Estimation-type results on the $k$-fractional Simpson-type integral inequalities and applications

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ABSTRACT
We establish a Simpson-type identity of multiparameter and certain Simpson-type inequalities via $k$-fractional integrals. Worth mentioning, the obtained inequalities in this article generalize some results presented by Set et al. [Simpson type integral inequalities for convex functions via Riemann–Liouville integrals. Filomat. 2017;31(14):4415–4420] and Sarikaya et al. [On new inequalities of Simpson’s type for $s$-convex functions. Comput Math Appl. 2010;60:2191–2199]. As applications, we also provide several inequalities for $k$-divergence measures and probability density functions. We expect that this study will be result in the new $k$-fractional integration explorations for Simpson-type inequalities.

1. Introduction
The following inequality is named the Simpson type integral inequality:

$$\int \left( h(t_1) + 4h \left( \frac{t_1 + t_2}{2} \right) + h(t_2) \right) dt = \frac{1}{2880} \|h''(t)\|_{\infty} (t_2 - t_1)^4,$$

where $h : [t_1, t_2] \to \mathbb{R}$ is a four-order differentiable mapping on $(t_1, t_2)$ and $\|h''(t)\|_{\infty} = \sup_{t \in [t_1, t_2]} |h''(t)| < \infty$.

Considering the Simpson type inequalities, many researches generalized and extended them. For example, Hsu et al. [1], Du et al. [2], Noor et al. [3], İşcan et al. [4] and Tunc et al. [5] obtained some Simpson type inequalities for differentiable mappings which are convex, extended $(s, m)$-convex, geometrically relative convex, $p$-quasi-convex mappings and $h$-convex, respectively. Further results involving the Simpson type inequality in question with applications to Riemann–Liouville fractional integrals have been explored out by some scholars, including Set et al. [6] and Hwang et al. [7] in the study of the Simpson type inequalities using convexity, as well as İşcan [8] in the study of the Simpson type inequalities using $s$-convexity. More details correspond to the Simpson type inequality and its extension, we refer to some articles by Hussain and Qaisar [9], Matloka [10], Qaisar et al. [11], Ujević [12] and Ul-Haq et al. [13].

Let us consider an $m$-invex set $A$. A set $A \subseteq \mathbb{R}^n$ is named $m$-invex set with respect to the mapping $\eta : A \times A \times (0, 1) \to \mathbb{R}^n$ for certain fixed $m \in (0, 1]$, if $m\theta_1 + \eta_2(\theta_1, \theta_1, m)$ holds, for all $\theta_1, \theta_2 \in A$ and $t \in [0, 1]$. A mapping $h : A \to \mathbb{R}$ is called generalized $(a, m)$-preinvex respecting $\eta$, if the following inequality

$$h(m\theta_1 + \eta_2(\theta_1, \theta_1, m)) \leq m(1 - t^a)h(\theta_1) + t^ah(\theta_2)$$

holds, for every $\theta_1, \theta_2 \in A$ and $t \in [0, 1]$.

Fractional calculus, as a very useful tool, shows its significance to implement differentiation and integration of real or complex number orders. This topic has attracted much attention from researchers who focus on the study of partial differential equations during the last few decades. For recent results related to this subject, we refer to some studies by Sohail et al. [14], Hameed et al. [15], and Khan et al. [16,17]. Among a lot of the fractional integral operators growed, the Riemann–Liouville fractional integral operator has been extensively studied, because of applications in many fields of sciences, such as differential equations, differential geometry and physics science. An important generalization of Riemann–Liouville fractional integrals was considered by Mubeen et al. in [18] which is named $k$-fractional integral operators.

Definition 1.1: Let $h \in L^1([t_1, t_2])$, the $k$-fractional integrals $k \int_{t_1}^{t_2} h(x)dx$ and $k \int_{t_1}^{t_2} h(x)dx$ of order $\mu > 0$ are
defined by
\[ k_J^\mu \tau h(x) = \frac{1}{k \Gamma_k(\mu)} \int_{t_1}^{x} (x - \lambda)^{\mu - 1} h(\lambda) \, d\lambda, \]
\[ (0 \leq t_1 < x < t_2) \tag{3} \]
and
\[ k_J^\mu \tau h(x) = \frac{1}{k \Gamma_k(\mu)} \int_{t_1}^{x} (\lambda - x)^{\mu - 1} h(\lambda) \, d\lambda, \]
\[ (0 \leq t_1 < x < t_2), \tag{4} \]
respectively, where \( k > 0 \) and \( \Gamma_k(\mu) \) is the \( k \)-gamma function, i.e. \( \Gamma_k(\mu) = \int_0^\infty x^{\mu - 1} e^{-\frac{x}{k}} \, dx \), with the properties \( \Gamma_k(\mu + k) = \mu \Gamma_k(\mu) \). Note that \( k_J^0 \tau h(x) = k_J^0 \tau h(x) = h(x) \).

