Abstract

Quantum span program algorithms for function evaluation commonly have reduced query complexity when promised that the input has a certain structure. We design a modified span program algorithm to show these speed-ups persist even without having a promise ahead of time, and we extend this approach to the more general problem of state conversion. For example, there is a span program algorithm that decides whether two vertices are connected in an $n$-vertex graph with $O(n^{3/2})$ queries in general, but with $O(\sqrt{kn})$ queries if promised that, if there is a path, there is one with at most $k$ edges. Our algorithm uses $\tilde{O}(\sqrt{kn})$ queries to solve this problem if there is a path with at most $k$ edges, without knowing $k$ ahead of time.

1 Introduction

Quantum algorithms often yield speed-ups when given a promise on the input. For example, if we know that there are $M$ marked items out of $N$, or no marked items at all, then Grover’s search can be run in time $O(\sqrt{N/M})$, rather than $O(\sqrt{N})$, the worst case runtime with a single marked item [1].

In the case of Grover’s algorithm, a series of results [7–9, 25] removed the promise: if there are $M$ marked items, there is a quantum search algorithm that runs in $O(\sqrt{N/M})$ time, even without knowing the number of marked items ahead of time. Most relevant for our work, several of these algorithms involve iteratively running Grover’s search with exponentially growing runtimes [7, 9] until a marked item is found.

Grover’s algorithm was one of the first quantum query algorithms discovered [14]. Since that time, a much more general framework for quantum query algorithms has been laid out. Span programs, and more generally, the dual of the general adversary bound and the filtered $\gamma_2$ norm, are frameworks for creating optimal query algorithms for function decision problems [23, 24] and nearly optimal algorithms for state conversion problems, in which the goal is to generate a quantum state based on an oracle and an input state [20]. Moreover, these frameworks are also useful in practice [3–6, 10, 12].

For many span program algorithms, analogous to multiple marked items in Grover’s search, there are features which, if promised they exist, allow for improvement over the worst case query complexity. For example, the span program version of Grover’s algorithm also runs faster with the promise of multiple marked items. Another well-studied example is the span program algorithm for $st$-connectivity. This algorithm uses $O(n^{3/2})$ queries on an $n$-vertex graph. However, if promised ahead of time that if there is a path, it has length at most $k$, then the problem can be solved with $O(\sqrt{kn})$ queries [4].

Our contribution is to speed up the generic span program and state conversion algorithms in the case that some speed-up inducing structure (such as multiple marked items, or a short path) is present, even without the promise on the structure ahead of time. One might expect that it is
trivial to remove the promise. Surely if an algorithm produces a correct result in shorter time when promised a property is present, then it should also produce a correct result in a shorter time without the promise if the property still holds? While this is true and these algorithms always output a result, even if run for a short time, the problem is that they don’t produce a flag of completion and their output can not be easily verified. Without a flag of completion or a promise of structure, it is impossible to be confident that the result is correct. Span program and state conversion algorithms differ from Grover’s algorithm in their lack of a flag; in Grover’s algorithm one can test with a single query whether the output is a marked item, thus flagging that the output of the algorithm is correct, and that the algorithm has run for a sufficiently long time. We note that when span program algorithms previously have claimed a speed-up with structure, they always included a promise, or they give the disclaimer that running the algorithm will give an incorrect result with high probability if the promise is not known ahead of time to be satisfied, e.g. [10, App. C.3].

To overcome this challenge, we use an approach that is similar to the iterative modifications to Grover’s algorithm; we run subroutines for exponentially increasing times, and we have novel ways to flag when the computation should halt. Our algorithms match the performance of algorithms with an optimal promise, up to logarithmic factors.

Based on prior work showing the presence of a speed-up for span program algorithms with a promise, our work immediately provides analogous speed-ups without the promise:

- For undirected st-connectivity described above, our algorithm determines whether there is a path from s to t in an n-vertex graph with $\tilde{O}(\sqrt{kn})$ queries if there is a path of length $k$, and if there is no path, the algorithm uses $O(\sqrt{n}c)$ queries, where $c$ is the size of the smallest cut between $s$ and $t$. In either case, $k$ and $c$ need not be known ahead of time.

- For an n-vertex undirected graph, we can determine if it is connected in $\tilde{O}(n\sqrt{R})$ queries, where $R$ is the average effective resistance and we can determine the graph is not connected in $\tilde{O}(\sqrt{n^3/\kappa})$ queries, where $\kappa$ is the number of components. These query complexities hold without knowing $R$ or $\kappa$ ahead of time. See Ref. [17] for the promise version of this problem.

- For cycle detection on an n-vertex undirected graph, whose promise version was analyzed in Ref. [12], if the circuit rank is $C$, then our algorithm will decide there is a cycle in $\tilde{O}(\sqrt{n^3/C})$ queries, while if there is no cycle and at most $\mu$ edges, the algorithm will decide there is no cycle in $\tilde{O}(\mu\sqrt{n})$ queries. This result holds without knowing $C$ or $\mu$ ahead of time.

- We can decide the winner of the NAND-tree [13] and similar games, whose promise version was analyzed in [18, 26], in $\tilde{O}(b^k)$ queries for a constant $b$ that depends on the details of the game, if the game satisfies the $k$-fault condition, a bound on the number of consequential decisions a player must make through the course of the game. The parameter $k$ need not be known ahead of time.

### 1.1 Directions for Future Work

In the original state conversion algorithm, to achieve an error of $\varepsilon$ in the output state (by some metric), the query complexity scales as $O(\varepsilon^{-2})$. However, in our result, the query complexity scales as $O(\varepsilon^{-5})$. In cases where accuracy must scale with the input size, this error term could overwhelm any advantage from our approach, and so it would be beneficial to improve this error scaling.

By iterating with exponentially increasing runtimes, we incur a logarithmic factor to manage errors. However, the fixed-point method for Grover’s algorithm [25] avoids this overhead. Perhaps fixed-point techniques could improve the performance of our algorithms.

Our result is similar to that of Ito and Jeffery [16], which estimates structure size (e.g. the number of marked items in the case of search) with queries proportional to the square root of the structure size. While there are similarities between our approaches, neither result seems to directly imply the other. Better understanding the relationship between these strategies could
lead to improved algorithms for determining properties of input structure for both span programs and state generation problems.

Finally, these algorithms would likely provide opportunities for proving significant quantum to classical speed-ups in average-case query complexity. Exponential and super-polynomial speed-ups for average case query complexity have been shown when the quantum algorithm can alert the user that the algorithm has terminated early [2, 22]. Our work significantly expands the set of algorithms for which such an early termination flag exists.

2 Preliminaries

Basic Notation: Let \([n]\) represent \(\{1, 2, \ldots, n\}\), and \(\log\) denotes base 2 logarithm. We denote a linear operator from the space \(V\) to the space \(U\) as \(L(V, U)\). We use \(I\) for the identity operator. (It will be clear from context which space \(I\) acts on.) Given a projection \(\Pi\), its complement is \(\bar{\Pi} = (I - \Pi)\). For a matrix \(M\), by \(M_{xy}\) or \((M)_{xy}\), we denote the element in the \(x\)th row and \(y\)th column of \(M\). By \(\hat{O}\), we denote big-O notation that ignores log factors. The \(l_2\)-norm of a vector \(|v\rangle\) is denoted by \(|||v||\rangle\). For any unitary \(U\), let \(P_\Theta(U)\) be the projection onto the eigenvectors of \(U\) with phase at most \(\Theta\). That is, \(P_\Theta(U)\) is the projection onto \(\text{span}\{|u\rangle : \langle u|U|u\rangle = e^{i\Theta}|u\rangle\text{ with }|\Theta| \leq \Theta\}\).

2.1 Quantum Algorithmic Building Blocks

In this paper, we consider quantum query algorithms, in which one has access to a unitary \(O_x\), called the oracle, which encodes a string \(x \in X\) for \(X \subseteq [q]^n\) and some parameter \(q \geq 2\). The oracle \(O_x\) acts on the Hilbert space \(\mathbb{C}^n \otimes \mathbb{C}^q\) as \(O_x|i\rangle|b\rangle = |i\rangle|x_i + b \text{ mod } q\rangle\), where \(x_i \in [q]\) is the \(i\)th element of \(x\).

Given \(O_x\), and with no knowledge of \(x\) ahead of time, except that \(x \in X\), we would like to perform a computation that depends on \(x\). The query complexity is the minimum number of uses of the oracle required such that for all \(x \in X\), the computation is successful with some desired probability of success.

Several of our key algorithmic subroutines are based around a parallelized version of phase estimation, as described in Ref. [21]. The idea of this parallel phase estimation algorithm is as follows: for a unitary \(U\), a precision \(\Theta > 0\), and an accuracy \(\epsilon > 0\), the circuit \(D(U)\) implements \(O(\log \frac{1}{\epsilon})\) copies of the phase estimation circuit on \(U\), each to precision \(O(\epsilon)\), that all measure the phase of a single state on the same input register. If \(U\) acts on a Hilbert Space \(\mathcal{H}\), then \(D(U)\) acts on the space \(\mathcal{H}_A \otimes (\mathbb{C}^2)^\otimes B\) for \(b = O(\log \frac{1}{\epsilon} \log \frac{1}{\Theta})\), where we’ve used \(A\) to label the register that stores the input state, and \(B\) to label the registers that store the results of the parallel phase estimations. \(D(U)\) is uniformly constructed in \(\epsilon\) and \(\Theta\), and its structure is independent of \(U\).

The circuit \(D(U)\) can be used for Phase Checking: checking if a state \(|\psi\rangle\) is close to an eigenvector of \(U\) that has eigenvalue close to 1. This is related to the probability of outcome \(|0\rangle_B\) after \(D(U)\) is applied to the state \(|\psi\rangle_A|0\rangle_B\). To characterize this probability, we define \(\Pi_0(U)\) to be the orthogonal projection onto the subspace of \(\mathcal{H}_A \otimes (\mathbb{C}^2)^\otimes B\) that \(D(U)\) maps to states with \(|0\rangle_B\) in the \(B\) register. That is, \(\Pi_0(U) = D(U)\dagger(I_A \otimes |0\rangle\langle 0|_B)D(U)\). (Since \(\Pi_0(U)\) depends on the choice of \(\Theta\) and \(\epsilon\) used in \(D(U)\), those values must be specified, if not clear from context, when discussing \(\Pi_0(U)\).) We now summarize relevant prior results for Phase Checking (see [11, 19, 21] for detailed analysis) in Lemma 1:

**Lemma 1** (Phase Checking). Let \(U\) be a unitary on a Hilbert Space \(\mathcal{H}\), and let \(\Theta, \epsilon > 0\). We call \(\Theta\) the precision and \(\epsilon\) the accuracy. Then there is a circuit \(D(U)\) that acts on the space \(\mathcal{H}_A \otimes (\mathbb{C}^2)^\otimes B\) for \(b = O(\log \frac{1}{\epsilon} \log \frac{1}{\Theta})\), and that uses \(O\left(\frac{1}{\Theta} \log \frac{1}{\epsilon}\right)\) calls to control-\(U\). Then for any state \(|\psi\rangle \in \mathcal{H}\)

- \(\|P_0(U)|\psi\rangle\|^2 \leq \|\Pi_0(U)|\langle \psi|_A|0\rangle_B\|^2 \leq \|P\bar{\Theta}(U)|\psi\rangle\|^2 + \epsilon\), and
- \(\|\Pi_0(U) (P\bar{\Theta}(U)|\psi\rangle)_A|0\rangle_B\|^2 \leq \epsilon\).
It is possible to modify the Phase Checking circuit by implementing $D(U)$ as described above, applying a $-1$ phase to the $A$ register if the $B$ register is not in the state $|0\rangle_B$, and then implementing $D(U)^\dagger$. We call this circuit Phase Reflection$^1$ and denote it as $R(U)$. Note that $R(U) = \Pi_0(U) - \Pi_0(U)$, where $R(U)$ and $\Pi_0(U)$ have the same implicit precision $\Theta$ and accuracy $\epsilon$. The following lemma summarizes prior results on relevant properties of Phase Reflection.

