MULTILINEAR $BMO$ ESTIMATES FOR THE COMMUTATORS 
OF MULTILINEAR FRACTIONAL MAXIMAL AND INTEGRAL 
OPERATORS ON THE PRODUCT GENERALIZED MORREY 
SPACES

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Abstract. In this paper, we establish multilinear $BMO$ estimates for commutators of multilinear fractional maximal and integral operators both on product generalized Morrey spaces and product generalized vanishing Morrey spaces, respectively. Similar results are still valid for commutators of multilinear maximal and singular integral operators.

1. Introduction and main results

Because of the need for study of the local behavior of solutions of second order elliptic partial differential equations (PDEs) and together with the now well-studied Sobolev Spaces, constitute a formidable three parameter family of spaces useful for proving regularity results for solutions to various PDEs, especially for non-linear elliptic systems, in 1938, Morrey [24] introduced the classical Morrey spaces $L^{p,\lambda}$ which naturally are generalizations of the classical Lebesgue spaces.

We will say that a function $f \in L^{p,\lambda} = L^{p,\lambda}(\mathbb{R}^n)$ if

\[
\sup_{x \in \mathbb{R}^n, r > 0} \left[ r^{-\lambda} \int_{B(x,r)} |f(y)|^p \, dy \right]^{1/p} < \infty.
\]

Here, $1 < p < \infty$ and $0 < \lambda < n$ and the quantity of (1.1) is the $(p, \lambda)$-Morrey norm, denoted by $\|f\|_{L^{p,\lambda}}$. In recent years, more and more researches focus on function spaces based on Morrey spaces to fill in some gaps in the theory of Morrey type spaces (see, for example, 10, 12, 13, 14, 15, 16, 18, 20, 26, 28, 32). Moreover, these spaces are useful in harmonic analysis and PDEs. But, this topic exceeds the scope of this paper. Thus, we omit the details here. On the other hand, the study of the operators of harmonic analysis in vanishing Morrey space, in fact has been almost not touched. A version of the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ where it is possible to approximate by "nice" functions is the so called vanishing Morrey space $V L^{p,\lambda}(\mathbb{R}^n)$ has been introduced by Vitanza in [29] and has been applied there to obtain a regularity result for elliptic PDEs. This is a subspace of functions in

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\[ L_{p,\lambda}(\mathbb{R}^n), \text{ which satisfies the condition } \]
\[
\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \left( \frac{t^{-\lambda}}{r^\frac{n}{p}} \int_{B(x,t)} |f(y)|^p \, dy \right)^{1/p} = 0, \]
where \( 1 < p < \infty \) and \( 0 < \lambda < n \) for brevity, so that

\[ V L_{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L_{p,\lambda}(\mathbb{R}^n) : \lim_{r \to 0} \sup_{x \in \mathbb{R}^n} t^{-\frac{\lambda}{p}} \| f \|_{L_p(B(x,t))} = 0 \right\}. \]

Later in [30] Vitanza has proved an existence theorem for a Dirichlet problem, under weaker assumptions than in [21] and a \( W^{3,2} \) regularity result assuming that the partial derivatives of the coefficients of the highest and lower order terms belong to vanishing Morrey spaces depending on the dimension. For the properties and applications of vanishing Morrey spaces, see also [1].

After studying Morrey spaces in detail, researchers have passed to the concept of generalized Morrey spaces. Firstly, motivated by the work of [24], Mizuhara [22] introduced generalized Morrey spaces \( M_{p,\varphi} \). Then, Guliyev [10] defined the generalized Morrey spaces \( M_{p,\varphi} \) with normalized norm as follows:

**Definition 1.** \([10]\) (generalized Morrey space) Let \( \varphi(x, r) \) be a positive measurable function on \( \mathbb{R}^n \times (0, \infty) \). If \( 0 < p < \infty \), then the generalized Morrey space \( M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n) \) is defined by

\[
\left\{ f \in L^\text{loc}_{p,\varphi}(\mathbb{R}^n) : \| f \|_{M_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} |B(x,r)|^{-\frac{1}{p}} \| f \|_{L_p(B(x,r))} < \infty \right\}.
\]

Obviously, the above definition recover the definition of \( L_{p,\lambda}(\mathbb{R}^n) \) if we choose \( \varphi(x, r) = r^{\frac{\lambda}{p}} \), that is

\[ L_{p,\lambda}(\mathbb{R}^n) = M_{p,\varphi}(\mathbb{R}^n) \big|_{\varphi(x, r) = r^{\frac{\lambda}{p}}}. \]

 Everywhere in the sequel we assume that \( \inf_{x \in \mathbb{R}^n, r > 0} \varphi(x, r) > 0 \) which makes the above spaces non-trivial, since the spaces of bounded functions are contained in these spaces. We point out that \( \varphi(x, r) \) is a measurable nonnegative function and no monotonicity type condition is imposed on these spaces.

In [10, 14, 18, 20, 22] and [28], the boundedness of the maximal operator and Calderón-Zygmund operator on the generalized Morrey spaces \( M_{p,\varphi} \) has been obtained, respectively.

For brevity, in the sequel we use the notations

\[
\mathcal{M}_{p,\varphi}(f; x, r) := \frac{|B(x,r)|^{-\frac{1}{p}} \| f \|_{L_p(B(x,r))}}{\varphi(x, r)}
\]

and

\[
\mathcal{M}^W_{p,\varphi}(f; x, r) := \frac{|B(x,r)|^{-\frac{1}{p}} \| f \|_{W_L p(B(x,r))}}{\varphi(x, r)}.
\]

In this paper, extending the definition of vanishing Morrey spaces [29], we introduce generalized vanishing Morrey spaces \( VM_{p,\varphi}(\mathbb{R}^n) \) with normalized norm in the following form.
Definition 2. (generalized vanishing Morrey space) The generalized vanishing Morrey space $V M_{p, \varphi}(\mathbb{R}^n)$ is defined by

$$\left\{ f \in \mathcal{M}_{p, \varphi}(\mathbb{R}^n) : \lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathcal{M}_{p, \varphi}(f; x, r) = 0 \right\}.$$ 

Everywhere in the sequel we assume that

$$\lim_{r \to 0} \inf_{x \in \mathbb{R}^n} \frac{1}{\varphi(x, r)} = 0,$$

and

$$\sup_{0 < r < \infty} \inf_{x \in \mathbb{R}^n} \frac{1}{\varphi(x, r)} < \infty,$$

which make the spaces $V M_{p, \varphi}(\mathbb{R}^n)$ non-trivial, because bounded functions with compact support belong to this space. The spaces $V M_{p, \varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{V M_{p, \varphi}} \equiv \|f\|_{\mathcal{M}_{p, \varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \mathcal{M}_{p, \varphi}(f; x, r).$$

The spaces $V M_{p, \varphi}(\mathbb{R}^n)$ are also closed subspaces of the Banach spaces $\mathcal{M}_{p, \varphi}(\mathbb{R}^n)$, which may be shown by standard means.

Furthermore, we have the following embeddings:

$$V M_{p, \varphi} \subset \mathcal{M}_{p, \varphi}, \quad \|f\|_{\mathcal{M}_{p, \varphi}} \leq \|f\|_{V M_{p, \varphi}}.$$ 

On the other hand, it is well known that, for the purpose of researching non-smoothness partial differential equation, mathematicians pay more attention to the singular integrals. Moreover, the fractional type operators and their weighted boundedness theory play important roles in harmonic analysis and other fields, and the multilinear operators arise in numerous situations involving product-like operations, see [2, 5, 6, 7, 8, 19, 23, 27] for instance.

First of all, we recall some basic properties and notations used in this paper. Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space of points $x = (x_1, \ldots, x_n)$ with norm $|x| = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2}$ and corresponding $m$-fold product spaces $(m \in \mathbb{N})$ be $(\mathbb{R}^n)^m = \mathbb{R}^n \times \cdots \times \mathbb{R}^n$. Let $B = B(x, r)$ denotes open ball centered at $x$ of radius $r$ for $x \in \mathbb{R}^n$ and $r > 0$ and $B^c(x, r)$ its complement. Also $|B(x, r)|$ is the Lebesgue measure of the ball $B(x, r)$ and $|B(x, r)| = v_n r^n$, where $v_n = |B(0, 1)|$. We also denote by $\overrightarrow{y} = (y_1, \ldots, y_m)$, $d\overrightarrow{y} = dy_1 \cdots dy_m$, and by $\overrightarrow{f}$ the $m$-tuple $(f_1, \ldots, f_m)$, $m, n$ the nonnegative integers with $n \geq 2, m \geq 1$.

