Kinetics of self-induced aggregation of Brownian particles: non-Markovian and non-Gaussian features

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In this paper we have studied a model for self-induced aggregation in Brownian particle incorporating the non-Markovian and non-Gaussian character of the associated random noise process. In this model the time evolution of each individual is guided by an over-damped Langevin equation of motion with a non-local drift resulting from the local unbalance distributions of the other individuals. Our simulation result shows that colored nose can induce the cluster formation even at large noise strength. Another observation is that critical noise strength grows very rapidly with increase of noise correlation time for Gaussian noise than non Gaussian one. However, at long time limit the cluster number in aggregation process decreases with time following a power law. The exponent in the power law increases remarkable for switching from Markovian to non Markovian noise process.

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I. INTRODUCTION

Formation of large spatial structure and clusters by the aggregation of small species joining each other constitutes a broad area of research in pure and applied sciences\(^1\)\(^-\)\(^3\). General properties of aggregation dynamics include microphysics of clouds and precipitation\(^1\), principle of polymer formation\(^2\), different types of ecological problems\(^1\), etc., to mention a few. The most important direction of the theory of aggregation dynamics leads us to the realm of biological systems in which cooperative activity among individuals usually involves social behaviours\(^4\)\(^-\)\(^5\). Still now its underlying mechanism in various biological systems remains unknown. A number of attempt have been made to established rigorous and quantitative basis of the emergence of cooperation among individuals\(^6\)\(^-\)\(^9\). The kinship theory is the first and important step in this direction which is based on genetic arguments\(^10\). But it does not account correctly the explanation of cooperation that involves among unrelated individuals (individuals having no common genes). On the other hand, according to some models of mathematical population biology and game theory aggregation is the result of short and long range interactions among different individuals\(^11\)\(^-\)\(^12\). According to these models the aggregation dynamics of social biological systems has the same statistical basis as that have been used in fluid dynamics and condensed matter physics.

The essential requirements for a general and simple theoretical description of aggregation dynamics in social communities involve the space of state (where each point represents the status of the individuals) and strategy\(^10\)\(^-\)\(^13\) (that is the rule according to which the individuals players decide to change their status in response to partial and complete information about the action of the other players). Based on game theoretical approach, Sigmund and Nowak\(^14\) assumed that cooperation to work in evolved social systems require the knowledge of reputation or status of their members (players). The reputation is denoted by a dynamical coordinate, \(S\) known as image score. The image score is assigned to each player which is the identity to the other member of community and indicates both wether an individual provides help and if he/she is worthy of being help. Recently, Cecconi et al\(^16\)\(^,\)\(^17\) studied the model suggested by Sigmund and Nowak\(^14\) to test the hypothesis of emergence of cooperation by indirect reciprocity among unrelated individuals. In their study\(^16\)\(^,\)\(^17\) they assumed nonlinear Fokker-Planck equation for the population of individuals with a certain image score. The equation had a non-local drift term that characterized the strategy. Based on the stochastic differential\(^18\)\(^,\)\(^19\) equations corresponding to a Brownian dynamics with drift induced by the spatial distribution of other random worker the authors have also studied the aggregation dynamics\(^20\). In this study they assumed that the random fluctuations correspond to the white Gaussian noise. However, the random fluctuations in the social communities problem are in general non thermal in origin. They may appear as a result of complicated inherent dynamics and therefore the noise of non thermal origin may be non Gaussian and correlated in characteristics. The correlated noise may have important role in the context of cooperation which is related to aggregation dynamics and others since correlation among the particles increases with increase of noise correlation time. Keeping in mind of this aspect we have extended the study of the aggregation dynamics of the self-induced model\(^20\) of the social biological system in non-Markovian and non-Gaussian limit. Our aim is to explore how the dynamics of aggregation to form cluster depends on non-Markovian and non-Gaussian properties of the noise process. Here it is to be noted that the noise of biological(non thermal) origin is non Gaussian in character. Recent experimental and theoretical studies in neural network and sensory systems\(^21\)\(^,\)\(^22\) offer

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strong indication that the noise sources in these systems could be non-Gaussian. The noise of biological origin in many cases is due to nonlinear dynamics which may be correlated and non-Gaussian in character, specifically, in the context biological evolution [21, 23]. The role of colored non-Gaussian noise in the barrier crossing dynamics, the stochastic resonance and complex network has been explored by several authors [24].

