Algebraic Geometry and Hofstadter Type Model

Shao-shiung Lin
Department of Mathematics,
Taiwan University
Taipei, Taiwan
(e-mail: lin@math.ntu.edu.tw)

Shi-shyr Roan
Institute of Mathematics
Academia Sinica
Taipei, Taiwan
(e-mail: maroan@ccvax.sinica.edu.tw)

Abstract

In this report, we study the algebraic geometry aspect of Hofstadter type models through the algebraic Bethe equation. In the diagonalization problem of certain Hofstadter type Hamiltonians, the Bethe equation is constructed by using the Baxter vectors on a high genus spectral curve. When the spectral variables lie on rational curves, we obtain the complete and explicit solutions of the polynomial Bethe equation; the relation with the Bethe ansatz of polynomial roots is discussed. Certain algebraic geometry properties of Bethe equation on the high genus algebraic curves are discussed in cooperation with the consideration of the physical model.

1 Introduction

It is known for the past decade that algebraic geometry has played a certain intriguing role in certain 2-dimensional solvable statistical lattice models, a notable example would be the chiral Potts $N$-state integrable model (see e.g., [1], [2] and references therein). In the note, we report the algebraic geometry aspect of another model of physical interest in solid state physics. In the early 90’s, motivated by the work of Wiegmann and Zabrodin [3] on the appearance of $U_q(sl_2)$ symmetry in problems of magnetic translation, Faddeev and Kashaev [4] pursued the diagonalization problem

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on the following Hamiltonian by the quantum transfer matrix method which was developed by the Leningrad school in the early eighties:

\[
H_{FK} = \mu(\alpha U + \alpha^{-1} U^{-1}) + \nu(\beta V + \beta^{-1} V^{-1}) + \rho(\gamma W + \gamma^{-1} W^{-1})
\]

where \(U, V, W\) are unitary operators with the Weyl commutation relation for a primitive \(N\)-th root of unity \(\omega\) and the \(N\)-th power identity property, \(UV = \omega VU, VW = \omega WV, WU = \omega UW\); \(U^N = V^N = W^N = 1\). As a special limit case for \(\rho = 0\), the model is reduced to the (rational flux) Hofstadter Hamiltonian, a model possessing several physical interpretations with the history which can trace back to the work of Peierls \([11]\) on Bloch electrons in metals with the presence of a constant external magnetic field. By the pioneering works of the 50s and 60s \([2]\) \([5]\) \([7]\) \([9]\) \([13]\), the role of magnetic translations was found, and it began a systematic study of this 2D lattice model. In 1976, Hofstadter \([8]\) found the butterfly figure of the spectral band versus the magnetic flux which exhibits a beautiful fractal picture. Here the phase of \(\omega\) represents the magnetic flux (per plaquette). In \([3]\), a general framework to determine the eigenvalues of certain quantum chains appeared in the transfer matrix was presented. The method relies on a special monodromy solution of the Yang-Baxter equation for the six-vertex \(R\)-matrix; this solution appeared also in the study of chiral Potts model \([3]\). For a finite size \(L\), the trace of the monodromy matrix gives rise to the transfer matrix acting on the quantum space \(\otimes^L \mathbf{C}^N\); while the Hofstadter type Hamiltonian \([1]\) can be realized in the case \(L = 3\). In general, the diagonalization problem of the transfer matrix can be formulated into the Bethe equation through the Baxter vector, visualized on a "spectral" curve associated to the corresponding model. In \([11]\), we presented a detailed and rigorous mathematical study on the Bethe equation associated to the Hofstadter type model. In particular, we obtained the complete solution of the Bethe equation for models with rational spectral curves for \(L \leq 3\), among which a special Hofstadter type of \(H_{FK}\) in \([3]\) is included, and further expended to all the other sectors. In this note, we explain the main results we have obtained in \([10]\); detailed derivations, as well as extended references to the literature, may be found in that work.

This paper is organized as follows. In Sect. 2, we first recall results in transfer matrix relevant to our discussion; then introduce the Bethe equation (or Baxter \(T-Q\) equation) through the Baxter vector on the spectral curve. In Sect. 3, we consider the case when the spectral data lie on rational curves and perform the mathematical derivation of the answer. We present the complete solutions of the Bethe polynomial equations of all sectors for \(L \leq 3\). In Sect. 4, we discuss the "degeneracy" relation between the Bethe solutions and the eigenspaces in the quantum space of the transfer matrix for \(L = 3\); also its connection with the usual Bethe ansatz technique in literature, in particular the result obtained in \([3]\). In Sect. 5, we describe the algebraic geometry properties of the high genus spectral curve arisen from the Hofstadter Hamiltonian.

