Thermal $\mathcal{N} = 4$ SYM theory as a 2D Coulomb gas

Sean A. Hartnoll\textsuperscript{1} and S. Prem Kumar\textsuperscript{2}

\textsuperscript{1} KITP, University of California Santa Barbara
CA 93106, USA
hartnoll@kitp.ucsb.edu

\textsuperscript{2} Department of Physics, University of Wales Swansea
Swansea, SA2 8PP, UK
s.p.kumar@swansea.ac.uk

Abstract

We consider $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with $SU(N)$ gauge group at large $N$ and at finite temperature on a spatial $S^3$. We show that, at finite weak ’t Hooft coupling, the theory is naturally described as a two dimensional Coulomb gas of complex eigenvalues of the Polyakov-Maldacena loop, valued on the cylinder. In the low temperature confined phase the eigenvalues condense onto a strip encircling the cylinder, while the high temperature deconfined phase is characterised by an ellipsoidal droplet of eigenvalues.
1 Introduction

1.1 Known results on the phase structure of thermal $\mathcal{N} = 4$ SYM

At present $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory provides the canonical example of an explicit gauge theory/string theory duality at large $N$ [1]. The extension of the correspondence to finite temperature [2] suggests that the duality may allow the phase structure of the theory to be fully mapped out as a function of temperature and coupling. From a gravitational perspective, the finite temperature theory furnishes a unique window into black hole physics. A key question, for instance, is whether stringy corrections fundamentally alter the nature of black holes in strongly curved spacetimes, or whether weakly coupled plasmas are qualitatively similar to black holes.

The determination of the phase structure of $\mathcal{N} = 4$ SYM theory at finite temperature on a spatial $S^3$ began with Witten’s identification [2, 3] of the first order Hawking-Page transition in the bulk as a strong coupling deconfinement transition. It was later noticed that a similar first order transition [4, 5] occurs at a critical temperature in the free field theory on $S^3$ at large $N$. In both of these transitions, at zero and strong coupling, the free energy of the system jumps from $O(1)$ to $O(N^2)$ at the critical temperature, as the theory deconfines. This suggests that the phase transitions should be identified via a phase boundary running from zero to strong coupling. In fact this boundary turned out to have some structure of its own, as is described in [5].

The deconfinement transition at weak coupling was given an elegant treatment in [5], where the theory was systematically shown to reduce to a unitary matrix model for the Polyakov loop. This is the holonomy of the $SU(N)$ gauge field around the Euclidean time circle of the thermal field theory

$$U = Pe^{i \oint A_0 d\tau}.$$  \hspace{1cm} (1)

In the low temperature confining phase $\langle \text{Tr} U^k \rangle = 0$, while at high temperatures these traces are nonvanishing. Such behaviour of these traces implies that, in the large $N$ limit, the distribution of eigenvalues of the Polyakov loop is uniform in the confining phase and nonuniform at high temperatures.

1.2 The role of scalar fields

This work was motivated by two open questions from previous treatments:

- The order parameter for the weak coupling deconfinement transition is the eigenvalue
distribution of the Polyakov loop. However, the order parameter which can be computed at strong coupling, and which is natural from the viewpoint of the dual string theory, is the Polyakov-Maldacena loop [6]. To compare the phases at strong and weak coupling, we would like to use the same order parameter.

- At zero 't Hooft coupling all fields except the zero mode of $A_0$ (or equivalently, the Polyakov loop), are massive and can be integrated out. This is possible because all the scalars have positive conformal mass term. However, we shall see that quantum effects at nonzero 't Hooft coupling $\lambda$ cause condensation of the scalar fields as well as $A_0$ [7].

These two questions are related. Recall that the Polyakov-Maldacena loop [6] is

$$W = P e^{\int [A_0 + i \Phi_J \theta^J(\tau)] d\tau},$$

where $\theta^J$ is a unit vector in $\mathbb{R}^6$. We can see that if $A_0$ and $\Phi_J$ have commuting nonvanishing vacuum expectation values, then the eigenvalue distribution of $iA_0 + \Phi_J$ is well defined and will be directly related to the computation of $\langle \text{tr} W^k \rangle$.

In this letter we study the eigenvalue distribution of $\Phi_J + iA_0$ at finite weak coupling as a function of temperature. This is a distribution on the complex cylinder $\mathbb{R} \times S^1$. The resulting two dimensional framework is the necessary generalisation to nonzero coupling of studies of the free theory, such as [5], which were phrased in terms of the eigenvalue distribution of $A_0$ on $S^1$.

