Deviation from the Standard Uncertainty Principle and the Dark Energy Problem

Shahram Jalalzadeh\textsuperscript{1}\*; Mohammad Ali Gorji\textsuperscript{2†} and Kourosh Nozari\textsuperscript{2‡}

\textsuperscript{1}Department of Physics, Shahid Beheshti University, G. C. Evin, Tehran 19839, Iran
\textsuperscript{2}Department of Physics, Faculty of Basic Sciences, University of Mazandaran, P. O. Box 47416-95447, Babolsar, Iran.

October 31, 2013

Abstract

Quantum fluctuations of a real massless scalar field are studied in the context of the Generalized Uncertainty Principle (GUP). The dynamical finite vacuum energy is found in spatially flat Friedmann-Robertson-Walker (FRW) spacetime which can be identified as dark energy to explain late time cosmic speed-up. The results show that a tiny deviation from the standard uncertainty principle is necessary on cosmological ground. By using the observational data we have constraint the GUP parameter even more stronger than ever.

Keywords: Generalized Uncertainty Principle, FRW cosmology, Vacuum energy

1 Introduction

The idea that the uncertainty principle is influenced by gravity has been suggested by many candidates of quantum gravity as well as string theory. The Generalized Uncertainty Principle (GUP) is an immediate way to impose the quantum gravity effects in ordinary quantum mechanics through deforming the usual Heisenberg uncertainty principle. Such a deformation has origin in the existence of a minimal measurable length which is predicted by quantum gravity proposal [1]. Furthermore, Doubly Special Relativity (DSR) theories predict an upper bound for the test particles’ momentum [2]. Inspired by DSR theories, the UV-regularized version of GUP has been proposed in [3] which supports the existence of a minimal length and also a maximal momentum. In one dimension, such deformed uncertainty relation can be written as [4],

\[
\Delta x \Delta p \geq \frac{1}{2} \left( 1 - \frac{2\alpha_0}{M_{P_1}} \langle p \rangle + \frac{4\alpha_0^2}{M_{P_1}^2} \langle p^2 \rangle \right),
\]

where $\alpha_0$ is a numerical factor and $M_{P_1}$ is the Planck mass [5]. The uncertainty relation [1] predicts the smallest uncertainty in position $\Delta x_{\text{min}} = 2\alpha_0 l_{P_1}$ and a maximum uncertainty in momentum measurement $\Delta p_{\text{max}} = M_{P_1}/2\alpha_0$. This maximal uncertainty in momentum measurement gives non-trivially an upper bound also for a test particle’s momentum. The GUP numerical factor $\alpha_0$ defines the quantum gravity scale. But, how much these effects are small? How one can detect these small corrections? Recently, some authors attempt to answer these questions. Authors in [6] studied some phenomenological aspects of quantum gravity in quantum mechanical systems and showed that GUP numerical factor cannot exceed the Electroweak scale $\alpha_0 \leq 10^{17}$. In [7], the effects of GUP on the transition rate of ultra cold neutrons in gravitational field have been studied and they found $\alpha_0 \leq 10^{29}$, which is weaker than the bound predicted in [6].

In this paper, we study the effects of the Generalized Uncertainty Principle (GUP) in cosmology. We show that quantum fluctuations of a real massless scalar field in Friedmann-Robertson-Walker (FRW) spacetime, naturally leads to the dynamical UV-regularized vacuum energy density in GUP framework. We consider the effects of this vacuum energy density on the expansion rate of the universe and we find some constraints on the GUP deformation parameter $\alpha_0$.

We stress that while GUP seems to be a UV correction of the standard uncertainty principle, but as we will show, it is necessary even at the late time for the renormalizability of the scalar field theory in a cosmological setup. In fact, as has been stated in Ref. [8], the existence of even an at present unmeasurably small uncertainty in position (for instance at about the Planck length) could have a drastic effect in field theory by rendering the theory to be ultraviolet finite (see also [11]).

*email: s-jalalzadeh@sbu.ac.ir
†email: m.gorji@umz.ac.ir
‡email: knozari@umz.ac.ir
2 Natural Cutoff

QFT predicts a divergent vacuum energy for quantum fields. The common way to resolve this problem is adopting a UV cutoff and renormalizing vacuum energy to the observed value. On the other hand, UV cutoff should be determined with quantum gravity theories. We will see that GUP naturally induces a UV cutoff in QFT.

Taking the vacuum expectation value of the energy-momentum tensor associated to the real massless scalar field $\phi(x)$, one finds the well known contribution of the field to the vacuum energy density,

$$\rho = \frac{1}{4\pi^2} \int_0^\infty dk \, k^3,$$

where $k$ is the wave vector and $k = |k|$. Putting a cutoff $\Lambda_c$, the vacuum energy density diverges quartically with cutoff $\rho \sim \Lambda_c^4$. Hence, one has to put a finite cutoff to get a finite vacuum energy density in QFT.

