The Polychronakos–Frahm spin chain of $BC_N$ type and Berry–Tabor’s conjecture

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We compute the partition function of the $su(m)$ Polychronakos–Frahm spin chain of $BC_N$ type by means of the freezing trick. We use this partition function to study several statistical properties of the spectrum, which turn out to be analogous to those of other spin chains of Haldane–Shastry type. In particular, we find that when the number of particles is sufficiently large the level density follows a Gaussian distribution with great accuracy. We also show that the distribution of (normalized) spacings between consecutive levels is of neither Poisson nor Wigner type, but is qualitatively similar to that of the original Haldane–Shastry spin chain. This suggests that spin chains of Haldane–Shastry type are exceptional integrable models, since they do not satisfy a well-known conjecture of Berry and Tabor according to which the spacings distribution of a generic integrable system should be Poissonian. We derive a simple analytic expression for the cumulative spacings distribution of the $BC_N$-type Polychronakos–Frahm chain using only a few essential properties of its spectrum, like the Gaussian character of the level density and the fact the energy levels are equally spaced. This expression is in excellent agreement with the numerical data and, moreover, there is strong evidence that it can also be applied to the Haldane–Shastry and the Polychronakos–Frahm spin chains.

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I. INTRODUCTION

Solvable spin chains often provide a natural setting for testing or modeling interesting physical phenomena and mathematical results in such disparate fields as fractional statistics, random matrix theory or orthogonal polynomials. Among these chains, those of Haldane–Shastry (HS) type occupy a distinguished position due to their remarkable integrability and solvability properties. The original chain of this type was independently introduced twenty years ago by Haldane and Shastry, in an attempt to construct a model whose ground state coincided with Gutzwiller’s variational wave function for the Hubbard model in the limit of large on-site interaction. In the original HS chain, the spins are equally-spaced on a circle and present pairwise interactions inversely proportional to their chord distance.

An essential feature of the spin chains of HS type is their close connection with the spin versions of the Calogero and Sutherland models, and their generalizations due to Olshanetsky and Perelomov. This observation—already pointed out by Shastry in his original paper—was elegantly formulated by Polychronakos in Ref. [10]. In the latter reference, the author showed that the original HS chain can be obtained from the spin Sutherland model in the strong coupling limit, in which the dynamical and spin degrees of freedom decouple, so that the particles “freeze” at the equilibrium positions of the scalar part of the potential. In this regime, the integrals of motion of the spin Sutherland model directly yield first integrals of the HS chain, thereby explaining its complete integrability. This procedure was also applied in [10] to construct a new integrable spin chain related to the original Calogero model. The spectrum of this chain was numerically studied by Frahm, who found that the levels are grouped in highly degenerate multiplets. In a subsequent publication, Polychronakos computed the partition function of this chain (usually referred to in the literature as the Polychronakos–Frahm chain) by the “freezing trick” described above. Interestingly, the partition function of the original HS chain was computed only very recently.

Both the HS and the PF (Polychronakos–Frahm) chains are obtained from the Sutherland and Calogero models associated with the $A_N$ root system in Olshanetsky and Perelomov’s approach. The $BC_N$ versions of both chains have also been studied in the literature. More precisely, the integrability of the PF chain of $BC_N$ type was established by Yamamoto and Tsuchiya using again the freezing trick. On the other hand, the partition function of the HS chain of $BC_N$ type was computed in closed form in Ref. [18]. The explicit knowledge of the partition function made it possible to study certain statistical properties of the spectrum of this chain. In particular, it was observed that for a large number of spins the level density is Gaussian. As a matter of fact, this property also holds for the original HS chain, as shown in Ref. [10]. The analysis of the distribution of the spacing between consecutive levels of the original HS chain was also undertaken in the latter reference. Rather unexpectedly, it was found that this distribution is not of Poisson type, as should be the case for a “generic” integrable model according to a long-standing conjecture of Berry and Tabor. This behavior has also been recently reported for a supersymmetric version of the HS...
chain 20.

The aim of this paper is twofold. In the first place, we shall compute in closed form the partition function of the PF chain of $BC_N$ type by means of the freezing trick. Using the partition function, we shall perform a numerical study of the density of levels and the distribution of the spacing between consecutive energies. We shall see that the level density is again Gaussian, and that the spacings distribution is analogous to that of the original HS chain. In particular, our results show that the distribution of spacings is neither Poissonian nor of Wigner type (characteristic of chaotic systems). We shall next derive a simple analytic expression for the cumulative spacings distribution, which reproduces the numerical data with much greater accuracy than the empiric formula proposed in Ref. [16]. In fact, we have strong numerical evidence that the new expression can also be applied to the HS and PF chains of $A_N$ type. In view of the Berry–Tabor conjecture, our results suggest that spin chains of HS type are exceptional among the class of integrable models.

II. THE FUNCTION PARTITION OF THE PF CHAIN OF $BC_N$ TYPE

The Hamiltonian of the (antiferromagnetic) su($m$) PF chain of $BC_N$ type is defined by

$$
\mathcal{H}^c = \sum_{i \neq j} \left[ \frac{1 + S_{ij}}{(\xi_i - \xi_j)^2} + \frac{1 + S_{ij}}{(\xi_i + \xi_j)^2} \right] + \beta \sum_i \left[ 1 - \xi_i S_i \right],
$$

(1)

where the sums run from 1 to $N$ (as always hereafter, unless otherwise stated), $\beta > 0$, $\epsilon = \pm 1$, $S_{ij}$ is the operator which permutes the $i$-th and $j$-th spins, $S_i$ is the operator reversing the $i$-th spin, and $S_{ij} = S_i S_j S_{ij}$. Note that the spin operators $S_{ij}$ and $S_i$ can be expressed in terms of the fundamental su($m$) spin generators $J_k^a$ at the site $k$ (with the normalization $\text{tr}(J_k^a J_k^a) = \frac{1}{2} \delta^{aa}$) as

$$
S_{ij} = \frac{1}{m} + 2 \sum_{a=1}^{m-1} J_i^a J_j^a, \quad S_i = \sqrt{2m} J_i^1.
$$

The chain sites $\xi_i$ are the coordinates of the unique minimum in $C = \{ x | 0 < x_1 < \cdots < x_N \}$ of the potential

$$
U(x) = \sum_{i \neq j} \left[ \frac{1}{(x_{ij})^2} + \frac{1}{(x_{ij})^2} \right] + \sum_i \frac{\beta^2}{x_i^2} + \frac{r^2}{4},
$$