Notations. The letters \(\mathbf{Z}, \mathbf{R}, \mathbf{C}\) will denote the ring of integers, real, complex numbers respectively, \(\mathbf{N} = \mathbf{Z}_{>0}, \mathbf{Z}_N = \mathbf{Z}/N\mathbf{Z}\). Throughout this report, \(\mathbf{N}\) will always denote an odd positive integer with \(M = \left\lceil \frac{N}{2} \right\rceil\): \(N = 2M + 1, M \geq 1\); \(\omega\) is a primitive \(N\)-th root of unity, and \(q := \omega^{\frac{1}{2}}\) with \(q^N = 1\), i.e., \(q = \omega^{M+1}\). An element \(v\) in the vector space \(\mathbf{C}^N\) is represented by a sequence of coordinates, \(v_k, k \in \mathbf{Z}\), with the \(N\)-periodic condition, \(v_k = v_{k+N}\), i.e., \(v = (v_k)_{k \in \mathbf{Z}_N}\). The standard basis of \(\mathbf{C}^N\) will be denoted by \(|k\rangle\), with the dual basis of \(\mathbf{C}^{N^*}\) by \(\langle k|\) for \(k \in \mathbf{Z}_N\).

For a positive integers \(n\), we denote \(\otimes^n \mathbf{C}^N\) the tensor product of \(n\)-copies of the vector space \(\mathbf{C}^N\). We use the notation of \(\rho\)-shifted factorials: \((a; \rho)_n = (1 - a)(1 - a\rho) \cdots (1 - a\rho^{n-1})\) for \(n \in \mathbf{N}\), and \((a; \rho)_0 = 1\).

\(^2\)It is also called as the "Baxter vacuum state" in other literature.
2 Transfer Matrix and the Bethe Equation

We consider the Weyl algebra generated by the operators $Z, X$ satisfying the Weyl commutation relation with the $N$-th power identity, $ZX = \omega XZ, Z^N = X^N = I,$ and denote $Y := ZX.$ In the canonical irreducible representation of the Weyl algebra, the operators $Z, X, Y$ act on $\mathbb{C}^N$ with the expressions: $Z(v)_k = \omega^k v_k,$ $X(v)_k = v_{k-1},$ $Y(v)_k = \omega^k v_{k-1}.$ It is known that the following $L$-operator for an element $h = [a : b : c : d]$ of the projective 3-space $\mathbb{P}_3$ with operator-entries acting on the quantum space $\mathbb{C}^N$,

$$L_h(x) = \begin{pmatrix} aY & xbX \\ xcz & d \end{pmatrix}, \quad x \in \mathbb{C},$$

possesses the intertwining property of the Yang-Baxter relation,

$$R(x/x')(L_h(x) \otimes 1)(1 \otimes L_h(x')) = (1 \otimes L_h(x'))(L_h(x) \otimes 1)R(x/x'), \quad (2)$$

where $R(x)$ is the matrix of a 2-tensor of the auxiliary space $\mathbb{C}^2$ with the following numerical expression,

$$R(x) = \begin{pmatrix} x\omega - x^{-1} & 0 & 0 & 0 \\ 0 & \omega(x - x^{-1}) & \omega - 1 & 0 \\ 0 & \omega - 1 & x - x^{-1} & 0 \\ 0 & 0 & 0 & x\omega - x^{-1} \end{pmatrix}.$$ 

By performing the matrix product on auxiliary spaces and the tensor product of quantum spaces, one has the $L$-operator associated to an element $\bar{h} = (h_0, \ldots, h_{L-1}) \in (\mathbb{P}_3)^L,$ $L_{\bar{h}}(x) = \bigotimes_{j=0}^{L-1} L_{h_j}(x),$ which again satisfies the relation (2). The entries of $L_{\bar{h}}(x)$ are operators of the quantum space $\mathbb{C}^N 
\otimes \mathbb{C}^N$, and its trace defines the commuting transfer matrices for $x \in \mathbb{C}, T_{\bar{h}}(x) = \text{tr}_{aux}(L_{\bar{h}}(x)).$ The transfer matrix $T_{\bar{h}}(x)$ can also be computed by changing $L_{h_j}$ to $\tilde{L}_{h_j}$ via a gauge transformation

$$:\tilde{L}_{h_j}(x, \xi, \xi_{j+1}) = A_jL_{h_j}(x)A_{j+1}^{-1}, 0 \leq j \leq L - 1,$$ 

with $A_j = \begin{pmatrix} 1 & \xi_j - 1 \\ 1 & \xi_j \end{pmatrix}$ and $A_L := A_0.$ One has

