Bayes-Optimal Entropy Pursuit for Active Choice-Based Preference Learning

Stephen N. Pallone
Peter I. Frazier
Shane G. Henderson

School of Operations Research and Information Engineering
290 Rhodes Hall, Cornell University
Ithaca, NY 14853, USA

Abstract

We analyze the problem of learning a single user’s preferences in an active learning setting, sequentially and adaptively querying the user over a finite time horizon. Learning is conducted via choice-based queries, where the user selects her preferred option among a small subset of offered alternatives. These queries have been shown to be a robust and efficient way to learn an individual’s preferences. We take a parametric approach and model the user’s preferences through a linear classifier, using a Bayesian prior to encode our current knowledge of this classifier. The rate at which we learn depends on the alternatives offered at every time epoch. Under certain noise assumptions, we show that the Bayes-optimal policy for maximally reducing entropy of the posterior distribution of this linear classifier is a greedy policy, and that this policy achieves a linear lower bound when alternatives can be constructed from the continuum. Further, we analyze a different metric called misclassification error, proving that the performance of the optimal policy that minimizes misclassification error is bounded below by a linear function of differential entropy. Lastly, we numerically compare the greedy entropy reduction policy with a knowledge gradient policy under a number of scenarios, examining their performance under both differential entropy and misclassification error.

Keywords: preferences, entropy, information theory, conjoint analysis, active learning

1. Introduction

The problem of preference learning is a well-studied and widely applicable area of study in the machine learning literature. Preference elicitation is by no means a new problem (Schapire and Singer, 1998), and is now ubiquitous in many different forms in nearly all subfields of machine learning. One such scenario is the active learning setting, where one sequentially and adaptively queries the user to most efficiently learn his or her preferences. In general, learning in an online setting can be more efficient than doing so in an offline supervised learning setting, which is consequential when queries are expensive. This is often the case for preference elicitation, where a user may not be inclined to answer too many questions. The ability to adaptively query the user with particular exemplars that facilitate learning to the labels of the rest is invaluable in the context of preference elicitation.
In particular, there is great interest in using choice-based queries to learn the preferences of an individual user. In this setting, a user is offered two or more alternatives and is asked to select the alternative he or she likes most. There are other types of responses that can assess one’s preferences among a set of alternatives, such as rating each of the items on a scale, or giving a full preference order for all alternatives in the set. However, choosing the most-preferred item in a given set is a natural task, and is a more robust measurement of preference than rating or fully-ranking items. For this reason, choice-based methods have been shown to work well in practice (see Louviere et al., 2000), and these are the types of queries we study. In this paper, we formulate the problem of sequential choice-based preference elicitation as a finite horizon adaptive learning problem.

The marketing community has long been focused on preference elicitation and isolating features that matter the most to consumers. In this field, *conjoint analysis* is a class of methods that attempts to learn these important features by offering users a subset of alternatives (Green and Srinivasan, 1978). Lately, there has been a push in the marketing community to design sequential methods that adaptively select the best subset of alternatives to offer the user. In the marketing research literature, this is referred to as adaptive choice-based conjoint analysis. In the past, geometrically-motivated heuristics have been used to adaptively choose questions (Toubia et al., 2004). These heuristics have since evolved to include probabilistic modeling that captures the uncertainty in user responses (Toubia et al., 2007).

These problems are also tackled by the active learning community. For instance, Maldonado et al. (2015) use existing support vector machine (SVM) technology to identify features users find important. In the context of preference elicitation in the active learning literature, there are two main approaches. The first is to take a non-parametric approach and infer a full preference ranking, labeling every pairwise combination of alternatives (Fürnkranz and Hüllermeier, 2003). The benefit to this approach is the generality offered by a non-parametric model and its ability to capture realistic noise. Viewing preference learning as a generalized binary search problem, Nowak (2011) proves exponential convergence in probability to the correct preferential ordering for all alternatives in a given set, and shows his algorithm is optimal to a constant factor. Unfortunately, this probabilistic upper bound is weakened by a coefficient that is quadratic in the total number of alternatives, and the running time of this optimal policy is proportional to the number of valid preferential orderings of all the alternatives. These issues are common for non-parametric ranking models. Using a statistical learning theoretic framework, Ailon (2012) develops an adaptive and computationally efficient algorithm to learn a ranking, but the performance guarantees are only asymptotic. In practice, one can only expect to ask a user a limited number of questions, and in this scenario, Yu et al. (2012) show that taking a Bayesian approach to optimally and adaptively selecting questions is indispensable to the task of learning preferences for a given user. In the search for finite-time results and provable bounds, we opt to learn a parametric model using a Bayesian approach. In particular, this paper focuses on a greedy policy that maximally reduces posterior entropy of a linear classifier, leveraging information theory to derive results pertaining to this policy.

Maximizing posterior entropy reduction has long been a suggested objective for learning algorithms (Lindley, 1956; Bernardo, 1979), especially within the context of active learning (MacKay, 1992). But even within this paradigm of preference elicitation, there is a variety of work that depends on the user response model. For example, Dzyabura and
Hauser (2011) study maximizing entropy reduction under different response heuristics, and Saure and Vielma (2016) uses ellipsoidal credibility regions to capture the current state of knowledge of a user’s preferences. Using an entropy-based objective function allows one to leverage existing results in information theory to derive theoretical finite-time guarantees (Jedynak et al., 2012). Most similar to our methodology, Brochu et al. (2010) and Houlsby et al. (2011) model a user’s utility function using a Gaussian process, updating the corresponding prior after each user response, and adaptively choose questions by minimizing an estimate of posterior entropy. However, while the response model is widely applicable and the method shows promise in practical situations, the lack of theoretical guarantees leaves much to be desired. Ideally, one would want concrete performance bounds for an entropy-based algorithm under a parameterized response model. In contrast, this paper proves information theoretic results in the context of adaptive choice-based preference elicitation for arbitrary feature-space dimension, leverages these results to derive bounds for performance, and shows that a greedy entropy reduction policy (hereafter referred to as entropy pursuit) optimally reduces posterior entropy of a linear classifier over the course of multiple choice-based questions. In particular, the main contributions of the paper are summarized as follows:

- In Section 2, we formally describe the response model for the user. For this response model, we prove a linear lower bound on the sequential entropy reduction over a finite number of questions in Section 3, and provide necessary and sufficient conditions for asking an optimal comparative question.

- Section 3.3 presents results showing that the linear lower bound can be attained by a greedy algorithm up to a multiplicative constant when we are allowed to fabricate alternatives (i.e., when the set of alternatives has a non-empty interior). Further, the bound is attained exactly with moderate conditions on the noise channel.

- Section 4 focuses on misclassification error, a more intuitive metric of measuring knowledge of a user’s preferences. In the context of this metric, we show a Fano-type lower bound on the optimal policy in terms of an increasing linear function of posterior differential entropy.

- Finally in Section 5, we provide numerical results demonstrating that entropy pursuit performs similarly to an alternative algorithm that greedily minimizes misclassification error. This is shown in a variety of scenarios and across both metrics. Taking into account the fact that entropy pursuit is far more computationally efficient than the alternative algorithm, we conclude that entropy pursuit should be preferred in practical applications.

2. Problem Specification

The alternatives $x^{(i)} \in \mathbb{R}^d$ are represented by $d$-dimensional feature vectors that encode all of their distinguishing aspects. Let $X$ be the set of all such alternatives. Assuming a linear utility model, each user has her own linear classifier $\theta \in \Theta \subset \mathbb{R}^d$ that encodes her preferences $^1$. At time epoch $k$, given $m$ alternatives $X_k = \{x^{(1)}_k, x^{(2)}_k, \ldots, x^{(m)}_k\} \in X^m$, the

\footnote{1. Throughout the paper, we use boldface to denote a random variable.}
user prefers to choose the alternative \( i \) that maximizes \( \theta^T x_k^{(i)} \). However, we do not observe this preference directly. Rather, we observe a signal influenced by a noise channel. In this case, the signal is the response we observe from the user.

Let \( Z = \{1, 2, \ldots, m\} \) denote the \( m \) possible alternatives. We define \( Z_k(X_k) \) to be the alternative that is consistent with our linear model after asking question \( X_k \), that is, \( Z_k(X_k) = \min \left\{ \arg\max_{i \in \mathbb{Z}} \theta^T x_k^{(i)} \right\} \). The minimum is just used as a tie-breaking rule; the specific rule is not important so long as it is deterministic. We do not observe \( Z_k(X_k) \), but rather observe a signal \( Y_k(X_k) \in \mathcal{Y} \), which depends on \( Z_k(X_k) \). We allow \( \mathcal{Y} \) to characterize any type of signal that can be received from posing questions in \( X \). In general, the density of the conditional distribution of \( Y_k(X_k) \) given \( Z_k(X_k) = z \) is denoted \( f(z) \). In this paper, we primarily consider the scenario in which \( \mathcal{Y} = Z = \{1, 2, \ldots, m\} \), where nature randomly perturbs \( Z_k(X_k) \) to some (possibly the same) element in \( Z \). In this scenario, the user’s response to the preferred alternative is the signal \( Y_k(X_k) \), which is observed in lieu of the model-consistent “true response” \( Z_k(X_k) \). In this case, we define a noise channel stochastic matrix \( P \) by setting \( P^{(z)} = f(z) \) to describe what is called a discrete noise channel.

One sequentially asks the user questions and learns from each of their responses. Accordingly, let \( \mathbb{P}_k \) be the probability measure conditioned on the \( \sigma \)-field generated by \( \mathcal{Y}_k = (Y_\ell(X_\ell) : 1 \leq \ell \leq k - 1) \). Similarly, let \( \mathcal{Y}_k = \{ Y_\ell(X_\ell) : 1 \leq \ell \leq k - 1 \} \) denote the history of user responses. As we update, we condition on the previous outcomes, and subsequently choose a question \( X_k \) that depends on all previous responses \( \mathcal{Y}_k \) from the user. Accordingly, let policy \( \pi \) return a comparative question \( X_k \in \mathcal{X}^m \) that depends on time epoch \( k \) and past response history \( \mathcal{Y}_k \). The selected question \( X_k \) may also depend on i.i.d. random uniform variables, allowing for stochastic policies. We denote the space of all such policies \( \pi \) as \( \Pi \). In this light, let \( \mathbb{E}^\pi \) be the expectation operator induced by policy \( \pi \).

In this paper, we consider a specific noise model, which is highlighted in the following assumptions.

**Noise Channel Assumptions** For every time epoch \( k \), signal \( Y_k(X_k) \) and true response \( Z_k(X_k) \) corresponding to comparative question \( X_k \), we assume

- **model-consistent response** \( Z_k(X_k) \) is a deterministic function of question \( X \) and linear classifier \( \theta \), and

- **given true response** \( Z_k(X_k) \), signal \( Y_k(X_k) \) is conditionally independent of linear classifier \( \theta \) and previous history \( \mathcal{Y}_k \), and

- **the conditional densities** \( f = \{ f(z) : z \in \mathbb{Z} \} \) differ from each other on a set of Lebesgue measure greater than zero.

The first two assumptions ensure that all the information regarding \( \theta \) is contained in some true response \( Z_k(X_k) \). In other words, the model assumes that no information about the linear classifier is lost if we focus on inferring the true response instead. The last assumption is focused on identifiability of the model: since we infer by observing a signal, it is critical that we can tell the conditional distributions of these signals apart, and the latter condition guarantees this.
One of the benefits this noise model provides is allowing us to easily update our beliefs of $\theta$. For a given question $X \in \mathcal{X}^m$ and true response $z \in \mathcal{Z}$, let

$$A^{(z)}(X) = \left\{ \theta \in \Theta : \begin{array}{l} \theta^T x^{(z)} \geq \theta^T x^{(i)} \quad \forall i > z \\ \theta^T x^{(z)} > \theta^T x^{(i)} \quad \forall i < z \end{array} \right\}. \tag{1}$$

These $m$ sets form a partition of $\Theta$ that depend on the question $X$ we ask at each time epoch, where each set $A^{(z)}$ corresponds to all linear classifiers $\theta$ that are consistent with the true response $Z = z$.

Let $\mu_k$ denote the prior measure of $\theta$ at time epoch $k$. Throughout the paper, we assume that $\mu_k$ is absolutely continuous with respect to $d$-dimensional Lebesgue measure, admitting a corresponding Lebesgue density $p_k$. At every epoch, we ask the user a comparative question that asks for the most preferred option in $X_k = \{x_1, x_2, \ldots, x_m\}$. We observe signal $Y_k(X_k)$, and accordingly update the prior.

