := \{ y \in \mathbb{R}^d : |\tilde{y} - \tilde{P}_1^{i_0}| < \sqrt{3}r_1, |y^{i_0} - \tilde{P}_1^{i_0}| < r_1 \}; \\
P_2 &:= Q + (\frac{5r_1}{\cos \theta_1}) e_{i_0} \quad \text{and} \quad D_2 := \{ y \in \mathbb{R}^d : |\tilde{y} - \tilde{P}_2^{i_0}| < \sqrt{3}r_1, |y^{i_0} - \tilde{P}_2^{i_0}| < r_1 \}.
\end{align*}
\]
Then

\[ D_1 \subset D_0^* \cap D \quad \text{and} \quad D_2 \subset D_Q(R_0, R_0) \setminus D_Q(R_0, r_1). \]

More precisely, for \( y \in D_1 \), since \( |y - P_1| \leq 2r_1, |y - Q| \geq |Q - P_1| - |P_1 - y| > 5r_1 - 2r_1 = 3r_1 \) and (2.32) indicates that \( D_1 \subset D_0^* \). Let \( B(O, R) \) be the interior ball tangent to \( Q \) with a center \( O \) and radius \( R \). We may assume that \( \Lambda > 2 \) so that \( r_1 < R/16 \). Since \( |Q - O| = R \) and \( \cos \theta_1 \in (\frac{1}{\sqrt{2}}, 1] \),

\[ |O - P_1|^2 = (|O - Q| \sin \theta_1)^2 + (|O - Q| \cos \theta_1 - 5r_1)^2 = R^2 - 10r_1R \cos \theta_1 + 25r_1^2 < (R - 2r_1)^2. \]

Therefore, for any \( y \in D_1 \),

\[ |O - y| \leq |O - P_1| + |P_1 - y| < (R - 2r_1) + 2r_1 = R, \quad \text{hence} \quad D_1 \subset B(O, R) \subset D. \]

Also for any \( y \in D_2 \) in \( CS_Q \), note that

\[ |\tilde{P}_2| = \sin \theta_1 \cdot \left( \frac{5r_1}{\cos \theta_1} \right) = 5r_1 \tan \theta_1 < 5r_1 \quad \text{and} \quad |P_2^d| = \cos \theta_1 \cdot \left( \frac{5r_1}{\cos \theta_1} \right) = 5r_1. \]

Then

\[ |\tilde{y}| \leq \left| \tilde{y} - \tilde{P}_2 \right| + \left| \tilde{P}_2 \right| < 2r_1 + 5r_1 = 7r_1 < R_0. \]

Also since \( 3r_1 < |P_2^d| - |y^d - P_2^d| \leq |y^d| \leq |y^d - P_2^d| + |P_2^d| < 7r_1 \) and the fact that \( r_1 < \frac{1}{\sqrt{2}} \),

\[ r_1 < 3r_1 - \Lambda(7r_1)^2 \leq \left| \rho_Q(y) \right| = \left| y^d \right| - \varphi_Q(\tilde{|y|}) < R_0 \]

therefore, \( D_2 \subset D_Q(R_0, R_0) \setminus D_Q(R_0, r_1) \). By (2.30) with \( D_1 \) and (2.31) with \( D_2 \), we obtain our conclusion (1) and (2), respectively.

Let \( z \in D_0 \) and \( y := z + le_{1\alpha} \in D_1 \) for some \( l \geq 0 \). Since \( z \in B(Q, \sqrt{3}r_1) \) by (2.32),

\[ 2r_1 < |Q^{1\alpha} - P_1^{1\alpha}| - |y^{1\alpha} - P_1^{1\alpha}| - |z^{1\alpha} - Q^{1\alpha}| \leq |y^{1\alpha} - z^{1\alpha}| = l \]

\[ \leq |Q^{1\alpha} - P_1^{1\alpha}| + |y^{1\alpha} - P_1^{1\alpha}| + |z^{1\alpha} - Q^{1\alpha}| < 5r_1 + r_1 + \sqrt{3}r_1 < 8r_1 < \frac{1}{\sqrt{2}}. \]

Note that \( \rho_Q(y) \geq y^d - \Lambda|\tilde{y}|^2 \), and with \( \theta_1 \) such that \( \sin \theta_1 < \frac{\pi}{2} < \cos \theta_1 \),

\[ y^d = z^d + l \cos \theta_1 \geq \frac{l}{\sqrt{2}}, \quad \Lambda|\tilde{y}|^2 = \Lambda(|\tilde{z}| + l \sin \theta_1)^2 \leq \Lambda\left(\frac{l}{\sqrt{2}} + \frac{l}{\sqrt{2}}\right)^2 < \frac{9}{16}l. \]

So by (2.6)

\[ \delta_D(y)\sqrt{1 + \Lambda^2} \geq \rho_Q(y) > l/16. \]

Therefore, using (1.2) and (1.3) with the above observations, (2.27) implies that

\[ E_x \left[ h_{R_0}(X_{T_D_0}) \right] \geq \kappa_0^{-1} E_x \left[ \int_{D_1} \int_{0}^{T_{D_0}} h_{R_0}(y) \nu^1(|X_t^{1\alpha} - y^{1\alpha}|) \, dt \, dy \right] \geq \kappa_0^{-1} E_x [\tau_{D_0}] \int_{2r_1}^{8r_1} V(l/16\sqrt{(1 + \Lambda^2)}) \nu^1(|l|) \, dl \geq c_1 E_x [\tau_{D_0}] C^{-1}_{21} \int_{2r_1}^{8r_1} V(l/16\sqrt{(1 + \Lambda^2)}) \nu^1(|l|) \, dl \geq C_2 E_x [\tau_{D_0}] V(\lambda^{-1} R_0)^{-1} \]

for some \( C_2 := C_2(\phi, \kappa_0, d) > 0 \). Here we also used the fact that \( V \) is non-decreasing, (2.2) and Proposition 2.1. Let \( \lambda_0 := \left( 2C_1 dC_{2,7}/C_2 \right)^{2/\alpha} \gamma 4\Lambda \geq 1 \). Using (2.3), \( V(\lambda^{-1} R_0) \leq V(\lambda_0^{-1} R_0) \leq C_\gamma \lambda_0^{-d/2} V(R_0) \) for any \( \lambda \geq \lambda_0 \). Then combining (2.30) and (2.33), we have that for \( \lambda \geq \lambda_0 \),

\[ V(\delta_D(x)) \geq (C_2 V(\lambda^{-1} R_0)^{-1} - dC_{2,7} V(R_0)^{-1}) E_x [\tau_{D_0}] \geq (C_2/2) V(\lambda^{-1} R_0)^{-1} E_x [\tau_{D_0}] \]

Thus, we have proved (1) with \( \lambda_0 \) and \( s = \lambda^{-1} R_0 \) where \( \lambda \geq \lambda_0 \).
For the second assertion, according to the definition of $h_{R_0}$, we first note that

$$
\mathbb{E}_x[h_{R_0}(X_{\tau_{D_0}})] \leq c_3 V(R_0) \mathbb{P}_x(X_{\tau_{D_0}} \in D).
$$

Combining this fact and Lemma 2.6 with $D_0 = D_Q(\lambda^{-1}R_0, \lambda^{-1}R_0)$, (2.31) together with (2.2) yields that for $x \in D_0$,

$$
V(\delta_D(x)) \leq c_4 \left( V(R_0) \mathbb{P}_x(X_{\tau_{D_0}} \in D) + V(R_0)^{-1} \mathbb{E}_x[\tau_{D_0}] \right)
\leq c_4 \left( C_{2.6} V(R_0) \phi(\lambda^{-1}R_0)^{-1} + V(R_0)^{-1} \mathbb{E}_x[\tau_{D_0}] \right)
\leq c_3 V(R_0) \left( V(\lambda^{-1}R_0)^{-2} + V(R_0)^{-2} \right) \mathbb{E}_x[\tau_{D_0}].
$$

Recall that

$$
P_2 = Q + \left( \frac{5r_1}{\cos \theta} \right) \varepsilon_i \text{ and } D_2 = \{ y \in \mathbb{R}^d : |\hat{y}^i - \hat{\rho}_2^i| < \sqrt{3}r_1, |y^i - P_2^i| < r_1 \}.
$$

For $z \in D_0$ and $y = z + le_i \in D_2$ with some $l > 0$,

$$
r_1 < |P_2^i - Q^i| - |Q^i - z^i| - |y^i - P_2^i| \leq |y^i - z^i| = l
\leq |P_2^i - Q^i| + |Q^i - z^i| + |y^i - P_2^i| \leq \frac{5r_1}{\cos \theta} + \sqrt{3}r_1 + r_1 \leq 10r_1.
$$

Therefore, by (1.2), (2.27), (2.2) and (2.3), we obtain that

$$
\mathbb{P}_x \left( X_{\tau_{D_0}} \in (D_Q(R_0, R_0) \setminus D_Q(R_0, r_1)) \right) \geq \mathbb{P}_x \left( X_{\tau_{D_0}} \in D_2 \right)
\geq \kappa_0^{-1} \mathbb{E}_x \left[ \int_{D_2} \int_0^{\tau_{D_0}} \nu^1(|X^i_t - y^i|) dt \, dy \right]
\geq c_6 \mathbb{E}[\tau_{D_0}] \int_{r_1}^{2r_1} \nu^1(||l||)dl
\geq c_7 \mathbb{E}[\tau_{D_0}] \int_{r_1}^{2r_1} \frac{1}{V(||l||)^2} \, dl \geq c_8 V(r_1)^{-2} \mathbb{E}[\tau_{D_0}] \cdot (2.35)
$$

Recall that $r_1 = \lambda^{-1}R_0$, combining (2.34), (2.35), and the fact that $V$ is non-decreasing, we conclude that for $\lambda \geq 4\Lambda$ and for $x \in B(Q, R_0/(2\lambda\sqrt{1 + \Lambda^2}))$,

$$
V(\delta_D(x)) \leq c_8 \mathbb{E}[\tau_{D_0}] \left( V(\lambda^{-1}R_0)^{-2} + V(R_0)^{-2} \right) \mathbb{E}[\tau_{D_0}] \cdot (2.34)
\leq c_0 V(R_0) \mathbb{P}_x \left( X_{\tau_{D_0}} \in (D_Q(R_0, R_0) \setminus D_Q(R_0, \lambda^{-1}R_0)) \right).
$$

Thus, we have proved (2) with $s = R_0$.

3. Regularity of Dirichlet heat kernel and upper bound estimate

Let $U \subset \mathbb{R}^d$ be an open set. For any $t > 0, x, y \in \mathbb{R}^d$, we consider the function

$$
p_U(t, x, y) := p(t, x, y) - \mathbb{E}_x[p(t - \tau_U, X_{\tau_U}, y) : \tau_U < t].
$$

It can be pointwisely defined and it is jointly measurable on $(0, \infty) \times (\mathbb{R}^d \setminus \partial U) \times (\mathbb{R}^d \setminus \partial U)$.

In Subsection 3.1, we obtain the regularity result of $p_U(t, x, y)$ in Theorem 3.4, which states that $p_U(t, x, y)$ admits a continuous refinement. In Subsection 3.2, we obtain the short time sharp upper bound for the Dirichlet heat kernel, which is the result in Theorem 1.1.

3.1. Regularity of Dirichlet heat kernel. Using the same methods as in the proof of [CZ95, Theorem 2.4], one obtains the following fundamental properties for $p_U(t, x, y)$.