**Lemma 2** (Phase Reflection [20, 21]). Let $U$ be a unitary on a Hilbert Space $\mathcal{H}$, and let $\Theta, \epsilon > 0$. We call $\Theta$ the precision and $\epsilon$ the accuracy. Then there is a circuit $R(U)$ that acts on the space $\mathcal{H}_A \otimes (\mathbb{C}^2)^{\otimes b}$ for $b = O(\log(1/\Theta)\log(1/\epsilon))$, and that uses $O\left(\frac{1}{\Theta} \log \frac{1}{\epsilon}\right)$ calls to control-$U$ and control-$U^\dagger$ such that

- $R(U) (|0\rangle_A |\psi\rangle_B) = (|0\rangle_A (|\psi\rangle_B + |\psi^\perp\rangle_B))$,
- $\|R(U) + I)(|\psi\rangle_B)\|_A |0\rangle_B \leq \epsilon$.

Furthermore, $R(U)$ is uniformly constructed in $\epsilon$ and $\Theta$, with structure independent of $U$.

Finally, we will use Amplitude Estimation [9] and the effective spectral gap lemma [20]:

**Lemma 3** (Amplitude Estimation [9]). Let $\delta > 0$, and let $A$ be a quantum circuit such that $A(|\psi\rangle) = \alpha_0 |\psi_0\rangle + \alpha_1 |\psi_1\rangle$. Then there is an algorithm that estimates $|\alpha_0|^2$ to additive error $\delta$ with success probability at least $1 - p$ using $O\left(\frac{1}{\delta^2p}\right)$ calls to $A$ and $A^\dagger$.

**Lemma 4** (Effective spectral gap lemma, [20]). Let $\Pi$ and $\Lambda$ be projections, and let $U = (2\Pi - I)/(2\Lambda - I)$ be the unitary that is the product of their associated reflections. If $\Lambda|w\rangle = 0$, then $\|P_\Theta(U)\Pi|w\rangle\| \leq \frac{2}{3}\|w\|$. 

### 2.2 Span Programs

**Definition 5** (Span Program). A span program is a tuple $P = (H, V, \tau, A)$ on $[q]^n$ where

1. $H$ is a direct sum of finite-dimensional inner product spaces: $H = H_1 \oplus H_2 \oplus \cdots \oplus H_n \oplus H_{\text{true}} \oplus H_{\text{false}}$, and for $j \in [n]$ and $a \in [q]$, we have $H_{j,a} \subseteq H_j$, such that $\sum_{a=1}^{q} H_{j,a} = H_j$.
2. $V$ is a vector space
3. $\tau \in V$ is a target vector, and
4. $A \in \mathcal{L}(H, V)$.

Given a string $x \in [q]^n$, we use $H(x)$ to denote the subspace $H_{1,x_1} \oplus \cdots \oplus H_{n,x_n} \oplus H_{\text{true}}$, and we denote by $\Pi_{H(x)}$ the orthogonal projection onto the space $H(x)$.

We use Definition 5 for span programs because it applies to both binary and non-binary inputs ($q \geq 2$). The definitions in Refs. [4, 10] only apply to binary inputs ($q = 2$).

**Definition 6** (Positive Witness). Given a span program $P = (H, V, \tau, A)$ on $[q]^n$ and $x \in [q]^m$, then $|w\rangle \in H(x)$ is a positive witness for $x$ in $P$ if $A|w\rangle = \tau$. If a positive witness exists for $x$, we define the witness size of $x$ in $P$ as

$$w(P, x) := \min \left\{ \|w\|^2 : |w\rangle \in H(x) \text{ and } A|w\rangle = \tau \right\}.$$  \hspace{1cm} (1)

We say that $|w\rangle \in H(x)$ is an optimal witness for $x$ if $\|w\|^2 = w(P, x)$ and $A|w\rangle = \tau$.

**Definition 7** (Negative Witness). Given a span program $P = (H, V, \tau, A)$ on $[q]^m$ and $x \in [q]^m$, then $\omega \in \mathcal{L}(V, \mathbb{R})$ is a negative witness for $x$ in $P$ if $\omega \tau = 1$ and $\omega A \Pi_{H(x)} = 0$. If a negative witness exists for $x$, we define the witness size of $x$ in $P$ as

$$w(P, x) := \min \left\{ \|\omega A\|^2 : \omega \in \mathcal{L}(V, \mathbb{R}), \omega A \Pi_{H(x)} = 0, \text{ and } \omega \tau = 1 \right\}.$$  \hspace{1cm} (2)

We say that $\omega$ is an optimal witness for $x$ if $\|\omega A\|^2 = w(P, x)$, $\omega A \Pi_{H(x)} = 0$, and $\omega \tau = 1$.

$^1$In Ref. [20], this procedure is referred to as “Phase Detection,” but since no measurement is made, and rather only a reflection is applied, we thought renaming this protocol as “Phase Reflection” would be more descriptive and easier to distinguish from “Phase Checking.” We apologize for any confusion this may cause when comparing to prior work!
Each $x \in [q]^m$ has a positive or negative witness (but not both), so $w(P, x)$ is well defined.

We say that a span program $P$ decides the function $f : X \subseteq [q]^n \rightarrow \{0, 1\}$ if each $x \in f^{-1}(1)$ has a positive witness in $P$, and each $x \in f^{-1}(0)$ has a negative witness in $P$. Then denote $W_+(P, f) = \max_{x \in f^{-1}(1)} w(P, x)$ and let $W_-(P, f) = \max_{x \in f^{-1}(0)} w(P, x)$.

Given a description of a span program that decides a function, one can use it to design a quantum query algorithm that evaluates the same function. The query complexity of the quantum algorithm depends on $W_+(P, f)$ and $W_-(P, f)$:

**Theorem 8** ([16, 23]). For $X \subseteq [q]^n$ and $f : X \rightarrow \{0, 1\}$, let $P$ be a span program that decides $f$. Then there is a quantum algorithm that for any $x \in X$, evaluates $f(x)$ with bounded error, and uses $O\left(\sqrt{W_+(P, f)W_-(P, f)}\right)$ queries to the oracle $O_x$.

Not only can any span program that decides a function $f$ be used to create a quantum query algorithm that decides $f$, but there is always a span program that creates an algorithm with asymptotically optimal query complexity [23, 24]. Thus when designing quantum query algorithms for function decision problems, it is sufficient to consider only span programs.

As noted in Ref. [16], one can scale and normalize a span program $P$ to create a new span program $P'$ such that:

- All positive and negative witnesses of $P'$ have value at least 1.
- $|W_+(P', f) - W_-(P', f)| \leq 1$.
- $W_+(P', f)W_-(P', f) = O(W_+(P, f)W_-(P, f))$

The first point is achieved by scaling the target vector, see [16, Definition 2.13], and the next points are achieved by applying [16, Theorem 2.14] with $\beta = (W_+(P, f)W_-(P, f))^{1/4}$. The final point ensures that the query complexity of the algorithm produced by the scaled and normalized span program is the same as the original span program.

Thus without loss of generality, we henceforward assume our span programs are scaled and normalized such that the maximum positive and negative witnesses have the same size, which we denote $W(P, f)$. When clear from context, we will drop the input parameters and refer to $W(P, f)$ as simply $W$.

Given this renormalization, we can restate Theorem 8 in a way that is more conducive for comparison with our results:

**Corollary 9.** For $X \subseteq [q]^n$, let $P$ be a (scaled and normalized) span program that decides $f : X \rightarrow \{0, 1\}$ with witness size $W = \max_{x \in X} w(P, x)$. Then there is a bounded error quantum algorithm that for every input $x \in X$ evaluates $f(x)$ and uses $O(W)$ queries to the oracle $O_x$.

We will also find it helpful to use a transformation that takes a span program $P$ that decides a function $f : X \rightarrow \{0, 1\}$ and creates a span program $P^\dagger$ that decides $\neg f$, the negation of $f$, without increasing witness sizes. While such a transformation is known for Boolean span programs [23], we show it exists for the span programs of Definition 5. (The proof can be found in Appendix A.)

**Lemma 10.** Given a span program $P = (H, V, \tau, A)$ on $[q]^m$ that decides a function $f : X \rightarrow \{0, 1\}$ for $X \subseteq [q]^m$, there exists a span program $P^\dagger = (H', V', \tau', A')$ such that $\forall x \in X$, $w(P, x) \geq w(P^\dagger, x)$, and $P^\dagger$ decides $\neg f$.

### 2.3 State Conversion and Filtered $\gamma_2$ Norm

In the state conversion problem, for $X \subseteq [q]^n$, we are given descriptions of sets of pure states $\{|\rho_x\rangle\}_{x \in X}$ and $\{|\sigma_x\rangle\}_{x \in X}$. (Moving forward, we will simply write $\{\rho_x\}$ and $\{\sigma_x\}$.) Then given access to an oracle for $x$, and the quantum state $|\rho_x\rangle$, the goal is to create a state $|\sigma'_x\rangle$ such that $\||\sigma'_x\rangle - |\sigma_x\rangle\|_2 \leq \varepsilon$. We call $\varepsilon$ the error of the state conversion procedure\(^2\).

\(^2\)We only consider what in Ref. [20] is called the coherent state conversion problem. We are concerned with algorithms rather than lower bounds, and the algorithm we describe also applies to the less restrictive non-coherent problem.
The filtered $\gamma_2$ norm is used in designing quantum algorithms for state conversion:

**Definition 11 (Filtered $\gamma_2$ norm).** Let $B$ be a matrix whose rows and columns are indexed by the elements of a set $X$. Let $Z = \{Z_1, \ldots, Z_m\}$ be a set of matrices whose rows and columns are indexed by the elements of $X$. Define $\gamma_2(B|Z)$ as

$$\gamma_2(B|Z) = \min_{m \in \mathbb{N}} \max \left\{ \max_{x \in X} \sum_j ||u_{xj}||^2, \max_{x \in X} \sum_j ||v_{yj}||^2 \right\}$$

$$\text{s.t. } \forall x, y \in X, B_{xy} = \sum_{j=1}^n (Z_j)_{xy} \langle u_{xj}, v_{yj} \rangle.$$ (3)

Let $\rho$ and $\sigma$ be the Gram matrices of the sets $\{|\rho_x\rangle\}$ and $\{|\sigma_x\rangle\}$, respectively. In other words, $\rho$ and $\sigma$ are matrices whose rows and columns are indexed by the elements of $X$ such that $\rho_{xy} = \langle \rho_x|\rho_y \rangle$, and $\sigma_{xy} = \langle \sigma_x|\sigma_y \rangle$. Let $\Delta = \{\Delta_1, \ldots, \Delta_n\}$ be a set of matrices whose rows and columns are indexed by the elements of $X$ such that the element in the $x$th row and $y$th column of $\Delta_j$ is 1 if the $j$th element of $x$ and $y$ differ, and 0 if they are the same: $(\Delta_j)_{xy} = 1 - \delta_{x,y}$.

