A weighted planar stochastic lattice with scale-free, small-world and multifractal properties

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We investigate a class of weighted planar stochastic lattice (WPSL1) created by random sequential nucleation of seed from which a crack is grown parallel to one of the sides of the chosen block and ceases to grow upon hitting another crack. It results in the partitioning of the square into contiguous and non-overlapping blocks. Interestingly, we find that the dynamics of WPSL1 is governed by infinitely many conservation laws and each of the conserved quantities, except the trivial conservation of total mass or area, is a multifractal measure. On the other hand, the dual of the lattice is a scale-free network as its degree distribution exhibits a power-law $P(k) \sim k^{-\gamma}$ with $\gamma = 4.13$. The network is also a small-world network as we find that (i) the total clustering coefficient $C$ is high and independent of the network size and (ii) the mean geodesic path length grows logarithmically with $N$. Besides, the clustering coefficient $C_k$ of the nodes which have degree $k$ decreases exactly as $2/(k-1)$ revealing that it is also a nested hierarchical network.

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I. INTRODUCTION

Structures, especially lattice-like structures, are important in physics as we often use them as a backbone while solving theoretical models to gain deeper insight into various physical phenomena. These structures, however, must have some universal properties such as the number of nearest neighbors, next nearest neighbors, symmetry etc. which are characteristic hallmarks of a given structure. The success of a theoretical model often depends not only on how good the model itself is but also on the properties of the backbone on which the model is applied. For instance, we use regular and ordered structures like Bravais lattice, honeycomb lattice, triangular lattice etc. as a backbone to solve models that help us knowing the properties of matter since life-less atoms assemble themselves in a regular and ordered structure in the solid-state phase. However, there are also models to study the spreading of contagious diseases, rumors, information, biological and computer virus that occur not through regular structure rather through random or scale-free and/or small-world network. Recently, topological network structures have been gaining popularity since many biologically motivated physics models on these structures are studied more than ever before [1, 2]. In fact, we are now living in a world which is facing a pandemic due to COVID-19 and it has resulted in a surge of research interest in epidemiology [3–8]. Thus, scale-free, small-world network as a backbone is more relevant now than ever.

Prior to 1998, topological network as a structure and its properties have been studied only by mathematicians. Two seminal works on scale-free and small-world network in the late 1990s have revolutionized the notion of network [9, 10]. Soon scientists have found that most real-life networks through which disease, rumor, messages, viruses etc. spread are neither completely regular nor completely random but rather scale-free in nature and often possess small-world properties too [11, 12]. One of the characteristics of nodes in the topological network is their degree $k$ defined by the number of other nodes to whom a given node is connected. In the case where nodes in the network have great many different degree $k$, it is often worth investigating the degree distribution $P(k)$ which quantifies the probability that a node picked at random have degree $k$. The hallmark of the Erdős-Rényi (ER) model is that $P(k)$ follows Poisson distribution suggesting that it is almost impossible to find nodes that have significantly higher or fewer links than the average degree $\langle k \rangle$. On the other hand, scale-free network exhibits a power-law $P(k) \sim k^{-\gamma}$ with exponent $\gamma$ whose value is mostly found approximately within $2$ and $4$ [13–18]. The mean geodesic path length $l$, the average of the shortest paths among all the connected pairs of nodes, is yet another quantity that characterizes the long-range properties of the network [19]. Watts and Strogatz in 1998 introduced the idea of clustering coefficient which can be considered as yet another characteristic of nodes in the network [10]. The clustering coefficient $C_i$ of a node $i$ is defined as the ratio of the number of edges $e_i$ that actually exist among its $k_i$ neighbors to the maximum possible edges $z_i = k_i(k_i - 1)/2$ that could exist among the same $k_i$ neighbors. The clustering co-efficient $C$ of the whole network is then obtained by averaging it over the entire network.

