Calculation of the exact Lyapunov exponent of affine Boole transformations using Feynman’s trick

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To the memory of Gabriella Vas

Abstract

We show an elementary way for calculation of the exact Lyapunov exponent of affine Boole transformations using a method for evaluation of integrals via differentiating under the integral sign, a trick originated to Richard Feynman.

1 Introduction and results

The theory and applications of Lyapunov exponents are important and extensively growing areas in mathematics and physics as well. Roughly speaking, Lyapunov exponent of a dynamical system can be considered as a measure of the degree of sensitive dependence on the initial conditions of the dynamical system in question. For several precise mathematical results on the relationships of Lyapunov exponents and sensitive dependence on initial conditions in the context of mappings of the unit interval into itself, see, e.g., Koçak and Palmer [12]. Lyapunov exponents play an important role in the theory of random matrices and random maps; in linear stochastic systems and stability theory; in random Schrödinger operators and wave propagation in random media; in nonlinear stochastic systems and stochastic flows on manifolds; and in chaos and phase transitions, see the lecture note of Arnold and Wihstutz [1]. The recent book of Barreira [2] contains applications of Lyapunov exponents in hyperbolicity, ergodic theory, and multifractal analysis as well.

We point out the fact that, in general, one can find it difficult to obtain an explicit formula for Lyapunov exponent of a given dynamical system, and hence in practice numerical procedures are applied. For example, very recently Miranda-Filho et al. [14] have numerically approximated the so-called largest Lyapunov exponent for the Vicsek model that describes the dynamics of a flock of self-propelled particles.

In this paper we focus on affine Boole transformations, and hence below we will recall the existing and corresponding literature only on the calculation of Lyapunov exponents of these
mappings. In the present paper, using Birkhoff’s ergodic theorem for the Newton iteration for the mapping \( x \mapsto (x-a)^2+b^2 \) (where \( a \) is a real number and \( b \) is a positive real number) and a method for evaluation of integrals via differentiating under the integral sign (a trick originated to Richard Feynman), we calculate the Lyapunov exponent of the corresponding affine Boole transformation \( x \mapsto \frac{b}{2} \left( \frac{x-a}{b} - \frac{1}{x-a} \right) + a \), which is known to be \( \ln(2) \). The result itself is widely known, but our proof technique seems to be new, that’s why we decided to write the present note.

Let \( \mathbb{Z}_+ \), \( \mathbb{N} \), \( \mathbb{R} \), \( \mathbb{R}_+ \) and \( \mathbb{C} \) denote the set of non-negative integers, positive integers, real numbers, non-negative real numbers, and complex numbers, respectively. The Borel \( \sigma \)-algebra on \( \mathbb{R} \) and \( (0, \infty) \) is denoted by \( \mathcal{B}(\mathbb{R}) \) and \( \mathcal{B}((0, \infty)) \), respectively. All the random variables are defined on a probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \). Equality in distribution is denoted by \( \overset{\text{D}}{=} \). A random variable \( \xi \) is said to have Cauchy distribution with parameter \((a, b)\), where \( a \in \mathbb{R} \) and \( b > 0 \), if \( \xi \) has a density function

\[
\frac{1}{b \pi \left((\frac{x-a}{b})^2 + 1\right)}, \quad x \in \mathbb{R};
\]

and the distribution of \( \xi \) is denoted by \( \mathbb{P}_{C(a,b)} \). In case of \( a = 0 \) and \( b = 1 \), \( \mathbb{P}_{C(a,b)} \) is called a standard Cauchy distribution. Note that if \( \xi \) has a standard Cauchy distribution, then \( a \xi + b \) has distribution \( \mathbb{P}_{C(a,b)} \) for any \( a \in \mathbb{R} \) and \( b > 0 \).

Extensions of Newton method for finding complex roots of polynomials has a long history, it goes back at least to the work of Cayley \[4, 5\]. Let \( a \in \mathbb{R} \) and \( b > 0 \). The Newton iteration for the function \( f: \mathbb{R} \to \mathbb{R}, \ f(x) := (x-a)^2 + b^2, \ x \in \mathbb{R}, \) takes the form

\[
(1.1) \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n-a)^2 + b^2}{2(x_n-a)} = \frac{b}{2} \left( \frac{x_n-a}{b} - \frac{1}{x_n-a} \right) + a
\]

for those \( n \in \mathbb{Z}_+ \) that satisfy \( x_n \neq a \). The two roots of the equation \( f(z) = 0, \ z \in \mathbb{C} \), are \( a \pm ib \), and hence the perpendicular bisector of these two roots is the real axis. By Cayley’s theorem, the Newton iteration for \( f \) started from a complex number having positive imaginary part converges to \( a + ib \), while started from a complex number having negative real part converges to \( a - ib \). Further, the Newton iteration for \( f \) started from a real number do not converge (see, e.g., Devaney \[2\] page 170), and below we recall an ergodic result (see Proposition \[11\]) which roughly speaking states that the arithmetic averages of appropriate functions of the Newton iterates in question converge.

