ON THE EXISTENCE AND UNIQUENESS OF AN INVERSE PROBLEM IN
EPIDEMIOLOGY

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Abstract. In this paper we introduce the functional framework and the necessary conditions
for the well-posedness of an inverse problem arising in the mathematical modeling of disease
transmission. The direct problem is given by an initial boundary value problem for a reaction
diffusion system. The inverse problem consists in the determination of the disease and recovery
transmission rates from observed measurement of the direct problem solution at the end time.
The unknowns of the inverse problem are coefficients of the reaction term. We formulate the
inverse problem as an optimization problem for an appropriate cost functional. Then, the existence
of solutions of the inverse problem is deduced by proving the existence of a minimizer for
the cost functional. Moreover, we establish the uniqueness up an additive constant of identifica-
tion problem. The uniqueness is a consequence of the first order necessary optimality condition
and a stability of the inverse problem unknowns with respect to the observations.

1. Introduction

The mathematical modeling of disease transmission is an active research area of mathe-
matical biology [1, 2, 4, 5, 10, 12–14, 18, 20, 22, 24, 25]. Nowadays, there are several approaches used to
construct the mathematical models in mathematical epidemiology. Despite the different kinds of
such models, and analogously to biochemical systems, we can distinguish five common steps in
the processes of modelling [7]: collection and analysis of experimental data and information on
the specific disease; selection of the mathematical theory to be used in the model formulation; the
mathematical analysis of well-posedness of the model; the calibration or parameter identification
of the model; and the model validation and refinement. Moreover, we note that the modeling is a
cyclical rather than a linear process: all assumptions made in the previous steps are reconsidered
and refined and upon completion of the modeling process. We can improve the model by introduc-
ing new hypotheses, design new experiments, made predictions and deep the analysis of each step.
Thus, in particular, we are interested in the analysis of calibration or parameter identification of
the model. To be more precise, the aim of this paper is to provide a framework to solve the inverse
problem arising in the step of model calibration by assuming that the mathematical model is an
initial boundary value problem for a reaction-diffusion system.

Let us precise the mathematical model or the direct problem. We consider that the infectious
diseases taken place in a bounded domain Ω ⊂ R^d (d = 1, 2, 3) and is described by an SIS
reaction-diffusion model, where the population density of susceptible and infected individuals
at time t and location x are given by S(x, t) and I(x, t), respectively. The diffusion matrix
is assumed to be equals to the identity. We assume that the infection process is given by the
interaction of susceptible and infected densities at the point x and time t via the “power law”
β(x)S^m(x, t)I^n(x, t), where β is the rate of disease transmission and m, n ∈ ]0, 1[ are some given
(fixed) parameters. The recovery process is represented by γ(x)I(x, t) with γ the rate of disease
recovery. Thus, the direct problem is defined as follows: Given the set of functions {β, γ, S_0, I_0}
find the functions S and I satisfying the following initial boundary value problem

\begin{align}
S_t - \Delta S &= -\beta(x)S^m I^n + \gamma(x)I, & \text{in } Q_T := \Omega \times [0, T], \\
I_t - \Delta I &= \beta(x)S^m I^n - \gamma(x)I, & \text{in } Q_T, \\
\nabla S \cdot n &= \nabla I \cdot n = 0, & \text{on } \Gamma := \partial \Omega \times [0, T],
\end{align}
where \( \partial \Omega \) is the boundary of \( \Omega \) and \( \mathbf{n} \) is the unit exterior normal to \( \partial \Omega \). The boundary conditions \((1.3)\) and the functions \( S_0 \) and \( I_0 \) models the initial conditions.

The inverse problem consists in the determination of the rate functions \( \beta \) and \( \gamma \) in the SIS model \((1.1)-(1.3)\) from observed measurement for \( S \) and \( I \) at \( t = T \) given by the functions \( \{ S_{obs}, I_{obs} \} \) defined on \( \Omega \). Then, we can define the inverse problem as follows: Given the set of functions \( \{ S_0, I_0, S_{obs}, I_{obs} \} \) defined on \( \Omega \), find the functions \( \beta \) and \( \gamma \) such that the solution \( S \) and \( I \) of initial boundary value problem \((1.1)-(1.3)\) satisfy the overspecified end condition \( S(x, T) = S_{obs}(x), \quad I(x, T) = I_{obs}(x) \) for \( x \in \Omega \). Indeed, in order to precise the analysis of the inverse problem, we consider an equivalent reformulation as the following optimization problem

\[
\inf J(\beta, \gamma) \quad \text{subject to } (S_{\beta, \gamma}, I_{\beta, \gamma}) \text{ solution of } (1.1)-(1.5),
\]

where

\[
J(\beta, \gamma) := \frac{1}{2} \left[ \| S(\cdot, T) - S_{obs} \|^2_{L^2(\Omega)} + \| I(\cdot, T) - I_{obs} \|^2_{L^2(\Omega)} \right] + \frac{\delta}{2} \left[ \| \nabla \beta \|^2_{L^2(\Omega)} + \| \nabla \gamma \|^2_{L^2(\Omega)} \right]
\]

is a functional defined on the admissible set

\[
U_{ad}(\Omega) = A(\Omega) \cap \left[ H^{[d/2]+1}(\Omega) \times H^{[d/2]+1}(\Omega) \right],
\]

\[
A(\Omega) = \left\{ (\beta, \gamma) \in C^0(\Omega) \times C^0(\Omega) : \quad \text{Ran}(\beta) \subseteq [0, B] \subset [0, 1], \quad \text{Ran}(\gamma) \subseteq [0, T] \subset [0, 1], \quad \nabla \beta, \nabla \gamma \in L^2(\Omega) \right\},
\]

and for an appropriate \( \delta > 0 \). Here, \( H^m(\Omega) \) and \( C^0(\Omega) \) denote the standard Sobolev and Hölder spaces \( W^{m,2}(\Omega) \) and \( C^{0,\alpha}(\Omega) \), respectively; and \( \text{Ran}(f) \) denote the range of function \( f \). The construction of \( U_{ad}(\Omega) \) was recently developed in [8] and also we note that \( U_{ad}(\Omega) = A(\Omega) \) when \( d = 1 \) and coincides with the admissible set considered by Xiang and Liu in [26].

The main result of this paper is the necessary conditions for the well-posedness theory of the inverse problem. More precisely, we prove the following theorem:

**Theorem 1.1.** Let us consider the notation

\[
\mathcal{U}(\Omega) = \left\{ (\beta, \gamma) \in U_{ad}(\Omega) : \quad \| \beta \|_{L^1(\Omega)} \text{ and } \| \gamma \|_{L^1(\Omega)} \text{ are constants} \right\}.
\]

Consider that the open bounded and convex set \( \Omega \) is such that \( \partial \Omega \) is \( C^1 \) and the initial conditions \( S_0 \) and \( I_0 \) are functions belong to \( C^{2,\alpha}(\Omega) \) and satisfy the inequalities

\[
S_0(x) \geq 0, \quad I_0(x) \geq 0, \quad \int_{\Omega} I_0(x) dx > 0, \quad S_0(x) + I_0(x) \geq \phi_0 > 0,
\]

on \( \Omega \), for some positive constant \( \phi_0 \). Moreover assume that the observation functions \( S_{obs} \) and \( I_{obs} \) are functions belong to \( L^2(\Omega) \). Then, there exists at least one solution of \((1.6)\) and there exist \( \Theta \in \mathbb{R}^+ \) such that the solution of \((1.6)\) is uniquely defined, up an additive constant, on \( \mathcal{U}(\Omega) \) for any regularization parameter \( \delta > \Theta \).

