Entropic uncertainty relations for extremal unravelings of super-operators

Alexey E Rastegin

Department of Theoretical Physics, Irkutsk State University, Gagarin Bv. 20, Irkutsk 664003, Russia
E-mail: rast@api.isu.ru

Received 29 June 2010, in final form 5 January 2011
Published 9 February 2011
Online at stacks.iop.org/JPhysA/44/095303

Abstract
A way to pose the entropic uncertainty principle for trace-preserving super-operators is presented. It is based on the notion of extremal unraveling of a super-operator. For a given input state, different effects of each unraveling result in some probability distribution at the output. As is shown, all Tsallis’ entropies of positive order as well as some of Rényi’s entropies of this distribution are minimized by the same unraveling of a super-operator. Entropic relations between a state ensemble and the generated density matrix are revisited in terms of both the adopted measures. Using Riesz’s theorem, we obtain two uncertainty relations for any pair of generalized resolutions of the identity in terms of the Rényi and Tsallis entropies. The inequality with Rényi’s entropies is an improvement of the previous one, whereas the inequality with Tsallis’ entropies is a new relation of a general form. The latter formulation is explicitly shown for a pair of complementary observables in a \(d\)-level system and for the angle and angular momentum. The derived general relations are immediately applied to extremal unravelings of two super-operators.

PACS numbers: 03.65.Ta, 03.67.-a, 02.10.Yn

1. Introduction

Since Heisenberg’s famous paper [1] was published, uncertainty relations have been the subject of extensive research [2, 3]. There are two well-known approaches to formulating the uncertainty principle. The first was initiated by Robertson [4] who showed that a product of the standard deviations of two observables is bounded from below. Here, we may run across some disputable topics such as the number–phase case [5]. Both the well-defined Hermitian operator of phase and the corresponding number–phase uncertainty relation have been fit within the Pegg–Barnett formalism [6]. The second approach is generally characterized by posing the uncertainty principle via information–theoretic terms especially via entropies [7, 8]. Although the first relation of such a kind was derived by Hirschman [9], a general statement
of the problem is examined in [10, 11]. Mutually unbiased bases [12, 13], the time–energy case [14] and tomographic processes [15] have been considered within an entropic approach as well. In the presence of quantum memory, the standard entropic formulation should be recast and strengthened [16].

Most of the known uncertainty relations deal with observables or, more generally, with POVM measurements. Nevertheless, there exist relations for unitary transformations [17] and non-Hermitian annihilation operator [18]. Both the measurement and unitary evolution are rather simple types of state change in quantum theory [19]. The formalism of quantum operations, or super-operators, is now a standard tool for treating quantum processes. In this work, we address how to formulate the uncertainty principle for super-operators. It turns out that one of the possible ways is naturally provided with the notion of extremal unraveling of a super-operator. For the Shannon entropy, this notion was examined in [20]. As an entropic measure, we will use the Tsallis entropy, which has found use in various physical problems (see references in [21]), and the Rényi entropy.

The paper is organized as follows. In section 2, the properties of the Tsallis entropies and super-operators are recalled. For a given input state and super-operator, we find the extremal unraveling that minimizes all the Tsallis entropies simultaneously. Relations between the ensemble entropy and the entropy of a generated density matrix are revisited. In section 3, we derive the uncertainty relation for two generalized resolutions of the identity in terms of Tsallis entropies. The previous result on the Rényi entropies is refined. The obtained entropic relations are directly used for the extremal unravelings of two super-operators. Section 4 concludes the paper with a summary of results.

2. Tsallis’ entropies and extremal unravelings

First, we briefly recall the definitions of used entropic measures (for a discussion and further references, see [22]). For real $\alpha > 0$ and $\alpha \neq 1$, we define the non-extensive $\alpha$-entropy of the probability distribution $\{p_i\}$ by [23]

$$H_\alpha(p_i) \triangleq (1 - \alpha)^{-1} \left( \sum_i p_i^\alpha - 1 \right). \quad (2.1)$$

This can be rewritten as $H_\alpha(p_i) = -\sum_i p_i^\alpha \ln_\alpha p_i$, where $\ln_\alpha x \equiv (x^{1-\alpha} - 1)/(1 - \alpha)$ is the $\alpha$-logarithmic function, defined for $\alpha \geq 0, \alpha \neq 1$ and $x \geq 0$. Quantity (2.1) will be referred to as ‘Tsallis $\alpha$-entropy’, though it was previously discussed by Havrda and Charvát [24]. In the limit $\alpha \to 1$, $\ln_\alpha x \to \ln x$ and quantity (2.1) recovers the Shannon entropy. We will also use the Rényi $\alpha$-entropy defined for $\alpha \neq 1$ as [25]

$$R_\alpha(p_i) \triangleq (1 - \alpha)^{-1} \ln \left( \sum_i p_i^\alpha \right). \quad (2.2)$$

The Rényi $\alpha$-entropy also coincides with the Shannon entropy in the limit $\alpha \to 1$. There are various forms of extrapolation between different entropies [26]. We will only need the equality

$$R_\alpha(p_i) = (1 - \alpha)^{-1} \ln(1 + (1 - \alpha)H_\alpha(p_i)),$$

which immediately follows from definitions (2.1) and (2.2).

In the following, some notation of linear algebra will be used. Let $\mathcal{H}$ be the finite-dimensional Hilbert space. For the two given vectors $\psi$ and $\varphi \in \mathcal{H}$, their inner product is denoted by $\langle \psi, \varphi \rangle$. For two operators $X$ and $Y$ on $\mathcal{H}$, we define the Hilbert–Schmidt inner product by [27]

$$\langle X, Y \rangle_{\text{hs}} \triangleq \text{tr}(X^*Y). \quad (2.4)$$
This inner product naturally induces the so-called Frobenius norm $\|X\|_F = \langle X, X \rangle_{\text{hs}}^{1/2}$. The Frobenius norm can be re-expressed as $\|X\|_F = \left( \sum_j |s_j(X)|^2 \right)^{1/2}$, where the singular values $s_j(X)$ are the eigenvalues of $X = \sqrt{X^*X}$. The largest singular value of $X$ gives the spectral norm $\|X\|_\infty$ of this operator [27].

