Smooth “creation” of an open universe in five dimensions

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Abstract

We present a non-singular instanton describing the creation of an open universe with a compactified extra dimension. The four dimensional section of this solution is a singular instanton of the type introduced by Hawking and Turok. The “singularity” is viewed in five dimensions as a smooth bubble of “nothing” which eats up a portion of spacetime as it expands. Flat space with a compact extra dimension is shown to be gravitationally metastable, but sufficiently long lived if the size of the extra dimension is large compared with the Planck length.

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Hawking and Turok have recently proposed that an open universe may be created from nothing [1]. This is a very interesting possibility since it would lead to open inflation without requiring a special form of the inflaton potential. This is an advantage over existing models, where part of the inflaton potential has to be suitable for tunneling in one region and suitable for slow roll in another. The price to pay, however, is that the instanton describing this process has a singular boundary (which is time-like in the Lorentzian region). Nevertheless, the Euclidean action of the instanton is integrable, and the boundary term is finite. Therefore, Hawking and Turok were able to proceed formally, and used their instanton to assign probabilities to different open universes.

Naturally, the use of singular instantons has raised some concern [2–4]. For instance, it has been argued [4] that information may flow in and out from the singularity, which would mean that predictions cannot be made in that spacetime. On closer examination, however, it turns out that the singularity behaves as a reflecting boundary for scalar and tensor cosmological perturbations [5,6]. Hence, the Cauchy problem seems to be well posed and the model is well suited for the quantization of small perturbations and for comparison with observations.

Even so, it is clear that singular instantons cannot be used without further justification. Indeed, Vilenkin has shown [3] that an instanton with the same singularity as Hawking and Turok’s would lead to the immediate decay of flat space, in contradiction with observations. Hence, there is some question as to whether singular instantons, even if integrable, can be used to describe the creation of an open universe.

On the other hand, Vilenkin’s argument provides an interesting clue towards fixing the problem. The unsuppressed decay of flat spacetime is due to the fact that singular instantons can have an arbitrarily small size, so that their action is as small as desired. It is therefore possible that if the model has a length scale below which Physics is different, the instability can be traded for metastability with a low decay rate.

In this paper, we present a non-singular five dimensional model where some of the nice properties of the singularity in four dimensions can be understood and where the unsuppressed decay of flat space is avoided. As we shall see, flat space is metastable, but its decay rate will be exponentially small provided that the size of the extra dimension is much larger than the Plack length.

**METASTABLE FLAT SPACETIME**

Let us show that flat space with an extra dimension is gravitationally metastable. It decays through the nucleation of bubbles of “nothing” which eat up spacetime as they expand.

The five-dimensional action for pure gravity is given by

$$S_E = -\frac{1}{16\pi G_5} \int \sqrt{\tilde{g}} \tilde{R} d^5x - \frac{1}{8\pi G_5} \int \sqrt{\tilde{g}} \tilde{K} d^4\xi,$$

where $G_5$ is the five-dimensional gravitational coupling, $\tilde{R}$ is the Ricci scalar and the last term is the integral over the boundary of the trace of the extrinsic curvature $\tilde{K}$. The tilde distinguishes five-dimensional quantities from their four-dimensional counterparts, which we shall encounter in the next section.
Taking an $O(4) \times U(1)$ symmetric ansatz for the metric
\[ d\tilde{s}^2 = d\tau^2 + R^2(\tau) dS^{(3)} + r^2(\tau) dy^2, \] (2)  
where $dS^{(3)} = (d\psi^2 + \sin^2 \psi d\Omega_2^2)$ is the metric on the three-sphere and $y$ is the coordinate in the fifth compact dimension, the equations of motion reduce to $\ddot{X} = 2k$ and $r\ddot{R} - \dot{r}\dot{R} = 0$, where $X \equiv R^2$, $k = 1$ is the spatial curvature of the 3-sphere and dots indicate derivative with respect to $\tau$. The first equation indicates that $R^2$ is quadratic in $\tau$. The second tells us that $r$ is proportional to $\dot{R}$. The constant of proportionality is unimportant, since it can be reabsorbed in a redefinition of $y$. Thus, the general solution is given by
\[ d\tilde{s}^2 = d\tau^2 + (\tau^2 + A^2) dS^{(3)} + \left(\frac{\tau^2}{\tau^2 + A^2}\right) dy^2. \] (3)  
In what follows, we shall take $A^2 > 0$. 