For any \( a \in \mathbb{R} \) and \( b > 0 \), let us introduce the function \( \varphi_{a,b}: \mathbb{R} \to \mathbb{R}, \)

\[
(1.2) \quad \varphi_{a,b}(x) := \begin{cases} \frac{b}{2} \left( \frac{x-a}{b} - \frac{1}{x-a} \right) + a = \frac{x}{2} + \frac{a}{2} - \frac{1}{2} \frac{x^2}{x-a} & \text{if } x \neq a, \\ a & \text{if } x = a. \end{cases}
\]

Note that \( \varphi_{a,b} \) is Borel measurable, \( \varphi_{a,b}(x) = b \varphi_{0,1}(\frac{x-a}{b}) + a, \ x \in \mathbb{R}, \) and one calls \( \varphi_{a,b} \) an affine Boole transformation. In the special case \( (a, b) = (0, 1) \), the map \( \varphi_{0,1} \) is called Boole
transformation. The origin of the notion of Boole transformation is the following formula due to G. Boole [3]:
\[
\int_{\mathbb{R}} f(x) \, dx = \int_{\mathbb{R}} f\left(x - \frac{1}{x}\right) \, dx
\]
for any integrable function \( f : \mathbb{R} \to \mathbb{R} \).

It is known that \( \varphi_{0,1} \) preserves the standard Cauchy distribution \( \mathbb{P}_{C(0,1)} \), see, e.g., Pitman and Williams [10] or Chin et al. [6]. It is also known that the Boole transformation \( \varphi_{0,1} \) is ergodic with respect to the standard Cauchy distribution, see, e.g., Prykarpatsky and Feldman [17] Theorem 2.2], Lee and Suriajaya [13 Theorem 3.1] or Chin et al. [6] Lemma 9.

We also have that for each \( a \in \mathbb{R} \) and \( b > 0 \), the map \( \varphi_{a,b} \) preserves \( \mathbb{P}_{C(a,b)} \), i.e., the Cauchy distribution with parameter \((a,b)\), and is ergodic with respect to \( \mathbb{P}_{C(a,b)} \). Indeed, since \( \varphi_{0,1} \) preserves \( \mathbb{P}_{C(0,1)} \), for each \( a \in \mathbb{R} \) and \( b > 0 \), we have \( \varphi_{a,b} \) preserves the Cauchy distribution with parameter \((a,b)\), i.e., \( \varphi_{a,b}(b\xi + a) \overset{D}{=} b\xi + a \), where \( \xi \) is a random variable having distribution \( \mathbb{P}_{C(0,1)} \). One can also easily check that \( \varphi_{a,b} \) is ergodic with respect to the Cauchy distribution with parameter \((a,b)\). Namely, we have to check that \( \mathbb{P}_{C(a,b)}(A) = 0 \) or \( \mathbb{P}_{C(a,b)}(\mathbb{R} \setminus A) = 0 \) (equivalently \( \mathbb{P}_{C(a,b)}(A) \in \{0,1\} \)) for any \( A \in \mathcal{B}(\mathbb{R}) \) with \( \varphi_{a,b}^{-1}(A) = A \). Note that
\[
\varphi_{a,b}^{-1}(A) = \{x \in \mathbb{R} : \varphi_{a,b}(x) \in A\} = \left\{x \in \mathbb{R} : b\varphi_{0,1}\left(\frac{x-a}{b}\right) + a \in A\right\}
\]
\[
= \left\{x \in \mathbb{R} : \varphi_{0,1}\left(\frac{x-a}{b}\right) \in \frac{A-a}{b}\right\}
\]
\[
= b\left\{\frac{x-a}{b} \in \mathbb{R} : \varphi_{0,1}\left(\frac{x-a}{b}\right) \in \frac{A-a}{b}\right\} + a
\]
\[
= b\left\{y \in \mathbb{R} : \varphi_{0,1}(y) \in \frac{A-a}{b}\right\} + a = b\varphi_{0,1}^{-1}\left(\frac{A-a}{b}\right) + a.
\]
Since \( \varphi_{a,b}^{-1}(A) = A \), we have \( \varphi_{0,1}^{-1}\left(\frac{A-a}{b}\right) = \frac{A-a}{b} \). Hence, using that \( \varphi_{0,1} \) is ergodic with respect to the standard Cauchy distribution \( \mathbb{P}_{C(0,1)} \), we have \( \mathbb{P}_{C(0,1)}\left(\frac{A-a}{b}\right) \in \{0,1\} \), i.e., if \( \xi \) has standard Cauchy distribution, then \( \mathbb{P}(\xi \in \frac{A-a}{b}) \in \{0,1\} \). Consequently, using that \( b\xi + a \) has Cauchy distribution with parameter \((a,b)\), we have \( \mathbb{P}(b\xi + a \in A) = \mathbb{P}_{C(a,b)}(A) \in \{0,1\} \), as desired.

Using that \( \varphi_{a,b} \) is measure preserving and ergodic with respect to the Cauchy distribution \( \mathbb{P}_{C(a,b)} \), by Birkhoff’s ergodic theorem (see, e.g., Durrett [5] Theorem 6.2.1] or Einsiedler and Ward [9] Theorem 2.30]), we have the following well-known result.

