Joint extreme values of $L$-functions

Kamalakshya Mahatab · Łukasz Pańkowski · Akshaa Vatwani

Received: 22 March 2020 / Accepted: 14 July 2022 / Published online: 16 August 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract
We consider $L$-functions $L_1, \ldots, L_k$ from the Selberg class which have polynomial Euler product and satisfy Selberg’s orthonormality condition. We show that on every vertical line $s = \sigma + it$ with $\sigma \in (1/2, 1)$, these $L$-functions simultaneously take large values of size $\exp\left(c \left(\frac{\log t}{\log \log t}\right)^{1-\sigma}\right)$ inside a small neighborhood. Our method extends to $\sigma = 1$ unconditionally, and to $\sigma = 1/2$ on the generalized Riemann hypothesis. We also obtain similar joint omega results for arguments of the given $L$-functions.

1 Introduction

A fundamental object of study in number theory is the Riemann zeta function $\zeta(s)$, whose zeros are intimately connected with the distribution of the prime numbers. A long standing conjecture in this context is the Riemann Hypothesis, which asserts that the nontrivial zeros of $\zeta(s)$ lie on the line $\text{Re}(s) = 1/2$. It is this conjecture that motivates the theory of distribution of values of $\zeta(s)$ on the critical line $\text{Re}(s) = 1/2$.

KM was supported by Grant 227768 of the Research Council of Norway, Project 1309940 of Finnish Academy and INSPIRE Faculty Award Program Ref. No. IFA19-MA134. ŁP was partially supported by the grant no. 2016/23/D/ST1/01149 from the National Science Centre. Part of this work was carried out when KM was visiting Adam Mickiewicz University, Poznań with support from a grant of ŁP and also when he was a Leibniz fellowship at MFO, Oberwolfach. AV was supported by the DST INSPIRE Faculty Award Program and grant no. ECR/2018/001566 from SERB-DST. KM and AV are deeply grateful to Adam Mickiewicz University for the kind hospitality during their visit.

✉ Akshaa Vatwani
akshaa.vatwani@iitgn.ac.in

Kamalakshya Mahatab
accessing.infinity@gmail.com

Łukasz Pańkowski
lpan@amu.edu.pl

1 Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur, West Bengal 721302, India
2 Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Uniwersytetu Poznańskiego 4, 61-614 Poznan, Poland
3 Department of Mathematics, Indian Institute of Technology Gandhinagar, Palaj, Gandhinagar, Gujarat 382355, India
The related problem of finding extreme values of the Riemann zeta-function on a given vertical line lying in the right half of the critical strip was first investigated by Titchmarsh in 1927 [33]. More precisely, he proved that for any fixed \( \sigma \in [1/2, 1) \) and any \( \epsilon > 0 \), we have

\[
\log |\zeta(\sigma + it)| = \Omega((\log t)^{1-\sigma-\epsilon})
\]
as \( t \) tends to infinity. This was the first result of this type which did not rely on assumption of the Riemann Hypothesis.

Interestingly, substantial development of this theory transpired only in the 70’s of the twentieth century due to the work of Levinson [23], Balasubramanian and Ramachandra [8], and Montgomery [25]. It is worth mentioning that by introducing a new technique based on Dirichlet’s theorem on homogeneous Diophantine approximation, Montgomery proved that for any fixed \( \sigma \in (1/2, 1) \), any real number \( \theta \) and every sufficiently large \( T \), there exists \( t \in [T^{(\sigma-1)/3}, T] \) such that

\[
\text{Re } e^{-\iota \theta} \log \zeta(\sigma + it) \geq \frac{1}{20} \left( \sigma - \frac{1}{2} \right)^{1/2} \frac{(\log T)^{1-\sigma}}{(\log \log T)^{\sigma}}. \tag{1}
\]

Moreover, he showed that under the Riemann Hypothesis, the above inequality can be extended to \( \sigma \in [1/2, 1) \) with a slightly better constant and better range of \( t \). Such a result was proved unconditionally in [8] for \( \sigma = 1/2 \) and \( \theta = 0 \).

Another important breakthrough was achieved very recently by Bondarenko and Seip in [11, 12]. Based on the resonance method introduced by Soundararajan in [31] and a connection between extreme values of the Riemann zeta-function and the so-called GCD sums (see [1, 20]), they showed that

\[
\max_{0 \leq t \leq T} |\zeta \left( \frac{1}{2} + it \right) | \geq \exp \left( (1 + o(1)) \sqrt{\frac{\log \log \log T}{\log \log T}} \right).
\]

Later the above result was optimized by de la Bretèche and Tenenbaum [15]. For \( \sigma = 1 \), it was proved by Aistleitner, Munsch and the first author [2] that

\[
\max_{T^{1/2} \leq t \leq T} |\zeta (1 + it)| \geq e^\gamma (\log \log T + \log \log \log T + O(1)).
\]

In view of the aforementioned results it is natural to ask if similar \( \Omega \)-results hold for other \( L \)-functions in number theory. A partial answer to this question was given by Balakrishnan [7], who proved such results for the Dedekind zeta-function. Sankaranarayanan and Sengupta [30] generalized Montgomery’s argument to obtain the analogue of (1) for a class of \( L \)-functions with real Dirichlet coefficients, satisfying certain analytic and arithmetic conditions. Recently, such results were obtained by Aistleitner and the second author [4] for \( L \)-functions belonging to the Selberg class having polynomial Euler product and satisfying Selberg’s orthonormality conjecture. Related results on large values of general \( L \)-functions on the \( \sigma = 1 \) line were also proved by the first author in joint work with Aistleitner, Munsch, Peyrot [3] and with Dixit [16]. In very recent work, Blomer, Fouvry, Kowalski, Michel, Miličević and Sawin [9, Theorem 7.2] have exhibited large central values for a product of two twisted \( L \)-functions lying in a certain family of \( L \)-functions with non-real coefficients.

Let us recall that the Selberg class \( S \) consists of meromorphic functions \( L(s) \) satisfying the following axioms.

(i) **Dirichlet series** \( L(s) \) can be expressed as a Dirichlet series

\[
L(s) = \sum_{n=1}^{\infty} \frac{a_L(n)}{n^s}.
\]
which is absolutely convergent in the region \( \text{Re}(s) > 1 \).

(ii) **Analytic continuation** There exists a non-negative integer \( m \), such that \( (s - 1)^m L(s) \) is an entire function of finite order.

