EVEN SETS OF NODES ON SEXTIC SURFACES

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Abstract. We determine the possible even sets of nodes on sextic surfaces in $\mathbb{P}^3$, showing in particular that their cardinalities are exactly the numbers in the set $\{24, 32, 40, 56\}$. We also show that all the possible cases admit an explicit description. The methods that we use are an interplay of coding theory and projective geometry on one hand, of homological and computer algebra on the other.

We give a detailed geometric construction for the new case of an even set of 56 nodes, but the ultimate verification of existence relies on computer calculations. Moreover, computer calculations have been used more than once in our research in order to get good guesses.

The construction gives a maximal family, unirational and of dimension 27, of nodal sextics with an even set of 56 nodes.

As in [Ca-Ca] (where the other cases were described) each such nodal surface $F$ is given as the determinant of a symmetric map $\varphi : \mathcal{E}^i \to \mathcal{E}$, for an appropriate vector bundle $\mathcal{E}$ depending on $F$. The first difficulty here is to show the existence of such vector bundles. This leads us to the investigation of a hitherto unknown moduli space of rank 6 vector bundles which we show to be birational to a moduli space of plane representations of cubic surfaces in $\mathbb{P}^3$. The resulting picture shows a very rich and interesting geometry. The main difficulty is to show the existence of "good" maps $\varphi$, and the interesting phenomenon which shows up is the following: the "moduli space" of such pairs $(\mathcal{E}, \varphi)$ is (against our initial hope) reducible, and for a general choice of $\mathcal{E}$ the determinant of $\varphi$ is the double of a cubic surface $G$. Only when the vector bundle $\mathcal{E}$ corresponds to a reducible cubic surface, then we get an extra component of the space of such pairs $(\mathcal{E}, \varphi)$, and a general choice in this component yields one of our desired nodal sextic surfaces.

Introduction

Let $F$ be a nodal surface in $\mathbb{P}^3$ of degree $d$: i.e., $F$ has only $\mu$ nodes (ordinary double points) $P_1, \ldots, P_\mu$ as singularities.

A natural and classical question is to ask for the maximum possible number of nodes $\mu(d)$ that such a surface $F$ can have.

The theory of projectively dual surfaces shows easily that $\mu(d) < \frac{1}{2}d(d - 1)^2$ for $d \geq 3$ and the slightly better inequality given by Bassett in 1907 (cf. [Bass]) was obtained using this method.

The function $\mu(d)$ is only known for $d \leq 6$, and for $d \leq 5$ one has an explicit description of the nodal surfaces which attain the maximum $\mu(d)$: the

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Cayley cubic, the Kummer quartics, and the Togliatti quintics (cf. [Cay1], [Cay2], [Kum], [Tog1], [Tog2], [Bea]).

An important tool to investigate the function $\mu(d)$ for small values of $d$ ($d \leq 17$), and to characterize the maximizing surfaces was introduced by Beauville in [Bea]: he attached a binary code to each nodal surface $F$ and used coding theory in order to show that $\mu(5) = 31$.

The method of using coding theory was later used by Jaffe and Ruberman in order to show (see [Ja-Ru]) that $\mu(6) = 65$, but their proof is not so short as the one by Beauville, partly because at that time a complete knowledge about the cardinality of an even set of nodes\footnote{We adopt here the terminology of [Ca-Ca] concerning the notion of even sets of nodes which was introduced in [Cat1]: namely, the strictly even sets of [Cat1] are called even sets, while the weakly even sets of [Cat1] are called half-even, or $1/2$-even set of nodes.} on a sextic was missing (the binary code consists of the even sets of nodes on $F$, introduced in [Cat1], where a complete classification of even sets for degree $d = 5$ was given).

Today we still ignore if the Barth 6-ics (see [Ba1]) are those which achieve the maximum $\mu(6) = 65$ and until now an explicit description of the possible even sets of nodes for sextic surfaces was missing. A general structure theorem for even and $1/2$-even sets was given in [Ca-Ca]; but the cases where the cardinality $t$ of an even set would be $> 40$ were excluded only as a consequence of a conceptual error which was pointed out to the authors by Duco van Straten. Thus the simple proof by J. Wahl (cf. [Wahl]) of $\mu(6) = 65$ also became invalid.

We rescue the situation here by showing the following

**Main Theorem A.** Let $F$ be a nodal surface of degree $d = 6$ in $\mathbb{P}^3$ with an even set of $t$ nodes. Then $t \in \{24, 32, 40, 56\}$. These four possibilities occur and can be explicitly described.

The situation is thus more complicated than for $d \leq 5$, the list of possible cardinalities $t$ is (cf. e.g. [Ca-Ca]):

\[
\begin{align*}
  d = 3 & \quad t = 4 \\
  d = 4 & \quad t \in \{8, 16\} \\
  d = 5 & \quad t \in \{16, 20\} \\
  d = 6 & \quad t \in \{24, 32, 40, 56\}
\end{align*}
\]

We first show in section 1 that the case of an even set of 64 nodes cannot exist. The simple new idea is to study the so called extended code (cf. e.g. [Cat2]) and we then use a mixture of geometric and coding theory arguments, as was done in the papers cited above, for instance in [Ja-Ru], where the case of an even set of 48 nodes was excluded.

We then proceed, using the structure theorem of [Ca-Ca], to construct explicit cases of sextics with an even set of 56 nodes.

The bulk of the paper is devoted to this purpose, and we get the following result.

**Main Theorem B.** There is a family of nodal sextic surfaces with 56 nodes forming an even set, parametrized by a smooth irreducible rational
variety $\Phi_0$ of dimension 33, whose image $\Xi_0$ is a unirational subvariety of
dimension 27 of the space of sextic surfaces. Moreover, the above family is
versal, thus $\Xi_0$ yields an irreducible component of the subvariety of nodal
sextic surfaces with 56 nodes.

The fact that a maximal family of nodal sextics with 56 nodes forming
an even set has dimension equal to 27 means that these nodes impose in-
dependent conditions on the space of sextic surfaces (cf. [Bu-Wa]). It is an
interesting question to find the smallest degree for which there exist even
sets of nodes failing to impose independent conditions.

As already mentioned, it follows from the more general result of [Ca-Ca]
that every even nodal set on a sextic surface $F$ occurs as the corank 2
degeneracy locus of a symmetric map $\varphi : E^\vee \rightarrow E$, for an appropriate vector
bundle $E$ depending on $F$.

The method to construct $E$ is based on a combination of Beilinson’s theo-
rem and of a revisitation of Horrocks’ correspondence due to Charles Walter
(cf. [Wal]), which was exploited in [Ca-Ca]. The bundle $E$ is constructed
starting from a submodule $M$ of the intermediate cohomology module $H^1(F)$
of the quadratic sheaf $F$ associated to the even set, and corresponding to
the choice of a Lagrangian subspace $U$ of $H^1(F(1))$. The choice of $M$ deter-
mines a unique vector bundle $E$, if a certain generality assumption (which
we call first assumption) is verified.

The construction is quite explicit if we make another generality assump-
tion, namely that the two nonzero degree components of the artinian graded
module $M$, the previously mentioned $U$ and another one denoted by $W$, both
have dimension equal to 3. If we denote as customary by $V$ the vector space
of linear forms on $\mathbb{P}^3$, then the module $M$ is completely determined by the
multiplication tensor $B \in U^\vee \otimes V^\vee \otimes W$ for $M$.

We then show that the tensor $B$ determines explicitly the bundle $E$ as the
kernel of an exact sequence

$$0 \rightarrow E \rightarrow U \otimes V \otimes O(1) \rightarrow (W \otimes O(1)) \oplus (U \otimes O(2)) \rightarrow 0$$

where the first component is precisely $B$, and the second is the standard
Euler map, here denoted by $\epsilon$.

Section 3 then ends by showing that the family of pairs $(E, \varphi)$ is
parametrized (non uniquely) by the following family of pairs

$$\mathcal{M}_{AB} := \{(B, A) \mid B \in U^\vee \otimes V^\vee \otimes W,\ A \in (U \otimes V) \otimes (U \otimes V) \otimes H^0(O_{\mathbb{P}^3}(2)), A = t A, (B, \epsilon) \cdot A = 0\}.$$ 

$\mathcal{M}_{AB}$ sits inside an affine space of dimension 816, and it is not possible even
for the computer to find the decomposition of $\mathcal{M}_{AB}$ into irreducible com-
ponents. It is clear that $\mathcal{M}_{AB}$ dominates the space of the above tensors $B$, and,
if $\mathcal{M}_{AB}$ were irreducible, one would obtain the sextic surfaces immediately
by a random choice.
However, for long time all the random choices would always give the double of a cubic surface $G$ as determinant of $\varphi$, and it looked like even sets with 56 nodes would not exist. We then tried to prove that this was indeed the case, and we had to find an explanation for the cubic surface $G$.

Now, it is classical that to a $3 \times 3 \times 4$ tensor $B$ one can associate a cubic surface in $\mathbb{P}^3$ by taking the determinant of the corresponding $3 \times 3$ matrix of linear forms on $\mathbb{P}^3$. However, in our case we get a cubic surface $G^*$ in the dual projective space $\mathbb{P}^3^\vee = \text{Proj}(V^\vee)$, together with two different realizations of $G^*$ as a blow up of a projective plane $\text{Proj}(U^\vee)$ (respectively, $\text{Proj}(W^\vee)$) in a set of 6 points. These are the points where the Hilbert-Burch $3 \times 4$ matrix of linear forms on $U$ drops rank by 1, and the rational map to $\mathbb{P}^3^\vee$ is given by the system of cubics through the 6 points, system which is generated by the determinants of the $4 \times 3 \times 3$ minors of the Hilbert Burch matrix.

One passes from one realization to the other simply by transposing the tensor, and we will call this the trivial involution for $3 \times 3 \times 4$ tensors: but what we have discovered, through geometry, is the existence of another involution for $3 \times 3 \times 4$ tensors, which we called the cross-product involution.

This second involution associates to a general tensor $B \in U^\vee \otimes V^\vee \otimes W$ another tensor $B' \in W^\vee \otimes V \otimes U'$, where $W' := \Lambda^2(W)$ and $U'$ is defined as the kernel of the map $\Lambda^2(W^\vee) \otimes V \to U^\vee \otimes W^\vee$ induced by contraction with $B$ (cf. 4.19 for the proof that we have indeed a birational involution).

In fact, to $B$ corresponds now a cubic surface $G \subset \mathbb{P}^3$, which is related to a general bundle $E$ through the existence of an exact sequence

$$0 \to 6\mathcal{O} \to \mathcal{E} \to \tau \to 0,$$

where $\tau$ is an invertible sheaf on the cubic surface $G$. One can see more precisely that $B$ determines a sheaf $\mathcal{G}$ on $G$ such that $\tau = \mathcal{G}^{\otimes 2}(-1)$.

We found in this way a nice explanation of the phenomenon pointed out by the computer: we got as determinant the surface $G$ counted twice, simply because, in view of the above exact sequence, for a smooth cubic surface (indeed, irreducible) all the symmetric endomorphisms $\phi \in H^0(S^2(E))$ are induced by the inclusion $S^2(6\mathcal{O}) \to S^2(\mathcal{E})$.

It was clear at this point that if $H^0(S^2(E))$ would always have dimension 21, then we could not get any nodal sextic surface of the desired type, but it was of course possible that the dimension could jump up for special surfaces $G$, and that our parameter space $\mathcal{M}_{A,B}$ would be reducible. As explained in section 7, a small computational simplification and the reduction to finite fields allowed to make many more random attempts, until the first sextic surface appeared. Since a determinantal approach predicts that the space of tensors $B$ for which the dimension of $H^0(S^2(E))$ jumps has codimension 7, it was only natural to guess that the case which works out is the case of tensors $B$ corresponding to reducible cubic surfaces. This guess turned out to be true.
The cross product involution can also be phrased as a duality theorem for a certain moduli spaces of vector bundles on $\mathbb{P}^3$. Namely, we prove the following

**Theorem C** Consider the moduli space $\mathcal{M}^s(6; 3,6,4)$ of simple rank 6 vector bundles $\mathcal{E}$ on $\mathbb{P}^3$ with Chern polynomial $1+3t+6t^2+4t^3$, and inside it the open set $\mathfrak{A}$ corresponding to the simple bundles with minimal cohomology, i.e., such that

1. $H^i(\mathcal{E}) = 0 \forall i \geq 1$
2. $H^i(\mathcal{E}(-1)) = 0 \forall i \neq 1$
3. $H^i(\mathcal{E}(-2)) = 0 \forall i \neq 1$.

Then $\mathfrak{A}$ is irreducible of dimension 19 and it is bimeromorphic to $\mathfrak{A}^0$, where $\mathfrak{A}^0$ is an open set of the G.I.T. quotient space of the projective space $\mathfrak{B}$ of tensors of type $(3,4,3)$, $\mathfrak{B} := \{ B \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W) \}$ by the natural action of $SL(W) \times SL(U)$.

Let moreover $[B] \in \mathfrak{A}^0$ be a general point: then to $[B]$ corresponds a vector bundle $\mathcal{E}_B$ on $\mathbb{P}^3$, and also a vector bundle $\mathcal{E}_B^*$ on $\mathbb{P}^3$, obtained from the direct construction applied to $G^*_B$ (cf. 5.2). $\mathcal{E}_B^*$ is the vector bundle $\mathcal{E}_B$, where $[B] \in \mathfrak{A}^0$ is obtained from $B$ via the cross product involution.

It would be interesting to further investigate the moduli space of Gieseker semistable rank 6 vector bundles $\mathcal{E}$ on $\mathbb{P}^3$ with Chern polynomial $1+3t+6t^2+4t^3$. In section 5 we make a first step in this direction proving the Mumford-Takemoto semistability of the general bundle in $\mathfrak{A}$.

Section 7 is devoted to a brief account of the random approach which we already mentioned, while the appendix contains the two Macaulay scripts which are needed for the ultimate verification of the existence of surfaces which have an even set of 56 distinct nodes as the only singularities.

### 1. Excluding via coding theory

Throughout this section $F$ will be a normal surface in $\mathbb{P}^3$ of degree $d$ having at most Rational Double Points as singularities, and possessing moreover $\mu$ nodes (ordinary double points) $P_1, \ldots P_\mu$ among its singularities.

We let $\pi : \tilde{F} \rightarrow F$ be the minimal resolution of the singularities of $F$. It is well known (cf. [13]) that $\tilde{F}$ is diffeomorphic to a smooth surface of degree $d$ in $\mathbb{P}^3$; in particular $\tilde{F}$ is simply connected and for its second Betti number we have $b_2(\tilde{F}) = d(d^2 - 4d + 6) - 2$.

We let $A_1, \ldots, A_\mu$ be the exceptional $(-2)$-curves ($\cong \mathbb{P}^1$) coming from the blow up of the nodes $P_1, \ldots P_\mu$, and we let $H$ be the full transform of a plane section of $F$.

Let $V$ be the $\mathbb{Z}/2$-vector space freely generated by the $A_i$’s and consider the map

$$\epsilon : V := \bigoplus_{i=1}^\mu (\mathbb{Z}/2)A_i \rightarrow H^2(\tilde{F}, \mathbb{Z}/2),$$

given by the reduction modulo two of the integral first Chern class of a divisor: $\epsilon(\Sigma_i a_i A_i) := c_1(\Sigma_i a_i A_i) \mod (2)$. Let $U$ be the image of $\epsilon$. 


Since $A_i \cdot A_j = -2\delta_{ij}, A_i \cdot H = 0, H^2 = d$, it follows that $U$ is an isotropic subspace of $V$, and since the intersection product modulo 2 is non degenerate its dimension does not exceed $b_2(\widetilde{F})/2$.

In the case where the surface has even degree $d \equiv 0(\text{mod} \, 2)$, we consider more generally $\tilde{V} := V \oplus \mathbb{Z}/2H$, $\tilde{\epsilon} : \tilde{V} \to H^2(\tilde{F}, \mathbb{Z}/2)$, and the corresponding isotropic subspace $\tilde{U} := \text{Im}(\tilde{\epsilon})$.

**Definition 1.1.** 1) The strict code $K$ associated to the nodal set $\{P_1, \ldots, P_\mu\}$ on the surface $F$ is the binary code $K := \ker(\epsilon)$.

2) If $d \equiv 0(\text{mod} \, 2)$ the enlarged code $\tilde{K}$ associated to the nodal set $\{P_1, \ldots, P_\mu\}$ on the surface $F$ is the binary code $\tilde{K} := \ker(\tilde{\epsilon})$.

By the above inequality for $\dim(U)$, we get

$$\dim K \geq \mu - \frac{1}{2} b_2(\tilde{F}) = \mu - \frac{1}{2} d(2^2 - 4d + 6) + 1, \quad \dim \tilde{K} \geq \mu - \frac{1}{2} d(d^2 - 4d + 6) + 2.$$  

**Remark 1.2.** By Miyaoka’s inequality (cf. [Miy]) $\mu \leq \frac{1}{6} d(d - 1)^2$, therefore only for $d \leq 17$ we get for sure a nontrivial code $K$ because $\dim K \geq \mu - \frac{1}{2} d(d^2 - 4d + 6) + 1$.

Notice that the notion of an even, respectively half-even, set of nodes can be derived from the coding-theory framework.

