Attractors with Vanishing Central Charge

S. Bellucci\textsuperscript{a}, A. Marrani\textsuperscript{a,b}, E. Orazi\textsuperscript{c} and A. Shcherbakov\textsuperscript{a}*

\textsuperscript{a} INFN - Laboratori Nazionali di Frascati, Via Enrico Fermi 40, 00044 Frascati, Italy
bellucci@lnf.infn.it, marrani@lnf.infn.it, ashcherb@lnf.infn.it

\textsuperscript{b} Museo Storico della Fisica e Centro Studi e Ricerche “Enrico Fermi” Via Panisperna 89A, 00184 Roma, Italy

\textsuperscript{c} Dipartimento di Fisica, Politecnico di Torino, Corso Duca degliAbruzzi 24, 10129 Torino, Italy
and INFN - Sezione di Torino, Italy emanuele.orazi@polito.it

Abstract

We consider the Attractor Equations of particular \( \mathcal{N} = 2, d = 4 \) supergravity models whose vector multiplets’ scalar manifold is endowed with homogeneous symmetric cubic special Kähler geometry, namely of the so-called \( st^2 \) and \( stu \) models. In this framework, we derive explicit expressions for the critical moduli corresponding to non-BPS attractors with vanishing \( \mathcal{N} = 2 \) central charge. Such formulæ hold for a generic black hole charge configuration, and they are obtained without formulating any \textit{ad hoc} simplifying assumption. We find that such attractors are related to the \( \frac{1}{2} \)-BPS ones by complex conjugation of some moduli. By uplifting to \( \mathcal{N} = 8, d = 4 \) supergravity, we give an interpretation of such a relation as an exchange of two of the four eigenvalues of the \( \mathcal{N} = 8 \) central charge matrix \( Z_{AB} \). We also consider non-BPS attractors with non-vanishing \( Z \); for peculiar charge configurations, we derive solutions violating the Ansatz usually formulated in literature. Finally, by group-theoretical considerations we relate Cayley’s hyperdeterminant (the invariant of the \( stu \) model) to the invariants of the \( st^2 \) and of the so-called \( t^3 \) model.

1 Introduction

The Attractor Mechanism in extremal black holes (BHs) \cite{11,2,3,4,5} has recently been investigated in depth, and various advances along such a line of research has been performed \cite{6}–\cite{39}.

The horizon geometry of extremal black holes in \( d = 4 \) space-time dimensions is the direct product of two spaces with non-vanishing, constant (and opposite) curvature, namely it is the Bertotti-Robinson geometry \cite{40,41,42}:

\[
AdS_2 \times S^2 = \frac{SO(1,2)}{SO(1,1)} \times \frac{SO(3)}{SO(2)}.
\] (1.1)

*On leave of absence from JINR, Dubna, Russia
In the framework of $\mathcal{N} = 2, d = 4$ supergravity, such an horizon geometry is associated to the maximal $\mathcal{N} = 2$ supersymmetry algebra $\mathfrak{psu}(1, 1 | 2)$, which is an interesting example of superalgebra containing not Poincaré nor semisimple groups, but direct products of simple groups as maximal bosonic subalgebra. Indeed, in this case the maximal bosonic subalgebra is $\mathfrak{so}(1, 2) \oplus \mathfrak{su}(2)$ (with related maximal spin bosonic subalgebra $\mathfrak{su}(1, 1) \oplus \mathfrak{su}(2)$), matching the corresponding bosonic isometry group of the Bertotti-Robinson metric [11].

In this context, the attractor configurations of the scalars at the event horizon of the BH have been recognized to fall into three distinct classes ([17, 21, 18]; see also [43] for a recent review):

1. the $\frac{1}{2}$-BPS class, known since [1, 2, 3, 4, 5], preserving four supersymmetries out of the eight pertaining to the asymptotical Minkowski space-time background;

2. the non-BPS class with non-vanishing $\mathcal{N} = 2$ central charge at the horizon, which does not preserve any supersymmetry at all;

3. the non-BPS class with vanishing $\mathcal{N} = 2$ central charge at the horizon; thus, for this class the complete breakdown of supersymmetry at the BH event horizon is associated with the lack of central extension of $\mathfrak{psu}(1, 1 | 2)$.

In this work we address the issue of the explicit determination of the non-BPS scalar configuration with vanishing $\mathcal{N} = 2$ central charge, in the framework of peculiar $\mathcal{N} = 2, d = 4$ supergravities coupled to $n_V = 2$ and 3 vector multiplets, namely for the so-called $st^2$ and $stu$ models [21].

Such models belong to the broad class of the so-called $d$-geometries ([44]; see Sect. 2 for elucidation), whose explicit $\frac{1}{2}$-BPS attractors, for generic BH charges and generic $n_V$, are known after [45] (the $stu$ model was previously investigated in [46]). Recently, in [10] the $d$-geometries with generic $n_V$ were reconsidered, and the non-BPS attractors with non-vanishing central charge were explicitly determined for a peculiar choice of BH charges.

In our investigation, we find that, for a general charge configuration, the non-supersymmetric attractors with vanishing $\mathcal{N} = 2$ central charge always violate the Ansatz used in [10]. Furthermore, due to the high symmetry of the scalar geometries analyzed, they turn out to be intimately related to the $\frac{1}{2}$-BPS attractors.

The plan of this work is as follows.

In Sect. 2 we briefly recall the foundations of the special Kähler (SK) geometry endowing the vector multiplets’ scalar manifold of $\mathcal{N} = 2, d = 4$ supergravity, focussing on the so-called SK $d$-geometries, and limiting ourselves to the sole quantities needed in the subsequent treatment. Sect. 3 is devoted to the $st^2$ model, the simplest symmetric model in which non-supersymmetric attractors with vanishing central charge appear; the violation of the Ansatz of [10] is pointed out, and the explicit form of such attractors, along with the relation with the well-known $\frac{1}{2}$-BPS ones, is derived. As a byproduct of our approach, we also obtain a one-parameter family of non-BPS attractors with non-vanishing central charge which violate the Ansatz of [10], showing that it actually implies some loss of generality in the context of SK $d$-geometries. In Sect. 4 we perform an analogous analysis in the $stu$ model [17, 46, 23]. We derive the explicit expression of non-supersymmetric attractors with vanishing central charge, and elucidate how they related with the supersymmetric ones. Finally, in Sect. 5 we relate, by simple group-theoretical considerations, Cayley’s hyperdeterminant to the quartic (and unique) invariants of the $U$-duality groups of the models $t^3$ [32] and $st^2$. Concluding remarks and an outlook can be found in the final Sect. 6.
2 Special Kähler Geometry

In this section we briefly recall some notions of the special Kähler (SK) geometry underlying the vector multiplets’ scalar manifold of $\mathcal{N} = 2, d = 4$ supergravity coupled to $n_V$ vector multiplets. Our treatment is far from exhaustive, as we introduce only the quantities needed in the subsequent computations (for the notation, explanation and extensive treatment, see e.g. [48] and Refs. therein).

Once a holomorphic prepotential function $F(X)$ of the sections $X^\Lambda = X^\Lambda(\Lambda = 0, 1, \ldots, n_V)$ is given, one can derive all the fundamental quantities in the framework of SK geometry. The Kähler potential and the corresponding moduli space metric are found to be

$$K = -\ln \left[ i \left( \bar{X}^\Lambda \partial_\Lambda F - X^\Lambda \partial_\Lambda \bar{F} \right) \right], \quad g_{ij} = \partial_i \partial_j K,$$

(2.1)

where the indices $i$ and $\bar{j}$ refer to the moduli $z^i$ and $\bar{z}^{\bar{j}}$ ($i, \bar{j} = 1, \ldots, n_V$ throughout), respectively. The covariantly holomorphic $\mathcal{N} = 2$ central charge function

$$Z(z, \bar{z}, p, q) = e^{\frac{1}{2}K(z, \bar{z})}W(z, p, q)$$

(2.2)

is given in terms of the holomorphic superpotential

$$W(z, p, q) = q_\Lambda X^\Lambda - p_\Lambda \partial_\Lambda F,$$

(2.3)

and it can be used to calculate the so-called BH effective potential [5]

$$V_{BH} = e^K \left[ g^{i\bar{j}} (\mathcal{D}_i W) \overline{\mathcal{D}}_{\bar{j}} W + W W \right] = |Z|^2 + g^{i\bar{j}} (\mathcal{D}_i Z) \overline{\mathcal{D}}_{\bar{j}} \overline{Z},$$

(2.4)

where $\mathcal{D}_i = \partial_i + \frac{1}{2} \partial_i K$ denotes the Kähler-covariant derivative acting on an object with holomorphic Kähler weight $p$.

