ADDITIVE TWISTS OF FOURIER COEFFICIENTS OF GL(3) MAASS FORMS

XIANNAN LI

(Communicated by Matthew A. Papanikolas)

Abstract. We prove cancellation in a sum of Fourier coefficients of a $GL(3)$ form $F$ twisted by additive characters, uniformly in the form $F$. Previously, this type of result was available only when $F$ is a symmetric square lift.

1. Introduction

Substantial work has been done in studying sums involving coefficients attached to various $L$-functions. A very classical example is the problem of estimating exponential sums of the form $\sum_{n \leq x} n^{it}$, which is related to subconvex bounds for the Riemann zeta function and to Dirichlet’s divisor problem. For more on this, see, for instance, Chapters V and XII in [14]. A vast literature also exists for the estimation of character sums. These are, among other things, related to subconvexity for Dirichlet $L$-functions and estimates for the least quadratic non-residue. See, for instance, Chapter 12 of [8].

The estimation of sums of coefficients twisted by additive characters is also classical. To be specific, we shall be interested in sums of the type

$$S = \sum_{n \leq N} a_n e(n\alpha)$$

where, as usual, $e(x) = e^{2\pi ix}$. Here $a_n$ may be the coefficients of certain $L$-functions or more general coefficients of arithmetic interest. This type of sum had already appeared in the work of Hardy and Littlewood [5] in 1914 and has been investigated extensively. See also the work of Montgomery and Vaughan [13].

In the case of automorphic forms on $GL_2(\mathbb{R})$, obtaining cancellation in $S$ is well understood when the $a_n$ are either the normalized Fourier coefficients of a modular form or a Maass form on the upper half plane. For instance, if $f(z) = \sum_n a(n)n^{k-1}e(nz)$ is a weight $k$ modular form, then it is not hard to prove that

$$S \ll_f N^{1/2}\log N,$$

and this is essentially the truth, as can be seen from the $L^2$ norm of $S = S(\alpha)$ for $\alpha \in [0, 1]$. (See Chapter 5 of [7].) Note that while the bound depends implicitly on $f$, it is uniform in $\alpha$, which is useful for applications towards proving the same bound for the sum of such coefficients restricted to any arithmetic progression.
Moreover, the proof for this case is fairly straightforward, depending only on an estimate for the size of $f(z)$.

Results on such sums in higher rank settings are quite recent and exhibit new features. S. D. Miller in [11] proved the first result and showed that
\[
\sum_{n \leq N} A(1, n) e(\alpha n) \ll_F N^{3/4 + \epsilon},
\]
where $A(m, n)$ are the Fourier coefficients of a cusp form $F$ on $GL(3, \mathbb{Z}) \backslash GL(3, \mathbb{R})$, where the result is uniform in $\alpha$, but the implied constant depends on the form $F$. In the same paper, he discusses the connection between such a bound and bounds on the second moment $\int_{-T}^{T} |L(1/2 + it)|^2 dt$, where $L(s) = L(s, F)$ is the $L$-function attached to $F$. The main tool used in this proof is Voronoi summation for $GL(3)$ developed by Miller and Schmid [12].

It is natural and sometimes desirable for applications to prove such a bound uniformly in $F$. In this direction, Xiaoqing Li and M. Young [10] prove a result in the special case where $F$ is a symmetric square lift of an $SL(2, \mathbb{Z})$ Hecke-Maass form. Their main result is
\[
\sum_{n \leq N} A(1, n) e(\alpha n) \ll N^{3/4 + \epsilon} \lambda_F(\Delta) D + \epsilon,
\]
where $\lambda_F(\Delta)$ is the analytic conductor of $L(s, F)$ and $D = 1/4$ assuming Ramanujan and $D = 1/3$ unconditionally. The proof is more intricate, depending on a careful technical analysis of exponential integrals which appear in Voronoi. An interesting new phenomenon which occurs in their work is the localization of the dual sum in very short intervals. It is for this reason that the Ramanujan conjecture becomes relevant.

The authors of [10] restrict their attention to the symmetric square case as a compromise between generality and difficulty. Symmetric square lifts are a thin subset of all $GL(3, \mathbb{Z}) \backslash GL(3, \mathbb{R})$ cusp forms, so it would be interesting to extend this result to general Maass forms. That is the focus of the present paper.

**Theorem 1.** Let $F$ be a tempered cusp form on $GL(3, \mathbb{Z}) \backslash GL(3, \mathbb{R})$ with Fourier coefficients $A(m, n)$ and $A(1, 1) = 1$, and Langlands parameters $\alpha_i$, $1 \leq i \leq 3$. Let $\mathfrak{C} = \prod_{i=1}^{3} (1 + |\alpha_i|)$. Then for any $\alpha \in \mathbb{R}$,
\[
\sum_{n \leq N} A(1, n) e(n\alpha) \ll_{\epsilon} N^{3/4 + \epsilon} \mathfrak{C} D,
\]
where we may take $D = 1/4$ assuming Ramanujan, and $D = 5/12$ unconditionally.

