Risk and parameter convergence of logistic regression

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Abstract

The logistic loss is strictly convex and does not attain its infimum; consequently the solutions of logistic regression are in general off at infinity. This work provides a convergence analysis of stochastic and batch gradient descent for logistic regression. Firstly, under the assumption of separability, stochastic gradient descent minimizes the population risk at rate $O\left(\frac{\ln(t)}{t}\right)$ with high probability. Secondly, with or without separability, batch gradient descent minimizes the empirical risk at rate $O\left(\frac{\ln(t)}{t}\right)$. Furthermore, parameter convergence can be characterized along a unique pair of complementary subspaces defined by the problem instance: one subspace along which strong convexity induces parameters to converge at rate $O\left(\frac{\ln(t)^2}{\sqrt{t}}\right)$, and its orthogonal complement along which separability induces parameters to converge in direction at rate $O\left(\frac{\ln \ln(t)}{\ln(t)}\right)$.

1 Overview

Logistic regression is the task of finding a vector $w \in \mathbb{R}^d$ which approximately minimizes the population or empirical logistic risk, namely

$$R_{\text{log}}(w) := E\left[\ell_{\text{log}}(\langle w, -yx \rangle)\right] \quad \text{or} \quad \hat{R}_{\text{log}}(w) := \frac{1}{n} \sum_{i=1}^{n} \ell_{\text{log}}(\langle w, -y_i x_i \rangle),$$

where $\ell_{\text{log}}(r) := \ln(1 + \exp(r))$ is the logistic loss.

A traditional way to minimize $R_{\text{log}}$ or $\hat{R}_{\text{log}}$ is to pick an arbitrary $w_0$, and from there recursively construct stochastic gradient descent (SGD) or gradient descent (GD) iterates $(w_j)_{j \geq 0}$ via $w_{j+1} := w_j - \eta_j g_j$, where $(\eta_j)_{j \geq 0}$ are step sizes, and $g_j := \ell_{\text{log}}'(\langle w_j, -y_j x_j \rangle)(-y_j x_j)$ for SGD or $g_j := \nabla \hat{R}_{\text{log}}(w_j)$ for GD.

A common assumption is that the distribution on $(x, y)$ is separable (Novikoff, 1962; Soudry et al., 2017). Under this assumption, the first result here is that SGD with constant step sizes minimizes the population risk $R_{\text{log}}$ at rate $\tilde{O}(1/t)$ with high probability.

**Theorem 1.1 (Simplification of Theorem 2.1).** Suppose there exists a unit vector $\bar{u}$ with $\langle \bar{u}, yx \rangle \geq \gamma$, and $|yx| \leq 1$ almost surely. Consider SGD iterates $(w_j)_{j \geq 0}$ as above, with $w_0 = 0$ and $\eta_j = 1$. Then for any $t \geq 1$, $|w_t| \leq O(\ln(t))$, and with probability at least $1 - \delta$, the average iterate $\hat{w}_t := t^{-1} \sum_{j < t} w_j$ satisfies

$$R_{\text{log}}(\hat{w}_t) \leq O\left(\frac{\ln(t)^2 + \ln(t) \ln(1/\delta)}{t}\right).$$

This result improves upon prior work in the following ways.

1. Since $\ell_{\text{log}}$ exhibits an exponential tail and Lipschitz continuity, standard analyses give a $\tilde{O}(1/\sqrt{t})$ rate of SGD with step size $\eta_j = 1/\sqrt{j+1}$, but only in expectation (Nesterov 2004; Bubeck 2015). With the $O(\ln(t))$ bound on $|w_t|$ given by Theorem 1.1, Azuma-Hoeffding’s inequality gives a high-probability $\tilde{O}(1/\sqrt{t})$ rate. However, since standard analyses do not ensure a vanishing risk with constant step sizes, they do not give a $\tilde{O}(1/t)$ rate.
2. The only other prior high-probability $O(1/t)$ rate is under strong convexity (e.g., [Rakhlin et al., 2012]): $|w_t - \hat{w}|^2 \leq O(1/t)$ with high probability, where $\hat{w}$ is the global optimum. However, $\ell_{\log}$ is not strongly convex on $\mathbb{R}$, and even in trivial cases the optimum of logistic regression may be off at infinity: suppose examples $(x_i)^n_{i=1}$ lie on the positive real line, and labels $(y_i)^n_{i=1}$ are all +1; then $\inf_{w \in \mathbb{R}} \hat{R}_{\log}(w) = 0$, but $\hat{R}_{\log}(w) > 0$ for every $w \in \mathbb{R}$. Furthermore, not only do the guarantees here avoid strong convexity; the proofs are different.

The proof of Theorem 1.1 is based on the perceptron proof, and might be of independent interest.

Now consider batch gradient descent on empirical risk. In this case, a stronger result is possible: no assumption is made (separability is dropped), and the parameters may be precisely characterized.

**Theorem 1.2** (Simplification of Theorems 3.1, 3.2 and 3.6). Let examples $((x_i, y_i))_{n=1}^n$ be given with $|x_i y_i| \leq 1$, along with a loss $\ell \in \{\ell_{\log}, \exp\}$, with corresponding empirical risk $\hat{R}$ as above. Consider GD iterates $(w_j)_{j \geq 0}$ as above, with $w_0 = 0$.

1. **(Convergence in risk.)** For any step sizes $\eta_j \leq 1$ and any $t \geq 1$,

$$\hat{R}(w_t) - \inf_{w \in \mathbb{R}^d} \hat{R}(w) = O\left(\frac{1}{t} + \frac{\ln(t)^2}{\sum_{j < t} \eta_j}\right) = \begin{cases} O\left(\ln(t)^2/t \right) & \eta_j = O(1), \\ O\left(\ln(t)^2/\sqrt{T}\right) & \eta_j = O(1/\sqrt{T+1}). \end{cases}$$

2. **(Convergence in parameters.)** There exists a unique subspace $S$, unique unit vector $\bar{u} \in S^\perp$, and unique vector $\bar{v} \in S$ which characterize convergence as follows. Fixing step size $\eta_j := 1/\sqrt{T+1}$, letting $\Pi_S$ denote orthogonal projection onto $S$, and defining optimal iterates $\hat{w}_t := \inf \{\hat{R}(w) : |w| \leq |w_t|\}$, if $\bar{v}^2 = O(n \ln(t))$, then

$$\begin{align*}
|\Pi_S w_t| &= \Theta(1) \quad \text{and} \quad \max \left\{ |\Pi_S w_t - \bar{v}|^2, |\Pi_S \bar{w}_t - \bar{v}|^2 \right\} = O\left(\frac{\ln(t)^2}{\sqrt{T}}\right), \\
|\Pi_{S^\perp} w_t| &= \Theta(\ln(t)) \quad \text{and} \quad \max \left\{ \frac{|w_t|}{|\bar{w}_t|} - \bar{u}^2, \frac{|\bar{w}_t|}{|\bar{w}_t|} - \bar{u}^2 \right\} = O\left(\frac{\ln \ln t}{\ln t}\right).
\end{align*}$$

In particular, $w_t/|w_t| \to \bar{u}$ and $\Pi_S w_t \to \bar{v}$.

To make sense of this, recall firstly the earlier discussion that the solutions are off at infinity in general. As will be made precise in Section 3.1 if examples are separable, then $S = \{0\}$, $\bar{v} = 0$, and $\bar{u}$ is the unique maximum margin separator. The GD iterates not only keep driving $\hat{R}$ to its infimum 0, but also closely track $\bar{u}$ in direction, meaning $w_t/|w_t| \to \bar{u}$. Note that this is a strong mode of convergence: $\hat{R}(w_t) = O(1/t)$ and $|w_t| = \Theta(\ln(t))$ do not exclude the possibility that there is always a large angle between $w_t/|w_t|$ and $\bar{u}$.

The full part 2 of Theorem 1.2 gives a characterization of parameter convergence in the general setting: the parameter space can be split into a subspace $S$ along which optimization is strongly convex, and its complement $S^\perp$ along which the problem behaves as though separable. While there does exist recent prior work on parameter convergence of gradient descent for logistic regression (Soudry et al., 2017; Gunasekar et al., 2018; Nacson et al., 2018), it assumes separability. The analysis essentially for free gives convergence not only to $(\bar{v}, \bar{u})$, but also to the optimal iterates $\hat{w}_t$. Consequently, logistic regression is implicitly regularized in a strong sense: it produces a sequence of iterates which closely track the outputs of a sequence of constraint optimization problems.

This paper is organized as follows.

**SGD population risk convergence (Section 2).** An online guarantee is given by a perceptron-style analysis, and transformed into a high-probability bound via a martingale Bernstein bound.

**Empirical samples structure (Section 3.1).** The first step of the analysis is to derive the unique complementary subspaces $(S, S^\perp)$, along with unique optimal direction $\bar{u} \in S^\perp$ and unique vector $\bar{v} \in S$. $(S, S^\perp)$ and $\bar{u}$ are unique given the data $((x_i, y_i))_{n=1}^n$ and are not affected by the choice of $\ell$ or optimization method; $\bar{v}$ depends on $\ell$ but not on the optimization method.

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GD empirical risk convergence (Section 3.2). With the structure out of the way, it is fairly easy to adapt convergence guarantees of smooth convex objectives with gradient descent via the choice of a comparison point $\bar{v} + \bar{u}$ ($\ln(t) / \gamma$), immediately yielding a rate of $O(1/t)$ to the (in general unattainable) infimal risk.

Parameter convergence (Section 3.3). Over $S$, the risk $R$ is strongly convex over bounded sublevel sets, and thus the preceding risk guarantee implies converges over $S$ via standard arguments.

To establish convergence over $S^\bot$ to the optimal direction $\bar{u}$, the key is to study not $R$ but instead $\ln R$, which more conveniently captures local smoothness (extreme flattening) of $R$. This proof goes through more easily for the exponential loss, which is the main reason why it appears in Theorem 1.2.

Open problems (Section 4) and appendices. Some open problems are discussed in Section 4. All missing proofs are in the appendices.

1.1 Related work

The analysis which gives $O(1/t)$ high-probability rate of SGD is inspired by the perceptron proof (Novikoff, 1962). Rakhlin et al. (2012) give a high-probability $O(1/t)$ bound on $|w_t - \bar{w}|^2$ in the strongly-convex case with $\bar{w}$ being the global optimum.

The structural result for the gradient descent part (cf. Section 3.1) is drawn from the literature on AdaBoost, which was originally stated for separable data (Freund and Schapire, 1997), but later adapted to general instances (Mukherjee et al., 2011; Telgarsky, 2012). This analysis revealed not only a problem structure which can be refined into the $(S, S^\bot)$ used here, but also the convergence to maximum margin solutions (Telgarsky, 2013). Since the structural analysis is independent of the optimization method, the key structural result (Theorem 3.1) can be partially found in prior work; the present version provides not only an elementary proof, but moreover differs by providing $(S, S^\bot)$ (and not just a partition of the data) and the subsequent construction of a unique $\bar{u}$ and its properties.

The empirical risk analysis has some connections to the AdaBoost literature, for instance when providing smoothness inequalities for $R$ (cf. Lemma 3.5). Some tools are borrowed from the convex optimization literature, for instance smoothness-based convergence proofs of gradient descent (Nesterov, 2004; Bubeck, 2015).

There is some work in online learning on optimization over unbounded sets, for instance bounds where the regret scales with the norm of the comparator (Orabona and Pal, 2016; Streeter and McMahan, 2012). By contrast, as the present work is not adversarial and instead has a fixed training set, part of the work (a consequence of Theorem 3.1) is the existence of a good, small comparator.

In parallel to the present work, a collection of papers have studied parameter convergence of gradient descent for logistic regression when the data is separable (Soudry et al., 2017; Gunasekar et al., 2018; Nacson et al., 2018). With separability, the closest of these works provides a rate that is the square of the one here (Soudry et al., 2017). The other works show that not just gradient descent but also steepest descent with other norms can lead to margin maximization (Gunasekar et al., 2018) (see also (Telgarsky, 2013)), and also that constructing loss functions with an explicit goal of margin maximization can lead to better rates (Nacson et al., 2018). Lastly, another parallel line of work develops a collection of condition numbers suitable for characterizing the convergence of logistic regression over unbounded parameter spaces (Freund et al., 2017).

