A Conjecture Regarding the Riemann Hypothesis

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Abstract

Numerical display of the behavior of t strings originating inside the critical strip for the Dirichlet Eta function provides strong visual evidence for why the Riemann hypothesis is most likely true. The t strings are generated by the action of the Dirichlet eta function on the unit interval of sigma for each fixed value of t. The t strings can exhibit at least three types of behavior. They do not intersect the origin, they can intersect the origin once at a point emanating from sigma equal 0.5, or they can intersect themselves. If only the first two possibilities occur the Riemann hypothesis is true. If the third possibility occurs and the intersection point is also the origin then the Riemann hypothesis is false. Heuristic numerical and visual evidence is presented that suggests the Riemann hypothesis is true. This is because of general very different behavior of t strings for sigma greater than or less than 0.5.
Introduction

The Riemann hypothesis remains one of the outstanding problems in analytic number theory after almost 160 years [1]. It is concerned with the distribution of prime numbers and is encapsulated by the Riemann zeta function and an equivalent expression written exclusively in terms of primes found by Euler [2] much earlier (for real \( s \)).

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}
\]

in which \( s = \sigma + ti \), a complex number with real part \( \sigma \) and imaginary part coefficient \( t \). This is the notation of Riemann. The expressions above are valid for \( \sigma > 1 \) and are absolutely convergent there. In [1] Riemann used analytic continuation to extend the zeta function into the regime \( 0 < \sigma < 1 \) (and into the regime \( \sigma < 0 \)) and found the Dirichlet eta function equation (conjectured by Euler in 1749)

\[
\eta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s} = (1 - 2^{1-s})\zeta(s)
\]

This result involves conditionally convergent series and must be handled with care. In the form

(1)

\[
\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s} = \frac{1}{1 - 2^{1-s}} \eta(s)
\]

it provides a way of computing the zeta function for \( 0 < \sigma < 1 \) in terms of the alternating eta series.

Also in [1] Riemann established the reflection formula for \( 0 < \sigma < 1 \)

(2)

\[
\zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1 - s) \zeta(1 - s)
\]
Combining Eq.(1) with Eq.(2) yields the reflection formula for the eta function

\[ \eta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1-s) \frac{1 - 2^{1-s}}{(1-2^s)} \eta(1-s) \]

There are zeroes of zeta and eta of three kinds. There are some so-called trivial zeroes for \( s = \sigma = -2, -4, \ldots \) created by the sine term. There are also trivial zeroes associated with the factor \( \frac{(1-2^{1-s})}{(1-2^s)} \). These are of the form \( 1 - i \frac{k2\pi}{\ln(2)} \) for integer \( k \). The remaining non-trivial zeroes of zeta are also non-trivial zeroes for eta and \textit{vis versa}. The critical strip defined by \( 0 < \sigma < 1 \) and \( -\infty < t < \infty \) contains these non-trivial zeroes. The Riemann hypothesis is: \( \sigma = 0.5 \) for all non-trivial zeroes. In 2004, Gourdon [3] verified the hypothesis for the first \( 10^{13} \) zeroes using an algorithm invented by Odlyzko [4].

**A modified reflection formula**

The reflection formulas for zeta and eta given in the introduction are most useful when considering non-trivial zeroes of zeta/eta. The trivial zeroes account for all zeroes due to factors other than zeta/eta in the reflection equations. The trivial zeroes with \( \sigma = 1 \) are on the edge of the critical strip. Here we will focus attention directly on the eta function. Two results will be of use later in the paper. The first result is

**Lemma 1:** if \( s_0 \) is a zero of eta then so is \( s_0^* \).

For general analytic functions, \( F \), it is very easy to construct a counter-example to the statement that if \( z \) is a zero of \( F \), then so is \( z^* \). The structure of eta, however, makes the statement of Lemma 1 true.

\[
\eta(s_0) = 0 \rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{s_0}} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{\sigma_0}} \exp[-it_0 \ln(n)]
\]

\[
= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{\sigma_0}} (\cos[t_0 \ln(n)]) - i \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{\sigma_0}} \sin[t_0 \ln(n)] = 0
\]
Since both the real and the imaginary parts must vanish separately, one is free to replace the $-i$ by $+i$ in the last line and that implies $\eta(s_0^*) = 0$.

