MINIMAL MASS BLOW-UP SOLUTIONS FOR NONLINEAR SCHRODINGER EQUATIONS
WITH A SINGULAR POTENTIAL

NAOKI MATSUI

Abstract. We consider the following nonlinear Schrödinger equation with an inverse-log potential:
\[
\frac{\partial u}{\partial t} + \Delta u + |u|^2 u + \frac{1}{|x|^{2\sigma}} \log |x| u = 0
\]
in \(\mathbb{R}^N\). From the classical argument, the solution with subcritical mass \(\|u\|_2 < \|Q\|_2\) is global and bounded in \(H^1(\mathbb{R}^N)\). Here, \(Q\) is the ground state of the mass-critical problem. Therefore, we are interested in the existence and behaviour of blow-up solutions for the threshold \(\|u_0\|_2 = \|Q\|_2\).

1. Introduction

We consider the following nonlinear Schrödinger equation with an inverse-log potential:
\[
\frac{\partial u}{\partial t} + \Delta u + |u|^2 u + \frac{1}{|x|^{2\sigma}} \log |x| u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,
\]
where \(N \in \mathbb{N}\) and \(\sigma \in \mathbb{R}\). It is well known that if
\[
0 < \sigma < \min \left\{ \frac{N}{2}, 1 \right\},
\]
then \((1)\) is locally well-posed in \(H^1(\mathbb{R}^N)\) from \cite{2} Proposition 3.2.2, Proposition 3.2.5, Theorem 3.3.9, and Proposition 4.2.3. This means that for any initial value \(u_0 \in H^1(\mathbb{R}^N)\), there exists a unique maximal solution \(u \in C((T_*, T^*), H^1(\mathbb{R}^N)) \cap C^1((T_*, T^*), H^{-1}(\mathbb{R}^N))\) for \((1)\) with \(u(0) = u_0\). Moreover, the mass (i.e., \(L^2\)-norm) and energy \(E\) of the solution \(u\) are conserved by the flow, where
\[
E(u) := \frac{1}{2} \|\nabla u\|^2_2 - \frac{1}{2 + \frac{\sigma}{N}} \|u\|_{2+\frac{\sigma}{N}}^2 + \frac{1}{2} \int_{\mathbb{R}^N} \frac{1}{|x|^{2\sigma}} \log |x| \|u(x)\|^2 dx.
\]
Furthermore, the blow-up alternative holds:
\[T^* < \infty \quad \text{implies} \quad \lim_{t \nearrow T^*} \|\nabla u(t)\|_2 = \infty.\]

We define \(\Sigma^k\) by
\[
\Sigma^k := \left\{ u \in H^k(\mathbb{R}^N) \mid |x|^k u \in L^2(\mathbb{R}^N) \right\}, \quad \|u\|_{2+k}^2 := \|\nabla u\|_{2+k}^2 + \|u\|^2_{2+k}.
\]
Particularly, \(\Sigma^1\) is called the virial space. If \(u_0 \in \Sigma^1\), then the solution \(u\) for \((1)\) with \(u(0) = u_0\) belongs to \(C((T_*, T^*), \Sigma^1)\) from \cite{2} Lemma 6.5.2.

Moreover, we consider the case
\[
0 < \sigma < \min \left\{ \frac{N}{4}, 1 \right\}.
\]
If \(u_0 \in H^2(\mathbb{R}^N)\), then the solution \(u\) for \((1)\) with \(u(0) = u_0\) belongs to \(C((T_*, T^*), H^2(\mathbb{R}^N)) \cap C^1((T_*, T^*), L^2(\mathbb{R}^N))\) and \(|x|\nabla u \in C((T_*, T^*), L^2(\mathbb{R}^N))\) from \cite{2} Theorem 5.3.1. Furthermore, if \(u_0 \in \Sigma^2\), then the solution \(u\) for \((1)\)
with \( u(0) = u_0 \) belongs to \( C((T_*, T^*), \Sigma^2) \cap C^1((T_*, T^*), L^2(\mathbb{R}^N)) \) and \(|x| \nabla u \in C((T_*, T^*), L^2(\mathbb{R}^N))\) from the same proof as in [2, Lemma 6.5.2].

1.1. Critical problem. Firstly, we describe the results regarding the mass-critical problem:

(4) \[ i \frac{\partial u}{\partial t} + \Delta u + |u|^\frac{4}{N} u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \]

In particular, (4) with \( \sigma = 0 \) is reduced to (1).

It is well known ([11] [13] [18]) that there exists a unique classical solution \( Q \) for

\[ -\Delta Q + Q - |Q|^\frac{4}{N} Q = 0, \quad Q \in H^1(\mathbb{R}^N), \quad Q > 0, \quad Q \text{ is radial}, \]

which is called the ground state. If \(|u|_2 = |Q|_2 \) \((|u|_2 < |Q|_2, \ |u|_2 > |Q|_2)\), we say that \( u \) has the critical mass (subcritical mass, supercritical mass, respectively).

We note that \( E_{\text{crit}}(Q) = 0 \), where \( E_{\text{crit}} \) is the energy with respect to (4). Moreover, the ground state \( Q \) attains the best constant in the Gagliardo-Nirenberg inequality

\[ \|v\|_{2^*}^{2^*} \leq \left( 1 + \frac{2}{N} \right) \left( \frac{|v|_2}{|Q|_2} \right)^\frac{4^*}{2} \|\nabla v\|_2^2 \quad \text{for} \ v \in H^1(\mathbb{R}^N). \]

Therefore, for all \( v \in H^1(\mathbb{R}^N) \),

\[ E_{\text{crit}}(v) \geq \frac{1}{2} \|\nabla v\|_2^2 \left( 1 - \left( \frac{|v|_2}{|Q|_2} \right)^\frac{4^*}{2} \right) \]

holds. This inequality and the mass and energy conservations imply that any subcritical mass solution for (4) is global and bounded in \( H^1(\mathbb{R}^N) \).

Regarding the critical mass case, we apply the pseudo-conformal transformation

\[ u(t, x) \mapsto \frac{1}{|t|^{\frac{N}{2}}} u \left( -\frac{1}{t}, \pm \frac{x}{t} \right) e^{i|\nabla u|^2}, \]

to the solitary wave solution \( u(t, x) := Q(x) e^{it} \). Then we obtain

\[ S(t, x) := \frac{1}{|t|^{\frac{N}{2}}} Q \left( \frac{x}{t} \right) e^{-\frac{4}{N} \cdot \frac{|\nabla u|^2}{|t|^2}}, \]

which is also a solution for (4) and satisfies

\[ \|S(t)\|_2 = |Q|_2, \quad \|\nabla S(t)\|_2 \sim \frac{1}{|t|} \quad (t \nearrow 0). \]

Namely, \( S \) is a minimal mass blow-up solution for (4). Moreover, \( S \) is the only finite time blow-up solution for (4) with critical mass, up to the symmetries of the flow (see [13]).

Regarding the supercritical mass case, there exists a solution \( u \) for (4) such that

\[ \|\nabla u(t)\|_2 \sim \sqrt{\frac{\log(\log|T^*-t|)}{T^*-t}} \quad (t \nearrow T^*) \]

(see [12] [23]).

1.2. Previous results. Le Coz, Martel, and Raphael [6] based on the methodology of [14] obtains the following results for

(5) \[ i \frac{\partial u}{\partial t} + \Delta u + |u|^\frac{4}{N} u + |u|^{p-1} u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N. \]

**Theorem 1.1** ([6] [11]). Let \( 1 < p < 1 + \frac{4}{N} \), and \( \pm = + \). Then for any energy level \( E_0 \in \mathbb{R} \), there exist \( t_0 < 0 \) and a radially symmetric initial value \( u_0 \in H^1(\mathbb{R}^N) \) with

\[ \|u_0\|_2 = |Q|_2, \quad E(u_0) = E_0 \]
such that the corresponding solution $u$ for (3) with $u(t_0) = u_0$ blows up at $t = 0$ with a blow-up rate of

$$\|\nabla u(t)\|_2 = \frac{C(p) + o_{t \to 0}(t)}{|t|^\sigma},$$

where $\sigma = \frac{4}{4 + N(p - 1)}$ and $C(p) > 0$.

**Theorem 1.2** (8). Let $1 < p < 1 + \frac{2}{N}$, and $\pm = -$. If an initial value has critical mass, then the corresponding solution for (3) with $u(0) = u_0$ is global and bounded in $H^1(\mathbb{R}^N)$.

8 obtains the following results for (6)

$$i \frac{\partial u}{\partial t} + \Delta u + |u|^{4} u + \frac{1}{|x|^{2\sigma}} u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

**Theorem 1.3** (8). Assume (3). Then for any energy level $E_0 \in \mathbb{R}$, there exist $t_0 < 0$ and a radially symmetric initial value $u_0 \in H^1(\mathbb{R}^N)$ with $\|u_0\|_2 = \|Q\|_2$, $E(u_0) = E_0$ such that the corresponding solution $u$ for (3) with $\pm = +$ and $u(t_0) = u_0$ blows up at $t = 0$. Moreover,

$$\left| \left| u(t) - \frac{1}{\lambda(t)^{\sigma}} P \left( t, \frac{x}{\lambda(t)} \right) e^{-i \frac{\lambda(t)}{\lambda(t) \sigma} |x|^2 + \gamma(t)} \right| \right|_{\Sigma^1} \to 0 \quad (t \nearrow 0)$$

holds for some blow-up profile $P$ and $C^1$ functions $\lambda : (t_0, 0) \to (0, \infty)$ and $b, \gamma : (t_0, 0) \to \mathbb{R}$ such that

$$P(t) \to Q \quad \text{in} \quad H^1(\mathbb{R}^N), \quad \lambda(t) = C_1(\sigma) |t|^{\frac{1}{1+\sigma}} (1 + o(1)), \quad b(t) = C_2(\sigma) |t|^{\frac{1}{1+\sigma}} (1 + o(1)), \quad \gamma(t)^{-1} = O \left( |t|^{\frac{1}{1+\sigma}} \right)$$

as $t \nearrow 0$.

On the other hand, the following result holds in (6) with $\pm = -$.

**Theorem 1.4** (8). Assume $N \geq 2$ and (2). If $u_0 \in H^1_{\text{rad}}(\mathbb{R}^N)$ such that $\|u_0\|_2 = \|Q\|_2$, the corresponding solution $u$ for (3) with $\pm = -$ and $u(0) = u_0$ is global and bounded in $H^1(\mathbb{R}^N)$.

1.3. **Main results.** It is immediately clear from the classical argument that all subcritical-mass solutions for (1) with (2) are global and bounded in $H^1(\mathbb{R}^N)$.

