ON THE PERTURBATION OF THE EXTREMAL SINGULAR VALUES OF A MATRIX AFTER APPENDING A COLUMN

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Abstract. We provide new bounds on the extreme singular values of a matrix obtained after appending a column vector to a given matrix. The proposed bounds improve upon the results obtained in [24]. Moreover, we present two applications of independent interest: a first one regarding the restricted isometry constant and the coherence in Compressed Sensing theory, and a second one concerning the perturbation of the algebraic connectivity of a graph after removing an edge.

1. Introduction

1.1. Framework. Let $d$ be an integer. Let $X \in \mathbb{R}^{d \times n}$ be a $d \times n$-matrix and let $x \in \mathbb{R}^d$ be column vector. We denote by a subscript $^t$ the transpose of vectors and matrices. There exist at least two ways to study the matrix $(x, X)$ obtained by appending the column vector $x$ to the matrix $X$:

(A1) Consider the matrix

$$A = \begin{bmatrix} x^t & x & x^t X \\ X^t & X & X^t X \end{bmatrix};$$

(A2) Consider the matrix

$$\tilde{A} = \begin{bmatrix} x & X \\ X^t & X \end{bmatrix} = XX^t + xx^t.$$

On one hand, one may study in (A1) the eigenvalues of the $(n + 1) \times (n + 1)$ hermitian matrix $A$, i.e. the matrix $X^tX$ augmented with an arrow matrix.

On the other hand, one will deal in (A2) with the eigenvalues of the $d \times d$ hermitian matrix $\tilde{A}$, which may be seen as a rank-one perturbation of $XX^t$. The matrices $A$ and $\tilde{A}$ have the same non-zeros eigenvalues, and in particular $\lambda_{\max}(A) = \lambda_{\max}(\tilde{A})$. Moreover, the singular values of the matrix $(x, X)$ are the square-root of the eigenvalues of the matrix $A$.

Equivalently, the problem of a rank-one perturbation can be rephrased as the one of controlling the perturbation of the singular values of a matrix after appending a column.

In the current paper, we study a slightly more general framework than (A1), that is the case of a matrix

$$A = \begin{bmatrix} c & a^t \\ a & M \end{bmatrix},$$

where $a \in \mathbb{R}^d$, $c \in \mathbb{R}$ and $M \in \mathbb{R}^{d \times d}$ is a symmetric matrix.

Our goal is to present new bounds on the extreme eigenvalues of $A$ as a function of the eigenvalues of $M$ and the norm of $a$, and we will focus on various applications. Indeed, this
problem occurs in a variety of contexts such as the perturbation analysis of covariance matrices in statistics [28], the study of the Restricted Isometry Constant in Compressed Sensing [9], spectral graph theory and edge deletion [8], control theory of complex networks [31], hitting time analysis for classical or quantum random walks [38], robust face recognition [32], wireless communications [34], communication theory and signal processing [38], numerical methods for partial differential equations [4], numerical analysis of bifurcations [17], among many applications.

Notice further that in (1.2) if \( M \) and \( A \) are positive definite, there exist \( X \in \mathbb{R}^{d \times n} \) and \( x \in \mathbb{R}^d \) such that \( M = X^tX \) and \( A \) can be written as in (1.1) due to the Cholesky decomposition.

1.2. Additional notations. The Kronecker symbol is denoted by \( \delta_{i,j} \), i.e. \( \delta_{i,j} = 1 \) if \( i = j \) and is equal to zero otherwise. For any symmetric matrix \( B \in \mathbb{R}^{d \times d} \) we will denote its eigenvalues by \( \lambda_1(B) \geq \cdots \geq \lambda_d(B) \). The largest eigenvalue will sometimes also be denoted by \( \lambda_{\text{max}}(B) \) and the smallest by \( \lambda_{\text{min}}(B) \). The smallest nonzero eigenvalue of \( B \) will be denoted by \( \lambda_{\text{min},>0}(B) \).

1.3. Plan of the paper. Section 2 is devoted to an overview of known results. Section 3 presents new upper and lower bounds for the extreme eigenvalues. Section translates some previous results in terms of operator norm together with a slight variation. Finally, Section 5 is concerned with the applications in Compressed sensing and graphs theory.

2. Previous results on eigenvalue perturbation

We now review some previous, old and recent results from matrix perturbation theory and apply them to our problem of appending a column.

Obtaining precise estimates on the eigenvalues of a sum of two matrices (say \( X + P \), considering \( P \) as a perturbation) is a very difficult task in general. Weyl’s and Horn’s inequalities for instance can be employed and these bounds can be improved when knowing that the perturbation \( P \) is small with respect to \( X \) (see e.g. [20, Chap. 6]). The whole point of the works [2] and [3], to name a few, is to understand how randomness can simplify this analysis.

2.1. Weyl’s inequalities. The reference [35] gives an overview of many inequalities on the eigenvalues of sums of symmetric (and Hermitian) matrices. The Weyl inequalities are given as follows:

**Theorem 2.1** (Weyl). Let \( B \) and \( B' \) be symmetric real matrices in \( \mathbb{R}^{d \times d} \) and let \( \lambda_j(B) \), \( j = 1, \ldots, d \), (resp. \( \lambda_j(B') \)), denote the eigenvalues of \( B \) (resp. \( B' \)). Then, we have

\[
\lambda_{i+j-1}(B + B') \leq \lambda_i(B) + \lambda_j(B'),
\]

whenever \( i, j \geq 1 \) and \( i + j - 1 \leq n \).

2.1.1. The arrowhead perturbation. Consider the case where we would like to control the largest eigenvalue of \( A \) with the eigenvalues of \( M = X^tX \). We have the following result.

**Proposition 2.2.** We have

\[
\lambda_1(A) \leq \max\{c, \lambda_1(M)\} + \|a\|_2.
\]

**Proof.** The Weyl inequalities for \( i, j = 1 \) gives that

\[
(2.3) \quad \lambda_1(A) \leq \lambda_1 \left( \begin{bmatrix} c & 0 \\ 0 & M \end{bmatrix} \right) + \lambda_1(E)
\]
with
\[ E = \begin{bmatrix} 0 & a^t \\ a & 0 \end{bmatrix}. \]

Moreover, using the variational representation of the maximum eigenvalue and the method of Lagrange multipliers, we have \( \lambda_1(E) = \|a\|_2 \). Combining this with (2.3), we obtain the desired result.