1.1 Proposition. For any \( a \in \mathbb{R} \) and \( b > 0 \), let us consider the affine Boole transformation \( \varphi_{a,b} \) given in \([1,2]\). Then for Lebesgue almost every \( x \in \mathbb{R} \) and for any Borel measurable function \( f : \mathbb{R} \to \mathbb{R} \) such that \( \int_{\mathbb{R}} |f(u)|/((u-a)^2/b^2 + 1) \, du < \infty \), we have
\[
\frac{1}{n} \sum_{k=0}^{n-1} f(\varphi_{a,b}^{(k)}(x)) \to \int_{\mathbb{R}} \frac{f(u)}{b\pi ((u-a)/b)^2 + 1} \, du = \mathbb{E}(f(\xi)) \quad \text{as} \quad n \to \infty,
\]
where $\varphi_{a,b}^{(k)}$ denotes the $k$-fold iteration of $\varphi_{a,b}$ for $k \in \mathbb{Z}_+$ with the convention $\varphi_{a,b}^{(0)}(x) := x$, $x \in \mathbb{R}$, and $\xi$ is a random variable having Cauchy distribution with parameter $(a, b)$.

In the next remark, we recall some existing literature related to Proposition 1.1. We point out that Proposition 1.1 is well-known and authors reproved it several times independently of each other.

1.2 Remark. (i). By Prykarpatsky and Feldman [17, Lemma 2.1 and Theorem 2.2], for any $\alpha \in \mathbb{R}$ and $\beta > 0$, the mapping $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\varphi(x) := \begin{cases} \frac{x}{2} + \alpha - \frac{\beta}{x - 2\alpha} & \text{if } x \neq 2\alpha, \\ 2\alpha & \text{if } x = 2\alpha, \end{cases}$$

is measure preserving and ergodic with respect to the Cauchy distribution $\mathbb{P}_{C(2\alpha, \sqrt{2} \beta)}$. One can easily check that $\varphi = \varphi_{2\alpha, \sqrt{2} \beta}$, so Proposition 1.1 can be applied to $\varphi$.

(ii). Proposition 1.1 is a special case of Theorem 4 in Ishitani and Ishitani [10]. In fact, Ishitani and Ishitani [10, Example 1] considered a generalized Boole transformation of the form $\mathbb{R} \setminus \{0\} \ni x \mapsto \alpha x - \frac{\beta}{x}$, where $\alpha \in (0, 1)$ and $\beta > 0$, and derived an analogue of Proposition 1.1.

(iii). Proposition 1.1 is a special case of the Theorem (Basic Ergodicity) in Umeno [18, page 165]. In fact, Umeno [18] proved similar results for a branch of transformations defined by the addition theorems for the tangent and cotangent functions such as for $\mathbb{R} \setminus \{\pm 1\} \ni x \mapsto 2x/(1 - x^2)$ or $\mathbb{R} \setminus \{\pm 1/\sqrt{3}\} \ni x \mapsto (3x - x^2)/(1 - 3x^2)$.

(iv). Proposition 1.1 is the same as Corollary 3.2 in Lee and Suriajaya [13]. Indeed, for $\alpha > 0$ and $\beta \in \mathbb{R}$, they consider the transformation $T_{\alpha, \beta} : \mathbb{R} \to \mathbb{R}$,

$$T_{\alpha, \beta}(x) := \begin{cases} \frac{\alpha}{x} \left(\frac{x + \beta}{\alpha} - \frac{\alpha}{x - \beta}\right) = \frac{x}{2} + \frac{\beta}{2} - \frac{\alpha}{x - \beta} & \text{if } x \neq \beta, \\ \beta & \text{if } x = \beta, \end{cases}$$

and hence $T_{\alpha, \beta} = \varphi_{\beta, \alpha}$.

(v). By choosing $a = 0$ and $b = 1$ in Proposition 1.1 we get back Proposition 6 in Chin et al. [6].

In Appendix B we give some applications of Proposition 1.1.

Given a function $g : \mathbb{R} \to \mathbb{R}$, for each $k \in \mathbb{Z}_+$, let $g^{(k)}$ be the $k$-fold iteration of $g$ with the convention $g^{(0)}(x) := x$, $x \in \mathbb{R}$.

1.3 Definition. Let $g : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable Lebesgue almost everywhere. Let $x_0 \in \mathbb{R}$ be such that $g'(x_k)$ is well-defined and nonzero for each $k \in \mathbb{Z}_+$, where $x_k := g^{(k)}(x_0)$, $k \in \mathbb{Z}_+$, and $g'(x_k)$ denotes the derivative of $g$ at $x_k$. Then the Lyapunov exponent of the orbit $(g^{(k)}(x_0))_{k \in \mathbb{Z}_+}$ is defined by

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln(|g'(x_k)|),$$

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provided that the limit exists.

Note that, under the conditions of Definition 1.3, if the limit \( \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln(|g'(x_k)|) \) exists, then, for each \( m \in \mathbb{N} \), we have the limit \( \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln(|g'(x_{k+m})|) \) exists as well and the two limits are equal. This can be interpreted in a way that the Lyapunov exponent is indeed a quantity corresponding to the orbit \((g^{(k)}(x_0))_{k \in \mathbb{Z}^+}\).

The next theorem states that the Lyapunov exponent of the affine Boole transformation \( \varphi_{a,b} \) (defined in (1.2)) is \( \ln(2) \), which follows, e.g., from Umeno and Okubo [19, Theorem 2].

