Nonuniversal power law spectra in turbulent systems

Vasil Bratanov, Frank Jenko, and David Hatch
Max-Planck-Institut für Plasmaphysik, EURATOM Association, 85748 Garching, Germany

Michael Wilczek
Department of Mechanical Engineering, The Johns Hopkins University, 3400 North Charles Street, Baltimore MD 21218, USA

Turbulence is generally associated with universal power law spectra in scale ranges without significant drive or damping. Although many examples of turbulent systems do not exhibit such an inertial range, power law spectra may still be observed. As a simple model for such situations, a modified version of the Kuramoto-Sivashinsky equation is studied. By means of semi-analytical and numerical studies, one finds power laws with nonuniversal exponents in the spectral range for which the ratio of nonlinear and linear time scales is (roughly) scale-independent.

Introduction. Turbulence can generally be described as spatio-temporal chaos in open systems, brought about by the nonlinear interaction of many degrees of freedom under out-of-equilibrium conditions. As such, it is ubiquitous in nature and in the laboratory, and represents a fundamental challenge to theoretical physics. Power law energy spectra constitute one of the most prominent features of such systems. A first prediction along those lines was provided for three-dimensional Navier-Stokes turbulence as early as 1941 by Kolmogorov.\[1\] The typical physical picture is that power laws emerge on scales where both energy injection and dissipation are negligible, i.e., in the so-called inertial range. Here, on the basis of dimensional analysis, the value of the spectral exponent is considered to be determined entirely by the nonlinear energy transfer rate, implying universality.

Interestingly, there exist numerous examples of turbulent systems which display (simple or broken) power laws even in the presence of multiscale drive and/or damping. These include, e.g., flows generated by space-filling fractal square grids [2], the mesoscale dynamics in dense bacterial suspensions [3], and turbulence in astrophysical [4] and laboratory [5] plasmas. At least in the latter case, numerical studies suggest that the observed power law exponents are not universal, however.\[2\] Instead, they appear to depend on the underlying linear physics of the system. This finding clearly calls for a theoretical understanding that can also help to interpret and guide experimental as well as numerical investigations.

In a previous investigation [6], a simple model for density fluctuation spectra in magnetized laboratory plasmas was proposed which is based on the notion of disparate-scale interactions between small-scale eddies and large-scale structures like mean or zonal flows, also taking into account effective linear drive and/or (eddy/Landau) damping. In this context, universal broken power laws with an exponential cutoff were predicted. In the present Letter, we consider an alternative scenario. It is shown that one may obtain nonuniversal power laws in a certain spectral range if the ratio of the relevant nonlinear and linear time scales is (roughly) scale-independent there.

Modified Kuramoto-Sivashinsky model. To enable a semi-analytical treatment, we will employ a modified version of one of the simplest models for spatio-temporal chaos and turbulence, the Kuramoto-Sivashinsky equation (KSE), which was originally put forward to describe turbulence in magnetized plasmas [7, 8], chemical reaction-diffusion processes [9], and flame front propagation [10]. In general, it can be used for the study of nonlinear, spatially extended systems driven far from thermodynamic equilibrium by long-wavelength instabilities in the presence of appropriate (translational, parity, and Galilean) symmetries, and subject to short-wavelength damping. In its one-dimensional form, it reads

\[ u_t = -uu_x - \mu u_{xx} - \nu u_{xxxx} \] (1)

for the velocity field \( u(x,t) \) with the positive parameters \( \mu \) and \( \nu \). The equation is supplemented by the periodic boundary condition \( u(L,t) = u(0,t) \) for all \( t \geq 0 \) and the initial condition \( u(x,0) = u_0(x) \). Considering only functions that belong to \( C^2(\Omega) \cap L^2(\Omega) \) ensures that the system has finite total kinetic energy. Eq. 1 can be rewritten in dimensionless units by substituting \( u \rightarrow \mu u/L, \ t \rightarrow tL^2/\mu, \ x \rightarrow Lx, \) and \( \nu \rightarrow L^2\mu \nu \). The non-dimensionalized form of the equation is the same as before, where simply \( \mu = 1 \). In this following, we keep the damping parameter \( \nu \) undetermined, but all quantitative results are obtained with \( \nu = 1 \). The second- and fourth-order spatial derivatives on the right-hand side of Eq. 1 provide an energy source and sink, respectively. Similar to three-dimensional Navier-Stokes turbulence, energy is injected on large scales and dissipated on small scales, with the nonlinear term providing the inter-scale transfer.

