The Pólya Urn: Limit Theorems, Pólya Divergence, Maximum Entropy and Maximum Probability

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Abstract—Sanov’s Theorem and the Conditional Limit Theorem (CoLT) are established for a multicolor Pólya Eggenberger urn sampling scheme, giving the Pólya divergence and the Pólya extension to the Maximum Relative Entropy (MaxEnt) method. Pólya MaxEnt includes the standard MaxEnt as a special case. The universality of standard MaxEnt - advocated by an axiomatic approach to inference for inverse problems - is challenged, in favor of a probabilistic approach based on CoLT and the Maximum Probability principle.

I. INTRODUCTION

Consider an urn containing \(a_i > 0\) balls of colors \(i, i = 1, 2, \ldots, m\); let \(m\) be finite. A single ball is drawn from the urn, recorded and then returned together with \(c \in \mathbb{Z}\) balls of the same color. Assuming \(-nc \leq \min(a_1, a_2, \ldots, a_m)\), the drawing is repeated \(n\) times. This sampling is known as the multicolor Pólya Eggenberger (PE) urn scheme; c.f. [13], [27], [17]. Let \(\nu^n \triangleq \frac{\pi^n}{n!}\) be the relative number of times a ball of color \(i\) is drawn in \(n\) drawings. The vector \(\nu^n \triangleq [\nu^n_1, \nu^n_2, \ldots, \nu^n_m]\) will be called type [10], or \(n\)-type where necessary to stress that it is induced by \(n\) drawings. Given the PE scheme, the probability \(p(\nu^n; q, c)\) that \(n\)-type \(\nu^n\) will be drawn is (c.f. [27], [17]):

\[
\pi(\nu^n; q, c) \triangleq \frac{n!}{\prod_{i=1}^{m} a_i n_i!} \frac{\prod_{i=1}^{\sum} a_i (q_i + c) \cdots (q_i + (n_i - 1)c)}{N(N + c) \cdots (N + (n - 1)c)},
\]

(1)

where \(N \triangleq \sum_{i=1}^{m} a_i\), and vector \(q\) consists of \(q_i \triangleq \frac{\nu^n_i}{n}\), \(i = 1, 2, \ldots, m\). The Pólya Eggenberger (PE) distribution (1) contains, as prominent special cases, the multinomial distribution for \(c = 0\) (i.e., random sampling; identically and independently distributed (iid) outcomes), the multivariate hypergeometric distribution for \(c = -1\) (i.e., sampling without replacement) and the multivariate negative hypergeometric distribution for \(c = 1\); c.f. [17].

Identify the set of possible colors (outcomes), states with support \(\mathcal{X}\) of a random variable \(X\). Following the notation of [10], let \(\mathcal{P}(\mathcal{X})\) be the set of all probability mass functions on \(\mathcal{X}\). Let \(\mathcal{P}_n(\mathcal{X})\) be the set of all possible \(n\)-types. Finally, let \(\Pi \subseteq \mathcal{P}(\mathcal{X})\) be the feasible set of distributions and \(\Pi_n \triangleq \Pi \cap \mathcal{P}_n(\mathcal{X})\). The aim of this work is to examine the Sanov Theorem for Pólya sampling (i.e., the large deviations behavior of \(\pi(\nu^n \in \Pi; q, c)\)), its associated Conditional Limit Theorem (CoLT) and Gibbs Conditioning Principle (GCP), and connections to the Maximum Probability (MaxProb) principle [3], [30], [14], [22], [23], [24]. The asymptotic investigations are conducted under the assumption that \(N, \beta \triangleq \frac{N}{n}\) and \(q\) may change with \(n\) in such a way that \(q(n) \to q \in \mathcal{P}(\mathcal{X})\) and \(\beta(n) \to \beta \in (0, 1)\) as \(n \to \infty\).

II. PÓLYA DIVERGENCE

Let \(\beta \in (0, 1), c \neq 0, p, q \in \mathcal{P}(\mathcal{X})\), and \(q + \beta c p \geq 0\). The Pólya divergence \(I(p \| q; \beta, c)\) of \(p\) with respect to \(q\) is given by:

\[
I(p \| q; \beta, c) \triangleq I(p \| q + \beta c p) + \frac{1}{\beta c} I(q \| q + \beta c p) + \frac{1 + \beta c}{\beta c} \log(1 + \beta c),
\]

where \(I(a \| b) \triangleq \sum_{i=1}^{m} a_i \log \frac{a_i}{b_i}\) is the Kullback Leibler (KL) divergence [21], with standard conventions. By the continuity argument, \(I(p \| q; \beta, 0) \triangleq I(p \| q)\). When convenient, \(I(p \| q; \beta, c)\) will be replaced by \(I(p \| q)\).

