On Central Binomial Series Related to $\zeta(4)$.

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March 17, 2022

Abstract

In this paper, we prove two related central binomial series identities: $B(4) = \sum_{n \geq 0} \frac{(2n)}{2^{4n}(2n+1)^3} = \frac{7\pi^3}{216}$ and $C(4) = \sum_{n \in \mathbb{N}} \frac{1}{n^4(\frac{2}{n})} = \frac{17\pi^4}{3240}$. Both series resist all the standard approaches used to evaluate other well-known series. To prove the first series identity, we will evaluate a log-sine integral that is equal to $B(4)$. Evaluating this log-sine integral will lead us to computing closed forms of polylogarithms evaluated at certain complex exponentials. To prove the second identity, we will evaluate a double integral that is equal to $C(4)$. Evaluating this double integral will lead us to computing several polylogarithmic integrals, one of which has a closed form that is a linear combination of $B(4)$ and $C(4)$. After proving these series identities, we evaluate several challenging logarithmic and polylogarithmic integrals, whose evaluations involve surprising appearances of integral representations of $B(4)$ and $C(4)$. We also provide an insight into the generalization of a modern double integral proof of Euler’s celebrated identity $\sum_{n \in \mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}$, in which we encounter an integral representation of $C(4)$.

1 Introduction

Consider the series

$$B(4) = \sum_{n \geq 0} \frac{(2n)}{2^{4n}(2n+1)^3}$$

(1.1)

$$C(4) = \sum_{n \in \mathbb{N}} \frac{1}{n^4(\frac{2}{n})}$$

(1.2)

Both series are known as central binomial series due to the appearance of the central binomial coefficient $\binom{2n}{n}$ in each summand. The values of each sum are

$$B(4) = \frac{7\pi^3}{216}$$

(1.3)

$$C(4) = \frac{17\pi^4}{3240}$$

(1.4)

but it is not known who first discovered these results. van Der Poorten [1] numerically conjectured (1.4), but the literature suggests either Comtet [2] or Lewin [3] had discovered (1.4) earlier.

Both $B(4)$ and $C(4)$ resist many of the standard techniques used to evaluate other series such as

$$\sum_{n \in \mathbb{N}} \frac{1}{n^{2k}}$$

for $k \in \mathbb{N}$. Such techniques include multiple integration, Fourier Series, and the Calculus of Residues (see all of [4–9]). Many authors, however, have studied intimate connections between central binomial series and log-sine integrals (see all of [10–17]). In particular, $B(4)$ and $C(4)$ are equal to the integrals

$$I = \int_0^{\hat{\pi}} \log^2(2\sin(t)) \, dt$$

(1.5)

$$J = \int_0^{\hat{\pi}} 8t \log^2(2\sin(t)) \, dt,$$

(1.6)
respectively. To evaluate either integral, one must rewrite $2 \sin(t)$ in terms of complex exponentials, expand out the integrand, and evaluate several logarithmic integrals. These integrals have closed forms in terms of polylogarithms, which are complex infinite series of the form

$$\text{Li}_k(z) = \sum_{n \in \mathbb{N}} \frac{z^n}{n^k},$$

where $k \in \mathbb{N}$ and $z \in \mathbb{C}$ satisfies $|z| \leq 1$. In particular, one will have to explicitly compute $\text{Li}_k(e^{i\theta})$ for certain $\theta \in \mathbb{R}$, extract real parts, and sum all such results together. This procedure is manageable for evaluating the integral $I$, but laborious for evaluating the integral $J$.

In this paper, we will prove the central binomial series identities (1.3) and (1.4). To prove (1.3), we will first show that the series $B(4)$ is equal to $I$. To do this, we will change variables $x = 2 \sin(t)$ on $I$, convert the transformed integrand into a binomial series, interchange sum and integral, and finally integrate term-by-term to recover $B(4)$. Then, we will show $I = \frac{7}{3240} \pi^4$ using the procedure described in the previous paragraph to evaluate $I$.

Next, we will prove (1.4) by showing that the series $C(4)$ is equal to the double integral

$$K = \int_0^1 \int_0^{1-x} \frac{2 \log^2(x+y)}{1-xy} \, dy \, dx$$

and evaluating $K$ directly. To show $C(4)$ is equal to $K$, we will change variables $x = (u+v)/2$, $y = (u-v)/2$ on $K$, integrate with respect to $v$, convert the resulting integrand into a special binomial series, interchange sum and integral, and finally integrate term-by-term to recover $C(4)$. Then, we will show $K = 17 \pi^4/3240$ by integrating the original double integral representation with respect to $y$, which will result in us evaluating three different polylogarithmic integrals. Two of these integrals have closed forms that are rational multiples of $\pi^4$, while the remaining integral has a closed form that is a linear combination of $B(4)$ and $C(4)$. Our proof of (1.4) does require elaborate integral calculations, but it is less laborious than the standard approach of evaluating the log-sine integral $J$.

After proving the central binomial series identities, we state and evaluate some difficult logarithmic and polylogarithmic integrals. In all of their evaluations, certain integral representations of $B(4)$ and $C(4)$ make surprising appearances in all of these integral evaluations. We conclude with a remark about the generalization of Apostol’s famous double integration solution [18] to proving

$$\sum_{n \in \mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}.$$ 

In particular, in mimicking Apostol’s method to prove

$$\sum_{n \in \mathbb{N}} \frac{1}{n^4} = \frac{\pi^4}{90},$$

we will encounter an integral representation of $C(4)$.

2 Preliminaries

This section will be dedicated to recalling standard definitions and results from complex variables and analytic number theory that we will need to prove our main central binomial series identities.

2.1 Complex Numbers

Recall a complex number $z \in \mathbb{C}$ is of the form

$$z = x + iy,$$  \hspace{1cm} (2.1)

where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$. We call $x$ and $y$ the real and imaginary parts of $z$, respectively, and write $x = \Re(z)$ and $y = \Im(z)$.

If $z \in \mathbb{C}$, we define its modulus to be the real number

$$|z| = \sqrt{\Re(z)^2 + \Im(z)^2},$$  \hspace{1cm} (2.2)
and if \( z \neq 0 \), we define its argument to be the real number
\[
\arg(z) = \tan^{-1}\left( \frac{\Im(z)}{\Re(z)} \right) \in [-\pi, \pi),
\]
(2.3)
and its complex logarithm to be the complex number
\[
\log(z) = \ln(|z|) + i \arg(z).
\]
(2.4)
We define a complex exponential to be a complex number of the form
\[
e^{i\theta} = \cos(\theta) + i \sin(\theta),
\]
(2.5)
where \( \theta \in \mathbb{R} \). With (2.5), we can easily deduce the following theorem.

**Theorem 2.1.1** (Euler’s Formulas). We have
\[
\cos(\theta) = \Re(e^{i\theta}) = \frac{e^{i\theta} + e^{-i\theta}}{2},
\]
(2.6)
\[
\sin(\theta) = \Im(e^{i\theta}) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.
\]
(2.7)
Using (2.7), we can derive another representation of the function \( \log(2 \sin(t)) \) for \( t \in (0, \pi) \) that we will need to complete the proof of our first central binomial series identity.

