Block Markov Superposition Transmission of RUN Codes
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Abstract—In this paper, we propose a simple procedure to construct (decodable) good codes with any given alphabet (of moderate size) for any given (rational) code rate to achieve any given target error performance (of interest) over the additive white Gaussian noise (AWGN) channels. We start with constructing codes over groups for any given code rates. This can be done in an extremely simple way if we ignore the error performance requirement for the time being. Actually, this can be satisfied by repetition (R) codes and uncoded (UN) transmission along with time-sharing technique. The resulting codes are simply referred to as RUN codes for convenience. The encoding/decoding algorithms for RUN codes are almost trivial. In addition, the performance can be easily analyzed. It is not difficult to imagine that a RUN code usually performs far away from the corresponding Shannon limit. Fortunately, the performance can be improved as required by spatially coupling the RUN codes via block Markov superposition transmission (BMST), resulting in the BMST-RUN codes. Simulation results show that the BMST-RUN codes perform well (within one dB away from Shannon limits) for a wide range of code rates and outperform the BMST with bit-interleaved coded modulation (BMST-BICM) scheme.

Index Terms—Block Markov superposition transmission (BMST), codes over groups, spatial coupling, time-sharing.

I. INTRODUCTION

Since the invention of turbo codes [1] and the rediscovery of low-density parity-check (LDPC) codes [2], many turbo/LDPC-like codes have been proposed in the past two decades. Among them, the convolutional LDPC codes [3], recast as spatially coupled LDPC (SC-LDPC) codes in [4], exhibit a threshold saturation phenomenon and were proved to have better performance than their block counterparts. In a certain sense, the terminology “spatial coupling” is more general, as can be interpreted as making connections among independent subgraphs, or equivalently, as introducing memory among successive independent transmissions. With this interpretation, braided block codes [5] and staircase codes [6], as the convolutional versions of (generalized) product codes, can be classified as spatially coupled codes. In [7], the spatially coupled version of turbo codes was proposed, whose belief propagation (BP) threshold is also better than that of the uncoupled ensemble.

 Recently, block Markov superposition transmission (BMST) [8–10] was proposed, which can also be viewed as the spatial coupling of generator matrices of short codes. The original BMST codes are defined over the binary field $F_2$. In [9], it has been pointed out that any code with fast encoding algorithms and soft-in soft-out (SISO) decoding algorithms can be taken as the basic code. For example, one can take the Hadamard transform (HT) coset codes as the basic codes, resulting in a class of multiple-rate codes with rates ranging from $1/2^p$ to $(2^p - 1)/2^p$, where $p$ is a positive integer [11, 12]. Even more flexibly, one can use the repetition and/or single-parity-check (SPC) codes as the basic codes to construct a class of multiple-rate codes with rates ranging from $1/N$ to $(N - 1)/N$, where $N > 1$ is an integer [13]. It has been verified by simulation that the construction approach is applicable not only to binary phase-shift keying (BPSK) modulation but also to bit-interleaved coded modulation (BICM) [14], spatial modulation [15], continuous phase modulation (CPM) [16], and intensity modulation in visible light communications (VLC) [17]

In this paper, we propose a procedure to construct codes over groups, which extends the construction of BMST-RSPC codes [13] in the following two aspects. First, we allow uncoded symbols occurring in the basic codes. Hence the encoding/decoding algorithms for the basic codes become simpler. Second, we derive a performance union bound for the repetition codes with any given signal mapping, which is critical for designing good BMST codes without invoking simulations. We will not argue that the BMST construction can always deliver better codes than other existing constructions,

Rather, we argue that the proposed one is more flexible in the sense that it applies to any given signal set (of moderate size), any given (rational) code rate and any target error performance (of interest). We start with constructing group codes, referred to as RUN codes, with any given rate by time-sharing between repetition (R) codes and/or uncoded (UN) transmission. By transmitting the RUN codes in the BMST manner, we can have a class of good codes (called BMST-RUN codes). The performance of a BMST-RUN code is closely related to the encoding memory and can be predicted analytically in the high signal-to-noise ratio (SNR) region with the aid of the readily-derived union bound. Simulation results show that the BMST-RUN codes can approach the Shannon limits at any given target error rate (of interest) in a wide range of code rates over the additive white Gaussian noise (AWGN) channels.

1Actually, compared with SC-LDPC codes, the BMST codes usually have a higher error floor. However, the existence of the high error floor is not a big issue since it can be lowered if necessary by increasing the encoding memory.
addition, compared with the BMST-BICM scheme, the BMST-RUN codes perform better in the low code rate region.