The instanton (3) is perfectly regular. At $\tau = 0$ there is a “polar” coordinate singularity in the $(\tau, y)$ plane, but the manifold is smooth and has no conical singularity if we take coordinate range in the fifth dimension as $0 \leq y < 2\pi A$. The size of the extra dimension is zero at $\tau = 0$ and goes to the constant value $A$ at large distances $\tau \to \infty$. Thus the instanton can be viewed as the direct product of a “cigar” times a three-sphere. The size of the three-sphere tends to a constant $A$ at $\tau = 0$, and grows linearly with $\tau$ at large distances, like it would in flat space. Thus our instanton is asymptotically flat.

Our solution (3) is analogous to Coleman and De Luccia’s instanton [8] describing the nucleation of a “true vacuum” bubble. The important difference is that here there is no “true vacuum” to speak of. The interior of the 3-sphere of radius $A$ at $\tau = 0$ contains no spacetime: it is a bubble of “nothing”. The evolution of the bubble after nucleation is given by the analytic continuation of (3) to the Lorentzian section. This is obtained by complexifying the angular coordinate $\psi \to (\pi/2) - i\hat{\psi}$, where $\hat{\psi}$ is real. With this, the 3-spheres become 2+1 dimensional time-like hyperboloids. The bubble grows with constant proper acceleration $A^{-1}$, eating up spacetime as it expands.

The nucleation rate can be estimated as [8]
\[ \Gamma \sim A^{-4} B^2 e^{-B}, \] (4)  
where $B = S_E - S_E^{\text{flat}}$ is the difference between the action of our instanton minus the action of flat space. Our instanton is a vacuum solution, so only the boundary term at infinity ($\tau \to \infty$) contributes to the action. (Clearly, there is no boundary term at $\tau = 0$, since the fifth dimension smoothly closes the manifold there.) This term can be expressed as the normal derivative of the volume of the boundary,
\[ S_E = \frac{-1}{8\pi G_5} \int \partial_\tau \sqrt{\gamma} d^3 S^{(3)} dy, \] (5)  
where $\sqrt{\gamma} = R^3 r = A^2 \tau + \tau^3$. The integral in (3) diverges in the limit $\tau \to \infty$, but this is remedied when we subtract the boundary term for flat space. The boundary has the

\[ ^1\text{A duality transformation takes } A^2 \to -A^2. \]
topology of $S^3 \times S^1$. This is the boundary of a flat space solution where $R$ is proportional to the distance to the origin and $r = const$. Thus, the trace of the extrinsic curvature is given by $3R^{-1}$ and we have

$$S_{E}^{\text{flat}} = \frac{-3}{8\pi G S} \int \sqrt{\gamma}R^{-1}d^3S \, dy.$$  \hfill (6)

In the limit $\tau \rightarrow \infty$ we obtain

$$B = \frac{\pi A^2}{8G}$$  \hfill (7)

where $G = G_5/(2\pi A)$ is Newton’s constant in four dimensions.

Thus, we find that even though flat space is metastable, the decay rate can be comfortably small provided that the size $A$ of the extra dimension is much larger than the Planck length. For instance, in the context of M-theory, this size is of order $10^2 l_p$, and the rate (4) would be unobservably small, even if we multiply it by the whole volume of our past light cone.

**SINGULARITY IN FOUR DIMENSIONS**

What is smooth in five dimensions may look singular in four. In this section we shall show that the solution given in the previous section can be cast as Vilenkin’s singular instanton [3]. The singularity is of the same form as the one in Hawking and Turok’s solution.

Compactifying the fifth dimension on a circle and using the ansatz

$$\tilde{g}_{AB} = e^{2\kappa \phi/3} \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & e^{-2\kappa \phi} \end{pmatrix},$$  \hfill (8)

where $\kappa = (12\pi G)^{1/2}$, the action (1) can be written as

$$S_E = \int \sqrt{g} \left[ \frac{1}{2} (\partial \phi)^2 - \frac{R}{16\pi G} \right] d^4x - \frac{1}{8\pi G} \int \sqrt{\gamma}Kd^3\xi.$$  \hfill (9)

This coincides with the action used in Ref. [3].