On the other hand, we recall that inverse problems in reaction-diffusion equations and systems have been addressed in the literature of the last decades, for instance [6,9,11,19,21,23,26]. In [6] the authors study the identification of \( q(x) \) in the equation \( u_t = \Delta u + q(x) u \) with Dirichlet boundary condition and from final measurement data \( u(x, T) \). They prove the existence of solutions and develop a numerical solution of the inverse problem by using an optimization problem. The authors of [9] consider the nonlinear reaction-diffusion equation \( u_t = \Delta u + p(x) f(u) \) with \( f \) a nonlinear function and study the identification of \( p \), getting some results for the existence and the local uniqueness. Now, in [23] the authors study the inverse problem for a reaction-diffusion system with a linear reaction term and obtain existence and local uniqueness of the inverse problem. More recently, in [26] the authors have studied the one-dimensional version of the inverse problem considered in this paper. They obtain a result for existence and local uniqueness of the solution by assuming that the infection process is modeled by a frequency-dependent transmission function.
instead of the power law function. Now, the articles [11][19][21] are focused on inverse problems in epidemic systems, but are of different type to that one considered in this paper. Thus, the Theorem [11] is an extension to the multidimensional global uniqueness framework of the one-dimensional and local uniqueness results obtain recently in [20].

The rest of the paper is organized in two sections. In section 2 we present some results for the direct problem solution, we introduce the adjoint state and the necessary optimality conditions, and prove a stability result. On section 3 we present the proof of Theorem [11].

2. Preliminary

2.1. Direct problem solution. The well-posedness of the direct problem (1.1)-(1.5) is given by the following result.

**Theorem 2.1.** Consider that \( \Omega, S_0 \) and \( I_0 \) satisfy the hypotheses of Theorem [11]. If \( (\beta, \alpha) \in C^\alpha(\overline{\Omega}) \times C^\alpha(\overline{\Omega}) \), the initial boundary value problem (1.1)-(1.5) admits a unique positive classical solution \((S, I)\), such that \( S \) and \( I \) are belong to \( C^{2+\alpha,1+\alpha/2}(\overline{Q}_T) \) and also \( S \) and \( I \) are bounded on \( \overline{Q}_T \), for any given \( T \in \mathbb{R}^+ \).

The existence and the uniqueness can be developed by the Shauder’s theory for parabolic equations [15][17]. Meanwhile, the positive behavior of the solution is a consequence of the maximum principle. Indeed, if we denote by \( N \) the total population, i.e. \( N(x,t) = S(x,t) + I(x,t) \). Then, from the system (1.1)-(1.5) we have that \( N \) satisfy the following initial boundary value problem

\[
\begin{align*}
N_t - \Delta N &= 0, & \text{in } Q_T, \\
\nabla N \cdot n &= 0, & \text{on } \Gamma, \\
N(x,0) &= S_0(x) + I_0(x), & \text{in } \Omega.
\end{align*}
\]

By the maximum principle of parabolic equations and the hypothesis [11][11] we have that \( N(x,t) \geq S_0(x) + I_0(x) \geq \phi_0 > 0 \) in \( Q_T \).

**Corollary 2.1.** Consider that \( \Omega, S_0 \) and \( I_0 \) satisfy the hypotheses of Theorem [21]. If \( (\beta, \alpha) \in C^\alpha(\overline{\Omega}) \times C^\alpha(\overline{\Omega}) \) and \((S, I)\) is the solution of the initial boundary value problem of (1.1)-(1.5), then the estimates \( 0 < S_m \leq S(x,t) \leq S_M, \) and \( 0 < I_m \leq I(x,t) \leq I_M, \) are valid on \( Q_T, \) for some strictly positive constants \( S_m, S_M, I_m, \) and \( I_M. \)

2.2. Adjoint System. Let us consider that \((\tilde{\beta}, \tilde{\gamma})\) is a solution of the optimal control problem [11][11] and \((\overline{S}, \overline{I})\) is the corresponding solution of (1.1)-(1.5) with \((\tilde{\beta}, \tilde{\gamma})\) instead of \((\beta, \gamma)\). Then we introduce \((p_1, p_2)\) the adjoint variables, i.e., the solution of the adjoint system which is given by the following backward boundary value problem

\[
\begin{align*}
(p_1)_t + \Delta p_1 &= m\tilde{\beta}(x)\overline{S}^{m-1}\overline{I}^{n-1}(p_1 - p_2), & \text{in } Q_T, \\
(p_2)_t + \Delta p_2 &= n\tilde{\gamma}(x)\overline{S}^{m-1}(p_1 - p_2) - \tilde{\gamma}(x)(p_1 - p_2), & \text{in } Q_T, \\
\nabla p_1 \cdot n &= \nabla p_2 \cdot n = 0, & \text{on } \Gamma, \\
p_1(x,T) &= \overline{S}(x,T) - S^{\text{obs}}(x), & \text{in } \Omega, \\
p_2(x,T) &= I(x,T) - I^{\text{obs}}(x), & \text{in } \Omega.
\end{align*}
\]

The existence of solutions for system (2.1)-(2.5) can be developed by similar arguments to a similar result presented in [3]. Now, for our purpose, we need some a priori estimates given on the following result.

**Lemma 2.1.** Consider that \( \Omega, S_0, I_0, S^{\text{obs}} \) and \( I^{\text{obs}} \), satisfy the hypotheses of Theorem [11]. Moreover, consider that \((\tilde{\beta}, \tilde{\gamma})\in U_{ad}\) is a solution of (1.1), and \((\overline{S}, \overline{I})\) is the solution of (1.1)-(1.5) with \((\tilde{\beta}, \tilde{\gamma})\) instead of \((\beta, \gamma)\). Then, the solution of the adjoint system (2.1)-(2.5) satisfy the following estimates

\[
\begin{align*}
\|p_1(\cdot, t)\|^2_{L^2(\Omega)} + \|p_2(\cdot, t)\|^2_{L^2(\Omega)} &\leq P_1, \quad (2.6) \\
\|p_1(\cdot, t)\|_{H^1(\Omega)} + \|p_2(\cdot, t)\|_{H^1(\Omega)} &\leq P_2, \quad (2.7)
\end{align*}
\]
\begin{align}
\|\Delta p_1(\cdot, t)\|_{L^2(\Omega)} + \|\Delta p_2(\cdot, t)\|_{L^2(\Omega)} & \leq P_3, \\
\|p_1(\cdot, t)\|_{L^\infty(\Omega)} + \|p_2(\cdot, t)\|_{L^\infty(\Omega)} & \leq P_4, \\
\|p_3(\cdot, t)\|_{L^\infty(\Omega)} & \leq P_5,
\end{align}

for \(t \in [0, T]\) and some positive constants \(P_1, \ldots, P_5\).

**Proof.** Let us consider the change of variable \(\tau = T - t\) for \(t \in [0, T]\) and also consider the notation \(w_i(x, \tau) = p_i(x, T - \tau), \ i = 1, 2, \ S^*(x, \tau) = S(x, T - \tau), \ I^*(x, \tau) = I(x, T - \tau)\).

