EXISTENCE AND UNIQUENESS OF GLOBAL KOOPMAN EIGENFUNCTIONS FOR STABLE FIXED POINTS AND PERIODIC ORBITS

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ABSTRACT. We consider $C^1$ dynamical systems having a globally attracting hyperbolic fixed point or periodic orbit and prove existence and uniqueness results for $C^k_{\text{loc}}$ globally defined linearizing semiconjugacies, of which Koopman eigenfunctions are a special case. Our main results both generalize and sharpen Sternberg’s $C^k$ linearization theorem for hyperbolic sinks, and in particular our corollaries include uniqueness statements for Sternberg linearizations and Floquet normal forms. Additional corollaries include existence and uniqueness results for $C^k_{\text{loc}}$ Koopman eigenfunctions, including a complete classification of $C^\infty$ eigenfunctions assuming a $C^\infty$ dynamical system with semisimple and nonresonant linearization. We give an intrinsic definition of “principal Koopman eigenfunctions” which generalizes the definition of Mohr and Mezić for linear systems, and which includes the notions of “isostables” and “isostable coordinates” appearing in work by Ermentrout, Mauroy, Mezić, Moehlis, Wilson, and others. Our main results yield existence and uniqueness theorems for the principal eigenfunctions and isostable coordinates and also show, e.g., that the (a priori non-unique) “pullback algebra” defined in [MM16b] is unique under certain conditions. We also discuss the limit used to define the “faster” isostable coordinates in [WE18, MWMM19] in light of our main results.

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1. Introduction

In this paper, we consider $C^1$ dynamical systems $\Phi: Q \times T \to Q$ having a globally attracting hyperbolic fixed point or periodic orbit. Here $Q$ is a smooth manifold and either $T = \mathbb{Z}$ or $T = \mathbb{R}$; when $T = \mathbb{R}$, a common example is that of $t \mapsto \Phi^t(x_0)$ being the solution to the initial value problem

$$\frac{d}{dt} x(t) = f(x(t)), \quad x(0) = x_0$$

determined by a complete $C^1$ vector field $f$ on $Q$. Our main contributions are existence and uniqueness results regarding globally defined $C^k_{\text{loc}}$ linearizing semiconjugacies $\psi: Q \to C^m$ which, by definition, make

\begin{itemize}
\item $\psi(\Phi^t(x)) = \psi(x)$ for all $x \in Q$ and $t \in T$
\item $\psi$ is smooth near the origin
\end{itemize}
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the diagram

\[
\begin{array}{c}
Q \xrightarrow{\Phi^t} Q \\
\downarrow \psi \quad \downarrow \psi \\
\mathbb{C}^m \xrightarrow{e^{tA}} \mathbb{C}^m
\end{array}
\]

commute for some \( A \in \mathbb{C}^{m \times m} \) and all \( t \in \mathbb{T} \). By \( C^{k,\alpha}_{\text{loc}} \) with \( k \in \mathbb{N}_{\geq 1} \) and \( 0 \leq \alpha \leq 1 \) we mean that \( \psi \in C^k(Q; \mathbb{C}^m) \) and that all \( k \)-th partial derivatives of \( \psi \) are locally \( \alpha \)-Hölder continuous in local coordinates.

Linearizing semiconjugacies are also known as linearizing factors or factor maps in the literature and can be viewed as a further generalization of the generalized Koopman eigenfunctions of [Mez19, KM19]. We note that such semiconjugacies are distinct from those in the diagram

\[
\begin{array}{c}
Q \xrightarrow{\Phi^t} Q \\
\downarrow K \quad \downarrow K \\
\mathbb{C}^m \xrightarrow{e^{tA}} \mathbb{C}^m
\end{array}
\]

obtained from (1) by flipping the vertical arrows. Existence results for semiconjugacies of the type in (2) were obtained by [CFdlL03a, CFdlL03b, CFdlL05] in the context of proving invariant manifold results using the parameterization method.

Our main result for the case of a globally attracting hyperbolic fixed point both generalizes and sharpens Sternberg’s linearization theorem [Ste57, Thms 2,3,4] which provides conditions ensuring the existence of a linearizing local \( C^k \) diffeomorphism defined on a neighborhood of the fixed point; the results of [LM13, EKR18] show that this local diffeomorphism can be extended to a global \( C^k \) diffeomorphism \( \psi: Q \to \mathbb{R}^n \subset \mathbb{C}^n \) making (1) commute. Under Sternberg’s conditions, a corollary of our main result is that this global linearizing diffeomorphism is uniquely determined by its derivative at the fixed point. Additionally, we sharpen Sternberg’s result from \( C^k \) to \( C^{k,\alpha}_{\text{loc}} \) linearizations. For the case of a globally attracting hyperbolic periodic orbit of a flow, our main result also yields a similar existence and uniqueness corollary for Floquet normal forms.

Our main results also imply several existence and uniqueness corollaries relevant to the “applied Koopmanism” literature, which has experienced a surge of interest initiated by [DJ99, MB04, Mez05]—motivated largely by data-driven applications—nearly a century after Koopman’s seminal work [Koo31].\(^1\) Our results yield precise conditions under which various quantities in this literature—including targets of numerical algorithms—exist and are unique, and are especially relevant to work on principal eigenfunctions and isostables for point attractors [MM16b, MMM13] and to work on isostable coordinates for periodic orbit attractors in [WM16, SKN17, WE18, MWMM19]. We give an intrinsic definition of principal eigenfunctions for nonlinear dynamical systems which generalizes the definition for linear systems in [MM16b]. We provide existence and uniqueness results for \( C^{k,\alpha}_{\text{loc}} \) principal eigenfunctions, and we also show that the (a priori non-unique) “pullback algebra” defined in [MM16b] is unique under certain conditions. For the case of periodic orbit attractors, principal eigenfunctions essentially coincide with the notion of isostable coordinates defined in [WE18, MWMM19], except that the definition in these references involves a limit which might not exist except for the “slowest” isostable coordinate. Our techniques shed light on this issue, and our results imply that this limit does in fact always exist for the “slowest” isostable coordinate if the dynamical system is at least smoother than \( C^{1,\alpha}_{\text{loc}} \) with \( \alpha > 0 \). In fact our results imply—assuming that there is a unique and algebraically simple “slowest” real Floquet multiplier—that any corresponding “slowest” \( C^1 \) isostable coordinate is always unique modulo scalar multiplication for a \( C^1 \) dynamical system, without the need for any nonresonance or spectral spread assumptions; furthermore, such a unique isostable coordinate always exists if the dynamics are at least smoother than \( C^{1,\alpha}_{\text{loc}} \) with \( \alpha > 0 \). Similarly, if instead there is a unique and algebraically simple “slowest” complex conjugate pair of Floquet multipliers, then any corresponding “slowest” complex conjugate pair of isostable coordinates are always unique modulo scalar

\(^1\)See, e.g., [BMM12, MM12, MMM13, LM13, MM14, GSZ15, Mez15, WKR15, MM16a, MM16b, BBPK16, Sur16, SB16, AM17a, AM17b, KKB17, Mez19, PBK18, KM18, KPM18, KM19, DG19, BRV19, DTK19, AT19].
multiplication for a $C^1$ dynamical system, and such a unique pair always exists if the dynamics are $C^{1,\alpha}_{\text{loc}}$ with $\alpha > 0$. As a final application of our main results, we give a complete classification of $C^\infty$ eigenfunctions for a $C^\infty$ dynamical system with semisimple (diagonalizable over $\mathbb{C}$) and nonresonant linearization, generalizing known results for analytic dynamics and analytic eigenfunctions [MMM13, Mez19].

The remainder of the paper is organized as follows. In §2 we state Theorems 1 and 2, our two main results, without proof. We also state a proposition on the uniqueness of linearizing factors which does not assume any nonresonance conditions. As applications we derive in §3 several results which are essentially corollaries of this proposition and the two main theorems. §3.1 contains existence and uniqueness theorems for global Sternberg linearizations and Floquet normal forms. In §3.2 we define principal Koopman eigenfunctions and isostable coordinates for nonlinear systems and show how Theorems 1 and 2 yield corresponding existence and uniqueness results. We then discuss the relationship between various notions defined in [MM16b] and our definitions, and we also discuss the convergence of the isostable coordinate limits in [WE18, MWMM19]. §3.3 contains our classification theorem for $C^\infty$ eigenfunctions of $C^\infty$ dynamical systems with a globally attracting hyperbolic fixed point or periodic orbit. Finally, §4 contains the proofs of Theorems 1 and 2.

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2. Main results

Before stating our main results, we give two definitions which are essentially asymmetric versions of some appearing in [Ste57, Sel85]. When discussing eigenvalues and eigenvectors of a linear map or matrix in the remainder of the paper, we are always discussing eigenvalues and eigenvectors of its complexification, although we do not always make this explicit.

Definition 1 ($((X, Y) \ k$-nonresonance). Let $X \in \mathbb{C}^{m \times m}$ and $Y \in \mathbb{C}^{n \times n}$ be matrices with eigenvalues $\mu_1, \ldots, \mu_m$ and $\lambda_1, \ldots, \lambda_n$, respectively, repeated with multiplicities. For any $k \in \mathbb{N}_{\geq 1}$, we say that $(X, Y)$ is $k$-nonresonant if, for any $i \in \{1, \ldots, m\}$ and any $m = (m_1, \ldots, m_n) \in \mathbb{N}_{\geq 0}^n$ satisfying $2 \leq m_1 + \cdots + m_n \leq k$,

$$\mu_i \neq \lambda_1^{m_1} \cdots \lambda_n^{m_n}.$$

(Note this condition vacuously holds if $k = 1$.) We say $(X, Y)$ is $\infty$-nonresonant if $(X, Y)$ is $k$-nonresonant for every $k \in \mathbb{N}_{\geq 1}$.

For the definition below, recall that the spectral radius $\rho(X)$ of a matrix is defined to be the largest modulus (absolute value) of the eigenvalues of (the complexification of) $X$.

Definition 2 ($((X, Y) \ spectral \ spread$). Let $X \in \text{GL}(m, \mathbb{C})$ and $Y \in \text{GL}(n, \mathbb{C})$ be invertible matrices with the spectral radius $\rho(Y)$ satisfying $\rho(Y) < 1$. We define the spectral spread $\nu(X, Y)$ to be

$$\nu(X, Y) := \max_{\mu \in \text{spec}(X)} \ln(\frac{\mu}{\lambda}) \nu_{\lambda \in \text{spec}(Y)} \ln(\frac{\mu}{\lambda})$$

Finally, here we recall the definition of $C^{k,\alpha}_{\text{loc}}$ functions.

Definition 3 ($C^{k,\alpha}_{\text{loc}}$ functions). Let $M, N$ smooth manifolds of dimensions $m$ and $n$, let $\psi \in C^k(M, N)$ a $C^k$ map $\psi: M \to N$ with $k \in \mathbb{N}_{\geq 0}$, and let $0 \leq \alpha \leq 1$. We will say that $\psi \in C^{k,\alpha}_{\text{loc}}(M, N)$ if for every $x \in M$ there exist charts $(U_1, \varphi_1)$ and $(U_2, \varphi_2)$ containing $x$ and $\psi(x)$ such that all $k$-th partial derivatives of $\varphi_2 \circ \psi \circ \varphi_1^{-1}$ are H"older continuous with exponent $\alpha$. If the domain and codomain $M$ and $N$ are clear from context, we will write $C^{k,\alpha}_{\text{loc}}$ instead of $C^{k,\alpha}_{\text{loc}}(M, N)$. 

Theorem 1. Let \( \nu(e^A, D_{x_0}\Phi^1) < k + \alpha \) of Theorem 1. Unwinding Definition 2, it follows that this condition is equivalent to every eigenvalue of \( D_{x_0}\Phi^1 \) (represented by an “\( \times \)” above) belonging to the open disk with radius given by raising the smallest modulus of the eigenvalues of \( e^A \) to the power \( \frac{1}{k+\alpha} \).

**Remark 1.** Using the chain rule and the fact that compositions and products of locally \( \alpha \)-Hölder continuous functions are again locally \( \alpha \)-Hölder, it follows that the property of being \( C^{k,\alpha}_{\text{loc}} \) on a manifold does not depend on the choice of charts in Definition 3.

**Notation.** Given a differentiable map \( F: M \to N \) between smooth manifolds, in the remainder of this paper we use the notation \( D_xF \) for the derivative of \( F \) at the point \( x \in M \). (Recall that the derivative \( D_xF: T_xM \to T_{F(x)}N \) is a linear map between tangent spaces [Lee13], which can be identified with the Jacobian of \( F \) evaluated at \( x \) in local coordinates.) In particular, given a dynamical system \( \Phi: Q \times T \to Q \) and fixed \( t \in T \), we write \( D_{\Phi^t}\Phi: T_xQ \to T_{\Phi^t(x)}Q \) for the derivative of the time-\( t \) map \( \Phi^t: Q \to Q \) at the point \( x \in Q \). Given \( i \geq 1 \), we similarly use the notation \( D^i_{\Phi^t}F \) for the \( i \)-th derivative of \( F \), which can be identified with the linear map \( D^i_{\Phi^t}F: (T_xM)^{\otimes i} \to T_{F(x)}N \) from the \( i \)-th tensor power \((T_xM)^{\otimes i}\) to \( T_xN \) represented in local coordinates by the \((1+i)\)-dimensional array of \( i \)-th partial derivatives of \( F \) evaluated at \( x \).

We now state our main results, Theorems 1 and 2, as well as Proposition 1. Figure 2 illustrates the condition \( \nu(e^A, D_{x_0}\Phi^1) < k + \alpha \) in Theorem 1 below.

**Theorem 1** (Existence and uniqueness of \( C^{k,\alpha}_{\text{loc}} \) global linearizing factors for a point attractor). Let \( \Phi: Q \times T \to Q \) be a \( C^1 \) dynamical system having a globally attracting hyperbolic fixed point \( x_0 \in Q \), where \( Q \) is a smooth manifold and either \( T = \mathbb{Z} \) or \( T = \mathbb{R} \). Let \( m \in \mathbb{N}_{\geq 1} \) and \( e^A \in \text{GL}(m, \mathbb{C}) \) have spectral radius \( \rho(e^A) < 1 \), and let the linear map \( B: T_{x_0}Q \to \mathbb{C}^m \) satisfy

\[
\forall t \in T: \quad BD_{x_0}\Phi^t = e^{tA}B.
\]

**Uniqueness.** Fix \( k \in \mathbb{N}_{\geq 1} \cup \{\infty\} \) and \( 0 \leq \alpha \leq 1 \), assume that \( (e^A, D_{x_0}\Phi^1) \) is \( k \)-nonresonant, and assume that either \( \nu(e^A, D_{x_0}\Phi^1) < k + \alpha \) or \( \nu(e^A, D_{x_0}\Phi^1) \leq k \). Then any \( \psi \in C^{k,\alpha}_{\text{loc}}(Q, \mathbb{C}^m) \) satisfying

\[
\psi \circ \Phi^1 = e^{A}\psi, \quad D_{x_0}\psi = B
\]

is unique, and if \( B: T_{x_0}Q \to \mathbb{R}^m \) and \( A \in \mathbb{R}^{m \times m} \) are real, then \( \psi: Q \to \mathbb{R}^m \subset \mathbb{C}^m \) is real.

**Existence.** If furthermore \( \Phi \in C^{k,\alpha}_{\text{loc}} \) and \( \nu(e^A, D_{x_0}\Phi^1) < k + \alpha \), then such a \( \psi \) exists and additionally satisfies

\[
\forall t \in T: \quad \psi \circ \Phi^t = e^{tA}\psi.
\]

In fact, if \( P \) is any “approximate linearizing factor” satisfying \( D_{x_0}P = B \) and

\[
P \circ \Phi^1 = e^{A}P + R
\]
with $D^i_{x_0}R = 0$ for all integers $0 \leq i < k + \alpha$, then
\begin{equation}
\psi = \lim_{t \to \infty} e^{-tA}P \circ \Phi_t,
\end{equation}
in the topology of $C^{k,\alpha}$-uniform convergence on compact subsets of $Q$.