The rest of this paper is organized as follows. In Section II, we take a brief review of the BMST technique. In Section III, we discuss constructing group codes with any given signal set and any given code rate. In Section IV, we propose the construction method of BMST-RUN codes and discuss the performance lower bound. In Section V, we give simulation results and make a performance comparison between the BMST-RUN codes and the BMST-BICM scheme. In Section VI, we conclude this paper.

II. REVIEW OF BINARY BMST CODES

A. Basics of BMST Codes

Binary BMST codes are convolutional codes with large constraint lengths [8,9]. Typically, a binary BMST code of memory $m$ consists of a short code (called the basic code) and at most $m + 1$ interleavers [10]. Let $C[n,k]$ be the basic code defined by a $k \times n$ generator matrix $G$ over the binary field $\mathbb{F}_2$. Denote $u^{(0)}, u^{(1)}, \ldots, u^{(L-1)}$ as $L$ blocks of data to be transmitted, where $u^{(t)} \in \mathbb{F}_2^k$ for $0 \leq t \leq L - 1$. Then, the encoding output $c^{(t)} \in \mathbb{F}_2^t$ at time $t$ can be expressed as [10]

$$c^{(t)} = u^{(t)}G\Pi_0 + u^{(t-1)}G\Pi_1 + \cdots + u^{(t-m)}G\Pi_m,$$

(1)

where $u^{(t)}$ is initialized to be $0 \in \mathbb{F}_2^k$ for $t < 0$ and $\Pi_0, \ldots, \Pi_m$ are $m + 1$ permutation matrices of order $n$. For $L \leq t \leq L + m - 1$, the zero message sequence $u^{(t)} = 0 \in \mathbb{F}_2^k$ is input into the encoder for termination. Then $c^{(t)}$ is mapped to a signal vector $s^{(t)}$ and transmitted over the channel, resulting in a received vector $y^{(t)}$.

At the receiver, the decoder executes the sliding-window decoding (SWD) algorithm to recover the transmitted data $u^{(0)}, \ldots, u^{(L-1)}$ [8,9]. Specifically, an SWD algorithm with a decoding delay $d$, the decoder takes $y^{(t)}, \ldots, y^{(t+d)}$ as inputs to recover $u^{(t)}$ at time $t$, which is similar to the window decoding (WD) of the SC-LDPC codes [18–20]. The structure of the BMST codes also admits a two-phase decoding (TPD) algorithm [10], which can be used to reduce the decoding delay and to predict the performance in the extremely low bit-error-rate (BER) region.

As discussed in [9], binary BMST codes have the following two attractive features.

1) Any code (linear or nonlinear) can be the basic code as long as it has fast encoding algorithms and SISO decoding algorithms.

2) Binary BMST codes have a simple genie-aided lower bound when transmitted over the AWGN channels using BPSK modulation, which shows that the maximum extra coding gain can approach $10\log_{10}(m+1)$ dB compared with the basic code. The tightness of this simple lower bound in the high SNR region under the SWD algorithm has been verified by both the simulation and the extrinsic information transfer (EXIT) chart analysis [21].

Based on the above two facts, a general procedure has been proposed for constructing capacity-approaching codes at any given target error rate [10]. Suppose that we want to construct a binary BMST code of rate $R$ at a target BER of $P_{\text{target}}$.
BMST-HT codes perform better than the RSPC codes by embedding the RSPC codes into the BMST system. As the time-sharing factor $\alpha$ changes from 0 to $N - 2$, the code rate varies from $(N - 1)/N$ to 1/N. A general framework. An example for $N = 5$.

class of multiple-rate codes with fixed code length were constructed [11, 12]. In [11, 12], it has been demonstrated that the construction method is effective and the BMST-HT codes perform well for a wide range of code rates. Fig. 2 shows the performance of the BMST-HT codes with the Cartesian products of the HT-coset codes $[8, K]^{1250}_+$ and the basic codes [12].

2) Repetition (R) Codes and/or Single-Parity-Check (SPC) Codes As Basic Codes: Although BMST-HT codes perform well, only codes of rate $K/2^p$ can be constructed. To make the construction more flexible, the R code $[N, 1]$ and/or the SPC code $[N, N - 1]$ are used as basic codes [13]. Fig. 3 shows the time-sharing of an R code $[N, 1]$ and an SPC code $[N, N - 1]$. As the time-sharing factor $\alpha$ varies from 0 to $N - 2$, we have a set of multiple-rate codes with rates ranging from $(N - 1)/N$ to 1/N. Then the performance can be improved as required by embedding the RSPC codes into the BMST system. Fig. 2 shows the performance of the BMST-RSPC codes with the Cartesian products of the RSPC codes $[80, 8K]^{125}_+$ as basic codes [13].

