The Limit Shape of a Stochastic Bulgarian Solitaire

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Abstract

We consider a stochastic version of Bulgarian solitaire: A number of cards are distributed in piles; in every round a new pile is formed by cards from the old piles, and each card is picked independently with a fixed probability. This game corresponds to a multi-square birth-and-death process on Young diagrams of integer partitions. We prove that this process converges in a strong sense to an exponential limit shape as the number of cards tends to infinity. Furthermore, we bound the probability of deviation from the limit shape and relate this to the number of rounds played in the solitaire.

Keywords: Bulgarian solitaire; birth-and-death process; limit shape; Young diagram; Markov chain

1 Introduction

The game of Bulgarian solitaire is played with a deck of n identical cards divided arbitrarily into several piles. A move consists of picking one card from each pile and letting these cards form a new pile. This move is repeated over and over again. If the total number of cards in the deck is a triangular number, i.e., \( n = 1 + 2 + \ldots + k \) for some \( k \), this process has an interesting

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Regardless of the initial configuration, a finite number of moves will lead to the stable configuration where there is one pile of each size from 1 up to $k$, see [3, 13]. This property motivated initial interest in Bulgarian solitaire in the early 1980’s, and it featured in a 1983 column by Martin Gardner in Scientific American. Later research have studied also other aspects of Bulgarian solitaire [1, 2, 6, 7, 10]. For information about the earlier history of the game (including its name), and a summary of subsequent research, see [9].

If $n$ is not a triangular number, a stable configuration does not exist but after at most $O(n)$ rounds the game will enter into a cycle. Moreover, all configurations of the cycle are “almost triangular” in the following sense: If $k = \max \{ j : 1 + 2 + \ldots + j \leq n \}$, then all the configurations in that cycle can be constructed from the triangular configuration $(k, k-1, \ldots, 1)$ by adding at most one card to each pile, and possibly adding one more pile of size 1. For exact formulations and more details, see [1, 2, 6, 7]. Obviously, in any configuration the set of pile sizes constitute an integer partition of $n$. Thus, as a sweeping statement we can say that the Bulgarian solitaire is drawn towards a triangular shape of the corresponding Young diagram.

Popov [12] considered a stochastic version of Bulgarian solitaire. In this version, a move in the game consists of forming a new pile by picking one card from each of a random selection of the old piles. Specifically, any given pile must release one card with a fixed probability $0 < p < 1$ and independently of the other piles. The resulting game is a discrete-time irreducible and aperiodic Markov chain on the space of all partitions of the number of cards $n$. Popov proved that this game too is drawn towards triangular configurations, in the sense that the stationary probability measure of the set of configurations that are close to triangular (with a slope that depends on $p$) is close to 1.

There are many other possible ways to formulate stochastic versions of Bulgarian solitaire, but to our knowledge no other possibility has been studied before. Here we consider a particularly natural version, where selection acts on cards rather than piles: When forming a new pile by picking cards from the old piles, every card is picked with a fixed probability $0 < p < 1$, independently of all other cards. Our aim is to show that this card-based stochastic Bulgarian solitaire is not drawn to triangular configurations but to an exponential shape. The intuitive reason for this difference in results is that the speed by which a pile loses its cards is constant in the pile-based solitaire but proportional to the current size of the pile in the card-based version.

The precise result comprises a theorem and its corollary. Theorem 1 says
that as the number of cards tends to infinity the probability of reaching a close to exponential shape when playing sufficiently long tends to 1. The outcome of a computer simulation of the process can be seen in Figure 1. In addition, we also consider the stationary distribution of the card-based stochastic Bulgarian solitaire regarded as a Markov chain. As a corollary, it turns out that the stationary probability measure of the set of partitions that have a close to exponential shape is close to 1. Compared to Popov’s treatment of the pile-based stochastic Bulgarian solitaire, our treatment of the card-based version has both similarities and novel elements.

Figure 1: The result of a computer simulation after 200 rounds of the card-based Bulgarian solitaire with \( n = 10^5 \) cards and \( p = 0.01 \) with scaling \( 1/p \), starting from a triangular configuration. The rough curve is the diagram-boundary function of the card configuration (defined in Section 2.1) and the smooth curve is the limit shape \( y = e^{-x} \).

2 Notation and Preliminaries

2.1 Representation by Partitions

Let \( \mathcal{P}(n) = \{ \lambda : \lambda \vdash n \} \) be the set of partitions of the integer \( n \) into integral parts \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0 \), where \( \ell = \ell(\lambda) \) is the number of parts of the partition \( \lambda \), i.e., \( \sum_{i=1}^{\ell} \lambda_i = n \). We write \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \). For \( i > \ell(\lambda) \) it will be convenient to define \( \lambda_i = 0 \).

We shall represent an integer partition \( \lambda \) by a Young diagram drawn as columns of squares in the first quadrant such that the \( i \)th column has height \( \lambda_i \). For example, the configuration of 12 cards in which there are five piles
of sizes 4, 4, 2, 1 and 1 corresponds to the partition \((4, 4, 2, 1, 1) \vdash 12\), which is represented by the left diagram in Figure 2.

![Diagram of \(\lambda\) and Function graph \(y = \partial \lambda(x)\)](image)

Figure 2: The partition \(\lambda = (4, 4, 2, 1, 1) \in \mathcal{P}(12)\).

