Boundedness of singular integral operators on weak Herz type spaces with variable exponent

Hongbin Wang¹ · Zongguang Liu²

Received: 21 November 2019 / Accepted: 29 April 2020 / Published online: 27 May 2020
© Tusi Mathematical Research Group (TMRG) 2020

Abstract
In this paper, the authors define the weak Herz spaces and the weak Herz-type Hardy spaces with variable exponent. As applications, the authors establish the boundedness for a class of singular integral operators including some critical cases.

Keywords Weak Herz spaces · Weak Herz-type Hardy spaces · Variable exponent · Boundedness · Singular integral operators

Mathematics Subject Classification 42B35 · 42B20 · 46E30

1 Introduction
The main purpose of this article is to introduce the weak Herz type spaces with variable exponent and give the boundedness for a class of singular integral operators. The classical weak Herz type spaces can be traced to the work of Hu, Lu and Yang [8, 9] which studied the boundedness for some operators. The classical weak Herz type spaces mainly include the weak Herz spaces and the weak Herz-type Hardy spaces. It is well-known that many important operators in harmonic analysis, such as Calderón–Zygmund operators and Ricci–Stein oscillatory singular integral operators, are not bounded on $L^1(\mathbb{R}^n)$, but map $L^1(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$. So the weak Lebesgue spaces $WL^p(\mathbb{R}^n)$ are very important in operator theory. The weak Herz spaces
are the analog of $WL^p(\mathbb{R}^n)$ in the setting of Herz spaces. Meanwhile, the weak Herz-type Hardy spaces are the suitable local version of the weak Hardy spaces $WH^p(\mathbb{R}^n)$. The classical weak Herz type spaces and related function spaces have interesting applications in studying boundedness of many operators and the theory of partial differential equations, for example [6, 12, 14, 17, 19].

On the other hand, due to their wide applications in partial differential equations with non-standard growth [7], electrorheological fluids [16] and image processing [1], the theory of function spaces with variable exponents has attracted a lot of attentions in recent years. Particularly, such theory has achieved great progresses after the notable work of Kováčik and Rákosník [13] in 1991 (see [3, 5] and the references therein). In 2010, Izuki [10] studied the Herz spaces with variable exponent and from [9, 10, 18, 20], we aim to develop the weak Herz spaces and the weak Herz-type Hardy spaces with variable exponent.

Recently, Yan et al. [20] introduced the variable weak Hardy spaces $WH^p(\mathbb{R}^n)$ and established some characterizations and the boundedness of some operators. Subsequently, Zhuo et al. [21] further studied the variable weak Hardy spaces.

Motivated by [9, 10, 18, 20], we aim to develop the weak Herz spaces and the weak Herz-type Hardy spaces in the setting of variable exponents. In Sect. 2, we first briefly recall some standard notations and lemmas in variable function spaces. Then the weak Herz spaces and the weak Herz-type Hardy spaces with variable exponent will be defined in this section. In Sect. 3, first we will prove the boundedness for a class of sublinear operators from $K^p_{q;(}) (\mathbb{R}^n)$ into the weak Herz spaces with variable exponent $WK^p_{q;(}(\mathbb{R}^n)$ (or $WK^p_{q;(}(\mathbb{R}^n)$). Then the boundedness of local Calderón–Zygmund type operators is bounded on $HK^p_{q;(}(\mathbb{R}^n)$ and from $HK^p_{q;(}(\mathbb{R}^n)$ into $WK^p_{q;(}(\mathbb{R}^n)$ (or $WK^p_{q;(}(\mathbb{R}^n)$) at the critical index will be established. Subsequently, we will show the standard Calderón–Zygmund operators is bounded on $HK^p_{q;(}(\mathbb{R}^n)$ and from $HK^p_{q;(}(\mathbb{R}^n)$ into the weak Herz-type Hardy spaces with variable exponent $WHK^p_{q;(}(\mathbb{R}^n)$ (or $WHK^p_{q;(}(\mathbb{R}^n)$) at the endpoint case.

In addition, we denote the Lebesgue measure and the characteristic function of a measurable set $A \subset \mathbb{R}^n$ by $|A|$ and $I_A$ respectively. The notation $f \approx g$ means that there exist constants $C_1, C_2 > 0$ such that $C_1 g \leq f \leq C_2 g$.

2 Weak Herz type spaces with variable exponent

In this section we first recall some basic definitions and properties of variable function spaces, and then introduce weak Herz spaces and weak Herz-type Hardy spaces with variable exponents. Let $E$ be a measurable set in $\mathbb{R}^n$ with $|E| > 0$. 

Birkhäuser
Definition 2.1 Let \( p(\cdot) : E \rightarrow [1, \infty) \) be a measurable function.

(i) The Lebesgue space with variable exponent \( L^{p(\cdot)}(E) \) is defined by

\[
L^{p(\cdot)}(E) := \left\{ f \text{ is measurable} : \int_E \left( \frac{|f(x)|}{\eta} \right)^{p(x)} \, dx < \infty \text{ for some constant } \eta > 0 \right\}.
\]

(ii) The space \( L^{p(\cdot)}_{\text{loc}}(E) \) is defined by

\[
L^{p(\cdot)}_{\text{loc}}(E) := \{ f : f \in L^{p(\cdot)}(F) \text{ for all compact subsets } F \subset E \}.
\]

\( L^{p(\cdot)}(E) \) is a Banach function space when it is equipped with the Luxemburg-Nakano norm

\[
\|f\|_{L^{p(\cdot)}(E)} = \inf \left\{ \eta > 0 : \int_E \left( \frac{|f(x)|}{\eta} \right)^{p(x)} \, dx \leq 1 \right\}.
\]

Define \( \mathcal{P}(E) \) to be the set of measurable functions \( p(\cdot) : E \rightarrow [1, \infty) \) such that

\[
p^- = \text{ess inf} \{ p(x) : x \in E \} > 1, \quad p^+ = \text{ess sup} \{ p(x) : x \in E \} < \infty.
\]

Denote \( p'(x) = p(x)/(p(x) - 1) \).

Let \( f \) be a locally integrable function. The Hardy–Littlewood maximal operator is defined by

\[
Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{B \cap E} |f(y)| \, dy,
\]

where the supremum is taken over all balls \( B \) containing \( x \). Let \( \mathcal{B}(E) \) be the set of \( p(\cdot) \in \mathcal{P}(E) \) such that \( M \) is bounded on \( L^{p(\cdot)}(E) \).

In variable \( L^p \) spaces there are some important lemmas as follows.

Lemma 2.1 [2] Given an open set \( E \subset \mathbb{R}^n \). If \( p(\cdot) \in \mathcal{P}(E) \) and satisfies

\[
|p(x) - p(y)| \leq \frac{C}{-\log(|x - y|)}, \quad |x - y| \leq 1/2
\]  

(2.1)

and

\[
|p(x) - p(y)| \leq \frac{C}{\log(|x| + e)}, \quad |y| \geq |x|,
\]

(2.2)

then \( p(\cdot) \in \mathcal{B}(E) \), that is the Hardy–Littlewood maximal operator \( M \) is bounded on \( L^{p(\cdot)}(E) \).