**Lemma 1** Suppose that the Noise Channel Assumptions hold. Then we can write the posterior $p_{k+1}$ as

$$p_{k+1}(\theta | Y_k(X_k) = y) = \left( \frac{\sum_{z \in \mathcal{Z}} \mathbb{I}(\theta \in A^{(z)}(X_k)) f^{(z)}(y)}{\sum_{z \in \mathcal{Z}} \mu_k(A^{(z)}(X_k)) f^{(z)}(y)} \right) p_k(\theta), \tag{2}$$

where $\mathbb{I}$ denotes the indicator function.

**Proof** Using Bayes’ rule, we see

$$p_{k+1}(\theta | Y_k(X_k) = y) \propto \mathbb{P}_k(Y_k(X_k) = y | \theta = \theta) \cdot p_k(\theta)$$

$$= \sum_{z \in \mathcal{Z}} \mathbb{P}_k(Y_k(X_k) = y | Z_k(X_k) = z, \theta = \theta) \cdot \mathbb{P}_k(Z_k(X_k) = z | \theta = \theta) \cdot p_k(\theta).$$

Now we use a property of $Y_k(X_k)$ and $Z_k(X_k)$ from the Noise Channel Assumptions, namely that $Y_k(X_k)$ and $\theta$ are conditionally independent given $Z_k(X_k)$. This implies

$$p_{k+1}(\theta | Y_k(X_k) = y) \propto \sum_{z \in \mathcal{Z}} \mathbb{P}_k(Y_k(X_k) = y | Z_k(X_k) = z) \cdot \mathbb{P}_k(Z_k(X_k) = z | \theta = \theta) \cdot p_k(\theta)$$

$$= \sum_{z \in \mathcal{Z}} f^{(z)}(y) \cdot \mathbb{I}(\theta \in A^{(z)}(X_k)) \cdot p_k(\theta),$$

where the last line is true because $Z_k(X_k)$ is a deterministic function of $\theta$ and $X_k$. Normalizing to ensure the density integrates to one gives the result. \hfill \blacksquare

The Noise Channel Assumptions allow us to easily update the prior on $\theta$. As we will see next, they also allow us to easily express the conditions required to maximize one-step entropy reduction.

### 3. Posterior Entropy

We focus on how we select the alternatives we offer to the user. First, we need to choose a metric to evaluate the effectiveness of each question. One option is to use a measure of
dispersion of the posterior distribution of $\theta$, and the objective is to decrease the amount of spread as much as possible with every question. Along these lines, we elect to use differential entropy for its tractability.

For a probability density $p$, the differential entropy of $p$ is defined as

$$H(p) = \int_{\Theta} -p(\theta) \log_2 p(\theta) \, d\theta.$$ 

For the entirety of this paper, all logarithms are base-2, implying that both Shannon and differential entropy are measured in bits. Because we ask the user multiple questions, it is important to incorporate the previous response history $Y_k$ when considering posterior entropy. Let $H_k$ be the entropy operator at time epoch $k$ such that $H_k(\theta) = H(\theta | Y_k)$, which takes into account all of the previous observation history $Y_k$. Occasionally, when looking at the performance of a policy $\pi$, we would want to randomize over all such histories. This is equivalent to the concept of conditional entropy, with $H^\pi(\theta | Y_k) = E^\pi [H_k(\theta)]$.

Throughout the paper, we represent discrete distributions as vectors. Accordingly, define $\Delta^m = \{u \in \mathbb{R}^m : \sum_z u(z) = 1, u \geq 0\}$ to be the set of discrete probability distributions over $m$ alternatives. For a probability distribution $u \in \Delta^m$, we define $h(u)$ to be the Shannon entropy of that discrete distribution, namely

$$h(u) = \sum_{z \in Z} -u(z) \log_2 u(z).$$

Here, we consider discrete probability distributions over the alternatives we offer, which is why distributions $u$ are indexed by $z \in Z$.

Since stochastic matrices are be used to model some noise channels, we develop similar notation for matrices. Let $\Delta^{m \times m}$ denote the set of $m \times m$ row-stochastic matrices. Similarly to how we defined the Shannon entropy of a vector, we define $h(P)$ as an $m$-vector with the Shannon entropies of the rows of $P$ as its components. In other words,

$$h(P)^{(z)} = \sum_{y \in Y} -P^{(zy)} \log_2 P^{(zy)}.$$

An important concept in information theory is mutual information, which measures the entropy reduction of a random variable when conditioning on another. It is natural to ask about the relationship between the information gain of $\theta$ and that of $Z^k(X_k)$ after observing signal $Y_k(X_k)$. Mutual information in this context is defined as

$$I_k(\theta; Y_k(X_k)) = H_k(\theta) - H_k(\theta | Y_k(X_k)). \quad (3)$$

One critical property of mutual information is that it is symmetric, or in other words, $I_k(\theta; Y_k(X_k)) = I_k(Y_k(X_k); \theta)$ (see Cover, 1991, p. 20). In the context of our model, this means that observing signal $Y_k(X_k)$ gives us the same amount of information about linear classifier $\theta$ as would observing the linear classifier would provide about the signal. This is one property we exploit throughout the paper, since the latter case only depends on the noise channel, which by assumption does not change over time. We show in Theorem 2 below that the Noise Channel Assumptions allow us to determine how the noise channel affects the posterior entropy of linear classifier $\theta$. 

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The first identity, given by (4), says that the noise provides an additive effect with respect to entropy, particularly because the noise does not depend on $\theta$ itself. The second identity, given by (5), highlights the fact that $Y_k(X_k)$ provides the same amount of information on the linear classifier $\theta$ as it does on the true answer $Z_k(X_k)$ for a given question. This means that the entropy of both $\theta$ and $Z_k(X_k)$ are reduced by the same number of bits when asking question $X_k$. Intuitively, asking the question that would gain the most clarity from a response would also do the same for the underlying linear classifier. This is formalized in Theorem 2 below.

**Theorem 2** The following information identities hold under the Noise Channel Assumptions for all time epochs $k$. The first is the **Noise Separation Equality**, namely

$$H_k(\theta | Y_k(X_k)) = H_k(\theta | Z_k(X_k)) + H_k(Z_k(X_k) | Y_k(X_k)), \tag{4}$$

and the **Noise Channel Information Equality**, given by

$$I_k(\theta; Y_k(X_k)) = I(Z_k(X_k); Y_k(X_k)), \tag{5}$$

where the latter term does not depend on response history $Y_k$.

**Proof** Using the symmetry of mutual information,

$$H_k(\theta | Y_k(X_k)) - H_k(\theta | Y_k(X_k), Z_k(X_k)) = H_k(Z_k(X_k) | Y_k(X_k)) - H_k(Z_k(X_k) | \theta, Y_k(X_k)).$$

Further, we know $H_k(\theta | Y_k(X_k), Z_k(X_k)) = H_k(\theta | Z_k(X_k))$ because $Y_k(X_k)$ and $\theta$ are conditionally independent given $Z_k(X_k)$. Also, since $Z_k(X_k)$ is a function of $\theta$ and $X_k$, it must be that $H_k(Z_k(X_k) | \theta, Y_k(X_k)) = 0$. Putting these together gives us the first identity.

To prove the second identity, we use the fact that

$$H_k(\theta | Z_k(X_k)) + H_k(Z_k(X_k)) = H_k(Z_k(X_k) | \theta) + H_k(\theta).$$

Again, $H_k(Z_k(X_k) | \theta) = 0$ because $Z_k(X_k)$ is a function of $\theta$ and $X_k$. This yields $H_k(\theta | Z_k(X_k)) = H_k(\theta) - H_k(Z_k(X_k)).$ Substitution into the first identity gives us

$$H_k(\theta) - H_k(\theta | Y_k(X_k)) = H_k(Z_k(X_k) | Y_k(X_k)),$$

which is (5), by definition of mutual information. Finally, by the Noise Channel Assumptions, signal $Y_k(X_k)$ is conditionally independent of history $Y_k$ given $Z_k(X_k)$, and therefore, $I_k(Z_k(X_k); Y_k(X_k)) = I(Z_k(X_k); Y_k(X_k))$.

The entropy pursuit policy is one that maximizes the reduction in entropy of the linear classifier, namely $I_k(\theta; Y_k(X_k)) = H_k(\theta) - H_k(\theta | Y_k(X_k))$, at each time epoch. We leverage the results from Theorem 2 to find conditions on questions that maximally reduce entropy in the linear classifier $\theta$. However, we first need to introduce some more notation.

For a noise channel parameterized by $f = \{f(z) : z \in \mathbb{Z}\}$, let $\varphi$ denote the function on domain $\Delta^m$ defined as

$$\varphi(u ; f) = H \left( \sum_{z \in \mathbb{Z}} u^{(z)} f^{(z)} \right) - \sum_{z \in \mathbb{Z}} u^{(z)} H \left( f^{(z)} \right). \tag{6}$$
We will show in Theorem 3 that (6) refers to the reduction in entropy from asking a question, where the argument \( u \in \Delta^m \) depends on the question. We define the channel capacity over noise channel \( f \), denoted \( C(f) \), to be the supremum of \( \varphi \) over this domain, namely

\[
C(f) = \sup_{u \in \Delta^m} \varphi(u; f), \tag{7}
\]

and this denotes the maximal amount of entropy reduction at every step. These can be similarly defined for a discrete noise channel. For a noise channel parameterized by transmission matrix \( P \), we define

\[
\varphi(u; P) = h(u^T P) - u^T h(P), \tag{8}
\]

and \( C(P) \) is correspondingly the supremum of \( \varphi(\cdot; P) \) in its first argument. In Theorem 3 below, we show that \( \varphi(u; f) \) is precisely the amount of entropy over linear classifiers \( \theta \) reduced by asking a question with respective predictive distribution \( u \) under noise channel \( f \).

**Theorem 3** For a given question \( X \in \mathbb{X}^m \), define \( u_k(X) \in \Delta^m \) such that \( u_k^{(z)}(X) = \mu_k(A^{(z)}(X)) \) for all \( z \in \mathbb{Z} \). Suppose that the Noise Channel Assumptions hold. Then for a fixed noise channel parameterized by \( f = \{f^{(z)} : z \in \mathbb{Z}\} \),

\[
I_k(\theta; Y_k(X_k)) = \varphi(u_k(X_k); f). \tag{9}
\]

Consequently, for all time epochs \( k \), we have

\[
\sup_{X_k \in \mathbb{X}^m} I_k(\theta; Y_k(X_k)) \leq C(f), \tag{10}
\]

and there exists \( u_* \in \Delta^m \) that attains the supremum. Moreover, if there exists some \( X_k \in \mathbb{X}^m \) such that \( u_k(X_k) = u_* \), then the upper bound is attained.

**Proof** We first use (5) from Theorem 2, namely that \( I_k(\theta; Y_k(X_k)) = I_k(Z_k(X_k); Y_k(X_k)) \). We use the fact that mutual information is symmetric, meaning that the entropy reduction in \( Z_k(X_k) \) while observing \( Y_k(X_k) \) is equal to that in \( Y_k(X_k) \) while observing \( Z_k(X_k) \). Putting this together with the definition of mutual information yields

\[
I_k(\theta; Y_k(X_k)) = I_k(Z_k(X_k); Y_k(X_k)) = H_k(Y_k(X_k)) - H_k(Y_k(X_k) | Z_k(X_k))
= H \left( \sum_{z \in \mathbb{Z}} \mathbb{P}_k(Z_k(X_k) = z) f^{(z)} \right) \tag{\ref{eq:mutual的信息}} - \sum_{z \in \mathbb{Z}} \mathbb{P}_k(Z_k(X_k) = z) H(f^{(z)})
= H \left( \sum_{z \in \mathbb{Z}} \mu_k(A^{(z)}(X_k)) f^{(z)} \right) \tag{\ref{eq:entropy的信息}} - \sum_{z \in \mathbb{Z}} \mu_k(A^{(z)}(X_k)) H(f^{(z)}),
\]

which is equal to \( \varphi(u_k(X_k); f) \), where \( u_k^{(z)}(X_k) = \mu_k(A^{(z)}(X_k)) \). Therefore, the optimization problem in (10) is equivalent to

\[
\sup_{X_k \in \mathbb{X}^m} \varphi(u_k(X_k); f).
\]
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Since \( \{u_k(X) : X \in \mathbb{X}^m\} \subseteq \Delta^m \), we can relax the above problem to

\[
\sup_{u \in \Delta^m} \varphi(u; f).
\]

It is known that mutual information is concave in its probability mass function (see Cover, 1991, p. 31), and strictly concave when the likelihood functions \( f(z) \) differ on a set of positive measure. Thus, for a fixed noise channel \( f \), \( \varphi(\cdot; f) \) is concave on \( \Delta^m \), a compact convex set, implying an optimal solution \( u_* \) exists and the optimal objective value \( C(f) > 0 \) is attained. Further, if we can construct some \( X_k \in \mathbb{X}^m \) such that \( \mu_k(\theta(z); X_k) = u^*_k(z) \) for every \( z \in \mathbb{Z} \), then the upper bound is attained.