Let $\overrightarrow{f} \in L_{p_1}^{\text{loc}}(\mathbb{R}^n) \times \cdots \times L_{p_m}^{\text{loc}}(\mathbb{R}^n)$. Then multi-sublinear fractional maximal operator $M_{\alpha}^{(m)}$ is defined by

$$M_{\alpha}^{(m)}(\overrightarrow{f})(x) = \sup_{t > 0} |B(x, t)|^\alpha \left[ \prod_{i=1}^{m} \frac{1}{|B(x, t)|} \int_{B(x, t)} |f_i(y_i)|^p \right] d\overrightarrow{y}, \quad 0 \leq \alpha < mn.$$ 

From definition, if $\alpha = 0$ then $M_{\alpha}^{(m)}$ is the multi-sublinear maximal operator $M^{(m)}$ and also; in the case of $m = 1$, $M_{\alpha}^{(m)}$ is the classical fractional maximal operator $M_{\alpha}$. 
The theory of multilinear Calderón-Zygmund singular integral operators, originated from the works of Coifman and Meyer’s [4], plays an important role in harmonic analysis. Its study has been attracting a lot of attention in the last few decades. A systematic analysis of many basic properties of such multilinear singular integral operators can be found in the articles by Coifman-Meyer [4], Grafakos-Torres [7, 8, 9], Chen et al. [2], Fu et al. [5].

Let $T^{(m)}$ ($m \in \mathbb{N}$) be a multilinear operator initially defined on the $m$-fold product of Schwartz spaces and taking values into the space of tempered distributions,

$$T^{(m)} : S(\mathbb{R}^n) \times \cdots \times S(\mathbb{R}^n) \to S(\mathbb{R}^n).$$

Following [7], recall that the $m$-linear Calderón-Zygmund operator $T^{(m)}$ ($m \in \mathbb{N}$) for test vector $\mathbf{f} = (f_1, \ldots, f_m)$ is defined by

$$T^{(m)}(\mathbf{f})(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) \left\{ \prod_{i=1}^m f_i(y_i) \right\} dy_1 \cdots dy_m, \quad x \notin \bigcap_{i=1}^m \text{supp} f_i,$$

where $K$ is an $m$-Calderón-Zygmund kernel which is a locally integrable function defined off the diagonal $y_i = y_1 = \cdots = y_m$ on $(\mathbb{R}^n)^{m+1}$ satisfying the following size estimate:

$$|K(x, y_1, \ldots, y_m)| \leq \frac{C}{|(x - y_1, \ldots, x - y_m)|^m},$$

for some $C > 0$ and some smoothness estimates, see [7 8 9] for details.

The result of Grafakos and Torres [7 9] shows that the multilinear Calderón-Zygmund operator is bounded on the product of Lebesgue spaces.

**Theorem 1.** [7 9] Let $T^{(m)}$ ($m \in \mathbb{N}$) be an $m$-linear Calderón-Zygmund operator. Then, for any numbers $1 \leq p_1, \ldots, p_m, p < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $T^{(m)}$ can be extended to a bounded operator from $L^{p_1} \times \cdots \times L^{p_m}$ into $L^p$, and bounded from $L^{1} \times \cdots \times L^{1}$ into $L^{\frac{1}{m}}$.

On the other hand, the multilinear fractional type operators are natural generalization of linear ones. Their earliest version was originated on the work of Grafakos [6] in 1992, in which he studied the multilinear maximal function and multilinear fractional integral defined by

$$M^{(m)}_\alpha(\mathbf{f})(x) = \sup_{r > 0} \frac{1}{r^{n-\alpha}} \int_{|y| < r} \left| \prod_{i=1}^m f_i(x - \theta_i y) \right| dy$$

and

$$I^{(m)}_\alpha(\mathbf{f})(x) = \int_{\mathbb{R}^n} \frac{1}{|y|^{n-\alpha}} \prod_{i=1}^m f_i(x - \theta_i y) dy,$$

where $\theta_i (i = 1, \ldots, m)$ are fixed distinct nonzero real numbers and $0 < \alpha < n$. We note that, if we simply take $m = 1$ and $\theta_i = 1$, then $M_\alpha$ and $I_\alpha$ are just the operators studied by Muckenhoupt and Wheeden in [25]. In this paper we deal with another kind of multilinear operator which was defined by Kenig and Stein [19] for $\mathbf{f} = (f_1, \ldots, f_m)$, which is called multilinear fractional integral operator as follows

$$I^{(m)}_\alpha(\mathbf{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{1}{|(x - y_1, \ldots, x - y_m)|^{m+n-\alpha}} \left\{ \prod_{i=1}^m f_i(y_i) \right\} d\mathbf{y},$$
Let us recall the definition of the space of \( \text{BMO} \).

**Definition 3.** [17] [18] The space \( \text{BMO}(\mathbb{R}^n) \) of functions of bounded mean oscillation consists of locally summable functions with finite semi-norm

\[
\|b\|_* \equiv \|b\|_{\text{BMO}} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |b(y) - b_{B(x, r)}| dy < \infty,
\]

where \( b_{B(x, r)} \) is the mean value of the function \( b \) on the ball \( B(x, r) \). The fact that precisely the mean value \( b_{B(x, r)} \) figures in (1.4) is inessential and one gets an equivalent seminorm if \( b_{B(x, r)} \) is replaced by an arbitrary constant \( c \):

\[
\|b\|_* \approx \sup \inf_{r > 0, c \in \mathbb{C}} \frac{1}{|B(x, r)|} \int_{B(x, r)} |b(y) - c| dy.
\]

Each bounded function \( b \in \text{BMO} \). Moreover, \( \text{BMO} \) contains unbounded functions, in fact \( \log|x| \) belongs to \( \text{BMO} \) but is not bounded, so \( L_\infty(\mathbb{R}^n) \subset \text{BMO}(\mathbb{R}^n) \).

In 1961 John and Nirenberg [17] established the following deep property of functions from \( \text{BMO} \).

**Theorem 2.** [17] If \( b \in \text{BMO}(\mathbb{R}^n) \) and \( B(x, r) \) is a ball, then

\[
\left| \{ x \in B(x, r) : |b(x) - b_{B(x, r)}| > \xi \} \right| \leq |B(x, r)| \exp \left( \frac{-\xi}{C\|b\|_*} \right), \quad \xi > 0,
\]

where \( C \) depends only on the dimension \( n \).
By Theorem 2, we can get the following results.

**Corollary 1.** \[17, 18\] Let \( b \in \text{BMO}(\mathbb{R}^n) \). Then, for any \( q > 1 \),

\[
\|b\|_\ast \approx \sup_{x \in \mathbb{R}^n, r > 0} \left( \frac{1}{|B(x, r)|} \int_{B(x, r)} |b(y) - b_{B(x, r)}|^p dy \right)^{\frac{1}{p}}
\]

is valid.

**Corollary 2.** \[17, 18\] Let \( b \in \text{BMO}(\mathbb{R}^n) \). Then there is a constant \( C > 0 \) such that

\[
|b_{B(x, r)} - b_{B(x, t)}| \leq C\|b\|_\ast \left(1 + \ln \frac{t}{r}\right)
\]

for \( 0 < 2r < t \), and for any \( q > 1 \), it is easy to see that

\[
\|b - (b)_B\|_{L_q(B)} \leq Cr^{\frac{n}{q}}\|b\|_\ast \left(1 + \ln \frac{t}{r}\right)
\]

where \( C \) is independent of \( b, x, r \) and \( t \).

Now inspired by Definition 3, we can give the definition of multilinear \( \text{BMO} \) (= \( \text{BMO}^m \)). Indeed in this paper we will consider a multilinear version (= multilinear \( \text{BMO} \) or \( \text{BMO}^m \)) of the \( \text{BMO} \).

**Definition 4.** We say that \( \vec{b} = (b_1, \ldots, b_m) \in \text{BMO}^m \) if

\[
\left\| \vec{b} \right\|_{\text{BMO}^m} = \sup_{x \in \mathbb{R}^n, r > 0} \prod_{i=1}^m \frac{1}{|B(x, r)|} \int_{B(x, r)} \left| b_i(y_i) - (b_i)_{B(x, r)} \right| dy_i < \infty,
\]

where

\[
(b_i)_{B(x, r)} = \frac{1}{|B(x, r)|} \int_{B(x, r)} b_i(y_i) dy_i.
\]

**Remark 1.** Notice that \( (\text{BMO})^m \) is contained in \( \text{BMO}^m \) and we have

\[
\left\| \vec{b} \right\|_{\text{BMO}^m} \leq \prod_{i=1}^m \|b_i\|_\ast,
\]

so

\( (\text{BMO})^m \subset \text{BMO}^m \)

is valid.

We now make some conventions. Throughout this paper, we use the symbol \( A \lesssim B \) to denote that there exists a positive constant \( C \) such that \( A \leq CB \). If \( A \lesssim B \) and \( B \lesssim A \), we then write \( A \approx B \) and say that \( A \) and \( B \) are equivalent. For a fixed \( p \in [1, \infty) \), \( p' \) denotes the dual or conjugate exponent of \( p \), namely, \( p' = \frac{p}{p-1} \) and we use the convention \( 1' = \infty \) and \( \infty' = 1 \).

