On $p$-adic quaternionic Eisenstein series

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Abstract

We show that certain $p$-adic Eisenstein series for quaternionic modular groups of degree 2 become “real” modular forms of level $p$ in almost all cases. To prove this, we introduce a $U(p)$ type operator. We also show that there exists a $p$-adic Eisenstein series of the above type that has transcendental coefficients. Former examples of $p$-adic Eisenstein series for Siegel and Hermitian modular groups are both rational (i.e., algebraic).

1 Introduction

Serre [12] first developed the theory of $p$-adic Eisenstein series and there have subsequently been many results in the field of $p$-adic modular forms. Several researchers have attempted to generalize the theory to modular forms with several variables. For example, we showed that a $p$-adic limit of a Siegel Eisenstein series becomes a “real” Siegel modular form (cf. [4]). The same result has also been proved for Hermitian modular forms (e.g., [11]).

In the present paper, we study $p$-adic limits of quaternionic Eisenstein series. This study has two principal aims. The first is to show that these $p$-adic limits become “real” modular forms of level $p$ for higher $p$-adical weights (Theorem 3.1). To prove this, we introduce a $U(p)$ type Hecke operator and study its properties; this is a similar method to that used by Böcherer for Siegel modular forms [2]. The second aim is to show that a strange phenomenon occurs for low $p$-adical weights; namely, there exists a transcendental $p$-adic Eisenstein series in the quaternionic case (Theorem 3.5).

2 Preliminaries

2.1 Notation and definitions

Let $\mathbb{H}$ be Hamiltonian quaternions and $\mathcal{O}$ the Hurwitz order (cf. [6]). The half-space of quaternions of degree $n$ is defined as

$$H(n; \mathbb{H}) := \{ Z = X + iY \mid X, Y \in Her_n(\mathbb{H}), Y > 0 \}.$$ 

Let $J_n := \begin{pmatrix} O_n & 1_n \\ -1_n & O_n \end{pmatrix}$. Then, the group of symplectic similitudes

$$\{ M \in M(2n, \mathbb{H}) \mid ^tMJ_nM = qJ_n \text{ for some positive } q \in \mathbb{R} \}$$

acts on $H(n; \mathbb{H})$ by

$$Z \mapsto M(Z) = (AZ + B)(CZ + D)^{-1}, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$
Let $\Gamma_n$ denote the modular group of quaternions of degree $n$ defined by
\[
G_n := \{ M \in M(2n, \mathbb{H}) \mid {}^t\overline{M}J_nM = J_n \},
\]
\[
\Gamma_n := \Gamma_n(\mathcal{O}) = M(2n, \mathcal{O}) \cap G_n.
\]

For a given $q \in \mathbb{N}$, the congruence subgroup $\Gamma_0^{(n)}(q)$ of $\Gamma_n$ is defined by
\[
\Gamma_0^{(n)}(q) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \mid C \equiv O_n \mod qM(n, \mathcal{O}) \right\}.
\]

In this subsection, $\Gamma$ always denotes either $\Gamma_n$ or $\Gamma_0^{(n)}(q)$.

Let $1 = e_1, e_2, e_3, e_4$ denote the canonical basis of $\mathbb{H}$, which is characterized by the identities
\[
e_4 = e_2e_3 = -e_3e_2, \quad e_2^2 = e_3^2 = -1.
\]

We consider the canonical isomorphism
\[
M(n, \mathbb{H}) \longrightarrow M(2n, \mathbb{C})
\]
given by $A = (\tilde{a}_{ij})$, where $\tilde{a} = (a_1 + a_2i, a_3 + a_4i, -a_3 + a_4i, a_1 - a_2i)$, if $a = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$ (cf. [6]).

We use the above isomorphism to define $\det(A)$ for $A \in M(n, \mathbb{H})$. For a similitude $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and a function $f : H(n; \mathbb{H}) \longrightarrow \mathbb{C}$, we define the slash operator $|_k$ by
\[
(f|_k M)(Z) = \det(M)^{\frac{k}{2}} \det(CZ + D)^{-k} f((AZ + B)(CZ + D)^{-1}).
\]

A holomorphic function $f : H(n; \mathbb{H}) \longrightarrow \mathbb{C}$ is called a quaternionic modular form of degree $n$ and weight $k$ for $\Gamma$ if $f$ satisfies
\[
(f|_k M)(Z) = f(Z),
\]
for all $M \in \Gamma$. (The cusp condition is required if $n = 1$.)