The main fact to retain from this inequality is that if \( x \) is orthogonal to all columns of \( X \), then \( a = 0 \) and the perturbation has no effect on the largest eigenvalue as long as \( c \leq \lambda_1(M) \).

2.1.2. The rank-one perturbation. If we only want to study the perturbation of the largest eigenvalue, then we can consider the rank-one perturbation described by (A2). In this case, Weyl’s bound gives the following result.

**Proposition 2.3.** We have
\[
\lambda_1(A) \leq \lambda_1(M) + \|x\|_2^2.
\]

**Proof.** Using that \( \lambda_1(A) = \lambda_1(\tilde{A}) \) and \( \lambda_1(M) = \lambda_1(\tilde{M}) \), we obtain from Theorem 2.1:
\[
\lambda_1(A) \leq \lambda_1(M) + \lambda_1(xx^t).
\]

Since \( \lambda_1(xx^t) = \|x\|_2^2 \), the conclusion follows.

The main drawback of this inequality is that it does not take into account the geometry of the problem and in particular the angle between \( X \) and the new vector \( x \) that we want to append to \( X \). This does not disqualify the rank-one perturbation approach to controlling the maximum eigenvalue as will be shown in Subsection 2.3.

2.2. An inequality of Li and Li. They prove a general inequality concerning the perturbation of eigenvalues under off-block diagonal perturbations. We specify their result, [24, Theorem 2], in our context:

\[
|\lambda_1(A) - \max(c, \lambda_1(M))| \leq \frac{2\|a\|^2}{\eta + \sqrt{\eta^2 + 4\|a\|^2}},
\]

with \( \eta = \min\{|c - \lambda_i(M)|, 1 \leq i \leq d\} \). In their paper, \( \tilde{\lambda}_1 \) is actually \( \max(c, \lambda_1(M)) \) here.

We refer to [24] and references therein for the history of such inequalities.

2.3. An inequality of Ipsen and Nadler. In [21], the authors propose a bound for the eigenvalues of \( \tilde{A} \) in the problem of rank one perturbation (A2). The following theorem is a corollary of their main result where we restrict our attention to the largest eigenvalue.

**Theorem 2.4.** Let \( \tilde{M} \in \mathbb{C}^{d \times d} \) denote an Hermitian matrix and let \( x \in \mathbb{C}^d \). Let \( V_1 \) (resp. \( V_2 \)) denote the eigenvector associated to the eigenvalue \( \lambda_1(\tilde{M}) \) (resp. \( \lambda_2(\tilde{M}) \)). Let \( \bar{A} = \tilde{M} + xx^t \).

Then
\[
\lambda_1(\tilde{M}) + \delta_{\min} \leq \lambda_1(\bar{A}) \leq \lambda_1(\tilde{M}) + \delta_{\max},
\]
with
\[
\delta_{\text{min}} = \frac{1}{2} \left( \|P_{((V_1, V_2))}(x)\|^2 - \text{gap}_2 + \sqrt{(\text{gap}_2 + \|P_{((V_1, V_2))}(x)\|^2)^2 - 4 \text{gap}_2 \|P_{((V_1)}(x)\|^2} \right)
\]
\[
\delta_{\text{max}} = \frac{1}{2} \left( \|x\|^2_2 - \text{gap}_2 + \sqrt{(\text{gap}_2 + \|x\|^2_2)^2 - 4 \text{gap}_2 \|P_{((V_1, V_2))}(x)\|^2} \right),
\]
where \((V_1, \ldots, V_j), 1 \leq i \leq j \leq d,\) denotes the vector space generated by \(V_1, \ldots, V_j\) and \(P_{((V_1, \ldots, V_j))}\) denotes the orthogonal projection onto this space, and
\[
\text{gap}_2 = \lambda_1(M) - \lambda_2(M).
\]

This inequality has been used in various applications such as control of complex systems [31], quantum information theory [15], communication theory and signal processing [38], numerical methods for partial differential equations [4]. One drawback of using this result in our context is that we have to know the spacing \(\text{gap}_2\) for the second eigenvalue. Moreover, the upper bound depends on \(\|x\|^2_2\) and does not take into account the scalar products of \(x\) with the columns of \(X\), which may lead to serious overestimation of the perturbation, especially in the case of random matrices.

3. Main results on the perturbation of the extreme singular values

In this section, we present and prove our main results. We improve the bound obtained from Weyl’s inequality over a non-trivial and useful range of perturbations. Moreover, our bound does not depend on the spacing \(\text{gap}_2\) unlike in [21].

3.1. The maximum eigenvalue. The following theorem provides sharp upper bounds for \(\lambda_{\text{max}}(A)\), and lower bounds on \(\lambda_{\text{min}}(A)\), depending on various informations on the sub-matrix \(M\) of \(A\). As discussed above, this problem has close relationships with our problem of appending a column to a given rectangular matrix, because \(\lambda_1(A) = \lambda_1(M)\).

**Theorem 3.1.** Let \(d\) be a positive integer and let \(M \in \mathbb{C}^{d \times d}\) be an Hermitian matrix, whose eigenvalues are \(\lambda_1 \geq \cdots \geq \lambda_d\) with corresponding eigenvectors \((V_1, \cdots, V_d)\). Set \(c \in \mathbb{R}, a \in \mathbb{C}^d\). Let \(A\) be given by \((3.3)\). Therefore:
\[
(3.5) \quad \frac{2\langle a, V_1 \rangle^2}{\eta_1 + \sqrt{\eta_1^2 + 4\langle a, V_1 \rangle^2}} \leq \lambda_1(A) - \max(c, \lambda_1) \leq \frac{2\|a\|^2}{\eta_1 + \sqrt{\eta_1^2 + 4\|a\|^2}},
\]
with
\[
\eta_1 = |c - \lambda_1|.
\]

