Weak Modular Product of Bipartite Graphs, Bicliques and Isomorphism

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Abstract

A 1978 theorem of Kozen states that two graphs on \( n \) vertices are isomorphic if and only if there is a clique of size \( n \) in the weak modular product between the two graphs. Restricting to bipartite graphs and considering complete bipartite subgraphs (bicliques) therein, we study the combinatorics of the weak modular product. We identify cases where isomorphism is tractable using this approach, which we call Isomorphism via Biclique Enumeration (IvBE). We find that IvBE is polynomial for bipartite \( 2K_2 \)-free graphs and quasi-polynomial for families of bipartite graphs, where the largest induced matching and largest induced crown graph grows slowly in \( n \), that is, \( O(\text{poly log } n) \). Furthermore, as expected a straightforward corollary of Kozen’s theorem and Lovász’s sandwich theorem is if the weak modular product between two graphs is perfect, then checking if the graphs are isomorphic is polynomial in \( n \). However, we show that for balanced, bipartite graphs this is only true in a few trivial cases. In doing so we define a new graph product on bipartite graphs, the very weak modular product. The results pertaining to bicliques in bipartite graphs proved here may be of independent interest.

1 Introduction

Graph products have been extensively studied and are of vast theoretical and practical interest, see, e.g., Hammack, Imrich and Klavžar [HIK11]. A common problem is to determine how graph invariants such as the independence number and clique number behave under the action of a particular graph product. For instance, a famous result of Lovász [Lov79] states that for graphs \( G \) and \( H \), \( \vartheta(G \boxtimes H) = \vartheta(G)\vartheta(H) \), where \( \vartheta(\cdot) \) denotes the Lovász number and \( \boxtimes \) is the strong graph product. There are three graph products that have received most attention in the literature: the aforementioned strong

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graph product, the direct (or tensor) product and the Cartesian product. These three graph products are the most studied as they satisfy the following: they are associative and projections onto the factors are weak homomorphisms. Loosely speaking, the second property means that the adjacency structure of the product graph allows one to infer the adjacency structure of the factor graphs.

One can consider graph products that do not satisfy the second property. Such a product is the weak modular product, whose adjacency structure also includes information about non-adjacency in the factor graphs. Interestingly, as originally proved by Kozen [Koz78], a clique of a certain size exists in the product graph if and only if the factors are isomorphic. Since the decision version of finding the clique number of a general graph is NP-complete, this result has largely been ignored in the literature with reference to graph isomorphism.

The graph isomorphism problem (GI) has been studied extensively for decades, but its complexity status remains unknown. Clearly, GI ∈ NP as one can check easily if a given candidate isomorphism preserves all adjacencies and non-adjacencies between the two graphs at hand. However, it is unlikely that GI is NP-complete as this would imply collapse of the polynomial hierarchy [GMW91]. The question of whether GI ∈ P remains open. There has been a considerable research effort to find a polynomial-time algorithm for GI, culminating recently in the recent quasi-polynomial algorithm by Babai [Bab15]. However, there are many classes of graph for which GI admits a polynomial-time algorithm, for instance graphs with a forbidden minor [Pon91, Gro10], including planar graphs and graphs of bounded genus. In practice, the approach of McKay and Piperno [MP14] works efficiently on almost all graphs and so efficiently solving GI ‘in the wild’ is all but solved.

Many of the recent advances in GI, including Babai’s recent breakthrough [Bab15] and the nauty/ traces programs of McKay and Piperno [MP14] use a group theoretic approach. In this paper we consider a combinatorial approach, which was the primary method for GI in the earlier days of its study, but also make use of contemporary ideas from outside this tradition. The combinatorial construction we consider is the weak modular product, mentioned earlier. This construction has been used in the pattern recognition community under the label association graph to solve graph matching problems [PSZ99, CFSV04]. Indeed, Pelillo [Pel99] uses a heuristic inspired by theoretical biology to find cliques in the weak modular product as an approach to inexact graph matching, a problem that one can interpret as an approximation to isomorphism. He provides computational evidence that this technique is tractable for this problem in certain regimes.

Here we examine the combinatorics of the weak modular product of bipartite graphs. Bipartite graphs are of particular interest, since deciding if two bipartite graphs are isomorphic is GI-complete [UTN05]. We study cliques in the weak modular product analytically, considering this as an approach for GI. We find that in the bipartite case, bicliques play a similar role to cliques in the weak modular product. Recall that a biclique is a fully connected bipartite subgraph. Finding bicliques in bipartite graphs
has garnered much interest in the bioinformatics community, since this can be applied to tasks such as identifying common gene-set associations [CL07] and integrating diverse functional genomics data [BJP+09]. In this work, we use results on finding bicliques in bipartite graphs commonly used by bioinformaticians.

In particular, we propose an algorithm, IvBE (Algorithm 1) for isomorphism of bipartite graphs based on Kozen’s theorem and counting bicliques. We show that this algorithm runs in polynomial time for bipartite $2K_2$-free graphs (Proposition 12) and in quasi-polynomial time for families of bipartite graphs where the largest induced matching and the largest induced crown graph grow polylogarithmically in the number of vertices (Proposition 11). Bipartite $2K_2$-free graphs are also known under a different name, difference graphs [HPS90]. A graph is a difference graph if every vertex $u_i$ can be assigned a real number $a_i$ and there exists a positive real number $T$ such that $i. |a_i| < T$ for all $i$ and $ii. u_i \sim u_j$ if and only if $|a_i - a_j| \geq T$. For this class of graphs, GI is already known to be polynomial, as it has bounded cliquewidth [LR04]. By a theorem of Courcelle, Makowsky and Rotics [CMR00], this implies existence of a linear-time algorithm for GI on difference graphs.

A direct corollary of the Lovász sandwich theorem gives us that the Lovász number of a perfect graph is the same as its clique number [Lov79]. Since the Lovász number can be computed in polynomial time, if the weak modular product of two graphs is perfect, testing if they are isomorphic is polynomial. To this end, we define a new graph product on bipartite graphs, the very weak modular product. We show that the very weak modular product of balanced bipartite graphs is perfect only in three trivial cases (Theorem 2) and as a corollary, the weak modular product is perfect only in these cases (Corollary 4). Recall that a bipartite graph is balanced when the size of its bipartition classes is equal. It is clear isomorphism of balanced bipartite graphs is equivalent to the general case by adding zero-degree vertices.

The paper is structured as follows. In Section 2 we formally define the weak modular product and re-prove Kozen’s theorem using modern notation. Section 3 contains some results on bipartite graphs that will be used throughout the paper. In Section 4 we prove results about bicliques in bipartite graphs which are needed for the main results of the paper. Section 5 presents the main results of the paper: the IvBE algorithm and the full characterisation of balanced bipartite graphs whose weak modular product is perfect. In Section 6 we find the number of bicliques for certain classes of bipartite graphs, including random bipartite graphs and the extremal cases.

## 2 The Weak Modular Product

A graph $G = (V(G), E(G))$ consists of a set $V(G)$ of $n$ vertices and a set of edges $E(G) \subseteq \{u_i, u_j \in V(G), u_i \neq u_j\}$. We consider only finite, undirected graphs. We write $u_i \sim u_j$ to denote that vertices $u_i$ and $u_j$ are adjacent.
We say the graph with vertex set $V(G) = V(G')$ and edge set $E(G) = E(G')$ if and only if $u_i \sim u_j$ in $G$ and $u_i \neq u_j$. The union of graphs $G$ and $H$ is the graph $G \cup H$ with vertex set $V(G \cup H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$. The disjoint union of graphs $G$ and $H$ is the graph $G \uplus H$ with vertex set $V(G \uplus H) = V(G) \uplus V(H)$ and edge set $E(G \uplus H) = E(G) \uplus E(H)$. For a graph $G$, we denote the disjoint union of $k$ copies of $G$ by $kG$.

A graph $G'$ is a subgraph of another graph $G$, $G' \subseteq G$, if and only if $V(G') \subseteq V(G)$, and $E(G') \subseteq E(G)$ and for all $\{u_i, u_j\} \in E(G')$, $u_i, u_j \in V(G')$. (2)

Suppose we have a subset of the vertices $U \subseteq V(G)$. An induced subgraph of $G$, $G[U]$, is the graph with vertex set $V(G[U]) = U$ and edge set

$$\{\{u_i, u_j\} \mid u_i, u_j \in U, \{u_i, u_j\} \in E(G)\}. \quad (3)$$

We say $G[U]$ is induced by $U \subseteq V(G)$.

We now define two closely related graph products. The tensor product of graphs $G$ and $H$, denoted by $G \otimes H$, has vertex set $V(G) \times V(H)$ and an edge $\{(u_i, v_j), (u_k, v_l)\}$ if and only if $\{u_i, u_k\} \in E(G)$ and $\{v_j, v_l\} \in E(H)$.

The weak modular product (see, e.g., Hammack, Imrich and Klavžar [HIK11]) of graphs $G$ and $H$, denoted by $G \nabla H$, has vertex set $V(G \nabla H) = V(G) \times V(H)$ and an edge $\{(u_i, v_j), (u_k, v_l)\}$ if and only if

1. either $\{u_i, u_k\} \in E(G)$ and $\{v_j, v_l\} \in E(H)$;
2. or $\{u_i, u_k\} \in E(\overline{G})$ and $\{v_j, v_l\} \in E(\overline{H})$.

The next statement is a direct consequence of the definitions of the weak modular product and the tensor product.

**Lemma 1** For graphs $G$ and $H$,

$$G \nabla H = G \otimes H \cup \overline{G} \otimes \overline{H}. \quad (4)$$

**Lemma 2** For graphs $G$ and $H$ on $n$ vertices, we have

$$A(G \nabla H) = A(G \otimes H) + A(\overline{G} \otimes \overline{H}). \quad (5)$$

**Proof.** Consider the adjacency matrix of $G \nabla H$, which by Eq. (1) and the definition of
the weak modular product has entries given by

\[ [A(G \nabla H)]_{(i-1)n+j,(k-1)n+l} = \begin{cases} 
1, & \text{either } \{u_i, u_k\} \in E(G) \text{ and } \{v_j, v_l\} \in E(H); \\
& \text{or } \{u_i, u_k\} \in E(\overline{G}) \text{ and } \{v_j, v_l\} \in E(\overline{H}); \\
0, & \text{otherwise}
\end{cases} \]  

(6)

for all \(1 \leq i, j, k, l \leq n\). For the adjacency matrices of the tensor product, we have

\[ [A(G \otimes H)]_{(i-1)n+j,(k-1)n+l} = \begin{cases} 
1, & \{u_i, u_k\} \in E(G) \text{ and } \{v_j, v_l\} \in E(H); \\
0, & \text{otherwise}
\end{cases} \]  

(7)

and

\[ [A(\overline{G} \otimes \overline{H})]_{(i-1)n+j,(k-1)n+l} = \begin{cases} 
1, & \{u_i, u_k\} \in E(\overline{G}) \text{ and } \{v_j, v_l\} \in E(\overline{H}); \\
0, & \text{otherwise},
\end{cases} \]  

(8)

for all \(1 \leq i, j, k, l \leq n\). Since there cannot be both an edge and a non-edge simultaneously between two vertices, taking the sum of \(A(G \otimes H)\) and \(A(\overline{G} \otimes \overline{H})\) gives Eq. 6, the desired result. □

Given graphs \(G\) and \(H\), we say that \(G\) and \(H\) are isomorphic, \(G \cong H\), whenever there is a bijection \(f : V(G) \rightarrow V(H)\) such that \(u_i \sim u_j\) if and only if \(f(u_i) \sim f(u_j)\) for every \(u_i, u_j \in V(G)\). A clique is a subset of the vertices of a graph such that every two distinct vertices in the clique are adjacent. The clique number of a graph \(G\), \(\omega(G)\), is the cardinality of its largest clique.

