Concentration inequalities for measures of a Boolean model

Fabian Gieringer* and Günter Last*

July 17, 2018

Abstract
We consider a Boolean model $Z$ driven by a Poisson particle process $\eta$ on a metric space $\mathcal{Y}$. We study the random variable $\rho(Z)$, where $\rho$ is a (deterministic) measure on $\mathcal{Y}$. Due to the interaction of overlapping particles, the distribution of $\rho(Z)$ cannot be described explicitly. In this note we derive concentration inequalities for $\rho(Z)$. To this end we first prove two concentration inequalities for functions of a Poisson process on a general phase space.

2000 Mathematics Subject Classification. 60D05, 60G55.
Key words and phrases. concentration inequality, Poisson process, Boolean model, covariance identity

1 Introduction
Let $\mathcal{Y}$ be a locally compact separable metric space and let $\mathcal{F}$ be the space of all closed subsets of $\mathcal{Y}$ equipped with a suitable $\sigma$-field. Let $\eta$ be a Poisson process on $\mathcal{F}$ with a $\sigma$-finite intensity measure $\Lambda$. If $\eta(K) > 0$, then we write $K \in \eta$ and say that $K$ is a particle of $\eta$. The Boolean model associated with $\eta$ is the random set $Z$ defined by the union of all particles, that is

$$Z := \bigcup_{K \in \eta} K.$$

Let $\rho$ be a measure on $\mathcal{Y}$ satisfying

$$\int_{\mathcal{F}} \rho(K) \Lambda(dK) < \infty.$$

Then $\rho(Z)$ is a finite random variable even though $Z$ might not be a random closed set in the sense of [15] [18].

The random set $Z$ is a fundamental model of stochastic geometry and continuum percolation; see [5] [15] [18]. Explicit formulae for the distribution of geometric functionals of the Boolean model are not available, even not in the simplest case of a stationary Boolean on $\mathbb{R}^d$ and $\rho = \lambda_d(\cdot \cap W)$ being the restriction of Lebesgue measure to a convex and compact set $W$. The

*fabian.gieringer@kit.edu, guenter.last@kit.edu, Karlsruhe Institute of Technology, Institute of Stochastics, 76128 Karlsruhe, Germany.
reason for the absence of such formulae is the interaction between the particles from \( \eta \) caused by overlapping. One way out are moment formulae and central limit theorems; see e.g. [10] and [13] Chapter 22. In this paper we will prove concentration inequalities of the form

\[
P(F - \mathbb{E}[F] \geq r) \leq \exp\left(\inf_{\xi \geq 0} \int_0^s v(u) \, du - sr\right), \quad r > 0,
\]

where the function \( v : [0, \infty) \to [0, \infty] \) is determined by \( \Lambda \) and \( \rho \). In the stationary Euclidean case such inequalities were first proved in [7]. Our bounds improve these results. Moreover, we generalize the setting of [7] in several ways. First, we study the Boolean model on a metric space \( \mathcal{Y} \) and not only on \( \mathbb{R}^d \). Second, we will allow that compact subsets of \( \mathcal{Y} \) are intersected by infinitely many Poisson particles. Hence, in general, the random set \( Z(t) \) is not closed and its boundary might have fractal properties. Roughly speaking, this means that we can allow for a \( \sigma \)-finite distribution of the typical grain. Closely related models of this type were introduced in [21], a seminal paper on fractal percolation, that was almost completely ignored in the later literature. Third, we consider general measures and not only the volume. Finally, our method allows to treat also Lipschitz functions of these measures.

Similarly as in [8,12] our approach is based on a covariance identity for square integrable Poisson functionals. In fact we first prove a concentration inequality for functions of a Poisson process on a general phase space. Using the log-Sobolev inequality, related concentration inequalities were derived in [2,1,19].

2 Concentration of Poisson functionals

Let \( (\mathcal{X}, \mathcal{X}) \) be a measurable space and let \( \Lambda \) be a \( \sigma \)-finite measure on \( \mathcal{X} \). Let \( \eta \) be a Poisson process on \( \mathcal{X} \) with intensity measure \( \Lambda \), defined over a probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \); see [13]. In particular, \( \eta \) is a point process, that is a measurable mapping from \( \Omega \) to the space \( \mathcal{N} = \mathcal{N}(\mathcal{X}) \) of all \( \sigma \)-finite measures with values in \( \mathbb{N}_0 := \{0, 1, 2, \ldots\} \), where \( \mathbb{N} \) is equipped with the smallest \( \sigma \)-field \( \mathcal{N} \) such that \( \mu \mapsto \mu(B) \) is measurable for all \( B \in \mathcal{X} \). The distribution of \( \eta \) is denoted by \( \Pi_\Lambda := \mathbb{P}(\eta \in \cdot) \). Since we are only interested in distributional properties of \( \eta \), Corollary 6.5 in [13] shows that it is no restriction of generality to assume that \( \eta \) is proper. This means that there exist random elements \( X_1, X_2, \ldots \) in \( \mathcal{X} \) and an \( \mathbb{N}_0 \)-valued random variable \( \kappa \) such that almost surely \( \eta = \sum_{n=1}^{\infty} \delta_{X_n} \).

Let \( 0 \leq t \leq 1 \) and \( Y_1, Y_2, \ldots \) be a sequence of independent random variables with distribution \((1 - t)\delta_0 + t\delta_1\), independent of \( \eta \). Define \( \eta_t := \sum_{n=1}^{\kappa} Y_n \delta_{X_n} \) as the \( t \)-thinning of \( \eta \). Then \( \eta_t \) and \( \eta - \eta_t \) are independent Poisson processes with intensity measures \( t\Lambda \) and \((1 - t)\Lambda\), respectively. Given \( x \in \mathcal{X} \) and a measurable function \( f : \mathcal{N} \to \mathbb{R} \), the difference operator \( D_x f \) is defined by

\[
D_x f(\mu) := f(\mu + \delta_x) - f(\mu), \quad \mu \in \mathcal{N}.
\]

This mapping is measurable since \((\mu, x) \mapsto \mu + \delta_x \) is measurable. We call a random variable \( F \) a Poisson functional if there is a measurable \( f : \mathcal{N} \to \mathbb{R} \) such that \( F = f(\eta) \) almost surely. In this case we define

\[
D_x F := D_x f(\eta), \quad x \in \mathcal{X},
\]

(which is almost surely, for \( \Lambda \)-almost all \( x \), independent of the choice of an admissible \( f \)) and further a mapping \( DF : \Omega \times \mathcal{X} \to \mathbb{R} \), given by \((\omega, x) \mapsto (D_x F)(\omega)\).
where the case $s < \theta$ is possible. Define

$$V_F(s) := \int_\mathbb{R} (e^{sD_F} - 1) \int_0^1 \int_\mathbb{N} D_x f(\eta_t + \mu) \Pi_{(1-\eta)\Lambda}(d\mu) \, dt \, \Lambda(dx), \quad 0 \leq s < s_F.$$  

The following bound for the cumulant-generating function is the main result of this section.

