DEFORMATION OF MATRIX-VALUED ORTHOGONAL POLYNOMIALS RELATED TO GELFAND PAIRS

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Abstract. In this paper we present a method to obtain deformations of families of matrix-valued orthogonal polynomials that are associated to the representation theory of compact Gelfand pairs. These polynomials have the Sturm-Liouville property in the sense that they are simultaneous eigenfunctions of a symmetric second order differential operator and we deform this operator accordingly so that the deformed families also have the Sturm-Liouville property. Our strategy is to deform the system of spherical functions that is related to the matrix-valued orthogonal polynomials and then check that the polynomial structure is respected by the deformation. Crucial in these considerations is the full spherical function $\Psi_0$, which relates the spherical functions to the polynomials. We prove an explicit formula for $\Psi_0$ in terms of Krawtchouk polynomials for the Gelfand pair $(SU(2) \times SU(2), \text{diag}(SU(2)))$. For the matrix-valued orthogonal polynomials associated to this pair, a deformation was already available by different methods and we show that our method gives same results using explicit knowledge of $\Psi_0$.

Furthermore we apply our method to some of the examples of size $2 \times 2$ for more general Gelfand pairs. We prove that the families related to the groups $SU(n)$ are deformations of one another. On the other hand, the families associated to the symplectic groups $Sp(n)$ give rise to a new family with an extra free parameter.

1. Introduction and statement of results

It is well known that the classical orthogonal polynomials are characterized by the property that their derivatives are also orthogonal polynomials, see e.g. [12]. One can exploit this characterization, for instance, to construct the Gegenbauer polynomials $C^{(\nu)}_n(y)$, for integer values of $\nu$, by repeated differentiation of the Chebyshev polynomials $U_n(y) = C^{(1)}_n(y)$. In this way, the orthogonality relations, differential equations and other properties of the Gegenbauer polynomials can be obtained from those of the Chebyshev polynomials. In this paper we deal with the analogous construction for matrix-valued orthogonal polynomials.

For $N \in \mathbb{N}$, let $W : (a, b) \to \mathbb{C}^{N \times N}$ be a positive definite smooth weight matrix with finite moments. We consider the sesqui-linear pairing defined for a pair of matrix polynomials $P, Q \in \mathbb{C}^{N \times N}[y]$ by the matrix

$$\langle P, Q \rangle_W = \int_I P(y)W(y)Q(y)^* \, dy,$$

where $Q(x)^*$ denotes the conjugate transpose of $Q(x)$. We say that $(P_d)_{d \in \mathbb{N}_0}$, $P_d \in \mathbb{C}^{N \times N}[x]$, $d \in \mathbb{N}_0$, is a sequence of matrix-valued orthogonal polynomials (MVOPs from now on) with respect to $W$ if

1. $\langle P_d, P_{d'} \rangle_W = 0$ if $d \neq d'$,
2. $\deg P_d = d$, for all $d \in \mathbb{N}_0$,
3. the leading coefficient of $P_d$ is invertible for all $d \in \mathbb{N}$.

A sequence of monic MVOPs can be obtained by applying the Gram–Schmidt process on the ordered basis $(I, yI, y^2I, \ldots)$, where $I$ denotes the $N \times N$ identity matrix.

We say that the differential operator $D : \mathbb{C}^{N \times N}[y] \to \mathbb{C}^{N \times N}[y]$ is symmetric with respect to the matrix weight $W$ if it satisfies $\langle PD, Q \rangle = \langle P, QD \rangle$ for all matrix polynomials $P, Q$. Any sequence $(P_d)_{d \in \mathbb{N}_0}$ of MVOPs with respect to $W$ is a basis of the space of matrix polynomials, so that the
symmetry condition is equivalent to
\[(P_d D, P_{d'}) = (P_d, P_{d'} D), \quad \text{for all } d, d' \in \mathbb{N}_0.\]  
(1.2)
A pair \((W, D)\) consisting of a matrix weight \(W\) together with a matrix-valued differential operator
\[D = \partial_y^2 F_2 + \partial_y F_1 + F_0,\]
where \(F_i\) is a polynomial of degree at most \(i\), which is symmetric with respect to \(\langle \cdot, \cdot \rangle\) is called a matrix-valued classical pair (MVCP from now on). Given a MVCP \((W, D)\), any sequence of MVOPs with respect to \(W\) is a family of simultaneous eigenfunctions of \(D\), see [11 Proposition 2.10].

**Definition 1.1.** A deformation of a MVCP \((W, D)\) is a family \((W^{(\kappa)}, D^{(\kappa)})_{\kappa \in K}\) of MVCPs, where \(K \subset \mathbb{R}\) is an open interval, so that \((W, D) = (W^{(\kappa_0)}, D^{(\kappa_0)})\) for some \(\kappa_0 \in K\).

We say that a deformation \((W^{(\kappa)}, D^{(\kappa)})\) allows for the shift \(\partial_y\), if for a family of MVOPs \((P^{(\kappa)}_d)_{\kappa \in K}\) of \(W^{(\kappa)}\), we have that \((\partial_y P^{(\kappa)}_d)\) is a family of MVOPs for \(W^{(\kappa+1)}\).

The goal of this paper is to present a method to find deformations that allow for the shift \(\partial_y\) of MVCPs \((W, D)\) that are associated to compact Gelfand pairs of rank one [15, 21]. We also present a Rodrigues type formula for these cases. We apply our method to some examples of size 2 \(\times\) 2 taken from [22], where we observe that some of the families actually fit in a single deformation, that moreover, allows for the shift \(\partial_y\).

An earlier result in this direction is the case studied in [17], where sequences of matrix-valued Chebyshev polynomials [15, 19] are deformed into matrix-analogues of Gegenbauer polynomials. However, those results are based on a decomposition for the weight matrix that is quite particular for the Gelfand pair \((\text{SU}(2) \times \text{SU}(2), \text{diag}(\text{SU}(2)))\). If we apply our method to this case, we obtain the same deformations.

**MVOPs associated to representation theory.** The theory of MVOPs dates back to the work of M.G. Krein in the 1940s. Since then, the theory has been developed and connected to different fields such as scattering theory and spectral analysis, see for instance [11, 5, 6, 7]. In [8] A. Durán raises the question of whether it is possible to construct a sequence of MVOPs together with a matrix valued differential operator which has the MVOPs as simultaneous eigenfunctions. The first results in this direction are given in [9] from the study of matrix-valued spherical functions on \(\text{SU}(3)/\text{U}(2)\). The link between matrix-valued spherical functions and MVOPs has been developed in several papers following [23, 9], see for instance [18, 19, 24]. This led to a uniform construction of MVOPs for compact Gelfand pairs \((G, K)\) of rank one [15, 21, 22]. The families of MVOPs obtained from representation theory have many interesting properties and, in some cases, can be described in great detail. Certain families of MVOP related to the pair \((\text{SU}(n+1), \text{U}(n))\) have been exploited to derive stochastic models, see [8, 10, 13]. The family described in Section 3 leads to models of continuous-time bivariate Markov processes which are analyzed in detail in [14].

In this paper we deal with the families of MVOPs obtained in [15, 21, 22]. We develop a method to deform these families that relies in a specific decomposition of the weight matrix and the differential operator, see (1.7) and (1.4). The same decomposition arises in families of MVOPs for compact Gelfand pairs (G, K) of rank one [15, 21]. We also present in several papers following [20, 9], see for instance [18, 19, 24]. This led to a uniform construction of MVOPs for compact Gelfand pairs \((G, K)\) of rank one [15, 21]. The families of MVOPs obtained from representation theory have many interesting properties and, in some cases, can be described in great detail. Certain families of MVOP related to the pair \((\text{SU}(n+1), \text{U}(n))\) have been exploited to derive stochastic models, see [8, 10, 13]. The family described in Section 3 leads to models of continuous-time bivariate Markov processes which are analyzed in detail in [14].
As a consequence of the Schur orthogonality relations, the full spherical functions \((\Psi_\mu^d)_d\) are pairwise orthogonal. More precisely, in \([15, 21]\) it is proven that

\[
\int_0^1 \Psi_\mu^d(y) T_\alpha^\mu(\Psi_\mu^d(y))^* (1 - y)^\alpha y^\beta \, dy = 0, \quad d \neq d', \tag{1.3}
\]

where \((1 - y)^\alpha y^\beta\) is the ordinary Jacobi weight on the interval \([0,1]\) that is associated to the Riemann symmetric space \(G/K\) and \(T_\alpha^\mu\) is a constant diagonal matrix. The spherical functions are eigenfunctions of the Casimir operator of the group \(G\). This implies that the full spherical function \(\Psi_\mu^d\) is an eigenfunction of a single variable differential operator \(\Omega^\mu\), the radial part of the Casimir operator. In other words there is an operator

\[
\Omega^\mu = y(1 - y)\partial_y^2 + a(y)\partial_y + F^\mu(y), \tag{1.4}
\]

such that

\[
\Psi_\mu^d(y) \Omega^\mu = y(1 - y)(\Psi_\mu^d)'(y) + a(y)(\Psi_\mu^d)'(y) + \Psi_\mu^d(y) F^\mu(y) = \Lambda_\mu^d \Psi_\mu^d(y), \tag{1.5}
\]

where \(\Lambda_\mu^d\) is a constant diagonal matrix, see for instance \([15]\) or \([22]\) Section 3]. Note that the expression \((1.3)\) is available for any irreducible \(K\)-representation, see e.g. \([25]\) Prop. 9.1.2.11]. The full spherical function \(\Psi_\mu^d\) of degree zero is also a solution to the first-order differential equation,

\[
y(1 - y)\partial_y \Psi_\mu^0(y) = (S^\mu + yR^\mu)\Psi_\mu^0(y), \tag{1.6}
\]

for certain constant matrices \(R^\mu\) and \(S^\mu\). \([22]\) Theorem 3.1]. Note that differentiating \((1.6)\) yields a matrix-valued hypergeometric differential equation for \(\Psi_\mu^d\) in the sense of \([23]\). Observe that \((1.6)\) can be seen as a differential operator acting on the left of \(\Psi_\mu^d\) while \((1.5)\) is an operator acting on the right. One of the consequences of \((1.6)\) is that \(\Psi_\mu^0\) is invertible on \((0, 1)\), see \([22]\) Cor. 3.4] so that

\[
P_\mu^d(y) = \Psi_\mu^d(y) (\Psi_\mu^0(y))^{-1},
\]

Now that the link between the spherical functions of type \(\mu\) and the corresponding MVOPs is clear, we allow ourselves to drop the index \(\mu\) from the notation. However, various constants and coefficients that occur later on, do depend on \(\mu\).