**Proposition 1** (Kozen [Koz78]). Let \(G\) and \(H\) be graphs on \(n\) vertices. Then \(\omega(G \nabla H) \leq n\). Moreover, \(\omega(G \nabla H) = n\) if and only if \(G \cong H\).

**Proof.** To see that there is no clique in \(G \nabla H\) larger than \(n\) consider the following. First lay the vertices of \(G \nabla H\) in an \(n \times n\) grid so that the vertex \((u_i, v_j)\) is in the same row as \((u_k, v_l)\) if \(i = k\), and in the same column if \(j = l\). Then by the definition of the weak modular product there can be no edges between vertices in the same row or in the same column. The vertices of an \(n\)-clique will occupy positions on the grid such that no two vertices are in the same row or column. This restriction means that no larger clique can exist, as there is no position in the grid where one can place a new vertex such that it does not share a row or column with any of the vertices already in the clique.

Now suppose there is an \(n\)-clique in \(G \nabla H\). The vertices \((u_i, v_j)\) in the clique represent the bijection \(u_i \mapsto v_j\) for all \(u_i \in V(G), v_j \in V(H)\), which we denote \(\sigma\). We can see that \(\sigma\) is an isomorphism because for all \(u_i, u_k \in V(G), \sigma(u_i) \sim \sigma(u_k)\) if and only if \(u_i \sim u_k\), from the definition of the weak modular product. For the converse, suppose that \(G \cong H\), with \(\sigma : V(G) \rightarrow V(H)\) an isomorphism. Then from the definition of the weak modular
product, we will have the collection of edges
\[
\{(u_i, \sigma(u_i)), (u_k, \sigma(u_k))\} \mid u_i, u_k \in V(G), u_k \neq u_i \subseteq E(G \nabla H).
\] (9)

This collection of edges induces an $n$-clique in $G \nabla H$, so an $n$-clique exists if and only if $G \cong H$. \hfill \Box

We will find the following well-known result about cliques in tensor products useful, and include a proof for completeness. We let $[k] = \{1, 2, \ldots, k\}$ for any $k \in \mathbb{N}$.

**Lemma 3** (Hammack, Imrich and Klavžar [HIK11, Exercise 26.1]). Let $G$ and $H$ be graphs. Then
\[
\omega(G \otimes H) = \min\{\omega(G), \omega(H)\}.
\] (10)

**Proof.** First, from the definition of the tensor product, for any clique in $G \otimes H$, the corresponding vertices in the factors form a clique, so we have $\omega(G \otimes H) \leq \omega(G)$ and $\omega(G \otimes H) \leq \omega(H)$. Now suppose without loss of generality $\omega(G) \leq \omega(H)$. Let $K_G \subseteq V(G)$ induce a maximum clique in $G$. Now, since $\omega(G) \leq \omega(H)$ by assumption there exists a subset of vertices $K_H \subseteq V(H)$ such that $|K_G| = |K_H|$. Then, by the definition of the tensor product, the vertices $K_G \times K_H \subseteq V(G \otimes H)$ induce a clique on $G \otimes H$. Thus we have $\omega(G \otimes H) \geq \omega(G)$ when $\omega(G) \leq \omega(H)$. A similar argument gives $\omega(G \otimes H) \geq \omega(H)$ when $\omega(H) \leq \omega(G)$ and the result follows. \hfill \Box

An independent set in a graph is a subset of the vertices such that no two vertices in the subset are adjacent. The independence number of a graph $G$, $\alpha(G)$, is the cardinality of its largest independent set. Since a clique in a graph $G$ corresponds to in independent set in $\overline{G}$, we have the following corollary.

**Corollary 1** For graphs $G$ and $H$, we have
\[
\omega(\overline{G} \otimes \overline{H}) = \min\{\alpha(G), \alpha(H)\}. \quad (11)
\]

## 3 Bipartite Graphs

A graph $G$ on $n$ vertices is said to be bipartite if $V(G) = V_0(G) \cup V_1(G)$ such that if $u_i \sim u_j$ then $u_i \in V_0(G)$ and $u_j \in V_1(G)$ or $u_i \in V_1(G)$ and $u_j \in V_0(G)$, for every $u_i, u_j \in V(G)$. The sets $V_0(G)$ and $V_1(G)$ are said to be the bipartition classes of $G$. We can also use a graph invariant to define bipartite graphs. The chromatic number of a graph $\chi(G)$ is the minimum number of colours for which every pair of adjacent vertices has a different colour. We then say $G$ is bipartite if and only if $\chi(G) = 2$.

The following well-known characterisation is useful to describe the structure of a bipartite graph. We denote by $M^T$ the transpose of a matrix $M$ and write $M^{(m,n)}$ to specify that $M$ has $m$ rows and $n$ columns, when this is not clear from the context. Let
Let \( G \) be a graph on \( n + m \) vertices. Then, \( G \) is bipartite if and only if there is an ordering of \( V(G) \) such that

\[
A(G) = \begin{bmatrix}
0^{(m \times m)} & A_G^{(m \times n)} \\
A_G^{(n \times m)} & 0^{(n \times n)}
\end{bmatrix},
\]

where \( A_G \) is an \( m \times n \) \( \{0, 1\} \)-matrix and \( 0^{(m \times m)} \) is the \( m \times m \) all zeros matrix. We call the matrix \( A_G \) the biadjacency matrix of a bipartite graph \( G \).

### 3.1 Tensor Products of Bipartite Graphs

We will need to make use of some results concerning tensor products of bipartite graphs. It will be illustrative to state the following established lemma.

**Lemma 4** (Hammack, Imrich and Klavžar [HIK11, Section 5.3]). Let \( G \) and \( H \) be bipartite graphs. Then, \( G \otimes H \) is bipartite. Moreover, \( G \otimes H \) is bipartite when only one of \( G \) or \( H \) is bipartite.

**Proposition 2** (Weichsel [Wei62]). Let \( G \) and \( H \) be connected, bipartite graphs. If at least one of \( G \) and \( H \) has an odd cycle, then \( G \otimes H \) is connected. If both \( G \) and \( H \) are bipartite, \( G \otimes H \) has exactly two connected components.

**Corollary 2** (Hammack, Imrich and Klavžar [HIK11, Corollary 5.10]). A tensor product of connected graphs is connected if and only if at most one of the factors is bipartite.

The two connected components of \( G \otimes H \) are the bipartite graphs \( B_{0,0,1,1} \) and \( B_{0,1,0,1} \), with vertex-sets \( V_0(G) \times V_0(H) \cup V_1(G) \times V_1(H) \) and \( V_0(G) \times V_1(H) \cup V_1(G) \times V_0(H) \), respectively. There is an ordering of their vertices such that they have the adjacency matrices

\[
A(B_{0,0,1,1}) = \begin{bmatrix}
A_G \otimes A_H \\
A_G^T \otimes A_H
\end{bmatrix} \quad \text{and} \quad A(B_{0,1,0,1}) = \begin{bmatrix}
A_G \otimes A_H^T \\
A_G^T \otimes A_H
\end{bmatrix}.
\]

A natural question to ask is under which conditions \( B_{0,0,1,1} \cong B_{0,1,0,1} \). Such a question was formulated by Jha, Klavžar and Zmazek [JKZ97] and answered by Hammack [Ham09]. An automorphism of a graph \( G \) is an isomorphism from \( V(G) \) into \( V(G) \). A bipartite graph is said to be symmetric if there is an automorphism that swaps the bipartition classes. Hammack showed that for \( G \) and \( H \) connected bipartite graphs, the two components of \( G \otimes H \) are isomorphic if and only if at least one of \( G \) or \( H \) is symmetric [Ham09].

### 3.2 Tensor Products of Cobipartite Graphs

Here we collect more statements that will prove to be useful later in the text, beginning with the definition of a cobipartite graph.

**Definition 1.** Cobipartite graphs A graph is cobipartite if it is the complement of a bipartite graph.
Observation 1. For a bipartite graph $G$, where $A(G)$ can be written as in Eq. (12), its (cobipartite) complement $\overline{G}$ can be written as

$$A(\overline{G}) = \begin{bmatrix} J^{(m \times m)} - I^{(m \times m)} & J^{(m \times n)} - A_G^{(m \times n)} \\ J^{(n \times m)} - A_G^{(n \times m)} & J^{n \times n} - I^{(n \times n)} \end{bmatrix},$$

(14)

where $J^{(m \times n)}$ is the $m \times n$ all-ones matrix and $I^{(n \times n)}$ is the $n \times n$ identity matrix.

Proof. Use Eq. (12) and the fact that $A(\overline{G}) = J - I - A(G)$.

Also, note that $J^{(n \times n)} - I^{(n \times n)} = A(K_n)$, where $K_n$ is the complete graph on $n$ vertices.

