The construction of finite solvable groups revisited

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We describe a new approach towards the systematic construction of finite groups up to isomorphism. This approach yields a practical algorithm for the construction of finite solvable groups up to isomorphism. We report on a GAP implementation of this method for finite solvable groups and exhibit some sample applications.

1 Introduction

The construction of all groups of a given order is an old and fundamental topic in finite group theory. Given an order $n$, the aim is to determine a list of groups of order $n$ so that every group of order $n$ is isomorphic to exactly one group in the list. There are many contributions to this topic in the literature. In the early history these are based on hand calculations; in more recent years algorithms have been developed for this purpose. We refer to [5] for a historic overview and a survey of the available algorithms.

Modern group construction algorithms distinguish three cases: nilpotent groups, solvable groups and non-solvable groups. Nilpotent groups are determined as direct products of $p$-groups and $p$-groups can be constructed using the $p$-group generation algorithm [21]. Solvable groups can be determined by the Frattini extension method [2] or the cyclic split extensions methods [3]. Non-solvable groups can be obtained via cyclic extensions of perfect groups as in [2] or via the method in [1].

The $p$-group generation algorithm has been used to determine the groups of order dividing $2^9$ [9] and the construction of the groups of order dividing $p^7$ for all primes $p$ is also based on it, see [20] and [23]. The algorithm can also be used to determine groups with special properties; for example, it is a main tool in the investigation of finite $p$-groups by coclass, see [16] for background, and in the construction of restricted Burnside groups, see [19] and [22]. The $p$-group generation algorithm reduces the isomorphism problem to an orbit-stabilizer calculation.
The Frattini extension method and the cyclic split extension method have been used to determine most solvable non-nilpotent groups of order at most 2000, see [5]. The cyclic split extension method applies to groups with normal Sylow subgroup and order of the form $p^n \cdot q$ for different primes $p$ and $q$ only, while the Frattini extension method applies to all non-nilpotent solvable groups. The Frattini extension method uses a random isomorphism test for the reduction to isomorphism types.

The central aim of this paper is to introduce a new approach towards the systematic construction of groups up to isomorphism. This new approach is particularly useful for finite solvable groups and we developed it in detail and implemented it in GAP [28] for this case. For finite $p$-groups, our new approach coincides with the approach of $p$-group generation. In particular, the new approach reduces the solution of the isomorphism problem to an orbit-stabilizer calculation.

We discuss some sample applications of our new approach. We determined (again) all solvable non-nilpotent groups of order at most 2000 and thus check the results available in the Small Groups Library [4] and we constructed (for the first time) the groups of order $2304 = 3^2 \cdot 2^8$. We believe that our new approach could also be useful in the experimental investigation of coclass theory for finite solvable groups as suggested in [13] or in the construction of other finite solvable groups with special properties.

2 The general approach of the algorithm

In this section we exhibit a top-level introduction towards our new approach. The central idea is to use induction along a certain series: the so-called $F$-central series. We first introduce and investigate this series. Throughout this section, let $G$ be a finite group.

Recall that the Fitting subgroup $F(G)$ is the maximal nilpotent normal subgroup of $G$. Define $\nu_0(G) = F(G)$ and let $\nu_{i+1}(G)$ be the smallest normal subgroup of $F(G)$ so that $\nu_i(G)/\nu_{i+1}(G)$ is a direct product of elementary abelian groups which is centralized by $F(G)$. Then we define the $F$-central series of $G$ as

$$G \geq \nu_0(G) \geq \nu_1(G) \geq \ldots .$$

The following lemma provides an alternative characterization of the terms of the $F$-central series. We omit its straightforward proof. For an integer $n$ with prime factorisation $n = p_1^{e_1} \cdots p_r^{e_r}$ for different primes $p_1, \ldots, p_r$ and exponents $e_1, \ldots, e_r \neq 0$, we call $p_1 \cdots p_r$ the core of $n$.

**Lemma 1.** Let $G$ be a finite group and let $k$ be the core of $|F(G)|$. Then

$$\nu_{i+1}(G) = [F(G), \nu_i(G)]\nu_i(G)^k \text{ for each } i \geq 0.$$
If $F(G)$ is a finite $p$-group, then the series $F(G) = \nu_0(G) \geq \nu_1(G) \geq \ldots$ coincides with the lower exponent-$p$ central series of $F(G)$. In general, the group $F(G)$ is nilpotent and the series $F_i(G) = \nu_0(G) \geq \nu_1(G) \geq \ldots$ is a central series of $F(G)$. Thus there exists an integer $c$ which $\nu_c(G) = \{1\}$. We call the smallest such integer the $F$-rank of $G$. Further, if $G$ has $F$-rank at least 1, then the order of the quotient $\nu_0(G)/\nu_1(G)$ is called the $F$-rank of $G$. The next two lemmas collect some elementary facts about the $F$-central series. Let $\phi(G)$ denote the Frattini subgroup of $G$.

**Lemma 2.** Let $G$ be a finite group.

(a) $\nu_i(G) = \phi(F(G)) \leq \phi(G)$.
(b) $\nu_i(G)$ is a characteristic subgroup of $G$ for each $i \geq 0$.
(c) $\nu_i(G/\nu_1(G)) = \nu_j(G)/\nu_i(G)$ for each $i \geq j \geq 0$ and $i \geq 1$.

**Proof.** (a) and (b) are elementary and we consider (c) only. Suppose $j = 0$, then we need to show $F_i(G/\nu_i(G)) = F_i(G)/\nu_i(G)$. Let $L \leq G$ be defined by $L/\nu_i(G) = F_i(G/\nu_i(G))$. As $F_i(G/\nu_i(G))$ is nilpotent, hence $L/\nu_i(G)$ is nilpotent. Additionally, $L$ is finite and normal in $G$, thus $L$ is nilpotent by Gaschütz’ theorem (see [25, 5.2.15]). In summary we conclude $L \leq F(G)$, and in fact $L = F(G)$, proving the claim. For $j > 0$ the result follows by induction using Lemma 1.

**Lemma 3.** Let $G$ be a finite group.

(a) $G$ is solvable if and only if $G/\nu_0(G)$ is solvable.
(b) $G$ is nilpotent if and only if $G/\nu_0(G)$ is trivial.
(c) If $G \neq \{1\}$ is solvable, then $\nu_0(G) \neq \{1\}$ and hence $G$ has $F$-rank at least 1.

**Proof.** All three items follow directly from the fact that $\nu_0(G) = F(G)$ is the maximal nilpotent normal subgroup of $G$.

**Remark 4.** Let $G$ be a finite group of $F$-class 0. Then $G$ does not have any non-trivial solvable normal subgroup. The structure of such groups and their construction up to isomorphism is well understood, see [25, p. 89f] for details.

Let $G$ be a finite group of $F$-class $c$. A group $H$ is a descendant of $G$ (and $G$ is the ancestor of $H$) if $H$ has $F$-class $c + 1$ and $H/\nu_c(H) \cong G$. Lemma 2 asserts that $G$ and $H$ have the same $F$-rank.

Our approach to construct finite groups uses the $F$-class and the $F$-rank as primary invariants. Its two central ingredients are the following algorithms.

(a) **Algorithm I**

Given a positive integer $\ell$, determine up to isomorphism all finite groups of $F$-class 1 and $F$-rank $\ell$.  

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Algorithm II

Given a finite group $G$, determine up to isomorphism all descendants of $G$.

Algorithm I and an iterated application of Algorithm II thus yield a method to determine up to isomorphism all finite groups of $F$-class $c > 1$ and $F$-rank $\ell$. If Algorithm I is restricted to determine solvable groups only, then this approach yields an algorithm to determine up to isomorphism all finite solvable groups of $F$-class $c$ and $F$-rank $\ell$, as the following remark asserts.

Remark 5. Let $H$ be a descendant of $G$. Then $G$ is solvable if and only if $H$ is solvable.

In the following Sections 3 and 4 we discuss methods for Algorithm I and II in more detail. In both cases we are particularly interested in the construction of solvable groups. We then briefly describe the GAP implementation of our method in Section 5 and we exhibit applications in Section 6.

3 Finite groups of $F$-class 1

In this section we describe an effective method to determine up to isomorphism the finite groups $G$ of $F$-class 1 and given $F$-rank $\ell$. Every such group $G$ has a Fitting subgroup $F(G)$ which is a direct product of elementary abelian groups and has order $\ell$. The following lemma analyzes the structure of such groups $G$ further.

