COMPUTING MINIMAL WEIERSTRASS EQUATIONS OF HYPERELLIPTIC CURVES

QING LIU

Abstract. We describe an algorithm for determining a minimal Weierstrass equation for hyperelliptic curves over principal ideal domains. When the curve has a rational Weierstrass point \( w_0 \), we also give a similar algorithm for determining the minimal Weierstrass equation with respect to \( w_0 \).

Tate’s algorithm [7] determines the reduction type of elliptic curves over discrete valuation rings with perfect residue field. In particular it determines the minimal Weierstrass equation. Over number fields with trivial class number, Laska [2] gave a faster method to determine the minimal Weierstrass equation.

More generally, for any hyperelliptic curve \( C \) of genus \( g \geq 1 \) over a discrete valuation ring, there is a natural notion of minimal Weierstrass equations. This question is studied in [5] and [3]. An algorithm for determining a minimal Weierstrass equation is sketched in [3]. See also [1], §4, for hyperelliptic curves of genus 1 (not necessarily elliptic). Here we deal with hyperelliptic curves of any genus \( g \geq 1 \) and also with those having a rational Weierstrass point (in which case there is a notion of minimal pointed Weierstrass equation as for elliptic curves, see [5] or below) over principal ideal domains. The aim of the present work is to make our algorithm completely explicit. It is now implemented over \( \mathbb{Z} \) in PARI release 2.15 ([6]) by B. Allombert.

Let us briefly present the content of this work. Let \( A \) be a principal ideal domain with perfect residue fields at its maximal ideals. Let \( C \) be a hyperelliptic curve of genus \( g \geq 1 \) over \( K := \text{Frac}(A) \), given by an integral Weierstrass equation

\[
y^2 + Q(x)y = P(x)
\]

2010 Mathematics Subject Classification. 11G30, 11G20.

I would like to thank Bill Allombert for clarifications regarding some computational aspects in this article and for pointing out related references. I would also like to thank the referees for their thorough reading. Thank you also to the referees and Bill Allombert for suggestions which led to improvements in the presentation of this manuscript.
with \( Q(x), P(x) \in A[x] \) such that \( \deg Q \leq g + 1 \) and \( \deg P \leq 2g + 2 \). Such an equation is said to be minimal, when \( A \) is local, if its discriminant \( \Delta \) ([3], §2) has the smallest valuation among all integral Weierstrass equations describing the same curve over \( K \). When \( A \) is global, the equation is said to be minimal if it is minimal at all localizations of \( A \) at maximal ideals. It is well known that such an equation exists because \( A \) is principal, see for instance [3], §3, Proposition 2. Note that for a given \( C \) there are finitely many minimal Weierstrass equations (up to the natural action of \( \text{GL}_2(A) \)), but in general they are not unique. For instance, an elliptic curve of type \( I_n \) (\( n \geq 1 \)) over a discrete valuation ring has \( n \) non-equivalent (Definition 1.1) minimal Weierstrass equations as (non pointed) hyperelliptic curve.

When \( C \) has a rational Weierstrass point \( w_0 \), an integral pointed Weierstrass equation of \((C, w_0)\) is an equation

\[
y^2 + Q(x)y = P(x)
\]

over \( A \) with \( \deg Q \leq g \), \( P(x) \) monic of degree \( 2g + 1 \) and such that \( w_0 \) is the pole of \( x \). A minimal pointed Weierstrass equation exists and is unique up to the transformations

\[
x = u^2x_1 + c, \quad y = u^{2g+1}y_1 + H(x)
\]

with \( u \in A^* \), \( c \in A \) and \( H(x) \in A[x] \). See Lemma 5.4.

The principle of the minimization (i.e. finding a minimal Weierstrass equation) we use is to successively minimize at a finite list of bad primes. At each bad prime \( p \), we first normalize the equation (i.e. find a Weierstrass equation defining a normal scheme). This is very simple at primes of odd residue characteristic, but requires an appropriate algorithm otherwise (Algorithm 6.1). Then the minimality at \( p \) is checked by computing the multiplicities \( \lambda(p_0) \) at very special rational points \( p_0 \) of the reduction mod \( p \) (see Proposition 4.3). These points correspond to roots of high order of some polynomials over the residue field at \( p \) (Lemma 4.5). If the minimality condition is not satisfied, a new candidate (normal) Weierstrass equation is given (§ 4.1) and we can restart the minimality checking.

To get a global minimal Weierstrass equation, we notice that our minimization process at a prime dividing \( 2 \) does not change the valuation of the discriminant of the initial equation at other primes. Similarly the minimization process at odd primes will keep unchanged the discriminant of the initial equation at the other odd primes, but it will affect the discriminant at primes dividing \( 2 \). Nevertheless we use an easy trick to combine minimal Weierstrass equations at odd primes and at primes dividing \( 2 \) (Lemma 4.7). The strategy is similar for pointed
Weierstrass equations. The procedure is then much simpler because we only use transformations of the form \( x = u^2 x_1 + c, \ y = u^{2g+1} y_1 + H(x) \).

Note that it is essential here to suppose \( A \) is principal (or at least that the primes dividing the discriminant of an initial equation over \( A \) are principal), as otherwise a global minimal Weierstrass equation may not exist.

In §2 and §3 we explain the process of normalization and the computation of the multiplicity \( \lambda \). In §4 and §5 we give the minimality criterion respectively for Weierstrass equations and pointed Weierstrass equations. Finally algorithms to find minimal (resp. pointed) Weierstrass equations are described in the last two sections.

**Notation** We denote by \( A \) a principal ideal domain with field of fractions \( K \) such that its residue fields at maximal ideals are perfect. Primes of \( A \) will be denoted by \( p \). When \( p \) is fixed, we denote by \( k = k(p) \) the residue field at \( p \), \( p = \text{char}(k) \) and \( \pi \in A \) a generator of \( p \).

For any \( H(x) \in A[x] \), its image in \( k[x] \) is denoted by \( \bar{H}(x) \).

The normalized valuation on \( K \) defined by \( p \) will be denoted by \( v_p \) or just \( v \) if there is no ambiguity.

An element \( a \in A \) is odd if \( a \neq 0 \) and \( aA + 2A = A \). We call \( p \) an odd prime if \( p + 2A = A \). Otherwise it is called an even prime.

1. **Weierstrass models**

**Definition 1.1** The projective scheme over \( A \) defined by Equation (1) is denoted by \( W \). It is obtained by gluing the affine schemes

\[
W_0 = \text{Spec } A[x, y]/(y^2 + Q(x)y - P(x))
\]

and

\[
W_\infty = \text{Spec } A[t, z]/(z^2 + t^{g+1}Q(t^{-1})z - t^{2g+2}P(t^{-1}))
\]

along the identification \( t = 1/x, z = y/x^{g+1} \). This scheme is integral and flat over \( A \), with generic fiber isomorphic to \( C \). This is a Weierstrass model of \( C \) over \( A \). It is said to be normal if \( W \) is a normal scheme. An isomorphism of Weierstrass models of \( C \) is an isomorphism of \( A \)-schemes compatible with the isomorphisms with \( C \).

If \( W \times_{\text{Spec } A} \text{Spec } A_p \) satisfies some property, then we say that \( W \) satisfies this property at \( p \).

We say that two Weierstrass equations of \( C \) over \( A \) are equivalent at \( p \) if the associated Weierstrass models are isomorphic over \( A_p \). This
implies that their discriminants have the same valuation at $p$ (see below).

**Definition 1.2** Let $F(x) = 4P(x) + Q(x)^2$ with leading coefficient $a$. Recall ([3], §2) that when $\text{char}(K) \neq 2$, the discriminant $\Delta$ of Equation (1) is given by

$$
\Delta = \begin{cases} 
2^{-4(g+1)} \text{disc}(F) & \text{if } \deg F = 2g + 2 \\
2^{-4(g+1)} a^2 \text{disc}(F) & \text{if } \deg F = 2g + 1.
\end{cases}
$$

So if $Q = 0$, then $\Delta = 2^{4g} \text{disc}(P)$.

Other integral Weierstrass equations of $C$ are obtained by change of variables

$$
x = ax_1 + b, \quad y = ey_1 + H(x_1)(cx_1 + d)^{g+1},
$$

with $a, b, c, d, e \in A$, $H(x_1) \in A[x_1]$, and $ad - bc, e \neq 0$. The corresponding Weierstrass models are isomorphic if and only if $ad - bc, e \in A^*$ (invertible).

The discriminant $\Delta_1$ of the equation with $x_1, y_1$ is given by

$$
\Delta_1 = e^{-4(2g+1)}(ad - bc)^2(2g+1)\Delta.
$$

2. **Normalization**

**Notation 2.1** Let $p$ be a prime of $A$. For any $H(x) = \sum a_i x^i \in A[x]$, we denote by

$$
v_p(H) := \min_{i} \{ v_p(a_i) \} \in \mathbb{N} \cup \{ +\infty \}
$$

(or just $v(H)$). This is the Gauss valuation on $K(x)$ with respect to the variable $x$ extending $v_p$.

Let $W$ be the Weierstrass model over $A$ defined by Equation (1).

