Algebraic Geometry

Stable bundles as Frobenius morphism direct image

Faisceaux stables en tant qu’images directes par le morphisme de Frobenius

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Let $X$ be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field $k$ of characteristic $p > 0$, and let $F : X \to X_1$ be the relative Frobenius morphism. We show that a vector bundle $E$ on $X_1$ is the direct image under $F$ of some stable bundle on $X$ if and only if the instability of $F^*E$ is equal to $(p-1)(2g-2)$.

RÉSUMÉ

Soient $X$ une courbe projective lisse de genre $g \geq 2$ définie sur un corps $k$ algébriquement clos de caractéristique $p > 0$, et $F : X \to X_1$ le morphisme de Frobenius relatif. On montre qu’un fibré vectoriel $E$ sur $X_1$ est l’image directe sous $F$ d’un certain fibré stable sur $X$ si et seulement si l’instabilité de $F^*E$ est égale à $(p-1)(2g-2)$.

1. Introduction

Let $X$ be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field $k$ of characteristic $p > 0$. The absolute Frobenius morphism $F_X : X \to X$ is induced by $O_X \to O_X$, $f \mapsto f^p$. Let $F : X \to X_1 := X \times_k k$ denote the relative Frobenius morphism over $k$. One of the themes is to study its action on the geometric objects on $X$. Recall that a vector bundle $E$ on a smooth projective curve is called semi-stable (resp. stable) if $\mu(E') \leq \mu(E)$ (resp. $\mu(E') < \mu(E)$) for any nontrivial proper subbundle $E' \subset E$, where $\mu(E)$ is the slope of $E$. It is known that $F_*$ preserves the stability of vector bundles (cf. [5]), but $F^*$ does not preserve the semi-stability of vector bundles (cf. [1] for example).

Semi-stable bundles are basic constituents of vector bundles in the sense that any bundle $E$ admits a unique filtration:

$$HN_i(E) : 0 = HN_0(E) \subset HN_1(E) \subset \cdots \subset HN_\ell(E) = E,$$

which is the so-called Harder–Narasimhan filtration, such that:

1. $gr_i^{HN}(E) := HN_i(E)/HN_{i-1}(E) (1 \leq i \leq \ell)$ are semi-stable;
2. $\mu(gr_1^{HN}(E)) > \mu(gr_2^{HN}(E)) > \cdots > \mu(gr_\ell^{HN}(E))$.

The rational number $I(E) := \mu(gr_1^{HN}(E)) - \mu(gr_\ell^{HN}(E))$, which measures how far a vector bundle is from being semi-stable, is called the instability of $E$. It is clear that $E$ is semi-stable if and only if $I(E) = 0$. 

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Given a semi-stable bundle $E$ on $X_1$, then $F^*E$ may not be semi-stable, so it is natural to consider the instability $I(F^*E)$. In [4, Theorem 3.1], the author proves $I(F^*E) \leq (\ell - 1)(2g - 2)$, where $\ell$ is the length of Harder–Narasimhan filtration of $F^*E$. If $E = F_*W$ where $W$ is a stable bundle on $X$, we know, by Sun’s theorem [5, Theorem 2.2], that $E$ is stable, the length of Harder–Narasimhan filtration of $F^*E$ is $p$ and $I(F^*E) = (p - 1)(2g - 2)$. Thus $I(F^*E) = (p - 1)(2g - 2)$ is a necessary condition for $E$ to be a direct image under Frobenius. In this short note, we show the following theorem:

**Theorem 1.** Let $E$ be a stable vector bundle on $X$. Then the following statements are equivalent:

1. There exists a stable bundle $W$ such that $E = F_*W$;
2. $I(F^*E) = (p - 1)(2g - 2)$.

The case $\text{rk} E = p$ was proved in [3]. Our observation is that the arguments in [3] together with Sun’s theorem imply the general case.

**2. Proof of the theorem**

Let $X$ be a smooth projective curve over an algebraically closed field $k$ with $\text{char}(k) = p > 0$. The absolute Frobenius morphism $F_X : X \to X$ is induced by the following homomorphism:

$$O_X \to O_X, \ f \mapsto f^p.$$ Let $F : X \to X_1 := X \times_k k$ denote the relative Frobenius morphism over $k$ that satisfies the following commutative diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{F} & X \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \xrightarrow{F_k} & \text{Spec}(k).
\end{array}
$$

For a vector bundle $E$ on $X$, the slope of $E$ is defined as

$$\mu(E) := \frac{\deg E}{\text{rk} E}$$

where $\text{rk} E$ (resp. $\deg E$) denotes the rank (resp. degree) of $E$. Then:

**Definition 1.** A vector bundle $E$ on $X$ is called semi-stable (resp. stable) if for any nontrivial proper subbundle $E' \subset E$, we have

$$\mu(E') \leq \text{(resp. <)} \mu(E).$$

**Theorem 2** (Harder–Narasimhan filtration). For any vector bundle $E$, there is a unique filtration:

$$HN_*(E) : 0 = HN_0(E) \subset HN_1(E) \subset \cdots \subset HN_\ell(E) = E,$$

which is called Harder–Narasimhan filtration, such that:

1. $\text{gr}_i^{\text{HN}}(E) := HN_i(E)/HN_{i-1}(E)$ ($1 \leq i \leq \ell$) are semi-stable;
2. $\mu(\text{gr}_1^{\text{HN}}(E)) > \mu(\text{gr}_2^{\text{HN}}(E)) > \cdots > \mu(\text{gr}_\ell^{\text{HN}}(E))$.