The second result is the known modified reflection formula [5]

Lemma 2: if $s_0$ is a zero of eta then so is $1 - s_0^*$.

If $s_0$ is a zero of eta then the eta reflection formula implies that $1 - s_0$ is also a zero of eta. By Lemma 1 this means that $1 - s_0^*$ is also a zero of eta.

The modified reflection formula can be expressed: if $\sigma_0 + t_0i$ is a zero of eta then so is $1 - \sigma_0 + t_0i$. These two expressions have the same imaginary parts and that is key to what follows.

**Visualization of the genesis of zeroes**

Displaying the behavior of eta visually greatly enhances our ability to understand how and why the non-trivial zeroes of eta (zeta) occur for $\sigma = 0.5$. We begin by restricting the argument of eta to the real axis of the critical strip ($t = 0$). For $\sigma \in (0, 1)$ we find $\eta(\sigma) \in (0.5, ln(2))$. The unit interval for $\sigma$ is compressed by $\eta$ into an interval of length 0.193.... Because $\eta$ is continuous and infinitely differentiable in both $\sigma$ and $t$, we expect that small variations in $t$ will create small variations in $\eta$. We know that the first zero of eta has an imaginary part slightly bigger than 14. Therefore we will begin by looking at what happens to a set of points for $\{\sigma, .02, .98, 0.02\}$ and for a fixed value of $t$. The bracketed sigma structure means sigma goes from .02 to .98 in steps of length 0.02. The choice of discrete values of $\sigma$ is imposed in order to keep the magnitude of the computation reasonable and to illustrate non-uniformity in the output. For the first figure below, the choice of spacing works pretty well and the sets of points for a fixed $t$ represent a string very well. Some discreteness does appear in the longer strings (larger $t$ values) and shows a non-uniform density.
All computations have used *Wolfram Mathematica* 12. They have been done by manipulating the output from the program

\[
\text{Table}[\text{DirichletEta}[\sigma + ti], \{\sigma, 0.02, 0.98, 0.02\}, \{t, 1, 14, 1\}]
\]

The manipulation referred to is for converting data for a table into plots in the complex plane for a collection of \( t \)-strings with different \( t \) values. This involves a few tricks with the parentheses.

There is a lot of information in this figure. Each string is labeled by \( t \) and is made up of the values of eta for fixed \( t \) and for 49 evenly spaced...
values of $\sigma$ with the 25$^{th}$ value equal to 0.5. There are 14 strings in the figure corresponding to 14 values of $t$ in $\{t, 1, 14, 1\}$. The $t = 1$ string is located near the 1 on the abscissa at about 10 o’clock. It is the shortest string in the plot, slightly larger than 0.193 (the arc length for the case $t = 0$). As we go clockwise we see the strings for $t = 2, 3, 4, 5,$ and 6. Clearly they are getting longer in arc length and are moving away from the origin. String 6 appears below the abscissa at about 4 o’clock. The points on the string are arranged by $\eta$ with the small values of $\sigma$ distal to the 1 on the abscissa and the larger values of $\sigma$ proximal to the 1 on the abscissa. Since the $\eta$ values for uniformly spaced $\sigma$’s are not uniformly spaced the $\sigma = 0.5$ value of $\eta$ is not halfway along the length of the string. By enlarging the figure the location of this special $\sigma$ can be found. The 9$^{th}$ and the 14$^{th}$ strings are close to each other just before 9 o’clock. The 9$^{th}$ string is the shorter of the two. It’s $\sigma = 1$ value is closest to the origin. It looks like a slight adjustment in the $t$ value would put the $\sigma = 1$ right on the origin. Indeed, $t = 9.0647\ldots$ does do so. This corresponds precisely with a trivial zero given above by $1 - i \frac{k2\pi}{\ln(2)}$ for the case of $k = 1$. Similarly the 14$^{th}$ string looks like an adjustment of its $t$ value would possibly put $\sigma = 0.5$ on the origin. Let us try $t = 14.134725\ldots$. This is plotted below in figure 2.
The lower string is the $t = 14$ string from the previous plot. The difference in apparent slope is the result of different aspect ratios in the two plots. The upper curve is for $t = 14.134725$, the imaginary part of first non-trivial zero for eta. Moreover, $\sigma = 0.5$ corresponds with the 25th dot in the 49 dot representation of the $\sigma$ interval (it can be located by counting from the left). As you can see the typical value for a dot is a real part in the tenths and an imaginary part in the tenths as well. However the 25th $\sigma$ dot has the eta value $1.62123 \times 10^{-6} - 2.6635 \times 10^{-7} i$. Since we have expressed the $t$ for the first zero of eta to only one part in $10^6$ we cannot expect to get “zero” to any better precision. The smaller values of $\sigma$ produce the points to the left in the figure and the larger values of $\sigma$ produce the points to the right in the figure. After it became clear that the trivial zeros on the edge of the critical strip did not cause problems the sigma range was changed to $\{\sigma, 0, 1, .05\}$. 