The matrix-valued polynomials \(P_d\) satisfy a three-term recurrence relation,

\[
yP_d(y) = A_d P_{d+1}(y) + B_d P_d(y) + C_d P_{d-1}(y),
\]

with \(A_d\) being an invertible diagonal matrix so that the leading coefficient of \(P_d\) is invertible for all \(d \in \mathbb{N}_0\), see \([15]\) Section 1]. Furthermore the orthogonality relations for the full spherical functions \((1.3)\) imply that the polynomials \(P_d\) are orthogonal with respect to the pairing

\[
(P, Q)_W = \int_0^1 P(y)W(y)Q(y)^* \, dy,
\]

where \(W\) is the weight matrix defined by

\[
W(y) = (1 - y)^\alpha y^\beta W_{pol}(y), \quad W_{pol}(y) = \Psi_0(y) T(\Psi_0(y))^*.
\tag{1.7}
\]

The first order differential equation \((1.6)\) implies that the polynomials \(P_d\) are eigenfunctions of the hypergeometric differential operator

\[
D = \Phi_0 \Omega (\Phi_0)^{-1} = y(y - 1)\partial_y^2 + \partial_y(C - yU) - V, \tag{1.8}
\]

where

\[
C = \frac{\lambda_1 m}{2\rho^2 (M - m)} = 2S, \quad U = 2R - \frac{\lambda_1}{\rho^2}, \quad V = -\frac{\Lambda_0}{\rho^2}.
\]

Here \(M, m, p, r\) are constants related with the pair \((G, K)\) and their values are given in \([22]\) Table 2] for the various cases. The differential operator \((1.8)\) is symmetric with respect to \(W\) so that the pair \((W, D)\) is a MVCP. This fact follows from the symmetry of the Casimir operator \(\Omega\) on \(G\).

**Deformation of MVOPs.** We summarize the discussion above as follows: we have two pairs, a pair \((w, \Omega)\) together with a sequence of full spherical functions \((\Psi_d)_d\) and a matrix valued
classical pair \((W,D)\) with a sequence of MVOPs \((P_d)\). These pairs are related by the function 
\[ W(y) = y^\alpha(1-y)^\beta \Psi_0(y) T(\Psi_0(y))^*, \quad D = \Psi_0 \Omega \Psi_0^{-1}, \]
\[ P_d = \Psi_d(\Psi_0)^{-1} \quad \forall d \in \mathbb{N}. \]
We shall refer to the functions, weights, differential operators on the spherical level and on the 

The Jacobi polynomials in a single variable can be given as a family of scalar valued orthogonal 

\[ \Psi \] is a MVCP, if 
\[ T^{(\kappa)}(y) > 0. \]
Remark 1.3. Using D3 we show that several examples of MVCPs that we obtained in [22] admit a deformation that allows for the shift $\partial_x$. Moreover, the deformations of the MVCPs of [18, 19] that we obtain, are the same, up to a constant, as those in [17].

Remark 1.4. Note that $W^{(\kappa)} > 0$ almost everywhere if and only if $T(\kappa) > 0$. The conclusion that $W^{(\kappa)} > 0$ almost everywhere for the case studied in [17], depends on the LDU-decomposition of the weight. The method that we propose has the important advantage that its nature is simpler than the deformation considered in [17], because we go back to the spherical level, where the weight and the differential operator are in some sense much simpler.

In fact, the matrix $T = T^0$ corresponding to $(G_0, K_0, \mu_0)$ is a diagonal matrix whose entries are dimensions of the irreducible $M_0$-modules that occur in the restriction of $\pi^{K_0}_{\mu_0}$ to $M_0$ (where $M_0 \subset K_0$ is a compact subgroup that we obtain after some choices, see [22]). In some of the examples, where the $T^{(k)}$ can be interpreted as the "$T^{(0)}$" for other triples $(G_k, K_k, \mu_k)$, our deformation is really an interpolation of all these dimensions.

The fact that we pass from the level of spherical functions to the level of polynomials, always using the same function $\Psi_0$, is a restriction to the theory. However, this restriction is justified by the examples, where we also have families $(G_k, K_k, \mu_k)_{k \in \mathbb{N}}$ of Gelfand pairs with specified $K_k$-type $\mu_k$, such that $\Psi^{(k)}_0 = \Psi_0^{(\mu_0)}$, for some $k_0 \in \mathbb{N}$.

A second restriction to the theory is given by the shape of the weight matrices on the spherical level, that we insist to be of the form (1.1) with $T^{(\kappa)} > 0$ diagonal. This is justified by the same argument as before, namely that this is what happens in the examples.

Outlook. It would be interesting to investigate the effect of other shift operators on the MVOPs in a single variable. A possible approach is to study the group theoretic interpretation of the shift operators for MVOPs with geometric parameters. This may also give more insight to the result of Cantero, Moral and Velázquez [2] on the level of the spherical functions.

It would also be interesting to determine the generators for the algebra of differential operators that have the spherical functions as eigenfunctions. If the dependence of these generators on the root multiplicities is understood, one should investigate whether the whole algebra may be deformed, instead of just one of the differential operators.

Organization of the paper. In Section 2 we will work out D1, D2 and derive equations on the level of the spherical functions that will in turn yield a deformation of the given MVCP. In view of D3 we explain how it can be verified that the deformation allows for the shift $\partial_y$. For the cases where this holds true we derive a Rodrigues type formula.

In Section 3 we apply our construction to the family of matrix-valued Chebyshev polynomials obtained in [18, 19]. In Section 4 we apply our new method to the examples of $2 \times 2$ matrix-valued orthogonal polynomials obtained in [22]. We see that the families related to the groups $SU(n)$ and $SO(n)$ are maximal in the sense that they are closed under our deformations. On the other hand, the families associated to the symplectic groups $Sp(n)$ give rise to a new families with an extra free parameter.

2. Deformation of MVCPs

In this section we carry out D1-D3 of the introduction. Let $(w, \Omega)$ and $\Psi_0$ be the input data coming from the representation theory of compact Gelfand pairs of rank one, and let $(W, D)$ be the MVCP associated to this data.

Proposition 2.1. Let $Q$ be a polynomial on $\mathbb{R}$ that satisfies
\[ y(1 - y)Q''(y) + Q'(y)(C - yU) - Q(y)V = \Lambda Q(y), \]
where $\Lambda$ is a constant matrix. Then the $k$-th derivative $\partial_y^k Q = Q^{(k)}$ satisfies
\[ y(1 - y)(Q^{(k)})''(y) + (Q^{(k)})'(y)(C + k - y(U + 2k)) - Q^{(k)}(y)(V + kU + k(k - 1)) = \Lambda Q^{(k)}(y). \]

Proof. The verification, which is based on induction over $k$, is left to the reader. □
The following Corollary follows immediately by analytic continuation and it settles D1 of our program.

**Corollary 2.2.** Given a differential operator \( D = y(1 - y) \frac{\partial^2}{\partial y^2} + \partial_y (C - yU) - V \), the differential operator \( D^{(\kappa)} = y(1 - y) \frac{\partial^2}{\partial y^2} + \partial_y (C^{(\kappa)} - yU^{(\kappa)}) - V^{(\kappa)} \) with \( C^{(\kappa)} = C + \kappa \), \( U^{(\kappa)} = U + 2\kappa \) and \( V^{(\kappa)} = V + \kappa U + \kappa(\kappa - 1) \), satisfies \( \partial \circ D^{(\kappa)} = D^{(\kappa + 1)} \circ \partial \) for all \( \kappa \geq 0 \).