**Theorem 2.1.2** (Log-Sine Expansion). If \( t \in (0, \pi) \), we have
\[
\log(2 \sin(t)) = \log(1 - e^{2it}) + \left( \frac{\pi}{2} - t \right) i.
\]
(2.8)
In particular,
\[
\Im(\log(2 \sin(t))) = 0.
\]
(2.9)
**Proof.** Rewriting \( 2 \sin(t) \) according to the complex exponential formula from (2.7) and splitting the logarithm apart, we get
\[
\log(2 \sin(t)) = \log\left( \frac{e^{it} - e^{-it}}{i} \right)
= \log\left( ie^{-it}(1 - e^{2it}) \right)
= \log(i) + \log(e^{-it}) + \log(1 - e^{2it})
= i \left( \frac{\pi}{2} - t \right) + \log(1 - e^{2it}).
\]
The second statement (2.9) is immediate from the fact that \( 2 \sin(t) \) is positive for any \( t \in (0, \pi) \), which implies \( \log(2 \sin(t)) \) is real.

\[
\Box
\]

2.2 Polylogarithms

The polylogarithm of order \( k \in \mathbb{N} \) is defined to be the series
\[
\text{Li}_k(z) = \sum_{n \in \mathbb{N}} \frac{z^n}{n^k},
\]
(2.10)
where \( z \in \mathbb{C} \) satisfies \( |z| \leq 1 \). In particular, when \( z = 1 \), we recover the well-known Zeta function, which is the series
\[
\zeta(k) = \text{Li}_k(1) = \sum_{n \in \mathbb{N}} \frac{1}{n^k},
\]
(2.11)
and when \( z = -1 \), we recover the well-known Eta function, which is the series
\[
\eta(k) = \text{Li}_k(-1) = \sum_{n \in \mathbb{N}} \frac{(-1)^n}{n^k}.
\]
(2.12)
From (2.10), we can deduce the following functional identities.
**Theorem 2.2.1** (Polylogarithmic Functional Identities). We have

\[
\frac{d}{dz} \text{Li}_k(z) = \frac{\text{Li}_{k-1}(z)}{z} \tag{2.13}
\]

\[
\text{Li}_{k+1}(z) = \int \frac{\text{Li}_k(t)}{t} \, dt \tag{2.14}
\]

\[
\text{Li}_k(-z) = \frac{\text{Li}_k(z^2)}{2^{k-1}} - \text{Li}_k(z) \tag{2.15}
\]

The differentiation formula in (2.13) and the integration formula in (2.14) are ones we will be repeatedly using throughout this paper for elaborate integral calculations.

The following theorem highlights a handy algebraic relation between \( \eta(k) \) and \( \zeta(k) \).

**Theorem 2.2.2** (Eta and Zeta Function Relation). We have

\[
\eta(k) = -\left(1 - \frac{1}{2^{k-1}}\right) \zeta(k). \tag{2.16}
\]

*Proof.* This follows from substituting \( z = 1 \) into (2.15). \( \square \)

The polylogarithm has some closed forms for certain complex \( z \) in the cases \( k = 1 \) and \( 2 \). We list the most important closed forms that we need for this paper.

**Theorem 2.2.3** (Polylogarithmic Closed Forms). We have

\[
\text{Li}_1(z) = -\log(1 - z), \quad z \neq 1 \tag{2.17}
\]

\[
\text{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6} \tag{2.18}
\]

\[
\text{Li}_2\left(\frac{1}{2}\right) = \frac{\zeta(2)}{2} - \frac{\log^2(2)}{2} \tag{2.19}
\]

\[
\text{Li}_4(1) = \zeta(4) = \frac{\pi^4}{90}. \tag{2.20}
\]

*Proof.* To prove (2.17), we evaluate the integral

\[
\int_0^z \frac{1}{1 - t} \, dt \tag{2.21}
\]

in two ways. On one hand, directly integrating (2.21) with elementary calculus gives

\[
\int_0^z \frac{1}{1 - t} \, dt = -\log(1 - z).
\]

On the other hand, we rewrite the integrand of (2.21) as a geometric series, and we get

\[
\int_0^z \frac{1}{1 - t} \, dt = \int_0^z \sum_{n \in \mathbb{N}} t^{n-1} \, dt
\]

\[
= \sum_{n \in \mathbb{N}} \int_0^z t^{n-1} \, dt
\]

\[
= \sum_{n \in \mathbb{N}} \frac{z^n}{n} = \text{Li}_1(z).
\]

Hence, equating these two representations of (2.21) yields (2.17).

To prove (2.19), we evaluate the integral

\[
\int_0^{\frac{1}{2}} -\frac{\log(1 - t)}{t} \, dt \tag{2.22}
\]
in two ways. Notice we can rewrite the numerator of the integrand using (2.17), and using the integration formula in (2.14), we get

\[ \int_0^{\frac{1}{2}} \frac{\log(1-t)}{t} \, dt = \int_0^{\frac{1}{2}} \frac{\log(t)}{1-t} \, dt = \text{Li}_2 \left( \frac{1}{2} \right). \]

On the other hand, we integrate (2.22) by parts. Let \( u = -\log(1-t) \) and \( dv = \frac{dt}{t} \). Then, \( v = \log(t) \) and \( du = \frac{dt}{1-t} \), and we get

\[ \int_0^{\frac{1}{2}} -\log(1-t) \cdot \frac{dt}{t} = -\log \left( \frac{1}{2} \right) - \int_0^{\frac{1}{2}} \frac{\log(1-t)}{1-t} \, dt \]

\[ = -\log \left( \frac{1}{2} \right) + \int_0^{\frac{1}{2}} \frac{\text{Li}_1(1-t)}{u} \, du \]

\[ = -\log \left( 2 \right) + \zeta(2) - \text{Li}_2 \left( \frac{1}{2} \right), \quad (2.23) \]

where (2.24) follows from substituting \( t = 1 - u \) into the integral term in (2.23). Hence, equating these two representations of (2.22) and rearranging terms yields (2.19).

The proofs of (2.18) and (2.20) are well-known results, for which one can consult any one of [5, 6, 8, 9, 18–22].

**Remark.** The identity (2.18) is commonly known as the **Basel Problem** in Analytic Number Theory.

See [3] or [23] for an extensive list of polylogarithmic identities.

We will need to explicitly evaluate \( \text{Li}_k(z) \) when \( z \) is a complex exponential. To do this, we identify the real and imaginary parts of \( \text{Li}_k(e^{i\theta}) \), which we call the **Clausen Functions**. In particular, the Clausen Cosine and Clausen Sine functions of order \( k \in \mathbb{N} \), are respectively defined to be the series

\[ C_k(\theta) = \mathbb{R}(\text{Li}_k(e^{i\theta})) = \sum_{n \in \mathbb{N}} \frac{\cos(n\theta)}{n^k}, \quad (2.25) \]

\[ S_k(\theta) = \mathbb{I}(\text{Li}_k(e^{i\theta})) = \sum_{n \in \mathbb{N}} \frac{\sin(n\theta)}{n^k}, \quad (2.26) \]

where again, \( \theta \in \mathbb{R} \).

We can explicitly evaluate these Clausen functions for certain values of \( \theta \) very easily.

