The $\mathbb{F}$-valued points of the algebra of strongly regular functions of a Kac-Moody group

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Abstract

Let $G_m$ resp. $G_f$ be the minimal resp. formal Kac-Moody group, associated to a symmetrizable generalized Cartan matrix, over a field $\mathbb{F}$ of characteristic 0. Let $\mathbb{F}[G_m]$ be the algebra of strongly regular functions on $G_m$.

We denote by $\widehat{G}_m$ resp. $\widehat{G}_f$ certain monoid completions of $G_m$ resp. $G_f$, build by using the faces of the Tits cone.

We show that there is an action of $\widehat{G}_f \times \widehat{G}_f$ on the spectrum of $\mathbb{F}$-valued points of $\mathbb{F}[G_m]$. As a $\widehat{G}_f \times \widehat{G}_f$-set it can be identified with a certain quotient of the $\widehat{G}_f \times \widehat{G}_f$-set $\widehat{G}_f \times \widehat{G}_f$, build by using $\widehat{G}_m$.

We prove a Birkhoff decomposition for the $\mathbb{F}$-valued points of $\mathbb{F}[G_m]$.

We describe the stratification of the spectrum of $\mathbb{F}$-valued points of $\mathbb{F}[G_m]$ in $G_f \times G_f$-orbits. We show that every orbit can be covered by suitably defined big cells.

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Introduction

The minimal Kac-Moody group $G_m$, which V. Kac and D. Peterson associated in $\mathbb{K} \mathbb{P} 2$ to a Kac-Moody algebra $\mathfrak{g}$ over a field $\mathbb{F}$ of characteristic 0, is a group analogue of a semisimple simply connected algebraic group.

For a symmetrizable minimal Kac-Moody group, Kac and Peterson defined and investigated in $\mathbb{K} \mathbb{P} 2$ the algebra of strongly regular functions $\mathbb{F}[G_m]$ on $G_m$. This algebra has many properties in common with the coordinate ring of a semisimple simply connected algebraic group. It is an integrally closed domain,
even a unique factorization domain. It admits a Peter and Weyl theorem, i.e.,

\[ \mathbb{F} [G_m] \cong \bigoplus_{\Lambda \in P^+} L^*(\Lambda) \otimes L(\Lambda) \]

as \( G_m \times G_m \)-modules. But the following things, which hold in the non-classical case, are different:

1) Assigning to every element of \( G_m \) its point evaluation, \( G_m \) embeds in the set of \( \mathbb{F} \)-valued points of \( \mathbb{F} [G_m] \), which we denote by \( \text{Specm} \mathbb{F} [G_m] \). But this map is not surjective.

2) There exists no comultiplication of \( \mathbb{F} [G_m] \), dual to the multiplication of \( G_m \). The situation is not too bad, left and right multiplications with elements of \( G_m \) induce comorphisms. A more serious difference, the inverse map of \( G_m \) does not induce a comorphism.

In particular there is no natural group structure, even no natural monoid structure on \( \text{Specm} \mathbb{F} [G_m] \).

Kac and Peterson posed the problem to determine \( \text{Specm} \mathbb{F} [G_m] \), or at least a certain part of it, \( [K,P 2] \).

The tensor category \( \mathcal{O}_{adm} \) of admissible modules of \( \mathcal{O} \) generalizes the category of finite dimensional representations of a semisimple Lie algebra, keeping the complete reducibility theorem.

In a way similar to the reconstruction process of the Tannaka-Krein duality, a suitable category of representations of a Lie algebra, together with a suitable category of duals, determines a monoid with coordinate ring. In some sense, this monoid is the biggest monoid acting reasonably on the representations. The coordinate ring is a coordinate ring of matrix coefficients.

We determined and investigated in [M 1] the monoid \( \widehat{G}_m \) corresponding to \( \mathcal{O}_{adm} \) and its category of restricted duals. Equipped with its coordinate ring \( \mathbb{F} [\widehat{G}_m] \) of matrix coefficients, the monoid \( \widehat{G}_m \) contains \( G_m \) as Zariski open, dense unit group. It has similar properties as a reductive algebraic monoid. But it is a purely non-classical phenomenon, its classical analogue is a semisimple, simply connected algebraic group. For generalizing results of classical invariant theory, this monoid is more fundamental than the Kac-Moody group itself. For its history in connection with Slodowy and Peterson we refer to the introduction of [M 1].