**Lemma 3.1.** For any $t > 0, A \in \mathcal{B}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$
\mathbb{P}_x(t < \tau_U : X_t \in A) = \int_A p_U(t, x, y) dy.
$$
\( p_U(t, x, y) \) is almost surely symmetric on \( \mathbb{R}^d \times \mathbb{R}^d \), that is, for any \( t > 0 \),
\[
  p_U(t, x, y) = p_U(t, y, x) \quad \text{for a.e. } x, y \in U \times U.
\]
Moreover, we have for any \( s, t > 0 \) and \( x \in U \),
\[
  p_U(t + s, x, y) = \int_{\mathbb{R}^d} p_U(t, x, z) p_U(s, z, y) dz \quad \text{for a.e. } y \in U.
\]

The above properties determine a semigroup, that is, for any non-negative measurable function \( f \) on \( U \) or \( f \in L^2(U) \cap L^\infty(U) \),
\[
  P_t^U f(x) := \int_U p_U(t, x, z) f(z) dz \quad \text{for } t > 0, \ x \in \mathbb{R}^d \setminus \partial U,
\]
and the heat semigroup is pointwisely defined on \( (0, \infty) \times (\mathbb{R}^d \setminus \partial U) \). Moreover, for any \( \lambda > 0 \), we define the \( \lambda \)-potential as follows:
\[
  U^\lambda f(x) = \mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t} f(X^U_t) dt \right] = \int_0^\infty e^{-\lambda t} P_t^U f(x) dt \quad \text{for } x \in \mathbb{R}^d \setminus \partial U,
\]
and it is also defined pointwisely on \( \mathbb{R}^d \setminus \partial U \). The following results are established in [KW22, Proposition 4.2].

**Proposition 3.2.** For any \( r > 0 \), there exist \( a_i := a_i(\phi, d) > 0, i = 1, 2 \) such that
\[
  a_1 \phi(r) \leq \mathbb{E}_x[\tau_{B(x, r)}] \leq a_2 \phi(r).
\]

A function \( h : \mathbb{R}^d \to \mathbb{R} \) is said to be harmonic in the open set \( D \) with respect to \( X \) if for every open set \( U \subset D \) whose closure is a compact subset of \( D \), \( \mathbb{E}_x[|h(X_{\tau_U})|] < \infty \) and
\[
  h(x) = \mathbb{E}_x[h(X_{\tau_U})] \quad \text{for every } x \in U.
\]
It is said that \( h \) is regular harmonic in \( D \) with respect to \( X \) if \( h \) is harmonic in \( D \) with respect to \( X \) and (3.1) holds for \( U = D \).

We now have the Hölder continuity of harmonic functions according to [KW22, Theorem 4.6].

For any \( x_0 \in \mathbb{R}^d \) and \( r \in (0, 1] \), suppose that \( h \) is harmonic in \( B(x_0, r) \) with respect to \( X \) and bounded in \( \mathbb{R}^d \). Then there exist constants \( a_3, \beta > 0 \) depending on \( \phi \) and \( d \) such that
\[
  |h(x) - h(y)| \leq a_3 \left( \frac{|x - y|}{r} \right)^\beta \|h\|_\infty \quad \text{for any } x, y \in B(x_0, r/2).
\]

**Proposition 3.3.** Let \( r \in (0, 1] \) and \( B(x_0, r) \subset U \), then there exists a constant \( c = c(\phi, \lambda, d) > 0 \) such that for any \( x, y \in B(x_0, r/2) \) and \( f \in L^\infty(U) \cap L^2(U) \),
\[
  \|U^\lambda f(x) - U^\lambda f(y)\| \leq c \left( \phi(r) + \frac{|x - y|}{r^\beta} \right) \|f\|_{L^\infty(U)}.
\]

**Proof.** For simplicity of notations, we write \( \tau_x := \tau_{B(x, r)} \). For any \( f \in L^\infty(U) \cap L^2(U) \), by the strong Markov property, we have
\[
  U^\lambda f(x) = \mathbb{E}_x \left[ \int_0^{\tau_x} e^{-\lambda t} f(X^U_t) dt \right] + \mathbb{E}_x \left[ (e^{-\lambda \tau_x} - 1) U^\lambda f(X^U_{\tau_x}) \right] + \mathbb{E}_x \left[ U^\lambda f(X^U_{\tau_x}) \right].
\]
We get similar expressions when \( x \) is replaced by \( y \). By Proposition 3.2, the mean value theorem, and the fact that \( \|U^\lambda f\|_\infty \leq \frac{1}{\lambda} \|f\|_\infty \), we obtain that
\[
  \|U^\lambda f(x) - U^\lambda f(y)\| \leq \left( \mathbb{E}_x[\tau_x] + \mathbb{E}_y[\tau_y] \right) \|f\|_\infty + \lambda \|U^\lambda f\|_\infty + \left| \mathbb{E}_x[U^\lambda f(X^U_{\tau_x})] - \mathbb{E}_y[U^\lambda f(X^U_{\tau_y})] \right| \leq 4a_2 \phi(r) \|f\|_\infty + \left| \mathbb{E}_x[U^\lambda f(X^U_{\tau_x})] - \mathbb{E}_y[U^\lambda f(X^U_{\tau_y})] \right|.
\]
Proposition 3.3: yield that, for fixed \( x, y \in B(x_0, r/2) \),

\[
\left| \mathbb{E}_x[U^\lambda f(X^U_{t_\ast})] - \mathbb{E}_y[U^\lambda f(X^U_{t_\ast})] \right| \leq a_3 \frac{|x - y|^\beta}{r^\beta} \|U^\lambda f\|_{\infty}.
\]

Combining the above estimates, we obtain that for any \( x, y \in B(x_0, r/2) \),

\[
|U^\lambda f(x) - U^\lambda f(y)| \leq c \left( \phi(r) + \frac{|x - y|^\beta}{r^\beta} \right) \|f\|_{L^\infty(U)}
\]

where \( c \) is a constant depending on \( \phi, d, \lambda \).

Since \( \{P^U_t, t > 0\} \) is a heat semigroup on \( L^2(U) \), according to spectral theory, there exists a spectral family \( \{E_\mu, \mu \in \mathbb{R}\} \) such that

\[
f = \int_0^\infty dE_\mu(f), \quad P^U_t f = \int_0^\infty e^{-\mu t} dE_\mu(f) \quad \text{and} \quad U^\lambda f = \int_0^\infty \frac{1}{\lambda + \mu} dE_\mu(f).
\]

For any \( f \in L^\infty(U) \cap L^2(U) \), set

\[
h := h(f) := \int_0^\infty (\lambda + \mu)e^{-\mu t} dE_\mu(f).
\]

For any \( g \in L^1(\mathbb{R}^d) \), by (1.5), we have that for any \( t > 0 \),

\[
\|P^U_t g\|_{\infty} = \left\| \int_U p_U(t, \cdot; z) g(z) dz \right\|_{\infty} \leq \left\| \int_U |p(t, t; z)| g(z) |dz| \right\|_{\infty} \leq c[\phi^{-1}(t)]^{-d} \|g\|_1.
\]

Correspondingly,

\[
\|P^U_t g\|_2 \leq \sqrt{\|P^U_t g\|_1 \|P^U_t g\|_{\infty}} \leq \sqrt{\|g\|_1 c[\phi^{-1}(t)]^{-d} \|g\|_1} \leq \sqrt{c[\phi^{-1}(t)]^{-d/2}} \|g\|_1.
\]

Thus, we conclude that \( P^U_t g \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \). Then by Cauchy-Schwartz inequality with the fact that \( \sup_{\mu > 0} (\lambda + \mu)e^{-\mu t/2} \leq 2(t^{-1} \lor \lambda) \),

\[
\langle h, g \rangle = \int_0^\infty (\lambda + \mu)e^{-\mu t} d\langle E_\mu(f), E_\mu(g) \rangle
\]

\[
\leq \left( \int_0^\infty (\lambda + \mu)e^{-\mu t} d\langle E_\mu(f), E_\mu(f) \rangle \right)^{1/2} \left( \int_0^\infty (\lambda + \mu)e^{-\mu t} d\langle E_\mu(g), E_\mu(g) \rangle \right)^{1/2}
\]

\[
\leq 2(t^{-1} \lor \lambda) \left( \int_0^\infty d\langle E_\mu(f), E_\mu(f) \rangle \right)^{1/2} \left( \int_0^\infty e^{-\mu t} d\langle E_\mu(g), E_\mu(g) \rangle \right)^{1/2}
\]

\[
= 2(t^{-1} \lor \lambda) \|f\|_{L^2(U)} \|P^U_t g\|_2 \leq c(t^{-1} \lor \lambda)[\phi^{-1}(t)]^{-d/2} \|f\|_{L^2(U)} \|g\|_1.
\]

Since we take \( g \in L^1(\mathbb{R}^d) \) to be arbitrary, we get

\[
\|h\|_{\infty} \leq c(t^{-1} \lor \lambda)[\phi^{-1}(t)]^{-d/2} \|f\|_{L^2(U)}.
\]

**Theorem 3.4.** For \( f \in L^2(U) \cap L^\infty(U) \), \( P^U_t f \) is equal a.e. to a function that is (locally) Hölder continuous. Hence, we can refine \( p_U(t, x, y) \) to be jointly (locally) Hölder continuous for any \( t > 0 \) and \( x, y \in U \).

**Proof.** The above discussion and Proposition 3.3 yield that, for fixed \( t > 0 \), \( r \in (0, 1] \) and \( B(x_0, r) \subset U \), there exists a constant \( c_1 \) such that for any \( x, y \in B(x_0, r/2) \),

\[
|P^U_t f(x) - P^U_t f(y)| = |U^\lambda h(x) - U^\lambda h(y)|
\]

\[
\leq c_1 \left( \phi(r) + \frac{|x - y|^\beta}{r^\beta} \right) \|h\|_{L^\infty(U)}
\]

\[
\leq c_1(t^{-1} \lor \lambda)[\phi^{-1}(t)]^{-d/2} \left( \phi(r) + \frac{|x - y|^\beta}{r^\beta} \right) \|f\|_{L^2(U)}.
\]
For any fixed compact set $K \subset U$ and any points $x, y \in K$, let $x_0 = x$ and $\delta_K = \frac{1}{4} (\text{dist}(K, \partial U) \wedge \text{dist}(K, \partial U)^2 \wedge 1)$. If $|x - y| < \delta_K$, then with $r := |x - y|^{1/2}$, we have

$$\frac{1}{2} \text{dist}(K, \partial U) \geq \sqrt{\delta_K} > |x - y|^{1/2} = r \geq 2|x - y|.$$ 

Then applying (3.2) with $x_0 = x$ and $r = |x - y|^{1/2}$, and (WS) with $r < 1$,

$$|P_t^f(x) - P_t^f(y)| \leq c_2 (t^{-1} \vee \lambda) |\phi^{-1}(t)|^{-d/2} \left(|x - y|^{d/2} + |x - y|^\beta/2\right) ||f||_{L^2(U)}.$$ 

If $|x - y| \geq \delta_K$, by the definition of $P_t^f$ and (1.5), we have that for any $t > 0$ and $z \in U$,

$$|P_t^f(x)| = \left| \int_U p_U(t, z, y) f(y) dy \right| \leq \sqrt{\int_U p_U(t, z, y)^2 dy} ||f||_{L^2(U)} \leq \sqrt{p(2t, z, z)} ||f||_{L^2(U)} \leq \sqrt{c_3 (\phi^{-1}(2t))^{-d/2}} ||f||_{L^2(U)}.$$

Hence, for $x, y \in K$ with $|x - y| \geq \delta_K$,

$$|P_t^f(x) - P_t^f(y)| \leq |P_t^f(x)| + |P_t^f(y)| \leq 2 \sqrt{c_3 (\phi^{-1}(2t))^{-d/2}} ||f||_{L^2(U)} \leq \sqrt{c_3 (\phi^{-1}(2t))^{-d/2}} \left( \frac{|x - y|^{d/2}}{\delta_K^{d/2}} + \frac{|x - y|^\beta/2}{\delta_K^{\beta/2}} \right) ||f||_{L^2(U)}.$$

We have our first assertion by combining the above two discussions and (WS), that is, for any compact set $K \subset U$ and $x, y \in K$,

$$|P_t^f(x) - P_t^f(y)| \leq c_4 (\phi, d, \text{dist}(K, \partial U)) |x - y|^{(d + \beta)/2} |\phi^{-1}(t)|^{-d/2} ||f||_{L^2(U)}. \quad (3.3)$$

Let $\tilde{f}(z) = p_U(t/2, z, y)$. According to Lemma 3.1, since

$$p_U(t, x, y) = \int_U p_U(t/2, z, x) p_U(t/2, z, y) dz = P_{t/2}^U \tilde{f}(x),$$

by (3.3) and (1.5), we have that for any compact set $K \subset U$ and $x, x', y \in K$,

$$|p_U(t, x, y) - p_U(t, x', y)| = |P_{t/2}^U \tilde{f}(x) - P_{t/2}^U \tilde{f}(x')| \leq c_4 |x - x'|^{(d + \beta)/2} |\phi^{-1}(t)|^{-d/2} ||p_U(t/2, \cdot, y)||_{L^2(U)} \leq c_5 |x - x'|^{(d + \beta)/2} |\phi^{-1}(t)|^{-d/2} \sqrt{p(t, y, y)} \leq c_6 |x - x'|^{(d + \beta)/2} |\phi^{-1}(t)|^{-d}.$$

With the symmetry property in Lemma 3.1, $p_U(t, x, y)$ is jointly Hölder continuous with constants independent of $x, y$ on $K \times K$, hence, it is jointly (locally) Hölder continuous on $U \times U$. 