**Theorem 2.2.4 (Clausen Closed Formulas).** For \( t \in (0, \pi) \), we have

\[ S_1(2t) = \frac{\pi}{2} - t \]

\[ C_2(2t) = \zeta(2) - \pi t + t^2 \]

\[ S_3(2t) = \frac{\pi^2 t}{3} - \pi t^2 + \frac{2t^3}{3} \]

\[ C_4(2t) = \zeta(4) - \frac{\pi^2 t^2}{3} + \frac{2\pi t^3}{3} - \frac{t^4}{3}. \]

**Proof.** On one hand, recall from (2.20) that \( \mathbb{I}(\log(2\sin(t))) = 0 \). On the other hand, recalling our representation of \( \log(2\sin(t)) \) from (2.8) and the closed form for \( \text{Li}_1(z) \) from (2.17), we have

\[ \mathbb{I}(\log(2\sin(t))) = \frac{\pi}{2} - t - \mathbb{I}(\text{Li}_1(e^{i2t})) \]

\[ = \frac{\pi}{2} - t - S_1(2t). \]

Equating our two representations for \( \mathbb{I}(\log(2\sin(t))) \) and rearranging terms proves the first statement (2.27).
To obtain the second statement in (2.28), we compute the integral
\[
\int_0^t S_1(2u) \, du
\] (2.31)
in two ways. On one hand using the original definition of \(S_1(2u)\) by substituting \(\theta = 2u\) and \(k = 1\) into (2.26), we have
\[
\int_0^t S_1(2u) \, du = \int_0^t \sum_{n \in \mathbb{N}} \frac{\sin(2nu)}{n} \, du
= \sum_{n \in \mathbb{N}} \int_0^t \frac{\sin(2nu)}{n} \, du
= \sum_{n \in \mathbb{N}} \frac{1 - \cos(2nt)}{2n^2} = \zeta(2) - C_2(2t).
\]
On the other hand, using (2.27), we have
\[
\int_0^t S_1(2u) \, du = \int_0^t \pi - u \, du
= \frac{\pi t}{2} - \frac{t^2}{2}.
\]
Equating both these representations of (2.31) and solving for \(C_2(2t)\) will yield (2.28).

Remark. We can in fact rederive \(\zeta(2) = \pi^2/6\) by substituting \(t = \pi/2\) into the formula for \(C_2(2t)\) in (2.28). The left hand side of (2.28) is \(\eta(2)\), while the right hand side of (2.28) is \(\zeta(2) - \pi^2/4\). The result follows from rewriting \(\eta(2) = -\zeta(2)/2\) by virtue of substituting \(k = 2\) into the algebraic formula for \(\eta(k)\) in (2.16) and rearranging terms. By mimicking this exact same argument by substituting \(t = \pi/2\) into the formula for \(C_4(2t)\) in (2.30), we can recover \(\zeta(4) = \pi^4/90\).

2.3 Beta Functions and Binomial Coefficients

We now recall the well-known Beta function and generalized binomial coefficients, which will appear all throughout the second central binomial series identity proof.

The **Beta function** is defined to be any one of the following two integrals
\[
B(u, v) = \int_0^1 x^{u-1}(1-x)^{v-1} \, dx = \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}} \, dt,
\] (2.32)
where \(u, v \in \mathbb{C}\) with \(\Re(u), \Re(v) > 0\). Note that the integrals in (2.32) are equal under the change of variables \(x = t/(1+t)\).

We define the **generalized binomial coefficient** to be the number
\[
\binom{\alpha}{k} = \frac{\alpha(\alpha-1) \ldots (\alpha-k+1)}{k!},
\] (2.33)
where \(\alpha \in \mathbb{C}\) and \(k \in \mathbb{N} \cup \{0\}\). In the case \(\alpha = 2k\), we call \(\binom{2k}{k}\) the **central binomial coefficient**.

There are some identities specifically involving the central binomial coefficient which we will need.

**Theorem 2.3.1** (Generalized Binomial Coefficient Identities). For \(n \in \mathbb{N} \cup \{0\}\), we have
\[
\binom{-1/2}{n} = (-1)^n \binom{2n}{n} \binom{2n}{n}
\]
(2.34)
\[
B\left(\frac{2n+1}{2}, \frac{2n+1}{2}\right) = \frac{\pi}{2^{2n}} \binom{2n}{n}
\]
(2.35)
\[
B(n, n) = \frac{2}{n \binom{2n}{n}}, \quad n > 0.
\] (2.36)
We recall the well-known binomial series expansion, which we will use to prove our first central binomial series identity.

**Theorem 2.3.2** (Binomial Series Expansion). If \( \alpha \in \mathbb{C} \) and \( x \in \mathbb{R} \) satisfies \( |x| < 1 \), we have

\[
(1 + x)^\alpha = \sum_{n \geq 0} \binom{\alpha}{n} x^n.
\] (2.37)

The next binomial series representation involves the arcsine function, which we will use to prove our second central binomial series identity.

**Theorem 2.3.3** (ArcSine Series Expansion). If \( x \in \mathbb{R} \) satisfies \( |x| < 2 \), we have

\[
\frac{\sin^{-1}\left(\frac{x}{\sqrt{4 - x^2}}\right)}{\sqrt{4 - x^2}} = \sum_{n \in \mathbb{N}} \frac{x^{2n-1}}{2n\binom{2n}{n}}.
\] (2.38)

**Proof.** We evaluate the integral

\[
\int_0^1 \frac{x}{4 - 4x^2(y - y^2)} \, dy
\] (2.39)

in two ways.

On one hand, we can complete the square in the denominator of the integrand and evaluate (2.39) directly as follows:

\[
\int_0^1 \frac{x}{4 - 4x^2(y - y^2)} \, dy = \int_0^1 \frac{1}{4x} \frac{1}{\left(\frac{4x}{4x^2} + (y - \frac{1}{2})^2\right)} \, dy
\] (2.40)

\[
= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{4x} \frac{1}{\left(\frac{4x}{4x^2} + u^2\right)} \, du
\] (2.41)

\[
= \frac{\tan^{-1}\left(\frac{u}{\sqrt{4 - x^2}}\right)}{\sqrt{4 - x^2}} = \sin^{-1}\left(\frac{x}{\sqrt{4 - x^2}}\right),
\]

where we substituted \( u = y - 1/2 \) into the right hand side of (2.40) to get (2.41). On the other hand, we evaluate (2.39) by rewriting the integrand as a geometric series, in which we get

\[
\int_0^1 \frac{x}{4 - 4x^2(y - y^2)} \, dy = \int_0^1 \sum_{n \in \mathbb{N}} \frac{x^{2n-1}y^{n-1}(1 - y)^{n-1}}{4} \, dy
\] (2.42)

\[
= \sum_{n \in \mathbb{N}} \int_0^1 \frac{x^{2n-1}y^{n-1}(1 - y)^{n-1}}{4} \, dy
\]

\[
= \sum_{n \in \mathbb{N}} \frac{x^{2n-1}B(n, n)}{4} = \sum_{n \in \mathbb{N}} \frac{x^{2n-1}}{2n\binom{2n}{n}}
\] (2.43)

where the interchanging of sum and integral in (2.42) is permitted by the Monotone Convergence theorem, and (2.43) follows from simplifying the summand using the formula for \( B(n, n) \) from (2.36). Thus, equating these two representations for (2.39) gives the desired result. \( \square \)

### 3 Evaluating \( B(4) \) with a Log-Sine Integral

We are now ready to prove our first identity (1.3), namely

\[
B(4) = \sum_{n \geq 0} \frac{\binom{2n}{n}}{2^n(2n + 1)^2} = \frac{7\pi^3}{216}
\]

using the log-sine integral

\[
I = \int_0^\frac{\pi}{2} \log^2(2 \sin(t)) \, dt.
\] (3.1)

We first establish that \( I = B(4) \).
Theorem 3.0.1. We have \( I = B(4) \).