Lemma 6. Let $G$ be a finite group of $F$-class 1 and $F$-rank $\ell$. Let $H$ be the maximal solvable normal subgroup of $G$.

(a) $F(G)$ is a direct product of elementary abelian groups and has order $\ell$.
(b) $H$ is solvable of $F$-class 1 and $F$-rank $\ell$ with $F(G) = F(H)$.
(c) $F(H)$ is self-centralizing in $H$, i.e. $C_H(F(H)) = F(H)$.

Proof. (a) As $G$ is $F$-class 1, it follows that $\nu_1(G) = \{1\}$ and thus $F(G) = \nu_0(G)$ is a direct product of elementary abelian groups and has order $\ell$.
(b) As $H$ is characteristic in $G$, it follows that $F(H)$ is normal in $G$. As $F(H)$ is nilpotent and $F(G)$ is the maximal nilpotent normal of $G$, this yields $F(H) \subseteq F(G)$. Conversely, $F(G) \leq H$ and thus $F(G)$ is a nilpotent normal subgroup of $H$. Hence $F(G) \leq F(H)$.
(c) This follows from [25, 5.4.4].

Hence a finite group $G$ of $F$-class 1 and $F$-rank $\ell$ can be constructed from a finite solvable normal subgroup $H$ of $F$-class 1 and $F$-rank $\ell$ and a quotient $G/H$ of $F$-class 0. We discuss the effective construction of the solvable groups of $F$-class 1 in more detail in the following.
3.1 The solvable case

Let $G$ be a finite solvable group of $F$-rank 1 and $F$-class $\ell$. Then Lemma 6(c) asserts $G$ can be written as an extension of $F(G)$ by a solvable subgroup $U \leq \text{Aut}(F(G))$. The isomorphism type of the group $F(G)$ is fully determined by $\ell$. We discuss the construction of the relevant subgroups $U$ of $\text{Aut}(F(G))$ in the following Section 3.1.1 and the determination of the corresponding extensions of $F(G)$ by $U$ in Section 3.1.2. In Section 3.1.3 we summarize the resulting algorithm to construct finite solvable groups of $F$-class 1.

3.1.1 The relevant subgroups

Let $A$ be the direct product of elementary abelian groups with $|A| = \ell$. Our aim is to determine up to conjugacy those solvable subgroups $U$ of $\text{Aut}(A)$ so that there exists an extension $G$ of $A$ by $U$ with $F(G) \cong A$. We first investigate these subgroups in more detail.

A subgroup $N$ of $\text{Aut}(A)$ centralizes a series through $A$ if there exists an $N$-invariant series $A = A_1 > A_2 > \ldots > A_l > A_{l+1} = \{1\}$ so that $N$ induces the identity on every quotient $A_i/A_{i+1}$. We say that a group $U \leq \text{Aut}(A)$ is $F$-relevant if none of its non-trivial normal subgroups centralizes a series through $A$.

Lemma 7. Let $A$ be the direct product of elementary abelian groups with $|A| = \ell$ and let $U \leq \text{Aut}(A)$. Then the following are equivalent:

(a) $U$ is $F$-relevant.
(b) Every extension $G$ of $A$ by $U$ satisfies $F(G) \cong A$.
(c) There exists an extension $G$ of $A$ by $U$ with $F(G) \cong A$.

Proof. (a) $\Rightarrow$ (b) Suppose that $U$ is $F$-relevant and let $G$ be an arbitrary extension of $A$ by $U$. We consider $A$ as normal subgroup of $G$. Then $A \leq F(G)$, as $A$ is abelian and thus nilpotent. Further, as $G/A$ corresponds to $U$, it follows that $F(G)/A$ corresponds to a normal subgroup $N$ of $U$. As $F(G)$ is nilpotent, $N$ centralizes a series through $A$. As $U$ is $F$-relevant, $N$ is trivial and hence $A = F(G)$ follows.

(b) $\Rightarrow$ (c) Trivial, since the split extension exists.

(c) $\Rightarrow$ (a) Let $N$ be a normal subgroup of $U$ which centralizes a series through $A$. Then $N$ is nilpotent, as it is also a subgroup of $\text{Aut}(A)$. Let $H$ be the normal subgroup of $G$ with $A \leq H$ and $H/A$ corresponding to $N$. Then $H$ is a nilpotent normal subgroup of $G$ and $H \leq F(G) = A$ follows. Thus $H = A$ and $N = \{1\}$. Hence $U$ is $F$-relevant.

Lemma 7 translates our aim in this subsection to a determination up to conjugacy of all solvable $F$-relevant subgroups of $\text{Aut}(A)$. As a first step towards this, we give an alternative description for the $F$-relevant subgroups of $\text{Aut}(A)$. Let $\ell = p_1^{d_1} \cdots p_r^{d_r}$ be
the prime factorization of $\ell$. We identify $A$ with $A_1 \times \cdots \times A_r$, where $A_i$ is elementary abelian of order $p_i^{d_i}$ for $1 \leq i \leq r$, and thus obtain

$$\text{Aut}(A) = GL(d_1, p_1) \times \cdots \times GL(d_r, p_r).$$

For a subgroup $U$ of $\text{Aut}(A)$ and $1 \leq i \leq r$ we write $\sigma_i(U) = U \cap GL(d_i, p_i)$ and we denote with $\pi_i(U)$ the projection of $U$ into $GL(d_i, p_i)$. Further, let $P(U) = \pi_1(U) \times \cdots \times \pi_r(U)$ and $S(U) = \sigma_1(U) \times \cdots \times \sigma_r(U)$. Then $U$ is a subdirect product in $P(U)$ with kernel $S(U)$. Figure 3.1.1 illustrates these subgroups for the case $r = 2$.

Next, we define $O(U) = O_{p_1}(\sigma_1(U)) \times \cdots \times O_{p_r}(\sigma_r(U))$, where $O_p(H)$ is the maximal normal $p$-subgroup of the group $H$. Thus we obtain the following series of subgroups

$$\{1\} \leq O(U) \leq S(U) \leq U \leq P(U) \leq \text{Aut}(A).$$

We note that $\sigma_i(U)$ is normal in $\pi_i(U)$ for $1 \leq i \leq r$ and thus $S(U)$ is normal in $P(U)$. Further, $O(U)$ is characteristic in $S(U)$ and thus normal in $P(U)$. The following lemma analyzes the situation further.

**Lemma 8.** Let $A$ be the direct product of elementary abelian groups with $|A| = \ell$ and let $U \leq \text{Aut}(A)$.

(a) $O(U)$ is the maximal normal subgroup of $U$ that centralizes a series through $A$.
(b) $U$ is $F$-relevant if and only if $O(U) = \{1\}$.

**Proof.** (b) follows directly from (a) and it remains to prove (a).
Consider the prime factorization $\ell = p_1^{d_1} \cdots p_r^{d_r}$. Suppose $i \in \{1, \ldots, r\}$ and set $U_i = \sigma_i(U)$. Then $U_i$ is normal in $U$ and $O_{p_i}(U_i)$ is characteristic in $U_i$. Thus $O_{p_i}(U_i)$ is normal in $U$. Further, $O_{p_i}(U_i)$ is a $p_i$-subgroup of $GL(d_i, p_i)$ and thus centralizes a series through $A_i$. This implies that $O(U)$ is a normal subgroup of $U$ centralizing a series through $A_i$.

Conversely, suppose $N$ is a normal subgroup of $U$ centralizing a series through $A_i$. For $i \in \{1, \ldots, r\}$ let $P_i$ denote the projection of $N$ into $GL(d_i, p_i)$. Then $P_i$ centralizes a series through $A_i$. Thus $P_i$ is a $p_i$-group. As all primes $p_1, \ldots, p_r$ are different, it follows that $N = P_1 \times \cdots \times P_r$ and $P_i \leq U_i$. Hence $P_i \leq O_{p_i}(U_i)$ and $N \leq O(U)$. This yields that $O(U)$ is the maximal normal subgroup of $U$ that centralizes a series through $A_i$. \qed

Hence our aim translates now to the determination up to conjugacy of all solvable subgroups $U$ of $Aut(A)$ with $O(U) = \{1\}$; additionally, we also determine their normalizers $N(U) = N_{Aut(A)}(U)$. If $U$ is solvable, then $P(U)$ is solvable. We use the following approach.