**Remark 2.** Definitions 1 and 2 are not independent. In particular, if $(X, Y)$ is $(\ell - 1)$-nonresonant and $\nu(X, Y) < \ell$, then it follows that $(X, Y)$ is $\infty$-nonresonant. Hence an equivalent statement of Theorem 1 could be obtained by replacing $k$-nonresonance $\infty$-resonance everywhere (alternatively, for the existence statement only $(k - 1)$-nonresonance need be assumed in the case $\alpha = 0$). We prefer to use the stronger-sounding statement of the theorem above since it makes it apparent that the set of matrix pairs $(e^A, D_{x_0}\Phi)$ satisfying its hypotheses are open in the space of all matrix pairs. Openness for $k < \infty$ is immediate, and openness for $k = \infty$ follows from the uniqueness statement and the fact that $\nu(e^A, D_{x_0}\Phi) < \infty$ is always finite.

**Remark 3.** The statement regarding the above limit actually holds without any nonresonance assumptions if such an approximate linearizing factor $P$ exists; see Lemma 5 in §4.1.2.

**Remark 4.** In the uniqueness portion of the above theorem and also later in this paper, the point of the condition that either
\[ \nu(e^A, D_{x_0}\Phi) < k + \alpha \quad \text{or} \quad \nu(e^A, D_{x_0}\Phi) \leq k \]
is to require that $\nu(e^A, D_{x_0}\Phi) < k + \alpha$ be strictly less than $\alpha$ except when $\alpha = 0$, in which case non-strict inequality is allowed to hold. This is relevant for, e.g., the case that $k = 1$ and $\alpha = 0$. Of course the “or” above is inclusive, i.e., we allow both inequalities to hold in the hypotheses of the above theorem and later results.

**Remark 5** (the $C^\infty$ case). In the case that $k = \infty$, the hypothesis $\nu(e^A, D_{x_0}\Phi) < k + \alpha$ becomes $\nu(e^A, D_{x_0}\Phi) < \infty$ which is automatically satisfied since $\nu(e^A, D_{x_0}\Phi)$ is always finite. Hence for the case $k = \infty$, no assumption is needed on the spectral spread in Theorem 1; we need only assume that $(e^A, D_{x_0}\Phi)$ is $\infty$-nonresonant. Similar remarks hold for all of the following results which include a condition of the form $\nu(\cdot, \cdot) < k + \alpha$.

**Remark 6** (sketch of the proof of the existence portion of Theorem 1). Here we sketch the proof of the existence statement of Theorem 1, which is somewhat more involved than the uniqueness proof. (The existence proof also yields uniqueness, but under stronger assumptions than the uniqueness statement in Theorem 1.) Since global asymptotic stability of $x_0$ implies that $Q$ is diffeomorphic to $\mathbb{R}^n$ [Wil67, Lem 2.1], we may assume that $Q = \mathbb{R}^n$ and $x_0 = 0$. For now we consider the case $k < \infty$. First, the $k$-nonresonance assumption implies that we can uniquely solve (7) order by order (in the sense of Taylor polynomials) for $P$ up to order $k$. Once we obtain a polynomial $P$ of sufficiently high order, we derive a fixed point equation for the high-order remainder term $\varphi$, where $\psi = P + \varphi$ is the desired linearizing factor. Given a sufficiently small, positively invariant, compact neighborhood $B$ containing the fixed point, the proof of Lemma 5 shows that the spectral spread condition $\nu(e^A, D_{x_0}\Phi) < k + \alpha$ implies that the restriction $\varphi|_B$ of the desired high-order term is the fixed point of a map $S : C^{k,\alpha}(B, C^m) \to C^{k,\alpha}(B, C^m)$ which is a contraction with respect to the standard $C^{k,\alpha}$ norm $\| \cdot \|_{k,\alpha}$ making $C^{k,\alpha}(B, C^m)$ a Banach space (note, however, that $\| \cdot \|_{k,\alpha}$ must be induced by an appropriate underlying adapted norm [CFdlL03a, Sec. A.1] on $\mathbb{R}^n$ to ensure that $S$ is a contraction). In fact, $S$ is the affine map defined by
\begin{equation}
S(\varphi|_B) := -P|_B + e^{-A} \left( P|_B + \varphi|_B \right) \circ \Phi^1.
\end{equation}
Hence we can obtain $\varphi|_B$ by the standard contraction mapping theorem, thereby obtaining a function $\psi|_B \in C^{k,\alpha}(B, C^m)$ satisfying $\psi|_B \circ \Phi^1|_B = e^A\psi|_B$. We then extend the domain of $\psi|_B$ using the globalization techniques of [LM13, EKR18] to obtain a globally defined function $\psi \in C^{k,\alpha}_{\text{loc}}(Q, C^m)$ satisfying $\psi \circ \Phi^1 = e^A\psi$.

Using an argument of Sternberg [Ste57, Lem. 4] in combination with the uniqueness statement of Theorem 1, we show that the function $\psi$ satisfies (6), i.e., $\psi$ is actually a linearizing factor for $\Phi^t$ for all $t \in \mathbb{R}$. We extend the result to the case that $k = \infty$ using a bootstrapping argument.
Remark 7 (a numerical consideration). Our proof of the existence portion of Theorem 1, outlined above, was inspired by Sternberg’s proof of his linearization theorem [Ste57, Thms 2, 3, 4] and also has strong similarities with the techniques used to prove the existence of semiconjugacies of the type (2) using the parameterization method [CFdlL03a, CFdlL03b, CFdlL05]. We repeat here an observation of [CFdlL03a, Sec. 3] and [CFdlL05, Rem. 5.5] which is also relevant for numerical computations of linearizing semiconjugacies of the type (1) (such as Koopman eigenfunctions) based on our proof of Theorem 1. Consider $P$ satisfying (7), $B$ and $S$ as in Remark 6, and an initial guess $\psi_0|B = P|B + \varphi_0|B$ for a local linearizing factor. If

$$\text{Lip}(S) \leq \kappa < 1 \quad \|S(\varphi_0|B) - \varphi_0|B\| \leq \delta,$$

then the standard proof of the contraction mapping theorem implies the estimate

(10) \[\|\varphi|_B - \varphi_0|_B\| \leq \delta/(1 - \kappa),\]

where $\varphi|_B$ is such that $\psi|_B = P|_B + \varphi|_B$ is the unique actual local linearizing factor. Thus equation (10) furnishes an upper bound on the distance between the initial guess $\varphi_0|_B$ and the true solution $\varphi|_B$, and can be used for a posteriori estimates in numerical analysis.

Theorem 1 gave conditions ensuring existence and uniqueness of linearizing factors under spectral spread and nonresonance conditions. Before stating Theorem 2, we state a result on the uniqueness of linearizing factors which does not assume any nonresonance conditions. Proposition 1 follows immediately from Lemma 3 (used to prove the uniqueness statement of Theorem 1) and the fact that $f$ is q-dimensional.

Proposition 1. Fix $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$ and $0 \leq \alpha \leq 1$, and let $\Phi: Q \times \mathbb{T} \to Q$ be a $C^{k,\alpha}_\text{loc}$ dynamical system having a globally attracting hyperbolic fixed point $x_0 \in Q$, where $Q$ is a smooth manifold and either $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$. Let $m \in \mathbb{N}_{\geq 1}$ and $e^A \in \text{GL}(m, \mathbb{C})$ have spectral radius $\rho(e^A) < 1$ and satisfy either $\nu(e^A, D_0 F) < k + \alpha$ or $\nu(e^A, D_0 F) \leq k$. Let $\varphi \in C^{k,\alpha}_\text{loc}(Q, \mathbb{C}^m)$ satisfy $D^i_{x_0}\varphi = 0$ for all $0 \leq i \leq k$ and

$$\varphi \circ \Phi^1 = e^A\varphi.$$

Then it follows that $\varphi \equiv 0$. In particular, if $\varphi = \psi_1 - \psi_2$, then

$$\psi_1 = \psi_2.$$

Theorem 2 (Existence and uniqueness of $C^{k,\alpha}_\text{loc}$ global linearizing factors for a limit cycle attractor). Fix $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$ and $0 \leq \alpha \leq 1$, and let $\Phi: Q \times \mathbb{R} \to Q$ be a $C^{k,\alpha}_\text{loc}$ flow having a globally attracting hyperbolic $\tau$-periodic orbit with image $\Gamma \subset Q$, where $Q$ is a smooth manifold. Fix $x_0 \in \Gamma$ and let $E^s_{x_0}$ denote the unique $E^s_{x_0}\Phi^\tau$-invariant subspace complementary to $T_{x_0}\Gamma$. Let $m \in \mathbb{N}_{\geq 1}$ and $e^A \in \text{GL}(m, \mathbb{C})$ have spectral radius $\rho(e^A) < 1$, and let the linear map $B: E^s_{x_0} \to \mathbb{C}^m$ satisfy

(11) \[BD^s_{x_0}\Phi^\tau|_{E^s_{x_0}} = e^{\tau A}B.\]

Uniqueness. Assume that $(e^A, D_{x_0}\Phi^\tau|_{E^s_{x_0}})$ is $k$-nonresonant, and assume that either $\nu(e^A, D_{x_0}\Phi^\tau|_{E^s_{x_0}}) < k + \alpha$ or $\nu(e^A, D_{x_0}\Phi^\tau|_{E^s_{x_0}}) \leq k$. Then any $\psi \in C^{k,\alpha}_\text{loc}(Q, \mathbb{C}^m)$ satisfying

(12) \[\forall t \in \mathbb{R}: \psi \circ \Phi^t = e^{t A}\psi, \quad D_{x_0}\psi|_{E^s_{x_0}} = B\]

is unique, and if $B: E^s_{x_0} \to \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times m}$ are real, then $\psi: Q \to \mathbb{R}^m \subset \mathbb{C}^m$ is real.

Existence. If furthermore $\nu(e^A, D_{x_0}\Phi^\tau|_{E^s_{x_0}}) < k + \alpha$, then such a unique $\psi$ exists.

3. Applications

In this section, we give some applications of Theorems 1 and 2 and Proposition 1. §3.1 contains results on Sternberg linearizations and Floquet normal forms. §3.2 gives applications to principal Koopman eigenfunctions and isostable coordinates. §3.3 contains the classification theorems for $C^\infty$ eigenfunctions of a $C^\infty$ dynamical system.
3.1. Sternberg linearizations and Floquet normal forms. The following result is an improved statement of Sternberg’s linearization theorem for hyperbolic sinks [Ste57, Thms 2,3,4]; it includes uniqueness of the linearizing conjugacy, it sharpens Sternberg’s $C^k$ linearization result to a $C^{k,\alpha}$ linearization result, and the linearization is globally defined on all of $Q$ rather than on some neighborhood of $x_0$. Our technique for globalizing the domain of the linearization is essentially the same as those used in [LM13, EKR18].

**Proposition 2** (Existence and uniqueness of global $C^{k,\alpha}$ Sternberg linearizations). Fix $k \in \mathbb{N}_{\geq 1} \cup \{ \infty \}$ and $0 \leq \alpha \leq 1$. Let $\Phi: Q \times \mathbb{T} \to Q$ be a $C^{k,\alpha}_{\text{loc}}$ dynamical system having a globally attracting hyperbolic fixed point $x_0 \in Q$, where $Q$ is a smooth manifold and either $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$. Assume that $\nu(D_{x_0} \Phi^t, D_{x_0} \Phi^1) < k + \alpha$, and assume that $(D_{x_0} \Phi^t, D_{x_0} \Phi^1)$ is $k$-nonresonant.

Then there exists a unique diffeomorphism $\psi \in C^{k,\alpha}_{\text{loc}}(Q, T_{x_0}Q)$ satisfying

$$\forall t \in \mathbb{T}: \psi \circ \Phi^t = D_{x_0} \Phi^t \psi, \quad D_{x_0} \psi = \text{id}_{T_{x_0}Q}. \quad (13)$$

(In writing $D_{x_0} \psi = \text{id}_{T_{x_0}Q}$, we are making the standard and canonical identification $T_{x_0}Q \cong T_{x_0}Q$.)

**Proof.** Identifying $T_{x_0}Q$ with $\mathbb{R}^n$ by choosing a basis, we apply Theorem 1 with $e^{tA} = D_{x_0} \Phi^t$ and $B = \text{id}_{T_{x_0}Q}$ to obtain a unique $\psi \in C^{k,\alpha}_{\text{loc}}(Q, T_{x_0}Q)$ satisfying (13) and $D_{x_0} \psi = \text{id}_{T_{x_0}Q}$. It remains only to show that $\psi$ is a diffeomorphism. To do this, we separately show that $\psi$ is injective, surjective, and a local diffeomorphism.

By continuity, $D_{x_0} \psi = \text{id}_{T_{x_0}Q}$ implies that $D_{x} \psi$ is invertible for all $x$ in some neighborhood $U \ni x_0$. Since $Q = \bigcup_{t \geq 0} \Phi^{-t}(U)$ by asymptotic stability of $x_0$, (13) and the chain rule imply that $D_{x} \psi$ is invertible for all $x \in Q$. Hence $\psi$ is a local diffeomorphism.

To see that $\psi$ is injective, let $U$ be a neighborhood of $x_0$ such that $\psi|_U: U \to \psi(U)$ is a diffeomorphism, and let $x, y \in Q$ be such that $\psi(x) = \psi(y)$. By asymptotic stability of $x_0$, there is $T > 0$ such that $\Phi^T(x), \Phi^T(y) \in U$, and (13) implies that $\psi \circ \Phi^T(x) = \psi \circ \Phi^T(y)$. Injectivity of $\psi|_U$ then implies that $\Phi^T(x) = \Phi^T(y)$, and injectivity of $\Phi^T$ then implies that $x = y$. Hence $\psi$ is injective.

To see that $\psi$ is surjective, fix any $y \in T_{x_0}Q$ and let the neighborhood $U$ be as in the last paragraph. Asymptotic stability of 0 for $D_{x_0} \Phi; T_{x_0}Q \times \mathbb{T} \to T_{x_0}Q$ implies that there is $T > 0$ such that $D_{x_0} \Phi^T y \in \psi(U)$, so there exists $x \in U$ with $D_{x_0} \Phi^T y = \psi(x)$. Hence $y = D_{x_0} \Phi^{-T} \psi(x) = \psi \circ \Phi^{-T}(x)$, where we have used (13). It follows that $\psi$ is surjective. This completes the proof. \hfill \Box

The following is an existence and uniqueness result for the $C^{k,\alpha}$ Floquet normal form of a stable hyperbolic periodic orbit of a flow. The result is proved using a combination of Proposition 2 and stable manifold theory [Fen74, Fen77, HPS77, dllW95] specialized to the theory of isochrons [Guc75].

**Proposition 3** (Existence and uniqueness of $C^{k,\alpha}_{\text{loc}}$ global Floquet normal forms). Fix $k \in \mathbb{N}_{\geq 1} \cup \{ \infty \}$ and $0 \leq \alpha \leq 1$. Let $\Phi: Q \times \mathbb{R} \to Q$ be a $C^{k,\alpha}_{\text{loc}}$ flow having a globally attracting hyperbolic $\tau$-periodic orbit with image $\Gamma \subset Q$, where $Q$ is a smooth manifold. Fix $x_0 \in \Gamma$ and let $E_{x_0}^{s,\tau} \subset T_{x_0}Q$ denote the unique $D_{x_0} \Phi^\tau$-invariant subspace complementary to $T_{x_0}\Gamma$. Assume that $\nu(D_{x_0} \Phi^\tau|_{E_{x_0}^s}, D_{x_0} \Phi^\tau|_{E_{x_0}^s}) < k + \alpha$, and assume that $(D_{x_0} \Phi^\tau|_{E_{x_0}^s}, D_{x_0} \Phi^\tau|_{E_{x_0}^s})$ is $k$-nonresonant.

Then if we write $D_{x_0} \Phi^\tau|_{E_{x_0}^s} = e^{rA}$ for some complex linear $A: (E_{x_0}^s)_C \to (E_{x_0}^s)_C$, there exists a unique proper $C^{k,\alpha}_{\text{loc}}$ embedding $\psi = (\psi_1, \psi_2): Q \to S^1 \times (E_{x_0}^s)_C$ such that $\psi(x_0) = 1$, $(D_{x_0} \psi_1)|_{E_{x_0}^s} = (E_{x_0}^s) \leftarrow (E_{x_0}^s)_C$, and

$$\forall t \in \mathbb{T}: \psi_1 \circ \Phi^t(x) = e^{2\pi i t} \psi_1(x), \quad \psi_2 \circ \Phi^t(x) = e^{tA} \psi_2(x). \quad (14)$$

where $S^1 \subset \mathbb{C}$ is the unit circle. If $A: E_{x_0}^s \to E_{x_0}^s$ is real, then $\psi_2 \in C^{k,\alpha}_{\text{loc}}(Q, E_{x_0}^s)$ is real, and the codomain-restricted map $\psi: Q \to S^1 \times (E_{x_0}^s)_C$ is a diffeomorphism.