Then, the adjoint system (2.11)-(2.12) can be rewritten as follows

\begin{align}
(w_1)_\tau - \Delta w_1 &= -m\bar{\beta}(x)(S^*)^{m-1}(I^*)^n(w_1 - w_2), \quad \text{in } Q_T, \\
(w_2)_\tau - \Delta w_2 &= -n\bar{\beta}(x)(S^*)^{m-1}(I^*)^{n-1}(w_1 - w_2) + \tilde{\gamma}(x)(w_1 - w_2), \quad \text{in } Q_T, \\
\nabla w_1 \cdot n &= \nabla w_2 \cdot n = 0, \quad \text{on } \Gamma, \\
(w_1(x, 0) = \vec{S}(x, T) - S^{obs}(x), \quad w_2(x, 0) = \vec{I}(x, T) - I^{obs}(x), \quad \text{in } \Omega.
\end{align}

Now, we get for \(w_i\) the estimates of the form (2.6), (2.7), and (2.9).

In order to prove (2.6), we multiply (2.10) by \(w_1\) and (2.11) by \(w_2\), integrate on \(\Omega\) and use the Green formula, to get

\[
\int_\Omega (w_1)_\tau w_1 \, dx + \int_\Omega (\nabla w_1)^2 \, dx = -m \int_\Omega \bar{\beta}(x)(S^*)^{m-1}(I^*)^n w_1^2 \, dx
\]

\[
+ m \int_\Omega \bar{\beta}(x)(S^*)^{m-1}(I^*)^n w_2 w_1 \, dx,
\]

\[
\int_\Omega (w_2)_\tau w_2 \, dx + \int_\Omega (\nabla w_2)^2 \, dx = - \int_\Omega [n\tilde{\beta}(x)(S^*)^{m}(I^*)^{n-1} - \tilde{\gamma}(x)] w_1 w_2 \, dx
\]

\[
+ \int_\Omega [n\tilde{\beta}(x)(S^*)^{m}(I^*)^{n-1} - \tilde{\gamma}(x)] w_2^2 \, dx,
\]

respectively. Then, adding the equalities, applying the Cauchy inequality, rearranging some terms, and applying the Corollary 2.1, we can deduce the following estimate

\[
\frac{1}{2} \frac{d}{dt} \left( \|w_1(\cdot, \tau)\|_{L^2(\Omega)}^2 + \|w_2(\cdot, \tau)\|_{L^2(\Omega)}^2 \right) + \|\nabla w_1(\cdot, \tau)\|_{L^2(\Omega)}^2 + \|\nabla w_2(\cdot, \tau)\|_{L^2(\Omega)}^2
\]

\[
\leq \hat{C} \left[ \|w_1(\cdot, \tau)\|_{L^2(\Omega)}^2 + \|w_2(\cdot, \tau)\|_{L^2(\Omega)}^2 \right],
\]

with

\[
\hat{C} = \max \left\{ \frac{3\hat{C}_1 + \hat{C}_2}{2}, \frac{\hat{C}_1 + 3\hat{C}_2}{2} \right\}, \quad \hat{C}_1 = \tilde{b} m S_m^{m-1} T_m, \quad \hat{C}_2 = \tilde{b} n S_m^{m} t_n^{m-1} + \tau.
\]

Then, from (2.14) and the Gronwall inequality, we obtain

\[
\|w_1(\cdot, \tau)\|_{L^2(\Omega)}^2 + \|w_2(\cdot, \tau)\|_{L^2(\Omega)}^2 \leq \left( \|w_1(\cdot, 0)\|_{L^2(\Omega)}^2 + \|w_2(\cdot, 0)\|_{L^2(\Omega)}^2 \right) e^{2\hat{C}T},
\]

which implies (2.6).

From (2.14) and (2.10), we have that

\[
\|\nabla w_1(\cdot, \tau)\|_{L^2(\Omega)}^2 + \|\nabla w_2(\cdot, \tau)\|_{L^2(\Omega)}^2 \leq \hat{C} e^{2\hat{C}T} \left( \|w_1(\cdot, 0)\|_{L^2(\Omega)}^2 + \|w_2(\cdot, 0)\|_{L^2(\Omega)}^2 \right)
\]

and by the definition of the norm of \(H^1_0(\Omega)\) we deduce the estimate (2.7).

On the other hand, using the fact that

\[
\int_\Omega (w_i)_\tau \nabla w_i \, dx = - \int_\Omega \nabla [(w_i)_\tau] \cdot \nabla w_i \, dx + \int_{\partial \Omega} (w_i)_\tau \nabla w_i \cdot n \, dS = \frac{1}{2} \frac{d}{dt} \|w_i(\cdot, \tau)\|_{L^2(\Omega)}^2,
\]

for \(i = 1, 2\). We note that, multiplying (2.11) by \(\Delta w_1\), multiplying (2.12) by \(\Delta w_2\), integrating on \(\Omega\), and adding the results, we deduce that

\[
\frac{1}{2} \frac{d}{dt} \left( \|w_1(\cdot, \tau)\|_{L^2(\Omega)}^2 + \|w_2(\cdot, \tau)\|_{L^2(\Omega)}^2 \right) + \|\Delta w_1(\cdot, \tau)\|_{L^2(\Omega)}^2 + \|\Delta w_2(\cdot, \tau)\|_{L^2(\Omega)}^2
\]

\[
\leq \hat{C} \left[ \|w_1(\cdot, \tau)\|_{L^2(\Omega)}^2 + \frac{1}{4\epsilon} \|\Delta w_1(\cdot, \tau)\|_{L^2(\Omega)}^2 + \|w_2(\cdot, \tau)\|_{L^2(\Omega)}^2 + \frac{1}{4\epsilon} \|\Delta w_2(\cdot, \tau)\|_{L^2(\Omega)}^2 \right]
\]

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Lemma 2.2. Necessary Optimality Conditions.

Let \((\tilde{\beta}, \tilde{\gamma})\) be the solution of the optimal control problem \((1.6), (\tilde{S}, \tilde{I})\) be the solution of \((1.1)-(1.5)\) with \((\tilde{\beta}, \tilde{\gamma})\) instead of \((\beta, \gamma)\) and \((p_1, p_2)\) the solution of the adjoint system \((2.1)-(2.4)\). Then, the inequality

\[
\int_{Q_T} \left[ \left( \beta - \tilde{\beta} \right) S^m I^n - \left( \gamma - \tilde{\gamma} \right) \tilde{I} \right] (p_2 - p_1) \, dx 
+ \delta \int_\Omega \left[ \nabla \beta \nabla \left( \beta - \tilde{\beta} \right) + \nabla \gamma \nabla \left( \gamma - \tilde{\gamma} \right) \right] \, dx \geq 0,
\]

is satisfied for any \((\beta, \gamma)\) \in U_{ad}.

Proof. Let us consider an arbitrary pair \((\tilde{\beta}, \tilde{\gamma})\) \in U_{ad} and introduce the notation

\[
\nu \frac{d\nu}{d\varepsilon} |_{\varepsilon=0} = \int_\Omega \left[ \left| S^\varepsilon(x, t) - S^{obs}(x) \right|^2 + \left| I^\varepsilon(x, t) - I^{obs}(x) \right|^2 \right] \, dx 
+ \delta \int_\Omega \left[ \nabla S^\varepsilon \cdot (\beta - \tilde{\beta}) + \nabla I^\varepsilon \cdot (\gamma - \tilde{\gamma}) \right] \, dx \geq 0,
\]

where \(\partial_\varepsilon S^\varepsilon\) and \(\partial_\varepsilon I^\varepsilon\) for \(\varepsilon = 0\) are calculated by analyzing the sensitivities of solutions for \((1.1)-(1.5)\) with respect to perturbations of \((\beta, \gamma)\).

