Random Construction of Riemann Surfaces

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In this paper, we address the following question: What does a typical compact Riemann surface of large genus look like geometrically?

By a Riemann surface, we mean an oriented surface with a complete, finite-area metric of constant curvature -1.

In the standard geometric picture of Riemann surfaces via Fenchel-Nielsen coordinates, it is difficult to keep track of global geometric quantities such as the first eigenvalue of the Laplacian, the injectivity radius, and the diameter. Here, we present a model for looking at Riemann surfaces based on 3-regular graphs, with which it is easier to control the global geometry.

The idea of using 3-regular graphs to study the first eigenvalue of Riemann surfaces originated in Buser ([Bu1], [Bu2]), who associated cubic graphs to Riemann surfaces as a tool for comparing the spectral geometry of surfaces with the spectral geometry of graphs. We introduce a somewhat different method, which associates to each 3-regular graph with an orientation a finite area Riemann surface.

To be more specific, if Γ is a finite 3-regular graph, an orientation O on Γ is a function which assigns to each vertex v of Γ a cyclic ordering of the edges emanating from v. In §2 below, we will show how, given a pair (Γ, O), we may associate to (Γ, O) two Riemann surfaces $S^O(Γ, O)$ and $S^C(Γ, O)$. $S^O(Γ, O)$ is constructed by associating an ideal hyperbolic triangle to each vertex of Γ, and gluing sides together according to the edges of the graph Γ and the orientation O. It is a finite-area Riemann surface with cusps.

The surface $S^C(Γ, O)$ is then the conformal compactification of $S^O(Γ, O)$.

It follows from a theorem of Belyi [Bel] (see [JS] for a discussion of Belyi's Theorem), that the surfaces $S^C(Γ, O)$ are dense in the space of all Riemann surfaces. Thus, the process of randomly selecting a Riemann surface can be modeled on the process of picking a finite 3-regular graph with orientation at random.

Since the pair (Γ, O) gives a description of $S^O(Γ, O)$ as an orbifold covering of $\mathbb{H}^2/PSL(2,\mathbb{Z})$, one can give a qualitative description of the global geometry of $S^O(Γ, Z)$ by a corresponding description of the pair (Γ, O). Thus, for example, the first eigenvalue of $S^O(Γ, O)$ will be large if and only if the first eigenvalue of Γ is large ([SGTC] and [VD]).

It is our observation here, building on the work of [PS], that the same will be true of $S^C(Γ, O)$, provided that $S^O(Γ, O)$ satisfies a “large cusps” condition, to be described in §1 below. This has a purely combinatorial interpretation in terms of the pair (Γ, O), and so can be analyzed with relative ease.
For $n$ a positive integer, let $\mathcal{F}_n^*$ denote the finite set of pairs $(\Gamma, \mathcal{O})$, where $\Gamma$ is a 3-regular graph on $2n$ vertices. We will endow $\mathcal{F}_n^*$ with a probability measure introduced and studied by Bollobás [Bol1], [Bol2], which we review in §3 below.

If $Q$ is a property of 3-regular graphs with orientation, denote by $\text{Prob}_n[Q]$ the probability that a pair $(\Gamma, \mathcal{O})$ picked from $\mathcal{F}_n^*$ has property $Q$.

Our main technical result, shown in §4 below, is:

**Theorem 0.1** As $n \to \infty$,

$$\text{Prob}_n[S^O(\Gamma, \mathcal{O})] \text{ satisfies the large cusps condition } \to 1.$$

We will use Theorem 0.1 in order to study geometric properties of the surfaces $S^C(\Gamma, \mathcal{O})$. To that end, we define the Cheeger constant $h(S)$ of a Riemann surface $S$ by the formula

$$h(S) = \inf_{C} \frac{\text{length}(C)}{\min[\text{area}(A), \text{area}(B)]},$$

where $C$ runs over (possibly disconnected) closed curves on $S$ which divide $S$ into two parts $A$ and $B$.