**Remark 2.** Let \( 0 < \alpha < mn \) and \( 1 < p_i < \infty \) with \( \frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}, \frac{1}{q} = \frac{1}{p'} - \frac{\alpha}{mn}, \)

\[
\frac{1}{q} = \sum_{i=1}^m \frac{1}{p_i} = \frac{1}{p'} - \frac{\alpha}{mn}
\]

and \( \vec{b} = (b_1, \ldots, b_m) \in (\text{BMO})^m \) for \( i = 1, \ldots, m \). Then, from
Corollary [3] it is easy to see that

\begin{equation}
\prod_{i=1}^{m} \| b_i - (b_i)_B \|_{L_{q_i}(B)} \leq C \prod_{i=1}^{m} |B(x,r)|^{\frac{1}{q_i}} \| b_i \|_{\ast} \left( 1 + \ln \frac{t}{r} \right),
\end{equation}

and

\begin{equation}
\prod_{i=1}^{m} \| b_i - (b_i)_B \|_{L_{q_i}(2B)} \leq \prod_{i=1}^{m} \left( \| b_i - (b_i)_B \|_{L_{q_i}(2B)} + \| (b_i)_B - (b_i)_B \|_{L_{q_i}(2B)} \right)
\end{equation}

On the other hand, Xu [31] has established the boundedness of the commutators generated by \( m \)-linear Calderón-Zygmund singular integrals and RBMO functions with nonhomogeneity on the product of Lebesgue space. Inspired by [2, 3, 7, 9, 27], commutators \( T^{(m)}_b \) generated by \( m \)-linear Calderón-Zygmund operators \( T^{(m)} \) and bounded mean oscillation functions \( \overrightarrow{b} = (b_1, \ldots, b_m) \) is given by

\[ T^{(m)}_b \left( \overrightarrow{f} \right)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) \left[ \prod_{i=1}^{m} [b_i(x) - b_i(y_i)] f_i(y_i) \right] d\overrightarrow{y}, \]

where \( K(x, y_1, \ldots, y_m) \) is a \( m \)-linear Calderón-Zygmund kernel, \( b_i \in (BMO)^{i}(\mathbb{R}^n) \) for \( i = 1, \ldots, m \). Note that \( T_b \) is the special case of \( T^{(m)}_b \) with taking \( m = 1 \). Similarly, let \( b_i \) \((i = 1, \ldots, m)\) be a locally integrable functions on \( \mathbb{R}^n \), then the commutators generated by \( m \)-linear fractional integral operators and \( \overrightarrow{b} = (b_1, \ldots, b_m) \) is given by

\[ I^{(m)}_{\alpha, \overrightarrow{b}} \left( \overrightarrow{f} \right)(x) = \int_{(\mathbb{R}^n)^m} \frac{1}{|[x-y_1, \ldots, x-y_m]|^{mn-\alpha}} \left[ \prod_{i=1}^{m} [b_i(x) - b_i(y_i)] f_i(y_i) \right] d\overrightarrow{y}, \]

where \( 0 < \alpha < mn \), and \( f_i \) \((i = 1, \ldots, m)\) are suitable functions.

The commutators of a class of multi-sublinear maximal operators corresponding to \( T^{(m)}_b \) and \( I^{(m)}_{\alpha, \overrightarrow{b}} \) \((m \in \mathbb{N})\) above are, respectively, defined by

\[ M^{(m)}_{\overrightarrow{b}} \left( \overrightarrow{f} \right)(x) = \sup_{t > 0} \left[ \prod_{i=1}^{m} \frac{1}{|B(x,t)|} \int_{B(x,t)} \left[ \| b_i(x) - b_i(y_i) \| f_i(y_i) \right] d\overrightarrow{y}, \]

and

\[ M^{(m)}_{\alpha, \overrightarrow{b}} \left( \overrightarrow{f} \right)(x) = \sup_{t > 0} |B(x,t)|^\frac{\alpha}{m} \left[ \prod_{i=1}^{m} \frac{1}{|B(x,t)|} \int_{B(x,t)} \left[ \| b_i(x) - b_i(y_i) \| f_i(y_i) \right] d\overrightarrow{y}, \quad 0 \leq \alpha < mn. \]

The following result is known.

**Lemma 1.** [27] (Strong bounds of \( I^{(m)}_{\alpha, \overrightarrow{b}} \)) Let \( 0 < \alpha_i < n \), \( 1 < p_1, \ldots, p_m < \infty \),

\[ \frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i}, \quad \alpha = \sum_{i=1}^{m} \alpha_i \text{ and } \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{m}. \]

Then there is \( C > 0 \) independent of \( \overrightarrow{f} \) and
such that
\[
\left\| T^{(m)}_{\vec{b},\alpha} \left( \overrightarrow{f} \right) \right\|_{L_q(\mathbb{R}^n)} \leq C \prod_{i=1}^{m} \|b_i\|_{*} \|f_i\|_{L_{p_i}(\mathbb{R}^n)}.
\]

Using the idea in the proof of Lemma 3.2 in [15], we can obtain the following corollary.

**Corollary 3.** (Strong bounds of $M^{(m)}_{\alpha, \vec{b}}$) Under the assumptions of Lemma 1, the operator $M^{(m)}_{\alpha, \vec{b}}$ is bounded from $L_{p_1}(\mathbb{R}^n) \times \cdots \times L_{p_m}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$. Moreover, we have
\[
\left\| M^{(m)}_{\alpha, \vec{b}} \left( \overrightarrow{f} \right) \right\|_{L_q(\mathbb{R}^n)} \leq C \prod_{i=1}^{m} \|b_i\|_{*} \|f_i\|_{L_{p_i}(\mathbb{R}^n)}.
\]

**Proof.** Set
\[
\bar{T}^{(m)}_{\vec{b},\alpha} (|f|) (x) = \int_{(\mathbb{R}^n)^m} \frac{1}{|x - y_1, \ldots, x - y_m|^m} \prod_{i=1}^{m} |b_i(x) - b_i(y_i)| |f_i(y_i)| \ d\vec{y} \quad 0 < \alpha < mn.
\]

It is easy to see that Lemma 1 is also hold for $\bar{T}^{(m)}_{\vec{b},\alpha}$. On the other hand, for any $t > 0$, we have
\[
\bar{T}^{(m)}_{\vec{b},\alpha} (|f|) (x) \geq \int_{(B(x,t))^m} \frac{1}{|x - y_1, \ldots, x - y_m|^m} \prod_{i=1}^{m} |b_i(x) - b_i(y_i)| |f_i(y_i)| \ d\vec{y} \geq \frac{1}{t^{mn-\alpha}} \int_{B(x,t)} \prod_{i=1}^{m} |b_i(x) - b_i(y_i)| |f_i(y_i)| \ d\vec{y}.
\]

Taking supremum over $t > 0$ in the above inequality, we get
\[
(1.10) \quad M^{(m)}_{\alpha, \vec{b}} \left( \overrightarrow{f} \right) (x) \leq C_{\alpha, \vec{b}}^{-1} \bar{T}^{(m)}_{\vec{b},\alpha} (|f|) (x) \quad C_{\alpha, \vec{b}} = |B(0,1)|^{-\frac{mn-\alpha}{\beta}}.
\]

As a simple corollary of Lemma 1 and Corollary 3, we can obtain the following result.

**Corollary 4.** (Strong bounds of $T^{(m)}_{\vec{b}}$ and $M^{(m)}_{\vec{b}}$) Let $1 < p_1, \ldots, p_m < \infty$ and $0 < p < \infty$ with $\frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i}$. Then there is $C > 0$ independent of $\vec{f}$ and $\vec{b}$ such that
\[
\left\| T^{(m)}_{\vec{b}} \left( \overrightarrow{f} \right) \right\|_{L_p(\mathbb{R}^n)} \leq C \prod_{i=1}^{m} \|b_i\|_{*} \|f_i\|_{L_{p_i}(\mathbb{R}^n)} \quad \left\| M^{(m)}_{\vec{b}} \left( \overrightarrow{f} \right) \right\|_{L_p(\mathbb{R}^n)} \leq C \prod_{i=1}^{m} \|b_i\|_{*} \|f_i\|_{L_{p_i}(\mathbb{R}^n)}.
\]

The purpose of this paper is to consider the mapping properties on $M_{p_1,\varphi_1} \times \cdots \times M_{p_m,\varphi_m}$ and $VM_{p_1,\varphi_1} \times \cdots \times VM_{p_m,\varphi_m}$ for the commutators of multilinear fractional maximal and integral operators, respectively. Similar results still hold for commutators of multilinear maximal and singular integral operators. Commutators
of multilinear fractional maximal and integral operators on product generalized Morrey spaces have not also been studied so far and this paper seems to be the first in this direction. Now, let us state the main results of this paper. Indeed our Morrey spaces have not also been studied so far and this paper seems to be the first in this direction. Now, let us state the main results of this paper. Indeed our Morrey spaces have not also been studied so far and this paper seems to be the first in this direction.

Corollary 5. Let \( (1.12) \)

\[
\sum_{i=1}^{m} \frac{1}{q_i} = \frac{\alpha}{n} \quad \text{and} \quad \vec{b} \in (BMO)^m(\mathbb{R}^n) \quad \text{for} \quad i = 1, \ldots, m. \quad \text{Let functions} \quad \varphi, \varphi_i : \mathbb{R}^n \times (0, \infty) \to (0, \infty) \quad (i = 1, \ldots, m) \quad \text{and} \quad (\varphi_1, \ldots, \varphi_m, \varphi) \quad \text{satisfies the condition}
\]

\[
(1.11)
\]

\[
\int_{r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^{m} \frac{\text{essinf}}{t} \prod_{i=1}^{m} \varphi_i(x, \tau)^{\tau^p} \quad dt \leq C \varphi(x, r),
\]

where \( C \) does not depend on \( x \in \mathbb{R}^n \) and \( r > 0 \).