The deformation \( (w^{(\kappa)}, \Omega^{(\kappa)}) \) of \( (w, \Omega) \) is of the form
\[
\Omega^{(\kappa)} = y(1 - y) \frac{\partial^2}{\partial y^2} + a^{(\kappa)}(y) \partial_y + F^{(\kappa)}(y), \quad w^{(\kappa)}(y) = y^{\beta^{(\kappa)}(1 - y)\alpha^{(\kappa)}T^{(\kappa)}},
\]
for certain functions \( a^{(\kappa)}(y), F^{(\kappa)}(y) \), a constant matrix \( T^{(\kappa)} \) and constants \( \alpha^{(\kappa)}, \beta^{(\kappa)} \) such that
\[
a^{(0)}(y) = a(y), \quad F^{(0)}(y) = F(y), \quad T^{(0)} = T, \quad \alpha(0) = \alpha, \quad \beta(0) = \beta.
\]
This leads to a deformed pair \( (W^{(\kappa)}, D^{(\kappa)}) = (\Psi_0 w^{(\kappa)} \Psi_0^*, \Psi_0 \Omega^{(\kappa)} \Psi_0^{-1}) \). To see how this deformation translates to the level of spherical functions, where it is easier to check the symmetry conditions, we use the following result.

**Proposition 2.3.** Let \( \Omega^{(\kappa)} \) be given by (2.2). The differential operator \( D^{(\kappa)} = \Psi_0 \Omega^{(\kappa)}(\Psi_0)^{-1} \) is a matrix-valued hypergeometric operator of the form
\[
D^{(\kappa)} = y(1 - y) \frac{\partial^2}{\partial y^2} + \partial_y (C^{(\kappa)} - yU^{(\kappa)}) - V^{(\kappa)},
\]
for some constant matrices \( C^{(\kappa)}, U^{(\kappa)}, V^{(\kappa)} \) if and only if \( a^{(\kappa)} \) and \( F^{(\kappa)} \) are given by
\[
a^{(\kappa)}(y) = \frac{-\Psi_0^{-1} V^{(\kappa)} \Psi_0 - y(1 - y) \Psi_0^{-1} \Psi_0^{-1} a^{(\kappa)} \Psi_0^{-1} \Psi_0}{y}, \quad F^{(\kappa)}(y) = \left( C^{(\kappa)} - yU^{(\kappa)} \right) - 2(S + yR),
\]
where \( S, R \) are the matrices given in (1.6).

**Proof.** By a straightforward computation, it is readily seen that \( \Psi_0^{-1} \Omega^{(\kappa)} \Psi_0 \) is the differential operator
\[
y(1 - y) \frac{\partial^2}{\partial y^2} + \partial_y \left[ 2y(1 - y) \Psi_0 \Psi_0^{-1} + a^{(\kappa)} \right] + \left[ y(1 - y) \Psi_0 \Psi_0^{-1} + a^{(\kappa)} \Psi_0 \Psi_0^{-1} + \Psi_0 F^{(\kappa)} \Psi_0^{-1} \right].
\]
This is a matrix-valued hypergeometric operator if and only if there exist constant matrices \( C^{(\kappa)}, U^{(\kappa)} \) and \( V^{(\kappa)} \) such that
\[
2y(1 - y) \Psi_0 \Psi_0^{-1} + a^{(\kappa)} = C^{(\kappa)} - yU^{(\kappa)}, \quad y(1 - y) \Psi_0 \Psi_0^{-1} + a^{(\kappa)} \Psi_0 \Psi_0^{-1} + \Psi_0 F^{(\kappa)} \Psi_0^{-1} = V^{(\kappa)}.
\]
In the first equation of (2.3) we use (1.6) to obtain
\[
2(S + yR) + a^{(\kappa)}(y) = (C^{(\kappa)} - yU^{(\kappa)}).
\]
Finally the second equation of (2.3) holds if and only if \( F^{(\kappa)} \) is given by (2.2). \( \square \)

As an immediate consequence of our calculations, we obtain the following result, which in particular shows that \( a^{(\kappa)} \) is a scalar function, whenever we deform \( D \) according to Corollary 2.2.

**Corollary 2.4.** The deformed differential operator is of the form
\[
D^{(\kappa)} = y(1 - y) \frac{\partial^2}{\partial y^2} + \partial_y (C^{(\kappa)} - yU^{(\kappa)}) - V^{(\kappa)},
\]
with \( C^{(\kappa)} = C + \kappa, U^{(\kappa)} = U + 2\kappa \) and \( V^{(\kappa)} = V + \kappa U + \kappa(\kappa - 1) \) if and only if
\[
a^{(\kappa)}(y) = a(y) + \kappa(1 - 2y), \quad F^{(\kappa)}(y) = F(y) + \kappa \Psi_0^{-1} (U + \kappa - 1) \Psi_0 - \kappa(1 - 2y) \Psi_0^{-1} \Psi_0.
\]

We proceed to investigate the symmetry relations for the deformed differential operator on the level of the spherical functions.

**Proposition 2.5.** Assume that the differential operator \( D^{(\kappa)} = \Psi_0 \Omega^{(\kappa)}(\Psi_0)^{-1} \) is of the form of Corollary 2.4, then \( D^{(\kappa)} \) is symmetric with respect to \( W^{(\kappa)} \) if and only if
\[
(y(1 - y) w^{(\kappa)}(y))^\dagger = a^{(\kappa)}(y) w^{(\kappa)}(y), \quad T^{(\kappa)} (F^{(\kappa)}(y))^\dagger = F^{(\kappa)}(y) T^{(\kappa)}.
\]
Proof. Let \((P_d)\) be a sequence of MVOPs with respect to \(W^{(\kappa)}\) and let \(\Psi^{(\kappa)} = P_d \Psi_0\). The operator \(D^{(\kappa)}\) is symmetric with respect to \(W^{(\kappa)}\) if and only if
\[
\int_0^1 (\Psi_d^{(\kappa)} \Omega^{(\kappa)}(y) T^{(\kappa)}(\Psi_d^{(\kappa)})^\ast (1-y)^{\alpha^{(\kappa)}} y^{\beta^{(\kappa)}}) dy = \int_0^1 (P_d D^{(\kappa)})(y) W^{(\kappa)}(y) (P_d)^\ast dy
\]
\[
= \int_0^1 P_d(y) W^{(\kappa)}(y) (P_d D^{(\kappa)})^\ast dy
\]
\[
= \int_0^1 \Psi_d^{(\kappa)}(y) T^{(\kappa)}(\Psi_d^{(\kappa)} \Omega^{(\kappa)})^\ast (1-y)^{\alpha^{(\kappa)}} y^{\beta^{(\kappa)}}) dy,
\]
where we used the symmetry condition \(1.2\) in the second and third equation. In other words, \(D^{(\kappa)}\) is symmetric with respect to \(W^{(\kappa)}\) if and only if
\[
\int_0^1 (\Psi_d^{(\kappa)} \Omega^{(\kappa)}(y) T^{(\kappa)}(\Psi_d^{(\kappa)})^\ast (1-y)^{\alpha^{(\kappa)}} y^{\beta^{(\kappa)}}) dy = 0.
\]

It follows from integration by parts that \((2.5)\) holds true for all \(D\). If \(D\) then the condition on the right of \((2.4)\) is equivalent to
\[
\int_0^1 \Psi_d^{(\kappa)}(y) T^{(\kappa)}(\Psi_d^{(\kappa)} \Omega^{(\kappa)})^\ast (1-y)^{\alpha^{(\kappa)}} y^{\beta^{(\kappa)}}) dy = 0.
\]

This completes the proof of the proposition.

\[\square\]

Remark 2.6. If \(F^{(\kappa)}\) is a tridiagonal matrix and \(F^{(\kappa)}_{i,i+1}/F^{(\kappa)}_{i+1,i}\) is constant for all \(i = 0, \ldots, N\), then the condition on the right of \((2.4)\) is equivalent to
\[
T^{(\kappa)} = \frac{F^{(\kappa)}_{i+1,i}}{F^{(\kappa)}_{i,i+1}} T^{(\kappa)}
\]
so that \(T^{(\kappa)}\) is determined up to a constant factor.

Note that \((2.4)\) for \(\kappa = 0\) implies \(\alpha(y) = 1 + \beta - (2 + \beta - \alpha)y\), so by \((2.4)\) we have \(\alpha^{(\kappa)}(y) = 1 + \beta + \kappa - (2 + 2\kappa + \beta - \alpha)y\). Again by \((2.4)\) we find \(\alpha^{(\kappa)} = \alpha + \kappa\) and \(\beta^{(\kappa)} = \beta + \kappa\).

We have now settled step \(D2\) of our program. The question remains, whether the deformation \((W^{(\kappa)}, D^{(\kappa)})\) allows for the shift \(\partial_y\). In other words, we need to determine whether the sequence of derivatives \((\partial_y P^{(\kappa)}_n)\) is orthogonal with respect to \(W^{(\kappa+1)}\). In [2] Cantero, Moral, Velázquez characterized the sequences of MVOPs whose derivatives are also orthogonal. If there exist matrix polynomials \(\Gamma^{(\kappa)}_2\) and \(\Gamma^{(\kappa)}_1\) of degree two and one respectively such that
\[
(W^{(\kappa)}(y) \Gamma^{(\kappa)}_2(y))^\prime = W^{(\kappa)}(y) \Gamma^{(\kappa)}_1(y),
\]
then the sequence of derivatives \((\partial_y P^{(\kappa)}_n)\) is orthogonal with respect to \(W^{(\kappa)}(y) \Gamma^{(\kappa)}_2\). In our case, using the expression \((1.9)\) for the weight matrices, we have that if
\[
\Gamma^{(\kappa)}_2(y) = y(1-y)(\Psi_0(y))^\prime (T^{(\kappa)})^{-1} T^{(\kappa+1)} \Psi_0(y)^\ast,
\]
\[
\Gamma^{(\kappa)}_1(y) = (W^{(\kappa)}(y))^{-1} (W^{(\kappa+1)}(y))^\prime
\]
are polynomials of degree two and one respectively, then sequence \((\partial_y P^{(\kappa)}_n)\) is orthogonal with respect to \(W^{(\kappa+1)} = W^{(\kappa)} \Gamma^{(\kappa)}_2\). We can rewrite \((2.7)\) in terms of differential operators, as stated in the following proposition which summarizes this discussion.