(iii) **Functional equation** \( L(s) \) satisfies the functional equation

\[
\Phi(s) = \theta \Phi(1 - s),
\]

where

\[
\Phi(s) := L(s) Q^s \prod_{j=1}^{k} \Gamma(\lambda_j s + \mu_j),
\]

\( |\theta| = 1, Q \in \mathbb{R}, \lambda_j \geq 0, \) and \( \mu_j \in \mathbb{C} \) with \( \text{Re } \mu_j \geq 0 \).

(iv) **Ramanujan hypothesis** For any \( \epsilon > 0 \),

\[
|a_F(n)| = O(n^\epsilon).
\]

(v) **Euler product** There is an Euler product of the form

\[
L(s) = \prod_{p \text{ prime}} L_p(s)
\]

for \( \text{Re}(s) > 1 \), where

\[
L_p(s) = \exp \left( \sum_{j=1}^{\infty} b_L(p^j) \frac{p^{js}}{p^{s}} \right)
\]

with \( b_L(p^j) \in \mathbb{C}, b_L(p^j) = O(p^{j\delta}) \) for some \( \delta < 1/2 \).

Given \( L \in \mathcal{S} \), one may associate to it an invariant called its degree \( d_L \), which is defined by

\[
d_L = 2 \sum_{j=1}^{k} \lambda_j.
\]

We note that in some cases it is convenient to assume a stronger axiom than (v), namely the so-called polynomial Euler product:

(v') **Polynomial Euler product** For \( \text{Re}(s) > 1 \), we have

\[
L(s) = \prod_{p \text{ prime}} \prod_{j=1}^{r_L} \left( 1 - \frac{\nu_{L,j}(p)}{p^s} \right)^{-1},
\]

where \( \nu_{L,j}(p) \) are complex numbers.

We shall denote by \( \mathcal{S}_{\text{poly}} \) the subclass of \( \mathcal{S} \) consisting of those \( L \)-functions that satisfy the axiom (v'). It is worth emphasizing that in all likelihood this stronger assumption does not exclude any interesting \( L \)-functions in number theory, since all known examples of elements from the Selberg class also satisfy the axiom (v'), at least under some widely believed conjectures. From our point of view, the most important consequence of (v') is the fact that in addition with the Ramanujan hypothesis (iv), it implies that \( |\nu_{L,j}(p)| \leq 1 \) (see [32, Lemma 2.2]). Therefore we have

\[
a_L(p) = b_L(p) = \sum_{j=1}^{r_L} \nu_{L,j}(p) \quad \text{and} \quad |a_L(p)| \leq r_L.
\]

Let us also recall that Selberg’s normality conjecture claims that

\[
\sum_{p \leq x} \frac{|a_L(p)|^2}{p} = \kappa_L \log \log x + O(1).
\]
In addition to this, it is conjectured that for any distinct primitive functions $L_1, L_2 \in S$, we have
\[ \sum_{p \leq x} \frac{a_{L_1}(p)\overline{a_{L_2}(p)}}{p} = O(1). \tag{3} \]

The assertions (2), (3) together comprise Selberg’s orthonormality conjecture.

It is quite remarkable that Montgomery’s approach as well as the resonance method do not seem (see [4]) strong enough to get $\Omega$-results even of the same size as in (1) for general $L$-functions, if the assumption that the Dirichlet coefficients are real is dropped. More precisely, as was shown in [4], the best known result for $L$-functions from $S_{\text{poly}}$ satisfying Selberg’s normality condition is
\[ \max_{\tau \in [T, 2T]} |L(\sigma + it)| \geq \exp \left( (C_L(\sigma) + o(1)) \frac{(\log T)^{1-\sigma}}{\log \log T} \theta(\sigma) \right), \]
where $C_L(\sigma)$ is an explicitly given positive constant, $\sigma \in [1/2, 1)$ is fixed, $T$ is sufficiently large, and $\theta(1/2) = 1/2, \theta(\sigma) = 1$ otherwise.

Let us consider $L_1, \ldots, L_n$ from $S_{\text{poly}}$, pairwise satisfying Selberg’s normality conjecture. The above discussion establishes that on the vertical segment $[\sigma_0 + iT, \sigma_0 + 2iT]$ lying in the strip $1/2 < \sigma_0 < 1$, the $L$-functions $L_i$ individually take extreme values of the size $\exp \left( (C_L(\sigma_0) \frac{(\log T)^{1-\sigma_0}}{\log \log T}) \right)$. The question we shall address in this article is whether $L_1, \ldots, L_n$ can take their extreme values simultaneously or at least close together, on the aforementioned vertical segment.

In order to expect existence of joint extreme values, one is inclined to assume some kind of independence between distinct $L$-functions. Indeed in order to prove such a result, we assume Selberg’s orthonormality conjecture, which seems to be the most natural condition of this type for the Selberg class. An interesting phenomenon that reiterates this was observed by Bombieri and Hejhal in [10], namely they showed the approximate statistical independence of any collection of $L$-functions under a stronger version of Selberg’s orthogonality condition.

Very recently it was proved by Lee, Nakamura and the second author [22] that under some natural assumptions on $L_1, \ldots, L_n \in \mathcal{S}$ and a stronger version of (2) and (3) for every pair $L_i, L_j, i \neq j$, the functions $L_1, \ldots, L_n \in \mathcal{S}$ are jointly universal in the Voronin sense. Roughly speaking, this means that any non-vanishing analytic functions $f_1(s), \ldots, f_n(s)$ can be approximated uniformly by certain shifts $L_1(s + i\tau), \ldots, L_n(s + i\tau)$ (see [22, Theorem 1.2]).

To summarize, we see that $L$-functions $L_1, \ldots, L_n$ from the Selberg class which pairwise satisfy Selberg’s orthogonality condition, are in some sense independent. Nonetheless, note that joint existence of extreme values which are away from the typical, does not follow as an immediate consequence of this independence. Our main result answers this question in the affirmative. More precisely, we show the existence of closely spaced extreme values on a given vertical segment, for any collection of $L$-functions in $S_{\text{poly}}$, satisfying Selberg’s condition. Our proof relies on a modification of Montgomery’s approach in [25], combined with an idea of Good [17], and yields joint omega results for $\log |L_i|$, as well as for $\arg L_i$, $i = 1, \ldots, n$.