**Algorithm** RelevantSolvableSubgroups($\ell$)

- Factorize $\ell = p_1^{d_1} \cdots p_r^{d_r}$.
- For $1 \leq i \leq r$ determine up to conjugacy all solvable subgroups $P_i$ of $GL(d_i, p_i)$ together with their normalizers $R_i = N_{GL(d_i, p_i)}(P_i)$. (See [12] Sec. 10.4 for algorithms for this purpose.)
- For each combination $P = P_1 \times \cdots \times P_r$, with normalizer $R = R_1 \times \cdots \times R_r$, determine up to conjugacy all subdirect products $U$ in $P$ together with their normalizers $N_R(U)$. (See [3] for an algorithm for this purpose.)
- Discard those subdirect products $U$ with $O(U) \neq \{1\}$ and return the remaining list of subgroups $U$ together with their normalizers $N_R(U)$.

In the special case $r = 1$ we only need to determine up to conjugacy those solvable subgroups $U$ of $GL(d_1, p_1)$ with $O_{p_1}(U) = \{1\}$.

**Theorem 9.** Algorithm RelevantSolvableSubgroups($\ell$) determines up to conjugacy all solvable $F$-relevant subgroups of $Aut(A)$ together with their normalizers in $Aut(A)$ where $A$ is the direct product of elementary abelian groups with $|A| = \ell$.

**Proof.** We continue to use the notation from the algorithm.

First we observe that $N_R(U) = N_{Aut(A)}(U)$. Let $g \in N_{Aut(A)}(U)$. As $Aut(A)$ is a direct product, it follows that $g$ also normalizes each $\sigma_i(U)$ and thus $P(U)$. Hence $g \in R$ and thus $g \in N_R(U)$. Thus $N_{Aut(A)}(U) \subseteq N_R(U)$. As $N_R(U) \subseteq N_{Aut(A)}(U)$ is obvious, the result follows.

Completeness: Let $U$ be a solvable subgroup of $Aut(A)$ with $O(U) = \{1\}$. We show that $U$ is conjugate to a subgroup in the returned list. First $\pi_i(U)$ is conjugate in $GL(d_i, p_i)$ to a subgroup $P_i$ as determined in the algorithm. Hence we may suppose without loss of generality that $\pi_i(U) = P_i$ for $1 \leq i \leq r$. Thus $P(U) = P$ with normalizer $R$ is
Let \( \delta \) be a positive integer, let \( A \) be the direct product of elementary abelian groups with \( |A| = \ell \) and let \( U \) be an \( F \)-relevant subgroup of \( \text{Aut}(A) \) with normalizer \( N(U) = N_{\text{Aut}(A)}(U) \). Our aim is to determine up to isomorphism all extensions of \( A \) by \( U \).

As described in [25, p. 315ff], each element \( \delta \) of the group of 2-cocycles \( Z^2(U, A) \) defines an extension \( G_\delta \) of \( A \) by \( U \) and each extension of \( A \) by \( U \) is isomorphic to a group of the form \( G_\delta \). Hence the group \( Z^2(U, A) \) allows the construction of a complete (but possibly redundant) set of isomorphism types of extensions of \( A \) by \( U \). It remains to solve the isomorphism problem.

The subgroup of 2-coboundaries \( B^2(U, A) \) in \( Z^2(U, A) \) reduces this isomorphism problem. If \( \delta \) and \( \lambda \) are two elements of \( Z^2(U, A) \) which belong to the same coset of \( B^2(U, A) \), then their corresponding extensions \( G_\delta \) and \( G_\lambda \) are isomorphic, see [25, p. 316].

Another reduction is induced by the normalizer \( N(U) \). This acts on \( Z^2(U, A) \) via \( (g(\delta))(u, v) := g(\delta(u^g, v^g)) \) for \( \delta \in Z^2(U, A) \), \( g \in N(U) \) and \( u, v \in U \). If \( \lambda \) and \( \delta \) are two elements of \( Z^2(U, A) \) which belong to the same orbit under \( N(U) \), then their corresponding extensions \( G_\delta \) and \( G_\lambda \) are isomorphic, see [24].

For \( \lambda \in Z^2(U, A) \) we write \( [\lambda] := \lambda + B^2(U, A) \in H^2(U, A) = Z^2(U, A)/B^2(U, A) \). The action of \( N(U) \) on \( Z^2(U, A) \) leaves \( B^2(U, A) \) setwise invariant and hence induces an action of \( N(U) \) on \( H^2(U, A) \). We denote \( g([\lambda]) := [g(\lambda)] \) for \( g \in N(U) \) and \( \lambda \in Z^2(U, A) \). Combining the reductions induced by \( B^2(U, A) \) and this action, we obtain that two elements \( [\delta] \) and \( [\lambda] \) in the same \( N(U) \)-orbit of \( H^2(U, A) \) yield isomorphic extensions \( G_\delta \cong G_\lambda \).

In the special case of \( F \)-relevant subgroups of \( \text{Aut}(A) \), we show in the following theorem that the action of \( N(U) \) on \( H^2(U, A) \) yields a full solution of the isomorphism problem.

**Theorem 10.** Let \( U \) be an \( F \)-relevant subgroup of \( \text{Aut}(A) \) and let \( \delta, \lambda \in Z^2(U, A) \). Then \( G_\delta \cong G_\lambda \) if and only if there exists an element \( g \in N(U) \) with \( g([\delta]) = [\lambda] \).
Proof. It suffices to show that an isomorphism \( \iota : G_\delta \to G_\lambda \) implies that \([\delta]\) and \([\lambda]\) are in the same \( N(U)\)-orbit.

We consider \( A \) as additive group and write the extension \( G_\delta \) as set \( \{(u,a) \mid u \in U, a \in A\} \) with multiplication \( (u,a)(v,b) = (uv, u(a) + b + \delta(u,v)) \). Let \( A_\delta = \{(1,a) \mid a \in A\} \leq G_\delta \). Then \( \iota \) maps \( A_\delta \) onto \( A_\lambda \) and hence there exists an automorphism \( \beta \in \text{Aut}(A) \) with \( \iota((1,a)) = (1, \beta(a)) \). Further, \( \iota \) induces an automorphism \( \alpha \) of the quotient \( G_\delta / A_\delta \cong U \cong G_\lambda / A_\lambda \) and thus \( \iota \) has the form

\[
\iota : G_\delta \to G_\lambda : (u,a) \mapsto (\alpha(u), \beta(a) + \iota(a, u))
\]

for some map \( t : A \times U \to A \) satisfying \( t(a,1) = 0 \). Evaluating the equation \( \iota((u,a)) = \iota((u,0))\iota((1,a)) = \iota((u,0))\iota(1, \beta(a)) \) shows that \( t(a, u) = t(0, u) \) for all \( u \in U \) and \( a \in A \).

We thus write \( t(u) = t(0, u) \) to shorten notation.

Evaluating \( \iota((1,a))\iota(u, 0) = \iota((u, u(\iota(a)))) \) shows that \( \alpha(u)(a) = (\beta u \beta^{-1})(a) \) for all \( u \in U \) and \( a \in A \). Hence \( \alpha(u) = \beta u \beta^{-1} \) for all \( u \in U \). Thus \( \beta \in N(U) \) and \( \alpha \) is induced by conjugation with \( \beta^{-1} \).

Let \( \mu : U \to A : u \mapsto t(\alpha^{-1}(u)) \). Then \( \mu(1) = t(1) = 0 \) and hence \( \mu \in C^1(U, A) \).

Let \( \sigma \in B^2(U, A) \) be the coboundary induced by \( \mu \), i.e. \( \sigma(u, v) := \mu(uv) - v(\mu(u)) - \mu(v) \). Evaluating \( \iota((u,0)(v,0)) = \iota(u,0)\iota(v,0) \) yields that \( \beta(\delta(u,v)) + \sigma(\alpha(u), \alpha(v)) = \lambda(\alpha(u), \alpha(v)) \) for every \( u, v \in U \).

Let \( \gamma = \beta \). Then \( \alpha^{-1}(u) = u^g \). Hence \( g(\delta(u^g, v^g)) + \sigma(u, v) = \lambda(u, v) \) for every \( u, v \in U \). Thus \( g([\delta]) = [\lambda] \) as desired. \( \square \)

We note (without proof) that dual to Theorem 10 one can also determine the automorphism groups of the considered extensions.

Theorem 11. Let \( U \) be an \( F \)-relevant subgroup of \( \text{Aut}(A) \) and let \( \delta \in Z^2(U, A) \). Then

\[
\eta : \text{Aut}(G_\delta) \to \text{Stab}_{N(U)}([\delta]) : \alpha \mapsto \alpha|_A
\]

is an epimorphism with \( \ker(\eta) \cong Z^1(U, A) \).