**Remark 1.**

1. The quality of the unconditional bound in our result is inferior to the unconditional bound in [10] due to the presence of functoriality results for $GL(2)$ which can be used for symmetric square lifts.
2. Here, $\mathfrak{C}$ is the usual analytic conductor for $L(1/2, F)$. It is the same size as $\max(|\lambda_1|, |\lambda_2|)$, where the $\lambda_i$s are the eigenvalues of the Laplace-Casimir operators as defined in §6 of [3].
3. As mentioned before, the work of Xiaoqing Li and Young [10] includes an analysis of very short sums in a range like $A \leq n \leq A + B$, where $B$ is small. One of the differences in the general case is that sometimes this short sum behaviour disappears because $A$ can also be very small. However, this
is balanced out by the matching properties of functions appearing in the integral transform.

Rather than bound the sum $\sum_{n \leq N} A(1, n)e(n\alpha)$ directly, it will be more convenient to bound a smooth version of that sum.

**Theorem 2.** Preserve notation as in Theorem 1. Let $w$ be a smooth function with support in $[N, 2N]$ and such that $w^{(j)}(y) \ll_j N^{-j}$ for all $j \geq 0$. Then

$$\sum_{n \geq 1} A(1, n)e(n\alpha)w(n) \ll \varepsilon N^{3/4+\varepsilon}C^D,$$

where we may take $D = 1/4$ assuming Ramanujan, and $D = 5/12$ unconditionally.

Theorem 1 follows from Theorem 2 by standard methods (see §9 of [10]). We now concentrate on proving Theorem 2.

### 2. Background and basic setup

We first briefly recall some facts about $SL_3(\mathbb{Z})$ Maass forms, referring to [3] for specifics. The Maass forms we consider are analogous to the classical Maass forms on the upper half plane. To be precise, we shall consider smooth functions $F$ on $GL(n, \mathbb{R})/(O(n, \mathbb{R}) \cdot \mathbb{R}^\times)$ which are invariant under the action of $SL(3, \mathbb{Z})$ and are eigenfunctions of the center of the universal enveloping algebra of $\mathfrak{gl}(3, \mathbb{R})$. In the familiar classical case, the center of the universal enveloping algebra of $\mathfrak{gl}(2, \mathbb{R})$ is generated by the hyperbolic Laplacian, and in our case is generated by two Laplace-Casimir operators, which are explicitly written in §6 of [3]. Further, we ask that $F$ satisfy a cuspidality condition, which is formally stated in 5.1.3 of [3].

Like classical Maass forms, $F$ has a Fourier expansion and is associated with an $L$-function. The Fourier coefficients of $F$ are naturally indexed by two parameters, and will be written as $A(m, n)$. We are interested in finding cancellation in $\sum_{n \geq 1} A(1, n)e(n\alpha)w(n)$, as in Theorem 2. The first step of this is to approximate $\alpha$ by a rational number so that we may use Voronoi summation.

To that end, write $\alpha = a/q + \theta$, where $(a, q) = 1$, $q \leq Q$ and $\theta \leq 2\pi/q$, possible by Dirichlet’s theorem on Diophantine approximation.\(^2\) We then apply Voronoi summation to

$$S = \sum_{n \leq N} A(1, n)e(an/q)\psi(n),$$

where $\psi(y) = e^{i\theta y}w(y)$.

The Voronoi summation formula for $GL(3)$ was first proven by Miller and Schmid [12], and re-proved by Goldfeld and Xiaoqing Li [4] using an alternate method. We first introduce some notation. Let

$$\tilde{\psi}(s) = \int_0^\infty \psi(x)x^s \frac{dx}{x}$$

and

$$\Psi_k(x) = \frac{1}{2\pi i} \int_{(\sigma)} (\pi^3 x)^{-s} \frac{\Gamma \left( \frac{1+s+a_1+k}{2} \right) \Gamma \left( \frac{1+s+a_2+k}{2} \right) \Gamma \left( \frac{1+s+a_3+k}{2} \right)}{\Gamma \left( \frac{-s-a_1+k}{2} \right) \Gamma \left( \frac{-s-a_2+k}{2} \right) \Gamma \left( \frac{-s-a_3+k}{2} \right)} \tilde{\psi}(-s) ds.$$\(^3\)

\(^2\)One can consider the universal enveloping algebra as the set of differential operators of all orders which are left invariant under the action of $GL(3, \mathbb{R})$.

