A Rohlin Type Theorem for Automorphisms of Certain Purely Infinite $C^*$-Algebras

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Abstract

We show a noncommutative Rohlin type theorem for automorphisms of a certain class of purely infinite simple $C^*$-algebras. This class consists of the purely infinite unital simple $C^*$-algebras which are in the bootstrap category $\mathcal{N}$ and have trivial $K_1$-groups.

1 Introduction

A noncommutative Rohlin type theorem is a fundamental tool for the classification theory of actions of operator algebras. This theorem was first introduced by A. Connes for single automorphisms (i.e. actions of $\mathbb{Z}$) of finite von Neumann algebras [3]. Subsequently it was extended for actions of more general groups [18, 19]. Also in the framework of $C^*$-algebras this type of theorem was established first for the UHF algebras [1, 8, 9] and recently for some AF, AT algebras and some purely infinite simple $C^*$-algebras [12, 13, 14]. In particular A. Kishimoto showed the Rohlin type theorem for automorphisms of the Cuntz algebras $O_n$ with $n$ finite [12]. Our first motivation of this paper is to obtain a similar result for the Cuntz algebra $O_\infty$. When $n$ is finite, the Rohlin property of the unital one-sided shift on the UHF algebra $M_n\otimes$ plays a crucial role to derive Rohlin projections from outer automorphisms of $O_n$. However for $O_\infty$, there does not seem to be a similar mechanism at work. But fortunately by the progress of the classification theory of purely infinite simple $C^*$-algebras due to E. Kirchberg, N.C. Phillips and M. Rørdam, the Cuntz algebras $O_n$, $n = 2, 3, \ldots, \infty$ (or more generally the purely infinite unital simple
C*-algebras which are in the bootstrap category \( \mathcal{N} \) and have trivial \( K_1 \)-groups) can be decomposed as the crossed products of unital AF algebras by proper (i.e. non-unital) corner endomorphisms \([10, 22, 23]\). Moreover these non-unital endomorphisms also have the Rohlin property like the unital one-sided shift on \( M_{\infty} \). We shall use these endomorphisms to derive Rohlin projections.

The content of this paper is as follows. In Section 2 we review several facts about the crossed products of C*-algebras by their endomorphisms, which are the starting point of our argument. The C*-algebras described in this section contain all the purely infinite unital simple C*-algebras in the bootstrap category \( \mathcal{N} \) having trivial \( K_1 \)-groups and the statements mainly come from M. Rørdam’s paper \([23]\) and the remarkable classification theory by E. Kirchberg and N.C. Phillips \([10, 22]\). In Section 3 we show a Rohlin type theorem for approximately inner automorphisms of the C*-algebras described in Section 2. Our claim is that for any such automorphism whose nonzero powers are all outer, it has the Rohlin property. We will meet a technical difficulty where we have to make Rohlin projections for the automorphism almost central. To overcome this difficulty we use the Rohlin property of the endomorphism which appears in the crossed product decomposition as stated above. Finally in Section 4 we present several examples of automorphisms which have the Rohlin property. Up to conjugacy these examples include well-known automorphisms of Cuntz algebras which are found in \([7, 17]\).

2 Crossed product decomposition

We start our argument with some definitions of key words which we use throughout this paper. For details we refer to \([20, 23]\).

**Definition 1** Let \( \alpha \) be a (unital or non-unital) endomorphism on a unital C*-algebra \( A \). Then \( \alpha \) is said to have the Rohlin property if for any \( M \in \mathbb{N} \), finite subset \( F \) of \( A \) and \( \varepsilon > 0 \), there exist projections \( e_0, \ldots, e_{M-1}, f_0, \ldots, f_M \) in \( A \) such that

\[
\sum_{i=0}^{M-1} e_i + \sum_{j=0}^{M} f_j = 1 ,
\]

\[
e_i \alpha(1) = \alpha(1)e_i , \quad f_j \alpha(1) = \alpha(1)f_j ,
\]

\[
\|e_ix - xe_i\| < \varepsilon , \quad \|f_jx - xf_j\| < \varepsilon ,
\]

\[
\|\alpha(e_i) - e_{i+1}\alpha(1)\| < \varepsilon , \quad \|\alpha(f_j) - f_{j+1}\alpha(1)\| < \varepsilon
\]

for \( i = 0, \ldots, M-1, j = 0, \ldots, M \) and all \( x \in F \), where \( e_M = e_0, f_{M+1} = f_0 \).

**Definition 2** An endomorphism \( \rho \) on a unital C*-algebra \( B \) is called a corner endomorphism if \( \rho \) is an isomorphism from \( B \) onto \( \rho(1)B\rho(1) \). A corner endomorphism \( \rho \) is called a proper corner endomorphism if \( \rho \) is non-unital. Let \( \rho \) be
a corner endomorphism on $B$. Then the crossed product $B \rtimes_{\rho} N$ is defined to be the universal $C^*$-algebra generated by a copy of $B$ and an isometry $s$ which implements $\rho$, that is, $\rho(b) = sb^*$ for all $b \in B$.

