The Romance of the Ising Model

Barry M. McCoy
State University of New York
Stony Brook, NY

Abstract
The essence of romance is mystery. In this talk, given in honor of the 60th birthday of Michio Jimbo, I will explore the meaning of this for the Ising model beginning in 1946 with Bruria Kaufman and Willis Lamb, continuing with the wedding by Jimbo and Miwa in 1980 of the Ising model with the Painlevé VI equation which had been first discovered by Picard in 1889. I will conclude with the current fascination of the magnetic susceptibility and explore some of the mysteries still outstanding.

1 Introduction

A search of Google books reveals that the observation

The essence of romance is mystery

has been made by many authors in many different ways and in many different contexts ranging from the literary to the scientific. But in all contexts romance betokens fascination and the Ising model has fascinated many people, including myself, for many decades and in spite of many breakthroughs and moments of understanding the mystery continues to this day. In this talk I will present some of the milestones of this romance.

2 Kaufman and Lamb

In his talk “The Ising model in two dimensions” [1] presented at the fifth Battelle Colloquium on Materials Science, held in Geneva and Gstaad, Switzerland, September 7-12, 1970, Lars Onsager wrote, following a discussion of his famous 1944 computation of the free energy [2] and a sketch of his 1945 proof of his conjectured spectrum of the transfer matrix,

“Before long, however, Bruria Kaufman had developed a much better strategy.

At Columbia University she first asked Willis E. Lamb to direct her work on order-disorder problems; but he was much too heavily engaged in an experimental effort, and I was asked to assume the responsibility. Unable to talk her out of the idea I suggested that she explore · · · By the summer of 1946 she had a beautifully compact computation of the partition function, bypassing all tedious detail.
By itself that was only a more elegant derivation of an old result but the approach looked powerful enough to produce a few more new ones. Very well, how about correlations?

The history of the Ising model from that time forth has been the study of these correlations.

But the deeper meaning of this passage from Onsager’s paper completely escaped me until many years later Rodney Baxter wrote to me concerning a typescript [3] that had been given to him which is certainly a draft of Onsager and Kaufman’s calculation of the spontaneous magnetization of the Ising model. Why in the world would Kaufman, who was creating pioneering mathematics, ask Lamb, an experimental physicist, to supervise her research? This question was brought into sharp focus when Baxter told me that he was going to contact her about the authorship of the typescript. She was then living in Tucson, Arizona with her husband, Willis Lamb.

So this is the first romance concerned with the Ising model. Both Bruria and Willis were married to other people in 1946 when Bruria asked Willis to be her research supervisor and he turned her down. But decades later, when Kaufman’s husband died in 1992, Lamb invited her to Tucson as a Visiting Scholar at the University of Arizona where he was a professor. In 1996, after his wife died, Willis and Bruria were married.

3 Correlations and form factors

The great understanding of Kaufman was that the Ising partition function could be written by use of fermionic methods as the sum of four Pfaffians [4] and that this fermionic method is powerful enough to write all correlation functions of the Ising model as determinants [5].

The Ising model is a system of “spins” $\sigma_{j,k}$ at row $j$ and column $k$ of a square lattice which take on the values $\sigma_{j,k} = \pm 1$ and interact with their nearest neighbors with the interaction energy

$$\mathcal{E} = - \sum_{j=-L^v}^{L^v} \sum_{k=-L^h}^{L^h} \left\{ E^h \sigma_{j,k} \sigma_{j+1,k} + E^v \sigma_{j,k} \sigma_{j+1,k} \right\}. \quad (1)$$

The correlation functions studied by Kaufman and Onsager are defined as

$$\langle \sigma_{0,0} \sigma_{M,N} \rangle = \lim_{L^v,L^h \to \infty} Z_{L^v,L^h}^{-1} \sum_{\sigma=\pm 1} \sigma_{0,0} \sigma_{M,N} e^{-\mathcal{E}/k_B T} \quad (2)$$

where $T$ is the temperature, $k_B$ is Boltzmann’s constant,

$$Z_{L^v,L^h} = \sum_{\sigma=\pm 1} e^{-\mathcal{E}/k_B T} \quad (3)$$

is the partition function and the sum $\sum_{\sigma=\pm 1}$ is over all values of the variables $\sigma_{j,k}$. 

2
The discovery of Kaufman and Onsager [5] is that the row and diagonal correlations can be written as a sum of two determinants. These are further simplified by Montroll, Potts and Ward [6] to a single determinant. The diagonal $\langle \sigma_{0,0}\sigma_{N,N} \rangle$ and the row correlations $\langle \sigma_{0,0}\sigma_{0,N} \rangle$ can both be written as $N \times N$ Toeplitz determinants

$$D_N = \begin{vmatrix} a_0 & a_{-1} & \cdots & a_{-N+1} \\ a_1 & a_0 & \cdots & a_{-N+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1} & a_{N-2} & \cdots & a_0 \end{vmatrix}$$

(4)

where

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-i n \theta} \phi(\theta)$$

(5)

with

$$\phi(\theta) = \left[ \frac{(1 - \alpha_1 e^{i\theta})(1 - \alpha_2 e^{-i\theta})}{(1 - \alpha_1 e^{-i\theta})(1 - \alpha_2 e^{i\theta})} \right]^{1/2}.$$  

(6)

For $\langle \sigma_{0,0}\sigma_{N,N} \rangle$

$$\alpha_1 = 0, \quad \alpha_2 = (\sinh 2E^v/k_BT \sinh 2E^h/k_BT)^{-1}$$

(7)

and for $\langle \sigma_{0,0}\sigma_{0,N} \rangle$

$$\alpha_1 = e^{-2E^v/k_BT} \tanh E^h/k_BT, \quad \alpha_2 = e^{-2E^v/k_BT} \coth E^h/k_BT$$

(8)

and the square roots are defined to be positive at $\theta = \pi$. These determinants are very efficient for the calculation of the correlations when $N$ is small.

However, when $N$ is large the determinantal representation is not an efficient method of calculation and a different representation must be found.

The first step in finding this new representation is the computation of the limiting value as $N \to \infty$

$$\lim_{N \to \infty} \langle \sigma_{0,0}\sigma_{0,N} \rangle = \lim_{N \to \infty} \langle \sigma_{0,0}\sigma_{N,N} \rangle = (1 - t)^{1/4}$$

(9)

with

$$t = (\sinh 2E^v/k_BT \sinh 2E^h/k_BT)^{-2},$$

(10)

which is valid for $0 \leq t \leq 1$. For $t > 1$ the limit vanishes. The value of $T$ for which $t = 1$ is called the critical temperature $T_c$. It is the evaluation of this limit for $\langle \sigma_{0,0}\sigma_{N,N} \rangle$ which is accomplished by Kaufman and Onsager in the manuscript recently published by Baxter [3].

The next step in the evaluation of the long distance behavior of the correlations was made in 1966 by Wu [7] who computed the first correction $f_{0,N}^{(2)}$ to $\langle \sigma_{0,0}\sigma_{N,N} \rangle$ as $N \to \infty$ for $\langle \sigma_{0,0}\sigma_{0,N} \rangle$ for $T < T_c$ as a two-dimensional integral and the leading behavior $f_{0,N}^{(1)}$ as $N \to \infty$ of $\langle \sigma_{0,0}\sigma_{0,N} \rangle$ for $T > T_c$ as a one-dimensional
integral. These are the first terms in what is now called the form factor expansion of the correlation functions, which for general \( M, N \) is written for \( T < T_c \) as

\[
\langle \sigma_{0,0} \sigma_{M,N} \rangle = (1 - t)^{1/4} \left\{ 1 + \sum_{n=1}^{\infty} f^{(2n)}_{M,N} \right\}
\]  

and for \( T > T_c \) as

\[
\langle \sigma_{0,0} \sigma_{M,N} \rangle = (1 - t)^{1/4} \sum_{n=0}^{\infty} f^{(2n+1)}_{M,N},
\]  

where for \( T > T_c \) we use the definition

\[
t = (\sinh 2E^v / k_B T \sinh 2E^h / k_B T)^2.
\]  

The derivation of the complete expansions (11) and (12) has its own interesting story. In 1976 Wu, McCoy, Tracy and Barouch [8] derived an expansion valid for all \( N \) of the correlations in the form for \( T < T_c \) of

\[
\langle \sigma_{0,0} \sigma_{M,N} \rangle = (1 - t)^{1/4} \exp \sum_{n=0}^{\infty} F^{(2n)}_{M,N}
\]  

and for \( T > T_c \)

\[
\langle \sigma_{0,0} \sigma_{M,N} \rangle = (1 - t)^{1/4} \sum_{n=0}^{\infty} G^{(2n+1)}_{M,N} \exp \sum_{n=0}^{\infty} \tilde{F}^{(2n)}_{M,N}
\]  

where \( F^{(2n)}_{M,N} \) and \( \tilde{F}^{(2n)}_{M,N} \) are \( 4n \) dimensional integrals and \( G^{(2n+1)}_{M,N} \) are \( 4n + 2 \) dimensional integrals. For all three functions half of the integrals may be executed by closing a contour integral on a pole. The forms (14) and (15) of the correlation functions are called the exponential forms.

The form factor expansions (11) and (12) are obtained from the exponential forms (14) and (15) by expanding the exponentials. For a few low values of \( n \) this was done in [8] in connection with the study of the magnetic susceptibility but the general results for the \( f^{(n)}_{M,N} \) were not given by Nickel [9] and [10] until 1999 and 2000.

A curious feature of the derivation given in [8] of (14) and (15) is that the method of [7] developed for the row correlation \( \langle \sigma_{0,0} \sigma_{0,N} \rangle \) is not used; instead the method used by Cheng and Wu [11] in the study of the leading terms of large separation behavior of the general correlation \( \langle \sigma_{0,0} \sigma_{M,N} \rangle \) is used. The original method [7] of Wu as applied to the correlations \( \langle \sigma_{0,0} \sigma_{0,N} \rangle \) and \( \langle \sigma_{0,0} \sigma_{N,N} \rangle \) was extended to all orders in 2007 by Lyberg and McCoy [12]. The results in [12] for the diagonal form factors \( f^{(n)}_{N,N}(t) \) are for \( T < T_c \)

\[
f^{(n)}_{N,N}(t) = \frac{t^n (N+n)}{(n!)^2 \pi^{2n}} \int_0^1 \prod_{k=1}^{2n} dx_k x_k^N \prod_{j=1}^{n} \left( \frac{(1 - tx_j)(x_j^{-1} - 1)}{(1 - tx_{j-1})(x_{j-1}^{-1} - 1)} \right)^{1/2}
\]  

4
for $T > T_c$

$$f_{N,N}^{(2n+1)}(t) = \frac{t^{(n+1)/2}N+n(n+1)}{n!(n+1)!\pi^{2n+1}} \int_0^1 \prod_{k=1}^{2n+1} dx_k \prod_{j=1}^{n+1} x_j^{-1} [(1-tx_j^{-1})(x_j^{-1} - 1)]^{-1/2} \prod_{j=1}^{n} x_j^{-1} [(1-tx_j^{-1})(x_j^{-1} - 1)]^{1/2} \prod_{1 \leq j < k \leq n+1} (x_j - x_k)^2 \prod_{1 \leq j < k \leq n} (x_j - x_k)^2.$$  \hspace{1cm} (17)

A closely related form for the row form factor $f_{0,N}^{(n)}$ is also obtained in [12]. The results (16) and (17) have the startling feature that in the diagonal case the $f_{N,N}$ do not manifestly reduce term by term to the corresponding functions obtained from [8]. The reconciliation of these two forms is one of the present mysteries of the Ising model.