The following key properties of the Pólya divergence are needed for later analyses.

1) Non-negativity. \(I(p \| q) \geq 0\), with equality if and only if (iff) \(p = q\).

2) Lower semicontinuity. \(I(p \| q)\) is lower semicontinuous in \(p, q\). If \(q_i > 0\), for \(i = 1, 2, \ldots, m\), then Pólya divergence is continuous in the pair \(p, q\).

3) Convexity in \(p, q\). For any \(\lambda \in [0, 1]\) and \(p, p', q, q'\), it holds that \(\lambda I(p \| q) + (1 - \lambda)I(p' \| q') \geq I(p + (1 - \lambda)p' \| q + (1 - \lambda)q')\).

4) Partition inequality. If \(A \triangleq \{A_1, A_2, \ldots, A_k\}\) is a partition of \(\mathcal{X}\) and \(p_A(j) \triangleq \sum_{i \in A_j} p_i, q_A(j) \triangleq \sum_{i \in A_j} q_i, j = 1, 2, \ldots, k\), then \(I(p \| q) \geq I(p_A \| q_A)\), with equality if \(p(i \in A_j) = q(i \in A_j), i \in A_j\), for each \(j\).

5) Pinsker inequality. If \(c \geq 0\), the total variation distance \(d(p, q) \triangleq \sum_{i=1}^{m} |p_i - q_i|\) is bounded as follows: \(I(p \| q) \geq \frac{1}{2(1 + \beta c^2)} d(p, q)\).

Proof: The properties can be established along standard lines (c.f. [11], [6], [19]). In particular, Properties 1, 3 and 4 follow from the log-sum inequality. We outline the proof of the Pinsker inequality for the Pólya divergence. Since
the partition inequality holds for Pólya divergence, from the standard argument (c.f. [6]) it is sufficient to consider the binary \( X \) with \( p \triangleq [p, 1-p] \) and \( q \triangleq [q, 1-q] \), such that \( p \geq q \), and find out under what restriction on \( \gamma \) the difference \( g(q) \triangleq I_\beta(p || q) - \gamma d^2(p, q) \) remains negative. Note that the difference is 0 for \( p = q \), by Property 1. The first derivative \( g'(q) = q'(q - p)(q - \frac{1}{\beta c q - p p} \log \frac{1+b \beta c q - p}{1-q} - \gamma) \). By assumption \( q \leq p \), \( q + \beta c p < p (1 + \beta c) \). If \( c \geq 0 \), \( y \triangleq \frac{1}{\beta c q - p p} \frac{1+b \beta c q - p}{1-q} < (1 + \beta c)(1-q) \) and \( y > 1 \), hence, in order to assure negativity of the derivative, \( 8 \gamma \leq \frac{4}{(1+\beta c)^2} \). Setting up \( \gamma = \frac{1}{2(1+\beta c)^2} \) establishes the bound. 

### III. Sanov Theorem for Pólya Sampling

Topological qualifiers are meant in topology induced on the \( m \)-dimensional simplex by the usual topology on \( \mathbb{R}^m \). Following [10], for a set \( \mathcal{P} \) of \( \mathcal{P}(X) \) and \( q \in \mathcal{P}(X) \), inf \( P \in \mathcal{P} \) \( I_\beta(p || q) \) is denoted by \( I_\beta(\mathcal{P} || q) \).

**Theorem 1 (Pólya Sanov Thm.):** Let \( \Pi \) be an open set. Let \( q(n) \rightarrow q, \beta(n) \rightarrow \beta \in (0, 1) \), as \( n \rightarrow \infty \). Then, for \( n \rightarrow \infty \),

\[
\frac{1}{n} \log \pi(\nu^n \in \Pi; q(n), c) = -I_\beta(\Pi || q), \tag{1}
\]

**Proof:** The Method of Types [10] approach to Large Deviations will be used.

For \( c = 0 \), the Sanov Theorem is already established, c.f. [26], [10], [7]. The rate function is just the KL divergence, i.e., \( I_\beta(\cdot || \cdot) \).

The case of \( c \neq 0 \) will be divided into two subcases: \( c > 0 \) and \( c < 0 \).