**Remark 3.2.**

- Inequality \((3.5)\) is sharp: the upper bound is reached when choosing \(M = \text{Id}, c = 1\) and any \(a\), so that \(\lambda_{\text{max}}(A) = 1 + \|a\|\);
- The upper bound in \((3.5)\) is better than \((2.4)\) since \(\eta_1 \geq \eta\). A typical example where the improvement holds is basically when \(c\) is one of the eigenvalues of \(M\) (i.e. \(\eta = 0\)). For instance, take \(c = 1, a^t = (\alpha, 0)\) and \(M = \text{diag}(2, 1)\). In particular, \(\eta_1 = 1\). An easy computation yields \(\lambda_1(A) = 3/2 + \sqrt{1/4 + \alpha^2}\) and then
\[
\lambda_1(A) - \lambda_1(M) = \sqrt{1/4 + \alpha^2} - 1/2 = \frac{2\alpha^2}{1 + \sqrt{1 + 4\alpha^2}},
\]
which is the upper bound in \((3.5)\), while the bound \((2.4)\) is simply the triangle inequality \(|\lambda_1(A) - \lambda_1(M)| \leq |\alpha|\).
• The lower bound in (3.5) is also better than (2.4) since we have:

$$\lambda_1(A) \geq \max(c, \lambda_1) + \frac{2\langle a, V_1 \rangle^2}{\eta_1 + \sqrt{\eta_1^2 + 4\langle a, V_1 \rangle^2}} \geq \max(c, \lambda_1) - \frac{2\|a\|^2}{\eta + \sqrt{\eta^2 + 4\|a\|^2}}.$$  

Our lower bound is in particular consistent with Cauchy interlacing theorem, which states that $$\lambda_1(A) \geq \lambda_1.$$  

• A great feature of Theorem 2 of Li and Li in [24] is that it holds for all eigenvalues and for block perturbations. We will treat the whole spectrum in a subsequent work. Moreover, generalizing our bounds for block perturbations may be an interesting perspective.

Proof. Let $$M = VDV^*$$ denote the eigenvalue decomposition of $$M$$, i.e. $$V = (V_1, \cdots, V_d)$$ where the $$V_i$$’s are the orthonormal eigenvectors of $$M$$ and $$D$$ is a diagonal matrix whose diagonal entries are the real eigenvalues $$\lambda_1 \geq \cdots \geq \lambda_d$$. We can write

$$A = \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} c & a^*V \\ V^*a & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & V^* \end{pmatrix},$$

and we set

$$B = \begin{pmatrix} c & b^* \\ b & D \end{pmatrix}, \quad b = V^*a,$$

where we use the notation $$b_j := \langle a, V_j \rangle$$. Therefore, $$A$$ and $$B$$ have the same spectra and in particular,

$$\lambda_{\max}(A) = \lambda_{\max}(B).$$

(3.6)

As in [16], we compute the characteristic polynomial of the arrow matrix $$B$$:

$$P_B(\lambda) = (c - \lambda) \prod_{i=1}^d (\lambda_i - \lambda) - \sum_{i=1}^d \prod_{j \neq i} (\lambda_j - \lambda)b_j^2.$$  

Let us define the function $$f$$ on $$\mathbb{R} \setminus \{\lambda_i, 1 \leq i \leq d\}$$ as

$$f(\lambda) := P_B(\lambda) \prod_{i=1}^d (\lambda_i - \lambda)^{-1} = c - \lambda + \sum_{j=1}^d \frac{b_j^2}{\lambda - \lambda_j},$$

which is decreasing on $$(-\infty, +\infty)$$ (even if $$b = 0$$).

We now assume that $$b_1 = \langle a, V_1 \rangle \neq 0$$. Thus $$\lim_{\lambda \to \lambda_1} f(\lambda) = +\infty$$. From $$\lim_{\lambda \to +\infty} f(\lambda) = -\infty$$, we then deduce that the continuous function $$f$$ has a unique root on $$(-\infty, +\infty)$$, that is

$$\lambda_{\max}(B) > \lambda_1.$$  

(3.7)

For all $$\lambda > \lambda_1$$, we have

$$f(\lambda) \leq c - \lambda + \frac{\|b\|^2}{\lambda - \lambda_1} := g(\lambda).$$  

For the same reasons as $$f$$, the function $$g$$ has a unique root $$\lambda^*$$ on $$(-\infty, +\infty)$$. Since $$f$$ is decreasing on $$(-\infty, +\infty)$$ and $$f(\lambda_{\max}(B)) = 0 = g(\lambda^*) \geq f(\lambda^*)$$, we deduce:

$$\lambda_{\max}(B) \leq \lambda^*.$$  

We have

$$(\lambda^* - c)(\lambda^* - \lambda_1) = \|b\|^2_2,$$
and thus \( \lambda^* \) is a root of the polynomial
\[
Q(x) = (x - c)(x - \lambda_1) - \|b\|^2 \\
= x^2 - (c + \lambda_1)x + c\lambda_1 - \|b\|^2.
\]
The discriminant of \( Q \) reads:
\[
\Delta = (c + \lambda_1)^2 - 4(c\lambda_1 - \|b\|^2) \\
= (c - \lambda_1)^2 + 4\|b\|^2 > 0.
\]
Since \( Q(\lambda_1) < 0 \) and the dominant coefficient of \( Q \) is positive, we deduce that \( \lambda^* \) is actually the greatest root of \( Q \). Hence, noting that \( \|b\| = \|a\| \),
\[
\lambda^* = \frac{c + \lambda_1}{2} + \frac{1}{2} \sqrt{(c - \lambda_1)^2 + 4\|a\|^2}.
\]
(3.8)

Assume that \( \langle a, V_1 \rangle \neq 0 \). In order to find a lower bound for \( \lambda_{\text{max}}(B) \), we perform the same reasoning by writing
\[
f(\lambda) \geq c - \lambda + \frac{\langle a, V_1 \rangle^2}{\lambda - \lambda_1},
\]
and considering the polynomial \((x - c)(x - \lambda_1) - \langle a, V_1 \rangle^2\).

Finally, we have:
\[
\frac{c + \lambda_1}{2} + \frac{1}{2} \sqrt{(c - \lambda_1)^2 + 4\langle a, V_1 \rangle^2} \leq \lambda_1(A) \leq \frac{c + \lambda_1}{2} + \frac{1}{2} \sqrt{(c - \lambda_1)^2 + 4\|a\|^2}.
\]
(3.9)

Set \( \eta_1 = |c - \lambda_1| \). Since
\[
2\max(\lambda_1, c) = c + \lambda_1 + \eta_1,
\]
we deduce
\[
\frac{1}{2} \left( \sqrt{\eta_1^2 + 4\langle a, V_1 \rangle^2} - \eta_1 \right) \leq \lambda_1(A) - \max(\lambda_1, c) \leq \frac{1}{2} \left( \sqrt{\eta_1^2 + 4\|a\|^2} - \eta_1 \right).
\]

Multiplying by the "conjugate quantity" yields the lower and the upper bounds in (3.5).