It will then follow from Theorem 0.1 that:

**Theorem 0.2** There exist constants $C_1, C_2, C_3,$ and $C_4$ such that, as $n \to \infty$:

(a) The first eigenvalue $\lambda_1(S^C(\Gamma, \mathcal{O}))$ satisfies

$$\text{Prob}_n[\lambda_1(S^C(\Gamma, \mathcal{O})) \geq C_1] \to 1.$$

(b) The Cheeger constant $h(S^C(\Gamma, \mathcal{O}))$ satisfies

$$\text{Prob}_n[h(S^C(\Gamma, \mathcal{O})) \geq C_2] \to 1.$$

(c) The shortest geodesic $\text{syst}(S^C(\Gamma, \mathcal{O}))$ satisfies

$$\text{Prob}_n[\text{syst}(S^C(\Gamma, \mathcal{O})) \geq C_3] \to 1.$$

(d) The diameter $\text{diam}(S^C(\Gamma, \mathcal{O}))$ satisfies

$$\text{Prob}_n[\text{diam}(S^C(\Gamma, \mathcal{O})) \leq C_4 \log(\text{genus}(S^C(\Gamma, \mathcal{O})))] \to 1.$$
Of these properties, (a) follows from (b) by Cheeger’s inequality \cite{Ch}, while (d) also follows from (b) and (c) and the following well-known argument: if $M$ is a manifold and $B(r_0)$ is the infimum over all points of $M$ of the volume of a ball of radius $r_0$, then

$$\text{diam}(M) \leq 2[r_0 + \frac{1}{h(M)} \log(\frac{\text{vol}(M)}{2B(r_0)})].$$

Property (b) for the surfaces $SO(\Gamma, O)$ will follow from the corresponding result on graphs \cite{Bol} together with \cite{SGTC} and \cite{VD}, while the passage from the surfaces $SO(\Gamma, O)$ to the surfaces $SC(\Gamma, O)$ will follow from Theorem 0.1. For properties (c) and (d), the translation from graphs to surfaces is not as simple, but the idea is similar.

The forms of (a), (b), and (d) are sharp, up to constants. Regarding (a), it follows from Cheng’s Theorem \cite{Chng} that a Riemann surface $S$ must have $\lambda_1(S) \leq 1/4 + \varepsilon$ for some $\varepsilon \to 0$ as genus($S$) $\to \infty$. The upper bound $h(S) \leq 1 + \varepsilon$, is well-known, and follows from a similar argument. The estimate $\text{diam}(S) \geq (\text{const}) \log(\text{genus}(S))$ follows from area considerations and Gauss-Bonnet.

The estimate in (c) is certainly not optimal, as there are Riemann surfaces whose injectivity radius grows like $(\text{const})[\log(\text{genus}(S))]$. Indeed, this occurs for the Platonic surfaces \cite{PS}, and also for congruence coverings of compact arithmetically defined surfaces. It follows from our analysis that, for a given constant $C_5$, there is a positive constant $C_6$ such that

$$\text{Prob}_n[\text{syst}(SC(\Gamma, O)) \geq C_5] \to C_6.$$ 

Thus, probability of selecting a surface having injectivity radius at least a given large number is asymptotically positive, but certainly not asymptotically 1.

In the language of \cite{SGNT}, Theorem 0.2 shows that, with probability $\to 1$, a typical Riemann surface is short, with a large first eigenvalue, but not necessarily fat.

The results that we use from \cite{PS} are qualitative rather than quantitative. However, they have been put in quantitative form in \cite{M}. In particular, it follows from \cite{M} that whenever in the following the condition of “cusps of length $\geq L$” is used, we may take $L = 7$.

The results of Theorem 0.2 were announced in \cite{BM}, under the weaker conclusion that properties (a)-(d) occur with positive probability, rather than probability $\to 1$ as $n \to \infty$. 

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1 Compactification of Riemann Surfaces

In this section, we review the connection between a finite-area Riemann surface and its conformal compactification.

Let $S^0$ be a Riemann surface with a complete finite area metric of curvature $-1$. Then $S^0$ has finitely many cusps neighborhoods $C^1, \ldots, C^k$, such that, for each $C^i$ there is an isometry

$$f_i : C^i \to C^{y_i}$$

for some $y_i$, where $C^{y_i}$ is the space

$$C^{y_i} = \left\{ z \in \mathbb{C} : \Im(z) \geq \frac{1}{y_i} \right\} / (z \sim z + 1),$$

endowed with the hyperbolic metric

$$ds^2 = \frac{1}{y_i^2} \left[ dx^2 + dy^2 \right].$$

The curve $h_i = f_i^{-1}(z : \Im(z) = \frac{1}{y_i})$ on $S^0$ is a closed horocycle on $C^i$ whose length is $y_0$. The length of the largest closed horocycle on the cusp $C^i$ is a measure of how large the cusp is.

