Symbolic Computation of Conservation Laws of Nonlinear Partial Differential Equations

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Outline

• A simple example of a conservation law
• The Korteweg-de Vries equation
• The Zakharov-Kuznetsov equation
• Methods for computing conservation laws
• Tools (variational calculus, differential geometry)
  • The variational derivative (testing exactness)
  • The homotopy operator (inverting $D_x$ and $\text{Div}$)
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• Demonstration of ConservationLawsMD.m
• Conclusions and future work
Additional examples

- Manakov-Santini system
- Camassa-Holm equation in (2+1) dimensions
- Khoklov-Zabolotskaya equation
- Shallow water wave model for atmosphere
- Kadomtsev-Petviashvili equation
- Potential Kadomtsev-Petviashvili equation
- Generalized Zakharov-Kuznetsov equation
Example of a Conservation Law in Traffic Flow

Modeling the density of cars (Bressan, 2009)

\[ u(x, t) \] density of cars on a highway (e.g., number of cars per km).

\[ s(u) \] mean (equilibrium) speed (e.g., in km per hour) of the cars. It depends on the density.
Change in number of cars in segment $[a, b]$ equals the difference between cars entering at $a$ and leaving at $b$ during time interval $[t_1, t_2]$:

$$\int_{a}^{b} \left( u(x, t_2) - u(x, t_1) \right) \, dx = \int_{t_1}^{t_2} \left( J(a, t) - J(b, t) \right) \, dt$$

$$\int_{a}^{b} \left( \int_{t_1}^{t_2} u_t(x, t) \, dt \right) \, dx = -\int_{t_1}^{t_2} \left( \int_{a}^{b} J_x(x, t) \, dx \right) \, dt$$

where $J(x, t) = u(x, t)s(u(x, t))$ is the traffic flow (e.g., in cars per hour) at location $x$ and time $t$. 
Then, \( \int_a^b \int_{t_1}^{t_2} (u_t + J_x) \, dt \, dx = 0 \) holds \( \forall (a, b, t_1, t_2) \).

This yields the conservation law:

\[
\begin{aligned}
  u_t + [s(u) u]_x &= 0 \\
  \text{or} \\
  D_t \rho + D_x J &= 0
\end{aligned}
\]

\( \rho = u \) is the conserved density;

\( J(u) = s(u) u \) is the associated flux.

A simple Lighthill-Whitham-Richards model:

\[
s(u) = s_{\text{max}} \left( 1 - \frac{u}{u_{\text{max}}} \right), \quad 0 \leq u \leq u_{\text{max}}
\]

\( s_{\text{max}} \) is posted road speed, \( u_{\text{max}} \) is the jam density.
Conservation Laws for Nonlinear PDEs

• System of evolution equations of order $M$

$$u_t = F(u^{(M)}(x))$$

with $u = (u, v, w, \ldots)$ and $x = (x, y, z)$.

• Conservation law in $(1+1)$-dimensions

$$D_t \rho + D_x J = 0$$

where the dot means evaluated on the PDE. Conserved density $\rho$ and flux $J$.

$$P = \int_{-\infty}^{\infty} \rho \, dx = \text{constant in time}$$

if $J$ vanishes at $\pm \infty$. 
• Conservation law in (2+1)-dimensions

\[ \frac{D_t \rho + \nabla \cdot \mathbf{J}}{D_t \rho + D_x J_1 + D_y J_2} \dot{=} 0 \]

Conserved density \( \rho \) and flux \( \mathbf{J} = (J_1, J_2) \).

• Conservation law in (3+1)-dimensions

\[ \frac{D_t \rho + \nabla \cdot \mathbf{J}}{D_t \rho + D_x J_1 + D_y J_2 + D_z J_3} \dot{=} 0 \]

Conserved density \( \rho \) and flux \( \mathbf{J} = (J_1, J_2, J_3) \).
• **Computing total derivatives:** $D_t, D_x, D_y, \ldots$

**Example:** If $f = t^2u^3u_x^2$ then

\[
D_t f = \frac{\partial f}{\partial t} + u_t \frac{\partial f}{\partial u} + u_{xt} \frac{\partial f}{\partial u_x} \\
= \frac{\partial f}{\partial t} + u_t \frac{\partial f}{\partial u} + u_{xt} \frac{\partial f}{\partial u_x} \\
= 2tu^3u_x^2 + 3t^2u^2u_tu_x^2 + 2t^2u^3u_xu_{xt}
\]
Reasons for Computing Conservation Laws

• Conservation of physical quantities (linear momentum, mass, energy, electric charge, ... ).

• Testing of complete integrability and application of Inverse Scattering Transform.

• Testing of numerical integrators.

• Study of quantitative and qualitative properties of PDEs (Hamiltonian structure, recursion operators, ... ).

• Verify the closure of a model.
Examples of PDEs with Conservation Laws

Example 1: KdV Equation

\[ \frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \] or

\[ u_t + 6uu_x + u_{xxx} = 0 \]

shallow water waves, ion-acoustic waves in plasmas

Diederik Korteweg  
Gustav de Vries
**Dilation Symmetry**

\[
 u_t + 6uu_x + u_{xxx} = 0
\]

has dilation (scaling) symmetry \( (x, t, u) \rightarrow \left( \frac{x}{\kappa}, \frac{t}{\kappa^3}, \kappa^2 u \right) \)

\( \kappa \) is an arbitrary parameter.

**Notion of weight:** \( W(x) = -1 \), thus, \( W(D_x) = 1 \).

\[
 W(t) = -3, \text{ hence, } W(D_t) = 3.
\]

\[
 W(u) = 2.
\]

**Notion of rank (total weight of a monomial).**

**Examples:** \( \text{Rank}(u^3) = \text{Rank}(3u_x^2) = 6. \)

\[
 \text{Rank}(u^3u_{xx}) = 10.
\]
Key Observation: Scaling Invariance

Every term in a density has the same fixed rank.

Every term in a flux has some other fixed rank.

The conservation law

\[
D_t \rho + D_x J \dot{} = 0
\]

is uniform in rank.