\(^3\)Q is a parameter to be determined later.
Write $\tilde{a}$ for the multiplicative inverse of $a$ modulo $q$. Further define

$$\Psi_{\pm}(x) = \frac{1}{2^{3/2}} \left( \Psi_0(x) \pm \frac{1}{i} \Psi_1(x) \right).$$

Then, by Voronoi summation [12], $\mathcal{S} = \mathcal{S}_+ + \mathcal{S}_-$, where

$$\mathcal{S}_\pm = q \sum_{n_1 | q n_2 \geq 1} A(n_2, n_1) S(\tilde{a}, \pm n_2; q/n_1) \Psi_\pm \left( \frac{n_2 n_1^2}{q^3} \right).$$

It is important to understand the dependence of the integral transforms $\Psi_k$ on the Langlands parameters $\alpha_i$ since this is where the dependence on the conductor arises. This forms the bulk of the proof. Before proceeding, we record a few basic results from [10]. First, by Lemma 4.1 of [10],

$$S \ll q^{3/2+\epsilon} \max_{\pm} \max_{d | q} \max_{n_1 | q/d} \sum_{n \geq 1} \left| A(n, 1) \right| \left| \Psi_{\pm} \left( \frac{n}{q^3} \right) \right|. $$

The presence of the parameters $d$ and $n_1$ are unimportant to the actual analysis. Without loss of generality, we will assume that $d = n_1 = 1$, which will simplify the cluttered notation; the other values of $d$ and $n_1$ can be bounded the same way. This reduces the problem of bounding $S$ to bounding

$$T_k = q^{3/2+\epsilon} \sum_{n \geq 1} \left| A(n, 1) \right| \left| \Psi_k \left( \frac{n}{q^3} \right) \right|. $$

2.1. A saddlepoint approximation. Write $s = \sigma + i\tau$ so that $\tilde{\psi}(s) = x^\sigma I$, where

$$I = \int_0^\infty \omega(x) e^{i\theta x} x^{i\tau} \frac{dx}{x}. $$

If the integral is oscillatory, then the saddlepoint method may be applied to evaluate $I$. We quote Lemma 5.1 from [10] for this purpose.

**Lemma 1.** With notation as above, if $|\tau| \geq 1$ and $|\theta N| \geq 1$, then

$$I = \sqrt{2\pi} \omega(-\tau/\theta) |\tau|^{-1/2} e^{i\tau \log |\tau|} e^{i\pi \text{sgn}(\theta)} + O(|\tau|^{-3/2}).$$

Further, if $|\tau| \geq |\theta N|^{1+\epsilon}$, then

$$I \ll_{A, \epsilon} |\tau|^{-A},$$

and if $|\tau| \leq |\theta N|^{1-\epsilon}$, then

$$I \ll_{A, \epsilon} |\theta N|^{-A}. $$

**Remark 2.** Also, we note that if $|\tau| \leq 1$, then $I \ll_{A} |\theta N|^{-A}$, and if $|\theta N| \leq 1$, then $I \ll_{A} (1 + |\tau|)^{-A}$.

We refer the reader to [10] for the proofs of the preceding statements.

In further analysis of the exponential integral, we will see that sometimes the sum is localized to very short intervals. We record the following easy lemma for convenience.

**Lemma 2.** Let $A \geq B > 0$. Then,

$$\sum_{A \leq n \leq A+B} \left| A(1, n) \right| n \ll \left( \frac{B}{A} \right)^{p} A^{\epsilon} c^{\epsilon},$$
where we have \( p = 1 \) if the Ramanujan conjecture holds, and \( p = 1/2 \) unconditionally.

**Proof.** If Ramanujan holds, then
\[
\sum_{A \leq n \leq A+B} \frac{|A(1,n)|}{n} \ll A^\epsilon \log \left( \frac{A+B}{A} \right),
\]
from which the conclusion follows. Otherwise, by Cauchy’s inequality,
\[
\sum_{A \leq n \leq A+B} \frac{|A(1,n)|}{n} \leq \left( \sum_{A \leq n \leq A+B} \frac{|A(1,n)|^2}{n} \right)^{1/2} \left( \sum_{A \leq n \leq A+B} \frac{1}{n} \right)^{1/2}
\[
\ll A^\epsilon \mathcal{C}^\epsilon \log \left( \frac{A+B}{A} \right)^{1/2},
\]
from which the claim follows. Here we have used that
\[
\sum_{A \leq n \leq A+B} \frac{|A(1,n)|^2}{n} \ll A^\epsilon \mathcal{C}^\epsilon,
\]
which follows by the convexity bound for Rankin-Selberg \( L \)-functions \( L(s, F \times \tilde{F}) \). Brumley \cite{1} proved this convexity bound for \( L(s, F) \) automorphic for \( GL(n) \) for \( n \leq 4 \) using recent progress in functoriality, and the author \cite{9} proved this for all \( n \) by a different method.

2.2. Preliminary cleaning. Let
\[
G(s) = \frac{\Gamma \left( \frac{s+k}{2} \right)}{\Gamma \left( \frac{1-s+k}{2} \right)}.
\]
The product of \( \Gamma \) factors which appear in the integral transform \( \Psi_k \) is \( \mathcal{G}(1+s) \), where
\[
\mathcal{G}(s) = G(s + \alpha_1)G(s + \alpha_2)G(s + \alpha_3)
\]
and the Langlands parameters \( \alpha_i \) satisfy \( \alpha_1 + \alpha_2 + \alpha_3 = 0 \), and \( \text{Re} \ \alpha_1 = \text{Re} \ \alpha_2 = \text{Re} \ \alpha_3 = 0 \) by temperedness. Thus set \( \alpha_j = ia_j \) for \( a_j \in \mathbb{R} \).