We denote by $M_k(\Gamma)$ the $\mathbb{C}$-vector space of all quaternionic modular forms of degree $n$ and weight $k$ for $\Gamma$. A modular form $f \in M_k(\Gamma)$ possesses a Fourier expansion of the form
\[
f(Z) = \sum_{0 \leq H \in Her_n(\mathcal{O})} a_f(H)e^{2\pi i \tau(H, Z)}, \quad Z \in H(n; \mathbb{H}),
\]
where $Her_n^*(\mathcal{O})$ denotes the dual lattice of $Her_n(\mathcal{O}) := \{ S \in M(n, \mathcal{O}) \mid \tau^*S = S \}$ with respect to the reduced trace form $\tau$ (cf. [6]). For simplicity, we put $q^H := e^{2\pi i \tau(H, Z)}$ for $H \in Her_n^*(\mathcal{O})$. Using this notation, we write the above Fourier expansion simply as $f = \sum_H a_f(H)q^H$.

For an even integer $k$, we consider the Eisenstein series
\[
E_k^{(n)}(Z) := \sum_{(AB) \in \Gamma_0 \setminus \Gamma_n} \det(CZ + D)^{-k}, \quad Z \in H(n; \mathbb{H}),
\]
where $\Gamma_0 := \left\{ \begin{pmatrix} A & B \\ O_n & D \end{pmatrix} \in \Gamma_n \right\}$. It is well known that this series belongs to $M_k(\Gamma_n)$ if $k > 4n - 2$. We call this series the quaternionic Eisenstein series of degree $n$ and weight $k$.

### 2.2 Fourier coefficients of Eisenstein series

In this section, we introduce an explicit formula for the Fourier coefficients of the degree 2 quaternionic Eisenstein series obtained by Krieg (cf. [7]).
Let $k > 6$ be an even integer and let

$$E_k^{(2)}(Z) = \sum_{0 \leq H \in \text{Her}_2^*(\mathcal{O})} a_k(H) e^{2\pi i\tau(H,Z)}$$

be the Fourier expansion of the degree 2 quaternionic Eisenstein series $E_k^{(2)}$. According to [7], we introduce an explicit formula for $a_k(H)$. Given $O_2 \neq H \in \text{Her}_2^*(\mathcal{O})$, the “greatest common divisor” of $H$ is given by

$$\varepsilon(H) := \max\{d \in \mathbb{N} \mid d^{-1}H \in \text{Her}_2^*(\mathcal{O})\}.$$

**Theorem 2.1** (Krieg [7]). Let $k > 6$ be even and $H \neq O_2$. Then, the Fourier coefficient $a_k(H)$ is given by:

$$a_k(H) = \sum_{0 < d \mid \varepsilon(H)} d^{k-1} \alpha^*(2\det(H)/d^2)$$

and

$$\alpha^*(\ell) = \begin{cases} 
-\frac{2k}{B_k} & \text{if } \ell = 0, \\
-\frac{4k(k-2)}{(2^{k-2}-1)B_k B_{k-2}}[\sigma_{k-3}(\ell) - 2^{k-2}k_{k-3}(\ell/4)] & \text{if } \ell \in \mathbb{N},
\end{cases}$$

where $B_m$ is the $m$-th Bernoulli number and

$$\sigma_k(m) := \begin{cases} 
0 & \text{if } m \notin \mathbb{N}, \\
\sum_{0 < d \mid m} d^k & \text{if } m \in \mathbb{N}.
\end{cases}$$

### 2.3 $U(p)$-operator

In the remainder of this paper, we assume that $p$ is an odd prime. For a formal power series of the form $F = \sum_H a_F(H)q^H$, we define a $U(p)$ type operator as

$$U(p) : F = \sum_H a_F(H)q^H \mapsto F|U(p) := \sum_H a_F(pH)q^H.$$

In particular, for a modular form $F \in M_k(\Gamma_0^{(n)}(p))$, we may regard $U(p)$ as a Hecke operator (cf. [2], [7]). We prove this in this section. More precisely, we prove that

**Proposition 2.2.** If $F \in M_k(\Gamma_0^{(n)}(p))$ then $F|U(p) \in M_k(\Gamma_0^{(n)}(p))$.

To prove this proposition, we introduce the following lemma.