Some recent results pertaining \( k \)-fractional integrals can be found in [19–21].

Here, via \( k \)-fractional integral operators, we obtain some estimation-type results of Simpson-type inequality in terms of a multi-parameter identity. We also consider the established inequalities applying to \( f \)-divergence measures and probability density functions.

2. Main results

Throughout this article, let \( \mathbb{N}^* \) be the set of all positive integers, and let \( A \subseteq \mathbb{R} \) be an open \( m \)-index subset respecting \( \eta : A \times A \times (0, 1] \rightarrow \mathbb{R} \setminus \{0\} \) for \( m \in \{0, 1\}, t_1, t_2 \in A, t_1 < t_2 \). Suppose that \( h : A \rightarrow \mathbb{R} \) is differentiable satisfying that \( h' \) is integrable on \([m\tau_1, m\tau_1 + \eta(t_2, t_1, m)]\). We also utilize the following notation:

\[ \Delta_{h, \eta}(\mu, k; n, m) \]
\[ := \frac{1}{6} \left[ h(m\tau_1) + h(m\tau_1 + \eta(t_2, t_1, m)) ight. \]
\[ + 2h\left( m\tau_1 + \frac{1}{n + 1} \eta(t_2, t_1, m) \right) \]
\[ + 2h\left( m\tau_1 + \frac{n}{n + 1} \eta(t_2, t_1, m) \right) \]
\[ - \frac{\Gamma_k(\mu + k)(n + 1)^{\frac{\mu}{k}}}{6n^{\frac{\mu}{k}}(t_2, t_1, m)} \]
\[ \times \left[ k_J^\mu (m\tau_1) h \right] \]
\[ + k_J^\mu (m\tau_1 + \eta(t_2, t_1, m)) h \left( m\tau_1 + \frac{n}{n + 1} \eta(t_2, t_1, m) \right) \]
\[ \times \frac{\Gamma_k(\mu + k)(n + 1)^{\frac{\mu}{k}}}{3n^{\frac{\mu}{k}}(t_2, t_1, m)} \]
\[ \times \left[ k_J^\mu (m\tau_1 + \frac{n}{n + 1} \eta(t_2, t_1, m)) h(4) \right] \]
\[ + k_J^\mu h - \Delta_{h, \eta}(\mu, k; n, m). \]

To prove main results, we give the following lemma.

**Lemma 2.1**: One has the following equality

\[ \Delta_{h, \eta}(\mu, k; n, m) = \frac{\eta(t_2, t_1, m)}{2(n + 1)} \int_0^1 \left[ \frac{2(1 - t) \tau - \eta(t_2, t_1, m)}{3} \right] h' \]
\[ \times \left( m\tau_1 + \frac{1 - t}{n + 1} \eta(t_2, t_1, m) \right) \, dt \]
\[ + \int_0^1 \left[ \frac{\tau - 2(1 - t) \tau}{3} \right] h' \]
\[ \times \left( m\tau_1 + \frac{n + t}{n + 1} \eta(t_2, t_1, m) \right) \, dt, \tag{6} \]

for \( k \)-fractional integrals with \( x \in [m\tau_1, m\tau_1 + \eta(t_2, t_1, m)] \), \( \mu > 0, k > 0 \) and \( n \in \mathbb{N}^* \).

**Proof**: Integration by parts, we have

\[ \ll_1 = \int_0^1 \left[ \frac{2(1 - t) \tau - \eta(t_2, t_1, m)}{3} \right] h' \]
\[ \times \left( m\tau_1 + \frac{1 - t}{n + 1} \eta(t_2, t_1, m) \right) \, dt \]
\[ = \frac{n + 1}{3\eta(t_2, t_1, m)} h(m\tau_1) + \frac{2(n + 1)}{3\eta(t_2, t_1, m)} h \]
\[ \times \left( m\tau_1 + \frac{1}{n + 1} \eta(t_2, t_1, m) \right) \]
\[ - \frac{\eta(t_2, t_1, m)}{3\eta(t_2, t_1, m)} \int_{m\tau_1}^{m\tau_1 + \frac{1}{n + 1} \eta(t_2, t_1, m)} h(x) \]
\[ \times \left( m\tau_1 + \frac{n + t}{n + 1} \eta(t_2, t_1, m) - x \right)^{\frac{\mu}{k} - 1} \, dx \]
\[ \times (x - m\tau_1)^{\frac{\mu}{k} - 1} \, dx \tag{7} \]

