TOPOLOGY OF IRREGULAR ISOMONODROMY TIMES ON A FIXED POINTED CURVE

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Abstract. We will define and study moduli spaces of deformations of irregular classes on Riemann surfaces, which provide an intrinsic viewpoint on the ‘times’ of irregular isomonodromy systems in general. Our aim is to study the deeper generalisation of the G-braid groups that occur as fundamental groups of such deformation spaces, with particular focus on the generalisation of the full G-braid groups.

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Introduction

Classically, the theory of isomonodromy constitutes a collection of nonlinear integrable differential equations, whose unknown is a (linear) meromorphic connection on a vector bundle over the Riemann sphere. Geometrically, these are flat Ehresmann connections on a bundle whose fibres are moduli spaces of such meromorphic connections.

The underlying deformation parameters, the ‘times’, have recently been given an intrinsic formulation, leading to a generalisation of the moduli of pointed curves (in any genus). This framework is especially useful when considering the generalised deformations, beyond the generic case: recall [JMU81] set up a theory of ‘generic’ isomonodromic deformations of meromorphic connections on vector bundles over a Riemann surface $\Sigma$, where the leading coefficient at each pole has distinct eigenvalues (building on [Bir13]; cf. [Mal83, BJL79]). This has been extended in two
directions: i) replacing vector bundles by principal $G$-bundles, leading in particular to the appearance of $G$-braid groups for complex reductive groups $G$ [Boa02], and ii) considering nongeneric admissible deformations, e.g. [Boa12, Boa14], where the (untwisted/unramified) irregular type of the connection is arbitrary, leading to cabled braid groups [DRT].

In particular the spaces of generalised monodromy data, the wild character varieties (a.k.a. wild Betti spaces), have been proved to form a local system of Poisson varieties [Boa14]¹

\[(1) \quad \mathcal{M}_0 \longrightarrow \mathcal{B},\]

over any space $\mathcal{B}$ of admissible deformations. These give a purely topological description of the nonlinear isomonodromy differential equations, via the Riemann–Hilbert–Birkhoff correspondence.

Our purpose in this paper is to study the fundamental groups of the base spaces $\mathcal{B}$ of such admissible deformations, the groups that will act by algebraic Poisson automorphisms on the wild character varieties (the fibres of (1)) from the parallel transport of the isomonodromy connection—i.e. the monodromy of the nonlinear differential equations. This builds on our previous paper [DRT], which used a fixed marking: here we will quotient by the Weyl group action and get to the full version of ‘wild’ mapping class groups, in analogy to forgetting the ordering of marked points on the underlying pointed curve.

This encompasses the much-studied case of regular singular connections, involving the complex character varieties, which is the entry point for the standard mapping-class- and braid-group-actions in classical/quantum 2d gauge theories—via deformations of pointed curves, e.g. [Koh87, Dri89, Mas03, And06] in the quantum case. The case of poles of order 2, however, has been extensively studied by various authors: there are relations to quantum groups [Boa02, Xu20, Xuc], and already there the boundary of the space of times has a rich structure (corresponding to the ‘coalescence’ of irregular times, cf. [CDG19, Xua, Xub]). In particular the simplest irregular singular case has been understood in rigorous analytic way, while this paper focuses on the algebro-geometric aspects of the general nongeneric case.

In this series of ‘local’ papers we fix the underlying pointed curve, and vary the rest of the wild Riemann surface structure [Boa14], i.e. the irregular types/classes, controlling principal parts of irregular singular connections beyond their (formal) residues. More precisely [DRT] constructs a fine moduli space of untwisted/unramified irregular types for the split Lie algebra $(\mathfrak{g}, \mathfrak{t}) := (\text{Lie} G, \text{Lie} T)$, where $T \subseteq G$ is a maximal (algebraic) torus, while here we consider irregular classes.

Recall in brief an untwisted irregular type $Q$ at a point $a \in \Sigma$ is the germ of a $t$-valued meromorphic function based there, defined up to holomorphic terms:

\[(2) \quad Q = \sum_{j=1}^{p} A_j z^{-j} \in t(z)/t[z], \quad A_j \in \mathfrak{t}, \]

in a local coordinate vanishing at the marked point. Then the Weyl group $W_{\theta} = N(T)/T$ acts on the left tensor factor of

\[t \otimes_{\mathbb{C}} (\mathbb{C}[z]/\mathbb{C}[z]) \simeq t(z)/t[z],\]

¹Basically speaking, a bundle of Poisson manifolds equipped with a complete flat connection: the (Betti) isomonodromy connection, a.k.a. the wild nonabelian Gauss–Manin connection.
and the irregular class underlying (2) is its projection \( \overline{Q} \) in the quotient, i.e. the Weyl-orbit through \( Q \) [Boa14, Rk. 10.6].

The important fact is the fibres of (1) only depend on the collection of irregular classes underlying the irregular types at each marked point, and thus any (admissible) space of irregular classes provides an intrinsic topological description of the corresponding isomonodromy times. In the generic case, where the leading coefficient of (2) is out of all root hyperplanes, the homotopy type of the deformation space brings about the G-braid group: in this paper we shall encounter a generalisation in the nongeneric case, which we relate to braid cabling in type \( A \).

**Layout of the paper and main results.** In § 1 we give the main definition: to a one-pointed (bare) wild Riemann surface \( \Sigma = (\Sigma, a, \overline{Q}) \) we associate a full/nonpure local ‘wild’ mapping class group \( \Gamma_{\Sigma} \) (WMCG), viz. the fundamental group of a space \( B_{\Sigma} \) of admissible deformations of the irregular class \( \overline{Q} \) (cf. Def. 1.1). The latter is a topological quotient of the (universal) admissible deformation space \( B_{Q} \) of \( Q \), where \( Q \) is any irregular type lifting \( \overline{Q} \).

In § 2 we describe the subgroup of \( W_{Q} \) preserving \( B_{Q} \subseteq \mathfrak{t}^{r} \), and further the quotient thereof that acts freely; the resulting subgroup is denoted \( W_{\mathfrak{gl}_{b}} \). The relevant statements are proven inductively along the sequence of fission/Levi (root) subsystems of \( \Phi_{Q} \) associated with \( Q \) (cf. [DRT]): first in the case where \( Q = A \zeta^{-1} \) has a single coefficient \( A \in \mathfrak{t} \) (in § 2.1), and then in the general case (in § 2.2).

**Theorem** (Cf. Thm. 2.1). The space \( B_{Q} \) is a Galois covering of \( B_{\overline{Q}} \) with \( \text{Gal}(B_{Q}, B_{\overline{Q}}) = W_{\mathfrak{gl}_{b}} \), so \( \Gamma_{\overline{Q}} \) is an extension of \( W_{\mathfrak{gl}_{b}} \) by the pure local WMCG.

In § 3 we describe all full/nonpure local WMCGs for the irreducible rank-2 root systems, after explaining the classification boils down to simple Lie algebras.

Finally in § 4 we explicitly describe the full/nonpure local WMCGs when \( \mathfrak{g} \in \{ \mathfrak{gl}_{n}(\mathbb{C}), \mathfrak{sl}_{n}(\mathbb{C}) \} \), in the nonabelian case \( n \geq 2 \). This means identifying the ‘effective’ subgroup of the Weyl group that controls the Galois covering (a Coxeter-type group), and then compute the fundamental group of the base (an Artin-type group). The inductive step \( Q = A \zeta^{-1} \) is in § 4.1, where we prove the following statement.

**Theorem** (Cf. Prop. 4.1, Cor. 4.1 and Prop. 4.2). The Weyl-stabiliser of \( B_{Q} \) is a direct product of wreath products of symmetric groups, and the effective quotient \( W_{\mathfrak{gl}_{b}} \) is a direct product of symmetric groups; then \( \Gamma_{\overline{Q}} \) is the subgroup of braids whose underlying permutation lies in \( W_{\mathfrak{gl}_{b}} \), and it is an extension of the latter by a pure braid group.

In the general case instead we introduce a family of trees \( (T, r) \) with some decoration, called ‘ranked’ fission trees, which depend on the choice of the irregular class \( \overline{Q} \) (cf. Def. 4.1, and compare with the unranked fission trees of [DRT, § 5]). Their automorphisms control the Coxeter-type groups in the general type-A case:

**Theorem** (Cf. Thm. 4.1 and Prop. 4.3). The automorphism group \( \text{Aut}(T, r) \) of the ranked fission tree is isomorphic to \( W_{\mathfrak{gl}_{b}} \).

Finally we attach a (full/nonpure) ‘cabled’ braid group \( \mathcal{B}(T, r) \) to any ranked fission tree, in Def. 4.4, with a recursive algorithm (along maximal subtrees): this relies on the operadic composition of the symmetric and braid group operads, extending the pure cabled braid group of [DRT]—which rests in turn on the pure braid group operad.

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1E.g. if \( \mathfrak{g} = GL_{n}(\mathbb{C}) \) we thus consider the coefficients \( A_{i} \) in (3) up to simultaneous permutations of their eigenvalues.
The main result of §4 is that the elements of type-A full/nonpure local WMCGs are precisely such ‘cabled’ braids, in a proof of the multilevel braiding conjecture that appears e.g. in [Ram12] (extending the proof in [DRT]).

**Theorem** (Cf. Thm. 4.2). There is a group isomorphism \( \Gamma_Q \simeq \mathcal{B}(T, r) \), where \((T, r)\) is the ranked fission tree associated with the irregular class \( \overline{Q} \).

All Lie algebras are defined over \( \mathbb{C} \); tensor products are \( \mathbb{C} \)-bilinear.

Some basic notions and conventions, used throughout the body of the paper, are collected in §A, while §B contains the proof of few lemmata. In §C we spell out the relation of wild Riemann surfaces with the much-studied Hamiltonian viewpoint on isomonodromic deformations.

The end of remarks/examples is signaled by a \( \triangle \).

### 1. Full/nonpure local WMCG

Let \( \Sigma \) be a Riemann surface, \( G \) a finite-dimensional connected reductive Lie group over \( \mathbb{C} \), \( g = \text{Lie}(G) \) its Lie algebra, \( T \subseteq G \) a maximal (algebraic) torus, and \( t = \text{Lie}(T) \subseteq g \) the associated Cartan subalgebra. Denote then \( \Phi_g = \Phi(g, t) \subseteq t^* \) the root system of the split Lie algebra \((g, t)\), and \( W_g = N(T)/T \) the Weyl group.

Choose a point \( a \in \Sigma \), and let

\[
Q \in t \otimes \mathcal{F}_{\Sigma, a}, \quad \mathcal{F}_{\Sigma, a} := \mathcal{F}_{\Sigma, a}/\hat{\mathcal{O}}_{\Sigma, a},
\]

be an untwisted irregular type based there, introducing the completed local ring \( \hat{\mathcal{O}}_{\Sigma, a} \) of the surface and its fraction field \( \mathcal{F}_{\Sigma, a} \). Recall if \( z \) is a local coordinate on \( \Sigma \) with \( z(a) = 0 \) then (3) becomes

\[
Q = \sum_{i=1}^{p} A_i z^{-i} \in z^{-1}t[z^{-1}] \simeq t[[z]]/t[[z]],
\]

for suitable coefficients \( A_i \in t \) and for an integer \( p \geq 1 \). (Hereafter we refer to this simply as an ‘irregular type’, always untwisted/unramified.)

As explained in the introduction, the moduli spaces attached to (3) (the de Rham/Betti spaces [Boa]) only depend on the Weyl-orbit of (3), denoted

\[
\overline{Q} \in (t \otimes \mathcal{F}_{\Sigma, a})/W_g.
\]

Here the Weyl group acts on the Cartan subalgebra—and trivially on the other tensor factor; the element (4) defines an irregular class, a.k.a. ‘bare’ irregular type [Boa14, Rem. 10.6] (cf. [BY, BDR] in the twisted case).

If \( Q \) is a ‘starting’ irregular type, then we have associated with it the (universal) admissible deformation space \( B_Q \) in [DRT]. In brief, this is the (connected) complex manifold \( B_Q = \prod_{i=1}^{p} B_{A_i} \), with

\[
B_{A_i} := \bigcap_{d_a < i} \text{Ker}(\alpha) \cap \bigcap_{d_a = i} (t \setminus \text{Ker}(\alpha)) \subseteq t,
\]

where

\[
d_a = \text{ord}(\alpha \circ Q), \quad \alpha \in \Phi_g,
\]

taking the pole order at \( \alpha \in \Sigma \).

Then the pure local wild mapping class group of the wild Riemann surface \((\Sigma, a, Q)\) is \( \Gamma_Q = \pi_1(B_{Q}, Q) \), i.e. the fundamental group of the (pointed) deformation space of \( Q \).

**Remark 1.1** (Terminology). The term ‘pure’ is reminiscent of pure braid groups, which are fundamental groups of configuration spaces of ordered points. Below we will instead consider the ‘full/nonpure’ case, which in turn is an analogue of the
fundamental group for unordered configurations. The latter configuration space arises by modding out the action of a symmetric group (permuting the points): then in our situation this is generalised by an action of the Weyl group.

The picture in the case of simple poles is as follows. Suppose \( O \subseteq g \) is a semisimple adjoint \( G \)-orbit, so that \( O \cap t \subseteq \emptyset \) is nonempty: a marking of \( O \) is the choice of a point \( A \in O \cap t \). In turn \( O \cap t \subseteq t \) then coincides with the \( W_g \)-orbit of the vector \( A \in t \), and the Weyl group acts by forgetting the choice of marking. This is essentially because semisimple orbits correspond bijectively to the quotient \( t/W_g \), cf. [CM93, §2].