**Lemma 2.3.** A complete set of representatives for the left cosets of

$$\Gamma_0^{(n)}(p) \left( \begin{array}{cc} O_n & -1_n \\ 1_n & O_n \end{array} \right) \Gamma_0^{(n)}(p)$$

is given by

$$\left\{ \left( \begin{array}{cc} O_n & -1_n \\ 1_n & T \end{array} \right) \mid T \in \text{Her}_n(\mathcal{O})/p\text{Her}_n(\mathcal{O}) \right\}.$$

**Proof of Lemma 2.3.** We set $\gamma_T := \left( \begin{array}{cc} O_n & -1_n \\ 1_n & T \end{array} \right)$ and prove

$$\Gamma_0^{(n)}(p) \left( \begin{array}{cc} O_n & -1_n \\ 1_n & O_n \end{array} \right) \Gamma_0^{(n)}(p) = \bigcup_{T \in \text{Her}_n(\mathcal{O})/p\text{Her}_n(\mathcal{O})} \Gamma_0^{(n)}(p) \gamma_T.$$
By decomposition
\[
\begin{pmatrix} O_n & -1_n \\ 1_n & T \end{pmatrix} = \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix} \begin{pmatrix} 1_n & T \\ O_n & 1_n \end{pmatrix},
\]
we easily see the inclusion
\[
\Gamma_0^{(n)}(p) \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix} \Gamma_0^{(n)}(p) \supset \bigcup_{T \in \text{Her}_n(O)/p\text{Her}_n(O)} \Gamma_0^{(n)}(p)\gamma_T.
\]

We shall prove the converse inclusion. Note that \( T \equiv T' \mod p\text{Her}_n(O) \) if and only if \( \Gamma_0^{(n)}(p)\gamma_T = \Gamma_0^{(n)}(p)\gamma_{T'} \). Hence, we have
\[
\bigcup_{T \in \text{Her}_n(O)/p\text{Her}_n(O)} \Gamma_0^{(n)}(p)\gamma_T = \bigcup_{T \in \text{Her}_n(O)} \Gamma_0^{(n)}(p)\gamma_T
\]
as a set. Again, by the decomposition (2.2), it suffices to show that, for any \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(p) \), there exists \( S \in \text{Her}_n(O) \) such that
\[
\begin{pmatrix} O_n & -1_n \\ 1_n & T \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} O_n & -1_n \\ 1_n & S \end{pmatrix}^{-1} \in \Gamma_0^{(n)}(p).
\]
A direct calculation shows that
\[
\begin{pmatrix} O_n & -1_n \\ 1_n & T \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} O_n & -1_n \\ 1_n & S \end{pmatrix}^{-1} = \begin{pmatrix} -CS + D \\ A + TC \end{pmatrix} \begin{pmatrix} -C \\ A + TC \end{pmatrix}.
\]
Hence, the proof is reduced to finding \( S \in \text{Her}_n(O) \) such that \( AS \equiv B + TD \mod pM(n, O) \) Recall that \( A^T \overline{T} - B^T \overline{C} = 1_n \) and hence \( A^T \overline{T} \equiv 1_n \mod pM(n, O) \). If we choose \( S \) as \( S := \overline{T}(B + TD) \), then \( AS \equiv B + TD \mod pM(n, O) \). To complete the proof, we need to show that \( S = \overline{T}(B + TD) \in \text{Her}_n(O) \). This assertion comes from the fact that \( \overline{T}B, \overline{T}TD \in \text{Her}_n(O) \).

We now return to the proof of Proposition 2.2.