**Definition 1.3.** A vector $v \in V$ is completely determined by its support $N_v := \{i \mid v_i = 1\}$. The cardinality of the support is called the weight of $v$ and denoted by $w(v) := \#N_v$.

By the universal coefficients theorem and Lefschetz’ (1,1) theorem the condition $v \in K$ is equivalent to the 2-divisibility of $\sum_{i \in N_v} A_i$ in $\text{Pic}(\tilde{F})$. We denote by $L$ a divisor on $\tilde{F}$ such that $2L \equiv \sum_{i \in N_v} A_i$. The class of $L$ in $\text{Pic}(\tilde{F})$ is uniquely determined, because $\text{Pic}(\tilde{F})$ has no torsion. We have then a finite double cover $\tilde{S}$ of $\tilde{F}$ branched exactly on the nodal curves $A_i$ such that $i \in N_v$, and moreover $f_*O_S = O_{\tilde{F}} \oplus O_{\tilde{F}}(-L)$. Correspondingly, we have a double cover $f : S \to F$, with $f_*O_S = O_F \oplus F$, ramified exactly in $\Delta := \{P_i \in F \mid i \in N_v\}$ (cf. [Cat1] and [Ca-Ca] for more details).

These sets $\Delta$ are called even sets of nodes (cf. [Cat1]).

Similarly, one defines a half-even set of nodes $\Delta$ by the condition that its associated word $\tilde{v} := (v_1, \ldots, v_\mu, 1)$, obtained by setting $v_i = 1 \iff P_i \in \Delta$, belongs to the enlarged code $\tilde{K}$. This condition is again equivalent to the existence of a divisor $L$ in $\text{Pic}(\tilde{F})$ with $2L \equiv \sum_{i \in N_v} A_i + H$.

We define the weight and support of $\tilde{v}$ as the weight and support of the word $v := (v_1, \ldots, v_\mu) \in V$ (these notions are different from the corresponding ones in coding theory).

Observe finally that $K = \tilde{K} \cap \{\tilde{v} | \tilde{v}_{\mu+1} = 0\}$.

As shown in [Cat1], Prop. 2.11 and Prop. 2.13, the geometric interpretation of even sets of nodes in terms of double coverings allows to give the
following restrictions for the cardinality $t$ of an even (resp.: half-even) set of nodes

**Proposition 1.4.** Let $t := w(u)$ be the weight of a code word $u$.

1. Assume $u \in K$: then $t \equiv 0 \ (4)$. Moreover, if $d$ is even, then $t \equiv 0 \ (8)$.
2. If ($d$ is even) $u \in \tilde{K} \setminus K$, then $t \equiv d(2d - 7)/2 \ (4)$. In particular, for $d = 6$, $t \equiv -1 \ (4)$.

**Corollary 1.5.** Let $d = 2(2k + 1)$ be twice an odd integer and assume that $K, \tilde{K}$ are the codes corresponding to an even set of nodes $\Delta$: then $\tilde{K} = K$.

**Proof.** Our assumption is that the code $K \subset \tilde{K}$ contains the vector $\mathbb{I}$ whose coordinates are all equal to 1, except the last which equals 0. If we have a vector $w \in \tilde{K} \setminus K$ and let $t$ be its weight, then the weight of ($\mathbb{I} + w$) $\in \tilde{K} \setminus K$ is congruent to $-t$ modulo (4). Since $t \equiv d(2d - 7)/2 \equiv 2k + 1 \ (4)$, then $-t \not\equiv d(2d - 7)/2 \ (4)$, contradicting (2) of the previous proposition. 

Q.E.D.

Let us examine by means of coding theory which even sets of nodes can occur on sextic nodal surfaces. The main result of this section is the following theorem.

**Theorem 1.6.** On a sextic normal surface $F$ with only Rational Double Points as singularities there does not exist an even set of nodes of cardinality $t = 64$.

In order to prove the theorem we first prove some preliminary results.

**Lemma 1.7.** Suppose that there exists an even set $\Delta$ of nodes of cardinality $t = 64$ on a normal sextic surface $F$.

Let $\gamma : F \dashrightarrow \tilde{F} \subset \mathbb{P}^3$ be the Gauss map of $F$, given by the partial derivatives $\left(\frac{\partial F}{\partial x_i}\right)$.

1) $\gamma$ corresponds to a linear subsystem $\mathcal{L}$ of $|5H - \sum_{i=1}^{t} A_i|$ on $\tilde{F}$ whose fixed part $\Phi$ is contained in the preimage of the singular points of $F$ which are not the nodes of $\Delta$.
2) Let $L$ be a divisor on $\tilde{F}$ with $2L \equiv \sum_{i=1}^{t} A_i$: then $H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(2H - L)) = 0$.

**Proof.** The first assertion follows since the zero locus of the partial derivatives $\left(\frac{\partial F}{\partial x_i}\right)$ on $F$ is exactly the singular locus of $F$, and at each node the partials $\left(\frac{\partial F}{\partial x_i}\right)$ define the maximal ideal. Thus $\gamma$ is a morphism around each $(-2)$-curve $A_i$, which is indeed embedded as a plane conic.

Assertion 2) is proven by contradiction. Assume in fact that $C \in |2H - L|$. We calculate now $C \cdot (5H - \sum_{i=1}^{t} A_i) = 60 - 64 = -4$.

However, $\Phi \cdot H = \Phi \cdot A_i = 0$ by our first assertion, whence $C \cdot (5H - \sum_{i=1}^{t} A_i)$ equals the intersection number of $C$ with the movable part of the linear system $\mathcal{L}$, which is obviously a non negative number.

We have obtained the desired contradiction.
Proposition 1.8. Suppose there exists an even set of nodes of cardinality 64 on a sextic normal surface with only Rational Double Points as singularities, and let $f : S \to F$ be the corresponding finite double cover. Then $h^1(S, O_S) = 5$.

Proof. We have

$$h^1(S, O_S) = h^1(F, F) = h^1(\tilde{F}, O_{\tilde{F}}(-L)).$$

Moreover, since $H \cdot L = 0$ one easily sees that $h^0(\tilde{F}, O_{\tilde{F}}(-L)) = 0$ and argues then that $h^2(\tilde{F}, O_{\tilde{F}}(-L)) = h^0(\tilde{F}, O_{\tilde{F}}(2H + L)) = h^0(\tilde{F}, O_{\tilde{F}}(2H - L) = 0$ by the previous lemma, the second equality following from the fact that every divisor in $|2H + L|$ contains $\sum_{i=1}^t A_i$.

Whence $-h^1(S, O_S) = -h^1(\tilde{F}, O_{\tilde{F}}(-L)) = \chi(\tilde{F}, O_{\tilde{F}}(-L)) = \chi(O_{\tilde{F}}) + 1/2(-L) \cdot (-L - 2H) = 11 - 16 = -5$.

Lemma 1.9. Suppose there exists an even set of nodes $\Delta$ of cardinality 64 on a sextic normal surface $F$ with only Rational Double Points as singularities. Let $K$, $\tilde{K}$ denote the corresponding binary codes. Then $\dim \tilde{K} = 12 > \dim K = 11$.

Proof. By the previous proposition the surface $S$, the finite double cover of $F$ ramified exactly along $\Delta$, has invariants $p_g = 10, q = 5, K_S^2 = 48, \chi(S) = 6$.

The corresponding (non minimal) smooth surface $\tilde{S}$, the double cover of $\tilde{F}$, has the same invariants as $S$.

By [Ben, Lemma 2] or [Ja-Ru, Thm 4.5] it follows that the code $K$ has dimension $b_1(\tilde{S}) + 1 = 11$. We already remarked that $\dim \tilde{K} \geq 65 - 53 = 12$, and it is obvious that $K \subset \tilde{K}$ has codimension at most 1.

Proof of Thm. 1.6. The previous lemma contradicts then corollary 1.5.

As an immediate consequence of Thm. 1.6 and [Ja-Ru, Sec. 7], we obtain the following.

Corollary 1.10. Let $F$ be a a sextic normal surface in $\mathbb{P}^3$ with only Rational Double Points as singularities with an even set of $t$ nodes. Then $t \in \{24,32,40,56\}$.

Proof. Since $t \equiv 0 (8)$, the inequality $t \leq 64$ follows from the classical inequalities of Bassett and of Miyaoka, and the case $t = 64$ was just excluded. In [Ca-Ca] it is shown that $t \geq 24$, and that the cases $t = 24,32,40$ do exist.

The non-existence of even sets of 48 nodes on nodal sextics is proven in [Ja-Ru, Sec. 7].

Q.E.D.
2. Cohomology modules and bundle symmetric maps

In this section, after recalling the main result of \cite{Ca-Ca}, namely the correspondence between even sets of nodes and bundle symmetric maps, we shall give bounds for the cohomology groups $H^i(F(j))$ of the quadratic sheaf $F$ associated to an even set of nodes $\Delta$.

We first recall the main result of \cite{Ca-Ca}, according to the following notation: $\delta \in \{0, 1\}$ and $\delta/2$-even stands for even if $\delta = 0$, respectively for half even if $\delta = 1$.

**Theorem 2.1** (\cite{Ca-Ca, Thm. (0.3)}). Let $\Delta$ be a $\delta/2$-even set of nodes on a normal surface $F$ of degree $d$, let $f : S \to F$ denote a corresponding double cover of $F$, let $F$ be the anti-invariant part of $f_*O_S$.

Then there exists a locally free sheaf $E$ on $\mathbb{P}^3$ and a symmetric map $\varphi$ yielding an exact sequence

\[(***) 0 \to E^\vee(-d-\delta) \xrightarrow{\tilde{\varphi}} E \to F \to 0.\]

In particular, $F = \{x \mid \det(\varphi(x)) = 0\}$, $\Delta = \{x \mid \operatorname{corank}(\varphi(x)) \geq 2\}$.

Conversely, assume that one is given an exact sequence as in $(**)$ with $\varphi$ symmetric, such that $F = \{x \mid \det(\varphi(x)) = 0\}$ is a normal surface and $\Delta := \{x \mid \operatorname{corank}(\varphi(x)) \geq 2\}$ is a reduced set of $t$ points: then $\Delta$ is a $\delta/2$-even set of nodes on $F$.

The ideal of the subscheme $\Delta$ is the second Fitting ideal of $\varphi$, i.e., on local trivializing affine sets for $E$, it is given by the determinants of the $(\operatorname{rk}E-1)$-minors of $\varphi$.

We briefly explain how the sheaf $E$ is explicitly constructed in \cite{Ca-Ca} by means of C. Walter’s interpretation of Horrocks’ correspondence (cf. \cite{Wal}).

Assume that the intermediate cohomology module $H^1_*(F, F) := \oplus_{i \in \mathbb{Z}} H^1(F, F(i))$ is known (it is an Artinian graded module over the polynomial ring of $\mathbb{P}^3$, $\mathcal{A} := \mathbb{C}[x_0, x_1, x_2, x_3]$).

One considers then the (Artinian) graded module

\[(2.1) M := U \bigoplus (\oplus_{i > (d-4)/2} H^1(F, F(i))),\]

where, if $d$ is even, $U$ is a Lagrangian subspace in the Serre self-dual cohomology space $H^1(F, F((d-4)/2))$, and $U := 0$ if $d$ is odd.

Recall that the first syzygy bundle $\operatorname{Sy}z^1(M)$ is obtained from a projective graded resolution of the module $M$ by free $\mathcal{A}$-modules

\[0 \to \mathcal{P}^4 \to \ldots \to P^1 \xrightarrow{\alpha_{1}} P^0 \xrightarrow{\alpha_{0}} M \to 0\]

as follows: the homomorphism $\alpha_0 : P^1 \to P^0$ induces a corresponding homomorphism $(\alpha_1)^\sim$ between the (Serre-) associated sheaves $(P^1)^\sim$ and $(P^0)^\sim$ and the first syzygy bundle of $M$ is defined as $\operatorname{Sy}z^1(M) := \operatorname{Ker}(\alpha_1^\sim)$.

One has a natural homomorphism $\operatorname{Sy}z^1(M) \to F$ (cf. \cite{Ca-Ca}, pages 240-1) induced by truncation, whence one gets a homomorphism $H^0_*(\operatorname{Sy}z^1(M)) \to H^0_*(F)$, which needs not be surjective.
The bundle $\mathcal{E}$ is then defined as the direct sum of line bundles, whose generators induce a minimal set of generators of the cokernel of $H^0(Syz^1(M)) \to H^0(\mathcal{F})$, in order that one obtains a surjection between $H^0(\mathbb{P}^3, \mathcal{E})$ and $H^0(F, \mathcal{F})$.

Thus a first important step is the one of determining the intermediate (Artinian) cohomology module $H^1(F, \mathcal{F}) := \bigoplus_{i \in \mathbb{Z}} H^1(F, \mathcal{F}(i))$, in particular one has to determine the possible dimensions of its graded pieces, i.e., the numbers $h^1(\mathcal{F}(i))$. Later on, when we want to impose the surjectivity of $H^0(\mathbb{P}^3, \mathcal{E}) \to H^0(F, \mathcal{F})$ it will also be important to determine the dimensions $h^0(\mathcal{F}(i))$.

In short, the first necessary task is to determine the possible values for the cohomology table $h^j(\mathcal{F}(i))$ of $\mathcal{F}$ (a priori just $\chi(\mathcal{F}(i))$ is known, and it is determined by the degree $d$ of $F$ and the number $t$ of nodes of $\Delta$).

Besides geometrical estimates, an important tool used in [Ca-Ca] is the Beilinson complex (cf. [Bei]) constructed from the cohomology table $h^j(\mathcal{F}(i))$.

**Remark 2.2.** It is well known that for any coherent sheaf $\mathcal{G}$ on $\mathbb{P}^n$ the complex $\mathcal{K}^i := \bigoplus_j (H^{i-j}(\mathbb{P}^n, \mathcal{G}(j)) \otimes \Omega^{-j}(-j))$, called Beilinson’s monad, has cohomology $H^i(\mathcal{K}^*)$ equal to $\mathcal{G}$ in degree $i = 0$ and 0 in all other degrees, cf. [Bei].

In the case of even sets of nodes on sextics, [Ca-Ca] classifies the sets of cardinalities $t = 24, 32, 40$. These cases are given by the following symmetric bundle maps (here and in the rest of the paper, we denote the sheaf $\mathcal{O}_{\mathbb{P}^3}$ simply by $\mathcal{O}$):

$$
t = 24 \quad \varphi : \mathcal{O}(-5) \oplus \mathcal{O}(-4) \to \mathcal{O}(-2) \oplus \mathcal{O}(-1)
$$

$$
t = 32 \quad \varphi : 3\mathcal{O}(-4) \to 3\mathcal{O}(-2)
$$

$$
t = 40 \quad \varphi : \Omega^2(-1) \oplus \mathcal{O}(-4) \to \Omega^1(-1) \oplus \mathcal{O}(-2).
$$

The cases of even sets of nodes with cardinality $t = 48$ and $t = 64$ are excluded by corollary (1.10). Thus the only remaining case, in order to complete the classification of even sets of nodes on sextic surfaces, is the case $t = 56$.

Let us then restrict ourselves now to the case $d = 6$ and $t = 56$, and let us consider the module $M := U \bigoplus (\oplus_{i>1} H^1(F, \mathcal{F}(i)))$.

We shall use the analysis done in [Ca-Ca] p. 254 for $\mathcal{F} |_H$, where $H$ is a smooth plane section of $F$: it shows that $h^0(F, \mathcal{F}(1)) = h^2(F, \mathcal{F}(1)) = 0$ (in loc. cit. it is shown that this holds unless $\mathcal{F} |_H$ is of type $(2, 4)$, but if $\mathcal{F} |_H$ of type $(2, 4)$ then $F$ is of type $(2, 4)$ too, in the sense of [Cat1] Thm. 2.2 and Thm. 2.16), and $t = 24$).

Hence we can assume that $h^0(F, \mathcal{F}(1)) = h^2(F, \mathcal{F}(1)) = 0$, so that by Riemann-Roch applied to $\tilde{F}$

$$
h^1(F, \mathcal{F}(1)) = -\chi(\mathcal{F}(1)) = -8 + \frac{t}{4} = 6.
$$
According to the notation in [Ca-Ca, p. 254], set \( \tau := h^1(F,F(1)) \), \( a := h^1(F,F) = h^1(F,F(2)) \), \( b := h^2(F,F) = h^0(F,F(2)) \), the equalities following by Serre duality. Our previous calculation yields \( \tau = 3 \).

The exact sequence

\[
\begin{array}{c}
H^0(F,F(1)) = 0 \to H^0(H,F(1)|_H) \to H^1(F,F) \to H^1(F,F(1)) \to \\
\to H^1(H,F(1)|_H) \to H^2(F,F) \to H^2(F,F(1)) \cong H^0(F,F(1)) = 0
\end{array}
\]  

(2.2)

gives the relation \( \chi(F(1)|_H) - a + 2\tau + b = 0 \). An application of Riemann-Roch on \( H \) yields \( \chi(F(1)|_H) = -3 \) and the above relation becomes \( 0 \leq b = a - 3 \).

Finally notice that (see [Ca-Ca, Formula 3.2, p. 248])

\[
H^1(F,F(-m)) \cong H^1(F,F(m+2))^\vee = 0, m > 0,
\]

and trivially also \( H^0(F,F(-m)) = 0, m > 0 \).