Then, for \( \sigma > -1 \), Stirling’s approximation gives
\[
G(1 + \sigma + it) \ll (1 + |t|)^{\sigma+1/2}
\]
so that
\[
\mathcal{G}(1 + \sigma + it) \ll \left( (1 + |t+a_1|)(1 + |t+a_2|)(1 + |t+a_3|) \right)^{\sigma+1/2}.
\]
Recalling that
\[
\mathcal{C} = (1 + |\alpha_1|)(1 + |\alpha_2|)(1 + |\alpha_3|),
\]
we have
\[
\mathcal{G}(1 + \sigma + it) \ll (\mathcal{C}(|t| + 1) + |t|^3)^{\sigma+1/2}.
\]
To see this, assume without loss of generality that \( |\alpha_1| \geq |\alpha_2| \geq |\alpha_3| \) and note that \( |t + a_1| \leq |t| + |a_1| \). The bound will then follow if we verify that \( |t|^2(1 + |a_1|) \ll |t|\mathcal{C} + |t|^3 \). If \( (1 + |a_1|) < |t| \), then we are done. Otherwise, since \( a_1 + a_2 + a_3 = 0 \), we have that \( |t| \leq (1 + |a_1|) \leq (1 + |a_2|)(1 + |a_3|) \), so that \( |t|^2(1 + |a_1|) \ll |t|\mathcal{C} \).
By Lemma [1] and Remark [2], $\tilde{\psi}(-\sigma + it) \ll A N^{-\sigma} \left(1 + \frac{|t|}{1 + |\theta N|^{1+\epsilon}}\right)^{-A}$. Thus, for $\sigma > -1$,
\[
\Psi_k(x) \ll_{\sigma,A} \int_{-\infty}^{\infty} (xN)^{-\sigma} \left(1 + \frac{|t|}{1 + |\theta N|^{1+\epsilon}}\right)^{-A} (C(|t| + 1) + |t|^3)^{\sigma + 1/2} (xN)^{-\sigma} (C(|\theta N| + 1) + |\theta N|^3)^{\sigma + 1/2} (1 + |\theta N|^\epsilon).
\]

(2.3)

We first record the following results on $\Psi_k(x)$.

**Lemma 3.** Let $U = C(|\theta N| + 1) + |\theta N|^3$.

1. If $xN \geq N^\epsilon U$, then $\Psi_k(x) \ll A N^{-A}$ for any $A > 0$.
2. If $|\theta N| \ll C^\epsilon$, then $\Psi_k(x) \ll C^{1/2+\epsilon}$.
3. Let $R_2 = \{ t \in \mathbb{R} : |t + a_i| \geq C^\epsilon \text{ for all } 1 \leq i \leq 3 \}$. Further, let
\[
f(t) = t \log \left(\frac{\pi^3 |t|}{e|\theta|}\right) - (t + a_1) \log \left(\frac{|t + a_1|}{2e}\right)
- (t + a_2) \log \left(\frac{|t + a_2|}{2e}\right) - (t + a_3) \log \left(\frac{|t + a_3|}{2e}\right).
\]

If $|\theta N| \geq C^\epsilon$, then there exists a smooth function $g(t)$ with support when $|t| \asymp |\theta N|$, satisfying $\frac{d}{dt} g(t) \ll_j |t|^{-1/2-j}$ such that
\[
\Psi_k(x) \ll \sqrt{xN} \int_{R_2} g(t) e^{if(t)} dt + \sqrt{xN} |\theta N|^{-1/2+\epsilon} C^\epsilon.
\]

**Proof.** If $xN \geq N^\epsilon U$, then shift contours to the right to see that the integral is $\ll_{\sigma,A} N^{-A}$ for any $A > 0$. Now, if $|\theta N| \ll C^\epsilon$, the desired bound follows from (2.3) upon setting $\sigma = 0$.

Hence assume that $|\theta N| \geq C^\epsilon$. We restrict our attention to the range $|\theta N|^{1-\epsilon} \leq |t| \leq |\theta N|^{1+\epsilon}$, since otherwise $\tilde{\psi}(s)$ is very small by Lemma [1]. Set $\sigma = -1/2$. Then by Lemma [1] we have that
\[
\tilde{\psi}(x) = \sqrt{2\pi x N} \omega \left(-\frac{t}{\theta}\right) |t|^{-1/2} e^{it \log \left(\frac{|t|}{2|\theta|}\right)} e^{i\frac{1}{2} \text{sgn}(\theta)} + O(|t|^{-3/2}).
\]

The contribution of $O(|t|^{-3/2})$ to the integral $\Phi_k(x)$ is
\[
\ll \sqrt{xN} \int_{|\theta N|^{1-\epsilon} \leq |t| \leq |\theta N|^{1+\epsilon}} |t|^{-3/2} dt \ll \sqrt{xN} |\theta N|^{-1/2+\epsilon}.
\]

We now seek to understand the contribution from the main term, which up to a constant factor is
\[
(2.4) \quad \sqrt{xN} \int_{-\infty}^{\infty} (x\pi^3)^{it} \omega \left(-\frac{t}{\theta}\right) |t|^{-1/2} e^{it \log \left(\frac{|t|}{2|\theta|}\right)} G(1 + \sigma - it) dt.
\]