In contrast, regarding critical mass in (1) with $\pm = -$, we obtain the following result:

**Theorem 1.5** (Existence of a minimal-mass blow-up solution). Assume (3). Then for any energy level $E_0 \in \mathbb{R}$, there exist $t_0 < 0$ and a radially symmetric initial value $u_0 \in H^1(\mathbb{R}^N)$ with $\|u_0\|_2 = \|Q\|_2$, $E(u_0) = E_0$ such that the corresponding solution $u$ for (1) with $\pm = -$ and $u(t_0) = u_0$ blows up at $t = 0$. Moreover,

$$\left| \left| u(t) - \frac{1}{\lambda(t)^{\sigma}} P \left( t, \frac{x}{\lambda(t)} \right) e^{-i \frac{\lambda(t)}{\lambda(t) \sigma} |x|^2 + \gamma(t)} \right| \right|_{\Sigma^1} \to 0 \quad (t \nearrow 0)$$

holds for some blow-up profile $P$ and $C^1$ functions $\lambda : (t_0, 0) \to (0, \infty)$ and $b, \gamma : (t_0, 0) \to \mathbb{R}$ such that

$$P(t) \to Q \quad \text{in} \quad H^1(\mathbb{R}^N), \quad \lambda(t) \approx |t|^{\frac{1}{1+\sigma}} \log |t| |t|^{\frac{1}{1+\sigma}}, \quad b(t) \approx |t|^{\frac{1}{1+\sigma}} \log |t| |t|^{\frac{1}{1+\sigma}}, \quad \gamma(t)^{-1} = O \left( |t|^{\frac{1}{1+\sigma}} \right)$$

as $t \nearrow 0$.

On the other hand, the following result holds in (1) with $\pm = +$.

**Theorem 1.6** (Non-existence of a radial minimal-mass blow-up solution). Assume $N \geq 2$ and (2). If $u_0 \in H^1_{\text{rad}}(\mathbb{R}^N)$ such that $\|u_0\|_2 = \|Q\|_2$, the corresponding solution $u$ for (1) with $\pm = +$ and $u(0) = u_0$ is global and bounded in $H^1(\mathbb{R}^N)$.

This result is proved in the same way as for Theorem 1.2 and Theorem 1.4.
We can expect that there exists a critical-mass blow-up solution such that the blow-up rate is 
\[ \frac{1}{|x|^{2\sigma}} \leq -\frac{1}{|x|^{2\sigma}} \log |x| \leq \frac{1}{|x|^{2(\sigma+\epsilon)}} \]
holds for any \( \epsilon > 0 \). The corresponding blow-up rates from Theorem 1.3 and Theorem 1.5 satisfy 
\[ |t|^{-\frac{4}{1+\sigma}} \gtrsim |t|^{-\frac{2}{1+\sigma}} \log |t|^{-\frac{2}{1+\sigma}} \gtrsim |t|^{-\frac{4}{1+\sigma}}. \]
This suggests that a large or small relationship between the strength of the potential’s singularities gives a large or small relationship for the blow-up rates.

This result suggests that it is possible to construct a critical-mass blow-up solution if an equation such as 
\[ 0 = i\frac{\partial v}{\partial s} + \Delta v - v + f(v) + g_1(\lambda)g_2(y,v) + \text{error terms} \]
can be obtained by separating \( \lambda \) from \( v \), as in [S]. In order to construct an blow-up solution, it is necessary to be able to obtain at least \( \lambda_{\text{app}}, b_{\text{app}} \) in Lemma 5.1 and define \( F \) in Lemma 5.4. Furthermore, in the case of \( g(\lambda) = O(\lambda^2) \), we can expect that there exists a critical-mass blow-up solution such that the blow-up rate is \( t^{-1} \).

2. Notations

In this section, we introduce the notation used in this paper.

Let 
\[ \mathbb{N} := \mathbb{Z}_{\geq 1}, \quad \mathbb{N}_0 := \mathbb{Z}_{\geq 0}. \]

We define
\[ (u,v)_2 := \text{Re} \int_{\mathbb{R}^N} u(x)\overline{v}(x)dx, \quad \|u\|_p := \left( \int_{\mathbb{R}^N} |u(x)|^pdx \right)^{\frac{1}{p}}, \]
\[ f(z) := |z|^\frac{4}{N}z, \quad F(z) := \frac{1}{2 + \frac{4}{N}}|z|^{2 + \frac{4}{N}} \quad \text{for } z \in \mathbb{C}. \]

By identifying \( \mathbb{C} \) with \( \mathbb{R}^2 \), we denote the differentials of \( f \) and \( F \) by \( df \) and \( dF \), respectively. We define
\[ \Lambda := \frac{N}{2} + x \cdot \nabla, \quad L_+ := -\Delta + 1 - \left( 1 + \frac{4}{N} \right) Q^\frac{4}{N}, \quad L_- := -\Delta + 1 - Q^\frac{4}{N}. \]

Namely, \( \Lambda \) is the generator of \( L^2 \)-scaling, and \( L_+ \) and \( L_- \) come from the linearised Schrödinger operator to close \( Q \). Then
\[ L_- Q = 0, \quad L_+ \Lambda Q = -2Q, \quad L_- |x|^2 Q = -4\Lambda Q, \quad L_+ \rho = |x|^2 Q, \quad L_- x Q = -\nabla Q \]
hold, where \( \rho \in \mathcal{S}(\mathbb{R}^N) \) is the unique radial solution for \( \rho = |x|^2 Q \). Note that there exist \( C_\alpha, \kappa_\alpha > 0 \) such that
\[ \left| \left( \frac{\partial}{\partial x} \right)^\alpha Q(x) \right| \leq C_\alpha Q(x), \quad \left| \left( \frac{\partial}{\partial x} \right)^\alpha \rho(x) \right| \leq C_\alpha (1 + |x|)^{\kappa_\alpha} Q(x). \]

for any multi-index \( \alpha \). Furthermore, there exists \( \mu > 0 \) such that for all \( u \in H_1^{\text{rad}}(\mathbb{R}^N) \),
\[ (L_+ \text{ Re } u, \text{ Re } u) + (L_- \text{ Im } u, \text{ Im } u) \geq \mu \|u\|^2_{H^1} - \frac{1}{\mu} \left( (\text{Re } u, Q)_2^2 + (\text{Re } u, |x|^2 Q)_2^2 + (\text{Im } u, \rho)_2^2 \right) \]
(e.g., see [11] [12] [13] [17]). We denote by \( \mathcal{Y} \) the set of functions \( g \in C^\infty(\mathbb{R}^N \setminus \{0\}) \cap C(\mathbb{R}^N) \cap H_1^{\text{rad}}(\mathbb{R}^N) \) such that
\[ \exists C_\alpha, \kappa_\alpha > 0, \quad |x| \geq 1 \Rightarrow \left| \left( \frac{\partial}{\partial x} \right)^\alpha g(x) \right| \leq C_\alpha (1 + |x|)^{\kappa_\alpha} Q(x) \]
for any multi-index \( \alpha \). Moreover, we defined by \( \mathcal{Y'} \) the set of functions \( g \in \mathcal{Y} \) such that
\[ \Lambda g \in H^1(\mathbb{R}^N) \cap C(\mathbb{R}^N). \]
Finally, we use the notation \(\lesssim\) and \(\gtrsim\) when the inequalities hold up to a positive constant. We also use the notation \(\approx\) when \(\lesssim\) and \(\gtrsim\) hold. Moreover, positive constants \(C\) and \(\varepsilon\) are sufficiently large and small, respectively.

3. CONSTRUCTION OF A BLOW-UP PROFILE

In this section, we construct a blow-up profile \(P\) and introduce a decomposition of functions based on the methodology in [6, 14].

Heuristically, we state the strategy. We look for a blow-up solution in the form of (8):

\[
u(t, x) = \frac{1}{\lambda(s)^{\alpha}} v(s, y) e^{-\frac{|\lambda(s)|y^2}{4} + \gamma(s)}, \quad y = \frac{x}{\lambda(s)}, \quad \frac{ds}{dt} = \frac{1}{\lambda(s)^2}, \]

where \(v\) satisfies

\[
0 = i \frac{\partial v}{\partial s} + \Delta v - v + f(v) - \lambda^\alpha \log \lambda \frac{1}{|y|^{2\sigma}} v - \lambda^\alpha \frac{1}{|y|^{2\sigma}} \log |y| v
- i \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) \lambda v + \left( 1 - \frac{\partial \gamma}{\partial s} \right) v + \left( \frac{\partial b}{\partial s} + b^2 \right) \frac{|y|^2}{4} v - \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) b \frac{|y|^2}{2} v,
\]

(8)

where \(\alpha = 2 - 2\sigma\). Since we look for a blow-up solution, it may hold that \(\lambda(s) \to 0\) as \(s \to \infty\). Therefore, it seems that \(\lambda^\alpha |y|^{-2\sigma} v\) is ignored. By ignoring \(\lambda^\alpha |y|^{-2\sigma} v\),

\[v(s, y) = Q(y), \quad \frac{1}{\lambda(s)} \frac{\partial \lambda}{\partial s} + b = 1 - \frac{\partial \gamma}{\partial s} \quad \text{and} \quad \frac{\partial b}{\partial s} + b^2 = 0\]

is a solution of (8). Accordingly, \(v\) is expected to be close to \(Q\). We now consider the case where \(\sigma = 0\), i.e., the critical problem. Then \(\lambda^2 v\) corresponds to the linear term with the constant coefficient and can be removed by an appropriate transformation. In other words, \(\lambda^2 v\) is a negligible term for the construction of minimal-mass blow-up solutions. This suggests that \(\alpha = 2\) may be the threshold for ignoring the term in the context of minimal-mass blow-up. Therefore, \(\lambda^\alpha |y|^{-2\sigma} v\) may become a non-negligible term if \(\alpha < 2\), i.e., \(\sigma > 0\). Also, (8) is difficult to solve explicitly. Consequently, we construct an approximate solution \(P\) that is close to \(Q\) and fully incorporates the effects of \(\lambda^\alpha |y|^{-2\sigma} v\), e.g., the singularity of the origin.

For \(K \in \mathbb{N}\), we define

\[\Sigma_K := \{ (j, k_1, k_2) \in \mathbb{N}_0^3 \mid j + k_1 + k_2 \leq K \} .\]

Proposition 3.1. Let \(K, K' \in \mathbb{N}\) be sufficiently large. Let \(\lambda(s) > 0\) and \(b(s) \in \mathbb{R}\) be \(C^1\) functions of \(s\) such that \(\lambda(s) + |b(s)| \ll 1\).