(2)

where $x_{ij}^\pm = x_i \pm x_j$ and $r^2 = \sum_i x_i^2$. The existence of this minimum follows from the fact that $U$ tends to $+\infty$ on the boundary of $C$ and as $r \to \infty$, and its uniqueness was established in Ref. [21] by expressing the potential $U$ in terms of the logarithm of the ground state of the $BC_N$ Calogero model

$$
H^c = -\sum_i \partial^2_{x_i} + a(a - 1) \sum_{i \neq j} \left[ \frac{1}{(x_{ij})^2} + \frac{1}{(x_{ij})^2} \right] + b(b - 1) \sum_i \frac{1}{x_i^2} + \frac{a^2}{4} r^2,
$$

(3)

with $b = \beta a$ and $a > 1/2$. Moreover, it can be shown 22 that $\xi_i = \sqrt{2y_i}$, where $y_i$ is the $i$-th zero of the generalized Laguerre polynomial $L_N^{\beta - 1}$. From this fact, one can infer [23] that for $N \gg \beta$ the density of sites (normalized to unity) $\rho_N(x)$ is given by the circular law

$$
\rho_N(x) = \frac{1}{2\pi N} \sqrt{8N - x^2}. \quad (4)
$$

Note that in this limit the sites' density is independent of $\beta$ and is qualitatively similar to that of the PF chain of $A_N$ type [14]. Integrating the previous equation, we obtain the implicit asymptotic relation

$$
4\pi k = \xi_k \sqrt{8N - \xi_k^2} + 8N \arcsin \left( \frac{\xi_k}{\sqrt{8N}} \right),
$$

valid also for $N \gg \beta$.

The spin chain [11] can be expressed in terms of the spin Calogero model of $BC_N$ type

$$
H' = -\sum_i \partial^2_{x_i} + a \sum_{i \neq j} \left[ \frac{a + S_{ij}}{(x_{ij})^2} + \frac{a + \tilde{S}_{ij}}{(x_{ij})^2} \right] + b \sum_i \frac{b - \epsilon S_i}{x_i^2} + \frac{a^2}{4} r^2 \quad (5)
$$

and its scalar reduction [3] as

$$
\tilde{H}' = H'^c + a \tilde{H}^c, \quad (6)
$$

where $\tilde{H}'$ is obtained from $H'$ replacing the chain sites $\xi$ by the particles' coordinates $x$. Since

$$
H' = -\sum_i \partial^2_{x_i} + a^2 U + O(a),
$$

when the coupling constant $a$ tends to infinity the particles in the spin dynamical model [17] concentrate at the coordinates of the minimum of the potential $U$, that is at the sites $\xi$ of the chain [11]. Thus, in the limit $a \to \infty$ the spin and dynamical degrees of freedom of the Hamiltonian [5] decouple, so that by Eq. [10] its eigenvalues are approximately given by

$$
E_{ij}^c \simeq E_{ij}^c + a \xi_j^2, \quad a \gg 1, \quad (7)
$$

where $E_{ij}^c$ and $\xi_j^2$ are two arbitrary eigenvalues of $H_{ij}^c$ and $H'$, respectively. The asymptotic relation [11] immediately yields the following exact formula for the partition function $Z^c$ of the chain [11]:

$$
Z^c(T) = \lim_{a \to \infty} \frac{Z^c(aT)}{Z^c(aT)}, \quad (8)
$$
where $Z^b$ and $Z^{sc}$ are the partition functions of $H^b$ and $H^{sc}$, respectively.

We shall next evaluate the partition function $Z^b$ of the chain (11) by computing the partition functions $Z^t$ and $Z^{sc}$ in Eq. (3). In order to determine the spectra of the corresponding Hamiltonians $H^b$ and $H^{sc}$, following Ref. [16] we introduce the auxiliary operator

$$H' = - \sum_i \partial^2_{x_i} + \sum_{i \neq j} \left[ \frac{a}{(x_{ij})^2} (a - K_{ij}) + \frac{a}{(a - K_{ij})^2} \right]$$

$$+ \sum_i \frac{b}{x_i} (b - K_i) + \frac{a^2}{4} r^2,$$

(9)

where $K_{ij}$ and $K_i$ are coordinate permutation and sign reversing operators, defined by

$$(K_{ij} f)(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) = f(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N),$$

$$(K_i f)(x_1, \ldots, x_i, \ldots, x_N) = f(x_1, \ldots, -x_i, \ldots, x_N),$$

and $K_{ij} = K_i K_j K_{ij}$. We then have the obvious relations

$$H' = H'\big|_{K_i, K_j \rightarrow S_i, S_j, K_{ij} \rightarrow \epsilon S_{ij}},$$

$$H^{sc} = H'\big|_{K_i, K_j \rightarrow \epsilon}.$$

On the other hand, the spectrum of $H'$ can be easily computed by noting that this operator can be written in terms of the rational Dunkl operators of $BC_N$ type [24]

$$J^-_i = \partial_{x_i} + a \sum_{j \neq i} \left[ \frac{1}{x_{ij}} (1 - K_{ij}) + \frac{1}{x_{ij}} (1 - K_{ij}) \right]$$

$$+ \frac{b}{x_i} (1 - K_i), \quad i = 1, \ldots, N,$$

(11)

as follows [25]:

$$H' = \mu \left[ - \sum_i (J^-)^2 + a \sum_i x_i \partial x_i + E_0 \right] \mu^{-1},$$

(12)

where

$$\mu(x) = e^{-\sum_i |x_i|^2} \prod_i \prod_{i < j} |x_i^2 - x_j^2|^a$$

(13)

is the ground state of the Hamiltonian (3) and

$$E_0 = Na \left( b + a (N - 1) \right).$$

(14)

Since the Dunkl operators [11] map any monomial $\prod_i x_i^{n_i}$ into a polynomial of total degree $n_1 + \cdots + n_N - 1$, by Eq. (12) the operator $H'$ is represented by an upper triangular matrix in the (non-orthonormal) basis with elements

$$\phi_n = \mu \prod_i x_i^{n_i}, \quad n \equiv (n_1, \ldots, n_N) \in (N \cup \{0\})^N,$$

ordered according to the total degree $|n| = n_1 + \cdots + n_N$ of the monomial part. More precisely,

$$H' \phi_n = E_n \phi_n + \sum_{|m| < |n|} c_{mn} \phi_m,$$

(16)

where

$$E_n = a |n| + E_0$$

(17)

and the coefficients $c_{mn}$ are real constants.

