Teacher Orchestrations of Transitions Within and Beyond Fractions Virtual Manipulatives

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Abstract
Incorporating visual representations, rather than strictly relying on symbolic representations, is a research-based strategy for supporting fraction learning. However, students must also make transitions between visual and symbolic fraction representations to apply the conceptual understanding they gained from visual representations to symbolic fraction computation. Virtual manipulatives (VMs) offer opportunities for supporting students in making these transitions, as many fraction VMs integrate visual and symbolic representations into one manipulative. Some VMs also dynamically link the representations, so learners can observe how changes to one representation impact the other. For these features to support students in transitioning among representations, teachers must orchestrate opportunities for students to use and reflect on their use of the features. This study examined how six fourth- and fifth-grade teachers orchestrated opportunities in lesson plans for students to make transitions among the visual and symbolic representations within and beyond fraction VMs. Results showed that teachers used two strategies for orchestrating these transitions: VM choice and direct teacher intervention. Implications of teachers’ uses of these strategies are discussed in terms of what kinds of transition opportunities were made available to students and what professional learning experiences could be needed to support teachers in orchestrating transitions.

Keywords Elementary teachers · Fractions · Virtual manipulatives · Lesson planning · Teacher orchestration

Introduction
While physical manipulatives have been a mainstay in elementary mathematics classrooms for many years (Carbonneau et al., 2013), virtual versions of these tools are becoming more popular (Moyer-Packenham & Bolyard, 2016). These virtual
Theoretical Background

In this section, I review research on fraction learning to establish the importance of supporting students in connecting—or making transitions between—visual and symbolic representations of fractions. Next, I present research and theory related to teaching and learning with physical and virtual manipulatives to provide framing for why VMs may serve as powerful and effective tools for supporting students in making connections between representations. Next, I clarify the ways in which the potential learning benefits of VMs are dependent on how the VMs are used by students, highlighting the critical role of the teacher in shaping how students will use VMs. Finally, I summarize what is currently known about how teachers plan to use VMs in their classrooms and how the current study adds to what is known.

Challenges in Fraction Learning

Fractions are a particularly difficult topic for mathematics teachers to teach and for students to learn. Students and teachers both tend to exit their schooling with
superficial understanding of fractions and fraction computation that rely heavily on part–whole conceptions of fractions and memorization of procedures without meaning (Fazio & Siegler, 2011). Many struggles and misunderstandings with fractions stem from students making incomplete and incorrect attempts to reconcile their developing knowledge of fractions with their existing knowledge of the properties of whole numbers (Stafylidou & Vosniadou, 2004).

When they are faced with symbolic fraction representations, students’ intuitive strategies for working with fractions illustrate their attempts to apply whole-number reasoning inappropriately. Common fraction comparison strategies used by students in late elementary and middle grades include attending only to the numerators and saying the fraction with the larger numerator is the larger fraction (Stafylidou & Vosniadou, 2004) or judging the relative size of fractions based on the size of the “gap” between the numerator and denominator (e.g., saying $\frac{4}{5}$ is greater than $\frac{4}{7}$ because 4 is closer to 5 than to 7; Clarke & Roche, 2009). A common mistake that students make with fraction addition is to add numerators and denominators (e.g., reasoning that $\frac{1}{2} + \frac{2}{5} = \frac{3}{7}$; Siegler et al., 2013). These strategies reflect attention to numerators and denominators as independent whole numbers, rather than to the relationship between the numerators and denominators. These errors also suggest that students are relying on procedural understanding without sufficient conceptual understanding (Hiebert & Lefevre, 1986).

The well-documented and persistent challenges in fraction teaching and learning naturally raise the question, what instructional strategies show promise for overcoming common misconceptions and bolstering students’ conceptual understanding of fractions? One promising strategy for minimizing students’ application of whole-number reasoning to fractions is to reduce reliance on symbolic fractions by incorporating other kinds of fraction representations into instruction (Rau & Matthews, 2017). In the next sub-section, I discuss research on the use of visual representations to support fraction learning.

**Visual Representations in Fraction Learning**

In this article, I use the term *fraction representation* to refer to observable depictions of the abstract mathematical concept of a fraction, existing external to an individual human mind (Lesh et al., 1987). Specifically, I use *symbolic representation* to refer to representations made up of numerals and other mathematical symbols, including fractions written with a vinculum (e.g., $\frac{1}{2}$) and number sentences and expressions that include such fractions. I use *visual representation* to refer both to physical manipulatives (e.g., plastic fraction bars) and to pictures or diagrams, both of which can be internalized as visual images (Lesh et al., 1987). Incorporation of visual representations into fraction instruction may reduce dependence on whole-number reasoning that is often elicited when students work with symbolic fractions. Evidence from studies of human cognition suggests that elementary students and adults process visual representations of fraction magnitude more easily than symbolic representations (Rau & Matthews, 2017). Additionally, a recent study suggested allowing students to create visual representations of fraction sums by concatenating...
the addends along a number line improved students’ abilities to make accurate estimates of the sums (Braithwaite & Siegler, 2021). More generally, using visual representations of fractions is a research-based strategy for helping students develop a conceptual understanding of fractions (Fazio & Siegler, 2011).

The importance of including visual representations in fraction instruction raises two important questions. First, what tools and resources teachers might use to incorporate visual representations into instruction? Second, what activities and experiences can support students in connecting their developing knowledge of the visual representations to symbolic representations of fractions? Theory and research on the use of manipulatives—a kind of visual representation—in mathematics instruction, discussed in the next sub-section, provides insight to these questions.

### Challenges in Learning with Manipulatives

Manipulatives are physical objects used in classrooms to help children learn mathematics (Uttal et al., 1997). Many of the earliest physical manipulatives were designed with the specific purpose of serving as tangible instantiations of abstract mathematical ideas. For example, Dienes designed base-10 blocks (and blocks of other bases) for the purpose of embodying the concept of place value, and physical manipulation of the blocks was intended to support students in constructing an isomorphic mental model of place value (Sriraman & Lesh, 2007).

Despite robust learning theories and frameworks underlying their use (e.g. Agrawal & Morin, 2016; Bruner, 1964; Sriraman & Lesh, 2007), research makes clear that incorporating manipulatives into mathematics instruction does not always lead to increases in learning (Puchner et al., 2008; Resnick & Omanson, 1986; Uttal et al., 1997). To learn with manipulatives, students need to actively construct connections between manipulatives and mathematical ideas (Ball, 1992; Nührenbörger & Steinbring, 2008; Puchner et al., 2008; Uttal et al., 2009). They must develop internal conceptualizations of their work with manipulatives that reflect the important mathematical ideas. If use of physical manipulatives is to support understanding of symbolic representations, students must also construct connections between manipulatives and symbolic representations (Uttal et al., 2009).

When applied to the fraction challenges discussed above, this means that introducing manipulatives into fraction instruction is not likely to help children overcome whole-number biases unless they construct connections between the manipulatives and symbolic fractions. I refer to this process of constructing connections between visual and symbolic representations of fractions as making a transition from one form of representation to the other. Transitions between representations require students to make conversions between visual and symbolic mathematics registers—a process distinct from manipulating representations within one register (Duval, 2006).

Unfortunately, research on how students and teachers use manipulatives suggests children do not spontaneously make these transitions from manipulatives to other, often symbolic, representations. When manipulatives are used in mathematics instruction, teachers usually bear the responsibility of supporting children in...
transitioning between visual and symbolic representations of mathematical ideas. Children’s grasp of representational relationships may be related to how those representations are introduced. For example, Uttal and colleagues (Uttal et al., 1997) presented evidence that children made stronger connections between manipulatives and mathematics (specifically, base-10 computation and fraction concepts) when the connections were introduced immediately. In successful programs, transitioning from manipulatives to mathematical symbols was presented to children as the first step in learning. Children were also not expected to transition between manipulatives and symbols without explicit, consistent instruction. For example, Resnick and Omanson (1986) had limited success supporting students in connecting base-10 block manipulations to symbolic algorithms for multidigit subtraction via mapping instruction, but they found that verbalization of connections may relate to stronger learning.

While it is not yet clear what kinds of instruction and support most effectively help children make the desired transitions between symbolic representations of fractions and physical objects designed to represent fractions visually, VMs may have features that help to facilitate this process. I discuss VMs in the next sub-section.

**Virtual Manipulatives**

Moyer-Packenham and Bolyard (2016) defined a virtual manipulative as “an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated, that presents opportunities for constructing mathematical knowledge” (p. 13). Many VMs are screen-based versions of the physical mathematics manipulatives found in many classrooms, such as base-10 blocks and fraction circle pieces. A meta-analysis revealed a moderate positive effect on learning for students using VMs over other instructional treatments and a small positive effect on learning for students using VMs over physical manipulatives (Moyer-Packenham & Westenskow, 2013). Moreover, several studies have shown that using VMs supported upper elementary students’ conceptual and procedural learning about fractions (Moyer-Packenham & Suh, 2012; Reimer & Moyer, 2005; Suh et al., 2005).

Sarama and Clements (2009) argued that VMs have affordances that explain what makes VMs good for learning. Specifically, they claimed VMs have affordances that promote the development of integrated-concrete knowledge, or knowledge of manipulatives that is connected to other mathematical ideas. In essence, they argued that VMs have affordances that may help children connect the manipulatives to underlying concepts—something they often do not do when using physical manipulatives. For example, VMs often enable the creation of quick, precise mathematical representations through features that automatically size fraction pieces or snap them together (Moyer-Packenham & Westenskow, 2013). These precise representations can help keep important mathematical properties at the forefront of students’ thinking. Many VMs also constrain how users can interact with the components—for example, only allowing users to move one counter at a time (Manches et al., 2010)—in ways that highlight particular mathematical ideas or strategies.
One affordance of VMs that may have utility for supporting students in making transitions between visual and symbolic representations is *linked representations* (Zbiek et al., 2007). VMs with linked representations have at least two representations of the same mathematical concept, such as a symbolic fraction and an area model for that fraction. With physical manipulatives, any links between visual and symbolic representations are necessarily static (such as when physical fraction circle pieces are labeled with symbolic unit fractions). In VMs, the representations can be dynamically connected such that when a student changes one representation, the other changes to match it.

An important aspect of linked representations is the feedback they provide to students (Sarama & Clements, 2009; Zbiek et al., 2007). For example, when students watch the size of the parts of a fraction area model decrease as they progressively increase the denominator of a symbolic fraction, they receive immediate, visual feedback about the impact of their changes. Whereas children working with physical manipulatives often do not connect the physical manipulations of the objects to symbolic manipulations, the automatic feedback from linked representations may make it more difficult for students to “overlook the consequences of their actions” (Sarama & Clements, 2009, p. 148). Moyer-Packenham and Westenskow (2013) argued that the feedback provided by linked representations may support students in reinterpreting representations when their initial ideas about the connections between them are incorrect or incomplete, ultimately supporting transitions between representations.