On the other hand, the modified uncertainty relation \( [x, p] = i (1 - \frac{\alpha_0}{M_{Pl}} p + \frac{2\alpha_0^2}{M_{Pl}^2} p^2) \delta_{ij} \), \( [x, x] = i \frac{\alpha_0}{M_{Pl}} \left( \frac{4\alpha_0}{M_{Pl}} - \frac{1}{p} \right) (p_i x_j - p_j x_i) \), \( [p_i, p_j] = 0 \).

The deformed density of states which is consistent with the deformed commutation relations \( [x, p] \) is obtained in appendix A as

$$\frac{1}{(2\pi)^D} \int_{-\infty}^{+\infty} d^D x \, d^D p \rightarrow \frac{1}{(2\pi)^D} \int_{\frac{M_{Pl}}{\alpha_0}}^{+\infty} d^D x \, d^D p \left( 1 - \frac{\alpha_0}{M_{Pl}} p + \frac{2\alpha_0^2}{M_{Pl}^2} p^2 \right)^{-D},$$

where $D$ is the number of degrees of freedom. In appendix A, we show that the deformed density of states \( \rho_{bare}(\alpha_0) \) is invariant under the time evolution and consequently the Liouville theorem is satisfied in the GUP framework. Neglecting the linear term $-\frac{\alpha_0}{M_{Pl}} p$ and identifying deformation parameters as $\beta = 2\alpha_0^2$, relation \( \rho_{bare}(\alpha_0) \) coincide with the result obtained in Ref. [9] which supports only the existence of minimal length, not the maximal momentum. Also, neglecting the quadratic term $\frac{2\alpha_0^2}{M_{Pl}^2} p^2$, relation \( \rho_{bare}(\alpha_0) \) is in agreement with the result obtained in Ref. [10] where the author calculated the density of states to first order of $\alpha_0$. The deformed state density \( \rho_{bare}(\alpha_0) \) gives the vacuum energy density of the real massless scalar field $\phi(x)$ in GUP framework as

$$\rho_{bare}(\alpha_0) = \frac{1}{4\pi^2} \int_0^{\frac{M_{Pl}}{\alpha_0}} dk \, k^3 \left( 1 - \frac{\alpha_0}{M_{Pl}} k + \frac{2\alpha_0^2}{M_{Pl}^2} k^2 \right)^{-3} = \frac{\zeta}{\alpha_0^3} \frac{M_{Pl}^4}{16\pi^2},$$

where $\zeta = \frac{192\sqrt{7}\arctan(1/\sqrt{7}) - 77}{1672} \simeq 0.077674$, is a numerical constant and $|p| = |k| = k$ in our units. One can recover relation \( \rho \) in the limit of $\alpha_0 \rightarrow 0$. Clearly, there is a maximum value for the wave numbers in GUP framework as $k_{max} = M_{Pl}/2\alpha_0$, and consequently the integral automatically converges. An interesting result of this section is that the vacuum energy density of the scalar field naturally rendered to be finite in the GUP framework. The vacuum energy density \( \rho_{bare}(\alpha_0) \) is the bare quantity and can be renormalized to the observed value by standard renormalization methods. But in this section, our aim was only to show that this is a finite quantity in the GUP framework.

3 Vacuum Energy in FRW Spacetime

We consider the zero-point quantum fluctuation of a real massless scalar field $\phi(x)$ in the spatially flat FRW spacetime. The mode expansion of the field is

$$\phi(x) = \int \frac{d^3k_c}{(2\pi)^3 \sqrt{2k_c}} \left( a_k \phi_k(t) e^{ik_c \cdot x} + a_k^\dagger \phi_k^*(t) e^{-ik_c \cdot x} \right),$$

where $k_c$ is the wave vector in the GUP framework. The vacuum energy density (7) is the bare quantity and can be renormalized to the observed result of this section is that the vacuum energy density of the scalar field naturally rendered to be finite in the GUP framework.
where \( k_c = k a(t) \) is the comoving momentum, \( a(t) \) is the scale factor which is the solution of the Friedmann equation and \( \phi_k(t) \) is determined by the Klein-Gordon equation in FRW spacetime,

\[
\phi_k'' + 2\frac{a'}{a} \phi_k' + k_c^2 \phi_k = 0,
\]

where a prime denotes derivative with respect to the conformal time, \( \eta = \int dt/a(t) \). The energy-momentum tensor for a minimally coupled real massless scalar field is \( T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi \), where \( g_{\mu\nu} = (-1, a^2 \delta_{ij}) \) is the metric of the spatially flat FRW spacetime. Taking the vacuum expectation value of the energy-momentum tensor, one finds the contribution of the scalar field to the vacuum energy density and pressure \(\rho, p\),

\[
\rho = \frac{1}{8\pi^2} \int dk k \left( |\phi_k|^2 + \frac{k_c^2}{a^2} |\phi_k|^2 \right),
\]

\[
p = \frac{1}{8\pi^2} \int dk k \left( |\phi_k|^2 - \frac{k_c^2}{3a^2} |\phi_k|^2 \right).
\]