Then, \( f^{(m)}_{\alpha, b} \) and \( M^{(m)}_{\alpha, b} \) \( (m \in \mathbb{N}) \) are bounded operators from product space \( M_{p_1, \varphi_1} \times \cdots \times M_{p_m, \varphi_m} \) to \( M_{p, \varphi} \). Moreover, we have

\[
(1.12) \quad \left\| f^{(m)}_{\alpha, b} (\vec{f}) \right\|_{M_{q, \varphi}} \leq \left\| \vec{b} \right\|_{BMO} \left\| f_{i} \right\|_{M_{p_i, \varphi_i}} \leq \prod_{i=1}^{m} \left\| b_i \right\|_{q_i} \left\| f_{i} \right\|_{M_{p_i, \varphi_i}},
\]

\[
(1.13) \quad \left\| M^{(m)}_{\alpha, b} (\vec{f}) \right\|_{M_{q, \varphi}} \leq \left\| \vec{b} \right\|_{BMO} \left\| f_{i} \right\|_{M_{p_i, \varphi_i}} \leq \prod_{i=1}^{m} \left\| b_i \right\|_{q_i} \left\| f_{i} \right\|_{M_{p_i, \varphi_i}}.
\]

Corollary 5. Let \( 1 < p_i < \infty \) and \( 0 < p < \infty \) with \( \frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i} \) and \( \vec{b} \in (BMO)^m(\mathbb{R}^n) \) for \( i = 1, \ldots, m. \) Let functions \( \varphi, \varphi_i : \mathbb{R}^n \times (0, \infty) \to (0, \infty) \) \( (i = 1, \ldots, m) \) and \( (\varphi_1, \ldots, \varphi_m, \varphi) \) satisfies the condition

\[
(1.11)
\]

\[
\int_{r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^{m} \frac{\text{essinf}}{t^{\tau^p}} \prod_{i=1}^{m} \varphi_i(x, \tau)^{\tau^p} \quad dt \leq C \varphi(x, r),
\]

where \( C \) does not depend on \( x \in \mathbb{R}^n \) and \( r > 0 \).

Then, \( T^{(m)}_{b} \) and \( M^{(m)}_{b} \) \( (m \in \mathbb{N}) \) are bounded operators from product space \( M_{p_1, \varphi_1} \times \cdots \times M_{p_m, \varphi_m} \) to \( M_{p, \varphi} \). Moreover, we have

\[
\left\| T^{(m)}_{b} (\vec{f}) \right\|_{M_{p, \varphi}} \leq \left\| \vec{b} \right\|_{BMO} \left\| f_{i} \right\|_{M_{p_i, \varphi_i}} \leq \prod_{i=1}^{m} \left\| b_i \right\|_{q_i} \left\| f_{i} \right\|_{M_{p_i, \varphi_i}},
\]

\[
\left\| M^{(m)}_{b} (\vec{f}) \right\|_{M_{p, \varphi}} \leq \left\| \vec{b} \right\|_{BMO} \left\| f_{i} \right\|_{M_{p_i, \varphi_i}} \leq \prod_{i=1}^{m} \left\| b_i \right\|_{q_i} \left\| f_{i} \right\|_{M_{p_i, \varphi_i}}.
\]

Our another main result is the following.
Theorem 4. Let $0 < \alpha < mn$ and $1 \leq p_i < \frac{mn}{\alpha}$ with $\frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i}$, $\frac{1}{q} = \sum_{i=1}^{m} \frac{1}{p_i} + \sum_{i=1}^{m} \frac{1}{q_i}$ and $\vec{b} \in (BMO)^m(\mathbb{R}^n)$ for $i = 1, \ldots, m$. Let functions $\varphi, \varphi_i : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ for every $i = 1, \ldots, m$ and $(\varphi_1, \ldots, \varphi_m, \varphi)$ satisfies conditions (L.2)-(L.3) and

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \prod_{i=1}^{m} \varphi_i(x, t) \frac{t^{\frac{n}{p}}}{\left(\frac{n}{p} - \sum_{i=1}^{m} \frac{1}{q_i}\right) + 1} \, dt \leq C_0 \varphi(x, r),$$

where $C_0$ does not depend on $x \in \mathbb{R}^n$ and $r > 0$.

$$\lim_{r \to 0} \inf_{x \in \mathbb{R}^n} \varphi(x, r) = 0$$

and

$$c_4 := \int_\delta^\infty \left(1 + \ln |t|\right)^m \sup_{x \in \mathbb{R}^n} \prod_{i=1}^{m} \varphi_i(x, t) \frac{t^{\frac{n}{p}}}{\left(\frac{n}{p} - \sum_{i=1}^{m} \frac{1}{q_i}\right) + 1} \, dt < \infty$$

for every $\delta > 0$.

Then, $I_{1, \varphi}^{(m)}$ and $M_{1, \varphi}^{(m)}$ ($m \in \mathbb{N}$) are bounded operators from product space $VM_{p_1, \varphi_1} \times \cdots \times VM_{p_m, \varphi_m}$ to $VM_{q, \varphi}$. Moreover, we have

$$\left\| I_{1, \varphi}^{(m)} \right\|_{VM_{q, \varphi}} \leq \left\| \vec{b} \right\|_{BMO} \left\| f_1 \right\|_{VM_{p_1, \varphi_1}} \leq \prod_{i=1}^{m} \left\| b_i \right\| \cdot \left\| f_i \right\|_{VM_{p_i, \varphi_i}},$$

$$\left\| M_{1, \varphi}^{(m)} \right\|_{VM_{q, \varphi}} \leq \left\| \vec{b} \right\|_{BMO} \left\| f_1 \right\|_{VM_{p_1, \varphi_1}} \leq \prod_{i=1}^{m} \left\| b_i \right\| \cdot \left\| f_i \right\|_{VM_{p_i, \varphi_i}}.$$

Corollary 6. Let $1 < p_i < \infty$ and $0 < p < \infty$ with $\frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i}$ and $\vec{b} \in (BMO)^m(\mathbb{R}^n)$ for $i = 1, \ldots, m$. Let functions $\varphi, \varphi_i : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ for every $i = 1, \ldots, m$ and $(\varphi_1, \ldots, \varphi_m, \varphi)$ satisfies conditions (L.2)-(L.3) and

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \prod_{i=1}^{m} \varphi_i(x, t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{p} + 1}} \, dt \leq C_0 \varphi(x, r),$$

where $C_0$ does not depend on $x \in \mathbb{R}^n$ and $r > 0$.

$$\lim_{r \to 0} \inf_{x \in \mathbb{R}^n} \varphi(x, r) = 0$$

and

$$c_4 := \int_\delta^\infty \left(1 + \ln |t|\right)^m \sup_{x \in \mathbb{R}^n} \prod_{i=1}^{m} \varphi_i(x, t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{p} + 1}} \, dt < \infty$$

for every $\delta > 0$. 
Lemma 2. Let \( x_0 \in \mathbb{R}^n \), \( 0 < \alpha < mn \) and \( 1 \leq p_i < \frac{mn}{\alpha} \) with \( \frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i} \), \( \frac{1}{q} = \sum_{i=1}^{m} \frac{1}{p_i} + \sum_{i=1}^{m} \frac{1}{q_i} - \frac{\alpha}{n} \) and \( \vec{b} \in (\text{BMO})^{m}(\mathbb{R}^n) \) for \( i = 1, \ldots, m \). Then the inequality
\[
\left\| T_{\vec{b}}^{(m)} (\vec{f}) \right\|_{L_{q}(B(x_0, r))} \lesssim \prod_{i=1}^{m} \| b_i \|_{\ast} \left( \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right) \frac{dt}{t} \right)^{1/n} \left( \sum_{i=1}^{m} \frac{1}{q_i} \right) + 1
\]
holds for any ball \( B(x_0, r) \) and for all \( \vec{f} \in I_{p_1}^{\text{loc}}(\mathbb{R}^n) \times \cdots \times I_{p_m}^{\text{loc}}(\mathbb{R}^n) \).

Proof. In order to simplify the proof, we consider only the situation when \( m = 2 \). Actually, a similar procedure works for all \( m \in \mathbb{N} \). Thus, without loss of generality, it is sufficient to show that the conclusion holds for \( I_{\alpha, \vec{b}}^{(2)} (\vec{f}) = I_{\alpha, (b_1, b_2)}^{(2)} (f_1, f_2) \).