**Proof.** Theorem 2 implies that a map $\psi_2 \in C^{k,\alpha}_{\text{loc}}(Q, (E_{x_0}^s)_C)$ satisfying the conclusions above exists. Letting $W_{x_0}^s$ denote the global strong stable manifold (isochron) through $x_0$, we have $T_{x_0}W_{x_0}^s = E_{x_0}^s$, and that $\Phi^\tau(W_{x_0}^s) = W_{x_0}^s$. Additionally, $W_{x_0}^s$ is a $C^{k,\alpha}_{\text{loc}}$ manifold since it is the stable manifold for the fixed point $x_0$. \hfill \Box
of the $C^{k,\alpha}_{\text{loc}}$ diffeomorphism $\Phi^r$ [dILW95, Thm 2.1]. Proposition 2 then implies that $\psi_z|_{W^s_{x_0}}: W^s_{x_0} \to E^s_{x_0} \subset (E^s_{x_0})^2$ is a diffeomorphism.\footnote{Strictly speaking, Proposition 2 was stated for smooth manifolds. Hence in order to apply Proposition 2 here (and also in the proofs of Theorems 2 and 4) we must first give $W^s_{x_0}$ a compatible $C^\infty$ structure, but this can always be done [Hir94, Thm 2.2.9], so we will not mention this anymore.}

Since the vector field generating $\Phi$ intersects $W^s_{x_0}$ transversely, a standard argument [HS74, p. 243] together with the $C^{k,\alpha}_{\text{loc}}$ implicit function theorem [Eld13, Cor. A.4] imply that a real-valued $C^{k,\alpha}_{\text{loc}}$ “time-to-imact $W^s_{x_0}$” function can be defined on a neighborhood of any point. Using these facts, one can show that the function $\psi_\theta: Q \to S^1$ defined via $\psi_\theta(W^s_{x_0}) \equiv 1$ and $\psi_\theta(\Phi^t(W^s_{x_0})) \equiv e^{2\pi i \frac{\tau}{A}}$ is a $C^{k,\alpha}_{\text{loc}}$ function. This function $\psi_\theta$ clearly satisfies $\psi_\theta(x_0) = 1$ and (14). $\psi_\theta$ is unique among all continuous functions satisfying these equalities, since if $\tilde{\psi}\theta$ is any other such function, then asymptotic stability of $\Gamma$ implies that the quotient $(\psi_\theta/\tilde{\psi}\theta)$ is constant on $Q$, and since $(\psi_\theta/\tilde{\psi}\theta)(x_0) = 1$ it follows that $\tilde{\psi}\theta \equiv \psi_\theta$.

Since the kernels $\ker(D_{x_0}\psi_z) = T_{x_0}\Gamma$ and $\ker(D_{x_0}\psi_\theta) = E^s_{x_0}$ are transverse, $\psi_\theta$ is an immersion, and $\psi$ is injective since the restriction of $\psi_z$ to any level set $W^s_{\psi_\theta(x_0)}$ of $\psi_\theta$ is the composition of injective maps $e^{tA} \circ \psi_z|_{W^s_{x_0}} \circ \Phi^{-t}|_{W^s_{\psi_\theta(x_0)}}$. $\psi$ is a $C^{k,\alpha}_{\text{loc}}$ embedding since, if $(\psi_z|_{W^s_{x_0}})^{-1}: E^s_{x_0} \to W^s_{x_0}$ is the inverse of $\psi_z|_{W^s_{x_0}}$, then $\psi^{-1}: \psi(Q) \to Q$

$$\psi^{-1}(\theta, z) := \Phi^{\arg(z)} \circ (\psi_z|_{W^s_{x_0}})^{-1}(e^{-2\pi i \frac{\tau}{A}} z)$$

is the $C^{k,\alpha}_{\text{loc}}$ inverse of $\psi$. Since it is clear from (15) that $\psi^{-1}$ is the restriction to $\psi(Q)$ of a continuous function $G: S^1 \times (E^s_{x_0})^2 \to Q$ (given by the same formula), it follows that $\psi$ is a proper map [Lee11, Prop. 4.93(e)]. That $\psi^{-1}$ is $C^{k,\alpha}_{\text{loc}}$ follows since $(\psi_\theta|_{W^s_{x_0}})^{-1}$ is, and because the definition of $\psi^{-1}$ is independent of the branch of $\arg(\cdot)$ used since $\Phi^r \circ (\psi_z|_{W^s_{x_0}})^{-1} \circ e^{-\tau A} = (\psi_z|_{W^s_{x_0}})^{-1}$. Hence $\psi$ is a proper $C^{k,\alpha}_{\text{loc}}$ embedding, and the same argument shows that $\psi$ is a diffeomorphism onto $S^1 \times E^s_{x_0}$ if $A$ is real. This completes the proof. \qed

3.2. Principal Koopman eigenfunctions, isostats, and isotostable coordinates. Given a $C^1$ dynamical system $\Phi: Q \times T \to T$, where $Q$ is a smooth manifold and either $T = \mathbb{Z}$ or $T = \mathbb{R}$, we say that $\psi: Q \to \mathbb{C}$ is a Koopman eigenfunction if $\psi$ is not identically zero and satisfies

$$\forall t \in T; \; \psi \circ \Phi^t = e^{\mu t} \psi$$

for some $\mu \in \mathbb{C}$. The following are intrinsic definitions of principal eigenfunctions and the principal algebra which extend the definitions for linear systems given in [MM16b, Def. 2.2–2.3]; a more detailed comparison is given later in Remark 11. The condition $|\psi|_M \equiv 0$ was motivated in part by the definition of a certain space $\mathcal{F}_{A_\prime}$ of functions in [MM16a, p. 3358].

Definition 4. Suppose that $\Phi$ has a distinguished, closed, invariant subset $M \subset Q$. We say that an eigenfunction $\psi \in C^1(Q)$ is a principal eigenfunction if

$$|\psi|_M \equiv 0 \quad \text{and} \quad \forall x \in M: D_x \psi \neq 0.$$ 

We define the $C^{k,\alpha}_{\text{loc}}$ principal algebra $A^{k,\alpha}_{\Phi}$ to be the complex subalgebra of $C^{k,\alpha}_{\text{loc}}(Q, \mathbb{C})$ generated by all $C^{k,\alpha}_{\text{loc}}$ principal eigenfunctions.

Remark 8. In the case that $\Phi|_{M \times T}$ is minimal (has no proper, closed, nonempty invariant subsets)—(17) can be replaced by the weaker condition

$$\exists x \in M: \psi(x) = 0 \quad \text{and} \quad \exists y \in M: D_y \psi \neq 0.$$ 

This will be the case in the sequel, in which we will consider only the cases that $M$ is either a fixed point or periodic orbit.

Differentiating (16) and using the chain rule immediately yields Propositions 4 and 5, which have appeared in the literature (see e.g. the proof of [MM16a, Prop. 2]). In these results, $d$ denotes the exterior derivative and $T^*_{x_0} Q$ denotes the cotangent space to $x_0$; $d\psi(x_0)$ corresponds to $D_x \psi$ after making the canonical identification $\mathbb{C} \cong T_{\psi(x_0)} \mathbb{C}$. 


Proposition 4. Let $x_0$ be a fixed point of the $C^1$ dynamical system $\Phi: \mathbb{T} \rightarrow Q$. If $\psi$ is a principal Koopman eigenfunction for $\Phi$ satisfying (16) with $\mu \in \mathbb{C}$, then for any $t \in \mathbb{T}$, it follows that $d\psi(x_0) \in (T^*x_0 Q)_\mathbb{C}$ is an eigenvector of the (complexified) adjoint $(D_{x_0} \Phi^\dagger)_*$ with eigenvalue $e^{\mu t}$.

Proposition 5. Let $\Gamma$ be the image of a $\tau$-periodic orbit of the $C^1$ dynamical system $\Phi: \mathbb{T} \rightarrow Q$. If $\psi$ is a principal Koopman eigenfunction for $\Phi$ satisfying (16) with $\mu \in \mathbb{C}$, then for any $x_0 \in \Gamma$, it follows that $d\psi(x_0) \in (T^*x_0 Q)_\mathbb{C}$ is an eigenvector of the (complexified) adjoint $(D_{x_0} \Phi^\dagger)_*$ with eigenvalue $e^{\mu \tau}$; in particular, $e^{\mu \tau}$ is a Floquet multiplier for $\Gamma$.

Notice that, for a dynamical system having a globally attracting compact invariant set $M$, any continuous eigenfunction satisfying (16) with $\mu \in \mathbb{C}$ must have $|e^\mu| \leq 1$. If this attracting set $M$ is furthermore a hyperbolic fixed point, then there is the stronger conclusion that either $\mu = 0$ or $|e^\mu| < 1$. With this observation, Proposition 6 below is now immediate from Theorem 1 and Proposition 4. We remind the reader of Remark 5 which points out that, in the case $k = \infty$, no spectral spread conditions are needed because all inequalities of the form $\nu(\cdot, \cdot) < \infty$ trivially hold.

Proposition 6 (Existence and uniqueness of $C^{k,\alpha}_{\text{loc}}$ Koopman eigenvalues and principal eigenfunctions for a point attractor). Let $\Phi: \mathbb{T} \rightarrow Q$ be a $C^1$ dynamical system having a globally attracting hyperbolic fixed point $x_0 \in Q$, where either $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$. Fix $k \in \mathbb{N}_{\geq 1} \cup \{+\infty\}$ and $0 \leq \alpha \leq 1$, fix $\mu \in \mathbb{C}$, and let $\psi_1 \in C^{k,\alpha}_{\text{loc}}(Q, \mathbb{C})$ be any Koopman eigenfunction satisfying (16) with $\mu \in \mathbb{C}$.

Uniqueness of Koopman eigenvalues and principal eigenfunctions. Assume that either $\nu(e^\mu, D_{x_0} \Phi^1) < k + \alpha$ or $\nu(e^\mu, D_{x_0} \Phi^1) \leq k$.

1. Then there exists $m = (m_1, \ldots, m_n) \in \mathbb{N}_{\geq 0}$ such that

$$e^\mu = e^{m \lambda},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $D_{x_0} \Phi^1$ repeated with multiplicities and $\lambda := (\lambda_1, \ldots, \lambda_n)$.

2. Additionally assume that $\psi_1$ is a principal eigenfunction so that $e^\mu \in \text{spec}(D_{x_0} \Phi^1)$, and assume that $(e^\mu, D_{x_0} \Phi^1)$ is $k$-nonresonant. Then $\psi_1$ is uniquely determined by $d\psi_1(x_0)$, and if $\mu$ and $d\psi_1(x_0)$ are real, then $\psi: \mathbb{T} \rightarrow \mathbb{R} \subseteq \mathbb{C}$ is real. In particular, if $e^\mu$ is an algebraically simple eigenvalue of the (complexification of) $D_{x_0} \Phi^1$ and if $\psi_2$ is any other principal eigenfunction satisfying (16) with the same $\mu$, then there exists $c \in \mathbb{C} \setminus \{0\}$ such that

$$\psi_1 = c\psi_2.$$

Existence of principal eigenfunctions. Assume that $\Phi \in C^{k,\alpha}_{\text{loc}}$ that $e^\mu \in \text{spec}(D_{x_0} \Phi^1)$, that $(e^\mu, D_{x_0} \Phi^1)$ is $k$-nonresonant, and that $\nu(e^\mu, D_{x_0} \Phi^1) < k + \alpha$. Let $w \in (T^*x_0 Q)_\mathbb{C}$ be any eigenvector of the (complexified) adjoint $(D_{x_0} \Phi^1)_*$ with eigenvalue $e^\mu$.

1. Then there exists a unique principal eigenfunction $\psi \in C^{k,\alpha}_{\text{loc}}(Q, \mathbb{C})$ satisfying (16) with $\mu$ and $w$.

2. In fact, if $P$ is any “approximate eigenfunction” satisfying $D_{x_0} P = w$ and

$$P \circ \Phi^1 = e^\mu P + R$$

with $D_{x_0} R = 0$ for all integers $0 \leq i < k + \alpha$, then

$$\psi = \lim_{t \rightarrow \infty} e^{-\mu t} P \circ \Phi^t,$$

in the topology of $C^{k,\alpha}_{\text{loc}}$-uniform convergence on compact subsets of $Q$.

Remark 9 (Laplace averages). Given $P: Q \rightarrow \mathbb{C}$ in the Koopman literature the Laplace average

$$\psi := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{-\mu t} P \circ \Phi^t dt$$

is used to produce a Koopman eigenfunction satisfying (16) with $\mu$ as long as the limit exists [MMM13, MM14]. Since convergence of the limit (20) clearly implies convergence of the Laplace average to the same limiting function, Proposition 6 gives sufficient conditions under which the Laplace average of $P$ exists and is equal to a unique $C^{k,\alpha}_{\text{loc}}$ principal eigenfunction satisfying $D_{x_0} P = w$. 
Remark 10 (Isostables and isostable coordinates). It follows from the discussion after [MMM13, Def. 2] that the definition of isostables given in that paper—for Φ having a globally attracting hyperbolic fixed point x0 with D_{x0} Φ^1 having a unique eigenvalue e^{μ1} (or complex conjugate pair of eigenvalues) of largest modulus—is equivalent to the following. Isostables as defined in [MMM13] are the level sets of the modulus |ψ_1| of a principal eigenfunction ψ_1 satisfying (16) with μ = μ1. Because e^{μ1} is the “slowest” eigenvalue of D_{x0} Φ^1, Proposition 6 implies that any such ψ_1 ∈ C^1(Q, C) is unique modulo scalar multiplication for a C^1 dynamical system without any further assumptions (since ν(e^{μ}, D_{x0} Φ^1) = 1), and such a unique eigenfunction exists if Φ ∈ C^{1,α}_{loc} for some α > 0 (since ν(e^{μ}, D_{x0} Φ^1) = 1 < 1 + α for any α > 0). Since the complex conjugate ψ_1 is a principal eigenfunction satisfying (16) with μ = μ1, it follows that the isostables as defined in [MMM13] are unique even if μ ∈ C \ R. A uniqueness proof for analytic isostables under the additional assumptions of (D_{x0} Φ^1, D_{x0} Φ^1) ∞-nonresonance and an analytic vector field was given in [MMM13]. For the special case that the eigenvalue of largest modulus is real, unique, and algebraically simple, in [MMM13, p. 23] these authors do point out that uniqueness of C^1 isostables follows from the fact that the isostables are the unique C^1 global strong stable manifolds—leaves of the unique global strong stable foliation—over an attracting, normally hyperbolic, 1-dimensional, inflowing invariant “slow manifold”, and this argument works even if the dynamical system is only C^1 (see [EK18] for detailed information on the global stable foliation of an inflowing invariant manifold). The slow manifold is itself generally non-unique without further assumptions, but this does not affect the isostable uniqueness argument. However, as pointed out in [MMM13, p. 23], this argument does not work when the eigenvalue of largest modulus is not real, because in this case the isostables can no longer be interpreted as strong stable manifolds (e.g., the relevant slow manifold is now 2-dimensional, so the dimension of the codimension-1 isostables is too large by 1).

For the case that T = R and Φ has an attracting hyperbolic periodic orbit, several authors have investigated various versions of isostable coordinates without restricting attention to the “slowest” isostable coordinate. The authors in [WM16, Eq. 5] defined a “finite-time” approximate version of isostable coordinates which provide an approximation of our principal eigenfunctions. Subsequently, [SKN17, Sec. 2] defined a version of “exact” isostable coordinates (termed amplitudes and phases) directly in terms of Koopman eigenfunctions, and in particular our Proposition 7 and Theorem 4 can be used to directly infer existence, uniqueness and regularity properties of these coordinates under relatively weak assumptions. It appears that [WE18, WMM19] intended to define a different version of “exact” isostable coordinates close in spirit to the approximate version in [WM16]. However, these definitions [WE18, WMM19, Eq. 24, Eq. 58] are given in terms of a limit which might not exist for principal eigenfunctions other than the “slowest”, as we show in Example 2 below. In any case, it appears that principal Koopman eigenfunctions provide a means for defining all of the isostable coordinates for a periodic orbit attractor which does not require such limits.