We note that when the scalar field propagates, the background is assumed to be fixed. In other words, the field has no effect on the matter, radiation or the cosmological constant in each epoch, but the scale factor in each case is different. So it is necessary to see how the different epochs with different scale factors affect the propagation of the field. Therefore, we have to compute the vacuum expectation values (Eqs. 10 and 11) in all of the three mentioned cases separately.

### 3.1 Radiation domination era (RD)

The positive frequency solution for the Klein-Gordon equation (9) during RD era is \( \phi_k = \frac{1}{a(\eta)} e^{-ik_c \eta} \). Plugging this solution into (10), gives

\[
\rho = \frac{1}{4\pi^2} \int_0^{\infty} dk k \left( k^2 + \frac{H^2}{2} \right),
\]

where \( H \) is the Hubble parameter in RD era. The first term in the right hand side of the above relation is nothing but the flat space contribution which we have obtained perviously in (2), and the second term comes from the curvature of the spacetime.

The energy has no well defined definition in general relativity, but there is a standard definition for an asymptotically flat spacetime, the so-called ADM energy [15]. The ADM definition of the energy associated to the spacetime with metric \( g_{\mu\nu} \) is \( E = \mathcal{H}(g_{\mu\nu}) - \mathcal{H}(g_{\mu\nu}) \), where \( \mathcal{H}(g_{\mu\nu}) \) is the Hamiltonian of the asymptotically flat spacetime and \( \eta_{\mu\nu} \) is the metric of the flat spacetime. Therefore, the main idea of the ADM proposal is that, the energy associated to the flat spacetime, doesn’t gravitate. Inspired by ADM prescription, one can discard the flat space contribution from the vacuum energy density [12] [16]

\[
\rho_{\text{bare}} = \frac{H^2}{8\pi^2} \int_0^{\infty} dk k.
\]

The vacuum energy has its origin only in the curvature of spacetime, but still it is a divergent quantity. Putting a cutoff \( \Lambda_c \), vacuum energy density diverges quadratically with cutoff \( \rho_{\text{bare}} \sim H^2 \Lambda_c^2 \). Using the deformed density of states [6], the vacuum energy density in GUP framework becomes

\[
\rho_{\text{bare}}(\alpha_0) = \frac{H^2}{8\pi^2} \int_0^{\frac{M_{Pl}}{\alpha_0}} dk \left( 1 - \frac{\alpha_0}{M_{Pl}} k + \frac{2\alpha_0^2}{M_{Pl}^2} k^2 \right)^{-3} = \frac{\sigma}{\alpha_0^2} \frac{H^2 M_{Pl}^2}{8\pi^2},
\]

where \( \sigma = \frac{96\sqrt{7}}{\pi \arctan(1/\sqrt{7})} + 13.3 \approx 0.16 \) is a numerical constant. Of course, the energy density [14] is the bare quantity and can be renormalized to observed value by adding counterterm. But the natural value of the energy density due to the zero-point fluctuation is given by

\[
\rho_Z(\alpha_0) = \epsilon \frac{\sigma}{\alpha_0^2} \frac{H^2 M_{Pl}^2}{8\pi^2},
\]

where \( \epsilon \) is the renormalization numerical factor of the order of unity. Note that there is no a priori reason for positivity of the vacuum energy [16]. Nevertheless, here we assume it to be positive definite. Plugging \( \phi_k = \frac{1}{a(\eta)} e^{-ik_c \eta} \) into relation (11) gives the pressure

\[
p = \frac{1}{4\pi^2} \int_0^{\infty} dk k \left( \frac{k^2}{3} + \frac{H^2}{2} \right),
\]

where \( \alpha \) denotes derivative with respect to the conformal time, \( \eta = \int dt/a(t) \). The energy-momentum tensor for a minimally coupled real massless scalar field is \( T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi \), where \( g_{\mu\nu} = (-1, a^2 \delta_{ij}) \) is the metric of the spatially flat FRW spacetime. Taking the vacuum expectation value of the energy-momentum tensor, one finds the contribution of the scalar field to the vacuum energy density and pressure \(\rho, p\),

\[
\rho = \frac{1}{8\pi^2} \int dk k \left( |\phi_k|^2 + \frac{k_c^2}{a^2} |\phi_k|^2 \right),
\]

\[
p = \frac{1}{8\pi^2} \int dk k \left( |\phi_k|^2 - \frac{k_c^2}{3a^2} |\phi_k|^2 \right).
\]
the first term is the flat space result and the second term is the correction due to the curvature of spacetime. Discarding flat space contribution and using the deformed density of states [6], the pressure becomes

\[ p_{\text{bare}}(\alpha_0) = \frac{H^2}{8\pi^2} \int_0^{\frac{M_{Pl}}{2\pi}} dk \left( 1 - \frac{\alpha_0}{M_{Pl}} k + \frac{2\alpha^2}{M_{Pl}^2} k^2 \right)^{-3} = \frac{\sigma}{\alpha_0} \frac{H^2 M_{Pl}^2}{8\pi^2}. \]  