$$\tilde{L}_{h_j}(x, \xi, \xi_{j+1}) = \begin{pmatrix} F_{h_j}(x, \xi_j - 1, \xi_{j+1}) & -F_{h_j}(x, \xi_j - 1, \xi_{j+1} - 1) \\ F_{h_j}(x, \xi_j, \xi_{j+1}) & -F_{h_j}(x, \xi_j, \xi_{j+1} - 1) \end{pmatrix},$$

where $F_{h}(x, \xi, \xi') := \xi' aY - xbX + \xi' xcz - \xi d.$ Hence $T_{\bar{h}}(x) = \text{tr}_{aux}(\tilde{L}_{\bar{h}}(x, \vec{\xi})), \vec{\xi} := (\xi_0, \ldots, \xi_{L-1})$ where

$$\tilde{L}_{\bar{h}}(x, \vec{\xi}) := \bigotimes_{j=0}^{L-1} \tilde{L}_{h_j}(x, \xi_j, \xi_{j+1}) = \begin{pmatrix} \tilde{L}_{\bar{h},1,1}(x, \vec{\xi}) & \tilde{L}_{\bar{h},1,2}(x, \vec{\xi}) \\ \tilde{L}_{\bar{h},2,1}(x, \vec{\xi}) & \tilde{L}_{\bar{h},2,2}(x, \vec{\xi}) \end{pmatrix}, \quad \xi_L := \xi_0.$$

We consider the variables $(x, \xi_0, \ldots, \xi_{L-1})$ in the following spectral curve,

$$C_{\bar{h}}: \quad \xi_j^N = (-1)^N \frac{\xi_{j+1}^N a_j^N - x^N b_j^N}{\xi_{j+1}^N x^N c_j^N - d_j^N}, \quad j = 0, \ldots, L - 1,$$ 

(3) and denote $p_j = (x, \xi_j, \xi_{j+1}).$ Then the operator $F_{h_j}(x, \xi_j, \xi_j)$ has 1-dimensional null space in $\mathbb{C}^N$ generated by the vector $|p_j\rangle$ with the form:

$$|0\rangle_{p_j} = 1, \quad \frac{\langle m|p_j\rangle}{\langle m - 1|p_j\rangle} = \frac{\xi_{j+1} a_j \omega^m - xb_j}{-\xi_j (\xi_{j+1} xc_j \omega^m - d_j)}.$$
The Baxter vector \( |p\rangle \) for \( p \in \mathcal{C}_\mathcal{H} \) is defined by \( |p\rangle := |p_0\rangle \otimes \cdots \otimes |p_{L-1}\rangle \in \otimes_{\mathbb{C}^N}^L \), which possesses the following property:

\[
\tilde{L}_{\mathcal{H},1,1}(x, \xi)|p\rangle = (\tau_- p) \Delta_-(p), \quad \tilde{L}_{\mathcal{H},2,2}(x, \xi)|p\rangle = (\tau_+ p) \Delta_+(p), \quad \tilde{L}_{\mathcal{H},2,1}(x, \xi)|p\rangle = 0,
\]

where \( \Delta_\pm \) are functions of \( \mathcal{C}_\mathcal{H} \) defined by \( \Delta_-(x, \xi) = \prod_{j=0}^{L-1} (d_j - x \xi_{j+1} + c_j), \Delta_+(x, \xi) = \prod_{j=0}^{L-1} \xi_j (a_j d_j - x^2 b_j c_j), \) and \( \tau_\pm \) are the automorphisms, \( \tau_\pm(x, \xi) = (q^{\pm 1} x, q^{\mp 1} \xi) \). It follows the important relation of the transfer matrix on the Baxter vector over the curve \( \mathcal{C}_\mathcal{H} \):

\[
T_\mathcal{H}(x)|p\rangle = (\tau_- p) \Delta_-(p) + (\tau_+ p) \Delta_+(p), \quad \text{for} \ p \in \mathcal{C}_\mathcal{H}.
\]

As \( T_\mathcal{H}(x) \) are commuting operators for \( x \in \mathbb{C} \), a common eigenvector \( \langle \varphi \rangle \) is a constant vector of \( \otimes_{\mathbb{C}^N}^L \) with an eigenvalue \( \Lambda(x) \in \mathbb{C}[x] \). Define the function \( Q(p) = \langle \varphi | p \rangle \) of \( \mathcal{C}_\mathcal{H} \), then it satisfies the following Bethe equation,

\[
\Lambda(x)Q(p) = Q(\tau_- p) \Delta_-(p) + Q(\tau_+ p) \Delta_+(p), \quad \text{for} \ p \in \mathcal{C}_\mathcal{H}.
\]