Proof of Proposition 2.2 Let \( F \in M_k(\Gamma_0^{(n)}(p)) \). From Lemma 2.3, we have
\[
F|\Gamma_0^{(n)}(p) \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix} \Gamma_0^{(n)}(p) = \sum_T F|_{k} \begin{pmatrix} O_n & -1_n \\ 1_n & T \end{pmatrix} = \sum_T F|_{W_p} \begin{pmatrix} 1_n & T \\ O_n & p1_n \end{pmatrix},
\]
where \( W_p \) is the Fricke involution
\[
F \mapsto F|W_p := F|_{k} \begin{pmatrix} O_n & -1_n \\ p1_n & O_n \end{pmatrix}.
\]
We see by the usual way that \( F|W_p \in M_k(\Gamma_0^{(n)}(p)) \). If we write \( G = F|W_p = \sum_H a_G(H)q^H \), then
\[
\sum_T F|W_p|_{k} \begin{pmatrix} 1_n & T \\ O_n & p1_n \end{pmatrix} = \sum_T G|_{k} \begin{pmatrix} 1_n & T \\ O_n & p1_n \end{pmatrix} = \sum_H \left( \sum_T e^{\frac{i}{p}\tau(H,T)} \right) a_G(H)e^{\frac{i}{p}\tau(H,Z)} = c \cdot G|U(p),
\]
where \( c := \sharp\text{Her}_n(O)/p\text{Her}_n(O) \) and the last equality follows from the following lemma.
Lemma 2.4. For fixed $H \in \text{Her}_{n}^{r}(\mathcal{O})$, we have
\[
\sum_{T} e^{2\pi i \tau(H,T)} = \begin{cases} 
0 & \text{if } H \not\in p\text{Her}_{n}^{r}(\mathcal{O}), \\
c & \text{if } H \in p\text{Her}_{n}^{r}(\mathcal{O}). 
\end{cases} 
\] (2.4)

Proof of Lemma 2.4. For $H \in \text{Her}_{n}^{r}(\mathcal{O})$, we define
\[
G(H) := \sum_{T \in \text{Her}_{n}(\mathcal{O})/p\text{Her}_{n}(\mathcal{O})} e^{2\pi i \tau(H,T)}. 
\]
This definition is independent of the choice of the representation $T$. Replacing $T$ by $T + S$, we obtain
\[
G(H) = G(H) e^{2\pi i \tau(H,S)}. 
\]
Hence, $G(H) = 0$ unless $e^{2\pi i \tau(H,S)} = 1$; i.e., $\tau(H,S) \in p\mathbb{Z}$. This implies $\tau(\frac{1}{p}H,S) \in \mathbb{Z}$ for all $S \in \text{Her}_{n}(\mathcal{O})$. The definition of a dual lattice yields
\[
\frac{1}{p}H \in \text{Her}_{n}^{r}(\mathcal{O}). 
\]
\[\square\]

From this lemma, we have
\[
F|\Gamma_{0}^{(n)}(p) \begin{pmatrix} O_{n} & -1_{n} \\ 1_{n} & O_{n} \end{pmatrix} \Gamma_{0}^{(n)}(p) = c \cdot F|W_{p}|U(p). 
\]
Hence, the action of $U(p)$ is described by the action of the double coset
\[
\Gamma_{0}^{(n)}(p) \begin{pmatrix} 1_{n} & O_{n} \\ O_{n} & p1_{n} \end{pmatrix} \Gamma_{0}^{(n)}(p). 
\]
Therefore, we have $F|U(p) \in M_{k}(\Gamma_{0}^{(n)}(p))$, which completes the proof of Proposition 2.2. \[\square\]

Remark 2.5. The proof of Lemma 2.4 is due to Krieg.

3 Main results

3.1 Modularity of $p$-adic Eisenstein series

In this subsection, we deal with a suitable constant multiple of the normalized quaternionic Eisenstein series
\[
G_{k} = G_{k}^{(2)} := (2^{k-2} - 1) \frac{B_{k}B_{k-2}}{4k(k-1)} E_{k}^{(2)} 
\]
and show that certain $p$-adic limits of this Eisenstein series are “real” modular forms for $\Gamma_{0}^{(2)}(p)$.

We write the Fourier expansion of $G_{k}$ as $G_{k} = \sum_{H} b_{k}(H) q^{H}$. We remark that
\[
b_{k}(O_{2}) = (2^{k-2} - 1) \frac{-B_{k}B_{k-2}}{4k(k-2)}. 
\]

For an odd prime $p$ we put
\[
G_{k}^{*} := \frac{-1}{1 + p^{k-3}} \left\{ p^{2(k-3)}(G_{k}|U(p) - p^{k-1}G_{k}) - (G_{k}|U(p) - p^{k-1}G_{k})|U(p) \right\} 
\]
\[\in M_{k}(\Gamma_{0}^{(n)}(p)), \]
where this modularity follows from Proposition 2.2. The first main theorem is
Theorem 3.1. Let $p$ be an odd prime and $k$ an even integer with $k \geq 4$. Define a sequence $\{k_m\}$ by
\[
k_m := k + (p - 1)p^{m-1}.
\]
Then, the corresponding sequence of Eisenstein series $\{G_{k_m}\}$ has a $p$-adic limit $G^*_k$ and we have
\[
\lim_{m \to \infty} G_{k_m} = G^*_k \in M_k(\Gamma_0^2(p)). \tag{3.1}
\]

Proof. The proof of (3.1) is reduced to show that $G^*_k$ is obtained by removing all $p$-factors of the Fourier coefficients of the quaternionic Eisenstein series.

