A common framework for some techniques in
Applied mathematics

by

Bilal Chanane

Department of Mathematics and Statistics
KFUPM, Dhahran 31261, Saudi Arabia
E-Mail: chanane@kfupm.edu.sa

Abstract
The objective in this paper is to demonstrate that four of the most used techniques in applied
mathematics, viz., Fourier series, Fourier transform, Laplace transform and the Fourier-Laplace
transform can be introduced using eigenvalue problems for first order differential operators with
discrete/continuous spectra.

Key Words Discrete spectrum, continuous spectrum, Fourier series, Fourier transform, Laplace
transform, Fourier-Laplace transform, eigenvalue problems, eigenvalue problems, eigenfunctions
expansion.

AMS subject classification 42A16, 42B05, 42A38, 44A10, 34L10

1 Introduction
A look at several applied/engineering mathematics textbooks (e.g., [1], [2], [3], [4], [5], [6] ) reveals
that four of the most important techniques in applied mathematics, viz., Fourier series, Fourier
transform, Laplace transform and Fourier-Laplace transform are introduced somewhat independ-
ently though some kind of limiting argument is used to go from Fourier series to Fourier transform.
More often the Fourier series is introduced as arising from a second order Sturm-Liouville prob-
lem. We note however, in some instances, its introduction through first order eigenvalue problem
is mentioned. It seems that no textbook has approached all four problems using the same method.
We shall present in the following sections these notions as arising from from first order eigenvalue
problems with discrete/continuous spectra. This is by no means a substitute for a more sophisti-
cated presentation involving higher mathematics to which students are not introduced until very
late in their curricula, if at all. Having said that, we think that the presentation using this ap-
proach allows one to deal in a unified manner and at an elementary level these fundamental tools
and it should be adequate for engineering and applied mathematics students.

2 The Fourier series
Consider the problem of finding the values of the parameter $\lambda$ for which the following problem

$$\begin{cases}
\frac{dy}{dx} = \lambda y, \quad x \in (-L, L) \\
y(-L) = y(L)
\end{cases} \quad (2.1)
$$

will have non trivial solutions, that is solutions that are not identically zero. Such a parameter
$\lambda$ is called an eigenvalue, while the corresponding non zero solution is called an eigenfunction
belonging to the eigenvalue $\lambda$. 
Theorem 1 The eigenvalue problem (2.1) has an infinite sequence of eigenvalues \( \lambda_k = k\pi/L, k \in \mathbb{Z} \), where \( \mathbb{Z} \) is the set of relative integers. The corresponding eigenfunctions \( y_k = \exp(-ik\pi x/L), k \in \mathbb{Z} \), are orthogonal with respect to the inner product \( \langle f, g \rangle = \int_{-L}^{L} f(x)g(x)dx \). The set of eigenfunctions is complete and any function \( f \) satisfying \( \int_{-L}^{L} |f(x)|^2 dx < \infty \), will have the expansion

\[
f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{-ik\pi x/L}
\]

where

\[
c_k = \frac{1}{2L} \int_{-L}^{L} f(x)e^{ik\pi x/L}dx, \ k \in \mathbb{Z}.
\]

The above series is called the complex Fourier series of \( f \).

Proof. From the differential equation we have \( dy/dx = -i\lambda y \) whose general solution is \( y = c \exp(-i\lambda x) \). Using the boundary condition, we get \( c \exp(i\lambda L) = c \exp(-i\lambda L) \) leading to \( 2ic \sin(\lambda L) = 0 \). Therefore, \( \lambda_k = k\pi/L, k \in \mathbb{Z} \) are the eigenvalues of the problem. The corresponding eigenfunctions are \( y_k = \exp(-ik\pi x/L), k \in \mathbb{Z} \). Note that we have taken the constant \( c = 1 \) as any non zero multiple of an eigenfunction is an eigenfunction itself. The eigenfunctions \( y_k, k \in \mathbb{Z} \), are orthogonal with respect to the inner product

\[
\langle f, g \rangle = \int_{-L}^{L} f(x)g(x)dx
\]

Indeed,

\[
\langle \exp(-i\frac{k\pi x}{L}), \exp(-i\frac{l\pi x}{L}) \rangle = \int_{-L}^{L} \exp(-i\frac{k\pi x}{L})\exp(i\frac{l\pi x}{L})dx
\]

\[
= \int_{-L}^{L} \exp(-i\frac{\pi x}{L}(k-l))dx
\]

\[
= \begin{cases} 
2L & \text{if } k = l \\
\frac{2L}{\pi(k-l)} \sin(\pi(k-l)) & \text{if } k \neq l
\end{cases}
\]

(2.3)

The set of eigenfunctions is complete since for if \( f \) is any function satisfying \( \int_{-L}^{L} |f(x)|^2 dx < \infty \) then \( \int_{-L}^{L} f(x)\exp(i\frac{k\pi x}{L})dx = 0 \) for all \( k \in \mathbb{Z} \) will imply \( f(x) = 0 \) for almost all \( x \in [-L, L] \).