Proposition 2.7. The following are equivalent.

1. The deformed pair \((W^{(\kappa)}, D^{(\kappa)})\) allows for the shift \(\partial_y\).
2. \(\Gamma^{(\kappa)}_2\) is a matrix polynomial of degree two and \(\Gamma^{(\kappa)}_1\) is a matrix polynomial of degree one.
3. The differential operator
\[
D^{(\kappa)}(\Gamma_2, \Gamma_1) = \frac{d^2}{dy^2} \Gamma^{(\kappa)}_2(y)^\ast + \frac{d}{dy} \Gamma^{(\kappa)}_1(y)^\ast
\]
is symmetric with respect to \(W^{(\kappa)}\) and has the polynomials \(P^{(\kappa)}_d\) as eigenfunctions.
Then

Theorem 2.9. Let

It follows from (2.10) that

Proof. This is analogous to [17, Corollary 2.5].

Corollary 2.10. The following integral formula holds,

where \( \gamma \) is a closed contour around \( y \).

Proof. The proof follows by applying Cauchy’s integral formula to (2.10).

Corollary 2.11. The following relation holds for the monic orthogonal polynomials \( Q_n \)

\[
G_n^{(\kappa)}(G_n^{(\kappa + 1)})^{-1} = (\partial_y Q_n^{(\kappa + 1)})(\Gamma_2^{(\kappa)})^* + Q_n^{(\kappa + 1)}(\Gamma_1^{(\kappa)})^*.
\] (2.11)

Proof. It follows from (2.10) that

\[
Q_n^{(\kappa + 1)}(y)W^{(\kappa + 1)}(y) = G_n^{(\kappa)}(\partial_y W^{(\kappa + n + 1)}(y)).
\]
If we take the derivative with respect to $y$ on both sides of this equation we obtain
\[ \partial_y (Q^{(k+1)}(y)W^{(k+1)}) = G_n^{(k)}(\partial_y^{k+1} W^{(k+1)}(y)) = G_n^{(k)}(G_n^{(k)}(\partial_y W^{(k+1)})). \] (2.12)

On the other hand we have
\[ \partial_y (Q_n^{(k+1)}W^{(k+1)}) = (\partial_y Q_n^{(k+1)})W^{(k+1)} + Q_n^{(k+1)}(\partial_y W^{(k+1)}). \] (2.13)

By combining (2.12) and (2.13) and multiplying by $(W^{(k)})^{-1}$ on the right, we obtain
\[ G_n^{(k)}(G_n^{(k)})^{-1} Q_n^{(k+1)} = (\partial_y Q_n^{(k+1)})W^{(k+1)}(W^{(k)})^{-1} + Q_n^{(k+1)}(\partial_y W^{(k+1)})(W^{(k)})^{-1}. \]

Using (2.8) and (2.9) gives the result. \( \square \)

Corollary 2.12. The matrices $\Gamma_2^{(k)}$ and $\Gamma_1^{(k)}$ can be written in terms of the monic polynomials $P_n^{(k)}$ and the coefficients $G_n^{(k)}$ in the following way
\[ (\Gamma_1^{(k)})^* = G_0^{(k)}(G_1^{(k)})^{-1} Q_1^{(k)}, \]
\[ (\Gamma_2^{(k)})^* = G_1^{(k)}(G_2^{(k)})^{-1} Q_2^{(k)} - G_0^{(k)}(G_1^{(k)})^{-1}(\Gamma_1^{(k)})^* (\Gamma_1^{(k)})^*. \]

Proof. The corollary follows by evaluating (2.11) in $n = 0$ and $n = 1$. \( \square \)

3. Matrix-valued Gegenbauer polynomials

The goal of this section is to deform the family of matrix-valued Chebyshev polynomials obtained in [18, 19] from the study of spherical functions associated to the pair $(G, K) = (SU(2) \times SU(2), \text{diag}(SU(2)))$. We will show that our construction is an alternative to the one given in [17] and provides a different factorization for the weight matrix. The key factorization in [17] is the LDU decomposition of the weight matrix, which plays a fundamental role, for instance to show that the weight matrix is positive definite, see [17, Corollary 2.5]. In our case, this is a direct consequence of the decomposition (1.11) of the deformed weight, as we noticed in Remark 1.3.

In this case, we have all the ingredients to perform the deformation explicitly. For $\ell \in \frac{1}{2} \mathbb{N}$, the full spherical functions $\Phi_d : [0, 4\pi] \to \mathbb{C}^{(2\ell+1) \times (2\ell+1)}$ were introduced in [18, Definition 2.2, Theorem 2.1 and (2.5)]. In particular, the full spherical function of degree zero is given by
\[ (\Phi_0(t))_{n,m} = \sum_{j_1=-\frac{\ell-n}{2}}^{\frac{\ell-n}{2}} \sum_{j_2=-\frac{\ell-n}{2}}^{\frac{\ell-n}{2}} \delta_{-\ell+m,j_1+j_2} \left( \begin{array}{c} n \\ j_1 + \frac{\ell-n}{2} \end{array} \right) \left( \begin{array}{c} 2\ell - n \\ j_2 + \frac{2\ell-n}{2} \end{array} \right) \left( \begin{array}{c} 2\ell \\ 2\ell - m \end{array} \right)^{-1} e^{ij_1j_2 t}. \] (3.1)

If we denote $\Psi_d(y) = \Phi_d(\arccos(1 - 2y))$, the full spherical polynomial of degree $d$ is given by
\[ P_d(y) = \Psi_d(y)(\Psi_0(y))^{-1}, \quad y \in [0, 1]. \]

For $\ell = 0$, it boils down to a $1 \times 1$ matrix and it is a multiple of the Chebyshev polynomial of degree $d$. The polynomials $(P_d)_{d}$ form a sequence of matrix-valued orthogonal polynomials with non-singular leading coefficients, see [18, Proposition 4.6], with respect to the sesqui-linear pairing
\[ \langle P,Q \rangle = \int_0^1 P(y)W(y)Q(y)^*dy, \quad W(y) = y^{1/2}(1-y)^{1/2}\Psi_0(y)\Psi_0(y)^*. \]

The full spherical functions $\Psi_d$ satisfy the following differential equation,
\[ \Psi_d(y) = y(1-y)\Psi'_{d}(y) + a(y)\Psi'_d(y) + \Psi_d(y) F(y) = \Lambda_d \Psi_d(y), \]
where $a(y) = \frac{3}{2} - 3y$, the eigenvalue is the diagonal matrix $(\Lambda_d)_{i,i} = -d(2\ell+2+d)i+i(2\ell-i)$, and
\begin{align*}
F(y) = \sum_{i=0}^{2\ell} \frac{2y(1-y)(\ell(\ell+2) - i^2 + 2i) - \ell(2i+1) + i^2}{2y(1-y)} E_{i,i} \\
+ \sum_{i=1}^{2\ell} i(2\ell-i+1)(1-2y)E_{i,i-1} + \sum_{i=0}^{2\ell-1} (i+1)(2\ell-i)(1-2y)E_{i,i+1}. \quad (3.2)
\end{align*}
The differential operator \( \Omega \) is the radial part of the Casimir operator on \( G \), see [19] Section 7.2. The function \( \Psi_0 \) satisfies the first order differential equation
\[
2y(1-y)\Psi_0'(y) + (S - \ell + 2y)\Psi_0(y) = 0,
\]
where \( S \) is the tridiagonal matrix
\[
S = \sum_{i=1}^{2\ell} \frac{i}{2} E_{i,i-1} + \sum_{i=0}^{2\ell-1} \frac{(2\ell - i)}{2} E_{i,i+1},
\]
see [19] Lemma 7.12. The polynomials \( P_d \) are joint eigenfunctions of the matrix-valued differential operators \( D \) and \( E \) given by
\[
D = y(1-y)\frac{d^2}{dy^2} + \left( \frac{d}{dy} \right) (C - yU) - V, \quad E = \left( \frac{d}{dy} \right) (yB_1 + B_0) + A_0,
\]
where the matrices \( C, U, V, B_0, B_1 \) and \( A_0 \) are given by
\[
C = -\sum_{i=0}^{2\ell} \frac{(2\ell - i)}{2} E_{i,i+1} + \sum_{i=0}^{2\ell} \frac{(2\ell + 3)}{2} E_{i,i} - \sum_{i=0}^{2\ell} \frac{i}{2} E_{i,i-1}, \quad U = (2\ell + 3)I,
\]
\[
V = -\sum_{i=0}^{2\ell} i(2\ell - i)E_{i,i}, \quad A_0 = \sum_{i=0}^{2\ell} \frac{(2\ell + 2)(i - 2\ell)}{2\ell} E_{i,i}, \quad B_1 = -\sum_{i=0}^{2\ell} \frac{(\ell - i)}{\ell} E_{i,i},
\]
\[
B_0 = -\sum_{i=0}^{2\ell} \frac{(2\ell - i)}{4\ell} E_{i,i+1} + \sum_{i=0}^{2\ell} \frac{(\ell - i)}{2\ell} E_{i,i} + \sum_{i=0}^{2\ell} \frac{i}{4\ell} E_{i,i-1}.
\]

The first order differential equation (3.3) can be used to derive a simple and compact expression for \( \Psi_0 \) which will be crucial in the forthcoming subsections. The proof is relegated to the Appendix. Let \( K \) be the constant matrix
\[
K_{i,j} = K_j(i) = K_j(i, 1/2, 2\ell), \quad i, j \in \{0, \ldots, 2\ell\},
\]
where \( K_j(x, p, N) \) are the Krawtchouk polynomials, see e.g. [16] §1.10. The orthogonality relations for the Krawtchouk polynomials give a simple inverse for the matrix \( K \), namely
\[
K^{-1} = 2^{-2\ell} M K M,
\]
where \( M \) is the diagonal matrix with entries \( M_{j,j} = (2\ell/j), j = 0, \ldots, 2\ell \).

