The Complementarity of Quantum Observables
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Figure 1: Scheme of a Mach-Zehnder interferometer.
Figure 2: Mach-Zehnder interferometer with a Kerr medium.
Figure 3: Expanded Mach-Zehnder interferometer.
The Complementarity of Quantum Observables: Theory and Experiments.

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ABSTRACT
Three notions of complementarity – operational, probabilistic, and value complementarity – are reanalysed with respect to the question of joint measurements and compared with reference to some examples of canonically conjugate observables. It is shown that the joint measurability of noncommuting observables is a consequence of the quantum formalism if unsharp observables are taken into account; a fact not in conflict with the idea of complementarity, which, in its strongest version, was originally formulated only for sharp observables. As an illustration of the general theory, the wave-particle duality of photons is analysed in terms of complementary path and interference observables and their unsharp joint measurability.

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1. Introduction. Recent advances in experimental quantum physics have made it possible to perform joint measurements of complementary observables of individual quantum objects, like neutrons or photons. Despite its apparently contradictory nature, this statement is in full harmony with the quantum theory, the theory of complementarity\(^1\), being, in fact, anticipated in the notion of coexistence.

The concept of complementarity has both probabilistic and operational contents. When these two aspects are clearly distinguished it becomes evident that probabilistically complementary observables may also be coexistent so that they can be measured together. This is exactly the situation encountered in recent experiments or in proposals for such experiments.

In this work we set out to reanalyse the notion of complementarity, emphasising from the outset the distinction between its probabilistic and operational versions. We show their interrelations and study their measurement-theoretical implications. Various canonically conjugate pairs of observables, including also some less known ones, are revisited and shown to be complementary. The general results are put to work in an analysis of new experiments displaying the wave-particle duality for single photons.

The *which path* experiments for photons and other quantum objects have remained an issue of intensive experimental and theoretical investigations throughout the history of quantum mechanics. The famous two-slit arrangement, well-known as a source of interference phenomena in classical light optics, was quickly recognised as an excellent illustration of the nonobjectivity of quantum observables of individual systems. The more recent quantum optical split-beam analogue provided by the Mach-Zehnder interferometer offered the possibility of actually realising the thought experiments invented by Bohr and Einstein in their attempts to demonstrate the idea of complementarity or to circumvent the measurement limitations due to the uncertainty relations. The wave-particle duality has been strikingly confirmed on the single-photon level in a series of recent experiments.\(^2\) These rather new experimental achievements – the possibility of investigating single quantum objects – were accompanied on the theoretical side by an elaboration and justification of an individual interpretation of quantum mechanics capable of appropriately dealing with such experiments. In such a framework we shall rederive the mutual exclusiveness of the particle and the wave behaviour, reflected in the two options of path determination and interference measurements. Moreover, we show that both aspects can be reconciled with each other where one does not require absolute certainty with respect to the path nor optimal interference contrast.

The mathematical description of these experiments is sufficiently simple as to allow for an exhaustive account of the physical situation. In fact, it turns out that there is not just one description but a variety of them, each yielding the same experimental figures but nevertheless leading to totally different mathematical representations and physical interpretations. In particular, we encounter illustrations of an instrumentalistic account, a phenomenological description and a realistic pic-
ture of physical experiments. In the first case one is only concerned with computing the counting frequencies, treating the whole experimental set-up as a "black box" with a variety of control parameters. The second type of account acknowledges that there is an input system that influences the black box, the measuring apparatus, and one may interpret the counting statistics with respect to this input system. In both cases varying the control parameters amounts to specifying another measurement. It is only in the third, realistic, account that the mathematical language utilised matches the wordings used by the experimenters in devising the set-up, carrying out the preparations and measurements, and interpreting the outcomes in terms of the prepared system. In effect, the other two approaches must also base their way of computing on a certain interpretation; they do introduce a splitting of the whole process into an observing and an observed part (measurement performed after a preparation); but the cut is placed at different locations, and this decision determines the ensuing mathematical picture. In addition, different degrees of reality are ascribed to the phenomena, ranging from mere measurement outcomes over highly contextual entities to something that may be considered as a kind of quantum object.

2. The framework. In order to allow for an appropriately rigorous exposition, we recall briefly the relevant features of quantum mechanics together with its measurement theory. In the Hilbert space formulation of quantum mechanics the description of a physical system $S$ is based on a (complex, separable) Hilbert space $\mathcal{H}$, with the inner product $\langle \cdot | \cdot \rangle$. We let $\mathcal{L}(\mathcal{H})$ denote the set of bounded operators on $\mathcal{H}$.

The basic concepts of observables and states will be adopted as dual pairs; observables being associated (and identified) with normalised positive operator valued (pov) measures $E : \mathcal{F} \to \mathcal{L}(\mathcal{H})$ on some measurable spaces $(\Omega, \mathcal{F})$ (the value spaces of the observables), whereas states are represented as (and identified with) positive bounded linear operators $T$ on $\mathcal{H}$ of trace one. The value space $(\Omega, \mathcal{F})$ of an observable is here taken to be the real Borel space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, a subspace of it, or a Cartesian product of such spaces.

The familiar concept of an observable as a self-adjoint operator is contained in the above formulation as a special case. Indeed if $E$ is a projection valued (pv) measure [i.e., $E(X) = E(X)^2$ for all $X \in \mathcal{F}$] and if its value space is $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then $E$ can be identified with a unique self-adjoint operator $A$ acting in $\mathcal{H}$. In this case we write $E^A$ to indicate that this measure is the spectral measure of $A$. Observables which are given as pv measures are called sharp observables, others will be referred to as unsharp observables. A particular class of states are the vector states, the one-dimensional projections generated by the unit vectors $\psi$ of $\mathcal{H}$. We denote them as $P[\psi]$ or $|\psi\rangle\langle \psi|$, where $P[\psi](\psi) := \langle \psi | \psi \rangle \psi$.

An immediate advantage of this formulation is that the measurement outcome probabilities, which form the empirical basis of quantum mechanics, are now directly available. Indeed any pair of an observable $E$ and a state $T$ defines a probability
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For which the minimal interpretation is adopted: the number \( p^E_T(X) \) is the probability that a measurement of the observable \( E \) leads to a result in the set \( X \) when the system is prepared in the state \( T \). If \( T \) is a vector state generated by a vector \( \varphi \), we write \( p^E_\varphi \) instead of \( p^E_{\varphi [\varphi]} \), and we recall that \( p^E_\varphi(X) = \langle \varphi | E(X) \varphi \rangle \). If \( E \equiv E^A \), we also write \( p^A_T \) for \( p^{E^A}_T \).

It may be appropriate to remark that the representation of an observable as a POV measure, without restricting it to be a PV measure, not only formally exhausts the probability structure of quantum theory but is even mandatory from the operational point of view. We shall have several occasions to see that the measurement outcome statistics collected under certain circumstances do define an observable not as a PV measure but only as a POV measure.

In analysing the measurement-theoretical content of the notion of complementarity we need to rely on some general results of the theory of measurement. Therefore we recall that in order to model a measurement of a given observable \( E \) one usually fixes a measuring apparatus \( A \), with its Hilbert space \( K \), an initial state \( T' \) of the apparatus, a pointer observable \( Z \), and a measurement coupling \( V \) (a state transformation of the compound object-apparatus system). If \( T \) is the initial state of the object system, then \( V(T \otimes T') \) is the object-apparatus state after the measurement. Denoting the corresponding reduced state of the apparatus as \( W \), one may pose the probability reproducibility condition as the minimal requirement for \( K, T', Z \) and \( V \) to constitute a measurement of \( E \): for any \( X \) and \( T \),

\[
p^E_T(X) = p^Z_W(\bar{X}),
\]

where \( \bar{X} \) is the value set of the pointer observable \( Z \) corresponding to the value set \( X \) of the measured observable \( E \). It is a basic result of the quantum theory of measurement that every observable can be measured in the above sense\(^4\). Moreover, for any observable \( E \) there are measurements, for which the initial state of the apparatus is a vector state, the pointer observable is a PV measure, and the measurement coupling is given by a unitary mapping. We shall give several examples of such measurements subsequently.

There is another important reading of Eq. (2). Assume that we are given a certain device characterised by a measurement coupling \( V \), a pointer observable \( Z \), and an initial state \( T' \) of the apparatus. Then for each value set \( X \) of the pointer observable, the mapping \( T \mapsto \text{tr} [V(T \otimes T') I \otimes Z(X)] \) is a positive bounded (by unity) linear functional on the state space of the object system. This means that the condition

\[
\text{tr}[TF^{T',Z,V}(X)] := \text{tr}[V(T \otimes T') I \otimes Z(X)]
\]

defines an observable \( F^{T',Z,V} \) of the measured system, with the same value space as the pointer observable. In order to obtain an observable with a given value space
(Ω, F) one needs to introduce an appropriate pointer function f : ΩZ → Ω assigning to a given pointer value the corresponding value of the observable. One then has the system observable FT′,Z,V,f for which
\[ p_{T}^{F_{T′,Z,V,f}}(X) := p_{T}^{F_{T′,Z,V}}(f^{-1}(X)) = p_{W}^{Z}(f^{-1}(X)) \] (4)
for all initial states T of the object system and for all value sets X ∈ F of the measured observable FT′,Z,V,f. The condition (2) for K, T′, Z, and V to constitute a measurement of E is thus equivalent to the requirement that the actually measured observable FT′,Z,V is E for some pointer function f, that is,
\[ F_{T′,Z,V,f} = E. \] (5)
This way of reading the probability reproducibility condition corresponds to the experimenters’ actual practice.

