Dissipation in systems of linear and nonlinear quantum scissors

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We analyze truncation of coherent states up to a single-photon Fock state by applying linear quantum scissors, utilizing the projection synthesis in a linear optical system, and nonlinear quantum scissors, implemented by periodically driven cavity with a Kerr medium. Dissipation effects on optical truncation are studied in the Langevin and master equation approaches. Formulas for the fidelity of lossy quantum scissors are found.

**Keywords:** Quantum state engineering, Kerr effect, qubit generation, finite-dimensional coherent states

I. INTRODUCTION

The breathtaking advances in quantum computation and quantum information processing in the last decade [1] have stimulated progress in quantum optical state generation and engineering [2]. Among various schemes, the proposal of Pegg, Phillips and Barnett [3, 4] of the optical state truncation via projection synthesis has attracted considerable interest [5–20], due to the simplicity of the scheme to generate and teleport ‘flying’ qubits defined as a running wave superpositions of zero- and single-photon states. The scheme is referred to as linear quantum scissors (LQS) since the coherent state entering the system is truncated in its Fock expansion to the first two terms (see, e.g., [22, 23]), the quantum-statistical properties of dissipative systems are usually treated in three ways, by applying (i) the Langevin (Langevin-Heisenberg) equations of motion with stochastic forces, (ii) the master equation for the density matrix, and (iii) the classical Fokker-Planck equation for quasiprobability distribution. In the next section we will apply the Langevin approach to describe dissipative LQS, while in section 3 we shall use the master equation approach to study dissipative NQS.

II. LOSSY LINEAR QUANTUM SCISSORS IN THE LANGEVIN APPROACH

The linear quantum scissors device of Pegg, Phillips and Barnett [3, 4] is a simple physical system for optical state truncation based only on linear optical elements (two beam splitters BS1 and BS2) and two photodetectors (D2 and D3) as depicted in figure 1. If the input modes $a_1$ and $a_2$ are in the single-photon and vacuum states, respectively, and one photon is detected at D2 but no photons at D3, then the lossless LQS device with 50/50 beam splitters truncates the input coherent state $|\alpha\rangle$ in mode $b_3$ to the following superposition of vacuum and single-photon states in mode $b_1$

$$|\alpha_{\text{trunc}}\rangle_{b_1} = N'_{c2e3}|10\rangle_{\psi_{\text{out}}} = \frac{|0\rangle_{b_1} + \alpha|1\rangle_{b_1}}{\sqrt{1 + |\alpha|^2}}$$

where $\alpha$ is the complex amplitude and $N'$ is a renormalization constant. The state $|\alpha\rangle_{b_1}$ is referred to as the truncated two-dimensional (or two-level) coherent state since it is the normalized superposition of the first two terms of Fock expansion of the Glauber coherent state. By introducing a new variable $\tilde{\alpha}$ such that $\cos(|\tilde{\alpha}|) = 1/\sqrt{1 + |\alpha|^2}$ and $\sin(|\tilde{\alpha}|) = |\alpha|/\sqrt{1 + |\alpha|^2}$, and $\varphi = \text{Arg}\alpha$, state $|\alpha\rangle_{b_1}$ can be rewritten as

$$|\alpha_{\text{trunc}}\rangle = \cos(|\tilde{\alpha}|)|0\rangle + e^{i\varphi} \sin(|\tilde{\alpha}|)|1\rangle$$

where, for brevity, subscript $b_1$ is skipped. If the $j$th ($j = 1, 2$) beam splitter has an arbitrary but real transmission
coefficient $t_j$ and an imaginary reflection coefficient $r_j$, then the LQS generates the state
\[
|\psi\rangle_{b1} = \frac{|r_1 t_2||0\rangle_{b1} + \alpha r_2 t_1|1\rangle_{b1}}{\sqrt{|r_1 t_2|^2 + |\alpha|^2|r_2 t_1|^2}}.
\] (3)

This state evolves into the truncated coherent state by assuming identical BSs ($r_1 = r_2$ and $t_1 = t_2$).

