Active set strategy-based sequential approximate programming for reliability-based design optimization

Xue An and Dongyan Shi

Abstract
To improve the evaluation efficiency of failure probability in RBDO models with uncertainty, many RIA-based, PMA-based methods have evolved as a powerful procedure, including the modified reliability index approach (MRIA), PMA two-level, PMA with sequential approximate programming (SAP). However, MRIA may encounter inefficiency and instability when applied to complex concave performance functions, and so does PMA two-level, not for PMA with SAP. The active set strategy-based SAP (ASS-based SAP) for PMA is proposed to accelerate computational efficiency through establishing an active set strategy and a deciding factor. The active set strategy defined by using inequality is to identify the feasible most probable target point (MPTP) in the inner loop. The decision factor integrates the reliability index and the active set strategy to quickly renew the active constraints in the outer loop. The reliability assessment and outer optimization are driven simultaneously, thereby the computational efficiency is strengthened. Numerical examples are compared with other reliability methods to demonstrate the excellent performance of the proposed method in efficiency and robustness. Results also show that the proposed method has the ability to solve complex RBDO problems.

Keywords
RBDO, uncertainties, reliability assessment, PMA

Introduction
Reliability-Based Design Optimization (RBDO) is proposed as the most outstanding design tool to tackle uncertainties in engineering and is widespread attention in various fields.1–4 RBDO specifies the uncertainty as a probability optimization model, which includes the minimum cost-minimizing function and probability constraints with random distribution information to attain an optimized design that meets the expected reliability level.5,6 However, the evaluation inefficiency and numerical convergence difficulties of large and complex highly nonlinear systems severely limit the application of RBDO in engineering.

Generally, methods for solving RBDO could be roughly divided into the decoupling method,7,8 two-level method,9–11 and single-loop method.12–14 Amongst, the two-level method is the most prioritized method due to its simpler and more reliable. Specifically, two-level methods involve double loops in each iteration: the inner loop promotes reliability analysis while deterministic optimization analysis is run according to the outer loop. The key step is the inner loop, where does a sub-optimization program solve by either the reliability index approach (RIA)15,16 or the performance measure approach (PMA).17,18 RIA is
essentially built on the simplified HL-RF algorithm\textsuperscript{19,20} to convert probability constraints into reliability index constraints and specify the most probable failure point (MPFP) of the limit failure surface.\textsuperscript{21} Conversely, PMA adopts the inverse reliability analysis to capture the most probable target point (MPTP), which is located on the target reliability surface by converting probability measures into performance measures.\textsuperscript{22} Therefore, PMA is unable to provide the information on the reliability index because it aims to minimize a complex constraint as the objective function and is subject to simple target reliability. However, a remarkable fact is that the solution of MPFP or MPTP must be numerically active and stable so that the process of evaluating probabilistic constraints can be effective. Otherwise, this sub-optimization procedure will aggravate the cost of probabilistic evaluations during the whole optimization process. For the outer loop, by means of the MPFP or MPTP, probabilistic constraints are converted to deterministic constraints according to the Taylor series expansion.\textsuperscript{22,23} This is the essential difference between the traditional RIA two-level and PMA two-level and has a property of nesting. Therefore, reducing the evaluation cost of probabilistic constraints could be started from both sub-optimization and outer optimization.

Initially, the traditional RIA two-level is deployed to search the MPFP and corresponding reliability indices, which often results in slow convergence and numerical singularities.\textsuperscript{5,24,25} Lin et al.\textsuperscript{9} developed the modified reliability index approach (MRIA) by revising the reliability indices and implementing it to accurately and stably find solutions for MPFP. However, it also inherited some inefficient features of the MPFP search when highly nonlinear performance functions are involved and furthermore proposed the hybrid reliability method.\textsuperscript{26} Thereafter, Cheng et al.\textsuperscript{27} recommended the RIA-based sequential approximate programming (SAP) to capture simultaneous convergence of reliability assessment and design optimization by resolving a series of sub-programming the approximate MPFP, while it also optimizes the tolerance synthesis problems.\textsuperscript{28} From the application examples, the SAP approach shows superior efficiency and robustness in comparison with the other reliability methods when assessing the reliability indexes. Alternatively, from the perspective of decoupling strategy, the traditional two-level optimization is completely converted into one-level optimization to remove nesting nature, but when it explores the multi-dimensional (i.e. multiple design variables) problem of the RBDO system, it still presents an unaffordable computational pressure.\textsuperscript{29,30}

Compared to RIA-based, PMA-based is comparatively more effective and robust due to its being slightly affected by the types of distribution for random variables.\textsuperscript{31,32} However, the iterative scheme of the traditional PMA two-level sometimes is susceptible to numerical instability and non-convergence issues when computing the MPTP for a concave performance function with highly nonlinear problems.\textsuperscript{33,34} For this reason, many PMA-based optimization schemes have been emerged recently to strengthen the efficiency of searching MPTP, such as a step length adjustment (SLA),\textsuperscript{35} hybrid chaos control (HCC) and modified chaos control (MCC),\textsuperscript{21} a relaxed mean value (RMV),\textsuperscript{36} etc. Although some efforts have been made to quickly search for suitable and efficient MPTP, the computational expense of PMA-based two-level is still heavy when complex performance functions are engaged.\textsuperscript{37} Similar to the RIA-based method, Yi et al.\textsuperscript{11} not only extended the framework of SAP strategy for the PMA-based probabilistic structural design optimization (PSDO) but also further studied the efficiency of SAP with PMA and conducted error analysis.\textsuperscript{38} SAP strategy discretizes the RBDO problem into a series of approximate subproblems by first-order Taylor expansion and solved it before reaching the optimum. The merit of SAP is that reliability analysis and structural optimization design are performed at the same time, which greatly reduces computational cost issues. Inspired by the literature presented, PMA with SAP is more promising in convergence and efficiency.

In this paper, we suggest the active set strategy-based SAP (ASS-based SAP) for PMA to strengthen the efficiency and robustness of evaluating probabilistic constraints for large-scale RBDO applications. The critical step is to establish an active set strategy in the loop, where the feasible MPTP can be selected. Meanwhile, the reliability index and active set strategy are integrated into the outer loop to define the decisive factor, which is used to update the active constraints.