The Attractor Mechanism in extremal BHs [1, 2, 3, 4, 5] yields that at the BH event horizon the moduli are stabilized in terms of the electric $q_0, q_i$ and magnetic charges $p^0, p^i$ of the BH as they are critical points of $V_{BH}$, i.e. they are solutions of the Attractor Equations (AEs) given by the criticality conditions of $V_{BH}$. Through the fundamental relations characterizing the special Kähler geometry the AEs can be cast in the following form:

$$\mathcal{D}_i V_{BH} = 2Z \mathcal{D}_i Z + iC_{ijk} g^{i\bar{j}} g^{k\bar{m}} (\mathcal{D}_i Z) \overline{\mathcal{D}}_{\bar{j}} \overline{Z},$$

(2.5)

where $C_{ijk}$ is the completely symmetric, covariantly holomorphic rank-3 tensor, defined as

$$C_{ijk} = e^K (\partial_i X^\Lambda) (\partial_j X^{\Sigma}) (\partial_k X^{\Xi}) \partial_\Xi \partial_\Sigma F_\Lambda(X).$$

(2.6)

Starting from the general structure of the criticality conditions (2.5) and assuming also the non-degeneracy (i.e. $V_{BH}|_{\partial V_{BH}=0} > 0$) condition, as mentioned above the critical points of $V_{BH}$ (i.e. the attractors of $\mathcal{N} = 2, d = 4$ supergravity) can be classified in three general classes (17, 21, 18; see also [43] for a recent review):

1. The supersymmetric $\frac{1}{2}$-BPS class, determined by the constraints

$$Z \neq 0, \quad \mathcal{D}_i Z = 0, \forall i.$$  

(2.7)

The corresponding horizon ADM squared mass [49] saturates the BPS bound [50]:

$$M^2_{ADM} = V_{BH}|_{cr} = |Z|_{cr}^2 > 0,$$

(2.8)

where the BH effective potential and central charge are evaluated at critical points.
2. The non-BPS $\mathcal{Z} \neq 0$ class, determined by the constraints

$$\mathcal{Z} \neq 0, \quad \mathcal{D}_i \mathcal{Z} \neq 0 \quad \text{at least for one value of } i. \quad (2.9)$$

By using the properties of SK geometry, the non-BPS $\mathcal{Z} \neq 0$ horizon ADM squared mass is found not to saturate the BPS bound:

$$M^2_{ADM} = V_{BH} \bigg|_{cr} = 4 |\mathcal{Z}|^2_{cr} \quad (2.10)$$

3. The non-BPS $\mathcal{Z} = 0$ class, determined by the constraints

$$\mathcal{Z} = 0, \quad \mathcal{D}_i \mathcal{Z} \neq 0 \quad \text{at least for one value of } i. \quad (2.11)$$

As far as $g^{\mathcal{Z}}$ is strictly positive-definite the non-BPS $\mathcal{Z} = 0$ horizon ADM squared mass does not saturate the BPS bound \cite{9, 14, 16}:

$$M^2_{ADM} = V_{BH} \bigg|_{cr} = g^{\mathcal{Z}} (\partial_i \mathcal{Z}) (\partial_j \mathcal{Z}) \bigg|_{cr} > |\mathcal{Z}|^2_{cr}. \quad (2.12)$$

In general, the $n_V$ complex nonlinear equations (2.5) are very difficult to solve. Some simplification can be achieved by switching off some of the magnetic or electric charges, and eventually by formulating suitable ad hoc assumptions. The treatment given in the next Sections concerns particular examples the so-called special Kähler $d$-geometries \cite{44}, based on cubic holomorphic prepotentials:

$$F(X) = \frac{1}{3!} d_{ijk} \frac{X^i X^j X^k}{X^0} = (X^0)^2 f(z), \quad f(z) \equiv \frac{1}{3!} d_{ijk} z^i z^j z^k, \quad (2.13)$$

where the special coordinates $X^\Lambda = (X^0, X^0 z^i)$ have been introduced. We will deal with $f(z)$, by further fixing the Kähler gauge such that $X^0 \equiv 1$. Special Kähler $d$-geometries naturally arise in the large volume limit of Calabi-Yau compactifications of type IIA superstrings. They include all special Kähler coset manifolds, of which symmetric spaces are in turn a peculiar case.

As mentioned in the Introduction, $\frac{1}{2}$-BPS solutions to AEs (2.5) for $d$-geometries are known after \cite{46} and \cite{45}. Some non-BPS $\mathcal{Z} \neq 0$ solutions to AEs (2.5) for $d$-geometries has been explicitly obtained for a charge configuration $q_i = 0$ in \cite{10}. Such solutions satisfy the peculiar Ansatz

$$z^i_{cr} (q, p) = p^i \lambda (q, p), \quad (2.14)$$

where $\lambda$ is a complex function of the considered set of BH charges. While general $\frac{1}{2}$-BPS solutions always satisfy (2.14), this is not the case for all non-BPS solutions, as we will show for the $st^2$ and $stu$ models. Therefore we refer to (2.14) as a “BPS Ansatz”.

As it will be demonstrated below, among non-BPS $\mathcal{Z} \neq 0$ solutions there are those obeying BPS Ansatz as well as those violating it, while non-BPS solutions with vanishing central charge never satisfy (2.14).

3 $st^2$ Model

In the so-called $st^2$ model the scalar manifold is the rank-2 homogeneous symmetric space $\left( \frac{SU(1,1)}{U(1)} \right)^2$ ($\dim \mathcal{C} = 2$), parameterized by the moduli $z^1 \equiv s$ and $z^2 \equiv t$ with negative imaginary parts \cite{51}. 


The corresponding $d = 5$ parent theory is based on the rank-1 real special manifold $SO(1, 1)$, which in turn can be uplifted to pure $\mathcal{N} = 2$, $d = 6$ supergravity. In special coordinates the prepotential reads $f = st^2$.

Let us mention that the $st^2$ can be obtained e.g. by a “$t = u$ degeneracy” of the $stu$ model (see Sect. 4), or as the element $n = -1$ of the reducible cubic sequence of homogeneous symmetric SK manifolds $\mathbb{SU}_n(1, 1)$ (see e.g. [21] and Refs. therein), or also as $\text{Solv}_{SK}(-1, 0)$, i.e. as the homogeneous symmetric $P = 0$ element of the homogeneous (non-symmetric, except for $P = 0$) sequence $\text{Solv}_{SK}(-1, P)$ [52]. As the $stu$ model, it is contained in all homogeneous (not necessarily symmetric) SK $d$-geometries.