2 Stochastic gradient descent for separable risk minimization

First, some notation. Suppose samples $((x_j, y_j))_{j \geq 0}$ are drawn i.i.d. from some probability distribution over $\mathbb{R}^d$ satisfying $|x_j| \leq 1$ and $y_j \in \{-1, +1\}$ almost surely, where $|v|^2 = \sum_i v_i^2$. Furthermore, suppose the distribution is separable: there exists a unit vector $\bar{u}$ such that $\langle \bar{u}, yx \rangle \geq \gamma$ almost surely. For convenience, denote $-yx$ by $z$, $-y_j x_j$ by $z_j$, whereby $|z| \leq 1$ and $\langle \bar{u}, z \rangle \leq -\gamma$ almost surely.

Given loss function $\ell : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$, define $R(w) := \mathbb{E} \left[ \ell(\langle w, z \rangle) \right]$. The focus of this section is on the logistic loss $\ell_{\log}(x) := \ln(1 + \exp(x))$ and the corresponding population risk $R_{\log}$. The results hold for similar losses, albeit with messier analysis.

Stochastic gradient descent here starts with $w_0 := 0$, and proceeds with $w_{j+1} := w_j - \eta_j g_j$, where $g_j := z_j \ell'(\langle w_j, z_j \rangle)$ denotes the stochastic gradient. $\eta_j$ is always in $[0, 1]$.  

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Theorem 2.1. For any $t \geq 1$, $|w_t| \leq 2^{\ln(t)}/\gamma + 2$, and with probability at least $1 - \delta$,

$$\sum_{j < t} \eta_j \mathcal{R}_{\log}(w_j) \leq 16 \left( \frac{\ln(t)}{\gamma} + 1 \right) \left( \frac{\ln(t)}{\gamma^2} + 1 + \ln \left( \frac{1}{\delta} \right) \right).$$

To prove Theorem 1.1 by Theorem 2.1, set $\eta_j = 1$ and note $t \mathcal{R}_{\log}(\hat{w}_t) \leq \sum_{j < t} \mathcal{R}_{\log}(w_j)$ by Jensen.

The proof centers upon a key quantity which is extracted from the perceptron convergence proof (Novikoff [1962]). Specifically, the perceptron convergence proof establishes bounds on the number of mistakes up through time $t$. As the perceptron algorithm is stochastic gradient descent on the ReLU loss $z \mapsto \max\{0, z\}$, the number of mistakes up through time $t$ is more generally the quantity $\sum_{j < t} \eta_j \ell'(\langle w_j, z_j \rangle) = \sum_{j < t} \eta_j \ell'(\langle w_j, z_j \rangle).$ Generalizing this, the proof here controls

$$\ell'_{<t} = \sum_{j < t} \eta_j \ell'(\langle w_j, z_j \rangle) \quad \text{and} \quad \mathcal{L}'_{<t} = \sum_{j < t} \eta_j \mathbb{E} \left[ \ell'(\langle w_j, z_j \rangle) | z_0, \ldots, z_{j-1} \right].$$

The proof of Theorem 2.1 contains three essential parts:

1. A perceptron-style analysis which controls simultaneously $|w_t|$ and $\ell'_{<t}$. In the standard perceptron proof, an upper bound on $|w_t|$ and a lower bound on $\langle w_t, \bar{u} \rangle$ are combined together to give the mistake bound. The new trick here is to combine an upper bound on $|w_t - u_t|$ and a lower bound on $\langle w_t - u_t, \bar{u} \rangle$, where $w_t = \bar{u} \ln(t)/\gamma$.

2. A martingale Bernstein bound to control $\mathcal{L}'_{<t} - \ell'_{<t}$, which is a martingale, can be controlled by analyzing its conditional variance and invoking a martingale Bernstein bound.

3. A high-probability bound on $\sum_{j < t} \eta_j \mathcal{R}_{\log}(w_j)$. This follows from the high-probability bound on $\mathcal{L}'_{<t}$, the bound on $|w_t|$, and the fact that $\ell_{\log}(x) \leq \ell'(x)(|x| + 2)$.

In more detail, the first step starts with an upper bound on $|w_t - u_t|$.

**Lemma 2.2.** Given a convex loss $\ell$ with $0 \leq \ell' \leq 1$ and $\ell' \leq \ell$, for any $w \in \mathbb{R}^d$ and $t \geq 1$,

$$|w_t - w|^2 \leq |w|^2 + 2 \sum_{j < t} \eta_j \ell'\langle w, z_j \rangle.$$

Suppose furthermore $\ell \leq \exp$, let $u_t = \bar{u} \ln(t)/\gamma$,

$$|w_t - u_t|^2 \leq \frac{\ln(t)^2}{\gamma^2} + 2.$$

The proof starts as usual with $|w_{j+1} - w|^2 = |w_j - w|^2 - 2\eta_j \langle g_j, w_j - w \rangle + \eta_j^2 |g_j|^2$, and then uses $\ell^2 \leq \ell' \leq \ell$.

A bound on $|w_t|$ follows immediately from Lemma 2.2.

**Lemma 2.3.** Suppose $\ell$ satisfies all conditions in Lemma 2.2. For any $t \geq 1$,

$$|w_t| \leq \frac{2 \ln(t) + 2}{\gamma}.$$

**Proof.** $|w_t| \leq |w_t| + |w_t - u_t| \leq \ln(t)/\gamma + \sqrt{\ln(t)/\gamma^2 + 2} \leq 2\ln(t)/\gamma + 2. \quad \square$

With an upper bound on $|w_t - u_t|$, the perceptron-style analysis further requires a lower bound on $\langle w_t - u_t, \bar{u} \rangle$. It follows from the separability assumption:

$$\langle w_t - u_t, \bar{u} \rangle = \langle w_t, \bar{u} \rangle - \langle u_t, \bar{u} \rangle = - \sum_{j < t} \eta_j \ell'\langle w_j, z_j \rangle \langle z_j, \bar{u} \rangle - \frac{\ln(t)}{\gamma} \geq - \gamma \ell'_{<t} - \frac{\ln(t)}{\gamma}.$$

Equation (2.4) together give an upper bound on $\ell'_{<t}$, which also leads to the second step.
Lemma 2.5. Suppose $\ell$ satisfies all conditions in Lemma 2.2. For any $t \geq 1$,
\[ \ell'_{\leq t} \leq \frac{2\ln(t)}{\gamma^2} + \frac{2}{\gamma}. \]
Furthermore, with probability at least $1 - \delta$,
\[ \mathcal{L}'_{\leq t} \leq \frac{4\ln(t)}{\gamma^2} + \frac{4}{\gamma} + 4\ln \left( \frac{1}{\delta} \right). \]

The key is to get a high-probability bound on $\mathcal{L}'_{\leq t} - \ell'_{\leq t}$. Azuma-Hoeffding’s inequality will introduce another $\sqrt{t}$ factor, which makes it impossible to show $O(1/t)$ rate of SGD. The key observation is that the Lipschitz continuity of $\ell$ allows a sharp control on the conditional variance, by which the martingale Bernstein bound (e.g., Theorem 1 in Beygelzimer et al. (2011)) gives Lemma 2.5.

Now to get a bound on $\sum_{j \leq t} \eta_j \hat{\mathcal{R}}(w_j)$ from a bound on $\mathcal{L}'_{\leq t}$, it is enough to bound the difference between $\ell'$ and $\ell$. Here is one such property of $\ell_{\log}$.

Lemma 2.6. For any $B \geq 0$, $x \leq B$, $\ell_{\log}(x) \leq \ell'_{\log}(x)(B + 2)$.

Lemmas 2.3, 2.5 and 2.6 together prove Theorem 2.1. Missing proofs are in Appendix A.

### 3 Gradient descent for non-separable empirical risk minimization

First, some notation. There are $n$ examples $((x_i, y_i))_{i=1}^n$, collected into a matrix $A \in \mathbb{R}^{n \times d}$, with $i$th row $A_i := -y_i x_i$. As above, it is assumed that $\max_i |A_i| = \max_i |y_i x_i| \leq 1$.

As in Theorem 1.2 and as will be elaborated in Section 3.1, the data matrix $A$ defines a unique division of $\mathbb{R}^d$ into a direct sum of subspaces $\mathbb{R}^d = S \oplus S^\perp$. The individual examples and corresponding rows of $A$ are either within $S$ or $S^\perp$, and without loss of generality the rows of $A$ may be permuted so that $A := \left[ \begin{array}{c} A_S \\ A_c \end{array} \right]$, where the rows of $A_S$ and $A_c$ are respectively within $S$ and $S^\perp$. Furthermore, let $\Pi_S$ and $\Pi_c$ respectively denote orthogonal projection onto $S$ and $S^\perp$, and define $A_\perp := \Pi_\perp A_c$.

Given loss function $\ell : \mathbb{R} \to \mathbb{R}_{\geq 0}$, define $\hat{\mathcal{R}}(w) := \sum_{i=1}^n \ell((w_i - y_i x_i)/n) = \sum_{i=1}^n \ell((A_i w)/n)$. For convenience, for vector $v \in \mathbb{R}^d$ where $k$ is arbitrary, define a coordinate-wise form $L(v) := \sum_{i=1}^k \ell(v_i)/n$, whereby $\hat{\mathcal{R}}(w) = L(A w) = \hat{\mathcal{R}}_S(w) + \hat{\mathcal{R}}_c(w)$, where (for any row vector $a$)

\[ \hat{\mathcal{R}}_S(w) := \frac{1}{n} \sum_{a \in A_S} \ell(a, w) = L(A_S w), \quad \text{and} \quad \hat{\mathcal{R}}_c(w) := \frac{1}{n} \sum_{a \in A_c} \ell(a, w) = L(A_c w), \]

and $\nabla \hat{\mathcal{R}}(w) = A^\top \nabla L(A w)$.

Gradient descent here starts with $w_0 := 0$, and thereafter set $w_{j+1} := w_j - \eta_j \nabla \hat{\mathcal{R}}(w_j)$ with $\eta_j \leq 1$. For convenience, define $\gamma_j := |\nabla \hat{\mathcal{R}}(w_j)|/\hat{\mathcal{R}}(w_j) = |\nabla (\ln \hat{\mathcal{R}})(w_j)|$ and $\hat{\eta}_j := \eta_j \hat{\mathcal{R}}(w_j)$, whereby $\eta_j |\nabla \hat{\mathcal{R}}(w_j)| = \hat{\eta}_j \gamma_j$. $\ln \hat{\mathcal{R}}$ will be crucial when analyzing parameter convergence over $S^\perp$.

### 3.1 Problem structure

The first step towards Theorem 1.2 is to pin down the problem structure: developing the subspaces $(S, S^\perp)$, the optimal point $\bar{v} \in S$, and the optimal direction $\bar{u} \in S^\perp$. The following theorem captures the properties needed to prove empirical risk and parameter convergence in later sections.

Theorem 3.1. There exists a unique partition of rows of $A$ into matrices $(A_S, A_c)$, and a corresponding pair of orthogonal subspaces $(S, S^\perp)$ where $S = \text{span}(A_S^\perp)$ satisfying properties below.

