Noncompact sigma-models:
Large $N$ expansion and thermodynamic limit

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Noncompact SO(1, $N$) sigma-models are studied in terms of their large $N$ expansion in a lattice formulation in dimensions $d \geq 2$. Explicit results for the spin and current two-point functions as well as for the Binder cumulant are presented to next to leading order on a finite lattice. The dynamically generated gap is negative and serves as a coupling-dependent infrared regulator which vanishes in the limit of infinite lattice size. The cancellation of infrared divergences in invariant correlation functions in this limit is nontrivial and is in $d = 2$ demonstrated by explicit computation for the above quantities. For the Binder cumulant the thermodynamic limit is finite and is given by $2/(N+1)$ in the order considered. Monte Carlo simulations suggest that the remainder is small or zero. The potential implications for “criticality” and “triviality” of the theories in the SO(1, $N$) invariant sector are discussed.
1. Introduction

In quantum field theories with nonabelian symmetries and dynamical mass generation the large $N$ expansion often provides a qualitatively correct and quantitatively reasonable description of the physics of the systems. Specifically in sigma-models with a compact global symmetry group the expansion is known to be an asymptotic expansion [1] and when slightly ad hoc applied to low orders at fixed small $N$ sometimes gives surprisingly accurate results, see e.g. [2, 3] for the renormalized coupling. In a lattice formulation one starts off on a finite lattice, the associated ‘finite volume’ mass gap then is uniformly bounded away from zero, and in the large $N$ series for invariant correlation functions the limit of infinite lattice size (also called the thermodynamic limit) can safely be taken termwise.

A study of the large $N$ expansion in sigma-models having a noncompact SO(1, $N$) internal symmetry group has been initiated in [4, 5]. A large value of $N$ in this case is also physically relevant for the granular limit of random hamiltonians describing disordered electrons with $N$ orbitals per site [6]. In a lattice formulation of the SO(1, $N$) sigma-models again a gap is dynamically generated in the large $N$ expansion, which is however negative and vanishes as the size of the lattice goes to infinity. Effectively the gap now acts as a subtle, coupling-dependent, infrared regulator and the technical problem consists in studying the ‘coordinated’ limit $V \to \infty$ of lattice sums of the form $\frac{1}{V^n} \sum_{k_1,\ldots,k_n} f_V(k_1,\ldots,k_n)$, where $f_V$ carries an explicit $V$-dependence via the gap. The sums associated with individual Feynman diagrams of the large $N$ expansion will typically diverge in the limit. The issue whether or not in the combinations entering invariant correlation functions the infrared divergences cancel is analogous to the one encountered in the perturbation theory of compact sigma-models [7, 8] and is the subject of the present paper. Since this issue is most critical in the two-dimensional systems we examine the limit specifically in this case, although our finite volume results are valid in all dimensions $d \geq 2$. We compute a number of physically interesting quantities to leading and subleading order and show that they indeed do have a well-defined thermodynamic limit. Concretely we consider the spin two-point function, the two-point function of the Noether current, and the Binder cumulant.

The Binder cumulant $U$ is defined in terms of the zero momentum limit of the connected four-point function. In massive scalar field theories it serves to define an intrinsic measure of the interaction strength and has been used to explore “triviality” issues. In a massless theory, like the systems considered here, there is no obvious reason why $U$ should have a finite thermodynamic limit. Somewhat surprisingly we find that $U$ does have a finite and nonzero limit to leading and subleading order, which is moreover independent of $\lambda$ and given by $2/(N+1)$. Supported also by Monte-Carlo simulations we conjecture that the infinite volume limit of the exact $U$ is also very close to $2/(N+1)$. The potential implications for “criticality” and “triviality” in the SO(1, $N$) invariant sector of the theory will be discussed in the conclusions.

The rest of the article is organized as follows. In the next section we review a result from
a previous paper \[5\] which on a finite lattice allows one to do large \(N\) computations in the simpler compact models and then transfer the results to the noncompact ones via a “large \(N\) correspondence”. This correspondence has an interesting interplay with the Schwinger-Dyson equations and instead of going through the (fairly routine) diagrammatic computations we merely present the results as solutions of the large \(N\) expanded Schwinger-Dyson equations with the correct ‘initial’ data. Expressions for the two- and four point functions to leading and subleading order are given (valid on a finite lattice in all dimensions \(d \geq 2\)), from which also the two-point function of the Noether current and the Binder cumulant can be obtained in the same order. The thermodynamic limit in \(d = 2\) of the local quantities and of the Binder cumulant are studied in Sections 4 and 5, respectively.

2. Large \(N\) expansions of compact and noncompact models

In compact sigma-models the large \(N\) expansion is a saddle point expansion based on a generating functional obtained by ‘dualizing’ the spins, i.e. by imposing the constraint via a Lagrange multiplier field and performing the Gaussians. The counterpart of this duality transformation is somewhat ill-defined in the noncompact models. The large \(N\) expansion can nevertheless be justified and on a finite lattice the expansion coefficients for invariant correlation functions can be inferred from those in the compact model \[5\]. This “large \(N\) correspondence” allows one to do computations in the compact model, where no gauge gauge-fixing is required, and the familiar framework can be used. Here we briefly summarize the correspondence and present explicit results for two and four-point functions in Section 3. The results of Sections 2 and 3 are valid in all dimensions \(d \geq 2\).

2.1 Definitions

Here we recall the notation and the definitions for the invariant correlation functions considered and their generating functionals. We consider the SO\((N + 1)\) spherical and the SO\((1, N)\) hyperbolic sigma-models in two dimensions with standard lattice action, defined on a hypercubic lattice \(\Lambda \subset \mathbb{Z}^d\) of volume \(V = |\Lambda| = L^d\). The dynamical variables (“spins”) will be denoted by \(n^a_x, x \in \Lambda, a = 0, \ldots, N\), in both cases, and periodic boundary conditions are assumed throughout \(n_{x+L\hat{\mu}} = n_x\). The constraint is \(n \cdot n = 1\) in both cases, but with different ‘dot’ products; namely \(a \cdot b := a^0b^0 + a^1b^1 + \ldots + a^N b^N\) in the compact model, and \(a \cdot b := a^0b^0 - a^1b^1 - \ldots - a^N b^N\) in the noncompact model. Clearly \(S^N = \{n \in \mathbb{R}^{N+1} \mid n \cdot n = 1\}\) is the \(N\)-sphere and \(H^N = \{n \in \mathbb{R}^{1:N} \mid n \cdot n = 1, n^0 > 0\}\) is
the upper half of the two-sheeted $N$-dimensional hyperboloid. The lattice actions are

$$S_\pm = \mp \beta \sum_{x, \mu} (n_x \cdot n_{x+\hat{\mu}} - 1) = \mp \frac{\beta}{2} \sum_x n_x \cdot (\Delta n)_x \geq 0,$$

(2.1)

where the upper sign refers to the compact model and the lower sign to the noncompact model. The laplacian is $\Delta_{xy} = -\sum_\mu [2\delta_{x,y} - \delta_{x,y+\hat{\mu}} - \delta_{x,y-\hat{\mu}}]$, as usual. We write

$$d\Omega_+ (n) = d^{N+1}n \delta(n \cdot n - 1),$$

$$d\Omega_- (n) = 2d^{N+1}n \delta(n \cdot n - 1)\theta(n^0),$$

(2.2)

for the invariant measure on $S^N$ and $H^N$, respectively. Further $\delta_\pm (n, n')$ is the invariant point measure on $S^N$, $H^N$, and $n^\dagger = (1, 0, \ldots, 0)$. Note that the measure $d\Omega_+ (n)$ is normalized while $H^N$ has infinite volume.

In the compact model we consider the generating functional,

$$\exp W_+ [H] = \mathcal{N} \int \prod_x d\Omega_+(n_x) \exp \left\{ - S_+ + \frac{1}{2} \sum_{x,y} H_{xy} (n_x \cdot n_y - 1) \right\},$$

(2.3)

where $H_{xy} \geq 0$ is a source field and the normalization $\mathcal{N}$ is such that $W[0] = 0$. For the noncompact model we consider the generating functional

$$\exp W_- [H] = \mathcal{N} \int \prod_x d\Omega_- (n_x) \delta_- (n_{x_0}, n^\dagger) \exp \left\{ - S_- + \frac{1}{2} \sum_{x,y} H_{xy} (n_x \cdot n_y - 1) \right\},$$

(2.4)

where now $H_{xy} < 0$ sources give damping exponentials, and one spin at site $x_0$ is fixed in order to make the generating functional well defined.

Connected $2r$ point functions are defined by

$$W_{\pm,r} [H] = \sum_{r \geq 1} \frac{1}{r! 2^r} W_{\pm,r} (x_1, y_1; \ldots; x_r, y_r) H_{x_1 y_1} \ldots H_{x_r y_r},$$

$$W_{\pm,r} (x_1, y_1; \ldots; x_r, y_r) := h_{x_1 y_1} \ldots h_{x_r y_r} W_{\pm} [H] \bigg|_{H=0}, \quad h_{xy} := \frac{\delta}{\delta H_{xy}}.$$

(2.5)

In particular $W_{\pm,1} (x, y) = \langle n_x \cdot n_y \rangle_\pm - 1$, $W_{\pm,2} (x_1, y_1; x_2, y_2) := \langle n_{x_1} \cdot n_{y_1} n_{x_2} \cdot n_{y_2} \rangle_\pm - \langle n_{x_1} \cdot n_{y_1} \rangle_\pm \langle n_{x_2} \cdot n_{y_2} \rangle_\pm$, where $\langle \rangle_\pm$ are the functional averages with respect to $\mathcal{N}^{-1} e^{-S_\pm}$. Note that $W_{\pm,r} (\ldots; x, x; \ldots) = 0$. 

3
2.2 The 1/N expansion

The goal in the following is to construct these invariant correlation functions in a large $N$ asymptotic expansion. That is, $\lambda := (N+1)/\beta$ is kept fixed and the coefficient functions $W_{\pm,r}$ in

$$W_{\pm,r}(x_1, \ldots, y_r) = \frac{\lambda^r}{(N+1)^{r-1}} \sum_{s=0}^{\infty} \frac{1}{(N+1)^s} W_{\pm,r}^{(s)}(x_1, \ldots, y_r),$$

are sought, with the understanding that the right hand side of (2.6) provides a valid asymptotic expansion of the exact $W_r$, initially on a finite lattice.

The diagrammatic algorithm for the computation of the coefficient functions $W_{\pm,r}^{(s)}$ is rather straightforward in the compact model, see e.g. [9]. From [1] it is also known to provide a valid asymptotic expansion. Direct computation of the functions $W_{\pm,r}^{(s)}$ in the noncompact model is also possible [5], although due to the gauge fixing the computations are considerably more tedious than in the compact model. In [10] it will be shown that this algorithm also provides a valid asymptotic expansion (2.6) for the $W_{-r}$.

One of the advantages of a lattice formulation for these systems is that there is an exact correspondence [5] between the functions $W_{\pm,r}^{(s)}$ in the noncompact model and their counterparts $W_{\pm,r}^{(s)}$ in the compact model, valid on a finite lattice in all dimensions $d \geq 2$:

(a) The coefficient functions $W_{\pm,r}$ are translation invariant and can be expressed in terms of $D_\pm(x) = D(x)|_{\omega=\omega_\pm}$, with $D(x)$ the free propagator of squared mass $\omega$ and with $\omega_\pm(\lambda, V)$ the solutions of the gap equations $\lambda D(0) = \pm 1$ discussed further in Subsection 3.1.

(b) For all $r \geq 1$, $s \geq 0$, there exists unique functionals $X_{s}^{r}[D](\lambda)$ of $D$ such that $W_{r,\pm}^{(s)} = X_{r}^{(s)}[D_\pm](\lambda)$ are the coefficients in the compact model and $W_{r,\pm}^{(s)} = (-1)^r X_{r}^{(s)}[D_-](-\lambda)$ are the coefficients in the noncompact model.

As a consequence the computations only have to be done in the compact model, and the result in the noncompact model can be obtained via (b). In the next section we will compute a subset of correlation functions, but instead of doing the computation using the 1/N Feynman rules we present the results and verify that they solve the associated Schwinger-Dyson equations.
3. Schwinger-Dyson Equations

In the compact model the Schwinger-Dyson equations for the functions (2.5) have, to our knowledge, first been formulated by M. Lüscher [11]. The derivation is readily extended to noncompact models and is reproduced in [5]. Here we just record the basic equations:

\[
\pm \beta \Delta_{\pm} \left[ h_{xy} W_{\pm} - h_{zx} W_{\pm} - h_{xz} h_{xy} W_{\pm} - (h_{zx} W_{\pm}) (h_{xy} W_{\pm}) \right]_{z=x} \\
+ \sum_{z \neq x} H_{xz} \left[ h_{xy} W_{\pm} - h_{zx} W_{\pm} - h_{xz} h_{xy} W_{\pm} - (h_{zx} W_{\pm}) (h_{xy} W_{\pm}) \right] \\
- N (1 - \delta_{xy}) (h_{xy} W_{\pm} + 1) = 0,
\]

(3.1)

where the upper sign corresponds to the compact model and the lower sign to the noncompact model. In terms of the multi-point functions (2.5) these equations amount to an infinite coupled system of nonlinear partial differential equations. As such boundary conditions have to be specified; without them even the exact equations (3.1) do not determine their solution uniquely, see [5] for a counter example. However, since (3.1) does not contain a closed equation for any of the \( W_r \), it is difficult to impose such boundary conditions in practice.

In contrast, the large \( N \) ansatz (2.6) effectively converts the \( W_r \) equations into a hierarchy which can be solved recursively and where ‘initial’ conditions can be specified. The recursion pattern for the \( W_r^{(s)} \), \( r + s > 1 \), functions is given in Fig. 1. To compute a given coefficient all quantities having arrows pointing towards it are needed.

\[
W_1^{(0)} \rightarrow W_2^{(0)} \rightarrow W_3^{(0)} \rightarrow \\
\downarrow \quad \downarrow \\
W_1^{(1)} \rightarrow W_2^{(1)} \rightarrow \\
\downarrow
\]

Fig. 1: Recursion pattern for the solution of the large \( N \) expanded SD equations.

The first few equations for the \( W_r^{(s)} \) are spelled out in Appendix A; a closed formula for the generic equation can also be given and used to show the recursion pattern in Fig. 1 by induction. The key assumption in our use of the large \( N \) expanded Schwinger-Dyson equations will be that at each recursion step in Fig. 1 there exists a solution and that the solution is unique. This assumption implies that, once \( W_1^{(0)} \) has been specified, there will be an infinite sequence of functions \( W_r^{(s)} \), \( r + s > 1 \), uniquely associated with it, which in turn determine the series (2.6) uniquely for each \( W_r \). The choice of \( W_1^{(0)} \) is ultimately determined by the physics problem one seeks to study; different choices are possible [5] for the same (initial) equation (A.1).
The existence and uniqueness of a solution at each recursion step of Fig. 1 is presumably difficult to establish directly from the equations. In terms of the underlying discretized functional integral (which solves the exact equation (3.1) by construction) existence and uniqueness of the solution to the recursive equations amount to the existence of a well-defined asymptotic expansion of the form (2.6). For the generating functionals $W_+ [H]$ and $W_- [H]$ the latter is guaranteed by the results of [11] and [10], respectively. This provides an indirect justification of the assumption stated in the preceding paragraph. It also provides the rationale for the procedure adopted in Subsections 3.2 and 3.3: we present the expressions for $W^{(s)}_{\pm,1}$ and $W^{(s)}_{\pm,2}$ to leading and subleading order ($s = 0, 1$) and claim that they solve the Schwinger-Dyson equations with the correct initial conditions, $W^{(0)}_{1,\pm}$, respectively. The required equations are tabulated in Appendix A. The verification that the $W^{(s)}_{\pm,r}$ presented indeed solve these equations is straightforward and is omitted. It is however far shorter than the diagrammatic computation in either the compact or the noncompact model.