(i) Existence of blow-up profile. For any \((j, k_1, k_2) \in \Sigma_{K+K'}\), there exist \(P^+_{1,j,k_1,k_2}, P^+_{2,j,k_1,k_2}, P^-_{1,j,k_1,k_2}, P^-_{2,j,k_1,k_2} \in \mathcal{V}', \beta_{1,j,k_1,k_2}, \beta_{2,j,k_1,k_2} \in \mathbb{R}\), and \(\Psi \in H^1(\mathbb{R}^N)\) such that \(P\) satisfies

\[
\frac{\partial P}{\partial s} + \Delta P - P + f(P) - \lambda^\alpha \log \lambda \frac{1}{|y|^{2\sigma}} P - \lambda^\alpha \frac{1}{|y|^{2\sigma}} \log |y| P + \theta \frac{|y|^2}{4} P = \Psi,
\]

where \(\alpha = 2 - 2\sigma\), and \(P\) and \(\theta\) are defined by

\[P(s, y) := Q(y) + \sum_{(j,k_1,k_2) \in \Sigma_{K+K'}} b(s)^{2j} (\lambda(s)^\alpha \log \lambda(s))^{k_1} \lambda(s)^{k_2} \alpha \left( \lambda(s)^\alpha \log \lambda(s) P^+_{1,j,k_1,k_2}(y) + \lambda(s)^\alpha P^+_{2,j,k_1,k_2}(y) \right)
+ \sum_{(j,k_1,k_2) \in \Sigma_{K+K'}} b(s)^{2j+1} (\lambda(s)^\alpha \log \lambda(s))^{k_1} \lambda(s)^{k_2} \alpha \left( \lambda(s)^\alpha \log \lambda(s) P^-_{1,j,k_1,k_2}(y) + \lambda(s)^\alpha P^-_{2,j,k_1,k_2}(y) \right),\]

\[\theta(s) := \sum_{(j,k_1,k_2) \in \Sigma_{K+K'}} b(s)^{2j} (\lambda(s)^\alpha \log \lambda(s))^{k_1} \lambda(s)^{k_2} \alpha (-\lambda(s)^\alpha \log \lambda(s) \beta_{1,j,k_1,k_2} + \lambda(s)^\alpha \beta_{2,j,k_1,k_2}) .\]

Moreover, for some sufficiently small \(\varepsilon' > 0\),

\[
\left\| e^{\varepsilon'|y|} \Psi \right\|_{H^1} \lesssim \lambda^\alpha |\log \lambda| \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) + \frac{\partial b}{\partial s} + b^2 + \theta + (b^2 + \lambda^\alpha |\log \lambda|)^{K+2}\]

holds.
(ii) **Mass and energy properties of blow-up profile.** Let define

\[ P_{\lambda,b,\gamma}(s,x) := \frac{1}{\lambda(s)^2} P \left( s, \frac{x}{\lambda(s)} \right) e^{-\frac{1}{\lambda(s)^2} \frac{|x|^2}{2} + \gamma(s)}. \]

Then

\[
\left| \frac{d}{ds} \| P_{\lambda,b,\gamma} \|_2^2 \right| \leq \lambda^\alpha |\log \lambda| \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) + \frac{\partial b}{\partial s} + b^2 - \theta \right) + (b^2 + \lambda^\alpha |\log \lambda|)^{K+2},
\]

\[
\left| \frac{d}{ds} E(P_{\lambda,b,\gamma}) \right| \leq \frac{1}{\lambda^2} \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) + \frac{\partial b}{\partial s} + b^2 - \theta \right) + (b^2 + \lambda^\alpha |\log \lambda|)^{K+2}
\]

hold. Moreover,

\[
\left| 8E(P_{\lambda,b,\gamma}) - \| y \| Q \|_2^2 \left( \frac{b^2}{\lambda^2} + \frac{2\beta_1}{2-\alpha} \lambda^{\alpha-2} \log \lambda - \beta_1' \lambda^{\alpha-2} \right) \right| \leq \frac{\lambda^\alpha |\log \lambda| (b^2 + \lambda^\alpha |\log \lambda|)}{\lambda^2}
\]

holds, where

\[
\beta_1 := \beta_{1,0,0,0} = \frac{4\sigma \| \cdot \|_{-\sigma} Q \|_2^2}{\| \cdot \| Q \|_2^2} > 0 \quad \beta_1' := \frac{4}{\| \| Q \|_2^2} \int_{\mathbb{R}^n} \frac{1}{|y|^{2\sigma}} \log |y| Q^2 \, dy.
\]

**proof.** See [6, 8] for details of proofs.

We prove (i). We set

\[
Z_1 := \sum_{(j,k_1,k_2) \in \Sigma_{K+K'}} b^{2j} (\lambda^\alpha \log \lambda)^{k_1} \lambda^{k_2\alpha} P_{1,j,k_1,k_2}^+ + i \sum_{(j,k_1,k_2) \in \Sigma_{K+K'}} b^{2j+1} (\lambda^\alpha \log \lambda)^{k_1} \lambda^{k_2\alpha} P_{1,j,k_1,k_2}^-,
\]

\[
Z_2 := \sum_{(j,k_1,k_2) \in \Sigma_{K+K'}} b^{2j} (\lambda^\alpha \log \lambda)^{k_1} \lambda^{k_2\alpha} P_{2,j,k_1,k_2}^+ + i \sum_{(j,k_1,k_2) \in \Sigma_{K+K'}} b^{2j+1} (\lambda^\alpha \log \lambda)^{k_1} \lambda^{k_2\alpha} P_{2,j,k_1,k_2}^-,
\]

Then \( P = Q + \lambda^\alpha \log \lambda Z_1 + \lambda^\alpha Z_2 \) holds. Moreover, let set

\[
\Theta(s) := \sum_{(j,k) \in \Sigma_{K+K'}} b(s)^{2j} (\lambda(s)^\alpha \log \lambda(s))^{k_1} \lambda(s)^{k_2\alpha} \left( \lambda(s)^\alpha \log \lambda(s)c_{1,j,k_1,k_2}^+ + \lambda(s)^\alpha c_{2,j,k_1,k_2}^+ \right),
\]

\[
\Phi := i \frac{\partial P}{\partial s} + \Delta P - P + f(P) - \lambda^\alpha \log \lambda \frac{1}{|y|^{2\sigma}} P - \lambda^\alpha \frac{1}{|y|^{2\sigma}} \log |y| P + \theta \frac{|y|^2}{4} P + \Theta Q,
\]

where \( P_{\bullet,j,k_1,k_2}^\pm \in \mathcal{Y} \) and \( \beta_{\bullet,j,k_1,k_2}, c_{\bullet,j,k_1,k_2}^+ \in \mathbb{R} \) are to be determined.
As in [8], there exist $F_{\pm,j_k,k_2}^\sigma$, $F_{\ast,j_k,k_2}^\log$, $F_{\pm,j_k,k_2}^\pm$, and $\Phi$ such that

$$i \frac{\partial P}{\partial s} + \Delta P - P + f(P) - \lambda^\sigma \log \lambda \frac{1}{|y|^{2\sigma}} P - \lambda^\sigma \frac{1}{|y|^{2\sigma}} \log |y| P + \theta \frac{|y|^2}{4} P + \Theta Q$$

$$= \sum_{(j,k_1,k_2) \in \Sigma_{\kappa + \kappa'}} b^{2j} (\lambda^\sigma \log \lambda)^{k_1+1} \lambda^{k_2}$$

$$\times \left( - L_+ P_{1,j_k,k_2}^+ - \beta_{1,j_k,k_2} \frac{|y|^2}{4} Q - \frac{1}{|y|^{2\sigma}} F_{1,j_k,k_2}^\sigma - \frac{1}{|y|^{2\sigma}} \log |y| F_{1,j_k,k_2}^\log + F_{1,j_k,k_2}^+ + c_{1,j_k,k_2}^+ \right)$$

$$+ \sum_{(j,k_1,k_2) \in \Sigma_{\kappa + \kappa'}} b^{2j} (\lambda^\sigma \log \lambda)^{k_1+1} \lambda^{k_2}$$

$$\times \left( - L_+ P_{2,j_k,k_2}^+ - \beta_{2,j_k,k_2} \frac{|y|^2}{4} Q - \frac{1}{|y|^{2\sigma}} F_{2,j_k,k_2}^\sigma - \frac{1}{|y|^{2\sigma}} \log |y| F_{2,j_k,k_2}^\log + F_{2,j_k,k_2}^+ + c_{2,j_k,k_2}^+ \right)$$

$$+ \sum_{(j,k_1,k_2) \in \Sigma_{\kappa + \kappa'}} b^{2j} (\lambda^\sigma \log \lambda)^{k_1+1} \lambda^{k_2}$$

$$\times \left( - L_+ P_{1,j_k,k_2}^- - (2j + (k_1 + k_2 + 1)\alpha) P_{1,j_k,k_2}^+ - \frac{1}{|y|^{2\sigma}} F_{1,j_k,k_2}^\sigma - \frac{1}{|y|^{2\sigma}} \log |y| F_{1,j_k,k_2}^\log - F_{1,j_k,k_2}^- \right)$$

$$+ \sum_{(j,k_1,k_2) \in \Sigma_{\kappa + \kappa'}} b^{2j} (\lambda^\sigma \log \lambda)^{k_1+1} \lambda^{k_2}$$

$$\times \left( - L_+ P_{2,j_k,k_2}^- - (2j + (k_1 + k_2 + 1)\alpha) P_{2,j_k,k_2}^+ - \frac{1}{|y|^{2\sigma}} F_{2,j_k,k_2}^\sigma - \frac{1}{|y|^{2\sigma}} \log |y| F_{2,j_k,k_2}^\log - F_{2,j_k,k_2}^- \right)$$

$$+ \Phi.$$
Moreover,
\[ \| e^{\gamma |y|} \Theta Q \|_{H^1} \lesssim (b^2 + \lambda^\alpha \log \lambda)^{K+2} \]
holds. Therefore, we have
\[ \| e^{\gamma |y|} \Psi \|_{H^1} \lesssim \lambda^\alpha \| \psi \|_{H^1} \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) \left( \frac{\partial b}{\partial s} + b^2 - \theta \right) + (b^2 + \lambda^\alpha)^{K+2}, \]
where \( \Psi := \Phi - \Theta Q. \)

The rest is the same as in [6, 8]. \( \Box \)

In the rest of this section, we construct solutions \( (P^+_{j,k}, P^-_{j,k}, \beta_{j,k}, c^+_{j,k}) \in \mathcal{Y}^2 \times \mathbb{R}^2 \) for systems \( (S_{j,k}) \) in the proof of Proposition 3.1.

**Proposition 3.2.** The system \( (S_{j,k}) \) has a solution \( (P^+_{j,k}, P^-_{j,k}, \beta_{j,k}, c^+_{j,k}) \in \mathcal{Y}^2 \times \mathbb{R}^2 \).

**proof.** We solve
\[
\begin{align*}
(S_{j,k}) \quad & \left\{ \begin{array}{l}
L_+ P^+_{1,j,k_1,k_2} + \beta_{1,j,k_1,k_2} \frac{|y|^2}{4} Q + \frac{1}{|y|^{2\sigma}} F_{1,j,k_1,k_2}^{\gamma,+,} + \frac{1}{|y|^{2\sigma}} \log |y| F_{1,j,k_1,k_2}^{\log,+,} - F^+_{1,j,k_1,k_2} - c^+_{1,j,k_1,k_2} Q = 0, \\
L_+ P^+_{2,j,k_1,k_2} - \beta_{2,j,k_1,k_2} \frac{|y|^2}{4} Q + \frac{1}{|y|^{2\sigma}} F_{2,j,k_1,k_2}^{\gamma,+,} + \frac{1}{|y|^{2\sigma}} \log |y| F_{2,j,k_1,k_2}^{\log,+,} - F^+_{2,j,k_1,k_2} - c^+_{2,j,k_1,k_2} Q = 0, \\
L_- P^-_{1,j,k_1,k_2} + (2j + (k_1 + k_2 + 1) \alpha) P^+_{1,j,k_1,k_2} + \frac{1}{|y|^{2\sigma}} F_{1,j,k_1,k_2}^{\gamma,-} + \frac{1}{|y|^{2\sigma}} \log |y| F_{1,j,k_1,k_2}^{\log,-} - F^+_{1,j,k_1,k_2} = 0, \\
L_- P^-_{2,j,k_1,k_2} + (2j + (k_1 + k_2 + 1) \alpha) P^+_{2,j,k_1,k_2} + \frac{1}{|y|^{2\sigma}} F_{2,j,k_1,k_2}^{\gamma,-} + \frac{1}{|y|^{2\sigma}} \log |y| F_{2,j,k_1,k_2}^{\log,-} - F^+_{2,j,k_1,k_2} = 0.
\end{array} \right.
\]
\[
(\tilde{S}_{j,k}) \quad & \left\{ \begin{array}{l}
L_+ \tilde{P}^+_{1,j,k_1,k_2} + \beta_{1,j,k_1,k_2} \frac{|y|^2}{4} Q + \frac{1}{|y|^{2\sigma}} F_{1,j,k_1,k_2}^{\gamma,+,} + \frac{1}{|y|^{2\sigma}} \log |y| F_{1,j,k_1,k_2}^{\log,+,} - F^+_{1,j,k_1,k_2} = 0, \\
L_+ \tilde{P}^+_{2,j,k_1,k_2} - \beta_{2,j,k_1,k_2} \frac{|y|^2}{4} Q + \frac{1}{|y|^{2\sigma}} F_{2,j,k_1,k_2}^{\gamma,+,} + \frac{1}{|y|^{2\sigma}} \log |y| F_{2,j,k_1,k_2}^{\log,+,} - F^+_{2,j,k_1,k_2} = 0, \\
L_- \tilde{P}^-_{1,j,k_1,k_2} + (2j + (k_1 + k_2 + 1) \alpha) \tilde{P}^+_{1,j,k_1,k_2} + \frac{1}{|y|^{2\sigma}} F_{1,j,k_1,k_2}^{\gamma,-} + \frac{1}{|y|^{2\sigma}} \log |y| F_{1,j,k_1,k_2}^{\log,-} - F^+_{1,j,k_1,k_2} = 0, \\
L_- \tilde{P}^-_{2,j,k_1,k_2} + (2j + (k_1 + k_2 + 1) \alpha) \tilde{P}^+_{2,j,k_1,k_2} + \frac{1}{|y|^{2\sigma}} F_{2,j,k_1,k_2}^{\gamma,-} + \frac{1}{|y|^{2\sigma}} \log |y| F_{2,j,k_1,k_2}^{\log,-} - F^+_{2,j,k_1,k_2} = 0.
\end{array} \right.
\]
For \((S_{j,k})\), we consider the following two systems:

\[
(\tilde{S}^\prime_{j,k}) \quad \left\{ \begin{array}{l}
P^+_{j,k_1,k_2} = -\frac{c^+_{j,k_1,k_2}}{2} \Lambda Q, \\
P^-_{j,k_1,k_2} = -\frac{c^-_{j,k_1,k_2}}{2} \frac{(2j + (k_1 + k_2 + 1) \alpha) c^+_{j,k_1,k_2}}{8} |y|^2 Q.
\end{array} \right.
\]

Then by applying \((\tilde{S}^\prime_{j,k})\) to a solution for \((\tilde{S}_{j,k})\), we obtain a solution for \((S_{j,k})\).

Firstly, we solve
\[
(\tilde{S}_{0,0}) \quad \left\{ \begin{array}{l}
L_+ \tilde{P}^+_{1,0,0,0} + \beta_{1,0,0,0} \frac{|y|^2}{4} Q + \frac{1}{|y|^{2\sigma}} Q = 0, \\
L_+ \tilde{P}^+_{2,0,0,0} - \beta_{2,0,0,0} \frac{|y|^2}{4} Q + \frac{1}{|y|^{2\sigma}} \log |y| Q = 0, \\
L_- \tilde{P}^-_{0,0,0,0} + \alpha \tilde{P}^+_{0,0,0,0} = 0.
\end{array} \right.
\]
For any \(\beta_{1,0,0,0} \in \mathbb{R}\), there exists a solution \(\tilde{P}^+_{1,0,0,0} \in \mathcal{Y}\). Let
\[
\beta_{1,0,0,0} := \frac{4\alpha \| \cdot \|_{-\sigma} Q \|_2^2}{\| \cdot \|_2^2}.
\]
Then since
\[ \left( \bar{P}^+_{1,0,0,0} \right)_2 = -\frac{1}{2} \left< L + \bar{P}^+_{1,0,0,0} \right| Q \right>_2 = -\frac{1}{2} \left( \frac{\beta_0}{4} \right) = 0, \]
there exists a solution \( \bar{P}^-_{1,0,0,0} \in \mathcal{Y} \). By taking \( \epsilon_{1,0,0,0} = 0 \), we obtain a solution \( (P^+_{1,0,0,0}, P^-_{1,0,0,0}, \beta_{1,0,0,0}, c^+_{1,0,0,0}) \in \mathcal{Y}^2 \times \mathbb{R}^2 \). Similarly, we obtain a solution \( (P^+_{2,0,0,0}, P^-_{2,0,0,0}, \beta_{2,0,0,0}, c^+_{2,0,0,0}) \in \mathcal{Y}^2 \times \mathbb{R}^2 \). Here, let \( H(j_0, k_1, k_2) \) denote by that
\[ \forall (j, k_1, k_2) \in \Sigma_{K+K'}, \quad k_2 < k_2 \text{ or } (k_2 = k_2 \text{ and } k_1 < k_1) \text{ or } (k_2 = k_2 \text{ and } k_1 = k_1 \text{ and } j < j_0) \]
\[ \Rightarrow (S_{j_1,k_1,k_2}) \text{ has a solution } (P^+_{j_1,k_1,k_2}, P^-_{j_1,k_1,k_2}, \beta_{j_1,k_1,k_2}, c^+_{j_1,k_1,k_2}) \in \mathcal{Y}^2 \times \mathbb{R}^2. \]
From the above discuss, \( H(1,0,0) \) is true. If \( H(j_0, k_1, k_2) \) is true, then \( P^\pm_{j_0,k_1,k_2} \) is defined and belongs to \( \mathcal{Y} \). Moreover, for any \( \beta_{j_0,k_1,k_2} \), there exists a solution \( \bar{P}^\pm_{j_0,k_1,k_2} \). Let be \( \beta_{j_0,k_1,k_2} \) such that
\[ \left( (2j + (k_1 + k_2 + 1)\alpha) \bar{P}^\pm_{j_1,k_1,k_2} + \frac{1}{|y|^{2\sigma}} F^\sigma_{j_1,k_1,k_2} + \frac{1}{|y|^{2\sigma}} \log |y| F^\log_{j_1,k_1,k_2} - F^-_{j_1,k_1,k_2} \right) = 0. \]
Then we obtain a solution \( \bar{P}^-_{j_0,k_1,k_2} \). Here, we define
\[
c^\pm_{j_0,k_1,k_2} := \begin{cases} 
\bar{P}^\pm_{j_0,k_1,k_2}(0) & (j_0 + k_1, 0 + k_2, 0) \neq K + 1), \\
0 & (j_0 + k_1, 0 + k_2, 0) = K + 1, \text{ and } \bar{P}^-_{j_0,k_1,k_2}(0) \neq 0), \\
1 & (j_0 + k_1, 0 + k_2, 0) = K + 1, \text{ and } \bar{P}^-_{j_0,k_1,k_2}(0) = 0), \\
0 & (j_0 + k_2, 0) \leq K), \\
0 & (j_0 + k_1, 0 + k_2, 0) = K + 1, \text{ and } \bar{P}^+_{j_0,k_1,k_2}(0) \neq 0), \\
1 & (j_0 + k_1, 0 + k_2, 0) = K + 1, \text{ and } \bar{P}^+_{j_0,k_1,k_2}(0) = 0), \\
2\bar{P}^+_{j_0,k_1,k_2}(0) & (j_0 + k_1, 0 + k_2, 0) \geq K + 2). 
\end{cases}
\]
Then we obtain a solution for \( (S_{j_0,k_1,k_2}) \). This means that \( H(j_0+1, k_1, k_2) \) is true if \( j_0 + k_1 + k_2, 0 \leq K + K' - 1, \\
H(0, k_1, 0 + 1, k_2) \) is true if \( j_0 + k_1, 0 + k_2, 0 = K + K' \) and \( k_1, 0 + k_2, 0 \leq K + K' - 1, \) and \( H(0, 0, k_2, 0) \) is true if \( j_0 + k_1, 0 + k_2, 0 = K + K' \). In particular, \( H(0, 0, K + K' + 1) \) means that for any \( (j, k_1, k_2) \in \Sigma_{K+K'} \), there exists a solution \( (P^+_{j_1,k_1,k_2}, P^-_{j_1,k_1,k_2}, \beta_{j_1,k_1,k_2}, c^\pm_{j_1,k_1,k_2}) \in \mathcal{Y} \times \mathbb{R}^2 \).

Furthermore, \( P^\pm_{j_1,k_1,k_2}(0) \neq 0 \) for \( j + k_2, 0 = K + 1 \) and \( P^\pm_{j_1,k_1,k_2}(0) = 0 \) for \( j + k_2, 0 \geq K + 2 \) hold. \hfill \Box

**Proposition 3.3.** For \( P^\pm_{j_1,k_1,k_2} \),
\[ \Delta P^\pm_{j_1,k_1,k_2} \in H^1(\mathbb{R}^N) \cap C(\mathbb{R}^N). \]

Namely, \( P^\pm_{j_1,k_1,k_2} \in \mathcal{Y}' \).

**proof.** For the proof, see \( [5] \). \hfill \Box

**Proposition 3.4.** For \( P^\pm_{0,k_1,k_2} \) with \( k_1 + k_2 = K + K' \),
\[ \frac{1}{r^2} P^\pm_{0,k_1,k_2} - \frac{1}{r} \frac{\partial P^\pm_{0,k_1,k_2}}{\partial r} \in L^{\infty}(\mathbb{R}^N), \]
where \( r = |y|, \)

**proof.** We prove only for \( P^+_{0,k_1,k_2} \). See \( [5] \) for details of proofs.

Let \( f_{k_1,k_2} := P^+_{1,0,k_1,k_2} \) for \( k_1, k_2 \in \mathbb{N}_0 \). Then \( f_{k_1,k_2}(0) \neq 0 \) if \( k_1 + k_2 = K + 1 \) and \( f_{k_1,k_2}(0) = 0 \) for \( k_1 + k_2 \geq K + 2 \) hold. Moreover, Let
\[ F_{k_1,k_2} := f_{k_1,k_2} - \left( \frac{4}{N} + 1 \right) \bar{Q} \frac{\beta_{1,0,k_1,k_2}}{4} \frac{|y|^2}{Q} - F^+_{1,0,k_1,k_2} - c^+_{1,0,k_1,k_2} Q. \]
Then
\[
\frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left( r^{N-1} \frac{\partial f_{k_1,k_2}}{\partial r} \right) = \frac{1}{r^{2\sigma}} f_{k_1-1,k_2} + \frac{\log r}{r^{2\sigma}} f_{k_1,k_2-1} + F_{k_1,k_2}
\]
holds. If \( k_2 = 0 \), then we obtain conclusion for \( P_{1,0,K+K',0}^+ \) as in [5].

If \( r^{-q}(\log r)^q f_{k_1-1,k_2} \) and \( r^{-q}(\log r)^q f_{k_1,k_2-1} \) converge to non-zero as \( r \to +0 \) for some \( q \in [0, 2\sigma] \) and \( q' \geq 0 \) or \( r^{-q}(\log r)^q f_{k_1-1,k_2} \) and \( q' \geq 0 \) or \( r^{-q}(\log r)^q f_{k_1,k_2-1} \) converge as \( r \to +0 \) for some \( q > 2\sigma \) and \( q' \geq 0 \), then \( r^{N-1} \frac{\partial f_{k+1}}{\partial r} \) converges to 0 as \( r \to +0 \).