To calculate the Fourier coefficients of $G^*_k$, we set
\[
F_k = G_k|U(p) - p^{k-1}G_k.
\]

We can then rewrite $G^*_k$ as
\[
G^*_k = -\frac{1}{1 + p^{k-3}}(p^{2(k-3)}F_k - F_k|U(p)).
\]

We write the Fourier expansions as
\[
G^*_k = \sum_H A_k(H)q^H, \quad F_k = \sum_H B_k(H)q^H.
\]

First, we calculate the constant term of $G^*_k$. Since
\[
b_k(O_2) = (2^{k-2} - 1)\frac{B_kB_{k-2}}{4k(k-2)},
\]
the constant term of $G^*_k$ becomes
\[
A_k(O_2) = -\frac{1}{1 + p^{k-3}}\left(p^{2(k-3)}(b_k(O_2) - p^{k-1}b_k(O_2)) - (b_k(O_2) - p^{k-1}b_k(O_2))\right)
\]
\[
= (1 - p^{k-1})(1 - p^{k-3})(2^{k-2} - 1)\frac{B_kB_{k-2}}{4k(k-2)}.
\]

Second, we calculate the coefficient $A_k(H)$ for $H$ with rank($H$) = 1.
\[
B_k(H) = b_k(pH) - p^{k-1}b_k(H)
\]
\[
= (2^{k-2} - 1)\frac{B_{k-2}}{2(k-2)} \left( \sum_{0<d|p} d^{k-1} - p^{k-1} \sum_{0<d|\varepsilon} d^{k-1} \right)
\]
\[
= (2^{k-2} - 1)\frac{B_{k-2}}{2(k-2)} \sigma^*_{k-1}(\varepsilon(H)),
\]
where $\sigma^*_m(N)$ is defined as
\[
\sigma^*_m(N) := \sum_{0<d|N} d^m.
\]

Note that $B_k(pH) = B_k(H)$ when rank($H$) = 1. Hence, we have
\[
A_k(H) = (1 - p^{k-3})(2^{k-2} - 1)\frac{B_{k-2}}{2(k-2)} \sigma^*_{k-1}(\varepsilon(H)).
\]
Finally, we consider the case $\text{rank}(H) = 2$. 

$$B_k(H) = b_k(pH) - p^{k-1}b_k(H)$$

$$= \sum_{0 < d \mid \epsilon(H)} d^{k-1} \left[ \sigma_{k-3} \left( 2p^2 \frac{\det H}{d^2} \right) - 2^{k-2} \sigma_{k-3} \left( 2p^2 \frac{\det H}{4d^2} \right) \right]$$

$$- p^{k-1} \sum_{0 < d \mid \epsilon(H)} d^{k-1} \left[ \sigma_{k-3} \left( 2p^2 \frac{\det H}{d^2} \right) - 2^{k-2} \sigma_{k-3} \left( 2p^2 \frac{\det H}{4d^2} \right) \right]$$

$$= \sum_{0 < d \mid \epsilon(H)} d^{k-1} \left[ \sigma_{k-3} \left( 2p^2 \frac{\det H}{d^2} \right) - 2^{k-2} \sigma_{k-3} \left( 2p^2 \frac{\det H}{4d^2} \right) \right].$$

Here, the last equality was obtained from the elemental property:

**Lemma 3.2.** Let $p$ be a prime and $N$ a positive integer. For a function $f : \mathbb{N} \to \mathbb{N}$, the following holds:

$$\sum_{0 < d \mid N} f(d) = \sum_{0 < d \mid N} f(d) + \sum_{0 < d \mid N} f(pd).$$