(17)

The bare vacuum energy density and pressure satisfy \( p_{\text{bare}}(\alpha_0) = \rho_{\text{bare}}(\alpha_0) \), but the counterterm can be chosen so that the renormalized vacuum energy density and pressure satisfy \( p_z = -\rho_z \), as usually one assumes [10]. An important issue should be explained at this stage: we set the equation of state to be

\[ p = -\rho \]

in relation (11) and subtracting flat space term, gives

\[ p_{\text{bare}} = \frac{1}{4\pi^2} \int_0^{\infty} dk \left( \frac{H^2}{3} + \frac{9H^4}{32k^2} \right). \]

(18)

Discarding flat space result, the vacuum energy density in GUP framework can be obtained through the deformed density of states [6]

\[ \rho_{\text{bare}}(\alpha_0) = \frac{H^2}{8\pi^2} \int_0^{\frac{M_{Pl}}{2\pi}} dk \left( 1 - \frac{\alpha_0}{M_{Pl}} k + \frac{2\alpha^2}{M_{Pl}^2} k^2 \right)^{-3} = \frac{\sigma}{\alpha_0} \frac{H^2 M_{Pl}^2}{8\pi^2}, \]

(19)

where, again \( \sigma \approx 0.16 \). Substituting solution \( \phi_k(\eta) = \frac{1}{\alpha} \left( 1 - \frac{i}{k_{c,\eta}} \right) e^{-ik_{c,\eta} \eta} \) in relation [11] and subtracting flat space term, gives

\[ p_{\text{bare}} = \frac{1}{4\pi^2} \int_0^{\infty} dk \left( \frac{H^2}{3} + \frac{9H^4}{32k^2} \right). \]

(20)

The first term diverges quadratically with UV cutoff \( H^2 \Lambda^2 \), but the second term diverges logarithmically with UV cutoff and also requires an IR cutoff. Putting IR cutoff \( H \), the second term produces a term of order \( H^4 \ln \Lambda_c / H \) which is negligible in the late time [10]. Using the deformed state density [6], the pressure becomes

\[ p_{\text{bare}}(\alpha_0) = \frac{H^2}{12\pi^2} \int_0^{\frac{M_{Pl}}{2\pi}} dk \left( 1 - \frac{\alpha_0}{M_{Pl}} k + \frac{2\alpha^2}{M_{Pl}^2} k^2 \right)^{-3} = \frac{\sigma}{\alpha_0} \frac{H^2 M_{Pl}^2}{12\pi^2}. \]

(21)

In particular one can choose the renormalized vacuum energy density and pressure so that \( p_z = -\rho_z \).

3.2 Matter domination era (MD)

The positive frequency solution of the equation (9) during MD era will be \( \phi_k(\eta) = \frac{1}{a} \left( 1 - \frac{i}{k_{c,\eta}} \right) e^{-ik_{c,\eta} \eta} \). Inserting this solution into the [11], gives the vacuum energy density as

\[ \rho \approx \frac{1}{4\pi^2} \int_0^{\infty} dk \left( k^2 + \frac{H^2}{2} \right). \]

(22)

As pervious sections, substituting positive frequency solution of the Klein-Gordon equation (9) in de Sitter space \( \phi_k(\eta) = \frac{1}{a} \left( 1 - \frac{i}{k_{c,\eta}} \right) e^{-ik_{c,\eta} \eta} \) into [10], gives the vacuum energy as

\[ \rho = \frac{1}{4\pi^2} \int_0^{\infty} dk \left( k^2 + \frac{H^2}{2} \right), \]

(23)

where \( H \) is the constant Hubble parameter in de Sitter space. The first term in the right hand side of the above equation is the flat space contribution which can be eliminated by the ADM prescription. Using the deformed density of states [6], the bare vacuum energy density in GUP framework becomes

\[ \rho_{\text{bare}}(\alpha_0) = \frac{H^2}{8\pi^2} \int_0^{\frac{M_{Pl}}{2\pi}} dk \left( 1 - \frac{\alpha_0}{M_{Pl}} k + \frac{2\alpha^2}{M_{Pl}^2} k^2 \right)^{-3} = \frac{\sigma}{\alpha_0} \frac{H^2 M_{Pl}^2}{8\pi^2}, \]