Let $\mathcal{N}$ be the smallest full subcategory of the separable nuclear $C^*$-algebras which contains the separable Type I $C^*$-algebras and is closed under strong Morita equivalence, inductive limits, extensions, and crossed products by $\mathbb{R}$ and by $\mathbb{Z}$. A simple unital $C^*$-algebra $A$, which has at least dimension two, is said to be purely infinite if for any nonzero positive element $a \in A$ there exists $x \in A$ such that $xax^* = 1$. For convenience let $\mathcal{A}$ denote the purely infinite unital simple $C^*$-algebras which are in the bootstrap category $\mathcal{N}$ and have trivial $K_1$-groups. According to Theorem 3.1, Proposition 3.7, Corollary 4.6 in [23] and to Theorem 4.2.4 in [22] we have the following theorem.

**Theorem 3** For any $C^*$-algebra $A$ in $\mathcal{A}$ there exist a unital simple AF algebra $B$ with a unique tracial state, unital finite-dimensional $C^*$-subalgebras $(B_N \mid N \in \mathbb{N})$ of $B$ and a proper corner endomorphism $\rho$ on $B$ with the Rohlin property such that

$$A \cong B \rtimes_{\rho} \mathbb{N},$$

$$B_N \subseteq B_{N+1}, \quad \bigcup_{N \in \mathbb{N}} B_N \text{ is dense } B,$$

$$\rho(B_N) \subseteq B_{N+1}, \quad pB_Np \subseteq \rho(B_{N+1})$$

for all $N \in \mathbb{N}$, where $p \equiv \rho(1) \neq 1$ and that $p$ is full in $B_1$, i.e. $p \in B_1$ and the linear hull of $B_1pB_1$ is $B_1$. Conversely every $C^*$-algebra arising as a crossed product algebra described above and having the trivial $K_1$-group is in $\mathcal{A}$.

Henceforth we let $A$ denote a $C^*$-algebra in $\mathcal{A}$ and let $B$, $(B_N \mid N \in \mathbb{N})$, $\rho$, $p$ be as in the statement of Theorem 3. Finally in this section we state some technical lemma needed later. Since $p$ is full in $B_1$ we have elements $a_1, \ldots, a_r$ in $B_1$ such that

$$\sum_{i=1}^r a_i p a_i^* = 1, \quad a_i p = a_i.$$

Let $s$ be an isometry in $A \cong B \rtimes_{\rho} \mathbb{N}$ which implements $\rho$. Define $\sigma(x) = \sum_{i=1}^r a_i sxs^* a_i^*$ for $x \in A$, then $\sigma$ has the following properties ([23, Lemma 6.3.]):

**Lemma 4**

(1) $\sigma \mid A \cap B_2'$ is a unital $*$-homomorphism.

(2) $\sigma(A \cap B_{N+1}') \subseteq A \cap B_N'$ for all $N \in \mathbb{N}$.

(3) $s^j xs'^j = \sigma^j(x)s^j s'^j = s^j s'^j \sigma^j(x)$ for all $j \in \mathbb{N}$, and $x \in A \cap B_{j+1}'$. 

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3 Rohlin type theorem

In this section we state the main theorem of this paper. That is

**Theorem 5** Let $A$ be a $C^*$-algebra in the class $\mathcal{A}$. For any approximately inner automorphism $\alpha$ of $A$ the following conditions are equivalent:

1. $\alpha^k$ is outer for any nonzero integer $k$.

2. $\alpha$ has the Rohlin property.

Here an automorphism of a $C^*$-algebra is said to be *approximately inner* if it can be approximated pointwise by inner automorphisms. It is clear that (2) implies (1). To show the converse we take several steps. Since $A$ is in $\mathcal{A}$ we use the notation appeared in the previous section. Suppose that (1) in Theorem 5 holds. The next three lemmas follow by the methods used in [6, 12].

**Lemma 6** Let $q$ be a projection in $A \cap B_2'$. Then

$$c(\alpha^k \sigma(q)) = c(\alpha^k(q))$$

for any $k \in \mathbb{Z}$, where $c(\cdot)$ denotes the central support in the enveloping von Neumann algebra $A^{**}$ of $A$.

**Proof.** Since $\alpha^k \sigma^{-k}$ is inner, we have that

$$c(\alpha^k \sigma(q)) = c(\alpha^k \sigma \alpha^{-k} \alpha^k(q)) \leq c(\alpha^k(q)) .$$

Since $\sigma(q)p = sqs^*$ by (3) of Lemma 4 we have

$$c(\alpha^k \sigma(q)) \geq c(\alpha^k(\sigma(q)p)) = c(\alpha^k(sqs)) = c(\alpha^k(q)).$$

This completes the proof. \qed

Let $\text{Proj}(A)$ denote the projections of a $C^*$-algebra $A$.