From the definition of \((S^\varepsilon, I^\varepsilon)\) and \((\tilde{S}, \tilde{I})\) we have that

\[
(S^\varepsilon)_t - \Delta S^\varepsilon = -\beta^\varepsilon(x)(S^\varepsilon)^m(I^\varepsilon)^n + \gamma^\varepsilon(x)I^\varepsilon, \quad \text{in } Q_T,
\]

\[
(I^\varepsilon)_t - \Delta I^\varepsilon = \beta^\varepsilon(x)(S^\varepsilon)^m(I^\varepsilon)^n - \gamma^\varepsilon(x)I^\varepsilon, \quad \text{in } Q_T,
\]

\[
\nabla S^\varepsilon \cdot n = \nabla I^\varepsilon \cdot n = 0, \quad \text{on } \Gamma,
\]

\[
S^\varepsilon(x, 0) = S_0(x), \quad I^\varepsilon(x, 0) = I_0(x), \quad \text{in } \Omega,
\]

and

\[
(\tilde{S})_t - \Delta \tilde{S} = -\tilde{\beta}(x)(\tilde{S})^m(\tilde{I})^n + \tilde{\gamma}(x)\tilde{I}, \quad \text{in } Q_T,
\]

\[
(\tilde{I})_t - \Delta \tilde{I} = \tilde{\beta}(x)(\tilde{S})^m(\tilde{I})^n - \tilde{\gamma}(x)\tilde{I}, \quad \text{in } Q_T.
\]
\[ \nabla \vec{S} \cdot \mathbf{n} = \nabla \vec{I} \cdot \mathbf{n} = 0, \quad \text{on } \Gamma, \quad (2.25) \]
\[ \vec{S}(x, 0) = S_0(x), \quad \vec{I}(x, 0) = I_0(x), \quad \text{in } \Omega. \quad (2.26) \]

Subtracting the system (2.23)-(2.26) from (2.19)-(2.22), dividing by \( \varepsilon \) and using the notation \((z_1^\varepsilon, z_2^\varepsilon) = \varepsilon^{-1} (S^\varepsilon - \bar{S}, I^\varepsilon - \bar{I})\), we deduce the following system

\[ (z_1^\varepsilon)_t - \Delta z_1^\varepsilon = -\beta^\varepsilon(x) \frac{[(S^\varepsilon)^m - (\bar{S})^m]}{S^\varepsilon - \bar{S}} (I^\varepsilon)x z_1^\varepsilon - \beta^\varepsilon(x)(\bar{S})^m \frac{[(I^\varepsilon)^n - (\bar{I})^n]}{I^\varepsilon - \bar{I}} z_2^\varepsilon, \quad \text{in } Q_T, \quad (2.27) \]
\[ (z_2^\varepsilon)_t - \Delta z_2^\varepsilon = \beta^\varepsilon(x) \frac{[(S^\varepsilon)^m - (\bar{S})^m]}{S^\varepsilon - \bar{S}} (I^\varepsilon)x z_1^\varepsilon + \beta^\varepsilon(x)(\bar{S})^m \frac{[(I^\varepsilon)^n - (\bar{I})^n]}{I^\varepsilon - \bar{I}} z_2^\varepsilon + (\hat{\beta} - \bar{\beta}) (\bar{S})^n (\bar{I})x z_2^\varepsilon - (\hat{\gamma} - \bar{\gamma}) \bar{I}, \quad \text{in } Q_T, \quad (2.28) \]
\[ \nabla z_1^\varepsilon \cdot \mathbf{n} = \nabla z_2^\varepsilon \cdot \mathbf{n} = 0, \quad \text{on } \Gamma, \quad (2.29) \]
\[ z_1^\varepsilon(x, 0) = z_2^\varepsilon(x, 0) = 0, \quad \text{in } \Omega. \quad (2.30) \]

Then, denoting by \((z_1, z_2)\) the limit of \((z_1^\varepsilon, z_2^\varepsilon)\) when \( \varepsilon \to 0 \), from (2.27)-(2.30), we deduce that

\[ (z_1)_t - \Delta z_1 = -m \beta(x)(\bar{S})^{m-1} (\bar{I})x z_1 - n \beta(x)(\bar{S})^m (\bar{I})^{n-1} z_2 \]
\[ - (\hat{\beta} - \bar{\beta})(\bar{S})^m (\bar{I})^n + \hat{\gamma}(\bar{S})^n (\bar{I})x z_2 - (\hat{\gamma} - \bar{\gamma}) \bar{I}, \quad \text{in } Q_T, \quad (2.31) \]
\[ (z_2)_t - \Delta z_2 = m \beta(x)(\bar{S})^{m-1} (\bar{I})x z_1 + n \beta(x)(\bar{S})^m (\bar{I})^{n-1} z_2 \]
\[ + (\hat{\beta} - \bar{\beta})(\bar{S})^m (\bar{I})^n - \hat{\gamma}(\bar{S})^n (\bar{I})x z_2 - (\hat{\gamma} - \bar{\gamma}) \bar{I}, \quad \text{in } Q_T, \quad (2.32) \]
\[ \nabla z_1 \cdot \mathbf{n} = \nabla z_2 \cdot \mathbf{n} = 0, \quad \text{on } \Gamma, \quad (2.33) \]
\[ z_1(x, 0) = z_2(x, 0) = 0, \quad \text{in } \Omega. \quad (2.34) \]

Thus, in (2.18) we have that

\[ \left. \frac{dJ_\varepsilon}{d\varepsilon} \right|_{\varepsilon=0} = \int_\Omega \left( |S^\varepsilon(\cdot, t) - S^{obs}(\cdot, t)| z_1(\cdot, t) + |I^\varepsilon(\cdot, t) - I^{obs}(\cdot, t)| z_2(\cdot, t) \right) dx \]
\[ + \delta \int_\Omega \left[ \nabla \beta \nabla (\hat{\beta} - \bar{\beta}) + \nabla \gamma \nabla (\hat{\gamma} - \bar{\gamma}) \right] dx \geq 0, \quad (2.35) \]

where \((z_1, z_2)\) is the solution of (2.31)-(2.34).

On the other hand, from (2.1)-(2.2) and (2.31)-(2.32), we deduce that

\[ \frac{\partial}{\partial t}(p_1 z_1 + p_2 z_2) = p_1 \Delta z_1 + p_2 \Delta z_2 - z_1 \Delta p_1 - z_2 \Delta p_2 + (\hat{\beta} - \bar{\beta}) \bar{S}^m \bar{I}^n (p_2 - p_1) - (\hat{\gamma} - \bar{\gamma}) \bar{I} (p_2 - p_1), \]

which implies that

\[ \int_{Q_T} \frac{\partial}{\partial t}(p_1 z_1 + p_2 z_2) dx dt = \int_{Q_T} (\hat{\beta} - \bar{\beta}) \bar{S}^m \bar{I}^n (p_2 - p_1) dx dt, \quad (2.36) \]

by integration on \( Q_T \). Moreover, we notice that

\[ \int_{Q_T} \frac{\partial}{\partial t}(p_1 z_1 + p_2 z_2) dx dt = \int_{Q_T} \left( p_1(x, T) z_1(x, T) + p_2(x, T) z_2(x, T) \right) dx \]
\[ = \int_{\Omega} \left( |\bar{S}(x, T) - S^{obs}(x)| z_1(x, T) + |\bar{I}(x, T) - I^{obs}(x)| z_2(x, T) \right) dx. \quad (2.37) \]

Then, from (2.36) and (2.37) we deduce that

\[ \int_{Q_T} (\hat{\beta} - \bar{\beta}) \bar{S}^m \bar{I}^n - (\hat{\gamma} - \bar{\gamma}) \bar{I} (p_2 - p_1) dx dt \]
\[ = \int_{\Omega} \left( |\bar{S}(x, T) - S^{obs}(x)| z_1(x, T) + |\bar{I}(x, T) - I^{obs}(x)| z_2(x, T) \right) dx. \quad (2.38) \]

We can conclude the proof of (2.17) by replacing (2.28) in the first term of (2.38). \qed
2.4. Some stability results.