In general, the transmission and reflection coefficients of a perfect BS obey the conditions $|t|^2 + |r|^2 = 1$ and $tr^* + tr = 0$, implied by the unitarity of BS transformation. By including dissipation, these conditions can be violated. Thus, the main goal of this section is to analyze the deterioration of the truncation process due to the noise introduced by lossy beam splitters and also by inefficient photodetectors. In the simplest approach, one can model the BS losses and finite detector efficiency by adding to our system additional beam splitters, then all components of the system (including the new BSs) can be assumed perfect. Here, we apply another standard approach of the quantum theory of damping based on the Langevin noise operators and components of the system (including the new BSs) can be assumed perfect. First, we apply another standard approach of the quantum theory of damping based on the Langevin noise operators. We follow the analyses of Barnett et al. and Villas-Bôas et al. The lossy BS1 transforms the input annihilation operators $\hat{a}_j$ into the output $\hat{b}_j$ as follows
\[
\begin{align*}
\hat{a}_1 &= r_1^* \hat{b}_1 + r_2^* \hat{b}_2 + \hat{L}_{a1}, \\
\hat{a}_2 &= r_1 \hat{b}_1 + t_1^* \hat{b}_2 + \hat{L}_{a2}
\end{align*}
\] (4)
where we use the notation of figure 1, and $\hat{L}_{a1}$ and $\hat{L}_{a2}$ are the Langevin noise (force) operators satisfying the following commutation relations
\[
\begin{align*}
[\hat{L}_{a1}, \hat{L}_{a1}^\dagger] &= [\hat{L}_{a2}, \hat{L}_{a2}^\dagger] = 1 - |t_1|^2 - |r_1|^2 \equiv \Gamma_1, \\
[\hat{L}_{a1}, \hat{L}_{a2}] &= [\hat{L}_{a2}, \hat{L}_{a1}] = -(t_1 r_1^* + t_1^* r_1) \equiv -\Omega_1.
\end{align*}
\] (5)

The transformation between the input ($\hat{b}_j$) and output ($\hat{c}_j$) annihilation operators of the lossy BS2 together with effect of finite efficiency ($\eta \equiv \eta_1 = \eta_2$) of detectors generalizes to
\[
\begin{align*}
\hat{b}_2 &= \sqrt{\eta_2^*} \hat{c}_2 + \sqrt{\eta_2} \hat{c}_3 + \hat{L}_{b2}, \\
\hat{b}_3 &= \sqrt{\eta_2} \hat{c}_2 + \sqrt{\eta_2^*} \hat{c}_3 + \hat{L}_{b3}
\end{align*}
\] (6)
where the Langevin noise operators $\hat{L}_{b2}$ and $\hat{L}_{b3}$ obey
\[
\begin{align*}
[\hat{L}_{b2}, \hat{L}_{b2}^\dagger] &= [\hat{L}_{b3}, \hat{L}_{b3}^\dagger] = \eta \Gamma_2 + (1 - \eta) \equiv x, \\
[\hat{L}_{b2}, \hat{L}_{b3}] &= [\hat{L}_{b3}, \hat{L}_{b2}^\dagger] = -\eta \Omega_2.
\end{align*}
\] (7)