The periodic boundary conditions suggest a representation of \( u(x,t) \) in terms of a Fourier series defined as

\[ u(x,t) = \sum_{n \in \mathbb{Z}} \hat{u}(k_n,t) e^{i k_n x}, \] (2)
where the wave numbers \( k_n = n(2\pi/L) \) are discrete and \( n \in \mathbb{Z} \). From the condition that \( u(x,t) \) is real, it follows that \( \bar{u}(k_n,t) = \bar{u}(-k_n,t) \) where the overbar denotes complex conjugation. Expressing Eq. (1) in terms of Fourier coefficients gives

\[
\tilde{u}_t(k_n) = -\frac{1}{2} ik_n \sum_{m \in \mathbb{Z}} \tilde{u}(k_n - k_m)\bar{u}(k_m) + (k_n^2 - \nu k_n^4)\tilde{u}(k_n),
\]

(3)

where we have suppressed the time dependence for the ease of notation. Linearly, each mode is characterized by the drive/damping rate \( \gamma = k_n^2 - \nu k_n^4 \). The nonlinear term does not inject or dissipate energy (i.e., summed over \( n \), it gives zero, but only redistributes it among the modes.

We now modify the linear term by the replacement

\[
k_n^2 - \nu k_n^4 \rightarrow (k_n^2 - \nu k_n^4)/(1 + bk_n^4),
\]

(4)

where \( b > 0 \), thus providing a constant damping rate of \( \nu/b \) in the high wavenumber limit. One motivation for such a modification comes from the (gyro-)kinetic theory of magnetized plasmas where the growth rates of linear instabilities tend to a negative constant for large perpendicular wavenumbers. Moreover, this is one of the simplest realizations of a controlled deviation from the classical inertial range. Note that the real-space representation of the modified linear term is well defined for all functions in the domain \( C^4(\Omega) \cap L^2(\Omega) \).

Energy transfer physics. To gain insight into the turbulent dynamics of Eq. (3), it is solved numerically, employing the Exponential Time Differencing fourth-order Runge-Kutta (ETDRK4) algorithm [12, 13] and changing the normalized system size to \( 32\pi \). We focus our investigations on the physics of the net nonlinear energy transfer. As it will turn out, the latter is dominated by nonlocal interactions in wavenumber space. Two neighboring high \( k \) modes exchange energy via the coupling to a third mode with \( k \sim 1 \). This can be quantified by introducing the scale disparity parameter \( S(k,p) = \max\{|k|,|p|,|k-p|\}/\min\{|k|,|p|,|k-p|\} \) defined in Refs. [14, 15]. We shall follow the literature and refer to interactions with small (large) values of \( S \) as local (nonlocal). In Ref. [15], the observation was made that in Burgers turbulence, the net energy transfer in the inertial and dissipation ranges is dominated by local interactions, similar to Navier-Stokes turbulence. Our numerical simulations show that this type of behavior carries over to the original KSE. To our knowledge, this has not been shown before. The modified KSE exhibits a completely different scenario, however. The function \( T(k_n,S) \), characterizing the energy transfer into mode \( k_n \) via triads with the scale disparity parameter \( S \) and defined over logarithmic \( S \)-bands like in Ref. [15], is displayed in Fig. 1 as a function of \( S/k_n \) for three different values of \( k_n \). In all three cases, one finds a strong peak at \( S/k_n \sim 1 \), implying that for Eq. (3), the net energy transfer at large wavenumbers is dominated by nonlocal interactions, with a \( k \sim 1 \) mode acting as kind of a catalyst. Nevertheless, the energy cascade itself is local. The relevant triadic interactions can be realized in two different ways: \( k_n \approx k_n \) and \( k_n-k_m \) small or \( k_n \) small and \( k_n-k_m \sim k_n \). Defining for convenience \( k_q = k_m - k_n \), the nonlinearity becomes \( k_n \sum_{q \in \mathbb{Z}} P(k_n,k_q) \) where the summand represents the triple correlation \( \Im(\bar{u}(k_n,t)\bar{u}(k_q,t)\bar{u}(k_n+k_q,t)) \).