**1.4 Theorem.** For each \( a \in \mathbb{R} \) and \( b > 0 \), the Lyapunov exponent of the affine Boole transformation \( \varphi_{a,b} \) defined in (1.2) is \( \ln(2) \), more precisely, we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln(|\varphi'_{a,b}(x_k)|) = \ln(2)
\]

for Lebesgue almost every \( x_0 \in \mathbb{R} \), where \( x_k := \varphi_{a,b}^{(k)}(x_0) \), \( k \in \mathbb{Z}^+ \), and \( \varphi_{a,b}^{(k)} \) denotes the \( k \)-fold iteration of \( \varphi_{a,b} \) for \( k \in \mathbb{Z}^+ \) with the convention \( \varphi_{a,b}^{(0)}(x) := x \), \( x \in \mathbb{R} \).

In the present paper we give an elementary and alternative proof of Theorem 1.4 compared to the existing ones available in the literature. Our proof is based on Proposition 1.1 and we calculate the corresponding integral appearing in Proposition 1.1 using a trick originated to Richard Feynman, namely, via differentiating under the integral sign. For a good reference on this method of calculating integrals, see Nahin [15, Chapter 3]. For historical fidelity we note that Umeno and Okubo [19, Theorem 2] calculated the Lyapunov exponent of a generalized Boole transformation \( \mathbb{R} \setminus \{0\} \ni x \mapsto \alpha x - \frac{\beta}{x} \), where \( \alpha \in (0,1) \) and \( \beta > 0 \) using Proposition 1.1. However, their proof technique is completely different, they used complex analysis for the calculation of the corresponding integral appearing in Proposition 1.1.

The remaining part of the paper is structured as follows. Section 2 contains a proof of Theorem 1.4 using Feynman’s trick mentioned earlier. In Appendix A we recall results on the continuity and differentiability of parametric integrals that we use in the proof of Theorem 1.4. In Appendix B we give some further applications of Proposition 1.1.

## 2 Proof of Theorem 1.4 using Feynman’s trick

First, we prove Theorem 1.4 in case of \((a,b) = (0,1)\). Note that the function \( \varphi_{0,1} \) is differentiable on \( \mathbb{R} \setminus \{0\} \), but it is not differentiable at 0 following from

\[
\lim_{x \to 0} \frac{\varphi_{0,1}(x) - \varphi_{0,1}(0)}{x - 0} = \lim_{x \to 0} \frac{1}{2} \left( \frac{x + \frac{1}{x}}{x - 0} \right) = \lim_{x \to 0} \frac{1}{2} \left( \frac{1}{x^2} \right) = -\infty.
\]

However, we can check that the sequence \((\varphi_{0,1}^{(k)}(x_k))_{k \in \mathbb{Z}^+} = (\varphi_{0,1}^{(k)}(\varphi_{0,1}^{(k)}(x_0)))_{k \in \mathbb{Z}^+}\) is well-defined for each \( x_0 \in \mathbb{R} \setminus N \), where \( N \) is a countable subset of \( \mathbb{R} \). For this it is enough to check
that the set

$$\{ x_0 \in \mathbb{R} : \varphi_{0,1}^{(k)}(x_0) = 0 \text{ for some } k \in \mathbb{Z}_+ \}$$

is countable. This set can be written in the form \( \bigcup_{k=1}^{\infty} A_k \), where

$$A_k := \{ x_0 \in \mathbb{R} : \varphi_{0,1}^{(\ell)}(x_0) = 0 \text{ for some } \ell \in \{0, 1, \ldots, k\} \}, \quad k \in \mathbb{N}.$$ 

Using that the union of countably many finite sets is countable, it is enough to verify that \( A_k \) is finite for each \( k \in \mathbb{N} \). We prove it by induction. We have \( \varphi_{0,1}^{(0)}(x) = 0 \) if and only if \( x = 0 \). Further, \( \varphi_{0,1}(x) = 0 \) if and only if \( x = 0 \) or \( \frac{1}{2} (x - \frac{1}{x}) = 0 \), \( x \neq 0 \), which holds if and only if \( x = \pm 1 \), and hence \( A_1 = \{-1, 0, 1\} \) being a finite set. Suppose that the sets \( A_1, \ldots, A_k \) are finite. Then \( A_{k+1} = A_k \cup \{ x_0 \in \mathbb{R} : \varphi_{0,1}^{(k+1)}(x_0) = 0 \} \), and the function

$$\mathbb{R} \setminus \bigcup_{\ell=1}^{k} A_\ell \ni x \mapsto \varphi_{0,1}^{(k+1)}(x) = \frac{1}{2} \left( \varphi_{0,1}^{(k)}(x) - \frac{1}{\varphi_{0,1}^{(k)}(x)} \right)$$

is a well-defined rational function such that its numerator is a polynomial of degree \( 2^{k+1} \) and its denominator is a polynomial of degree \( 2^{k+1} - 1 \). Consequently, the equation \( \varphi_{0,1}^{(k+1)}(x) = 0 \), \( x \in \mathbb{R} \setminus \bigcup_{\ell=1}^{k} A_\ell \), has at most \( 2^{k+1} \) (i.e., finitely many) solutions due to the fundamental theorem of algebra. Hence we have \( A_{k+1} \) is finite, as desired.