In $\eta$ and $\Omega_j = t_j r_j^* + t_j^* r_j$ and $\Gamma_j = 1 - |t_j|^2 - |r_j|^2$ are the $j$th beam splitter phase and amplitude dissipation coefficients, respectively, which vanish for perfect beam splitters. For simplicity, we assume that the BSs are identical ($r_1 = r_2 \equiv r$, $t_1 = t_2 \equiv t$) and they cause only amplitude damping ($\Gamma \equiv \Gamma_1 = \Gamma_2 \neq 0$) without introducing phase noise ($\Omega_1 = \Omega_2 = 0$). By applying the transformations and for the input state $|\psi\rangle_{a12} = |1\rangle_{a1} |0\rangle_{a2} |\alpha\rangle_{b3}$ and performing the conditional measurement (projection synthesis) on modes $c_2$ and $c_3$ (as shown in figure 1), one finds that the state of the output mode $b_1$ of the LQS is entangled with the environment as follows
\[
\begin{align*}
|\psi\rangle_{b1E} &= N'' \langle c_{2-3}|0\rangle_{\text{out}}^{|b12-3E} \\
&= N\langle 0|_{b1} |\text{A}_0\rangle_E + \alpha |1|_{b1} |\text{A}_1\rangle_E
\end{align*}
\] (8)
where we write compactly the environmental states as
\[
\begin{align*}
|\text{A}_0\rangle_E &= \sqrt{\eta r} |t + \alpha \hat{L}_{b2}^\dagger| \exp(\alpha \hat{L}_{b3}^\dagger) |0\rangle_E, \\
|\text{A}_1\rangle_E &= \sqrt{\eta r} \exp(\alpha \hat{L}_{b3}^\dagger) |0\rangle_E
\end{align*}
\] (9)
and the normalization $N$ is given by
\[
N = \langle \eta |r^2 |\alpha|^2 (|t|^2(|\alpha|^2 - 1) + |r|^2 x + \Gamma')^{-1/2}
\] (10)
and $N''$ is a renormalization constant. The fidelity of the output state (8) of the lossy LQS to a desired perfectly truncated state, given by (11), can be calculated from
\[
F = ||b_1 \langle \alpha_{\text{trunc}} |\psi\rangle_{b1E}||^2
\] (11)
which leads us to the following relation
\[
F = N''^2 |\eta r^2 \exp(x|\alpha|^2)
\times \left(|t|^2(|\alpha|^2 + 1) + |\alpha|^2 \Gamma + |r|^2 x + \Gamma\right) - 1/2
\] (12)
where the normalization $N$ is given by (13). By defining $R = 1/|\alpha|^2$, equation (12) can be simplified to
\[
F = 1 - \frac{\eta \Gamma + (1 - \eta)|r|^2 + \Gamma}{(1 + |\alpha|^2)(1 - \eta)|r|^2 + (1 + R)}
\] (13)
where $x = \eta \Gamma + (1 - \eta)$. In a special case for 50/50 BSs, $|r|^2 = |t|^2$, our solution simplifies to that of Villas-Bôas et al. (20), note that the corresponding fidelity in (7) is misprinted). By neglecting losses caused by beam splitters, solution (13) is further reduced to the well-known Pegg-Phillips-Barnett fidelity
\[
F = 1 - \frac{|\alpha|^2(1 - \eta)}{(1 + |\alpha|^2)(1 + |\alpha|^2(2 - \eta))}
\] (14)
By assuming also perfect detectors, the fidelity becomes unity, as expected.

**III. LOSSY NONLINEAR QUANTUM SCISSORS IN THE MASTER EQUATION APPROACH**

In nonlinear quantum scissors scheme, schematically depicted in figure 2, a cavity mode is pumped by an external classical pulsed laser field, described by Hamiltonian $\hat{H}_K$, and is interacting with a Kerr medium, described by Hamiltonian $\hat{H}_{NL}$. Thus, the whole system Hamiltonian is given by
\[
\hat{H} = \hat{H}_0 + \hat{H}_{NL} + \hat{H}_K
\] (15)
where \( \hat{H}_{NL} = \hbar \frac{\kappa}{2} (\hat{a}^\dagger)^2 \hat{a}^2 \),

\[
\hat{H}_K = \hbar \epsilon (\hat{a}^\dagger + \hat{a}) \sum_{k = -\infty}^{\infty} \delta (t - kT_K)
\]