In the limits of low and high temperature the eigenvalue distribution is given precisely by a 2D Coulomb gas in an external potential. At intermediate temperatures, the interaction potential has deviations from the Coulomb law. We compute the distributions analytically in these limits: the results are constant density droplets that are shown in figures 1 and 2 below.

2 The two dimensional Coulomb gas

2.1 One loop effective potential

To understand the phase structure of the theory at finite, weak 't Hooft coupling we need the quantum effective potential generated at finite temperature. Such an effective potential can be a function of all homogeneous modes in the theory on $S^3 \times S^1$. In particular it will depend on the zero modes of the six scalar fields of the $\mathcal{N} = 4$ theory and the Polyakov loop
(or $A_0$). Fortunately, this potential has very recently been computed at one loop order [7]. Several consistency checks were also performed in that paper. There are three contributions: the classical term and then the bosonic and fermionic fluctuation determinants obtained by integrating out the respective Kaluza-Klein harmonics on $S^3 \times S^1$. The potential turns out to be a gauge invariant function depending only on the eigenvalues $\{\phi_{Jp}\}$ of the adjoint scalars and those of the time component of the gauge field $\{\theta_p\}$

$$S_{\text{eff}}[\varphi_p, \theta_p] = S^{(0)} + S^{(1)}_b + S^{(1)}_f.$$  \hspace{1cm} (3)

The classical term is

$$S^{(0)} = \beta R\pi^2 N \frac{\lambda}{\lambda} \sum_{p=0}^{N-1} \varphi_{p}^2,$$  \hspace{1cm} (4)

which is just the scalar mass term due to conformal coupling to the curvature of the $S^3$ of radius $R$. Furthermore, $\lambda = g^2_{YM} N \ll 1$ is the 't Hooft coupling and $T = \frac{1}{\beta}$, the temperature. The reason it is possible to have simultaneously diagonal scalar and $A_0$ background values can be traced to the $\mathcal{N} = 4$ classical potential and the fact that it allows a Coulomb branch moduli space on $\mathbb{R}^{3,1}$. In the presence of diagonal background fields, the off-diagonal fluctuations are massive and appear quadratically in the action. This allows them to be consistently integrated out with vanishing expectation values.

Integrating out the gauge fields, scalar and ghost fluctuations yields the bosonic contribution to the one loop potential

$$S^{(1)}_b = \sum_{p,q=0}^{N-1} \left( \beta C_b(\varphi_{pq}) - \frac{1}{2} \log \left[ \cosh \beta \sqrt{\varphi_{pq}^2 - \cos \theta_{pq}} \right] - \frac{1}{2} \log 2 \right)$$

$$+ \sum_{\ell=0}^{\infty} (2\ell + 3)(2\ell + 1) \log \left[ 1 - e^{-\beta \sqrt{(\ell+1)^2 R^{-2} + \varphi_{pq}^2 + i\theta_{pq}}} \right] + \text{c.c.},$$  \hspace{1cm} (5)

and

$$S^{(1)}_f = \sum_{p,q=0}^{N-1} \left( \beta C_f(\varphi_{pq}) - \sum_{\ell=1}^{\infty} 4\ell(\ell + 1) \log \left[ 1 + e^{-\beta \sqrt{(\ell+1)^2 R^{-2} + \varphi_{pq}^2 + i\theta_{pq}}} \right] + \text{c.c.} \right).$$  \hspace{1cm} (6)

In these expressions $\theta_{pq} = \theta_p - \theta_q$ and

$$\varphi_{p}^2 = \sum_{J=1}^{6} \phi_{Jp}^2, \quad \varphi_{pq}^2 = \sum_{J=1}^{6} (\phi_{Jp} - \phi_{Jq})^2.$$  \hspace{1cm} (7)

The functions $C_b$ and $C_f$ are the bosonic and fermionic Casimir energy contributions in the presence of background expectation values for the adjoint scalar fields [7]. Their sum
yields the zero temperature one loop effective potential of $\mathcal{N} = 4$ theory on $S^3$. For present purposes it will suffice to know that this potential has the expansion
\[
C_b(\varphi_{pq}) + C_t(\varphi_{pq}) = \frac{1}{R} \left[ \frac{3}{16} + (R\varphi_{pq})^2 (\log 2 - \frac{1}{4}) + \mathcal{O}(R^4 \varphi_{pq}^4) + \cdots \right]
\]
(8)
near the origin.