Proof. Substituting \( x = 2 \sin(t) \) into (3.1), we get

\[
I = \int_0^1 \frac{\log^2(x)}{2\sqrt{1 - x^2}} \, dx
\]

(3.2)

\[
= \int_0^1 \sum_{n \geq 0} \left( \frac{-1}{2n} \right) \left( \frac{-1}{n} \right) \frac{x^{2n} \log^2(x)}{2} \, dx
\]

(3.3)

\[
= \sum_{n \geq 0} \left( \frac{1}{4} \right)^{2n} \left( \frac{-1}{n} \right) \int_0^1 \frac{x^{2n} \log^2(x)}{2} \, dx
\]

(3.4)

\[
= \sum_{n \geq 0} \left( \frac{1}{4} \right)^{2n} \left( \frac{-1}{n} \right) \frac{1}{(2n + 1)^3}
\]

(3.5)

\[
= \sum_{n \geq 0} \left( \frac{1}{2} \right)^{4n} \left( \frac{2n}{n} \right) \frac{1}{(2n + 1)^3} = B(4),
\]

(3.6)

where (3.3) follows from rewriting the integrand (3.2) using the binomial series expansion formula in (2.37) and simplifying the resulting summand, and the interchanging of sum and integral in (3.4) is permitted by the Monotone Convergence Theorem. Finally, (3.5) follows evaluating the integral inside the summand of (3.4) with repeated integration by parts, and (3.6) follows from rewriting and simplifying the summand with the formula for \((-1/2)^n\) from (2.34).

We now evaluate \( I \) directly. Using our formula for \( \log(2 \sin(t)) \) from (2.8) to expand the integrand of (3.1) and using linearity of the integral, we get

\[
I = \int_0^{\pi/6} \left( \log(1 - e^{2it}) + \left( \frac{\pi}{2} - t \right)i \right)^2 \, dt
\]

(3.7)

We denote the three integrals in (3.7) as:

\[
I_1 = \int_0^{\pi/3} \left( \frac{\pi}{2} - t \right)^2 \, dt
\]

(3.8)

\[
I_2 = \int_0^{\pi/3} i(\pi - 2t) \log(1 - e^{2it}) \, dt
\]

(3.9)

\[
I_3 = \int_0^{\pi/3} \log^2(1 - e^{2it}) \, dt.
\]

(3.10)

Since \( I \) is a real number, it suffices to compute just the real parts of \( I_1, I_2, I_3 \) and sum those results together.

We will start with \( I_1 \), which is easy to evaluate with elementary calculus.

Theorem 3.0.2. We have

\[
\Re(I_1) = I_1 = -\frac{19\pi^3}{648}.
\]

(3.11)

Proof. Substituting \( x = \frac{\pi}{2} - t \) into (3.8), we have

\[
I_1 = \int_{\pi/2}^{\pi/3} x^2 \, dx = -\frac{19\pi^3}{648}.
\]

(3.12)

Using the Clausen Function closed formulas from Theorem [2.2.4] we can obtain \( \Re(I_2) \).
Theorem 3.0.3. We have

\[ \Re(I_2) = \frac{19\pi^3}{324}. \]  

(3.12)

Proof. We compute the integral of the real part of the integrand in (3.9) and see

\[ \Re(I_2) = \int_0^{\frac{\pi}{6}} \Re \left( i(\pi - 2t) \log(1 - e^{2it}) \right) \, dt \]

\[ = \int_0^{\frac{\pi}{6}} \Re \left( -i(\pi - 2t) \text{Li}_1(e^{2it}) \right) \, dt \]  

(3.13)

\[ = \int_0^{\frac{\pi}{6}} (\pi - 2t) S_1(2t) \, dt \]  

(3.14)

\[ = \int_0^{\frac{\pi}{6}} 2 \left( \frac{\pi}{2} - t \right)^2 \, dt \]  

(3.15)

\[ = -2I_1 = \frac{19\pi^3}{324}, \]

where (3.13) follows from substituting \( z = e^{2it} \) into the closed form of \( \text{Li}_1(z) \) from (2.17), and (3.15) follows from rewriting and the integrand of (3.14) with the closed form of \( S_1(2t) \) from (2.27).

Now, we need to evaluate \( I_3 \) in order to obtain \( \Re(I_3) \). Before we proceed to evaluate \( I_3 \), we need to use the following antiderivative identity, whose proof will require repeated integration by parts.

Theorem 3.0.4. For \( z \in \mathbb{C} \) with \( 0 < |z| < 1 \), we have

\[ \int \frac{\log^2(1 - z)}{z} \, dz = \log(z) \log^2(1 - z) + 2\text{Li}_2(1 - z) \log(1 - z) - 2\text{Li}_3(1 - z). \]  

(3.16)

Proof. Notice that the left hand side of (3.16) is equal to

\[ \int \frac{\text{Li}_1(z)^2}{z} \, dz. \]  

(3.17)

by (2.17). We now integrate (3.17) by parts. Let \( u = \text{Li}_1(z)^2 \) and \( dv = \frac{dz}{z} \). Then, \( v = \log(z) \) and applying the differentiation formula from (2.13) together with the Chain Rule, we get \( du = -\frac{2\text{Li}_1(z)}{1-z} \, dz \). Thus,

\[ \int \frac{\text{Li}_1(z)^2}{z} \, dz = \text{Li}_1(z)^2 \log(z) - \int \frac{2\log(z)\text{Li}_1(z)}{1-z} \, dz \]

\[ = \text{Li}_1(z)^2 \log(z) + \int \frac{2\text{Li}_1(1-z)\text{Li}_1(z)}{1-z} \, dz. \]  

(3.18)

Now integrating the integral term in (3.18) by parts with \( u = 2\text{Li}_1(z) \) and \( dv = \frac{\text{Li}_1(1-z)}{1-z} \, dz \), we have \( v = -\text{Li}_2(1-z) \) and \( du = \frac{2}{1-z} \, dz \), and

\[ \int \frac{\text{Li}_1(z)^2}{z} \, dz = \text{Li}_1(z)^2 \log(z) - 2\text{Li}_1(z)\text{Li}_2(1-z) + \int \frac{2\text{Li}_2(1-z)}{1-z} \, dz \]

\[ = \text{Li}_1(z)^2 \log(z) - 2\text{Li}_1(z)\text{Li}_2(1-z) - 2\text{Li}_3(1-z). \]  

(3.19)

The result follows upon replacing \( \text{Li}_1(z) \) with \( -\log(1-z) \) in all terms of (3.19).

With this antiderivative formula, we can evaluate \( I_3 \) and extract its real part.

Theorem 3.0.5. We have

\[ \Re(I_3) = \frac{\pi^3}{324}. \]
Proof. Substituting \( z = e^{2it} \) into (3.10), we get

\[
I_3 = \int_1^{\infty} \frac{\log^2(1 - z)}{2iz} \, dx. \tag{3.20}
\]

Letting \( \psi(z) \) be the antiderivative expression on the right hand side of (3.16), we can use the Fundamental Theorem of Calculus to see that

\[
\Re(I_3) = \Re \left( \frac{\psi(e^{i\pi/3})}{2i} - \lim_{z \to 1} \frac{\psi(z)}{2i} \right) - 0 \tag{3.21}
\]

\[
= -\frac{\pi^3}{54} - \frac{\pi}{3} C_2 \left( \frac{\pi}{3} \right) + S_3 \left( \frac{\pi}{3} \right) \tag{3.22}
\]

\[
= -\frac{\pi^3}{54} - \frac{\pi}{108} + \frac{5\pi^3}{162} = -\frac{\pi^3}{324} \tag{3.23}
\]

where (3.23) follows from simplifying the second and third terms of (3.22) by substituting \( t = \pi/6 \) into the formulas for \( C_2(2t) \) and \( S_3(2t) \) from (2.28) and (2.29), respectively.

