Domain wall partition functions and KP

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Abstract. We observe that the partition function of the six-vertex model on a finite square lattice with domain wall boundary conditions is (a restriction of) a KP $\tau$ function and express it as an expectation value of charged free fermions (up to an overall normalization).

Keywords: classical integrability, integrable spin chains (vertex models), quantum integrability (Bethe ansatz), solvable lattice models

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To Professor T Miwa on his 60th birthday.
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Introduction

In [1], Korepin introduced domain wall boundary conditions for the six-vertex model on an \( N \times N \) square lattice, and obtained a set of conditions that determine the corresponding partition function \( Z_N \). In [2], Izergin proposed a determinant expression for \( Z_N \), and showed that it satisfies Korepin’s conditions. In [3], Lascoux rewrote \( Z_N \) as a determinant of a product of two non-square matrices. In this paper we observe that, given Lascoux’s form, \( Z_N \) is (a restriction of) a KP \( \tau \) function (for all values of the crossing parameter) and rewrite it as an expectation value of charged free fermions.

We believe that the connection between \( Z_N \) and KP is interesting for two reasons. Firstly, it connects a widely studied quantum integrable model, namely the \( XXZ \) spin-\( 1/2 \) Heisenberg chain, to an equally widely studied classical integrable hierarchy, namely KP. Secondly, and more concretely, \( Z_N \) is a building block and a starting point in the algebraic Bethe ansatz approach to the correlation functions of the \( XXZ \) spin-\( 1/2 \) Heisenberg chain [4]. The fact that \( Z_N \) is a KP \( \tau \) function for all values of the crossing parameter gives us reason to expect that KP will play a role in further studies of the correlation functions in this spin chain model.

Conceptually speaking, this is not entirely a new idea, as the nonlinear Schrödinger equation plays a crucial role in computing the asymptotics of correlation functions in the Bose gas model and of the \( XXZ \) spin-\( 1/2 \) Heisenberg chain at the free fermion point [4]. Our result points to KP as the corresponding hierarchy at general values of the crossing parameter.

In section 1, we recall basic definitions related to the six-vertex model and domain wall boundary conditions, including Korepin’s conditions, and in section 2, we recall Izergin’s determinant expression for \( Z_N \), followed by Lascoux’s expression, and observe that the latter is (a restriction of) a KP \( \tau \) function. In section 3, we propose an expectation value of KP charged free fermions \( F_{N}^{\text{free}} \) that by construction is a KP \( \tau \) function, and show that (under suitable restrictions of the KP time variables) \( F_{N}^{\text{free}} \) becomes equal to \( Z_N \) (up to an overall normalization). In section 4, we include a number of remarks.
1.1. Oriented lattice lines and rapidity variables

Consider a square lattice with $N$ horizontal lines (rows) and $N$ vertical lines (columns) that intersect at $N^2$ points, and assign the $i$th horizontal line an orientation from left to right and a rapidity $x_i$, and the $j$th vertical line an orientation from bottom to top and a rapidity $y_j$, as in figure 1.

1.2. Arrows, weights and a crossing parameter

Assign each line segment an arrow that can point in either direction, and define the vertex $v_{ij}$ as the intersection point of the $i$th horizontal line and the $j$th vertical line, the four line segments attached to this intersection point and the arrows on these segments. Assign $v_{ij}$ a weight $w_{ij}$ that depends on the orientations of its arrows, the rapidities $x_i$ and $y_j$ and a crossing parameter $\mu$ that is the same for all vertices.

1.3. The six-vertex model

Since an arrow on a line segment can point in either direction, there are $2^4 = 16$ possible (types of) vertices. In the six-vertex model, the weights of all vertices, except six of them, are zero [5]. The six vertices with non-zero weights form three pairs of equal-weight vertices. They are shown in figure 2.

In the notation of figure 2, the weights of the six-vertex model are

$$w_{ij}^a = \sinh(-x_i + y_j + \mu) \quad w_{ij}^b = \sinh(-x_i + y_j) \quad w_{ij}^c = \sinh \mu$$

which satisfy the Yang–Baxter equations [5].

Figure 1. A square lattice with oriented lines and rapidity variables. The line orientations are indicated by white arrows.
1.4. Domain wall boundary conditions

When all arrows on the left and right boundaries point inwards, and all arrows on the upper and lower boundaries point outwards, as in figure 3, we obtain domain wall boundary conditions. The inner arrows remain free. Different inner arrow orientations lead to different lattice configurations that together constitute the statistical states of the model.
1.5. Domain wall partition functions

The weight of a lattice configuration is the product of the weights \(w_{ij}\) of its vertices \(v_{ij}\). The weighted sum over all lattice configurations, on an \(N \times N\) lattice with fixed domain wall boundary conditions, is a domain wall partition function \(Z_N\):

\[
Z_N \equiv Z_N(x, y, \mu) := \sum_{\text{all allowed configurations}} \prod_{\text{all vertices}} w_{ij}(x_i, y_j, \mu).
\]  

1.6. Korepin’s conditions

In [1], Korepin obtained four conditions that uniquely determine \(Z_N(x, y, \mu)\). These are:

1. \(Z_N\) is a symmetric function in the \(\{x\}\) and in the \(\{y\}\) rapidity variables.
2. \(Z_N\) is a trigonometric polynomial of degree \((N - 1)\) in any rapidity variable.
3. Setting \(x_1 = y_1 + \mu\), \(Z_N\) satisfies the recursion relation

\[
Z_N|_{x_1=y_1+\mu} = \left( \prod_{i=2}^{N} \sinh(-x_i + y_1) \right) \sinh \mu \left( \prod_{j=2}^{N} \sinh(-x_1 + y_j) \right) Z_{(N-1)}(\hat{x}_1, \hat{y}_1, \mu)
\]

where \(\hat{x}_1, \hat{y}_1\) indicate that \(x_1\) and \(y_1\) are missing from the sets \(\{x\}\) and \(\{y\}\).
4. \(Z_1 = \sinh \mu\).

We refer the reader to [1,4] for details and proofs.