The outline of this article is organized along these lines: details of two fundamental RBDO approaches are specified in Section 2. The different iterative schemes derived from two basic approaches are discussed in Section 3. In Section 4, the process of deriving the proposed method is presented. In Section 5, three complex RBDO problems are shown to demonstrate the efficiency and robustness of the proposed method. Finally, the conclusions are drawn in Section 6.

**Basic formulation of RBDO**

RBDO\textsuperscript{39} has evolved into a powerful design tool, which can explore the best design within the desired reliability level while meeting the constraints. Its basic mathematical expression is

\[
\begin{align*}
\min \quad & z(\tilde{d}) \\
\text{s.t.} \quad & P[G_i(\bar{X}, \tilde{d}) \geq 0] \leq P_{f_i}, \quad i = 1, 2, ..., m \\
& \tilde{d}^l \leq \tilde{d} \leq \tilde{d}^u
\end{align*}
\]
where \( \vec{d} \) denotes the design variable vector of the system, which is also the mean vector of the random variable vector \( \vec{X} \), and varies between the lower bound \( \vec{d}^l \) and the upper bound \( \vec{d}^u \). \( \vec{X} \) is used to model uncontrolled uncertainties in the system with independent and uncorrelated characteristics. \( z \) indicates the minimum cost function in terms of \( \vec{d} \). The limit state function or performance function is denoted by \( G_i(\vec{X}, \vec{d}) \) and \( G_i(\vec{X}, \vec{d}) \geq 0 \) specifies the failure region. Then, the probabilistic constraint \( P(G_i(\vec{X}, \vec{d}) \geq 0) \leq P^*_i \) describes that the failure probability of dissatisfying the \( i \)-th limit state function should be less than the acceptable failure probability, \( P^*_i \). The total number of \( G_i(\vec{X}, \vec{d}) \) is presented in \( m \).

In statistics, the random uncertainties are assessed by the probability model, where the cumulative distribution function (CDF), \( F_{G_i}(\cdot) \) is exercised to calculate the failure probability

\[
P_f = P[G_i(\vec{X}, \vec{d}) \geq 0] = F_{G_i}(0, \vec{d})
\]

where \( f_X(\vec{x}) \) states the joint probability density function (JPDF) of \( \vec{X} \). \( \beta_i^* \) stands for the target reliability index, which can be derived from \( P^*_i \) and \( \Phi \) means the standard CDF.

According to equation (2), the failure probability of the systems requires to be evaluated for each constraint where multiple integrations are involved. As we all know, it may be heavily costly to perform multidimensional integration throughout the coverage of JPDF. Thus, the probabilistic constraints of RBDO can be further assessed in two alternative forms

\[
\beta_i(\vec{d}) = (-\Phi^{-1}(F_{G_i}(0, \vec{d}))) \geq \beta_i^*
\]

\[
G_i(\vec{d}) = F_{G_i}^{-1}(\Phi(-\beta_i^*), \vec{d}) \geq 0
\]

where \( \beta_i(\vec{d}) \) indicates the evaluated reliability index and \( G_i(\vec{d}) \) stands for the probabilistic performance measure.

The probabilistic constraint in equation (1) is calculated by applying equation (3) with the reliability index, the generating RIA is

\[
\min z(\vec{d})
\]

\[
s.t. \quad -\beta_i(\vec{d}) \leq -\beta_i^*, \quad i = 1, ..., m
\]

\[
\vec{d}^l \leq \vec{d} \leq \vec{d}^u
\]

Similarly, equation (4) in terms of the performance measure function is utilized to compute the probabilistic constraint in equation (1), then, the PMA is derived

\[
\min \quad \beta_{HL} = \|\vec{u}_i\|
\]

\[
s.t. \quad G_i(\vec{d}, \vec{u}_i) = 0
\]

where \( \vec{u}_i \) presents the MPFP, which has the highest probability of failure and further away from origin indicates less failure likelihood.

However, the conventional RIA has been acknowledged as intolerable numerical instability and inability to converge to the global optimum in searching for MPFP.9,17

**MRIA two-level**

MRIA\(^9\) outperforms RIA in convergence and stability for computing the MPFP owing to a new reliability index \( \beta_{M,i} \) as

\[
\beta_{M,i} = \vec{u}_i^\top \nabla_u G(\vec{d}, \vec{u}_i) \| \nabla_u G(\vec{d}, \vec{u}_i) \|^{-1}
\]
where \( \nabla_u g(\tilde{d}, \tilde{u}^i) \) refers to the gradient vector of \( i \)-th performance function at \( \tilde{u}^i \). Thus, MRIA can be mathematically given as

\[
\begin{align*}
\min & \quad \beta_{M,i} = \tilde{u}^i, \\
\text{s.t.} & \quad G(\tilde{d}, \tilde{u}_i) = 0
\end{align*}
\tag{9}
\]

Combining equation (9) and equation (5), MRIA-based two-level is represented by

\[
\begin{align*}
\min & \quad C(\tilde{d}) \\
\text{s.t.} & \quad -(\min \beta_{M,i}(\tilde{d}, \tilde{u}^i) | G(\tilde{d}, \tilde{u}) = 0) \leq -\beta'_i(\tilde{d}), i = 1, \ldots, m \\
& \quad \tilde{d} \leq \tilde{d} \leq \tilde{d}'
\end{align*}
\tag{10}
\]

Here, the first level is that the inner loop executes equation (9) with the initial value \( \tilde{u}^0 = 0 \), which is used to calculate MPFP and corresponds to the reliability index. Another level is the optimization of the outer loop for the design variable in the physical space (\( \bar{X} \)-space), doing equation (10). Thus, equation (10) is called MRIA two-level.