Then the topology of each semisimple orbit is determined by the centraliser of its marked point, i.e. the Levi factor of a parabolic subgroup of \( G \), and this yields a partition of \( t/W_g \). In particular there is a bulk consisting of \( t_{reg}/W_g \), i.e. the generic/regular semisimple elements, and its fundamental group is the full/nonpure \( g \)-braid group (cf. §A). In the nongeneric case one instead finds more complicated hyperplane arrangements (possibly noncrystallographic [DRT, Rem]), and more complicated reflection groups; so in turn more complicated fundamental groups.

Finally we take this one step further, to get to the theory of wild/irregular singularities, considering more than semisimple orbits inside \( g \). Rather we consider ‘very good’ orbits inside the (nonreductive) Lie algebra \( g_p = g[z]/z^p g[z] \), for an integer \( p \geq 1 \) as above, cf. [Boa17, §1.5]. Choosing an irregular type means fixing a marking of such orbits, and here we get rid of this choice to get the intrinsic spaces of (local) isomonodromic deformations, whose fundamental group is then the ‘full/nonpure’ (local) wild mapping class group.

Concretely, two deformations of \( Q \) are equivalent if they lie in the same \( W_g \)-orbit inside \( t \otimes \mathcal{R}_{\Sigma,a} \), in which case they define the same irregular class. This leads to admissible deformations of the ‘starting’ irregular class \( Q \), and to the main definition.

**Definition 1.1.** The full/nonpure local wild mapping class group of the (bare) wild Riemann surface \( \Sigma = (\Sigma, a, Q) \) is

\[
\Gamma_{\Sigma} := \pi_1(B_{\Sigma}/Q),
\]

where \( B_{\Sigma} = B_Q/\sim \) is the topological quotient with respect to the equivalence relation

\[
Q_1 \sim Q_2 \text{ if } W_g Q_1 = W_g Q_2 \subseteq t \otimes \mathcal{R}_{\Sigma,a}.
\]

(Hereafter we will simply refer to (7) as the ‘WMCG’, always full/nonpure and local.)

Note \( \Gamma_{\Sigma} \) depends on the root system \( \Phi_g \subseteq t^\vee \), and the tuple of integers \( d = (d_\alpha)_{\alpha \in \Phi_g} \). It is in general larger than its pure counterpart, as some nonclosed paths in \( B_Q \) may become loops in \( B_{\Sigma} \).

**Remark 1.2.** The space \( B_Q \) itself depends on the irregular type \( Q \), not just on the underlying irregular class. However if \( w \in W_g \) then \( B_{wQ} = B_{Q^w} \), and (7) only depends on \( \overline{Q} \).

**Remark 1.3 (Intrinsic definition).** Note one also has

\[
B_Q = \bigcap_{\alpha \in \Phi_g} \left\{ Q' = \sum_{i=1}^p A_i z^{-i} \mid \text{ord}(\alpha \circ Q') = d_\alpha \right\} \subseteq t \otimes \mathcal{R}_{\Sigma,a},
\]
using the notation of (6), and the factorisation (5) follows (cf. [DRT, § 1]). This gives another viewpoint on how the (germs of) meromorphic functions \( q_\alpha = \alpha \circ Q \) determine the space.\(^3\)

In turn, the pole orders of the \( q_\alpha \) are well defined up to local biholomorphisms of \( \Sigma \) which fix the marked point \( a \in \Sigma \). Hence the integers \( d_\alpha \) only depend on the element \( Q \in t \otimes \mathcal{T}_L \), and not on the identifications

\[
\hat{\Omega}_{\Sigma,a} \simeq \mathbb{C}[z], \quad \mathcal{K}_{\Sigma,a} \simeq \mathbb{C}(z), \quad \mathcal{T}_{\Sigma,a} \simeq \mathbb{C}(z)/\mathbb{C}[z].
\]

Hence the deformation spaces \( B_Q \) and \( B_{\Sigma} \) are independent of a choice of local coordinate vanishing at the marked point. \( \triangle \)

Now the Weyl action does not preserve \( B_Q \) in the nongeneric case, i.e. the case where \( \Lambda_r \) is not regular, so we first need to describe the subset

\[
W_q Q \cap B_Q \subseteq W_q Q,
\]

and further understand the Weyl-stabiliser of the irregular type. (This is already visible in the tame case \( p = 1 \), cf. Rk. 1.1 above.)

2. Weyl group fission

2.1. Inductive step. We first consider the case of a single coefficient, i.e. \( Q = A z^{-1} \). In the general case the irregular type is transformed along the diagonal Weyl action on each coefficient.

Choose then \( A \in t \), and let \( h = 3_g(A) \subseteq g \) be the centraliser: it is the (reductive) Levi factor of a parabolic subalgebra of \( g \). The associated deformation space (5) becomes

\[
B_A = \text{Ker}(\Phi_h) \cap \bigcap_{\Phi_h \setminus \Phi_q} (t \setminus \text{Ker}(\alpha)) \subseteq \text{Ker}(\Phi_h),
\]

where

\[
\text{Ker}(\Phi_h) = \{ A \in t \mid \langle \alpha | A \rangle = 0 \text{ for } \alpha \in \Phi_h \} = \bigcap_{\Phi_h} \text{Ker}(\alpha) \subseteq t.
\]

For later use we set \( U := \text{Ker}(\Phi_h) \).

Now if \( w \in \text{Stab}_{W_q}(B_A) \subseteq W_q \) then certainly \( w A \in B_A \), but the converse is true. To state this let \( O_A \subseteq B_A \) be the orbit of \( A \) under the action of \( \text{Stab}_{W_q}(B_A) \); then:

**Lemma 2.1.** One has \( \text{Stab}_{W_q}(U) = \text{Stab}_{W_q}(B_A) \), and \( (W_q A) \cap B_A = O_A \).

**Proof.** The Weyl group permutes the root hyperplanes via

\[
w(\text{Ker}(\alpha)) = \text{Ker}(w \alpha), \quad w \in W_q, \alpha \in \Phi_q,
\]

i.e. along the permutation of the roots. (Recall we identify \( W(\Phi_q) \subseteq \text{GL}(t^\vee) \) and \( W(\Phi_g) \subseteq \text{GL}(t) \), cf. § A.) Hence \( w(B_A) \subseteq B_A \) if and only if \( w \in W_q \) preserves the partition \( \Phi_q = \Phi_h \cup (\Phi_g \setminus \Phi_h) \), by (9). In turn this is equivalent to \( w(\Phi_h) \subseteq \Phi_h \), proving the first statement.

Analogously if \( w A \in B_A \) then \( w \) preserves the above partition, whence the inclusion \( (W_q A) \cap B_A \subseteq O_A \)—and the opposite one is tautological. \( \square \)

Thus the restriction of orbits to the deformation space is controlled by the setwise stabiliser of \( U \subseteq t \).

**Remark 2.1.** The extremal cases are \( A = 0 \), in which case \( U = t \) and \( \text{Stab}_{W_q}(U) = 1 \); and \( A \in \text{reg} \), in which case \( U = \{ 0 \} \) and \( \text{Stab}_{W_q}(U) = W_q \). \( \triangle \)

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\(^3\)These are exponential factors of local fundamental solutions for the linear differential equations associated with the meromorphic connection with irregular type \( Q \), featuring in the Stokes data of the connection.
Now the Weyl group $W_b = W(\Phi_b) \subseteq W_g$ of the Levi factor lies in the setwise stabiliser of $U$, but in general the inclusion is proper. Namely by definition
\[ \operatorname{Ker}(\alpha) = \operatorname{Ker}(\sigma\alpha - 1) \subseteq t, \quad \alpha \in \Phi_g, \]
and the subgroup $W_b$ is generated by the reflections along the hyperplanes of the subsystem $\Phi_b \subseteq \Phi_g$; hence automatically any element of $W_b$ acts trivially on $U = \operatorname{Ker}(\Phi_b)$, i.e.
\[ W_b \subseteq (W_g)_U \subseteq \operatorname{Stab}_{W_g}(U). \]
However it is possible to show the first inclusion is an equality, and more precisely that $W_b$ is the (maximal) parabolic subgroup fixing the given semisimple element $A \in t$.

**Lemma 2.2.** One has $W_b = (W_g)_U = (W_g)_A$.

**Proof.** In principle
\[ (W_g)_U \subseteq (W_g)_B \subseteq (W_g)_A, \]
since $A \in B_A \subseteq U$, so it is enough to show the inclusion $(W_g)_A \subseteq W_b$, i.e. that any element of $W_g$ fixing $A$ lies in the Weyl group of $\Phi_b$.

To this end recall the Lie-group-theoretic definition of the Weyl group is
\[ W_b = N_H(T)/T \subseteq N_G(T)/T = W_g, \]
using the normalisers $N_H(T) \subseteq N_G(T)$ of the given maximal torus, where $H \subseteq G$ is the (connected, reductive) subgroup integrating $h \subseteq g$. Hence an element $w \in W_g$ such that $w(A) = A$ corresponds to an element $g \in N_G(T)$—defined up to the $T$-action—such that $\operatorname{Ad}_g(A) = A$: this means
\[ g \in N_G(T) \cap H = N_H(T), \]
whence $w \in W_b$.

Finally we have an identification $O_A \simeq W_g/A$, introducing the quotient group
\[ (10) \quad W_{g/b} := \operatorname{Stab}_{W_g}(U)/W_b. \]
Denote then $B_{\overline{A}}$ the topological quotient (of $B_A$) of Def. 1.1, in the present case where $Q = Az^{-1}$.

**Proposition 2.1.** The fundamental group $\pi_1(B_{\overline{A}}, O_A)$ is an extension of (10) by $\pi_1(B_A, A)$.

**Proof.** The above discussion yields a homeomorphisms
\[ (11) \quad B_{\overline{A}} \simeq B_A/W_{g/b}, \]
by which we identify $O_A$ as an element of $B_{\overline{A}}$.

Now by construction the $W_{g/b}$-action on $B_A$ is free: the stabilisers of all points are conjugated, and by Lem. 2.2 the stabiliser of the base point is trivial. Moreover the action is automatically properly discontinuous ($W_{g/b}$ is finite), and the spaces involved are Hausdorff; hence every point of $B_A$ has a neighbourhood $O \subseteq B_A$ such that $\overline{W}_1(O) \cap \overline{W}_2(O) \neq \varnothing$ implies $\overline{W}_1 = \overline{W}_2 \in W_{g/b}$.

It follows that the canonical projection $p : B_A \to B_{\overline{A}}$ is a Galois covering, with automorphisms provided by the monodromy action of $W_{g/b}$. The choice of the base point $A \in p^{-1}(O_A) \subseteq B_A$ in the fibre yields an identification $p^{-1}(O_A) \simeq W_{g/b}$ between the torsor and the group, so there is a (principle) fibre bundle
\[ W_{g/b} \to B_A \to B_{\overline{A}}. \]

Then the resulting exact sequence of homotopy groups contains the short sequence
\[ (12) \quad 1 \to \pi_1(B_A, A) \to \pi_1(B_{\overline{A}}, \overline{A}) \to W_{g/b} \to 1, \]
identifying \( \pi_0(\mathcal{W}_{\mathfrak{g}/\mathfrak{h}}) \simeq \mathcal{W}_{\mathfrak{g}/\mathfrak{h}} \) (the connecting map is then the monodromy action at the base point). \(\square\)

**Example 2.1.** For example if \( \mathfrak{g} \in \mathfrak{t}_{\text{reg}} \) then \( \mathcal{W}_{\mathfrak{g}} \) is trivial, so \( \mathcal{W}_{\mathfrak{g}/\mathfrak{h}} = \mathcal{W}_{\mathfrak{g}} \). Hence (11) generalises the (full/nonpure) Hecke proper. On the opposite end, let us set (13) associates to any irregular type

\[
\mathcal{H} = \left\{ \text{Ker}(\alpha) \cap \mathcal{U} = \text{Ker}(\alpha|_{\mathcal{U}}) \mid \alpha \in \Phi_{\mathfrak{g}} \setminus \Phi_{\mathfrak{h}} \right\} \subseteq \mathcal{P}(\mathcal{U}^\prime):
\]

however this is not the case in general.

For example if \( \Phi_{\mathfrak{h}} = \mathfrak{A}_1 \subseteq \mathfrak{A}_2 = \Phi_{\mathfrak{g}} \) then (10) is trivial (see §3), while the reduced arrangement is of type \( \mathfrak{A}_1 \)—so has a Weyl group of order 2. The point is there are reflections of (13) which do not come as restrictions of elements in \( \text{Stab}_{W_{\mathfrak{g}}}(\mathcal{U}) \) (see §4). \(\triangle\)

2.2. **General case.** The results of the previous section can be used inductively to deal with the general case, i.e. with an arbitrary irregular type. Indeed, recall from [DRT] the usage of fission (which is an in the title of [Boa09, Boa14]): this associates to any irregular type \( Q = \sum_{i=1}^{\mathfrak{p}} \mathfrak{A}_i z^{-1} \) an increasing sequence of (Levi) root subsystems

\[
\Phi_{\mathfrak{h}_1} \subseteq \cdots \subseteq \Phi_{\mathfrak{h}_{\mathfrak{p}+1}} := \Phi_{\mathfrak{g}},
\]

as follows.