These diagonal form factor integrals, which on the surface may appear to be indigestible, have proven to have many very special properties.

1) All the integrals in (16) and (17) reduce at $t = 0$ to a product of two special cases of the celebrated Selberg integral [13]

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} t_i^{a-1}(1-t_i)^{b-1} \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2\gamma} dt_1 \cdots dt_n. \hspace{1cm} (18)$$

2) In [14] it was discovered by Maple calculations the $f_{N,N}^{(n)}$ satisfy Fuchsian differential equation with a factorized “Russian doll” structure

$$F_{2n} f_{N,N}^{(n)} = 0 \quad \text{with} \quad F_{2n} = L_{2n+1}(N) \cdots L_3(N) \cdot L_1(N) \hspace{1cm} (19)$$

$$F_{2n+1} f_{N,N}^{(2n+1)} = 0 \quad \text{with} \quad F_{2n+1} = L_{2n+2}(N) \cdots L_4(N) \cdot L_2(N) \hspace{1cm} (20)$$

where $L_j(N)$ are linear differential operators of order $j$.

3) It was also discovered in [14] by Maple calculations that the operators $F_n$ have in addition a direct sum decomposition

$$F_{2n} = M_{2n+1}(N) \oplus \cdots \oplus M_3(N) \oplus M_1(N) \hspace{1cm} (21)$$

$$F_{2n+1} = M_{2n+2}(N) \oplus \cdots \oplus M_4(N) \oplus M_2(N) \hspace{1cm} (22)$$

4) Furthermore, the $f_{N,N}^{(n)}(t)$ have a factorization property first found in [14] by computer computations and proven for $n = 1, 2, 3$ in [15] that

$$f_{N,N}^{(2n)}(t) = \sum_{m=0}^{n-1} K_m^{(2n)}(N) \cdot f_{N,N}^{(2m)}(t) + \sum_{m=0}^{2n} C_{m}^{(2n)}(N;t) \cdot F_{N}^{2n-m} \cdot F_{N}^{m}(23)$$

\[5\]
\[
\frac{f^{(2n+1)}_{N,N}(t)}{t^{N/2}} = \sum_{m=0}^{n-1} K_m^{(2n+1)}(N) \cdot \frac{f^{(2m+1)}_{N,N}(t)}{t^{N/2}} + \sum_{m=0}^{2n+1} C_m^{(2n+1)}(N; t) \cdot F_{N}^{2n+1-m} \cdot F_{N+1}^m,
\]

where \( F_N \) is the hypergeometric function

\[
F_N = \, _2F_1(1/2, N + 1/2; N + 1; t),
\]

and \( f^{(0)}_{N,N} = 1 \). The \( K_m^{(n)}(N) \) depend only on \( N \) and we note in particular that

\[
K_0^{(3)}(0) = \frac{1}{6}, \quad K_1^{(3)}(0) = \frac{1}{3}, \quad K_0^{(5)}(0) = \frac{1}{120}, \quad K_1^{(5)}(0) = \frac{1}{2}, \quad K_0^{(6)}(0) = 0, \quad K_1^{(6)}(0) = -\frac{2}{45}, \quad K_2^{(6)}(0) = \frac{2}{3}.
\]

The \( C_m^{(n)}(N; t) \) are polynomials in \( t \) of degree for \( N \geq 1 \)

\[
\deg C_m^{(2n)}(N; t) = \deg C_m^{(2n+1)}(N; t) = n \cdot (2N + 1),
\]

with \( C_m^{(n)}(N; t) \approx t^m \) as \( t \to 0 \) which have the palindromic property

\[
C_m^{(2n)}(N; t) = t^{n(2N+1)+m} \cdot C_m^{(2n)}(N; 1/t),
\]

\[
C_m^{(2n+1)}(N; t) = t^{n(2N+1)+m} \cdot C_m^{(2n+1)}(N; 1/t).
\]

Explicit formulas for the polynomials \( C_m^{(n)}(N; t) \) have been obtained in [15] for \( n = 1, 2, 3 \) and conjectured for \( n = 4 \). For example \( K_0^{(2)} = N/2 \) and

\[
C_m^{(2)}(N; t) = (-1)^{m+1} \frac{N}{2} \binom{m}{2} \left( \frac{(2N+1)^2}{4N(N+1)} \right)^m \sum_{n=0}^{2N+1-m} c_m^{(2)}(N; t)^n t^n,
\]

where for \( 0 \leq n \leq N - 1 \)

\[
c_{2n}^{(2)}(N) = c_{2; 2N-1-n}^{(2)}(N) = \sum_{k=0}^{n} a_k(N)a_{n-k}(N),
\]

\[
c_{1; 2N-n}^{(2)}(N) = \sum_{k=0}^{n} a_k(N)a_{n-k}(N + 1),
\]

and for \( 0 \leq n \leq N \)

\[
c_{0; N}^{(2)}(N) = c_{0; 2N+1-n}^{(2)}(N) = c_{2; n}^{(2)}(N + 1),
\]

and

\[
c_{1; N}^{(2)}(N) = \left( \frac{(1/2)_N}{N!} \right)^2 \{1 + 2NH_N(1/2)\}
\]

(37)
where
\[ a_n(N) = \frac{(1/2)_N(1/2 - N)_n}{(1 - N)_n n!} \] (38)
and
\[ H_N(1/2) = \sum_{k=0}^{N-1} \frac{1}{k + 1/2} \] (39)

It is certainly true (but not yet proven) that the factorizations (23) and (24) hold for all \( f^{(n)}_{N,N} \). The computations in [15] are based on Fuchsian differential equations for the \( f^{(n)}_{N,N}(t) \). For \( n = 4 \) the order of these equations is 20. These equations have a direct sum decomposition into operators which are homomorphic to symmetric powers and products of the operator which annihilates the hypergeometric function \( F_N \).

It is furthermore very suggestive that this factorization property has been previously seen in the correlation functions of the XXZ model [16]-[22].

The final property of the form factors to be discussed can best be illustrated by making a “lambda extension”, first introduced in [23], of the expansions (11) and (12) by defining
\[ C_- (M, N; \lambda) = (1 - t)^{1/4} \{ 1 + \sum_{n=1}^{\infty} \lambda^{2n} f^{(2n)}_{M,N} \} \] (40)
and
\[ C_+ (M, N; \lambda) = (1 - t)^{1/4} \sum_{n=0}^{\infty} \lambda^{2n} f^{(2n+1)}_{M,N}, \] (41)
which reduce to the Ising correlations below and above \( T_c \) when \( \lambda = 1 \). By use of a remarkable set of relations presented by Orrick, Nickel, Guttmann and Perk [24] in 2001 for small values of \( M \) and \( N \), these lambda extensions can be written in terms of theta functions [25]
\[ \theta_3(u; q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2nu \] (42)
\[ \theta_2(u; q) = 2q^{1/4} \sum_{n=0}^{\infty} q^{n(n+1)} \cos [(2n + 1)u] = q^{1/4} e^{iu} \theta_3(u + \pi \tau/2; q) \] (43)
and their derivatives
\[ \frac{d}{du} \theta_n(u; q) = \theta'_n(u; q) \] (44)
where \( t^{1/2} = k \) is the modulus of elliptic functions which is related to the nome \( q \) by
\[ q = e^{-\pi K'(t^{1/2})/K(t^{1/2})} \] (45)
and
\[ K(t^{1/2}) = \frac{\pi}{2} {}_2 F_1(1/2, 1/2; t) \] (46)
is the complete elliptic integral of the first kind with $K'(t^{1/2}) = K((1 - t)^{1/2})$

The simplest example given in [14] is for the low temperature case with $M = N = 0$

$$C_-(0, 0; \lambda) = \frac{\theta_3(u; q)}{\theta_3(0; q)} \quad \text{where} \quad \lambda = \cos u. \quad (47)$$

For the special values $\lambda = \cos(\pi m/n)$ we find that $C_-(0, 0; \lambda)$ and $t$ satisfy an algebraic equation. Calling $C_-(0, 0; \lambda) = \tau$, it is seen in [14] that for $\lambda = \cos \pi/3$

$$16\tau^{12} - 16\tau^8 - 8(t - 1)\tau^3 + t(1 - t) = 0, \quad (48)$$

which is a curve of genus one. For $\lambda = \cos(\pi/4)$,

$$16\tau^{16} + 16(t - 1)\tau^8 + t^2(t - 1) = 0 \quad (49)$$

is a curve of genus three which has the simple algebraic expression

$$C_-(0, 0; \cos(\pi/4)) = 2^{-1/4}(1 - t)^{1/16}[1 + (1 - t)^{1/2}]^{1/4} \quad (50)$$

Further results in this direction are [14]

$$C_-(1, 1; \cos(\pi/4)) = 2^{-3/4}(1 - t)^{1/16}[1 + (1 - t)^{1/2}]^{3/4} \quad (51)$$

$$C_-(2, 2; \cos(\pi/4)) = 2^{-5/4}(1 - t)^{1/16}[1 + (1 - t)^{1/2}]^{5/4}[5 - (1 - t)^{1/2}]^{3/4} \quad (52)$$

Further results which follow from [24] are given in [26]

$$C_+(0, 0; \lambda) = \frac{\theta_2(u; q)}{\theta_2(0; q)} \quad (53)$$

$$C_-(1, 1; \lambda) = -\frac{\theta_3(u; q)}{\sin u \theta_2(0; q) \theta_2^2(0; q)} \quad (54)$$

$$C_+(1, 1; \lambda) = -\frac{\theta_3(u; q)}{\sin u \theta_3(0; q) \theta_3^2(0; q)} \quad (55)$$

where is to be noted (for $N = 0, 1$) that $C_+(N, N; \lambda)$ is obtained from $C_-(N, N; \lambda)$ by the interchange $\theta_2 \leftrightarrow \theta_3$.

Many further results for various low values of $M, N$ remain (in the tradition of Kaufman and Onsager) to be published by the authors of [14].

