Linear Superposition for a Large Number of Nonlinear Equations

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Abstract:
We demonstrate a kind of linear superposition for a large number of nonlinear equations, both continuum and discrete. In particular, we show that whenever a nonlinear equation admits solutions in terms of Jacobi elliptic functions \( cn(x, m) \) and \( dn(x, m) \), then it also admits solutions in terms of their sum as well as difference, i.e. \( dn(x, m) \pm \sqrt{m} \, cn(x, m) \). Further, we also show that whenever a nonlinear equation admits a solution in terms of \( dn^2(x, m) \), it also has solutions in terms of \( \sqrt{m} \, cn(x, m) \, dn(x, m) \) even though \( cn(x, m) \, dn(x, m) \) is not a solution of that nonlinear equation. Finally, we obtain similar superposed solutions in coupled theories.
**Introduction:** Linear superposition principle is one of the hallmarks of linear theories which does not hold good in nonlinear theories because of the nonlinear term. For example, even if two solutions are known for a nonlinear theory, their superposition is in general not a solution of the nonlinear theory. The purpose of this letter is to point out a kind of superposition for a large number of nonlinear equations. In particular, there are several nonlinear equations, both discrete and continuum, which are known to admit exact periodic solutions in terms of Jacobi elliptic functions $\text{cn}(x, m)$ as well as $\text{dn}(x, m)$, where $m$ denotes the modulus of the elliptic function $[1]$. Many of these solutions have found application in several areas of physics $[2, 3]$. In particular, we have examined a large number of nonlinear equations, both continuum and discrete, which admit both $\text{cn}(x, m)$ and $\text{dn}(x, m)$ solutions and find that in all these cases even $\text{dn}(x, m) \pm \sqrt{m} \text{cn}(x, m)$ is also an exact solution. We have also examined a number of coupled field theories and have obtained superposed coupled solutions of the form $\text{dn} \pm \sqrt{m} \text{cn}$ in both the fields.

Further, we have also examined a number of continuum field theories which admit $\text{dn}^2(x, m)$ as a solution and find that such theories also admit $\text{dn}^2(x, m) \pm \sqrt{m} \text{cn}(x, m) \text{dn}(x, m)$ as solutions. While this cannot be treated as a linear superposition of two solutions (since $\text{cn}(x, m) \text{dn}(x, m)$ is not a solution of these models), we find it rather remarkable that such solutions exist, without exception, in a number of continuum field theories (including several coupled models) that we have looked at so far.

In this letter we only discuss a few selected examples, several other examples will be given elsewhere in a longer version. To begin with, we discuss one continuum (quadratic-cubic nonlinear Schrödinger equation or QCNLS) and one discrete model (saturated discrete nonlinear Schrödinger equation or DNLS) both of which admit $\text{cn}(x, m)$ as well as $\text{dn}(x, m)$ as solutions and show that these models also admit $\text{dn}(x, m) \pm \sqrt{m} \text{cn}(x, m)$ as solutions. Note that stable kink $[4]$ and chaotic soliton solutions under nonlinearity management $[5]$ are known for the QCNLS equation which arises in such diverse physical systems as nonlinear optics, chemical kinetics, matter-radiation interactions and mathematical ecology $[4, 5]$. The DNLS equation arises in the context of optical fibre communication $[6]$. We then discuss a coupled NLS-MKdV model and show that it has $\text{dn}(x, m) \pm \sqrt{m} \text{cn}(x, m)$ type coupled solutions in both the fields. Here MKdV refers to the modified KdV equation $[2]$.

Subsequently, we discuss the Korteweg-de Vries (KdV) equation which is known to admit $\text{dn}^2(x, m)$ as
a solution and show that this model also admits $\text{dn}^2(x, m) \pm \sqrt{m} \text{cn}(x, m) \text{dn}(x, m)$ as solutions even though $\text{cn}(x, m) \text{dn}(x, m)$ is not a solution of the KdV equation. We also show that the coupled NLS-MKdV model, not only admits $\text{dn}(x, m) \pm \sqrt{m} \text{cn}(x, m)$ but also $\text{dn}^2(x, m) \pm \sqrt{m} \text{cn}(x, m) \text{dn}(x, m)$ type solutions in both the fields. Further, we show that a coupled NLS-KdV model has superposed solutions of the form $\text{dn}(x, m) \pm \sqrt{m} \text{cn}(x, m) - \text{dn}^2(x, m)$.