II. THE MODEL

Consider a system consists of $N$ individuals which change their $x_i$ state according to majority rule. $x_i$ denote the reputation score of $i^{th}$ member or the position in a possible chemotaxis description or some other amplitude characterizing the role of individual within population biology framework. We assumed that each individual changes $x_i$ by the following stochastic equation of motion.

$$x_j = v(x_j) + \eta(t)$$

(1)

$v$ denotes the drift, which is given by the following expression

$$v(x_i) = \lambda \frac{w_+(x_i, t) - w_-(x_i, t)}{w_+(x_i, t) - w_-(x_i, t)}$$

(2)

where $w_\pm$ are defined by

$$w_\pm(x) = \sum_j \Theta[\pm(x_j - x_i)] \exp(-\alpha |x_j - x_i|)$$

(3)

$\Theta$ is the unitary step function. The equations (1) and (2) reveal that the velocity at which an individual decides to move to the left or to the right depends on the difference $w_+ - w_-$. The magnitude of $w_\pm$ depends on the exponential weight with a coefficient $\alpha = 1/r_0$, $r_0$ specify the range up to which one individual still perceives the presence and the influence of the other member of the group. So according to the above description aggregation is the result from preferably migration of individuals depending on the sign of $v$. The term $w_+ - w_-$ in the denominator of Eq.(2) present the normalization factor and so the velocity bounded within $[-\lambda, \lambda]$.

We assume that $\eta(t)$ in the above equation is colored noise which may be of Gaussian or non-Gaussian type depending on the situation. The non-Gaussian noise can be generated from the solution of the following Langevin equation [25]

$$\dot{\eta} = -\frac{1}{\tau} \frac{d}{d\eta} V_p(\eta) + \frac{\sqrt{D}}{\tau} \zeta(t)$$

(4)

$\zeta(t)$ being a standard Gaussian noise of zero mean and its two-time correlation given by

$$\langle \zeta(t)\zeta(t') \rangle = 2\delta(t - t')$$

(5)

and

$$V_p(\eta) = \frac{D}{\tau(p-1)} \ln[1 + \alpha_1(p-1)\eta^2/2].$$

(6)

Here the form for noise $\eta$ allows us to control the departure from the Gaussian behavior easily by changing a single parameter $p$. $D$ and $\tau$ are noise parameters related to the noise intensity and the correlation time of $\eta$. The parameter $\alpha_1$ in Eq.(6) is defined as

$$\alpha_1 = \frac{\tau}{D}.$$ 

(7)

Now we consider two different situations. For $p = 1$ Eq.(4) becomes

$$\dot{\eta} = -\frac{\eta}{\tau} + \frac{\sqrt{D}}{\tau} \zeta(t).$$

(8)

i. e., the time evolution equation of Ornstein-Uhlenbeck noise process [24] for which the correlation function $\langle \eta(t)\eta(0) \rangle$ decays exponentially

$$\langle \eta(t)\eta(0) \rangle = D/\tau^2 \exp(-t/\tau).$$

(9)

Thus $\tau$ is the correlation time of the Ornstein-Uhlenbeck noise. In the next step we consider a situation where $p > 1$. For $p > 1$ the stationary properties of the noise $\eta$, including the time correlation function, have been studied in [26] and here we summarize the main results. The stationary probability distribution is given by

$$P(\eta) = \frac{1}{Z_p} \left[ 1 + \alpha_1(p-1)\eta^2/2 \right]^{-\gamma/2}.$$ 

(10)
where $Z_p$ is the normalization factor and given by

$$Z_p = \int_{-\infty}^{\infty} d\eta \left[ 1 + \alpha_1(p-1)\eta^2 \right]^{-\frac{1}{2}}$$

$$= \frac{\pi}{\sqrt{\alpha_1(p-1)}} \frac{\Gamma(1/(p-1) - 1/2)}{\Gamma(1/(p-1))} \ \ , \ (11)$$

$\Gamma$ indicates the Gamma function. This distribution can be normalized only for $p < 3$. Since the above distribution function is an even function of $\eta$, the first moment, $\langle \eta \rangle$, is always equal to zero, and the second moment is given by

$$\langle \eta^2 \rangle = \frac{2D}{\tau(1-p)} \ \ , \ (12)$$

which is finite only for $p < 5/3$. Furthermore, for $p < 1$, the distribution has a cut-off and it is only defined for

$$|\eta| < \eta_c = \sqrt{\frac{2D}{\tau(1-p)}} \ \ . \ (13)$$

Finally, the correlation time of non-Gaussian noise $\tau$ of the stationary regime of the process $\eta(t)$ diverges near $p = 5/3$ and it can be approximated over the whole range of values of $p$ as