We consider the sequence of nested centralisers of the coefficients of \( Q \), starting from the leading one. Hence

\[
\mathcal{H}_{\mathfrak{p}} = \left\{ \mathfrak{g} \in \mathcal{G} \mid \text{Ad}_{\mathfrak{g}}(\mathfrak{A}_{\mathfrak{p}}) = \mathfrak{A}_{\mathfrak{p}} \right\} \subseteq \mathcal{G},
\]

which is a connected reductive subgroup. Then further

\[
\mathcal{H}_{\mathfrak{p}-1} = \left\{ \mathfrak{g} \in \mathcal{H}_{\mathfrak{p}} \mid \text{Ad}_{\mathfrak{g}}(\mathfrak{A}_{\mathfrak{p}-1}) = \mathfrak{A}_{\mathfrak{p}-1} \right\} \subseteq \mathcal{H}_{\mathfrak{p}},
\]

which coincides with the set of elements of \( \mathcal{G} \) stabilising both \( \mathfrak{A}_{\mathfrak{p}-1} \) and \( \mathfrak{A}_{\mathfrak{p}} \). Thus in general \( \mathcal{H}_i = \text{Stab}_{\mathcal{G}}(\mathfrak{A}_i, \ldots, \mathfrak{A}_{\mathfrak{p}}) \), for \( i \in \{1, \ldots, \mathfrak{p}\} \), and at the bottom we find the subgroup \( \mathcal{H}_1 \subseteq \mathcal{G} \) stabilising the whole of \( Q \)—which appears in the quasi-Hamiltonian quotients of [Boa14]. By construction \( \mathcal{T} \subseteq \mathcal{H}_1 \), and the inclusion may be proper. On the opposite end, let us set \( \mathcal{H}_{\mathfrak{p}+1} := \mathcal{G} \) for the sake of uniform notation.

**Remark 2.3 (Fission = breaking).** This means the structure group of a principal \( \mathcal{G} \)-bundle is ‘broken’, from \( \mathcal{G} \) down to a reductive subgroup, at any marked point where a connection has an irregular singularity. \(\triangle\)

Now set \( \mathfrak{h}_1 = \text{Lie}(\mathcal{H}_1) \), i.e.

\[
\mathfrak{h}_1 = \left\{ X \in \mathfrak{g} \mid \text{ad}_X(\mathfrak{A}_1) = \cdots = \text{ad}_X(\mathfrak{A}_{\mathfrak{p}}) = 0 \right\} \subseteq \mathfrak{g},
\]

which is a reductive Lie subalgebra. In particular \( \mathfrak{h}_1 \) is the centraliser of \( Q \), and the previous section corresponds to the case where \( \mathfrak{p} = 1 \)—with \( \mathfrak{h} = \mathfrak{h}_1 \). Finally, since \( \mathfrak{t} \subseteq \mathfrak{h}_1 \) (and it is a Cartan subalgebra there), we can consider the corresponding root system \( \Phi_{\mathfrak{h}_1} = \Phi(\mathfrak{h}_1, \mathfrak{t}) \subseteq \Phi_{\mathfrak{g}} \), leading to (14).

Now there is a corresponding filtration of Weyl (sub)groups

\[
\mathcal{W}_{\mathfrak{h}_1} \subseteq \cdots \subseteq \mathcal{W}_{\mathfrak{h}_{\mathfrak{p}+1}} = \mathcal{W}_{\mathfrak{g}},
\]

and one of the main results of [DRT] establishes that the deformation space \( \mathcal{B}_Q \) is a product, with each factor \( \mathcal{B}_{\mathfrak{A}_i} \subseteq \mathfrak{t} \) determined as in (9) by the fission \( \Phi_{\mathfrak{h}_i} \subseteq \Phi_{\mathfrak{h}_{i+1}} \).
(it is the space of admissible deformations of $A_i \in \mathfrak{t}$). Then $W_\mathfrak{g}$ acts diagonally on $B_Q \subseteq t^\mathfrak{p}$.

Now $w(Q) \in B_Q$ means that $w(A_i) \in B_{A_i}$ for $i \in \{1, \ldots, p\}$, and this condition can be described recursively using Lem. 2.1. To this end define a sequence of subgroups

$$W_1 \subseteq \cdots \subseteq W_p \subseteq W_\mathfrak{g},$$

as follows. Set as above $U_i := \text{Ker}(\Phi_{h_i}),$ and then

$$W_p := \text{Stab}_{W_\mathfrak{g}}(U_p), \quad W_{i-1} := \text{Stab}_{W_i}(U_{i-1}) \subseteq W_i, \quad i \in \{2, \ldots, p\}.$$  

Denote then $\mathcal{O}_Q$ the orbit of the irregular type under the action of the smallest group $W_1 \subseteq W_\mathfrak{g}$.

**Lemma 2.3.** One has $W_{i-1} = \text{Stab}_{W_i}(B_{A_i})$ for $i \in \{1, \ldots, p\}$, and $(W_\mathfrak{g} \cap B_Q) \cap B_Q = \emptyset_Q$.

**Proof.** First $w(A_p) \in B_{A_p}$ if and only if $w \in W_p$, and the first statement has been proven in Lem. 2.1—for $i = p$.

Then we can replace $(\mathfrak{h}_p, \mathfrak{g})$ with $(\mathfrak{h}_{p-1}, \mathfrak{h}_p)$, and repeat the same construction: we need $w \in W_p$ such that $w(A_{p-1}) \in B_{A_{p-1}}$, where

$$B_{A_{p-1}} = U_{p-1} \cap \bigcap_{\Phi_{h_p} \setminus \Phi_{h_{p-1}}} (t \setminus \text{Ker}(\alpha)) \subseteq t,$$

by (9). Reasoning as in the proof of Lem. 2.1 this requires $w(\Phi_{h_{p-1}}) \subseteq \Phi_{h_{p-1}}$, which is equivalent to preserving the partition $\Phi_{h_p} = \Phi_{h_{p-1}} \cup (\Phi_{h_p} \setminus \Phi_{h_{p-1}})$, since by (recurrence) hypothesis $w(\Phi_{h_p}) \subseteq \Phi_{h_p}$. Hence $w \in W_{p-1}$ and

$$W_{p-1} = \text{Stab}_{W_\mathfrak{g}}(B_{A_{p-1}}) \subseteq W_p.$$  

Descending until $i = 1$ shows that $w(Q) \in B_Q$ if and only if $w \in \bigcap_i W_i = W_1$, and proves the first statement—inductively. $\square$

Note $W_1 \subseteq W_\mathfrak{g}$ is determined by the flag of kernels

$$U = (t \supseteq U_1 \supseteq \cdots \supseteq U_p),$$

by (15). Indeed consider the (parabolic) stabiliser of (16), within $W_\mathfrak{g}$, i.e.

$$\text{Stab}_{W_\mathfrak{g}}(U) := \bigcap_i \text{Stab}_{W_\mathfrak{g}}(U_i) \subseteq W_\mathfrak{g},$$

which coincides with $W_\mathfrak{g} \cap \text{Stab}_{GL(1)}(U)$; then:

**Lemma 2.4.** One has $W_1 = \text{Stab}_{W_\mathfrak{g}}(U)$.

**Proof.** Postponed to B. $\square$

Thus the restriction of orbits to $B_Q$ is controlled by the action of the setwise Weyl-stabiliser of the kernel flag (16), generalising the inductive step.

**Remark 2.4.** Beware however it is not true in general that $\text{Stab}_{W_\mathfrak{g}}(U_1) = \text{Stab}_{W_\mathfrak{g}}(B_{A_1})$: e.g. for the fission $\varnothing = \Phi_{h_1} \subseteq \Phi_{h_2} \subseteq \Phi_\mathfrak{g}$ one has $U_1 = t$, so $\text{Stab}_{W_\mathfrak{g}}(U_1) = W_\mathfrak{g}$; but

$$B_{A_1} = t \setminus \bigcup_{\Phi_{h_2}} \text{Ker}(\alpha) \subseteq t,$$

which is not stabilised by $W_\mathfrak{g}$ if $h_2 \subseteq \mathfrak{g}$ is a proper Lie subalgebra. $\triangle$

Analogously we can identify the subgroup fixing the irregular type.

**Lemma 2.5.** One has $W_{h_1} = (W_\mathfrak{g} \cap U_1) = (W_\mathfrak{g} \cap Q)$.\footnote{Note in the rightmost identity we consider two different actions of $W_\mathfrak{g}$: the former is an action on $t$, the latter on $t \otimes \mathcal{F}_{\Sigma, \alpha}$.}
Proof. By definition \( w(Q) = Q \) if and only if \( w(A_i) = A_i \) for \( i \in \{1, \ldots, p\} \), i.e. \( w \in \bigcap_{i} (W_{\theta})_{A_i} \).

Now the argument of Lem. 2.2 yields the inductive step for the proof of the identity
\[
\bigcap_{j \leq i \leq p} (W_{\theta})_{A_i} = (W_{\theta})_{(A_1, \ldots, A_p)} = W_{b_1} \subseteq W_{\theta}, \quad j \in \{1, \ldots, p\},
\]
whence
\[
(W_{\theta})_Q = W_{b_1} \subseteq W_{\theta}.
\]

On the other hand
\[
W_{b_1} = \bigcap_i W_{b_i} \subseteq \bigcap_i (W_{\theta})_{u_i} = (W_{\theta})_{u_i},
\]

since \( U_1 = \bigcup_i U_i \subseteq t \), and \( W_{b_i} \) acts as the identity on \( U_i = \text{Ker} \Phi_{b_i} \).

Finally if \( w \) acts as the identity on \( U_1 \) then it also fixes (pointwise) \( B_{A_i} \subseteq U_1 \) for \( i \in \{1, \ldots, p\} \), thus
\[
(W_{\theta})_{U_1} \subseteq (W_{\theta})_{B_Q} \subseteq (W_{\theta})_Q,
\]
proving the remaining inclusion. \( \square \)

In particular by Lemmata 2.4 and 2.5 we also have an inclusion \( W_{b_1} \subseteq W_i \), since
\[
W_{b_1} = (W_{\theta})_{U_i} \subseteq \text{Stab}_{W_{\theta}}(U_1), \quad j \in \{i, \ldots, p\},
\]
as \( U_i = \bigcup_{j \geq i} U_j \subseteq t \).

It follows that \( W_{b_1} \) is a normal subgroup of \( \text{Stab}_{W_{\theta}}(U_1) \), hence a fortiori of (17), and we consider again the quotient group
\[
(W_{\theta})_Q = W_{b_1} \subseteq W_{\theta},
\]

Note the numerator of (18) depends on the whole sequence \( h = (h_1, \ldots, h_p) \), while the denominator only depends on the last term—the pointwise stabiliser of a flag/filtration only depend on its union/sum, contrary to the setwise stabiliser.

**Example 2.2 (Complete fission and generic case).** In particular if the fission is ‘complete’, which means that \( H_1 = \text{Stab}_{G}(Q) = T \subseteq G \) is the maximal torus, then \( \Phi_{h_1} = \emptyset \); in this case \( U_1 = t \) and \( W_{b_1} \) is trivial, so \( W_{g/b} \simeq \text{Stab}_{W_{\theta}}(U) \).

If further \( A_p \in t_{\text{reg}} \), then \( U \) is stationary at \( t \), and \( \text{Stab}_{W_{\theta}}(U) = W_{\theta} \). \( \triangle \)

Finally by construction there is a topological identification \( B_{Q} \simeq B_{Q}/W_{g/b} \), and the same argument of the proof of Prop. 2.1 yields the following.

**Theorem 2.1.** The projection \( B_{Q} \twoheadrightarrow B_{Q} \) is a Galois covering, and \( \Gamma_{Q} \) is an extension of \( W_{g/b} \) by \( \Gamma_{Q} \).

Of course if \( W_1 = W_{b_1} \), then \( B_{Q} = B_{T} \), in which case the WMCG is pure (and has been studied in [DRT]).

3. Low-rank examples

Analogously to the pure case, we provide examples of WMCGs for low-rank Lie algebras, after proving we can reduce to the simple case.
3.1. **Reduction to the simple case.** Suppose \( g = \bigoplus_i \mathfrak{z}_i \) is a decomposition of \( g \) into mutually commuting ideals. Choose then a root subsystem \( \Phi \subseteq \Phi_g \).

Introduce \( t_i := t \cap \mathfrak{z}_i \) (a Cartan subalgebra of \( \mathfrak{z}_i \)), and let \( \Phi_{\mathfrak{z}_i} = \Phi(t_i, t_i) \subseteq \Phi_g \) be the associated root system; this way there are two other decompositions:

\[
t = \bigoplus_i t_i, \quad \Phi_g = \bigoplus_i \Phi_{\mathfrak{z}_i}.
\]

Further let \( \Phi^{(i)} := \Phi \cap \Phi_{\mathfrak{z}_i} \), which is a root subsystem of \( \Phi_{\mathfrak{z}_i} \).

Then one can show [DRT] the deformation space \((9)\) splits as a product \( B_\Theta = \prod_i (B_{\Theta_i}) \), where

\[
(\text{B}_\Theta)_i = \operatorname{Ker}(\Phi^{(i)}) \cap \bigcap_{\Phi_{\mathfrak{z}_i} \setminus \Phi^{(i)}} (t_i \setminus \operatorname{Ker}(\alpha)) \subseteq t_i.
\]

Finally the Weyl group also splits as a product \( W_\Theta = \prod_i W_{\mathfrak{z}_i} \), where the \( i \)-th factor (the Weyl group of \( \Phi_{\mathfrak{z}_i} \)) acts trivially on the complementary direct summands [Bou68, Ch. 6, §1.2]. It follows that every \( W_\Theta \)-orbit (inside \( t \)) splits as a product of \( W_{\mathfrak{z}_i} \)-orbits (inside \( t_i \)), so the previous discussion of setwise/pointwise stabilisers can be carried over factorwise, and:

**Corollary 3.1.** The deformation space \( B_\Theta / \Sigma \) decomposes as a (topological) product \( \prod_i (B_{\Theta_i}) / \Theta_i \), where \( (B_{\Theta_i})_i \) is the topological quotient of \( (19) \) with respect to the equivalence relation \( (8) \) — with \( W_\Theta \), replacing \( W_\Theta \).