By the definition of \( T_\mathcal{H}(x), \Lambda(x) \), one easily see that \( T_\mathcal{H}(x) \) is an operator-coefficient even \( x \)-polynomial of degree \( 2[\frac{L}{2}] \) with the constant term \( T_0 = \prod_{j=0}^{L-1} a_j \otimes Y + \prod_{j=0}^{L-1} d_j \). Hence the polynomial \( \Lambda(x) \) in (\ref{eq:Lambda}) is an even function of degree \( \leq 2[\frac{L}{2}] \) with \( \Lambda(0) = q^k \prod_{j=0}^{L-1} a_j + \prod_{j=0}^{L-1} d_j \) for some \( l \in \mathbb{Z}_N \). For \( L = 3 \), we have \( T_\mathcal{H}(x) = T_0 + x^2 T_2 \) where

\[
T_2 = \quad b_0 c_1 a_2 X \otimes Z \otimes Y + a_0 b_1 c_2 Y \otimes X \otimes Z + c_0 a_1 b_2 Z \otimes Y \otimes X
+ c_0 b_1 d_2 Z \otimes X \otimes I + d_0 c_1 b_2 I \otimes Z \otimes X + b_0 d_1 c_2 X \otimes I \otimes Z.
\]

The above \( T_2 \) can be put into the form of the Hofstadter type Hamiltonian \( \otimes_{\mathbb{Z}_N}^L \mathbb{Z}_N \).

In the equation (\ref{eq:Lambda}), \( Q(p) \) is a rational function of \( \mathcal{C}_\mathcal{H} \) with zeros and poles. Hence the understanding of the Bethe solutions of (\ref{eq:Bethe}) relies heavily on the function theory of \( \mathcal{C}_\mathcal{H} \), and the algebraic geometry of the curve inevitably plays a key role on the complexity of the problem.

### 3 The Rational Degenerated Bethe Equation

In this section, we consider the case when the spectral curve \( \mathcal{C}_\mathcal{H} \) degenerates into an union of rational curves under the conditions: \( a_j = q^{-1} d_j, b_j = q^{-1} c_j \) for \( j = 0, \ldots, L - 1 \). By replacing \( c_j, d_j \) by \( \xi_j, 1 \), we assume \( d_j = 1 \) for all \( j \) with the parameter \( c_j \)s to be generic. In this case, \( \mathcal{C}_\mathcal{H} \) is the union of disjoint copies of the \( x \)-(complex) line , containing the following \( \tau_\pm \)-invariant subset of \( \mathcal{C}_\mathcal{H} \) which will be sufficient for the discussion of Bethe equation,

\[
\mathcal{C} := \{(x, \xi_0, \ldots, \xi_{L-1})|\xi_0 = \cdots = \xi_{L-1} = q^l, l \in \mathbb{Z}_N\}.
\]

We shall make the identification \( \mathcal{C} = \mathbb{P}^1 \times \mathbb{Z}_N \) via \( (x, q^l) \leftrightarrow (x, l) \). The automorphisms \( \tau_\pm \) on \( \mathcal{C} \) become \( \tau_\pm(x, l) = (q^{\pm 1} x, q^l - 1) \), by which the action (\ref{eq:action}) of \( T(x) := T_\mathcal{H}(x) \) on the Baxter vector \( |x, l\rangle \) now takes the form,

\[
T(x)|x, l\rangle = |q^{-1} x, l - 1\rangle \Delta_-(x, l) + |qx, l - 1\rangle \Delta_+(x, l),
\]

where \( \Delta_\pm \) are the rational functions of \( x; \Delta_-(x, l) = \prod_{j=0}^{L-1} (1 - x c_j q^j), \Delta_+(x, l) = \prod_{j=0}^{L-1} \frac{1-x^2 c_j^2 q^{-l}}{1-x c_j q^{-l}} \).

Furthermore, one can express the Baxter vector \( |x, l\rangle \) over the curve \( \mathcal{C} \) in the component-form
with deg. Here the bold letter $k$ denotes a multi-index vector $k = (k_0, \ldots, k_L)$ for $k_j \in \mathbb{Z}_N$ with the square-length of $k$ defined by $|k|^2 := \sum_{j=0}^{L} k_j^2$. Each ratio-term in the above right hand side is given by a non-negative representative for each element in $\mathbb{Z}_N$ appeared in the formula. We have the following result on the Bethe equation and its connection with the transfer matrix $T(x)$:

**Theorem 1** Denote $f^c, f^o$ the functions on $\mathbb{C}$, $f^c(x, 2n) = \prod_{j=0}^{L-1} \frac{(x c_j q^{1-2} \omega^{-1})_{k_j}}{(x c_j q^{2\omega})_{k_j}}$, and $f^o(x, 2n + 1) = \prod_{j=0}^{L-1} \frac{(x c_j q^{-1} \omega^{-1})_{n+1}}{(x c_j q^{\omega})_{n+1}}$. For $x \in \mathbb{P}^1$, $l \in \mathbb{Z}_N$, we define the following vectors in $\otimes \mathbb{C}^N$,

\[
|x|^f = \sum_{n=0}^{N-1} |x, 2n\rangle f^c(x, 2n) \omega^{jn}, \quad |x|^o = \sum_{n=0}^{N-1} |x, 2n + 1\rangle f^o(x, 2n + 1) \omega^{jn} ,
\]

\[
|x|^T = |x|^f q^{-1} u(qx) + |x|^o u(x); \quad \text{where } u(x) := \prod_{j=0}^{L-1} (1 - x N c_j^N)(x c_j q^2)^2_M .
\]

Then

(i) $|x|^T u(qx) = |x|^f q^T u(x)$, or equivalently, $|x|^T = 2q^{-1} |x|^f u(qx) = 2|x|^o u(x)$.

(ii) The $T(x)$-transform on $|x|^T$ is given by

\[
q^{-1} T(x) |x|^T = |q^{-1} x|^T \Lambda(x, -1) + |q x|^T \Lambda(x, 0) , \quad l \in \mathbb{Z}_N .
\]

(iii) For a common eigenvector $\langle \varphi |$ of $T(x)$ with the eigenvalue $\Lambda(x)$, the function $Q_l^+(x) := \langle \varphi | x|^T \rangle$ and $\Lambda(x)$ are polynomials with the properties: $\text{deg} Q_l^+(x) \leq (3M + 1) L$, $\text{deg} \Lambda(x) \leq 2 \left( \frac{L}{2} \right)$, $\Lambda(x) = \Lambda(-x)$, $\Lambda(0) = q^2 + 1$, and the following Bethe equation holds:

\[
q^{-1} \Lambda(x) Q_l^+(x) = \prod_{j=0}^{L-1} (1 - x c_j q^{-1}) Q_l^+(x q^{-1}) + \prod_{j=0}^{L-1} (1 + x c_j) Q_l^+(x q) .
\]

Furthermore for $0 \leq m \leq M$, $Q_m^+(x), Q_{N-m}^+(x)$ are elements in $x^m \prod_{j=0}^{L-1} (1 - x N c_j^N) \mathbb{C}[x]$.

By (iii) of the above theorem, the equation $(8)$ for the sector $m, N - m$ can be combined into a single one. For the rest of this report the letter $m$ will always denote an integer between 0 and $M$: $0 \leq m \leq M$. By introducing the polynomials $\Lambda_m(x), Q(x)$ via the relation,

\[
(\Lambda_m(x), x^m \prod_{j=0}^{L-1} (1 - x N c_j^N) Q(x)) = (q^{-m} \Lambda(x), Q_m^+(x)), \quad (q^m \Lambda(x), Q_{N-m}^+(x)) ,
\]

the equation $(8)$ for $l = m, N - m$ becomes the following polynomial equation of $Q(x), \Lambda_m(x)$:

\[
\Lambda_m(x) Q(x) = q^{-m} \prod_{j=0}^{L-1} (1 - x c_j q^{-1}) Q(x q^{-1}) + q^m \prod_{j=0}^{L-1} (1 + x c_j) Q(x q) ,
\]

with $\text{deg} Q(x) \leq ML - m$, $\text{deg} \Lambda_m(x) \leq 2 \left( \frac{L}{2} \right)$. $\Lambda_m(x) = \Lambda_m(-x), \Lambda_m(0) = q^m + q^{-m}$. The general mathematical problem will be to determine the solution space of the Bethe equation $(8)$ for a given positive integer $L$.

For $L = 1, 2$, we have the following result.

**Theorem 2** (i) For $L = 1$, we have $\Lambda_m(x) = q^m + q^{-m}$ and the solutions $Q_m(x)$ of $(8)$ form an one-dimensional vector space generated by the following polynomial of degree $M - m$,

\[
B_m(x) = 1 + \sum_{j=1}^{M-m} \left( \frac{q^{m+i-1} - q^{-m-i}}{q^m + q^{-m} - q^{-m-i} - q^{m+i}} \right) (x c_0)^j .
\]
(II) For \( L = 2 \), the equation (11) has a non-trivial solution \( Q_m(x) \) if and only if \( \deg Q_m(x) = M - m + m' \) for \( 0 \leq m' \leq M \). For each such \( m' \), the eigenvalue \( \Lambda_m(x) \) in (11) is equal to \( \Lambda_{m,m'}(x) := q^k(q^{m'-1} + q^{-m'-2})x^2c_0c_1 + q^m + q^{-m}, \) and the corresponding solutions of \( Q_m(x) \) form an one-dimensional space generated by a polynomial \( B_{m,m'}(x) \) of degree \( M - m + m' \) with \( B_{m,m'}(0) = 1 \).