We just consider the case \( p_i > 1 \) for \( i = 1, 2 \). For any \( x_0 \in \mathbb{R}^n \), set \( B = B(x_0, r) \) for the ball centered at \( x_0 \) and of radius \( r \) and \( 2B = B(x_0, 2r) \). Indeed, we also decompose \( f_i \) as \( f_i (y_i) = f_i (y_i) \chi_{2B} + f_i (y_i) \chi_{(2B)^c} \) for \( i = 1, 2 \). And, we write \( f_1 = f_1^0 + f_1^\infty \) and \( f_2 = f_2^0 + f_2^\infty \), where \( f_i^0 = f_i \chi_{2B}, \ f_i^\infty = f_i \chi_{(2B)^c} \), for \( i = 1, 2 \). Thus, we have
\[
\left\| I_{\alpha, (b_1, b_2)}^{(2)} (f_1, f_2) \right\|_{L_{q}(B(x_0, r))} \leq \left\| I_{\alpha, (b_1, b_2)}^{(2)} (f_1^0, f_2^0) \right\|_{L_{q}(B(x_0, r))} + \left\| I_{\alpha, (b_1, b_2)}^{(2)} (f_1^0, f_2^\infty) \right\|_{L_{q}(B(x_0, r))} + \left\| I_{\alpha, (b_1, b_2)}^{(2)} (f_1^\infty, f_2^0) \right\|_{L_{q}(B(x_0, r))} + \left\| I_{\alpha, (b_1, b_2)}^{(2)} (f_1^\infty, f_2^\infty) \right\|_{L_{q}(B(x_0, r))}
= F_1 + F_2 + F_3 + F_4.
\]
Firstly, we use the boundedness of $I_{\alpha,(b_1,b_2)}^{(2)}$ from $L_{p_1} \times L_{p_2}$ into $L_q$ (see Lemma 1) to estimate $F_1$, and we obtain

$$F_1 = \left\| I_{\alpha,(b_1,b_2)}^{(2)} \left( f_1^0, f_2^\infty \right) \right\|_{L_q(B(x_0,r))} \lesssim \prod_{i=1}^2 \left\| b_i \right\|_{L_{p_i}(2B)} \int_{2r}^\infty \frac{dt}{t^{\frac{n}{q}+1}}$$

$$\lesssim r^\frac{n}{2} \prod_{i=1}^2 \left\| b_i \right\|_{L_{p_i}(2B)} \int_0^\infty \frac{dt}{t^{\frac{n}{q}+1}}$$

$$\lesssim \prod_{i=1}^2 \left\| b_i \right\|_{L_{p_i}(2B)} \int_0^\infty \left(1 + \ln \frac{t}{r} \right)^2 \prod_{i=1}^2 \left\| f_i \right\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$ 

Secondly, for $F_2 = \left\| I_{\alpha,(b_1,b_2)}^{(2)} \left( f_1^0, f_2^\infty \right) \right\|_{L_q(B(x_0,r))}$, we decompose it into four parts as follows:

$$F_2 \lesssim \left\| \left[ (b_1 - \{b_1\}_B) \right] \left[ (b_2 - \{b_2\}_B) \right] I_{\alpha}^{(2)} \left( f_1^0, f_2^\infty \right) \right\|_{L_q(B(x_0,r))}$$

$$+ \left\| \left[ (b_1 - \{b_1\}_B) \right] I_{\alpha}^{(2)} \left[ (b_2 - \{b_2\}_B) f_2^\infty \right] \right\|_{L_q(B(x_0,r))}$$

$$+ \left\| \left[ (b_2 - \{b_2\}_B) \right] I_{\alpha}^{(2)} \left[ (b_1 - \{b_1\}_B) f_1^0, f_2^\infty \right] \right\|_{L_q(B(x_0,r))}$$

$$+ \left\| I_{\alpha}^{(2)} \left[ (b_1 - \{b_1\}_B) f_1^0, (b_2 - \{b_2\}_B) f_2^\infty \right] \right\|_{L_q(B(x_0,r))}.$$

Let $1 < p_1, p_2 < \frac{2n}{n-2}$, such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{2}$, and $\frac{1}{q} = \frac{1}{p} + \frac{1}{2}$. Then, using Hölder’s inequality and by (1.8) we have

$$F_{21} = \left\| \left[ (b_1 - \{b_1\}_B) \right] \left[ (b_2 - \{b_2\}_B) \right] I_{\alpha}^{(2)} \left( f_1^0, f_2^\infty \right) \right\|_{L_q(B(x_0,r))}$$

$$\lesssim \left\| (b_1 - \{b_1\}_B) \right\|_{L_{q_1}(B(x_0,r))} \left\| (b_2 - \{b_2\}_B) \right\|_{L_{q_2}(B(x_0,r))} \left\| I_{\alpha}^{(2)} \left( f_1^0, f_2^\infty \right) \right\|_{L_{q_1}(B(x_0,r))}$$

$$\lesssim \left\| (b_1 - \{b_1\}_B) \right\|_{L_{q_1}(B(x_0,r))} \left\| (b_2 - \{b_2\}_B) \right\|_{L_{q_2}(B(x_0,r))} \left\| I_{\alpha}^{(2)} \left( f_1^0, f_2^\infty \right) \right\|_{L_{q_1}(B(x_0,r))}$$

$$= \prod_{i=1}^2 \left\| b_i \right\|_{L_{p_i}(B(x_0,r))} \left(1 + \ln \frac{t}{r} \right)^{\frac{n}{q_1} + \frac{n}{q_2}} \prod_{i=1}^2 \left\| f_i \right\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$
where in the second inequality we have used the following fact:

It is clear that \(|(x_0 - y_1, x_0 - y_2)|^{2n-\alpha} \geq |x_0 - y_2|^{2n-\alpha}\). By Hölder’s inequality, we have

\[
\left| I^{(2)}_{\alpha} \left( f_1^0, f_2^\infty \right)(x) \right| \lesssim \int_{\mathbb{R}^n} \frac{|f_1^0(y_1)| |f_2^\infty(y_2)|}{(x - y_1, x - y_2)^{2n-\alpha}} dy_1 dy_2
\]

where \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \). Thus, the inequality

\[
\left\| I^{(2)}_{\alpha} \left( f_1^0, f_2^\infty \right) \right\|_{L^\infty(B(x_0, r))} \lesssim r^{\frac{\alpha}{n}} \prod_{i=1}^2 \|f_i\|_{L^{q_i}(B(x_0, t))} \frac{dt}{t^{\frac{\alpha}{n}+1}}
\]

is valid.

On the other hand, for the estimates used in \( F_{22}, F_{23} \), we have to prove the below inequality:

(2.2)

\[
\left| I^{(2)}_{\alpha} \left[ f_1^0, (b_2 \cdot (b_2)_B) f_2^\infty \right](x) \right| \lesssim \|b_2\|_{2} \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^{\frac{\alpha}{n}} \prod_{i=1}^2 \|f_i\|_{L^{q_i}(B(x_0, t))} \frac{dt}{t^{\frac{\alpha}{n}+1}}
\]

To estimate \( F_{22} \), the following inequality

\[
\left| I^{(2)}_{\alpha} \left[ f_1^0, (b_2 \cdot (b_2)_B) f_2^\infty \right](x) \right| \lesssim \int_{2B} |f_1(y_1)| dy_1 \int_{(2B)^c} \frac{|b_2(y_2) - (b_2)_B| |f_2(y_2)|}{|x_0 - y_2|^{2n-\alpha}} dy_2
\]

is satisfied. It’s obvious that

(2.3)

\[
\int_{2B} |f_1(y_1)| dy_1 \lesssim \|f_1\|_{L^{p_1}(2B)} |2B|^{1-\frac{1}{p_1}}
\]
and using Hölder’s inequality and by (1.6) and (1.7) we have

\[
\int_{(2B)^c} \frac{|b_2(y_2) - (b_2)_B| |f_2(y_2)|}{|x_0 - y_2|^{2n-\alpha}} dy_2 \\
\lesssim \int_{(2B)^c} |b_2(y_2) - (b_2)_{B(x_0,t)}| |f_2(y_2)| \left[ \int_{|x_0 - y_2|}^{\infty} \frac{dt}{t^{2n-\alpha+1}} \right] dy_2 \\
\lesssim \int_{2r}^{\infty} \left( b_2(y_2) - (b_2)_{B(x_0,t)} \right) \left( f_2 \right)_{L_p(B(x_0,t))} \left| B(x_0,t) \right|^{1-\frac{1}{p_2}} \frac{dt}{t^{2n-\alpha+1}} \\
+ \int_{2r}^{\infty} \left( b_2(y_2) - (b_2)_{B(x_0,t)} \right) \left( f_2 \right)_{L_p(B(x_0,t))} \left| B(x_0,t) \right|^{1-\frac{1}{p_2}} \frac{dt}{t^{2n-\alpha+1}} \\
\lesssim \|b_2\|_* \int_{2r}^{\infty} \left( 1 + \frac{t}{r} \right)^2 \left| B(x_0,t) \right|^{\frac{1}{p_2}} \left( f_2 \right)_{L_p(B(x_0,t))} \left| B(x_0,t) \right|^{1-\frac{1}{p_2}} \frac{dt}{t^{2n-\alpha+1}} \\
+ \|b_2\|_* \int_{2r}^{\infty} \left( 1 + \frac{t}{r} \right)^2 \left| B(x_0,t) \right|^{\frac{1}{p_2}} \left( f_2 \right)_{L_p(B(x_0,t))} \left| B(x_0,t) \right|^{1-\frac{1}{p_2}} \frac{dt}{t^{2n-\alpha+1}} \\
\lesssim \|b_2\|_* \int_{2r}^{\infty} \left( 1 + \frac{t}{r} \right)^2 \left( f_2 \right)_{L_p(B(x_0,t))} \frac{dt}{t^{n\left(1+\frac{1}{p_2}\right)+1-\alpha}}.
\]