{2}
The question that shouts out from this account is why does the string intersect the origin at $\sigma = 0.5$?! This is where the modified reflection formula comes into play.

“Theorem:”

A string labeled by $t$ can intersect the origin only for $\sigma = 0.5$.

Suppose that a string labeled by $t'$ does intersect the origin for the value $\sigma'$. Therefore $\sigma' + t'i$ is a zero of eta. The modified reflection formula states that $1 - \sigma' + t'i$ is also a zero. Since these two zeros have the same $t'$ they are on the same string. No string can intersect a point (the origin) more than once. Thus the two points must be the same point: $\sigma' + t'i = 1 - \sigma' + t'i$ which implies $\sigma' = 0.5$.

There is one possibility that would invalidate this conclusion. What happens if the string intersects itself? That is suppose that $\sigma' + t'i$ and $1 - \sigma' + t'i$ are zeroes of eta and $\sigma' \neq 0.5$. Then the string intersects itself at the origin producing a loop there. Thus the Riemann hypothesis would be false. To investigate this possibility further we will have to look at the behavior of strings for larger values of $t$, because for the small values exhibited in the figures so far there is no reason to suspect self-intersections. However beware of unwarranted expectations.

**Complexity of strings for larger $t$**

Let us begin our study of large $t$ strings with the value $t = 267653395648.8475231278$. This has 22 digits, 12 to the left of the decimal point and 10 to the right. We have it on good authority that this is the imaginary part of a zero [4]. If we compute the eta values for all 49 $\sigma$ points and for this value of $t$ then we get
Note the scale. The arc length is more than 14,000. The point distal to the origin is the eta value for $\sigma = .02$. After the first 7 discrete points the remaining 42 points are a smear. By restricting $\sigma$ to $\{\sigma, .42, .58, .02\}$ we obtain a blow-up of the region around the origin. The point at the origin has the value $0.000939283 + 0.00431777i$ which is incredibly small compared to the other values of points in the plot.
The small values of $\sigma$ make up the points on the left branch whereas the large values of $\sigma$ make up the points on the right branch. Unlike before it appears that the large $\sigma$ values of eta are receding from the origin like the small $\sigma$ values but in a different direction. This is incorrect as can be seen by including more points. If we include $\sigma$ values {$\sigma,.38,.98,.02$} then we get
The right branch does not go very far out but instead turns inward making the whole plot a kind of logarithmic spiral. You can count the 24 dots to the right of the origin. On the left there are 22 more dots unseen because they are too far out for the scale of this plot (they can be seen in fig.\{3\} for this value of $t$).