We have shown that entropy reduction of the posterior of \( \theta \) depends only on the implied predictive distribution of a given question and structure of the noise channel. If we are free to fabricate alternatives to achieve the optimal predictive distribution, then we reduce the entropy of the posterior by a fixed amount \( C(f) \) at every time epoch. Perhaps the most surprising aspect of this result is the fact that the history \( Y_k \) plays no role in the amount of entropy reduction, which is important for showing that entropy pursuit is an optimal policy for reducing entropy over several questions.

In practice, one can usually ask more than one question, and it is natural to ask if there is an extension that gives us a bound on the posterior entropy after asking several questions. Using the results in Theorem 3, we can derive an analogous lower bound for this case.

**Corollary 4** For a given policy \( \pi \in \Pi \), we can write the entropy of linear classifier \( \theta \) after \( K \) time epochs as

\[
H(\theta) - H^\pi(\theta | Y_K) = \mathbb{E}_\pi \left[ \sum_{k=1}^{K} \varphi(u_k(X_k) ; f) \right], \tag{11}
\]

and a lower bound for the differential entropy of \( \theta \) after asking \( K \) questions is given below by

\[
\inf_{\pi \in \Pi} H^\pi(\theta | Y_K) \geq H(\theta) - K \cdot C(f). \tag{12}
\]

Further, if for a given policy \( \pi \) and history \( Y_k \) indicates that comparative question \( X_k \) should be posed to the user, then the lower bound is attained if and only if \( u_k(X_k) = u_* \) for every \( k = 1, 2, \ldots, K \), as defined in Theorem 3. Thus, entropy pursuit is an optimal policy.

**Proof** Using the information chain rule, we can write the entropy reduction for a generic policy \( \pi \in \Pi \) as

\[
H(\theta) - H^\pi(\theta | Y_K) = I^\pi(\theta; Y_K)
\]

\[
= \sum_{k=1}^{K} \mathbb{E}_\pi \left[ I_k(\theta; Y_k(X_k)) \right] \leq K \cdot C(f),
\]

where the last inequality comes directly from Theorem 3, and the upper bound is attained if and only if \( u_k(X_k) = u_* \) for every \( k = 1, 2, \ldots, K \). This coincides with the entropy pursuit policy. \( \blacksquare \)
Essentially, Corollary 4 shows that the greedy entropy reduction policy is, in fact, the optimal policy over any time horizon. However, there is still an important element that is missing: how can we ensure that there exists some alternative that satisfies the entropy pursuit criteria? We address this important concern in Section 3.3.

3.1 Optimality Conditions for Predictive Distribution

Because of the properties of entropy, the noise channel function $\varphi$ has a lot of structure. We use this structure to find conditions for a non-degenerate optimal predictive distribution $u^*$ as well as derive sensitivity results that allow the optimality gap of a close-to-optimal predictive distribution to be estimated.

Before we prove structural results for the channel equation $\varphi$, some more information theoretic notation should be introduced. Given two densities $f^{(i)}$ and $f^{(j)}$, the cross entropy of these two densities is defined as

$$H(f^{(i)}, f^{(j)}) = \int_Y -f^{(i)}(y) \log_2 f^{(j)}(y) \, dy.$$ 

Using the definition of cross entropy, the Kullback-Leibler divergence between two densities $f^{(i)}$ and $f^{(j)}$ is defined as

$$\text{KL}(f^{(i)} \parallel f^{(j)}) = H(f^{(i)}, f^{(j)}) - H(f^{(i)}).$$

Kullback-Leibler divergence is a tractable way of measuring the difference of two densities. An interesting property of Kullback-Leibler divergence is that for any densities $f^{(i)}$ and $f^{(j)}$,

$$\text{KL}(f^{(i)} \parallel f^{(j)}) \geq 0,$$

with equality if and only if $f^{(i)} = f^{(j)}$ almost surely. Kullback-Leibler divergence plays a crucial role the first-order information for the channel equation $\varphi$.

We now derive results that express the gradient and Hessian of $\varphi$ in terms of the noise channel, which can either be parameterized by $f$ in the case of a density, or by a fixed transmission matrix $P$ in the discrete noise channel case. For these results to hold, we require the cross entropy $H(f^{(i)}, f^{(j)})$ to be bounded in magnitude for all $i, j \in \mathbb{Z}$, which is an entirely reasonable assumption.

**Lemma 5** For a fixed noise channel characterized by $f = \{f^{(z)} : z \in \mathbb{Z}\}$, if the cross entropy terms $H(f^{(i)}, f^{(j)})$ are bounded for all $i, j \in \mathbb{Z}$, then the first and second partial derivatives of $\varphi$ with respect to $u$ are given by

$$\frac{\partial \varphi(u; f)}{\partial u^{(z)}} = \text{KL} \left( f^{(z)} \parallel \sum_{i \in \mathbb{Z}} u^{(i)} f^{(i)} \right) - \xi,$$

$$\frac{\partial^2 \varphi(u; f)}{\partial u^{(z)} \partial u^{(w)}} = -\xi \int_Y \frac{f^{(z)}(y) f^{(w)}(y)}{\sum_{i \in \mathbb{Z}} u^{(i)} f^{(i)}(y)} \, dy,$$

where $\xi = \log_2 e$, and $\text{KL}(\cdot \parallel \cdot)$ is the Kullback-Leibler Divergence.

In particular, if a discrete noise channel is parameterized by transmission matrix $P$, the gradient and Hessian matrix of $\varphi$ can be respectively expressed as

$$\nabla_u \varphi(u; P) = -P \log_2 (P^T u) - h(P) - \xi e,$$

$$\nabla^2_u \varphi(u; P) = -\xi P \left( \text{diag} (u^T P) \right)^{-1} P^T,$$
where the logarithm is taken component-wise.

**Proof** We first prove the result in the more general case when the noise channel is parameterized by \( f \). From the definition of \( \varphi \),

\[
\varphi(u; f) = \int_Y - \left( \sum_{i \in Z} u(i) f^{(i)}(y) \right) \log_2 \left( \sum_{i \in Z} u(i) f^{(i)}(y) \right) dy - \sum_{i \in Z} u(i) H(f^{(i)}).
\]

Since \( t \mapsto -\log t \) is convex, by Jensen’s inequality, \( H(f^{(z)}, \sum_i u(i) f^{(i)}) \leq \sum_i u(i) H(f^{(z)}, f^{(i)}) \), which is bounded. By the Dominated Convergence Theorem, we can switch differentiation and integration operators, and thus,

\[
\frac{\partial}{\partial u(z)} \varphi(u; f) = \int_Y - f^{(z)}(y) \log_2 \left( \sum_{i \in Z} u(i) f^{(i)}(y) \right) dy - \xi - H(f^{(z)})
\]

\[
= \text{KL} \left( f^{(z)} \parallel \sum_{i \in Z} u(i) f^{(i)} \right) - \xi.
\]

Concerning the second partial derivative, Kullback-Leibler divergence is always non-negative, and therefore, Monotone Convergence Theorem again allows us to switch integration and differentiation, yielding

\[
\frac{\partial^2 \varphi(u; f)}{\partial u(z) \partial u(w)} = -\xi \int_Y f^{(z)}(y) f^{(w)}(y) dy.
\]

For the discrete noise channel case, the proof is analogous to above, using Equation (8). Vectorizing yields

\[
\nabla_u \varphi(u; P) = -P \left( \log_2 (P^T u) + \xi e \right) - h(P)
\]

\[
= -P \log_2 (P^T u) - h(P) - \xi e.
\]

Similarly, the discrete noise channel analogue for the second derivative is

\[
\frac{\partial^2 \varphi(u; P)}{\partial u(z) \partial u(w)} = -\xi \sum_{y \in Y} \frac{P^{(zy)} P^{(wy)}}{\sum_{i \in Z} u(i) P^{(i) y}},
\]

and vectorizing gives us the Hessian matrix.

One can now use the results in Lemma 5 to find conditions for an optimal predictive distribution for a noise channel parameterized either by densities \( f = \{ f^{(z)} : z \in Z \} \) or transmission matrix \( P \). There has been much research on how to find the optimal predictive distribution \( u_* \) given a noise channel, as in Gallager (1968). Generally, there are two methods for finding this quantity. The first relies on solving a constrained concave maximization problem by using a first-order method. The other involves using the Karush-Kuhn-Tucker conditions necessary for an optimal solution (see Gallager, 1968, p. 91 for proof).
Theorem 6 (Gallager) Given a noise channel parameterized by \( f = \{ f(z) : z \in \mathbb{Z} \} \), the optimal predictive distribution \( u^* \) satisfies

\[
\text{KL} \left( f(z) \left\| \sum_{i \in \mathbb{Z}} u(i) f(i) \right\| \right) = \begin{cases} C(f) & u^*_z > 0 \\ < C(f) & u^*_z = 0, \end{cases}
\]

where \( C(f) \) is the channel capacity.

The difficulty in solving this problem comes from determining whether or not \( u^*_z > 0 \). In the context of preference elicitation, when fixing the number of offered alternatives \( m \), it is critical for every alternative to contribute to reducing uncertainty. However, having a noise channel where \( u^*_z = 0 \) implies that it is more efficient to learn without offering alternative \( z \).

To be specific, we say that a noise channel parameterized by \( f = \{ f(z) : z \in \mathbb{Z} \} \) is admissible if there exists some \( f^* \in \text{Int (Hull}(f)) \) such that for all \( z \in \mathbb{Z} \),

\[
\text{KL} \left( f(z) \left\| f^* \right\| \right) = C
\]

for some \( C > 0 \). Otherwise, we say the noise channel is inadmissible. Admissibility is equivalent the existence of a predictive distribution \( u^*_z > 0 \) where all \( m \) alternatives are used to learn a user’s preferences. For pairwise comparisons, any noise channel where \( f^{(1)} \) and \( f^{(2)} \) differ on a set of non-zero Lebesgue measure is admissible. Otherwise, for \( m > 2 \), there are situations when \( u^*_z = 0 \) for some \( z \in \mathbb{Z} \), and Lemma 7 provides one of them. In particular, if one density \( f^{(z)} \) is a convex combination of any of the others, then the optimal predictive distribution will always have \( u^*_z = 0 \).

Lemma 7 Suppose the noise channel is parameterized by densities \( f = \{ f(z) : z \in \mathbb{Z} \} \), and its corresponding optimal predictive distribution is \( u^* \). If there exists \( \lambda^{(i)} \geq 0 \) for \( i \neq z \) such that \( \sum_{i \neq z} \lambda^{(i)} = 1 \) and \( f(z)(y) = \sum_{i \neq z} \lambda^{(i)} f^{(i)}(y) \) for all \( y \in \mathcal{Y} \), then \( u^*_z = 0 \).

Proof Suppose \( f^{(z)} = \sum_{i \neq z} \lambda^{(i)} f^{(i)} \). Take any \( u \in \Delta^m \) such that \( u(z) > 0 \). We will construct a \( \tilde{u} \in \Delta^m \) such that \( \tilde{u}(z) = 0 \) and \( \varphi(\tilde{u}; f) > \varphi(u; f) \). Define \( \tilde{u} \) as

\[
\tilde{u}(i) = \begin{cases} u^{(i)} + \lambda^{(i)} u^*(z) & i \neq z \\ 0 & i = z. \end{cases}
\]
It is easy to verify that \( \sum_i \tilde{u}^{(i)} f^{(i)} = \sum_i u^{(i)} f^{(i)} \). But since entropy is strictly concave, we have \( H (f^{(z)}) > \sum_{i \neq z} \lambda^{(i)} f^{(i)} \). Consequently,

\[
\varphi(u ; f) = H \left( \sum_{i \in \mathbb{Z}} u^{(i)} f^{(i)} \right) - \sum_{i \in \mathbb{Z}} u^{(i)} H \left( f^{(i)} \right)
\]

\[
= H \left( \sum_{i \neq z} \tilde{u}^{(i)} f^{(i)} \right) - \sum_{i \neq z} u^{(i)} H \left( f^{(i)} \right) - \sum_{i \neq z} u^{(z)} H \left( f^{(z)} \right)
\]

\[
< H \left( \sum_{i \neq z} \tilde{u}^{(i)} f^{(i)} \right) - \sum_{i \neq z} u^{(i)} H \left( f^{(i)} \right) - \sum_{i \neq z} u^{(z)} \sum_{i \neq z} \lambda^{(i)} H \left( f^{(i)} \right)
\]

\[
= H \left( \sum_{i \neq z} \tilde{u}^{(i)} f^{(i)} \right) - \sum_{i \neq z} \tilde{u}^{(i)} H \left( f^{(i)} \right) = \varphi(\tilde{u} ; f),
\]

and therefore, one can always increase the objective value of \( \varphi \) by setting \( u^{(z)} = 0 \). ■

Of course, there are other cases where the predictive distribution \( u_* \) is not strictly positive for every \( z \in \mathbb{Z} \). For example, even if one of the densities is an approximate convex combination, the optimal predictive distribution would likely still have \( u_*^{(z)} = 0 \). In general, there is no easy condition to check whether or not \( u_* > 0 \). However, our problem assumes \( m \) is relatively small, and so it is simpler to find \( u_* \) and confirm the channel is admissible. In the case of a discrete noise channel, Shannon and Weaver (1948) gave an efficient way to do this by solving a relaxed version of the concave maximization problem, provided that the transmission matrix \( P \) is invertible.