Compared with RIA, MRIA two-level could provide a reliable and correct solution while rejecting the shortcomings of RIA, however, the efficiency of searching for the MPFP problems has always been the bottleneck of MRIA two-level.\textsuperscript{26}

PMA two-level

PMA two-level\textsuperscript{38} is similar to MRIA two-level, however, the simple difference is that its inner loop entails the inverse reliability analysis in the \( \bar{U} \)-space which is explicit as

\[
\begin{align*}
\min & \quad G(\bar{u}) \\
\text{s.t.} & \quad ||\bar{u}|| = \beta'_i
\end{align*}
\tag{11}
\]

where \( \bar{u}^i \) is the solution of equation (11) is labeled as MPTP with the prescribed reliability index \( \beta'_i \). By integrating equations (11) and (6), thus, the classical two-level based PMA is defined

\[
\begin{align*}
\min & \quad z(\tilde{d}) \\
\text{s.t.} & \quad -(\min G(\tilde{d}, \tilde{u}) | ||\tilde{u}|| = \beta'_i) \leq 0, i = 1, 2, \ldots, m
\end{align*}
\tag{12}
\]

The iteration of equation (12) usually starts with the origin (i.e. \( \bar{u}^0 = 0 \)) in \( \bar{U} \)-space, which is the initial estimation of MPFP and always keeps starting from the origin at each sub-step. The fixed iterative point of the concave performance measure function incurs that the boundary solution so that the point cannot attain correctly the global optimal design point of equation (12), although PMA two-level exhibits high performance for both efficiency and robustness.\textsuperscript{33}

PMA with SAP

Yi and Cheng\textsuperscript{38} successfully proposed the PMA with SAP method is represented by

\[
\begin{align*}
\min & \quad z^{(k)}(\tilde{d}) \\
\text{s.t.} & \quad C^{(k)} = -G^{(k)}(\tilde{d}) \leq 0 (k = 1, 2, \ldots), (i = 1, \ldots, m) \\
& \quad \tilde{d}^i \leq \tilde{d} \leq \tilde{d}^i \leq \tilde{d}'
\end{align*}
\tag{13}
\]

where \( G^{(k)}(\tilde{d}) \) denotes the approximate function, which is computed by the linear Taylor’s expansion of \( G(\tilde{d}) \) in terms of \( \tilde{d} \) at the current design point \( \tilde{d}^{(k)} \), that is,

\[
G^{(k)}(\tilde{d}) = G(\tilde{d}^{(k-1)}) + \nabla_d G(\tilde{d}^{(k-1)})^T(\tilde{d} - \tilde{d}^{(k-1)}) \tag{14}
\]

where \( k \) denotes the \( k \)-th sub-programming step of the outer structural optimization. Equation (13) is rewritten linear constraint as

\[
\begin{align*}
\min & \quad z^{(k)}(\tilde{d}) \\
\text{s.t.} & \quad C^{(k)} = -G^{(k)}(\tilde{d}^{(k-1)}) - \nabla_d G^{(k)}(\tilde{d}^{(k-1)})^T(\tilde{d} - \tilde{d}^{(k-1)}) \leq 0 \\
& \quad (k = 1, 2, \ldots), (i = 1, \ldots, m) \\
& \quad \tilde{d} \leq \tilde{d}^i \leq \tilde{d} \leq \tilde{d}^i \leq \tilde{d}'
\end{align*}
\tag{15}
\]

where \( C^{(k)} \) means the \( k \)-th deterministic constraint for the outer loop. For the item \( G^{(k)}(\tilde{d}^{(k-1)}) = G(\tilde{d}^{(k-1)}, \tilde{u}^{k-1}) \), a solution to equation (11) requires to be called repeatedly with the following several iterations

\[
\begin{align*}
\tilde{u}^{k} = -\beta'_i \cdot \frac{\nabla_u G^{(k-1)}(\tilde{d}^{k-1}, \tilde{u}^{k-1})}{\nabla_G(\tilde{d}^{k-1}, \tilde{u}^{k-1})} \\
G^{(k)}(\tilde{d}, \tilde{u}) = G^{(k)} (\tilde{d}^{k}, \tilde{u}^{k}) = G^{(k)} (\tilde{d}^{k-1}, \tilde{u}^{k-1}) \\
= G^{(k)} (\tilde{d}^{k-1}, \tilde{u}^{k-1}) - \beta'_i \frac{\nabla_u G^{(k-1)}(\tilde{d}^{k-1}, \tilde{u}^{k-1})}{\nabla_G(\tilde{d}^{k-1}, \tilde{u}^{k-1})}
\end{align*}
\tag{16}
\]

where \( \tilde{u}^{k} \) and \( \tilde{u}^{k-1} \) stand for the MPTP in the current and previous step, respectively. PMA with SAP is a two-level algorithm in nature. The MPTP obtained is preserved in the previous step (i.e. \( \tilde{u}^{k-1} \)) and hired as the starting estimation of MPTP in the next step. Therefore, it efficiently accomplishes the synchronous convergence of reliability analysis and structural optimization. This differs from the typical PMA two-level, which \( \tilde{u}^{0} = 0 \) is appointed as the starting estimate of optimized MPTP for any sub-programming step.
The proposed method

Iterative strategy of the proposed method

To further cut down the number of function evaluations without losing robustness, a new efficient and robust method-based SAP for PMA is proffered in this part, via establishing the active set strategy in the inner loop while defining the deciding factors in the outer loop.

For the inner loop, an active set strategy by using inequality is equipped, which is expressed as

$$ A = \left\{ \frac{\nabla f(x)}{\nabla f(x)} \mid 6\sigma < 0, \frac{\nabla f(x)}{\nabla f(x)} \in R \right\} \quad (17) $$

where $A$ is equivalent to the interval threshold to filter and obtain the feasible MPTP. Thus, a new reliability index is refined after screening the MPTP as

$$ \beta(d, \tilde{u}_{MPTP}) = \begin{cases} \tilde{u}_{f}(d), & \frac{\nabla G(x, \tilde{u}_{MPTP})}{\nabla G(x, \tilde{u}_{MPTP})} \in A, \text{ feasible} \\ \text{o.w.,} & \tilde{u}_{f}(d) \in A, \text{ infeasible} \end{cases} \quad (18) $$

where $\tilde{u}_{f}(d)$ denotes effective MPTP obtained from the equation (11) and equation (17), $\frac{\nabla G(x, \tilde{u}_{MPTP})}{\nabla G(x, \tilde{u}_{MPTP})}$ expresses the gradient vector of $G(x, \tilde{u}_{MPTP})$ at $\tilde{u}_{f}(d)$.