Earlier in 2010 we proposed a weighted planar stochastic lattice (WPSL), which we now regard as WPSL2, and have shown that despite the coordination number disorder and the block size disorder it still exhibits some order [20, 21]. Clearly, the construction process of WPSL2 would result in a fractal if one of the blocks with certain rules were removed from the system like in the construction of Cantor set or in Sierpinsky carpet [22]. In one dimension, such systems are fractal and exhibit simple scaling but in higher dimensions they are multifractal and exhibit multiscaling [23, 24]. Recently, we have...
shown that the emergence of stochastic fractal is also accompanied by a conservation law which is reminiscent of Noether’s theorem \[25, 26\]. This is also true for multifractal. In the case of WPSL2, it is governed by infinitely many conservation laws and each non-trivial conserved quantity namely conservation of total area, if the value of \(\mu_i\) is distributed in the \(i\)th block then the total content \(M = \sum_i \mu_i\) is unevenly distributed across the square (initiator). We have shown that such uneven distribution can be best quantified as multifractal. The support is a two dimensional Euclidian space where the conserved quantities are distributed. Each of the non-trivial conserved quantities gives rise to multifractal spectrum. The strongest criticism about that model was, however, that the exponent of the degree distribution \(\gamma\) was far too high compared to the most empirical data or theoretical model has so far predicted. Besides, we do not yet know whether it possesses small-world properties or not. The impact of such scale-free and small-world properties has already been proved to leave its signature through distinct results which were otherwise unexpected \[27–29\].

In this article, we propose a close variant of the weighted planar stochastic lattice. Like WPSL2, it too describes the partition of a square into contiguous and mutually exclusive rectangular blocks. Here at each time step, a seed is nucleated randomly on a square whereby a crack parallel to one of the sides of the square is grown and then ceases to grow upon hitting another crack. We name this as WPSL1 as it is created by one crack at each time step and WPSL2 when it is created by two orthogonal cracks at each time step. We show that WPSL1 is governed by infinitely many conservation laws and each non-trivial conserved quantity is a multifractal measure. The dual of the WPSL1, obtained by replacing the center of each block with a node and the common border between two blocks by an edge connecting the corresponding nodes, can be described as a network. We can regard WPSL1 as growing by sequential addition of one node and WPSL2 as a network growing by sequential addition of a group of three nodes \[20\]. We show that the dual of the WPSL1 too self-organizes into a scale-free network but with exponent much smaller than that of WPSL2. Finally, we study the clustering coefficient and mean geodesic path length and find that the dual of both WPSL1 and WPSL2 are nested hierarchical network and WPSL1 is also truly small world.

The organization of the rest of this paper is as follows. In section II, we revisit the topological and geometric properties of weighted planar stochastic lattice with two cracks (WPSL2) to coordinate with the newly proposed lattice. In section III, we propose a stochastic planar lattice with one crack which we call WPSL1 and describe the algorithm. In section IV, various topological properties of WPSL1 are discussed and we show that its dual is a scale-free network with exponent of the degree distribution much lower than that of the WPSL2. In section V, the various geometric properties of WPSL1 is explored and we show that it is a multi-multifractal like WPSL2. In section VI, the small world properties of WPSL1 and WPSL2 are discussed and compared. Finally, results are discussed and conclusions drawn in section VII.

II. WEIGHTED PLANAR STOCHASTIC LATTICE BY TWO CRACKS: WPSL2

In 2010 we proposed a weighted planar stochastic lattice that provided many non-trivial topological and geometric properties \[20\]. We considered that the substrate is a square of unit area and at each time step a seed is nucleated from which two orthogonal partitioning lines, parallel to the sides of the substrate, are grown until intercepted by existing lines. It results in partitioning the square into ever smaller mutually exclusive rectangular blocks. The condition is that, the higher the area of a block, the higher is the probability that the seed will be nucleated in it and divide it into four smaller blocks since seeds are sown at random on the substrate. The resulting lattice, which we now call weighted planar stochastic lattice or in short WPSL2, where the factor 2 refers to the fact that two mutually perpendicular partitioning lines are used to divide blocks randomly into four smaller blocks, has much richer properties than the square and kinetic square lattice \[20, 21\]. It provides an awe-inspiring perspective of intriguing and rich pattern of blocks which have different sizes and have great many different number of neighbors with whom they share common border. Such seemingly disordered lattice will have no place in physics unless there is some order and emergent behaviors from the statistical perspective. One of the emergent behavior is that the coordination number distribution function of WPSL2 follows inverse power-law \[20\]. It immediately implies that the degree distribution of the network corresponding to its dual follows the same inverse power-law \(P(k) \sim k^{-\gamma}\) with the same exponent \(\gamma = 5.66\). We have also shown that it is governed by infinitely many conservation laws and one of conserved quantity can be used as multifractal measure so that the size disorder of the lattice can be quantified by multifractal scaling. Later Davieen and Hassan have shown that each of the infinitely many conservation laws is actually a multifractal measure and hence WPSL2 is a multi-multifractal \[21\].