Though in this paper we shall be concerned with the large $N$ expansion exclusively, let us add that the Schwinger-Dyson equations (3.4) can also be subjected to a perturbative expansion, i.e. the ansatz (2.6) is replaced with one in terms of powers of $1/\beta$. The recursive pattern determining the coefficients is similar to that in Fig. 1, but the differential part of the equations now involves a linear differential operator with constant coefficients. This has been used by Lüscher [11] to show directly from the equations that each recursion step has at most one solution. The existence of a solution of course follows from the diagrammatic algorithm as described (on the lattice) by Hasenfratz [12]. Once the results are known the perturbatively expanded Schwinger-Dyson equations provide an efficient way to verify them.

3.1 2– and 4– functions to leading order

The leading order two-point functions $W^{(0)}_{\pm,1}$ provide the starting point for the recursion. They have to solve Eq. (A.1) but in order to pick a specific solution further information has to be added. The appropriate solutions turn out to be given by

$$W^{(0)}_{\pm,1}(x, y) = \pm D_\pm (x - y) - \lambda^{-1}, \quad (3.2)$$

with

$$D_\pm (x) = \frac{1}{V} \sum_p \frac{e^{ipx}}{E_p + \omega_\pm}, \quad (3.3)$$

where the sum goes over $p = \frac{2\pi}{L} (n_1, \ldots, n_d)$, $n_\mu = 0, 1, \ldots, L - 1$ and $E_p = \sum_{\mu=1}^d \hat{p}_\mu^2$ with
\[\hat{p}_\mu = 2 \sin \frac{p_\mu}{2}.\] Further \(\omega_\pm\) are the particular solutions to the ‘gap equations’

\[\pm \lambda^{-1} = D_\pm(0) = \frac{1}{V} \sum_p \frac{1}{E_p + \omega_\pm},\]  

(3.4)

obeying \(\omega_+ > 0\) in the compact model and \(0 > \omega_- > -\frac{4}{2d+1} \sin^2 \pi/L\) in the noncompact model [5]. The rationale for the choice of these solutions of (A.1) is that their properties are necessary for the stability of the expansions, see [1] for the compact and [10] for the noncompact model. All \(W_{\pm, r}, r + s > 1\), are then in principle uniquely determined by the \(W_{\pm, 1}\), and we shall simply present the solutions of the associated Schwinger-Dyson equations.

The solution of (A.3) for the 4–point function in leading order is

\[W_{\pm, 2}(x_1, y_1; x_2, y_2) = D_\pm(x_1 - x_2)D_\pm(y_1 - y_2) + D_\pm(x_1 - y_2)D_\pm(y_1 - x_2)\]

\[-2 \sum_{u, v} D_\pm(x_1 - u)D_\pm(y_1 - u)\Delta_\pm(u - v)D_\pm(v - x_2)D_\pm(v - y_2),\]  

(3.5)

where \(\Delta_\pm(u - v)\) is defined by

\[\sum_u D_\pm(x - u)^2 \Delta_\pm(u - v) = \delta_{xv},\]  

(3.6)

so that

\[\Delta_\pm(u) = \frac{1}{V} \sum_k e^{iku} \Pi_\pm(k),\]

\[\Pi_\pm(k) := \frac{1}{V} \sum_p \frac{1}{(E_p + \omega_\pm)(E_{k+p} + \omega_\pm)}.\]  

(3.7)

Obviously in the compact model \(\Pi_+(k) > 0\) for all \(k\). In contrast in the noncompact model one has \(\Pi_-(k) < 0, k \neq 0\) and \(\Pi_-(0) > 0\), a fact of importance for the validity of the large \(N\) expansion [3]. \(\Delta_\pm(u)\) is the propagator of the auxiliary field in the functional treatment of the \(1/N\) expansion, and correspondingly the last term in (3.5) corresponds to a tree diagram with an intermediate auxiliary field propagator.

### 3.2 2– and 4–point functions to next-to-leading order

The solution of (A.2) for the 2–point function in next-to-leading order is

\[W^{(1)}_{\pm, 1}(x, y) = -2q_\pm \frac{\partial}{\partial \omega_\pm} W^{(0)}_{\pm, 1}(x, y)\]

\[\mp 2 \sum_{u, v} D_\pm(x - u)D_\pm(u - v)\Delta_\pm(u - v)D_\pm(v - y),\]  

(3.8)
\[ q_\pm = \frac{1}{\Pi_\pm(0)} \sum_{u,v} D_\pm(u) \Delta_\pm(u-v) D_\pm(v) \]
\[ = \frac{1}{\Pi_\pm(0)V^2} \sum_p \sum_q \frac{1}{(E_p + \omega_\pm)^2 (E_p + q + \omega_\pm) \Pi_\pm(q)}. \]  
(3.9)

and the partial derivative \( \frac{\partial}{\partial \omega_\pm} \) means \( \pm \lambda^3 \Pi_\pm(0) \frac{\partial}{\partial \lambda} \) (at fixed volume). In particular
\[ \mp \frac{\partial}{\partial \omega_\pm} W^{(0)}_{\pm,1}(x, y) = D_{\pm,2}(x - y) \]
\[ := \sum_w D_\pm(w - x) D_\pm(w - y) = \frac{1}{V} \sum_p \frac{e^{ip(x-y)}}{(E_p + \omega_\pm)^2}. \]  
(3.10)

Note \( \Pi_\pm(0) = D_{\pm,2}(0) \). The first term on the rhs of (3.8) corresponds to the tadpole diagram and the second term to the non-trivial self-energy diagram.

Finally the solution for the next-to-leading 4-point function is given by
\[ W_{\pm,2}^{(1)}(x_1, y_1; x_2, y_2) = -2q_{\pm} \frac{\partial}{\partial \omega_\pm} W^{(0)}_{\pm,2}(x_1, y_1; x_2, y_2) \]
\[ -2 \sum_{u,v,w,z} W^{(0)}_{\pm,2}(x_1, y_1; u, v) D_{\pm}(u - w) \Delta_\pm(u - w) D_{\pm}^{-1}(v - z) W^{(0)}_{\pm,2}(x_2, y_2; w, z) \]
\[ +2 \sum_{u,v,w,z} W^{(0)}_{\pm,2}(x_1, y_1; u, v) D_{\pm}(u - v) \Delta_\pm(u - w) \Delta_\pm(v - z) D_{\pm}(w - z) W^{(0)}_{\pm,2}(x_2, y_2; w, z) \]
\[ -\sum_{u,v} W^{(0)}_{\pm,2}(x_1, y_1; u, v) \Delta_\pm(u - v) W^{(0)}_{\pm,2}(x_2, y_2; u, v). \]  
(3.11)

Again the separate contributions in (3.11) become more apparent when drawn as corresponding Feynman diagrams.

### 3.3 Two-point current correlation function

In both models the Noether currents are given by
\[ J^{ab}_{\mu}(x) = \beta \left[ n_x^a \partial_\mu n_x^b - n_x^b \partial_\mu n_x^a \right], \quad 0 \leq a, b \leq N. \]  
(3.12)

The invariant two-point function of these currents is
\[ J_{\pm,\mu\nu}(x, y) := \left\{ \begin{array}{l} \sum_{a,b} \langle J^{ab}_{\mu}(x) J^{ab}_{\mu}(y) \rangle_+ , \\ \sum_{a,b,c,d} \langle \eta_{ac} \eta_{bd} J^{ab}_{\mu}(x) J^{cd}_{\nu}(y) \rangle_- . \end{array} \right. \]  
(3.13)
It obeys the Ward identity

$$
\sum_\mu \partial^*_\mu J_{\pm,\mu\nu}(x, y) = \pm 2N \beta E_{\pm}(\delta_{x,y} - \delta_{x,y+\hat{\nu}}),
$$

(3.14)

where \(\partial^*_\mu f(x) = f(x) - f(x - \hat{\mu})\) and in the noncompact case (3.14) holds for \(x \neq x_0\) only. Further

$$
E_{\pm} = \langle n_x \cdot n_{x+\hat{\nu}} \rangle_{\pm},
$$

(3.15)

is independent of \(x\) because of translation invariance. One way of obtaining (3.14) is by specializing the ‘pre-Schwinger-Dyson’ equation Eq. (3.39) of [5] to \(O = J_{\nu}(y)\) and using the completeness relations Eq. (3.38) in [5].

The two–point function can be expressed in terms of the 2– and 4–point functions of the spins according to

$$
J_{\pm,\mu\nu}(x, y) = 2\beta^2 \left[ W_{\pm,2}(x, y; x + \hat{\mu}, y + \hat{\nu}) - W_{\pm,2}(x, y + \hat{\nu}; x + \hat{\mu}, y) 
+ W_{\pm,1}(x, y)W_{\pm,1}(x + \hat{\mu}, y + \hat{\nu}) - W_{\pm,1}(x, y + \hat{\nu})W_{\pm,1}(x + \hat{\mu}, y) 
+ W_{\pm,1}(x, y) + W_{\pm,1}(x + \hat{\mu}, y + \hat{\nu}) - W_{\pm,1}(x, y + \hat{\nu}) - W_{\pm,1}(x + \hat{\mu}, y) \right].
$$

(3.16)

The current correlation function has accordingly a \(1/N\) expansion of the form:

$$
J_{\pm,\mu\nu}(x, y) = 2N(N + 1) \sum_{s \geq 0} \frac{1}{(N + 1)^s} J^{(s)}_{\pm,\mu\nu}(x, y).
$$

(3.17)

In the lowest order we have

$$
J^{(0)}_{\pm,\mu\nu}(x, y) = \lambda^{-1} \left[ W^{(0)}_{\pm,1}(x, y) + W^{(0)}_{\pm,1}(x + \hat{\mu}, y + \hat{\nu}) - W^{(0)}_{\pm,1}(x, y + \hat{\nu}) - W^{(0)}_{\pm,1}(x + \hat{\mu}, y) 
+ W^{(0)}_{\pm,1}(x, y)W^{(0)}_{\pm,1}(x + \hat{\mu}, y + \hat{\nu}) - W^{(0)}_{\pm,1}(x, y + \hat{\nu})W^{(0)}_{\pm,1}(x + \hat{\mu}, y) \right].
$$

(3.18)

Inserting the solution (3.2) one gets

$$
J^{(0)}_{\pm,\mu\nu}(x, y) = D_{\pm}(x - y)D_{\pm}(x + \hat{\mu} - y - \hat{\nu}) - D_{\pm}(x - y - \hat{\nu})D_{\pm}(x + \hat{\mu} - y),
$$

(3.19)

the Fourier transform of which is

$$
\tilde{J}^{(0)}_{\pm,\mu\nu}(q) = \sum_x e^{-iqx} J^{(0)}_{\pm,\mu\nu}(x, 0)
$$

$$
= \exp \left( \frac{i}{2} [q_\mu - q_\nu] \right) \frac{2}{V} \sum_p \frac{\sin(p + q/2)_\mu \sin(p + q/2)_\nu}{(E_p + \omega_\pm)(E_{p+q} + \omega_\pm)}. \quad (3.20)
$$
It is seen to satisfy

$$\sum_{\mu} (1 - e^{-i q\mu}) J_{\pm,\mu}^{(0)}(q) = (1 - e^{-i q\nu}) D_{\pm}(\nu), \quad (3.21)$$

which is the Ward identity \((3.14)\) to lowest order \(1/N\).

In next order we have

$$J_{\pm,\mu}^{(1)}(x, y) = J_{\pm,\mu}^{(0)}(x, y) + W_{\pm,2}^{(0)}(x, y; x + \hat{\mu}, y + \hat{\nu}) - W_{\pm,2}^{(0)}(x, y + \hat{\nu}; x + \hat{\mu}, y)$$

$$+ W_{\pm,1}^{(1)}(x, y) \left[ W_{\pm,1}^{(0)}(x + \hat{\mu}, y + \lambda^{-1}) + W_{\pm,1}^{(1)}(x + \hat{\mu}, y + \hat{\nu}) \right] W_{\pm,1}^{(0)}(x, y + \lambda^{-1})$$

$$- W_{\pm,1}^{(1)}(x, y + \hat{\nu}) \left[ W_{\pm,1}^{(0)}(x + \hat{\mu}, y + \lambda^{-1}) - W_{\pm,1}^{(1)}(x + \hat{\mu}, y) \right] W_{\pm,1}^{(0)}(x + \hat{\nu} + \lambda^{-1})$$

\((3.22)\)

Using the solutions \((3.2), (3.5), (3.8)\), this becomes

$$J_{\pm,\mu}^{(1)}(x, y) = 2 q_{\pm} \left[ D_{\pm,2}(x - y) D_{\pm}(x + \hat{\mu} - y - \hat{\nu}) + D_{\pm}(x - y) D_{\pm,2}(x + \hat{\mu} - y - \hat{\nu}) \right]$$

$$- D_{\pm,2}(x + \hat{\mu} - y) D_{\pm}(x - y - \hat{\nu}) - D_{\pm}(x + \hat{\mu} - y) D_{\pm,2}(x - y - \hat{\nu})$$

$$- 2 \sum_{w} \Delta_{\pm}(w) D_{\pm}(w) \left[ D_{\pm,2}(y - x + w) D_{\pm}(x - y + \hat{\mu} - \hat{\nu}) \right]$$

$$+ D_{\pm,2}(y - x + \hat{\nu} - \hat{\mu} + w) D_{\pm}(x - y) - D_{\pm,2}(y - x - \hat{\mu} + w) D_{\pm}(x - y - \hat{\nu})$$

$$- D_{\pm,2}(y - x + \hat{\nu} + w) D_{\pm}(x - y + \hat{\mu})$$

$$- 2 \sum_{u,v} D_{\pm}(x - u) D_{\pm}(y - u) \Delta_{\pm}(u - v) D_{\pm}(x + \hat{\mu} - v) D_{\pm}(y + \hat{\nu} - v)$$

$$+ 2 \sum_{u,v} D_{\pm}(x - u) D_{\pm}(y + \hat{\nu} - u) \Delta_{\pm}(u - v) D_{\pm}(x + \hat{\mu} - v) D_{\pm}(y - v) \right]. \quad (3.23)$$

Its Fourier transform is simpler in form:

$$\tilde{J}_{\pm,\mu}^{(1)}(q) = - \exp \left( \frac{i}{2} [q_{\mu} - q_{\nu}] \right) \left[ X_{\pm,1,\mu\nu}(q) + X_{\pm,2,\mu\nu}(q) + X_{\pm,3,\mu\nu}(q) \right], \quad (3.24)$$
\[
X_{\pm,1\mu\nu}(q) = \frac{8q_1}{V} \sum_p \frac{\sin(p + q/2)_\mu \sin(p + q/2)_\nu}{(E_p + \omega_\pm)^2 (E_{p+q} + \omega_\pm)},
\]
\[
X_{\pm,2\mu\nu}(q) = \frac{8}{V^2} \sum_{p_1, p_2} \frac{\sin(p_1 + q/2)_\mu \sin(p_1 + q/2)_\nu}{(E_{p_1} + \omega_\pm)^2 (E_{p_1+q} + \omega_\pm) (E_{p_2} + \omega_\pm) \Pi_\pm(p_1 - p_2)},
\tag{3.25}
\]
\[
X_{\pm,3\mu\nu}(q) = \frac{4}{V^2} \sum_{p_1, p_2} \frac{\sin(p_1 + q/2)_\mu \sin(p_2 + q/2)_\nu}{(E_{p_1} + \omega_\pm) (E_{p_1+q} + \omega_\pm) (E_{p_2} + \omega_\pm) (E_{p_2+q} + \omega_\pm) \Pi_\pm(p_1 - p_2)}.
\]

It can be checked to satisfy the Ward identity in the next order in the large \(N\) expansion.