For the outer loop, a new decision factor is defined according to $A$ and $\beta(d, \tilde{u}_{MPTP})$ to renew the active constraint, which is specified as

$$ \lambda : \tilde{u}_{f}(d) \in A \cup \beta(d, \tilde{u}_{MPTP}) \leq \beta', \quad C^{(k)} = \frac{d^{(k-1)}}{\nabla f(x)} - \nabla f(x) \tilde{u}^{(k-1)} \quad (19) $$

where $\lambda$ is the deciding factor. Since the limit state function $G(x, \tilde{u})$ is an implicit form with respect to $\frac{\nabla f(x) \tilde{u}^{(k-1)}}{\nabla f(x) \tilde{u}^{(k-1)}}$ and $\nabla f(x) \tilde{u}^{(k-1)}$ necessitates the adoption of the numerical finite-difference method. It is represented by

$$ \nabla f(x) \frac{\tilde{u}^{(k-1)}}{\nabla f(x) \tilde{u}^{(k-1)}} = \frac{G(d^{(k-1)}) + \delta - G(d^{(k-1)})}{\delta}, \delta = 10^{-5} $$

$$ \nabla f(x) \tilde{u}^{(k-1)} = \frac{G(d^{(k-1)}) + \delta - G(d^{(k-1)})}{\delta}, \delta = 10^{-5} \quad (20) $$

where $\delta$ states the differential step size.

Thus, the inner reliability analysis and outer optimization will run simultaneously at a faster speed than PMA with SAP. It is noteworthy that $6\sigma$ is suggested to filter the MPTP in equation (17) because the $6\sigma$ level design sufficiently considers the uncertainties in engineering and has good robustness and reliability. As shown in Figure 2, the $6\sigma$ level design has a 99.999998% reliability level is far better than any $\sigma$ level design. Therefore, the $6\sigma$ level is selected to search for MPTP efficiently and robustly.

Framework of the proposed method

The proposed method is known as the active set strategy-based SAP (ASS-SAP) and its flowchart is depicted in Figure 3. The specific steps of the ASS-SAP are listed below to measure the probabilistic constraints in RBDO:

1. Given $k = 1$, $\tilde{d}^{(0)}$, $\tilde{u}^{(0)}$, and $\varepsilon$ (stopping criterion).
2. Transfer random variables to $\tilde{U}$-space by $\tilde{U}^{(k)} = T(\tilde{x}^{(k)})$.
3. Execute the inner loop by equation (11) and equation (17) to search for effective MPTP, $\tilde{u}_{f}^{(MPTP)}$.
4. Transfer random variables to $\tilde{x}$-space by $\tilde{x}^{(k)} = T^{-1}(\tilde{U}^{(k)})$ and perform the outer loop by equation (19) for the deterministic constraints. Calculate $\beta(d, \tilde{u}_{f}^{(MPTP)})$ using equation (18). If the deciding factor $\lambda$ is met, the current constraint is active, otherwise, the second deterministic constraint replaces the former one and continues to identify whether the constraint is active or not.
5. if the convergence criterion $||\tilde{d}^{(k+1)} - \tilde{d}^{(k)}||/||\tilde{d}^{(k+1)}|| \leq \varepsilon$ is satisfied, then end the computation and print the optimum $\tilde{d}$; else, set $k = k + 1$ and return step (2).
Examples

The ASS-SAP is integrated into two-level to analyze its performance when faced with complex RBDO problems. To better perceive the superiority of the recommended algorithm, two common mathematical and practical engineering examples are tested.

Example 1

This nonlinear mathematical problem is abstracted from the literature, whose RBDO model is written as

\[
\begin{align*}
\text{Min } & \quad z(\tilde{d}) = \tilde{d}_1 + \tilde{d}_2 \\
\text{s.t. } & \quad P[G_1 = 1 - (\bar{X}_1^2 + \bar{X}_2^2)/10 > 0] \leq P_f \\
& \quad P[G_2 = 1 - \left(\frac{(\bar{X}_1 + \bar{X}_2 - 5)^2}{30} + \frac{(\bar{X}_1 - \bar{X}_2 - 12)^2}{120}\right) > 0] \leq P_f \\
& \quad P[G_3 = 1 - 80/(\bar{X}_1^2 + 8\bar{X}_2 + 5) > 0] \leq P_f.
\end{align*}
\]

where \(\bar{X}_1\) and \(\bar{X}_2\) are mutually independent and compiled four different types of random distributions: Normal, Lognormal, Gumbel, and Uniform. The initial designs are located at \(\tilde{d}^{(0)} = [5., 5.]^T\) and \(\tilde{u}^{(0)} = [0., 0.]^T\).

The outcomes computed for all approaches are brief in Table 1. The MCS is employed to evaluate the violation conditions of probability constraints by providing 10^6 samples size and offered in Table 2, where results have confirmed the failure probabilities \((P_f)\) of the system are acceptable, which agree well with the target failure probability \((P_f)\). Also, for the three constraints, \(G_1, G_2\) denote active constraints, while \(G_3\) indicating an inactive constraint.

In Table 1, the values in each row comprise the number of iterations and FEs, then, the minimum cost function and the optimal design point, respectively, denote as 5/195 6.7318/(3.4409, 3.2909). It is stated that all algorithms can accurately arrive at the identical optimal design point at the same initial settings and convergence criterion \((\varepsilon = 10^{-5})\) for different distribution types, but the proposed method (ASS-SAP) requires the lowest iterations and FEs. Moreover, MRIA two-level suffers.
from local optima and difficult convergence issues for lognormal and uniform distribution. PMA two-level also experienced the same difficulty of convergence for uniform distribution. In comparison, the PMA with SAP approach appears to have good robustness and convergence while also being less reliant on stochastic distribution types. ASS-based SAP has better adaptability than MRIA two-level and PMA two-level but is also more efficient than PMA with SAP since it significantly reduced FEs and iterations while guaranteeing similar accuracy to PMA with SAP.