Note that we have referred to $t = 267653395648.8475231278$ as a large value of $t$. That is a matter of perspective. Clearly there are still larger values with zillions of digits. What do the strings look like for such larger values, that are still finite and therefore not really so big in any absolute sense. Does the logarithmic spiral like structure seen above develop more turns as $t$ increases but remain non-self-intersecting, implying that the Riemann hypothesis is true, or do self-intersections begin to occur, at the origin, for some large threshold value of $t$, implying that the Riemann hypothesis is false?
Observation of self-crossing t-strings

If it could be proved that t-strings are never self-crossing then the Riemann hypothesis (RH) is true, following from the theorem above. All my attempts at a proof failed. Finally the contrary position was adopted and it did not take long to find a counter-example. From Odlyzko’s tables [4] of large t’s we chose $t = 267653395648.8475231278$ to use in figs.3-5. In fig.1 above where small values of $t = 1, 2 \ldots 14$, are looked at the t-strings are slightly curved. For the large Odlyzko t values, such as the one we have selected, one end of the t-string is tightly wound (the end emanating from $0.5 < \sigma < 1$). The other end is very long and nearly straight. By moving t slightly away from its value as the imaginary part of an eta zero we get a crossing:
The value of $t$ is 267653395648.83. The left branch (the end emanating from $0 < \sigma < 0.5$) shoots down to $-199897 - 605732 \, i$ that is too big to be plotted simultaneously with the loop crossing that occurs for large $\sigma$\’s. Note that for the overall scale of this $t$-string the crossing occurs rather close to the origin. The dots correspond with choosing $\sigma$ in $\{\sigma, 0.495, 1, 0.005\}$. This plot only uses $\{\sigma, 0.495, 1, 0.005\}$ so that the long tail does not dominate the plot. There are 102 dots. The dot next to the ordinate at -1.34 is the dot for $\sigma = 0.5$. Clearly the crossing is for two different values of $\sigma$. Indeed the crossing points are both for values of $\sigma$ that are bigger than $\sigma = 0.5$, and, therefore, not obeying the reflection formula. Moreover, the use of Odlyzko\’s zero, unchanged, clearly shows that the origin is crossed for $\sigma = 0.5$. See figure {5}. The spacing of the dots is 4 times coarser in fig.{5} compared to fig.{6}. The increase in the value of $t$ for this figure (an eta zero), needed to get the value for $t$ in the preceding figure (a crossing), is just 0.0175231278 out of $2.6765339564883 \times 10^{11}$. However, for this value of $t$ (a zero) there is no longer a self-crossing of the $t$-string.

After making a more extensive search for examples of crossings it was found that $t$ needn\’t be so large. For $t = 231.61$ a tight hairpin crossing occurs. Fig.{7} shows this feature for $\{\sigma, 0.5, 1, 0.005\}$. This crossing value, $t = 231.61$, is about midway between the imaginary parts of two zeros: 231.250188700 and 231.987235253. However at the $t$ values for the two zeros there are no crossings.
Another example, fig.{8}, of a robust crossing for small $t$ is for $t = 357.60$, about half way between the imaginary parts of two zeros: 357.151302252 and 357.952685102. The features are the same as above, \{\sigma, .5, 1, .005\}. However at the $t$ values for the two zeros there are no crossings.
A search was made to find the first example (smallest $t$) of a self-intersecting string. Such a string with a hairpin loop was found for $t = 111.46345$. To make it visible the $\sigma$ range had to be restricted to $\{\sigma, 0.57, 0.83, 0.001\}$. The result is
The turning point is at $0.509843 + 0.156075i$ and the intersection point closing the loop is at $0.5273(3) + 0.16012(1)i$. As a reminder of how much magnification is needed the result for the calculation plotted above with no magnification (no restrictions on $\sigma$) is
One can see that something is happening in the lower left but not what it is. An eight-fold linear magnification or a sixty four-fold areal magnification is needed. For the record, 111.46345… is almost halfway between the two smallest Lehmer-pair numbers, 111.02953554 and 111.87465918 [6, 7].