Again, we take an $O(4) \times U(1)$ symmetric ansatz for the metric $g_{\mu\nu}dx^\mu dx^\nu = d\sigma^2 + b^2(\sigma)dS^{(3)}$ and for $\phi = \phi(\sigma)$. With this ansatz, the field equations reduce to

$$\phi'' + \frac{3b'}{b} \phi' = 0$$  \hfill (10)

$$\left(\frac{b'}{b}\right)^2 = \frac{4\pi G}{3} \phi^2 + \frac{1}{b^2},$$  \hfill (11)

where primes stand for derivatives with respect to $\sigma$. From the four dimensional point of view, the instanton (3) corresponds to

$$b = \tau^{1/2}(A^2 + \tau^2)^{1/4}$$  \hfill (12)
\[
\phi = -\frac{3}{4\kappa} \ln \left( \frac{\tau^2}{A^2 + \tau^2} \right).
\]

(13)

Using \(|d\tau/d\sigma| = (A^2 + \tau^2)^{1/4}\tau^{-1/2}\), it is straightforward to check that \((12-13)\) satisfy the field equations \((10-11)\).

At large \(\tau\), we have \(\sigma \approx \tau\). Therefore \(b \approx \sigma\) and \(\phi \approx C/(2\sigma^2)\), where

\[
C = 3A^2/2\kappa,
\]

(14)

Near \(\tau = 0\) we have \(\tau^3/2 \approx (3A^{1/2}/2)(\sigma - \sigma_f)\), where \(\sigma_f\) is a constant. Substituting in \((12)\) we find that \(\phi \approx -\kappa^{-1}\ln(\sigma - \sigma_f) + \text{const.}\), and \(b^3 \approx C\kappa(\sigma - \sigma_f)\). Therefore, our regular instanton looks exactly like Vilenkin’s singular solution when viewed in four dimensions.

**SINGULAR INSTANTONS AND THE BOUNDARY TERM**

As mentioned above, the instanton \((3)\) looks singular at \(\sigma = \sigma_f\) when viewed in four dimensions. It was generally believed \((3,1,2)\) that the contribution of the singularity to the Euclidean action should be given by the Hawking-Gibbons boundary term [i.e., the last term in \((9)\)] evaluated at the singular boundary. For the solution given above, this action is given by

\[
S_{GH} = 3\pi^2\kappa^{-1}C = \frac{3\pi A^2}{8G}
\]

(15)

However, this procedure is not very well motivated within the framework of the “no-boundary” proposal, where the boundary is not supposed to be there in the first place. Indeed, the naive result \((15)\) is off by a factor of 3 of the correct result given in \((7)\).

In ref. \((5)\) we suggested that, like most time-like singularities, the one in Hawking and Turok’s solution should be regularized with matter. In this way, the singular instanton can be viewed as the limit of a family of no boundary solutions where both the scalar field and the geometry are regular. When this is done, it is found that the contribution of the singularity to the Euclidean action is given by 1/3 of the Gibbons-Hawking term at the singular boundary, in agreement with the result \((7)\).

One may worry that boundary contributions in four and five dimensions do not add up to be the same. This is not the case. The difference between \((8)\) and \((15)\) is due to the fact that in four dimensions a new boundary was added at \(\tau = 0\). It can be easily checked that if the point \(\tau = 0\) is removed from the smooth manifold described by \((3)\), there is a new boundary term which accounts for the difference. However, such ad-hoc operation would be unjustifiable. If the point \(\tau = 0\) is excised, nothing links the parameter \(A\) with the asymptotic radius of the extra dimension. As a result, the parameter can be chosen as small as desired and the Euclidean action can be made arbitrarily small, which would mean that the decay of flat spacetime would be unsuppressed.

**OPEN INFLATION**

In order to find inflationary solutions, a potential must be added to the action \((9)\). In the context of Kaluza-Klein theories, it is believed that the dilaton will be stabilized by an
effective potential generated by quantum corrections. It is possible that this same potential may drive inflation in the appropriate range. Here, we shall not attempt to enter into a detailed discussion of this possibility. Instead, we note that a cosmological constant in the five dimensional theory acts as an exponential potential for \( \phi \) in the four dimensional theory. Denoting by \( \Lambda \) the cosmological constant in five dimensions, the potential in four dimensions is given by

\[
V(\phi) = \lambda e^{2\kappa \phi/3}.
\]  

Here, \( \lambda = 2\pi A \Lambda \), where \( A \) is the coordinate radius of the extra dimension. The rest of the four-dimensional action (14) remains unchanged by this addition. A potential of the form (16) drives power-law inflation [9], with power exponent \( p = 3 \) [10].