**Theorem 3.1.** For any \( \ell \in \frac{1}{2} \mathbb{N} \), we have
\[
\Psi_0(y) = K \Upsilon(y) K,
\]
where \( K \) is the constant matrix given by (3.5) and \( \Upsilon \) is the diagonal matrix
\[
\Upsilon(y)_{j,j} = (-1)^{\frac{\nu}{2}}\left(\frac{2\ell}{j}\right) y^{\frac{\nu}{2}} (1-y)^{\frac{2\ell-j}{2}}.
\]

### 3.1. The deformation
In this subsection we apply Theorem 1.2 to obtain a deformation of the pair \((W, D)\). Since we want to deform matrix-valued Chebyshev polynomials into matrix-valued Gegenbauer polynomials, we shift \( \kappa = \nu - 1 \) in order to match the standard convention for Gegenbauer polynomials. In this way, our deformed polynomials \( P_{d(\nu)} \) coincide with the polynomials \( P_d \) for \( \nu = 1 \). We take
\[
a^{(\nu)}(y) = \frac{1}{2} + \nu - y(2\nu + 1), \quad F^{(\nu)} = F - (\nu - 1)(2\ell + \nu + 1) - (\nu - 1)(1 - 2y)\Psi_0^{-1}\Psi'_0,
\]
where \( F \) is given in (3.2). We follow the method described in Section 2. It follows from the explicit expression in Theorem 3.1 that
\[
\Psi_0^{-1}\Psi'_0 = -\frac{1}{2y(1-y)}(S^* - \ell + 2\ell y).
\]
Therefore we have
\[
F^{(\nu)}(y) = F(y) - (\nu - 1)(2\ell + \nu + 1) + \frac{(\nu - 1)(1 - 2y)}{2y(1-y)}(S^* - \ell + 2\ell y).
\]
Note that $F^{(v)}$ is a tridiagonal matrix. Therefore we can use Remark 2.6 to obtain a diagonal matrix $T^{(v)}$ as in Theorem 1.2.

**Lemma 3.2.** Let $T^{(v)}$ be the diagonal matrix

$$T^{(v)}_{i,i} = \binom{2\ell}{i} \frac{(\nu)_i}{(\nu + 2\ell - i)_i},$$

for $i = 0, \ldots, \lfloor \ell \rfloor$ and $T_{i,i} = T_{2\ell-i,2\ell-i}$. Then

$$T^{(v)}(F^{(v)}(y))^* = F^{(v)}(y) T^{(v)}.$$

**Proof.** It follows from (3.7) and (3.2) that

$F^{(v)}_{i,i-1} = (2\ell - i + 1)(\nu - i + 1) \frac{(1 - 2y)}{y(1 - y)}$, \quad $F^{(v)}_{i,i+1} = (i + 1)(2\ell + \nu - i - 1) \frac{(1 - 2y)}{y(1 - y)}$.

By Remark 2.6 if we define $T^{(v)}$ as the diagonal matrix

$$T^{(v)}_{i+1,i+1} = \frac{F^{(v)}_{i+1,i+1}}{F^{(v)}_{i,i+1}} T^{(v)}_{i,i} = \frac{(2\ell - i + 1)(\nu - i - 1)}{i(2\ell + \kappa - 1)} T^{(v)}_{i,i},$$

then the condition in Theorem 1.2 holds true. This completes the proof of the lemma.

**Corollary 3.3.** If $T^{(v)}$ is as in Lemma 3.2 then the pair

$$W^{(v)}(y) = (1 - y)^{\nu-1/2} y^{\nu-1/2} \Psi_0(y) T^{(v)} \Psi_0(y)^*,$$

$$D^{(v)} = y(1 - y) \partial_y^2 + \partial_y(C + \nu - 1 - y(U + 2\nu + 1)) - (V + (\nu - 1)U + (\nu - 1)(\nu - 2)),$$

is a MVCP.

**Remark 3.4.** Observe that the differential operator $2D^{(v)}$ is, up to a the change of variables $x = 1 - 2y$ and a multiple of the identity, the differential operator $D^{(v)}$ in [17, Theorem 2.3]. In the following subsection we will show that the weight matrix $W^{(v)}$ is also closely related to the weight matrix introduced in [17, Definition 2.1].

**Remark 3.5.** In [24] the authors construct families of matrix-valued orthogonal polynomials of size $2 \times 2$ and $3 \times 3$ from the study of spherical functions of fundamental $K$-types associated with the pair $(G, K) = (SO(n + 1), SO(n))$. If we restrict our deformed weight $W^{(v)}$ to the size 3 × 3, it can be matched with the one given in [24, Section 9.2] by the identification $\kappa = 1/(l + 1)$.

### 3.2. The shift operator $\partial_y$.

The goal of this subsection is to prove that the pair $(W^{(v)}, D^{(v)})$ allows for the shift $\partial_y$. For this we will show that $\Gamma_2^{(v)}$ and $\Gamma_1^{(v)}$ defined as in (2.3) and (2.5) are matrix-valued polynomials of degree two and one respectively. Then by Proposition 2.7 if $(Q^{(v)}_n)_n$ is the sequence of monic orthogonal polynomials with respect to $W^{(v)}$, we have that $\partial_y Q^{(v)}_d = dQ^{(v+1)}_d$, since the sequence of monic orthogonal polynomials is unique. The proof will follow from the explicit expression of $\Psi_0$ given in Theorem 3.1 and involves manipulation of Krawtchouk polynomials. Some of the formulas for Krawtchouk polynomials that are necessary in the proof are collected in the Appendix.

**Proposition 3.6.** The functions $\Gamma_2^{(v)}$ and $\Gamma_1^{(v)}$ introduced in (2.3) and (2.5) are matrix polynomials of degree two and one respectively.

**Proof.** First we compute the constant matrix $\Delta = 2^{-2l} MKM(T^{(v)})^{-1} T^{(v+1)} K$. Note that the $i$-th diagonal element of $(T^{(v)})^{-1} T^{(v+1)}$ is $(\nu + 2\ell - i)(\nu + i)/(\nu(\nu + 2\ell))$, so that

$$\Delta_{k,j} = \frac{2^{-2l}}{\nu(\nu + 2\ell)} \sum_{i=0}^{2\ell} \binom{2\ell}{i} \binom{2\ell}{k} (\nu + 2\ell - i)(\nu + i) K_i(k) K_j(i).$$
It follows from the relations (A.9), (A.10) and (A.11), that

\[
\nu(\nu + 2\ell) \Delta = - \sum_{k=0}^{N} \frac{(k+1)(k+2)}{4} E_{k,k+2} + \sum_{k=0}^{2} \frac{(\ell(\ell-1/2) + k(k/2 - \ell) + \nu(\nu + 2\ell)) E_{k,k}}{4} - \sum_{k=0}^{2(\ell-k+1)(2\ell-k+2)} E_{k,k-2}. \tag{3.9}
\]

First we show that \( \Gamma_2^{(\nu)} \) is a polynomial of degree two. It follows from Proposition 6.1 that

\[
\Gamma_2^{(\nu)}(y) = y(1 - y) K^{-1} \Upsilon(y)^{-1} \Delta \Upsilon(y) K.
\]

It follows directly from the explicit expressions of \( \Delta \) and \( \Upsilon \) that \( y(1 - y) \Upsilon(y)^{-1} \Delta \Upsilon(y) \) is a polynomial of degree two and, therefore, so is \( \Gamma_2^{(\nu)} \).