Definition 1.1 $S^0$ has cusps of length $\geq L$ if we may choose all the $C^i$’s disjoint, with $y_i \geq L$ for all $i = 1, \ldots, k$.

Given a finite-area Riemann surface $S^0$, there is a unique compact Riemann surface $S^C$ and finitely many points $p_1, \ldots, p_k$ of $S^C$ such that $S^0$ is conformally equivalent to $S^C - \{p_1, \ldots, p_k\}$. $S^C$ may be constructed from $S^0$ by observing that each cusp neighborhood $C^i$ is conformally equivalent to a punctured disk. One may fill in this puncture conformally and then reglue the disk to obtain the conformal structure on the closed surface $S^C$. By the Uniformization Theorem, there is a unique constant curvature metric which agrees with this conformal structure.
It is natural to raise the question of the relationship between the constant curvature metric on $S^O$ and the constant curvature metric on $S^C$. In general, the relationship need not be close. For instance, the surface $S^C$ need not carry a hyperbolic metric even when $S^O$ carries one. However, it is shown in Theorem 2.1 of [PS] that, if $S^O$ has cusps of length $\geq L$ for a suitably large $L$, then $S^C$ will carry a hyperbolic metric, and indeed this metric will be very closely related to the hyperbolic metric on $S^O$.

More precisely, we have:

**Theorem 1.1 (PS)** For every $\varepsilon$, there exists numbers $L, r, \text{ and } y$ such that, if the cusps of $S^O$ have length $\geq L$, then, outside the union of cusp neighborhoods $U = \cup_{i=1}^k f^{-1}_i (C^y) \subset S^O$ of the cusps $C^i$, and $V = \cup_{i=1}^k B_r(p_i) \subset S^C$, the metrics $ds^2_C$ and $ds^2_O$ satisfy

$$\frac{1}{1 + \varepsilon} ds^2_O \leq ds^2_C \leq (1 + \varepsilon) ds^2_O.$$ 

The proof of this theorem is based on the Ahlfors Schwarz Lemma [A]. The idea of the proof is to build on the compact surface an intermediate metric $ds^2_{int}$ with curvature close to the curvature of the metric on the open surface, and to use the Ahlfors-Schwarz Lemma to compare this metric to the constant curvature metric. The large cusps condition enters precisely here, by
giving the metric $ds^2_{\text{int}}$ sufficient time to evolve from the hyperbolic metric on the ball to the hyperbolic metric on the cusp, while keeping curvature close to constant.

It was shown in [PS] that this result may be employed to show that, under the assumption of large cusps, the surfaces $S^O$ and $S^C$ share a number of global geometric properties.

**Theorem 1.2 ([PS])** For every $\varepsilon$, there exists an $L$ such that, if $S^O$ has cusps of length $\geq L$, then

(a) the Cheeger constants $h(S^O)$ and $h(S^C)$ satisfy

$$\frac{1}{(1+\varepsilon)}h(S^O) \leq h(S^C) \leq (1+\varepsilon)h(S^O).$$

(b) the shortest closed geodesics $\text{syst}(S^O)$ and $\text{syst}(S^C)$ satisfy

$$\frac{1}{(1+\varepsilon)}\text{syst}(S^O) \leq \text{syst}(S^C).$$

We do not obtain an inequality of the form

$$\text{syst}(S^C) \leq (\text{const})\text{syst}(S^O),$$

in (b), because it may happen that the shortest closed geodesic on $S^O$ becomes homotopically trivial on $S^C$.

## 2 Surfaces and 3-Regular Graphs

Let $\Gamma$ be a finite 3-regular graph. An orientation $O$ on the graph is the assignment, for each vertex $v$ of $\Gamma$, of a cyclic ordering of the three edges emanating from $v$. If $\Gamma$ has $2n$ vertices, then clearly there are $2^{2n}$ orientations on $\Gamma$.

