SUBEXPONENTIAL GROWTH RATES IN FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This paper determines the rate of growth to infinity of a scalar autonomous nonlinear functional differential equation with finite delay, where the right hand side is a positive continuous linear functional of \( f(x) \). We assume \( f \) grows sublinearly, and is such that solutions should exhibit growth faster than polynomial, but slower than exponential. Under some technical conditions on \( f \), it is shown that the solution of the functional differential equation is asymptotic to that of an auxiliary autonomous ordinary differential equation with righthand side proportional to \( f \) (with the constant of proportionality equal to the mass of the finite measure associated with the linear functional), provided \( f \) grows more slowly than \( l(x) = x / \log x \). This linear–logarithmic growth rate is also shown to be critical: if \( f \) grows more rapidly than \( l(x) \), the ODE dominates the FDE; if \( f \) is asymptotic to a constant multiple of \( l(x) \), the FDE and ODE grow at the same rate, modulo a constant non–unit factor.

1. Introduction. In this paper, the growth rate to infinity of positive solutions of nonlinear autonomous functional differential equations of the form

\[
x'(t) = \int_{[-\tau,0]} \mu(ds)f(x(t + s)), \quad t > 0, \quad x_0 = \psi \in C([-\tau,0];(0,\infty)),
\]

is studied. Here \( \tau > 0 \) and \( \mu \) is a positive finite Borel measure on \([−\tau,0]\) (so by definition \( \mu(E) \in [0,\infty) \) for all Borel sets \( E \subseteq [−\tau,0] \), and \( \mu([-\tau,0]) = M \in (0,\infty) \)). If \( f \) is positive, by the Riesz representation theorem, (1) is equivalent to \( x'(t) = L(|f(x)|_t), \quad t > 0 \) where \( L \) is a positive continuous linear functional from \( C([−\tau,0];\mathbb{R}^+) \) to \( \mathbb{R}^+ \). Uniqueness of a continuous solution of (1) is guaranteed by asking that \( f \) is continuously differentiable (see e.g. [4] for existence results and properties of measures); positivity of solutions is guaranteed by the positivity of \( \mu \) and of \( f \) on \([0,\infty) \). Non–explosion of solutions in finite time, as well as subexponential growth to infinity of solutions (in the sense that \( \log x(t)/t \to 0 \) as \( t \to \infty \)) arises because \( f'(x) \to 0 \) as \( x \to \infty \). Precise asymptotic results are obtained by asking that \( f \) or \( f' \) belong to the class of regularly varying functions (see [2]). Recall that a measurable function \( g : (0,\infty) \to (0,\infty) \) is regularly varying at infinity with index \( \beta \in \mathbb{R} \) if \( g(\lambda t)/g(t) \to \lambda^\beta \) as \( t \to \infty \), for every \( \lambda > 0 \). We write \( g \in RV_{\infty}(\beta) \).

In the case when \( f \) grows to infinity slightly slower than linearly (in the sense that \( f \in RV_{\infty}(\beta) \) for \( \beta < 1 \)), it is known when \( \mu(ds) = \delta_{(0)}(ds) + \lambda \delta_{[-\tau]}(ds) \), that the rate of...
growth of solutions of (1) and of 
\[ y'(t) = Mf(y(t)), \quad t > 0; y(0) = y_0 > 0 \] 
(2) 
with \( M = 1 + \lambda \) is the same, in the sense that \( x(t)/y(t) \to 1 \) as \( t \to \infty \) (see [1]). The non–delay equation (2) can be considered as a special type of equation (1) in which all the mass \( M \) of \( \mu \) is concentrated at 0. On the other hand, if \( f \) is linear, collapsing the mass of \( \mu \) to zero will grant different rates of (exponential) growth to solutions of (1) and (2). Therefore, the phenomenon that solutions of (2) yield the growth rate of those of (1) ceases for some critical rate of growth of \( f \) faster than functions in \( \text{RV}_\infty(\beta) \) for \( \beta < 1 \), but slower than linear. This suggests that the critical growth rate may be captured by a function \( f \) in \( \text{RV}_\infty(1) \) but with \( f(x)/x \to 0 \) (or \( f'(x) \to 0 \)) as \( x \to \infty \).

In our main result here (Theorem 2.2), we show that the critical rate of growth is \( O(x/\log x) \): more precisely, if 
\[ \lambda := \lim_{x \to \infty} \frac{f(x)}{x/\log(x)} \in [0, \infty], \] 
(3) 
then \( x(t)/y(t) \to \exp(-\lambda \int_{[-\tau,0]} |s|\mu(ds)) \) as \( t \to \infty \), provided \( f \) is ultimately increasing and \( f' \in \text{RV}_\infty(0) \), a hypothesis stronger than, but implying \( f \in \text{RV}_\infty(1) \). In proving Theorem 2.2, we find that \( F(x(t))/t \to M \) as \( t \to \infty \) where 
\[ F(x) = \int_1^x \frac{1}{f(u)}du, \quad x > 0 \] 
(4) 
and similarly \( F(y(t))/t \to M \) as \( t \to \infty \). Therefore our result identifies a subtle distinction in the growth rates of \( x \) and \( y \), which are in some sense close.