The case \( \langle a, V_1 \rangle = 0 \) can be treated by standard continuity arguments: consider a continuous \( \varepsilon \mapsto a(\varepsilon) \) such that for all \( \varepsilon > 0, \langle a(\varepsilon), V_1 \rangle \neq 0 \) and \( a(0) = a \). Ones then writes (3.3) for \( \varepsilon > 0 \) and passes to the limit as \( \varepsilon \to 0 \).

\( \Box \)

**Corollary 3.3.** In particular, the following simple perturbation bounds hold:
\[
\lambda_1(A) \leq \max(c, \lambda_1) + \|a\|_2
\]
(3.10)
\[
\lambda_1(A) \leq \max(c, \lambda_1) + \frac{\|a\|_2^2}{|\lambda_1 - c|},
\]
(3.11)

*Proof.* Inequality (3.10) (resp. (3.11)) follows from (3.5) by using \( \eta_1 \geq 0 \) (resp. \( \|a\| \geq 0 \)). \( \Box \)
3.2. Perturbation of the smallest nonzero eigenvalue.

**Theorem 3.4.** Let \( d \) be a positive integer and let \( M \in \mathbb{C}^{d \times d} \) be an Hermitian matrix, whose eigenvalues are \( \lambda_1 \geq \cdots \geq \lambda_d \) with corresponding eigenvectors \( (V_1, \ldots, V_d) \). Set \( c \in \mathbb{R}, a \in \mathbb{C}^d \). Let \( A \) be given by (1.2). Assume that \( M \) has rank \( r \leq d \). Therefore:

\[
\lambda_{r+1}(A) \geq \min(c, \lambda_r) - \frac{2\|a\|^2}{\eta_r + \sqrt{\eta_r^2 + 4\|a\|^2}},
\]

with

\[
\eta_r = |c - \lambda_r|.
\]

**Proof.** In the case where \( r = d \), Inequality (3.12) immediately follows from applying (3.10) to the matrix \( -A \).

Assume now that \( r < d \). The eigenvalues of \( M \) are \( \lambda_1 \geq \cdots \geq \lambda_r \geq \lambda_{r+1} = \cdots = \lambda_d = 0 \). We use the same reduction for the matrix \( A \) as in the proof of Theorem 3.1. If \( a \neq 0 \), we have that \( b \neq 0 \) and so there exists \( j_0 \) such that \( b_{j_0} \neq 0 \). Again, we consider the function \( f \), defined on \( \mathbb{R} \setminus \{\lambda_i, 1 \leq i \leq d\} \) by

\[
f(\lambda) := c - \lambda + \sum_{j=1}^{d} \frac{b_j^2}{\lambda - \lambda_j},
\]

which is decreasing on \((\lambda_{r+1}, \lambda_r)\). Since \( \lim_{\lambda \downarrow \lambda_{r+1}} f(\lambda) = +\infty \) and \( \lim_{\lambda \uparrow \lambda_r} f(\lambda) = -\infty \), we deduce that the continuous function \( f \) has a unique root on \((\lambda_{r+1}, \lambda_r)\), which is \( \lambda_{r+1}(A) \). We have

\[
f(\lambda) := c - \lambda + \sum_{j=1}^{r} \frac{b_j^2}{\lambda - \lambda_j} + \sum_{j=r+1}^{d} \frac{b_j^2}{\lambda},
\]

For all \( \lambda \) s.t. \( 0 = \lambda_{r+1} < \lambda < \lambda_r \), we have

\[
f(\lambda) \geq c - \lambda + \frac{\sum_{j=1}^{r} b_j^2}{\lambda - \lambda_r} + \frac{\sum_{j=r+1}^{d} b_j^2}{\lambda} := g(\lambda).
\]

The function \( g \) has a unique root \( \lambda^* \) in \((-\infty, \lambda_r)\).

Since \( f \) is decreasing on \((0 = \lambda_{r+1}, \lambda_r)\) and \( f(\lambda_{r+1}(B)) = 0 = g(\lambda^*) \leq f(\lambda^*) \), we deduce that:

\[
\lambda_{r+1}(A) \geq \lambda^*.
\]

Let us now bound \( \lambda^* \) from below. We have

\[
(c - \lambda^*) (\lambda_r - \lambda^*) = \sum_{j=1}^{d} b_j^2 = \|b\|^2 = \|a\|^2.
\]

Therefore

\[
\lambda^* \geq \lambda_r - \|a\|^2 = 0.
\]
and thus,
\[
\lambda^* = \frac{1}{2} \left( c + \lambda_r - \sqrt{(c + \lambda_r)^2 - 4c \lambda r + 4\|a\|^2} \right),
\]
(3.18)
since one easily checks that the other root is greater than \( \lambda_r \). Expanding the term \((c + \lambda_r)^2\) inside the square root and simplifying the resulting expression, we get
\[
\lambda^* = \frac{1}{2} \left( c + \lambda_r - \sqrt{(c - \lambda_r)^2 + 4\|a\|^2} \right)
\]
(3.19) and
\[
\lambda^* - \min(c, \lambda_r) \geq \frac{1}{2} \left( \eta_r - \sqrt{\eta_r^2 + 4\|a\|^2} \right),
\]
(3.20)
with
\[
\eta_r = |c - \lambda_r|.
\]
The desired result then follows by multiplying by the "conjugate quantity".

Corollary 3.5. In particular, the following simple perturbation bounds hold:
\[
\lambda_{r+1}(A) \geq \min(c, \lambda_r) - \|a\|_2
\]
(3.21)
\[
\lambda_{r+1}(A) \geq \min(c, \lambda_r) - \frac{\|a\|^2}{c - \lambda_r}
\]
(3.22)
Proof. Inequality (3.21) (resp. (3.22)) follows from (3.12) by using \( \eta_r \geq 0 \) (resp. \( \|a\| \geq 0 \)).

4. Bounds on the perturbation of the operator norm

We provide here three bounds on the operator norm: the first and second inequalities are easy consequences of Theorem 3.1, the third one is based on a new trick.