and the free Hamiltonian of the system is \( \hat{H}_0 = \hbar \omega_0 \hat{a}^\dagger \hat{a} \). In Eq. 16, \( \hat{a} \) is the annihilation operator for a cavity mode at frequency \( \omega_0 \), and \( \kappa \) is the nonlinear coupling proportional to the third-order susceptibility of the Kerr medium. In Ref. 14, Dirac \( \delta \) functions describe external ultra-short light pulses (kicks); real parameter \( \epsilon \) is the strength of the interaction of the cavity mode with the external field; \( T_K \) is the period of free evolution between the kicks. The truncation process in the system, given by 15, occurs if (i) \( T_K \gg T_{\text{round-trip}} \gg 2\pi/\omega \), where \( \omega \) is the light frequency and \( T_{\text{round-trip}} \) is the round-trip time of the light in the cavity, and (ii) the kicks are much weaker than the Kerr nonlinear interaction, \( \epsilon \ll \kappa \). As shown in Refs. 14 16, the state generated by the NQS is a two-dimensional coherent state 27 28 of the form

\[
|\alpha_{\text{trunc}}\rangle \approx \cos(|\bar{\alpha}|)|0\rangle - i \sin(|\bar{\alpha}|)|1\rangle
\]

where \( \bar{\alpha} = -i\kappa \epsilon \). Dissipation of the NQS system is modelled by its coupling to a reservoir of oscillators (heat bath) described by the Hamiltonian

\[
\hat{H} = \hat{H}_S + \hat{H}_R + \hat{H}_{SR},
\]

\[
\hat{H}_{SR} = \hbar \sum_j (g_{j} \hat{a}_{j}^\dagger \hat{b}_{j} + g_{j}^* \hat{b}_{j}^\dagger \hat{a}_{j})
\]

where \( \hat{H}_S \) is given by 14 and \( \hat{H}_R = \hbar \sum_j \chi_j \hat{b}_{j}^\dagger \hat{b}_{j} \) is the free Hamiltonian of the reservoir, where \( \hat{b}_{j} \) is the boson annihilation operator of the \( j \)th reservoir oscillator. By applying the standard methods of the quantum theory of damping 22, one finds that the NQS evolution between the kicks is governed under the Markov approximation by the following master equation in the interaction picture

\[
\frac{\partial}{\partial t} \hat{\rho} = -i \frac{\kappa}{2} \{ (\hat{a}^\dagger)^2 \hat{a}^2, \hat{\rho} \} - \frac{\gamma}{2} (\hat{a}^\dagger \hat{a}, \hat{\rho} + \text{h.c.}) + \gamma n [\hat{a}^\dagger, [\hat{\rho}, \hat{a}]]
\]

(21)

where \( \gamma \) is the damping constant and \( n \) is the mean number of thermal photons, \( n = \{ \exp[\hbar \omega/(k_B T)] - 1 \}^{-1} \), at the reservoir temperature \( T \), where \( k_B \) is the Boltzmann constant. Let the kick be applied at time \( t_K \), then the solution of Eq. 21 for any time \( t \) after \( t_K \) but before moment \( t_K + T_K \) is the same as the solution for the ordinary damped anharmonic oscillator 24 31 32 with the initial state given at time \( t_K \). We can write the solution compactly as \( \rho_{nm} = |\langle n | \hat{\rho} | m \rangle| \):

\[
\rho_{nm}(t_K + t) = \exp \left[ \frac{\gamma t}{2} + i(n - m) \kappa t \right] E_t^{n+m+1}(t)
\]

\[
\times \sum_{l=0}^{\infty} \rho_{n+l,m+l}(t_K) \sqrt{C_{n+l}^{m+l} + 1} \overline{g}_n(t)
\]

\[
\times F[-n - m, l + 1; \frac{4n(n + 1)}{\Delta_x^2} \sin^2 t_t - m]
\]

(22)

where \( F \) is the hypergeometric function, \( C_n^m \) are binomial coefficients, \( t_t = \gamma \Delta_x t / 2 \) and

\[
\overline{g}_n(t) = \frac{2(n + 1)}{\Omega_x + \Delta_x \coth t_t},
\]

\[
E_t(t) = \frac{\Delta_x}{\Omega_x \sinh t_t + \Delta_x \cosh t_t}
\]