Let us apply Proposition 1.1 with \( a = 0, \ b = 1 \), and the Borel measurable function \( f : \mathbb{R} \to \mathbb{R} \),

$$f(x) := \begin{cases} \ln(|\varphi'_{0,1}(x)|) = \ln \left( \frac{1}{2} \left( 1 + \frac{1}{x^2} \right) \right) & \text{if } x \in \mathbb{R} \setminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases}$$

Note that we cannot define \( f \) to be \( \ln(|\varphi'_{0,1}(x)|) \) for all \( x \in \mathbb{R} \), since, as we noted, the function \( \varphi_{0,1} \) is not differentiable at \( 0 \). Using that

$$\int_{\mathbb{R}} |f(x)| \pi(1 + x^2) \, dx = \int_{(-\infty,0)} \frac{|\ln \left( \frac{1}{2} \left( 1 + \frac{1}{x^2} \right) \right)|}{\pi(1 + x^2)} \, dx + \int_{(0,\infty)} \frac{|\ln \left( \frac{1}{2} \left( 1 + \frac{1}{x^2} \right) \right)|}{\pi(1 + x^2)} \, dx$$

$$= 2 \int_{(0,\infty)} \frac{|\ln \left( \frac{1}{2} \left( 1 + \frac{1}{x^2} \right) \right)|}{\pi(1 + x^2)} \, dx$$

$$\leq 2 \ln(2) \int_{(0,\infty)} \frac{1}{\pi(1 + x^2)} \, dx + 2 \int_{(0,\infty)} \frac{\ln \left( 1 + \frac{1}{x^2} \right)}{\pi(1 + x^2)} \, dx$$

$$= \ln(2) + 2 \int_{(0,\infty)} \frac{\ln \left( 1 + \frac{1}{x^2} \right)}{\pi(1 + x^2)} \, dx,$$

in order to have right to apply Proposition 1.1 we need to check that

$$\int_{(0,\infty)} \frac{\ln \left( 1 + \frac{1}{x^2} \right)}{\pi(1 + x^2)} \, dx < \infty.$$
We calculate the integral in (2.3) via differentiating under the integral sign, a trick originated to Richard Feynman. Let $G : [0, 1] \to \mathbb{R}$ be given by

$$G(t) := \int_{(0, \infty)} \frac{\ln \left(1 + \frac{t}{x^2}\right)}{\pi(1 + x^2)} \, dx, \quad t \in [0, 1].$$

Then we need to calculate $G(1)$, and note that $G(0) = 0$. Let $\varepsilon \in (0, 1)$ be arbitrary, and let us apply Theorem A.2 with the following choices

$$I := (\varepsilon, 1), \quad X := (0, \infty), \quad A := B((0, \infty)), \quad \mu := \text{Lebesgue measure on } A,$$

(2.4)

$$g(x, t) := \frac{\ln \left(1 + \frac{t}{x^2}\right)}{\pi(1 + x^2)}, \quad x \in (0, \infty), \quad t \in (\varepsilon, 1).$$

Next we check the conditions of Theorem A.2. For any $t \in (\varepsilon, 1)$, by partial integration,

$$0 \leq \int_{(0, \infty)} \frac{\ln \left(1 + \frac{t}{x^2}\right)}{\pi(1 + x^2)} \, dx \leq \frac{1}{\pi} \int_{(0, \infty)} \ln \left(1 + \frac{t}{x^2}\right) \, dx$$

$$= \lim_{x \to \infty} \frac{x}{\pi} \ln \left(1 + \frac{t}{x^2}\right) - \lim_{x \to 0} \frac{x}{\pi} \ln \left(1 + \frac{t}{x^2}\right) + 2t \int_{(0, \infty)} \frac{1}{\pi(x^2 + t)} \, dx$$

$$= 2 \int_{(0, \infty)} \frac{1}{\pi((x/\sqrt{t})^2 + 1)} \, dx = 2\sqrt{t} \int_{(0, \infty)} \frac{1}{\pi(y^2 + 1)} \, dy = \sqrt{t} < \infty,$$

since, by L'Hospital's rule,

$$\lim_{x \to \infty} x \ln \left(1 + \frac{t}{x^2}\right) = \lim_{x \to \infty} \frac{\ln \left(1 + \frac{t}{x^2}\right)}{\frac{1}{x}} = \lim_{x \to \infty} \frac{2t}{x + \frac{t}{x}} = 0$$

and

$$\lim_{x \to 0} x \ln \left(1 + \frac{t}{x^2}\right) = \lim_{x \to 0} \frac{2t}{x + \frac{t}{x}} = 0.$$

Hence condition (i) of Theorem A.2 is satisfied.

For all $x \in (0, \infty)$, the map $(\varepsilon, 1) \ni t \mapsto -\frac{\ln(1 + \frac{t}{x^2})}{\pi(1 + x^2)}$ is differentiable with derivative

$$(\varepsilon, 1) \ni t \mapsto \frac{1}{\pi(1 + x^2)(t + x^2)} = \partial_2 g(x, t),$$

and hence condition (ii) of Theorem A.2 is satisfied.