Hence,

\[
\text{Rank}(\rho) + \text{Rank}(D_t) = \text{Rank}(J) + \text{Rank}(D_x)
\]
• First six (of infinitely many) conservation laws:

\[
D_t(u) + D_x \left( 3u^2 + u_{xx} \right) \dot{=} 0
\]

\[
D_t(u^2) + D_x \left( 4u^3 - u_x^2 + 2uu_{xx} \right) \dot{=} 0
\]

\[
D_t \left( u^3 - \frac{1}{2}u_x^2 \right) + D_x \left( \frac{9}{2}u^4 - 6uu_x^2 + 3u^2u_{xx} + \frac{1}{2}u_x^2 - u_xu_{xxx} \right) \dot{=} 0
\]

\[
D_t \left( u^4 - 2uu_x^2 + \frac{1}{5}u_{xx}^2 \right) + D_x \left( \frac{24}{5}u^5 - 18uu_x^2 + 4u^3u_{xx} + 2u_x^2u_{xx} + \frac{16}{5}uu_x^2u_{xx} - 4u_xu_{xxx} - \frac{1}{5}u_x^2 - \frac{2}{5}u_xu_{xx}u_{4x} \right) \dot{=} 0
\]
\[ D_t \left( u^5 - 5 u^2 u_x^2 + u u_{xx}^2 - \frac{1}{14} u_{xxx}^2 \right) \]
\[ + D_x \left( 5 u^6 - 40 u^3 u_x^2 - \ldots - \frac{1}{7} u_{xxx} u_{5x} \right) \dot{=} 0 \]
\[ D_t \left( u^6 - 10 u^3 u_x^2 - \frac{5}{6} u_x^4 + 3 u^2 u_{xx}^2 \right. \]
\[ \left. + \frac{10}{21} u_{xxx}^2 - \frac{3}{7} u u_{xxx}^2 + \frac{1}{42} u_{4x}^2 \right) \]
\[ + D_x \left( \frac{36}{7} u^7 - 75 u^4 u_x^2 - \ldots + \frac{1}{21} u_{4x} u_{6x} \right) \dot{=} 0 \]

- Third conservation law: Gerald Whitham, 1965
- Fourth and fifth: Norman Zabusky, 1965-66
- Seventh (sixth thru tenth): Robert Miura, 1966
Robert Miura
• Conservation law explicitly dependent on $t$ and $x$:

$$
D_t \left( tu^2 - \frac{1}{3} xu \right) \\
+ D_x \left( 4tu^3 - xu^2 + \frac{1}{3} u_x - tu_x^2 + 2tuu_{xx} - \frac{1}{3} xu_{xx} \right) = 0
$$
• First five: IBM 7094 computer with FORMAC (1966) → storage space problem!

IBM 7094 Computer
First eleven densities: Control Data Computer CDC-6600 computer (2.2 seconds) → large integers problem!
Example 2: The Zakharov-Kuznetsov Equation

\[ u_t + \alpha uu_x + \beta (u_{xx} + u_{yy})_x = 0 \]

models ion-sound solitons in a low pressure uniform magnetized plasma.

• Conservation laws:

\[
D_t(u) + D_x \left( \frac{\alpha}{2} u^2 + \beta u_{xx} \right) + D_y \left( \beta u_{xy} \right) = 0
\]

\[
D_t(u^2) + D_x \left( \frac{2\alpha}{3} u^3 - \beta (u_x^2 - u_y^2) + 2\beta u (u_{xx} + u_{yy}) \right) + D_y \left( -2\beta u_x u_y \right) = 0
\]
• More conservation laws (ZK equation):

\[
D_t \left( u^3 - \frac{3\beta}{\alpha} (u_x^2 + u_y^2) \right) + D_x \left( 3u^2 \left( \frac{\alpha}{4} u^2 + \beta u_{xx} \right) - 6\beta u (u_x^2 + u_y^2) \right) + \frac{3\beta^2}{\alpha} (u_{xx}^2 - u_{yy}^2) - \frac{6\beta^2}{\alpha} (u_x (u_{xxx} + u_{xyy}) + u_y (u_{xxy} + u_{yyy})) \right) \\
+ D_y \left( 3\beta u_x^2 u_y + \frac{6\beta^2}{\alpha} u_x u_y (u_{xx} + u_{yy}) \right) = 0
\]

\[
D_t \left( tu^2 - \frac{2}{\alpha} x u \right) + D_x \left( t \left( \frac{2\alpha}{3} u^3 - \beta (u_x^2 - u_y^2) + 2\beta u (u_{xx} + u_{yy}) \right) - x(u^2 + \frac{2\beta}{\alpha} u_{xx}) + \frac{2\beta}{\alpha} u_x \right) + D_y \left( - 2\beta (t u_x u_y + \frac{1}{\alpha} x u_{xy}) \right) = 0
\]
Methods for Computing Conservation Laws

• Use the Lax pair \( L \) and \( A \), satisfying \([L, A] = 0\).
  If \( L = D_x + U, \ A = D_t + V \) then \( V_x - U_t + [U, V] = 0 \).
  \( \hat{L} = TLT^{-1} \) gives the densities, \( \hat{A} = TAT^{-1} \) gives the fluxes.

• Use Noether’s theorem (Lagrangian formulation) to generate conservation laws from symmetries (Ovsiannikov, Olver, Mahomed, Kara, etc.).

• Integrating factor methods (Anderson, Bluman, Anco, Cheviakov, Mason, Naz, etc.) require solving ODEs (or PDEs).
Proposed Algorithmic Method

- Density is linear combination of scaling invariant terms (in the jet space) with undetermined coefficients.

- Compute $D_t \rho$ with total derivative operator.

- Use variational derivative (Euler operator) to express exactness.

- Solve a (parametrized) linear system to find the undetermined coefficients.

- Use the homotopy operator to compute the flux (invert $D_x$ or $\text{Div}$).
• Work with linearly independent pieces in finite dimensional spaces.

• Use linear algebra, calculus, and variational calculus (algorithmic).