**Theorem 2.2.** Let $F = f(\eta)$ be Poisson functionals such that $F, G \in L^2(\mathbb{P})$ and $DF, DG \in L^2(\mathbb{P} \otimes \Lambda)$. Then

$$\log \mathbb{E}[e^{s(F-\mathbb{E}[F])}] \leq \inf_{0 < \theta < 1} \frac{\theta}{1 - \theta} \left( \int_0^s \frac{1}{s} \log \mathbb{E}[e^{sV_f(u)/\theta}] - \frac{1}{u} \log \mathbb{E}[e^{uF}] \right) du + s \mathbb{E}[F].$$

**Proof.** We combine the idea of the proof of Lemma 3.1 in [9] (see also the proof of Theorem 1 in [8]) with Lemma 11 in Massart [14]. Let $\theta \in (0, 1)$ and $s \in (0, s_F)$ be such that $\mathbb{E}[e^{sV_f(u)/\theta}] < \infty$ and let $u \in (0, s]$. Since $u < s_F$, we can use the covariance identity (2.1) to obtain that

$$\text{Cov}(F, e^{uF}) = \mathbb{E} \left[ \int_\mathbb{R} \int_0^1 \int_\mathbb{N} D_x f(\eta_t + \mu) \Pi_{(1-\eta)\Lambda}(d\mu) \, dt \, \Lambda(dx) \right] = \mathbb{E}[e^{uF} V_F(u)].$$

Now, Lemma 11 of Massart [14] applied to $V_F(u)/\theta$ and $F$ yields

$$\frac{\mathbb{E}[e^{uF} V_F(u)]}{\mathbb{E}[e^{uF}]} \leq \frac{\theta \mathbb{E}[e^{uF}]}{\mathbb{E}[e^{uF}]} + \frac{\theta}{u} \log \mathbb{E}[e^{V_f(u)/\theta}] - \frac{\theta}{u} \log \mathbb{E}[e^{uF}].$$

The combination of the last two displays leads to the inequality

$$\frac{\mathbb{E}[F e^{uF}]}{\mathbb{E}[e^{uF}]} - \mathbb{E}[F] = \frac{\text{Cov}(F, e^{uF})}{\mathbb{E}[e^{uF}]} \leq \frac{\theta \mathbb{E}[e^{uF}]}{\mathbb{E}[e^{uF}]} + \frac{\theta}{u} \log \mathbb{E}[e^{V_f(u)/\theta}] - \frac{\theta}{u} \log \mathbb{E}[e^{uF}]$$

and a simple rearrangement yields

$$\frac{\mathbb{E}[F e^{uF}]}{\mathbb{E}[e^{uF}]} \leq \frac{\theta}{u(1 - \theta)} \left( \frac{\theta}{u} \mathbb{E}[F] + \log \mathbb{E}[e^{V_f(u)/\theta}] - \log \mathbb{E}[e^{uF}] \right).$$

3
Setting \( h(t) := \log \mathbb{E}[e^{tF}] \) and \( g_u(t) := \log \mathbb{E}[e^{V_F(u)}], \ t \geq 0 \), we have
\[
 h(s) = h(0) + \int_0^s h'(u) \, du = \int_0^s \frac{\mathbb{E}[F e^{tF}]}{\mathbb{E}[e^{uF}]} \, du 
 \leq \int_0^s \frac{\theta}{u(1-\theta)} \left( \frac{u}{\theta} \mathbb{E}[F] + g_u(u/\theta) - h(u) \right) \, du.
\]
By \( g_u(0) = 0 \) and the convexity of \( g_u \), we have \( g_u(\frac{u}{s}t) \leq \frac{u}{s} g_u(t) \) for \( t > 0 \), thus
\[
 h(s) \leq \frac{s}{1-\theta} \mathbb{E}[F] + \frac{\theta}{1-\theta} \int_0^s \left( \frac{1}{s} g_u(s/\theta) - \frac{1}{u} h(u) \right) \, du.
\]
From \( \log \mathbb{E}[e^{s(F-\mathbb{E}[F])}] = h(s) - s \mathbb{E}[F] \) and the preceding inequality, (2.2) follows. Using Jensen’s inequality, this simplifies to (2.3).

\[\Box\]

**Theorem 2.2** and the well-known Chernoff bound (see [4])
\[
 \mathbb{P}(F - \mathbb{E}[F] \geq r) \leq \inf_{s > 0} \frac{\mathbb{E}[e^{s(F-\mathbb{E}[F])}]}{e^{sr}}, \quad r \geq 0,
\]
(a direct consequence of Markov’s inequality) imply a concentration inequality. If \( V_F(\cdot) \) has a deterministic bound, this inequality can be simplified as follows.

**Corollary 2.3.** Let \( F = f(\eta) \in L^2(\mathbb{P}) \) and assume that \( v: [0, s_F] \rightarrow \mathbb{R} \) is a measurable function such that almost surely \( V_F(s) \leq v(s) \) for each \( s \in [0, s_F] \). Then,
\[
 \mathbb{P}(F - \mathbb{E}[F] \geq r) \leq \exp \left( \inf_{0 < s < s_F} \int_0^s v(u) \, du - sr \right), \quad r \geq 0.
\]

**Proof.** Let \( r \geq 0 \) and \( s \in (0, s_F) \). By the Chernoff bound (2.4), inequality (2.3) and assumption \( V_F(s) \leq v(s) \), we get
\[
 \mathbb{P}(F - \mathbb{E}[F] \geq r) \leq \mathbb{E}[e^{s(F-\mathbb{E}[F])}] e^{-sr} \leq \exp \left( \inf_{0 < s < s_F} \frac{1}{1-\theta} \int_0^s v(u) \, du - sr \right).
\]
We have \( \int_0^s v(u) \, du \geq 0 \) since the contrary would lead to \( \mathbb{P}(F - \mathbb{E}[F] \geq 0) \leq 0 \) which is obviously wrong. Since \( \inf_{0 < \theta < 1} (1-\theta)^{-1} = 1 \), we obtain that
\[
 \mathbb{P}(F - \mathbb{E}[F] \geq r) \leq \exp \left( \int_0^s v(u) \, du - sr \right), \quad s \in [0, s_F),
\]
and hence the assertion. \[\Box\]

**Remark 2.4.** Concentration inequalities for the lower tail can be derived analogously. Under the obvious integrability assumptions, the bounds (2.2) and (2.3) hold again upon replacing \( F \) by \( -F \) and \( V_F \) by \( V_{-F} \). Thus, by the Chernoff bound \( \mathbb{P}(F - \mathbb{E}[F] \leq -r) \leq \inf_{s > 0} \mathbb{E}[e^{-s(F-\mathbb{E}[F])}] e^{-sr}, \quad r \geq 0 \), a result analogous to Corollary 2.3 gives a bound for the lower tail when \( V_{-F} \) has a deterministic bound. Hence, all results relying on Corollary 2.3 can be given for the lower tail as well.

Our next result was motivated by a question in [11] whether the Mecke formula (cf. [13]) can be combined with the covariance identity to yield reasonable concentration inequalities.
Theorem 2.5. Let \( F = f(\eta) \in L^2(\mathbb{P}) \) be such that \( DF \geq 0 \) holds \((\mathbb{P} \otimes \Lambda)\)-almost everywhere. Assume further that there exist a measurable function \( g : \mathcal{X} \to [0, \infty) \) and constants \( a > 0 \) and \( b \geq 0 \) such that a.s.

\[
\int_0^1 \int_N D_x f(\eta_t + \mu) \Pi_{1-\eta} d\mu \, dt \leq g(x), \quad \Lambda\text{-a.e. } x \in \mathcal{X}, \tag{2.6}
\]

and

\[
\int D_x f(\eta - \delta_x) g(x) \, dx \leq aF + b. \tag{2.7}
\]

Then

\[
\mathbb{P}(F - \mathbb{E}[F] \geq r) \leq \exp \left( \inf_{0 < s < a^{-1} \wedge \delta_x} \left( - \frac{bs}{a} - s \mathbb{E}[F] - \left( \frac{\mathbb{E}[F]}{a} + \frac{b}{a^2} \right) \log(1 - as) - sr \right) \right). \tag{2.8}
\]

In particular, if \( a^{-1} \leq s_F \), we have

\[
\mathbb{P}(F - \mathbb{E}[F] \geq r) \leq \exp \left( - \frac{1}{a} \left( r + \left( \mathbb{E}[F] + \frac{b}{a} \right) \log \left( \frac{\mathbb{E}[F] + b/a}{r + \mathbb{E}[F] + b/a} \right) \right) \right). \tag{2.9}
\]

Proof. Let \( 0 < u \leq s < a^{-1} \wedge s_F \). By the covariance identity (2.1) and assumption (2.6), we have

\[
\text{Cov}(F, e^{uF}) \leq \mathbb{E} \left[ \int D_x e^{uF} g(x) \Lambda(dx) \right].
\]