Remark 11 (Relationship to the principal eigenfunctions, principal algebras, and pullback algebras of [MM16b]). Given a nonlinear dynamical system Φ: Q × T → Q with a globally attracting hyperbolic fixed point x0, Mohr and Mezić defined the principal eigenfunctions for the associated linearization D_{x0} Φ: T_{x0} Q × T → T_{x0} Q to be those of the form v ↦ w(v) where w ∈ (T^*_{x0} Q)_C is an eigenvector of the complexified adjoint (D_{x0} Φ^1)^*: (T^*_{x0} Q)_C → (T^*_{x0} Q)_C [MM16b, Def. 2.2], and they defined the principal algebra A_{D_{x0} Φ^1} to be the subalgebra of C^0(T_{x0} Q, C) generated by the principal eigenfunctions [MM16b, Def. 2.3]. Mohr and Mezić do not define principal eigenfunctions or the principal algebra for the nonlinear system itself but, given a topological conjugacy τ: T_{x0} Q → Q between Φ and D_{x0} Φ, they define the pullback algebra

(21) \( A_{D_{x0} Φ^1} \circ τ^{-1} := \{ ϕ \circ τ^{-1}: ϕ ∈ A_{D_{x0} Φ^1} \} \).

Assuming that Φ ∈ C^{k,α}_{loc}, Proposition 6 implies that the relationship between the concepts in our Definition 4 and those of [MM16b] is as follows. If (D_{x0} Φ^1, D_{x0} Φ^1) is k-nonresonant and either ν(D_{x0} Φ^1, D_{x0} Φ^1) < k + α or ν(D_{x0} Φ^1, D_{x0} Φ^1) ≤ k, our principal eigenfunctions for D_{x0} Φ^1 coincide precisely with their principal eigenfunctions w ∈ (T^*_{x0} Q)_C. This implies that our principal algebra A_{D_{x0} Φ^1} coincides with their A_{D_{x0} Φ^1}. Next, notice that the pullback algebra (21) is generated by the functions w ∘ τ^{-1} where w ∈ (T^*_{x0} Q)_C.
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Proposition 7

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Uniqueness of Koopman eigenvalues.

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Additionally assume that \(\psi_1\) is a principal eigenfunction so that \(e^{\mu t} \in \text{spec}(D_{x_0} \Phi^t|_{E_{x_0}})\), and assume

that \((e^{\mu t}, D_{x_0} \Phi^t|_{E_{x_0}})\) is \(k\)-nonresonant. Then \(\psi_1\) is uniquely determined by \(d\psi_1(x_0)\), and if \(\mu\) and \(d\psi_1(x_0)\) are real, then \(\psi: Q \to \mathbb{R} \subset \mathbb{C}\) is real. In particular, if \(\mu\) is an algebraically simple eigenvalue of (the complexification of) \(D_{x_0} \Phi^1\) and if \(\psi_2\) is any other principal eigenfunction satisfying

(16) with the same \(\mu\), then there exists \(c \in \mathbb{C} \setminus \{0\}\) such that

\(\psi_1 = c\psi_2\).

Existence of principal eigenfunctions. Assume that \(e^{\mu t} \in \text{spec}(D_{x_0} \Phi^t|_{E_{x_0}})\), that \((e^{\mu t}, D_{x_0} \Phi^t|_{E_{x_0}})\) is \(k\)-nonresonant, and that \(\nu(e^{\mu t}, D_{x_0} \Phi^t|_{E_{x_0}}) < k + \alpha\). Let \(w \in (E_{x_0})_x^*\) be any eigenvector of the (complexified) adjoint \((D_{x_0} \Phi^t|_{E_{x_0}})^*\) with eigenvalue \(e^{\mu t}\). Then there exists a unique principal eigenfunction

\(\psi_1 \in \mathcal{C}_{0}^{k,\alpha}(Q, \mathbb{C})\) for \(\Phi\) satisfying

(16) with \(\mu\) and \(\mathbb{T} = \mathbb{R}\) and satisfying \(d\psi_1(x_0)|_{E_{x_0}} = w\).

A well-known example of Sternberg shows that, even for an analytic diffeomorphism \(\Phi^1\) of the plane having the globally attracting fixed point 0, there need not exist a \(C^2\) principal eigenfunction corresponding to \(e^{\mu} \in \text{spec}(D_0 \Phi^1)\) if \((e^{\mu}, D_0 \Phi^1)\) is not \(2\)-nonresonant [Ste57, p. 812]. Concentrating now on the issue of uniqueness of principal eigenfunctions, the following example shows that our nonresonance and spectral spread conditions are both necessary for the uniqueness statements of Propositions 6 and 7.

Example 1 (Uniqueness of principal eigenfunctions). Consider \(\Phi = (\Phi_1, \Phi_2): \mathbb{R}^2 \times \mathbb{T} \to \mathbb{R}^2\) defined by

\[
\Phi_1(x, y) = e^{-t}x \\
\Phi_2(x, y) = e^{-(k+\alpha)t}y
\]
where \( k \in \mathbb{N}_{\geq 1}, 0 \leq \alpha \leq 1 \), and either \( T = \mathbb{Z} \) or \( T = \mathbb{R} \). \( \Phi \) is a linear dynamical system, so clearly the eigenvalues of \( D_0 \Phi^1 \) are \( e^{-1} \) and \( e^{-(k+\alpha)} \). Furthermore, for any irrational \( \alpha \in [0, 1] \), \( (e^{-(k+\alpha)}, D_0 \Phi^1) \) is \( \infty \)-nonresonant. However, if we define \( \sigma_\alpha(x) := |x| \) for \( \alpha > 0 \) and \( \sigma_\alpha(x) := x \) for \( \alpha = 0 \), then for any \( k \in \mathbb{N}_{\geq 1} \) and \( \alpha \in [0, 1] \) both

\[
h_1(x, y) := y
\]

and

\[
h_2(x, y) := y + \sigma_\alpha(x)^{k+\alpha}
\]

are \( C^{k,\alpha} \) principal eigenfunctions satisfying (16) with the same value

\[
\mu = -(k + \alpha).
\]

In particular this shows that \( C^{k,\alpha}_{\text{loc}} \) principal Koopman eigenfunctions are not unique (modulo scalar multiplication) even if the \( \infty \)-nonresonance condition is satisfied. Since here \( h_2 \in C^{k,\alpha}_{\text{loc}}(\mathbb{R}^2, \mathbb{C}) \) and \( \nu(e^{-(k+\alpha)}, D_0 \Phi^1) = k + \alpha \), this shows that the spectral spread condition \( \nu(e^{-(k+\alpha)}, D_0 \Phi^1) < k + \alpha \) is both necessary and sharp for the principal eigenfunction uniqueness statement of Proposition 6 to hold in the case that \( \alpha > 0 \). (Note that Proposition 6 does imply that \( C^{k,\alpha}_{\text{loc}} \) principal eigenfunctions are unique for any \( k' + \alpha' > k + \alpha \).) If instead \( k = 1 \) and \( \alpha = 0 \), then \( h_1 \) and \( h_2 \) are both \( C^1 \) eigenfunctions satisfying (16) with the same value \( \mu = -(k + \alpha) \), but now these eigenfunctions are distinguished by their derivatives at the origin, which is consistent with the case that \( \nu(e^{-1}, D_0 \Phi^1) = 1 \) in the uniqueness statement of Proposition 6. On the other hand, if \( k = 2 \) and \( \alpha = 0 \) so that \( (e^{-2}, D_0 \Phi^1) \) is not \( 2 \)-nonresonant, (23) and (24) show that analytic eigenfunctions are not unique despite the fact that the spectral spread certainly satisfies \( \nu(e^{-2}, D_0 \Phi^1) < \infty \). Hence the nonresonance condition is also necessary for the principal eigenfunction uniqueness statement of Proposition 6 to hold. Finally, by taking \( T = \mathbb{R} \), changing the state space \( \mathbb{R}^2 \) above to \( \mathbb{R}^2 \times S^1 \), and prescribing \( S^1 \) with the decoupled dynamics \( \Phi^j_t(x, y, \theta) := \theta + t \mod 2\pi \) yields an example showing that the spectral spread and nonresonance conditions are both necessary for the uniqueness statement in Proposition 7 to hold as well.

**Example 2** (Existence of the eigenfunction (20) and isostable coordinates). Existence of the limit in (20) is not automatic if the “approximate eigenfunction” \( P \) is not an approximation to sufficiently high order. In fact fix \( k \in \mathbb{N}_{\geq 1}, \alpha \in [0, 1] \), and \( r, \epsilon \in \mathbb{R}_{>0} \). Define \( \sigma_\alpha(x) := |x| \) for \( \alpha > 0 \) and \( \sigma_\alpha(x) := x \) for \( \alpha = 0 \), and consider the \( C^{k,\alpha}_{\text{loc}} \) dynamical system \( \Phi: \mathbb{R}^2 \times T \to \mathbb{R}^2 \) defined by

\[
\Phi^1(x, y) = e^{-t}x
\]

\[
\Phi^j(x, y) = e^{-rt}(y - \epsilon \sigma_\alpha(x)^{k+\alpha}) + \epsilon e^{-(k+\alpha)t} \sigma_\alpha(x)^{k+\alpha},
\]

where either \( T = \mathbb{Z} \) or \( T = \mathbb{R} \). To see that \( \Phi \) is indeed a dynamical system (i.e., that \( \Phi \) satisfies the group property \( \Phi^{t+s} = \Phi^t \circ \Phi^s \)), define the diagonal linear system \( \Phi^j_t(x, y) = (e^{-t}x, e^{-rt}y) \) and the \( C^{k,\alpha} \) diffeomorphism \( H: \mathbb{R}^2 \to \mathbb{R}^2 \) via \( H(x, y) := (x, y + \epsilon \sigma_\alpha(x)^{k+\alpha}) \), and note that \( \Phi^j = H \circ \Phi^j \circ H^{-1} \). In other words, \( \Phi \) is obtained from a diagonal linear dynamical system via a \( C^{k,\alpha} \) change of coordinates; note also that this change of coordinates can be made arbitrarily close to the identity by taking \( \epsilon \) arbitrarily small. We note that \( \nu(e^{-r}, D_0 \Phi^1) = r \) and that, for any choice of \( \epsilon \), the analytic function \( P(x, y) := y \) satisfies

\[
P \circ \Phi^1 = e^{-r}P + R
\]

where \( D^j_{(0,0)}R = 0 \) for all integers \( 0 \leq j < k + \alpha \). However,

\[
\lim_{t \to \infty} e^{rt}P \circ \Phi^j(x, y) = y - \epsilon \sigma_\alpha(x)^{k+\alpha} + \epsilon \sigma_\alpha(x)^{k+\alpha} \lim_{t \to \infty} e^{(r-(k+\alpha))t}
\]

\[
\begin{cases}
 y - \epsilon \sigma_\alpha(x)^{k+\alpha} & 0 < r < k + \alpha \\
 y & r = k + \alpha \\
 y & r > k + \alpha
\end{cases}
\]

for any \( x \neq 0 \) and \( \epsilon > 0 \). We see that the limit (26) diverges whenever \( \nu(e^{-r}, D_0 \Phi^1) = r > k + \alpha \), but the limit converges when \( r \leq k + \alpha \). For the case that \( r \) is not an integer, this is consistent with Proposition
6 which guarantees that the limit converges if \( \Phi \in C_{\text{loc}}^{k,\alpha} \), if \( \nu(e^{-r},D_0\Phi^1) < k + \alpha \), and if \( (e^{-r},D_0\Phi^1) \) is \( k \)-nonresonant. When \( r \) is not an integer and \( r = k + \alpha \), convergence is also guaranteed by Proposition 6 for this specific example, because then (i) \( (e^{-r},D_0\Phi^1) \) is \( \infty \)-nonresonant, (ii) \( \Phi \) is linear and hence \( C^\infty \), and (iii) Proposition 6 guarantees that this limit always exists if \( \Phi \in C^\infty \) and \( (e^{-r},D_0\Phi^1) \) is \( \infty \)-nonresonant because the spectral spread condition \( \nu(e^{-r},D_0\Phi^1) < \infty \) always holds. As alluded to in Remark 3, the preceding reasoning can actually be applied even without the assumption that \( r \) is not an integer if Lemma 5 is used instead of Proposition 6 as the tool of inference. We emphasize that the divergence in (26) is associated purely with the spectral spread condition since, e.g., we can choose \( r > k + \alpha \) as the tool of inference.

Note that by taking \( T = \mathbb{R} \), changing the state space \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \times S^1 \), and prescribing \( S^1 \) with the decoupled dynamics \( \Phi^2(x,y,\theta) := \theta + t \mod 2\pi \) yields a corresponding example with a globally attracting hyperbolic periodic orbit \( \{(0,0)\} \times S^1 \). In this case, for this example [WE18, MWMM19, Eq. 24, Eq. 58] would attempt to define the isostable coordinate (principal eigenfunction in our terminology) \( \psi_2 \) satisfying (16) with \( \mu_2 := -r \) via the limit (26), but (26) shows that this limit does not exist if \( r > k + \alpha \). This phenomenon should be compared with the explanation in the preceding paragraph based on our general results.

3.3. Classification of all \( C^\infty \) Koopman eigenfunctions.

**Notation.** To improve the readability of Theorems 3 and 4 below, we introduce the following multi-index notation. We define an \( n \)-dimensional multi-index to be an \( n \)-tuple \( i = (i_1, \ldots, i_n) \in \mathbb{N}^n_0 \) of nonnegative integers, and define its sum to be \( |i| := i_1 + \cdots + i_n \). For a multi-index \( i \in \mathbb{N}^n_0 \) and \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \), we define \( z[i] := z_1^{i_1} \cdots z_n^{i_n} \). Given a \( \mathbb{C}^n \)-valued function \( \psi = (\psi_1, \ldots, \psi_n) : Q \to \mathbb{C}^n \), we define \( \psi[i] : Q \to \mathbb{C} \) via \( \psi[i](x) := (\psi(x))[i] \) for all \( x \in Q \). We also define the complex conjugate of \( \psi = (\psi_1, \ldots, \psi_n) \) element-wise:

\[
\bar{\psi} := (\bar{\psi}_1, \ldots, \bar{\psi}_n).
\]

**Theorem 3** (Classification of all \( C^\infty \) eigenfunctions for a point attractor). Let \( \Phi : Q \times T \to Q \) be a \( C^\infty \) dynamical system having a globally attracting hyperbolic fixed point \( x_0 \in Q \), where either \( T = \mathbb{Z} \) or \( T = \mathbb{R} \). Assume that \( D_{x_0}\Phi^1 \) is semisimple and that \( (D_{x_0}\Phi^1, D_{x_0}\Phi^1) \) is \( \infty \)-nonresonant.

Letting \( n = \dim(Q) \), it follows that there exists an \( n \)-tuple

\[
\psi = (\psi_1, \ldots, \psi_n)
\]

of \( C^\infty \) principal eigenfunctions such that every \( C^\infty \) Koopman eigenfunction \( \varphi \) is a (finite) sum of scalar multiples of products of the \( \psi_i \) and their complex conjugates \( \bar{\psi}_i \):

\[
\varphi = \sum_{|i| + |\ell| \leq k} c_{i,\ell} \psi[i] \bar{\psi}[\ell]
\]

for some \( k \in \mathbb{N}_{\geq 1} \) and some coefficients \( c_{i,\ell} \in \mathbb{C} \).