Corollary 4.1. Let \( d \) be an integer, \( a \in \mathbb{C}^d \), \( c \in \mathbb{R} \) and let \( M \in \mathbb{C}^{d \times d} \) be an Hermitian matrix. Let \( A \) be given by (1.2). Then the following inequalities hold:
\[
\|A\| \leq \max(c, \|M\|) + \|a\|_2
\]
(4.23)
\[
\|A\| \leq \|M\| + \frac{\|a\|^2}{\|M\| - c}, \quad \text{if } c \leq \lambda_{\max}(M)
\]
(4.24)
\[
\|A\| \leq \|M\| + \frac{|c|}{2} + \frac{\|a\|^2 + c^2/8}{\|M\|}
\]
(4.25)
Remark 4.2. Notice that (4.24) is better than (4.23) if
\[
\|a\| \leq \|M\| - c,
\]
and that (4.25) is better than (4.23) if
\[
\frac{c}{2} + \frac{\|a\|^2 + c^2/8}{\|M\|} \leq \|a\|.
\]
Proof. We obtain (4.23) by applying (3.10) with $-A$ and by noticing that $\lambda_{\text{max}}(A) \leq \|A\|$.

Now assume that $c \leq \lambda_{\text{max}}(M)$. We bound $\Delta$ as:

$$\sqrt{\Delta} \leq \sqrt{\left(\|M\| - c\right)^2 + 4\|a\|^2},$$

and then

$$2\lambda^* \leq 2\|M\| + \frac{2\|a\|^2}{\|M\| - c},$$

which yields (4.24).

To prove (4.25), we now consider, instead of $B$,

$$B' = \begin{pmatrix} c & b^t & b^t \\ b & D & 0 \\ b & 0 & -D \end{pmatrix}. \quad (4.26)$$

Since the operator norm increases by adding elements to a matrix, we obtain

$$\|A\| \leq \|B'\|. \quad (4.27)$$

The functions $f, g$ in (3.13) are now replaced resp. by,

$$\tilde{f}(\lambda) = c - \lambda + \sum_{j=1}^d b_j^2 \left( \frac{1}{\lambda - \lambda_j} + \frac{1}{\lambda + \lambda_j} \right) = c - \lambda + \sum_{j=1}^d b_j^2 \frac{2\lambda}{\lambda^2 - \lambda_j^2},$$

$$\tilde{g}(\lambda) = c - \lambda + \|b\|^2 \frac{2\lambda}{\lambda^2 - \|M\|^2}, \quad \lambda > \|M\|.$$

If $c \leq 0$ then

$$\tilde{f}(\lambda) \leq \tilde{g}(\lambda) \leq \lambda + \|b\|^2 \frac{2\lambda}{\lambda^2 - \|M\|^2} := h(\lambda).$$

Let $x^*$ be a root of $h$. As previously, $\tilde{f}(\lambda_{\text{max}}(\tilde{B})) = 0 = h(x^*) \geq \tilde{f}(x^*)$, and then

$$\lambda_{\text{max}}(\tilde{B}) \leq x^*.$$

But $x^*$ is less than the greatest root of the polynomial $x \mapsto x^2 - \|M\|^2 + 2\|b\|^2$, that is:

$$x^* \leq \sqrt{\|M\|^2 + 2\|b\|^2}. \quad (4.28)$$

If $c > 0$, we notice that

$$\left(\lambda^2 - \|M\|^2\right)(c - \lambda) + 2\lambda\|b\|^2 = -\lambda^3 + c\lambda^2 + (2\|b\|^2 + \|M\|^2)\lambda - c\|M\|^2$$

$$\leq -\lambda^3 + c\lambda^2 + (2\|b\|^2 + \|M\|^2)\lambda,$$

and we set

$$R(x) = x^2 - cx - (2\|b\|^2 + \|M\|^2).$$
The greatest root $x^*$ of $R$ reads:

$$
\begin{align*}
x^* & \leq \frac{c}{2} + \sqrt{\frac{c^2}{4} + \|M\|^2 + 2\|b\|^2} \\
& \leq \frac{c}{2} + \sqrt{1 + \frac{2\|b\|^2 + c^2/4}{\|M\|^2}} \\
& \leq \frac{c}{2} + \|M\| + \frac{\|b\|^2 + c^2/8}{\|M\|}.
\end{align*}
$$

Repeating the analysis with $-A$ yields (4.25) as desired. \(\square\)

5. Applications

As already mentioned in the introduction, perturbations bounds on the extreme eigenvalues have many applications in science and engineering and some references were proposed. In this section, we focus two more applications where quadratic inequalities as the upper bound (3.5) can yield some improvements in the order of magnitude for the perturbed system.

5.1. Restricted isometry constant and coherence in Compressed Sensing.

5.1.1. General framework. The purpose of Compressed Sensing (CS) is to study the various possible strategies for constructing efficient sensors allowing the recovery of very sparse signals in a high dimensional space (See e.g. the pioneering work of Candès, Romberg and Tao [13]).

The possibility of building such types of sensors was first discovered through simulations in the study of Magnetic Resonance Imaging, where sparsity in a certain dictionary was used in order to reconstruct the signal from much fewer measurements than was previously imagined. Since then, Compressed Sensing has found many applications as can be seen from the blog "Nuit Blanche" maintained by Igor Caron.

The problem can be expressed mathematically as the one of solving the linear system

$$
y = X\beta + \sigma\varepsilon
$$

in the variable $\beta$, where $X \in \mathbb{R}^{n \times p}$, $\sigma \in \mathbb{R}_+$, and $\varepsilon$ is a random noise. A major breakthrough occurred in late 2005-early 2006 when [13], [12], [11] and [10] appeared. One of the main discoveries contained in these works is that the vector $\beta$ can be recovered exactly even when $p$ is much larger than $n$ and $n$ is as small as a constant times $s \log(p/s)$. The assumptions initially required that $\sigma = 0$ and $\beta$ is $s$-sparse and the results were obtained for most $X$ drawn with i.i.d. components with standard gaussian or ±1-Bernoulli distribution. It was then obtained in [11] and [14] that the support of $\beta$ can be exactly recovered in the noisy case $\sigma > 0$ when $n$ is roughly of the same order. A basic property, which emerged from the analysis as a tool for proving the reconstructibility from few measurements, is the Restricted Isometry Property, which requires that all the submatrices $X_T$ have their singular values in the interval $[1 - \rho, 1 + \rho]$ for some constant $\rho \in (0, 1/2)$. Several authors [36], [37] and [14] subsequently noticed that, assuming the columns of $X$ to be $\ell_2$-normalized, most submatrices $X_T$ obtained by selecting the columns indexed by $T$ with $|T|$ such that