2. \(Z_N\) is (a restriction of) a KP \(\tau\) function

2.1. Izergin’s determinant expression

Working in terms of the variables \(u_i = e^{2x_i}\), \(v_i = e^{2y_i}\), and \(q = e^{-2\mu}\), Izergin’s determinant expression is

\[
Z_N(u, v, q) = c_N \frac{\prod_{i,j=1}^{N} (u_i - v_j)(qu_i - v_j)}{\prod_{1 \leq i < j \leq N} (u_i - u_j)(v_j - v_i)} \det \left[ \frac{1}{(u_i - v_j)(qu_i - v_j)} \right]_{i,j=1}^{N}
\]  

where

\[
c_N = (q - 1)^N \prod_{i=1}^{N} u_i^{1/2} v_i^{1/2}.
\]

The virtue of Izergin’s expression is that it is straightforward to show that it satisfies Korepin’s conditions [2,4].

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2.2. Lascoux’s determinant expression

In [3], Lascoux showed that Izergin’s expression for $Z_N$ can be rewritten as

$$Z_N(u, v, q) = c_N \det \left[ \sum_{j=1}^{2N-1} \alpha_{i,j}(u) \beta_{j,k}(v,q) \right]_{i,k=1}^N$$  \hspace{1cm} (5)

where

$$\alpha_{i,j}(u) = e_{-i+j}(u), \quad \beta_{j,k}(v,q) = \frac{q^{j-k+1} - q^{k-1}}{q - 1} e_{j+k+N-1}(-v)$$  \hspace{1cm} (6)

and $h_n(u)$ and $e_n(v)$ are the $n$th homogeneous and the $n$th elementary symmetric functions, respectively [6]. For a proof that Lascoux’s expression is equivalent to Izergin’s, see [3]. As we will see shortly, the virtue of Lascoux’s expression is that it is straightforward to show that it is (a restriction of) a KP $\tau$ function.

2.3. Remarks on notation

In $\alpha_{i,j}$, $\beta_{i,j}$, and other matrix elements, $i$ is the row and $j$ is the column index. We will omit writing the dependence on the parameter $q$ explicitly, as it is always there.

2.4. Partitions

We denote the partition of an integer $|\lambda|$ into parts $\lambda_1 \geq \lambda_2 \geq \cdots$, by $[\lambda] = [\lambda_1, \lambda_2, \ldots, \lambda_L]$, where $L$ is the number of non-zero parts. $m^n$ in $[\ldots, m^n, \ldots]$ indicates that $[\lambda]$ has $n$ parts of length $m$. Further, $m^0$ indicates that there are no parts of length $m$ and $0^n$ stands for $n$ parts of length 0.

2.5. Young diagrams

Consider the Young diagram $\lambda$ of a partition $[\lambda]$ as in figure 4. The cells are labelled from top to bottom and from left to right. The cell with coordinates $(i, j)$, $i, j \geq 1$, is $c_{ij}$. The number of cells $c_{11}, c_{22}, \ldots$ on the main diagonal is $d$.

2.6. Hooks

Given a cell $c_{ij}$, we define a hook $h_{ij}$ as the union of $c_{ij}$, the set of all cells below it and all cells to the right of it. We say that the hook $h_{ij}$ has a corner at $c_{ij}$. The diagram on the left-hand side of figure 5 has six hooks, one for each cell. The right-hand side of figure 5 is the hook corresponding to the cell $c_{(2,1)}$.

2.7. Horizontal $a$-parts and vertical $b$-parts

We decompose a hook $h_{ij}$ into a horizontal $a$-part $a_{ij}$ of length $|a_{ij}|$ that consists of all cells to the right of $c_{ij}$, and a vertical $b$-part $b_{ij}$ of length $|b_{ij}|$ that consists of $c_{ij}$ and all cells below it. Hence the allowed $a$-part lengths are $0, 1, 2, \ldots$, while the allowed $b$-part lengths are $1, 2, \ldots$. The length of $h_{ij}$ will be indicated by $|h_{ij}|$ as well, so that

$$|h_{ij}| = |a_{ij}| + |b_{ij}|.$$  \hspace{1cm} (7)
2.8. Frobenius coordinates of $\lambda$

Decomposing each hook $h_{ii}$, $i = 1, \ldots, d$, that has a corner at a cell on the main diagonal of $\lambda$ into the corner cell $c_{ii}$, the set of $m_i$ cells to the right of $c_{ii}$, and $n_i$ cells below it, we obtain $2d$ integers $\{m_i, n_j\}$ which define the Frobenius coordinates of $\lambda$.

2.9. Expanding $Z_N$ in terms of Schur functions

To make contact with KP theory and eventually obtain an expression for $Z_N$ as a vacuum expectation value of free fermions, we start by expanding Lascoux’s expression in terms of Schur functions. Using (6) and the Cauchy–Binet identity in (5), we obtain

$$Z_N(u, v) = c_N \sum_{1 \leq j_1 < \cdots < j_N \leq 2N - 1} \det[h_{-i+j_1}(u)]_{i,l=1}^{N} \det[\beta_{j_1,k}(v)]_{l,k=1}^{N}.$$  \hspace{1cm} (8)

Rewriting $j_l$ as $j_l = \lambda_{(N+1)-l} + l$, then reversing the order of all rows and all columns in the first matrix and transposing it, the sum on the right-hand side of (8) becomes

$$\sum_{0 \leq \lambda_N \leq \cdots \leq \lambda_1 \leq N - 1} \det[h_{\lambda_{N}+l}(u)]_{i,l=1}^{N} \det[\beta_{\lambda_{(N+1)-l}+l,k}(v)]_{l,k=1}^{N}.$$  \hspace{1cm} (9)
Finally, using the definition of Schur functions
\[ S_\lambda(u) = \det[h_{\lambda_i+i}(u)]_{i,j=1}^N \]  
and (6) to write
\[ c_\lambda^{(N)}(v) = \det[\beta(\lambda_{\text{conj}}) - l_{\text{conj}} + (N+1)k(v)]_{l,k=1}^N \]
\[ = \det \left[ q^{l_{\text{conj}} - k + N + 2} - q^{k-1} - c^{-(\lambda_{\text{conj}}) + l_{\text{conj}} + k - 2}(-v) \right]_{l,k=1}^N \]
where
\[ l_{\text{conj}} = (N + 1) - l \]
we end up with
\[ Z_N(u, v) = c_N \sum_{\lambda \subseteq [(N-1)N]} c_\lambda^{(N)}(v)S_\lambda(u). \]

Since the summation in (13) is over partitions of at most \( N \) parts, and there are \( N \) independent \( u \)-variables, and \( N \) independent \( v \)-variables, there are no terms in the expansion that trivially vanish due to lack of sufficiently many independent variables.