The problem now is to find the eigenvalue distribution that minimises the effective potential (3).

2.2 Instability of the ring configuration and short distance repulsion

Previous works have considered condensation of the gauge field eigenvalues $\theta_p$ while assuming that the scalar eigenvalues remained zero: $\phi_{Jp} = 0$. We will instead look for general distributions in which both sets of eigenvalues can condense. Let us introduce the complex field
\[
z_p = \beta \phi_p + i \theta_p.
\]
(9)
The $z_p$ live on a cylinder: $\Re z_p$ ranges from $-\infty$ to $+\infty$ but $\Im z_p$ is periodic with range $\pi$.

In our definition of the complex variables $z_p$ and in what follows we will omit $SO(6)$ vector indices on $\phi_{Jp}$ and think of it as the radial mode in $SO(6)$ space. A non-zero radial mode implies symmetry breaking. This is possible on $S^3 \times S^1$ due to the large $N$ limit which plays the role of the thermodynamic limit [7]. Hence we set $\varphi_p = \phi_p$ and $\varphi_{pq} = \phi_p - \phi_q$.

The first statement we can make is that although $\phi_p = 0$ is always a solution, call it the ring configuration, it is never a stable configuration at finite 't Hooft coupling. This is due to a strong repulsive force between the eigenvalues at short distances, originating from the first line of (5). The term may be rewritten as
\[
\frac{1}{2} \left( \log 2 + \log \left[ \cosh \beta \sqrt{\varphi_{pq}^2 - \cos \theta_{pq}} \right] \right) = \log 2 + \log |\sinh z_{pq}|.
\]
(10)
This term reduces to the familiar $\log \sin \frac{\theta_{pq}}{2}$ when $\phi_p = 0$. As $z_p \to z_q$ we see that the eigenvalues experience a 2D Coulombic repulsion with potential $-\log |z_p - z_q|$. At large $N$ there is no other term in the action (3) that can compete with this term at short distances and therefore the eigenvalues always spread out into two dimensions. Let us see this explicitly.
2.3 Low temperature limit

Consider first the low temperature limit of the confining phase. That is, take \( \frac{R}{\beta} \ll 1 \). The action (3) at low temperatures becomes

\[
S_{\beta < 1} = N^2 \left[ \frac{3\beta}{16R} - \log 2 \right] + \frac{N\pi^2 R}{\beta \lambda} \sum_{p=0}^{N-1} (z_p + \bar{z}_p)^2 - \sum_{p,q=0}^{N-1} \log |\sinh z_{pq}| .
\]  

(11)

As well as the terms in the action which are exponentially suppressed at low temperatures, we have also dropped all the Casimir energy terms from (5) and (6). This will be valid if \( R\phi_{J_p} \ll 1 \). We will see a posteriori that this is true for the solution we are about to present.

The action in (11) is an elegant reformulation of the problem as a Coulomb gas in two dimensions with an external potential. This system is closely related to normal matrix models. The interaction potential \( \log |\sinh z_{pq}| \) is precisely the Coulomb potential on a cylinder.

In writing down the equations of motion following from (11) it is convenient to pass to the continuum large \( N \) limit. This is done by introducing an eigenvalue distribution

\[
\sum_{q=0}^{N-1} \rightarrow N \int d^2z \rho(z,\bar{z}) .
\]  

(12)

The continuum limit implies the normalisation condition\(^1\)

\[
\int d^2z \rho(z,\bar{z}) = 1 ,
\]  

(13)

while the equations of motion become

\[
z + \bar{z} = \frac{\beta \lambda}{2\pi^2 R} \int d^2z' \rho(z',\bar{z}') \coth(z - z') .
\]  

(14)

As is standard for these types of system, we can immediately derive from (14) that the eigenvalue distribution is everywhere either a fixed constant, or it is zero. This follows from acting on (14) with \( \partial \), and gives

\[
\rho(z,\bar{z}) = \begin{cases} 
\frac{\pi R}{\beta \lambda} & \text{if} \quad z \in B \\
0 & \text{if} \quad z \notin B 
\end{cases}. 
\]  

(15)