For all $t \in (\varepsilon, 1)$, we have

$$|\partial_2 g(x, t)| \leq \frac{1}{\pi(1 + x^2)t} \leq \frac{1}{\pi \varepsilon(1 + x^2)}, \quad x \in (0, \infty),$$

where $\frac{1}{\pi \varepsilon(1 + x^2)}, \quad x \in (0, \infty)$, is integrable on $(0, \infty)$ with respect to the Lebesgue measure, so condition (iii) of Theorem A.2 is satisfied.
Thus we can apply Theorem A.2 and we have

\[ G'(t) = \int_{(0,\infty)} \frac{1}{\pi(1 + x^2)(t + x^2)} \, dx, \quad t \in (\varepsilon, 1). \]

Using that

\[ \frac{1}{(1 + x^2)(t + x^2)} = \frac{1}{1 - t} \left( -\frac{1}{1 + x^2} + \frac{1}{t + x^2} \right), \quad t \in (\varepsilon, 1), \quad x \in \mathbb{R}, \]

we have

\[
\int_{(0,\infty)} \frac{1}{\pi(1 + x^2)(t + x^2)} \, dx = \frac{1}{\pi(1 - t)} \int_{(0,\infty)} \left( -\frac{1}{1 + x^2} + \frac{1}{t + x^2} \right) \, dx
\]

\[ = -\frac{1}{2(1 - t)} + \frac{1}{\pi t(1 - t)} \int_{(0,\infty)} \frac{1}{1 + y^2} \sqrt{t} \, dy
\]

\[ = -\frac{1}{2(1 - t)} + \frac{1}{2\sqrt{t}(1 - t)} = \frac{1}{2\sqrt{t}(1 + \sqrt{t})}, \quad t \in (\varepsilon, 1). \]

Consequently,

\[ G'(t) = \frac{1}{2\sqrt{t}(1 + \sqrt{t})}, \quad t \in (\varepsilon, 1), \]

and hence

\[ G(1) - G(\varepsilon) = \int_{\varepsilon}^{1} \frac{1}{2\sqrt{t}(1 + \sqrt{t})} \, dt. \]

Then, by substituting \( \sqrt{t} = z \), we have

\[ (2.6) \quad G(1) - G(\varepsilon) = \int_{\sqrt{\varepsilon}}^{1} \frac{1}{2z(1 + z)} \, dz = \int_{\sqrt{\varepsilon}}^{1} \frac{1}{1 + z} \, dz = \ln(2) - \ln(1 + \sqrt{\varepsilon}). \]

Using Theorem A.1 we check that the function \( G \) is continuous at 0. Let us apply Theorem A.1 with the following choices: \( E := [0, 1] \), \( d \) is the usual Euclidean metric, \( t_0 := 0 \), \( X \), \( A \), and \( \mu \) are given in (2.4), and \( g \) is also given in (2.4) with the extension of its domain to \((0, \infty) \times [0, 1]\). Then condition (i) of Theorem A.1 holds (it follows from (2.5) in case of \( t \in (0, 1) \), and from \( g(x, 0) = 0 \), \( x \in (0, \infty) \), in case of \( t = 0 \)). Condition (ii) of Theorem A.1 readily holds. Condition (iii) of Theorem A.1 holds as well, since

\[ \left| \frac{\ln \left(1 + \frac{t}{x^2}\right)}{\pi(1 + x^2)} \right| \leq \ln \left(1 + \frac{t}{x^2}\right) \leq \ln \left(1 + \frac{1}{x^2}\right), \quad t \in [0, 1], \quad x \in (0, \infty), \]

and, similarly to (2.5), we have

\[ \int_{(0,\infty)} \ln \left(1 + \frac{1}{x^2}\right) \, dx = \pi < \infty. \]
Thus we can apply Theorem A.1 and we have the continuity of $G$ at 0.

Consequently, by taking the limit as $\varepsilon \downarrow 0$ in (2.6), and using $G(0) = 0$, we have
\begin{equation}
G(1) = G(1) - G(0) = \ln(2) - \ln(1) = \ln(2),
\end{equation}
and especially, we have (2.3).

Hence we can apply Proposition 1.1 with $a = 0$, $b = 1$ and the function $f$ given in (2.2), and we get
\begin{equation}
\frac{1}{n} \sum_{k=0}^{n-1} f(\phi_{0,1}(x_0)) \to \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x)}{1 + x^2} \, dx \quad \text{as } n \to \infty
\end{equation}
for Lebesgue almost every $x_0 \in \mathbb{R}$. Since, by (2.7),
\begin{equation}
\int_{\mathbb{R}} \frac{f(x)}{1 + x^2} \, dx = 2 \int_{(0, \infty)} \frac{\ln \left(\frac{1}{2} \left(1 + \frac{1}{x^2}\right)\right)}{\pi(1 + x^2)} \, dx = 2 \left( -\ln(2) \int_{0}^{\infty} \frac{1}{\pi(1 + x^2)} \, dx + G(1) \right)
\end{equation}
\begin{equation*}
= 2 \left( -\frac{\ln(2)}{2} + G(1) \right) = \ln(2),
\end{equation*}
we have
\begin{equation}
\frac{1}{n} \sum_{k=0}^{n-1} f(\phi_{0,1}(x_0)) \to \ln(2) \quad \text{as } n \to \infty
\end{equation}
for Lebesgue almost every $x_0 \in \mathbb{R}$. Using (2.1), we have
\begin{equation}
\frac{1}{n} \sum_{k=0}^{n-1} \ln(|\phi'_{a,b}(x_k)|) \to \ln(2) \quad \text{as } n \to \infty
\end{equation}
for Lebesgue almost every $x_0 \in \mathbb{R}$, where $x_k := \phi_{a,b}^{(k)}(x_0)$, $k \in \mathbb{Z}_+$, as desired.