Fig. 2. The required SNRs for the BMST-HT codes with $N = 8$ and the BMST-RSPC codes with $N = 10$ to achieve the BER of $10^{-5}$ with BPSK modulation over the AWGN channels, where both schemes are with a sub-block length of 10000.

$$N - 2 = \alpha + \beta$$

(a)

$$3 = \alpha + \beta$$

(b)

Fig. 3. The time-sharing of an R code $[N, 1]$ and an SPC code $[N, N - 1]$. As the time-sharing factor $\alpha$ changes from 0 to $N - 2$, the code rate varies from $(N - 1)/N$ to 1/N. (a) A general framework. (b) An example for $N = 5$.

C. Applications of Binary BMST Codes

Binary BMST codes can also be combined with high-order modulations and transmitted over other channels. In [14], high-order modulations are serially concatenated to the binary BMST encoder, resulting in the BMST-BICM scheme. The BMST-BICM scheme performs well over both the AWGN channels and the Rayleigh channels with a small coherence period. In [15], the encoded sequence of a binary BMST code is transmitted with the spatial modulation over the Rayleigh channels. In [16], the minimum-shift keying (MSK) modulation is combined with the binary BMST codes and the performance lower bound is analyzed. In [17], binary BMST codes are applied to the indoor visible light communications. All these examples show that the BMST construction is effective for a wide range of code rates over different types of channels. All delivered codes behave similarly, such as having error propagation in the low SNR region, having near-capacity performance in the waterfall region and matching the genie-aided bound in the error-floor region.

III. RUN CODES OVER GROUPS

A. System Model and Notations

Consider a symbol set $M = \{0, 1, \cdots, q - 1\}$ and an $\ell$-dimensional signal constellation $A \subset \mathbb{R}^\ell$ of size $q$. The symbol set $M$ can be treated as a group by defining the operation $u \oplus w = (u + w) \mod q$ for $u, w \in M$. Let $\varphi$ be a (fixed) one-to-one mapping $\varphi : M \rightarrow A$. Let $u \in M$ be a symbol to be transmitted. For the convenience of performance analysis, instead of transmitting $\varphi(u)$ directly, we transmit a signal $s = \varphi(u \oplus w)$, where $w$ is a sample of a uniformly distributed random variable over $M$ and assumed to be known at the receiver. The received signal $y = s + z$, where $+$ denotes the component-wise addition over $\mathbb{R}^\ell$ and $z$ is an $\ell$-dimensional sample from a white Gaussian noise process with each dimension having mean zero and variance $\sigma^2$. The SNR is defined as

$$\text{SNR} = \frac{\sum_{s \in A} ||s||^2}{\ell \sigma^2 q},$$

where $||s||^2$ is the squared Euclidean norm of $s$.

In this paper, for a discrete random variable $V$ over a finite set $\mathcal{V}$, we denote its a priori message and extrinsic message as $P_V^{\text{a}}(v), v \in \mathcal{V}$ and $P_V^{\text{e}}(v), v \in \mathcal{V}$, respectively. A SISO decoding is a process that takes a priori messages as inputs and delivers extrinsic messages as outputs. We assume that the information messages are independent and uniformly distributed (i.i.d.) over $M$.

B. Repetition (R) codes

Fig. 4 shows the transmission of a message $u$ for $N$ times over the AWGN channels.

![Fig. 4. A message $u$ is encoded into $v = (u, \cdots, u)$ and transmitted over the AWGN channels.](image-url)
1) **Encoding**: The encoder of an R code $\mathcal{C}[N, 1]$ over $\mathcal{M}$ takes as input a single symbol $u \in \mathcal{M}$ and delivers as output an $N$-dimensional vector $v = (v_0, \ldots, v_{N-1}) = (u, \ldots, u)$.

2) **Mapping**: The $j$-th component $v_j$ of the codeword $v$ is mapped to the signal $s_j = \varphi(v_j \oplus w_j)$ for $j = 0, \ldots, N - 1$, where $w = (w_0, \ldots, w_{N-1})$ is a random vector sampled from an i.u.d. process over $\mathcal{M}$.

3) **Demapping**: Let $y = (y_0, \ldots, y_{N-1})$ be the received signal vector corresponding to the codeword $v$. The a priori messages input to the decoder are computed as

$$P_{Y_j}(v) \propto \exp \left( - \frac{||y_j - \varphi(v \oplus w_j)||^2}{2\sigma^2} \right), v \in \mathcal{M}$$

for $j = 0, \ldots, N - 1$.

4) **Decoding**: The SISO decoding algorithm computes the a posteriori messages

$$P_{\alpha}^\gamma(u) \propto \prod_{0 \leq \ell \leq N-1} P_{Yj}(u), u \in \mathcal{M}$$

for making decisions and the extrinsic messages

$$P_{\eta}^\gamma(v) \propto \prod_{0 \leq \ell \leq N-1, \ell \neq j} P_{Yj}(v), v \in \mathcal{M}$$

for $j = 0, \ldots, N - 1$ for iteratively decoding when coupled with other sub-systems.