and

\[ \ll_2 = \int_0^1 \left[ \frac{\tau - 2(1 - t) \tau}{3} \right] h' \]
\[ \times \left( m\tau_1 + \frac{n + t}{n + 1} \eta(t_2, t_1, m) \right) \, dt \]
\[ = \frac{n + 1}{3\eta(t_2, t_1, m)} h(m\tau_1 + \eta(t_2, t_1, m)) \]
\[ + \frac{2(n + 1)}{3\eta(t_2, t_1, m)} h \left( m\tau_1 + \frac{n}{n + 1} \eta(t_2, t_1, m) \right) \]
\[ - \frac{\eta(t_2, t_1, m)}{3\eta(t_2, t_1, m)} \int_{m\tau_1 + \frac{n}{n + 1} \eta(t_2, t_1, m)}^{m\tau_1 + \frac{n}{n + 1} \eta(t_2, t_1, m)} h(x) \]
\[ \times \left( x - m\tau_1 - \frac{n}{n + 1} \eta(t_2, t_1, m) \right)^{\frac{\mu}{k} - 1} \, dx \]
\[
- \frac{2 \Gamma(n + 1)}{3} \int_{\mathbf{R}^d} f(x) \, dx \quad \text{and}
\]
\[
\frac{1}{k \Gamma(\mu)} \int_{m_1 + \eta(t_2, t_1, m)}^{m_1 + \eta(t_2, t_1, m)} h(x) \, dx
\]
\[
\times \left( m_1 + \eta(t_2, t_1, m) \right)^{\mu - 1} \, dx,
\]

and
\[
\int_{m_1 + \eta(t_2, t_1, m)}^{m_1 + \eta(t_2, t_1, m)} h(x) \, dx
\]
\[
\times \left( m_1 + \eta(t_2, t_1, m) \right)^{\mu - 1} \, dx,
\]

which completes the proof.

**Remark 2.1:** In Lemma 2.1, taking \( \eta(t_2, t_1, m) = t_2 - m_1 \) with \( m = 1 \) and \( k = 1 \), we have Lemma 2.1 in [6].

**Theorem 2.1:** For \( \rho > 1 \) with \( \varphi^{-1} + \rho^{-1} = 1 \), if \( |h'(x)|^\rho \) is generalized \((\alpha, m)\)-preinvex on \( A \), then the following inequality with \( \mu > 0 \) and \( k > 0 \) holds:

\[
|\Delta_{h,\alpha}(\mu, k; n, m)| \leq \frac{|\eta(t_2, t_1, m)|}{2(n + 1)}
\]
\[
\times \left( \int_{0}^{1} \left[ 2(1 - t)^{\frac{n}{\rho}} - \frac{n}{n + 1} \right] dt \right)^{\frac{1}{\rho}}
\]
\[
\times \left\{ \left[ \left( 1 - \frac{1}{n + 1} \right)^{\alpha \rho} \right] m |h'(t_1)|^\rho
\]
\[
+ \frac{1}{(n + 1)^\alpha (\alpha + 1)} \right\}.
\]

**Proof:** Using Lemma 2.1, the Hölder integral inequality and the generalized \((\alpha, m)\)-preinvexity of \( |h'(x)|^\rho \), we have

\[
|\Delta_{h,\alpha}(\mu, k; n, m)|
\]
\[
\leq \frac{|\eta(t_2, t_1, m)|}{2(n + 1)} \left[ \int_{0}^{1} \left[ 2(1 - t)^{\frac{n}{\rho}} - \frac{n}{n + 1} \right] dt \right]^{\frac{1}{\rho}}.
\]

If \( \mu > 0 \) and \( k > 0 \), then we have

\[
\left( \int_{0}^{1} \left[ 2(1 - t)^{\frac{n}{\rho}} - \frac{n}{n + 1} \right] dt \right)^{\frac{1}{\rho}}
\]
\[
\times \left\{ \left[ \left( 1 - \frac{1}{n + 1} \right)^{\alpha \rho} \right] m |h'(t_1)|^\rho
\]
\[
+ \frac{1}{(n + 1)^\alpha (\alpha + 1)} \right\}.
\]
Remark 2.2: In Theorem 2.1, taking \( \eta(t_2, t_1, m) = t_2 - m t_1 \) with \( m = 1 \) and \( k = 1 = \alpha \), we have Theorem 2.3 in [6].

Corollary 2.1: In Theorem 2.1, taking \( \eta(t_2, t_1, m) = t_2 - m t_1 \) with \( m = 1 \) and choosing \( n = 1 = \alpha \) with \( \mu = 1 = k \), we have the following inequalities

\[
\left| \frac{1}{6} \left[ h(t_1) + 4 h \left( \frac{t_1 + t_2}{2} \right) + h(t_2) \right] - \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} h(x) dx \right| \\
\leq \frac{t_2 - t_1}{4} \left[ \frac{1 + 2^\alpha + 1}{3^{\alpha + 1} (\alpha + 1)} \right]^\frac{1}{\alpha} \times \left[ \frac{3}{4} |h'(t_1)|^\alpha + \frac{1}{4} |h'(t_2)|^\alpha \right]^\frac{1}{\alpha} \\
+ \left( \frac{1}{4} |h'(t_1)|^\alpha + \frac{3}{4} |h'(t_2)|^\alpha \right)^\frac{1}{\alpha}. \tag{18}
\]