A quantum measurement is described by a ‘generalized resolution of the identity’ (or by a ‘positive operator-valued measure’). This is a set $\{M_i\}$ of positive semidefinite operators obeying the completeness relation $\sum_i M_i = \mathbb{1}$, where $\mathbb{1}$ is the identity [28]. In some respects, quantum measurements themselves can be treated as the object of information processing [29, 30]. Consider a linear map $\mathcal{S}$ that takes linear operators on $\mathcal{H}$ to linear operators on $\mathcal{H}'$, and also satisfies the conditions of trace preservation and complete positivity. Following [31], this map will be referred to as ‘super-operator’. Each super-operator has a Kraus representation, namely [19, 31]

$$S(\rho) = \sum_i A_i \rho A_i^\dagger,$$

(2.5)

where the Kraus operators $A_i$ map the input space $\mathcal{H}$ to the output space $\mathcal{H}'$ and obey $\sum_i A_i A_i^\dagger = \mathbb{1}$ (the preservation of the trace). Representations of such a kind are never unique [27]. For the given super-operator $\mathcal{S}$, there are many sets $\mathcal{A}$ that unravel the input space $\mathcal{H}$ to the output space $\mathcal{H}'$ and obey (2.5). In [20], each concrete set $\mathcal{A} = \{A_i\}$ resulting in (2.5) is called an ‘unraveling’ of the super-operator $\mathcal{S}$. This terminology is due to Carmichael [32] who introduced this word for a representation of the master equation. It is well known that two Kraus representations of the same super-operator are related as

$$B_i = \sum_j A_j u_{ji},$$

(2.6)

where the matrix $U = \{[u_{ij}]\}$ is unitary [31]. (We assume in (2.6) that unravelings $\mathcal{A}$ and $\mathcal{B}$ have the same cardinality by adding zero operators, if needed.) For the given density operator $\rho$ and super-operator unraveling $\mathcal{A} = \{A_i\}$, we introduce the matrix

$$\Pi(\mathcal{A}|\rho) \triangleq [[(A_i\sqrt{\rho}, A_i\sqrt{\rho})_{\text{hs}}]] = [[\text{tr}(A_i^\dagger A_i \rho)]] .$$

(2.7)

The diagonal element $p_i = \text{tr}(A_i^\dagger A_i \rho)$ is clearly positive and gives the $i$th effect probability. Then, the entropies $H_a(\mathcal{A}|\rho)$ and $R_a(\mathcal{A}|\rho)$ are merely defined by (2.1) and (2.2), respectively. By definition, the matrix $\Pi(\mathcal{A}|\rho)$ is Hermitian. Suppose that the two sets $\mathcal{A} = \{A_i\}$ and $\mathcal{B} = \{B_j\}$ fulfill (2.6). Using the properties of the inner product, we have

$$\langle B_j \sqrt{\rho}, B_k \sqrt{\rho} \rangle_{\text{hs}} = \sum_{jl} u^*_{jl} u_{lk} \langle A_j \sqrt{\rho}, A_i \sqrt{\rho} \rangle_{\text{hs}},$$

(2.8)

or $\Pi(\mathcal{B}|\rho) = U^\dagger \Pi(\mathcal{A}|\rho) U$ as the matrix relation. That is, if the sets $\mathcal{A}$ and $\mathcal{B}$ are both unravelings of the same super-operator, then the matrices $\Pi(\mathcal{A}|\rho)$ and $\Pi(\mathcal{B}|\rho)$ are unitarily similar. Due to Hermiticity, all the matrices of a kind $\Pi(\mathcal{A}|\rho)$ assigned to one and the same super-operator are unitarily similar to a unique (up to permutations) diagonal matrix $D = \text{diag}(\lambda_1, \lambda_2, \ldots)$, where the $\lambda_i$’s and perhaps zeros are the eigenvalues of each of these matrices. So any matrix $\Pi(\mathcal{A}|\rho)$ is positive semidefinite.

For the given unraveling $\mathcal{A} = \{A_i\}$, we obtain the concrete matrix $\Pi(\mathcal{A}|\rho)$ and diagonalize it through a unitary transformation $V^\dagger \Pi(\mathcal{A}|\rho) V = D$. Let us define a specific unraveling $\mathcal{A}_i^{(ex)}$ related to the given $\mathcal{A}$ as

$$A_i^{(ex)} = \sum_j A_j v_{ji},$$

(2.9)

where the unitary matrix $V$ diagonalizes $\Pi(\mathcal{A}|\rho)$. It turns out that the unraveling (2.9) enjoys the extremality property with respect to all the Tsallis $\alpha$-entropies for $\alpha \in (0; \infty)$ and the Rényi $\alpha$-entropies for $\alpha \in (0; 1)$. 

Theorem 1. For a given density operator $\rho$ and super-operator $S$, each unraveling $\mathcal{U}$ of the $S$ satisfies
\[
H_{\alpha}((\alpha\mathcal{U}^{\text{ex}})|\rho) \leq H_{\alpha}((\alpha\mathcal{U})|\rho) \quad \forall \alpha \in (0; \infty),
\] (2.10)
\[
R_{\alpha}((\alpha\mathcal{U}^{\text{ex}})|\rho) \leq R_{\alpha}((\alpha\mathcal{U})|\rho) \quad \forall \alpha \in (0; 1),
\] (2.11)
where the extremal unraveling $\alpha\mathcal{U}^{\text{ex}}$ is defined by formula (2.9).

Proof. We firstly note that $\Pi(\alpha\mathcal{U}^{\text{ex}})|\rho) = \mathcal{D}$ with the probabilities $\lambda_j$ of effects. In view of $\Pi((\alpha\mathcal{U})|\rho) = \mathcal{V} \mathcal{D} \mathcal{V}^\dagger$, the probabilities $p_i = \text{tr}(A_i \mathcal{A}_i \rho)$ of different effects of the unraveling $\mathcal{U}$ are related to the $\lambda_j$’s by
\[
p_i = \sum_j v_{ij} \lambda_j v_{ij}^\dagger = \sum_j s_{ij} \lambda_j ,
\] (2.12)
where $s_{ij} = v_{ij} v_{ij}^\dagger$. The matrix $\mathcal{S} = [s_{ij}]$ is unistochastic, whence $\sum_j s_{ij} = 1$ for all $j$ and $\sum_j s_{ij} = 1$ for all $i$. We now use these relations and an obvious fact that for $\alpha > 0$, the function $h_{\alpha}(x) = (x^\alpha - x)/(1 - \alpha)$ is concave in the range $x \in [0; 1]$. According to Jensen’s inequality (see, e.g., [33]), there holds
\[
H_{\alpha}((\alpha\mathcal{U})|\rho) = \sum_i h_{\alpha}\left(\sum_j s_{ij} \lambda_j \right) \geq \sum_i \sum_j s_{ij} h_{\alpha}(\lambda_j) = \sum_j h_{\alpha}(\lambda_j) = H_{\alpha}((\alpha\mathcal{U}^{\text{ex}})|\rho).
\] (2.13)
This completes the proof for (2.10). Further, we note that for $\alpha < 1$ the function $(1 - \alpha)^{-1}\ln(1 + (1 - \alpha)x)$ is increasing in the range $x \in [0; \infty)$. Combining the equality (2.3) with (2.10) then gives (2.11). □