Applying the Mecke formula and the elementary bound \( e^z - 1 \leq ze^z, \ z \in \mathbb{R} \), yields

\[
\text{Cov}(F, e^{uF}) \leq \mathbb{E} \left[ \int D_x e^{uF(\eta - \delta_x)} g(x) \, dx \right] = \mathbb{E} \left[ \int e^{uF(\eta - \delta_x)} (e^{uD_x f(\eta - \delta_x)} - 1) g(x) \, dx \right]
\]

\[
\leq u \mathbb{E} \left[ \int e^{uF(\eta)} D_x f(\eta - \delta_x) g(x) \, dx \right].
\]

Assumption (2.7) yields that \( \text{Cov}(F, e^{uF}) \leq ua \mathbb{E}[F e^{uF}] + ub \mathbb{E}[e^{uF}] \). By a rearrangement, we obtain

\[
\frac{\mathbb{E}[F e^{uF}]}{\mathbb{E}[e^{uF}]} \leq \frac{\mathbb{E}[F] + ub}{1 - au}.
\]

Setting \( h(u) := \log \mathbb{E}[e^{uF}] \), we get \( h'(u) \leq (\mathbb{E}[F] + ub)/(1 - au) \) and consequently, by integration and \( h(0) = 0 \), the bound \( \log \mathbb{E}[e^{uF}] \leq -bs/a - (\mathbb{E}[F]/a + b/a^2) \log(1 - as) \). Using the Chernoff bound (2.4), we obtain (2.8). The second assertion (2.9) then follows by optimising with \( s = r/(ar + a\mathbb{E}[F] + b) \). This choice of \( s \) is at most \( a^{-1} \), since \( F + b/a \geq 0 \) a.s. by assumption (2.7), and the case \( \mathbb{E}[F] + b/a = 0 \) can be ruled out as this implies \( DF \equiv 0 \) and therefore a trivial concentration. \( \qed \)
3 General Boolean models

In this section we consider a locally compact separable Hausdorff space $\mathbb{Y}$ together with the Borel $\sigma$-field $\mathcal{Y}$. Let $\mathcal{F} \equiv \mathcal{F}(\mathbb{Y})$ denote the class of closed subsets of $\mathbb{Y}$ equipped with the Fell topology generated by the sets $\{F \in \mathcal{F} : F \cap C = \emptyset\}$ and $\{F \in \mathcal{F} : F \cap G \neq \emptyset\}$ for arbitrary compact sets $C \subset \mathbb{Y}$ and open sets $G \subset \mathbb{Y}$; see \cite{15,13}. The associated Borel $\sigma$-field is denoted by $\mathcal{B}(\mathcal{F})$.

Let $\rho$ be a measure on $\mathbb{Y}$ and let $\mathcal{F}' \in \mathcal{B}(\mathcal{F})$ be such that $\rho$ is finite on $\mathcal{F}'$ and $K \mapsto \rho(K)$ is measurable on $\mathcal{F}'$. We also assume that $\mathcal{F}'$ is closed under (finite) unions and equip $\mathcal{F}'$ with the trace $\sigma$-field $\{B \cap \mathcal{F}' : B \in \mathcal{B}(\mathcal{F})\}$.

Let $\Lambda$ be a $\sigma$-finite intensity measure on $\mathcal{F}'$ satisfying

$$\int \rho(K) \Lambda(dK) < \infty. \tag{3.1}$$

Let $(B_n)_{n \in \mathbb{N}}$ be such that $B_n \uparrow \mathcal{F}'$ and $\Lambda(B_n) < \infty$ for all $n \in \mathbb{N}$. Let $N_l$ denote the measurable set of all $\mu \in \mathbb{N} = \mathbb{N}(\mathcal{F}')$ satisfying $\mu(B_n) < \infty$ for each $n \in \mathbb{N}$. Note that $\mathbb{P}(\eta \in N_l) = \Pi_{\Lambda}(N_l) = 1$. Define

$$Z(\mu) := \bigcup_{K \in \mu} K, \quad \mu \in N_l, \tag{3.2}$$

and $Z(\mu) := \emptyset$ for $\mu \notin \mathbb{N} \setminus N_l$. The random set $Z := Z(\eta)$ is called the Boolean model governed by $\eta$. Now consider the function $f : \mathbb{N} \to [0, \infty)$ given by

$$f(\mu) := 1\{\rho(Z(\mu)) < \infty \} \rho(Z(\mu)),$$

with the convention $0 \cdot \infty := 0$. Our goal is to obtain a concentration inequality for

$$F := f(\eta).$$

Campbell’s formula (Proposition 2.7 in \cite{13}) and assumption (3.1) show that $\int \rho \, d\eta < \infty$ a.s., so that the sub-additivity of $\rho$ shows that $\rho(Z) < \infty$ a.s. In particular $\mathbb{P}(F = \rho(Z)) = 1$. However, we need to check that $\rho(Z)$ is a random variable. This follows from the assumption on $\rho$ and the next lemma.

**Lemma 3.1.** The mapping $(x, \mu) \mapsto 1\{x \in Z(\mu)\}$ is measurable on $\mathbb{Y} \times \mathbb{N}$. Furthermore, for each $K \in \mathcal{Y}$ with $\rho(K) < \infty$, the mapping $\mu \mapsto \rho(Z(\mu) \cap K)$ is measurable on $\mathbb{N}$. Finally $\mu \mapsto \rho(Z(\mu))$ is measurable on $\mathbb{N}$.

**Proof.** By Theorem 1-2 in \cite{15}, $\mathcal{F}$ is a compact and separable Hausdorff space and hence $\mathcal{F}'$ (equipped with the trace $\sigma$-field) is a Borel space. By \cite{13} Proposition 6.2] (see also the proof of \cite{13} Proposition 6.3]), there exist measurable functions $\pi_n : \mathbb{N} \to \mathcal{F}'$ such that

$$\mu = \sum_{n=1}^{\mu(\mathcal{F}')} \delta_{\pi_n(\mu)}, \quad \mu \in N_l.$$

This shows that $Z(\mu) = \bigcup_{n=1}^{\mu(\mathcal{F}')} \pi_n(\mu)$ for each $\mu \in N_l$. By \cite{15} Theorem 2-5-1], the mapping $(K, x) \mapsto 1\{x \in K\}$ is measurable on $\mathcal{F} \times \mathbb{Y}$. Since

$$1\{x \notin Z(\mu)\} = \prod_{n=1}^{\mu(\mathcal{F}')} 1\{x \notin \pi_n(\mu)\}, \quad \mu \in N_l,$$
this proves the first assertion. The second assertion follows from Fubini’s theorem.

By monotone convergence, the third assertion follows, once we have shown that

$$
\mu \mapsto \rho\left( \bigcup_{m=1}^{\min[n(\mathcal{F}'),n]} \pi_m(\mu) \right)
$$

is measurable for each $n \in \mathbb{N}$. By [15, Corollary 1-2-1] the mapping $(L, L') \mapsto L \cup L'$ is measurable on $\mathcal{F} \times \mathcal{F}$ and hence also on $\mathcal{F}' \times \mathcal{F}'$. Since $\mathcal{F}'$ is closed under unions, it follows that $\mu \mapsto \bigcup_{m=1}^{\min[n(\mathcal{F}'),n]} \pi_m(\mu)$ is a measurable mapping from $\mathcal{N}_L$ to $\mathcal{F}'$. Since $L \mapsto \rho(L)$ is measurable on $\mathcal{F}'$, the final assertion now follows. □

Let $t \in [0, 1]$. We now compute the probability that a point $y \in \mathcal{Y}$ lies inside the Boolean model of intensity measure $\lambda$. We use the notation

$$
\mathcal{F}'_y := \{ K \in \mathcal{F}' : y \in K \}, \quad y \in \mathcal{Y}.
$$

Since $(K, y) \mapsto 1[y \in K]$ is $\mathcal{B}(\mathcal{F}) \otimes \mathcal{Y}$-measurable, this is a measurable set. The first definition in (3.2) and the defining properties of a Poisson process yield

$$
\int 1[y \in Z(\mu)] \Pi_\lambda(d\mu) = 1 - \mathbb{P}(\eta(\mathcal{F}'_y) = 0) = 1 - \exp(-t\lambda(\mathcal{F}'_y)).
$$