For \( L = 3 \), this is the case related to the Hamiltonian (11). We consider the \( N \times N \) matrix,

\[
A = \begin{pmatrix}
\delta'_{N-1} & u'_{N-1} & 0 & \cdots & 0 & 0 \\
u'_{N-2} & \delta'_{N-2} & u'_{N-2} & 0 & \cdots & 0 \\
w'_{N-3} & v'_{N-3} & \delta'_{N-3} & u'_{N-3} & 0 & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & v'_1 & \delta'_1 & u'_1 \\
0 & \cdots & 0 & w'_0 & v'_0 & \delta'_0 \\
\end{pmatrix}
\]  

with the entries defined by \( u'_k = q^{k+\frac{1}{2}} + q^{-k-\frac{1}{2}} - q^m - q^{-m}, \) \( v'_k = (q^{k+\frac{1}{2}} - q^{-k-\frac{1}{2}})(c_0 + c_1 + c_2), \) \( \delta'_k = (q^{k+\frac{1}{2}} + q^{-k-\frac{1}{2}})(c_0c_1 + c_1c_2 + c_2c_0), \) \( w'_k = (q^{k-\frac{1}{2}} - q^{-k-\frac{1}{2}})c_0c_1c_2. \) Then one can derive the following result.

**Theorem 3** For \( L = 3 \), the condition of the eigenvalue \( \Lambda_m(x) = \lambda_m x^2 + q^m + q^{-m}, \) \( 0 \leq m \leq M \), with a non-trivial solution \( Q_m(x) \) in the equation (11) is determined by the solution of \( \det(A - \lambda_m) = 0 \), where \( A \) is the matrix defined by (11). For each such \( \Lambda_m(x) \), there exists an unique (up to constants) non-trivial polynomial solution \( Q_m(x) \) of (11) with the degree \( Q_m(x) \) equal to \( 3M - m \) and \( Q_m(0) \neq 0 \).

### 4 The Degeneracy and Bethe Ansatz Relation of Roots of Bethe Polynomial

We first discuss the degeneracy relation of eigenspaces of the transform matrix \( T(x) \) in \( L \otimes \mathbb{C}^{N^*} \) with respect to the Bethe solutions obtained in the previous section. As before, we denote \( \Lambda(x) \) the eigenvalues of \( T(x) \), whose constant term is given by \( T_0 = D + 1, \) where \( \mathbb{L} := q^{-L} \otimes \mathbb{Y}; \) hence \( \Lambda(0) = q^L + 1. \) For \( l \in \mathbb{Z}_N \), we denote \( \mathcal{E}_l \) the \( N^{L-1} \)-dimensional eigenspace of \( L \otimes \mathbb{C}^{N^*} \) of the operator \( D \) with the eigenvalue \( q^L. \) For \( 0 \leq m \leq M \), the equation (11) describes the relation of \( \Lambda(x) \) and its eigenfunctions with \( \Lambda(0) = q^{2m} + 1 \) or \( q^{2(N-m)} + 1. \) We now consider the case for \( L = 3, \) where \( T_2 \) in (11) is now expressed by

\[
T_2 = q^{-2}(c_0c_1 X \otimes Z \otimes Y + c_1c_2 Y \otimes X \otimes Z + c_0c_2 Z \otimes Y \otimes X) \\
+ q^{-1}(c_0c_1 Z \otimes X \otimes I + c_1c_2 I \otimes Z \otimes X + c_0c_2 X \otimes I \otimes Z).
\]

We have \( qD = (Z \otimes X \otimes I)(X \otimes I \otimes Z)/(I \otimes Z \otimes X) \). We shall denote \( \mathcal{O}_3 \) the operator algebra generated by the tensors of \( X, Y, Z, I \) appeared in the above expression of \( T_2 \). Then \( \mathcal{O}_3 \) commutes with \( D \), hence one obtains a \( \mathcal{O}_3 \)-representation on \( \mathbb{E}_3 \) for each \( l \). With the identification, \( U = D^{-1/2} Z \otimes X \otimes I, \) \( V = D^{-1/2} X \otimes I \otimes Z, \) \( \mathcal{O}_3 \) is generated by \( U, V \) which satisfy the Weyl relation \( UV = \omega VU \) and the \( N \)-th power identity. Hence \( \mathcal{O}_3 \) is the Heisenberg algebra and contains \( D \) as a central element. Then \( qD^{-1} T_2 \) has the following expression,