As an illustration of the above ideas consider a toy model of a measurement of an observable represented by a discrete self-adjoint operator \( A = \sum a_{k} P_{k} \), with the eigenvalues \( a_{k} \) and the eigenprojections \( P_{k} \). We aim at measuring \( A \) by monitoring the position of a particle (apparatus) confined to moving on a line; the Hilbert space of the apparatus is thus \( \mathcal{K} = L^{2}(R, dq) \), and the pointer observable is taken to be the position \( Q_{A} \) of the apparatus. An appropriate (unitary) measurement coupling is then \( U = \exp \left( -iA \otimes P_{A} \right) \), which correlates the observable to be measured with shifts in the position of the apparatus. Let \( P[\phi] \) be the initial apparatus state. If \( P[\varphi] \) is an initial object state, then \( P[U(\varphi \otimes \phi)] = P[\sum a_{k} \varphi \otimes \phi_{k}] \) is the final object-apparatus state, where \( \phi_{k}(x) = e^{-i\lambda a_{k}} P_{A} \phi(x) = \phi(x - \lambda a_{k}) \) (in the position representation). Assuming that the spacing between the eigenvalues \( a_{k} \) is greater than \( \delta \) and that \( \phi \) is supported in \( (-\frac{\delta}{2}, \frac{\delta}{2}) \), then the \( \phi_{k} \) are supported in the mutually disjoint sets \( I_{k} = (\lambda a_{k} - \frac{\delta}{2}, \lambda a_{k} + \frac{\delta}{2}) \). With an appropriate choice of the pointer function \( f \) the condition (2) is always fulfilled, that is, for each \( a_{k} \) and for all \( \varphi \),
\[ p_{\varphi}^{A}(a_{k}) = p_{W}^{Q_{A}}(f^{-1}(a_{k})) = p_{W}^{Q_{A}}(I_{k}), \] (6)
where the final apparatus state takes now the form \( W = \sum p_{A}^{F}(a_{k}) P[\phi_{k}] \). This canonical measurement model, which indeed serves as an \( A \)-measurement, has some peculiar features: it is value reproducible, of the first kind, and repeatable.3

As another example of this type of model, we consider a case where the actually measured observable is not the intended one. Assume that we wish to measure (a component of) the position \( Q \) of the object system with the above method, using the coupling \( U = e^{-iA Q \otimes P_{A}} \) to correlate the object position \( Q \) with the apparatus position \( Q_{A} \) serving as the pointer observable. In the position representations, the initial object-apparatus wave function \( \Psi(q, x) = \varphi(q) \phi(x) \) is transformed into \( \Psi'(q, x) = (\exp(-iA Q \otimes P_{A})) \Psi(q, x) = \varphi(q) \phi(x - \lambda q) \), which allows one to determine the actually measured observable:
\[ \langle \Psi' | I \otimes E_{Q}^{Q_{A}}(\lambda X) \Psi' \rangle = \int_{R} dq |\varphi(q)|^{2} \int_{R} dq' \lambda |\phi(\lambda(q' - q))|^{2} I_{X}(q') dq' \]
\[ = \langle \varphi | I_{X} * f(Q) \varphi \rangle =: \langle \varphi | E_{Q}^{f}(X) \varphi \rangle, \] (7)
where $I_X * f$ denotes the convolution of the characteristic function $I_X$ of the set $X$ with the confidence function $f$, which is determined by the coupling constant $\lambda$ and the initial state of the apparatus: $f(q) = \lambda |\phi(-\lambda q)|^2$. The actually measured observable is therefore an unsharp position $E^{Q,f} : X \mapsto I_X * f (Q)$ and not the sharp position $Q$ (with the spectral measure $E^Q$). This model was already discussed by von Neumann\(^5\), though at that time the notion of a pov measure was not yet available for a definite conclusion.

Another important concept of measurement theory which we shall apply subsequently is that of a state transformer. It describes the possible state changes of the system under a measurement. Consider a measurement of an observable $E$ given by $V$, $T'$, and $Z$. For any $X$, there exists a (non-normalised) state $\mathcal{I}(X)(T)$ such that the conditional expectation $\text{tr}[V(T \otimes T') B \otimes Z(\bar{X})]$ for any bounded self-adjoint operator $B$ acting in $\mathcal{H}$ can be expressed as $\text{tr}[\mathcal{I}(X)(T) B]$. In agreement with Eq. (2) the trace (norm) of this state operator is $\text{tr}[\mathcal{I}(X)(T)] = p^E_T(X)$. According to its definition, $\mathcal{I}(X)(T)$ is (modulo normalisation) the state of the system after the measurement on the condition that the pointer observable $Z$ has the value $\bar{X}$ after the measurement. In particular, $\mathcal{I}(\Omega)(T)$ is the state of the system after the measurement on the plain condition that the measurement has been performed. By construction, the maps $\mathcal{I}(X) : T \mapsto \mathcal{I}(X)(T)$ are linear, positive transformations on the trace class and satisfy $\text{tr}[\mathcal{I}(X)(T)] \leq \text{tr}[T]$ for all state operators. Such transformations are called state transformations. Hence the state transformer $\mathcal{I}$ is a state transformation valued measure $X \mapsto \mathcal{I}(X)$. Various properties of measurements are conveniently described in terms of the associated state transformer. For instance, a measurement of an observable $E$ is of the first kind if it does not change the outcome statistics, that is, $\text{tr}[T E(X)] = \text{tr}[\mathcal{I}(\Omega)(T) E(X)]$ for all $X$ and $T$, and it is repeatable if its repeated application does not lead to a new result, that is, $\text{tr}[\mathcal{I}(X)(T)] = \text{tr}[\mathcal{I}(X)^2(T)]$ for all $X$ and $T$. In the first of the two examples above the state transformer is of the form

$$\mathcal{I}^A(a_k)(T) = P_k T P_k,$$

which is clearly repeatable, and thus also of the first kind. In the second example one gets

$$\mathcal{I}^{Q,f}(X)(T) = \int_X A_q T A^*_q dq,$$

with

$$A_q = \sqrt{\lambda} \phi(-\lambda(q - X)),$$

which is still of the first kind but no longer repeatable. In fact, as is well known, no continuous observable admits a repeatable measurement.\(^4\)

3. Coexistence. A key concept in the characterisation of quantum mechanical observables is that of the coexistence of observables which serves to describe the
possibility of measuring together two or more observables. Its rudimentary formal expression is the commutativity of self-adjoint operators. For our study of complementarity it will be useful to recall briefly both the probabilistic and the measurement-theoretical aspects of coexistence.

An observable $E : \mathcal{F} \to \mathcal{L}(\mathcal{H})$ is a representation of a class of measurement procedures in the sense that it associates with any state $T$ the probability $p^E_T(X)$ for the occurrence of an outcome $X \in \mathcal{F}$. For a pair of observables $E_1$, $E_2$ the question may be raised as to whether their outcome statistics $p^E_{T_1}(X)$ and $p^E_{T_2}(Y)$ can be collected within one common measurement procedure for arbitrary states $T$ and sets $X \in \mathcal{F}_1$, $Y \in \mathcal{F}_2$. Thus one is asking for the existence of a third observable whose statistics contain those of $E_1$ and $E_2$. We say that observables $E_1$ and $E_2$ are coexistent whenever this is the case. More explicitly, and with a slight generalisation, a collection of observables $E_i, i \in \mathbf{I}$, is coexistent if there is an observable $E$ such that for each $i \in \mathbf{I}$ and for each $X \in \mathcal{F}_i$ there is a $Z \in \mathcal{F}$ such that

$$p^E_{T_i}(X) = p^E_T(Z)$$

for all states $T$. In other words, a set of observables $E_i, i \in \mathbf{I}$, is coexistent if there is an observable $E$ such that the ranges $R(E_i) := \{E_i(X) | X \in \mathcal{F}_i\}$ of all $E_i$ are contained in that of $E$, that is, $\bigcup_{i \in \mathbf{I}} R(E_i) \subseteq R(E)$. Observable $E$ is called a joint observable for the $E_i$.

In concrete applications a joint observable $E$ for a coexistent pair $E_1$, $E_2$ will usually be constructed on the product space $\Omega := \Omega_1 \times \Omega_2$ of the two outcome spaces, with $\mathcal{F}$ being some $\sigma$-algebra on $\Omega$ such that $X \times \Omega_2 \in \mathcal{F}$, $\Omega_1 \times Y \in \mathcal{F}$ for all $X \in \mathcal{F}_1$, $Y \in \mathcal{F}_2$. Thus, if $\mathcal{F}_i = \mathcal{B}(\mathbb{R})$, then $\mathcal{F}$ will be conveniently chosen as $\mathcal{B}(\mathbb{R}^2)$. The observables $E_1$, $E_2$ are then recovered from $E$ as its marginal observables: $E_1(X) = E(X \times \Omega_2)$, $E_2(Y) = E(\Omega_1 \times Y)$.

In order to measure jointly two observables $E_1$ and $E_2$ of a system $\mathcal{S}$ there must be a single measurement process which allows one to collect the measurement outcome statistics of both observables. In the language of measurement theory this means that there is an apparatus, initially in a state $T'$, a pointer observable $Z$, and a measurement coupling $V$ such that for any initial state $T$ of $\mathcal{S}$,

$$E_1 = F^{T',Z,V,f_1},$$
$$E_2 = F^{T',Z,V,f_2},$$

where $f_1 : \Omega_Z \to \Omega_1$ and $f_2 : \Omega_Z \to \Omega_2$ are suitable pointer functions. This shows immediately that $E_1$ and $E_2$ are coexistent, being in fact coarse-grained versions of the actually measured observable $F^{T',Z,V}$. On the other hand, if two observables $E_1$ and $E_2$ are coexistent, with a joint observable $E$, then clearly any measurement of $E$ is a joint measurement of $E_1$ and $E_2$ in the sense just described.

There is still another intuitive idea of joint measurability which refers to the possibility of performing order independent sequential measurements. Indeed assume that a measurement of $E_1$, performed on a system initially in state $T$, is
followed by a measurement of $E_2$. Then $\text{tr}[\mathcal{I}_1(X)(T)E_2(Y)] = \text{tr}[\mathcal{I}_2(Y)\mathcal{I}_1(X)(T)]$ is the joint probability for obtaining results in $X$ for the $E_1$-measurement and in $Y$ for the $E_2$-measurement. It may happen that such sequential probabilities are order independent, that is, for all $X, Y$ and $T$, $\text{tr}[\mathcal{I}_2(Y)\mathcal{I}_1(X)(T)] = \text{tr}[\mathcal{I}_1(X)\mathcal{I}_2(Y)(T)]$. In that case we say that the involved sequential measurements of $E_1$ and $E_2$ are order independent. Any two observables are coexistent whenever they admit order independent sequential measurements. Yet it appears that the existence of order independent sequential measurements is not guaranteed by coexistence.

4. Complementarity. The mutual exclusiveness expressed by the term complementarity refers to the possibilities of predicting measurement outcomes as well as to the value determinations. Both of these aspects were discussed by Bohr and Pauli. We shall review below two formalisations, referred to as (measurement-theoretical) complementarity and probabilistic complementarity. The former implies noncoexistence. In the case of sharp observables the two formulations are equivalent. However, for unsharp observables the measurement-theoretical notion of complementarity is stronger than the probabilistic one. It is due to this fact that probabilistically complementary observables can be coexistent; and this explains the sense in which it is possible to speak of simultaneous measurements of complementary observables, like position and momentum, or path and interference observables.