Example 2
This high nonlinear problem, extracted from\(^2\) and described with the below RBDO formulation

\[
\begin{align*}
\text{Min} & \quad z(\bar{d}) = -\frac{(\bar{d}_1 + \bar{d}_2 - 10)^2}{30} - \frac{(\bar{d}_1 - \bar{d}_2 + 10)^2}{120} \\
\text{s.t.} & \quad P(G_1 = (\bar{X}_1^2 + \bar{X}_2)/20 - 1 > 0) \leq P_f' \\
& \quad P(G_2 = 1 - (\bar{Y} - 6)^2 - (\bar{Y} - 6)^4 + 0.6(\bar{Y} - 6)^6 - Z > 0) \leq P_f' \\
& \quad P(G_3 = 80/(\bar{X}_2^2 + 8\bar{X}_2 + 5)>0) \leq P_f' \\
& \quad \bar{Y} = 0.9063\bar{X}_1 + 0.4226\bar{X}_2, \bar{Z} = 0.4226\bar{X}_1 - 0.9063\bar{X}_2 \\
& \quad \bar{d}_1, \bar{d}_2 \in [0, 10]., P_f' = 0.13\% \\
\end{align*}
\]

This model contains two random variables \(\bar{X}_N \sim \mathcal{N}(\bar{d}_N, 0.3^2), N = 1, 2\), complied with a mutually independent normal distribution and three probabilistic constraints, here, \(G_2\) is a highly nonlinear concave function. The purpose of this example is to find the

---

Table 1. Comparison results Example 1.

| Type     | MRIA two-level \(^a\) | PMA two-level \(^b\) | PMA with SAP \(^b\) | ASS-based SAP |
|----------|------------------------|----------------------|----------------------|--------------|
| Normal   | 5/195                  | 7/443                | 7/202                | 7/195        |
|          | 6.7318 (3.4409, 3.2909) | 6.7318 (3.4409, 3.2909) | 6.7318 (3.4409, 3.2909) | 6.7318 (3.4409, 3.2909) |
| Lognormal| Local optimum          | 8/229                | 6/414                | 4/140        |
|          | 6.5914 (3.4022, 3.1893) | 6.5914 (3.4022, 3.1893) | 6.5914 (3.4022, 3.1893) | 6.5914 (3.4022, 3.1893) |
| Gumbel   | 9/132                  | 9/92                 | 6.2932 (3.2804, 3.0128) | 6.2932 (3.2804, 3.0128) |
| Uniform  | –                      | –                    | 6/83                 | 5/55         |
|          | –                      | –                    | 6.3252 (3.3436, 2.9816) | 6.3252 (3.3436, 2.9816) |

\(^a\)The method is extracted from Lin et al.\(^9\)  
\(^b\)The method is taken from Yi and Cheng.\(^38\)

Table 2. Reliability at the optimum by MCS in Example 1.

| Type     | MRIA two-level\(^a\) \((P_f)\) | PMA two-level\(^b\) \((P_f)\) | PMA with SAP\(^b\) \((P_f)\) | ASS-based SAP \((P_f)\) |
|----------|---------------------------------|---------------------------------|---------------------------------|------------------------|
| Normal   | 0.1435/0.1071/0                 | 0.1432/0.1134/0                 | 0.1436/0.1086/0                 | 0.1477/0.1137/0        |
| Lognormal| –                               | 0.1299/0.1039/0                 | 0.1336/0.1007/0                 | 0.1312/0.1091/0        |
| Gumbel   | 0.1342/0.0859/0                 | 0.1100/0.0898/0                 | 0.1110/0.0914/0                 | 0.1096/0.0878/0        |
| Uniform  | –                               | –                               | 0.0561/0.0543/0                 | 0.0551/0.0548/0        |

\(^a\)The method is extracted from Lin et al.\(^9\)  
\(^b\)The method is taken from Yi and Cheng.\(^38\)

Table 3. Comparison results for Example 2.

| Approach  | Cost | Optimal          | Iter. | FEs  |
|-----------|------|------------------|-------|------|
| PMA two-level\(^a\) | –     | Periodic-4       | –     | –    |
| MRIA two-level\(^b\) | –1.7228 | [4.5572, 1.9689] | 11    | 332  |
| PMA with SAP\(^b\) | –1.7228 | [4.5572, 1.9689] | 7     | 175  |
| ASS-based SAP | –1.7228 | [4.5572, 1.9689] | 6     | 130  |

The bold numbers are FEs obtained by ASS-based SAP for the convergence.  
\(^a\)The method is extracted from Lin et al.\(^9\)  
\(^b\)The method is taken from Yi and Cheng.\(^38\)
optimum, which satisfies the maximum permissible failure probability $P_f = 0.13\%$ under different stopping criteria ($\varepsilon$, i.e. $10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$). See the comparison results in Table 3.

From Table 3, all the approaches except for the PMA two-level could successfully and stably reach the same minimum objective value $z = 1.7228$ and the optimal design point $\ddot{d} = [4.5572, 1.9689]$, which shows a good agreement with the results from. In terms of FEs, the ASS-based SAP is the most efficient with only 130 FEs and 6 Iterations (Iter.), PMA with SAP is next as it requires 175 FEs and 7 iterations, MRIA two-level is the most inefficient thanks to 332 FEs and 11 iterations. Moreover, the iterative histories of ASS-based SAP are mapped for first, second, and final iteration sixth in Figure 4 for better cognition.