Lemma 3.2. We have that

$$
D_K \mu = \rho(K) - \rho(Z(\mu) \cap K), \quad \Pi_\lambda \otimes \lambda \mbox{-a.e. } (\mu, K).
$$

Proof. For each $\mu \in \mathcal{N}_L$ and each $K \in \mathcal{F}'$ we have that

$$
D_K \mu = \rho(Z(\mu + \delta_K)) - \rho(Z(\mu)) = \rho(Z(\mu + K)) - \rho(Z(\mu)).
$$

Since $\Pi_\lambda(\{\mu \in \mathcal{N}_L : \rho(Z(\mu)) < \infty\}) = 1$ we can use the additivity of $\rho$ to conclude the proof. □

Lemma 3.3. Let $t \in [0, 1]$ and $K \in \mathcal{F}'$. Then $\mathbb{P}$-a.s.

$$
\int D_K \mu \Pi_{(1-t)\lambda}(d\mu) \leq \int 1[y \in K] \exp(-t\lambda(\mathcal{F}'_y)) \rho(dy).
$$

Proof. It follows from Lemma 3.2 and the superposition theorem for Poisson processes that $\mathbb{P}$-a.s.

$$
D_K \mu \leq \rho(K) - \rho(Z(\mu) \cap K), \quad \Pi_{(1-t)\lambda} \otimes \lambda \mbox{-a.e. } (\mu, K).
$$

Furthermore,

$$
\int \rho(Z(\mu) \cap K) \Pi_{(1-t)\lambda}(d\mu) = \int \int 1[y \in Z(\mu)] 1[y \in K] \Pi_{(1-t)\lambda}(d\mu) \rho(dy)
$$

$$
= \rho(K) - \int 1[y \in K] \exp(-t\lambda(\mathcal{F}'_y)) \rho(dy),
$$

where we have used (3.4). The assertion now follows. □
Define
\[
\rho^\ast(K) := \int_K \tau(\Lambda(F'_y)) \rho(dy), \quad K \in F',
\]
where \(\tau : [0, \infty) \to [0, 1]\) is given by \(\tau(z) := (1 - e^{-z})/z\) for \(z \in (0, \infty)\), \(\tau(0) := \lim_{z \to 0} \tau(z) = 1\) and \(\tau(\infty) := \lim_{z \to \infty} \tau(z) = 0\). We define a measure \(\nu^\ast\) on \([0, \infty)\) by
\[
\nu^\ast := \int_{F'} 1_{\rho(K) \in \cdot} \frac{\rho'(K)}{\rho(K)} \Lambda(dK),
\]
with the convention \(0/0 := 0\) and another measure \(\nu\) on \([0, \infty)\) by
\[
\nu := \int_{F'} 1_{\rho(K) > 0, \rho(K) \in \cdot} \Lambda(dK).
\]
Naturally, our concentration inequalities require the constant \(c_0\) to be positive.

The function \(\phi : [0, \infty) \to \mathbb{R}\), defined by \(\phi(z) := e^z - 1 - z\), plays an important role in the sequel.

**Theorem 3.4.** Assume that \((\text{3.1})\) holds and that \(s_0 > 0\), where \(s_0\) is given by \((\text{3.6})\). Then the Poisson functional \(F = \rho(Z)\) satisfies
\[
\mathbb{P}(F - \mathbb{E}[F] \geq r) \leq \exp \left( \inf_{0 < s < s_0} \left( \int \phi(su) \nu^\ast(du) - sr \right) \right), \quad r > 0.
\]

**Proof.** We write \(\eta = \sum_{k=1}^{\eta(F')} \delta_{Z_k}\). Let \(n \in \mathbb{N}\). Then \(\eta_n := \sum_{k=1}^{\eta(F')} 1_{\rho(Z_k) \leq n} \delta_{Z_k}\) is a Poisson process with intensity measure \(\Lambda_n(dK) := 1_{\rho(K) \leq n} \Lambda(dK)\). Define \(Z_n := Z(\eta_n)\) and \(F_n := \rho(Z_n)\). We wish to apply Corollary 2.3 to the pair \((\eta_n, F_n)\). We start by checking the integrability properties of the Poisson functional \(F_n\). First, we obtain from Lemma 3.2 that
\[
\mathbb{E} \left[ \int (D_k F_n)^2 \Lambda_n(dK) \right] \leq \int 1_{\rho \leq n} u^2 \nu(du) \leq n \int u \nu(du)
\]
which is finite by \((\text{3.1})\). Since \(\mathbb{E}[F_n] < \infty\) the Poincaré inequality (see \([13, \text{Exercise 18.2}]\)) shows that \(\mathbb{E}[F^2] < \infty\). Secondly, we have for each \(s \geq 0\) that
\[
\mathbb{E}[e^{2sF_n}] \leq \mathbb{E} \left[ \exp \left[ 2s \int \rho(K) \eta_n(dK) \right] \right] = \exp \left[ \int \left( e^{2s\rho(K)} - 1 \right) \Lambda_n(dK) \right],
\]
where we have used a well-known formula for Poisson processes (see e.g. \([13]\)). The integral in the exponent is dominated by a multiple (depending on \(n\) and \(s\)) of \(\int 1_{\rho \leq n} u \nu(du)\) and hence finite. Thirdly, we have
\[
\mathbb{E} \left[ \int (D_k e^{sF_n})^2 \Lambda_n(dK) \right] = \mathbb{E} \left[ \left( e^{sF_n} (e^{sD_k F_n} - 1) \right)^2 \right] \Lambda_n(dK)
\]
\[
\leq \mathbb{E}[e^{2sF_r}] \int (e^{s\rho(K)} - 1)^2 \Lambda_n(dK).
\]

This is finite, since \( \int 1[u \leq n](e^{su} - 1)^2 \nu(du) \) is bounded by a multiple of \( \int u \nu(du) \).

By Lemma 3.2 and Lemma 3.3 (applied with \((\eta_n, F_n)\) in place of \((\eta, F)\),
\[
V_{F_n}(s) \leq \int_{F'} (e^{s\rho(K)} - 1) \int_{F'} 1[y \in K] \int_0^1 \exp(-(1-t)\Lambda_n(F'_y)) dt \rho(dy) \Lambda_n(dK) = \int_{F'} (e^{s\rho(K)} - 1) \int_K \tau(\Lambda_n(F'_y)) \rho(dy) \Lambda_n(dK) =: h_n(s).
\]

Let \( \rho^*_n(K) := \int_K \tau(\Lambda_n(F'_y)) \rho(dy) \). Then we have
\[
\int_0^1 h_n(u) du = \int_{F'} 1[\rho(K) > 0] \rho^*_n(K) \int_0^s (e^{s\rho(K)} - 1) du \Lambda_n(dK) = \int 1[\rho(K) > 0] \rho^*_n(K) \left( \frac{e^{s\rho(K)} - 1}{\rho(K)} - s \right) \Lambda_n(dK) = \int \frac{\rho^*_n(K)}{\rho(K)} \phi(s\rho(K)) \Lambda_n(dK).
\]

For each \( r > 0 \) we now obtain from (2.5) that
\[
\mathbb{P}(F_n - \mathbb{E}[F_n] \geq r) \leq \exp \left( \int 1[\rho(K) \leq n] \frac{\rho^*_n(K)}{\rho(K)} \phi(s\rho(K)) \Lambda(dK) - sr \right). \tag{3.8}
\]

As \( n \to \infty \) we have \( Z_n \uparrow Z \) and hence \( F_n \uparrow F \). Monotone convergence implies \( \mathbb{E}[F_n] \to \mathbb{E}[F] \). We now assume that \( s \in (0, s_0) \) and assert that
\[
\lim_{n \to \infty} \int 1[\rho(K) \leq n] \frac{\rho^*_n(K)}{\rho(K)} \phi(s\rho(K)) \Lambda(dK) = \int \frac{\rho^*(K)}{\rho(K)} \phi(s\rho(K)) \Lambda(dK). \tag{3.9}
\]