Therefore,

$$A_k(H) = \frac{-1}{1 + p^{k-3}} \left( p^{2(k-3)} B_k(pH) - B_k(H) \right)$$

$$= \frac{-1}{1 + p^{k-3}} \left( p^{2(k-3)} \sum_{0 < d \mid \epsilon(H)} d^{k-1} \left[ \sigma_{k-3} \left( 2p^2 \frac{\det H}{d^2} \right) - 2^{k-2} \sigma_{k-3} \left( 2p^2 \frac{\det H}{4d^2} \right) \right] \right)$$

$$- \sum_{0 < d \mid \epsilon(H)} d^{k-1} \left[ \sigma_{k-3} \left( 2p^2 \frac{\det H}{d^2} \right) - 2^{k-2} \sigma_{k-3} \left( 2p^2 \frac{\det H}{4d^2} \right) \right].$$

By repeatedly applying Lemma 3.2 we obtain

$$p^{2m} \sigma_m(N) - \sigma_m(p^2 N) = - (1 + p^m) \sum_{0 < d \mid N} d^m.$$ 

From this, we have

$$A_k(H) = \sum_{0 < d \mid \epsilon(H)} d^{k-1} \left[ \sigma_{k-3}^* \left( 2 \frac{\det H}{d^2} \right) - 2^{k-2} \sigma_{k-3}^* \left( 2 \frac{\det H}{4d^2} \right) \right].$$

Summarizing these calculations, we obtain the following formula:

**Proposition 3.3.** The following holds:

$$A_k(H) = \begin{cases} (1 - p^{k-1})(1 - p^{k-3})(2^{k-2} - 1) \frac{B_k B_{k-2}}{4k(k-2)}, & \text{if } H = O_2, \\ (1 - p^{k-3})(2^{k-2} - 1) \frac{B_{k-2}}{2(k-2)} \sigma_{k-1}(\epsilon(H)), & \text{if } \text{rank}(H) = 1, \\ \sum_{0 < d \mid \epsilon(H)} d^{k-1} \left[ \sigma_{k-3}^* \left( 2 \frac{\det H}{d^2} \right) - 2^{k-2} \sigma_{k-3}^* \left( 2 \frac{\det H}{4d^2} \right) \right], & \text{if } \text{rank}(H) = 2. \end{cases}$$
On the other hand,
\[
b_{km}(H) = \begin{cases} 
(2^{km-2} - 1) \frac{-B_{km}B_{km-2}}{4km(km-2)}, & \text{if } H = O_2, \\
(2^{km-2} - 1) \frac{B_{km-2}}{2(km-2)} \sigma_{km-1}(\varepsilon(H)), & \text{if } \operatorname{rank}(H) = 1, \\
\sum_{0 < d < \varepsilon(H)} d^{km-1} \left[ \sigma_{km-3} \left( \frac{2 \det H}{d^2} \right) - 2^{km-2} \sigma_{km-3} \left( \frac{2 \det H}{d^2} \right) \right], & \text{if } \operatorname{rank}(H) = 2.
\end{cases}
\]

Combining these formulas and the Kummer congruence, we can prove that
\[
\lim_{m \to \infty} b_{km}(H) = A_k(H)
\]
for all \( H \in \text{Her}_2^\ast(O) \). This completes the proof of Theorem 3.1.

**Remark 3.4.** Following Hida [3], our \( G^\ast_k \) can be \( p \)-adic analytically interpolated with respect to the weight.

### 3.2 Transcendental \( p \)-adic Eisenstein series

As we have seen in the previous section, under certain conditions, a \( p \)-adic limit of a quaternionic Eisenstein series becomes a “real” modular form with rational Fourier coefficients. This also holds for Siegel Eisenstein and Hermitian Eisenstein series. More precisely, they coincide with the genus theta series (cf. [4], [11]). In these cases (Siegel, Hermitian cases), the \( p \)-adic Eisenstein series is algebraic. We shall show that there exists an example of a transcendental \( p \)-adic Eisenstein series for quaternionic modular forms.

The second main theorem is

**Theorem 3.5.** Let \( p \) be an odd prime and \( \{k_m\} \) the sequence defined by
\[
k_m := 2 + (p - 1)p^{m-1}.
\]

Then, the \( p \)-adic Eisenstein series \( \widetilde{E} = \lim_{m \to \infty} E^{(2)}_{k_m} \) is transcendental; namely, \( \widetilde{E} \) has transcendental coefficients where \( E^{(2)}_k \) is the normalized quaternionic Eisenstein series of degree 2 defined in (2.1).