Table 4 presents the important effects of different convergence criteria ($\varepsilon$) on the efficiency for the PMA two-level, MRIA two-level, PMA with SAP, and the proposed ASS-based SAP. The PMA with SAP seems to be sensitive to convergence criteria. The smaller the convergence criteria, the lower the computational efficiency, that is the more FEs are required. However, the MRIA two-level and ASS-based SAP seem to be insensitive to the convergence criteria, and even with smaller stepping criteria, they still maintain a stable convergence efficiency and guarantee a global solution. Despite that, the ASS-based SAP method can capture the converged solution more quickly than MRIA two-level under the same settings. The ASS-based SAP terminates at the seventh iteration owing to the stopping criteria ($\varepsilon$) with 142 FEs while the MRIA two-level requires 338 FEs. PMA with SAP is great faster than MRIA two-level, although PMA with SAP is slightly sensitive to convergence criteria. As seen, compared with all methods, the suggested ASS-based SAP not only has good convergence ability but also has high efficiency for highly nonlinear constraints with concave and convex functions. At the same time, it again demonstrated that the SAP-based method has wider applicability in solving RBDO problems.

In addition, the MCS is implemented to assess the violation conditions of three constraints around the optimum based on $10^6$ samples and tabulated in Table 5, where is presented that all failure probability of constraints around the optimal solution conforms to the given failure probability. Besides, it also hints that the MCS fails to evaluate the probabilistic constraint $G_3$ as it is invalid, while $G_1$ and $G_2$ are active. Beyond that, the MCS results have once again revealed that the proposed ASS-based SAP has similar accuracy as MRIA two-level and PMA with SAP.

Example 3
A practical engineering example is on the speed reducer design issue including seven random variables, $\bar{X}_N, N = 1, \ldots, 7$ and 11 probabilistic constraints, $G_i, i = 1, \ldots, 11$ to test and compare the performance of all reliability methods, which is extracted from and as depicted in Figure 5. Each random variable complies with the independent normal distribution, and its stochastic properties are referred to in Table 6 for details. The weight of the speed reducer is the minimizing cost function while the desired reliability index is $\beta_l = 3.0$.
Table 5. Constraints are assessed by MCS for all methods with different stopping criteria.

| Method                        | $\varepsilon = 10^{-3}$ | $\varepsilon = 10^{-4}$ | $\varepsilon = 10^{-5}$ | $\varepsilon = 10^{-6}$ |
|-------------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| PMA two-level\(^a\)           | 0.1520/0.0648/0          | 0.1512/0.0643/0          | 0.1529/0.0636/0          | 0.1528/0.0636/0          |
| MRIA two-level\(^b\)          | 0.1490/0.0647/0          | 0.1513/0.0620/0          | 0.1515/0.0649/0          | 0.1507/0.0636/0          |
| PMA with SAP\(^b\)            | 0.1513/0.0636/0          | 0.1510/0.0620/0          | 0.1512/0.0650/0          | 0.1516/0.0658/0          |
| ASS-based SAP\(^b\)           | 0.1513/0.0636/0          | 0.1510/0.0620/0          | 0.1512/0.0650/0          | 0.1516/0.0658/0          |

\(^a\)The method is extracted from Lin et al.\(^9\)

\(^b\)The method is from Yi and Cheng.\(^38\)

Besides, the physical meaning of probabilistic constraints is detailed in Zhu et al.\(^2\). Then, the mode of RBDO for the speed reducer is formulated as

$$
\begin{align*}
\text{Min} & \quad z(\bar{d}) = 0.7854\bar{d}_1\bar{d}_2^2 \\
& \quad (3.3333\bar{d}_2^2 + 14.9334\bar{d}_3 - 43.0934) \\
& \quad - 1.508\bar{d}_1(\bar{d}_6^2 + \bar{d}_7^2) \\
& \quad + 7.477(\bar{d}_6^2 + \bar{d}_7^2) + 0.7854(\bar{d}_4\bar{d}_6^2 + \bar{d}_5\bar{d}_7^2)
\end{align*}
$$

s.t. :

\[
\begin{align*}
P \left[ G_1 = \frac{27}{\bar{X}_1\bar{X}_2\bar{X}_3} - 1 > 0 \right] & \leq \Phi(-3.0), \\
P \left[ G_2 = \frac{397.5}{\bar{X}_1\bar{X}_2\bar{X}_3} - 1 > 0 \right] & \leq \Phi(-3.0), \\
P \left[ G_3 = \frac{1.93\bar{X}_1^3}{\bar{X}_2\bar{X}_3\bar{X}_4} - 1 > 0 \right] & \leq \Phi(-3.0), \\
P \left[ G_4 = \frac{1.93\bar{X}_1^3}{\bar{X}_2\bar{X}_3\bar{X}_4} - 1 > 0 \right] & \leq \Phi(-3.0) \\
P \left[ G_5 = \frac{\sqrt{\frac{2483}{\bar{X}_4\bar{X}_5}}} {0.13\bar{X}_5} + 16.9 \times 10^6 - 1100 > 0 \right] & \leq \Phi(-3.0) \\
P \left[ G_6 = \frac{\sqrt{\frac{2483}{\bar{X}_4\bar{X}_5}}} {0.13\bar{X}_5} - 850 > 0 \right] & \leq \Phi(-3.0) \\
P \left[ G_7 = \bar{X}_2\bar{X}_3 - 40 > 0 \right] & \leq \Phi(-3.0), \\
P \left[ G_8 = 5 - \frac{\bar{X}_1}{\bar{X}_2} > 0 \right] & \leq \Phi(-3.0), \\
P \left[ G_9 = \frac{\bar{X}_1}{\bar{X}_2} - 12 > 0 \right] & \leq \Phi(-3.0), \\
P \left[ G_{10} = \frac{1.5\bar{X}_6 + 1.9}{\bar{X}_4} - 1 > 0 \right] & \leq \Phi(-3.0),
\end{align*}
\]

This multiple design constraints example has been solved by Zhu et al.\(^2\) and is also studied in this paper. The results are added in Table 7.

Table 7 shows a summary of the results (i.e. optimum, cost, iterations, FE) computed by MRIA two-level, PMA two-level, PMA with SAP, and ASS-based SAP, where the approaches consistently converge to the same desired design point of $[3.5765, 0.7000, 17.0000, 7.3000, 7.7527, 3.3584, 5.3004]$ and the minimum cost function of 3036.0 with the acceptable failure probabilities. As far as FE are concerned, it could be observed that among these methods, the ASS-based SAP incurs a total of 7 iterations and 548 FE are required to satisfy the stopping criterion $\varepsilon = 10^{-5}$, which is the fewest FE than others evaluated for active constraints.