Now we turn to prove Theorem 1.4 in the general case $(a, b) \in \mathbb{R} \times (0, \infty)$. By induction, one can check that
\begin{equation*}
\phi_{a,b}^{(k)}(y_0) = b \phi_{0,1}^{(k)} \left( \frac{y_0 - a}{b} \right) + a, \quad k \in \mathbb{Z}_+.
\end{equation*}
Hence, by (2.1), we have the set
\begin{equation*}
\left\{ y_0 \in \mathbb{R} : \phi_{a,b}^{(k)}(y_0) = a \quad \text{for some } k \in \mathbb{Z}_+ \right\} = \left\{ y_0 \in \mathbb{R} : \phi_{0,1}^{(k)} \left( \frac{y_0 - a}{b} \right) = 0 \quad \text{for some } k \in \mathbb{Z}_+ \right\}
\end{equation*}
is countable. Therefore, by (2.9), we get
\begin{equation*}
\frac{1}{n} \sum_{k=0}^{n-1} \ln(|\phi_{a,b}^{(k)}(y_k)|) = \frac{1}{n} \sum_{k=0}^{n-1} \ln \left( |\phi_{0,1}^{(k)} \left( \frac{y_k - a}{b} \right)| \right) = \frac{1}{n} \sum_{k=0}^{n-1} \ln \left( \left| \phi_{0,1}^{(k)} \left( \phi_{0,1}^{(k)} \left( \frac{y_0 - a}{b} \right) \right) \right| \right)
\end{equation*}
\begin{equation*}
\quad \to \ln(2)
\end{equation*}
as $n \to \infty$ for Lebesgue almost every $y_0 \in \mathbb{R}$, where $y_k := \phi_{a,b}^{(k)}(y_0)$, $k \in \mathbb{Z}_+$, as desired. \hfill \Box

**Appendix**
A Continuity and differentiation of parametric integrals

We recall two results on the continuity and differentiability of parametric integrals of functions of two variables where the integration is taken with respect to one of the variables, see, e.g., Klenke [11, Theorems 6.27 and 6.28]. Given a measure space \((X, \mathcal{A}, \mu)\), a function \(f : X \to \mathbb{R}\) is said to be in \(L^1(X, \mathcal{A}, \mu)\) (for short in \(L^1(\mu)\)) if it is \((\mathcal{A}, B(\mathbb{R}))\)-measurable and \(\int_X |f(x)| \mu(dx) < \infty\).

A.1 Theorem. (continuity of parametric integrals) Let \((X, \mathcal{A}, \mu)\) be a \(\sigma\)-finite measure space, \((E, d)\) be a metric space, \(t_0 \in E\), and let \(g : X \times E \to \mathbb{R}\) be a map with the following properties:

(i) for any \(t \in E\), the map \(X \ni x \mapsto g(x, t)\) is in \(L^1(\mu)\),
(ii) for \(\mu\)-almost every \(x \in X\), the map \(E \ni t \mapsto g(x, t)\) is continuous at \(t_0\),
(iii) there is a map \(h : X \to \mathbb{R}\) such that \(h \geq 0\), \(h \in L^1(\mu)\) and

\[|g(x, t)| \leq h(x) \quad \mu\text{-a.e. } x \in X \text{ for all } t \in E.\]

Then the map \(E \ni t \mapsto \int_X g(x, t) \mu(dx)\) is continuous at \(t_0\).

A.2 Theorem. (differentiability of parametric integrals) Let \((X, \mathcal{A}, \mu)\) be a \(\sigma\)-finite measure space, \(I\) be a nontrivial open interval (having at least two different points), and let \(g : X \times I \to \mathbb{R}\) be a map with the following properties:

(i) for any \(t \in I\), the map \(X \ni x \mapsto g(x, t)\) is in \(L^1(\mu)\),
(ii) for \(\mu\)-almost every \(x \in X\), the map \(I \ni t \mapsto g(x, t)\) is differentiable with derivative denoted by \(I \ni t \mapsto \partial_2 g(x, t)\),
(iii) there is a map \(h : X \to \mathbb{R}\) such that \(h \geq 0\), \(h \in L^1(\mu)\) and

\[|\partial_2 g(x, t)| \leq h(x) \quad \mu\text{-a.e. } x \in X \text{ for all } t \in I.\]

Then the map \(G : I \to \mathbb{R}, \ G(t) := \int_X g(x, t) \mu(dx), \ t \in I\), is differentiable with derivative

\[G'(t) = \int_X \partial_2 g(x, t) \mu(dx), \quad t \in I.\]