5) **Complexity**: Both the encoding/mapping and the demapping/decoding have linear computational complexity per coded symbol.

6) **Performance**: Let $\hat{u}$ denote the hard decision output. The performance is measured by the symbol-error-rate (SER) $\text{SER} \triangleq \Pr \{ \hat{U} \neq U \} = \sum_{u \in \mathcal{M}} \frac{1}{q} \Pr \{ \hat{U} \neq U | U = u \}$. Define $e = \hat{u} \oplus u$, where $\oplus$ denotes the subtraction under modulo-$q$ operation. Due to the existence of the random vector $w$, the performance is irrelevant to the transmitted symbol $u$. We define

$$D_e(X) = \sum_{w \in \mathcal{M}} \frac{1}{q} X ||\varphi(w) - \varphi(e \oplus w)||^2$$

as the average Euclidean distance enumerating function (EDEF) corresponding to the error $e$, where $X$ is a dummy variable. Then the average EDEF $B_{\delta}^{(N)}(X)$ for the R code $\mathcal{C}[N, 1]$ over all messages $u$ and all possible vectors $w$ can be computed as

$$B_{\delta}^{(N)}(X) = \sum_{\delta \in \mathcal{M}} \sum_{w \in \mathcal{M}^N} \frac{1}{q^N} \sum_{\delta \in \mathcal{M}} \frac{1}{q} X^{\sum_{j=0}^{N-1} ||\varphi(u \oplus w_j) - \varphi(u \oplus e \oplus w_j)||^2}$$

where $B_{\delta}^{(N)}$ denotes the average number of signal pairs $(s, \hat{s})$ with Euclidean distance $\delta$, $s = (\varphi(u \oplus w_0), \ldots, \varphi(u \oplus w_{N-1}))$ and $\hat{s} = (\varphi(u \oplus w_0), \ldots, \varphi(\hat{u} \oplus w_{N-1}))$. The performance under the mapping $\varphi$ can be upper-bounded by the union bound as

$$\text{SER} = f_{\varphi, N}(\text{SNR}) \leq \sum_{\delta > 0} B_{\delta}^{(N)}(X)Q \left( \frac{\delta}{2\sigma} \right),$$

where $Q \left( \frac{\delta}{2\sigma} \right)$ is the pair-wise error probability with $Q(x) \triangleq \int_{x}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp \left( - \frac{z^2}{2} \right) dz$.

From the above derivation, we can see that the performance bounds of the R codes are related to the mapping $\varphi$. In this paper, we consider as examples the BPSK, the signal set $\{-1, 0, +1\}$ (denoted as 3-ary pulse amplitude modulation (3-}
PAM), 4-PAM, 8-ary phase-shift keying (8-PSK) modulation, or 16-ary quadrature amplitude modulation (16-QAM), which are depicted in Fig. 5 along with mappings denoted by \( \varphi_0, \ldots, \varphi_7 \) as specified in the figure. Fig. 6 and Fig. 7 show performance bounds for some R codes defined with the considered constellations. From the figures, we have the following observations.

1) The performance gap between the code \( C[N, 1] \) and the uncoded transmission, when measured by the SNR instead of \( E_b/N_0 \), is roughly \( 10 \log_{10}(N) \) dB.

2) Given a signal constellation, mappings that are universally good for all R codes may not exist. For example, as shown in Fig. 7, \( \varphi_2 \) is better than \( \varphi_3 \) for rate \( 1/63 \) (\( N = 63 \)) but becomes worse for rate \( 1/7 \) (\( N = 7 \)).

C. Time-Sharing

With repetition codes over groups, we are able to implement code rates \( \frac{1}{N} \) for any given integer \( N \geq 1 \). To implement other code rates, we turn to the time-sharing technique. To be precise, let \( R = \frac{P}{Q} \) be the target rate. There must exist a unique \( N \geq 1 \) such that \( \frac{1}{N+1} < \frac{P}{Q} \leq \frac{1}{N} \). Then we can implement a code by time-sharing between the code \( C[N+1, 1] \) and the code \( C[N, 1] \), which is equivalent to encoding \( \alpha P \) information symbols with the code \( C[N+1, 1] \) and the remaining \( (1-\alpha)P \) symbols with the code \( C[N, 1] \), where \( \alpha = \frac{1}{R} - N \) is the time-sharing factor. Apparently, to construct codes with rate \( R > \frac{1}{2} \), we need time-sharing between the code \( C[2, 1] \) and the uncoded transmission. For this reason, we call this class of codes as RUN codes, which consist of the R codes and codes obtained by time-sharing between the R codes and/or the uncoded transmission. We denote a RUN code of rate \( \frac{P}{Q} \) as \( C_{\text{RUN}}[Q, P] \). Replacing in Fig. 4 the R codes with the RUN codes, we then have a coding system that can transmit messages with any given code rate over any given signal set.