\[
c_0c_1(U + U^{-1}) + c_0c_2(V + V^{-1}) + c_1c_2(qD^{5/2}UV + q^{-1}D^{-5/2}V^{-1}U^{-1}). \quad (11)
\]
The above Hamiltonian is the same as $H_{FK}$ with $W=q^{-1}D^{-5/2}V^{-1}U^{-1}$, $\alpha = \beta = \gamma = 1$. Our conclusion on the sector $m = M$ is equivalent to that in [3] as it becomes clearer later on. There is an unique (up to equivalence) non-trivial irreducible representation of $O_3$, denoted by $C^N_\rho$, which is of dimension $N$. For each $l$, $E^l_3$ is equivalent to $N$-copies of $C^N_\rho$ as $O_3$-modules: $E^l_3 \simeq N C^N_\rho$. For $0 \leq m \leq M$, we consider the space $E^l_3$ with $q^l = q^{2m}$. The evaluation of $E^l_3$ on $|x\rangle_{\pm m}$ gives rise to a $N$-dimensional kernel in $E^l_3$. By Theorem 3, there are $N$ polynomials $Q_m(x)$ of degree $3M - m$ as solutions of (9) with the corresponding $N$ distinct eigenvalues $\Lambda_m(x)$. The $N$-dimensional vector space spanned by those $Q_m(x)$s becomes a realization of the irreducible representation $C^N_\rho$ for the Heisenberg algebra $O_3$.

Now we discuss the relation between the Bethe equation (9) and the usual Bethe ansatz formulation in literature. For $0 \leq m \leq M$, a solution $Q_m(x)$ in (9) always have the property $Q_m(0) \neq 0$ by Theorem 3, hence one has the form $Q_m(x) = \prod_{l=1}^{3M-m} (x - \frac{1}{z_l})$ with $z_l \in C^*$. By setting $x = z_l^{-1}$ in (9), we obtain the following relation among $z_l$s, which is called the Bethe ansatz relation,

$$q^{m+\frac{3}{2}} \prod_{j=0}^{2} \frac{z_l + c_j}{q z_l - c_j} = \prod_{n=1,n \neq l}^{3M-m} \frac{q z_l - z_n}{z_l - q z_n}, \quad 1 \leq l \leq 3M - m.$$ 

For the sector $m = M$, the comparison of the $x^2$-coefficient of (9) yields the expression of eigenvalue,

$$\lambda_M = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})s_2 + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})s_1 \sum_{n=1}^{2M} z_n + (q^{\frac{3}{2}} + q^{-\frac{3}{2}} - q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \sum_{l<n} z_l z_n.$$

With the substitution, $\mu = q^{\frac{1}{2}}c_1^{-1}$, $\nu = q^{\frac{1}{2}}c_2^{-1}$, $\rho = q^{\frac{1}{2}}c_2^{-1}$, the above expression coincides with (5.27) in [3]. Note that the Bethe ansatz relation can be shown to be equivalent to the Bethe equation (9) for the sector $M$. However, the parallel statement is no longer true for other sectors $m \neq M$, i.e., it does exist some non-physical Bethe ansatz solutions in the above form, while not corresponding to any polynomial solution of Bethe equation (9). Some example can be found in the $(M-1)$-sector.

### 5 High Genus Curves for the Hofstadter Model

We are now going back to the general situation in Sect. 2. Note that the values $\xi_l^N$’s of the curve $C_l$ in (9) are determined by $\xi_0^N$ and $x^N$, denoted by $y = x^N$, $\eta = \xi_0^N$. The variables $(y, \eta)$ defines the curve which is a double cover of $y$-line,

$$B_h : \quad C_h(y)\eta^2 + (A_h(y) - D_h(y))\eta - B_h(y) = 0$$

where the functions $A_h, B_h, C_h, D_h$ are the following matrix elements,

$$\begin{pmatrix} -A_h(y) & B_h(y) \\ C_h(y) & -D_h(y) \end{pmatrix} := \prod_{j=0}^{L-1} \begin{pmatrix} -a_j^N & yb_j^N \\ yc_j^N & -d_j^N \end{pmatrix}.$$ 