B Some applications of Proposition 1.1

We give some applications of Proposition 1.1. Recall that for each \(a \in \mathbb{R}\) and \(b > 0\), \(\varphi_{a,b}^{(k)}\) denotes the \(k\)-fold iteration of \(\varphi_{a,b}\) for \(k \in \mathbb{Z}_+\) with the convention \(\varphi_{a,b}^{(0)}(x) := x, \ x \in \mathbb{R}\).
B.1 Example. Let \( a \in \mathbb{R} \) and \( b > 0 \). If \( f(u) := b\pi \left( \frac{u-a}{b} \right)^2 + 1 \), \( u \in \mathbb{R} \), then

\[
\int_{\mathbb{R}} \frac{|f(u)|}{b\pi \left( \left( \frac{u-a}{b} \right)^2 + 1 \right)} \, du = \int_{\mathbb{R}} \frac{f(u)}{b\pi \left( \left( \frac{u-a}{b} \right)^2 + 1 \right)} \, du = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \, du = 1,
\]

so, by Proposition \ref{prop:1.1} for Lebesgue almost every \( x \in \mathbb{R} \), we have

\[
\frac{1}{n} \sum_{k=0}^{n-1} f(\varphi_{a,b}(k)(x)) \to 1 \quad \text{as} \quad n \to \infty.
\]

\[\square\]

B.2 Example. Let \( a \in \mathbb{R} \) and \( b := 1 \). If \( f(u) := \pi((u-a)^2 + 1)u \frac{1}{\sqrt{2\pi}} e^{-\frac{(u-a)^2}{2}} \), \( u \in \mathbb{R} \), then using \( |u| \leq u^2 + 1 \), \( u \in \mathbb{R} \), we have

\[
\int_{\mathbb{R}} \frac{|f(u)|}{\pi((u-a)^2 + 1)} \, du = \int_{\mathbb{R}} \frac{|f(u)|}{\pi((u-a)^2 + 1)} \, du = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(u-a)^2}{2}} \, du \leq E((\zeta + a)^2 + 1) = a^2 + 2,
\]

where \( \zeta \) is a standard normally distributed random variable, and

\[
\int_{\mathbb{R}} \frac{f(u)}{\pi((u-a)^2 + 1)} \, du = \int_{\mathbb{R}} \frac{f(u)}{\pi((u-a)^2 + 1)} \, du = \int_{\mathbb{R}} u \frac{1}{\sqrt{2\pi}} e^{-\frac{(u-a)^2}{2}} \, du = a.
\]

Hence, by Proposition \ref{prop:1.1} for Lebesgue almost every \( x \in \mathbb{R} \), we have

\[
\frac{1}{n} \sum_{k=0}^{n-1} f(\varphi_{a,1}(k)(x)) \to a \quad \text{as} \quad n \to \infty.
\]

\[\square\]

B.3 Example. Let \( a \in \mathbb{R} \), \( b := 1 \), and \( \eta \) be an absolutely continuous random variable with a density function \( h_\eta \) such that \( E(\eta^2) < \infty \). If \( f(u) := \pi(u^2 + 1)h_\eta(u) \), \( u \in \mathbb{R} \), then

\[
\int_{\mathbb{R}} \frac{|f(u)|}{\pi((u-a)^2 + 1)} \, du = \int_{\mathbb{R}} \frac{f(u)}{\pi((u-a)^2 + 1)} \, du = \int_{\mathbb{R}} \frac{1 + \eta^2}{1 + (\eta-a)^2} h_\eta(u) \, du = E \left( \frac{1 + \eta^2}{1 + (\eta-a)^2} \right) \leq E(1 + \eta^2) < \infty.
\]

Hence, by Proposition \ref{prop:1.1} for Lebesgue almost every \( x \in \mathbb{R} \), we have

\[
\frac{1}{n} \sum_{k=0}^{n-1} f(\varphi_{a,1}(k)(x)) \to E \left( \frac{1 + \eta^2}{1 + (\eta-a)^2} \right) \quad \text{as} \quad n \to \infty.
\]
Here the function $\mathbb{R} \ni a \mapsto \frac{1}{\pi \mathbb{E}(1+\eta^2)} \mathbb{E}\left(\frac{1+\eta^2}{1+(\eta-a)^2}\right)$ is a bounded density function, since it is Borel measurable, non-negative and, by Fubini’s theorem,

$$
\int_{-\infty}^{\infty} \mathbb{E}\left(\frac{1+\eta^2}{1+(\eta-a)^2}\right) da = \int_{-\infty}^{\infty} \left(\int_{\Omega} \frac{1+\eta(\omega)^2}{1+(\eta(\omega)-a)^2} \mathbb{P}(d\omega)\right) da
$$

$$
= \int_{\Omega} \left(\int_{-\infty}^{\infty} \frac{1+\eta(\omega)^2}{1+(\eta(\omega)-a)^2} da\right) \mathbb{P}(d\omega)
$$

$$
= \int_{\Omega} (1+\eta(\omega)^2) \left(\lim_{a \to \infty} \left(-\arctan(\eta(\omega)-a)\right) + \lim_{a \to -\infty} \left(\arctan(\eta(\omega)-a)\right)\right) \mathbb{P}(d\omega)
$$

$$
= \pi \int_{\Omega} (1+\eta(\omega)^2) \mathbb{P}(d\omega) = \pi \mathbb{E}(1+\eta^2).
$$

Further,

$$
\frac{1}{\pi \mathbb{E}(1+\eta^2)} \mathbb{E}\left(\frac{1+\eta^2}{1+(\eta-a)^2}\right) \leq \frac{1}{\pi}, \quad a \in \mathbb{R}.
$$

\[\Box\]

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