The question is, can the two fundamental mechanisms, the growth and the preferential attachment rule, also be found responsible for the self-organization of the WPSL2 into a power-law degree distribution? Indeed, it was found that both the ingredients are present in WPSL2. However, while the presence of growth mechanism is inherent to the definition of the model, the preferential attachment rule is not as straightforward as that. For instance, at each time step a triad (three nodes linked by two edges) joins the existing network and hence it is
not static rather grows by sequential addition of a triad. It is interesting to note that it is not the node or the block which is picked preferentially with respect to area to gain links rather its neighbors. Yet, it embodies the preferential attachment rule since the higher the number of links (neighbors) a given node (block) has, the higher is the chance that one of its neighbors will be picked and as a result will gain links with incoming nodes (blocks). However, the small world properties, that are the clustering coefficient and the geodesic path length of WPSL2 have not been explored in any previous endeavors.

III. WEIGHTED PLANAR STOCHASTIC LATTICE BY A SINGLE CRACK: WPSL1

What if we further modify the generator of WPSL2 so that it now divides the initiator into two blocks either horizontally or vertically with equal a priori probability? It describes a process whereby a seed is nucleated randomly at each time step on the initiator and upon nucleation a single line grows either horizontally or vertically with equal probability until intercepted by already existing lines. Perhaps an exact algorithm rather than a mere definition can better describe the model. In step one, the generator divides the initiator, say a square of unit area, either horizontally or vertically into two smaller blocks at random. We then label the top or left block by $a_1$ if divided horizontally or vertically respectively and the other block as $a_2$. In each step thereafter only one block is picked preferentially with respect to their area (which we also refer to as the fitness parameter) and then it is divided randomly into two blocks in the same fashion i.e. either horizontally or vertically. In general, the $j$th step of the algorithm can be described as follows.

(i) Subdivide the interval $[0, 1]$ into $j$ subintervals of size $[0, a_1]$, $[a_1, a_1 + a_2]$, ..., $[\sum_{i=1}^{j-1} a_i, 1]$ each of which represents the blocks labeled by their areas $a_1, a_2, ..., a_j$ respectively.

(ii) Generate a random number $R$ from the interval $[0, 1]$ and find which of the $j$ sub-intervals contains this $R$. The corresponding block it represents, say the $k$th block of area $a_k$, is picked.

(iii) Calculate the length $x_k$ and the width $y_k$ of this block and keep note of the coordinate of the lower-left corner of the $k$th block, say it is $(x_{low}, y_{low})$.

(iv) Generate two random numbers $x_R$ and $y_R$ from $[0, x_k]$ and $[0, y_k]$ respectively and hence the point $(x_{R} + x_{low}, y_{R} + y_{low})$ mimics a random point chosen in the block $k$.

(v) Generate a random number $p$ within $[0, 1]$.

(vi) If $p < 0.5$ then draw a vertical line else a horizontal line through the point $(x_{R} + x_{low}, y_{R} + y_{low})$ to divide it into two smaller rectangular blocks. The label $a_k$ is now redundant and hence it can be reused.

(vii) Label the left or top of the two newly created blocks as $a_k$ depending whether a vertical or horizontal line is drawn respectively and the remaining block is then labeled as $a_{j+1}$.

(viii) Increase time by one unit and repeat the steps (i) - (vii) ad infinitum.

We name the resulting weighted planar stochastic lattice as WPSL1. A snapshot of this lattice too provides an awe-inspiring perspective of intriguing and rich pattern of blocks and its blocks also have different sizes and have great many different number of neighbors with whom they share common border, see Fig. (1). The number

\[ \text{FIG. 1: Snapshots of (a) WPSL1 (b) WPSL2 where both snapshots contain 4999 cells.} \]
of blocks $N(t)$ in WPSL1 grows linearly as $N(t) = 1 + \alpha t$ with $\alpha = 1$ while in the case of WPSL2 it grows in the same fashion except we have $\alpha = 3$. It implies that the dual of the WPSL1 grows by sequential addition of a monad or a single node while the dual of the WPSL2 grows by sequential addition of triad or a group of three nodes already linked by two edges.