Proof: The second inequality is obtained by using the fact that \( \sum_{k=1}^{n} (\xi_k + \gamma_k)^2 \leq \sum_{k=1}^{n} (\xi_k)^2 + \sum_{k=1}^{n} (\gamma_k)^2 \) for \( 0 \leq s \leq 1 \), \( \xi_1, \xi_2, \xi_3, \ldots, \xi_n \geq 0; \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n \geq 0. \)

In the next theorem, we use the following functions.

(1) The beta function,

\[
\beta(u, v) = \frac{\Gamma(u) \Gamma(v)}{\Gamma(u + v)} = \int_0^1 t^{u-1} (1 - t)^{v-1} dt, \quad u, v > 0. \tag{19}
\]

(2) The incomplete beta function,

\[
\beta(\lambda; u, v) = \int_0^\lambda t^{u-1} (1 - t)^{v-1} dt, \quad 0 < \lambda < 1, u, v > 0. \tag{20}
\]

Theorem 2.2: If \( |h'(x)|^\rho \) for \( \rho \geq 1 \) is generalized \((\alpha, m)\)-preinvex on \( A \), then the following inequality with \( \mu > 0 \) and \( k > 0 \) holds:

\[
|\Delta_{h, \alpha}(\mu, k; n, m)| \leq \frac{|\eta(t_2, t_1, m)|}{2(n + 1)} \left[ \frac{1}{\mu} \right]^{\frac{1}{\alpha}} \times \left[ \left[ \frac{\kappa_0 - \kappa_1}{m} |h'(t_1)|^\rho \right] + \left[ \kappa_1 |h'(t_2)|^\rho \right]^{\frac{1}{\alpha}} \right]^{\frac{1}{\alpha}} + \left[ \kappa_1 m |h'(t_1)|^\rho + (2^{1-\alpha} \kappa_0 - \kappa_1) \times |h'(t_2)|^\rho \right]^{\frac{1}{\alpha}}. \tag{21}
\]

where

\[
\kappa_0 := \frac{3 - 2 \left( \frac{2^{1-\alpha}}{2^{\alpha + 1}} \right)^{n+1} - 4 \left( 1 - \frac{2^{1-\alpha}}{2^{\alpha + 1}} \right)^{n+1}}{3 \left( \frac{2}{2^{\alpha + 1}} \right)} \tag{22}
\]

and

\[
\kappa_1 := \frac{2 - 4 \left( 1 - \frac{2^{1-\alpha}}{2^{\alpha + 1}} \right)^{n+1}}{3(n + 1)^\alpha \left( \frac{2}{2^{\alpha + 1}} \right) + \frac{1}{3(n + 1)^\alpha}} \times \left[ \beta \left( \frac{\mu}{k} + 1, \alpha + 1 \right) \times \left( 2 \frac{2^{1-\alpha}}{2^{\alpha + 1}} \frac{\mu}{k} + 1, \alpha + 1 \right) \right]. \tag{23}
\]
Proof: Suppose that $\rho = 1$. From Lemma 2.1 and using the generalized $(\alpha, m)$-preinversity of $|h'(x)|$, we have
\[
|\Delta_{h,y}(\mu, k; n, m)| \\
\leq |\eta(t_2, t_1, m)| \int_0^1 \frac{(1 - t)^{\frac{\alpha}{m}} - t^{\frac{\alpha}{m}}}{3} \left( m|h'(t_1)| + (\frac{n + t}{n + 1})^{\alpha} |h'(t_2)| \right) dt \\
\leq m|h'(t_1)| \int_0^1 \frac{(1 - t)^{\frac{\alpha}{m}} - t^{\frac{\alpha}{m}}}{3} \left( m|h'(t_1)| + (\frac{n + t}{n + 1})^{\alpha} |h'(t_2)| \right) dt \\
+ |h'(t_2)| \int_0^1 \frac{(1 - t)^{\frac{\alpha}{m}} - t^{\frac{\alpha}{m}}}{3} \left( m|h'(t_1)| + (\frac{n + t}{n + 1})^{\alpha} |h'(t_2)| \right) dt.
\] (24)

From (24) we get the desired inequality (21) for $\rho = 1$, since
\[
\int_0^1 \frac{(1 - t)^{\frac{\alpha}{m}} - t^{\frac{\alpha}{m}}}{3} \left( m|h'(t_1)| + (\frac{n + t}{n + 1})^{\alpha} |h'(t_2)| \right) dt \\
\leq m|h'(t_1)| \int_0^1 \frac{(1 - t)^{\frac{\alpha}{m}} - t^{\frac{\alpha}{m}}}{3} \left( 1 - \left( \frac{n + t}{n + 1} \right)^{\alpha} \right) dt \\
+ |h'(t_2)| \int_0^1 \frac{(1 - t)^{\frac{\alpha}{m}} - t^{\frac{\alpha}{m}}}{3} \left( 1 - \left( \frac{n + t}{n + 1} \right)^{\alpha} \right) dt
\] (25)