In particular the factor corresponding to the centre \( Z_\Theta \subseteq \Theta \) is contractible, and can be removed; and further if \( g \) is semisimple then the WMCG is a direct product of the groups associated with its simple ideals.

**Hereafter we thus assume that \( g \) be simple!**

**Remark 3.1.** In this case all Cartan subalgebras \( \mathfrak{t} \subseteq \mathfrak{g} \) are conjugated by (inner) Lie-algebra automorphisms of \( \mathfrak{g} \), which in turn induces homeomorphisms of the resulting deformation spaces \( B_\Theta \). Hence \( \Gamma_\Theta \) does not depend on the choice of the Cartan subalgebra, and so in turn neither does \( \Gamma_\Theta \). △

3.2. **Rank one.** If \( \mathrm{rk}(\mathfrak{g}) = 1 \) then the only possible nontrivial fission \( \Phi_h \subseteq \Phi_g \) is the ‘generic’ one \( \emptyset \subseteq \Phi_g \), so \( \Gamma_\Theta \) is either trivial or isomorphic to the \( g \)-braid group. This is of type \( A_1 \), i.e. the braid group on 2-strands, and \( (12) \) becomes

\[
1 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1,
\]

with \( \Gamma_\Theta \simeq \mathbb{Z} \simeq \Gamma_\Theta \), matching up with a particular case of \( (35) \):

\[
1 \rightarrow \text{PBr}_2 \rightarrow \text{Br}_2 \rightarrow S_2 \rightarrow 1.
\]

Equivalently up to homotopy we have \( B_\Theta \simeq S^1 \), and the arrow \( B_\Theta \rightarrow B_\Theta \) is the 2-sheeted covering of the circle over itself.

3.3. **Rank 2.** Suppose now \( \mathrm{rk}(\mathfrak{g}) = 2 \): since \( g \) is simple then \( \Phi_g \) is isomorphic to \( A_2, B_2/C_2 \) or \( G_2 \), and \( W_\Theta \) is isomorphic to \( \text{Dih}_3 \simeq S_3, \text{Dih}_4 \) or \( \text{Dih}_6 \) respectively (i.e. the symmetries of a triangle, a square, or a hexagon). Here \( \text{Dih}_n \) denotes the dihedral group of order \( 2n \), for an integer \( n \geq 1 \)—i.e. we use the ‘geometric’ convention rather than the ‘algebraic’ one.

Let \( \Delta_g \subseteq \Phi_g \) be a base of simple roots. The generic fission is \( \emptyset \subseteq \Phi_g \), in which case we obtain the \( g \)-braid group, while the nongeneric (incomplete) fission is \( \Phi_h \subseteq \Phi_g \), with \( \Phi_h = \{ \pm \theta \} \) for some \( \theta \in \Delta_g \); this corresponds to the deformation space \( B_\Theta = \mathbb{C} \setminus \{ 0 \} \). With the usual notation we find \( U = \operatorname{Ker}(\theta) \) and \( W_\Theta \simeq \mathbb{Z}/2\mathbb{Z} \), and we must describe \( \text{Stab}_{W_\Theta}(U) \subseteq W_\Theta \)—acting on \( B_\Theta \). This is the same as the
setwise stabiliser of the line $C \theta \subseteq t'$ for the dual action, and the difference among the three types is due to the parity of the corresponding dihedral group.

Namely for type A the Weyl group yields the standard permutation action of $S_3$ on $C^3 \supseteq t'$—identified with the standard dual Cartan subalgebra for $\mathfrak{gl}(C) \supseteq \mathfrak{sl}(C)$. Then the only nontrivial permutation fixing the line generated by either simple root is the associated (simple) reflection. It follows that $\text{Stab}_{W_g}(U) = W_h$, so (10) is trivial and the WMCG is pure: it is thus infinite cyclic.

For type B the long roots are vertices of a square centered at the origin of $t' \simeq \mathbb{R}^2$, while the short roots are vertices of a smaller square obtained by taking midpoints of each side:

![Square Diagram]

The Weyl group acts by preserving both squares, and operates as the group of their symmetries. In both cases a diagonal is fixed by the subgroup generated by the (simple) reflection along the corresponding axis, but also by a rotation of $\pi$. This means the stabiliser is always the Klein four-group $K_4 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, hence (10) becomes

$$W_{gh} \simeq \mathbb{Z}/2\mathbb{Z},$$

acting as the antipode on the punctured plane, and $B_Q \rightarrow B_{\mathfrak{Q}}$ is again a two-sheeted covering of the circle onto itself (up to homotopy equivalence). In particular $\Gamma_Q$ is infinite cyclic.

Finally type G yields an analogous situation. Long/short roots assemble into two Weyl-invariant hexagons in the real plane, and the action of the Klein group (within $\text{Dih}_6$) fixes any given diagonal within each hexagon:

![Hexagon Diagram]

Then we can extend to the complete (nongeneric) fission $\varnothing = \Phi \subseteq \Phi_h \subseteq \Phi_g$, with the middle term as above. The associated kernel flag is $U = (t \supset t \supset \ker(\theta))$, so the setwise stabiliser stays the same; but this time the irregular type is centralised by the maximal torus only, so $W_{gh}$ will either have order 2 (for type A), or be isomorphic to the Klein group (for type B/C and G). The result is a covering

$$\mathbb{C}^* \times \mathbb{C}^* \simeq B_Q \rightarrow B_{\mathfrak{Q}}, \quad \mathbb{C}^* = \mathbb{C} \setminus \{0\},$$

with either 2 or 4 sheets, and $\Gamma_Q$ is an extension of the monodromy group by $\Gamma_{\mathfrak{Q}} \simeq \mathbb{Z}/2\mathbb{Z}$.

4. Type A

In this section we will explicitly describe WMCGs for the special/general linear Lie algebras, in full generality, building on [DRT].

Let $n \geq 2$ be an integer and $\mathfrak{g} = \mathfrak{sl}_n(C)$. The Weyl group $W_g \simeq S_n$ acts naturally on $V \equiv \mathbb{C}^n$, and we will use the vector representation $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V) \simeq \mathfrak{gl}_n(C)$. 
Remark 4.1 (General linear description). Using the basis we identify $V$ with the standard Cartan subalgebra of $\mathfrak{g}(V)$, so $t = V \cap \mathfrak{g}$ (the standard Cartan subalgebra of $\mathfrak{g}$) becomes the subspace of $n$-tuples of points of the complex plane with vanishing barycentre.

The resulting inclusion $\mathbb{C}^{n-1} \simeq t \hookrightarrow V$ induces a homotopy equivalence

$$t_{\text{reg}} \simeq \text{Conf}_n = \mathbb{C}^n \setminus \bigcup_{1 \leq i < j \leq n} H_{ij},$$

using the notation of (33), which moreover is compatible with the Weyl group action. Hence there is a second homotopy equivalence

$$t_{\text{reg}}/W_\Phi \simeq \text{UConf}_n,$$

in the notation of (34), and whenever useful we will work within the general linear Lie algebra. △

4.1. Inductive step. If $\Phi_h \subseteq \Phi_\mathfrak{g} = \Lambda_{n-1}$ is a Levi subsystem, we have an associated $J$-partition $\underline{n} = \prod_{j \in J} I_j$, and

$$\Phi_h \simeq \bigoplus_{j \in J} A_{|I_j|-1} \subseteq \Lambda_{n-1},$$

with the usual convention that $A_0 = \emptyset$ (cf. [DRT] and § A). Namely for $i \in \underline{n}$ we set

$$I_i := \{i\} \cup \{ j \in \underline{n} \mid \pm \alpha_{ij} \in \Phi_h \} \subseteq \underline{n}, \quad \alpha_{ij} = e_j^\vee - e_i^\vee \in t^\vee,$$

where $e_j^\vee \in V^\vee$ is an element of the canonical dual basis, and

$$J := \{ \min(I_i) \mid i \in \underline{n} \} \subseteq \underline{n}.$$

The Weyl group of $\mathfrak{h}$ thus comes with a natural factorisation

$$W_\mathfrak{h} \simeq \prod_{j \in J} S_{I_j} \subseteq S_{\underline{n}} = W_\mathfrak{g},$$

with trivial factors corresponding to the trivial components of $\Phi_\mathfrak{h}$. The setwise stabiliser of $U = \ker(\Delta_\mathfrak{h}) \subseteq t$ is bigger in general, since we can also permute components of $\Phi_\mathfrak{h}$ of the same rank.

To state this precisely consider two nonempty finite sets $I$ and $K$, and suppose $I = \bigsqcup_{k \in K} I_k$ is a $K$-partition of $I$ with parts $I_k \subseteq I$ of equal cardinality $m \geq 1$—so $|I| = m|K|$. Then the symmetric group $S_I$ contains the subgroup

$$N := \{ \tau \in S_I \mid \tau(I_k) \subseteq I_k \text{ for } k \in K \} \simeq (S_m)^{|K|},$$

which stabilises all parts (and permutes their elements). If $I$ has a total order then there is a ‘complementary’ subgroup $P \subseteq S_I$, which permutes all parts (fixing their elements): more precisely, if $I_k = (i^{(k)}_1, \ldots, i^{(k)}_m) \subseteq I$ for $k \in K$, then any element of $\sigma \in S_K \simeq P$ acts as

$$\sigma : i^{(k)}_j \mapsto i^{(\sigma_k)}_j \in I, \quad j \in \{1, \ldots, m\}.$$

By construction $N \cap P = 1$ inside $S_I$, and $P$ acts on $N$ by conjugation.

Lemma 4.1 (Cf. [Yau], Lem. 3.2.8). If $\tau = \prod_{k} \tau^{(k)} \in N$, with $\tau^{(k)} \in S_{I_k} \simeq S_m$, then

$$\sigma \tau \sigma^{-1} = \prod_k \tau^{(\sigma_k^{-1})} \in N,$$

$$\sigma \in P.$$
Hence we have an inner semidirect product $P \rtimes N \subseteq S_1$, and it follows that $(P \rtimes N)/N \cong P$ canonically. Equivalently the outer semidirect product of $P$ and $N$, with respect to the action (20), comes with a natural group embedding $P \rtimes N \hookrightarrow S_1$. This latter is also the wreath product $S_K \wr S_m \hookrightarrow S_1$, cf. §A.5.

Remark 4.2. One has
\[ |P \rtimes N| = (m!)^{|K|}/|K|! \leq (m|K|)! = |S_1|, \]
with strict inequality if $1 < m < |I|$. Thus the embedding $P \rtimes N \hookrightarrow S_1$ is proper for nontrivial partitions of $I$.

Let us apply this to the present situation: for an integer $i \geq 1$ denote
\[ K_i = \left\{ j \in J \left| |I_j| = i \right. \right\} \subseteq J. \]
If $i \geq 2$, the integer $|K_i| \geq 0$ is thus the multiplicity of $A_{i-1}$ as an irreducible component of $\Phi_h$. Instead for $i = 1$ one has a natural bijection
\[ K_1 \to \left\{ i \in \mathbb{N} \left| \pm \alpha_i^j \notin \Phi_h \text{ for any } j \right. \right\}, \quad I_j = \{ i \} \mapsto i. \]
The subgroups $P_i \cong S_{K_i}$ and $N_i \cong (S_i)^{|K_i|}$ of $S_j$ are defined as above—note $J$ inherits a natural total order as a subset of $\mathbb{N}$.

Proposition 4.1. There is a canonical group isomorphism
\[ (21) \quad \text{Stab}_W(g) \cong \prod_{i \geq 0} (S_{K_i} \wr S_i) \subseteq W_g. \]

Proof. The statement is the algebraic rewriting of the following claim: the setwise stabiliser of $U = \ker(\Delta_h)$ is the subgroup of $W_g = S_n$ that permutes parts $I_j \subseteq \mathbb{N}$ of the same cardinality, and that further permutes the elements within each part. By the above discussion this yields the direct product (21)—as permutations of disjoint parts commute.

To prove the claim, recall the `extended’ kernel $\widetilde{\ker}(\Delta_h) \subseteq V$ (in the general linear case) is defined by the condition that the coordinate of any vector are equal within each part $C^i \subseteq V$, so its setwise stabiliser is given by the above condition. Thus to conclude it is enough to show that the setwise Weyl-stabiliser of the ‘essential’ kernel $U = \widetilde{\ker}(\Delta_h) \cap t$ (in the special linear case) is the same; but by construction $\ker(\Delta_h) = U \oplus \mathbb{C}1 \subseteq \mathfrak{g}(V)$, and the centre is fixed (pointwise) by $W_g$ (cf. Rem. 4.1). \hfill \Box

Corollary 4.1. One has $W_{g|\mathfrak{h}} \cong \prod_{i \geq 0} S_{K_i}$.

Proof. By definition $W_{g|\mathfrak{h}}$ is the quotient (10), which is readily computed in this case using Prop. 4.1 and the factorisation
\[ (22) \quad W_h \cong \prod_{i \neq 1} (S_i)^{|K_i|}. \]

Remark 4.3. This corresponds to the fact that $W_{g|\mathfrak{h}}$ is naturally identified with the subgroup permuting parts of equal cardinality, fixing elements within each part. In particular the exact group sequence
\[ 1 \longrightarrow W_h \longrightarrow \text{Stab}_W(U) \longrightarrow W_{g|\mathfrak{h}} \longrightarrow 1 \]
splits. (This underlies a more general statement about reflection groups.)

\(^{5}\)This is a particular example of application of the operadic composition of the symmetric group operad, cf. [Yau, §3.1] and see below.
Further (22) is naturally a subgroup of the Weyl group of the ‘reduced’ root system
\[ \Phi_{\hat{\rho}U} = \{ \alpha|_{\hat{\rho}} \mid \alpha \in \Phi_{\hat{\rho}} \} \simeq \Lambda_{\hat{\rho}U}, \]
viz. a subgroup of ‘admissible’ permutations—inside \( F \).