Let \( \sigma_0 = \sigma''_0 := 0 \) and \( C_{k_1,k_2} := f_{k_1,k_2}(0) \) for \( k_1 + k_2 = K + 1 \). Moreover, let
\[
k := k_1 + k_2 - K - 1,
\]
\[
\sigma_{k+1} := \begin{cases} 1 - \sigma + \sigma_k & (\sigma_k < \sigma) \\ 1 & (\sigma_k \geq \sigma) \end{cases},
\]
\[
\sigma'_k := \begin{cases} \sigma'_k-1 + 1 & (\sigma_k-1 < \sigma) \\ 0 & (\sigma_k-1 \geq \sigma) \end{cases},
\]
\[
C_{k_1,k_2} := \begin{cases} C_{k_1,k_2-1} \frac{2^{\sigma_k}(N-2(\sigma-\sigma_k-1))}{\sigma_1} & (\sigma_k-1 < \sigma) \\ \sigma_1 C_{k_1,k_2-1}+F_{k_1,k_2}(0) & (\sigma_k-1 = \sigma) \\ \frac{F_{k_1,k_2}(0)}{2} & (\sigma_k-1 > \sigma) \end{cases}.
\]

In particular, if \( \sigma_k-1 < \sigma \), then \( C_{k_1,k_2} \neq 0 \). Then
\[
\lim_{r \to +0} \frac{1}{r^{2\sigma_k}(\log r)^{\sigma_k}} f_{k_1,k_2}(r) = C_{k_1,k_2}
\]
holds. For \( k = 0 \), it clearly holds. Moreover, for \( k \geq 1 \),
\[
\lim_{r \to +0} \frac{1}{r^{2\sigma_k-1}(\log r)^{\sigma_k}} \frac{\partial f_{k_1,k_2}}{\partial r}(r) = 2\sigma_k C_{k_1,k_2}
\]
holds. Consequently, we obtain Proposition [3.4] if \( K' \) is sufficiently large.

\[\square\]

4. Decomposition of Functions

The parameters \( \tilde{\lambda}, \tilde{b}, \tilde{\gamma} \) to be used for modulation are obtained by the following lemma:

**Lemma 4.1** (Decomposition). For any sufficiently small constants \( \tilde{t}, \tilde{\tau} > 0 \), there exist a sufficiently small \( \delta > 0 \) such that the following statement holds. Let \( I \) be an interval. We assume that \( u \in C(I, H^1(\mathbb{R}^N)) \cap C^1(I, H^{-1}(\mathbb{R}^N)) \) satisfies
\[
\forall \ t \in I, \ \left\| \lambda(t) \frac{\partial}{\partial t} u(t, \lambda(t) y) e^{i\gamma(t)} - Q \right\|_{H^1} < \delta
\]
for some functions \( \lambda : I \to (0, \tilde{t}) \) and \( \gamma : I \to \mathbb{R} \). Then there exist unique functions \( \tilde{\lambda} : I \to (0, \infty), \tilde{b} : I \to \mathbb{R}, \) and \( \tilde{\gamma} : I \to \mathbb{R}/2\pi \mathbb{Z} \) such that
\[
u(t, x) = \frac{1}{\lambda(t)^{\frac{N}{2}}} (P + \tilde{\xi}) \left( t, \frac{x}{\lambda(t)} \right) e^{-i \left( \frac{\tilde{\xi}}{\lambda(t)} - \tilde{b}(t) + \tilde{\gamma}(t) \right) x},
\]
\[
\left| \frac{\tilde{\lambda}(t)}{\lambda(t)} - 1 \right| + |\tilde{b}(t)| + |\tilde{\gamma}(t) - \gamma(t)|_{\mathbb{R}/2\pi \mathbb{Z}} < \tilde{\tau}
\]
hold, where \( | \cdot |_{\mathbb{R}/2\pi \mathbb{Z}} \) is defined by
\[
|c|_{\mathbb{R}/2\pi \mathbb{Z}} := \inf_{m \in \mathbb{Z}} |c + 2\pi m|,
\]
and that \( \tilde{\xi} \) satisfies the orthogonal conditions
\[
(\tilde{\xi}, \hat{\nu})_2 = (\tilde{\xi}, |y|^2 P)_2 = (\tilde{\xi}, \hat{\rho})_2 = 0
\]
on \( I \). In particular, \( \tilde{\lambda}, \tilde{b}, \) and \( \tilde{\gamma} \) are \( C^1 \) functions and independent of \( \lambda \) and \( \gamma \).
5. Approximate blow-up law

In this section, we describe the initial values and the approximation functions of the parameters $\lambda$ and $b$ in the decomposition.

We expect the parameters $\lambda$ and $b$ in the decomposition to approximately satisfy

$$\frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b = \frac{\partial b}{\partial s} + b^2 - \theta = 0.$$  

Therefore, the approximation functions $\lambda_{\text{app}}$ and $b_{\text{app}}$ of the parameters $\lambda$ and $b$ will be determined by the following lemma:

Let $\lambda_0 > 0$ be sufficiently small and define $J$ by

$$J(\lambda) := \int_{\lambda}^{\lambda_0} \frac{1}{\mu^{\frac{2}{2} + 1}} \sqrt{-\frac{2\beta_1}{(2-\alpha)^2} + \frac{2\beta_2}{2-\alpha} - \frac{2\beta_1}{2-\alpha} \log \mu} \, d\mu$$

for any $\lambda \in (0, \lambda_0)$, where $\beta_2 := \beta_{2,0,0,0}$. In particular, since

$$\sigma = \frac{2 - \alpha}{2}, \quad \beta_1 = \frac{4\alpha}{\gamma} \cdot \frac{\|\cdot\|_2^2}{\|\cdot\|_2^2}, \quad \beta_2 = \frac{4}{\|\cdot\|_2^2} \left( \sigma \int_{\mathbb{R}^N} \frac{1}{\|y\|^{2\sigma}} \log |y|Q^2 \, dy - \frac{1}{2}\|\cdot\|_2^2 \right),$$

$$\beta_1' = \frac{4}{\|\cdot\|_2^2} \int_{\mathbb{R}^N} \frac{1}{\|y\|^{2\sigma}} \log |y|Q^2 \, dy.$$  

we obtain

$$\beta_1' = -\frac{2\beta_1}{(2-\alpha)^2} + \frac{2\beta_2}{2-\alpha}.$$  

**Lemma 5.1.** Let

$$\lambda_{\text{app}}(s) := J^{-1}(s), \quad b_{\text{app}}(s) := \lambda_{\text{app}}(s) \frac{2\beta_1}{2-\alpha} \log \lambda_{\text{app}}(s).$$

Then $(\lambda_{\text{app}}, b_{\text{app}})$ is a solution for

$$\frac{\partial b}{\partial s} + b^2 + \beta_1 \lambda^\alpha \log \lambda - \beta_2 \lambda^\alpha = 0, \quad \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b = 0$$

in $s > 0$.

**proof.** This result is proven via direct calculation. For the construction of $(\lambda_{\text{app}}, b_{\text{app}})$, see [6]. \hfill \Box

For the sake of the discussion that follows, we will introduce some properties of the Lambert function $W_{-1}$. For definition of $W_{-1}$, see [1].

**Corollary 5.2 ([2]).** For any $u > 0$,

$$(1 - \epsilon)u - \frac{2}{\epsilon} < -W_{-1}(-e^{-u-1}) - 1 - \sqrt{2u} < u$$

holds. In particular, for any $z \in (0, \frac{1}{\epsilon})$,

$$-(1 - \epsilon) \log z + \sqrt{2(-\log z - 1)} + \frac{2}{\epsilon} < -W_{-1}(-z) < -\log z + \sqrt{2(-\log z - 1)}$$

holds.

**Lemma 5.3.** Let $\lambda$ be sufficiently small. Then

$$J(\lambda)^{-1} = \frac{2}{\alpha} \sqrt{\frac{2\beta_1}{2-\alpha}} \lambda^\alpha \sqrt{\log \lambda} \left( 1 + O\left( \frac{1}{\log \lambda} \right) \right)$$

holds. In addition,

$$\left| \frac{\alpha^2}{4} \left( -\frac{2\beta_1}{2-\alpha} \lambda^\alpha \log \lambda + \beta_1' \lambda^\alpha \right) - J(\lambda)^{-2} \right| \lesssim \lambda^\alpha$$

holds.
Moreover, let \( s \) be sufficiently large. Then
\[
 s^{-2} = J(\lambda_{\text{app}}(s))^{-2} = \frac{4}{\alpha^2} \lambda_{\text{app}}(s)^{\alpha} \log \lambda_{\text{app}}(s)(1 + o(1)),
\]
(16) 
\[
\lambda_{\text{app}}(s)^\alpha \log \lambda_{\text{app}}(s) \approx b_{\text{app}}(s)^2 \approx s^{-2}, \quad \lambda_{\text{app}}(s) \approx s^{-\frac{\alpha}{2}} \log s^{-\frac{\alpha}{2}}
\]
hold.

**proof.** Since \( \lambda_0 \) is sufficiently small,
\[
\mu^{\frac{\alpha}{2} + 1} \sqrt{\frac{\beta_1}{2 - \alpha} \log \mu} \leq \mu^{\frac{\alpha}{2} + 1} \sqrt{-\frac{2\beta_1}{(2 - \alpha)^2} + \frac{2\beta_2}{2 - \alpha} - \frac{2\beta_1}{2 - \alpha} \log \mu} \leq \mu^{\frac{\alpha}{2} + 1} \sqrt{\frac{4\beta_1}{2 - \alpha} \log \mu}
\]
holds for any \( \mu \in (0, \lambda_0) \). Therefore, we obtain
\[
\frac{\alpha}{2} \sqrt{\frac{4\beta_1}{2 - \alpha} \lambda^{\frac{\alpha}{2}}} \sqrt{\log \lambda} \leq J(\lambda) \leq \frac{\alpha}{4} \sqrt{\frac{\beta_1}{2 - \alpha} \lambda^\alpha \log \lambda}
\]
for any sufficiently small \( \lambda \).