### 3.4 The Binder cumulant

In scalar field theories with a mass gap a renormalized 4-point coupling is defined in terms of the Binder cumulant

\[
U := 1 + \frac{2}{N+1} - \frac{\langle (\Sigma \cdot \Sigma)^2 \rangle}{\langle (\Sigma \cdot \Sigma)^2 \rangle} = -\frac{1}{\langle (\Sigma \cdot \Sigma)^2 \rangle} \sum_{x_1,x_2,y_1,y_2} \langle n_{x_1} \cdot n_{y_1}, n_{x_2} \cdot n_{y_2} \rangle_c.
\tag{3.26}
\]

Here \(\Sigma^a = \sum_x n_i^a\) and

\[
\langle n_{x_1} \cdot n_{y_1}, n_{x_2} \cdot n_{y_2} \rangle_c = W_2(x_1, y_1; x_2, y_2) - \frac{1}{N+1} \langle n_{x_1} \cdot n_{x_2} \rangle \langle n_{y_1} \cdot n_{y_2} \rangle - \frac{1}{N+1} \langle n_{x_1} \cdot n_{y_2} \rangle \langle n_{y_1} \cdot n_{x_2} \rangle.
\tag{3.27}
\]

is the usual connected 4-point function, related to the previously used second \(H\)-moment as indicated. In terms of \(W_1\) and \(W_2\) the Binder cumulant reads

\[
U_{\pm} = \frac{2}{N+1} - \frac{\sum_{x_1,x_2,y_1,y_2} W_{\pm,2}(x_1, y_1; x_2, y_2)}{[\sum_{x,y} (W_{\pm,1}(x, y) + 1)]^2}.
\tag{3.28}
\]

The Binder cumulant has accordingly a large \(N\) expansion of the form

\[
U_{\pm} = \sum_{s=0} \frac{1}{(N+1)^{s+1}} U_{\pm,s}(\lambda, V).
\tag{3.29}
\]

The coefficients can obviously be expanded in terms of the coefficients of the 2- and 4-point functions summed over all arguments:

\[
w_{\pm,s}(\lambda, V) := \sum_{x_1,x_2,y_1,y_2} W_{\pm,2}^{(s)}(x_1, y_1; x_2, y_2),
\]
\[
\sigma_{\pm,s}(\lambda, V) := \sum_{x,y} \left[ W_{\pm,1}^{(s)}(x, y) + \lambda^{-1} \delta_{s0} \right].
\tag{3.30}
\]
The two lowest orders

\[ U_{\pm,0} = 2 - \sigma_{\pm,0}^2 w_{\pm,0}, \]
\[ U_{\pm,1} = -\sigma_{\pm,0}^2 w_{\pm,1} + 2\sigma_{\pm,1} \sigma_{\pm,0}^3 w_{\pm,0}, \] (3.31)

involve only functions already computed in the previous subsections. We have

\[ \sigma_{\pm,0} = \pm \frac{V}{\omega_{\pm}}, \]
\[ w_{\pm,0} = 2\sigma_{\pm,0}^2 - \frac{2V}{\omega_{\pm}^2 \Pi_{\pm}(0)}, \] (3.32)

and

\[ \sigma_{\pm,1} = \pm \frac{2V}{\omega_{\pm}^2}[q_{\pm} - \sum_x \Delta_{\pm}(x) D_{\pm}(x)], \]
\[ w_{\pm,1} = -2q_{\pm} \frac{\partial}{\partial \omega_{\pm}} w_{\pm,0} + 2 \sum_{u,v,z,w} r_{\pm}(u,v)r_{\pm}(w,z) \]
\[ \times \Delta_{\pm}(u-w)[-D_{\pm}(u-w)D_{\pm}^{-1}(v-z) + D_{\pm}(u-v)\Delta_{\pm}(v-z)D_{\pm}(w-z)] \]
\[ - \sum_{u,v} r_{\pm}(u,v)^2 \Delta_{\pm}(u-v), \] (3.33)

where

\[ r_{\pm}(x,y) := \sum_{z,w} W_{\pm,2}^{(0)}(z,w;x,y) = \frac{2}{\omega_{\pm}^2} \left[ 1 - \Pi_{\pm}(0)^{-1} D_{\pm,2}(x-y) \right], \] (3.34)
4. TD limit of spin and current two-point functions

Up to this point we have been considering both the compact and noncompact models in arbitrary dimensions \( d \geq 2 \). In the following we restrict attention to \( d = 2 \). Also numerous articles have dealt with the \( 1/N \) expansion of the compact model so we will in this and in the next section restrict attention to the noncompact case and drop the minus \((-\) \) suffix on all functions.

The results summarized in Section 2 seemingly suggest a simple relation between the compact and the noncompact models. It is important to stress, however, that the relations hold only on a finite lattice and for invariant correlators. Physical quantities arising after taking the limit of infinite lattice size (thermodynamic limit) turn out to be very different in both systems. We will illustrate this fact by studying the thermodynamic (TD) limit of the coefficients in the \( 1/N \) expansions of the correlators computed in the last section in the noncompact model. The very existence of the limit is non-trivial in this case because \( \omega \to 0 \) as \( V \to \infty \), specifically

\[
\omega = \omega_-(\lambda, V) \sim -\frac{4\pi}{V \ln V} + O\left(\frac{1}{V \ln^2 V}\right). \tag{4.1}
\]

This means a ‘coordinated’ limit of lattice sums of the form \( \frac{1}{V} \sum k_1, \ldots, k_n f_V(k_1, \ldots, k_n) \) has to be taken, where \( f_V \) via \( \omega \) carries an explicit \( V \)-dependence. The gap equation (3.4) effectively acts as a subtle infrared regulator whose usefulness is underlined by the result summarized in Subsection 2.2. As mentioned, the sums associated with individual Feynman diagrams will typically diverge in the limit. The issue is whether the infrared divergences cancel in the \( W_r^{(s)} \) and the quantities computed in terms of them.

In this section we discuss the limit of the spin and current two–point functions; the limit of the Binder cumulant is computed in Section 5.

4.1 TD limit of the spin two–point function

In the leading order the 2–point function has an infinite volume limit

\[
-W_1^{(0)}(x, 0) = \frac{1}{V} \sum_{p \neq 0} \frac{e^{ipx} - 1}{E_p + \omega} \quad \longrightarrow \quad \mathcal{D}(x) := \int_p \frac{e^{ipx} - 1}{E_p}, \tag{4.2}
\]

where here and in the following \( \int_p \) means integration over the Brillouin zone \( \int_0^{2\pi} \frac{d^2 p}{(2\pi)^2} \). The infinite volume lattice propagator \( \mathcal{D}(x) \) is a remarkable function which has been discussed in detail by Shin [13]. At every lattice point it is given by an expression
of the form $r_1(x) + r_2(x)/\pi$ where $r_i$ are rational numbers. As $|x| \to \infty$ it diverges logarithmically:

$$D(x) \sim -\frac{1}{4\pi} \left( \ln x^2 + 2\gamma + 3 \ln 2 \right) + O(|x|^{-2}),$$  \hspace{1cm} (4.3)

where $\gamma \approx 0.577$ is Euler’s constant.

The next-to-leading term is given by

$$W^{(1)}_1(x, 0) = \frac{1}{V} \sum_{p \neq 0} (e^{ipx} - 1) W^{(1)}_1(p),$$  \hspace{1cm} (4.4)

with

$$W^{(1)}_1(p) = \frac{2}{(E_p + \omega)^2} \left[ -q + \frac{1}{V} \sum_k \frac{1}{\Pi(k)(E_{p-k} + \omega)} \right],$$  \hspace{1cm} (4.5)

and $q = q_-$ as in (3.9). For $p = 0$ the TD limit of $W^{(1)}_1(p)$ does not exist (similarly to the situation for the leading order): $W^{(1)}_1(0) \sim V \ln V \ln \ln V$, reflecting the fact that $W_1(x, 0)$ is an increasing function of distance $|x|$.

For $p \neq 0$, however the limit exists. To see this we first note that $\Pi(p)$ has a TD limit for $p \neq 0$. Indeed using one insertion of (5.17b) below and the gap equation one can rewrite $\Pi(p)$ as

$$\Pi(p) = -\frac{1}{(E_p + 2\omega)} \left[ \frac{2}{\lambda} + J(p) \right], \hspace{1cm} p \neq 0,$$

$$J(p) := \frac{1}{V} \sum_k \frac{E_k + E_{p-k} - E_p}{(E_k + \omega)(E_{p-k} + \omega)}. \hspace{1cm} (4.6)$$

Throughout we often use the symbol $J$ to denote lattice sums which give rise to convergent integrals over the Brillouin zone upon taking the infinite volume limit. The limit $\Pi_\infty(p)$ of $\Pi(p)$ is then given by

$$\Pi_\infty(p) = -\frac{1}{E_p v(p)}, \hspace{1cm} p \neq 0, \hspace{1cm} v(p) := \left[ \frac{2}{\lambda} + J_\infty(p) \right]^{-1}, \hspace{1cm} (4.7)$$

with

$$J_\infty(p) = \int_k \frac{E_k + E_{p-k} - E_p}{E_k E_{p-k}}. \hspace{1cm} (4.8)$$

The properties of the function $J_\infty(p)$ will be important later on; we mostly need:

$$J_\infty(p) \geq 0 \hspace{1cm} \text{for} \hspace{1cm} p \neq 0, \hspace{1cm} J_\infty((\pi, \pi)) = 0, \hspace{1cm} (4.9)$$

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and the behavior for \( p \to 0 \)

\[
J_\infty(p) = -\frac{1}{2\pi} [\ln p^2 - 5 \ln 2] + O(p^2 \ln p^2). \tag{4.10}
\]

Eq. (4.10) follows from [13], the positivity in (4.9) follows from Appendix A of [5]. A direct way to see \( J_\infty(p) \geq 0 \) is by performing one of the integrations explicitly. This leads to the integral representation

\[
J_\infty(p) = \frac{1}{2} \int_0^{2\pi} \frac{dk_1}{2\pi} \left[ -\text{th} \frac{a}{2} - \text{th} \frac{b}{2} + \frac{\text{sh}(a + b)}{\text{sh}a \text{sh}b} \left( \frac{4\text{sh}^2 \left( \frac{a + b}{2} \right) - \hat{p}_1^2}{4\text{sh}^2 \left( \frac{a + b}{2} \right) + \hat{p}_2^2} \right) \right], \tag{4.11}
\]

where \( a, b > 0 \) are determined by \( \text{sh} \frac{a}{2} = |\sin \frac{k_1}{2}|, \text{sh} \frac{b}{2} = |\sin \frac{k_1 - p_1}{2}|. \) For the numerator of the last term in the integrand one then has

\[
4 \sinh^2 \left( \frac{a + b}{2} \right) - \hat{p}_1^2 = 8 \left[ \sinh \frac{a}{2} \sinh \frac{b}{2} \cosh \left( \frac{a + b}{2} \right) + \sin \left( \frac{k_1}{2} \right) \sin \left( \frac{k_1 - p_1}{2} \right) \cos \left( \frac{p_1}{2} \right) \right]
\]

\[
= 8 \sinh \frac{a}{2} \sinh \frac{b}{2} \left[ \cosh \left( \frac{a + b}{2} \right) + \epsilon \cos \left( \frac{p_1}{2} \right) \right], \tag{4.12}
\]

where \( \epsilon = \pm 1. \) For fixed \( p_1, k_1 \) the integrand of (4.11) therefore is a monotonically decreasing function of \( p_2 \) for \( 0 < p_2 < \pi. \) By symmetry the same must hold with the roles of \( p_1 \) and \( p_2 \) interchanged, so that \( J_\infty(p) \geq J_\infty((\pi, \pi)) = 0, \) for all \( p \neq 0. \)

Returning to (4.5) we rewrite the sum as

\[
\frac{1}{V} \sum_k \frac{1}{\Pi(k)(E_{p-k} + \omega)} = J_\Pi(p) - \frac{1}{\lambda \Pi(p)}, \tag{4.13}
\]

\[
J_\Pi(p) := \frac{1}{2V} \sum_k \frac{1}{E_k + \omega} \left( \frac{1}{\Pi(p-k)} + \frac{1}{\Pi(p+k)} - \frac{2}{\Pi(p)} \right).
\]

Then using the properties of the \( \Pi \) function (4.9) and (4.10), it follows that \( J_\Pi \) has a finite TD limit which we denote by \( J_{\Pi,\infty}. \) In Section 5 we show that also the TD limit of \( q \) exists,

\[
q_\infty = -\int_k v(k) = J_{\Pi,\infty}(0). \tag{4.14}
\]

Putting the results together one sees that the limit of (4.5) is

\[
W_{1,\infty}^{(1)}(p) = \frac{2}{E_p} \left[ j(p) + \lambda^{-1} v(p) \right], \quad p \neq 0, \tag{4.15}
\]

\[
j(p) := \frac{1}{E_p} \left[ J_{\Pi,\infty}(p) - J_{\Pi,\infty}(0) \right].
\]
From (4.13) one also gets the ‘sum rule’

$$\int_p E_p W^{(1)}_{1,\infty}(p) = 2 + \frac{2}{\lambda} q_\infty,$$

(4.16)

while \( \int_p W^{(1)}_{1,\infty}(p) = 0 = W^{(1)}_{1,\infty}(x, x) \), as required.

It is instructive to compare now the continuum (i.e. small \( ap \) behavior where \( a \) is the lattice spacing) of \( W^{(1)}_{1,\infty}(p) \) with its counterpart in the compact model. In Appendix B the small \( p \) asymptotics of \( j(p) \) is determined. In terms of the (non-universal) constants \( g_2(\lambda), g_3(\lambda) \) the small \( p \) behavior comes out as

$$E_p W^{(1)}_{1,\infty}(p) \sim -\ln \left(-\ln(p^2/T)\right) + 2g_2(\lambda) + \frac{2g_3(\lambda) - 4\pi/\lambda}{\ln(p^2/T)} + O\left(\left[\ln(p^2/T)\right]^{-2}\right) \tag{4.17}$$

where

$$T = 32 \exp \left(\frac{4\pi}{\lambda}\right). \tag{4.18}$$

Eqs. (4.17), (4.18) illustrate in particular the nonperturbative nature of the large \( N \) expansion in the noncompact model, despite the fact that in infinite volume the expansion is effectively performed with respect to massless fields, as it is the case in perturbation theory. The infrared regulator \( \omega(\lambda, V) \) clearly works very differently from a constant ‘small mass’ regulator.