4 Jimbo, Miwa and Painlevé

The immediate object of the computation of the leading term in the form factor expansion by Wu [7] for the row correlation $\langle \sigma_0, 0 \sigma_0, N \rangle$ and by Cheng and Wu [11] for the general case $\langle \sigma_0, 0 \sigma_M, N \rangle$ was to compute the leading behavior of the correlations functions for large separations $R = (M^2 + N^2)^{1/2}$. They found that for $T < T_c$ the correlation decays to the limiting value [9] as

$$\langle \sigma_0, 0 \sigma_M, N \rangle \sim (1 - t)^{1/4} \left\{1 - \frac{C_-(T)}{R^2} e^{-R/\xi(T)} \right\} \quad (56)$$
and vanishes for $T > T_c$ as

$$\langle \sigma_{0,0} \sigma_{M,N} \rangle \sim (1 - t)^{1/4} \frac{C_+(T)}{R^{1/2}} e^{-R/\xi_+(T)}, \quad (57)$$

where, in addition to depending on the temperature $T$, the $R$ independent quantities $C_+(T)$ and $\xi_+(t)$ depend on the ratio $M/N$. It is found in $[7]$ and $[11]$ as $t \to 1$ that

$$\xi_+(T) \sim \frac{A_+}{1 - t}, \quad (58)$$
$$C_+(T) \sim \frac{A_+}{(1 - t)^{1/2}}, \quad (59)$$

and

$$C_-(T) \sim \frac{A_-}{(1 - t)^2}. \quad (60)$$

where again the amplitudes $A_+ \xi$ and $A_+ \xi$ depend on the ratio $M/N$. Neither of these asymptotic leading terms reduces to the result valid for $T = T_c$ (i.e. $t = 1$) where in $[7]$ Wu found that the diagonal correlation has the leading behavior for large $N$

$$\langle \sigma_{0,0} \sigma_{N,N} \rangle \sim \frac{A_{T_c} N^{1/4}}{N^{3/4}} \quad (61)$$

and

$$A_{T_c} = 2^{1/12} e^{3\zeta'(-1)} \quad (62)$$

with $\zeta'(-1)$ the derivative of Riemann’s zeta function at $-1$.

The history of the result (61) is romantic in its own way. In the original 1949 paper of $[5]$ there is a remark that the diagonal correlation vanishes “slowly”. In 1959 Fisher $[27]$ derived the exponent $1/4$ and remarked in footnote 8 that

Onsager, private communication, has derived exact expressions for the correlations along the main diagonal $\cdots$

This computation was never published and perhaps there is another typescript out there waiting to be discovered.

Wu $[7]$ also found the large $N$ behavior of the row correlation $\langle \sigma_{0,0} \sigma_{0,N} \rangle$, which has the same dependence on $N$ as (61) but with an amplitude

$$A_{row} = A_{T_c} (\cosh 2E_h/k_B T_c)^{1/4}. \quad (63)$$

The first purpose of the paper $[8]$ was to connect the three different asymptotic behaviors $[56], [57]$ and (61) by defining an interpolating function, traditionally called a scaling function,

$$G_\pm (r) = \lim_{M,N \to \infty, t \to 1} (1 - t)^{-1/4} \langle \sigma_{0,0} \sigma_{M,N} \rangle \quad (64)$$

with

$$\left[ \frac{\sinh 2E_h/k_B T_c}{\sinh 2E^n/k_B T_c} \right]^{1/2} M^2 + \left[ \frac{\sinh 2E^n/k_B T_c}{\sinh 2E_h/k_B T_c} \right]^{1/2} N^2 \quad (1 - t) = r \quad \text{fixed.} \quad (65)$$
For this purpose the exponential representation of the correlation functions was derived. When the scaling function was computed it was discovered that $G_{\pm}(r)$ is expressed in terms of a Painlevé equation of the third kind

$$\frac{d^2 \eta}{d\theta^2} = \frac{1}{\eta} \left( \frac{d\eta}{d\theta} \right)^2 - \frac{1}{\theta} \frac{d\eta}{d\theta} + \eta^3 - \eta^{-1}$$

as

$$G_{\pm}(r) = \frac{1 \mp \eta(r/2)}{2\eta(r/2)^{1/2}} \exp \frac{1}{4} \int_{r/2}^{\infty} d\theta \eta \left( \frac{1 - \eta^2}{\eta} \right)^2 - \eta'(2\theta)^2$$

with the boundary condition

$$\eta(\theta) \sim 1 - \frac{2}{\pi} \lambda K_0(2\theta) \text{ as } \theta \to \infty,$$

where $K_0(2\theta)$ is the modified Bessel function and $\lambda = 1$.

This result was first announced in [28] and [29]. Two different proofs were given. The first, in [8], is based on Myers’ work [30] on the scattering of electromagnetic radiation from a strip and the second [23] is based on a direct manipulation of the exponential representation in the scaling limit.

It is at this point that I first learned of the existence of Sato, Miwa and Jimbo when in 1977 I received in the mail (how long ago it was that papers were sent by mail) a letter by the three of them with title “Studies on holonomic quantum fields II” [31] which generalized several of the results of [8] and made clear the relation of the Painlevé III equation with the massive Dirac equation. This letter was followed by many more where the only change in the title was that the Roman numeral was different and by a series of 5 papers with the title “Holonomic quantum field theory” [32]. These papers culminated in the groundbreaking paper “Studies on holonomic quantum fields XVII” [33] where it is derived that the diagonal Ising correlation function for a general temperature on the lattice and not in the scaling limit satisfies the sigma form of the Painlevé VI equation

$$\left( t(t-1) \frac{d^2 \sigma}{dt^2} \right)^2 = N^2 \left( t(t-1) \frac{d\sigma}{dt} - \sigma \right)^2 - 4 \frac{d\sigma}{dt} \left( t(t-1) \frac{d\sigma}{dt} - \sigma - 1/4 \right) \left( t \frac{d\sigma}{dt} - \sigma \right).$$

The diagonal correlation is related to $\sigma$ for $T > T_c$ by

$$\sigma(t) = t(t-1) \cdot \frac{d}{dt} \log \langle \sigma_{0,0} \sigma_{N,N} \rangle - 1/4,$$

with the boundary condition at $t = 0$ of

$$\langle \sigma_{0,0} \sigma_{N,N} \rangle = t^{N/2} \frac{(1/2)_N}{N!} + O(t^{1+N/2}),$$

and for $T < T_c$ by

$$\sigma(t) = t(t-1) \cdot \frac{d}{dt} \log \langle \sigma_{0,0} \sigma_{N,N} \rangle - t/4.$$
with the boundary condition

\[
\langle \sigma_0, 0; \sigma_{N,N} \rangle = (1 - t)^{1/4} \{ 1 - \frac{t^{N+1}}{2N+1} \left( \frac{(1/2)^{N+1}}{(N+1)!} \right)^2 + O(t^{N+2}) \} \tag{73}
\]

where \((a)_N = a(a+1) \cdots (a+N-1)\) for \(1 \leq N\) and \((a)_0 = 1\) is Poisson's symbol. These boundary conditions are obtained from the leading terms of (16) and (17) as \(t \to 0\). Furthermore the lambda extensions (40) and (41) satisfy the same Painlevé VI equation (69) where the \(\lambda\) appears as a boundary condition.

The six Painlevé equations have a long history \[34, 35\]. They are defined as those second-order nonlinear equations the location of whose branch points and essential singularities (but not poles) are independent of the boundary conditions and which cannot be reduced to simpler functions. Painlevé obtained three of these equations \[36\] and Gambier \[37\] obtained the remaining three including the PVI equation which in the general case has four parameters. However, the specific case of Painlevé VI needed for the Ising model (69) had already been obtained by Picard \[38\] in 1889. Subsequent to the discovery that this PVI equation characterizes the diagonal Ising model, this equation has appeared in many contexts \[39-41\] ranging from Poncelet polygons to mirror symmetry. The sigma form of the Painlevé equations was first obtained by Okamoto \[42\].

5 The susceptibility

The second purpose of the paper \[8\] was to begin the study of the magnetic susceptibility at zero magnetic field \(\chi(T)\), which is computed in terms of the correlation functions as

\[
k_B T \chi(T) = \sum_{M=-\infty}^{\infty} \sum_{N=-\infty}^{\infty} \{ \langle \sigma_{0,0} \sigma_{M,N} \rangle - M^2 \}, \tag{74}
\]

where \(M^2\) is the square of the spontaneous magnetization which was given in \[9\]. In order to evaluate the sums in (74) the exponential forms (14) and (15) which were the basis of computing the Painlevé III equation cannot be used and instead the exponentials must be expanded into the form factor representations (11) and (12). Using these forms the sums over \(M\) and \(N\) are easily evaluated as geometric series and the susceptibility is written as the infinite sum of \(n\) “particle” contributions

\[
k_B T \chi_+(T) = (1 - t)^{1/4} t^{-1/4} \sum_{j=0}^{\infty} \chi^{(2j+1)}(T) \text{ for } T > T_c \tag{75}
\]

\[
k_B T \chi_-(T) = (1 - t)^{1/4} \sum_{j=1}^{\infty} \chi^{(2j)}(T) \text{ for } T < T_c. \tag{76}
\]

In \[8\] the terms \(\chi^{(n)}(T)\) for \(n = 1, 2, 3, 4\) were studied. In the scaling limit the scaled \(\chi^{(n)}(T)\) for general \(n\) were given by Nappi \[13\] in 1978. For arbitrary
temperature the results in the isotropic case were obtained by Nickel \[9\] and \[10\] and for \(E_v \neq E_h\) in \[24\]

\[
\chi^{(j)}(T) = \frac{\cot^j \alpha}{j!} \int_{-\pi}^{\pi} \frac{d\omega_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{d\omega_{j-1}}{2\pi} \left( \prod_{n=1}^{j-1} \frac{1}{\sinh \gamma_n} \right) H^{(j)} \frac{1 + \prod_{n=1}^{j} x_n}{1 - \prod_{n=1}^{j} x_n},
\]

with

\[
x_n = \cot^2 \alpha \left[ \xi - \cos \omega_n - \sqrt{(\xi - \cos \omega_n)^2 - (\cot \alpha)^{-4}} \right]
\]

\[
\sinh \gamma_n = \cot^2 \alpha \sqrt{(\xi - \cos \omega_n)^2 - (\cot \alpha)^{-4}},
\]

where

\[
\cot \alpha = \sqrt{s_h/s_v}
\]

\[
\xi = (1 + s_h^{-2})^{1/2}(1 + s_v^2)^{1/2}
\]

\[
s_v = \sinh 2E_v/k_BT \quad s_h = \sinh 2E_h/k_BT
\]

\[
H^{(j)} = \left( \prod_{1 \leq i < k \leq j} h_{ik} \right)^2
\]

with

\[
h_{ik} = \cot \alpha \frac{\sin \frac{1}{2}(\omega_i - \omega_k)}{\sinh \frac{1}{2}(\gamma_i - \gamma_k)} = \frac{1}{\cot \alpha} \frac{\sin \frac{1}{2}(\gamma_i - \gamma_k)}{\sin \frac{1}{2}(\omega_i + \omega_k)},
\]

and \(\omega_j\) is defined in terms of the remaining \(\omega_i\) from \(\omega_1 + \cdots + \omega_j = 0 \mod 2\pi\). We note in particular that for \(E_v = E_h\)

\[
\chi^{(1)}(t) = \frac{t^{1/4}}{(1 - t^{1/4})^2}
\]

with \(t\) given by \[13\] and

\[
\chi^{(2)}(t) = \frac{(1 + t)E(t^{1/2}) - (1 - t)K(t^{1/2})}{3\pi(1 - t^{1/2})(1 - t)}
\]

with \(t\) given by \[10\].