We shall now construct a basis of the Hilbert space of the Hamiltonian $H'$ in which this operator is also represented by an upper triangular matrix. To this end, let us define by $\Lambda^c$ the projector on states antisymmetric under simultaneous permutations of spatial and spin coordinates, and with parity $\epsilon$ under sign reversals of coordinates and spins. If

$$|s\rangle \equiv |s_1, \ldots, s_N\rangle, \quad s_i = -M, -M + 1, \ldots, M \equiv \frac{m-1}{2},$$

denotes a state of the $su(m)$ spin basis, the functions

$$\psi_{n,s}(x) = \Lambda^c (\phi_n(x)|s\rangle),$$

(18)

form a basis of the Hilbert space of the Hamiltonian $H^c$ provided that:

(i) $n_1 \geq \cdots \geq n_N$.

(ii) $s_i > s_j$ whenever $n_i = n_j$ and $i < j$.

(iii) $s_i \geq 0$ for all $i$, and $s_i > 0$ if $(-1)^{n_i} = -\epsilon$.

The first two conditions are a consequence of the antisymmetry of the states (15) under particle permutations, while the last condition is due to the fact that these states must have parity $\epsilon$ under sign reversals. Since $K_{ij} \Lambda^c = -S_{ij} \Lambda^c$ and $K_i \Lambda^c = \epsilon S_i \Lambda^c$, it follows that $H' \Lambda^c = H^c \Lambda^c$. Using this identity and the fact that $H'$ obviously commutes with $\Lambda^c$, from Eq. (16) we easily obtain

$$H^c \psi_{n,s} = H' \psi_{n,s} = \Lambda^c (|H' \phi_n|s\rangle)$$

$$= E_n \psi_{n,s} + \sum_{|m| < |n|} c_{mn} \psi_{m,s}.$$
where the spin degeneracy factor is given by

\[ Z^{sc}(aT) = \sum_{k_1 \geq \cdots \geq k_N \geq 0} q^{2|k|}, \]

where \( q = e^{-1/(knT)} \). The previous sum can be evaluated by expressing it in terms of the differences \( p_i = k_i - k_{i+1} \), \( i = 1, \ldots, N - 1 \), with \( p_N \equiv k_N \). Since \( k_j = \sum_{i=j}^{n} p_i \), we easily obtain

\[ Z^{sc}(aT) = \sum_{p_1, \ldots, p_N \geq 0} q^{2 \sum_{i=1}^{N} p_i} = \sum_{p_1, \ldots, p_N \geq 0} q^{\sum_{i=1}^{N} ip_i} = \prod_{i=1}^{N} \sum_{p_i \geq 0} (q^{2i})^{p_i} = \prod_{i=1}^{N} (1 - q^{2i})^{-1}. \]  

In order to compute the partition function of the spin Hamiltonian \( H^s \), we shall first assume that \( m \) is even, so that condition (iii) simplifies to

\( (iii') \quad s_i > 0 \) for all \( i \).

As neither the value of \( E^*_{n,\epsilon} \) nor conditions (i), (ii) and (iii') depend on \( \epsilon \), in this case the partition functions \( Z^s \) and \( Z^r \) do not contribute to the partition function \( Z^* \). With this convention, the partition function of the scalar Hamiltonian \( H^{sc} \) is given by

\[ Z(aT) = \sum_{\nu \in \mathcal{P}_N} d(\nu) \sum_{k_1 \geq \cdots \geq k_r \geq 0} q^{\sum_{i=1}^{r} \nu_i k_i} = \sum_{\nu \in \mathcal{P}_N} d(\nu) \sum_{p_1, \ldots, p_{r-1} \geq 0} q^{\sum_{i=1}^{r} \nu_i p_i} = \sum_{\nu \in \mathcal{P}_N} d(\nu) \prod_{i=1}^{r} q^{\nu_i}, \]

where

\[ N_j = \sum_{i=1}^{j} \nu_i. \]

From Eqs. (8), (21) and (22) we finally obtain the following explicit expression for the partition function of the \( su(m) \) PF chain of \( BCN \) type in the case of even \( m \):

\[ Z(T) = q^{-N} \prod_{i} \left(1 - q^{2i}\right) \sum_{\nu \in \mathcal{P}_N} d(\nu) \prod_{j=1}^{r} q^{N_j}, \]  

where \( \ell(\nu) = r \) is the number of components of the multiindex \( \nu \). For instance, for spin 1/2 we have \( \nu_i = 1 \) for all \( i \), and therefore \( r = N, \nu = 1 \) and \( N_j = j \), so that the previous formula simplifies to

\[ Z(T) = q^{N(N-1)} \prod_{i} (1 + q^i), \quad m = 2. \]  

Thus, for spin 1/2 the spectrum of the chain is given by

\[ \mathcal{E}_j = \frac{1}{2} N(N-1) + j, \quad j = 0, 1, \ldots, \frac{1}{2} N(N+1), \]  

and the degeneracy of the energy \( \mathcal{E}_j \) is the number \( Q_N(j) \) of partitions of the integer \( j \) into distinct parts no larger than \( N \) (with \( Q_N(0) = 1 \)). For \( j \leq N \) this number coincides with the number \( Q(j) \) of partitions of \( j \) into distinct parts, which has been extensively studied in the mathematical literature. It is also interesting to observe that the partition function \( (23) \) is closely related to Ramanujan’s fifth order mock theta function

\[ \psi_1(q) = \sum_{N=0}^{\infty} q^{N} Z_N(q), \]

where \( Z_N(q) \) denotes the RHS of Eq. (23).

Equation (20) shows that for spin 1/2 the chain is equivalent to a system of \( N \) species of noninteracting
fermions (with vacuum energy $E_0 = N(N - 1)/2$), whose effective Hamiltonian is given by

$$H_{\text{eff}} = E_0 + \sum_{i=1}^{N} E_i b_i^0 b_i.$$ 

Here $b_i$ (resp. $b_i^0$) is the annihilation (resp. creation) operator of the $i$-th species of fermion, and $E_i = \mathfrak{i} t$ its energy. A similar result was obtained in Ref. [28] for the supersymmetric $\text{su}(1|1)$ (ferromagnetic) HS chain, although in the latter case the energy of the $i$-th fermion is $E_i = \mathfrak{i}(N - i)$ (the dispersion relation of the original Haldane–Shastry chain).