Since the rank of \( F \) at the generic point of \( \mathbb{P}^3 \) is 0, a computation of the ranks of all the terms of the Beilinson’s monad of \( F(3) \) yields the relation \( 4b + 6\tau - 4a - c = 0 \), i.e. \( c = 12 - 2\tau = 6 \).

Therefore, the Beilinson table of \( F(3) \) is:

| \( i \) | \( j \) |
|---|---|
| \( c = 6 \) | \( b \) | 0 | 0 | 0 |
| 0 | \( b + 3 \) | 2\( \tau = 6 \) | \( b + 3 \) | 0 |
| 0 | 0 | 0 | \( b \) | \( c = 6 \) |

**Proposition 2.3.** Let \( F \) be a nodal surface of degree 6 with \( t \) nodes and with an even set \( \Delta \) of \( t = 56 \) nodes and \( L \in \text{Pic}(\tilde{F}) \) the corresponding divisor such that \( \sum_{i \in N_\Delta} A_i = 2L \). Then \( b := h^0(2H - L) \leq 1 \).

**Proof.** Assume \( h^0(2H - L) \geq 2 \) and write \(|2H - L| = |M| + |\Psi| \), where \( \Psi \) is the fixed part of the linear system \(|2H - L| \). Let \( L \) be the Gaussian linear subsystem of \( 5H - \sum_{i=1}^7 A_i \) and \( C \) any effective divisor in \(|2H - L| \).

Since \( F \) is nodal it follows by our previous argument that \( L \) is free from base points, hence for any effective divisor \( C' \leq C \) we have \( C'L \leq CL = 60 - t = 4 \).

Observe that by [Ja-Ru] the number \( \overline{t} \) of nodes of \( F \) satisfies \( \overline{t} \leq 65 \) : since \( L^2 \geq 150 - 2\overline{t} \geq 20 \), the Index Theorem ensures that \((C')^2 \leq 0 \). In particular, it follows that \( M^2 = 0 \).

Since the dual surface \( F^\vee \) is birational to \( F \) and therefore it is still of general type, \( \gamma(M) \) has degree at least 4.

Hence, by the previous calculation, \( ML = 4 \) and \( \Psi L = 0 \), the general curve in \(|M| \) is irreducible (hence, smooth) and maps \( 1 : 1 \) to a quartic.

Since the arithmetic genus of \( M \) is at least 2, it follows that \( M \) maps \( 1 : 1 \) to a plane quartic, and therefore its arithmetic genus is at most 3.
But then $4 \geq 2p_a(M) - 2 = M \cdot (M + K_F) = M \cdot K_F = 2M \cdot H$, i.e. $M \cdot H \leq 2$, a contradiction since $F$ is of general type.

We can summarize the above results in the following statement.

**Theorem 2.4.** Let $F$ be a nodal surface of degree 6 with an even set of 56 nodes. Then $M$ is an Artinian module of length 2 with Hilbert function $(\tau, a) = (3, 3)$ or $(3, 4)$.

**Proof.** By hypothesis $\tau = 3$. Prop. (2.3) yields $b = h^1(F, F(2)) \leq 1$. Therefore only the following two cases are possible: $b = 0, a = 3$ or $b = 1, a = 4$.

We shall treat in the sequel only the first case, for the second we limit ourselves to parenthetically posing the following

**Problem 2.5.** Let $F$ be a normal sextic surface in $\mathbb{P}^3$ with an even set of 56 nodes. Can the module $M := \bigoplus (\oplus i > 1 H^1(F, F(i)))$ have Hilbert function $(3, 4)$?

**Remark 2.6.** This case cannot be excluded by coding theory since there exists a 9-dimensional code $K \subset (\mathbb{Z}/2\mathbb{Z})^{56}$ with weights $(24, 32, 56)$.

**Proof.** This code is constructed as follows: consider a code $U \subset (\mathbb{Z}/2\mathbb{Z})^{51}$ of dimension 8 and weights $(24, 32)$, and let $e \in (\mathbb{Z}/2\mathbb{Z})^{56}$ be the vector with all coordinates equal to 1. It suffices to define $K$ as the span of $U$ and $e$.

3. **Hilbert function $(3, 3)$: general features.**

We shall assume, throughout the rest of the paper, that we have an even set $\Delta$ of 56 nodes, and that $b = 0$, i.e., $a = 3$ (cf. [21]). In other terms, the Artinian module $M$ has dimension $\tau = 3$ in degree 1, dimension $a = 3$ in degree 2, and 0 in degree $\neq 1, 2$.

By Theorem (2.1) (applied to $F(3)$ instead of $F$, i.e., we replace the previous $E$ by its twist $E(3)$) we have a resolution of $F(3)$ of the form:

$$0 \to E^\vee \xrightarrow{\varphi} E \to F(3) \to 0.$$ (3.1)

In this setting, the symmetric map $\varphi$ appearing in the resolution of $F(3)$ belongs to $H^0(\mathbb{P}^3, S^2(E)) \subset Hom(E^\vee, E)$.
Definition 3.1. Throughout the rest of the paper we denote by $U$ a given Lagrangian 3-dimensional subspace of $H^1(F(1))$, and we denote by $W$ the 3-dimensional space $W := H^1(F(2))$.

Moreover, we shall denote by $V$ the four dimensional vector space $V := H^0(O(1))$. Later on, more generally, $V$ shall denote a four dimensional vector space and we shall often continue to denote $\text{Proj}(V)$ by $\mathbb{P}^3$.

Thus we shall have

Remark 3.2. Beilinson’s theorem and the cohomology table for $F(2)$, which we described above, implies that $E(-1)$ is obtained by adding a direct sum of line bundles to

$$E'(-1) := \ker(U \otimes \Omega^1(1) \cong 3\Omega^1(1) \to W \otimes O \cong 3O),$$

and that (since the Beilinson’s complex has no cohomology in degree $\neq 0$) the above map is surjective: hence $E'$ is a vector bundle with $\text{rk}(E') = 6$.

Consider now the Euler sequence

$$0 \to \Omega^1(1) \to V \otimes O \cong 4O \to O(1) \to 0.$$ (3.2)

It implies that $h^0(\Omega^1(1)) = 0$ and $h^0(\Omega^1(2)) = 6$, thus $h^0(E'(-1)) = 0$ and, since Beilinson’s Table for $F(3)$ implies that $h^1(E') = 0$, we infer that $h^0(E') = 3 \times 6 - 4 \times 3 = 6$.

On the other hand, Beilinson’s complex for $F(3)$ yields an exact sequence:

$$0 \to 3O(-4) \to 6O^2(2) \to 3\Omega^1(1) \oplus 6O \to F(3) \to 0,$$

and we make the following simplifying

FIRST ASSUMPTION: $F$ is generated in degree 3 and the linear map $H^0(E') \to H^0(F(3))$ is an isomorphism.

Proposition 3.3. According to the previous notation, the above first assumption implies that $E = E'$, equivalently, that rank $(E) = 6$. More precisely, it means that there exists a homomorphism $\beta : U \otimes \Omega^1(2) \cong 3\Omega^1(2) \to W \otimes O(1) \cong 3O(1)$ such that $E = \ker\beta$ and that we have an exact sequence

$$0 \to E \to U \otimes \Omega^1(2) \cong 3\Omega^1(2) \xrightarrow{\beta} W \otimes O(1) \cong 3O(1) \to 0.$$ (3.3)

Conversely, if $E$ is obtained in this way, it is a rank 6 bundle with an intermediate cohomology module $M$ with the required Hilbert function of type $(3,3)$. Moreover $H^0(E'^\vee) = 0$.

Proof. If $F$ is generated in degree 3 there is an exact sequence

$$0 \to \tilde{G} \to 6O \to F(3) \to 0,$$

where $h^0(\tilde{G}) = h^1(\tilde{G}) = 0$ (cf. Beilinson’s table).

Dualizing the sequence $0 \to E' \to 3\Omega^1(2) \xrightarrow{\beta} 3O(1) \to 0$ yields

$$0 \to 3O(-1) \to 3T(-2) \to E'^\vee \to 0.$$ (3.3)

Thus $h^0(E'^\vee) = 0$. 

Assume now that $H^0(\mathcal{E}') \to H^0(\mathcal{F}(3))$ is not surjective. Then, since by our assumption $H^0(\mathcal{E}') \to H^0(\mathcal{F}(3))$ is surjective, $\mathcal{E}$ will be obtained from $\mathcal{E}'$ by adding a direct sum of line bundles $\mathcal{O}(-m)$ where $m > 0$. This leads however to a contradiction, since then $\mathcal{O}(m)$ is a direct summand of $\mathcal{E}^\vee$ but it cannot embed in $\mathcal{E}$ since $H^0(\mathcal{E}'(-1)) = 0$ implies that $\text{Hom}(\mathcal{O}(m), \mathcal{E}) = H^0(\mathcal{E}(-m)) = H^0(\mathcal{E}'(-m)) = 0$.

For the converse, we simply observe that if (3.3) holds, then $H^1(\mathcal{E}(-2)) \cong 3H^1(\Omega^1)$, $H^1(\mathcal{E}(-1)) \cong 3H^0(\mathcal{O})$. Since $\mathcal{E}' \cong \mathcal{E}$ follows rightaway that $H^0(\mathcal{E}^\vee) = 0$. Q.E.D.

Therefore we get the following exact commutative diagram:

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & \mathcal{E}^\vee & \mathcal{E} & \mathcal{F}(3) & 0 \\
& & & \downarrow & \downarrow & \downarrow & \\
& & & U^\vee \otimes T(-2) & U \otimes \Omega^1(2) & & \\
& \varphi & & & & \beta & \\
& W^\vee \otimes \mathcal{O}(-1) & & W \otimes \mathcal{O}(1) & & \\
& & 0 & 0 & & \\
\end{array}
\]

and the map $\varphi$ yields, by composition, a homomorphism $\Phi \in \text{Hom}(U^\vee \otimes T(-2), U \otimes \Omega^1(2))$ which is symmetric since $\varphi$ is symmetric. Conversely, such a homomorphism $\Phi$ determines $\varphi$ if and only if $\beta \Phi = \Phi \beta = 0$: since however we choose $\Phi$ symmetric the two conditions are equivalent to each other.

A more concrete way to setup the parameter space for such vector bundles is to replace $\text{Hom}(T(-2), \Omega^1(2))$ via matrices of polynomials, as follows.

Recall that $V := H^0(\mathbb{P}^3, \Omega(1))$ is the space of linear forms on $\mathbb{P}^3$. Applying $\text{Hom}(-, \mathcal{O})$ to the Euler sequence and tensoring by $\mathcal{O}(1)$ yields, since $\text{Hom}(\mathcal{O}(2), \mathcal{O}(1)) = 0$, $\text{Ext}^1(\mathcal{O}(2), \mathcal{O}(1)) = 0$,

\begin{equation}
(3.4) \quad \text{Hom}(\mathcal{O}^1(2), \mathcal{O}(1)) \cong \text{Hom}(V \otimes \mathcal{O}(1), \mathcal{O}(1)).
\end{equation}

Thus the map $\beta$ factors through a map $B : U \otimes (V \otimes \mathcal{O}(1)) \to W \otimes \mathcal{O}(1)$ and the sheaf map $B$ is surjective. This surjectivity is obviously equivalent to $H^0(B(-1)) : U \otimes V \to W$ being surjective. In the sequel we shall often identify the sheaf map $B$ with the corresponding tensor $H^0(B(-1)) \in U^\vee \otimes V^\vee \otimes W$.

Let $\epsilon$ be the tensor product of the identity map of the isotropic subspace $U$ with the evaluation map $V \otimes \mathcal{O} \to \mathcal{O}(1)$. Then one sees easily that
\[ \mathcal{E} = \ker \beta = \ker B \cap \ker \epsilon, \text{ the short exact sequence becomes} \]

\[ (3.5) \quad 0 \to \mathcal{E} \to U \otimes V \otimes \mathcal{O}(1) \xrightarrow{(B, \epsilon)} (W \otimes \mathcal{O}(1)) \oplus (U \otimes \mathcal{O}(2)) \to 0, \]

and the previous diagram is replaced by:

\[
\begin{array}{ccc}
0 & \to & \mathcal{E}^\vee \\
\uparrow & & \downarrow \varphi \\
U^\vee \otimes V^\vee \otimes \mathcal{O}(-1)) & \cong & 12\mathcal{O}(-1) \\
\uparrow \eta(B, \epsilon) & & \downarrow A \\
(W^\vee \otimes \mathcal{O}(-1)) \oplus (U^\vee \otimes \mathcal{O}(-2)) & \cong & (W \otimes \mathcal{O}(1)) \oplus (U \otimes \mathcal{O}(2)) \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{E} \\
\uparrow & & \downarrow \\
F(3) & \to & 0
\end{array}
\]

where by a similar token to the one before the map \( \varphi \) yields a symmetric matrix \( A \in \text{Mat}(12 \times 12, \text{Hom}(\mathcal{O}(-1), \mathcal{O}(1))) \) and conversely such a matrix determines \( \varphi \) if and only if \( (B, \epsilon) \cdot A = 0 \).

Thus we obtain, as a parameter space for the symmetric resolutions of \( F(3) \) satisfying the open condition given by the main assumption, the variety of pairs

\[ \mathfrak{M}_{AB} := \{ (B, A) \mid B \in \text{Mat}(3 \times 12, \mathbb{C}), A \in \text{Mat}(12 \times 12, H^0(\mathcal{O}_{\mathbb{P}^3}(2))), A = A, (B, \epsilon) \cdot A = 0 \}. \]

As a matter of fact, the equation of the surface \( F \) will then be given as the G.C.D. of the determinants of the \( 6 \times 6 \) minors of the matrix \( A \), whereas the even set of nodes \( \Delta \) will be found to be given by the ideal of the determinants of the \( 5 \times 5 \) minors of the matrix \( A \).

A direct but complicated calculation shows that for a general choice of the parameter \( B \) (determining the bundle \( \mathcal{E} \)) the solution space for the \( A \)'s (yielding the symmetric map \( \varphi \)) has positive dimension.

However, by computer algebra, one checks that a random choice of \( B \) and a random choice of \( A \) do not give a sextic surface with 56 nodes, but the square of a cubic surface. Notice that the condition that a given pair \((B, A)\) yields a sextic with 56 nodes is an open condition, and therefore there can exist sextics with an even set of 56 nodes and satisfying the main assumption only if the above parameter space is reducible: we shall later show that this is indeed the case.

We finish this section by remarking that \( B \) is the multiplication matrix of the module \( H^1(\mathcal{E}) \) (i.e., the matrix of the only part of the multiplication map which is not a priori trivial).
Remark 3.4. The cohomology exact sequence associated to the following
twist of \( (3.5) \), namely:
\[
0 \to \mathcal{E}(-2) \to U \otimes V \otimes \mathcal{O}(-1) \xrightarrow{(B, \epsilon)} (W \otimes \mathcal{O}(-1)) \oplus (U \otimes \mathcal{O}) \to 0
\]
yields a canonical isomorphism \( U \cong H^1(\mathcal{E}(-2)) \).

Since there is a canonical isomorphism
\[
H^0(\epsilon(-1)) : U \otimes V \to U \otimes H^0(\mathcal{O}(1)),
\]
the projection of \( W \oplus (U \otimes V) \to W \) induces an isomorphism of \( H^1(\mathcal{E}(-1)) = \text{Coker} H^0((B \oplus \epsilon)(-1)) \) with \( W \), such that the map \( B : U \otimes V \to W \) corresponds to the multiplication map of the cohomology module \( H^1_*(\mathcal{E}) \).

Remark 3.5. The condition that the linear map \( B \) has maximal rank = 3 (which, as we observed, follows from the first assumption) is obviously equivalent to the condition that the module \( M \) is generated in degree \(-2\). On the other hand, it also implies that there is an exact sequence
\[
0 \to \mathcal{E} \to 9\mathcal{O}(1) \xrightarrow{b} 3\mathcal{O}(2) \to 0.
\]

Remark 3.6. Let \( F \) be a nodal sextic surface in \( \mathbb{P}^3 \) with an even set of 56 nodes, and assume that \( h^1(F, F(1)) = 6 \). One may ask whether there is a Lagrangian subspace \( U \) such that the module \( M = U \oplus W := U \oplus H^1(F, F(2)) \), which has Hilbert function \((3,3)\), is generated by \( U \).

A necessary condition is that, setting \( \tilde{W} := H^1(F, F(1)) \), \( W \otimes V \to W \) be surjective. In turn this is equivalent to the pairing \( H^1(\mathcal{O}_S) \times H^0(f^*\mathcal{O}_F(1)) \to H^1(f^*\mathcal{O}_F(1)) \) being non degenerate in the first factor.

We proceed in the next section with the analysis of the vector bundles corresponding to a general choice of \( B \), giving a geometrical explanation of the phenomenon of which the computer made us aware.

4. General Bundles and cubic surfaces

Main purpose of this section is to describe the beautiful geometry which relates the main component of the moduli space of our vector bundles with given intermediate cohomology module \( M \) and the space of cubic surfaces viewed as blow ups of the projective plane in six points.