$$
|T| \leq \frac{p}{\log p} \frac{C}{\|X\|^2}
$$

(5.28)
for some constant $C$, have their singular values in the interval $[1 - \rho, 1 + \rho]$ for some constant $\rho \in (0, 1/2)$. Recall that the coherence $\mu(X)$ is defined by

$$\mu(X) = \max_{i \neq j'} |X_i^T X_{j'}|.$$ 

This latter property can be interpreted in a probabilistic setting: let $T$ be a random subset of $\{1, \ldots, n\}$ drawn with uniform distribution over all subsets with cardinal bounded from above as in (5.28). Then, with high probability, $\|X_i^T X_T - I\| \leq \rho$.

5.1.2. Perturbation of the singular values. When an additional column is appended to the matrix $X$, one may wonder what is the impact of this operation on the localisation of the extreme singular values of all submatrices with $s$ columns which can be extracted from the resulting matrix. Notice that appending just one column to $X$ results in creating $p! / (s - 1)! (p - s + 1)!$ additional submatrices. Therefore, having a flexible bound on the perturbation of the extreme eigenvalues may be a valuable tool in practice. Another situation where perturbation has to be precisely controlled is when one wants to study the random variable $\|X_i^T X_T - I\|$ using the tools of modern concentration of measure theory [6]. Indeed, after a 'Poissonization' trick has been employed as in Claim (3.29) p.2173 in [14], one may study the problem on a product space for which the celebrated theorem of Talagrand or recent variants by Boucheron, Lugosi and Massart can be used. However, for such concentration theorems to be relevant, one also needs precise perturbation bounds on the extreme singular values.

Let us consider the case where one uses a fixed design matrix $X$ and $T$ is obtained by selecting $s$ columns uniformly at random. Then, Lemma 3.6 in [14] implies that

$$\mathbb{P} \left( \|X_i^T X_j\|^2 \geq \frac{s}{p} \|X\|^2 + t \right) \leq 2 \exp \left( \frac{t^2}{2\mu^2(X)(s\|X\|^2/p + t/3)} \right)$$

and thus, using (5.28), one easily obtains that

(5.29) $$\|X_i^T X_j\|^2 \leq \frac{1}{4\log(p)}$$

with probability at least $1 - 2e^{-\frac{9}{64\mu^2(X)\log(p)}}$ if $C \leq 1/8$. Assuming that the coherence is of the order of $1/ \log(p)$, one obtains that (5.29) holds with high probability. Thus, using inequality (3.11), one obtains a perturbation of the order of $\log(p)^{-1/2}$ of the maximum eigenvalue of $X_i^T X_T$. On the other hand, if one is interested in the perturbation with norm already larger than $\sqrt{1 + \rho}$, (3.11) gives a perturbation of the norm of the order $\rho^{-1} \log(p)^{-1}$ which is significantly smaller and, as one might check in the assumptions of Theorem 5 in [5], is the right order of magnitude for obtaining the desired concentration of measure for this problem.

5.2. Perturbation of the algebraic connectivity of a graph by removing an edge. Another application of spectral perturbation is in hypergraph theory.

5.2.1. The Laplacian of a graph. The $G = (V, E)$ denote an oriented graph with vertex set $V$ and edge set $E$. In such a graph, each edge $e$ has a positive end and a negative end. We say that two vertices are adjacent if they are ends of the same edge. The indicence matrix $I_G$ associated to $G$ is the matrix whose rows are indexed by the vertices and the columns are
indexed by the oriented edges. The \((i, j)\)-entry of \(I_G\) is

\[
I_G(i, j) = \begin{cases} 
+1 & \text{if vertex } i \text{ is the positive end of edge } j \\
-1 & \text{if vertex } i \text{ is the negative end of edge } j \\
0 & \text{otherwise.}
\end{cases}
\]

The adjacency matrix \(A_G\) is the matrix whose rows and columns are indexed by the vertices. The \((i, i')\)-entry of \(A_G\) is

\[
A_G(i, i') = \begin{cases} 
+1 & \text{if vertex } i \text{ and vertex } i' \text{ are adjacent} \\
0 & \text{otherwise.}
\end{cases}
\]

The degree vector of \(G\) is the vector \(d_G\) where \(d_G(i)\) is the number of edges of \(G\) to which vertex \(i\) is an end. The Laplacian matrix of \(G\) is the matrix \(L_G\) defined by

\[
L_G = D(d_G) - A_G,
\]

and the following well known identity holds

\[
L_G = I_G I_G^t.
\]

If \(G\) is not oriented, the degree vector and the adjacency matrix are defined in exactly the same way and any arbitrary orientation of the edges of \(G\) will of course provide the same result. Notice that \(L_G\) is positive semi-definite and that 0 is always an eigenvalue of \(L_G\). If the second smallest eigenvalue is nonzero, then the graph \(G\) is connected. This second smallest eigenvalue is very important for the study of various graphs and is called the algebraic connectivity of \(G\) or Fiedler’s value of \(G\). We will denote the algebraic connectivity by \(a(G)\). The eigenvalues of the Laplacian of a graph have been the subject of intense research for many years and is connected to various fields of pure and applied mathematics like expander families [19], geometry of Banach spaces [1], Markov chains [7], clustering [25], to name just a few.

5.2.2. Edge deletion and the algebraic connectivity. We now turn to the problem of controlling the impact of deleting an edge on the algebraic connectivity of \(L\). The complement of a graph is the graph obtained by putting an edge between every non-adjacent couple of vertices and by deleting all edges already present in the graph before this operation. It is well known [26] that

\[
a(G) \geq n - \lambda_1(G^c).
\]

Thus, controlling the effect of adding an edge to the complement of a graph allows to control the effect of deleting an edge of the graph on the algebraic connectivity.