### 5.1 The amplitude of the susceptibility divergence

The study of the susceptibility from the form factor expansions was initiated in 1973 in \[28\] where it was demonstrated that as \(T \to T_c \pm\) the susceptibility diverges as

\[
k_B T \chi(T) \sim C_{\pm} \left| \frac{s^{-1} - s}{2} \right|^{-7/4} \sqrt{2}.
\]

where in the isotropic case

\[
s = \sinh 2E/k_BT.
\]
The constants $C_-$ and $C_+$ are different and are given as infinite series

\[ C_- = \sum_{n=1}^{\infty} C^{(2n)} \quad C_+ = \sum_{n=0}^{\infty} C^{(2n+1)} \]  

(89)

where the $C^{(n)}$ are $n$-fold integrals coming from the form factor expansion and have been studied both numerically for $n = 1, \ldots, 5$ [28], [8]. The first term in each of (89) has been analytically evaluated in [28], [8]

\[ C^{(1)} = 1 \quad C^{(2)} = \frac{1}{12\pi} \]  

(90)

and the next leading term was evaluated by Tracy [44] as

\[ C^{(3)} = \frac{1}{2\pi^2} \left( \frac{\pi^2}{3} + 2 - 3\sqrt{3}\text{Cl}_2(\pi/3) \right) \]  

(91)

where

\[ \text{Cl}_2(\theta) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2} \]  

(92)

is Clausen’s function and

\[ C^{(4)} = \frac{1}{16\pi^3} \left( \frac{4\pi^2}{9} - \frac{1}{6} - \frac{7}{2}\zeta(3) \right). \]  

(93)

In the tradition of Onsager and Kaufman [3] the details are only in an unpublished typescript. A curious feature of these results is that the ratio $C_+/C_-$ is found to be closely approximated by $12\pi$ and the second terms are approximately three orders of magnitude less than the leading term. The study of the constants $C_-$ and $C_+$ has been continued by high precision numerical computations [24] and the most recent evaluation [45] in 2011 is to an incredible 104 places. This is one of the most precisely determined constants in all of mathematical physics.

However, the $\chi^{(n)}(w)$ have singularities at other points besides $\sinh E/k_B T = \pm 1$ and the determination of the analytic properties of the magnetic susceptibility as a function of temperature has become the most challenging problem in the field.

### 5.2 Nickel singularities and the natural boundary conjecture

The first studies of analytic properties after the initial computations of [8] were made in 1999 [9] and 2000 [10] when Nickel demonstrated for the isotropic case $E^v = E^h = E$ that the integrals (44) $\chi^{(n)}$ have singularities in the complex $T$ plane on the curve

\[ |\sinh 2E/k_B T| = 1, \]  

(94)
which is the same curve on which the four Pfaffians of Kaufman’s original evaluation [4] of the Ising partition function vanish. This was extended to the general case $E^v \neq E^h$ in [24] where the singularities of $\chi^{(n)}(T)$ are at

$$\cosh\frac{2E^v}{k_BT} \cosh\frac{2E^h}{k_BT}$$

$$- \sinh\frac{2E^h}{k_BT} \cos(2\pi j/n) - \sinh\frac{2E^v}{k_BT} \cos(2\pi k/n) = 0. \quad (95)$$

with

$$0 \leq j, k \leq \left\lfloor \frac{n}{2} \right\rfloor, \ j = k = 0 \text{ excluded} \quad (96)$$

where $\lfloor x \rfloor$ is the integer part of $x$ and for $n$ even $j + k = n/2$ is also excluded. In terms of the variable used in [46]-[55] for the isotropic lattice with $s$ given by

$$w^{-1} = 2(s + s^{-1}) \quad (97)$$

these singularities for $n = 3, 4, 5, 6$ are given in Table 5.2

| $n$ | $w$          |
|-----|-------------|
| 3   | $-1/2, 1$   |
| 4   | $\pm 1/2$   |
| 5   | $-1, -\frac{1+i\sqrt{3}}{2}, \frac{3+i\sqrt{3}}{2}$ |
| 6   | $\pm 1, \pm 1/3$ |

where we note that $\sinh 2E/k_BT$ is real for $-1/4 \leq w \leq 1/4$ and is complex with $|\sinh 2E/k_BT| = 1$ for $1/4 < |w|$. If we call $\epsilon$ the deviation from the singular temperatures $T^{(j)}_{m,m'}$ determined by (95), then for $T > T_c$ the singularity in $\chi^{(2j+1)}(T)$ is

$$\epsilon^{2j+1} \ln \epsilon \quad (98)$$

and for $T < T_c$ the singularity in $\chi^{(2j)}(T)$ is

$$\epsilon^{2j^2-3/2}. \quad (99)$$

It is striking that the number of singularities increases with $n$ and becomes dense in the limit $n \to \infty$. This feature led Nickel to the conclusion that unless cancellations occur there will be a natural boundary in the susceptibility $\chi(T)$ in the complex $T$ plane at the location (95). The existence of a natural boundary in the complex temperature plane is not contemplated in the scaling theory of critical phenomena.

### 5.3 Fuchsian equations

The next step in the study of the susceptibility was begun in 2005 [46] and has continued in the series of papers [47]-[55]. In these papers exact Fuchsian differential equations for the $\chi^{(n)}(T)$ in the isotropic case $E^v = E^h$ are determined
by use of Maple by first expanding the integrals in an appropriate variable such as $w$ or $w^2$ and then using Maple programs which obtain ODE’s from these series. The resulting differential equations have very special properties such as being globally nilpotent [52] which allow for extensive analysis to be carried out. These studies have uncovered several new and important features of the susceptibility; namely that the $\chi^{(n)}(w)$ have a direct sum decomposition and that they have further singularities beyond those of [54].

5.3.1 Direct sum decompositions

In [49] and [54] it is shown for $1 \leq n \leq 6$ that $\chi^{(n)}(w)$ have the same direct sum decomposition seen already in the diagonal form factors

$$\chi^{(2n)}(w) = \sum_{m=1}^{n-1} K_m^{(2n)} \chi^{(2m)}(w) + \Omega^{(2n)}(w)$$

$$\chi^{(2n+1)}(w) = \sum_{m=1}^{n-1} K_m^{(2n+1)} \chi^{(2m+1)}(w) + \Omega^{(2n+1)}(w)$$

where the $\Omega^{(n)}(w)$ satisfy Fuchsian equations of order $m$

$$L_m^{(n)} \cdot \Omega^{(n)} = 0$$

with

$$n \quad 3 \quad 4 \quad 5 \quad 6$$

$$m \quad 6 \quad 8 \quad 29 \quad 46$$

The $K_j^{(n)}$ are constants which for $n = 3, 4, 5, 6$ coincide with the values of $K_m^{(n)}(0)$ given in [26]-[29].

The operators in (102) factorize further. For $L_6^{(3)}$ and $L_8^{(4)}$ we have we have

$$L_6^{(3)} = L_3^{(3)} \cdot L_2^{(3)} \cdot L_1^{(3)}$$

and

$$L_8^{(4)} = L_4^{(4)} \cdot L_1^{(4)} \cdot \left( L_{1:;a}^{(4)} \oplus L_{1:k}^{(4)} \oplus L_{1:e}^{(4)} \right)$$

where the numeral in the subscript indicates the order of the operators which are given in [48] and [49]. The operator $L_{29}^{(5)}$ has been found in [51], [53] and [55] to have the factorization

$$L_{29}^{(5)} = L_5^{(5)} \cdot L_{12}^{(5)} \cdot L_1^{(5)} \cdot L_{11}^{(5)}$$

where $L_{11}^{(5)}$ has the further direct sum decomposition (A.1) of [53]

$$L_{11}^{(5)} = (Z_2 \cdot N_1) \oplus V_2 \oplus (F_3 \cdot F_2 \cdot L_1^*)$$
Similarly in (56) and (57) of [54] the operator $L_{46}^{(6)}$ is shown to have the decomposition

$$L_{46}^{(6)} = L_6^{(6)} \cdot L_{23}^{(6)} \cdot L_{17}^{(6)}$$

(108)

where $L_{17}^{(6)}$ has a direct sum decomposition into the sum of four operators but the possible reducibility of $L_{23}^{(6)}$ has not yet been determined due to computational complexity.

### 5.3.2 Singularities

The location of the singularities of the operators $L_m^{(n)}$ are obtained by examining the roots of the polynomial multiplying the highest derivative $d^m/dw^m$ and this analysis shows that there are further singularities beyond the singularities at $w = \pm 1/4$, $\infty$ and the Nickel singularities (95).

In [47] that the differential equation for $\chi^{(3)}(w)$ admits additional singularities at

$$w = \frac{-3 \pm i\sqrt{7}}{8}$$

(109)

which correspond to

$$s = \frac{-1 \pm i\sqrt{7}}{4}, \quad |s| = \frac{1}{\sqrt{2}}$$

(110)

$$s = \frac{-1 \pm i\sqrt{7}}{2}, \quad |s| = \sqrt{2}$$

(111)

where we note the the singularity at (110) is inside the unit circle $|s|=1$ and thus cannot appear in the principle sheet of the integral for $\chi^{(3)}$ which is analytic for $|s| < 1$.

There are no additional singularities in $\chi^{(4)}(w)$ and the singularities of $\chi^{(5)}(w)$ are shown in (34) of [51] to be at the roots of following polynomial

$$w^{33}(1-4w)^2(1+4w)^{16}(1-w)^2(1+2w)^4(1+3w+4w^2)$$

$$(1+w)(1-3w+w^2)(1+2w-4w^2)$$

$$(1-w-3w^2+4w^3)(1+8w+20w^2+15w^3+4w^4)(1-7w+5w^2-4w^3)$$

$$(1+4w+8w^2)(1-2w)$$

(112)

The singularities located by the roots of the first line in (112) are identical with the location of singularities of $\chi^{(3)}$ and the roots of the second line are the Nickel singularities of of $\chi^{(5)}$. Most of the remaining singularities correspond to complex values of $s$ not on $|s| = 1$.