We shall now evaluate the matrix $A(\overline{G}) \otimes A(\overline{H})$, for bipartite graphs $G$ and $H$. Let the adjacency matrices $A(G)$ and $A(H)$ be as in Eq. (11), with $A_G$ being an $m_G \times n_G$ matrix and $A_H$ being an $m_H \times n_H$ matrix. Using Eq. (13), we can write the adjacency matrix of $G \otimes H$ as

$$A(G \otimes H) = \begin{bmatrix} A(B_{0,0,1,1}) & A_G \otimes A_H \\ A(B_{0,1,1,0}) & A_G^T \otimes A_H^T \end{bmatrix}.$$

(15)

Note that even if $G$ and $H$ are not connected, we can still write the adjacency matrix of $G \otimes H$ in this way. The adjacency matrix of $\overline{G} \otimes \overline{H}$, using Observation 1 and the same vertex labelling as in Eq. (15), is

$$A(\overline{G}) \otimes A(\overline{H}) = \begin{bmatrix} A(K_{n_G}) \otimes A(K_{m_H}) & (J - A_G) \otimes (J - A_H) & A(K_{n_G}) \otimes (J - A_H) & (J - A_G) \otimes A(K_{m_H}) \\ (J - A_G^T) \otimes (J - A_H^T) & A(K_{n_G}) \otimes A(K_{n_H}) & (J - A_G^T) \otimes A(K_{n_H}) & A(K_{m_H}) \otimes (J - A_H^T) \\ A(K_{m_H}) \otimes (J - A_H) & (J - A_G^T) \otimes A(K_{m_H}) & A(K_{m_H}) \otimes A(K_{m_H}) & (J - A_G^T) \otimes (J - A_H) \\ (J - A_G) \otimes A(K_{m_H}) & A(K_{n_G}) \otimes (J - A_H^T) & (J - A_G) \otimes (J - A_H^T) & A(K_{n_G}) \otimes A(K_{m_H}) \end{bmatrix}.$$

(16)

It is useful to note that the matrices $A(G \otimes H)$ and $A(\overline{G} \otimes H)$ (under the vertex labelling employed in Eq. (15)) have the block structure

$$\begin{bmatrix} m_G m_H \times m_G m_H & m_G m_H \times n_G n_H & m_G m_H \times m_G n_H & m_G m_H \times n_G m_H \\ n_G n_H \times m_G m_H & n_G n_H \times n_G n_H & n_G n_H \times m_G n_H & n_G n_H \times n_G m_H \\ m_G m_H \times n_G n_H & n_G n_H \times n_G m_H & m_G n_H \times m_G n_H & m_G n_H \times n_G m_H \\ m_G m_H \times n_G m_H & n_G m_H \times n_G m_H & n_G m_H \times n_G m_H & n_G m_H \times n_G m_H \end{bmatrix},$$

(17)

where the entries of the array in Eq. (17) are the sizes of the respective blocks in Eqs. (15) and (16).

We note that the diagonal blocks of $A(\overline{G} \otimes H)$ are the adjacency matrices of the graphs $K_m \otimes K_n$ for different values of $m$ and $n$. These graphs are in fact the complement of
the \( m \times n \) rook’s graph \( R(m,n) \), the graph whose vertices are the tiles on an \( m \times n \) chessboard and whose edges represent the legal moves of a rook.

**Lemma 5** (Hon et. al. [HKL+13]) Let \( m, n \in \mathbb{N} \). Then

\[
K_m \otimes K_n \cong R(m,n).
\]  

We therefore call the graph \( K_m \otimes K_n \) a \( m \times n \) co-rook.

**Lemma 6** Let \( m, n \in \mathbb{N} \), with \( r_{\min} = \min\{m,n\} \) and \( r_{\max} = \max\{m,n\} \). Then, \( \omega(R(m,n)) = r_{\min} \). Furthermore, there are \( r_{\max}/(r_{\max} - r_{\min})! \) \( r_{\min} \)-cliques in \( R(m,n) \).

**Proof.** The first statement is a direct consequence of Lemma 3 and Lemma 5. For the second, from the definition of a co-rook, we count the non-attacking positions of \( r_{\min} \) rooks on an \( m \times n \) chessboard to count the number of \( r_{\min} \)-cliques in \( R(m,n) \). Without loss of generality, assume \( m \leq n \). Start with an empty board and move down the rows from the top. In the first row, there are \( n \) tiles where one can place a rook. In the second row, there are now \( n - 1 \) allowed tiles, so as to avoid the column the first rook has been placed in. Continuing this procedure, by the \( m^{th} \) row, there are \( (n - m + 1) \) allowed positions for a rook to be placed in. Thus in total we have \( n!/(n-m)! \) possible non-attacking configurations in which to place the \( m \) rooks, each of which induces an \( m \)-clique in \( R(m,n) \). By a similar argument for the case in which \( m \geq n \), we get the desired result. \( \square \)

### 4 Bicliques in Bipartite Graphs

Suppose we have a bipartite graph \( G \) with bipartition classes \( V_0(G) \) and \( V_1(G) \) with cardinalities \( m \) and \( n \) respectively. If \( E(G) \) is such that every vertex in \( V_0(G) \) is adjacent to every vertex in \( V_1(G) \) then we call \( G \) a **complete bipartite graph**, \( K_{m,n} \). A \( (m',n') \)-biclique in a graph is a complete bipartite subgraph, \( K_{m',n'} \), where \( m', n' \in \mathbb{N} \). We say a biclique has size \( k \) if it contains \( k \) vertices. The problem of deciding whether or not there exists a biclique of certain size in a bipartite graph has been discussed in Garey and Johnson [GJ79, GT24], where they comment that this is solvable in polynomial time. We prove this formally in Proposition 3, but first we require some additional notions. A biclique is **maximal** if it is not contained in any other biclique. If a graph \( G \) has \( k \) maximal bicliques, we say that \( \beta(G) = k \), where \( \beta(G) \) is the **maximal biclique number** of \( G \).

Bicliques in bipartite graphs can be interpreted as an analogue to cliques in general graphs, in that the definitions are the same, modulo the constraint in the bipartite case that no edges exist within a given bipartition class. In a similar way, it will be useful to define a bipartite analogue of the graph complement. Following Giakoumakis and Vanherpe [GV97], for a bipartite graph \( G \) with bipartition classes \( V_0(G) \) and \( V_1(G) \), its
bi-complement, $\overline{G}^{\text{bip}}$ is the graph with $V(\overline{G}^{\text{bip}}) = V(G)$ and

$$E(\overline{G}^{\text{bip}}) = \{\{u_i, u_j\} \mid u_i \in V_0(G), u_j \in V_1(G), \{u_i, u_j\} \notin E(G)\}.$$  \hspace{1cm} (19)

**Observation 2.** For $G$ a bipartite graph, the vertices comprising an independent set in $G$ form a biclique in $G$ and $\overline{G}^{\text{bip}}$.

Given a graph $G$, a set of nodes $S \subseteq V(G)$ is called a *vertex cover* if for every edge $\{u_i, u_j\} \in E(G)$, either $u_i \in S$, $u_j \in S$, or both. A *minimum vertex cover* is a vertex cover of smallest possible cardinality. Furthermore, $S \subseteq V(G)$ is an independent set in $G$ if and only if $V(G) \setminus S$ is a vertex cover of $G$.

A *matching* in a graph $G$ is a set of edges $M \subseteq E(G)$ such that no edges share a common vertex. We say $M$ is a *maximum matching* if it is of maximum possible cardinality and $M$ is a *perfect matching* if every vertex in $V(G)$ belongs to an edge in $M$. Furthermore, $M$ is an *induced matching* if it occurs as an induced subgraph of $G$. König established the relationship between maximum matchings and minimum vertex covers in bipartite graphs in his famous theorem [Kön31]. Namely, in any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover. Moreover, the proof of König’s Theorem provides a way of constructing the minimum vertex cover from the maximum matching.

We are now in a position to prove existence of a polynomial-time algorithm to find bicliques of a given size in bipartite graphs.

**Proposition 3** Let $G$ be a bipartite graph and let $k \leq |V(G)|$ be a positive integer. Then there is a polynomial-time algorithm to decide whether there exists a biclique of size $k$ in $G$.

**Proof.** Since $S$ is a vertex cover, $V(G) \setminus S$ is an independent set. By Observation 2, an independent set in $G$ is a biclique in $\overline{G}^{\text{bip}}$. We also have

$$\overline{G}^{\text{bip}} = G$$  \hspace{1cm} (20)

from the definition of the bi-complement. Using the above, a biclique of size $|V(G)| - k'$ exists in $G$ if there is a vertex cover of size $k'$ in $\overline{G}^{\text{bip}}$. If we find the minimum vertex cover of $\overline{G}^{\text{bip}}$, we can check if a biclique of size $k$ or more exists in $G$ by checking if $|S| \leq k$, where $S$ is the minimum vertex cover.

By König’s Theorem, in a bipartite graph the number of edges in the maximum matching is the same as the number of vertices in the minimum vertex cover. Finding the maximum matching of a bipartite graph has a polynomial time algorithm, thus we have the desired result.

There are numerous efficient algorithms for finding the maximum bipartite matching on a bipartite graph. Let $G$ be a bipartite graph and $n := |V(G)|$ and $e := |E(G)|$. Hopcroft and Karp’s [HK71] celebrated algorithm finds the maximum matching in $G$ in
$O(\sqrt{n})$ time. For relatively sparse graphs, this is still the best bound. In the case of a dense graph, i.e. when $e$ approaches $n^2$, Mucha and Sandowski [MS04] provide an $O(n^\omega)$ algorithm, where $\omega \leq 2.3727$ is the exponent of matrix multiplication. Madry [Mad13] gave a $O(n^{10/7})$ algorithm.

**Lemma 7** Let $m_G, n_G, m_H, n_H \in \mathbb{N}$. Then,

$$K_{m_G, n_G} \otimes K_{m_H, n_H} \cong K_{m_G m_H, n_G n_H} \uplus K_{m_G n_H, n_G m_H}.$$  

**Proof.** Let $G = K_{m_G, n_G}$ and $H = K_{m_H, n_H}$. Then we have $|V_0(G)| = m_G$, $|V_1(G)| = n_G$, $|V_0(H)| = m_H$ and $|V_1(H)| = n_H$. By Proposition 2, since $G$ and $H$ are connected and bipartite, $G \otimes H$ has two connected components $B_{0,0,1,1}$ and $B_{0,1,1,0}$ with bipartition classes $(V_0(G) \times V_0(H), V_1(G) \times V_1(H))$ and $(V_0(G) \times V_1(H), V_1(G) \times V_0(H))$, respectively, so we have

$$G \otimes H = B_{0,0,1,1} \uplus B_{0,1,1,0}. \tag{22}$$

We now argue that $B_{0,0,1,1} \cong K_{m_G m_H, n_G n_H}$. An edge $\{u_i, u_j\} \in E(G)$ exists for all $u_i \in V_0(G)$, $u_j \in V_1(G)$ since $G$ is complete bipartite. Similarly, an edge $\{v_k, v_l\} \in E(H)$ exists for all $v_k \in V_0(H)$, $v_l \in V_1(H)$. By the definition of the tensor product, we then have $\{w_{(i,k)}, w_{(j,l)}\} \in E(B_{0,0,1,1})$ for all $\{(i,j,k,l) \mid u_i \in V_0(G), v_k \in V_0(H), v_k \in V_1(H), v_l \in V_1(H)\}$. This is a complete bipartite graph with bipartition classes $(V_0(G) \times V_0(H), V_1(G) \times V_1(H))$, which is isomorphic to $K_{m_G m_H, n_G n_H}$ as the first bipartition has $|V_0(G)||V_0(H)| = m_G m_H$ vertices and the second has $|V_1(G)||V_1(H)| = n_G n_H$. A similar argument shows that $B_{0,1,1,0} \cong K_{m_G n_H, n_G m_H}$. The statement then follows after substituting into Eq. (22).