Here \( B \) is a subset of the complex cylinder. In other words, the eigenvalue distribution takes the form of a uniform density ‘droplet’ on the cylinder. The equations of motion (14) therefore imply that we need to find a domain \( B \) such that

\[
z + \bar{z} = \frac{1}{2\pi} \int_B d^2z' \coth(z - z') .
\]  

(16)

\(^1\)We work with the convention that \( d^2z = 2dxdy \), where \( z = x + iy \).
From symmetry considerations it is not difficult to see what form the droplet has to take. The external force acting on the eigenvalues acts parallel to the real axis and pushes the eigenvalues towards the imaginary axis. The repulsive force between the eigenvalues must balance this effect to lead to a finite width band. This is illustrated in figure 1 below. We will show momentarily that a band of constant width $2A$ does indeed solve (16), for all $A$. Given this fact, the normalisation condition (13) determines the width $A$ of the band to be

$$A = \frac{\beta \lambda}{4\pi^2 R}. \quad (17)$$

**Figure 1:** The low temperature distribution of eigenvalues in a band of width $2A = \frac{\beta \lambda}{2\pi^2 R}$ on the $z$ cylinder.

The equation of motion (16) is checked by doing the double integral. Set $z' = x + iy$ so that the integral becomes

$$\frac{1}{2\pi} \int_B d^2 z' \coth(z - z') = \frac{1}{\pi} \int_{-A}^{A} dx \int_{0}^{\pi} dy \coth(z - x - iy)$$

$$= \int_{-A}^{(z + \bar{z})/2} dx - \int_{(z + \bar{z})/2}^{A} dx$$

$$= z + \bar{z}. \quad (18)$$

In going from the first to the second line here, one needs to be careful to keep track of the branch cut in log sinh.

Given the band solution, we can justify the low temperature approximations made to obtain the action (11). We see that $R\phi \sim \lambda \ll 1$. It follows that evaluated on the solution, the Casimir energy terms are suppressed by a power of the 't Hooft coupling relative to the terms kept in (11). It furthermore follows that because the spread $R\Delta \phi \sim \lambda$ makes no reference to the temperature $\beta$, the condensate has a finite zero temperature limit.
Consistent with confinement, and with strong coupling results, the band solution yields 
\( \langle \text{Tr} W^k \rangle = \langle \text{Tr} U^k \rangle = 0 \) due to cylindrical symmetry of the distribution which causes all 
phases to average out to zero.

Having established that the finite band solves the equations of motion, we can check 
that indeed the configuration has lower action than the ring \( z + \bar{z} = 0 \), which we argued 
was unstable. Evaluating the action (11) on the solution in the continuum limit gives

\[
S_{\text{band}} = \frac{N^2 \beta}{R} \left[ \frac{3}{16} - \frac{\lambda}{12\pi^2} \right].
\]  

(19)

The ring solution at low temperatures is uniformly distributed along the ring, \( \theta_p = 2\pi p/N \),
which gives the action

\[
S_{\text{ring}} = \frac{N^2 \beta}{R} \frac{3}{16} > S_{\text{band}}.
\]  

(20)

2.4 Deconfinement transition and high temperature limit

If we raise the temperature and consider the effects of the exponentially suppressed terms 
in (5) and (6), then we find that the constant density band solution we have just described 
remains a solution. This occurs because the action may be expanded in terms of \( \cos(n\theta_{pq}) \) 
factors, which give zero when summed over the uniform distribution on the \( \theta \) circle.

The behaviour of these sums of exponentials is identical to that described in [5] for the 
Polyakov loop matrix model. Let us recall from that work that when \( \beta/R \) becomes of order 
one, then although the uniform distribution in the \( \theta \) direction remains a solution, it is no 
longer the absolute minimum. This is the deconfinement temperature. The presence of 
\( R\varphi_{pq} \) in the exponent, the new ingredient in our computations, will not change this story 
qualitatively. It will however play a role in determining the order of the transition, c.f. [5].

Above the deconfinement transition, the force between eigenvalues in the circle direction 
becomes attractive and the eigenvalue distribution develops a gap along the \( \theta \) circle. We 
postpone a careful discussion of the behaviour of eigenvalue droplets near the transition for 
future work [9]. The story is complicated by likely non uniformities in the \( \phi \) direction.

