Constraint stabilization of two-wheeled sleigh

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Abstract. In this paper a problem of constraint stabilization of a two-wheeled sleigh is considered. This problem is solved with the help of the Chaplygin’s approach, in which Lagrange equations of the second kind are modified with respect to the nonholonomic constraints. For the obtained equations we define the functions of reaction forces of constraints with respect to their stabilization. During the numerical integration some of the stabilization parameters are defining at each step of the summation. This gives an advantage in comparison with the classical stabilization approach.

1. Introduction

The main problem of the classical mechanics is to find a solution of motion equations that are often can be written as a system of ordinary differential equations. But there are only few systems that we can solve by obtaining an analytic solution. In general the methods of numerical integration are used to obtain a specific result. Most of them are realized with the help of the summation of deference schemes. But when we have a constraint mechanical system deviations from the constraints will accumulate and lead to the numerical solution instability. So J. Baumgarte suggested to modify the functions of reaction forces by equating the derivatives of the constraints to the linear form of themselves [1]. The optimal range of values of linear form coefficients allow us to control the accumulation of the deviations. The problem of determining this range was considered in [2–4]. The constraint stabilization method is applied to the specific types of nonholonomic systems, called Chaplygin’s. When the structure of constraint equations and a Lagrange function allow us to separate independent variables from dependent ones Lagrange equations of the second kind can be modified so they can include constraint terms in them [5]. In this paper we consider a problem of defining linear form stabilization coefficients for the Chaplygin nonholonomic system.

2. Motion Equations

2.1. Chaplygin systems

Let us consider the motion of a nonholonomic system defined by the set of generalized coordinates \( q^1, \ldots, q^n \) and generalized velocities \( v^1 = dq^1/dt = \dot{q}^1, \ldots, v^n = \dot{q}^1 \). If the structure of nonholonomic constraint equations allows us to divide a set of generalized velocities into dependent and independent parts and the Lagrange function and external forces do not depend on the dependent coordinates then a nonholonomic system is called a Chaplygin system. For example, for the constraint equation

\[
\dot{q}^i = b^i_a(q^1, \ldots, q^n)q^a,
\]
the nonholonomic system can be considered as Chaplygin’s. Here and elsewhere we consider the summation under the repeating indexes. Latin letters \( i, j, k, \ldots \) stand for the summation under the dependent variables \((q^{p+1}, \ldots, q^n)\) and letters \( a, b, c, \ldots \) for the summations under independent ones \((q^1, \ldots, q^p)\). So there are only \( p \) independent variables due to the fact that there are \( n - p \) constraint equations (1).

It has been shown that the motion equations for the Chaplygin systems can be written in the following form:

\[
\frac{d}{dt} \frac{\partial \mathcal{L}^*}{\partial \dot{q}^a} - \frac{\partial \mathcal{L}^*}{\partial q^a} + \left( \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right)^* \left( b_i^c q^a - b_i^c q^a \right) \dot{q}^c = 0, \tag{2}
\]

where the asterisk means the operation of defining the dependent velocities through the independent ones due to the equation (1).

Let us consider that the Lagrange function of the mechanical system with constraint equations (1) takes form

\[
L = \frac{1}{2} \sum_{r,s=1}^n g_{rs}(q^1, \ldots, q^p)q^r \dot{q}^s - U(q^1, \ldots, q^p). \tag{3}
\]

So it is just represented as a sum of the kinetic energy \( T \) and a potential energy \( U \). As we can see both of these functions do not depend on the coordinates \((q^{p+1}, \ldots, q^n)\) then such nonholonomic system can be treated Chaplygin’s.

After applying the asterisk operation to the Lagrange function (3) we will obtain a modified function

\[
L^* = \frac{1}{2} g_{ab}(q^1, \ldots, q^p)\dot{q}^a \dot{q}^b - U(q^1, \ldots, q^p), \tag{4}
\]

where the bilinear coefficients \( g_{ab}^* \) are determined as follows:

\[
g_{ab}^* = g_{ab} + g_{ib} b_a^i + g_{ia} b^i_b + g_{ij} b^i_a b^j_b.
\]

So we can see that these bilinear coefficients that can be interpreted as metric tensor for Chaplygin system define through the linear coefficients \( b^i_b \) of the nonholonomic constraint equations (1). Motion equations (2) for the Lagrange function (4) take form

\[
\begin{aligned}
g_{ac}^* \ddot{q}^a + \dddot{c}_{ab}^* q^a q^b &= 0, \\
\dddot{c}_{ab}^* &= \Gamma^*_{cab} + \left( b_i^c q^a - b_i^c q^a \right) \left( g_{ia} + g_{ij} b^j_a \right), \\
\Gamma^*_{cab} &= \frac{1}{2} \left( g_{ac} q^b + g_{bc} q^a - g_{ab} q^c \right).
\end{aligned} \tag{5}
\]