Then the query complexity of state conversion is characterized as follows:

**Theorem 12 ([20]).** Given $X \subseteq [q]^n$, and sets of states $\{|\rho_x\rangle\}$ and $\{|\sigma_x\rangle\}$ for each $x \in X$, with respective Gram matrices $\rho$ and $\sigma$, the query complexity of state conversion with error $\varepsilon$ is $O\left(\gamma_2(\rho - \sigma|\Delta) \log(1/\varepsilon) \right)$.

Any set of vectors that satisfies the constraints of Eq. (4) for $\gamma_2(\rho - \sigma|\Delta)$ can be used to create an algorithm to solve the state conversion problem from $\{|\rho_x\rangle\}$ and $\{|\sigma_x\rangle\}$, although it will not necessarily be optimal in terms of query complexity; its query complexity will depend on the value of the objective function in Eq. (3) for that set of vectors [20]. We call such a set of vectors a *converting vector set* from $\rho$ to $\sigma$.

**Definition 13 (Converting vector set).** We say a set of vectors $P = \{|u_{xj}\rangle, |v_{yj}\rangle\}$ converts $\rho$ to $\sigma$ if it satisfies the constraints of Eq. (4) for $B = \rho - \sigma$, and $Z = \Delta$. We call such a $P$ a converting vector set from $\rho$ to $\sigma$.

Note that for a converting vector set $P = \{|u_{xj}\rangle, |v_{yj}\rangle\}$ from $\rho$ to $\sigma$, given the constraints of Eq. (4) with $Z_j$ set to $\Delta_j$, we have that $\forall x, y \in X$,

$$\rho_{xy} - \sigma_{xy} = \sum_{j:yj \neq yj} \langle u_{xj}|v_{yj} \rangle.$$ (5)

Analogous to witness sizes in span programs, we define a notion of witness sizes for converting vector sets:

**Definition 14 (Converting vector set witness sizes).** Given a converting vector set $P = \{|u_{xj}\rangle, |v_{yj}\rangle\}$, we define the witness sizes of $P$ as

$$w_+(P, x) := \sum_j ||u_{xj}||^2 \quad \text{positive witness size for } x \text{ in } P$$

$$w_-(P, x) := \sum_j ||v_{yj}||^2 \quad \text{negative witness size for } x \text{ in } P$$

$$W(P) := \max_{x \in X} \{\max\{w_+(P, x), w_-(P, x)\}\} \quad \text{witness size of } P$$ (6)

If $P$ is a converting vector set from $\rho$ to $\sigma$, then the value of the objective function in Eq. (3) equals $W(P)$, and thus the query complexity of converting from $\rho$ to $\sigma$ using $P$ is $O\left(W(P) \log(1/\varepsilon) \right)$. When clear from context, we refer to $W(P)$ as $W$.

The following two lemmas (whose proofs can be found in Appendix A), provide some useful transformations for converting vector sets:
Lemma 15. If \( \mathcal{P} = \{\{u_{xj}\}\}, \{v_{yj}\}\} \) converts \( \rho \) to \( \sigma \), then there is a complementary converting vector set \( \mathcal{P}' = \{\{u_{xj}'\}\}, \{v_{yj}'\}\} \) that also converts \( \rho \) to \( \sigma \), such that for all \( x \in X \) and for all \( j \in [n] \), we have \( w_+(\mathcal{P}, x) = w_+(\mathcal{P}', x) \), and \( w_-(\mathcal{P}, x) = w_-(\mathcal{P}', x) \); the complement exchanges the values of the positive and negative witness sizes.

Lemma 16. Given \( \mathcal{P} = \{\{u_{xj}\}\}, \{v_{yj}\}\} \) that converts \( \rho \) to \( \sigma \), there exists a normalized converting vector set \( \mathcal{P}' = \{\{u_{xj}'\}\}, \{v_{yj}'\}\} \) from \( \rho \) to \( \sigma \) such that \( W(\mathcal{P}') \leq W(\mathcal{P}) \) and \( \max_{x \in X} w_+(\mathcal{P}', x) = \max_{x \in X} w_+(\mathcal{P}', x) \).

Thus without loss of generality, we henceforward assume our converting vector sets are normalized such that the maximum positive and negative witness sizes are the same. In particular, given Lemma 16, and using the algorithm for state conversion [20], we can restate Theorem 12 in a way that will be more conducive to comparing to our result:

Corollary 17. Let \( \mathcal{P} \) be a (normalized) converting vector set from \( \rho \) to \( \sigma \) with witness size \( W = \min \{\max_{x \in X} w_+(\mathcal{P}, \chi), \max_{x \in X} w_-(\mathcal{P}, \chi)\} \). Then there is a quantum algorithm that on every input \( x \in X \) converts \( |\rho_x\rangle \) to \( |\sigma_x\rangle \) with error \( \varepsilon \) and uses \( \tilde{O}(W/\varepsilon^2) \) queries to the oracle \( O_x \).

We will make use of the unit vectors \( \{|\mu_i\rangle\}_{i \in [q]} \) and \( \{|\nu_i\rangle\}_{i \in [q]} \), which are commonly seen in dual adversary algorithms like state-conversion:

\[
|\mu_i\rangle = -\alpha|i\rangle + \sqrt{\frac{1 - \alpha^2}{q-1}} \sum_{j \neq i} |j\rangle, \quad |\nu_i\rangle = \sqrt{1 - \alpha^2}|i\rangle + \frac{\alpha}{\sqrt{q-1}} \sum_{j \neq i} |j\rangle, \tag{7}
\]

where \( \alpha = \sqrt{1/2 - \sqrt{q-1}}/q \). These states have the property that \( \langle \mu_i | \nu_j \rangle = \frac{q}{2(q-1)} (1 - \delta_{i,j}) \).

3 Function Evaluation

In this section, we prove:

Theorem 18. For \( X \subseteq [q]^n \), let \( P \) be a (scaled and normalized) span program that decides \( f : X \to \{0,1\} \) with witness size \( W = \max_{x \in X} w(P, \chi) \). Then there is a quantum algorithm that for any \( x \in X \) evaluates \( f(x) \) with probability \( 1 - O(\delta) \) and uses \( \tilde{O}\left(\sqrt{w(P, x)}W \log(1/\delta)\right) \) queries to the oracle \( O_x \).

The algorithm might use more than the stated number of queries or output the incorrect value, but the probability of either or both of these events occurring is \( O(\delta) \).

Comparing Theorem 18 to Corollary 9, we see that in the worst case, when we have an input \( x \) where \( w(P, x) = \max_{x \in X} w(P, \chi) \), the performance of our algorithm is the same, up to log factors, as the standard span program algorithm. However, when we have an instance \( x \) with a smaller value of \( w(P, x) \), then our algorithm has improved query complexity, without having to know anything ahead of time about the size of \( w(P, x) \).

Our algorithm makes use of a subroutine that is similar to the standard span program algorithm, but which will almost never output 1 unless the function has value 1, while it might output 0 even if the value of the function is 1. In other words, the subroutine has a negligible probability of a false positive, but a potentially large probability of a false negative.

We repeatedly run this subroutine while exponentially increasing a parameter \( \alpha \). As \( \alpha \) gets larger, the probability of a false negative decreases, while the runtime of the algorithm increases. If we see a negative outcome at an intermediate round, since we do not know if this value is a true negative or false negative, we continue running. We stop when we get a positive outcome, or when \( \alpha \) reaches its maximum possible value, at which point the probability of getting a false negative is also negligible.

Given a span program \( P = (H, V, \tau, A) \) on \([q]^n\), let \( \hat{H} = H \oplus \text{span}\{\hat{0}\} \), and \( \hat{H}(x) = H(x) \oplus \text{span}\{\hat{0}\} \), where \( \hat{0} \) is orthogonal to \( H \) and \( V \). Then we define the linear operator \( \hat{A}^\alpha \in \mathcal{L}(H, V) \) as

\[
\hat{A}^\alpha = \frac{1}{\alpha} |\tau\rangle \langle \hat{0}| + A. \tag{8}
\]
Let $\Lambda^o \in \mathcal{L}(\hat{H}, \hat{H})$ be the orthogonal projection onto the kernel of $\hat{A}^o$, and let $\Pi_x \in \mathcal{L}(\hat{H}, \hat{H})$ be the projection onto $\hat{H}(x)$. Finally, let $U(P, x, \alpha) = (2\Pi_x - I)(2\Lambda^o - I)$. Note that $2\Pi_x - I$ can be implemented with two applications of $O_x$ [16, Lemma 3.1], and $2\Lambda^o - I$ can be implemented without any applications of $O_x$.

Algorithm 1:

1. **Input**: Error tolerance $\delta$, span program $P$ that decides a function $f$, oracle $O_x$

2. **Output**: $f(x)$ with probability $1 - O(\delta)$

   1. $N \leftarrow 9 \log \left( \left\lceil \log 3W \right\rceil / \delta \right) / 2$; $\epsilon \leftarrow 1/9$

   2. for $i \geq 0$ to $\lceil \log 3W \rceil$ do

      3. $\alpha \leftarrow 2^i$

      4. Repeat $N$ times: Phase Checking of $U(P, x, \alpha)$ on $|0\rangle_A |0\rangle_B$ with error $\epsilon$, precision $\sqrt{\frac{\epsilon}{\alpha W}}$

      5. if Measure $|0\rangle$ in register $B$ at least $N/2$ times then return 1

      6. Repeat $N$ times: Phase Checking of $U(P^t, x, \alpha)$ on $|0\rangle_A |0\rangle_B$ with error $\epsilon$, precision $\sqrt{\frac{\epsilon}{\alpha W}}$

      7. if Measure $|0\rangle$ in register $B$ at least $N/2$ times then return 0

8. return 1 // with low probability the algorithm makes a random guess

We will show Alg. 1 solves the function decision problem correctly and with the desired complexity. We use the following lemma, which we prove in Appendix B, to analyze the calls to Phase Checking (see Lemma 1) in Lines 4 and 6 of Alg. 1, which then allows us to prove Theorem 18.

**Lemma 19.** Let the span program $P$ decide the function $f$ with witness size $W$, and let $C \geq 2$. Then for Phase Checking with unitary $U(P, \alpha)$ on the state $|0\rangle_A |0\rangle_B$ with error $\epsilon$ and precision $\Theta = \sqrt{\frac{\epsilon}{\alpha W}}$.

1. If $f(x) = 1$, and $\alpha^2 \geq Cw(P, x)$, for a large enough constant $C$, then the probability of measuring the $B$ register to be in the state $|0\rangle_B$ is at least $1 - 1/C$.