The predictions of measurement outcomes for two observables are mutually exclusive if probability equal to one for some outcome of one observable entails that none of the outcomes of the other one can be predicted with certainty. Observables $E_1$ and $E_2$ are called probabilistically complementary if they share the following property:

$$\begin{align*}
\text{if } p^E_{T_1}(X) &= 1, \text{ then } 0 < p^E_{T_2}(Y) < 1, \\
\text{if } p^E_{T_2}(Y) &= 1, \text{ then } 0 < p^E_{T_1}(X) < 1,
\end{align*}$$

for any state $T$ and for all bounded sets $X \in \mathcal{F}_1$ and $Y \in \mathcal{F}_2$ for which $E_1(X) \neq I \neq E_2(Y)$.

Assume that for observables $E_1$ and $E_2$ the probabilities $p^E_{T_1}(X)$ and $p^E_{T_2}(Y)$ both equal one for some state $T$ and some sets $X$ and $Y$. This is equivalent to saying that $E_1(X)\varphi = \varphi$ and $E_2(Y)\varphi = \varphi$ for some unit vector $\varphi$, that is, $P[\varphi] \leq E_1(X)$ and $P[\varphi] \leq E_2(Y)$. Hence in that case $E_1(X)$ and $E_2(Y)$ have a positive lower bound. If these effects are projection operators, then $E_1(X)\varphi = \varphi$ and $E_2(Y)\varphi = \varphi$ holds exactly when $\varphi \in E_1(X)(\mathcal{H}) \cap E_2(Y)(\mathcal{H})$. Treating similarly the two other cases excluded by (13), [i.e., $p^E_{T_1}(X) = 1$ and $p^E_{T_2}(Y) = 0$, or $p^E_{T_1}(X) = 0$ and $p^E_{T_2}(Y) = 1$] one observes, first of all, that in the case of sharp observables the probabilistic complementarity of a given pair of observables is equivalent to the
disjointness of their spectral projections:

\[
E_1(X) \land E_2(Y) = O,
E_1(X) \land E_2(\Omega_2 \setminus Y) = O,
E_1(\Omega_1 \setminus X) \land E_2(Y) = O,
\]

for all bounded sets \(X\) and \(Y\) for which \(O \neq E_1(X) \neq I\) and \(O \neq E_2(Y) \neq I\).

The above considerations also show that for unsharp observables condition (14) always implies (13), but the reverse implication need not hold. We take condition (14) as the formal definition of the complementarity of observables \(E_1\) and \(E_2\).

Thus we note that any two complementary observables are also probabilistically complementary, but not necessarily vice versa.

Position and momentum form the most prominent pair of complementary observables. Yet, among the unsharp position and momentum pairs, which are all probabilistically complementary, there are coexistent pairs, thus breaking the complementarity. We return to this example subsequently.

We mention in passing that in the literature one still finds another version of complementarity which we shall term value complementarity. This refers to the case when certain predictability of some value of \(E_1\) implies that all the values of \(E_2\) are equally likely. This concept is not rigorously applicable in the case of continuous observables as they have no proper eigenstates. Nevertheless it is applied also to position and momentum in the intuitive sense that, for example, a sharp momentum (plane wave) state goes along with a uniform position distribution. Examples of value complementary observables are (the Cartesian components of the) position and momentum of a particle in a trap, mutually orthogonal spin components of a spin-\(\frac{1}{2}\) object, the canonically conjugate spin and spin phase, or the number and phase observable. As is evident, value complementarity implies probabilistic complementarity. On the other hand, the examples to be studied below show that value complementarity and measurement-theoretical complementarity are logically independent concepts.

Let us imagine a pair of observables \(E_1\) and \(E_2\) which are probabilistically complementary but coexistent, so that they can be measured together. Let \(E\) be a joint observable, so that for any \(X \in \mathcal{F}_1\) and \(Y \in \mathcal{F}_2\) one has \(E_1(X) = E(\bar{X})\) and \(E_2(Y) = E(\bar{Y})\) for some \(X \in \mathcal{F}, \bar{Y} \in \mathcal{F}\). We assume that there exists a repeatable joint measurement of \(E_1\) and \(E_2\), with the ensuing \(E\)-compatible state transformer \(I\). Let \(X\) and \(Y\) be any two bounded sets for which \(O \neq E_1(X) \neq I\) and \(O \neq E_2(Y) \neq I\), and let \(T\) be a state for which \(p_{E_2}^T(Y) \neq 0\). If \(T_Y := I(\bar{Y})(T)/p_{E_2}^T(Y)\), then the repeatability implies \(p_{E_1}^{T_Y}(X) = 1\) as well as

\[
p_{E_1}^{T_Y}(X) = \text{tr}[I(X)(T_Y)] = p_{E_2}^T(Y)^{-1} \text{tr}[I(X \cap Y)(T)].
\]

If \(\text{tr}[I(X \cap Y)T] = 0\), then \(p_{E_2}^T(Y) = 1\) and \(p_{E_1}^{T_Y}(X) = 0\), which is excluded by the second line of Eq. (13). On the other hand, if \(\text{tr}[I(X \cap Y)T] \neq 0\), then in the state
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$T' := I(\bar{X} \cap \bar{Y}) T / p_{T'}^E(\bar{X} \cap \bar{Y})$ one has $p_{T'}^{E_1}(X) = 1$ and $p_{T'}^{E_2}(Y) = 1$, which is again excluded by (13). It follows that $I$ cannot be repeatable, and we have the following result.

A. PROPOSITION. Probabilistically complementary observables do not admit any repeatable joint measurements.

Consider next a pair of complementary observables $E_1$ and $E_2$. Conditions (14) imply that these observables cannot be coexistent. Indeed, assuming they were coexistent, let $E$ be a joint observable. By the additivity of measures, and with the above notations, $E(\bar{X} \cup \bar{Y}) + E(\bar{X} \cap \bar{Y}) = E(\bar{Y}) + E(\bar{X}) = E_1(X) + E_2(Y)$, for all $X$ and $Y$. If $X$ and $Y$ are bounded sets for which $O \neq E_1(X) \neq I$, and $O \neq E_2(Y) \neq I$, then, by the first line of Eq. (14), $E(\bar{X} \cap \bar{Y}) = O$. Therefore, $E_1(X) = E(\bar{X} \cup \bar{Y}) - E_2(Y) \leq I - E_2(Y)$, which contradicts the second line of (14). Hence $E_1$ and $E_2$ cannot be coexistent, and we have proved the following.

B. PROPOSITION. Complementary observables do not admit any joint measurements. Neither do they have any order independent sequential measurements.

Let $I_1$ and $I_2$ be state transformers compatible with $E_1$ and $E_2$, respectively. Assume that for some sets $X$ and $Y$ there is a state transformation $\Phi$ such that $\Phi \leq I_1(X)$ and $\Phi \leq I_2(Y)$. Such a state transformation is a test of the effects $E_1(X)$ and $E_2(Y)$. It would allow one to construct a measurement (state transformer) which provides some probabilistic information on both $E_1(X)$ and $E_2(Y)$. We say that two state transformers are mutually exclusive if no such test exists, that is, if

\begin{align}
I_1(X) \land I_2(Y) &= 0, \\
I_1(X) \land I_2(\Omega_2 \setminus Y) &= 0, \\
I_1(\Omega_1 \setminus X) \land I_2(Y) &= 0, \\
\end{align}

for all bounded sets $X$ and $Y$ for which $O \neq I_1(X) \neq I_1(\Omega_1)$ and $O \neq I_2(Y) \neq I_2(\Omega_2)$. The following is then a measurement-theoretical characterisation of the complementarity.

C. COMPLEMENTARITY. Two observables are complementary if and only if their associated state transformers are mutually exclusive.

This result sharpens the preceding one.

D. PROPOSITION. Noncoexistent observables do not admit any joint measurements. Complementary observables do not admit any joint tests.

With this result we close our general account of complementarity and we present some examples next. Sections 6 and 7 provide then an analysis of the wave-particle duality of photons in view of these results.
5. Examples.

a) Position - momentum pairs. The canonically conjugate position and momentum, \( Q \) and \( P \), are the basic observables of any localisable object. They are Fourier equivalent physical quantities and can be represented as a Schrödinger couple. This fundamental connection, which results from the Galilei covariance of the localisation observable, is the root of the familiar coupling properties known for these observables. In view of their relevance to the joint measurability of position and momentum, we collect some of those features here.

The basic coupling properties are, of course, the commutation relation,

\[
QP - PQ = iI,
\]

which holds on a dense domain, and the uncertainty relation,

\[
\Delta(Q, \varphi) \Delta(P, \varphi) \geq \frac{1}{2},
\]

which holds for all unit vectors \( \varphi \).

The Fourier connection \( P = U_F^{-1}QU_F \) extends also to the spectral measures, so that one has \( E^P(Y) = U_F^{-1}E^Q(Y)U_F \) for all \( Y \in \mathcal{B}(\mathbb{R}) \). Thus, if \( E^Q(X)\varphi = \varphi \) and \( E^P(Y)\varphi = \varphi \) for some vector \( \varphi \) and bounded sets \( X, Y \), then, using the Schrödinger representation, the function \( \varphi \) vanishes (almost everywhere) in \( \mathbb{R} \setminus X \) and its Fourier transform \( \hat{\varphi} \) is an analytic function that vanishes (almost everywhere) in \( \mathbb{R} \setminus Y \). Such a function vanishes everywhere according to the identity theorem for analytic functions. Therefore, the following well-known relations for the spectral projections of a Schrödinger couple \( (Q, P) \) are obtained:

\[
E^Q(X) \land E^P(Y) = 0,
\]

\[
E^Q(X) \land E^P(\mathbb{R} \setminus Y) = 0,
\]

\[
E^Q(\mathbb{R} \setminus X) \land E^P(Y) = 0,
\]

for all bounded \( X, Y \in \mathcal{B}(\mathbb{R}) \). These relations show the complementarity of \( Q \) and \( P \) both in the sense of (13) and (14). It may be of interest to note that \( E^Q(\mathbb{R} \setminus X) \land E^P(\mathbb{R} \setminus Y) \neq 0 \), for all bounded \( X \) and \( Y \).

The uncertainty relations (18) as well as the complementarity (19) of \( Q \) and \( P \) imply their total noncommutativity: there are no vectors with respect to which \( Q \) and \( P \) commute, that is,

\[
\{ \varphi \in \mathcal{H} \mid E^Q(X)E^P(Y)\varphi = E^P(Y)E^Q(X)\varphi, \; X, Y \in \mathcal{B}(\mathbb{R}) \} = \{0\}.
\]

This is obvious from (19) but it follows also from (18). Interestingly, in spite of their total noncommutativity, \( Q \) and \( P \) do have mutually commuting spectral projections. Indeed

\[
E^Q(X)E^P(Y) = E^P(Y)E^Q(X)
\]
whenever $X$ and $Y$ are periodic (Borel) sets satisfying $X = X + 2\pi/a$ and $Y = Y + a$. This is a special case of the fact that $Q$ and $P$ admit coexistent coarse-grainings $f(Q)$ and $g(P)$ for periodic functions $f$ and $g$, that is,

$$f(Q)g(P) = g(P)f(Q), \quad (22)$$

exactly when $f$ and $g$ are both periodic functions with minimal periods $\alpha, \beta$ satisfying $2\pi/\alpha\beta \in \mathbb{Z} \setminus \{0\}$.