**Theorem 1.1** Let $\sigma_0 \in (1/2, 1)$ be fixed and $L_1(s), \ldots, L_k(s)$ be distinct elements of $S_{\text{poly}}$, whose Dirichlet coefficients $a_{L_j}$ $(j = 1, 2, \ldots, k)$ satisfy the following strong version of

\[ \sum_{p \leq x} \frac{a_{L_1}(p)\overline{a_{L_2}(p)}}{p} = O(1). \]
Selberg’s orthonormality condition

$$\sum_{p \leq x} a_{L_i}(p)a_{L_j}(p) = \begin{cases} \kappa_j \text{Li}(x) + O\left(\frac{x}{\log^4 x}\right) & \text{if } i = j, \\ O\left(\frac{x}{\log^4 x}\right) & \text{if } i \neq j, \end{cases}$$

for suitable $\kappa_j > 0$. Here and throughout the paper $A$ denotes an arbitrary large positive constant, not necessarily the same at each appearance.

Moreover, let us assume that there exists $\delta > 0$ such that, for $j = 1, \ldots, k$, we have

$$N_{L_j}(\sigma_0, T) := \#\{\rho = \beta + i\gamma : L_j(\rho) = 0, \beta \geq \sigma_0, \gamma \in [0, T]\} \ll T^{1-\delta}.$$  

(5)

Then for every real $\theta_1, \ldots, \theta_k$ and for sufficiently large $T$, there are $t_1, \ldots, t_k \in [T, 2T]$ such that

$$\text{Re } e^{-i\theta_j} \log L_j(\sigma_0 + it_j) \gg \frac{(\log T)^{1-\alpha_0}}{\log \log T}, \quad j = 1, \ldots, k$$

with $|t_i - t_j| \leq 2(\log T)^{(1+\alpha_0)/2}(\log \log T)^{1/2}$ for all $i, j \in \{1, \ldots, k\}$.

While this result gives us closely spaced extreme values, it is natural to ask for simultaneous extreme values, with all the $t_j$ being equal. The main obstacle towards this in our method stems from Lemma 2.1 which is stated in Sect. 2. This lemma allows us to choose $t_j$ separately for each $L$-function with a precision of the order $\tau$, but does not allow any more control over the choosing of these $t_j$’s.

The assumption (4) implies Selberg’s original orthogonality condition (2) and (3). Nevertheless, it is quite likely that (4) is fulfilled by all $L$-functions. We refer to [22, Section 4] for a detailed discussion of this matter. At this point, we only mention that there is a grand hypothesis that each $L$-function from the Selberg class can be defined as a suitable automorphic $L$-function. So far, all known automorphic $L$-functions satisfying Selberg’s orthonormality conjecture do fulfill (4), thus giving some evidence for the truth of this conjecture.

We make some remarks here about the condition (5). Proving such a zero-density estimate for a generic $L$-function is non-trivial, except when $\sigma_0$ is close to 1. In [21], Kaczorowski and Perelli showed that given $L \in S$ and $\epsilon > 0$, there exists a constant $C > 0$ such that

$$N_L(\sigma, T) \ll_{\epsilon} T^{C(1-\sigma)+\epsilon},$$

as $T \rightarrow \infty$ uniformly for $1/2 \leq \sigma \leq 1$. Their proof allows for $C = 4d_L + 12$, where $d_L$ is the degree of $L$. Assuming an upper bound for the mean-square on the critical line, Mukhopadhyay and Koytyada [26] obtained better zero-density estimates for certain functions in the Selberg class, when $2/3 < \sigma < 1$. Refinements of such results have been proved for specific $L$-functions, for instance by Akbary and Trudgian [5]. Moreover, the zero-density estimate in the form we require in (5) is known for many examples of $L$-functions like the Riemann zeta-function, Dirichlet $L$-functions, Dedekind zeta-functions, etc (see for example Section 4 of [30]).

Let us note that putting $\theta_1 = \cdots = \theta_k = 0$ in Theorem 1.1 yields that in every interval $[T, 2T]$ there is a short interval containing $t_1, \ldots, t_k$ such that $|L_1(\sigma_0 + it_1), \ldots, L_k(\sigma_0 + it_k)|$ are at least of size $\exp\left(c\frac{(\log T)^{1-\alpha_0}}{\log \log T}\right)$ for some positive constant $c$. On the other hand, taking $\theta_1 = \cdots = \theta_k = \pi$ we have that the absolute values of the given $L$-functions can be very small in a close neighborhood, namely at most of size $\exp\left(-c\frac{(\log T)^{1-\alpha_0}}{\log \log T}\right)$. Interestingly, one can put some $\theta_j$’s as 0, while others as $\pi$, to show that the values taken by $L$-functions satisfying (4) are indeed independent, since in a very close neighborhood some of them might
take large values while others take extremely small values. Similarly, one can consider the case when $\theta_j = \pm \frac{\pi}{2}$ to deduce similar results on joint extreme values of arguments of the given $L$-functions.

The proof of Theorem 1.1 is given in Sect. 3. We may extend Theorem 1.1 to the case $\sigma_0 = 1$ to show joint large values of $L_1, \ldots, L_k$ of size $O(\log \log T)$ and to $\sigma_0 = 1/2$ under the generalized Riemann hypothesis to show joint large values of size $\exp\left(\frac{c (\log T)^{1/2}}{\log \log T}\right)$ for some constant $c > 0$. The former involves recalculating Lemma 2.2 for $\sigma_0 = 1$ and choosing $M$ and $\rho$ as in Section 5 of [4]. We note that the assumption of the analogue of the Riemann hypothesis is natural when one would like to follow Montgomery’s approach, as we partially do (see, for example, p. 512 of [23]). More precisely, in order to prove our theorem for $\sigma_0 = 1/2$ we need to assume that there is $\delta > 0$ such that $N_L(\sigma, T) \ll T^{1-\delta}$ for every $\sigma > 1/2$. Since this is not even known for the Riemann zeta-function, it is more natural to assume the analogue of the Riemann Hypothesis for our given $L$-functions, which implies that one may take $\delta = 1$.

It is also possible to further generalize our theorem to a more generic version which includes Dirichlet polynomials having coefficients as random variables [24, 29], and also includes other $L$-functions which are not in the Selberg class. Another possible direction for further research is to investigate when two $L$-functions have a high probability of taking large values in a small neighbourhood by combining our techniques with recent developments from [6, 19, 27]. We relegate these problems to future work.