2. If $f(x) = 0$, the probability of measuring the $B$ register in the state $|0\rangle_B$ is at most $3\epsilon$.

Note that if $f(x) = 1$ and $Cw(P, x) > \alpha^2$, Lemma 19 makes no claims about the output. However, since our algorithm can handle false negatives, as discussed previously, this is acceptable. Without more information about the phase gap of the unitary we are running Phase Checking on, it seems difficult to obtain information about this regime.

To prove Lemma 19, we use an analysis that mirrors the Boolean function decision algorithm of Belovs and Reichardt [4, Section 5.2] and Cade et al. [10, Section C.2] and the dual adversary algorithm of Reichardt [24, Algorithm 1]. Our approach differs from these previous algorithms in the addition of a parameter that controls the precision of our phase estimation; we do not always run phase estimation with a precision that is as high as those in previous works, which is what causes our false negatives. This general algorithmic approach has not (to the best of our knowledge\(^3\)) been applied to the non-Boolean span program of Definition 5, so while not surprising that it works in this setting, our analysis in Appendix B may be of independent interest for other applications.

**Proof of Theorem 18.** We analyze Alg. 1.

\(^3\)Jeffery and Ito [16] also design a function decision algorithm for non-Boolean span programs, but it has a few differences from our approach and from that of Refs. [4, 10]; for example, the initial state of Jeffery and Ito’s algorithm might require significant time to prepare, while our initial state can be prepared in $O(1)$ time.
Consider the case that \( f(x) = 1 \). Let \( T = \lceil \log \sqrt{3W(P, x)} \rceil \). We analyze the probability that the algorithm outputs 1 within the first \( T \) iterations of the for loop. This is the probability that the algorithm doesn’t output 0 in any of the first \( T - 1 \) rounds (not outputting a 0 includes the events of outputting a 1 or continuing to the next round), times the probability it outputs at 1 in the \( T \)th round. Note that because \( w(P, x) \leq W \), and the for loop can repeat up to \( \lceil \log \sqrt{3W} \rceil \) times, it is possible to have \( T \) iterations.

At the \( T \)th round,
\[
\alpha = 2^{\lceil \log \sqrt{3w(P, x)} \rceil} \geq \sqrt{3w(P, x)}.
\] (9)

Since \( \alpha^2 \geq 3w(P, x) \), by Lemma 19, the probability of measuring \( |0 \rangle \) at a single repetition of Phase Checking is at least \( \frac{2}{3} \), so the probability of seeing at least \( N/2 \) results with outcome \( |0 \rangle \) and outputting 1 is at least
\[
1 - 2^{-2\log eN/9} > 1 - 2^{-2N/9}
\] (10)

using Hoeffding’s inequality [15] for the binomial distribution.

The probability of not outputting a 0 at any previous round depends on Phase Checking with \( U(P^1, x, \alpha) \). From Lemma 10, \( P^1 \) decides the function \( \neg f \), and since \( f(x) = 1 \), we have \( \neg f(x) = 0 \). By Lemma 19, each time we run Phase Checking of \( U(P^1, x, \alpha) \), there is at most a \( 3e = 1/3 \) probability of measuring \( |0 \rangle \). Thus the probability of seeing at least \( N/2 \) results with outcome \( |0 \rangle \) and outputting 0 is at most \( 2^{-2N/9} \). Thus, even if we assume a worst case situation where we never output a 1 in the first \( T - 1 \) rounds, the probability that we do not output a 0 over the first \( T - 1 \) rounds is at least
\[
\left(1 - 2^{-2N/9}\right)^{T-1} > 1 - T2^{-2N/9}.
\] (11)

Thus our total success probability is at least
\[
\left(1 - T2^{-2N/9}\right)\left(1 - 2^{-2N/9}\right) = 1 - O(T2^{-2N/9}) = 1 - O(\delta),
\] (12)

where we’ve used that \( T \leq \lceil \log \sqrt{3W} \rceil \) and \( N = 9\log\left(\lceil \log \sqrt{3W} \rceil /\delta \right) / 2 \).

For the case that \( f(x) = 0 \), we have \( \neg f(x) = 1 \), so since \( P^1 \) decides \( \neg f \) (by Lemma 10), we can use the same analysis as in the case of \( f(x) = 1 \), with \( w(P, x) \) replaced by \( w(P^1, x) \). By Lemma 10, we have \( w(P^1, x) \leq w(P, x) \), so the same analysis of the success probability will apply.

Now to analyze the query complexity. We’ve argued that with probability \( 1 - O(\delta) \), the algorithm terminates within \( T \) rounds. In the \( i \)th round, by Lemma 1, the number of queries required to run a single repetition of Phase Checking is \( O \left( 2^i \sqrt{W} \right) \). Over the \( N \) repetitions, we have that the \( i \)th round uses \( O \left( 2^i \sqrt{W} \log \log(W)/\delta \right) \) queries. Summing over the rounds from \( i = 0 \) to \( T = \lceil \log \sqrt{3w(P, x)} \rceil \) and using the geometric series formula, we find that the total number of queries is
\[
O \left( \sqrt{w(P, x)W} \log \log(W)/\delta \right) = \tilde{O} \left( \sqrt{w(P, x)W} \log(1/\delta) \right).
\] (13)

\[\square\]

### 3.1 Application to st-connectivity

As an example, we apply Theorem 18 to the problem of \( st \)-connectivity on an \( n \) vertex graph. There is a span program \( P \) such that (after scaling), for inputs \( x \) where there is a path from \( s \) to \( t \), \( w(P, x) = R_{s,t}(x)\sqrt{n} \) where \( R_{s,t}(x) \) is the effective resistance from \( s \) to \( t \) on the subgraph induced by \( x \), and for inputs \( x \) where there is not a path from \( s \) to \( t \), \( w(P, x) = C_{s,t}(x)\sqrt{n} \), where \( C_{s,t}(x) \) is the effective capacitance between \( s \) and \( t \) [4, 17]. In an \( n \)-vertex graph, the
effective resistance is at most $n$, and the effective capacitance is at most $n^2$. Thus if we desire a bounded error algorithm, by Theorem 18, we can determine whether or not there is a path with $\tilde{O}(\sqrt{R_{st}(x)n^2})$ queries if there is a path, and $\tilde{O}(\sqrt{C_{st}(x)n})$ queries if there is not a path. The effective resistance is at most the shortest path between two vertices, and the effective capacitance is at most the smallest cut between two vertices. Thus our algorithm determines whether or not there is a path from $s$ to $t$ with $\tilde{O}(\sqrt{kn})$ queries if there is a path of length $k$, and if there is no path, the algorithm uses $\tilde{O}(\sqrt{nq})$ queries, where $c$ is the size of the smallest cut between $s$ and $t$. Importantly, one does not need to know bounds on $k$ or $c$ ahead of time to achieve this query complexity.

The analysis of the other examples listed in Section 1 is similar.

4 State Conversion

In this section, we prove the following result regarding quantum state conversion:

**Theorem 20.** Let $P$ be a (normalized) converting vector set from $\rho$ to $\sigma$ with witness size $W = \min\{\max_{\chi \in X} w_+(\rho, \chi), \max_{\chi \in X} w_-(\rho, \chi)\}$. Then there is quantum algorithm that for any $x \in X$ with probability $1 - p$ converts $|\rho_x\rangle$ to $|\sigma_x\rangle$ with error $\varepsilon$ and uses $\tilde{O}\left(\sqrt{\min\{w_+(P, x), w_-(P, x)\}} W / (\varepsilon^5 p)\right)$ queries to the oracle $O_x$.

Comparing Theorem 20 with Corollary 17, and for a moment ignoring the scaling with $\varepsilon$ and $p$, we see that in the worst case, when we have an input $x$ where $\min\{w_+(P, x), w_-(P, x)\} = \min\{\max_{\chi \in X} w_+(P, \chi), \max_{\chi \in X} w_-(P, \chi)\}$ the performance of our algorithm is the same, up to log factors, as the standard state conversion algorithm. However, when we have an instance $x$ with a smaller value of $w_\pm(P, x)$, then our algorithm has improved query complexity, without having to know anything about the witness size of our input ahead of time.

Our algorithm has worse scaling in $\varepsilon$ than Corollary 17, but when $\varepsilon$ is a constant this is a non-issue. We also note that while the $\tilde{O}$ in Corollary 17 hides logarithmic factors in $\varepsilon$, in Theorem 20 it hides logarithmic factors in both $\varepsilon$ and $W$. We believe that it should be possible to improve the the $1/p$ factor in the complexity to $\log(1/p)$.

The problem of state conversion is a more general problem than function evaluation, and it can be used to solve the function evaluation problem. However, because of the worse scaling with $\varepsilon$ in Theorem 20, we considered function evaluation separately (see Section 3).

There might be some inputs $x \in X$ for which the problem of converting from $|\rho_x\rangle$ to $|\sigma_x\rangle$ is less difficult, and thus requires fewer queries. However if we run the standard state conversion algorithm for less than the worst-case queries, we can not tell whether the computation has completed, since the output is a state $|\sigma_x\rangle$ where $x$ is unknown, and any measurement will collapse the state. This contrasts with function evaluation, where at least there was a measurement at the end of the computation.

Thus instead of repeatedly running the standard state conversion algorithm for increasingly longer times, as with function decision, we instead use an initial probing protocol that we repeatedly run for increasingly longer times. This probing subroutine helps us determine how long we need to run the main state conversion algorithm for to guarantee success.

In the following we use most of the notation conventions of Ref. [20] for clarity. For $X \subseteq \{q\}^n$, let $P = (\{|u_x\rangle\}, \{|v_y\rangle\})$ be a converting vector set from $\rho$ to $\sigma$, where for all $x \in X$, the states $|\rho_x\rangle$ and $|\sigma_x\rangle$ are in the Hilbert space $H$. For all $x \in X$, define $|t_x\rangle, |\psi_x\rangle \in (\mathbb{C}^2 \otimes H) \oplus (\mathbb{C}^n \otimes \mathbb{C}^q \otimes \mathbb{C}^m)$ as

$$|t_x\rangle = \frac{1}{\sqrt{2}} (|0\rangle|\rho_x\rangle \pm |1\rangle|\sigma_x\rangle), \quad \text{and} \quad |\psi_x\rangle = \sqrt{\frac{\varepsilon}{\alpha}} |t_x\rangle - \sum_{j \in [n]} |j\rangle |\mu_{xj}\rangle |u_{xj}\rangle, \quad (14)$$

where $|\mu_{xj}\rangle$ is from Eq. (7), and $\alpha$ is a parameter analogous to the parameter $\alpha$ in Eq. (8). We will choose $\varepsilon$ to achieve a desired accuracy of $\varepsilon$ in our state conversion procedure. Set $A^\alpha\varepsilon$ to equal the projection onto the orthogonal complement of the span of the vectors $\{|\psi_x\rangle\}_{x \in X}$, and set
\[ \Pi_x = I - \sum_{j \in [n]} |j \rangle \langle j| \otimes |\mu_x,j \rangle \rangle_{\mathcal{C}^n} \]. Finally, we set \( \mathcal{U}(\mathcal{P}, x, \alpha, \hat{\epsilon}) = (2\Pi_x - I)(2\mathcal{A}^\alpha,\hat{\epsilon} - I) \).