Position and momentum are complementary observables. All of their measurements are mutually exclusive. Therefore, they cannot be measured or even tested together. In spite of this impossibility, Heisenberg boldly maintained that the position and momentum of a particle can be determined simultaneously provided that the measuring inaccuracies are in accordance with the uncertainty relations. Today such joint position-momentum measurements are indeed reported to be feasible. We complete our discussion of this example by reviewing the measurement-theoretical model which anticipated this possibility.

In Sect. 2 we showed that a measurement scheme with the coupling $U = e^{-i\lambda Q \otimes P_A}$ does define an unsharp position observable $E^{Q,f}$. Similarly, the coupling $U = e^{i\mu P \otimes P_A}$ yields, via the position monitoring of the apparatus, a measurement of unsharp momentum $E^{P,g}$, where the confidence function $g$ depends again on the coupling constant and on the initial apparatus state. The question then is whether such unsharp position and momentum observables could be coexistent. A first positive answer is immediately obtained by showing that combining any two of such position and momentum measurements does lead to a measurement scheme which defines a joint observable for a pair of unsharp position and momentum observables. To be explicit, consider a measuring apparatus containing two initially independent probe particles 1 and 2 prepared in a state $P[\phi_1 \otimes \phi_2]$. Assume that the object system is coupled to the apparatus via the interaction $U = \exp(-i\lambda Q \otimes P_1 \otimes I_2 + i\mu P \otimes I_1 \otimes P_2)$. If the positions of the two parts of the apparatus are monitored, that is, if the pointer observable is taken to be $Q_1 \otimes Q_2$, then the system observable $F \equiv F^{P[\phi_1 \otimes \phi_2], U, Q_1 \otimes P_2}$ which is measured by this scheme has an unsharp position and unsharp momentum as its marginal observables: $F(X \times R) = E^{Q,f}(X), F(R \times Y) = E^{P,g}(Y)$ for all $X$ and $Y$. It is remarkable that in this case the confidence functions $\hat{f}$ and $\hat{g}$ become automatically coupled. They become Fourier related, that is, they are of the form $\hat{f}(q) = \langle q | T_0 | q \rangle$ and $\hat{g}(p) = \langle p | T_0 | p \rangle$ for some positive operator $T_0$ of trace one. $T_0$ being a state operator, it follows that the distributions $\hat{f}$ and $\hat{g}$ do fulfil the inequality

$$\Delta(\hat{f}) \cdot \Delta(\hat{g}) \geq \frac{1}{2}. \quad (23)$$

In this way Heisenberg’s claim on the joint measurability of complementary position and momentum observables is corroborated both in terms of a precise notion of joint observables as well as by means of a quantum mechanical measurement model.
There is no need to go into further details of the question posed. It suffices to mention that any pair of unsharp position and momentum observables \( E^{Q,f} \) and \( E^{P,g} \) are coexistent whenever the confidence functions are Fourier related.\(^{21}\). Being coexistent observables, \( E^{Q,f} \) and \( E^{P,g} \) cannot be complementary. This can also be confirmed directly since for any (Borel) subsets \( X \) and \( Y \) of \( \mathbb{R} \)

\[
\begin{align*}
F(X \times Y) & \leq F(X \times \mathbb{R}) = E^{Q,f}(X), \\
F(X \times Y) & \leq F(\mathbb{R} \times Y) = E^{P,g}(Y).
\end{align*}
\]

Assume that for some bounded sets \( X,Y \) the probabilities \( \text{tr}[TE^{Q,f}(X)] \) and \( \text{tr}[TE^{P,g}(Y)] \) both equal one for some vector state \( T = P[\varphi] \). This would imply that both \( \varphi \) and \( \hat{\varphi} \) have bounded supports, which is impossible for any vector state \( P[\varphi] \). Therefore \( E^{Q,f} \) and \( E^{P,g} \) are probabilistically complementary observables, whether coexistent or not.

In view of the experimental advances in the trapping of individual particles it may be of interest to remark that the canonically conjugate position and momentum observables of a particle in a box are complementary even though they are clearly not a Schrödinger couple. Avoiding any technical details, we note only that the Cartesian components \( Q_k, P_k \) of the position and momentum of a particle confined to a cube (of unit length, say) are complementary and hence also probabilistically complementary.\(^{16}\) Since \( P_k \) is now discrete, the value complementarity of the pair \( Q_k, P_k \) is readily verified. Indeed if \( \varphi_n \) is an eigenvector of \( P_k \), then the conjugate position distribution is uniform:

\[
\langle \varphi_n | E^{Q_k}((0,x)] \varphi_n \rangle = x \text{ for all } 0 \leq x \leq 1.
\]

\textbf{b) Components of spin.} Any spin observable \( s_{\hat{a}} := \hat{a} \cdot \sigma \) of a spin-\( \frac{1}{2} \) object is given by its spectral projections, the sharp spin properties

\[
E(\pm \hat{a}) = \frac{1}{2} (I \pm \hat{a} \cdot \sigma),
\]

where \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) are the Pauli spin operators and \( \hat{a} \in \mathbb{R}^3 \) is a unit vector. Such observables are mutually complementary, and also value complementary whenever they are related to mutually orthogonal directions.

Rotation covariant families of two-valued unsharp spin observables are generated by the unsharp spin properties, the operators of the form

\[
F(\pm \hat{a}) = \frac{1}{2} (I \pm \hat{a} \cdot \sigma), \quad \|\hat{a}\| \leq 1.
\]

Such observables arise as smeared versions of sharp spin observables \( s_{\hat{a}} \), the degree of smearing being determined by the parameter \( \|\hat{a}\| \).\(^{22}\)

Pairs of unsharp observables of the form (26) are probabilistically complementary but in general not complementary. We may derive a simple geometric criterion for the coexistence of such observables. To ensure the coexistence of the unsharp spin properties \( F(\hat{a}_1) \) and \( F(\hat{a}_2) \), one needs to find an operator \( G = \gamma F(\hat{c}) \) such that

\[
\begin{align*}
O & \leq \gamma F(\hat{c}), \\
\gamma F(\hat{c}) & \leq F(\hat{a}_1), \\
\gamma F(\hat{c}) & \leq F(\hat{a}_2), \\
F(\hat{a}_1) + F(\hat{a}_2) - \gamma F(\hat{c}) & \leq I.
\end{align*}
\]
Taking into account that $\alpha F(a) \leq \beta F(b)$ is equivalent to $\|\beta b - \alpha a\| \leq \beta - \alpha$, this system of inequalities is equivalent to the following one:

$$\begin{align*}
\|\gamma c\| &\leq \gamma, \\
\|a_1 - \gamma c\| &\leq 1 - \gamma, \\
\|a_2 - \gamma c\| &\leq 1 - \gamma, \\
\|a_1 + a_2 - \gamma c\| &\leq \gamma.
\end{align*}$$

(28)

Let us denote by $S(a, r)$ the closed ball with radius $r$ and centre point $a$. Then (28) can be rewritten as follows:

$$\gamma c \in S(a_1, 1 - \gamma) \cap S(a_2, 1 - \gamma) \cap S(a_1 + a_2, \gamma) \cap S(o, \gamma).$$

(29)

The intersection of the first two balls is nonempty exactly when $\gamma \leq 1 - \frac{1}{2} \|a_1 - a_2\|$, while the intersection of the last two balls is nonempty if and only if $\gamma \geq \frac{3}{2} \|a_1 + a_2\|$. Thus the coexistence of $F(a_1)$ and $F(a_2)$ implies the inequality $\frac{1}{2} \|a_1 + a_2\| \leq 1 - \frac{1}{2} \|a_1 - a_2\|$. Conversely, the validity of this relation entails the existence of some $\gamma$ satisfying the preceding two inequalities. In turn, these ensure that $c_o := \frac{1}{2}(a_1 + a_2)$ is in the intersection of all four balls. Taking for $\gamma$ the norm of this vector and defining $c := c_o/\|c_o\|$, one has satisfied (29). We have thus established the following result.

E. Proposition. The unsharp spin properties $F(a_1)$ and $F(a_2)$ are coexistent if and only if

$$\|a_1 + a_2\| + \|a_1 - a_2\| \leq 2.$$  

(30)

If $\|a_1\| = 1$, say, so that $F(a_1)$ is a sharp spin property, then (30) is fulfilled exactly when $a_2 = \pm \|a_2\| a_1$, that is, $F(a_2)$ commutes with $F(a_1)$. This confirms the general result that the coexistence of two observables amounts to their commutativity if one of them is a sharp observable. For the coexistence of two noncommuting unsharp observables a sufficient degree of unsharpness is required so that the values of $\|a_1\|, \|a_2\|$ must not be too close to unity.

A joint observable for a pair of coexistent unsharp spin properties $F(a_1), F(a_2)$ can easily be constructed. For example, the following set of operators will do:

$$G_{ik} := \alpha_{ik} F\left(\frac{1}{\alpha_{ik}} \frac{1}{2} (a_i + a_k)\right)$$

(31)

where $a_i$ is one of $a_1, a_\bar{1} = -a_1$, and $a_k$ is either $a_2$ or $a_\bar{2} = -a_2$. The factor $\alpha_{ik}$ can be taken to be $\frac{1}{2}(1 + a_i \cdot a_k)$. It is readily verified that (30) ensures the positivity of all $G_{ik}$ and that, for example, $F(a_1) = G_{12} + G_{12}$.

c) Spin and spin phase. The conjugate pair of a spin and spin phase constitute another example of probabilistically complementary observables which may or may not be complementary. A spin phase conjugate to $s_3 = \hat{e}_3 \cdot \sigma$, say, can easily be constructed from the polar decomposition of the raising operator $s_+ := s_1 + i s_2$. The construction applies to any spin $s$, so that we consider the spin Hilbert
space $H_s = \mathbb{C}^{2s+1}$ with a complete orthonormal set of $s_3$-eigenvectors $|m\rangle$, $m = -s, -s+1, \ldots, s-1, s$. From the polar decomposition $s_+ = B |s_+\rangle$, with $|s_+|^2 = \sqrt{(s_1 - is_2)(s_1 + is_2)}$, one easily computes that the partial isometry $B$ has the structure $B = \sum_{m=-s}^{s-1} |m+1\rangle \langle m|$. This shows that $B$ is a contraction with the norm $\|B\| = 1$. Hence there is a unique POV measure $S$ such that

$$B^n = \int_0^{2\pi} e^{i\alpha} S(d\alpha),$$

for each $n = 0, 1, 2, \ldots$. We call $S$ the spin phase. The commutation relation $[B, s_3] = B$ implies that $e^{-i\alpha s_3} B e^{i\alpha s_3} = e^{-i\alpha} B$, $\alpha \in [0,2\pi]$. Due to the uniqueness of the POV measure $S$, the spin phase satisfies the covariance condition

$$e^{-i\alpha s_3} S(X) e^{i\alpha s_3} = S(X + \alpha),$$

modulo $2\pi$ for arbitrary $X \in \mathcal{B}[0,2\pi]$. The explicit structure of $S$ can easily be obtained.