Let us consider now the case of odd $m$. In this case, it is convenient to slightly modify condition (i) above by first grouping the components of $\mathbf{n}$ with the same parity and then ordering separately the even and odd components. In other words, we shall write $\mathbf{n} = (\mathbf{n}_e, \mathbf{n}_o)$, where

$$\mathbf{n}_e = \left( \nu_1 (2k_1), \ldots, \nu_r (2k_r) \right),$$

$$\mathbf{n}_o = \left( \nu_{r+1} (2k_{r+1} + 1), \ldots, \nu_{2r} (2k_{2r} + 1) \right),$$

and

$$k_1 > \cdots > k_r \geq 0, \quad k_{r+1} > \cdots > k_{2r} \geq 0.$$

By conditions (ii) and (iii), the spin degeneracy factor is now

$$d^r_s = \prod_{i=1}^{s} \left( \frac{m+\nu_i}{\nu_i} \right) \cdot \prod_{s+1}^{r} \left( \frac{m-\nu_i}{\nu_i} \right) \equiv d^r_s(\nu).$$

Calling

$$\tilde{N}_j = \sum_{i=s+1}^{j} \nu_i, \quad j = s + 1, \ldots, r,$$

and proceeding as before, we obtain

$$Z^e(aT) = \sum_{\nu \in \mathcal{P}_N} \sum_{s=0}^{r} d^r_s(\nu) \sum_{k_1 > \cdots > k_r \geq 0} \sum_{k_{r+1} > \cdots > k_{2r} \geq 0} \nu_{(2k_i)} \left[ \sum_{k_1 > \cdots > k_r \geq 0} \frac{2\nu_i k_i}{\nu_i} \right]$$

$$\times \left[ \sum_{k_{r+1} > \cdots > k_{2r} \geq 0} \frac{2\nu_i k_i}{\nu_i} \right]$$

$$= \sum_{\nu \in \mathcal{P}_N} \sum_{s=0}^{r} d^r_s(\nu) q^{N_s} \prod_{j=1}^{\ell(\nu)} q^{2N_j} \left( \prod_{j=s+1}^{r} \frac{1}{1 - q^{2N_j}} \right).$$

Substituting the previous expression and [21] into [8], we immediately deduce the following explicit formula for the partition functions of the $\text{su}(m)$ PF chain of $BC_N$ type for odd $m$:

$$Z^e(T) = \prod_{i=0}^{N_e} \prod_{\nu} d^r_s(\nu) q^{-1} \prod_{j=1}^{\ell(\nu)} q^{2N_j} \prod_{s=1}^{r} \frac{1}{1 - q^{2N_j}} \cdot \prod_{j=s+1}^{r} \frac{1}{1 - q^{2N_j}}.$$

Although we have chosen, for definiteness, to study the antiferromagnetic chain [1], a similar analysis can be performed for its ferromagnetic counterpart

$$\mathcal{H}_F^t = \sum_{i \neq j} \left[ \frac{1 - \hat{S}_{ij}}{(\xi_i - \xi_j)^2 + (\xi_i + \xi_j)^2} + \beta \sum_{j} \frac{1 - \epsilon S_j}{\xi_i^2} \right].$$

Since now

$$H_{F}^t = H_{F}^t\big|_{K_{ij} \to S_{ij}, K_{r} \to \epsilon S_{i}},$$

we must replace the operator $\Lambda'$ in Eq. (18) by the projector on states symmetric under simultaneous permutations of the particles’ spatial and spin coordinates, and with parity $\epsilon$ under sign reversal of coordinates and spin. Hence condition (ii) above on the basis states $\psi_{n,s}$ should now read

$$\text{(ii') } s_i \geq s_j \text{ whenever } n_i = n_j \text{ and } i < j.$$

As a result, the degeneracy factors $d(\nu)$ and $d^e(\nu)$ in Eqs. [25] and [29] should be replaced by their “bosonic” versions

$$d_{F}(\nu) = \prod_{i=1}^{r} \left( \frac{m + \nu_i - 1}{\nu_i} \right),$$

$$d_{F}^e(\nu) = \prod_{i=1}^{s} \left( \frac{m + \nu_i - 1}{\nu_i} \right) \cdot \prod_{s+1}^{r} \left( \frac{m - \nu_i - 1}{\nu_i} \right).$$

Therefore the partition function of the ferromagnetic $\text{su}(m)$ PF chain of $BC_N$ type [31] is still given by Eq. [25] (for even $m$) or [29] (for odd $m$), but with $d(\nu)$ and $d^e(\nu)$ replaced respectively by $d_{F}(\nu)$ and $d_{F}^e(\nu)$.

On the other hand, the chains [11] and [33] are obviously related by

$$\mathcal{H}_F + \mathcal{H}^{-t} = \left( \sum_{i \neq j} \frac{1}{[(\xi_i - \xi_j)^2 + (\xi_i + \xi_j)^2]} \right) + \beta \sum_{i} \xi_i^{-2}.$$

The RHS of this equation clearly coincides with the largest eigenvalue $E_{\text{max}}$ of the antiferromagnetic chains $\mathcal{H}^t$, whose corresponding eigenvectors are the spin states symmetric under permutations and with parity $\epsilon$ under spin reversal. This eigenvalue is most easily computed for the spin 1/2 chains, since in this case the spectrum is explicitly given in Eq. [27]. We thus obtain

$$E_{\text{max}} = \frac{1}{2} N(N - 1) + \frac{1}{2} N(N + 1) = N^2,$$
so that
\[ \mathcal{H}_e = N^2 - \mathcal{H}^e. \]

Hence the partition functions \( Z_e \) and \( Z_f \) of \( \mathcal{H}_e \) and \( \mathcal{H}_f \) satisfy the remarkable identity
\[ Z_f(q) = q^{N^2} Z_e(q^{-1}). \]

This is a manifestation of the boson-fermion duality discussed in detail in Ref. [20] for the \( \text{su}(m|n) \) supersymmetric HS spin chain, since the ferromagnetic (resp. antiferromagnetic) chain can be regarded as purely bosonic (resp. fermionic). For instance, using the latter identity and Eq. (25) we easily obtain the following expression for the partition function of the ferromagnetic spin 1/2 chains:
\[ Z_F(T) = \prod_i (1 + q^i), \quad m = 2. \] (34)

(Note that, as in the antiferromagnetic case, \( Z_F \) is actually independent of \( \epsilon \) for even \( m \).) This is, again, the partition function of a system of \( N \) species of free fermions of energy \( E_i = i \), but now the vacuum energy vanishes.