**Lemma 2.2** (Normalization away from 2). Suppose that $\text{char}(K) \neq 2$. Let $e^2$ be a biggest odd square factor of the content $\text{cont}(4P + Q^2)$ of $4P + Q^2$. Then the equation

$$
z^2 = e^{-2}(4P(x) + Q^2(x))
$$

defines the normalization of $W$ at all odd primes of $A$.

**Proof.** See [3], Lemme 2(d), p. 4582. $\square$

**Lemma 2.3** (Normalization at even primes). Let $p$ be an even prime of $A$.

(1) If one of the following conditions is satisfied, then $W$ is normal at $p$:
(a) \( v(Q) = 0 \) (then \( W_k \) is reduced);
(b) \( v(Q) > 0 \) and \( P(x) \) is not a square in \( k[x] \);
(c) \( v(Q) > 0 \) and \( v(P) = 1 \).

(2) If the pair \( Q(x), P(x) \) satisfies one of the above conditions, then the pair \( x^{g+1}Q(1/x), x^{2g+2}P(1/x) \) satisfies the same condition.

(3) If the pair \( Q(x), P(x) \) satisfies none of the conditions of (1), there exists a change of variables
\[
y = \pi^n y_1 + H(x)
\]
with \( n \geq 0 \) and \( H(x) \in A[x] \) such that \( Q_1(x) := \pi^{-n}(Q - 2H) \), \( P_1(x) := \pi^{-2n}(P + QH - H^2) \) belong to \( A[x] \) and satisfy one of the conditions of (1). The new equation
\[
y_1^2 + Q_1(x)y_1 = P_1(x)
\]
with \( y_1 = \pi^{-n}(y + H) \), then defines the normalization of \( W \).

Proof. (1) The normality under (a) or (b) holds by [3], Lemme 2(a)-(b), p. 4582. Use the same lemma, Part (c) when Condition (c) is satisfied.

(2) is straightforward.

(3) The construction of \( y_1 \) is given in Algorithm 6.1. \( \square \)

Remark 2.4 (1) The transformation in Lemma 2.3(3) does not affect the discriminant at other primes than \( p \), as the new discriminant is the former discriminant divided by a power of \( \pi \). On the other hands, the normalization process at odd primes in Lemma 2.2 multiplies the discriminant by \( (2e^{-1})^{4(2g+1)} \), therefore it does modify its even part.

(2) The converse of Lemma 2.3(1) is false. For example the equation \( y^2 = x^6 + 1 \) over \( \mathbb{Z}_2 \) is normal (even minimal), but it does not satisfy any of the conditions of 2.3(1). However the converse may fail only when \( v(Q) > 0 \) and \( P \) is a non-zero square in \( k[x] \) (indeed \( W \) normal implies that if \( v(Q) > 0 \) then \( v(P) \leq 1 \)). In this case the transformation at step (2.b) of Algorithm 6.1 provides a pair \( Q_1, P_1 \) for \( W \) satisfying 2.3(1.c).

3. Multiplicity \( \lambda \)

Fix a prime \( p \) of \( A \). Recall that \( v = v_p \), \( k = k(p) \) and \( p \) is the characteristic of \( k \).

Definition 3.1 Let \( H(x) \in A[x] \), let \( c \in A \). We write \( H(x) \) in the form (Taylor expansion at \( c \)):
\[
H(x) = \sum_i a_{c,i}(x - c)^i,
\]
and define

\[ \mu_c(H) = \min \{v(a_{c,i}) + i\} = v(H(\pi x + c)). \]

The map \( \mu_c \) depends only on the class of \( c \) modulo \( \pi \). We have \( \mu_c(0) = +\infty \) and, if \( H \neq 0 \), then \( \mu_c(H) \) is the biggest integer \( m \) such that \( H(x) \in (x-c, \pi)^m A[x] \). In fact \( \mu_c \) is the restriction to \( A[x] \) of the Gauss valuation on \( K(x) \) with respect to the variable \( \pi x + c \). Thus we have

\[ \mu_c(H_1 H_2) = \mu_c(H_1) + \mu_c(H_2), \quad \mu_c(H_1 + H_2) \geq \min \{\mu_c(H_1), \mu_c(H_2)\}. \]

For any pair of polynomials \( Q, P \in A[x] \), denote by

\[ \lambda_c(Q, P) = \min \{2\mu_c(Q), \mu_c(P)\}. \]

As \( \mu_c \) is a valuation, when \( p \neq 2 \) we have

\[ \lambda_c(Q, P) = \lambda_c(Q, 4P) \leq \mu_c(4P + Q^2) \]

If we denote by \( \text{ord}_c \) the vanishing order of \( H(x) \in k[x] \) at \( c \), then it follows immediately from the definition that

\[ \text{ord}_c H \geq \mu_c(H). \]

**Definition 3.2** ([3], Définition 10, p. 4589, case \( r = 1 \)) Let \( p_0 \in W_0(k) \) (see Definition 1.1) be a rational point. Let \( \bar{c} = x(p_0) \in k \) be the \( x \)-coordinate of \( p_0 \) for some \( c \in A \). We define the multiplicity \( \lambda(p_0) \) by

\[ \lambda(p_0) = \max \{\lambda_c(Q - 2H, P + QH - H^2) \mid H(x) \in A[x]\} \]

(we have \((y + H)^2 + (Q - 2H)(y + H) = P + QH - H^2\).)

The multiplicity of the pole of \( x \) in \( W_k \) is defined as the multiplicity at 0 of \( z^2 + x^{g+1}Q(1/x) = x^{2g+2}P(1/x) \).

Note that

\[ \lambda(p_0) = \mu_c(4P + Q^2), \quad \text{if} \quad p \neq 2. \]

Indeed, \( \lambda(p_0) \leq \mu_c(4P + Q^2) \) by Inequality (7), and the inverse inequality holds by taking \( H = Q/2 \) in (9).

The next lemma explains how to compute \( \lambda(p_0) \) and, starting with a suitable equation, how to find a new pair \( Q, P \) such that \( \lambda(p_0) = \lambda_c(Q, P) \). This is partly sketched in [3], bottom of page 4590.

**Lemma 3.3.** Suppose that \( p = 2 \), \( W \) is normal at \( p \), and that the equation

\[ y^2 + Q(x)y = P(x) \]

satisfies Lemma 2.3(1).

(1) If \( 2\mu_c(Q) \leq \mu_c(P) \), then \( \lambda(p_0) = 2\mu_c(Q) = \lambda_c(Q, P) \).
(2) Suppose $2\mu_c(Q) > \mu_c(P)$.
   (a) If $\mu_c(P)$ is odd, then $\lambda(p_0) = \mu_c(P) = \lambda_c(Q, P)$.
   (b) Suppose $\mu_c(P) = 2r \geq 0$ is even. Write $P(x) = \sum_i a_{c,i}(x - c)^i$.
      (i) If $\mu_c(P) = v(a_{c,i}) + i$ for some odd $i$, then $\lambda(p_0) = 
       \mu_c(P) = \lambda_c(Q, P)$.
      (ii) Otherwise, let $\lambda_c(Q - 2H_0, P + QH_0 - H_0^2) > \lambda_c(Q, P)$.

We have $\lambda(p_0) \leq 2g + 3$. Moreover, if $\lambda_c(Q, P) < \lambda(p_0)$, then there exists a new pair $Q_0, P_0 \in A[x]$ for $W$ such that
   (a) $Q_0 \equiv Q \mod 2$ and $P_0 - P$ is congruent to a square in $k[x]$ modulo $(\pi, Q)$;
   (b) in the case $v(Q) > 0$ and $v(P) = 1$, we have $Q_0 \equiv Q \mod 2\pi$,
       $P_0 - P \equiv 0 \mod \pi^2$;
   (c) The pair $Q_0, P_0$ satisfies the same condition in Lemma 2.3(1) as $Q, P$;
   (d) $\lambda_c(Q_0, P_0) = \lambda(p_0)$.

Proof. (1) We have $Q \neq 0$ because $\mu_c(0) = +\infty$. Let $H \in A[x]$. If $\mu_c(Q - 2H) \leq \mu_c(Q)$, then
   $\lambda_c(Q - 2H, P + QH - H^2) \leq 2\mu_c(Q - 2H) \leq 2\mu_c(Q) = \lambda_c(Q, P)$.

Suppose $\mu_c(Q - 2H) > \mu_c(Q)$. Then $\mu_c(2H) = \mu_c(Q)$, so $\mu_c(H) = 
\mu_c(Q) - v(2)$. As $\mu_c(P) \geq 2\mu_c(Q)$, $\mu_c(QH) = 2\mu_c(Q) - v(2)$, $\mu_c(H^2) = 2\mu_c(Q) - 2v(2)$
we have $\mu_c(P + QH - H^2) = 2\mu_c(Q) - 2v(2) < 2\mu_c(Q - 2H)$ and
   $\lambda_c(Q - 2H, P + QH - H^2) = 2\mu_c(Q) - 2v(2) < 2\mu_c(Q) = \lambda_c(Q, P)$.

So $\lambda(p_0) = \lambda_c(Q, P)$.