Now, let \( f \) be such that \( \int_{-L}^{L} |f(x)|^2 dx < \infty \), then it has the eigenfunctions expansion,

\[
f(x) \sim \sum_{k=-\infty}^{\infty} c_k \exp(-i\frac{k\pi x}{L})
\]

Multiplying by \( \exp(i\frac{l\pi x}{L}) \) and integrating both sides with respect to \( x \) from \(-L\) to \( L\), we get

\[
\int_{-L}^{L} f(x)\exp(i\frac{l\pi x}{L})dx = 2Lc_l
\]

that is,

\[
c_k = \frac{1}{2L} \int_{-L}^{L} f(x)\exp(i\frac{k\pi x}{L})dx, \ k \in \mathbb{Z}
\]

(2.4)

which ends the proof. ■
A connection with second order Sturm-Liouville problems goes like this. Applying \( id/dx \) to both sides of (2.1), we get

\[
-i\frac{d^2 y}{dx^2} = \lambda i \frac{dy}{dx} = \lambda^2 y.
\]

Since \( y(-L) = y(L) \) we have

\[
\int_{-L}^{L} \left| \frac{dy}{dx} \right|^2 dx = 0.
\]

That is

\[
\frac{d}{dx}(y(-L)) = \frac{d}{dx}(y(L)).
\]

Thus,

\[
\begin{cases}
-\frac{d^2 y}{dx^2} = \lambda^2 y, & x \in (-L, L) \\
y(-L) = y(L) \\
\frac{dy}{dx}(-L) = \frac{dy}{dx}(L)
\end{cases}
\]

(2.5)

a second order Sturm-Liouville problem which is usually taken as a point of departure for the introduction of Fourier series. Its eigenvalues \( \lambda_k^2 \) are just the square of the eigenvalues \( \lambda_k \) of (2.1),

\[
\lambda_k^2 = \left( \frac{k\pi}{L} \right)^2, \quad k \geq 0
\]

and the corresponding eigenfunctions are

\[
1, \left\{ \cos \left( \frac{k\pi x}{L} \right), \sin \left( \frac{k\pi x}{L} \right) \right\}_{k \geq 1}
\]

Noting that, \( c_{-k} = \overline{c_k} \), \( k \geq 1 \), one obtains the real Fourier series as,

\[
f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left\{ a_k \cos \left( \frac{k\pi x}{L} \right) + b_k \sin \left( \frac{k\pi x}{L} \right) \right\}
\]

(2.6)

where,

\[
a_k = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{k\pi x}{L} \right) dx, \quad b_k = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{k\pi x}{L} \right) dx.
\]

(2.7)

3 The Fourier transform

Consider the eigenvalue problem,

\[
i \frac{dy}{dx} = \lambda y, \quad x \in \mathbb{R}
\]

(3.1)

From the differential equation we get \( dy/dx = -i\lambda y \) whose solution is \( y = c \exp(-i\lambda x) \). To have a non trivial solution we need \( c \neq 0 \). We may as well take \( c = 1 \) since any other non zero solution is just a multiple of this one. Thus,

\[
y_\lambda(x) = \exp(-i\lambda x)
\]

(3.2)

We notice that this function is not square integrable because no matter what the value of \( \lambda \) is,

\[
\int_{-\infty}^{\infty} |y_\lambda(x)|^2 dx = \int_{-\infty}^{\infty} |\exp(-i\lambda x)|^2 dx = \int_{-\infty}^{\infty} \exp(2x \text{Im} \lambda) dx = \infty
\]

therefore, the operator \( i \frac{d}{dx} \) has no eigenvalue as such. However, we shall introduce the concepts of continuum eigenvalue and continuum eigenfunction \([1]\).