Now we consider only the case: $L = 3$, $a_0 = d_0 = 0$, $b_0 = c_0 = 1$, with generic $h_1, h_2$. The expression of $T(x)$ is given by

$$T(x) = x^2(c_1a_2X \otimes Z \otimes Y + a_1b_2Z \otimes Y \otimes X + b_1d_2Z \otimes X \otimes I + d_1c_2X \otimes I \otimes Z),$$

equivalently, $x^{-2}D_{\mp}T(x)$ is equal to the Hofstadter Hamiltonian $\left[\begin{array}{cc} 0 & U \\ V & 0 \end{array}\right]$ with $U = D^{-1/2}Z \otimes X \otimes I$, $V = D^{-1/2}X \otimes I \otimes Z$ and $\mu, \nu, \alpha, \beta$ related to $h_1, h_2$ by $\mu^2 = qb_1c_1a_2d_2$, $\alpha^2 = q^{-1}b_1c_1^{-1}a_2^{-1}d_2$, \ldots
\[ v^2 = qa_1b_1c_1, \quad \beta^2 = q^{-1}a_1^{-1}d_1b_1^{-1}c_2. \] By factoring out the \( y \)-component of \( B_{\hbar} \), the main irreducible component of \( B_{\hbar} \) is the curve,

\[
B : (y^2b_1^Nc_1^N + a_1N a_2^N)\eta^2 + (a_1N b_1^N + b_1N d_1^N - c_1N a_2^N - d_1N c_2^N)y\eta - (y^2c_1N b_2^N + d_1N d_2^N) = 0,
\]

which is an elliptic curve as a double-cover of the \( y \)-line. For the curve \( C_{\hbar} \), the variables \( \xi_0 \) and \( \xi_1 \) are related by \( \xi_0^{N-N} = \xi_1^{-N} \), which implies that \( C_{\hbar} \) can be identified with \( W \times \mathbb{Z}_N \) where \( W \) is a genus \( 6N^3 - 6N^2 + 1 \) curve with the following equation in the variable \( p = (x, \xi_0, \xi_2) \),

\[
W : \xi_0^{-N} = \frac{-\xi_0^{N} a_1^N + x^N b_1^N}{x^N \xi_0^{N} c_1^N - d_1^N}, \quad \xi_2^N = \frac{-\xi_0^{N} a_2^N + x^N b_2^N}{x^N \xi_0^{N} c_2^N - d_2^N}.
\]

By averaging the Baxter vectors \( |p, s \rangle \) of \( C_{\hbar} \) over an element \( p \) of \( W \), \( |p\rangle := \frac{1}{N} \sum_{s=0}^{N-1} |p, s\rangle q^{s^2} \), which defines the Baxter vector on \( W \). Furthermore, the transfer matrix can be descended to one on \( W \) with the following relation,

\[
x^{-2}T(x)|p\rangle = |\tau_-(p)\rangle \Delta_-(p) + |\tau_+(p)\rangle \Delta_+(p),
\]

where \( \Delta_{\pm} \) are the functions on \( W: \Delta_-(x, \xi_0, \xi_2) = \frac{\xi_0^{2}(a_1d_1-x^2b_1c_1)(a_2d_2-x^2b_2c_2)}{x(\xi_0^2-a_1-a_2)}, \quad \Delta_+(x, \xi_0, \xi_2) = \frac{\xi_0^{2}(a_1d_1-x^2b_1c_1)(a_2d_2-x^2b_2c_2)}{x(\xi_0^2-a_1-a_2)}\). For an eigenvector \( \langle \varphi \rangle \in \otimes^3 \mathbb{C}^{N^*} \) of \( x^{-2}T(x) \), the eigenvalue is a scalar \( \lambda \in \mathbb{C} \), and the function \( Q(p) := \langle \varphi | p \rangle \) of \( W \) satisfies the Bethe equation: \( \lambda Q(p) = Q(\tau_-(p))\Delta_-(p) + Q(\tau_+(p))\Delta_+(p) \), where \( \tau_{\pm} \) are the transformations of \( W \) with the same expression as before, but only in the coordinates \( (x, \xi_0, \xi_2) \). Consider the \( D \)-eigenspace decomposition of \( \otimes^3 \mathbb{C}^{N^*} = \bigoplus_{l \in \mathbb{Z}_N} E_3^l \). The evaluation of the Baxter vector over \( W \) gives rise to the following linear transformation, \( \varepsilon_l : E_3^l \rightarrow \{ \text{rational functions of } W \} \) with \( \varepsilon_l(v)(p) := \langle v | p \rangle \), for \( l \in \mathbb{Z}_N \). One has the following result.

**Theorem 4** For \( l \in \mathbb{Z}_N \), the linear map \( \varepsilon_l \) is injective, hence it induces an identification of \( E_3^l \) with a \( N^2 \)-dimensional functional space of \( W \).

By the discussion in Sect. 4, as the Heisenberg algebra \( O^3 \) representations, \( E_3^l \) is equivalent to \( N \) copies of the standard one. Hence it induces an \( O^3 \)-module structure on the function space \( \varepsilon_l(E_3^l) \), induced by the one of \( E_3^l \) by above theorem. The mathematical structure of the functional space \( \varepsilon_l(E_3^l) \) in terms of the divisor theory of Riemann surfaces in corporation with the interpretation of Heisenberg algebra representation remains an algebraic geometry problem for further study.

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