(2) Suppose that $\lambda(p_0) > \lambda_c(Q, P)$. So there exists $H \in A[x]$ such that
   $\lambda_c(Q - 2H, P + QH - H^2) > \lambda_c(Q, P) = \mu_c(P)$.

We have $P + QH - H^2 = P + (Q - 2H)H + H^2$. This implies that
   $\mu_c(P) = 2\mu_c(H) = 2r \in 2\mathbb{N}$ (hence (2.a) is proved) and $\mu_c(P + H^2) >$
2r. Then
\[ \pi^{-2r}P(\pi x + c) \equiv -(\pi^{-r}H(\pi x + c))^2 \mod \pi. \]
Therefore, if \( \lambda(p_0) > \mu_c(P) \), then \( \mu_c(P) = \lambda_c(Q, P) \) is only reached by terms of even degrees \( a_c(x) \). This proves (2.b.i).

(2.b.ii) Now we have \( \mu_c(P) = 2r \) and \( \mu_c(P) < v(a_{c,i}) + i \) for all odd \( i \). By construction we see that \( \mu_c(H_0) = r, \mu_c(P - H_0^2) > 2r \), and that \( v(H_0) = 0 \) if and only if \( r \leq g + 1 \) and \( v(a_{2r}) = 0 \). As \( \mu_c(QH_0) > 2r \)
and
\[ 2\mu_c(Q - 2H_0) \geq \min\{2\mu_c(Q), 2(r + v(2))\} > 2r, \]
we have \( \lambda_c(Q - 2H_0, P + QH_0 - H_0^2) > \lambda_c(Q, P) \).

(3) In the case (2.b.ii) we let temporarily \( Q_0 = Q - 2H_0, P_0 = P + QH_0 - H_0^2 \). We may need to modify them later. Property (a) is satisfied by construction. We will first prove the inequality \( \lambda_c(Q, P) \leq 2g + 3 \) and, in the case (2.b.ii), the same inequality for \( Q_0, P_0 \) and the property (b). Property (c) is a direct consequence of (a) and (b).

Notice that from the construction, we have \( \mu_c(F) \leq v(F) + \deg F \)
for all \( F(x) \in A[x] \).

(3.1) If \( v(Q) = 0, \mu_c(Q) \leq \deg Q \leq g + 1 \). Thus \( \lambda_e(Q, P) \leq 2g + 2 \).
In the case (2.b.ii), \( v(Q_0) = 0, \) and \( \mu_c(Q_0) \leq \deg Q_0 \leq g + 1 \) because \( \deg H_0 \leq g + 1 \) by construction. Hence \( \lambda_c(Q_0, P_0) \leq 2g + 2 \).

(3.2) Suppose now that \( v(Q) > 0 \). If \( v(P) = 0 \), then \( \bar{P}(x) \notin k[x^2] \) by hypothesis and \( \mu_c(P) \leq \deg P \leq 2g + 2 \). Moreover, in the case (2.b.ii),
\[ P_0 = P + QH_0 - H_0^2 \equiv P - H_0^2 \not\equiv 0 \mod \pi, \]
and \( \mu_c(P_0) \leq 2g + 2 \) as well.

(3.3) Suppose that \( v(Q) > 0 \) and \( v(P) = 1 \). Then \( \mu_c(P) \leq 1 + \deg P \leq 2g + 3 \). In the case (2.b.ii), \( v(H_0) > 0 \) because \( v(a_i) = 0 \) for all \( i \). So \( Q_0 - Q = 2H_0 \equiv 0 \mod 2\pi \) and \( v(P_0 - P) \geq 2 \). Thus \( v(P_0) = 1 \) and \( \mu_c(P_0) \leq \deg P_0 + 1 \leq 2g + 3 \).

To prove (3.d), if \( \lambda(p_0) = \lambda_e(Q_0, P_0) \) then we are done. Otherwise, we repeat the same operations with \( Q_0, P_0 \). As the \( \lambda_c \) increases strictly, we will end-up with a pair having \( \lambda_c \) equal to \( \lambda(p_0) \).

\textbf{Lemma 3.4.} Keep the assumptions and notation of Lemma 3.3 and suppose that \( \lambda(p_0) = \lambda_e(Q, P) \). Let \( r = [\lambda(p_0)/2], x_1 = \pi^{-1}(x - c), \) and
\[ Q_1(x_1) = \pi^{-r}Q(\pi x_1 + c), \quad P_1(x_1) = \pi^{-2r}P(\pi x_1 + c) \in A[x_1]. \]
Then the equation
\[ z^2 + Q_1(x_1)z = P_1(x_1) \]
\((z = y/\pi^r)\) defines a Weierstrass model \( W(p_0) \) of \( C \), normal at \( p \) with the pair \((Q_1, P_1)\) satisfying Lemma 2.2(1).
Proof. We have $v(Q_1) = \mu_c(Q) - r$ and $v(P_1) = \mu_c(P) - 2r$. Let us distinguish three cases.

(a) If $\lambda(p_0) = 2r = 2\mu_c(Q)$, then $v(Q_1) = 0$;
(b) If $\lambda(p_0) = 2r = \mu_c(P) < 2\mu_c(Q)$, then by (2.b), there exists an odd index $i_0$ such that $v(a_{i_0}) + i_0 = \mu_c(P)$. This implies that $\bar{P}_1(x)$ has a non-zero odd degree term. In particular $\bar{P}_1(x) \notin k[x^2]$;
(c) If $\lambda(p_0) = 2r + 1$. Then $\mu_c(Q) > r$, $\mu_c(P) = 2 + 1$ and $v(Q_1) > 0$, $v(P_1) = 1$.

So the pair $(Q_1, P_1)$ satisfies Lemma 2.3(1) and $W(p_0)$ is normal. □

4. Minimality criterion

We fix a prime $p$ of $A$. We will assume that $W$, defined by Equation (1), is normal at $p$. Moreover, if $p$ is even, we suppose that the pair $(Q, P)$ satisfies Lemma 2.3(1).

**Notation 4.1** We let $\epsilon(W) = 0$ if $W_k$ is a reduced scheme, and $\epsilon(W) = 1$ otherwise. If necessary it will be denoted by $\epsilon_p(W)$.

Under the above conditions, we have $\epsilon(W) = \min\{v(Q), v(P)\}$ if $p$ is even, and $\epsilon(W) = v(F) - 2[v(F)/2]$ if $p$ is odd and $F = 4P + Q^2$.

4.1. Dilatation. Let $p_0 \in W(k)$. Let $W(p_0)$ be the model defined by Equation (11). See also a more geometrical description in [3], Définition 12, p. 4592.

**Remark 4.2** Let $p_0 \in W(k)$.

(a) The birational map $W(p_0) \rightarrow W$ is an isomorphism at all primes different from $p$.

(b) Denote by $\Delta_W$ the discriminant of Equation (1) and $\Delta_{W(p_0)}$ that Equation (11). Then

$$v(\Delta_{W(p_0)}) - v(\Delta_W) = 2(2g + 1)(g + 1 - 2[\lambda(p_0)/2])$$

([3], Lemme 9(a), p. 4593).

(c) We have

$$\epsilon(W(p_0)) = \lambda(p_0) - 2[\lambda(p_0)/2].$$

(If $p$ is even, this is contained in the proof of Lemma 3.4).

4.2. Minimality criterion. The minimality of $W$ at $p$ can be determined by looking at the multiplicity $\lambda(p_0)$ at rational points of $W(k)$.

**Proposition 4.3** (Minimality criterion). Let $p$ be a prime of $A$. Suppose that $W$, defined by

$$y^2 + Q(x)y = P(x),$$

is normal at $p$. 
(1) If for all \( p_0 \in W(k) \) we have \( \lambda(p_0) \leq g + 1 \), then \( W \) is minimal at \( p \). The converse is true if \( g \) is even.\(^1\)

(2) Suppose \( g \) is odd and there exists \( p_0 \in W(k) \) with \( \lambda(p_0) \geq g + 2 \).

(a) If \( \lambda(p_0) \geq g + 3 \), then \( W \) is not minimal at \( p \). More precisely, 
\[ v(\Delta_{W(p_0)}) < v(\Delta_W). \]

(b) If \( \lambda(p_0) = g + 2 \) and \( \epsilon(W) = 1 \), then \( W \) is minimal at \( p \).

(c) If \( \lambda(p_0) = g + 2 \) and \( \epsilon(W) = 0 \), then \( W \) has the same discriminant as \( W(p_0) \) with \( \epsilon(W(p_0)) = 1 \). The model \( W(p_0) \) (hence \( W \)) is minimal at \( p \) if and only if for all \( q \in W(p_0)(k) \), we have \( \lambda(q) \leq g + 2 \).