1. (Separable part.) If $A_c$ is nonempty, then there exists a unique unit vector $\bar{u} \in S^\perp$ with $A_S \bar{u} = 0$ and $A_c \bar{u} = A_\perp \bar{u} \leq -\gamma$ where
\[ \gamma := -\min \left\{ \max_i (A_i^\perp u)_i : |u| = 1 \right\} = \min \left\{ |A_i^\perp q| : q \geq 0, \sum_i q_i = 1 \right\} > 0, \]

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and \( \bar{u} \) is the unique optimum to the primal optimization problem, and satisfies \( \bar{u} = -A_1^\top \bar{q}/\gamma \) for every dual optimum \( \bar{q} \). Moreover, if \( \lim_{z \to -\infty} \ell(z) = 0 \) and \( \ell \geq 0 \), then

\[
0 = \inf_{u \in S^\perp} L(A_c u) = \lim_{r \to \infty} L(rA_c \bar{u}), \quad \inf_{w \in \mathbb{R}^d} L(Aw) = \inf_{v \in S} L(A_S v) + \inf_{u \in S^\perp} L(A_c u).
\]

2. (Strongly convex part.) If \( \ell \) is twice continuously differentiable with \( \ell'' > 0 \), \( \ell \geq 0 \), and \( \lim_{z \to -\infty} \ell(z) = 0 \), then \( \mathcal{R}_S = L \circ A_S \) has compact sublevel sets over \( S \), is strongly convex on those sublevel sets, and admits a unique minimizer \( \hat{v} \in S \) with \( \inf_{w \in \mathbb{R}^d} L(Aw) = \inf_{v \in S} L(A_S v) = L(A_S \hat{v}) \).

To develop an intuition for \((S, S^\perp)\), first consider a familiar setting: separable data. As in Figure 1a the separable setting supposes the existence of a unit vector \( \bar{u} \) and a positive scalar \( \gamma \) so that \( \langle \bar{u}, y, x_i \rangle \geq \gamma \) for all \( i \); equivalently, \( A\bar{u} \leq -\gamma \). This quantity \( \gamma \) can be chosen by maximization: \( \gamma := \min_i \langle \bar{u}, y, x_i \rangle \). This quantity is the margin between the separator corresponding to \( \bar{u} \), and the closest data point. It is natural to choose \( \bar{u} \) by maximizing this margin.

In the notation of Theorem 3.1 in this separable setting, \( S = \{0\} \), \( S^\perp = \mathbb{R}^d \), and \( A = A_c = A_\perp \). Part 1 of Theorem 3.1 captures basic properties of the margin. The proof is based on convex duality, the additional uniqueness of \( \bar{u} \) being a consequence of the curvature of the 2-norm ball.

Now consider another familiar setting: strong convexity. Rather than assuming a strongly convex objective or adding a strongly convex regularizer, strong convexity arises here as a consequence of the data configuration. For example, suppose as depicted in Figure 1b that every unit vector \( \bar{v} \) has mixed signs. Then for each unit vector \( \bar{v} \) there is a row \( (\bar{v}, v) > 0 \), and thus \( L(rAv) \geq \ell(r \langle (\bar{v}, v) \rangle)^\top \infty \) as \( r \to \infty \). From here it follows that \( L \circ A \) has bounded sublevel sets, which gives the other properties in part 2 of Theorem 3.1 assuming a pure instance: \( A = A_S \), and if \( A \) has rank \( d \), \( S = \mathbb{R}^d \) and \( S^\perp = \{0\} \).

Besides the pure separable case \((A = A_c)\) and pure strongly convex case \((A = A_S)\), a third possibility can exist; see Figure 1c. In words, a simple way to construct non-separable non-strongly-convex data is to choose two orthogonal subspaces, and embed a separable instance in one and a strongly convex instance in the other.

In general, instances can fail to be so simple; see Figure 1d. Despite this, general instances can still be partitioned into a separable part and a strongly convex part. Consider the following procedure to isolate a separable component of \( A \). For each row \( a_i \), add it to \( A_c \) if there exists \( u_i \) so that \( A_u \leq 0 \) but \( \langle a_i, u_i \rangle < 0 \); otherwise add it to \( A_S \). Let \( u := \sum_i u_i \), it follows that \( A_c u < 0 \) whereas \( A_S u = 0 \). Therefore, regardless of what happens over \( A_S \), it is always possible to add in a large multiple of \( u \) and do arbitrarily well over \( A_c \).

This idea is used in the full proof of Theorem 3.1 in Appendix 3.2

### 3.2 Empirical risk convergence

Gradient descent decreases the empirical risk as follows.

**Theorem 3.2.** For \( \ell \in \{\ell_{\log}, \ell_{\exp}\} \) and \( t \geq 1 \),

\[
\hat{R}(w_t) - \inf_w R(w) \leq \frac{\exp [(\bar{v})]}{t} + \frac{|\bar{v}| + \ln(t)^2/\gamma^2}{2 \sum_{j \leq t} \eta_j}.
\]
Fixing step sizes in Theorem 3.2 provides a proof for the first part of Theorem 1.2. The proof of Theorem 3.2 has three main steps.

1. A standard smoothness-based gradient descent guarantee (cf. Lemma 3.3).

2. A useful comparison point to feed into the preceding gradient descent bound (cf. Lemma 3.4): the choice \( \bar{\nu} = \bar{\nu}(\ln(t)/\gamma) \), made possible by Theorem 3.1.

3. Smoothness estimates for \( \hat{R} \) when \( \ell \in \{ \ell_{\log}, \ell_{\exp} \} \) (cf. Lemma 3.5).

In more detail, the first step, a standard smoothness-based gradient descent guarantee (Bubeck 2013; Nesterov 2004), is as follows.

**Lemma 3.3.** Suppose \( f \) is convex, and there exists \( \beta \geq 0 \) so that \( \eta_j \beta \leq 1 \) and gradient iterates \((w_0, \ldots, w_t)\) with \( w_{t+1} := w_t - \eta_j \nabla f(w_j) \) satisfy

\[
    f(w_{j+1}) \leq f(w_j) - \eta_j \left(1 - \frac{\eta_j \beta}{2}\right) \|\nabla f(w_j)\|^2.
\]

Then for any \( z \in \mathbb{R}^d \),

\[
    \left(2 \sum_{j<\ell} \eta_j \right) (f(w_\ell) - f(z)) \leq 2 \sum_{j<\ell} \eta_j (f(w_{j+1}) - f(z)) \leq |w_0 - w|^2 - |w_\ell - w|^2.
\]

The second step produces a reference point \( z \) to plug into Lemma 3.3 provided by Theorem 3.1.

**Lemma 3.4.** For \( \ell \in \{ \ell_{\log}, \ell_{\exp} \} \) and \( t \geq 1 \), \( z := \bar{\nu} = \bar{\nu}(\ln(t)/\gamma) \) satisfies \( |z|^2 = |\bar{\nu}|^2 + \ln(t)^2/\gamma^2 \) and

\[
    \hat{R}(z) \leq \inf_w \hat{R}(w) + \frac{\exp(|\bar{\nu}|)}{\ell}.
\]

Lastly, the smoothness guarantee on \( \hat{R} \). Although \( \ell_{\log} \) is smooth, this proof gives a refined smoothness where step-\( j \) is \( \hat{R}(w_j) \)-smooth; this refinement will be essential when proving parameter convergence. This proof is based on the convergence guarantee for AdaBoost (Schapire and Freund 2012).

**Lemma 3.5.** Suppose \( \ell \) is convex, \( \ell' \leq \ell \leq \ell'' \), and \( \eta_j \hat{R}(w_j) \leq 1 \), whereby \( \bar{\eta}_j := \eta_j \hat{R}(w_j) \leq 1 \).

\[
    \hat{R}(w_{j+1}) \leq \hat{R}(w_j) - \eta_j \left(1 - \frac{\eta_j \hat{R}(w_j)}{2}\right) \|\nabla \hat{R}(w_j)\|^2 = \hat{R}(w_j) \left(1 - \bar{\eta}_j (1 - \bar{\eta}_j/2)\gamma_j^2\right),
\]

and thus

\[
    \ln \hat{R}(w_\ell) \leq \ln \hat{R}(w_0) - \sum_{j<\ell} \bar{\eta}_j (1 - \bar{\eta}_j/2)\gamma_j^2.
\]

Combining these pieces now leads to a proof of Theorem 3.2 given in full in Appendix C.

### 3.3 Parameter convergence

First define optimal iterates \( \bar{w}_t := \arg\min \{ \hat{R}(w) : |w| \leq |w_t| \} \). As in Theorem 1.2, parameter convergence gives convergence to \( \bar{v} \in S \) over the strongly convex part \( S (\Pi_S w_t \to \bar{v} \text{ and } \Pi_S \bar{w}_t \to \bar{v}) \), and convergence in direction (convergence of the normalized iterates) to \( \bar{u} \in S^\perp \) over the separable part \( S^\perp (w_t/|w_t| \to \bar{u} \text{ and } \bar{w}_t/|\bar{w}_t| \to \bar{u}) \). In more detail, the convergence rates are as follows.

**Theorem 3.6.** Let loss \( \ell \in \{ \ell_{\exp}, \ell_{\log} \} \) be given. Suppose \( \eta_j = 1/\sqrt{j + 1}, t \geq 5, \) and \( \sqrt{t/\ln(t)} \geq n(1 + R) / \gamma^2 \), where \( R := \sup_{j<\ell} \|\Pi_S w_j\| = O(1) \). Then \( \|\Pi_S w_t\| = \Theta(1), \|\Pi_S \bar{w}_t\| = \Theta(\ln(t)), \)

\[
    \max \left\{ \|\Pi_S w_t - \bar{v}\|^2, \|\Pi_S \bar{w}_t - \bar{v}\|^2 \right\} = O \left(\frac{\ln(t)^2}{\sqrt{t}}\right),
\]

\[
    \max \left\{ \frac{w_t}{|w_t|} - \bar{u}, \frac{\bar{w}_t}{|\bar{w}_t|} - \bar{u} \right\} = O \left(\frac{\ln n - \ln |w_t|}{|w_t| \gamma^2}\right) = O \left(\frac{\ln n + \ln n}{\gamma^2 \ln(t)}\right).
\]
Convergence over $S$ is a quick consequence of strong convexity. By Theorem 3.1, $\tilde{R}_S(\bar{v}) = \inf_w \tilde{R}(w)$, and $\tilde{R}_S$ is strongly convex on sublevel sets over $S$. Let $\lambda$ denote the modulus of strong convexity of $\tilde{R}_S$ on the 1-sublevel set,

$$\|\Pi_S w_t - \bar{v}\|^2 \leq \frac{2}{\lambda} \left( \tilde{R}_S(w_t) - \tilde{R}_S(\bar{v}) \right) \leq \frac{2}{\lambda} \left( \tilde{R}(w_t) - \inf_w \tilde{R}(w) \right).$$

Convergence over $S$ then follows from Theorem 3.2.

To show convergence over $S^\perp$, start with the separable case (i.e., $A = A_c = A_\perp$). It is enough to show $\langle w_t / |w_t|, \bar{u} \rangle$ is close to 1. Applying the primal-dual property in Theorem 3.1 and Fenchel-Young,

$$\left\langle \frac{w_t}{|w_t|}, \bar{u} \right\rangle = -\left\langle \frac{w_t}{|w_t|}, A^T \bar{q} \right\rangle = -\left( \frac{\langle A w_t, \bar{q} \rangle}{|w_t| \gamma} \right) \geq -\frac{\ln \exp(A w_t)}{|w_t| \gamma} - \frac{g^*(q)}{|w_t| \gamma},$$

where $g^*(q) = \ln n + \sum_{i=1}^n q_i \ln q_i \leq \ln n$.

Since $L(A w_t) \to 0$ while there is no finite optimum, $|w_t| \to \infty$, and thus the second term converges to 0. For the first term, in $L_{exp}(A w_t)$ can be upper bounded by Lemma 3.5 yielding an expression which will cancel with the denominator since $|w_t| \leq \sum_{j=1}^n q_j \gamma_j$. To get an explicit convergence rate, just note $|w_t| = \Theta(\ln(t))$; since $|A_c| \leq 1$, $|w_t|$ has to be large enough to make the empirical risk as small as guaranteed by Theorem 3.2.

To handle the general case, the following techniques have to be applied.