Since (1) can be written, with \( M = \int_{[-\tau,0]} \mu(ds) \), as 
\[ x'(t) = Mf(x(t)) - \int_{[-\tau,0]} \mu(ds)\{f(x(t+s)) - f(x(t))\} =: Mf(x(t)) - \delta(t), \] 
(5) 
we can view (1) as a perturbation of (2), and if the perturbed term \( \delta \) (which will be positive for large \( t \), by the monotonicity of \( x \) and \( f \)) is small relative to \( Mf(x(t)) \), we may expect \( x(t)/y(t) \) to tend to a finite limit. This is in the spirit of a Hartman–Wintner type–result (see [6, Cor X.16.4], [5]), so to gain insight into the asymptotic behaviour of (1), we prove a nonlinear Hartman–Wintner theorem (Theorem 2.1), comparing the growth rate of the differential equations \( x'(t) = Mf(x(t)) - \epsilon(x(t)) \) and \( y'(t) = Mf(y(t)) \), where \( f(x)/x \to 0 \) and \( \epsilon(x)/f(x) \to 0 \) as \( x \to \infty \). Under an integral condition on \( \epsilon \), we can show that \( x(t)/y(t) \) tends to zero, unity or a non–trivial non–unit limit. Even though the result is for a simple scalar ODE, we were unable to find in the literature a result of this type. Furthermore, we believe this result is of independent interest, and can show, when allied with an analysis of the asymptotic behaviour of \( \delta \), that Theorem 2.1 identifies the critical growth rate of \( f \) in (3) for the FDE (1), and predicts accurately that \( x(t)/y(t) \to \exp(-\lambda \int_{[-\tau,0]} |s|\mu(ds)) \) as \( t \to \infty \). For these reasons, the result is presented and proven here. We note of course, that there is a huge literature in asymptotic integration and Hartman–Wintner type–results in determining the asymptotic behaviour of nonlinear functional differential equations; some excellent, representative, papers include [3, 7, 8].

2. Results. We start by proving a scalar, sublinear type of Hartman–Wintner theorem.

**Theorem 2.1.** Suppose \( f \in \text{RV}_\infty(1) \) is an increasing function, that \( f(x) - \epsilon(x) > 0 \) for all \( x > 0 \) and that the following limits hold as \( x \to \infty \):
\[ 0 < \frac{\epsilon(x)}{f(x)} \to 0; \quad \frac{f(x)}{x} \to 0. \]
If $f$ and $\epsilon$ are continuous, and $x$ and $y$ are the continuous solutions of

\[ y'(t) = f(y(t)), \; t > 0, \; y(0) > 0; \quad x'(t) = f(x(t)) - \epsilon(x(t)), \; t > 0, \; x(0) > 0, \]

and there is $\mu \in [0, \infty)$ such that

\[ \lim_{x \to \infty} \frac{f(x)}{x} \int_0^x \frac{\epsilon(u)}{f^2(u)} du = \mu, \quad (6) \]

then

\[ \lim_{t \to \infty} \frac{x(t)}{y(t)} = e^{-\mu}. \]

Proof. The increasing, invertible functions

\[ F(x) = \int_1^x \frac{1}{f(u)} du; \quad \Phi(x) = \int_1^x \frac{1}{f(u) - \epsilon(u)} du, \]

are both well defined and we then have that

\[ F(y(t)) = F(y(0)) + t; \quad \Phi(x(t)) = \Phi(x(0)) + t. \]

Hence

\[ y(t) = F^{-1}(F(y(0)) + t); \quad x(t) = \Phi^{-1}(\Phi(x(0)) + t). \]

Since $(\epsilon(x) + f(x))/x \to 0$, $f(x)/x \to 0$ as $x \to \infty$, we have that $y(t) \sim F^{-1}(t)$ and $x(t) \sim \Phi^{-1}(t)$ as $t \to \infty$. Therefore it is sufficient to prove that $\Phi^{-1}(t)/F^{-1}(t) \to e^{-\mu}$ as $t \to \infty$. Define the function

\[ \Psi(x) = F(x) - \Phi(x) = -\int_1^x \frac{\epsilon(u)}{f^2(u) - f(u)\epsilon(u)} du. \]

By hypothesis, we then have $\Psi(x) \sim -\mu[x/f(x)]$ as $x \to \infty$. Now let $\Psi(x) = [\epsilon(x) - \mu]x/f(x)$, where $\epsilon(x) \to 0$ as $x \to \infty$. Thus, since $\Phi$ is invertible, $x = F(\Phi^{-1}(x)) - \Psi(\Phi^{-1}(x))$. For a fixed $x > 0$, $y = \Phi^{-1}(x)$ is the unique solution to $\delta_x(y) = 0$, where $\delta_x(y) = F(y) - \Psi(y) - x = F(y) - [\epsilon(y) - \mu]y/f(y) - x$. Suppose $K < 1$ and let $z = KF^{-1}(x)$. Thus $x = F(z/K)$ and

\[ \delta_x(z) = F(z) - \frac{\epsilon(z) - \mu}{f(z)} - F(z/K) = \int_{z/K}^z \frac{1}{f(u)} du - \frac{\epsilon(z) - \mu}{f(z)}. \]