We may think of an orientation on a 3-regular graph in the following way: suppose one were to walk along the graph. Then, when one approaches a vertex, the orientation allows one to distinguish between a left-hand turn and a right-hand turn at the vertex. Thus, any path on $\Gamma$ beginning at a vertex $v_0$ may be described by picking an initial direction and a series of $L$’s (signalling a left-hand turn) and $R$’s (signalling a right-hand turn).
To a pair \((\Gamma, \mathcal{O})\) we will associate two Riemann surfaces \(S^O(\Gamma, \mathcal{O})\) and \(S^C(\Gamma, \mathcal{O})\) as follows: We begin by considering the ideal hyperbolic triangle \(T\) with vertices \(0, 1, \) and \(\infty\) shown in Figure 2 below. The solid lines in Figure 2 are geodesics joining the points \(i, i + 1,\) and \(\frac{i + 1}{2}\) with the point \(\frac{i + 1}{2}\sqrt{3}\), while the dotted lines are horocycles of length 1 joining pairs of points from the set \(\{i, i + 1, \frac{i + 1}{2}\}\). We may think of these points as “midpoints” of the corresponding sides, even though they are of infinite length. We may also think of the three solid lines as segments of a graph surrounding a vertex. We then give them the cyclic ordering \((i, i + 1, \frac{i + 1}{2})\).

![Figure 2: The marked ideal triangle \(T\)](image)

We may now construct the surface \(S^O(\Gamma, \mathcal{O})\) from \((\Gamma, \mathcal{O})\) by placing on each vertex \(v\) of \(\Gamma\) a copy of \(T\), so that the cyclic ordering of the segments in \(T\) agrees with the orientation at the vertex \(v\) in \(\Gamma\). If two vertices of \(\Gamma\) are joined by an edge, we glue the two copies of \(T\) along the corresponding sides subject to the following two conditions:

(a) the midpoints of the two sides are glued together,

and

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(b) the gluing preserves the orientation of the two copies of $T$.

The conditions (a) and (b) determine the gluing uniquely. It is easily seen that the surface $S^O(\Gamma, \mathcal{O})$ is a complete Riemann surface with finite area equal to $2\pi n$, where $2n$ is the number of vertices of $\Gamma$.

The surface $S^C(\Gamma, \mathcal{O})$ is then the conformal compactification of $S^O(\Gamma, \mathcal{O})$.

In the remainder of this section, we will describe how to read off many geometric properties of the surfaces $S^O(\Gamma, \mathcal{O})$ and $S^C(\Gamma, \mathcal{O})$ from the combinatorics of the pair $(\Gamma, \mathcal{O})$.

We begin with the observation that the topology of $S^O$ is easy to reconstruct from $(\Gamma, \mathcal{O})$. We will need the following

**Definition 2.1** A left-hand-turn path on $(\Gamma, \mathcal{O})$ is a closed path on $\Gamma$ such that, at each vertex, the path turns left in the orientation $\mathcal{O}$.

Traveling on a path on $\Gamma$ which always turns left describes a path on $S^O(\Gamma, \mathcal{O})$ which travels around a cusp. Indeed, if we set $l = l(\Gamma, \mathcal{O})$ to be the number of disjoint left-hand-turn paths, then the topology of $S^O(\Gamma, \mathcal{O})$ is easily describable in terms of $l$ and $n$. Indeed, the graph $\Gamma$ divides $S^O(\Gamma, \mathcal{O})$ into $l$ regions, each bordered by a left-hand-turn path and containing one cusp in its interior. From this, we can immediately read off the signature of $S^O(\Gamma, \mathcal{O})$ by the Euler characteristic. The genus of $S^O(\Gamma, \mathcal{O})$ is given by

$$\text{genus} = 1 + \frac{n - l}{2},$$

and the number of cusps is $l$.

Note that the usual orientation on the 3-regular graph which is the 1-skeleton of the cube contains six left-hand-turn paths, giving that the associated surface is a sphere with six punctures, while a choice of a different orientation on this graph can have either two, four, or six left-hand-turn paths, so that the associated surface can have genus 0, 1, or 2. Thus, the topology of $S^O(\Gamma, \mathcal{O})$ is heavily dependent on the choice of $\mathcal{O}$.