1. In the non-separable case, Lemma 3.5 is not enough to ensure the required cancellation. Lemma 3.5 helps in the separable case because it captures the correct local smoothness (i.e., $\tilde{R}$ is $\tilde{R}(w_j)$-smooth near $w_j$). In the non-separable non-strongly-convex case (i.e., both $A_S$ and $A_c$ are non-empty), while $\tilde{R}(w_j) \geq \inf_w \tilde{R}(w) > 0$, $\Pi_\perp w_t$ is unbounded and experiences near-flat smoothness, and thus Lemma 3.5 does not give enough smoothness over $S^\perp$. The key step here is to replace the appearance of $\tilde{R}$ in eq. (3.7) with $\tilde{R}^\perp(w_t) - \inf_w \tilde{R}(w)$, and adapt Lemma 3.5 to control $\ln \left( \tilde{R}(w_t) - \inf_w \tilde{R}(w) \right)$ (cf. Lemma D.6).

2. $\ell_{\log}$ is handled via its proximity to $\ell_{exp}$. It is always true that $\ell'_{\log}(x) \leq \ell_{\log}(x) \leq \ell_{exp}(x)$, and when $x < 0$ and $|x|$ is large, $\ell'_{\log}(x) \approx \ell_{\log}(x) \approx \ell_{exp}(x)$ (cf. Lemma D.1). More specifically, after applying Fenchel-Young, the adapted iteration guarantee Lemma D.6 is applied from some $t_0$ to $t$, where $t_0$ is large enough so that $\ell_{\log}$ is close enough to $\ell_{exp}$. However, this step will lead to an additional $|w_t|$ term in the numerator.

3. The above $|w_t|$ is handled via a general result that $|w_t| = \Theta(\ln(t))$ (cf. Lemma D.2). The proof of the upper bound is similar to the proof of Lemma 2.3 with a careful treatment of the interplay between $A_S$ and $A_c$. The proof of the lower bound is similar as above: $|w_t|$ has to be large enough to minimize the empirical risk as guaranteed by Theorem 3.2.

4 Open problems

**SGD risk convergence without separability.** In the convergence rate of SGD (cf. Theorem 1.1), it is assumed that the underlying data distribution is separable. What about the non-separable case? Since the perceptron-style analysis relies on a positive margin, it is not clear how to adapt the analysis in Section 2.

**SGD parameter convergence.** With or without separability, does SGD exhibit parameter convergence in the same sense as GD (cf. Theorem 1.2)? Given $t$th SGD iterate $w_t$ and $t'$th GD iterate $v_t$, how large is the angle between the directions $w_t / |w_t|$ and $v_t / |v_t|$?

**Improved parameter convergence guarantees for GD.** Give a separable finite sample satisfying a few further conditions, Soudry et al. (2017) show that gradient descent decreases $|w_t / |w_t| - \bar{u}|^2$ at the rate $O(1/\ln(t)^2)$. Is it possible to prove such a rate for non-separable data?
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A Omitted proofs from Section 2

Proof of Lemma 2.5. By Cauchy-Schwarz inequality, for any \( w \in \mathbb{R}^d \),
\[
|w_{j+1} - w|^2 = |w_j - w|^2 - 2\eta_j \langle g_j, w_j - w \rangle + \eta_j^2 |g_j|^2 \\
\leq |w_j - w|^2 - 2\eta_j \left( \ell(\langle w_j, z_j \rangle) - \ell(\langle w, z_j \rangle) \right) + \eta_j^2 |g_j|^2. \tag{A.1}
\]
Furthermore, since \( \eta_j \leq 1 \), \( |z_j| \leq 1 \), \( 0 \leq \ell' \leq 1 \), and \( \ell' \leq \ell \),
\[
\eta_j^2 |g_j|^2 = \eta_j^2 |z_j \ell'(\langle w_j, z_j \rangle)|^2 \\
\leq \eta_j \ell'(\langle w_j, z_j \rangle) \\
\leq \eta_j \ell(\langle w_j, z_j \rangle). \tag{A.2}
\]
Combining eq. (A.1) and eq. (A.2),
\[
|w_{j+1} - w|^2 \leq |w_j - w|^2 - 2\eta_j \left( \ell(\langle w_j, z_j \rangle) - \ell(\langle w, z_j \rangle) \right) + \eta_j \ell(\langle w_j, z_j \rangle) \\
= |w_j - w|^2 - \eta_j \ell(\langle w_j, z_j \rangle) + 2\eta_j \ell(\langle w, z_j \rangle) \\
\leq |w_j - w|^2 + 2\eta_j \ell(\langle w, z_j \rangle), \tag{A.3}
\]
since \( \ell \geq 0 \). Summing eq. (A.3) from 0 to \( t-1 \), the first claim of Lemma 2.2 gets proved.

By the separability assumption, \( \langle \bar{u}, z_j \rangle \leq -\gamma \). Therefore \( \langle u_t, z_j \rangle \leq -\ln(t) \), and since \( \ell \leq \exp \),
\[
2 \sum_{j<t} \eta_j \ell(\langle u_t, z_j \rangle) \leq 2 \sum_{j<t} \eta_j \exp(-\ln(t)) \leq 2 \sum_{j<t} \eta_j \leq 2.
\]

The following martingale Bernstein bound is used in the proof of Lemma 2.5

Lemma A.4 ([Beygelzimer et al., 2011] Theorem 1). Let \((M_t, \mathcal{F}_t)_{t \geq 0}\) denote a martingale with \( M_0 = 0 \) and \( \mathcal{F}_0 \) be the trivial \(\sigma\)-algebra. Let \((\Delta_t)_{t \geq 1}\) denote the corresponding martingale difference sequence. Assume \(\Delta_t \leq R\) a.s., and let
\[
V_t = \sum_{j=1}^t E \left[ |\Delta_j^2|^{\mathcal{F}_{j-1}} \right]
\]
denote the sequence of conditional variance. Then for any \( 0 < \delta < 1 \), with probability at least \( 1 - \delta \),
\[
M_t \leq \frac{V_t}{R} (e - 2) + R \ln \left( \frac{1}{\delta} \right).
\]

Proof of Lemma 2.5. By Cauchy-Schwarz inequality,
\[
\langle w_t - u_t, \bar{u} \rangle \leq |w_t - u_t| |\bar{u}| = |w_t - u_t|. \tag{A.5}
\]

Plugging Lemma 2.2 and eq. (2.4) into eq. (A.5) gives
\[
\gamma'_{<t} - \frac{\ln(t)}{\gamma} \leq \sqrt{\left( \frac{\ln(t)}{\gamma^2} \right)^2 + 2} \leq \frac{\ln(t)}{\gamma} + 2,
\]
and thus
\[
\ell'_{<t} \leq \frac{2\ln(t)}{\gamma^2} + \frac{2}{\gamma}. \tag{A.5}
\]

For the high-probability bound, note that \( \ell'_{<t} - \ell'_{<t} \) is a martingale w.r.t. the filtration \( \mathcal{F}_t = \sigma(z_0, \ldots, z_{t-1}) \).
Let
\[
\ell' = \eta \ell'(\langle w_t, z_t \rangle) \quad \text{and} \quad \mathcal{L}' = \eta E \left[ \ell'(\langle w_t, z_t \rangle) | z_{0:t-1} \right],
\]
where \( \eta > 0 \) is a parameter.
where \( z_{0,t-1} \) means \( \sigma(z_0, \ldots, z_{t-1}) \). Then \( \mathcal{L}'_t - \ell'_t \) gives the martingale difference sequence.

Since \( 0 \leq \ell'_t \leq 1, \eta_t \leq 1 \),

\[
\mathcal{L}'_t - \ell'_t = \eta_t \mathbb{E} \left[ \ell'(\langle w_t, z_t \rangle) \big| z_{0,t-1} \right] - \eta_t \ell'(\langle w_t, z_t \rangle) \leq \eta_t \cdot 1 - \eta_t \cdot 0 \leq 1 \leq 2. \tag{A.6}
\]

Furthermore, \( 0 \leq \ell'_t \leq 1 \),

\[
\mathbb{E} \left[ (\mathcal{L}'_t - \ell'_t)^2 \big| z_{0,t-1} \right] = (\mathcal{L}'_t)^2 - 2 \mathcal{L}'_t \mathbb{E} \left[ \ell'_t \big| z_{0,t-1} \right] + \mathbb{E} \left[ (\ell'_t)^2 \big| z_{0,t-1} \right] \\
= -(\mathcal{L}'_t)^2 + \mathbb{E} \left[ (\ell'_t)^2 \big| z_{0,t-1} \right] \\
\leq \mathbb{E} \left[ (\ell'_t)^2 \big| z_{0,t-1} \right] \\
\leq \mathbb{E} \left[ \ell'_t \big| z_{0,t-1} \right] \\
= \mathcal{L}'_t. \tag{A.7}
\]

Plugging eq. (A.6) and eq. (A.7) into Lemma [A.4], with probability at least \( 1 - \delta \),

\[
\mathcal{L}_{<t}' - \ell_{<t}' \leq \mathcal{L}_{<t}' - \frac{e^r - 2}{2} + 2 \log \left( \frac{1}{\delta} \right),
\]

which gives the high-probability bound of Lemma [2.5] after rearrangement.

\textit{Proof of Lemma 2.6}. Notice

\[
\left( \frac{\ell'_x(x)}{\ell''_x(x)} \right)' = \frac{\ell'_x(x)^2 - \ell'_x(x)\ell''_x(x)}{\ell'_x(x)^2} = \frac{e^x - \ln(1 + e^x)}{\ell'_x(x)^2(1 + e^x)^2} e^x \\
\geq 0,
\]

and thus \( \ell'_x/\ell''_x \) is non-decreasing. Given \( x \leq B \) with \( B \geq 0 \),

\[
\frac{\ell'_x(x)}{\ell''_x(x)} \leq \frac{\ell'_x(B)}{\ell''_x(B)}.
\]

To finish,

\[
\ell'_x(B) = \ell'_x(0) + \left( \ell'_x(B) - \ell'_x(0) \right) \\
= \frac{\ell'_x(0)}{\ell''_x(0)} \ell'_x(B) + \int_0^B \ell'_x(x) dx \\
= \frac{\ell'_x(0)}{\ell''_x(0)} \ell'_x(B) + \int_0^B e^x dx \\
\leq (2 \ln 2) \ell'_x(B) + 2 \ell'_x(B) \\
\leq \ell'_x(B) (B + 2).
\]

\textit{Proof of Theorem 2.1}. In this proof, \( \ell = \ell'_x \). Let \( z_{0,j-1} = \sigma(z_0, \ldots, z_{j-1}) \). By Lemma 2.6

\[
\sum_{j < t} \eta_j \mathcal{R}(w_j) = \sum_{j < t} \eta_j \mathbb{E} \left[ \ell(\langle w_j, z_j \rangle) \big| z_{0,j-1} \right] \\
\leq \sum_{j < t} \eta_j \mathbb{E} \left[ \ell'(\langle w_j, z_j \rangle) \big| w_j, z_j \big| z_{0,j-1} \right] \tag{A.8} \\
\leq \sum_{j < t} \eta_j \mathbb{E} \left[ \ell'(\langle w_j, z_j \rangle) \big| w_j \big| z_{0,j-1} \right].
\]
Moreover, there exists a unique nonzero primal optimum \( \bar{u} \), but then the unit vector \( u \) is unique since the objective value will only decrease by increasing the length. Consequently, suppose \( u \) and moreover every primal-dual optimal pair \( (\bar{u}, \bar{q}) \), which means \( \bar{u} = -A^\top \bar{q}/\gamma \).

Proof of Lemma B.1. To start, note \( \gamma > 0 \) since there exists \( u \) with \( A_u < 0 \).

Continuing, for convenience define simplex \( \Delta := \{ q \in \mathbb{R}^n : \sum_i q_i = 1 \} \), and convex indicator \( \iota_K(z) = \infty \cdot 1[z \in K] \). With this notation, note the Fenchel conjugates

\[
\iota^*_\Delta(v) = \sup_{q \in \Delta} \langle v, q \rangle = \max_i v_i,
\]

\[
(\| \cdot \|_2)^*(q) = \iota_{\| \cdot \|_2}(q).
\]

Combining this with the Fenchel-Rockafellar duality theorem (Borwein and Lewis 2000, Theorem 3.3.5, Exercise 3.3.9.f),

\[
\min |A^\top q|_2 + \iota_\Delta(q) = \max -\iota_{\| \cdot \|_2}(u) - \iota^*_\Delta(-A_u)
\]

\[
= \max \left\{ -\max_i (-A_u)_i : |u|_2 \leq 1 \right\}
\]

\[
= -\min \left\{ \max_i (A_u)_i : |u|_2 \leq 1 \right\},
\]

and moreover every primal-dual optimal pair \( (\bar{u}, \bar{q}) \) satisfies \( A^\top \bar{q} \in \partial \left( \iota_{\| \cdot \|_2} \right)(-\bar{u}) \), which means \( \bar{u} = -A^\top \bar{q}/\gamma \).