When we speak of shapes of integer partitions we shall mean the shape of the boundary of the Young diagram drawn in this way. To this end, for any partition \(\lambda\), define its \textit{diagram-boundary function} as the nonnegative, integer-valued, weakly decreasing and piecewise constant function \(\partial \lambda : \mathbb{R}_{\geq 0} \to \mathbb{N}\) given by

\[
\partial \lambda(x) = \lambda\floor{x} + 1.
\]

For example, the right diagram in Figure 2 depicts the function graph \(y = \partial \lambda(x)\) for \(\lambda = (4, 4, 2, 1, 1)\). (Note that, since we defined \(\lambda_i = 0\) for \(i > \ell(\lambda)\), we have \(\partial \lambda(x) = 0\) for \(x \geq \ell(\lambda)\).)

As \(n\) grows we need to rescale the diagram to achieve any limiting behaviour. Following [4] and [14], we do this so that the area of the rescaled diagram is always 1. We say that we employ the \textit{scaling factor} \(a > 0\) if all row lengths are multiplied by \(1/a\) and all column heights are multiplied by \(a/n\). Thus, given a partition \(\lambda\), define the \textit{\(a\)-rescaled} diagram-boundary function of \(\lambda\) as the positive, real-valued, weakly decreasing and piecewise constant function \(\partial^a \lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\) given by

\[
\partial^a \lambda(x) = \frac{a}{n} \partial \lambda(ax) = \frac{a}{n} \lambda\floor{ax} + 1.
\]

(1)
2.2 Limit Shapes of Birth-and-Death Processes on Young Diagrams

Eriksson and Sjöstrand [4] studied limit shapes of birth-and-death processes on Young diagrams where in every step a single square dies and a single square is born. Every move of the card-based stochastic Bulgarian solitaire moves several cards and the game can therefore be viewed as a multi-square birth-and-death process on Young diagrams. Thus our present study extends the family of processes studied in [4].

For each positive integer \( n \), let \( \nu^{(n)} \) be a probability distribution on \( \mathcal{P}(n) \) obtained from the card-based stochastic Bulgarian solitaire. We are interested in finding a sequence \( \{ a_n \} \) of scaling factors such that the rescaled diagrams approach a limit shape \( \phi \) in probability as \( n \) grows to infinity. The precise meaning of this is that

\[
\lim_{n \to \infty} \nu^{(n)} \{ \lambda \in \mathcal{P}(n) : \| \partial^{a_n} \lambda(x) - \phi(x) \|_{\infty} < \varepsilon \} = 1.
\]

(2)

where \( \| \cdot \|_{\infty} \) denotes the max-norm \( \| f \|_{\infty} = \sup \{ |f(x)| : x \geq 0 \} \). We remark that Vershik [14] and Erlihson and Granovsky [5] use a weaker condition for convergence towards a limit shape, namely that

\[
\lim_{n \to \infty} \nu^{(n)} \left\{ \lambda \in \mathcal{P}(n) : \sup_{x \in [a,b]} |\partial^{a_n} \lambda(x) - \phi(x)| < \varepsilon \right\} = 1
\]

should hold for any compact interval \([a,b]\), and any \( \varepsilon > 0 \). Yakubovich [15] and Eriksson and Sjöstrand [4] use an even weaker condition:

\[
\lim_{n \to \infty} \nu^{(n)} \{ \lambda \in \mathcal{P}(n) : |\partial^{a_n} \lambda(x) - \phi(x)| < \varepsilon \} = 1
\]

for all \( x > 0 \) and all \( \varepsilon > 0 \).

2.3 Representation by Weak Compositions

It will sometimes be convenient to consider a configuration of \( n \) cards as a weak integer composition, by which we mean an infinite sequence \( \alpha = (\alpha_1, \alpha_2, \ldots) \), not necessarily decreasing, of nonnegative integers adding up to \( n \). Let \( W(n) \) be the set of weak compositions of the integer \( n \).

We define the diagram, the boundary function \( \partial \alpha \), and the rescaled boundary function \( \partial^{a_n} \alpha \) of a weak integer composition \( \alpha \) in exact analogy to the way we defined them for integer partitions in Section 2.1. For example, the diagram of \( \alpha = (3, 0, 2, 4, 1, 0, 0, \ldots) \) and the corresponding function graph \( y = \partial \alpha(x) \) are shown in Figure 3.
2.4 A Necessary Lemma

In the proof of the limit shape result, Theorem 1, we need to be able to keep track of individual piles of cards. We shall therefore order piles by time of creation rather than by size, which means that configurations are represented by weak integer compositions rather than integer partitions. It will turn out that the rescaled diagram-boundary function of these compositions tends to a limit function as the game is played and as the number of cards tends to infinity. However, in the end we want to express the limit-shape result in terms of diagram-boundary functions of integer partitions, not weak integer compositions. For any \( \alpha \in \mathcal{W}(n) \), define the operator \( \text{ord} \) as the ordering operator that sorts the parts of \( \alpha \) in descending order. If we omit trailing zeros, \( \text{ord} \alpha \) is an integer partition of \( n \). For example, if \( \alpha = (3, 0, 2, 4, 1, 0, \ldots) \), then \( \text{ord} \alpha = (4, 3, 2, 2) \). In this section, we prove an important lemma which says that such sorting of the piles by size does not harm the convergence to a limiting shape.

The proof of the lemma will use some basic theory of symmetric-decreasing rearrangements, see [8] or [11, Ch. 3]. For any measurable function \( f: \mathbb{R} \to \mathbb{R}_{\geq 0} \) such that \( \lim_{x \to \pm \infty} f(x) = 0 \), there is a unique function \( f^*: \mathbb{R} \to \mathbb{R}_{\geq 0} \), called the symmetric-decreasing rearrangement of \( f \), with the following properties:

- \( f^* \) is symmetric, that is, \( f^*(-x) = f^*(x) \) for all \( x \),
- \( f^* \) is weakly decreasing on the interval \([0, \infty)\),

Figure 3: The composition \( \alpha = (3, 0, 2, 4, 1, 0, \ldots) \in \mathcal{W}(10) \).
• $f^*$ and $f$ are equimeasurable, that is,
\[
\mathcal{L}(\{x : f(x) > t\}) = \mathcal{L}(\{x : f^*(x) > t\})
\]
for all $t > 0$, where $\mathcal{L}$ denotes the Lebesgue measure,
• $f^*$ is lower semicontinuous.