Finally, we briefly mention possible reasons why such a linear superposition works in several nonlinear theories.

**dn $\pm \sqrt{m} \text{cn}$ as Exact Solutions**

We first discuss the quadratic-cubic nonlinear Schrödinger equation (QCNLSE) and then the discrete NLS equation (DNLSE) and show the existence of such superposed solutions in both the cases.

**QCNLSE:** We start with the quadratic-cubic nonlinear Schrödinger equation

$$iu_t + u_{xx} + g_1 |u|u + g_2 |u|^2u = 0.$$  \hspace{1cm} (1)

One of the exact moving periodic solutions to this equation is

$$u = (A \text{dn} [\beta (x - vt + \delta_1), m] + B) \exp[-i(\omega t - kx + \delta)],$$  \hspace{1cm} (2)

provided

$$g_2 A^2 = 2 \beta^2, \quad g_2 B^2 = (2 - m) \beta^2, \quad g_1 = -3Bg_2, \quad \omega = k^2 + 2(2 - m) \beta^2, \quad v = 2k.$$  \hspace{1cm} (3)

Here $\delta, \delta_1$ are two arbitrary constants.

Similarly, another exact moving periodic solution to the QCNLSE, Eq. (1), is

$$u = (A \sqrt{m} \text{cn} [\beta (x - vt + \delta_1), m] + B) \exp[-i(\omega t - kx + \delta)],$$  \hspace{1cm} (4)

provided

$$g_2 A^2 = 2 \beta^2, \quad g_2 B^2 = (2m - 1) \beta^2, \quad g_1 = -3Bg_2, \quad \omega = k^2 + 2(2m - 1) \beta^2, \quad v = 2k.$$  \hspace{1cm} (5)

Notice that the $\text{cn}(x, m)$ solution only exists if $1/2 < m \leq 1$. We now show that, remarkably, even though $\text{cn}(x, m)$ solution does not exist if $m \leq 1/2$, a linear superposition of $\text{cn}(x, m)$ and $\text{dn}(x, m)$ is still an exact solution over the entire range $0 < m \leq 1$, i.e.

$$u = \left( \frac{A}{2} \text{dn} [\beta (x - vt + \delta_1), m] + \frac{D}{2} \sqrt{m} \text{cn} [\beta (x - vt + \delta_1), m] + B \right) \exp[-i(\omega t - kx + \delta_2)],$$  \hspace{1cm} (6)
is an exact solution to the QCNLSE, Eq. (1), provided

\[ D = \pm A, \quad g_2 A^2 = 2\beta^2, \quad g_2 B^2 = (1 + m)/2\beta^2, \]
\[ g_1 = -3B g_2, \quad \omega = k^2 + (1 + m)\beta^2, \quad v = 2k. \quad (7) \]

Here \( cn(x, m) \) and \( sn(x, m) \) are periodic functions with period \( 4K(m) \), \( dn(x, m) \) is a periodic function with period \( 2K(m) \), with \( K(m) \) being the complete elliptic integral of the first kind [1]. It is worth noting that the frequency \( \omega \) of the three solutions (i.e. \( cn, dn \) and \( dn \pm \sqrt{m} cn \)) is different except at \( m = 1 \). We thus have two new periodic solutions of QCNLSE depending on if \( D = A \) or \( D = -A \).

Few remarks are in order here which are in fact valid for all the models (both continuum and discrete) that admit such solutions.

1. Both the solutions \( dn + \sqrt{m} cn \) and \( dn - \sqrt{m} cn \) exist for the same values of the parameters.

2. In the limit \( m = 1 \), the three solutions \( dn, cn \) as well as \( dn + \sqrt{m} cn \) go over to the well known pulse (i.e. sech) solution.

3. In all the continuum models admitting such solutions, the factors of \( 2 - m \) or \( 2m - 1 \) which appear in the \( dn \) and \( cn \) solutions, get replaced by the factor of \( (1 + m)/2 \) in the \( dn \pm \sqrt{m} cn \) solutions.