What Lemma 7 means in practice is that taking the tensor product of two complete bipartite graphs with $(m_G + n_G)$ and $(m_H + n_H)$ vertices respectively yields two disconnected complete bipartite graphs with $(m_G m_H + n_G n_H)$ and $(m_G n_H + n_G m_H)$ vertices respectively. Thus, tensor products of graphs containing bicliques will certainly contain two bicliques of larger size than in the factors. We can make the following statement about bicliques in tensor products.

**Lemma 8** Let $G$ and $H$ be bipartite graphs. Then

$$\beta(G \otimes H) = 2\beta(G)\beta(H). \tag{23}$$

**Proof.** Let $K_G$ and $K_H$ be maximal bicliques in $G$ and $H$ respectively. By Lemma 7, there will be two bicliques in $G \otimes H$ arising from $K_G$ and $K_H$. Both of these bicliques are maximal by the following: Suppose we increase the size of one of the bicliques in $G \otimes H$ by adding a vertex. This implies that there is an additional vertex in both $K_G$ and $K_H$ from the definition of the tensor product, but since $K_G$ and $K_H$ are both maximal, we have a contradiction.

Now let $K$ be a biclique in $G \otimes H$. From Lemma 7, we see that $K$ originates from a biclique $K_G$ in $G$ and a biclique $K_H$ in $H$. If $K_G$ and $K_H$ are both maximal, then
from the above, $K$ is maximal. If either $K_G$ or $K_H$ are non-maximal, then we have a contradiction, as increasing the size of $K_G$ of $K_H$ will increase the size of $K$, from the definition of the tensor product. Thus, $G \otimes H$ has exactly two maximal bicliques arising from a given pair of maximal bicliques in its factors, and has no additional maximal bicliques. The result follows. \hfill \square

To prove the next proposition, we introduce the notion of the neighbourhood of a set of vertices in a graph. For a subset $S \subseteq V(G)$ of the vertices of a graph $G$, the \textit{neighbourhood} of $S$, $N_G(S)$, is the set of vertices adjacent to all vertices in $S$, i.e. $N_G(S) = \{u_i \mid \{u_j, u_i\} \in E \text{ for all } u_j \in S\}$. When the context makes it clear, we omit the subscript and say $N(S)$.

**Proposition 4** Let $G$ and $H$ be bipartite graphs with $V_0(G) = V_0(H) := V_0$ and $V_1(G) = V_1(H) := V_1$. Then,

$$\beta(G \cup H) < (\beta(G) + 1)(\beta(H) + 1).$$

**Proof.** First we note that any maximal biclique in $G$ or $H$ will be induced by a subset of vertices in $V_0$ and its neighbourhood. In a slight abuse of notation, we can say without loss of generality that $S \subseteq V_0$ \textit{induces} a given maximal biclique, where it is understood that $S \cup N(S)$ induces the maximal biclique. Now let $k := \beta(G)$ and $k' := \beta(H)$ for brevity. Suppose the sets $S_1, S_2, \ldots, S_k \subseteq V_0$ induce all of the maximal bicliques in $G$ and the sets $S'_{1}, S'_{2}, \ldots, S'_{k'} \subseteq V_0$ induce all of the maximal bicliques in $H$.

We make the following claim: Any maximal biclique in $G \cup H$ is induced on a set of vertices of the form $S_i, S'_j$ or $S_i \cup S'_j$, for some $i \in [k], j \in [k']$. Since there are $k + k' + kk'$ sets of this form, the proposition follows as $k + k' + kk' < (k + 1)(k' + 1)$.

It remains to prove the above claim. Call $K \subseteq V_0$ the set of vertices which induces some maximal biclique in $G \cup H$. Note that $K$ must be a superset of a maximal biclique in either $G$ or $H$, that is, either $K = S_i \cup X$ for some $i \in [k]$ or $K = S'_j \cup X$ for some $j \in [k']$, where $X \subseteq V_0$. Suppose without loss of generality that $K = S_i \cup X$, for some $i \in [k]$ and some $X \subseteq V_0$ such that there is no $j \in [k']$ for which $X = S'_j$, that is, $X$ does not induce a maximal biclique in $H$. Since $K$ induces a biclique in $G \cup H$, we have that $N_G(S_i) \cap N_H(X) \neq \emptyset$. Now since $X$ does not induce a maximal biclique in $H$ by supposition, there exists a subset of vertices $X' \subseteq V_0 \setminus X$ such that $N_H(X) \subseteq N_H(X \cup X')$ and we have $|N_G(S_i) \cap N_H(X \cup X')| \geq |N_G(S_i) \cap N_H(X)|$. Thus, the set of vertices $K \cup X'$ induces a biclique in $G \cup H$. This gives a contradiction if $X$ is non-empty, since by assumption $K$ induces a maximal biclique in $G \cup H$. So for $K = S_i \cup X$, if $X \neq \emptyset$, then $X = S'_j$ for some $j \in [k']$ and if $X = \emptyset$, then $K = S_i$. Using a similar argument, when $K = S'_j \cup X$, either $X = \emptyset$ or $X = S_i$ for some $i \in [k]$. Thus, the claim holds and the proposition follows. \hfill \square

We now define $\tilde{a}(G)$ as the number of maximal independent sets in a graph $G$. An independent set $S$ is maximal if there is no additional vertex $u \in V(G) \setminus S$ that can be included in $S$, such that $S \cup \{u\}$ is an independent set. To consider the number of
maximal biclique in \( \overline{G}^{\text{bip}} \), we can use the relationship between maximal independent sets in \( G \) and maximal bicliques in \( \overline{G}^{\text{bip}} \).

**Lemma 9** Let \( G \) be a bipartite graph. Then,

\[
\beta(\overline{G}^{\text{bip}}) \leq \overline{\alpha}(G) \leq \beta(\overline{G}^{\text{bip}}) + 2. \tag{25}
\]

**Proof.** Suppose the subset of vertices \( U \subseteq V(G) \) induces a maximal biclique in \( \overline{G}^{\text{bip}} \). Then, from the definition of the bi-complement, \( U \) forms a maximal independent set in \( G \), as there is no vertex \( u_i \in V(G) \setminus U \) such that \( U \cup \{u_i\} \) is a maximal biclique in \( \overline{G}^{\text{bip}} \). Thus, to every maximal biclique in \( \overline{G}^{\text{bip}} \) there corresponds a maximal independent set in \( G \). There can be up to two additional maximal independent sets in \( G \), arising when either \( V_0(G) \) or \( V_1(G) \) are not contained in maximal bicliques in \( \overline{G}^{\text{bip}} \), that is, if there are no zero-degree vertices in \( G \) in one or both of the bipartition classes. \( \square \)

## 5 Bicliques and Isomorphism

We call a bipartite graph balanced when the cardinalities of the bipartition classes are the same. It will be convenient to talk about balanced, bipartite graphs on \( 2n \) vertices, such that each bipartition class has \( n \) vertices. We can do so without loss of generality as we can simply add degree-zero vertices to the smaller bipartition class of a bipartite graph until it is balanced. To simplify the discussion, for bipartite graphs \( G \) and \( H \) with bipartition classes \( (V_0(G), V_1(G)) \) and \( (V_0(H), V_1(H)) \), respectively, we denote by \( \mathcal{U} \) the set \( (V_0(G) \times V_0(H)) \cup (V_1(G) \times V_1(H)) \) and \( \mathcal{V} \) the set \( (V_0(G) \times V_1(H)) \cup (V_1(G) \times V_0(H)) \).

Then we have the following result.

**Theorem 1** Let \( G \) and \( H \) be balanced bipartite graphs on \( 2n \) vertices, and \( \overline{R}_0 \) and \( \overline{R}_1 \) be co-rooms in \( G \nabla H \), with both being subgraphs of \( (G \nabla H)[\mathcal{U}] \) or both being subgraphs of \( (G \nabla H)[\mathcal{V}] \). Then, there is a biclique of size \( 2n \) joining an \( n \)-clique in \( \overline{R}_0 \) and an \( n \)-clique in \( \overline{R}_1 \) if and only if \( G \cong H \).

**Proof.** We consider the case when \( \overline{R}_0 \) and \( \overline{R}_1 \) are subgraphs of \( (G \nabla H)[\mathcal{U}] \). The same argument applies for the \( (G \nabla H)[\mathcal{V}] \) case, upon interchange of \( \mathcal{U} \) with \( \mathcal{V} \), and \( V_0(H) \) with \( V_1(H) \).

We have \( \overline{R}_0 \subseteq (G \nabla H)[V_0(G) \times V_0(H)] \) and \( \overline{R}_1 \subseteq (G \nabla H)[V_1(G) \times V_1(H)] \) from the discussion in Section 3.1. Two vertices \( (u_i, v_j) \) and \( (u_k, v_l) \) are adjacent in \( \overline{R}_0 \) when \( u_i \neq u_k \) and \( v_j \neq v_l \), for \( u_i, u_k \in V_0(G), v_j, v_l \in V_0(H) \). Similarly, two vertices \( (u_i, v_j) \) and \( (u_k, v_l) \) are adjacent in \( \overline{R}_1 \) when \( u_i \neq u_k \) and \( v_j \neq v_l \), for \( u_i, u_k \in V_1(G), v_j, v_l \in V_1(H) \). Let \( \sigma_0 : V_0(G) \rightarrow V_0(H) \) and \( \sigma_1 : V_1(G) \rightarrow V_1(H) \) be bijections and let \( f : v_j \mapsto j \) for all \( v_j \in V(H) \). Then, the \( n \)-cliques in \( \overline{R}_0 \), for a given \( \sigma_0 \) will be induced on the vertices

\[
\{(u_i, \sigma_0(u_i)) \mid u_i \in V_0(G)\}. \tag{26}
\]
Similarly, the \( n \)-cliques in \( \overline{R}_1 \), for a given \( \sigma_1 \) will be induced on

\[
\{(u_j, \sigma_1(u_j)) \mid u_j \in V_1(G)\}. \tag{27}
\]

So a given pair of \( n \)-cliques, one in \( \overline{R}_0 \) and one in \( \overline{R}_1 \), can be indexed by the pair of bijections \((\sigma_0, \sigma_1)\).