By using this unique filtration of $E$, an invariant $I(E)$ of $E$, which is called the instability of $E$ was introduced (see [5] and [4]). It is a rational number and measures how far is $E$ from being semi-stable.

**Definition 2.** Let $\mu_{\max}(E) = \mu(\text{gr}_1^{\text{HN}}(E)), \mu_{\min}(E) = \mu(\text{gr}_\ell^{\text{HN}}(E))$. Then the instability of $E$ is defined to be

$$I(E) := \mu_{\max}(E) - \mu_{\min}(E).$$

It is easy to see that a vector bundle $E$ is semi-stable if and only if $I(E) = 0$.

For any semi-stable bundle $E$, let

$$HN_*(F^*E) : 0 = HN_0(F^*E) \subset HN_1(F^*E) \subset \cdots \subset HN_\ell(F^*E) = F^*E$$

be the Harder–Narasimhan filtration of $F^*E$. Then we have the following lemma, which is implicit in [3].
Lemma 1. For any semi-stable bundle $E$, we have

$$
\mu_{\text{max}}(F^*E) \leq p \cdot \mu(E) + (p - 1)(g - 1);
$$

$$
\mu_{\text{min}}(F^*E) \geq p \cdot \mu(E) - (p - 1)(g - 1),
$$

and if $l(F^*E) = \mu_{\text{max}}(F^*E) - \mu_{\text{min}}(F^*E) = (p - 1)(2g - 2)$. Then

$$
\mu_{\text{max}}(F^*E) = p \cdot \mu(E) + (p - 1)(g - 1);
$$

$$
\mu_{\text{min}}(F^*E) = p \cdot \mu(E) - (p - 1)(g - 1).
$$

Now we prove our theorem by using this Lemma 1, the canonical filtration on the vector bundle $V = F^*F_s W$ and Sun’s theorem on the stability of Frobenius’ direct images.

Proof of Theorem 1. $(1) \Rightarrow (2)$. In [2, Section 5.3], there is a canonical filtration on the vector bundle $V = F^*F_s W$:

$$
0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell - 1} \subset V_\ell \subset \cdots \subset V_{p - 1} \subset V_p = V,
$$

which is indeed the Harder–Narasimhan filtration on $V$, and satisfies

$$
V_\ell / V_{\ell - 1} \cong (V_{\ell + 1} / V_\ell) \otimes \Omega^1_X
$$

for $1 \leq \ell \leq p - 1$, and $V_p / V_{p - 1} \cong W$. So $\mu(V_p / V_{p - 1}) = \mu(W)$, $\mu(V_0 / V_1) = \mu(W) + (p - 1)(2g - 2)$, and now the result is clear.

$(2) \Rightarrow (1)$. Since $l(F^*E) = (p - 1)(2g - 2)$, we have $\mu_{\text{max}}(F^*E) = p \cdot \mu(E) + (p - 1)(g - 1)$, $\mu_{\text{min}}(F^*E) = p \cdot \mu(E) - (p - 1)(g - 1)$ by Lemma 1. We consider the surjection:

$$
F^*E \to gr^\text{HN}_\ell(F^*E).
$$

The bundle $gr^\text{HN}_\ell(F^*E)$ is semi-stable of slope $\mu_{\text{min}}(F^*E)$. Replacing $gr^\text{HN}_\ell(F^*E)$ by a stable graded piece $W$ in the Jordan–Hölder filtration of $gr^\text{HN}_\ell(F^*E)$, we have a surjection:

$$
F^*E \to W,
$$

where $W$ is a stable bundle of slope $\mu(W) = \mu_{\text{min}}(F^*E) = p \cdot \mu(E) - (p - 1)(g - 1)$. By adjunction, we have a nontrivial morphism:

$$
\psi : E \to F_s W.
$$

By Sun’s theorem (cf. [5, Theorem 2.2]), we know that $F_s W$ is a stable bundle of slope:

$$
\mu(F_s W) = \frac{\mu(W)}{p} + \frac{(p - 1)(g - 1)}{p} = \mu(E).
$$

Thus $\psi$ induce an isomorphism:

$$
E \cong F_s W. \quad \square
$$

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