**Proof.** By Proposition 2, there exists a proper \( C^\infty \) embedding \( Q \hookrightarrow \mathbb{C}^n \) which maps \( Q \) diffeomorphically onto an \( \mathbb{R} \)-linear subspace, maps \( x_0 \) to 0, and semiconjugates \( \Phi \) to the diagonal linear flow \( \Theta^t(z_1, \ldots, z_n) = (e^{\lambda_1 t} z_1, \ldots, e^{\lambda_n t} z_n) \). Identifying \( Q \) with its image under the embedding, we may view \( Q \) as a \( \Theta \)-invariant submanifold of \( \mathbb{C}^n \) and \( \Phi = \Theta|_{Q \times T} \). Let \( \varphi \in C^\infty(Q, \mathbb{C}) \) be any \( C^\infty \) Koopman eigenfunction satisfying (16) with \( \mu \in \mathbb{C} \). Write \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \). For any \( k \in \mathbb{N}_{\geq 1} \), by Taylor’s theorem we may write

\[
\varphi(z) = \sum_{|i| + |\ell| \leq k} c_{i,\ell} z[i] \bar{z}[\ell] + R_k(z) := P_k(z) + R_k(z)
\]

where \( R_k(0) = D_0R_k = \cdots = D_k^0R_k \) and the coefficients \( c_{i,\ell} \) are independent of \( k \). Defining \( \lambda := (\lambda_1, \ldots, \lambda_n) \) and writing the eigenfunction equation \( \varphi \circ \Phi^1 = e^\mu \varphi \) in terms of the expansion (28) yields

\[
\sum_{|i| + |\ell| \leq k} e^{i\lambda + \ell} \lambda c_{i,\ell} z[i] \bar{z}[\ell] + R_k \circ \Phi^1(z) = \sum_{|i| + |\ell| \leq k} e^{\mu} c_{i,\ell} z[i] \bar{z}[\ell] + R_k(z).
\]

Equating coefficients of \( z[i] \bar{z}[\ell] \) implies that we must have \( c_{i,\ell} = 0 \) whenever \( e^\mu \neq e^{i\lambda + \ell} \), which implies that \( P_k \) is equal to a sum of products of the principal eigenfunctions of the form \( \bar{\psi}_j(z) := z_j \bar{\psi}_j(z) = \bar{z}_j \).
such that each such product $z^{[i]}z^{[ℓ]}$ is itself an eigenfunction satisfying (16) with $μ$, and therefore $P_k$ is also an eigenfunction satisfying (16) with $μ$. It follows that the same is true of $R_k = ψ - P_k$. If we choose $k \in \mathbb{N}_{\geq 1}$ sufficiently large so that $k > ν(μ, D_0Φ^1)$, Proposition 1 implies that $R_k \equiv 0$, so it follows that $ψ = P_k$. This completes the proof. □

For a globally attracting hyperbolic $τ$-periodic orbit of a $C^{k,α}_{\text{loc}}$ flow with image $Γ$, let $W_{x_0}^s$ be the global strong stable manifold (isochron) through the point $x_0 \in Γ$. As discussed in the proof of Proposition 3, there is a unique (modulo scalar multiplication) continuous eigenfunction satisfying (16) with $μ = \frac{2π}{τ}$ and $T = \mathbb{R}$, and this eigenfunction is in fact $C^∞$ for a $C^∞$ flow. In the theorem below, let $ψ_θ$ be the unique such eigenfunction satisfying $ψ_θ|W_{x_0}^s \equiv 1$, where $W_{x_0}^s$ is the global strong stable manifold (isochron) through the point $x_0$ in the theorem statement. Explicitly, $ψ_θ$ is given by

$$ψ_θ|W_{x_0}^s = e^{i2πt}$$

for all $t \in \mathbb{R}$. This defines $ψ_θ$ on all of $Q$ since $Q = \bigcup_{t \in \mathbb{R}} W_{φ(t(x_0))}^s$, and the definition makes sense since $W_{φ^jτ(x_0)}^s = W_{x_0}^s$ for all $j \in \mathbb{Z}$.

**Theorem 4** (Classification of all $C^∞$ eigenfunctions for a limit cycle attractor). Let $Φ: Q \times \mathbb{R} \to Q$ be a $C^{∞}$ dynamical system having a globally attracting hyperbolic $τ$-periodic orbit with image $Γ \subset Q$. Fix $x_0 \in Γ$ and denote by $E_{x_0}^s$ the unique $τ$-invariant subspace complementary to $T_{x_0}Γ$. Assume that $D_{x_0}Φ^τ|E_{x_0}^s$ is semisimple and that $(D_{x_0}Φ^τ|E_{x_0}^s, D_{x_0}Φ|E_{x_0}^s)$ is $∞$-nonresonant.

Letting $n + 1 = \text{dim}(Q)$, it follows that there exists an $n$-tuple

$$ψ = (ψ_1, \ldots, ψ_n)$$

of $C^∞$ principal eigenfunctions such that every $C^∞$ Koopman eigenfunction $φ$ is a (finite) sum of scalar multiples of products of integer powers of $ψ_θ$ with products of the $ψ_i$ and their complex conjugates $\bar{ψ}_i$:

$$φ = \sum_{|ℓ|+|m| \leq k} c_{ℓ,m}ψ_{[ℓ]}\bar{ψ}_{[m]}ψ_θ^{jτ,m}$$

for some $k \in \mathbb{N}_{\geq 1}$, some coefficients $c_{ℓ,m} \in \mathbb{C}$, and $jτ,m \in \mathbb{Z}$.

**Proof.** Let $W_{x_0}^s$ be the $C^∞$ global strong stable manifold through $x_0$. We remind the reader of the facts $Q = \bigcup_{t \in \mathbb{R}} W_{φ(x_0)}^s \subset W_{φ(x_0)}^s = Φ^τ(W_{x_0}^s)$ which are implicitly used in the remainder of the proof.

First, we note that every eigenfunction $χ \in C^∞(W_{x_0}^s, \mathbb{C})$ of $F^τ(x) := Φ^τ|W_{x_0}^s(x)$ satisfying (16) with $μ \in \mathbb{C}$ and $T = \mathbb{R}$ admits a unique extension to an eigenfunction $\tilde{χ} \in C^∞(Q, \mathbb{C})$ of $Φ$ satisfying (16) with $μ$ and $T = \mathbb{R}$; this unique extension $\tilde{χ}$ is defined via

$$\tilde{χ}|W_{φ^τ(x_0)}^s = e^{-μτ}χ \circ Φ^τ|W_{φ^τ(x_0)}^s$$

for all $t \in \mathbb{R}$. $χ$ is a principal eigenfunction if and only if its extension $\tilde{χ}$ is.

Next, let $φ \in C^∞(Q, \mathbb{C})$ be an eigenfunction satisfying (16) with $μ$ and $T = \mathbb{R}$. Theorem 3 implies that $φ|W_{x_0}^s$ is equal to a sum of products of principal eigenfunctions $χ_1, \ldots, χ_n, \bar{χ}_1, \ldots, \bar{χ}_n$ of $Φ^τ|W_{x_0}^s$ of the form:

$$φ|W_{x_0}^s = \sum_{|ℓ|+|m| \leq k} c_{ℓ,m}χ_{[ℓ]}\bar{χ}_{[m]}$$

for some $k \in \mathbb{N}_{\geq 1}$, where $χ = (χ_1, \ldots, χ_n)$. Let $λ = (λ_1, \ldots, λ_n) \in \mathbb{C}^n$ be such that each $χ_j$ satisfies $χ_j \circ Φ^τ|W_{x_0}^s = e^{λ_jτ}χ_j$. The proof of Theorem 3 showed that

$$e^{μτ} = e^{(ℓλ_m+\bar{λ})τ}$$

for all $ℓ, m \in \mathbb{N}_{≥ 1}$ such that $c_{ℓ,m} \neq 0$, so for such $ℓ, m$ we have

$$μ = ℓ \cdot λ + m \cdot \bar{λ} + \frac{2π}{τ}jℓ,m$$
for some $j_{t, m} \in \mathbb{Z}$. By the previous paragraph, we may uniquely write $\chi = \psi|_{\mathbb{W}_{z_0}^s} = (\psi_1|_{\mathbb{W}_{z_0}^s}, \ldots, \psi_n|_{\mathbb{W}_{z_0}^s})$ for principal eigenfunctions $\psi_i$ of $\Phi$ satisfying (16) with $\lambda_i$ and $T = \mathbb{R}$.

Using (31), (33), and the extension formula (30), we obtain

$$\psi|_{\mathbb{W}_{\Phi^{-t}(z_0)}^s} = \sum_{|\ell|+|m| \leq k} c_{\ell, m} e^{-\mu t} \cdot (\chi^{[\ell]}{\bar{\chi}}^{[m]}) \circ \Phi^t|_{\mathbb{W}_{\Phi^{-t}(z_0)}^s}$$

for all $t \in \mathbb{R}$ as desired. To obtain the last equality we used the fact that $\psi|_{\mathbb{W}_{z_0}^s} \equiv 1$, so the extension formula (30) implies that $\psi|_{\mathbb{W}_{\Phi^{-t}(z_0)}^s} \equiv e^{-i2\pi t}$ and hence also $\left(\psi^j_{\Phi^{-t}(z_0)}\right) \equiv e^{-i(2\pi j_{t, m})t}$. This completes the proof.

4. Proofs of the main results

4.1. Proof of Theorem 1. In this section we prove Theorem 1, which we repeat here for convenience.

**Theorem 1** (Existence and uniqueness of $C^{k, \alpha}_{\text{loc}}$ global linearizing factors for a point attractor). Let $\Phi: \mathbb{T} \to Q$ be a $C^1$ dynamical system having a globally attracting hyperbolic fixed point $x_0 \in Q$, where $Q$ is a smooth manifold and either $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$. Let $m \in \mathbb{N}_{\geq 1}$ and $e^A \in \GL(m, \mathbb{C})$ have spectral radius $\rho(e^A) < 1$, and let the linear map $B: \mathbb{T} \times Q \to \mathbb{C}^m$ satisfy

$$\forall t \in \mathbb{T}: B \mathbb{D}_{x_0} \Phi^t = e^{tA}B. \tag{5}$$

**Uniqueness.** Fix $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$ and $0 \leq \alpha \leq 1$, assume that $(e^A, \mathbb{D}_{x_0} \Phi^1)$ is $k$-nonresonant, and assume that either $\nu(e^A, \mathbb{D}_{x_0} \Phi^1) < k + \alpha$ or $\nu(e^A, \mathbb{D}_{x_0} \Phi^1) \leq k$. Then any $\psi \in C^{k, \alpha}_{\text{loc}}(Q, \mathbb{C}^m)$ satisfying

$$\psi \circ \Phi^1 = e^A \psi, \quad \mathbb{D}_{x_0} \psi = B$$

is unique, and if $B: \mathbb{T} \times Q \to \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times m}$ are real, then $\psi: \mathbb{T} \to \mathbb{R}^m \subset \mathbb{C}^m$ is real.

**Existence.** If furthermore $\Phi \in C^{k, \alpha}_{\text{loc}}$ and $\nu(e^A, \mathbb{D}_{x_0} \Phi^1) < k + \alpha$, then such a $\psi$ exists and additionally satisfies

$$\forall t \in \mathbb{T}: \psi \circ \Phi^t = e^{tA} \psi. \tag{6}$$

In fact, if $P$ is any “approximate linearizing factor” satisfying $\mathbb{D}_{x_0} P = B$ and

$$P \circ \Phi^1 = e^A P + R \tag{7}$$

with $\mathbb{D}_{x_0}^i R = 0$ for all integers $0 \leq i < k + \alpha$, then

$$\psi = \lim_{t \to \infty} e^{-tA} P \circ \Phi^t, \tag{8}$$

in the topology of $C^{k, \alpha}$-uniform convergence on compact subsets of $Q$.

We prove the uniqueness and existence portions of Theorem 1 in the following §4.1.1 and §4.1.2, respectively.
4.1.1. Proof of uniqueness. In this section, we prove the uniqueness portion of Theorem 1. The proof of uniqueness consists of an algebraic part and an analytic part. The algebraic portion is carried out in Lemmas 1 and 2, and the analytic portion is carried out in Lemma 3.

Lemma 1. Let \( k \in \mathbb{N}_{\geq 1} \cup \{\infty\} \) and \( X \in \mathbb{C}^{m \times m} \), and \( Y \in \mathbb{R}^{n \times n} \) be such that \((X,Y)\) is \( k\)-nonresonant. For all \( 1 < \ell < k \), let \( \mathcal{L}((\mathbb{R}^n)^\otimes \ell, \mathbb{C}^m) \) denote the space of linear maps from the \( \ell \)-fold tensor product \((\mathbb{R}^n)^\otimes \ell \) to \( \mathbb{C}^m \), and define the linear operator

\[
(34) \quad T_i : \mathcal{L}((\mathbb{R}^n)^\otimes \ell, \mathbb{C}^m) \to \mathcal{L}((\mathbb{R}^n)^\otimes \ell, \mathbb{C}^m), \quad T_i(P) := PY^\otimes \ell - XP.
\]

(By this formula we mean that \( T_i(P) \) acts on tensors \( \tau \in (\mathbb{R}^n)^\otimes \ell \) via \( \tau \mapsto P(Y^\otimes \ell(\tau)) - XP(\tau) \).

Then for all \( 1 < \ell \leq k \), \( T_i \) is a linear isomorphism. (The conclusion holds vacuously if \( k = 1 \).

Proof. Let \( \lambda_1, \ldots, \lambda_n \) and \( \mu_1, \ldots, \mu_m \) respectively be the eigenvalues of \( Y \) and \( X \) repeated with multiplicity. First assume that \( Y \) and \( X \) are both semisimple, i.e., diagonalizable over \( \mathbb{C} \). Identifying \( Y \) with its complexification, let \( e_1, \ldots, e_n \in \mathbb{C}^n \) be a basis of eigenvectors for \( Y \) and let \( e_1, \ldots, e_n \in (\mathbb{C}^n)^* \) be the associated dual basis. Let \( f_1, \ldots, f_m \in \mathbb{C}^m \) be a basis of eigenvectors for \( X \). Fix any integer \( i \) with \( 1 < i \leq k \), any \( p \in \{1, \ldots, m\} \), and any multi-indices \( \ell, j \in \mathbb{N}_{\geq 1} \); defining \( e^{\otimes [\ell]} := e^{\otimes 1} \otimes \cdots \otimes e^{\otimes \ell} \) and similarly for \( e^{\otimes [j]} \), we compute

\[
T_i (f_p \otimes e^{\otimes [\ell]}) \cdot e^{\otimes [j]} = \lambda_{j_1} \cdots \lambda_{j_n} \cdot (e^{\otimes [\ell]} \cdot e^{\otimes [j]}) f_p - \mu_p \cdot (e^{\otimes [\ell]} \cdot e^{\otimes [j]}) f_p
\]

(no summation implied), where the multi-index Kronecker delta \( \delta_\ell^j = 1 \) and \( \delta_\ell^j = 0 \) if \( \ell \neq j \). Hence the \( f_p \otimes e^{\otimes [\ell]} \) are eigenvectors of \( T_i \) with eigenvalues \( (\lambda_{j_1} \cdots \lambda_{j_n} - \mu_p) \), and dimension counting implies that these are all of the eigenvector/eigenvalue pairs. The \( k\)-nonresonance assumption implies that none of these eigenvalues are zero, so \( T_i \) is invertible.

Since the operator \( T_i \) depends continuously on the matrices \( X \) and \( Y \), since eigenvalues of a matrix depend continuously on the matrix, and since semisimple matrices are dense, it follows by continuity that the eigenvalues of \( T_i \) are all of the form \( (\lambda_{j_1} \cdots \lambda_{j_n} - \mu_p) \neq 0 \) even if one or both of \( X \) and \( Y \) are not semisimple (c.f. [Nel70, p. 37]). Hence \( T_i \) is still invertible in the case of general \( X \) and \( Y \). \( \square \)

Lemma 2. Let \( F \in C^1(\mathbb{R}^n, \mathbb{R}^n) \) have the origin as a fixed point. Let \( k \in \mathbb{N}_{\geq 1} \cup \{\infty\} \) and \( X \in \mathbb{C}^{m \times m} \) be such that \((X,D_0F)\) is \( k\)-nonresonant. Assume that \( \psi \in C^k(\mathbb{R}^n, \mathbb{C}^m) \) satisfies \( D_0 \psi = 0 \) and

\[
(35) \quad \psi \circ F = X \psi.
\]

Then it follows that

\[
D_0^i \psi = 0.
\]

for all \( 1 < i \leq k \). (The conclusion holds vacuously if \( k = 1 \).)

Remark 12. We can restate the conclusion of Lemma 2 in the language of jets [Hir94, GS85, BK94]. If \( \psi \) is a linearizing factor such that the 1-jet \( j_0^1(\psi - \psi(0)) = 0 \), then automatically the \( k\)-jet \( j_0^k(\psi - \psi(0)) = 0 \).