• Implement the algorithm in Mathematica.
Tools from the Calculus of Variations

Differential Topology and Differential Geometry

• **Definition:**
  
  A differential function \( f \) is a exact iff \( f = \text{Div} \mathbf{F} \).

  Special case (1D): \( f = D_x F \).

• **Question:** How can one test that \( f = \text{Div} \mathbf{F} \) ?

• **Theorem (exactness test):**

  \[ f = \text{Div} \mathbf{F} \text{ iff } \mathcal{L}_{\mathbf{u}(j)}(x)f \equiv 0, \quad j = 1, 2, \ldots, N. \]

  \( N \) is the number of dependent variables.

  The Euler operator annihilates divergences...
• Euler operator in 1D (variable \( u(x) \)):

\[
\mathcal{L}_u(x) = \sum_{k=0}^{M} (-D_x)^k \frac{\partial}{\partial u_{kx}} \\
= \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} - D_x^3 \frac{\partial}{\partial u_{xxx}} + \cdots
\]

• Euler operator in 2D (variable \( u(x, y) \)):

\[
\mathcal{L}_u(x, y) = \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} (-D_x)^k (-D_y)^\ell \frac{\partial}{\partial u_{kx \ell y}} \\
= \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} \\
+ D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_y \frac{\partial}{\partial u_{xy}} + D_y^2 \frac{\partial}{\partial u_{yy}} - D_x^3 \frac{\partial}{\partial u_{xxx}} + \cdots
\]
Application: Testing Exactness

Example:

\[ f = 8v_x v_{xx} - u_x^3 \sin u + 2u_x u_{xx} \cos u - 6vv_x \cos u + 3u_x v^2 \sin u \]

where \( u(x) \) and \( v(x) \)

- \( f \) is exact
- After integration by parts (by hand):

\[
F = \int f \, dx = 4v_x^2 + u_x^2 \cos u - 3v^2 \cos u
\]
• **Exactness test with Euler operator:**

\[ f = 8v_x v_{xx} - u_x^3 \sin u + 2u_x u_{xx} \cos u - 6v v_x \cos u + 3u_x v^2 \sin u \]

\[
\mathcal{L}_{u(x)} f = \frac{\partial f}{\partial u} - D_x \frac{\partial f}{\partial u_x} + D_x^2 \frac{\partial f}{\partial u_{xx}} \equiv 0
\]

\[
\mathcal{L}_{v(x)} f = \frac{\partial f}{\partial v} - D_x \frac{\partial f}{\partial v_x} + D_x^2 \frac{\partial f}{\partial v_{xx}} \equiv 0
\]
• Question: How can one compute $F = \text{Div}^{-1} f$?

• Theorem (integration by parts):
  
  - In 1D: If $f$ is exact then
    \[
    F = D_x^{-1} f = \int f \, dx = \mathcal{H}_u(x)f
    \]
  
  - In 2D: If $f$ is a divergence then
    \[
    F = \text{Div}^{-1} f = (\mathcal{H}^{(x)}_{u(x,y)} f, \mathcal{H}^{(y)}_{u(x,y)} f)
    \]

The homotopy operator inverts total derivatives and divergences!
Two continuous functions are called homotopic if one can be “continuously deformed” into the other. Such a deformation is called a homotopy between the two functions.

\[ T(u_0, u) = u_0 + \lambda(u - u_0) = (1 - \lambda)u_0 + \lambda u \]

If \( u_0 = 0 \) then simply replace \( u \) by \( \lambda u \) since \( T(0, u) = \lambda u \).
• Homotopy Operator in 1D (variable $x$):

$$
H_{u(x)} f = \int_0^1 \sum_{j=1}^N (I_{u(j)} f)[\lambda u] \frac{d\lambda}{\lambda}
$$

with integrand

$$
I_{u(j)} f = \sum_{k=1}^{M_{x(j)}} \left( \sum_{i=0}^{k-1} u_{i,x}^{(j)} (-D_x)^{k-(i+1)} \right) \frac{\partial f}{\partial u_{k,x}^{(j)}}
$$

$(I_{u(j)} f)[\lambda u]$ means that in $I_{u(j)} f$ one replaces $u \rightarrow \lambda u, \ u_x \rightarrow \lambda u_x$, etc.

More general: $u \rightarrow \lambda(u - u_0) + u_0$

$u_x \rightarrow \lambda(u_x - u_{x0}) + u_{x0}$ etc.
• Homotopy Operator in 2D (variables $x$ and $y$):

$$
\mathcal{H}_{u(x,y)}^{(x)} f = \int_0^1 \sum_{j=1}^N (I_{u(j)}^{(x)} f)[\lambda u] \frac{d\lambda}{\lambda}
$$

$$
\mathcal{H}_{u(x,y)}^{(y)} f = \int_0^1 \sum_{j=1}^N (I_{u(j)}^{(y)} f)[\lambda u] \frac{d\lambda}{\lambda}
$$

where for dependent variable $u(x, y)$

$$
\mathcal{I}_{u}^{(x)} f = \sum_{k=1}^{M_x} \sum_{\ell=0}^{M_y} \left( \sum_{i=0}^{k-1} \sum_{j=0}^{\ell} u_{ixjy} \frac{(i+j)(k+\ell-i-j-1)}{(k-i-1)(k+i)} \right) \left( (-D_x)^{k-i-1} (-D_y)^{\ell-j} \right) \frac{\partial f}{\partial u_{kx \ell y}}
$$

$$
\mathcal{I}_{u}^{(y)} f = \sum_{k=1}^{M_x} \sum_{\ell=0}^{M_y} \left( \sum_{i=0}^{k-1} \sum_{j=0}^{\ell} u_{ixjy} \frac{(i+j)(k+\ell-i-j-1)}{(k-i-1)(k+i)} \right) \left( (-D_x)^{k-i-1} (-D_y)^{\ell-j} \right) \frac{\partial f}{\partial u_{kx \ell y}}
$$
Application 1: The KdV Equation
\[ u_t + 6uu_x + u_{xxx} = 0 \]

**Step 1:** Compute the dilation symmetry

Set \((x, t, u) \rightarrow (\frac{x}{\kappa}, \frac{t}{\kappa^a}, \kappa^b u) = (\tilde{x}, \tilde{t}, \tilde{u})\)

Apply change of variables (chain rule)

\[ \kappa^{-(a+b)} \tilde{u}_\tilde{t} + \kappa^{-(2b+1)} \tilde{u}_\tilde{x} + \kappa^{-(b+3)} \tilde{u}_{3\tilde{x}} = 0 \]

Solve \(a + b = 2b + 1 = b + 3\).