**Lemma 7** Let $l, m$ and $N$ be nonnegative integers with $N \geq l + m + 2$ and let $k$ be a nonzero integer. Then for any nonzero projection $e$ in $A \cap B_N'$,

$$\inf\{ ||qa^k\sigma^l(q)|| | q \in \text{Proj}(e(A \cap B_N')e) \setminus \{0\} \} = 0 .$$

**Proof.** First we show the lemma when $l, m = 0$. Assume that

$$\delta \equiv \inf\{ ||qa^k(q)|| | q \in \text{Proj}(e(A \cap B_N')e) \setminus \{0\} \} > 0 .$$

(1)

Let $(e_{s,t}^{(j)} | j = 1, \ldots, J_N; s, t = 1, \ldots, d_j)$ be a system of matrix units for $B_N \cong \bigoplus_{j=1}^{J_N} M_{d_j} (\mathbb{C})$ and set $p^{(j)} = \sum_{s=1}^{d_j} e_{s,s}^{(j)}$. We may assume that $e \in p^{(j)} A p^{(j)}$ for some $j$, which satisfies $ee_{1,1}^{(j)} \neq 0$. Then it is easily verified that the set

$$\{ q | q \in \text{Proj}(A) \setminus \{0\}, q \leq ee_{1,1}^{(j)} \}$$


is equal to the set
\[ \{ qe^{(j)}_{1,1} | q \in \text{Proj}(e(A \cap B_{N'})e) \setminus \{0\} \} . \]

Combining this with the fact that \( \alpha^k \) is outer, we obtain
\[
\inf \{ \|qe^{(j)}_{1,1}a\alpha^k(qe^{(j)}_{1,1})\| | q \in \text{Proj}(e(A \cap B_{N'})e) \setminus \{0\} \} = 0 .
\]

for any \( a \in A \setminus \{0\} \) by virtue of [1] Lemma 1.1. Here we have unitaries \( v_1, \ldots, v_d \) in \( B_N \) such that
\[
\sum_{s=1}^{d_j} v_s e^{(j)}_{1,1} v_s^* = p^{(j)} .
\]

Then for any nonzero projection \( q \) in \( e(A \cap B_{N'})e \),
\[
qa^{k}(q) = q p^{(j)} \alpha(q p^{(j)}) = \sum_{s,t} q v_s e^{(j)}_{1,1} v_t^* \alpha^k(q v_s e^{(j)}_{1,1} v_t^*) = \sum_{s,t} v_s q e^{(j)}_{1,1} v_t^* \alpha^k(v_s) \alpha^k(q e^{(j)}_{1,1}) \alpha^k(v_t^* .
\]

By the first assumption (1) this implies, for some \( s, t \)
\[
\|q e^{(j)}_{1,1} v_t^* \alpha^k(v_s) \alpha^k(q e^{(j)}_{1,1}) \| \geq \frac{\delta}{d_j} .
\]

But this inequality contradicts (2). Thus we arrive at the result when \( l, m = 0 \). For general \( l \) and \( m \), using the above result we obtain a projection \( p_1 \) in \( e(A^{**} \cap B_{N'})e \) such that \( p_1 \) is minimal in \( A^{**} \) and \( c(p_1) c(\alpha^k(p_1)) = 0 \). Set \( p_2 = \sigma^m(p_1) \), then by Lemma 3
\[
c(\alpha^k \sigma^l(p_2)) = c(\alpha^k \sigma^{l+m}(p_1)) = c(\alpha^k(p_1)) ,
\]
\[
c(p_2) = c(\sigma^m(p_1)) = c(p_1) .
\]

Approximating \( p_1 \) by projections in \( e(A \cap B_{N'})e \) we obtain the result. \( \square \)

Lemma 8 Let \( K, L \) and \( N \) be positive integers with \( N \geq K + L + 2 \) and let \( \varepsilon > 0 \). Then there exists a nonzero projection \( e \) in \( A \cap B_{N'} \) such that
\[
[e] = 0 \quad \text{in } K_0(A \cap B_{N'})
\]
\[
\|a^{k_1} c^{l_1}(e) \cdot a^{k_2} c^{l_2}(e)\| < \varepsilon
\]
for \( k_1, k_2 = 0, \ldots, K \) and \( l_1, l_2 = 0, \ldots, L \) with \( (k_1, l_1) \neq (k_2, l_2) \).
Proof. From the Rohlin property of $\rho$, we have a nonzero projection $p_1$ in $A \cap B_{N'}$ such that

$$\|p_1 \sigma^l(p_1)\| < \varepsilon$$

for $l = 1, \ldots, L$. Using Lemma 7 with $m = 0$, we find a nonzero projection $p_2$ in $p_1(A \cap B_{N'})p_1$ such that

$$\|p_2 \sigma^k \sigma^l(p_2)\| < \varepsilon$$

for $k = \pm 1, \ldots, \pm K$ and $l = 0, \ldots, L$. Again using Lemma 7 with $m = 1$, we find a nonzero projection $p_3$ in $p_2(A \cap B_{N'})p_2$ such that

$$\|\sigma(p_3) \sigma^k \sigma^{l+1}(p_3)\| < \varepsilon$$

for any $k$ and $l$ as above. Repeating this application of Lemma 7 until $m = L$ we obtain a nonzero projection $p_{L+2}$ in $A \cap B_{N'}$ such that

$$\|\sigma^j(p_{L+2}) \sigma^k \sigma^{l+j}(p_{L+2})\| < \varepsilon$$

for $j = 0, \ldots, L$, $k = \pm 1, \ldots, \pm K$ and $l = 0, \ldots, L$. Since $A$ is purely infinite we can find a nonzero projection $e$ in $p_{L+2}(A \cap B_{N'})p_{L+2}$ such that $[e] = 0$ in $K_0(A \cap B_{N'})$. This projection $e$ satisfies the required condition.