Using Lemma 2.1 and the Hölder’s integral inequality in the following way, we have
\[
|\Delta_{h,y}(\mu, k; n, m)| \\
\leq \int_0^1 \frac{(1 - t)^{\frac{\alpha}{m}} - t^{\frac{\alpha}{m}}}{3} \left( 1 - \left( \frac{n + t}{n + 1} \right)^{\alpha} \right) dt
\] (26)

Considering the generalized $(\alpha, m)$-preinversity of $|h'(x)|^\rho$, we get
\[
|\Delta_{h,y}(\mu, k; n, m)| \\
\leq |\eta(t_2, t_1, m)| \int_0^1 \frac{(1 - t)^{\frac{\alpha}{m}} - t^{\frac{\alpha}{m}}}{3} \left( 1 - \left( \frac{n + t}{n + 1} \right)^{\alpha} \right) dt
\] (29)
A combination of (25), (26), (27) and (28) into (30)

\[
\tau
\]

Theorem 2.2 in [6]. Furthermore, choosing

\[
\alpha = \left| \int_{0}^{1} \frac{2(1-t)^{\frac{\mu}{2}} - t^{\frac{\mu}{2}}}{3} dt \right|
\]

(31)

A combination of (25), (26), (27) and (28) into (30) with (31) gives the desired result in (21) for \(\rho > 1\). This ends the proof. \(\blacksquare\)

**Corollary 2.2:** In Theorem 2.2, taking \(\eta(t_{2}, t_{1}, m) = t_{2} - m t_{1}\) with \(m = 1\), and choosing \(n = 1 = \alpha\) with \(\mu = 1 = k\), we have

\[
\left\{ \begin{array}{l}
\frac{1}{6} \left[ h(t_{1}) + 4h \left( \frac{t_{1} + t_{2}}{2} \right) + h(t_{2}) \right] \\
- \frac{1}{t_{2} - t_{1}} \int_{t_{1}}^{t_{2}} h(x) dx \\
\leq \frac{t_{2} - t_{1}}{4} \left[ \left( \frac{61}{4 \cdot 3^{4}} |h'(t_{1})|^{\rho} \right)^{\frac{1}{\rho}} + \left( \frac{29}{4 \cdot 3^{4}} |h'(t_{2})|^{\rho} \right)^{\frac{1}{\rho}} \right] \\
\end{array} \right.
\]

(32)

**Remark 2.3:** In Theorem 2.2, taking \(\eta(t_{2}, t_{1}, m) = t_{2} - m t_{1}\) with \(m = 1\), \(\rho = 1\) and \(k = 1 = \alpha\), we have Theorem 2.2 in [6]. Furthermore, choosing \(\mu = 1 = n\), one has Corollary 1 in [22].

For obtaining further estimation-type results, we next deal with the boundedness and the Lipschitzian condition of \(h'\).

**Theorem 2.3:** If there exist constants \(r < R\) satisfying that \(-\infty < r \leq h'(x) \leq R < \infty\) for all \(x \in [t_{1}, t_{2}]\), then the following inequality

\[
|\Delta_{h,\eta}(\mu, k; n, m)| \leq \left\{ \begin{array}{l}
\frac{(R - r)\eta(t_{2}, t_{1}, m)}{2(n + 1)} \left| \int_{0}^{1} \frac{2(1-t)^{\frac{\mu}{2}} - t^{\frac{\mu}{2}}}{3} dt \right| \\
\times \left[ \frac{2^{k}}{2^{k+1}} \right]^{\frac{\mu+1}{\mu}} - 4 \left( 1 - \frac{2^{k}}{2^{k+1}} \right) \left( \frac{\mu+1}{\mu} \right) \\
\end{array} \right.
\]

(33)

holds with \(\mu > 0, k > 0, n \in \mathbb{N}^{+}\) and \(m \in (0, 1]\).