Indeed, we have for each \( y \in Y \) that \( \lim_{n \to \infty} \Lambda_n(F'_y) = \Lambda(F'_y) \), and since \( \tau(\cdot) \leq 1 \) we obtain for each \( K \in F'_y \) from dominated convergence that \( \lim_{n \to \infty} \rho^*_n(K) = \rho^*(K) \). Hence (3.9) follows from dominated convergence once we have shown that \( \int \phi(s\rho(K)) \Lambda(dK) \) is finite. By assumption (3.6) it is sufficient to show that
\[
\int 1[u \leq 1] \phi(su) \nu(du) < \infty. \tag{3.10}
\]

For \( u \in [0, 1] \) the definition of \( \phi \) implies that \( \phi(su) \leq u\phi(s) \) and (3.10) follows. Let \( \varepsilon > 0 \) such that \( r - \varepsilon > 0 \). Fatou’s Lemma shows that
\[
\mathbb{P}(F - \mathbb{E}[F] > r - \varepsilon) \leq \liminf_{n \to \infty} \mathbb{P}(F_n - \mathbb{E}[F_n] > r - \varepsilon) \leq \exp \left( \int \frac{\rho^*(K)}{\rho(K)} \phi(s\rho(K)) \Lambda(dK) - s(r - \varepsilon) \right),
\]

where we have used (3.9) and (3.8) to obtain the second inequality. Letting \( \varepsilon \to 0 \), we obtain the asserted concentration inequality (3.7).
Theorem [3.4] can be generalized to Lipschitz functions of $F$. Recall that a function $T : [0, \infty) \to \mathbb{R}$ is Lipschitz with Lipschitz constant $c_T \geq 0$ if $|T(u) - T(v)| \leq c_T|u - v|$ for all $u, v \geq 0$.

**Theorem 3.5.** Let the assumptions of Theorem [3.4] be satisfied and let $T : [0, \infty) \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant $c_T > 0$. Then the Poisson functional $G := T(\rho(Z))$ satisfies

$$\mathbb{P}(G - \mathbb{E}[G] \geq r) \leq \exp\left( - \int_{0<s<s_0/c_T} \inf_{0<s<s_0/c_T} \left( \int \phi(c_T s u) \nu'(du) - sr \right) \right), \quad r > 0.$$  

**Proof.** We generalize the proof of Theorem [3.4]. Let $G_n := T(F_n), n \in \mathbb{N}$. We first note that

$$|D_K G_n| = |T(f(\eta_n + \delta_K) - T(f(\eta_n))| \leq c_T D_K F_n, \quad K \in \mathcal{T}'.$$

Since $|G_n| \leq |T(0)| + c_T F_n$, we can use the first part of the above proof to conclude that the pair $(G_n, \eta_n)$ satisfies the assumptions of Corollary [2.3] with $s_0/c_T$ in place of $s_0$. Using (3.11) and the inequality $|e^u - 1| \leq e^{su} - 1, u \in \mathbb{R}$, we now obtain for all $s \geq 0$ that

$$V_{G_n}(s) \leq |V_{G_n}(s)| \leq \int_{\mathcal{T}'} (e^{sc_T \nu(K)} - 1) \int_K c_T \tau(\Lambda_n(F_y)) \rho(dy) \Lambda_n(dK) := h_n(s).$$

Since

$$\int_0^s h_n(u) \, du = \int \frac{\nu(K)}{\rho(K)} \phi(s c_T \nu(K)) \Lambda_n(dK),$$

we can finish the proof as before. \qed

In the remainder of the paper we shall work with Theorem [3.4] and not with its more general version. However, all results can be formulated for Lipschitz functions of $\rho(Z)$.

Define a function $h : [0, \infty) \to [0, \infty)$ by

$$h(s) := \int_0^\infty u(e^{su} - 1) \nu^*(du), \quad s \geq 0.$$  

(3.12)

If $\nu^* = 0$, then $h \equiv 0$. Otherwise $h$ is finite and strictly increasing on $[0, s_0)$. Let $h^{-1} : [0, \infty) \to [0, \infty]$ denote the generalized inverse of $h$, defined by

$$h^{-1}(u) := \inf\{s \geq 0 : h(s) \geq u\}, \quad u \geq 0,$$

where $\inf \emptyset := \infty$. If $\nu^* = 0$, then $h^{-1} \equiv \infty$ on $(0, \infty)$. Otherwise $h^{-1}$ is strictly increasing and continuous on $[0, h(s_0^-))$, where $h(s_0^-) := \lim_{s \downarrow s_0} h(s)$.

**Theorem 3.6.** Under the assumptions of Theorem [3.4]

$$\mathbb{P}(F - \mathbb{E}[F] \geq r) \leq \exp\left( - \int_{0}^{r} h^{-1}(u) \, du \right), \quad r \in (0, h(s_0^-)).$$

**Proof.** If $\int \rho(K) \Lambda(dK) = 0$, then $F \equiv 0$ and the result is trivial. Hence we can assume that $\int \rho(K) \Lambda(dK) > 0$. We next show that then $\nu^*(0, \infty) > 0$. By definition of $\nu^*$ it is sufficient to
show for each \( K \in \mathcal{F}' \) that \( \rho^*(K) > 0 \) whenever \( \rho(K) > 0 \). Since \( \tau > 0 \) on \([0, \infty)\), it is sufficient to prove that \( \Lambda(\mathcal{F}_y) < \infty \) for \( \rho \)-a.e. \( y \in K \). But this follows from

\[
\int K \Lambda(\mathcal{F}_y) \rho(dy) = \int 1[y \in L]1[y \in K] \Lambda(dL) \rho(dy) = \int \rho(K \cap L) \Lambda(dL),
\]

which is finite by (3.1).

Since \( \nu^*(0, \infty) > 0 \) we obtain for each \( s \in (0, s_0) \) that

\[
\frac{d}{ds} \int \phi(su) \nu^*(du) = h(s) \in (0, \infty), \quad \frac{d}{ds} h(s) = \int u^2 e^{su} \nu^*(du) \in (0, \infty).
\]

In view of Theorem 3.4 the proof can now be finished as that of [8, Theorem 1].

\[ \square \]

**Remark 3.7.** Proposition 3.2 in [19] implies (3.7) with \( \nu \) instead of \( \nu^* \). Since \( \nu^* \leq \nu \), our result improves this inequality. The larger \( y \mapsto \Lambda(\mathcal{F}_y) \) the larger the improvement. Recall from (3.4) that \( \mathbb{P}(y \in Z) = 1 - \exp \left(-\Lambda(\mathcal{F}_y) \right) \) is the probability that the point \( y \in \mathcal{Y} \) is covered by \( Z \). Our concentration inequality takes into account these covering probabilities and hence the overlapping of distinct grains.

In the sequel we use the function \( \psi: [0, \infty) \to (-\infty, \infty] \), defined by

\[
\psi(z) := 1 - \frac{1}{z} (1 + z) \log(1 + z), \quad z > 0,
\]

and \( \psi(0) := \infty \). We also define

\[
m_i := \int_0^\infty u^i \nu^*(du), \quad i \in \{0, 1, 2\}.
\]

The proof of the following corollary of Theorem 3.6 is similar to that of [8, Corollary 1].

**Corollary 3.8.** Assume that (5.1) holds and that \( \nu^* \neq 0 \). Assume also that there is some \( a > 0 \) such that \( \rho(K) \leq a \) for \( \Lambda \)-a.e. \( K \in \mathcal{F}' \). Then we have for each \( i \in \{0, 1, 2\} \) that

\[
\mathbb{P}(F - \mathbb{E}[F] \geq r) \leq \exp \left[ \frac{r}{a} \psi \left( \frac{d^{i-1}K}{m_i} \right) \right], \quad r > 0.
\]

**Proof.** We first note that \( h(s_0-) = \infty \). This follows by \( \nu^* \neq 0 \) once we have shown that \( s_0 = \infty \). To this end, let \( s > 0 \). Then, we have \( \int 1[\rho(K) > 1] e^{\psi(K)} \Lambda(dK) \leq e^{sa} \int \rho(K) \Lambda(dK) \) which is finite by (3.1).