Now we prove that \( \Gamma_1^{(\nu)} \) is a polynomial of degree one. Note that

\[
K \Gamma_1^{(\nu)}(y) K^{-1} = -\frac{(2y-1)(\nu+1)}{2} \Upsilon(y)^{-1} \Delta \Upsilon(y) - 2^{-2\ell-1} \Upsilon(y)^{-1} M K M N K \Upsilon(y) - \frac{\ell(2y-1)}{2} \Upsilon(y)^{-1} \Delta \Upsilon(y) + y(1-y) \Upsilon(y)^{-1} \Delta \Upsilon(y)', \tag{3.10}
\]

where \( N = (T^{(\nu)})^{-1} S^* T^{(\nu+1)} \). A simple computation shows that \( N \) is the tridiagonal matrix given by

\[
N_{i,i-1} = \frac{i(\nu + 2\ell - i)(\nu + 2\ell - i + 1)}{2\nu(\nu + 2\ell)}, \quad N_{i,i} = 0, \quad N_{i,i+1} = \frac{(2\ell - i)(\nu + i)(\nu + i + 1)}{2\nu(\nu + 2\ell)}.
\]

Using the explicit expressions of \( N \) and \( K \) we obtain:

\[
2^{-2\ell} (MKMNK)_{k,j} = \frac{(2\ell)}{2\nu(\nu + 2\ell)} \sum_{i=0}^{2\ell} \left( \frac{2\ell}{i} \right) [(2\ell - i)(\nu + i)(\nu + i + 1) K_k(i)K_j(i + 1) + i(2\ell + \nu - i)(2\ell + \nu - i + 1) K_k(i)K_j(i - 1)]
\]

In the formula above, we replace the terms \( (2\ell - i)K_j(i + 1) \) and \( iK_j(i - 1) \) using Lemma A.4 and we get an expression that is be evaluated using (A.10) and (A.11). We obtain

\[
2^{-2\ell-1} (MKMNK)_{k,j} = \frac{(2\nu + 3\ell - k + 1)(k+1)(k+2)}{8\nu(\nu + 2\ell)} \delta_{j,k+2} + \frac{(k - \ell)(k^2 - 2\ell k - \ell + 1 - 4\nu\ell - 2\nu^2 - 2\ell^2)}{4\nu(\nu + 2\ell)} \delta_{j,k-2}.
\]

It follows from (3.10) and the equation above that \( (K \Gamma_1^{(\nu)}(y) K^{-1})_{i,j} = 0 \) unless \( j = k - 2, k, k + 2 \). Moreover, a straightforward computation using the explicit expressions of \( \Delta, \Upsilon \) and \( 2^{-2\ell} MKMNK \) shows that \( (K \Gamma_1^{(\nu)}(y) K^{-1})_{i,j} \) is a polynomial of degree one. This completes the proof of the proposition.

In order to relate the deformed family \( W^{(\nu)} \) with the family introduced in [17] we need the following corollary.
Corollary 3.7. The polynomial $\Gamma_2^{(\nu)}$ is given explicitly by

$$\frac{4\kappa(\kappa + 2\ell)}{\ell^2} \Gamma_2^{(\nu)}(x) = (1 - 2y)^2 \sum_{i=0}^{2\ell} \frac{(\ell - i)^2}{\ell^2} E_{i,i} - 4y(1 - y)(\ell + \nu)^2$$

$$+ (1 - 2y) \sum_{i=1}^{2\ell} \frac{(i - 1 - 2\ell)(2\ell - 2i + 1)}{2\ell^2} E_{i,i-1} + (1 - 2y) \sum_{i=0}^{2\ell-1} \frac{(i + 1)(2\ell - 2i - 1)}{2\ell^2} E_{i,i+1}$$

$$+ \sum_{i=2}^{2\ell} \frac{(2\ell - i + 2)(2\ell - i + 1)}{4\ell^2} E_{i,i-2} + \sum_{i=0}^{2\ell} \frac{-i(2\ell - i + 1) - (2\ell - i)(i + 1)}{4\ell^2} E_{i,i}$$

$$+ \sum_{i=0}^{2\ell-2} \frac{(i + 2)(i + 1)}{4\ell^2} E_{i,i+2}.$$

Remark 3.8. Observe that, up to the change of variables $x = 1 - 2y$ and a constant $4\kappa(\kappa + 2\ell)/\ell^2$, the matrix $\Gamma_2$ coincides with the polynomial of degree two $\Phi$ given in [17 (4.9)].

Proof. The corollary is proven by a straightforward computation. From the proof of Proposition 3.6 we have that

$$K \Gamma_2^{(\nu)}(y) = y(1 - y) \tilde{\Upsilon}(y)^{-1} \Delta \tilde{\Upsilon}(y) K. \tag{3.11}$$

Now the proof follows by a tedious but direct verification of (3.11). Using the explicit expression of $\Delta$ given in (3.9) the right hand side of (3.11) becomes

$$y(1 - y) \tilde{\Upsilon}(y)^{-1} \Delta \tilde{\Upsilon}(y) = \sum_{j=2}^{2\ell} y^j \frac{(2\ell - j)(2\ell - j - 1)}{2\ell^2} \frac{(2\ell - j)}{2\ell^2} E_{j,j+2}$$

$$+ \sum_{j=0}^{2\ell} y(1 - y) \frac{(\ell(\ell - 1/2) + j(j/2 - \ell) + \nu(\nu + 2\ell))}{\nu(\nu + 2\ell)} E_{j,j} + \sum_{j=0}^{2\ell-2} (1 - y)^2 \frac{j(j - 1)}{4\nu(\nu + 2\ell)} E_{j,j-2}.$$

Therefore, both sides of (3.11) are polynomials of degree two. The proof of the corollary follows by showing that the coefficients of degree 0,1,2 on the left and right hand sides of (3.11) coincide. We include a sketch of the proof for the coefficient of degree two and the other cases are analogous. The coefficient of $y^2$ of the $(j,k)$-th entry of (3.11) is given by

$$-(\nu + j)(\nu + 2\ell - j)K_j(k) = \frac{j(j - 1)}{4} K_k(j - 2) - (\ell(\ell - 1/2) + j(j/2 - \ell) + \nu(\nu + 2\ell))K_k(j)$$

$$+ \frac{(2\ell - j - 1)(2\ell - j)}{4} K_k(j + 2).$$

Now we apply (A.12) and (A.13) twice on the first and last terms of the right hand side, and the three term recurrence relation on the left hand side and the middle term of the right hand side. The equation above becomes an expression involving Krawtchouk polynomials $K_j(k)$ for $h = k - 2, k - 1, k, k + 1, k + 2$ with coefficients which are independent of $j$. The equation is then verified by checking the coefficients of the Krawtchouk polynomials of different degrees. The remaining cases follow in a similar way.

Recall that our deformed family of weight matrices $W^{(\nu)}$ coincides with the one given in [17 Definition 2.1] for $\nu = 1$. Since $W^{(\nu + 1)} = W^{(\nu)} \Gamma_2^{(\nu)}$, see (2.8), in view of [17 Theorem 2.4], it follows that the weight matrix in [17 Definition 2.1] coincides up to a constant with $W^{(\nu)}$ for all integer values of $\nu$. From a continuation argument we conclude that the two families of weights coincide up to constant multiple.

Remark 3.9. Note that the operator $E^{(\nu)} = E + \nu(A_0 + B_1)$ satisfies $\partial \circ E^{(\nu)} = E^{(\nu + 1)} \circ \partial$. Since $[E^{(\nu + 1)}, D^{(\nu + 1)}] \circ \partial = \partial \circ [E^{(\nu)}, D^{(\nu)}]$, and $[E, D] = 0$ see [18, 19], it follows that $[E^{(\nu)}, D^{(\nu)}] = 0$
for all $\nu \in \mathbb{N}$. Moreover for any $\nu \geq 0$ and any smooth $\mathbb{C}^{2\ell+1}$-valued function $F$ we have
\[
F(D^{(\nu)}E^{(\nu)} - E^{(\nu)}D^{(\nu)}) = \nu(1 - 2y)(\partial_y F) E + \nu(FE)(U + \nu - 1) \\
+ \nu(FD)(A_0 + B_1) - \nu(F(A_0 + B_1)) D - \nu(1 - 2y)\partial_y \nu(FE) - \nu(FE)(U + \nu - 1). \tag{3.12}
\]
Now (3.12) is a polynomial function in $\nu$ which is zero for infinitely many values of $\nu$ and thus it is zero for all $\nu$. Therefore we have
\[
[E^{(\nu)}, D^{(\nu)}] = 0, \quad \text{for all } \nu > 0.
\]
Since $E^{(\nu)}$ commutes with $D^{(\nu)}$, it follows that the monic orthogonal polynomials $P_n^{(\nu)}$ are eigenfunctions of $E^{(\nu)}$, i.e.
\[
P_n^{(\nu)}(y) = \Lambda_n(E^{(\nu)}) P_n^{(\nu)}, \quad \text{for all } n \in \mathbb{N},
\]
where $\Lambda_n(E^{(\nu)}) = nB_1 + A_0 + \nu(B_1 + A_0)$. Observe that since the eigenvalues $\Lambda_n(E^{(\nu)})$ are diagonal with real entries, and thus Hermitian, the differential operator $E^{(\nu)}$ is symmetric with respect to $W^{(\nu+1)}$, see \cite{11} Corollary 4.5. Observe that the differential operator $E^{(\nu)}$ coincides with the differential operators in \cite{17} Corollary 4.1.

4. Examples of dimension two

In this section we apply our method to two of the examples given in \cite{22} [1]. These examples are related to the Gelfand pairs $(SU(n+1), U(n))$ and $(USp(2n), USp(2n-2) \times USp(2))$ and correspond to matrix-valued orthogonal polynomials of Jacobi type.