Now suppose there is a biclique \( K_{n,n}^{(\sigma_0,\sigma_1)} \) between the \( n \)-cliques indexed by \( \sigma_0 \) and \( \sigma_1 \) respectively, that is, we have

\[
(u_i, \sigma_0(u_i)) \sim (u_j, \sigma_1(u_j)) \quad \text{for all} \quad u_i \in V_0(G), \ u_j \in V_1(G). \tag{28}
\]

The graph \( K_{n,n}^{(\sigma_0,\sigma_1)} \cup \overline{R}_0 \cup \overline{R}_1 \) forms a \( 2n \)-clique since every vertex is connected to every other vertex. By Proposition 1, \( G \) and \( H \) are thus isomorphic.

Now suppose \( G \) and \( H \) are isomorphic with \( \sigma : V(G) \to V(H) \) being an isomorphism. Then either \( V_0(G) \to V_0(H) \) and \( V_1(G) \to V_1(H) \) under \( \sigma \), or \( V_0(G) \to V_1(H) \) and \( V_1(G) \to V_0(H) \). Without loss of generality, we can assume that we have the first instance as we can just relabel the bipartition classes for this to be true. Let \( \sigma_0 : V_0(G) \to V_0(H) \) and \( \sigma_1 : V_1(G) \to V_1(H) \) be the bijective mappings that act as \( \sigma \) does on the domains \( V_0(G) \) and \( V_1(G) \) respectively, that is, \( \sigma_0(u_i) = \sigma(u_i) \) and \( \sigma_1(u_j) = \sigma(u_j) \) for all \( u_i \in V_0(G), \ u_j \in V_1(G) \). From the definition of the weak modular product, we have \((u_i, \sigma_0(u_i)) \sim (u_j, \sigma_1(u_j))\) in \( G \nabla H \) if and only if either

- \( u_i \sim u_j \) in \( G \) and \( \sigma_0(u_i) \sim \sigma_1(u_j) \) in \( H \) for \( u_i \in V_0(G), \ u_j \in V_1(G) \);
- or \( u_i \sim u_j \) in \( G \) and \( \sigma_0(u_i) \sim \sigma_1(u_j) \) in \( H \) for \( u_i \in V_0(G), \ u_j \in V_1(G) \).

Now \( u_i \sim u_j \) and \( \sigma_0(u_i) \sim \sigma_1(u_j) \) for all \( u_i \in V_0(G), \ u_j \in V_1(G) \) as \( G \) and \( H \) are bipartite, so we have \((u_i, \sigma_0(u_i)) \sim (u_j, \sigma_1(u_j))\) for all \( u_i \in V_0(G), \ u_j \in V_1(G) \). The graph induced by this collection of edges has two disjoint vertex subsets, \( \{(u_i, \sigma_0(u_i))\} \) and \( \{(u_j, \sigma_1(u_j))\} \), each containing \( n \) vertices, where each vertex in a given subset is connected to every vertex in the other subset. By definition, this is a biclique of size \( 2n \), which we call \( K_{n,n}^{(\sigma_0,\sigma_1)} \). From Eqs. (26) and (27), we can see that the bipartition classes of \( K_{n,n}^{(\sigma_0,\sigma_1)} \) induce \( n \)-cliques in \( \overline{R}_0 \) and \( \overline{R}_1 \) respectively. \( \square \)

**Corollary 3** Let \( G \) and \( H \) be balanced, bipartite graphs on \( 2n \) vertices. An edge in \( G \nabla H \) contributes to a \( 2n \)-clique in \( G \nabla H \) if and only if it is in \((G \nabla H)[U] \cup (G \nabla H)[V]\).

**Proof.** We can see from the proof of Theorem 1 that any \( 2n \)-clique will either be in the graph \((G \nabla H)[U]\) or \((G \nabla H)[V]\). Thus any edge in a \( 2n \)-clique will be in the disjoint union of these two graphs. \( \square \)

Theorem 1 and Corollary 3 motivate the definition of a new graph product, defined on balanced bipartite graphs, where a clique of a certain size is a certificate for isomorphism for the two graphs.
**Definition 2.** *(Very Weak Modular Product).* Let $G$ and $H$ be balanced, bipartite graphs on $2n$ vertices, with bipartition classes $(V_0(G), V_1(G))$ and $(V_0(H), V_1(H))$, respectively. Then the *very weak modular product*, $G \triangledown H$, is the graph with vertex set $V(G \triangledown H) = V(G) \times V(H)$ and edges $\{(u_i, v_j), (u_k, v_l)\} \in E(G \triangledown H)$ if and only if, for all $1 \leq i, j, k, l \leq n$

1. either $\{u_i, u_k\} \in E(G)$ and $\{v_j, v_l\} \in E(H)$;
2. or $\{u_i, u_k\} \in E(\overline{G})$ and $\{v_j, v_l\} \in E(\overline{H})$;
3. and either $(u_i, v_j), (u_k, v_l) \in \mathcal{U}$ or $(u_i, v_j), (u_k, v_l) \in \mathcal{V}$,

where as before, $\mathcal{U} = V_0(G) \times V_0(H) \cup V_1(G) \times V_1(H)$ and $\mathcal{V} = V_0(G) \times V_1(H) \cup V_1(G) \times V_0(H)$.

We now consider the adjacency matrix of $G \triangledown H$. First we fix the labelling of the vertices of $G$ and $H$ such that the adjacency matrix $A(G \triangledown H)$ uses the same vertex labelling as Eqs. (15) and (16). Thus $A(G \triangledown H)$ is a block diagonal matrix, with blocks $A^{(\mathcal{U})}_{G \triangledown H}$ and $A^{(\mathcal{V})}_{G \triangledown H}$ (in that order), given by

$$A^{(\mathcal{U})}_{G \triangledown H} = \begin{bmatrix} A(K_n) \otimes A(K_n) & A_G \otimes A_H + (J - A_G) \otimes (J - A_H) \\ A_G^T \otimes A_H^T + (J - A_G^T) \otimes (J - A_H^T) & A(K_n) \otimes A(K_n) \end{bmatrix}$$

$$A^{(\mathcal{V})}_{G \triangledown H} = \begin{bmatrix} A(K_n) \otimes A(K_n) & A_G^T \otimes A_H + (J - A_G^T) \otimes (J - A_H^T) \\ A_G \otimes A_H^T + (J - A_G) \otimes (J - A_H) & A(K_n) \otimes A(K_n) \end{bmatrix}$$

These blocks represent the graphs $(G \triangledown H)[\mathcal{U}]$ and $(G \triangledown H)[\mathcal{V}]$, so we have the following observations.

**Observation 3.** For $G$ and $H$ balanced bipartite graphs on $2n$ vertices,

$$G \triangledown H \cong (G \triangledown H)[\mathcal{U}] \cup (G \triangledown H)[\mathcal{V}].$$

**Observation 4.** The graphs $(G \triangledown H)[\mathcal{U}]$ and $(G \triangledown H)[\mathcal{V}]$ are connected, that is, $G \triangledown H$ has exactly two connected components.

The previous observation can be deduced from Lemma 2 and the adjacency matrix of $G \triangledown H$.

**Observation 5.** Let $\overline{R}_1$ and $\overline{R}_2$ be the co-rooks in a connected component of $G \triangledown H$. There is a $2n$-clique between an $n$-clique in $\overline{R}_1$ and an $n$-clique in $\overline{R}_2$ if and only if $G \cong H$.

Observation 5 directly follows from Theorem 1. Each connected component of the graph $G \triangledown H$ is the disjoint union of two co-rooks $\overline{R}(n^2, n^2)$, with the edges of a bipartite graph joining the vertices of the two co-rooks. Note that a biclique of size $2n$ can exist.
Figure 1: Illustration of the case in which the very weak modular product between two balanced, bipartite, non-isomorphic graphs on $2n = 6$ vertices has bicliques of size $2n$ between pairs of co-rooks, but not between cliques in these co-rooks. The vertices comprising the bicliques are circled.

Figure 2: Two balanced, bipartite graphs on $n = 2$ vertices for which the very weak modular product between them does not contain $K_{n,n}$ as a subgraph.

between the co-rooks of two balanced, bipartite graphs $G$ and $H$, for which $G \nRightarrow H$, just not between $n$-cliques in the co-rooks. An example of this is shown in Figure 1.

On the other hand, one might be lead to think that there might always be a biclique of size $2n$ in $G \nlor H$.

**Proposition 5** Let $B_{n,n}$ be the set of balanced, bipartite graphs on $2n$ vertices. Then there exist $G, H \in B_{n,n}$ such that neither of the connected components of $G \nlor H$ contains $K_{n,n}$ as a subgraph.

An example of such $G$ and $H$ is shown Figure 2.
5.1 Perfect Graphs and the Very Weak Modular Product

A graph is \textit{perfect} if the chromatic number of every induced subgraph equals the clique number of the subgraph. Lovász’s famous ‘sandwich theorem’ states that for any graph \( G \), \( \omega(G) \leq \vartheta(G) \leq \chi(G) \), where \( \vartheta(G) \) is the Lovász number of the graph \( G \), which can be computed in polynomial time [Lov79]. Since for a perfect graph \( G \), \( \omega(G) = \chi(G) \), by the sandwich theorem we have that \( \omega(G) = \vartheta(G) \). Now suppose for certain graphs \( G \) and \( H \) on \( n \) vertices, their weak modular product \( G \nabla H \) was perfect. Then, deciding if \( G \cong H \) would be easy, by virtue of the Lovász number giving us the maximum clique in \( G \nabla H \) in time polynomial in \( n \). In the same way, for balanced bipartite graphs \( G \) and \( H \) whose very weak modular product \( G \nabla H \) were perfect, deciding if \( G \cong H \) would be easy. In Theorem 2 below we characterise the cases in which this is true, which turn out to be trivial.

We first need the following. The co-rook \( \overline{R}(m, n) \) is in fact perfect [HKL⁺13], as is any bipartite graph, thus \( G \nabla H \) is the union of perfect graphs. Cameron, Edmonds and Lovász [CEL86] characterised the conditions under which the union of perfect graphs is perfect, which we state without proof.

\textbf{Proposition 6} (Cameron, Edmonds and Lovász [CEL86, Theorem 1’]) Let \( G_1 \) and \( G_2 \) be perfect graphs and \( G := G_1 \cup G_2 \) be their union with \( V(G_1) = V(G_2) = V(G) \). Suppose that for any \( u_i, u_j, u_k \in V(G) \), \( \{u_i, u_j\} \in E(G_1) \) and \( \{u_j, u_k\} \in E(G_2) \) implies that \( \{u_i, u_k\} \in E(G) \). Then, \( G \) is perfect.