Discussion

Had we been able to prove that every $t$-string never has a self-crossing then we would have a proof of the RH. That these curves can self-intersect has been demonstrated by examples. This is not equivalent to a disproof of the RH. Only if we can demonstrate a self-intersection coincidently at the origin as well, would we have a disproof (a zero and two values of $\sigma \neq 0.5$). So far we have an example of a self-crossing loop formation tantalizingly close to the origin from the viewpoint of the entire string’s length scale. The crossing in fig. {6}, for large $t$, is located with a
real part of size 0.9 and an imaginary part of size $-1.15$, whereas the arc-length of the entire $t$-string (not shown) is of order $6.3 \times 10^5$. Typically the small values of $\sigma$ are transformed by eta into a very long tail with only slight overall curvature, and certainly no crossings. The effect of eta on large values of $\sigma$ is to produce a tightly wound string of very limited areal extent and sometimes with crossings.

If these loop formations never occurred a proof of RH already would have been achieved. A string and a point (the origin) can only coincide at one point of the string. With the possibility of loops and the possibility that a loop forming self-intersection occurs at the origin the possibility that RH is false is confronted. However it is demonstrated here that the behavior of strings follows the following (unproved) rules:

1) Strings are labeled by a continuous variable, $t$. Except for isolated distinct values of $t$ corresponding to the imaginary parts of zeros none of these strings has a point at the origin. This is just the definition of a zero.

2) Strings without loops can have a point at the origin at most once and if they do have such a point then the string crossing point is for $\sigma = 0.5$ as was proved above.

3) Strings can grow very long and have two dominate domains. The domain for $\sigma < 0.5$ gets very long, is only slightly deviating from straight and does not deviate from straightness so much near $\sigma = 0.5$ as to form a loop there. The domain for $\sigma > 0.5$ is tightly wound up into structures of very small areal extent. Even though there are self-intersections they never occur at the origin. Instead a two point intersection takes place for two points both with $\sigma > 0.5$. This is inconsistent with the modified reflection principle for zeros. Whenever a loop is formed, nearby values of $t$ do not form loops but instead consist of bends and turns of the string. These take many forms but they all occur in the $\sigma > 0.5$ domain, in proximity to the loop. Every part of 3) needs proof.

These rules need to be understood and proved, or perhaps corrected and modified. Nevertheless they permit a clear visualization of what probably makes RH true. A string and the origin only meet at a single point.

Appendix
In this appendix a variety of t-strings formed for t in the range 1 to 67.6, with various spacing’s from 1 to 0.1 are exhibited in order to justify the rules presented above. The $\sigma - t$ ranges label the figures. This is followed by a discussion of the figure contents. Then the figure is presented.
There exists an imaginary part of a trivial zero at $t = 9.0647 \ldots = \frac{2\pi}{\ln(2)}$.