It only remains to show that \( \bar{u} \) is unique. Since \( \gamma > 0 \), necessarily any primal optimum has unit length, since the objective value will only decrease by increasing the length. Consequently, suppose \( u_1 \) and \( u_2 \) are two primal optimal unit vectors. Then \( u_3 := (u_1 + u_2)/2 \) would satisfy

\[
\max_i (A_u u_3)_i = \frac{1}{2} \max_i (A_u u_1 + A_u u_2)_i \leq \frac{1}{2} \left( \max_i (A_u u_1) + \max_j (A_u u_2)_j \right) = \max_i (A_u u_1)_i,
\]

but then the unit vector \( u_4 := u_3/|u_3| \) would have \( |u_4| > |u_3| \) when \( u_1 \neq u_2 \), and thus \( \max_i (A_u u_4) < \max_i (A_u u_3) = \max_i (A_u u_1) \), a contradiction.

B Omitted proofs from Section 3.1

First, the primal-dual characterization of margin is proved.

**Lemma B.1.** Suppose \( A_u \) has \( n_c > 0 \) rows and there exists \( u \) with \( A_u u < 0 \). Then

\[
\gamma := -\min \{ \max_i (A_u u)_i : |u| = 1 \} = \min \left\{ |A^\top q| : q \geq 0, \sum_i q_i = 1 \right\} > 0.
\]

Moreover there exists a unique nonzero primal optimum \( \bar{u} \), and every dual optimum \( \bar{q} \) satisfies \( \bar{u} = -A^\top \bar{q}/\gamma \).

Proof of Lemma B.1. To start, note \( \gamma > 0 \) since there exists \( u \) with \( A_u u < 0 \).

Combining this with the Fenchel-Rockafellar duality theorem (Borwein and Lewis 2000, Theorem 3.3.5, Exercise 3.3.9.f),

\[
\sum_{j < t} \eta_j R(w_j) \leq \sum_{j < t} \eta_j E \left[ \ell'((w_j, z_j)) | w_j | | z_{0,j-1} \right]
\]

\[
\leq \left( \frac{2 \ln(t)}{\gamma} + 4 \right) \sum_{j < t} \eta_j E \left[ \ell'((w_j, z_j)) | z_{0,j-1} \right]
\]

\[
= \left( \frac{2 \ln(t)}{\gamma} + 4 \right) L'<_t
\]

\[
\leq \left( \frac{2 \ln(t)}{\gamma} + 4 \right) \left( \frac{4 \ln(t)}{\gamma^2} + \frac{1}{\gamma} + \ln \left( \frac{1}{\delta} \right) \right)
\]

\[
\leq 16 \left( \frac{\ln(t)}{\gamma} + 1 \right) \left( \frac{\ln(t)}{\gamma^2} + \frac{1}{\gamma} + \ln \left( \frac{1}{\delta} \right) \right).
\]

\[
\square
\]
The proof of Theorem 3.1 follows.

Proof of Theorem 3.1. Partition the rows of $A$ into $A_c$ and $A_S$ as follows. For each row $i$, put it in $A_c$ if there exists $u_i$ so that $Au_i \leq 0$ (coordinate-wise) and $(Au_i)_i < 0$; otherwise, when no such $u_i$ exists, add this row to $A_S$. Define $S := \text{span}(A_S)$, the linear span of the rows of $A_S$. This has the following consequences.

- To start, $S^\perp = \text{span}(A_S^\perp) \supset \ker(A)$.
- For each row $i$ of $A_c$, the corresponding $u_i$ has $A_Su_i = 0$, since otherwise $Au_i \leq 0$ implies there would be a negative coordinate of $A_Su_i$, and this row should be in $A_c$ not $A_S$. Combining this with the preceding point, $u_i \in \ker(A_S) = S^\perp$. Define $\tilde{u} := \sum_i u_i \in S^\perp$, whereby $A_c\tilde{u} < 0$ and $A_S\tilde{u} = 0$. Lastly, $\tilde{u} \in \ker(A_S)$ implies moreover that $A_c\tilde{u} = A_c\tilde{u} < 0$. As such, when $A_c$ has a positive number of rows, Lemma B.1 can be applied, resulting in the desired unique $\tilde{u} = -A_c^\perp \tilde{q} / \gamma \in S^\perp$ with $\gamma > 0$.
- $S$, $S^\perp$, $A_S$, and $A_c$, and $\tilde{u}$ are unique and constructed from $A$ alone, with no dependence on $\ell$.
- First consider part 1 of Theorem 3.1. If $A_c$ is empty, there is nothing to show, thus suppose $A_c$ is nonempty. Since $\lim_{z \to -\infty} \ell(z) = 0$, 
  
  \[
  0 \leq \inf_{w \in \mathbb{R}^d} L(A_c w) \leq \inf_{w \in S^\perp} L(A_c w) \leq \inf_{u \in S^\perp} L(A_c u) \leq \lim_{r \to \infty} L(r \cdot A_c \tilde{u}) = 0.
  \]

  Since these inequalities start and end with 0, they are equalities, and $\inf_{w \in \mathbb{R}^d} = \inf_{u \in S^\perp} L(A_c u) = 0$. Moreover,

  \[
  \begin{align*}
  \inf_{w \in \mathbb{R}^d} L(Aw) &= \inf_{u \in S^\perp} \left( L(A_S(u+v)) + L(A_c(u+v)) \right) \\
  &= \inf_{u \in S} \left( L(A_Sv) + \inf_{u \in S^\perp} L(A_c(u+v)) \right) \\
  &\leq \inf_{v \in S} \left( L(A_Sv) + \inf_{r > 0} L(A_c(r\tilde{u} + v)) \right) \\
  &= \left( \inf_{v \in S} L(A_Sv) \right) \leq \inf_{w \in \mathbb{R}^d} L(Aw),
  \end{align*}
  \]

  which again is in fact a chain of equalities.
- Now consider part 2 of Theorem 3.1. If $A_S$ is empty, there is nothing to show, therefore suppose $A_S$ is nonempty. For every $v \in S$ with $|v| > 0$, there exists a row $a$ of $A_S$ such that $\langle a, v \rangle > 0$. To see this, suppose contradictorily that $A_Sv \leq 0$. It cannot hold that $A_Sv = 0$, since $v \neq 0$ and $\ker(A_S) \subseteq S^\perp$. This means $A_Sv < 0$ and moreover $(A_Sv)_i < 0$ for some $i$. But since $A\tilde{u} \leq 0$ and $A_c\tilde{u} < 0$, then for a sufficiently large $r > 0$, $A(v + r\tilde{u}) \leq 0$ and $(A_S(v + r\tilde{u}))_j < 0$, which means row $j$ of $A_S$ should have been in $A_c$, a contradiction.
- Consider any $v \in S \setminus \{0\}$. By the preceding point, there exists a row $a$ of $A_S$ such that $\langle a, v \rangle > 0$. Since $\ell(0) > 0$ (because $\ell'' > 0$) and $\lim_{z \to -\infty} = 0$, there exists $r > 0$ so that $\ell(-r \langle a, v \rangle) = \ell(0)/2$. By convexity, for any $t > 0$, setting $\alpha := r/(t + r)$ and noting $\alpha \langle a, tv \rangle + (1 - \alpha) \langle a, -rv \rangle = 0$,

  \[
  \alpha \ell(t \langle a, v \rangle) \geq \ell(0) - (1 - \alpha)\ell(-r \langle a, v \rangle) = \left( \frac{1 + \alpha}{2} \right) \ell(0),
  \]

  thus $\ell(t \langle a, v \rangle) \geq \left( \frac{1 + \alpha}{2r} \right) \ell(0)$, and

  \[
  \begin{align*}
  \lim_{t \to \infty} \frac{L(t Av) - L(0)}{t} &\geq \lim_{t \to \infty} \frac{\ell(t \langle a, v \rangle) - n\ell(0)}{nt} \\
  &\geq \lim_{t \to \infty} \frac{\ell(0)}{2r} \left( \frac{(t + 2r) - 2nr}{nt} \right) \\
  &\geq \frac{\ell(0)}{2r} > 0.
  \end{align*}
  \]

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Consequently, $L \circ A$ has compact sublevel sets over $S$ ([Hiriart-Urruty and Lemaréchal 2001, Proposition B.3.2.4]).

- Note $\nabla^2 L(v) = \text{diag}(\ell''(v_1), \ldots, \ell''(v_n))$. Moreover, since $\ker(A) \subseteq S^\perp$, then the image $B_0 := \{Av : v \in S, |v| = 1\}$ over the surface of the ball in $S$ through $A$ is a collection of vectors with positive length. Thus for any compact subset $S_0 \subseteq S$,
  \[
  \inf_{v_1 \in S_0} \inf_{v_2 \in S, |v_2| = 1} v_2^T \nabla^2 (L \circ A)(v_1)v_2 = \inf_{v_1 \in S_0} \inf_{v_2 \in S, |v_2| = 1} (Av_2)^T \nabla^2 L(Av_1)(Av_2)
  \]
  \[
  = \inf_{v_1 \in S_0} \inf_{v_2 \in B_0} v_2^T \nabla^2 L(Av_1)v_2
  \]
  \[
  \geq \inf_{v_1 \in S_0} |v_3|^2 \min_i \ell''((v_1)_i) > 0,
  \]
  the final inequality since the minimization is of a continuous function over a compact set, thus attained at some point, and the infimand is positive over the domain. Consequently, $L \circ A$ is strongly convex over compact subsets of $S$.

- Since $L \circ A$ is strongly convex over $S$ and moreover has bounded sublevel sets over $S$, it attains a unique optimum over $S$.

$\blacksquare$

C  Omitted proofs from Section 3.2

To start, note how the three key lemmas provided in the main text lead to a proof of Theorem 3.2.

Proof of Theorem 3.2. Note that by Lemma 3.5 for $\ell \in \{\ell_{\log}, \ell_{\exp}\}$, since $\hat{\ell}(w_0) \leq 1$, choosing $\eta_j \leq 1$ will ensure $\eta_j \hat{\ell}(w_j) \leq 1$ and $\hat{\ell}(w_{j+1}) \leq \hat{\ell}(w_j) \leq 1$. As a result, $\hat{\ell}_{\exp}$ and $\hat{\ell}_{\log}$ satisfy conditions in Lemma 3.3 with $\beta = 1$. Thus, by Lemma 3.3 for any $z \in \mathbb{R}^d$,

\[
2 \left( \sum_{j < t} \eta_j \right) (\hat{\ell}(w_j) - \hat{\ell}(z)) \leq |w_0 - z|^2 - |w_t - z|^2 \leq |z|^2.
\]

Consequently, by the choice $z := \bar{v} + \bar{u}(\ln(t)/\gamma)$ and Lemma 3.4

\[
\hat{\ell}(w_t) \leq \hat{\ell}(z) + \frac{|z|^2}{2} \leq \inf_w \hat{\ell}(w) + \frac{\exp(|\bar{v}|)}{t} + \frac{|\bar{v}|^2 + \ln(t)^2/\gamma^2}{2 \sum_{j < t} \eta_j}.
\]

$\blacksquare$

To fill out the proof, first comes the smoothness-based risk guarantee.