Proof. (1) and (2.b) follow from [3], Corollaire 2, p. 4594 and Lemme 9(c), p. 4593. (2.a) and the first part of (2.c) follow from the equality (12). To finish the proof of (2.c), as \( \lambda(p_0) = g + 2 \) is odd, we have \( \epsilon(W(p_0)) = 1 \) by Remark 4.2(3). The pole of \( x \) in \( W(p_0) \) has multiplicity \( \lambda = g + 1 \) by op. cit., Lemme 9(b), p. 4593. This finishes the proof by (1) and (2.b). \( \square \)

Remark 4.4 Let us say that a multiplicity \( \lambda \) is small if \( \lambda \leq g + 1 \), or if \( g \) is odd and \( \lambda = g + 2 \) with \( \epsilon = 1 \) (Conditions (1) or (2.b) in Proposition 4.3). We say it is medium if \( g \) is odd, \( \lambda = g + 2 \) with \( \epsilon = 0 \) (Condition (2.c)). Otherwise we say it is big: \( \lambda \geq g + 2 \) and \( g \) is even or \( \lambda \geq g + 3 \) and \( g \) is odd.

(1) Proposition 4.3 then can be rephrased as following:

(i) If all rational points of \( W(k) \) have small multiplicities, then \( W \) is minimal;

(ii) if there is a rational point \( p_0 \in W(k) \) with big multiplicity, then \( W \) is not minimal and \( v(\Delta_{W(p_0)}) < v(\Delta_W) \);

(iii) if there is a rational point \( p_0 \in W(k) \) with medium multiplicity, then we work with \( W(p_0) \) whose discriminant has the same valuation as \( W \). But \( W(p_0) \) has no rational point with medium multiplicity because \( \epsilon(W(p_0)) = 1 \).

(2) During the minimization process (6.7 and 6.8), once we encounter a rational point \( p_0 \) with big or medium multiplicity, we work with \( W(p_0) \) defined by Equation (11) in Lemma 3.4. The points at \( \infty \) in \( W(p_0) \) corresponding to \( x_1 = \infty \) has small multiplicities (\( \lambda \leq g + 1 \)). This follows from [3], Lemme 9(b), page 4593 by case-by-case analysis. These points at infinity are denoted by \( p'_0 \) in loc. cit.

\(^1\)Let \( g \geq 1 \) be odd and let \( p > 2 \). Consider the equation \( y^2 = px^{2g+1} + p^{g+2} \) over \( \mathbb{Z}_p \). Then \( \epsilon = 1 \). For the point \( x = y = p = 0 \), we have \( \lambda = g + 2 \). By (2.b), this equation is minimal. But \( \lambda > g + 1 \). So in (1) the converse does not hold for odd \( g \) in general.
Therefore in the next loops of the algorithm we do not have to deal with the points at infinity.

(3) Using \[3\], Lemme 7(f), pages 4589-4590, one can show that if there is more than one point in \(W(k)\) with big or medium multiplicities, then there are exactly 2 such points \(p_0, p_1\). Moreover, \(g\) must be even, \(\epsilon(W) = 1\), \(\lambda(p_i) = g + 2\) and the \(W(p_i)\)’s are then minimal at \(p\).

The minimality criterion \[1.3\] needs a priori to compute the multiplicity \(\lambda\) for all points in \(W(k)\). The next lemma explains that it is only necessary to do it for at most 2 points of \(W(k)\) and how to find them.

**Lemma 4.5.** Keep the notation of the above proposition. Suppose further that when \(p\) is even, \((Q, P)\) satisfies Lemma 2.3(1). Let \(p_0 \in W_0(k)\) be such that \(\lambda(p_0) \geq g + 2\) and denote by \(\epsilon = \epsilon(W)\). Let \(\bar{c} = x(p_0)\).

1. If \(p \neq 2\), then
   \[
   \text{ord}_{\epsilon} (\pi^{-\epsilon}(4P + Q^2)) \geq g + 2 - \epsilon.
   \]

2. Suppose \(p = 2\).
   
   (i) If \(\bar{Q} \neq 0\), then
   
   \[\text{ord}_{\epsilon}(\bar{Q}) \geq (g + 2)/2.\]
   
   (ii) If \(\bar{Q} = 0\) and \(\bar{P} \notin k[x^2]\), then
   
   \[\text{ord}_{\epsilon}(\bar{P}') \geq g + 1,\]
   where \(P'\) is the derivative of \(P(x)\).
   
   (iii) If \(\epsilon = 1\), then
   
   \[\text{ord}_{\epsilon}(\pi^{-1}Q) \geq g/2, \quad \text{ord}_{\epsilon}(\pi^{-1}P) \geq g + 1.\]

*Proof.* (1) follows from Inequality \[3\].

(2) Suppose \(\lambda_c(Q, P) < \lambda(p_0)\). Then we are in the the case (2.b.ii) of Lemma 3.3. Let \((Q_0, P_0)\) be the pair given by Lemma 3.3(3). Then the properties (3.a)-(3.b) there imply immediately that in the computations of the vanishing orders we can replace \((Q, P)\) by \((Q_0, P_0)\). Therefore we can suppose \(\lambda_c(Q, P) = \lambda(p_0)\). Again by Inequality \[3\]

\[2 \text{ord}_{\epsilon}(\bar{Q}) \geq 2\mu_c(Q) \geq g + 2, \quad \text{ord}_{\epsilon}(\bar{P}) \geq \mu_c(P) \geq g + 2.\]

This proves (i) and (ii). When \(\epsilon = 1\), the same proof works by noting that \(\mu_c(F) = \mu_c(\pi F) - 1\) for any \(F \in A[x]\).

The next proposition is just a more explicit transcription of \[3\], Corollaire 2(b), p. 4594, plus Lemme 9(c), p. 4593 for (2.a)). The parenthetical sentence in (2.b) below follows from *op. cit.*, Lemme 7(f), p.
4595-4596. These results rely on the invariant denoted by $\lambda'(p_0)$ which is the maximum of the $\lambda(p)$’s for all rational points $p \in W(p_0)(k)$ except the points at infinity ($x_1 = \infty$ in the notation of Lemma 3.4).

**Proposition 4.6** (Uniqueness criterion). The following properties are true.

1. If for all $p_0 \in W(k)$ we have $\lambda(p_0) \leq g$, then $W$ is the unique minimal model at $p$. The converse is true if $g$ is odd.

2. Suppose $g$ is even and there exists $p_0 \in W(k)$ with $\lambda(p_0) = g + 1$.
   
   (a) If $\epsilon(W) = 1$, and if $W$ is minimal, then $W$ is the unique minimal model at $p$.
   
   (b) Suppose $\epsilon(W) = 0$, then $W$ is the unique minimal model at $p$ if and only if for all $p_0 \in W(k)$ with $\lambda(p_0) = g + 1$ (there are at most two such points), we have $\lambda(q) \leq g + 1$ for all $q \in W(p_0)(k)$.

The next lemma allows to construct a global equation from local equations at odd primes and even primes.

**Lemma 4.7** (Combining local equations). Suppose $\text{char}(K) = 0$. Let

\[ y^2 + Q(x)y = P(x) \]

be an equation of $C$ over $A$ with $\deg Q \leq g + 1$ and $\deg P \leq 2g + 2$, and let

\[ z^2 = F_1(x_1) \]

be another equation of $C$ over $A$ obtained by the change of variables

\[ x = \frac{ax_1 + b}{cx_1 + d}, \quad 2y + Q(x) = \frac{ez}{(cx_1 + d)^{g+1}} \]

with coefficients in $A$ and such that $e, ad - bc$ are odd. Let $m \in A$ be such that

\[ 4m \equiv 1 \pmod{e} \]

(take $m = 0$ if $e \in A^*$). Let

\[ y = \frac{ey_1}{(cx_1 + d)^{g+1}}, \]

\[ Q_1(x_1) = e^{-1}(1 - 4m)(cx_1 + d)^{g+1}Q(x) \]

and

\[ P_1(x_1) = e^{-2}(cx_1 + d)^{2g+2}(P(x) + (2m - 4m^2)Q(x)^2). \]

Then

\[ y_1^2 + Q_1(x_1)y_1 = P_1(x_1) \]

\[ ^2 \text{this hypothesis was omitted in the published version.} \]
is an equation of $C$ over $A$, equivalent to Equation (13) (Definition 1.1) at even primes and to Equation (14) at odd primes.

Proof. It is clear that $Q_1(x_1) \in A[x_1]$. We have

$$4P_1(x_1) + Q_1(x_1)^2 = F_1(x_1),$$

so $4P_1(x_1) \in A[x_1]$. But by construction $P_1(x_1) \in A_e[x_1]$, where $A_e$ is the localization of $A$ with respect to the positive powers of $e$, thus $P_1(x_1) \in A[x_1]$. Moreover the above relation implies that Equation (15) is equivalent to Equation (14) at odd primes. As $e$ and $ad - bc$ are odd, hence invertible at even primes, Equation (15) is equivalent to Equation (13) at even primes. □

5. Pointed minimal Weierstrass equations

Suppose that $C$ has a rational Weierstrass point $w_0$. A pointed Weierstrass equation of $(C, w_0)$ over $A$ is an equation

$$y^2 + Q(x)y = P(x)$$

over $A$ with deg $Q \leq g$ and $P(x)$ monic of deg $P = 2g + 1$, and $w_0$ is the pole of $x$. By suitably scaling an initial equation of $C$ over $K$ with $x$ having its pole at $w_0$, one can always obtain such an equation. The associated model is automatically normal with reduced fiber at all primes $p$ (see § 2).