Let us now turn to the solution of the model. This is straightforward in five dimensions. With the ansatz (4) the equations of motion reduce to \( \ddot{X} + \alpha X = 2k \), and \( r \propto \dot{R} \), where \( X \equiv R^2 \), \( \alpha \equiv 2\Lambda/3 \), \( k = 1 \) for the three-sphere and the dot indicates derivative with respect to \( \tau \). The only solution which is regular everywhere is given by

\[
R = 2\alpha^{-1/2} \sin(\alpha^{1/2} \tau/2)
\]  

\[
r = 2\alpha^{-1/2} A^{-1} \cos(\alpha^{1/2} \tau/2).
\]

The parameter \( A \) is irrelevant: it is eliminated by changing the fifth coordinate to an angular coordinate \( \theta_5 = y/A \) which runs from 0 to \( 2\pi \). The solution (17) has an interesting topology. Near \( \tau = 0 \) we have \( \sigma \approx \tau \), \( \phi \approx (3\sigma^2/4\kappa A) \), and \( b \approx \sigma \). Therefore, the solution is perfectly regular from the four-dimensional point of view too. However, near \( \tau = \tilde{A}\pi/2 \) we have \( \tau - (\tilde{A}\pi/2) \propto (\sigma_f - \sigma)^{2/3} \), where \( \sigma_f = A(2/\pi)^{1/2} \Gamma^2(3/4) \) is the value of the coordinate \( \sigma \) at \( \tau = \tilde{A}\pi/2 \), and \( b^3 \approx \tilde{C}(\sigma_f - \sigma) \), \( \phi \approx -\kappa^{-1} \ln(\sigma_f - \sigma) + \text{const} \). Here \( \tilde{C} = 3\tilde{A}^2/2\kappa \). Therefore, the singularity is of the same form as the one in Hawking and Turok’s model. Notice that the four dimensional “scale factor” \( b(\sigma) \) goes to zero at \( \sigma_f \) too. The evolution of the universe after nucleation is given by the analytic continuation of our instanton through the complexification \( \psi \to (\pi/2) - i\dot{\psi} \). The resulting chart covers all of the spacetime, except for the the future light cone from the center of symmetry \( \sigma = 0 \). The open chart covering this region is obtained by taking \( \sigma \to i\tilde{\sigma} \) and \( \psi \to -i\tilde{\psi} \), and it represents the inflating open universe from where the observable universe would emerge.

**CONCLUSIONS**

We have constructed a smooth five-dimensional instanton representing the creation of an open universe. When reduced to four dimensions, the solution looks like a singular instanton
of the type introduced by Hawking and Turok. Although our construction seems to validate the use of such instantons, some words of caution must be said.

The smooth instanton presented above exists only when the value of inflaton at \(\sigma = 0\) is \(\phi = 0\). Otherwise, there will be a conical singularity at the point where the fifth dimension closes. The same would happen for a more general potential: the fifth dimension would only close smoothly for a particular value of \(\phi\) at \(\sigma = 0\). Hence, it appears that these instantons cannot produce a range of values of the density parameter \(\Omega\). This casts some doubt on the method used in [1] to find the probability distribution for \(\Omega\). On the positive side, it is still true that one can obtain an open universe without requiring a special form of the inflaton potential. We should add that there are more “conventional” ways of obtaining a range of values of \(\Omega\) in theories where one field undergoes a first order phase transition and a second field is responsible for slow roll inflation inside the nucleated bubbles [12]. In such models, one finds that a range of values of \(\Omega\) occurs inside of each nucleated bubble [13], and depending on parameters of the model, it is not unlikely for an observer to measure the density parameter in the range \((1 - \Omega)/\Omega \sim 1\) [14].

We have also found that flat space with a compact extra dimension is gravitationally metastable. It decays through nucleation of bubbles of “nothing” which eat up spacetime as they expand. The nucleation rate is unobservably small provided that the size of the extra dimension is large compared with the Planck length.

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