Let $\mathcal{H} := L^2([0,2\pi], \frac{d\alpha}{2\pi})$ and consider the following complete orthogonal system of normalised vectors $\psi_m \in \mathcal{H}$, $\alpha \mapsto \psi_m(\alpha) = e^{im\alpha}$. The map

$$W_s : H_s \rightarrow \mathcal{H}, \quad \psi \mapsto \sum_{m=-s}^{s} \langle m | \psi \rangle \psi_m$$

is an isometry from $H_s$ on to the $(2s+1)$-dimensional subspace $\mathcal{H}_s := \text{span}\{\psi_m | m = -s, \ldots, s\}$ of $\mathcal{H}$. In $\mathcal{H}_s$ the operator $B$ is the Neumark projection of the unitary operator $\tilde{B}_o = \sum_{m=-\infty}^{\infty} \langle \psi_{m+1} | \psi_m \rangle |m\rangle \langle m|$ on $\mathcal{H}$. Since $(\tilde{B}_o \psi)(\alpha) = e^{i\alpha} \psi(\alpha)$, $\psi \in \mathcal{H}$, the spectral measure of $\tilde{B}_o$ is the canonical one $X \mapsto E(X)$, with $E(X) \psi = I_X \psi$. Thus one gets:

$$S(X) = \sum_{m,n=-s}^{s} \langle \psi_m | E(X) \psi_n \rangle |m\rangle \langle n| = \sum_{m,n=-s}^{s} \int_X e^{i(n-m)\alpha} |m\rangle \langle n| \frac{d\alpha}{2\pi}. \quad (35)$$

It is important to note that $S$ itself is no PV measure. The space $\mathcal{H}_s$ consists of finite linear combinations of trigonometric functions. The idempotency $S(X) = S(X)^2$ would demand that $I_X \xi \in \mathcal{H}_s$ whenever $\xi \in \mathcal{H}_s$. This is impossible since such functions cannot be represented as finite linear combinations of sine and cosine functions.

For the same reasons the spin phase cannot be “localised” since for any proper subset $X$ of $[0,2\pi]$ (with $0 < \int_X d\alpha < 2\pi$) and all $\xi \in \mathcal{H}_s$ one has

$$0 < \langle \xi | E(X) \xi \rangle < 1. \quad (36)$$

Thus the pair $(s_3, S)$ is trivially probabilistically complementary. It is also value complementary since in any spin eigenstate $|m\rangle$ the spin phase is uniformly distributed: $\langle m | S([0,\alpha]) m \rangle = \frac{\alpha}{2\pi}$. On the other hand, while these observables are
noncoexistent, they are not complementary: for any spin state $|m⟩$ there is a positive number $\lambda < 1$ such that $\lambda |m⟩⟨m|$ is a lower bound to $|m⟩⟨m|$ as well as to any (nonzero) $S(X)$. This follows from a result as according to which $\lambda P[\varphi]$ is a lower bound of an operator $A$, $O \leq A \leq I$, for some positive $\lambda$ exactly when $\varphi$ is in the range of the square root of $A$; in the present case of $A = S(X)$ this range is the whole Hilbert space due to the validity of (36) for any state.

d) Number and phase. It may be noted that similar results are obtained for the conjugate pair of photon number and the phase of a single-mode electromagnetic field. The only formal difference consists in the fact that the number operator $N = a^*a$ is semibounded. The phase, defined as a POVM measure $M$ via the polar decomposition of the annihilation operator $a = B\sqrt{N}$ analogously to (32), is shift covariant under the group generated by $N$. Again the phase is not localisable so that the pair $(N, M)$ is probabilistically complementary; moreover it is value complementary as the phase is uniformly distributed in any eigenstate of the number.

6. Photon split-beam experiments. Using the tools and concepts developed in Sections 2-4, we analyse the photon split-beam experiments performed by a Mach-Zehnder interferometer. Figure 1 shows the scheme of such a device consisting of two beam splitters $BS_1$ and $BS_2$, with transparencies $\varepsilon_1$ and $\varepsilon_2$, reflecting mirrors $M_1$ and $M_2$, and a phase shifter $PS(\delta)$ allowing for the variation of the path difference between the two arms of the interferometer. The detectors $D_1$ and $D_2$ are assumed to register the number of photons $N_1 = \sum n_1 |n_1⟩⟨n_1|$ and $N_2 = \sum n_2 |n_2⟩⟨n_2|$ emerging from the second beam splitter, when a photon pulse is impinging on the first one. We shall take the incoming photon pulse to be a single-mode field in a state $T$, and we let $N_a = a^*a$ denote its number operator. The first beam splitter effects a coupling of this mode with an idle single-mode field, with $N_b = b^*b$. In the Schrödinger picture one may identify $N_1$ and $N_2$ with $N_a$ and $N_b$, respectively. It will be useful to consider first arbitrary states $T'$ of the $b$-mode, and only later fix it to be idle, $T' = |0⟩⟨0|$. The action of a beam splitter $BS$ with transparency $\varepsilon$ can be described by means of the unitary operator:

$$U_\alpha = \exp(\bar{\alpha}a \otimes b^* - \alpha a^* \otimes b),$$

with $\alpha = |\alpha|e^{i\theta}$, $\cos |\alpha| = \sqrt{\varepsilon}$. The phase shifter $PS(\delta)$ acts according to

$$V_\delta = e^{i\delta N_a \otimes I}.$$  

Therefore, if $T \otimes T'$ is the initial state of the two-mode field entering the interferometer, then the state of the field emerging from the second beam splitter is

$$W := U_\beta V_\delta U_\alpha (T \otimes T') U^*_\alpha V^*_\delta U^*_\beta.$$
where $\alpha = |\alpha| e^{i\theta_1}, \cos |\alpha| = \sqrt{\epsilon_1}$, and $\beta = |\beta| e^{i\theta_2}, \cos |\beta| = \sqrt{\epsilon_2}$. The probability of detecting $n_1$ photons in $D_1$ and $n_2$ photons in $D_2$ is thus

$$p_W^{N_1 \otimes N_2}(n_1, n_2) = \langle n_1, n_2 | W | n_1, n_2 \rangle. \quad (40)$$

[Figure 1. Scheme of a Mach-Zehnder interferometer.]

In this reading the measured observable is the two-mode number observable $N_1 \otimes N_2$, and the measured system is the output field emerging from the interferometer. The field’s passage through the interferometer is treated as an indivisible part of the preparation of the phenomenon to be observed, and no attempt is made at analysing the various stages of this process. This view corresponds to a positivistic attitude according to which quantum theory is merely an instrument for calculating the statistics of measurement outcomes. All adjustable variables, the two-mode input state, the transparencies, and the phase shift, are treated on equal footing as control parameters of the black box determining the preparation of the output state $W$, which is subjected to a counting measurement.

The counting probability (40) can equivalently be written as a measurement outcome probability with respect to the input state $T \otimes T'$ of the two-mode field,

$$p_T^E(n_1, n_2) \equiv \text{tr}[T \otimes T' E(n_1, n_2)] := p_W^{N_1 \otimes N_2}(n_1, n_2), \quad (41)$$

the observable being now

$$E(n_1, n_2) := U_\alpha^* V_\delta^* U_\beta^*(|n_1\rangle\langle n_1| \otimes |n_2\rangle\langle n_2|) U_\beta V_\delta U_\alpha. \quad (42)$$

Here the interferometer is taken as a part of the measuring device, which now serves to yield information on the input state. Accordingly, the set of variables mentioned above is split into two parts, the input state representing the preparation, and the interferometer parameters belonging to the measurement. In order to realise wave or particle phenomena, one must take into account, in this view, all the details of the experiment, including the measuring system parameters. The ‘particle’ or ‘wave’ cannot be described as an entity existing independently of the constituting measurement context. In fact the input state alone does not determine whether the field passing the interferometer behaves like a particle or a wave.

To work out the explicit form of Eq. (42), we note first that $V_\delta U_\alpha = U_{\alpha'} V_\delta$, with $\alpha' = e^{i\delta} \alpha$. Next, observe that the operators $a \otimes b^*, a^* \otimes b$, and $\frac{1}{2} (N_a \otimes I - I \otimes N_b)$ satisfy the standard commutation relations of the generators of the group $SU(2)$. Therefore, for any $\alpha$ and $\beta$, there is a $\gamma$ such that

$$U_\alpha U_\beta = e^{\frac{i}{2} f(\alpha, \beta)(N_a \otimes I - I \otimes N_b)} U_\gamma. \quad (43)$$

This allows one to write the observable $E$ as

$$E(n_1, n_2) = E^\epsilon(n_1, n_1) := U_\gamma^* |n_1\rangle \langle n_1| \otimes |n_2\rangle\langle n_2| U_\gamma, \quad (44)$$
for an appropriate $\gamma$. It follows that the Mach-Zehnder interferometer acts like a single beam splitter with transparency $\varepsilon$, where $\sqrt{\varepsilon} = \cos |\gamma|$, and $\gamma = \gamma(\alpha, \beta, \delta)$. The explicit form of $\gamma$, and thus of the effective transparency $\varepsilon$ can be most easily calculated by evaluating, for example, the single-photon probability $p^E_{|10\rangle}(1, 0)$, which is done below.