The subleading logarithmic terms in (4.17) are difficult to determine on the lattice. Using a continuum cutoff instead and cutoff-normalized continuum momenta

$$\frac{1}{V} \sum_k \rightarrow \int \frac{d^2k}{(2\pi)^2} \theta(\Lambda^2 - k^2), \quad [-\pi, \pi] \ni p_{\text{latt}} \rightarrow p_c := \frac{\sqrt{32p}}{\Lambda}, \tag{4.19}$$

the continuum counterpart of (4.15) can be found in closed form:

$$E_{p_c} W_{1,c}(p_c) = -\frac{2T}{p_c^2} \text{Li}\left(\frac{p_c^2}{T}\right) - 1 + \frac{4\pi}{\lambda} \frac{1}{\ln(T/p_c^2)} \int_0^{32/\nu^2} ds \frac{\ln s}{|1 - s| \ln(T/p_c^2) - \ln s}, \tag{4.20}$$

where \( \text{Li}(x) = \int_0^x ds / \ln s \). The integral in the last term is singularity free as the divergence of the \( 1/|1 - s| \) factor at \( s = 1 \) is removed by the \( \ln s \), and \( \ln(T/p_c^2) > \ln s \) holds over the entire range of the integration. The asymptotic expansion of (4.20) comes out...
\[ E_{p_c} W_{1,c}(p_c) \sim -\ln \left( -\frac{\lambda}{4\pi} \ln(p_c^2/T) \right) + \frac{2}{\ln(T/p_c^2)} + \sum_{n \geq 0} c_n \frac{1}{\ln(T/p_c^2)} + O(p_c^2), \]

\[ c_n := (-1)^{n+1} \Gamma(n+2)[2 - ((-1)^{n+1} + 1)\zeta(n+2)]. \quad (4.21) \]

We add some remarks. First, note that (4.21) contains factorially growing terms both with oscillating signs (Borel-summable) and with constant phase (non-Borel-summable). Second, (4.21) is an expansion in terms of the running coupling

\[ \frac{\alpha(p)}{2\pi} := \frac{1}{\ln(T/p_c^2)} = \frac{\lambda}{4\pi} \ln \frac{\Lambda^2}{p^2}, \quad (4.22) \]

which can be re-expanded in terms of positive powers of the bare coupling \( \lambda \). Comparing the result with its counterpart in the compact model one may verify that both perturbative expansions are related simply by flipping the sign of \( \lambda \); see [5] for a general proof of this “perturbative correspondence”. Third, the expressions (4.17), (4.21) do not suggest a nontrivial continuum limit reachable merely by a multiplicative field and a coupling renormalization. For example defining a renormalized coupling \( \lambda_r \) by

\[ \Lambda^2 e^{-4\pi/\lambda} = \mu^2 e^{-4\pi/\lambda_r}, \]

the infinite cutoff limit can only be taken by allowing negative bare couplings. For all \( \lambda > 0 \) both \( \lambda_r \) and \( \alpha(p) \) vanish for \( \Lambda \to \infty \). Finally, the expressions (4.20), (4.21) can also directly be compared with their analogues in the compact model, see [14] for the former. In the compact model the bare mass gap \( m_0^2 = \Lambda^2 e^{-4\pi/\lambda} \) enters and for \( m_0^2/p^2 = 32e^{-4\pi/\lambda}/p_c^2 \ll 1 \) (i.e. small \( \lambda \) at fixed \( p_c \)) expressions for the self energy \( E_{p_c} W_{1,c}(p_c) \) are obtained related to (4.20), (4.21) formally by flipping the sign of \( \lambda \). The attempt to take the sign flip beyond the asymptotic small \( \lambda \) expansion, however, would on the bare level produce a “hyperviolet” mass scale \( \Lambda^2 e^{+4\pi/\lambda} \), difficult to interpret. In contrast, in a lattice formulation the exact “large \( N \) correspondence” summarized in Subsection 2.2 exists.

### 4.2 TD limit of the Noether current two–point function

To obtain the infinite volume limit of the leading order contribution to the current correlation function (3.20) we decompose it according to:

\[ \tilde{j}^{(0)}_{\mu\nu}(q) = 2 \exp \left( \frac{i}{2} [q_\mu - q_\nu] \right) [A_{\mu\nu}(q) + \sin(q_\mu/2) \sin(q_\nu/2)\Pi(q)], \quad (4.23) \]
with

$$A_{\mu\nu}(q) = \frac{1}{V} \sum_p \frac{\sin(p + q/2)_\mu \sin(p + q/2)_\nu - \sin(q_\mu/2) \sin(q_\nu/2)}{(E_p + \omega)(E_{p+q} + \omega)}.$$  (4.24)

Now we have seen in the last subsection that $\Pi(q)$, $q \neq 0$, has a finite limit, and so obviously does $A_{\mu\nu}(q)$:

$$A_{\mu\nu}(q) \to A_{\infty,\mu\nu}(q) = \int \frac{1}{E_p E_{p+q}} \sin(p + q/2)_\mu \sin(p + q/2)_\nu - \sin(q_\mu/2) \sin(q_\nu/2),$$  (4.25)

which is independent of $\lambda$. Thus the infinite volume limit of $J^{(0)}_{\mu\nu}(q)$ exists.

In the next order we have the result (3.24). We now show consecutively: (i) that $X_3$ has a finite TD limit, and (ii) that $X_1 + X_2$ has a finite TD limit.

(i) Noting the identity

$$\sum_p \frac{\sin(p + q/2)_\mu}{(E_p + \omega)(E_{p+q} + \omega)} = 0,$$  (4.26)

we can write $X_3$ in (3.25) as

$$X_{3,\mu\nu}(q) = \frac{4}{V^2} \sum_{p_1, p_2} \frac{\sin(p_1 + q/2)_\mu \sin(p_2 + q/2)_\nu Y(p_1, p_2)}{(E_{p_1} + \omega)(E_{p_1+q} + \omega)(E_{p_2} + \omega)(E_{p_2+q} + \omega)},$$  (4.27)

where

$$Y(p_1, p_2) = \frac{1}{\Pi(p_1 - p_2)} - \frac{1}{\Pi(p_1)} - \frac{1}{\Pi(p_2)},$$  (4.28)

which vanishes when either $p_1 = 0$ or $p_2 = 0$. Then we break up $X_3$

$$X_{3,\mu\nu} = \frac{1}{(E_q + \omega)^2} [X_{4,\mu\nu} - X_{5,\mu\nu} - X_{5,\nu\mu} - X_{6,\mu\nu}],$$  (4.29)

with

$$X_{4,\mu\nu}(q) = \frac{4}{V^2} \sum_{p_1, p_2} \frac{\sin(p_1 + q/2)_\mu \sin(p_2 + q/2)_\nu Y(p_1, p_2)(E_q - E_{p_1})(E_q - E_{p_2})}{(E_{p_1} + \omega)(E_{p_1+q} + \omega)(E_{p_2} + \omega)(E_{p_2+q} + \omega)},$$

$$X_{5,\mu\nu}(q) = \frac{4}{V^2} \sum_{p_1, p_2} \frac{\sin(p_1 + q/2)_\mu \sin(p_2 + q/2)_\nu Y(p_1, p_2)(E_q - E_{p_1})}{(E_{p_1} + \omega)(E_{p_1+q} + \omega)(E_{p_2} + \omega)(E_{p_2+q} + \omega)},$$  (4.30)

$$X_{6,\mu\nu}(q) = \frac{4}{V^2} \sum_{p_1, p_2} \frac{\sin(p_1 + q/2)_\mu \sin(p_2 + q/2)_\nu Y(p_1, p_2)}{(E_{p_1+q} + \omega)(E_{p_2+q} + \omega)}.$$
Now the limit of $X_4$ exists. Next for $X_5$ we write

$$X_{5;\mu\nu}(q) = X_{7;\mu\nu}(q) + X_{8;\mu\nu}(q), \quad (4.31)$$

with

$$X_{7;\mu\nu}(q) = \frac{4}{V^2} \sum_{p_1, p_2} \sin (p_1 + q/2)_\mu \sin (p_2 + q/2)_\nu \left[ Y(p_1, p_2) - Y(p_1, -q) \right] \frac{(E_q - E_{p_1})}{(E_{p_1} + \omega)(E_{p_1+q} + \omega)(E_{p_2+q} + \omega)},$$

$$X_{8;\mu\nu}(q) = r_\nu(q) \frac{1}{V} \sum_p \sin (p + q/2)_\mu Y(p, -q) \left( E_q - E_{p} \right) \frac{1}{(E_p + \omega)(E_{p+q} + \omega)}, \quad (4.32)$$

where (using the gap equation)

$$r_\mu(q) := \frac{1}{V} \sum_p \sin (p + q/2)_\mu \frac{1}{E_{p+q} + \omega} = \frac{1}{4} \left[ 1 + (4 + \omega) \lambda^{-1} \right] \sin(q_\mu/2). \quad (4.33)$$

It is clear that both $X_7, X_8$ and so also $X_5$ have finite limits. Finally we have

$$X_{6;\mu\nu}(q) = X_{9;\mu\nu}(q) + X_{10;\mu\nu}(q) + X_{10;\nu\mu}(q) + 4Y(q, q)r_\mu(q)r_\nu(q), \quad (4.34)$$

with

$$X_{9;\mu\nu}(q) = \frac{4}{V^2} \sum_{p_1, p_2} \sin (p_1 + q/2)_\mu \sin (p_2 + q/2)_\nu \times \frac{[Y(p_1, p_2) - Y(p_1, -q) - Y(p_2, -q) + Y(q, q)]}{(E_{p_1+q} + \omega)(E_{p_2+q} + \omega)} \quad (4.35)$$

$$X_{10;\mu\nu}(q) = r_\nu(q) \frac{4}{V} \sum_p \sin (p + q/2)_\mu \left[ Y(p, -q) - Y(q, q) \right] \frac{1}{E_{p+q} + \omega}. \quad (4.36)$$

It follows that $X_8$ has a limit, and the demonstration that $X_3$ has a finite TD limit is complete.

(ii) We first rewrite the sum

$$X_{1;\mu\nu} + X_{2;\mu\nu} = -8q_\tau A_{2;\mu\nu} + X_{11;\mu\nu}, \quad (4.37)$$

with

$$A_{2;\mu\nu}(q) = \frac{1}{V} \sum_p \frac{F_{\mu\nu}(p, q)}{(E_p + \omega)^2},$$

$$X_{11;\mu\nu}(q) = \frac{8}{V^2} \sum_{p_1, p_2} \frac{F_{\mu\nu}(p_1, q)}{\Pi(p_1 - p_2)(E_{p_1} + \omega)^2(E_{p_2} + \omega)}. \quad (4.37)$$
Here
\[ F_{\mu\nu}(p, q) = \frac{\sin(p + q/2)_{\mu} \sin(p + q/2)_{\nu}}{E_{p+q} + \omega} - \frac{\sin(q/2)_{\mu} \sin(q/2)_{\nu}}{E_{q} + \omega}. \] (4.38)

To proceed we again write \( X_{11} \) as a sum of terms:

\[ X_{11;\mu\nu}(q) = X_{12;\mu\nu}(q) + X_{13;\mu\nu}(q) + 8f_{1} A_{2;\mu\nu}(q), \] (4.39)

where

\[
X_{12;\mu\nu}(q) = \frac{8}{V^2} \sum_{p_1, p_2} \frac{Y(p_1, p_2) F_{\mu\nu}(p_1, q)}{(E_{p_1} + \omega)^2 (E_{p_2} + \omega)},
\]

\[
X_{13;\mu\nu}(q) = -\frac{8\lambda^{-1}}{V} \sum_{p} \frac{F_{\mu\nu}(p, q)}{\Pi(p) (E_{p} + \omega)^2}, \] (4.40)

and \((f_2 \text{ will appear in the next section})\):

\[
f_s := \frac{1}{V} \sum_{p} \frac{1}{\Pi(p) (E_{p} + \omega)^s}. \] (4.41)

We still need to consider the limits of \( X_{12;\mu\nu}, X_{13;\mu\nu}, A_{2;\mu\nu} \) (actually the contribution in \( X_1 + X_2 \) involving \( A_2 \) has a coefficient proportional to \( q - f_1 \) which vanishes in the TD limit). We note that these three functions can be written in the form \( T_{2;\mu\nu}[g](q) \) where functions \( T_{s;\mu\nu}[g](q) \) are defined by

\[
T_{s;\mu\nu}[g](q) = \frac{1}{V} \sum_{p} \frac{g(p) F_{\mu\nu}(p, q)}{(E_{p} + \omega)^s}, \] (4.42)

with \( g(p) \) a regular periodic function with \( g(p) = g(-p) \) and finite at \( p = 0 \). Now for functions of the type \( T_1 \) we have

\[
T_{1;\mu\nu}[g](q) = \frac{1}{V} \sum_{p} \frac{g(p) F_{\mu\nu}(p, q)}{E_{p} + \omega}
= \frac{1}{E_{q} + 2\omega} \left\{ \frac{1}{V} \sum_{p} \frac{g(p) F_{\mu\nu}(p, q)(E_{q} - E_{p} - E_{p+q})}{E_{p} + \omega} \right. \\
+ \frac{1}{V} \sum_{p} \frac{g(p) F_{\mu\nu}(p, q)(E_{p+q} + \omega)}{E_{p} + \omega} \\
\left. + \frac{1}{V} \sum_{p} [g(p) - g(q)] F_{\mu\nu}(p, q) + g(q) \frac{1}{V} \sum_{p} F_{\mu\nu}(p, q) \right\}, \] (4.43)
which has a finite limit since $\frac{1}{V} \sum_p F_{\mu\nu}(p, q)$ does, as can easily be seen using the gap equation. Next

$$T_{2,\mu\nu}(g)(q) = \frac{1}{V} \sum_p \frac{g(p) F_{\mu\nu}(p, q)}{(E_p + \omega)^2}$$

$$= \frac{1}{E_q + 2\omega} \{ T_{1,\mu\nu}(g)(q) + T_{3,\mu\nu}(g)(q) + T_{4,\mu\nu}(g)(q) \} .$$  \hspace{1cm} (4.44)

Here

$$T_{3,\mu\nu}(g)(q) = \frac{1}{V} \sum_p \frac{g(p) F_{\mu\nu}(p, q)(E_q - E_p - E_{p+q})}{(E_p + \omega)^2} ,$$

$$T_{4,\mu\nu}(g)(q) = \frac{1}{(E_q + \omega)} \frac{1}{V} \sum_p \frac{g(p) \overline{F}_{\mu\nu}(p, q)}{(E_p + \omega)^2} ,$$  \hspace{1cm} (4.45)

with

$$\overline{F}_{\mu\nu}(p, q) := (E_q + \omega)(E_{p+q} + \omega) F_{\mu\nu}(p, q) .$$  \hspace{1cm} (4.46)

Decomposing $T_3$ further gives

$$T_{3,\mu\nu}(g)(q) = \frac{1}{E_q + \omega} \{ T_{5,\mu\nu}(g)(q) + T_{6,\mu\nu}(g)(q) \} ,$$  \hspace{1cm} (4.47)

with

$$T_{5,\mu\nu}(g)(q) = \frac{1}{V} \sum_p \frac{g(p) F_{\mu\nu}(p, q)(E_q - E_p - E_{p+q})(E_q - E_{p+q})}{(E_p + \omega)^2} ,$$

$$T_{6,\mu\nu}(g)(q) = \frac{1}{E_q + \omega} \frac{1}{V} \sum_p \frac{g(p) \overline{F}_{\mu\nu}(p, q)(E_q - E_p - E_{p+q})}{(E_p + \omega)^2} ,$$  \hspace{1cm} (4.48)

where $T_5$ clearly has a TD limit. Now

$$\bar{F}_{\mu\nu}(p, q) = \bar{F}_{-\mu\nu}(p, q) + \bar{F}_{+\mu\nu}(p, q) ,$$  \hspace{1cm} (4.49)

with $F_{\pm}(-p, q) = \pm F_{\pm}(p, q)$:

$$\bar{F}_{-\mu\nu}(p, q) = (E_q + \omega) [\sin p\mu \cos p\nu \cos(q/2)_\mu \sin(q/2)_\nu + (\mu \leftrightarrow \nu)]$$

$$-2 \sum_{\rho} \sin p\rho \cos q\rho \sin(q/2)_\mu \sin(q/2)_\nu ,$$  \hspace{1cm} (4.50a)

$$\bar{F}_{+\mu\nu}(p, q) = (E_q + \omega) \sin p\mu \sin p\nu \cos(q/2)_\mu \cos(q/2)_\nu$$

$$-\frac{1}{2} \sin(q/2)_\mu \sin(q/2)_\nu \left[ (E_q + \omega)(\hat{p}_\mu^2 + \hat{p}_\nu^2) - \frac{1}{2} \hat{p}_\mu^2 \hat{p}_\nu^2 \right] + 2E_p - \sum_{\rho} \hat{p}_\rho^2 \hat{p}_\rho^2 .$$  \hspace{1cm} (4.50b)
So
\[ T_{6;\mu\nu}[g](q) = \frac{1}{E_q + \omega} \{ T_{7;\mu\nu}[g](q) + T_{8;\mu\nu}[g](q) + g(0)T_{9;\mu\nu}(q) \} , \]  
(4.51)

with
\[ T_{7;\mu\nu}[g](q) = \frac{1}{2} \sum_\rho \hat{q}_\rho \frac{1}{V} \sum_p \frac{g(p)\bar{F}_{+;\mu\nu}(p, q)p_\rho^2}{(E_p + \omega)^2} , \]
\[ T_{8;\mu\nu}[g](q) = -2 \sum_\rho \cos q_\rho \frac{1}{V} \sum_p \frac{[g(p) - g(0)]\bar{F}_{-;\mu\nu}(p, q) \sin p_\rho}{(E_p + \omega)^2} , \]  
(4.52)
\[ T_{9;\mu\nu}(q) = -2 \sum_\rho \cos q_\rho \frac{1}{V} \sum_p \frac{\bar{F}_{-;\mu\nu}(p, q) \sin p_\rho}{(E_p + \omega)^2} . \]

All of these quantities have a finite limit, in particular \( T_{9;\mu\nu}(q) \) on account of the gap equation. A similar decomposition can be performed for \( T_{4;\mu\nu}(q) \), showing that it likewise has a TD limit. So \( T_{2;\mu\nu}[g](q) \) has a TD limit. It follows that \( X_{12;\mu\nu}, X_{13;\mu\nu}, A_{2;\mu\nu} \) all have infinite volume limits.