Putting everything together, we have

\[
I = \Re(I_1) + \Re(I_2) + \Re(I_3)
\]

\[
= -\frac{19\pi^3}{648} + \frac{19\pi^3}{324} + \frac{\pi^3}{324} = \frac{7\pi^3}{216}.
\]

Hence,

\[
B(4) = \sum_{n \geq 0} \frac{\binom{2n}{n}}{2^{4n}(2n+1)^3} = \frac{7\pi^3}{216}.
\]

4 Evaluating \( C(4) \) with a Double Integral

We now prove our second identity (1.4), namely

\[
C(4) = \sum_{n \in \mathbb{N}} \frac{1}{n^4} \binom{2n}{n} = \frac{17\pi^4}{3240},
\]

using the double integral

\[
K = \int_0^1 \int_0^{1-x} \frac{2 \log^2(x+y)}{1-xy} \, dy \, dx. \tag{4.1}
\]

We first establish that \( K = C(4) \).

**Theorem 4.0.1.** We have \( K = C(4) \).
Proof. Substituting \( x = \frac{1 + u^2}{2} \) and \( y = \frac{u - u^3}{2} \) into (4.1) and using the Change of Variables formula, we get

\[
K = \int_0^1 \int_{-u}^u \frac{4 \log^2(u)}{4 - u^2 + v^2} \, dv \, du
\]

\[
= \int_0^1 8 \tan^{-1} \left( \frac{u}{\sqrt{4 - u^2}} \right) \log^2(u) \, du
\]

\[
= \int_0^1 8 \sin^{-1} \left( \frac{u}{2} \right) \log^2(u) \, du
\]

\[
= \int_0^1 \sum_{n \in \mathbb{N}} \frac{4}{n^2 n} u^{2n-1} \log^2(u) \, du
\]

\[
= \sum_{n \in \mathbb{N}} \frac{1}{n^4} \int_0^1 u^{2n-1} \log^2(u) \, du
\]

(4.2)

where (4.3) follows from rewriting the integrand in (4.2) using the arcsine series representation from (2.30) and simplifying the resulting summand, and the interchanging of sum and integral in (4.3) is permitted by the Monotone Convergence Theorem. Finally, (4.5) follows from repeated integration by parts of the integral inside the sum of (4.3) and simplifying the resulting summand. \( \square \)

We now evaluate the original double integral representation of \( K \) in (4.1) directly. We make the substitution \( y = \frac{1-u(1+x^2)}{x} \) on (4.1) to see that

\[
K = \int_0^1 \int_{-u}^u \frac{2 \log^2 \left( \frac{1 + u^2}{x} \right) (1 - u)}{ux} \, du \, dx
\]

\[
= \int_0^1 \int_{1+u^2}^{1+u^2} \frac{2 \left( \log \left( \frac{1 + x^2}{x} \right) + \log(1 - u) \right)^2}{ux} \, du \, dx
\]

\[
= \int_0^1 \int_{1+u^2}^{1+u^2} \frac{2 \log^2 \left( \frac{1 + x^2}{x} \right) + 4 \log \left( \frac{1 + x^2}{x} \right) \log(1 - u) + 2 \log^2(1 - u)}{ux} \, du \, dx
\]

\[
= \int_0^1 \int_{1+u^2}^{1+u^2} \frac{2 \log^2 \left( \frac{1 + x^2}{x} \right) - 4 \log \left( \frac{1 + x^2}{x} \right) \operatorname{Li}_1(u) + 2 \log^2(1 - u)}{ux} \, du \, dx
\]

\[
= \int_0^1 \frac{2 \log^2(x) \log (1 + x^2)}{x} \, dx + \int_0^1 \frac{4 \operatorname{Li}_3 \left( \frac{x^2}{1 + x^2} \right)}{x} \, dx + \int_0^1 \frac{4 \log(x) \operatorname{Li}_2 \left( \frac{x^2}{1 + x^2} \right) - 4 \operatorname{Li}_3 \left( \frac{x^2}{1 + x^2} \right)}{x} \, dx,
\]

(4.6)

(4.7)

where (4.7) follows from carrying out the inner integral of (4.6) with respect to \( u \) by using \( \int \frac{du}{u} = \log(u) \), along with the integration formula in (2.14) and the antiderivative formula (3.16) and simplifying the result with the Fundamental Theorem of Calculus. We now denote the integrals in (4.7) as:

\[
K_1 = \int_0^1 \frac{2 \log^2(x) \log (1 + x^2)}{x} \, dx
\]

(4.8)

\[
K_2 = \int_0^1 \frac{4 \operatorname{Li}_3 \left( \frac{x^2}{1 + x^2} \right)}{x} \, dx
\]

(4.9)

\[
K_3 = \int_0^1 \frac{4 \log(x) \operatorname{Li}_2 \left( \frac{x^2}{1 + x^2} \right) - 4 \operatorname{Li}_3 \left( \frac{x^2}{1 + x^2} \right)}{x} \, dx.
\]

(4.10)

We evaluate \( K_1, K_2, \) and \( K_3 \).

\( K_1 \) is the easiest to evaluate by using the definition of the polylogarithm.
Theorem 4.0.2. We have
\[ K_1 = -\frac{7\pi^4}{1440}. \] (4.11)

Proof. Substituting \( x = \sqrt{u} \) into \( 4.8 \) and observing that \( \log(1 + u) = -\text{Li}_1(-u) \), we have
\[ K_1 = \int_0^1 \frac{\log^2(u)\text{Li}_1(-u)}{4u} \, dx \] (4.12)
\[ = \int_0^1 \sum_{n \in \mathbb{N}} \frac{(-1)^n}{4n} u^{n-1} \log^2(u) \, du \] (4.13)
\[ = \sum_{n \in \mathbb{N}} \frac{(-1)^n}{4n} \int_0^1 u^{n-1} \log^2(u) \, du \] (4.14)
\[ = \sum_{n \in \mathbb{N}} \frac{(-1)^n}{2n^4} \] (4.15)
\[ = \frac{\eta(4)}{2} = -\frac{7}{16} \zeta(4) = -\frac{7\pi^4}{1440}. \] (4.16)

where \( 4.13 \) follows from rewriting the integrand in \( 4.12 \) by substituting \( z = -u \) into the definition of \( \text{Li}_1(z) \) in \( 2.10 \) and simplifying the resulting summand, and the interchanging of sum and integral in \( 4.14 \) is permitted by the Lebesgue Dominated Convergence Theorem. Finally, \( 4.15 \) follows from evaluating the integral inside the sum of \( 4.14 \) by repeated integration by parts and simplifying the resulting summand, and \( 4.16 \) follows from substituting \( k = 4 \) into the algebraic formula for \( \eta(k) \) stated in \( 2.16 \). \( \square \)

\( K_2 \) is also easy to evaluate and will require us to invoke the central binomial coefficient identities from Theorem 2.3.1.

Theorem 4.0.3. We have
\[ K_2 = \frac{7\pi^4}{216} + \frac{C(4)}{4}. \] (4.17)

Proof. Substituting \( x = \sqrt{u} \) into \( 4.9 \), we have
\[ K_2 = \int_0^1 \frac{2\text{Li}_3 \left( \frac{\sqrt{u}}{1+u} \right)}{u} \, du. \] (4.18)

Further substituting \( u \mapsto 1/u \) shows that
\[ K_2 = \int_1^\infty \frac{2\text{Li}_3 \left( \frac{\sqrt{u}}{1+u} \right)}{u} \, du. \] (4.19)
Hence, adding the representations (4.18) and (4.19) and dividing by 2, we get

\[
K_2 = \int_0^\infty \frac{\text{Li}_3 \left( \frac{x}{1+n} \right)}{u} \, du
\]

(4.20)

\[= \int_0^\infty \sum_{n \in \mathbb{N}} \frac{1}{n^2} \frac{u^2}{(1+u)^n} \, du
\]

(4.21)

\[= \sum_{n \in \mathbb{N}} \int_0^\infty \frac{1}{n^2} \frac{u^2}{(1+u)^n} \, du
\]

(4.22)

\[= \sum_{n \in \mathbb{N}} \frac{\text{B} \left( \frac{2n+1}{2} \right)}{n^2}
\]

(4.23)

\[= \sum_{n \geq 0} \frac{\pi (2n)}{4} \frac{(2n+1)^3}{4} + \sum_{n \in \mathbb{N}} \frac{1}{(2n)^4}
\]

(4.24)

\[= \pi \text{B}(4) + \frac{C(4)}{4} = \frac{7\pi^4}{216} + \frac{C(4)}{4},
\]

(4.25)

where (4.24) follows from rewriting the integrand in (4.20) by substituting \( z = \frac{x}{1+n} \) into the definition of \( \text{Li}_3(z) \) from (2.11), and the interchanging of sum and integral in (4.22) is permitted by the Monotone Convergence Theorem. Then (4.24) follows from splitting the series (4.23) into a sum of odd terms and a sum of even terms. Finally, (4.25) follows from simplifying the summands in (4.24) using the formulas for \( \text{B}(2n+1)/2, (2n+1)/2 \) from (2.35) and \( \text{B}(n,n) \) from (2.36).