The coordinate ring \( \mathbb{F} [\widehat{G}_m] \) is isomorphic to the algebra of strongly regular functions \( \mathbb{F} [G_m] \) by the restriction map. Therefore the monoid \( \widehat{G}_m \) embeds in \( \text{Specm} \mathbb{F} [G_m] \), but also this map is not surjective.

To investigate the \( \mathbb{F} \)-valued points of \( \mathbb{F} [G_m] \), we define and investigate a monoid \( \widehat{G}_f \), which is build in a similar way as \( \widehat{G}_m \), but the minimal Kac-Moody group \( G_m \) replaced by the formal Kac-Moody group \( G_f \). In a subsequent paper we will prove that this monoid corresponds to \( \mathcal{O}_{adm} \) and its category of full duals.

In this paper we obtain the following description of \( \text{Specm} \mathbb{F} [G_m] \): We get, in a natural way, an action \( \pi \) of \( \widehat{G}_f \times \widehat{G}_f \) on \( \mathbb{F} [G_m] \) by homomorphisms of algebras.
Therefore we also obtain a $\tilde{G}_f \times \tilde{G}_f$-action on the spectrum of $\mathbb{F}$-valued points of $\mathbb{F}[G_m]$ from the right. Composing the evaluation map at the unit of $G_m$ with $\pi$, we get a $\tilde{G}_f \times \tilde{G}_f$-equivariant map

$$\circ : \tilde{G}_f \times \tilde{G}_f \to \text{Specm} \mathbb{F}[G_m]$$

$$(x, y) \mapsto x \circ y .$$

We show that this map factors to a $\tilde{G}_f \times \tilde{G}_f$-equivariant bijection with a quotient set of $\tilde{G}_f \times \tilde{G}_f$, which is obtained as follows: The Chevalley involution of $G_m$ extends to an involution $*$ of $\hat{G}_m$. We factor $\tilde{G}_f \times \tilde{G}_f$ by the $\hat{G}_f \times \hat{G}_f$-equivariant equivalence relation generated by

$$(x, zy) \sim (z^* x, y) , \quad x, y \in \hat{G}_f , \quad z \in \hat{G}_m .$$

Due to this description, we can use structural properties of $\hat{G}_m$ and $\hat{G}_f$ to prove properties of the $\tilde{G}_f \times \tilde{G}_f$-space Specm $\mathbb{F}[G]$. In particular we prove the Birkhoff decomposition

$$\text{Specm} \mathbb{F}[G_m] = \bigcup_{w \in \hat{W}} B_f \circ \hat{w} B_f .$$

Here $B_f$ is the formal Borel group, and the Weyl monoid $\hat{W}$ is a certain monoid containing the Weyl group.

We determine the stratification of Specm $\mathbb{F}[G]$ in $G_f \times G_f$-orbits:

$$\text{Specm} \mathbb{F}[G_m] = \bigcup_{\Theta \text{ special}} G_f \circ e(R(\Theta)) G_f .$$

Here $\{ e(R(\Theta)) | \Theta \text{ special } \}$ is a finite set of certain idempotents of $\hat{G}_m$. We show that each orbit is locally closed and irreducible. We determine the closure relation of the orbits. We describe the big cell $B_f \circ e(R(\Theta)) B_f$ of the orbit $G_f \circ e(R(\Theta)) G_f$, and also the covering of $G_f \circ e(R(\Theta)) G_f$ by this big cell. We give stratified transversal slices to the orbits.

The following work is in relation to these results:

Let $(M, C[M])$ be a connected reductive algebraic monoid. Denote by $G$ its reductive unit group. Let $T$ be a maximal torus of $G$. Let $B$ and $B^-$ be opposite Borel subgroups containing $T$.