3.2. The upper bound. We first obtain a key lemma, Lemma 3.5, that serves as a guideline to obtain the upper bound for $p_U(t, x, y)$. Together with Theorem 2.8 and (1.5), then we derive the upper bound with one boundary decay, see Proposition 3.7. Finally, we prove Theorem 1.1.

**Lemma 3.5.** Let $U \subset \mathbb{R}^d$ be an open set and $U_1, U_3 \subset U$ be disjoint open sets with dist($U_1, U_3$) > 0. Define $U_2 := U \setminus (U_1 \cup U_3)$. Then for any $t > 0$, $x \in U_1$ and $y \in U_3$,

$$p_U(t, x, y) \leq \mathbb{P}_x (X_{\tau_{U_1}} \in U_2) \sup_{t/2 < s < t, z \in U_2} p_U(s, z, y) + \frac{2}{t} \mathbb{E}_x [\tau_{U_1}] \sup_{z \in U_1} p_U(t/2, z, y) \quad (3.4)$$

$$+ \int_0^{t/2} \int_{U_1} p_U(s, x, u) \left( \sum_{i=1}^d \int_{h \in \mathbb{R}} p_U(t - s, u + e_i h, y) dh \cdot \sup_{u + e_i h \in U_3} J(u, u + e_i h) \right) duds.$$
Proof. Let \( x \in U_1 \) and \( f \) be a non-negative function in \( L^1(U) \cap L^\infty(U) \). By the strong Markov property of \( X \), we first obtain

\[
P^U_t f(x) = \mathbb{E}_x[f(X_t); t < \tau_U] = \mathbb{E}_x[f(X_t); t < \tau_{U_1}] + \mathbb{E}_x[f(X^U_t); \tau_{U_1} < t]
\]  

(3.5)

\[
P^U_t f(x) = \mathbb{E}_x[P^U_{t-\tau_{U_1}} f(X_{\tau_{U_1}}); \tau_{U_1} < t, X_{\tau_{U_1}} \in U_2] + \mathbb{E}_x[P^U_{t-\tau_{U_1}} f(X_{\tau_{U_1}}); \tau_{U_1} < t, X_{\tau_{U_1}} \in U_3]
\]

\[
\leq P^U_{t-\tau_{U_1}} f(x) + \sup_{s < t, x \in U_2} \mathbb{P}_x[X_{\tau_{U_1}} \in U_2] + \mathbb{E}_x[P^U_{t-\tau_{U_1}} f(X_{\tau_{U_1}}); \tau_{U_1} < t, X_{\tau_{U_1}} \in U_3].
\]

By the Lévy system of \( X \) in (2.27), we have

\[
\mathbb{E}_x[P^U_{t-\tau_{U_1}} f(X_{\tau_{U_1}}); \tau_{U_1} < t, X_{\tau_{U_1}} \in U_3] = \int_0^t \int_{U_1} \int_{U_1} \left( \sum_{i=1}^d \int_{\{h \in \mathbb{R}^d; u + e_i h \in U_3\}} P^U_{t-s} f(u + e_i h)J(u, u + e_i h) dh \right) duds
\]

(3.6)

Letting \( f(\cdot) = p_U(t, \cdot, y) \) in (3.5) and (3.6), the semigroup property implies that for any \( y \in U_3 \),

\[
p_U(2t, x, y) \leq \int_{\mathbb{R}^d} p_U(t, x, z)p_U(t, z, y) dz + \sup_{t < s < 2t, x \in U_2} p_U(s, z, y) \mathbb{P}_x[X_{\tau_{U_1}} \in U_2] + \int_0^t \int_{U_1} \int_{U_1} \left( \sum_{i=1}^d \int_{\{h \in \mathbb{R}^d; u + e_i h \in U_3\}} p_U(2t - s, u + e_i h, y) dh \sup_{u \in U_1} J(u, u + e_i h) \right) duds.
\]

(3.7)

Since the first term in (3.7) satisfies

\[
\int_{\mathbb{R}^d} p_U(t, x, z)p_U(t, z, y) dz \leq \sup_{z \in U_1} p_U(t, z, y) \mathbb{P}_x(\tau_{U_1} > t) \leq \sup_{z \in U_1} p_U(t, z, y) \frac{\mathbb{E}_x[\tau_{U_1}]}{t},
\]

we achieve the result by replacing \( 2t \) by \( t \) in (3.7).

\[\]  

Lemma 3.6. For any \( T > 0 \), there exists a constant \( C_{3.6} = C_{3.6}(\phi, \kappa_0, \kappa_1, R, \Lambda, T, \eta, d) > 0 \) such that for any \( x \in D \) and \( t \leq T \),

\[
\mathbb{P}_x(\tau_D \geq t) \leq C_{3.6} \left( 1 + \frac{V(\delta_D(x))}{\sqrt{t}} \right).
\]

Proof. Let \( \alpha_T := T^{-1}[V(R/(4\lambda_0))^2] \), where \( \lambda_0 \geq 1 \) is the constant from Theorem 2.8. By the definition of \( (R, \Lambda) \), we may assume that \( R < 1 \) and \( \Lambda > 2 \). Also it is enough to consider the case that \( V(2\Lambda \cdot \delta_D(x)) < \sqrt{\alpha_T} \), so that \( \delta_D(x) < V^{-1}(\sqrt{\alpha_T})/\sqrt{1 + \Lambda^2} \leq R/(4\lambda_0) < 1/4 \). Now let \( r_t := V^{-1}(\sqrt{\alpha_T}) \) and \( Q \in \partial D \) such that \( \delta_D(x) = |x - Q| \). Applying Lemma 2.6 with \( U := D_Q(2r_t, 2r_t) \),

\[
\mathbb{P}_x(\tau_D \geq t) = \mathbb{P}_x(\tau_U \geq t, \tau_D = \tau_U) + \mathbb{P}_x(\tau_D > \tau_U, \tau_D \geq t) \leq \mathbb{P}_x(\tau_U \geq t) + \mathbb{P}_x(X_{\tau_U} \in D) \leq \frac{\mathbb{E}_x[\tau_U]}{t} + C_{2.6} \frac{\mathbb{E}_x[\tau_U]}{\phi(2r_t)}.
\]

Note that by (2.2) and (2.3), \( \phi(2r_t) \approx [V(2r_t)]^\frac{d}{2} \approx \alpha_T t \). Applying Theorem 2.8(1) yields that

\[
\mathbb{P}_x(\tau_D \geq t) \leq c_1(1 + C_{2.6}^{-1} \alpha_T) C_{2.8.1} \frac{V(2r_t) V(\delta_D(x))}{t} \leq c_2 \frac{V(\delta_D(x))}{\sqrt{t}}.
\]

\[\]  

Proposition 3.7. For any \( T > 0 \), there exists a constant \( C_{3.7} = C_{3.7}(\phi, \kappa_0, \kappa_1, R, \Lambda, T, \eta, d) > 0 \) such that for any \( x, y \in D \) and \( t \leq T \),

\[
p_D(t, x, y) \leq C_{3.7} \Psi(t, x)p(t, x, y),
\]
where $\Psi(t, x)$ is given in (1.6).

**Proof.** For any fixed $T > 0$, let $t \in (0, T]$. We choose and fix a constant $a < a_T := T^{-1}[V(R/(4\lambda_0))]^2$ such that $r_t := V^{-1}(\sqrt{a}) < 1$ for all $t \in [0, T]$. It suffices to show the result for $\delta_D(x) \leq r_t/2\sqrt{1 + N^2}$. We deal with two cases separately.

**Case 1.** For all $i \in \{1, \ldots, d\}$, let $|x^i - y^i| < 4r_t$. By the semigroup property, (1.5) and Lemma 3.6, we have

$$p_D(t, x, y) \leq \sup_{z \in D} p_D(t/2, z, y) \int_D p_D(t/2, x, z)dz \leq c_1[\phi^{-1}(t/2)]^{-d}p_x(TD > t/2) \leq c_2[\phi^{-1}(t/2)]^{-d} \left(1 \wedge \frac{V(\delta_D(x))}{\sqrt{t}}\right).$$

Since $|x^i - y^i| < 4r_t \approx \phi^{-1}(t)$, according to (1.5) and (2.2), we have our desired result.

**Case 2.** There exists $i \in \{1, \ldots, d\}$ such that $|x^i - y^i| \geq 4r_t$. Consider the set of indices of the coordinate for $x, y$ that

$$E := \{i \in \{1, \ldots, d\} : |x^i - y^i| \geq 4r_t\}.$$ 

Let $Q \in \partial D$ such that $\delta_D(x) = |x - Q|$. Define $U_1 := D_Q(r_t, r_t)$ and $U_3 := \{z \in D : \text{there exists } i \in E \text{ such that } |z^i - x^i| > |x^i - y^i|/2\}$. Then dist$(U_1, U_3) > 0$. Let $U_2 := D \setminus (U_1 \cup U_3)$. To obtain our assertion we will estimate the right-hand side of (3.4). Note that $x \in U_1$, $y \in U_3$ and if $i \notin E$, then

$$|x^i - y^i| < 4V^{-1}(\sqrt{a}) \approx \phi^{-1}(t).$$

For any $z \in U_2$, if $i \in E$,

$$|z^i - y^i| \geq |x^i - y^i| - |z^i - x^i| \geq |x^i - y^i|/2.$$ 

Hence, by (1.5), Lemma 2.6 and Theorem 2.8, we have

$$P_x(X_{Tv_1} \in U_2) \sup_{t/2 < s < t, z \in U_2} p_U(s, z, y) \leq P_x(X_{Tv_1} \in D) \sup_{t/2 < s < t, z \in U_2} p(s, z, y) \leq C_{2.6} \frac{\mathbb{E}_x[\tau_{U_1}]}{\phi(r)} \sup_{t/2 < s < t, z \in U_2} \left\{c_3[\phi^{-1}(s)]^{-d} \prod_{i \in E} \left(1 \wedge \frac{s\phi^{-1}(s)}{|z^i - y^i|}\right) \prod_{i \notin E} 1 \right\} \leq c_4 \frac{V(\delta_D(x))}{\sqrt{t}}[\phi^{-1}(t)]^{-d} \prod_{i = 1}^d \left(1 \wedge \frac{t\phi^{-1}(t)}{|x^i - y^i|}\right).$$

(3.8)

For the second term of (3.4), let $z \in U_1$. Then $|z^i - x^i| < 2r_t$, and for $i \in E$,

$$|z^i - y^i| \geq |x^i - y^i| - |z^i - x^i| \geq |x^i - y^i| - 2r_t \geq |x^i - y^i|/2.$$ 