1) Encoding: Let \( u \in \mathcal{M}^P \) be the message sequence. The encoder of the code \( C_{\text{RUN}}[Q, P] \) encodes the left-most \( \alpha P \) symbols of \( u \) into \( \alpha P \) codewords of \( C[N+1, 1] \) and the remaining symbols into \((1-\alpha)P\) codewords of \( C[N, 1] \).

2) Decoding: The decoding is equivalent to decoding separately \( \alpha P \) codewords of \( C[N+1, 1] \) and \((1-\alpha)P\) codewords of \( C[N, 1] \).

3) Complexity: Both the encoding/mapping and the demapping/decoding have the same complexity as the R codes.

4) Performance: The performance of the RUN code of rate \( R = \frac{P}{Q} \) is given by

\[
\text{SER} = \alpha \cdot f_{\varphi, N+1}(\text{SNR}) + (1-\alpha) \cdot f_{\varphi, N}(\text{SNR}),
\]

which can be upper-bounded with the aid of (9).

Not surprisingly, the performances of the RUN codes are far away from the corresponding Shannon limits (more than 5 dB) at the SER lower than \( 10^{-2} \).

IV. BMST OVER GROUPS

A. BMST Codes with RUN Codes As Basic Codes

We have constructed a class of codes called RUN codes with any given code rate over groups. However, the RUN codes perform far away from the Shannon limits, as evidenced by the examples in Fig. 6 and Fig. 7. To remedy this, we transmit the RUN codes in the BMST manner as inspired by the fact that, as pointed out in [9], any short code can be embedded into the BMST system to obtain extra coding gain in the low error-rate region. The resulted codes are referred to as BMST-RUN codes. More precisely, we use the \( B \)-fold Cartesian product of the RUN code \( C_{\text{RUN}}[Q, P] \) (denoted as \( C_{\text{RUN}}[Q, P]^B \)) as the basic code. Fig. 8 shows the encoding structure of a BMST-RUN code with memory \( m \), where \( \text{RUN} \) represents the basic encoder, \( \Pi_1, \ldots, \Pi_m \) represents \( m \) symbol-wise interleavers, \((\oplus)\) represents the superposition with modulo-\( q \) addition, and \( \varphi \) represents the mapping \( \varphi \). Let \( u^{(t)} \in \mathcal{M}^{PB} \) and \( v^{(t)} \in \mathcal{M}^{QB} \) be the information sequence and the corresponding codeword of the code \( C_{\text{RUN}}[Q, P]^B \) at time \( t \), respectively. Then the sub-codeword \( c^{(t)} \) can be expressed as

\[
c^{(t)} = u^{(t)} \oplus u^{(t,1)} \oplus \cdots \oplus u^{(t,m)},
\]

where \( \oplus \) denotes the symbol-wise modulo-\( q \) addition, \( v^{(t)} = 0 \in \mathcal{M}^{QB} \) for \( t < 0 \) and \( u^{(t,i)} \) is the interleaved version of


is given by

\[ v^{(t-i)} \] by the \( i \)-th interleave \( \Pi_i \) for \( i = 1, \ldots, m \). Then \( c^{(t)} \) is mapped to the signal vector \( s^{(t)} \in A^{QB} \) symbol-by-symbol and input to the AWGN channels. After every \( L \) sub-blocks of information sequence, we terminate the encoding by inputting \( m \) all-zero sequences \( u^{(t)} = 0 \in M^{PB}(L \leq t \leq L + m - 1) \) to the encoder. The termination will cause a code rate loss. However, the rate loss can be negligible as \( L \) is large enough.

B. Choice of Encoding Memory

The critical parameter for BMST-RUN codes to approach the Shannon limits at a given target SER is the encoding memory \( m \), which can be determined by the genie-aided lower bound. Essentially the same as for the binary BMST codes [9], the genie-aided bound for a BMST-RUN code can be easily derived by assuming all but one sub-blocks \( \{u^{(t)} \mid 0 \leq i \leq L - 1, i \neq t \} \) are known at the receiver. With this assumption, the genie-aided system becomes an equivalent system that transmits the basic RUN codeword \( m \) + 1 times. Hence the performance of the genie-aided system is the same as the original system \( C[(N + 1)(m + 1), 1] \) and the code \( C[N(m + 1), 1] \). As a result, the genie-aided bound under a mapping \( \varphi \) is given by

\[
\text{SER} = f_{\text{BMST-RUN}}(\text{SNR}, m) \geq f_{\text{genie}}(\text{SNR}, m) \\
= \alpha \cdot \frac{1}{\varphi(N+1)(m+1)} (\text{SNR}) + (1 - \alpha) \cdot \frac{1}{\varphi[N(m+1)]} (\text{SNR}),
\]

(12)

which can be approximated using the union bound in the high SNR region.