4. On the other hand, in the discrete theories admitting such solutions (see below), the factor of \( dn(\beta, m) \) or \( cn(\beta, m) \) appearing in \( dn \) and \( cn \) solutions, gets replaced by a factor of \( [dn(\beta, m) + cn(\beta, m)]/2 \) in the \( dn \pm \sqrt{m} cn \) solutions.

5. In view of the above two points, either the frequency \( \omega \) or the velocity \( v \) (or could be even both) are different for \( dn, cn \) and \( dn \pm \sqrt{m} cn \) solutions except at \( m = 1 \).

**Saturated Discrete NLS Equation:** We now consider a discrete nonlinear equation which admits both \( dn \) and \( cn \) solutions and show that it also has superposed solutions \( dn \pm \sqrt{m} cn \). In particular, we consider saturated DNLS equation which has received great attention in the context of optical fibre communication [6]

\[ i du_n/dt + [u_{n+1} + u_{n-1}] + \frac{\nu |u_n|^2}{1 + |u_n|^2} u_n = 0. \quad (8) \]
One well known periodic solution to this equation is

\[ u_n = A \text{dn}[\beta(n + \delta_1), m] e^{-i(\omega t + \delta)} , \]

(9)

provided

\[ A^2 \text{cs}^2(\beta, m) = 1, \ \beta = \frac{2K(m)}{N_p}, \ \omega = -\nu = -2 \frac{\text{dn}(\beta, m)}{\text{cn}^2(\beta, m)} . \]

(10)

The other solution is

\[ u_n = A \sqrt{m} \text{cn}[\beta(n + \delta_1), m] e^{-i(\omega t + \delta)} , \]

(11)

provided

\[ A^2 \text{ds}^2(\beta, m) = 1, \ \beta = \frac{4K(m)}{N_p}, \ \omega = -\nu = -2 \frac{\text{cn}(\beta, m)}{\text{dn}^2(\beta, m)} . \]

(12)

Remarkably, even a linear superposition of the two is also an exact solution to the saturable DNLS Eq. [8], i.e. it is easily shown using the recently derived identities for the Jacobi elliptic functions [8] that

\[ u_n = \left( \frac{A}{2} \text{dn}[\beta(n + \delta_1), m] + \frac{B}{2} \sqrt{m} \text{cn}[\beta(n + \delta_1), m] \right) e^{-i(\omega t + \delta)} , \]

(13)

is also an exact solution to Eq. [8] provided

\[ B = \pm A, \ A^2 [\text{cs}(\beta, m) + \text{ds}(\beta, m)]^2 = 4, \ \beta = \frac{4K(m)}{N_p}, \ \omega = -\nu = -\frac{4}{\text{dn}(\beta, m) + \text{cn}(\beta, m)} . \]

(14)

Here \( N_p \) is the spatial period of the system, \( \text{cs}(\beta, m) = \text{cn}(\beta, m)/\text{sn}(\beta, m) \) and \( \text{ds}(\beta, m) = \text{dn}(\beta, m)/\text{sn}(\beta, m) \).

As remarked earlier, observe that if we replace \( \text{dn}(\beta, m) \) and \( \text{cn}(\beta, m) \) by \( [\text{dn}(\beta, m) + \text{cn}(\beta, m)]/2 \) in relations (10) and (12), we recover relations (13). Further, the frequency \( \omega \) of the three solutions \( \text{dn}, \text{cn} \) and \( \text{dn} \pm \sqrt{m} \text{cn} \) is different except at \( m = 1 \).

**Coupled NLS-MKdV Model:** We now show that the coupled NLS-MKdV model admits \( \text{dn} \pm \sqrt{m} \text{cn} \) type solution in both the fields.