Hence

\[ \frac{f(z)}{z} \delta_x(z) = \mu - \epsilon(z) + \int_{1/K}^1 \frac{f(z)}{f(\alpha z)} d\alpha, \]

and, since $\epsilon(z) \to 0$, we have

\[ \lim_{z \to \infty} \frac{f(KF^{-1}(x))}{KF^{-1}(x)} \delta_x(KF^{-1}(x)) = \lim_{z \to \infty} \left( \int_{1/K}^1 \frac{f(z)}{f(\alpha z)} d\alpha \right) + \mu. \quad (7) \]

The function $\tilde{f}(x) := 1/f(x) \in RV_{\infty}(-1)$ and we then have

\[ \int_{1/K}^1 \frac{f(z)}{f(\alpha z)} d\alpha = \int_{1/K}^1 \frac{f(\alpha z)}{f(z)} d\alpha = \int_{1/K}^1 \left( \frac{f(z)}{f(\alpha z)} - \alpha^{-1} \right) d\alpha + \int_{1/K}^1 \alpha^{-1} d\alpha. \]

Applying the Uniform Convergence Theorem for Regularly Varying functions (Theorem 1.5.2 in [2]) we return to (7) to conclude that

\[ \lim_{x \to \infty} \frac{f(KF^{-1}(x))}{KF^{-1}(x)} \delta_x(KF^{-1}(x)) = \int_{1/K}^1 \frac{1}{\alpha} d\alpha + \mu = \log(K) + \mu. \]
If $\mu > 0$, then the above limit is positive for $K > e^{-\mu}$ and negative for $K < e^{-\mu}$. Now let $\epsilon \in (0, e^{\mu} - 1) \cap (0, 1)$ be arbitrary and consider

$$\lim_{x \to \infty} \frac{f(e^{-\mu}(1 - \epsilon)F^{-1}(x))}{e^{-\mu}(1 - \epsilon)F^{-1}(x)} \delta_x(e^{-\mu}(1 - \epsilon)F^{-1}(x)) = \log(1 - \epsilon) < 0.$$  

Similarly, we obtain

$$\lim_{x \to \infty} \frac{f(e^{-\mu}(1 + \epsilon)F^{-1}(x))}{e^{-\mu}(1 + \epsilon)F^{-1}(x)} \delta_x(e^{-\mu}(1 + \epsilon)F^{-1}(x)) > 0.$$  

Therefore, there exist $x_1(\epsilon)$ and $x_2(\epsilon)$ such that for all $x \geq x^* := \max(x_1(\epsilon), x_2(\epsilon))$

$$\delta_x(e^{-\mu}(1 - \epsilon)F^{-1}(x)) < 0; \quad \delta_x(e^{-\mu}(1 + \epsilon)F^{-1}(x)) > 0.$$  

However, $\delta_x(y) = 0$ if and only if $y = \Phi^{-1}(x)$, and thus for all $x \geq x^*$ we have

$$e^{-\mu}(1 - \epsilon)F^{-1}(x) < \Phi^{-1}(x) < e^{-\mu}(1 + \epsilon)F^{-1}(x).$$

This allows us to conclude that

$$\lim_{x \to \infty} \frac{\Phi^{-1}(x)}{F^{-1}(x)} = e^{-\mu}, \ \mu \in (0, \infty).$$

In the case when $\mu = 0$ we note that since $\epsilon(x) > 0$ we have $F(x) < \Phi(x)$ and therefore $F^{-1}(x) > \Phi^{-1}(x)$. Hence we may immediately conclude that

$$\limsup_{t \to \infty} \frac{\Phi^{-1}(t)}{F^{-1}(t)} \leq 1.$$  

Recalling that $F(x) = \Phi(x) + \Psi(x)$ we have that $t = \Phi(F^{-1}(t)) + \Psi(F^{-1}(t)).$ Thus

$$F^{-1}(t) = \Phi^{-1}(t - \Psi(F^{-1}(t))) = u(t - \Psi(F^{-1}(t))),$$

where $u(t) = \Phi^{-1}(t)$ and obeys $u'(t) = f(u(t)) - \epsilon(u(t)), \ u(0) = 1$. Next we write

$$\frac{\Phi^{-1}(t)}{F^{-1}(t)} = \frac{\Phi^{-1}(t)}{\Phi^{-1}(t - \Psi(F^{-1}(t)))} = \frac{u(t)}{u(t - \Psi(F^{-1}(t)))}. \quad (8)$$