Assigning to every element of $M$ its point evaluation, $M$ identifies with the $C$-valued points of $C[M]$. L. Renner showed in [Re 2], that $M$ admits Bruhat decompositions. Due to the existence of a longest element of the Weyl group, these are equivalent to the Birkhoff decompositions. In particular $M = \bigcup_{r \in R} B^- r B$, where $R$ is the Renner monoid. Due to the work of M. Putcha and L. Renner in [Pu 2], [Re 1] is the decomposition $M = \bigcup_{\lambda \in \Lambda} G \lambda G$. The set $\Lambda$ is a certain set of idempotents, a cross-section lattice, which has been introduced by Putcha in [Pu 1]. Renner
found and described in [Re 2] the big cell $B^eB$ in $GeG$.

The full spectrum of the algebra of strongly regular functions has been used by M. Kashiwara in [Kas] for his infinite dimensional algebraic geometric approach to the flag variety of a Kac-Moody group. In contrary to Kac and Peterson, he constructs the algebra of strongly regular functions without using the minimal Kac-Moody group. He uses the coalgebra structure of the universal enveloping algebra of $g$, to construct the algebra of strongly regular functions as a certain subalgebra of the corresponding dual algebra. Kashiwara defines an open subscheme of the full spectrum, which has a countable covering by suitably defined big cells. To obtain his flag variety, Kashiwara factors the subscheme by the action of $B_f \times \{1\}$.

Taking $F = \mathbb{C}$, there exists a real unitary form $K$ of $G_m$, which coincides with the compact form in the classical case. D. Pickrell conjectures in [Pic] certain $K$-biinvariant resp. $K$-invariant measures for Kac-Moody groups. He proves the existence of these measures in the affine case.

Important for the construction of the biinvariant measures is a $G_f \times G_f$-space $G_f \times_{G_m} G_f$. He equips this space with a proalgebraic complex manifold structure, using a covering of big cells as an atlas. He also equips $G_f \times_{G_m} G_f$ with an algebra of matrix coefficients, isomorphic to the algebra of strongly regular functions. Important for the construction of the invariant measures is the flag $G_f$-space $\{1\} \times B_f \backslash G_f \times_{G_m} G_f$, also equipped with a proalgebraic manifold structure.

It is not difficult to see, that the $\mathbb{C}$-valued points of the subscheme of Kashiwara, as well as the $G_f \times G_f$-space $G_f \times_{G_m} G_f$ of Pickrell identify with the biggest $G_f \times G_f$-orbit of Specm $\mathbb{C}[G_m]$. In a subsequent paper, we will investigate the completed flag varieties in the setting of Kashiwara and in the setting of Pickrell. Presumably the extended Bruhat order of [M 2] will be important.

Pickrell also proved a Birkhoff decomposition for $G_f \times_{G_m} G_f$. He noted, that in the affine case, many interesting completions of $G_m$ corresponding to loop groups are embedded in the set $G_f \times_{G_m} G_f$. The Birkhoff decomposition of $G_f \times_{G_m} G_f$ induces the Birkhoff decomposition of these completions.

Now certain functional analytical closures, for example the closure used by Peterson in [K,P 4] for his KAK-decomposition, are also embedded in the full set Specm $\mathbb{C}[G_m]$. Hopefully this will help to make their structure more explicit.

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1 Preliminaries

In this section we collect some basic facts about Kac-Moody algebras, minimal and formal Kac-Moody groups, the algebra of strongly regular functions, and the corresponding monoid completion, which are used later.

One aim is to introduce our notation. Another aim is to put these things, which can be found in the literature, on equal footing appropriate for our goals.

The minimal Kac-Moody group, given in \([K,P 1], [K,P 3]\), corresponds to the derived Kac-Moody algebra. We work with a slightly enlarged group corresponding to the full Kac-Moody algebra as in \([K,P 1], [Mo, Pi]\). The algebra of strongly regular functions on this group is slightly larger, than the algebra of \([K,P 2]\). We introduce this algebra as the restricted coordinate ring of a monoid.