For \(e = (u, v)\), with \(u, v \in V(G)\), let \(G^c + e\) denote the graph obtained from \(G^c\) by appending the edge \(e\). Let \(i_e\) denote the column vector obtained by setting the component indexed by \(u\) to -1 and the component indexed by \(v\) to +1, and by setting all other components to zero. Since the Laplacian matrix \(L_{G^c}\) admits a factorization analogous to (5.30), we obtain that \(L_{G^c}\) can be written in the form \((1.2)\) with \(c = 2\) and \(a = I_{G^c}i_e\).

In many fields, it is very important to study the robustness of the graph topology to structural perturbations. For instance, the study of food webs has been of growing interest in the recent years [33]. As is well known, predation habits evolve with time as a consequence of landscape changes and competition. The world wide web is also an interesting application of graph theory and the formation and perturbation of communities is a topic of growing interest [29]. Communication systems are also often viewed as an interesting application of
graph theory. In these examples, as in many other from ecology, social sciences, wireless communications, genetics, etc, one is often interested in predicting the impact on topology of removing or adding an edge, a vertex or of various other modifications of the structure, as measured by a relevant index such as the algebraic connectivity.

5.2.3. Controllability of complex networks. In [31], the following model was proposed. One considers a set of $N$ $n$-dimensional oscillators governed by a system of nonlinear differential equations. Moreover, we assume that each oscillator is coupled with a restricted set of other oscillators. This coupling relationship can be efficiently described using a graph where the vertices are indexed by the oscillators and there is an edge between two oscillators if they are coupled. The overall dynamical system is given by the following set of differential equations

\begin{equation}
\begin{aligned}
x'_i(t) &= f(x_i(t)) - \sigma B \sum_{j=1}^{N} l_{ij} x_j(t) + u_i(t), \quad t \geq t_0.
\end{aligned}
\end{equation}

$i = 1, \ldots, N$, where $x_i(t) \in \mathbb{R}^n$ is the state of the $i$th oscillator, $\sigma$ is a positive real number, $B \in \mathbb{R}^{n \times n}$, $f : \mathbb{R} \mapsto \mathbb{R}$ describes the dynamics of each oscillator, $L = (l_{ij})_{i,j=1,\ldots,N}$ is the graph Laplacian of the underlying graph, and $u_i(t)$, $i = 1, \ldots, N$ are the controls. For the system to be well defined, we have to specify some initial conditions $x_i(t_0) = x_{i0}$ for $i = 1, \ldots, N$.

Assume that we have a reference trajectory $s(t)$, $t \geq t_0$ satisfying the differential equation

\begin{equation}
\begin{aligned}
s'(t) &= f(s(t)).
\end{aligned}
\end{equation}

We want to control the system using a limited number of nodes. The selected nodes are called the "pinned nodes". For this purpose, we use a linear feedback law of the form

\begin{equation}
\begin{aligned}
u_i(t) &= p_i K e_i(t),
\end{aligned}
\end{equation}

where $e_i(t) = s(t) - x_i(t)$, $K$ is a feedback gain matrix, and where

\begin{equation}
\begin{aligned}
p_i &= \begin{cases} 1 & \text{if node } i \text{ is pinned} \\ 0 & \text{otherwise}. \end{cases}
\end{aligned}
\end{equation}

Let $P$ denote the diagonal matrix with diagonal vector $p_1, \ldots, p_N$.

The authors then give the definition of (global pinning-) controllability (based on Lyapunov stability criteria):

**Definition 5.1.** We say that the system (5.32) is controllable if the error dynamical system $e := (e_i(t))_{1 \leq i \leq N}$ is Lyapunov stable around the origin, i.e. there exists a positive definite function $V$ such that $\frac{d}{dt} V(e(t)) < 0$ when $e(0) \neq 0$.

The following result, [31, Corollary 5], provide a sufficient condition for a system to be controllable:

**Proposition 5.2** ([31]). Assume that $f$ is such that there exists a bounded matrix $F_{\xi\tilde{\xi}}$, whose coefficients depend on $\xi$ and $\tilde{\xi}$, which satisfies

\begin{equation}
\begin{aligned}
F_{\xi\tilde{\xi}}(\xi - \tilde{\xi}) &= f(\xi) - f(\tilde{\xi}), \quad \xi, \tilde{\xi} \in \mathbb{R}^n.
\end{aligned}
\end{equation}

Let $Q \in \mathbb{R}^{n \times n}$ be a positive definite matrix such that

\begin{equation}
\begin{aligned}
Q K + K^t Q^t &= \kappa (Q B + B^t Q^t) \\
(Q B + B^t Q^t) &\succeq 0.
\end{aligned}
\end{equation}

Let $Q$ denote the diagonal matrix with diagonal vector $p_1, \ldots, p_N$.
and

\[
\frac{1}{2} \lambda_N (\sigma L + \kappa P) \lambda_n \left( QB + B^t Q^t \right) > \sup_{\xi, \tilde{\xi}} \| F_{\xi, \tilde{\xi}} \| \| Q \|.
\]

Then the system is controllable.

Many systems of interest satisfy the constraint specified by (5.33); see [22]. This proposition is very useful for node selection via the matrix \( P \). Indeed, assume that \( Q \) is selected, then one may try to maximise \( \lambda_N (\sigma L + \kappa P) \) as a function of \( P \), under the constraint that no more than \( r \) nodes can be pinned. This is a combinatorial problem that can be relaxed using semi-definite programming or various heuristics [18].

Using Theorem 3.1, we are in position for stating an easy controllability condition in the spirit of [31, Corollary 7], based on the algebraic connectivity of the graph, the number of pinned nodes, the coupling strength and the feedback gain.

**Proposition 5.3.** Let \( Q \in \mathbb{R}^{n \times n} \) be a positive definite symmetric matrix that satisfies

\[
QK + K^t Q^t = \kappa \left( QB + B^t Q^t \right)
\]

and assume that

\[
\| F_{\xi, \tilde{\xi}} \| < \frac{\sigma \lambda_{\min > 0} (L) \lambda_{\min} \left( QB + B^t Q^t \right)}{2 \| Q \|}.
\]

If \( \kappa \) satisfies

\[
\kappa \geq \frac{\sum_{i=1}^r \deg_i}{\sigma \lambda_{\min > 0} (L)} - \frac{\| F_{\xi, \tilde{\xi}} \| \| Q \|}{\lambda_{\min} \left( QB + B^t Q^t \right)} + \sigma \lambda_{\min > 0} (L),
\]

then the system is controllable.