4.1. Case a1. Let us consider the Gelfand pair $(G, K) = (SU(n+1), U(n))$. This example is studied in \cite{22} Page 7, case a1]. For $n \geq 2$ we have $\alpha = n-1$, $\beta = 0$ and two free parameters $1 \leq i \leq n-1$ and $m \in \mathbb{N}$. We have the following expressions,
\[
\Psi_0^{(n,m,i)}(y) = y^{\frac{n+1}{2}} \left( \frac{1}{y^{n+1}} - \frac{1}{i^{n+1}} \right), \quad T = \begin{pmatrix} 1 & 0 \\ 0 & n-i \end{pmatrix}, \tag{4.1}
\]
where we indicate the free parameters as superscripts. Moreover, we have $a(y) = 1 - y(n+1)$ and $b(y) = \sqrt{y}$.

and the differential operator $D = y(1 - y)\partial_y^2 + \partial_y, (C - yU) - V$ is determined by
\[
C = \begin{pmatrix} \frac{m}{2} & 0 \\ 0 & \frac{n}{2} \end{pmatrix}, \quad U = \begin{pmatrix} n + m + 2 & 0 \\ 1 & n + m + 3 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 \\ 0 & n + m + 1 - i \end{pmatrix}.
\]
As in Theorem \cite{12} we define $F^{(\kappa)} = F + \kappa \Psi_0^{-1}(U + \kappa - 1) \Psi_0 - \kappa(1 - 2y)\Psi_0^{-1}\Psi_0'$. By a straightforward computation we verify that
\[
F^{(\kappa)}(y)_{0,1} = -\sqrt{y}(\kappa + i), \quad F^{(\kappa)}(y)_{1,0} = \sqrt{y}(-n + i),
\]
so that by Remark \cite{23} the matrix $T^{(\kappa)} = \begin{pmatrix} 1 & 0 \\ 0 & n+1+i \end{pmatrix}$, is a solution to (1.10). Therefore Theorem \cite{12} implies that $(W^{(n,m,i,\kappa)}, D^{(n,m,i,\kappa)})$ with
\[
W^{(n,m,i,\kappa)}(y) = y^n(1 - y)^{n-1+i} \Psi_0^{(n,m,i,\kappa)}(y) T^{(\kappa)}(\Psi_0^{(n,m,i,\kappa)}(y))^* \tag{4.2}
\]
\[
D^{(n,m,i,\kappa)}(y) = y(1 - y)\partial_y^2 + \partial_y(C + \kappa - y(U + 2\kappa)) - (V + \kappa U + \kappa(\kappa - 1)),
\]
is a MVCP. It is a straightforward computation that the functions
\[
\Gamma_2^{(\kappa)} = (\Psi_0(y))^{-1}(T^{(\kappa)})^{-1}T^{(\kappa)}\Psi_0(y)^*, \quad \Gamma_1^{(\kappa+1)} = (W^{(n,m,i,\kappa)}(y))^{-1}(W^{(n,m,i,\kappa)}(y)+1)^*.
\]
are matrix-valued polynomials of degrees two and one respectively. We omit this computation. It then follows that Proposition 2.7 and Theorem 2.9 apply to this case. Therefore the monic orthogonal polynomials \( (Q_d^{(\kappa)})_d \) with respect to \( W^{(\kappa)} \) satisfy
\[
\partial_y Q_d^{(n,m,1,\kappa)}(y) = d Q_d^{(n,m,1,\kappa+1)}(y),
\]
and the following Rodrigues formula holds
\[
Q_d(y) = G_d \left( \partial_y^d W^{(n,m,1,\kappa+d)} \right) \left( W^{(n,m,1,\kappa)} \right)^{-1},
\]
for certain constant matrix \( G_d \).

**Remark 4.1.** For nonnegative integer values of \( \kappa \), it follows directly from the form of \( T^{(\kappa)} \) and (12) that \( W^{(n,m,1,\kappa)} = W^{(n+k,m+k,i+\kappa)} \) so that
\[
\partial_y Q_d^{(n,m,1)}(y) = d Q_d^{(n+1,m+1,i+1)}(y).
\]

**Remark 4.2.** The case \( m \in \mathbb{Z}_{<0} \) is given in [22] Page 7, case a2]. The matrix \( \Psi_0 = \Psi_0^{m<0} \) is given by
\[
\Psi_0^{m<0}(y) = \left( \begin{array}{c}
\frac{y^2}{y^{n-1}} \\
\frac{y^2}{y^{n-2}} \\
\vdots \\
\frac{y^2}{y^{n-m}}
\end{array} \right) = \left( \begin{array}{c}
0 \\
1 \\
0 \\
1
\end{array} \right) \Psi_0^{(n,m-1,n-i)}(y) \left( \begin{array}{c}
0 \\
1 \\
0 \\
1
\end{array} \right),
\]
where \( \Psi_0^{(n,m-1,n-i)} \) is given in (11). The case a2 is therefore contained into the case a1.

4.2. **Case c1.** Now we consider the Gelfand pair \( (G, K) = (\text{Sp}(2n), \text{Sp}(2n-2) \times \text{Sp}(2)) \). This example is studied in [22] Page 7, case a1]. For \( n \geq 3 \) we have \( \alpha = 2n - 3, \beta = 1 \). We have the following expressions
\[
\Psi_0(y) = \left( \begin{array}{c}
\sqrt{y} \\
\sqrt{y}
\end{array} \right) \frac{1}{\sqrt{n-1}}, \quad T = \left( \begin{array}{cc}
1 & 0 \\
0 & n - 2
\end{array} \right).
\]
We also have \( a(y) = 2 - 2yn \),
\[
F(t) = \left( \begin{array}{c}
4n^2 - 4y^2 + y^2 - 18y + 3 \\
4y^{(n-1)} \\
2y^{(n-2)} \\
y - 1
\end{array} \right), \quad -\frac{2y}{y - 1},
\]
and the matrix-valued differential operator \( D = y(1 - y) \partial_y^2 + \partial_y, (C - yU) - V \), where
\[
C = \left( \begin{array}{cc}
2n - 1 \\
1
\end{array} \right), \quad U = \left( \begin{array}{cc}
2n + 1 & 0 \\
-1 & 2n + 2
\end{array} \right), \quad V = \left( \begin{array}{cc}
0 & 0 \\
0 & 2n - 2
\end{array} \right).
\]
If we define \( F^{(\kappa)} \) as in Theorem 1.2, a straightforward verification shows that the unique solution \( T^{(\kappa)} \) to (1.10) normalized by \( T^{(0)} = T \) is given by
\[
T^{(\kappa)} = \left( \begin{array}{cc}
1 & 0 \\
0 & \frac{2n-2}{\kappa + 2}
\end{array} \right).
\]
Now Theorem 1.2 says that \( (W^{(\kappa)}, D^{(\kappa)}) \) with
\[
W^{(\kappa)}(y) = y^{\alpha + \kappa}(1 - y)^{\beta + \kappa} \Psi_0(y) T^{(\kappa)} \Psi_0(y)^*,
\]
\[
= y^{(\kappa+1)}(1 - y)^{\kappa+2n-3} \left( \begin{array}{cc}
\frac{2yn - 2 + y}{2 \kappa} \\
\frac{2yn - 2}{2 \kappa}
\end{array} \right)
\]
\[
D^{(\kappa)} = y(1 - y) \partial_y^2 + \partial_y(C + \kappa - y(U + 2n)) - (V + \kappa U + \kappa(\kappa - 1)),
\]
is a MVCP. Moreover, if we compute explicitly the functions $\Gamma_2^{(\kappa)}$ and $\Gamma_1^{(\kappa)}$ given in (2.8) and (2.8) respectively, we obtain
\[
\Gamma_2^{(\kappa)} = y^2 \left( \begin{array}{cc}
-1 & -(3+\kappa)^{-1} \\
0 & -2+\kappa \\
\end{array} \right) + y \left( \begin{array}{c}
\frac{3n+1+2\kappa+\kappa^2}{(3+\kappa)(n-1)} \\
\frac{1}{(3+\kappa)(n-1)} \\
\frac{4n+4\kappa+2\kappa^2n}{(3+\kappa)(n-1)} \\
\frac{4n+4\kappa+2\kappa^2n}{(3+\kappa)(n-1)} \\
\frac{3+2\kappa}{(3+\kappa)(n-1)} \\
\frac{3+2\kappa}{(3+\kappa)(n-1)} \\
\end{array} \right),
\]
\[
\Gamma_1^{(\kappa)} = y \left( \begin{array}{cc}
-2n - 1 - 2\kappa \\
0 \\
\end{array} \right) - \frac{1+2\kappa}{3+\kappa} + \frac{-3-\kappa^2+6n-2\kappa+4\kappa n+\kappa^2n}{(3+\kappa)(n-1)} \\
+ \frac{(n-2)2n+1+2\kappa}{(3+\kappa)(n-1)} \\
+ \frac{\kappa^2+7n+6n-8\kappa-10-\kappa^2}{(3+\kappa)(n-1)} \\
\end{array} \right),
\]
so that $\Gamma_2^{(\kappa)}$ is a polynomial of degree two and $\Gamma_1^{(\kappa)}$ is a polynomial of degree one. Therefore, it follows from Proposition 2.7 that the sequence of derivatives $(\partial_y F_2^{(\kappa)})$ is orthogonal with respect to $W^{(\kappa+1)}$.

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Appendix A.

Here we give a proof of the explicit expression for the function $\Psi_0$ in terms of Krawtchouk polynomials. We will use that $\Psi_0$ is a solution to the equation (A.3) and the fact that the matrix $S$ can be diagonalized explicitly. Note that the columns of $K$ are precisely the different eigenvectors of the matrix $S$ from (A.3). More precisely we have
\[
S \cdot K = \text{diag}(-\ell, -\ell+1, \ldots, \ell-1, \ell) \cdot K,
\]
see [19] Lemma 4.3 for the proof. Observe that the columns of $\Psi_0$ are solutions to the first order differential equation
\[
y(1-y) F'(y) + \frac{1}{2} (S-\ell+2\ell y) F(y) = 0, \quad (A.1)
\]

Proposition A.1. For $j = 0, 1, \ldots, 2\ell$, let $v_j$ be the eigenvector of $S$ with eigenvalue $-\ell-j$. Then
\[
F_j(y) = y^\ell (1-y)^{-\ell-j} v_j,
\]
is a solution of (A.1). Moreover, \(\{F_j\}_{j=0}^{2\ell}\) is a basis of solutions of (A.1).