For a given set of BH charges ($q_0, q_1, q_2, p^0, p^1, p^2$) the superpotential, BH potential, contravariant metric and $C$-tensor are respectively given by:

$$W(s, t) = q_0 + q_1 s + q_2 t + p^0 s t^2 - p^1 t^2 - 2p^2 st,$$

$$V_{BH} = \frac{i}{2(s - \bar{s})(t - \bar{t})^2} \left[ |W(s, t)| + |W(s, \bar{t})| + (p^1 - p^0 s)(t - \bar{t})^2 \right] \cdot [W(s, t) + W(s, \bar{t}) + (p^1 - p^0 s)(t - \bar{t})^2] + 2(|W(s, t)|^2 + |W(s, \bar{t})|^2)$$

$$g^{ij} = -\text{diag} \left( (s - \bar{s})^2, \frac{1}{2}(t - \bar{t})^2 \right), \quad C_{stt} = \frac{2i}{(s - \bar{s})(t - \bar{t})^2}.$$

Correspondingly, the two complex AEs have the form

$$\begin{aligned}
&W(s, t) W(s, \bar{t}) + W(s, t) + (p^1 - p^0 s)(t - \bar{t})^2] + \\
&W(s, \bar{t}) W(s, t) + W(s, \bar{t}) + (p^1 - p^0 s)(t - \bar{t})^2] + \\
+W(s, \bar{t}) W(s, \bar{t}) + W(s, \bar{t}) + (p^1 - p^0 s)(t - \bar{t})^2] = 0.
\end{aligned}$$

### 3.1 Magnetic charge configuration

By putting $p^0 = 0$ and $q_1 = q_2 = 0$, the solutions of AEs (3.2) can be found to be ($s \equiv s_1 + is_2$ and $t \equiv t_1 + it_2$):

$$\begin{aligned}
&s_1 = t_1 = 0, \quad s_2 = \pm \sqrt{\frac{q_0 p^1}{p^2}}, \quad t_2 = -\sqrt{\frac{q_0}{p^1}}, \quad q_0 p^1 > 0, \quad p^2 \leq 0; \\
&s_1 = t_1 = 0, \quad s_2 = \pm \sqrt{-\frac{q_0 p^1}{p^2}}, \quad t_2 = -\sqrt{-\frac{q_0}{p^1}}, \quad q_0 p^1 < 0, \quad p^2 \leq 0.
\end{aligned}$$

Some of these solutions are known since [45] ($\frac{1}{2}$-BPS) and [11] (non-BPS $\mathcal{Z} \neq 0$ satisfying the Ansatz (2.11)).

The analysis of the first set of solutions (3.3) yields the following results:
| sign chosen for $s_2$ | class | $BPS$ Ansatz satisfied | BH charge domain |
|----------------------|-------|------------------------|-----------------|
| 1                   | +     | non-BPS $\mathcal{Z} = 0$ | $q_0 p^1 > 0, \ p^2 < 0$ |
| 2                   | -     | $\frac{1}{2}$-BPS      | yes             | $q_0 p^1 > 0, \ p^2 > 0$ |

Such solutions are stable, since the corresponding critical Hessian matrix has all strictly positive eigenvalues. The non-BPS $\mathcal{Z} = 0$ solution 2 turns out to violate the $BPS$ Ansatz (2.11) by a sign.

On the other hand, the analysis of the possible combinations of signs within the second set of solutions (3.4) yield the following results:

| sign chosen for $s_2$ | class | $BPS$ Ansatz | BH charge domain |
|----------------------|-------|--------------|-----------------|
| 3                   | -     | non-BPS $\mathcal{Z} \neq 0$ | yes             | $q_0 p^1 < 0, \ p^2 > 0$ |
| 4                   | +     | non-BPS $\mathcal{Z} \neq 0$ | no              | $q_0 p^1 < 0, \ p^2 < 0$ |

Such solutions belong to the class non-BPS $\mathcal{Z} \neq 0$. In agreement with [10], the corresponding real form of the $4 \times 4$ Hessian matrix has three strictly positive and one vanishing eigenvalue. As found in [36], such a massless Hessian mode is actually a “flat” direction of $V_{BH}$, spanning the non-BPS $\mathcal{Z} \neq 0$ moduli space $SO(1, 1)$, which is the real special manifold ($dim_{\mathbb{R}} = 1$) of the corresponding $d = 5$ supergravity theory (see Table 2 of [36]). Thus, solutions 3 and 4 are stable, up to a “flat” direction to all orders. Solution 4 violates $BPS$ Ansatz by a sign, and thus it shows that such an Ansatz implies some loss of generality, also when considering only the class of non-BPS $\mathcal{Z} \neq 0$ critical points of $V_{BH}$ in SK $d$-geometries within peculiar BH charge configurations.

 Beside the sets (3.3) and (3.4), the only other set of solutions of AE (3.2) is given by the following non-BPS $\mathcal{Z} \neq 0$ family, parameterized by $t_1 \equiv \text{Re} \ t$:

$$s_1 = -\frac{2q_0 p^1 t_1}{p^2 (q_0 - p^1 t_1^2)}, \quad s_2 = -\frac{q_0 + p^1 t_1^2}{q_0 - p^1 t_1^2} \sqrt{-\frac{q_0 p^1}{(p^2)^2}}, \quad t_2 = -\sqrt{-\frac{q_0}{p^1} - t_1^2}$$

(3.5)

and defined for $q_0 p^1 < 0, \ t_1^2 < -q_0 / p^1$. Let us also notice that in the limit $t_1 \to 0$ the solutions (3.5) coincide with the (3.3).

For all sets (3.3), (3.4) and (3.5) the minimal value of the BH effective potential is

$$V_{BH} \big|_{cr} = \frac{S_{BH}}{\pi} = 2\sqrt{|q_0 p^1| \ (p^2)^2}.$$  

(3.6)

### 3.2 General charge configuration

Let us now consider the $st^2$ AE (3.2) for a generic BH charge configuration. By using the geometrical data (3.1), such Eqs. can be recast in the following form:

$$\begin{align*}
\partial_s V_{BH} &= 2\overline{\mathcal{Z}} D_s \mathcal{Z} + i C_{ss} (g^tt)^2 (\overline{D_t \mathcal{Z}})^2 = 0; \\
\partial_t V_{BH} &= 2\overline{\mathcal{Z}} D_t \mathcal{Z} + 2i C_{stt} g^{s\bar{t}} g^{tt} (\overline{D_s \mathcal{Z}}) \overline{D_t \mathcal{Z}} = 0.
\end{align*}$$

(3.7)
Let us consider the non-BPS $Z = 0$ class of attractors. It is clear then that the only possibility for the BH effective potential to have an extremum is the following one:

$$Z = 0, \quad \mathcal{D}_t Z = 0.$$ 

The corresponding AEs read

$$\begin{align*}
q_0 + q_1 s + q_2 t + p^0 s t^2 - p^1 t^2 - 2p^2 s t = 0, \\
2q_0 + 2q_1 s + q_2 (t + \bar{t}) + 2p^0 st\bar{t} - 2p^1 t\bar{t} - 2p^2 s(t + \bar{t}) = 0,
\end{align*}$$

with solutions

$$s = \frac{p^1 q_1 - 2p^2 q_0 \pm i\sqrt{\mathcal{J}_4}}{2[(p^2)^2 - p^0 q_1]}, \quad t = \frac{p^1 q_1 - 2p^2 q_2 \mp i\sqrt{\mathcal{J}_4}}{2p^1 p^2 - p^0 q_2},$$

$$\mathcal{J}_4 = \mathcal{J}_4(p, q) = -(p^0 q_0 + p^1 q_1)^2 + (2p^1 p^2 - p^0 q_2)(2p^2 q_0 + q_1 q_2).$$

The newly appeared expression for $\mathcal{J}_4$ is nothing but unique quartic invariant of the $U$-duality group $SU(1,1)^2$ in the symplectic charge basis. The value of the entropy is given by

$$S = \pi \sqrt{\mathcal{J}_4(p, q)}.$$  

Let us note that sign twist yields that the critical value of the moduli (3.9) with $Z = 0$ relate to the corresponding ones with $Z \neq 0$ by the complex conjugation of some of them.

The BH charge configurations supporting the non-BPS $Z = 0$ attractors of the $s t^2$ model satisfy the following constraints (for an analysis of the supporting BH charge orbits see [21]):

$$\mathcal{J}_{4,ST^2}(p, q) > 0, \quad (p^2)^2 - p^0 q_1 \leq 0, \quad 2p^1 p^2 - p^0 q_2 \geq 0.$$  

## 4 $stu$ Model

In the so-called $stu$ model the scalar manifold is the rank-3 homogeneous symmetric space $\left(\frac{SU(1,1)}{U(1)}\right)^3$ ($dim_{\mathbb{C}} = 3$), parameterized by the moduli $z^1 \equiv s, z^2 \equiv t$ and $z^3 \equiv u$ with negative imaginary parts [51]. The corresponding $d = 5$ parent theory is based on the rank-2 real special manifold $(SO(1,1))^2$. In special coordinates the prepotential reads $f = stu$.