IV. TOPOLOGICAL PROPERTIES OF WPSL1

We will first investigate the property of coordination numbers of the blocks with time. The number of blocks and the coordination numbers evolve with each step. If we define each step of the construction process as one unit of time, then we find a relation between the coordination number of the blocks with time. Note that, while calculating the coordination number, we are taking periodic boundary condition in account. The total number of blocks $N(t)$ grows with time linearly as $N(t) = 1 + \alpha t$ with $\alpha = 1$. Now, the number of blocks having coordination number $k$ is given by $N_k(t)$ which also varies linearly with time as the plot of $N_k(t)$ vs $t$ results in a straight line passing through the origin as shown in Fig. 2. So, we can write $N_k = m_k t$. The ratio between $N_k(t)$ and $N(t)$ is then given by $\rho_k(t) = N_k(t)/N(t) = m_k t$ if we consider $N(t) \sim t$ in the long time limit. So the coordination number distribution $\rho_k(t)$ is independent of time for a fixed $k$ and can be evaluated by determining the slope of the resulting straight line of the plot of $N_k(t)$ vs $t$. However, for a fixed time, $\rho_k(t)$ becomes a function of coordination number $k$ and we simply denote it by $\rho(k)$.

We can find the coordination number distribution $\rho(k)$ for a fixed time by finding the fraction of blocks having a coordination number exactly equal to $k$. Now, we can consider the blocks of the lattice as a node at the center of each block and the nearest neighboring blocks as other nodes connected by links. So, we can map the WPSL1 as a network, which is the dual of WPSL1 and we call it DWPSL1. It can be easily concluded that the degree distribution $P(k)$ of this network will be equivalent to the coordination number distribution $\rho(k)$ of WPSL1. In Fig. 3, we plot the natural logarithm of the degree distribution $\ln(P(k))$ vs $\ln(k)$. The resulting plot is a straight line with a slope equals to $\gamma - 1 = 3.13$. This implies that the degree distribution obeys a power law given by,

$$P(k) \sim k^{-\gamma},$$

with $\gamma = 4.13$. This is much less than that of WPSL2 which was 5.66. Also, the exponent found for WPSL1 is much closer to the real life networks than WPSL2. An interesting fact is that, this exponent is almost equal to the exponent of the degree distribution of the electric power grid, which is $\gamma = 4$. It is worth mentioning that the nodes of the electric power grid is spatially embedded just like the nodes of WPSL1.

One of the interesting features of the degree distribution plot is that it has a fat-tail, which is a significant characteristic of a scale free network. This means that there are few nodes with a very high number of connected neighbors, which are called hubs. These hubs play an
important role for the spreading of physical quantities through a network. However, a heavy or fat tail in the degree distribution makes finding the exponent $\gamma$ more problematic. To solve it we must invoke the idea of cumulative distribution $P(k' > k)$ and plot $\ln(P(k' > k))$ vs $\ln(k)$. The cumulative degree distribution is related to the degree distribution by the following law,

$$P(k) = \frac{-dP(k' > k)}{dk}.$$  \hspace{1cm} (2)

The plot of the cumulative distribution is expected to smooth out the fat-tail and result in a straight line in the infinite size limit. But, for the finite size of the network, the plot is a straight line up to a limit and then falls off. However, a straight line can be approximated in the finite size limit which has a slope greater than the slope of normal degree distribution plot by an additive factor of 1. Indeed, by plotting the cumulative degree distribution (see the inset of Fig. 3) we can approximate a straight line which has a slope $-3.13$. So, it is evident from the plot that the degree distribution decays following a power law $P(k) \sim k^{-\gamma}$ with $\gamma = 3.13 + 1 = 4.13$.