Minimal pointed Weierstrass equations of $C$ are defined in a similar way to the non-pointed case. They are studied in [5]. For a given $(C, w_0)$, the minimal pointed Weierstrass equation exists and is unique up to the transformations described in Lemma 5.1 below. See also [5] and [4], Corollary 5.2. The next lemma is stated in [4], Remark after Definition 2.1.

**Lemma 5.1.** Fix a prime $p$ of $A$. Let $R = A_p$. Let

$$y_1^2 + Q_1(x_1)y_1 = P_1(x_1)$$

be another pointed Weierstrass equation of $(C, w_0)$ with discriminant $\Delta_1$ such that $v_p(\Delta_1) \leq v_p(\Delta)$. Then there exist $u, c \in R, H(x) \in R[x]$ of degree $\leq g$ such that

$$x = u^2x_1 + c, \quad y = u^{2g+1}y_1 + H(x),$$

and we have $\Delta_1 = u^{-4g(2g+1)}\Delta$.

Proof. We only have to complete the proof of the integrality statement: $c \in R, H(x) \in R[x]$ (left to the reader in [5]). Suppose that $c \notin R$. 

Then \( c^{-1} \in \pi R. \) As
\[
\frac{1}{x} - c^{-1} = -(c^{-1}u)^2x', \quad \text{where } x' = \frac{x_1}{(c^{-1}u)^2x_1 + 1},
\]
by [4], Lemma 5.1, we have \( v(\Delta) < v(\Delta_1) \) (with the notation of op. cit., \( d = 2v(c^{-1}u) > 0 \) and the point \( p \) is the pole of \( x \) in \( W_k \) which is a smooth point). So \( c \in R. \)

It remains to prove that \( H(x) \in R[x] \) and \( \deg H \leq g. \) Without loss of generality we can suppose that \( c = 0. \) We then have
\[
P(x) + Q(x)H(x) + H(x)^2 = u^{2(2g+1)}P_1(x_1) = u^{2(2g+1)}P_1(x/u^2) \in R[x].
\]
This implies that \( H(x) \in R[x] \) using the Gauss valuation on \( K(x) \) associated to \( v_p. \) Finally, the same equality implies that \( \deg H \leq g. \) \( \square \)

**Corollary 5.2.** Let \( y^2 + Q(x)y = P(x) \) be a pointed Weierstrass equation of \( (C, w_0). \) Let \( W \) be the associated Weierstrass model. Then the equation is not pointed-minimal at \( p \) if and only if there exist \( p_0 \in W_0(k) \) such that \( \lambda(p_0) = 2g + 1 \) and \( q_0 \in W(p_0)(k) \) such that \( \lambda(q_0) = 2g + 2. \) The model \( W(p_0)(q_0) \) is then a pointed model and
\[
v(\Delta_{W(p_0)(q_0)}) = v(\Delta) - 4g(2g + 1).
\]

**Proof.** Suppose that the equation is not pointed-minimal at \( p. \) By Lemma 5.1, there exists a pointed equation over \( A_p \)
\[
y_1^2 + Q_1(x_1)y_1 = P_1(x_1)
\]
with \( x = \pi^{2r}x_1 + c \) for some \( r \geq 1 \) and \( y = \pi^{r(2g+1)}y_1 + H(x). \) Approximating elements of \( A_p \) by that of \( A, \) one can suppose that all coefficients belong to \( A. \) Indeed, let \( a \in A, U(x) \in A[x] \) be such that
\[
c - a \in \pi^{2r}A_p, \quad H(x) - U(x) \in \pi^{r(2g+1)}A_p[x].
\]
Consider the change of variables \( x_2 = x_1 + \pi^{-2r}(c - a) \) and \( y_2 = y_1 + \pi^{r(2g+1)}(H - U) \) over \( A_p. \) We then have \( x - \pi^{2r}x_2 = a \in A \) and \( y - \pi^{r(2g+1)}y_2 = U(x) \in A[x]. \)

Translating \( x \) by \( c \) and replacing \( y \) with \( y - H(x), \) we can suppose that \( c = 0 \) and \( y = \pi^{r(2g+1)}y_1. \) This implies that
\[
Q(x) = \pi^{r(2g+1)}Q_1(x/\pi^{2r}), \quad P(x) = \pi^{2r(2g+1)}P_1(x/\pi^{2r}).
\]
So \( \mu_0(Q) \geq g + 1 \) and \( \mu_0(P) = 2g + 1 \) is odd. Therefore if \( p_0 \in W(k) \) is the zero of \( x \) (meaning that \( p_0 \) is the unique point whose \( x \)-coordinate is zero), then \( \lambda(p_0) = 2g + 1 \) (Lemma 3.3 (2.a)). An equation of \( W(p_0) \) is then
\[
y^2 + \pi^{-g}Q(\pi x_0)y_0 = \pi^{-2g}P(\pi x_0)
\]
with \( x = \pi x_0 \) and \( y = \pi^g y_0. \) Using the relations (16), we see that as elements of \( A[x_0], \) we have \( \mu_0(\pi^{-g}Q(\pi x_0)) \geq g + 1, \) and \( \mu_0(\pi^{-2g}P(\pi x_0)) = \)
2g + 2 is reached at the odd degree \( i = 2g + 1 \). Let \( q_0 \in W(p_0)(k) \) be the zero of \( x_0 \). Then \( \lambda(q_0) = 2g + 2 \) by Lemma 3.3 (1) and (2.b.i).

Let us prove the converse. Under the hypothesis of the lemma, it is enough to show that \( W(p_0)(q_0) \) is defined by a pointed Weierstrass equation. Its discriminant is given by Remark 4.2 (2). Let \( \bar{c} \) be the clear coordinate of \( p_0 \). If \( \min\{2\mu_c(Q), \mu_c(P)\} < \lambda(p_0) \), we modify \( Q, P \) as in Lemma 5.1 with \( 0 \). As the coefficient of degree 2 of \( P(x) \) is zero, we see by construction that deg \( H_0 \leq g \). Repeating the algorithm if necessary, we get a new pair \( (Q, P) \) such that \( \min\{2\mu_c(Q), \mu_c(P)\} = \lambda(p_0) \), \( \deg Q \leq g \) and \( P(x) \) is monic of degree \( 2g + 1 \).

An equation of \( W(p_0) \) is

\[
y_0^2 + \pi Q_0(x_0)y_0 = \pi P_0(x_0)
\]

with

\[
Q_0(x_0) = \pi^{-g-1}Q(\pi x_0 + c), \quad P_0(x_0) = \pi^{-2g-1}P(\pi x_0 + c)
\]

and \( P_0(x_0) \) is monic. Let \( \bar{c}_1 \) be the \( x_0 \)-coordinate of \( q_0 \). Again, if \( \lambda(q_0) \) is not reached by the pair \( \pi Q_0, \pi P_0 \), we see in the construction of Lemma 3.3 (2.b.ii) that \( H_0(x_0) \) is divisible by \( \pi \) and has degree \( \leq g \).

So we get a new pair \( \pi Q_1(x_0), \pi P_1(x_0) \) reaching \( \lambda(q_0) \) and such that deg \( Q_1 \leq g \) and \( P_1(x_0) \) is monic of degree \( 2g + 1 \). Then \( W(p_0)(q_0) \) is defined by the equation

\[
y^2 + \pi^{-g}Q_1(\pi x_1 + c_1)y = \pi^{-2g+1}P_1(\pi x_1 + c_1)
\]

which is a pointed Weierstrass equation. \( \square \)

**Remark 5.3** Similarly to Lemma 3.3 (3), because \( \deg Q \leq g \) and \( P(x) \) is monic of degree \( 2g + 1 \), one can show that for all \( p_0 \in W_0(k) \) (resp. \( q_0 \in W(p_0)(q_0)(k) \), \( \lambda(p_0) \leq 2g + 1 \) (resp. \( \lambda(q_0) \leq 2g + 2 \)). Moreover, by Remark 4.4 there is at most one such point.

**Remark 5.4** Suppose \( p \) is odd. It defines an absolute value \( |.|_p \) on \( K \).

Fix an algebraic closure \( K^{alg} \) of \( K \) and an extension of \( |.|_p \) to it. Then a pointed Weierstrass equation \( y^2 = P(x) \) is pointed-minimal at \( p \) if and only if there is no disc centered in \( A \) of radius \( \leq |\pi|^2_p \) containing all roots of \( P(x) \) in \( K^{alg} \).

Indeed, if such a disc, centered in some \( c \in A \) exists, then the equation of variables \( x = \pi^2 x_1 + c \), \( y = \pi^{2g+1}y_1 \), leads to a pointed Weierstrass equation of discriminant \( \pi^{-4g(2g+1)} \Delta \). Conversely, if \( y^2 = P(x) \) is not pointed-minimal, then the minimal one is given by a change of variables as in Lemma 5.1 with \( H(x) = 0 \) and \( v(u) > 0 \). As \( A \) is dense in \( A_p \), one can take \( c \in A \). Translating \( x \) by \( c \) we can suppose that \( x = u^2 x_1 \).
As \( P(u^2 x_1) \in u^{2(2g+1)} A_p[x_1] \). This implies that the roots \( \alpha \in K^{\text{alg}} \) of \( P(x) \) all have \( |\alpha|_p \leq |u|^2 \leq |\pi|^2 \).