### 6 Diagonal susceptibility

The integrals (77) for the $n$ particle contribution to the susceptibility $\chi^{(n)}(T)$ are quite complex and the Maple-based studies cannot be extended much beyond
their present limits. Therefore it would be of great utility if a simpler set of integrals could be found which would still incorporate all significant analytic features of the the $\chi^{(n)}$. Several such simplified modifications of the integrals have been studied \[50\] but by far the most natural case is to restrict the two dimensional sum over the lattice positions $M, N$ in (74) to the lattice diagonal $M = N$ and thus to consider the susceptibility that will result if a magnetic field is applied only to the diagonal

$$k_B T \chi_d(t) = \sum_{N=-\infty}^{\infty} \{ \langle \sigma_{0,0} \sigma_{N,N} \rangle - M^2 \},$$

(113)

where the dependence on $T$ is now for all $E^v$ and $E^h$ in terms of the single variable $t$ defined by \[10\] for $T < T_c$ and by \[13\] for $T > T_c$.

This diagonal susceptibility has been studied in \[56\] and \[57\] and has been found to have the remarkable simplification over the bulk susceptibility that all singularities of the differential equations are at $s = 0, \infty$ and $|s| = 1$. There are no other complex singularities for $|s| \neq 1$ such as appear in $\chi^{(n)}(t)$. Furthermore $\chi_d^{(3)}(t)$ and $\chi_d^{(4)}(t)$ have been found to be explicitly expressed in terms of generalized hypergeometric functions $p+1 F_p$.

### 6.1 Integral representations

From the integral expressions for $f_{N,N}(t)$ of \[12\] and \[14\], we find in \[56\] the expansion for $T < T_c$

$$k_B T \chi_d^-(t) = (1-t)^{1/4} \sum_{n=1}^{\infty} \chi_d^{(2n)}(t)$$

(114)

and for $T > T_c$

$$k_B T \chi_d^+(t) = (1-t)^{1/4} \sum_{n=0}^{\infty} \chi_d^{(2n+1)}(t),$$

(115)

where

$$\chi_d^{(2n)}(t) = \frac{t^n}{(n!)^2} \frac{1}{\pi^2 n} \cdot \int_0^1 \cdots \int_0^1 \prod_{k=1}^{2n} dx_k \cdot \frac{1 + t^n x_1 \cdots x_{2n}}{1 - t^n x_1 \cdots x_{2n}}$$

$$\times \prod_{j=1}^{n} \left( \frac{x_{2j-1} (1 - x_{2j}) (1 - tx_{2j})}{x_{2j} (1 - x_{2j-1}) (1 - tx_{2j-1})} \right)^{1/2} \prod_{1 \leq j \leq n} \prod_{1 \leq k \leq n} (1 - tx_{2j-1} x_{2k})^{-2}$$

$$\times \prod_{1 \leq j < k \leq n} (x_{2j-1} - x_{2k-1})^2 (x_{2j} - x_{2k})^2$$

(116)

and for $T > T_c$

$$\chi_d^{(2n+1)}(t) = \frac{t^{n(n+1)}}{\pi^{2n+1} n! (n+1)!} \cdot \int_0^1 \cdots \int_0^1 \prod_{k=1}^{2n+1} dx_k$$

(117)
\begin{align*}
&\times \frac{1 + t^{n+1/2} x_1 \cdots x_{2n+1}}{1 - t^{n+1/2} x_1 \cdots x_{2n+1}} \cdot \prod_{j=1}^{n} \left(1 - x_{2j} \right) \left(1 - t x_{2j} \cdot x_{2j} \right)^{1/2} \\
&\times \prod_{j=1}^{n+1} \left(1 - x_{2j-1} \right) \left(1 - t x_{2j-1} \cdot x_{2j-1} \right)^{-1/2} \\
&\times \prod_{1 \leq j < k \leq n+1} \left(1 - t x_{2j-1} x_{2k} \right)^{-2} \\
&\times \prod_{1 \leq j < k \leq n} \left(1 - t x_{2j-1} \cdot x_{2k} \right)^{-2} \\
&\times \prod_{1 \leq j < k \leq n+1} \left(x_{2j-1} - x_{2k-1} \right)^2 \\
&\prod_{1 \leq j < k \leq n} \left(x_{2j} - x_{2k} \right)^{2}. \quad (117)
\end{align*}

The expressions (116) and (117) are, indeed, much simpler than the corresponding expressions for \( \chi^{(n)} \) given in (77). In particular

\[ \chi_d^{(1)}(t) = \frac{1}{1 - t^{1/2}} \quad (118) \]

and

\[ \chi_d^{(2)}(t) = \frac{t}{4(1 - t)} \quad (119) \]

which are simpler than (85) and (86) respectively. Most noticeable is that \( \chi_d^{(2)}(w) \) in (86) has a logarithmic singularity at \( t = 1 \) \((w = 1/4)\) while \( \chi_d^{(2)}(t) \) in (119) does not.

### 6.2 Root of unity singularities

In addition to the singularity at \( t = 1 \) it is straightforward to see from the integral expressions (116) and (117) that \( \chi_d^{(2n)}(t) \) has singularities at

\[ t_0^n = 1 \quad (120) \]

of the form

\[ \epsilon^{2n^2 - 1} \ln \epsilon \quad (121) \]

and \( \chi_d^{(2n+1)}(t) \) has singularities

\[ t_0^{n+1/2} = 1 \quad (122) \]

of the form

\[ \epsilon^{(n+1)^2 - 1/2} \quad (123) \]

where \( \epsilon \) is the deviation from \( t_0 \). These are the analogues for the diagonal susceptibility of the Nickel singularities of the bulk susceptibility \( \chi^{(n)} \) of (95).
6.3 Direct sum decomposition

The \( \chi_d^{(n)}(t) \) have the same direct sum decomposition seen already in the diagonal form factors and \( \chi^{(n)}(w) \)

\[
\chi_d^{(2n)}(t) = \sum_{j=1}^{n-1} K_{d,j}^{(2n)} \chi_d^{(2j)}(t) + \Omega_d^{(2n)}(t)
\]

\[
\chi_d^{(2n+1)}(t) = \sum_{j=1}^{n-1} K_{d,j}^{(2n+1)} \chi_d^{(2j+1)}(t) + \Omega_d^{(2n+1)}(t)
\]

where \( K_{d,j}^{(n)} \) are constants. However, unlike \( \chi^{(n)}(w) \), the operators \( L_d^{(n)} \) which annihilate \( \Omega_d^{(n)}(t) \) have a further direct sum decomposition

\[
L_{d,5}^{(3)} = L_{d,2}^{(3)} + L_{d,3}^{(3)} \quad \text{and} \quad L_{d,7}^{(4)} = L_{d,3}^{(4)} + L_{d,4}^{(4)}
\]

6.4 Results for \( \chi_d^{(3)}(t) \)

For \( \chi_d^{(3)}(t) \) we explicitly find by combining \[52\] and \[56\] and setting \( x = t^{1/2} \) that

\[
\chi_d^{(3)}(x) = \frac{1}{3} \chi_{d;1}^{(3)}(x) + \frac{1}{2} \chi_{d;2}^{(3)}(x) - \frac{1}{6} \chi_{d;3}^{(3)}(x)
\]

where

\[
\chi_{d;1}^{(3)}(x) = \frac{1}{1-x} = \chi_d^{(1)}(x)
\]

\[
\chi_{d;2}^{(3)}(x) = \frac{1}{(1-x)^3} 2F_1(1/2,-1/2;1;x^2) - \frac{1}{1-x^2} 2F_1(1/2,1/2;1;x^2)
\]

and

\[
\chi_{d;3}^{(3)}(x) = \frac{(1+2x)(x+2)}{(1-x)(x^2+x+1)} \left[ F(1/6, 1/3; 1; Q)^2 + \frac{2Q}{9} F(1/6, 1/3; 1; Q)(F(7/6, 4/3; 2; Q) \right]
\]

with

\[
Q = \frac{27}{4} \frac{(1+x)^2 x^2}{(x^2 + x + 1)^3}
\]

where we note that

\[
1 - Q = \frac{(1-x)^2(1+2x)(2+x)^2}{4(1+x+x^2)^3}
\]

From \[127\] and \[128\] we see that in \[125\] we have \( K_{1}^{(1)} = 1/3 \).

We see from \[117\] that \( \chi_d^{(3)}(x) \) vanishes when \( x \to 0 \) as \( x^4 \). However, \( \chi_{d;1}^{(3)}(x) \) and \( \chi_{d;3}^{(3)}(x) \) are constant as \( x \to 0 \) and \( \chi_{d;2}^{(3)}(x) \) vanishes linearly in \( x \). The three
constants in \([127]\) are determined by matching with the \(x^4\) behavior of \(\chi_d^{(3)}\) and this requires that the three constants will solve a set of five (overdetermined) linear equations.

As \(x \to 1\) we find that \(\chi_d^{(3)}\) diverges as

\[
\chi_d^{(3)}(x) = \frac{1}{1 - x} \left( \frac{1}{3} - \frac{5\pi}{18\Gamma^2(5/6)\Gamma^2(2/3)} + \frac{4\pi}{\Gamma^2(1/6)\Gamma^2(1/3)} \right) = \frac{0.016329 \cdots}{1 - x}
\]

Furthermore \(\chi_d^{(3)}(x)\) has an additional singularity at \(x \to e^{\pm 2\pi i/3}\) which, to leading order is

\[
\chi_d^{(3)} \text{sing} \to \frac{3^{4/5}16}{35\pi} e^{\pm 5\pi i/12} e^{7/2}
\]

### 6.5 Results for \(\chi_d^{(4)}(t)\)

These results have been extended in [56] and [57] to \(\chi_d^{(4)}(t)\) where it is shown that

\[
\chi_d^{(4)}(t) = \frac{1}{23} \chi_d^{(4)}(t) + \frac{1}{3} \cdot \frac{1}{23} \chi_d^{(4)}(t) - \frac{1}{23} \chi_d^{(4)}(t)
\]

where

\[
\chi_d^{(4)}(t) = \chi_d^{(2)}(t)
\]

\[
\chi_d^{(4)}(t) = \frac{1 + t}{(1 - t)^2} F_1(1/2, -1/2; 1; t)^2 - 2 F_1(1/2, 1/2; 1, t)^2
\]

\[
- \frac{2t}{1 - t} F(1/2, 1/2; 1; t) F(1/2, -1/2; 1, t)
\]

and

\[
\chi_d^{(4)}(t) = A_3 \cdot 4 F_3([1/2, 1/2, 1/2, 1/2]; [1, 1, 1] t^2)
\]

with

\[
A_3 = 2(1 + t) t^3 D_t^3 + \frac{16t^2 - t - 11}{3} t^2 d_t^2
\]

\[
+ \frac{13t^2 - 4t - 11}{3} t D_t + t
\]

The singular behavior as \(t \to 1\) of \(\chi_d^{(4)}(t)\) and \(\chi_d^{(4)}\) is easily obtained and the singularity of \(\chi_d^{(4)}\) at \(t = 1\) is obtained by use of the analytic continuation formula of Bühring [61]. The final result [57] is that as \(t \to 1\)

\[
\chi^{(4)}(t) \to \frac{1}{8(1 - t)} \left( 1 - \frac{1}{3\pi^2} [64 + 16(3I_1 - 4I_2)] \right) + \frac{7}{16\pi^2} \ln \frac{16}{1 - t} - \frac{1}{16\pi^2} \ln^2 \frac{16}{1 - t}
\]
where
\[ 3I_1 - 4I_2 = -2.2128121 \ldots \] (141)

has been given to 100 digits.