The formal Kac-Moody group has been constructed in \([Sl]\), starting with a realization, glueing parabolic subgroups of finite type, which are equipped with a proalgebraic structure. We only need the formal Kac-Moody group corresponding to a simply connected minimal free realization, and we introduce this group by a representation theoretic construction.

All the material stated in this subsection about Kac-Moody algebras can be found in the books \([K]\) (most results also valid for a field of characteristic zero with the same proofs), \([Mo, P1]\), about the minimal Kac-Moody group in \([K,P 1], [K,P 3]\), \([Mo, P1]\), about the formal Kac-Moody group in \([Sl]\), about the algebra of strongly regular functions in \([K,P 2]\), about the faces of the Tits cone in \([Loo, Sl]\), \([M 1]\), and about the monoid completion of the minimal Kac-Moody group in \([M 1]\).

We denote by \(\mathbb{N} = \mathbb{Z}^+, \mathbb{Q}^+, \) resp. \(\mathbb{R}^+\) the sets of strictly positive numbers of \(\mathbb{Z}, \mathbb{Q}, \) resp. \(\mathbb{R},\) and the sets \(\mathbb{N}_0 = \mathbb{Z}_0^+, \mathbb{Q}_0^+, \mathbb{R}_0^+\) contain, in addition, the zero.

In the whole paper, \(\mathbb{F}\) is a field of characteristic 0 and \(\mathbb{F}^\times\) its group of units.

**Generalized Cartan matrices:** Starting point for the construction of a Kac-Moody algebra, and its associated simply connected minimal and formal Kac-Moody groups is a *generalized Cartan matrix*, which is a matrix \(A = (a_{ij}) \in M_n(\mathbb{Z})\) with \(a_{ii} = 2, a_{ij} \leq 0\) for all \(i \neq j,\) and \(a_{ij} = 0\) if and only if \(a_{ji} = 0.\) Denote by \(l\) the rank of \(A,\) and set \(I := \{1, 2, \ldots, n\} .\)

For the properties of the generalized Cartan matrices, in particular their classification, we refer to the book \([K]\). In this paper we assume \(A\) to be symmetriz-
Realizations: A simply connected minimal free realization of $A$ consists of dual free $\mathbb{Z}$-modules $H$, $P$ of rank $2n - l$, and linear independent sets $\Pi_\vee = \{h_1, \ldots, h_n\} \subseteq H$, $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subseteq P$ such that $\alpha_i(h_j) = a_{ji}$, $i, j = 1, \ldots, n$. Furthermore there exist (non-uniquely determined) fundamental dominant weights $\Lambda_1, \ldots, \Lambda_n \in P$ such that $\Lambda_i(h_j) = \delta_{ij}$, $i, j = 1, \ldots, n$. $P$ is called the weight lattice, and $Q := \mathbb{Z}$-span $\{\alpha_i \mid i \in I\}$ the root lattice. Set $Q_0^+ := \mathbb{Z}_0^+$-span $\{\alpha_i \mid i \in I\}$, and $Q^\perp := Q_0^\perp \setminus \{0\}$.

We fix a system of fundamental dominant weights $\Lambda_1, \ldots, \Lambda_n$, and extend $h_1, \ldots, h_n \in H$, $\Lambda_1, \ldots, \Lambda_n \in P$ to a pair of dual bases $h_1, \ldots, h_{2n-l} \in H$, $\Lambda_1, \ldots, \Lambda_{2n-l} \in P$. We set $H_{\text{rest}} := \mathbb{Z}$-span $\{h_i \mid i = n + 1, \ldots, 2n - l\}$.