Despite the promise of VM affordances, it is important to note that using VMs may lead to gains or losses of learning opportunities, depending on the ways in which they are used (Cohen et al., 2003). Linked representations provide feedback about how representations relate (Zbiek et al., 2007), for example, but a student must manipulate a VM in particular ways for the linked representations to be evident. Moreover, even if those manipulations take place, the feedback will not support students in transitioning between the representations unless they interpret the feedback in terms of how the representations are connected. Students could merely use the automated connection between the linked representations to copy down “pairs” of related representations (e.g., a visual model and symbolic fraction for $\frac{3}{4}$), without making sense of how the representations show aspects of the same mathematical idea and relate to each other. In such a situation, the student would be offloading the intellectual work of generating related representations to the VM and potentially losing the opportunities to make connections for themselves.

Even assuming students notice feedback from linked representations and utilize it to make connections between visual and symbolic representations, their work is not finished. They will have made a transition *within* the digital environment of the VM, but if students are to carry this new way of understanding the connections between visual and symbolic representations into later work with paper-and-pencil symbolic representations (e.g., Geraniou & Mavrikis, 2015), they must also make a transition *beyond* the digital environment. It is not clear how VMs would support students in making those transitions. In short, VM affordances may offer some support to students in connecting manipulatives to symbols but making full transitions *beyond* the VM is a multistep process in which linked representations within VMs only facilitate one step.
This line of reasoning establishes the need for research on how teachers make sense of VMs and make plans to use them in their classrooms. VMs have affordances that may support students in making transitions between visual and symbolic representations (Sarama & Clements, 2009; Zbiek et al., 2007), but only if features reflecting those affordances are utilized in learning experiences that support students in making a transition from a visual representation in the VM’s environment to a symbolic representation beyond the environment, potentially after making an intermediate transition to a symbolic representation within the environment. In brief, how the VM is used by students (e.g., what tasks the VMs are used to support students in completing) will matter. Teachers are key players in determining how educational resources are used (Cohen et al., 2003). To understand fully the potential for VMs to support students’ development of conceptual understanding of fractions, the field needs more information about how teachers plan to create opportunities for students to use VMs, whether and how they consider how to support both transitions within and beyond the VMs, and what additional support teachers may need to use VMs to their full potential. In the next section, I summarize what is currently known about the teacher’s role in shaping how students interact with VMs in anticipation of describing how the current study builds on that work.

The Teacher’s Role in Students’ Use of VMs and Other Instructional Resources

While the instructional resources available to teachers can impact the nature of instruction in classrooms, the ways in which teachers take up resources also affect the resulting instruction (e.g., Cohen et al., 2003). One framework for considering the mutual influence of teachers and resources—particularly technology resources—on instruction is that of instrumental orchestration (Drijvers et al., 2010, 2013). Instrumental orchestration is based on the theory of instrumentation, which says that as a student learns to use a technology tool for doing mathematics, the student develops schemes of use, or ways of using the tool to accomplish a mathematical task. The tool itself and the students’ schemes together form an instrument, which is a construct distinct from either the tool or the user alone. When teachers plan and enact instruction using technology tools, they necessarily influence the ways students view the tools and the schemes they develop. The ways the teacher sets up the classroom and instruction are called instrumental orchestrations. In this study, I was specifically interested in the strategies that teachers used to plan for and frame how students would interact with VMs, potentially shaping how the VMs develop into instruments that include attention to connections between visual and symbolic representations. Within this article, I refer to these strategies as orchestration strategies.

Studies have identified at least two orchestration strategies teachers use to affect how students interact with VMs, whether students encounter or make use of affordances, and whether and how students make sense of their actions (Hansen et al., 2016; Moyer-Packenham et al., 2008). First, teachers’ choices of tasks for students to complete with VMs will shape which features they use. Second, the questions teachers ask and the comments they make during instruction will guide how students interpret their interactions with VMs. For example, Hansen and colleagues (Hansen...
et al., 2016) studied how a teacher used the VM *Fractions Lab* in his classroom. This VM had a built-in constraint: The VM would not show a symbolic sum unless students first partitioned area models of the addends to create like denominators. The teacher chose to sequence the fraction addition problems presented to a student such that the student solved a like-denominator problem first, then a problem where one denominator was a multiple of the other, then a problem where both denominators had to be changed. This task choice affected how the student interacted with and interpreted the constraint because it set up an opportunity for the teacher to prompt the student to think about how partitioning the area models could help her find an equivalent addition problem with the same sum.

Relatedly, Trgalová and Rousson (2017) conducted a case study of one kindergarten teacher using an enumeration app designed to support students’ counting skills. The teacher examined the several modes of exploration available in the app to determine how he could vary aspects of the counting tasks, including whether or not students could move the counted objects and whether the objects were identical. Thinking through the task options available in the app helped the teacher consider the various aspects of counting students may need support with, and to tailor experiences for different groups of students. Moreover, the teacher supplemented students’ individual work on the app with one-on-one teacher guidance and whole-group discussions of students’ strategies.

While these studies (Hansen et al., 2016; Trgalová & Rousson, 2017) illustrate how teachers applied the two orchestration strategies (selecting tasks and intervening with questions and comments) as they used a VM in instruction, the studies did not attend specifically to how the teachers used the orchestration strategies to support students in making transitions from visual to symbolic representations or the kinds of transition opportunities each strategy makes available to students. Moreover, in each of these studies, teachers were asked to use a specific VM. Less attention has been paid to whether and how teachers use these or other orchestration strategies when they select their own VMs. With the wide variety of free VMs available (Moyer-Packenham & Bolyard, 2016) and the practice of utilizing Web-based resources in instruction becoming more common (Webel et al., 2015), examining teachers’ reasoning when incorporating VMs of their own choosing into their lessons is a worthy pursuit.

A few studies focused on the reasons that teachers choose particular technology tools and found teachers often place an emphasis on choosing tools they are already familiar with (Moyer-Packenham et al., 2008) or believe will be engaging for students (Johnston & Suh, 2009; Suh, 2016). At least one study examined the broad purposes for which teachers used VMs, finding teachers utilized them to allow students to investigate concepts and solidify skills (Moyer-Packenham et al., 2008). However, no studies specifically examined the orchestration strategies teachers use to shape students’ interactions with VMs during lessons with VMs teachers chose. It remains unclear whether and how teachers coordinate their choice of VM with the instructional goals of their lessons.

Generalized frameworks and other guidance already exist to support teachers in using task choice and questioning techniques to orchestrate opportunities for engage in high-level mathematical thinking, including tasks and questions related to making
transitions between representations (e.g., Drake et al., 2015; Smith & Stein, 1998; Sullivan et al., 2015). Although the frameworks do not relate specifically to using VMs or other technology tools in instruction, some of the guidance could be applied to selecting tasks or asking questions that facilitate students making transitions within or beyond representations in VMs. For example, the math task framework (Smith & Stein, 1998) highlights opportunities to make connections among representations as a characteristic of tasks with higher-level cognitive demand. Drake and colleagues’ (Drake et al., 2015) strategies for opening curriculum spaces include suggestions for teachers to cut lesson components that tell, direct, or show students how to approach problems and instead focus on using prompts that allow for multiple solution strategies shown with multiple representations. To date, little research has focused on connecting these frameworks for supporting implementation of high-level tasks to teachers’ use of technology tools.

**Current Study and Research Question**

In this study, I aim to extend the current work that demonstrated how teachers use task choice and direct intervention through questioning and comments as orchestration strategies for shaping student interactions with VMs (Hansen et al., 2016; Trgalová & Rousson, 2017). Specifically, I examine whether and how teachers planned to apply these or other orchestration strategies to create opportunities for students to make transitions between visual and symbolic representations of fractions using VMs—in particular, the multiple, sometimes linked, representations embedded in VMs (Sarama & Clements, 2009).

I connect the study to other existing work about teachers’ strategies for selecting VMs (Johnston & Suh, 2009; Moyer-Packenham et al., 2008) by allowing teachers to select which VM to use in their lesson plans. Building on Suh’s (2016) suggestion that connecting VM features to frameworks for supporting high-level mathematics thinking could support teachers in more effectively choosing and implementing technology tools, I explore how six elementary teachers planned to use fraction VMs after being introduced to their features and potential benefits in professional development. The results of this study will inform the kinds of additional professional support teachers may need to use VMs to support students in making transitions between visual and symbolic representations of fractions.

This study addresses the following research question:

*How, if at all, do elementary teachers plan to orchestrate opportunities for students to make transitions between visual and symbolic representations of fractions in lesson plans involving teacher-selected virtual fractions manipulatives?*

**Methods**

**Study Context**

This article is part of a broader study focused on teacher thinking about VMs across multiple contexts. The participants in the study attended three 1-h, online professional learning sessions that focused on introducing teachers to affordances of VMs
and how they might be useful in overcoming common challenges in fraction teaching and learning. During the sessions, teachers worked in groups to compare sets of fraction VMs, discuss their differences, and brainstorm how certain features could make each VM useful for certain learning goals. For example, a fraction circle VM that allows students to overlap the pieces may be useful for supporting student thinking about fraction equivalence. By contrast, a fraction circle VM that does not allow pieces to overlap, but rather automatically snaps pieces together without leaving gaps or overlaps, may be more useful for supporting student thinking about fraction addition. The learning goal for the sessions was for teachers to connect the features of VMs to the challenges and common misconceptions (e.g., a student tendency to add numerators and denominators) they had encountered during fraction instruction in the past.

During the final session, I shared a classroom anecdote (Ball, 1992) about students who thought about solving fraction problems with manipulatives as an entirely separate process from solving fraction problems symbolically. This launched a discussion about how teachers might encourage students to think about how their work with VMs is connected to the underlying mathematical ideas and their symbolic representations. Based on prior research, I specifically prompted teachers to consider how they could foster high-level student thinking, either through task selection (e.g., do not just generate multiple examples of equivalent fractions, instead look for and explain patterns in the examples) or through direct intervention and questioning (e.g., why does it make sense that the dot on the number line moves to the right when you shade in a new piece in the area model?).

After the professional learning sessions, teachers participated in a semi-structured interview where they freely explored a VM that was not introduced in the professional learning and used it to solve problems. All the teachers were invited to participate in an additional phase of the study where they would plan lessons using fraction VMs. This article focuses on the analysis of the teacher lesson planning data, which is described more fully below.

Participants

Thirteen teachers participated in the broader study, but only six chose to continue into the lesson planning phase. A summary of location, grade level, and years of teaching experience of the teacher participants is in Table 1. Data collection took place during the height of the COVID-19 pandemic, when many teachers in the United States (U.S.) were teaching remotely at least part-time. Information about each participant’s teaching circumstances at the time of the study is included in the last column. Each teacher received a $300 stipend for their participation.