At the root of unit singularity \( t = -1 \) the leading singular behavior is
\[ \chi_d^{(4)} \rightarrow \frac{1}{26880} (1 + t)^7 \ln(1 + t) \] (142)

6.6 \( \chi_d^{(5)}(t) \)

The ODE satisfied by \( \chi_d^{(5)}(x) \) has been studied in [57] modulo a large prime. It is found that the minimal order ODE is of order 19 and that the operator \( L_{d;19} \) has the decomposition
\[ L_{d;19}^{(5)} = L_{d;2}^{(3)} \oplus L_{d;17}^{(15)} \] (143)

with
\[ L_{d;17}^{(5)} = L_{d;6}^{(5)} \cdot L_{d;11}^{(5)} \] (144)

where \( L_{d;2}^{(3)} \) annihilates \( \chi_{d;2}^{(3)}(x) \) and \( L_{d;17}^{(15)} \) has singularities at \( x = 0, \infty, 1, -1, x_3 = e^{\pm 2\pi i/3}, x_5 = e^{\pm 4\pi i/5}, x_7 = e^{\pm 6\pi i/7} \) where the non-integer exponents at \( x_3 \) are \( 5/2, 7/2, 7/2 \) and at \( x_5 \) are \( 23/2 \).

It has been further found that
\[ L_{d;11}^{(5)} = L_{d;3}^{(15)} \oplus L_{d;1}^{(15)} \oplus \left( W_{d;1}^{(5)} \cdot U_{d;1}^{(5)} \right) \oplus \left( L_{d;4}^{(5)} \cdot V_{d;1}^{(5)} \cdot U_{d;1}^{(5)} \right) \] (145)

where \( L_{d;m}^{(3)} \) annihilates \( \chi_{d;m}^{(3)} \) and the remaining operators in this decomposition are all given in [57].

6.7 Singularities and cancellations

By examining the integral representations for \( \chi^{(n)}(w) \) (77) and \( \chi_d^{(n)}(t) \) (110, 117) it is clear that these integrals have no singularities for \( |\sinh 2E/k_B T| < 1 \). The singularities at \( |\sinh 2E/k_B T| < 1 \) of the differential equations for \( \chi^{(n)}(w) \) will only appear in analytic continuations of the integral in the complex plane of the variable \( w \). The corresponding differential equations for \( \chi_d^{(n)}(t) \) are significantly simpler because they have singularities only at \( t = 0, \infty \) and \( |t| = 1 \).

It remains to discuss the singularities in the differential equations which do lie on \( |\sinh 2E/k_B T| = 1 \) and to give an explanation for the observation that the singularities of the ODEs for \( \chi^{(n-2m)}(w) \) and \( \chi_d^{(n-2m)}(t) \) are also singularities of \( \chi^{(n)}(w) \) and \( \chi^{(n)}(t) \) respectively even though the integrands are singular only at the points given by (95) for \( \chi^{(n)}(w) \) and by (120) and (122) for \( \chi_d^{(n)}(t) \).

The resolution of this is easily seen for \( \chi_d^{(n)}(t) \). By an examination of the integrals (110) and (117) we see that there are paths of analytic continuation possible in the complex \( t \) plane where the contour of integration must be deformed past the pole at
\[ 1 - tx_{2j}x_{2k+1} = 0 \] (146)
and the residue at that pole will reduce the denominators in $\chi^{2n}_d(t)$ and $\chi^{(2n+1)}_d(t)$ from

$$1 - t^n x_1 \cdots x_{2n}$$

and

$$1 - t^{n+1/2} x_1 \cdots x_{2n+1}$$

(147)

(148)

to the denominators in $\chi^{(2n-2)}_d(t)$ and $\chi^{(2n-1)}_d(t)$ respectively with $n \to n-1$ and two less integration variables. Therefore, the singularities of $\chi^{(n-2m)}_d(t)$ will not appear on the principle sheet of the integral which is analytic at $t = 0$ but only on analytic continuations to non-physical branches. The similar phenomenon occurs for $\chi^{(n)}(w)$.

It remains to reconcile this non appearance of the singularities of $\chi^{(n-2m)}_d(t)$ in the physical sheet of $\chi^{(n)}_d(t)$ with the direct sum decompositions (124) and (125). This will be accomplished by showing that the term $\Omega^{(n)}_d(t)$ has singularities which exactly cancel the singularities on $\chi^{(n)}_d(t)$. This requires the solution of a global connection problem which has not yet been explicitly done even though from the examination of the original integral the resulting exact cancellation must hold.

7 Conclusion

Now that we have summarized the known features of the Ising correlations, form factors and susceptibility we can proceed to discuss what is not known. This is the fascinating, mysterious and thus romantic part of the subject.

7.1 Conformal and quantum field theory

One of the most important features of the Ising model is that the scaling limit satisfies all the axioms for a massive Euclidean quantum field theory and that at $T = T_c$ the long range correlations are those of a conformal field theory with central charge $c = 1/2$. This is in fact the earliest conformal field theory known and from this beginning a vast new field of mathematics and physics has been developed in the last 30 years. However, the Ising model is much more than a conformal field theory because we have a vast number of results for $T \neq T_c$ which are the simplest examples of properties of massive Euclidean quantum field theories. Part of the romance is the exploration of how these Ising results can be used to extend massless conformal field theories into the massive region.

7.2 Form factors, exponential forms and amplitudes

The derivation [12] of the exponential and form factor expansion for the diagonal Ising correlation is much more general than this special case. Indeed in [12] it is proven that every Toeplitz determinant (4) with a generating function $\phi(\xi)$ such that $\ln \phi(\xi)$ is continuous and periodic on $|\xi| = 1$ has both an exponential
and a form factor expansion. Furthermore these Toeplitz determinants are also expressible as Fredholm determinants \[58\] (at times in several different ways \[59\]). Consequently the Ising computations have subsequently been extended to several very important problems including the seminal work on the one dimensional impenetrable Bose gas and on random matrices by Jimbo, Miwa, Mori and Sato \[60\].

To illustrate the differences between the form factor and the exponential representation of the correlation functions, we consider the computation by Tracy \[62\] of the constant \(A_T\) of \(62\). In the scaling limit the scaled correlations in the general case where \(E^v \neq E^h\) depend only on the single variable \(r\) \[65\]. Therefore we can restrict attention to the scaled form of the diagonal correlation \(\langle \sigma_{0,0} \sigma_{N,N} \rangle\) and consider the lambda extension of the scaling form of the exponential form \(144\) which we write as

\[
G_-(r; \lambda) = \exp \sum_{n=1}^{\infty} \lambda^{2n} g^{(2n)}(r),
\]

where

\[
g^{(2n)}(r) = \lim_{\text{scaling}} F^{(2n)}_{N,N}(t),
\]

which depends on the single variable \(r\) instead of the two independent variables \(N\) and \(t\). Tracy finds that, as \(r \to 0\),

\[
g^{(2n)}(r) = -\alpha_n \ln r + \beta_n + o(1).
\]

Therefore, defining the lambda dependent sums

\[
\alpha(\lambda) = \sum_{n=1}^{\infty} \lambda^{2n} \alpha_n \quad \beta(\lambda) = \sum_{n=1}^{\infty} \lambda^{2n} \beta_n,
\]

we find

\[
G_-(r; \lambda) \sim \exp\{-\alpha(\lambda) \ln r + \beta(\lambda)\} = e^{\beta(\lambda) / r^{\alpha(\lambda)}}.
\]

In the Ising case where \(\lambda = 1\) the functions specialize to

\[
\alpha(1) = 1/4 \quad \beta(1) = \ln A
\]

where \(A\) is the constant in \(151\).

If, however, instead of the scaled exponential form we define the scaled limit of the form factors \(f^{(2n)}_{N,N}(t)\) as

\[
\tilde{f}^{(2n)}(r) = \lim_{\text{scaling}} f^{(2n)}_{N,N}(t),
\]

then as \(r \to 0\)

\[
\tilde{f}^{(2n)}(r) = \sum_{k=0}^{n} a_k^{(2n)} \ln^k r + o(1).
\]
Thus, in order for \(153\) to agree with the \(r \to 0\) behavior of the form factor expansion, we need

\[
e^{\beta(\lambda)} \left/ r^{\alpha(\lambda)} \right. = \left[ \sum_{k=0}^{\infty} \frac{1}{k!} \left( \ln r \sum_{n=1}^{\infty} \lambda^{2n} \alpha_n \right)^k \right] \left[ \sum_{k=0}^{\infty} \frac{1}{k!} \left( \sum_{n=1}^{\infty} \lambda^{2n} \beta_n \right)^k \right] \]

\[
= 1 + \sum_{n=1}^{\infty} \lambda^{2n} \sum_{k=0}^{n} \tilde{a}_k^{(2n)} \ln^k r
\]

(157)

to hold term by term for each power \(\lambda^{2n}\). The requires an infinite number of identities between the \(\tilde{a}_k^{(2n)}\).

As an additional remark we note that if we rewrite the integral (16) for \(f^{(2n)}_{N,N}\) as a contour integral, rescale the variables \(x_k\) by \(x_k = t^{-1/2}y_k\) and then send \(y_{2k} \to 1/y_{2k}\) we see that as \(t \to 1\) the integral has logarithmic divergences as in (156). The amplitudes \(a_n\) are closely related to the special case with \(\rho = 1\) of the integral found by Dotsenko and Fateev \([63]\) in their study of four point correlations in conformal field theories with central charge \(c \leq 1\).

\[
I_{n,m}(\alpha, \beta; \rho) = \frac{1}{n!m!} \prod_{i=1}^{n} \int_{0}^{1} dt_i \prod_{i=1}^{m} \int_{0}^{1} d\tau_i (1 - \tau_i)^{\beta} P \prod_{i<j}^{n,m} \left( t_i - t_j \right)^2 \prod_{i<j}^{n,m} \left( \tau_i - \tau_j \right)^2
\]

(158)

where \(P\) indicates the principal value and

\[
\alpha' = -\rho' \alpha, \quad \beta' = -\rho' \beta, \quad \rho' = \rho^{-1}
\]

(159)

## 7.3 Exponentiation

Form factor expansions exist for many massive models of quantum field theory including sine-Gordon and the non-linear sigma model \([64]\) and similar form factor expansions exist \([65],[66]\) for the XXZ model on a chain of finite length

\[
H_{XXZ} = -\sum_{j=1}^{L} \{ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^x \sigma_{j+1}^x + H \sigma_j^z \}
\]

(160)

Moreover the Feynman expansion of amplitudes in quantum field theory is also what we have called here a form factor expansion. In all of the models there are limiting cases where series of multiple dimensional integrals expand to series in powers of logarithms which need to be summed. However, unlike the Ising correlation functions these form factor expansions do not come from either Toeplitz or Fredholm determinants and thus the exponentiation methods of Ising correlations are not applicable.