Lemma 2.3. Consider that the sets of functions \{S, I, p_1, p_2\} and \{\hat{S}, \hat{I}, \hat{p}_1, \hat{p}_2\} are solutions to the systems (1.1)-(1.5) and (2.4)-(2.5) with the data \{\beta, \gamma, S_{obs}, I_{obs}\} and \{\beta, \gamma, \hat{S}_{obs}, \hat{I}_{obs}\}, respectively. Then, there exist the positive constants \Psi_1, i = 1, 2, 3 such that the estimates

\[
\| (\hat{S} - S)(\cdot, t) \|^2_{L^2(\Omega)} + \| (\hat{I} - I)(\cdot, t) \|^2_{L^2(\Omega)} \leq \Psi_1 \left( \| \hat{\beta} - \beta \|^2_{L^2(\Omega)} + \| \hat{\gamma} - \gamma \|^2_{L^2(\Omega)} \right),
\]

\[
\| (\hat{p}_1 - p_1)(\cdot, t) \|^2_{L^2(\Omega)} + \| (\hat{p}_2 - p_2)(\cdot, t) \|^2_{L^2(\Omega)} \leq \Psi_2 \left( \| \hat{\beta} - \beta \|^2_{L^2(\Omega)} + \| \hat{\gamma} - \gamma \|^2_{L^2(\Omega)} \right)
\]

\[
+ \Psi_3 \left( \| \hat{S}_{obs} - S_{obs} \|^2_{L^2(\Omega)} + \| \hat{I}_{obs} - I_{obs} \|^2_{L^2(\Omega)} \right)
\]

holds for any \( t \in [0, T] \).

Proof. For the sake of simplicity of the presentation, we introduce the following notations

\[
\delta S = \hat{S} - S, \quad \delta I = \hat{I} - I, \quad \delta p_1 = \hat{p}_1 - p_1, \quad \delta p_2 = \hat{p}_2 - p_2, \quad \delta \beta = \hat{\beta} - \beta, \quad \delta \gamma = \hat{\gamma} - \gamma.
\]

Then, from the system (1.1)-(1.5) for (S, I) and (\hat{S}, \hat{I}) we have that (\delta S, \delta I) satisfy the system

\[
(\delta S)_t - \Delta (\delta S) = -\hat{\beta}(x) \left[ (\hat{S})^m (\hat{I})^n - (S)^m (I)^n \right]
\]

\[
- \delta \beta(x)(\hat{S})^m (\hat{I})^n + \delta \gamma(x) \delta I + \delta \gamma(x) I, \quad \text{in } Q_T, \quad (2.41)
\]

\[
(\delta I)_t - \Delta (\delta I) = \hat{\beta}(x) \left[ (\hat{S})^m (\hat{I})^n - (S)^m (I)^n \right]
\]

\[
+ \delta \beta(x)(\hat{S})^m (\hat{I})^n - \delta \gamma(x) \delta I - \delta \gamma(x) I, \quad \text{in } Q_T, \quad (2.42)
\]

\[
\nabla (\delta S) \cdot n = \nabla (\delta I) \cdot n = 0, \quad \text{on } \Gamma, \quad (2.43)
\]

\[
(\delta S)(x, 0) = (\delta I)(x, 0) = 0, \quad \text{in } \Omega. \quad (2.44)
\]

Similarly, from the adjoint system (2.4)-(2.5), we deduce that (\delta p_1, \delta p_2) is solution of the system

\[
(\delta p_1)_t + \Delta (\delta p_1) = m \hat{\beta}(x)(\hat{S})^{m-1} (\hat{I})^n (\hat{p}_1 - p_2) - m \beta(x)(S)^{m-1} (I)^n (p_1 - p_2), \quad \text{in } Q_T, \quad (2.45)
\]

\[
(\delta p_2)_t + \Delta (\delta p_2) = n \hat{\beta}(x)(\hat{S})^m (\hat{I})^{n-1} (\hat{p}_1 - p_2) - \hat{\gamma}(x)(\hat{p}_1 - p_2)
\]

\[
- n \beta(x)S^m I^{n-1} (p_1 - p_2) + \gamma(x)(p_1 - p_2), \quad \text{in } Q_T, \quad (2.46)
\]

\[
\nabla (\delta p_1) \cdot n = \nabla (\delta p_2) \cdot n = 0, \quad \text{on } \Gamma, \quad (2.47)
\]

\[
(\delta p_1)(x, T) = \delta S(x, T) - \left( \hat{S}_{obs}(x) - S_{obs}(x) \right), \quad \text{in } \Omega, \quad (2.48)
\]

\[
(\delta p_2)(x, T) = \delta I(x, T) - \left( \hat{I}_{obs}(x) - I_{obs}(x) \right), \quad \text{in } \Omega. \quad (2.49)
\]

Then, the proofs of (2.39) and (2.40) are reduced to get estimations for the systems (2.41)-(2.44) and (2.45)-(2.49), respectively.

In order to prove (2.39), we test the equations (2.41) and (2.42) by \delta S and \delta I, respectively. Then, adding the results we get

\[
\frac{1}{2} \frac{d}{dt} \left( \| \delta S(\cdot, t) \|^2_{L^2(\Omega)} + \| \delta I(\cdot, t) \|^2_{L^2(\Omega)} \right) + \| \nabla (\delta S)(\cdot, t) \|^2_{L^2(\Omega)} + \| \nabla (\delta I)(\cdot, t) \|^2_{L^2(\Omega)}
\]

\[
\leq \int_{\Omega} |\hat{\beta}(x)||\hat{S}^m \hat{I}^n - S^m I^n| \delta S \, dx + \int_{\Omega} |\delta \beta(x)||\hat{S}^m \hat{I}^n| \delta S \, dx + \int_{\Omega} |\delta \gamma(x)||\delta I||\delta S| \, dx
\]

\[
+ \int_{\Omega} |\hat{\beta}(x)||I||\delta S| \, dx + \int_{\Omega} |\delta \beta(x)||\hat{S}^m \hat{I}^n - S^m I^n| \delta I \, dx + \int_{\Omega} |\delta \beta(x)||\hat{S}^m \hat{I}^n| \delta I \, dx
\]

\[
+ \int_{\Omega} |\gamma(x)||\delta I|^2 \, dx + \int_{\Omega} |\delta \gamma(x)||I||\delta S| \, dx
\]

\[
= \sum_{j=1}^{8} I_j, \quad (2.50)
\]
where $I_j$ are defined by each term. Now, using the Corollary 2.1 to get that

$$|\hat{S}^m \hat{I}^n - S^m I^n| = |\hat{S}^m \hat{I}^n - \hat{S}^m I^n + \hat{S}^m I^n - S^m I^n|$$

$$= |\hat{S}^m \int_I u^{n-1} du + I^n m \int_S u^{m-1} du|$$

$$\leq n|\hat{S}| \int_I I^{n-1} du + m|I^n| \int_S S^{m-1} du,$$

$$\leq n S^m M_n |I^n - I| + m S^{m-1} |S - S|,$$  \hspace{1cm} (2.51)

we proceed to get the appropriate bounds for $I_j$. Indeed, by the Cauchy inequality and (2.51), we have that $I_1$ can be bounded as follows

$$I_1 \leq \frac{n b S_{M_n}^{m-1}}{2} \left( \int_\Omega |\delta I|^2 d\mathbf{x} + \int_\Omega |\delta S|^2 d\mathbf{x} \right) + m \frac{b S_{m-1}^{m-1}}{2} \int_\Omega |\delta S|^2 d\mathbf{x}.$$  \hspace{1cm} (2.52)