Let \( A \) be the direct product of elementary abelian groups with \( |A| = \ell \) and let \( U \leq \text{Aut}(A) \) be an \( F \)-relevant solvable subgroup. The following algorithm determines all extensions \( G \) of \( A \) by \( U \) up to isomorphism together with \( \text{Aut}(G) \).

Algorithm RelevantExtensions( \( U, A \) )

- Compute \( H^2(U, A) \) and \( Z^1(U, A) \).
- Determine the orbits and stabilizers of \( N(U) \) on \( H^2(U, A) \).
- For each orbit representative \( [\delta] \in H^2(U, A) \), compute a corresponding extension \( G_\delta \).
- Determine \( \text{Aut}(G_\delta) \) from the stabilizer of \( [\delta] \) in \( N(U) \) and \( Z^1(U, A) \).
Let $A = A_1 \times \cdots \times A_r$ with $A_i$ elementary abelian of order $p_i^{d_i}$. Then $H^2(U, A) = H^2(U, A_1) \times \cdots \times H^2(U, A_r)$ and, similarly, $Z^1(U, A) = Z^1(U, A_1) \times \cdots \times Z^1(U, A_r)$. The cohomology groups $H^2(U, A_i)$ and $Z^1(U, A_i)$ can be computed with the algorithms in [12] for polycyclic groups.

### 3.1.3 Algorithm I for solvable groups

We now combine the results of this section to an algorithm to determine all finite solvable groups of $F$-class 1 and $F$-rank $\ell$.

**Algorithm** $\text{SolvableGroupsOfFClass1}(\ell)$

- Let $A$ be the direct product of elementary abelian groups with $|A| = \ell$.
- Determine up to conjugacy all $F$-relevant finite solvable subgroups $U$ in $\text{Aut}(A)$ together with their normalizers $N(U)$; see Section 3.1.1.
- For each such $U$, determine all extensions $G$ of $A$ by $U$ up to isomorphism together with their automorphism groups $\text{Aut}(G)$; see Section 3.1.2.

**Theorem 12.** Algorithm $\text{SolvableGroupsOfFClass1}$ determines a complete and irredudant set of finite solvable groups $G$ with $F$-class 1 and $F$-rank $\ell$.

**Proof.** First we observe that every group $G$ in the output of Algorithm I is of the required type. As $G$ is an extension of $A$ by $U$, it follows that it is finite solvable. Lemma 7 asserts that $G$ has $F$-class 1 and $F(G) \cong A$. Hence $G$ has $F$-rank $\ell$.

Completeness: Let $G$ be a finite solvable group of $F$-class 1 with $F$-rank $\ell$. Then $G$ is an extension of $A$ by some $V \leq \text{Aut}(A)$ by Lemma 8. The group $V$ is $F$-relevant by Lemma 9. Hence $V$ is conjugate $\text{Aut}(A)$ to some subgroup $U$ determined in Step 1 of Algorithm I. We can consider $G$ as extension of $A$ by $U$. Using Theorem 10 it follows that $G$ is isomorphic to a group in the output of Algorithm I.

Non-redundancy: Suppose $G_1$ and $G_2$ in the output of Algorithm I are isomorphic. Let $G_i$ be an extension of $A$ by $U_i$ for $i = 1, 2$. Then $U_1$ and $U_2$ are conjugate in $\text{Aut}(A)$. We write $U = U_1$ and assume that $G_2$ is also an extension of $A$ by $U$, as the conjugation from $U$ to $U_2$ induces an isomorphism of extensions. Using Theorem 10 we now obtain that $G_1 = G_2$.

### 3.1.4 An example

We determine the groups of order $96 = 2^5 \cdot 3$ with $F$-class 1. Each such group is solvable and has $F$-rank $\ell \in \{2^n \cdot 3^m \mid 0 \leq n \leq 5, 0 \leq m \leq 1\}$.

**Case 1:** $\ell = 2^n$ with $0 \leq n \leq 5$. Let $A$ be the elementary abelian group of order $\ell$. Then $\text{Aut}(A) = \text{GL}(n, 2)$. The relevant subgroups $U$ of $\text{Aut}(A)$ are the solvable subgroups of $\text{GL}(n, 2)$ of order $2^{5-n} \cdot 3$ with $O_2(U) = \{1\}$. The following table lists the
possible isomorphism types for $U$ and their value $n$, the number of conjugacy classes in $\text{Aut}(A)$ with this isomorphism type of subgroup $U$ and the relevant extensions with this isomorphism type of group $U$; the extensions are described by their number in the small groups library.

| $n$ | $U$  | # of classes | extensions                      |
|-----|------|--------------|---------------------------------|
| 4   | $S_3$ | 3            | (96, 194), (96, 195), (96, 226), (96, 227) |
|     | $C_3$ | 2            | (96, 228), (96, 229)            |

**Case 2:** $\ell = 2^n \cdot 3$ with $0 \leq n \leq 5$. Let $A$ be the group $C_2^n \cdot C_3$. Then $\text{Aut}(A) = \text{GL}(n, 2) \times \text{GL}(1, 3)$. The relevant subgroups of $\text{Aut}(A)$ are the solvable subgroups $U$ of $\text{Aut}(A)$ of order $2^{5-n}$ with $O(U) = \{1\}$. Note that $\text{GL}(4, 2)$ has two conjugacy classes of subgroups $V$ of order 2. Each of these determine $V \times \text{GL}(1, 3) \cong C_2^2$ and thus yield three conjugacy classes of subgroups $U$ of order 2 in $\text{Aut}(A)$. Of these, two satisfy $O(U) = \{1\}$. The following table lists the possibilities for $U$.

| $n$ | $U$  | # of classes | extensions                      |
|-----|------|--------------|---------------------------------|
| 4   | $C_2$ | 4            | (96, 159), (96, 160), (96, 218), (96, 219), (96, 230) |
| 5   | $\{1\}$ | 1            | (96, 231)                        |

4 Descendants

In this section we describe an effective method for Algorithm II. Thus given a finite group $G$ of $F$-class $c \geq 1$, our aim is to determine the descendants of $G$ up to isomorphism. Again, we are particularly interested in the computation of descendants of finite solvable groups.

We give a brief overview on our method: Its first step is the construction of a certain covering group $G^*$ of $G$, see Subsection 4.1. This will have the property that every descendant of $G$ is isomorphic to a quotient of $G^*$. Further, it will turn out that the isomorphism problem for descendants of $G$ translates to an orbit-problem for the action of a certain group of automorphisms on certain subgroups of $G^*$. Hence the covering group construction yields an effective solution of the isomorphism problem for descendants.

In the following subsections we discuss the details of this method. In the final subsection 4.6 we summarize our resulting method for Algorithm II.
4.1 Covering groups of finite groups

Let $G$ be a finite group of $F$-rank $\ell = p_1^{d_1} \cdots p_r^{d_r}$ and let $k = p_1 \cdots p_r$ be the core of $\ell$. Let $g = \{g_1, \ldots, g_n\}$ be an arbitrary generating set of $G$ and let $F$ be free on $n$ generators $\{f_1, \ldots, f_n\}$. Let $\mu : F \to G$ be the homomorphism induced by mapping $f_i$ to $g_i$ for $1 \leq i \leq n$. Denote $R = \ker(\mu)$ and let $L$ be the full preimage of $F(G)$ under $\mu$. Then we define

$$K(G) := [R, L]R^k$$

and $G^* := F/K(G)$ and $M(G) := R/K(G)$. We call $G^*$ a covering group of $G$ with multiplicator $M(G)$ and covering kernel $K(G)$ and we denote $\mu$ as the presentation epimorphism of $G$ associated with $G^*$. Further, for $p \in \{p_1, \ldots, p_r\}$ we define

$$G^*_p := F/[R, L]R^p$$

and $M_p(G) := R/[R, L]R^p$. We call $G^*_p$ a $p$-covering group of $G$ with $p$-multiplicator $M_p(G)$. The group $R/R' R^p$ is an elementary abelian $p$-group of rank $1 + (n - 1)|G|$ by Schreier’s theorem. The $p$-multiplicator $M_p(G)$ is isomorphic to $K_p/[K_p, L]$. By construction, $M(G) = M_{p_1}(G) \times \ldots \times M_{p_r}(G)$. This implies that the covering group $G^*$ is a subdirect product of the $p$-covering groups $G^*_{p_1}, \ldots, G^*_{p_r}$ with amalgamated factor group $G$.

**Theorem 13.** The isomorphism type of $G^*_p$ depends only on $G$, the number of generators $n$ and the prime $p$, but not on the presentation $\mu$. Similarly, the isomorphism type of $G^*$ depends only on $G$ and $n$, but not on $\mu$.