Lemma 9 Let $N$ be a positive integer and let $\{ p^{(j)} \ | \ j = 1, \ldots, J_N \}$ be the set of minimal central projections in $B_N$ with $\sum_{j=1}^{J_N} p^{(j)} = 1$. Then there exist positive integers $N_1 \geq N$ which satisfy the following condition:

$$p^{(j)} \sigma^l(q) p^{(j)} \neq 0$$

for all integer $l \geq 0$, nonzero projection $q$ in $A \cap B_{N_1+1}'$ and $j = 1, \ldots, J_N$.

Proof. By the simplicity of $B$ we choose $N_1 \leq N$ such that the central support of $p^{(j)}$ in $B_{N_1}$ is 1 for all $j$. Then for any nonzero projection $q \in A \cap B_{N_1+1}'$, it follows that $\sigma^l(q) p^{(j)} \neq 0$ since $\sigma^l(q) \in A \cap B_{N_1}'$. This completes the proof.

The next lemma says that we almost find Rohlin projections if we drop the condition that the sum of the projections is 1.

Lemma 10 Let $M, N$ be positive integers and let $\varepsilon > 0$. Then there exist mutually orthogonal nonzero projections $e_0, \ldots, e_{M-1}$ in $A$ such that

$$\|\sigma(e_i) - e_{i+1}\| < \varepsilon,$$

$$e_i \in B_{N_i}, \quad \|e_i s - se_i\| < \varepsilon$$

for $i = 0, \ldots, M - 1$, where $e_M = e_0$. 

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Proof. Let $N_0$ be a positive integer which we shall make very large later. Take the minimal central projections $\{p^{(j)}| j = 1, \ldots, J_{2N_0}\}$ in $B_{2N_0}$ such that $\sum_{j=1}^{J_{2N_0}} p^{(j)} = 1$. Using Lemma 8 with $N = 2N_0$, we find positive integers $N_1 \geq 2N_0$ which satisfies the condition in Lemma 8. Let $N_2, m$ be positive integers with $N_2 \gg N_0, N_1$ and let $\varepsilon_2 > 0$. From Lemma 8 we obtain mutually orthogonal nonzero projections $E(k, l)$ $(k = 0, \ldots, mM - 1, l = 0, \ldots, N_0 - 1)$ in $A$ such that

$$\|E(k, l) - \alpha^k \sigma^l(E(0, 0))\| < \varepsilon_2,$$

$$E(0, l) \in B_{N_2}',$$

$$[E(0, l)] = 0 \text{ in } K_0(A \cap B_{N_2}')$$

for any $k, l$. Furthermore by the property of $N_1$, we may assume that

$$p^{(j)} E(0, 0) p^{(j)}, p^{(j)} E(0, 1) p^{(j)} \neq 0$$

for each $j$. Thus noting that $[E(0, 0)] = [E(0, 1)] = 0$ in $K_0(A \cap B_{N_2}')$, we have a partial isometry $w_1$ in $A \cap B_{2N_0}'$ such that

$$w_1^* w_1 = E(0, 0), \quad w_1 w_1^* = E(0, 1).$$

Since $\|\sigma(E(0, l)) - E(0, l + 1)\| < 2\varepsilon_2$, if $\varepsilon_2$ is sufficiently small then we can find a unitary $u_1$ in $A \cap B_{2N_0}'$ such that

$$\|u_1 - 1\| < 10N_0\varepsilon_2,$$

$$Ad u_1 \circ \sigma(E(0, l)) = E(0, l + 1)$$

for $l = 0, 1, \ldots, N_0 - 2$. Indeed $u_1$ is taken as follows. Set

$$x = \sum_{l=0}^{N_0-2} E(0, l + 1) \sigma(E(0, l))$$

$$+ (1 - \sum_{l=1}^{N_0 - 1} E(0, l))(1 - \sum_{l=0}^{N_0-2} \sigma(E(0, l))).$$

Then $x \in A \cap B_{N_2-1}'$ and

$$x - 1 = \sum_{l=0}^{N_0-2} E(0, l + 1)\{\sigma(E(0, l)) - E(0, l + 1)\}$$

$$+ (1 - \sum_{l=1}^{N_0 - 1} E(0, l))(- \sum_{l=0}^{N_0-2} \sigma(E(0, l))).$$

Thus

$$\|x - 1\| \leq 2\varepsilon_2 + 2\varepsilon_2(N_0 - 1) = 2N_0\varepsilon_2,$$
\[ \|xx^* - 1\| \leq 6N_0\varepsilon_2 . \]