Lemma 2.2 [13] (Generalized Hölder’s inequality) Given an open set \( E \subset \mathbb{R}^n \) and let \( p(\cdot) \in \mathcal{P}(E) \). If \( f \in L^{p(\cdot)}(E) \) and \( g \in L^{p'(\cdot)}(E) \), then \( fg \) is integrable on \( E \) and
\[
\int_E |f(x)g(x)|dx \leq r_p \|f\|_{L^{p'}(E)} \|g\|_{L^{p'}(E)},
\]

where
\[
r_p = 1 + 1/p^--1/p^+.
\]

**Lemma 2.3** [10] Let \(q(\cdot) \in \mathcal{B}(\mathbb{R}^n)\). Then there exists a positive constant \(C\) such that for all balls \(B\) in \(\mathbb{R}^n\) and all measurable subsets \(S \subset B\),
\[
\frac{\|X_B\|_{L^{p'}(\mathbb{R}^n)}}{\|X_S\|_{L^{p'}(\mathbb{R}^n)}} \leq C \frac{|B|}{|S|}, \quad \frac{\|X_S\|_{L^{p'}(\mathbb{R}^n)}}{\|X_B\|_{L^{p'}(\mathbb{R}^n)}} \leq C \left( \frac{|S|}{|B|} \right)^{\delta_1} \quad \text{and} \quad \frac{\|X_S\|_{L^{p'}(\mathbb{R}^n)}}{\|X_B\|_{L^{p'}(\mathbb{R}^n)}} \leq C \left( \frac{|S|}{|B|} \right)^{\delta_2},
\]

where \(0 < \delta_1, \delta_2 < 1\) depend on \(q(\cdot)\).

Throughout this paper \(\delta_2\) is the same as in Lemma 2.3.

**Lemma 2.4** [10] Suppose \(q(\cdot) \in \mathcal{B}(\mathbb{R}^n)\). Then there exists a positive constant \(C\) such that for all balls \(B\) in \(\mathbb{R}^n\),
\[
\frac{1}{|B|} \|X_B\|_{L^{p'}(\mathbb{R}^n)} \|X_B\|_{L^{p'}(\mathbb{R}^n)} \leq C.
\]

**Lemma 2.5** [3] Given \(E\) and \(q(\cdot) \in \mathcal{P}(E)\), let \(f : E \times E \to \mathbb{R}\) be a measurable function (with respect to product measure) such that for almost every \(y \in E, f(\cdot, y) \in L^{q(\cdot)}(E)\). Then
\[
\left\| \int_E f(\cdot, y)dy \right\|_{L^{q(\cdot)}(E)} \leq C \int_E \|f(\cdot, y)\|_{L^{q(\cdot)}(E)}dy.
\]

Now we recall the definition of the Herz-type spaces with variable exponent. Let \(B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}\) and \(A_k = B_k \setminus B_{k-1}\) for \(k \in \mathbb{Z}\). Denote \(\mathbb{Z}_+\) and \(\mathbb{N}\) as the sets of all positive and non-negative integers, \(\chi_k = \chi_{A_k}\) for \(k \in \mathbb{Z}\), \(\tilde{\chi}_k = \chi_k\) if \(k \in \mathbb{Z}_+\) and \(\tilde{\chi}_0 = \chi_{B_0}\).

**Definition 2.2** [10] Let \(\alpha \in \mathbb{R}, 0 < p \leq \infty\) and \(q(\cdot) \in \mathcal{P}(\mathbb{R}^n)\). The homogeneous Herz space with variable exponent \(K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)\) is defined by
\[
K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},
\]

where
\[
\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{kap} \|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p}.
\]

The non-homogeneous Herz space with variable exponent \(K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)\) is defined by
\[ K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) = \{ f \in L_{\text{loc}}^{q(\cdot)}(\mathbb{R}^n) : \| f \|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} < \infty \}, \]

where

\[
\| f \|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} = \left\{ \sum_{k=0}^{\infty} 2^{kp} \| f \varphi_k \|_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p}.
\]

In [18], the authors gave the definition of Herz-type Hardy space with variable exponent \( H\hat{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) \) and the atomic decomposition characterizations. \( \mathcal{S}(\mathbb{R}^n) \) denotes the space of Schwartz functions, and \( \mathcal{S}'(\mathbb{R}^n) \) denotes the dual space of \( \mathcal{S}(\mathbb{R}^n) \). Let \( G_N(f)(x) \) be the grand maximal function of \( f(x) \) defined by

\[ G_N(f)(x) = \sup_{\phi \in \mathcal{A}_N} |\phi_N^*(f)(x)|, \]

where \( \mathcal{A}_N = \{ \phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta| \leq N} |x^\alpha D^\beta \phi(x)| \leq 1 \} \) and \( N > n + 1 \), \( \phi_N^* \) is the non-tangential maximal operator defined by

\[ \phi_N^*(f)(x) = \sup_{|y-x| < t} |\phi_t * f(y)| \]

with \( \phi_t(x) = t^{-n} \phi(x/t) \).

**Definition 2.3** [18] Let \( \alpha \in \mathbb{R}, 0 < p < \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) and \( N > n + 1 \).

(i) The homogeneous Herz-type Hardy space with variable exponent \( H\hat{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) \) is defined by

\[ H\hat{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : G_N(f) \in K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) \right\} \]

and \( \| f \|_{H\hat{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} = \| G_N(f) \|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \).

(ii) The non-homogeneous Herz-type Hardy space with variable exponent \( H\hat{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) \) is defined by

\[ H\hat{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : G_N(f) \in K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) \right\} \]

and \( \| f \|_{H\hat{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} = \| G_N(f) \|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \).

For \( \alpha \in \mathbb{R} \) we denote by \( [\alpha] \) the largest integer less than or equal to \( \alpha \).

**Definition 2.4** [18] Let \( n\delta_2 \leq \alpha < \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \), and non-negative integer \( s \geq [\alpha - n\delta_2] \).

(i) A function \( a(x) \) on \( \mathbb{R}^n \) is said to be a central \((\alpha, q(\cdot))-\)atom, if it satisfies

1. \( \text{supp} \, a \subset B(0, r) = \{ x \in \mathbb{R}^n : |x| < r \} \).
2. \( \| a \|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq |B(0, r)|^{-\alpha/n} \).
3. \( \int_{\mathbb{R}^n} a(x) x^\beta dx = 0, \forall \beta \in \mathbb{Z}_+^n \) with \( |\beta| \leq s \).
(ii) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot))$-atom of restricted type, if it satisfies the conditions (2), (3) as above and

$$\supp a \subset B(0, r), r \geq 1.$$ 

If $r = 2^k$ for some $k \in \mathbb{Z}$ in Definition 2.4, then the corresponding central $(\alpha, q(\cdot))$-atom is called a dyadic central $(\alpha, q(\cdot))$-atom.

**Lemma 2.6** [18] Let $n\delta_2 \leq \alpha < \infty$, $0 < p < \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then $f \in HK^{\alpha, p}_{q(\cdot)}(\mathbb{R}^n)$ (or $HK^{\alpha, p}_{q(\cdot)}(\mathbb{R}^n)$) if and only if $f$ satisfies the conditions (2), (3) as above and

$$f = \sum_{k = -\infty}^{\infty} \lambda_k a_k \left( \text{or} \sum_{k = 0}^{\infty} \lambda_k a_k \right),$$

in the sense of $\mathcal{S'}(\mathbb{R}^n)$, where each $a_k$ is a central $(\alpha, q(\cdot))$-atom (or central $(\alpha, q(\cdot))$-atom of restricted type) with support contained in $B_k$ and $\sum_{k = -\infty}^{0} |\lambda_k|^p < \infty$ (or $\sum_{k = 0}^{\infty} |\lambda_k|^p < \infty$). Moreover,

$$\|f\|_{HK^{\alpha, p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left( \sum_{k = -\infty}^{\infty} |\lambda_k|^p \right)^{1/p} \quad \text{or} \quad \|f\|_{HK^{\alpha, p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left( \sum_{k = 0}^{\infty} |\lambda_k|^p \right)^{1/p},$$

where the infimum is taken over all the decompositions of $f$ as above.