This is the $k = 1$ case. The rightmost sigma dot ($\sigma = 1$) of the ninth string counting clockwise from the shortest string almost crosses the origin. Increasing $t$ from 9 to 9.0647\ldots will make that happen. The maximum $t$ is 11 which is not large enough to produce a nontrivial zero whose imaginary part must be larger than 14. Proceeding clockwise from $t = 1$ to $t = 11$ illustrates the points made above about string location, length, and dot density. The seventh string is the first to regain at least unit arc length.
[[\{\sigma, 0, 1, .05\}, \{t, 11, 15, .5\}]
There is an imaginary part of a zero for \(t\) between 14 and 14.5. The eleventh sigma (\(\sigma = 0.5\)) almost crosses the origin. Compared to the figure above the spacing of the \(t\) values has changed from 1 to .5. The \(t = 11\) string was the last string in the figure above and is the first string in this figure. The strings with \(t = 14\) and \(t = 14.5\) are near 9 o’clock. A string between them will cross the origin for \(\sigma = 0.5\). The value for this string is \(t = 14.134725 \ldots\)
Another trivial zero occurs for $t = 18.1294 \ldots$ ($\sigma = 1$). This is the $k = 2$ case. The last string in this figure is for $t = 18$. If this value of $t$ is increased by 0.1294… then the $\sigma = 1$ end will sit on the origin.
There is a non-trivial zero for $t = 21.0220$ ... The eleventh sigma dot of the last string ($t = 21$) almost crosses the origin ($\sigma = 0.5$). A slight increase and it will do so.
There is the same zero as above near $t = 21$. It appears evident that the string’s $\sigma = 1$ ends are focused on a center of rotation although this center appears to move around as well. Why all strings appear as nearly straight rays for $\sigma < 0.5$ remains to be understood. The $t$ spacing is now .2.
There are no zeros in this $t$ range. Smaller increments in $t$ fill up one rotation of a circle with a given number of strings. Mysterious centers of rotation are evident. For example in the first figure, for $t$ running from 1 to 11 separated by ones, 1.33 rotations of a circle were produced, whereas in the figure above 11 strings separated by 0.2 produced $5/8$ of a circle. The area of the first figure is roughly $4 \times 5$ and the area of this figure is roughly $6 \times 8$. This increase reflects the increase in string lengths.
There is a nontrivial zero for $t = 25.010 \ldots$ The eleventh dot of the center string ($t = 25$) is almost right on the origin. The center of rotation is contracting and moving. The $t = 26$ string is shorter than the $t = 24$ string. In the long run length increases with increasing $t$. 
A trivial zero for $t = 27.1941 \ldots (k = 3)$ is apparent as the $\sigma = 1$ dot of the seventh string ($t = 27.2$) from the lower right crosses the origin. The strings are getting longer again.
There is a nontrivial zero at $t = 30.424 \ldots$ just above the eleventh dot ($\sigma = 0.5$) of the eighth string, counting clockwise from 3 o’clock. The nonuniformity of the distribution of dots caused by the action of Eta is increasing. The dense end is always the $\sigma = 1$ end.
[\{\sigma, 0, 1, .05\}, \{t, 31, 33, .2\}]
There is a nontrivial zero at \(t = 32.935\) ... This is just below the eleventh dot of the \(t = 33\) string (at 9 o’clock).
There is a trivial zero at $t = 36.2588 \ldots (k = 4)$. This is very near the $\sigma = 1$ end of the seventh string counting clockwise from 5 o’clock. Notice that the curvature of the $\sigma = 1$ ends of strings 36.8 and 37 are greater than in any earlier figure.
This is a magnification of the $\sigma = 1$ end of the strings in the figure above for $\sigma$ greater or equal to 0.6.
There is a nontrivial zero at $t = 37.586$ ... Counting clockwise from the string at 3 o’clock, the fourth string is for $t = 37.6$ and the eleventh dot of this string is right at the origin. Something dramatic seems to be happening at the $\sigma = 1$ ends in this region of the strings.
Cutting out the small values of $\sigma$ enlarges this region.
A strange configuration appears at this magnification. By reducing the maximum $t$ from 39 to 38 more magnification is achieved. Three string ends are seen together. Their curvatures are a little bit bigger than for other strings and their ends are very close together. The first two strings have opposite curvatures.
There is a nontrivial zero in this range with $t = 37.586$ ... Having cut $\sigma$ off at 0.6 above eliminated the origin crossing for this zero. By adding $\sigma = 0.5$ back in the crossing can be seen close to the origin after Eta acts.
In this range there is the nontrivial zero with $t = 40.918 \ldots$ This is about halfway between the $t$ equals 40.8 and 41 strings. Indeed the eleventh dot ($\sigma = 0.5$) for $t = 40.8$ is just to the left of the ordinate and the eleventh dot for $t = 41$ is just to the right of the ordinate.
There is a nontrivial zero for $t = 43.327 \ldots$. The tenth dot from the left end of string 43.4 is on the ordinate just above the origin. If Eta were evaluated for 43.327 \ldots instead of 43.4 then the eleventh dot would be at the origin as required. Note that $t$ is increasing less and less as the strings go around a circle. The strings are longer and the lack of uniformity in dot density is becoming more extreme. The centers of rotation continue to move around.
Only a single trivial zero is in this range. It has $t = 45.3235 \ldots$ (k = 5). Interpolating this value into the figure between 45.2 and 45.4 it is clear that the $\sigma = 1$ end of the string will be at the origin.
There is a nontrivial zero with \( t = 48.005 \) ... The eleventh dot (counting from the left) of the third string counting clockwise is right on the origin. It is continuing to appear that the organizing centers of groups of the strings are always associated with \( \sigma = 1 \) ends of strings.
There is a nontrivial zero with $t = 49.773 \ldots$ in this range. The eleventh dot of the fifth string ($t = 49.8$) counting clockwise is almost on the origin.
\[\{\sigma, 0, 1, .05\}, \{t, 50.4, 52, .2\}\]
In this range there are no zeros.
[\{\sigma, 0, 1, .05\}, \{t, 52, 53.4, .2\}]
There is a zero with \(t = 52.970\) ... Counting clockwise the sixth string
\((t = 53)\) at 12 o’clock has its eleventh dot (counting down) just to the right
of the origin. Distinct dots appear in these plots for \(\sigma\) less than or equal to
0.5 and dots are forming an indistinct smear for \(\sigma\) greater than 0.5.
There is a trivial zero with $t = 54.3882 \ldots (k = 6)$. Counting clockwise the sixth string with $t = 54.4$ has the Eta value for $\sigma = 1$ right at the origin.
There is a nontrivial zero with $t = 56.446$ ... This is just beyond the range for $t$ in this figure and will be discussed in the next figure. Notice here how tight the center of rotation has become. This is partly because the $t$-string lengths are increasing and partly because it is novel behavior.
$[\{\sigma, 0, 1, .05\}, \{t, 56.4, 58, .2\}]$