Proof. We will prove the lemma by induction on \( i \). The base case, \( D_0^0 \psi = D_0 \psi = 0 \), is one of the hypotheses of the lemma. For the inductive step, assume that \( D_0^i \psi = \cdots = D_0^i \psi = 0 \) for an integer \( i \) satisfying \( 1 \leq i \leq k - 1 \). Differentiating \((35) \ (i+1) \) times using the chain rule and the inductive hypothesis, we obtain

\[
(36) \quad \left(D_0^{(i+1)} \psi\right)(D_0 F)^{\otimes (i+1)} - XD_0^{(i+1)} \psi = T_{i+1}(D_0^{(i+1)} \psi) = 0,
\]

where the linear operator \( T_{i+1} : \mathcal{L}((\mathbb{R}^n)^{\otimes (i+1)}, \mathbb{C}^m) \to \mathcal{L}((\mathbb{R}^n)^{\otimes (i+1)}, \mathbb{C}^m) \) is as defined in Lemma 1 (taking \( Y := D_0 F \)). In deriving \( (36) \) we have used the fact that symmetric tensors are completely determined by

3The "higher-order chain rule," also known as Faà di Bruno’s formula, gives a general expression for higher-order derivatives of the composition of two functions (see [Jac14] for an exposition). Our inductive hypothesis implies that every term in Faà di Bruno’s formula is zero except for those appearing in \((36)\); however, it is easy to deduce \((36)\) directly without using the full strength of this formula.

4Note that here \((D_0 F)^{\otimes (i+1)}\) denotes the tensor product of \( D_0 F \) with itself \( i \in \mathbb{N}_{\geq 1} \) times, and is distinct from the multi-index notation \((\cdot)^{[i]}\) used in Lemma 1.
their action on tensors of the form $\psi^\otimes i$ \cite[Thm 1]{thomases14}. Lemma 1 implies that $T_{(i+1)}$ is invertible, so (36) implies that $D_0^{(i+1)} \psi = 0$. This completes the inductive step and the proof. \qed

**Lemma 3.** Let $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ be a diffeomorphism such that the origin is a globally attracting hyperbolic fixed point for the dynamical system defined by iterating $F$. Fix $k \in \mathbb{N} \cup \{\infty\}$ and $0 \leq \alpha \leq 1$. Let $e^A \in \text{GL}(m, \mathbb{C})$ have spectral radius $\rho(e^A) < 1$ and satisfy either $\nu(e^A, D_0 F) < k + \alpha$ or $\nu(e^A, D_0 F) \leq k$. Assume $\psi \in C_{\text{loc}}^{k,\alpha}(\mathbb{R}^n, \mathbb{R}^m)$ satisfies

\begin{align*}
\psi \circ F &= e^A \psi \\
). \end{align*}

and

\begin{align*}
D_0^i \psi &= 0
\end{align*}

for all $1 \leq i \leq k$. Then $\psi \equiv 0$.

**Proof.** We first observe that since (i) $0$ is asymptotically stable for the iterated dynamical system defined by $F$, (ii) $\psi$ is continuous, and (iii) $\rho(e^A) < 1$, it follows that $\psi(0) = 0$ since

\begin{equation}
0 = \lim_{n \to \infty} e^{nA} \psi(x_0) = \lim_{n \to \infty} \psi(F^n(x_0)) = \psi(0)
\end{equation}

for any $x_0 \in \mathbb{R}^n \setminus \{0\}$. The second equality follows from (37).

For the remainder of the proof, define $x_j := F^j(x_0)$ for $j \in \mathbb{N}$, and choose $r \in \mathbb{R}$ as follows: (i) if $\alpha = 0$ define $r := k$, and (ii) if $\alpha > 0$ define $r \in \mathbb{R}$ to be any number satisfying $\nu(e^A, D_0 F) < r < k + \alpha$. Taylor’s theorem for $C_{\text{loc}}^{k,\alpha}$ functions \cite[p. 162]{dljo99} says that

\begin{equation}
\psi(x) = \sum_{i=0}^k D_0^i \psi \cdot x^\otimes i + R(x),
\end{equation}

where $\lim_{x \to 0} \frac{R(x)}{\|x\|^r} = 0$. Equations (38) and (39) imply that all of the terms in the sum above vanish, so we obtain $\psi = R$. Using (37) it follows that $e^{jA} \psi = \psi \circ F^j = R \circ F^j$, and since $x_j = F^j(x_0)$ we obtain

\begin{equation}
e^{jA} \psi(x_0) = R(x_j), \quad \lim_{x \to 0} \frac{R(x)}{\|x\|^r} = 0.
\end{equation}

Noting that, as $j \to \infty$, $x_j$ approaches the origin tangent to the generalized eigenspace $E_\lambda$ corresponding to some eigenvalue $\lambda$ of $D_0 F$, it follows that $\frac{|\lambda|^r}{\|x_j\|^r} \to C \neq 0$.\footnote{If $\Phi \in C^2$ this follows from Hartman’s $C^1$ linearization theorem; for the general case that $\Phi \in C^1$, this follows from the pseudohyperbolic versions of the (un)stable and center-(un)stable manifold theorems [HPS77, Ch. 5].} Dividing both sides of (40) by $\|x_j\|^r$, multiplying by $1 = \frac{|\lambda|^r}{|\lambda|^r}$ and taking the limit as $j \to \infty$ therefore yields

\begin{equation}
\lim_{j \to \infty} e^{jA} \psi(x_0) = \lim_{j \to \infty} \left( \frac{|\lambda|}{\|x_j\|} \right)^r \left( \frac{e^{jA}}{|\lambda|^r} \right) \psi(x_0) = C^r \lim_{j \to \infty} \left( \frac{e^{jA}}{|\lambda|^r} \right) \psi(x_0) = 0.
\end{equation}

But

\begin{equation}
r \geq \nu(e^A, D_0 F) := \max_{\alpha \in \text{spec}(e^A)} \frac{\ln(|\alpha|)}{\ln(|\beta|)}
\end{equation}

implies that all eigenvalues $\alpha$ of $e^A$ satisfy $\ln(|\alpha|) \geq r \ln(|\lambda|)$, with the inequality flipping since all eigenvalues of $D_0 F$ have modulus smaller than one (note that the inequality is actually strict in the case $\alpha > 0$). Exponentiating, this implies that all eigenvalues $\alpha$ of $e^A$ satisfy $|\alpha| \geq |\lambda|^r$, and therefore all eigenvalues of $e^{A \over |\lambda|^r}$ have modulus greater than or equal to $1$.\footnote{The desire for this conclusion was part of what motivated our definition of the spectral spread $\nu(\cdot, \cdot, \cdot)$.} Hence the diagonal entries in the (upper triangular) Jordan normal form of $(e^{A \over |\lambda|^r})^j$ are bounded below by $1$, so if $\psi(x_0) \neq 0$ then at least one component of $(e^{A \over |\lambda|^r})^j \psi(x_0)$ with respect to the Jordan basis is bounded below uniformly in $j$. It follows that (41) holds if and only if $\psi(x_0) = 0$. Since $x_0 \in \mathbb{R}^n \setminus \{0\}$ was arbitrary and since we already obtained $\psi(0) = 0$ in (39), it follows that $\psi \equiv 0$ on $\mathbb{R}^n$. This completes the proof. \qed
Using Lemmas 2 and 3, we now prove the uniqueness portion of Theorem 1.

Proof of the uniqueness portion of Theorem 1. Since $x_0$ is globally asymptotically stable, the Brown-Storming theorem [Wil67, Lem 2.1] implies that there is a diffeomorphism $Q \approx \mathbb{R}^n$ sending $x_0$ to 0 where $n = \text{dim}(Q)$, so we may assume that $Q = \mathbb{R}^n$ and $x_0 = 0$.

Define the diffeomorphism $F := \Phi^1$ to be the time-1 map. Let $\psi_1$ and $\psi_2$ be two functions satisfying $D_0 \psi_i = B$ and $\psi_i \circ F = e^{A} \psi_i$ for $i = 1, 2$. Then $\psi := \psi_1 - \psi_2$ satisfies $D_0 \psi = 0$ and $\psi \circ F = e^{A} \psi$. Lemma 2 implies that $D_0 \psi = 0$ for all $1 \leq i \leq k$, and Lemma 3 then implies that $\psi_1 - \psi_2 = \psi \equiv 0$. If $A$ and $B$ are real, then we can define $\psi_2 := \psi_1$ to be the complex conjugate of $\psi_1$, and so the preceding implies that $\psi_1 = \psi_1$; hence $\psi_1$ is real if $A$ and $B$ are real. This completes the proof of the uniqueness statement of Theorem 1. 

\[\Box\]

4.1.2. Proof of existence. In this section, we prove the existence portion of Theorem 1. As with the proof of the uniqueness portion, the proof consists of an algebraic part and an analytic part. The techniques we use in the existence proof are similar to those used in [Ste57, CFdIL03a]. The algebraic portion of our proof is carried out in Lemma 4, and the analytic portion is carried out in Lemma 5.

Lemma 4 (Existence and uniqueness of approximate polynomial linearizing factors for diffeomorphisms). Fix $k \in \mathbb{N}_{\geq 1}$ and let $F \in C^k(\mathbb{R}^n, \mathbb{R}^n)$ have the origin as a fixed point. Let $X \in \mathbb{C}^{m\times m}$ be such that $(X, D_0 F)$ is $k$-nonresonant, and assume $B \in \mathbb{C}^{m\times n}$ satisfies

$$BD_0 F = XB.$$

Then there exists a unique degree-$k$ symmetric polynomial $P : \mathbb{R}^n \rightarrow \mathbb{C}^m$ vanishing at 0 such that $D_0 P = B$ and such that

$$P \circ F = XP + R,$$

where $R$ satisfies $D_0 R = 0$ for all $0 \leq i \leq k$. Furthermore, if $X \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{m\times n}$ are real, then this unique polynomial $P : \mathbb{R}^n \rightarrow \mathbb{R}^m \subset \mathbb{C}^m$ is real.

Remark 13. We prove Lemma 4 for the case of finite $k$ only and rely on a bootstrapping method to prove the existence portion for the case $k = \infty$ at the end of this section. We could have proved a $C^\infty$ version of Lemma 4 (at least the existence part) using the fact that every formal power series comprises the derivatives of some $C^\infty$ function [Nel70, p. 34], but choose not to do so.

Proof. By Lemma 1, the linear operator

$$T_i : \mathcal{L}(\mathbb{R}^n)^{\otimes i}, \mathbb{C}^m \rightarrow \mathcal{L}(\mathbb{R}^n)^{\otimes i}, \mathbb{C}^m), \quad T_i(P_i) := P_i(D_0 F)^{\otimes i} - XP_i$$

is invertible for all $1 < i \leq k$. Denoting by Sym$^i(\mathbb{R}^n) \subset (\mathbb{R}^n)^{\otimes i}$ the linear subspace of fully symmetric $i$-tensors (the $i$-th symmetric power), symmetry of the tensor $(D_0 F)^{\otimes i}$ also implies that $T_i$ restricts to a well-defined automorphism of $\mathcal{L}(\text{Sym}^i(\mathbb{R}^n), \mathbb{C}^m)$.

By Taylor’s theorem we may write $F$ as a degree-$k$ polynomial plus remainder: $F(x) = \sum_{i=1}^{k} F_i \cdot x^{\otimes i} + R_1$, where $F_1 = D_0 F$ and $\lim_{x \rightarrow 0} \frac{R_i(x)}{\|x\|^i} = 0$. Defining $F_{\otimes [j]} := F_{j_1} \otimes \cdots \otimes F_{j_\ell}$ for any multi-index $j \in \mathbb{N}_{\ell \geq 1}$, we may therefore write (42) as

$$\sum_{\ell=1}^{k} \sum_{j \in \mathbb{N}_{\ell \geq 1}} P_{\ell} \cdot F_{\otimes [j]} \cdot x^{\otimes \ell} = X \sum_{\ell=1}^{k} P_{\ell} \cdot x^{\otimes \ell} + R_2 \cdot x^{\otimes \ell},$$

where $j = |(j_1, \ldots, j_\ell)| = \sum_{i=1}^{\ell} j_i$, $P(x) = \sum_{\ell=1}^{k} P_{\ell} \cdot x^{\otimes \ell}$, $P_1 = B$, and $\lim_{x \rightarrow 0} \frac{R_2(x)}{\|x\|^\ell} = 0$. If we require that all tensors $P_\ell$ are symmetric then all tensors appearing in (44) are symmetric, and since symmetric tensors

\[\vdots\]

\[\vdots\]

\[\vdots\]
are completely determined by their values on all vectors of the form $x^\otimes i$ \cite[Thm 1]{Tho14}, this implies that

\begin{equation}
\sum_{\ell=1}^{k} \sum_{j \in \mathbb{N} \setminus 1 \atop |j| \leq k} P_{\ell} \cdot F_{\otimes [j]} = X \sum_{\ell=1}^{k} P_{\ell} + R_{2}.
\end{equation}

Since $BD_{0}F = XB$ and $P_{1} = D_{0}P_{1} = B$ by assumption, an inductive argument implies that equation (45) holds for some suitable $R_{2}$ if and only if

\begin{equation}
\sum_{\ell=1}^{i-1} \sum_{j \in \mathbb{N} \setminus 1 \atop |j| = 1} P_{\ell} \cdot F_{\otimes [j]} = XP_{i} - P_{i}(D_{0}F)^{\otimes i}.
\end{equation}

for all $0 \leq i \leq k$ (the induction is on $i$, and the base case $0 = XB - BD_{0}F$ is one of our hypotheses). Since the left side of (46) belongs to $\mathcal{L}(\text{Sym}^{i}(\mathbb{R}^{n}), \mathbb{C}^{m})$ and involves only $P_{\ell}$ for $\ell < i$, and since $T_{i}$ is invertible, we can can inductively solve for $P_{i}$ using (46). Since additionally $T_{i}|_{\mathcal{L}(\text{Sym}^{i}(\mathbb{R}^{n}), \mathbb{C}^{m})}$ is a well-defined automorphism of $\mathcal{L}(\text{Sym}^{i}(\mathbb{R}^{n}), \mathbb{C}^{m})$ as discussed above, it follows that each $P_{i} \in \mathcal{L}(\text{Sym}^{i}(\mathbb{R}^{n}), \mathbb{C}^{m})$ which is compatible with our earlier stipulation that each $P_{i}$ be symmetric (which we used in obtaining equation (45) from (44)).

Finally, assume that $X \in \mathbb{R}^{m \times m}$ is real, and assume by induction that $B = P_{1}, P_{2}, \ldots, P_{i-1}$ are real. Taking the complex conjugate of (46), we find that $P_{i}$ solves (46) if and only if its complex conjugate $\overline{P_{i}}$ solves (46). Invertibility of $T_{i}$ thus implies that $P_{i} = \overline{P_{i}}$, so $P_{i} : \mathbb{R}^{n} \to \mathbb{R}^{m}$ and hence also $P = \sum_{i=1}^{k} P_{i}$ are real. This completes the proof. \hfill $\square$

Lemma 5 (Making approximate linearizing factors exact). Fix $k \in \mathbb{N} \setminus 1 \cup \{\infty\}$, $0 \leq \alpha \leq 1$, and let $F : \mathbb{R}^{n} \to \mathbb{R}^{n}$ be a $C^{k,\alpha}_{\text{loc}}$ diffeomorphism such that the origin is a globally attracting hyperbolic fixed point for the dynamical system defined by iterating $F$. Let $e^{A} \in \text{GL}(m, \mathbb{C})$ satisfy $\nu(e^{A}, D_{0}F) < k + \alpha$, and assume that there exists $P \in C^{k,\alpha}_{\text{loc}}(\mathbb{R}^{n}, \mathbb{C}^{m})$ such that

$$
P \circ F = e^{A}P + R,
$$

where $R \in C^{k,\alpha}_{\text{loc}}(\mathbb{R}^{n}, \mathbb{C}^{m})$ satisfies $D_{0}^{i}R = 0$ for all integers $0 \leq i < k + \alpha$ (note the case $\alpha = 0$).