Solution: \(a = 3\) and \(b = 2\)

\((x, t, u) \rightarrow (\frac{x}{\kappa}, \frac{t}{\kappa^3}, \kappa^2 u)\)
Compute the density of selected rank, say, 6.

- **Step 2: Determine the form of the density**

List powers of \( u \), up to rank 6: \([u, u^2, u^3]\)

Differentiate with respect to \( x \) to increase the rank

- \( u \) has weight 2 \( \rightarrow \) apply \( D_x^4 \)
- \( u^2 \) has weight 4 \( \rightarrow \) apply \( D_x^2 \)
- \( u^3 \) has weight 6 \( \rightarrow \) no derivatives needed
Apply the $D_x$ derivatives

Remove total and highest derivative terms:

\[ D_x^4 u \rightarrow \{u_{4x}\} \rightarrow \text{empty list} \]
\[ D_x^2 u^2 \rightarrow \{u_x^2, uu_{xx}\} \rightarrow \{u_x^2\} \]

since $uu_{xx} = (uu_x)_x - u_x^2$

\[ D_x^0 u^3 \rightarrow \{u^3\} \rightarrow \{u^3\} \]

Linearly combine the “building blocks”

Candidate density: $\rho = c_1u^3 + c_2u_x^2$
Step 3: Compute the coefficients $c_i$

Compute

$$D_t \rho = \frac{\partial \rho}{\partial t} + \rho'(u)[u_t]$$

$$= \frac{\partial \rho}{\partial t} + \sum_{k=0}^{M} \frac{\partial \rho}{\partial u_{kx}} D_x^k u_t$$

$$= (3c_1u^2I + 2c_2u_x D_x) u_t$$

Substitute $u_t$ by $-(6uu_x + u_{xxx})$

$$E = -D_t \rho = (3c_1u^2I + 2c_2u_x D_x)(6uu_x + u_{xxx})$$

$$= 18c_1u^3u_x + 12c_2u_x^3 + 12c_2uu_xu_{xx}$$

$$+ 3c_1u^2u_{xxx} + 2c_2u_x u_{4x}$$
Apply the Euler operator (variational derivative)

$$\mathcal{L}_u(x) = \frac{\delta}{\delta u} = \sum_{k=0}^{m} (-D_x)^k \frac{\partial}{\partial u_{kx}}$$

Here, $E$ has order $m = 4$, thus

$$\mathcal{L}_u(x) E = \frac{\partial E}{\partial u} - D_x \frac{\partial E}{\partial u_x} + D_x^2 \frac{\partial E}{\partial u_{xx}} - D_x^3 \frac{\partial E}{\partial u_{3x}} + D_x^4 \frac{\partial E}{\partial u_{4x}}$$

$$= -18(c_1 + 2c_2) u_x u_{xx}$$

This term must vanish!

So, $c_2 = -\frac{1}{2} c_1$. Set $c_1 = 1$ then $c_2 = -\frac{1}{2}$.

Hence, the final form density is

$$\rho = u^3 - \frac{1}{2} u_x^2$$
• **Step 4: Compute the flux** $J$

**Method 1: Integrate by parts (simple cases)**

Now,

$$E = 18u^3u_x + 3u^2u_{xxx} - 6u_x^3 - 6uu_xu_{xx} - u_xu_{xxxx}$$

Integration of $D_xJ = E$ yields the flux

$$J = \frac{9}{2}u^4 - 6uu_x^2 + 3u^2u_{xx} + \frac{1}{2}u_{xx}^2 - u_xu_{xxxx}$$
Method 2: Use the homotopy operator

\[ J = D_x^{-1} E = \int E \, dx = \mathcal{H}_{u(x)} E = \int_0^1 (I_u E)[\lambda u] \frac{d\lambda}{\lambda} \]

with integrand

\[ I_u E = \sum_{k=1}^{M} \left( \sum_{i=0}^{k-1} u_{ix} (-D_x)^{k-(i+1)} \right) \frac{\partial E}{\partial u_{kx}} \]
Here $M = 4$, thus

\[
I_u E = (uI) \left( \frac{\partial E}{\partial u_x} \right) + (u_x I - u D_x) \left( \frac{\partial E}{\partial u_{xx}} \right) \\
+ (u_{xx} I - u_x D_x + u D_x^2) \left( \frac{\partial E}{\partial u_{xxx}} \right) \\
+ (u_{xxx} I - u_{xx} D_x + u_x D_x^2 - u D_x^3) \left( \frac{\partial E}{\partial u_{4x}} \right) \\
= (uI)(18u^3 + 18u_x^2 - 6uu_{xx} - u_{xxxx}) \\
+ (u_x I - u D_x)(-6uu_x) \\
+ (u_{xx} I - u_x D_x + u D_x^2)(3u^2) \\
+ (u_{xxx} I - u_{xx} D_x + u_x D_x^2 - u D_x^3)(-u_x) \\
= 18u^4 - 18uu_x^2 + 9u^2u_{xx} + u_{xx}^2 - 2u_x u_{xxx}
\]

Note: correct terms but incorrect coefficients!
Finally,

\[ J = \mathcal{H}_u(x) E = \int_0^1 (I_u E)[\lambda u] \frac{d\lambda}{\lambda} \]

\[ = \int_0^1 \left( 18\lambda^3 u^4 - 18\lambda^2 uu_x^2 + 9\lambda^2 u^2 u_{xx} + \lambda u_{xx}^2 - 2\lambda uu_x u_{xxx} \right) d\lambda \]

\[ = \frac{9}{2} u^4 - 6uu_x^2 + 3u^2 u_{xx} + \frac{1}{2} u_{xx}^2 - uu_x u_{xxx} \]