2.10. Expanding \( Z_N \) in terms of character polynomials

Next, following [7], we introduce the power sum variables
\[ t_n = \frac{1}{n} \sum_{i=1}^N u_i^n, \quad n = 1, 2, \ldots \]
and the polynomials \( p_n(t) \):
\[ p_n(t) = \sum_{n_1 + 2n_2 + 3n_3 + \ldots = n} \frac{t_1^{n_1} t_2^{n_2} t_3^{n_3} \cdots}{n_1! n_2! n_3! \cdots} \]
where \( p_i(t) = 0 \) for \( i < 0 \). Next, we define the polynomials \( h_{m,n}(t) \):
\[ h_{m,n}(t) := (-1)^n \sum_{k=0}^N p_{k+m+1}(t)p_{n-k}(-t) \]
and finally the character polynomials \( \chi_\lambda(t) \):
\[ \chi_\lambda(t) := \det[h_{m_i,n_j}(t)]_{i,j=1}^d \]
where \( \{m_i, n_j\} \) are the Frobenius coordinates of \( \lambda \), and the upper limit \( d \) is the number of cells on the main diagonal. In terms of character polynomials and the \( t \)-variables of (14) (and by slightly abusing notation), we rewrite \( Z_N(u, v) \) as
\[ Z_N(t, v) = c_N \sum_{\lambda \subseteq [(N-1)^N]} c_\lambda^{(N)}(v)\chi_\lambda(t) \]
where \( c_\lambda^{(N)}(v) \) was defined in (11). We conclude that \( Z_N \) can be expanded in terms of character polynomials, with coefficients that are minors of a determinant such that (by construction) they satisfy Plücker relations [7]. If the \( t \)-variables were all independent, then \( Z_N(t, v) \) would be identically a KP \( \tau \) function. We refer the reader to [7] for details and proofs.

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2.11. Restricted t-variables

$Z_N(u, v)$ is a function of $N$ independent $u$-variables. On the other hand, from the definition of the character polynomials, it easy to show that $Z_N(t, v)$ requires in general more than $N$ $t$-variables. Since the latter are power sums of the $N$ $u$-variables, they cannot be all independent. Let us write $Z_N^{\text{rest}}(t, v)$ to emphasize the fact that the $t$-variables are restricted to be power sums of a (generally) smaller number of $u$-variables. This restriction spoils the interpretation of the $t$-variables as KP time variables, and of $Z_N^{\text{rest}}(t, v)$ as a KP $\tau$ function.

2.12. Free t-variables

We can formally consider $Z_N^{\text{rest}}(t, v)$ as our starting point, take the $t$-variables to be independent, write it as $Z_N^{\text{free}}(t, v)$ and think of $Z_N^{\text{rest}}(t, v)$ as a restriction of $Z_N^{\text{free}}(t, v)$ obtained by setting the $t$-variables to power sums of the $Nu$-variables using (14). Viewed this way, $Z_N^{\text{free}}(t, v)$ is a KP $\tau$ function, and $Z_N^{\text{rest}}(t, v) = Z_N(u, v)$, is a restriction of a KP $\tau$ function.

3. Fermionic expression for $Z_N$

Given that any KP $\tau$ function can written as the vacuum expectation value of exponentials in neutral bilinears in KP charged fermions\(^1\) [7], we wish to write $Z_N^{\text{free}}$ as such. Our plan is to propose a vacuum expectation value $F_N^{\text{free}}(t, v)$, where we use the superscript $\text{free}$ to emphasize that the $t$-variables are independent, that by construction is a KP $\tau$ function, then show that $F_N^{\text{free}}(t, v) = Z_N^{\text{free}}(t, v)$. Hence restricting the $t$-variables in $F_N^{\text{free}}(t, v)$ to be powers sums of $Nu$-variables, one recovers $Z_N^{\text{rest}}(t, v)$, and $Z_N(u, v)$, as an expectation value of charged free fermions.

3.1. Charged free fermions

Consider the free fermion operators $\{\psi_n, \psi_n^\ast\}$, $n \in \mathbb{Z}$, with charges $\{+1, -1\}$ and energies $n$. They generate a Clifford algebra over $\mathbb{C}$ defined by the anticommutation relations

\[
\begin{align*}
[\psi_m, \psi_n]_+ &= 0, \\
[\psi_m^\ast, \psi_n^\ast]_+ &= 0, \\
[\psi_m, \psi_n^\ast]_+ &= \delta_{m,n}
\end{align*}
\]

\(\forall m, n \in \mathbb{Z}.\) (19)

3.2. Vacuum states

We define the vacuum states $\langle 0 |$ and $| 0 \rangle$ by the actions

\[
\begin{align*}
\langle 0 | \psi_n &= \psi_m | 0 \rangle = 0, \\
| 0 \rangle \psi_m^\ast &= \psi_n^\ast | 0 \rangle = 0,
\end{align*}
\]

\(\forall m < 0, \quad n \geq 0\) (20)

and we adopt the inner product normalization

\[
\langle 0 | 0 \rangle = 1.
\]

\(^1\) By this, we mean linear combinations of terms of the form $\psi_m \psi_n^\ast$. 

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3.3. Creation and annihilation operators

We refer to the operators that annihilate the vacuum state \(|0\rangle\), \(\psi'_m|0\rangle = 0\), \(m = -1, -2, \ldots\), and \(\psi'_n|0\rangle = 0\), \(n = 0, 1, 2, \ldots\), as annihilation operators, and to the rest, \(\psi'_m|0\rangle \neq 0\), \(m = 0, 1, 2, \ldots\), and \(\psi'_n|0\rangle \neq 0\), \(n = -1, -2, \ldots\), as creation operators.

3.4. Normal ordering

The normal-ordered product is defined, as usual, by placing annihilation operators to the right of creation operators:

\[ :\psi_i\psi_j^* := \psi_i\psi_j^* - \langle 0|\psi_i\psi_j^*|0\rangle. \]

3.5. The Heisenberg algebra

Consider the operators \(H_m\),

\[ H_m := \sum_{j \in \mathbb{Z}} \psi_j\psi_{j+m}^*, \quad m \in \mathbb{Z} \]  

(22)

that together with the central element 1 form a Heisenberg algebra

\[ [H_m, H_n] = m\delta_{m+n,0}, \quad \forall m, n \in \mathbb{Z}. \]  

(23)

3.6. The Hamiltonian

Using \(H_m\), \(m = 1, 2, 3, \ldots\), we define the Hamiltonian

\[ H(t) := \sum_{m=1}^{\infty} t_m H_m. \]  

(24)

3.7. Fermion bilinears

Given a partition \(\lambda\), we can associate with each hook \(h_{ij}\) a bilinear \(\psi_{a_{ij}}^*\psi_{-b_{ij}}^*\) with net charge zero. Both fermions in the bilinear are creation operators

\[ \psi_{a_{ij}}^*\psi_{-b_{ij}}^* = -\psi_{-b_{ij}}^*\psi_{a_{ij}}\]  

(25)

and normal ordering is unnecessary.