**Proof.** We follow the same steps as for the proof of Corollary 7 in [31]. We assume without loss of generality that the first \( r \) nodes are the pinned nodes. We may write \( P \) as

\[
P = \sum_{i=1}^r e_i e_i^t,
\]

where \( e_i \) is the \( i \)th member of the canonical basis of \( \mathbb{R}^N \), i.e. \( e_i(j) = \delta_{i,j} \). We will try to compare \( \lambda_N (\sigma L + \kappa P) \) with \( \lambda_N (\sigma L) \) and use Proposition 5.2 to obtain a sufficient condition for controllability based on \( L \), i.e. the topology of the network. For this purpose, let us notice recall that \( L \) can be written as

\[
L = I \cdot I^t,
\]

where \( I \) is the incidence matrix of any directed graph obtained from the system’s graph by assigning an arbitrary sign to the edges [8]. Of course \( L \) will not depend on the chosen assignment. Using this factorization of \( L \), we obtain that

\[
\sigma L + \kappa \sum_{i=1}^r e_i e_i^t = \left[ \sqrt{\kappa} \ e_r, \ldots, \sqrt{\kappa} \ e_1, \sqrt{\sigma} I \right] \left[ \sqrt{\kappa} \ e_r, \ldots, \sqrt{\kappa} \ e_1, \sqrt{\sigma} I \right]^t.
\]
Moreover, \( \lambda_{\text{min}>0}(\sigma L + \kappa P) \) can be expressed easily as the smallest nonzero eigenvalue of the \( r^{th} \) term of a sequence of matrices with shape (1.2) for which we can use Theorem 3.4 iteratively. Indeed, we have

\[
\lambda_{\text{min}>0}(\sigma L + \kappa e_1) = \lambda_{\text{min}>0}\left(\left[\sqrt{\kappa} e_1, \sqrt{\kappa} I\right] [\sqrt{\kappa} e_1, \sqrt{\kappa} I^T]\right).
\]

Let us denote by \( x \) the vector \( \sqrt{\kappa} e_1 \) and by \( X \) the matrix \( [\sqrt{\kappa} I, \sqrt{\kappa} I^T] \). Then, we have that

\[
\left[\sqrt{\kappa} e_1, \sqrt{\kappa} I\right] [\sqrt{\kappa} e_1, \sqrt{\kappa} I^T] = \begin{bmatrix} x^T x & x^T X \\ X^T x & X^T X \end{bmatrix}.
\]

Therefore, Theorem 3.4 gives

\[
\lambda_{\text{min}>0}(\sigma L + \kappa e_1) \geq \sigma \lambda_{\text{min}>0}(L) - \frac{\text{deg}_1}{(\kappa - \sigma \lambda_{\text{min}>0}(L))},
\]

where \( \text{deg}_1 \) is the degree of node number 1.

Let us now consider \( \lambda_{\text{min}>0}(\sigma L + \kappa e_1 + \delta_2 e_2) \). We have that

\[
\lambda_{\text{min}>0}(\sigma L + \kappa e_1 + \delta_2 e_2) = \lambda_{\text{min}>0}\left(\left[\sqrt{\kappa} e_2, \sqrt{\kappa} e_1, \sqrt{\kappa} I\right] [\sqrt{\kappa} e_2, \sqrt{\kappa} e_1, \sqrt{\kappa} I^T]\right).
\]

Let us denote by \( x \) the vector \( \sqrt{\kappa} e_2 \) and by \( X \) the matrix \( [\sqrt{\kappa} I, \sqrt{\kappa} I^T] \). Then, we have that

\[
\left[\sqrt{\kappa} e_2, \sqrt{\kappa} e_1, \sqrt{\kappa} I\right] [\sqrt{\kappa} e_2, \sqrt{\kappa} e_1, \sqrt{\kappa} I^T] = \begin{bmatrix} x^T x & x^T X \\ X^T x & X^T X \end{bmatrix},
\]

and using Theorem 3.4 again, we obtain

\[
\lambda_{\text{min}>0}(\sigma L + \kappa e_1 e_1^T + \kappa e_2 e_2^T) \geq \lambda_{\text{min}>0}(\sigma L + \kappa e_1 e_1^T) - \frac{\text{deg}_2}{(\kappa - \sigma \lambda_{\text{min}>0}(L))}.
\]

Since \( \lambda_{\text{min}>0}(\sigma L + \kappa e_1 e_1^T) \leq \lambda_{\text{min}>0}(\sigma L) \), we thus obtain

\[
\lambda_{\text{min}>0}(\sigma L + \kappa e_1 e_1^T + \kappa e_2 e_2^T) \geq \lambda_{\text{min}>0}(\sigma L + \kappa e_1 e_1^T) - \frac{\text{deg}_2}{(\kappa - \sigma \lambda_{\text{min}>0}(L))}.
\]

We can repeat the same argument \( r \) times and obtain

\[
\lambda_{\text{min}>0}(\sigma L + \kappa P) \geq \sigma \lambda_{\text{min}>0}(L) - \frac{\sum_{i=1}^{r} \text{deg}_i}{(\kappa - \sigma \lambda_{\text{min}>0}(L))}.
\]

Finally, by Proposition 5.2 we know that the following constraint is sufficient for preserving controllability

\[
\lambda_{\text{min}>0}\left(\sigma L + \kappa \sum_{i=1}^{r} e_i e_i^T\right) \geq \frac{2 \|F \xi\| \|Q\|}{\lambda_{\text{min}}(QB + B'Q')},
\]

By (5.36), it is sufficient to guarantee the controllability of our system to impose

\[
\sigma \lambda_{\text{min}>0}(L) - \frac{\sum_{i=1}^{r} \text{deg}_i}{(\kappa - \sigma \lambda_{\text{min}>0}(L))} \geq \frac{2 \|F \xi\| \|Q\|}{\lambda_{\text{min}}(QB + B'Q')}.
\]

Then, combining (5.37) with (5.36) implies that

\[
\kappa \geq \frac{\sum_{i=1}^{r} \text{deg}_i}{(\sigma \lambda_{\text{min}>0}(L) - \frac{2 \|F \xi\| \|Q\|}{\lambda_{\text{min}}(QB + B'Q')})} + \sigma \lambda_{\text{min}>0}(L)
\]

is a sufficient condition for controllability.

\qed
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