The following inequalities are needed:

i) \( n \log n - n \leq \log n! \leq (n+1) \log n - n \), valid for \( n > 6 \),

ii) \( (b + \frac{1}{2}) \log b - (a + \frac{1}{2}) \log a < \log G(b) - \log G(a) < (b - \frac{1}{2}) \log b - (a - \frac{1}{2}) \log a < (b - a) \), \( 0 < a < b \); due to [18].

For \( c \neq 0 \) and \( N(c,n) \notin (\mathbb{Z}^-)^m \), formula (1) can equivalently be expressed as [17]:

\[
\pi(\nu^n; q, c) = \frac{n!}{\prod_{i=1}^{m} (\frac{N(c,n) + n_i}{c + n})} \prod_{i=1}^{m} \frac{\Gamma(\frac{N(c,n) + n_i}{c + n})}{\Gamma(\frac{N(c,n)}{c + n})}, \tag{2}
\]

where \( \Gamma(\cdot) \) is the Gamma function.

Let \( c > 0 \). Note then that the other restrictions under which (1) and (2) are equivalent are not active, since \( -nc \leq \min(\alpha_0, \alpha_2, \ldots, \alpha_m) \). Applying the inequalities i), ii) to (2), \( \frac{1}{n} \log \pi(\nu^n, q(n), c) \) is, for \( n > 6 \), bounded from above by \( U_n \) and from below by \( L_n \):

\[
U_n = -I(\nu^n || q(n); \beta(n), c) + \frac{m+1}{n} \log n + \frac{1}{2n} (\log(1 + \beta(n)c) - \frac{1}{2n} \left( \sum_{i=1}^{m} \log \frac{q_i(n) + \beta(n)c}{q_i(n)} \right)),
\]

\[
L_n = -I(\nu^n || q(n); \beta(n), c) + \frac{1}{n} \left( \log(1 + \beta(n)c) - \frac{1}{2n} \left( \sum_{i=1}^{m} \log \frac{q_i(n) + \beta(n)c}{q_i(n)} \right) \right). \tag{3}
\]

To establish \( L_n \) the standard “trick” of binding \( \sum_{i=1}^{m} \log \nu^n_i \) from above by \( -\sum_{i=1}^{m} \log m \) was used, in addition to i) and ii). The non-divergence terms will be denoted \( u_n, l_n \), respectively.

Let the cardinality of a set \( \mathcal{A} \) be denoted \( |\mathcal{A}| \). \( |\mathcal{P}(X)| \leq (n + 1)^m \); c.f. [10]. Thus,

\[
-I(\Pi \ || q(n); \beta(n), c) + l_n \leq \frac{1}{n} \log \pi(\nu^n \in \Pi; q(n), c) \leq \frac{m \log(n + 1)}{n} - I(\Pi \ || q(n); \beta(n), c) + u_n.
\]

Since \( m \) is finite, all terms other than \( I(\cdot) \) converge to zero as \( n \rightarrow \infty \). By assumption, \( q(n) \rightarrow q, \beta(n) \rightarrow \beta \in (0, 1) \). Also, by assumption, \( q_i(n) > 0 \), for all \( i \), thus \( I(\cdot || \cdot; \beta(n), c) \) is continuous. \( L \) is assumed to be open. Thus \( I(\Pi \ || q(n); \beta(n), c) \rightarrow I(\Pi \ || q; \beta, c) \) as \( n \rightarrow \infty \).

For \( c \neq 0 \) and \( N(c,n) \notin (\mathbb{Z}^-)^m \), formula (1) can equivalently be expressed as:

\[
\pi(\nu^n; q, c) = \frac{n!}{\prod_{i=1}^{m} (\Gamma(\frac{N(c,n) + n_i}{c + n}) \prod_{i=1}^{m} \Gamma(\frac{N(c,n) + n_i}{c + n})},
\]

\[
\Gamma(\frac{N(c,n)}{c + n}) - \Gamma(\frac{N(c,n)}{c + n} - \frac{n}{c + n}) \tag{3}
\]