In this case the field equations are given by

\[ iu_t + u_{xx} + gu|u|^2u + \alpha uv^2 = 0, \]

\[ v_t + v_{xxx} + 6v^2v_x + \gamma v(|u|^2)_x = 0 . \]

(15)
These coupled equations admit four solutions with $u$ being either $cn$ or $dn$ (multiplied by an exponential) and similarly $v$ can be either $cn$ or $dn$. For example, one of the solutions is

$$u(x, t) = A \exp[-i(\omega t - kx + \delta)] \text{dn}[\beta(x - ct + \delta_1), m],$$

$$v(x, t) = B \sqrt{m} \text{cn}[\beta(x - ct + \delta_1), m],$$

provided

$$c = 2k = (2m - 1)\beta^2, \quad \omega = k^2 - (2 - m)\beta^2 + (1 - m)\alpha B^2,$$

$$gA^2 + \alpha B^2 = 2\beta^2, \quad \gamma A^2 + 3B^2 = 3\beta^2.$$  

Remarkably, even a linear superposition

$$u(x, t) = \frac{1}{2} \exp[-i(\omega t - kx + \delta)] \left( A \text{dn}[\beta(x - ct + \delta_1), m] 
+ D \sqrt{m} \text{cn}[\beta(x - ct + \delta_1), m] \right),$$

$$v(x, t) = \frac{1}{2} \left( B \text{dn}[\beta(x - ct + \delta_1), m] 
+ F \sqrt{m} \text{cn}[\beta(x - ct + \delta_1), m] \right),$$

is an exact solution of coupled Eqs. (15) provided

$$c = 2k = (1 + m)\beta^2/2, \quad \omega = k - 2, \quad D = \pm A, \quad F = \pm B,$$

while $A, B$ are given by Eq. (18). Note that the signs of $D = \pm A$ and $F = \pm B$ are correlated.

$\text{dn}^2 \pm \sqrt{m} \text{cn} \text{dn}$ superposed Solutions

We now discuss one example of a continuum field theory which admits $\text{dn}^2$ as a solution and show that the same model also admits $\text{dn}^2 \pm \sqrt{m} \text{cn} \text{dn}$ as solutions, even though $\text{cn} \text{dn}$ is not a solution of the model.

KdV Equation: It is well known that one of the exact solutions to the KdV equation is

$$u_t + u_{xxx} + guu_x = 0,$$

is

$$u = A \text{dn}^2[\beta(x - vt + \delta_1), m],$$

provided

$$gA = 12\beta^2, \quad v = 4(2 - m)\beta^2.$$
Remarkably, even a linear superposition, i.e.
\[ u = \frac{1}{2} \left( \text{Adn}^2[\beta(x - vt + \delta_1), m] + B \sqrt{m} \text{cn}[\beta(x - vt + \delta_1), m] \text{dn}[\beta(x - vt + \delta_1), m] \right), \]  
(24)
is an exact solution to the KdV Eq. (21) provided
\[ B = \pm A, \quad gA = 12\beta^2, \quad v = (5 - m)\beta^2. \]  
(25)

We thus have two new periodic solutions of KdV Eq. (21) depending on if \( B = A \) or \( B = -A \).

Several remarks are in order here which are in fact valid for all the solutions of the form \( \text{dn}^2 \pm \sqrt{m} \text{cn} \text{dn} \).

1. We find that all the models which admit \( \text{dn}^2 \) as a solution, also admit \( \text{dn}^2 \pm \sqrt{m} \text{cn} \text{dn} \) as a solution even though \( \sqrt{m} \text{cn} \text{dn} \) is not a solution of these models and that both of the new solutions exist for the same values of the parameters.

2. In the limit \( m = 1 \), the two solutions \( \text{dn}^2 \) and \( \text{dn}^2 + \sqrt{m} \text{cn} \text{dn} \) go over to the well known pulse (i.e. \( \text{sech}^2 \)) solution while \( \text{dn}^2 - \sqrt{m} \text{cn} \text{dn} \) solution goes over to the vacuum solution \( u = 0 \).

3. The factors of \( 2 - m \) and \( \sqrt{1 - m + m^2} \) which appear in the \( \text{dn}^2 \) solution, get replaced by the factor of \( (5 - m)/4 \) and \( \sqrt{1 + 14m + m^2}/4 \), respectively, in the \( \text{dn}^2 \pm \sqrt{m} \text{cn} \text{dn} \) solutions (see below). As a result the velocity \( v \) or frequency \( \omega \) (or even both) for the solutions \( \text{dn}^2 \) and \( \text{dn}^2 \pm \sqrt{m} \text{cn} \text{dn} \) are different (except at \( m = 1 \)).