**Final form of the flux:**

\[ J = \frac{9}{2} u^4 - 6uu_x^2 + 3u^2 u_{xx} + \frac{1}{2} u_{xx}^2 - uu_x u_{xxx} \]
Application 2: Zakharov-Kuznetsov Equation

\[
 u_t + \alpha uu_x + \beta(u_{xx} + u_{yy})_x = 0
\]

• Step 1: Compute the dilation invariance

ZK equation is invariant under scaling symmetry

\[
 (t, x, y, u) \rightarrow \left( \frac{t}{\kappa^3}, \frac{x}{\kappa}, \frac{y}{\kappa}, \kappa^2 u \right) = (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u})
\]

\(\kappa\) is an arbitrary parameter.

• Hence, the weights of the variables are

\[
 W(u) = 2, \quad W(D_t) = 3, \quad W(D_x) = 1, \quad W(D_y) = 1.
\]
A conservation law is invariant under the scaling symmetry of the PDE.

\[ W(u) = 2, \quad W(D_t) = 3, \quad W(D_x) = 1, \quad W(D_y) = 1. \]

For example,

\[
\begin{align*}
D_t \left( u^3 - \frac{3\beta}{\alpha} (u_x^2 + u_y^2) \right) + D_x \left( 3u^2 \left( \frac{\alpha}{4} u^2 + \beta u_{xx} \right) - 6\beta u(u_x^2 + u_y^2) \right) \\
+ \frac{3\beta^2}{\alpha} (u_{xx}^2 - u_{xy}^2) - \frac{6\beta^2}{\alpha} (u_x(u_{xxx} + u_{xyy}) + u_y(u_{xxy} + u_{yyy})) \\
+ D_y \left( 3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy}(u_x + u_{yy}) \right) = 0
\end{align*}
\]

\[ \text{Rank } (\rho) = 6, \quad \text{Rank } (J) = 8. \]
\[ \text{Rank } (\text{conservation law}) = 9. \]
Compute the density of selected \textbf{rank}, say, 6.

- \textbf{Step 2: Construct the candidate density}

For example, construct a density of rank 6.

Make a list of all terms with rank 6:

\( \{u^3, u_x^2, uu_xx, u_y^2, uu_yy, ux uy, uu_xy, u_4x, u_3xy, u_2x2y, ux3y, u_4y\} \)

Remove divergences and divergence-equivalent terms.

\textbf{Candidate density of rank 6:}

\[
\rho = c_1 u^3 + c_2 u_x^2 + c_3 u_y^2 + c_4 ux uy
\]
Step 3: Compute the undetermined coefficients

Compute

\[ D_t \rho = \frac{\partial \rho}{\partial t} + \rho'(u)[u_t] \]

\[ = \frac{\partial \rho}{\partial t} + \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} \frac{\partial \rho}{\partial u_{kx \ell y}} D_x^k D_y^\ell u_t \]

\[ = \left( 3c_1 u^2 I + 2c_2 u_x D_x + 2c_3 u_y D_y + c_4 (u_y D_x + u_x D_y) \right) u_t \]

Substitute \( u_t = -\left( \alpha uu_x + \beta (u_{xx} + u_{yy}) \right) \).
\[ E = -D_t \rho = 3c_1 u^2 (\alpha u u_x + \beta (u_{xx} + u_{xy})_x) + 2c_2 u_x (\alpha u u_x + \beta (u_{xx} + u_{yy})_x)_x + 2c_3 u_y (\alpha u u_x + \beta (u_{xx} + u_{yy})_x)_x + \beta (u_{xx} + u_{yy})_y + c_4 (u_y (\alpha u u_x + \beta (u_{xx} + u_{yy})_x)_x + u_x (\alpha u u_x + \beta (u_{xx} + u_{yy})_x)_y) \]

Apply the Euler operator (variational derivative)

\[ \mathcal{L}_u(x,y) E = \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} (-D_x)^k (-D_y)^\ell \frac{\partial E}{\partial u_{kx \ell y}} \]

\[ = -2 \left( (3c_1 \beta + c_3 \alpha) u_x u_{yy} + 2(3c_1 \beta + c_3 \alpha) u_y u_{xy} + 2c_4 \alpha u_x u_{xy} + c_4 \alpha u_y u_{xx} + 3(3c_1 \beta + c_2 \alpha) u_x u_{xx} \right) \equiv 0 \]
Solve a parameterized linear system for the $c_i$:

$$3c_1\beta + c_3\alpha = 0, \quad c_4\alpha = 0, \quad 3c_1\beta + c_2\alpha = 0$$

Solution:

$$c_1 = 1, \quad c_2 = -\frac{3\beta}{\alpha}, \quad c_3 = -\frac{3\beta}{\alpha}, \quad c_4 = 0$$

Substitute the solution into the candidate density

$$\rho = c_1 u^3 + c_2 u_x^2 + c_3 u_y^2 + c_4 u_x u_y$$

Final density of rank 6:

$$\rho = u^3 - \frac{3\beta}{\alpha} (u_x^2 + u_y^2)$$
Step 4: Compute the flux

Use the **homotopy operator** to invert \( \text{Div} \):

\[
J = \text{Div}^{-1} E = \left( \mathcal{H}^{(x)}_{u(x,y)} E, \mathcal{H}^{(y)}_{u(x,y)} E \right)
\]

where

\[
\mathcal{H}^{(x)}_{u(x,y)} E = \int_0^1 (I^{(x)}_u E)[\lambda u] \frac{d\lambda}{\lambda}
\]

with

\[
\mathcal{I}^{(x)}_u E = \sum_{k=1}^{M_x} \sum_{\ell=0}^{M_y} \left( \sum_{i=0}^{k-1} \sum_{j=0}^{\ell} u_{ixjy} \frac{(i+j)(k+\ell-i-j-1)}{(k+i-1)(k+\ell)} \right) (-D_x)^{k-i-1} (-D_y)^{\ell-j} \frac{\partial E}{\partial u_{kx\elly}}
\]