2 Auxiliary lemmas

The following lemma allows us to estimate a given $L$-function by a suitable Dirichlet polynomial. Henceforth, we let $p$ denote a prime number.

**Lemma 2.1** Let $L(s) \in S_{\text{poly}}$ be defined by the Dirichlet series $\sum_{n \geq 1} a_L(n) n^{-s}$ for $\Re s > 1$. Suppose that $1/2 \leq \sigma_0 \leq 1$, $\tau_0 \geq 15$, $\tau = \tau(\tau_0) = O(\tau_0)$, and $L(s)$ does not vanish on $\{a + ib : a \geq \sigma_0, |b - \tau| \leq 2\tau\}$. Then for any real $\theta$ and $\rho > 2$ we have

$$\Re e^{-i\theta} \log L(s_0 + it) \geq \frac{1}{2} \Re \sum_{|\log(p/\rho)| \leq 1} \frac{a_L(p)}{p^{s_0}} \left(1 - \left|\log \frac{p}{\rho}\right|\right) + O\left(1 + \rho \frac{\tau + \log \tau_0}{\tau^2}\right),$$

for some $t \in [-\tau, \tau]$, as $T \to \infty$.

A version of the above lemma for the Riemann zeta function was used by Montgomery [25] to demonstrate large values. Later, various generalizations of Montgomery’s lemma have appeared in [17], [28] and [14] to exhibit different properties of $L$-functions. Since our formulation of the lemma is slightly different from that in the existing literature, we briefly reprove the lemma below.

**Proof** From [28, Lemma 4.1] with $\alpha = 1/2$, we obtain for any real $\zeta$,

$$\frac{2}{\pi} \int_{-\tau}^{\tau} \log L(s_0 + it) \left(\frac{\sin(t/2)}{t}\right)^2 e^{i\zeta t} dt = \sum_p \sum_{k \geq 1} \sum_{j=1}^{r_k} \frac{v_{L,j}(p)^k}{kp^{s_0}} \max(0, 1 - |\zeta - k \log p|) + O(e^{\frac{1+\log \log T}{\tau^2}}).$$

Springer
Now, we use the last equation for \( x = -\log \rho, 0, \log \rho \), and multiply the resulting equations by \( \frac{1}{2} e^{-i\theta}, 1, \frac{1}{2} e^{i\theta} \), respectively. Adding these formulae up, taking into account the elementary estimates
\[
\int_{-\tau}^{\tau} \left( \frac{\sin t/2}{t} \right)^2 (1 + \cos(\theta + t \log \rho)) \, dt \leq \frac{\pi}{2},
\]
and
\[
\sum_p \sum_{k \geq 2} \sum_{j=1}^{r_k} \frac{v_j(p)^k}{kp^{k\sigma_0}} \max(0, 1 - |\log \rho - k \log p|) \ll \sum_{2 \leq k \leq \log(e \rho)} \sum_{1 \leq p \leq e \rho} \frac{1}{p^{k\sigma_0}} \ll 1,
\]
proves the lemma.

\[\square\]

**Lemma 2.2** Let \( \sigma_0 \in (1/2, 1) \) be fixed and \( L_1(s), \ldots, L_k(s) \) be distinct elements of \( S_{\text{poly}} \), defined by the Dirichlet series \( \sum_{n \geq 1} a_{L_j}(n)n^{-s} \) for \( \text{Re} \, s > 1, j = 1, 2, \ldots, k \), whose coefficients satisfy (4). Let \( \rho > 2 \) and \( \Delta_\rho \) denote the set of tuples \( z = (z_\rho)_{e^{-1} \rho \leq p \leq e \rho} \) with \( z_\rho \in \mathbb{C}, |z_\rho| \leq 1, \) and \( w_p = (1 - |\log(p/\rho)|) \). Moreover, suppose that \( g_j(z), j = 1, 2, \ldots, k \) are functions on \( \Delta_\rho \) defined by
\[
g_j(z) = \sum_{e^{-1} \rho \leq p \leq e \rho} \frac{a_{L_j}(p)w_p}{p^{\sigma_0}} z_p,
\]
and \( \xi_1, \ldots, \xi_k \) are arbitrary complex numbers satisfying
\[
|\xi_j| \leq c_0 \frac{\min j k_j}{\max j r_{L_j}} \rho^{1-\sigma_0} / \log \rho, \quad (j = 1, \ldots, k),
\]
where \( c_0 \) is an arbitrary fixed constant such that \( c_0 < C_{\sigma_0} := \left( \frac{2 \sinh((1-\sigma_0)/2)}{1-\sigma_0} \right)^2 \). Then the system of equations
\[
g_j(z) = \xi_j, \quad (j = 1, \ldots, k) \tag{6}
\]
has a solution \( z \) in \( \Delta_\rho \) for all sufficiently large \( \rho \).

**Proof** We will first show that the system (6) has a solution in \( \Delta_\rho \), if and only if for arbitrary \( l_j \in \mathbb{C} \), there exists \( z \in \Delta_\rho \) such that
\[
\sum_{j=1}^{k} l_j g_j(z) = \sum_{j=1}^{k} l_j \xi_j. \tag{7}
\]
Trivially, every solution of the system (6) is a solution of (7). In order to prove the other direction, we apply the Hahn–Banach separation theorem to the singleton set \( \mathcal{A} = \{ (\xi_1, \ldots, \xi_k) \} \subseteq \mathbb{C}^k \) and the set \( \mathcal{B} = \{ (g_1(z), \ldots, g_k(z)) | z \in \Delta_\rho \} \), each of which is convex in \( \mathbb{C}^k \). If the sets \( \mathcal{A}, \mathcal{B} \) are disjoint, then there exists a continuous linear map \( T : \mathbb{C}^k \to \mathbb{C} \) and \( a, b \in \mathbb{R} \) such that
\[
\text{Re} \left( T(g_1(z), \ldots, g_k(z)) \right) < a < b < \text{Re} \left( T(\xi_1, \ldots, \xi_k) \right), \quad \text{for all } z \in \Delta_\rho.
\]
The existence of a solution of (7) for arbitrary complex numbers \( l_j \) implies that there cannot exist such a map \( T \). Thus, the sets \( A \) and \( B \) cannot be disjoint, giving us a solution to the system (6).