The geometry of the cusps can also be read off from $(\Gamma, \mathcal{O})$. To that end, we observe that the horocycles on the various copies of $T$ fit together to form a system of disjoint closed horocycles about the cusps of $S^O(\Gamma, \mathcal{O})$. We call this system of horocycles the *canonical horocycles* of $S^O(\Gamma, \mathcal{O})$. The length of each closed horocycle in this set is precisely the length of the corresponding left-hand-turn path, since the length of the horocycle joining $i$ to $i + 1$ has
length 1. Thus, $S^O(\Gamma, \mathcal{O})$ has cusps of length $\geq L$ if the length of any left-hand turn path on $(\Gamma, \mathcal{O})$ is at least $L$.

The converse to this is not true, as it is possible to choose a system of horocycles other than the canonical horocycles such that the length of the shortest horocycle for the new system is larger than the length of the shortest canonical horocycle. We will return to this idea in §4 below.

The Cheeger constant $h(S^O(\Gamma, \mathcal{O}))$ can be estimated in terms of the graph $(\Gamma, \mathcal{O})$ as well. Recall that, by analogy with the Cheeger constant of a manifold, the Cheeger constant $h(\Gamma)$ of a graph $\Gamma$ is given by

$$h(\Gamma) = \inf_{E} \frac{\#(E)}{\min(\#(A), \#(B))},$$

where $E$ is a collection of edges such that $\Gamma - E$ disconnects into two components $A$ and $B$, and $\#(A)$ (resp. $\#(B)$) is the number of vertices in $A$ (resp. $B$).

Then we have

**Theorem 2.1 ([SGTC], [VD])** There are positive constants $C_1$ and $C_2$ such that

$$C_1 h(\Gamma) \leq h(S^O(\Gamma, \mathcal{O})) \leq C_2 h(\Gamma)$$

for all finite 3-regular graphs $\Gamma$.

In effect, the pair $(\Gamma, \mathcal{O})$ describes $S^O(\Gamma, \mathcal{O})$ as an orbifold covering space of the orbifold $\mathbb{H}^2/\text{PSL}(2, \mathbb{Z})$. The behavior of the Cheeger constant of a finite covering of a compact manifold in terms of the graph of a covering is described in [SGTC]. In the present case, the base manifold is not compact, but rather a finite-area Riemann surface (with singularities). The additional complication which this difficulty presents is solved in [VD].

This gives Theorem 2.1.

Note in particular that the quantity $h(\Gamma)$ of Theorem 2.1 depends only on $\Gamma$ and not on $\mathcal{O}$.

The geodesics of $S^O(\Gamma, \mathcal{O})$ are also describable in terms of $(\Gamma, \mathcal{O})$. To explain this, let $\mathcal{L}$ and $\mathcal{R}$ denote the matrices

$$\mathcal{L} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$
A closed path $P$ of length $k$ on the graph may be described by starting at a midpoint of an edge, and then giving a sequence $(w_1, \ldots, w_k)$, where each $w_i$ is either $l$ or $r$, signifying a left or right turn at the upcoming vertex. We then consider the matrix

$$M_P = W_1 \ldots W_k,$$

where $W_j = L$ if $w_j = l$ and $W_j = R$ if $w_j = r$.

The closed path $P$ on $\Gamma$ is then homotopic to a closed geodesic $\gamma(P)$ on $S^O(\Gamma, O)$ whose length $\text{length}(\gamma(P))$ is given by

$$2 \cosh\left(\frac{\text{length}(\gamma(P))}{2}\right) = \text{tr}(M_P).$$

Note that $\text{length}(\gamma(P))$ depends very strongly on $O$. Indeed, if the path $P$ contains only left-hand turns, then $\text{length}(\gamma(P)) = 0$, and if $\gamma(P)$ is a path of length $r$ containing precisely one right-hand turn, then

$$\text{length}(\gamma(P)) = 2 \log \left(\frac{(1 + r) + \sqrt{(1 + r)^2 - 4}}{2}\right) \sim 2 \log(1 + r),$$

and hence grows linearly in $\log(r)$. On the other hand, if the path $P$ of length $r$ consists of alternating left- and right-hand turns, then

$$\text{length}(\gamma(P)) = r \log\left(\frac{3 + \sqrt{5}}{2}\right),$$

which is linear in $r$.

We now consider the description of $S^C(\Gamma, O)$ in terms of $(\Gamma, O)$. We will carry out this description under the assumption that the cusps of $S^O(\Gamma, O)$ are large.