Similar formulas for \( \mathcal{H}^{(y)}_{u(x,y)} E \) and \( \mathcal{I}^{(y)}_u E \).
Let \( A = \alpha uu_x + \beta (u_{xxx} + u_{xyy}) \) so that

\[
E = 3u^2 A - \frac{6\beta}{\alpha} u_x A_x - \frac{6\beta}{\alpha} u_y A_y
\]

Then,

\[
J = \left( \mathcal{H}_{u(x,y)}^{(x)} E, \mathcal{H}_{u(x,y)}^{(y)} E \right)
\]

\[
= \left( \frac{3\alpha}{4} u^4 + \beta u^2 (3u_{xx} + 2u_{yy}) - 2\beta u (3u_x^2 + u_y^2)
\right.

+ \frac{3\beta^2}{4\alpha} u (u_{2x2y} + u_{4y}) - \frac{\beta^2}{\alpha} u_x \left( \frac{7}{2} u_{xyy} + 6u_{xxx} \right)

- \frac{\beta^2}{\alpha} u_y (4u_{xxy} + \frac{3}{2} u_{yyy}) + \frac{\beta^2}{\alpha} (3u_{xx}^2 + \frac{5}{2} u_{xy}^2 + \frac{3}{4} u_{yy}^2)

+ \frac{5\beta^2}{4\alpha} u_{xx} u_{yy}, \quad \beta u^2 u_x - 4\beta uu_x u_y

- \frac{3\beta^2}{4\alpha} u (u_{x3y} + u_{3xy}) - \frac{\beta^2}{4\alpha} u_x \left( 13u_{xyy} + 3u_{yyy} \right)

- \frac{5\beta^2}{4\alpha} u_y (u_{xxx} + 3u_{xyy}) + \frac{9\beta^2}{4\alpha} u_{xy} (u_{xx} + u_{yy}) \right)
\]
However, $\text{Div}^{-1}E$ is not unique.

Indeed, $\mathbf{J} = \tilde{\mathbf{J}} + \mathbf{K}$, where $\mathbf{K} = (D_y\theta, -D_x\theta)$ is a curl term.

For example,

$$\theta = 2\beta u^2 u_y + \frac{\beta^2}{4\alpha} \left( 3u(u_{xxy} + u_{yyy}) + 10u_x u_{xy} + 5u_y(3u_{yy} + u_{xx}) \right)$$

Shorter flux:

$$\tilde{\mathbf{J}} = \mathbf{J} - \mathbf{K}$$

$$= \left( 3u^2\left( \frac{\alpha}{4} u^2 + \beta u_{xx} \right) - 6\beta u(u_x^2 + u_y^2) + \frac{3\beta^2}{\alpha} \left( u_{xx}^2 - u_{yy}^2 \right) \right. $$

$$- \frac{6\beta^2}{\alpha} \left( u_x(u_{xxx} + u_{xyy}) + u_y(u_{xxy} + u_{yyy}) \right),$$

$$3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy}(u_{xx} + u_{yy}) \right)$$
Software Demonstration

Software packages in Mathematica

Codes are available via the Internet:
URL: http://inside.mines.edu/~whereman/
Conclusions and Future Work

• The power of Euler and homotopy operators:
  ▶ Testing exactness
  ▶ Integration by parts: $D_x^{-1}$ and $\text{Div}^{-1}$

• Integration of non-exact expressions

Example: $f = u_x v + uv_x + u^2 u_{xx}$

$$\int f \, dx = uv + \int u^2 u_{xx} \, dx$$

• Use other homotopy formulas (moving terms amongst the components of the flux; prevent curl terms)
• Broader class of PDEs (beyond evolution type)

Example: short pulse equation (nonlinear optics)

\[ u_{xt} = u + (u^3)_{xx} = u + 6uu_x^2 + 3u^2u_{xx} \]

with non-polynomial conservation law

\[ D_t \left( \sqrt{1 + 6u_x^2} \right) - D_x \left( 3u^2 \sqrt{1 + 6u_x^2} \right) = 0 \]

• Continue the implementation in Mathematica

• Software: http://inside.mines.edu/~whereman
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Thank You

Additional examples on next slides
Additional Examples

• **Manakov-Santini system**

\[
\begin{align*}
    u_{tx} + u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y &= 0 \\
    v_{tx} + v_{yy} + uv_{xx} + v_x v_{xy} - v_y v_{xx} &= 0
\end{align*}
\]

• **Conservation laws for Manakov-Santini system:**

\[
\begin{align*}
    D_t \left( f u_x v_x \right) + D_x \left( f (uu_x v_x - u_x v_x v_y - u_y v_y) \\ - f' y (u_t + uu_x - u_x v_y) \right) + D_y \left( f (u_x v_y + u_y v_x + u_x v_x^2) \\ + f' (u - yu_y - yu_x v_x) \right) &= 0
\end{align*}
\]

where \( f = f(t) \) is arbitrary.
Conservation laws – continued:

\[
D_t\left(f(2u + v_x^2 - yu_x v_x)\right) + D_x\left(f (u^2 + uv_x^2 + u_y v
- v_y^2 - v_x v_y - y(uu_x v_x - u_x v_x v_y - u_y v_y))
- f'y(v_t + uv_x - v_x v_y) + (f' - 2fx)y^2(u_t + uu_x - u_x v_y)\right)
\]

\[
+ D_y\left(f(v_x^3 + 2v_x v_y - u_x v - y(u_x v_x^2 + u_x v_y + u_y v_x))
+ f'(v - y(2u + v_y + v_x^2)) + (f'y^2 - 2fx)(u_x v_x + u_y)\right) = 0
\]

where \( f = f(t) \) is arbitrary.