\( K_3 \) is the hardest to evaluate. For this, we need an antiderivative identity whose proof requires several tedious integration by parts. We simply quote the following result from Mathematica.

**Theorem 4.0.4.** We have

\[
\int \log \left( \frac{1-z}{z} \right) \frac{\text{Li}_2(z) - 2\text{Li}_3(z)}{z^2 - z} \, dz
\]

\[= \frac{1}{4} \log^4(1-z) - \frac{1}{2} \log(z) \log^3(1-z) + \frac{1}{2} \log^2(z) \log^2(1-z) + \text{Li}_2(z)^2
\]

\[+ \frac{1}{2} \text{Li}_2(1-z) \log^2(1-z) + \frac{1}{2} \text{Li}_2(z) \log^2(1-z) + \text{Li}_2(z) \log^2(z)
\]

\[- \text{Li}_2(z) \log(z) \log(1-z) + \text{Li}_2 \left( \frac{z}{z-1} \right) \log^2(1-z) + \text{Li}_2 \left( \frac{z}{z-1} \right) \log^2(z)
\]

\[- 2\text{Li}_2 \left( \frac{z}{z-1} \right) \log(z) \log(1-z) + \text{Li}_3(1-z) \log(1-z) + 2 \text{Li}_3(z) \log(1-z)
\]

\[- \text{Li}_3(z) \log(z) + 2\text{Li}_3 \left( \frac{z}{z-1} \right) \log(1-z) - 2\text{Li}_3 \left( \frac{z}{z-1} \right) \log(z) + \text{Li}_4(1-z)
\]

\[- \text{Li}_4(z) + 2\text{Li}_4 \left( \frac{z}{z-1} \right).
\]

(4.26)

**Theorem 4.0.5.** We have

\[K_3 = -\frac{17\pi^4}{720}.
\]

(4.27)

**Proof.** Substituting \( x = \sqrt{1-z} \) into (4.10), we get

\[
K_3 = \int_0^\frac{1}{2} \log \left( \frac{\sqrt{1-z}}{z} \right) \frac{\text{Li}_2(z) - 2\text{Li}_3(z)}{z^2 - z} \, dz.
\]

(4.28)
Letting $\psi(z)$ be the expression on the right hand side of (4.26), we get by the Fundamental Theorem of Calculus that

$$K_3 = \psi\left(\frac{1}{2}\right) - \lim_{z \to 1} \psi(z) = 2\text{Li}_4(-1) + \text{Li}_2\left(\frac{1}{2}\right)^2 + \log^2(2) \text{Li}_2\left(\frac{1}{2}\right) + \frac{\log^2(2)}{4} - \text{Li}_4(1)$$

(4.29)

$$= -\frac{7\pi^4}{360} + \frac{\pi^2}{12} - \frac{\log^2(2)}{2} + \log^2(2) \left(\frac{\pi^2}{12} - \frac{\log^2(2)}{2}\right) + \frac{\log^2(2)}{4} - \frac{\pi^4}{90}$$

(4.30)

$$= -\frac{7\pi^4}{360} + \frac{\pi^4}{144} - \frac{\pi^4}{90} = -\frac{17\pi^4}{720},$$

where (4.30) follows from simplifying (4.29) using the closed formula for $\text{Li}_2(1/2)$ from (2.19).

**Remark.** There may be a connection between $K_3$ and the identity

$$\sum_{n \in \mathbb{N}} H_n^2 n^{-2} = \frac{17\pi^4}{360},$$

as studied in [17], where $H_n$ is the $n$-th harmonic number defined by

$$H_n = \sum_{m=1}^{n} \frac{1}{m}.$$

Thus, we have

$$K = K_1 + K_2 + K_3 = -\frac{7\pi^4}{1440} + \frac{7\pi^4}{216} + \frac{C(4)}{4} - \frac{17\pi^4}{720}$$

$$= \frac{17\pi^4}{4320} + \frac{K}{4},$$

which implies $K = \frac{17\pi^4}{3240}$ upon rearranging terms. Hence, we have

$$\sum_{n \in \mathbb{N}} \frac{1}{n^4(2n^2)} = \frac{17\pi^4}{3240}.$$

### 5 Evaluating Some Challenging Polylogarithmic Integrals

We now evaluate some challenging polylogarithmic integrals whose calculations which surprisingly require us to know $B(4)$ and $C(4)$.

Before we state these integrals, we give integral representations of $B(4)$ and of $C(4)$ which will appear in subsequent evaluations.

**Theorem 5.0.1.** We have

$$B(4) = \int_{0}^{1} \frac{\log^2(x)}{2\sqrt{1-x^2}} \, dx = \int_{0}^{1} \frac{-2\sin^{-1}\left(\frac{x}{2}\right)\log(x)}{x} \, dx$$

(5.1)

$$C(4) = \int_{0}^{1} \frac{8\sin^{-1}\left(\frac{x}{2}\right)\log(x)}{\sqrt{4-x^2}} \, dx = \int_{0}^{1} \frac{\text{Li}_3(x-x^2)}{x} \, dx = \int_{0}^{\infty} \frac{\text{Li}_3\left(\frac{x}{(1+x)^2}\right)}{2x} \, dx$$

(5.2)

**Proof.** We already showed that substituting $x = 2\sin(t)$ into the integral $I$ from (3.1) recovers the first integral in (5.1). Integrating by parts using $u = \log^2(x)$ and $dv = \frac{dx}{2\sqrt{1-x^2}}$ recovers the second integral in (5.1).
We already showed that substituting \( x = (u + v)/2, \ y = (u - v)/2 \) into the double integral \( K \) from (4.1) recovers the first integral and integrating with respect to \( v \) yields the first integral in (5.2). Using the definition of \( \text{Li}_3(z) \), we see the second integral in (5.2) is

\[
\int_0^1 \frac{\text{Li}_3(x - x^2)}{x} \, dx = \int_0^1 \sum_{n \in \mathbb{N}} \frac{x^n(1-x)^n}{n^3} \, dx
\]

\[
= \sum_{n \in \mathbb{N}} \int_0^1 \frac{x^{n-1}(1-x)^{n-1}}{n^3} \, dx
\]

\[
= \sum_{n \in \mathbb{N}} \frac{B(n,n)}{n^3}
\]

\[
= \frac{1}{n^4} (\frac{2n}{n}) = C(4).
\]

Then substitute \( x = \frac{1}{1+\sqrt{z}} \) in the second integral to get the third integral in (5.2).