Hence, by (2.3) and (2.4), it follows that:

\[
\left| I_n^{(2)} \left[ f_1^0, (b_2 (\cdot) - (b_2)_B) f_2^\infty \right] (x) \right| \\
\lesssim \|b_2\|_* \left( f_1 \right)_{L_p(2B)} \left( f_2 \right)_{L_p(B(x_0,t))} \frac{dt}{t^{n\left(1+\frac{1}{p_2}\right)+1-\alpha}} \\
\lesssim \|b_2\|_* \left( 1 + \frac{t}{r} \right)^2 \prod_{i=1}^{2} \left( f_i \right)_{L_p(B(x_0,t))} \frac{dt}{t^{n\left(1+\frac{1}{p_2}\right)+1-\alpha}}.
\]
This completes the proof of inequality (2.2). Thus, let $1 < \tau < \infty$, such that $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then, using Hölder’s inequality and from (2.2) and (1.7), we get

$$F_{22} = \left\| \left( b_1 - \{b_1\}_B \right) f_{\alpha}^0 \left[ f_1^0, \left( b_2 - \{b_2\}_B \right) f_2^\infty \right] \right\|_{L_q(B(x_0, r))} \leq \left\| b_1 - \{b_1\}_B \right\|_{L_{q_1}(B)} \left\| f_{\alpha}^0 \left[ f_1^0, \left( b_2 - \{b_2\}_B \right) f_2^\infty \right] \right\|_{L_r(B)} \leq \prod_{i=1}^2 \left\| b_i \right\|_* |B(x_0, r)|^{\frac{1}{q_1} + \frac{1}{q_2}} \times \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right) \prod_{i=1}^2 \left\| f_i \right\|_{L_{p_i}(B(x_0, t))} \frac{dt}{t^{n \left( \frac{1}{q_1} + \frac{1}{q_2} \right) + 1}}.$$

Similarly, $F_{23}$ has the same estimate above, here we omit the details, thus the inequality

$$F_{23} = \left\| \left( b_2 - \{b_2\}_B \right) f_{\alpha}^0 \left[ \left( b_1 - \{b_1\}_B \right) f_1^0, f_2^\infty \right] \right\|_{L_q(B(x_0, r))} \leq \prod_{i=1}^2 \left\| b_i \right\|_* |B(x_0, r)|^{\frac{1}{q_1} + \frac{1}{q_2}} \times \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right) \prod_{i=1}^2 \left\| f_i \right\|_{L_{p_i}(B(x_0, t))} \frac{dt}{t^{n \left( \frac{1}{q_1} + \frac{1}{q_2} \right) + 1}}$$

is valid.

Now we turn to estimate $F_{24}$. Similar to (2.2), we have to prove the following estimate for $F_{24}$:

$$(2.5) \quad \left| f_{\alpha}^0 \left[ \left( b_1 - \{b_1\}_B \right) f_1^0, \left( b_2 - \{b_2\}_B \right) f_2^\infty \right] (x) \right| \leq \prod_{i=1}^2 \left\| b_i \right\|_* \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right) \prod_{i=1}^2 \left\| f_i \right\|_{L_{p_i}(B(x_0, t))} \frac{dt}{t^{n \left( \frac{1}{q_1} + \frac{1}{q_2} \right) + 1}}.$$

Firstly, the following inequality

$$\left| f_{\alpha}^0 \left[ \left( b_1 - \{b_1\}_B \right) f_1^0, \left( b_2 - \{b_2\}_B \right) f_2^\infty \right] (x) \right| \leq \int_{2B} \left| b_1 (y_1) - \{b_1\}_B \right| \left| f_1 (y_1) \right| dy_1 \int_{(2B)^c} \frac{|b_2 (y_2) - \{b_2\}_B | \left| f_2 (y_2) \right|}{|x_0 - y_2|^{2n-\alpha}} dy_2$$

is valid.

It’s obvious that from Hölder’s inequality and (1.7)

$$(2.6) \quad \int_{2B} \left| b_1 (y_1) - \{b_1\}_B \right| \left| f_1 (y_1) \right| dy_1 \leq \left\| b_1 \right\|_* \left| B(x_0, r) \right|^{1 - \frac{n}{q_1}} \left\| f_1 \right\|_{L_{p_1}(2B)}.$$

Then, by (2.4) and (2.6) we have

$$\left| f_{\alpha}^0 \left[ \left( b_1 - \{b_1\}_B \right) f_1^0, \left( b_2 - \{b_2\}_B \right) f_2^\infty \right] (x) \right| \leq \prod_{i=1}^2 \left\| b_i \right\|_* \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right) \prod_{i=1}^2 \left\| f_i \right\|_{L_{p_i}(B(x_0, t))} \frac{dt}{t^{n \left( \frac{1}{q_1} + \frac{1}{q_2} \right) + 1}}.$$

Now, let us estimate $F$. Then, by Hölder’s inequality and (1.8), we get

$$F \lesssim \prod_{i=1}^{2} \|b_i\|_\alpha^\frac{1}{q} \left( 1 + \ln \frac{t}{r} \right)^\frac{2}{q} \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} t^{\frac{1}{q} - \left( \frac{1}{q_1} + \frac{1}{q_2} \right)} + 1.$$  

Considering estimates $F_{21}, F_{22}, F_{23}, F_{24}$ together, we get the desired conclusion

$$F_2 = \left\| I_{\alpha}(b, b_2) (f_1^\infty, f_2^\infty) \right\|_{L_q(B(x_0,r))} \lesssim \prod_{i=1}^{2} \|b_i\|_\alpha^\frac{1}{q} \left( 1 + \ln \frac{t}{r} \right)^\frac{2}{q} \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} t^{\frac{1}{q} - \left( \frac{1}{q_1} + \frac{1}{q_2} \right)} + 1.$$  

At last, we consider the last term $F_4 = \left\| I_{\alpha, b, (b_1, b_2)} (f_1^\infty, f_2^\infty) \right\|_{L_q(B(x_0,r))}$. We split $F_4$ in the following way:

$$F_4 \leq F_{41} + F_{42} + F_{43} + F_{44},$$  

where

$$F_{41} = \left\| (b_1 - (b_1)_B) (b_2 - (b_2)_B) I_{\alpha}^2 (f_1^\infty, f_2^\infty) \right\|_{L_q(B)} \!, \quad F_{42} = \left\| (b_1 - (b_1)_B) I_{\alpha}^2 (f_1^\infty, (b_2 - (b_2)_B) f_2^\infty) \right\|_{L_q(B)} \!, \quad F_{43} = \left\| (b_2 - (b_2)_B) I_{\alpha}^2 ((b_1 - (b_1)_B) f_1^\infty, f_2^\infty) \right\|_{L_q(B)} \!, \quad F_{44} = \left\| I_{\alpha}^2 ((b_1 - (b_1)_B) f_1^\infty, (b_2 - (b_2)_B) f_2^\infty) \right\|_{L_q(B)} .$$  

Now, let us estimate $F_{41}, F_{42}, F_{43}, F_{44}$ respectively.

For the term $F_{41}$, let $1 < r < \infty$, such that $\frac{1}{q} = \left( \frac{1}{q_1} + \frac{1}{q_2} \right) + \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{\tau}$. Then, by Hölder’s inequality and (1.8), we get

$$F_{41} = \left\| (b_1 - (b_1)_B) (b_2 - (b_2)_B) I_{\alpha}^2 (f_1^\infty, f_2^\infty) \right\|_{L_q(B)} \! \lesssim \|b_1 - (b_1)_B\|_{L_{q_1}(B)} \|b_2 - (b_2)_B\|_{L_{q_2}(B)} \left\| I_{\alpha}^2 (f_1^\infty, f_2^\infty) \right\|_{L_{1/\tau}(B)} \lesssim \prod_{i=1}^{2} \|b_i\|_\alpha^\frac{1}{q_1} \left( 1 + \ln \frac{t}{r} \right)^\frac{2}{q_1} \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} t^{\frac{1}{q_1} - \left( \frac{1}{q_1} + \frac{1}{q_2} \right)} + 1.$$  

where in the second inequality we have used the following fact:
Noting that 

\[ (x_0 - y_1, x_0 - y_2) \geq |x_0 - y_1|^{n - \frac{2}{p}} |x_0 - y_2|^{n - \frac{2}{q}} \]

and by Hölder’s inequality, we get

\[
\left| I^{(2)}_\alpha (f_1^\infty, f_2^\infty) (x) \right| \\
\lesssim \int \int \left| f_1 (y_1) \chi_{(2B)^c} (x) \right| \left| f_2 (y_2) \chi_{(B)^c} (x) \right| \frac{dy_1 dy_2}{(|x_0 - y_1, x_0 - y_2|^{n - \alpha})}
\]

Moreover, for \( p_1, p_2 \in [1, \infty) \) the inequality

\[
\left| I^{(2)}_\alpha (f_1^\infty, f_2^\infty) (x) \right| \\
\lesssim \sum_{j=1}^{\infty} \prod_{i=1}^2 \int \frac{|f_i (y_i)|}{|x_0 - y_i|^{n - \frac{2}{p_i}}} dy_i
\]

\[
\lesssim \sum_{j=1}^{\infty} \prod_{i=1}^2 (2^j r)^{-n + \frac{2}{p_i}} \int |f_i (y_i)| dy_i
\]

\[
\lesssim \sum_{j=1}^{\infty} \prod_{i=1}^2 (2^j r)^{-n + \frac{2}{p_i}} \int |f_i (y_i)| dy_i
\]

\[
\lesssim \sum_{j=1}^{2^j + 2} \prod_{i=1}^2 \| f_i \|_{L_{p_i}(2^j B)} \| 2^j B \|^{1 - \frac{1}{p_i}} dt
\]

\[
\lesssim \sum_{j=1}^{2^j + 2} \prod_{i=1}^2 \| f_i \|_{L_{p_i}(B(x_0, t))} \| B (x_0, t) \|^{1 - \frac{1}{p_i}} \frac{dt}{t^{2n + 1 - \alpha}}
\]

\[
\lesssim \int \prod_{i=1}^2 \| f_i \|_{L_{p_i}(B(x_0, t))} \| B (x_0, t) \|^{2 - \left( \frac{1}{p_1} + \frac{1}{p_2} \right)} \frac{dt}{t^{2n + 1 - \alpha}}
\]

\[
\lesssim \int \prod_{i=1}^2 \frac{dt}{t^{1 + \frac{1}{p_i}}}
\]

is valid.