Over the years an immense effort has been made to sum the form factor series of logarithms. In quantum field theory this starts with the classic 1939
paper of Bloch and Nordsiek [67] on resummation of infrared divergences in quantum electrodynamics. A second example is the Regge theory of the 60’s and 70’s an 2nth order Feynman diagram expansion of a four point scattering amplitude is shown to diverge as the energy \( s \to \infty \) with a fixed momentum transfer \( t \) as

\[
g^{2n} \alpha(t)^n \frac{\ln^n s}{n!}
\]

This is a “leading” log approximation and is analagous to the Ising case if only the first term in the series \( \alpha_n \) for \( \alpha_n \) is retained. More recently there has been a great deal of work on quantum chromodynamics [69] where many non leading terms summed by the use of an ingenius decomposition of the multidimensional integrals.

In the theory of integrable systems a great deal of effort has been devoted to compute the long range asymptotic behavior of the correlations of the XXZ model in the massless region \(-1 < \Delta < 1\) from multiple integral representations. One method is presented in [70] which shows how to modify the Fredholm determinant form which holds for \( \Delta = 0 \) by suitably picking out the important pieces of the multiple integrals. This has led to the computation of both the exponents and the amplitude of the long range behavior of the correlations when the \( H \neq 0 \). The study of the correlations from the form factors is begun in [65]–[66] with more results announced to be forth coming. A full exploration of the relation of these subjects is beyond the scope of this article.

7.4 Short distance versus scaling terms

In the \( n \)-particle expansions of the full (76), (75) and the diagonal (114), (115) susceptibility the \( \chi^{(n)}(t) \) and the \( \chi^{(n)}_d(t) \) will (for \( n \geq 3 \) ) have terms which contain powers of \( \ln t \). From this it might be inferred that the susceptibility will contain terms of the form \((1-t)^{1/4+p} \ln^q(1-t)\). However, from the extensive calculations on long low and high temperature series expansions made in [24] and [45] such terms do not appear to exist. Instead the susceptibility is conjectured to have the form for \( t \to 1 \) of

\[
k_B T \chi(t)_\pm = (1-t)^{-7/4} \sum_{j=0}^\infty C_{j,\pm}^j (1-t)^j + \sum_{q=0}^\infty \sum_{p=0}^{[\sqrt{q}]} b_{p,q}^\pm (1-t)^q \ln^p (1-t)\] (162)

The first term is called the “scaling function”. The second term is called the “background” or “short distance” term and is numerically obtained by summing correlation functions instead of form factors. In [45] it is stated that the “scaling function” is determined by conformal field theory while for the “short distance” term here is “no explicit prediction”. In [24] the belief is stated that the separation into “scaling” and “short distance” parts is “tantamount to the scaling argument that in the critical region there is a single length scale proportional to \((1-t)^{-\nu}\) with \( \nu = 1 \). It would be highly desirable if this distinction between “scaling” and “short distance” terms could be made precise and if both terms could be obtained by use of the form factor expansion alone.
7.5 Natural boundaries and $\lambda$ extensions

Perhaps the most perplexing question concerning the relation of the Ising model on a lattice with the scaling field theory limit is the existence of the natural boundary in the susceptibility implied by the singularities (98), (99) found by Nickel [9], [10]. The magnetic susceptibility is the second derivative of the free energy with respect to an external magnetic field $H$ interacting with the spins as $-H \sum_{j,k} \sigma_{j,k}$. In the scaling limit the Ising model in a magnetic field is also a field theory and the analyticity properties of this field theory have been extensively studied by Fonseca and Zamolodchikov [71] with the conclusion that there is no natural boundary. How can this be reconciled with the computations of [9] and [10]?

The existence of the natural boundary suggested by Nickel in [9] and [10] rests on the accumulation of the singularities (98) and (99) and the assumption that there is no cancellation. However, for this argument to hold we need to be able to show that the limit of $t$ approaching the location of the supposed natural boundary (95) will commute with the infinite sum over the $n$ particle contributions $\chi^{(n)}(T)$ in (75) and (76). Since the natural boundary does not exist if only a finite number of the $\chi^{(n)}(T)$ are included this interchange need to be investigated. It is also possible that the existence of a natural boundary could depend on the value of $\lambda$ in the lambda extensions of (75) and (76)

$$k_B T \chi_+(T; \lambda) = (1 - t)^{-1/4} \sum_{j=0}^{\infty} \lambda^{2j} \chi^{(2j+1)}(T) \text{ for } T > T_c \quad (163)$$

$$k_B T \chi_-(T; \lambda) = (1 - t)^{-1/4} \sum_{j=1}^{\infty} \lambda^{2j} \chi^{(2j)}(T) \text{ for } T < T_c. \quad (164)$$

These possibilities remain to be investigated.

7.6 Row correlations

All of the results obtained for the diagonal correlation, which depend on the single variable $t$, can be extended to the row correlation, which depends on the two variables $\alpha_1$ and $\alpha_2$ in a symmetric fashion [5]. In particular it has been pointed out to me by Jean-Marie Maillard and Nicholas Witte in private conversations that the Painlevé VI results of Jimbo and Miwa [33] can be extended to a two variable Garnier system [72]. However, this system must possess some most interesting properties because one of the most important properties of the Ising model is the fact that, when these two variables are rewritten as

$$k = \sinh 2E^v/k_B T \sinh 2E^h/k_B T \text{ and } r = \frac{\sinh 2E^v/k_B T}{\sinh 2E^h/k_B T}, \quad (165)$$

the dependence on $k$ (the modulus of the elliptic functions) and the anisotropy ratio $r$ which is related to the spectral variable of the star triangle equation [73] is dramatically different. These results for Garnier systems have also yet to be obtained.

26
8 Romance versus Understanding

In a lecture given in Melbourne in January 2006 [74], I gave the following definition of “understanding”

No one can be said to understand a paper unless he is able to generalize the paper.

This definition is open to criticism on at least two grounds. Firstly the use of the word “he” has a sexist implication which is neither appropriate nor intended. Secondly, there are surely subjects which are fully understood where further generalization is pointless. An illustration of this are the laws of thermodynamics which have been fully understood by physicists for many decades (even if they are not accepted by the overwhelming majority of voters and politicians).

However, precisely because thermodynamics is fully understood, it has lost the mystery it had at the time of Gibbs, Boltzmann and Ehrenfest. This illustrates the great truth that understanding is the enemy of romance because once the mysteries are understood the romance dies.

Fortunately for romance, there are many mysteries of the Ising model which are far from being understood. The romantic in me says that, even when these mysteries have been understood, the understanding of the mysteries will generate new mysteries and the romance of the Ising model will be everlasting.

Acknowledgments

In my long running romance with the Ising model I have profited greatly from the help many people. In particular I want to thank M. Assis, H. Au-Yang, R.J. Baxter, V.V. Bazhanov, P.J. Forrester, M. Jimbo, J-M. Maillet, J-M. Maillet, T. Miwa, W. Orrick, J.H.H.Perk, C.A.Tracy, N. Witte, and T.T. Wu for their wisdom and inspiration.

References

[1] L. Onsager, in “Critical Phenomena in Alloys, Magnets, and Superconductors” ed. R.E. Mills, E. Ascher and R.I. Jaffe, (McGraw-Hill Book Company 1971), pp. 3-12.

[2] L. Onsager, Crystal Statistics I. A two-dimensional model with an order-disorder transition, Phys. Rev. 65 (1944) 117–149.

[3] R.J. Baxter, Onsager and Kaufman’s calculation of the spontaneous magnetization of the Ising model, J. Stat. Phys. 145 (2011) 518-548.

[4] B.Kaufman, Crystal Statistics II. Partition function evaluated by spinor analysis, Phys. Rev. 76 (1949) 1232-1243.

[5] B. Kaufman and L. Onsager, Crystal Statistics III. Short range order in a binary Ising lattice, Phys. Rev. 76 (1949) 1244-1252.
[6] E. Montroll, R.B. Potts and J.C. Ward, Correlations and spontaneous magnetization of the two dimensional Ising model, J. Math. Phys. 4 (1963) 308-322.

[7] T.T. Wu, Theory of Toeplitz determinants and the spin correlations of the two-dimensional Ising model, Phys. Rev. 149 (1966) 380-401.

[8] T. T. Wu, B. M. McCoy, C. A. Tracy and E. Barouch, Spin-spin correlation functions for the two-dimensional Ising model: Exact theory in the scaling region, Phys. Rev. B 13 (1976) 316-374.

[9] B. Nickel, On the singularity of the 2D Ising model susceptibility, J. Phys. A 32 (1999) 3889-3906.

[10] B. Nickel, Addendum to “On the singularity of the 2D Ising model susceptibility”, J. Phys. A 33 (2000) 1693-1711.

[11] H. Cheng and T.T Wu, Theory of Toeplitz determinants and the spin correlations of the two-dimensional Ising model III, Phys. Rev. 164 (1967) 719-735.

[12] I. Lyberg and B.M.McCoy, Form factor expansion of the row and diagonal correlation functions of the two dimensional Ising model, J. Phys. A 40 (2007) 3329-3346.

[13] A. Selberg, Bemerkninger om et multipelt integral, Nork. Mat. Tidsskr 24 (1944) 71-78. For a current review of the many applications see P.J. Forrester and S.O. Warnaar, The importance of the Selberg integral, Bull. of the Am. Math. Soc. 45 (2008) 489-534.

[14] S. Boukraa, S. Hassani, J.-M. Maillard, B.M.McCoy, W.P.Orrick and N.Zenne, Holonomy of Ising model form factors, J. Phys. A40 (2007) 75-112.

[15] M. Assis, J-M. Maillard and B.M.McCoy, Factorization of the Ising model form factors, J. Phys. A 44 (2011) 305004.

[16] H.E. Boos and V.E. Korepin, Quantum spin chains and Riemann zeta function with odd arguments, J. Phys. A 34 (2001) 5311–5316.

[17] H.E. Boos, V.E. Korepin, Y. Nishiyama and M. Shiroishi, Quantum correlations and number theory, J. Phys. A 35 (2002) 4443–4451.

[18] J. Sato, M. Shiroishi, and M. Takahashi, Correlation functions of the spin-1/2 antiferromagnetic Heisenberg chain: exact calculation via the generating function, Nucl. Phys. B729 (2005) 441–466.