6. Minimization algorithm

We start with a Weierstrass equation

\[
y^2 + Q(x)y = P(x)
\]

of \( C \) over \( A \), with discriminant \( \Delta \) and \( \deg Q \leq g + 1, \deg P \leq 2g + 2 \). Note that the formula (3) implies that if \( v_p(\Delta) < 2(2g + 1) \) (resp. \( v_p(\Delta) < 4(2g + 1) \)) if \( g \) is even (resp. if \( g \) is odd), then the equation is minimal at \( p \).

Let us describe the minimization algorithm. It consists in minimizing successively at even primes dividing \( \Delta \), then at odd primes dividing \( \Delta \), and finally we globalize in \( \S 6.6 \) using Lemma 4.7. The local minimization is done step by step as follows. Start with an integral Weierstrass equation and a prime \( p \) dividing \( \Delta \).

(i) First normalize the equation at \( p \) (Algorithm 6.1).

(ii) Candidates for rational points away from \( x = \infty \) having big or medium multiplicities are found by Algorithm 6.4 and 6.5. For even primes, whether the points at infinity might have big or medium multiplicities are checked in Test 6.6 (after inverting \( x \) to reduce to the points with \( x = 0 \)). For odd primes this task is much simpler and is included directly in 6.8(I).

(iii) For a candidate \( p_0 \) found above, the actual multiplicity is then computed in Algorithm 6.2 for even primes (no need of algorithm at odd primes). Test 6.3 tells us whether \( \lambda \) is small or not. If yes we move to the next candidate.

(iv) As soon as a rational point \( p_0 \) with big \( \lambda \) is found, we consider a new normal Weierstrass equation, corresponding to \( W(p_0) \) (Lemma 3.4). Then we start again the algorithm with this new equation. In the new equation, the points at infinity always have small multiplicities.

We have \( v(\Delta_{W(p_0)}) \leq v(\Delta_W) \), with equality if and only \( \lambda(p_0) \) is medium. In this case, either \( W(p_0) \) (and hence \( W \)) is minimal, or there exists a rational point \( p_1 \in W(p_0)(k) \) with big multiplicity, so that after the next change of variables, the valuation of the discriminant will decrease strictly. After finitely many loops we get in Step (x).

(v) If there is no point with big or medium multiplicity, then the equation is minimal at \( p \).
6.1. Normalization algorithm.

Algorithm 6.1 (Lemma 2.3)

Input: an even prime \( p \) of \( A \), a pair of polynomials \( Q, P \) as in Equation (17), and \( e_0 \in A \) (corresponding to the \( e \) in Equation (2) for even primes.)

Output: a new pair \( Q, P \) defining a Weierstrass model normal at \( p \) satisfying Lemma 2.3(1), and the new \( e_0 \).

1. If \( v(Q) = 0 \), go to (3).
2. Otherwise,
   - (a) if \( v(P) = 0 \), and \( \overline{P} \notin k[x^2] \), go to (3);
   - (b) if \( v(P) = 0 \) and \( \overline{P}(x) \in k[x^2] \). Let \( H(x) = \sum_{0 \leq i \leq g+1} b_i x^i \) with \( \overline{b_i} = \overline{a_{2i}} \) (\( k \) is perfect of characteristic 2). Then \( Q \leftarrow Q - 2H, P \leftarrow P + QH - H^2 \).
     - (The new pair satisfies \( v(Q), v(P) > 0 \).)
   - (c) if \( v(P) = 1 \), go to (3);
   - (d) if \( v(P) \geq 2 \), let \( r = \left[ \frac{1}{2} \min\{2v(Q), v(P)\} \right] \). Then \( e_0 \leftarrow \pi^r e_0, Q \leftarrow \pi^{-r} Q, P \leftarrow \pi^{-2r} P \).
     - Restart at (1).
3. Output \( Q, P, e_0 \) and \( \epsilon_p = \min\{v(Q), v(P)\} \).

The algorithm terminates because when we need to restart the loop (only at the step (2.d)), the discriminant of the new equation is equal to the previous one divided by \( \pi^{4r(2g+1)} \) with \( r > 0 \).

Recall that at odd primes, the normalization of \( y^2 = F(x) \) consists just in dividing both sides by a biggest odd square of \( \text{cont}(F) \).

6.2. Computing the multiplicity \( \lambda \). Let \( p \) be a prime of \( A \), let \( p_0 = (\overline{c}, \overline{d}) \) be a solution of Equation (17) mod \( p \) with \( c, d \in A \). We want to compute \( \lambda(p_0) \) (see Definition 3.2). See also Definition 3.1 for the notation \( \mu_c \). Recall that \( \lambda(p_0) = \mu_c(4P + Q^2) \) if \( p \) is odd.

Algorithm 6.2 (Lemma 3.3)

Input: an even prime \( p \), a pair \( Q, P \) satisfying Lemma 2.3(1) at \( p \) and \( \overline{c} \in k \).

Output: the multiplicity \( \lambda(p_0) \) and new pair \( Q, P \) such that \( \lambda(p_0) = \min\{2\mu_c(Q), \mu_c(P)\} \).

1. Compute \( \mu_c(Q), \mu_c(P) \);
2. if \( 2\mu_c(Q) \leq \mu_c(P) \), then \( \lambda(p_0) = 2\mu_c(Q) \), go to (3);
3. if \( \mu_c(P) \) is odd, then \( \lambda(p_0) = \mu_c(P) \), go to (3);
(4) write $P(x) = \sum_i a_{c,i}(x - c)^i$. If $\mu_c(P) = v(a_{c,i}) + i$ for some odd $i$, then $\lambda(p_0) = \mu_c(P)$, go to [6].

(5) set $H_0(x) = \sum_{i \leq r} e_{c,i}(x - c)^i$ where the sum runs through the indexes such that $v(a_{c,2i}) + 2i = 2r$ and where $e_{c,i} \in \pi^{r-i}A$ satisfy

$$(\pi^{1-r}e_{c,i})^2 + \pi^{2i-2r}a_{c,2i} \equiv 0 \mod \pi.$$ 

Then

$$Q \leftarrow Q - 2H_0, \quad P \leftarrow P + QH_0 - H_0^2.$$ 

Go back to (1).

(6) Output $Q, P$ and $\lambda = \lambda(p_0)$. This algorithm does not change $\epsilon_p$ and the conditions in Lemma 2.3(1). The test below will say if the multiplicity $\lambda(p_0)$ is small (true) or too big (false) in which case our equation is (probably) not minimal at $p$.

**Test 6.3 (Remark 4.4)**

**Input:** a multiplicity $\lambda$ and $\epsilon_p$.

**Output:** true or false.

1. If $\lambda \leq g + 1$, output true ($\lambda$ is small);
2. if $g$ is even, output false ($\lambda$ is big);
3. if $\lambda \geq g + 3$, output false ($\lambda$ is big);
4. if $\epsilon_p = 1$, output true ($\lambda$ is small);
5. output false ($\lambda$ is medium).

6.3. Finding $p_0$ with big or medium $\lambda$. To use the minimality criterion (Proposition 4.3), we only have to compute $\lambda$ for the $\bar{c}$'s given by the algorithms below.

**Algorithm 6.4 (Lemma 4.5(2))**

**Input:** An even prime $p$ of $A$, a pair $Q, P$ satisfying Lemma 2.3(1).

**Output:** $\epsilon_p$ and the elements $\bar{c} \in k$ such that the corresponding $\lambda(p_0)$ may not be small. There are at most two such $\bar{c}$'s.

1. Output $\epsilon_p = \min\{v(Q), v(P)\}$.
2. If $\bar{Q} \neq 0$, output the $\bar{c} \in k$ such that $\text{ord}_\bar{c}(\bar{Q}) \geq (g + 2)/2$.
3. If $\bar{Q} = 0$ and $\bar{P} \notin k[x^2]$, output the $\bar{c} \in k$ such that $\text{ord}_\bar{c}(\bar{P'}) \geq g + 1$

where $P'$ is the derivative of $P$. 
(4) If \( \epsilon_p = 1 \) (so \( Q = P = 0 \)), output the \( \bar{c} \in k \) such that
\[
\text{ord}_e(\pi^{-1}Q) \geq g/2 \quad \text{and} \quad \text{ord}_e(\pi^{-1}P) \geq g + 1.
\]

**Algorithm 6.5 (Lemma 4.5(1))**

*Input:* An odd prime \( p \) of \( A \) and \( F \in A[x] \) such that \( v(F) \leq 1 \).

*Output:* \( \epsilon_p \) and the elements \( \bar{c} \in k \) such that the corresponding \( \lambda(p_0) \) may not be small. There are at most two such \( \bar{c} \)'s.

1. Output \( \epsilon_p = v(F) \).
2. Output the zeros in \( k \) of \( \pi - \epsilon_p F(x) \) of order \( \geq g + 2 - \epsilon_p \).