The reflection \(2\Pi_x - I\) can be implemented with two applications of \(O_x\) [20], and the reflection \((2\mathcal{A}^\alpha,\hat{\epsilon} - I)\) is independent of \(x\) and so requires no queries.

Algorithm 2:

**Input**: Converting vector set \(\mathcal{P}\) from \(\rho\) to \(\sigma\) with witness size \(W\), failure probability \(p\), error \(\epsilon\), oracle \(O_x\), initial state \(|\rho_x\rangle\)

**Output**: \(\hat{\sigma}_x\) such that \(|\langle \hat{\sigma}_x | - 1\rangle |\sigma_x\rangle |0⟩| \leq \epsilon\)

/* Probing Stage */

1. \(\hat{\epsilon} \leftarrow \epsilon^2/9\)
2. for \(i = 0\) to \([\log W]\) do
   3. \(\alpha \leftarrow 2^i\)
   4. for \(\mathcal{P}' \in \{\mathcal{P}, \mathcal{P}^C\}\) do
      5. \(\mathcal{A} \leftarrow D(\mathcal{U}(\mathcal{P}', x, \alpha, \hat{\epsilon}))\) (Lemma 1) to precision \(\hat{\epsilon}^{3/2}/\sqrt{\alpha W}\) and accuracy \(\hat{\epsilon}^2\)
      6. \(\hat{a} \leftarrow \text{Amplitude Estimation} \) (Lemma 3) of probability of outcome \(|0⟩⟩_B\) in register \(B\)
      7. \(\text{if } \hat{a} - 1/2 > -\frac{\epsilon}{2} \text{ then } \text{Continue to State Conversion Stage} \)

/* State Conversion Stage */

8. Apply \( \mathcal{R}(\mathcal{U}(\mathcal{P}', x, \alpha, \hat{\epsilon}))\) (Lemma 2) with precision \(\hat{\epsilon}^{3/2}/\sqrt{\alpha W}\) and accuracy \(\hat{\epsilon}^2\) to \(|\langle 0 |\rho_x⟩⟩_A |0⟩⟩_B\) and output the result

The idea of our approach, given in Alg. 2, is that when we apply Phase Reflection of \( \mathcal{U}(\mathcal{P}', x, \alpha, \hat{\epsilon}) \) in Line 8 to \(|\langle 0 |\rho_x⟩⟩_A |0⟩⟩_B = \frac{1}{\sqrt{2}}(|t_{x+}⟩⟩_A |0⟩⟩_B + |t_{x-}⟩⟩_A |0⟩⟩_B)\), we want \(|t_{x+}⟩⟩_A |0⟩⟩_B\) to pick up a \(+1\) phase, and \(|t_{x-}⟩⟩_A |0⟩⟩_B\) to pick up a \(-1\) phase. If this happened perfectly, we would have the desired state \(|(1⟩⟩ |\sigma_x⟩⟩⟩_A |0⟩⟩_B\). In Lemmas 21 to 24 we derive results that show that in the State Conversion stage of Alg. 2, \(|t_{x+}⟩⟩_A |0⟩⟩B\) will mostly pick up a \(+1\) phase, and \(|t_{x-}⟩⟩_A |0⟩⟩_B\) will mostly pick up a \(-1\) phase, resulting in a state close to \(|(1⟩⟩ |\sigma_x⟩⟨⟩⟩_A |0⟩⟩_B\).

In Lemmas 21 to 24, let \(\mathcal{P} = \{|w_{x]}\}, \{|w_{y}\}\}\) be a converting vector set from \(\rho\) to \(\sigma\) with \(W = W(\mathcal{P})\). When we write \(\Pi_0(\mathcal{U}(\mathcal{P}, x, \alpha, \hat{\epsilon}))\), \(\Pi_0(\mathcal{U}(\mathcal{P}, x, \alpha, \hat{\epsilon}))\) or \(\mathcal{R}(\mathcal{U}(\mathcal{P}, x, \alpha, \hat{\epsilon}))\) it refers to Phase Checking/Reflection on \(\mathcal{U}(\mathcal{P}, x, \alpha, \hat{\epsilon})\) with precision \(\hat{\epsilon}^{3/2}/\sqrt{\alpha W}\) and accuracy \(\hat{\epsilon}^2\).

**Lemma 21.** If \( \theta = \hat{\epsilon}^{3/2}/\sqrt{\alpha W}\), then \(\|\Pi_0(\mathcal{U}(\mathcal{P}, x, \alpha, \hat{\epsilon}))|t_{x-}\rangle\|^2 \leq \frac{4}{\theta^2}\).

**Lemma 22.** If \(\alpha \geq w_+(\mathcal{P}, x)\), then \(\|\Pi_0(\mathcal{U}(\mathcal{P}, x, \alpha, \hat{\epsilon}))(|0⟩⟩_A |0⟩⟩_B\|^2 \geq \frac{1}{2} (1 - 5\hat{\epsilon}).\)

**Lemma 23.** If \(\|\Pi_0(\mathcal{U}(\mathcal{P}, x, \alpha, \hat{\epsilon}))|0⟩⟩_A |0⟩⟩_B\|^2 \geq 1/2 - 3\hat{\epsilon}\), then \(\|\Pi_0(\mathcal{U}(\mathcal{P}, x, \alpha, \hat{\epsilon}))|t_{x+}\rangle|0⟩⟩_B\|^2 \leq 10\hat{\epsilon}\).

**Lemma 24.** If \(\|\Pi_0(\mathcal{U}(\mathcal{P}, x, \alpha, \hat{\epsilon}))|0⟩⟩_A |0⟩⟩_B\|^2 \geq 1/2 - 3\hat{\epsilon}\), then \(\|\Pi_0(\mathcal{U}(\mathcal{P}, x, \alpha, \hat{\epsilon}))|t_{x-}\rangle|0⟩⟩_B\|^2 \leq 6\sqrt{\hat{\epsilon}}\).

Lemma 21 ensures that the \(|t_{x-}\rangle\) portion of the state will mostly pick up a \(-1\) phase. The proof closely follows [20, Claim 4.5]. Lemma 22 ensures the Probing Routine halts when \(\alpha\) reaches the witness size. The proof follows [20, Claim 4.4] to show that \(\|\Pi_0|t_{x+}\rangle\|^2\) is close to 1 (which also ensures that the \(|t_{x+}\rangle\) portion of the state will mostly pick up a \(+1\) phase, and then with a little bit of work and Lemma 21, we can prove that \(\Pi_0(\mathcal{U}(\mathcal{P}, x, \alpha, \hat{\epsilon}))|0⟩⟩_A |0⟩⟩_B\|^2\) is close to \(1/2\). With Lemma 23, we bound the amount of \(|t_{x+}\rangle\) that can pick up an incorrect phase of \(-1\). The proof uses the triangle inequality and our previous lemmas. Lemma 24 uses Lemma 23 and Lemma 21 to show that after a successful Probing Stage, the State Conversion stage of Alg. 2 produces a state close to our target. It is our version of [20, Proposition 4.6], proven using slightly weaker bounds. See Appendix B for proofs of these Lemmas.

We now use Lemma 22 and Lemma 24 to prove Theorem 20:
Proof of Theorem 20. We analyze Alg. 2. With probability \( p \), the Probing Stage will stop with assignments of \( \alpha \) and \( \mathcal{P}' \) such that \( \| \Pi_0(\mathcal{U}(\mathcal{P}', x, \alpha, \hat{\varepsilon})) \|_2 \leq \| \Pi_0(\mathcal{U}(\mathcal{P}', x, \alpha, \hat{\varepsilon})) \|_2 \geq \hat{\varepsilon} \). This is because during the Probing Stage, we increase until it is larger than \( W \), so we eventually have \( \alpha \geq \min \{ w_+(\mathcal{P}, x), w_+(\mathcal{P}^C, x) \} \), which by Lemma 22, implies \( \| \Pi_0(\mathcal{U}(\mathcal{P}', x, \alpha, \hat{\varepsilon})) \|_2 \geq \frac{1}{2} - \frac{5/2}{\hat{\varepsilon}} \) for \( \mathcal{P}' = \mathcal{P} \) if \( \alpha \geq w_+(\mathcal{P}, x) \), and \( \mathcal{P}' = \mathcal{P}^C \) if \( \alpha \geq w_-(\mathcal{P}, x) \). This ends the Probing Stage since amplitude amplification estimates the value of \( \| \Pi_0(\mathcal{U}(\mathcal{P}', x, \alpha, \hat{\varepsilon})) \|_2 \geq \frac{1}{2} - \frac{5/2}{\hat{\varepsilon}} \). Finally, as there are \( O(\log W) \) rounds, and each fails with probability \( O(p/\log W) \), the probability that all rounds of amplitude amplification are successful is \( O(p) \).

The Probing Stage may stop before \( \alpha \geq \min \{ w_+(\mathcal{P}, x), w_+(\mathcal{P}^C, x) \} \), but (assuming no errors in this stage), it is guaranteed to stop when \( \| \Pi_0(\mathcal{U}(\mathcal{P}', x, \alpha, \hat{\varepsilon})) \|_2 \geq \frac{1}{2} - \frac{5/2}{\hat{\varepsilon}} \), since the value of \( \| \Pi_0(\mathcal{U}(\mathcal{P}', x, \alpha, \hat{\varepsilon})) \|_2 \) will be within \( \frac{1}{4} \hat{\varepsilon} \) of its estimated value by our parameter choices for amplitude estimation.

Applying Lemma 24, since \( \| \Pi_0(\mathcal{U}) \|_2 \| \rho_x \|_A \|_B \geq 1/2 - 3\hat{\varepsilon} \), the State Conversion stage produces a state \( |\tilde{\sigma} \rangle = R(\mathcal{U}(\mathcal{P}', x, \alpha, \hat{\varepsilon}))(\| \rho_x \|_A \|_B \) such that

\[
\| |\tilde{\sigma} \rangle - (1)(|\sigma_x \rangle)_A \|_B \leq 3\sqrt{\hat{\varepsilon}} = \varepsilon.
\]

Thus the algorithm is correct with the stated success probability and accuracy.