(23)
where \( \sigma = \frac{96 \sqrt{7} \arctan(1/\sqrt{7}) + 133}{13 \sqrt{2}} \approx 0.16 \) is a numerical constant. Plugging \( \phi_k(\eta) = \frac{1}{a} (1 - \frac{i}{\kappa \eta}) e^{-i \kappa \eta} \) into the relation (11), gives the pressure as follows

\[
p = \frac{1}{4 \pi^2} \int_0^\infty dk \frac{k^2}{3 - H^2}.
\]

the first term is the flat space result and the second term is the correction due to the curvature of spacetime. Discarding the flat space contribution and using the deformed density of states (6), the pressure becomes

\[
p_{\text{bare}}(a_0) = -\frac{H^2}{24 \pi^2} \int_0^{M_{\text{Pl}}} dk \left( 1 - \frac{a_0}{M_{\text{Pl}}} k + \frac{2a_0^2 - M_{\text{Pl}}^2}{M_{\text{Pl}}^2} k^2 \right)^{-3} = -\frac{\sigma H^2 M_{\text{Pl}}^2}{24 \pi^2}.
\]

The bare vacuum energy density and pressure satisfy \( p_{\text{bare}}(a_0) = -1/3 \rho_{\text{bare}}(a_0) \), but again the counterterm can be chosen so that the renormalized vacuum energy density and pressure satisfy the relation \( p = -\rho \) (see for instance [17] and [18]).

4 Dark Energy

The main outcome of the previous sections is that quantum fluctuations of a real massless scalar field in FRW spacetime lead to the natural dynamical vacuum energy

\[
\rho_z = \epsilon \frac{\sigma}{a_0^3} \frac{H^2(t) M_{\text{Pl}}^2}{8 \pi^2}.
\]

Note that in some sense this relation is similar to the result obtained in the context of the Holographic dark energy model [19] [20]. Nevertheless, since we are going to include an explicit interaction between the vacuum energy and other sources of the energy-momentum, a deviation from the pure Holographic setup occurs in our case. Once again, in this interacting scenario to preserve the general covariance we set the equation of state for matter contribution \( \rho \).

One can define cosmological parameter \( \Omega_z \) as

\[
\Omega_z = \frac{\rho_z}{\rho_c} = \epsilon \frac{\sigma}{3 \pi a_0^3},
\]

where \( \rho_c = \frac{H^2}{8 \pi G} \) is the critical energy density and we have used \( M_{\text{Pl}}^2 = G^{-1} \). One is tempted to identify this vacuum energy as the dark energy responsible for the late time cosmic speed-up. But this is not actually the case since time dependence of the dark energy and cold dark matter (CDM) cannot be the same from observational grounds (see [16] for more details). In which follows we model a scenario that contains a constant contribution of \( \rho_{\Lambda} \) with unknown origin, a time-dependent dark energy contribution \( \rho_z \) interacting with dark matter contribution \( \rho_m \). This model is usually dubbed as \( \Lambda \)CDM after [16]. Note that the time dependence of the dark energy comes just from \( \rho_z \). With \( p_z = -\rho_z, \rho_{\Lambda} = -\rho_{\Lambda} = -\frac{\Lambda}{8 \pi G} \) and the contribution of the dark energy as \( \rho_{\text{DE}} = \rho_z + \rho_{\Lambda} \), we have

\[
H^2(t) = \frac{8 \pi G}{3} (\rho_m + \rho_z + \rho_{\Lambda}),
\]

\[
\dot{H} + H^2 = -\frac{4 \pi G}{3} \rho_m + \frac{8 \pi G}{3} (\rho_z + \rho_{\Lambda}),
\]

and the Bianchi identities give the conservation equations as

\[
\dot{\rho}_m + \dot{\rho}_z + 3H \rho_m = 0, \quad \dot{\rho}_{\Lambda} = 0.
\]

Using Eq. (27) and combining equations (28) and (29) one finds

\[
\dot{H} = -\frac{3(1 - \Omega_z)}{2} H^2 + \frac{\Lambda}{2}.
\]

Integrating the above relation gives the Hubble parameter as

\[
H = \left( \frac{\Lambda}{3(1 - \Omega_z)} \right)^{\frac{1}{2}} \frac{1 + e^{-\sqrt{3(1 - \Omega_z)} \Lambda t}}{1 - e^{-\sqrt{3(1 - \Omega_z)} \Lambda t}}.
\]
Table 1: Bounds on $\Omega_z$ from different sources of cosmological observations and corresponding values for the GUP numerical factor $\alpha_0$. In all of these results, we suppose $\epsilon \sim \mathcal{O}(1)$.