\textbf{Theorem 2} Let \( G \) and \( H \) be balanced, bipartite graphs on \( 2n \) vertices, with \( n \geq 2 \). Then, \( G \nabla H \) is perfect if and only if either

1. \( G \) and \( H \) are both the complete bipartite graph \( K_{n,n} \);
2. \( G \) and \( H \) are both the empty graph \( \overline{K}_{2n} \);
3. One of \( G \), \( H \) is the complete bipartite graph \( K_{n,n} \) and the other the empty graph \( \overline{K}_{2n} \).

\textit{Proof.} Each connected component of the graph \( G \nabla H \) is comprised of the disjoint union of two co-rooks, \( \overline{R}(n^2, n^2) \) with the edges of a bipartite graph \( B_{n^2,n^2} \), adjoining the vertices of the co-rooks. We show that a connected component of \( G \nabla H \) is perfect if and only if \( B_{n^2,n^2} \cong K_{n^2,n^2} \) or \( B_{n^2,n^2} \cong \overline{K}_{2n^2} \). Then we show that \( B_{n^2,n^2} \cong K_{n^2,n^2} \) if and only if \( G, H \cong K_{2n} \) and \( B_{n^2,n^2} \cong \overline{K}_{2n^2} \) if and only if one of \( G \), \( H \) is isomorphic to \( K_{n,n} \) and the other \( \overline{K}_{2n} \). Taken together, the theorem is thus proved, as a graph whose connected components are perfect is perfect.

Take one of the connected components of \( G \nabla H \) and call the first co-rook \( \overline{R}_1 \) and the second \( \overline{R}_2 \). The bipartition classes of \( B_{n^2,n^2} \) are the vertex sets of \( \overline{R}_1 \) and \( \overline{R}_2 \). Suppose we have a vertex \( (u_i, v_j) \) in \( \overline{R}_1 \), adjacent to a vertex \( (u_k, v_l) \) in \( \overline{R}_2 \), that is, \( \{(u_i, v_j), (u_k, v_l)\} \) is an edge in \( B_{n^2,n^2} \). Now \( (u_k, v_l) \sim (u_{k'}, v_{l'}) \) in \( \overline{R}_2 \), for all \( k' \neq k \) and \( l' \neq l \) from the definition of a co-rook. By Proposition 6, for \( G \nabla H \) to be perfect, we
must have \((u_i, v_j) \sim (u_{i'}, v_{j'})\) in \(B_{n^2,n^2}\), for all \(1 \leq k', l' \leq n\) such that \(k' \neq k\) and \(l' \neq l\), since the only edges connecting vertices in \(\overline{R}_1\) and \(\overline{R}_2\) are in \(B_{n^2,n^2}\). Now for each of the \((u_{k'}, v_{l'})\), again by Proposition 6 \((u_i, v_j)\) will have to be adjacent to all of the neighbours of \((u_{k'}, v_{l'})\) to ensure perfection of \(G \vee H\). Since \(\overline{R}_2\) is connected by Lemma 2, continuing recursively we see that if \((u_i, v_j)\) in \(\overline{R}_1\) is adjacent to a single vertex \((u_k, v_l)\) in \(\overline{R}_2\), it must be adjacent to all of the vertices in \(\overline{R}_2\) for \(G \vee H\) to be perfect. By symmetry, if a vertex in \(\overline{R}_2\) is adjacent to \(\overline{R}_1\), it must be adjacent to all of the vertices in \(\overline{R}_1\) for \(G \vee H\) to be perfect. Thus any \(B_{n^2,n^2}\) with non-empty edge set must be a complete bipartite graph for \(G \vee H\) to be perfect. The disjoint union of perfect graphs is perfect so if the edge set of \(B_{n^2,n^2}\) is empty, \(G \vee H\) is perfect.

We now show that \(B_{n^2,n^2} \cong K_{n^2,n^2}\) if and only if \(G, H \cong K_{n,n}\) or \(G, H \cong \overline{K}_{2n}\) and \(B_{n^2,n^2} \cong \overline{K}_{2n}^2\) if and only if one of \(G, H\) is isomorphic to \(K_{n,n}\) and the other \(\overline{K}_{2n}\). Inspection of Eq. (29) shows that proving the statement is equivalent to proving that

\[
A_G \otimes A_H + (J^{(n \times n)} - A_G) \otimes (J^{(n \times n)} - A_H) = \begin{cases} 
J^{(n^2 \times n^2)}, & \text{iff } A_G = A_H = J^{(n \times n)}, \\
0^{(n^2 \times n^2)}, & \text{iff } A_G = J^{(n \times n)}, A_H = 0^{(n \times n)}, \\
or A_G = A_H = 0^{(n \times n)}; & \text{iff } A_G = 0^{(n \times n)}, A_H = J^{(n \times n)}, 
\end{cases}
\]

with an equivalent statement applying to the matrix

\[
A_G^T \otimes A_H + (J^{(n \times n)} - A_G^T) \otimes (J^{(n \times n)} - A_H),
\]

where \(A_G\) and \(A_H\) are the biadjacency matrices of \(G\) and \(H\) respectively. We consider the first case without loss of generality. Sufficiency can be straightforwardly seen from Eq. (31), so we show necessity. Call the left hand side of Eq. (31) \(M_H\), which can be written in block form as

\[
\begin{bmatrix}
[A_G]_{1,1} A_H + (1 - [A_G]_{1,1}) (J^{(n \times n)} - A_H) & \cdots & [A_G]_{1,n} A_H + (1 - [A_G]_{1,n}) (J^{(n \times n)} - A_H) \\
\vdots & \ddots & \vdots \\
[A_G]_{n,1} A_H + (1 - [A_G]_{n,1}) (J^{(n \times n)} - A_H) & \cdots & [A_G]_{n,n} A_H + (1 - [A_G]_{n,n}) (J^{(n \times n)} - A_H)
\end{bmatrix}.
\]

Consider the \((i, j)\)th block of \(M_H\),

\[
[A_G]_{i,j} A_H + (1 - [A_G]_{i,j}) (J^{(n \times n)} - A_H) = \begin{cases} 
A_H, & \text{if } [A_G]_{i,j} = 1; \\
J^{(n \times n)} - A_H, & \text{if } [A_G]_{i,j} = 0.
\end{cases}
\]

Observe that all the elements of \(M_H\) can only be the same if all of the elements of \(A_G\) are the same, as otherwise \(M_H\) would be a block matrix comprised of differing blocks. Now suppose all the elements of \(A_G\) are equal. Then for all of the elements of \(M_H\) to be equal, all of the elements of \(A_H\) have to be the same as each other, since \(M_H\) is made up of repeating blocks of either \(A_H\), or \(J^{(n \times n)} - A_H\). Together, we have that all of the elements
of $M_U$ are the same only if all of the elements in $A_G$ are equal and all of the elements in $A_H$ are equal. From Eq. (34) we can see that $M_U$ is the all-ones matrix only if all of the elements of both $A_G$ and $A_H$ are the same, i.e. are both all-ones or both all-zeroes, otherwise $M_U$ will be the all-zeroes matrix. Thus we have that $B_{n^2,n^2} \cong K_{n^2,n^2}$ if and only if $G, H \cong K_{n,n}$ or $G, H \cong \overline{K}_{2n}$, and $B_{n^2,n^2} \cong \overline{K}_{2n^2}$ if and only if one of $G, H$ is $K_{n,n}$ and the other $\overline{K}_{2n}$. □

**Corollary 4** Let $G$ and $H$ be balanced, bipartite graphs on $2n$ vertices, with $n \geq 2$. Then, $G \nabla H$ is perfect if and only if either

1. $G$ and $H$ are both the complete bipartite graph $K_{n,n}$;
2. $G$ and $H$ are both the empty graph $\overline{K}_{2n}$;
3. One of $G, H$ is the complete bipartite graph $K_{n,n}$ and the other the empty graph $\overline{K}_{2n}$.

**Proof.** Since $G \nabla H \subset G \nabla H$, the only possible cases where $G \nabla H$ can be perfect are the cases when $G \nabla H$ is perfect, which are the three cases given. We show that $G \nabla H$ is perfect in each of these cases. For case 1., using Eqs. (15) and (16), one can see that $A(G \nabla H) = A(G \nabla H)$ implying $G \nabla H \cong G \nabla H$, which is perfect. For both cases 2. and 3. (in case 3. assume without loss of generality that $G = K_{n,n}$ and $H = \overline{K}_{2n}$) one can readily verify using Eqs. (15) and (16) that $A(G \nabla H) = [1 1] \otimes A(X)$, where $A(X) = A((G \nabla H)[U]) = A((G \nabla H)[V])$. This corresponds to taking the disjoint union $2X$, calling the connected components $X_1$ and $X_2$, then drawing edges $\{u_i, v_j\}$ for all $u_i \in V(X_1), v_j \in V(X_2)$ such that $u_i \sim u_j$ (or equivalently $v_i \sim v_j$). Since $X_1$ and $X_2$ are perfect, using Proposition 6 we have the result. □

### 5.2 Isomorphism via Biclique Enumeration: The Algorithm

We present our algorithm for isomorphism of bipartite graphs, Isomorphism via Biclique Enumeration (IvBE), shown in Algorithm 1. We use the MBEA algorithm of Zhang et al. [ZPR+14] for enumerating all of the maximal bicliques of a bipartite graph $G$, which we denote $\text{mbea}(G)$.

**Proposition 7** [ZPR+14, Theorem 2] Let $G$ be a bipartite graph with $|E(G)| = e$. There is an algorithm, $\text{mbea}(G)$, that finds all the maximal bicliques in $G$ with runtime $O(e \beta(G))$.

**Theorem 3** Let $G$ and $H$ be balanced, bipartite graphs on $2n$ vertices, each having $e$ edges. Then, given $G$ and $H$ as input, IvBE (Algorithm 1) returns an isomorphism $\sigma : V(G) \to V(H)$ if $G \cong H$ and returns NOT ISOMORPHIC otherwise. Moreover, IvBE runs in time $O(\beta(\Omega)e\bar{n}^2)$, where $\Omega := G \otimes H \cup \overline{G_{\text{bip}}} \otimes \overline{H_{\text{bip}}}$.
Algorithm 1 Isomorphism via Biclique Enumeration (IvBE)

**Input:** Balanced, bipartite graphs on $2n$ vertices $G$ and $H$, with $V(G) = \{u_1, \ldots, u_{2n}\}$ and $V(H) = \{v_1, \ldots, v_{2n}\}$.

**Output:** A bijection $\sigma : V(G) \rightarrow V(H)$ if $G \cong H$, or the certificate NOT ISOMORPHIC if $G \not\cong H$.