For the Shannon entropy, a problem of ‘minimal’ unraveling was considered in [34] and later in [20]. Diagonalizing the matrix $\Pi((\alpha\mathcal{U})|\rho)$ is actually equivalent to the extreme condition that has been derived by the method of Lagrange’s multipliers in [20]. Latter reasons local in spirit are quite complemented by the above proof based on the concavity. Our treatment allows us to find extremal unravelings easily from (2.9). Thus, for the prescribed state $\rho$, all the Tsallis entropies of order $\alpha \in (0; \infty)$ and the Rényi entropies of order $\alpha \in (0; 1)$ are minimized by one and the same unraveling of a given super-operator. However, this unraveling does not minimize other Rényi entropies in general. An unraveling extremal with respect to Rényi’s entropy of order $\alpha \geq 1$ may also be dependent on $\alpha$. A search for such an unraveling seems to be difficult since the Rényi entropy of such an order is not purely convex nor purely concave.

Relations (2.10) and (2.11) can be put in the context of a state ensemble having a prescribed density operator. In line with (2.1) and (2.2), we introduce the quantum Tsallis and Rényi entropies of a density matrix $\rho$ by
\[
H_{\alpha}(\rho) \triangleq (1 - \alpha)^{-1}\text{tr}(\rho^\alpha - \rho) , \quad R_{\alpha}(\rho) \triangleq (1 - \alpha)^{-1}\ln(\text{tr}(\rho^\alpha)).
\] (2.14)
In the limit $\alpha \to 1$, both the expressions coincides with the von Neumann entropy $-\text{tr}(\rho \ln \rho)$. Let $\{p_i, \psi_i\}$ be an ensemble of normalized pure states, $(\psi_i , \psi_j) = 1$ and $\sum_i p_i = 1$, which leads to the density operator $\rho$, namely
\[
\sum_i p_i \psi_i \psi_i^\dagger = \rho = \sum_j \lambda_j \varphi_j \varphi_j^\dagger.
\] (2.15)
In general, the states $\psi_i$ are not mutually orthogonal. The right-hand side of (2.15) poses the spectral decomposition of $\rho$, so that the vectors $\varphi_j$ form an orthonormal basis in $\mathcal{H}$. The ensemble classification theorem states that [35]
\[
\sqrt{p_i} \psi_i = \sum_j u_{ij} \sqrt{\lambda_j} \varphi_j
\] (2.16)
for some unitary matrix $[u_{ij}]$. It follows from (2.16) and $\langle \varphi_j, \varphi_k \rangle = \delta_{jk}$ that $p_i = \sum_j s_{ij} \lambda_j$, where $s_{ij} = u_{ij}^* u_{ij}$ are elements of a unistochastic matrix. Changing notation in formula (2.13) appropriately, we finally obtain

$$H_\alpha(\rho) = H_\alpha(\lambda) \leq H_\alpha(p_i) \quad (0 < \alpha), \quad R_\alpha(\rho) = R_\alpha(\lambda) \leq R_\alpha(p_i) \quad (0 < \alpha < 1).$$

(2.17)

In the limit $\alpha \to 1$, both the inequalities are reduced to $-\text{tr}(\rho \ln \rho) \leq H_1(p_i)$. The latter is a known relation between the von Neumann entropy of the generated density operator and the Shannon entropy of an ensemble [36]. For the Tsallis entropies, this result can be proceeded to ensembles of mixed states. Let an ensemble $\{p_i, \omega_i\}$ of normalized density operators, $\text{tr}(\omega_i) = 1$ and $\sum_i p_i = 1$, give rise to a density operator

$$\rho = \sum_i p_i \omega_i = \sum_{ij} p_i v_{ij} \varphi_{ij} \varphi_{ij}^\dagger,$$

(2.18)

where the spectral decomposition $\omega_i = \sum_j v_{ij} \varphi_{ij} \varphi_{ij}^\dagger$. Applying the first inequality of (2.17) to the right-hand side of (2.18), we have

$$H_\alpha(\rho) \leq -\sum_{ij} p_i^\alpha v_{ij}^\alpha \ln \alpha(p_i v_{ij}) = -\sum_{ij} p_i^\alpha v_{ij}^\alpha (\ln \alpha v_{ij} + v_{ij}^{1-\alpha} \ln \alpha p_i) = -\sum_i p_i^\alpha \sum_j v_{ij}^\alpha \ln \alpha v_{ij} - \sum_i p_i^\alpha \ln \alpha p_i.$$

Here, we used the identity $\ln \alpha(xy) = \ln \alpha y + y^{1-\alpha} \ln \alpha x$. Adding a consequence of concavity of the function $h_{\alpha}(x) = (x^\alpha - x)/(1 - \alpha)$, we finally write

$$\sum_i p_i H_\alpha(\omega_i) \leq H_\alpha(\rho) \leq \sum_i p_i^\alpha H_\alpha(\omega_i) + H_\alpha(p_i).$$

(2.19)

The inequality on the left can be shown as follows. For any concave function $f(x)$, the functional $\text{tr}(f(X))$ is also concave on Hermitian $X$ (for details, see section III in [37]). We further note that $H_\alpha(\rho) = \text{tr}(h_{\alpha}(\rho))$. In the limit $\alpha \to 1$, the inequalities (2.19) are reduced to the well-known bounds on the von Neumann entropy of a state mixture (for instance, see [36]). It seems that such a treatment fails in the case of Rényi’s entropies. Indeed, the Tsallis $\alpha$-entropy does enjoy so-called strong additivity of degree $\alpha$, whereas the Rényi $\alpha$-entropy does not [22].