Let \( i \in \{0, 1, 2\} \). In the case \( i = 0 \) we can assume that \( m_0 = \nu^*([0, \infty)) > 0 \). (Otherwise there is nothing to prove.) It is easy to see that \( h(s) \leq m_i a^{1-i} (e^{sa} - 1) \) for all \( s > 0 \); cf. the proof of [8, Corollary 1] for the case \( i = 2 \). Therefore

\[
h^{-1}(r) \geq \frac{1}{a} \log \left( 1 + \frac{ra^{i-1}}{m_i} \right), \quad r > 0.
\]

Since

\[
-\int_0^r \log(1 + zu) du = r \psi(z), \quad r > 0,
\]

for each \( z > 0 \), we deduce the assertion from Theorem 3.6. \[ \square \]
Example 3.9. In this example we specialize the setting of this section to the case $\mathbb{Y} = \mathbb{R}^d$ for some fixed integer $d \in \mathbb{N}$. We set $\mathcal{F} := \mathcal{F}(\mathbb{R}^d)$. We fix $r \in [0, d]$ and let $\lambda_r$ denote the $r$-dimensional Hausdorff measure on $\mathbb{R}^d$. Let $\mathcal{F}'$ denote the set of all $C \in \mathcal{F}$ such that $\lambda_r(C \cap \cdot)$ is a locally finite measure on $\mathbb{R}^d$. By [20, Corollary 2.1.5], we have that $K \mapsto \rho(K)$ is measurable on $\mathcal{F}'$, so that the pair $(\mathcal{F}', \rho)$ satisfies the general assumptions of this section (with $\mathcal{F}' = \mathcal{F}'$).

4 Stationary Boolean models

In this section we consider the setting of Example 3.9 in the case $r = d$. We let $\eta$ be a Poisson process on the space $\mathcal{F}^d$ of all closed sets $K \subset \mathbb{R}^d$ with $\lambda_d(K) < \infty$. We assume here that the intensity measure $\Lambda$ of $\eta$ is of the translation invariant form

$$\Lambda = \int_{\mathcal{F}^d} \int_{\mathbb{R}^d} 1\{K + x \in \cdot\} dx \mathbb{Q}(dK), \quad (4.1)$$

where $K + x := \{y + x : y \in K\}$ and $\mathbb{Q}$ is a $\sigma$-finite measure on $\mathcal{F}^d$ satisfying

$$0 < \gamma_1 := \int \lambda_d(K) \mathbb{Q}(dK) < \infty. \quad (4.2)$$

Example 4.1. Let $\mathbb{Q}_0$ be a probability measure on $\mathcal{F}^d$ satisfying $\int \lambda_d(K) \mathbb{Q}_0(dK) < \infty$ and let $\rho_0$ be a measure on $(0, \infty)$ such that $\int_0^\infty r^d \rho_0(dr) < \infty$. Assume that

$$\mathbb{Q} = \iint 1\{rK \in \cdot\} \rho_0(dr) \mathbb{Q}_0(dK).$$

Then

$$\int \lambda_d(K) \mathbb{Q}(dK) = \iint \lambda_d(rK) \rho_0(dr) \mathbb{Q}_0(dK) = \int r^d \rho_0(dr) \int \lambda_d(K) \mathbb{Q}_0(dK) < \infty.$$

We fix a closed set $W \subset \mathbb{R}^d$ with positive finite volume and derive concentration inequalities for the Poisson functional

$$F = \lambda_d(Z \cap W),$$

where $Z(\mu), \mu \in \mathbb{N}$, is given by (3.2) and the $\sigma$-finiteness of $\Lambda$ will be checked below. We do this by applying the results of the previous section in the case $\rho := \lambda_d(W \cap \cdot)$. Let

$$p := 1 - e^{-\gamma_i}.$$

By (3.4), we have $p = \mathbb{P}(0 \in Z)$. Moreover, Fubini’s theorem and (4.3) below imply that

$$\mathbb{E}[F] = p \lambda_d(W),$$

so that $p$ is the volume fraction of $Z$.

Theorem 4.2. Assume that (4.2) holds. Then the Poisson functional $F = \lambda_d(Z \cap W)$ satisfies

$$\mathbb{P}(F - \mathbb{E}[F] \geq r) \leq \exp \left( \inf_{s > 0} \left( \frac{p}{\gamma_1} \int \int \phi(s \lambda_d((K + x) \cap W)) dx \mathbb{Q}(dK) - sr \right) \right), \quad r > 0.$$
Proof. We wish to apply Theorem \[3.4\] in the case \( \rho = \lambda_d(W \cap \cdot) \).

Set \( \mathcal{Q}'(dK) := \mathbf{1}\{\lambda_d(K) > 0\} \mathcal{Q}(dK) \) and \( \Lambda'(dK) := \mathbf{1}\{\lambda_d(K) > 0\} \Lambda(dK) \) and let \( \eta' \) be a Poisson process with intensity measure \( \Lambda' \). Then \( \lambda_d(Z(\eta') \cap W) \) has the same distribution as \( \lambda_d(Z(\eta) \cap W) \). Hence we can assume without loss of generality that \( \lambda_d(K) > 0 \) for \( \mathcal{Q} \)-a.e. \( K \). In particular \( \Lambda \) is then \( \sigma \)-finite.

For each Borel set \( K \subset \mathbb{R}^d \) we have that

\[
\int \lambda_d(W \cap (K + x)) \, dx = \iint \mathbf{1}\{y \in K + x\} \mathbf{1}\{y \in W\} \, dy \, dx
\]

By Fubini’s theorem and \( (4.1) \) we obtain that

\[
\int \lambda_d(W \cap K) \, \Lambda(dK) = \lambda_d(W) \int \lambda_d(K) \, \mathcal{Q}(dK),
\]

so that \( (4.2) \) implies assumption \( (3.1) \).

Let \( s > 0 \). Then

\[
\iint \mathbf{1}\{\lambda_d((K + x) \cap W) > 1\} e^{\lambda_d((K + x) \cap W)} \, dx \, \mathcal{Q}(dK) \\
\leq e^{\lambda_d(W)} \iint \lambda_d((K + x) \cap W) \, dx \, \mathcal{Q}(dK) \\
= e^{\lambda_d(W)} \lambda_d(W) \int \lambda_d(K) \, \mathcal{Q}(dK) < \infty.
\]

Therefore we have \( s_0 = \infty \), where \( s_0 \) is given by \( (3.6) \).

As at \( (3.3) \) we define \( \mathcal{F}^d_x := \{ K \in \mathcal{F}^d : x \in K \} \) for \( x \in \mathbb{R}^d \). From \( (4.1) \) we obtain that

\[
\Lambda(\mathcal{F}^d_x) = \Lambda(\mathcal{F}^d_0) = \int \lambda_d(K) \, \mathcal{Q}(dK) = \gamma_1.
\]

Hence \( \tau(\Lambda(\mathcal{F}^d_x)) = p/\gamma_1 \). Therefore the measure \( \nu^* \) defined by \( (3.5) \) is given by

\[
\nu^* = \frac{p}{\gamma_1} \int \mathbf{1}\{\lambda_d(K \cap W) \in \cdot\} \, \Lambda(dK).
\]

Hence Theorem \[3.4\] implies the assertion. \( \square \)

The right-hand side of the concentration inequality provided by Theorem \[4.2\] is of a rather complicated form. In the sequel we shall derive more explicit versions. We use the function \( \psi \) defined by \( (3.13) \) and the constant

\[
\gamma_2 := \int \lambda_d(K)^2 \, \mathcal{Q}(dK).
\]