If the state $T'$ of the second mode is kept fixed, one may view the $a$-mode alone as the input system. This step is necessary if one wants to represent the idea that a light pulse, or even one photon, coming from one source is subjected to an interferometric measurement. In that view the counting statistics (40) defines an observable $F^{T', \varepsilon}$ of this mode such that for any $T$ and for all $n_1, n_2$ one has

$$\text{tr}[T F^{T', \varepsilon}(n_1, n_2)] := \text{tr}[T \otimes T' E^\varepsilon(n_1, n_2)].$$

(45)

If $T'$ is a vector state, then this observable is just the Neumark projection of $E^\varepsilon$ with the projection $I \otimes T'$:

$$F^{T', \varepsilon}(n_1, n_2) \equiv I \otimes T' E^\varepsilon(n_1, n_2) I \otimes T'.$$

(46)

Finally, taking the second mode to be in the vacuum state, the explicit form of this observable is obtained by a simple computation:

$$F^{0; \varepsilon}(n_1, n_2) = \frac{(n_1 + n_2)!}{n_1! n_2!} \varepsilon^{n_1} (1 - \varepsilon)^{n_2} |n_1 + n_2\rangle \langle n_1 + n_2|.$$

(47)

The counting statistics of a single mode light field sent through an interferometer defines thus an observable of this field. This observable is a PV measure exactly when $\varepsilon = 0$, or $\varepsilon = 1$. Any choice of the parameter $\varepsilon$ refers to a different experimental arrangement consisting of $BS_1$, $BS_2$, and $PS(\delta)$. As a rule, they all give rise to different observables $F^{0; \varepsilon}$, which are thus mutually exclusive in the trivial sense that any choice of $\varepsilon_1$, $\varepsilon_2$, and $\delta$ excludes another one. However, these observables all are mutually commuting. Nevertheless they reflect on a phenomenological level the wave-particle duality of a single photon, as will become clear below. We call this description phenomenological as the production of the wave or the particle phenomena is described in terms of the instrumental tools, without making reference to the behaviour of the observed entity.

The marginals $F^{0; \varepsilon}_i$, $i = 1, 2$, of the observable $F^{0; \varepsilon}$ are found to be of the form

$$F^{0; \varepsilon}_1(n) = \sum_{m=n}^{\infty} \binom{m}{n} \varepsilon^n (1 - \varepsilon)^{m-n} |m\rangle \langle m|,$$

$$F^{0; \varepsilon}_2(n) = F^{0; 1-\varepsilon}_1(n).$$

(48)

They are mutually commuting observables representing unsharp versions of the number observable $N_a$. 
We consider next the case of the incoming $\alpha$-mode being prepared in a number state $T = |n\rangle\langle n|$. The probabilities (45) obtain then the simple form
\[
p_{F_0;\varepsilon}^{(n_1, n_2)}(n_1, n_2) = \langle n | F_0;\varepsilon(n_1, n_2)|n\rangle = \frac{(n_1+n_2)!}{n_1!n_2!} \varepsilon^{n_1}(1-\varepsilon)^{n_2} \delta_{n_1+n_2, n}.
\] (49)

From these probabilities the conservation of photon number (energy) in the interferometer is manifest: $n$ input photons give rise to a total of $n = n_1 + n_2$ counts in the two detectors.

We are now ready to discuss the wave-particle duality for a single photon input $T = |1\rangle\langle 1|$. Formula (49) gives
\[
\begin{align*}
\langle 1 | F_0;\varepsilon(1, 0)|1\rangle &= \varepsilon, \quad (50a) \\
\langle 1 | F_0;\varepsilon(0, 1)|1\rangle &= 1 - \varepsilon, \quad (50b) \\
\langle 1 | F_0;\varepsilon(n_1, n_2)|1\rangle &= 0 \text{ if } n_1 + n_2 \neq 1. \quad (50c)
\end{align*}
\]

The anticoincidence of counts expressed in (50c) is an indication of the corpuscular nature of photons: one single photon cannot make two detectors fire. In this case the dependence of $\varepsilon$ on the parameters of the interferometer is easily determined:
\[
\varepsilon = \varepsilon_1\varepsilon_2 + (1 - \varepsilon_1)(1 - \varepsilon_2) + 2\sqrt{\varepsilon_1(1-\varepsilon_1)\varepsilon_2(1-\varepsilon_2)} \cos(\vartheta_2 - \vartheta_1 - \delta). \quad (51)
\]

There are three cases of special interest: $\varepsilon_1$ variable, $\varepsilon_2 = 1$; $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$; and $\varepsilon_1 = \frac{1}{2}$, $\varepsilon_2$ variable. The first choice gives $\varepsilon = \varepsilon_1$, the second $\varepsilon = \cos^2\left(\frac{1}{2}(\vartheta_2 - \vartheta_1 + \delta)\right)$, and the third $\varepsilon = \frac{1}{2}\left[1 + 2\sqrt{\varepsilon_2(1-\varepsilon_2)} \cos(\vartheta_2 - \vartheta_1 - \delta)\right] = \frac{1}{2}\left[(1 + 2\sqrt{\varepsilon_2(1-\varepsilon_2)} \cos\delta\right]$, where the last equality is under the assumption that $\vartheta_1 = \vartheta_2$. Some of these cases have been investigated experimentally, confirming thus the predicted quantum mechanical probabilities (50).\(^2\)

The experiment with $\varepsilon_2 = 1$ would allow one to decide from a single count event whether $\varepsilon_1$ was 1 or 0 if one of these values was given. This is interpreted as the calibration for a path measurement. If $\varepsilon_1$ differs from these values, then, of course, the notion of a path taken by the photon is meaningless. But the very ability of the device to detect the path (if it was fixed) destroys any interference. Next, the statistics obtained in the case $\varepsilon_2 = \frac{1}{2}$ reproduce the expected interference pattern resulting from many runs of this single-photon experiment. More precisely, the interference disappears if $\varepsilon_1$ is 0 or 1, which corresponds to the situation where the photon is forced to take exactly one path. In this sense a precise fixing of the path again destroys the interference. Maximal path indeterminacy, $\varepsilon_1 = \frac{1}{2}$, gives rise to optimal interference, while there is no way of getting any information on the path when $\varepsilon_2 = \frac{1}{2}$. In this way we recover the wave-particle duality in Bohr’s complementarity interpretation. There are mutually exclusive options for both, the preparation, as well as the registration, of path or wave behaviour.
In a recent experiment\textsuperscript{28}, a modified Mach-Zehnder interferometer, with $\varepsilon_1 = \frac{1}{2}$ fixed and variable $\varepsilon_2$, was introduced for testing the detection probability

$$
\frac{1}{2} \left[ 1 + 2 \sqrt{\varepsilon_2 (1 - \varepsilon_2) \cos \delta} \right].
$$

This experiment was interpreted as providing simultaneous information on the two complementary properties of a photon. Indeed, letting $\varepsilon_2$ vary from $\frac{1}{2}$ to 1, one recognises that the interference fades away gradually from the pattern with optimal contrast, $\cos^2 \left( \frac{1}{2} (\vartheta_2 - \vartheta_1 + \delta) \right)$, to no interference at all, $\frac{1}{2}$. The experimentally realised case $\varepsilon_2 = 0.994$ still leads to a recognisable interference pattern $(\varepsilon = \frac{1}{2}(1 + 0.154 \cos \delta))$ even though there is, loosely speaking, already a high (84%) confidence on the path of the photon. In a suitable measure, this situation was characterised by ascribing 98.2% particle nature and 1.8% wave nature to the photon. Unfortunately, in this experiment\textsuperscript{28} the incoming light pulses originated from a laser so that no genuine single photon situation was guaranteed. That is, the intensity was low enough to ensure, with high probability, the presence of only one photon in the interferometer, but the detection was not sensitive to single counts.

The analysis carried out so far rephrases the common view that the detection statistics of single-photon Mach-Zehnder interferometry exhibit both the wave-particle duality as well as the unsharp wave-particle behaviour for single photons. Note, however, that the language used here goes beyond the formal description that could be given in terms of the observables $F^{0;\varepsilon}$. An account based solely on the latter is phenomenological in the sense that the relevant photon observables $F^{0;\varepsilon}$, which pertain to the object under investigation, are mutually commutative; hence on the object level there is no complementarity. Only the various statistics for single photon input states display the \emph{complementary phenomena} in question. We shall now change our point of view to show that the same statistics can also be interpreted on the basis of \emph{complementary observables}.

To this end we redefine again the cut to be placed in the experimental set-up of Figure 1. Instead of taking the whole interferometer together with the detectors as the registration device, we consider the first beam splitter $BS_1$ and the phase shifter $PS(\delta)$ as parts of the preparation device. The object system is therefore the two-mode field prepared in a state

$$
S := V_\delta U_\alpha (T \otimes T') U^*_\alpha V^*_\delta.
$$

The detection statistics (40) can then be written as

$$
p_S^{E^\beta}(n_1, n_2) = p_W^{N_1 \otimes N_2}(n_1, n_2),
$$

for the observable $E^\beta$,

$$
E^\beta(n_1, n_2) := U^*_\beta |n_1\rangle \langle n_1| \otimes |n_2\rangle \langle n_2| U_\beta.
$$
We restrict our attention to the single photon case, $T = |1⟩⟨1|$, $T' = |0⟩⟨0|$, so that the possible initial states of the two-mode field are the vector states

$$\psi_{\alpha, \delta} := V_\delta U_\alpha |10⟩ = \sqrt{\xi_1} |10⟩ + e^{-i(\theta_1 + \delta)} \sqrt{1 - \xi_1} |01⟩,$$

(55)

where, for instance, $|10⟩ = |1⟩ \otimes |0⟩$. Let $P_{10}$ and $P_{01}$ denote the one dimensional projections $|1⟩⟨1| \otimes |0⟩⟨0|$ and $|0⟩⟨0| \otimes |1⟩⟨1|$ of the two-mode Fock space. Then $P_{10} + P_{01}$ projects on to the two dimensional subspace of the vectors (55) which we take to represent the object system to be investigated. Due to the number conservation under the unitary map $U_\beta$ the projection operators (54) commute with $P_{10} + P_{01}$, so that the following operators define a PV measure on the state space of the object system:

$$F^{1,0;\varepsilon_2}(n_1, n_2) := (P_{10} + P_{01}) E^{\beta}(n_1, n_2) (P_{10} + P_{01}).$$

(56)

For the states (55) the observables $E^{\beta}$ and $F^{1,0;\varepsilon_2}$ have the same expectations,

$$\langle \psi_{\alpha, \delta} | F^{1,0;\varepsilon_2}(n_1, n_2) \psi_{\alpha, \delta} \rangle = \langle \psi_{\alpha, \delta} | E^{\beta}(n_1, n_2) \psi_{\alpha, \delta} \rangle,$$

(57)

for all $n_1, n_2$ and for each $\alpha$ and $\delta$. In particular, this means that the observable $F^{1,0;\varepsilon_2}$ is in fact determined by the detection statistics.