**Test 6.6 (Lemma 4.5(2))**

*Input:* same as in Algorithm 6.4.

*Output:* true, if \( 0 \) belongs to the list returned by Algorithm 6.4 and false otherwise.

1. If \( \bar{Q} \neq 0 \), output true if
\[
\text{ord}_0(\bar{Q}) \geq (g + 2)/2.
\]
and false otherwise.
2. If \( \bar{Q} = 0 \) and \( \bar{P} \notin k[x^2] \), output true if
\[
\text{ord}_0(\bar{P}') \geq g + 1,
\]
and false otherwise.
3. If \( \epsilon_p = 1 \), output true if
\[
\text{ord}_0(\pi^{-1}Q) \geq g/2 \quad \text{and} \quad \text{ord}_0(\pi^{-1}P) \geq g + 1
\]
and false otherwise.

6.4. Minimization at even primes.

**Algorithm 6.7**

*Input:* Equation (17).

*Output:* A new equation minimal at even primes together with the change of variables on \( x \) given by a matrix \( M_0 \in M_{2 \times 2}(A) \), and the multiplicative factor \( e_0 \in A \) in the change of variables on \( y \).

The algorithm will produce a new equation \( y_0^2 + Q_0(x_0)y_0 = P_0(x_0) \) with \( x = M_0x_0, \ y = (e_0x_0 + H(x_0))/(c_0x_0 + d_0)^{g+1} \), where \( (c_0, d_0) \) is the second row of \( M_0 \), for some \( H(x_0) \in A[x_0] \).

Let \( p_1, \ldots, p_m \) be the even primes dividing the discriminant \( \Delta \) of Equation (17). We start with \( i = 1, M_0 = I_2 \in \text{Gl}_2(A), e_0 = 1 \).

(I) Let \( p = p_i \). Run Algorithm 6.1. If \( v(\Delta) < 2(2g + 1) \) (for even \( g \)) or \( v(\Delta) < 4(2g + 1) \) (for odd \( g \)), goto (IV).
(II) Let $Q_\infty(x) = x^{g+1}Q(1/x)$ and $P_\infty(x) = x^{2g+2}P(1/x)$. Run Test 6.6 with the pair $Q_\infty(x), P_\infty(x)$.
(a) Go to (III) if we get false.
(b) Otherwise, run Algorithm 6.2 for the pair $Q_\infty(x), P_\infty(x)$ at $\tilde{c} = 0$.
(c) Run Test 6.3. If we get true, go to (III).
(d) Otherwise, set $r = \lfloor \lambda/2 \rfloor$, $Q(x) \leftarrow \pi^{-r}Q_\infty(\pi x)$, $P(x) \leftarrow \pi^{-2r}P_\infty(\pi x)$.
   where $Q_\infty(x), P_\infty(x)$ are the new polynomials given at (b),
   $e_0 \leftarrow e_0 \pi^r$, $M_0 \leftarrow M_0 \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$.
   Go to (III).
(III) Run Algorithm 6.4.
   (a) Pick the first $\tilde{c}$. Run Algorithm 6.2 for this $\tilde{c}$. Run Test 6.3.
      If we get true, go back to (a) with the next $\tilde{c}$. If there is no $\tilde{c}$ left, go to (IV).
   (b) As soon as we get false for some $\tilde{c}$, set $r = \lfloor \lambda/2 \rfloor$. Then
      $Q(x) \leftarrow \pi^{-r}Q(\pi x + c)$, $P(x) \leftarrow \pi^{-2r}P(\pi x + c)$.
      $e_0 \leftarrow e_0 \pi^r$, $M_0 \leftarrow M_0 \begin{pmatrix} \pi & c \\ 0 & 1 \end{pmatrix}$.
      Go back to (III).
(IV) If $i < m$, then $i \leftarrow i + 1$ and go back to (I). Otherwise output $Q, P, e_0$ and $M_0$.

6.5. Minimization at odd primes. Suppose $A$ has both even and odd primes. Let

$$F(x) = 4P(x) + Q(x)^2$$

where $P, Q$ are the polynomials returned by Algorithm 6.7.

Algorithm 6.8
Input: $F(x)$ as above.
Output: An equation minimal at all odd primes of $A$, together with the changes of variables on $x$ and $y$ leading to the new equation.

Number the odd primes $p_1, \ldots, p_n$ dividing the discriminant $\Delta$ of Equation (17). Start with $M_1 = I_2 \in \text{Gl}_2(A)$, $e_1 = 1$ and $i = 1$.

(I) (Normalization at all odd primes) Let $s^2$ be a greatest odd square dividing $\text{cont}(F)$.

$$F(x) \leftarrow s^{-2}F(x), \quad e_1 \leftarrow e_1 s.$$
(II) Let \( p = p_i \). If \( v(\Delta) < 2(2g + 1) \) (for even \( g \)) or \( v(\Delta) < 4(2g + 1) \) (for odd \( g \)), goto (IV). Otherwise, let \( \epsilon_p = v(\text{cont}(F)) \). If
\[
\deg \pi^{-\epsilon_p} F(x) \geq g + 1 + \epsilon_p,
\]
go to (III). Otherwise, compute \( \lambda := \mu_0(x^{2g+2}F(1/x)) \).
(a) Run Test 6.3. If we get true, go to (III).
(b) Otherwise, set \( r = \lfloor \lambda/2 \rfloor \). Then
\[
F(x) \leftarrow \pi^{-2r} (\pi x)^{2g+2} F(1/(\pi x))
\]
\[
e_1 \leftarrow \pi^r e_1, \quad M_1 \leftarrow M_1 \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}.
\]
Go to (III).

(III) Run Algorithm 6.5.
(a) Pick the first \( \bar{c} \) and compute \( \lambda := \mu_0(F) \). Run Test 6.3. If we get true, pass to the next \( \bar{c} \) and restart at (a). If there is no \( \bar{c} \) left, go to (IV).
(b) Otherwise we get false for some \( \bar{c} \). Set \( r = \lfloor \lambda/2 \rfloor \). Then
\[
F(x) \leftarrow \pi^{-2r} (\pi x + c)
\]
\[
e_1 \leftarrow \pi^r e_1, \quad M_1 \leftarrow M_1 \begin{pmatrix} \pi & c \\ 0 & 1 \end{pmatrix}.
\]
Go back to (III).

(IV) If \( i < n \), then \( i \leftarrow i + 1 \) and go back to (II). Otherwise output \( M_1, e_1 \) and \( F(x) \).

6.6. Final step. We give a minimal equation of \( C \) over \( A \).
(1) If there are only even primes dividing the initial \( \Delta \), Algorithm 6.7 already returned a global minimal equation of \( C \) over \( A \).
(2) Suppose there are odd primes dividing the initial \( \Delta \). Let \( Q_0, P_0 \) denote the pair \( Q, P \) returned by Algorithm 6.7 and let \( e_1 \in A \) and \( M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in M_{2 \times 2}(A) \) be returned by Algorithm 6.8.
Note that \( e_1 \) and \( \det M_1 \) are odd, and \( e_1 \not\in A^* \). Let \( m \in A \) be such that
\[
4m \equiv 1 \mod e_1.
\]
For computations, \( m \) should be chosen as small as possible, for whatever measure of the size. Let
\[
Q_1(x_1) = e_1^{-1}(1 - 4m)(c_1 x_1 + d_1)^{g+1} Q_0(x)
\]
and
\[
P_1(x_1) = e_1^{-2}(c_1 x_1 + d_1)^{2g+2} (P_0(x) + (2m - 4m^2)Q_0(x)^2)
\]
where \( x = (a_1x_1 + b_1)/(c_1x_1 + d_1) \). Then
\[
y_1^2 + Q_1(x_1)y_1 = P_1(x_1)
\]
is a minimal equation of \( C \) over \( A \).

**Remark 6.9** Let \( y^2 + Q(x)y = P(x) \) be the equation we start with. The change of variables to the minimal equation above is given as follows:
\[
x = \frac{ax_1 + b}{cx_1 + d}, \quad y = \frac{ey_1 + H(x_1)}{(cx_1 + d)^{g+1}}
\]
where
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = M_0M_1, \quad e = e_0e_1,
\]
and \( H(x_1) \in A[x_1] \) is determined by
\[
2H(x_1) = eQ_1(x_1) - (cx_1 + d)^{g+1}Q((ax_1 + b)/(cx_1 + d)),
\]
by comparing the traces of \( y \) and of \( y_1 \) in the extension \( K(x)[y] \) over \( K(x) \).

The minimal discriminant \( \Delta_{\text{min}} \) is given by
\[
\Delta_{\text{min}} = e^{-4(2g+1)}(ad - bc)^{2(g+1)(2g+1)}\Delta
\]
in terms of the initial discriminant \( \Delta \) of Equation (17).