Now we analyze the query complexity. From our previous discussion, the Probing Stage stops by the round when \( \alpha \) is at least \( \min \{ w_+(\mathcal{P}, x), w_+(\mathcal{P}^C, x) \} \). At the ith round of the Probing Stage, phase estimation with precision \( \varepsilon^{3/2}/\sqrt{2W} \) and accuracy \( \varepsilon^2 \), which \( O \left( \frac{\sqrt{2W}}{\varepsilon^{3/2}} \log \left( \frac{1}{\varepsilon^{3/2}} \right) \right) \) queries is implemented implement \( O \left( \frac{\log W}{\varepsilon^p} \right) \) times inside the amplitude estimation subroutine. Thus the queries used by the Probing Stage of the algorithm is

\[
\sum_{i=0}^{\log[\min \{ w_+(\mathcal{P}, x), w_+(\mathcal{P}^C, x) \}]} O \left( \frac{\sqrt{2W}}{\varepsilon^{3/2}} \log \left( \frac{1}{\varepsilon^{3/2}} \right) \frac{\log W}{\varepsilon^p} \right) = \tilde{O} \left( \frac{\sqrt{\min \{ w_+(\mathcal{P}, x), w_+(\mathcal{P}^C, x) \} W}}{\varepsilon^{5/2} p} \right).
\]

Finally, the Phase Reflection stage of the algorithm has precision \( \varepsilon^{3/2}/\sqrt{2W} \) and accuracy \( \varepsilon^2 \), so uses another \( O \left( \sqrt{\min \{ w_+(\mathcal{P}, x), w_+(\mathcal{P}^C, x) \} W} \log(1/\hat{\varepsilon}^2/p) \right) \) queries. Thus the probing state dominates, and the total number of queries used by the algorithm is (using Lemma 15) \( \tilde{O} \left( \sqrt{\min \{ w_+(\mathcal{P}, x), w_-(\mathcal{P}, x) \} W} / (\varepsilon^5 p) \right) \), where we’ve used that \( \xi = \varepsilon^2/9 \) (Line 2 of Alg. 2). \( \square \)

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References

[1] Dorit Aharonov. Quantum Computation. Annual Reviews of Computational Physics VI, pages 259–346, 1999. doi: 10.1142/9789812815569_0007.

[2] Andris Ambainis and Ronald De Wolf. Average-case quantum query complexity. Journal of Physics A: Mathematical and General, 34(35):6741, 2001.
[3] Salman Beigi and Leila Taghavi. Quantum Speedup Based on Classical Decision Trees. \textit{arXiv:1905.13095}, 2019.

[4] A. Belovs and B.W. Reichardt. Span programs and quantum algorithms for st-connectivity and claw detection. \textit{Lecture Notes in Computer Science}, 7501 LNCS:193–204, 2012. doi: 10.1007/978-3-642-33090-2_18.

[5] Aleksandrs Belovs. Span programs for functions with constant-sized 1-certificates: Extended abstract. In \textit{Proceedings of the Forty-Fourth Annual ACM Symposium on Theory of Computing}, STOC ’12, pages 77–84, 2012. doi: 10.1145/2213977.2213985.

[6] Aleksandrs Belovs and Ansis Rosmanis. Tight Quantum Lower Bound for Approximate Counting with Quantum States. \textit{arXiv:2002.06879}, 2020.

[7] Michel Boyer, Gilles Brassard, Peter Høyer, and Alain Tapp. Tight Bounds on Quantum Searching. \textit{Fortschritte der Physik}, 46(4-5):493–505, 1998. doi: 10.1002/(SICI)1521-3978(199806)46:4/5<493::AID-PROP493>3.0.CO;2-P.

[8] Gilles Brassard, Peter Høyer, and Alain Tapp. Quantum counting. In \textit{Automata, Languages and Programming}, pages 820–831, 1998. doi: 10.1007/BFb0055105.

[9] Gilles Brassard, Peter Høyer, Michele Mosca, and Alain Tapp. Quantum Amplitude Amplification and Estimation. \textit{arXiv:quant-ph/0005055}, 2000.

[10] Chris Cade, Ashley Montanaro, and Aleksandrs Belovs. Time and space efficient quantum algorithms for detecting cycles and testing bipartiteness. \textit{Quantum Information \\& Computation}, 18(1-2):18–50, 2018.

[11] R. Cleve, A. Ekert, C. Macchiavello, and M. Mosca. Quantum algorithms revisited. \textit{Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences}, 454(1969):339–354, 1998. doi: 10.1098/rspa.1998.0164.

[12] Kai DeLorenzo, Shelby Kimmel, and R. Teal Witter. Applications of the Quantum Algorithm for st-Connectivity. In \textit{14th Conference on the Theory of Quantum Computation, Communication and Cryptography (TQC 2019)}, volume 135, pages 6:1–6:14, 2019.

[13] Edward Farhi, Jeffrey Goldstone, and Sam Gutmann. A quantum algorithm for the hamiltonian nand tree. \textit{arXiv preprint quant-ph/0702144}, 2007.

[14] Lov K. Grover. Quantum Mechanics Helps in Searching for a Needle in a Haystack. \textit{Physical Review Letters}, 79(2):325–328, 1997. doi: 10.1103/PhysRevLett.79.325.

[15] Wassily Hoeffding. Probability inequalities for sums of bounded random variables. \textit{Journal of the American Statistical Association}, 58(301):13–30, 1963. doi: 10.1080/01621459.1963.10500830. URL \texttt{http://www.tandfonline.com/doi/abs/10.1080/01621459.1963.10500830}.

[16] Tsuyoshi Ito and Stacey Jeffery. Approximate Span Programs. \textit{Algorithmica}, 81(6):2158–2195, 2019. doi: 10.1007/s00453-018-0527-1.

[17] Michael Jarret, Stacey Jeffery, Shelby Kimmel, and Alvaro Piedrafita. Quantum Algorithms for Connectivity and Related Problems. In \textit{26th Annual European Symposium on Algorithms (ESA 2018)}, volume 112 of \textit{Leibniz International Proceedings in Informatics (LIPIcs)}, pages 49:1–49:13, 2018. doi: 10.4230/LIPIcs.ESA.2018.49.

[18] Shelby Kimmel. Quantum adversary (upper) bound. \textit{Chicago Journal of Theoretical Computer Science}, 2013(4), 2011.
\[ \Lambda_H \]

Finally, set

\[ \omega \]

\textbf{Proof.} We first define \( H', \) starting from \( H'_{j,a} : \)
\[ H'_{j,a} = \text{span}\{ |v\rangle : |v\rangle \in H_j \text{ and } |v\rangle \in H_{j,a}^\perp \}, \] (16)
where \( H_{j,a}^\perp \) is the orthogonal complement of \( H_{j,a} \). We define \( H'_j = \sum_{a \in [q]} H'_{j,a} \), and \( H_{\text{true}}' = H_{\text{false}}' \) and \( H_{\text{false}} = H_{\text{true}} \). Then
\[ H' = H'_1 \oplus H'_2 \cdots \oplus H'_n \oplus H_{\text{false}}' \oplus H_{\text{false}}. \] (17)

Let \( |\tilde{0}\rangle \) be a vector that is orthogonal to \( H \) and \( V \), and define \( V' = H \oplus \text{span}\{ |\tilde{0}\rangle \} \) and \( \tau' = |\tilde{0}\rangle \).

Finally, set
\[ A' = |\tilde{0}\rangle\langle w| + \Pi_H \Lambda_A, \] (18)
where \( \Lambda_A \) is the projection onto the kernel of \( A \), \( \Pi_H \) is the projection onto \( H \), and
\[ |w_0\rangle = \arg \min_{|v\rangle \in H; A|v\rangle = \tau} \| |v\rangle \|. \] (19)

Let \( x \in X \) be an input with \( f(x) = 1 \), so \( x \) has a positive witness \( |w\rangle \) in \( P \). We will show \( \omega' = |\tilde{0}\rangle + |w\rangle \) is a negative witness for \( x \) in \( P' \). Note \( \omega' \tau' = 1 \), and also,
\[ \omega' A' = (|\tilde{0}\rangle + |w\rangle)(|\tilde{0}\rangle\langle w_0| + \Pi_H \Lambda_A) \] (20)
\[ = \langle w_0| + \langle w| \Pi_H \Lambda_A \] (21)
\[ = \langle w_0| + \langle w| \Lambda_A \] (22)
\[ = \langle w|, \] (23)

\[ A \quad \text{Proofs from Section 2} \]

\textbf{Lemma 10.} Given a span program \( P = (H,V,\tau,A) \) on \([q]^m\) that decides a function \( f : X \rightarrow \{0,1\} \) for \( X \in [q]^m \), there exists a span program \( P' = (H',V',\tau',A') \) such that \( \forall x \in X, w(P,x) \geq w(P',x) \), and \( P' \) decides \( \neg f \).

\textbf{Proof.} We first define \( H' \), starting from \( H'_{j,a} : \)
\[ H'_{j,a} = \text{span}\{ |v\rangle : |v\rangle \in H_j \text{ and } |v\rangle \in H_{j,a}^\perp \}, \] (16)
where \( H_{j,a}^\perp \) is the orthogonal complement of \( H_{j,a} \). We define \( H'_j = \sum_{a \in [q]} H'_{j,a} \), and \( H_{\text{true}}' = H_{\text{false}}' \) and \( H_{\text{false}} = H_{\text{true}} \). Then
\[ H' = H'_1 \oplus H'_2 \cdots \oplus H'_n \oplus H_{\text{false}}' \oplus H_{\text{false}}. \] (17)

Let \( |\tilde{0}\rangle \) be a vector that is orthogonal to \( H \) and \( V \), and define \( V' = H \oplus \text{span}\{ |\tilde{0}\rangle \} \) and \( \tau' = |\tilde{0}\rangle \).

Finally, set
\[ A' = |\tilde{0}\rangle\langle w| + \Pi_H \Lambda_A, \] (18)
where \( \Lambda_A \) is the projection onto the kernel of \( A \), \( \Pi_H \) is the projection onto \( H \), and
\[ |w_0\rangle = \arg \min_{|v\rangle \in H; A|v\rangle = \tau} \| |v\rangle \|. \] (19)

Let \( x \in X \) be an input with \( f(x) = 1 \), so \( x \) has a positive witness \( |w\rangle \) in \( P \). We will show \( \omega' = |\tilde{0}\rangle + |w\rangle \) is a negative witness for \( x \) in \( P' \). Note \( \omega' \tau' = 1 \), and also,
\[ \omega' A' = (|\tilde{0}\rangle + |w\rangle)(|\tilde{0}\rangle\langle w_0| + \Pi_H \Lambda_A) \] (20)
\[ = \langle w_0| + \langle w| \Pi_H \Lambda_A \] (21)
\[ = \langle w_0| + \langle w| \Lambda_A \] (22)
\[ = \langle w|, \] (23)
where in the second line, we have used that \( \langle 0 \rangle \Pi_H A_A = 0 \) and \( \langle w \rangle 0 \) because \( 0 \) is orthogonal to \( H \). The final line follows from \( 16, \text{Definition 2.12} \), which showed that every positive witness can be written as \( |w\rangle = |w_0\rangle + |w^\perp\rangle \), where \( |w^\perp\rangle \) is in the kernel of \( A \) and \( |w_0\rangle \) is orthogonal to the kernel of \( A \).

Then \( \langle w \rangle \Pi_H(x) = 0 \), because \( |w\rangle \in H(x) \), and \( H'(x) \) is orthogonal to \( H(x) \), so \( \omega' \) is a negative witness for \( x \) in \( P^1 \). Also, \( ||\omega' A||^2 = ||w||^2 \), so the witness size of this negative witness in \( P^1 \) is the same as the corresponding positive witness in \( P \).

If \( f(x) = 0 \), there is a negative witness \( \omega \) for \( x \) in \( P \). Consider \( |w'\rangle = (\omega A)^\dagger \). Then

\[
A'|w'\rangle = (\langle 0 \rangle|w_0\rangle + \Pi_H A_A)(\omega A)^\dagger
= \langle 0 \rangle(\omega A|w_0\rangle)^\dagger
= \langle 0 \rangle(\omega \tau)^\dagger
= \langle 0 \rangle
\]

where in the first line, we’ve used that \( (\omega A)^\dagger \) is orthogonal to the kernel of \( A \). Also, \( \Pi_H(x)(\omega A)^\dagger = 0 \), so \( |w'\rangle \in H'(x) \). This means \( |w'\rangle \) is a positive witness for \( x \) in \( P^1 \). Also, \( ||w'\rangle^2 = ||\omega A||^2 \), so the witness size of this positive witness in \( P^1 \) is the same as the corresponding negative witness in \( P \).