Stirling’s approximation gives us that
\[
G(1 + \sigma - it) = e^{-it \log |t|/2e} \left(c_0 + \frac{c_1}{|t|} + \ldots + O \left(\frac{1}{|t|^A}\right)\right),
\]
where the $c_i$ are absolute constants. We split the integral in (2.4) into two ranges $R_1$ and $R_2$, where $R_1 = \{ t \in \mathbb{R} : |t + a_i| < C^\epsilon \text{ for some } 1 \leq i \leq 3 \}$ and $R_2$ is the
complement of $R_1$. The contribution of $R_1$ gives $\ll \sqrt{xN|\theta N|^{-1/2}}c^\varepsilon$. For $R_2$, we use Stirling’s approximation for $G$ to get that (2.4) can be rewritten as

$$\sqrt{xN} \int_{R_2} g(t)e^{if(t)}dt,$$

as desired. □

3. Proof of Theorem 2

We will be deriving various bounds for $\Psi_k(x)$ in this section, and it will be convenient to record the contributions these make to $T_k$ below. Note that $x = \frac{n}{q^3}$ and $U^\varepsilon \ll (QN)^\varepsilon$. Since Theorem 2 is trivial otherwise, we also assume that $C^\varepsilon \ll N^\varepsilon$.

\begin{equation}
q^{3/2+\varepsilon} \sum_{xN \leq U^\varepsilon} \frac{|A(n,1)|}{n} \left( c^{1/2} + \sqrt{U}|\theta N|^{-1/2} + |\theta N|^{3/2} \right) 
\ll (QN)^\varepsilon \left( Q^{3/2}c^{1/2} + q^{3/2} \left( c^{1/2} \sqrt{|\theta N| + 1} + |\theta N|^{3/2} \right) |\theta N|^{-1/2} + q^{3/2}|\theta N|^{3/2} \right) 
\ll (QN)^\varepsilon \left( Q^{3/2}c^{1/2} + \left( \frac{N}{Q} \right)^{3/2} \right).
\end{equation}

In particular, we see that the contribution of the terms from parts (1) and (2) of Lemma 3 and from the error term from part (3) of Lemma 3 to $T_k$ is bounded by the above. Let

$$J = \sqrt{xN} \int_{R_2} g(t)e^{if(t)}dt,$$

with notation as in Lemma 3. In order to prove cancellation in this integral, our first step is to record some expressions for $f'(t)$ and $f''(t)$.

Without loss of generality, assume that $|a_1| \geq |a_2| \geq |a_3|$. Note that $a_1 \asymp a_2$, so $c^{1/3} \leq a_1 \leq c^{1/2}$. For future use, let

\begin{equation}
C(t) := \prod_i (t + a_i) = t^3 - (a_1^2 - a_2a_3)t + a_1a_2a_3,
\end{equation}

since $\sum_i a_i = 0$. For $|\theta N| \asymp |t|$, we have that

\begin{equation}
C(t) \ll (|\theta N| + 1)(|\theta N|^2 + c).
\end{equation}

Further, after some calculations,

\begin{equation}
f'(t) = \log \left( \frac{8\pi^3 xN|t|}{|\theta N| \prod_i |t + a_i|} \right) = \log \left( \frac{8\pi^3 xN|t|}{|\theta NC(t)|} \right).
\end{equation}
Using the fact that $\sum_{i} a_i = 0$,
\begin{equation}
(3.5) \quad f''(t) = \frac{1}{t} - \sum_{i} \frac{1}{t + a_i} = \frac{\prod_{i}(t + a_i) - t \left( 3t^2 + \sum_{i<j} a_ia_j \right)}{t \prod_{i}(t + a_i)} = \frac{t^3 + t \left( \sum_{i<j} a_ia_j \right) + \prod_{i} a_i - 3t^3 - t \left( \sum_{i<j} a_ia_j \right)}{t \prod_{i}(t + a_i)} = \frac{\prod_{i} a_i - 2t^3}{t \prod_{i}(t + a_i)} = \frac{\prod_{i} a_i - 2t^3}{tC(t)}.
\end{equation}

We now consider different ranges of $|\theta N|$. We consider the case $|\theta N| \gg \mathcal{C}^{1/2-\epsilon}$ in §3.1, $\mathcal{C}^{1/3} \leq |\theta N| \leq \mathcal{C}^{1/2-\epsilon}$ in §3.2, and $\mathcal{C}^\epsilon < |\theta N| < \mathcal{C}^{1/3}$ in §3.3.