Let \( c < 0 \). Note then that the other restrictions under which (1) and (3) are equivalent are not active, since \( -nc \leq \min(\alpha_0, \alpha_2, \ldots, \alpha_m) \). Applying the inequalities i), ii) to (3), the probability \( \frac{1}{n} \log \pi(\nu^n; q(n)) \) can be bounded by \( U_n \) from above and by \( L_n \) from below, as \( L_n = -A(\nu^n || q(n); \beta(n), c) + l_n \), \( U_n = -A(\nu^n || q(n); \beta(n), c) + u_n \), where

\[
A(\nu^n || q(n); \beta(n), c) \triangleq \sum_{i=1}^{m} \frac{\log \nu_i^n + 1 + \beta(n)c}{\beta(n)c}.
\]

\[
\cdot \log \left( -\frac{1 + \beta(n)c}{\beta(n)c} + \frac{1}{n} \right) = \log \left( \frac{1 + \beta(n)c}{\beta(n)c} + \frac{1}{n} \right) + \frac{m}{n} \log \left( -\frac{q_i(n) + \beta(n)c}{\beta(n)c} + 1 \right) - \frac{m}{n} \log \left( -\frac{q_i(n) + \beta(n)c}{\beta(n)c} + 1 \right)
\]

\[+ \frac{m}{n} \log \left( -\frac{q_i(n) + \beta(n)c}{\beta(n)c} + 1 \right).
\]

and \( l_n, u_n \) stand for terms that converge to 0 as \( n \rightarrow \infty \).

Using the same argument as for \( c > 0 \), \( \frac{1}{n} \log \pi(\nu^n \in \Pi; q(n), c) \) is bounded

\[
-A(\Pi \ || q(n); \beta(n), c) + l_n \leq \frac{1}{n} \log \pi(\nu^n \in \Pi; q(n), c) \leq \frac{m \log(n + 1)}{n} - A(\Pi \ || q(n); \beta(n), c) + u_n.
\]

Since \( m \) is finite, the terms other than \( A(\cdot) \) converge to zero, for \( n \rightarrow \infty \). Since \( A(\cdot || \cdot; \beta(n), c) \) is continuous, the argument used above (case of \( c > 0 \)) implies that \( A(\Pi \ || q(n); \beta(n), c) \rightarrow I(\Pi \ || q; \beta, c) \), as \( n \rightarrow \infty \).

### IV. Pólya Conditional Limit Theorem

The Pólya information projection \( \hat{p}(\beta, c) \) (Pólya I-projection, or \( I_\beta \)-projection, for short) of \( q \) on \( \Pi \) is defined
as \( \hat{p}(β, c) \equiv \arg\inf_{p ∈ \Pi} I_β^c(p || q) \). The standard I-projection [11] is the special \((c = 0)\)-case of the Pólya I-projection.

The Pólya Conditional Limit Theorem (CoLT) is an important consequence of the Pólya Sanov Theorem.

**Theorem 2 (Pólya CoLT):** Let \( q(n) → q, β(n) → β \in (0, 1) \), as \( n → ∞ \). Let \( Π \) be a convex, closed set. Let \( \hat{p}(β, c) \) be the \( I_β^c \)-projection of \( q \) on \( Π \). Let \( B(q, c) \) be the \( ε \)-ball defined by the total variation metric, centered at \( q \). Then for any \( ε > 0 \),

\[
\lim_{n→∞} \pi(ν^n \in B(\hat{p}(β, c), ε) | ν^n \in Π; q(n), c) = 1.
\]

**Proof:** Let \( B^C(\hat{p}(β, c), ε) \equiv \mathcal{P}(X) \setminus B(\hat{p}(β, c), ε) \). Apply Pólya Sanov Theorem to \( \pi(ν^n \in B^C(·) | ν^n \in Π; q(n), c) = \pi(ν^n \in B^C(·)) \). The decay rate \( I_β^c(B^C(·) || q) - I_β^c(Π || q) > 0 \).

Since \( Π \) is, by assumption, convex and closed, by convexity of Pólya information projection (Property 3) there is unique \( I_β^c \)-projection of \( q \) on \( Π \). Types thus asymptotically concentrate on it.

Pólya CoLT has the same interpretation as the standard, iid-case, CoLT (see [30], [29], [28], [4], [9]): types induced by PE sampling, asymptotically conditionally (on the event \( ν^n ∈ Π \)) concentrate on the Pólya information projection \( \hat{p}(β, c) \) of \( q \) on \( Π \).

Setting \( Π = \mathcal{P}(X) \) reduces Pólya CoLT into its special case: the Law of Large Numbers for PE sampling.

**V. Further Results**

By means of the Pólya Sanov Theorem and the bounds used for its proof, three additional results can be obtained.