**Theorem 8 (Shannon)** For a discrete noise channel parameterized by a non-singular transmission matrix \( P \), let

\[
u = \frac{\exp (-\xi^{-1} P^{-1} h(P))}{e^T \exp (-\xi^{-1} P^{-1} h(P))},
\]

where the exponential is taken component-wise. If there exists \( u > 0 \) such that \( u^T P = v^T \), then \( u \in \text{Int}(\Delta^m) \) is the optimal predictive distribution, meaning that \( \nabla_u \varphi(u ; P) = \beta e \) for some \( \beta \in \mathbb{R} \), and \( \varphi(u_* ; P) = C(P) \), and the noise channel is admissible. Otherwise, then there exists some \( z \in \mathbb{Z} \) such that \( u^{(z)} = 0 \), and the noise channel is inadmissible.

**Proof** Using (8) and Lagrangian relaxation,

\[
\sup_{u : e^T u = 1} \varphi(u ; P) = \sup_{u : e^T u = 1} h(u^T P) - u^T h(P)
\]

\[
= \sup_{u \in \mathbb{R}^m} \inf_{\lambda \in \mathbb{R}} h(u^T P) - u^T h(P) - \lambda (e^T u - 1)
\]

Differentiating with respect to \( u \) and setting equal to zero yields

\[
-P \log_2 (P^T u) - h(P) + \xi e - \lambda e = 0,
\]
and since $P$ is invertible,

$$-\log_2 \left( P^T u \right) = P^{-1} h(P) + (\lambda - \xi) e,$$

since $Pe = e$ for all stochastic matrices $P$. Algebra yields

$$P^T u = \exp \left( -\xi^{-1} P^{-1} h(P) + (\lambda/\xi - 1) e \right)$$

$$= \Lambda \cdot \exp \left( -\xi^{-1} P^{-1} h(P) \right),$$

where $\Lambda = \exp \left( \lambda/\xi - 1 \right)$ is some positive constant. We require $e^T u = 1$, and if $u^T P = v^T$, it must be that

$$u^T e = u^T P e = v^T e,$$

implying that $e^T u = 1$ if and only if $e^T v = 1$. Hence, $\Lambda$ is a normalizing constant that allows $v^T P = 1$. Thus, we can set $v$ as in (13), and now it is clear that $v \in \Delta^m$. We can invert $P$ to find an explicit form for $u$, but $P^{-T} v$ is only feasible for the original optimization problem if it is non-negative. However, if there exists some $u \in \Delta^m$ such that $u^T P = v^T$, then the optimal solution to the relaxed problem is feasible for the original optimization problem, proving the theorem.

If there does not exist some $u \geq 0$ that satisfied $u^T P = v^T$ for $v$ defined in (13), then the non-negativity constraint would be tight, and $u^*(z) = 0$ for some $z \in Z$. In this case, the noise channel is inadmissible, because it implies asking the optimal question under entropy pursuit would assign zero probability to one of the alternatives being the model consistent answer, and thus posits a question of strictly less than $m$ alternatives to the user.

The condition of $P$ being non-singular has an enlightening interpretation. Having a non-singular transmission matrix implies there would be no two distinct predictive distributions for $Z_k(X_k)$ that yield the same predictive distribution over $Y_k(X_k)$. This is critical for the model to be identifiable, and prevents the previous problem of having one row of $P$ being a convex combination of other rows. The non-singular condition is reasonable in practice: it is easy to verify that matrices in the form $P = \alpha I + (1 - \alpha) v e^T$ for some $v \in \Delta^m$ is invertible if and only if $\alpha > 0$. Transmission matrices of this type are fairly reasonable: with probability $\alpha$, the user selects the “true response,” and with probability $(1 - \alpha)$, the user selects from discrete distribution $v$, regardless of $Z_k(X_k)$. The symmetric noise channel is a special case of this. In general, if one models $P = \alpha I + (1 - \alpha) S$, where $S$ is an $m \times m$ stochastic matrix, then $P$ is non-singular if and only if $-\alpha/(1 - \alpha)$ is not an eigenvalue of $S$, which guarantees that $P$ is invertible when $\alpha > 1/2$. Nevertheless, regardless of whether or not $P$ is singular, it is relatively easy to check the admissibility of a noise channel, and consequently conclude whether or not it is a good modeling choice for the purpose of preference elicitation.

### 3.2 Sensitivity Analysis

In reality, we cannot always fabricate alternatives so that the predictive distribution is exactly optimal. In many instances, the set of alternatives $X$ is finite. This prevents us from choosing an $X_k$ such that $u_k(X_k) = u_*$ exactly. But if we can find a question that has a predictive distribution that is sufficiently close to optimal, then we can reduce the entropy
at a rate that is close to the channel capacity. Below, we elaborate on our definition of sufficiently close by showing $\varphi$ is strongly concave, using the Hessian to construct quadratic upper and lower bounds on the objective function $\varphi$.

**Theorem 9** If there exists $u_\ast \in \Delta^m$ such that $u_\ast > 0$ and $\varphi(u_\ast ; f) = C(f)$ (i.e., if the noise channel is admissible), then there exist constants $0 \leq r(f) \leq R(f)$ such that

$$ r(f) \cdot \|u - u_\ast\|^2 \leq C(f) - \varphi(u ; f) \leq R(f) \cdot \|u - u_\ast\|^2. $$

Further, suppose transmission matrix $P$ encoding a discrete noise channel is non-singular, and has minimum probability $\kappa_1 = \min_{zy} P^{(zy)}> 0$, maximum probability $\kappa_2 = \max_{zy} P^{(zy)}$, channel capacity $C(P)$ and distribution $u_\ast$ such that $\varphi(u_\ast ; P) = C(P)$. If $u_\ast > 0$, we have

$$ \frac{\xi}{2\kappa_2} \| (u - u_\ast)^T P \|^2 \leq C(P) - \varphi(u ; P) \leq \frac{\xi}{2\kappa_1} \| (u - u_\ast)^T P \|^2 $$

for all $u \in \Delta^m$, with $\xi = \log_2 e$.

**Proof** The $(z, w)$ component of $-\nabla^2\varphi(\cdot ; f)$ is lower bounded by

$$ \int_Y \frac{f^{(z)}(y) f^{(w)}(y)}{\sum_{i \in \mathbb{Z}} u^{(i)} f^{(i)}(y)} \, dy \geq \frac{1}{\max_{i \in \mathbb{Z}, y \in Y} f^{(i)}(y)} \int_Y f^{(z)}(y) f^{(w)}(y) \, dy, \quad (14) $$

since the denominator can be upper bounded. Let $M$ denote the $m \times m$ matrix with its $(z, w)$ component equal to the right-most term in (14) above. Since it can be written as a Gram matrix for an integral product space, $M$ is positive semi-definite, and it is clear that $M \preceq -\nabla^2 \varphi(u ; f)$ for all $u \in \Delta^m$. Correspondingly, let $r(f)$ be the smallest eigenvalue of $M$.

For an upper bound, we employ a different approach. Let $q_R(u) = C - (R/2) \|u - u_\ast\|^2$ denote the implied quadratic lower bound to $\varphi$. It is clear that $q_R(u) \geq 0$ if and only if $\|u - u_\ast\| \leq \sqrt{2C/R}$. Since $\varphi$ is a non-negative function, we only need to find $R$ so that $q_R$ is a lower bound when $q_R(u) > 0$. Consider

$$ \inf_R R \quad \text{s.t.} \quad q_R(u) \leq \varphi(u ; f) \quad \forall u : \|u - u_\ast\| < \sqrt{2C/R}. $$

The problem is feasible since $\nabla^2 \varphi$ is continuous about $u_\ast$, and hence, there exists an $R$ sufficiently large such that $q_R$ is a lower bound of $\varphi$ in a small neighborhood around $u_\ast$. The problem is obviously bounded since the optimal value must be greater than $r(f)$. Now let $R(f)$ denote the optimal value to the problem above. Taylor expanding about $u_\ast$ yields

$$ r(f) \cdot \|u - u_\ast\|^2 \leq C(f) - \varphi(u ; f) + \nabla_u \varphi(u_\ast ; f)^T (u - u_\ast) \leq R(f) \cdot \|u - u_\ast\|^2. $$

But since $u_\ast > 0$, optimality requires $\nabla_u \varphi(u_\ast ; f) = \beta e$ for some $\beta \in \mathbb{R}$. Since $u$ and $u_\ast$ are both probability distributions,

$$ \nabla_u \varphi(u_\ast ; f)^T (u - u_\ast) = \beta e^T (u - u_\ast) = 0, $$
and hence the lower and upper bounds hold.

The proof for the discrete noise channel case is similar, with the exception being that we can easily find constants that satisfy the quadratic lower and upper bounds of the optimality gap \( C(P) - \varphi(u_\star; P) \). We observe the elements of \( u^T P \) are lower bounded by \( \kappa_1 = \min_{zy} P_{zy} \) and upper bounded by \( \kappa_2 = \max_{zy} P_{zy} \). Therefore, for all \( u \in \Delta^m \),

\[
\xi \kappa_1^{-1} P P^T \leq \nabla^2 \varphi(u; P) \leq \xi \kappa_2^{-1} P P^T.
\]

Lemma 5 implies that \( \nabla_u \varphi(u_\star | P) = \beta e \) since \( u_\star > 0 \). Thus, Taylor expansion about \( u_\star \) yields

\[ \xi \kappa_1 \| (u - u_\star)^T P \|^2 \leq (C(f) - \varphi(u; P) + \nabla_u \varphi(u_\star; P)^T (u - u_\star)) \leq \xi \kappa_2 \| (u - u_\star)^T P \|^2. \]

Lastly, since both \( u_\star \) and \( u \) are distributions, their components sum to one, implying \( \nabla_u \varphi(u; P)^T (u - u_\star) = 0 \). The result directly follows.

This gives us explicit bounds on the entropy reduction in terms of the \( L_2 \) distance of a question’s predictive distribution from the optimal predictive distribution. In theory, this allows us to enumerate through all questions in \( X^m \) and select that whose predictive distribution is closest to optimal, although this is difficult when the size of \( X \) is large.

### 3.2.1 Symmetric Noise Channel

A symmetric noise channel is a special case of a discrete noise channel, where the transmission matrix entries only depend on whether or not \( y = z \). There are many instances where in a moment of indecision, the user can select an alternative uniformly at random, especially when she does not have a strong opinion on any of the presented alternatives. A symmetric noise channel useful for modeling situations when \( m \) is relatively small; if \( m \) is large, with the offered alternatives being presented as a list, the positioning in the list might have an effect on the user’s response. However, if the number of alternatives in the comparative question is small, the ordering should not matter, and a symmetric noise channel would be a reasonable modeling choice.

One way to parameterize a symmetric noise channel is by representing the transmission matrices as \( P_\alpha = \alpha I + (1 - \alpha)(1/m) ee^T \), where \( e \) is a vector of all ones, and \( \alpha \in [0, 1] \). There are other scenarios including symmetric noise channels that allow \( P^{(zy)} > P^{(zz)} \) for \( y \neq z \), but these situations would be particularly pessimistic from the perspective of learning, so we opt to exclude these noise channels from our definition. Since \( \varphi(\cdot; P_\alpha) \) is concave and now symmetric in its first argument, choosing \( u_\star^{(z)} = 1/m \) for every \( z \in Z \) is an optimal solution. Thus, we want to choose the question \( X_k \) so that the user is equally likely to choose any of the offered alternatives.