3.8. Boson–fermion correspondence

It is possible to realize fermionic Fock space expressions as elements in the polynomial ring \(\mathbb{C}[t_1, t_2, \ldots] \) [7]. For the purposes of this work, all we need is that fact that the character polynomial \(\chi_\lambda(t)\) can be generated as follows:

\[ \langle 0|e^{H(t)}\psi_{-b_1}^* \cdots \psi_{-b_d}^* \psi_{a_1} \cdots \psi_{a_1}^*|0\rangle = (-1)^{b_1 + \cdots + b_d} \chi_\lambda(t) \]  

(26)

where \(a_d < \cdots < a_1\) and \(b_d < \cdots < b_1\) are the \(a\)-part and \(b\)-part lengths of the hooks associated with cells \(c_{11}, c_{22}, \ldots\) on the main diagonal of \(\lambda\), and \(d\) is the number of these cells. In terms of Frobenius coordinates, \(a_i = m_i\) and \(b_j = n_j + 1\).
3.9. Partitions and fermion monomials

Consider the fermion monomial consisting of \( d \psi \) and \( d \psi^* \) fermions, \( \psi_{-b_1}^* \cdots \psi_{-b_d}^* \psi_{a_d} \cdots \psi_{a_1} \), where \( a_d < \cdots < a_1 \) and \( b_d < \cdots < b_1 \) as on the left-hand side of (26). The fermions in the monomial are ordered according to length and correspond to the \( a \)-parts and \( b \)-parts of the partition \( \lambda \) on the right-hand side of (26), with the longer \( b \)-parts to the left and the longer \( a \)-parts to the right. We refer to such a monomial associated with a partition \( \lambda \) as a partition-ordered, or \( \lambda \)-ordered monomial. The partition in figure 4 has \( d = 3 \), as a set of \( b \)-parts \( \{4, 3, 1\} \) and a set of \( a \)-parts \( \{3, 1, 0\} \). The corresponding monomial is \( \psi_{-4}^* \psi_{-3}^* \psi_{-1}^* \psi_{0} \psi_{1} \psi_{3} \).

3.10. A fermion expectation value

Consider the vacuum expectation value

\[
F_{N}^{\text{free}}(t, v) = \langle 0 | e^{H(t)} e^{X_{a}^{(N)}} e^{X_{b}^{(N)}} \cdots e^{X_{N-2}^{(N)}} | 0 \rangle
\]  

(27)

where

\[
X_{a}^{(N)} = \sum_{b=1}^{N} (-1)^{b} d_{[a+1, b-1]}^{(N)} \psi_{-b}^* \psi_{a}, \quad a = 0, 1, \ldots, N - 2
\]

(28)

where \( 1^0 \) inside a partition [\( \lambda \)] stands for a trivial, length-0 part, and

\[
d_{\lambda}^{(N)} = c_{\lambda}^{(N)}/c_{\phi}^{(N)}.
\]

The coefficient \( c_{\lambda}^{(N)} \) was defined in (11), and the (infinitely many) \( t \)-parameters in (27) are taken as independent variables rather than power sums of the (finitely many) \( u \)-parameters. Since \( F_{N}^{\text{free}} \) is an expectation value of exponentials of neutral bilinears in \( \{\psi_n, \psi_m^*\} \), which are generators of \( gl(\infty) \), followed by the time evolution operator \( e^{H(t)} \), it is by construction a KP \( \tau \) function [7]. Since all fermions in the linear combinations of bilinears in (28) are creation operators, there is no need for normal ordering these terms.

3.11. A combinatorial interpretation

The exponentials that appear in \( F_{N}^{\text{free}} \) in (27) have the following interpretation. Consider a rectangular partition of \( N \) rows and \( (N - 1) \) columns. For example, for \( N = 3 \), see the diagram on the left-hand side of figure 5. For the \( j \)th column, with \( N \) cells of \( a \)-part length \( a = (N - 1) - j \), we have an exponential \( X_{a}^{(N)} \), \( 0 \leq a \leq (N - 2) \), which is a sum of \( N \) terms. The \( b \)th term is of the form \( (-1)^{b} d_{[h]}^{(N)} \psi_{-b}^* \psi_{a} \), where \( [h] \) is the partition corresponding to the hook that has a corner at that cell, and \( a \) and \( b \) are the \( a \)-part and \( b \)-part lengths of that hook.

3.12. \( F_{N} \) is proportional to \( Z_{N} \)

In the rest of this work, we show that

\[
c_{N} c_{\phi}^{(N)}(v) F_{N}^{\text{free}}(t, v) = Z_{N}^{\text{free}}(t, v).
\]

(30)

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3.13. The normalization factor

Expanding the exponentials in (27) and using $\langle 0 | 0 \rangle = 1$, the character polynomial expansion of $F_N^{\text{free}}$ starts with 1. On the other hand, since $\chi_0(t) = 1$, the character polynomial expansion of $Z_N$ in (18) starts with the $t$-independent term, $c_N c_0^{(N)}(v)$. We choose to keep the definition of the vertex weights as in (1), and show that $F_N$ and $Z_N$ are equal up to a factor of $c_N c_0^{(N)}(v)$.