Proof of Lemma 3.3. By conditions of Lemma 3.3

\[
f(w_{j+1}) \leq f(w_j) - \eta_j |\nabla f(w_j)|^2 + \frac{\eta_j^2 \beta}{2} |\nabla f(w_j)|^2
\]

\[
\leq f(w_j) - \eta_j |\nabla f(w_j)|^2 + \frac{\eta_j}{2} |\nabla f(w_j)|^2
\]

\[
= f(w_j) - \frac{\eta_j}{2} |\nabla f(w_j)|^2,
\]

and thus

\[
\eta_j^2 |\nabla f(w_j)|^2 \leq 2\eta_j \left( f(w_j) - f(w_{j+1}) \right).
\]
As a result,
\[ |w_{j+1} - z|^2 = |w_j - z|^2 + 2\eta_j \langle \nabla f(w_j), z - w_j \rangle + \eta_j^2 |\nabla f(w_j)|^2 \]
\[ \leq |w_j - z|^2 + 2\eta_j (f(z) - f(w_j)) + 2\eta_j (f(w_j) - f(w_{j+1})) \]
\[ = |w_j - z|^2 + 2\eta_j (f(z) - f(w_{j+1})). \]

Summing this inequality over \( j \in \{0, \ldots, t\} - 1 \) and rearranging gives the bound. \( \square \)

**Proof of Lemma C.2.** By Theorem 3.1 and since \( \ell_{\text{log}} \leq \ell_{\exp} \) and \( |A_i| \leq 1 \),
\[ L(Az) = L(Aw_{\bar{v}}) + L(A \nu) \leq \inf_w L(Aw) + \exp \left( \langle |\bar{v}| - \ln(t) \rangle = \inf_w L(Aw) + \frac{\exp(\langle |\bar{v}| \rangle)}{t}. \]

With that out of the way, the remainder of this subsection establishes smoothness properties of \( \hat{R} \). For convenience, for the rest of this subsection define \( w' := w - \eta \nabla \hat{R}(w) \). Additionally, suppose throughout that \( \ell \) is twice differentiable.

**Lemma C.1.** For any \( w \in \mathbb{R}^d \),
\[ \hat{R}(w') \leq \hat{R}(w) - \eta |\nabla \hat{R}(w)|^2 + \frac{\eta^2}{2} |\nabla \hat{R}(w)|^2 \max_{v \in [w, w']} \sum_i \ell''(A_i; v)/n. \]

**Proof.** By Taylor expansion,
\[ \hat{R}(w') \leq \hat{R}(w) - \eta |\nabla \hat{R}(w)|^2 + \frac{1}{2} \max_{v \in [w, w']} \sum_i (A_i(w - w'))^2 \ell''(A_i; v)/n. \]

By Hölder’s inequality,
\[ \max_{v \in [w, w']} \sum_i (A_i(w - w'))^2 \ell''(A_i; v) \leq \max_{v \in [w, w']} |A(w - w')|^2 \sum_i \ell''(A_i; v). \]

Since \( \max_i |A_i| \leq 1 \),
\[ |A(w - w')|^2 = \eta^2 |A \nabla \hat{R}(w)|^2 \]
\[ = \eta^2 \max_{v} \langle A_i, \nabla \hat{R}(w) \rangle^2 \]
\[ \leq \eta^2 \max_{v} |A_i|^2 |\nabla \hat{R}(w)|^2 \]
\[ \leq \eta^2 |\nabla \hat{R}(w)|^2. \]
Thus
\[ \hat{R}(w') \leq \hat{R}(w) - \eta |\nabla \hat{R}(w)|^2 + \frac{\eta^2}{2} |\nabla \hat{R}(w)|^2 \max_{v \in [w, w']} \sum_i \ell''(A_i; v)/n. \]

**Lemma C.2.** Suppose \( \ell', \ell'' \leq \ell \) and \( \ell \) is convex. Then, for any \( w \in \mathbb{R}^d \),
\[ \max_{v \in [w, w']} \sum_i \ell''(A_i; v)/n \leq \max \left\{ \hat{R}(w), \hat{R}(w') \right\}. \]

Define \( \hat{\eta} := \eta \hat{R}(w) \) and suppose \( \hat{\eta} \leq 1 \); then \( \hat{R}(w') \leq \hat{R}(w) \) and
\[ \hat{R}(w') \leq \hat{R}(w) \left( 1 - \hat{\eta}(1 - \hat{\eta}/2) \frac{|\nabla \hat{R}(w)|^2}{\hat{R}(w)^2} \right). \]
Proof. Since $\ell'' \leq \ell$ and $\ell$ is convex,

$$\max_{v \in [w, w']} \sum_i \ell''(A_i v)/n \leq \max_{v \in [w, w']} \sum_i \ell(A_i v)/n = \max_{v \in [w, w']} \hat{R}(v) = \max \{ \hat{R}(w), \hat{R}(w') \}.$$ 

Combining this, the choice of $\eta$, and Lemma D.1,

$$\hat{R}(w') \leq \hat{R}(w) - \eta |\nabla \hat{R}(w)|^2 + \frac{\eta^2}{2} |\nabla \hat{R}(w)|^2 \max \{ \hat{R}(w), \hat{R}(w') \}$$

$$= \hat{R}(w) - \hat{\eta} |\nabla \hat{R}(w)|^2 \left( 1 - \frac{\hat{\eta}}{2} \max \{ \hat{R}(w), \hat{R}(w') \} \right).$$

As a final simplification, suppose $\hat{R}(w') > \hat{R}(w)$; since $\hat{\eta} \leq 1$ and $\ell'' \leq \ell$ and $\max_i |A_i| \leq 1$,

$$\frac{\hat{R}(w')}{\hat{R}(w)} - 1 \leq \frac{\hat{\eta} |\nabla \hat{R}(w)|^2}{\hat{R}(w)} \left( \frac{\hat{\eta} \hat{R}(w')}{2 \hat{R}(w)} - 1 \right) \leq \hat{\eta} \left( \frac{\hat{\eta} \hat{R}(w')}{2 \hat{R}(w)} - 1 \right) \leq \frac{\hat{\eta} \hat{R}(w')}{2 \hat{R}(w)} - 1 \leq \frac{\hat{\eta} \hat{R}(w')}{2 \hat{R}(w)} - 1,$$

a contradiction. Therefore $\hat{R}(w') \leq \hat{R}(w)$, which in turn implies

$$\hat{R}(w') \leq \hat{R}(w) - \hat{\eta} |\nabla \hat{R}(w)|^2 \left( 1 - \frac{\hat{\eta}}{2} \right).$$

Together, these pieces prove the desired smoothness inequality.

Proof of Lemma 3.5. For any $j \leq t$, by Lemma C.2 and the definition of $\gamma_j$,

$$\hat{R}(w_{j+1}) \leq \hat{R}(w_j) \left( 1 - \hat{\eta}_j (1 - \hat{\gamma}_j/2) |\nabla \hat{R}(w_j)|^2/\hat{R}(w_j)^2 \right) = \hat{R}(w_j) \left( 1 - \hat{\eta}_j (1 - \hat{\gamma}_j/2) \gamma_j^2 \right).$$

Applying this recursively gives the bound.

D Omitted proofs from Section 3.3

D.1 Convergence over $S$

Lemma D.1. Let $\ell \in \{ \ell_{\exp}, \ell_{\log} \}$, and $\lambda$ denote the modulus of strong convexity of $\hat{R}_S$ over the 1-sublevel set (guaranteed positive by Theorem 3.7). With step sizes $\eta_j = 1/\sqrt{j} + 1$, for any $t \geq 1$,

$$\max \left\{ |\Pi_S w_t - \bar{v}|^2, |\Pi_S \bar{w}_t - \bar{v}|^2 \right\} \leq \frac{2}{\lambda} \min \left\{ 1, \frac{\exp(\frac{1}{t}) + |\bar{v}|^2 + \ln(t)^2/\gamma^2}{2 \sum_{j < t} \eta_j} \right\}.$$ 

Proof. By Theorem 3.1 $\hat{R}_S(\bar{v}) = \inf_w \hat{R}(w)$. Thus, by strong convexity, for $z \in \{ w_t, \bar{w}_t \}$ (whereby $\hat{R}(z) \leq \hat{R}(w_t)$),

$$|\Pi_S z - \bar{v}|^2 \leq \frac{2}{\lambda} \left( \hat{R}_S(z) - \hat{R}_S(\bar{v}) \right) \leq \frac{2}{\lambda} \left( \hat{R}(w_t) - \inf_w \hat{R}(w) \right).$$

The bound follows by noting $\hat{R}(w_t) \leq \hat{R}(w_0) \leq 1$, and alternatively invoking in Theorem 3.2. 


D.2 Bounding $|\Pi_\perp w_t|$  

Firstly, upper and lower bounds on $|w_t|$ is given.

**Lemma D.2.** Suppose $A_c$ is nonempty, $\ell \in \{\ell_{\text{exp}}, \ell_{\text{log}}\}$. Define $R := \sup_{j < t} |\Pi_\perp w_j - w_j|$, where Lemma D.1 guarantees $R = O(1)$. Let $n_c > 0$ denote the number of rows in $A_c$. For any $t \geq 1$,

$$ |\Pi_\perp w_t| \leq \max \left\{ \frac{2 \ln(t) + 2R}{\gamma^2}, 2 \right\}, $$

$$ |\Pi_\perp w_t| \geq \min \left\{ \ln(t) - \ln(2) - |\bar{v}|, \ln \left( \sum_{j=0}^{t-1} \eta_j \right) - \ln \left( |\bar{v}|^2 + \ln(t)^2 / \gamma^2 \right) \right\} - R + \ln \left( \ln(2) - \ln(n/n_c) \right). $$

The upper bound proof starts with a smoothness-based convergence guarantee with sensitivity to $A_c$.

**Lemma D.3.** Let any $\ell \in \{\ell_{\text{exp}}, \ell_{\text{log}}\}$ be given, and suppose $\eta_j \leq 1$. For any $u \in S_\perp$ and $t \geq 1$,

$$ |\Pi_\perp w_t - u|^2 \leq |u|^2 + 2 + \sum_{j < t} 2\eta_j \left( \tilde{R}_c(u) - \tilde{R}_c(w_j) \right) + \sum_{j < t} 2\eta_j \left( \nabla \tilde{R}_c(w_j), w_j - \Pi_\perp w_j \right). $$

**Proof.** Fix any $u \in S_\perp$. Expanding the square,

$$ |\Pi_\perp w_{j+1} - u|^2 = |\Pi_\perp w_j - u|^2 + 2\eta_j \left( \Pi_\perp \nabla \tilde{R}(w_j), u - \Pi_\perp w_j \right) + \eta_j^2 \left| \Pi_\perp \nabla \tilde{R}(w_j) \right|^2, $$

whose two key terms can be bounded as

$$ \left( \Pi_\perp \nabla \tilde{R}(w_j), u - \Pi_\perp w_j \right) = \left( \nabla \tilde{R}(w_j), u - \Pi_\perp w_j \right) $$

$$ = \left( \nabla \tilde{R}(w_j), u - w_j \right) + \left( \nabla \tilde{R}(w_j), w_j - \Pi_\perp w_j \right) $$

$$ \leq \tilde{R}_c(u) - \tilde{R}_c(w_j) + \left( \nabla \tilde{R}(w_j), w_j - \Pi_\perp w_j \right), $$

$$ \eta_j^2 \left| \Pi_\perp \nabla \tilde{R}(w_j) \right|^2 \leq \eta_j \left| \nabla \tilde{R}(w_j) \right|^2 $$

$$ \leq 2 \left( \tilde{R}(w_j) - \tilde{R}(w_{j+1}) \right), $$

the last inequality making use of smoothness, namely Lemma 3.5. Therefore

$$ |\Pi_\perp w_{j+1} - u|^2 \leq |\Pi_\perp w_j - u|^2 + 2\eta_j \left( \tilde{R}_c(u) - \tilde{R}_c(w_j) + \left( \nabla \tilde{R}_c(w_j), w_j - \Pi_\perp w_j \right) \right) $$

$$ + 2 \left( \tilde{R}(w_j) - \tilde{R}(w_{j+1}) \right). $$

Applying $\sum_{j < t}$ to both sides and canceling terms yields

$$ |\Pi_\perp w_t - u|^2 \leq |u|^2 + 2 + \sum_{j < t} 2\eta_j \left( \tilde{R}_c(u) - \tilde{R}_c(w_j) \right) + \sum_{j < t} 2\eta_j \left( \nabla \tilde{R}_c(w_j), w_j - \Pi_\perp w_j \right) $$

as desired. \hfill $\square$

**Proof of upper bound in Lemma D.2.** For a fixed $t \geq 1$, define

$$ u := \frac{\ln(t)}{\gamma} \bar{u}, \quad \ell_{\tilde{c}} := \sum_{j < t} \eta_j |\nabla L(A_c w_j)|_1, \quad R := \sup_{j < t} |\Pi_\perp w_j - w_j| \leq |\bar{v}| + \sqrt{\frac{2}{\lambda}}, $$

where the last inequality comes from Lemma D.1.
The strategy of the proof is to rewrite various quantities in Lemma D.3 with \(\ell'_{<t}\), which after applying Lemma D.3 cancel nicely to obtain an upper bound on \(\ell'_{<t}\). This in turn completes the proof, since

\[
|\Pi w_t| \leq \sum_{j<t} \eta_j |\Pi \nabla R(w_j)| \leq \sum_{j<t} \eta_j |\nabla R_c(w_j)| \leq \sum_{j<t} \eta_j |\nabla L(A_c w_j)|_1 = \ell'_{<t}.
\]