**Definition 2** A number \( \lambda \) is said to be a continuum eigenvalue for an operator \( M \) if there exists a sequence of functions \( y_n \) in the domain of \( M \) such that the ratio \( \frac{\|M - \lambda y_n\|}{\|y_n\|} \) converges to zero as \( n \) goes to \( \infty \). If the functions \( y_n \) converge pointwise to a function \( y \), then \( y \) is called a continuum eigenfunction of \( M \) corresponding to \( \lambda \). We say then that \( \lambda \) belongs to the continuous spectrum of \( M \).
Remark 3 If the convergence of \( y_n \) is in the sense of the space to \( y \), \( \lambda \) would be an eigenvalue and \( y \) a corresponding eigenfunction. In that case we say that \( \lambda \) belongs to the discrete spectrum of \( M \).

Returning to the eigenvalue problem \((3.1)\), we claim,

**Theorem 4** The eigenvalue problem \((3.1)\) has a continuous spectrum given by \( \mathbb{R} \). Corresponding to the continuum eigenvalue \( \lambda \in \mathbb{R} \), we associate the continuum eigenfunction \( y_\lambda(x) = \exp(-i\lambda x) \).

The set of continuum eigenfunctions is complete and any function \( f \) satisfying \( \int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty \), will have the representation

\[
  f(x) \sim \int_{\mathbb{R}} F(\lambda) \exp(-i\lambda x) \, d\lambda
\]

where

\[
  F(\lambda) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \exp(i\lambda x) \, dx.
\]

\( F \) is called the Fourier transform of \( f \) and \( f \) the inverse Fourier transform of \( F \).

**Proof.** We have here a continuous spectrum given by \( \mathbb{R} \). As for the orthogonality we have,

\[
  \int_{\mathbb{R}} y_\lambda(x)y_\mu(x) \, dx = \int_{\mathbb{R}} \exp(-i\lambda x) \exp(-i\mu x) \, dx
\]

\[
  = \int_{\mathbb{R}} \exp(-i\lambda x) \exp(i\mu x) \, dx
\]

\[
  = \int_{\mathbb{R}} \exp(-i(\lambda - \mu)x) \, dx
\]

\[
  = 2\pi \delta(\lambda - \mu)
\]

where we have made use of the property of the Dirac delta (generalized) function \( \delta \),

\[
  \delta(a) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp(-iax) \, dx
\]

Thus, for \( f \) such that \( \int_{\mathbb{R}} |f(x)|^2 \, dx < \infty \) we have the representation,

\[
  f(x) = \int_{\mathbb{R}} F(\lambda) \exp(-i\lambda x) \, d\lambda
\]

from which we get, after multiplication by \( \exp(i\mu x) \) and integration with respect to \( x \) from \(-\infty\) to \(+\infty\),

\[
  \int_{\mathbb{R}} f(x) \exp(i\mu x) \, dx = \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} F(\lambda) \exp(-i\lambda x) \, d\lambda \right\} \exp(i\mu x) \, dx
\]

\[
  = \int_{\mathbb{R}} F(\lambda) \left\{ \int_{\mathbb{R}} \exp(-i(\lambda - \mu)x) \, dx \right\} \, d\lambda
\]

\[
  = \int_{\mathbb{R}} F(\lambda) \{2\pi \delta(\lambda - \mu)\} \, d\lambda
\]

\[
  = 2\pi F(\mu)
\]

Here we used the property

\[
  \int_{\mathbb{R}} g(z) \delta(z) \, dz = g(0)
\]

Thus,

\[
  F(\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \exp(i\mu x) \, dx
\]

which concludes the proof.
4 The Laplace transform

Consider the problem of finding the values of the parameter \( \lambda \) for which the following problem will have non trivial solutions,

\[
i \left( \frac{dy}{dx} - \sigma y \right) = \lambda y, \quad x \in [0, \infty)
\]  

(4.1)

We have \( \frac{dy}{dx} = (\sigma - i\lambda)y \) thus, a non trivial solution is

\[
y_\lambda = \exp((\sigma - i\lambda)x)
\]

(4.2)

for any \( \lambda \in \mathbb{R} \) and any other non trivial solution is a multiple of this one.