Next we will give the definitions of the weak Herz spaces and the weak Herz-type Hardy spaces with variable exponents.

**Definition 2.5** Let $\alpha \in \mathbb{R}$, $0 < p \leq \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. A measurable function $f(x)$ on $\mathbb{R}^n$ is said to belong to the homogeneous weak Herz space with variable exponent $WK^{\alpha, p}_{q(\cdot)}(\mathbb{R}^n)$, if

$$\|f\|_{WK^{\alpha, p}_{q(\cdot)}(\mathbb{R}^n)} = \sup_{\lambda > 0} \left\{ \sum_{k = -\infty}^{\infty} 2^{k\alpha p} \|\chi_{[x \in A_k : |f(x)| > \lambda]}\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p} < \infty,$$

where the usual modification is made when $p = \infty$.

A measurable function $f(x)$ on $\mathbb{R}^n$ is said to belong to the non-homogeneous weak Herz space with variable exponent $WK^{\alpha, p}_{q(\cdot)}(\mathbb{R}^n)$, if

$$\|f\|_{WK^{\alpha, p}_{q(\cdot)}(\mathbb{R}^n)} = \sup_{\lambda > 0} \left\{ \sum_{k = 0}^{\infty} 2^{k\alpha p} \|\chi_{[x \in A_k : |f(x)| > \lambda]}\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p} < \infty,$$

where the usual modification is made when $p = \infty$.

**Definition 2.6** Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $N > n + 1$. We define the spaces
\[
WHK_{q,C}^{\alpha,p}(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : G_N(f)(x) \in WK_{q,C}^{\alpha,p}(\mathbb{R}^n) \}
\]
and
\[
WHK_{q,C}^{\alpha,p}(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : G_N(f)(x) \in WK_{q,C}^{\alpha,p}(\mathbb{R}^n) \}.
\]

Moreover, we define that
\[
\|f\|_{WHK_{q,C}^{\alpha,p}(\mathbb{R}^n)} = \|G_N(f)\|_{WK_{q,C}^{\alpha,p}(\mathbb{R}^n)} \quad \text{and} \quad \|f\|_{WK_{q,C}^{\alpha,p}(\mathbb{R}^n)} = \|G_N(f)\|_{WK_{q,C}^{\alpha,p}(\mathbb{R}^n)}.
\]

**Remark 2.1** If \( q(\cdot) = q \) is a constant in Definitions 2.5 and 2.6, then we can easily get the classical cases of constant exponent.

**Remark 2.2** If \( \alpha = 0 \), then \( WL^{q(\cdot)}(\mathbb{R}^n) \subseteq WK_{q,C}^{0,p}(\mathbb{R}^n) \), where \( WL^{q(\cdot)}(\mathbb{R}^n) \) is the weak Lebesgue space with variable exponent and
\[
\|f\|_{WL^{q(\cdot)}(\mathbb{R}^n)} = \sup_{\lambda > 0} \lambda \| \chi_{\{x \in A_\lambda : |f(x)| > \lambda\}} \|_{L^{q(\cdot)}(\mathbb{R}^n)} < \infty.
\]

### 3 Boundedness of some singular integral operators

First we establish the boundedness for a class of operators on Herz type spaces with variable exponent. Let \( T \) be an operator defined on a linear space of all almost everywhere finite measurable functions on \( \mathbb{R}^n \). Then \( T \) is called a sublinear operator if
\[
|T(f_1 + f_2)(x)| \leq |Tf_1(x)| + |Tf_2(x)| \quad \text{and} \quad |T(\lambda f)| = |\lambda| |Tf|, \quad \lambda \in \mathbb{C}.
\]

In [20, Theorem 3.1], the authors proved that \( T \) is bounded on \( WL^{q(\cdot)}(\mathbb{R}^n) \) when it satisfies some conditions. Now we give the corresponding result about the sublinear operator \( T \) on Herz type spaces with variable exponent.

**Theorem 3.1** Let \( 0 < \alpha < n\delta_2, q(\cdot) \in \mathcal{B}(\mathbb{R}^n), p \in (0, 1] \). If \( T \) is a sublinear operator and bounded on \( WL^{q(\cdot)}(\mathbb{R}^n) \) satisfying
\[
|Tf(x)| \leq C|x|^{-n} \int_{\mathbb{R}^n} |f(y)| dy \tag{3.1}
\]
for any \( f \in L_1^{q(C)}(\mathbb{R}^n) \), \( \text{supp} \ f \subseteq B(0, r) \) and \( x \notin B(0, 2r) \), then \( T \) maps continuously \( K_{q,C}^{\alpha,p}(\mathbb{R}^n) \) (or \( K_{q,C}^{0,p}(\mathbb{R}^n) \)) into \( WK_{q,C}^{\alpha,p}(\mathbb{R}^n) \) (or \( WK_{q,C}^{0,p}(\mathbb{R}^n) \)).

**Proof** We only prove the homogeneous case. The non-homogeneous case can be proved in the same way. For any \( k \in \mathbb{Z} \), we decompose \( f \) into
\[
f(x) = f(x) \chi_{\{|x| \leq 2^{k-3}\}}(x) + f(x) \chi_{\{|x| > 2^{k-3}\}}(x) = f_1(x) + f_2(x).
\]
Then \( |Tf(x)| \leq |Tf_1(x)| + |Tf_2(x)| \), and

\begin{flushright}
\text{Birkhäuser}
\end{flushright}
\[
\|Tf\|_{W^a_p(\mathbb{R}^n)} = \sup_{\lambda > 0} \left\{ \sum_{k=-\infty}^{\infty} 2^{kap} \| \chi_{\{x \in A_k : |Tf(x)| > \lambda\}} \|_{L^p(\mathbb{R}^n)}^p \right\}^{1/p}
\]

\[
\leq C \sup_{\lambda > 0} \left\{ \sum_{k=-\infty}^{\infty} 2^{kap} \| \chi_{\{x \in A_k : |Tf(x)| > \lambda^{1/2} \}} \|_{L^p(\mathbb{R}^n)}^p \right\}^{1/p}
\]

\[
+ C \sup_{\lambda > 0} \left\{ \sum_{k=-\infty}^{\infty} 2^{kap} \| \chi_{\{x \in A_k : |Tf(x)| > \lambda^{1/2} \}} \|_{L^p(\mathbb{R}^n)}^p \right\}^{1/p}
\]

\[= I_1 + I_2.\]