**Proof.** Let $p \in \{p_1, \ldots, p_r\}$. By [10, Satz 1] we obtain that the isomorphism type of the group $F/R' R^p$ depends on $G$ and on $n$, but not on $\mu$. As $G^*_p = F/[R, L]R^p \cong (F/R' R^p)/([R, L]R^p/R' R^p)$, the result for the $p$-covering group follows. Using that $G^*$ is a subdirect product of $G^*_{p_1}, \ldots, G^*_{p_r}$, we now obtain the same result for the covering group $G^*$.

4.2 Covering groups of finite solvable groups

If $G$ is solvable, then $G^*$ is solvable. Thus $G^*$ can be described by a polycyclic presentation in this case and this type of presentation allows effective computations with $G^*$. In this section we describe how a polycyclic presentation of $G^*$ can be determined.

Let $\mu : F \to G : f_i \mapsto g_i$ be the presentation epimorphism associated with $G^*$ with $R = \ker(\mu)$. Then $R/R'$ is free abelian on $(n - 1)|G| + 1$ generators and $G$ acts by conjugation on $R/R'$. We consider each prime $p$ dividing the $F$-rank $\ell$ in turn and use the Gaussian elimination algorithm to determine a basis for $[R, L]R^p/R'$. This allows us to read off a basis for the quotient $(R/R')/([R, L]R^p/R') \cong R/[R, L]R^p \cong M_p(G)$. This
information allows us to extend a polycyclic presentation of $G$ to a polycyclic presentation of $G^*_p$. In turn, this allows us to determine a polycyclic presentation for $G^*$ as subdirect product of $G^*_p, \ldots, G^*_p$.

We usually use a minimal generating set for $G$ to determine $G^*$. We note that this can be determined effectively as for example described in [17].

4.3 Automorphisms of covering groups

Let $G$ be a finite group with covering group $G^*$ and multiplicator $M = M(G)$. Let $\text{Aut}_M(G^*)$ denote the group of automorphisms of $G^*$ which leaves $M$ setwise invariant. Every $\alpha \in \text{Aut}_M(G^*)$ induces an automorphism $\overline{\alpha}$ of $G^*/M$ via $\overline{\alpha}(gM) = \alpha(g)M$. We identify $G$ with $G^*/M$ and thus $\overline{\alpha}$ with an automorphism of $G$ and obtain the following homomorphisms:

$$
\eta_G : \text{Aut}_M(G^*) \to \text{Aut}(G) : \alpha \mapsto \overline{\alpha},
$$

$$
\eta_M : \text{Aut}_M(G^*) \to \text{Aut}(M) : \alpha \mapsto \alpha|_M.
$$

We use these homomorphisms to investigate the structure of $\text{Aut}_M(G^*)$. Let $W = \ker(\eta_G)$ and $V = \ker(\eta_G) \cap \ker(\eta_M)$. Then $W$ contains those automorphisms of $G^*$ that induce the identity map on the quotient $G^*/M \cong G$ and $V$ contains those automorphisms that induce the identity map on $G^*/M$ and on $M$. The group $\text{Aut}_M(G^*)$ has the normal series

$$
\text{Aut}_M(G^*) \triangleright W \triangleright V \triangleright \{id\}.
$$

Let $\gamma : G \to \text{Aut}(M) : g \mapsto \gamma_g$ denote the conjugation action of $G$ on $M$ in $G^*$ and let $C$ be the centralizer in $\text{Aut}(M)$ of the image of $\gamma$. The natural action of $C$ on $M$ induces an action of $C$ on $Z^2(G, M)$ via $c(\delta)(g, h) = c(\delta(g, h))$. This action leaves $B^2(G, M)$ invariant and hence induces an action on $H^2(G, M)$ via $c([\delta]) = [c(\delta)]$. Let $\epsilon$ denote an element of $Z^2(G, M)$ which defines $G^*$ as extension of $M$ by $G$. Recall that $[\epsilon] = \epsilon + B^2(G, M) \in H^2(G, M)$.

**Theorem 14.** Let $G$ be a finite group with covering group $G^*$ and multiplicator $M$.

(a) $\eta_G$ is surjective and thus $\text{Aut}_M(G^*)/W \cong \text{Aut}(G)$.

(b) $W/V \cong \eta_M(W) = \text{Stab}_C([\epsilon])$.

(c) $V \cong Z^1(G, M)$.

**Proof.** (a) Let $\mu : F \to G : f_i \mapsto g_i$ be the presentation epimorphism associated with $G^*$ and let $K(G)$ be the corresponding covering kernel so that $G^* = F/K(G)$. Let $g^*_i = f_iK(G)$ for $1 \leq i \leq n$. Then $g^*_1, \ldots, g^*_n$ generates $G^*$ and $\varphi : G^* \to G : g^*_i \mapsto g_i$ is a natural epimorphism from $G^*$ to $G$. 

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Let $\beta \in \text{Aut}(G)$. Then $\{\beta(g_1), \ldots, \beta(g_n)\}$ is another generating set of $G$. By Gaschütz’ theorem \cite[ Satz 1]{G}, there exists a generating set $h_1, \ldots, h_n$ of $G^\ast$ so that $\phi(h_i) = \beta(g_i)$ for $1 \leq i \leq n$.

Let $f_i \in F$ with $f_i K(G) = h_i$ for $1 \leq i \leq n$ and define the homomorphism $\rho : F \to F$ via $\rho(f_i) = f_i^\ast$. Then $\rho$ induces $\beta$ via $\mu$ on $G$. Thus $\rho(R) \subseteq R$ and $\rho(L) \subseteq L$. Hence $\rho(K(G)) \subseteq K(G)$ and $\rho$ induces a homomorphism $\alpha : G^\ast \to G^\ast$. By construction, $\alpha(g_i^\ast) = h_i$ for $1 \leq i \leq n$ and hence $\alpha$ is surjective. As $G^\ast$ is finite, this yields that $\alpha \in \text{Aut}(G^\ast)$.

Then, this implies that $\alpha$ is a preimage of $\beta$ under $\eta_G$.

(b) Clearly $W/V \cong \eta_M(W)$. It remains to show that $\eta_M(W) = \text{Stab}_C([\epsilon])$. We use the ideas of \cite{24} for this purpose. First, we recall that the group of compatible pairs $\text{Comp}(G, M)$ is defined as $\{(\alpha, \beta) \in \text{Aut}(G) \times \text{Aut}(M) \mid \overline{g} = \overline{\beta^{-1}g\beta} \text{ for } g \in G\}$. Next, we recall from \cite{24} that $\text{Comp}(G, M)$ acts on $H^2(G, M)$ via $\delta(g, h)(\alpha, \beta) = \delta(g^{\alpha^{-1}}, h^{\alpha^{-1}})^\beta$ for $\delta \in Z^2(G, M)$, $(\alpha, \beta) \in \text{Comp}(G, M)$ and $g, h \in G$. Let $\eta : \text{Aut}(G^\ast) \to \text{Aut}(G) \times \text{Aut}(M) : \alpha \mapsto (\eta_G(\alpha), \eta_M(\alpha))$.

Then \cite{24} shows that $\text{im}(\eta) = \text{Stab}_{\text{Comp}(G, M)}([\epsilon]) \subseteq \text{Comp}(G, M)$.

Let $\alpha \in W$. Then the definition of $W$ implies that $\eta(\alpha) = (\text{id}, \beta) \in \text{Comp}(G, M)$ for some $\beta \in \text{Aut}(M)$. The definitions of $C$ and $\text{Comp}(G, M)$ imply that $\beta \in C$. Hence $\eta(W) \leq \{\text{id}\} \times C \subseteq \text{Comp}(G, M)$. Thus $\eta(W) = \text{Stab}_{\{\text{id}\} \times C}([\epsilon])$. It remains to note that $\eta(W) = \{\text{id}\} \times \eta_M(W)$ and that the action of $C$ on $H^2(G, M)$ as defined above coincides with the action of $\{\text{id}\} \times C$ as subgroup of $\text{Comp}(G, M)$. This yields the desired result.

(c) Is folklore (see e.g. \cite[Exercises 11.4]{25}).

Theorem \cite{14} and its proof are constructive and indicate a method to determine $\text{Aut}_M(G^\ast)$ explicitly. First, $Z^1(G, M)$ and thus $V$ can be computed readily as described in \cite[Section 7.6.1]{12}. Then, a preimage for a generator of $\text{Aut}(G)$ under $\eta_G$ can be obtained as described in the proof of Theorem \cite{14}. It remains to determine generators for the image $\eta_M(W)$ together with preimages under $\eta_M$ for each generator. We discuss this in more detail.