If a determinant of the matrix \( (g_{ac}^*) \) is nonzero then we can raise an index of the tensor \( \dddot{c}_{ab}^d = \dddot{c}_{ab}^d \dddot{c}_{ab}^d \). The obtained tensor is an analog of the affine connection in a nonholonomic basis. The torsion tensor \( \dddot{c}_{ab}^d = \dddot{c}_{ab}^d - \dddot{c}_{ba}^d \) is nonzero for the Chaplygin systems due to equation (5). So this fact defines the properties of a nonholonomic motion.

2.2. Reaction forces

To solve the problem of the constraint stabilization of a Chaplygin system we have to introduce an extra set of holonomic and nonholonomic constraints:

\[
\begin{aligned}
f_\mu(q^1, \ldots, q^p) &= 0, \\
h_\alpha(q^1, \ldots, q^p, v^1, \ldots, v^p) &= 0,
\end{aligned} \tag{6}
\]

where Greek indexes \( \mu, \nu = 1, \ldots, m_1, \alpha, \beta = 1, \ldots, m_2, \) \( m_1 + m_2 = r < p \) stand for the summation under constraint terms.

To take into account the set of constraints (6) we have to introduce extra terms of the external generalized forces in (5) which are called the reaction forces. These forces can be defined with the help of the Lagrange multipliers \( (\lambda_1, \ldots, \lambda_r) \). So we consider them as follows:

\[
R_a = \lambda_\mu \frac{\partial f_\mu}{\partial q^a} + \lambda_{m_1 + a} \frac{\partial h_\alpha}{\partial v^a}.
\]
Now we have to construct a system of algebraic equations to determine the Lagrange multipliers as functions of generalized coordinates and velocities so the reaction forces will be defined. For this we must consider second full time derivative of holonomic constraints \( f_\mu \) and first derivative of nonholonomic constraints \( h_\alpha \). The obtained equations combined with the motion equations (5) compose a system of algebraic equations with respect to generalized accelerations and Lagrange multipliers:

\[
\begin{cases}
\ddot{f}_\mu(q^1, ..., q^p) = 0, \\
\dot{h}_\alpha(q^1, ..., q^p, v^1, ..., v^p) = 0, \\
\ddot{q}^a = Q^a + R^a,
\end{cases}
\]

where \( Q^a = -\Gamma^a_{bc} q^b \dot{q}^c \).

Due to system (7) a solution for Lagrange multipliers can be found from the following system of \( r \times r \) algebraic equations:

\[
\begin{align*}
\frac{\partial f_\mu}{\partial q^a} \lambda_\nu \frac{\partial f_\nu}{\partial q^a} + \frac{\partial f_\mu}{\partial q^a} \dot{m}_{1+a} \frac{\partial h_\alpha}{\partial q^a} + \frac{\partial f_\mu}{\partial q^a} Q^a + \frac{\partial^2 f_\mu}{\partial q^a \partial q^b} \ddot{q}^a \dot{q}^b &= 0, \\
\frac{\partial h_\alpha}{\partial q^a} \lambda_\nu \frac{\partial f_\nu}{\partial q^a} + \frac{\partial f_\mu}{\partial q^a} \dot{m}_{1+a} \frac{\partial h_\beta}{\partial q^a} - \frac{\partial h_\alpha}{\partial q^a} Q^a &= 0.
\end{align*}
\]

If this system is nondegenerate then Lagrange multipliers can be uniquely defined as functions of generalized coordinates and velocities which determines the functions of the reaction forces.

2.3. Constraint stabilization

After determining the reaction forces we consider the following system of equations:

\[
\begin{align*}
\ddot{q}^a &= Q^a + R^a, \\
\dot{f}_\mu(q^1, ..., q^p) &= 0, \\
\dot{h}_\alpha(q^1, ..., q^p, v^1, ..., v^p) &= 0.
\end{align*}
\]

To find its solution we can apply some methods of a numerical integration.

A deference scheme of the numerical integration can be realized with the help of the program cycle. But we have to be careful, because when we deal with the multiple repeating operations of the summation, deviations from the constraints, caused mostly by rounding errors, accumulate. And it leads to the numerical solution instability.