Next, since
\[
\frac{1}{2} \sqrt{\frac{4\beta_1}{2 - \alpha} \lambda^{\frac{\alpha}{2}}} \sqrt{\log \lambda} = \int_\lambda^{\lambda_0} \left( \frac{1}{\sqrt{\frac{2\beta_1}{2 - \alpha} \mu^{\frac{\alpha}{2} + 1} \log \mu} - \frac{1}{\sqrt{\frac{2\beta_1}{2 - \alpha} \mu^{\frac{\alpha}{2} + 1} \log \mu}} \right) d\mu + \frac{1}{2} \sqrt{\frac{2\beta_1}{2 - \alpha} \lambda^{\frac{\alpha}{2}}} \sqrt{\log \lambda_0}
\]
we obtain
\[
J(\lambda) = \frac{1}{\frac{\lambda}{2} \sqrt{\frac{2\beta_1}{2 - \alpha} \lambda^{\frac{\alpha}{2}}} \sqrt{\log \lambda}} = \int_\lambda^{\lambda_0} \frac{1}{\mu^{\frac{\alpha}{2} + 1} \log \mu} + \int_\lambda^{\lambda_0} \frac{\beta_1}{2} \sqrt{\frac{2\beta_1}{2 - \alpha} \log \mu} \left( \sqrt{\frac{2\beta_1}{2 - \alpha} \log \mu} + \beta_1 \sqrt{\frac{2\beta_1}{2 - \alpha} \log \mu} \right) d\mu + \frac{1}{\frac{\lambda}{2} \sqrt{\frac{2\beta_1}{2 - \alpha} \lambda^{\frac{\alpha}{2}}} \sqrt{\log \lambda}}
\]
Therefore,
\[
J(\lambda) \leq \frac{1}{\frac{\lambda}{2} \sqrt{\frac{2\beta_1}{2 - \alpha} \lambda^{\frac{\alpha}{2}}} \sqrt{\log \lambda}} \leq \int_\lambda^{\lambda_0} \frac{1}{\mu^{\frac{\alpha}{2} + 1} \log \mu} d\mu + \frac{1}{\frac{\lambda}{2} \sqrt{\frac{2\beta_1}{2 - \alpha} \lambda^{\frac{\alpha}{2}}} \sqrt{\log \lambda_0}}
\]
Accordingly,
\[
\left| J(\lambda)^{1} - \frac{2}{\alpha} \sqrt{\frac{2\beta_1}{2 - \alpha} \lambda^{\frac{\alpha}{2}}} \sqrt{\log \lambda} \right| \leq \lambda^\alpha |\log \lambda| \frac{1}{\lambda^{\frac{\alpha}{2}} |\log \lambda|} = \lambda^\frac{\alpha}{2} \sqrt{\log \lambda} \frac{1}{\log \lambda}
\]
Since \( \lambda_{\text{app}}(s) \to 0 \) as \( s \to \infty \),
\[
 s^{-1} = J(\lambda_{\text{app}}(s))^{-1} = \frac{2}{\alpha} \sqrt{\frac{2\beta_1}{2 - \alpha} \lambda_{\text{app}}(s)^{\frac{\alpha}{2}}} \sqrt{\log \lambda_{\text{app}}(s)}(1 + o(1))
\]
for any sufficiently large \( s \). Therefore,
\[
\frac{1}{2} s^{-2} \leq \frac{4}{\alpha^2} \lambda_{\text{app}}(s)^{\alpha} |\log \lambda_{\text{app}}(s)| \leq 2s^{-2}
\]
and
\[ \frac{\beta_1}{2 - \alpha} \log \lambda_{\text{app}}(s) \leq b_{\text{app}}(s)^2 \leq \frac{4\beta_1}{2 - \alpha} \log \lambda_{\text{app}}(s) \]
hold.

Finally, since
\[ -C_1 s^{-2} \leq \lambda_{\text{app}}(s) \leq \log \lambda_{\text{app}}(s) = e^{\log \lambda_{\text{app}}(s)} = W^{-1}(\log \lambda_{\text{app}}(s)) \leq -C_2 s^{-2}, \]
we obtain
\[ W_{-1}(C_2 s^{-2}) \leq \log \lambda_{\text{app}}(s) \leq W_{-1}(C_1 s^{-2}). \]
Since \( e^{W(z)} = \frac{z}{W(z)} \), we obtain
\[ -C_2 s^{-2} \leq \lambda_{\text{app}}(s) \leq -C_2 s^{-2} \]
\[ W_{-1}(C_2 s^{-2}) \leq \lambda_{\text{app}}(s) \leq W_{-1}(C_2 s^{-2}). \]
Since
\[ \lambda_{\text{app}}(s) \approx s^{-\frac{2}{\alpha}} (\log s)^{-\frac{d}{\alpha}}. \]

Furthermore, the following lemma determines \( \lambda(s_1) \) and \( b(s_1) \) for a given energy level \( E_0 \) and a sufficiently large \( s_1 \).

**Lemma 5.4.** Let define \( C_0 := \frac{8E_0}{\|\phi\|_2^2} \) and \( 0 < \lambda_0 \ll 1 \) such that \( -\frac{2\beta_1}{2 - \alpha} \log \lambda_0 + \beta_1' + C_0 \lambda_0^{2-\alpha} > 0 \). For \( \lambda \in (0, \lambda_0) \), we set
\[ \mathcal{F}(\lambda) := \int_{\lambda}^{\lambda_0} \frac{1}{\mu^{\frac{d}{\alpha}+1}} \sqrt{\frac{2\beta_1}{2 - \alpha} \log \mu + \beta_1' + C_0 \mu^{2-\alpha}} d\mu. \]
Then for any \( s_1 \gg 1 \), there exist \( b_1, \lambda_1 > 0 \) such that
\[ \left| \frac{s_1^{-1}}{J(\lambda_1)^{-1}} - 1 \right| + \left| \frac{b_1}{b_{\text{app}}(s_1)} - 1 \right| \leq s_1^{-\frac{d}{\alpha}} |\log s_1| + s_1^{-\frac{2d}{\alpha}} |\log s_1|, \quad \mathcal{F}(\lambda_1) = s_1, \quad E(P_{\lambda_1, b_1}) = E_0. \]
Moreover,
\[ |\mathcal{F}(\lambda) - J(\lambda)| \lesssim \lambda^{-\frac{d}{\alpha}} + \lambda^{2-\frac{2d}{\alpha}} \]
holds.

**Proof.** The method of choosing \( \lambda_1 \) and \( b_1 \) and the estimate of \( \mathcal{F} \) and \( b_1 \) are the same as in [6] [8].

Since
\[ s_1 = J(\lambda_{\text{app}}(s_1)), \quad J(\lambda)^{-1} \approx \lambda^{\frac{d}{\alpha}} \sqrt{|\log \lambda|}, \]
we obtain
\[ \left| \frac{J(\lambda_{\text{app}}(s_1))}{J(\lambda_1)} - 1 \right| \lesssim \lambda_1^{\frac{d}{\alpha}} |\log \lambda_1| + \lambda_1^{2-\alpha} \sqrt{|\log \lambda_1|}. \]
Moreover, since
\[ s_1 = \mathcal{F}(\lambda_1) \lesssim \int_{\lambda}^{\lambda_0} \frac{1}{\mu^{\frac{d}{\alpha}+1}|\log \mu|} d\mu \leq \int_{\lambda}^{\lambda_0} \frac{1}{\mu^{\frac{d}{\alpha}+1}} d\mu \leq \lambda_1^{-\frac{d}{\alpha}} \]
we obtain
\[ \lambda_1 \lesssim s_1^{-\frac{d}{\alpha}}. \]
Consequently, we obtain the conclusion. \( \square \)
6. Uniformity estimates for decomposition

In this section, we estimate modulation terms.

We define
\[ t_{\text{app}}(s) := - \int_s^{\infty} \lambda_{\text{app}}(\mu)^2 d\mu. \]

For \( t_1 < 0 \) such that is sufficiently close to 0, we define
\[ s_1 := t_{\text{app}}^{-1}(t_1). \]

Additionally, let \( \lambda_1 \) and \( b_1 \) be given in Lemma 5.4 for \( s_1 \) and \( \gamma_1 = 0 \). Let \( u \) be the solution for (11) with \( \pm = + \) with an initial value
\[ u(t_1, x) := P_{\lambda_1, b_1, 0}(x). \]

Then since \( u \) satisfies the assumption ofLemma 4.1 in a neighbourhood of \( t_1 \), there exists a decomposition
\((\bar{\lambda}_t, \bar{b}_t, \gamma_t, \bar{\xi}_t)\) such that (13) holds in a neighbourhood \( I \) of \( t_1 \). The rescaled time \( s_{t_1} \) is defined by
\[ s_{t_1}(t) := s_1 - \int_t^{t_1} \frac{1}{\lambda_{t_1}(\tau)} d\tau. \]

Then we define an inverse function \( s_{t_1}^{-1} : s_{t_1}(I) \to I \). Moreover, we define
\[ t_{t_1} := s_{t_1}^{-1}, \quad \lambda_{t_1}(s) := \bar{\lambda}(t_{t_1}(s)), \quad b_{t_1}(s) := \bar{b}(t_{t_1}(s)), \quad \gamma_{t_1}(s) := \gamma(t_{t_1}(s)), \quad \xi_{t_1}(s, y) := \bar{\xi}(t_{t_1}(s), y). \]

For the sake of clarity in notation, we often omit the subscript \( t_1 \). In particular, it should be noted that \( u \in C((T_*, T^*), \Sigma^2(\mathbb{R}^N)) \) and \( |x| \nabla u \in C((T_*, T^*), L^2(\mathbb{R}^N)) \). Furthermore, let \( I_{t_1} \) be the maximal interval such that a decomposition as (13) is obtained and we define
\[ J_{s_1} := s(I_{t_1}). \]

Additionally, let \( s_0 \) (\( \leq s_1 \)) be sufficiently large and let
\[ s' := \max\{s_0, \inf J_{s_1}\}. \]

Let \( s_* \) be defined by
\[ s_* := \inf\{\sigma \in (s', s_1] \mid (17) \text{ holds on } [\sigma, s_1]\}, \]

where
\[ \|\xi(s)\|_{H^1}^2 + b(s)^2 \|y/\xi(s)\|_2^2 < s^{-2K}, \quad \left| \frac{s^{-1}}{J(\lambda(s))^{-1}} - 1 \right| + \left| \frac{b(s)}{b_{\text{app}}(s)} - 1 \right| < (\log s)^{-\frac{1}{2}}. \]

Finally, we define
\[ \text{Mod}(s) := \left( 1 + \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b, \frac{\partial b}{\partial s} + b^2 - \theta, 1 - \frac{\partial \gamma}{\partial s} \right). \]

The goal of this section is to estimate of \( \text{Mod}(s) \).

**Lemma 6.1.** On \((s_*, s_1)\),
\[ \lambda(s)^{\alpha} |\log \lambda(s)| \approx s^{-2}, \quad b(s) \approx s^{-1} \]

holds.

**proof.** For \( b \), it is clear from (10) and (17).

From Lemma 4.1 we may assume \( |\lambda(s)| \leq \frac{1}{2} \). Since
\[ \left| \frac{J(\lambda(s))^{-1}}{s^{-1}} - 1 \right| \lesssim (\log s)^{-\frac{1}{2}}, \]

we obtain
\[ |J(\lambda(s))^{-1}| \lesssim s^{-1} \left( 1 + (\log s)^{-\frac{1}{2}} \right). \]

Moreover, since
\[ \lambda(s)^{\alpha} \lesssim \lambda(s)^{\alpha} |\log \lambda(s)| \lesssim s^{-2} \left( 1 + (\log s)^{-\frac{1}{2}} \right)^2 \]
Lemma 7.1 \[ \text{in Section 8.} \]

holds. Moreover, let \( \lambda \) hold. Therefore, we obtain from (15), we have \( \lambda = o(1) \) and

\[ \frac{4}{\alpha^2} \frac{2\beta_1}{2 - \alpha} \lambda(s)^\alpha |\log \lambda(s)| = J(\lambda(s))^{-2}(1 + o(1)). \]

Next, since

\[ \frac{J(\lambda(s))^{-2}}{s^{-2}} - 1 \lesssim (\log s)^{-\frac{1}{2}}, \quad \text{i.e., } |J(\lambda(s))^{-2} - s^{-2}| \lesssim s^{-2}(\log s)^{-\frac{1}{4}}, \]

we obtain

\[ \frac{4}{\alpha^2} \frac{2\beta_1}{2 - \alpha} \lambda(s)^\alpha |\log \lambda(s)| - s^{-2} \lesssim J(\lambda(s))^{-2}o(1) + s^{-2}(\log s)^{-\frac{1}{4}} = o(s^{-2}). \]

Therefore, we obtain

\[ \frac{4}{\alpha^2} \frac{2\beta_1}{2 - \alpha} \lambda(s)^\alpha |\log \lambda(s)| = s^{-2} (1 + o(1)). \]

\[ \square \]

7. Modified energy function

We proceed with a modified version of the technique presented in Le Coz, Martel, and Raphaël [6] and Raphaël and Szeftel [14]. Let \( m > 0 \) be sufficiently large and define

\[ H(s, \varepsilon) := \frac{1}{2} \|\varepsilon\|^2_{H^1} + b^2 \|y|\varepsilon\|^2_2 - \int_{\mathbb{R}^N} (F(P + \varepsilon) - F(P) - dF(P)(\varepsilon)) \, dy \]

\[ + \frac{1}{2} \lambda^o \log \lambda \int_{\mathbb{R}^N} \frac{1}{|y|^{2\sigma}} |\varepsilon|^2 \, dy + \frac{1}{2} \lambda^o \int_{\mathbb{R}^N} \frac{1}{|y|^{2\sigma}} \log |y| |\varepsilon|^2 \, dy, \]

\[ S(s, \varepsilon) := \frac{1}{\lambda^m} H(s, \varepsilon). \]

In this section, we obtain upper and lower estimates of \( S \) and a lower estimate of the time derivative of \( S \) for use in Section 8.