Thus, it suffices to show that (7) has a solution in \( \Delta_p \) for arbitrary \( l_j \in \mathbb{C} \). Using the definition of \( g_j \), it follows that every complex number of modulus

\[
\leq \sum_{e^{-\rho} \leq p \leq e^\rho} w_p p^{-\sigma_0} \left| \sum_{j=1}^{k} l_j a_{L_j}(p) \right|
\]

can be represented by the left hand side of (7) as \( z \) runs over all the tuples in \( \Delta_p \). Since

\[
\left| \sum_{j=1}^{k} l_j \xi_j \right| \leq \sum_{j=1}^{k} |l_j| \| \xi_j \|,
\]

it is enough to show that

\[
\sum_{j=1}^{k} |l_j| \| \xi_j \| \leq \sum_{e^{-\rho} \leq p \leq e^\rho} w_p p^{-\sigma_0} \left| \sum_{j=1}^{k} l_j a_{L_j}(p) \right|.
\]

We have

\[
\sum_{e^{-\rho} \leq p \leq e^\rho} w_p p^{-\sigma_0} \left| \sum_{j=1}^{k} l_j a_{L_j}(p) \right| \leq \sum_{j=1}^{k} |l_j| \sum_{e^{-\rho} \leq p \leq e^\rho} w_p p^{-\sigma_0} \left| a_{L_j}(p) \right|^{2}.
\]

By partial summation, one can easily show that (4) implies that

\[
\sum_{p \leq x} |a_{L_j}(p)|^2 \log p = \kappa_j x + O\left(\frac{x \log \log A}{x}\right).
\]

Therefore, we have

\[
\sum_{a \rho \leq p \leq b \rho} \frac{|a_{L_j}(p)|^2}{p^{\sigma_0}} = \kappa_j \frac{\rho^{1-\sigma_0}}{(1-\sigma_0) \log \rho} (b^{1-\sigma_0} - a^{1-\sigma_0})
\]

\[
+ \kappa_j \frac{\rho^{1-\sigma_0}}{(1-\sigma_0) \log^2 \rho} (b^{1-\sigma_0}((1-\sigma_0)^{-1} - \log b)
\]

\[
- a^{1-\sigma_0}((1-\sigma_0)^{-1} - \log a))
\]

\[
+ O\left(\frac{\rho^{1-\sigma_0}}{\log^3 A \rho}\right).
\]
Joint extreme values of... 1185

\[ \sum_{a \rho \leq p \leq b \rho} \frac{|a L_j(p)|^2 \log p}{p^{\sigma_0}} = \kappa_j \rho^{1-\sigma_0} \left( b^{1-\sigma_0} - a^{1-\sigma_0} \right) + O\left( \frac{\rho^{1-\sigma_0}}{\log^{A} \rho} \right), \]

where \( a < b \) are arbitrary given positive constants. Now, by splitting the sum \( \sum_{e^{-1} \rho \leq p \leq e \rho} \) into \( \sum_{e^{-1} \rho \leq p \leq \rho} \) and \( \sum_{\rho \leq p \leq e \rho} \), and applying the above estimates, one can easily get after short calculations, that

\[ \sum_{e^{-1} \rho \leq p \leq \rho} \frac{|a L_j(p)|^2 w_p}{p^{\sigma_0}} = C_{\sigma_0} \kappa_j \rho^{1-\sigma_0} + O\left( \frac{\rho^{1-\sigma_0}}{\log^{A} \rho} \right). \] (11)

Similarly, by (4) we get for \( i \neq j \),

\[ \sum_{e^{-1} \rho \leq p \leq e \rho} \frac{a L_i(p) a L_j(p)}{p^{\sigma_0}} w_p = O\left( \frac{\rho^{1-\sigma_0}}{\log^{A} \rho} \right). \]

so the latter sum on the right hand side of (10) is

\[ O\left( \frac{\rho^{1-\sigma_0}}{\log^{A} \rho} \right) \sum_{i \neq j} l_i l_j = O\left( \frac{\rho^{1-\sigma_0}}{\log^{A} \rho} \right) \sum_{j=1}^{k} |l_j|^2. \]

Hence we see that for every \( \varepsilon > 0 \) and sufficiently large \( \rho \) we have

\[ \sum_{e^{-1} \rho \leq p \leq e \rho} w_p p^{-\sigma} \left| \sum_{j=1}^{k} l_j a L_j(p) \right| \geq \left( C_{\sigma_0} - \varepsilon \right) \min_{j} \kappa_j \rho^{1-\sigma_0} \sum_{j=1}^{k} |l_j|^2 \]

\[ \geq \frac{C_{\sigma_0} - \varepsilon}{k} \min_{j} \kappa_j \rho^{1-\sigma_0} \left( \sum_{j=1}^{k} |l_j| \right)^2. \]

This proves (8) as required, since \( |\xi_j| \leq c_0 \frac{\min_{j} \kappa_j \rho^{1-\sigma}}{\max_{j} r_{L_j} \log \rho} \) for all \( j \) and \( c_0 < C_{\sigma_0} \). \( \Box \)

We will use the following lemma due to Chen. Henceforth, \( \| \cdot \| \) will denote the distance to the nearest integer.

**Lemma 2.3** (Chen [13]) Let \( \lambda_1, \ldots, \lambda_n \) and \( \alpha_1, \ldots, \alpha_n \) be real numbers and assume the following property: for all integers \( u_1, \ldots, u_n \) with \( |u_j| \leq M \), the assertion

\[ u_1 \lambda_1 + \cdots + u_n \lambda_n = 0 \]

implies that \( u_1 \alpha_1 + \cdots + u_n \alpha_n \) is an integer. Then for all positive real numbers \( \delta_1, \ldots, \delta_n \) and for \( T_1 < T_2 \), we have

\[ \inf_{t \in [T_1, T_2]} \sum_{j=1}^{n} \delta_j \| \lambda_j t - \alpha_j \|^2 \leq \frac{\Delta}{4} \sin^2 \left( \frac{\pi}{2(M + 1)} \right) + \frac{\Delta M^n}{4\pi(T_2 - T_1)\Lambda}, \]

where

\[ \Delta = \sum_{j=1}^{n} \delta_j \]
and
\[ \Lambda = \min \left\{ |u_1 \lambda_1 + \cdots + u_n \lambda_n| : u_j \in \mathbb{Z}, |u_j| \leq M, \sum_{j=1}^{n} \lambda_j u_j \neq 0 \right\}. \]