Let us preliminarily observe that, if the main assumption is satisfied, the vector bundle \( \mathcal{E} \) is determined by the matrix \( B \), hence we have an irreducible parameter space for our vector bundles, and each open condition, if verified at some point, is verified by the generic bundle \( \mathcal{E} \).

Next, we have a surjection \( H^0(\mathcal{E}) \to H^0(\mathcal{F}(3)) \) and we have seen that both spaces are 6-dimensional, whence we get a homomorphism \( \iota : 6\mathcal{O} \to \mathcal{E} \). We make the

SECOND ASSUMPTION:
1) \( \iota : 6\mathcal{O} \to \mathcal{E} \) is injective, whence an exact sequence:
\[
0 \to 6\mathcal{O} \to \mathcal{E} \to \tau \to 0,
\]
2) the torsion sheaf \( \tau \) is \( \mathcal{O}_G \)-invertible, where \( G \) is the divisor of \( \Lambda^6(i) \).

**Lemma 4.1.** Let a vector bundle \( \mathcal{E} \) be given as in 3.3 or as in 3.5 Then its total Chern class is

\[
(4.2) \quad c(\mathcal{E})(t) = 1 + 3t + 6t^2 + 4t^3.
\]

In particular, if the second assumption is satisfied, the divisor \( G \) is a cubic surface.

*Proof.* The sheaf \( \tau \) has Chern polynomial \( c(\tau) = c(\mathcal{E}) = c(\Omega^1(2))^3c(\mathcal{O}(1))^{-3} = (c(\Omega(1))^4c(\mathcal{O}(2))^{-1})^3c(\mathcal{O}(1))^{-3} = (1 + t)^3(1 + 2t)^{-3} = (1 + 9t + 36t^2 + 84t^3)(1 - 6t + 24t^2 - 80t^3) = 1 + 3t + 6t^2 + 4t^3. \)

\( \square \)

**Remark 4.2.** Observe that the space \( H^0(S^2(\mathcal{E})) \) of symmetric morphisms from \( \mathcal{E}' \) to \( \mathcal{E} \) contains \( H^0(S^2(\mathcal{O})) \) since to \( \tilde{\alpha} \in H^0(S^2(\mathcal{O})) \) corresponds \( \alpha := \mathcal{O}^\vee \tilde{\alpha} \). For these morphisms one has \( \det \alpha = \det \tilde{\alpha} (\det \mathcal{O})^2 \), whence in this case \( \text{div}(\det(\alpha)) = 2G \), and not a sextic surface.

The next lemmas are meant to investigate the question: when does one have equality \( h^0(S^2(\mathcal{E})) = 21 \), i.e., when is \( H^0(S^2(\mathcal{E})) = H^0(S^2(\mathcal{O})) \)?

In order to answer this question, it is convenient first to analyse the geometry and the cohomology of the invertible sheaf \( \tau \) on \( G \).

**Remark 4.3.** Even without assuming \( \tau \) to be \( \mathcal{O}_G \)-invertible, set \( \tau' = \mathcal{E}xtr^1(\tau, \mathcal{O}) \): then the dual of the previous exact sequence 4.4 gives

\[
(4.3) \quad 0 \to \mathcal{E}' \to 6\mathcal{O} \to \tau' \to 0.
\]

and we have:

1. By (4.4) clearly holds \( H^0(\tau) = H^1(\tau) = H^2(\tau) = 0 \).
2. From (4.3) and \( h^i(\mathcal{E}') \cong h^{3-i}(\mathcal{E}(-4)) \) we get \( h^0(\tau') = 6 \), \( H^1(\tau') = H^2(\tau') = 0 \).
3. Since by definition \( \tau' = \mathcal{E}xtr^1(\tau, \mathcal{O}) \), applying the functor \( \text{Hom}(\tau, -) \) to the exact sequence \( 0 \to \mathcal{O} \to \mathcal{O}(3) \to \mathcal{O}_G(3) \to 0 \) we get \( \tau' = \text{Hom}(\tau, \mathcal{O}_G(3)) \). Therefore, if \( \tau = \mathcal{O}_G(D) \), then \( \tau' = \mathcal{O}_G(3H - D) \).

Since \( h^i(D) = 0 \forall i \), \( h^0(3H - D) = 6 \), \( h^i(3H - D) = 0 \) for \( i = 1, 2 \), follows by Riemann Roch that \( D^2 + DH = -2 \) and \( 10 = 36 - 7DH + D^2 \). Therefore \( HD = 3, D^2 = -5 \).

By setting \( \Delta := D + H \), it turns out that \( \Delta H = 6, \Delta^2 + \Delta K_G = -2 \).

**Lemma 4.4.** Assume that \( G \) is a smooth cubic surface: then there exists a realization of \( G \) as the image of the plane under the system \( [3L - \sum_{i=1}^6 E_i] \) of plane cubics through six points, such that either \( \Delta \equiv 2L \), i.e., \( \Delta \) corresponds to the conics in the plane, or (up to permutations of the six points) \( \Delta \equiv 3L - 2E_1 - E_2 \).
Proof. Observe preliminarly that if \(|H| = |3L - \sum_1^6 E_i|\) is such a planar realization of a cubic surface, then another one is obtained via a standard Cremona transformations centered at three of the points \(P_i\) corresponding to the \(-1\)-curves \(E_i\).

In fact, if \(L' := 2L - E_1 - E_2 - E_3\), then
\[
H = 3L' - (L - E_1 - E_2) - (L - E_1 - E_3) - (L - E_2 - E_3) - E_4 - E_5 - E_6.
\]
We have \(0 = H^2(D) = H^0(-D - H)\) and a fortiori \(H^2(\Delta) = H^0(-D - 2H) = 0\). It follows that \(|\Delta|\) has \(h^0(\Delta) \geq 6\), \(\Delta^2 = 4\), and the arithmetic genus \(p_a(\Delta) = 0\).

Since \(0 = H^1(D) = H^1(K_G - D)\) implies \(H^1(-H - D) = H^1(\Delta)\), it follows that \(\Delta\) is connected.

Hence we have a representation \(\Delta \equiv nL - \sum_1^6 a_i E_i\), where the \(a_i\)'s are non negative and we assume \(a_1 \geq a_2 \geq \cdots \geq a_6\).

We have: \(\Delta^2 = 4 = n^2 - \sum_1^6 a_i^2\), \(\Delta \cdot H = 6 = 3n - \sum_1^6 a_i\), i.e.
\[
(4.4) \quad n^2 = \sum_1^6 a_i^2 + 4, \quad 3n = \sum_1^6 a_i + 6.
\]

We want to show that, after a suitable sequence of standard Cremona transformations, \(\Delta \equiv 2L\) or \(\Delta \equiv 3L - 2E_1 - E_2\). By \(4.4\) we have \(n \geq 2\) and for \(n = 2, 3\) \(\Delta\) is as claimed. Hence the claim is that there exists a sequence of standard Cremona transformations which makes \(|\Delta|\) have degree \(n \leq 3\).

By applying \(|2L - E_1 - E_2 - E_3|\) we get a new system \(\Delta'\) with degree \(n' = 2n - a_1 - a_2 - a_3\).

By our ordering choice for the \(a_i\)'s, we have
\[
a_1 + a_2 + a_3 \geq \frac{1}{2} \sum_{i=1}^6 a_i = 3n/2 - 3,
\]
with strict inequality unless all \(a_i\)'s are equal. We study this latter case first:

Sublemma. In the previous setting, \(a_1 = a_2 = \ldots = a_6\) if and only if \(n = 2\) and \(a_1 = a_2 = \ldots = a_6 = 0\) or \(n = 10\) and \(a_1 = a_2 = \ldots = a_6 = 4\).

Proof. The statement follows immediately by defining \(a := a_1 = a_2 = \ldots = a_6\) and using both conditions of \(4.4\) \(n = 2a + 2, n^2 = 6a^2 + 4\), which imply \(8a = 2a^2\).

The previous inequality gives:
\[
n' \leq \frac{n}{2} + 3 \leq n \quad \text{for } n \geq 6,
\]
and \(n' < n\) for \(n \geq 6\) unless \(n' = n = 6\) and \(a_1 = a_2 = \ldots = a_6\), which has no solution by the above sublemma. We conclude that after suitable Cremona transformations \(n \leq 5\).
If $n = 5$, then $n' \leq 5/2 + 3$, i.e., $n' \leq 5$. Moreover, if also $n' = 5$, then $a_1 + a_2 + a_3 = 5$ and using again we obtain $a_4 + a_5 + a_6 = 4$. But then $a_1 = a_6 + 1$ and we easily get a contradiction since then $a_2 = a_3 = a_4 = a_5$ and they either equal $a_1$ or $a_6$. Hence, after a suitable Cremona transformation, we can always reduce to the case $n \leq 4$.

Let now $n = 4$. Using we get \[ \sum_{i=1}^{6} a_i = 6 \text{ and } \sum_{i=1}^{6} a_i(a_i-1) = 6. \]

We have the following two possibilities: $a_1 = 3, a_2 = a_3 = a_4 = 1, a_5 = a_6 = 0$ or $a_1 = a_2 = a_3 = 2, a_4 = a_5 = a_6 = 0$. In both cases we have that $a_1 + a_2 + a_3 \geq 5$, and therefore $n' \leq 3$.

Q.E.D.

Remark 4.5. The complete linear system $\Delta$ has as image in $\mathbb{P}^5$ either the Veronese embedding of $\mathbb{P}^2$, or the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ through $H^0(O_{\mathbb{P}^1 \times \mathbb{P}^1}(1,2))$. In both case we have a surface of minimal degree $(=4)$.

Thus we have concluded that either $D = 2L - H = -L + \sum E_i$, or $D = 3L - 2E_1 - E_2 - H = -E_1 + \sum E_i$.

Corollary 4.6. $H^i(O_G(2D)) = 0$ for $i = 0, 2$, $h^1(O_G(2D)) = 6$.

Proof. The second part follows from the first by Riemann Roch, for the first it suffices to intersect with $L$ (using Serre duality in the case of $H^2(O_G(2D))$).

We are now ready to show that the smoothness assumption for the cubic surface $G$ implies that all symmetric morphisms from $E^\vee$ to $E$ factor through and are induced by symmetric morphisms from $6O$ to $6O$, whence their determinant is a double cubic, instead of a nodal sextic (cf. Rem. 4.2).

Lemma 4.7. Let

\[ 0 \to F \to E \to \tau \to 0 \]

be a locally free resolution of a coherent torsion sheaf $\tau$, which is $O_G$-invertible on a divisor $G$.

Then we have an exact sequence

\[ 0 \to \Lambda^2 F \to F \otimes E \to S^2 E \to \tau \otimes \tau \to 0 \]

and a monad

\[ 0 \to S^2 F \to F \otimes E \to \Lambda^2 E \to 0 \]

whose cohomology in the middle is exactly $\text{Tor}^1(\tau, \tau)$.

Proof. Recall that locally, by our assumption, we can write:

$E = Oe_1 \oplus Oe_2 \oplus \ldots \oplus Oe_r$, $F = Ox_1 \oplus Ox_2 \oplus \ldots \oplus Ox_r$,

where $x$ is a local equation for $G$.

Since $\text{Tor}^1(B, B') = 0$ if $B$ is locally free, we obtain the following commutative diagram with exact rows and columns:
Hence the composite map \( E \otimes E \to \tau \otimes E \to \tau \otimes \tau \) is surjective and has kernel generated by \((F \otimes E) \oplus (E \otimes F)\), as the diagram shows. Let \( K \) denote the kernel of the map \((F \otimes E) \oplus (E \otimes F) \to E \otimes E\). \( K \) contains the image of \( F \otimes F \) through the injective map \((id \otimes -id) \otimes i\), where \( i : F \to E \) is the inclusion.

Therefore we get the complex
\[
0 \to F \otimes F \to (F \otimes E) \oplus (E \otimes F) \to E \otimes E \to \tau \otimes \tau \to 0,
\]
extact except possibly at \((F \otimes E) \oplus (E \otimes F) \to E \otimes E \to \tau \otimes \tau \to 0\).

Now let \( K_1 \) be the inverse image of \( Tor^1(\tau, \tau) \) in \( F \otimes E \) via the short exact sequence \( 0 \to F \otimes F \to F \otimes E \to F \otimes \tau \to 0 \).

We claim that \( K \cong K_1 \). In fact, if \( h_1 + (-h_2) \in K \), the diagram shows that \( h_1 \in K_1 \) and moreover that \( h_2 \) is uniquely determined. Conversely, if \( h_1 \in K_1 \), then there is an (unique) element \( h_2 \in E \otimes F \) with \( h_1 = h_2 \in E \otimes E \).

This implies that \( K/F \otimes F \cong Tor^1(\tau, \tau) \).

Locally, \( K_1 \) is generated by \( xe_1 \otimes e_1 \mod F \otimes F \). Thus \( K/F \otimes F \) is generated by \((xe_1 \otimes e_1) \oplus (-e_1 \otimes xe_1) \in (F \otimes E) \oplus (E \otimes F)\), which is an antisymmetric tensor, and we are done.

\[\square\]

**Corollary 4.8.** According to the previous notation, assume that \( G \) is a smooth cubic surface. Then \( h^0(S^2E) = 21, h^1(S^2E) = 6, h^2(S^2E) = h^3(S^2E) = 0 \).

The same conclusion \( h^0(S^2E) = 21 \) holds if more generally \( E \) verifies the first and second assumption and \( H^0(\tau \otimes 2) = 0 \).
Proof. Split the long exact sequence (4.5) into
\[ 0 \rightarrow 15\mathcal{O} \rightarrow 6\mathcal{E} \rightarrow \mathcal{H} \rightarrow 0, \]
\[ 0 \rightarrow \mathcal{H} \rightarrow S^2\mathcal{E} \rightarrow \tau^{\otimes 2} \rightarrow 0. \]

Recall from the construction of \( \mathcal{E} \) that \( h^0(\mathcal{E}) = 6 \) and \( h^1(\mathcal{E}) = 0 \) for \( i = 1, 2, 3 \).

The first corresponding long exact cohomology sequence yields \( h^0(\mathcal{H}) = 36 - 15 = 21 \) and \( h^i(\mathcal{H}) = 0 \) for \( i = 1, 2, 3 \).

Therefore it suffices to observe that if \( G \) is smooth, by the previous corollary, one has \( h^0(\tau^{\otimes 2}) = h^2(\tau^{\otimes 2}) = 0, h^1(\tau^{\otimes 2}) = 6 \). The rest is straightforward.

Recall now that the vector bundle \( \mathcal{E} \), provided that the main assumption and the second assumption are satisfied, produces an invertible sheaf \( \tau \) on a cubic surface \( G \); conversely, given such a sheaf \( \tau \), one can construct \( \mathcal{E} \) as an extension of \( 6\mathcal{O} \) and \( \tau \) as in 4.1.

Setting as before \( \tau' := \mathcal{E}xt^1(\tau, \mathcal{O}) \), we see that such an extension is parametrized by \( Ext^1(\tau, 6\mathcal{O}) = H^0(6\mathcal{E}xt^1(\tau, \mathcal{O})) \cong \mathbb{C}^{36} \), if, as in remark 4.3, we have \( h^0(\tau') = 6 \).

Lemma 4.9. Assume that \( h^0(\mathcal{E}^\vee) = 0 \) (cf. the proof of proposition 3.2), and that \( \mathcal{E} \) is an extension as in 4.1, then the extension class in \( Ext^1(\tau, 6\mathcal{O}) = H^0(6\mathcal{E}xt^1(\tau, \mathcal{O})) \cong \mathbb{C}^6 \otimes \mathbb{C}^6 \) is a rank 6 tensor (we shall refer to this statement by saying that the extension does not partially split).

In particular, \( \mathcal{E} \) is then uniquely determined up to isomorphism.

Proof. The extensions which yield vector bundles form an open set.

We canonically view these extension classes as \( Hom(H^0(\tau'), H^0(6\mathcal{O})) = Hom(H^0(\tau'), \mathbb{C}^6) \) through the coboundary map of the corresponding exact sequence. We have then an action of \( \text{GL}(6, \mathbb{C}) \) as a group of automorphisms of \( 6\mathcal{O} \), which induces an action on \( Hom(H^0(\tau'), H^0(6\mathcal{O})) = Hom(H^0(\tau'), \mathbb{C}^6) \) which is immediately identified to the composition of the corresponding linear maps.

The extensions which yield vector bundles form an open set, which contains an open dense orbit, on which this action is free, namely, the tensors of rank = 6.

If the rank of the tensor corresponding to an extension is \( r < 6 \), it follows that the extension is obtained from an extension \( 0 \rightarrow r\mathcal{O} \rightarrow \mathcal{E}' \rightarrow \tau \rightarrow 0 \) taking then a direct sum with \( (6-r)\mathcal{O} \); but then \( (6-r)\mathcal{O} \) is a direct summand of \( \mathcal{E}^\vee \), a contradiction.

Corollary 4.10. \( \mathcal{E} \) as above 4.9 is a vector bundle if \( H^0(\tau') \) has no base points.

Proof. Our hypothesis shows that in each point of \( G \) the local extension class is non zero, hence it yields a locally free sheaf. 

\[ \square \]
Let us now show that the second case contemplated in Lemma 4.4 does not occur, since it produces a vector bundle $\mathcal{E}$ with a different intermediate cohomology as the one we require.