Lemma 7.1 (Estimates of \( S \)). For \( s \in (s_*, s_1] \),

\[ \|\varepsilon\|^2_{H^1} + b^2 \|y|\varepsilon\|^2_2 + O(s^{-2(K+2)}) \lesssim H(s, \varepsilon) \lesssim \|\varepsilon\|^2_{H^1} + b^2 \|y|\varepsilon\|^2_2 \]

hold. Moreover,

\[ \frac{1}{\lambda^m} \left( \|\varepsilon\|^2_{H^1} + b^2 \|y|\varepsilon\|^2_2 + O(s^{-2(K+2)}) \right) \lesssim S(s, \varepsilon) \lesssim \frac{1}{\lambda^m} \left( \|\varepsilon\|^2_{H^1} + b^2 \|y|\varepsilon\|^2_2 \right) \]

hold.

Lemma 7.2 (Derivative of \( S \) in time). For \( s \in (s_*, s_1] \),

\[ \frac{d}{ds} H(s, \varepsilon(s)) \gtrsim -b \left( \|\varepsilon\|^2_{H^1} + b^2 \|y|\varepsilon\|^2_2 \right) + O(s^{-2(K+2)}) \]

holds. Moreover,

\[ \frac{d}{ds} S(s, \varepsilon(s)) \gtrsim \frac{b}{\lambda^m} \left( \|\varepsilon\|^2_{H^1} + b^2 \|y|\varepsilon\|^2_2 + O(s^{-2(K+3)}) \right) \]

holds for \( s \in (s_*, s_1] \).

For the proofs, see [6, 8].
8. Bootstrap

In this section, we use the estimates obtained in Section 7 and the bootstrap to establish the estimates of the parameters. Namely, we confirm \( s \in (s_0, s_1) \).

**Lemma 8.1** (Re-estimation). For \( s \in (s_0, s_1) \),

\[
\|\varepsilon(s)\|^2_{H^1} + b(s)^2 \|y|\varepsilon(s)\|^2_2 \lesssim s^{-2(K+2)},
\]

\[
\left| \frac{1}{J(\lambda(s)) - 1} - \frac{b(s)}{b_{app}(s)} - 1 \right| \lesssim (\log s)^{-1}
\]

holds.

**Proof.** We can prove (18) as in [6].

Next, since

\[
F \lambda \log \lambda - \beta' \lambda^\alpha - C_0 \lambda^2
\]

we have

\[
\left| b^2 + \frac{2\beta_1}{2 - \alpha} \lambda^\alpha \log \lambda - \beta' \lambda^\alpha - C_0 \lambda^2 \right|
\]

\[
\lesssim \lambda^2 \left( \left| \frac{b^2}{\lambda^2} + \frac{2\beta_1}{2 - \alpha} \lambda^{-2} \log \lambda - \beta' \lambda^{-2} \right| + \frac{8}{\|y\|_Q^2} |E(P_{\lambda,\beta,\gamma})| + \frac{8}{\|y\|_Q^2} |E(P_{\lambda,\beta,\gamma}) - E_0| \right)
\]

\[
\lesssim s^{-4}.
\]

From the definition of \( \mathcal{F} \), we have

\[
|E(P_{\lambda,\beta,\gamma}(s)) - E_0| \leq \int_{s_1}^s \frac{d}{ds} E(P_{\lambda,\beta,\gamma}(s)) d\tau \leq \int_{s_1}^{s_1} \tau^{-(K+2)+\frac{2}{\alpha}} (\log s)^{\frac{2}{\alpha}} d\tau \lesssim s^{-(K+1)+\frac{2}{\alpha}},
\]

where \( \mathcal{F}(s) := \mathcal{F}(\lambda(s)) \). Indeed, since

\[
|\mathcal{F}'(s) - 1| \lesssim s^{-2},
\]

where \( \mathcal{F}(s) := \mathcal{F}(\lambda(s)) \). Thus, since

\[
|s - \mathcal{F}(\lambda(s))| \lesssim s^{-1}
\]

holds since \( \mathcal{F}(\lambda(s_1)) = s_1 \). Accordingly, since

\[
|s - J(\lambda(s))| \leq |s - \mathcal{F}(\lambda(s))| + |J(\lambda(s)) - \mathcal{F}(\lambda(s))| \lesssim s^{\frac{1}{2}} (\log s)^{\frac{1}{2}} + s^{2 - \frac{1}{\alpha}} (\log s)^{\frac{2}{\alpha} - \frac{3}{\alpha}},
\]

we obtain

\[
\left| \frac{1}{J(\lambda(s)) - 1} - \frac{b(s)}{b_{app}(s)} - 1 \right| \lesssim s^{-\frac{1}{2}} (\log s)^{\frac{1}{2}} + s^{2 - \frac{1}{\alpha}} (\log s)^{\frac{2}{\alpha} - \frac{3}{\alpha}}.
\]

Finally, from (20), we have

\[
\left| b(s)^2 - b_{app}(s)^2 \right| \lesssim s^{-4} + \lambda(s)^2 + \frac{4}{\alpha^2} \left( -\frac{2\beta_1}{2 - \alpha} \lambda(s)^\alpha \log \lambda(s) + \beta' \lambda(s)^\alpha \right) - J(\lambda(s))^{-2}
\]

\[
+ \left| J(\lambda(s))^{-2} - J(\lambda_{app}(s))^{-2} \right| + \frac{4}{\alpha^2} \left( -\frac{2\beta_1}{2 - \alpha} \lambda_{app}(s)^\alpha \log \lambda_{app}(s) + \beta' \lambda_{app}(s)^\alpha \right) - J(\lambda_{app}(s))^{-2}
\]

\[
\lesssim s^{-4} + s^{\frac{1}{\alpha}} (\log s)^{-\frac{1}{\alpha}} + s^{-2} (\log s)^{-1} + s^{-2} \left( s^{\frac{1}{\alpha}} (\log s)^{\frac{1}{2}} + s^{2 - \frac{1}{\alpha}} (\log s)^{\frac{2}{\alpha} - \frac{3}{\alpha}} \right)
\]
and
\[ \left| \frac{b(s)}{\tilde{b}_{app}(s)} - 1 \right| \lesssim (\log s)^{-1} + s^{-\frac{1}{2}}(\log s)^{\frac{1}{2}} + s^{2-\frac{\alpha}{2}}(\log s)^{\frac{1}{2}-\frac{\alpha}{2}}. \]

Consequently, we obtain \([19]\).

Lemma 8.2. If \( s_0 \) is sufficiently large, then \( s_* = s' = s_0 \).

proof. This result is proven from Lemma 8.1 and the definitions of \( s_* \) and \( s' \). See [7] for details of the proof. \( \square \)

9. Conversion of estimates

In this section, we rewrite the uniform estimates obtained for the time variable \( s \) in Lemma 8.1 into uniform estimates for the time variable \( t \).

Lemma 9.1 (Interval). If \( s_0 \) is sufficiently large, then there is \( t_0 < 0 \) that is sufficiently close to 0 such that for \( t_1 \in (t_0, 0) \),
\[ [t_0, t_1] \subset s_1^{-1}([s_0, s_1]), \quad |t_{app} - s_1^{-1}(t)| \lesssim s_1(t) - s_1^{-1}(t) \sim (\log s_1(t))^{-\frac{1}{2}}. \]

holds.

proof. Since \( t_1(s_1) = t_1 = t_{app}(s_1) \), we have
\[ t_{app}(s) - t_1(s) = t_1(s_1) - t_1(s) = (t_{app}(s) - t_{app}(s_1)) + t_1(s_1) - t_{app}(s_1) \]
\[ = \int_{s_1}^{s} \lambda_{app}(\tau) (\lambda_{t_1}(\tau) - \lambda_{app}(\tau)) \left( \frac{\lambda_{t_1}(\tau)}{\lambda_{app}(\tau)} + 1 \right) d\tau. \]

Since \( J^{-1} \) is \( C^1 \) function on \( J((0, \lambda_0)) \),
\[ |\lambda_{t_1}(\tau) - \lambda_{app}(\tau)| \leq \frac{\tau}{|J^{-1}(J(\lambda_{t_1}(\tau)) + \xi (J(\lambda_{app}(\tau)) - J(\lambda_{t_1}(\tau)))))| \bigg| J(\lambda_{t_1}(\tau))^{-1} - 1 \bigg| \]
for some \( \xi \in [0, 1] \). Since
\[ \left| \frac{1}{J'(\lambda)} \right| = \lambda^{\frac{1}{2}+1} \sqrt{-\frac{2\beta_1}{(2-\alpha)^2} + \frac{2\beta_2}{2-\alpha} - \frac{2\beta_1}{2-\alpha} \log \lambda} \lesssim \lambda^{\frac{1}{2}+1} \sqrt{\log \lambda} \]
and
\[ C_1 \tau \leq J(\lambda_{t_1}(\tau)) + \xi (J(\lambda_{app}(\tau)) - J(\lambda_{t_1}(\tau))) \leq C_2 \tau, \]
we obtain
\[ J^{-1}(C_1 \tau) \leq J^{-1}(J(\lambda_{t_1}(\tau)) + \xi (J(\lambda_{app}(\tau)) - J(\lambda_{t_1}(\tau)))) \leq J^{-1}(C_1 \tau). \]
Moreover, since
\[ J^{-1}(C\tau) = \lambda_{app}(C\tau) \approx (C\tau)^{-\frac{\alpha}{2}} (\log(C\tau))^{-\frac{\alpha}{2}} \approx \tau^{-\frac{\alpha}{2}} (\log \tau)^{-\frac{\alpha}{2}} \approx \lambda_{app}(\tau) \]
from Lemma 8.3, we obtain
\[ \left| J'(J^{-1}(J(\lambda_{t_1}(\tau)) + \xi (J(\lambda_{app}(\tau)) - J(\lambda_{t_1}(\tau)))))) \bigg| \lesssim \tau \lambda_{app}(\tau)^{\frac{1}{2}+1} \sqrt{\log \lambda_{app}(\tau)} \lesssim \lambda_{app}(\tau). \]
Therefore, we obtain
\[ |t_{app}(s) - t_1(s)| \lesssim s^{-\frac{1-\alpha}{2} \alpha} (\log s)^{-\frac{1}{2}}. \]