There are three additional conservation laws.
• (2+1)-dimensional Camassa-Holm equation

\[
(\alpha u_t + \kappa u_x - u_{txx} + 3\beta uu_x - 2u_xu_{xx} - uu_{xxx})_x + u_{yy} = 0
\]

Interchange \( t \) with \( y \)

\[
(\alpha u_y + \kappa u_x - u_{xyy} + 3\beta uu_x - 2u_xu_{xx} - uu_{xxx})_x + u_{tt} = 0
\]

Set \( v = u_t \) to get

\[
\begin{align*}
u_t &= v \\
\nu_t &= -\alpha u_{xy} - \kappa uu_x + u_{3xy} - 3\beta u_x^2 - 3\beta uu_{xx} + 2u_{xx}^2 \\
&\quad + 3u_xu_{xxx} + uu_{4x}
\end{align*}
\]
Conservation laws for the Camassa-Holm equation

\[\begin{align*}
Dt(fu) + Dx & \left( \frac{1}{\alpha} f \left( \frac{3\beta}{2} u^2 + \kappa u - \frac{1}{2} u_x^2 - uu_{xx} - u_{tx} \right) \\
& + \left( \frac{1}{2} f'y^2 - \frac{1}{\alpha} fx \right) (\alpha u_t + \kappa u_x + 3\beta uu_x - 2u_xu_{xx} - uu_{xxx} \\
& - u_{txx}) \right) + Dy \left( \left( \frac{1}{2} f'y^2 - \frac{1}{\alpha} fx \right) u_y - f'y u \right) = 0
\end{align*}\]

\[\begin{align*}
Dt(fyu) + Dx & \left( \frac{1}{\alpha} fy \left( \frac{3\beta}{2} u^2 + \kappa u - \frac{1}{2} u_x^2 - uu_{xx} - u_{tx} \right) \\
& + \left( \frac{1}{6} f'y^2 - \frac{1}{\alpha} fx \right) (\alpha u_t + \kappa u_x + 3\beta uu_x - 2u_xu_{xx} - uu_{xxx} \\
& - u_{txx}) \right) + Dy \left( \left( \frac{1}{6} f'y^2 - \frac{1}{\alpha} fx \right) u_y + \left( \frac{1}{\alpha} fx - \frac{1}{2} f'y^2 \right) u \right) = 0
\end{align*}\]

where \( f = f(t) \) is an arbitrary function.
Khoklov-Zabolotskaya equation describes e.g., sound waves in nonlinear media

\[ (u_t - uu_x)_x - u_{yy} - u_{zz} = 0 \]

Conservation law:

\[
D_t\left(fu\right) - D_x\left(\frac{1}{2} fu^2 + (fx + g)(u_t - uu_x)\right) \\
+ D_y\left((fx + g)u_y - (fyx + g_y)u\right) \\
+ D_z\left((fx + g)u_z - (fzx + g_z)u\right) = 0
\]

under the constraints \( \Delta f = 0 \) and \( \Delta g = f_t \)
where \( f = f(t, y, z) \) and \( g = g(t, y, z) \).
• Shallow water wave model (atmosphere)

\[ u_t + (u \cdot \nabla) u + 2 \Omega \times u + \nabla (\theta h) - \frac{1}{2} h \nabla \theta = 0 \]

\[ \theta_t + u \cdot (\nabla \theta) = 0 \]

\[ h_t + \nabla \cdot (uh) = 0 \]

where \( u(x, y, t), \theta(x, y, t) \) and \( h(x, y, t) \).

• In components:

\[ u_t + uu_x + vu_y - 2 \Omega v + \frac{1}{2} h \theta_x + \theta h_x = 0 \]

\[ v_t + uv_x + vv_y + 2 \Omega u + \frac{1}{2} h \theta_y + \theta h_y = 0 \]

\[ \theta_t + u \theta_x + v \theta_y = 0 \]

\[ h_t + hu_x + uh_x + hv_y + vh_y = 0 \]
• First few conservation laws of SWW model:

\[ \rho_{(1)} = h \]

\[ \rho_{(2)} = h \theta \]

\[ \rho_{(3)} = h \theta^2 \]

\[ \rho_{(4)} = h \left( u^2 + v^2 + h\theta \right) \]

\[ \rho_{(5)} = \theta \left( 2\Omega + v_x - u_y \right) \]

\[ \mathbf{J}^{(1)} = h \begin{pmatrix} u \\ v \end{pmatrix} \]

\[ \mathbf{J}^{(2)} = h \theta \begin{pmatrix} u \\ v \end{pmatrix} \]

\[ \mathbf{J}^{(3)} = h \theta^2 \begin{pmatrix} u \\ v \end{pmatrix} \]

\[ \mathbf{J}^{(4)} = h \begin{pmatrix} u (u^2 + v^2 + 2h\theta) \\ v (v^2 + u^2 + 2h\theta) \end{pmatrix} \]

\[ \mathbf{J}^{(5)} = \frac{1}{2} \theta \begin{pmatrix} 4\Omega u - 2uu_y + 2uv_x - h\theta_y \\ 4\Omega v + 2uv_x - 2vu_y + h\theta_x \end{pmatrix} \]
More general conservation laws for SWW model:

\[ \frac{D}{Dt}(f(\theta)h) + \frac{D}{Dx}(f(\theta)hu) + \frac{D}{Dy}(f(\theta)hv) = 0 \]

\[ \frac{D}{Dt}(g(\theta)(2\Omega + v_x - u_x)) \]

\[ + \frac{D}{Dx}\left(\frac{1}{2}g(\theta)(4\Omega u - 2uu_y + 2uv_x - h\theta_y)\right) \]

\[ + \frac{D}{Dy}\left(\frac{1}{2}g(\theta)(4\Omega v - 2u_yv + 2vv_x + h\theta_x)\right) = 0 \]

for any functions \( f(\theta) \) and \( g(\theta) \).
• Kadomtsev-Petviashvili (KP) equation

\[(u_t + \alpha uu_x + u_{xxx})_x + \sigma^2 u_{yy} = 0\]

parameter \(\alpha \in \mathbb{R}\) and \(\sigma^2 = \pm 1\).