To summarize, we have shown that the TD limit of the contribution to the two leading orders in the \( 1/N \) expansion of the current correlator \( J_{\mu\nu}^{(s)}, s = 0, 1 \), exists.

5. TD limit of the Binder cumulant

In the following we evaluate the infinite volume limit of \( U \) in the 2-dimensional noncompact model to sub-leading order, i.e. the coefficients \( U_0, U_1 \) in

\[ U(\lambda, V) = \frac{U_0(\lambda, V)}{N+1} + \frac{U_1(\lambda, V)}{(N+1)^2} + O\left(\frac{1}{(N+1)^3}\right) , \]  
(5.1)

are computed for large volumes. Somewhat surprisingly we will find that the limit \( V \rightarrow \infty \) exists and is independent of \( \lambda \)!

In Subsection 5.1 the large volume asymptotics will be evaluated analytically. As a test of the estimates and in order to have finite volume results to compare Monte-Carlo data with, we also directly evaluated the multiple lattice sums numerically up to \( L = 1024 \). The results are reported in Subsection 5.2.

Finally, to preclude that the large \( N \) results are misleading we performed a Monte-Carlo study of \( U(\lambda, V) \) up to \( L = 384 \) for \( N = 8 \).
5.1 Analytical analysis of large $V$ asymptotics

Evaluation of the leading order coefficient is straightforward. From (3.31) and (3.32) one obtains

$$U_0(\lambda, V) = \frac{2}{V \omega^2 \Pi(0)}. \quad (5.2)$$

Now the TD limit of $\omega \Pi(0)$ can be evaluated from

$$- \omega \Pi(0) = \frac{1}{\lambda} + \frac{1}{V} \sum_{k \neq 0} \frac{E_k}{(E_k + \omega)^2} = \frac{1}{4\pi} \ln V + \frac{1}{\lambda} + a(\lambda) + O\left(\frac{1}{\ln V}\right), \quad (5.3)$$

where $a(\lambda) > 0$. Note that (taking the $\lambda$-derivative of the gap equation (3.4)) this translates into a large volume asymptotics for $\omega = \omega_-$ of the form

$$V \omega(\lambda, V) = -\frac{4\pi}{\ln V} + \frac{(4\pi)^2}{\lambda} \frac{1}{\ln^2 V} - \frac{(4\pi)^3}{\lambda^2} \left(1 - \lambda^2 \int ds \frac{a(s)}{s^2}\right) \frac{1}{\ln^3 V} + O\left(\frac{1}{\ln^4 V}\right). \quad (5.4)$$

From (5.3) one has

$$U_0(\lambda, V) = 2 - \frac{8\pi a(\lambda)}{\ln V} + O\left(\frac{1}{\ln^2 V}\right),$$

$$U_0(\lambda, \infty) = 2, \quad (5.5)$$

in stark contrast to the compact model (as discussed in the conclusions).

The discussion of the sub-leading order is more involved and is best done in Fourier space. We prepare the following auxiliary functions

$$\Pi_{st}(p) = \frac{1}{V} \sum_k \frac{1}{(E_k + \omega)^s(E_{p-k} + \omega)^t}, \quad s, t \geq 1, \quad (5.6)$$

where $\Pi(p) := \Pi_1(p) = \tilde{\Delta}(p)^{-1}$ is the inverse of the Fourier transform of $\Delta(x)$. Further we shall need in addition to (4.11)

$$f_{st} = \frac{1}{V^2} \sum_{p,k} \frac{1}{\Pi(k)(E_p + \omega)^s(E_{p-k} + \omega)^t} = \frac{1}{V} \sum_k \frac{\Pi_{st}(k)}{\Pi(k)}. \quad (5.7a)$$

Note

$$f_{21} = \Pi(0)q. \quad (5.8)$$
The expression (3.29) for $U_1$ we break up according to

$$U_1 = U_{11} + U_{12},$$

$$U_{11} = 2\sigma_1\sigma_0^{-3} w_0,$$

$$U_{12} = -\sigma_0^{-2} w_1 = -\frac{1}{\omega^2 V^2} [A + B + C + D].$$

Using $\sigma_1 = 2V\omega^{-2}(-q + f_1)$ and $w_0 = 2V\omega^{-2}(V - \omega^{-2}\Pi(0)^{-1})$ one has

$$U_{11} = \frac{8}{\omega}(q - f_1) \left(1 - \frac{1}{\omega^2 V\Pi(0)}\right).$$

In $U_{12}$ the term $A$ comes from the $\partial w_0 / \partial \omega$ derivative in $w_1$ and is given by

$$A = -2q\omega^4 \frac{\partial}{\partial \omega} \left\{ \frac{2V}{\omega^2} \left(V - \frac{1}{\omega^2\Pi(0)}\right) \right\} = 8qV^2 \left\{ \omega - \frac{2}{V\omega\Pi(0)} + \frac{\Pi_2(0)}{V\Pi(0)^2} \right\}.$$  

The term $B$ corresponds to the $W_2^{(0)} D\Delta D^{-1}W_2^{(0)}$ piece in $w_1$ and is given by

$$B = -8 \left(V\omega - \frac{2}{\omega\Pi(0)}\right)V f_1 - \frac{8V}{\Pi(0)^2} f_{31}.$$

The term $C$ arises from the $W_2^{(0)} D\Delta DW_2^{(0)}$ structure and reads

$$C = 8 \sum_k \frac{1}{\Pi(k)^2} \left[ \frac{\Pi_2(k)}{\Pi(0)} - \frac{1}{E_k + \omega} \right]^2.$$  

Finally $D$ comes from the $W_2^{(0)} \Delta W_2^{(0)}$ term

$$\frac{1}{V^2\omega^2} D = -\frac{4}{V\omega^2\Pi(0)} \left[1 - 2f_2 + \frac{1}{\Pi(0)^2} f_{22}\right].$$

In the $V \rightarrow \infty$ limit certain terms are superficially divergent, so it is best to combine terms where such divergences cancel. Specifically we rewrite $A + B$ as

$$\frac{1}{\omega^2 V^2} (A + B) = 2U_{11} + \frac{8}{\omega} (q - f_1) \left(\frac{\Pi_2(0)}{V\omega\Pi(0)^2} - 1\right) + \frac{8}{V\omega^2\Pi(0)^2} \frac{1}{\Pi(0)} (f_1\Pi_2(0) - f_{31}).$$
Further useful combinations to discuss the limit turn out to be
\begin{align*}
q - f_1 &= \frac{1}{V} \sum_k \frac{1}{\Pi(k)} \left[ \frac{\Pi_2(k)}{\Pi(0)} - \frac{1}{E_k + \omega} \right], \quad (5.16a) \\
\frac{1}{\Pi(0)} f_{22} - 2 f_2 &= \frac{1}{V} \sum_k \frac{1}{\Pi(k)} \left[ \frac{\Pi_{22}(k)}{\Pi(0)} - \frac{2}{(E_k + \omega)^2} \right], \quad (5.16b) \\
f_{31} - \Pi_2(0) f_1 &= \frac{1}{V} \sum_k \frac{1}{\Pi(k)} \left[ \Pi_3(k) - \frac{\Pi_2(0)}{E_k + \omega} \right]. \quad (5.16c)
\end{align*}

For the evaluation of the TD limit then mainly the combinations in square brackets in (5.16) have to be studied, for large volumes. A naive replacement of the lattice sums by integrals over the Brillouin zone with \( \omega = 0 \) would produce infrared divergent integrals. The strategy in the following will be to evaluate the large volume asymptotics of the functions \( \Pi_s \), \( s = 1, 2, 3 \) and \( \Pi_{22} \) by repeated insertion of one of the following decompositions of unity
\begin{align*}
1 &= \frac{1}{E_p + \omega} \left[ E_p - E_{p-k} + (E_{p-k} + \omega) \right], \quad (5.17a) \\
1 &= \frac{1}{E_p + 2 \omega} \left[ E_p - E_k - E_{p-k} + (E_{p-k} + \omega) + (E_k + \omega) \right], \quad (5.17b)
\end{align*}
until terms corresponding to infrared convergent integrals over the Brillouin zone arise. The volume and the momentum dependence of the additional pieces picked up in the process (which may diverge as \( V \to \infty \)) can then be studied analytically. For \( \Pi_1 = \Pi \) itself only one insertion of (5.17b) was needed and no divergent piece arose, see (4.6).

Proceeding similarly we derive the relations
\begin{align*}
\Pi_{st}(p) &= \frac{1}{(E_p + 2 \omega)} \left[ X_{st}(p) + \Pi_{(s-1)t}(p) + \Pi_{s(t-1)}(p) \right], \quad s, t \geq 1, \\
X_{st}(p) &= \frac{1}{V} \sum_k \frac{E_p - E_k - E_{p-k}}{(E_k + \omega)^s(E_{p-k} + \omega)^t}. \quad (5.18)
\end{align*}

Applying it for the case \( s = 2, t = 1 \) (and noting \( \Pi_{s0}(p) = \Pi_{s-1}(0) \)) we get
\begin{align*}
\Pi_2(p) &= \frac{1}{E_p + 2 \omega} \left[ X_{21}(p) + \Pi(p) + \Pi(0) \right]. \quad (5.19)
\end{align*}

This can be rewritten as
\begin{align*}
\frac{1}{\Pi(p)} \left[ \frac{\Pi_2(p)}{\Pi(0)} - \frac{1}{E_p + \omega} \right] &= \frac{1}{\Pi(0)} \left[ \frac{1}{E_p + 2 \omega} + \frac{X_{21}(p)}{(E_p + 2 \omega)\Pi(p)} \right] - \frac{\omega}{\Pi(p)(E_p + \omega)(E_p + 2 \omega)}, \quad (5.20)
\end{align*}
from which \( q - f_1 \) can be evaluated. In a first step one finds

\[
\Pi(0)(q - f_1 + \omega \tilde{f}_2) = \frac{1}{V \omega} \left( \frac{\omega \Pi(0)}{\Pi(0)} - 1 \right) \\
+ \frac{1}{V} \sum_{k \neq 0} \frac{1}{E_k + 2\omega} + \frac{1}{V} \sum_k \frac{J_3(k)}{(E_k + \omega)^2}.
\] (5.21)

Here

\[
\tilde{f}_2 = \frac{1}{V} \sum_{k \neq 0} \frac{1}{\Pi(k)(E_k + \omega)(E_k + 2\omega)},
\]

\[
J_3(p) = \frac{1}{V} \sum_{k \neq 0} \frac{E_k - E_p - E_{p-k}}{\Pi(k)(E_{p-k} + \omega)(E_k + 2\omega)}.
\] (5.22)

The function \( J_3(p) \) has a finite limit which for small \( p \) behaves as

\[
J_{3,\infty}(p) = E_p \left[ j_3 + O(1/\ln p^2) \right],
\]

\[
j_3 = \int_k \frac{v(k)}{E_k^2} \cos k_1 - \cos k_2^2.
\] (5.23)

This can be used to show that

\[
\Pi(0) \omega f_2 = \frac{1}{8\pi} \ln \ln V \ln V + \frac{q_2}{4\pi} \ln V + O(\ln \ln V),
\]

\[
\Pi(0)(q - f_1) = -\frac{1}{8\pi} \ln \ln V \ln V + \frac{q_1}{4\pi} \ln V + O(\ln \ln V),
\] (5.24)

where the constants are related by

\[
q_1 + q_2 = j_3.
\] (5.25)

Indeed, using the fact that \( E_k \Pi(k) \) has a finite limit which scales like \( -\frac{1}{2\pi} \ln k^2/T \) for \( k^2 \to 0 \), one readily verifies the first equation. Further

\[
1 - \frac{\omega \Pi(0)}{\Pi(0)} = O(\frac{1}{\ln^2 V}),
\]

\[
\tilde{f}_2 + \frac{1}{V \omega^2 \Pi(0)} = f_2 + O(\frac{1}{\ln^2 V}),
\]

\[
\frac{1}{V} \sum_{k \neq 0} \frac{1}{E_k + 2\omega} = -\frac{1}{V \omega} - \frac{1}{\lambda} + O(\frac{1}{\ln V}),
\]

\[
\frac{1}{V} \sum_k \frac{J_3(k)}{(E_k + \omega)^2} = \frac{1}{4\pi} j_3 \ln V + O(1).
\] (5.26)
Inserted into (5.21) gives

\[ \Pi(0)[q - f_1 + \omega f_2] = j_3 \frac{1}{4\pi} \ln V + O(\ln \ln V), \]  

(5.27)

and hence the first equation in (5.24). Combined with (5.3) we arrive at

\[ U_{11} = 16\pi a(\lambda) \left( \frac{\ln \ln V}{\ln V} \right) + O\left( \frac{\ln \ln V}{\ln^2 V} \right), \]  

(5.28)

using

\[ 1 - \frac{\Pi^2(0)}{\omega \Pi(0)^2} = 1 - \frac{1}{V \omega \Pi(0)^2} + O\left( \frac{1}{\ln V} \right). \]  

(5.29)

Since \( U_1 = U_{11} - (A + B + C + D)/(\omega^2 V^2) \) one concludes from (5.28) and (5.15) that the softly decaying \( U_{11} \) terms in \( U_1 \) cancel.