With Theorem 2.8(1) and (1.5), we arrive at

$$\frac{2}{t} \mathbb{E}_x[\tau_{U_1}] \sup_{z \in U_1} p_U(t/2, z, y) \leq C_{2.8.1} \frac{2}{t} V(r_t)V(\delta_D(x)) \cdot C_1[\phi^{-1}(t/2)]^{-d} \prod_{i \in E} \left(1 \wedge \frac{t\phi^{-1}(t/2)}{|x^i - y^i|}\right) \prod_{i \notin E} 1$$

$$\leq c_5 \frac{V(\delta_D(x))}{\sqrt{t}}[\phi^{-1}(t)]^{-d} \prod_{i = 1}^d \left(1 \wedge \frac{t\phi^{-1}(t)}{|x^i - y^i|}\right).$$

(3.9)
where the last inequality follows from $|x^i - y^i| < 4V^{-1}(\sqrt{a}) \lesssim \phi^{-1}(t)$ for $i \not\in E$. For the last term in (3.4), we first note that for $u \in U_1$, $u + e_i h$ cannot be in $U_3$ if $i \not\in E$. So

$$\sup_{u \in U_1, u + e_i h \in U_3, i \not\in E} J(u, u + e_i h) = 0. \quad (3.10)$$

If $u \in U_1$ and $u + e_i h \in U_3$ where $i \in E$,

$$|h| \geq |u^i + h - x^i| - |x^i - Q^i| - |u^i| \geq |x^i - y^i|/2 - 3r_t/2 \geq |x^i - y^i|/8.$$\

Therefore, by (1.2), (1.3) and (WS),

$$\sup_{u \in U_1, u + e_i h \in U_3} J(u, u + e_i h) \leq \frac{C 8^{1 + \pi \kappa_0}}{|x^i - y^i| \phi(|x^i - y^i|)} \leq c_6 \frac{1}{t \phi^{-1}(t)} \left(1 + \frac{t \phi^{-1}(t)}{|x^i - y^i| \phi(|x^i - y^i|)} \right).$$

Note that for $u \in U_1$ with $k \in E \setminus \{i\}$, $|u^k - y^k| \geq |x^k - y^k| - |u^k - x^k| \geq |x^k - y^k| - 2r_t \geq |x^k - y^k|/2$. Then by (1.5), for $u \in U_1$ and $s \in (t/2, t)$,

$$\sum_{i \in E} \left(\sup_{u \in U_1} J(u, u + e_i h) \cdot \int_{\mathbb{R}} p(s, u + e_i h, y) dh\right) \leq \sum_{i \in E} \left(c_6 \frac{t \phi^{-1}(t)}{1 + \frac{t \phi^{-1}(t)}{|x^i - y^i| \phi(|x^i - y^i|)} \cdot \frac{2^{1 + \pi \kappa_0}}{|x^k - y^k| \phi(|x^k - y^k|)} \int_{\mathbb{R}} \left([\phi^{-1}(s)]^{-1} \wedge \frac{s}{|h| \phi(|h|)}\right) dh\right) \leq c_7 t^{-1} [\phi^{-1}(t)]^{-d} \prod_{k=1}^d \left(1 + \frac{t \phi^{-1}(t)}{|x^k - y^k| \phi(|x^k - y^k|)} \right). \quad (3.11)$$

For the last inequality, we used the fact that $\int_{\mathbb{R}} \left([\phi^{-1}(s)]^{-1} \wedge \frac{s}{|h| \phi(|h|)}\right) dh < \infty$ and $|x^k - y^k| < 4V^{-1}(\sqrt{a})$ for $k \not\in E$. By (1.5) and Theorem 2.8 with (3.10) and (3.11), we obtain

$$\int_{0}^{1/2} \int_{U_1} p_{U_1}(s, x, u) \left(\sum_{i=1}^{d} \sup_{h \in \mathbb{R}; u + e_i h \in U_3} J(u, u + e_i h) \cdot \int_{\mathbb{R}} p(t - s, u + e_i h, y) dh\right) duds \leq \int_{t/2}^{t} \int_{U_1} p_{U_1}(t - s, x, u) \left(\sum_{i \in E} \sup_{u \in U_1} J(u, u + e_i h) \cdot \int_{\mathbb{R}} p(s, u + e_i h, y) dh\right) duds \leq c_8 p(t, x) t^{-1} \int_{0}^{t/2} \mathbb{P}_x(\tau_{U_1} > s) ds \leq c_8 p(t, x) t^{-1} \mathbb{E}_x[\tau_{U_1}] \leq c_8 C_{2.8.1} t^{-1} V(r_t) V(\delta_D(x)) p(t, x, y) \leq c_8 \frac{V(\delta_D(x))}{\sqrt{t}} p(t, x, y). \quad (3.12)$$

Applying (3.8), (3.9), (3.12) and (1.5) to Lemma 3.5, it gives the result.

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** By Proposition 3.7, the semigroup and symmetry property of Dirichlet heat kernel $p_D(t, x, y)$ yield that

$$p_D(t, x, y) = \int_{\mathbb{R}^d} p_D(t/2, x, z) p_D(t/2, z, y) dz.$$
\[ \leq C_3^2 \Psi(t/2, x) \Psi(t/2, y) \int_{\mathbb{R}^d} p(t/2, x, z) p(t/2, z, y) dz \]
\[ = C_2 \Psi(t/2, x) \Psi(t/2, y) p(t, x, y). \]

4. Dirichlet lower bound estimate

In this section, we first obtain an interior near-diagonal lower bound of \( p_U(t, x, y) \) on an open set \( U \), see Proposition 4.2. Then under the condition \((D_\gamma)\) for \( \gamma \in (0, 1] \), we discuss a preliminary lower bound in Proposition 4.5. The boundary decay result in Lemma 4.6 plays a vital role in proving the sharp lower bound estimate for the Dirichlet heat kernel for small time in Theorem 1.2(1).

4.1. The preliminary lower bound estimate. For any fixed \( r > 0, x_0 \in \mathbb{R}^d \), let \( B := B(x_0, r) \). In the following, we discuss the lower bound for \( p_B(t, x, y) \).

**Proposition 4.1.** For any \( \zeta_1 > 0 \), there exist \( \zeta_2 := \zeta_2(\phi, \zeta_1, d), \zeta_3 := \zeta_3(\phi, \zeta_1, d) > 0 \) such that for any \( t \leq \phi(\zeta_1 r) \) and for any \( x, y \in B(x_0, \zeta_2 \phi^{-1}(t)) \subset B \),
\[ p_B(t, x, y) \geq \zeta_3 |\phi^{-1}(t)|^{-d}. \]

**Proof.** For any \( t > 0 \), \( 0 \leq h \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \), by strong Markov property, we have that
\[ P_t h(x) = P_t^B h(x) + \mathbb{E}_x [P_{t-\tau_B} h(X_{\tau_B}); t > \tau_B] \leq P_t^B h(x) + \sup_{s \in (0, t)} \sup_{z \in B^c} P_s h(z). \quad (4.1) \]

Let \( f, g \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) be nonnegative functions. Replacing \( t \) by \( t/2 \), and \( h \) by \( P_{t/2} f, P_{t/2} g \) in (4.1), respectively yield that
\[ P_{t/2} P_{t/2} f(x) \leq P_{t/2}^B P_{t/2} f(x) + \sup_{s \in (0, t/2)} \sup_{z \in B^c} P_s P_{t/2} f(z); \]
\[ P_{t/2} P_{t/2} g(x) \leq P_{t/2}^B g(x) + \sup_{s \in (0, t/2)} \sup_{z \in B^c} P_s P_{t/2} g(z). \]

Hence, we obtain that
\[ \langle P_t f, g \rangle \leq \langle P_{t/2}^B P_{t/2} f, g \rangle + \sup_{s \in (0, t/2)} \sup_{z \in B^c} P_s P_{t/2} f(z) \|g\|_{L^1} \]
\[ = \langle f, P_{t/2} P_{t/2}^B g \rangle + \sup_{s \in (0, t/2)} \sup_{z \in B^c} P_s f(z) \|g\|_{L^1} \]
\[ \leq \langle f, P_t^B g \rangle + \sup_{s \in (0, t/2)} \sup_{z \in B^c} P_s P_{t/2}^B g(z) \|f\|_{L^1} + \sup_{s \in (0, t/2)} \sup_{z \in B^c} P_s f(z) \|g\|_{L^1} \]
\[ \leq \langle f, P_t^B g \rangle + \sup_{s \in (t/2, t)} \sup_{z \in B^c} P_s g(z) \|f\|_{L^1} + \sup_{s \in (t/2, t)} \sup_{z \in B^c} P_s f(z) \|g\|_{L^1}. \]

For any Borel set \( E \subset B \), the above inequality holds for all nonnegative functions \( f, g \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) with \( \text{Supp}(f), \text{Supp}(g) \subset E \). So for any \( t > 0 \) and \( x, y, z \in E \),
\[ p(t, x, y) \leq p_B(t, x, y) + 2 \sup_{s \in (t/2, t)} \sup_{w \in E, z \in B^c} p(s, w, z). \quad (4.2) \]

Let \( E = B(x_0, (r \wedge \phi^{-1}(t))/2) \). For any \( t \leq \phi(\zeta r) \), \( w \in E \) and \( z \in B^c \), \( |w - z| \geq \frac{r}{2} \geq \frac{\phi^{-1}(t)}{2\zeta} \), and there exists \( 1 \leq k \leq d \) such that
\[ |w^k - z^k| \geq \frac{\phi^{-1}(t)}{2\zeta}. \]

Now applying (4.2) with (1.5) and (WS), for any \( x, y \in B(x_0, (r \wedge \phi^{-1}(t))/2) \)
\[ p_B(t, x, y) \geq C_1 |\phi^{-1}(t)|^{-d} - 2 \sup_{s \in (t/2, t)} \sup_{w \in E, z \in B^c} p(s, w, z) \]
If $\phi < 1$, then $B$ according to $x$

The semigroup property implies that for any $\phi^{-1}(t)$, we can observe the two cases, $\phi^{-1}(t) > 1$ and $\phi^{-1}(t) < 1$.

$Case 1.$ If $\phi^{-1}(t) < 1$, we take $\phi^{-1}(t) = \frac{1}{2}(\phi^{-1}(t) \wedge \phi^{-1}(t))$.

$Case 2.$ If $\phi^{-1}(t) > 1$, we then take $\phi^{-1}(t) = \frac{1}{2}(\phi^{-1}(t) \wedge \phi^{-1}(t))$.

Thus, $B(x_0, \zeta_2 \phi^{-1}(t)) < B(x_0, (r \wedge \phi^{-1}(t))/2)$, the result gets proved by letting $\zeta = \frac{\zeta}{\zeta_1}$.

For any $t < \phi(r)$, we first have $B(x_0, \zeta_2 \phi^{-1}(t)) < B(x_0, (r \wedge \phi^{-1}(t))/2)$ since $\zeta_2 < 1$ and

$\zeta_2 \phi^{-1}(t) \leq \frac{\zeta}{\zeta_1}(1 < \zeta_1) \zeta_1 r \leq \frac{1}{2} \zeta_1^1 r \leq \frac{1}{2} r$.