In the case of $I_2, I_3$ and $I_4$, we get

$$I_2 \leq \frac{1}{2} \frac{b S_{M_n}^{m-1}}{S_{M_n}^{m-1}} \left( \int_\Omega |\delta \beta|^2 d\mathbf{x} + \int_\Omega |\delta S|^2 d\mathbf{x} \right),$$

$$I_3 \leq \frac{\tau}{2} \left( \int_\Omega |\delta I|^2 d\mathbf{x} + \int_\Omega |\delta S|^2 d\mathbf{x} \right),$$

$$I_4 \leq \frac{1}{2} \left( \int_\Omega |\delta \gamma|^2 d\mathbf{x} + \int_\Omega |\delta S|^2 d\mathbf{x} \right).$$

Similarly, we deduce that

$$I_5 \leq n \frac{b S_{M_n}^{m-1}}{2} \left( \int_\Omega |\delta I|^2 d\mathbf{x} + \int_\Omega |\delta S|^2 d\mathbf{x} \right) + m \frac{b S_{m-1}^{m-1}}{2} \int_\Omega |\delta S|^2 d\mathbf{x},$$

$$I_6 \leq \frac{1}{2} \frac{S_{M_n}^{m-1}}{S_{M_n}^{m-1}} \left( \int_\Omega |\delta \beta|^2 d\mathbf{x} + \int_\Omega |\delta I|^2 d\mathbf{x} \right),$$

$$I_7 \leq \tau \int_\Omega |\delta I|^2 d\mathbf{x},$$

$$I_8 \leq \frac{1}{2} \|S\|_{L^2(\Omega)} \left( \int_\Omega |\delta \gamma|^2 d\mathbf{x} + \int_\Omega |\delta I|^2 d\mathbf{x} \right).$$

Thus, from the estimates of $I_j$ and (2.50) we have that

$$\frac{d}{d t} \left( \|\delta S(\cdot), t\|_{L^2(\Omega)}^2 + \|\delta I(\cdot), t\|_{L^2(\Omega)}^2 \right) + 2 \left( \|\nabla(\delta S)(\cdot), t\|_{L^2(\Omega)}^2 + \|\nabla(\delta I)(\cdot), t\|_{L^2(\Omega)}^2 \right)$$

$$\leq D_1 \left( \|\delta S(t), t\|_{L^2(\Omega)}^2 + \|\delta I(t), t\|_{L^2(\Omega)}^2 \right) + D_2 \left( \|\delta \beta(t), t\|_{L^2(\Omega)}^2 + \|\delta \gamma(t), t\|_{L^2(\Omega)}^2 \right),$$

where $D_1 = 2 \hat{C} + \|S\|_{L^2(\Omega)}^2 + \|S\|_{L^2(\Omega)}^2$. Then, applying the Gronwall inequality, we deduce that

$$\|\delta S(\cdot), t\|_{L^2(\Omega)}^2 + \|\delta I(\cdot), t\|_{L^2(\Omega)}^2 \leq e^{D_1 T} \left( \|\delta S_0\|_{L^2(\Omega)}^2 + \|\delta I_0\|_{L^2(\Omega)}^2 \right) + D_2 T \left( \|\delta \beta\|_{L^2(\Omega)}^2 + \|\delta \gamma\|_{L^2(\Omega)}^2 \right),$$

which implies (2.39) by using (2.43).

The proof of (2.39) is developed as follows. We can easily prove that the algebraic identity

$$\hat{\zeta} \hat{A} (\hat{p}_1 - \hat{p}_2) - \zeta A (p_1 - p_2)$$

$$= \left( \hat{\zeta} - \zeta \right) \hat{A} p_1 + \zeta \left( \hat{A} - A \right) p_1 - \left( \hat{\zeta} - \zeta \right) \hat{A} \delta p_1 - \zeta \left( \hat{A} - A \right) \delta p_2 - \zeta \delta p_2$$

is valid. Now, in particular, by selecting $(\hat{\zeta}, \zeta, \hat{A}, A) = \left( \hat{\beta}, \beta, m(\hat{S})^{m-1}(\hat{I})^n, m(S)^{m-1}(I)^n \right)$, we have that (2.52) implies that the right hand sides of equation (2.45) can be rewritten as follows

$$m \beta \left( \hat{S}^{m-1}(\hat{I})^n (\hat{p}_1 - \hat{p}_2) - m \beta \left( S^{m-1}(I)^n \right) (p_1 - p_2) \right)$$

$$= m \delta \beta \left( \hat{S}^{m-1}(\hat{I})^n \hat{p}_1 + m \beta \left( \hat{S}^{m-1}(\hat{I})^n - (S)^{m-1}(I)^n \right) \hat{p}_1 \right.$$

$$+ m \beta \left( S^{m-1}(I)^n \delta p_1 - \delta \beta \left( S^{m-1}(I)^n \right) \delta p_2 \right.$$}

$$- m \beta \left[ \left( \hat{S}^{m-1}(\hat{I})^n - (S)^{m-1}(I)^n \right) \delta p_2 - m \beta \left( S^{m-1}(I)^n \right) \delta p_2 \right].$$

(2.53)
Then, by testing \( (2.46) \) by \( \delta p_1 \) and using \((2.53)\), we get
\[
\frac{1}{2} \frac{d}{dt} \|\delta p_1(\cdot, t)\|_{L^2(\Omega)}^2 = \|\nabla (\delta p_1)(\cdot, t)\|_{L^2(\Omega)}^2 + \int_\Omega m \beta (\hat{S})^{-m-1}(\hat{I})^n \delta_{\hat{I}}^1 \delta p_1 \, dx + \int_\Omega m \beta \left[ (\hat{S})^{-m-1}(\hat{I})^n - (S)^{-m-1}(I)^n \right]^2 \delta_{\hat{I}}^1 \delta p_1 \, dx
\]
\[
+ \int_\Omega m \beta (S)^{-m-1}(I)^n (\delta p_1)^2 \, dx - \int_\Omega m \beta (\hat{S})^{-m-1}(\hat{I})^n \hat{p}_2 \delta p_1 \, dx - \int_\Omega m \beta (S)^{-m-1}(I)^n \delta p_1 \delta p_2 \, dx.
\]
From Lemma 2.1 and Corollary 2.1 by using similar arguments to \((2.51)\), and the Cauchy inequality we have that
\[
-\frac{1}{2} \frac{d}{dt} \|\delta p_1(\cdot, t)\|_{L^2(\Omega)}^2 + \|\nabla \delta p_1(\cdot, t)\|_{L^2(\Omega)}^2 \leq \max \{ P_4, P_5 \} \left( m S_m^{-1} \| \delta p_1(\cdot, t) \|_{L^2(\Omega)}^2 + \| \delta \beta \|_{L^2(\Omega)}^2 \right)
\]
\[
+ m \beta \delta S_m^{-1} \| \delta p_1(\cdot, t) \|_{L^2(\Omega)}^2 + \| \delta I(\cdot, t) \|_{L^2(\Omega)}^2
\]
\[
+ m |m| - 1 \| \delta S_m^{-2} \|_{L^2(\Omega)}^2 \| \delta p_1(\cdot, t) \|_{L^2(\Omega)}^2 + \| \delta S(\cdot, t) \|_{L^2(\Omega)}^2 \}
\]
\[
+ \frac{m \beta}{2} S_m^{-1} \| \delta p_1(\cdot, t) \|_{L^2(\Omega)}^2 + \| \delta p_2(\cdot, t) \|_{L^2(\Omega)}^2 \).
\] (2.54)