Now by the Mean Value Theorem there exists $\theta \in [0, 1]$ such that

$$u(t - \Psi(F^{-1}(t))) = u(t) - u(t) \Psi(F^{-1}(1))(t)\Psi(F^{-1}(t))$$

$$= u(t) - \Psi(F^{-1}(t))[f(u(t) - \Psi(F^{-1}(t))) - \epsilon(u(t - \theta, \Psi(F^{-1}(t)))].$$

Taking care to note that $\Psi(F^{-1}(t))|\epsilon(u(t - \theta, \Psi(F^{-1}(t)))) < 0$ we have the estimate

$$u(t - \Psi(F^{-1}(t))) \leq u(t) - \Psi(F^{-1}(1))f(u(t - \Psi(F^{-1}(t))))$$

$$= \Phi^{-1}(t) - \Psi(F^{-1}(t))f(F^{-1}(t)).$$

Putting this into (8) yields

$$\frac{\Phi^{-1}(t)}{F^{-1}(t)} \geq \frac{\Phi^{-1}(t)}{\Phi^{-1}(t) - \Psi(F^{-1}(t))f(F^{-1}(t))} = \frac{1}{1 - \Psi(F^{-1}(t))f(F^{-1}(t)).}$$

Now let $\mu(t) = \Psi(F^{-1}(t))f(F^{-1}(t))/F^{-1}(t) < 0$. Thus $\mu(t)F^{-1}(t)/\Phi^{-1}(t) = \Psi(F^{-1}(t))f(F^{-1}(t))/\Phi^{-1}(t)$. Hence

$$\frac{\Phi^{-1}(t)}{F^{-1}(t)} \geq \frac{1}{1 - \mu(t)\frac{F^{-1}(t)}{\Phi^{-1}(t)).}$$

Now multiply across by the strictly positive number $1 - \mu(t)$

$$F^{-1}(t)/\Phi^{-1}(t)$$

to obtain $\Phi^{-1}(t)/F^{-1}(t) \geq 1 + \mu(t)$. By hypothesis, $\mu(t) \to 0$ as $t \to \infty$ and we have

$$\lim_{t \to \infty} \Phi^{-1}(t)/F^{-1}(t) \geq 1.$$  

Combining this with the limit superior gives the conclusion for $\mu = 0$. \hfill \Box
Proof. 

Remark 2. Our next result confirms that this intuition is in fact correct. 

Theorem 2.2. Let \( f(x) > 0 \) for all \( x > 0 \), \( f'(x) > 0 \) for all \( x > x_1 \), \( f'(x) \to 0 \) as \( x \to \infty \), and \( f' \in RV_\infty(0) \). If \( \tau > 0 \), \( f \) obeys (3), and \( \mu \in M([-\tau,0];\mathbb{R}^+) \) is a positive finite Borel measure, the unique continuous solution \( x \) of (1) obeys 

\[
\lim_{t \to \infty} \frac{x(t)}{F^{-1}(Mt)} = e^{-\mu} = e^{-\lambda C}, 
\]

where \( F \) is given by (4), \( M := \int_{[-\tau,0]} \mu(ds) \) and \( C := \int_{[-\tau,0]} |s| \mu(ds) \).

Remark 2. We note that under these hypotheses we have \( f(x)/x \to 0 \) as \( x \to \infty \). Since \( f \) is ultimately increasing it must either have a finite limit or tend to infinity as \( x \to \infty \). In the former case, \( x'(t) \) tends to a finite limit, and (9) is trivially true.

Proof. Our hypotheses on \( \psi \) and the positivity of \( f \) immediately yield that \( x(t) \to \infty \) as \( t \to \infty \). Thus there exists \( T_1 \) such that \( x(t) > x_1 \) for all \( t \geq T_1 \). Letting \( t > T_1 + \tau \), and noting that \( t \to x(t) \) is increasing on \([0,\infty)\) we have 

\[
0 < x'(t) = \int_{[-\tau,0]} \mu(ds)f(x(t+s)) < \int_{[-\tau,0]} \mu(ds)f(x(t)) < Mf(x(t)), \quad t > T_1 + \tau.
\]

This means that \( x'(t)/x(t) \to 0 \) as \( t \to \infty \). Furthermore, for \( t > T_1 + \tau \), \( f(x(t+s)) > f(x(t-\tau)) \) for \( s \in [-\tau,0] \). Thus \( x'(t) > Mf(x(t-\tau)) \), \( t > T_1 + \tau \). Applying the Mean Value Theorem to the continuous function \( f \circ x \) for each \( t > T_1 + \tau \) there exists \( \theta_t \in [0,\tau] \) such that \( f(x(t)) = f(x(t-\tau)) + f'(x(t-\theta_t))\tau \). Combining this identity with the fact that \( f'(x) \to 0 \) as \( t \to \infty \), we see that \( f(x(t-\tau))/f(x(t)) \to 1 \) as \( t \to \infty \). Hence \( \lim_{t \to \infty} x'(t)/f(x(t)) = M \).