For the terms \( F_{12}, F_{13} \), similar to the estimates used for \( f_2^\infty (2.2) \), we have to prove

the following inequality:

\[
\left| I^{(2)}_\alpha (f_1^\infty, f_2^\infty) (x) \right| \\
\lesssim \| f_2^\infty \|_{L_r (B(x_0, r))} \int \frac{dt}{t^{1 + \frac{1}{p_i}}}
\]

(2.7)
Noting that \(|(x_0 - y_1, x_0 - y_2)|^{2n-\alpha} \geq |x_0 - y_1|^{n-\frac{\alpha}{2}} |x_0 - y_2|^{n-\frac{\alpha}{2}}\), we get

\[
I^{(2)}_\alpha \left[ f_1^\infty, (b_2 - (b_2)_B) f_2^\infty \right](x) \lesssim \int_{\mathbb{R}^n} \frac{|b_2(y_2) - (b_2)_B| |f_1(y_1)| |f_2(y_2)|}{|x_0 - y_1|^{n-\frac{\alpha}{2}} |x_0 - y_2|^{n-\frac{\alpha}{2}}} dy_1 dy_2 \\
\lesssim \int_{(2B)^c} \int |b_2(y_2) - (b_2)_B| |f_1(y_1)| |f_2(y_2)| dy_1 dy_2 \\
\lesssim \sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} |f_1(y_1)| dy_1 \int_{2^{j+1}B \setminus 2^j B} |b_2(y_2) - (b_2)_B| |f_2(y_2)| dy_2 \\
\lesssim \sum_{j=1}^{\infty} (2^j r)^{-2n+\alpha} \int_{2^{j+1}B} |f_1(y_1)| dy_1 \int_{2^{j+1}B} |b_2(y_2) - (b_2)_B| |f_2(y_2)| dy_2.
\]

On the other hand, it’s obvious that

\[
\int_{2^{j+1}B} |f_1(y_1)| dy_1 \leq \|f_1\|_{L_{p_1}(2^{j+1}B)} |2^{j+1}B|^{1 - \frac{1}{p_1}}
\]

and using Hölder’s inequality and by (1.6) and (1.7)

\[
\int_{2^{j+1}B} |b_2(y_2) - (b_2)_B| |f_2(y_2)| dy_2 \\
\leq \|b_2 - (b_2)_B\|_{L_{p_2}(2^{j+1}B)} \|f_2\|_{L_{p_2}(2^{j+1}B)} |2^{j+1}B|^{1 - \frac{1}{p_2}} \\
+ \|b_2\|_{L_{p_2}(2^{j+1}B)} \|f_2\|_{L_{p_2}(2^{j+1}B)} |2^{j+1}B|^{1 - \frac{1}{p_2}} \\
\lesssim \|b_2\|_{L_{p_2}(2^{j+1}B)} \left( 1 + \ln \frac{2^{j+1}r}{r} \right) \|f_2\|_{L_{p_2}(2^{j+1}B)} |2^{j+1}B|^{1 - \frac{1}{p_2}} \\
+ \|b_2\|_{L_{p_2}(2^{j+1}B)} \left( 1 + \ln \frac{2^{j+1}r}{r} \right) |2^{j+1}B| \|f_2\|_{L_{p_2}(2^{j+1}B)} |2^{j+1}B|^{1 - \frac{1}{p_2}} \\
\lesssim \|b_2\|_{L_{p_2}(2^{j+1}B)} \left( 1 + \ln \frac{2^{j+1}r}{r} \right)^2 |2^{j+1}B|^{1 - \frac{1}{p_2}} \|f_2\|_{L_{p_2}(2^{j+1}B)}.
\]
Hence, by (2.9) and (2.10), it follows that:

\[
\left| f^{(2)}_\alpha \left[ f_1^\infty, (b_2 - (b_2)_B) f_2^\infty \right] (x) \right| \\
\lesssim \sum_{j=1}^{\infty} (2^j r)^{-2n+\alpha} \int_{2^{j+1}B} |f_1(y_1)| \, dy_1 \int_{2^{j+1}B} |b_2(y_2) - (b_2)_B| \, |f_2(y_2)| \, dy_2
\]

\[
\lesssim \|b_2\| \sum_{j=1}^{\infty} (2^j r)^{-2n+\alpha} \left( 1 + \ln \left( \frac{2^{j+1} r}{r} \right) \right)^2 |2^{j+1} B|^{2 - \left( \frac{1}{p_1} + \frac{1}{p_2} \right)} \prod_{i=1}^{2} \|f_i\|_{L^p_i, (2^{j+1}B)}
\]

\[
\lesssim \|b_2\| \int_{2r}^{\infty} \left( 1 + \ln \left( \frac{t}{r} \right) \right)^2 |B(x_0, t)|^{2 - \left( \frac{1}{p_1} + \frac{1}{p_2} \right)} \prod_{i=1}^{2} \|f_i\|_{L^p_i(B(x_0, t))} \frac{dt}{t^{2n+\alpha+1}}
\]

This completes the proof of (2.3).

Now we turn to estimate $F_{43}$. Let $1 < \tau < \infty$, such that $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{\tau}$. Then, by Hölder’s inequality, (1.7) and (2.8), we obtain

\[
F_{43} = \left\| (b_2 - (b_2)_B) I^{(2)}_\alpha \left[ f_1^\infty, (b_2 - (b_2)_B) f_2^\infty \right] \right\|_{L^q(B)}
\]

\[
\lesssim \|b_2 - (b_2)_B\|_{L^{q_1}(B)} \left\| I^{(2)}_\alpha \left[ f_1^\infty, (b_2 - (b_2)_B) f_2^\infty \right] \right\|_{L^\tau(B)}
\]

\[
\lesssim \prod_{i=1}^{2} \|b_i\|_{L^{q_i}} \int_{2r}^{\infty} \left( 1 + \ln \left( \frac{t}{r} \right) \right)^2 \prod_{i=1}^{2} \|f_i\|_{L^p_i(B(x_0, t))} \frac{dt}{t^{n+\alpha}}
\]

Similarly, $F_{43}$ has the same estimate above, here we omit the details, thus the inequality

\[
F_{43} = \left\| (b_2 - (b_2)_B) I^{(2)}_\alpha \left[ (b_2 - (b_2)_B) f_1^\infty, f_2^\infty \right] \right\|_{L^q(B)}
\]

\[
\lesssim \prod_{i=1}^{2} \|b_i\|_{L^{q_i}} \int_{2r}^{\infty} \left( 1 + \ln \left( \frac{t}{r} \right) \right)^2 \prod_{i=1}^{2} \|f_i\|_{L^p_i(B(x_0, t))} \frac{dt}{t^{n+\alpha}}
\]

is valid.
Finally, to estimate $F_{44}$, similar to the estimate of (2.8), we have 

$$\left| I^2_\alpha \left( (b_1 - (b_2)_B) f_1^\infty, (b_2 - (b_2)_B) f_2^\infty \right) (x) \right|$$

$$\lesssim \sum_{j=1}^{\infty} (2^j r)^{-2n+\alpha} \int_{2^{j+1} B} |b_1 (y_1) - (b_1)_B| |f_1 (y_1)| dy_1 \int_{2^{j+1} B} |b_2 (y_2) - (b_2)_B| |f_2 (y_2)| dy_2$$

$$\lesssim \prod_{i=1}^{2} \|b_i\|_r \int_{2r}^{\infty} \left( 1 + \ln \left( \frac{t}{r} \right) \right)^{\frac{n}{2}} \prod_{i=1}^{2} \|f_i\|_{L^p(B(x_0,t))} \frac{dt}{t^{n \left( \frac{1}{q} - \left( \frac{1}{q_1} + \frac{1}{q_2} \right) \right) + 1}}.$$ 

Thus, we have 

$$F_{44} = \left\| I^2_\alpha \left( (b_1 - (b_1)_B) f_1^\infty, (b_2 - (b_2)_B) f_2^\infty \right) \right\|_{L^q(B)}$$

$$\lesssim \prod_{i=1}^{2} \|b_i\|_r \int_{2r}^{\infty} \left( 1 + \ln \left( \frac{t}{r} \right) \right)^{\frac{n}{2}} \prod_{i=1}^{2} \|f_i\|_{L^p(B(x_0,t))} \frac{dt}{t^{n \left( \frac{1}{q} - \left( \frac{1}{q_1} + \frac{1}{q_2} \right) \right) + 1}}.$$ 

By the estimates of $F_{4j}$ above, where $j = 1, 2, 3, 4$, we know that 

$$F_4 = \left\| I^2_\alpha, (b_1, b_2) (f_1^\infty, f_2^\infty) \right\|_{L^q(B(x_0,t))} \lesssim \prod_{i=1}^{2} \|b_i\|_r \int_{2r}^{\infty} \left( 1 + \ln \left( \frac{t}{r} \right) \right)^{\frac{n}{2}} \prod_{i=1}^{2} \|f_i\|_{L^p(B(x_0,t))} \frac{dt}{t^{n \left( \frac{1}{q} - \left( \frac{1}{q_1} + \frac{1}{q_2} \right) \right) + 1}}.$$ 

Consequently, combining all the estimates for $F_1, F_2, F_3, F_4$, we complete the proof of Lemma 2. 