1. Construct the graph $\Omega := G \odot H \cup \overline{G}^\text{bip} \odot \overline{H}^\text{bip}$, where $V(\Omega) = V(G) \times V(H)$.
2. Enumerate the maximal bicliques $B_\Omega \leftarrow \text{mbea}(\Omega)$.
3. if $\{B \in B_\Omega : |V_0(B)| = |V_1(B)| = n\} = \emptyset$ then
   4. return NOT ISOMORPHIC
   5. end if
   6. for all $B \in B_\Omega : |V_0(B)| = |V_1(B)| = n$ do
      7. if $\text{is\_co\_rook\_clique}(\Omega, V_0(B)) \land \text{is\_co\_rook\_clique}(\Omega, V_1(B))$ then
         8. Instantiate bijection $\sigma : V(G) \rightarrow V(H)$.
         9. for all $i$ such that $(u_i, v_j) \in V_0(B)$ do
            10. Add $\sigma : u_i \mapsto v_j$, where $u_i \in V(G)$, $v_j \in V(H)$.
         11. end for
         12. for all $i$ such that $(u_i, v_j) \in V_1(B)$ do
            13. Add $\sigma : u_i \mapsto v_j$, where $u_i \in V(G)$, $v_j \in V(H)$.
         14. end for
         15. return $\sigma$.
      16. else
         17. return NOT ISOMORPHIC
      18. end if
   19. end for

1. function $\text{is\_co\_rook\_clique}(\Omega, S \subseteq V(\Omega))$
2. if $(\forall (u_i, v_j), (u_k, v_l) \in S \text{ s.t. } (u_i, v_j) \neq (u_k, v_l), \{(u_i, v_j), (u_k, v_l)\} \in E(G \odot H))$ then
   3. return TRUE
4. else
   5. return FALSE
6. end if
7. end function

**Proof.** First, observe that $G \odot H$ can be produced by taking the union of $\Omega$ with co-rooks on the bipartition classes of the two connected components of $\Omega$, that is, $G \odot H = \Omega \cup 4R(n, n) = \Omega \cup 4(K_n \odot K_n)$ under the appropriate vertex labelling. Recall from Observation 5 that $G \cong H$ if and only if there is a $(n, n)$-biclique joining the $n$-cliques in the co-rooks of a connected component of $G \odot H$.

First, suppose $G \cong H$ with $\sigma' : V(G) \rightarrow V(H)$ an isomorphism. Then by Observa-
tion 5, at line 7 in Algorithm 1 the \((n, n)\)-biclique \(B\) encoding \(\sigma'\) induces \(n\)-cliques in the co-rooks within \(G \triangledown H\). The lines 8-15 then write down and return \(\sigma'\) as the bijection \(\sigma\). Now suppose \(G \not\cong H\). Either we have the situation as in Figure 2 where there are no \((n, n)\)-bicliques in \(\Omega\) and the algorithm returns NOT ISOMORPHIC at line 4. More commonly there exist \((n, n)\)-bicliques in \(\Omega\), but they do not satisfy the necessary and sufficient conditions of Observation 5 for isomorphism. In this case, Algorithm 1 returns NOT ISOMORPHIC at line 17. The function is_co-rook_clique checks that under the chosen vertex labelling, a set \(S\) of vertices in \(\Omega\) induces a clique in a co-rook of \(G \triangledown H\). The if statement is equivalent to checking pairwise adjacency of the vertices \(S\) in \(G \triangledown H\), which is polynomial in \(|S| = n\) in the cases where the function is used.

The runtime of the algorithm is \(O(\beta(\Omega)en^2)\), with the \(O(\beta(\Omega)e)\) factor arising from enumerating the maximal bicliques in \(\Omega\) using MBEA(\(\Omega\)) and the \(O(n^2)\) factor from the is_co-rook_clique function.

\section{Counting Maximal Bicliques in Bipartite Graphs}

We now consider graphs for which we can directly enumerate the number of maximal bicliques. For example, consider the classes of bipartite graph for which the number of maximal bicliques is polynomial in the number of vertices. In such an instance, we know that we can find all the maximal bicliques in polynomial time, by Proposition 7. We characterise the number of maximal bicliques in certain graphs here, considering first the extremal cases.

\subsection{Extremal Cases}

For \(n \geq 2\), define the graph \(B_n\), where for even \(n\), \(B_n\) is the graph \(K_{\frac{n}{2}, \frac{n}{2}}\) with a perfect matching removed. For odd \(n\), \(B_n\) is the graph \(B_{n-1}\) with an additional vertex added, which is adjacent to all of the vertices in one of the bipartition classes of \(B_{n-1}\). We give examples of \(B_n\) in Figure 3 for clarification. In addition, if \(n\) is even we call \(B_n\) a crown graph. We then have the following result.

\begin{proposition}
Let \(G\) be a bipartite graph on \(n \geq 4\) vertices. Then,
\[
\beta(G) \leq \begin{cases} 
2\left\lfloor \frac{n}{2} \right\rfloor - 2, & \text{n even;} \\
2\left\lfloor \frac{n}{2} \right\rfloor - 1, & \text{n odd.}
\end{cases}
\] 
\tag{35}

Moreover, this bound is saturated if and only if \(G \cong B_n\).
\end{proposition}

\begin{proof}
We proceed by induction on \(n\), where we shall assume the hypothesis for even \(n\), then show that the hypothesis is consequently true for \(n + 1\) and \(n + 2\). As in the proof of Proposition 4, any biclique in \(G\) can be uniquely described as being induced on a subset of vertices \(S \subseteq V_0(G)\), as the associated biclique will be induced on the vertices

\end{proof}
Suppose \( n = 2s \), where \( s \geq 2 \), and \( B_n \) is the unique graph on \( n \) vertices with the maximum possible number of bicliques, \( 2 \lfloor \frac{n}{2} \rfloor - 2 \). Now \( 2 \lfloor \frac{n}{2} \rfloor = 2 \lfloor \frac{n}{2} \rfloor \) since \( n \) is even. Let us now add a vertex \( u_1 \) to this graph (without loss of generality, add it to \( V_1(B_n) \)). We now call this graph \( H \) and wish to increase its number of maximal bicliques by 1. Since in this graph the subsets \( S \cup N_H(S) \) for all \( \emptyset \subset S \subset V_0(H) \) already induce maximal bicliques, the only maximal biclique we can add is \( V_1(B_n) \). We now add a vertex \( S \cup \emptyset \subset V \) that maximises the number of maximal bicliques is \( \subseteq N \). Since in this graph the subsets \( \emptyset \subset S \subset V \) and elements of \( V \) are placed, including \( \{u_2, u_1\} \). iii. Fewer than \( n \) edges are placed.

- **Case i.** Suppose every possible edge between \( u_2 \) and every element of \( V_1(H) \) is added. Now, \( u_2 \) is added to every existing maximal biclique since it is adjacent to every vertex in \( V_1(H) \), so our graph has \( 2^{n+1} - 1 \) maximal bicliques and our bound for \( n + 2 \) is not saturated.

- **Case ii.** Suppose we draw the edge \( \{u_1, u_2\} \) and \( n - 1 \) further edges between \( u_2 \) and elements of \( V_1(H) \). Now, similar to in case i., \( u_1 \) is added to every maximal biclique induced on subsets of \( V_0(H) \), so our graph has \( 2^{n+1} - 1 \) maximal bicliques and does not saturate the bound. The \( n - 1 \) additional edges only selectively add \( u_2 \) to already existing maximal bicliques, so do not increase \( \beta(H) \).

- **Case iii.** Suppose fewer than \( n \) edges are drawn between \( u_2 \) and \( V_1(H) \). Then there will be at least one subset \( S \) satisfying \( \{u_2\} \subset S \subset V_0(H) \) for which \( N_H(S) = \emptyset \) as there will be no edge between \( u_2 \) and \( N_H(S) \setminus \{u_2\} \), for \( N_H(S) \setminus \{u_2\} \).

Figure 3: The graphs \( B_8, B_9 \) and \( T_9 \).
a single vertex. Since not every subset $\emptyset \subset S \subset V_0(H)$ induces a maximal biclique the bound is not saturated.

This also shows that for even $n$ the maximum $\beta(G) = 2^{\lfloor \frac{n}{2} \rfloor} - 2$, unlike in the odd $n$ case. We now have that only the graph $B_{n+2}$ saturates the bound for $n+2$ vertices, as required. It is straightforward to check the inductive hypothesis for 4 vertices, i.e. that $B_4 \cong 2K_2$ is the only graph that saturates the bound, and the result follows. \hfill \Box

We note that for an $n$-vertex bipartite graph $G$, Prisner [Pri00] provides an upper bound for $\tilde{\alpha}(G)$ of $\frac{2n}{n^2}$, with $B_n$ for even $n$ being the extremal case. This is consistent with Proposition 8 because his definition of a biclique allows bicliques with empty bipartition classes and he does not consider the odd $n$ case, since it does not saturate his bound.

We can also characterise the bipartite graphs which maximise the number of maximal independent sets in a graph, $\tilde{\alpha}(G)$. For odd $n$, define $T_n$ as the graph obtained by taking $\lfloor \frac{n}{2} \rfloor K_2$, then adding an additional vertex, which is adjacent to exactly one vertex in each $K_2$. An example is shown in Figure 3. Now for $n \geq 2$ and $0 \leq r \leq \lfloor \frac{n}{2} \rfloor$ define the graph

$$B'_{n,r} = \begin{cases} \frac{n}{2} K_2, & n \text{ even;} \\ rK_2 \uplus T_2(\frac{n}{2} - r) + 1, & n \text{ odd.} \end{cases}$$

Liu gave [Lin93] the following bound on $\tilde{\alpha}(G)$ for bipartite $G$.

**Proposition 9** [Lin93, Theorem 2.1]. Let $G$ be a bipartite graph on $n$ vertices. Then,

$$\tilde{\alpha}(G) \leq 2^{\lfloor \frac{n}{2} \rfloor}.$$  \hfill (37)

Moreover, this bound is saturated if and only if $G \cong B'_{n,r}$ for any $0 \leq r \leq \lfloor \frac{n}{2} \rfloor$.

Observe that $B_n^{\text{bip}} \cong B'_{n,r}$ for even $n$ and $B_n^{\text{bip}} \cong B'_{n,0}$ for odd $n$.