A method of constraint stabilization can be applied to control the accumulation of these deviations. According to it the second and the first derivatives in (7) should be equated to the linear form of the constraints themselves. Such as \( \ddot{f}_\mu = -k^\nu_\gamma \dot{f}_\nu - l^\nu_\gamma f_\nu \) and \( \dot{h}_\alpha = -m^\nu_\gamma h_\beta \). These equations are called the equations of perturbation and the linear coefficients - perturbation parameters. The main problem of the constraint stabilization is to determine the optimal range of the values of the perturbation parameters so the numerical solution will be stable in relation to the constraint equations.

Due to the method of constraint stabilization the equations (8) can be rewritten in the form

\[
\begin{align*}
\frac{\partial f_\mu}{\partial q^a} \lambda_\nu \frac{\partial f_\nu}{\partial q^a} + \frac{\partial f_\mu}{\partial q^a} \dot{m}_{1+a} \frac{\partial h_\alpha}{\partial q^a} + \frac{\partial f_\mu}{\partial q^a} Q^a + \frac{\partial^2 f_\mu}{\partial q^a \partial q^b} \ddot{q}^a \dot{q}^b &= -k^\nu_\gamma \dot{f}_\nu - l^\nu_\gamma f_\nu, \\
\frac{\partial h_\alpha}{\partial q^a} \lambda_\nu \frac{\partial f_\nu}{\partial q^a} + \frac{\partial f_\mu}{\partial q^a} \dot{m}_{1+a} \frac{\partial h_\beta}{\partial q^a} - \frac{\partial h_\alpha}{\partial q^a} Q^a &= -m^\nu_\gamma h_\beta.
\end{align*}
\]

The reaction forces now also depend on perturbation parameters like the Lagrange multipliers. So at each step of the numerical summation stabilization term of the reaction forces should control values of the deviations.

Let us consider Euler’s first order difference scheme. For the system (8) it can be written in the form

\[
\begin{align*}
q^a_{(n+1)} &= q^a_{(n)} + h v^a_{(n)}, \\
v^a_{(n+1)} &= v^a_{(n)} + h(R^a_{(n)} - Q^a_{(n)}), \\
t^a_{(n+1)} &= t^a_{(n)} + h,
\end{align*}
\]

where indexes in the round brackets stand for the number of the step of the summation.

To avoid some bulky expressions let’s write the perturbation equations in the matrix form:
Let us consider that sleigh axis, of inertia of the wheel also about the vertical axis, 2 wheels correspondingly, the symmetry axis and the projection of plane. Lagrange function and nonholonomic constraints are as follows: As an example we consider the well known mechanical problem of two wheeled sleigh on an inclined plane. Lagrange function and nonholonomic constraints are as follows:

$$\dot{f} = t \begin{pmatrix} f_1 \\ \vdots \\ f_{m_1} \\ g_1 \\ \vdots \\ g_{m_1} \\ h_1 \\ \vdots \\ h_{m_2} \end{pmatrix} = K f,$$

where \( f_\mu = \dot{f}_\mu \) and elements of the matrix \( K \) represent the perturbation parameters. Let us expand the vector function \( f(q_{(n+1)}, v_{(n+1)}) \) to the Taylor series of the powers \( h \):

$$f(q_{(n+1)}, v_{(n+1)}) = f(q_{(n)}, v_{(n)}) + h \frac{df(q_{(n+1)}, v_{(n+1)})}{dh} \bigg|_{h=0} + \frac{h^2}{2} f^{(2)},$$

where \( f^{(2)} = \frac{d^2 f(q_{(n+1)}, v_{(n+1)})}{dh^2} \bigg|_{h=0} \) is a vector of integral reminder terms in the form of a mean value theorem.

Such as \( df(q_{(n+1)}, v_{(n+1)})/dh \bigg|_{h=0} = f(q, v) t \big|_{(q, v) = (q_{(n)}, v_{(n)})} = K f(q_{(n)}, v_{(n)}) \). We can estimate the value of the deviations at \( (n + 1) \) step if we consider that at \( n \) step the following inequality takes place: \( |f_{(n)}| < \varepsilon \), where \( \varepsilon \) is chosen small control parameter and \( |f_{(n)}| \) is a norm of a vector in Euclidean space. So if we consider triangle inequality to our expansion we can obtain the following result:

$$|f_{(n+1)}| \leq |I_{2m_1+m_2} + hK|\varepsilon + \frac{h^2}{2} |f^{(2)}|,$$

(11)

where \( I_{2m_1+m_2} \) is a unit matrix, and \( |I_{2m_1+m_2} + hK| \) is an \( L_{2,1} \) matrix norm.