Equation (25) for the partition function of the antiferromagnetic chains with even \( m \) can be easily simplified to
\[ Z(T) = \prod_i \left( 1 + q^i \right) \sum_{\nu \in P_N} d(\nu) \prod_{j=1}^{\ell(\nu)} \prod_{N_j} \left( 1 - q^{N_j} \right), \]

where the positive integers \( N_j \) are defined by
\[ \{N_1, \ldots, N_{N-\ell(\nu)}\} = \{1, \ldots, N-1\} - \{N_1, \ldots, N_{\ell(\nu)-1}\}. \]

The sum in the RHS is easily recognized as the partition function \( Z(A)(T; m/2) \) of the \( \text{su}(m/2) \) (antiferromagnetic) PF chain of \( A_N \) type [20]. We thus obtain the remarkable factorization
\[ Z(T; m) = Z_F(T; 2) \cdot Z(A)(T; m/2), \quad m \in 2\mathbb{N}, \] (35)

where the second argument in \( Z \) and \( Z_F \) denotes the number of internal degrees of freedom. Replacing \( d(\nu) \) by \( d_F(\nu) \) in Eq. (25) we obtain a similar factorization for the partition function of the ferromagnetic chains:
\[ Z_F(T; m) = Z_F(T; 2) \cdot Z_F(A)(T; m/2), \quad m \in 2\mathbb{N}. \] (36)

Thus, for even \( m \) the PF chains of \( BC_N \) type [11] and [31] can be described by an effective model of two simpler noninteracting chains. This remarkable property, which to the best of our knowledge is unique among the class of chains of Haldane–Shastry type, certainly deserves further investigation.

III. SPACINGS DISTRIBUTION AND THE BERRY–TABOR CONJECTURE

For fixed values of the number of particles \( N \) and the internal degrees of freedom \( m \), it is straightforward to obtain the spectrum of the chain (11) by expanding in powers of \( q \) the expressions (25) or (30) for its partition function. In this way, we have been able to compute the spectrum of the latter chain for relatively large values of \( N \) (for instance, up to \( N = 22 \) for \( m = 3 \)). Our calculations conclusively show that the spectrum consists of a set of consecutive integers. For even \( m \), this observation follows immediately from the expression (34)–(35) and the fact that the energies of the PF chain of \( A_N \) type are also consecutive integers. For odd \( m \) we have been unable to deduce this property from Eq. (30) for the partition function, although we have verified it numerically for many different values of \( N \) and \( m \).

Our computations also evidence that for \( N \gg 10 \) the level density (normalized to unity) can be approximated with great accuracy by a normal distribution
\[ g(\mathcal{E}) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(\mathcal{E} - \mu)^2}{2\sigma^2}}, \] (37)

where \( \mu \) and \( \sigma \) are respectively the mean and the variance of the energy. For instance, in Fig. 1 we compare the cumulative level density
\[ F(\mathcal{E}) = m^{-N} \sum_{i \in \mathcal{E}, \xi \in \mathcal{E}} d_i, \]

where \( \mathcal{E}_i \) is the \( i \)-th energy and \( d_i \) its degeneracy, with the cumulative Gaussian density
\[ G(\mathcal{E}) = \int_{-\infty}^{\mathcal{E}} g(\mathcal{E}') \, d\mathcal{E}' = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{\mathcal{E}_i - \mu}{\sqrt{2\sigma}} \right) \right] \] (38)

for \( N = 15 \) and \( m = 2 \). Note, in this respect, that

![FIG. 1: Cumulative distribution functions \( F(\mathcal{E}) \) (at its discontinuity points) and \( G(\mathcal{E}) \) (continuous red line) for \( N = 15 \) and spin 1/2.](image-url)
the approximately Gaussian character of the level density has already been verified for other chains of HS type, like the original Haldane–Shastry chain \[16\], its $su(m|n)$ supersymmetric version \[20\], and the HS spin chain of $BC_N$ type \[18\].

The mean energy $\mu$ and its standard deviation $\sigma^2$, which by the previous discussion characterize the approximate level density of the chain \[1\] for large $N$, can be computed in closed form. Indeed, in Appendix A we show that

\[
\mu = \frac{1}{2} \left(1 + \frac{1}{m}\right) N^2 - \frac{N}{2m} \left(1 + \epsilon p\right),
\]

\[
\sigma^2 = \frac{N}{36} \left(4N^2 + 6N - 1\right) \left(1 - \frac{1}{m^2}\right) + \frac{N \left(1 - p\right)}{4m^2},
\]

where $p \in \{0, 1\}$ is the parity of $m$. Thus, when $N$ tends to infinity $\mu$ and $\sigma^2$ respectively diverge as $N^2$ and $N^3$, as for the original Polychronakos–Frahm chain \[39\]. By contrast, it is known that $\mu \sim N^3$ and $\sigma^2 \sim N^5$ for the trigonometric HS chains of both $AN$ \[16\] and $BC_N$ \[18\] types. It is also interesting to observe that the standard deviation of the energy is independent of $\epsilon$ even for odd $m$, when the spectrum does depend on $\epsilon$ according to the previous section’s results on the partition function.

We have next studied the probability density $p(s)$ of the spacing $s$ between consecutive (unfolded) levels of the chain \[1\]. For many important integrable systems it is known that $p(s)$ is Poissonian \[30, 31\], in agreement with a well-known conjecture of Berry and Tabor \[19\]. On the other hand, it has been recently shown that for the HS chain of $AN$ type \[16\] (and its supersymmetric extension \[20\]) the cumulative density $P(s) = \int_0^s p(x)\,dx$ is well approximated by an empiric law of the form

\[
P(s) = \left(\frac{s}{s_{\text{max}}}\right)^{\alpha} \left[1 - \gamma \left(1 - \frac{s}{s_{\text{max}}}\right)^{\beta}\right],
\]

where $s_{\text{max}}$ is the largest normalized spacing and $\alpha, \beta$ are adjustable parameters in the interval $(0, 1)$. The parameter $\gamma$ is fixed by requiring that the average spacing be equal to 1, with the result

\[
\gamma = \left(\frac{1}{s_{\text{max}}} - \frac{\alpha}{\alpha + 1}\right) / B(\alpha + 1, \beta + 1),
\]

where $B$ is Euler’s Beta function. Thus, the cumulative density of spacings for the HS chains of $AN$ type follows neither Poisson’s nor Wigner’s law

\[
P(s) = 1 - e^{-s^2/4},
\]

characteristic of a chaotic system. Our aim is to ascertain whether the cumulative density of spacings for the PF chain of $BC_N$ type \[1\] resembles that of the $AN$-type HS chain, or is rather Poissonian as expected for a generic integrable model.