**Lemma 4.11.** The second case in Lemma 4.4 does not occur, else the associated vector bundle $\mathcal{E}$ would then have $h^2(\mathcal{E}(-3)) = 1 \neq b = 0$.

**Proof.** Assume that $D = \sum_3^6 E_i - E_1$: then the linear system $|2H - D| = [6L - E_1 - 2E_2 - 3(\sum_3^6 E_i)]$ has bigger dimension than the expected dimension $27 - 28 = -1$, since it contains an effective divisor, $|2L - E_1 - (\sum_3^6 E_i)| + 2|2L - E_2 - (\sum_3^6 E_i)|$. This amounts to the nonvanishing of the cohomology group $H^2(-3H + D) = H^2(\tau(-3))$.

From the exact sequence we infer that $h^2(\mathcal{E}(-3)) = 1$, whereas we assumed throughout that $h^2(\mathcal{E}(-3)) := b = 0$, a contradiction.

We assume now that $G$ is a smooth cubic surface, and that $\tau$ is an invertible sheaf on $G$, corresponding to the divisor class $-L + \sum_1^6 E_i$.

Consider now the associated vector bundle $\mathcal{E}$: we want to verify that $\mathcal{E}$ has the required cohomology table, i.e., we want to calculate the dimensions $h^i(\mathcal{E}(-n))$ for $n = 0, 1, 2, 3$. This will allow us to verify that there are bundles $\mathcal{E}$ which satisfy the main and the second assumption.

**Lemma 4.12.** Let $G$ be a smooth cubic surface, and let $\tau$ be the invertible sheaf on $G$ corresponding to the divisor class $-L + \sum_1^6 E_i$: then the associated vector bundle $\mathcal{E}$ has the required cohomology table.

**Proof.** Observe that

- clearly $h^0(\tau(-n)) = h^0(D - nH) = h^0(-1 + 3n)L + (n - 1)\sum_1^6 E_i) = 0$, for $n = 0, 1, 2, 3$.
- clearly $h^2(\tau(-n)) = h^0((n-1)H - D) = h^0((3n - 2)L - n\sum_1^6 E_i) = 0$, for $n = 0, 1, 2$, since a quartic with 6 double points is a union of 4 lines.
- $h^1(\tau(-3)) = h^1(-3H + D) = h^1(-10L - 4\sum_1^6 E_i) = 0$ by Ramanujam's vanishing trick for regular surfaces, since the linear system $|10L - 4\sum_1^6 E_i|$ contains a reduced and connected divisor, namely $Q_1 + \ldots + Q_5 + E_6$, where $Q_i \in 2L - \sum_1^6 E_j + E_i$.
- since $\chi(\tau(-n)) = 1 + \frac{1}{2}(D - nH)(D - (n - 1)H) = \frac{1}{2}n(n - 3)$, we have also $h^1(\tau) = 0$, $h^1(\tau(-1)) = h^1(\tau(-2)) = 3$, $h^2(\tau(-3)) = 0$.

Q.E.D.

We have seen how to a linear map $B : V \otimes U \rightarrow W$, where $V = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ denotes the space of linear forms, $U$ is a fixed isotropic subspace in $H^1(\mathcal{F}(1))$, $W := H^1(\mathcal{F}(2))$, corresponds a homomorphism of vector bundles $B : V \otimes U \otimes \mathcal{O} \rightarrow W \otimes \mathcal{O}$, inducing $\beta : U \otimes \Omega^1(2) \rightarrow W \otimes \mathcal{O}(1)$, whence finally a vector bundle $\mathcal{E} := \ker(\beta)$ if $\beta$ is surjective.

The second assumption yields a cubic surface $G \subset \text{Proj}(V)$ and an invertible sheaf $\tau$ on $G$. If $G$ is smooth, the invertible sheaf $\tau(1)$ yields then
a birational morphism onto a Veronese surface, whence represents $G$ as the blow up of a projective plane $\mathbb{P}^2$ in a subscheme $\zeta$ consisting of six points, and as the image of $\mathbb{P}^2$ through the linear system of cubic curves passing through $\zeta$. The Hilbert-Burch theorem allows us to make an explicit construction which goes in the opposite direction.

**Remark 4.13.** Let $U', W'$ be 3-dimensional vector spaces.

Consider a $3 \times 3 \times 4$ tensor $\hat{B} \in (U' \otimes V \otimes W'^\vee)$ and assume that the corresponding sheaf homomorphism $\hat{B} : W' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^2}$ yields an exact sequence

\[(4.7)\] \[0 \to W' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\hat{B}} V \otimes \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(3) \to \mathcal{O}_{\zeta}(3) \to 0\]

which is the Hilbert-Burch resolution of a codimension 2 subscheme $\zeta$ of length 6.

We obtain a canonical isomorphism $V \cong H^0(I_{\zeta}(3))$ and we let $G \subset \text{Proj}(V)$ be the image of $\mathbb{P}^2$ via the rational map $\psi$ associated to $V$. Under the above assumption on $\hat{B}$, and if moreover $\zeta$ is a local complete intersection, $G$ is a normal cubic surface, and if we define $\mathcal{G} := (\psi_* (\mathcal{O}_1))$, then we have an exact sequence on $\text{Proj}(V)$:

\[(4.8)\] \[0 \to W' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\hat{B}} U' \otimes \mathcal{O} \to \mathcal{G} \to 0.\]

**Proof.** Let us continue to use our previous notation: $\mathcal{O}_G(L) = \mathcal{G}$, and $\mathcal{O}_G(H) = \mathcal{O}_G(1)$, so that our explicit choice of $\hat{B}$ provides a canonical basis $x_0, x_1, x_2$ of $H^0(G)$, and a canonical basis $y_0, \ldots, y_3$ of $V = H^0(I_{\zeta}(3))$ provided by the cubic polynomials $\gamma_i(x)$ which are the coordinates of $\Lambda^3(\hat{B})$; in other words, $(\gamma_0(x), \ldots, \gamma_3(x))$ are the $3 \times 3$ minors of $\hat{B}$, taking with alternate signs, and the Hilbert Burch sequence amounts to giving the relations

$$\sum_{k,i} (\hat{B})_{k,j}^{i} x_i \gamma_k(x) = 0.$$ 

In turn, we view these as relations on $G$, $\sum_{k,i} (\hat{B})_{k,j}^{i} x_i y_k = 0$.

We define now the $3 \times 3$ matrix $B$ as:

\[(4.9)\] \[B_{i,j} := \sum_{k} (\hat{B})_{k,j}^{i} y_k.\]

Let us still denote by $\mathcal{B}$ the morphism in $\text{Hom}(W' \otimes V^\vee, U')$ represented by the above matrix with respect to the chosen bases: $\mathcal{B}$ is another way of representing the $3 \times 3 \times 4$ tensor $\hat{B} \in (U' \otimes V \otimes W'^\vee)$ and induces then the exact sequence on $\text{Proj}(V)$.

\[\square\]

Under the more general assumption that $\mathcal{B}$ never drops rank by 2, $\mathcal{G}$ is an invertible sheaf on a cubic surface $G$, and $\mathcal{G} = \mathcal{O}_G(L)$, with $h^0(L) = 3$.

Conversely, given an exact sequence of the space $H^0(G)$, since $\mathcal{G}$ is generated by global sections, yields a morphism $\pi : G \to \text{Proj}(H^0(G))$. We
calculate the Euler characteristic of \( G \):
\[
\chi(G(n)) = 3 \left[ \binom{n+3}{3} - \binom{n+2}{3} \right] = \frac{1}{2}(3n^2 + 9n + 6).
\]

If the cubic surface \( G \) is normal, and we let \( G' \) be its minimal resolution, the Hilbert Polynomial of \( \mathcal{O}_{G'}(L) \) is equal to \( \chi(L + nH) = 1 + \frac{1}{2}(L + nH)(L + (n + 1)H) = 1 + \frac{1}{2}(L^2 + LH) + \frac{3}{2}n^2 + n(\frac{3}{2} + LH) \) and therefore \( LH = 3, L^2 = 1 \). Since \( L^2 = 1 \) \( \pi \) is a birational morphism, and since \( L \) has genus 0 and degree three, \( G' \) is the birational image of \( \mathbb{P}^2 \) under a linear system of plane cubics \( H^0(I_\zeta) \), where \( \zeta \) is a length six zero-dimensional subscheme.

**Proposition 4.14.** More generally, assume that \( G \) is an invertible sheaf on a (non necessarily irreducible) cubic surface \( G \), given by \( I_\delta \), then the Cartier divisor \( L \) has degree 3, and the morphism \( \pi \) yields a birational morphism of a component of the surface \( G \) onto the plane.

Assume that \( G \) is irreducible and that \( \tau := G^{\otimes 2}(-1) \): then \( H^0(\tau^{\otimes 2}) = 0 \).

**Proof.** It suffices to consider two general divisors \( L_1, L_2 \) in the linear system \( |L| \), and to consider the Hilbert polynomials of the sheaves appearing in the two exact sequences
\[
0 \to \mathcal{O}_G(nH) \to \mathcal{O}_G(nH + L_1) \to \mathcal{O}_{L_1}(nH + L_1) \to 0
\]
\[
0 \to \mathcal{O}_{L_2}(nH) \to \mathcal{O}_{L_2}(nH + L_1) \to \mathcal{O}_{L_1 \cap L_2}(nH + L_1) \to 0.
\]
The conclusion is that \( H^0(\mathcal{O}_{L_1 \cap L_2}(nH + L_1)) = 1 \forall n >> 0 \), thus \( L_1 \) and \( L_2 \) meet transversally in a smooth point.

Let us now assume that \( G \) is irreducible, and consider the inverse of the birational morphism \( \pi \). We can factor it as a sequence of blow ups \( \sigma : Y \to \mathbb{P}^2 \) followed by a projection \( p : Y \to G \), so to it corresponds a sublinear system of a complete linear system on \( Y \), which reads out on the plane as \( |H| = |bL - \sum_i a_iE_i| \). Here, \( b = H \cdot L \), and if \( b \geq 3 \), clearly \( L \cdot (4L - 2H) < 0 \), hence \( |4L - 2H| = \emptyset \) and our desired vanishing is proven.

If instead \( b \leq 2 \), since \( \dim |H| \geq 3 \), it follows that \( H = 2L - E \) and \( G \) is a linear projection of the cubic scroll \( Y \subset \mathbb{P}^4 \) with centre a point in \( \mathbb{P}^4 \setminus Y \).

We claim however that this case does not occur.

Essentially, since otherwise \( \mathcal{G} = p_*(\mathcal{O}_Y(L)) \) would not be invertible.

As an alternative argument, observe that the factorization \( \sigma = \pi \circ p \) is not possible, since \( \sigma \) is an isomorphism on the complement of the line \( E \subset Y \), while the inverse image of the double curve of \( G \) is a conic (possibly reducible) contained in \( Y \).

Q.E.D.

**Remark 4.15.** The case where \( G \) is a a linear projection of the cubic scroll yields two sheaves \( \mathcal{G} \) which are not invertible.

As it is well known, every point in \( \mathbb{P}^4 \) lies in one of the planes spanned by the conics of the system \( L \). If we project from a point in \( \mathbb{P}^4 \setminus Y \), this conic maps two to one to the double line of the cubic \( G \).
Such a plane is said to be **special** if the conic splits into two lines $E + F$, $F \equiv L - E$.

In the non special case, we may assume without loss of generality that the conic corresponds to the line $z = 0$ in the plane, that the blown up point is the point $x = y = 0$, and that the linear system mapping to $G$ is generated by $(zx := x_0, zy := x_1, x^2 := x_2, y^2 := x_3)$. In this case one sees that the matrix $B$ is

$$B = \begin{pmatrix} x_0 & 0 & x_1 \\ 0 & x_1 & -x_0 \\ -x_2 & -x_3 & 0 \end{pmatrix},$$

and that $G$ is then the cubic of equation $-x_0^2x_3 + x_2x_1^2 = 0$.

In the special case, we may again assume that the blown up point is the point $x = y = 0$, we assume that the line $F$ is the proper transform of $x = 0$, and that the linear system mapping to $G$ is generated by $(zx, zy, x^2, xy, y^2)$ it corresponds to projection from the point $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \in \mathbb{P}^4 \setminus Y$.

In this case one sees that the matrix $B$ is

$$B = \begin{pmatrix} x_3 & -x_2 & -x_0 \\ -x_1 & 0 & x_3 \\ 0 & x_1 & -x_1 \end{pmatrix},$$

and that $G$ is then the cubic of equation $-x_3^3 + x_1^2x_2 + x_1x_2x_3 = 0$.

**Definition 4.16.** We define now the **direct construction** of the bundle $E$ relying on our results above.

Consider a sheaf $G$ defined by an exact sequence as in 4.8, and which is invertible on a cubic surface $G$ (i.e., at each point $y \in \mathbb{P}^3 \ \text{rank}(G \otimes \mathcal{O}_y) \leq 1$).

Define $\tau := G^{\otimes 2}(-1)$ and let $E$ be a vector bundle which is an extension of $6\mathcal{O}$ by $\tau$ as in 4.7 (here and elsewhere, $\mathcal{O} := \mathcal{O}_{\text{Proj}(V)}$).

**Proposition 4.17.** $E$ as above is unique up to isomorphism in the following cases:

1. if $G$ is a smooth cubic surface.
2. if $G$ is reducible to the union of a plane $T$ and a smooth quadric $Q$ intersecting transversally.

**Proof.** As before, it suffices to show that $\dim \text{Ext}^1(\tau, \mathcal{O}) = 6$.

Now, $\text{Ext}^1(\tau, \mathcal{O}) = H^0(\text{Ext}^1(\tau, \mathcal{O}))$ and the exact sequence

$$0 \to \mathcal{O}(-3) \to \mathcal{O} \to \mathcal{O}_G \to 0$$

yields

$$0 \to \text{Hom}(\tau, \mathcal{O}_G) \to \text{Ext}^1(\tau, \mathcal{O}(-3)) \to \text{Ext}^1(\tau, \mathcal{O})$$

where the last map is 0 on $G$. Hence, $\text{Ext}^1(\tau, \mathcal{O}) \cong \text{Hom}(\tau, \mathcal{O}_G(3)) = \tau^3(3)$.

Using the previous notation for the Cartier divisors corresponding to $G$ and $\mathcal{O}_G(1)$, we want to show that $h^0(\mathcal{O}_G(4H - 2L) = 6$. 


Assume first that $G$ is a smooth cubic surface: then by Riemann Roch it suffices to show the vanishing of the first cohomology group $h^1(O_G(4H - 2L)) = 0$.

We argue as before using Ramanujam’s vanishing theorem, since

$$h^1(O_G(4H - 2L)) = h^1(O_G(-5H + 2L)) = h^1(O_G(-13L + 5 \sum E_i)),$$

and $|13L - 5 \sum E_i| \supset |10L - 4 \sum E_i| + |H|$ contains a reduced and connected divisor.

In the second case, observe that there is no birational morphism of a smooth quadric $Q$ onto the plane, thus $G$ defines $\pi$ which is an isomorphism on the plane, and has degree zero on $Q$.

Since we know that $\pi$ is an embedding of $Q \cap T$, $\pi|_Q$ is the projection of $Q = \mathbb{P}^1 \times \mathbb{P}^1$ on the second factor, followed by the isomorphism of $\mathbb{P}^1$ onto $Q \cap T$.

Since $\tau^\vee(3)|_T = O_T(2)$, $\tau^\vee(3)|_Q = O_Q(4,0)$, and $H^0(\tau^\vee(3)|_T = O_T(2)) \to H^0(\tau^\vee(3)|_Q = O_Q(4,0))$ is surjective we obtain

$$\dim Ext^1(\tau, O) = h^0(\tau^\vee(3)) = 6.$$

Q.E.D.

Remark 4.18. Indeed the above proof shows that if $G = T \cup Q$, with $Q$ a smooth quadric, and $G$ is invertible, then necessarily $T$ and $Q$ intersect transversally.

We observe now that Lemma 4.4 provides a resolution of $\tau := G \otimes (-1)$ starting from 4.8. We take the second symmetric power of the sequence (4.8) and we obtain a resolution:

$$(4.10) 0 \to \Lambda^2 W' \otimes O(-2) \xrightarrow{-B(-1)} (U' \otimes W') \otimes O(-1) \xrightarrow{\tilde{B}} S^2 U' \otimes O \to \tau(1) \cong G \otimes 0$$

where $-B$ is the contraction given by the composition of the natural inclusion from $(\Lambda^2 W') \otimes O(-1)$ to $(W' \otimes W') \otimes O(-1)$ with the map $B \otimes id_{W'}(-1)$, while $\tilde{B}$ is the composition of $id_{U'} \otimes B$ with the surjection $(U' \otimes U') \otimes O$ to $(S^2 U') \otimes O$.

Consider now the exact sequence defining $E$

$$0 \to 6O \to E \to \tau \to 0,$$

and the above projective resolution of $\tau$: by the mapping cone construction (cf. e.g. [Eis], pages 650-651) we obtain a projective resolution of $E$:

$$(4.11) 0 \to \Lambda^2 W' \otimes O(-3) \xrightarrow{-B(-2)} U' \otimes W' \otimes O(-2) \xrightarrow{(\tilde{B}(-1), \lambda)} 6O \oplus (S^2 U' \otimes O(-1)) \to E \to 0.$$
and the above map $\tilde{B} : W' \to U' \otimes V$. Let $\mathcal{E}$ be the unique sheaf given by $B$ (cf. page 11).