Theorem 5.0.2. We have

\[
\int_0^1 \frac{x \sin^{-1} \left( \frac{x}{2} \right) \log(x)}{x^2 - 1} \, dx = \frac{5\pi^3}{1296}
\]

(5.3)

\[
\int_0^1 \frac{x \cos^{-1} \left( \frac{x}{2} \right) \log(x)}{x^2 - 1} \, dx = \frac{11\pi^3}{648}
\]

(5.4)

Proof. We will calculate the triple integral

\[
\int_0^1 \int_0^1 \int_0^1 \frac{x^2 y}{\sqrt{4 - x^2 \sqrt{4 - x^2 y^2}} \sqrt{4 - x^2 y^2}} \, dz \, dy \, dx
\]

(5.5)

in two ways.

Integrating (5.5) directly, we see it is equal to

\[
\int_0^1 \int_0^1 \frac{\sin^{-1} \left( \frac{x}{2} \right)}{\sqrt{4 - x^2 \sqrt{4 - x^2 y^2}}} \, dy \, dx = \int_0^1 \frac{\left( \sin^{-1} \left( \frac{x}{2} \right) \right)^2}{\sqrt{4 - x^2}} \, dx
\]

\[
= \frac{\pi^3}{1296}.
\]

On the other hand, we reverse the order of integration in (5.5) and integrate with respect to \( y \) first. We have

\[
\int_0^1 \int_0^1 \int_0^1 \frac{x^2 y}{\sqrt{4 - x^2 \sqrt{4 - x^2 y^2}} \sqrt{4 - x^2 y^2}} \, dy \, dz \, dx = \int_0^1 \int_0^1 \frac{-\log \left( \frac{x^2 \sqrt{4 - x^2 y^2} + \sqrt{4 - x^2}}{2 + 2z} \right)}{z \sqrt{4 - x^2}} \, dz \, dx.
\]

(5.6)

We carry out the inner integral on the right hand side of (5.6) using integration by parts. We let \( u = -\log \left( \frac{x^2 \sqrt{4 - x^2 y^2} + \sqrt{4 - x^2}}{2 + 2z} \right) \) and\( dv = \frac{dx}{z \sqrt{4 - x^2}} \). Then \( v = \frac{\log(x)}{\sqrt{4 - x^2}} \) and upon differentiating \( u \) with respect to \( z \), we get

\[
du = \frac{4 + x^2 z - \sqrt{4 - x^2 \sqrt{4 - x^2 z^2}}}{(z + 1)\sqrt{4 - x^2 z^2} \left( \sqrt{4 - x^2 z^2} + z \sqrt{4 - x^2} \right)} \, dz
\]

(5.7)

\[
= \frac{1 - \sqrt{4 - x^2 z^2}}{1 - z^2} \, dz,
\]

(5.8)

where (5.8) follows from multiplying top and bottom of (5.7) by the quantity \( \sqrt{4 - x^2 z^2} - z \sqrt{4 - x^2} \). As
a result, (5.6) becomes

\[
\int_0^1 \int_0^1 \frac{x^2 y}{\sqrt{4 - x^2} \sqrt{4 - x^2 y^2} \sqrt{4 - x^2 y^2 z^2}} \, dy \, dz \, dx = \int_0^1 \int_0^1 \left( \frac{1}{\sqrt{4 - x^2}} - \frac{1}{\sqrt{4 - x^2 y^2 z^2}} \right) \log(z) \frac{dz}{z^2 - 1} \, dx
\]

\[
= \int_0^1 \int_0^1 \left( \frac{1}{\sqrt{4 - x^2}} - \frac{1}{\sqrt{4 - x^2 y^2 z^2}} \right) \log(z) \frac{dx}{z^2 - 1}
\]

\[
= \int_0^1 \left( \pi z - 6 \sin^{-1} \left( \frac{z}{2} \right) \right) \log(z) \frac{dz}{6z(z^2 - 1)}
\]

\[
= \int_0^1 \frac{\pi \log(z)}{6(z^2 - 1)} \, dz - \int_0^1 \frac{\log(z) \sin^{-1} \left( \frac{z}{2} \right)}{z(z^2 - 1)} \, dz
\]

\[
= \int_0^1 \frac{\pi \log(z)}{6(z^2 - 1)} \, dz + \int_0^1 \frac{\log(z) \sin^{-1} \left( \frac{z}{2} \right)}{z} \, dz
\]

\[
- \int_0^1 \frac{z \sin^{-1} \left( \frac{z}{2} \right) \log(z)}{z^2 - 1} \, dz
\]

\[
= \int_0^1 \frac{\pi \log(z)}{6(z^2 - 1)} \, dz - \frac{B(4)}{2} - \int_0^1 \frac{z \sin^{-1} \left( \frac{z}{2} \right) \log(z)}{z^2 - 1} \, dz,
\]

(5.9)

(5.10)

(5.11)

where (5.10) follows from using partial fractions in the second integral term of (5.9).

Now, the first integral term of (5.11) is

\[
\int_0^1 \frac{\pi \log(z)}{6(z^2 - 1)} \, dz = -\int_0^1 \sum_{n \geq 0} \frac{\pi}{6} z^{2n} \log(z) \, dz
\]

\[
= -\sum_{n \geq 0} \int_0^1 \frac{\pi}{6} z^{2n} \log(z) \, dz
\]

\[
= \sum_{n \geq 0} \frac{\pi}{6(2n + 1)^2}
\]

\[
= \frac{\pi}{6} \left( \zeta(2) - \eta(2) \right) = \frac{\pi^3}{48}.
\]

Equating the two representations we obtained for (5.5) and rearranging terms, we get that

\[
\int_0^1 \frac{z \sin^{-1} \left( \frac{z}{2} \right) \log(z)}{z^2 - 1} \, dz = \frac{\pi^3}{48} - \frac{\pi^3}{1296} - \frac{B(4)}{2} = \frac{\pi^3}{48} - \frac{\pi^3}{1296} - \frac{7\pi^3}{432} = \frac{5\pi^3}{1296},
\]

which proves (5.3).

Recalling \( \sin^{-1}(z/2) + \cos^{-1}(z/2) = \pi/2 \), we have

\[
\int_0^1 \frac{z \cos^{-1} \left( \frac{z}{2} \right) \log(z)}{z^2 - 1} \, dz = \int_0^1 \frac{\zeta(z) \log(z)}{z^2 - 1} \, dz - \int_0^1 \frac{z \sin^{-1} \left( \frac{z}{2} \right) \log(z)}{z^2 - 1} \, dz
\]

\[
= \int_0^1 \frac{\pi \log(u)}{4(u - 1)} \, du - \int_0^1 \frac{z \sin^{-1} \left( \frac{z}{2} \right) \log(z)}{z^2 - 1} \, dz
\]

\[
= \int_0^1 \frac{\pi \text{Li}(1 - u)}{4(1 - u)} \, du - \int_0^1 \frac{z \sin^{-1} \left( \frac{z}{2} \right) \log(z)}{z^2 - 1} \, dz
\]

\[
= \frac{\pi^3}{48} - \frac{5\pi^3}{1296} = \frac{11\pi^3}{648},
\]

where we obtained the first integral term of (5.13) by substituting \( z = \sqrt{u} \) into the first integral term of (5.12). This proves (5.4).
The following theorems require us to know \( C(4) \).