We proceed with the evaluation of \( C \). To this end a more detailed evaluation of \( X_{21} \) defined in (5.18) is needed. In a first step one obtains

\[
(E_p + \omega)^2 X_{21}(p) = \left\{ (E_p + \omega)\Omega(0) \sum_{\mu} \cos^2 \frac{p_{\mu}}{2} + 2 \sum_{\mu} \sin^2 \frac{p_{\mu}}{2} \sum_{k} \frac{(\sin k_1 + \sin k_2)^2}{(E_k + \omega)^2} \right\} 
\]

\[ + \left\{ \frac{E_p + \omega}{\lambda} \sum_{\mu} \cos^2 \frac{p_{\mu}}{2} + J_2(p) + \sum_{\mu} \cos^2 \frac{p_{\mu}}{2} \cos \frac{p_{\mu}}{2} \sum_{k} \frac{E_k^2}{(E_k + \omega)^2} \right\} 
\]

\[- 64 \sin^2 \frac{p_1 - p_2}{2} \sin^2 \frac{p_1 + p_2}{2} \frac{1}{V} \sum_{k} \frac{(\sin k_1 + \sin k_2)^2}{(E_k + \omega)^2} \right\}, \]  

(5.30)

where

\[ J_2(p) = \frac{1}{V} \sum_{k} \frac{(E_p - E_k - E_{p-k})(E_p - E_{p-k})}{(E_k + \omega)^2(E_{p-k} + \omega)}. \]

\[ -2 \leq J_{2,\infty}(p) \leq 0, \quad J_{2,\infty}(p) = -2 + j_2 p^2 + O(p^4), \quad j_2 \approx 0.93. \]  

(5.31)

As \( V \to \infty \) the first curly bracket diverges logarithmically while the second one is convergent. Using

\[ \frac{1}{V} \sum_{k} \frac{(\sin k_1 \sin k_2)^2}{(E_k + \omega)^2} = \frac{1}{32\pi} + O\left( \frac{1}{V} \right), \]

\[ \frac{1}{V} \sum_{k} \frac{(\sin k_1 + \sin k_2)^2}{(E_k + \omega)^2} = \frac{1}{4\pi} \ln V + a(\lambda) - \frac{1}{4} + \frac{1}{4\pi} + O\left( \frac{1}{\ln V} \right), \]

\[ \frac{1}{V} \sum_{k} \frac{E_k^2}{(E_k + \omega)^2} = 1 + O\left( \frac{1}{V} \right), \]  

(5.32)
with \( a(\lambda) \) as in (5.3) one finds

\[
X_{21}(p) = -\frac{1}{E_p^2} (\cos p_1 - \cos p_2)^2 \left[ \frac{1}{4\pi} \ln V + a(\lambda) + \frac{1}{2\pi} \right] + \frac{1}{E_p^2} \left[ J_2(\infty)(p) + \frac{1}{2} \sum \mu (\cos p_\mu + \cos 2p_\mu) + \frac{1}{2\pi} \sum \mu \sin^2 p_\mu \right] + O\left( \frac{1}{\ln V} \right).
\]

For small momenta this behaves like

\[
X_{21}(p) = -\left[ \frac{1}{4\pi} \ln V + a(\lambda) + \frac{1}{2\pi} \right] \left[ \frac{1}{4} - \left( \frac{p_1}{p_2} + \frac{p_2}{p_1} \right)^{-2} + o(p_1, p_2) \right] + \frac{1}{p^2} \left( \frac{j_2}{4} - \frac{5}{4} + \frac{1}{2\pi} + o(p_1, p_2) \right).
\]

For the evaluation of \( C \) it is useful to rewrite (5.18) for \( p \neq 0 \) as

\[
1 + \Pi(p) \left[ \frac{\Pi_2(p)}{\Pi(0)} - \frac{1}{E_p + \omega} \right] = \frac{1}{(E_p + \omega)\Pi(0)} \left[ 1 - \frac{\omega \Pi_2(p)}{\Pi(p)} + \frac{X_{21}(p)}{\Pi(p)} \right],
\]

\[
1 - \frac{\omega \Pi_2(p)}{\Pi(p)} + \frac{X_{21}(p)}{\Pi(p)} = \frac{E_p}{E_p + 2\omega} X_2(p) + \frac{X_{21}(p)}{\Pi(p)} \frac{E_p + \omega}{E_p + 2\omega},
\]

where

\[
X_2(p) := 1 + \frac{\omega}{E_p \Pi(p)} (\Pi(p) - \Pi(0)).
\]

In the first term the fact that the numerator is proportional to \( E_p \) is important for the eventual decay properties of \( C \). Further \( X_2(p) \) is bounded by a function of order \( \ln V \) in the volume and it has at most logarithmic singularities for \( p \to 0 \). The latter is consistent with

\[
\omega \Pi_2(p) \sim \frac{\omega \Pi(0)}{E_p \Pi(p)} + O(1/V),
\]

where the \( O(1/V) \) piece comes from \( \omega X_{21}(p) \).

Inserting (5.35) into (5.13) gives

\[
C = \frac{8}{\omega^2 \Pi(0)^2} \left( 1 - \frac{\omega \Pi_2(0)}{\Pi(0)} \right)^2 + \frac{8V}{\Pi(0)^2 V} \sum_{k \neq 0} \frac{1}{(E_k + \omega)^2} \left[ 1 - \frac{\omega \Pi_2(k)}{\Pi(k)} + \frac{X_{21}(k)}{\Pi(k)} \right]^2.
\]

Using the known behavior of the constituent functions for large \( V \) and small momenta, one can verify a decay of the form\(^1\)

\[
\frac{C}{\omega^2 V^2} = O\left( \frac{1}{\ln^2 V} \right) + O\left( \frac{1}{\ln^3 V} \right).
\]

\(^1\)Here and later on we indicate the form of the sub-leading term, without however (in a slight abuse of the \( O \) symbol) presupposing that its coefficient is nonzero.
We proceed with the $D$ term, where $\Pi_{22}(p)$ enters. A useful representation is

$$\Pi_{22}(p) = \frac{2}{(E_p + 2\omega)^2}[\Pi(p) + \Pi(0)] + \frac{6}{(E_p + 2\omega)^2}X_{21}(p)$$

$$+ \frac{1}{(E_p + 2\omega)^3}[J_4(p) + 2J(p) - 2X_1(p)], \quad (5.40a)$$

$$J_4(p) = \frac{1}{V} \sum_k \frac{(E_p - E_{p-k} - E_k)^3}{(E_k + \omega)^2(E_{p-k} + \omega)^2}, \quad (5.40b)$$

$$X_1(p) = \frac{1}{V} \sum_k \frac{E_p - E_k - E_{p-k}}{(E_k + \omega)^2}. \quad (5.40c)$$

This can be used to determine the large $V$ behavior of the term $D$ in (5.14). We begin by separating the zero mode in the relevant combination

$$\frac{1}{\Pi(0)} f_{22} - 2 f_2 + 1 = \left(1 - \frac{1}{\omega^2 V^2 \Pi(0)}\right) - \frac{1}{V \omega^2 \Pi(0)} \left(1 - \frac{\omega^2 \Pi_{22}(0)}{\Pi(0)}\right)$$

$$+ \frac{1}{V} \sum_{k \neq 0} \frac{1}{\Pi(k)} \left[\frac{\Pi_{22}(k)}{\Pi(0)} - \frac{2}{(E_k + \omega)^2}\right]. \quad (5.41)$$

On account of

$$1 - \frac{1}{\omega^2 V \Pi(0)} = O\left(\frac{1}{\ln V}\right),$$

$$1 - \frac{\omega^2 \Pi_{22}(0)}{\Pi(0)} = 1 - \frac{\omega \Pi_2(0)}{\Pi(0)} + O\left(\frac{1}{\ln^3 V}\right),$$

the zero mode pieces are $O(1/\ln V)$. For the last term on the right hand side of (5.41) we introduce the shorthand $S_1 + S_2$. Upon insertion of (5.40) we write $S_1$ for the part coming from the $\Pi(p) + \Pi(0)$ piece in (5.40) and $S_2$ for the rest,

$$S_1 = \frac{2}{\Pi(0)} \frac{1}{V} \sum_{k \neq 0} \frac{1}{(E_k + 2\omega)^2} + \frac{2}{V} \sum_{k \neq 0} \frac{1}{\Pi(k)} \left(\frac{1}{(E_k + 2\omega)^2} - \frac{1}{(E_k + \omega)^2}\right). \quad (5.42)$$

The term $S_2$ reads

$$S_2 = \frac{1}{\Pi(0)} \frac{1}{V} \sum_{k \neq 0} \frac{1}{\Pi(k)(E_k + 2\omega)^2}[J_4(k) + 2J(k) - 2X_1(k)]$$

$$+ \frac{6}{\Pi(0)} \frac{1}{V} \sum_{k \neq 0} \frac{X_{21}(k)}{\Pi(k)(E_k + 2\omega)^2}. \quad (5.43)$$
and is checked to behave as
\[ S_2 = O\left( \frac{1}{\ln^2 V} \right) + O\left( \frac{1}{\ln^3 V} \right). \]  
(5.44)

Together
\[ \frac{D}{\omega^2 V^2} = O\left( \frac{1}{\ln V} \right) + O\left( \frac{1}{\ln^2 V} \right). \]  
(5.45)

It remains to consider \( A + B \). In view of (5.15) and (5.28) we know
\[ \frac{A + B}{V^2 \omega^2} = U_{11} + O\left( \frac{\ln \ln V}{\ln^2 V} \right) + \frac{8}{V \omega^2 \Pi(0) \Pi(0)} [f_1 \Pi_2(0) - f_{31}], \]  
(5.46)

so that only \( f_{31} - \Pi_2(0) f_1 \) is still needed. Using (5.18) for the case \( s = 3, t = 1 \) we obtain
\[ f_{31} - \Pi_2(0) f_1 = \frac{1}{\omega^2 \Pi(0)} \left[ \omega^2 \Pi_3(0) - \omega \Pi_2(0) \right] + \frac{1}{V} \sum_{k \neq 0} \frac{X_{31}(k)}{\Pi(k)(E_k + 2\omega)} + \frac{1}{V} \sum_{k \neq 0} \frac{\Pi_2(k)}{\Pi(k)(E_k + 2\omega)} - \omega \Pi_2(0) \frac{1}{V} \sum_{k \neq 0} \frac{1}{\Pi(k)(E_k + 2\omega)(E_k + \omega)}. \]  
(5.47)

Since
\[ \omega^2 \Pi_3(0) - \omega \Pi_2(0) = O(V^2), \]  
(5.48)

the zero mode piece scales like \( O(V/\ln V) \). For the second term we observe
\[ \frac{1}{V} \sum_{k \neq 0} \frac{X_{31}(k)}{\Pi(k)(E_k + 2\omega)} = \frac{1}{V} \sum_{p} \frac{J_3(p)}{(E_p + \omega)^3}, \]  
(5.49)

with \( J_3 \) as in (5.22). As a consequence this term in (5.47) scales like \( O(V) \) for large \( V \). In the last two terms we insert (5.19) to get
\[ \frac{1}{V} \sum_{k \neq 0} \frac{1}{(E_k + 2\omega)^2} \left[ 1 + \frac{X_{21}(k)}{\Pi(k)} \right] + \frac{1}{V} \sum_{k \neq 0} \frac{1}{\Pi(k)(E_k + 2\omega)^2} \left[ \Pi(0) - \omega \Pi_2(0) \frac{E_k + 2\omega}{E_k + \omega} \right]. \]  
(5.50)

The very first term is \( O(V \ln V) \), the one involving \( X_{21}(p) \) is \( O(V/\ln V) \), and the last one is \( O(V \ln \ln V) \). Together
\[ f_{31} - \Pi_2(0) f_1 = O(V \ln V) + O(V \ln \ln V). \]  
(5.51)

For \( A + B \) this results in
\[ \frac{1}{V^2 \omega^2} (A + B) = U_{11} + O\left( \frac{1}{\ln V} \right) + O\left( \frac{\ln \ln V}{\ln^2 V} \right). \]  
(5.52)
Combining (5.9) with (5.32), (5.39) and (5.45) we arrive at the conclusion:

\[ U_1(\lambda, V) = O\left(\frac{1}{\ln V}\right) + O\left(\frac{\ln \ln V}{\ln^2 V}\right) + O\left(\frac{1}{\ln^2 V}\right), \]

\[ U_1(\lambda, \infty) = 0. \]  

(5.53)

For the TD limit of the full Binder cumulant the result (5.53) amounts to

\[ U(\lambda, \infty) = \frac{2}{N+1} + O\left(\frac{1}{(N+1)^3}\right). \]  

(5.54)

This result will be backed by Monte-Carlo simulations in Subsection 5.2. Potential implications for “criticality” and “triviality” of the theories are discussed in the conclusions.

5.2 Direct evaluation of lattice sums

Both as a check on the previous analysis and in order to have finite volume data to compare Monte-Carlo data with, we also evaluated the lattice sums defining \( U_1 \) and several other quantities numerically up to \( L = 1024 \). Since \( O(L^4) \) terms have to be summed and both very small (e.g. \( \omega \)) and very large numbers (e.g. \( \Pi(0) \)) enter high precision is needed. The summations were performed to 96 bit (26 significant figures) accuracy using the publicly available arbitrary precision MPFR library (www.mpfr.org) and for moderate \( L \) also with Mathematica.

The results were found to vary with \( \lambda \) such that for smaller \( \lambda \) the presumed large \( V \) asymptotics sets in later. Below we present the results for \( \lambda = 3 \); qualitatively those for other \( \lambda \) values are similar. Due to the predicted occurrence of very slowly varying terms (e.g. of \( \ln \ln V / \ln V \) type) one cannot expect that the genuine large \( V \) asymptotics can be unambiguously probed by direct summation. Nevertheless two or three parameter fits of the \( L \leq 1024 \) sums to the expected decay form are generally convincing. Table 1 summarizes results for \( \omega, \Pi(0) \) and some slowly varying quantities entering \( U_{11} \). Here \( U_{11} \) is defined in Eq. (5.10), \( q - f_1 \) is evaluated directly from (3.9), (4.41) and from (5.16a), \( f_2 \) is defined in (4.41). For example the leading asymptotics \( \omega \Pi(0) \sim -\frac{1}{4\pi} \ln V \) and the coefficients in

\[ - f_2 \sim \frac{1}{2} \ln \ln V \sim \frac{1}{\omega} (q - f_1), \]

(5.55)
come out well in fits to the data.

Table 2 presents results for the terms used in the breakup of \( U_1 \), see Eq. (5.9), and the final result for \( U_1 \). The column for \( A + B \) again illustrates the need for high precision, as individually \( A \) and \( B \) are \( 2 - 5 \) orders of magnitudes larger that their sum. It also
Table 1: Quantities entering $U_{11}$ for $\lambda = 3$; all given digits are significant.

Table 2: Quantities contributing to $U_1$ for $\lambda = 3$; all given digits are significant.

Finally we present a fit of the $U_1$ data to the predicted decay form in (5.53).

5.3 MC results for $U$

Since the large $N$ expansion is only an asymptotic expansion the higher order coefficients in (3.29) are not bound to be small, even in finite volume. At any given $N$ the truncated series could in principle misrepresent the exact $U(\lambda, V)$. In order to preclude this possibility we estimated $U(\lambda, V)$ via Monte-Carlo simulations.

We have chosen to simulate a SO(1,8) theory at $\lambda = 3$ on lattices of linear dimensions $L = 32, 64, 128, 256, 384$. The simulations were performed in a fixed spin gauge (the spin at the origin was held fixed). The variable spins were updated by a Metropolis procedure tuned to achieve a roughly 50% acceptance rate. Equilibration and autocorrelation times for various observables have an enormous range: non-gauge-invariant observables in particular (e.g. $\langle n^0 \rangle$) require extremely long runs, and on larger lattices fail to reach
equilibrium even after billions of Monte-Carlo sweeps. The situation is much better for
gauge-invariant observables, such as $\Sigma \cdot \Sigma$ entering $U_1$, see (3.26). The fluctuations in
this latter quantity determine the Binder cumulant, and are typically stable after a few
million sweeps. Results for the quantity $\frac{2}{N+1} - U$ versus lattice size are shown in Fig. 2.
This quantity is not monotonic, but reaches a maximum near $L = 64$ and then decreases
quite rapidly. The decrease appears to be faster than the log-type decay found for $U_1$ in
Subsection 5.1, suggesting that the termwise large $V$ asymptotics of the large $N$ series
(when formally treated as convergent) sums to a power-like large volume decay.

In summary, the numerical evidence suggests that the large $N$ contributions to the
Binder cumulant beyond leading order (i.e. \( U_0(\lambda, V) \)) may indeed vanish in the thermodynamic limit.

6. Conclusions

Noncompact SO(1, N) sigma-models are expected to be massless in contrast to their compact counterparts. The infrared problem therefore is nontrivial, especially in dimension \( d = 2 \), and the goal of the present paper has been to gain computational control over the limit of vanishing infrared regulator. The large \( N \) expansion is well suited for this; in a lattice formulation the dynamically generated gap is negative and serves as a coupling dependent infrared regulator which vanishes in the limit of infinite lattice size. The cancellation of infrared divergences has been demonstrated in \( d = 2 \) by explicit computation of a number of physically interesting quantities defined in terms of invariant correlation functions: the spin and current two-point functions as well as the Binder cumulant, all to next to leading order. A complementary result is [15] where a noninvariant observable was shown to have a finite thermodynamic limit in \( d \geq 3 \) beyond large \( N \). In \( d = 2 \) we expect that a ‘large \( N \)’ counterpart of David’s theorem [7] can be established, showing that infrared divergences cancel termwise in the large \( N \) expansion of invariant correlation functions to all orders.