In particular, if $f$ is a symmetric function that is weakly decreasing and right-continuous on $[0, \infty)$ and tends to 0 at infinity, then $f^* = f$.

**Lemma 1.** Let $\alpha \in \mathcal{W}(n)$ be a weak composition of $n$, let $a > 0$ be any scaling factor and let $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be a right-continuous and weakly decreasing function such that $f(x) \rightarrow 0$ as $x \rightarrow \infty$. The rescaled diagram-boundary functions before and after sorting of the weak composition satisfy the inequality
\[
\|\partial^a \text{ord} \alpha - f\|_{\infty} \leq \|\partial^a \alpha - f\|_{\infty}.
\]

**Proof.** The intuition of the lemma should be obvious from Figure 4. To be able to use the standard machinery of symmetric rearrangements, we consider the functions $f$, $\partial^a \alpha$, and $\partial^a \text{ord} \alpha$ as being defined on the complete real axis by letting $f(x) = f(|x|)$ and analogously for $\partial^a \alpha$, and $\partial^a \text{ord} \alpha$.

Since $f(x) \rightarrow 0$ as $x \rightarrow \infty$, its symmetric-decreasing rearrangement $f^*$ is defined, and, since $f$ is weakly decreasing and lower semicontinuous, we have $f^* = f$. Similarly, $\partial^a \text{ord} \alpha(x) \rightarrow 0$ as $x \rightarrow \infty$ and is weakly decreasing, so $(\partial^a \text{ord} \alpha)^* = \partial^a \text{ord} \alpha$. Moreover, $(\partial^a \alpha)^* = \partial^a \text{ord} \alpha$ must hold because the operator $\text{ord}$ sorts the composition parts in descending order.

Now, since symmetric rearrangements decrease $L^p$-distances for any $1 \leq p \leq \infty$ (see for example [11], Section 3.4), we obtain
\[
\|\partial^a \text{ord} \alpha - f\|_{\infty} = \|(\partial^a \alpha)^* - f^*\|_{\infty} \leq \|\partial^a \alpha - f\|_{\infty}.
\]

\[\square\]

**Corollary 1.** For any distribution $\rho^{(n)}$ on $\mathcal{W}(n)$, define a corresponding distribution $\nu^{(n)}$ on $\mathcal{P}(n)$ by
\[
\nu^{(n)}(\lambda) = \sum_{\substack{\alpha \in \mathcal{W}(n) \\ \text{ord} \alpha = \lambda}} \rho^{(n)}(\alpha).
\]

If $\phi$ is a limit shape of $\rho^{(n)}$ on $\mathcal{W}(n)$ with some scaling $\{a_n\}$, then $\phi$ is also a limit shape of $\nu^{(n)}$ on $\mathcal{P}(n)$ with the same scaling.
Proof. That $\phi$ is a limit shape of the distribution $\rho^{(n)}$ on $W(n)$ with scaling $\{a_n\}$ means that

$$\lim_{n \to \infty} \rho^{(n)} \{ \alpha \in W(n) : \|\partial^{a_n} \alpha - \phi\|_{\infty} < \varepsilon \} = 1.$$ 

By virtue of Lemma 1 we can replace $\alpha$ with $\text{ord} \alpha$ in this formula:

$$\lim_{n \to \infty} \rho^{(n)} \{ \alpha \in W(n) : \|\partial^{a_n} \text{ord} \alpha - \phi\|_{\infty} < \varepsilon \} = 1. \quad (4)$$

The set of weak compositions $A := \{ \alpha \in W(n) : \|\partial^{a_n} \text{ord} \alpha - \phi\|_{\infty} < \varepsilon \}$ can be written as a disjoint union of equivalence classes with respect to sorting:

$$A = \bigcup_{\lambda \in L} \{ \alpha \in W(n) : \text{ord} \alpha = \lambda \}.$$

where $L = \{ \lambda \in P(n) : \|\partial^{a_n} \lambda - \phi\|_{\infty} < \varepsilon \}$. The $\rho^{(n)}$-probability measure of
A is

\[ \rho^{(n)}(A) = \rho^{(n)} \left( \bigcup_{\lambda \in L} \{ \alpha \in W(n) : \text{ord} \alpha = \lambda \} \right) \]
\[ = \sum_{\lambda \in L} \rho^{(n)} \{ \alpha \in W(n) : \text{ord} \alpha = \lambda \} \]
\[ = \sum_{\lambda \in L} \sum_{\alpha \in W(n) \atop \text{ord} \alpha = \lambda} \rho^{(n)}(\alpha) \]
\[ = \sum_{\lambda \in L} \rho^{(n)}(\lambda) \quad \text{(by (3))} \]
\[ = \nu^{(n)}(L). \]

From (4) we have that \( \lim_{n \to \infty} \rho^{(n)}(A) = 1 \), and we can conclude that also \( \lim_{n \to \infty} \nu^{(n)}(L) = 1 \), that is,

\[ \lim_{n \to \infty} \nu^{(n)} \{ \lambda \in P(n) : \| \partial^{a_n} \lambda - \phi \|_\infty < \varepsilon \} = 1. \]

This means that \( \phi \) is a limit shape of the distribution \( \nu^{(n)} \) on \( P(n) \).