Now we define $n = n(r) := \left[ \frac{\phi(r)}{\phi(r)} \right] + 1$,

where $[x]$ represents the largest integer that is less than or equal to $x$. Note that by (WS), we have two-sided bounds for $n$ that is independent of $r$,

$$\zeta \left( \frac{\zeta_1}{\zeta} \right)^{1/n} \leq n \leq \frac{\zeta_1}{\zeta} + 1. \quad (4.3)$$

The semigroup property implies that for any $t < \phi(r)$ and $x, y \in B(x_0, \zeta_2 \phi^{-1}(t)) =: B_0$,

$$p_B(t, x, y) = \int_{B_0} \cdots \int_{B_0} p_B(\xi_n, x, z_1) p_B(\xi_n, z_1, z_2) \cdots p_B(\xi_n, z_n, y) dz_1 dz_2 \cdots dz_n$$

$$\geq \int_{B_0} \cdots \int_{B_0} p_B(\xi_n, x, z_1) p_B(\xi_n, z_1, z_2) \cdots p_B(\xi_n, z_n, y) dz_1 dz_2 \cdots dz_n$$

$$= \left( \frac{c_1}{2} \right)^n |B(x_0, \zeta_2 \phi^{-1}(t))|^{n-1}$$

$$\geq \left( \frac{c_1}{2} \right)^n \left( \frac{\pi^{d/2}}{\Gamma(1 + \frac{d}{2})} \right)^{n-1} \zeta_2^{d(n-1)} |\phi^{-1}(t)|^{d(n-1)}$$

where, according to (4.3), $\zeta$ depends on $\phi, \zeta_1, d$.

As an immediate application of Proposition 4.1, we arrive at the following interior near-diagonal lower bound for the Dirichlet heat kernel.

**Proposition 4.2.** For any open set $U \subset \mathbb{R}^d$ and any $\zeta_1 > 0$, there exist constants $\zeta_2 := \zeta_2(\phi, \zeta_1, d), \zeta_3 := \zeta_3(\phi, \zeta_1, d) > 0$ such that for any $(t, x, y) \in (0, \infty) \times U \times U$ with $d_U(x) \wedge d_U(y) \geq \zeta_1^{-1} \phi^{-1}(t)$ and $|x - y| < \zeta_2 \phi^{-1}(t)$, we have

$$p_U(t, x, y) \geq \zeta_3 |\phi^{-1}(t)|^{-d}.$$
Proposition 4.1. There exists \( \zeta_2, \zeta_3 > 0 \) such that
\[
p_B(t, y, z) \geq \zeta_3[\phi^{-1}(t)]^{-d}
\]
for any \( y, z \in B(x, \zeta_2 \phi^{-1}(t)) \). Thus, for any \( x, y \in U \) with \( \delta_U(x) \wedge \delta_U(y) \geq \zeta_1 \phi^{-1}(t) \) and \( |x - y| < \zeta_2 \phi^{-1}(t) \),
\[
y \in B(x, \zeta_2 \phi^{-1}(t)) \subset B \subset U,
\]
and we have \( p_U(t, x, y) \geq p_B(t, x, y) \geq \zeta_3[\phi^{-1}(t)]^{-d} \).

Lemma 4.3. Let \( a, b \) and \( T \) be positive constants. Then there exists a constant \( C_{4.3} = C_{4.3}(a, b, \phi, T, d) > 0 \) such that for every \( t \leq T \),
\[
\inf_{y \in B(x, b \phi^{-1}(t))} \mathbb{P}_y(\tau_{B(x, 2b \phi^{-1}(t))} > at) \geq C_{4.3}.
\]

Proof. For any \( y \in B(x, b \phi^{-1}(t)) \),
\[
B(y, b \phi^{-1}(t)) \subset B(x, 2b \phi^{-1}(t)).
\]
By Proposition 4.1 with \( r = b \phi^{-1}(T) \), \( \zeta_1 = r^{-1} \phi^{-1}(aT) \) and \( B := B(y, b \phi^{-1}(t)) \) there exist \( \zeta_2 = \zeta_2(a, b, \phi, T, d), \zeta_3 = \zeta_3(a, b, \phi, T, d) > 0 \) such that \( B(y, \zeta_2 \phi^{-1}(at)) \subset B \) and
\[
p_B(at, y, z) \geq \zeta_3[\phi^{-1}(at)]^{-d}
\]
for \( z \in B(y, \zeta_2 \phi^{-1}(at)) \).

Hence, for any \( y \in B(x, b \phi^{-1}(t)) \),
\[
\mathbb{P}_y(\tau_{B(x, 2b \phi^{-1}(t))} > at) = \int_{B(x, 2b \phi^{-1}(t))} p_B(x, b \phi^{-1}(t))(at, y, u)du
\]
\[
\geq \int_B p_B(at, y, u)dy \geq \int_{B(y, \zeta_2 \phi^{-1}(at))} p_B(at, y, u)du
\]
\[
\geq \zeta_3[\phi^{-1}(at)]^{-d}|B(y, \zeta_2 \phi^{-1}(at))| = \zeta_3 \zeta_2^d |B(0, 1)| > 0.
\]

The following result is obtained in [KW22, Proposition 4.1].

Proposition 4.4. For any \( A > 0 \) and \( B \in (0, 1) \), there exists \( \varrho = \varrho(A, B, \phi) \in (0, 2^{-1}) \) such that for every \( r > 0 \) and \( x \in \mathbb{R}^d \),
\[
\mathbb{P}_x(\tau_{B(x, Ar)} < \varrho \phi(r)) = \mathbb{P}_x \left( \sup_{s \leq \varrho \phi(r)} |X_s - X_0| > Ar \right) \leq B.
\]

Now we prove the preliminary lower bound of Dirichlet heat kernel, and this gives the sharp interior lower bound.

Proposition 4.5. Assume that \( U \) is an open set satisfying condition \( (D_\gamma) \) for some \( \gamma \in (0, 1] \). For any \( a, t > 0 \) and \( x, y \in U \) with \( \delta_U(x) \wedge \delta_U(y) \geq a \phi^{-1}(t) \), there exists a constant \( C_{4.5} = C_{4.5}(a, \phi, \kappa_0, \gamma, d) > 0 \) such that
\[
p_U(t, x, y) \geq C_{4.5} p(t, x, y).
\]

Proof. With \( \delta_U(x) \wedge \delta_U(y) \geq a \phi^{-1}(t) \), by Proposition 4.2, there exists \( \zeta_2 = \zeta_2(a, \phi, d), \zeta_3 = \zeta_3(a, \phi, d) > 0 \) such that if \( |x - y| < \zeta_2 \phi^{-1}(t) \),
\[
p_U(t, x, y) \geq \zeta_3[\phi^{-1}(t)]^{-d}.
\]

It suffices to consider the case that \( \delta_U(x) \wedge \delta_U(y) \geq a \phi^{-1}(t) \) and \( |x - y| \geq \zeta_2 \phi^{-1}(t) \). Without loss of generality, we assume \( \zeta_2 \leq a \). Now let \( 0 < b \leq \zeta_2 / 2 \) and \( 0 < \rho < 1 \) be constants that are determined later. Then by the semigroup property,
\[
p_U(t, x, y) = \int_U p_U(\rho t, x, z)p_U((1 - \rho) t, z, y)dz \geq \int_{B(y, b \phi^{-1}(t))} p_U(\rho t, x, z)p_U((1 - \rho) t, z, y)dz
\]

\[
\geq C_{4.5} \int_{B(y, b \phi^{-1}(t))} p_U(\rho t, x, z)p_U((1 - \rho) t, z, y)dz.
\]

\[
\geq C_{4.5} p(t, x, y).
\]
\[ \geq \inf_{z \in B(y, b\phi^{-1}(t))} p_U((1 - \rho)t, z, y)P_x \left(X^U_{\rho t} \in B(y, b\phi^{-1}(t))\right). \] (4.4)

Note that for \( z \in B(y, b\phi^{-1}(t)) \), \( \delta_U(z) \geq (a - b)\phi^{-1}(t) \geq \frac{a}{2}\phi^{-1}(t) \). Then by Proposition 4.2, there exists \( \tilde{\zeta}_2 = \tilde{\zeta}_2(a, d, \phi) > 0 \) such that
\[ |y - z| \leq b\phi^{-1}(t) < \tilde{\zeta}_2^{1/\alpha}(1 - \rho)^{1/\alpha}\phi^{-1}(t) < \tilde{\zeta}_2\phi^{-1}((1 - \rho)t), \] (4.5)
for some small \( b, \rho > 0 \), there exists \( \tilde{\zeta}_3 = \tilde{\zeta}_3(a, d, \phi) > 0 \) satisfying that
\[ \inf_{z \in B(y, b\phi^{-1}(t))} p_U((1 - \rho)t, z, y) \geq \tilde{\zeta}_3(\phi^{-1}((1 - \rho)t))^{-d} \geq \tilde{\zeta}_3C^{d/\alpha}(1 - \rho)^{-d/\alpha}\phi^{-1}(t)^{-d}. \] (4.6)

Since \( U \) satisfies condition (\( D_\gamma \)) and \( \delta_U(x) \wedge \delta_U(y) \geq a\phi^{-1}(t) \), there exists a permutation \( \{i_1, \ldots, i_d\} \) of \( \{1, \ldots, d\} \) such that
\[ B(\xi_{(k)}), \gamma a\phi^{-1}(t)) \subset U, \quad \text{for } k = 1, \ldots, d. \]
Choose \( b \) small enough such that
\[ b < \gamma a/2. \] (4.7)

For any \( k = 1, \ldots, d \), define \( r_k := \frac{b\phi^{-1}(t)}{2^d - k\sqrt{d}} \) and
\[ Q_k := Q(\xi_{(k)}, r_k) \subset B_k := B(\xi_{(k)}, \sqrt{d}r_k) \subset U, \]
where \( Q(x, r) \) is a cube centered at \( x \) with side length \( r \). By the semigroup property,
\[ P_x \left(X^U_{\rho t/d} \in B(y, b\phi^{-1}(t))\right) \]
\[ \geq \int_{Q_d} \cdots \int_{Q_1} p_U \left(\frac{\rho t}{d}, x, z_1\right) p_U \left(\frac{\rho t}{d}, z_1, z_2\right) \cdots p_U \left(\frac{\rho t}{d}, z_{d-1}, z_d\right) dz_1 \cdots dz_d \]
\[ \geq P_x \left(X^U_{\rho t/d} \in Q_1\right) \prod_{k=1}^{d-1} \inf_{z_k \in Q_k} P_{z_k} \left(X^U_{\rho t/d} \in Q_{k+1}\right). \] (4.8)

Let \( r := b\phi^{-1}(t)/\sqrt{d} \). Then \( r_k/2 \geq 2^{-d}r \) for any \( 1 \leq k \leq d - 1 \) and \( t \leq C_d^{d/2} b^{-\alpha} \phi(r) \) by (\( WS \)). Therefore for any \( w \in Q_k \),
\[ P_w \left(\tau_{B(w, r_k/2)} \geq \frac{\rho t}{2d}\right) \geq P_w \left(\tau_{B(w, 2^{-d}r)} \geq \frac{\rho t}{2d}\right) \geq P_w \left(\tau_{B(w, 2^{-d}r)} \geq \frac{\rho C_d h^{d/2 - 1}}{2b^{\alpha}} \phi(r)\right), \]
and by Proposition 4.4, we can select \( \rho \) small enough such that
\[ P_w \left(\tau_{B(w, r_k/2)} \geq \frac{\rho t}{2d}\right) \geq \frac{1}{2}. \] (4.9)

Let \( \overline{Q}_k := Q(\xi_{(k)}, r_k/2) \). By (4.9), strong Markov property and Lévy system in (2.27), for any \( v \in Q_k \), we have that
\[ P_v \left(X_{\rho t/d} \in Q_{k+1}\right) \geq 2^{-1}P_v \left(X \text{ hits } \overline{Q}_{k+1} \text{ by time } \frac{\rho t}{2d}\right) \]
\[ \geq c_1 P_v \left[ \int_0^{2\rho t/\alpha} \int_{Q_{k+1}} \frac{1}{X_s - u|\phi(|X_s - u|)|} m(du) ds \right], \]
where \( m(du) \) is the measure on \( \sum_{i=1}^d \mathbb{R} \) restricted only on each coordinate. By the same discussion as (4.9),
\[ E_v \left[ \frac{\rho t}{2d} \wedge \tau_{Q(v, r_k)} \right] \geq \frac{\rho t}{2d} P_v \left(\tau_{Q(v, r_k)} > \frac{\rho t}{2d}\right) \geq \frac{\rho t}{4d}. \]
Suppose that $|x^{i,k+1} - y^{i,k+1}| \geq r_k$. Then for any $w \in Q(\xi_{(k)}, 2r_k)$ and $u \in Q(\xi_{(k+1)}, 2r_k)$, by (1.8),

$$|w^{i,k+1} - u^{i,k+1}| \leq |x^{i,k+1} - y^{i,k+1}| + |x^{i,k+1} - w^{i,k+1}| + |y^{i,k+1} - u^{i,k+1}| = |x^{i,k+1} - y^{i,k+1}| + |\xi^{i,k+1} - u^{i,k+1}| + |\xi^{i,k+1} - u^{i,k+1}| \leq 5|x^{i,k+1} - y^{i,k+1}|.$$