Using the $WL^q(\mathbb{R}^n)$-boundedness of $T$ and $0 < p \leq 1$, we have

\[
I_2 \leq C \left\{ \sum_{k=-\infty}^{\infty} 2^{kap} \| T_{f_2} \chi_k \|_{WL^q(\mathbb{R}^n)}^p \right\}^{1/p}
\]

\[
\leq C \left\{ \sum_{k=-\infty}^{\infty} 2^{kap} \left( \sum_{l=-\infty}^{\infty} \| f \chi_l \|_{L^q(\mathbb{R}^n)} \right)^p \right\}^{1/p}
\]

\[
\leq C \left\{ \sum_{l=-\infty}^{\infty} \| f \chi_l \|_{L^q(\mathbb{R}^n)}^p \left( \sum_{k=-\infty}^{\infty} 2^{kap} \right) \right\}^{1/p}
\]

\[
\leq C \left\{ \sum_{l=-\infty}^{\infty} 2^{lap} \| f \chi_l \|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p}
\]

\[= C \| f \|_{K^a_p(\mathbb{R}^n)}.\]

For $I_1$, noting that $x \in A_k$ and $\text{supp } f_1 \subset \{ x \in \mathbb{R}^n : |x| \leq 2^{k-3} \}$, and by (3.1) and the generalized Hölder inequality, we have

\[
|Tf_1(x)| \leq C|x|^{-n} \| f_1 \|_{L^1(\mathbb{R}^n)}
\]

\[
\leq C 2^{-kn} \sum_{l=-\infty}^{\infty} \| f \chi_l \|_{L^1(\mathbb{R}^n)}
\]

\[
\leq C 2^{-kn} \sum_{l=-\infty}^{\infty} \| f \chi_l \|_{L^q(\mathbb{R}^n)} \| \chi_l \|_{L^{q'}(\mathbb{R}^n)}.\]

So by Lemmas 2.3 and 2.4 we have
This completes the proof of Theorem 3.1.

\[ \| \chi_{\{x \in A_k : |Tf(x)| > \frac{1}{2} \}} \|_{L^p(\mathbb{R}^n)} \leq \| \chi_{\{x \in A_k : C^{2-kn} \sum_{k=0}^{2} \| f \chi_k \|_{L^p(\mathbb{R}^n)} \| \chi \|_{L^p(\mathbb{R}^n)} > \frac{1}{2} \}} \|_{L^p(\mathbb{R}^n)} \leq C 2^{2-kn} \lambda^{-1} \sum_{l=-\infty}^{k-3} \| f \chi_l \|_{L^p(\mathbb{R}^n)} \| \chi \|_{L^p(\mathbb{R}^n)} \leq C 2^{2-kn} \lambda^{-1} \sum_{l=-\infty}^{k-3} \| f \chi_l \|_{L^p(\mathbb{R}^n)} \| \chi \|_{L^p(\mathbb{R}^n)} \leq C \lambda^{-1} \sum_{l=-\infty}^{k-3} \| f \chi_l \|_{L^p(\mathbb{R}^n)} \| \chi \|_{L^p(\mathbb{R}^n)} \leq 2^{(l-k)n \delta_2} . \]

Thus by \( 0 < p \leq 1 \) and \( 0 < \alpha < n \delta_2 \), we obtain

\[ I_1 = C \sup_{\lambda > 0} \lambda \left\{ \sum_{k=0}^{2^{kn}} \sum_{l=-\infty}^{k-3} \| f \chi_l \|_{L^p(\mathbb{R}^n)} \| \chi \|_{L^p(\mathbb{R}^n)} \right\} \leq C \left\{ \sum_{k=0}^{2^{kn}} \sum_{l=-\infty}^{k-3} \| f \chi_l \|_{L^p(\mathbb{R}^n)} \| \chi \|_{L^p(\mathbb{R}^n)} \leq 2^{2ap} \left( \sum_{k=0}^{2^{kn}} \| f \chi_l \|_{L^p(\mathbb{R}^n)} \right)^{1/p} \right\} \leq C \left\{ \sum_{l=-\infty}^{2^{kn}} \| f \chi_l \|_{L^p(\mathbb{R}^n)} \right\} \leq C \| f \|_{K^{a,p}_q(\mathbb{R}^n)} . \]

This completes the proof of Theorem 3.1.

\[ |Tf(x)| \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^a} dy, \quad x \notin \text{supp} f, \quad (3.2) \]

then the conclusions of Theorem 3.1 are still true.

Remark 3.1 If the condition \( (3.1) \) is substituted by the following condition

\[ \| f \|_{K^{a,p}_q(\mathbb{R}^n)} \]

then the conclusions of Theorem 3.1 are still true.

Remark 3.2 The condition \( (3.1) \) is very weak and many classical operators satisfy \( (3.1) \), such as Calderón–Zygmund operators, multipliers and oscillatory singular integrals, Bochner-Riesz operators at the critical index and so on.

Remark 3.3 Similar to the method of [9], we can easily obtain that the Riesz transforms cannot map continuously \( K^{a,p}_q(\mathbb{R}^n) \) (or \( K^{a,p}_q(\mathbb{R}^n) \)) into \( W^{a,p}_q(\mathbb{R}^n) \) (or \( W^{a,p}_q(\mathbb{R}^n) \)) if \( \alpha \geq n \delta_2 \). Thus we are more interested in the case \( \alpha \geq n \delta_2 \).

Now we turn our attention to the behaviours of local Calderón–Zygmund type operators on the Herz-type Hardy spaces with variable exponent.
Theorem 3.2 Let $T$ defined by $Tf(x) = \int_{\mathbb{R}^n} k(x,y)f(y)dy$ be a linear and continuous operator on $L^2(\mathbb{R}^n)$. Suppose that the distribution kernel of $T$ coincides in the complement of the diagonal with a locally integrable function $k(x,y)$ which satisfies

$$\sup_{y \in B_k} \| [(k(\cdot, y) - k(\cdot, 0))X_l]_{L^2(\mathbb{R}^n)} \leq C 2^{(k-l)\delta - l\beta} \| X_l \|_{L^2(\mathbb{R}^n)}$$

(3.3)

for $k, l \in \mathbb{Z}$, some $\delta \in (0, 1]$ and some $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Suppose that $T$ can be extended to a continuous operator on $L^q(\cdot)(\mathbb{R}^n)$. If $n\delta_2 \leq \alpha < n\delta_2 + \delta$ and $0 < p \leq \infty$, then $T$ maps continuously $H^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ into $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Remark 3.4 If the condition (3.3) is true only for $k \in \mathbb{N}$ and $l \in \mathbb{Z}_+$, then the operator $T$ in Theorem 3.2 maps continuously $H^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ into $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Remark 3.5 Assuming more regularity on the kernel $k(x,y)$, we can extend Theorem 3.2 to a larger range of $\alpha$.

Proof of Theorem 3.2 First we suppose that $0 < p \leq 1$. In this case, we only need to prove $\| T_a \|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \leq C$, where $a_k$ is a dyadic central $(\alpha, q(\cdot))$-atom with the support $B_k$ and $C$ is independent of $k$. Write

$$\| T_a \|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \leq C \left\{ \sum_{l = -\infty}^{k+3} 2^{l\alpha p} \| (T_a) X_l \|_{L^p(\mathbb{R}^n)} \right\}^{1/p} + C \left\{ \sum_{l = k+4}^{\infty} 2^{l\alpha p} \| (T_a) X_l \|_{L^p(\mathbb{R}^n)} \right\}^{1/p} = J_1 + J_2.$$

Since $T$ is bounded on $L^q(\cdot)(\mathbb{R}^n)$, by Definition 2.4 we have

$$J_1 \leq C \left\{ \sum_{l = -\infty}^{k+3} 2^{l\alpha p} \| T_a \|_{L^p(\mathbb{R}^n)} \right\}^{1/p} \leq C \left\{ \sum_{l = -\infty}^{k+3} 2^{l\alpha p} \| a_k \|_{L^p(\mathbb{R}^n)} \right\}^{1/p} \leq C 2^{-k\alpha} \left\{ \sum_{l = -\infty}^{k+3} 2^{l\alpha} \right\}^{1/p} \leq C,$$

where $C$ is independent of $k$.