### 3. Entropic uncertainty relations

To obtain entropic relations, we will use a version of Riesz’s theorem (see theorem 297 in [33]). Let $x \in \mathbb{C}^n$ be $n$-tuple of complex numbers $x_j$ and let $T = [t_{ij}]$. Define $\eta$ to be maximum of $|t_{ij}|$, i.e. $\eta = \max(|t_{ij}| : 1 \leq i \leq m, 1 \leq j \leq n)$. The fixed matrix $T$ describes a linear transformation $\mathbb{C}^n \to \mathbb{C}^m$. That is, to each $x$ we assign $m$-tuple $y \in \mathbb{C}^m$ with elements

$$y_i(x) = \sum_{j=1}^m t_{ij} x_j \quad (i = 1, \ldots, m).$$

(3.1)

For $b \geq 1$, the $l_b$ norm of vector $x$ is $\|x\|_b = (\sum_j |x_j|^b)^{1/b}$. For $\beta \geq 1/2$, we use a like function $\|q\|_\beta = (\sum_j q_j^\beta)^{1/\beta}$ of the probability distribution $q = \{q_j\}$, though it is not a norm for $\beta < 1$. Riesz’s theorem is formulated as follows.

**Lemma 1.** Suppose the matrix $T$ satisfies $\|y\|_2 \leq \|x\|_2$ for all $x \in \mathbb{C}^n$, $1/a + 1/b = 1$ and $1 < b < 2$; then,

$$\|y\|_\alpha \leq \eta^{(2-b)/b} \|x\|_b.$$

(3.2)
We will now obtain an improved version of the statement emerged in [38]. To each
generalized resolution \( \mathcal{M} = \{ M_i \} \) and a given mixed state \( \rho \), we assign the probabilistic vector \( p = \{ p_i \} \) with elements
\( p_i = \text{tr}(M_i \rho) = \| M_i^{1/2} \sqrt{p} \|_F^2 \). Another resolution \( \mathcal{N} = \{ N_j \} \) is assigned by the vector \( q \) with elements
\( q_j = \text{tr}(N_j \rho) = \| N_j^{1/2} \sqrt{p} \|_F^2 \).

**Lemma 2.** For any two resolutions \( \mathcal{M} = \{ M_i \} \) and \( \mathcal{N} = \{ N_j \} \) of the identity and the given
density operator \( \rho \), there holds
\[
\| p \|_a \leq g(\mathcal{M}, \mathcal{N}|\rho) 2^{(1-\beta)/\beta} \| q \|_b,
\] (3.3)
where \( 1/\alpha + 1/\beta = 2 \), \( 1/2 < \beta < 1 \), and the function \( g(\mathcal{M}, \mathcal{N}|\rho) \) is defined by
\[
g(\mathcal{M}, \mathcal{N}|\rho) \triangleq \max\{|p_i q_j|^{-1/2} |\text{tr}(M_i N_j \rho) : p_i \neq 0, q_j \neq 0\|.
\] (3.4)

**Proof.** We first consider the case when both the resolutions \( \mathcal{M} = \{ M_i \} \) and \( \mathcal{N} = \{ N_j \} \) are
orthogonal. For those values of labels \( i \) and \( j \) that satisfy \( \text{tr}(M_i \rho) \neq 0 \) and \( \text{tr}(N_j \rho) \neq 0 \), we put (generally non-Hermitian) operators
\[
\omega_i = \| M_i^{1/2} \sqrt{p} \|_F^{-1} M_i \sqrt{p}, \quad \theta_j = \| N_j^{1/2} \sqrt{p} \|_F^{-1} N_j \sqrt{p}.
\] (3.5)
It is clear that \( \| \omega_i \|_F = 1 \) and \( \| \theta_j \|_F = 1 \). Since the resolutions \( \{ M_i \} \) and \( \{ N_j \} \) are orthogonal, we further have \( \langle \omega_i, \omega_j \rangle_{\text{hs}} = \delta_{ik} \) and \( \langle \theta_i, \theta_j \rangle_{\text{hs}} = \delta_{jk} \). The matrix elements of the transformation \( T \) are then defined by
\[
t_{ij} = \langle \omega_i, \theta_j \rangle_{\text{hs}} = \| M_i^{1/2} \sqrt{p} \|_F^{-1} \| N_j^{1/2} \sqrt{p} \|_F^{-1} \langle M_i^{1/2} \sqrt{p}, N_j^{1/2} \sqrt{p} \rangle_{\text{hs}}.
\] (3.6)
We now rewrite (3.1) as \( y_i(x) = \langle \omega_i, \sigma \rangle_{\text{hs}} \), where \( \sigma = \sum_j x_j \theta_j \) by definition. Due to
\( \| \sigma \|_F^2 = \sum_j |x_j|_2 \) and \( \sigma = \sum_j y_i \omega_i + \sigma \), where \( \langle \omega_i, \sigma \rangle_{\text{hs}} = 0 \) for all \( i \), we have
\( \| \sum_j y_i \omega_i \|_F^2 \leq \| \sigma \|_F^2 \) and the precondition of lemma 1 here. We apply (3.2) to the special values \( y'_j = \| M_i^{1/2} \sqrt{p} \|_F \) and \( x'_j = \| N_j^{1/2} \sqrt{p} \|_F \). By the completeness relation,
\[
1 \| \sqrt{p} \|_a = \sum_j y'_i \omega_i = \sum_j x'_j \theta_j,
\] whence \( y'_j = \sum_j (\langle \omega_i, \theta_j \rangle_{\text{hs}} x'_j \), i.e. the values \( y'_j \) are related to \( x'_j \) via the transformation \( T \) too. Since
the operators \( M_i \) and \( N_j \) are projective, \( p_i = |y'_i|_2^2 \) and \( q_j = |x'_j|_2^2 \), whence \( \| p \|_a = \| y' \|_a^2 \) and
\( \| q \|_b = \| x' \|_b^2 \) with \( \alpha = a/2 \) and \( \beta = b/2 \). The statement of lemma 1 then results in the inequality
\( \| p \|_a^{1/2} \leq \max\{|\langle \omega_i, \theta_j \rangle_{\text{hs}}|^{(1-\beta)/\beta} \| q \|_b^{1/2} \}, \) under the conditions \( 1/\alpha + 1/\beta = 2 \) and \( 1/2 < \beta < 1 \). Noting that the maximum of modulus of (3.6) is actually \( g(\mathcal{M}, \mathcal{N}|\rho) \), we resolve the case when the resolutions \( \mathcal{M} \) and \( \mathcal{N} \) are both orthogonal. To generalize (3.3) to the case of arbitrary two resolutions, we will use the method proposed in [39] and further developed in [38]. In the extended space \( \mathcal{H} \oplus \mathcal{K} \), the resolutions \( \mathcal{M} \) and \( \mathcal{N} \) are realized as new
resolutions \( \tilde{\mathcal{M}} \) and \( \tilde{\mathcal{N}} \), respectively, and the \( \tilde{\mathcal{M}} \) is now orthogonal by Naimark’s extension (see, e.g., section 5.1 in [27]). It can be made in such a way that for any density matrix \( \rho \) on
\( \mathcal{H} \), \( \text{tr}(\mathcal{M}|\rho) = \text{tr}(\tilde{\mathcal{M}}|\tilde{\rho}) \), \( \text{tr}(\mathcal{N}|\rho) = \text{tr}(\tilde{\mathcal{N}}|\tilde{\rho}) \), and \( \text{tr}(\mathcal{M} N_j |\rho) = \text{tr}(\tilde{\mathcal{M}} \tilde{N}_j |\tilde{\rho}) \) (the \( \tilde{\rho} \) is built from \( \rho \) by adding zero rows and columns). Hence, we have the same values of entropies [38] and
\( g(\mathcal{M}, \mathcal{N}|\rho) = g(\tilde{\mathcal{M}}, \tilde{\mathcal{N}}|\tilde{\rho}) \). By a similar extension of \( \tilde{\mathcal{N}} \), the question is quite reduced to the above case of two orthogonal resolutions. \( \Box \)