We have proved that \( w(P, x) \geq w(P^1, x) \) for all \( x \), and that if \( x \) has a positive witness in \( P \), it has a negative witness in \( P^1 \), and vice versa. This shows that \( P^1 \) does indeed decide \(-f\). \( \square \)

**Lemma 15.** If \( \mathcal{P} = \{\{u_{xj}\}, \{v_{yj}\}\} \) converts \( \rho \) to \( \sigma \), then there is a complementary converting vector set \( \mathcal{P}^C = \{(u_{xj}^C), \{v_{yj}^C\}\} \) that also converts \( \rho \) to \( \sigma \), such that for all \( x \in X \) and for all \( j \in [n] \), we have \( w_+(\mathcal{P}, x) = w_-(\mathcal{P}^C, x) \), and \( w_-(\mathcal{P}, x) = w_+(\mathcal{P}^C, x) \); the complement exchanges the values of the positive and negative witness sizes.

**Proof.** We will prove the more general result showing that this holds when the matrix \( B \) in Definition 11 is Hermitian, and the matrices \( Z \) are symmetric (as is true when \( B = \rho - \sigma \) and \( Z = \Delta \) for a converting vector set). For all \( x \in X \) and \( j \in [n] \), define

\[
|u_{xj}^C\rangle = |v_{xj}\rangle, \quad \text{and} \quad |v_{xj}^C\rangle = |u_{xj}\rangle.
\]

Note \( (u_{xj}^C|v_{yj}^C) = (u_{yj}|v_{xj})^* \) and \( (Z_j)_{xy} = (Z_j)_{yx} \) by our symmetric assumption. Since \( \mathcal{P} \) satisfies the constraints of Definition 11,

\[
\sum_j (Z_j)_{xy} \langle u_{yj}^C|v_{xj}^C\rangle = \sum_j (Z_j)_{yx} \langle u_{yj}|v_{xj}\rangle^* = \left( \sum_j (Z_j)_{yx} \langle u_{yj}|v_{xj}\rangle \right)^* = B_{xy}^* = B_{yx}.
\]

Thus the complementary vectors satisfy the same constraints from Definition 5, and thus produce the same optimal value in Eq. (3). However, now \( w_+(\mathcal{P}, x) = w_-(\mathcal{P}^C, x) \), and \( w_-(\mathcal{P}, x) = w_+(\mathcal{P}^C, x) \). \( \square \)

**Lemma 16.** Given \( \mathcal{P} = \{\{u_{xj}\}, \{v_{yj}\}\} \) that converts \( \rho \) to \( \sigma \), there exists a normalized converting vector set \( \mathcal{P}' = \{\{u'_{xj}\}, \{v'_{yj}\}\} \) from \( \rho \) to \( \sigma \) such that \( W(\mathcal{P}') \leq W(\mathcal{P}) \) and \( \max_{x\in X} w_+(\mathcal{P}', x) = \max_{x\in X} w_+(\mathcal{P}, x) \) and \( \max_{x\in X} w_-(\mathcal{P}', x) = \max_{x\in X} w_-(\mathcal{P}, x) \).

**Proof.** For all \( x \in X \) and \( j \in [n] \), set

\[
|u'_{xj}\rangle = \frac{\max_x w_-(\mathcal{P}, x)}{\max_x w_+(\mathcal{P}, x)} |u_{xj}\rangle, \quad |v'_{yj}\rangle = \frac{\max_x w_+(\mathcal{P}, x)}{\max_x w_-(\mathcal{P}, x)} |v_{yj}\rangle.
\]

It is straightforward to verify that \( \mathcal{P}' \) is still a converting vector set from \( \rho \) to \( \sigma \), and that \( \max_x w_+(\mathcal{P}', x) = \max_x w_+(\mathcal{P}, x) \) and \( \max_x w_-(\mathcal{P}', x) = \max_x w_-(\mathcal{P}, x) \), and so is at most the maximum of either term. \( \square \)
B Proofs from Section 3 and Section 4

Lemma 19. Let the span program \( P \) decide the function \( f \) with witness size \( W \), and let \( C \geq 2 \). Then for Phase Checking with unitary \( U(P, \alpha) \) on the state \( |0\rangle_A |0\rangle_B \) with error \( \epsilon \) and precision \( \Theta = \sqrt{\frac{\alpha^2}{W}} \).

1. If \( f(x) = 1 \), and \( \alpha^2 \geq Cw(P, x) \), for a large enough constant \( C \), then the probability of measuring the B register to be in the state \( |0\rangle_B \) is at least \( 1 - 1/C \).

2. If \( f(x) = 0 \), the probability of measuring the B register in the state \( |0\rangle_B \) is at most \( 3\epsilon \).

Proof of Lemma 19.

Part 1: Since \( f(x) = 1 \), there is an optimal witness \( |w\rangle \in H(x) \) for \( x \). Then set \( |u\rangle \in \tilde{H}(x) \) to be \( |u\rangle = \alpha |0\rangle - |w\rangle \). Clearly \( \Pi_x |u\rangle = |u\rangle \), but also, \( |u\rangle \) is in the kernel of \( \tilde{A}^\alpha \), because \( \tilde{A}^\alpha |u\rangle = |\tau\rangle - |\tau\rangle = 0 \). Thus \( \tilde{A}^\alpha |u\rangle = |u\rangle \), and so \( U(P, x, \alpha)|u\rangle = |u\rangle \); \( |u\rangle \) is a 1-valued eigenvector of \( U(P, x, \alpha) \).

We perform phase estimation on the state \( |0\rangle \), so the probability of measuring the state \( |0\rangle_B \) in the phase register is at least \( 1 \) (by Lemma 1), the overlap of \( |0\rangle \) and (normalized) \( |u\rangle \). This is

\[
\frac{|\langle 0 | u \rangle|^2}{||u||^2} = \frac{\alpha^2}{\alpha^2 + ||w||^2} = \frac{1}{1 + \frac{w(P, x)}{\alpha^2}}.
\]

(31)

Using our assumption that \( w(P, x) \leq \alpha^2/C \), and a Taylor series expansion for \( C \geq 2 \), the probability that we measure the state \( |0\rangle_B \) is at least \( 1 - 1/C \).

Part 2: Since \( f(x) = 0 \), there is an optimal negative witness \( \omega \) for \( x \), and we set \( |v\rangle \in \tilde{H} \) to be \( |v\rangle = \alpha (\omega \tilde{A}^\alpha)^\dagger \). By Definition 7, \( \omega \tau = 1 \), so \( |v\rangle = (|0\rangle + \omega A)^\dagger \). Again, from Definition 7, \( \omega \Pi_H(x) = 0 \), so we have \( \Pi_x |v\rangle = |0\rangle \).

Then when we perform phase estimation of the unitary \( U(P, x, \alpha) \) to some precision \( \Theta \) with error \( \epsilon \) on state \( |0\rangle \), by Lemma 1, we will measure \( |0\rangle_B \) in the phase register with probability at most

\[
||P_\Theta |0\rangle||^2 + \epsilon = ||P_\Theta \Pi_x |v\rangle||^2 + \epsilon,
\]

(32)

where throughout this proof, \( P_\Theta \) is understood to be \( P_\Theta(U(P, x, \alpha)) \).

Now \( |v\rangle \) is orthogonal to the kernel of \( \tilde{A}^\alpha \). (To see this, note that if \( |k\rangle \) is in the kernel of \( \tilde{A}^\alpha \), then \( \langle k | k \rangle = \alpha \omega \tilde{A}^\alpha |k\rangle = 0 \).) Applying Lemma 4, and setting \( \Theta = \sqrt{\frac{\alpha^2}{W}} \), we have

\[
||P_\Theta \Pi_x |v\rangle||^2 \leq \frac{\epsilon}{\alpha^2 W} |||v\rangle||^2.
\]

(33)

To bound \( |||v\rangle||^2 \), we observe that

\[
|||v\rangle||^2 = ||\langle 0 | + \alpha \omega A ||^2 = 1 + \alpha^2 w(P, x) \leq 2\alpha^2 W.
\]

(34)

Plugging Eqs. (33) and (34) into Eq. (32), we find that the probability of measuring \( |0\rangle_B \) in the phase register is at most \( 3\epsilon \), as claimed.

In Lemmas 21 to 24, to reduce notation, we use the following simplified conventions. Let \( \mathcal{P} = \{ |u_{ij}\rangle \}, \{ |v_{ij}\rangle \} \) be a converting vector set from \( \rho \) to \( \sigma \) and let \( W = W(\mathcal{P}) \). Let \( U = U(P, x, \alpha, \hat{\xi}) \). We denote \( P_\Theta(U) \) and \( \Pi_0(U) \) as \( P_\Theta \) and \( \Pi_0 \) respectively, where \( \Pi_0 \) refers to the projection onto the subspace that is mapped to the \( |0\rangle \) phase register when phase estimation on \( U \) is run with precision \( \hat{\xi}^{3/2}/\sqrt{\alpha W} \) and accuracy \( \hat{\xi}^2 \). Likewise \( R \) refers to \( R(U) \) (Phase Reflection) with precision \( \hat{\xi}^{3/2}/\sqrt{\alpha W} \) and accuracy \( \hat{\xi}^2 \). When we write \( \langle 0 | |0\rangle \rangle \), note that the first \( |0\rangle \) is a single qubit register, while the final \( |0\rangle \) is an \( O \left( \log \frac{\sqrt{\alpha W}}{\hat{\xi}^{3/2}} \log \frac{1}{\hat{\xi}^2} \right) \) qubit register. Likewise the final \( |0\rangle \) in expressions like \( |t_{x^+}\rangle |0\rangle \) is an \( O \left( \log \frac{\sqrt{\alpha W}}{\hat{\xi}^{3/2}} \log \frac{1}{\hat{\xi}^2} \right) \) qubit register.