For $J_2$, using (3.3), Lemmas 2.3, 2.4, 2.5 and the generalized Hölder inequality we have
\begin{align*}
\|(Tak)x\|_{L^q(\mathbb{R}^n)} & \leq C \int_{B_k} \| [k(\cdot, y) - k(\cdot, 0)] x_l \|_{L^q(\mathbb{R}^n)} |a_k(y)| dy \\
& \leq C 2^{(k-\delta-k\alpha)l} \| x_l \|_{L^q(\mathbb{R}^n)} \| a_k \|_{L^{\infty}(\mathbb{R}^n)} \| x_k \|_{L^q(\mathbb{R}^n)} \\
& \leq C 2^{(k-\delta-k\alpha)l} \| x_k \|_{L^q(\mathbb{R}^n)} \| x_k \|_{L^q(\mathbb{R}^n)} \\
& \leq C 2^{(k-\delta+n\delta_2-k\alpha)}.
\end{align*}

Thus by $\alpha < n\delta_2 + \delta$, we obtain

$$J_2 \leq C \left\{ \sum_{l=k+4}^{\infty} 2^{(l-k)(\alpha-n\delta_2)p} \right\}^{1/p} \leq C,$$

where $C$ is independent of $k$. This finishes the proof for the case $0 < p \leq 1$. Now let $1 < p < \infty$ and $f \in H^{a,p}_{q,(\mathbb{R}^n)}$. By Lemma 2.6 we get $f = \sum_{k=-\infty}^{\infty} \hat{f}_k a_k$, where $\|f\|_{H^{a,p}_{q,(\mathbb{R}^n)}} \approx \inf(\sum_{k=-\infty}^{\infty} |\hat{f}_k|^{p})^{1/p}$ (the infimum is taken over all the decompositions of $f$ as above), and $a_k$ is a dyadic central $(\alpha, q(\cdot))$-atom with the support $B_k$. Write

$$\|Tf\|_{L^q_{\psi,p}(\mathbb{R}^n)} \leq C \left\{ \sum_{l=-\infty}^{\infty} 2^{\text{lap}} \left( \sum_{k=-\infty}^{l-4} |\hat{f}_k| \| (Tak)x_l \|_{L^q(\mathbb{R}^n)} \right)^p \right\}^{1/p} + C \left\{ \sum_{l=-\infty}^{\infty} 2^{\text{lap}} \left( \sum_{k=-\infty}^{\infty} |\hat{f}_k| \| (Tak)x_l \|_{L^q(\mathbb{R}^n)} \right)^p \right\}^{1/p} = U_1 + U_2.$$

For $U_2$, by the Hölder inequality and the fact that $T$ is bounded on $L^q(\mathbb{R}^n)$, we have

$$U_2 \leq C \left\{ \sum_{l=-\infty}^{\infty} 2^{\text{lap}} \left( \sum_{k=-\infty}^{\infty} |\hat{f}_k| \| (Tak)x_l \|_{L^q(\mathbb{R}^n)} \right)^p \right\}^{1/p} \leq C \left\{ \sum_{l=-\infty}^{\infty} 2^{\text{lap}} \left( \sum_{k=-\infty}^{\infty} |\hat{f}_k| 2^{-ka} \right)^p \right\}^{1/p} \leq C \left\{ \sum_{l=-\infty}^{\infty} 2^{\text{lap}/2} \left( \sum_{k=-\infty}^{\infty} |\hat{f}_k| 2^{-kap/2} \right)^p \right\}^{1/p} \leq C \left\{ \sum_{k=-\infty}^{\infty} |\hat{f}_k|^p \right\}^{1/p} \leq C \|f\|_{H^{a,p}_{q,(\mathbb{R}^n)}}.$$

On the other hand, noting that $\alpha < n\delta_2 + \delta$. So by (3.3), Lemmas 2.3, 2.4, 2.5 and the generalized Hölder inequality we have
When \( \alpha = n\delta_2 + \delta \), we have the following result.

**Theorem 3.3** Let \( T \) defined by \( Tf(x) = \int_{\mathbb{R}^n} k(x, y) f(y) dy \) be a linear and bounded operator on \( L^2(\mathbb{R}^n) \). Suppose that the distribution kernel of \( T \) coincides in the complement of the diagonal with a locally integrable function \( k(x, y) \) satisfying

\[
|k(x, y) - k(x, 0)| \leq C \frac{|y|^\delta}{|x|^{n+\delta}}, \quad (3.4)
\]

if \( 2|y| < |x| \), for some \( \delta \in (0, 1) \). Suppose that \( T \) is bounded on \( L^{q(\cdot)}(\mathbb{R}^n) \) for some \( q(\cdot) \in \mathscr{B}(\mathbb{R}^n) \), \( \alpha = n\delta_2 + \delta \) and \( 0 < p \leq 1 \), then \( T \) maps continuously \( HK^{a,p,q(\cdot)}(\mathbb{R}^n) \) into \( WK^{a,p,q(\cdot)}(\mathbb{R}^n) \) (or \( WK^{a,p,q(\cdot)}(\mathbb{R}^n) \)).

**Proof** We only prove the homogeneous case. Let \( \alpha = n\delta_2 + \delta \) and \( f \in HK^{a,p,q(\cdot)}(\mathbb{R}^n) \). By Lemma 2.6 we get \( f = \sum_{l=-\infty}^{\infty} \lambda_l a_l \), where \( \|f\|_{HK^{a,p,q(\cdot)}(\mathbb{R}^n)} \approx \inf(\sum_{l=-\infty}^{\infty} |\lambda_l|^p)^{1/p} \) (the infimum is taken over all the decompositions of \( f \) as above), and \( a_l \) is a dyadic central \((\alpha, q(\cdot))\)-atom with the support \( B_l \). Given \( \lambda > 0 \), we can write

\[
\lambda \left\{ \sum_{k=-\infty}^{\infty} 2^{kap} \| \chi_{\{x \in A_k : |Tf(x)| > l\}} \|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p} \leq C \lambda \left\{ \sum_{k=-\infty}^{\infty} 2^{kap} \| \chi_{\{x \in A_k : |\sum_{l=-\infty}^{l-4} \lambda_l T_0(x)| > l/2 \}} \|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p}
\]

+ \( C \lambda \left\{ \sum_{k=-\infty}^{\infty} 2^{kap} \| \chi_{\{x \in A_k : |\sum_{l=-4}^{l-3} \lambda_l T_0(x)| > l/2 \}} \|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p} = CV_1 + CV_2.