A regime that is again analytically tractable is the high temperature limit, \( \frac{\beta}{R} \gg 1 \). From 
previous work [5], our intuition is that at high temperatures the eigenvalues will clump into 
a small droplet. Let us take, simultaneously to the high temperature limit, the limit in 
which \( \theta_p, \beta \phi_q \ll 1 \) in the action (3). Similarly to before, we will verify a posteriori that this 
condition is satisfied by the solution. It is convenient to again introduce the complex \( z_p \) as
we did in (9). The action in these limits becomes

\[
S_{TR \gg 1} = \frac{N \pi^2 R}{\beta \lambda} \sum_{p=0}^{N-1} (z_p + \bar{z}_p)^2 - \sum_{p,q=0}^{N-1} \log |z_{pq}| + \frac{\pi^2 R^3}{\beta^2} \sum_{p,q=0}^{N-1} \left[ -z_{pq}^2 - \bar{z}_{pq}^2 + 6|z_{pq}|^2 \right].
\] (21)

Although the first two terms in this expression are naively subleading, they become important depending on the ’t Hooft coupling and proximity of the eigenvalues. We now take the continuum limit as we did at low temperatures (12). The same arguments as before imply that the eigenvalues form a droplet \( B \) of constant density

\[
\rho(z, \bar{z}) = \frac{\pi R}{\beta \lambda} \left[ 1 + \frac{6R^2 \lambda}{\beta^2} \right] \text{ if } z \in B.
\] (22)

The equation of motion becomes

\[
\tanh \mu z + \bar{z} = \frac{1}{2\pi} \int_B d^2z' \frac{1}{z - z'},
\] (23)

where we defined

\[
\tanh \mu = \frac{1 - \frac{2R^2 \lambda}{\beta^2}}{1 + \frac{6R^2 \lambda}{\beta^2}}.
\] (24)

The equation (23) has been simplified by the observation that if the symmetry \( z \rightarrow -z \) is unbroken, then we must have \( \sum_q z_q = \sum_q \bar{z}_q = 0 \) on the solution.

Before solving the equation of motion, the normalisation condition (13) tells us that the area of the droplet on the complex plane will be

\[
\text{Area}(B) = \frac{1}{12\pi^2} \frac{\beta^3}{R^3} \frac{1}{1 + \frac{\beta^2}{6R^2 \lambda}}.
\] (25)

This area goes to zero as \( \beta/R \rightarrow 0 \) at high temperatures, and therefore our initial assumption on \( \phi_p \) and \( \theta_p \) was justified.

The droplet shape for the Coulomb gas problem (23) is known to be an ellipse [8]. This is illustrated in figure 2 below. The semi-axes of the ellipse are of length \( re^{-\mu} \) along the real axis and \( re^{\mu} \) along the imaginary axis, where \( \mu \) is given in (24) and \( \pi r^2 \) equals the area (25). The eccentricity is therefore controlled by the ratio \( \lambda R^2 / \beta^2 \). Interestingly, it is precisely this ratio which controls the validity of perturbation theory at finite temperature on \( S^3 \). The one loop effective potential and results we have deduced from it are valid for all temperatures \( T \lesssim \frac{R^{-1}}{\sqrt{\lambda}} \) [7].

3 Discussion

In this letter we have shown that weakly coupled \( \mathcal{N} = 4 \) SYM theory at finite temperature on a spatial \( S^3 \) is described at large \( N \) by an eigenvalue distribution on the complex cylinder.
Remarkably, these eigenvalues are naturally associated to the Polyakov-Maldacena loop, the order parameter known to emerge at strong coupling. At high and low temperatures we determined the distribution by solving an associated 2D Coulomb gas problem. The deconfinement transition of the theory on $S^3$ corresponds to a ‘band-to-droplet’ transition on the cylinder. A key point we emphasised is that the eigenvalues of the scalars as well as $A_0$ condense. Outstanding questions are the implications of scalar condensates for the dual string theory and the relevance of the ‘band-to-droplet’ picture for the emergent Big Black Hole phase at high temperatures.

It is of immediate interest to understand the droplet picture in the vicinity of the phase transition, and also whether the strongly coupled theory may be formulated in the Coulomb gas language. This could lead to a new approach for counting black hole microstates. Some progress studying higher traces of the Polyakov-Maldacena loops at strong coupling was recently made [10]. The Coulomb gas framework we have presented has a description in terms of fermions, suggesting a connection to a different droplet picture of certain BPS states recently developed for the zero temperature theory [11, 12].

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