The $stu$ model exhibits the noteworthy triality symmetry, in which all three moduli $s, t$ and $u$ are on the same footing and it has been studied in [17, 46, 23, 33, 34, 36]. Due to this symmetry all expressions acquire quite elegant form.

It can be obtained as the element $n = 0$ of the reducible cubic sequence of homogeneous symmetric SK manifolds $\frac{SU(1,1)^c}{U(1)} \otimes \frac{SO(2,2+n)}{SO(2) \times SO(2+n)}$ (see e.g. [21] and Refs. therein). Furthermore, it is contained in all homogeneous (not necessarily symmetric) SK $d$-geometries.

For a general BH charge configuration the superpotential, BH effective potential, contravariant metric and $C$-tensor are respectively given by:

$$W(s, t, u) = q_0 + q_1 s + q_2 t + q_3 u + p^0 stu - p^1 tu - p^2 su - p^3 st,$$

$$V_{BH} = -i \frac{|W(s, t, u)|^2 + |W(s, t, u)|^2 + |W(s, t, u)|^2 + |W(s, t, u)|^2}{(s - \bar{s})(t - \bar{t})(u - \bar{u})},$$

$$g^{ij} = -\text{diag}((s - \bar{s})^2, (t - \bar{t})^2, (u - \bar{u})^2), \quad C_{stu} = -\left(\frac{s - \bar{s}(t - \bar{t})(u - \bar{u})}{(s - \bar{s})(t - \bar{t})(u - \bar{u})}\right)^i.$$
Correspondingly, the three complex AEs have the form

\[
\begin{align*}
W(s, \bar{t}, u) W(s, t, u) - 2W(s, \bar{t}, u) W(s, \bar{t}, \bar{u}) &= 0, \\
W(s, \bar{t}, u) W(s, \bar{t}, u) - 2W(s, \bar{t}, u) W(s, t, \bar{u}) &= 0, \\
W(s, t, \bar{u}) W(s, t, \bar{u}) - 2W(s, t, \bar{u}) W(s, \bar{t}, \bar{u}) &= 0,
\end{align*}
\]  

(4.2)

reflecting the above mentioned triality symmetry of the model.

### 4.1 Magnetic charge configuration

Attractor equations (4.2) are a system of nonlinear equations of sixth order. Therefore their solving in a general case constitutes a certain problem. Let us start by putting \( p_0 = 0 \) and \( q_1 = q_2 = q_3 = 0 \).

All \( \frac{1}{2} \)-BPS and some non-BPS \( \mathcal{Z} \neq 0 \) solutions can be obtained using BPS Ansatz \([10, 45, 46]\). Using a hint dropped by the \( st^2 \) model, we will search only for solutions to the AEs (4.2) of the form:

\[
z^i_{cr} (q, p) = \pm p^i \lambda (q, p),
\]  

(4.3)

with a choice of signs depending on \( i \). By choosing the same sign for all \( i \), one recovers the BPS Ansatz (2.14). One should stress that solutions of the form (4.3) for the magnetic charge configuration were as well derived in [23].

By triality symmetry, without loss of generality we can assume

\[
s = p^1 \lambda, \quad t = -p^2 \lambda, \quad u = p^3 \lambda.
\]  

(4.4)

Consequently, by solving the \( stu \) AEs (4.2) with \( p_0 = 0 \), \( q_1 = q_2 = q_3 = 0 \) and assuming (4.4), one gets for \( \lambda (q_0, p^1 p^2, p^3) \) the same solutions as above, namely:

1: \[ \left\{ \begin{array}{l}
\lambda_1 = 0, \\
\lambda_2 = \pm \sqrt{\frac{q_0}{p^1 p^2 p^3}},
\end{array} \right. \quad q_0 p^1 p^2 p^3 > 0,
\]

2: \[ \left\{ \begin{array}{l}
\lambda_1 = 0, \\
\lambda_2 = \pm \sqrt{-\frac{q_0}{p^1 p^2 p^3}},
\end{array} \right. \quad q_0 p^1 p^2 p^3 < 0.
\]  

(4.5)

Solution 1 is non-BPS \( \mathcal{Z} = 0 \), and it is stable (no Hessian massless modes \([33, 36]\)). Solution 2 is non-BPS \( \mathcal{Z} \neq 0 \), with superpotential \( W = 2q_0 \). The corresponding Hessian matrix splits \([10]\) in four massive and two massless modes and all the consideration done above for the solution 2 hold here, as well.

For all solutions the minimal value of the BH effective potential is

\[
V_{BH}^{\text{cr}} = \left. \frac{1}{\pi} S_{BH} \right| = 2 \sqrt{|q_0 p^1 p^2 p^3|}.
\]  

(4.6)

### 4.2 General Charge Configuration

Let us now consider the \( stu \) AEs (4.2) for a generic BH charge configuration. By using the geometrical data (4.1), such equations can be recast in the following form:

\[
\begin{align*}
\partial_s V_{BH} &= 2 \overline{Z} D_s Z + i C_{stu} g^{t \bar{u}} \left( \overline{D_t Z} \right) \overline{D_{\bar{u}} Z} = 0, \\
\partial_t V_{BH} &= 2 \overline{Z} D_t Z + i C_{stu} g^{s \bar{u}} \left( \overline{D_s Z} \right) \overline{D_{\bar{u}} Z} = 0, \\
\partial_u V_{BH} &= 2 \overline{Z} D_u Z + i C_{stu} g^{s t} \left( \overline{D_s Z} \right) \overline{D_t Z} = 0,
\end{align*}
\]  

(4.7)
Now it is clear that to find a zero central charge solution to these equations it is enough to make equal to zero two out of three components of the covariant derivative vector $\mathcal{D}_i Z$. Since the triality symmetry one can choose, for example,

$$\mathcal{D}_t Z = \mathcal{D}_u Z = 0.$$  

Therefore the total set of AEs read as follows

$$W(s, t, u) = W(s, \bar{t}, u) = W(s, t, \bar{u}) = 0$$  \hspace{1cm} \text{(4.8)}

and has the following solution

$$s = \frac{p^i q_i - 2 p^1 q_1 \pm i \sqrt{J_4}}{2 (p^2 p^3 - p^0 q_1)}, \quad t = \frac{p^i q_i - 2 p^2 q_2 \mp i \sqrt{J_4}}{2 (p^1 p^3 - p^0 q_2)}, \quad u = \frac{p^i q_i - 2 p^3 q_3 \mp i \sqrt{J_4}}{2 (p^1 p^2 - p^0 q_3)}. \hspace{1cm} \text{(4.9)}$$

The expression for the entropy is the same as for the $st^2$ model \cite{62, 63}, but with quartic invariant of the $U$-duality group $(SU(1,1))^3$ now given by

$$J_{stu} (p, q) \equiv - (p \cdot q)^2 + 4 (p^i q_i p^2 q_2 + p^1 q_1 p^3 q_3 + p^2 q_2 p^3 q_3) - 4 p^0 q_1 q_2 q_3 + 4 q_0 p^1 p^2 p^3. \hspace{1cm} \text{(4.10)}$$

In \cite{53} this invariant was recognized to be nothing but (the opposite of) the so-called Cayley’s hyperdeterminant \cite{54}. This fact leads to interesting developments, relating the physics of extremal BHs to quantum information theory \cite{55}-\cite{60}.

As it occurs in the $st^2$ model, for the $stu$ model non-BPS $Z = 0$ solutions are related to those with $Z \neq 0$ by the complex conjugations of some moduli.