The operators of Eq. (56) read as follows:

$$F^{1,0;\varepsilon_2}(1,0) = E^{\beta}(1,0) = P \left[ U_\beta^* |10⟩ \right]$$

$$= P \left[ \sqrt{\xi_2} |10⟩ + e^{-i\theta_2} \sqrt{1 - \xi_2} |01⟩ \right],$$

(58a)

$$F^{1,0;\varepsilon_2}(0,1) = E^{\beta}(0,1) = P \left[ U_\beta^* |01⟩ \right]$$

$$= P \left[ \sqrt{1 - \xi_2} |10⟩ - e^{-i\theta_2} \sqrt{\xi_2} |01⟩ \right],$$

(58b)

$$F^{1,0;\varepsilon_2}(n_1, n_2) = O \text{ if } n_1 + n_2 \neq 1.$$  

(58c)

Also, for any $\alpha$ and $\delta$

$$\langle \psi_{\alpha, \delta} | F^{1,0;\varepsilon_2}(1,0) \psi_{\alpha, \delta} \rangle = \varepsilon,$$

(59a)

$$\langle \psi_{\alpha, \delta} | F^{1,0;\varepsilon_2}(0,1) \psi_{\alpha, \delta} \rangle = 1 - \varepsilon,$$

(59b)

$$\langle \psi_{\alpha, \delta} | F^{1,0;\varepsilon_2}(n_1, n_2) \psi_{\alpha, \delta} \rangle = 0 \text{ if } n_1 + n_2 \neq 1,$$

(59c)

with $\varepsilon$ given by Eq. (51).

On the level of a statistical description the change of viewpoint has brought nothing new; the measurement outcome probabilities (59) are, as they should be, the same as those of Eq. (50), or just the detection statistics (40). There is, however, an essentially new aspect in the description. All the photon observables $F^{1,0;\varepsilon_2}$ are mutually complementary in the strong measurement-theoretical sense: these
observables cannot be measured or even tested together. In particular, the observables \( F^{1,0;1} \) and \( F^{1,0;\frac{1}{2}} \) associated with the extreme choices \( \varepsilon_2 = 1 \) and \( \varepsilon_2 = \frac{1}{2} \) are complementary path and interference observables,

\[
F^{1,0;1}(1,0) = P_{10}, \tag{60a}
\]
\[
F^{1,0;\frac{1}{2}}(1,0) = P \left[ \frac{1}{\sqrt{2}} | 10 \rangle + \frac{1}{\sqrt{2}} e^{-i\vartheta_2} | 01 \rangle \right]. \tag{60b}
\]

In Ref. 28 a measurement of \( F^{1,0;\varepsilon} \) was interpreted as a joint unsharp determination of the complementary path and interference observables \( F^{1,0;1} \) and \( F^{1,0;\frac{1}{2}} \). While the observable \( F^{1,0;\varepsilon} \) does entail probabilistic information on the two complementary observables, it is a sharp observable, and cannot therefore be considered to represent an unsharp joint measurement in the sense of the general point of view adopted in this work. In the next section a proposal is (re)analysed which does lead to such a joint measurement, as can be read off from the ensuing POV measures.

7. Photon interference with a QND path determination. We describe next an experimental scheme in which a Mach-Zehnder interferometer is again used for measuring an interference observable; but this time an additional component is added that is capable of carrying out a nondemolishing path determination. The experimental set-up is sketched out in Figure 2.

[Figure 2. Mach-Zehnder interferometer with a Kerr medium.]

The new component is a Kerr medium placed in the second arm of the interferometer, which will couple the \( b \)-mode field with a single mode probe field according to the interaction

\[
U_K = I_1 \otimes e^{-i\lambda(N_2 \otimes N_3)}, \tag{61}
\]

where \( N_3 = c^* c \) is the number observable of the probe field. This coupling will not change the number of photons of the interferometer field, but it will affect the phase of the probe field. Therefore, analysing the latter with a phase sensitive detector \( D_K \) yields information on the number of photons in the second arm of the interferometer. At the same time the detectors \( D_1 \) and \( D_2 \) register the number of photons \( N_1 = a^* a \) and \( N_2 = b^* b \) emerging from the second beam splitter, exhibiting thereby the possible interference pattern. Such a scheme was proposed recently,\(^{29}\) with the quadrature observable \( \frac{1}{2}(c^* + c) \) employed as the readout observable for the probe mode. In fact any phase observable conjugate to \( N_3 \) would do as well.

Let \( T, |0\rangle \langle 0|, \) and \( T' \) be the input states of the incoming photon pulse, the \( b \)-mode, and the probe field. One is again facing the task of splitting the whole experiment into an observed and an observing part, or into a preparation and a registration. Taken as a whole, the state of the total field will change in the Mach-Zehnder-Kerr apparatus according to

\[
T \otimes |0\rangle \langle 0| \otimes T' \mapsto W := U_{\beta} U_K V_{\delta} U_\alpha (T \otimes |0\rangle \langle 0| \otimes T') U^*_\alpha V^*_\delta U^*_K U^*_\beta. \tag{62}
\]
The probability of detecting \( n \) photons in the counter \( D_1 \) and reading a value in a set \( X \) in the homodyne detector \( D_K \) is therefore

\[
p_{\mathcal{W}}^{N_1 \otimes I_2 \otimes E}(n, 1, X) = \text{tr}[W |n\rangle\langle n| \otimes I_2 \otimes E(X)],
\]  

where \( E \) is some (phase sensitive) readout observable of the third mode (such as the spectral measure of a quadrature component). For simplicity, we have omitted now the detector \( D_2 \), since the statistics of \( D_1 \) is already sufficient for indicating the possible interference phenomenon.

The detection statistics (63) can again be interpreted in various ways with respect to the input state of an appropriate system. We consider the incoming photon pulse, the \( a \)-mode, as the input system. The statistics (63) define then, for each initial state \( T' \) of the probe field, an observable \( A^{0,T'} \) associated with the \( a \)-mode field such that for all initial states \( T \) of that mode and for all \( D_1 \)-values \( n \) and \( D_K \)-values \( X \),

\[
p_{T}^{A^{0,T'}}(n, X) := p_{\mathcal{W}}^{N_1 \otimes I_2 \otimes E}(n, 1, X).
\]

This observable is an ‘enriched’ version of the first marginal \( F_1^{0;\varepsilon} \), Eq. (48), of the observable (47). It depends on the whole variety of the “apparatus parameters” \(|0\rangle\langle 0|, T', \alpha, \beta, \delta, \) and \( \lambda \). Unlike \( F_1^{0;\varepsilon} \), the observable \( A^{0,T'} \) contains direct information on the dual aspects of a single photon. To see this, we determine this observable by specifying the beam splitters to be semitransparent (\( \varepsilon_1 = \varepsilon_2 = \frac{1}{2} \)) with \( \vartheta_1 = \vartheta_2 = \frac{\pi}{2} \), the case where one expects optimal interference. The observable \( A^{0,T'} \) is now found by direct computation:

\[
A^{0,T'}(n, X) = \sum_{m=0}^{\infty} \frac{(m+n)!}{m!n!} |m+n\rangle\langle m+n| \text{tr}\left[ T' \sin^n\left(\frac{\delta}{2}I_3 - \frac{\lambda}{2}N_3\right) \cos^m\left(\frac{\delta}{2}I_3 - \frac{\lambda}{2}N_3\right) e^{-i(m+n)\frac{\delta}{2}N_3} E(X) e^{i(m+n)\frac{\delta}{2}N_3} \sin^n\left(\frac{\delta}{2}I_3 - \frac{\lambda}{2}N_3\right) \cos^m\left(\frac{\delta}{2}I_3 - \frac{\lambda}{2}N_3\right) \right].
\]

One may go on to determine the marginal observables of \( A^{0,T'} \). The first one is obtained by putting \( X = R \):

\[
A^{0,T'}_1(n) = \sum_{m=0}^{\infty} \frac{(m+n)!}{m!n!} |m+n\rangle\langle m+n| \text{tr}\left[ T' \sin^{2n}\left(\frac{\delta}{2}I_3 - \frac{\lambda}{2}N_3\right) \cos^{2m}\left(\frac{\delta}{2}I_3 - \frac{\lambda}{2}N_3\right) \right],
\]

showing that \( A^{0,T'}_1 \) is an unsharp number observable. The second marginal observable \( X \mapsto A^{0,T'}_2(X) := \sum_n A^{0,T'}(n, X) \) is also directly obtained from (65) but it is less straightforward to exhibit a simplified expression for it. However, this is not
needed here since our primary interest is in the case of the single photon input state $T = |1\rangle\langle 1|$. For this state all the relevant probabilities are easily computed:

$$p_{|1\rangle}^{A,\lambda}(0, X) = tr \left[T' \cos (\frac{\lambda}{2} I_3 - \frac{\lambda}{2} N_3) e^{-i \lambda N_3} E(X) e^{i \lambda N_3} \cos (\frac{\lambda}{2} I_3 - \frac{\lambda}{2} N_3) \right], \quad (67a)$$

$$p_{|1\rangle}^{A,\lambda}(1, X) = tr \left[T' \sin (\frac{\lambda}{2} I_3 - \frac{\lambda}{2} N_3) e^{-i \lambda N_3} E(X) e^{i \lambda N_3} \sin (\frac{\lambda}{2} I_3 - \frac{\lambda}{2} N_3) \right], \quad (67b)$$

$$p_{|1\rangle}^{A,\lambda}(0, R) = p_{|1\rangle}^{A,\lambda}(0) = tr \left[T' \cos^2 (\frac{\lambda}{2} I_3 - \frac{\lambda}{2} N_3) \right], \quad (68a)$$

$$p_{|1\rangle}^{A,\lambda}(1, R) = p_{|1\rangle}^{A,\lambda}(1) = tr \left[T' \sin^2 (\frac{\lambda}{2} I_3 - \frac{\lambda}{2} N_3) \right], \quad (68b)$$

$$p_{|1\rangle}^{A,\lambda}(N, X) = p_{|1\rangle}^{A,\lambda}(0, X) + p_{|1\rangle}^{A,\lambda}(1, X) = \frac{1}{2} tr \left[T' E(X) \right] + \frac{1}{2} tr \left[T' e^{-i \lambda N_3} E(X) e^{i \lambda N_3} \right]. \quad (69)$$

As a consistency check one may first observe that for $\lambda = 0$ the probabilities (68) are just the single photon counting probabilities in a Mach-Zehnder interferometer with $\varphi_1 = \varphi_2 = \frac{\pi}{2}$ and $\varphi_1 = \varphi_2 = \frac{\pi}{2}$. The introduction of the Kerr medium ($\lambda \neq 0$) in the second arm of the interferometer affects these probabilities with a $T'$-dependent phase shift. Moreover, the detector $D_K$ allows one to collect now an additional single-mode statistics (69) which contains information on the path of the photon. It must be emphasised that the $\alpha$-mode observable $A_\alpha^{\lambda,\lambda'}$ and the ensuing single photon probability measures $p_{|1\rangle}^{A_\alpha^{\lambda,\lambda'}}$ do depend on the state of the probe field $T'$ as well as on the path-indicating observable $E$. Till now no properties of $T'$ or $E$ are used, and the problem is to choose these “control parameters” in such a way that the probabilities (68-69) would display a good interference pattern together with a reliable path determination. Clearly, if $T'$ is a number state or $E$ is compatible with the number observable $N_3$ there will be no path information available from (69).