The power-law degree distribution $P(k)$ has been found in many seemingly unrelated real life and man-made networks. It suggests that there must exist some common underlying mechanisms for which disparate systems behave in such a remarkably similar fashion 17. Barabási and Albert in 1999 addressed exactly this and found that the growth and the preferential attachment rule are the main factors behind the emergence of such power-law degree distribution 8. Indeed, the dual of the WPSL network too grows with time. The dual of the WPSL2 grows by addition of a group of three nodes which are already linked by two edges. On the other hand, the dual of WPSL1, grows by addition of one new node. However, in either case the node which is picked at random is not the one which gains links rather its neighbors gain link. It means the higher the number of nearest neighbors, the higher the probability the block has to gain new neighbors vis-a-vis link. Using this idea, in 2017 we proposed a mediation-driven attachment (MDA) rule to construct Barabási-Albert like network 30. At a glance, it may seem that MDA defies the PA rule but a deeper look suggests that it embodies the intuitive idea of the PA rule, albeit in disguise. The MDA rule is in fact not only preferential but it also can be super-preferential in some cases.

\section*{V. GEOMETRIC PROPERTIES OF WPSL1}

Now, we look into the geometric properties of WPSL1 in an attempt to check if there exists some rules and order. We can also use this model to describe the kinetics of planar fragmentation in an attempt to understand the extent of influence of size and shape when both are considered dynamical variables 31-34. The distribution function $f(x, y, t)$, that describes concentration of blocks of sides having length $x$ and width $y$ of WPSL1, evolves as

$$\frac{\partial f(x, y, t)}{\partial t} = -xyf(x, y, t) + \frac{1}{2} \times 2y \int_{x}^{\infty} f(x_1, y, t)dx_1 + \frac{1}{2} \times 2x \int_{y}^{\infty} f(x, y_1, t)dy_1.$$  \hspace{1cm} (3)

The first term on the right accounts for the loss of blocks of sides $x$ and $y$ due to nucleation of seed. The pre-factor $xy$ here implies that the seeds are nucleated on the block chosen preferentially according to the area of the existing blocks. The second term on the right describes placing a cut vertically with probability $1/2$ on a side $x_1 > x$ of block of sides $x_1$ and $y$ to form two blocks, hence the factor $2$, of sides $(x, y)$ and $(x_1 - x, y)$. Similarly, the third term represents the gain of blocks of sides $x$ an $y$ upon placing a cut horizontally with probability $1/2$ on a block of sides $x$ and $y_1 > y$ to form two blocks of sides $(x, y)$ and $(x, y_1 - y)$.

We find it a formidable task to solve Eq. 3 to find solution for $f(x, y, t)$. Instead, we introduce the 2-tuple Mellin transform of $f(x, y, t)$ given by

$$M(m, n; t) = \int_{0}^{\infty} \int_{0}^{\infty} x^{m-1}y^{n-1}f(x, y, t)dxdy,$$  \hspace{1cm} (4)

which describes moment at fixed $m$ and $n$. Incorporating
It can be re-written as,

\[ \frac{dM(m, n; t)}{dt} = -\left(\frac{\alpha_+ + \alpha_-}{mn}\right) M(m + 1, n + 1; t) \]  

(5)

It can be re-written as,

\[ \frac{dM(m, n; t)}{dt} = -\left(\frac{\alpha_+ + \alpha_-}{mn}\right) M(m + 1, n + 1; t) \]  

(6)

where

\[ \alpha_\pm = \frac{m + n - 2}{2} \pm \sqrt{\frac{(m-n)^2}{4} + 1}. \]  

(7)

We can now iterate Eq. (5) to find various derivatives of \( M(m, n; t) \) which are given by,

\[ \frac{d^j M(m, n; t)}{dt^j} = (-1)^j \prod_{k=1}^j \frac{(\alpha_+ + k - 1)(\alpha_- + k - 1)}{(m + k - 1)(n + k - 1)} M(m + k, n + k; t) \]  

(8)

These derivatives can now be used in Taylor series expansion of \( M(m, n; t) \) about \( t = 0 \) along with using mono-disperse initial condition which gives

\[ M(m, n; t) = 2F_2(\alpha_+, \alpha_-; m, n; -t), \]

(9)

where \( 2F_2 \) is the generalized hypergeometric function [35].