Proceeding with this plan, first note

\[
|\Pi w_t - u| \geq (\Pi w_t - u, \bar{u})
\]

\[
= \left< -\sum_{j<t} \eta_j \Pi \nabla \hat{R}(w_j), \bar{u} \right> - (u, \bar{u})
\]

\[
= \sum_{j<t} \eta_j \left< -A^\top_{\ell} \nabla L(A_c w_j), \bar{u} \right> - \frac{\ln(t)}{\gamma}
\]

\[
= \sum_{j<t} \eta_j |\nabla L(A_c w_j)|_1 \left< -A^\top_{\ell} \frac{\nabla L(A_c w_j)}{|\nabla L(A_c w_j)|_1}, \bar{u} \right> - \frac{\ln(t)}{\gamma}
\]

\[
\geq \sum_{j<t} \eta_j |\nabla L(A_c w_j)|_1 \gamma - \frac{\ln(t)}{\gamma}
\]

\[
= \gamma \ell'_{<t} - \frac{\ln(t)}{\gamma},
\]

and since \(\ell' \leq \ell\)

\[
\sum_{j<t} \eta_j \hat{R}_c(w_j) = \sum_{j<t} \eta_j L(A_c w_j)
\]

\[
\geq \sum_{j<t} \eta_j |\nabla L(A_c w_j)|_1
\]

\[
= \ell'_{<t},
\]

and

\[
\sum_{j<t} \eta_j \left< \nabla \hat{R}_c(w_j), w_j - \Pi w_j \right> \leq \sum_{j<t} \eta_j \left| \nabla \hat{R}_c(w_j) \right| |w_j - \Pi w_j|
\]

\[
\leq \sum_{j<t} \eta_j |\nabla L(A_c w_j)|_1 \left| A^\top_{\ell} \frac{\nabla L(A_c w_j)}{|\nabla L(A_c w_j)|_1} \right| R
\]

\[
\leq R \ell'_{<t}.
\]

Combining these terms with Lemma D.3

\[
2\ell'_{<t} + \left( \gamma \ell'_{<t} - \ln(t)/\gamma \right)^2 \leq \sum_{j<t} 2\eta_j \hat{R}_c(w_j) + |\Pi w_t - u|^2
\]

\[
\leq |u|^2 + \sum_{j<t} 2\eta_j \left< \nabla \hat{R}_c(w_j), w_j - \Pi w_j \right> + \sum_{j<t} 2\eta_j \hat{R}_c(u) + 2
\]

\[
\leq \ln(t)^2 \gamma^2 + 2R \ell'_{<t} + \frac{2}{\gamma} \sum_{j<t} \eta_j + 2
\]

\[
\leq \ln(t)^2 \gamma^2 + 2R \ell'_{<t} + 4.
\]

Equivalently,

\[
2\ell'_{<t} + \gamma^2 (\ell'_{<t})^2 \leq 2 \ln(t) \ell'_{<t} + 2R \ell'_{<t} + 4,
\]
which implies 
\[ \ell'_{<t} \leq \max \left\{ \frac{2 \ln(t) + 2R}{\gamma^2}, 2 \right\}. \]

**Proof of lower bound in Lemma D.2.** First note
\[
\frac{n_c}{n} \exp(-|\Pi \perp w_t|) \leq \frac{n_c}{n} \ell(-|\Pi \perp w_t|)/\ln 2
\leq L(A_c \Pi \perp w_t)/\ln 2
= L(A_c w_t - A_c(w_t - \Pi \perp w_t))/\ln 2
= L(A_c w_t - A_c \Pi S w_t)/\ln 2.
\]

If \( \ell = \ell_{\text{exp}} \), then \( L(A_c w_t + R) = e^R L(A_c w_t) \). Otherwise, by Bernoulli’s inequality,
\[
\sum_i \ell_{\log}(A_c w_t) + R = \sum_i \ln \left(1 + e^R \exp((A_c w_t)_i)\right) \leq \sum_i e^R \ln \left(1 + \exp((A_c w_t)_i)\right).
\]

Combining these steps, and invoking Theorem 3.1
\[
\frac{n_c}{n} \ln(2) \exp(-|\Pi \perp w_t|) \leq \exp(R) L(A_c w_t)
= \exp(R) \left(L(A w_t) - L(A_S w_t)\right) \leq \exp(R) \left(L(A w_t) - \inf_w L(A_w)\right).
\]

By Theorem 3.2
\[
\ln \left(\mathcal{R}(w_t) - \inf_w \mathcal{R}(w)\right) \leq \ln \max \left\{ \frac{2 \exp \left(|\bar{v}| \right)}{t}, \frac{|\bar{v}|^2 + \ln(t)^2/\gamma^2}{\sum_{j=0}^{t-1} \eta_j} \right\}
\leq \max \left\{ \ln(2) + |\bar{v}| - \ln(t), \ln \left(|\bar{v}|^2 + \ln(t)^2/\gamma^2\right) - \ln \left(\sum_{j=0}^{t-1} \eta_j\right) \right\}.
\]

Together,
\[
|\Pi \perp w_t| \geq - \ln \left(\mathcal{R}(A w_t) - \inf_w \mathcal{R}(A w)\right) - R + \ln(2) - \ln(n/n_c)
\geq \min \left\{ - \ln(2) - |\bar{v}| + \ln(t), - \ln \left(|\bar{v}|^2 + \ln(t)^2/\gamma^2\right) + \ln \left(\sum_{j=0}^{t-1} \eta_j\right) \right\}
- R + \ln(2) - \ln(n/n_c).
\]

**D.3 Proximity between \( \ell_{\log} \) and \( \ell_{\exp} \)**

**Lemma D.4.** For any \( 0 < \epsilon \leq 1, \ell \in \{ \ell_{\text{exp}}, \ell_{\log} \} \), if \( \ell(z) \leq \epsilon \), then
\[
\frac{\ell'(z)}{\ell(z)} \geq 1 - \epsilon \quad \text{and} \quad \frac{\ell_{\exp}(z)}{\ell(z)} \leq 2.
\]
Proof. The claims are immediate for $\ell = \ell_{\exp}$, thus consider $\ell = \ell_{\log}$. First note that $r \mapsto (e^r - 1)/r$ is increasing and not smaller than 1 when $r \geq 0$. Now set $r := \ell_{\log}(z)$, whereby $\ell'_{\log}(z) = e^r/(1 + e^r) = (e^r - 1)/e^r$. Suppose $r \leq \epsilon$; since exp(·) lies above its tangents, then $1 - \epsilon \leq 1 - r \leq e^{-r}$, and

$$\frac{\ell'_{\log}(z)}{\ell_{\log}(z)} = \frac{e^r - 1}{re^r} \geq \frac{1}{e^r} \geq 1 - \epsilon.$$

For $\ell_{\exp}(z) \leq 2\ell_{\log}(z)$, note

$$\frac{\ell_{\exp}(z)}{\ell_{\log}(z)} = \frac{e^r - 1}{r}$$

is increasing for $r = \ell_{\log}(z) > 0$, and $e - 1 < 2$. $\square$

### D.4 Parameter convergence

For convenience, let $\hat{R}^*$ denote $\inf_w \hat{R}(w)$.

**Lemma D.5.** Let $\ell \in \{\ell_{\exp}, \ell_{\log}\}$. For any $0 < \epsilon \leq 1$, $t \geq 1$, and any $w$ with $\hat{R}(w) - \hat{R}^* \leq \hat{R}(w_t) - \hat{R}^* \leq \epsilon/n$,

$$\frac{\langle \tilde{u}, w \rangle}{|\tilde{u}| \cdot |w|} \geq \frac{-\ln(\hat{R}(w_t) - \hat{R}^*)}{\gamma |w|} - \frac{\ln 2 + g^*(\hat{q}) + |\Pi_S(w)|}{\gamma |w|}.$$

Note the appearance of the additional cross term $|\Pi_S(w)|$; by Lemma D.1 this term is bounded.

**Proof.** First note that

$$A_\perp w = A_c \Pi_\perp w = A_c w + A_c (\Pi_\perp w - w) = A_c w - A_c \Pi_S w,$$

thus

$$\langle \tilde{q}, A_\perp w \rangle = \langle \tilde{q}, A_c w \rangle - \langle \tilde{q}, A_c \Pi_S(w) \rangle \leq \langle \tilde{q}, A_c w \rangle + |q|_1 |A_c \Pi_S(w)| \leq \langle \tilde{q}, A_c w \rangle + |\Pi_S(w)|.$$

Furthermore, since $\hat{R}_c(w) \leq \hat{R}(w) - \hat{R}^* \leq \epsilon/n$, for each row vector $a \in A_c$, $\ell(\langle a, w \rangle) \leq \epsilon$, and thus by Lemma D.4

$$\ell_{\exp}(\langle a, w \rangle) \leq 2\ell(\langle a, w \rangle) \quad \text{and} \quad \hat{R}_{c,\exp}(w) \leq 2\hat{R}_c(w).$$

Thus

$$\frac{\langle \tilde{u}, w \rangle}{|\tilde{u}| \cdot |w|} = -\frac{\langle A_\perp \tilde{q}, w \rangle}{\gamma |w|} = -\frac{\langle \tilde{q}, A_\perp w \rangle}{\gamma |w|} = -\frac{\langle \tilde{q}, A_c w \rangle}{\gamma |w|} - \frac{|\Pi_S(w)|}{\gamma |w|} \geq -\frac{\ln(\hat{R}_{c,\exp}(w) - g^*(\tilde{q}))}{\gamma |w|} - \frac{|\Pi_S(w)|}{\gamma |w|} \geq -\frac{\ln \hat{R}_c(w) - \ln 2 - g^*(\tilde{q})}{\gamma |w|} - \frac{|\Pi_S(w)|}{\gamma |w|} \geq -\frac{\ln(\hat{R}(w) - \hat{R}^*) - \ln 2 - g^*(\tilde{q})}{\gamma |w|} - \frac{|\Pi_S(w)|}{\gamma |w|}. \quad \square$$

Next, the adjustment of Lemma 3.5 to upper bounding $\ln(\hat{R}(w_t) - \hat{R}^*)$, which leads to an upper bound with $\gamma \gamma_j$ rather than $\gamma_j^2$, and the necessary cancellation.
Lemma D.6. Suppose $\ell \in \{\ell_{\text{log}}, \ell_{\text{exp}}\}$ and $\eta_j \leq 1$ (meaning $\eta_j R(w_j) \leq 1$). Also suppose that $j$ is large enough such that $R(w_j) - R^* \leq \min \left\{ \epsilon/n, \lambda (1 - r)/2 \right\}$ for some $\epsilon, r \in (0, 1)$, where $\lambda$ is the strong convexity modulus of $R_S$ on the 1-sublevel set. Then

$$R(w_{j+1}) - R^* \leq \left( R(w_j) - R^* \right) \exp \left( -r(1 - \epsilon) \gamma_j \eta_j \left( 1 - \eta_j/2 \right) \right).$$

Moreover, if there exists a sequence $(w_j)_{j=0}^{t-1}$ such that the above condition holds, then

$$\hat{R}(w_t) - R^* \leq \left( \hat{R}(w_0) - R^* \right) \exp \left( -r(1 - \epsilon) \gamma \sum_{j=0}^{t-1} \eta_j \left( 1 - \eta_j/2 \right) \gamma_j \right).$$

Proof. The first inequality implies the second via the same direct induction in Lemma 3.5, so consider the first inequality.