Note that at the late time the Hubble parameter tends to the de Sitter constant value $H(t \to \infty) \simeq \sqrt{\frac{\Lambda_{eff}}{3}}$, where $\Lambda_{eff} = \Lambda - \Omega_z$. The scale factor at the late time is $a(t) \propto e^{\sqrt{\frac{\Lambda_{eff}}{3}}t}$.

Using the relations (31) and (32), we obtain the deceleration parameter $q$ as

$$q = -1 - \frac{\dot{H}}{H^2} = -1 + \frac{6(1 - \Omega_z)e^{-\sqrt{3(1-\Omega_z)}\Lambda t}}{1 + e^{-\sqrt{3(1-\Omega_z)}\Lambda t}}$$

(33)

Again, at the late time we attain an accelerating phase of expansion. The deceleration parameter should be positive for a universe dominated by matter and therefore

$$\lim_{t \to 0} q \simeq \frac{2 - 6\Omega_z}{4} > 0 \rightarrow \Omega_z < \frac{1}{3},$$

(34)

which imposes a lower bound on the GUP numerical factor $\alpha_0$ as $\alpha_0 > 0.2283$, where we have set $\epsilon \sim \mathcal{O}(1)$. This is a strong constraint on the GUP parameter and means that quantum gravity is inevitable even at large scales and at late time! From another perspective, while deviation from the standard prescription is so small, this result shows that modification of the standard Heisenberg uncertainty principle is also inevitable.

Integrating the relation (32) gives the scale factor of the model

$$a(t) = c_1 e^{-\sqrt{\frac{\Lambda_{eff}}{3}}t} \left[ 2 \left( 1 - e^{\sqrt{3(1-\Omega_z)}\Lambda t} \right) \right]^{\frac{2}{3(1-\Omega_z)}}$$

(35)

where $c_1$ is the constant of integration. Using the relation (27) in (28) one can find $\rho_z = \frac{\Omega_z}{1-\Omega_z} (\rho_M + \rho_\Lambda)$ which leads to the relation

$$\dot{\rho}_z = \frac{\Omega_z}{1-\Omega_z} \dot{\rho}_M,$$

(36)

plugging this relation into (30) and integrating gives

$$\rho_M(z) = \rho_M(0) (1 + z)^{3(1-\Omega_z)},$$

(37)

and for the vacuum energy density gives

$$\rho_\Lambda(z) = \rho_\Lambda(0) (1 + z)^{3(1-\Omega_z)},$$

(38)

$$\rho_{DE}(z) = \rho_z + \rho_\Lambda = \frac{\Omega_z}{1-\Omega_z} \left( \rho_M(0) (1 + z)^{3(1-\Omega_z)} + \rho_\Lambda \right).$$

(39)

In table 1, we have shown the bounds on $\Omega_z$ from different observational probes and we obtained the corresponding values for the GUP numerical factor $\alpha_0$. We note that the values of $\Omega_z$ used in this table are from Ref. [16] and we have supposed $\epsilon \sim \mathcal{O}(1)$. It is important to note also that the combined data set CMB+BAO+SNIa best fit for $\alpha_0$ gives a result of order of unity that is far more stronger than the bounds obtained in [6] and [7].
5 Conclusions

Quantum gravity proposal provides some corrections to the standard uncertainty principle which is called the Generalized Uncertainty Principle. There is a free parameter $\alpha_0$ in this theory which determines the fundamental length of quantum gravity $\alpha_0 l_P$. It is widely believed that Planck length is the fundamental length and consequently $\alpha_0$ is of the order of unity $\alpha_0 \sim 1$. In fact, $\alpha_0$ should be fixed via experiments. Recently, some upper bounds for the $\alpha_0$ has been obtained in some quantum mechanical phenomena. In this paper, we proposed a possible relation between $\alpha_0$ and cosmological observations. We have studied quantum fluctuations of a real massless scalar field in the spatially flat FRW spacetime within the GUP framework. We have shown that the vacuum energy density of the field naturally gets finite value in this framework. The constraint on GUP numerical parameter obtained in this paper is very tighter than those obtained in Refs. [6, 7]. The lower bound for $\alpha_0$ shows that a very small deviation from uncertainty principle is necessary on cosmological grounds.

Acknowledgement

We would like to thanks an anonymous referee for his/her very valuable comments.