Otherwise, $|x^{i,k+1} - y^{i,k+1}| \leq r_k$, $|u^{i,k+1} - u^{i,k+1}| \leq 5r_k$. Therefore, for any $v \in Q_k$ and $k \in \{1, \ldots, d - 1\}$,

$$\mathbb{P}_v \left(X_{\mu/d} \in Q_{k+1}\right) \geq c_2 \mathbb{E}_v \left[ \int_t^{t + \tau Q(v, r_k)} \left( \frac{1}{t} \wedge \frac{\phi^{-1}(t)}{|x^{i,k+1} - y^{i,k+1}|} \right) \right] \geq c_3 \left( 1 \wedge \frac{t \phi^{-1}(t)}{|x^{i,k+1} - y^{i,k+1}|(x^{i,k+1} - y^{i,k+1})} \right).$$

(4.10)

In a similar way to obtain (4.10), we have that

$$\mathbb{P}_x \left(X_{\mu/d} \in Q_1\right) \geq c_4 \left( 1 \wedge \frac{t \phi^{-1}(t)}{|x^{i,k+1} - y^{i,k+1}|(x^{i,k+1} - y^{i,k+1})} \right).$$

(4.11)

By choosing $b, \rho$ small enough that (4.5), (4.7) and (4.9) are satisfied, and by plugging (4.10)–(4.11) into (4.8), we arrive at our result by (4.6) and (4.8) with (4.4).

4.2. The lower bound. We now obtain the boundary decay using Theorem 2.8 and Lemma 4.3. For fixed $T > 0$, define

$$\hat{\alpha}_T := \hat{\alpha}_{T,R} := \frac{R}{\sqrt{1 + \Lambda^2} \phi^{-1}(T)} \quad \text{and} \quad \hat{\tau}_T := \hat{\alpha}_T \phi^{-1}(t) \text{ for any } t \in [0, T].$$

By (2.2), there exists $C_0$ such that $\frac{V(\xi^i)^2}{C_0} \leq C_0$. Using $C_{2.8.1}, C_{2.8.2}$ and $C_V$, let

$$\lambda := 8\Lambda \sqrt{1 + \Lambda^2} \vee \left( \frac{6C_{2.8.1}C_V}{C_0C_{2.8.2}} \right)^{2/\alpha} \quad \text{and} \quad \lambda_1 := 2\lambda \sqrt{1 + \Lambda^2}.$$  

(4.12)

For any $x \in D$, let $Q_x \in \partial D$ such that $\delta_D(x) = |x - Q_x|$, and define sets

$$U_x := D_{Q_x}(\hat{\tau}_t, \hat{\tau}_t) \setminus D_{Q_x}(\hat{\tau}_t, \lambda^{-1}\hat{\tau}_t)$$

$$E_x := \left\{ \begin{array}{ll}
B(x, \lambda^{-1}\hat{\tau}_t), & \text{if } x \notin D_{Q_x}(\lambda^{-1}\hat{\tau}_t, \lambda^{-1}\hat{\tau}_t); \\
\cup_{x \in E_x} B(z, \lambda^{-1}\hat{\tau}_t), & \text{if } x \in D_{Q_x}(\lambda^{-1}\hat{\tau}_t, \lambda^{-1}\hat{\tau}_t). \end{array} \right.$$  

(4.13)

Lemma 4.6. For any $T > 0$, there exists a constant $C_{4.6} = C_{4.6}(\phi, \kappa_0, \kappa_1, \eta, R, \Lambda, T, d) > 0$ such that for any $t \leq T$ and $x \in D$,

$$\int_{E_x} p_D(t/3, x, z)dz \geq C_{4.6} \Psi(t, x)$$

where $E_x$ is the set defined in (4.13).

Proof. By the definition of $(R, \Lambda)$, we may assume that $\Lambda > 2$. So for fixed $T > 0$,

$$\hat{\alpha}_T := \hat{\alpha}_{T,R} := \frac{R}{\sqrt{1 + \Lambda^2} \phi^{-1}(T)} \leq \frac{R}{2 \phi^{-1}(T)},$$

and $\hat{\tau}_t < R/2$ for $t \leq T$. If $x \notin D_{Q_x}(\lambda^{-1}\hat{\tau}_t, \lambda^{-1}\hat{\tau}_t)$, then $\delta_D(x) \geq \lambda^{-1}\hat{\tau}_t$. Let $B_{x} := B(x, \lambda^{-1}\hat{\tau}_t)$, then by Lemma 4.3,

$$\int_{E_x} p_D(t/3, x, z)dz \geq \int_{B_x} p_D(x, t/3, x, z)dz = \mathbb{P}_x(\tau_{B_x} > t/3) \geq c_1.$$  

If $x \in D_{Q_x}(\lambda^{-1}\hat{\tau}_t, \lambda^{-1}\hat{\tau}_t)$, that is, $\delta_D(x) < \lambda^{-1}\hat{\tau}_t$, Markov property implies that

$$\int_{E_x} p_D(t/3, x, z)dz = \mathbb{P}_x(X_{t/3}^D \in E_x) \geq \mathbb{P}_x \left( \sigma_{U_x}^D < t/3, \tau_{E_x} \circ \theta_{U_x}^D > t/3 \right)$$

where $U_x$ is the set defined in (4.13).
$\geq \inf_{z \in U_x} P_z(\tau_{E_x} > t/3) P_x(\sigma^D_{U_x} < t/3)$  \hspace{1cm} (4.14)

where $\sigma^D_{U_x} := \inf\{s \geq 0 : X^D_s \in U_x\}$ is the first hitting time of $U_x$. For any $z \in U_x$, $\delta_D(z) \geq \lambda^{-1}\hat{r}_t$ and $B_z := B(z, \lambda^{-1}\hat{r}_t) \subset E_x \subset D$, so Lemma 4.3 implies that

$$\inf_{z \in U_x} P_z(\tau_{E_x} > t/3) \geq \inf_{z \in U_x} P_z(\tau_{B_z} > t/3) \geq c_2.$$  \hspace{1cm} (4.15)

Applying Theorem 2.8 with $s = \lambda^{-1}\hat{r}_t$ and with $s = \hat{r}_t$ for $\delta_D(x) \leq \lambda^{-1}\hat{r}_t$, respectively, we obtain

$$E_x[\tau_{D_{Q_x}(\lambda^{-1}\hat{r}_t, \lambda^{-1}\hat{r}_t)}] \leq C_{2.8.1} V(\lambda^{-1}\hat{r}_t) V(\delta_D(x)) \leq C_{2.8.1} C_V \lambda^{-3/2} V(\hat{r}_t) V(\delta_D(x))$$

$$P_x\left(X_{\tau_{D_{Q_x}(\lambda^{-1}\hat{r}_t, \lambda^{-1}\hat{r}_t)}} \in D_{Q_x}(\hat{r}_t, \hat{r}_t) \setminus D_{Q_x}(\hat{r}_t, \lambda^{-1}\hat{r}_t)\right) \geq C_{2.8.2} V(\delta_D(x)) \frac{V(\hat{r}_t)}{\lambda^{3/2}} - \frac{3 C_{2.8.1} C_V V(\hat{r}_t) V(\delta_D(x))}{\lambda^{3/2}}.$$  \hspace{1cm} (4.16)

The last inequality holds by (4.12). Since $V(\delta_D(x)) \asymp \sqrt{\phi(\delta_D(x))}$ and $V(\hat{r}_t) \asymp \sqrt{t}$ by (2.2), we conclude that

$$P_x(\sigma^D_{U_x} < t/3) \geq c_3 \frac{\sqrt{\phi(\delta_D(x))}}{t}.$$  \hspace{1cm} (4.16)

Therefore, we arrive our result by plugging (4.15) and (4.16) into (4.14).

Now we are ready to prove Theorem 1.2(1).

Proof of Theorem 1.2(1). By the semigroup property and Lemma 4.6, for any $x, y \in D$,

$$p_D(t, x, y) = \int_{D \times D} p_D(t/3, x, u)p_D(t/3, u, v)p_D(t/3, v, y)dudv$$

$$\geq \int_{E_x \times E_y} p_D(t/3, x, u)p_D(t/3, u, v)p_D(t/3, v, y)dudv$$

$$\geq \inf_{u \in E_x, v \in E_y} p_D(t/3, u, v) \int_{E_x} p_D(t/3, x, u)du \cdot \int_{E_y} p_D(t/3, v, y)dv$$

$$\geq C^2_{4.6} \Psi(t, x) \inf_{u \in E_x, v \in E_y} p_D(t/3, u, v) \geq C_{4.6}^2 \Psi(t, x) \inf_{u \in E_x, v \in E_y} p_D(t/3, u, v)$$  \hspace{1cm} (4.17)

where $E_x, E_y$ are defined in (1.4). Note that for $(u, v) \in E_x \times E_y$, $|x^i - u^i| \vee |y^i - v^i| \leq 3\hat{r}_t$ for all $i \in \{1, \ldots, d\}$. If $|x^i - y^i| \leq 3\hat{r}_t$, then $|u^i - v^i| \leq |x^i - u^i| + |x^i - y^i| + |y^i - v^i| \leq 9\hat{r}_t$. Using (WS), $\phi(0^i - v^i) \leq \phi(0^i - t) \times \phi(\phi^{-1}(t)) = t$, and so

$$|u^i - v^i| \phi(|u^i - v^i|) \leq c_1 t \phi^{-1}(t).$$  \hspace{1cm} (4.18)

If $|x^i - y^i| > 3\hat{r}_t$, then $|u^i - v^i| \leq 3|x^i - y^i|$, and

$$|u^i - v^i| \phi(|u^i - v^i|) \leq c_2 |x^i - y^i| \phi(|x^i - y^i|).$$  \hspace{1cm} (4.19)
Recall that \( \rho_Q(w) := w^d - \varphi_Q(\tilde{w}) \) for \( w \in D \). Then by (2.6) and the fact that \( \lambda_1 \geq 2\lambda \), for any \((u, v) \in E_x \times E_y\),
\[
\delta_D(u) \land \delta_D(v) \geq \frac{\rho_{Q_x}(u) \land \rho_{Q_y}(v)}{\sqrt{1 + \Lambda^2}} \geq \frac{(\lambda_1^{-1} - \lambda_1^{-1})\hat{c}_t}{\sqrt{1 + \Lambda^2}} \geq \frac{\hat{a}_T}{2\lambda \sqrt{1 + \Lambda^2}}\phi^{-1}(t).
\]

Applying Proposition 4.5 with \( a = \frac{\hat{a}_T}{2\lambda \sqrt{1 + \Lambda^2}} \), and then combining (1.5) with (4.18)–(4.19), we obtain the lower bound of the last part in (4.17) as follows:
\[
\inf_{u \in E_x, v \in E_y} p_D(t/3, u, v) \geq C_{4.5} \inf_{u \in E_x, v \in E_y} p(t/3, u, v)
\geq C_{4.5} \frac{C_1^{-1}[\phi^{-1}(t)]^{-d}}{\lambda} \int_{\mathbb{R}^d} (1 \land \frac{t\phi^{-1}(t)}{|u - v|}) \prod_{i=1}^d \left( 1 \land \frac{t\phi^{-1}(t)}{|x_i - y_i|} \right) \geq c_3 \rho(t, x, y).
\]

\[\square\]

5. Large time Dirichlet heat kernel estimates

In this section, we assume \( D \) is a bounded \( C^{1,1} \) open set satisfying condition (\( D_\gamma \)) for some \( \gamma \in (0, 1] \), and the jumping kernel \( J \) satisfies (1.3), (K\(_\gamma\)), (SD). We aim to prove Theorem 1.2(2) by using the on-diagonal estimate for large time in Lemma 5.1.