The nontrivial nature of the solution given by Eq. (9) can be best appreciated by considering its behavior in the limit \( t \to \infty \). To this end, we find \( M(m, n; t) \) decays following a power-law

\[ M(m, n; t) = \frac{\Gamma(m)\Gamma(n)\Gamma(\alpha_+ + \alpha_-)}{\Gamma(\alpha_+)\Gamma(m - \alpha_-)\Gamma(n - \alpha_-)} t^{-\alpha_-}. \]  

(10)

It immediately implies that

\[ M(m + 1, 1/m + 1; t) = \text{const.} \quad \forall \ t > 0, \]  

(11)

or if the blocks are labeled as \( i = 1, 2, ..., N \) then the WPSL obeys infinitely many conservation laws i.e.,

\[ \sum_{i} x_i^m y_i^{1/m} = \text{const.}. \]  

(12)

which includes the trivial conservation law that is conservation of total mass or area \( M(2, 2; t) = 1 \).

We performed extensive numerical simulation to measure \( M(m + 1, 1/m + 1; t) \) using Eq. (12). We show the plot of this in the inset of Fig. (4) as a function of \( t \) for \( m = 2, 3 \) and 4 for a single realization. The moment for each \( m \) value remains constant with time albeit the numerical values are different for every different values of \( m \). Moreover, for a given value of \( m \) the numerical value of the conserved quantity is different for every single realization although it remains constant with time. The moment \( M(m + 1, 1/m + 1; t) \) for \( m = 2 \) is plotted as a function of \( t \) in Fig. (3) for four independent realizations, which confirms the above-mentioned statement. Interestingly, the ensemble average value of \( M(m + 1, 1/m + 1; t) \) is always equal to 1 which is shown in Fig. (3) for \( m = 2, 3 \) and 4 where data is averaged over 4000 independent realizations. This is not a surprise, since with some algebra the coefficient of \( t^{-\alpha_-} \) in Eq. (10) turns out to be 1 i.e., \( M(m + 1, 1/m + 1; t) = 1 \) \( \forall \ m > 0 \).

We shall now invoke the idea of multifractality and show that each of the non-trivial conserved quantities (except the trivial conserved quantity namely the conservation of total area) is distributed in the lattice in such a fashion that each of them can be best described as a multifractal. We assume that \( p_i = x_i^m y_i^{1/m} \) is the probability that the \( i \)th block contains the total measure \( \sum_i x_i^m y_i^{1/m} \). We can now construct the partition function

\[ Z_q = \sum_i p_i^q \]  

(13)

of the probability \( p_i \) and comparing it with the definition of the two-tuple Mellin transform of \( f(x, y, t) \) we find that

\[ Z_q = M(mq + 1, q/m + 1; t). \]  

(14)

According to Eq. (10), the asymptotic solution for the partition function therefore is

\[ Z_q(t) \sim t^{\frac{(m - 1/m)^2 q^2 + 4 - (m + 1/m)q}{2}}. \]  

(15)
FIG. 6: Plot of the $f(\alpha)$ spectrum for $m = 2, 3$ and 4. The maximum of each plot occurs at $q = 0$ and the value is always equal to 2. It indicates that the Hausdorff-Besicovitch dimension of the support is 2.

Measuring it using the square root of the mean area
\[ \delta(t) = \sqrt{\frac{M(2, 2; t)}{M(1, 1; t)}} \sim t^{-1/2}, \]
(16)
as an yard-stick we find $Z_q$ decays following a power-law
\[ Z_q(\delta) \sim \delta^{-\tau(q)}, \]
(17)where the mass exponent
\[ \tau(q) = \sqrt{(m - 1/m)^2 q^2 + 4} - (m + 1/m)q. \]
(18)
The mass exponent $\tau(q)$ must possess the properties such that $\tau(0)$ is the Hausdorff-Besicovitch dimension of the support and $\tau(1) = 0$ required by the normalization of the probabilities $p_i$, which in our case is clearly true since $\tau(0) = 2$ and $\tau(1) = 0 \ [38, 39]$.