A The Deformed Density of States and the Liouville Theorem

In this Appendix, we consider time evolution of the deformed density of states [6] and we show that Liouville theorem is satisfied in the GUP framework. The classical limit of the deformed commutation relations [3], [4] and [5] can be obtained by replacing the operators with their classical counterparts and Dirac commutators with Poisson brackets as $\{ , \} \rightarrow \{ , \}$. In $D$-dimensions, the deformed Poisson algebra is given by [4]

$$\{ x_i, p_j \} = (1 - \alpha p + 2\alpha^2 p^2)\delta_{ij},$$
$$\{ x_i, x_j \} = \alpha(4\alpha - \frac{1}{p}) (p_i x_j - p_j x_i),$$
$$\{ p_i, p_j \} = 0,$$  \hspace{1cm} (A-1)

where we have defined $\alpha = \frac{\alpha_0}{M_P l_P}$. The deformation to the phase space due to the deformed commutation relations (A-1) can be obtained through a general transformation in the corresponding phase space which deforms the phase space volume as [21]

$$\frac{d^D x d^D p}{J},$$  \hspace{1cm} (A-2)

where $J(x, p)$ is the Jacobian of the transformation which can be expressed in terms of the Poisson brackets as [22]

$$J = \frac{1}{2^D D!} \sum_{i_1...i_{2D}=1}^{2D} \varepsilon_{i_1...i_{2D}} \{ J_{i_1}, J_{i_2} \}...\{ J_{i_{2D-1}}, J_{i_{2D}} \},$$  \hspace{1cm} (A-3)

where $\varepsilon$ is the Levi-Civita symbol and $J_i$ denotes the new phase space variables so that for odd $i$ it is a coordinate and for even $i$ it is a conjugate momentum. In the Jacobian (A-3), the coordinate-coordinate Poisson brackets are always multiplied by the momentum-momentum Poisson brackets. So, the non-zero coordinate-coordinate Poisson brackets have no contribution in the Jacobian because the momenta commutes through relation (A-1). Consequently, the Jacobian (A-3) simplifies to [21]

$$J = \prod_{i=1}^{D} \{ x_i, p_i \} = \left( 1 - \alpha p + 2\alpha^2 p^2 \right)^D.$$  \hspace{1cm} (A-4)

Using the above Jacobian in relation (A-2) gives the deformed phase space volume in the GUP framework

$$\frac{d^D x d^D p}{\left( 1 - \alpha p + 2\alpha^2 p^2 \right)^D}.$$  \hspace{1cm} (A-5)

In the next step we consider the time evolution of the deformed phase space volume (A-5).

The classical equations of motion can be represented with Poisson brackets in the Hamiltonian formalism as

$$\dot{x}_i = -\{ x_i, p_j \} \frac{\partial H}{\partial p_j} + \{ x_i, x_j \} \frac{\partial H}{\partial x_j},$$
$$\dot{p}_i = -\{ x_j, p_i \} \frac{\partial H}{\partial x_j},$$  \hspace{1cm} (A-6)
where $\mathcal{H}(x, p)$ is the Hamiltonian of the system. Consider an infinitesimal change for the phase space variables under time evolution

$$\begin{align*}
x'_i &= x_i + \delta x_i, \\
p'_i &= p_i + \delta p_i.
\end{align*}$$

The dynamics of $\delta x_i$ and $\delta p_i$ is given by relations (A-8)

$$\delta x_i = \{x_i, p_j\} \frac{\partial \mathcal{H}}{\partial p_j} \delta t + \{x_i, x_j\} \frac{\partial \mathcal{H}}{\partial x_j} \delta t,$$

$$\delta p_i = -\{x_j, p_i\} \frac{\partial \mathcal{H}}{\partial x_j} \delta t.$$  

An infinitesimal phase space volume evolves through relations (A-7) as

$$d^Dx' d^Dp' = \left| \frac{\partial (x'_i, p'_i)}{\partial (x_i, p_i)} \right| d^Dx d^Dp.$$  

The Jacobian can be obtained by using the relations (A-7) and (A-8). Up to the first order of $\delta t$, we have

$$\left| \frac{\partial (x'_i, p'_i)}{\partial (x_i, p_i)} \right| = 1 + \left( \frac{\partial \mathcal{H}}{\partial x_i} - \frac{\partial \mathcal{H}}{\partial p_i} \right) \frac{\partial \mathcal{H}}{\partial x_j} \delta t$$

$$= 1 - \alpha D (4\alpha - \frac{1}{p}) p_j \frac{\partial \mathcal{H}}{\partial x_j} \delta t.$$  

where we have used the relations (A-1). Substituting the above relation in the relation (A-9) gives

$$d^Dx' d^Dp' = \left( 1 - \alpha D (4\alpha - \frac{1}{p}) p_j \frac{\partial \mathcal{H}}{\partial x_j} \delta t \right) d^Dx d^Dp.$$  