Here again, we notice that this function is not square integrable because no matter what the value of \( \lambda \) is,

\[
\int_{\mathbb{R}} |y_\lambda(x)|^2 \, dx = \int_{\mathbb{R}} |\exp((\sigma - i\lambda)x)|^2 \, dx = \int_{\mathbb{R}} \exp(2x \{\sigma + \text{Im} \lambda\}) \, dx = \infty
\]

therefore, the operator \( i \left( \frac{dy}{dx} - \sigma y \right) \) has no eigenvalue as such. We claim,

**Theorem 5** The eigenvalue problem (4.1) has a continuous spectrum given by \( \mathbb{R} \). Corresponding to the continuum eigenvalue \( \lambda \in \mathbb{R} \), we associate the continuum eigenfunction \( y_\lambda = \exp(\sigma - i\lambda)x \). The set of continuum eigenfunctions is complete and any function \( f \) satisfying \( \int_{-\infty}^{\infty} e^{-2\sigma x} |f(x)|^2 \, dx < \infty \), will have the representation

\[
f(x) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda)e^{(\sigma - i\lambda)x} \, d\lambda
\]

where

\[
F(\lambda) \sim \int_{0}^{\infty} f(x)e^{(\sigma + i\lambda)x} \, dx.
\]

**Proof.** We have here again a continuous spectrum given by \( \mathbb{R} \). Any two continuum eigenfunctions \( y_\lambda \) and \( y_\mu \) are orthogonal with respect to the weight \( w(x) = \exp(-2\sigma x) \) over \( [0, \infty) \). Indeed,

\[
\int_{0}^{\infty} e^{-2\sigma x} y_\lambda(x) y_\mu(x) \, dx = \int_{0}^{\infty} e^{-2\sigma x} e^{(\sigma - i\lambda)x} e^{(\sigma + i\mu)x} \, dx
\]

\[
= \int_{0}^{\infty} e^{-i(\lambda - \mu)x} \, dx
\]

\[
= 2\pi\delta(\lambda - \mu).
\]

(4.3)

Let \( f \) be such that

\[
\int_{0}^{\infty} |f(x)|^2 e^{-2\sigma x} \, dx < \infty.
\]

(4.4)

We have,

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda)e^{(\sigma - i\lambda)x} \, d\lambda.
\]

(4.5)

We have taken the factor \( \frac{1}{2\pi} \) for convenience and compatibility with known results. Multiplying by \( \exp(-2\sigma x) \exp((\sigma + i\mu)x) \) and integrating with respect to \( x \) from 0 to \( \infty \), we get

\[
\int_{0}^{\infty} f(x) \exp((-\sigma + i\mu)x) \, dx = \int_{0}^{\infty} \exp((-\sigma + i\mu)x) \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) \exp((\sigma - i\lambda)x) \, d\lambda \, dx
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda)2\pi\delta(\lambda - \mu) \, d\lambda
\]

\[
= F(\mu)
\]

(4.6)
that is,
\[
F(\lambda) = \int_0^\infty f(x)e^{(-\sigma + i\lambda)x}dx. \quad (4.7)
\]

If we let \( s = \sigma - i\lambda \) and denote \( \hat{f}(s) = F(\lambda) \), we get,
\[
\hat{f}(s) = \int_0^\infty f(x)e^{-sx}dx. \quad (4.8)
\]
Now, \( ds = -id\lambda \) so that
\[
f(x) = \frac{1}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \hat{f}(s)e^{sx}ids \quad (4.9)
\]
leading to,
\[
f(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \hat{f}(s)e^{sx}ds \quad (4.10)
\]
Form the above development we can see that for \( \hat{f} \) to exist, \( f \) has to satisfy
\[
\int_0^\infty |f(x)|^2 e^{-2\sigma x}dx < \infty \quad (4.11)
\]
Thus there exists \( M > 0 \) and \( c > 0 \) such that \( |f(x)|^2 e^{-2\sigma x} < M^2 \) for all \( x > c \). That is,
\[
|f(x)| < Me^{\sigma x}, \text{ for all } x > c \quad (4.12)
\]
We say that \( f \) is of exponential type and \( \sigma \) is called the abscissa of convergence. Hence, we introduce,

**Definition 6** Let \( f \) be of exponential type, then the function \( \hat{f} \) defined by,
\[
\hat{f}(s) = \int_0^\infty f(x)e^{-sx}dx
\]
is called the Laplace transform of \( f \). Furthermore,
\[
f(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \hat{f}(s)e^{sx}ds
\]
gives the inverse Laplace transform of \( \hat{f} \).