Lemma 5.1. Suppose \( D \subset \mathbb{R}^d \) is a bounded open set. Then there are two positive constants \( c_i = c_i(\phi, \text{diam}(D), d), i = 1, 2 \) such that
\[
p_D(t, x, y) \leq c_1 e^{-c_2 t}, \quad \text{for } (t, x, y) \in (1, \infty) \times D \times D.
\]

Proof. By the semigroup property and (1.5), for all \((t, x, y) \in (1, \infty) \times D \times D\),
\[
p_D(t, x, y) = \int_D p_D(t - 1, x, z)p_D(1, z, y)dz \leq C_1[\phi^{-1}(1)]^{-d} \int_D p_D(t - 1, x, z)dz \quad (5.1)
\]
and for \( t \in (n, n + 1] \cap (2, \infty) \), there is \( n \in \mathbb{N} \) and
\[
\int_D p_D(t - 1, x, z)dz = \int_{D^{n+1}} p_D(1, x, x_1) \cdots p_D(1, x_{n-1}, x_n) p_D(t - n, x_n, z)dz =: c_4(\phi, d_0).
\]

Let \( d_0 := \text{diam}(D) \) and \( p_0(1, x^1, y^1) \) be the transition density for \( X^1 \). Since \( \{|z| \geq d_0\} \supset \{z = (z^1, \ldots, z^d) : |z^1| \geq d_0 \setminus 1\} \), by (1.5) and (WS), for \( x \in D \),
\[
\mathbb{P}_x(\tau_D \leq 1) \geq \mathbb{P}_x(X_1 \in D^{c}) = \int_{D^{c}} p(1, x, z)dz \geq \int_{\{|z| \geq d_0\}} p(1, 0, z)dz \geq \int_{\{|z| \geq d_0\}} p_0(1, 0, z)dz = 2 \int_{d_0}^{\infty} ds \frac{ds}{s\phi(s)} =: c_4(\phi, d_0)
\]
so that
\[
\sup_{x \in D} \int_D p_D(1, x, y)dy = \sup_{x \in D} \mathbb{P}_x(\tau_D > 1) \leq (1 - c_4) =: c_5 < 1
\]
and (5.2) is bounded as follows: \( t \in (n, n + 1] \cap (2, \infty) \)
\[
\int_D p_D(t - 1, x, z)dz \leq c_0 \int_D p_D(t - n - 1, x_n, z)dz \leq c_0^n \leq c_5^{-1} \cdot t_0 = c_5^{-1} \cdot e^{-t\ln c_5^{-1}}.
\]

Clearly,
\[
\int_D p_D(t - 1, x, z)dz \leq 1 \leq e \cdot e^{-t}, \quad \text{for } t \in (1, 2).
\]
By letting $c_6 := \ln(c_5^{-1} \land e)$, we have that
\[
\int_D p_D(t-1, x, z) dz \leq e \cdot e^{-t} \land c_5^{-1} \cdot e^{-t \ln c_5^{-1}} \leq (e \lor c_5^{-1}) e^{-c_6t}, \quad \text{for } (t, x) \in (1, \infty) \times D.
\]
Therefore, we obtain our result plugging the above in (5.1) with $c_1 := c_3(e \lor c_5^{-1})$ and $c_2 = c_4$.

Now we are ready to prove Theorem 1.2(2).

**Proof of Theorem 1.2(2).** Let $D$ be bounded and $\mathcal{L}_D$ be the infinitesimal generator of the semigroup $\{P^D_t\}$ on $L^2(D)$. For each $t > 0$, by (1.7), the heat kernel $p_D(t, x, y)$ is bounded in $D \times D$ and so $\mathcal{L}_D$ is a Hilbert-Schmidt operator which is a compact operator. Therefore there exist the first eigenvalue $\lambda^D := -\sup(\sigma(\mathcal{L}_D))$ of multiplicity 1 and the corresponding eigenfunction $f_D$ with unit $L^2$ norm, $\|f_D\|_{L^2(D)} = 1$. Since $\{P^D_t\}$ is the semigroup corresponding to $\mathcal{L}_D$,
\[
f_D(x) = e^{\lambda^D} P^D_1 f_D(x) = e^{\lambda^D} \int_D p_D(t, x, y) f_D(y) dy.
\]
We obtain the upper bound of (5.3) when $t = \frac{1}{4}$ using (1.5), (1.7) with $T = 1$ and the Hölder inequality that for all $x \in D$
\[
f_D(x) \leq c e^{\frac{1}{2} \lambda^D} \left(1 \land 2 \sqrt{\phi(\delta_D(x))} \right) \int_D p_\frac{1}{4}, x, y f_D(y) dy
\]
\[
\leq 2 c e^{\frac{1}{2} \lambda^D} \left(1 \land \sqrt{\phi(\delta_D(x))} \right) \int_D p_\frac{1}{4}, x, y^2 dy \cdot \|f_D\|_{L^2(D)}
\]
\[
\leq 2 c e^{\frac{1}{2} \lambda^D} \left(1 \land \sqrt{\phi(\delta_D(x))} \right) \sqrt{p_\frac{1}{4}, x, x}
\]
\[
\leq 2 c e^{\frac{1}{2} \lambda^D} \left(1 \land \sqrt{\phi(\delta_D(x))} \right) [\phi^{-1}(2^{-1})]^{-d/2} =: c^* e^{\frac{1}{4} \lambda^D} \left(1 \land \sqrt{\phi(\delta_D(x))} \right).
\]
for some $c^* = c^*(\kappa_0, \kappa_1, \eta, \phi, R, \Lambda, \gamma, d) > 0$. On the other hand, if $t = 1$ in (5.3), (1.5) and (1.9) with $T = 1$ yields that for all $x \in D$
\[
f_D(x) \geq c e^{\lambda^D} \left(1 \land \sqrt{\phi(\delta_D(x))} \right) \int_D \left(1 \land \sqrt{\phi(\delta_D(y))} \right) p(1, x, y) f_D(y) dy
\]
\[
\geq c e^{\lambda^D} \left(1 \land \sqrt{\phi(\delta_D(x))} \right) \left(1 \land \frac{1}{d_0^2 \phi(d_0)^d} \right) \int_D \left(1 \land \sqrt{\phi(\delta_D(y))} \right) f_D(y) dy
\]
\[
=: c_* e^{\lambda^D} \left(1 \land \sqrt{\phi(\delta_D(x))} \right) \int_D \left(1 \land \sqrt{\phi(\delta_D(y))} \right) f_D(y) dy
\]
for some $c_* = c_4(\kappa_0, \kappa_1, \eta, \phi, R, \Lambda, \gamma, d_0, d) > 0$ where $d_0 := \text{diam}(D)$. Combining these two inequalities, since $\|f_D\|_{L^2(D)} = 1$, we have that for any $x \in D$,
\[
f_D(x) \geq c_* e^{\frac{3}{4} \lambda^D} \left(1 \land \sqrt{\phi(\delta_D(x))} \right) \int_D f_D(y)^2 dy = \frac{c_*}{c^*} e^{\frac{3}{4} \lambda^D} \left(1 \land \sqrt{\phi(\delta_D(x))} \right)
\]
By (5.4) and (5.6), we have that
\[
c_* (c^*)^{-1} e^{\frac{3}{4} \lambda^D} \leq e^{\frac{3}{4} \lambda^D} \text{ which implies } \lambda^D \leq 2 \ln((c^*)^2/c_*) < \infty,
\]
and this prove the upper bound of (1.10). By (5.4)–(5.5) and (5.7), for $x \in D$,
\[
c_0^{-1} \left(1 \land \sqrt{\phi(\delta_D(x))} \right) \leq f_D(x) \leq c_0 \left(1 \land \sqrt{\phi(\delta_D(x))} \right)
\]
for some $c_0 = c_0(\kappa_0, \kappa_1, \eta, \phi, R, \Lambda, \gamma, d_0, d) \geq 1$. From (5.3),
\[
1 = \|f_D\|_{L^2(D)}^2 = e^{\lambda^D} \int_{D \times D} f_D(x) p_D(t, x, y) f_D(y) dx dy
\]
and so that 
\[ \int_{D \times D} f_D(x)p_D(t, x, y)f_D(y)dx dy = e^{-t\lambda_D}. \]
Therefore (5.8) implies that for \( t > 0, \)
\[ c_0^{-2}e^{-t\lambda_D} \leq \int_{D \times D} \left( 1 + \sqrt{\phi(\delta_D(x))} \right) p_D(t, x, y) \left( 1 + \sqrt{\phi(\delta_D(y))} \right) dx dy \leq c_0^2e^{-t\lambda_D}. \]  
(5.9)
For the lower bound of (1.10), by Lemma 5.1, there exists \( c_i = c_i(\phi, d_0, d) > 0, i = 1, 2 \) such that 
\[ p_D(t, x, y) \leq c_1e^{-ct}, \quad \text{for} \quad (t, x, y) \in (0, \infty) \times D \times D. \]
Combining this with (5.9), for all \( t > 0, \)
\[ c_0^{-2}e^{-t\lambda_D} \leq \int_{D \times D} \left( 1 + \sqrt{\phi(\delta_D(x))} \right) p_D(t, x, y) \left( 1 + \sqrt{\phi(\delta_D(y))} \right) dx dy \leq c_1e^{-ct}\int_{D \times D} \left( 1 + \sqrt{\phi(\delta_D(x))} \right) \left( 1 + \sqrt{\phi(\delta_D(y))} \right) dx dy \leq c_1e^{-ct}|D|^2. \]
Therefore, we have 
\[ e^{(c_2-\lambda_D)t} \leq (c_0)^2|D|^2 < \infty \quad \text{which implies} \quad \lambda_D \geq c_2. \]
Now we will prove (1.11). For any \( T > 0, \) let \( t_0 := T/4. \) By (1.7) and (1.9), there is a constant \( c_3 = c_3(\kappa_0, \kappa_1, \eta, \phi, R, T, \Lambda, \gamma, d) \geq 1 \) such that for \((u, v) \in D \times D, \)
\[ p_D(t_0, u, v) \leq c_3[\phi^{-1}(t_0)]^{-d}t_0^{-1} \left( 1 + \sqrt{\phi(\delta_D(u))} \right) \left( 1 + \sqrt{\phi(\delta_D(v))} \right), \]  
(5.10)
\[ p_D(t_0, u, v) \geq c_3^{-1} \left( \phi^{-1}(t_0) \right)^{d} \left( 1 + \sqrt{\phi(\delta_D(u))} \right) \left( 1 + \sqrt{\phi(\delta_D(v))} \right). \]  
(5.11)
By the semigroup property with (5.9) and (5.10), for all \((t, x, y) \in (T, \infty) \times D \times D, \)
\[ p_D(t, x, y) = \int_{D \times D} p_D(t_0, x, u)p_D(t - 2t_0, u, v)p_D(t_0, v, y)dudv \leq \int_{D \times D} \left( 1 + \sqrt{\phi(\delta_D(u))} \right) p_D(t - 2t_0, u, v) \left( 1 + \sqrt{\phi(\delta_D(v))} \right) dudv \leq (c_0c_3)^2e^{-(t-2t_0)\lambda_D} \left( \phi^{-1}(t_0) \right)^{-2d}t_0^{-2} \sqrt{\phi(\delta_D(x))}\sqrt{\phi(\delta_D(y))}. \]  
(5.12)
Similarly, using (5.9) and (5.11), for all \((t, x, y) \in (T, \infty) \times D \times D, \) since \( D \) is bounded,
\[ p_D(t, x, y) \geq c_3^{-2} \left( \phi^{-1}(t_0) \right)^{2d} \sqrt{\phi(\delta_D(x))}\sqrt{\phi(\delta_D(y))} \cdot \int_{D \times D} \left( 1 + \sqrt{\phi(\delta_D(u))} \right) p_D(t - 2t_0, u, v) \left( 1 + \sqrt{\phi(\delta_D(v))} \right) dudv \geq (c_0c_3)^2e^{-(t-2t_0)\lambda_D} \left( \phi^{-1}(t_0) \right)^{2d}t_0^{-2} \sqrt{\phi(\delta_D(x))}\sqrt{\phi(\delta_D(y))}. \]  
(5.13)
Therefore, we conclude (1.11) by (5.12) with (5.7) and (5.13.