Using simple algebra (see the proof of proposition 3 in [38]), inequality (3.3) can be rewritten as
\[
R_\alpha(\mathcal{M}|\rho) + R_\beta(\mathcal{N}|\rho) \geq -2 \ln g(\mathcal{M}, \mathcal{N}|\rho).
\] (3.7)
For a pure state, this relation in terms the Rényi entropies coincides with the one deduced in a previous work [38]. Indeed, the Cauchy–Schwarz inequality for the Hilbert–Schmidt inner product directly shows the modulus of (3.6) does not exceed 1. The task is solved in the appendix, and the sum $H_\alpha(p_i) + H_\beta(q_j)$ cannot be arbitrarily small because of constraint (3.3). We rewrite this sum as

$$H_\alpha(p_i) + H_\beta(q_j) = \phi(\xi, \zeta) = \frac{\xi - 1}{1 - \alpha} + \frac{\zeta - 1}{1 - \beta}$$

(3.9)

in terms of the variables $\xi = \sum_i p_i^\alpha = \|p\|_\alpha^\alpha$ and $\zeta = \sum_j q_j^\beta = \|q\|_\beta^\beta$ and the function $\phi(\xi, \zeta)$. Assuming $\alpha > 1 > \beta$, we obviously have $\xi \leq 1$ and $\zeta \geq 1$. Adding (3.3) in the form $\xi \geq \gamma \phi^{\alpha/\beta}$, where $\gamma = g(M, N|\rho)^{-2(1-\beta)}$, we have arrived at a task of minimizing $\phi(\xi, \zeta)$ under the above conditions. It is important here that $g(M, N|\rho) \leq 1$ and, therefore, $\gamma \geq 1$. Indeed, the Cauchy–Schwarz inequality for the Hilbert–Schmidt inner product directly shows that the modulus of (3.6) does not exceed 1. The task is solved in the appendix, and the sum $H_\alpha(p_i) + H_\beta(q_j)$ cannot be less than

$$\phi(\xi_0, 1) = \frac{\gamma^{-\alpha/\beta} - 1}{1 - \alpha} = \frac{g(M, N|\rho)^{2\alpha(1-\beta)/\beta} - 1}{1 - \alpha} = \ln_\mu(g(M, N|\rho)^{-2})$$

where we used $(1 - \beta)/\beta = (\alpha - 1)/\alpha$ due to $1/\alpha + 1/\beta = 2$. By $g(M, N|\rho) = g(N, M|\rho)$, we claim the following.

**Theorem 2.** For any two resolutions $M = \{M_i\}$ and $N = \{N_j\}$ of the identity and the given density operator $\rho$, there holds

$$H_\alpha(M|\rho) + H_\beta(N|\rho) \geq \ln_\mu(g(M, N|\rho)^{-2}),$$

(3.10)

where $1/\alpha + 1/\beta = 2$ and $\mu = \max(\alpha, \beta)$.

Inequality (3.10) gives the uncertainty relation in terms of the Tsallis entropies for two generalized measurements. Note that relations in terms of the Tsallis entropies have been considered for particular cases of the position and momentum [41] and the spin-1/2 components [42, 43]. It is easy to see that the uncertainty relations (3.7) and (3.10) become trivial in the deterministic classical case. Any impure state can be represented by a nontrivial ensemble of pure ones. So it is assigned by some indeterminacy additional to purely quantum indeterminacy in each of the pure states from such an ensemble. Therefore, the completely deterministic situation occurs when all the involved operators are diagonal in the same basis and the state is just pure. In this case, we obviously have $g = 1$, as needed.

It is of some interest to obtain a state-independent version of (3.10). Since the function $\ln_\mu \times$ is increasing for $\mu > 0$, $g(M, N|\rho) \leq f(M, N|\rho)$ and [38]

$$f(M, N|\rho) \leq \max \left\{ \left\| M_i^{1/2} N_j^{1/2} \right\|_\infty : M_i \in M, N_j \in N \right\} \equiv \tilde{f}(M, N),$$

(3.11)

we have the state-independent bound

$$H_\alpha(M|\rho) + H_\beta(N|\rho) \geq \ln_\mu(\tilde{f}(M, N)^{-2}).$$

(3.12)
Note that $\tilde{f}(\mathcal{M}, \mathcal{N}) \leq 1$ is provided by $\|M_1^{1/2}N_2^{1/2}\|_{\infty}^2 \leq \|M\|_{\infty} \|N\|_{\infty} \leq 1$. Here, the inequality on the left is a Cauchy–Schwarz inequality for ordinary matrix products and the spectral norm (see, e.g., result (4.50) in [44]); the inequality on the right follows from the completeness relation. The inequality $\tilde{f}(\mathcal{M}, \mathcal{N}) \leq 1$ is saturated if and only if for some two elements $M_0$ and $N_0$ and nonzero $\varphi_0 \in \mathcal{H}$, there hold $M_0\varphi_0 = \varphi_0$ and $N_0\varphi_0 = \varphi_0$ simultaneously. In other words, $\tilde{f}(\mathcal{M}, \mathcal{N}) = 1$ implies that there exist those operators $M_0 \in \mathcal{M}$ and $N_0 \in \mathcal{N}$ that act as commuting projectors in a nonempty subspace $\mathcal{K}_0 \subset \mathcal{H}$.