In the case of symmetric noise, we can easily calculate the channel capacity using (8), yielding

\[
C(P_\alpha) = \log_2 m - h \left( \alpha e^{(1)} + (1 - \alpha)(1/m) e \right),
\]

(15)

where \( e^{(1)} \) is an \( m \)-vector with its first component equal to one, and all others equal to zero. The concavity of \( h \) gives a crude upper bound for the channel capacity, namely \( C(P_\alpha) \leq \alpha \log_2 m \). Comparatively, under no noise, one can reduce the entropy of the posterior of the
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linear classifier by \( \log_2 m \) bits at every time epoch. There is an intuitive explanation for this result. With noise level \( \alpha \), we only observe the model-consistent response with probability \( \alpha \) at each step. Even under the best case scenario of knowing which responses were model-consistent and which were a random draw, the expected number of bits of reduced posterior entropy at each step would only be \( \alpha \log_2 m \). In fact, the expected entropy reduction in reality is lower than this because we do not know which responses are informative of linear classifier \( \theta \).

Because the symmetric noise channel is a special case of a discrete noise channel, we can leverage the results from Theorem 9 to derive symmetric noise channel sensitivity bounds.

**Corollary 10** Suppose we have a symmetric noise channel parameterized by \( P_\alpha \), where \( P_\alpha = \alpha I + (1 - \alpha)(1/m) ee^T \), implying that \( u^{(z)}_* = 1/m \) for all \( z \). Then

\[
\frac{\xi \alpha^2}{2(\alpha + (1 - \alpha)(1/m))} \|u - u_*\|^2 \leq C(P_\alpha) - \varphi(u; P_\alpha) \leq \frac{\xi \alpha^2}{2(1 - \alpha)(1/m)} \|u - u_*\|^2
\]

for all \( u \in \Delta^m \).

**Proof** We start with the bounds from Theorem 9 and further refine. The off-diagonal entries of \( P_\alpha \), by our parameterization of symmetric noise channel, are its smallest elements, and therefore, \( \kappa_1 = (1 - \alpha)(1/m) \). Similarly, the diagonal entries of \( P_\alpha \) are the largest elements, and so \( \kappa_2 = \alpha + (1 - \alpha)(1/m) \). Lastly, one can easily verify \( (u - u_*)^T P = \alpha (u - u_*)^T \).

We return to the symmetric noise channel case in Section 3.3, where we show that in the theoretical case of allowing fabrication of alternatives, a subset of alternatives can always be constructed to achieve a uniform predictive distribution regardless of the prior, and hence the optimal rate of entropy reduction can always be achieved.

### 3.3 Selection of Alternatives from the Continuum

Now that we have results relating the predictive distribution \( u_k(X_k) \) to the entropy reduction in the linear classifier \( \theta \), we now explore how we can appropriately choose alternatives \( X_k \) at every time epoch that yield a desirable predictive distribution.

We first focus on the easier case where we can construct alternatives to ask any comparative questions we desire. For a set of \( m \) alternatives \( (x^{(1)}, \ldots, x^{(m)}) = X \in \mathbb{X}^m \) and a prior probability measure \( \mu \), the characteristic polytopes \( A^{(1)}(X), \ldots, A^{(m)}(X) \) determine the predictive probabilities. Each set \( A^{(z)}(X) \) composed of constraints \( \theta^T (x^{(z)} - x^{(i)}) \geq 0 \) for \( i \neq z \) (ignoring strictness vs. non-strictness of inequalities). Thus, for the set of alternatives \( X \) to have full expressiveness with respect to our model, one must be able to choose alternatives so that \( x^{(i)} - x^{(j)} \) can take any direction in \( \mathbb{R}^d \). A reasonable and sufficient condition for the interior of \( X \) to be non-empty. When this is the case, we can always choose alternatives such that the relative direction between any two can take any value. This is what we refer to as the continuum regime.

In most practical situations, the set of alternatives is finite, and such construction is not possible. However, this assumption is more mathematically tractable and allows us to give conditions for when we can ask questions that yield a desirable predictive distribution, and
consequently maximize entropy reduction. We return to the more realistic assumption of a
finite alternative set later in Section 5.

Consider using pairwise comparisons, i.e., when \( m = 2 \). Is it true that regardless of the
noise channel and the prior distribution of \( \theta \) that we can select a question \( X_k \) that achieves
the optimal predictive distribution \( u_k(X_k) \)? A simple example proves otherwise. Suppose
\textit{a priori}, the linear classifier \( \theta \) is normally distributed with zero mean and an identity
covariance matrix. Because the distribution is symmetric about the origin, regardless of
the hyperplane we select, exactly \( 1/2 \) of the probabilistic mass lies on either side of the
hyperplane. This is the desirable outcome when the noise channel is symmetric, but suppose
this were not the case. For example, if the noise channel required \( 2/3 \) of the probabilistic
mass on one side of the hyperplane, there is no way to achieve this.

This issue is related to a certain metric called \textit{halfspace depth}, first defined by Tukey
(1975) and later refined by Donoho and Gasko (1992). The halfspace depth at a point \( \eta \in \mathbb{R}^d \)
refers to the minimum probabilistic mass able to be partitioned to one side of a
hyperplane centered at \( \eta \). In this paper, we only consider the case where the cutting plane
is centered at the origin, and need only to consider the case where \( \eta = 0 \). Hence, let

\[
\delta(\mu_k) = \inf_{v \neq 0} \mu_k \left( \{ \theta : \theta^T v \geq 0 \} \right).
\] (16)

In our previous example, the halfspace depth of the origin was equal to \( 1/2 \), and therefore,
there were no hyperplanes that could partition less than \( 1/2 \) of the probabilistic mass on a
side of a hyperplane.

The question now is whether we can choose a hyperplane such that \( u_k(z)(X_k) = u_z(z) \)
for any \( u_z(z) \in [\delta(\mu_k), 1 - \delta(\mu_k)] \). We first prove an intuitive result regarding the continuity
of probabilistic mass of a halfspace with respect to the cutting plane. One can imagine
rotating a hyperplane about the origin, and since the probability measure has a density
with respect to Lebesgue measure, there will not be any sudden jumps in probabilistic mass
on either side of the hyperplane.

\textbf{Lemma 11} \textit{If probability measure \( \mu \) is absolutely continuous with respect to Lebesgue mea-
sure, then the mapping \( v \mapsto \mu \left( \{ \theta \in \Theta : \theta^T v \geq 0 \} \right) \) is continuous.}

\textbf{Proof} Suppose we have a sequence \( (v_j : j \geq 0) \) in \( \mathbb{R}^d \setminus \{0\} \) such that \( v_j \to v \). The functions \( \mathbb{I}(\{ \theta : \theta^T v_j \geq 0 \}) \) converge to \( \mathbb{I}(\theta : \theta^T v \geq 0) \) almost surely. Taking expectations and using
Figure 2: Selection of alternatives $x^{(1)}$ and $x^{(2)}$. Since $x^{(1)}$ lies in the interior of $X$, it is always possible to choose $x^{(2)}$ so that $x^{(1)} - x^{(2)}$ has the same direction as $v$.

Dominated Convergence Theorem gives the result.  

Lemma 11 enables us to find conditions under which we can ask a question $X_k$ that yields a desirable predictive distribution $u_k(X_k)$. In particular, Corollary 12 uses a variant of the intermediate value theorem.

**Corollary 12** Suppose $u_* > 0$ and $\text{Int}(X) \neq \emptyset$. Then there exists $X_k = (x_1, x_2) \in X^2$ such that $u_k(X_k) = u_*$ if and only if $\max u_* \leq 1 - \delta(\mu_k)$.

**Proof** Take any $v \in C = \{w \in \mathbb{R}^d : \|w\| = 1\}$, where $\mu_k(\{\theta : \theta^T v \geq 0\}) = \delta(\mu_k)$. Now let $v' = -v$, and since $\mu_k$ is absolutely continuous with respect to Lebesgue measure, $\mu_k(\theta^T v' \geq 0) = \mu_k(\theta^T v' > 0) = 1 - \delta(\mu_k)$. Also, $C$ is connected, and $w \mapsto \mu_k(\{\theta : \theta^T w \geq 0\})$ is a continuous mapping: it follows that the image of any path from $v$ to $v'$ must also be connected. But the image is a subset of the real line, and therefore must be an interval. Lastly, $C$ is a compact set, implying that the endpoints of this interval are attainable, and so the image of any such path is equal to $[\delta(\mu_k), 1 - \delta(\mu_k)]$.

To recover the two alternatives, first select a vector $w \in \mathbb{R}^d$ such that $\mu_k(\{\theta : \theta^T w \geq 0\}) = u^{(1)}$. Choose $x^{(1)} \in \text{Int}(X)$, and subsequently choose $x^{(2)} = x^{(1)} - cw$, where $c > 0$ is a positive scalar that ensures $x^{(2)} \in X$. Finally, let $X_k = (x^{(1)}, x^{(2)})$.

To prove the converse statement, suppose $\max u_* > 1 - \delta(\mu_k)$. Then by definition of halfspace depth, $\min u_* \notin \{\mu_k(\{\theta : \theta^T v \geq 0\}) : v \neq 0\}$. Thus, there does not exist a hyperplane that can separate $\mathbb{R}^d$ into two halfspaces with probabilistic mass $u_*$.

Can we draw a similar conclusion if we offer more more than two alternatives at each time epoch? The mass partition problem becomes increasingly complex when greater than two alternatives are included. Since the sets $A(z)(X)$ correspond to convex polyhedral cones, the problem becomes that of finding a partition of $m$ convex polyhedral cones, or a polyhedral $m$-fan as it is known in the computational geometry literature, that attains the prescribed probabilistic mass $u_*$. There are a number of results pertaining to convex equipartitions and extensions of the Borsuk-Ulam Theorem, most notably the Ham Sandwich Theorem. Despite this, to the best of our knowledge, there is no result for general mass partitions of convex polyhedral $m$-fans in the computational geometry literature. For this reason, we prove such a result here: that one can construct a polyhedral $m$-fan with the corresponding predictive distribution $u_*$ if the measure $\mu$ is such that $\max u_* < 1 - \delta(\mu)$. 

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Unlike the previous case that focused on pairwise comparisons, the inequality is strict. One of the reasons this is the case is because of the specific structure of the polyhedral cones in our problem. Since \( A(z)(x) \) corresponds to the linear classifier in which the dot product with alternative \( z \) is maximal, these polyhedral cones cannot be halfspaces unless the predictive probability for some alternative equals zero, which we do not allow. Thus, we enforce the additional constraint that each \( A(z) \) is a salient convex polyhedral cone, meaning that it does not contain a linear subspace.

To prove the result, we first show the result in the case of two dimensions: constructing the polyhedral \( m \)-fan, then deriving the feature vectors for the corresponding alternatives. This result is then generalized to the case of any dimension by using a projection argument.

**Lemma 13** Suppose \( d = 2 \) and \( m > 2 \). If \( \max \{ u_* \} < 1 - \delta(\mu) \), then there exists a two-dimensional polyhedral \( m \)-fan characterized by polyhedral cones \( (A(z) : z \in \mathbb{Z}) \) such that \( \mu(A(z)) = u_*^{(z)} \) for all \( z \in \mathbb{Z} \).

**Proof** Without loss of generality we can assume \( \|\theta\| = 1 \), and in the case of two dimensions that is equivalent to \( \theta \) being parameterized by the interval \([0, 2\pi)\) on the unit circle. For an interval \( I \) measuring angles in radians, let

\[
\text{Cone}(I) = \left\{ \left( \begin{array}{c} r \cos \eta \\ r \sin \eta \end{array} \right) : \eta \in I, \ r > 0 \right\}.
\]

Accordingly, let \( \mu^C \) be a measure defined on the unit circle such that \( \mu^C(I) = \mu(\text{Cone}(I)) \) for every Lebesgue-measurable interval on \([0, 2\pi)\). This implies that \( \delta(\mu^C) = \delta(\mu) \). For radian angles \( \eta^{(1)} < \eta^{(2)} < \cdots < \eta^{(m+1)} = \eta^{(1)} + 2\pi \), we define

\[
B^{(z)} = \begin{cases} 
\left[ \eta^{(1)}, \eta^{(2)} \right] & z = 1 \\
\left( \eta^{(z)}, \eta^{(z+1)} \right] & z = 2, \ldots, m - 1 \\
\left( \eta^{(m)}, \eta^{(m+1)} \right) & z = m
\end{cases}
\]

for all \( z \in \mathbb{Z} \). The asymmetry with respect to sets being \( B^{(z)} \) closed or open is due to the definition of \( A(z) \) in (1). For each \( B^{(z)} \) to correspond to a convex set strictly contained in a halfspace, we require \( \eta^{(z+1)} - \eta^{(z)} < \pi \). Our objective is to appropriately select the angles \( (\eta^{(z)} : z \in \mathbb{Z}) \) so that \( \mu^C(B^{(z)}) = u^{(z)} \). It suffices to consider only two cases.