On the whole there is a Galois covering \( B_Q \to B_T \) with \( \prod_{i \geq 0}(|K_i|!^\times) \) sheets, and to go further let us work within \( gl(V) \supseteq g \). Recall from [DRT] that there is a canonical vector space isomorphism \( \tilde{U} := \ker(\Delta_h) \simeq C^J \), and by Rem. 4.1 the \( W_g \)-equivariant inclusion \( U \hookrightarrow \tilde{U} \) yields homotopy equivalences
\[ B_Q \simeq \text{Conf}|J|, \quad B_Q/W_g \simeq U\text{Conf}|J|, \]
with fundamental groups \( \text{PBr}|J| \) and \( \text{Br}|J| \), respectively. What we have here is an ‘intermediate’ covering, since it is only \( W_g \) that acts (freely) on \( B_Q \).

To simplify the notation let us then contemplate the following abstract situation. For an integer \( d \geq 1 \) consider the ordered configuration space \( Y_d := \text{Conf}d \subseteq C^d \), as well as an \( I \)-partition \( \varphi : d \to I \) with parts \( I_i := \varphi^{-1}(i) \subseteq d \) for \( i \in I \). Then there is a natural group embedding
\[ S_\varphi := \prod_I S_{I_i} \hookrightarrow S_d, \]
obtained by juxtaposing permutations, and we let \( X_\varphi := Y_d/S_\varphi \) (the ‘semiordered’ configuration space): this is the space of configurations of \( d = \sum |I_i| \) points in the complex plane, such that 2 of them are indistinguishable if they lie within the same part of the \( I \)-partition.

To identify the fundamental group recall there is an ‘augmentation’ group morphism \( p_d : \text{Br}_d \to S_d \) with kernel \( \pi_1(Y_d) = \text{PBr}_d \subseteq \text{Br}_d \).

**Proposition 4.2.** There is a group isomorphism
\[ \pi_1(X_\varphi) \simeq \text{Br}_\varphi \subseteq \text{Br}_d, \quad \text{Br}_\varphi := p_d^{-1}(S_\varphi), \]
and \( \text{Br}_\varphi \) is an extension of \( S_\varphi \) by \( \text{PBr}_d \).

Here \( \text{Br}_\varphi \) is thus the ‘semipure’ braid group of the partition, i.e. the group of braids whose underlying permutation lies within \( S_\varphi \subseteq S_d \).

**Proof.** There are Galois coverings \( Y_d \to X_d := U\text{Conf}_d \) and \( Y_d \to X_\varphi \), and it follows the induced map \( X_\varphi \to X_d \) is a covering (with \( [S_d : S_\varphi] \) sheets). Up to identifying groups and torsors after a suitable choice of base points, this yields a commutative diagram of pointed topological spaces, with (principle) fibre bundles in each row:
\[
\begin{array}{ccc}
S_\varphi & \longrightarrow & Y_d \\
\downarrow & & \downarrow \\
S_d & \longrightarrow & X_d
\end{array}
\]

In turn this leads to a morphism of (short) exact group sequences, proving the statement:
\[
\begin{array}{ccc}
1 & \longrightarrow & \text{PBr}_d \\
\downarrow & \text{Br}_d & \downarrow p_d \\
\pi_1(X_\varphi) & \longrightarrow & S_\varphi \\
\downarrow & & \downarrow \\
1 & \longrightarrow & 1
\end{array}
\]

Note indeed \( \text{Ker}(p_d|_{\text{Br}_\varphi}) = \text{Ker}(p_d) \cap \text{Br}_\varphi = \text{PBr}_d \). □
In our situation we thus find a group isomorphism
\[ \pi_1(B_{\mathbf{FR}}) \simeq \text{Br}_\varphi \subseteq \text{Br}_{|J|}, \]
where \( \varphi : J \to I \subseteq \mathbb{Z}_{\geq 0} \) is the \( I \)-partition obtained from \( J = \coprod_{i \geq 0} K_i \) by removing the empty parts.

**Remark 4.4.** The extreme cases are \( S_\varphi = 1 \), where \( X_\varphi = Y_d \), and \( S_\varphi = S_d \), where \( X_\varphi = X_d \) is the (fully) unordered configuration space. In our setting this means either no two irreducible components of \( \Phi_h \) have the same rank, or conversely they all have the same rank—respectively.

**Remark 4.5.** There is also a different subgroup of \( \text{Br}_d \) associated with the partition and projecting onto \( S_\varphi \), namely \( \prod_1 \text{Br}_1 \hookrightarrow \text{Br}_d \): this is the subgroup obtained by juxtaposing \(|I|\) braids, each on \(|I|\) strands. However in general the inclusion \( \prod_1 \text{Br}_1 \subseteq \text{Br}_\varphi \) is proper. E.g. \( \text{Br}_1 \times \text{Br}_1 \subseteq \text{Br}_2 \) is trivial, while
\[ p_2^{-1}(S_1 \times S_1) = p_2^{-1}(1) = \text{PBr}_2 \simeq \mathbb{Z}. \]

This simple example shows the fundamental group of the semiordered configuration space is not just the direct product of the corresponding braid groups. Indeed it is possible two points in different parts braid across each other (along a loop in \( X_\varphi \)), provided the are not swapped by the underlying permutation of the overall braid.

**Remark 4.6.** By the Galois correspondence, the isomorphism class of the covering \( X_\varphi \to X_d \) matches up with the conjugacy class of a subgroup of \( \text{Br}_d = \pi_1(X_d) \).

This is precisely the conjugacy class of \( \text{Br}_\varphi \subseteq \text{Br}_d \), which is generically nontrivial, as \( S_\varphi \subseteq S_d \) is generically not a normal subgroup.

### 4.2. General case: ranked fission trees

Suppose now to have an increasing filtration
\[ \Phi_{h_1} \subseteq \cdots \subseteq \Phi_{h_p} \subseteq \Phi_{h_{p+1}} := A_{n-1}. \]
of Levi subsystems. As in [DRT] this corresponds to a ‘fission’ tree \( T = (T_0, \Phi) \) of height \( p \geq 1 \) (cf. § A). The set \( J_1 = J_{h_1} \) is as above, for \( l \in p_2 \) and then we add a tree root at level \( p + 1 \). By definition \( \Phi(i) = j \in J_{h_{i+1}} \) means that the irreducible component of \( h_1 \subseteq h_{i+1} \) corresponding to \( i \in J_1 \) lies within the irreducible component of \( h_{i+1} \) corresponding to \( j \).

This was enough to encode \( \Gamma_Q \), while in the full/nonpure case we must retain more data, according to the results of the previous section.

**Definition 4.1 (Ranked fission tree).** A ranked fission tree is a fission tree \( T = (T_0, \Phi) \) equipped with a rank function \( r : T_0 \to \mathbb{Z}_{\geq 1} \), i.e. a function satisfying
\[ r(i) = \sum_{\Phi(i) = j} r(j), \quad i \in T_0. \]

Then \( r(T) := r(\ast) \geq 1 \) is the rank of the tree.

This means to each node we attach a positive rank, which equals the sum of its child-nodes'. In particular \( \sum_{J_1} r(i) = r(T) \), independently of the level, and the rank function is determined by assigning ranks to the leaves—i.e. by \( r|_{J_1} \in \mathbb{Z}_{\geq 1}^{|J_1|}. \)

The algorithm to associate a ranked fission tree \((T, r)\) to (14) is the following: the underlying fission tree is constructed as in [DRT], and we further set \( r(i) = k + 1 \) if the node \( i \in J_1 \) corresponds to a type-A irreducible rank-k component of \( \Phi_{h_{i+1}} \). Working within the general linear Lie algebra, this is the same as setting \( r(i) = k \)

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*All spaces involved are (locally) path-connected and semi-locally simply-connected [Hat02, Thm. 1.38].*
if $i$ corresponds to an irreducible component isomorphic to $\Phi_{gl_n(C)}$—including $\Phi_{gl_1(C)} = \emptyset$. It follows that $r(T) = n$ if we work within $gl_n(C)$.

By construction the Weyl group of $h_1 \subseteq g$ comes with a canonical group isomorphism

$$W_{h_1} \simeq \prod_{i \in J_1} S_{r(l)}, \quad l \in p,$$

and to construct the stabiliser of the kernel flag in terms of the tree we introduce the following.

**Definition 4.2.** An *isomorphism* $(T_0, \Phi, r) \rightarrow (T'_0, \Phi', r')$ of ranked fission trees is a bijection $f : T_0 \rightarrow T'_0$ matching roots, and such that there are commutative diagrams:

$$\begin{array}{ccc}
T_0 \setminus \{\ast\} & \xrightarrow{f} & T'_0 \setminus \{\ast\} \\
\Phi & \downarrow & \Phi' \\
T_0 & \xrightarrow{f} & T'_0
\end{array}$$

and

$$\begin{array}{ccc}
T_0 & \xrightarrow{f} & T'_0 \\
\{\ast\} & \searrow & \{\ast\} \\
& \nearrow & \\
T_0 & \xrightarrow{f} & T'_0
\end{array}.$$

An *automorphism* of $(T_0, \Phi, r)$ is an isomorphism $(T_0, \Phi, r) \rightarrow (T_0, \Phi, r)$; their group is denoted $\text{Aut}(T, r)$.

This restricts the usual notion of isomorphism of (rooted) trees, by further asking that ranks be preserved. Note by definition an automorphism preserves the nodes at each level, and is uniquely determined by the image of the leaves.

### 4.3. General case: reflection groups

It is possible to compute the automorphism group of the tree recursively, and in turn this will control the monodromy action of the Galois covering $B_Q \rightarrow B_{\mathcal{T}} = B_Q / W_{gl}.$

Choose then a ranked fission tree $(T, r)$, and note its subtrees are equipped with restricted rank functions. In particular let $\mathcal{T} = \mathcal{T}(T, r)$ be the set of (ranked) maximal proper subtrees, i.e. the subtrees of $T$ rooted at each child-node of the root, and choose a complete set of representatives $\mathcal{J} \subseteq \mathcal{T}$ of isomorphism classes. Finally denote $n(t) \geq 1$ the cardinality of the isomorphism class of any maximal proper subtree $t \in \mathcal{T}$.

**Definition 4.3.** The *extended* automorphism group $\tilde{\text{Aut}}(T, r)$ of the ranked fission tree $(T, r)$ is defined recursively by

$$\tilde{\text{Aut}}(T, r) = \prod_{t \in \mathcal{J}} S_{n(t)} \cdot \tilde{\text{Aut}}(t, r_0), \quad r_0 := r|_{t_0},$$

with basis $\tilde{\text{Aut}}(i, r(i)) := S_{r(i)}$ for $i \in J_1$.

A priori (24) depends on the choice of $\tilde{\mathcal{J}} \subseteq \mathcal{J}$, but the following identification in particular shows it does not.

**Theorem 4.1.** One has $\tilde{\text{Aut}}(T, r) = \text{Stab}_{W_{gl}}(U)$. Further the automorphism group $\text{Aut}(T, r)$ is obtained recursively as in (24) but with recursion basis $\text{Aut}(i, r(i)) := S_i$ for $i \in J_1$—the trivial group.

**Proof.** The first item can be proven by induction on $p \geq 1$. If $p = 1$ then a maximal proper subtree is a leaf, so $\mathcal{J} = J_1$; then two leaves are isomorphic (as ranked trees)

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7This choice is more natural since the (nonsemisimple) rank of the general linear Lie algebra controls the Weyl/braid groups without shifts.
if and only if they have the same rank. Hence for $i \in J_1$ the integer $n(i) \geq 1$ is the number of rank-$r(i)$ leaves, and in this case

$$\widetilde{\text{Aut}}(T, r) = \prod_{i \in J_1} S_{n(i)} \ltimes S_{r(i)} \subseteq S_{r(T)},$$

where $\tilde{J}_1 \subseteq J_1$ is a set of representatives of leaves—of all possible ranks. The result follows from Prop. 4.1.

Now let $p \geq 2$. By the induction hypothesis, the bases of the wreath products in (24) are the setwise stabilisers of the deformation space of the ‘sub-irregular types’ corresponding to the eigenspaces of the leading coefficient. In addition to that, we are then permuting isomorphic maximal proper subtrees, i.e. eigenspaces of the leading coefficient whose nested decomposition (into eigenspaces for the subleading coefficients) plays a symmetric role: this yields the whole of $\text{Stab}_W(\{U\})$, as any other permutation of the eigenvalues of the leading coefficient moves the irregular type out of the space of admissible deformations.

The second item is a straightforward extension from the unranked case, and can also be proven recursively on $p \geq 1$. If $p = 1$ an automorphism is the data a permutation of the leaves which matches up ranks. Hence in this case

$$\text{Aut}(T, r) = \prod_{i \in \tilde{J}_1} S_{n(i)} = \prod_{i \in J_1} S_{n(i)} \ltimes S_1,$$

using the above notation. (Note $S_{n(i)} \subseteq S_{J_1}$ is naturally identified with the symmetric group of rank-$r(i)$ leaves.)

Now let $p \geq 2$. By the induction hypothesis the bases of the wreath products in (24) are the automorphism groups of the maximal proper ranked subtrees. In addition to that, we are then permuting isomorphic maximal proper subtrees: this yields the whole of $\text{Aut}(T, r)$, as any other permutation of child-nodes of the root, bringing along the corresponding subtrees, cannot restrict to an isomorphism of these latter.

\begin{example}
The most symmetric example is that in which $r$ is constant at each level. In this case any automorphism of the underlying tree $T = \langle T_0, \phi \rangle$ preserves the rank function.