For each \( t > T_1 + \tau \) and \( s \in [-\tau,0] \) there is a \( \theta_{t,s} \in [s,0] \subset [-\tau,0] \) such that 

\[
0 < f(x(t)) - f(x(t + s)) = (f \circ x)'(t - \theta_{t,s})s = f'(x(t - \theta_{t,s}))x'(t - \theta_{t,s})\vert s \vert 
= (f f')(x(t - \theta_{t,s}))x'(t - \theta_{t,s}) \frac{x(t - \theta_{t,s})}{f(x(t - \theta_{t,s}))} \vert s \vert.
\]

Now for every \( \epsilon \in (0,1/2) \) there exists \( T_2(\epsilon) > 0 \) such that 

\[
M(1 - \epsilon) < \frac{x'(t)}{f(x(t))} < M, \quad \text{for all } t > T_2.
\]

\( t > T_2 + \tau \) implies \( t - \theta_{t,s} > T_2 \) and hence 

\[
M(1 - \epsilon) < \frac{x'(t - \theta_{t,s})}{f(x(t - \theta_{t,s}))} < M, \quad \text{for all } t > T_2 + \tau.
\]
Next $x(t - \tau) < x(t - \theta_{t,s}) < x(t)$ for $t > T_1 + \tau$, $s \in [-\tau, 0]$. Since $x'(t)/x(t) \to 0$ as $t \to \infty$, there is $T_3(\epsilon)$ such that $x(t - \tau)/x(t) > 1 - \epsilon$ for all $t > T_3(\epsilon) + \tau$. Let $T_4 := T_1 + T_2 + T_3 + \tau$. Hence $(1 - \epsilon)x(t) < x(t - \theta_{t,s}) < x(t)$, $s \in [-\tau, 0]$, $t > T_4$. Then with $\lambda_{t,s} := x(t - \theta_{t,s})/x(t)$ we have $\lambda_{t,s} \in [1 - \epsilon, 1]$. Now let $t > T_4 + \epsilon$, so

$$-1 + \frac{(f')(x(t - \theta_{t,s}))}{(f')(x(t))} = \frac{(f')'(\lambda_{t,s}x(t))}{(f')'(x(t))} - \lambda_{t,s} + \lambda_{t,s} - 1.$$ 

Hence for $s \in [-\tau, 0]$, $t > T_4$, we have

$$\left| \frac{(f')(x(t - \theta_{t,s})))}{(f')(x(t))} - 1 \right| \leq \left| \frac{(f')'(\lambda_{t,s}x(t))}{(f')'(x(t))} - \lambda_{t,s} \right| + |\lambda_{t,s} - 1|$$

$$\leq \sup_{\lambda \in [1/2 - \epsilon, 1]} \left| \frac{(f')'(\lambda_{t,s}x(t))}{(f')'(x(t))} - \lambda_{t,s} \right| + \epsilon.$$ 

Therefore

$$\lim_{x \to \infty} \sup_{\lambda \in [1/2 - \epsilon, 1]} \left| \frac{(f')'(\lambda_{t,s}x(t))}{(f')'(x(t))} - \lambda_{t,s} \right| = 0.$$ 

Hence, for every $\epsilon \in (0, 1/2)$, there is $x_2(\epsilon) > 0$ such that $x > x_2(\epsilon)$ implies

$$\sup_{\lambda \in [1/2 - \epsilon, 1]} \left| \frac{(f')'(\lambda_{t,s}x(t))}{(f')'(x(t))} - \lambda_{t,s} \right| < \epsilon, x > x_2(\epsilon).$$

Let $x(t) > x_2(\epsilon)$ for $t > T_5(\epsilon)$ and set $T_6 := T_4 + T_5$. Then for all $s \in [-\tau, 0]$, $t \geq T_6$,

$$\left| \frac{(f')(x(t - \theta_{t,s}))}{(f')(x(t))} - 1 \right| \leq 2\epsilon, t \geq T_6.$$ 

Thus for $t > T_6$

$$\int_{-\tau}^{t-s} d\mu(dT_1(s,t)) + \int_{-\tau}^{t-s} \mu(dT_2(s,t)).$$ 

Hence $\delta(t) := \int_{-\tau}^{t} \mu(dT_1(s,t)) + \int_{-\tau}^{t} \mu(dT_2(s,t)).$ For $t \geq T_6$, we have

$$\int_{-\tau}^{t} \mu(dT_1(s,t)) \leq 2\epsilon \int_{-\tau}^{t} |s|\mu(dT_2(s,t)), M(1 - \epsilon) \int_{-\tau}^{t} (f')(x(t)).$$ 