Recall that $M = M_{p_1} \times \cdots \times M_{p_r}$, where $M_{p_i}$ is elementary abelian of rank $e_i$, say. Thus

$$\text{Aut}(M) = \text{GL}(e_1, p_1) \times \cdots \times \text{GL}(e_r, p_r).$$

Further, the group $G^\ast$ is the subdirect product of $G_{p_1}^\ast, \ldots, G_{p_r}^\ast$ and each $G_{p_i}^\ast$ is an extension of $M_{p_i}$ with $G$. Let $C_{p_i}$ denote the centralizer of the conjugation action of $G_{p_i}$ on $M_{p_i}$. Next, note that

$$Z^2(G, M) = Z^2(G, M_{p_1}) \times \cdots \times Z^2(G, M_{p_r}).$$

If $\epsilon \in Z^2(G, M)$ defines $G^\ast$ as extension of $M$ by $G$ and $\epsilon = \epsilon_{p_1} + \cdots + \epsilon_{p_r}$ with $\epsilon_{p_i} \in Z^2(G, M_{p_i})$, then $\epsilon_{p_i}$ defines $G_{p_i}^\ast$ as extension of $M_{p_i}$ by $G$. Let $[\epsilon_{p_i}] = \epsilon_{p_i} + B^2(G, M_{p_i})$ for $1 \leq i \leq r$. 

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The following lemma reduces the determination of \( \text{Stab}_C([e]) \) to the elementary abelian direct factors of \( M \).

**Lemma 15.** Let \( W_{p_i} = \eta_M(W) \cap \text{GL}(e_i, p_i) \) for \( 1 \leq i \leq r \).

(a) \( \eta_M(W) = W_{p_1} \times \cdots \times W_{p_r} \).

(b) \( W_{p_i} = \text{Stab}_{C_{p_i}}([e_i]) \) for \( 1 \leq i \leq r \).

**Proof.** Let \( C \) be the centralizer in \( \text{Aut}(M) \) of the action of \( G^* \) on \( M \). The definition of a centralizer induces that \( C \) is the direct product of the groups \( C_{p_1}, \ldots, C_{p_r} \). This yields that \( \eta_M(W) \) is the direct product of the stabilizers \( \text{Stab}_{C_{p_i}}([e_i]) \) as required. \( \square \)

Let \( p \in \{p_1, \ldots, p_r\} \) and let \( e \) be the rank of \( M_p \). The centralizer \( C_p \) of the action of \( G^* \) on \( M_p \) can be determined by the following steps:

- Determine the subalgebra \( L \) of \( F_p^{\times \times} \) centralizing the action of \( G^* \) on \( M_p \).
- Determine the unit group \( U(L) \) using the radical series of \( L \) and Wedderburn’s theorem for each quotient of this series as described in [28].

Given generators for \( C_p \) one can then construct generators for the stabilizer \( \text{Stab}_{C_p}([e_i]) \).

The stabilizer is usually obtained by concurrently constructing the orbit of \([e_i] \). This is time-consuming if the orbit is large. Thus is it useful to reduce \( C_p \) to a subgroup containing the stabilizer \textit{a priori}. The following lemma exhibits a method for this purpose.

**Lemma 16.** Let \( p \in \{p_1, \ldots, p_r\} \) and let \( S \) be a Sylow \( p \)-subgroup of \( G^* \). Then \( S \) centralizes a series \( M_p = M_{p,1} > \cdots > M_{p,s} = \{1\} \) through \( M_p \). Let \( T_p = S_p^r S_p^0 \) and \( T_{p,i} = (T_p \cap M_{p,i}) M_{p,i+1}/M_{p,i+1} \). Then \( W_p \) acts trivially on \( T_{p,i} \) for \( 1 \leq i \leq s \).

**Proof.** Each element \( w \in W_{p_i} \) defines an automorphism \( \beta \) of \( G^*_p \) that centralizes \( G \cong G^*_p/M_p \). Note that \( M_p \leq S_p \) by construction and \( \beta \) centralizes \( S_p/M_p \). Let \( a, b \in S_p \).

Then \( \beta(a) = am \) and \( \beta(b) = bn \) for certain \( m, n \in M_p \). Thus \( \beta([a, b]) = \beta(a) \beta(b) = [am, bn] = [a, b] \), as \( M_p \) is central in \( S_p \). Hence \( \beta \) induces the identity on \( S_p^r \). Similarly, \( \beta(a^p) = \beta(a)^p = (am)^p = a^p \) and thus \( \beta \) induces the identity on \( S_p^0 \). This yields the desired result. \( \square \)

### 4.4 Allowable subgroups

Let \( G \) be a finite group of \( F \)-class \( c \) with covering group \( G^* \) and multiplicator \( M \). Then \( G^* \) is a finite group of \( F \)-class \( c \) or \( c + 1 \). We define \( N := \nu_c(G^*) \) and call it the \textit{nucleus} of \( G \). An \textit{allowable subgroup} \( U \) of \( G^* \) is a proper subgroup of \( M \) which is normal in \( G^* \) and satisfies \( M = NU \).

**Lemma 17.** If \( U \) is an allowable subgroup of \( G^* \), then \( G^*/U \) is a descendant of \( G \).
We consider the isomorphism problem for descendants. For this purpose we consider Theorem 18.

**Theorem 18.** Let \( H \) be an arbitrary descendant of \( G \). Then \( H \cong G^*/U \) for an allowable subgroup \( U \).

**Proof.** Let \( \mu : F \rightarrow G : f_i \rightarrow g_i \) be the presentation epimorphism associated with \( G^* \). Let \( R = \ker(\mu) \) and let \( K(G) \) be the corresponding covering kernel so that \( G^* = F/K(G) \).

As \( H \) is a descendant of \( G \), there exists an epimorphism \( \kappa : H \rightarrow G \) with kernel \( \nu_c(H) \).

Let \( e = \{e_1, \ldots, e_n\} \) be a subset of \( H \) such that \( \kappa(e_i) = g_i \) for \( 1 \leq i \leq n \). Then \( e \) generates \( H \), since by Lemma 18 (a) we have \( \nu_c(H) \leq \phi(H) \) for \( c > 0 \). Let \( \lambda : F \rightarrow H \) the be homomorphism with \( \lambda(f_i) = e_i \) for \( 1 \leq i \leq n \). Then \( \mu = \kappa \circ \lambda \) by construction.

As \( \kappa \) has kernel \( \nu_c(H) \), it maps \( F(H) \) onto \( F(G) = F(H/\nu_c(H)) \). It follows that \( \lambda \) maps \( L \) onto \( F(H) \) and \( R \) onto \( \nu_c(H) \). Therefore \( K(G) \) is mapped to \( \nu_{c+1}(H) = \{1\} \) and thus \( K(G) \) is contained in the kernel of \( \lambda \). As \( G^* = F/K(G) \), we obtain that \( \lambda \) induces a map (which we also denote by \( \lambda \)) from \( G^* \) onto \( H \).

Let \( U \) be the kernel of \( \lambda \). It remains to show that \( U \) is an allowable subgroup of \( G^* \). First, \( U \) is a normal subgroup of \( G^* \). Next, \( U \) is contained in \( M \), as \( \lambda \circ \kappa \) is an epimorphism on \( G \) with kernel \( M \). Then \( U \) is a proper subgroup of \( M \), as \( H \) has larger \( F \)-class than \( G \). Finally, \( UN = M \), as \( \lambda(N) = \nu_c(H) = \lambda(M) \).

For a subgroup \( U \) of \( M \) let \( U_p = U \cap M_p \) and note that \( U_p \) is the Sylow \( p \)-subgroup of \( U \). If \( N \) is the nucleus of \( M \) and \( U \) is an allowable subgroup, then \( U_pN_p = M_p \) and hence \( U_p \) is a supplement to \( N_p \) in \( M_p \). Conversely, each allowable subgroup can be described as direct product of its Sylow subgroups and hence can be composed from supplements to \( N_p \) in \( M_p \).

### 4.5 The isomorphism problem

Again, let \( G \) be a finite group with covering group \( G^* \), multiplicator \( M \) and nucleus \( N \). We consider the isomorphism problem for descendants. For this purpose we consider the action of \( \text{Aut}_M(G^*) \) on the set of all allowable subgroups. We first show that this action is well-defined and then we prove that it solves the isomorphism problem for descendants.