Data Generation

After the professional learning sessions, each teacher participant received written directions and a lesson planning template. The template was a simplified version of the Thinking Through a Lesson Protocol (Smith et al., 2008). This protocol places the
| Pseudonym | Location within the U.S | Grade level (student ages) | Years of experience | Teaching circumstances at the time of study |
|-----------|-------------------------|---------------------------|---------------------|---------------------------------------------|
| Colleen   | New England             | 5 (10 years)              | 3                   | Some students face-to-face, some online     |
|           |                         |                           |                     | Self-contained classroom                    |
| Tammy     | New England             | 4 (9 years)               | 28                  | Face-to-face, then switched to remote        |
|           |                         |                           |                     | Self-contained classroom                    |
| Sheila    | Midwest                 | 3–5 (8–10 years)          | 6                   | Fully remote                                |
|           |                         |                           |                     | Teaching math only                          |
| Nancy     | New England             | 4 (9 years)               | 18                  | Fully remote                                |
|           |                         |                           |                     | Teaching math, social studies, and science  |
| Janice    | Midwest                 | 4 (9 years)               | 2                   | Some students face-to-face, some online     |
|           |                         |                           |                     | Self-contained classroom                    |
| Karla     | New England             | 4 (9 years)               | 14                  | Fully remote                                |
|           |                         |                           |                     | Self-contained classroom                    |
focus of teachers’ attention on the tasks that students will work on and how they might approach them, as well as what the teachers will do to support students in completing the tasks. I chose this protocol to prompt teachers’ thinking about both the tasks they might pose and the questions they might ask students as they worked, corresponding to the two orchestration strategies teachers used with VMs in prior research (Hansen et al., 2016; Trgalová & Rousson, 2017). The directions asked teachers to choose a VM from a list provided at the end of the professional learning, specify the context in which they expected to teach the lesson (e.g., remotely or in person), and choose a learning goal about fraction equivalence, comparison, or addition and subtraction. Teachers could either adapt a lesson from their curriculum materials to suit the VM they chose or create an entirely new lesson. The teachers returned these lesson plans to me within 6 weeks of receiving the lesson planning directions.

After I received the written lesson plans from each teacher, I planned an artifact-based interview (Brennan & Resnick, 2012)—with the lesson plan as the artifact—to support teachers in recalling and restating the thinking they did as they created the lesson plan. I developed a personalized interview protocol based on what was written in the plan. I used a set of common questions to guide the structure of each interview, but also included specific questions to probe teachers’ thinking about the details articulated in their written plan. A sample lesson plan interview protocol is included in Appendix 1. My goal was to understand why each teacher chose the VM she did and how she expected students to use it in the lesson (including the tasks students would do). These interviews took place within a week of receiving teachers’ written plans and lasted 30–90 min. After each interview, I made annotations on the associated lesson plan and sent this annotated version to each teacher and invited her to comment. This served as a form of member-checking (Tracy, 2010).

After each lesson planning interview, I developed three short vignettes of situations that could arise when teachers taught their lesson. My goal was to pose situations that would stimulate the kind of teacher thinking that might happen in the moment-to-moment teaching of a lesson. (It was not possible to collect classroom data due to the research restrictions in place during the COVID-19 pandemic.) Each vignette was based on one of the following:

- The sample student strategies the teacher described in her lesson plan (and, when necessary for me to understand them, elaborated in her lesson plan interview);
- Common fraction misconceptions and difficulties highlighted by teachers in the professional learning sessions or in the fraction research literature;
- Issues that might be introduced by the features of the VM, when student strategies with the VM could diverge from what they might do with physical manipulatives or drawings.

A sample vignette is included in Appendix 2 with its rationale. I presented these vignettes to teachers in a task-based interview (Maher & Sigley, 2020). To simulate the common teaching task of making sense of and responding to student thinking, I shared my screen and demonstrated what a student might do with the VM while narrating what they might say. After posing each situation, I asked teachers the following questions:
• What do you think is going on with the students’ thinking?
• What would you do next?
  o (if needed) What might you suggest the student try with the VM?
• Do you think this issue might have come up even if students were not using the VM? Why or why not?
• What is useful about the VM for addressing this issue?

These 20–30-min interviews took place within a few days of the lesson plan interviews.
All interviews were conducted using video-conferencing software. Video (including shared screens) and audio were recorded with permission from the participants. I used a paid transcription service to create verbatim transcripts of the audio. I also created a second transcript for each interview, which I called the “cursor transcript,” that reflected what was happening on the shared screen. These cursor transcripts made it possible to code screen activity (i.e., what the participants were doing with the VM) as well as verbal utterances.

Analysis

Analytical Approach

I utilized a cross-case analysis technique (Stake, 2006) to analyze the transcripts of the six participating teachers’ lesson plan interviews and student response interviews. I treated each teacher as a case and considered each pair of interviews—that is, an individual teacher’s lesson plan interview paired with her student response interview—as two means of insight into one teacher’s reasoning about VMs and the teacher’s role in orchestrating opportunities for students to transition between visual and symbolic representations of fractions. I used a coding technique guided by the professional noticing framework (described further below; Jacobs et al., 2010) to identify excerpts of the lesson plan and student response interviews where each teacher provided evidence of how they planned to orchestrate student opportunities to make transitions between visual and symbolic fraction representations. After examining each individual teachers’ thinking on this topic, I compared cases in order to identify and describe generalized orchestration strategies and how teachers used them alone or in combination.

Coding Software

To code the interview data, I used Transana, a video coding program that coordinates video with written transcripts and allows coding across multiple transcripts. Codes are applied to data by selecting sections of one or more transcripts and applying a keyword (Transana’s language for codes). This creates a clip or a section of the video along with its transcript sections.
Coding Technique

My approach to coding the interview data was guided by the professional noticing framework (Jacobs et al., 2010), which explicates three related processes in which teachers engage as they make sense of instructional artifacts (such as VMs or descriptions of student activity, as in this study’s vignettes). First, teachers attend to particular aspects of the artifacts (and often do not attend to others). Second, they interpret the aspects to which they attend in order to construct meaning for what they notice. Third, they decide how to respond to what they notice (e.g., decide how to use a tool in instruction or what problems to pose next to students). For this analysis, my goal was to capture and describe episodes of teachers creating opportunities for students to transition between visual and symbolic representations of fractions. Thus, I coded for teachers attending to visual and symbolic fraction representations, interpreting the relationship between those representations and the potential importance of students making sense of that relationship, and deciding how to respond in terms of how they might orchestrate opportunities for students to transition between the visual and symbolic representations.

My first round of coding focused on teachers’ attention to representations of fractions. Because creating visual fraction representations—specifically, circle models, bar models, or both—was the primary function of each VM, almost all interview passages contained evidence of a teacher attending to a visual representation. As such, my first round of coding focused on identifying episodes where teachers attended to a symbolic representation. These episodes featured teachers discussing at least one of three topics: (1) a VM feature that showed a symbolic representation, such as a symbolic label; (2) how they (or students) might use an annotation feature in the VM, such as a pen tool or text tool, to create a symbolic representation; and (3) how they might write, or have students write, symbolic representations outside of the VM environment. I also included episodes where one teacher specifically discussed choosing a VM that did not show any symbolic representations or labels. Examples of interview excerpts containing each of these three types of evidence of attention to a symbolic representation are given in Table 2.

I next examined these episodes of attention to symbolic representations for evidence of the teachers interpreting those symbolic representations in relation to the visual representations in the VM. I first coded the excerpts according to whether there was a mention both of visual and of symbolic representations. For the excerpts with references to both representation types, I looked for evidence of teachers relating the representations to each other. This evidence typically consisted of (1) explaining potential benefits of students being able to see visual and symbolic representations together or (2) describing a teaching move related to prompting students to attend to the two types of representations together. Examples of each type of interpretation evidence are in Table 2, along with an example of an excerpt that included mention of both representations but no evidence relating them to each other.

Finally, having identified all the interview excerpts where a teacher provided evidence of considering relationships between visual and symbolic representations or fractions, I moved to considering the deciding how to respond part of the professional noticing framework. I reviewed these excerpts to develop codes for the
| Participant | Excerpt (evidence of attention in italic; evidence of interpretation in bold-italic) | Attention evidence type | Interpretation evidence type |
|-------------|----------------------------------------------------------------------------------|-------------------------|-----------------------------|
| Colleen     | I like that, as they click through, the dot moves and the fraction. It shows them the number, it shows them that point, moving along the number line. And it gives them a visual down here. So I just think that there’s a lot of great visuals for them to see and connect all of the concepts together | Reference to a VM feature that showed a symbolic representation | Explaining benefits of students seeing the visual and symbolic representations together |
| Janice      | Then I would tell [students], “Okay, label it.” [using her cursor to identify “it” as a visual representation and writing a symbolic fraction with a pen feature.] We’re not good at labeling right now | Reference to using an annotation feature to create a symbolic representation | Teaching move related to prompting students to connect visual and symbolic representations |
| Sheila      | Since the tool is showing the model, and they [students] are writing the fractions, I’m hoping that they’re going to start noticing those patterns like we said, 1/2, 2/4, where the numerator is half the denominator | Reference to having students write symbolic representations outside of the VM environment | None; references visual and symbolic representations, but not in relation to each other |
particular kinds of responses I was interested in for this study: orchestration strategies for shaping students’ use of VMs to foster opportunities to make transitions between visual and symbolic representations. While developing these codes, I began by considering whether the two orchestration strategies discussed in previous VM research, *task choice* and *direct intervention*, were used by the teachers in this study. I coded any excerpt where a teacher described prompting students to take a particular action (e.g., Janice’s directive to “label it” in Table 2) or consider a particular question that would prompt attention to connections between symbolic and visual representations as examples of teachers using a direct intervention orchestration strategy.

I did not code any episodes where the teacher used task choice as a strategy for orchestrating opportunities for students to transition between representations. I noticed that rather than focusing on choosing or adapting tasks, teachers often described how they believed the nature of the tool itself would orchestrate opportunities for students to notice relationships between representations (e.g., Colleen’s comment that her VM allowed students to “see and connect all of the concepts” in Table 2). This led me to consider tool choice, rather than task choice, as a potential orchestration strategy. I wrote a brief analytic memo (Maxwell, 2012) about this possibility and, after stepping away from the data for a short time, returned to the coding task and applied the two orchestration strategies (direct intervention and tool choice) as a coding scheme corresponding to the *deciding how to respond* part of the noticing framework. These codes fit the data well.

After finishing the coding process, I returned to examining each teacher as a case. I looked at each teacher’s use (or lack of use) of each orchestration strategy to understand better their overall pattern of thinking about how to orchestrate opportunities for students to transition between visual and symbolic representations. Making comparisons between cases allowed me to uncover differences in which strategies each teacher used, how they applied the strategies, and whether in the transitions they used, the strategies to orchestrate were transitions within or transitions beyond. To finish the analysis, I wrote case summaries for each teacher with attention to how she used the orchestration strategies, using comparisons to other cases to aid in my description.