**Theorem 2.2** Assume that the cusps of $S^O(\Gamma, O)$ have length $\geq L = L(\varepsilon)$. Then there exist constants $C_1, C_2, C_3, C_4,$ and $C_5$ depending only on $L$ such that:

(a) The Cheeger constant $h(S^C(\Gamma, O))$ satisfies

$$C_1 h(\Gamma) \leq h(S^C(\Gamma, O)) \leq C_2 h(\Gamma).$$

(b) The shortest closed geodesic $\text{syst}(S^C(\Gamma, O))$ satisfies

$$\text{syst}(S^C(\Gamma, O)) \geq C_3 \log(1 + \text{syst}(\Gamma)) \geq C_4,$$

where $\text{syst}(\Gamma)$ is the girth of the graph $\Gamma$. 

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(c) The genus of $S^C(\Gamma,\mathcal{O})$ satisfies
\[ \text{genus}(S^C(\Gamma,\mathcal{O})) \geq C_5\#(\Gamma). \]

**Proof:** (a) follows from Theorem 1.2 (a) and Theorem 2.1. (b) follows from Theorem 1.2 (b) and the calculation of lengths of geodesics on $S^O(\Gamma,\mathcal{O})$. (c) follows from the formula for the genus of $S^O(\Gamma,\mathcal{O})$, which is also the genus of $S^C(\Gamma,\mathcal{O})$, together with the simple observation that if each cusp in $S^O(\Gamma,\mathcal{O})$ is bounded by a horocycle of length at least $L$, then the number of such cusps is bounded by $\frac{1}{L}[\text{area}(S^O(\Gamma,\mathcal{O}))]$, since $L$ is the area inside a horocycle of length $L$.

### 3 The Bollobás Model

In this section we discuss a model, due to Bollobás, for studying the process of randomly selecting a 3-regular graph.

The problem of putting a probability measure on the set of 3-regular graphs on $2n$ vertices would not appear at first sight to be difficult, since this is a finite set. It has, however, proven problematic to find as model which is amenable to meaningful calculation, and a number of different models have been proposed and studied, each with its own benefits and drawbacks. See Janson [Jan] for a discussion and comparison of the different models.

We will use a model introduced by Bollobás ([Bol1], [Bol2]). Bollobás considered the problem for $k$-regular graphs, $k$ arbitrary, but we will need only the case $k = 3$. This model has the advantage that calculations of an asymptotic character (as $n \to \infty$) can be carried out with relative ease.

For each $n$, let $\mathcal{F}_n$ denote the finite set of 3-regular graphs on $2n$ vertices, and let $\mathcal{F}_n^*$ denote the set of 3-regular graphs with orientation. We put a probability measure on $\mathcal{F}_n$ and $\mathcal{F}_n^*$ in the following way: we consider a hat with $6n$ balls, each ball labeled with an integer between 1 and $2n$, each integer occurring three times. We then build a graph at random by selecting pairs of balls from the hat, without replacement. If at step $i$ the numbers $b_i$ and $c_i$ are selected, we add to the graph an edge joining $b_i$ and $c_i$. An orientation at vertex $b$ is determined by the cyclic order in which edges are added to the vertex $b$.

We will need two results of Bollobás concerning this model. The first result concerns the Cheeger constant of a graph:
Theorem 3.1 ([Bol2]) There is a constant $C > 0$ such that the probability of a graph $\Gamma$ chosen randomly from $F_n$ having Cheeger constant $h(\Gamma)$ greater than $C$ satisfies

$$\text{Prob}_n[h(\Gamma) > C] \to 1 \text{ as } n \to \infty.$$ 

Bollobás gives numerical estimates showing that $C > 2/11$.

To state the second result, we recall the notion of an asymptotic Poisson distribution:

**Definition 3.1** (a) A random variable $X$ which takes values in the natural numbers $\mathbb{Z}^+$ is a Poisson distribution with mean $\mu$ if

$$\text{Prob}(X = k) = e^{-\mu} \frac{\mu^k}{k!}.$$ 

The mean $\mu$ is the expected value of $X$.

(b) Let $\{X^n\}$ be a family of random variables on the probability spaces $\{P_n\}$.

The $\{X^n\}$ are asymptotic Poisson distributions as $n \to \infty$ if there exists $\mu$ such that

$$\lim_{n \to \infty} \text{Prob}(X^n = k) = e^{-\mu} \frac{\mu^k}{k!}$$

for all $k$.