For \( V_2 \), by the \( L^{q(\cdot)}(\mathbb{R}^n) \)-boundedness of \( T \), we have
\[
V_2 \leq C \left\{ \sum_{k=-\infty}^{\infty} 2^{k} \left| \sum_{l=k-3}^{\infty} |\lambda_l T_\alpha| \chi_k \right|^p \right\}^{1/p} \\
\leq C \left\{ \sum_{k=-\infty}^{\infty} 2^{k} \left( \sum_{l=k-3}^{\infty} |\lambda_l| \|a_l\|_{L^p(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
\leq C \left\{ \sum_{k=-\infty}^{\infty} \sum_{l=k-3}^{\infty} |\lambda_l|^p 2^{(k-l)p} \right\}^{1/p} \\
\leq C \left\{ \sum_{l=-\infty}^{\infty} |\lambda_l|^p \sum_{k=-\infty}^{l+3} 2^{(k-l)p} \right\}^{1/p} \\
\leq C \left\{ \sum_{l=-\infty}^{\infty} |\lambda_l|^p \right\}^{1/p} \leq C \|f\|_{HK^{p,s}(\mathbb{R}^n)}.
\]

To estimate \(V_1\), we notice that if \(x \in A_k\) and \(k \geq l + 4\), then by (3.4), \(\alpha = n\delta_2 + \delta\), Lemma 2.3 and the generalized Hölder inequality we have

\[
|T_\alpha(x)| \leq \int_{\mathbb{R}^n} |k(x, y) - k(x, 0)| |a_l(y)| dy \\
\leq C 2^{l - k(n + \delta)} \int_{\mathbb{R}^n} |a_l(y)| dy \\
\leq C 2^{l - k(n + \delta)} \|\chi_l\|_{L^p(\mathbb{R}^n)} \|X_k\|_{L^{p^*}(\mathbb{R}^n)} \\
\leq C 2^{(\delta - \alpha) - k(n + \delta)} \|\chi_l\|_{L^p(\mathbb{R}^n)} \|X_k\|_{L^{p^*}(\mathbb{R}^n)} \\
= C 2^{-(k + \delta + n\delta_2)} \|\chi_l\|_{L^{p^*}(\mathbb{R}^n)}.
\]

So by \(0 < p \leq 1\) we have

\[
\left| \sum_{l=-\infty}^{k-4} \lambda_l T_\alpha(x) \right| \leq C 2^{-(k + \delta + n\delta_2)} \|\chi_k\|_{L^{p^*}(\mathbb{R}^n)} \sum_{l=-\infty}^{k-4} |\lambda_l| \\
\leq C 2^{-(k + \delta + n\delta_2)} \|\chi_k\|_{L^{p^*}(\mathbb{R}^n)} \left( \sum_{l=-\infty}^{k-4} |\lambda_l|^p \right)^{1/p} \\
\leq C_0 2^{-(k + \delta + n\delta_2)} \|\chi_k\|_{L^{p^*}(\mathbb{R}^n)} \|f\|_{HK^{p,s}(\mathbb{R}^n)}.
\]

If \(|x \in A_k : | \sum_{l=-\infty}^{k-4} \lambda_l T_\alpha(x) | > \frac{\lambda}{2} | \neq 0\), then

\[
\lambda \leq C_0 2^{-(k + \delta + n\delta_2)} \|\chi_k\|_{L^{p^*}(\mathbb{R}^n)} \|f\|_{HK^{p,s}(\mathbb{R}^n)}.
\]

For any given \(\lambda > 0\), let \(k_{\lambda}\) be the maximal positive integer such that

\[
2^{k_{\lambda}(n + \delta + n\delta_2)} \|\chi_{k_{\lambda}}\|_{L^{p^*}(\mathbb{R}^n)}^{-1} \leq 2C_0 \lambda^{-1} \|f\|_{HK^{p,s}(\mathbb{R}^n)}.
\]

Thus by \(\alpha = n\delta_2 + \delta\) and Lemma 2.4 we obtain

\[\Box\text{Birkhäuser}\]
By Lemma 2.6 we get

\[ \|Tf(x)\|_{L^q(\mathbb{R}^n)} \leq C \lambda \left\{ \sum_{k=-\infty}^{k_1} 2^{\lambda p} \|\chi_k\|_{L^p(\mathbb{R}^n)} \right\}^{1/p} \]

This completes the proof of Theorem 3.3.

**Remark 3.6** Similar to the method in the proof of Theorem 3.3, we can extend the result of Theorem 3.1 to the case \( \alpha = n\delta_2 \). Here we omit the details.

If the operator \( T \) in Theorem 3.2 is an operator of convolution type, then we can obtain the following stronger conclusion.

**Theorem 3.4** Let \( T \) defined by \( Tf(x) = p.v.(k * f)(x) \) be a bounded operator on \( L^{q(\cdot)}(\mathbb{R}^n) \) for some \( q(\cdot) \in \mathcal{B}(\mathbb{R}^n) \), and for some \( \delta \in (0, 1] \), the kernel \( k \) satisfies

\[ |k(x - y) - k(x)| \leq C \frac{|y|^\delta}{|x|^{n+\delta}}, \quad \text{if } |x| > 2|y|. \quad (3.5) \]

If \( n\delta_2 \leq \alpha < n\delta_2 + \delta \) and \( 0 < p < \infty \), then \( T \) can be extended to a bounded operator on \( HK^a_{q(\cdot)}(\mathbb{R}^n) \) (or \( HK^a_{q(\cdot)}(\mathbb{R}^n) \)).

Using the atomic and molecular theory for the spaces \( HK^a_{q(\cdot)}(\mathbb{R}^n) \) and \( HK^a_{q(\cdot)}(\mathbb{R}^n) \), we can prove Theorem 3.4 by a standard procedure (see [18]). Here we omit the details.

For the operator \( T \) in Theorem 3.4, we have the following result which is stronger than Theorem 3.3 with the end case \( \alpha = n\delta_2 + \delta \).

**Theorem 3.5** Let \( T \) be the same as in Theorem 3.4 with \( \delta \in (0, 1) \). If \( \alpha = n\delta_2 + \delta \), \( q(\cdot) \in \mathcal{B}(\mathbb{R}^n) \) and \( 0 < p < \infty \), then \( T \) maps continuously \( HK^a_{q(\cdot)}(\mathbb{R}^n) \) (or \( HK^a_{q(\cdot)}(\mathbb{R}^n) \)) into \( WHK^a_{q(\cdot)}(\mathbb{R}^n) \) (or \( WHK^a_{q(\cdot)}(\mathbb{R}^n) \)).

**Proof** We only prove the homogeneous case. Let \( \alpha = n\delta_2 + \delta \) and \( f \in HK^a_{q(\cdot)}(\mathbb{R}^n) \). By Lemma 2.6 we get \( f = \sum_{k=-\infty}^{\infty} \lambda_k d_k \), where \( \|f\|_{HK^a_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \sum_{k=-\infty}^{\infty} |\lambda_k|^p \) (the infimum is taken over all the decompositions of \( f \) as above), and \( d_k \) is a dyadic central \( (\alpha, q(\cdot)) \)-atom with the support \( B_k \). For a fixed \( \lambda > 0 \), we can write
Now we consider the term \( \sum_{j=-\infty}^{\infty} 2^{ijp} \| x_{x \in A_j : G_N(T_f)(x) > \lambda_j} \|_{L^p(R^n)} \)

\[
\lambda \left\{ \sum_{j=-\infty}^{\infty} 2^{ijp} \| x_{x \in A_j : \sum_{k=j-3}^{\infty} |\lambda_k| G_N(T_a_k)(x) > \lambda_j} \|_{L^p(R^n)} \right\}^{1/p} \\
\leq C \lambda \left\{ \sum_{j=-\infty}^{\infty} 2^{ijp} \| x_{x \in A_j : \sum_{k=j-3}^{\infty} |\lambda_k| G_N(T_a_k)(x) > \lambda_j} \|_{L^p(R^n)} \right\}^{1/p} \\
+ C \lambda \left\{ \sum_{j=-\infty}^{\infty} 2^{ijp} \| x_{x \in A_j : \sum_{k=j-3}^{\infty} |\lambda_k| G_N(T_a_k)(x) > \lambda_j} \|_{L^p(R^n)} \right\}^{1/p} \\
= E_1 + E_2.
\]