**Lemma 2.4** Let \( \sigma_0 \in (1/2, 1) \) be fixed and \( L_j(s) \in S_{\text{poly}}, j = 1, 2, \ldots, k \) be defined by the Dirichlet series \( \sum_{n \geq 1} a_{L_j}(n)n^{-s} \) for \( \Re s > 1 \), whose coefficients satisfy the first equality of (4). Moreover, let \( \theta_p, e^{-1} \rho < p \leq \epsilon \rho \), be any real numbers and \( w_p = 1 - |\log(p/\rho)| \). Then for every positive constant \( c' \) and every interval \( T^{(l)} = [T + (l - 1)T^\mu, T + lT^\mu] \) with \( 0 < \mu < 1, 1 \leq l \leq T^{1-\mu} \) there is some \( t_0 \in T^{(l)} \) such that
\[ \max_{1 \leq j \leq k} \left| \sum_{e^{-1} \rho \leq p \leq \epsilon \rho} \frac{a_{L_j}(p)}{p^{\sigma_0 + i t_0}} w_p - \sum_{e^{-1} \rho \leq p \leq \epsilon \rho} \frac{a_{L_j}(p)e^{-2\pi i \theta_p}}{p^{\sigma_0}} w_p \right| \leq c' \rho^{1-\sigma_0} \log \rho, \quad (T \to \infty), \]
where \( \rho = \frac{1}{2M_\epsilon} \log T, c' \) is a constant satisfying \( c' \mu > 2 \sinh(1) \), and \( M \) is a sufficiently large constant that depends on \( c' \).

**Proof** Using the Cauchy–Schwartz inequality and the fact that \( w_p^2 \leq w_p \), we have
\[ \left| \sum_{e^{-1} \rho \leq p \leq \epsilon \rho} \frac{a_{L_j}(p)w_p e^{-2\pi i \theta_p}}{p^{\sigma_0}} - \sum_{e^{-1} \rho \leq p \leq \epsilon \rho} \frac{a_{L_j}(p)w_p}{p^{\sigma_0} + i t_0} \right| \leq \left| \sum_{e^{-1} \rho \leq p \leq \epsilon \rho} \frac{a_{L_j}(p)w_p}{p^{\sigma_0} + i t_0} \right| \left| e^{-2\pi i \left( \theta_p - \frac{t_0 \log p}{2\pi} \right)} - 1 \right| \leq \left( \sum_{e^{-1} \rho \leq p \leq \epsilon \rho} \left| \frac{a_{L_j}(p)w_p}{p^{\sigma_0}} \right|^2 \right)^{1/2} \left( \sum_{e^{-1} \rho \leq p \leq \epsilon \rho} \left| \frac{1}{p^{\sigma_0}} \left| \theta_p - \frac{t_0 \log p}{2\pi} \right| \right|^2 \right)^{1/2} \]
\[ \leq \left( \sum_{e^{-1} \rho \leq p \leq \epsilon \rho} \frac{|a_{L_j}(p)|^2 w_p}{p^{\sigma_0}} \right)^{1/2} \left( \sum_{e^{-1} \rho \leq p \leq \epsilon \rho} \frac{1}{p^{\sigma_0}} \left| \theta_p - \frac{t_0 \log p}{2\pi} \right|^2 \right)^{1/2} \]
(12)
For the first sum above, we have from (11) that
\[ \sum_{e^{-1} \rho \leq p \leq \epsilon \rho} \frac{|a_{L_j}(p)|^2 w_p}{p^{\sigma_0}} \ll \rho^{1-\sigma_0} \log \rho \]
(13)
For the latter sum on the right hand side of (12), we shall apply Lemma 2.3 with \( T_1 \) and \( T_2 \) being the endpoints of the interval \( T^{(l)} \) and
\[ \lambda_j = \frac{\log p_j}{2\pi}, \quad \alpha_j = \theta_{p_j}, \quad \text{and} \quad \delta_j = \frac{1}{p_j^{\sigma_0}}, \]
where \( p_1, \ldots, p_n \) run over all primes lying in the interval \( \left[ e^{-1} \rho, \epsilon \rho \right] \). Note that the hypothesis of Lemma 2.3 is satisfied due to linear independence of the logarithms of primes. In particular if \( u_1, \ldots, u_n \) are integers with \( |u_j| \leq M \), then \( \sum_{j=1}^{n} u_j \lambda_j = 0 \) implies that each \( u_j \) must be zero, irrespective of the value of \( M \). This seems to indicate that we do not need to utilize the full strength of Chen’s lemma here.
In order to estimate $\Lambda$, note that $|\sum_{\epsilon^{-1} \rho \leq p_j \leq \epsilon \rho} u_j \log p_j|$ is minimum when the $u_j$’s are chosen to be integers such that $|\sum_{\epsilon^{-1} \rho \leq p_j \leq \epsilon \rho u_j} u_j \log p_j|$ is close to $\left|\sum_{\epsilon^{-1} \rho \leq p_j \leq \epsilon \rho} u_j \log p_j \right|$. Without loss of generality we may assume that $\prod_{\epsilon^{-1} \rho \leq p_j \leq \epsilon \rho} P^{-u_j}_{p_j} < \prod_{\epsilon^{-1} \rho \leq p_j \leq \epsilon \rho} P^{u_j}_{p_j}$. Therefore

$$\left|\sum_{\epsilon^{-1} \rho \leq p_j \leq \epsilon \rho} u_j \log p_j \right| = \log \left( \frac{\prod_{\epsilon^{-1} \rho \leq p_j \leq \epsilon \rho} P^{-u_j}_{p_j}}{\prod_{\epsilon^{-1} \rho \leq p_j \leq \epsilon \rho \geq 0} P^{u_j}_{p_j}} \right) \gg \log \left( 1 + \frac{1}{\prod_{\epsilon^{-1} \rho \leq p_j \leq \epsilon \rho} P^{u_j}_{p_j}} \right).$$

Since

$$\prod_{\epsilon^{-1} \rho \leq p_j \leq \epsilon \rho \geq 0} P^{u_j}_{p_j} \ll \exp \left( M \sum_{\epsilon^{-1} \rho \leq p_j \leq \epsilon \rho} \log p \right) \leq \exp \left( M(2 \sinh(1) + o(1))\rho \right),$$

we have the lower bound

$$\Lambda \gg \exp(-M(2 \sinh(1) + o(1))\rho).$$

By Lemma 2.3, we have

$$\inf_{t \in T^{(1)}} \sum_{\epsilon^{-1} \rho \leq p \leq \epsilon \rho} \frac{1}{p^{\sigma_0}} \left\| \theta_p - \frac{t \log p}{2\pi} \right\|^2 \ll \Delta \left( \frac{1}{(M + 1)^2} + \frac{\exp(M(2 \sinh(1) + o(1))\rho)}{T^{\mu}} \right).$$