Equation be written as a conservation law

\[D_t(u_x) + D_x(\alpha uu_x + u_{xxx}) + D_y(\sigma^2 u_y) = 0.\]

Exchange \(y\) and \(t\) and set \(u_t = v\)

\[
\begin{align*}
  u_t &= v \\
  v_t &= -\frac{1}{\sigma^2}(u_{xy} + \alpha u^2_x + \alpha uu_{xx} + u_{xxxx})
\end{align*}
\]
• Examples of conservation laws for KP equation (explicitly dependent on $t, x,$ and $y$)

\[ D_t (xu_x) + D_x \left( 3u^2 - u_{xx} - 6xuu_x + xu_{xxx} \right) + D_y (\alpha xu_y) = 0 \]

\[ D_t (yu_x) + D_x \left( y(\alpha uu_x + u_{xxx}) \right) + D_y \left( \sigma^2 (yu_y - u) \right) = 0 \]

\[ D_t \left( \sqrt{t}u \right) + D_x \left( \frac{\alpha}{2} \sqrt{t}u^2 + \sqrt{t}u_{xx} + \frac{\sigma^2 y^2}{4\sqrt{t}}u_t + \frac{\sigma^2 y^2}{4\sqrt{t}}u_{xxx} \right) \]

\[ + \frac{\alpha \sigma^2 y^2}{4\sqrt{t}} uu_x - x\sqrt{t}u_t - \alpha x\sqrt{t}uu_x - x\sqrt{t}u_{xxx} \]

\[ + D_y \left( x\sqrt{t}u_y + \frac{y^2 u_y}{4\sqrt{t}} - \frac{yu}{2\sqrt{t}} \right) = 0 \]
• More general conservation laws for KP equation:

\[
D_t(fu) + D_x \left( f \left( \frac{\alpha}{2} u^2 + u_{xx} \right) \right) \\
+ \left( \frac{\sigma^2}{2} f' y^2 - fx \right) (u_t + \alpha uu_x + u_{3x}) \right) \\
+ D_y \left( \left( \frac{1}{2} f' y^2 - \sigma^2fx \right) u_y - f'yu \right) = 0
\]

\[
D_t(fyu) + D_x \left( fy \left( \frac{\alpha}{2} u^2 + u_{xx} \right) \right) \\
+ y \left( \frac{\sigma^2}{6} f' y^2 - fx \right) (u_t + \alpha uu_x + u_{3x}) \right) \\
+ D_y \left( y \left( \frac{1}{6} f' y^2 - \sigma^2 fx \right) u_y + \left( \sigma^2 fx - \frac{1}{2} f' y^2 \right) u \right) = 0
\]

where \( f(t) \) is arbitrary function.
• Potential KP equation

Replace \( u \) by \( u_x \) and integrate with respect to \( x \).

\[
\begin{align*}
  u_{xt} + \alpha u_x u_{xx} + u_{xxxx} + \sigma^2 u_{yy} &= 0
\end{align*}
\]

• Examples of conservation laws

(not explicitly dependent on \( x, y, t \)):

\[
\begin{align*}
  D_t (u_x) + D_x \left( \frac{\alpha}{2} u_x^2 + u_{xxx} \right) + D_y \left( \sigma^2 u_y \right) &= 0 \\
  D_t (u_x^2) + D_x \left( \frac{2\alpha}{3} u_x^3 - u_x^2 + 2u_x u_{xxx} - \sigma^2 u_{yy} \right) + D_y \left( 2\sigma^2 u_x u_y \right) &= 0
\end{align*}
\]
Conservation laws for pKP equation – continued:

\[ D_t(u_x u_y) + D_x(\alpha u_x^2 u_y + u_t u_y + 2u_{xxx} u_y - 2u_{xx} u_{xy}) \]
\[ + D_y\left(\sigma^2 u_y^2 - \frac{1}{3} u_x^3 - u_t u_x + u_{xx}^2\right) = 0 \]

\[ D_t\left(2\alpha uu_x u_{xx} + 3uu_{4x} - 3\sigma^2 u_y^2\right) + D_x\left(2\alpha u_t u_x^2 + 3u_t^2 \right. \]
\[ - 2\alpha uu_x u_{tx} - 3u_{tx} u_{xx} + 3u_t u_{xxx} + 3u_x u_{txx} - 3uu_t u_{xxx}\right) \]
\[ + D_y\left(6\sigma^2 u_t u_y\right) = 0 \]

Various generalizations exist.
• Generalized Zakharov-Kuznetsov equation

$$u_t + \alpha u^n u_x + \beta (u_{xx} + u_{yy})_x = 0$$

where $n$ is rational, $n \neq 0$.

Conservation laws:

$$D_t(u) + D_x \left( \frac{\alpha}{n+1} u^{n+1} + \beta u_{xx} \right) + D_y (\beta u_{xy}) = 0$$

$$D_t(u^2) + D_x \left( \frac{2\alpha}{n+2} u^{n+2} - \beta (u_x^2 - u_y^2) + 2\beta u (u_{xx} + u_{yy}) \right) + D_y \left( -2\beta u_x u_y \right) = 0$$
• Third conservation law for gZK equation:

\[
D_t\left(u^{n+2} - \frac{(n+1)(n+2)\beta}{2\alpha}(u_x^2 + u_y^2)\right) \\
+ D_x\left(\frac{(n+2)\alpha}{2(n+1)}u^{2(n+1)} + (n + 2)\beta u^{n+1}u_{xx}\right) \\
- (n + 1)(n + 2)\beta u^n(u_x^2 + u_y^2) + \frac{(n+1)(n+2)\beta^2}{2\alpha}(u_{xx}^2 - u_{yy}^2) \\
- \frac{(n+1)(n+2)\beta^2}{\alpha}\left(u_x(u_{xxx} + u_{xyy}) + u_y(u_{xxy} + u_{yyy})\right) \\
+ D_y\left((n + 2)\beta u^{n+1}u_{xy} + \frac{(n+1)(n+2)\beta^2}{\alpha}u_{xy}(u_{xx} + u_{yy})\right) = 0.
\]