The BH charge configurations supporting the non-BPS $Z = 0$ attractors of the $stu$ model satisfy the following constraints (for an analysis of the supporting BH charge orbits see \cite{21}):

$$J_{stu} (p, q) > 0, \hspace{0.5cm} p^2 p^3 - p^0 q_1 \leq 0, \hspace{0.5cm} p^1 p^3 - p^0 q_2 \geq 0, \hspace{0.5cm} p^1 p^2 - p^0 q_3 \geq 0. \hspace{1cm} \text{(4.11)}$$

It should be noted that the $stu$ AEs \cite{17} can be cast in the following form \cite{19}:

$$z_1 z_2 + \bar{z}_3 \bar{z}_4 = 0, \hspace{0.5cm} z_1 z_3 + \bar{z}_2 \bar{z}_4 = 0, \hspace{0.5cm} z_1 z_4 + \bar{z}_2 \bar{z}_3 = 0, \hspace{1cm} \text{(4.12)}$$

The $stu$ AEs in the form \cite{17} are identical to the AEs of $\mathcal{N} = 8, d = 4$ supergravity, where $z_i$ are the four complex eigenvalues of the central charge matrix $Z_{AB}$ ($A, B = 1, \ldots, 8$), which can be skew-diagonalized \cite{19, 21}. $\mathcal{N} = 2 \frac{1}{2}$-BPS and $\mathcal{N} = 2$ non-BPS $Z = 0$ solutions, both having $J_4 > 0$, are lifted to $\mathcal{N} = 8 \frac{1}{2}$-BPS solutions, with positive quartic Cartan-Cremmer-Julia invariant of the fundamental representation \cite{56} of the $\mathcal{N} = 8, d = 4$ $U$-duality group $E_{7(7)}$ \cite{62, 63}. On the other hand, $\mathcal{N} = 2$ non-BPS $Z \neq 0$ solutions $J_4 < 0$, are lifted to $\mathcal{N} = 8$ non-BPS solutions, with, with the negative quartic Cartan-Cremmer-Julia invariant. Such relations among $\mathcal{N} = 2$ and $\mathcal{N} = 8$ attractors have been analyzed in \cite{64, 65} for the BPS case and in \cite{33} for the non-BPS case.

One should recall that the moduli $s, t$ and $u$ can be interchanged through triality symmetry; the explicit signs in the relations \cite{19} is due to the “polarization” introduced by picking $\mathcal{D}_s Z \neq 0$ in solving the non-BPS $Z = 0$ AEs \cite{18}. From an $\mathcal{N} = 8$ perspective, $\mathcal{N} = 2$ non-BPS $Z = 0$ attractors are originated from $\mathcal{N} = 2 \frac{1}{2}$-BPS ones by simply exchanging two eigenvalues of the skew-diagonal matrix $Z_{AB}$, namely $z_1$ and $z_2$ in the conventions used above. Thus, the exchanging of two eigenvalues of the $\mathcal{N} = 8$ central charge matrix leads to complex conjugation of the descendant attractor solutions in the $\mathcal{N} = 2$ supergravity obtained by performing a consistent supersymmetry truncation $\mathcal{N} = 8 \rightarrow \mathcal{N} = 2$. 

9
5 Cayley’s Hyperdeterminant and its relation with $\mathcal{J}_{4,t^3}$ and $\mathcal{J}_{4,stu^2}$

In this Section we reconsider Cayley’s hyperdeterminant, and point out its relation with the invariant $\mathcal{J}_{4,t^3}$ (of the $U$-duality group $SU(1,1)$) of the so-called $t^3$ model [32] and with the invariant $\mathcal{J}_{4,stu^2}$ of the $stu^2$ model, given by Eq. (3.10).

Let us start by recalling once again that the $stu$ model is based on the rank-3 homogeneous symmetric reducible SK manifold $\frac{G_{4,stu}}{H_{4,stu}} = \left( \frac{SU(1,1)}{U(1)} \right)^3$. It also holds that $SU(1,1) \sim SL(2,\mathbb{R}) \sim Sp(2,\mathbb{R})$, and that the symplectic group of the $stu$ model is $Sp(2n_V + 2,\mathbb{R}) = Sp(8,\mathbb{R})$. The embedding of the $U$-duality group $G_{4,stu} = (SU(1,1))^3$ into $Sp(8,\mathbb{R})$ is encoded by the spinor $\Psi_{\alpha\beta\gamma}$, sitting in a real, spin $s = \frac{3}{2}$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$-representation of $G_{4,stu}$, with $\alpha, \beta, \gamma = 0, 1$. $\Psi_{\alpha\beta\gamma}$ has no particular properties of symmetry on its indices, thus it may get 24 elements of the cubic reducible sequence (see Eq. (2.8) of [55]).

Thus, $\{\Psi_{\alpha\beta\gamma}\}_{\alpha,\beta,\gamma = 0,1}$ is a basis for the BH charges, other than the symplectic one (5.1). The relation between such two basis is given in by Eq. (3.5) of [55].

The quartic (and unique) invariant of the $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$-representation of $G_{4,stu}$ can be defined as (see Eq. (2.8) of [55])

$$\mathcal{J}_{4,stu} = \frac{1}{2}\Psi_{\alpha_1\alpha_2\alpha_3}\Psi_{\beta_1\beta_2\beta_3}\Psi_{\gamma_1\gamma_2\gamma_3}\Psi_{\delta_1\delta_2\delta_3}\epsilon^{\alpha_1\beta_1\epsilon^{\alpha_2\beta_2}\epsilon^{\gamma_1\gamma_2}\epsilon^{\alpha_3\gamma_3}\epsilon^{\beta_3\delta_3}},$$

(5.2)

where $\epsilon$ is the metric of $SU(1,1)$. In particular, notice that no quadratic invariant of $G_{4,stu}$ exists, because $\Psi_{\alpha_1\alpha_2\alpha_3}\Psi_{\beta_1\beta_2\beta_3}\epsilon^{\alpha_1\beta_1}\epsilon^{\alpha_2\beta_2}\epsilon^{\alpha_3\beta_3} = 0$. In the symplectic basis $\mathcal{J}_{4,stu}$ can be rewritten as given by Eq. (4.10). Thus, $\mathcal{J}_{4,stu}$ can actually be recognized [53] as the opposite of the so-called Cayley’s hyperdeterminant [54]:

$$\mathcal{J}_{4,stu} = -Det(\Psi).$$

(5.3)

A manifestly $SO(2, 2)$-invariant form of $\mathcal{J}_{4,stu}$ can be obtained by going to the so-called hatted symplectic basis:

$$\hat{Q}_{stu} \equiv (\hat{q}_0, \hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{p}_0^\dagger, \hat{p}_1^\dagger, \hat{p}_2^\dagger, \hat{p}_3^\dagger)_{stu} = (\hat{q}_\Lambda, \hat{p}_\Lambda^A)_{stu,\Lambda = 0, 1, 2, 3},$$

(5.4)

whose relation with the symplectic basis (5.1) is given by Eq. (59) of [46]. Notice that $\hat{q}_\Lambda$ and $\hat{p}_\Lambda^A$ fit two distinct copies of the same vector representation of $SO(2, 2)$. In the $\hat{Q}$-basis $\mathcal{J}_{4,stu}$ reads (see Eq. (63) of [46])

$$\mathcal{J}_{4,stu} = (\hat{p}_\Lambda^A)^2 (\hat{q}_\Lambda^A)^2 - (\hat{p}_\Lambda \cdot \hat{q}_\Lambda)^2,$$

(5.5)

where $(\hat{p}_\Lambda)^2 \equiv (\hat{p}_0^\dagger)^2 + (\hat{p}_1^\dagger)^2 - (\hat{p}_2^\dagger)^2 - (\hat{p}_3^\dagger)^2$, and analogously for $(\hat{q}_\Lambda)^2$.