**Validity Strategies**

I used two strategies to support the validity of my findings. First, as part of the larger project from which this data was drawn, I engaged a second coder to identify parts of the interviews in which teachers attended to specific VM features (including the labeling features that served as one way of identifying episodes of potential orchestration of student consideration of visual and symbolic representations). Overall reliability on the feature coding was 92%. Second, as a check on my interpretation of teachers’ thinking, I compared my descriptions of their thinking about connecting visual and symbolic representation in their lesson planning with evidence in their VM exploration interview (which occurred after the professional learning sessions but before the lesson planning exercise; see “Study Context” section). I did not
discover any discrepancies in how an individual teacher thought about this issue in exploration versus lesson planning.

Results

My analysis of the six participants’ orchestrations of transitions between visual and symbolic fraction representations resulted in two orchestration strategies and variations in how the teachers applied them. Specifically, the two strategies teachers used to orchestrate transitions were (1) tool choice and (2) direct teacher intervention. Comparison of the cases revealed differences in (1) using the strategies to orchestrate transitions within the VM (Colleen, Sheila, Tammy), beyond the VM (Nancy, Janice), or both (Karla); (2) applying the tool choice strategy by choosing a VM with linked representations (Colleen, Sheila, Tammy), static labels (Karla), or no symbolic representations at all (Nancy); and (3) whether their orchestration strategies created opportunities for students to hear or verbalize connections between representations. The six cases are summarized in Table 3 with attention to these differences across cases.

This section begins by describing each of the six cases in detail and continues with summaries of each of the three differences among the cases.

Colleen: Orchestrating Transitions Within Through Tool Choice and Teacher Intervention

The first teacher case was Colleen, who mentioned both tool choice and teacher intervention as strategies for orchestrating student opportunities to make transitions between visual and symbolic representations of fractions. Colleen’s overall approach can be described as taking advantage of the tool’s features to orchestrate opportunities for students to make transitions between visual and symbolic representations within the VM environment, but also acknowledging students may need prompting to attend to these features. I begin this sub-section (and each of the remaining cases) with brief discussions of the teacher’s chosen VM and lesson focus. Then, I present evidence of the teacher’s use of each orchestration strategy.

VM and Lesson Overview

Colleen used a VM called the NCTM Equivalent Fractions tool to plan her lesson about generating equivalent fractions. When a user opened the VM, shown at the left of Fig. 1, three square area models are visible on the screen, one at the top showing a fraction and two at the bottom starting off blank. Between the top and bottom area models is a number line with three dots on it corresponding to the fractions on each area model. The colors of the number line dots and labels correspond to the colors of the area models (the top model is red, the bottom right model is blue, and the bottom left model is green). As a user partitions and adds shading to one of the bottom models, the corresponding dot on the number line moves. The user’s goal is to create two
| Teacher | Use of tool choice strategy | Use of direct intervention strategy |
|---------|----------------------------|------------------------------------|
| Colleen | Chose a VM with linked representations to orchestrate transitions within the VM environment | Planned to intervene to orchestrate transitions within the VM by drawing student attention to features |
| Karla   | Chose a VM with symbolic labels to orchestrate transitions within and beyond the VM environment | Planned to intervene to orchestrate transitions both within and beyond the VM by drawing student attention to features |
| Nancy   | Chose a VM without symbolic representations to orchestrate transitions beyond | Planned to intervene to orchestrate transitions beyond the VM by recording symbolic representations and modeling connecting language |
| Janice  | Minimal evidence of using this strategy; chose a VM with linked representations, but did not expect students to use them | Planned to intervene to support transitions beyond the VM by prompting the creation of symbolic representations and other annotations |
| Sheila  | Chose a VM with linked representations, but only mentioned them passively | No evidence of plans to intervene to support transitions |
| Tammy   | Chose VMs with linked representations and symbolic labels, but only mentioned them passively | Minimal evidence of plans to intervene to support transitions |
fractions on the bottom models that are equivalent to the fraction shown on the top model. Sliders allow the user to partition the models horizontally and vertically and clicking adds or removes shading from a part of the model. Figure 1 shows what the screen might look like after a user partitions and shades the blue and green (bottom) area models to create fractions equivalent to the red (top) model. Colleen planned to have her students build their understanding of the multiplication rule for equivalent fractions by experimenting with the VM and looking for patterns.

**Using Tool Choice to Orchestrate Transitions Within**

Colleen mentioned the dynamic labels on the number line in the NCTM Equivalent Fractions tool as she described her reasons for choosing it: “I like that, as they click through, the dot moves and the fraction. It shows them the number [on the label], it shows them the point, moving along the number line.” Colleen was more interested in the dynamic labels than the moving dots for the purposes of the lesson she planned. She described students attending to the position of the dots on the number line as a “side goal” of her lesson. She thought student attention to the labels on the number line was more important than their attention to the position of the dots because students had struggled to connect area models to symbolic fractions in the past:

I want them to see the picture but also see the numbers, see the fractions. Because I feel like sometimes kids are able to color in a model. But then, I say, or I could, going back to the adding and subtracting that we just did, I could say, “What’s $\frac{3}{8} + \frac{2}{8}$?” And they would color it in on a model. And then, some of them wouldn’t be able to tell me the answer.

This comment suggests Colleen saw supporting students in transitioning between visual and symbolic fraction representations as an ongoing goal in her instruction (if not the main goal of her lesson). She cited students’ inability to discuss one representation in terms of another (i.e., use a symbolic fraction to name a visual sum) as
evidence they were not successfully making those transitions. Colleen was looking for her students to show evidence of building connections between visual and symbolic representations of fractions, and that she saw the dynamic labels as a feature that might support students in this. Because the VM allowed the students to “see the numbers, see the fractions” as they manipulated in the area models, Colleen believed the VM itself was orchestrating opportunities for students to transition between visual and symbolic representations within the VM environment.

Intervening to Orchestrate Transitions Within by Drawing Student Attention to Features

In both her lesson plan and student response interviews, Colleen acknowledged that, even though the NCTM Equivalent Fractions app showed connected visual and symbolic representations of fractions, students might not attend to both or think meaningfully about either one. She described some trial-and-error strategies students might use to generate equivalent fractions by just clicking around in the area models until they saw the points on the number line coincide. When I presented a vignette in her student response interview of a student using such a trial-and-error strategy, her response suggested she planned to intervene and push students to connect the area models to the symbolic fractions on the labels. She first said she would encourage students to look for numeric patterns in the symbolic fractions:

But if they’re just clicking through, I would point out like, “Okay, what did you notice happened, when you were clicking through when you could stop, what do you notice about the numerator, denominator or whatever?” See if they could make those connections. And then they probably would say, “Oh, well, eight is double four. So, that ... And six is double three.” So, they probably would be able to do it.

She also acknowledged that directing attention to the symbolic fractions might lead students to ignore the area models and just focus on the multiplication rule, but she had a plan for that possibility as well: “But then again, that kind of leads just straight into the rule rather than them being able to see. But afterwards I would talk about looking at the sizes.” In general, throughout her interview, she encouraged student attention to both representations, suggesting she intended to intervene to support students in making transitioning between representations—or at the least, to prevent students from relying on only one representation. She saw an important role for herself in ensuring students had opportunities to transition between the representations.

Karla: Orchestrating Transitions Within and Beyond Via Tool Choice and Teacher Intervention

The second teacher case was Karla, who also mentioned both tool choice and teacher intervention as strategies for orchestrating student opportunities to make transitions between visual and symbolic representations of fractions. Karla’s overall approach
was similar to Colleen’s in that she intended to take advantage of a VM’s features to orchestrate opportunities for students to make transitions but knew she may have to intervene to ensure this happened. Karla’s descriptions of her strategies differed from Colleen’s, however, because she explicitly described supporting students in making transitions both within the VM environment and beyond it.

VM and Lesson Overview

Karla used the Toy Theater Fraction Circles to plan her lesson focused on generating sets of fractions that sum to 1. When a user loads this VM, brightly colored fraction pieces, each labeled with unit fraction, are visible on the screen (see Fig. 1, right). Just above the pieces are two blank circles into which the pieces can be dragged. When pieces are dragged into the circles, they snap into position. The first piece is positioned to start at the top of the circle (aligned with the dotted radius present in each circle) and subsequent pieces snap into place next to existing pieces. The pieces do not overlap and cannot be placed with gaps between them. Rather, the pieces “push” each other out of the way, and pieces that will not fit into the space left in the circle simply disappear. Karla planned to ask her students to generate examples of how a whole could be divided into parts (while meeting the constraints of a contextualized problem) by dragging fraction pieces into the circles.

Using Tool Choice to Orchestrate Transitions Within and Beyond

Karla’s lesson centered on a task that required students to partition one whole into different-size parts to solve a contextualized problem about a queen sharing her kingdom. In her lesson planning interview, Karla described the symbolic labels on the Toy Theater Fraction Circles (see Fig. 2) as a key reason she chose the VM, even though it did not offer a different feature she wanted: “I would have liked that this allowed some transparency, but I needed it to have the names of the fraction on it because … we’re doing the correspondence between adding them to one.” By “doing the correspondence between adding them to one,” Karla meant that she intended for students to use the VM generate ways for the queen to share her kingdom among three daughters, such as the solution shown in Fig. 3, and then write a corresponding number sentence (e.g., \(\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1\)) outside of the VM environment. Referring to the solution in Fig. 3, Karla explained:

If we line that up and say \(\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1\) whole, [the number sentence] is sort of generated for them. And then I think we have to get into that conversation about, okay, \(\frac{1}{2}\) is how many \(\frac{1}{4}\)s? And then say, okay, well that’s \(\frac{1}{4} + \frac{1}{4} + \frac{2}{4} = \frac{4}{4} = 1\) whole. I think it’s easy to see that way, that if they had to use the other [VM, which does not have the labels] and figure out what the fraction was in their head and then talk about adding it, I thought that was too many levels of thinking because this is complicated enough that they didn’t need that extra piece.

Karla had taught the lesson in the past without a VM, and instead had students use drawings to create their partitioned kingdoms. She explained that it was common for
students to draw something like Fig. 3, but label the smaller sections as $\frac{1}{3}$ rather than $\frac{1}{4}$. Students struggled with coordinating the size of the piece in relation to the whole (a visual fraction representation) with an appropriate fraction label:
They’ll draw something and divide it whatever way they want. I think when the kids drew it, they would draw the half here and then say a third and a third, and it would be done. Not paying any attention at all to the fraction, just thinking about pieces. “Oh, that’s three pieces now”, do you know what I mean? But when they have this in front of them, and they put that in, now they see it’s a fourth. When they’re drawing, they don’t see that, and they would just write a third in there.