3.14. Example: $N = 3$

Before we prove (30) for general $N$, we verify it in the simple but non-trivial case of $N = 3$. Omitting the superscripts (3) to simplify the notation, we have

\[
\begin{align*}
\lambda &= \det \left[ \frac{q^{\lambda_{ii}+i-j+1} - q^{i-j}}{q - 1} e^{-\lambda_{ii}-(i+j)} - i+j+2(-v) \right]_{i,j=1}^3 \\
X_0 &= -d_{[1,0,0]}\psi^*_{-1}\psi_0 + d_{[1,1,0]}\psi^*_{-2}\psi_0 - d_{[1,1,1]}\psi^*_{-3}\psi_0 \\
X_1 &= -d_{[2,0,0]}\psi^*_{-1}\psi_1 + d_{[2,1,0]}\psi^*_{-2}\psi_1 - d_{[2,1,1]}\psi^*_{-3}\psi_1
\end{align*}
\]

where $d_\lambda$ was defined in (29). Notice that in this case, the relevant partition is shown on the left-hand side of figure 5, and the bilinears in $X_0$ and $X_1$ correspond to the hooks of the cells in the two columns. This leads to

\[
e^{X_0}e^{X_1} = 1 - d_{[1,0,0]}\psi^*_{-1}\psi_0 + d_{[1,1,0]}\psi^*_{-2}\psi_0 - d_{[1,1,1]}\psi^*_{-3}\psi_0 \\
- d_{[2,0,0]}\psi^*_{-1}\psi_1 + d_{[2,1,0]}\psi^*_{-2}\psi_1 - d_{[2,1,1]}\psi^*_{-3}\psi_1 \\
+ (d_{[1,1,0]}d_{[2,0,0]} - d_{[1,0,0]}d_{[2,1,0]})\psi^*_{-2}\psi^*_{-1}\psi_0\psi_1 \\
+ (d_{[1,1,0]}d_{[2,1,0]} - d_{[1,0,0]}d_{[2,1,1]})\psi^*_{-3}\psi^*_{-2}\psi_0\psi_1.
\]

Consider the six fermion bilinears on the right-hand side of (32). Using the boson–fermion correspondence, each of these bilinears corresponds to a partition $[\lambda]$ (which in the case of bilinears is a single hook), and its coefficient $d_{[\lambda]}$ corresponds to the same partition $[\lambda]$. If the same is true for the three fermion quartic terms, then from (18), $F_3$ would be proportional to $Z_3$. To put the coefficients of the quartic terms in the right form, we need to show that

\[
\begin{align*}
d_{[1,1,0]}d_{[2,0,0]} - d_{[1,0,0]}d_{[2,1,0]} &= -d_{[2,2,0]} \\
 d_{[1,0,0]}d_{[2,1,1]} - d_{[1,1,1]}d_{[2,0,0]} &= d_{[2,2,1]} \\
 d_{[1,1,1]}d_{[2,1,0]} - d_{[1,0,0]}d_{[2,1,1]} &= -d_{[2,2,2]}
\end{align*}
\]

3.15. Plücker relations

Equations (33)–(35) are examples of Plücker relations, which are identities involving bilinears in $n \times n$ minors of a $2n \times 2n$ matrix with zero determinant. In this case, $n = 3$. doi:10.1088/1742-5468/2009/03/P03017
For details, see [8]. Using the notation
\[ c_\lambda \equiv c_{[\lambda_1,\lambda_2,\lambda_3]} = \det[\gamma_{\lambda_3+1}, \gamma_{\lambda_2+2}, \gamma_{\lambda_1+3}] = |\gamma_{\lambda_3+1}, \gamma_{\lambda_2+2}, \gamma_{\lambda_1+3}| \] (36)
where \( \gamma_b \) is the three-component column vector
\[ \gamma_b = \left( \frac{q^{b-a+1} - q^{a-1}}{q-1} e^{-b+a+2}(-v) \right)_{a=1,2,3} \] (37)
we consider the 6 \times 6 determinant
\[ \begin{vmatrix} \gamma_{\mu_1} & \gamma_{\mu_2} & \gamma_{\nu_1} & \gamma_{\nu_2} & \gamma_{\nu_3} & \gamma_{\nu_4} \\ 0 & 0 & \gamma_{\nu_1} & \gamma_{\nu_2} & \gamma_{\nu_3} & \gamma_{\nu_4} \end{vmatrix} = 0 \] (38)
and Laplace expand it as the sum of bilinears in 3 \times 3 determinants
\[ |\gamma_{\mu_1}, \gamma_{\mu_2}, \gamma_{\nu_1}| |\gamma_{\nu_2}, \gamma_{\nu_3}, \gamma_{\nu_4}| - |\gamma_{\mu_1}, \gamma_{\mu_2}, \gamma_{\nu_2}| |\gamma_{\nu_1}, \gamma_{\nu_3}, \gamma_{\nu_4}| + |\gamma_{\mu_1}, \gamma_{\mu_2}, \gamma_{\nu_3}| |\gamma_{\nu_1}, \gamma_{\nu_2}, \gamma_{\nu_4}| - |\gamma_{\mu_1}, \gamma_{\mu_2}, \gamma_{\nu_4}| |\gamma_{\nu_1}, \gamma_{\nu_2}, \gamma_{\nu_3}| = 0. \] (39)

We can use (39) to generate the Plücker relations that we need by suitably choosing the column vectors \( \gamma_b \).

### 3.16. The \( N = 3 \) Plücker relations

The required Plücker relation, defined as in (41), has a term \( d_{[\lambda]}d_{[\phi]} \), where \( d_{[\phi]} = 1 \), and \( \lambda = [2,2,0] \). From (36) we see that the bilinear that we need is
\[ |\gamma_{\lambda_3+1}, \gamma_{\lambda_2+2}, \gamma_{\lambda_1+3}| = |\gamma_{\lambda_3+1}, \gamma_{\lambda_2+2}, \gamma_{\lambda_1+3}|. \] This dictates the choice of parameters
\[ (\mu_1, \mu_2, \nu_1, \nu_2, \nu_3, \nu_4) = (1, 4, 5, 1, 2, 3). \] (40)

Using (40) in (39), and the antisymmetry property of determinants, we obtain
\[ |\gamma_{1,4}, \gamma_{5}| |\gamma_{1}, \gamma_{2}, \gamma_{3}| - |\gamma_{1}, \gamma_{3}, \gamma_{5}| |\gamma_{1}, \gamma_{2}, \gamma_{4}| = -|\gamma_{1}, \gamma_{3}, \gamma_{4}| |\gamma_{1}, \gamma_{2}, \gamma_{5}| \] (41)
which, following (29) and (36), is the Plücker relation that we need to prove (33).

Equations (34)–(35) are obtained by setting \((\mu_1, \mu_2, \nu_1, \nu_2, \nu_3, \nu_4)\) to \((2, 4, 5, 1, 2, 3)\), then to \((3, 4, 5, 1, 2, 3)\) in (39). Finally, using (26), we obtain
\[ c_\phi(0)e^{H(t)}e^{X_0}e^{X_1}|0\rangle = \sum_{\lambda \subseteq [2,2,2]} c_\lambda(v) \chi_\lambda(t). \] (42)

The right-hand side of (42) is Lascoux’s form expanded as in (18). The above example shows that the proof of (30) relies on using Plücker relations to simplify the coefficients of expansions of the exponentials that appear in (27).