3.1. $|\theta N| \gg \mathcal{C}^{1/2-\epsilon}$. Here, since we may assume that $|t| \asymp |\theta N|$, we have $|t| \gg \mathcal{C}^{1/2-\epsilon}$. In this case, since $t^3 \gg \mathcal{C}^{3/2-\epsilon}$ and $|\prod_{i} a_i| \leq \mathcal{C}$,
\[ f''(t) \asymp \frac{t^2}{C(t)} \gg \frac{t^2}{t^3} \mathcal{C}^{-\epsilon}, \]
by (3.2). Thus $|f''(t)| \gg \frac{1}{|\theta N|^{1/2}}$ and by Lemma 5.1.3 of [6],
\[ \int_{a}^{b} g(t)e^{\mathcal{I}(t)}dt \ll \frac{1}{|\theta N|^{1/2}} \sqrt{|\theta N|^{\mathcal{C}}} \ll |\theta N|^{\epsilon}. \]
Thus the contribution to $T_k$ is bounded by
\[ \ll q^{3/2+\epsilon} \sum_{xN \leq U} \frac{|a(n)|}{n} \sqrt{xN}|\theta N|^{\epsilon} \ll q^{3/2+\epsilon} \sqrt{UN} \ll q^{3/2+\epsilon} N^{\epsilon}|\theta N|^{3/2}, \]
where we have used $|\theta N| \gg \mathcal{C}^{1/2-\epsilon}$ to see that $U \ll |\theta N|^{3\mathcal{C}} \ll |\theta N|^{3N^{\epsilon}}$. The latter is bounded by (3.1).

3.2. $\mathcal{C}^{1/3} \leq |\theta N| \leq \mathcal{C}^{1/2-\epsilon}$. Let
\[ \Delta = xN - \frac{|\theta N| |\prod_{i} (t + a_i)|}{8\pi^3 |t|}. \]
We may write
\[ f'(t) = \log \left( 1 + \frac{8\pi^3 |t| \Delta}{|\theta N| |\prod_{i} (t + a_i)|} \right). \]
Now, if $xN \neq \frac{|\theta N| \prod_{i} (t + a_i)}{|t|}$, we are done, since then $f'(t) \gg 1$, and $J \ll \frac{\sqrt{xN}}{|\theta N|^{1/2-\epsilon}}$ by Lemma 5.1.2 of [6]. Then the contribution to $T_k$ is $\ll q^{3/2+\epsilon} \frac{\sqrt{U}}{|\theta N|^{1/2-\epsilon}}$, which is bounded as in (3.1).

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Thus, assume that
\[ xN \asymp \frac{|\theta N| \prod_i |t + a_i|}{|t|} \asymp |C(t)| \ll |\theta N| \mathcal{C}, \]
for some \( t \asymp |\theta N| \), where we have used (3.3). Note that \( f'(t) \asymp \frac{\Delta}{C(t)} \). We proceed differently according to the size of \( f'(t) \).

3.2.1. \( f'(t) \) is small. Suppose that
\[ \Delta \ll |\theta N|^3, \]
for some \( t \asymp |\theta N| \). In this case \( \prod_i a_i - 2t^3 \asymp t^3 \), since \( t^3 \geq \mathcal{C} \geq \prod_i |a_i| \). Then
\[ f''(t) \asymp \frac{t^2}{C(t)}. \]

Now for \( M \geq 1 \), let \( I_M = C^{-1}([M; 2M]) \cup (-2M, -M]) \)
and
\[ J_M = \sqrt{xN} \int_{I_M} g(t) e^{if(t)} dt. \]

Trivially, \( I_M \) is always a union of 6 intervals or less. From Lemma 3 by the definition of \( R_2 \), \( C(t) \gg \mathcal{C} \) for \( t \in R_2 \), so we may assume that \( M \gg \mathcal{C} \). Fix \( M \), and assume that \( xN \asymp \frac{|\theta N| C(t)}{|t|} \asymp C(t) \times M \) for \( t \in I_M \), since otherwise we have that \( f'(t) \gg 1 \) and \( J_M \ll \frac{1}{|\theta N|} \) by Lemma 5.1.2 of [6] as before. In particular, we need consider only one value of \( M \) in the sequel. In this case \( f''(t) \asymp \frac{|\theta N|^2}{M} \), so
\[ J_M \ll \frac{\sqrt{xN}}{\sqrt{|\theta N|} |\theta N|^{3/2}} = \frac{xN M}{|\theta N|^{3/2}} \]
by Lemma 5.1.3 in [6].

The contribution of this to \( T_k \) is bounded by
\[ S_M \quad := \quad q^{3/2 + \epsilon} \sum_{A \leq n \leq A + B} \frac{|a(n)| \sqrt{xN M}}{n |\theta N|^{3/2}} \ll \frac{M N^\epsilon}{|\theta N|^{3/2}} q^{3/2 + \epsilon} \left( \frac{B}{A} \right)^p, \]
by Lemma 2 where \( p = 1/2 \) unconditionally, and \( p = 1 \) on Ramamuan. Since \( xN \asymp M, A \asymp \frac{q^3 M}{N} \), and \( B \ll \frac{|\theta N|^3 q^3}{N} \). For \( p = 1 \), \( S_M \ll \frac{M N^\epsilon}{|\theta N|^{3/2}} q^{3/2 + \epsilon} |\theta N|^3 |\theta N|^{3/2+\epsilon}, \) which is bounded by the right hand side of (3.1). Unconditionally, when \( p = 1/2, \)
\[ S_M \ll M^{1/2} q^{3/2+\epsilon} \ll N^\epsilon \sqrt{|\theta N|} q^{3/2+\epsilon} \leq Q^\epsilon N^\epsilon \sqrt{|\theta N|}. \]