### 3. Proofs of the main results

Now we are ready to return to the proofs of Theorems 3 and 4.

#### 3.1. Proof of Theorem 3.

**Proof.** To prove Theorem 3, we will use the following relationship between essential supremum and essential infimum

$$(\text{essinf}_{x \in E} f(x))^{-1} = \text{esssup}_{x \in E} \frac{1}{f(x)}$$

where $f$ is any real-valued nonnegative function and measurable on $E$ (see [33], page 143). Indeed, we consider (1.12) firstly.
Since \( \tilde{f} \in M_{p_1, \varphi_1} \times \cdots \times M_{p_m, \varphi_m} \), by (3.3) and the non-decreasing, with respect to \( t \), of the norm \( \prod_{i=1}^{m} \| f_i \|_{L_{p_i}(B(x,t))} \), we get

\[
\prod_{i=1}^{m} \| f_i \|_{L_{p_i}(B(x,t))} \leq \essinf_{0 < t < \tau < \infty} \prod_{i=1}^{m} \varphi_i(x, \tau) t^{\frac{n}{p_i}} \\
\leq \essinf_{t < \tau < \infty} \prod_{i=1}^{m} \varphi_i(x, \tau) t^{\frac{n}{p_i}} \leq \prod_{i=1}^{m} \| f_i \|_{M_{p_i, \varphi_i}}.
\]

(3.2)

For \( 1 < p_1, \ldots, p_m < \infty \), since \( (\varphi_1, \ldots, \varphi_m, \varphi) \) satisfies (1.11) and by (3.2), we have

\[
\int_{r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^{m} \prod_{i=1}^{m} \| f_i \|_{L_{p_i}(B(x,t))} dt \leq \essinf_{t < \tau < \infty} \prod_{i=1}^{m} \varphi_i(x, \tau) t^{\frac{n}{p_i}} \int_{r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^{m} \prod_{i=1}^{m} \| f_i \|_{M_{p_i, \varphi_i}} \varphi(x, r) dt
\]

\[
\leq C \prod_{i=1}^{m} \| f_i \|_{M_{p_i, \varphi_i}} \int_{r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^{m} \essinf_{t < \tau < \infty} \prod_{i=1}^{m} \varphi_i(x, \tau) t^{\frac{n}{p_i}} dt \leq C \prod_{i=1}^{m} \| f_i \|_{M_{p_i, \varphi_i}} \varphi(x, r).
\]

(3.3)

Then by (2.1) and (3.3), we get

\[
\| f^{(m)}_{\alpha, b}(\vec{f}) \|_{M_{q, \varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} \| B(x, r)^{-\frac{1}{q}} \|_{L_{q}(B(x,r))} \| f^{(m)}_{\alpha, b}(\vec{f}) \|_{M_{q, \varphi}}
\]

\[
\lesssim \prod_{i=1}^{m} \| b_i \|_{s} \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} \int_{r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^{m} \prod_{i=1}^{m} \| f_i \|_{L_{p_i}(B(x_0,t))} dt \leq \prod_{i=1}^{m} \| b_i \|_{s} \| f_i \|_{M_{p_i, \varphi_i}}.
\]

Thus we obtain (1.12).

The conclusion of (1.13) is a direct consequence of (1.10) and (1.12). Indeed, from the process proving (1.12), it is easy to see that the conclusions of (1.12) also
hold for $f^{(m)}_{\beta, \alpha}$. Combining this with (1.10), we can immediately obtain (1.13), which completes the proof.

3.2. Proof of Theorem 4

Proof. Since the inequalities (1.17) and (1.18) hold by Theorem 3, we only have to prove the implication

$$(3.4)$$

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} r^{-\frac{2}{n}} \prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x,r))} = 0 \implies \lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \frac{r^{-\frac{2}{n}} \left\| I_{\alpha, \beta}^{(m)} \left( \frac{f}{r} \right) \right\|_{L_q(B(x,r))}}{\varphi(x,r)} = 0.$$  

To show that

$$\sup_{x \in \mathbb{R}^n} r^{-\frac{2}{n}} \left\| I_{\alpha, \beta}^{(m)} \left( \frac{f}{r} \right) \right\|_{L_q(B(x,r))} \varphi(x,r) < \varepsilon$$

for small $r$, we use the estimate (2.1):

$$r^{-\frac{2}{n}} \left\| I_{\alpha, \beta}^{(m)} \left( \frac{f}{r} \right) \right\|_{L_q(B(x,r))} \varphi(x,r) \leq \sup_{x \in \mathbb{R}^n} \prod_{i=1}^{m} \|b_i\|_{*} \int_{r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^{m} \prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x,t))} \frac{dt}{t^{\frac{1}{q} - \frac{m}{n} \sum_{i=1}^{m} \frac{1}{q_i} + 1}}.$$  

We take $r < \delta_0$, where $\delta_0$ is small enough and split the integration:

$$(3.5)$$

$$r^{-\frac{2}{n}} \left\| I_{\alpha, \beta}^{(m)} \left( \frac{f}{r} \right) \right\|_{L_q(B(x,r))} \varphi(x,r) \leq C \left[ I_{\delta_0}(x,r) + J_{\delta_0}(x,r) \right],$$  

where $\delta_0 > 0$ (we may take $\delta_0 < 1$), and

$$I_{\delta_0}(x,r) := \prod_{i=1}^{m} \|b_i\|_{*} \delta_0 \int_{r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^{m} \prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x,t))} \frac{dt}{t^{\frac{1}{q} - \frac{m}{n} \sum_{i=1}^{m} \frac{1}{q_i} + 1}},$$  

and

$$J_{\delta_0}(x,r) := \prod_{i=1}^{m} \|b_i\|_{*} \int_{\delta_0}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^{m} \prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x,t))} \frac{dt}{t^{\frac{1}{q} - \frac{m}{n} \sum_{i=1}^{m} \frac{1}{q_i} + 1}}.$$  

and $r < \delta_0$. Now we can choose any fixed $\delta_0 > 0$ such that

$$\sup_{x \in \mathbb{R}^n} \frac{t^{-\frac{2}{n}} \prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x,t))}}{\prod_{i=1}^{m} \varphi_i(x,t)} < \frac{\varepsilon}{2CC_0}, \quad t \leq \delta_0,$$  

for small $r$. We have thus proved Theorem 4.
where \( C \) and \( C_0 \) are constants from (1.14) and (3.5), which is possible since \( \vec{f} \in VM_{p_1,\varphi_1} \times \cdots \times VM_{p_m,\varphi_m} \). This allows to estimate the first term uniformly in \( r \in (0, \delta_0) \):
\[
\prod_{i=1}^{m} \|b_i\|_s \sup_{x \in \mathbb{R}^n} CI_{\delta_0} (x, r) < \frac{\varepsilon}{2}, \quad 0 < r < \delta_0
\]
by (1.14).

For the second term, writing \( 1 + \ln \frac{t}{r} \leq 1 + |\ln t| + \ln \frac{1}{r} \), by the choice of \( r \) sufficiently small because of the conditions (1.15) we obtain
\[
J_{\delta_0} (x, r) \leq c_{\delta_0} + \tilde{c}_{\delta_0} \ln \frac{1}{r} \prod_{i=1}^{m} \|b_i\|_s \|f_i\|_{VM_{p_i,\varphi_i}},
\]
where \( c_{\delta_0} \) is the constant from (1.10) with \( \delta = \delta_0 \) and \( \tilde{c}_{\delta_0} \) is a similar constant with omitted logarithmic factor in the integrand. Then, by (1.16) we can choose small enough \( r \) such that
\[
\sup_{x \in \mathbb{R}^n} J_{\delta_0} (x, r) < \frac{\varepsilon}{2},
\]
which completes the proof of (3.4).

For \( M_{(m)}^{(m)} \), we can also use the same method to obtain the desired result, but we omit the details. Therefore, the proof of Theorem 4 is completed. \( \square \)

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