**Lemma 19.** Let \( U \) be an allowable subgroup of \( G^* \) and let \( \alpha \in \text{Aut}_M(G^*) \). Then \( U^\alpha \) is an allowable subgroup of \( G^* \).
Proof. If $U$ is normal in $G^*$, then $U^\alpha$ is normal as well. Next, the nucleus $N$ is characteristic in $G^*$ and hence invariant under $\alpha$. Therefore $NU^\alpha = (NU)^\alpha = M^\alpha = M$ and thus $U^\alpha$ is a supplement to $N$ in $M$. \hfill \Box

Theorem 20. Let $U_1, U_2$ be two allowable subgroups of $G^*$. Then $G^*/U_1 \cong G^*/U_2$ if and only if there exists $\alpha \in \text{Aut}_M(G^*)$ which maps $U_1$ onto $U_2$.

Proof. If an $\alpha \in \text{Aut}_M(G^*)$ exists with $U_1^\alpha = U_2$, then it induces the isomorphism $G^*/U_1 \to G^*/U_2 : gU_1 \mapsto \alpha(g)U_2$. It remains to prove the converse. Let $\mu : F \to G : f_i \mapsto g_i$ be the presentation epimorphism associated with $G^*$. Let $R = \ker(\mu)$ and let $K(G)$ be the covering kernel so that $G^* = F/K(G)$. Let $\rho : F \to G^*$ be the natural homomorphism onto the factor group $G^*$.

Let $\beta : G^*/U_1 \to G^*/U_2$ be an isomorphism. For $i = 1, 2$, let $V_i \leq F$ be the full preimage of $U_i$ under $\rho$. Then $G^*/U_i$ is naturally isomorphic to $F/V_i$ for $i = 1, 2$ and thus $\beta$ induces an isomorphism $\beta : F/V_1 \to F/V_2$.

For each $1 \leq i \leq n$ choose $y_i \in F$ with $\hat{\beta}(f_i V_1) = y_i V_2$. Thus $G^*/U_2 \cong F/V_2 = \langle y_1 V_2, \ldots, y_n V_2 \rangle$. As $U_2 = V_2/K$ is finite and $G^*$ is finitely generated, we can apply [11 Satz 1]. Thus there exist $z_1, \ldots, z_n \in F$ with $z_i V_2 = y_i V_2$ for $1 \leq i \leq n$ and $G^* = \langle z_1 K, \ldots, z_n K \rangle$. Now we define the group homomorphism

$$\sigma : F \to G^*$$

given by $\sigma(f_i) = z_i K$ for $1 \leq i \leq n$.

Then $\sigma$ is surjective by construction, and for all $g \in F$ we have $\beta(\rho(g)U_1) = \sigma(g)U_2$. Therefore $\sigma(R)U_2 = \beta(\rho(R)U_1) = \beta(M/U_1) = M/U_2$, since $\beta$ maps $M/U_1 = \nu_\cdot(G^*/U_1)$ to $M/U_2 = \nu_\cdot(G^*/U_2)$. Hence $\sigma(R) \leq M = R/K$. Similarly, $\sigma$ maps $L$ into $L/K$. This yields that $\sigma$ maps $K = [R, L]R^k$ into $K/K = 1$, hence $K \leq \ker(\sigma)$. Since $G^* = F/K$ is finite, $\ker(\sigma) = K$. Thus $\sigma$ induces the automorphism

$$\alpha : G^* \to G^*$$

given by $\alpha(f, K) = z_i K$ for $1 \leq i \leq n$.

By construction, $\alpha$ maps $M$ onto $M$ and thus is an element of $\text{Aut}_M(G^*)$. Further, $\alpha$ maps $U_1$ onto $U_2$ and hence is an automorphism of the desired form. \hfill \Box

Similar to Theorem 20 one can determine the automorphism group of a descendant. The proof of the following theorem is a variation on the proof of Theorem 20 and is thus omitted.

Theorem 21. Let $H = G^*/U$ be an arbitrary descendant of $G$. Then there exists the natural epimorphism: $\text{Stab}_{\text{Aut}_M(G^*)}(U) \to \text{Aut}(H) : \alpha \mapsto \alpha_{G^*/U}$. 

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4.6 Algorithm II for solvable groups

In this section we summarize our method to determine the descendants for a finite solvable group. The main reason for restricting this part of the algorithm to finite solvable groups is that this type of groups can be defined by a polycyclic presentation and this, in turn, allows effective computations with the considered groups.

Algorithm Descendants( $G$ )
- Determine $G^*$ with multiplicator $M$ and nucleus $N$, see Section 4.1.
- Determine the set $\mathcal{L}$ of allowable subgroups of $G^*$.
- If $|\mathcal{L}| \leq 1$, then return $\{G^*/U \mid U \in \mathcal{L}\}$ and $\{\text{Aut}(G^*/U) \mid U \in \mathcal{L}\}$.
- Determine $\text{Aut}_M(G^*)$ from $\text{Aut}(G)$, see Section 4.3.
- Determine orbits and stabilizers for the action of $\text{Aut}_M(G^*)$ on $\mathcal{L}$.
- For each orbit representative $U$ determine $H = G^*/U$ and $\text{Aut}(H)$, see Theorem 21.
- Return the resulting list of descendants $H$ with their automorphism groups.

In the following, we discuss improvements to the Algorithm Descendants. Recall that the multiplicator $M$ is a direct product of elementary abelian groups $M = M_{p_1} \times \cdots \times M_{p_r}$. A major improvement can be obtained by reducing as many computations as possible to each of the direct factors $M_{p_i}$ and use linear algebra to compute with each such factor.

The factorisation of $M$ induces that the nucleus $N$ and each allowable subgroup $U$ splits as well into a similar direct product and $U_{p_i}$ is a supplement to $N_{p_i}$ in $M_{p_i}$. Thus if $\mathcal{L}_{p_i}$ is the set of proper supplements to $N_{p_i}$ in $M_{p_i}$, then it follows that

$$\mathcal{L} = \{U_1 \times \cdots \times U_r \mid U_i \in \mathcal{L}_{p_i} \text{ for } 1 \leq i \leq r\}.$$ 

Let $\eta_M : \text{Aut}_M(G^*) \to \text{Aut}(M) : \alpha \mapsto \alpha|_M$ the homomorphism induced by the action of $\text{Aut}_M(G^*)$ on $M$. Then all except the last step of Algorithm Descendants uses as acting group $\Gamma = \text{im}(\eta_M) \leq \text{Aut}(M)$ only and not $\text{Aut}_M(G^*)$ itself. We note that

$$\text{Aut}(M) = \text{Aut}(M_{p_1}) \times \cdots \times \text{Aut}(M_{p_r}),$$

and thus $\Gamma$ is a subgroup of this direct product. Let $\Gamma_i = \Gamma \cap \text{Aut}(M_{p_i})$ for $1 \leq i \leq r$. Then

$$\Delta := \Gamma_1 \times \cdots \times \Gamma_r \leq \Gamma.$$ 

The computation of orbits under the action of the direct product $\Delta$ splits into the computation of the orbits in each direct factor. Hence, instead of one computation with a long orbit, we obtain $r$ orbit computations with shorter orbits. This induces a significant reduction for Algorithm Descendants.

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However, the examples in Section 4.7 show that $\Delta$ can be a proper subgroup of $\Gamma$ and $\Gamma$ can act non-trivially on $\Delta$-orbits. Thus we cannot reduce the computation of descendants to the Sylow subgroups of $M$ completely. However, we determine and use the subgroup $\Delta$ of $\Gamma$ to reduce the arising orbit computations. For this purpose it is useful to observe that the image of $W$ in $\text{Aut}(M)$ is a subgroup of $\Delta$, see Section 4.3.