We split the above resolution \[\text{(4.11)}\] of $\mathcal{E}$ into two short exact sequences, denoting by $\mathcal{K}$ the image of $(\mathcal{B}(-1), \lambda)$. This gives $H^1_\mathcal{K}(\mathcal{E}) \cong H^2_\mathcal{K}(\mathcal{K}) = \ker[H^2_\mathcal{K}(\Lambda^2 W' \otimes \mathcal{O}(-3)) \to H^3_\mathcal{K}(U' \otimes W' \otimes \mathcal{O}(-2))]$. Fixed these isomorphisms, we can perform the following identifications: $U \cong \ker[\sim \mathcal{B} : \Lambda^2 W' \otimes V^\vee \to U' \otimes W']$, $W = \Lambda^2 W'$, and the multiplication map $B$ is given as the composition, in the following diagram

\[
\begin{array}{ccc}
U \otimes V & \xrightarrow{B} & W = \Lambda^2 W' \\
\downarrow & & \downarrow \\
\Lambda^2 W' \otimes V^\vee \otimes V & \sim & V
\end{array}
\]

of the natural inclusion with the natural contraction $\sim : V^\vee \otimes V \to \mathbb{C}$.

One can obtain the above factorization also in the following alternative way: Beilinson’s complex yields the following short exact sequence:

\[
0 \to U \otimes \Omega^2(2) \to W \otimes \Omega^1(1) \oplus 6 \mathcal{O} \to \mathcal{E} \to 0,
\]

where $U$ (resp. $W$) denotes as usual $H^1(\mathcal{E}(-2))$ (resp. $H^1(\mathcal{E}(-1))$).

From the above we get:

\[
(4.12) \quad 0 \to U \otimes \Omega^2(-2) \to (U \otimes V \otimes \Omega(-1)) \oplus (6 \mathcal{O}) \to W \otimes \Omega^2(-3) \oplus (W \otimes \Lambda^3 V \otimes \Omega(-2)) \oplus (W \otimes \Lambda^2 V \otimes \mathcal{O}(-1)) \to \mathcal{E} \to 0.
\]

Comparing the two sequences, we obtain the following identifications:

- $W' \cong \Lambda^2 W$,
- $U \cong \ker[\sim \mathcal{B} : \Lambda^2 W' \otimes V^\vee \to U' \otimes W']$,
- $U' \otimes W' \cong (W \otimes \Lambda^3 V)/U$,
- $S^2 U' \cong (W \otimes \Lambda^3 V)/(U \otimes V)$.

Based on the above considerations we give the following

**Definition-Proposition 4.19.**

*The Cross-Product-Involution on Tensors of type $3 \times 3 \times 4$ is given as follows:*

- to a 5-uple $(U', W', V, \delta', \mathcal{B})$, where $U', W'$ are 3-dimensional vector spaces, $V$ is a 4-dimensional vector space, $\mathcal{B} \in \text{Hom}(W' \otimes V^\vee, U') = W'^\vee \otimes V \otimes U'$, $\delta' : \Lambda^3 W' \cong \mathbb{C}$ an isomorphism, we associate the 5-uple $(U, W, V', \delta, B)$, where:

  1. $W := \Lambda^2 W'$ and, since then $W$ is canonically isomorphic to $W'^\vee$, by the duality $W' \otimes \Lambda^2 W' \to \mathbb{C}$ induced by $\delta'$, we let $\delta := \delta'^\vee$,
  2. $U := \ker[\tilde{B} : \Lambda^2 W' \otimes V^\vee \to U' \otimes W']$, where $\tilde{B}$ is defined as above through the contraction $\sim \mathcal{B}$;
  3. $B \in \text{Hom}(U \otimes V, W) = W \otimes V^\vee \otimes U'$, is the composition of the inclusion $U \otimes V \hookrightarrow \Lambda^2 W' \otimes V^\vee \otimes V$ with the natural contraction $\sim$. 
The dimension of $U$ is equal to three if we make the

**MAIN ASSUMPTION:** $\tilde{B}$ is surjective (this in turn obviously implies the injectivity of the map $B : U^\vee \to V \otimes W$).

The cross-product involution is then defined through the associated 5-uple on the open set of tensors satisfying the main assumption, and it is involutive whenever the composition is defined.

**Proof.** Given a 5-uple $(U', W', V, \delta, B)$, let $(U, W, V^\vee, \delta, B)$ be its corresponding 5-uple, to which corresponds a third 5-uple $(U'', W'', V, \delta, B'')$. We have: $B \in W^\vee \otimes V \otimes U'$, $B \in W \otimes V^\vee \otimes U'$, $B'' \in W \otimes V \otimes (U'')^\vee$.

We claim that there exists a canonical isomorphism $U' = (U'')^\vee$, equivalently, a canonical isomorphism $U'' = (U')^\vee$.

To show this, we shall first observe that both spaces can canonically be regarded as subspaces of $W \otimes V$, and then, since both spaces have the same dimension (for $U''$, this is a consequence of the hypothesis that $(U', W, V^\vee, \delta, B)$ also satisfies the main assumption), it will suffice to show that $(U')^\vee \subset U''$.

$U''$ is the kernel of $\neg B : \Lambda^2 W' \otimes V \to U^\vee \otimes W'$. Identifying $\Lambda^2 W'$ with $W$, the previous map becomes $\neg B : W \otimes V \to U^\vee \otimes W$. We consider now $B$ as the map $B : U^\vee \to V \otimes W$. It suffices now to show $(\neg B) \circ B(U^\vee) = 0$, i.e., by dualizing, that $B^\vee \circ (\neg B)^\vee (U \otimes W) = 0$. This is a consequence of the commutativity of the following diagram:

\[
\begin{array}{ccc}
U \otimes W & \xrightarrow{\neg B} & V^\vee \otimes W^\vee \\
\downarrow & & \downarrow \\
(U^\vee \otimes V^\vee) \otimes W & \xrightarrow{\neg B \otimes \text{id}_W} & (U^\vee \otimes W') \otimes W \\
\end{array}
\]

\[\square\]

**Remark 4.20.** The two tensors considered in remark 4.15, whose respective determinants yield the two non normal irreducible cubics (which are not projectively equivalent) satisfy the main assumption. But the crossproduct involution constructs out of them two tensors which do no longer satisfy the main assumption, and which are projectively equivalent:

\[
\begin{pmatrix}
x'_2 & x'_3 & 0 \\
0 & x'_2 & x'_3 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
x''_0 & x''_2 & 0 \\
x''_2 & 0 & x''_3
\end{pmatrix}.
\]

**5. Semistability of $E$ and Moduli**

In this section we try to have a broader outlook at the vector bundles $E$ that we are considering in this paper. We shall show that their explicit geometric construction lends itself to construct a natural moduli space $A^0$ for them. Since moduli space for vector bundles have been constructed in great generality by Maruyama, it seems natural to investigate their Gieseker stability (we refer to [O-S-S] and especially to [Hu-Le] as general references).
We conjecture that our bundles are Gieseker stable, but unfortunately for the time being we only managed to prove their slope (Mumford-Takemoto) semistability.

We are however able to prove that our vector bundles are simple, and we observe then (cf. Theorem 2.1 of [Kob]) that moduli spaces of simple vector bundles exist as (possibly non Hausdorff) complex analytic spaces.

We show indeed that the above moduli space exists as an algebraic variety. More precisely that, under a suitable open condition, we can construct a G.I.T. quotient $\mathfrak{M}^0$ which is a coarse moduli space.

Recall that, by 4.1 a vector bundle $E$ be given as in 3.3 or as in 3.5 has total Chern class
\begin{equation}
(5.1) \quad c(E)(t) = 1 + 3t + 6t^2 + 4t^3.
\end{equation}

The next lemma will lead to a characterization of the vector bundles obtained from the direct construction as an open set in any family of vector bundles with the above Chern polynomial.

**Lemma 5.1.** Consider a rank 6 vector bundle of $E$ with total Chern class
\begin{equation*}
c(E)(t) = 1 + 3t + 6t^2 + 4t^3,
\end{equation*}
and assume that $H^i(E) = 0 \forall i \geq 1$: then $h^0(E) = 6$.

**Proof.** Under these assumptions $h^0(E) = \chi(E)$, $\chi(E)$ is determined from the Riemann Roch theorem, and we know that there are vector bundles $E$ for which $H^i(E) = 0 \forall i \geq 1$ and $h^0(E) = 6$ (cf. lemma 4.12).

**Proposition 5.2.** Consider a rank 6 vector bundle of $E$ with total Chern class $1 + 3t + 6t^2 + 4t^3$, such that

1. $h^0(E) = 6$
2. the 6 sections generate a rank 6 trivial subsheaf with quotient $\tau$
3. $h^0(E^\vee) = 0$
4. $E$ is a subbundle of $3\Omega^1(2)$.

Then $E$ is slope-semistable.

**Proof.** Let $E''$ be a destabilizing subsheaf of rank $r \leq 5$ and maximal slope $\mu = d/r$ : without loss of generality we may assume that $E''$ is is a saturated reflexive subsheaf, and similarly $\tilde{E} := E'' \cap 6\mathcal{O}$ is a saturated reflexive subsheaf of $6\mathcal{O}$. 

\[
\begin{array}{c}
0 \rightarrow 6\mathcal{O} \rightarrow E \rightarrow \tau \rightarrow 0 \\
\uparrow \quad \uparrow \quad \uparrow \\
0 \rightarrow \tilde{E} \rightarrow E'' \rightarrow \tau'' \rightarrow 0 \\
\uparrow \quad \uparrow \quad \uparrow \\
0 \quad 0
\end{array}
\]
The slope $\mu(\mathcal{E})$ equals $1/2$. On the other hand, by hypothesis 4 and since $\Omega^1(2)$ is a stable bundle (cf. 1.2.6 b, page 167 of [O-S-S]), the slope of $\mathcal{E}''$ is at most $2/3$, and $< 2/3$ unless $\mathcal{E}'' \cong \Omega^1(2)$.

**CLAIM:** $\mathcal{E}$ contains no subsheaf isomorphic to $\Omega^1(2)$.

**Proof of the claim:** $h^0(\Omega^1(2)) = 6 = h^0(\mathcal{E})$, thus this calculation contradicts hypothesis 2.

We have that $d := c_1(\mathcal{E}''') = c_1(\tilde{\mathcal{E}}) + c_1(\tau'')$, and $\tau'' \subset \tau$ is a coherent subsheaf supported on a divisor, thus, $c_1(\tau'') \leq c_1(\tau) = 3$.

On the other hand, $c_1(\tilde{\mathcal{E}}) \leq 0$, and if equality holds, then $\tilde{\mathcal{E}} \cong r\mathcal{O}$.

Hence, $1 \leq d \leq 3$, and we have

$$\frac{2}{3} > \mu = \frac{d}{r} > \frac{1}{2} \iff 4d > 2r > 3d.$$

These inequalities leave open only the case $d = 3, r = 5$.

We show that this case does not exist.

In fact, otherwise we consider the quotient by the subbundle $\tilde{\mathcal{E}} \cong r\mathcal{O}$. By hypothesis 3, and the proof of lemma 4.9 we see that $\mathcal{E}/\tilde{\mathcal{E}}$ is an extension corresponding to a tensor of maximal rank, hence it yields a vector bundle $\mathcal{V}$ (cf. corollary 4.10).

Since the torsion sheaf $\tau'' \subset \mathcal{V}$, we obtain $\tau'' = 0$, hence $d \leq 0$, absurd.

Q.E.D.

**Remark 5.3.** The possible exceptions to slope-stability, in view of the inequalities

$$\frac{2}{3} > \mu = \frac{d}{r} \geq \frac{1}{2} \iff 4d \geq 2r > 3d$$

are:

1. $d = 1, r = 2$
2. $d = 2, r = 4$.

Matei Toma pointed out how case 2. could be excluded using Bogomolov’s inequality for stable bundles, as done in Lemma 3.1 of his paper [Toma]. The case $r = 2, c_1(\tilde{\mathcal{E}}) = -2$ seems as of now the most difficult case.

Observe that slope-stability of $\mathcal{E}$ implies Gieseker stability of $\mathcal{E}$, which in turn implies that there is a point in the moduli space of Gieseker semistable bundles corresponding to the isomorphism class of $\mathcal{E}$.

**Lemma 5.4.** Let $\mathcal{E}$ be a vector bundle as in (3.3) with $h^0(\mathcal{E}) = 6$ (equivalently, $h^1(\mathcal{E}) = 0$) and verifying the second assumption. Then $\text{hom}(\mathcal{E}, \mathcal{E}) = 1$, i.e., $\mathcal{E}$ is simple.

**Proof.** We consider the exact sequence

$$0 \to \text{Hom}(\mathcal{E}, 6\mathcal{O}) \to \text{Hom}(\mathcal{E}, \mathcal{E}) \to \text{Hom}(\mathcal{E}, \tau) \to \text{Ext}^1(\mathcal{E}, \mathcal{O}).$$

We have $\text{Ext}^1(\mathcal{E}, \mathcal{O}) \cong H^1(\mathcal{E}^\vee) \cong H^2(\mathcal{E}(-4))$ and from the exact sequence (3.3) we infer $H^2(\mathcal{E}(-4)) = 0$. Since $\text{Hom}(\mathcal{E}, 6\mathcal{O}) = 0$ by proposition 3.2, it follows that $\text{Hom}(\mathcal{E}, \mathcal{E}) \cong \text{Hom}(\mathcal{E}, \tau)$.

We compute $\text{hom}(\mathcal{E}, \tau)$ by considering the exact sequence

$$0 \to \text{Hom}(\tau, \tau) \to \text{Hom}(\mathcal{E}, \tau) \to \text{Hom}(6\mathcal{O}, \tau).$$
Indeed $\text{hom}(\mathcal{O}, \tau) = h^0(\tau) = 0$ (since $h^0(\mathcal{E}) = 6$) and, since $\tau$ is $\mathcal{O}_G$-invertible, we have $\text{hom}(\tau, \tau) = 1$.

Lemma 5.5. Let $\mathcal{E}$ be a simple vector bundle of rank 6, with Chern classes $c_1(\mathcal{E}) = 3, c_2(\mathcal{E}) = 6, c_3(\mathcal{E}) = 4$. Then the local dimension of the moduli space $\mathcal{M}^s(6; 3, 6, 4)$ of simple vector bundles at the point corresponding to $\mathcal{E}$ is at least 19.

Proof. The moduli space of simple vector bundles exists (cf. [Kob], Theorem 2.1) and it is well known that the local dimension is at least equal to the expected dimension $h^1(\mathcal{E}^\vee \otimes \mathcal{E}) - h^2(\mathcal{E}^\vee \otimes \mathcal{E})$. On the other hand, $\mathcal{E}$ simple means that $h^0(\mathcal{E}^\vee \otimes \mathcal{E}) = 1$, hence follows also that $h^3(\mathcal{E}^\vee \otimes \mathcal{E}) = 0$, since by Serre duality $h^3(\mathcal{E}^\vee \otimes \mathcal{E}) = h^0(\mathcal{E}^\vee \otimes \mathcal{E}(-4)) = 0$.

Thus the expected dimension equals $-\chi(\mathcal{E}^\vee \otimes \mathcal{E}) + 1$ and there remains to calculate $-\chi(\mathcal{E}^\vee \otimes \mathcal{E})$. This can be easily calculated in the case where we have an exact sequence $0 \to \mathcal{E} \to \mathcal{E} \to \mathcal{O}(1) \to 0$. We omit the rest of the easy calculation.

In the following theorem we shall phrase the geometric meaning of the cross product involution in terms of a birational duality of moduli space of vector bundles, $\mathfrak{A}^0$ on $\mathbb{P}^3$, $\mathfrak{A}^0$ on $\mathbb{P}^3^\vee$.

Theorem C Consider the moduli space $\mathcal{M}^s(6; 3, 6, 4)$ of rank 6 simple vector bundles $\mathcal{E}$ on $\mathbb{P}^3$ with Chern polynomial $1 + 3t + 6t^2 + 4t^3$, and inside it the open set $\mathfrak{A}$ corresponding to the simple bundles with minimal cohomology, i.e., such that

1. $H^i(\mathcal{E}) = 0 \quad \forall i \geq 1$
2. $H^i(\mathcal{E}(-1)) = 0 \quad \forall i \neq 1$
3. $H^i(\mathcal{E}(-2)) = 0 \quad \forall i \neq 1$.

Then $\mathfrak{A}$ is irreducible of dimension 19 and it is bimeromorphic to $\mathfrak{A}^0$, where $\mathfrak{A}^0$ is an open set of the G.I.T. quotient space of the projective space $\mathfrak{B}$ of tensors of type $(3, 4, 3)$, $\mathfrak{B} := \{B \in \mathbb{P}(U^\vee \otimes V \otimes W)\}$ by the natural action of $SL(W) \times SL(U)$ (recall that $U, W$ are two fixed vector spaces of dimension 3, while $V = H^0(\mathbb{P}^3, \mathcal{O}(1))$).