On the other hand, if $E$ is any phase observable conjugate to the number observable $N_3$, characterised by the covariance property $e^{-i \lambda N_3} E(X) e^{i \lambda N_3} = E(X - \lambda)$, then there is a possibility of obtaining information about the path: a photon traversing through the second arm of the interferometer leaves a trace in the statistics of the second marginal $A_\alpha^{\lambda,\lambda'}$ collected at the detector $D_K$. In Ref. 29 $E$ was chosen to be the observable $\frac{1}{2}(c + c^*)$ and $T'$ a coherent or squeezed state.

We may again describe the whole experiment on the basis of the ‘realistic’ cut introduced in the preceding subsection, where the first beam splitter and the phase shifter belong to the preparation device. We assume from the outset that the $b$-mode is idle so that the prepared state $S$ of the $(a, b)$-mode field is

$$S := V_\delta U_\alpha \left( T \otimes |0\rangle\langle 0| \right) U_\alpha^* V_\delta^*. \quad (70)$$

From the counting statistics (63) one then obtains a unique observable $E^{T'; \varphi_2}$ of the $(a, b)$-mode field for any fixed input state $T'$ of the probe mode:

$$p_{|S\rangle}^{E^{T'; \varphi_2}}(n, X) = tr \left[ S \otimes T' U_K U_\beta^* (|n\rangle\langle n| \otimes I_2 \otimes E(X)) U_\beta U_K \right]. \quad (71)$$
For the explicit determination of this observable we shall consider only a single photon input state $T = |1\rangle\langle 1|$ so that the object system is given again by the subspace of states $\psi_{\alpha,\delta}$ from Eq. (55). On that state space the observable $E^{T':\varepsilon_2}$ is reduced to the following:

$$F^{T':\varepsilon_2}(n, X) := (P_{10} + P_{01}) E^{T':\varepsilon_2}(n, X) (P_{10} + P_{01}). \quad (72)$$

Evaluation of Eq. (35) yields

$$E^{T':\varepsilon_2}(n, X) = \sum_{n_1, m_1, n_2, m_2} |n_1, n_2\rangle\langle m_1, m_2|$$

$$\langle n_1, n_2| U_\varepsilon^* (|n\rangle\langle n| \otimes I_2) U_\varepsilon |m_1, m_2\rangle \text{tr} [T' e^{i\lambda n_1 N_3} E(X) e^{-i\lambda m_2 N_3}] .$$

The observable (72) is obtained by simply carrying out the above sum under the constraint $n_1 + n_2 = m_1 + m_2 = 1$:

$$F^{T':\varepsilon_2}(n, X) = |10\rangle\langle 10| [\varepsilon_2 \delta_{n_1} + (1 - \varepsilon_2) \delta_{n_0}] \text{tr} [T' E(X)]$$

$$+ |01\rangle\langle 01| [(1 - \varepsilon_2) \delta_{n_1} + \varepsilon_2 \delta_{n_0}] \text{tr} [T' e^{i\lambda n_1 N_3} E(X) e^{-i\lambda n_2 N_3}]$$

$$+ |10\rangle\langle 01| \sqrt{\varepsilon_2 (1 - \varepsilon_2)} e^{i\vartheta_2} (\delta_{n_1} - \delta_{n_0}) \text{tr} [T' E(X) e^{-i\lambda n_3}]$$

$$+ |01\rangle\langle 10| \sqrt{\varepsilon_2 (1 - \varepsilon_2)} e^{-i\vartheta_2} (\delta_{n_1} - \delta_{n_0}) \text{tr} [T' e^{i\lambda n_3} E(X)].$$

It is instructive to determine the marginal observables $F_1^{T':\varepsilon_2}(n) := F^{T':\varepsilon_2}(n, R)$ and $F_2^{T':\varepsilon_2}(X) := \sum_n F^{T':\varepsilon_2}(n, X)$:

$$F_1^{T':\varepsilon_2}(1) = |10\rangle\langle 10| \varepsilon_2 + |01\rangle\langle 01| (1 - \varepsilon_2)$$

$$+ |10\rangle\langle 01| \sqrt{\varepsilon_2 (1 - \varepsilon_2)} e^{i\vartheta_2} \text{tr} [T' e^{-i\lambda n_3}]$$

$$+ |01\rangle\langle 10| \sqrt{\varepsilon_2 (1 - \varepsilon_2)} e^{-i\vartheta_2} \text{tr} [T' e^{i\lambda n_3}], \quad (75a)$$

$$F_1^{T':\varepsilon_2}(0) = |10\rangle\langle 10| (1 - \varepsilon_2) + |01\rangle\langle 01| \varepsilon_2$$

$$+ |10\rangle\langle 01| \sqrt{\varepsilon_2 (1 - \varepsilon_2)} e^{i\vartheta_2} \text{tr} [T' e^{-i\lambda n_3}]$$

$$+ |01\rangle\langle 10| \sqrt{\varepsilon_2 (1 - \varepsilon_2)} e^{-i\vartheta_2} \text{tr} [T' e^{i\lambda n_3}], \quad (75b)$$

$$F_2^{T':\varepsilon_2}(X) = |10\rangle\langle 10| \text{tr} [T' E(X)] + |01\rangle\langle 01| \text{tr} [T' e^{i\lambda n_3} E(X) e^{-i\lambda n_3}]. \quad (76)$$

It is obvious that irrespective of the choice of $T'$ and the observable $E$, the first marginal is a smeared interference observable, while the second one is a smeared path observable. This shows that the device serves its purpose of establishing a joint measurement of these complementary properties. With reference to the latter marginal it should be noted that neither $T'$ nor $E$ may commute with $N_3$ since otherwise the phase sensitivity is lost. One may proceed to analyse Eqs. (75, 76), showing that high path confidence will lead to low interference contrast, and vice
versa. Indeed, an optimal interference would be obtained if $\varepsilon_2 = \frac{1}{2}$ and if the numbers $\text{tr} \left[ e^{\pm i \lambda N_3 T'} \right]$ were equal to unity. But this requires $T'$ to be an eigenstate of $N_3$, which destroys the path measurement. On the other hand, imagine that $E$ is a phase observable, $X = [0, \frac{\pi}{2}]$, and $T'$ is chosen such that the number $\text{tr} [T' E(X)]$ is close to unity. This can be achieved, for example, for a coherent state with a large amplitude. Such states have a slowly varying number distribution so that the modulus of the complex numbers $\text{tr} \left[ e^{\pm i \lambda N_3 T'} \right]$ is small compared to unity. But this is to say that the first marginal observable is also ‘close’ to a path observable, thus yielding only low interference contrast.

There exists another scheme of a joint measurement of complementary path and interference observables that is based solely on mirrors, beam splitters and phase shifters, as they were used in the original Mach-Zehnder interferometer (Figure 3).

![Figure 3. Expanded Mach-Zehnder interferometer](image)

Regarding again the first beam splitter and the first phase shifter as parts of the preparation device, the residual elements constitute the measuring instrument which now has four detectors. An explicit analysis of this experiment with respect to a single-photon input has been carried out elsewhere,$^{30}$ so that here we may restrict ourselves to a nontechnical summary. The detection statistics again give rise to a unique observable of the object, which now has four outcomes. One may combine pairs of these outcomes to add them up to three two-valued marginal observables. By a suitable adjustment of the system parameters it is possible to ensure that two of these observables are path and interference observables; moreover, one may achieve the result that the full detection statistics uniquely determine the prepared state, that is, the measurement can be managed to be informationally complete. In particular, the statistics would allow one to find out the values of $\alpha$ and $\delta$.

### 8. Conclusion

In this paper we have elucidated the main measurement-theoretical implications of the various versions of complementarity. On the basis of an exact formalisation we have confirmed the common view that complementarity in its strongest sense implies the noncoexistence of the observables in question. Probabilistic complementarity, which is equivalent to the measurement-theoretical version as long as only sharp observables are involved, was found to persist for some pairs of unsharp observables even when these are coexistent. In this way the apparent contradiction between the possibility of joint measurements of “complementary” pairs of observables is resolved by taking into account unsharp observables: a complementary pair of sharp observables can be replaced with a pair of smeared versions which are then coexistent though still probabilistically complementary.

Starting from a quantum optical framework we have reconstructed the common description of single-photon interferometer experiments in a language which associates complementary path and interference observables with the mutually exclusive options of detecting particle or wave phenomena. Furthermore, it was found
that introducing additional elements into the Mach-Zehnder interferometer yields the possibility of jointly measuring the path and observing interference patterns, in one single experiment. The ensuing statistics is that of a joint observable for a co-existent pair of unsharp path and unsharp interference observables. Bohr’s original conception of complementarity as referring to a strict mutual exclusiveness is thus confirmed for pairs of sharp observables. Einstein’s attempt to evade the complementarity verdict can be carried out in some sense, but there is a price: lifting the measurement-theoretical complementarity must be paid for by introducing a certain degree of unsharpness, so that increasing confidence in the path determination goes along with decreasing interference contrast, and vice versa.

Finally it may be worth noting that there is not just one unique way of giving a theoretical account of one and the same experiment. What is regarded as part of physical reality depends appreciably on the physicists’ decision to place the Heisenberg cut between preparation and measurement. In the present example it becomes very obvious that quite different descriptions may arise. The crucial point is that some decision has to be made and that each corresponds to adopting a certain interpretation. In our view it seems natural to adopt the most realistic picture which is anyway used in practice as it serves as the best guide for physical intuition.

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9. We recall that for two bounded self-adjoint operators $A$ and $B$ the ordering $A \leq B$ is defined as $\langle \varphi | A \varphi \rangle \leq \langle \varphi | B \varphi \rangle$ for all $\varphi \in \mathcal{H}$.
10. For two bounded positive operators $A, B$ the greatest lower bound may or may not exist among the positive operators. If it does exist it is denoted $A \wedge B$.
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