Then there exists a unique $\varphi \in C^{k,\alpha}_{\text{loc}}(\mathbb{R}^{n}, \mathbb{C}^{m})$ such that $D_{0}^{i}\varphi = 0$ for all integers $0 \leq i < k + \alpha$ and such that $\psi := P + \varphi$ satisfies

$$
\psi \circ F = e^{A}\psi.
$$

In fact,

$$
\psi = \lim_{j \to \infty} e^{-jA}P \circ F^{j}
$$

in the topology of $C^{k,\alpha}$-uniform convergence on compact subsets of $\mathbb{R}^{n}$. Furthermore, if $e^{A} \in \text{GL}(m, \mathbb{R})$ is real and $P \in C^{k,\alpha}_{\text{loc}}(\mathbb{R}^{n}, \mathbb{R}^{m})$ is real, then $\varphi, \psi : \mathbb{R}^{n} \to \mathbb{R}^{m} \subset \mathbb{C}^{m}$ are real.

Remark 14. The uniqueness statement in Lemma 5 follows from the proof of the uniqueness statement of Theorem 1 proved in §4.1.1, but we include a self-contained proof below because the methods used to prove existence naturally yield the uniqueness statement of Lemma 5. However, the hypotheses $F \in C^{k,\alpha}_{\text{loc}}$ and $\nu(e^{A}, D_{0}F) < k + \alpha$ assumed here are stronger than needed for the uniqueness statement of Theorem 1, with the latter condition being stronger if $\alpha = 0$.

Proof. We first assume that $k$ is finite, and delay consideration of the case $k = \infty$ until the end of the proof.

Adapted norms. Later in the proof we will require that the following bound on operator norms holds (needed following (57)):

\begin{equation}
\|e^{-A}\|\|D_{0}F\|^{k+\alpha} < 1.
\end{equation}
Due to our assumption that $\nu(e^A, D_0F) < k + \alpha$, this bound can always be made to hold by using an appropriate choice of “adapted” norms (which induce the operator norms) on the underlying vector spaces $\mathbb{R}^n$ and $\mathbb{C}^m$, and so we may (and do) assume that (47) holds in the remainder of the proof.

But first we argue that such norms can indeed be chosen. Let $\lambda \in \text{spec}(D_0F)$ and $\mu \in \text{spec}(e^A)$ be the eigenvalues of $D_0F$ and $e^A$ with largest and smallest modulus, respectively. For any $\kappa > 0$, there exist adapted norms (both denoted by $\|\cdot\|$) on $\mathbb{R}^n$ and $\mathbb{C}^m$ having the property that the induced operator norms $\|e^A\|$ and $\|D_0F\|$ satisfy [HS74, p. 279–280], [CFdlLO03a, Sec. A.1]:

$$\|D_0F\| - |\lambda| \leq \kappa, \quad \|e^A\| - \frac{1}{|\mu|} \leq \kappa. \quad (48)$$

Now since $\ln(|\mu|) =: \nu(e^A, D_0F) < k + \alpha$ and since $|\lambda| < 1$, it follows that $|\lambda| |\mu|^{-1} < 1$. The inequalities (48) implies that $\|e^A\|\|D_0F\|^{k+\alpha} \approx |\lambda|^{k+\alpha}|\mu|^{-\alpha}$ if $\kappa$ is small, so choosing $\kappa$ sufficiently small yields (47) as claimed.

For later use we also note that (48) implies that $D_0F$ is a strict contraction if $\kappa$ is small enough since $|\lambda| < 1$ for all $\lambda \in \text{spec}(D_0F)$, which in turn implies that

$$F(B) \subset B \quad (49)$$

if $B \subset \mathbb{R}^n$ is a sufficiently small ball centered at the origin [HS74, p. 281].

Definition of function spaces. Let $B \subset \mathbb{R}^n$ be a closed ball centered at the origin. Given any Banach space $X$, let $C^k(B, X)$ be the space of $C^k$ functions $G \in C^k(B, X)$ equipped with the standard norm

$$\|G\|_k := \sum_{i=0}^k \sup_{x \in B} \|D_x^i G\|,$$

making $C^k(B, X)$ into a Banach space [diLO99]. Similarly, for a Banach space $Y$, we define the $\alpha$-Hölder constant $[H]_\alpha$ of a map $H : B \to Y$ via

$$[H]_\alpha := \sup_{x,y \in B, x \neq y} \frac{\|H(x) - H(y)\|}{\|x - y\|^\alpha},$$

and for $\alpha > 0$ we let $C^{k,\alpha}(B, X)$ be the space of $C^k$ functions $G \in C^{k,\alpha}(B, X)$ whose $k$-th derivative is uniformly $\alpha$-Hölder continuous equipped with the standard norm

$$\|G\|_{k,\alpha} := \|G\|_k + [D^k G]_\alpha$$

making $C^{k,\alpha}(B, X)$ into a Banach space [diLO99]. For $\alpha = 0$, we identify $C^{k,\alpha}(B, X)$ with $C^k(B, X)$ and make the special definition

$$\|\cdot\|_{k,0} := \|\cdot\|_k.$$

Let $F \subset C^{k,\alpha}(B, \mathbb{C}^m)$ denote the subspace of functions $\varphi$ such that $D_0\varphi = 0$ for all integers $0 \leq i < k + \alpha$; $F$ is a closed linear subspace of $C^{k,\alpha}(B, \mathbb{C}^m)$, hence also a Banach space.

Preliminary estimates. By the definition of $F$ and the mean value theorem it follows that, for any $\epsilon > 0$, if the radius of $B$ is sufficiently small then for any $\varphi \in F$:

$$\|\varphi\|_{k-1} \leq \epsilon \|D^k \varphi\|_0$$

and, if $\alpha > 0$,

$$\|\varphi\|_{k-1,\alpha} + \|D^k \varphi\|_0 \leq \epsilon [D^k \varphi]_\alpha,$$

$$\|\varphi\|_{k,\alpha} \leq (1 + \epsilon) [D^k \varphi]_\alpha. \quad (51)$$

Defining a linear contraction mapping on $F$. Recall that $F : \mathbb{R}^n \to \mathbb{R}^n$ is the diffeomorphism from the statement of the lemma. By (49), all sufficiently small closed balls $B \subset \mathbb{R}^n$ centered at the origin satisfy

---

8 Different $C^k$ and $C^{k,\alpha}$ norms are actually used in [diLO99], namely $\sup_k \sup_{x \in B} \|D_x^k G\|$ and $\sup(\sup_k \sup_{x \in B} \|D_x^k G\|, |G|_\alpha)$, but these two norms are equivalent to the corresponding norms we have chosen.
\( F(B) \subset B \). Additionally, since \( F \in C^{k,\alpha}_\text{loc}(\mathbb{R}^n, \mathbb{R}^n) \) and \( B \) is compact, \( F|_B \in C^{k,\alpha}(B, \mathbb{R}^n) \). It follows that there is a well-defined linear operator \( T: C^{k,\alpha}(B, \mathbb{C}^m) \rightarrow C^{k,\alpha}(B, \mathbb{C}^m) \) given by

\[
T(\varphi) := e^{-A}\varphi \circ F.
\]

Note that \( T(\mathcal{F}) \subset \mathcal{F} \), so that \( \mathcal{F} \) is an invariant subspace for \( T \). We claim that there is a choice of \( B \) so that \( T|_\mathcal{F}: \mathcal{F} \rightarrow \mathcal{F} \) is a contraction with constant \( \beta < 1 \):

\[
\|T(\varphi)\|_{k,\alpha} \leq \beta\|\varphi\|_{k,\alpha}.
\]

To see this, we give an argument essentially due to Sternberg, but which generalizes the proof of [Ste57, Thm 2] to the case of linearizing semiflows and to the \( C^{k,\alpha} \) setting. Using the notation \( D_x^{\otimes [j]}F := D_x^j F \otimes \cdots \otimes D_x^j F \) for a multi-index \( j \in \mathbb{N}_1 \), we compute

\[
D^k_x(T(\varphi)) = e^{-A}D^k_xF(\varphi) \cdot (D_xF)^{\otimes k} + e^{-A} \sum_{i=1}^{k-1} \sum_{\|j\|_1 \leq k} C_{i,j} D^i_xF(\varphi) \cdot D^j_xF,
\]

where the integer coefficients \( C_{i,j} \in \mathbb{N}_1 \) are combinatorially determined by Faà di Bruno’s formula for the “higher-order chain rule” [Jac14] and are therefore independent of \( F \). We choose \( B \) sufficiently small that its diameter is less than one, and we note that there exists a constant \( N_0 \) such that

\[
\sup_{\|x\| \leq 1} \sum_{i=1}^{k-1} \sum_{\|j\|_1 \leq k} C_{i,j} \|D^j_xF\| < N_0.
\]

Using (50) and (55) to bound the sum in (54), it follows that

\[
\|D^kT(\varphi)\|_0 \leq \|e^{-A}\|\|D^kF\|_0 + \epsilon\|D^k\varphi\|_0.
\]

For the case that \( \alpha > 0 \), we will now use (51) to obtain a bound on \( [D^k_xT(\varphi)]_\alpha \) analogous to (56). In order to do this, we use the estimate \( \|x \mapsto D^k_xF_\alpha(\varphi) \leq \|D^k\varphi\|_\alpha \|DF\|_0^0 \) and the product rule \( \|fg\|_\alpha \leq \|f\|_\alpha |g| + |f\|_\alpha \|g\|_0 \) for Hölder constants (see, e.g., [Eld13, Lem 1.19]) to bound the first term of (54) by

\[
\|e^{-A}\| \left( \|D^k\varphi\|_\alpha \|DF\|_0^{k+\alpha} + \|D^k\varphi\|_0 \cdot k\|DF\|_\alpha \|DF\|_0^{k-1} \right) \leq \|e^{-A}\| \|DF\|_0^{k+\alpha} + \epsilon k\|DF\|_\alpha \|DF\|_0^{k-1},
\]

where we have used (51) to bound the second term in parentheses. Next, we use (55), the product rule for Hölder constants again, and for \( 1 \leq i \leq k-1 \) the estimates \( \|x \mapsto D^i_xF_\alpha(\varphi) \leq \|D^i\varphi\|_\alpha \|DF\|_0^{i} \leq \epsilon \|D^i\varphi\|_\alpha \|DF\|_0^{i} \) to bound the second term of (54) by \( \epsilon N_0 \|e^{-A}\| \|D^k\varphi\|_\alpha \). This last estimate we used follows from (51) and the fact that we are requiring \( B \) to have diameter less than one, so that \( \|D^i\varphi\|_\alpha \leq \|D^{i+1}\varphi\|_0 \leq \epsilon \|D^i\varphi\|_\alpha \). We finally obtain

\[
[D^k_xT(\varphi)]_\alpha \leq \|e^{-A}\| \left( \|DF\|_0^{k+\alpha} + \epsilon k\|DF\|_\alpha \|DF\|_0^{k-1} + \epsilon N_0 \right) \|D^k\varphi\|_\alpha.
\]

The estimate (47) and continuity imply that \( \|e^{-A}\|\|DF\|_0^{k+\alpha} < 1 \) if \( B \) is sufficiently small. Hence if \( \epsilon \) is sufficiently small, the quantities respectively multiplying \( \|D^k\varphi\|_\alpha \) and \( \|D^k\varphi\|_\alpha \) in (56) and (57) will be bounded above by some positive constant \( \beta' < 1 \). The discussion preceding (50) and (51) implies that we can indeed take \( \epsilon \) this small after possibly further shrinking \( B \), so it follows that \( \|D^kT(\varphi)\|_0 < \beta' \|D^k\varphi\|_0 \) and, if \( \alpha > 0 \), also \( [D^k_xT(\varphi)]_\alpha \leq \beta' \|D^k\varphi\|_\alpha \). We therefore obtain a contraction estimate on the highest derivative and Hölder constant only. However, we can combine this observation with the second inequalities from each of the two displays (50) and (51) to obtain in both cases \( \alpha = 0 \) and \( \alpha > 0 \) contractions on \( \alpha \)-Hölder functions.

\[
\|T(\varphi)\|_{k,\alpha} \leq (1 + \epsilon)\beta'\|\varphi\|_{k,\alpha}.
\]

That \( D^kT(\varphi) \) is \( \alpha \)-Hölder follows from the chain rule, the fact that \( F \) and the first \( k-1 \) derivatives of \( \varphi \) are \( C^1 \) and hence Lipschitz, the fact that the composition of a bounded \( \alpha \)-Hölder function with a bounded Lipschitz function is again \( \alpha \)-Hölder, and the fact that the product of bounded \( \alpha \)-Hölder functions is again \( \alpha \)-Hölder (see, e.g., [Eld13, Lem 1.19]).
(This technique for the case $\alpha = 0$ was also used in the proof of [Ste57, Thm 2].) Define $\beta := (1 + \epsilon)\beta'$. Since $\beta' < 1$, if necessary we may shrink $B$ further to ensure that $\epsilon$ is sufficiently small that $\beta < 1$. This shows that $T|_{\mathcal{F}}$ is a contraction and completes the proof of (53).

Existence and uniqueness of a linearizing factor defined on $B$. We will now find a locally-defined linearizing factor $\tilde{\psi} \in C^{k,\alpha}(B, \mathbb{C}^m)$ of the form $\tilde{\psi} = P|_B + \tilde{\varphi}$, where $\tilde{\varphi} \in \mathcal{F}$ and $P: \mathbb{R}^n \to \mathbb{C}^m$ is as in the statement of the lemma. By definition, $\tilde{\psi}$ is linearizing if and only if $\tilde{\psi} = e^{-A}\tilde{\psi} \circ F = T(\tilde{\psi})$, so we need to solve the equation $P|_B + \tilde{\varphi} = T(P|_B + \tilde{\varphi})$ for $\tilde{\varphi}$. (We are writing $P|_B$ rather than $P$ because $\tilde{\varphi}$ is a function with domain $B$ rather than $\mathbb{R}^n$, and also because $T$ is a linear operator defined on functions with domain $B$.) Since $T$ is linear, after rearranging we see that this amounts to solving

\begin{equation}
\left(\text{id}_{C^{k,\alpha}(B, \mathbb{C}^m)} - T\right)\tilde{\varphi} = T(P|_B) - P|_B.
\end{equation}

One of the assumptions of the lemma is that $\left(P \circ F - e^{A}P\right)|_B \in \mathcal{F}$, and this implies that the right hand side of (59) belongs to $\mathcal{F}$ since $e^{-A} \cdot \mathcal{F} \subset \mathcal{F}$. Since $T(\mathcal{F}) \subset \mathcal{F}$, it follows that we may rewrite (59) as

\begin{equation}
(\text{id}_{\mathcal{F}} - T|_{\mathcal{F}})\tilde{\varphi} = T(P|_B) - P|_B.
\end{equation}

We showed earlier that $T|_{\mathcal{F}}$ is a strict contraction, i.e., its operator norm satisfies $\|T|_{\mathcal{F}}\|_{k,\alpha} < 1$. It follows that $(\text{id}_{\mathcal{F}} - T|_{\mathcal{F}})$ has a bounded inverse given by the corresponding Neumann series, so that (60) has a unique solution $\tilde{\varphi}$ given by

\begin{equation}
\tilde{\varphi} = (\text{id}_{\mathcal{F}} - T|_{\mathcal{F}})^{-1} \cdot (T(P|_B) - P|_B) = \sum_{n=0}^{\infty} (T|_{\mathcal{F}})^n \cdot (T(P|_B) - P|_B).
\end{equation}

Extension to a unique global linearizing factor. Since $x_0$ is globally asymptotically stable and since $B$ is positively invariant, for every $x \in B$ there exists $j(x) \in \mathbb{N}_{\geq 0}$ such that, for all $j > j(x)$, $F^j(x) \in \text{int}(B)$. If $j$ is large enough that $F^j(x) \in \text{int}(B)$ and $\ell > j$, then

\begin{equation}
e^{-\ell A} \psi \circ F^\ell |_B = e^{-j A} \left( e^{(\ell-j) A} \psi \circ F^{(\ell-j)} \right) |_B \in \mathcal{F}.
\end{equation}

so there is a well-defined map $\psi: \mathbb{R}^n \to \mathbb{C}^m$ given by

\begin{equation}
\psi(x) := e^{-j A} \tilde{\psi} \circ F^j(x),
\end{equation}

where $j \in \mathbb{N}_{\geq 0}$ is any nonnegative integer sufficiently large that $F^j(x) \in \text{int}(B)$. Clearly $\psi \circ F = e^{A} \psi$. If $x \in \mathbb{R}^n$ and $F^j(x) \in \text{int}(B)$, then $x$ has a neighborhood $U$ with $F^j(U) \subset \text{int}(B)$ by continuity, so $\psi|_U$ is given by (62) with $j$ constant on $U$. By the chain rule, this shows that $\psi \in C^{k,\alpha}_{\text{loc}}(\mathbb{R}^n)$. Clearly $\psi$ and $\varphi := \tilde{\psi} - P$ are uniquely determined by $\psi|_B = P|_B + \varphi|_B = P|_B + \tilde{\varphi}$ which is in turn uniquely determined by $\tilde{\varphi}$, and since $\tilde{\varphi} = \varphi|_B$ is unique it follows that $\varphi$ and $\psi$ are also unique. If $A \in \mathbb{R}^{m \times m}$ and $P \in C^{k,\alpha}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ are real, then the complex conjugate $\overline{\tilde{\psi}} = P + \tilde{\varphi}$, also satisfies $\overline{\tilde{\psi}} \circ F = e^{A} \overline{\tilde{\psi}}$, so uniqueness implies that $\tilde{\psi} = \psi$ and hence $\psi: \mathbb{R}^n \to \mathbb{R}^m$ is real.