If moreover $T$ is a complete $m$-ary tree, viz. if all interior nodes have $m \geq 1$ child-nodes, then simply

$$\text{Aut}(T, r) \simeq S_m \ltimes \cdots \ltimes S_m,$$

the $p$-fold wreath power—recall this example of wreath product is associative. The extended group instead is

$$\widetilde{\text{Aut}}(T, r) \simeq (S_m)^p \ltimes S_r,$$

where $r \geq 1$ is the rank of any leaf.

On the opposite end $r$ is injective at each level, so the group $\text{Aut}(T, r)$ is trivial, and $\widetilde{\text{Aut}}(T, r) \simeq \prod_{i \in J_1} S_{r(i)} \subseteq S_{r(T)}$. In this case the WMCG is pure. \end{example}

Now by (recursive) construction $\text{Aut}(T, r) \subseteq \widetilde{\text{Aut}}(T, r)$ is a subgroup, so by Thm. 4.1 it can be identified with a subgroup of the kernel-flag stabiliser.

Indeed choose $f \in \text{Aut}(T, r)$, so by definition $f : T_0 \rightarrow T_0$ yields rank-preserving permutations $f_l : f_{l_0} \in \tilde{J}_l$ of the nodes at each level $l \in p + 1$. In particular $f_1$ permutes subsets of leaves (of constant rank), and we can map it to an element of
\( W_\theta \simeq S_{\tau(T)} \) along the group embedding
\[
\prod_{J_1} S_{n(t)} \hookrightarrow \prod_{J_1} S_{n(t)} \cap S_{\tau(T)},
\]
keeping the notation of the proof of Thm. 4.1. This yields an injective group morphism \( \tilde{\tau} : \text{Aut}(T, r) \hookrightarrow \text{Stab}_{W_\theta}(U) \) — since \( f \) is determined by \( f_1 \).

By construction the image of \( \tilde{\tau} \) is disjoint from \( W_{h_1} = \prod_{J_1} S_{\tau(t)} \) (using (23)), and acts on it by conjugation, so there is a second group embedding
\[
(25) \quad \tilde{\tau} : \text{Aut}(T, p) \ltimes W_{h_1} \longrightarrow \text{Aut}(T, r).
\]

Proposition 4.3. The group morphism (25) is surjective.

Hence the exact group sequence
\[
1 \longrightarrow W_{h_1} \longrightarrow \text{Stab}_{W_\theta}(U) \longrightarrow W_{g \cdot h} \longrightarrow 1
\]
splits, generalising the recursive step.

Proof. It is equivalent to show that \( W_{h_1} \subseteq \text{Aut}(T, r) \) is a normal subgroup, and that
\[
\text{Aut}(T, r)/W_{h_1} \simeq \text{Aut}(T, r).
\]

This can be proven recursively on \( p \geq 1 \), the base being the content of Cor. 4.1.

If \( p \geq 2 \) consider a maximal proper subtree \( t \in T \): its leaves yield a subset \( J_1(t) \subseteq J_1 \), and there is a partition
\[
J_1 = \prod_T J_1(t),
\]
of the leaves of \( T \). Accordingly the centraliser of the irregular type splits as
\[
W_{h_1} \simeq \prod_T W_{h_1(t)} = \prod_T (W_{h_1(t)})^{n(t)},
\]
where \( W_{h_1(t)} = \prod_{J_1(t)} S_{\tau(t)} \) is the Weyl group of the Lie algebra \( h_1 \cap gl_{\tau(t)}(C) \subseteq h_1 \), and in turn \( gl_{\tau(t)}(C) \subseteq gl_{\tau(T)}(C) \) matches up with the eigenspace of the leading coefficient corresponding to the root of the subtree \( t \).

Hence, using the decomposition (24) (and that direct products and quotients commute), the result follows from Lem. 4.2 below; indeed in particular
\[
\text{Aut}(T, r)/W_{h_1} \simeq \left( \prod_T S_{n(t)} \cap \text{Aut}(t, r) \right)/W_{h_1} = \prod_T \left( S_{n(t)} \cap \text{Aut}(t, r)/W_{h_1(t)} \right)
\]
by the recursive hypothesis — and definition of the automorphism group. \( \square \)

Lemma 4.2. Let \( m \geq 0 \) be an integer and \( P \) a group, and choose a normal subgroup \( N \subseteq P \).

Then \( 1 = N \subseteq S_m \cap P \) is a normal subgroup, and in this identification there is a canonical group isomorphism
\[
(26) \quad (S_m \cap P)/N \cong S_m \cap (P/N).
\]

Proof. Postponed to § B. \( \square \)

Hence in brief there is an explicit (finite) algorithm to compute the ‘effective’ subquotient of the Weyl group acting freely on the deformation space of any type-A irregular type, to yield the deformation space of the associated irregular class.
Example 4.2. Let us look at the examples of type-A irregular type considered in [DRT], which all had pure WMCG isomorphic to $\mathrm{PBr}_2 \times \mathrm{PBr}_3 \times \mathrm{PBr}_4$. We will see their associated stabilisers are not all isomorphic.

Let us work with the Lie group $G = \mathrm{SL}_9(\mathbb{C})$; an irregular type $Q$ is given by a polynomial in the variable $x = z^{-1}$, whose coefficients are traceless diagonal matrices of size 9.

First consider

$$Q = A_1 x + A_2 x^2 + A_3 x^3, \quad A_i \in \mathfrak{sl}_9(\mathbb{C}),$$

with

$$A_1 = \begin{diagonal} 4, 3, 2, 1, 0, -1, -2, -3, -4 \end{diagonal},$$

$$A_2 = \begin{diagonal} 4, 4, 3, 2, 1, 0, -3, -4, -7 \end{diagonal},$$

$$A_3 = \begin{diagonal} 2, 2, 2, 1, 1, 0, 0, 0, -7 \end{diagonal}.$$

The corresponding ranked fission tree $(T, r)$ is drawn below, indicating the rank of each node within the corresponding vertex:

```
  9
 / \  \
 3  6
 / \ / \ \
2  1 2  1
 / \ / \ / \
1  1 1  1
```

From the recursive algorithm we get

$$\text{Aut}(T, r) = S_2 \times (S_2 \wr S_3).$$

This is the same as the automorphism group of the ranked tree for the irregular type $Q = A_1 x + A_2 x^2$, with

$$A_1 = \begin{diagonal} 4, 3, 2, 1, 0, -1, -2, -3, -4 \end{diagonal}, \quad A_2 = \begin{diagonal} 4, 1, 1, 0, 0, 0, -2, -2, -2 \end{diagonal} \in \mathfrak{sl}_9(\mathbb{C}).$$

Indeed in that case the ranked fission tree is as follows:

```
  9
 / \  \
 3  6
 / \ / \ \
2  1 2  1
 / \ / \ / \
1  1 1  1
```

Finally let us consider $Q = A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4$, with

$$A_1 = \begin{diagonal} 4, 3, 2, 1, 0, -1, -2, -3, -4 \end{diagonal},$$

$$A_2 = \begin{diagonal} 4, 4, 3, 2, 1, 0, -3, -4, -7 \end{diagonal},$$

$$A_3 = \begin{diagonal} 2, 2, 2, 2, 1, 0, -3, -3, -3 \end{diagonal},$$

$$A_4 = \begin{diagonal} 1, 1, 1, 1, 1, 1, 0, -2, -4 \end{diagonal}.$$

The ranked fission tree is then:
Its automorphism group is now $\text{Aut}(T, r) = S_3 \times S_3^2$, which is not the same as in the previous two cases. This is not contradictory: $\Gamma_Q$ only depends on the whole set of unordered configuration spaces attached to the (unranked) fission tree, while $\Gamma_{\Delta}$ also depends on their positions in the tree.

4.4. General case: (cabled) braid group. Write now $B_Q = \prod_{i=1}^r B_i$ the deformation space, in the decomposition associated with the fission tree (as in [DRT]). This means $B_i \subseteq C^{I_i}$ is a product of ordered configuration spaces (= type-$\Lambda$ root-hyperplane complements), attached to the nodes at the above level $I_{j+1}$: namely

$$B_i = \prod_{j+1}^{I_i} \text{Conf}_{k_i} \subseteq C^{I_i}, \quad k_i = |\phi^{-1}(i)| \geq 1,$$

counting the number of child-nodes. Thus globally

$$B_Q = \prod_{I_0} \text{Conf}_{k_i} \subseteq C^{T_0 \setminus \{\ast\}}, \quad \text{and} \quad \pi_1(B_Q) \simeq \prod_{I_0} \text{PBr}_{k_i}.$$

Now recall from [DRT, § 6] that the cabling of pure braid group operad, viz. the operadic composition

$$\gamma^{\text{PBr}} : \text{PBr}_n \times \prod_{i=1}^n \text{PBr}_{k_i} \longrightarrow \text{PBr}_k, \quad k = \sum_{i=1}^n k_i,$$

for $n, k_1, \ldots, k_n \geq 0$, yields a group embedding $\pi_1(B_Q) \hookrightarrow \text{PBr}_1$. More precisely recursive cabling along the (unranked) fission tree $T = (T_0, \phi)$ leads to the pure ‘cabled braid group’ $\text{PBr}(T) \subseteq \text{PBr}_{I_0}$ of the tree, and there is a group isomorphism $\pi_1(B_Q) \simeq \text{PBr}(T)$. The point is that in the pure case one finds a ‘noncrossed’ group/action operad, which in particular implies (28) is a group morphism equipping the domain with the direct product structure.

Here instead we naturally encountered the operadic composition of the symmetric group operad $\mathcal{S} = (S_\bullet, 1 \in S_1, \gamma^\mathcal{S})$, viz. the function

$$\gamma^\mathcal{S} : S_n \times \prod_{i=1}^n S_{k_i} \longrightarrow S_k, \quad (\sigma, \tau) \longmapsto \gamma^\mathcal{S}(\sigma; \tau),$$

for $\sigma \in S_n$ and $\tau = \prod_{i=1}^n \tau^{(i)}$. Its definition is a generalisation of the above construction to arbitrary partitions $k = \prod_{i=1}^n I_i$, where $k_i = |I_i|$. Namely there is a ‘block permutation’ operation

$$S_n \longrightarrow S_k, \quad \sigma \longmapsto \sigma(k_1, \ldots, k_n),$$

which consists in the permutation of all parts by fixing their elements, and then

$$\gamma^\mathcal{S}(\sigma; \tau) := \sigma(k_1, \ldots, k_n) \cdot \tau \in S_k,$$

\footnote{See op. cit. for an explanation of terminology, due to the nested braiding of eigenspaces for the coefficients of the irregular type.}
with tacit use of the natural group embedding $\prod_i S_{i_1} \hookrightarrow S_k$ on the right factor.

**Lemma 4.3.** If $k_1 = \cdots = k_n$, then (29) is an injective morphism, equipping the domain with the semidirect product structure.

**Proof.** Postponed to B. \hfill \square

**Remark 4.7.** This means the operadic compositions yields in particular group embeddings $S_n \rtimes S_m \hookrightarrow S_{mn}$, which were used above. \hfill \triangle

The recursive definition (24) now becomes

$$\text{Aut}(T, r) = \prod_{r \in \mathcal{T}} \gamma^{r} \left( S_{n(1)} \times \text{Aut}(t, r_0)^{n(1)} \right) \subseteq S_{j_1}, \quad r_0 = r|_{t_0} ,$$

starting again from the trivial group at each leaf. Finally, this can reformulated to exhibit the relation with braid groups.

**Lemma 4.4.** Let $\{ P_i \}_{i \in I}$ and $\{ N_i \}_{i \in I}$ be finite collections of groups, and $\rho_i : P_i \to \text{Aut}(N_i)$ group morphisms. Then there is a canonical group isomorphism

$$P \ltimes N \simeq \prod_{i \in I} P_i \ltimes N_i, \quad P = \prod_{i \in I} P_i, \quad N = \prod_{i \in I} N_i ,$$

using the production action on the left-hand side:

$$\rho : P \longrightarrow \prod_{i \in I} \text{Aut}(N_i) \subseteq \text{Aut}(N), \quad \rho = \prod_{i \in I} \rho_i .$$

**Proof.** Postponed to § B. \hfill \square

It follows that (30) is equivalent to the recursive definition

$$\text{Aut}(T, r) = S_{\varphi} \times \prod_{r \in \mathcal{T}} \text{Aut}(T, r_0)^{n(1)} ,$$

introducing the $\mathcal{T}$-partition $\varphi : J_p \to \mathcal{T}$ induced from the isomorphism classes of maximal proper subtrees; this means $S_{\varphi} = \prod_{r \in \mathcal{T}} S_{n(1)} \subseteq S_{J_p}$.

The expression (31) clarifies the natural definition of an analogous (full/nonpure) 'cabled' braid groups: one should ‘lift’ this through the (augmentation) operad morphism $p : \mathcal{B} \to \mathcal{S}$, where $\mathcal{B} = (\text{Br}_*, 1 \in \text{Br}_1, \gamma^{\mathcal{B}})$ is the (full/nonpure) braid group operad.

**Definition 4.4.** The **cabled braid group** of the ranked tree $(T, r)$ is the group recursively defined by

$$\mathcal{B}(T, r) = \mathcal{B}_{\varphi} \times \prod_{r \in \mathcal{T}} \mathcal{B}(t, r_0)^{n(1)} \subseteq \mathcal{B}_{J_1} ,$$

with basis $\mathcal{B}(i, r(i)) = \text{Br}_1$, for $i \in J_1$, using the semipure braid group of Prop. 4.2.

Here $\mathcal{B}_{\varphi} \subseteq \mathcal{B}_{|J_p|}$ is the subgroup corresponding to the braiding of the maximal proper subtrees (i.e. the equal-dimensional eigenspaces of the leading coefficient), acting by conjugation of the cabled braid group of any such subtree.