In order to compare the spacings distributions of spectra with different level densities, it is necessary to transform the “raw” spectrum by applying what is known as the unfolding mapping \[32\]. This mapping is defined by decomposing the cumulative level density $F(\mathcal{E})$ as the sum of a fluctuating part $F_{\text{fl}}(\mathcal{E})$ and a continuous part $\eta(\mathcal{E})$, which is then used to transform each energy $\mathcal{E}_i$, $i = 1, \ldots, n$, into an unfolded energy $\eta_i = \eta(\mathcal{E}_i)$. In this way one obtains a uniformly distributed spectrum \[\{\eta_i\}_{i=1}^n\], regardless of the initial level density. One finally considers the normalized spacings $s_i = (\eta_{i+1} - \eta_i)/\Delta$, where $\Delta = (\eta_n - \eta_1)/(n - 1)$ is the mean spacing of the unfolded energies, so that \[\{s_i\}_{i=1}^n\] has unit mean.

By the above discussion, in our case we can take the unfolding mapping $\eta(\mathcal{E})$ as the cumulative Gaussian distribution \[38\], with parameters $\mu$ and $\sigma$ respectively given by \[39\] and \[40\]. Just as for the level density, in order to compare the discrete distribution function $p(s)$ with a continuous distribution it is more convenient to work with the cumulative spacings distribution $P(s)$. Our computations show that for a wide range of values of $N, m$ and $\epsilon = \pm 1$, the distribution $P(s)$ is well approximated by the empiric law \[11\] with suitable values of $\alpha$ and $\beta$. For instance, for $N = 20$ and $m = 2$ the largest spacing $s_{\text{max}} = 3.13$, and the least-squares fit parameters $\alpha$ and $\beta$ are respectively 0.29 and 0.24, with a mean square error of 6.0$\times$10$^{-4}$ (see Fig. 2). Thus the PF spin chain of $BC_N$ type behaves in this respect as the HS chain of $AN$ type, and unlike most known integrable systems. In fact, we have also studied the spacings’ distribution of the original ($AN$-type) PF chain, obtaining completely similar results \[35\]. These results (and also those of Ref. \[20\]) suggest that a spacings distribution qualitatively similar to the empiric law \[11\] is characteristic of all spin chains of HS type.

Our next objective is to explain this characteristic behavior of the cumulative spacings distribution $P(s)$ of the chain \[1\] using only a few essential properties of its spectrum. We shall find a simple analytic expression without
any adjustable parameters approximating $P(s)$ even better than the empiric law \[^{[1]}\]. Moreover, we have strong numerical evidence that the new expression also provides a very accurate approximation to the cumulative spacings distribution of the original HS and PF chains.

Consider, to begin with, any spectrum $\mathcal{E}_1 \equiv \mathcal{E}_{\min} < \cdots < \mathcal{E}_n \equiv \mathcal{E}_{\max}$ obeying the following conditions:

(i) The energies are equispaced, i.e., $\mathcal{E}_{i+1} - \mathcal{E}_i = \delta$ for $i = 1, \ldots, n-1$.

(ii) The level density (normalized to unity) is approximately Gaussian, cf. Eq. \[^{[37]}\].

(iii) $\mathcal{E}_n - \mu, \mu - \mathcal{E}_1 \gg \sigma$.

(iv) $\mathcal{E}_1$ and $\mathcal{E}_n$ are approximately symmetric with respect to $\mu$, namely $|\mathcal{E}_1 + \mathcal{E}_n - 2\mu| \ll \mathcal{E}_n - \mathcal{E}_1$.

As discussed above, the spectrum of the chain \[^{[1]}\] satisfies the first condition with $\delta = 1$, while condition (ii) holds for sufficiently large $N$. As to the third condition, from Eqs. \[^{[33]}, [39], \[40], \[31]\] and \[^{[32]}\] it follows that both $(\mathcal{E}_n - \mu)/\sigma$ and $(\mu - \mathcal{E}_1)/\sigma$ grow as $N^{-1/2}$ when $N \to \infty$. The last condition is also satisfied for large $N$, since by the equations just quoted $|\mathcal{E}_1 + \mathcal{E}_n - 2\mu| = O(N)$ while $\mathcal{E}_n - \mathcal{E}_1 = O(N^2)$.

From conditions (i) and (ii) it follows that

$$\eta_{n+1} - \eta_n = G(\mathcal{E}_{n+1}) - G(\mathcal{E}_1) \approx G'(\mathcal{E}_1)\delta = g(\mathcal{E}_1)\delta.$$ 

On the other hand, by condition (iii) we have

$$\eta_n = G(\mathcal{E}_n) \approx 1, \quad \eta_1 = G(\mathcal{E}_1) \approx 0,$$

so that $\Delta = 1/(n-1)$. Thus

$$s_i = \frac{\eta_{i+1} - \eta_i}{\Delta} \approx W g(\mathcal{E}_i) = \frac{W}{\sqrt{2\pi}\sigma} e^{-\frac{(\mathcal{E}_i - \mu)^2}{2\sigma^2}}, \quad (43)$$

where

$$W \equiv (n-1)\delta = \mathcal{E}_n - \mathcal{E}_1 \quad (44)$$

on account of the first condition. The cumulative probability density $P(s)$ is by definition the quotient of the number of normalized spacings $s_i \leq s$ by the total number of spacings, that is,

$$P(s) = \frac{\#(s_i \leq s)}{n-1}. \quad (45)$$

By Eq. \[^{[43]}\],

$$\#(s_i \leq s) = \#(\mathcal{E}_1 \leq \mathcal{E}_i \leq \mathcal{E}_-) + \#(\mathcal{E}_+ \leq \mathcal{E}_i < \mathcal{E}_n), \quad (46)$$

where

$$\mathcal{E}_\pm = \mu \pm \sqrt{2}\sigma \sqrt{\log \left( \frac{s_{\max}}{s} \right)} \quad (46)$$

are the roots of the equation $s = W g(\mathcal{E})$ expressed in terms of the maximum normalized spacing

$$s_{\max} = \frac{W}{\sqrt{2\pi}\sigma}. \quad (47)$$

Using the first condition to estimate the RHS of Eq. \[^{[45]}\] we easily obtain

$$P(s) \approx \frac{1}{W} \left[ \max(\mathcal{E}_- - \mathcal{E}_1, 0) + \max(\mathcal{E}_n - \mathcal{E}_+, 0) \right]. \quad (48)$$