(c) The families $\{X^n\}$ are asymptotically independent Poisson distributions if, for each $i$, the random variables $X^n_i$ tend to a Poisson distribution $X_i$ as $n \to \infty$, and if the variables $X_i$ are independent.

A well-known example of an asymptotic Poisson distribution is given by the hatcheck lady who returns hats in a random fashion to the $n$ guests at a party. The random variable $X^n$ which is the number of guests who receive the correct hat is asymptotically Poisson with mean 1 as $n \to \infty$.

Bollobás proves:

**Theorem 3.2 ([Bol1])** Let $X_i$ denote the number of closed paths in $\Gamma$ of length $i$. Then the random variables $X_i$ on $F_n$ are asymptotically independent Poisson distributions with means

$$\lambda_i = \frac{2^i}{2i}.$$
In our case we have an additional structure on the graph—"the orientation". We are distinguishing between short paths that agree with the orientation and those that do not. To do so we look at all the possible orientations on a given closed path of length $i$. There are $2^i$ possible orientations, but only two yield a left hand path. Therefore we get:

**Corollary 3.1** Let $Y_i$ be the random variable on $\mathcal{F}_n^*$ which associates to $(\Gamma, O)$ the number of left-hand turn paths of length $i$. Then the $Y_i$ are asymptotically independent Poisson distributions with means

$$\mu_i = \frac{1}{i}.$$

Theorem 3.2 and Corollary 3.1 imply that short geodesics and small cusps will occur with positive probability in the surfaces $S^O(\Gamma, O)$, asymptotically as $n \to \infty$. One would expect on the grounds of asymptotic independence that as $n \to \infty$, these phenomena appear far apart. The following elementary lemma makes this expectation precise (compare [Bol1], Theorem 32):

**Lemma 3.1** For fixed numbers $l_1, l_2,$ and $d$, let $Q_n(l_1, l_2, d)$ denote the probability that a graph picked from $\mathcal{F}_n$ (resp. $\mathcal{F}_n^*$) has closed paths $\gamma_1$ and $\gamma_2$ of length $l_1$ and $l_2$ respectively, which are distance $d$ apart.

Then

$$Q_n(l_1, l_2, d) \to 0 \text{ as } n \to \infty.$$

**Proof:** We first observe that, since the statement is independent of the orientation $O$, we may restrict our attention to picking from $\mathcal{F}_n$. We will show that, for every $\varepsilon$, for $n$ sufficiently large, we have that

$$Q_n(l_1, l_2, d) < \varepsilon.$$

Since the number of closed paths of length $l_1$ are asymptotically Poisson distributed by Theorem 3.2, given $\varepsilon_1$, we may find $N(\varepsilon_1)$ such that with probability $> 1 - \varepsilon_1$, the number of closed paths of length $l_1$ is less than $N(\varepsilon_1)$.

Now let $\gamma$ be a closed path in $\Gamma$ of length $l_1$. We consider the $l_1 \cdot 2^{d + \lfloor \frac{l_2}{2} \rfloor - 1}$ vertices which are at distance at most $d + \lfloor \frac{l_2}{2} \rfloor$ from $\gamma$. When $n$ is large compared to $l_1 \cdot 2^{d + \lfloor \frac{l_2}{2} \rfloor}$, with probability $\to 1$ as $n \to \infty$, no vertex in this set will have been selected twice. This implies that there will be no closed path of length $l_2$ at distance $d$ from $\gamma$.

Applying this estimate to each of the $< N(\varepsilon_1)$ closed paths of length $l_1$ then gives the lemma.
4 Large Cusps

In this section, we complete the proof of Theorem 0.1. We will reformulate it in the following way:

**Theorem 0.1A**  Given \( L \), as \( n \to \infty \), we have

\[
\text{Prob}_n[S^O(\Gamma, \mathcal{O}) \text{ has cusps of length } \geq L] \to 1.
\]

**Proof:** We begin the proof by calculating the probability that the canonical horocycles of \( S^O(\Gamma, \mathcal{O}) \) all have length \( \geq L \). This is precisely the probability that all the random variables \( Y_i \) of Corollary 3.1 have the value 0, for \( 0 < i < L \). By Corollary 3.1, this is asymptotically

\[
e^{-\sum_{i=0}^{L-1} \frac{i}{i}} \sim e^{-\gamma (L-1)^{-1}},
\]

where \( \gamma \) is Euler’s constant.