**A. Pólya Gibbs Conditioning Principle**

For the iid sampling there is a claim, stronger than the Conditional Limit Theorem, known as Gibbs Conditioning Principle (GCP); c.f. [7], [10], [12]. Alongside of its proof [10], the following Gibbs Conditioning Principle for PE sampling can be established.

**Theorem 3 (Pólya GCP):** Let \( q(n) → q, β(n) → β \in (0, 1) \), as \( n → ∞ \). Let \( Π \) be a convex, closed set. Let \( \hat{p}(β, c) \) be the \( I_β^c \)-projection of \( q \) on \( Π \). Then for a fixed \( t \),

\[
\lim_{n→∞} \pi(X_1 = x_1, \ldots, X_t = x_t | ν^n ∈ Π; q(n), c) = \prod_{i=1}^t \hat{p}(β, c).
\]

Loosely put, asymptotically, conditionally upon the event \( ν^n ∈ Π \), a fixed-length sequence of drawn colors behaves as if it was identically and independently drawn from the Pólya information projection \( \hat{p}(β, c) \) of \( q \) on \( Π \).

Its \( t = 1 \) special case can be established for \( c = 0 \) by means of the Pythagoras property of the \( I \)-projection and the Pinsker inequality; see [6]. This approach does not carry on to \( c \neq 0 \), as the Pythagoras property does not hold for \( I_β^c \)-projection with \( c \neq 0 \).

**B. Maximum Probability - Maximum Entropy Correspondence**

Let \( ν^n(β, c) \equiv \arg\sup_{p ∈ Π_n} \pi(ν^n; q, c) \) be the Pólya \( μ \)-projection (\( μ_β^c \)-projection, for short) of \( q \) on \( Π_n \); i.e., the supremum-probable \( n \)-type in \( Π_n \). Using the \( U_n, L_n \) bounds (c.f. proof of Pólya Sanov Theorem), the asymptotic identity of Pólya \( μ \)-projections and Pólya I-projections, can be established along the lines of [15].

**Theorem 4 (MaxProb/MaxEnt):** Let \( q(n) → q, β(n) → β \in (0, 1) \), as \( n → ∞ \). Let \( M_n(β(n), c) \) be a set of all \( μ_β^c(n) \)-projections of \( q(n) \) on \( Π_n \). Let \( I_β^c \) be a set of all \( I_β^c \)-projections of \( q \) on \( Π \). Then, for \( n → ∞ \), \( M_n(β(n), c) = I(β, c) \).

This permits a deeper interpretation of the Pólya CoLT. Informally: types, asymptotically conditionally (upon \( ν^n ∈ Π \)) concentrate on the most probable type. Even more loosely put, the most probable is asymptotically conditionally the only possible.

The asymptotic identity of Pólya \( μ \)-projections and Pólya I-projections is illustrated by the following Example.

**Example 1:** Let \( X = \{1, 2, 3, 4\} \), i.e., there are four colors, associated with the numbers. Let \( Π = \{ p : \sum_{i=1}^4 p_i x_i = 3.2, \sum_{i=1}^m p_i = 1 \} \). Let \( n = 10, 50, 100, 1000 \) and \( N(n) = 100, 500, 1000, 10000 \), so that \( β_3 = 0.1 \), for all considered \( n \).

Let \( q(n) = q = \{21, 25, 31, 23\}/100 \) for all \( n \). For each \( n \), let \( c \in \{-2, 1, 0, 1, 5, 10\} \). The Table in Appendix A contains the Pólya \( μ \)-projection \( ν^n \) of \( q(n) \) on \( Π_n \). In the last block of the Table, the Pólya I-projection \( \hat{p} \) of \( q \) on \( Π \) is presented, for each considered \( c \).

**C. Pólya Conditional Equi-concentration of Types**

The Conditional Equi-concentration of Types (CET) on I-projections, an extension of CoLT to the case of \( Π \) admitting more than one I-projection, is discussed at [15]. Similarly, CET holds also for Pólya I-projections.

**VI. Applications and Implications**

Pólya CoLT has similar applications (and implications) as the standard iid CoLT (see [9], [15]), but holds in a broader context of PE sampling, which encapsulates the iid one. We will briefly discuss two applications.

**A. Pólya MaxEnt**

The Boltzmann Pólya Inverse Problem (BPIP) contains the Boltzmann Jaynes Inverse Problem [15] as its special, iid-case. BPIP is constituted by the information-pentad \( \{X, Nq, n, c, Π\} \) under which the objective is to select a type (one or more) from \( Π \). Three examples of BPIP are below.