Let moreover $[B] \in \mathfrak{A}^0$ be a general point: then to $[B]$ corresponds a vector bundle $\mathcal{E}_B$ on $\mathbb{P}^3$, and also a vector bundle $\mathcal{E}_B^\vee$ on $\mathbb{P}^3^\vee$, obtained from the direct construction applied to $\mathcal{G}_B^\vee$ (cf. 5.2). $\mathcal{E}_B^\vee$ is the vector bundle $\mathcal{E}_B$, where $[B] \in \mathfrak{A}_B^0$ is obtained from $B$ via the cross product involution.

Proof. To any such tensor $B$ we tautologically associate two linear maps which we denote by the same symbol,

$B : U \otimes V \to W, \quad B : U \otimes V \otimes \mathcal{O}(1) \to W \otimes \mathcal{O}(1)$

and using the Euler sequence we define a coherent sheaf $\mathcal{E}$ on $\mathbb{P}^3$ as a kernel, exactly as in the exact sequence 3.5 (except that surjectivity holds only for $B$ general).
As we already saw in [3.4], this is equivalent to giving $\mathcal{E}$ as the kernel of a homomorphism $\beta$ as in [3.3]. Observe that $GL(W) \times GL(U)$ acts on the vector space of such tensors, preserving the isomorphism class of the sheaf thus obtained.

We define $\mathcal{B}'$ as the open set in $\mathcal{B}$ where $\beta$ is surjective (thus $\mathcal{E}$ is a rank 6 bundle) and $h^0(\mathcal{E}) = 6$. Both conditions amount to the surjectivity of $h^0(\beta) = h^0(B, \epsilon)$, cf. [3.5]. We further define $\mathcal{B}''$ as the smaller open set where the second assumption is verified, and we observe then that lemma 5.4 ensures the existence of a morphism $\mathcal{B}'' \to \mathfrak{A}$ which factors through the action of $SL(W) \times SL(U)$.

Since we want to construct a G.I.T. quotient of an open set of $\mathcal{B}$, we let $\mathcal{B}^*$ the open set of tensors $B$ whose determinant defines a cubic surface $G^* \subset P^3$, i.e., we have an exact sequence on $P^3$ of the form (set $O_* := O_{P^3}$)

$$0 \to U \otimes O_*(-1) \xrightarrow{B} W \otimes O_* \to G^* \to 0. \tag{5.2}$$

Since the determinant map is obviously $SL(W) \times SL(U)$-invariant, the tensors in $\mathcal{B}^*$ are automatically semistable points for the $SL(W) \times SL(U)$-action, by virtue of the criterion of Hilbert-Mumford.

Observe now that the maximal torus $\mathbb{C}^* \times \mathbb{C}^*$ of $GL(W) \times GL(U)$ acts trivially on $\mathcal{B}$, thus we get an effective action of $SL(W) \times SL(U)$ only upon dividing by a finite group $K' \cong (\mathbb{Z}/3)^2$.

We claim that $(SL(W) \times SL(U))/K'$ acts freely on the open subset $\mathcal{B}^{**} \subset \mathcal{B}^*$, $\mathcal{B}^{**} = \{ B \in \mathcal{B}^* | \text{End}(G^*) = \mathbb{C} \}$.

This is clear since the stabilizer of $B$ corresponds uniquely to the group of automorphisms of $G^*$, and any such automorphism acts on $W \cong H^0(G^*)$, and induces a unique automorphism of $U$ in view of the exact sequence 5.2. But every automorphism is multiplication by a constant, thus it yields an element in $K'$.

We want to show that the orbits are closed. But the orbits are contained in the fibres of the determinant map: thus, it suffices to show that, fixed the cubic surface $G^*$, if we have a 1-parameter family where $G_t \cong G_1$ for $t \neq 0$, then also $G_0 \cong G_1$.

This holds on the smaller open set $\mathcal{B}^{***} \subset \mathcal{B}^{**}$ consisting of the tensors such that the cubic surface $G^*$ is smooth: since then $G_0$ is invertible, and the Picard group of $G^*$ is discrete.

We have proven that $\mathcal{B}^{***}$ consists of stable points, and observe that the condition $\text{End}(G^*) = \mathbb{C}$ holds if $G^*$ is $O_{G^*}$-invertible, or it is torsion free and $G^*$ is normal. Therefore the open set $\mathcal{B}^{st}$ of stable points is nonempty.

We define $\mathfrak{A}^0$ as the open set of the G.I.T. quotient corresponding to $\mathcal{B}^{st} \cap \mathcal{B}''$.

The fact that $\mathfrak{A}$ is irreducible follows since every bundle $\mathcal{E}$ in $\mathfrak{A}$ has a cohomology table which (by Beilinson’s theorem, as in the beginning of section 3) implies that $\mathcal{E}$ is obtained from a tensor $B$ in the open subset...
\( \mathfrak{B}^0 \subset \mathfrak{B} \) consisting of those \( B \) for which the corresponding bundle \( E \) is simple (note that \( \mathfrak{B}^0 \supset \mathfrak{B}'' \)).

Now, \( \dim \mathfrak{A}^0 = 19 \), while \( \dim \mathfrak{A} \geq 19 \) by 5.5; we only need to observe that if \( [B], [B'] \in \mathfrak{A}^0 \) and two bundles \( E_B \) and \( E_{B'} \) are isomorphic, then the corresponding tensors \( B, B' \) are \( \text{GL}(U) \times \text{GL}(W) \) equivalent, since they express the multiplication matrix for the intermediate cohomology module \( H^1_*(\mathcal{E}) \). Thus \( [B] = [B'] \in \mathfrak{A}^0 \).

It follows on the one side that \( \mathfrak{A}^0 \) parametrizes isomorphism classes of bundles, and on the other side \( \mathfrak{A}^0 \) maps bijectively to an open set in \( \mathfrak{A} \), in particular \( \dim \mathfrak{A} = 19 \), since \( \mathfrak{A} \) is irreducible.

\[ \text{Q.E.D.} \]

6. The explicit unirational family

We have up to now studied extensively the vector bundles \( \mathcal{E} \) such that an even set of 56 nodes on a sextic surface \( F \) should come from a symmetric homorphism associated to a section of \( S^2(\mathcal{E}) \).

We have however almost shown, because of corollary 4.8 and of proposition 4.14 that all such sections have as determinant the square of a cubic surface \( G \), if the cubic \( G \) appearing in the direct construction is an irreducible cubic.

It seems therefore only natural to try to see what happens for a reducible cubic, hoping that then \( h^0(S^2\mathcal{E}) > 21 \).

We assume henceforth that \( G \) is the union of a smooth quadric \( Q \) with a plane \( T \) intersecting transversally. We have already observed in the proof of proposition 4.17 that in this case there is a unique choice for \( \mathcal{G} \), likewise for \( \mathcal{E} \).

**Lemma 6.1.** If \( G \) is the union of a smooth quadric \( Q \) and of a plane \( T \) which intersect transversally, then \( h^0(\mathcal{G} \otimes 2) = 1 \) and \( h^0(S^2\mathcal{E}) = 22 \).

**Proof.** In this case the sheaf \( \mathcal{G} \) corresponds to the sheaf \( \mathcal{O}_Q(0,2) \) on \( Q \) and to \( \mathcal{O}_T(1) \) on \( T \).

Therefore \( \tau := \mathcal{G} \otimes 2(-1) \) corresponds to \( \mathcal{O}_Q(-1,3) \) on \( Q \) and to \( \mathcal{O}_P(1) \) on \( P \).

Thus the sections of \( H^0(\mathcal{G} \otimes 2) \) vanish identically on \( Q \) and correspond to section of \( \mathcal{O}_T(2) \) vanishing on \( Q \cap T \).

The second statement follows then from the proof of corollary 4.8. \( \square \)

We also remark that, since we assume \( Q \cap T \) is smooth, \( G \) is unique up to projective equivalence.

We shall now give an explicit tensor \( B_0 \) whose associated sheaf is the unique \( \mathcal{G} \) on a reducible cubic of the form \( T \cup Q \), where \( Q \) and \( Q \cap T \) are smooth, and compute explicitly that the tensor corresponding to the
unique \( \mathcal{E} \) gotten from the direct construction is again \( B \): this will allow us to calculate explicitly the determinant of a generic symmetric map \( \mathcal{E}^\vee \to \mathcal{E} \), and to show that it is a nodal sextic.

**Lemma 6.2.** Consider the following \( 3 \times 3 \times 4 \) tensor \( B_0 \):

\[
B_0 = \begin{pmatrix}
0 & -x_3 & x_2 \\
x_3 & 0 & -x_1 \\
-x_2 & x_1 & 0
\end{pmatrix} + x_0 \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

The sheaf \( \mathcal{G}_0 \) associated to \( B_0 \) is an invertible sheaf on the reducible cubic \( G_0 := \{ x_0(\sum_i x_i^2) = 0 \} \).

Its class in \( \mathcal{O}^0 \) is invariant under the cross product involution. More precisely, the vector bundle \( \mathcal{E}_0 \) obtained from \( B_0 \) via the direct construction has the required cohomology table and it has again \( B_0 \) as multiplication matrix for the intermediate cohomology module \( M_0 := H^1_*(\mathcal{E}_0) \).

Moreover \( h^0(S^2\mathcal{E}_0) = 22 \).

**Proof.** The determinant of \( B_0 \) equals \( G_0 \). On the plane \( x_0 = 0 \) the Pfaffians are \( x_1 = x_2 = x_3 = 0 \) and \( \mu \) has rank 2. Elsewhere \( G_0 \) is smooth, whence \( \mathcal{G} \) is everywhere invertible.

We show now that its class is invariant under the cross product involution. Denote by \( W' \) and \( U' \) the vector spaces of rows and columns of \( B_0 \).

Then \( U \) is defined as the kernel of the composition \( \sim B_0 : \Lambda^2 W' \otimes V^\vee \to (W' \otimes V^\vee) \otimes W' \to U' \otimes W' \). Let \( e_1, e_2, e_3 \) (resp. \( f_1, f_2, f_3 \)) be the standard basis of \( W' \cong \mathbb{C}^3 \) (resp. \( U' \)). Then we choose \( e_2 \wedge e_3, e_1 \wedge e_3, e_1 \wedge e_2 \) as basis of \( \Lambda^2 W' \subset W' \otimes W' \) and \( f_1 \otimes e_1, f_2 \otimes e_1, \ldots, e_3 \otimes f_3 \) as basis of \( U' \otimes W' \) and, viewing the map \( \sim B_0 \) as a \( 9 \times 3 \) matrix of linear forms on the space \( V^\vee \), we have:

\[
\tilde{B}_0 = \begin{pmatrix}
0 & (B_0)^3 & -(B_0)^2 \\
-(B_0)^3 & 0 & (B_0)^1 \\
(B_0)^2 & -(B_0)^1 & 0
\end{pmatrix},
\]

where \( (B_0)^i \) denotes the \( i \)-th column of \( B_0 \). The kernel \( U \) is then:

\[
U = \{ u = \sum a_i x_i^\vee | a_i \in \mathbb{C}^3, \tilde{B}_0 u = 0 \} = \langle \begin{pmatrix} x_0^\vee \\ x_3^\vee \\ -x_2^\vee \end{pmatrix}, \begin{pmatrix} -x_3^\vee \\ x_0^\vee \\ -x_1^\vee \end{pmatrix}, \begin{pmatrix} x_2^\vee \\ x_1^\vee \\ x_0^\vee \end{pmatrix} \rangle.
\]

A rapid inspection shows that, if we choose the above bases and the dual basis of \( V^\vee \), the \( 3 \times 3 \times 4 \) tensor that we obtain is exactly identical to \( B_0 \).

The direct construction gives then a unique vector bundle \( \mathcal{E} \) (cf. proposition 4.17), and we claim that its cohomology table is the required one. As in the proof of lemma 4.12 it suffices to calculate the dimensions of the cohomology groups \( h^i(\tau(\sim)) \), for \( 0 \leq n \leq 3 \) and to show their vanishing for \( i = 0, i = 2 \).

We use the exact sequence

\[
0 \to \tau \to \mathcal{O}_{\mathbb{P}^1(2)}(1) \to \mathcal{O}_{\mathbb{P}^1(2)}(1) \to 0
\]
which is soon seen to be exact on global sections, hence the case \( i = 0 \) follows right away. For the case \( i = 2 \) we observe that in the exact cohomology sequence

\[
0 \to H^1(\tau(-n)) \to H^1(O_Q(-1-n, 3-n)) \overset{\psi}{\to} H^1(O_{Q\cap T}(1-n)) \to H^2(\tau(-n)) \to 0
\]

\( \psi \) is surjective, since its cokernel is isomorphic to \( H^2(O_Q(-2-n, 2-n)) \) whose dimension equals \( h^0(O_Q(n, n-4)) = 0 \) (since \( n < 4 \)).

The last claim follows from lemma 6.1.

Q.E.D.

Remark 6.3. Consider the invertible sheaf given by:

\[(6.2) 0 \to 3O(-1) \xrightarrow{B_0} 3O \to \mathcal{G}_0 \to 0.\]

We have that \( h^0(\mathcal{G}_0) = 3 \) and the associated morphism \( \pi \) from \( G_0 \) to \( \mathbb{P}^2 \) is determined by the rational map given by the entries of any column of \( \text{Ad}(B_0) \):

\[
\text{Ad}(B_0) = \begin{pmatrix}
x_1^2 + x_0^2 & x_2^2 + x_0^2 & x_3^2 + x_0^2 \\
x_1x_2 + x_0x_3 & x_2x_3 + x_0x_1 & x_3^2 + x_0^2
\end{pmatrix}.
\]

We see right away that \( \pi \) is the identity on the plane \( T \), and the projection along one ruling from the quadric \( Q \) to \( Q \cap T \).

Proposition 6.4. For a section \( \phi \in H^0(S^2E_0) \), denote by \( \varphi \in \text{Hom}(E^\vee, E) \) its associated symmetric morphism. Then, for \( \phi \) general, \( F := \{x | \det(\varphi) = 0\} \) is a nodal sextic surface with, as singularities, exactly an even set of 56 nodes \( \Delta = \{x | \text{Corank}(\varphi) = 2\} \).

Proof. The required computations were performed and can be performed and verified by using the computer-algebra system [Gr-St] over a finite field, or over \( \mathbb{Q} \) (cf. script I in the Appendix).

The first step is to compute explicitly the fibre over \( B_0 \) inside the variety of pairs \( \mathfrak{M}_{AB} \) (cf. 3.6), i.e., the vector space of symmetric matrices \( A \in \text{Mat}(12 \times 12, H^0(O_{\mathbb{P}^3}(2))) \) satisfying the equation \( (B_0, \epsilon) \cdot A = 0 \).

Step two: for a random \( A \) in such a fibre one computes the G.C.D. of two (different) \( 6 \times 6 \) minors of \( A \): if the G.C.D. has degree 6, then it is the equation of the sextic \( F \).

Step three: one verifies with the jacobian criterion that the singular locus of \( F \) consists exactly of a 0-dimensional subscheme of length 56.

Step four: one verifies that the ideal sheaf of the singular locus is a radical ideal. Then the singularities are just a set of nodes.

A further (but not absolutely necessary) check consists in verifying that the scheme \( \Delta \) coincides with the subscheme formed by those 56 reduced points: this can be performed by computing a set of \( 5 \times 5 \) minors of \( A \) sufficient in order to generate the ideal of the 56 points.

\( \square \)
Remark 6.5. Since the space of reducible cubic surfaces has dimension 12 (9 + 3), we obtain an explicit family parametrized by a rational variety $\Phi_0$ of dimension $33 = 21 + 12$.

Proof. We simply construct a parameter space by choosing a 12-dimensional subgroup $H \subset \mathbb{P}GL(V)$ such that the orbit of $G_0$ dominates the space of reducible cubics, and then we take as parameter space $H \times \mathbb{P}(H^0(S^2(E)))$.

Then to the pair $(g, \phi_0)$ corresponds the vector bundle $g^*E := E_{gG_0}$ and the section $g^*(\phi_0)$, and, correspondingly, the sextic surface $g^*(\text{det}(\varphi) = 0)$.

□

Lemma 6.6. The morphism $\Phi_0 \to \mathbb{P}(S^6(V))$ associating to $(g, \varphi)$ the corresponding nodal sextic $F = \text{det}(\varphi)$ has fibres of dimension 6.

Proof. Recall first that we have already shown that the surface $g(G_0)$ uniquely determines a vector bundle $E$ and conversely.

Second, observe that if $F$ has exactly 56 nodes then $F$ determines the quadratic sheaf $\mathcal{F}$ uniquely (observe moreover that $\mathcal{F}$ has only constant automorphisms).

Suppose that there are two different vector bundles $E, E'$ and respective morphisms $\varphi, \varphi'$ forming exact sequences as in theorem 2.1 which define isomorphic cokernels $\mathcal{F}, \mathcal{F}'$.

By abuse of notation we identify $\mathcal{F}'$ with $\mathcal{F}$ and assume that we have $\gamma : E \to \mathcal{F}, \gamma' : E' \to \mathcal{F}$ inducing such isomorphisms of the respective cokernels with $\mathcal{F}$.