Corollary 4.3. Assume that \( (4.2) \) holds and that \( a > 0 \) is such that \( \lambda_d(K \cap W) \leq a \) for \( \Lambda \)-a.e. \( K \in \mathcal{F}^d \). Then \( F = \lambda_d(Z \cap W) \) satisfies

\[
\mathbb{P}(F - \mathbb{E}[F] \geq r) \leq \exp \left[ \frac{r}{a} \psi \left( \frac{r}{p \lambda_d(W)} \right) \right], \quad r > 0,
\]

\[
\mathbb{P}(F - \mathbb{E}[F] \geq r) \leq \exp \left[ \frac{r}{a} \psi \left( \frac{d \gamma_1 r}{p \lambda_d(W) \gamma_2} \right) \right], \quad r > 0.
\]
Proof. We can apply Corollary 3.8. Using (4.4) and (4.1) we obtain that
\[ m_1 = \frac{P}{\gamma_1} \int \lambda_d((K + x) \cap W) \, dx \, \mathbb{Q}(dK) = p \lambda_d(W). \]
Inequality (4.5) now follows from the case \( i = 1 \) of Corollary 3.8. Similarly we obtain that
\[ m_2 = \frac{P}{\gamma_1} \int \lambda_d((K + x) \cap W)^2 \, dx \, \mathbb{Q}(dK) \]
\[ \leq \frac{P}{\gamma_1} \int \lambda_d(K) \int \lambda_d((K + x) \cap W) \, dx \, \mathbb{Q}(dK) = \frac{P}{\gamma_1} \lambda_d(W) \int \lambda_d(K)^2 \, \mathbb{Q}(dK). \]
Since \( \psi \) is decreasing, the inequality (4.6) follows from the case \( i = 2 \) of Corollary 3.8. □

Remark 4.4. Suppose there exists \( a > 0 \) such that \( \lambda_d(K) \leq a \) for \( \mathbb{Q} \)-a.e. \( K \in \mathcal{F}^d \). Then (4.6) is superior to (4.5). If there exist \( \gamma > 0 \) and \( K_0 \in \mathcal{F}^d \) with \( \lambda_d(K_0) > 0 \) such that \( \mathbb{Q} = \gamma \delta_{K_0} \), then both inequalities yield
\[ \mathbb{P}(F - \mathbb{E}[F] \geq r) \leq \exp \left[ \frac{r}{\lambda_d(K_0)} \psi \left( \frac{r}{p \lambda_d(W)} \right) \right], \quad r > 0. \] (4.7)

Remark 4.5. The estimate (4.7) is quite sharp. This can be seen by a comparison with the concentration of a Poisson distributed random variable \( X \). Lemma 1.2 in [16] provides the bound
\[ \mathbb{P}(X - \mathbb{E}[X] \geq r) \leq \exp \left( r \cdot \psi(r/\mathbb{E}[X]) \right), \quad r > 0. \] Furthermore, only a slight modification of the last bound leads to the exact asymptotic \( \mathbb{P}(X - \mathbb{E}[X] \geq r) \sim (2\pi(\mathbb{E}[X] + r))^{-1/2} \exp \left( r \cdot \psi(r/\mathbb{E}[X]) \right) \), as \( r \to \infty \), see page 1225 in [8].

Remark 4.6. Choosing \( a = \lambda_d(W) \) in (4.5) yields
\[ \mathbb{P}(F - \mathbb{E}[F] \geq r) \leq \exp \left[ \frac{r}{\lambda_d(W)} \psi \left( \frac{r}{p \lambda_d(W)} \right) \right], \quad r > 0. \] (4.8)
The advantage of this result is that it holds under the only assumption (4.2). The disadvantage is the occurrence of \( \lambda_d(W)^{-1} \) as a factor of \( r \) outside the logarithmic term. This is in contrast with the situation in Remark 4.4.

If \( \lambda_d(\cdot) \) is not essentially bounded w.r.t. \( \mathbb{Q} \), we need an exponential moment assumption on \( \mathbb{Q} \) to improve (4.8) at least partially. Define a function \( \bar{h} : [0, \infty) \to [0, \infty] \) by
\[ \bar{h}(s) := \int \lambda_d(K)(e^{s \lambda_d(K)} - 1) \, \mathbb{Q}(dK), \quad s \geq 0. \] (4.9)

Corollary 4.7. Assume that (4.2) holds. Then the Poisson functional \( F = \lambda_d(Z \cap W) \) satisfies
\[ \mathbb{P}(F - \mathbb{E}[F] \geq r) \leq \exp \left( - \int_0^r \bar{h}^{-1} \left( \frac{\gamma_1 u}{p \lambda_d(W)} \right) du \right), \quad r > 0. \]

Proof. We wish to apply Theorem 3.6. Recall the definition (3.12) of the function \( h \). By (4.4) we have for each \( s \geq 0 \) that
\[ h(s) = \frac{P}{\gamma_1} \int \lambda_d((K + x) \cap W)(e^{s \lambda_d((K + x) \cap W)} - 1) \, dx \, \mathbb{Q}(dK) \]

14
\[ \frac{p}{\gamma_1} \int \lambda_d((K + x) \cap W)(e^{s\lambda_d(K)} - 1) \, dx \leq Q(dK) = \frac{p \lambda_d(W)}{\gamma_1} \tilde{h}(s). \]

Hence we have for each \( r > 0 \) that \( h^{-1}(r) \geq \tilde{h}^{-1}(\gamma_1 r/(p \lambda_d(W))) \), so that Theorem 3.6 and the identity \( s_0 = \infty \) imply the assertion once we have shown that \( \lim_{s \to s_0} \tilde{h}(s) = \infty \). But this follows from \( \nu'((0, \infty)) > 0 \) (a consequence of \( \gamma_1 > 0 \)).

We illustrate Corollary 4.7 with two examples. Let \( c := \gamma_1/(p \lambda_d(W)) \).

**Example 4.8.** Assume that \( Q((K : \lambda_d(K) \in du)) \leq \alpha u^{-1} e^{-\beta u} \, du \), where \( \alpha, \beta > 0 \). On the right-hand side we have here the Lévy measure of the gamma distribution with shape parameter \( \alpha > 0 \) and rate parameter \( \beta > 0 \); see e.g. [13, Example 15.6]. For instance, this assumption is satisfied with \( \alpha = 1/d \) if \( Q \leq \int 1_{r K_0} \cdot r^{-1} e^{-\beta r} \, dr \), where \( K_0 \in F^d \) has \( \lambda_d(K_0) = 1 \). Let \( s \in (0, \beta) \). A simple calculation shows that \( \tilde{h}(s) \leq \alpha/(\beta - s) - \alpha/\beta \), so that

\[ \tilde{h}^{-1}(r) \geq \beta - \frac{\alpha \beta}{\beta r + \alpha}, \quad r > 0. \]

It follows that

\[ \int_0^r \tilde{h}^{-1}(u) \, du \geq \beta r - \alpha \log \left(1 + \frac{\beta r}{\alpha}\right), \quad r > 0. \]

Therefore we obtain from Corollary 4.7

\[ \mathbb{P}(F - \mathbb{E}[F] \geq r) \leq \exp \left(-\beta r + \frac{\alpha}{c} \log \left(1 + \frac{\beta c r}{\alpha}\right)\right), \quad r > 0. \]

**Example 4.9.** Assume that \( Q((K : \lambda_d(K) \in du)) \leq \beta^\alpha / \Gamma(\alpha) u^{\alpha-1} e^{-\beta u} \, du \), where \( \alpha, \beta > 0 \). On the right-hand side we have here the gamma distribution with shape parameter \( \alpha \) and scale parameter \( \beta \). A similar calculation as in Example 4.8 yields \( \tilde{h}(s) \leq \alpha (\beta - s)^{\alpha-1} - \alpha/\beta \) for \( s \in (0, \beta) \), so that

\[ \tilde{h}^{-1}(r) \geq \beta - \left(\frac{\alpha \beta^{\alpha+1}}{\beta r + \alpha}\right)^{(\alpha+1)}, \quad r > 0, \]

and finally, by Corollary 4.7

\[ \mathbb{P}(F - \mathbb{E}[F] \geq r) \leq \exp \left(-\beta r + \frac{\alpha + 1}{c} \left(\frac{\alpha + \beta c r}{\alpha}\right)^{\alpha/(\alpha+1)} - 1\right). \]