**2.5 The Stationary Measure**

The card-based stochastic Bulgarian solitaire can be regarded as a Markov chain with state-space \( P(n) \). Let us denote it by \((\lambda^{(0)}, \lambda^{(1)}, \ldots)\). This Markov chain is aperiodic and irreducible. It is irreducible because, starting from any state, an arbitrary \( \lambda \in P(n) \) can be reached in \( \ell(\lambda) \) moves as follows. In the first move, choose cards for the last column in \( \lambda \); in the next move, choose (among remaining cards) the cards for the next-to-the-last column in \( \lambda \); repeat until all the \( \ell(\lambda) \)th columns have been chosen. All these selections of cards have probability \( > 0 \), hence the irreducibility. It is aperiodic because all states are aperiodic: There is always a positive probability to choose zero cards in a move and hence remain in the same state.

It is well known that a finite state-space irreducible Markov chain has a unique stationary distribution \( \pi \), and that, if it also is aperiodic, the distribution converges to \( \pi \) starting from any initial state. We denote by \( \pi_{p,n} \) the stationary measure of the Markov chain \((\lambda^{(0)}, \lambda^{(1)}, \ldots)\) on \( P(n) \) given by card-based stochastic Bulgarian solitaire.
3 The Limit Shape Result

We are now ready to state the limit shape result. If we view the solitaire as a process on Young diagrams, Theorem 1 says that after a sufficiently large number $m$ of rounds the $(1/p)$-rescaled boundary function of the diagram will resemble the exponential shape $e^{-x}$ with high probability.

**Theorem 1.** For each positive integer $n$, pick a probability $p \in (0, 1)$, a (possibly random) initial configuration $\lambda(0) \in \mathcal{P}(n)$ and let $(\lambda(0), \lambda(1), \ldots)$ be the Markov chain defined by the card-based stochastic Bulgarian solitaire with $n$ cards and probability $p$ for a card to be chosen into a new pile.

Suppose that $p$ as a function of $n$ has the asymptotical properties that

- $p \to 0$ as $n \to \infty$ and
- $p = \omega(\frac{\log n}{n})$, i.e. $p/\frac{\log n}{n} \to \infty$ as $n \to \infty$.

Then, for any $m = m(n) > \frac{\varepsilon^2}{2np}$, the probability distribution for the resulting diagram $\lambda(m)$ after playing $m$ rounds has the limit shape $g(x) = e^{-x}$ with scaling $1/p$. In fact, for any $\varepsilon > 0$ we have

$$P\left(\|\partial^{1/p}\lambda(m) - g\|_{\infty} \leq \varepsilon\right) \geq 1 - \exp\left[-\frac{\varepsilon^2}{2 + \varepsilon} np(1 - o(1))\right].$$

As a simple consequence of Theorem 1, the stationary distribution of the card-based Bulgarian solitaire also has the limit shape $e^{-x}$.

**Corollary 2.** Let $p$ be a probability, dependent on $n$, with the same asymptotical properties as in Theorem 1 and let $\pi_{p,n}$ denote the stationary measure of the Markov chain $(\lambda(0), \lambda(1), \ldots)$ on $\mathcal{P}(n)$ defined by the card-based stochastic Bulgarian solitaire with $n$ cards and probability $p$ for choosing a card to the new pile. Then $\pi_{p,n}$ has the limit shape $g(x) = e^{-x}$ with scaling $1/p$. In fact, for any $\varepsilon > 0$ we have

$$\pi_{p,n}\left(\{\lambda \in \mathcal{P}(n) : \|\partial^{1/p}\lambda - g\|_{\infty} \leq \varepsilon\}\right) \geq 1 - \exp\left[-\frac{\varepsilon^2}{2 + \varepsilon} np(1 - o(1))\right].$$

**Proof.** Since $\pi_{p,n}$ is the stationary distribution, if we start with a partition $\lambda^{(0)}$ sampled from $\pi_{p,n}$ and play $m$ rounds, the resulting partition $\lambda^{(m)}$ will also be sampled from $\pi_{p,n}$. Thus, the corollary follows from Theorem 1 by choosing $\lambda^{(0)}$ as a stochastic partition sampled from the stationary distribution.

$\square$
Let us discuss what happens if \( p \) does not fulfil the requirements in the theorem. If \( p \) is bounded away from zero, then the scaling \( 1/p \) is bounded and hence cannot transform the jumpy boundary diagrams into a continuous limit shape. On the other hand, if \( p \) tends to zero too fast, the pile sizes will be small and their random fluctuations will be large. For instance, the new pile after each round has a size drawn from the binomial distribution \( \text{Bin}(n,p) \) with relative standard deviation \( \sim 1/\sqrt{np} \).

4 Some Propositions

The proof of Theorem 1 will rely on a number of general analytic and probabilistic observations that we present in this section.

**Proposition 1.** The function \( f(p) = (1-p)^{1/p} \) is decreasing for \( 0 < p < 1 \).

**Proof.** The Taylor expansion at \( p = 0 \) of \( \log f(p) = \frac{1}{p} \log(1-p) \) is

\[
-1 - \frac{p}{2} - \frac{p^2}{3} - \cdots ,
\]

which is clearly decreasing in \( p \) for \( 0 < p < 1 \).

**Corollary 3.** For \( 0 < y < 1 \) and \( n \geq 1 \),

\[
(1-y)^n \geq 1 - ny \tag{5}
\]

**Proof.** If \( ny \geq 1 \), the inequality is trivially true, since the right-hand side is always nonpositive. If \( ny < 1 \), then we use Proposition 1 to conclude that \((1-y)^{1/y} \geq (1-ny)^{1/(ny)}\). Raising to the power \( ny \) yields \((1-y)^n \geq 1 - ny\).

The next observation is that the well-known convergence of \((1 - \frac{1}{n})^{nx}\) to \( e^{-x} \) as \( n \to \infty \) is uniform on \( x \geq 0 \). Setting \( p = \frac{1}{n} \), we state this result in the following way.