Now, from \((2.62)\), by selecting \((\hat{\zeta}, \hat{\zeta}, \hat{\alpha}, \hat{\alpha}) = \left( \beta, \beta, n(\hat{S})^{-m-1}, n(S)^{-m}(I)^{-n} \right)\) and \((\hat{\zeta}, \hat{\zeta}, \hat{\alpha}, \hat{\alpha}) = \left( \gamma, \gamma, 1, 1 \right)\), we can rewritten the right hand side of equation \((2.46)\). Then, testing \((2.45)\) by \( \delta p_2 \) and using similar arguments we get a similar estimate to \((2.54)\). Thus, we have that there exist the positive constants \(E_i, i = 1, 2, 3\), such that
\[
-\frac{d}{dt} \left( \| \delta p_1(\cdot, t) \|_{L^2(\Omega)}^2 \right) + \| \nabla \delta p_2(\cdot, t) \|_{L^2(\Omega)}^2 \leq E_1 \left( \| \delta p_1(\cdot, t) \|_{L^2(\Omega)}^2 + \| \delta p_2(\cdot, t) \|_{L^2(\Omega)}^2 \right)
\]
\[
+ E_2 \left( \| \delta S(\cdot, t) \|_{L^2(\Omega)}^2 + \| \delta I(\cdot, t) \|_{L^2(\Omega)}^2 \right) + E_3 \left( \| \delta \beta \|_{L^2(\Omega)}^2 + \| \delta \gamma \|_{L^2(\Omega)}^2 \right).
\]

Applying the estimate \((2.39)\) and rearranging some terms we deduce that
\[
-\frac{d}{dt} \left( e^{-\int_1^T} \left[ \| \delta p_1(\cdot, t) \|_{L^2(\Omega)}^2 + \| \delta p_2(\cdot, t) \|_{L^2(\Omega)}^2 \right] \right) \leq (E_2 \Psi_1 + E_3) \left( \| \delta \beta \|_{L^2(\Omega)}^2 + \| \delta \gamma \|_{L^2(\Omega)}^2 \right),
\]
and integrating on \([t, T]\) we have that
\[
e^{-\int_1^T} \left[ \| \delta p_1(\cdot, t) \|_{L^2(\Omega)}^2 + \| \delta p_2(\cdot, t) \|_{L^2(\Omega)}^2 \right] \leq e^{E_1 T} \left[ \| \delta p_1(\cdot, T) \|_{L^2(\Omega)}^2 + \| \delta p_2(\cdot, T) \|_{L^2(\Omega)}^2 \right]
\]
\[
+ T (E_2 \Psi_1 + E_3) e^{C_1 T} \left( \| \delta \beta \|_{L^2(\Omega)}^2 + \| \delta \gamma \|_{L^2(\Omega)}^2 \right).
\]

Hence, we can deduce \((2.40)\) by application of the end condition \((2.49)\). □

3. Proof of Theorem 1.1

Existence. We can prove the existence by considering the standard strategy of a minimizing sequence and using the appropriate compactness inclusions. Indeed, we clearly note that \(U_{ad}(\Omega) \neq \emptyset\) and \(J(\beta, \gamma)\) is bounded for any \((\beta, \gamma) \in U_{ad}(\Omega)\). Then, we can consider that \((\beta_n, \gamma_n)\) is a minimizing sequence of \(J\). Then, the compact embedding \(H^{[d/2] + 1}(\Omega) \subset C^\alpha(\Omega)\) for \(\alpha \in [0, 1/2]\), implies that the minimizing sequence \((\beta_n, \gamma_n)\) is bounded in the strong topology of \(C^\alpha(\Omega) \times C^\alpha(\Omega)\) for all \(\alpha \in [0, 1/2]\), since there exists a positive constant \(C\) (independent of \(\beta, \gamma\) and \(n\)) such that
\[
\| \beta_n \|_{C^\alpha(\Omega)} + \| \gamma_n \|_{C^\alpha(\Omega)} \leq C \left( \| \beta_n \|_{H^{[d/2] + 1}(\Omega)} + \| \gamma_n \|_{H^{[d/2] + 1}(\Omega)} \right), \quad \forall \alpha \in [0, 1/2].
\]
Notice that the right hand is bounded by the fact that $\beta_n, \gamma_n \in H^{d/2+1}(\Omega)$, see the definition of $U_{ad}(\Omega)$ given on (1.8). Now, let us denote by $(S_n, I_n)$ the solution of the initial boundary value problem (1.1)-(1.5) corresponding to $(\beta_n, \gamma_n)$. Then, by considering the fact that $\{(\beta_n, \gamma_n)\}$ is belong to $C^\alpha(\Omega) \times C^\alpha(\Omega)$ for all $\alpha \in [0, 1/2]$, by Theorem 2.11 we have that $S_n$ and $I_n$ are belong to the Hölder space $C^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega)$ and also $\{(S_n, I_n)\}$ is a bounded sequence in the strong topology of $C^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega)$ for all $\alpha \in [0, 1/2]$. Thus, the boundedness of the minimizing sequence and the corresponding sequence $(S_n, I_n)$, implies that there exist

$$
(\bar{\beta}, \bar{\gamma}) \in \left[ C^{1/2}(\Omega) \times C^{1/2}(\Omega) \right] \cap U_{ad}(\Omega),
$$

and the subsequences again labeled by $\{(\beta_n, \gamma_n)\}$ and $\{(S_n, I_n)\}$ such that

$$
\begin{align*}
\beta_n & \to \bar{\beta}, \quad \gamma_n \to \bar{\gamma} \quad \text{uniformly on } C^\alpha(\Omega), \\
S_n & \to \bar{S}, \quad I_n \to \bar{T} \quad \text{uniformly on } C^{\alpha/2}(\Omega) \cap C^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega),
\end{align*}
$$

Moreover, we can deduce that $(\bar{S}, \bar{T})$ is the solution of the initial boundary value problem (1.1)-(1.5) corresponding to the coefficients $(\bar{\beta}, \bar{\gamma})$. Hence, by Lebesgue’s dominated convergence theorem, the weak lower-semicontinuity of $L^2$ norm, and the definition of the minimizing sequence, we have that

$$
J(\beta, \gamma) \leq \lim_{n \to \infty} J(\beta_n, \gamma_n) = \inf_{(\beta, \gamma) \in U_{ad}(\Omega)} J(\beta, \gamma).
$$

Then, $(\bar{\beta}, \bar{\gamma})$ is a solution of (1.6) and the prove of existence is concluded.