**Proof.** We calculate \( \tilde{a}(H) := \lim_{m \to \infty} a_{km}(H) \) at \( H = \left( \frac{1}{2}, \frac{\varepsilon(H)}{2} \right) \in \text{Her}_2^\ast(O) \). The convergence for general \( H \) is proved similarly.

It follows from Theorem 2.1 that
\[
a_{km}(H) = -\frac{4km(km-2)}{(2^{km-2} - 1)B_{km}B_{km-2}}.
\]

(We note that \( \varepsilon(H) = 1 \) and \( \det(H) = \frac{p}{2} \).) We rewrite the right-hand side as
\[
-4 \cdot \frac{2 + (p - 1)p^{m-1}}{B_{2 + (p - 1)p^{m-1}}} \cdot \frac{1}{B_{(p-1)p^{m-1}}} \cdot \frac{p^m}{2^{(p-1)p^{m-1}} - 1} \cdot \frac{p - 1}{p}
\]
and calculate the \( p \)-adic limit separately:

(i) \( \lim_{m \to \infty} \frac{2 + (p - 1)p^{m-1}}{B_{2 + (p - 1)p^{m-1}}} = \frac{B_2}{2} = \frac{1}{12} \).

This is a consequence of the Kummer congruence.

(ii) \( \lim_{m \to \infty} B_{(p-1)p^{m-1}} = \frac{p - 1}{p} \).
This identity comes from the fact that the residue of the \( p \)-adic \( L \)-function \( L_p(s, \chi^0) \) at \( s = 0 \) is just \( 1 - \frac{1}{p} \).

\[
(iii) \quad \lim_{m \to \infty} \frac{2(p-1)p^{m-1} - 1}{p^m} = \frac{\log_p(2^{p-1})}{p},
\]

where \( \log_p \) is the \( p \)-adic logarithmic function defined by

\[
\log_p(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots , \quad (|x|_p < 1).
\]

Leopoldt’s formula [9] states that

\[
\lim_{m \to \infty} \frac{x^{p^m} - 1}{p^m} = \log_p(x).
\]

if \( |x - 1|_p < 1 \). This implies that

\[
\lim_{m \to \infty} \frac{2(p-1)p^{m-1} - 1}{p^m} = \frac{1}{p} \cdot \log_p(2^{p-1}).
\]

Combining these formulas, we obtain

\[
\tilde{a}(H) = \lim_{m \to \infty} a_{k_m}(H) = \frac{-48p}{\log_p(2^{p-1})}.
\]

We shall show that \( \log_p(2^{p-1}) \) is transcendental. Let \( \exp_p \) be the \( p \)-adic exponential function defined by

\[
\exp_p(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots , \quad (|x|_p < p^{\frac{1}{p-1}}).
\]

It is known that if \( |x|_p < p^{\frac{1}{p-1}} \), then

\[
\exp_p(\log_p(1 + x)) = 1 + x, \quad (e.g., [3]).
\]

To prove the transcendency of \( \log_p(2^{p-1}) \), we use the following theorem by Mahler:

**Theorem 3.6** (Mahler [10]). Let \( \mathbb{C}_p \) be the completion of the algebraic closure of \( \mathbb{Q}_p \). For any algebraic over \( \mathbb{Q} \) \( p \)-adic number \( \alpha \in \mathbb{C}_p \) with \( 0 < |\alpha|_p < p^{\frac{1}{p-1}} \), the quantity \( \exp_p(\alpha) \) is transcendental.

We note that \( |x|_p < p^{\frac{1}{p-1}} \) is equivalent to \( |x|_p < 1 \) for odd prime \( p \) (e.g., [3], p.114). We put \( \alpha = 2^{p-1} - 1 \). Since \( |\alpha|_p < 1 \), we have

\[
\exp_p(\log_p(1 + \alpha)) = 1 + \alpha = 2^{p-1}.
\]

The right-hand side is obviously algebraic. Hence, by Mahler’s theorem, \( \log_p(1 + \alpha) = \log_p(2^{p-1}) \) must be transcendental. Thus, we can prove the transcendency of \( \tilde{a}(H) \) at \( H = \left( \frac{1}{2}, \frac{2}{1} \right) \).

This completes the proof of Theorem 3.5.

**Remark 3.7.** By the above proof, we see that all coefficients \( \tilde{a}(H) \) corresponding to \( H \) with rank 2 are transcendental. However, \( \tilde{a}(H) \) for \( H \) with rank(\( H \)) \( \leq 1 \) are rational.

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