**Proposition 2.** The convergences

\[
(1-p)^{x/p} \to e^{-x} \quad \text{and} \quad (6)
\]

\[
(1-p)^x \to e^{-px} \quad \text{(7)}
\]

hold uniformly on the interval \( x \in [0, \infty) \) as \( p \to 0 \).
Proof. Let \( f(p) = (1-p)^{1/p} \). By Proposition 1, the function \(-\log f(p)\) tends to 1 from above as \( p \downarrow 0 \), and hence for any \( y \in [0, 1] \), we have \( y^{-\log f(p)} \sim y \) as \( p \downarrow 0 \). Since \([0, 1]\) is a compact set, Dini’s Theorem can be applied. Thus \( y^{-\log f(p)} \to y \) uniformly on \( y \in [0, 1] \) as \( p \to 0 \). Substituting \( y = e^{-x} \) yields (6) while instead substituting \( y = e^{-px} \) yields (7).

We shall also need the following formulation of Chernoff bounds.

**Proposition 3.** Let \( X_1, X_2, \ldots, X_n \) be independent random variables with \( X_i \in \{0, 1\} \) for all \( i = 1, 2, \ldots, n \). Let \( X = \sum_{i=1}^{n} X_i \) and \( \mu = EX = \sum_{i=1}^{n} EX_i \). Then, for any \( \eta \geq 0 \),

\[
P(X \geq (1 + \eta)\mu) \leq \exp \left( -\frac{\eta^2}{2 + \eta} \mu \right) \quad \text{and} \quad (8)
\]

\[
P(X \leq (1 - \eta)\mu) \leq \exp \left( -\frac{\eta^2}{2} \mu \right). \quad (9)
\]

**Proof.** Set \( p_i = E(X_i) = P(X_i = 1) \). We start with the bound (8) on the upper tail. Consider the random variable \( e^{tX} \) for a real parameter \( t > 0 \) (whose value will be fixed later). Since \( X \geq 0 \) and \( t > 0 \), we have

\[
P(X \geq (1 + \eta)\mu) = P \left[ e^{tX} \geq e^{(1+\eta)t\mu} \right].
\]

Now we use Markov’s inequality\(^1\) on \( e^{tX} \):

\[
P(X \geq (1 + \eta)\mu) = P \left[ e^{tX} \geq e^{(1+\eta)t\mu} \right] \leq \frac{E(e^{tX})}{e^{(1+\eta)t\mu}}. \quad (10)
\]

\(^1\)For any random variable \( X \geq 0 \) and \( a > 0 \), we have \( P(X \geq a) \leq \frac{E(X)}{a} \).
We will now establish an upper bound on \( E(e^{tX}) \).

\[
E(e^{tX}) = E(e^{t\sum_{i=1}^{n} X_i}) = E \left[ \prod_{i=1}^{n} e^{tX_i} \right] \tag{11}
\]

\[
= \prod_{i=1}^{n} E(e^{tX_i}) \quad \text{(by independence)}
\]

\[
= \prod_{i=1}^{n} (e^{t0} P(X_i = 0) + e^{t1} P(X_i = 1)) \quad \text{(since } X_i \in \{0, 1\})
\]

\[
= \prod_{i=1}^{n} (1 + p_i(e^t - 1)) \quad \text{(since } p_i = P(X_i = 1))
\]

\[
\leq \prod_{i=1}^{n} \exp (p_i(e^t - 1)) \quad \text{(as } 1 + x \leq e^x \forall x \in \mathbb{R})
\]

\[
= \exp \left( \sum_{i=1}^{n} p_i(e^t - 1) \right)
\]

\[
= \exp (e^t - 1)\mu \quad \text{(as } \sum_{i=1}^{n} p_i = \mu) \tag{12}
\]

Combining the resulting inequality \( E(e^{tX}) \leq \exp(e^t - 1)\mu \) with \(10\), we get

\[
P(X \geq (1 + \eta)\mu) \leq \frac{\exp(e^t - 1)\mu}{\exp(1 + \eta)t\mu} = \exp \left[ (e^t - 1 - (1 + \eta)t) \mu \right]. \tag{13}
\]

In order to make this bound as tight as possible, we will now find the value of \( t \) that minimizes the expression in the right hand side of \(13\). To do this, consider the function \( f : \mathbb{R} \to \mathbb{R}, f(t) = \exp \left[ (e^t - 1 - (1 + \eta)t) \mu \right] \). Note that \( f(t) > 0 \) and that \( \frac{df}{dt} f(t) = \mu(e^t - 1 - \eta) f(t), \) vanishing for \( t = \log(1+\eta) \).

Together with the fact that \( \frac{d^2 f}{dt^2} f(t) = \mu (e^t - 1 - \eta)^2 + e^t \) \( f(t) > 0, \) we conclude that the right hand side of \(13\) is minimized for \( t = \log(1+\eta) \).

Substituting this into \(13\), we obtain

\[
P(X \geq (1 + \eta)\mu) \leq \exp \left[ \left( e^{\log(1+\eta)} - 1 - (1 + \eta) \log(1 + \eta) \right) \mu \right] = (\exp [\eta - (1 + \eta) \log(1 + \eta)])^\mu. \tag{14}
\]

Now we use the fact that for all \( \eta > 0 \) we have \( \log(1 + \eta) \geq \frac{2\eta}{2+\eta} \). To see this, note that for the function \( g(x) = \log(1 + x) - \frac{2x}{2+x} \) we have \( \frac{d}{dx} g(x) = \).
\[
\frac{x^2}{(1+x)(2+x)^2} \geq 0 \text{ for } x \geq 0 \text{ and } g(0) = 0. \text{ Using this in (14) yields }
\]
\[
P(X \geq (1 + \eta)\mu) \leq \left( \exp \left[ \eta - (1 + \eta) \frac{2\eta}{2 + \eta} \right] \right)^\mu
\]
\[
= \exp \left( -\frac{\eta^2}{2 + \eta} \right)
\]
which proves (8).