**Example 2:** A network containing \( i = 1, 2, \ldots, m \), critical nodes, each with \( α_i \) branches, is accessed by users. Each time a connection to node \( i \) is accessed, \( c \) new connections to this node are established (\( c = -1 \) indicates simple congestion, and \( c < -1 \), accelerated congestion). Assuming that the number of transactions \( n \) is known, the objective is to select a type of connections to the nodes from \( Π = \mathcal{P}_n(X) \).

**Example 3:** A stock exchange is established with \( N \) equiprobable shares, with \( α_i \) of share \( i \). After a trade in stock \( i \), \( c \) new shares are issued in it (\( c < 0 \) indicates withdrawal of
The transactions are constrained by the mean value of trades in a given period. Given the feasible set \( \Pi_n \) of \( n \)-transactions determined by the mean value of trades, and the other above described information, the objective is to select an \( n \)-type of transactions.

**Example 4:** Let \( n \) out of \( N \) quantum mechanics particles be distributed among \( m \) energy levels according to the PE sampling scheme with initial distribution \( q \) and parameter \( c \). Let instead of the actual energy distribution (\( n \)-type) only the mean value of energy of \( n \) particles be available. Given this information, the objective is to select an \( n \)-type from the feasible set.

BPPIP is under-determined and in this sense is an ill-posed inverse problem. The indeterminacy of the problem translates into a multitude of possible methods for its solution. From an infinite set of possible methods of solving BPPIP, such a method has to be selected that, for \( n \to \infty \), does not violate Pólya CoLT. Clearly, selection of the Pólya \( I \)-projection of \( q \) on \( \Pi_n \) satisfies the above requirement of asymptotic consistency. This selection scheme could reasonably be called the Pólya Maximum Relative Entropy (MaxEnt) method, where the Pólya relative entropy is defined as the negative of the Pólya divergence; 

\[
H(p \parallel q; \beta, c) \triangleq -I(p \parallel q; \beta, c).
\]

Note that from the point of view of maximization over \( p \), the Pólya relative entropy effectively reduces to:

\[
- \sum p_i \log p_i + \sum \left( p_i + \frac{q_i}{\beta c} \right) \log (q_i + \beta c p_i).
\]

The other way for solving/regularizing BPPIP that is asymptotically consistent is Maximum Probability (MaxProb), that selects the \( \mu^*_q \)-projection of \( q \) on \( \Pi_n \). By Pólya MaxProb/MaxEnt the two methods asymptotically coincide, but for finite \( n \) they make, in general, a different choice.

The feasible set \( \Pi \) can for instance (as in the above Examples 3 and 4) be formed by moment-consistency constraints 

\[
\Pi = \left\{ p : \sum_{j=1}^{m} p_i u_j(x_i) = a_j, j = 0, 1, 2, \ldots, J \right\},
\]

where \( u_j(\cdot) \) is a given real-valued function, \( u_0(\cdot) \triangleq 1 \); \( a_j \) is given number, \( a_0 \triangleq 1 \); such a feasible set is also known as the linear family of distributions. The Pólya \( I \)-projection of \( q \) on the linear family of distributions \( \Pi \) is then implicitly given by:

\[
\hat{p}_i(\beta, c) = \frac{q_i e^{-\sum_{j=0}^{J} \lambda_j u_j(x_i)}}{1 - \beta ce^{-\sum_{j=0}^{J} \lambda_j u_j(x_i)}}.
\]  

**1) Distribution of anyons:** Not surprisingly, for \( c = 0 \), the probability distribution (4) turns into the familiar exponential (Maxwell-Boltzmann) form of the \( I \)-projection on the linear family; [11]. For \( c = -1 \), the distribution gives the Fermi Dirac distribution whilst for \( c = +1 \), the distribution gives the Bose Einstein distribution. These are generalizations of the standard Bose Einstein and Fermi Dirac distributions, in the sense that a general (not necessarily uniform) sampling distribution \( q \) is assumed [25]. In this respect it is worth recalling the Example 4 and noting that the PE distribution (4) contains an ansatz distribution of quantum-mechanical anyons (i.e., particles with properties intermediate between those of bosons and fermions; [32]) proposed at [1] as its special (uniform \( q \)) case. This in our view, provides both a probabilistic underpinning of the Acharya & Narayana Swamy [1] distribution of anyons as well as its extension to the non-uniform sampling case. Further discussion will be given elsewhere [25].