(23)

with \( \Delta_x = \sqrt{\Omega_x^2 - 4n(n + 1)} \) and \( \Omega_x = 1 + 2n + i \kappa \epsilon / \gamma \). By assuming the reservoir to be at zero temperature, the solution 22 reduces to 30 33 34

\[
\rho_{nm}(t_K + \tau) = \exp \left[ i(n - m) \frac{\tau}{2} f_n^{(n+m)/2}(\tau) \right]
\]

\[
\times \sum_{l=0}^{\infty} \rho_{n+l,m+l}(t_K) \sqrt{C_{n+l}^{m+l}} \left( \frac{\lambda [1 - f_n^{(n+m)}(\tau)]}{\lambda + i(n - m)} \right)^l
\]

(24)

where \( \tau \) is the scaled time given by \( \tau = \kappa t \), so \( t_K = \kappa t_K \). Moreover, \( \lambda = \gamma / \kappa \), and \( f_x(\tau) = \exp[-(\lambda + i \tau) \tau] \). For a lossless anharmonic oscillator, i.e., for \( \lambda = 0 \), the solution 24 further simplifies to

\[
\rho_{nm}(t_K + \tau) = \exp \left[ i(n - m - 1) \frac{\tau}{2} \right] \rho_{nm}(t_K)
\]

(25)

Solution 22 describes the evolution of the NQS between the kicks only. On the other hand, the evolution at each kick is given by

\[
\lim_{\delta \to 0} \langle n | \hat{\rho}(t_K + \delta) | m \rangle
\]

\[
= \lim_{\delta \to 0} \sum_{n',m'} U_{nm} \langle n' | \hat{\rho}(t_K - \delta) | m' \rangle U_{mm'}^*
\]

(26)
where

\[ U_{nm} = \langle n| \hat{U} | m \rangle = \langle n | \exp[-i\epsilon(\hat{a}^\dagger + \hat{a})] | m \rangle \] (27)

in analogy to the Milburn-Holmes transformation for the pulsed parametric amplifier with a Kerr nonlinearity [34]. By observing that \( \hat{U} \) is the displacement operator \( \hat{U} = \exp[-i\epsilon(\hat{a}^\dagger + \hat{a})] = D(-i\epsilon) \), we can use the well-known Cahill-Glauber [35] formulas leading for \( n \geq m \) to

\[ U_{nm} = e^{-\epsilon^2/2} \sqrt{\frac{m!}{n!}} (-i\epsilon)^{n-m} L_{m-n}^m(\epsilon^2) \] (28)

and for \( n < m \) to

\[ U_{nm} = e^{-\epsilon^2/2} \sqrt{\frac{n!}{m!}} (i\epsilon)^{m-n} L_{n-m}^m(\epsilon^2) \] (29)

where \( L_n^m(z) \) is an associated Laguerre polynomial. Thus, we have a complete solution to describe the effects of dissipation on, in particular, the truncation fidelity after the \( k \)th kick, which is given by

\[ \bar{F}(t) = \langle \alpha_{\text{trunc}} | \rho(t) | \alpha_{\text{trunc}} \rangle \] (30)

where the perfectly truncated state \( |\alpha\rangle_{\text{NQS}} \) was applied according to [18].

**IV. CONCLUSIONS**

We studied dissipative quantum scissors systems for truncation of a Glauber (infinite-dimensional) coherent state to a superposition of vacuum and single-photon Fock states (two-dimensional coherent state). We have contrasted the Pegg-Phillips-Barnett quantum scissors based on linear optical elements and the Leoński-Tanaś quantum scissors comprising nonlinear Kerr medium. We analyzed the effects of dissipation on truncation fidelity in the linear scissors within the Langevin noise operator approach and in the nonlinear system in the master equation approach.

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