Like Colleen, Karla had noticed a lack of connections being developed for her students between symbolic and visual representations of fractions (in this case, part sizes and their labels). For Karla, the VM labels, together with the exact sizing of the pieces in relation to the whole, acted as a scaffold to support students through the multiple levels of thinking it took to go from a partitioned whole, to determining what fraction each piece of the whole represented (a transition within), to writing a number sentence that expressed the sum of the parts being equal to 1 (a transition beyond).

Understanding that the fractions into which a whole is partitioned must sum to 1 was a main learning goal of her lesson, and understanding how piece sizes and symbolic fractions relate was a key to that learning goal. Karla felt the VM labels would push students to think about that size-to-symbol correspondence. She believed the tool would help to orchestrate opportunities for students to make fluid transitions between piece sizes and symbolic fractions. Although she did not discuss listening for students to describe the connections between the fraction labels and piece sizes verbally, it was clear she hoped students would make transitions between the representations in their own minds as they worked.

The above excerpts from Karla’s interview demonstrate that she chose the Toy Theater Fraction Circles in part because she thought their features would support students in making the transitions needed to complete this lesson’s task. However, several times during her interviews, Karla mentioned that she wished the Toy Theater Fraction Circles had an option to turn off the symbolic labels. She made it clear that the labels were useful for the particular lesson she was planning, but also that there would be other situations where she would not want students to see labels, although she never elaborated on what those situations might be. This suggests Karla’s intention to use tool choice as a way to orchestrate transitions was task-dependent. This may also have been the case for Colleen, but Colleen’s interview did not provide evidence either way.

Intervening to Orchestrate Transitions Within and Beyond via Drawing Student Attention to Features

Similar to Colleen, even as Karla appreciated the potential of the VM’s labels for supporting student thinking about how the size of the pieces contributed to the whole, she also acknowledged that the labels would not be beneficial unless students attended to them as they worked:

I hope they’re looking at the fractions. I do, I hope that they’re just not … You hope that they’re paying attention to which fractions actually fit and which
don’t. And, “Well, the $\frac{1}{5}$ doesn’t fit in there”, or whatever, and thinking about that rather than just quickly popping pieces in and seeing what works, which definitely will happen, I know it will. But I think we’ll talk a little bit about thinking as we work and things like that before we begin.

While the labels were important to her, Karla acknowledged their presence would not solve every problem and that students would need to be pushed to attend to them as they worked. Like Colleen, she chose a tool she thought would support her orchestration of opportunities for students to transition between visual and symbolic representations, but also acknowledged her own role in intervening to make sure students thought about the connections.

**Nancy: Avoiding a Tool with Symbolic Representations and Intervening to Orchestrate Transitions Beyond**

The third teacher case was Nancy, who used both strategies to orchestrate transitions between representations, but applied the tool choice strategy differently from Colleen and Karla. While Colleen and Karla discussed the presence of visual and symbolic representations in their VMs as facilitating students’ transitions, Nancy believed the presence of symbolic representations may lead students to skip over the important steps of meaningful connecting symbolic representations to visual representations. Because she chose a tool with no embedded symbolic representations, all the transitions she planned to orchestrate between the visual representations created with the VM and symbolic representations were transitions beyond the VM environment. She described intervening to support these transitions by writing number sentences to record comparisons students would make using the VM.

**VM and Lesson Overview**

Nancy planned a lesson on generating equivalent fractions using the Fraction Circles manipulative from the Phoenix College Mathematics Blog (see Fig. 4). When the VM is loaded, a user sees a set of fraction circle pieces on the left. When a full circle is considered the whole, the pieces represent $1$ whole, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{10}$, and $\frac{1}{12}$, respectively. The pieces are labeled with the first letter of their color (but not with their fraction names), and the labels can be turned off via a button. The pieces can be dragged onto a blank circle on the right side of the screen called a “drop zone.” A user can add up to three more drop zones to the canvas via buttons on the bottom of the screen. As pieces are dragged into the drop zones, they “snap” into a position adjacent to other same-color pieces (if there are any already in the drop zone). Small gray dots on the edges of the pieces allow a user to freely rotate them to a new orientation within the circle. The pieces can also be placed on top of each other, and a user can choose to make pieces transparent so the pieces underneath can be seen. Nancy planned to have her students use this VM to generate sets of equivalent fractions.
Choosing a Tool Without Symbolic Representations to Orchestrate Transitions Beyond

The Phoenix College Fraction Circles do not have symbolic fraction labels on the pieces. There are also no other features that might support students in creating symbolic representations to accompany the visual representations they create with the circle pieces. While Nancy did not explicitly say she chose this VM because the pieces did not show fraction labels, she said she liked that the pieces were not labeled because if students had to figure out what each piece was worth for themselves, they would build “a little bit deeper understanding” of why each piece represented the fraction it did:

If they had it labeled, they wouldn’t necessarily have to understand why it was labeled that. But if they have to stack them up and fit them in the circle themselves and then they go, “Aha, there is five of them. So that means fifths, this is one fifth.” They then have that association that it’s a fifth because I put five of them to make a whole circle. But if it just said fifth, there would be a lot of kids who would just be like, “Oh yeah, that’s a fifth.” They wouldn’t have any clue what they meant.

This excerpt shows that Nancy did think it was important for students to make transitions between the size of the VM’s pieces (a visual representation) and
symbolic fractions (and their verbal names), but, unlike Colleen and Karla, she did not choose a VM with symbolic labels embedded for the purpose of supporting students in making those transitions. Rather, she thought labeling features might make students less likely to make the transitions. She implied students’ verbal mentions of fraction names read from labels would not be evidence of them making meaningful transitions between piece size and symbolic fractions because they would not know what the labels meant.

Intervening to Orchestrate Transitions Beyond by Recording Symbolic Comparisons

To support students in connecting pieces to fraction values, Nancy planned to start her lesson with a discussion of how students could fill a whole to determine what fraction each piece represented. She did not rely on a feature of the tool to support these transitions—she positioned herself as the facilitator of the opportunities for students to make transitions. Nancy also described other ways she would intervene to support students in transitioning from the visual Phoenix College Fraction Circles to symbolic fractions. Nancy’s responses to the student vignettes all involved some combination of making suggestions of how students could manipulate the fraction pieces while she connected their work to symbolic fractions she would record with some kind of annotation tool (if she were teaching digitally) or on a whiteboard (if she were face to face with students). For example, after she helped me (as a hypothetical student) use the fraction circle pieces to “prove” that \( \frac{4}{6} \) was equivalent to \( \frac{2}{3} \) but \( \frac{3}{5} \) was not equivalent to either, Nancy said, “So this is where I would probably grab a pen tool. I would want them to see the fraction \( \frac{2}{3} \) written next to the fraction \( \frac{4}{6} \) to see if they could make any connections. … I definitely want to start connecting the actual notation to the visual.”

Her annotations were number sentences (equations or inequalities, depending on the pair of fractions), and she connected the symbols to the fraction pieces the students were working with and encouraged them to look for patterns. She modeled verbally describing the colored fraction pieces in terms of their fraction names while writing the symbols: “The next bigger piece that you tried was the teal ones, and it was three teal sixths. And then you tried the dark blue ones: four eighths.” She did not specifically say she hoped students would give verbal evidence building connections between visual and symbolic representations, but her consistent modeling of this language suggests she was hoping students would also use such language to make connections.

Janice: Minimal Use of Symbolic Representations Within the Tool and Orchestrating Transitions Beyond Via Labeling and Annotations

The fourth teacher case was Janice, who used teacher intervention (but not tool choice) as a strategy for orchestrating student opportunities to make transitions between visual and symbolic representations of fractions. She did not proactively avoid choosing a tool with embedded labels, but she made only one reference to the labels in her VM and did not mention them as part of her reason for choosing
the VM or expect students to rely on them. Thus, any transitions she orchestrated were transitions beyond. While Nancy’s direct interventions for supporting students in making transitions to symbolic representations beyond the VM relied on writing symbolic fractions and number sentences, Janice also used other annotations as part of her teacher interventions to support transitions.

**VM and Lesson Overview**

Janice used the Math Learning Center (MLC) Fractions app to plan her lesson about converting fractions greater than 1 to mixed numbers. When first opening this VM, a user sees a mostly blank screen and a blue tool bar of buttons at the bottom. A user presses buttons to add a circular or bar-shaped fraction model to the screen, then uses a pop-up keypad or a free-response textbox to choose a number of parts into which the model will be divided. (See Fig. 5.) After choosing a number of parts, a user can use the other buttons on the bottom of the screen to add colored fill to parts of the models, toggle numeric fraction labels on and off, rotate or duplicate models, or add annotations to the screen with text and free-write tools. See Fig. 5 for a model with fill (red) and examples of text (the inequality written in blue) and free-write (the curve around the shaded part of the circle marked in green) annotations. Users can freely drag models around the screen and overlap and resize them. Janice planned to ask her students to use the VM to represent fractions greater than 1, such as $\frac{15}{5}$ or $\frac{18}{4}$, and use the representations to convert the fractions to mixed or whole numbers.

**Minimal Evidence of Choosing a Tool to Orchestrate Transitions**

Although Janice’s chosen VM, the MLC Fractions app, did have a feature allowing symbolic fraction labels to be toggled on and off, Janice mentioned this feature only once during her lesson planning and student response interviews. Specifically, Janice said she might use the symbolic labels in the VM to help students realize

![Fig. 5 The MLC Fractions app](image)
the difference between $\frac{15}{5}$ and $\frac{5}{15}$. In her written lesson plan, she mentioned she anticipated one mistake students might make when initially trying to represent $\frac{15}{5}$ would be to create a representation of $\frac{5}{15}$. When I asked her how she might respond if this occurred, she said she would have students turn on the labels so they could see they had not represented $\frac{15}{5}$ and make sense of what fraction they had represented.

This is one example of Janice using a tool feature to orchestrate an opportunity for students to transition from visual to symbolic representations. However, her use of the tool to support an orchestration of a transition was limited to one specific prompt from me (as the interviewer) and was in the context of a directive to the student from her (as a teacher). She did not bring up the VM’s automated symbolic labels as a reason for choosing the VM. Rather, when I asked her why she chose it, Janice spoke about the VM’s overall flexibility in how many wholes could be added to the screen and how many denominators could be used. In brief, Janice’s opportunistic use of her VM’s symbolic labels to support connecting visual and symbolic representations suggests she saw opportunities to use the tool’s features as supports for facilitating students’ transitions, but these features did not guide her VM selection.