[19] K.Sakai, M. Shiroishi, Y. Nishiyama, and M. Takahashi, Third neighbor correlators of the spin-1/2 Heisenberg antiferromagnet, Phys. Rev. E67 (2003) 065101-(1-4).
[20] H.E. Boos, M. Shiroishi, and M. Takahashi, First principle approach to correlation functions of the spin-1/2 Heisenberg chain: fourth neighbor correlations, Nucl. Phys. B 712 (2005) 573–599.

[21] J. Sato and M Shiroishi, Fifth-neighbor spin-spin correlator for the antiferromagnetic Heisenberg chain, J. Phys. A39 (2005) L405-L411.

[22] G. Kato, M. Shiroishi, M. Takahashi and K. Sakai, Third-neighbor and other four-point functions of spin-1/2 XXZ chain, J. Phys. A 37 (2004) 5097-5123.

[23] B.M. McCoy, C.A. Tracy and T.T.Wu, Painlevé equations of the third kind, J. Math. Phys. 18 (1977) 1058-1092.

[24] W.P. Orrick, B.G. Nickel, A.J. Guttmann, J.H.H. Perk, The susceptibility if the square lattice Ising model: new developments, J. Stat. Phys. 102 (2001) 795-841.

[25] E.T. Whittaker and G.N. Watson, “A course of Modern Analysis”, fourth edition, (Cambridge 1963).

[26] V.V. Mangazeev and A.J. Guttmann, Form factor expansions in the 2D Ising model and PainlevéVI, Nucl. Phys. B838 (2010) 391-412.

[27] M.E. Fisher, The susceptibility of the plane Ising model, Physica 25A (1959) 521-524.

[28] E. Barouch, B.M. McCoy and T.T. Wu, Zero-field susceptibility of the two dimensional Ising model near $T_c$, Phys. Rev. Letts. 31 (1973) 1409-1411.

[29] C.A. Tracy and B.M. McCoy, Neutron scattering and the correlations of the Ising model near $T_c$, Phys. Rev. Letts. 31 (1973) 1500-1504.

[30] J.M. Myers, Wave scattering and the geometry of a strip, J. Math. Phys. 6 (1965) 1839-1846.

[31] M. Sato, T. Miwa and M. Jimbo, Studies on holonomic quantum fields II, Proc. Jpn. Acad. 53A (1977) 147-152.

[32] M. Sato, T. Miwa and M. Jimbo, Holonomic quantum field theory, Pub. RIMS 14 (1978) 223-267; 15 (1079) 201-278; 15 (1979) 577-629; 15 (1978) 871-972; 16 (1980) 531-584.

[33] M. Jimbo and T. Miwa, Studies on holonomic quantum fields XVII, Proc. Jpn. Acad. 56A (1980) 405; and 57A (1981) 347.

[34] E.L. Ince, “Ordinary differential equations”. (Dover Publications, New York, 1956).

[35] K. Iwasaki, H. Kimura, S. Shimomura and M. Yoshida, “From Gauss to Painlevé”, Friedr. Vieweg and Sohn Verlagsgesellschaft mbH, Braunschweig 1991.
[36] P. Painlevé, Sur les équations différentielles du second ordre et d’ordre supérieur dont l’intégrale général est uniform, Acta Math. 25 (1902) 1-85.

[37] B. Gambier, Sur les équations différentielles du second ordre et du premier degré dont l’intégrale général est a point point critiques fixes, Acta Math. 33 (1910) 1-55.

[38] E. Picard, Mémoire sur la théorie des fonctions algébriques de deux variables, Journal de Liouville 5 (1889).

[39] N.J. Hitchin, Poncelet Polygons and the Painlevé equations in Geometry and analysis (Bombay, 1992) 151-185, Tata Inst. Fund. Res., Bombay 1995.

[40] Yu. I. Manin, Sixth Painlevé equation, Universal elliptic curve, and mirror of P^2, AMS Transl. (2) vol. 186 (1998) 131-151.

[41] M. Mazzocco, Picard and Chazy solutions to the Painlevé VI equation, Math. Ann. 321 (2001) 157-195.

[42] K. Okamoto, Japan. J. Math. 5, No.1 (in Japanese); Ann. Math. Pura. Appl 146(1987) 337-381.

[43] C.R. Nappi, Nuovo Cimento 44A (1978) 392.

[44] C.A. Tracy, Painlevé transcendents and scaling functions of the two dimensional Ising model, in Nonlinear Equations in Physics and Mathematics, ed. A.O. Barut, D. Riedel Publ. Co., Dortedrecht, Holland, (1978) 378-380.

[45] Y. Chan, A.J.Guttmann, B.G. Nickel and J.H.H.Perk, The Ising susceptibility scaling function, J. Stat, Phys. 145 (2011) 549-590.

[46] N. Zenine, S. Boukraa, S. Hassani, J.M. Maillard, The Fuchsian differential equation of the square lattice Ising \( \chi^{(3)} \) susceptibility, J. Phys. A: Math. Gen. 37 (2004) 9651-9668.

[47] N. Zenine, S. Boukraa, S. Hassani, J.M. Maillard, Square lattice Ising model susceptibility: Series expansion method and differential equation for \( \chi^{(3)} \), J. Phys. A: Math. Gen. 38 (2005) 1875-1899.

[48] N. Zenine, S. Boukraa, S. Hassani, J.M. Maillard, Ising model susceptibility; The Fuchsian equation for \( \chi^{(4)} \) and its factorization properties, J. Phys. A: Math. Gen. 38 (2005) 4149-4173.

[49] N. Zenine, S. Boukraa, S. Hassani and J-M. Maillard, (2005), Square lattice Ising model susceptibility: connection matrices and singular behavior of \( \chi^{(3)} \) and \( \chi^{(4)} \), J. Phys. A 38 9439-9474.

[50] S. Boukraa, S. Hassani, J-M. Maillard, and N. Zenine, Landau Singularities and singularities of holonomic integrals of the Ising class, J. Phys. A40 (2007) 2583-2614.
[51] S. Boukraa, A.J. Guttmann, S. Hassani, I. Jensen, J-M. Maillard, B. Nickel and N. Zenine, Experimental mathematics on the magnetic susceptibility of the square lattice Ising model, J. Phys. A41 (2008) 455202 (51pp).

[52] A. Bostan, S. Boukraa, S. Hassani, J-M. Maillard J-A Weil and N. Zenine, Globally nilpotent differential operators and the square Ising model, J. Phys. A42 (2009) 125206 (50pp).

[53] A. Bostan, S. Boukraa, A.J. Guttmann, S. Hassani, I. Jensen, J-M. Maillard and N. Zenine, High order Fuchsian equations for the square lattice Ising model: $\tilde{\chi}(5)$, J. Phys. A42 (2009) 275209 (32pp).

[54] S. Boukraa, S. Hassani, I. Jensen, J-M. Maillard, and N. Zenine, High order Fuchsian equations for the square lattice Ising model: $\tilde{\chi}(6)$, J. Phys. A43 (2010) 115201 (22pp).

[55] B. Nickel, I. Jensen, S. Boukraa, A.J. Gutmann, S. Hassani, J-M Maillard and N. Zenine, Square lattice Ising model $\tilde{\chi}(5)$ ODE in exact arithmetic, J. Phys. 43 (2010) 195205.

[56] S. Boukraa, S. Hassani, J-M. Maillard, B.M. McCoy, J-A. Weil and N. Zenine, The diagonal Ising susceptibility, J. Phys. A40 (2007) 8219-8236.

[57] M. Assis, S. Boukraa, S. Hassani, M. van Hoeij, J-M. Maillard and B.M. McCoy, Diagonal Ising susceptibility: elliptic integrals, modular forms and Calabi-Yau equations, J. Phys. A (submitted) arXiv 1110.1705.

[58] A. Borodin and A. Okounkov, A Freholm determinant formula for Toeplitz determinants, Integr. equ. oper. theory 37 (Birkhäuser Verlag, Basel 2000) 3867-396.

[59] N.S. Witte and P.J. Forrester, Fredholm determinant evaluations of the Ising model diagonal correlations and their $\lambda$ generalization, arXiv:1105.4389v1.

[60] M. Jimbo, T. Miwa, Y. Mori and M. Sato, Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent, Physica 1A (1980) 80-158.

[61] W. Bühring, Generalized hypergeometric functions at unit argument, Proc. Am. math. Soc. 114 (1992) 145-153.

[62] C.A. Tracy, Asymptotics of a $\tau$-function arising in the two-dimensional Ising model, Commun. Math. Phys. 142 (1991) 297-311.

[63] Vl. S. Dotsenko and V.A. Fateev, Four-point correlation functions and the operator algebra in 2D conformal invariant theories with central charge $C \leq 1$, Nucl. Phys/ B521[fs13] (1985) 691-734.
[64] F.A. Smirnov, “Form factors in completely integrable models of quantum field theory”. Advanced Series in Mathematical Physics 14, (World Scientific, Singapore, 1992).

[65] N. Kitanine, K.K. Kozlowski, J.M. Maillet, N.A. Slavnov and V. Terras, On the thermodynamic limit of form factors in the massless XXZ Heisenberg chain, J. Math. Phys. 50:095209 (2009).

[66] K. Kitanine, K.K. Kozlowski, J.M.Maillet, N.A. Slavnov and V. Terras, Thermodynamics limit of particle-hole form factors in the masless XXZ Heisenberg chain, arXiv:1003.4557.

[67] F. Bloch and A. Nordsieck, Note on the radiation field of the electron, Phys. Rev. 52 (1937) 54-59.

[68] The literature on this is vast. For my contribution see B.M. McCoy and T.T. Wu, Theory of fermion exchange in massive quantum electrodynamics at high energy, Phys. Rev. D13 369-512.

[69] This literature is also vast. For example see A. Sen, Asymptotic behavior of the fixed-angle on-shell quark scattering amplitudes in non-Abelian gauge theories, Phys. Rev. D28 (1983) 860-875;
J.C. Collins, Sudakov form-factors, in Adv. Ser. Direct. High Energy Phys. 5 (1989) 573-614 (A. Mueller ed. World Scientific, Singapore);
G.F. Sterman and M. E. Tejeda-Yeomans, Multiloop amplitudes and resummation, Phys. Lett. B552 (2003) 48-56.

[70] K. Kitanine, K.K. Kozlowski, J.M.Maillet, N.A. Slavnov and V. Terras, Algebraic Bethe ansatz approach to the asymptotic behavior of correlation functions, J. Stat. Mech. (2009) P04003.

[71] P. Fonseca and A. Zamolodchikov, Ising field theory in a magnetic field; analytic properties of the free energy, J. Stat. Phys. 110 (2002) 527-590.

[72] For a modern exposition of Garnier systems see 35.

[73] R. J. Baxter, Exactly Solved Models in Statistical Mechanics, Academic Press. London (1982).

[74] B.M. McCoy, The meaning of understanding,