**Remark 4.10.** Examples 4.8 and 4.9 are geometrically quite different. Assume in the second example that \( Q((K : \lambda_d(K) \in du)) = \beta^\alpha / \Gamma(\alpha) u^{\alpha-1} e^{-\beta u} \, du \). Then each bounded set contains a finite number of grain centers (at least under a weak regularity assumption on \( Q \)). Ignoring overlapping, each grain contributes a gamma distributed volume. Assume in the first example that \( Q((K : \lambda_d(K) \in du)) = u^{-1} e^{-\beta u} \, du \). Then each measurable set \( B \subset \mathbb{R}^d \) with \( 0 < \lambda_d(B) < \infty \) contains infinitely many grain centers. However, the sum of the volumes of balls centered in \( B \) follows a gamma distribution with scale parameter \( \beta \) and shape parameter \( \lambda_d(B) \); see [13, Example 15.6]. Roughly speaking, \( Z \cap W \) might be interpreted as a finite union of random sets whose volumes are approximately gamma distributed. This might explain that the leading terms in both concentration inequalities are the same.
Our bounds of Corollary 4.3 improve significantly Theorem 3 in [7] which deals with the stationary Boolean model in \( \mathbb{R}^d \) and which assumes \( \mathbb{Q} \) to be a probability measure. The tail bound in [7] is only of order \( \exp(-O(r)) \) and therefore not able to reproduce the tail behaviour of the Poisson distribution in the special setting of Remark 4.4. Further, the constants we use arise naturally from the model and are much less involved than the ones in [7]. Moreover, we do not require that the moment-generating function of \( \lambda_d(Z_0) \) exists but only make the milder moment assumptions \( \gamma_1 < \infty \), respectively \( \gamma_2 < \infty \).

We note that the general concentration inequalities derived in [1] can be applied to some configurations of the stationary Boolean model in \( \mathbb{R}^d \), too. At least in the case of bounded grains, this application already improves the correspondent result of [7]. However, the functionals considered in [1] appear unable to incorporate the volume fraction. To be more precise, in the setting of Corollary 4.3, the bound (4.6) is superior to the result

\[
P(F - \mathbb{E}[F] \geq r) \leq \exp \left( \frac{r}{\gamma_1} \cdot \psi \left( \frac{ar}{\gamma_2 \lambda_d(W)} \right) \right), \quad r \geq 0,
\]

obtained from Corollary 3.3 in [1] by the bound

\[
\int (D_K F)^2 \Lambda(dK) \leq \int \int ((K + x) \cap W)^2 dx \mathbb{Q}(dK) \leq \lambda_d(W) \gamma_2.
\]

Finally, we apply Theorem 2.5.

**Proposition 4.11.** Assume that (4.2) holds. Then

\[
P(F - \mathbb{E}[F] \geq r) \leq \exp \left( -\frac{\gamma_1}{p \lambda_d(W)} \left( r + \mathbb{E}[F] \log \left( \frac{\mathbb{E}[F]}{r + \mathbb{E}[F]} \right) \right) \right), \quad r > 0.
\]  

(4.11)

**Proof.** By Lemma 3.3 and equation (4.3), we have, for \( \Lambda \)-a.e. \( K \in \mathcal{F}^d \),

\[
\int_0^1 \int_N D_K f(\eta_t + \mu) \Pi(1-\eta) \Lambda(d\mu) dt \leq \rho(K \cap W) \frac{p}{\gamma_1} \leq \lambda_d(W) \frac{p}{\gamma_1}.
\]

Using the properness of \( \eta \), we also get the bound

\[
\int D_K f(\eta - \delta_K) \eta(dK) = \sum_{K \in \eta} D_K f(\eta - \delta_K) = \sum_{K \in \eta} \lambda_d((K \cap W) \setminus \bigcup_{L \in \eta - \delta_K} L) \leq F.
\]

The assertion now follows from Theorem 2.5 using the same truncation method as in the proof of Theorem 3.4. \( \square \)

We note that \( \int D_K f(\eta - \delta_K) \eta(dK) \) actually equals the volume of the set of points which are covered by exactly one grain. Thus, as the Mecke formula allows us to employ the functional \( \int D_K f(\eta - \delta_K) \eta(dK) \), we are equipped with a finer tool to respect the interplay between the grains of \( Z \).

**Example 4.12.** Let \( \mathbb{Q}((K : \lambda_d(K) \in du)) = \beta e^{-\beta u} du \), that is the volume of the typical grain is exponentially distributed. The larger \( \beta \) (and therefore the smaller \( p \)) the better is the specific bound (4.10) in comparison to the general bound (4.11). If \( \beta > 0.14 \), i.e. \( p < 0.9992 \), then (4.10) outplays (4.11) uniformly. If \( \beta < 0.13 \), it is the other way round. Between, (4.11) might be better only for small values of \( r \). Comparing the more general bound (4.8) with (4.11), we see the same principle. The latter wins when \( p \) is large.
Acknowledgment: We wish to thank S. Bachmann and G. Peccati for discussing with us an early version of their paper [1]. This work was supported by the German Research Foundation (DFG) through the research unit “Geometry and Physics of Spatial Random System” under the grant LA 965/6-2.

References

[1] Bachmann, S. and Peccati, G. (2016). Concentration bounds for geometric Poisson functionals: Logarithmic Sobolev inequalities revisited. *Electron. J. Probab.* **21**, 1-44.

[2] Boucheron, S., Lugosi, G. and Massart, P. (2003). Concentration inequalities using the entropy method. *Ann. Probab.* **31**, 1583-1614.

[3] Boucheron, S., Lugosi, G. and Massart, P. (2013). *Concentration Inequalities: A Nonasymptotic Theory of Independence*. Oxford University Press.

[4] Chernoff, H. (1952). A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. *Ann. Math. Statist.* **23**, 493-507.

[5] Chiu, S.N., Stoyan, D., Kendall, W.S. and Mecke, J. (2013). *Stochastic Geometry and its Applications*. Third Edition, Wiley, Chichester.

[6] Eichelsbacher, P., Raič, M. and Schreiber, T. (2015). Moderate deviations for stabilizing functionals in geometric probability. *Ann. Inst. Henri Poincaré Probab. Stat.* **51**, 89-128.

[7] Heinrich, L. (2005). Large deviations of the empirical volume fraction for stationary Poisson grain models. *Ann. Appl. Probab.* **15**, 392-420.

[8] Houdré, C. (2002). Remarks on deviation inequalities for functions of infinitely divisible random vectors. *Ann. Probab.* **30**, 1223-1237.

[9] Houdré, C. and Privault, N. (2002). Concentration and deviation inequalities in infinite dimensions via covariance representations. *Bernoulli* **8**, 697-720.

[10] Hug, D., Last, G. and Schulte, M. (2016). Second order properties and central limit theorems for geometric functionals of Boolean models. *Ann. Appl. Probab.* **26**, 73-135.

[11] Kallenberg, O. (2002). *Foundations of Modern Probability*. Second Edition, Springer, New York.

[12] Last, G. and Penrose, M.D. (2011). Poisson process Fock space representation, chaos expansion and covariance inequalities. *Probab. Theory Related Fields* **150**, 663-690.

[13] Last, G. and Penrose, M.D. (2017). *Lectures on the Poisson Process*. Cambridge University Press. To appear. [http://www.math.kit.edu/stoch/~last/seite/lectures_on_the_poisson_process/de](http://www.math.kit.edu/stoch/~last/seite/lectures_on_the_poisson_process/de)

[14] Massart, P. (2000). About the constants in Talagrand’s concentration inequalities for empirical processes. *Ann. Probab.* **28**, 863-884.
[15] Matheron, G. (1975). *Random Sets and Integral Geometry*. Wiley, London.

[16] Penrose, M.D. (2003). *Random Geometric Graphs*. Oxford University Press, Oxford.

[17] Privault, N. (2009). *Stochastic Analysis in Discrete and Continuous Settings: With Normal Martingales*. Springer, Heidelberg.

[18] Schneider, R. and Weil, W. (2008). *Stochastic and Integral Geometry*. Springer, Berlin.

[19] Wu, L. (2000). A new modified logarithmic Sobolev inequality for Poisson point processes and several applications. *Probab. Theory Related Fields* 118, 427-438.

[20] Zähle, M. (1982). Random processes of Hausdorff rectifiable closed sets. *Math. Nachr.* 108, 49-72.

[21] Zähle, U. (1984). Random fractals generated by random cutouts. *Math. Nachr.* 116, 27-52.