So let \( u_1 = (xx^*)^{-\frac{1}{2}}x \) then \( u_1 \) is a desired unitary in \( A \cap B_{N_2-1'} \). Let \( \sigma_1 = Ad u_1 \circ \sigma \) and define

\[
E_{i,j} = \begin{cases} 
\sigma_1^{i-1}(w_1)\sigma_1^{i-2}(w_1)\cdots\sigma_1^1(w_1) & (i > j) \\
E_{(0,i)} & (i = j) \\
\sigma_1^1(w_1)^*\sigma_1^{i+1}(w_1)^*\cdots\sigma_1^{j-1}(w_1)^* & (i < j)
\end{cases}
\]

Then we can easily verify that \( (E_{i,j} | i, j = 0, \ldots, N_0 - 1) \) forms a system of matrix units in \( A \cap B_{N_0+2'} \). Furthermore define

\[
E_0 = \frac{1}{N_0} \sum_{i,j=0}^{N_0-1} E_{i,j}.
\]

Then \( E_0 \) is a non-zero projection in \( A \cap B_{N_0+2'} \). Noting that \( \sigma_1(E_{i,j}) = E_{i+1,j+1} \) we have

\[
\|\sigma_1(E_0) - E_0\| = \left\| \frac{1}{N_0} \sum_{i=0}^{N_0-1} \sigma_1(E_{i,N_0-1}) + \frac{1}{N_0} \sum_{i=0}^{N_0-2} \sigma_1(E_{i,N_0-1}) - \frac{1}{N_0} \sum_{i=0}^{N_0-1} \sigma_1(E_{0,i}) \right\|
\leq \frac{4}{\sqrt{N_0}} .
\]

Thus

\[
\|\sigma(E_0) - E_0\| \leq \|\sigma_1(E_0) - E_0\| + 2\|u_1 - 1\|
\leq \frac{4}{\sqrt{N_0}} + 20N_0\varepsilon_2 .
\]

Accordingly

\[
\|sE_0 - E_0s\| \leq \|sE_0s^* - E_0s^*\| = \|\sigma(E_0) - E_0\|
\leq \frac{4}{\sqrt{N_0}} + 20N_0\varepsilon_2 .
\]

Therefore if we make \( N_0 \) sufficiently large and \( \varepsilon_2 \) sufficiently small with \( \varepsilon_2 \ll N_0^{-1} \), then \( E_0 \) becomes almost central in \( A \). Consequently, for any positive integer \( N_3 \) and \( \varepsilon_3 > 0 \), we obtain mutually orthogonal projections \( E_0, \ldots, E_{mM-1} \) in \( A \cap B_{N_3'} \) such that

\[
\|E_k s - sE_k\| < \varepsilon_3 ,
\]

\[
\|\alpha^k(E_0) - E_k\| < \varepsilon_3
\]

(3)
for $k = 0, \ldots, mM - 1$. In particular taking $N_3$ sufficiently large and $\varepsilon_3$ sufficiently small, we can make $\sigma(E_k) = \sum_{i=1}^r a_isE_k^*a_i$ very close to $E_k$. Thus we have partial isometries $u_2, v_2$ in $A \cap B_{N_3-1}'$ such that

$$\text{Ad } u_2(\sigma(E_0)) = E_0,$$

$$\|u_2 - E_0\| \leq 2\|\sigma(E_0) - E_0\| \leq 2(\sum_{i=1}^r \|a_i\|)\varepsilon_3,$$

$$\text{Ad } v_2(\sigma(E_1)) = E_1,$$

$$\|v_2 - E_1\| \leq 2\|\sigma(E_1) - E_1\| \leq 2(\sum_{i=1}^r \|a_i\|)\varepsilon_3.$$ 

Furthermore by Lemma 11 below we may assume that there is a partial isometry $w_2$ in $A \cap B_{N_3-1}'$ such that $w_2^*w_2 = E_0$ and $w_2w_2^* = E_1$. Set $w_3 = v_2\sigma(w_2)u_2^*$ then $w_3^*w_2$ is a unitary in $E_0(A \cap B_{N_3-1}')E_0$. Since $K_1(A) = 0$, $w_3^*w_2$ is homotopic to $E_0$ in $E_0(A \cap B_{N_3-1}')E_0$ from [33, Lemma 2.3.] and [23, Lemma 6.6.]. By virtue of the stability of $\text{Ad } u_2 \circ \sigma$ ([23, Lemma 6.4., 6.5.]), if we make $N_3$ sufficiently large and $\varepsilon_3$ sufficiently small for any positive integer $N_4$ and $\varepsilon_4 > 0$, we obtain a unitary $y$ in $E_0(A \cap B_{N_4}')E_0$ such that

$$\|w_3^*w_2 - (\text{Ad } u_2 \circ \sigma)(y)y^*\| < \varepsilon_4.$$

Thus

$$\|w_2y - v_2\sigma(w_2y)u_2^*\| < \varepsilon_4.$$ 

Set $W = w_2y$ then $W$ is a partial isometry in $A \cap B_{N_4}'$ from $E_0$ onto $E_1$ which satisfies that

$$\|W - \sigma(W)\| \leq \|W - v_2\sigma(W)u_2^*\| + \|(v_2 - E_1)\sigma(W)u_2^*\| + \|E_1\sigma(W)(u_2 - E_0)^*\| + \|E_1\sigma(W)E_0 - \sigma(E_1)\sigma(W)\sigma(E_0)\| \leq \varepsilon_4 + 4(\sum_{i=1}^r \|a_i\|)\varepsilon_3 + 2(\sum_{i=1}^r \|a_i\|)\varepsilon_3 \leq 2\varepsilon_4.$$ 