We now show that the same coupled NLS-MKdV model as given by Eq. (15), also admits \( \text{dn}^2 \) as well as \( \text{dn}^2 \pm \sqrt{m} \text{cn} \text{dn} \) type solutions in both the fields.

It is easily shown that the coupled system as given by Eq. (15) admits an exact solution
\[ u = \left( \text{Adn}^2[\beta(x - ct + \delta_1), m] + F \right) \exp[-i(\omega t - kx + \delta)], \]
\[ v = B \text{dn}^2[\beta(x - ct + \delta_1), m] + D, \]  
(26)
provided
\[ \alpha = 3/2, \quad \gamma A^2 = -3B^2, \quad \gamma = 2g < 0, \quad (z - y)B^2 = 2\beta^2, \]
\[ c = 2k = 4[(2 - m) + 3z]\beta^2, \quad \omega = k^2 - [4(2 - m) + 9y + 3z]\beta^2, \]
\[ z = \frac{D}{B}, \quad y = \frac{F}{A} = \frac{[-(2 - m) \pm \sqrt{1 - m + m^2}]}{3}. \]  
(27)
Note that $A\text{dn}^2 + B$ is not a solution of either the NLS or MKdV uncoupled field equations even though the coupled system admits such a solution.

Remarkably, even a linear superposition, i.e.

$$u = \left( F + \frac{A}{2}\text{dn}^2[\beta(x - ct + \delta_1), m] \right)$$
$$+ \frac{G}{2}\sqrt{m}\text{cn}[\beta(x - ct + \delta_1), m]\text{dn}[\beta(x - ct + \delta_1), m] \exp[-i(\omega t - kx + \delta)],$$
$$v = D + \frac{B}{2}\text{dn}^2[\beta(x - ct + \delta_1), m]$$
$$+ \frac{H}{2}\sqrt{m}\text{cn}[\beta(x - ct + \delta_1), m]\text{dn}[\beta(x - ct + \delta_1), m],$$

is an exact solution to Eq. (15) provided

$$G = \pm A, \quad H = \pm B, \quad c = 2k = [5 - m + 12z]\beta^2,$$
$$\omega = k^2 - [(5 - m) + 9y + 3z]\beta^2, \quad y = \frac{F}{A} = \frac{-(5 - m) \pm \sqrt{1 + 14m + m^2}}{12},$$

while rest of the relations are exactly the same as those given by Eq. (27). Note that the signs of $G = \pm A$ and $H = \pm B$ are correlated. It is worth reminding once again that neither $A\text{dn}^2 + B$ nor $A\sqrt{m}\text{cn}\text{dn}$ is an exact solution of either NLS or MKdV field equations.

**Coupled NLS-KdV Fields:** Finally, let us consider the following coupled KdV-NLS field equations

$$iu_t + u_{xx} + g|u|^2u + \alpha uv = 0,$$
$$v_t + v_{xxx} + 6vv_x + \gamma v(|u|^2)_x = 0,$$

where $u$ and $v$ are the NLS and KdV fields, respectively. These equations admit coupled solutions of the form $\text{dn} - \text{dn}^2$ and $\text{cn} - \text{dn}^2$. For example, it is easily checked that

$$u(x, t) = A\exp[-i(\omega t - kx + \delta_1)]\text{dn}[\beta(x - ct + \delta), m],$$
$$v(x, t) = B\text{dn}^2[\beta(x - ct + \delta_1), m] + D,$$

is an exact solution to the coupled Eq. (30) provided

$$gA^2 + \alpha B = 2\beta^2, \quad \gamma A^2 + 6B = 12\beta^2,$$
Remarkably, the same model also admits interesting superposed solutions of the form $\text{dn} \pm \sqrt{m} \text{cn} - \text{dn}^2 \pm \sqrt{m} \text{cn} \text{dn}$. In particular, it is easily checked that