In fact, we can replace the latter approximation to $P(s)$ by the simpler one

$$P(s) \approx \frac{1}{W} (\mathcal{E}_- - \mathcal{E}_1 + \mathcal{E}_n - \mathcal{E}_+), \quad (49)$$

since the error involved is bounded by

$$\frac{1}{W} |\mathcal{E}_1 + \mathcal{E}_n - 2\mu|,$$

which is vanishingly small by condition (iv). Substituting the explicit expression \[^{[46]}\] for $\mathcal{E}_\pm$ into Eq. \[^{[45]}\] and using \[^{[44]}\] and \[^{[47]}\] we finally obtain

$$P(s) \approx 1 - \frac{2}{\sqrt{2\pi} s_{\max}} \sqrt{\log \left( \frac{s_{\max}}{s} \right)}. \quad (50)$$

The RHS of this remarkable expression depends only on the quantity $s_{\max}$, which for the PF chain of $BC_\nu$ type is completely determined as a function of $N$ and $m$ by Eqs. \[^{[35]}, [34], \[47]\] and \[^{[31]-[32]}\]. In particular, for large $N$ we have the asymptotic expression

$$s_{\max} = \frac{3}{\sqrt{2\pi}} \sqrt{\frac{m-1}{m+1}} N^{1/2} + O(N^{-1/2}). \quad (51)$$

Our numerical computations indicate that Eq. \[^{[50]}\] is in excellent agreement with the data for a broad range of values of $N$, $m$ and $\epsilon = \pm 1$, providing much greater accuracy than the empiric formula \[^{[1]}\]. For instance, for $N = 20$ and $m = 2$ we have $s_{\max} = 6\sqrt{35}/(41\pi) \simeq 3.12765$, which differs from the numerically computed maximum spacing by $4.5 \times 10^{-4}$. In Fig. \[^{[3]}\] we compare the corresponding cumulative spacings distribution with its approximation \[^{[50]}\] using the above value of $s_{\max}$. The mean square error is in this case $2.2 \times 10^{-5}$, smaller than that of the empiric law \[^{[1]}\] by more than an order of magnitude.

A natural question in view of these results is to what extent the approximation \[^{[50]}\] is applicable to other spin chains of HS type. For the PF chain of $A_N$ type, one can check that the spectrum satisfies conditions (i)–(iv) of this section, and in fact we have verified that \[^{[50]}\] holds with remarkable accuracy in this case \[^{[35]}\]. The situation is less clear for the original HS chain, whose spectrum is certainly not equispaced \[^{[1]}\]. Nevertheless, our computations show that the formula \[^{[50]}\] still fits the numerical data much better than our previous approximation \[^{[1]}\], a fact clearly deserving further study. As an illustration, in Fig. \[^{[4]}\] we compare the cumulative spacings distribution of the original HS chain with its approximations \[^{[1]}\] and \[^{[50]}\] for $N = 25$ and $m = 2$. It is apparent that the new expression \[^{[50]}\] provides a more accurate approximation to the numerical data than the empiric formula \[^{[1]}\] (their respective mean square errors are $1.8 \times 10^{-5}$ and $5.8 \times 10^{-4}$).
of the Laguerre polynomial $L_{N}^{\beta - 1}$ as follows:

$$\sum_{i \neq j} (h_{ij} + \tilde{h}_{ij}) = \sum_{i \neq j} \frac{2y_{i}}{(y_{i} - y_{j})^{2}}, \quad \sum_{i} h_{i} = \frac{\beta}{2} \sum_{i} \frac{1}{y_{i}}.$$  

(A2)

The latter sums can be easily computed using the following identities satisfied by the zeros $y_{i}$, which can be found in Ref. [33]:

$$\sum_{j, j \neq i} \frac{2y_{i}}{y_{i} - y_{j}} = y_{i} - \beta,$$  

(A3)

$$\sum_{j, j \neq i} \frac{12y_{i}^{2}}{(y_{i} - y_{j})^{2}} = -y_{i}^{2} + 2(2N + \beta)y_{i} - \beta(\beta + 4).$$  

(A4)

Indeed, from the first of these identities we easily obtain

$$\sum_{i} y_{i} = N(N + \beta - 1), \quad \sum_{i} \frac{1}{y_{i}} = \frac{N}{\beta},$$  

(A5)

so that, by Eq. (A4),

$$\sum_{i \neq j} \frac{2y_{i}}{(y_{i} - y_{j})^{2}} = \frac{1}{6}(\sum_{i} y_{i} + 2N(2N + \beta)) - \beta(\beta + 4) \sum_{i} \frac{1}{y_{i}} = \frac{1}{2} N(N - 1).$$

Combining the last two equations with (A1) and (A2) we immediately arrive at Eq. (39) for the level density $\mu$.

Turning now to the standard deviation of the energy $\sigma^{2}$, from Eqs. (66)–(68) in Ref. [18] we have

$$\sigma^{2} = \frac{\text{tr}\, H_{e}^{2}}{mN} - \mu^{2} = \left(1 - \frac{1}{m^{2}}\right) \left(2 \sum_{i \neq j} (h_{ij}^{2} + \tilde{h}_{ij}^{2}) + \sum_{i} h_{i}^{2}\right) + \frac{4}{m^{2}} (1 - p) \left(\frac{1}{4} \sum_{i} h_{i}^{2} - \sum_{i \neq j} h_{ij} \tilde{h}_{ij}\right).$$  

(A6)

All of the sums appearing in the latter expression can be readily evaluated. Indeed, we have

$$\sum_{i} h_{i}^{2} = \frac{\beta^{2}}{4} \frac{1}{y_{i}^{2}} = \frac{N(N + \beta)}{4(\beta + 1)},$$  

(A7)

$$\sum_{i \neq j} h_{ij} \tilde{h}_{ij} = \frac{1}{4} \sum_{i \neq j} \frac{1}{(y_{i} - y_{j})^{2}} = \frac{N(N - 1)}{16(\beta + 1)},$$  

(A8)

where we have used Eqs. (15) and (17) from Ref. [33]. On the other hand,

$$\sum_{i \neq j} (h_{ij}^{2} + \tilde{h}_{ij}^{2}) = \frac{1}{2} \sum_{i \neq j} \frac{y_{i}^{2} + y_{j}^{2} + 6y_{i}y_{j}}{(y_{i} - y_{j})^{4}}$$

$$= 2 \sum_{i \neq j} \frac{y_{i}^{2} + y_{j}^{2}}{(y_{i} - y_{j})^{2}} - \frac{3}{2} \sum_{i \neq j} \frac{1}{(y_{i} - y_{j})^{2}}$$

$$= 4 \sum_{i \neq j} \frac{y_{i}^{2}}{(y_{i} - y_{j})^{4}} - \frac{3N(N - 1)}{8(\beta + 1)}.$$  