### 4.7 Examples

Let $G_1$ be the symmetric group on four points, $G_2$ the group $(36, 3)$ from the small groups library and $G_3$ the group $(72, 22)$. The following table lists the orders of the multiplicators and nucleuses of these groups with respect to minimal generating sets of the underlying groups. The table then exhibits the descendants of these groups. In the cases of $G_1$ and $G_2$ we list the descendants explicitly by their number in the small groups library. In the case of $G_3$ we describe them by their stepsizes $\sigma$, where the stepsize of a descendant $H$ is $\sigma = |H|/|G_3| = |H|/72$. Note that an entry $\sigma^x$ means that there are $x$ descendants of stepsize $\sigma$.

| group | $|M|$ | $|N|$ | descendants |
|------|------|------|-------------|
| $G_1$ | $2^8$ | $2^4$ | (48, 28), (48, 29), (96, 64), (192, 180), (192, 181) |
| $G_2$ | $2^5 \cdot 3^3$ | $2^4 \cdot 3$ | (72, 3), (144, 3), (288, 3), (108, 3), (216, 3), (432, 3), (864, 3) |
| $G_3$ | $2^5 \cdot 3^6$ | $2 \cdot 3^3$ | $2^4, 3^2, 6^2, 9^2, 18^2, 27, 54^2$ |

For $G_2$ and $G_3$ the multiplicators are direct products of their Sylow 2- and 3-subgroups. We note that in the notation of Section 4.6, the group $\Delta$ is a proper subgroup of $\Gamma$ in both cases. This has an impact on the descendant computation in the case of $G_3$: there are three descendants of stepsize 2, two descendants of stepsize 3, but $7 > 2 \cdot 3$ descendants of stepsize $2 \cdot 3$. This shows that the orbit-stabilizer computation used in Algorithm Descendants does not fully reduce to the Sylow subgroups of the multiplicator.

### 5 Implementation

We have implemented our methods to construct finite solvable groups as GAP Package GroupExt [15]. Our implementation uses a variety of algorithms from GAP and other packages. In particular, it uses the following.

- The FGA package [27] for computations in free groups.
- The Polycyclic package [7] for computations in polycyclic groups.
- The AutPGroup package [8] to determine the automorphism groups of finite $p$-groups.
- The genss package [18] for computing stabilizers using a randomised Schreier-Sims algorithm.
6 The groups of a given order

Our implementations of Algorithms I and II for solvable groups can be used to determine the solvable non-nilpotent groups of a given order $o$. By induction, we assume that all solvable non-nilpotent groups of order properly dividing $o$ are given. We then proceed in two steps:

1. Determine all solvable non-nilpotent groups of order $o$ and $F$-class 1 up to isomorphism; For this purpose consider every proper divisor $\ell$ of $o$ and determine the solvable groups of order $o$, $F$-class 1 and $F$-rank $\ell$.

2. For every solvable non-nilpotent group $G$ of order $s$, where $s$ is a proper divisor of $o$, determine the descendants of $G$ of order $o$.

We have used this to determine (again) all solvable non-nilpotent groups of order at most 2000 as available in the small groups library. This provides the first independent check for the correctness of the Small Groups library.

6.1 The groups of order 2304

We have used the algorithm described in this paper to determine (for the first time) the groups of order $2304 = 2^8 \cdot 3^2$. We provide some summary information on these groups in this section. The groups themselves will be made available as part of a forthcoming GAP package [14].

The nilpotent groups of order 2304 can be obtained readily as direct products of $p$-groups and thus they can be considered as available; There are 112184 of them. In the following we concentrate on the non-nilpotent groups of order 2304. Every such group is solvable by Burnside’s pq-Theorem.

We first consider the non-nilpotent groups of $F$-class 1. Table 1 lists their possible $F$-ranks $\ell$ and for each $\ell$ the number of solvable groups of order 2304, $F$-class 1 and $F$-rank $\ell$. We omit those divisors $\ell$ of $o$ which do not lead to any groups. In summary, there are 1953 non-nilpotent groups of order 2304 of $F$-class 1. Note that $C_2^3 \times C_3^2$ has order 2304, $F$-class 1 and $F$-rank 2304 but is abelian and hence not included.
Next we consider the groups of $F$-class $c > 1$. To determine these, we have to consider each group of order properly dividing $2^{304}$ and determine descendants of order $2^{304}$. As a preliminary step, it is useful to determine $a \text{ priori}$ which groups of a given divisor $s$ of $o$ could have descendants of order $o$. The following lemma is highly useful for this purpose.

**Lemma 22.** Let $G$ be a non-nilpotent group of order $2^a \cdot 3$ such that $G$ has descendants of order $2^a \cdot 9$. Then $F(G) \cong C_2^{a-1} \times C_3$, and $G/O_2(G) \cong S_3$.

**Proof.** Since $G$ has a descendant of order $3 \cdot |G|$, its $F$-central rank is divisible by 3, hence $|F(G)|$ is divisible by 3. But then $F(G)$ contains a 3-Sylow subgroup $P_3 \cong C_3$ of $G$. Since $F(G)$ is nilpotent, it splits into a direct product of $P_3$ and a 2-group. Since $F(G)$ is normal in $G$, also $P_3$ is normal in $G$. Hence $O_3(G) = P_3 \cong C_3$.

By Sylow’s theorem, the number of 2-Sylow subgroups in $G$ is one or three. If there was only one, then it would be normal and $G$ would be nilpotent. Hence there are three and the conjugation action of $G$ on the set of its 2-Sylow subgroups yields a homomorphism into $S_3$ whose kernel is a normal 2-subgroup of $G$. Since the kernel is not a full 2-Sylow subgroup (else $G$ would be nilpotent), it must have index 6 in $G$ and thus order $2^{a-1}$. Clearly, this kernel coincides with $O_2(G)$. Therefore $G/O_2(G) \cong S_3$.

It remains to show that $O_2(G) \cong C_2^{a-1}$. For this, suppose that $H$ is a descendant of $G$ of order $2^a \cdot 9$ and $F$-central class $c$. Thus $H/\nu_c(H) \cong G$, hence $\nu_c(H) \cong C_3$. Since the $F$-central class of $H$ is larger than that of $G$, the only way this is possible is if $c = 2$ and the preimage of $P_3$ in $H$ is isomorphic to $C_9$. But then $G$ has $F$-central class 1 and $F(G)$ is a direct product of elementary abelian subgroups.

We can apply this to the 1090235 groups of order $768 = 2^7 \cdot 3$ to determine those which may have descendants of order 2304. It turns out that only 8 groups remain, each having precisely one descendant of the order 2304.

| F-central rank | # of groups |
|---------------|-------------|
| $32 = 2^5$    | 8           |
| $64 = 2^6$    | 37          |
| $128 = 2^7$   | 28          |
| $144 = 2^4 \cdot 3^2$ | 193       |
| $192 = 2^6 \cdot 3$ | 208       |
| $256 = 2^8$   | 9           |
| $288 = 2^5 \cdot 3^2$ | 834       |
| $384 = 2^7 \cdot 3$ | 54        |
| $576 = 2^6 \cdot 3^2$ | 558       |
| $768 = 2^8 \cdot 3$ | 8         |
| $1134 = 2^7 \cdot 3^2$ | 16        |

**Table 1:** Groups of F-central class 1
A summary of the descendant computation is given in Table 2.

| order | # groups | # non-nilpotent | # grps w/ descendants | # descendants |
|-------|----------|----------------|------------------------|--------------|
| 6     | 2        | 1              | 0                      | 0            |
| 12    | 5        | 3              | 0                      | 0            |
| 18    | 5        | 3              | 0                      | 0            |
| 24    | 15       | 10             | 0                      | 0            |
| 36    | 14       | 10             | 0                      | 0            |
| 48    | 52       | 38             | 4                      | 34210        |
| 72    | 50       | 40             | 2                      | 6            |
| 96    | 231      | 180            | 5                      | 728926       |
| 144   | 197      | 169            | 21                     | 68945        |
| 192   | 1543     | 1276           | 6                      | 24889        |
| 288   | 1045     | 943            | 116                    | 10835672     |
| 384   | 20169    | 17841          | 7                      | 426          |
| 576   | 8681     | 8147           | 865                    | 1980937      |
| 768   | 1090235  | 1034143        | 8                      | 8            |
| 1152  | 157877   | 153221         | 47848                  | 1967974      |

Table 2: Groups of order 2304 and $F$-class $c > 1$.

In total there are $112184 + 1953 + 15641993 = 15756130$ isomorphism types of groups of order 2304. In comparison, there are 10494213 groups of order 512.

Table 3 contains the top-ten among the groups of order properly dividing $o$; that is, those groups with the ten largest numbers of descendants. It is interesting to note that over 56% of the groups of order 2304 are descendants of a single group.

| group      | descendants |
|------------|-------------|
| (288, 1040)| 8937790     |
| (576, 8590)| 707578      |
| (96, 230)  | 696554      |
| (288, 1043)| 696554      |
| (288, 1044)| 696554      |
| (576, 8588)| 203006      |
| (288, 976) | 160928      |
| (576, 8582)| 131664      |
| (576, 8675)| 120310      |
| (576, 8589)| 110292      |

Table 3: Top ten groups in terms of number of descendants.
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