On the other hand, the Legendre transformation of the mass exponent $\tau(q)$ by using the Lipschitz-Hölder exponent $\alpha(q)$ is given by,
\[ f(\alpha) = \tau(q) + \alpha q, \]
(19)with
\[ \frac{d\tau}{dq} = -\alpha(q), \]
(20)which gives,
\[ \alpha(q) = (m + 1/m) - \frac{q(m - 1/m)^2}{\sqrt{(m - 1/m)^2 q^2 + 4}}. \]
(21)

VI. SMALL WORLD PROPERTIES OF WPSL1 AND WPSL2

It has been found that besides the power-law degree distribution, much of the man-made and natural networks have two more striking characteristic features in common \[40\]. First, the clustering coefficient $C$, that quantifies the probability that two nodes having a common neighbor are also neighbors of each other, is high and independent of system size in the large size limit. Second, the mean geodesic path length $l$, that describes the shortest mean path between two arbitrary nodes, is minuscule size as it increases logarithmically with system size. These two properties are also known as the benchmark of the small-world phenomena. The first theoretical model that can capture the small-world network features was proposed by Watts and Strogatz (WS) in...
1998. However, this model can describe two of the three characteristic features of the real-world networks with the exception of one, namely the power-law degree distribution.

We shall now look into the clustering coefficient, also known as transitivity in the social network, that describes how likely it is that two friends of a given individual are also friends of each other. Mathematically, the clustering coefficient $C_i$ of a node $i$ of the network is defined as the ratio of the number of edges $e_i$ among its $k_i$ neighbors and the number of potential edges $\frac{k_i(k_i-1)}{2}$ among the same neighbors that can exist i.e.,

$$C_i = \frac{e_i}{\frac{k_i(k_i-1)}{2}}.$$  \hfill (23)

The average of this probability over all the nodes in the network is called the clustering coefficient $C$ of the network and hence

$$C = \frac{1}{N} \sum_i^N C_i.$$  \hfill (24)

Except the tree network where $C = 0$, the clustering coefficient $C$ is often non-zero and often it starts high and then decreases like in the Barabási-Albert network. However, sometimes $C$ also has high value and retain the same value independent of the system size. The plots of clustering coefficient for WPSL1 and WPSL2 are shown in Fig. 8 as a function of lattice size $N$. The mean clustering coefficient for WPSL1 and WPSL2 are 0.48 and 0.367 and they are independent of the system size.

Yet another quantity of interest is to find the clustering coefficient $C_k$ of the nodes which have degree $k$ and check if it decays following a power-law or not as a function of $k$. Many network models consist of numerous small communities that form larger communities, which may again combine into even larger communities; this process can continue up to several stages. The quantitative measure of this nested hierarchical community is provided by finding the dependence of the clustering coefficient $C_k$ on the node which have degree $k$. Interestingly, we find nested hierarchical communities as $C_k$ of WPSL1 decreases exactly as

$$C_k = \frac{2}{k-1},$$  \hfill (25)

and hence in the large $k$ limit it decays following a power-law $C_k \sim k^{-\nu}$ with $\nu = 1$ as shown in Fig. 8a. The same result, in the large $k$ limit, is also true for WPSL2 as shown in Fig. 8b.

Now we may look at the mean geodesic distance of WPSL1 and WPSL2. It is defined as the average of the shortest paths among all the connected pairs of nodes in the network i.e.,

$$l = \frac{1}{2} \frac{1}{\binom{N}{2}} \sum_{i,j} d_{ij} = \frac{1}{N(N-1)} \sum_{i,j} d_{ij},$$  \hfill (26)

where $d_{ij}$ is the shortest path length between node $i$ and $j$ and factor $1/2$ is included to take care of double counting. Generally, for regular lattice, the geodesic distance scales as $l \sim N^{1/d}$, where $d$ is the dimension of the support of the lattice. Note that the logarithmically slow growth of $l$ is regarded as one of the benchmark of the small-world phenomena as it embodies the highly counter-intuitive idea that everyone in the world is on the average only six links of acquaintance away from everyone else. The idea of six degrees of separation was first conceived by Hungarian author F. Karinthy and later proved experimentally by S. Milgram [41].