Finally we can prove that (32) is the correct definition, i.e. that indeed this is the group controlling the topology of admissible deformations of type-A irregular classes.

**Theorem 4.2.** There is a group isomorphism $\pi_{\mathcal{B}}(\mathcal{B}_{\mathcal{B}}) \simeq \mathcal{B}(T, r)$, and $\mathcal{B}(T, r)$ is an extension of $\text{Aut}(T, r)$ by $\mathcal{B}(T)$. 
Proof. By Prop. 4.3 the projection $B_Q \to B_{\mathcal{T}}$ amounts to the Galois covering over the quotient $B_Q / \text{Aut}(T, r)$, so the claimed group extension will follow from the first statement—as $\mathcal{PB}(T) \simeq \eta_1(B_Q)$, cf. [DRT].

The first statement instead can be proven by induction on $p \geq 1$. If $p = 1$ then $\mathcal{PB}(T) = \mathcal{PB}_{1, r}$ and $\mathcal{PB}(T, r) \subseteq \mathcal{PB}_{1, r}$ is the 'semipure' braid group of partition of the leaves into equal-rank nodes: the result then follows from Prop. 4.2.

Now suppose $p \geq 2$, and write $B(T)$ the space determined by the (unranked) tree as in (27). By definition

$$B(T) = \text{Conf}_{[r]} \times \prod_{t \in \mathcal{T}} B(t, r_0)^{n(t)},$$

with the usual notation for representatives of maximal proper subtrees, and for the cardinality of their isomorphism classes. Then for $t \in \mathcal{T}$ the base of the wreath product $S_n(t)^t \text{Aut}(t, r_0)$ acts on the rightmost factor, while $S_n(t)$ is naturally a subgroup of permutations of the child-nodes of the roots—permuting the (isomorphic) subtrees rooted there.

Assume first $\mathcal{T} = \{ t \}$ is a singleton, i.e. all maximal proper subtrees are isomorphic, and let $n = |r_p|$. Then simply $B(T) = \text{Conf}_n \times B(t)^n$, and accordingly $\text{Aut}(T, r) = S_n \times \text{Aut}(t, r_0)$ by (24).

Now we have a natural surjective map $B(T) \to U \text{Conf}_n$, composing the canonical projection $B(T) \to \text{Conf}_n$ with the Galois covering $\text{Conf}_n \to U \text{Conf}_n$, and there is also a Galois covering $B(T) \to B(T, r) = B(T) / \text{Aut}(T, r)$. By construction the former factorises through the latter, so there is a commutative triangle of topological spaces:

$$\begin{array}{ccc}
B(T) & \xrightarrow{p} & U \text{Conf}_n \\
\downarrow \pi & & \downarrow \pi \\
B(T, r) & \xrightarrow{} & U \text{Conf}_n
\end{array}$$

The difference from the case $p = 1$ is that the arrows onto the unordered configuration space are not coverings, but rather (locally trivial) fibre bundles with positive-dimensional fibres. For $\pi$ this is clear (it is the composition of a trivial bundle and a locally trivial one), while for $\mathcal{PB}$ it can be proven as follows. If $O \subseteq U \text{Conf}_n$ is an open trivialising set for $\pi$, then $\pi^{-1}(O) = \tilde{O} \times B(t)^n$, where $\tilde{O} \simeq S_n \times \mathbb{U}$ is the preimage of $O$ under the standard Galois covering, and

$$\pi^{-1}(O) = p(\pi^{-1}(O)) = (\tilde{O} \times B(t)^n) / \text{Aut}(T, r).$$

Now the latter quotient can be taken in two steps: first the action of the base yields

$$(\tilde{O} \times B(t)^n) / (1 : \text{Aut}(t, r_0)) \simeq \tilde{O} \times B(t, r_0)^n,$$

and then the space $O \times B(t, r_0)^n \subseteq O \times S_n \times B(t, r_0)^n \simeq \tilde{O} \times B(t, r_0)^n$ is a slice for the action of the 'complement' subgroup $S_n \times 1$. In conclusion $\pi^{-1}(O) \simeq O \times B(t, r_0)^n$,

proving we have a locally trivial fibre bundle

$$B(t, r_0)^n \longrightarrow B(T, r) \xrightarrow{\pi} U \text{Conf}_n.$$ 

This yields the exact group sequence

$$1 \longrightarrow \mathcal{PB}(T, r_0)^n \longrightarrow \pi_1(B(T, r)) \xrightarrow{\eta_1(\mathcal{T})} \text{Br}_n \longrightarrow 1,$$

by the recursive hypothesis, using that $U \text{Conf}_n$ is a $\eta_1([K, 1]$-space and that fibres are connected. Moreover any continuous function $U \text{Conf}_n \to B(t, r_0)^n$ yields a
global section, so in conclusion there is a semidirect product decomposition
\[ \pi_1(\mathcal{B}(T, r)) \cong \text{Br}_n \triangleleft \mathcal{B}(t, r_0) \cong \text{Br}_n \times \mathcal{B}(t, r_0)^n, \]
in accordance with the recursive definition (32).

Finally consider the general case where \( \widetilde{T} \) is not a singleton. Then we can generalise the above argument using the composition of projections
\[ \mathcal{B}(T) \to T \to X_\varphi = \text{Conf}_n / S_\varphi, \]
on to the semiordered configuration space, where \( \varphi : J_\pi \to \widetilde{T} \) is as above. Again a 2-step quotient (over any open trivialising subspace \( O \subseteq X_\varphi \)) can be taken with respect to the actions of the subgroups
\[ \prod_{\mathcal{J}} (1 \triangleleft \text{Aut}(t, r_0)^n), \quad \prod_{\mathcal{J}} (S_{n(t)} / 1) \subseteq \text{Aut}(T, r), \]
whose (inner) product gives the whole of \( \text{Aut}(T, r) \) in view of Lem. 4.4. The analogous commutative triangle of topological spaces then yields the fibre bundle
\[ \prod_{\mathcal{J}} \mathcal{B}(t, r_0)^n_{\pi(t)} \hookrightarrow \mathcal{B}(T, r) \xrightarrow{\pi} X_\varphi, \]
whence the exact group sequence
\[ 1 \to \prod_{\mathcal{J}} \mathcal{B}(t, r_0)^n_{\pi(t)} \to \pi_1(\mathcal{B}(T, r)) \xrightarrow{\pi_1(\pi)} \text{Br}_n \to 1, \]
using the recursive hypothesis, Prop. 4.2, the fact that fibres are connected, and that \( X_\varphi \) is a covering of a \( K(\pi, 1) \)-space—so it is also a \( K(\pi, 1) \). Again this has global sections, since there are continuous maps \( X_\varphi \to \prod_{\mathcal{J}} \mathcal{B}(t, r_0)^n_{\pi(t)}, \) proving the statement.

\[ \square \]

Remark 4.8. The auxiliary fibre bundle
\[ S_n \times \mathcal{B}(t)^n \hookrightarrow \mathcal{B}(T) \xrightarrow{\pi} \text{UConf}_n, \]
which appears in the above proof when \( \mathcal{J} = \{ t \} \), yields the following exact group sequence:
\[ 1 \to \mathcal{B}(T)^n \to \mathcal{B}(T) \xrightarrow{\pi_1[\pi]} \text{Br}_n \to S_n \to 1. \]
This is recovering the fact that \( \mathcal{B}(T)^n / \mathcal{B}(t)^n \cong \text{PBr}_n \), considering the direct product over the nodes of the (unranked) fission tree. \( \square \)

Outlook

There is a ‘twisted’ version of irregular types/classes [BY, BY20, Dou]; and there exists ‘global’ deformations of (twisted, bare) wild Riemann surfaces, defined as in [DRT]; we plan to consider these elsewhere.\(^9\)

A further question is to obtain explicit expressions for the actions of the wild mapping class groups on wild character varieties, laying the ground to investigate the corresponding Poisson/symplectic dynamics. E.g., it is known that the algebraic solutions of Painlevé VI correspond to monodromy representations with finite orbits under the braid group action [DM00, Boa05], so it is natural to ask whether a similar phenomenon holds in the wild case.

Finally recall [DRT] constructed a fine moduli scheme \( \mathcal{M}_4^p \) of irregular types of bounded pole order \( p \in \mathbb{Z}_{\geq 1} \), and with given pole order \( \mathcal{d}_x \in \{ 0, \ldots, p \} \) after evaluation at each root \( \alpha \in \Phi_\varphi \)—for any complex reductive group \( G \). In particular

\[ \text{The twisted local case, both pure and full/nonpure, in type A, has now been studied in [BDR].} \]
The action of a permutation \( C \) on \( I \) is right-to-left. More generally if \( I \) is a finite set we denote \( S_I \) its symmetric group of permutations, so \( S_I \) acts on \( C \) in a natural way. We consider the subgroup permuting all parts, and further the elements within each part: this is the (restricted) wreath product

\[ S_J \wr S_m = S_J \wr S_m, \]

with respect to the action of \( S_J \) given by

\[ \sigma \cdot \tau = \prod_{j \in J} \tau^{(\sigma_j^{-1})}, \quad \sigma \in S_J, \quad \tau = \prod_{j \in J} \tau^{(j)} \in (S_m)^{|J|}, \]

where \( \tau^{(j)} \in S_{|J|} \simeq S_m \) for \( j \in J \). Elements of \( S_J \wr S_m \) are then written \( (\sigma; \tau) \).

**Weyl actions.** Let \((V, \Phi)\) be a root system in the finite-dimensional complex vector space \( V \), and \( W = W(\Phi) \subseteq \text{GL}(V) \) the Weyl group. If \( S \subseteq V \) is a subset, its **setwise** Weyl-stabiliser is the subgroup

\[ \text{Stab}_W(S) = \{ w \in W \mid w(S) \subseteq S \} \subseteq W, \]

and its **pointwise** Weyl-stabiliser is the subgroup

\[ W_S = \{ w \in W \mid S \subseteq \text{Ker}(w - \text{Id}_V) \} \subseteq \text{Stab}_W(S). \]

The latter is a **parabolic** subgroup of \( W \)—thinking of the Weyl group as a reflection group—and is a normal subgroup of the former. Clearly \( W_S = W_{CS} \), and if \( CS_1 \subseteq CS_2 \) then \( W_{S_1} \subseteq W_{S_2} \); further if \( U_1, U_2 \subseteq V \) are subspaces then

\[ W_{U_1 + U_2} = W_{U_1} \cap W_{U_2} \subseteq W. \]

On the other hand, tautologically, if \( W_I = \text{Stab}_W(U_I) \) then

\[ \text{Stab}_{W_I}(U_2) = W_I \cap \text{Stab}_W(U_2) = W_2 \cap \text{Stab}_W(U_1) = \text{Stab}_{W_2}(U_1). \]
We identify $W$ with the Weyl group $W(\Phi^\vee) \subseteq \text{GL}(V^\vee)$ for the dual/inverse root system, via $w \mapsto w^{-1}$ [Bou68, Ch. VI, § 1.1].

**Braid groups.** For an integer $n \geq 0$ we denote $\text{PBr}_n$ the *pure braid group* on $n$ strands—so $\text{PBr}_0$ and $\text{PBr}_1$ are trivial. It is the fundamental group of the space

$$\text{Conf}_n = \text{Conf}_n(\mathbb{C}) := \mathbb{C}^n \setminus \bigcup_{1 \leq i \neq j \leq n} H_{ij},$$

where

$$H_{ij} = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n | z_i = z_j \} \subseteq \mathbb{C}^n.$$

In particular $\text{Conf}_1 = \mathbb{C}$, and in general this yields the space of *ordered* configurations of $n$ points in the complex plane. The symmetric group $S_n$ acts naturally on (33), and the projection

$$\text{Conf}_n \longrightarrow U\text{Conf}_n := \text{Conf}_n / S_n,$$

to the space of *unordered* configurations, is a Galois covering. The (full/nonpure) *braid group* is $\text{Br}_n = \pi_1(U\text{Conf}_n)$, and the associated exact group sequence

$$\begin{array}{c}
1 \longrightarrow \text{PBr}_n \longrightarrow \text{Br}_n \longrightarrow \mathbb{Z}_n \longrightarrow 1
\end{array}$$

corresponds to the braid group ‘augmentation’, i.e. the morphism taking the permutation underlying the braiding of the $n$ strands [Art47].

More generally, for a (reductive) split Lie algebra $(g, t)$ we consider the root-hyperplane complement

$$t_{\text{reg}} = t \setminus \bigcup_{\alpha \in \Phi_\theta} \text{Ker}(\alpha) \subseteq t, \quad \Phi_\theta = \Phi(g, t),$$

generalising (33) in type $A$, and $\text{PBr}_\theta = \pi_1(t_{\text{reg}})$ is the *pure* $g$-braid group, a.k.a. the generalised (Artin–Tits) braid group of type $g$ [Bri71, BS72, Del72, Bri73]. The Weyl group $W_\theta = W(\Phi_\theta)$ acts freely on $t_{\text{reg}}$, and $t_{\text{reg}} \rightarrow t_{\text{reg}}/W_\theta$ is a Galois covering. Then $\text{Br}_\theta = \pi_1(t_{\text{reg}})$ is the *full/nonpure* $g$-braid group, and there is an exact group sequence generalising (35):

$$\begin{array}{c}
1 \longrightarrow \text{PBr}_\theta \longrightarrow \text{Br}_\theta \longrightarrow W_\theta \longrightarrow 1.
\end{array}$$

**Trees.** A (finite) tree $T = (T_0, \Phi)$ of height $p \geq 1$ is the data of a finite set $T_0$ with a partition $T_0 = \coprod_{j=1}^{p+1} J_j$, such $J_{p+1} = \{ * \}$ is a singleton, and a function $\Phi : T_0 \setminus \{ * \} \rightarrow T_0$ such that $\Phi(J_l) \subseteq J_{l+1}$ for $l \in \{ 1, \ldots, p \}$. The elements of $T_0$ and the *nodes* of the tree, and $\Phi(i)$ is the *parent-node* of $i \in T_0 \setminus \{ * \}$—so $* \in J_{p+1}$ is the root, while $J_1 \subseteq T_0$ contains the *leaves*. Conversely $\Phi^{-1}(i) \subseteq T_0$ is the set of *child-nodes* of $i \in T_0$.