Proof. This follows by replacing $F_j$ in (A.1). The set $\{F_j\}_{j=0}^{2\ell}$ is basis of solutions since $v_j$ are all linearly independent. \qed

A simple calculation shows that
\[
F_j \left( (1-\cos(t))/2 \right) = 2^{-2\ell} (-1)^j 2^j e^{-\ell t} (e^{\ell t} - 1)^j (e^{\ell t} + 1)^{2\ell-j} v_j, \quad (A.2)
\]

Remark A.2. It follows directly from the previous proposition that each entry of $F_j(\cos t)$ is a linear combination of $\{e^{-i\ell t}, e^{-i(\ell-1) t}, \ldots, e^{i\ell t}\}$. Moreover, for all $j = 0, 1, \ldots, 2\ell$, we have
\[
F_j \left( (1-\cos(t))/2 \right) = 2^{-2\ell} (-1)^j 2^j e^{-\ell t} + \sum_{k=1}^{2\ell} a_k e^{i(-\ell+k) t} v_j, \quad (A.3)
\]
for certain coefficients $a_k$.

Lemma A.3. Let $\Phi_0$ be given by (3.1). Then
\[
\Phi_0 = M^{-1} e^{-i\ell t} + \sum_{k=1}^{2\ell} A_k e^{i(-\ell+k) t}, \quad M = \sum_{j=0}^{2\ell} \binom{2\ell}{j} E_{j,j},
\]
where $A_k, k = 1, \ldots, 2\ell$ are $(2\ell+1) \times (2\ell+1)$ matrices.
Proof. The proof follows directly from (3.1).

Proof of Theorem 3.1. Since \( \Psi_0 \) is a solution of (A.1) and \( \{F_j\} \) is a basis of solutions, every column of \( \Psi_0 \) is a linear combination of the functions \( F_j \), \( j = 0, \ldots, 2\ell \). If we denote by \( \Psi_0^{(k)} \) the \( k \)-th column of \( \Psi_0 \), then there exist constants \( a_0^{(k)}, \ldots, a_{2\ell}^{(k)} \) such that

\[
\Psi_0^{(k)}(y) = a_0^{(k)}F_0(y) + \cdots + a_{2\ell}^{(k)}F_{2\ell}(y).
\]

If we make the change of variables variable \( \cos t = 1 - 2y \), using that \( \Psi_0^{(k)}(\arccos(1 - 2y)) = \Phi_0^{(k)}(t) \), it follows from (A.2) that

\[
\Phi_0^{(k)}(t) = 2^{-2\ell}(-1)^{\ell}e^{-it}a_0^{(k)}(e^{it} + 1)^{2\ell}v_0 + \cdots + (-1)^{\ell}a_{2\ell}^{(k)}(e^{it}1)^{2\ell}v_{2\ell}.
\]  

(A.4)

If we look at the coefficient of \( e^{-it} \) on both sides of (A.4), using (A.3) and Lemma A.3, we obtain the following equation,

\[
\left(2\ell\atop k\right)^{-1}e_k = 2^{-2\ell}\sum_{j=0}^{2\ell}a_j^{(k)}(-1)^{k\ell}v_j.
\]  

(A.5)

where \( e_k \) is the standard basis vector. Let \( \Gamma \) be the diagonal matrix given by \( \Gamma_{j,j} = (-1)^{\ell} \). Then (A.5) can be written in matrix form as

\[
\left(2\ell\atop k\right)^{-1}e_k = 2^{-2\ell}K\Gamma(a_0^{(k)}, \ldots, a_{2\ell}^{(k)})^t.
\]

Then it follows that

\[
(a_0^{(k)}, \ldots, a_{2\ell}^{(k)})^t = 2^\ell\left(2\ell\atop 2\ell - k\right)^{-1}\Gamma^{-1}K^{-1}e_k.
\]

In matrix form, (A.3) can be written as

\[
\tilde{\Phi}_0^{(k)}(t) = K\tilde{\Upsilon}(t)M(a_0^{(k)}, \ldots, a_{2\ell}^{(k)})^t,
\]

(A.6)

where \( \tilde{\Upsilon} \) is the diagonal matrix

\[
\tilde{\Upsilon}(t)_{j,j} = 2^{-\ell}(-1)^{\ell}j\left(2\ell\atop i\right)e^{-it}(e^{it}1)^j(e^{it}1)^{2\ell-j}.
\]

If we replace this expression in (A.6) we obtain

\[
\tilde{\Phi}_0^{(k)}(t) = 2^{2\ell}\left(2\ell\atop 2\ell - k\right)^{-1}K\tilde{\Upsilon}(t)\Gamma^{-1}K^{-1}e_k,
\]

which leads to

\[
\Psi_0^{(k)}(y) = 2^{2\ell}K\Upsilon(y)MK^{-1}M^{-1}e_k.
\]

(A.7)

The column vector (A.7) is precisely the \( k \)-th column of (3.5). This completes the proof of the theorem. \( \square \)

We proceed to collect results and formulas about Krawtchouk polynomials that we use in Section 3. We will need the following relations

\[
\sum_{i=0}^{2\ell}\left(2\ell\atop i\right)K_n(i)K_m(i) = \delta_{n,m}2^{2\ell}\left(2\ell\atop n\right)^{-1},
\]

(A.8)

\[-iK_n(i) = \frac{\frac{n}{2} - n}{2}K_{n+1}(i) - \frac{\frac{n}{2} - n}{2}K_{n-1}(i).
\]

(A.9)

Using the three-term recurrence relation (A.9) and orthogonality (A.8), we obtain

\[
\sum_{i=0}^{2\ell}\left(2\ell\atop i\right)iK_k(i)K_j(i) = 2^{2\ell}\left(2\ell\atop k\right)^{-1}\begin{cases} 
-\frac{k+1}{2}, & j = k+1, \\
\ell, & j = k, \\
\frac{-2k+1}{2}, & j = k-1, \\
0, & \text{otherwise},
\end{cases}
\]

(A.10)
and
\[
\sum_{i=0}^{2\ell} \binom{2\ell}{i} i^2 K_k(i) K_j(i) = 2^{2\ell} \binom{2\ell}{k}^{-1} \begin{cases} 
(k+1)(k+2)/4, & j = k + 2, \\
-\ell(k+1), & j = k + 1, \\
(\ell(\ell+1/2) + k(\ell-k/2))/4, & j = k, \\
-\ell(2\ell-k+1), & j = k - 1, \\
(2\ell-k+1)(2\ell-k+2)/4, & j = k - 2, \\
0, & \text{otherwise}
\end{cases}
\]

Lemma A.4. The following recurrence relations hold:
\[
(2\ell - i) K_k(i + 1) = \frac{1}{2} (2\ell - k) K_{k+1}(i) + (\ell - k) K_k(i) - \frac{1}{2} k K_{k-1}(i), \tag{A.12}
\]
\[
i K_k(i - 1) = -\frac{1}{2} (2\ell - k) K_{k+1}(i) + (\ell - k) K_k(i) + \frac{1}{2} k K_{k-1}(i). \tag{A.13}
\]

Proof. Note that we only have to prove one of the relations, because the relations add up to the difference equation \[16, (1.10.5)]. Alternatively, replace \(i\) by \(2\ell - i\) in the first equation and apply the basic relation \(\genfrac{[}{]}{0pt}{}{a,b}{c} = (1-z)^{-a} \genfrac{[}{]}{0pt}{}{a-c-b}{c} ; z/(z-1)\) specialized in \(z = 2\). The basic relations
\[
(2\ell - i) \genfrac{[}{]}{0pt}{}{2\ell}{-2\ell} ; z + (i + k - 2\ell) \genfrac{[}{]}{0pt}{}{-i-k}{-2\ell} ; z = -k(z-1) \genfrac{[}{]}{0pt}{}{-i-k+1}{-2\ell} ; z,
\]
\[
(2\ell - k) \genfrac{[}{]}{0pt}{}{-i-k-1}{-2\ell} ; z + (2k - 2\ell + (i - k)z) \genfrac{[}{]}{0pt}{}{-i-k}{-2\ell} ; z = -k(z-1) \genfrac{[}{]}{0pt}{}{-i-k+1}{-2\ell} ; z,
\]
imply the result. Indeed, subtracting one half times the second from the first yields
\[
(2\ell - i) \genfrac{[}{]}{0pt}{}{-i-k}{-2\ell} ; z = \frac{1}{2} (2\ell - k) \genfrac{[}{]}{0pt}{}{-i-k-1}{-2\ell} ; z + \frac{1}{2} z k + (\frac{1}{2} z - 1)i \genfrac{[}{]}{0pt}{}{-i-k}{-2\ell} ; z \frac{1}{2} \genfrac{[}{]}{0pt}{}{-i-k+1}{-2\ell} ; z,
\]
which specializes to the desired equation for \(z = 2\). \(\square\)

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