Otherwise, the entropic bound in relation (3.12) is nontrivial. So, we have extended the relation, conjectured in [45] and later proved in [7], to the Tsallis entropies and general quantum measurements. Let us consider two concrete examples of specific interest.

**Example 1.** The first example is a pair of complementary observables in a $d$-level system (for the Rényi formulation, see [40]). Let complex amplitudes $\tilde{c}_k$ and $c_l$ be connected by the discrete Fourier transform

$$\tilde{c}_k = \frac{1}{\sqrt{d}} \sum_{l=1}^{d} e^{2\pi i k l / d} c_l, \quad (3.13)$$

and the corresponding probability distributions $p_k = |\tilde{c}_k|^2$ and $q_l = |c_l|^2$. Transformation (3.13) is the ‘canonical’ example that leads to complementary observables [45]. It follows from $\|\tilde{c}\|_a = \|c\|_b$ and (3.2) that

$$\|c\|_a \leq \left( \frac{1}{\sqrt{d}} \right)^{(2-b)/b} \|\tilde{c}\|_b, \quad \|\tilde{c}\|_a \leq \left( \frac{1}{\sqrt{d}} \right)^{(2-b)/b} \|c\|_b, \quad (3.14)$$

where $1/a + 1/b = 1$ and $1 < b < 2$. Squaring, we obtain $\|p\|_a \leq (1/d)^{(1-\beta)/\beta} \|p\|_\beta$, $\|q\|_a \leq (1/d)^{(1-\beta)/\beta} \|q\|_\beta$ under the conditions on $\alpha$ and $\beta$ from lemma 2. In its symmetric form, the uncertainty relation in terms of the Tsallis entropies is written as

$$H_\alpha(p_k) + H_\beta(q_l) \geq \ln \mu d, \quad (3.15)$$

where $1/\alpha + 1/\beta = 2$ and $\mu = \max\{\alpha, \beta\}$. For $a = b = 2$, the inequalities (3.14) turn out into the equalities. So the relation $H_\alpha(p_k) + H_\beta(q_l) \geq \ln d$ is sharp in a sense that it can be saturated. For instance, this is the case when either of the distributions $\{p_k\}$ and $\{q_l\}$ is uniform (and other is deterministic). Say, $|\tilde{c}_k| = 1/\sqrt{d}$ for all $k$ and all the $c_l$’s, except one, are zeros. For such a choice, the inequalities (3.14) are also saturated with $a = \infty$ and $b = 1$.

But this case is rather formal, since $H_\alpha \to 0$ in the limit $\alpha \to \infty$. To obtain the strongest uncertainty relation, the exact $(b, a)$-norm of transformation (3.13) should be found for any pair of conjugate indices $a$ and $b$.

**Example 2.** The angle $\varphi$ and the angular momentum $J_z$ can similarly be treated. Taking one and the same size $\delta \varphi$ for all the angular bins (i.e. the ratio $2\pi/\delta \varphi$ is a strictly positive integer), we introduce probabilities

$$p_k = \int_{k \delta \varphi}^{(k+1)\delta \varphi} d\varphi |\Psi(\varphi)|^2, \quad q_l = |c_l|^2, \quad (3.16)$$

where the coefficients $c_l$’s are related to the expansion $\Psi(\varphi) = (2\pi)^{-1/2} \sum_{l=\infty}^{\infty} c_l \exp(il \varphi)$, with respect to the eigenstates of the $J_z$. Using theorem 192 of [33] for integral means and assuming $\beta < 1 < \alpha$, we have

$$\frac{1}{\delta \varphi} \int_{k \delta \varphi}^{(k+1)\delta \varphi} d\varphi |\Psi(\varphi)|^2 \beta \leq \left( \frac{1}{\delta \varphi} \int_{k \delta \varphi}^{(k+1)\delta \varphi} d\varphi |\Psi(\varphi)|^2 \right)^\beta \quad (3.17)$$
and the reversed inequality with $\alpha$ instead of $\beta$. Summing these inequalities with respect to $k$ and then raising them to the powers $1/\beta$ and $1/\alpha$, respectively, we finally write

$$\|\Psi\|_b^\beta \leq \delta \varphi(1-\beta)/\beta \|P\|_b \quad \text{and} \quad \delta \varphi(1-\alpha)/\alpha \|\Psi\|_a \leq \|\Psi\|_a^\alpha,$$

where the norm $\|\Psi\|_b = \left(\int_\varphi^{2\pi} \varphi \, |\Psi(\varphi)|^b \right)^{1/b}$ and $b = 2\beta, \alpha = 2\alpha$. Combining relations (3.18) with the Young–Hausdorff inequalities (see, e.g., section 8.17 in [33]), which are viewed in our notation as

$$\|c\|_a \leq \left(\frac{1}{\sqrt{2\pi}}\right)^{(2-b)/b} \|\Psi\|_b, \quad \|\Psi\|_a \leq \left(\frac{1}{\sqrt{2\pi}}\right)^{(2-b)/b} \|c\|_b,$$

we obtain $\|q\|_a \leq (\delta \varphi/2\pi)^{(1-\beta)/\beta} \|P\|_b$ and $\|P\|_a \leq (\delta \varphi/2\pi)^{(1-\beta)/\beta} \|q\|_b$. So, there holds $H_\alpha(p_1) + H_\beta(q_1) \geq \ln_\alpha(2\pi/\delta \varphi)$, where $1/\alpha + 1/\beta = 2$ and $\mu = \max(\alpha, \beta)$. When $\mu \to 1$, this new inequality in terms of Tsallis’ entropies coincides with the relation in terms of Rényi’s entropies deduced in [40]. Another way to pose entropic uncertainty relations for the observables $\varphi$ and $J_\alpha$ is expressed in terms of the entropy

$$H_\alpha(w) = (1 - \alpha)^{-1}\left(\|w\|_a^{\alpha} - 1\right) = (1 - \alpha)^{-1}\left(\int_0^{2\pi} w(\varphi)^\alpha \, d\varphi - 1\right),$$

where the probability density function $w(\varphi) = |\Psi(\varphi)|^2$. Here, the uncertainty relations follow directly from the Young–Hausdorff inequalities (3.19) themselves. Taking $\alpha > 1 > \beta$, we then obtain

$$H_\alpha(w) + H_\beta(q_1) \geq \ln_\alpha(2\pi).$$

(3.21)