The Weyl group, the Tits cone and its faces: Identify $H$ and $P$ with the corresponding sublattices of the following vector spaces over $\mathbb{F}$:

$$h := h_\mathbb{F} := H \otimes_\mathbb{Z} \mathbb{F} \quad , \quad h^* := h_\mathbb{F}^* := P \otimes_\mathbb{Z} \mathbb{F}.$$ 

$h^*$ is interpreted as the dual of $h$. Order the elements of $h^*$ by $\lambda \leq \lambda'$ if and only if $\lambda' - \lambda \in Q_0^+$. Choose a symmetric matrix $B \in M_n(\mathbb{Q})$ and a diagonal matrix $D = \text{diag}(\epsilon_1, \ldots, \epsilon_n)$, $\epsilon_1, \ldots, \epsilon_n \in Q^\perp$, such that $A = DB$. Define a nondegenerate symmetric bilinear form on $h$ by:

$$(h_i \mid h) = (h \mid h_i) := \alpha_i(h) \epsilon_i \quad i \in I, \quad h \in h,$n

$$(h' \mid h'') := 0 \quad h', h'' \in h_{\text{rest}} := H_{\text{rest}} \otimes \mathbb{F}.$$ 

Denote the induced nondegenerate symmetric form on $h^*$ also by $(\mid )$.

The Weyl group $W = W(A)$ is the Coxeter group with generators $\sigma_i$, $i \in I$, and relations

$$\sigma_i^2 = 1 \quad (i \in I) \quad , \quad (\sigma_i \sigma_j)^{m_{ij}} = 1 \quad (i, j \in I, \ i \neq j).$$

The $m_{ij}$ are given by:

$$\begin{array}{cccc}
    a_{ij}a_{ji} & 0 & 1 & 2 & 3 & 4 & 6 \\
    m_{ij} & 2 & 3 & 4 & 6 & \text{no relation between } \sigma_i \text{ and } \sigma_j
\end{array}$$

The Weyl group $W$ acts faithfully and contragrediently on $h$ and $h^*$ by

$$\sigma_i h := h - \alpha_i(h) h_i \quad i \in I, \quad h \in h,$n

$$\sigma_i \lambda := \lambda - \lambda(h_i) \alpha_i \quad i \in I, \quad \lambda \in h^*,$n

leaving the lattices $H$, $Q$, $P$, and the forms invariant.

$\Delta_{\text{re}} := W\{\alpha_i \mid i \in I\}$ is called the set of real roots, and $\Delta_{\text{re}}^\vee := W\{h_i \mid i \in I\}$ the set of real coroots. The map $\alpha_i \mapsto h_i$, $i \in I$, can be extended to a $W$-equivariant bijection $\alpha \mapsto h_{\alpha}$.

To illustrate the action of $W$ on $h_{\mathbb{F}}^*$ geometrically, for $J \subseteq I$ define

$$F_J := \{\lambda \in h_{\mathbb{F}}^* \mid \lambda(h_i) = 0 \text{ for } i \in J, \ \lambda(h_i) > 0 \text{ for } i \in I \setminus J\} ,$$

$$\overline{F}_J := \{\lambda \in h_{\mathbb{F}}^* \mid \lambda(h_i) = 0 \text{ for } i \in J, \ \lambda(h_i) \geq 0 \text{ for } i \in I \setminus J\} .$$
\( \mathcal{T}_J \) is a finitely generated convex cone with relative interior \( F_J \). The parabolic subgroup \( \mathcal{W}_J \) of \( \mathcal{W} \) is the stabilizer of every element \( \lambda \in F_J \). For \( \sigma \in \mathcal{W} \) call \( \sigma F_J \) a facet of type \( J \).

The fundamental chamber \( \mathcal{C} := \{ \lambda \in h_\mathbb{R}^n \mid \lambda(h_i) \geq 0 \text{ for } i \in I \} \) is a fundamental region for the action of \( \mathcal{W} \) on the convex cone \( X := \mathcal{W} \mathcal{C} \), which is called the Tits cone. The partition \( \mathcal{C} = \bigcup_{J \subseteq I} F_J \) induces a \( \mathcal{W} \)-invariant partition of \( X \) into facets.