Therefore we see that $\lim_{t \to \infty} \delta(t)/M \int_{-\tau}^{t} |s|\mu(dT_2(s,t)) = 1$. We note from (5) that $x'(t) = Mf(x(t)) - \delta(t)$. Therefore, for $t \geq T_6$, our previous estimates yield

$$x'(t) < Mf(x(t)) - MC(1 - \epsilon)(f')'(x(t)),$$ 

$$x'(t) > Mf(x(t)) - MC(1 + \epsilon)(f')(x(t)).$$ 

Hence, with $x_3(\epsilon) = x(T_0(\epsilon))$, we have

$$\int_{x_3(\epsilon)}^{x(t)} \frac{du}{f(u) - C(1 - \epsilon)f(u)f'(u)} = \int_{T_0}^{t} \frac{x(s)}{f(x(s)) - C(1 - \epsilon)f(x(s))f'(x(s))} ds \leq M(t - T_0),$$
Define and similarly
\[ \int_{x_3(e)}^{x(t)} \frac{du}{f(u) - C(1 + \epsilon)f(u)f'(u)} = \int_{x_3}^{t} \frac{x'(s)}{f(x(s)) - C(1 + \epsilon)f(x(s))f'(x(s))} ds \geq M(t - T_6). \]

For convenience of notation we define the functions
\[ \Phi_+(x) := \int_{x_3(e)}^{x(t)} \frac{du}{f(u) - C(1 - \epsilon)f(u)f'(u)}, \]
\[ \Phi_-(x) := \int_{x_3(e)}^{x(t)} \frac{du}{f(u) - C(1 + \epsilon)f(u)f'(u)}. \]

Thus, for \( t \geq T_6, x(t) \leq \Phi_+^{-1}(M(t - T_6)) \) and \( x(t) \geq \Phi_+^{-1}(M(t - T_6)) \). Define \( y_{\pm\epsilon}(t) = \phi_{\pm\epsilon}(y_{\pm\epsilon}(t)), t > 0, y_{\pm\epsilon}(0) = x_3, \) where \( \phi_{\pm\epsilon}(x) = f(x) - C(1 \pm \epsilon)f(x)f'(x) \). Then
\[ y_{\epsilon}(t) = \Phi_+^{-1}(t); \quad y_{-\epsilon}(t) = \Phi_-^{-1}(t). \]

Since \( \phi_{\pm\epsilon}(x)/x \to 0 \) as \( x \to \infty \), it follows that \( y_{\pm\epsilon}(t)/y_{\pm\epsilon}(t) \to 0 \) as \( t \to \infty \). Thus for any \( c \in \mathbb{R}, \lim_{t \to \infty} y_{\pm\epsilon}(t - c)/y_{\pm\epsilon}(t) = 1, \) and therefore
\[ \lim_{t \to \infty} \frac{\Phi_+^{-1}(Mt - MT_6)}{\Phi_\pm^{-1}(Mt)} = 1. \]

A short calculation reveals that
\[ \Phi_+(x) = F(x) - F(x_3) + \int_{x_3}^{x} \frac{C(1 - \epsilon)f'(u)}{f(u) - C(1 - \epsilon)f(u)f'(u)} du. \]

Similarly
\[ \Phi_-(x) = F(x) - F(x_3) + \int_{x_3}^{x} \frac{C(1 + \epsilon)f'(u)}{f(u) - C(1 - \epsilon)f(u)f'(u)} du. \]

Define
\[ \Psi_\pm(x) = \int_{x_3}^{x} \frac{C(1 \pm \epsilon)f'(u)}{f(u) - C(1 \pm \epsilon)f(u)f'(u)} du, \]
so that \( \Phi_\pm(x) = F(x) - F(x_3) + \Psi_\pm(x) \). Next we apply L’Hôpital’s rule to compute
\[ \lim_{x \to \infty} \frac{\Psi_+(x)}{C(1 \pm \epsilon) \log(f(x))} = \lim_{x \to \infty} \frac{\int_{x_3}^{x} \frac{f'(u)}{f(u)} \frac{1}{C(1 \pm \epsilon)f'(u)} du}{\log(f(x))} = 1. \]

Now fix \( x \) and let \( \delta_+(y) := F(y) - F(x_3) + \Psi(y) - x \). We note that \( \delta_+(\Phi_+^{-1}(x)) = 0 \) and that \( \delta_+ \) is continuous.
\[ \delta_+(y) = \frac{1}{f(y)} + \frac{C(1 - \epsilon)f'(y)}{f(y)C(1 - \epsilon)f'(y)} = \frac{1}{f(y)[1 - (1 - \epsilon)f'(y)]} > 0. \]

Therefore, \( \delta_+(y) = 0 \) if and only if \( y = \Phi_+^{-1}(x) \). Now with \( z = KF^{-1}(x) \) calculate
\[ \delta_+(z) = F(z) - F(z/K) - F(x_3) + \Psi_+(z) = \int_{z/K}^{z} \frac{du}{f(u)} - F(x_3) + \Psi_+(z). \]

Now \( f' \in RV_\infty(0) \) implies that \( f \in RV_\infty(1) \) and this gives us that
\[ \int_{z/K}^{z} \frac{du}{f(u)} \sim \frac{z}{f(z)} \log(K) \text{ as } z \to \infty. \]