$$\tau_p \simeq 2\tau/(5-3p) \ \ . \ (14)$$

Clearly, when $p_i \rightarrow 1$, we recover the limit of $\eta_i$ being a Gaussian colored noise, the Ornstein-Uhlenbeck process since in this limit the term in the square bracket of Eq.(5) can be written as

$$1 + \alpha_1(p-1)\eta^2 = \exp(\alpha_1(p-1)\eta^2/2) \ \ , \ (15)$$

and therefore Eq.(10) becomes

$$P(\eta) = \frac{1}{Z_1} \exp(-\alpha_1 \eta^2/2) \ \ , \ (16)$$

with

$$Z_1 = \sqrt{\frac{\pi}{\alpha_1}} \ \ . \ (17)$$

Here we would like to note that Eq.(12) shows that for a given external noise strength $D$ and noise correlation time $\tau$ the variance of the non-Gaussian is higher than that of the Gaussian noise for $p > 1$, i.e.

$$\langle \eta_p^2 \rangle > \langle \eta^2 \rangle \ \ . \ (18)$$

Similarly Eq.(2.14) implies that $\tau_p > \tau$ for $p > 1$.

III. RESULT AND DISCUSSION

Based on the above mentioned model we have numerically investigated the aggregation dynamics. To follow the dynamics of the each individuals present in the system we solve $N$ number coupled stochastic equations (Eq.(2.1)) along with the equation for noise process (2.4) simultaneously using standard Heun’s algorithm. $N$ refers to the number of individuals present in the system. A very small time step($\Delta t$) of 0.01 for numerical integration has been used. For the initial coordinates we have assumed that at $t = 0$ all the particles are uniformly distributed in the space. However, to exhibit the role of noise strength, noise correlation time and other noise parameters on the cluster formation dynamics we plot coordinates of all the particles vs. time, $t$ in Fig.1. It shows that the cluster formation tendency as well as size of the cluster increases as the noise strength increases up to a critical value (Figs.1(a,b)). After the critical value of the noise strength all the individual particles exhibits normal Brownian motion (Fig.1c). But if we increase the noise correlation time then even at high noise strength cluster formation is possible. It has been shown in the Figs.1(d,e). Thus colored noise can induce the cluster formation. Further more, Fig.1f demonstrates that there may be a phase transition from the clustered state to the state where particles are uniformly distributed over the space if one switches from Gaussian to non Gaussian noise(Fig.1f). These observations can be rationalize in the following way. The cluster formation is a result of cooperation among the particles. By virtue of diffusion in the presence of noise the particle which is far apparant from the nucleation center of the strong cluster may enter into the cluster zone. Thus noise accelerates to form strong and bigger cluster. But if the noise strength is very high then the diffusion dominates over the cooperative effect and we observe the phase where particle almost uniformly distributed over the space instead of cluster formation. Now we come to the point how noise correlation time can induce the cluster formation. With increase of noise correlation time the variance of the noise decreases.
and thereby diffusion of the particle suppressed as the non Markovian nature of the noise grows. Not only that, the noise correlation time strongly effects the drift term of the dynamics \([27, 28]\). The drift term in the present problem accounts the extent of cooperation among the particles which leads to cluster formation. Thus it is apparent in the cluster formation dynamics that the colored noise induced nucleation is a result of the extension of the cooperation and the correlation among the particles as well as suppression of diffusion nature of the particles with increase of noise correlation time. Finally, we consider how the cluster formation is suppressed by the non Gaussian noise. For a given noise strength the variance of the non gaussian noise is much higher than the Gaussian noise(see Eq.()). As a result of that for a given noise strength the diffusion may dominates over cooperative effect for non Gaussian noise and there is no cluster formation. But for the same noise strength the cluster formation may be possible for Gaussian noise due to weak diffusion compared to non Gaussian noise. Thus there may exist phase transition phenomenon if one switches from non-Gaussian to Gaussian noise or vice versa.