### 3.17. Plan of the proof of equation (30)

Firstly, we expand the product of two exponentials in terms of neutral monomials in the fermions, with coefficients that can be simplified using Plücker relations, then we use induction to show that this can be done for the product of \( N \) exponentials. Finally we use boson–fermion correspondence to obtain Lascoux’s form of \( Z_N \).

\[ \text{doi:10.1088/1742-5468/2009/03/P03017} \]

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3.18. Remark on notation

In the following, we consider only the general $N$ case, so we omit the superscripts $(N)$, and write $X_a$ instead of $X_a^{(N)}$, and so on.

3.19. Expanding two exponentials

For $0 \leq a_2 < a_1 \leq (N-2)$, consider the product

$$e^{X_{a_2}}e^{X_{a_1}} = 1 + X_{a_2} + X_{a_1} + X_{a_2}X_{a_1} \quad (43)$$

where

$$X_{a_1} = \sum_{b=1}^{N} (-1)^b d_{[a_1+1,1,b-1]} \psi^*_{-b} \psi_{a_1}$$

$$X_{a_2} = \sum_{b=1}^{N} (-1)^b d_{[a_2+1,1,b-1]} \psi^*_{-b} \psi_{a_2} \quad (44)$$

$$X_{a_2}X_{a_1} = \sum_{b_1,b_2=1}^{N} (-1)^{b_1+b_2} d_{[a_2+1,1,(b_2-1)]} d_{[a_1+1,1,(b_1-1)]} \psi^*_{-b_2} \psi_{a_2} \psi^*_{-b_1} \psi_{a_1}.$$  

Note that while we know that $a_2 < a_1$, we can have $b_2 < b_1$ and $b_1 < b_2$. We account for these two possibilities by rewriting the cross-term as

$$X_{a_2}X_{a_1} = \left( \sum_{1 \leq b_2 < b_1 \leq N} (-1)^{b_1+b_2} d_{[a_2+1,1,(b_2-1)]} d_{[a_1+1,1,(b_1-1)]} \psi^*_{-b_2} \psi_{a_2} \psi^*_{-b_1} \psi_{a_1} \right) \times \psi^*_{-b_1} \psi^*_{-b_2} \psi_{a_2} \psi_{a_1}. \quad (45)$$

Relabelling the indices in the second summation, so that $b_1 \to b_2$ and $b_2 \to b_1$, and using the anticommutation relations to put the product of fermion operators in the form used in (26), the right-hand side of (45) becomes

$$\sum_{1 \leq b_2 < b_1 \leq N} (-1)^{b_1+b_2} d_{[a_2+1,1,(b_2-1)]} d_{[a_1+1,1,(b_1-1)]} - d_{[a_2+1,1,(b_1-1)]} d_{[a_1+1,1,(b_2-1)]} \psi^*_{-b_2} \psi_{a_2} \psi^*_{-b_1} \psi_{a_1} \quad (46)$$

and we see that once again we need to simplify the coefficients. Following the same ideas as were used in the $N = 3$ example above, we write

$$c_\lambda = |\gamma(\lambda_N) + 1, \gamma(\lambda_{N-1}) + 2, \ldots, \gamma(\lambda_1) + N| \quad (47)$$

where $\gamma_b$ is the $N$-component column vector

$$\gamma_b = \left( \frac{q^{b-a+1} - q^{a-1}}{q-1} \right)_{a=1, \ldots, N}.$$  

and then consider the $2N \times 2N$ determinant expression

$$\begin{vmatrix} \gamma_{\mu_1} & \cdots & \gamma_{\mu_{N-1}} & \gamma_{\nu_1} & \gamma_{\nu_2} & \cdots & \gamma_{\mu_{N+1}} \\ 0 & \cdots & 0 & \gamma_{\nu_1} & \gamma_{\nu_2} & \cdots & \gamma_{\nu_{N+1}} \end{vmatrix} = 0 \quad (49)$$

$$\text{doi:10.1088/1742-5468/2009/03/P03017} \quad 15$$
and Laplace expand it as a sum of bilinears in $N \times N$ determinants
\[
\sum_{p=1}^{N+1} (-1)^{p+1} |\gamma_{\mu_1} \cdots \gamma_{\mu_{N-1}} \gamma_{\nu_p}| |\gamma_{\nu_1} \cdots \gamma_{\nu_{N+1}}| = 0
\]  
(50)

where $\hat{\gamma}$ indicates a missing term.

### 3.20. The general $N$ Plücker relation

The procedure outlined for the $N = 3$ case in section 3.16 can be extended to general $N$. From (46), the Plücker relation that we need contains a bilinear term $d_{[\lambda]} d_{[\phi]}$, where $\lambda$ has two $a$-parts, $a_1 > a_2$, and two $b$-parts, $b_1 > b_2$, so
\[
\lambda = [a_1 + 1, a_2 + 2, 2^{b_2 - 2}, 1^{b_1 - b_2 - 1}, 0^{N - b_1}]
\]
(51)

where we have indicated the zero-length elements as well to make the total length of the partition $N$. We prepare a $2N$-element sequence $\Sigma^0$:
\[
\Sigma^0 = [1, 2, \ldots, N|1, 2, \ldots, N]
\]
(52)

where the vertical bar $|\Sigma$ in (52) emphasizes the structure of $\Sigma^0$ as a concatenation of a left sequence $\Sigma^0_{\text{left}} = [1, 2, \ldots, N]$ and a right sequence $\Sigma^0_{\text{right}} = [1, 2, \ldots, N]$. Given $\lambda = [\lambda_1, \ldots, \lambda_N]$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$, we modify $\Sigma^0_{\text{left}}$ as follows. We add $\lambda_1$ to the largest integer $(N)$ in $\Sigma^0_{\text{left}}$, $\lambda_2$ to the next largest $(N - 1)$, and so forth. $\Sigma^0_{\text{right}}$ remains unmodified. This generates a new sequence $\Sigma$. Choosing the parameters in the $2N$-element sequence $(\mu_1, \ldots, \mu_{N-1}, \nu_1, \ldots, \nu_{N+1})$ by identifying them with the corresponding integers in $\Sigma$, we end up with
\[
\begin{align*}
(\mu_1, \ldots, \mu_{N-b_1}) &= (1, \ldots, N - b_1) \\
(\mu_{N-b_1+1}, \ldots, \mu_{N-b_2-1}) &= (N-b_1 + 2, \ldots, N - b_2) \\
(\mu_{N-b_2}, \ldots, \mu_{N-2}) &= (N - b_2 + 2, \ldots, N) \\
(\mu_{N-1}, \nu_1) &= (N + a_2 + 1, N + a_1 + 1) \\
(\nu_2, \ldots, \nu_{N+1}) &= (1, \ldots, N)
\end{align*}
\]  
(53)