**Proof:** From Lemma 2.1, one has

\[
\Delta_{h,\eta}(\mu, k; n, m) = \eta(t_{2}, t_{1}, m) \left\{ \int_{0}^{1} \frac{2(1-t)^{\frac{\mu}{2}} - t^{\frac{\mu}{2}}}{3} dt \right\} \\
\times \left[ \frac{2^{k}}{2^{k+1}} \right]^{\frac{\mu+1}{\mu}} - 4 \left( 1 - \frac{2^{k}}{2^{k+1}} \right) \left( \frac{\mu+1}{\mu} \right) \\
\times \left( \frac{r + R}{2} \right) \left( \frac{r + R}{2} \right) \left[ \frac{2^{k}}{2^{k+1}} \right]^{\frac{\mu+1}{\mu}} - 4 \left( 1 - \frac{2^{k}}{2^{k+1}} \right) \left( \frac{\mu+1}{\mu} \right)
\]

(34)

Utilizing the fact that \(r - \frac{r + R}{2} \leq h'(m t_{1} + \frac{1 - t}{n + 1} \eta(t_{2}, t_{1}, m)) - \frac{r + R}{2} \leq R - \frac{r + R}{2}\), one has

\[
\left| h' \left( m t_{1} + \frac{1 - t}{n + 1} \eta(t_{2}, t_{1}, m) \right) \right| - \frac{r + R}{2} \leq \frac{R - r}{2}.
\]

(35)

Similarly,

\[
\left| h' \left( m t_{1} + \frac{n + t}{n + 1} \eta(t_{2}, t_{1}, m) \right) \right| - \frac{r + R}{2} \leq \frac{R - r}{2}.
\]

(36)

Therefore

\[
|\Delta_{h,\eta}(\mu, k; n, m)| \leq \left\{ \begin{array}{l}
\frac{(R - r)\eta(t_{2}, t_{1}, m)}{2(n + 1)} \left| \int_{0}^{1} \frac{2(1-t)^{\frac{\mu}{2}} - t^{\frac{\mu}{2}}}{3} dt \right| \\
\times \left[ \frac{2^{k}}{2^{k+1}} \right]^{\frac{\mu+1}{\mu}} - 4 \left( 1 - \frac{2^{k}}{2^{k+1}} \right) \left( \frac{\mu+1}{\mu} \right) \\
\end{array} \right.
\]

(33)
\[ (R - r) |\eta(t_2, t_1, m)| \leq 4 \left( \frac{4}{2 + \frac{1}{m+1}} \right) + \frac{1}{3} \beta \left( \frac{2}{2 + \frac{1}{m+1}} \right) \left( \mu + 1, 2 \right) \]

Therefore
\[ |\Delta_{h,\eta}(\mu, k; n, m)| \leq \frac{\eta(t_2, t_1, m)}{2(n+1)} \int_0^1 \left| \frac{2(1 - t) t_2 - t_1}{3} \right| \, dt \]

This ends the proof.

**Theorem 2.4:** If \( h' \) satisfies Lipschitz condition on \([t_1, t_2]\) for certain \( \mathcal{L} > 0 \), then the following inequality holds with \( \mu > 0, k > 0, n \in \mathbb{N}^+ \) and \( m \in (0, 1] \),

\[
\left| \Delta_{h,\eta}(\mu, k; n, m) \right| \leq \frac{\mathcal{L}^2(t_2, t_1, m)}{2(n+1)^2} \left( \begin{array}{c} (n - 1) \left[ 3 - 2 \left( \frac{\mu}{2 + \frac{1}{m+1}} \right)^{\frac{\mu+1}{2}} \right] \\ -4 \left( \frac{\mu}{2 + \frac{1}{m+1}} \right)^{\frac{\mu+1}{2}} \end{array} \right) + \mathbb{R}_2.
\]

where
\[
\mathbb{R}_2 := -4 \left( \frac{\mu}{2 + \frac{1}{m+1}} \right)^{\frac{\mu+2}{2}} - 2 + \frac{8}{3} \beta \left( \frac{\mu}{2 + \frac{1}{m+1}} + 1, 2 \right).
\]

**Proof:** From Lemma 2.1, we get
\[
\Delta_{h,\eta}(\mu, k; n, m) = \frac{\eta(t_2, t_1, m)}{2(n+1)} \int_0^1 \frac{2(1 - t) t_2 - t_1}{3} \, dt \times \left( h' \left( m t_1 + \frac{1 - t}{n+1} \eta(t_2, t_1, m) \right) \right) - h' \left( m t_1 + \frac{n + t}{n+1} \eta(t_2, t_1, m) \right) \, dt.
\]

Since \( h' \) satisfies Lipschitz condition on \([t_1, t_2]\), for certain \( \mathcal{L} > 0 \), we have
\[
\left| h' \left( m t_1 + \frac{1 - t}{n+1} \eta(t_2, t_1, m) \right) \right| \leq \mathcal{L} |\eta(t_2, t_1, m)| \left( \frac{2t + n - 1}{n+1} \right).
\]

Therefore
\[
\left| \Delta_{h,\eta}(\mu, k; n, m) \right| \leq \frac{\eta(t_2, t_1, m)}{2(n+1)} \int_0^1 \frac{2(1 - t) t_2 - t_1}{3} \, dt \times \left( h' \left( m t_1 + \frac{1 - t}{n+1} \eta(t_2, t_1, m) \right) \right) - h' \left( m t_1 + \frac{n + t}{n+1} \eta(t_2, t_1, m) \right) \, dt.
\]

The proof is completed.