The Young–Hausdorff inequalities (3.19) are saturated if and only if $\Psi(\varphi) = (2\pi)^{-1/2} \exp(i m \varphi)$ and $c_l = 0$ for all $l \neq m$ (see, e.g., point (2.25) of chapter XII in [46]). So the entropic relation (3.21) is sharp in a sense that it is always saturated for one of eigenfunctions of $J_\alpha$. In practice, relation (3.21) could be used for $\delta \varphi \ll 1$.

Finally, we consider the case of extremal unravelings within the general formulation (3.10). For the extremal unravelings $\mathcal{C}_\varphi^\text{(ex)}$ and $\mathcal{B}_\varphi^\text{(ex)}$ of the super-operators $S_A$ and $S_B$, we obtain the entropic uncertainty relation

$$H_\alpha(\mathcal{C}_\varphi^\text{(ex)} | \rho) + H_\beta(\mathcal{B}_\varphi^\text{(ex)} | \rho) \geq \ln_\alpha \left(g(\mathcal{C}_\varphi^\text{(ex)} | \rho, \mathcal{B}_\varphi^\text{(ex)} | \rho)^{-2}\right),$$

(3.22)

where the $g(\mathcal{C}_\varphi^\text{(ex)} | \rho, \mathcal{B}_\varphi^\text{(ex)} | \rho)$ is put by (3.4) with $M_l = A_l A_l$ and $N_q = B_j B_j$ in terms of the Kraus operators $A_l \in \mathcal{C}_\varphi^\text{(ex)}$ and $B_j \in \mathcal{B}_\varphi^\text{(ex)}$. We can also write the uncertainty relation for unravelings extremal with respect to the Rényi entropies. Using (3.7), for $\alpha > 1$ we obtain the relation

$$R_\alpha(\mathcal{C}_\varphi^\text{(ex)} | \rho) + R_\beta(\mathcal{B}_\varphi^\text{(ex)} | \rho) \geq -2 \ln g(\mathcal{C}_\varphi^\text{(ex)} | \rho, \mathcal{B}_\varphi^\text{(ex)} | \rho),$$

(3.23)

where $\mathcal{C}_\varphi^\text{(ex)} | \rho$ denotes the unraveling of $S_A$ extremal for the Rényi entropy of order $\alpha > 1$. This unraveling differs from the one given by (2.9) and also depends on the parameter $\alpha$ in general. A search of explicit analytic expression for $\mathcal{C}_\varphi^\text{(ex)} | \rho$ seems to be complicated, because convexity/concavity properties cannot be used here. Nevertheless, the entropic relation in terms of the Rényi entropies holds, as a mathematical inequality at least.

4. Conclusion

We have examined the unraveling (i.e. the concrete set of Kraus operators) of a super-operator that is extremal with respect to all the Tsallis entropies of a positive order and the Rényi
entropies of an order $0 < \alpha < 1$. This general result is formally posed in theorem 1. If one of unrollings is given explicitly, then the relevant minimizing unrolling is easily calculated by diagonalizing a certain Hermitian matrix. The known relation between the Shannon entropy of an ensemble of pure states and the von Neumann entropy of the risen density operator is extended to both the Tsallis and Rényi entropies of the mentioned orders. The Tsallis entropy allows further extension to a mixture of density operators (see the bounds (2.9)). Due to Riesz’s theorem, there exists an inequality between certain functions of the probability distributions generated by two resolutions of the identity (see lemma 2). This inequality gives an origin for the uncertainty relations, given by (3.7) for the Rényi entropies and by theorem 2 for the Tsallis entropies. The general formulation (3.10) is a new result in the topic. It has been illustrated within the two interesting cases (see examples 1 and 2). The described approach does not lead always to the strongest relations, though sometimes the entropic bounds are really achievable for arbitrary parameters. Using Riesz’s theorem, we do not yield the best possible relation for all cases. Only an upper estimate on $(b, a)$-norms of transformation (3.1) is provided by (3.2). The claim can also be seen in the example of the Fourier transform, for which the exact values of norms were derived by Beckner [47]. Using Riesz’s theorem [7], we have arrived at a weaker Hirschman’s relation [9]. Finding the exact values of these norms for the given transformation is a separate hard problem. The Rényi form (3.7) and the Tsallis form (3.10) of entropic uncertainty relations are independent inequalities based on lemma 2. Which of them may be more suitable in practice? In order for a measure to be experimentally robust, it should enjoy the stability property. This issue was inspired by Lesche [48] who argued that the Rényi entropy is unstable to small deformations of the state. In contrast, the Tsallis entropy itself [49] and its partial sums [50] do enjoy the stability property. From this viewpoint, the Tsallis formulation (3.10) seems to be more appropriate. In formulae (3.22) and (3.23), both entropic uncertainty relations are naturally recast for the extremal unrollings of two given trace-preserving super-operators. It would be interesting to consider the presented relations along lines of the important paper [16], in which applications of entropic uncertainty relations to quantum information processing are examined.

Acknowledgment

The author is grateful to the anonymous referees for constructive remarks.