When proving the bound (9) on the lower tail, the argument is very similar. We use Markov’s inequality on the random variable \( e^{-tX} \) instead:
\[
P(X \leq (1 - \eta)\mu) = P\left[ e^{-tX} \geq e^{-(1-\eta)t\mu} \right] \leq \frac{E(e^{-tX})}{e^{-(1-\eta)t\mu}}. \quad (15)
\]
Using the established inequality (12) for \(-t\) gives \(E(e^{-tX}) \leq \exp(e^{-t} - 1)\mu\), which when used in (15) gives
\[
P(X \leq (1 - \eta)\mu) \leq \frac{\exp(e^{-t} - 1)\mu}{\exp(-(1-\eta)t\mu)} = \exp \left[ (e^{-t} - 1 + (1 - \eta)t) \mu \right]. \quad (16)
\]
Using the same method as in the proof of (8), we find that the right hand side in (16) is minimized for \( t = -\log(1 - \eta) \), which substituted back into (16) gives
\[
P(X \leq (1 - \eta)\mu) \leq \exp \left[ (-\eta - (1 - \eta) \log(1 - \eta)) \mu \right].
\]
Now consider the function \( g(x) = x + (1 - x) \log(1 - x) - \frac{x^2}{2} \) on \( x < 1 \). Note that \( \frac{d}{dx}g(x) = -x - \log(1 - x) \geq 0 \) and \( g(0) = 0 \). Hence, \(-\eta - (1 - \eta) \log(1 - \eta) \leq -\frac{\eta^2}{2} \) for \( \eta < 1 \). It follows that
\[
P(X \leq (1 - \eta)\mu) \leq \exp \left( -\frac{\eta^2}{2} \mu \right),
\]
which proves (9). This concludes the proof of Proposition 3. \( \square \)

**Corollary 4.** Let \( X_1, X_2, \ldots, X_n \) be independent random variables with \( X_i \in \{0, 1\} \) for all \( i = 1, 2, \ldots, n \). Let \( X = \sum_{i=1}^n X_i \) and \( \mu = EX = \sum_{i=1}^n EX_i \). Then, for any \( \gamma \geq 0 \),
\[
P(|X - \mu| \geq \gamma) \leq 2 \exp \left( -\frac{\gamma^2}{2\mu + \gamma} \right).
\]
Proof. Note that (8) and (9) in Proposition 3 are bounds on the relative deviation $\eta$ from the expected value $\mu$. By setting $\gamma = \eta \mu > 0$, we can rewrite these bounds in terms of the absolute deviation $\gamma$ instead:

\[
P(X \geq \mu + \gamma) \leq \exp \left( -\frac{(\gamma/\mu)^2}{2 + \gamma/\mu} \right) \quad \text{and} \quad P(X \leq \mu - \gamma) \leq \exp \left( -\frac{(\gamma/\mu)^2}{2} \right)
\]

Now, since $\exp \left( -\frac{\gamma^2}{2\mu} \right) \leq \exp \left( -\frac{\gamma^2}{2\mu + \gamma} \right)$, by summing the above equations, the result follows. $\square$

5 Proof of the Limit Shape Result

Below follows the proof of Theorem 1.

Proof. Let $0 < \varepsilon_1 < \varepsilon$ and define the function $f(x) = \frac{x^2}{2 + x}$. Note that $nf(\varepsilon_1) < nf(\varepsilon)$. For $m > \lfloor nf(\varepsilon_1) \rfloor$, playing $m$ rounds of the solitaire from the initial state $\lambda^{(0)}$ is equivalent to playing $\lfloor nf(\varepsilon_1) \rfloor$ rounds from the initial state $\lambda^{(m-\lfloor nf(\varepsilon_1) \rfloor)}$. Without loss of generality, we can therefore assume $m = \lfloor nf(\varepsilon_1) \rfloor$.

In order to keep track of the piles, we put each pile in a bowl and line up the bowls in a row on the table. In each round of the game, the new (possibly empty) pile is put in a new bowl to the left of all old bowls. Let $\alpha_k$ be the number of cards in the $k$th bowl from the left after $m$ rounds (i.e., $\alpha_1$ is always the number of cards in the most recently formed pile). For convenience we put infinitely many empty bowls to the right. This way, a configuration of cards corresponds to a weak integer composition $\alpha = (\alpha_1, \alpha_2, \ldots)$.

In our analysis of the number of cards $\alpha_k$ in the $k$th bowl, we shall consider two regimes of $k$: the first regime is $k \leq m$ and the second regime is $k > m$. In the first regime, Corollary 4 will be used to bound the probability that $|\alpha_k - E\alpha_k|$ is large. In the second regime we shall prove that the probability is small that there are any cards at all in the bowls.