(A9)
while from [34, Thm. 5.1] it follows that

\[
720 \sum_{i \neq j} \frac{y_i^2}{(y_i - y_j)^4} = \sum_i y_i^2 - 4(2N + \beta) \sum_i y_i \\
+ 2N(8N^2 + 2(4N - 1)\beta + 3\beta^2) \\
- 4(2N + \beta)(\beta^2 - 2\beta - 18) \sum_i \frac{1}{y_i} \\
+ \beta(\beta^3 - 4\beta^2 - 104\beta - 144) \sum_i \frac{1}{y_i^4}.
\] (A10)

All the sums in the right-hand side of the latter expression have already been evaluated in Eqs. (A5) and (A7), except the first one. In order to compute this sum, we multiply Eq. (A5) by \( y_i \) and sum over \( i \), obtaining

\[
\sum_{i \neq j} \frac{2y_i^2}{y_i - y_j} = \sum_{i \neq j} \frac{y_i^2 - y_j^2}{y_i - y_j} = \sum_i (y_i + y_j) \\
= 2(N - 1) \sum_i y_i = \sum_i y_i^2 - \beta \sum_i y_i,
\]

and hence, by Eq. (A5),

\[
\sum_i y_i^2 = N(N + \beta - 1)(2N + \beta - 2).
\] (A11)

Substituting the value of the latter sum in Eq. (A10) and using (A5) and (A7), we obtain

\[
\sum_{i \neq j} \frac{y_i^2}{(y_i - y_j)^4} = \frac{N(N - 1)((2N + 5)\beta + 2N + 14)}{144(\beta + 1)}.
\] (A12)

Equation (A10) now follows by inserting (A7), (A8) and (A9)-(A12) into Eq. (A6).

APPENDIX B: COMPUTATION OF THE MINIMUM ENERGY

In this appendix we shall obtain an explicit expression for the minimum energy \( \mathcal{E}_{\text{min}}^c \) of the spin chain (1). Our starting point is Eq. (7), which implies that \( \mathcal{E}_{\text{min}}^c \) is given by

\[
\mathcal{E}_{\text{min}}^c = \lim_{\alpha \to \infty} \frac{1}{\alpha} (\mathcal{E}_{\text{min}}^c - \mathcal{E}_{\text{min}}^c),
\]

in terms of the minimum energies \( \mathcal{E}_{\text{min}}^c \) and \( \mathcal{E}_{\text{min}}^c \) of the scalar and spin dynamical models (3) and (5), respectively. By the discussion in Section 11 (cf. Eq. (19)), \( \mathcal{E}_{\text{min}}^c \) is the minimum value of \( |n| \), where \( n \) is any multiindex compatible with conditions (i)–(iii) in the latter section. From these conditions it follows that the multiindex \( n \) minimizing \( |n| \) is given by

\[
n = (l, \ldots, k, k - 1, \ldots, k - 1, \ldots, 0, \ldots, 0),
\]

where \( 0 \leq l < l_k \) and \( l_i \) is given by

\[
l_i = \begin{cases} 
\frac{m}{2}, & \text{m even} \\
\frac{m - \epsilon}{2}, & \text{m odd and } i \\
\frac{m - \epsilon}{2}, & \text{m and } i \text{ odd}.
\end{cases}
\]

In view of the above expression, it is convenient to treat separately the cases of even and odd \( m \). For even \( m \), we have \( l_i = \frac{m}{2} \) for all \( i \), so that

\[
k = \lfloor 2N \big/ m \rfloor, \quad l = N \mod \frac{m}{2},
\]

and

\[
\mathcal{E}_{\text{min}}^c = \frac{m}{2} \sum_{i=1}^{k-1} i + lk = \frac{m}{4} k(k - 1) + lk \\
= \frac{N^2}{m} - \frac{N}{2} + \frac{l(m - 2l)}{2m} \quad \text{(m even)}.
\] (B1)

Suppose now that \( m \) is odd. If \( k = 2j + 1 \) is an even number, then \( l_0 = l_2 = \cdots = l_{2j-2} = \frac{m + \epsilon}{2}, l_1 = l_3 = \cdots = l_{2j-1} = \frac{m - \epsilon}{2} \) and thus \( N = jm + l \), so that

\[
j = \lfloor N \big/ m \rfloor, \quad l = (N \mod m) < l_{2j} = \frac{m + \epsilon}{2}.
\]

The minimum energy in this case is thus given by

\[
\mathcal{E}_{\text{min}}^c = \frac{m + \epsilon}{2} \sum_{i=0}^{j-1} i + \frac{m - \epsilon}{2} \sum_{i=0}^{j-1} (2i + 1) + 2jl \\
= mj(j - 1) + j \frac{m - \epsilon}{2} + 2jl \\
= \frac{N^2}{m} - \frac{N(m + \epsilon)}{2m} + \frac{l(m + \epsilon - 2l)}{2m}.
\]

On the other hand, if \( k = 2j + 1 \) is odd then \( l_0 = l_2 = \cdots = l_{2j} = \frac{m + \epsilon}{2}, l_1 = l_3 = \cdots = l_{2j-1} = \frac{m - \epsilon}{2} \) and thus \( N = jm + l + \frac{m + \epsilon}{2} \), with \( 0 \leq l < l_{2j+1} = \frac{m + \epsilon}{2} \). Calling \( l' = l + \frac{m + \epsilon}{2} \) we have

\[
j = \lfloor N \big/ m \rfloor, \quad l' = (N \mod m) \geq \frac{m + \epsilon}{2},
\]

and

\[
\mathcal{E}_{\text{min}}^c = \frac{m + \epsilon}{2} \sum_{i=0}^{j} 2i + \frac{m - \epsilon}{2} \sum_{i=0}^{j-1} (2i + 1) \\
+ (2j + 1) \left( l' - \frac{m + \epsilon}{2} \right) \\
= mj(j - 1) + j \frac{m - \epsilon}{2} + 2jl' + l' - \frac{m + \epsilon}{2} \\
= \frac{N^2}{m} - \frac{N(m + \epsilon)}{2m} + \frac{(l - m)(m + \epsilon - 2l)}{2m}.
\]

Hence we can express the minimum energy for odd \( m \) in a unified way as

\[
\mathcal{E}_{\text{min}}^c = \frac{N^2}{m} - \frac{N(m + \epsilon)}{2m} + \frac{1}{2m} \left( l - m \theta(2l - m - \epsilon) \right) \\
\times (m + \epsilon - 2l) \quad \text{(m odd)},
\] (B2)
where

\[ l = N \mod m \]

and \( \theta(x) = 1 \) for \( x \geq 0 \) and 0 otherwise.

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