**Remark 2.4:** As several special cases of Theorems 2.3 and 2.4 above, some sub-results can be deduced by taking different mappings \( \eta \) and special parameter values for \( \mu, k, n \) and \( m \).

**3. Applications**

**3.1. f-divergence measures**

Let the set \( \phi \) and the \( \sigma \) -finite measure \( \mu \) be given, and let the set of all probability densities on \( \mu \) to be defined on \( \Omega := \{ p : \phi \to \mathbb{R}, p(x) > 0, \int_\phi p(x) \, d\mu(x) = 1 \} \).

Let \( f : (0, \infty) \to \mathbb{R} \) be given mapping and consider \( D_f(p, q) \) to be defined by
\[
D_f(p, q) := \int_\phi p(x) f \left( \frac{q(x)}{p(x)} \right) d\mu(x), \quad p, q \in \Omega.
\]
If \( f \) is convex, then (43) is called as the Csiszár \( f \)-divergence.

Consider the following Hermite-Hadamard (HH) divergence
\[
D_{\text{HH}}(p, q) := \int_{\Omega} p(x) \left( \int_{\Omega} f(t) \frac{q(t)}{p(t)} \frac{dt}{1} \right) d\mu(x), \quad p, q \in \Omega,
\]
where \( f \) is convex on \((0, \infty)\) with \( f(1) = 0 \). Note that \( D_{\text{HH}}'(p; q) \geq 0 \) with the equality holds if and only if \( p = q \).

**Proposition 3.1:** Let all assumptions of Corollary 2.1 hold with \( \mathcal{A} = (0, \infty) \) and \( f(1) = 0 \), if \( p, q \in \Omega \), then the following inequality holds:
\[
\begin{align*}
&\left[ \frac{1}{6} D_2(p, q) + 4 \int_{\Phi_1} p(x)f \left( \frac{p(x) + q(x)}{2p(x)} \right) d\mu(x) \right] \\
&- \left[ \frac{1 + 2^{1+1}}{3^{1+1}(3+1)} \right]^{\frac{1}{2}} \left[ \left( \frac{3}{4} \right)^{\frac{1}{2}} + \left( \frac{1}{4} \right)^{\frac{1}{2}} \right] \\
&\times \left[ |f'(1)| \int_{\Phi_1} |q(x) - p(x)| d\mu(x) \right] \\
&+ \int_{\Phi_1} |q(x) - p(x)| \left| f' \left( \frac{q(x)}{p(x)} \right) \right| d\mu(x). \quad (45)
\end{align*}
\]

**Proof:** Let \( \Phi_1 = \{ x \in \phi : q(x) > p(x) \} \), \( \Phi_2 = \{ x \in \phi : q(x) < p(x) \} \) and \( \Phi_3 = \{ x \in \phi : q(x) = p(x) \} \).

Obviously, if \( x \in \Phi_3 \), then equality holds in (45). Now if \( x \in \Phi_1 \), then using Corollary 2.1 for \( \tau_1 = 1 \), \( \tau_2 = \frac{q(x)}{p(x)} \), multiplying both sides of the obtained results by \( p(x) \) and then integrating on \( \Phi_1 \), we have
\[
\begin{align*}
&\left[ \frac{1}{6} \int_{\Phi_1} p(x)f \left( \frac{p(x) + q(x)}{2p(x)} \right) d\mu(x) \right] \\
&+ \int_{\Phi_1} p(x)f \left( \frac{q(x)}{p(x)} \right) d\mu(x) \\
&- \int_{\Phi_1} p(x) \frac{q(t)}{p(t)} \frac{dt}{1} d\mu(x) \\
&\leq \left[ \frac{1 + 2^{1+1}}{3^{1+1}(3+1)} \right]^{\frac{1}{2}} \left[ \left( \frac{3}{4} \right)^{\frac{1}{2}} + \left( \frac{1}{4} \right)^{\frac{1}{2}} \right] \\
&\times \left[ |f'(1)| \int_{\Phi_1} |q(x) - p(x)| d\mu(x) \right] \\
&+ \int_{\Phi_1} |q(x) - p(x)| \left| f' \left( \frac{q(x)}{p(x)} \right) \right| d\mu(x). \quad (46)
\end{align*}
\]