**Example 6.10** Consider the equation
\[
y^2 + 2^4 \cdot 11 \cdot 13y = 5^6 \cdot 17^3 \cdot x^5
\]
over \( \mathbb{Q} \). It defines a genus 2 curve of discriminant
\[
\Delta = 2^{32} \cdot 5^{41} \cdot 11^8 \cdot 13^8 \cdot 17^{18}.
\]
The command `hyperellminimalmodel` in [6] gives a minimal equation over \( \mathbb{Z} \):
\[
z^2 + t^3z = 1477440t^6 + 20t,
\]
with the changes of variables:
\[
x = \frac{2^2}{5 \cdot 17t}, \quad y = \frac{2^4 \cdot 5^3 \cdot 17^2 z + 2^4 \cdot 3^5 \cdot 5^4 \cdot 17^2 t}{(5 \cdot 17t)^3}
\]
with minimal discriminant equal to \( 2^{12} \cdot 5^{11} \cdot 11^8 \cdot 13^8 \cdot 17^8 \). In particular, the initial equation is minimal away from 2, 5 and 17.

**Remark 6.11** Our algorithm always terminates. The total number of iterations is roughly bounded above by \( \sum_p v_p(\Delta) \) where \( \Delta \) is the discriminant of the initial equation.
Remark 6.12 If in our algorithm we input an arbitrary pair of polynomials $Q(x), P(x) \in A[x]$, we must check if $y^2 + Q(x)y = P(x)$ defines a smooth hyperelliptic curve $C$ and, if necessary, construct a new Weierstrass equation with polynomials $Q_0, P_0$ such that $\deg Q_0 \leq g(C) + 1$ and $\deg P_0 \leq 2g(C) + 2$.

Suppose for simplicity that $\text{char}(K) \neq 2$. Then the smoothness is detected by the non-vanishing of $\text{disc}(4P + Q^2)$. Suppose from now on that this is the case. Then $g = g(C)$ is given by $g = \lfloor (d - 1)/2 \rfloor$ where $d := \deg(4P + Q^2)$.

Now if $\deg Q > g + 1$, then $\deg P = 2\deg Q > 2g + 2$. Write $Q(x) = Q_0(x) + 2E(x)$ with $\deg Q_0 \leq g + 1$ and $x^{g+2} | E$. Let

$$y = y_0 - E(x).$$

Then

$$y_0^2 + Q_0(x)y_0 = P_0(x)$$

where $P_0 = P + QE - E^2$ satisfies $\deg P_0 \leq 2g + 2$, is a new equation of $C$ satisfying the requirement on the degrees of $Q_0, P_0$.

It reminds to show that $Q_0, P_0 \in A[x]$. It is enough to show that $E \in A[x]$. Consider a basis $\{1, z\}$ of the integral closure of $A[x]$ in $K(x, y) = K(C)$ with $\text{Tr}_{K(C)/K(x)}(z) \in A[x]$ of degree $\leq g + 1$. See [3], Lemme (1.b), page 4579. Then $y = ez + H(x)$ with $e \in A$ and $H(x) \in A[x]$. So

$$Q(x) = -\text{Tr}_{K(C)/K(x)}(y) = -e\text{Tr}_{K(C)/K(x)}(z) - 2H(x)$$

and $-E$ consists in the terms of degree $\geq g + 2$ in $H(x)$. Therefore $E \in A[x]$.

7. Minimization algorithm for pointed Weierstrass equations

Fix $(C, w_0)$ as in §5 and let

$$y^2 + Q(x)y = P(x)$$

be a pointed Weierstrass equation of $(C, w_0)$ over $A$ of discriminant $\Delta$ (see the beginning of §6). By Lemma 5.1 if $v(\Delta) < 4g(2g + 1)$, then the equation is pointed-minimal at $p$.

Algorithm 7.1 (Even primes)

Input: The above equation.

Output: A new pointed Weierstrass equation, minimal at even primes, and the change of variables $L_0(x)$ on $x$.

Let $p_1, \ldots, p_n$ be the even prime divisors of $\Delta$. Start with $i = 1$. Let $L_0(x) = x$. 
(I) Let $p = p_i$. If $v(\Delta) < 4g(g+1)$, go to (III).
(II) If $Q \neq 0$ or if $P(x)$ is not a $(2g+1)$th power, go to (III). Otherwise, $\bar{P}(x) = (x - \bar{c})^{2g+1}$ for some $c_1 \in A$. Run Algorithm 6.2 at $\bar{c}_1$.
(a) If $\lambda < 2g+1$, go to (III).
(b) Otherwise we have $\lambda = 2g+1$. Let $Q_1(x) = \pi^{-g}Q(\pi x + c_1), P_1(x) = \pi^{-2g}P(\pi x + c_1)$.
   (i) If $\bar{P}_1(x)$ is not a $(2g+1)$th power, go to (III).
   (ii) Otherwise let $\bar{P}_1(x) = (x - \bar{c})^{2g+1}$ for some $c \in A$. Run Algorithm 6.2 for the pair $Q_1, P_1$ at $\bar{c}$. If $\lambda < 2g+2$, go to (III).
   (iii) Otherwise

\[ Q(x) \leftarrow \pi^{-(g+1)}Q_1(\pi x + c), \quad P(x) \leftarrow \pi^{-(2g+2)}P_1(\pi x + c), \]
\[ L_0(x) \leftarrow \pi^2L_0(x) + \pi c + c_1. \]

Go back to (II).
(III) If $i < n$, then $i \leftarrow i + 1$, go back to (I). Otherwise output $Q, P$ and $L_0$.

Algorithm 7.2 (Odd primes) Suppose there are even and odd primes in $A$.

Input: The pair $Q, P$ returned by Algorithm 7.1

Output: An equation minimal at all odd primes of $A$, together with the changes of variables leading to the new equation.

Let
\[ F = 4P + Q^2. \]
(It has leading coefficient equal to 4, but this does not matter.) Let $p_1, \ldots, p_m$ be the odd prime divisors of $\Delta$. Let $L_1(x) = x$, let $i = 1$.
(I) Let $p = p_i$. If $v(\Delta) < 4g(g+1)$, go to (III).
(II) If $\bar{F}$ has no root of order $2g+1$, go to (III). Otherwise, let $\bar{c} \in k$ be the root of $\bar{F}$. Write
\[ F(x) = \sum_i a_{c,i}(x - \bar{c})^i. \]
Let
\[ \theta = \min \left\{ \frac{v(a_{c,i})}{(2g+1) - i} \mid 0 \leq i \leq 2g \right\}. \]
Let $r = [\theta/2]$.
(a) If $r = 0$, go to (III).
(b) Otherwise,
\[ F(x) \leftarrow \pi^{-2r(2g+1)}F(\pi^{2r}x + c), \quad y \leftarrow \pi^{-r(2g+1)} \]
\[ L_1(x) \leftarrow \pi^{2r}L_1(x) + c \]
and restart at (II).

(III) If \( i < m \), then \( i \leftarrow i + 1 \) and go back to (I). Otherwise output \( F(x) \) and \( L_1(x) \). The equation \( z^2 = \frac{1}{4}F(x) \) is pointed-minimal at all odd primes.

**Final step.** Suppose that \( A \) has even and odd primes. Denote by \( Q_0, P_0 \) be the pair \( Q, P \) returned by Algorithm 7.1 and let \( L_1(x) = u_1^2x + c_1 \) be returned by Algorithm 7.2. Then \( u_1 \) is odd. If \( u_1 \in A^* \), then \( y_0^2 + Q_0(x)y_0 = P_0(x) \) is pointed-minimal over \( A \). Suppose \( u_1 \notin A^* \). Let \( m \in A \) be such that
\[ 4m \equiv 1 \mod u_1^{2g+1}. \]
Then if we define
\[ Q_1(x_1) := u_1^{-(2g+1)}(1 - 4m)Q_0(u_1^2x_1 + c_1) \in A[x_1], \]
\[ P_1(x_1) := u_1^{-2(2g+1)}(P_0(u_1^2x_1 + c_1) + (2m - 4m^2)Q_0(u_1^2x_1 + b)^2) \in A[x_1] \]
we have that
\[ y_1^2 + Q_1(x_1) y_1 = P_1(x_1) \]
is a pointed-minimal equation of \( (C, w_0) \) over \( A \).

**Remark 7.3** Let \( y^2 + Q(x)y = P(x) \) be the pointed equation we start with, of discriminant \( \Delta \). The changes of variables to the above minimal pointed equation is given as follows:
\[ x = u^2x_1 + c, \quad y = u^{2g+1}y_1 + H(x_1) \]
where
\[ u^2x_1 + c = L_0(L_1(x_1)) \]
(choose any \( u \) satisfying the above equation), and \( H(x_1) \in A[x_1] \) is determined by
\[ 2H(x_1) = u^{2g+1}Q_1(x_1) - Q(u^2x_1 + c), \]
by comparing the traces of \( y \) and of \( y_1 \) in the extension \( K(x)[y] \) over \( K(x) \). The minimal pointed discriminant \( \Delta_{\text{min}} \) is given by
\[ \Delta_{\text{min}} = u^{-4g(2g+1)}\Delta. \]

8. **Statements**

No external dataset is used.

No conflicts of interest to declare.
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Université de Bordeaux, Institut de Mathématiques de Bordeaux, CNRS UMR 5251, 33405 Talence, France

Orcid iD 0000-0001-6884-139X

Email address: Qing.Liu@math.u-bordeaux.fr