Since the $stu$ model is the $n = 0$ element of the cubic reducible sequence $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,2+n)}{SO(2) \otimes SO(2n)}$ ($n_V = n + 3, n \in \mathbb{N} \cup \{0, -1\}$) of homogeneous symmetric SK manifolds, it is easy find that the manifestly $SO(2, 2 + n)$-invariant of the quartic (and unique) invariant of the $U$-duality group $SU(1,1) \otimes SO(2, 2 + n)$ reads as follows:

$$\mathcal{J}_{4, SU(1,1) \otimes SO(2, 2 + n)} = (\hat{p}_n^\dagger)^2 (\hat{q}_n^\dagger)^2 - (\hat{p}_n \cdot \hat{q}_n)^2,$$

(5.6)
where the scalar products are now taken in $SO(2,2+n)$:

$$\langle \hat{p} \cdot \hat{q} \rangle_n \equiv \sum_{\Lambda=0}^{n+3} p^\Lambda q_\Lambda, \quad (\hat{p})_n^2 \equiv (\hat{p}^0)^2 + (\hat{p}^1)^2 - \ldots - (\hat{p}^{n+3})^2,$$

and analogously for $\langle \hat{q} \cdot \hat{p} \rangle_n^2$. In this case, $\hat{q}_\Lambda$ and $\hat{p}^\Lambda$ fit two distinct copies of the same vector representation of $SO(2,2+n)$. Clearly, $\mathcal{J}_{4,stu} |_{_{\nu=0}} = \mathcal{J}_{4,stu}$. It should also be recalled that $G_{4,stu}$ in general acts linearly only in the basis $\{\Psi_{\alpha\beta\gamma}\}_{\alpha,\beta,\gamma=0,1}^3$, where each index is related to a different factor $SU(1,1)$ inside $G_{4,stu}$.

Let us now consider the quartic (and unique) invariants of the $U$-duality groups of the $t^3$ model [32], based on the rank-1 homogeneous symmetric SK manifold $SU(1,1)$, and of the $st^2$ model, based on the rank-2 homogeneous symmetric SK manifold $SU(1,1)$/$\mathbb{U}(1)$, and performing the following identifications and rescaling $\eta^3$.

In the symplectic basis

$$Q_{t^3} \equiv (q_0, q_1, p^0, p^1)_{t^3} = (q_{\Lambda}, p^\Lambda)_{t^3, \Lambda=0,1},$$

the quartic (and unique) invariant of the $U$-duality group $SU(1,1)$ of the $t^3$ model, denoted by $\mathcal{J}_{4,t^3}$, reads [32]

$$\mathcal{J}_{4,t^3}(p,q) \equiv -\frac{4}{9} p^0(q_1)^3 + \frac{1}{3} (p^1)^2 (q_1)^2 - (p^0)^2 (q_0)^2 - 2p^0q_0p^1q_1 + \frac{4}{3} (p^1)^3 q_0. \quad (5.9)$$

On the other hand, in the symplectic basis

$$Q_{st^2} \equiv (q_0, q_1, q_2, p^0, p^1, p^2)_{st^2} = (q_{\Lambda}, p^\Lambda)_{st^2, \Lambda=0,1,2},$$

the quartic (and unique) invariant of the $U$-duality group $(SU(1,1))^2$ of the $st^2$ model, denoted by $\mathcal{J}_{4,stu}$, is given by Eq. (5.10).

The basis $\{\Psi_{\alpha\beta\gamma}\}_{\alpha,\beta,\gamma=0,1}^3$ for the BH charges of the $stu$ model allows one to relate $\mathcal{J}_{4,stu}$ with $\mathcal{J}_{4,t^3}$ and $\mathcal{J}_{4,st^2}$.

The relation between $\mathcal{J}_{4,stu}$ and $\mathcal{J}_{4,t^3}$ can be simply achieved by totally symmetrizing the 3-index spinor $\Psi_{\alpha\beta\gamma}$:

$$stu : \Psi_{\alpha\beta\gamma} \rightarrow t^3 : \Psi_{(\alpha\beta\gamma)}, \quad (5.11)$$

yielding $p^1 = p^2 = p^3$ and $q_1 = q_2 = q_3$ in the $stu$ model. In such a way, the reduced symplectic BH charge vector of the $stu$ model becomes

$$\tilde{Q}_{stu} \equiv Q_{stu} |_{p^0=p^2=p^3,q_1=q_2=q_3} = (q_0, q_1, p^0, p^1)_{stu} \quad (5.12)$$

In the (semi)classical limit of real, large BH charges, one assumes in general $\tilde{Q}_{stu}$ and $Q_{t^3}$ to be related by the following anisotropic rescaling

$$\langle q_0, q_1, p^0, p^1 \rangle_{stu} = (\eta_1 q_0, \eta_2 q_1, \eta_3 p^0, \eta_4 p^1)_{t^3}, \quad \eta_1, \eta_2, \eta_3, \eta_4 \in \mathbb{R}. \quad (5.13)$$

By comparing $\mathcal{J}_{4,stu} |_{p^0=p^2=p^3,q_1=q_2=q_3}$ with $\mathcal{J}_{4,t^3}$, one obtains $\eta_1 = (\eta_3)^{-1}$, $\eta_2 = (9\eta_3)^{-3}$ and $\eta_4 = (\eta_3/3)^{3/3}$. Thus, in the symplectic basis, $\mathcal{J}_{4,t^3}$ can be obtained from $\mathcal{J}_{4,stu}$ by starting from $Q_{stu}$ and performing the following identifications and rescalings ($\eta_3 \in \mathbb{R}$):

$$\left\{ \begin{array}{l}
  q_{0,stu} = (\eta_3)^{-1} q_{0,t^3}; \quad q_{1,stu} = q_{2,stu} = q_{3,stu} = (9\eta_3)^{-3/3} q_{1,t^3}; \\
  p_{stu}^0 = \eta_3 p_{t^3}^0; \quad p_{stu}^1 = (\eta_3/3)^{1/3} p_{t^3}^1.
\end{array} \right. \quad (5.14)$$

11
On the other hand, the relation between $J_{4,stu}$ and $J_{4,stu^2}$ can be simply achieved by symmetrizing the 3-index spinor $\Psi_{\alpha\beta\gamma}$ only on two indices, say (by triality symmetry, without loss of generality) the first two ones:

$$stu : \Psi_{\alpha\beta\gamma} \longrightarrow st^2 : \Psi_{(\alpha\beta)\gamma}, \quad (5.15)$$

yielding $p^2 = p^3$ and $q_2 = q_3$ in the $stu$ model. In such a way, the reduced symplectic BH charge vector of the $stu$ model becomes

$$\tilde{Q}_{stu} \equiv Q_{stu}|_{p^2=p^3, q_2=q_3} = (q_0, q_1, q_2, p^0, p^1, p^2)_{stu} \quad (5.16)$$

As above, in the (semi)classical limit of real, large BH charges, one assumes in general $\tilde{Q}_{stu}$ and $Q_{stu^2}$ to be related by the following anisotropic rescaling

$$(q_0, q_1, q_2, p^0, p^1, p^2)_{stu} = (\theta_1 q_0, \theta_2 q_1, \theta_3 q_2, \theta_4 p^0, \theta_5 p^1, \theta_6 p^2)_{stu^2}, \ \theta_1, \ldots, \theta_6 \in \mathbb{R}. \quad (5.17)$$

By comparing $J_{4,stu}|_{p^2=p^3, q_2=q_3}$ with $J_{4,stu^2}$, one obtains $\theta_1 = (\theta_4)^{-1}$, $\theta_2 = (\theta_5)^{-1}$, $\theta_3, \theta_5, \theta_6 = \pm \frac{1}{2} (\theta_5/\theta_4)^{1/2}$ and $\theta_6, \theta_5, \theta_6 = \pm (\theta_1/\theta_5)^{1/2}$. Thus, in the symplectic basis, $J_{4,stu^2}$ can be obtained from $J_{4,stu}$ by starting from $Q_{stu}$ and performing the following identifications and rescalings ($\theta_4, \theta_5 \in \mathbb{R}, \theta_4 \theta_5 > 0$):

$$\begin{cases}
q_0_{stu} = (\theta_4)^{-1} q_{0,stu^2}; & q_{1,stu} = (\theta_5)^{-1} q_{1,stu^2}; & q_{2,stu} = q_{3,stu} = \pm \frac{1}{2} (\theta_5/\theta_4)^{1/2} q_{2,stu^2}; \\
p^0_{stu} = \theta_4 p^0_{stu^2}; & p^1_{stu} = \theta_5 p^1_{stu^2}; & p^2_{stu} = p^2_{stu^2} = \pm (\theta_4/\theta_5)^{1/2} p^2_{stu^2}. 
\end{cases} \quad (5.18)$$

The positions (5.14) and (5.18) on the $stu$ BH charges are the simplest ones determining the degeneracy of the quartic invariant $J_{4,stu}$ (which is actually the opposite of the Cayley’s hyperdeterminant $Det (\Psi)$) into $J_{4,t^3}$ and $J_{4,stu^2}$, respectively. Such positions have a nice interpretation as symmetrization of indices in the basis $\{\Psi_{\alpha\beta\gamma}\}_{\alpha,\beta,\gamma=0,1}$ for the $stu$ BH charges, because they respectively correspond to Eqs. (5.11) and (5.15).