Given a signal set \( A \) of size \( q \), a rate \( R = P/Q \) and a target SER \( p_{\text{target}} \), we can construct a good BMST-RUN code using the following steps.

1) Construct the code \( C_{\text{RUN}}[Q, P]^B \) over the modulo-\( q \) group by finding \( N \) such that \( \frac{1}{\alpha} < \frac{P}{Q} < \frac{1}{\varphi} \) and determining the time-sharing factor \( \alpha \) between the R code \( [N + 1, 1] \) and the R code \( [N, 1] \). To approach the Shannon limit and to avoid error propagation, we usually choose \( B \) such that \( QB \geq 1000 \).

2) Find the Shannon limit \( \gamma_{\text{lim}} \) under the signal set \( A \) corresponding to the rate \( R \).

3) Find a mapping \( \varphi \) and an encoding memory \( m \) such that \( m \) is the minimum integer satisfying \( f_{\text{genie}}(\gamma_{\text{lim}}, m) \leq p_{\text{target}} \).

4) Generate \( m \) interleavers of size \( QB \) uniformly at random.

Remark. Step 3) can be executed in a brute-force way by searching over all possible one-to-one mappings (\( q! \) in total) from \( M \) to \( A \). This can be speeded up if some symmetrical structures of the signal constellations are considered.

C. Decoding of BMST-RUN Codes

A BMST-RUN code can be decoded by an SWD algorithm with a decoding delay \( d \) over its normal graph, which is similar to that of the binary BMST codes [9]. Fig. 9 shows the unified (high-level) normal graph of a BMST-RUN code with \( L = 4 \) and \( m = 2 \). The normal graph can also be divided into layers, each of which consists of four types of nodes. These nodes represent similar constraints to those for binary BMST codes and have similar message processing as outlined below.

- The process at the node RUN is the SISO decoding of the RUN codes as described in Section III-B.
- The process at the node + can be implemented in the same way as the message processing at a generic variable node of an LDPC code (binary or non-binary).
- The process at the node = can be implemented in the same way as the message processing at a generic check node of an LDPC code (binary or non-binary).
- The process at the node ⊙ is the same as the original one, which interleaves or deinterleaves the input messages.

Upon the arrival of the received vector \( y^{(t)} \) (corresponding to the sub-block \( c^{(t)} \) at time \( t \), the SWD algorithm takes as inputs the a posteriori probabilities (APPs) corresponding to \( C^{(t)} \) and uses the APPs corresponding to \( C^{(t-d)}, \ldots, C^{(0)} \) to recover \( u^{(t-d)} \), where the computation of APPs is similar to (4). After \( u^{(t-d)} \) is recovered, the decoder discards \( y^{(t-d)} \) and slides one layer of the normal graph to the “right” to recover \( u^{(t-d+1)} \) with \( y^{(t+1)} \) received.

V. EXAMPLES OF BMST-RUN CODES

In this section, we present simulation results of several BMST-RUN codes over the AWGN channels, where code parameters are shown in Table I. For all simulations, the encoder terminates every \( L = 1000 \) sub-blocks and the decoder executes the SWD algorithm with a maximum iteration number 18. Without specification, the decoding delay \( d \) of the SWD algorithm is set to be \( 3m \).

A. BMST-RUN Codes with One-Dimensional Signal Sets

Consider BMST-RUN codes of rates \( \frac{K}{Q}(K = 1, \ldots, 7) \) defined with BPSK modulation to approach the Shannon limits at the SER of \( 10^{-5} \). Fig. 10 shows the required SNRs for the BMST-RUN codes to achieve the SER of \( 10^{-5} \). Also shown in Fig. 10 is the channel capacity curve with i.u.d. inputs. It can be seen that the gaps between the required SNRs and the Shannon limits are within 1 dB for all rates.