Accordingly we have

$$\|sW - Ws\| = \|sW^*s - Wss^*\| = \|(\sigma(W) - W)p\| \leq 2\varepsilon_4,$$

$$\|s^*W - Ws^*\| = \|ss^*W - sWs^*\| = \|p(\sigma(W) - W)\| \leq 2\varepsilon_4.$$ 

Therefore $W$ is almost central in $A$ when $N_4$ is very large and $\varepsilon_4$ is very small. On the other hand, by [3] we have a unitary $u_3$ in $A$ such that

$$\|u_3 - 1\| < 10mM\varepsilon_3,$$

$$\text{Ad } u_3 \circ \sigma(E_k) = E_{k+1}.$$
for \( k = 0, \ldots, mM - 1 \). Let \( \alpha_1 = \text{Ad} u_3 \circ \alpha \) and define

\[
f_{i,j} = \begin{cases} 
\alpha_1^{i-1}(W)\alpha_1^{i-2}(W) \cdots \alpha_1^{j}(W) & (i > j) \\
E_i & (i = j) \\
\alpha_1^{i}(W)^*\alpha_1^{i+1}(W)^* \cdots \alpha_1^{j-1}(W)^* & (i < j)
\end{cases}.
\]

Then \( (f_{i,j} | i, j = 0, \ldots, mM - 1) \) forms a system of matrix units. Furthermore define

\[
F_i = \frac{1}{m} \sum_{k,l=0}^{m-1} f_{i+kM,i+M}
\]

for \( i = 0, \ldots, M - 1 \). It is easy to verify that \( F_0, \ldots, F_{M-1} \) are mutually orthogonal and satisfy that

\[
\alpha_1(F_i) = F_{i+1}, \quad \|\alpha_1(F_{M-1}) - F_0\| < \frac{4}{\sqrt{m}}
\]

for \( i = 0, \ldots, M - 1 \). Hence if we make \( N_4, m \) sufficiently large and \( \varepsilon_4 \) sufficiently small, we can obtain projections \( (e_i | i = 0, \ldots, M - 1) \) which satisfy the required condition except that \( e_i \in B_{N'} \). But \( e_i \)'s are almost central, therefore by [2, Theorem 5.3] we have desired projections after a small inner perturbation.

**Lemma 11** Let \( \alpha \) be an approximately inner automorphism of a unital purely infinite \( C^* \)-algebra \( A \). If \( (p_j | j \in \mathbb{N}) \) is a uniformly central sequence of projections in \( A \) then for any \( \varepsilon > 0 \) and any unital finite dimensional \( C^* \)-subalgebra \( F \) of \( A \) there exist a \( j \in \mathbb{N} \) and a partial isometry \( w \) in \( A \) such that

\[
w^*w = p_j, \quad \|ww^* - \alpha(p_j)\| < \varepsilon, \quad \|wx - wx\| \leq \varepsilon \|x\|
\]

for any \( x \in F \).

**Proof.** Since \( \alpha \) is approximately inner and \( F \) is finite dimensional, the restriction of \( \alpha \) to \( F \) is inner i.e. there exists a unitary \( u \) in \( A \) such that \( \alpha \restriction F = \text{Ad} u \restriction F \). From uniform centrality of \( (p_j) \), we find a sufficiently large \( j \in \mathbb{N} \) such that

\[
\|\alpha(p_j)u - u\alpha(p_j)\| < \varepsilon, \quad \|p_jx - xp_j\| \leq \varepsilon \|x\|
\]

for any \( x \in F \). For these \( u \) and \( p_j \), since \( \alpha \) is approximate inner, we have a unitary \( v \) in \( A \) such that

\[
\|(\text{Ad} u^* \circ \alpha - \text{Ad} v)x\| \leq \varepsilon \|x\|
\]

for any \( x \in F \cup \{p_j\} \). Since \( \text{Ad} u^* \circ \alpha \restriction F \) is the identity, it follows that

\[
\|x - vxv^*\| \leq \varepsilon \|x\|
\]
for any $x \in F$. If we set $w = vp_j$, we have $w^*w = p_j$, $\|wx - xw\| \leq 2\varepsilon\|x\|$ for any $x \in F$ and
\[
\|ww^* - \alpha(p_j)\| = \|(Ad v - Ad u^* \circ \alpha)(p_j)\| + \|(Ad u^* \circ \alpha - \alpha)(p_j)\| \leq \varepsilon + \varepsilon.
\]
This completes the proof.