**Case 1:** \( \max \{ u_* \} < \delta(\mu) \)

This is the simpler case, since all the probabilities from the predictive distribution are strictly smaller than any halfspace measure. Arbitrarily choose \( \eta^{(1)} \). Now we want to choose \( \eta^{(2)} \in (\eta^{(1)}, \eta^{(1)} + \pi) \) so that the interval contains prescribed measure \( u_*^{(1)} \). The function \( \eta^{(2)} \mapsto \mu^C\left( [\eta^{(1)}, \eta^{(2)}] \right) \) is monotonically increasing, continuous, and takes values on \((0, \delta(\mu)]\). Since \( u_*^{(2)} \leq \max u_* \leq \delta(\mu_k) \), the Intermediate Value Theorem allows us to choose \( \eta^{(2)} \) so the interval has measure \( u_*^{(1)} \). Continue in this way until all such angles \( \eta^{(z)} \) are attained.

**Case 2:** \( \max \{ u_* \} \in [\delta(\mu), 1 - \delta(\mu)] \)
Here, it is necessary to define the set $B^z$ corresponding to the largest predictive probability $u_*$ first, and that with the smallest second. Without loss of generality, suppose $u_*^{(1)} = \max u_*$ and $u_*^{(2)} = \min u_*$. Then choose $\eta^{(1)}$ such that

$$\mu^C \left( [\eta^{(1)}, \eta^{(1)} + \pi] \right) \in (\max u_*, \max u_* + \min u_*) .$$

This is possible because

$$(\delta(\mu), 1 - \delta(\mu)) \cap (\max u_*, \max u_* + \min u_*) \neq \emptyset,$$

due to the assumptions that $\max u_* \in [\delta(\mu), 1 - \delta(\mu))$ and $\min u_* > 0$. Now define $\eta^{(2)}$ such that $\mu^C[\eta^{(1)}, \eta^{(2)} + \pi] > \min u_*$. Then by the Intermediate Value Theorem, there exists some $\eta^{(3)}$ such that $\mu^C(\eta^{(2)}, \eta^{(3)}) = \min u_*$. Otherwise, suppose that $\mu^C(\eta^{(m+1)} - \pi, \eta^{(m+1)}) > \min u_*$. Again, by the Intermediate Value Theorem, we can find $\eta^{(m)}$ less than $\pi$ radians from $\eta^{(m+1)}$ such that $\mu^C(\eta^{(m)}), \eta^{(m+1)}) = \min u_*$.

We claim that these are the only two possibilities. By way of contradiction, suppose that neither of these scenarios are true; in other words,

$$\mu^C[\eta^{(2)}, \eta^{(2)} + \pi] \leq \min u_*$$

$$\mu^C[\eta^{(m+1)} - \pi, \eta^{(m+1)}] \leq \min u_* .$$

We can decompose these intervals into non-overlapping parts. Define

$$a = \mu^C(\eta^{(2)} + \pi, \eta^{(1)} + 2\pi]$$

$$b = \mu^C(\eta^{(2)}, \eta^{(1)} + \pi]$$

$$c = \mu^C(\eta^{(1)} + \pi, \eta^{(2)} + \pi).$$

Suppose that $\max\{a, b\} + c \leq \min u_*$. The measure of the union of the three intervals $1 - \max u_* = a + b + c$, which implies $1 - \max u_* \leq \min u_* + \min\{a, b\}$. Finally, since
the smallest component of \( u_* \) must be smaller in magnitude than the sum of the other non-maximal components,

\[
\max\{a, b\} + c \leq \min u_* \leq 1 - \max u_* - \min u_* \leq \min\{a, b\},
\]

implying among other things that \( b = \min u_* \) in this scenario. However, this is a contradiction, since we originally chose \( \eta^{(1)} \) such that \( b + c < \max u_* + \min u_* \) due to (17). Therefore, this scenario is not possible, and we can always find an interval with probabilistic mass strictly greater than \( \min u_* \) directly adjacent to an interval with maximal probabilistic mass.

In all cases, we have defined the first two intervals, and the remaining unallocated region of the unit circle is strictly contained in an interval of width less than \( \pi \) radians. Thus, one can easily define a partition as in Case 1, and every subsequent interval would necessarily have to have length strictly less than \( \pi \) radians. To recover the convex cones, let \( A^{(z)} = \text{Cone}(B^{(z)}) \) for every \( z \in \mathbb{Z} \), and it is clear that \( A^{(z)} \) contains the desired probabilistic mass.

Lemma 13 gives a way to construct polyhedral fans with the desired probabilistic mass. We are interested in finding a set of alternatives that represents this polyhedral fan, and this is exactly what Theorem 14 does in the two-dimensional case. The critical condition required is for the set of alternatives \( \mathbb{X} \) to have non-empty interior.

**Theorem 14** Suppose \( d = 2 \) and \( m > 2 \). Then given a measure \( \mu \) that is absolutely continuous with respect to Lebesgue measure and an optimal predictive distribution \( u_* \), if \( \text{Int}(\mathbb{X}) \neq \emptyset \) and \( \max u_* < 1 - \delta(\mu) \), then there exists \( X \in \mathbb{X}^m \) such that \( u(X) = u_* \).

**Proof** First, use Lemma 13 to construct a polyhedral fan with the correct probabilistic weights. Using the angles \( \eta^{(1)}, \ldots, \eta^{(m)} \) constructed in the Lemma, we can define separating hyperplanes \( v^{(1)}, \ldots, v^{(m)} \) by setting \( v^{(z)} = (-\sin \eta^{(z)}, \cos \eta^{(z)}) \). Then we have

\[
\bar{A}^{(z)} = \left\{ \theta : \frac{\theta^T v^{(z)}}{\theta^T v^{(z+1)}} > 0, \frac{\theta^T v^{(z+1)}}{\theta^T v^{(z)}} \leq 0 \right\}.
\]

The goal now is to define the alternatives. First, choose \( x^{(1)} \in \text{Int}(\mathbb{X}) \). Now define \( x^{(z+1)} = x^{(z)} + e^{(z+1)}u^{(z+1)} \), where \( e^{(z+1)} > 0 \) is a positive scaling that ensures \( x^{(z+1)} \in \text{Int}(\mathbb{X}) \) if \( x^{(z)} \in \text{Int}(\mathbb{X}) \). Now we can equivalently write

\[
\tilde{A}^{(z)} = \left\{ \theta : \frac{\theta^T (x^{(z)} - x^{(z-1)})}{\theta^T (x^{(z)} - x^{(z+1)})} > 0 \right\}.
\]

Let \( X = (x^{(1)}, \ldots, x^{(m)}) \). It remains to show that \( A^{(z)}(X) = \tilde{A}^{(z)} \). Because \( A^{(z)}(X) \) has the same linear inequalities as \( \tilde{A}^{(z)} \), it is clear that \( \tilde{A}^{(z)}(X) \subseteq \tilde{A}^{(z)} \) for all \( z \). Now suppose there exists some \( \theta \in \tilde{A}^{(z)} \). Since \( \tilde{A}^{(z)} : z \in \mathbb{Z} \) is a partition of \( \mathbb{R}^2 \), it is clear that \( \theta \notin A^{(z')} \) for \( z' \neq z \), and thus, \( \theta \notin \tilde{A}^{(z')} \). Since \( \tilde{A}^{(z)}(X) : z \in \mathbb{Z} \) is also a partition of \( \mathbb{R}^2 \), it must be that \( \theta \in A^{(z)}(X) \). This directly implies \( \tilde{A}^{(z)} = A^{(z)}(X) \), and so \( u^{(z)}(X) = \mu(A^{(z)}) = u_*^{(z)} \).
Figure 4: Iterative selection of alternatives. Since each \( x^{(z)} \) is in the interior of \( X \), it is always possible to select \( x^{(z+1)} \) to maintain a specific direction for \( x^{(z+1)} - x^{(z)} \).

Theorem 14 shows that in the case of two dimensions, a set of \( m \) alternatives can be generated to ensure that the entropy of the posterior distribution of \( \theta \) maximally decreases. This result can be generalized to arbitrary dimension by selecting a two dimensional subspace and leveraging the previous result.

**Theorem 15** Suppose \( \text{Int}(X) \neq \emptyset \) and \( u_* > 0 \). If \( \max u_* < 1 - \delta(\mu_k) \), then there exists \( X_k = (x^{(1)}, x^{(2)}, \ldots, x^{(m)}) \in X^m \) such that \( u_k(X_k) = u_* \). Further, if \( \max u_* > 1 - \delta(\mu_k) \), then finding such a question is not possible.

**Proof** We begin by proving the last claim of the theorem. Since any \( A^{(z)}(X_k) \) can be contained by a halfspace centered at the origin, and since all such halfspaces have probabilistic mass less than or equal to \( 1 - \delta(\mu_k) \), then we must have \( \mu_k(A^{(z)}(X)) \leq 1 - \delta(\mu_k) \) for every \( z \in \mathbb{Z} \) and for every \( X_k \in X^m \).

Now we show the main result of the theorem. There exists some \( \beta \in \mathbb{R}^m \setminus \{0\} \) such that \( \mu_k \{ \theta : \theta^T \beta \geq 0 \} = \delta(\mu_k) \), since \( \mu_k \) has density \( p_k \) and is continuous. Let \( \mathcal{H} = \{ \theta : \theta^T \beta = 0 \} \) denote the hyperplane. Now choose a two-dimensional subspace \( L \) such that \( L \perp \mathcal{H} \). For \( \nu \in L \), define density \( p_k^L \) as

\[
p_k^L(\nu) = \int_{\omega \in L^\perp} p_k(\nu + \omega) \lambda_{d-2}(d\omega),
\]

where \( \lambda_{d-2} \) is \((d - 2)\)-Lebesgue measure, and let \( \mu_k^L \) denote measure induced by density \( p_k^L \). For \( \beta \in L \), we have

\[
\mu_k^L \{ \nu \in L : \nu^T \beta \geq 0 \} = \int_{\nu \in L : \nu^T \beta \geq 0} p_k^L(\nu) \lambda_2(d\nu)
\]

\[
= \int_{\nu \in L : \nu^T \beta \geq 0} \int_{\omega \in L^\perp} p_k(\nu + \omega) \lambda_{m-2}(d\omega) \lambda_2(d\nu)
\]

\[
= \int_{(\nu, \omega) \in (L \times L^\perp) : (\nu + \omega)^T \beta \geq 0} p_k(\nu + \omega) \lambda_d(d\nu \times d\omega)
\]

\[
= \int_{\theta : \theta^T \beta \geq 0} p_k(\theta) \lambda(d\theta) = \mu_k \{ \theta : \theta^T \beta \geq 0 \}.
\]

Thus, \( \mu_k^L \) is consistent with \( \mu_k \). In particular, \( \mu_k^L \{ \theta : \theta^T \beta \geq 0 \} = \delta(\mu_k) \), and thus \( \delta(\mu_k^L) \leq \delta(\mu_k) \), meaning we can use the previous Theorem 14 to find an appropriate comparative question.
In particular, let $\gamma_1$ and $\gamma_2$ denote two orthogonal $d$-vectors that span $L$, and let $\Gamma \in \mathbb{R}^{d \times 2}$ contain $\gamma_1$ and $\gamma_2$ as its columns. To use the Theorem 14, we pass $\mu_k^L \circ \Gamma$, and to convert the resulting question $X_k^L = (x^{(1)}, \ldots, x^{(m)})$ back into $d$-dimensional space, take $X_k = (\Gamma x^{(1)}, \ldots, \Gamma x^{(m)})$.

Theorem 15 provides one possible construction for a question $X_k$ that gives a desirable predictive distribution, although there may be others. However, it is clear that if the halfspace depth $\delta(\mu_k)$ is too large, it will not always be possible to find a question that can yield the optimal predictive distribution, even if we can construct questions in the continuum. But while it may not be possible to maximally reduce the entropy of the posterior distribution, we may choose a question $X_k$ that can still reduce entropy by a constant amount at each time epoch.

We conclude the section by showing that entropy in the linear classifier $\theta$ can be reduced linearly, even if not optimally.