Consider BMST-RUN codes of rates \( \frac{K}{Q}(K = 1, \ldots, 6) \) defined with 3-PAM to approach the Shannon limits at the SER
TABLE I
CONSTRUCTION EXAMPLES OF BMST−RUN CODES

| A | α | B | P_{target} | γ|lim (dB) | m | \varphi^* |
|---|---|---|-----------|---|---------|---|---------|
| BPSK | \frac{1}{2} | (\frac{1}{4}, \frac{1}{4}) | 0 | 1250 | 10^{-5} | -7.2 | 11 \varphi_0 |
| BPSK | \frac{1}{2} | (\frac{1}{4}, \frac{1}{4}) | 0 | 1250 | 10^{-5} | -3.8 | 10 \varphi_0 |
| BPSK | \frac{1}{2} | (\frac{1}{4}, \frac{1}{4}) | \frac{1}{2} | 1250 | 10^{-5} | -1.6 | 11 \varphi_0 |
| BPSK | \frac{1}{2} | (\frac{1}{4}, \frac{1}{4}) | 0 | 1250 | 10^{-5} | 0.2 | 8 \varphi_0 |
| BPSK | \frac{3}{4} | (\frac{1}{4}, \frac{1}{4}) | \frac{1}{2} | 1250 | 10^{-5} | 1.8 | 10 \varphi_0 |
| BPSK | \frac{3}{4} | (\frac{1}{4}, \frac{1}{4}) | \frac{1}{2} | 1250 | 10^{-5} | 3.4 | 7 \varphi_0 |
| 3-PAM | \frac{1}{2} | (\frac{1}{4}, \frac{1}{2}) | 0 | 300 | 10^{-4} | -4.3 | 7 \varphi_1 |
| 3-PAM | \frac{1}{2} | (\frac{1}{4}, \frac{1}{2}) | \frac{1}{2} | 300 | 10^{-4} | -0.5 | 6 \varphi_1 |
| 3-PAM | \frac{1}{2} | (\frac{1}{4}, \frac{1}{2}) | \frac{1}{2} | 300 | 10^{-4} | 2.1 | 6 \varphi_1 |
| 3-PAM | \frac{1}{2} | (\frac{1}{4}, \frac{1}{2}) | \frac{1}{2} | 300 | 10^{-4} | 4.4 | 6 \varphi_1 |
| 3-PAM | \frac{1}{2} | (\frac{1}{4}, \frac{1}{2}) | \frac{1}{2} | 300 | 10^{-4} | 6.5 | 5 \varphi_1 |
| 3-PAM | \frac{1}{2} | (\frac{1}{4}, \frac{1}{2}) | \frac{1}{2} | 300 | 10^{-4} | 8.8 | 3 \varphi_1 |
| 4-PAM | \frac{1}{2} | (\frac{1}{4}, \frac{1}{4}) | 0 | 200 | 10^{-4} | -3.1 | 9 \varphi_3 |
| 4-PAM | \frac{1}{2} | (\frac{1}{4}, \frac{1}{4}) | \frac{1}{2} | 200 | 10^{-4} | 0.9 | 8 \varphi_3 |
| 4-PAM | \frac{1}{2} | (\frac{1}{4}, \frac{1}{4}) | \frac{1}{2} | 200 | 10^{-4} | 3.8 | 6 \varphi_3 |
| 4-PAM | \frac{1}{2} | (\frac{1}{4}, \frac{1}{4}) | \frac{1}{2} | 200 | 10^{-4} | 6.3 | 7 \varphi_3 |
| 4-PAM | \frac{1}{2} | (\frac{1}{4}, \frac{1}{4}) | \frac{1}{2} | 200 | 10^{-4} | 8.7 | 5 \varphi_3 |
| 4-PAM | \frac{1}{2} | (\frac{1}{4}, \frac{1}{4}) | \frac{1}{2} | 200 | 10^{-4} | 11.2 | 3 \varphi_3 |
| 8-PSK | \frac{1}{2} | (\frac{1}{4}, \frac{1}{4}) | 0 | 150 | 10^{-4} | -2.8 | 6 \varphi_5 |
| 8-PSK | \frac{1}{2} | (\frac{1}{4}, \frac{1}{4}) | \frac{1}{2} | 150 | 10^{-4} | 1.3 | 6 \varphi_5 |
| 8-PSK | \frac{1}{2} | (\frac{1}{4}, \frac{1}{4}) | \frac{1}{2} | 150 | 10^{-4} | 4.7 | 6 \varphi_5 |
| 8-PSK | \frac{1}{2} | (\frac{1}{4}, \frac{1}{4}) | \frac{1}{2} | 150 | 10^{-4} | 8.1 | 4 \varphi_5 |
| 16-QAM | \frac{3}{4} | (\frac{1}{4}, \frac{1}{4}) | \frac{1}{2} | 4 | 10^{-3} | 12.7 | 2 \varphi_7 |

* The mappings in this table are the same as those specified in Fig. 5. Notice that \varphi_3 is not optimized but just for comparison in Section V-C. Also notice that \varphi_7 may not be the optimal one since we have not found efficient ways to search it.