Lemma 21. If \( \Theta = \hat{\xi}^{3/2}/\sqrt{\alpha W} \), then \( ||P_\Theta(U(P, x, \alpha, \hat{\xi}^2))|t_{x-}\rangle ||^2 \leq \frac{\hat{\xi}^2}{2} \).

\[
\begin{align*}
\text{Lemma 21.} & \quad \text{If} \quad \Theta = \hat{\xi}^{3/2}/\sqrt{\alpha W}, \quad \text{then} \quad ||P_\Theta(U(P, x, \alpha, \hat{\xi}))|t_{x-}\rangle ||^2 \leq \frac{\hat{\xi}^2}{2}. \\
\end{align*}
\]
Proof. Let $|w⟩ = \sqrt{\frac{2}{\alpha}}|ψ_{x,α,ξ}⟩$, so $\Lambda^{α,ξ}|w⟩ = 0$ and $Π_x|w⟩ = |t_x⟩$. Applying Lemma 4, we have

$$\|P_0|t_x⟩\| = \|P_0Π_x|w⟩\| \leq \frac{\sqrt{2}}{4}\|w\|.$$  \hspace{1cm} (35)

Now

$$\|w\| = \frac{α}{ξ}\left(\frac{θ}{α}(t_{x-} - Σ_j⟨u_{xj}|u_{xj}\rangle)\right) \leq 1 + \frac{αW}{ξ^2}. \hspace{1cm} (36)$$

Combining Eqs. (35) and (36), and setting $Θ = \frac{ξ^{3/2}}{\sqrt{αW}}$, we have that

$$\|P_0|t_x⟩\| \leq \frac{ξ^3}{4αW} \left(1 + \frac{αW}{ξ^2}\right) \leq \frac{ξ^2}{2}. \hspace{1cm} (37)$$

\hfill \Box

Lemma 22. If $α ≥ w_+(P, x)$, then $\|Π_0(Π(P, x, α, ξ))(|0⟩|Ψ_α⟩)A|0⟩B\| ≥ \frac{1}{2}(1 - 5ξ).$

Proof. We first prove that $\|P_0|t_x⟩\|^2 ≥ 1 - ξ$, by following [20, Claim 4.4]. Consider the state

$$|φ⟩ = |t_x⟩ \pm \sqrt{\frac{2(k - 1)}{α}} \sum_j |j⟩⟨ν_{xj}|v_{xj}⟩.$$  \hspace{1cm} (38)

Note that for all $|ψ_{y,α,ξ}⟩$, because $⟨y_−|t_x⟩ = \frac{1}{4}((ρ_0|ρ_x⟩ - ⟨σ_y|σ_x⟩)$, and also $\sum_j \sum_{j:j≠y_j} ⟨u_{yj}|v_{xj}⟩ = (ρ_y|ρ_x⟩ - (σ_y|σ_x⟩$ (see Eq. (5))

$$⟨y_−|φ⟩ = \sqrt{\frac{2(k - 1)}{α}}t_{y−} \sum_j \sum_{j:j≠y_j} ⟨u_{yj}|v_{xj}⟩ = 0. \hspace{1cm} (39)$$

Because $|φ⟩$ is orthogonal to all of the $|ψ_{y,α,ξ}⟩$, we have $Λ^{α,ξ}|φ⟩ = |φ⟩$. Also, $Π_x|φ⟩ = |φ⟩$ since $Π_x|t_x⟩ = |t_x⟩$ and $⟨η_{xj}|ν_{xj}⟩ = 0$ for every $j$. Thus $P_0|φ⟩ = |φ⟩$. Note

$$⟨φ|φ⟩ = 1 + \frac{ξ}{4α} \frac{4(k - 1)^2}{α^2}w_+(P, x) < 1 + ξ, \hspace{1cm} (40)$$

because of our assumption that $α ≥ w_+(P, x).$ Also, $⟨t_x|φ⟩ = 1$, so

$$\|P_0|t_x⟩\| > \frac{1}{1 + ξ} > 1 - ξ. \hspace{1cm} (41)$$

Then by Lemma 1, we have $\|P_0|t_x⟩\|^2 ≤ \|Π_0|t_x⟩|0⟩\|^2$, so

$$\|Π_0|t_x⟩|0⟩\|^2 > 1 - ξ. \hspace{1cm} (42)$$

Now we turn to analyzing $\|Π_0|0⟩|Ψ_α⟩⟩$. Using the triangle inequality, we have

$$\|Π_0|0⟩|Ψ_α⟩⟩ \geq \frac{1}{\sqrt{2}}\left(\|Π_0|t_+⟩|0⟩\| - \|Π_0|t_−⟩|0⟩\|\right). \hspace{1cm} (43)$$

Inserting $I = P_0 + \mathcal{P}_θ$ with $Θ = \frac{ξ^{3/2}}{\sqrt{αW}}$ into the right-most term and using the triangle inequality again, we have

$$\|Π_0|t_−⟩|0⟩\| ≤ \|Π_0(P_0|t_−⟩)|0⟩\| + \|Π_0(\mathcal{P}_θ|t_−⟩)|0⟩\|. \hspace{1cm} (44)$$

Now

$$\|Π_0(P_0|t_−⟩)|0⟩\| ≤ \|P_0|t_−⟩\| ≤ \frac{ξ}{\sqrt{2}}, \hspace{1cm} (45)$$

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by Lemma 21. Since we are running Phase Checking to precision $\bar{\varepsilon}^{3/2}/\sqrt{\alpha W}$ and accuracy $\bar{\varepsilon}^2$, from Lemma 1 we get
\[
\|\Pi_0(\tilde{P}_\Theta | t_{x-})|0\| \leq \bar{\varepsilon}.
\]
(46)

Plugging Eqs. (42) and (44) to (46) into Eq. (43), we obtain
\[
\|\Pi_0|0\rangle|\rho_x\rangle|0\| \geq \frac{1}{\sqrt{2}} \left( \sqrt{1 - \bar{\varepsilon}^2} - \bar{\varepsilon} \left( 1 + \frac{1}{\sqrt{2}} \right) \right)
\]
(47)

Using a series expansion, we find
\[
\|\Pi_0|0\rangle|\rho_x\rangle|0\| > \frac{1}{2} - \left( \frac{3}{2} + \frac{1}{\sqrt{2}} \right) \bar{\varepsilon} > \frac{1}{2} - \frac{5}{2} \bar{\varepsilon}.
\]
(48)

\[\square\]

Lemma 23. If $\|\Pi_0(U(P, x, \alpha, \bar{\varepsilon}))|0\rangle|\rho_x\rangle|0\| \geq 1/2 - 3\bar{\varepsilon}$, then $\|\Pi_0(U(P, x, \alpha, \bar{\varepsilon}))|t_{x+}|0\| \leq 10\bar{\varepsilon}$.

Proof. Starting from our assumption, writing $|0\rangle|\rho_x\rangle$ in terms of $|t_{x+}\rangle$ and $|t_{x+}\rangle$, and using the triangle inequality, we have
\[
\frac{1}{2} - 3\bar{\varepsilon} \leq \frac{1}{2} \|\Pi_0(|t_{x+}\rangle + |t_{x-}\rangle)|0\| \leq \frac{1}{2} \left( \|\Pi_0|t_{x+}\rangle|0\| + \|\Pi_0|t_{x-}\rangle|0\| \right)^2.
\]
(49)

We first bound the term $\|\Pi_0|t_{x-}\rangle|0\|$. Inserting the identity operator $I = P_\Theta + \tilde{P}_\Theta$ for $\Theta = \bar{\varepsilon}^{3/2}/\sqrt{\alpha W}$, we have
\[
\|\Pi_0|t_{x-}\rangle|0\| = \|\Pi_0 \left( (P_\Theta + \tilde{P}_\Theta) |t_{x-}\rangle \right) |0\|
\leq \|P_\Theta|t_{x-}\rangle\| + \|\Pi_0 \left( \tilde{P}_\Theta |t_{x-}\rangle \right) |0\|
\leq \frac{\bar{\varepsilon}}{\sqrt{2}} + \bar{\varepsilon} \leq 2\bar{\varepsilon}.
\]
(50)

We have
\[
1 - 6\bar{\varepsilon} \leq (\|\Pi_0|t_{x+}\rangle|0\| + 2\bar{\varepsilon})^2.
\]
(51)

Rearranging, we find:
\[
(\sqrt{1 - 6\bar{\varepsilon}} - 2\bar{\varepsilon})^2 \leq \|\Pi_0|t_{x+}\rangle|0\|^2.
\]
(52)

Since $\|\Pi_0|t_{x+}\rangle|0\|^2 + \|\Pi_0|t_{x+}\rangle|0\|^2 = 1$, we have
\[
\|\Pi_0|t_{x+}\rangle|0\|^2 \leq 1 - (\sqrt{1 - 6\bar{\varepsilon}} - 2\bar{\varepsilon})^2 < 10\bar{\varepsilon}.
\]
(53)

Lemma 24. If $\|\Pi_0(U(P, x, \alpha, \bar{\varepsilon}))|0\rangle|\rho_x\rangle|A|0\rangle_B\| \geq 1/2 - 3\bar{\varepsilon}$, then $\|R(U(P, x, \alpha, \bar{\varepsilon}))|0\rangle|\rho_x\rangle|A|0\rangle_B - |(1)|\sigma_x\rangle|A|0\rangle_B\| \leq 6\sqrt{\bar{\varepsilon}}$

Proof of Lemma 24. We have
\[
\|R|0\rangle|\rho_x\rangle|0\rangle - |1\rangle|\sigma_x\rangle|0\rangle\| = \frac{1}{\sqrt{2}} \|R(|t_{x+}\rangle + |t_{x-}\rangle)|0\rangle - (|t_{x+}\rangle - |t_{x-}\rangle)|0\rangle\|
\leq \frac{1}{\sqrt{2}} \|(|R - I)|t_{x+}\rangle|0\rangle\| + \frac{1}{\sqrt{2}} \|(|R + I)|t_{x-}\rangle|0\rangle\|.
\]
(54)
In first term, we can replace $R$ with $\Pi_0 - \overline{\Pi}_0$ (as described above Lemma 2), and we can insert $I = \Pi_0 + \overline{\Pi}_0$ to get
\[
\frac{1}{\sqrt{2}}\|(R - I)t_{x+}\|0\| = \frac{1}{\sqrt{2}}\|((\Pi_0 - \overline{\Pi}_0) - I)(\Pi_0 + \overline{\Pi}_0)t_{x+}\|0\|
\] (58)

Using the fact that $\Pi_0$ and $\overline{\Pi}_0$ are orthogonal, this simplifies to
\[
\frac{2}{\sqrt{2}}\|\Pi_0|t_{x+}\|0\| \leq 2\sqrt{5\varepsilon},
\] (59)

by Lemma 23.

In the second term, we insert $I = P_\Theta + \overline{P}_\Theta$ and use the triangle inequality to get
\[
\frac{1}{\sqrt{2}}\|(R + I)t_{x-}\|0\| \leq \frac{1}{\sqrt{2}}\|(R + I)(P_\Theta|t_{x-}\rangle)\|0\| + \frac{1}{\sqrt{2}}\|(R + I)(\overline{P}_\Theta|t_{x-}\rangle)\|0\|.
\] (60)

By Lemma 2, $\frac{1}{\sqrt{2}}\|(R + I)(\overline{P}_\Theta|t_{x-}\rangle)\|0\| \leq \frac{\varepsilon^2}{\sqrt{2}}$, and
\[
\frac{1}{\sqrt{2}}\|(R + I)(P_\Theta|t_{x-}\rangle)\|0\| \leq \frac{2}{\sqrt{2}}\|P_\Theta|t_{x-}\rangle\| \leq \hat{\varepsilon}.
\] (61)

by a triangle inequality and Lemma 21.

Combining all of these bounds together, we have
\[
\|R|0\rangle\rho_x\|0\| - |1\rangle|\sigma_x\|0\| \leq 2\sqrt{5\varepsilon} + \hat{\varepsilon} + \frac{\varepsilon^2}{\sqrt{2}} < 6\sqrt{\varepsilon}.
\] (62)