The above discussion implies that there is a critical noise strength \(D_c\) above which cluster formation is not possible for the given parameter set(Figs. 1(a)-1(c)). One can determine it numerically from the plot of cluster (it is defined as a set of particles within a given cutoff distance \(\epsilon\). \(\epsilon\) is called resolution.) number \(N_c\) vs. time \(t\). To demonstrate this we have plotted \(N_c\) vs. \(t\) in Fig.2 for the particle density \(\rho = 4.0\). It shows that for the given parameter set for the Fig.3 the \(D_c\) is very much close to 0.70 since cluster number remains almost close to its initial value. However, to explore its dependence on the noise correlation time we have plotted \(D_c\) vs. noise correlation time in Fig.3. It exhibits that the value of \(D_c\) changes rapidly for the Gaussian noise than the non Gaussian one. This is because of the higher variance for former than later for the given noise strength \(D\). For the same reason the value of \(D_c\) in general is higher for Gaussian noise than that of non Gaussian noise. It is to be noted here that the Fig.3 is a nice demonstration regarding the dependence of phase transition on the noise strength\((D)\) and the noise correlation time \((\tau)\). The parameter regime below the dashed curve is corresponding to the clustered phase for the Gaussian noise. Similarly the points below the solid curve represents this phase for the non Gaussian noise.

In the next step we have investigated the variation of cluster number at stationary state with noise correlation time \((\tau)\) and plotted in Fig.4. It exhibits that the cluster number rapidly falls with \(\tau\) for the Gaussian noise than the non Gaussian noise. As the correlation as well as the cooperation among the particles increases with increase of \(\tau\) the strong and bigger cluster is formed for larger noise correlation time. Because of that the cluster number reduces for both the Gaussian and non Gaussian noises as tow grows. The slow decrease of cluster number for non Gaussian noise compared to Gaussian noise is due to higher noise variance (after certain value of noise variance it is difficult to form cluster) for former than later.

Finally, we come to consider one of the important aspects of the present study. How the aggregation kinetics depends on the noise correlation time? To this aim we have investigated the aggregation processes through a set of simulation on a system involving \(10^4\) particles. Our observation shows (Fig.5) that both in the Markovian and non-Markovian limits cluster number rapidly decreases with time at short time regime and after that it slowly varies. However, For a detail analysis of the influence of the noise correlation and non-Gaussian parameter on the rate of aggregation processes we set an approximate

![FIG. 3: (Color online)This figure present the critical parameter regime for noise correlation and noise strength for different values of the non-Gaussian parameters. Other system parameters for the plots are \(\alpha = 1, \lambda = 1\).](image)

![FIG. 4: This plot present the variation of the cluster number as a function of the correlation time of the noise for different values of the non-Gaussian parameter(p). The other parameter sets are \(\lambda = 1, \alpha = 1, \rho = 4, \epsilon = 0.25\).](image)
FIG. 5: The plot presents the variation of the cluster number as a function of time at long time limit for different values of the noise correlation time (τ) and the non-Gaussian parameter (p). The other parameter sets are λ = 1, α = 1, ρ = 1, ε = 0.25. In the inset same plot which covers both the short time as well as long time limits.

power law of the following form.

\[ N_c(t) \sim t^{-z}. \]  

This type of algebraic decay law is valid for all τ and p values provided t is long enough (see the inset of the Fig.5). From Fig.5 it is surprising to note that the exponent z in the power law increases more than fifty percent compared to Markovian case for τ = 1.0. Thus it puts further evidence to consider that the noise correlation time enhances the cooperative effect in the aggregation dynamics through modification of drift term as well as reducing the noise variance for the large noise strength case. Before going to leave this section we mention that the exponent z is little bit smaller for colored non Gaussian noise compared to Gaussian colored noise as a result of higher noise variance for former than later for the given noise strength.

IV. CONCLUSION

Based on the numerical simulation of stochastic dynamics associated with colored non-Gaussian noise we have studied the aggregation kinetics of an self-induced model. Such type of self-induced model is used to deals with a certain class of problems in social biological problem. In this modelled, each individuals model is used to deals with a certain class of problems in social biological problem. In this modelled, each individual is represented by a Brownian particle, which undergoes a drift velocity depending upon the population imbalance perceived by a single individual between left and right. Through a set of numerical simulation we have examined the effect of the noise correlation and non-Gaussian character of the noise in the self-induced aggregation dynamics. Our main observation includes the following points:

(i) The colored noise can induce aggregation even at large noise strength (Fig.1). The aggregation may disappear if one switches from Gaussian to non Gaussian noise.

(ii) The critical noise strength rapidly increases with noise correlation for non Gaussian noise than the non gaussian one.

(iii) The cluster formation kinetics follows a power law for the variation of cluster number with time like, \( N_c(t) \sim t^{-z} \) at long time. The exponent z remarkable increases for non Markovian case.

We hope that our observations will be useful to understand the aggregation processes in the social biological system.

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