![Fig. 10](image-url)  
Fig. 10. The required SNRs to achieve the SER of $10^{-5}$ for the BMST-RUN codes with the codes $C_{BMST-RUN}[Q, P] = 1250$ as basic codes defined with BPSK modulation.

![Fig. 11](image-url)  
Fig. 11. Performances of the BMST-RUN codes with the codes $C_{BMST-RUN}[Q, P] = 1250$ as basic codes defined with 3-PAM.

![Fig. 12](image-url)  
Fig. 12. Performances of the BMST-RUN codes with the codes $C_{BMST-RUN}[Q, P] = 1250$ as basic codes defined with 8-PSK modulation.

lower than $10^{-4}$ at the SNR within 1 dB away from the corresponding Shannon limits, which is similar to the BPSK modulation case.

B. BMST-RUN Codes with Two-Dimensional Signal Sets

Consider BMST-RUN codes of rates $\frac{2}{5}(K = 1, \ldots, 4)$ defined with 8-PSK modulation to approach the Shannon limits at the SER of $10^{-4}$. Fig. 12 shows the SER performance curves for all codes together with their lower bounds and the corresponding Shannon limits.

Consider a BMST-RUN code of rate $\frac{2}{5}$ defined with 16-QAM (under the mapping \varphi_7 in Fig. 5) to approach the Shannon limit at the SER of $10^{-3}$, where an encoding memory $m = 2$ is required. The SER performance curves with decoding delays $d = 6$ and $20$ together with the lower bound and the Shannon limit are shown in Fig. 13. Since a large fraction of information symbols ($\frac{2}{25}$) are encoded in the basic code, a large decoding delay $d = 10m = 20$ is required...
Performance of the BMST-RUN codes with the codes $C_{\text{RUN}}[255, 239]^4$ as the basic code defined with 16-QAM, where the mapping is $\varphi_7$ in Fig. 5.

Performance of the BMST-RUN codes with the codes $C_{\text{RUN}}[7, K]^{200}(K = 1, \ldots, 6)$ over the modulo-4 group and the BMST-BICM scheme with the codes $C_{\text{RUN}}[7, K]^{400}(K = 1, \ldots, 6)$ over $\mathbb{F}_2$ as basic codes, where both schemes are under 4-PAM with the mapping $\varphi_3$ in Fig. 5.

In the following, we give an example to show that the BMST-RUN codes can perform better than the BMST-BICM scheme. To make a fair comparison, we have the following settings.

- For the BMST-BICM scheme, the basic codes are the RUN codes $[7, K]^{400}(K = 1, \ldots, 6)$ over $\mathbb{F}_2$, while for the BMST-RUN codes, the basic codes are the RUN codes $[7, K]^{200}(K = 1, \ldots, 6)$ over the modulo-4 group. Such setting ensures that both schemes have the same sub-block length 2800 in bits.
- Both the BMST-RUN codes and the BMST-BICM scheme use the 4-PAM with the mapping $\varphi_3$.
- For a specific code rate, the BMST-BICM scheme has the same encoding memory and the same decoding delay as the BMST-RUN code. The encoding memories are presented in Table I, while the decoding delay is set to be $3m$ for an encoding memory $m$.

Since the performance of the BMST-BICM scheme cannot be measured in SER, we compare the performance in BER. Fig. 14 shows the BER performance curves for both the BMST-RUN codes (denoted as “RUN”) and the BMST-BICM scheme (denoted as “BICM”) together with the Shannon limits. Fig. 15 shows the required SNRs to achieve the BER of $10^{-4}$ for both the BMST-RUN codes and the BMST-BICM scheme together with capacity curve of 4-PAM under i.u.d. inputs. From these two figures, we have the following observations.

- With the same encoding memory and decoding delay, the BMST-RUN codes achieve a lower BER than the BMST-BICM scheme for all considered code rates.
- The BMST-RUN codes perform better than the BMST-BICM scheme in the lower code rate region and have a similar performance as the BMST-BICM scheme in the high code rate region.
VI. CONCLUSIONS

In this paper, by combining the block Markov superposition transmission (BMST) with the RUN codes over groups, we have proposed a simple scheme called BMST-RUN codes to approach the Shannon limits at any target symbol-error-rate (SER) with any given (rational) rate over any alphabet (of moderate size). We have also derived the genie-aided lower bound for the BMST-RUN codes. Simulation results have shown that the BMST-RUN codes have a similar behavior to the binary BMST codes and have good performance for a wide range of code rates over the AWGN channels. Compared with the BMST with bit-interleaved coded modulation (BMST-BICM) scheme, the BMST-RUN codes are more flexible, which can be combined with signal sets of any size. In addition, with the same encoding memory, the BMST-RUN codes have a better performance than the BMST-BICM scheme under the same decoding latency.

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