Appendix. Minimum of the function

To obtain a lower bound on the sum of Tsallis entropies, we find the minimal value of function (3.9) in the domain $D$ such that $0 \leq \xi \leq 1$, $1 \leq \zeta < \infty$ and $\gamma \xi^{\beta/\alpha} \leq \zeta$. When $\gamma > 1$, the curve $\zeta = \gamma \xi^{\beta/\alpha}$ cuts off the down-right corner of the rectangle $((\xi, \zeta) : 0 \leq \xi \leq 1 , 1 \leq \zeta < \infty)$ and herewith the point $(1, 1)$ in which $\phi = 0$ (see figure A1). In the interior of $D$, we have

$$\frac{\partial \phi}{\partial \xi} = \frac{1}{1-\alpha} < 0, \quad \frac{\partial \phi}{\partial \zeta} = \frac{1}{1-\beta} > 0,$$

(A.1)

due to $\alpha > 1 > \beta$. So the minimum is reached on the boundary of the domain $D$. Using (A.1), the task is merely reduced to minimizing $\phi(\xi, \zeta)$ on a segment of the curve $C : \zeta = \gamma \xi^{\beta/\alpha}$ between the point $(\xi_0, 1)$, where $\xi_0 = \gamma^{-\alpha/\beta}$, and the point $(1, \gamma)$. Substituting $\xi = (\xi/\xi_0)^{\beta/\alpha}$ in (3.9) and differentiating with respect to $\xi$, we obtain

$$\frac{1}{1-\alpha} + \frac{\beta}{\alpha(1-\beta)}(\frac{\xi}{\xi_0})^{\beta/\alpha} = \frac{1}{\alpha-1} \left( \frac{1}{\xi} \left( \frac{\xi}{\xi_0} \right)^{\beta/\alpha} - 1 \right),$$

(A.2)
Figure A1. The domain $D$ in which we find the minimum of the function $\phi(\xi, \zeta)$ defined by (3.9).

where we used $\beta/(1 - \beta) = \alpha/(\alpha - 1)$ because of $1/\alpha + 1/\beta = 2$. When $\xi_0 < 1$, quantity (A.2) is strictly positive for $\xi_0 \leq \xi \leq 1$ and the minimal value is $\phi(\xi_0, 1) = (\xi_0 - 1)/(1 - \alpha)$ too.

References

[1] Heisenberg W 1927 Z. Phys. 43 172  
[2] Hall M J W 1999 Phys. Rev. A 59 2602  
[3] Busch P, Heinonen T and Lahti P J 2007 Phys. Rep. 452 155  
[4] Robertson H P 1929 Phys. Rev. 34 163  
[5] Lynch R 1995 Phys. Rep. 256 357  
[6] Barnett S M and Pegg D T 1989 J. Mod. Opt. 36 7  
[7] Maassen H and Uffink J B M 1988 Phys. Rev. Lett. 60 1103  
[8] Wehner S and Winter A 2010 New J. Phys. 12 025009  
[9] Hirschman I I 1957 Am. J. Math. 79 152  
[10] Manojka B 1974 Int. J. Theor. Phys. 11 73  
[11] Deutsch D 1983 Phys. Rev. Lett. 50 631  
[12] Ivanovic I D 1992 J. Phys. A: Math. Gen. 25 L363  
[13] Sanchez J 1993 Phys. Lett. A 173 233  
[14] Hall M J W 2008 J. Phys. A: Math. Theor. 41 255301  
[15] Man'ko M A 2009 J. Russ. Laser Res. 30 514  
[16] Berta M, Christandl M, Colbeck R, Renes J M and Renner R 2009 The uncertainty principle in the presence of quantum memory arXiv:0909.0950[quant-ph]  
[17] Massar S and Spindel P 2008 Phys. Rev. Lett. 100 190401  
[18] Urizar-Lanz I and Tóth G 2010 Phys. Rev. A 81 052108  
[19] Kraus K 1983 States, Effects, and Operations: Fundamental Notions of Quantum Theory (Lecture Notes in Physics vol 190) (Berlin: Springer)  
[20] Il'ichev L V 2003 JETP 96 982  
[21] http://tsallis.cat.cbpf.br/biblio.htm  
[22] Aczél J and Daróczy Z 1975 On Measures of Information and Their Characterizations (New York: Academic)  
[23] Tsallis C 1988 J. Stat. Phys. 32 479  
[24] Havrda J and Charvát F 1967 Kybernetika (Prague) 3 30  
[25] Rényi A 1961 On measures of entropy and information Proc. 4th Berkeley Symp. on Mathematical Statistics and Probability (Berkeley, CA: University of California Press) pp 547–61  
[26] Życzkowski K 2003 Open Syst. Inf. Dyn. 10 297  
[27] Watrous J 2008 Theory of Quantum Information (Computer Science vol 798) (Ontario, Canada: University of Waterloo) (available at http://www.cs.uwaterloo.ca/~watrous/quant-info/)
[28] Holevo A S 1982 Probabilistic and Statistical Aspects of Quantum Theory (Statistics and Probability vol 1) (Amsterdam: North-Holland)

[29] Ferraro A, Galbiati M and Paris M G A 2006 J. Phys. A: Math. Gen. 39 L219

[30] Rastegin A E 2010 Quantum Inf. Comput. 10 0971

[31] Preskill J 1998 Quantum Computation and Information (Lecture Notes in Physics vol 229) (California: California Institute of Technology) (available at http://www.theory.caltech.edu/people/preskill/ph229/)

[32] Carmichael H J 1993 An Open Systems Approach to Quantum Optics (Lecture Notes in Physics vol 18) (Berlin: Springer)

[33] Hardy G H, Littlewood J E and Polya G 1934 Inequalities (London: Cambridge University Press)

[34] Breslin J K and Milburn G J 1997 J. Mod. Opt. 44 2469

[35] Hughston L P, Jozsa R and Wootters W K 1993 Phys. Lett. A 183 14

[36] Wehrl A 1978 Rev. Mod. Phys. 50 221

[37] Rastegin A E 2010 Fano type quantum inequalities in terms of q-entropies arXiv:1010.1811[quant-ph]

[38] Rastegin A E 2010 J. Phys. A: Math. Theor. 43 155302

[39] Krishna M and Parthasarathy K R 2002 Sankhya A 64 842

[40] Bialynicki-Birula I 2006 Phys. Rev. A 74 052101

[41] Rajagopal A K 1995 Phys. Lett. A 205 32

[42] Majernik V and Majernikova E 2001 Reg. Math. Phys. 47 381

[43] Majernik V, Majernikova E and Sheyko S 2003 Cent. Eur. J. Phys. 3 393

[44] Zhan X 2002 Matrix Inequalities (Berlin: Springer)

[45] Kraus K 1987 Phys. Rev. D 35 3070

[46] Zygmund A 1959 Trigonometric Series vol 2 (Cambridge: Cambridge University Press)

[47] Beckner W 1975 Ann. Math. 102 159

[48] Lesche B 1982 J. Stat. Phys. 27 419

[49] Abe S 2002 Phys. Rev. E 66 046134

[50] Rastegin A E 2010 Lett. Math. Phys. 94 229