(14)

where $\Delta = \sum_{\epsilon^{-1} \rho \leq p \leq \epsilon \rho} p^{-\sigma_0}$. Since $\rho = \log T/(2Mc)$ and $c\mu > 2 \sinh(1)$, we see that the term in parenthesis on the right hand side of (14) is

$$\ll \frac{1}{(M + 1)^2} + \exp(2M\rho(2 \sinh(1) + o(1) - c\mu))$$

$$= \frac{1}{(M + 1)^2} + o(1).$$

Since $\Delta \ll \frac{\rho^{1-\sigma_0}}{\log \rho}$, we have obtained that there is $t_0 \in T^{(1)}$ such that the expression in (12) is

$$\ll \frac{1}{M + 1} \left( \sum_{\epsilon^{-1} \rho \leq p \leq \epsilon \rho} \frac{|a_{L_j}(p)|^2 w_p}{p^{\sigma_0}} \right)^{1/2} \left( \frac{\rho^{1-\sigma_0}}{\log \rho} \right)^{1/2}. $$

(15)

Recalling the inequality (12), the bound (13) and taking $M$ sufficiently large (depending on $c'$), the proof is complete. \qed
3 Proof of Theorem 1.1

First, let us choose $\mu < \delta$ and define the intervals $T^{(l)} = [T + (l - 1)T^\mu, T + lT^\mu]$, $1 \leq l \leq T^{1-\mu}$. By (5), we see that for sufficiently large $T$, there exists an index $l_0$, with $2 \leq l_0 \leq T^{1-\mu} - 1$, such that none of the $L_j(s)$’s vanish on

$$\{a + ib : a \geq \sigma_0, b \in T^{(l_0-1)} \cup T^{(l_0)} \cup T^{(l_0+1)}\}.$$  

Put $\tau = (\log T)^{(1+\sigma_0)/2}(\log \log T)^{1/2}$ and $\rho = (\log T)/(2Mc)$, where $c$ is the positive constant defined in Lemma 2.4 and $M$ will be chosen later. Then using Lemma 2.1 gives for any real numbers $\theta_1, \ldots, \theta_k$,

$$\Re e^{-i\theta_j} \log L_j(s_0 + i t_j') \geq \frac{1}{2} \Re \sum_{|\log(p/\rho)| \leq 1} \frac{aL_j(p)}{p^{\sigma_0}} w_p + O \left( \frac{(\log T)^{1-\sigma_0}}{M \log \log T} \right), \tag{16}$$

for arbitrary given $t_0 \in T^{(l_0)}$ and some $t_1', \ldots, t_k' \in [-\tau, \tau]$, where $w_p = 1 - \left| \log \frac{p}{\rho} \right|$.

Let $\theta_p, \rho e^{-1} < p \leq \rho e$, be real numbers to be chosen suitably later. By Lemma 2.4, we obtain that for some $t_0 \in T^{(l_0)}$, the right hand side of (16) equals

$$\frac{1}{2} \Re \sum_{|\log(p/\rho)| \leq 1} \frac{aL_j(p) w_p}{p^{\sigma_0}} e^{-2\pi i \theta_p} + E(T), \tag{17}$$

for some $t_0 \in T^{(l_0)}$, where $E(T) \leq c' \left( \frac{(\log T)^{1-\sigma_0}}{\log \log T} \right)$ for arbitrary given positive constant $c'$.

To complete the proof, we will show that the factors $e^{-2\pi i \theta_p}$ can be replaced by arbitrary complex numbers of modulus $\leq 1$, following which we can apply Lemma 2.2. In order to see this, we shall use the following lemma due to Good (see Lemma 6 of [16]).

**Lemma 3.1** Let $k, m$ be positive integers with $k \leq m$. Let $a_{j\ell}$ and $b_j, 1 \leq j \leq k, 1 \leq \ell \leq m$ denote complex numbers. Suppose that the system of equations

$$\sum_{\ell=1}^{m} a_{j\ell} z_{\ell} = b_j, \quad 1 \leq j \leq k, \tag{18}$$

has a solution $(z_1, \ldots, z_m) \in \mathbb{C}^m$ with $|z_{\ell}| \leq 1$ for $1 \leq \ell \leq m$. Then the system (18) also has a solution $(z_1', \ldots, z_m') \in \mathbb{C}^m$ with $|z_{\ell}'| \leq 1$ for $1 \leq \ell \leq m$, such that $|z_{\ell}'| = 1$ holds for at least $m - k$ positive integers $\ell \leq m$.

One can easily show that applying the above lemma to the system

$$\sum_{\rho e^{-1} < p \leq \rho \epsilon} \frac{aL_j(p) w_p}{p^{\sigma_0}} z_p = b_j, \quad 1 \leq j \leq k$$

implies that if this system has a solution in complex numbers $z_p, \rho e^{-1} < p \leq \rho \epsilon$, with $|z_p| \leq 1$, then one can also find a solution $z'_p, \rho e^{-1} < p \leq \rho \epsilon$ and $|z'_p| \leq 1$, of this system for which the strict inequality $|z'_p| < 1$ holds for at most $k$ primes $p$. This allows us to conclude that for any given complex numbers $z_p, \rho e^{-1} \leq p \leq \rho \epsilon$, with $|z_p| \leq 1$, there are $z'_p = e^{-2\pi i \theta_p}, \rho e^{-1} \leq p \leq \rho \epsilon$, with suitable real $\theta_p$ such that

$$\sum_{\rho e^{-1} < p \leq \rho \epsilon} \frac{aL_j(p) w_p}{p^{\sigma_0}} z_p = \sum_{\rho e^{-1} < p \leq \rho \epsilon} \frac{aL_j(p) w_p}{p^{\sigma_0}} e^{-2\pi i \theta_p} + O \left( \rho^{-\sigma_0} \right), \tag{19}$$

\textcopyright Springer
Joint extreme values of...