In Table 8, MCS is employed to simulate the reliability indices of each probabilistic constraint at the optimum utilizing $3 \times 10^6$ samples and the outcomes obtained are almost identical to the desired reliability index ($\beta_1 = 3$). And again, implied that all methods have similar accuracy. Another, the probability constraints $G_6$ and $G_8$ are inactive, while the remaining six are active. Therefore, ASS-based SAP not only has prominent convergence ability but also has a steady and efficient performance for reliability assessment in multiple design variables and multiple constraints.

**Conclusion**

The prevailing iterative algorithms for MRIA two-level and PMA two-level reveal the shortcomings of inefficacy and instability to a certain extent when exploring convergence efficiency of concave performance functions, however, PMA with SAP method appears the opposite performance by performing reliability analysis.
Table 6. Stochastic parameters in Example 3.

| Variables | Mean values | Standard deviation | Description               |
|-----------|-------------|--------------------|---------------------------|
| $x_1$     | $d_1$       | 0.005              | The width of the gear     |
| $x_2$     | $d_2$       | 0.005              | The module of gear        |
| $x_3$     | $d_3$       | 0.005              | The number of pinion teeth|
| $x_4$     | $d_4$       | 0.005              | Bearing distance of shaft 1|
| $x_5$     | $d_5$       | 0.005              | Bearing distance of shaft 2|
| $x_6$     | $d_6$       | 0.005              | The diameter of shaft 1   |
| $x_7$     | $d_7$       | 0.005              | The diameter of shaft 2   |

Table 7. Comparison results in Example 3.

| Methods               | Cost  | Optimal                          | Iter. | FEs  |
|-----------------------|-------|----------------------------------|-------|------|
| MRIA two-level         | 3036.0| $[3.5765, 0.7000, 17.0000, 7.3000, 7.7527, 3.3584, 5.3004]$ | 8     | 627  |
| PMA two-level          | 3036.0| $[3.5765, 0.7000, 17.0000, 7.3000, 7.7527, 3.3584, 5.3004]$ | 7     | 561  |
| PMA with SAP           | 3036.0| $[3.5765, 0.7000, 17.0000, 7.3000, 7.7527, 3.3584, 5.3004]$ | 7     | 570  |
| ASS-based SAP          | 3036.0| $[3.5765, 0.7000, 17.0000, 7.3000, 7.7527, 3.3584, 5.3004]$ | 7     | 548  |

*aThe method is extracted from Lin et al.*

*bThe method is taken from Yi and Cheng.*

Table 8. Reliability evaluation at the optimum by MCS in Example 3.

| Methods               | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ | $\beta_7$ | $\beta_8$ | $\beta_9$ | $\beta_{10}$ | $\beta_{11}$ |
|-----------------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|---------------|---------------|
| MRIA two-level         | $\infty$  | $\infty$  | $\infty$  | $\infty$  | 2.9977     | 3.0786     | $\infty$  | 2.9845     | $\infty$  | $\infty$     | 2.9760        |
| PMA two-level          | $\infty$  | $\infty$  | $\infty$  | $\infty$  | 3.0357     | 3.0729     | $\infty$  | 3.0564     | $\infty$  | $\infty$     | 3.0511        |
| PMA with SAP           | $\infty$  | $\infty$  | $\infty$  | $\infty$  | 3.0115     | 3.0068     | $\infty$  | 3.0068     | $\infty$  | $\infty$     | 3.0068        |
| ASS-based SAP          | $\infty$  | $\infty$  | $\infty$  | $\infty$  | 2.9760     | 3.029      | $\infty$  | 2.9977     | $\infty$  | $\infty$     | 3.1086        |

*aThe method is extracted from Lin et al.*

*bThe method is taken from Yi and Cheng.*

Figure 5. A speed reducer.
and design optimization simultaneously. Considering the better performance of PMA with SAP, a simple and robust approach is further developed to strengthen the efficiency of PMA with SAP, which is named the active set strategy-based SAP (ASS-base SAP). In the framework of ASS-base SAP, an active set using inequality is equipped in the inner loop to robustly and effectively perform reliability analysis and obtain the feasible MPTP. Meanwhile, the decisive factor is defined according to the reliability index and active set strategy in the outer loop, which is used to quickly renew the active constraints. Since both the inner reliability analysis and outer optimization are carried out concurrently at a faster convergence speed than PMA with SAP, thereby, the efficiency is enhanced remarkably.

Several numerical examples, including RBDO numerical problems and a practical engineering case, are quoted to verify the robustness and efficiency of the proposed method. The comparative study of RBDO problems has shown that the ASS-based SAP achieves the most efficiency and has better robustness than the traditional ones (MRIA two-level, PMA two-level, PMA with SAP). Therefore, the proposed method is a more efficient and robust PMA-based for structural reliability analysis method. it can also be concluded that the proposed method has good applicability for solving complex RBDO problems of ongoing research topics, but does not perform well for the uniformly distributed problem.

**Declaration of conflicting interests**

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

**Funding**

The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: This work was supported by the National Natural Science Foundation of China [grant number 51679056].

**ORCID iD**

Xue An https://orcid.org/0000-0001-6010-4432

**Data availability**

The data and code developed for this study are available from the corresponding author on reasonable request.

**References**

1. Ma J, Wriggers P, Gao W, et al. Reliability-based optimization of trusses with random parameters under dynamic loads. *Comput Mech* 2011; 47: 627–640.

2. Zhu S-P, Keshtegar B, Trung N-T, et al. Reliability-based structural design optimization: hybridized conjugate mean value approach. *Eng Comput* 2021; 37: 381–394.

3. das Neves Carneiro G and Conceição António C. Reliability-based robust design optimization with the Reliability Index Approach applied to composite laminate structures. *Compos Struct* 2019; 209: 844–855.

4. Jensen HA, Valdebenito MA and Schueller GI. An efficient reliability-based optimization scheme for uncertain linear systems subject to general Gaussian excitation. *Comput Methods Appl Mech Eng* 2008; 198: 72–87.