**Theorem 16** Suppose $\mathcal{X} = \mathbb{R}^d$, and let $\sigma_k = (\max\{u_*\} - (1 - \delta(\mu_k)) + \epsilon)^+$. Then the following upper bound holds when $m = 2$ for $\epsilon = 0$ and when $m > 2$ for arbitrarily small $\epsilon > 0$.

$$
K \cdot C(f) - \sup_{\pi} \mathbb{E}_\pi^\pi[I(\theta; \mathcal{Y}_k)] \leq r(f) \left(1 + \frac{1}{m - 1}\right) \sum_{k=1}^{K} \mathbb{E}_\pi^\pi[\sigma_k^2] \\
\leq K \cdot r(f) \left(1 + \frac{1}{m - 1}\right) \left((\max\{u_*\} - 1/2 + \epsilon)^+\right)^2.
$$

**Proof** We start with the case when $m > 2$. Fix any small $\epsilon > 0$. Let $z' = \arg\max\{u_*\}$. We write equality because in the cases where $\sigma_k > 0$, the maximum component is unique; otherwise, when $\sigma_k = 0$, the choice of $z'$ is irrelevant. We construct an “approximate predictive distribution” $\bar{u}_k$ such that

$$
\bar{u}_k^{(z)} = \begin{cases} 
  u_*^{(z)} - \sigma_k & z = z' \\
  u_*^{(z)} + \sigma_k/(m - 1) & z \neq z'
\end{cases}
$$

This new vector $\bar{u}_k$ is the projection of $u_*$ onto the set $\{u \in \Delta^n : \max\{u\} \leq (1 - \delta(\mu_k)) - \epsilon\}$. This “approximate predictive distribution” is chosen to minimize the $L_2$ distance from optimal $u_*$, and therefore maximize entropy reduction.

One can show that $\max\{|\bar{u}_k| < 1 - \delta(\mu_k)$, and $\|\bar{u}_k - u_*\|^2 \leq \sigma_k^2 (1 + 1/(m - 1))$. Now we can construct $X_k$ such that $u_k(X_k) = \bar{u}_k$ at every step, which is possible by Theorem 15.
since \( \bar{\mu} > 0 \) and \( \max \{ \bar{\mu} \} < 1 - \delta(\mu_k) \). Now we use Theorem 9 to show

\[
K \cdot C(f) - \sup_{\pi} I^\pi(\theta; Y_k) \leq \sum_{k=1}^{K} (C(f) - \mathbb{E}_k^\pi [\varphi(u_k(\bar{X}_k); f)])
\]

\[
= \sum_{k=1}^{K} (C(f) - \mathbb{E}_k^\pi [\varphi(\bar{u}_k; f)])
\]

\[
\leq \sum_{k=1}^{K} r(f) \mathbb{E}_k^\pi [||\bar{u}_k - u_*||^2]
\]

\[
= r(f) \left( 1 + \frac{1}{m-1} \right) \sum_{k=1}^{K} \mathbb{E}_k^\pi [\sigma_k^2].
\]

And since \( 1 - \delta(\mu_k) \geq 1/2 \), it follows that \( \sigma_k \leq (\max \{ u_* \} - 1/2 + \epsilon)^+ \), regardless of \( \mu_k \).

The proof is analogous for the \( m = 2 \) case: the only change required is to set \( \epsilon = 0 \), because by Corollary 12, we can find a question if \( \max \{ u_* \} \leq 1 - \delta(\pi_k) \), where the inequality need not be strict.

Putting Theorem 16 together with Corollary 4 shows that if the alternative set \( X \) has non-empty interior, the expected differential entropy of linear classifier \( \theta \) can be reduced at a linear rate, and this is optimal up to a constant factor.

Finally, recall in Section 3.2.1 we defined the case of a symmetric noise channel. There, \( u^*(z) = 1/m \) for all \( z \in Z \). In the pairwise comparison case, \( \max u_* = 1/2 \leq 1 - \delta(\mu) \) for all measures \( \mu \). In the multiple comparison case, \( \max u_* = 1/m < 1/2 \leq 1 - \delta(\mu) \) for all measures \( \mu \). Thus, regardless of \( m \), in the continuum setting, a set of alternatives can always be constructed to achieve a uniform predictive distribution, and therefore optimally reduce posterior entropy.

4. Misclassification Error

The entropy pursuit policy itself is intuitive, especially when the noise channel is symmetric. However, differential entropy as a metric for measuring knowledge of the user’s preferences is not intuitive. One way to measure the extent of our knowledge about a user’s preferences is testing ourselves using a randomly chosen question and estimating the answer after observing a response from the user. This probability we get the answer wrong called misclassification error.

Specifically, we sequentially ask questions \( X_k \) and observe signals \( Y_k(X_k) \) at time epochs \( k = 1, \ldots, K \). After the last question, we are then posed with an evaluation question. The evaluation question will be an \( n \)-way comparison between randomly chosen alternatives, where \( n \) can differ from \( m \). Denote the evaluation question as \( S_K \in X^n \), where a particular evaluation question \( S_K = (s^{(1)}, \ldots, s^{(n)}) \). The evaluation question is chosen at random according to some unknown distribution. Denote the model-consistent answer as \( W_K(S_K) = \min \{ \arg \max_{w \in W} \theta^T s^w \} \), where the minimum serves as a tie-breaking rule. The goal is to use history \( Y_K \) and the question \( S_K \) to predict \( W_K(S_K) \). Let \( \hat{W}_K \) denote the
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candidate answer that depends on the chosen evaluation question response history. Then our goal for the adaptive problem is to find a policy that minimizes

$$\mathcal{E}_K^* = \mathbb{E}[\inf_{\hat{W}_K \in \mathcal{W}} \mathbb{P}(W_K(S_K) \neq \hat{W}_K \mid Y_K, S_K)],$$

(18)

and one can do this by adaptively selecting the best question $X_k$ that will allow us to learn enough about the user’s preferences to correctly answer evaluation question $S_k$ with high certainty. Let $\mathcal{E}_K^* = \inf_{\pi} \mathcal{E}_K^\pi$ be the misclassification error under the optimal policy, assuming it is attained.

We make several reasonable assumptions on the dependence of $S_K$ and $W_K(S_K)$ with the model-consistent response $Z_k(X_k)$ and signal $Y_k(X_k)$ from the learning question $X_k$.

**Evaluation Question Assumptions** For evaluation question $S_K = S_K$ and corresponding model-consistent answer $W_K(S_K)$, we assume

- Evaluation question $S_K = S_K$ is selected randomly from $\mathbb{X}^n$, independent from all else, and

- For all such questions $S_K$, signal $Y_k(X_k)$ and model-consistent answer $W_K(S_K)$ are conditionally independent given $Z_k(X_k)$ for all $k = 1, \ldots, K$.

In practice, solving the fully adaptive problem is intractable, and instead, one can use a knowledge gradient policy to approach this problem. This is equivalent to solving a greedy version of the problem where we are evaluated at every step. In other words, after observing signal $Y_k(X_k)$, we are posed with answering a randomly selected evaluation question $S_k$, with no concern about any future evaluation. Every question in the sequence $(S_k : k = 1, \ldots, K)$ is selected i.i.d. and follows the Evaluation Question Assumptions. The knowledge gradient policy chooses $X_k$ such that at every time epoch $k$ it solves

$$\mathcal{E}_K^{KG} = \inf_{X_k \in \mathbb{X}^n} \mathbb{E}[\inf_{\hat{W}_k \in \mathcal{W}} \mathbb{P}(W_k(S_k) \neq \hat{W}_k \mid Y_k, S_k)].$$

Obviously, $\mathcal{E}_K^{KG} \geq \mathcal{E}_K^*$ for all $k$, since knowledge gradient cannot perform strictly better than the fully adaptive optimal policy. It would be beneficial to know how wide the gap is, and this can be done by finding a lower bound on the optimal misclassification error. Information theory provides a way to do this, and in the next sections, we will show a lower bound in terms of the entropy reduction of the underlying linear classifier, and that posterior entropy reduction is necessary to achieve misclassification error reduction.

### 4.1 An Interactive Approach

It would be helpful to have an analogue to Theorem 2 so we can relate the posterior Shannon entropy of the answer $W(S)$ of evaluation question $S$ to the answer $Z_k(X_k)$ of initial question $X_k$. It turns out that information content in observing signal $Y_k(X_k)$ to infer answer $W_k$ is related to a concept in information theory called interaction information. In
Similarly, we define Conditional Interaction Information as
$$I_k(W(S); Y_k; Z_k; S) = I_k(W(S); Y_k | Z_k) - I_k(W(S); Y_k)$$
$$= I_k(Y_k; Z_k | W(S)) - I_k(Y_k; Z_k)$$
$$= I_k(Z_k; W(S) | Y_k) - I_k(Z_k; W_k).$$

Similarly, we define Conditional Interaction Information as
$$I_k(W(S); Y_k; Z_k | S) = \mathbb{E} [I_k(W(S); Y_k; Z_k | S = S)].$$

Interaction information tells us the relationship between three random variables in terms of the redundancy in information content. In general, this quantity can be positive or negative. If the interaction information between three random variables is negative, then one does not learn as much from an observation when already knowing the outcome of another. This is the more natural and relevant case in the context of misclassification error.

In particular, the goal is to ask questions so that the observations can provide the maximum amount of information on the answer to an unknown evaluation question. Theorem 17 decomposes this problem into an equivalent formulation using interaction information, for which we seek to maximize the amount of redundancy between the chosen questions $X_k$ and the unknown evaluation question $S$.

**Theorem 17** Under the Noise Channel Assumptions and Evaluation Question Assumptions, we have
$$I_k(W(S); Y_k(X_k) | S) = I_k(W(S); Y_k(X_k); Z_k(X_k) | S) \leq I_k(Y_k(X_k); Z_k(X_k)). \quad (19)$$

**Proof** First, we use the fact that conditional mutual information is symmetric (Cover, 1991, p. 22) to get
$$H_k(Y_k(X_k) | Z_k(X_k), W(S), S) + H_k(Z_k(X_k) | W(S), S)$$
$$= H_k(Z_k(X_k) | Y_k(X_k), W(S), S) + H_k(Y_k(X_k) | W(S), S),$$
and using the Evaluation Question Assumptions, we see that the first term is equal to $H_k(Y_k(X_k) | Z_k(X_k))$, giving us
$$H_k(Y_k(X_k) | W(S), S) = H_k(Y_k(X_k) | Z_k(X_k)) + H_k(Z_k(X_k) | W(S), S)$$
$$- H_k(Z_k(X_k) | Y_k(X_k), W(S), S).$$
Subtracting both sides of the above equation from $H(Y_k(X_k))$ gives us
$$I_k(W(S); Y_k(X_k) | S) = I_k(Y_k(X_k); Z_k(X_k)) - I_k(Y_k(X_k); Z_k(X_k) | W(S), S) \quad (20)$$
Now since $S$ is independent of $Y_k(X_k)$ and $Z_k(X_k)$, we have $I(Y_k(X_k); Z_k(X_k)) = I(Y_k(X_k); Z_k(X_k) | S)$, and the equality in (19) directly follows. The inequality is because the last term in (20) is non-negative, due to the properties of mutual information. \qed

As previously mentioned, interaction information does not have to be non-negative. Here, the equality in (19) implies that the interaction information is non-negative since $I_k(W(S); Y_k(X_k) | S)$ is always non-negative. This means that when we ask question $X_k$, observing signal $Y_k(X_k)$ yields less information than also knowing the true answer $W(S)$ to another question $S$, an intuitive result. We use Theorem 17 to relate $I_k(W_k(S_k); Y_k(X_k) | S_k)$ to $I_k(\theta; Y_k(X_k))$.
4.2 Lower Bound on Misclassification Error

We now would like to relate misclassification error to the entropy of the posterior distribution of the linear classifier $\theta$. Theorem 18 shows that regardless of the estimator $\hat{W}_k$, one cannot reduce misclassification error without bound unless the posterior entropy of $\theta$ is reduced as well. This is due to an important tool in information theory called Fano’s Inequality.