$$
\begin{align*}
\text{u}(x, t) &= \frac{1}{2} \exp[-i(\omega t - kx + \delta)] \left( A \text{dn}[\beta(x - ct + \delta_1), m] \\
&+ B \text{cn}[\beta(x - ct + \delta_1), m] \right), \\
\text{v}(x, t) &= \frac{B}{2} \text{dn}^2[\beta(x - ct + \delta_1), m] \\
&+ \frac{F}{2} \sqrt{m} \text{cn}[\beta(x - ct + \delta_1), m] \text{dn}[\beta(x - ct + \delta_1), m] + D,
\end{align*}
$$

is an exact solution to the coupled field equations (30) provided

$$
\begin{align*}
z &= \frac{D}{B}, \quad H = \pm A, \quad F = \pm B, \quad c = 2k = [(5 - m) + 12z]^{\beta^2}, \\
\omega &= k^2 - \frac{(1 + m)\beta^2}{2} - \frac{\alpha}{4}[(1 - m) + 4z]B,
\end{align*}
$$

and further $A, B$ are again given by Eq. (32).

**Conclusion.** In this letter we have shown that a kind of linear superposition holds good in the case of quadratic-cubic NLSE and saturated discrete NLS equations in the sense that these models not only admit $\text{dn}(x, m)$, $\text{cn}(x, m)$ but even $\text{dn}(x, m) \pm \sqrt{m} \text{cn}(x, m)$ type superposed solutions. In fact, by now we have examined a large number of both discrete and continuum nonlinear equations such as NLS, MKdV, mixed KdV-MKdV system, $\lambda \phi^4$ field theory, Ablowitz-Ladik equation, saturable discrete NLSE, discrete $\lambda \phi^4$ field theories, among others, and in all these cases we have obtained such superposed solutions. Furthermore, even in several coupled theories such as NLS-MKdV we have obtained $\text{dn}(x, m) \pm \sqrt{m} \text{cn}(x, m)$ type coupled solution in both the fields.

Further, we have shown that for KdV equation, which is known to admit a periodic solution in terms of $\text{dn}^2(x, m)$, also admits a solution of the form $\text{dn}^2(x, m) \pm \sqrt{m} \text{cn}(x, m) \text{dn}(x, m)$. By now, we have examined a number of nonlinear equations like quadratic NLSE and in all these cases we have obtained such solutions. Furthermore, in several coupled theories like NLS-MKdV, quadratic NLS-KdV, and others too we have obtained $\text{dn}^2(x, m) \pm \sqrt{m} \text{cn}(x, m) \text{dn}(x, m)$ type solutions. We have also looked at the NLS-KdV coupled system and obtained mixed solutions of the form $\text{dn}(x, m) \pm \sqrt{m} \text{cn}(x, m) - \text{dn}^2(x, m) \pm \sqrt{m} \text{cn}(x, m) \text{dn}(x, m)$. 

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In addition, we have examined several other coupled equations such as MKdV-KdV, NLS-quadratic NLS, MKdV-quadratic NLS, among others, and there too we have obtained such superposed solutions.

Many of the nonlinear equations that we have looked at have found wide application in several interesting physical situations [1,2,3,6,7] including fibre optics, chemical kinetics and mathematical ecology. It would be worthwhile to enquire if the new solutions that we have obtained have some physical relevance, e.g., in optical fibre communication and related contexts [6]. As a first step in that direction, it is important to examine the (linear and nonlinear) stability of these newly found solutions in the various models.

We would like to reemphasize that what we have obtained is only a kind of linear superposition and not the full superposition as obtained in the linear theories. However, we find it remarkable that in spite of the nonlinear terms, even a kind of linear superposition holds good in these nonlinear models.

What could be the possible reason why such a linear superposition is possible? We surmise mathematically that the main reason is that $cn(x, m)$ and $dn(x, m)$ functions are quite similar and both of them as well as their derivatives are identical at $m = 1$. This is in contrast to the $sn(x, m)$ elliptic function which is different from both $cn(x, m)$ and $dn(x, m)$ functions at any value of $m$ and that is why $cn(x, m) \pm sn(x, m)$ superposition does not seem to work. We have checked it in the various examples mentioned above. Perhaps there is a deeper physical reason to all these findings which needs to be explored.

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