![FIG. 1: Net energy transfer into mode \( k_n \) via triads with the scale disparity parameter \( S \) as a function of \( S \) normalized to \( k_n \).](color online)
considering the numerical results mentioned before, we have
the following picture of the energy transfer in Fourier
space. A large mode \( k_n \) receives energy (on average)
mainly from the mode \( k_n - k_d \), where \( k_d \) is a relatively
small wave number in the drive range that mediates the
transfer. Part of this energy is dissipated and the rest is
forwarded primarily to the mode \( k_n + k_d \) again via \( k_d \).
The first term in Eq. (7) has to balance the energy dis-
sipated by the \( k_n \) mode which is the difference between
the energy received by \( k_n \) and the one given by \( k_n \).

\[
\sum_{q \in \mathbb{Z}} f_P(k_q) \approx \frac{1}{\Delta k} \int_{-\infty}^{+\infty} f_P(k_q) dk_q = -\frac{\Phi(\xi)}{\Delta k} (E(k_n + k_d) - E(k_n - k_d)),
\]

where \( \Phi(\xi) = \int k_q \psi(k_q) dk_q = -\Phi(-\xi) \). Considering
that \( k_d \approx 1/\sqrt{2} \ll k_n \) we have \( E(k_n - k_d) - E(k_n +
\]

\[
E(k) = \mathcal{E}_0 \exp \left( \frac{\lambda}{\sqrt{b}} \arctan(\sqrt{b}k^2) - \frac{\lambda\nu}{2b} \ln(1 + bk^4) \right)
\]

with \( \mathcal{E}_0 \) a constant of integration. In the limit of large
wave numbers the second term in the exponent dominates
and leads to

\[
E(k) = \mathcal{E}_0 k^{-2\lambda\nu/b},
\]

where \( \mathcal{E}_0 \) is a constant. This is a power law spectrum
with a nonuniversal scaling exponent. The latter is pro-
tional to the high-wavenumber damping rate \( \nu/b \).

An analytically convenient form for \( \psi \) is \( \psi(k_q) = a_1 e^{(k_q - a_2)/a_3} \) where \( a_1, a_2 \) and \( a_3 \) are free parameters.
Their values may be determined by a fit to the numerical
data which is shown with blue crosses in Fig. 2. One can
easily check that this particular choice for \( \psi \) gives for the
ratio between the maximum and the absolute value of the
minimum

\[
\frac{f_P(-k_d)}{|f_P(k_d)|} \approx \frac{(1 + r e^{4k_d a_2/a_3})}{(1 + r e^{-4k_d a_2/a_3})} e^{-4k_d a_2/a_3},
\]

where \( r = E(k_n - k_d)/E(k_n + k_d) \). A least squares fit gives
\( a_1 \approx 0.1403, a_2 \approx 0.2578 \) and \( a_3 \approx 0.7564 \) which leads to
\( f_P(-k_d)/|f_P(k_d)| \approx 1.217 \). The corresponding numerical
value is 1.184 and the good agreement signifies that the
particular form of \( \psi \) chosen captures well the important
asymmetry of the triple correlation. Consistency checks.
To check for consistency, we also computed numerically
the energy spectra for different values of the damping
rate \( \nu/b \). As can be seen in Fig. 3 one can thus confirm
that a constant high-\( k \) damping rate leads to an energy
spectrum in the form of a power law (in contrast to the
FIG. 3: Fit of a power law (blue line) to the high-\textit{k} end of the energy spectrum (red line) for $b = 0.036$. (color online)

standard KSE, which displays an exponential fall-off), and that the associated spectral exponents are indeed proportional to the damping rate. According to a linear fit to the data in Fig. 4 one obtains $\lambda \approx 0.25$, whereas the fitting procedure in the context of Fig. 2 yields a slightly larger value of $\lambda \approx 0.4$. The reason for this is that the area enclosed by the ragged curve (which is essential for computing the precise value of the energy transfer) is nearly 1.6 times smaller than the area under the red curve in Fig. 2. Taking this correction into account, the two approaches agree very well, providing a consistent overall picture.

FIG. 4: The exponent $\delta = 2\lambda \nu / b$ in the power law $E(k) \propto k^{-\delta}$ as a function of the damping rate $1/b$ for $\nu = 1$. (color online)

Summary. Motivated by the fact that many turbulent systems in nature as well as in the laboratory exhibit power law spectra even in the absence of a clean inertial range, we studied as a simple model system a modified version of the Kuramoto-Sivashinsky equation, with a constant high-\textit{k} damping rate. Via semi-analytical and numerical studies, we demonstrated the existence of power laws with nonuniversal scaling exponents in the spectral range for which the ratio of nonlinear and linear time scales is (roughly) scale-independent. Such situations may arise in various physical systems with multiscale drive and/or damping, including, in particular, magnetized laboratory plasmas. In this context, the present work provides a plausible explanation for the observation of nonuniversal power laws in numerical studies.

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