To discuss our result for the Binder cumulant let us first recall the situation in the compact model. In the notation of Subsection 3.4 one has there \( 1/\Pi_+(0) \sim 4\pi \omega_+ \), in the thermodynamic (and continuum) limit, so that \( V U = 8\pi/[(N+1)\omega_+] \). Taking \( \xi = 1/\sqrt{\omega_+} \) as the definition of the correlation length, this gives the familiar result for the renormalized coupling \( g_r = V U/\xi^2 = 8\pi/(N+1) \), to leading order. See also ref. [3] for a direct continuum computation to sub-leading order, with the result \( (N+1)g_r = 8\pi[1 - 0.602033/(N+1)] + O(1/(N+1)^2) \).

In the noncompact model the zero momentum limits of invariant correlation functions are expected to diverge in the thermodynamic limit. Indeed, mostly this reflects the fact that they are increasing functions of the lattice distance (recall \( n_x \cdot n_y \geq 1 \), always). Our results of Section 5 suggest however that the ratio entering \( U \) is finite, independent of \( \lambda \), and very close to \( 2/(N+1) \).

One can view this result as a manifestation of a “concentration of measure” phenomenon. For the 1D lattice model with \( L \) sites it was shown in [16] that the functional measure has support mostly on configurations boosted by an amount increasing at least powerlike with \( L \). In the thermodynamic limit the measure (or mean) is therefore concentrated ‘at infinity’, i.e. in the disc model of the hyperbolic geometry at the boundary of the disc. Though not proven in dimensions \( d > 1 \) it is very plausible that a similar concentration phenomenon will hold for the \( d \)-dimensional functional measures. Indeed our result
on the Binder cumulant can be put into this context: First note that in terms of the normalized average spins $\sigma^a := \Sigma^a / \sqrt{\Sigma \cdot \Sigma}$, with $\Sigma^a = \sum_x n_x^a$, one has

$$\text{Var}(\sigma^2) := \langle (\sigma \cdot \sigma - \langle \sigma \cdot \sigma \rangle)^2 \rangle = \frac{2}{N + 1} - U \geq 0. \quad (6.1)$$

In the thermodynamic limit $\text{Var}(\sigma^2)$ has been argued to vanish, which is natural if the components of $\sigma^a$ are typically very large rendering the relative fluctuations ensuring $\langle \sigma \cdot \sigma \rangle = 1$ and $\langle (\sigma \cdot \sigma)^2 \rangle \approx 1$, negligible. Indeed to leading order of the large $N$ expansion one finds $\langle \sigma^0 \rangle \sim \sqrt{\ln V}$. Note that the constant $2/(N+1)$ can be interpreted as the value of $U$ in a constant configuration and that the indefinite dot product is crucial here.

Alternatively $2/(N + 1) - U$ is given by the ratio of the susceptibilities defined from the partially connected 4-point and 2-point functions $W_2$ and $W_1$, respectively. Both diverge as $V \to \infty$, but the ratios entering the large $N$ expansion of $U$ (viz $w_1/\sigma_0^2$, $w_2/\sigma_0^2$, see Eqs. (3.28) – (3.31)) vanish in the thermodynamic limit. This is compatible with a genuine factorization of the 4-point function but does not entail it. (Since the connected four point function $\langle n_{x_1} \cdot n_{y_1} n_{x_2} \cdot n_{y_2} \rangle_c$ entering (3.20) does not take into account the nonzero one-point functions, the fact that it must be non-zero is only indirectly relevant for this.)

Concerning local quantities, the analysis of the TD limit for the subleading term of the spin-two point function in Subsection 4.1 does not suggest the existence of a nontrivial limit as the UV cutoff is removed. Positive bare couplings are required for the large $N$ series to be an asymptotic expansion, in which case a naturally defined renormalized coupling vanishes as the UV cutoff is removed. The situation should be similar for the two-point function of the Noether current.

Together, our results may be taken as an indication for “triviality” of the theory in the sector comprising SO(1, $N$) invariant observables. If corroborated beyond the large $N$ expansion this would be of significance in a number of other contexts, e.g. for a class of Kaluza-Klein theories or for the widely studied systems with AdS$_5 \times$ S$_5$ target spaces. The focus on invariant observables is certainly natural from the viewpoint of the compact models. In the context of the Osterwalder-Schrader reconstruction [16, 4], however, invariant correlators are not ideal for the non-compact systems, and the situation may well be be different when noninvariant observables are considered.

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Appendix A: Leading and next-to-leading order SD equations

Here we tabulate the first few of the hierarchy of Schwinger-Dyson equations for the large N coefficients $W_r^{(s)}$, $r+s > 1$. They can be obtained e.g. by first converting (3.1) into a system of equations for the exact $W_r$ and then inserting the large N ansatz (2.6). The ensuing recursive structure is summarized in Fig. 1.

The leading 2-point function (3.2) satisfies
\[
[\pm \Delta_x - \lambda] W^{(0)}_{\pm,1}(x, y) \mp \left[ \lambda W^{(0)}_{\pm,1}(x, y) + 1 \right] \Delta_x W^{(0)}_{\pm,1}(x, z) \big|_{z=x} = 1 - \delta_{xy} . \tag{A.1}
\]

In the next order we have
\[
\pm [\Delta_x - \omega_{\pm}] W^{(1)}_{\pm,2}(x, y) - \lambda D_{\pm}(x, y) \Delta_x W^{(1)}_{\pm,1}(x, z) \big|_{z=x} \\
= -(1 - \delta_{xy}) \left[ \lambda W^{(0)}_{\pm,1}(x, y) + 1 \right] \pm \lambda \Delta_x W^{(0)}_{\pm,2}(x, z; y, y) \big|_{z=x} , \tag{A.2}
\]

where we have used the solution (3.2) to the leading order equation (A.1) to simplify some terms.

We see that to solve (A.2) we first need to solve the equation for the leading order 4–point function:
\[
\pm (\Delta_{x_1} - \omega_{\pm}) W^{(0)}_{\pm,2}(x_1, y_1; x_2, y_2) - \lambda W^{(0)}_{\pm,1}(x_1 - y_1) \Delta_{x_1} W^{(0)}_{\pm,2}(x_1, z; x_2, y_2) \big|_{z=x_1} \\
= \mp [\delta_{x_1 x_2} D_{\pm}(y_1 - y_2) + \delta_{x_1 y_2} D_{\pm}(y_1 - x_2)] \\
+ \lambda [\delta_{x_1 x_2} + \delta_{x_1 y_2}] D_{\pm}(x_1 - y_1) D_{\pm}(x_2 - y_2) . \tag{A.3}
\]

The solution for $W^{(0)}_{\pm,2}$ is given in (3.3), from which can verify that $W^{(1)}_{\pm,1}$ in (3.8) solves (A.2).

In the next order the equation for the 4–point function is
\[
\pm [\Delta_{x_1} - \omega_{\pm}] W^{(1)}_{\pm,2}(x_1, y_1; x_2, y_2) - \lambda D_{\pm}(x_1 - y_1) \Delta_{x_1} W^{(1)}_{\pm,2}(x_1, z; x_2, y_2) \big|_{z=x_1} \\
= \pm \lambda \Delta_{x_1} W^{(0)}_{\pm,3}(x_1, u; v, y_1; x_2, y_2) \big|_{u=v=x_1} \tag{A.4} \\
\pm \lambda W^{(0)}_{\pm,2}(x_1, y_1; x_2, y_2) \Delta_{x_1} W^{(1)}_{\pm,1}(x_1, z) \big|_{z=x_1} \\
\pm \lambda W^{(1)}_{\pm,1}(x_1, y_1) \Delta_{x_1} W^{(0)}_{\pm,2}(x_1, z; x_2, y_2) \big|_{z=x_1} \\
- \lambda W^{(0)}_{\pm,2}(x_1, y_1; x_2, y_2) - \delta_{x_1 x_2} W^{(1)}_{\pm,1}(y_1, y_2) - \delta_{x_1 y_2} W^{(1)}_{\pm,1}(x_2, y_1) \\
+ \lambda \delta_{x_1 x_2} + \delta_{x_1 y_2} \left\{ W^{(0)}_{\pm,2}(x_1, y_1; x_2, y_2) \\
\pm D_{\pm}(x_2 - y_2) W^{(1)}_{\pm,1}(x_1, y_1) \pm D_{\pm}(x_1 - y_1) W^{(1)}_{\pm,1}(x_2, y_2) \right\} .
\]
One sees the pattern summarized in Fig. 1 emerging, in that the solution of (A.4) requires knowledge of the leading order 6–point function. The latter satisfies the equation:

\[ \pm (\Delta_{x_1} - \omega_\pm) W_{\pm,3}^{(0)}(x_1, y_1; x_2, y_2; x_3, y_3) \]

\[ - \lambda D_\pm (x_1 - y_1) \Delta_{x_1} W_{\pm,3}^{(0)}(x_1, z; x_2, y_2; x_3, y_3) |_{z=x_1} \]

\[ = \pm W_{\pm,2}^{(0)}(x_1, y_1; x_2, y_2) \Delta_{x_1} W_{\pm,2}^{(0)}(x_1, z; x_3, y_3) |_{z=x_1} \]

\[ \pm W_{\pm,2}^{(0)}(x_1, y_1; x_3, y_3) \Delta_{x_1} W_{\pm,2}^{(0)}(x_1, z; x_2, y_2) |_{z=x_1} \]

\[ - \delta_{x_1 x_2} W_{\pm,2}^{(0)}(y_1, y_2; x_3, y_3) - \delta_{x_1 x_3} W_{\pm,2}^{(0)}(y_1, y_3; x_2, y_2) \]

\[ - \delta_{x_1 y_2} W_{\pm,2}^{(0)}(y_1, x_2; x_3, y_3) - \delta_{x_1 y_3} W_{\pm,2}^{(0)}(y_1, x_3; x_2, y_2) \]

\[ \pm \lambda [\delta_{x_1 x_2} + \delta_{x_1 y_2}] [ D_\pm (x_1 - y_1) W_{\pm,2}^{(0)}(x_2, y_2; x_3, y_3) + D_\pm (x_2 - y_2) W_{\pm,2}^{(0)}(x_1, y_1; x_3, y_3) ] \]

\[ \pm \lambda [\delta_{x_1 x_3} + \delta_{x_1 y_3}] [ D_\pm (x_1 - y_1) W_{\pm,2}^{(0)}(x_2, y_2; x_3, y_3) + D_\pm (x_3 - y_3) W_{\pm,2}^{(0)}(x_1, y_1; x_2, y_2) ] , \]

where all the functions on the rhs are known from solutions of (A.1) and (A.3). The solution is simply given by

\[ W_{\pm,3}^{(0)}(x_1, y_1; x_2, y_2; x_3, y_3) = \pm \sum_{w_{1, w_{2, w_{3}}, z_1, z_2, z_3}} \sum_{x_1, y_1, z_1, z_2, z_3} \times W_{\pm,2}^{(0)}(x_1, y_1; w_1, z_1) W_{\pm,2}^{(0)}(x_2, y_2; w_2, z_2) W_{\pm,2}^{(0)}(x_3, y_3; w_3, z_3) \]

\[ \times D_\pm^{-1}(w_1 - z_2) D_\pm^{-1}(w_2 - z_3) D_\pm^{-1}(w_3 - z_1) . \quad (A.6) \]

Using (A.6) one can verify that \( W_{\pm,2}^{(1)} \) as given in (3.11) solves (A.4).
Appendix B: Continuum limit behavior of \( j(p) \)

In this appendix we consider the small \( p \) behavior of the function \( j(p) \) entering the spin two-point function (4.15) to subleading order. We have

\[
J_{\Pi \infty}(p) - J_{\Pi \infty}(0) = E_p j_1(p) + \sum_\mu \sin p_\mu j_{2,\mu}(p) + \sum_\mu \hat{k}_\mu^2 j_{3,\mu}(p),
\]  

(B.1)

with

\[
j_1(p) = - \int \frac{1}{E_{k-p}} [v(k) - v(p)] ,
\]

\[
j_{2,\mu}(p) = \int \frac{\sin k_\mu}{E_k} [v(p - k) - v(p + k)] ,
\]

\[
j_{3,\mu}(p) = \frac{1}{4} \int \frac{\hat{k}_\mu^2}{E_k} [v(p - k) + v(p + k)] ,
\]

where \( v(p) \) is defined in (4.7) and \( \hat{k}_\mu = 2 \sin \frac{k_\mu}{2} \), as usual. Note first \( j_{3,\mu}(p) \) is non-singular at \( p = 0 \):

\[
j_{3,\mu}(0) = - \frac{1}{4} J_{\Pi \infty}(0) .
\]

Next

\[
j_{2,\mu}(p) = - 2 \int \frac{\sin(k - p)_\mu}{E_{k-p}} [v(k) - v(p)]
\]
\[
= - \sin p_\mu [2 j_1(p) + j_{4,\mu}(p)] - 2 \cos p_\mu j_{5,\mu}(p) ,
\]

(B.2)

with

\[
j_{4,\mu}(p) = \int \frac{\hat{k}_\mu^2}{E_{k-p}} [v(k) - v(p)] ,
\]

\[
j_{5,\mu}(p) = \int \frac{\sin k_\mu}{E_{k-p}} [v(k) - v(p)] = 2 \sum_\nu \sin p_\nu j_{6,\nu}(p) ,
\]

\[
j_{6,\mu\nu}(p) = \int \frac{\sin k_\mu \sin k_\nu}{E_{k-p} E_{k+p}} [v(k) - v(p)] .
\]

(B.5)

Noting

\[
j_{4,\nu}(0) = - \frac{1}{2} J_{\Pi \infty}(0) ,
\]

we have for small \( p^2 \):

\[
j(p) \sim - j_1(p) - 4 \frac{\sin p_\mu \sin p_\nu}{E_p} j_{6,\mu\nu}(p) + \frac{1}{4} J_{\Pi \infty}(0) + O(1/\ln p^2) .
\]

(B.7)
From (4.10) we have

$$v(p) = \alpha(p) + O(p^2), \quad \alpha(p) := -\frac{2\pi}{\ln(p^2/T)}, \quad (B.8)$$

with $T$ defined in (4.18). We now consider two corresponding integrals

$$\tilde{j}_1(p) = -\int_{k}^{\infty} \frac{\theta(c^2 - k^2)}{(k - p)^2} [\alpha(k) - \alpha(p)],$$

$$\tilde{j}_6;\mu\nu(p) = \int_{k}^{\infty} \frac{\theta(c^2 - k^2)k_{\mu}k_{\nu}}{(k - p)^2(k + p)^2} [\alpha(k) - \alpha(p)], \quad (B.9)$$

where $\int_{k}^{\infty} d^2k/(2\pi)^2$ and $c$ is a momentum cutoff $T > c^2 > p^2$. These give the leading small $p^2$ contribution because

$$j_1(p) - \tilde{j}_1(p) = v_1(c) + O(1/\ln p^2),$$

$$j_6;\mu\nu(p) - \tilde{j}_6;\mu\nu(p) = -\frac{1}{2} \delta_{\mu\nu} \left[v_1(c) + \frac{1}{4} v_2\right] + O(1/\ln p^2), \quad (B.10)$$

with

$$v_1(c) = -\int_{k}^{\infty} \left[\frac{v(k)}{E_k} \prod_{\mu} \theta(\pi - |k_{\mu}|) - \frac{\alpha(k)}{k^2} \theta(c^2 - k^2)\right],$$

$$v_2 = \int_{k}^{\infty} \sum_{\mu} \frac{k_{\mu}^4 v(k)}{E_k^2}. \quad (B.11)$$