**Intervening to Orchestrate Transitions Beyond by Making Annotations**

Janice’s mentions of symbolic representations were similar to Nancy’s. Throughout her lesson plan interview, Janice repeatedly used her VM’s pencil tool to write symbolic representations to accompany the visual models she created—especially to write whole or mixed numbers to accompany visual representations of fractions greater than 1 (or suggest that students do so). For example, after she modeled how she expected students to represent $\frac{15}{3}$ with the VM’s bar models, Janice said, “Then I would tell them, ‘Okay, label it’. We’re not good at labeling right now.” As she said this, she used the pencil tool to write $\frac{15}{5}$. This shows how Janice intended to intervene to orchestrate transitions from visual to symbolic representations—students must make such a transition in order to “label it.” She indicated she would direct students to create symbolic labels for their VM-created visual representations three times across her student response and lesson planning interviews.

Janice also mentioned some other actions she would take to more directly focus students’ attention on connections between visual and symbolic representations. In addition to using the MLC Fractions app’s pencil tool to write out symbolic representations of fractions, Janice also used it to make other annotations on the visual representations. For example, when showing me how she would demonstrate using the VM to convert $\frac{18}{4}$ to a mixed number, she made a mark grouping the four full fraction strips together before writing 4, then circled the $\frac{2}{4}$ fraction strip before adding $\frac{2}{4}$ to her annotation (see Fig. 6). This is another way Janice saw herself playing a direct role in orchestrating students’ transitions from visual to symbolic representations. She also planned to involve students in connecting the parts of the visual representation to corresponding parts of the symbolic representation by asking questions such as, “Okay, where’s the two fourths?” By answering these questions, students would be providing evidence of making transitions between the representations.
Sheila: Minimal Attention to Orchestrating Transitions Through Tool Choice Only

The fifth teacher case was Sheila, who provided little evidence of using any orchestration strategies to facilitate opportunities for students to make transitions between visual and symbolic representations. She made one comment that suggested the nature of the connection between the symbolic and visual representations in her VM which was one reason she chose it. However, she did not describe much about how she expected the tool to support students in making connections. Unlike the four cases described so far, Sheila did not describe intervening to ensure students made transitions between visual and symbolic representations of fractions. She did not seem to view these transitions as a learning goal of her lesson.

VM and Lesson Overview

Sheila used the Toy Theater Fraction Bars to plan her lesson on generating and seeing patterns within equivalent fractions. When a user loads this VM, a fraction bar is visible on the bottom of the screen (see Fig. 7). Just above this fraction bar, a symbolic fraction is shown with arrow buttons on either side of the numerator and denominator. When a user presses these buttons, the numerator or denominator of the fraction changes accordingly, and the number of total parts and shaded parts in the fraction bar at the bottom of the screen also changes to correspond to the symbolic fraction. A user can drag the fraction bar into one of three stacked slots at the top of the screen. Figure 5 shows what the VM looks like after a user drags a $\frac{2}{3}$ bar into the top slot and a $\frac{2}{6}$ bar into the middle slot. Sheila planned to have her students use the VM to generate sets of equivalent fractions, then use their written list of fractions to derive the multiplication rule for equivalent fractions.
Some Evidence of Choosing a Tool to Orchestrate Transitions

Sheila mentioned the symbolic fraction representation embedded in the Toy Theater Fraction Bars, and the way it corresponded to the visual representation, as one of the reasons she chose to use this VM:

Some of [the other VMs] had the fractions labeled and some of them didn’t. And this with this one, you can see, this is $\frac{2}{5}$, and now I’m moving $\frac{2}{5}$. So even though it doesn’t stay labeled, I’m still seeing that fraction together. Whereas the ones where you build and put the pieces in, you’re not seeing $\frac{2}{5}$. You see $\frac{1}{5}$ plus $\frac{1}{5}$ plus … You know what I’m saying? So, I felt like that was why I liked this one, even though it doesn’t stay labeled.

Sheila seemed to appreciate, in particular, that the VM’s embedded symbolic representation allowed for the “label” to show non-unit fractions, and so students did not have to think of non-unit fractions as being composed of unit fractions. This suggests she hoped the nature of the VM would support students in transitioning between symbolic and visual representations of fractions. This was the extent of Sheila’s discussion of the connection between the symbolic fraction and the visual models, however. She never mentioned hoping that students would show evidence of making transitions by talking about the connection between the symbolic and visual representations.

No Evidence of Intervening to Orchestrate Transitions

Sheila did not discuss teaching moves she would make to more explicitly orchestrate opportunities for students to transition between visual and symbolic representations,
even when I (as the interviewer) opened a specific opportunity for her to talk about doing so. When she described how she imagined the lesson would proceed, she said students would write down lists of equivalent fractions they found by looking at the “common lines” on the VM. She later clarified that by “common lines,” she meant the places where the partitions in the fraction bars aligned. As students recorded symbolic fractions in their notebooks, she said she hoped students would, “start noticing or seeing those patterns like we said, $\frac{1}{2}$, $\frac{2}{4}$, where the numerator is half the denominator or that you can multiply to get from two to four, three to six.” When I followed up to ask if she thought students would be seeing those patterns in the VM’s visual representations as well as in their lists of symbolic fractions, Sheila replied:

I could see how the manipulative does show that as well, because there are two blocks here that make up this $\frac{1}{3}$ and two here that make up this $\frac{1}{3}$. I definitely think the student could see that pattern as well. I guess I didn’t initially think of that because that’s not what I’m drawn to. I would be drawn to seeing the numbers, but a student could see that pattern. … There are always independent thinkers. Students who think very differently about math and especially fractions because it’s different than other math. I would say for the majority of them, they would not see that pattern, or the majority of the students that I’ve had in the past would not have seen it.

This comment clarifies that although she expected students to use the VM to generate their lists of equivalent fractions, she was not expecting the subsequent pattern-finding to connect back to the visual representation. Rather, although she acknowledged students might see the pattern in the fraction bars, she did not expect them to do so and did not intend to push for that connection. Her goal was focused on students using models to generate equivalent fractions (with the VM doing the work of matching the symbolic fractions to the models) and then moving on to look for patterns in the symbolic representations.

**Tammy: Minimal Attention to Orchestrating Transitions Via Both Strategies**

The last teacher case was Tammy, who provided minimal evidence of using each orchestration strategy to facilitate opportunities for students to make transitions between visual and symbolic representations. She described features of the VM that may support transitions from visual to symbolic representations as one of her reasons for choosing her VMs, but did not emphasize this idea. She also described one way she might intervene to support transitions, but only in response to one specific prompt from me (as the interviewer). Like Sheila, Tammy did not seem to view these transitions as a learning goal of her lesson.

**VM and Lesson Overview**

Tammy made the Toy Theater Fraction Bars (see Fig. 7), along with the Toy Theater Fraction Circles (see Fig. 2), available during her lesson. She planned to ask students to use any methods and tools they wished (with the VMs available but not
required) to solve equal-sharing problems. The intent of the lesson was for student strategies to lead to answers expressed in different ways (e.g., $\frac{3}{2}$, $\frac{6}{4}$, or $1\frac{1}{2}$) and use the answers to discuss whether the answers were equivalent.

**Choosing Tools to Orchestrate Transitions**

Tammy said the symbolic labels on the Toy Theater Fraction Circles were one of the reasons she chose this VM over other fraction circle VMs. She explained, “With the circles, I think it’s important to have [labels] because you can’t actually see the chunks like you can in the bars.” She felt that because the bars showed the whole fully partitioned, students could more easily determine how much a piece was worth, but, in a circle, they could not easily count the pieces to determine this. Tammy also mentioned the symbolic labeling feature of the Toy Theater Fraction Bars in her student response interview. She explained that in the past, she noticed the ways her students were highly dependent on referencing colors of manipulatives as they described their work with fractions, and this concerned her since students would not have colorful manipulatives to work with when they took standardized assessments.

She wanted to make sure students could make their own drawings for fractions (eventually) and transition from those drawings to symbolic fractions. The symbolic fraction labels available in the fraction bars VM, in Tammy’s opinion, might facilitate students in making those transitions: “So I think that a benefit to this manipulative is actually seeing the number and seeing the fractions as opposed to just seeing color.” Like Sheila, however, she never mentioned hoping that students would show evidence of making transitions by talking about the connection between the symbolic and visual representations.

**Minimal Use of Direct Intervention as an Orchestration Strategy**

Unlike Sheila, there were no episodes in Tammy’s data where she explicitly said she did not expect students to make transitions between visual and symbolic representations or did not intend to intervene to make sure they did so. Even so, while Tammy occasionally mentioned how symbolic labeling features on each of the VMs could support students in transitioning between a visual representation and a symbolic fraction, she did not elaborate on this issue as something she would support or be paying attention to or orchestrating in her lesson—with one exception. In her student response interview, Tammy mentioned her belief that the symbolic labels on the Toy Theater Fraction Circles might help students overcome the common misconception that a larger denominator should mean a larger piece.

She thought that if students generated a solution to an equal-sharing problem using small pieces such as 12ths, that would serve as a good opportunity for her, as the teacher, to point out, “Okay, you see how that number is getting bigger, but every time that number gets bigger, the piece gets smaller and smaller, and smaller, and smaller?” This is an example of Tammy intending to intervene to help students build connections between piece size and symbolic fraction names. However, Tammy brought this up in response to a solution I presented in a vignette as a hypothetical
Summary of Orchestration Strategies Across Teachers

Teachers in this study used two strategies for orchestrating transitions between representations: tool choice and direct intervention. Five teachers used the tool choice strategy. Four teachers applied the strategy by choosing a tool with embedded symbolic representations. The fifth teacher, Nancy, applied the strategy by choosing a VM without symbolic representations, as she believed this would better support students in constructing their own connections between visual and symbolic representations of fractions.

Five teachers used the latter strategy, direct intervention, to orchestrate transitions to at least some extent. They applied this strategy by directing students’ attention, asking them questions, or modeling language that connected visual representations to formal fraction language. All of these interventions were intended, in part, to orchestrate opportunities for students to make transitions between visual and symbolic representations of fractions.

Summary of Cross-Case Differences

This sub-section summarizes three important differences among the teachers’ use of orchestration strategies.

Differences in Application of Strategies and Their Relationship to Transitions Within Versus Beyond

The differences in which of the two orchestration strategies teachers used and how the strategies were applied corresponded to differences in whether they orchestrated opportunities for students to make transitions within the digital environment of the VM or beyond the digital environment of the VM. For example, because Sheila focused only on tool choice to support transitions between visual and symbolic representations, her students’ opportunities to make transitions were exclusively based on seeing the symbolic fraction labels in proximity to the visual models within the digital environment. Two other teachers, Colleen and Tammy, used both orchestration strategies, but their orchestration strategies also focused only on transitions within the VM because their application of the direct intervention strategy focused on prompting students to examine the VMs and not to create any representations outside the VM environment.