Making use of Lemmas C.1 and C.2 and proceeding as in Lemma C.2,

$$R(w_{j+1}) - R^* \leq R(w_j) - R^* - \eta_j |\nabla \hat{R}(w_j)|^2 + \frac{\eta_j^2 \hat{R}(w_j)}{2} |\nabla \hat{R}(w_j)|^2$$

$$\leq \left( R(w_j) - R^* \right) \left( 1 - \frac{\eta_j |\nabla \hat{R}(w_j)|^2}{R(w_j) - R^*} \right) \left( 1 - \frac{\eta_j R(w_j)/2}{R(w_j) - R^*} \right)$$

$$= \left( R(w_j) - R^* \right) \left( 1 - \frac{|\nabla \hat{R}(w_j)|}{R(w_j) - R^*} \cdot \frac{\eta_j R(w_j)|\nabla \hat{R}(w_j)|}{R(w_j)} \right) \left( 1 - \frac{\eta_j R(w_j)/2}{R(w_j) - R^*} \right)$$

$$\leq \left( R(w_j) - R^* \right) \left( 1 - \frac{|\nabla \hat{R}(w_j)|}{R(w_j) - R^*} \cdot \eta_j \gamma_j \left( 1 - \eta_j/2 \right) \right). \quad (D.7)$$

Next it will be shown, by analyzing two cases, that

$$\frac{|\nabla \hat{R}(w_j)|}{R(w_j) - R^*} \geq r \gamma (1 - \epsilon). \quad (D.8)$$

In the following, for notational simplicity let $w$ denote $w_j$.

- Suppose $\hat{R}_w(w) < r \left( \hat{R}(w) - \hat{R}^* \right)$. Consequently,

$$\hat{R}_S(w) - R^* > (1 - r) \left( \hat{R}(w) - \hat{R}^* \right).$$

Then, since $4 \left( \hat{R}(w) - \hat{R}^* \right) \leq 2\lambda (1 - r),$

$$\frac{|\nabla \hat{R}(w)|}{\hat{R}(w) - \hat{R}^*} \geq \frac{|\nabla \hat{R}_w(w)| + |\nabla \hat{R}_S(w)|}{\hat{R}(w) - \hat{R}^*}$$

$$\geq \frac{-\hat{R}_w(w) + \sqrt{2\lambda (\hat{R}_S(w) - \hat{R}^*)}}{\hat{R}(w) - \hat{R}^*}$$

$$> \frac{-r(\hat{R}(w) - \hat{R}^*) + \sqrt{2\lambda (1 - r)(\hat{R}(w) - \hat{R}^*)}}{\hat{R}(w) - \hat{R}^*}$$

$$\geq \frac{(2 - r)(\hat{R}(w) - \hat{R}^*)}{\hat{R}(w) - \hat{R}^*}$$

$$\geq 1 \geq r \gamma.$$
• Otherwise, suppose $\hat{R}_c(w) \geq r \left( \hat{R}(w) - \hat{R}^* \right)$. Using an expression inspired by a general analysis of AdaBoost (Mukherjee et al., 2011 Lemma 16 of journal version),

$$|\nabla \hat{R}(w)| \geq \langle -\bar{u}, \nabla \hat{R}(w) \rangle = \langle -A\bar{u}, \nabla L(A\bar{w}) \rangle = \langle -\bar{c}, \nabla L(A\bar{w}) \rangle \geq \gamma |\nabla L(A\bar{w})|_1.$$ 

Since $\hat{R}_c(w) \leq \hat{R}(w) - \hat{R}^* \leq \epsilon/n$, for each row vector $a \in A_c$, $\ell(a, w) \leq \epsilon$, and thus by Lemma D.4

$$\frac{\ell'(\langle a, w \rangle)}{\ell(\langle a, w \rangle)} \geq 1 - \epsilon.$$

As a result,

$$|\nabla \hat{R}(w)| \geq \gamma |\nabla L(A\bar{w})|_1 = \gamma L(A\bar{w}) \frac{|\nabla L(A\bar{w})|}{L(A\bar{w})} \geq \gamma \hat{R}_c(w)(1 - \epsilon).$$

Thus

$$\frac{|\nabla \hat{R}(w)|}{\hat{R}(w) - \hat{R}^*} \geq \frac{\gamma(1 - \epsilon)\hat{R}_c(w)}{\hat{R}(w) - \hat{R}^*} \geq \frac{\gamma(1 - \epsilon) \left( r \left( \hat{R}(w) - \hat{R}^* \right) \right)}{\hat{R}(w) - \hat{R}^*} = r\gamma(1 - \epsilon).$$

Combining eq. (D.7) with eq. (D.8),

$$\hat{R}(w_j) - \hat{R}^* \leq \left( \hat{R}(w_j) - \hat{R}^* \right) \left( 1 - (1 - \epsilon)\gamma \bar{v}_j \left( 1 - \bar{v}_j/2 \right) \right).$$

A combination of Lemma D.6 and Lemma D.5 gives the following intermediate inequality.

**Lemma D.9.** Let $\ell \in \{\ell_{\log}, \ell_{\exp}\}$ and $0 < \epsilon \leq 1$ be given, and select $t_0$ so that $\mathcal{R}(w_{t_0}) - \mathcal{R} \leq \epsilon/n$. Then for any $t \geq t_0$ and any $w$ such that $\mathcal{R}(w) \leq \mathcal{R}(w_t)$,

$$\frac{\langle \bar{u}, w \rangle}{\| \bar{u} \| \| w \|} \geq \frac{r(1 - \epsilon) \sum_{j=t_0}^{t-1} \bar{v}_j (1 - \bar{v}_j/2) \gamma j}{\| w \|} - \frac{\ln 2 + g^*(\bar{q})}{\gamma \| w \|} - \frac{\| \Pi_S(w) \|}{\gamma \| w \|}.$$

**Proof.** By Lemma 3.5, since $\eta_j \leq 1$, the loss decreases at each step, and thus for any $t \geq t_0$, $\mathcal{R}(w_t) \leq \epsilon/n$. Combining Lemma D.5 and Lemma D.6

$$\frac{\langle \bar{u}, w \rangle}{\| \bar{u} \| \| w \|} \geq \frac{-\ln (\mathcal{R}(w) - \hat{R})}{\gamma \| w \|} - \frac{\ln 2 + g^*(\bar{q}) + \| \Pi_S(w) \|}{\gamma \| w \|}.$$

$$\geq \frac{r(1 - \epsilon) \sum_{j=t_0}^{t-1} \bar{v}_j (1 - \bar{v}_j/2) \gamma j}{\| w \|} - \frac{\ln (\mathcal{R}(w_{t_0}) - \hat{R})}{\gamma \| w \|} - \frac{\ln 2 + g^*(\bar{q}) + \| \Pi_S(w) \|}{\gamma \| w \|}.$$
By Theorem 3.2 and the choice of step sizes, it is enough to require

\[
\frac{\exp(|\bar{v}|)}{t_0} \leq \min \left\{ \frac{\epsilon}{6n}, \frac{\lambda \epsilon}{12} \right\}, \quad \frac{|\bar{v}|^2 + \ln(t_0)^2 / \gamma^2}{2\gamma^2 \sqrt{t_0}} \leq \min \left\{ \frac{\epsilon}{6n}, \frac{\lambda \epsilon}{12} \right\}, \quad \frac{1}{2t_0 + 1} \leq \frac{\epsilon}{3}.
\]

Therefore, choosing \( t_0 = \tilde{O} \left( \frac{n^2}{\epsilon^2} \right) \) suffices.

Invoking Lemma D.9 with the above choice for \( w \in \{ w_t, \hat{w}_t \} \),

\[
\frac{1}{2} \frac{w}{|w_t|} - \tilde{u}^2 = 1 - \frac{\langle \tilde{u}, w \rangle}{|\tilde{u}| \cdot |w_t|}
\leq 1 - \frac{r(1 - \epsilon/3) \sum_{j=0}^{t-1} \hat{y}_j (1 - \hat{y}_j/2) \gamma_j}{|w_t|} + \frac{\ln 2}{\gamma |w_t|} + \frac{|\Pi S(w)|}{\gamma |w_t|}
\leq 1 - \frac{(1 - \epsilon/3)(1 - \epsilon/3) \sum_{j=0}^{t-1} \hat{y}_j (1 - \epsilon/3) \gamma_j}{|w_t|} + \frac{\ln 2}{\gamma |w_t|} + \frac{|\Pi S(w)|}{\gamma |w_t|}
\leq 1 - \frac{(1 - \epsilon) (|w_{t0}| + \sum_{j=0}^{t-1} \hat{y}_j \gamma_j)}{|w_t|} + (1 - \epsilon) \frac{|w_{t0}|}{|w_t|} + \frac{\ln 2}{\gamma |w_t|} + \frac{|\Pi S(w)|}{\gamma |w_t|}
\leq \epsilon + \frac{|w_{t0}|}{|w_t|} + \frac{\ln 2}{\gamma |w_t|} + \frac{|\Pi S(w)|}{\gamma |w_t|}.
\]

(D.11)

Suppose \( t \geq 5 \) and \( \sqrt{t} / \ln^3 t \geq n(1 + R) / \gamma^4 \), where \( R = \sup_{t < T} |\Pi_{\perp} w_j - w_j| = \mathcal{O}(1) \) is introduced in Lemma D.2. As will be shown momentarily, \( t \) satisfies eq. (D.10) with \( \epsilon \leq C / |w_{t0}| \) for some constant \( C \), and therefore this choice of \( \epsilon \) can be plugged into eq. (D.11). To see this, note that Theorem 3.2 gives

\[
\mathcal{R}(w_t) - \tilde{\mathcal{R}} \leq \frac{\exp(|\bar{v}|)}{t} + \frac{|\bar{v}|^2 + \ln(t)^2}{2\gamma^2 \sqrt{t}}
\leq \frac{\exp(|\bar{v}|)}{\sqrt{t} / \ln(t)^2} + \frac{|\bar{v}|^2}{2\gamma^2 \sqrt{t} / \ln(t)^2} + \frac{\ln(t)^2}{2\gamma^2 \sqrt{t}}
\leq \frac{|\bar{v}|^2}{n(1 + R) \ln t} + \frac{|\bar{v}|^2}{2n(1 + R) \ln t} + \frac{\gamma^2}{2n(1 + R) \ln t}
\leq C_1 \frac{|w_{t0}|}{n \cdot 4(1 + R) \ln t} \leq C_1 \frac{1}{n |w_t|},
\]

where the last line uses Lemma D.2. It can be shown similarly that other parts of eq. (D.10) hold.

Continuing with Equation (D.11) but using \( \epsilon \leq C / |w_{t0}| \) and upper bounding \( |w_{t0}| \) via Lemma D.2

\[
\left| \frac{w_t}{|w_t|} - \tilde{u} \right|^2 \leq \mathcal{O} \left( \frac{\ln n + \ln \left( 1/|w_t| \right)}{|w_t|^2} \right).
\]

Lastly, controlling the denominator with the lower bound on \( |w_t| \) in Lemma D.2

\[
\left| \frac{w_t}{|w_t|} - \tilde{u} \right|^2 \leq \mathcal{O} \left( \frac{\ln n + \ln \ln t}{\gamma^2 \ln t} \right).
\]

\[\Box\]

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