**Uniqueness.** We prove the uniqueness by using adequately the stability result of Lemma 2.18 and the necessary optimality condition of Lemma 2.17. To be more precise, let us consider that the sets of functions $\{S, I, p_1, p_2\}$ and $\{\bar{S}, \bar{I}, \bar{p}_1, \bar{p}_2\}$ are solutions to the systems (1.1)-(1.5) and (1.1)-(1.5) with the data $\{\beta, \gamma, S_{obs}, I_{obs}\}$ and $\{\bar{\beta}, \bar{\gamma}, \bar{S}_{obs}, \bar{I}_{obs}\}$, respectively. From Lemma 2.17 and the hypothesis that $(\beta, \gamma)$ and $\{(\bar{\beta}, \bar{\gamma})\}$ are solutions of (1.6), we have that the following inequalities

$$
\begin{align*}
\int_{Q_T} \left[ (\bar{\beta} - \beta) S^m I^n - (\bar{\gamma} - \gamma) I \right] (p_2 - p_1) \, dx \, dt \\
+ \delta \int_{\Omega} \left[ \nabla \beta \nabla (\bar{\beta} - \beta) + \nabla \gamma \nabla (\bar{\gamma} - \gamma) \right] \, dx \geq 0, \quad \forall (\beta, \gamma) \in U_{ad},
\end{align*}
$$

$$
\begin{align*}
\int_{Q_T} \left[ (\bar{\beta} - \beta) \hat{S}^n I^n - (\bar{\gamma} - \gamma) \hat{I} \right] (\hat{p}_2 - \hat{p}_1) \, dx \, dt \\
+ \delta \int_{\Omega} \left[ \nabla \beta \nabla (\bar{\beta} - \beta) + \nabla \gamma \nabla (\bar{\gamma} - \gamma) \right] \, dx \geq 0, \quad \forall (\beta, \gamma) \in U_{ad},
\end{align*}
$$

are satisfied, respectively. In particular, selecting $(\hat{\beta}, \hat{\gamma}) = (\bar{\beta}, \bar{\gamma})$ in (3.4) and $(\hat{\beta}, \hat{\gamma}) = (\beta, \gamma)$ in (3.5), and adding both inequalities, we get

$$
\delta \left[ \| \nabla (\bar{\beta} - \beta) \|_{L^2(\Omega)}^2 + \| \nabla (\bar{\gamma} - \gamma) \|_{L^2(\Omega)}^2 \right] \leq \int_{Q_T} \left| \bar{\beta} - \beta \right| \left| \hat{S}^n I^n (\hat{p}_2 - \hat{p}_1) - S^m I^n (p_2 - p_1) \right| \, dx \, dt
$$

$$
+ \int_{Q_T} \left| \bar{\gamma} - \gamma \right| \left| \hat{I} (\hat{p}_2 - \hat{p}_1) - I (p_2 - p_1) \right| \, dx \, dt := I_1 + I_2.
$$

Now, from (2.51), (2.52), Corollary 2.1, Lemma 2.1 and the Cauchy inequality, we observe that

$$
I_1 \leq \int_{Q_T} |\beta - \bar{\beta}| \left| \hat{S}^n I^n - S^m I^n \right| ||p_1|| \, dx \, dt + \int_{Q_T} |\beta - \bar{\beta}| \left| \hat{S}^n I^n - S^m I^n \right| ||p_2|| \, dx \, dt
$$

$$
+ \int_{Q_T} |\beta - \bar{\beta}| \left| S^m I^n \right| ||p_1 - p_1|| \, dx \, dt + \int_{Q_T} |\beta - \bar{\beta}| \left| S^m I^n \right| ||p_2 - p_2|| \, dx \, dt
$$

$$
\leq \frac{\nu}{2} \frac{c_{M} m_{M}}{m_{M} - 1} \max \left\{ P_1, P_2 \right\} \left( T \| \bar{\beta} - \beta \|_{L^2(\Omega)}^2 + \int_0^T \| \hat{I}(\cdot, t) - I(\cdot, t) \|_{L^2(\Omega)}^2 \, dt \right)\right)
\[
\begin{aligned}
&+ \frac{m}{2} \| S \|^2_{L^2(\Omega)} \max \left\{ P_4, P_5 \right\} \left( T \| \hat{\beta} - \beta \|^2_{L^2(\Omega)} + \int_0^T \| \hat{S}(\cdot, t) - S(\cdot, t) \|^2_{L^2(\Omega)} dt \right) \\
&+ \frac{m}{2} m^{m-1}_M \left( 2T \| \hat{\beta} - \beta \|^2_{L^2(\Omega)} + \int_0^T \| \hat{p}_1 - p_1(\cdot, t) \|^2_{L^2(\Omega)} dt + \int_0^T \| \hat{p}_2 - p_2(\cdot, t) \|^2_{L^2(\Omega)} dt \right)
\end{aligned}
\]

and

\[
I_2 \leq \max \left\{ P_4, P_5 \right\} \left( T \| \hat{\gamma} - \gamma \|^2_{L^2(\Omega)} + \int_0^T \| \hat{I}(\cdot, t) - I(\cdot, t) \|^2_{L^2(\Omega)} dt \right) + \| M \| T \| \hat{\gamma} - \gamma \|^2_{L^2(\Omega)} + \int_0^T \| \hat{p}_1 - p_1(\cdot, t) \|^2_{L^2(\Omega)} dt \right).
\]

From Lemma 2.3 and the estimates of \( I_1 \) and \( I_2 \) in (3.6), we have that

\[
\delta \left[ \| \nabla (\hat{\beta} - \beta) \|^2_{L^2(\Omega)} + \| \nabla (\hat{\gamma} - \gamma) \|^2_{L^2(\Omega)} \right] 
\leq \Upsilon_1 \left[ \| \hat{\beta} - \beta \|^2_{L^2(\Omega)} + \| \hat{\gamma} - \gamma \|^2_{L^2(\Omega)} \right] + \Upsilon_2 \left[ \| \hat{S}^{\text{obs}} - S^{\text{obs}} \|^2_{L^2(\Omega)} + \| \hat{I}^{\text{obs}} - I^{\text{obs}} \|^2_{L^2(\Omega)} \right],
\]

where

\[
\Upsilon_1 = \left( \frac{m}{2} m^{m-1}_M + \frac{m}{2} m^{m-1}_{m-1} M + 1 \right) (1 + \Psi_1) \max \left\{ P_4, P_5 \right\} + \left( \frac{m}{2} m^{m} + M + M \right) (1 + \Psi_2) T,
\]

\[
\Upsilon_2 = \left( \frac{m}{2} m^{m} + M \right) \Psi_3 T.
\]

Now, considering that \((\hat{\beta}, \hat{\gamma}), (\beta, \gamma) \in U(\Omega)\), by the generalized Poincaré inequality, we have that

\[
\| \hat{\beta} - \beta \|^2_{L^2(\Omega)} + \| \hat{\gamma} - \gamma \|^2_{L^2(\Omega)} 
\leq C_{\text{pois}} \left( \| \nabla (\hat{\beta} - \beta) \|^2_{L^2(\Omega)} + \| \nabla (\hat{\gamma} - \gamma) \|^2_{L^2(\Omega)} \right) + \| \hat{\beta} - \beta \|^2_{L^2(\Omega)} + \| \hat{\gamma} - \gamma \|^2_{L^2(\Omega)}
= C_{\text{pois}} \left( \| \nabla (\hat{\beta} - \beta) \|^2_{L^2(\Omega)} + \| \nabla (\hat{\gamma} - \gamma) \|^2_{L^2(\Omega)} \right).
\]

Then, in (3.7), we have that

\[
\left( \delta - \Upsilon_2 C_{\text{pois}} \right) \left[ \| \nabla (\hat{\beta} - \beta) \|^2_{L^2(\Omega)} + \| \nabla (\hat{\gamma} - \gamma) \|^2_{L^2(\Omega)} \right] \leq \Upsilon_2 \left[ \| \hat{S}^{\text{obs}} - S^{\text{obs}} \|^2_{L^2(\Omega)} + \| \hat{I}^{\text{obs}} - I^{\text{obs}} \|^2_{L^2(\Omega)} \right].
\]

Thus, selecting \( \Theta = \Upsilon_2 C_{\text{pois}} \) we deduce the uniqueness up an additive constant.

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