Relatedly, because two other teachers, Nancy and Janice, did not consider the presence of symbolic representations within the VM as important (or even advantageous), the transitions they orchestrated were mostly from the visual representations in the VMs to symbolic representations beyond the VMs. Because only visual
representations were available in their VMs, it was not possible for students to make transitions within the digital environment.

The remaining teacher, Karla, was the only participant to orchestrate transitions both within and beyond the VM environment, and she did so by coordinating her use of both orchestration strategies. Karla chose her VM in large part because she believed it would support students in making transitions between proportionally sized pieces of a whole and the symbolic fractions representing those pieces. With the VM to better support the transition within the VM, from visual representation to symbolic, Karla also said she believed students would be able to transition beyond the digital environment, to write a number sentence for their solutions (e.g., \( \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 \)). Karla believed the transition within that was supported by the design of the VM would act as a scaffold for the transition beyond.

**Differences in Use of the Tool Choice Strategy**

Five of the six teachers in this study used the tool choice orchestration strategy, but there were two noteworthy differences in how the teachers applied the strategy. First, teachers varied according to whether the presence of linked representations (Zbiek et al., 2007), a particular affordance of VMs, related to their choice of VM and how the VM would support their orchestrations of transitions between visual and symbolic fraction representations. Only Colleen specifically mentioned the dynamic links as part of her tool choice; she described the dynamic updating of the labels on the moving dots on the number line as a reason she chose to use the NCTM Equivalent Fractions tool. Sheila and Tammy planned with the Toy Theater Fraction Bars, which has a dynamic link between the manipulable symbolic fraction and the fraction strip on the bottom of the screen. However, neither of them discussed this link with reference to its dynamic nature.

Second, there was also a noteworthy difference between Nancy’s use of the tool choice orchestration strategy and the other teachers. Specifically, while the other four teachers used the tool choice strategy to select a VM with symbolic representations embedded, Nancy choose a tool without symbolic representations, saying she felt a labeled VM might allow students to skip over the work of making meaningful connections between the visual and symbolic fractions. Other teachers said it was useful for students to see visual and symbolic representations together, but Nancy did not believe having the labels available would support students in making transitions.

**Use of the Direct Intervention Strategy and Differences in Emphasis on Verbalization of Connections**

The five teachers who used the teacher intervention orchestration strategy somehow referenced the importance of students describing the connections between visual and symbolic representations of fractions. Colleen, Karla, and Nancy all noted students’ inability to verbalize connections in the past and framed this as a problem they were trying as teachers to overcome. Colleen, Janice, and Tammy had plans to question students about how visual and symbolic representations were connected.
Nancy planned to model using language that connected the visual representations’ colors to symbolic fraction language (e.g., “three teal sixths”).

The only one teacher, who did not use the teacher intervention strategy but rather orchestrated opportunities for transitions using only their choice of VM (Sheila), did not emphasize the importance of students verbalizing their transitions. It is unclear why Sheila did not emphasize this point. Regardless of the reason, the absence of opportunities to provide verbalize connections in the lesson where Sheila orchestrated transitions only through VM choice suggests a limitation in the role of the digital tools themselves in orchestrating transitions.

Discussion

In this study, I examined how six upper elementary school teachers planned to orchestrate opportunities for students to make transitions between visual and symbolic representations of fractions in lessons involving fraction virtual manipulatives. My analysis articulated two strategies teachers used to orchestrate visual-symbolic transitions: through choice of VM and through direct intervention as students worked. The six teachers in this study varied in the following: (1) which strategies they used and to what extent, (2) whether they used the strategies to support transitions within or beyond the VMs, (3) whether and how they attended to the potential and pitfalls of VMs’ symbolic representations when they chose VMs, (4) whether and how they provided opportunities to hear or verbalize connections between representations via the direct intervention strategies. These differences have implications for the kinds of transitions between representations students are likely to make as they work with VMs and how robust the learning they take from these transitions might be. I discuss these implications in this section, with particular attention to theorizing how the results suggest what professional learning experiences might be needed to help teachers use VMs to support students in making transitions between visual and symbolic representations of fractions.

Orchestration Strategies

Prior research examining the teacher’s role in shaping students’ work with virtual manipulatives (e.g., Hansen et al., 2016; Trgalová & Rousson, 2017) suggests at least two strategies teachers can use to orchestrate learning opportunities for students: through choice of task, and through intervention with students as they work (e.g., asking questions or making suggestions in the moment). Five of the six teachers in this study use the direct intervention strategy to some extent (see Table 3). However, there was no evidence of teachers using task choice to shape students’ interactions with the VMs, for the purpose of orchestrating transitions or otherwise. Based on the presence of task choice as an orchestration strategy in prior research, the professional learning sessions included an activity that engaged teachers in considering how they might adapt tasks to take advantage of helpful VM features and the
instructions for the lesson planning task placed focus on selecting and adapting a task.

Moreover, the interview protocol contained a question to probe whether incorporating a VM into a lesson impacted the task teachers planned to have students complete. All teachers said no when asked this question. Those who elaborated said that incorporating the VM might give students more support or a different perspective on completing the task, but the task itself would be quite similar to what they would pose if students were working with physical manipulatives or paper and pencil. Although existing case studies have documented how teachers’ task choices can shape students’ interactions with VMs (Hansen et al., 2016; Trgalová & Rousson, 2017), the results of this study suggest that introduction of a VM does not necessarily influence teachers’ choice of tasks—especially when teachers are asked to choose their own VMs rather than use a particular VM chosen by researchers. Allowing teachers to choose their own VM, as in this study, prompted teachers to browse through their existing repertoire of lessons and tasks and choose one that seemed a good fit with the available VMs.

Choosing a VM, rather than choosing a task, was the second strategy teachers used to orchestrate opportunities for students to make transitions between representations. Five teachers chose a specific VM based, in part, on features (or lack of features, in Nancy’s case) they believed would support students in making those transitions. Use of this orchestration strategy is likely a reflection of how I posed the lesson planning part of the study to teachers: I asked them to choose a VM off a list to use in a fractions lesson. I allowed teachers in this study to select their own VMs because I believed this framing of the task would better simulate teachers’ everyday teaching conditions, where they are often given district-mandated curricula but have access to many free supplementary tools on the Internet (Webel et al., 2015).

The emergence of VM choice as a strategy for orchestrating transitions is a reflection of the procedure used in the study, but it may also suggest that teachers’ strategies for this work are related to the aspects of their lessons over which teachers feel the most agency. Teachers often do not feel they have agency to make significant changes to their mathematics curriculum materials (McClain et al., 2009; Rich, 2021a). In these circumstances, task changes may feel like too strong a deviation from school- or district-mandated curriculum guides or pacing schedules, whereas providing a new tool for students to use, such as a VM, feels like a minor change within their realm of agency. Further research on the teachers’ role in orchestrating transitions between fraction representations—in the context of VM use or otherwise—would benefit from more directly examining how the constraints on lesson planning, imposed by the researchers or by teachers’ broader professional context, are related to the strategies that teachers use.

Although teachers in this study did not use task selection as a strategy for shaping students’ interactions with VMs, several of them did consider the task carefully as they chose which VM to use. For example, Karla chose to use the Toy Theater Fraction Circles because she thought they had supports that were well aligned with the struggles her students had in the past with her task. Janice chose to use the MLC Fractions app because it allowed students to create multiple wholes—a necessary feature when representing fractions greater than 1. Frameworks and other guidance
already exist to support teachers in selecting and implementing high-quality tasks (e.g. Drake et al., 2015; Smith & Stein, 1998; Sullivan et al., 2015), but less attention has been giving to developing teacher learning experiences for selecting tools that could support students in engaging in tasks at a high level—for example, by ensuring they have opportunities to transition among multiple representations. The need for such experiences is likely to become more pressing as more and more digital tools become available to teachers.

In sum, the results of this study suggest the need for design and study of professional learning experiences that introduce and provide practice with all three potential orchestration strategies: task choice or adaptation, tool choice, and direct intervention. None of these strategies was used by teachers universally, and there were differences in the level of attention teachers gave to using the strategies to intentionally and meaningfully orchestrating transitions between visual and symbolic representations of fractions—a learning experience that is critical for students in overcoming whole-number biases in their work with symbolic fractions (Siegler et al., 2013), in using manipulatives meaningfully (Uttal et al., 2009), and generally in negotiating transitions across mathematical registers (Duval, 2006). Professional development could use examples of application of each strategy from this and other research studies (Hansen et al., 2016; Trgalová & Rousson, 2017) as exemplars. Such experiences could also encourage teachers to coordinate their choice of tools and tasks with learning goals related to making transitions rather than issues exclusively related to student engagement (which has been shown to guide teachers’ tool selection in other work; Johnston & Suh, 2009).

Transitions Within Versus Beyond Digital Environments

The expansive and dynamic nature of the environments in which digital tools for mathematics are embedded opens several possibilities for the kinds of transitions students might make among representations. Two of these are transitions within the digital environment and transitions beyond the digital environment. Transitions within the environment are a new and potentially helpful experience made available by technology (Zbiek et al., 2007), but these transitions within may need to be followed by transitions beyond if students are to carry what they gain from digital environments into later learning activities (Geraniou & Mavrikis, 2015). The differences in the ways teachers applied the orchestration strategies were associated with differences in the opportunities they orchestrated for students to make the transition between visual and symbolic fraction representations within versus beyond the VM environment. Five teachers orchestrated opportunities for only one type of transition or the other, and only one teacher (Karla) orchestrated opportunities for both.

While lessons focusing on one kind of transition may well be valuable, each type of transition has its limitations in isolation. Students who experience making transitions within a digital environment may not easily apply the knowledge gained outside the environment. Students using digital tools that do not embed multiple representations—thereby focusing only on transitions beyond—may be missing out on a richness of mathematics experience made possible by the digital environment.
The most durable learning experiences may be those that allow students’ opportunities to transition between representations within digital environments and subsequently support them in making transitions beyond—much like Karla imagined the transitions she planned to orchestrate.

The variation in the transitions teachers in this study planned to orchestrate suggests a need for further research on the benefits or challenges in providing students opportunities to make transitions alone or in sequence. Is it beneficial for students to make transitions within a digital environment before making transitions beyond? Under what conditions is this the case? How should transition experiences be timed and in coordination of what experiences? Answers to these questions could help to inform professional development experiences for teachers designed to support effective lesson design that subsequently supports students in making transitions between representations. Suh (2016) described a series of assignments for pre-service teachers that supported them in selecting and sequencing activities with digital tools in relation to learning trajectories. This series of assignments could be used as a model for helping teachers think about designing progressions of transition experiences, when evidence of effective progressions is available.