Thus, we have

\[ \text{Re} e^{-i\theta} \log L_j(\sigma_0 + it + i t_j') \gg \sum_{|\log(p/\rho)| \leq 1} \frac{a L_j(p) w_p}{p^{\sigma_0}} z_p + E(T) + O \left( \frac{(\log T)^{1-\sigma_0}}{M \log \log T} \right). \]

So, applying Lemma 2.2 with \( \xi_1, \ldots, \xi_k \) being positive real numbers of size \( \rho_1^{1-\sigma_0}/\log \rho \) and taking sufficiently large \( M \) (notice that \( c' \) tends to 0 as \( M \) tends to \( \infty \)) completes the proof, since \( \rho_1^{1-\sigma_0}/\log \rho \approx \frac{(\log T)^{1-\sigma_0}}{M^{1-\sigma_0} \log \log T} \).

Acknowledgements The authors are thankful to the anonymous referee for insightful comments and suggestions which improved the quality of the paper.

References

1. Aistleitner, C.: Lower bounds for the maximum of the Riemann zeta function along vertical lines. Math. Ann. 365(1-2), 473–496 (2016)
2. Aistleitner, C., Mahatab, K., Munsch, M.: Extreme values of the Riemann zeta function on the 1-line. Int. Math. Res. Not. IMRN 22, 6924–6932 (2019)
3. Aistleitner, C., Mahatab, K., Munsch, M., Peyrot, A.: On large values of \( L(\sigma, \chi) \). Q. J. Math. 70(3), 831–848 (2019)
4. Aistleitner, C., Pańkowski, Ł: Large values of \( L \)-functions from the Selberg class. J. Math. Anal. Appl. 446(1), 345–364 (2017)
5. Akbary, A., Trudgian, T.S.: A log-free zero-density estimate and small gaps in coefficients of \( L \)-functions Int. Math. Res. Not. IMRN 12, 4242–4268 (2015)
6. Arguin, L.P., Belius, D., Bourgade, P., Radziwiłł, M., Soundararajan, K.: Maximum of the Riemann zeta function on a short interval of the critical line. Commun. Pure Appl. Math. 72(3), 500–535 (2019)
7. Balakrishnan, U.: Extreme values of the Dedekind zeta function. Acta Arith. 46, 199–209 (1986)
8. Balasubramanian, R., Ramachandra, K.: On the frequency of Titchmarsh’s phenomenon for \( \zeta(s) \)-III. Proc. Indian Acad. Sci. 86, 341–351 (1977)
9. Blomer, V., Fouvry, É., Michel, P., Miličević, D., Sawin, W.: The second moment theory of families of \( L \)-functions. arXiv:1804.01450
10. Bombieri, E., Hejhal, D.A.: On the distribution of zeros of linear combinations of Euler products. Duke Math. J. 80, 821–862 (1995)
11. Bondarenko, A., Seip, K.: Large greatest common divisor sums and extreme values of the Riemann zeta function. Duke Math. J. 166(9), 1685–1701 (2017)
12. Bondarenko, A., Seip, K.: Extreme values of the Riemann zeta function and its argument. Math. Ann. 372(3-4), 999–1015 (2018)
13. Chen, Y.-G.: The best quantitative Kronecker’s theorem (2). J. Lond. Math. Soc. 61(3), 691–705 (2000)
14. Chirre, A., Mahatab, K.: Large oscillations of the argument of the Riemann zeta function-member. Bull. Lond. Math. Soc. 53(6), 1776–1785 (2021)
15. de la Bretèche, R., Tenenbaum, G.: Sommes de Gál et applications (3). Proc. Lond. Math. Soc. 119(1), 104–134 (2019)
16. Dixit, A.B., Mahatab, K.: Large values of \( L \)-functions on the 1-line Bull. Aust. Math. Soc. 103(2), 230–243 (2021)
17. Good, A.: On the distribution of the values of Riemann’s zeta function. Acta Arith. 38(4), 347–388 (1980/1981)
18. Granville, A., Soundararajan, K.: Extreme values of \( |\zeta(1 + it)| \). In The Riemann zeta function and related themes: papers in honour of Professor K. Ramachandra, Ramanujan Math. Soc. Lect. Notes Ser., vol. 2, pp. 65–80. Ramanujan Math. Soc., Myros (2006)
19. Harper, A.: The Riemann zeta function in short intervals [after Najnudel, and Arguin, Belius, Bourgade, Radziwiłł, and Soundararajan]. Séminaire N. BOURBAKI, 71e année, no. 1159 (2018–2019)
20. Hilberdink, T.: An arithmetical mapping and applications to \( \Omega \)-results for the Riemann zeta function. Acta Arith. 139(4), 341–367 (2009)
21. Kaczorowski, J., Perelli, A.: On the prime number theorem for the Selberg class. Arch. Math. (Basel) 80(3), 255–263 (2003)
22. Lee, Y., Nakamura, T., Pańkowski, Ł: Selberg’s orthonormality conjecture and joint universality of $L$-functions. Math. Z. 286, 1–18 (2017)
23. Levinson, N.: $\Omega$-theorems for the Riemann zeta-function. Acta Arith. 20, 317–330 (1972)
24. Mahatab, K.: 6-th norm of a steinhaus chaos. Int. J. Number Theory. arXiv: 1710.08201 (to appear)
25. Montgomery, H.L.: Extreme values of the Riemann zeta function. Comment. Math. Helv. 52(4), 511–518 (1977)
26. Mukhopadhyay, A., Kotyada, S.: A zero density estimate for the Selberg class. Int. J. Number Theory 3(2), 263–273 (2007)
27. Najnudel, J.: On the extreme values of the Riemann zeta function on random intervals of the critical line. Probab. Theory Relat. Fields 172(1–2), 387–452 (2018)
28. Pańkowski, Ł, Steuding, J.: Extreme values of $L$-functions from the Selberg class. Int. J. Number Theory 9(5), 1113–1124 (2013)
29. Saksman, E., Seip, K.: Some open questions in analysis for Dirichlet series. Recent progress on operator theory and approximation in spaces of analytic functions. Contemp. Math. 679, 179–191 (2016)
30. Sankaranarayanan, A., Sengupta, J.: Omega theorems for a class of $L$-functions. Funct. Approx. 36, 119–131 (2006)
31. Soundararajan, K.: Extreme values of zeta and $L$-functions. Math. Ann. 342(2), 467–486 (2008)
32. Steuding, J.: Value-Distribution of $L$-functions. Springer, Berlin (2007)
33. Titchmarsh, E.C.: On an inequality satisfied by the zeta-function of Riemann. Proc. Lond. Math. Soc. 2(28), 70–80 (1928)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.