## 6 Conclusion

In this work we derived the explicit non-BPS $Z = 0$ solutions to Attractor Equations for generic BH charge configurations in two homogeneous symmetric $N = 2, d = 4$ supergravity models, namely the so-called $st^2$ and $stu$ models.

Such non-supersymmetric solutions with vanishing central charge were so far unknown, for two main reasons:

1. the non-BPS $Z = 0$ class of attractors is not present in the so-called $t^3$ model (so far, the only one in which the AEs were explicitly and completely solved for generic BH charge configurations [32]),

2. the non-BPS $Z = 0$ class of attractors does not satisfy the so-called BPS Ansatz. Indeed, it may be shown that the non-BPS $Z = 0$ constraints (2.11) are not consistent with the assumption (2.14) in the framework of a generic cubic SK geometry.

Furthermore, for the considered models we found that the non-BPS $Z = 0$ critical moduli are (partially) related by complex conjugation to their $t^3$ BPS counterparts, whose explicit form for a generic SK $d$-geometry was known since [46] and [45]. Moreover, such a relation has a nice $N = 8$
interpretation in terms of exchange of two of the four eigenvalues of the $\mathcal{N} = 8$ central charge matrix $Z_{AB}$.

It is worth pointing out that the non-BPS $Z = 0$ attractors of $\mathcal{N} = 2$, $d = 4$ ungauged supergravity might have noteworthy phenomenological implications, because they correspond to stabilized, purely charge-dependent configurations of the scalars at the BH event horizon (with geometry $AdS_2 \times S^2$) for which the horizon maximal $\mathcal{N} = 2$ supersymmetry algebra $\text{psu}(1,1|2)$ is not centrally extended.

A possible field of application of the analytical formulæ (3.9) and (4.9) for non-BPS $Z = 0$ critical moduli in the $st^2$ and $stu$ models is the recently established connection between the thermodynamics of extremal BHs in supergravity and quantum information theory (also concerning, for homogeneous symmetric supergravities, the so-called Jordan algebras; see e.g. [21], [66], [67] and Refs. therein). As pointed out in Sect. [4] such a relation stems out from the observation, firstly made in [53], that the quartic (and unique) invariant $J_{4,stu}(p,q)$ of the $U$-duality group $(SU(1,1))^3$ in the symplectic charge basis is nothing but the Cayley’s hyperdeterminant [54]. In this sense, the explicit relations (derived in Sect. [3] between Cayley’s hyperdeterminant and the quartic invariants of the $U$-duality groups of the $t^3$ [32] and $st^2$ models might be relevant for further developments. Interesting advances along this line of research have been achieved in the last months [55]-[61], and hopefully many others are to come in next future.

Acknowledgments

We would like to thank S. Ferrara for enlightening and fruitful discussions.

The work of S.B., E.O and A.S. has been supported in part by the European Community Human Potential Program under contract MRTN-CT-2004-005104 “Constituents, fundamental forces and symmetries of the universe”.

The work of A.M. has been supported by a Junior Grant of the “Enrico Fermi” Center, Rome, in association with INFN Frascati National Laboratories.

References

[1] S. Ferrara, R. Kallosh and A. Strominger, $\mathcal{N}= 2$ Extremal Black Holes, Phys. Rev. D52, 5412 (1995), hep-th/9508072.

[2] A. Strominger, Macroscopic Entropy of $\mathcal{N}= 2$ Extremal Black Holes, Phys. Lett. B383, 39 (1996), hep-th/9602111.

[3] S. Ferrara and R. Kallosh, Supersymmetry and Attractors, Phys. Rev. D54, 1514 (1996), hep-th/9602136.

[4] S. Ferrara and R. Kallosh, Universality of Supersymmetric Attractors, Phys. Rev. D54, 1525 (1996), hep-th/9603090.

[5] S. Ferrara, G. W. Gibbons and R. Kallosh, Black Holes and Critical Points in Moduli Space, Nucl. Phys. B500, 75 (1997), hep-th/9702103.

[6] A. Sen, Black Hole Entropy Function and the Attractor Mechanism in Higher Derivative Gravity, JHEP 09, 038 (2005), hep-th/0506177.
K. Goldstein, N. Iizuka, R. P. Jena and S. P. Trivedi, *Non-Supersymmetric Attractors*, Phys. Rev. **D72**, 124021 (2005), hep-th/0507096.

A. Sen, *Entropy Function for Heterotic Black Holes*, JHEP **03**, 008 (2006), hep-th/0508042.

R. Kallosh, *New Attractors*, JHEP **0512**, 022 (2005), hep-th/0510024.

P. K. Tripathy and S. P. Trivedi, *Non-Supersymmetric Attractors in String Theory*, JHEP **0603**, 022 (2006), hep-th/0511117.

A. Giryavets, *New Attractors and Area Codes*, JHEP **0603**, 020 (2006), hep-th/0511215.

R. Kallosh, *New Attractors and Area Codes*, JHEP **0603**, 022 (2006), hep-th/0511215.

M. Alishahiha and H. Ebrahim, *Non-supersymmetric attractors and entropy function*, JHEP **0603**, 003 (2006), hep-th/0601016.

R. Kallosh, N. Sivanandam and M. Sorosh, *The Non-BPS Black Hole Attractor Equation*, JHEP **0603**, 060 (2006), hep-th/0602005.

B. Chandrasekhar, S. Parvizi, A. Tavanfar and H. Yavartanoo, *Non-supersymmetric attractors in $R^2$ gravities*, JHEP **0608**, 004 (2006), hep-th/0602022.

J. P. Hsu, A. Maloney and A. Tomasiello, *Black Hole Attractors and Pure Spinors*, JHEP **0609**, 048 (2006), hep-th/0602142.

S. Bellucci, S. Ferrara and A. Marrani, *On some properties of the Attractor Equations*, Phys. Lett. **B635**, 172 (2006), hep-th/0602161.

S. Bellucci, S. Ferrara and A. Marrani, *Supersymmetric Mechanics. Vol.2: The Attractor Mechanism and Space-Time Singularities* (LNP **701**, Springer-Verlag, Heidelberg, 2006).

S. Ferrara and R. Kallosh, *On $\mathcal{N}=8$ attractors*, Phys. Rev. D **73**, 125005 (2006), hep-th/0603247.

M. Alishahiha and H. Ebrahim, *New attractor, Entropy Function and Black Hole Partition Function*, JHEP **0611**, 017 (2006), hep-th/0605279.

S. Bellucci, S. Ferrara, M. Günyaydin and A. Marrani, *Charge Orbits of Symmetric Special Geometries and Attractors*, Int. J. Mod. Phys. **A21**, 5043 (2006), hep-th/0606209.

D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen and S. P. Trivedi, *Rotating Attractors*, JHEP **0610**, 058 (2006), hep-th/0606244.

R. Kallosh, N. Sivanandam and M. Sorosh, *Exact Attractive non-BPS STU Black Holes*, Phys. Rev. **D74**, 065008 (2006), hep-th/0606263.

P. Kaura and A. Misra, *On the Existence of Non-Supersymmetric Black Hole Attractors for Two-Parameter Calabi-Yau’s and Attractor Equations*, hep-th/0607132.