Convergence to the global linearizing conjugacy. We now complete the proof of the lemma by proving the sole remaining claim that $e^{-j A} P \circ F^j \to \psi$ with $C^{k,\alpha}$-uniform convergence on compact subsets of $\mathbb{R}^n$. To do this, we first inspect the finite truncations of the infinite series in (61). We see that, since

\begin{equation}
\sum_{n=0}^{j} (T|_{\mathcal{F}})^n \cdot (T(P|_B) - P|_B) = \sum_{n=0}^{j} T^{n+1}(P|_B) - T^n(P|_B) = T^{j+1}(P|_B) - P|_B
\end{equation}

for each $j \in \mathbb{N}_{\geq 1}$, taking the limit $j \to \infty$ shows that the series in (61) is equal to $-P|_B + \lim_{j \to \infty} T^j(P|_B)$. In other words,

\begin{equation}
\tilde{\psi} = \lim_{j \to \infty} e^{-j A} P \circ F^j |_B
\end{equation}

with convergence in $C^{k,\alpha}(B, \mathbb{C}^m)$.

Next, let $K \subset \mathbb{R}^n$ be any positively invariant compact subset. Since 0 is globally asymptotically stable and since $B$ contains a neighborhood of 0, there exists $j_0 > 0$ such that $F^j(K) \subset B$ for all $j > j_0$. We
compute
\[
\lim_{j \to \infty} e^{-jA}P \circ F^j |_{K} = \lim_{j \to \infty} e^{-jA} \left( e^{-jA}P \circ F^j \right) |_{K} = e^{-jA} \left( \lim_{j \to \infty} e^{-jA}P \circ F^j \right) |_{K} = e^{-jA} \psi |_{K} = \psi |_{K},
\]
with convergence in $C^{k,\alpha}(K, \mathbb{C}^m)$. Since we are considering convergence in $C^{k,\alpha}(K, \mathbb{C}^m)$ — rather than, e.g., merely pointwise convergence — it is not obvious that we can move the limit inside the parentheses to obtain the second equality. The reason this is valid is that composition maps of the form $g \mapsto f \circ g \circ h$ (for $f, h$ fixed, all maps $C^{k,\alpha}$) are continuous with respect to the $C^{k,\alpha}$-normed topologies [dILO99, Prop. 6.1, Prop. 6.2 (iii)]. Since every compact subset of $\mathbb{R}^n$ in contained in some positively invariant compact subset $K$ (e.g., a sublevel set of a Lyapunov function), this completes the proof for the case $k < \infty$.

Consideration of the case $k = \infty$. For the case $k = \infty$, repeating the proof above for any $k' < \infty$ such that $\nu(e^{A}, D_0 F) < k'$ yields unique $C^{k'}$ functions $\varphi : \mathbb{R}^n \to \mathbb{C}^m$ and $\psi : \mathbb{R}^n \to \mathbb{C}^m$ that satisfy the hypotheses of Theorem 1 (such as $\psi \circ F \circ \Phi$), this completes the proof for the case $k < \infty$.

Using Lemmas 4 and 5, we now complete the proof of Theorem 1 by proving the existence portion of its statement.

Proof of the existence portion of Theorem 1. As in the proof of the uniqueness portion of Theorem 1 at the end of §4.1.1, we may assume that $Q = \mathbb{R}^n$ and $x_0 = 0$. We first consider the case that $T = \mathbb{Z}$, and define the time-1 map $F := \Phi^1$.

First suppose that $k < \infty$. Lemma 4 implies that there exists a polynomial $P$ such that $D_0 P = B$ and $P \circ F = e^{A}P + R$, where $R \in C^{k,\alpha}_{\text{loc}}(\mathbb{R}^n, \mathbb{C}^m)$ satisfies $D_0^i R = 0$ for all integers $0 \leq i < k + 1$. Furthermore, $P$ and $R$ are real if $e^{A}$ and $B$ are real. Lemma 5 then implies that there exists $\varphi \in C^{k,\alpha}_{\text{loc}}(\mathbb{R}^n, \mathbb{C}^m)$ such that $\psi = P + \varphi \in C^{k,\alpha}_{\text{loc}}(\mathbb{R}^n, \mathbb{C}^m)$ satisfies $D_0 \psi = B$, $\psi \circ F = e^{A} \psi$, and $\psi$ is arbitrary it follows that $\psi \in C^{k,\alpha}_{\text{loc}}(\mathbb{R}^n, \mathbb{C}^m)$ such that $\psi \circ F = e^{A} \psi$, $e^{-jA} \psi |_{K} \psi C^{k,\alpha}_{\text{loc}}$-uniformly on compact subsets for any $\tilde{P}$ satisfying the hypotheses of Theorem 1 (such as $P$), and that $\psi$ is real if $A$ and $B$ are real. This completes the proof for the case $k < \infty$.

Now suppose that $k = \infty$. Repeating the proof above for finite $k' > \nu(e^{A}, D_0)$ yields $\psi \in C^{k'}(\mathbb{R}^n, \mathbb{C}^m)$ satisfying $D_0 \psi = B$ and $\psi \circ F = e^{A} \psi$. The proof of the uniqueness portion of Theorem 1 in §4.1.1 implies that $\psi$ is independent of $k'$ that $\psi \circ F = e^{A} \psi$, so since $k'$ is arbitrary it follows that $\psi \in C^{\infty}$. Additionally, by Lemma 5 we have that $e^{-jA} \tilde{P} \circ \Phi^j |_{K} \psi C^{\infty}$-uniformly on compact subsets for any $\tilde{P}$ satisfying the hypotheses of Theorem 1. This completes the proof for the case $T = \mathbb{Z}$.

It remains only to consider the case that $T = \mathbb{R}$, i.e., the case that $\Phi$ is a flow. By the proof of the case $T = \mathbb{Z}$, there exists $\tilde{\psi} \in C^{k,\alpha}_{\text{loc}}(\mathbb{R}^n, \mathbb{C}^m)$ satisfying $D_0 \tilde{\psi} = B$ and $\tilde{\psi} \circ \Phi^j = e^{jA} \tilde{\psi}$ for all $j \in \mathbb{Z}$. By adapting a technique of Sternberg [Ste57, Lem 4], from $\tilde{\psi}$ we will construct a map $\psi \in C^{k,\alpha}_{\text{loc}}(\mathbb{R}^n, \mathbb{C}^m)$ satisfying $D_0 \psi = B$ and $\psi \circ \Phi^t = e^{tA} \psi$ for all $t \in \mathbb{R}$. In fact define
\[
(64) \quad \psi := \int_0^1 e^{-sA} \tilde{\psi} \circ \Phi^s ds.
\]

10Actually we can find $P$ such that $D_0^i R = 0$ for all integers $0 \leq i < k$, with the only difference arising when $\alpha = 0$. However, we do not need this in the following.
By Leibniz’s rule for differentiating under the integral sign and basic estimates, \( \psi \in C_{\text{loc}}^{k,\alpha}(\mathbb{R}^n, \mathbb{C}^m) \), and using the assumption (5) we have that

\[
D_0 \psi = \int_0^1 e^{-sA} B D_0 \Phi^s \, ds = \int_0^1 B \, ds = B.
\]

To prove that \( \psi \circ \Phi^t = e^tA \psi \) for all \( t \in \mathbb{R} \), we compute

\[
\psi \circ \Phi^t = \int_0^1 e^{-sA} \tilde{\psi} \circ \Phi^{s+t} \, ds = \int_t^{1+t} e^{(t-s)A} \tilde{\psi} \circ \Phi^s \, ds
= e^{tA} \int_t^1 e^{-sA} \tilde{\psi} \circ \Phi^s \, ds + e^{tA} \int_1^{1+t} e^{-sA} \tilde{\psi} \circ \Phi^s \, ds
= e^{tA} \int_t^1 e^{-sA} \tilde{\psi} \circ \Phi^s \, ds + e^{tA} \int_1^{1+t} e^{-sA} \left( e^{A} \tilde{\psi} \circ \Phi^{-1} \right) \circ \Phi^s \, ds
= e^{tA} \int_0^1 e^{-sA} \tilde{\psi} \circ \Phi^s \, ds
= e^{tA} \psi
\]

as desired. Since \( \psi \) satisfies \( \psi \circ \Phi^1 = e^A \psi \), the uniqueness result for the case \( T = Z \) actually implies that \( \psi = \tilde{\psi} \).

Letting \( K \subset \mathbb{R}^n \) be any positively invariant compact subset, the map \( G: [0, 1] \times C^{k,\alpha}(K, \mathbb{C}^m) \to C^{k,\alpha}(K, \mathbb{C}^m) \) given by \( G(r, \varphi) := e^{-rA} \varphi \circ \Phi^r \big|_K \) is continuous and satisfies \( (r, \psi) \to \psi \) for all \( r \in [0, 1] \), so compactness of \([0, 1]\) implies that for every neighborhood \( V \subset C^{k,\alpha}(K, \mathbb{C}^m) \) of \( \psi \) there is a smaller neighborhood \( U \subset V \) of \( \psi \) such that \( G([0, 1] \times U) \subset V \), i.e., \( e^{-rA} \varphi \circ \Phi^r \big|_K \in V \) for every \( \varphi \in U \) and \( r \in [0, 1] \). Fix any \( P \in C^{k,\alpha}_{\text{loc}}(\mathbb{R}^n, \mathbb{C}^m) \) satisfying the hypothesis (7) of Theorem 1. By the proof for the case \( T = Z \) there exists \( N \in \mathbb{N}_{\geq 0} \) such that, for all \( j > N \), \( e^{-jA} P \circ \Phi^j \big|_K \in U \). By the definition of \( U \) it follows that \( e^{-tA} P \circ \Phi^t \big|_K \subset V \) for all \( t > N \). Since the neighborhood \( V \ni \psi = \tilde{\psi} \) was arbitrary, this implies that

\[
\psi \big|_K = \lim_{t \to \infty} e^{-tA} P \circ \Phi^t \big|_K
\]

in \( C^{k,\alpha}(K, \mathbb{C}^m) \). Since every compact subset of \( \mathbb{R}^n \) is contained in some positively invariant compact subset \( K \), this proves that \( e^{-tA} P \circ \Phi^t \big|_K \to \psi \) in the topology of \( C^{k,\alpha} \)-uniform convergence on compact subsets. This completes the proof of Theorem 1.

\[\square\]

### 4.2. Proof of Theorem 2

In this section we prove Theorem 2, which we repeat here for convenience. This proof invokes Theorem 1 and is much shorter because of this.

**Theorem 2** (Existence and uniqueness of \( C^{k,\alpha}_{\text{loc}} \) global linearizing factors for a limit cycle attractor). Fix \( k \in \mathbb{N}_{\geq 1} \cup \{ \infty \} \) and \( 0 \leq \alpha \leq 1 \), and let \( \Phi: Q \times \mathbb{R} \to Q \) be a \( C^{k,\alpha}_{\text{loc}} \) flow having a globally attracting hyperbolic \( \tau \)-periodic orbit with image \( \Gamma \subset Q \), where \( Q \) is a smooth manifold. Fix \( x_0 \in \Gamma \) and let \( E_{x_0}^s \) denote the unique \( D_{x_0} \Phi^T \)-invariant subspace complementary to \( T_{x_0} \Gamma \). Let \( m \in \mathbb{N}_{\geq 1} \) and \( e^{rA} \in \text{GL}(m, \mathbb{C}) \) have spectral radius \( \rho(e^{rA}) < 1 \), and let the linear map \( B: E_{x_0}^s \to \mathbb{C}^m \) satisfy

\[
BD_{x_0} \Phi^T \big|_{E_{x_0}^s} = e^{rA} B.
\]

**Uniqueness.** Assume that \( (e^{rA}, D_{x_0} \Phi^T \big|_{E_{x_0}^s}) \) is \( k \)-nonresonant, and assume that either \( \nu(e^{rA}, D_{x_0} \Phi^T \big|_{E_{x_0}^s}) < k + \alpha \) or \( \nu(e^{rA}, D_{x_0} \Phi^T \big|_{E_{x_0}^s}) \leq k \). Then any \( \psi \in C^{k,\alpha}_{\text{loc}}(Q, \mathbb{C}^m) \) satisfying

\[
\forall t \in \mathbb{R}: \psi \circ \Phi^t = e^{tA} \psi, \quad D_{x_0} \psi \big|_{E_{x_0}^s} = B
\]

is unique, and if \( B: E_{x_0}^s \to \mathbb{R}^m \) and \( A \in \mathbb{R}^{m \times m} \) are real, then \( \psi: Q \to \mathbb{R}^m \subset \mathbb{C}^m \) is real.

**Existence.** If furthermore \( \nu(e^{rA}, D_{x_0} \Phi^T \big|_{E_{x_0}^s}) < k + \alpha \), then such a unique \( \psi \) exists.

**Proof.** Let \( W_{x_0}^s \) be the global strong stable manifold (isochron) through \( x_0 \) \([\text{EKR18}]\). Since \( W_{x_0}^s \) is the stable manifold for the fixed point \( x_0 \) of the \( C^{k,\alpha}_{\text{loc}} \) diffeomorphism \( \Phi^r \), it follows that \( W_{x_0}^s \) is a \( C^{k,\alpha}_{\text{loc}} \) manifold \([\text{dILW95}, \text{Thm 2.1}]\).
After identifying $E_{s_{0}}^{s}$ with $\mathbb{R}^{n}$, the uniqueness portion of Theorem 1 applied to $\psi|_{W_{x_{0}}^{s}}$ implies that $\psi|_{W_{x_{0}}^{s}}$ is unique for any $\psi$ satisfying the uniqueness hypotheses, and furthermore $\psi|_{W_{x_{0}}^{s}}$ is real if $A$ and $B$ are real. Since $\psi$ is uniquely determined by $\psi|_{W_{x_{0}}^{s}}$ and (12) (which is true because $\mathcal{Q} = \bigcup_{t \in \mathbb{R}} \Phi^{t}(W_{x_{0}}^{s})$), this implies that $\psi$ is unique and that $\psi$ is real if $A$ and $B$ are real. This completes the proof of the uniqueness statement of Theorem 2.

Under the existence hypotheses, the existence portion of Theorem 1 similarly implies that there exists a unique $\varphi \in C_{\text{loc}}^{k,\alpha}(W_{x_{0}}^{s}, \mathbb{C}^{n})$ satisfying (11) and
\begin{equation}
\forall j \in \mathbb{Z}: \varphi \circ \Phi^{j}\tau|_{W_{x_{0}}^{s}} = e^{j\tau A} \varphi.
\end{equation}
The unique extension of $\varphi$ to a function $\psi: \mathcal{Q} \rightarrow \mathbb{C}^{m}$ satisfying (12) is given by
\begin{equation}
\forall t \in \mathbb{R}: \psi|_{\Phi^{\tau}W_{x_{0}}^{s}} := e^{-tA} \varphi \circ \Phi^{t}\tau|_{W_{x_{0}}^{s}}.
\end{equation}
$\psi$ is well-defined because $\Phi^{\tau}(W_{x_{0}}^{s}) = W_{x_{0}}^{s}$ and $e^{-tA} \varphi \circ \Phi^{-\tau}|_{W_{x_{0}}^{s}} = \varphi$ by (65). This completes the proof. \hfill $\Box$

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