For \( E_2 \), by the \( L^{p,q}(R^n) \)-boundedness of \( G_N \) and \( T \), we have

\[
E_2 \leq C \left\{ \sum_{j=-\infty}^{\infty} 2^{ijp} \left( \sum_{k=j-3}^{\infty} |\lambda_k| \| G_N(T_a_k) \|_{L^p(R^n)} \right) \right\}^{1/p} \\
\leq C \left\{ \sum_{j=-\infty}^{\infty} 2^{ijp} \left( \sum_{k=j-3}^{\infty} |\lambda_k| \| G_N(T_a_k) \|_{L^p(R^n)} \right) \right\}^{1/p} \\
\leq C \left\{ \sum_{j=-\infty}^{\infty} 2^{ijp} \left( \sum_{k=j-3}^{\infty} 2^{-kq_p |\lambda_k|^p} \right) \right\}^{1/p} \\
\leq C \left\{ \sum_{k=-\infty}^{\infty} |\lambda_k|^p \right\} \leq C \| f \|_{HK^{q,p}_0(R^n)}.
\]

Now we consider the term \( E_1 \). We want to obtain the pointwise estimate for \( G_N(T_a_k)(x) \) with \( x \in A_j \) and \( k \leq j - 4 \). Note that \( \int_{R^n} T_a_k(x) dx = 0 \). Let \( |x - y| < t \) and write

\[
|(T_a_k \ast \phi_t)(y)| \leq \int_{R^n} \left| \int_{B_t} a_k(v) k(w-v) dv \right| t^{-n} |\phi(y-w) - \phi(y)| dw \\
\leq \int_{|w| < 2^{j+1}} \left| \int_{B_t} a_k(v) k(w-v) dv \right| t^{-n} |\phi(y-w) - \phi(y)| dw \\
+ \int_{2^{j+1} \leq |w| < |x|/2} \left| \int_{B_t} a_k(v) k(w-v) dv \right| t^{-n} |\phi(y-w) - \phi(y)| dw \\
+ \int_{|w| \geq |x|/2} \left| \int_{B_t} a_k(v) k(w-v) dv \right| t^{-n} |\phi(y-w) - \phi(y)| dw \\
=: F_1 + F_2 + F_3.
\]

For \( F_1 \), by Lemma 2.3, the generalized Hölder inequality and the mean value theorem, we obtain
\[ F_1 \leq C \| T a_k \|_{L^q(\mathbb{R}^n)} t^{-n} \left\| \left( \phi \left( \frac{y - \cdot}{t} \right) - \phi \left( \frac{\cdot}{t} \right) \right) \chi_{B_{k+1}} \right\|_{L^q(\mathbb{R}^n)} \\
\leq C \| a_k \|_{L^q(\mathbb{R}^n)} t^{-n} \left\| \sup_{|\beta| = 1} D^\beta \phi \left( \frac{y - \theta \cdot}{t} \right) \right\|_{L^q(\mathbb{R}^n)} | \theta |^{n+1} \chi_{B_k} \\\n\leq C \| a_k \|_{L^q(\mathbb{R}^n)} \left\| \left( |x - y| + |y - \theta \cdot| \right)^{n+1} \right\|_{L^q(\mathbb{R}^n)} \\
\leq C 2^{-a} \frac{1}{|x|^{n+1}} \left\| \chi_{B_k} \right\|_{L^q(\mathbb{R}^n)} \\
\leq C 2^{-a+1} \frac{1}{|x|^{n+\delta}} \left\| \chi_{B_k} \right\|_{L^q(\mathbb{R}^n)},
\]

where \( 0 < \theta < 1 \). For \( F_2 \), by (3.5) and the vanishing condition of \( a_k \) we have

\[ F_2 = \int_{2^{k+1} \leq |x| < 2^k} \left\| \int_{B_k} a_k(v) (k(w - v) - k(w)) dv \right\|_{L^q(\mathbb{R}^n)} t^{-n} \left\| \phi \left( \frac{y - w}{t} \right) - \phi \left( \frac{\cdot}{t} \right) \right\|_{L^q(\mathbb{R}^n)} dw \\
\leq C \int_{2^{k+1} \leq |x| < 2^k} \left( \int_{B_k} |a_k(v)| |(k(w - v) - k(w))| dv \right) \left\| \phi \left( \frac{y - w}{t} \right) - \phi \left( \frac{\cdot}{t} \right) \right\|_{L^q(\mathbb{R}^n)} |w| \, dw \\
\leq C 2^{k\delta - a} \| a_k \|_{L^q(\mathbb{R}^n)} \| \chi_{B_k} \|_{L^q(\mathbb{R}^n)} \frac{1}{|x|^{n+1}} \int_{2^{k+1} \leq |x| < 2^k} |w|^{1-n-\delta} \, dw \\
\leq C 2^{k\delta - a} \| \chi_{B_k} \|_{L^q(\mathbb{R}^n)} \frac{1}{|x|^{n+1}} \left\| \chi_{B_k} \right\|_{L^q(\mathbb{R}^n)},
\]

where \( 0 < \theta < 1 \). By (3.5) and the vanishing condition of \( a_k \) we can obtain

\[ F_3 \leq \int_{|w| \geq |x|/2} \left\| \int_{B_k} a_k(v)(k(w - v) - k(w)) dv \right\|_{L^q(\mathbb{R}^n)} t^{-n} \left( \left\| \phi \left( \frac{y - w}{t} \right) \right\|_{L^q(\mathbb{R}^n)} + \left\| \phi \left( \frac{\cdot}{t} \right) \right\|_{L^q(\mathbb{R}^n)} \right) \, dw \\
\leq C 2^{k\delta - a} \| \chi_{B_k} \|_{L^q(\mathbb{R}^n)} \int_{|w| \geq |x|/2} |w|^{-n-\delta} t^{-n} \left( \left\| \phi \left( \frac{y - w}{t} \right) \right\|_{L^q(\mathbb{R}^n)} + \left\| \phi \left( \frac{\cdot}{t} \right) \right\|_{L^q(\mathbb{R}^n)} \right) \, dw \\
\leq C 2^{k\delta - a} \| \chi_{B_k} \|_{L^q(\mathbb{R}^n)} \left( |x|^{-n-\delta} + |x|^{-n} \int_{|w| \geq |x|/2} |w|^{-n-\delta} \, dw \right) \\
\leq C 2^{k\delta - a} \frac{1}{|x|^{n+\delta}} \left\| \chi_{B_k} \right\|_{L^q(\mathbb{R}^n)},
\]

where we have invoked the fact that \( t + |y| > |x - y| + |y| > |x| \). Thus, if \( x \in A_j \) and \( k \leq j - 4 \), then we have

\[ G_N(Ta_k)(x) \leq C 2^{k\delta - a} \frac{1}{|x|^{n+\delta}} \left\| \chi_{B_k} \right\|_{L^q(\mathbb{R}^n)}.
\]

Therefore, we obtain
Let \( j \leq \sum_{k=-\infty}^{j-4} |\lambda_k| G_N(T\alpha_k)(x) \leq C \sum_{k=-\infty}^{j-4} |\lambda_k| 2^{k^\delta-\alpha} \frac{1}{|x|^{\alpha+\delta}} \| \chi_{B_k} \|_{L^p((\mathbb{R}^n))} \).