G. L. Cardoso, V. Grass, D. Lüst and J. Perz, *Extremal non-BPS Black Holes and Entropy Extremization*, JHEP **0609**, 078 (2006), hep-th/0607202.
[26] S. Bellucci, S. Ferrara, A. Marrani and A. Yeranyan, *Mirror Fermat Calabi-Yau Threefolds and Landau-Ginzburg Black Hole Attractors*, hep-th/0608091.

[27] G.L. Cardoso, B. de Wit and S. Mahapatra, *Black hole entropy functions and attractor equations*, hep-th/0612225.

[28] R. D’Auria, S. Ferrara and M. Trigiante, *Critical points of the Black-Hole potential for homogeneous special geometries*, hep-th/0701090.

[29] S. Bellucci, S. Ferrara and A. Marrani, *Attractor Horizon Geometries of Extremal Black Holes*, Contribution to the Proceedings of the XVII SIGRAV Conference, 4–7 September 2006, Turin, Italy, hep-th/0702019.

[30] A. Ceresole and G. Dall’Agata, *Flow Equations for Non-BPS Extremal Black Holes*, JHEP 0703, 110 (2007), hep-th/0702088.

[31] L. Andrianopoli, R. D’Auria, S. Ferrara and M. Trigiante, *Black Hole Attractors in $\mathcal{N}=1$ Supergravity*, hep-th/0703178.

[32] K. Saraikin and C. Vafa, *Non-supersymmetric Black Holes and Topological Strings*, hep-th/0703214.

[33] S. Ferrara and A. Marrani, $\mathcal{N}=8$ non-BPS Attractors, Fixed Scalars and Magic Supergravities, ArXiV:0705.3866.

[34] S. Nampuri, P. K. Tripathy and S. P. Trivedi, *On The Stability of Non-Supersymmetric Attractors in String Theory*, ArXiV:0705.4554.

[35] L. Andrianopoli, R. D’Auria, E. Orazi, M. Trigiante, *First Order Description of Black Holes in Moduli Space*, ArXiV:0706.0712.

[36] S. Ferrara and A. Marrani, *On the Moduli Space of non-BPS Attractors for $\mathcal{N}=2$ Symmetric Manifolds*, ArXiV:0706.1667, accepted for publication in Phys. Lett. B.

[37] G. L. Cardoso, A. Ceresole, G. Dall’Agata, J. M. Oberreuter, J. Perz, *First-order flow equations for extremal black holes in very special geometry*, ArXiV:0706.3373.

[38] A. Misra and P. Shukla, *Area codes*, large volume (non-)perturbative alpha-prime and instanton: Corrected non-supersymmetric (A)dS minimum, the ‘inverse problem’ and ‘fake superpotentials’ for multiple-singular-loci-two-parameter Calabi-Yau’s, ArXiV:0707.0105.

[39] A. Ceresole, S. Ferrara and A. Marrani, *4d/5d Correspondence for the Black Hole Potential and its Critical Points*, ArXiV:0707.0964.

[40] T. Levi-Civita, R.C. Acad. Lincei 26, 519 (1917).

[41] B. Bertotti, *Uniform Electromagnetic Field in the Theory of General Relativity*, Phys. Rev. 116, 1331 (1959).

[42] I. Robinson, Bull. Acad. Polon. 7, 351 (1959).
[43] L. Andrianopoli, R. D’Auria, S. Ferrara and M. Trigiante, *Extremal Black Holes in Supergravity*, in: “String Theory and Fundamental Interactions”, M. Gasperini and J. Maharana eds. (LNP, Springer, Berlin-Heidelberg, 2007), hep-th/0611345.

[44] B. de Wit, F. Vanderseypen and A. Van Proeyen, *Symmetry Structures of Special Geometries*, Nucl. Phys. **B400**, 463 (1993), hep-th/9210068.

[45] M. Shmakova, *Calabi-Yau black holes*, Phys. Rev. **D56**, 540 (1997), hep-th/9612076.

[46] K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova and W. K. Wong, *STU Black Holes and String Triality*, Phys. Rev. **D54**, 6293 (1996), hep-th/9608059.

[47] M. J. Duff, J. T. Liu and J. Rahmfeld, *Four-dimensional string/string/string triality*, Nucl. Phys. **B459**, 125 (1996), hep-th/9508094.

[48] A. Ceresole, R. D’Auria and S. Ferrara, *The Symplectic Structure of $\mathcal{N}=2$ Supergravity and Its Central Extension*, Talk given at ICTP Trieste Conference on Physical and Mathematical Implications of Mirror Symmetry in String Theory, Trieste, Italy, 5-9 June 1995, Nucl. Phys. Proc. Suppl. **46** (1996), hep-th/9509160.

[49] R. Arnowitt, S. Deser and C. W. Misner, *The Dynamics of General Relativity*, in: “Gravitation: an Introduction to Current Research”, L. Witten ed. (Wiley, New York, 1962).

[50] G. W. Gibbons and C. M. Hull, *A Bogomol’ny Bound for General Relativity and Solitons in $\mathcal{N}=2$ Supergravity*, Phys. Lett. **B109**, 190 (1982).

[51] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (Dover Publications, 2006).

[52] P. Fré, F. Gargiulo, J. Rosseel, K. Rulik, M. Trigiante and A. Van Proeyen, *Tits-Satake projections of homogeneous special geometries*, Class. Quant. Grav. **24**, 27 (2007), hep-th/0606173.

[53] M.J. Duff, *String triality, black hole entropy and Cayley’s hyperdeterminant*, hep-th/0601134.

[54] A. Cayley, *On the theory of linear transformations*, Camb. Math. J. **4**, 193 (1845).

[55] R. Kallosh and A. Linde, *Strings, Black Holes and Quantum Information*, Phys. Rev. **D73**, 104033 (2006), hep-th/0602061.

[56] P. Lévy, *Stringy Black Holes and the Geometry of the Entanglement*, Phys. Rev. **D74**, 024030 (2006), hep-th/0603136.

[57] M.J. Duff and S. Ferrara, *$E_7$ and the tripartite entanglement of seven qubits*, quant-ph/0609227.

[58] P. Lévy, *Strings, black holes, the tripartite entanglement of seven qubits and the Fano plane*, hep-th/0610314.

[59] M.J. Duff and S. Ferrara, *Black hole entropy and quantum information*, hep-th/0612036.

[60] M.J. Duff and S. Ferrara, *$E_6$ and the and the bipartite entanglement of three qutrits*, ArXiV:0704.0507.
[61] P. Lévay, *Black Holes, Attractors and Multipartite Entanglement*, lectures given at the School on Attractor Mechanism 2007 (SAM07), 18-22 June 2007, INFN-LNF, Frascati, Italy; to be published on the School Proceedings.

[62] E. Cartan, *Œuvres complètes* (Editions du Centre National de la Recherche Scientifique, Paris, 1984).

[63] E. Cremmer and B. Julia, *The SO(8) Supergravity*, Nucl. Phys. **B159**, 141 (1979).

[64] L. Andrianopoli, R. D’Auria and S. Ferrara, *U-Invariants, Black-Hole Entropy and Fixed Scalars*, Phys. Lett. **B403**, 12 (1997), hep-th/9703156.

[65] L. Andrianopoli, R. D’Auria and S. Ferrara, *Flat Symplectic Bundles of $\mathcal{N}$-Extended Supergravities, Central Charges and Black-Hole Entropy*, Lectures given at the 5th Winter School on Mathematical Physics held at the Asia Pacific Center for Theoretical Physics, Seul (Korea), Feb. 1997, hep-th/9707203.

[66] M. Rios, *Jordan Algebras and Extremal Black Holes*, based on talk given at the 26th International Colloquium on Group Theoretical Methods in Physics, hep-th/0703238.

[67] S. Ferrara, E. G. Gimon and R. Kallosh, *Magic supergravities, $\mathcal{N} = 8$ and black hole composites*, Phys. Rev. **D74**, 125018 (2006), hep-th/0606211.