### 6.2 Maximal Bicliques in Random Bipartite Graphs

We can also consider the number of maximal bicliques in random bipartite graphs. We define a probability distribution over bipartite graphs on $m + n$ vertices as follows: Consider the independent random variables $X_{i,j}$ taking the values

$$X_{i,j} = \begin{cases} 1, & \text{with probability } p; \\ 0, & \text{with probability } 1 - p. \end{cases}$$

for $i \in [m], \ j \in [n]$ and $p \in [0, 1]$. A bipartite graph $G$ drawn from the distribution $G(m, n, p)$ has $|V_0(G)| = m$, $|V_1(G)| = n$ and edge set $\{\{u_i, v_j\} \mid u_i \in V_0(G), \ v_j \in V_1(G), \ X_{i,j} = 1\}$. A bipartite graph $G$ drawn from $G(m, n, p)$, is an Erdős-Rényi bipartite graph, which we denote by $G \sim G(m, n, p)$. We have the following result.
Lemma 10 Let \( m, n \in \mathbb{N} \) and \( p \in [0, 1] \). Then,

\[
E_{G \sim \mathcal{G}(m, n, p)}[\beta(G)] = \sum_{k=1}^{m} \sum_{l=1}^{n} p^{kl} (1 - p^l)^{m-k} (1 - p^k)^{n-l} \binom{m}{k} \binom{n}{l}.
\] (39)

Proof. First, let \( \text{ind}(A) \) be the indicator variable for some event \( A \). We then make the observation that

\[
E_{G \sim \mathcal{G}(m, n, p)}[\beta(G)] = \sum_{\emptyset \subseteq T \subseteq \mathcal{V}_0(G)} \sum_{\emptyset \subseteq U \subseteq \mathcal{V}_1(G)} E[\text{ind}(T \cup U \text{ is a maximal biclique})]
= \sum_{\emptyset \subseteq T \subseteq \mathcal{V}_0(G)} \sum_{\emptyset \subseteq U \subseteq \mathcal{V}_1(G)} \Pr[T \cup U \text{ is a maximal biclique}]
= \sum_{\emptyset \subseteq T \subseteq \mathcal{V}_0(G)} \Pr[T \cup U \text{ is a biclique}] \cdot \Pr[T \cup U \text{ is maximal | } T \cup U \text{ is a biclique}].
\] (40)

Suppose we have \( G \sim \mathcal{G}(m, n, p) \), \( \emptyset \subset T \subseteq \mathcal{V}_0(G) \) and \( \emptyset \subset U \subseteq \mathcal{V}_1(G) \), with \(|T| = k\) and \(|U| = l\). Then,

\[
\Pr[T \cup U \text{ is a biclique}] = p^{kl},
\] (41)

from the definition of \( G \) and the fact that there are \( k \cdot l \) possible edges in \( T \cup U \). We also have

\[
\Pr[T \cup U \text{ is maximal | } T \cup U \text{ is a biclique}] = 1 - \Pr[T \cup U \text{ is not maximal | } T \cup U \text{ is a biclique}].
\] (42)

The statement that \( T \cup U \) is not maximal is equivalent to saying that there exist vertices either \( u_i \in \mathcal{V}_0(G) \setminus T \) or \( v_j \in \mathcal{V}_1(G) \setminus U \) that can be added to \( T \cup U \) and it remain a biclique. We have using the inclusion-exclusion principle,

\[
\Pr[T \cup U \text{ is not maximal | } T \cup U \text{ is a biclique}]
= \Pr\left[ \bigcup_{u_i \in \mathcal{V}_0(G) \setminus (T \cup U)} \{u_i\} \cup T \cup U \text{ is a biclique} \right]
= \sum_{\emptyset \subseteq S \subseteq \mathcal{V}(G) \setminus (T \cup U)} (-1)^{|S|+1} \Pr[S \cup T \cup U \text{ is a biclique}]
= \sum_{\emptyset \subseteq T' \subseteq \mathcal{V}_0(G) \setminus T} \sum_{\emptyset \subseteq U' \subseteq \mathcal{V}_1(G) \setminus U} (-1)^{|T'|+|U'|+1} p^{|T'\cup U'|+|U'|} + 1
= \sum_{i=0}^{m-k} \sum_{j=0}^{n-l} (-1)^{i+j+1} \binom{m-k}{i} \binom{n-l}{j} p^{i+j+k} + 1.
\] (43)
We thus have from Eq. (42) that

\[
\Pr [T \cup U \text{ is maximal } | T \cup U \text{ is a biclique}] = \sum_{i=0}^{m-k} \sum_{j=0}^{n-l} (-1)^{i+j} \binom{m-k}{i} \binom{n-l}{j} p^{i+jk}
\]

\[
= \left( \sum_{i=0}^{m-k} (-1)^i \binom{m-k}{i} p^i \right) \cdot \left( \sum_{j=0}^{n-l} (-1)^j \binom{n-l}{j} p^j \right)
\]

\[
= (1 - p^i)^{m-k} \cdot (1 - p^j)^{n-l}
\]

Substituting into Eq. (40) along with Eq. (41) yields

\[
E_{G \sim G_{\{m,n,p\}}} [\beta(G)] = \sum_{\emptyset \subseteq T \subseteq V_0(G)} \sum_{\emptyset \subseteq U \subseteq V_1(G)} p^{|T||U|} (1 - p^{|U|})^{m-|T|} \cdot (1 - p^{|T|})^{n-|U|}
\]

\[
= \sum_{k=1}^{m} \sum_{l=1}^{n} p^k (1 - p^l)^{m-k} (1 - p^k)^{n-l} \binom{m}{k} \binom{n}{l}.
\]

(45)

6.3 Graphs with Polynomial Number of Maximal Bicliques and IvBE

Prisner [Pri00] provides a sufficient condition for the number of maximal bicliques in a graph to be polynomial.

**Proposition 10** [Pri00, Theorem 3.3] For every integer \( j \), if a bipartite graph \( G \) does not contain any induced \( B_{2j} \), then it contains at most \( (|V_0(G)||V_1(G)|)^{j-1} \) maximal bicliques.

**Corollary 5** For every integer \( j \), if a bipartite graph \( G \) does not contain any induced \( jK_2 \), then it contains at most \( (|V_0(G)||V_1(G)|)^{j-1} + 2 \) maximal independent sets.

**Proof.** The condition that \( G \) does not contain any induced \( jK_2 \) is equivalent to the condition that \( G_{\text{bip}} \) does not contain any induced \( B_{2j} \). If \( G_{\text{bip}} \) does not contain any induced \( B_{2j} \), then by Proposition 10 \( G_{\text{bip}} \) contains at most \( (|V_0(G)||V_1(G)|)^{j-1} \) maximal bicliques. Using Lemma 9 gives the result. \[\square\]

Observe that the condition in Corollary 5 is equivalent to saying that \( G \) does not contain an induced matching on \( 2j \) vertices and the condition in Proposition 10 is equivalent to saying that \( G \) does not contain an induced crown on \( 2j \) vertices. For a graph \( G \), let \( m^{\text{ind}}(G) \) be the largest induced matching in \( G \) and let \( c^{\text{ind}}(G) \) be the largest induced crown in \( G \). We are now in a position to state the following result.

**Proposition 11** Let \( \Gamma \) be the class of balanced, bipartite graphs that satisfy the following: For any \( G \in \Gamma \), \( m^{\text{ind}}(G) = O(\text{poly log}(n)) \) and \( c^{\text{ind}}(G) = O(\text{poly log}(n)) \), where \( n = |V(G)| \). Then, for any \( G, H \in \Gamma \) with \( |V(G)| = |V(H)| = 2n \), deciding if \( G \cong H \) takes time \( 2^{O(\text{poly log}(n))} \), using the IvBE algorithm.
Proof. We claim that there are $2^{O(\text{poly log}(n))}$ maximal bicliques in $\Omega := G \otimes H \cup G^{\text{bip}} \otimes \overline{H}^{\text{bip}}$, where $V(\Omega) = V(G \vee H)$. Using IvBE we can check if $G \cong H$ in time $O(n^2 e) \cdot 2^{O(\text{poly log}(n))}$. Since $e = O(n^2)$ we have runtime $O(n^4) \cdot 2^{O(\text{poly log}(n))} = 2^{O(\text{poly log}(n))}$.

Now it remains to justify the earlier claim that $\beta(\Omega) = 2^{O(\text{poly log}(n))}$. Proposition 10, Corollary 5 and Lemma 9 give us that $G$, $G^{\text{bip}}$, $H$ and $H^{\text{bip}}$ each have a quasi-polynomial number of maximal bicliques, as $(n^2)O(\text{poly log}(n)) = n^{O(\text{poly log}(n))} = 2^{O(\text{poly log}(n))}$. Lemma 8 ensures that there are a quasi-polynomial number of maximal bicliques in $G \otimes H$ and $G^{\text{bip}} \otimes H^{\text{bip}}$. From Proposition 4, the number of maximal bicliques in $G \otimes H \cup G^{\text{bip}} \otimes H^{\text{bip}} = \Omega$ is quasi-polynomial in $n$. Thus, the claim is proven and the result follows.

Proposition 10 and Corollary 5 suggest a class of graphs for which the IvBE algorithm is polynomial.

**Proposition 12** Let $\Gamma$ be the class of $2K_2$-free balanced, bipartite graphs, that is, the class of balanced, bipartite graphs whose members do not contain $2K_2$ as an induced subgraph. Then, for any $G, H \in \Gamma$ such that $|V(G)| = |V(H)| = n$, IvBE decides if $G \cong H$ in time polynomial in $n$.

**Proof.** From Proposition 10 and Corollary 5, we have that the number of maximal bicliques and maximal independent sets in $G$ and $H$ are polynomial (in fact quadratic) in $n$, since $2K_2 \cong B_4$. We also have from Lemma 9, Lemma 8 and Proposition 4 that the graph $\Omega = G \otimes H \cup G^{\text{bip}} \otimes H^{\text{bip}}$ has a polynomial number of maximal bicliques. Since the runtime of IvBE has runtime $O(\beta(\Omega)en^2)$ from Theorem 3 and $e = O(n^2)$, the result follows.

### 7 Conclusion

We have seen that investigating the combinatorics of the weak modular product of bipartite graphs can lead to some insights into using the clique number for isomorphism. Some open questions remain. For instance, here we have characterised the cases where the weak modular product of bipartite graphs is perfect. It would be interesting to extend this characterisation to general graphs and/or to other graph subclasses, as using theLovász number of the weak modular product graph would yield an efficient algorithm for GI in these cases.

Furthermore, we have approached the clique number question itself purely from a combinatorial viewpoint. It may prove useful to look into approximation schemes for the clique number in conjunction with the weak modular product to investigate GI. It is known that for all $\epsilon > 0$, there can be no polynomial time algorithm that approximates the maximum clique of a general graph on $n$ vertices to within a factor better than $O(n^{1-\epsilon})$ unless $P = NP$ [ZD06]. However, there is an $O(\sqrt{n})$-approximation algorithm for the dual problem of finding the independence number, when specialised to the union of perfect graphs [CPRS10]. The very weak modular product of bipartite graphs constitutes a union of perfect graphs so a modification of this algorithm could be a starting point.
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