Now $t = 56.446 \ldots$ is in the exhibited range. It is close to the first string in the figure at about 7 o’clock. The eleventh dot is closest to the origin. The difference in dot density between $\sigma < 0.5$ and $\sigma > 0.5$ is growing more dramatic.
There is a nontrivial zero at $t = 59.347 \ldots$ This is near the eighth string at $t = 59.4$. Counting down the dots the eleventh dot is just above and to the right of the origin.
There is another nontrivial zero at $t = 60.831 \ldots$ This time there is a complicated morass and the plot has been cut off slightly to the right of $-2$ on the abscissa. From the figure above it is seen that the $t = 59.6$ string is the most vertical one. Counting around clockwise to the seventh string ($t = 60.8$) shows it coming in from the left a bit below the origin. The dot just below the origin is the eighth dot from the left but three more dots of this string have been cut off. So it is the eleventh dot after all.
This figure is a magnification of the region nearer to the origin created by cutting off $\sigma < 0.5$. This makes it clearer where the string ends are and how the curvature is changing from that of the first three strings to that of the last five.
There are no zeros in this range. The $\sigma < 0.5$ dots have been cut off in order to magnify the behavior near the origin. Why these strings are so nearly straight has not been explained.
There is a trivial zero for \( t = 63.4529 \ldots (k = 7) \). The \( t = 63.4 \) string is close to this value of \( t \) and is the sixth string counting clockwise. The \( \sigma = 1 \) dot for this string is just below the origin. Again the \( \sigma < 0.5 \) dots have been omitted.
There is a nontrivial zero in this range with \( t = 65.112 \) ... This is almost half way between the \( t = 65 \) and the \( t = 65.2 \) strings in the figure. It is clear that a string there would cross the origin although perhaps one more dot is needed to reach that far. That means using a sigma range of \([\sigma, .45, 1, .05]\). In addition the strings are getting so far apart that there is overlap of strings.
To eliminate the overlap in the previous figure the range for $t$ is shortened and the spacing between $t$ values is halved. A twelfth dot proves to be unnecessary and the eleventh string counting clockwise has its eleventh dot just below the origin.
There is a nontrivial zero at $t = 67.079$ ... The eighth string, counting clockwise, is the $t = 67.1$ string and its eleventh dot is just above the origin.
The full range of $\sigma$ values is restored. Because the small $\sigma$ 's create long tails on the strings it is difficult to see the details near the $\sigma = 1$ ends of the strings. While the figure above has an area of roughly $2 \times 2.5$ the figure here is roughly $12 \times 12$. For larger $t$ values, such as 7000, the difference in scale between the $\sigma > 0.5$ and $\sigma < 0.5$ regions of the figures is much greater, as was shown in [1].
References

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