Similarly, \( \Psi_+(z) \sim C(1 - \epsilon) \log(f(z)) \sim C(1 - \epsilon) \log(z) \) as \( z \to \infty \). Suppose that \( \lambda \in (0, \infty) \) in (3). Therefore \( \int_{z/K}^{z} du/f(u) \sim (1/\lambda) \log(z) \log(K) \). Thus
\[ \lim_{z \to \infty} \frac{\delta_+(KF^{-1}(x))}{\log(K)} = \frac{1}{\lambda} \log(K) + C(1 - \epsilon). \]
Let $K_+(\epsilon) = (1 + \epsilon)e^{-\lambda C(1-\epsilon)}$, $K_-(\epsilon) = (1 - \epsilon)e^{-\lambda C(1-\epsilon)}$. This implies
\[
\lim_{x \to \infty} \frac{\delta_+(K_+(\epsilon)F^{-1}(x))}{\log(K_+(\epsilon)F^{-1}(x))} = \log(1 + \epsilon) > 0,
\]
\[
\lim_{x \to \infty} \frac{\delta_-(K_-(\epsilon)F^{-1}(x))}{\log(K_-(\epsilon)F^{-1}(x))} = \log(1 - \epsilon) < 0.
\]
Notice that $\lim_{\epsilon \to 0^+} K_+(\epsilon) = e^{-\lambda C}$. Thus for every $\eta \in (0, 1)$ there is $x(\eta, \epsilon) > 0$ such that $x > x(\eta, \epsilon)$ implies
\[
\frac{\delta_+(K_+(\epsilon)F^{-1}(x))}{\log(K_+(\epsilon)F^{-1}(x))} > (1 - \eta)\log(1 + \epsilon).
\]
Let $x_4(\epsilon) = x(1/2, \epsilon)$. Then $\delta_+(K_+(\epsilon)F^{-1}(x)) > 0$ for all $x > x_4(\epsilon)$. Similarly, for every $\eta \in (0, 1)$, there is $x(\eta, \epsilon) > 0$ such that $x > x(\eta, \epsilon)$ implies
\[
\frac{\delta_-(K_-(\epsilon)F^{-1}(x))}{\log(K_-(\epsilon)F^{-1}(x))} < (1 - \eta)\log(1 - \epsilon).
\]
Let $x_5(\epsilon) = x(1/2, \epsilon)$. Thus $\delta_+(K_-(\epsilon)F^{-1}(x)) < 0$ for all $x > x_5(\epsilon)$. Let $x_6 = \max(x_4, x_5)$ and then $\delta_+(K_+(\epsilon)F^{-1}(x)) > 0 > \delta_+(K_-(\epsilon)F^{-1}(x))$, $x > x_6$. Thus $\Phi_+^{-1}(x) \in (K_-(\epsilon)F^{-1}(x), K_+(\epsilon)F^{-1}(x))$, $x > x_6$. Let $t > T_8$, $M t > x_6$. Then
\[
\frac{x(t)}{F^{-1}(Mt)} \leq \frac{\Phi_+^{-1}(M(t - T_8))}{\Phi_+^{-1}(Mt)} \leq \frac{\Phi_+^{-1}(M(t - T_8))}{\Phi_+^{-1}(Mt)} K_+(\epsilon).
\]
Therefore $\limsup_{t \to \infty} x(t)F^{-1}(Mt) \leq K_+(\epsilon)$, and letting $\epsilon \to 0^+$ yields
\[
\limsup_{t \to \infty} \frac{x(t)}{F^{-1}(Mt)} \leq e^{-\lambda C}.
\]
Similarly we define $\delta_-(y) = F(y) - F(x_3) + \Psi_-(y) - x$, and $\Phi_+^{-1}(x)$ is the unique solution to $\delta_-(y) = 0$. An exactly analogous calculation to the above case yields
\[
\frac{x(t)}{F^{-1}(Mt)} \geq \frac{\Phi_-^{-1}(M(t - T_8))}{\Phi_-^{-1}(Mt)} K_-(\epsilon).
\]
Therefore taking the liminf as $t \to \infty$ and letting $\epsilon \to 0^+$ we obtain
\[
\liminf_{t \to \infty} \frac{x(t)}{F^{-1}(Mt)} \geq e^{-\lambda C}.
\]
When $\lambda = +\infty$ the proof is almost identical up to (14), which becomes
\[
\lim_{x \to \infty} \frac{\delta_+(K F^{-1}(x))}{\log(K F^{-1}(x))} = (1 - \epsilon) > 0.
\]
Letting $\epsilon = 1/4$, for all fixed $K < 1$, and $x > x^*(K)$, we have $\delta_{1/4}(KF^{-1}(x)) > 0$ and hence $\Phi_{1/4}(x) < KF^{-1}(x)$ for all $x > x^*(K)$. Therefore, similarly to (15), we obtain $\limsup_{t \to \infty} x(t)/F^{-1}(Mt) \leq K$, for any $K < 1$. Sending $K \to 0^+$ and combining this with the trivial lower bound of zero on the liminf then yields
\[
\lim_{t \to \infty} \frac{x(t)}{F^{-1}(Mt)} = 0.
\]
When $\lambda = 0$ the argument necessarily differs slightly since the leading order asymptotics of $\delta_+(z)$ are now given by $z/f(z)$. Consequently, (14) is replaced by
\[
\lim_{z \to \infty} \frac{\delta_+(z)}{z/f(z)} = \lim_{z \to \infty} \frac{\delta_+(K F^{-1}(x))}{KF^{-1}(x)/f(K F^{-1}(x))} = \log(K).
\]
Now define $K_+(\epsilon)$ and $K_-(\epsilon)$ and use the above limit to obtain $(1 - \epsilon)F^{-1}(x) < \Phi^{-1}_x < (1 + \epsilon)F^{-1}(x)$, $x > x^*(\epsilon)$. Proceeding as in (15) and letting $\epsilon \to 0^+$ gives