Similarly, we obtain
\[ |t_{app}(s)| \approx s^{-\frac{1-\alpha}{2} \alpha} (\log s)^{-\frac{1}{2}}. \]

Moreover, there exists \( t_0 \) from Lemma 8.2. \( \square \)

Corollary 9.2.
\[ s_1(t) \approx |t|^{-\frac{1-\alpha}{2} \alpha} |\log |t||^{-\frac{1}{2}}. \]
\textbf{proof.} From Lemma 9.1

\[ |t_{\text{app}}(s_1(t))| - C_1 s_1(t) - \frac{4 - \alpha}{\alpha} (\log s_1(t)) - \frac{\alpha}{\alpha} \leq |t| \leq |t_{\text{app}}(s_1(t))| + C_2 s_1(t) - \frac{4 - \alpha}{\alpha} (\log s_1(t)) - \frac{\alpha}{\alpha} \]

and

\[ |t_{\text{app}}(s_1(t))| \approx s_1(t) - \frac{4 - \alpha}{\alpha} (\log s_1(t)) - \frac{\alpha}{\alpha} \]

hold. Since

\[ |t| \approx s_1(t) - \frac{4 - \alpha}{\alpha} (\log s_1(t)) - \frac{\alpha}{\alpha} \]

we obtain

\[ C_1 |t|^{- \frac{\alpha}{\alpha}} \leq s_1(t) - \frac{4 - \alpha}{\alpha} (\log s_1(t)) - \frac{\alpha}{\alpha} = W_0^{-1} (\log s_1(t)) - \frac{\alpha}{\alpha} \leq C_2 |t|^{- \frac{\alpha}{\alpha}}. \]

Therefore,

\[ W_0(C_1 |t|^{- \frac{\alpha}{\alpha}}) \leq \log s_1(t) - \frac{4 - \alpha}{\alpha} \leq W_0(C_2 |t|^{- \frac{\alpha}{\alpha}}). \]

Moreover, since \( e^{W(z)} = \frac{\tilde{z}}{W(z)} \),

\[ \frac{C_1 |t|^{- \frac{\alpha}{\alpha}}}{W_0(C_1 |t|^{- \frac{\alpha}{\alpha}})} \leq s_1(t) - \frac{4 - \alpha}{\alpha} \leq \frac{C_2 |t|^{- \frac{\alpha}{\alpha}}}{W_0(C_2 |t|^{- \frac{\alpha}{\alpha}})}. \]

Since \( W_0(z) \approx \log z \) for sufficiently large \( z \), we obtain

\[ \frac{C_1 |t|^{- \frac{\alpha}{\alpha}}}{C_1 |\log |t||} \leq s_1(t) - \frac{4 - \alpha}{\alpha} \leq \frac{C_2 |t|^{- \frac{\alpha}{\alpha}}}{C_1 |\log |t||}. \]

Consequently, we obtain conclusion. \( \square \)

\textbf{Lemma 9.3 (Conversion of estimates).} For \( t \in [t_0, t_1] \),

\[ \tilde{\lambda}_1(t) \approx |t|^{- \frac{\alpha}{\alpha}} |\log |t|| - \frac{\alpha}{\alpha}, \quad \tilde{b}_1(t) \approx |t|^{- \frac{\alpha}{\alpha}} |\log |t|| - \frac{\alpha}{\alpha}, \]

\[ \| \tilde{\varepsilon}_1(t) \|_{H^1} \lesssim |t|^{\frac{2 \alpha}{\alpha}} |\log |t|| - \frac{\alpha}{\alpha}, \quad \| |y| \tilde{\varepsilon}_1(t) \|_2 \lesssim |t|^{\frac{2 (K - 1)}{\alpha} - \frac{\alpha}{\alpha}} |\log |t|| - \frac{2 (K - 1)}{\alpha} \]

\textbf{proof.} From Lemma 8.1 Lemma 6.1 and Corollary 9.2 it is proven. \( \square \)

10. Proof of Theorem 1.5

In this section, we complete the proof of Theorem 1.5. See [8] for details of proof.

\textbf{proof of Theorem 1.5.} Let \( (t_n)_{n \in \mathbb{N}} \subset (t_0, 0) \) be a monotonically increasing sequence such that \( \lim_{n \to \infty} t_n = 0 \). For each \( n \in \mathbb{N} \), \( u_n \) is the solution for (1) with \( \pm = - \) and with an initial value

\[ u_n(t_n, x) := P_{\lambda_1, t_n, b_1, 0}(x) \]

at \( t_n \), where \( b_1, 0 \) and \( \lambda_1, 0 \) are given by Lemma 5.4 for \( t_n \).

According to Lemma 4.1 with an initial value \( \tilde{\tau}_n(t_n) = 0 \), there exists a decomposition

\[ u_n(t, x) = \frac{1}{\tilde{\lambda}_n(t)} e^{-i \tilde{\varepsilon}_n(t) \left( t, \frac{x}{\lambda_n(t)} \right) + \tilde{\tau}_n(t)} \left( t, \frac{x}{\lambda_n(t)} \right) a_n(t, x). \]

Then \( (u_n(t_0))_{n \in \mathbb{N}} \) is bounded in \( \Sigma^1 \). Therefore, up to a subsequence, there exists \( u_\infty(t_0) \in \Sigma^1 \) such that

\[ u_n(t_0) \to u_\infty(t_0) \quad \text{in} \quad \Sigma^1, \quad u_n(t_0) \to u_\infty(t_0) \quad \text{in} \quad L^2(\mathbb{R}^N) \quad (n \to \infty), \]

see [8] for details.

Let \( u_\infty \) be the solution for (1) with \( \pm = + \) and an initial value \( u_\infty(t_0) \), and let \( T^* \) be the supremum of the maximal existence interval of \( u_\infty \). Moreover, we define \( T := \min(0, T^*) \). Then for any \( T' \in [t_0, T], [t_0, T'] \subset [t_0, t_n] \) if \( n \) is sufficiently large. Then there exist \( \tau_0(T', t_0) > 0 \) such that

\[ \sup_{n \geq \tau_0} \| u_n \|_{L^\infty([t_0, T'], \Sigma^1)} \leq C(T', t_0) \]

holds. Therefore,

\[ u_n \to u_\infty \quad \text{in} \quad C([t_0, T'], L^2(\mathbb{R}^N)) \quad (n \to \infty) \]
holds (see [7]). In particular, \( u_n(t) \to u_{\infty}(t) \) in \( \Sigma^1 \) for any \( t \in [t_0, T) \). Furthermore, from the mass conservation, we have

\[
\|u_{\infty}(t)\|_2 = \|u_{\infty}(t_0)\|_2 = \lim_{n \to \infty} \|u_n(t_0)\|_2 = \lim_{n \to \infty} \|u_n(t_n)\|_2 = \lim_{n \to \infty} \|P(t_n)\|_2 = \|Q\|_2.
\]

Based on weak convergence in \( H^1(\mathbb{R}^N) \) and Lemma 4.1 we decompose \( u_{\infty} \) to

\[
u_{\infty}(t,x) = \frac{1}{\lambda_{\infty}(t)^{\frac{2}{4}}}(P + \tilde{\varepsilon}_{\infty}) \left( t, \frac{x}{\lambda_{\infty}(t)} \right) e^{-\frac{i\lambda_{\infty}(t) - 2}{2} 4\tilde{\gamma}_{\infty}(t)},
\]

where an initial value of \( \tilde{\gamma}_{\infty} \) is \( \gamma_{\infty}(t_0) \in (t_0^{\frac{1}{2}} - \pi, t_0^{\frac{1}{2}} + \pi) \cap \tilde{\gamma}(u_{\infty}(t_0)) \). Furthermore, for any \( t \in [t_0, T) \), as \( n \to \infty \),

\[
\lambda_{n}(t) \to \tilde{\lambda}_{\infty}(t), \quad \tilde{b}_{n}(t) \to \tilde{b}_{\infty}(t), \quad e^{i\tilde{\gamma}_{n}(t)} \to e^{i\tilde{\gamma}_{\infty}(t)}, \quad \tilde{\varepsilon}_{n}(t) \to \tilde{\varepsilon}_{\infty}(t)
\]

hold. Consequently, from the uniform estimate in Lemma 9.3 as \( n \to \infty \), we have

\[
\tilde{\lambda}_{\infty}(t) \approx \|t\|^{\frac{2}{2\sigma}} \log \|t\|^{\frac{1}{2\sigma}}, \quad \tilde{b}_{\infty}(t) \approx \|t\|^{\frac{2}{2\sigma}} \log \|t\|^{\frac{1}{2\sigma}}, \quad \|\|g\|_{\tilde{\varepsilon}_{\infty}(t)}\|_2 \lesssim \|t\|^{\frac{(K-1)}{2-\alpha}} \log \|t\|^{\frac{2(K-1)}{2-\alpha}}
\]

Consequently, we obtain that \( u \) converges to the blow-up profile in \( \Sigma^1 \).

Finally, we check energy of \( u_{\infty} \). Since

\[
E(u_{n}) - E\left( P_{\lambda_{n}, \tilde{b}_{n}, \tilde{\gamma}_{n}} \right) = \int_{0}^{1} \left( E'(P_{\lambda_{n}, \tilde{b}_{n}, \tilde{\gamma}_{n}} + \tau \varepsilon_{\lambda_{n}, \tilde{b}_{n}, \tilde{\gamma}_{n}}) \right) d\tau
\]

and \( E'(w) = -\Delta w - |w|^2 - 2\alpha |x|^\alpha \log |x|w \), we have

\[
E(u_{n}) - E\left( P_{\lambda_{n}, \tilde{b}_{n}, \tilde{\gamma}_{n}} \right) = O\left( \frac{1}{\lambda_{n}^{\alpha}} \|\varepsilon_{\infty}\|_{H^1} \right) = O\left( \|t\|^{\frac{4K-4}{2-\alpha}} \log \|t\|^{\frac{2K-4}{2-\alpha}} \right).
\]

Similarly, we have

\[
E(u_{\infty}) - E\left( P_{\lambda_{\infty}, \tilde{b}_{\infty}, \tilde{\gamma}_{\infty}} \right) = O\left( \frac{1}{\lambda_{\infty}^{\alpha}} \|\varepsilon_{\infty}\|_{H^1} \right) = O\left( \|t\|^{\frac{4K-4}{2-\alpha}} \log \|t\|^{\frac{2K-4}{2-\alpha}} \right).
\]

From the continuity of \( E \), we have

\[
\lim_{n \to \infty} E\left( P_{\lambda_{n}, \tilde{b}_{n}, \tilde{\gamma}_{n}} \right) = E\left( P_{\lambda_{\infty}, \tilde{b}_{\infty}, \tilde{\gamma}_{\infty}} \right)
\]

and from the conservation of energy,

\[
E(u_{n}) = E(u_{n}(t_n)) = E\left( P_{\lambda_{1,n}, \tilde{b}_{1,n}, \tilde{\gamma}_{1,n,0}} \right) = E_0.
\]

Therefore, we have

\[
E(u_{\infty}) = E_0 + o_1(t_{\infty}) = E_0 + o(1)
\]

and since \( E(u_{\infty}) \) is constant for \( t \), \( E(u_{\infty}) = E_0 \).

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(N. Matsui) Department of Mathematics, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

Email address, N. Matsui: 1120703@ed.tus.ac.jp