5. Yang M, Zhang D and Han X. New efficient and robust method for structural reliability analysis and its application in reliability-based design optimization. *Comput Methods Appl Mech Eng* 2020; 366: 113018.

6. Yaseen ZM and Keshtegar B. Limited descent-based mean value method for inverse reliability analysis. *Eng Comput* 2019; 35: 1237–1249.

7. Meng Z, Zhou H, Li G, et al. A decoupled approach for non-probabilistic reliability-based design optimization. *Comput Struct* 2016; 175: 65–73.

8. Torii AJ, Lopez RH and F. Miguel LF. A general RBDO decoupling approach for different reliability analysis methods. *Struct Multidiscipl Optim* 2016; 54: 317–332.

9. Ting Lin P, Chang Gea H and Jaluria Y. A modified reliability index approach for reliability-based design optimization. *J Mech Des* 2011; 133: 044501.

10. Keshtegar B, Hao P and Meng Z. A self-adaptive modified chaos control method for reliability-based design optimization. *Struct Multidiscipl Optim* 2017; 55: 63–75.

11. Yi P, Cheng G and Jiang L. A sequential approximate programming strategy for performance-measure-based probabilistic structural design optimization. *Struct Saf* 2008; 30: 91–109.

12. Jiang C, Qiu H, Gao L, et al. An adaptive hybrid single-loop method for reliability-based design optimization using iterative control strategy. *Struct Multidiscipl Optim* 2017; 56: 1271–1286.

13. Keshtegar B and Hao P. Enhanced single-loop method for efficient reliability-based design optimization with complex constraints. *Struct Multidiscipl Optim* 2018; 57: 1731–1747.

14. Liang J, Mourelatos ZP and Tu J. A single-loop method for reliability-based design optimisation. *IJPD* 2008; 5: 76.

15. Grandhi RV and Wang L. Reliability-based structural optimization using improved two-point adaptive nonlinear approximations. *Finite Elem Anal Des* 1998; 29: 35–48.

16. Nikolaidis E and Burdisso R. Reliability based optimization: A safety index approach. *Comput Struct* 1988; 28: 781–788.

17. Tu J, Choi KK and Park YH. A new study on reliability-based design optimization. *J Mech Des* 1999; 121: 557–564.

18. Lee J-O, Yang YS and Ruy W-S. A comparative study on reliability-index and target-performance-based probabilistic structural design optimization. *Comput Struct* 2002; 80: 257–269.
19. Hasofer AM and Lind NC. Exact and invariant second-Moment Code format. *J Eng Mech Div* 1974; 100: 111–121.
20. Rackwitz R and Flessler B. Structural reliability under combined random load sequences. *Comput Struct* 1978; 9: 489–494.
21. Meng Z, Li G, Wang BP, et al. A hybrid chaos control approach of the performance measure functions for reliability-based design optimization. *Comput Struct* 2015; 146: 32–43.
22. Aoues Y and Chateauneuf A. Benchmark study of numerical methods for reliability-based design optimization. *Struct Multidiscipl Optim* 2010; 41: 277–294.
23. Oza K and Gea HC. Two-Level approximation method for reliability-based design optimization. In: *Proceedings of the ASME 2004 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, Salt Lake City, Utah, 2004, pp.885891. New York: American Society of Mechanical Engineers: Digital Collection.
24. Meng Z, Pu Y and Zhou H. Adaptive stability transformation method of chaos control for first order reliability method. *Eng Comput* 2018; 34: 671–683.
25. Yang D, Li G and Cheng G. Convergence analysis of first order reliability method using chaos theory. *Comput Struct* 2006; 84: 563–571.
26. Lin PT, Jaluria Y and Gea HC. A hybrid reliability approach for reliability-based design optimization. In: *Volume 1: 36th design automation conference, Parts A and B*, Montreal, Quebec, Canada, 2010, pp. 1099–107. ASMEDC.
27. Cheng G, Xu L and Jiang L. A sequential approximate programming strategy for reliability-based structural optimization. *Comput Struct* 2006; 84: 1353–1367.
28. Xu L, Cheng G and Yi P. Tolerance synthesis by a new method for system reliability-based optimization. *Eng Optim* 2005; 37: 717–732.
29. Chen Z, Qiu H, Gao L, et al. An optimal shifting vector approach for efficient probabilistic design. *Struct Multidiscipl Optim* 2013; 47: 905–920.
30. Shan S and Wang GG. Reliable design space and complete single-loop reliability-based design optimization. *Reliab Eng Syst Saf* 2008; 93: 1218–1230.
31. Youn BD, Choi KK and Du L. Enriched performance measure approach for reliability-based design optimization. *AIAA Journal* 2005; 43: 874–884.
32. Yang D and Yi P. Chaos control of performance measure approach for evaluation of probabilistic constraints. *Struct Multidiscipl Optim* 2009; 38: 83–92.
33. Youn BD, Choi KK and Park YH. Hybrid Analysis Method for reliability-based design optimization. *J Mech Des* 2003; 125: 221–232.
34. Youn BD and Choi KK. An investigation of nonlinearity of reliability-based design optimization approaches. *J Mech Des* 2004; 126: 403–411.
35. Yi P and Zhu Z. Step length adjustment iterative algorithm for inverse reliability analysis. *Struct Multidiscipl Optim* 2016; 54: 999–1009.
36. Keshtegar B and Lee I. Relaxed performance measure approach for reliability-based design optimization. *Struct Multidiscipl Optim* 2016; 54: 1439–1454.
37. Li X, Meng Z, Chen G, et al. A hybrid self-adjusted single-loop approach for reliability-based design optimization. *Struct Multidiscipl Optim* 2019; 60: 1867–1885.
38. Yi P and Cheng G. Further study on efficiency of sequential approximate programming for probabilistic structural design optimization. *Struct Multidiscipl Optim* 2008; 35: 509–522.
39. Keshtegar B. A modified mean value of performance measure approach for reliability-based design optimization. *Arab J Sci Eng* 2017; 42: 1093–1101.