First using

$$\int_{0}^{2\pi} d\phi \frac{1}{(t + \cos \phi)} = \frac{2\pi}{\sqrt{t^2 - 1}}, \quad t^2 > 1, \quad (B.12)$$

we can do the angular integrations in the $\tilde{j}$ functions, setting without loss of generality $p \mapsto (p, 0):$

$$\tilde{j}_1(p) = \frac{1}{2} \int_{0}^{c^2} dx \frac{1}{|x - p^2|} \left[\frac{1}{\ln(x/T)} - \frac{1}{\ln(p^2/T)}\right],$$

$$\tilde{j}_6;00(p) = \frac{1}{8p^2} \int_{0}^{c^2} dx \left[1 - \frac{x + p^2}{|x - p^2|}\right] \left[\frac{1}{\ln(x/T)} - \frac{1}{\ln(p^2/T)}\right]. \quad (B.13)$$

Noting $c^2 > p^2$ we obtain

$$\tilde{j}_1(p) = -\frac{1}{2 \ln(p^2/T)} \left[S_1(p^2, T) + S_2(p^2, T) + S_3(p^2, c^2, T)\right], \quad (B.14)$$
with

\[ S_1(p^2, T) = \int_0^1 dy \frac{\ln y}{(1-y) \ln(p^2y/T)}, \]
\[ S_2(p^2, T) = \int_0^1 dy \frac{\ln(1+y)}{y \ln(1+y)p^2/T}, \]
\[ S_3(p^2, c^2, T) = \int_{c^2/p^2}^{c^2/p^2-1} dy \frac{\ln(1+y)}{y \ln(1+y)p^2/T}. \]

(B.15)

Now for small \( p^2 \)

\[ S_1(p^2, T) \sim \frac{1}{\ln(p^2/T)} \sum_{n=0}^{\infty} \frac{s_1^{(n)}}{[\ln(p^2/T)]^n}, \]
\[ S_2(p^2, T) \sim \frac{1}{\ln(p^2/T)} \sum_{n=0}^{\infty} \frac{s_2^{(n)}}{[\ln(p^2/T)]^n}, \]

(B.16)

with

\[ s_1^{(n)} = (-1)^n \int_0^1 dy \frac{\ln[y]^{n+1}}{(1-y)}, \]
\[ s_2^{(n)} = (-1)^n \int_0^1 dy \frac{\ln[(1+y) y]^{n+1}}{y}, \]

(B.17)

giving \( s_1^{(0)} = -\frac{\pi^2}{6}, s_2^{(0)} = \frac{\pi^2}{12}, \ldots \)

Next

\[ S_3(p^2, c^2, T) = S_4(p^2, c^2, T) + S_5(p^2, c^2, T), \quad c^2 < T. \]

(B.18)

Here

\[ S_4(p^2, c^2, T) = \int_{c^2/p^2}^{c^2/p^2-1} dy \frac{\ln(1+y)}{(1+y) \ln(1+y)p^2/T} = \int_{\ln(2c^2/p^2)}^{c^2/p^2} \frac{\ln(z)}{x \ln(zp^2/T)} dx, \]
\[ = \int_{\ln(2c^2/p^2)}^{\ln(2c^2/p^2)} \frac{x}{x + \ln(p^2/T)} = -\ln(2c^2/p^2) + \ln(p^2/T) \ln \frac{\ln(2c^2/p^2)}{\ln(c^2/T)}, \]

(B.19)

and

\[ S_5(p^2, c^2, T) = \int_1^{c^2/p^2-1} dy \frac{\ln(1+y)}{y(1+y) \ln(1+y)p^2/T} \sim \frac{1}{\ln(p^2/T)} \sum_{n=0}^{\infty} \frac{s_5^{(n)}}{[\ln(p^2/T)]^n}, \]

(B.20)
with
\[ s_5^{(n)} = (-1)^n \int_1^\infty \frac{dy}{y^{1+y}} \frac{\ln(1+y)^{n+1}}{y}, \]
(B.21)
giving \( s_5^{(0)} = \frac{\pi^2}{12} + \frac{1}{2} (\ln 2)^2 \ldots \). So by (B.10) and (B.14)
\[ j_1(p) \sim -\frac{1}{2} \ln (-\ln(2p^2/T)) + g_1 + O (\ln(p^2/T)^{-1}) \]
\[ g_1 = v_1(c) + \frac{1}{2} \ln (-\ln(c^2/T)) + \frac{1}{2}, \]
(B.22)
which is independent of \( c \). Similarly
\[ \tilde{j}_{6,00}(p) = \frac{1}{4 \ln(p^2/T)} \left[ S_6(p^2,T) + S_2(p^2,T) + S_3(p^2,c^2,T) \right], \]
(B.23)
with
\[ S_6(p^2,T) = \int_0^1 dy \frac{y \ln y}{(1-y) \ln(p^2y/T)} \sim \frac{1}{\ln(p^2/T)} \sum_{n=0}^{\infty} S_6^{(n)} \frac{1}{[\ln(p^2/T)]^n}, \]
(B.24)
where
\[ s_6^{(n)} = (-1)^n \int_0^1 dy \frac{y^{[\ln y]^{n+1}}}{(1-y)}, \]
(B.25)
giving \( s_6^{(0)} = 1 - \frac{\pi^2}{6} \), \ldots
Putting all the results together we obtain (4.17) with
\[ g_2 = g_1 + \frac{1}{2} v_2 + \frac{1}{4} J_{11\infty}(0). \]
(B.26)
Appendix C: Large $N$ with two auxiliary fields

The results for the large $N$ expanded correlation functions in the noncompact model have in Section 3 been obtained via the large $N$ correspondence summarized in Subsection 2.2. Direct large $N$ computations in the noncompact model can be based on the following generating functional \[5\]

$$
\exp W_d[H] = \exp \left\{ -\frac{1}{2} \sum_{x,y} H_{xy} \right\} N \int \prod_{x \neq x_0} d\alpha_x \exp \left\{ -(N+1)S_-[\alpha, H] \right\},
$$

$$
S_-[\alpha, H] = \frac{1}{2} \text{Tr} \ln \hat{A} + i \sum_{x \neq x_0} \alpha_x - \frac{1}{2\lambda} (\hat{A}^{-1})^{-1}_{x_0},
$$

$$
A_{xy} = -\Delta_{xy} + 2i\lambda\alpha_x \delta_{xy} + \frac{\lambda}{N+1} H_{xy} = \tilde{A}_{xy} + 2i\lambda\delta_{xy} \delta_{xx_0} \alpha_{x_0} . \tag{C.1}
$$

Here $\hat{A}$ is the matrix obtained by deleting the $x_0$-th row and column of $A$ or $\tilde{A}$. The formal expansion based on (C.1) is not a valid saddle point expansion but it does produce the correct expansion coefficients and is related to its counterpart $W^d[H]$ in the compact model by the involution $\alpha_x \mapsto -\alpha_x, \lambda \mapsto -\lambda$. The functional (C.1) thus provides a simple heuristic way to understand the large $N$ correspondence. In contrast to the compact model, however, $W^d[H]$ is not equivalent to the original generating functional $W_-[H]$.

Here we outline how (C.1) can formally be obtained from the formulation of the large $N$ expansion with two auxiliary fields introduced in [4]. We begin by dualizing the ‘spatial’ spin components $\vec{n}_x, x \in \Lambda$, as one would do in the compact model. Indeed, the $N$ spatial components $\vec{n}_x, x \neq x_0$, enter (2.4) with the ‘good’ sign; their ‘dualization’ gives

$$
\exp W_-[H] = \exp \left\{ -\frac{1}{2} \sum_{x,y} H_{xy} \right\} \int \prod_x d\vec{n}_x^0 \delta(n_x^0 - 1) \tag{C.2}
$$

$$
\times \int \prod_{x \neq x_0} d\alpha_x \exp \left\{ -\frac{N}{2} \text{Tr} \ln \hat{A} - i(N+1) \sum_{x \neq x_0} \alpha_x \right\} \exp \left\{ +\frac{N+1}{2\lambda} \sum_{x,y} n_x^0 \tilde{A}_{xy} n_y^0 \right\} .
$$

Here $\hat{A}$ arises due to the constrained Gaussian integration. Note the small but crucial differences to the compact model [5]: only $N$ copies of $\text{Tr} \ln \hat{A}$ occur so far and the sign of the $\sum_{x \neq x_0} \alpha_x$ term is flipped, as is the sign of the $H_{xy}$ term in $A_{xy}$. Most importantly the kinetic term in the last exponential has the wrong sign, which is why one cannot naively interchange the order of the integrations over $n_x^0$ and $\alpha_x, x \neq x_0$. To proceed we assume that in a large $N$ expansion the replacement

$$
\int \prod_x d\vec{n}_x^0 \delta(n_x^0 - 1) = (-)^{|\Lambda|/2} \int \prod_x d\eta_x^0 \delta(\eta_x), \quad n_x^0 = \tilde{n}_x + i\eta_x , \tag{C.3}
$$
is legitimate, for certain “saddle point” configurations \( \bar{n}_x \), with \( \bar{n}_{x_0} = 1 \). With this replacement the kinetic term acquires the good sign. After the additional re-routing \( \alpha_x = -i\omega_x/(2\lambda) + \xi_x \) the saddle point conditions \( \partial S/\partial \eta_x = \partial S/\partial \xi_x = 0 \) lead to

\[
\bar{n}_x^2 = 1 + \lambda \hat{D}_{xx}, \quad (\Delta \bar{n})_x = \omega_x \bar{n}_x, \quad x \neq x_0.
\]

(See Eq. (5.18) of [4], with \( \omega_x = 2\lambda \bar{n}_x \), and correcting the sign in the second formula). Here \( D_{xy} = (M^{-1})_{xy} \) with \( M_{xy} := -\Delta_{xy} + \delta_{xy} \omega_x \). Since \( -\Delta \) is a positive operator it follows from the second equation in (C.4) that the position dependent \( \omega_x \) must be predominantly negative: \( 0 \leq -\sum_x \bar{n}_x (\Delta \bar{n})_x = -\sum_x \omega_x \bar{n}_x^2 \).

To leading order the spin two-point function is given by [4]

\[
\langle n_x \cdot n_y \rangle_{f.s.} = \bar{n}_x \bar{n}_y - \lambda \hat{D}_{x,y},
\]

\[
\hat{D}_{x,y} := D_{x,y} - \frac{D_{x,x_0} D_{y,x_0}}{D_{x_0,x_0}},
\]

where we write momentarily \( \langle \rangle_{f.s.} \) for the average computed with the fixed spin measure in (2.4). The quantity \( \bar{n}_x \) then is the nonzero expectation value \( \langle n_{x_0}^0 \rangle_{f.s.} \) to leading order in \( 1/(N + 1) \).

To make contact to the gap equation (3.4) in Subsection 3.1 we now first replace (C.4) by a simpler gap equation with constant \( \omega \) and \( \bar{n} \),

\[
\bar{n}^2 - \lambda D'(0) = 1, \quad \bar{n}^2 \omega = -\frac{\lambda}{V}, \quad \text{with} \quad D'(x) := \frac{1}{V} \sum_{p \neq 0} \frac{e^{ip \cdot x}}{E(p) + \omega}.
\]

These are the saddle point conditions arising from a translation invariant gauge fixing of the functional integral (see Eq. (5.6) of [4]) and imply \( -\lambda D(0) = 1 \), in accordance with (3.4). Given a solution \( \omega, \bar{n} \) of (C.6) we claim that

\[
\bar{n}_x := -\lambda D(x - x_0), \quad \omega_x := \omega + \lambda \delta_{x,x_0},
\]

is a solution of (C.4). The equation \( (\Delta \bar{n})_x = (\omega + \lambda \delta_{x,x_0}) \bar{n}_x \) is checked using \( -\lambda D(0) = 1 \).

To verify the first equation in (C.6) it suffices to observe that

\[
\hat{D}_{x,y} = D(x - y) - D(x - x_0) D(y - x_0)/D(0).
\]

This can be seen as follows: suppose that invertible matrices \( M \) and \( \widetilde{M} \) are related by \( M_{xy} = \widetilde{M}_{xy} - c \delta_{xy} \delta_{x_0,x} \). Then the inverse of \( M \) is related to the inverse of \( \widetilde{M} \) by

\[
(M^{-1})_{xy} = (\widetilde{M}^{-1})_{xy} + \frac{c}{1 - c(M^{-1})_{x_0,x_0} (\widetilde{M}^{-1})_{x_0,y_0}} (\widetilde{M}^{-1})_{x_0,x_0} (\widetilde{M}^{-1})_{y_0,x_0}.
\]
Applied to \( M_{xy} = -\Delta_{xy} + \omega_x \delta_{xy} = \tilde{M}_{xy} + \lambda \delta_{xy} \delta_{x,x_0} \), for \( \omega_x = \omega + \lambda \delta_{x,x_0} \), without yet assuming the gap equation (C.6), this gives first

\[
D_{xy} = D(x - y) - \frac{\lambda}{1 + \lambda D(0)} D(x - x_0) D(y - x_0), \tag{C.10}
\]

and then (C.8) from (C.5). In particular no pole occurs for the quantities in (C.8) at \( \lambda D(0) = -1 \). Using (C.8) and \( \lambda D(0) = -1 \), it follows \( 1 + \lambda \tilde{D}_{x,x} = \lambda^2 D(x - x_0)^2 = \bar{n}_x^2 \), while the sign in (C.7) is fixed by \( -\lambda D(x) \geq 1 \).

Inserting (C.7) into (C.5) we arrive at

\[
\langle n_x \cdot n_y \rangle_{\text{f.s.}} = -\lambda D(x - y) = \bar{n}_x^2 - \lambda D'(x - y) = \langle n_x \cdot n_y \rangle_{\text{trans}}, \tag{C.11}
\]

where the right hand side coincides with the spin two-point function computed to leading order in \( 1/(N+1) \) in the translation invariant gauge [4] and with that of Subsection 3.1.

In summary, the leading order results with two auxiliary fields and the two gauge fixings considered (fixed spin and translation invariant gauge) are related via (C.7). Both saddle point equations (C.4) and (C.6) imply the version \(-\lambda D(0) = 1 \) used here, but in addition provide the interpretation of \( \bar{n}_x = \langle n^0_x \rangle_{\text{f.s.}} \) and \( \bar{n} = \langle n^0_x \rangle_{\text{trans}} \), as the averages of \( n^0_x \) with respect to the respective gauge-fixed functional measures. Note that \( \bar{n}_x \) approaches \( \bar{n}^2 \) as \( |x - x_0| \) becomes large and that \( \sum_x \bar{n}_x = V \bar{n}^2 \). Although the invariant two-point functions coincide to leading order in the two gauges, the results for the noninvariant quantity \( \langle n^0_x \rangle \) are very different, \( \langle n^0_x \rangle_{\text{f.s.}} = \langle n^0_x \rangle_{\text{trans}}^2 \).

Equipped with this interpretation of \( \bar{n}_x \), we return to (C.2). Subject to the assumption (C.3) one can proceed by interchanging the order of integrations, which results in the Gaussian

\[
\int \prod_x d\eta_x \delta(\eta_{x_0}) \exp \left\{ -\frac{N + 1}{2\lambda} \sum_{x,y} \eta_x \tilde{A}_{xy} \eta_y + \frac{N + 1}{\lambda} \sum_x \eta_x \sum_y \tilde{A}_{xy} \tilde{n}_y \right\} = \text{Const} (\det \tilde{A})^{-1/2} \exp \left\{ \frac{N + 1}{2\lambda} \left[ n^2_{x_0} (\tilde{A}^{-1})^{-1}_{x_0 x_0} - \sum_{x,y} \tilde{n}_x \tilde{A}_{xy} \tilde{n}_y \right] \right\}. \tag{C.12}
\]

Using also \( \tilde{n}_{x_0} = 1 \) and substituting back into (C.2) we arrive at (C.1).
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