**Theorem 18** For any policy $\pi$, a lower bound for the misclassification error under that policy is given by

$$\varepsilon_k^\pi \geq \frac{H(W_k(S_k) \mid S_k) - I^\pi(\theta; Y_k)}{\log_2 n} - 1.$$ 

**Proof** Suppose we have a fixed question $S_k = S_k$, and let $\hat{W}_k$ be any estimator of $W_k(S_k)$ that is a function of history $Y_k$ and known assessment question $S_k$. By Fano’s inequality, (Cover, 1991, p. 39), we have

$$P_k(W_k(S_k) \neq \hat{W}_k \mid S_k = S_k) \geq \frac{H_k(W_k(S_k) \mid S_k = S_k) - 1}{\log_2 n}.$$ 

(21)

Taking an expectation over possible assessment questions and past history yields

$$E^\pi \left[ P_k(W_k(S_k) \neq \hat{W}_k \mid S_k) \right] \geq \frac{H^\pi(W_k(S_k) \mid Y_k, S_k) - 1}{\log_2 n},$$

(22)

where the right side holds because of the definition of conditional entropy. Now we use the upper bound on $I_k(W_k(S_k) \mid Y_k(X_k) \mid S_k)$ from Theorem 17 to show

$$H^\pi(W_k(S_k) \mid Y_k, S_k) = H(W_k(S_k) \mid S_k) - I^\pi(W_k(S_k) ; Y_k \mid S_k)$$

$$= H(W_k(S_k) \mid S_k) - E^\pi \left[ \sum_{\ell=1}^{k} I_\ell(W_\ell(S_\ell) ; Y_\ell(X_\ell)) \right]$$

$$\leq H(W_k(S_k) \mid S_k) - E^\pi \left[ \sum_{\ell=1}^{k} I_\ell(Z_\ell(X_\ell) ; Y_\ell(X_\ell)) \right]$$

$$= H(W_k(S_k) \mid S_k) - E^\pi \left[ \sum_{\ell=1}^{k} I_\ell(\theta ; Y_\ell(X_\ell)) \right]$$

$$= H(W_k(S_k) \mid S_k) - I^\pi(\theta ; Y_k),$$

where the penultimate equality is from Theorem 2. Thus, we get

$$E^\pi \left[ P_k(W_k(S_k) \neq \hat{W}_k \mid Y_k(X_k), S_k) \right] \geq \frac{H(W_k(S_k) \mid S_k) - I^\pi(\theta ; Y_k) - 1}{\log_2 n},$$

(23)

and the result follows.

The bound does not provide any insight if $H(W_k(S_k) \mid Y_k, S_k) < 1$ since the lower bound would be negative. This is most problematic when $n = 2$, in which case, the Shannon entropy of $W_k$ is bounded above by one bit. However, if the conditional entropy of $W_k(S_k)$
after observing signal $Y_k(X_k)$ is still significantly large, the misclassification error will not be reduced past a certain threshold.

There are some interesting conclusions that can be drawn from the lower bound. First, $H(W_k(S_k) \mid S_k)$ can be viewed as a constant that describes the problem complexity, representing the expected entropy of evaluation question $S_k$. The lower bound is a linear function with respect to the mutual information of linear classifier $\theta$ and the observation history $Y_k$.

We can use this result to bound both the knowledge gradient policy and the fully adaptive optimal policy from below. Corollary 19 below leverages Theorem 18 to estimate the optimality gap of knowledge gradient from the optimal policy.

**Corollary 19** Under noise channel $f$ with channel capacity $C(f)$, the optimal misclassification error under the optimal policy after asking $k$ comparative questions is bounded by

$$
\frac{H(W_k(S_k) \mid S_k) - C(f) \cdot k - 1}{\log_2 n} \leq \mathcal{E}_k^* \leq \mathcal{E}_k^{KG}.
$$

Of course, there is a fairly significant gap in the lower bound, since the misclassification errors are non-negative, and yet the lower bound is linear. The gap comes from the second inequality in (19), and this upper bound essentially throws out the redundant information about possible evaluation questions learned by previous user responses. Nonetheless, it tells us that posterior entropy reduction is necessary for misclassification error reduction.

5. Computational Results

In the following subsections, we present computational results from simulated responses using vectorizations of real alternatives. Section 5.1 discusses our approach and methodology for the numerical experiments, and Section 5.2 gives the results of the computational studies and provides insights regarding the performance of the entropy pursuit and knowledge gradient policies.

5.1 Methodology

As an alternative space, we use the 13,108 academic papers on arXiv.org from the condensed matter archive written in 2014. The information retrieval literature is rife with different methods on how to represent a document as a vector, including bag of words, term frequency inverse document frequency (Salton and McGill, 1986), and word2vec (Goldberg and Levy, 2014), along with many others (for an overview of such methods, see Raghavan and Schütze, 2008). In practice, the method for vectorizing the alternatives is critical; if the vectors do not sufficiently represent the alternatives, any recommendation system or preference elicitation algorithm will have trouble. For the numerical experiments, we elected to use a vector representation derived from Latent Dirichlet Allocation (LDA) as described by Blei, Ng, and Jordan (2003). The resulting feature vectors are low-dimensional and dense. Since we cannot compute the posterior distribution analytically, we resort to sampling instead, and the low-dimensional LDA vectors allow for more efficient sampling.

With any method that utilizes Bayesian inference, it is important to have enough structure that allows for an efficient sampling scheme from the resulting posterior distributions. The benefit of having the simple update of up-weighting and down-weighting polytopes is
that the sampling scheme becomes quite easy. We use a hit-and-run sampler as described and analyzed by Lovász and Vempala (2003) that chooses a direction uniformly from the unit sphere, then samples from the one-dimensional conditional distribution of the next point lying on that line. Now, re-weighting polytopes turns into re-weighting line segments. If it is easy sample points from the conditional distribution of lying on a given line, hit-and-run is an efficient way of sampling. We use a multivariate normally distributed prior because it allows for both computational tractability for sampling from this conditional distribution as well as a natural representation of prior information.

To select the hyperparameters for the prior, we sample academic papers and fit a multivariate normal distribution to this sample. Assuming users’ linear classifiers have the same form and interpretation as a vector representation is not reasonable in general. However, in the case of academic papers, authors are also readers, and so the content in which the users are interested is closely related to the content they produce. Therefore, in this situation, it is reasonable to assume that a user’s parameterization of preferences lives in the same space as the parameterization of the feature set. This is not necessarily the case for other types of alternatives, and even if it were, using feature vectors to model preference vectors may not be the best choice. That being said, there are many ways to initialize the prior. If one has a history of past user interaction with alternatives, one could estimate the linear preference vector for each user using an expectation maximization scheme, and fit a mixed normal prior to the empirical distribution of estimated linear classifiers, as done by Chen and Frazier (2016). Since the focus here is to compare the performance of the two algorithms of interest, our choice for initializing the prior is sufficient.

5.2 Cross-Metric Policy Comparison

We first compare the entropy pursuit and knowledge gradient policies while varying the number of presented alternatives. Due to the large set of alternatives, it is computationally intractable to choose questions that optimally follow either policy, so alternatives are subsampled from \(X\) and we approximate both policies using the alternatives from the subsample. If \(N\) alternatives are subsampled, then the approximate entropy pursuit policy requires exhaustively optimizing over combinations of alternatives (permutations if the noise channel not symmetric), and hence will require maximizing over \(\binom{N}{m}\) subsets. On the other hand, the knowledge gradient policy requires comparing \(\binom{N}{m}\) informative questions \(X\) with \(\binom{N}{n}\) assessment questions \(S\), and thus requires estimating \(\binom{N}{m}\binom{N}{n}\) quantities. Already, this implies that if the computational budget per question is fixed for both algorithms, one can afford a polynomially larger subsample for entropy pursuit than for knowledge gradient. For example, in the case where \(m = n = 2\), a computational budget that allows a subsample of \(N = 15\) alternatives for the knowledge gradient policy would allow the entropy pursuit policy a subsample size of \(N' = 149\). However, rather than fixing a computational budget for both policies at each step, we allow both policies the same number of subsamples, setting \(N = 15\) for both policies and all sets of parameters. We do this to allow for a more straightforward comparison of the two policies, although further computational studies should study their performance under a fixed computational budget. Lastly, the numerical study in this paper fixes \(n = 2\). We make this decision because any larger values
Figure 5: Comparison of average performance of the entropy pursuit and knowledge gradient policies under a symmetric noise channel ($\alpha = 0.7$), simulated and averaged with 100 sample paths. Estimates are accurate to $\pm 0.007$ for misclassification error and $\pm 0.06$ bits for entropy.

of $n$ will make the computations prohibitively expensive, and it is not clear that larger values of $n$ will provide any additional benefit.

Figure 5 compares the entropy pursuit and knowledge gradient policies by varying $m$ and fixing other parameters to reflect a low-noise, low prior information scenario. As expected, each algorithm performs better on their respective metrics for a fixed number of provided alternatives $m$. However, a more surprising conclusion is the similarity in performance of the two algorithms for any fixed $m$ for both metrics. This suggests that the price to pay for switching from the knowledge gradient policy to the entropy pursuit policy is small compared to the gain in computational efficiency. In fact, if the computational budget for each question were fixed, one would be able to subsample many more alternatives to compute the entropy pursuit policy compared to the knowledge gradient policy, and it is very likely the former would out-perform the latter in this setting. To see if this occurrence takes place in other scenarios, such as those with higher noise and a more informative prior, one can consult Figure 6. Again, for all the different parameter settings, both policies perform similarly.

Another interesting aspect of the computational results are the effects of the parameters on the performance of the two policies. Differential entropy predictably decreases faster when more alternatives are presented to the user. In the case of a symmetric noise channel, increasing $m$ only increases the channel capacity for a fixed noise level $\alpha$. From the perspective of minimizing posterior entropy, this makes sense because offering more alternatives at each time epoch should theoretically allow one to refine the posterior distribution faster. However, in reality, the noise channel most likely varies with the number of offered alternatives $m$, where the quality of the noise channel degrades as $m$ grows. In the most
Figure 6: Comparison of the entropy pursuit and knowledge gradient policies under a symmetric noise channel for various levels of noise and prior information, simulated and averaged with 100 sample paths. Estimates are accurate to ±0.007 for misclassification error and ±0.06 bits for entropy.

extreme example, offering too many alternatives to the user will result in a phenomenon called “decision paralysis,” where the user’s responses will not contain useful information about her preferences. In this case, the model is not capturing the added uncertainty, and focusing on posterior entropy as a performance metric may be misleading.

In contrast, the knowledge gradient policy captures this intuition, since pairwise comparisons decrease misclassification error faster than three-way comparisons in the cases of high noise or a highly informative prior. In fact, three-way comparisons only prevail in a low-noise, low prior information scenario, which is fairly optimistic. Both policies under three-way comparisons were aggressive, and in the high-noise case, they fail to learn anything at all about the user’s preferences. In practice, it will be necessary to estimate parameters for the noise channel in order to choose the correct value of $m$. For now, it suffices to say that pairwise comparisons are robust and reliable.
6. Conclusion

In this paper, we analyze the problem of eliciting a given user’s preferences by adaptively querying the user with choice-based questions. We formulate this problem in a sequential active learning setting, where a user’s preferences are governed by an unknown linear classifier, and the observed responses are perturbed by noise. We assume the underlying observation model where noise does not depend on the underlying preferences. Under this regime, we show that the differential entropy of the posterior distribution of this linear classifier can be reduced linearly with respect to the number of questions posed. Further, there exists an optimal predictive distribution that allows this optimal linear rate to be attained. We provide sensitivity results that show the entropy reduction is close to maximal when the actual predictive distribution of a given question is close to optimal in $L_2$ distance.

On the problem of appropriately choosing the alternatives: when the set of alternatives has non-empty interior, we provide a construction to find a question that achieves the linear lower bound to a constant multiplicative factor, and exactly for predictive distributions when $\max\{u^*_a\} = 1/2$ for pairwise comparisons or $\max\{u^*_a\} < 1/2$ for multi-way comparisons. When the set of alternatives is large but finite, we have demonstrated through simulation experiments that one can find questions that consistently yield a linear decrease in differential entropy, and this rate is reasonably close to optimal.

In addition to focusing on differential entropy, we consider misclassification error as an alternative metric that more intuitively captures the knowledge one has for a user’s preferences. Using Fano’s inequality, a classic result in the field of information theory, we show the performance of the optimal policy with respect to this metric is bounded below by a linear function in posterior entropy, suggesting a relationship between entropy-based and misclassification error-based policies. Our computational results largely confirm this, as the entropy pursuit policy and the knowledge gradient policy perform similarly in a variety of scenarios. For this reason, and the fact that the knowledge gradient requires a significantly larger computational budget, entropy pursuit is preferred for adaptive choice-based active preference learning.

Although the paper assumes that the number of alternatives $m$ is constant with respect to time, this can be relaxed with a word of caution. From the perspective of entropy, it is always beneficial to increase $m$, which can be misleading. Thus, if $m$ is allowed to vary with time, one should not use entropy pursuit to choose $m$, and should use another method to select the number of alternatives to present to the user. This may be done by fixing a static sequence $m_k$ in advance, or the parameter could be adjusted adaptively by another policy in tandem with entropy pursuit. Both approaches would most likely require extensive precomputation, since the geometry of the space of alternatives would heavily affect any policy governing $m$. Similar is the case of when a suitable prior for the user is not known. In practice, this would also dictate the need for a preprocessing step, perhaps fitting a Gaussian mixture to a population of estimated linear classifiers (Chen and Frazier, 2016, see). Regardless, this paper motivates the use of entropy pursuit in adaptive choice-based preference elicitation, as well as the study of its effectiveness using historical user responses and experimentation.

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