where ‘...’ between two integers $m$ and $n$ stands for the integers $m + 1, m + 2, \ldots, n - 1$, that increase by 1 at a time. Using (50), the above choice of parameters leads to the Plücker relation
\[
[\Sigma, \gamma_{\nu_{N+a_1+1}}]\gamma_1, \ldots, \gamma_N] + (-1)^{N-b_1}[\Sigma, \gamma_{\nu_{b_1+1}}]\gamma_1, \ldots, \gamma_{N-b_1+1}, \gamma_1, \ldots, \gamma_{N-b_1+1}, \gamma_N] \\
+ (-1)^{N-b_2}[\Sigma, \gamma_{\nu_{b_2+1}}]\gamma_1, \ldots, \gamma_{N-b_2+1}, \gamma_1, \ldots, \gamma_{N-b_2+1}, \gamma_N] = 0
\]  
(54)

where $\Sigma$ in (54) is the sequence
\[
\Sigma = \{\gamma_1, \ldots, \gamma_{N-b_1}, \gamma_{N-b_1+2}, \ldots, \gamma_{N-b_2}, \gamma_{N-b_2+2}, \ldots, \gamma_N, \gamma_{N+a_2+1}\}
\]
(55)

Using (36), (54) can be written as
\[
c_{[a_2+1, 1^{b_2-1}]} c_{[a_1+1, 1^{b_1-1}]} - c_{[a_2+1, 1^{b_1-1}]} c_{[a_1+1, 1^{b_2-1}]} = C_{[a_1+2, 2^{b_2-1}, 1^{b_1-2}]}.
\]
(56)

Substituting (56) in the second line of (45), we obtain
\[
X_{a_2} X_{a_1} = \sum_{1 \leq b_2 < b_1 \leq N} (-1)^{b_2+b_1} d_{[a_1+1, a_2+2, 2^{b_2-1}, 1^{b_1-2}] } \psi_{-b_1}^{\ast} \psi_{-b_2}^{\ast} \psi_{a_2} \psi_{a_1}.
\]
(57)
3.21. Expanding more than two exponentials

We wish to show that for all $0 \leq a_k < \cdots < a_1 \leq N - 2$, we have

$$X_{a_k} \cdots X_{a_1} = \sum_{1 \leq b_k < \cdots < b_1 \leq N} (-1)^{b_1 + \cdots + b_k} d_{\lambda(d=k)} \psi^*_{b_1} \cdots \psi^*_{b_k} \psi_{a_k} \cdots \psi_{a_1}$$  \hspace{1cm} (58)

where

$$\lambda(d=k) = [a_1 + 1, a_2 + 2, \ldots, a_k + k, b_{k-1} - b_k, \ldots, b_1 - b_2, 0^{N-b_1}]$$  \hspace{1cm} (59)

is the partition, with $k$ cells on the main diagonal, associated with the fermion monomial on the right-hand side of (58).

We start by assuming that (58) holds for some value of $k$, and show that it also holds for $k + 1$. Since it holds for $k = 2$, it holds for all $k$. Consider the expression

$$X_{a_k} \cdots X_{a_1} X_{a_0} = \sum_{1 \leq b_k < \cdots < b_1 \leq N} \sum_{b_0 = 1}^{N} (-1)^{b_0 + b_1 + \cdots + b_k}$$

$$\times d_{\lambda(d=k)} d_{[a_0 + 1, 1^{a_0-1}]} \psi^*_{b_1} \cdots \psi^*_{b_k} \psi_{a_k} \cdots \psi_{a_1} \psi_{a_0}$$  \hspace{1cm} (60)

for $0 \leq a_k < \cdots < a_1 < a_0 \leq (N - 2)$, where $\lambda(d=k)$ is still the same partition as in (58), and rewrite the summation as

$$\sum_{1 \leq b_k < \cdots < b_1 \leq N} \sum_{b_0 = 1}^{N} (-1)^{b_0 + b_1 + \cdots + b_k}$$

$$\times d_{\lambda(d=k)} d_{[a_0 + 1, 1^{a_0-1}]} \psi^*_{b_1} \cdots \psi^*_{b_k} \psi_{a_k} \cdots \psi_{a_1} \psi_{a_0}$$  \hspace{1cm} (61)

Ordering the fermions in each summation, so that they are in the same order as in (26), and relabelling the indices, we obtain

$$\sum_{1 \leq b_k < \cdots < b_0 \leq N} (-1)^{b_0 + \cdots + b_k} \left( \sum_{p=0}^{k} (-1)^p d_{\lambda(d=k|p)} d_{[a_0 + 1, 1^{b_0-1}]} \psi^*_{b_0} \cdots \psi^*_{b_k} \psi_{a_k} \cdots \psi_{a_0} \right)$$  \hspace{1cm} (62)

where $\lambda(d=k|p)$ stands for $\lambda(d=k)$, but the $b$-part that used to be $b_0$ is now omitted and the $b$-part $b_0$ is added in the correct position. For example, if $\lambda(d=6)$ has $b$-parts $\{b_0 < b_1 < b_2 < b_3 < b_4 < b_5\}$, then $\lambda(d=6|3)$ has $\{b_0 < b_1 < b_2 < b_3 < b_4\}$. $\lambda(d=k|0)$ indicates that no changes were made.