**Appendix B. Missing proofs**

**Proof of Lem. 2.4.** We can recursively prove that

$$W_i = \bigcap_{1 \leq j \leq p} \text{Stab}_{W_\theta}(U_j), \quad i \in \{ 1, \ldots, p \}.$$  

The base $i = p$ is tautological, and then

$$W_{i-1} = \{ w \in W_i \mid w(U_{i-1}) \subseteq U_{i-1} \} = \{ w \in W_\theta \mid w(U_j) \subseteq U_j \text{ for } j \geq i-1 \},$$

using (15) and the recursive hypothesis. \qed
Proof of Lem. 4.2. By definition $S_m \downarrow P = S_m \times P^m$, with respect to the natural permutation action $S_m \rightarrow \text{Aut}(P^m)$. Then $1 \downarrow N = 1 \times N^m \subseteq S_m \times P^m$, and it is normal since it is normalised by $1 \downarrow P = 1 \times P^m$ and stabilised by the permutation action. Hence the quotient on the left-hand of (26) is well defined.

Now there is an induced action $S_m \rightarrow \text{Aut}(Q^m)$, where $Q = P \downarrow N$, and finally the natural surjective group morphism $S_m \downarrow P \rightarrow S_m \downarrow Q$ vanishes on $1 \downarrow N$. □

Proof of Lem. 4.3. The compatibility with the product follows from (20) (which in turn is equivalent to the action-operad axiom for $\otimes$ [Yau, Eq. 4.1.2]), and from the fact that the block permutation operation is a group morphism in this case.

Injectivity follows from the identity

$$S_n \langle k \rangle \cap (S_k)^n = 1 \subseteq S_{nk},$$

where $S_n \langle k \rangle \subseteq S_{nk}$ is the image of the block permutation operation $S_n \rightarrow S_{nk}$. □

Proof of Lem. 4.4. There is a natural bijection

$$\prod_i (p_i, n_i) \mapsto \left( \prod_i p_i \prod_i n_i \right),$$

between the underlying sets, and one can show it is compatible with the semidirect multiplication.

Indeed choose elements $p_i', p_i \in P_i$ and $n_i', n_i \in N_i$ for $i \in I$, so that

$$\prod_i (p_i', n_i) \prod_i (p_i, n_i) = \prod_i (p_i', n_i) \bullet_i (p_i, n_i)$$

$$= \prod_i (p_i' p_i, p_i n_i') \in \prod_i P_i \times N_i,$$

which is mapped to $\left( \prod_i p_i' p_i, \prod_i p_i n_i' n_i \right) \in P \times N$. Conversely

$$\left( \prod_i p_i' \prod_i n_i' \right) \bullet \left( \prod_i p_i, \prod_i n_i \right) = \left( \prod_i p_i' \prod_i p_i, \rho(\prod_i p_i) \prod_i n_i' \prod_i n_i \right) \in P \times N,$$

which coincides with the above—using the product action and the direct product multiplication. □

Appendix C. Relations to isomonodromy systems

On the other side of the Riemann–Hilbert–Birkhoff correspondence there is a Poisson fibre bundle analogous to (1), viz.

$$\mathcal{M}_{\text{IR}} \xrightarrow{\pi} B,$$

whose fibres (the de Rham spaces) are moduli spaces of irregular singular algebraic connections on principal $G$-bundles. This is equipped with the pullback (flat, nonlinear) isomonodromy connection; see in particular [Boa01, Fig. 1], which spells out the picture of the wild nonabelian Gauß–Manin connection on the Betti side (extending the tame case [Sim94a, Sim94b]).

Now one can choose a local trivialisation of (36), i.e. an isomorphism of fibre bundles

$$\mathcal{M}_{\text{IR}}|_O = \pi^{-1}(O) \xrightarrow{\cong} O \times M$$

$$\xrightarrow{\pi} O \xrightarrow{p_1} P_1,$$

over an open subspace $O \subseteq B$, for a fixed Poisson manifold $(M, \{\cdot, \cdot\})$. Then the isomonodromy connection (on the upper-left corner of (37)) can be given by
explicit nonlinear first-order partial differential equations in local coordinates \( t = (t_1, \ldots, t_d) \) on \( O \), where \( d = \dim(B) \), for local sections over the trivialising locus.

Moreover the difference between the isomonodromy connection and the trivial connection (on the upper-right corner of (37)) can be ‘integrated’\(^{10} \) to a nonautonomous Hamiltonian system

\[
H = (H_1, \ldots, H_d) : M \times O \longrightarrow \mathbb{C}^d.
\]

In this Hamiltonian viewpoint, the symplectic nature of isomonodromic deformations is equivalent to the integrability of (38). The latter amounts to the identities

\[
\{ H_i, H_j \} + \frac{\partial H_i}{\partial t_j} - \frac{\partial H_j}{\partial t_i} = 0, \quad i, j \in \{1, \ldots, d\}.
\]

Hence the local coordinates become times of isomonodromic deformations over \( O \subseteq B \): but in principle they are not intrinsically associated with isomonodromic deformations, contrary to the flat Ehresmann connections on (1) and (36) (one needs a choice of ‘initial’ trivialisation [Boa01, Rk. 7.1]; cf. [Yam19]).

Examples of such isomonodromy systems abound, with far-reaching applications already in the genus-zero case, famously encompassing (generalisations of) the Painlevé equations [Oka87a, Oka87b, Oka86, BCR18, CGL19] and the Schlesinger system [Sch05] (cf. also [Miw81]). The (Harnad-)dual version of the Schlesinger system, on the other side of the Fourier–Laplace transform [Har94, Yam16], was considered in [Boa02], and the combination of Schlesinger and its dual yield the system of Jimbo–Miwa–Môri–Sato [JMMS80]. Recall the Painlevé property of the JMMS equations was studied in [Miw81].

Note also that [Har94] links previous papers about isospectral deformations [AHP88, AHH90] to the isomonodromic deformations of JMMS. However this hardly the whole story: a rigorous treatment of the degeneration of the isomonodromic deformations of JMMS into a combination of the isospectral deformations and the Whitham dynamics is still open, cf. [Tak98] for a related conjecture, and [Xuc] in the quantum case—with applications to quantum groups and canonical bases. In particular the ‘cactus’ groups naturally appear; these are fundamental groups of certain compactifications of the real points in the base space, controlling the asymptotics zones of the nonlinear isomonodromy equations (analogous of the braid groups encountered here).

Finally a generalisation of all the above was derived in [Boa12]. This latter setup brings about nongeneric isomonodromic deformations, considering connections with several levels, which extend examples of the seminal paper [JMU81]. (Recall the set of levels is the set of nonzero pole orders of the irregular types, evaluated at each root.) Importantly this is more symmetric than op. cit., which in turn is one of our main motivations for studying the ‘deeper’ nongeneric case: in particular in [Boa12] the group \( SL_2(C) \) acts on the bundle of de Rham spaces via automorphisms of the 1-dimensional Weyl algebra, and contains the Fourier–Laplace transform as the element \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \).

Importantly all these Hamiltonian systems have far-reaching applications in mathematical physics, notably in integrable hierarchies of differential equations such as KdV [Ger], and in 2d conformal field theory after quantisation (e.g. [NS10]).

\(^{10}\)The difference of the corresponding horizontal distributions—inside \( TO \times TM \rightrightarrows Tp. TO \)—is given by O-dependent (vertical) vector fields \( X_i : O \times M \rightarrow TM \) on the fibre, and one has \( dH_i = \langle \omega, X_i \rangle \) (cf. [Boa12, § 5]).
Yam11, AGT10, Gai13, FR23], opening to ‘irregular’ conformal blocks and the AGT correspondence). More precisely the quantisation of the Schlesinger system leads to the Knizhnik–Zamolodchikov connection (KZ) [Res92, Har96], while the quantisation of the dual Schlesinger system (as in [Boa02]) leads to the Casimir connection of de Concini/Millson–Toledano Laredo [MTL05]. The quantisation of the JMM systems, generalising the above and recovering the connection of Felder–Markov–Tarasov–Varchenko [FMTV00], was considered in [Rem19]; further op. cit. constructed a quantisation of the more general ‘simply-laced’ systems of [Boa12] (this is resumed in the table in the introduction of [Rem19]); cf. [Yam22] for a different construction of ‘quantum’ simply-laced isomonodromy systems, and [NS10, NS11, GMR] for a ‘confluence’ viewpoint on the quantisation of irregular singularities.

Finally encoding the irregular moduli in the base curve, and constructing bundles over their (admissible) deformations, is also helpful for the quantisation of the extended ‘classical’ symmetries of isomonodromy systems. In particular the quantised $\text{SL}_2(\mathbb{C})$-symmetries [Rem20] generalise the Howe duality [Bau], which in turn were used in [TL02] to compute the monodromy of the Casimir connection in terms of that of KZ (cf. also [TV02]): this latter example of generic ‘quantum’ monodromy action brings about the $G$-braid groups which we generalise in this series of papers.

Appendix D. List of some nonstandard notation (in rough order of appearance)

| Symbol | Description |
|--------|-------------|
| $\Sigma$ | Riemann surface |
| $G$ | connected complex reductive Lie group |
| $\mathcal{M}_g$ | Poisson/symplectic fibration of Betti spaces |
| $g$ | Lie algebra of $G$ |
| $t$ | Cartan subalgebra of $g$ |
| $T$ | maximal (algebraic) torus in $G$ |
| $Q$ | irregular type |
| $a$ | point of $\Sigma$ |
| $A_i$ | coefficients of $Q$ |
| $W_\mathfrak{g}$ | Weyl group of $(\mathfrak{g}, t)$ |
| $\mathcal{W}$ | irregular class underlying $Q$ |
| $\Sigma$ | wild Riemann surface |
| $\mathcal{B}_Q$ | space of admissible deformations of $Q$ |
| $\mathcal{B}_{\Lambda_i}$ | space of admissible deformations of $\Lambda_i$ |
| $\mathcal{B}_T$ | space of admissible deformations of $T$ |
| $W_\mathfrak{g}/\mathfrak{l}$ | subquotient of $W_\mathfrak{g}$ acting freely on $\mathcal{B}_Q$ |
| $(T, r)$ | ranked fission tree |
| $\text{Aut}(T, r)$ | automorphisms of $(T, r)$ |
| $\mathcal{B}(T, r)$ | full/nonpure cabled braid group of $(T, r)$ |
| $\Phi_{\mathfrak{g}}$ | root system of $(\mathfrak{g}, t)$ |
| $\mathcal{G}_{\Sigma, a}$ | completed local ring of $\Sigma$ at $a$ |
| $\mathcal{K}_{\Sigma, a}$ | fraction field of $\mathcal{G}_{\Sigma, a}$ |
| $\mathcal{Q}_{\Sigma, a}$ | quotient of $\mathcal{K}_{\Sigma, a}$ modulo $\mathcal{G}_{\Sigma, a}$ |
| $d_\alpha$ | pole order of $q_\alpha = (\alpha \otimes 1)Q$ |
| $d$ | tuple of the $d_\alpha$ |
| $\Gamma_Q$ | pure local WMCG |
| $\Gamma_{\mathcal{W}}$ | full/nonpure local WMCG |
| $h_i$ | nested stabilisers of $A_1, \ldots, A_p$ |
$H_i$ connected subgroup of $G$ with Lie algebra $h_i$

$\Phi_{h_i}$ Levi subsystem of $\Phi_h$

$W_{h_i}$ Weyl group of $(h_i, t)$

$U_i$ intersection of the root hyperplanes of $\Phi_{h_i}$

$W_i$ nested setwise stabilisers of the $U_i$

$\text{Stab}_{W_i}(B_{A_i})$ setwise stabiliser of $B_{A_i}$ in $W_i$

$U$ flag of the subspaces $U_i$ in $t$

$(W_\theta)_{U_i}$ pointwise stabiliser of $U_i$ in $W_\theta$

$PBr_n$ pure braid group on $n$ strands

$Br_n$ full/nonpure braid group on $n$ strands

$Dih_n$ dihedral group of order $2n$

$\Delta_\phi$ base of simple roots for $\Phi_\phi$

$\text{Conf}_n$ configuration space of $n$ ordered points in $\mathbb{C}$

$U\text{Conf}_n$ configuration space of $n$ unordered points in $\mathbb{C}$

$I$ group of permutations of a set $I$

$I_i$ parts of $I$ defined by a root subsystem of $A_{n-1}$

$J$ index set for the parts $I_i$

$K_i$ collection of parts $I_j$ with $i > 0$ elements

$S_\phi$ group of permutations preserving a partition $\phi$

$p_n$ augmentation group morphism of $Br_n$

$Br_\phi$ group of braids with underlying permutations in $S_\phi$

$T$ fission tree

$J_i$ levels of $T$

$T_0$ nodes of $T$

$\Phi$ parent-node function of $T$

$r_0$ restriction of $r$ to $t_0$

$\mathcal{S}$ symmetric braid group operad

$\gamma$ composition of $\mathcal{S}$

$B$ full/nonpure braid group operad

$\gamma_B$ composition of $B$

$\mathcal{P}_T$ pure cabled braid group of $T$

$\mathcal{M}_{\text{fR}}$ Poisson/symplectic fibration of de Rham spaces

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