6. Green function estimates

We first obtain the weak scaling condition for the inverse function \( \phi^{-1} \) by (WS) that
\[ \left( \frac{t}{s^{\alpha}} \right)^{1/\alpha} \leq \frac{\phi^{-1}(t)}{\phi^{-1}(s)} \leq \left( \frac{t}{s^{\alpha}} \right)^{1/\alpha} \quad \text{for} \quad 0 < s \leq t. \]  
(6.1)
Proof of Theorem 1.3. When $d = 1$, we obtain the result by [GKK20, Theorem 1.6]. So we only consider when $d \geq 2$. Suppose that $x, y \in D$ satisfying that $\min_{i \in \{1, \ldots, d\}} |x^i - y^i| > 0$. Let $r_1 := \min_{i \in \{1, \ldots, d\}} |x^i - y^i|, r_2 := \max_{i \in \{1, \ldots, d\}} |x^i - y^i|$ and $T := \phi(\text{diam}(D))$. We may assume that $r_1 < r_2$, and the Green function is decomposed according to the range of $t > 0$ that

$$
G_D(x, y) = \int_0^\infty p_D(t, x, y)dt
= \int_0^{\phi(r_1)} p_D(t, x, y)dt + \int_{\phi(r_1)}^{\phi(r_2)} p_D(t, x, y)dt + \int_{\phi(r_2)}^T p_D(t, x, y)dt + \int_T^\infty p_D(t, x, y)dt
=: I_0 + I_1 + I_2 + I_3.
$$

We first note that (1.11) in Theorem 1.2(2) implies

$$
I_3 = \int_T^\infty p_D(t, x, y)dt \asymp \sqrt{\phi(\delta_D(x))}\sqrt{\phi(\delta_D(y))}.
$$

We need to prove that there exists $c_0 > 0$ such that

$$
I_0 \asymp 1 \cdot \sqrt{\frac{\phi(\delta_D(x))}{\phi(r_1)}} \cdot \sqrt{\frac{\phi(\delta_D(y))}{\phi(r_1)}} \cdot \phi(r_1)^{d+1} \prod_{i=1}^d \frac{1}{|x^i - y^i|\phi(|x^i - y^i|)},
$$

$$
I_1 \leq c_0 \cdot \sqrt{\frac{\phi(\delta_D(x))}{\phi(r_2)}} \cdot \sqrt{\frac{\phi(\delta_D(y))}{\phi(r_2)}} \cdot \phi(r_2)^{d+1} \prod_{i=1}^d \frac{1}{|x^i - y^i|\phi(|x^i - y^i|)},
$$

$$
I_2 \leq c_0 \cdot \sqrt{\frac{\phi(\delta_D(x))}{\phi(r_2)}} \cdot \sqrt{\frac{\phi(\delta_D(y))}{\phi(r_2)}} \cdot \phi(r_2)^{d+1} \prod_{i=1}^d \frac{1}{|x^i - y^i|\phi(|x^i - y^i|)}.
$$

Asymptotic estimates of $I_0$: For $0 < t < \phi(r_1) \leq \phi(|x^i - y^i|) < T, i = 1, \ldots, d$, by letting $v_1 := \frac{\phi(r_1)}{t} \geq 1$, we have that

$$
\prod_{i=1}^d \left( \phi^{-1}(t) \wedge \frac{t}{|x^i - y^i|\phi(|x^i - y^i|)} \right) dt \asymp \prod_{i=1}^d \frac{t}{|x^i - y^i|\phi(|x^i - y^i|)} dt
= \prod_{i=1}^d v_1 |x^i - y^i|\phi(|x^i - y^i|) \left( -\frac{\phi(r_1)}{v_1^2} \right) dv_1.
$$

By change of variables with $v_1 = \frac{\phi(r_1)}{t} \geq 1$, Theorem 1.1, Theorem 1.2(1) and (6.1) imply that there exists $c_1 > 1$ such that

$$
c_1^{-1} \phi(r_1)^{d+1} \prod_{i=1}^d \frac{1}{|x^i - y^i|\phi(|x^i - y^i|)} \left( 1 \wedge \sqrt{\frac{\phi(\delta_D(x))}{\phi(r_1)}} \right) \cdot \sqrt{\frac{\phi(\delta_D(y))}{\phi(r_1)}}
\leq \phi(r_1)^{d+1} \prod_{i=1}^d \frac{1}{|x^i - y^i|\phi(|x^i - y^i|)} \left( 1 \wedge \sqrt{\frac{\phi(\delta_D(x))}{\phi(r_1)}} \right) \cdot \sqrt{\frac{\phi(\delta_D(y))}{\phi(r_1)}} \int_1^\infty v_1^{-d-2} dv_1
\leq I_0 \asymp \int_1^\infty \left( 1 \wedge \frac{v_1 \phi(\delta_D(x))}{\phi(r_1)} \right) \cdot \sqrt{\frac{\phi(\delta_D(y))}{\phi(r_1)}} \prod_{i=1}^d \frac{\phi(r_1)}{v_1 |x^i - y^i|\phi(|x^i - y^i|)} dv_1
\leq \phi(r_1)^{d+1} \prod_{i=1}^d \frac{1}{|x^i - y^i|\phi(|x^i - y^i|)} \left( 1 \wedge \sqrt{\frac{\phi(\delta_D(x))}{\phi(r_1)}} \right) \cdot \sqrt{\frac{\phi(\delta_D(y))}{\phi(r_1)}} \int_1^\infty v_1^{-d-1} dv_1
\leq c_1 \phi(r_1)^{d+1} \prod_{i=1}^d \frac{1}{|x^i - y^i|\phi(|x^i - y^i|)} \left( 1 \wedge \sqrt{\frac{\phi(\delta_D(x))}{\phi(r_2)}} \right) \cdot \sqrt{\frac{\phi(\delta_D(y))}{\phi(r_2)}}.
$$
Theorem 1.1 implies that, i.e., decomposing the time variable as 0

\[
\prod_{i=1}^{d} \left( \phi^{-1}(t) \right)^{-1} \frac{t}{|x^i - y^i| \phi(|x^i - y^i|)} dt \leq \prod_{i=1}^{d} \frac{\phi(r_2)}{v_2 |x^i - y^i| \phi(|x^i - y^i|)} \left( -\frac{\phi(r_2)}{v_2^2} \right) dv_2.
\]

Then since \( v_2 > 1 \), Theorem 1.1 implies that

\[
I_1 \leq \int_{\phi(r_2)/r_2}^{1} \left( 1 + \frac{\sqrt{v_2 \phi(\delta_D(x))}}{\sqrt{\phi(r_2)}} \right)^d \prod_{i=1}^{d} \frac{\phi(r_2)}{v_2 |x^i - y^i| \phi(|x^i - y^i|)} \left( -\frac{\phi(r_2)}{v_2^2} \right) dv_2
\]

\[
\leq c_2 \left( 1 + \frac{\sqrt{\phi(\delta_D(x))}}{\sqrt{\phi(r_2)}} \right)^d \prod_{i=1}^{d} \frac{1}{|x^i - y^i| \phi(|x^i - y^i|)} \phi(r_2)^{d+1} \prod_{i=1}^{d} \frac{1}{|x^i - y^i| \phi(|x^i - y^i|)}.
\]

Upper bound estimate of \( I_2 \): For \( \phi(r_2) \leq t < T \),

\[
\prod_{i=1}^{d} \left( \phi^{-1}(t) \right)^{-1} \frac{t}{|x^i - y^i| \phi(|x^i - y^i|)} dt \simeq [\phi^{-1}(t)]^{-d} dt = \left( \frac{r_2}{r_2 \phi^{-1}(t)} \right)^d dt.
\]

Let \( u_2 := \frac{\phi(r_2)}{r_2} \leq 1 \). Then \( r_2 = \phi^{-1}(u_2 t) \). By the change of variable with \( u_2 = \frac{\phi(r_2)}{r_2} \), (6.1) and the fact that \( r_i \leq r_{i+1} \) and \( \phi(r_i) \leq \phi(r_{i+1}) \), Theorem 1.1 implies that

\[
I_k \approx \int_{\phi(r_2)/T}^{1} \left( \frac{\phi^{-1}(u_2 t)}{r_2 \phi^{-1}(t)} \right)^d \left( 1 + \frac{\sqrt{u_2 \phi(\delta_D(x))}}{\sqrt{\phi(r_2)}} \right)^d \left( -\frac{\phi(r_2)}{u_2^2} \right) du_2
\]

\[
\leq c_3 \left( \frac{\phi(r_2)}{r_2} \right)^d \left( 1 + \frac{\sqrt{\phi(\delta_D(x))}}{\sqrt{\phi(r_2)}} \right)^d \prod_{i=1}^{d} \frac{1}{u_2 |x^i - y^i| \phi(|x^i - y^i|)}
\]

\[
\leq c_4 \left( 1 + \frac{\sqrt{\phi(\delta_D(x))}}{\sqrt{\phi(r_2)}} \right)^d \left( 1 + \frac{\sqrt{\phi(\delta_D(y))}}{\sqrt{\phi(r_2)}} \right)^d \prod_{i=1}^{d} \frac{1}{|x^i - y^i| \phi(|x^i - y^i|)}.
\]

Remark 6.1. Suppose that \( x^i \neq y^i \) for all \( i \in \{1, \ldots, d\} \) and \( |x^{i_1} - y^{i_1}| \neq |x^{i_2} - y^{i_2}| \) for \( i_1, i_2 \in E \subset \{1, 2, \ldots, d\} \). Let \( \sigma : \{1, \ldots, d\} \rightarrow \{1, \ldots, d\} \) be a permutation to obtain \( 0 < |x^{\sigma(i)} - y^{\sigma(i)}| < |x^{i_1} - y^{i_1}| \) and \( r_{\sigma(i)} := |x^{\sigma(i)} - y^{\sigma(i)}|, i \in E \). Using the similar proof as in Theorem 1.3, i.e., decomposing the time variable as \( 0 < \phi(r_{\sigma(i)}) < \phi(r_{\sigma(i)}) < T < \infty \), when \( d \geq 2 \), we can obtain a more accurate upper bound that

\[
G_D(x, y) \leq c \sum_{i \in E} \left( 1 + \frac{\sqrt{\phi(\delta_D(x))}}{\sqrt{\phi(|x^i - y^i|)}} \right) \left( 1 + \frac{\sqrt{\phi(\delta_D(y))}}{\sqrt{\phi(|x^i - y^i|)}} \right) \prod_{i=1}^{d} \frac{1}{|x^i - y^i| \phi(|x^i - y^i|)}
\]

\[
\times \prod_{i=1}^{d} \left( 1 + \frac{\sqrt{\phi(\delta_D(x))}}{\sqrt{\phi(|x^i - y^i|)}} \right) \left( 1 + \frac{\sqrt{\phi(\delta_D(y))}}{\sqrt{\phi(|x^i - y^i|)}} \right) \prod_{i=1}^{d} \frac{1}{|x^i - y^i| \phi(|x^i - y^i|)}.
\]
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