3.2.2. \( f'(t) \) is large. Here, suppose that \( Y \leq \Delta < 2Y \) for some \( Y \geq |\theta N|^3 \). Then
\[ |f'(t)| \asymp \frac{|t| \Delta}{|\theta N| |C(t)|} \asymp \frac{Y}{|C(t)|}. \]
Again split the integral into $J_M$ as before. Note that $J_M \ll \frac{M}{Y\sqrt{\theta N}}$ by Lemma 5.1.2 of [6]. Then for $A \asymp \frac{q^3}{N} M$ and $B \asymp \frac{q^3}{N} Y$, we have

\[
S_M \ll q^{3/2+\varepsilon} \sum_{A \leq n \leq A+B} \frac{|a(n)|}{n} \sqrt{M} \frac{M}{Y\sqrt{\theta N}}
\ll N^\varepsilon q^{3/2+\varepsilon} \frac{M^{3/2}}{Y\sqrt{\theta N}} \left( \frac{Y}{M} \right)^p,
\]

by Lemma 2. Assuming Ramanujan, we have $p = 1$. Since $M \leq C|\theta N|$, $S_M \ll q^{3/2+\varepsilon} \frac{M^{1/2}}{\sqrt{\theta N}} \ll q^{3/2+\varepsilon} \sqrt{C}$, which is bounded by (3.1).

Unconditionally we have $p = 1/2$. Using that $Y \geq |\theta N|^{3/2}, |\theta N| \geq C^{1/3}$, we have

(3.7) \[
S_M \ll q^{3/2+\varepsilon} N^\varepsilon \frac{M^{3/2}}{Y\sqrt{\theta N}} \left( \frac{Y}{M} \right)^{1/2}
\ll N^\varepsilon q^{3/2} \frac{C}{|\theta N|} \leq N^\varepsilon q^{3/2} C^{2/3}.
\]

3.3. $C^\varepsilon < |\theta N| < C^{1/3}$. If $xN \neq \frac{|\theta N|C(t)}{t} \asymp C(t)$ for $t \asymp |\theta N|$, then we are done as before, since then $f'(t) \gg 1$. Hence assume that $xN \asymp C(t)$ for some $t \asymp |\theta N|$. Define $J_M$ as in the last section. If $M \neq xN$, we are similarly done, so assume that $C(t) \asymp M \asymp xN$. We split this into two cases.

3.3.1. $\Delta \ll \frac{C(t)}{|\theta N|}$. The trivial bound gives $J_M \ll \sqrt{xN}|\theta N|^{1/2+\varepsilon}$, which contributes

\[
\ll q^{3/2+\varepsilon} \sum_{A \leq n \leq A+B} \frac{|a(n)|}{n} \sqrt{M}|\theta N|^{1/2+\varepsilon}
\ll N^\varepsilon q^{3/2+\varepsilon} \sqrt{M}|\theta N|^{1/2+\varepsilon} \left( \frac{B}{A} \right)^p,
\]

by Lemma 2 where $A \asymp C(t) \frac{q^3}{N}$ and $B \ll \frac{C(t)}{\sqrt{|\theta N|}} \frac{q^3}{N}$. Say that $p = 1$. Using $M \ll |\theta N|C$, this leads to $S \ll N^\varepsilon C^{1/2} q^{3/2+\varepsilon} |\theta N|^{\varepsilon}$, which is bounded by (3.1).

In the unconditional case, $p = 1/2$, so we have

(3.8) \[
q^{3/2+\varepsilon} \sqrt{M} \ll N^\varepsilon Q^\varepsilon \sqrt{NqC}.
\]

3.3.2. $\Delta \gg \frac{C(t)}{|\theta N|}$. Here we again split the range for $\Delta$ into dyadic intervals. Let $Y < \Delta \leq 2Y$. We have that $f'(t) \asymp \frac{\Delta}{C(t)} \asymp \frac{Y}{M}$. Thus

\[
J_M \ll \frac{1}{\sqrt{\theta N}} \frac{M}{Y}.
\]

Then

\[
S_M \ll q^{3/2+\varepsilon} \sum_{A \leq n \leq A+B} \frac{|a(n)|}{n} \frac{\sqrt{M}}{\sqrt{\theta N}} \frac{M}{Y} \ll N^\varepsilon q^{3/2+\varepsilon} \frac{\sqrt{M}}{\sqrt{\theta N}} \frac{M}{Y} \left( \frac{B}{A} \right)^p,
\]
where $A \asymp \frac{q^3}{N^2} M$ and $B \asymp \frac{q^3}{N^2} Y$. Thus for $p = 1$,

$$S_M \ll N^\epsilon q^{3/2+\epsilon} \frac{1}{\sqrt{\theta N}} \sqrt{M} \ll \frac{N^\epsilon}{\sqrt{\theta N}} q^{3/2+\epsilon} \sqrt{C|\theta N|} \ll N^\epsilon \sqrt{C} Q^{3/2+\epsilon},$$

which is bounded by (3.1).