If \(|x \in A_j : \sum_{k=-\infty}^{j-4} |\lambda_k| G_N(T\alpha_k)(x) > \lambda| \neq 0\), then by Lemma 2.3 and \( \alpha = n\delta_2 + \delta \) we have

\[
\lambda < C \sum_{k=-\infty}^{j-4} |\lambda_k| 2^{k^\delta-\alpha} \frac{1}{|x|^{\alpha+\delta}} \| \chi_{B_k} \|_{L^p((\mathbb{R}^n))} \\
\leq C \sum_{k=-\infty}^{j-4} |\lambda_k| 2^{j(n+\delta)} 2^{-j(n+\delta)} \| \chi_{B_k} \|_{L^p((\mathbb{R}^n))} \\
\leq C 2^{-j(n+\alpha)} \| \chi_{B_k} \|_{L^p((\mathbb{R}^n))} \sum_{k=-\infty}^{j-4} |\lambda_k| \\
\leq C 2^{-j(n+\alpha)} \| \chi_{B_k} \|_{L^p((\mathbb{R}^n))} \| f \|_{HK_{q,p}^{a,p}((\mathbb{R}^n))}.
\]

Let \( j_\lambda \) be the maximal integer such that the above inequality holds. Then by Lemma 2.4 we have

\[
E_1 \leq C \lambda \left( \sum_{j=-\infty}^{j_\lambda} 2^{j\alpha p} \| \chi_{B_j} \|_{L^p((\mathbb{R}^n))}^p \right)^{1/p} \\
\leq C \lambda \left( \sum_{j=-\infty}^{j_\lambda} 2^{j\alpha p} 2^{j\alpha p} \| \chi_{B_j} \|_{L^p((\mathbb{R}^n))}^p \right)^{1/p} \\
\leq C \| f \|_{HK_{q,p}^{a,p}((\mathbb{R}^n))}.
\]

This finishes the proof of Theorem 3.5.

In general, we have the following theorems. The proofs are similar. Here we omit the details.

**Theorem 3.6** Let \( q(\cdot) \in \mathcal{D}(\mathbb{R}^n), \alpha \geq n\delta_2, s = [\alpha + n\delta_2] \) and \( \varepsilon_0 = \alpha + n\delta_2 - s \). Suppose \( Tf(x) = \text{p.v.}(k * f)(x) \) is bounded on \( L^{q(\cdot)}((\mathbb{R}^n)) \) and the kernel \( k \) satisfies

\[
|D^J k(x - y)| < C|y|^\epsilon |x|^{n-\varepsilon}
\]

for all multi-index \( J \) with \( |J| = s \), some \( \epsilon > \epsilon_0 \) and \( |x| > 2|y| \). If \( 0 < p < \infty \), then \( T \) can be extended to a bounded operator on \( HK_{q,p}^{a,p}((\mathbb{R}^n)) \).

**Theorem 3.7** Let \( T \) be the same as in Theorem 3.6 with \( \epsilon = \epsilon_0 \) and \( \alpha > n\delta_2 \). If \( 0 < p \leq 1 \), then \( T \) maps continuously \( HK_{q,p}^{a,p}((\mathbb{R}^n)) \) into \( WHK_{q,p}^{a,p}((\mathbb{R}^n)) \).

**Acknowledgements** The authors are very grateful to the referees for their valuable comments. This work was supported by National Natural Science Foundation of China (Grant Nos. 11926343, 11926342, 11926342).
References

1. Chen, Y., Levin, S., Rao, M.: Variable exponent, linear growth functionals in image restoration. SIAM J. Appl. Math. 66, 1383–1406 (2006)
2. Cruz-Uribe, D., Fiorenza, A., Neugebauer, C.: The maximal function on variable $L^p$ spaces. Ann. Acad. Sci. Fenn. Math. 28, 223–238 (2003)
3. Cruz-Uribe, D., Fiorenza, A.: Variable Lebesgue Spaces: Foundations and Harmonic Analysis (Applied and Numerical Harmonic Analysis). Springer, Heidelberg (2013)
4. Cruz-Uribe, D., Wang, L.: Variable Hardy spaces. Indiana Univ. Math. J. 63, 447–493 (2014)
5. Diening, L., Harjulehto, P., Hästö, P., Růžička, M.: Lebesgue and Sobolev spaces with variable exponents. Lecture Notes in Math, vol. 2017. Springer, Heidelberg (2011)
6. Ferreira, L., Pérez-López, J.: Besov-weak-Herz spaces and global solutions for Navier–Stokes equations. Pacific J. Math. 296, 57–77 (2018)
7. Harjulehto, P., Hästö, P., Le, U.V., Nuortio, M.: Overview of differential equations with non-standard growth. Nonlinear Anal. 72, 4551–4574 (2010)
8. Hu, G., Lu, S., Yang, D.: The weak Herz spaces. J. Beijing Normal Univ. (Natur. Sci.) 33, 27–34 (1997)
9. Hu, G., Lu, S., Yang, D.: The applications of weak Herz spaces. Adv. Math. (China) 26, 417–428 (1997)
10. Izuki, M.: Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization. Anal. Math. 36, 33–50 (2010)
11. Kempka, H., Vybíral, J.: Lorentz spaces with variable exponents. Math. Nachr. 287, 938–954 (2014)
12. Komori, Y.: Weak type estimates for Calderón–Zygmund operators on Herz spaces at critical indexes. Math. Nachr. 259, 42–50 (2003)
13. Kováčik, O., Rákosník, J.: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czechoslovak Math. J. 41, 592–618 (1991)
14. Liu, L.: The inequalities of commutators on weak Herz spaces. J. Korean Math. Soc. 39, 899–912 (2002)
15. Nakai, E., Sawano, Y.: Hardy spaces with variable exponents and generalized Campanato spaces. J. Funct. Anal. 262, 3665–3748 (2012)
16. Růžička, M.: Electrorheological fluids: modeling and mathematical theory. Springer, Berlin (2000)
17. Tsutsui, Y.: The Navier–Stokes equations and weak Herz spaces. Adv. Differ. Equ. 16, 1049–1085 (2011)
18. Wang, H., Liu, Z.: The Herz-type Hardy spaces with variable exponent and their applications. Taiwanese J. Math. 16, 1363–1389 (2012)
19. Wang, H.: Some estimates of intrinsic square functions on weighted Herz-type Hardy spaces. J. Inequal. Appl. 20(62), 22 (2015)
20. Yan, X., Yang, D., Yuan, W., Zhuo, C.: Variable weak Hardy spaces and their applications. J. Funct. Anal. 271, 2822–2887 (2016)
21. Zhuo, C., Yang, D., Yuan, W.: Interpolation between $H^{p(.)}(\mathbb{R}^n)$ and $L^{\infty}(\mathbb{R}^n)$: real method. J. Geom. Anal. 28, 2288–2311 (2018)