Regime 1: $k \leq m$. In order for a specific card to be in the $k$th bowl (i) it must have been picked in the round when the bowl was created, for which the probability is $p$, and (ii) it must have stayed in that bowl and not been picked in the following $k-1$ rounds, for which the probability is $(1-p)^{k-1}$. Thus,

\[
\alpha_k \in \text{Bin} \left( n, p(1-p)^{k-1} \right) \quad \text{and hence} \quad E\alpha_k = np(1-p)^{k-1}. \quad (17)
\]
Since \( \alpha_k \) is a sum of \( n \) stochastic 0/1-variables (1 with probability \( p \) and 0 with probability \( 1 - p \)), we can use Corollary 4 with \( X = \alpha_k \) and \( \gamma = \varepsilon_1 np \) to obtain

\[
P(|\alpha_k - E\alpha_k| \geq \varepsilon_1 np) \leq 2 \exp \left( -\frac{\varepsilon_1^2}{2(1 - p)^{k-1}} + \varepsilon_1 np \right)
\]

\[
\leq 2 \exp (-f(\varepsilon_1)np) .
\] (18)

We shall now bound the probability that \(|\alpha_k - E\alpha_k| \geq \varepsilon_1 np\) for at least one \( k \) in the entire first regime (i.e., for at least one value of \( k \leq m = \lfloor nf(\varepsilon_1) \rfloor \)) by summing \( m \) terms of the type in the right hand side of (18). This is particularly easy to do as these terms are independent of \( k \).

\[
P(\exists k \leq m : |\alpha_k - E\alpha_k| \geq \varepsilon_1 np)
\]

\[
\leq \sum_{k=1}^{m} P(|\alpha_k - E\alpha_k| \geq \varepsilon_1 np)
\]

\[
\leq m \cdot 2 \exp (-f(\varepsilon_1)np) \quad \text{(by (18))}
\]

\[
\leq 2nf(\varepsilon_1) \exp (-f(\varepsilon_1)np) \quad \text{(since } m = \lfloor nf(\varepsilon_1) \rfloor < nf(\varepsilon_1))
\]

\[
= \exp \left[ -f(\varepsilon_1)np \left( 1 - \frac{\log(2nf(\varepsilon_1))}{f(\varepsilon_1)np} \right) \right]
\]

\[
= \exp \left[ -f(\varepsilon_1)np(1 - o(1)) \right] \quad \text{(since } p = \omega \left( \frac{\log n}{n} \right) \text{)}
\]

\[
= P_{\varepsilon_1},
\]

where \( o(1) \) is with respect to \( n \to \infty \), and where we introduced the symbol \( P_{\varepsilon_1} \) as a shorthand for \( \exp \left[ -f(\varepsilon_1)np(1 - o(1)) \right] \). Therefore, for the complementary event, we have

\[
P(\forall k \leq m : |\alpha_k - E\alpha_k| < \varepsilon_1 np) > 1 - P_{\varepsilon_1} .
\] (19)

Regime 2: \( k > m \). If all \( n \) cards have been picked at least once after \( m \) rounds, this regime is empty, i.e. \( \alpha_k = 0 \) for all \( k > m \). The probability for this is \((1 - (1 - p)^m)^n\), since the probability that a specific card has not been
picked after $m$ rounds is $(1 - p)^m$. Thus,

\[
P(\forall k > m : \alpha_k = 0) = (1 - (1 - p)^m)^n
\]

\[
\geq 1 - n(1 - p)^m
\]

(by Cor. 3)

\[
\geq 1 - ne^{-mp}
\]

(since $e^{-p} \geq 1 - p$)

\[
= 1 - n \exp\left(-|nf(\varepsilon_1)|p\right)
\]

\[
= 1 - \exp\left[\log n - |nf(\varepsilon_1)|p\right]
\]

\[
\geq 1 - \exp\left[\log n - (nf(\varepsilon_1) - 1)p\right]
\]

\[
= 1 - \exp\left[-f(\varepsilon_1)n p \left(1 - \frac{1}{f(\varepsilon_1)} \left[\frac{\log n}{np} + \frac{1}{n}\right]\right)\right]
\]

\[
= 1 - \exp\left[-f(\varepsilon_1)n p (1 - o(1))\right]
\]

(since $p = \omega\left(\frac{\log n}{n}\right)$)

\[
= 1 - P_{\varepsilon_1}
\]

(20)

Combining the results (19) and (20) from the two regimes, we see that $P_{\varepsilon_1}$ bounds both the probability that any pile in the first regime deviates from its expected size with more than $\varepsilon_1 np$ and the probability that any bowl in the second regime is nonempty. Left to do is to investigate how these pile sizes relate to the limiting function. (Note that we still have not rescaled the pile sizes.) Again, we treat the two regimes separately.

**Regime 1:** $k < m$. By the uniform convergence (7) in Proposition 2, for any $\varepsilon_2 > 0$ and for sufficiently small $p$, we have

\[
|np(1 - p)^{k-1} - np e^{-p(k-1)}| < np \varepsilon_2
\]

(21)

for all $k \geq 1$. Since we chose $\varepsilon_1 < \varepsilon$, we can choose $\varepsilon_2 > 0$ such that $\varepsilon_1 + \varepsilon_2 < \varepsilon$. Then, for sufficiently small $p$ (i.e., for sufficiently large $n$) we have with probability at least $1 - P_{\varepsilon_1}$ (initially using the triangle inequality),

\[
|\alpha_k - np e^{-p(k-1)}| \leq |\alpha_k - E\alpha_k| + |E\alpha_k - np e^{-p(k-1)}|
\]

\[
= |\alpha_k - E\alpha_k| + |np(1 - p)^{k-1} - np e^{-p(k-1)}| \quad \text{ (by (17))}
\]

\[
\leq np \varepsilon_1 + np \varepsilon_2 \quad \text{ (by (19), (21))}
\]

\[
= np(\varepsilon_1 + \varepsilon_2)
\]

(22)

**Regime 2:** $k > m = \lfloor nf(\varepsilon_1)\rfloor$. In this regime, for sufficiently large $n$
(i.e., for sufficiently small \( p \)) with probability at least \( 1 - P_{\varepsilon_1} \), we have
\[
|\alpha_k - np e^{-p(k-1)}| \leq |\alpha_k| + |np e^{-p(k-1)}| \\
= 0 + np e^{-p(k-1)} \quad \text{(by (20))} \\
< np e^{-p(nf(\varepsilon_1)-1)} \quad \text{(since } k \in \mathbb{Z}, k > nf(\varepsilon_1)\text{)} \\
= np e^{-pzf(\varepsilon_1)\varepsilon_1} \\
\leq np(\varepsilon_1 + \varepsilon_2), \quad (23)
\]
where the last inequality follows from the assumption \( p = \omega \left( \frac{\log n}{n} \right) \). (Namely, \( np \to \infty \) as \( n \to \infty \), so \( e^{p} e^{-pf(\varepsilon_1)} \) is eventually smaller than \( \varepsilon_1 + \varepsilon_2 \).