$$\limsup_{t \to \infty} \frac{x(t)}{F^{-1}(Mt)} \leq 1.$$ 

The argument for the liminf with $\lambda = 0$ can be obtained with the same modification to the argument in the $\lambda \in (0, \infty)$ case. \hfill \Box

**Remark 3.** Scrutinising the start of the proof, we see that if $f(x) > 0$ for all $x \geq 0$, $f'(x) > 0$ for all $x > x_1$ and $f'(x) \to 0$ as $x \to \infty$, we can show, without using the other hypotheses on $f$, that $F(x(t))/t \to M$ as $t \to \infty$. This can be inferred directly from the limit $x'(t)/f(x(t)) \to M$ as $t \to \infty$. The limit $F(x(t))/t \to M$ does not yield information on the behaviour of $x(t)/F^{-1}(Mt)$ because when $f \in RV_\infty(1)$, the function $F^{-1}$ is rapidly varying at infinity. Therefore, the rest of the proof of Theorem 2.2 yields more refined information on the growth rate of solutions of (1).

**Remark 4.** It is worthwhile to mention that with a slight modification of the above argument the hypothesis that $f'$ be regularly varying in this Theorem can be omitted entirely in the case when $\lambda = 0$.

3. **Example.** A simple example of an $f$ obeying the hypotheses of Theorem 2.2 is to take $f(x) = (x + 1)/\log^\alpha(2 + x)$, for $\alpha > 0$. Clearly $f(x) > 0$ for $x > 0$ and

$$f'(x) = \frac{1}{\log^\alpha(2 + x)} \left(1 - \frac{(1 + x)\alpha}{(2 + x)\log(2 + x)}\right) > 0, \quad x > e^\alpha - 2.$$ 

It is easy to see that $f'(x) \to 0$ as $x \to \infty$ and that $f' \in RV_\infty(0)$. We also have that $\lambda$ in (3) is 0, 1, or $+\infty$ according as to whether $\alpha$ is greater than, equal to, or less than, unity. Making a substitution and splitting the resulting integral gives

$$F(x) = \frac{1}{1 + \alpha} \log^{\alpha+1}(x + 2) - \frac{3^{\alpha+1}}{\alpha + 1} + \int_3^x \frac{w^\alpha}{e^\alpha - 1} \, dw.$$ 

From here we readily derive that

$$F(x) \sim \frac{1}{1 + \alpha} \log^{\alpha+1}(x), \quad F^{-1}(x) \sim e^{(\alpha+1)x^{1/\alpha+1}}, \quad \text{as } x \to \infty.$$ 

We note once more that $F^{-1}$ is rapidly varying at infinity so the rate of growth here is indeed subexponential but faster than any power function.

4. **Further Work.** In this short section, we suggest some further developments of the main result. The hypothesis (3) is clearly crucial: and for subexponential growth, the conditions that $f'(x) > 0$ and tends to zero are natural and mild. However, granted these three hypotheses, one might expect to be able to relax the regular variation hypothesis on $f$, because in the case $\lambda \in (0, \infty)$, they imply $\log f(x)/\log x \to 1$ as $x \to \infty$, which is satisfied by any $f \in RV_\infty(1)$. It is also tempting to conjecture that Theorem 2.2 also applies to analogous convolution Volterra equations where the measure $\mu$ is now supported on $[0, \infty)$. In the Volterra case, it would be especially interesting, in the light of (9), to contrast the cases where $C = \int_{[0, \infty)} s\mu(ds)$ is finite and infinite. Finally, if we view Theorem 2.2 as a Hartman–Wintner type–result, which yields exact asymptotic behaviour, it is also natural to ask if there are results which give less precise estimate of the rate of growth under weaker restrictions on $f$. In this direction, estimates of the form $F(x(t))/t \to M$ as $t \to \infty$, which are weaker than (9), are acceptable. We seek in a later work to investigate these three questions.
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