Proof of Theorem 5.
We have already shown in Lemma 10 that we almost have Rohlin projections except that the sum of the projections is 1. To derive genuine Rohlin projections, we can exactly follow the method of Proof of Theorem 3.1 in [12], replacing almost $\Phi$-invariance there by almost commutativity with $B_N \cup \{s, s^*\}$ as in Lemma 10. In this process the number of towers of projections increases from one to two as in Definition 1. We have thus proved the theorem.

4 Examples
In this section we present several examples of automorphisms which have the Rohlin property. Let $A$ be a $C^*$-algebra in $A$ and let $B \rtimes_\rho \mathbb{N}$ be a crossed product decomposition of $A$ as in Section 2. By the universality of the crossed product we have the dual action $\hat{\rho}$ of $\mathbb{T}$ on $B \rtimes_\rho \mathbb{N}$, that is, we define $\hat{\rho}$ by the formulas:
\[
\hat{\rho}(b) = b, \quad \hat{\rho}_\lambda(s) = \lambda s \quad \text{for all } b \in B, \lambda \in \mathbb{T}.
\]
Using the universality similarly for an automorphism $\alpha$ of $B$ with $\alpha \circ \rho = \rho \circ \alpha$, we define an automorphism $\hat{\alpha}$ of $B \rtimes_\rho \mathbb{N}$ by $\hat{\alpha}(b) = \alpha(b)$ for all $b \in B$ and by $\hat{\alpha}(s) = s$. Clearly $\hat{\alpha}$ commutes with each $\hat{\rho}_\lambda$ from the definition. Then we have

Proposition 12 An automorphism $\hat{\alpha} \circ \hat{\rho}_\lambda$ of $A \cong B \rtimes_\rho \mathbb{N}$ is approximately inner for any $\lambda \in \mathbb{T}$, and one has the following:

1. If $\alpha$ is the identity mapping on $A$ then $\hat{\alpha} \circ \hat{\rho}_\lambda = \hat{\rho}_\lambda$ is outer for any $\lambda \in \mathbb{T} \setminus \{0\}$.
2. If $\alpha$ is outer (as an automorphism of $B$) then $\hat{\alpha} \circ \hat{\rho}_\lambda$ is outer for any $\lambda \in \mathbb{T}$.
3. If $\alpha$ is inner then $\hat{\alpha} \circ \hat{\rho}_\lambda$ are inner for at most a countable number of $\lambda \in \mathbb{T}$.

Therefore in any case $\hat{\alpha} \circ \hat{\rho}_\lambda$ have the Rohlin property for an uncountable number of $\lambda \in \mathbb{T}$.

Proof. The dual action $\hat{\rho}$ of $\mathbb{T}$ on $B \rtimes_\rho \mathbb{N}$ is strongly continuous. Hence $\hat{\rho}_\lambda$ is approximately inner by virtue of Rørdam’s classification theorem [23, Theorem 6.12.]. Since $A$ is isomorphic to a corner of $(B \otimes \mathbb{K}) \rtimes_\beta \mathbb{Z}$ for some automorphism.
Finally we show (3). Suppose that \( \tilde{\alpha} \) is inner, that is, there is a unitary \( v \) in \( B \rtimes \rho \mathbb{N} \) such that \( \tilde{\alpha} \circ \tilde{\rho}_\lambda = Ad v \). Then since \( \tilde{\alpha} \) commutes with \( \tilde{\rho}_\lambda \) we have that \( (\tilde{\alpha} \circ \tilde{\rho}_\lambda) \circ \tilde{\rho}_\mu = \tilde{\rho}_\mu \circ (\tilde{\alpha} \circ \tilde{\rho}_\lambda) \), and that

\[
Ad v \circ \tilde{\rho}_\mu = \tilde{\rho}_\mu \circ Ad v = Ad \tilde{\rho}_\mu (v) \circ \tilde{\rho}_\mu
\]

for any \( \mu \in \mathbb{T} \). Since \( B \rtimes \rho \mathbb{N} \) is simple it follows that there exists a scalar \( c_\mu \) in \( \mathbb{T} \) with \( v^* \tilde{\rho}_\mu (v) = c_\mu 1 \). It is easily checked that \( c_{\mu \nu} = c_\mu c_\nu \) for all \( \mu, \nu \in \mathbb{T} \), thus there is an integer \( k \) satisfying \( c_\mu = \mu^k \) for all \( \mu \in \mathbb{T} \). Accordingly

\[
\tilde{\rho}_\mu (v s^{k}) = \mu^k v \rho^{k} s^{k} = v s^{k}
\]

for any \( \mu \in \mathbb{T} \), hence \( v s^{k} \in B \). This implies that \( k = 0 \) because \( B \) has no proper coisometry. Therefore \( v \in B \) and \( \alpha = Ad v \) is inner. This proves (2).