Combining (22) and (23) and dividing by \( np \) we obtain
\[
P \left( \forall k > 0 : \frac{\alpha_k}{np} - e^{-p(k-1)} \leq \varepsilon_1 + \varepsilon_2 \right) \geq 1 - P_{\varepsilon_1}. \quad (24)
\]

Now, we would like to say something about the \( (1/p) \)-rescaled boundary function \( \partial^{1/p} \alpha(x) = \frac{1}{np} \alpha_{\left\lfloor x/p \right\rfloor} + 1 \) for all real \( x \geq 0 \), and not only about the pile sizes \( \alpha_k \) themselves for integers \( k > 0 \). To this end, first define \( \varepsilon_3 > 0 \) such that \( \varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \). Then, using the triangle inequality, we have
\[
\frac{1}{np} \alpha_{\left\lfloor x/p \right\rfloor + 1} - e^{-x} \leq \frac{1}{np} \alpha_{\left\lfloor x/p \right\rfloor + 1} - e^{-\left\lfloor x/p \right\rfloor p} + e^{-\left\lfloor x/p \right\rfloor p} - e^{-x} \leq (\varepsilon_1 + \varepsilon_2) + (e^{-\left\lfloor x/p \right\rfloor p} - e^{-x}) \quad \text{(by (24) and since } \left\lfloor x/p \right\rfloor p \leq x\text{)} \\
\leq (\varepsilon_1 + \varepsilon_2) + e^{-(x-p)} - e^{-x} \quad \text{(since } \left\lfloor x/p \right\rfloor p > x-p\text{)} \\
\leq (\varepsilon_1 + \varepsilon_2) + (e^{p-1}) \quad \text{(since } e^{-x} \leq 1\text{)} \\
\leq (\varepsilon_1 + \varepsilon_2) + \varepsilon_3 \quad \text{(since } p \to 0 \text{ as } n \to \infty\text{)} \\
= \varepsilon. \quad (25)
\]

Since the bound \( \varepsilon \) in (25) is independent of \( x \), we conclude that it holds for all \( x \geq 0 \), and since also \( \frac{1}{np} \alpha_{\left\lfloor x/p \right\rfloor + 1} = \partial^{1/p} \alpha(x) \), we can write (25) as
\[
P := P \left( \left\| \partial^{1/p} \alpha - g \right\|_{\infty} \leq \varepsilon \right) \geq 1 - P_{\varepsilon_1}.
\]
In other words, \( \lim \inf_{n \to \infty} \frac{-\log(1-P)}{np} \geq f(\varepsilon_1) \) and, since this holds for any \( \varepsilon_1 < \varepsilon \), the continuity of the function \( f \) yields that \( \lim \inf_{n \to \infty} -\frac{\log(1-P)}{np} \geq f(\varepsilon) \). Thus,
\[
P \left( \left\| \partial^{1/p} \alpha - g \right\|_{\infty} \leq \varepsilon \right) \geq 1 - P_{\varepsilon} = 1 - \exp \left[ -f(\varepsilon) np (1 - o(1)) \right].
\]
An application of Lemma 1 concludes the proof of Theorem 1. \( \square \)
6 Discussion

Popov initiated the study of stochastic versions of Bulgarian solitaire [12]. Here we compare our work with his. Like the original deterministic game, Popov’s stochastic version was “pile-based” in that each old pile was independently picked with a fixed probability $p$ to release a card to the new pile. In contrast, we here studied a “card-based” version, where every single card was independently picked with the same probability $p$. Interestingly, this modification radically changed the outcome of the process from a triangular limit shape to an exponential limit shape. From an analytical point of view, a crucial difference is that the card-based version allowed us to study each pile independently of previously formed piles. To capitalise on this feature we represented game configurations by weak integer compositions. We proved asymptotic results about the shape of these compositions and thanks to Lemma 1 these results would hold a fortiori after sorting compositions to obtain integer partitions.

By formulating our results in terms of a limit shape of integer partitions we connected to an important literature by Vershik and others. However, in that literature the starting point is typically a given probability distribution on integer partitions, see [14]. Here the starting point is instead a stochastic process on Young diagrams. Often a stochastic process defines a unique stationary probability distribution, which in turn will define a limit shape. This was the case in our Corollary 2 as well as in previous work on birth-and-death processes on Young diagrams by Eriksson and Sjöstrand [4]. Although Popov did not connect his result to Vershik’s theory of limit shapes of integer partitions, his result too can be interpreted as a triangular limit shape of this kind being obtained under the stationary probability distribution of the pile-based stochastic Bulgarian solitaire.

A novel feature of the present work is that for a stochastic process on the partitions of a fixed integer $n$ we were able to prove a limit shape result that is genuinely about the process instead of just using its stationary distribution: Theorem 1 states, for a given number of cards, a relation between how many moves are played and the probability that this sequence of moves will end up close to an exponential shape. This is a kind of question that could be asked for any stochastic process on the partitions of a fixed integer $n$, such as those in [12] and [4].
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