**2) Limitation of axiomatic approach to linear inverse problems:** We would like to stress that for \( c \neq 0 \) the standard MaxEnt method [16], [20] (i.e., selection of \( I \)-projection of \( q \) on \( \Pi_n \)), when applied to BPPIP, does not satisfy the requirement of asymptotic consistency. Thus, although the standard MaxEnt is advocated by an axiomatic approach as the logically consistent way of solving ill-posed inverse problems with \( \Pi \) defined by the moment-consistency constraints (c.f. [8], [9]), the method, when applied under PE sampling with \( c \neq 0 \), violates the Pólya Conditional Limit Theorem. This reveals a limitation of the axiomatic approach to inference in the inverse problems context.

**B. Rare events simulation**

Pólya CoLT and Pólya GCP can be used for rare events simulation in the context of PE sampling, in the same way that the standard CoLT and GCP are used in the iid sampling; c.f. [5].

**VII. SUMMARY**

The standard Conditional Limit Theorem (CoLT) [6] for iid sampling provides a probabilistic justification (c.f. [30], [9]) of MaxEnt method in the context of so-called Boltzmann Jaynes Inverse Problem (BJIP), [15]. In [14] it was suggested that MaxEnt can be viewed as an asymptotic instance of the MaxProb method, which under the limited information available to the BJIP, selects the type (i.e., the empirical distribution) with the highest probability of occurrence, from the given data-sampling distribution. It was proposed in [30], [23], [24], that MaxProb can be considered in a broader context; in particular under sampling schemes other than the random (i.e., iid) sampling. There it was also pointed out that every sampling scheme might be associated with its own instance of MaxProb and its own relative entropy maximization method. For a particular sampling scheme (or, probabilistic question of certain form, in general) and adjoint inverse problem, the relevant entropy maximization can be discovered by considering the associated CoLT. The relevant CoLT, in turn, provides probabilistic justification of the associated relative entropy maximization method in the context of the inverse problem. Motivated by these observations, in this work we have established CoLT for the PE sampling scheme and discussed some of its consequences and applications.

**VIII. NOTES ON LITERATURE**

An early physics-motivated work that extends Boltzmann’s Maximum Probability principle, steps into the direction of Sanov Theorem for non-iid sampling and contains a few views ahead of its time is Vincze’s [30]; see also [31]. For sampling without replacement (i.e., \( c = -1 \))-case of PE sampling), the Sanov Theorem was established by [12]. A communications channel with Pólya noise has been considered at [2].
TABLE I
MAXPROB TO PÓLYA MAXENT CONVERGENCE

| n=10  | c=0  | 0.2   | 0.4   | 0.4   |
|------|------|-------|-------|-------|
| c=1  | 0.2  | 0.4   | 0.4   | 0.4   |
| c=2  | 0    | 0.2   | 0.4   | 0.4   |
|------|------|-------|-------|-------|
| c=5  | 0.1  | 0.1   | 0.3   | 0.5   |
| c=10 | 0.2  | 0.4   | 0.4   | 0.4   |
| n=50 | c=0  | 0.06  | 0.14  | 0.34  | 0.46  |
| c=1  | 0.06 | 0.14  | 0.34  | 0.46  |
| c=2  | 0.06 | 0.14  | 0.34  | 0.46  |
| c=5  | 0.06 | 0.14  | 0.34  | 0.46  |
| c=10 | 0.06 | 0.14  | 0.34  | 0.46  |
| n=100| c=0  | 0.05  | 0.15  | 0.35  | 0.45  |
| c=1  | 0.05 | 0.15  | 0.35  | 0.45  |
| c=2  | 0.06 | 0.14  | 0.34  | 0.46  |
| c=5  | 0.07 | 0.14  | 0.31  | 0.48  |
| c=10 | 0.07 | 0.14  | 0.31  | 0.48  |
| n=1000| c=0 | 0.05628 | 0.14179 | 0.34759 | 0.45434 |
| c=1 | 0.05624 | 0.14085 | 0.33108 | 0.46566 |
| c=2 | 0.06214 | 0.14085 | 0.33108 | 0.46566 |
| c=5 | 0.07014 | 0.14030 | 0.31010 | 0.48628 |
| c=10 | 0.07357 | 0.13913 | 0.30102 | 0.48628 |