Interestingly, for WPSL2 we find that the mean geodesic distance scales as $l \sim N^{1/4}$ and find a straight line with a slope 1.00 as shown in the Fig. 9a. This is reminiscent of the geodesic path...
length of $d$ dimensional regular hypercubic lattice where $l$ grows as $N^{1/d}$ \cite{4}. It suggests that WPSL2 is not really a small-world. However, for WPSL1, we find that mean geodesic distance scales as $l \sim \ln(N)$. We plot the mean geodesic distance as a function of $\ln(N)$ which results in a straight line with a slope 1.37, as shown in Fig. (b). It means that the mean geodesic path length grows much slowly for WPSL1 than WPSL2. Thus, high clustering co-efficient and logarithmically small geodesic mean path length makes the WPSL1 a perfect small-world network.

VII. SUMMARY AND CONCLUSIONS

In this article, we have first briefly revisited a class of weighted planar stochastic lattice which we re-named here as WPSL2. This lattice have rich topological and geometric properties since its dual is a scale-free network and geometrically it is a multi-multifractal. However, one of the critique of WPSL2 though is that the exponent $\gamma = 5.66$ of the degree distribution is far too high compared to that of the most real-life or man-made networks. Also, the small world properties of WPSL2 have never been investigated. To this end, we have found that it has small clustering coefficient, compared to WPSL1, which is independent of system size but the mean geodesic path length grows following a power-law thus revealing that it is not really a small-world. We proposed yet another weighted planar stochastic lattice namely WPSL1 in which the generator divides the initiator randomly either horizontally or vertically into two blocks with equal probability instead of four blocks and it is once again applied sequentially exactly in the same way as we have done for WPSL2. It has been shown analytically that the dynamics of the system is governed by infinitely many non-trivial conservation laws and we have verified it numerically too. We have shown that the ensemble average numerical value of each conserved quantity, although different for each independent realization, is always equal to one. Except the trivial mass conservation, each of the non-trivial conserved quantities is distributed in the lattice in such a fashion that the distribution can be best understood as multifractal. Since there are infinitely many conserved quantities for both WPSL1 and WPSL2, there are infinitely many multifractal measures too.

Interestingly, the dual of the weighted planar stochastic lattice can be mapped as a network if we replace the center of each cell by node and common boarder between cells as the link between the two nodes. The corresponding network can also be regarded as the dual of the WPSL. Furthermore, We have found that like WPSL2, the dual of the WPSL1 too self-organize into a scale-free network and its exponent $\gamma$ is much closer to the most real-life and man-made network than that of the WPSL2. We have found that the clustering coefficient $C$ for both the cases are quite high and remain independent of the system size. However, $C$ of the WPSL1 is much higher than that of the WPSL2. Besides, the clustering co-efficient $C_k$ of the nodes which have degree $k$ decreases inversely with $k$ in the large $k$ limit in both cases suggesting that they are also hierarchical networks. On the other hand, we have found that the mean geodesic path length in WPSL1 is miniscule in size as it grows logarithmically with system size ($l \sim \ln(N)$). However, the mean geodesic path length for WPSL2 grows algebraically with system size ($l \sim N^{1/4}$) reminiscent of that of hypercubic lattice. It means that WPSL1 truly is a small world network but not WPSL2.

To summarize, we have proposed a lattice, namely WPSL1, that has many interesting properties. Firstly, it has infinitely many conserved quantities each of which is a multifractal measure. Its dual is a scale-free network as its degree distribution exhibits a power-law with exponent quite close to natural and man-made scale-free networks. It is also a small-world as its mean geodesic path length grows logarithmically with system size and its clustering co-efficient is significantly high ($C = 0.48$). Besides, we find that the clustering coefficient $C_k$ of the nodes which have degree $k$ decays inversely with $k$ revealing nested hierarchical community. All these properties make it a lattice that can potentially be used as a skeleton to study diffusion, percolation etc. One of the central prediction of thermal and probabilistic models for continuous phase transition is that their critical behaviors depend only on the dimension of the lattice or skeleton and not on the specific choice of lattice. For instance, the critical exponents of percolation on all kind of two dimensional lattices are found to belong to the same universality class. However, recently we have shown this is no longer the case if the lattice, namely WPSL2, has a power-law degree distribution. Now that we know WPSL1 too is not only scale-free but also hierarchical small-world network with exponent of the degree distribution much lower than that of WPSL2, it would be interesting to investigate the nature of universality class of percolation on WPSL1. We hope to do this in our future endeavor.

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