Similarly if \( x \in \Phi_2 \), then using Corollary 2.1 for \( \tau_1 = \frac{q(x)}{p(x)} \) and \( \tau_2 = 1 \), multiplying both sides of the obtained results by \( p(x) \) and then integrating over \( \Phi_2 \), we get
\[
\begin{align*}
&\left[ \frac{1}{6} \int_{\Phi_2} p(x)f \left( \frac{p(x) + q(x)}{2p(x)} \right) d\mu(x) \right] \\
&+ \int_{\Phi_2} p(x)f \left( \frac{q(x)}{p(x)} \right) d\mu(x) \\
&- \int_{\Phi_2} p(x) \frac{q(t)}{p(t)} \frac{dt}{1} d\mu(x) \\
&\leq \left[ \frac{1 + 2^{1+1}}{3^{1+1}(3+1)} \right]^{\frac{1}{2}} \left[ \left( \frac{3}{4} \right)^{\frac{1}{2}} + \left( \frac{1}{4} \right)^{\frac{1}{2}} \right] \\
&\times \left[ |f'(1)| \int_{\Phi_2} |p(x) - q(x)| d\mu(x) \right] \\
&+ \int_{\Phi_2} |p(x) - q(x)| \left| f' \left( \frac{q(x)}{p(x)} \right) \right| d\mu(x). \quad (47)
\end{align*}
\]

Adding inequalities (46) and (47), and utilizing triangular inequality, we get the result.

**Proposition 3.2:** Let all assumptions of Corollary 2.2 hold with \( \mathcal{A} = (0, \infty) \) and \( f(1) = 0 \), if \( p, q \in \Omega \), then the following inequality holds:
\[
\begin{align*}
&\left[ \frac{1}{6} D_2(p, q) + 4 \int_{\phi} p(x)f \left( \frac{p(x) + q(x)}{2p(x)} \right) d\mu(x) \right] \\
&- D_{\text{HH}}(p, q) \\
&\leq \left( \frac{5}{18} \right)^{1-\frac{1}{2}} \left[ \left( \frac{61}{324} \right)^{\frac{1}{2}} + \left( \frac{29}{324} \right)^{\frac{1}{2}} \right] \\
&\times \left[ |f'(1)| \int_{\phi} \left| \frac{q(x) - p(x)}{4} \right| d\mu(x) \right] \\
&+ \int_{\phi} \left| \frac{q(x) - p(x)}{4} \right| \left| f' \left( \frac{q(x)}{p(x)} \right) \right| d\mu(x). \quad (48)
\end{align*}
\]

**Proof:** The proof is similar as to that of Proposition 3.1 but replace Corollary 2.1 with Corollary 2.2.

### 3.2. Probability density functions

Let \( g : [\tau_1, \tau_2] \to [0, 1] \) be the probability density mapping of a continuous random variable \( X \) with the cumulative distribution mapping
\[
F(x) = \Pr(X \leq x) = \int_{\tau_1}^{x} g(t) \, dt. \quad (49)
\]

Using the fact that \( E(X) = \int_{\tau_1}^{\tau_2} t \, dF(t) = \tau_2 - \int_{\tau_1}^{\tau_2} F(t) \, dt \), we have the following results.

**Proposition 3.3:** By Corollary 2.1, we get the inequality
\[
\begin{align*}
&\left[ \frac{1}{6} \left( 4 \Pr \left( \frac{\tau_1 + \tau_2}{2} \right) + 1 \right) - \frac{1}{\tau_2 - \tau_1} (E(X) - \tau_2) \right] \\
&\leq \frac{\tau_2 - \tau_1}{4} \left[ \left( \frac{1 + 2^{1+1}}{3^{1+1}(3+1)} \right)^{\frac{1}{2}} \left( \frac{3}{4} \right)^{\frac{1}{2}} + \left( \frac{1}{4} \right)^{\frac{1}{2}} \right] \\
&\times \left[ |g(\tau_1)| + |g(\tau_2)| \right]. \quad (50)
\end{align*}
\]
Proposition 3.4: By Corollary 2.2, we get the inequality
\[ \frac{1}{6} \left[ 4\mathbb{P}(X \leq \frac{\tau_1 + \tau_2}{2}) + 1 \right] = \frac{1}{\tau_2 - \tau_1}(\tau_2 - (\mathbb{E}(X)) \right] \leq \frac{(\tau_2 - \tau_1)}{4} \left[ \left( \frac{61}{4 \cdot 3^4} \right)^{\frac{3}{2}} + \left( \frac{29}{4 \cdot 3^4} \right)^{\frac{3}{2}} \right] \times (g(\tau_1) + g(\tau_2)). \]  
\[ (51) \]
Specially, taking \( \rho = 1 \), we have
\[ \frac{1}{6} \left[ 4\mathbb{P}(X \leq \frac{\tau_1 + \tau_2}{2}) + 1 \right] = \frac{1}{\tau_2 - \tau_1}(\tau_2 - (\mathbb{E}(X)) \right] \leq \frac{5(\tau_2 - \tau_1)}{72} (|g(\tau_1)| + |g(\tau_2)|). \]  
\[ (52) \]

Remark 3.1: Applications can be provided in terms of the obtained results to special means, and we omit the details.

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