To simplify the sum in large brackets in (62), we consider (50), and, following the same procedure as before, use the evaluations

$$(\mu_1, \ldots, \mu_{N-b_0}) = (1, \ldots, N - b_0)$$

$$(\mu_{N-b_0+1}, \ldots, \mu_{N-b_1-1}) = (N - b_0 + 2, \ldots, N - b_1)$$

$$(\mu_{N-b_1}, \ldots, \mu_{N-b_2-2}) = (N - b_1 + 2, \ldots, N - b_2)$$

$$(\mu_{N-b_2-1}, \ldots, \mu_{N-b_3-3}) = (N - b_2 + 2, \ldots, N - b_3)$$

$$\vdots$$

$$(\mu_{N-b_{k-1}}, \ldots, \mu_{N-(k+1)}) = (N - b_k + 2, \ldots, N)$$

$$(\mu_{N-k}, \ldots, \mu_{N-1}, \nu_1) = (N + a_k + 1, \ldots, N + a_1 + 1, N + a_0 + 1)$$

$$\nu_2, \ldots, \nu_{N+1} = (1, \ldots, N).$$

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The required Plücker relation is
\[
|\Sigma, \gamma_{N+\alpha_0+1}\rangle \langle \gamma_1, \ldots, \gamma_N| \\
+ (-1)^{N-b_0+1}|\Sigma, \gamma_{N-b_0+1}\rangle \langle \gamma_{N+\alpha_0+1}, \ldots, \hat{\gamma}_{N-b_0+1}, \ldots, \gamma_N| \\
+ (-1)^{N-b_1+1}|\Sigma, \gamma_{N-b_1+1}\rangle \langle \gamma_{N+\alpha_0+1}, \ldots, \hat{\gamma}_{N-b_1+1}, \ldots, \gamma_N| \\
+ \cdots \\
+ (-1)^{N-b_k+1}|\Sigma, \gamma_{N-b_k+1}\rangle \langle \gamma_{N+\alpha_0+1}, \ldots, \hat{\gamma}_{N-b_k+1}, \ldots, \gamma_N| = 0
\]
(64)
where the sequence \(\Sigma\) in (64) is obtained as follows. Consider the sets of indices in (63) from top to bottom except the very last, \((1, \ldots, N)\), and order them from left to right, so that the topmost \((1, \ldots, N-b_0)\) becomes the leftmost and \((N+a_k+1, \ldots, N+1, N+a_0+1)\) becomes the rightmost. Concatenate them and remove the rightmost element \((N+a_0+1)\). Use the resulting \((N-1)\)-element sequence in the same order to generate the indices of \(\Sigma\) in (64). Using (36), (64) can be rewritten as
\[
\sum_{p=0}^{k} (-1)^p C_{\lambda(d=k)p} C_{[a_0+1,1^{b_p-1}]} = C_\phi C_{\lambda(d=k+1)}
\]
(65)
where \(\lambda_{(d=k+1)}\) is the partition with \((k+1)\) cells on the main diagonal
\[
\lambda_{(d=k+1)} = [a_0 + 1, a_1 + 2, \ldots, a_k + (k+1), (k+1)^{b_k-1}, k^{b_k-1-b_k-1}, \ldots, 1^{b_0-b_1-1}].
\]
(66)
Relabelling \(a_i\) and \(b_j\) so that the indices take values from 1 to \((k+1)\), rather than from 0 to \(k\), such that \(a_1 > a_2 > \cdots > a_{k+1}\) and \(b_1 > b_2 > \cdots > b_{k+1}\), we obtain
\[
X_{a_{k+1}} \cdots X_{a_1} = \sum_{1 \leq b_{k+1} < \cdots < b_1 \leq N} (-1)^{b_1+\cdots+b_{k+1}} d_{\lambda_{(d=k+1)}} \psi_{-b_1}^* \cdots \psi_{-b_{k+1}}^* \psi_{a_{k+1}} \cdots \psi_{a_1}
\]
(67)
where \(\lambda_{(d=k+1)}\) now stands for the partition that corresponds to the partition-ordered monomial adjacent to it. This completes the proof of (58).

3.22. Expanding \((N-1)\) exponentials

Using (58), we obtain
\[
e^{X_0} \cdots e^{X_{N-2}} = 1 + \sum_{d=1}^{N-1} \sum_{0 \leq a_d < \cdots < a_1 \leq N-2} X_{a_d} \cdots X_{a_1}
\]
\[
= 1 + \sum_{d=1}^{N-1} \sum_{0 \leq a_d < \cdots < a_1 \leq (N-2)} (-1)^{b_1+\cdots+b_d} d_{\lambda_d} \psi_{-b_1}^* \cdots \psi_{-b_d}^* \psi_{a_d} \cdots \psi_{a_1}.
\]
(68)
From the remarks following (26), we note that the terms on the right-hand side of (68) generate every fermionic expression that corresponds to any partition that fits in the rectangle \([ (N-1)^N \]. Additionally, (58) shows that the fermionic expressions are

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accompanied by the required coefficient with the required sign. Thus
\[
\langle 0 | e^{H(t)} e^{X_0} \ldots e^{X_{N-2}} | 0 \rangle = d_\phi(v) \chi_\phi(t) \\
+ \sum_{k=1}^{N-1} \sum_{0 \leq a_k < \cdots < a_1 \leq N-1} \sum_{1 \leq b_k < \cdots < b_1 \leq N} \lambda_k(v) \chi_{\lambda_k}(t) \\
= \sum_{\lambda \subseteq (N-1)^N} \lambda(t). \quad (69)
\]

Multiplying both sides of (69) by \( c_N \), we obtain (30).

4. Discussion

The result of this paper is that a central object in the algebraic Bethe ansatz (ABA) approach to the six-vertex model (or more precisely the XXZ spin chain), that is the domain wall partition function \( Z_N \), is (a restriction of) a KP \( \tau \) function, for all values of the crossing parameter. In a sense, what we do is to propose a KP \( \tau \) function that depends on infinitely many time variables, then restrict the time variables to being power sums of exponentials of the ABA auxiliary space rapidities of \( Z_N \) to obtain Lascoux’s determinant expression of the latter.

We also obtained an expression for \( Z_N \) as an expectation value of exponentials of bilinears in the KP fermions. We observe that the bilinears that appear in this expression have a particularly simple combinatorial interpretation. This leads us to believe that analogous expressions can be found for other interesting objects in the algebraic Bethe ansatz approach to the correlation functions in the XXZ spin-\( \frac{1}{2} \) Heisenberg chain, such as the scalar products, the so-called elementary building blocks of the correlation functions, and at least the simplest correlation functions [9, 10]. In particular we hope that the connection with KP will simplify the study of the asymptotics of the correlation functions, just as the connection with the nonlinear Schrödinger equation did in the case of the Bose gas and the XXZ spin chain at the free fermion point [4]. We hope to report on these in future publications.

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