**5 The Fourier-Laplace transform**

Consider the problem of finding the values of the parameters \( \mu \) and \( \lambda \) for which the following problem will have non trivial solutions,
\[
\begin{aligned}
i \frac{\partial y}{\partial x} &= \lambda y \\
i \frac{\partial y}{\partial t} - \sigma y &= \mu y
\end{aligned} \quad (5.1)
\]
\((x, t) \in (-\infty, \infty) \times (0, \infty) \). We claim,
Theorem 7. The eigenvalue problem (5.1) has a continuous spectrum given by \( \mathbb{R}^2 \). Corresponding to the continuum eigenvalue \((\lambda, \mu) \in \mathbb{R}^2\), we associate the continuum eigenfunction \( y_{\lambda,\mu}(x,t) = \exp(-i\lambda x + (\sigma - i\mu)t) \). These continuum eigenfunctions are orthogonal with respect to the inner product \( < f, g > = \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-2\pi t} f(x)g(t)dxdt \). The set of continuum eigenfunctions is complete and any function \( f \) satisfying \( \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-2\pi t} |f(x,t)|^2 dxdt < \infty \), will have the representation

\[
 f(x,t) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\lambda, \mu) e^{-i\lambda x + \mu t} d\lambda d\mu
\]

where

\[
 F(\lambda, \mu) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} f(x,t) e^{i\lambda x - \mu t} dxdt.
\]

F is called the Fourier-Laplace transform of \( f \) and \( f \) is called the inverse Fourier-Laplace transform of \( F \).

Proof. The first differential equation in (5.1) gives \( y(x,t) = \exp(-i\lambda x)u(t) \). Replacing into the second differential equation gives, \( u'(t) = (\sigma - i\mu)u(t) \), that is \( u(t) = c \exp((\sigma - i\mu)t) \). Thus,

\[
 y_{\lambda,\mu}(x,t) = \exp(-i\lambda x + (\sigma - i\mu)t)
\]

where we have taken without loss of generality, \( c = 1 \), is a continuum eigenfunction corresponding to the continuum eigenvalue \((\lambda, \mu) \) in the continuous spectrum \( \mathbb{R}^2 \). Any two continuum eigenfunctions \( y_{\lambda,\mu}(x,t) \) and \( y_{\lambda',\mu'}(x,t) \) are orthogonal with respect to the inner product

\[
 < f, g > = \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-2\pi t} f(x)g(t)dxdt.
\]

Indeed,

\[
 < y_{\lambda,\mu}, y_{\lambda',\mu'} > = \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-2\pi t} y_{\lambda,\mu}(x,t) y_{\lambda',\mu'}(x,t) dxdt
\]

\[
 = \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-2\pi t} \exp(-i\lambda x + (\sigma - i\mu)t) \exp(i\lambda' x + (\sigma + i\mu')t) dxdt
\]

\[
 = \int_{-\infty}^{\infty} \int_{0}^{\infty} \exp(-i(\lambda - \lambda')x - i(\mu - \mu')t) dxdt
\]

\[
 = (2\pi)^2 \delta(\lambda - \lambda') \delta(\mu - \mu').
\]

If \( f \) is such that \( \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-2\pi t} |f(x,t)|^2 dxdt < \infty \) then,

\[
 f(x,t) \sim \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{F}(\lambda, \mu) e^{-i\lambda x + (\sigma - i\mu)t} d\lambda d\mu
\]

Multiplying both sides by \( e^{-2\pi t} \exp(i\lambda' x + (\sigma + i\mu')t) \) we get after integration with respect to \((x,t)\) over \( \mathbb{R} \times \mathbb{R}_+ \),

\[
 \tilde{F}(\lambda, \mu) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} f(x,t) e^{i\lambda x + (\sigma + i\mu)t} dxdt
\]

Let \( s = -(\sigma + i\mu) \) and \( F(\lambda, s) = \tilde{F}(\lambda, \mu) \), we have,

\[
 F(\lambda, s) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} f(x,t) e^{i\lambda x - st} dxdt
\]

and

\[
 f(x,t) \sim \frac{1}{2\pi} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\lambda, \mu) e^{-i\lambda x + \mu t} d\lambda d\mu
\]

which ends the proof. \( \blacksquare \)
6 Conclusion

In this paper we have provided a common framework to deal with four of the most used techniques in applied mathematics, viz., Fourier series, Fourier transform, Laplace transform and Fourier-Laplace transform. It has been shown that they arise from first order eigenvalue problems with discrete/continuous spectra. We believe that the approach is worth presenting in an introductory course on applied/engineering mathematics.

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