A first question is whether there do exist lifts $\alpha : E' \to E$ and $\beta : E' \to E'$ of the identity $\text{id}_F$ on $F$ such that the following diagram commutes

$$(6.3) \quad 0 \to E' \xrightarrow{\alpha} E \xrightarrow{\gamma} \mathcal{F} \to 0.$$

If such an $\alpha$ exists, then necessarily the submodule $M := \text{Im}(H^1_*(\gamma))$ of $H^1_*(\mathcal{F})$ equals the submodule $M' := \text{Im}(H^1_*(\gamma'))$.

Assume now that $M' = M$: then since any automorphism of $M$ lifts to an isomorphism of two minimal resolutions of $M$, this automorphism induces an isomorphism $\alpha$ of the respective first syzygy bundles, here $E$, resp. $E'$ (cf. the first page of section 2). We can henceforth assume that if $M = M'$, then $E = E'$, and then, using $\text{Hom}(E, E) = \mathbb{C}$ (cf. lemma 5.4) we get that $\alpha$ is the multiplication by a constant, necessarily $\neq 0$.

From the exact sequence

$$0 \to \text{Hom}(E', E') \to \text{Hom}(E', E) \xrightarrow{\gamma_*} \text{Hom}(E', \mathcal{F}),$$

it follows that there exists a unique homomorphism $\beta$ making the diagram commute. Again, using $\text{Hom}(E, E) = \mathbb{C}$, we get that $\beta$ is the multiplication
by a non-zero constant and we have thus shown that if \( M = M' \) then the sections \( \phi \) and \( \phi' \) are proportional.

On the other hand, the choice of \( M \) is completely determined by the choice of a Lagrangian subspace \( U \) of the 6-dimensional space \( H^1(F(1)) \), and we saw that for each choice of \( U \) there is a bundle \( \mathcal{E} \) and a \( \phi \) yielding an exact sequence as in theorem 2.1 with \( M \) equal to the image of \( (H^1_\mathcal{E}) \).

We are done, since the dimension of the Lagrangian Grassmannian \( LGr(3, 6) \) equals 6.

Q.E.D.

We want to show that the explicit unirational family that we constructed is locally maximal. To this purpose, observe that to a pair \( (g, \phi_0) \) corresponds a vector bundle \( g^*(\mathcal{E}_0) \) and a section \( g^*(\phi_0) \), but more precisely a tensor \( g^*(B) \) and a matrix of quadratic forms \( A_{g,\phi_0} \) representing \( g^*(\phi_0) \) as in 3.5.

Thus \( \Phi_0 \) maps in a generically finite way to the variety of pairs \( \mathcal{M}_{AB} \) (cf. 3.6) and we can consider the \( GL(U) \times GL(W) \)-orbit of its image.

Observe then that we obtain an irreducible algebraic set \( \Psi_0 \) of dimension \( 33 + 1 + 9 + 9 - 1 = 51 \).

The following lemma shows that \( \Psi_0 \) is indeed a component of \( \mathcal{M}_{AB} \).

**Lemma 6.7.** Let \( (B_0, A_0) \in \mathcal{M}_{AB} \) be a general point of the fibre over \( B_0 \). Then the tangent space to \( \mathcal{M}_{AB} \) at the point \( (B_0, A_0) \) has dimension 51.

**Proof.** Fixed the pair \( (B_0, A_0) \in \mathcal{M}_{AB} \), we search the solutions for a generic pair \( (B, A) \in \text{Mat}(3, 12, \mathbb{C}) \times \text{Mat}_{\text{Sym}}(12, 12, H^0(\mathcal{O}_{\mathbb{P}^3}(2))) \) of the equations

\[
(B_0 + tB, \epsilon)(A_0 + tA) \equiv 0 \mod t^2.
\]

The above equation is equivalent to the two equations \( BA_0 + B_0A = 0, \epsilon A = 0 \), and we have to compute the space of solutions.

Again we perform the computation by means of computer-algebra over a finite field, it suffices to choose a point \( A_0 \) at random for which the tangent space has dimension 51. The computation works out successfully (cf. script II in the Appendix).

We can now summarize the result of the construction of the above explicit family:

**Main Theorem B.** There is a family of nodal sextic surfaces with 56 nodes forming an even set, parametrized by a smooth irreducible rational variety \( \Phi_0 \) of dimension 33, whose image \( \Xi_0 \) is a unirational subvariety of dimension 27 of the space of sextic surfaces. Moreover, the above family is versal, thus \( \Xi_0 \) yields an irreducible component of the subvariety of nodal sextic surfaces with 56 nodes.

**Proof.** The first assertions were proven in the sequel between lemma 6.1 and lemma 6.6
Let $\Xi$ be the subvariety of nodal sextic surfaces with 56 nodes: since the property that the set of nodes is even is a topological property (cf. for instance [Cat1], [Cat2]), it follows that there is an open and closed set $\Xi' \subset \Xi$ such that for $F \in \Xi'$ the set of 56 nodes is even. We need only to prove that $\Xi_0 \subset \Xi'$ is open.

But $\Xi'$ contains the open set $\Xi''$ such that, for $F \in \Xi''$, $H^1(F(2))$ has dimension 3 and the first assumption is verified. We can form a variety $\Psi'$ consisting of quadruples $(F, U, B, \phi)$ where:

i) $F \in \Xi'$ is a sextic surface,

ii) $U \subset H^1(F(1))$ is a Lagrangian subspace, and, $M$ being the intermediate cohomology submodule of $H^1_1(F)$ determined by the choice of $U$ as in 2.1,

iii) $B$ is a multiplication tensor for $M$ (depending on two choice of bases)

iv) if $E$ is the unique vector bundle determined by $B$ as in 3.5, $\phi$ is a section of the vector bundle $S^2(E)$ such that $\det(\phi) = F$.

Then we see that the map $\Psi' \to \mathfrak{M}_{A,B}$ is an embedding. Now lemma 6.7 shows that $\Psi_0 \subset \Psi'$ is open, and we are done.

It is a natural question to ask if the above is the unique irreducible component of the subvariety of nodal sextic surfaces with 56 nodes forming an even set. For this purpose one should first settle the case of Hilbert function $(3,4)$ for $M$.

7. The random approach

Let $M$ be a variety defined over a finite field of order $q$ and let $M_0 \subset M$ be a subvariety of codimension $k$. The random approach consists in finding a point in $M_0$ by choosing at random points in $M$. Since the probability of hitting a point of $M_0$ is $q^{-k}$ it is evident that this method is only successful if the computational time to decide wether a point of $M$ actually belongs to $M_0$ is small enough (cf. [Sch]).

In this section we show how this method was applied to find the first examples of sextic surfaces with an even set of 56 nodes.

Let $A$ denote the coordinate ring of $\mathbb{P}^3$ and let $B$ be the multiplication matrix of the intermediate cohomology module $M$. If $B$ is general, since $E$ is a syzygy bundle (cf. section 3), it follows (cf. 3.5 and 4.11) that $M$ has a resolution of the form

$$0 \leftarrow M \leftarrow 3A[2] \leftarrow 9A[1] \leftarrow 6A \oplus 6A[-1] \leftarrow 9A[-2] \leftarrow 3A[-3] \leftarrow 0.$$

In an analogous way to the one followed after the exact sequence 3.5 we get that the symmetric morphisms $\varphi : E^* \to E$ are exactly induced by the
symmetric morphisms \( a : 9\mathcal{O}(-1) \to 9\mathcal{O}(1) \) such that \( b \circ a = 0 \), according to the following diagram:

\[
\begin{array}{ccc}
0 & \xleftarrow{3\mathcal{O}(2)} & 9\mathcal{O}(1) \\
& \downarrow{0} & \downarrow{a} \\
0 & \xleftarrow{3\mathcal{O}(-2)} & 9\mathcal{O}(-1) \\
& \downarrow{b} & \downarrow{\phi} \\
& & \mathcal{E}^\vee 
\end{array}
\]

It is clear that the replacement of \((A,B)\) with \((a,b)\) reduces the memory required for computations.

Repeated random choices of \( b \) allow to find an \( \mathcal{E} \) with:

\[
h^0(S^2\mathcal{E}) = \dim\{a : 9\mathcal{O}(-1) \to 9\mathcal{O}(1) \mid a = t, b \circ a = 0\} \geq 22.
\]

This property leads to the definition of \( \mathcal{M} \) and \( \mathcal{M}_0 \).

**Definition 7.1.** Let \( \mathcal{M} \) be the Zariski open set

\[
\mathcal{M} := \{b : 9\mathcal{O}(1) \to 3\mathcal{O}(2) \mid \text{coker } b \text{ has a resolution as in 7.1 and } \}
\]

\[
i : 6\mathcal{O} \to \mathcal{E} := Syz_1(M) \text{ is injective} \}
\]

\[
\mathcal{M}_0 := \{b \in \mathcal{M} \mid h^0(S^2\mathcal{E}) \geq 22\}
\]

We already remarked that \( \mathcal{M} \) is nonempty. A resolution for \( S^2\mathcal{E} \) is provided by the following lemma.

**Lemma 7.2.** If \( 0 \to A \to B \to C \to \mathcal{E} \to 0 \) is an exact sequence of locally free sheaves, then the following sequence is also exact:

\[
0 \to S^2A \to A \otimes B \to \Lambda^2B \oplus (A \otimes C) \to B \otimes C \to S^2C \to S^2\mathcal{E} \to 0.
\]

**Proof.** By hypothesis we have \( 0 \to (B/A) \to C \to \mathcal{E} \to 0 \). Therefore we get \( 0 \to \Lambda^2(B/A) \to (B/A) \otimes C \to S^2C \to S^2\mathcal{E} \to 0 \). Resolutions for \( \Lambda^2(B/A) \) and \( (B/A) \otimes C \) are standard, respectively \( 0 \to S^2A \to A \otimes B \to \Lambda^2B \to \Lambda^2(B/A) \to 0 \) and \( 0 \to A \otimes C \to B \otimes C \to (B/A) \otimes C \to 0 \). The resolution for \( S^2\mathcal{E} \) stated in the lemma is the mapping cone of the previous resolutions. \( \square \)

Hence it was guessed that the “good” locus has codimension 7:

**Proposition 7.3.** The condition \( h^0(S^2\mathcal{E}) \geq 22 \) is expected to hold on a codimension 7 algebraic subset of \( \mathcal{M} \).

**Proof.** By applying the previous Lemma to a minimal free resolution of \( \mathcal{E} \) we get a (non necessarily minimal) free resolution of \( S^2\mathcal{E} \):

\[
(7.2)
\]

\[
0 \to 6\mathcal{O}(-6) \to 27\mathcal{O}(-5) \xrightarrow{2} 18\mathcal{O}(-3) \xrightarrow{2} \oplus \frac{54\mathcal{O}(-3)}{18\mathcal{O}(-3)} \oplus \frac{36\mathcal{O}(-1)}{21\mathcal{O}} \oplus 36\mathcal{O}(-4) \xrightarrow{r} S^2\mathcal{E} \to 0.
\]
Denote by $K_i$ the image of the map $r_i$, split the above exact sequence into short exact sequences and look at the associated cohomology exact sequences. From

$$0 \to H^0(K_1) \to H^0(21\mathcal{O}) \xrightarrow{S^2(\epsilon)} H^0(S^2\mathcal{E}) \to H^1(K_1) \to 0 :$$

and since $S^2,\epsilon$ is injective ($\epsilon : 6\mathcal{O} \to \mathcal{E}$ being injective) we get, using also the other cohomology sequences, $0 = H^0(K_1) \cong H^1(K_2) \cong H^2(K_3)$ and $H^0(S^2\mathcal{E})/H^0(21\mathcal{O}) \cong H^3K_1 \cong H^2K_2$.

We have also the short exact sequence

$$H^2(K_3) = 0 \to H^3(6\mathcal{O}(-6)) \to H^3(27\mathcal{O}(-5)) \to H^3(K_3) \to 0,$$

hence $H^3(K_3)$ has dimension 48.

Finally, the exact sequence:

$$0 \to H^2(K_2) \to H^3(K_3) \xrightarrow{\alpha} H^3(54\mathcal{O}(-4)) \to H^3(K_2) \to 0,$$

since $H^3(K_2) \cong H^2(K_1) \cong H^1(S^2\mathcal{E})$ yields:

$$\text{(7.3)} \quad 0 \to \frac{H^0(S^2\mathcal{E})}{H^0(21\mathcal{O})} \to \mathbb{C}^{48} \xrightarrow{\alpha} \mathbb{C}^{54} \to H^1(S^2\mathcal{E}) \to 0.$$

Therefore the condition that $H^0(S^2\mathcal{E}) \cong H^0(21\mathcal{O})$ is equivalent to the condition that the linear map $\alpha$ has maximal rank, and since we know that this happens in general, the condition $h^0(S^2\mathcal{E}) \geq 22$ holds in a determinantal subscheme of $\mathbb{M}$ of expected codimension $54 - 48 + 1 = 7$.

Let $\mathbb{M}_{ab}$ be the variety, analogous to $\mathbb{M}_{AB}$ (cf. 3.6), of pairs $(b,a)$ such that $a = t^4 a, ab = 0$. Computations similar to the ones in lemma 6.7 verified over a finite field that in a random point $(b_0, a_0)$ the variety of pairs $\mathbb{M}_{ab}$ is smooth of dimension 123.

A standard argument then ensures the existence of a lift of the pair $(b_0, a_0)$ from a finite field to a number field (cf. [Sch]).

This random approach, and the remark that the space of reducible cubic surfaces is a codimension 7 subvariety of the projective space of cubic surfaces led then to the explicit family constructed in the previous section.

8. Appendix: Macaulay2 scripts

Script I:

```plaintext
kk=ZZ/101
R=kk[x_0..x_3]
Pa=kk[a_1..a_78]
RP=R**Pa

--Step 0: the matrix (B,\epsilon)
use R
B=matrix{{x_0,-x_3,x_2},{x_3,x_0,-x_1},{-x_2,x_1,x_0}}
vars R
betti (B=diff(vars R,B)**R^{-1})
```
betti (epsilon=(vars R)**id_(R^3))
betti (B1=B1|epsilon)
betti res coker B1
betti res prune coker B1

--Step 1: computing the 22-dim morphisms sol
use RP
A=genericSymmetricMatrix(RP,a_1,12);
equation=flatten (substitute(B1,RP)*A);
equation=substitute(diff(substitute(vars Pa,RP),transpose equation),R);
betti equation
betti (sol=syz(equation,DegreeLimit=>1)) --22

--Step 2: taking a random morphism A1 and computing the surface
A1=flatten (sol*random(R^22,R^1));
A1=substitute(substitute(A,Pa),A1);
B1*A1==0 --verification that it is really in the fiber
m1=random(R^6,R^12)*A1*random(R^12,-2,R^6,-2);
m2=random(R^6,R^12)*A1*random(R^12,-2,R^6,-2);
F=gcd(det(m1),det(m2));
degree F --6

--Step 3:
decompose ideal F; --sestica irreducible
SingF=ideal(jacobian ideal F)+ideal F;
degree SingF,codim SingF --(56, 3)

--Step 4:
SingFsat=saturate SingF; --equiv. to: SingFsat=saturate(SingF,ideal vars R);
SingFsat==radical SingFsat --true

--Further check:
n1=random(R^5,R^12)*A1*random(R^12,-2,R^5,-2);
I=ideal det n1
i=1
while (i<50) do (
 n1=random(R^5,R^12)*A1*random(R^12,-2,R^5,-2);
 I=I+ideal det n1;
 i=i+1;
);
codim I, degree I --(3,56)
I=ideal mingens I;
I=saturate I;
I==SingFsat --true

Script II:
kk=ZZ/101
R=kk[x_0..x_3]
Pa=kk[a_1..a_78]
Pb=kk[b_1..b_36]
P=kk[b_1..b_36,a_1..a_78]
RP=R**P
use R
\[ B = \begin{bmatrix} x_0, -x_3, x_2, x_3, x_0, -x_1, -x_2, x_1, x_0 \end{bmatrix} \]

vars R
betti (B = diff(vars R, B)**R^{-1})
betti (epsilon = (vars R)**id_(R^3))
betti (B1 = B||epsilon)
betti res coker presB1
betti res prune coker presB1
use RP
A = genericSymmetricMatrix(RP, a_1, 12)
equation = flatten (substitute(B1, RP)*A);
equation = substitute(diff(substitute(vars Pa, RP), transpose equation), R);
betti equation
betti (sol = syz(equation, DegreeLimit => 1)) --22
A1 = flatten (sol*random(R^22, R^1));
A1 = substitute(substitute(A, Pa), A1);

use P
betti (Bgen = map(P^{3:1}, P^{12:0}, genericMatrix(P, b_1, 3, 12)))
betti (B2 = substitute(Bgen, RP)||substitute(epsilon, RP))
A2 = A
equation = substitute(B1, RP)*A2-B2*substitute(A1, RP);
eq1 = flatten equation^{0..2}; eq2 = flatten equation^{3..5};
param = substitute(vars Pb, RP)|substitute(vars Pa, RP);
betti (eq1 = substitute(diff(param, transpose eq1), R))
betti (eq1 = map(R^{36:2}, R^{36:0, 78:2}, eq1))
eq2 = substitute(diff(param, transpose eq2), R);
betti (eq2 = map(R^{36:3}, R^{36:3, 78:2}, eq2))
betti (eq = eq1||eq2)
isHomogeneous eq
betti (sol = syz (eq, DegreeLimit => 0)) --51

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