On the other hand, we should consider the time evolution of the term on the denominator of the relation (A-8). Using relations (A-8) and (A-9), to first order of $\delta t$, we have

$$p'' = \sum_i p'_i p'_i = p^2 - 2(1 - \alpha p + 2\alpha^2 p^2) p_j \frac{\partial \mathcal{H}}{\partial x_j} \delta t,$$  

$$p' = \sqrt{p''} = p - (1 - \alpha p + 2\alpha^2 p^2) p_j \frac{\partial \mathcal{H}}{\partial x_j} \delta t,$$  

where we have used the fact that $\delta t$ is small. Using the relations (A-12) and (A-13) we have

$$1 - \alpha p' + 2\alpha^2 p'^2 = (1 - \alpha p + 2\alpha^2 p^2) \left( 1 - \alpha (4\alpha - \frac{1}{p}) p_j \frac{\partial \mathcal{H}}{\partial x_j} \delta t \right),$$  

which gives the result

$$(1 - \alpha p' + 2\alpha^2 p'^2)^D = (1 - \alpha p + 2\alpha^2 p^2)^D \left( 1 - \alpha D (4\alpha - \frac{1}{p}) p_j \frac{\partial \mathcal{H}}{\partial x_j} \delta t \right).$$  

Using the relations (A-11) and (A-15) we find

$$\frac{d^Dx' d^Dp'}{(1 - \alpha p' + 2\alpha^2 p'^2)^D} = \frac{d^Dx d^Dp}{(1 - \alpha p + 2\alpha^2 p^2)^D},$$  

which ensures that the phase space volume (A-5) is invariant under time evolution and consequently the Liouville theorem is satisfied in this setup. Also, it is important to note that the deformed algebra (A-1) induces the maximal momentum (UV cutoff) as $p_{\text{max}} = 1/2\alpha = M_{\nu}/2\alpha_0$. So the range of integrals in the deformed phase space with phase space volume (A-5) should be changed as has been indicated in Ref. [4]. Taking this results into account and using the invariant phase space volume (A-5), one can obtain the deformed density of states [6].
References

[1] D. Amati, M. Ciafaloni and G. Veneziano, Phys. Lett. B 216 (1989) 41
M. Maggiore, Phys. Lett. B 319 (1993) 83
L. Garay, Int. J. Mod. Phys. A 10 (1995) 145
A. Kempf, G. Mangano and R. B. Mann, Phys. Rev. D 52 (1995) 1108.

[2] G. A. Camelia, Int. J. Mod. Phys. D 11 (2000) 35
G. A. Camelia, Nature 418 (2002) 34
J. Magueijo and L. Smolin, Phys. Rev. Lett. 88 (2002) 190403.

[3] A. F. Ali, S. Das and E. C. Vagenas, Phys. Lett. B 678 (2009) 497.

[4] K. Nozari and A. Etemadi, Phys. Rev. D 85 (2012) 104029.

[5] We work in unit $\hbar = c = 1$, where $\hbar$ and $c$ are Planck constant and speed of light in vacuum, respectively.

[6] S. Das and E. C. Vagenas, Phys. Rev. Lett. 101 (2008) 221301.

[7] P. Pedram, K. Nozari and S. H. Taheri, JHEP 1103 (2011) 093.

[8] A. Kempf and G. Mangano, Phys. Rev. D 55 (1997) 7909.

[9] L. N. Chang, D. Minic, N. Okamura and T. Takeuchi, Phys. Rev. D 65 (2002) 125027.

[10] A. F. Ali, Class. Quantum Grav. 28 (2011) 065013.

[11] Y. S. Myung, Phys. Lett. B 679 (2009) 491
S. L. Cherkaas and V. L. Kalashnikov, JCAP 0701 (2007) 028
B. Vakili, Phys. Rev. D 77 (2008) 044023.

[12] L. Parker and S. A. Fulling, Phys. Rev. D 9 (1974) 341
S. A. Fulling and L. Parker, Ann. Phys. (N.Y.) 87 (1974) 176.

[13] In fact, one cannot always define the positive and negative frequency modes for the solutions of the Klein-Gordon equation in curved spacetime. Nevertheless, in FRW spacetime, it is possible in the limit of $\eta \to -\infty$ (see [14]).

[14] S. A. Fulling, Gen. Rel. Grav. 10 (1979) 807.

[15] R. L. Arnowitt, S. Deser and C. W. Misner, Gravitation: An Introduction to Current Research, edited by L. Witten (1962) (Wiley: New York).

[16] M. Maggiore, Phys. Rev. D 83 (2011) 063514.

[17] E. Kh. Akhmedov, arXiv:hep-th/0204048.

[18] T. Padmanabhan, Gen. Rel. Grav. 40 (2008) 529.

[19] M. Li, Phys. Lett. B 603 (2004) 1.

[20] H. Kim, H. W. Lee and Y. S. Myung, Phys. Lett. B 632 (2006) 605.

[21] B. Vakili and M. A. Gorji, J. Stat. Mech. (2012) P10013
A. Bina, S. Jalalzadeh and A. Moslehi, Phys. Rev. D 81 (2010) 023528.

[22] T. Fityo, Phys. Lett. A 372 (2008) 5872.