Finally we show (3). Suppose that \( \lambda_1, \lambda_2 \in \mathbb{T} \) with \( \lambda_1 \neq \lambda_2 \) and that \( \tilde{\alpha} \circ \tilde{\rho}_{\lambda_1}, \tilde{\alpha} \circ \tilde{\rho}_{\lambda_2} \) is inner. There are unitaries \( v_1, v_2 \) in \( B \rtimes \rho \mathbb{N} \) such that \( \tilde{\alpha} \circ \tilde{\rho}_{\lambda_i} = Ad v_i \), \( i = 1, 2 \). Then \( \lambda_i s = v_i s v_i^* \) and it follows that

\[
s^k v_i s^{k} = \lambda_i^k s^k s^{k} v_i s^k s^{k}
\]

for all \( k \in \mathbb{N} \). Thus

\[
\| v_1 - v_2 \| \geq \| s^k s^{k}(v_1 - v_2) s^{k} \| = \| \lambda_1^k s^k v_1 s^{k} - \lambda_2^k s^k v_2 s^{k} \|
\]

\[
= \| \lambda_1^k v_1 - \lambda_2^k v_2 \| = \| (\lambda_1 \lambda_2)^k v_1 v_2^* - 1 \|.
\]

Here it is obvious that \( \| (\lambda_1 \lambda_2)^k v_1 v_2^* - 1 \| \geq 1 \) for some \( k \), so \( \| v_1 - v_2 \| \geq 1 \). Therefore \( \tilde{\alpha} \circ \tilde{\rho}_\lambda \) are inner for at most a countable number of \( \lambda \in \mathbb{T} \) since \( B \) is separable. We have shown (3), thereby completing the proof.

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References

[1] O. Bratteli, D. E. Evans and A. Kishimoto, The Rohlin property for quasi-free automorphisms of the Fermion algebra, Proc. London. Math. Soc. (3)71(1995), 675–694.

[2] E. Christensen, Near inclusion of C*-algebras, Acta Math. 144(1980), 249–265.
[3] A. Connes, *Outer conjugacy class of automorphisms of factors*, Ann. Sci. Ec. Norm. Sup. 8(1975), 383–420.

[4] J. Cuntz, *Simple C*-algebras generated by isometries*, Comm. Math. Phys. 57(1977), 173–185.

[5] G. A. Elliott, D. E. Evans and A. Kishimoto, *Outer conjugacy classes of trace scaling automorphisms of stable UHF algebras*, preprint.

[6] D. E. Evans, and A. Kishimoto, *Trace scaling automorphisms of certain stable AF algebras*, preprint.

[7] M. Enomoto, H. Takahara and Y. Watatani, *Automorphisms on Cuntz algebras*, Math. Japonica 24(1979), 231–234.

[8] R. H. Herman and A. Ocneanu, *Stability for integer actions on UHF C*-algebras*, J. Func. Anal. 59(1984), 132–144.

[9] R. H. Herman and A. Ocneanu, *Spectral analysis for automorphisms of UHF C*-algebras*, J. Func. Anal. 66(1986), 1–10.

[10] E. Kirchberg, *The classification of purely infinite C*-algebras using Kasparov’s theory*, in preparation.

[11] A. Kishimoto, *Outer automorphisms and reduced crossed products of simple C*-algebras*, Comm. Math. Phys. 81(1981), 429–435.

[12] A. Kishimoto, *The Rohlin property for shifts on UHF algebras and automorphisms of Cuntz algebras*, J. Func. Anal. (to appear).

[13] A. Kishimoto, *The Rohlin property for automorphisms of UHF algebras*, J. reine angew. Math. 465(1995), 183–196.

[14] A. Kishimoto, *Automorphisms of AT algebras with the Rohlin property*, preprint.

[15] H. Lin, *Approximation by normal elements with finite spectra in C*-algebras of real rank zero*, Pacific J. Math. 173(1996), 443–489.

[16] H. Lin and N. C. Phillips, *Approximate unitary equivalence of homomorphisms from O∞*. J. reine angew. Math. 464(1995), 173–186.

[17] K. Matsumoto and J. Tomiyama, *Outer automorphisms on Cuntz algebras*, Bull. London Math. Soc. 25(1993), 64–66.

[18] A. Ocneanu, *A Rohlin type theorem for groups acting on von Neumann algebras*, Topics in Modern Operator Theory, Birkhäuser Verlag, (1981), 247–258.
[19] A. Ocneanu, *Actions of Discrete Amenable Groups on von Neumann Algebras*, Lec. Note in Math. 1138, Springer Verlag, (1985).

[20] W. Paschke, *The crossed product of a C*-algebra by an endomorphism*, Proc. Amer. Math. Soc. 80(1980), 113–118.

[21] G. K. Pedersen, *C*-algebras and their automorphism groups, Academic Press, (1979).

[22] N. C. Phillips, *A classification theorem for nuclear purely infinite simple C*-algebras*, preprint.

[23] M. Rørdam, *Classification of certain infinite simple C*-algebras*, J. Func. Anal. 131(1995), 415–458.

[24] J. Rosenberg and C. Schochet, *The Künneth theorem and the universal coefficient theorem for Kasparov’s generalized K-functor*, Duke Math. J. 55(1987), 431–474.