**Tool Choice: Potential and Pitfalls of Dynamic and Static Links Between VM Representations**

Many scholars have suggested that an advantage of VMs is not merely that they can embed multiple representations within the same environment, but also that those representations can be dynamically linked such that one representation changes in response to changes in the other (Moyer-Packenham & Westenskow, 2013; Sarama & Clements, 2009; Zbiek et al., 2007). One potential benefit of these dynamic links is the feedback they provide to students; dynamic links make it difficult to overlook the impact that changes in one representation make on the other (Zbiek et al., 2007). Feedback from dynamically changing representations may stimulate student thinking that leads to making meaningful transitions between visual and symbolic representations. On the other hand, both static and dynamic links have the potential to eliminate opportunities for students to transition between representations; because the connections are made within the tool, students may better be able to avoid constructing the connections for themselves. Much depends on how a teacher shapes students’ use of or attention to the links between representations.

Most of the teachers in this study did not make use of linked representations, a VM feature that is theorized to have great potential in supporting students in connecting representations (Zbiek et al., 2007), to orchestrate transitions. Although several teachers chose VMs with linked representations and several also planned to intervene in students’ work with VMs, only one teacher, Colleen, coordinated these two strategies in a way that would draw students’ attention to feedback from the linked representations. This was despite the fact that all six teachers attended 3 h of professional development that included specific attention to linked representations.
and how they might be useful in student learning. This suggests a need for further investigation of how teachers might be supported in learning how to take advantage of dynamic links between representations to support student thinking and learning and the development of connections between representations.

While only Colleen considered and attempted to take advantage of the potential of linked representations to support students in making transitions, several of the teachers provided some evidence of having considered the potential pitfalls of having students use VMs with symbolic representations embedded—whether they were static labels or dynamically linked symbolic fractions. Nancy is the teacher that most directly discussed the possibility that the presence of symbolic labels within VMs could actually reduce the opportunities students have to make transitions by relieving students of the cognitive work needed to generate connections. Nancy applied the tool choice orchestration strategy to select a VM without symbolic representations embedded as a way to create opportunities for students to connect the piece sizes to fraction symbols. Two other teachers, Colleen and Karla, showed attention to the potential loss of opportunities to make transitions by using direct intervention to guide students to make sense of the connections between representations. On the other hand, Sheila and Tammy did not seem concerned with the idea that the symbolic representations within the VMs could allow students to skirt the work of generating connections between symbolic and visual representations for themselves.

The wide variation in teachers’ levels and forms of attention to the potential and pitfalls of the symbolic representations embedded within VMs suggest a need for professional development experiences that support teachers in critically considering this issue. One idea is to relate this issue to develop teachers’ thinking about the task choice orchestration strategy (which did not come up in this study). Teachers could benefit from support in understanding how students might be guided to interpret feedback from linked representations in meaningful ways. Zbiek and colleagues (2007) argued that students could use feedback from linked representations to revise their initial interpretations of representations, which may be incorrect or incomplete. However, if the tasks students are given to complete using VMs do not emphasize the understanding of the representations itself as a goal, students may be less inclined to think about the feedback in these terms.

Prior work provides support for the approach of connecting task choice strategies to the issue of harnessing the potential of linked representations. Anderson-Pence and Moyer-Packenham (2016) found that students who used VMs with linked representations made more generalizations than students who used VMs without linked representations, but the tasks students were given to solve with VMs were different for each tool and “adapted from tool-specific lesson explorations” (p. 10) provided on each VM’s website. Thus, it is quite possible that the increased number of generalizations related not only to the linked representations, but to the ways the tasks shaped students’ interactions with and interpretations of the feedback provided by those linked representations.

Many teachers may need help developing such tasks. Using the math task framework (Smith & Stein, 1998) to guide teachers’ thinking specifically about how different tasks shape the potential and pitfalls of linked representations and static symbolic labels to create or remove opportunities for students to transition between
visual and symbolic representations of fractions could be a helpful activity for professional learning. While teachers are still learning to develop high-level tasks that both (a) utilize VMs with embedded symbolic labels and (b) provide opportunities for students to construct their own connections between representations, they also might be guided to consider the reasons that choosing a VM without symbolic representations could be advantageous for certain learning goals.

Direct Intervention and Creating Opportunities for Students to Show Evidence of Making Transitions

While it is likely possible for students to make transitions among visual and symbolic representations of fractions without explaining the connections aloud, eliciting verbal evidence of students transitioning from one representation to another is important for at least two reasons: first, verbalization of connections between manipulatives and symbols may support learning of the underlying mathematical concepts (Resnick & Omanson, 1986), and second, asking students to verbalize their thinking is one way for teachers to formatively assess students’ understanding of the connections between representations.

An interesting result of this study is that opportunities for students to verbalize connections between representations were more prevalent in lessons where teachers used the direct intervention strategy. Specifically, teachers who applied the direct intervention orchestration strategy planned to support students in verbalizing connections by asking questions, guiding discussions, or modeling connective language. This result further illustrates the critical role teachers play in encouraging students to verbalize their understanding of relationships between representations. Consideration of task choice or task adaption orchestration strategies within professional development experiences could attend to this issue by encouraging teachers to include explanation and justification—and not just generation of answers—as part of the tasks that students complete with VMs. Consideration of the direct intervention strategy could also encourage the use of high-level questions as students work.

Conclusion

This study explored how teachers orchestrated opportunities for students to use fraction VMs to make transitions between visual and symbolic fraction representations. All six teachers mentioned at least one strategy for orchestrating these transitions, but they varied in how much attention they gave these transitions in their lesson plans, whether their strategies would support transitions within and/or beyond the VM environment, the opportunities they gave students to hear or verbalize connections, and whether and how they took advantage of linked representations.

These results lay the foundation for future work in how to support teachers in orchestrating high-quality opportunities for students to make transitions from the visual fraction representations at the heart of VMs to symbolic representations within or beyond the VMs. In particular, the results of this study suggest that professional learning experiences designed to support teachers’ effective use of VMs
in fraction instruction should (1) include attention to three orchestration strategies (task choice, tool choice, and direct intervention), (2) guide teachers in sequencing opportunities for students to make transitions within and beyond the VM environment, (3) help teachers to consider the relationship between the potential of linked representations and symbolic labels and the various ways students might use them in completing tasks, and (4) encourage teachers to provide opportunities for students to hear or verbalize connections between representations.

This study has several limitations. First, the data collection and analysis occurred during the height of the COVID-19 pandemic. The prevalence of online teaching and other school safety precautions during this time prevented me from collecting classroom data, which left me unable to examine how teachers thought about or used the VMs in the context of their classroom practice. The student response interviews were intended to serve as a substitute for data about how teachers might think about VMs as they responded to students in real time. This data was helpful, but not adequate as a substitute for classroom data. Second, the design of the study may have had an impact on the orchestration strategies that emerged—in particular, the presence of the tool-choice strategy.

While I intended the study procedure to mimic the processes teachers use in their classrooms for integrating VMs into their teaching, a descriptive study of how teachers use VMs without researcher interference would help to remove bias in the emergence of orchestration strategies. Finally, the six participants all volunteered to be part of the study, which also likely introduced biases into the data. All the teachers had at least nominal interest in developing their mathematics teaching skills and exploring the potential of virtual manipulatives. In that sense, the results of this study may represent a best-case scenario of how teachers might think about and use fraction VMs.

Appendix 1: Sample lesson plan interview protocol

The questions common across participants are shown in italic. The questions in Roman font were specific to the participant.

1. What is the general structure of your remote lessons?
   a. How long do you typically spend?

2. What is the learning goal for your lesson? What do you hope kids will take away?
   a. Have they seen any equivalent fractions at all? What representations? Do they grasp the “different numbers, but same amount” idea?
   b. Do students know the rule for equivalent fractions yet? Do you hope they’ll derive it themselves?
   c. What did you mean when you said students would see the order of the fractions but it would not be the focus?
3. What virtual manipulative did you choose?
   a. Show it to me and tell me what you like about it.
      1. What do you mean by multiple visuals of the same number?
      2. Why do you like the circle and square options?
      3. Why do you like the equivalent fractions being displayed on the side?
   b. Is there anything you do not like about it?
   c. Why did you choose this particular virtual manipulative for your lesson?
   d. Did you choose your lesson to go with the VM, or choose the VM to go with your lesson?

4. Tell me more about your task. How will you pose it to students? What will their goal be?
   a. When you say they’ll discuss their thoughts as a group about $\frac{3}{8}$, what do you mean? Thoughts about how to start making something equivalent? Predictions of what an equivalent fraction might look like?, etc.
   b. What do you mean when you say they’ll discuss their ideas with a partner?, e.g. do you expect the ideas to be about strategies for creating equivalent fractions with the tool? Things they notice about the list on the side?, etc.

5. What are some things you expect students to do with the VM as they work on your task?
   a. Show me what you mean when you say they might go up one space at a time.
   b. Why might that be helpful? Why might that not be helpful?
   c. Do you have any other ideas of what they might do with the manipulative?
   d. What would you say to a student who skipped over the manipulative?

6. How do you hope the VM will support students’ thinking? What parts of the tool do you hope or intend for them to pay attention to?

7. Why did you choose to have the kids work in small groups?
   a. Will you group them by ability? Why?

8. What might you expect students to say to each of your prompts?
   • Can you give an example of what you mean by a more specific guiding question?

9. How might you use the manipulative in your concluding discussion? E.g. will you show your screen? Have a student do so?

10. How do you think you might have taught this lesson differently if you did not have this VM?
    a. Would the goals have changed?
    b. Would the tasks have changed?
Appendix 2: Sample vignette for student response interviews

**Teacher:** Colleen  **VM:** NCTM Equivalent Fractions (no longer available on-line)

![Digital Experiences in Mathematics Education (2023) 9:163–208](image)

**Vignette**

Imagine that after students use the tool to find equivalent fractions for $\frac{3}{8}$, a student makes $\frac{6}{16}$ as shown in the blue model. Then, they try to make $24$ths using just vertical lines but find that they cannot do so. A partner says they can make $24$ths if they divide the area model both ways (horizontally and vertically), as shown in the green model. A student fills in nine $24$ths, but then says, “This can’t be right. The number line says they are the same but I know equivalent fractions show the same amount. The green can’t be the same as the blue and red because it has that piece sticking out. It should be even like the others.”

**Rationale**

- *Sample strategies from plan:* This is an example that came up in Colleen’s lesson plan interview. She did not seem to be bothered by it or think about possible student responses.
- *Common misconceptions:* The CCSS identifies the idea that same-size pieces do not necessarily have to be the same shape as a standard, albeit for a grade level below 4–5. This suggests students may trip up on this idea.
- *Typical approaches vs VMs:* Many times, when using drawings, lessons focus on children sub-dividing existing parts, suggesting that the idea that equivalent fraction representations have to look the same might be reinforced. The VM makes this new issue more likely to come up.
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Data Availability  The data that support the findings of this study are not available because the data contain images and information that could compromise research participants’ anonymity.

Declarations

Competing Interests  The author declares no competing interests.

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