For $p = 1/2$, we get

$$S_M \ll N^\epsilon q^{3/2+\epsilon} \sqrt{M} \ll \frac{Q}{\sqrt{\theta N}} \left( \frac{M}{Y} \right)^{1/2}.$$ 

Since $Y \gg \frac{M}{\sqrt{\theta N}}$, this leads to

$$(3.9) \quad S_M \ll q^{3/2+\epsilon} \sqrt{M} \ll Q^\epsilon \sqrt{NqC}. \tag{3.9}$$

3.4. Conclusion. From (3.1) and the sections above, we have that on Ramanujan,

$$T_k \ll (QN)^\epsilon \left( Q^{3/2} C \epsilon^{1/2} + NQ^{-1/2} + \left( \frac{N}{Q} \right)^{3/2} \right) \ll N^{3/4+\epsilon} C^{1/4},$$

upon setting $Q = \frac{N^{1/2}}{C^{1/2}}$.

By (3.1), (3.6), (3.7), (3.8) and (3.9), we have that the unconditional bound has two extra terms so that for $Q = \frac{N^{1/2}}{C^{1/2}}$,

$$T_k \ll N^\epsilon Q^\epsilon \sqrt{NqC} + N^\epsilon Q^{3/2} C^{2/3} + N^{3/4+\epsilon} C^{1/4} \ll N^{3/4+\epsilon} C^{5/12}.$$ 

ACKNOWLEDGEMENTS

The author would like to thank Xiaoqing Li for commenting on a preprint of this paper. He would also like to express his gratitude to the referee for useful editorial comments.

REFERENCES

[1] Farrell Brumley, Second order average estimates on local data of cusp forms, Arch. Math. (Basel) 87 (2006), no. 1, 19–32, DOI 10.1007/s00013-005-1632-3. MR2246403 [2007b:11060]

[2] Harold Davenport, Multiplicative number theory, 3rd ed., revised and with a preface by Hugh L. Montgomery, Graduate Texts in Mathematics, vol. 74, Springer-Verlag, New York, 2000. MR1790423 [2001f:11001]

[3] Dorian Goldfeld, Automorphic forms and L-functions for the group $GL(n, \mathbb{R})$, with an appendix by Kevin A. Broughan, Cambridge Studies in Advanced Mathematics, vol. 99, Cambridge University Press, Cambridge, 2006. MR2254662 [2008d:11046]

[4] Dorian Goldfeld and Xiaoqing Li, Voronoi formulas on $GL(n, \mathbb{R})$, with an appendix by Kevin A. Broughan, Cambridge Studies in Advanced Mathematics, vol. 99, Cambridge University Press, Cambridge, 2006. MR2254662 [2008d:11046]

[5] G. H. Hardy and J. E. Littlewood, Some problems of diophantine approximation, Acta Math. 37 (1914), no. 1, 193–239, DOI 10.1007/BF02401834. MR1555099

[6] M. N. Huxley, Area, lattice points, and exponential sums, London Mathematical Society Monographs. New Series, vol. 13, The Clarendon Press, Oxford University Press, New York, 1996. MR1420620 [97g:11088]

[7] Henryk Iwaniec, Topics in classical automorphic forms, Graduate Studies in Mathematics, vol. 17, American Mathematical Society, Providence, RI, 1997. MR1474964 [98c:11051]

[8] Henryk Iwaniec and Emmanuel Kowalski, Analytic number theory, American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004. MR2061214 [2005h:11005]

[9] Xiannan Li, Upper bounds on L-functions at the edge of the critical strip, Int. Math. Res. Not. IMRN 4 (2010), 727–755, DOI 10.1093/imrn/rnp148. MR2595006 [2011a:11160]
[10] Xiaoqing Li and Matthew P. Young, *Additive twists of Fourier coefficients of symmetric-square lifts*, J. Number Theory 132 (2012), no. 7, 1626–1640, DOI 10.1016/j.jnt.2011.12.017. MR2903173

[11] Stephen D. Miller, *Cancellation in additively twisted sums on GL(n)*, Amer. J. Math. 128 (2006), no. 3, 699–729. MR2230922 (2007k:11078)

[12] Stephen D. Miller and Wilfried Schmid, *Automorphic distributions, L-functions, and Voronoi summation for GL(3)*, Ann. of Math. (2) 164 (2006), no. 2, 423–488, DOI 10.4007/annals.2006.164.423. MR2247965 (2007j:11065)

[13] H. L. Montgomery and R. C. Vaughan, *Exponential sums with multiplicative coefficients*, Invent. Math. 43 (1977), no. 1, 69–82. MR0457371 (56 #15579)

[14] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed., edited and with a preface by D. R. Heath-Brown, The Clarendon Press, Oxford University Press, New York, 1986. MR882550 (88c:11049)

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, Illinois 61801

Current address: Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX2 6GG, United Kingdom