If we want the norm of deviation vector to be also less than \( \varepsilon \) the following equality should take place:

$$|I_{2m_1+m_2} + hK| \leq 1 - \frac{h^2}{2\varepsilon} |f^{(2)}|.$$

(12)

Formula (12) defines the range of the values of perturbation parameters at which the values of constraint deviations do not increase.

3. Stabilization of two-wheeled sleigh

As an example we consider the well-known mechanical problem of two-wheeled sleigh on an inclined plane. Lagrange function and nonholonomic constraints are as follows:

$$L = \frac{m}{2} (x^2 + y^2) + m_0 \dot{\theta} (y \cos \theta - x \sin \theta) + \frac{1}{2} \dot{\theta}^2 + \frac{c}{2} (\dot{\varphi}_1^2 + \dot{\varphi}_2^2),$$

$$\dot{x} \cos \theta + \dot{y} \sin \theta + b \dot{\theta} + a \dot{\varphi}_1 = 0,$$

$$\dot{x} \cos \theta + \dot{y} \sin \theta - b \dot{\theta} + a \dot{\varphi}_2 = 0,$$

$$\dot{y} \cos \theta - \dot{x} \sin \theta = 0,$$

(13)

where \( m = m_0 + 2m_i \), \( m_0 \) – mass of the body, \( m_i \) – mass of the wheel, \((x, y)\) are the coordinates of the intersection point \( M(x, y) \) between the symmetry axis and the wheel axle, \( \theta \) is an angle between the symmetry axis and the projection of \( Ox \) axis on the inclined plane, \( \varphi_1 \) and \( \varphi_2 \) are rotation angle of wheels correspondingly, \( l \) is a distance between \( M(x, y) \) and a mass center of the sleigh, \( J = m_0 k_0^2 + 2m_i b^2 + 2A \) – moment of inertia about the vertical axis passing through the point \( M(x, y) \), \( k_0 \) – radius of inertia of the wheel also about the vertical axis, \( A \) – moment of inertia of a wheel about the diameter axis, \( C \) – axial moment of inertia of the wheel, \( a \) – radius of the wheel, \( b \) – its length of a semiaxis.

Let us consider that sleigh should follow to a particular trajectory - an extra holonomic constraint:

$$y = y(x) \quad \text{or} \quad f(x, \theta) = \tan \theta - \frac{dy}{dx} = 0.$$

(14)

The structure of the Lagrange function (13) and the constraint equations (14) allow us to separate independent coordinates \((x, \theta)\) and dependent ones \((y, \varphi_1, \varphi_2)\). So the problem can be solved with the
help of Chaplygin’s approach. The motion equations (9) with respect to the constraint stabilization take form

\[
\begin{align*}
\frac{m\ddot{x}}{\cos^2\theta} + \frac{m\dot{x}\tan\theta}{\cos^2\theta} - m_0l\dot{\theta}^2 &= -\lambda \frac{dy}{dx}, \\
\ddot{f} + \frac{m\ddot{x}\tan\theta}{\cos^2\theta} - m_0l\dot{\theta}^2 &= \frac{\lambda}{\cos^2\theta}, \\
\ddot{f} + k_2\ddot{f} + k_1f &= 0,
\end{align*}
\]

(15)

where \(m = m + 2\epsilon\), \(f = J + 2\epsilon\frac{b^2}{a^2}\). The dependent variables are determined through the constraint equations in (13).

We consider deference first order scheme (10) to obtain a stable numerical solution. To control the value of constraint deviations we also consider the stabilization terms in (15). There are only two perturbation parameters in them so we can specifically chose, for example, the value of \(k_2\) and determine the value of \(k_1\) with the help of the inequality (12). It be written in the form:

\[
(1 + h^2k_1^2) + (1 - hk_2)^2 = 1 - \frac{h^2}{2\epsilon} |f_{(n+1)}^{(2)}|^2 + \delta,
\]

(16)

where \(\delta\) is a chosen parameter. So at each step of the numerical the value of the perturbation parameter \(k_{1(n)}\) can be considered self-defining.

If \(\gamma(x) = A\sin(kx)\) and an initial data is chosen corresponding to the constraints, numerical solution will lead to the following result

![Figure 1. Value of deviation dependence on time.](attachment:image.png)

Basically, we are not interested in the trajectory curve because we have established it as sinusoid. But the plot of deviation value dependence on time can be quite illustrative for this problem. So the self-defining value of a particular perturbation parameter allows us to provide the minimum value of deviations.

References

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