Hence the lemma is proved if we replace the conclusion “probability \( \to 1 \)” with the weaker conclusion “probability \( \geq (\text{const}) > 0 \) as \( n \to \infty \).” This is a sufficiently strong version of the lemma to obtain the results announced in [BM].

We now show how to obtain the stronger results of Theorem 0.1A. To that end, suppose that the cusp \( C_0 \) of the surface \( S^O(\Gamma, \mathcal{O}) \) has a canonical horocycle of length \( < L \). We would like to choose a larger horocycle about this cusp.

There are two obstructions to choosing such a larger horocycle. The first obstruction is that as we increase the length of the horocycle about \( C_0 \), it may cease to be injective. This will happen if there is a short closed geodesic in the neighborhood of \( C_0 \).

The second obstruction is that, as we increase the length of the canonical horocycle about \( C_0 \), we must decrease the lengths of the horocycles of nearby cusps in order to keep the interiors of the horocycles disjoint. When we decrease the length of a nearby horocycle, it may then cease to have length \( \geq L \).

Both of these considerations are handled by Lemma 3.1. Indeed, both of these obstructions arise from the possibility that in the graph \( \Gamma \), there may be a short closed path close to the left-hand-turn path corresponding to \( C_0 \). According to Lemma 3.1, the probability of this occurring is asymptotically 0.
This argument is illustrated in Figure 3 below. The cusp in question lies between the vertical lines $x = 0$ and $x = 2$, and the canonical horocycle, the line $y = 1$, has length 2. We increase its length to 8 by lowering the horocycle to the line $y = 1/4$. This will be possible if none of the points $x + iy$ with $0 < x < 2, y > 1/4$ are identified in the surface (the first obstruction), and if all the horocycles which meet the line $y = 1/4$ have images in the surface which are sufficiently long (the second obstruction).

This concludes the proof of the lemma.

\begin{figure}[!h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Increasing the size of horocycle}
\end{figure}

References

[A] Lars V. Ahlfors, *An extension of Schwarz’s lemma*, Trans. Amer. Math. Soc. **43** (1938), no. 3, 359–364.

[Bel] G. V. Belyi, *Galois extensions of a maximal cyclotomic field*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 2, 267–276, 479.

[Bol1] Béla Bollobás, *Random graphs*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1985.

[Bol2] ______, *The isoperimetric number of random regular graphs*, European J. Combin. **9** (1988), no. 3, 241–244.
[SGNT] Robert Brooks, *Some relations between spectral geometry and number theory*, Topology ’90 (Columbus, OH, 1990) (Berlin), de Gruyter, Berlin, 1992, pp. 61–75.

[SGTC] ______, *The spectral geometry of a tower of coverings*, J. Diff. Geom., 23 (1986) 97–107.

[PS] ______, *Platonic surfaces*, Comment. Math. Helv. 74 (1999), no. 1, 156–170.

[VD] ______, *Some remarks on volume and diameter of Riemannian manifolds*, J. Diff. Geom. 27 (1988), 81–86.

[BM] Robert Brooks and Eran Makover, *The first eigenvalue of a Riemann surface*, Electron. Res. Announc. Amer. Math. Soc. 5 (1999), 76–81 (electronic).

[Bu1] Peter Buser, *Cubic graphs and the first eigenvalue of a Riemann surface*, Math. Z. 162 (1978), no. 1, 87–99.

[Bu2] ______, *On the bipartition of graphs*, Discrete Appl. Math. 9 (1984), no. 1, 105–109.

[Ch] Jeff Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, Problems in analysis (Princeton, N. J), vol. 1970, Papers dedicated to Salomon Bochner, 1969, Princeton Univ. Press, 1970, pp. 195–199.

[Chng] S.Y. Cheng, *Eigenvalue comparison theorems and its geometric applications*, Math. Zeit. 143 (1975), 289–297.

[Jan] Svante Janson, *Random regular graphs: asymptotic distributions and contiguity*, Combin. Probab. Comput. 4 (1995), no. 4, 369–405.

[JS] Gareth Jones and David Singerman, *Belyi functions, hypermaps and Galois groups*, Bull. London Math. Soc. 28 (1996), no. 6, 561–590.

[M] Dan Mangoubi, M. Sc. thesis, Technion, in preparation.