**Theorem 5.0.3.** We have

\[
\int_0^1 \frac{4 \text{Li}_3 \left( -\frac{x}{1+x^2} \right)}{x} \, dx = \frac{403 \pi^4}{12960}. \tag{5.14}
\]

**Proof.** Mimicking the same exact argument we used to obtain the alternative integral representation of \( K_2 \) from (4.10), namely (4.20), we get

\[
\int_0^1 \frac{4 \text{Li}_3 \left( -\frac{x}{1+x^2} \right)}{x} \, dx = \int_0^\infty \frac{\text{Li}_3 \left( -\frac{u}{1+u^2} \right)}{u} \, du = C(4) - K_2
\]

\[
= -\pi B(4) + \frac{C(4)}{4}
\]

\[
= \frac{7 \pi^4}{216} + \frac{17 \pi^4}{12960} = \frac{403 \pi^4}{12960},
\]

where (5.15) follows from substituting \( z = \frac{u}{1+u} \) into the formula for \( \text{Li}_3(-z) \) from (2.15).

**Theorem 5.0.4.** We have

\[
\int_0^1 \frac{\log^2(1 - x) \log(1 - x + x^2) + 2 \log(1 - x) \text{Li}_2(x - x^2)}{x} \, dx = -\frac{2 \pi^4}{243}. \tag{5.16}
\]

**Proof.** We will calculate the double integral

\[
\int_0^1 \int_0^{1-x} \frac{\log^2(x)}{1-xy} \, dy \, dx
\]

in two ways.

Carrying out the integration with respect to \( y \), we see (5.17) becomes

\[
\int_0^1 \int_0^{1-x} \frac{\log^2(x)}{1-xy} \, dy \, dx = \int_0^1 \frac{\log^2(x) \log(1 - x + x^2)}{x} \, dx
\]

\[
= \int_0^1 \frac{\log^2(x) \log(1 + x^3)}{x} \, dx + \int_0^1 \frac{\log^2(x) \log(1 + x)}{x} \, dx
\]

\[
= \int_0^1 -\frac{8 \log^2(u) \log(1 + u^2)}{27u} \, du + \int_0^1 \frac{8 \log^2(u) \log(1 + u^2)}{u} \, du
\]

\[
= -\frac{208}{54} K_1 = \frac{91 \pi^4}{4860},
\]

where the first integral in (5.19) follows from substituting \( x = u^{2/3} \) in the first integral on the right hand side of (5.18) and the second integral in (5.19) follows from substituting \( x = u^{1/3} \) in the second integral on the right hand side of (5.18).

On the other hand, reversing the order of integration in (5.17), we get (5.17) is equal to

\[
\int_0^1 \int_0^{1-y} \frac{\log^2(x)}{1-xy} \, dx \, dy = \int_0^1 \int_1^{1+y+x^2} \frac{\log^2 \left( \frac{1+y}{t} \right)}{ty} \, dt \, dy
\]

\[
= \int_0^1 \int_1^{1+y+x^2} \frac{\log^2(1-t) - 2 \log(1-t) \log(y) + \log^2(y)}{ty} \, dt \, dy
\]

\[
= \int_0^1 \frac{\log^2(1-y) \log(1-y + y^2) - 2 \log(1-y) \text{Li}_2(y - y^2) + 2 \text{Li}_3(y - y^2)}{y} \, dy
\]

\[
= \int_0^1 \frac{\log^2(1-y) \log(1-y + y^2) - 2 \log(1-y) \text{Li}_2(y - y^2)}{y} \, dy + 2C(4),
\]

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where we made the substitution $x = (1 - t)/y$ into the left hand side of (5.20) to get the right hand side (5.20), and (5.22) follows from evaluating the inner integral of (5.21) with respect to $t$ using (2.14) and (3.16).

Hence, equating these two representations of the double integral (5.17) and rearranging terms, we get

$$
\int_0^1 \frac{\log^2(1 - y) \log(1 - y + y^2) + 2 \log(1 - y) \text{Li}_2(y - y^2)}{y} \, dy = 2C(4) - \frac{91\pi^4}{4860}
$$

$$
= \frac{17\pi^4}{1620} - \frac{91\pi^4}{4860} = -\frac{2\pi^4}{243}.
$$

6 A Remark About Apostol’s $\zeta(2)$ Proof

We conclude this paper by highlighting an insight in Apostol’s double integration proof [18] of $\zeta(2) = \pi^2/6$ and comment on its extension to showing $\zeta(4) = \pi^4/90$.

Apostol uses the double integral

$$
L_2 = \int_0^1 \int_0^1 \frac{1}{1 - xy} \, dy \, dx.
$$

(6.1)

On one hand, converting the integrand into a geometric and interchanging sum and double integration shows that $L_2$ is equal to the series $\zeta(2)$. On the other hand, making the change of variables $x = (u + v)/2$, $y = (u - v)/2$ shows that (6.1) becomes

$$
L_2 = \int_0^1 \int_0^1 \frac{2}{4 - u^2 + v^2} \, dv \, du
$$

$$
= \int_0^1 \int_0^u \frac{2}{4 - u^2 + v^2} \, dv \, du + \int_0^1 \int_{u-2}^{2-u} \frac{2}{4 - u^2 + v^2} \, dv \, du
$$

$$
= \int_0^1 \frac{4 \tan^{-1} \left( \frac{\sqrt{4 - u}}{\sqrt{4 - u^2}} \right)}{\sqrt{4 - u^2}} \, du + \int_1^2 \frac{4 \tan^{-1} \left( \frac{\sqrt{4 - u^2}}{\sqrt{4 - u^2}} \right)}{\sqrt{4 - u^2}} \, du
$$

$$
= \int_0^1 \frac{4 \sin^{-1} \left( \frac{u}{2} \right)}{\sqrt{4 - u^2}} \, du + \int_1^2 \frac{4 \tan^{-1} \left( \frac{\sqrt{4 - u^2}}{\sqrt{4 - u^2}} \right)}{\sqrt{4 - u^2}} \, du
$$

$$
= \pi^2 + \pi^2 - \pi^2
$$

$$
= \frac{\pi^2}{18}.
$$

We note that the first integral in (6.2) is a central binomial series in disguise. Recalling our arcsine binomial series expansion from (2.38), we can follow the usual steps of rewriting the integrand of the first integral term of (6.2) using (2.38), interchanging sum and integral, and integrating term-by-term to obtain the identity

$$
\sum_{n \in \mathbb{N}} \frac{1}{n^2 \binom{n}{2}} = \frac{\pi^2}{18}.
$$

Something similar happens if we mimic Apostol’s method on the quadruple integral

$$
L_4 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{1}{1 - xyzw} \, dx \, dy \, dz \, dw.
$$

(6.3)

It can be seen that $L_4 = \zeta(4)$ by rewriting the integrand as a geometric and interchanging sum and
we repeat our proof of double integral term in (6.6). But neither are elementary integrals to evaluate this time around. If $B$ second integral in (6.5).

Thus, if we are to prove $\zeta(4) = \pi^4/90$ this way, we would need to know $C(4)$ and the value of the double integral term in (6.6). But neither are elementary integrals to evaluate this time around. If we repeat our proof of $C(4) = 17\pi^2/3240$ without assuming that $\zeta(4) = \pi^4/90$ but still assuming that $B(4) = 7\pi^3/216$, we will get

$$C(4) = \frac{4}{3} \pi B(4) - \frac{41}{12} \zeta(4) = \frac{7\pi^4}{162} - \frac{41}{12} \zeta(4).$$

But we still need to show that

$$-\int_1^2 \int_0^{16 \tan^{-1} \left( \frac{(2-u)\sqrt{1-s^2}}{\sqrt{1-s^2}+u^2} \right)} \frac{\log(s)}{\sqrt{1-s^2}} ds du = \frac{19\pi^4}{3240} = \frac{19}{36} \zeta(4).$$

At this time of writing, we do not know a proof of this without assuming $C(4) = 17\pi^4/3240$ and $\zeta(4) = \pi^4/90$.

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