A set \( \Theta \subseteq I \) is called special, if either \( \Theta = \emptyset \), or else all connected components of the generalized Cartan submatrix \( (a_{ij})_{i,j \in \Theta} \) are of non-finite type. Set \( \Theta^\perp := \{ i \in I \mid a_{ij} = 0 \text{ for all } j \in \Theta \} \). Every face of the Tits cone \( X \) is \( \mathcal{W} \)-conjugate to exactly one of the faces

\[
R(\Theta) := X \cap \{ \lambda \in h_\mathbb{R}^n \mid \lambda(h_i) = 0 \text{ for all } i \in \Theta \} = \mathcal{W}_{\Theta^\perp} \mathcal{F}_\Theta \quad , \Theta \text{ special}.
\]

The parabolic subgroup \( \mathcal{W}_{\Theta^\perp} \) is the pointwise stabilizer of \( R(\Theta) \), and the parabolic subgroup \( \mathcal{W}_{\Theta^\perp,\Theta} \) is the stabilizer of the set \( R(\Theta) \) as a whole.

The relative interior of \( R(\Theta) \) is given by the union of the facets \( \sigma F_{\Theta^\perp,\Theta} \), where \( \sigma \in \mathcal{W}_{\Theta^\perp} \), and \( \Theta^\perp \) is a subset of \( \Theta^\perp \), which is either empty, or else for which all connected components of \( (a_{ij})_{i,j \in \Theta} \) are of finite type.

**The Kac-Moody algebra:** The Kac-Moody algebra \( g = g(A) \) is the Lie algebra over \( \mathbb{F} \) generated by the abelian Lie algebra \( h \) and \( 2n \) elements \( e_i, f_i, (i \in I) \), with the following relations, which hold for any \( i,j \in I, h \in h \):

\[
[e_i, f_j] = \delta_{ij} h_i , \quad [h, e_i] = \alpha_i(h) e_i , \quad [h, f_i] = -\alpha_i(h) f_i , \quad (ad e_i)^{1-\alpha_{ij}} e_j = (ad f_j)^{1-\alpha_{ij}} f_j = 0 \quad (i \neq j) .
\]

The Chevalley involution \( * : g \to g \) is the involutive anti-automorphism determined by \( e_i^* = f_i, f_i^* = e_i, h^* = h, (i \in I, h \in h) \).

The nondegenerate symmetric bilinear form \( (|) \) on \( h \) extends uniquely to a nondegenerate symmetric invariant bilinear form \( (|) \) on \( g \). We have the root space decomposition

\[
g = \bigoplus_{x \in h^*} g_x \quad \text{where} \quad g_x := \{ x \in g \mid [h, x] = \alpha(h) x \quad \text{for all} \quad h \in h \} .
\]

In particular \( g_0 = h \), \( g_{\alpha_i} = \mathbb{F} e_i \), and \( g_{-\alpha_i} = \mathbb{F} f_i \), \( i \in I \).

The set of roots \( \Delta := \{ \alpha \in h^* \setminus \{0\} \mid g_\alpha \neq \{0\} \} \) is invariant under the Weyl group, \( \Delta = -\Delta \), and \( \Delta \) spans the root lattice \( Q \). We have \( \Delta_{re} \subseteq \Delta \), and \( \Delta_{im} := \Delta \setminus \Delta_{re} \) is called the set of imaginary roots.

\( \Delta, \Delta_{re}, \text{and} \Delta_{im} \) decompose into the disjoint union of the sets of positive and negative roots \( \Delta^\pm := \Delta \cap Q^\pm \), \( \Delta_{re}^\pm := \Delta_{re} \cap Q^\pm \), \( \Delta_{im}^\pm := \Delta_{im} \cap Q^\pm \).

There is the triangular decomposition \( g = n^- \oplus h \oplus n^+ \), where \( n^\pm := \bigoplus_{\alpha \in \Delta^\pm} g_\alpha \).

**Irreducible highest weight representations:** For every \( \Lambda \in h^* \) there exists, unique up to isomorphism, an irreducible representation \( (L(\Lambda), \pi_\Lambda) \) of \( g \) with highest weight \( \Lambda \). It is \( h \)-diagonalizable, and we denote its set of weights by \( P(\Lambda) \).
Any such representation carries a nondegenerate symmetric bilinear form $\langle \cdot \mid \cdot \rangle : L(\Lambda) \times L(\Lambda) \to \mathbb{F}$ which is contravariant, i.e., $\langle v \mid wx \rangle = \langle x^*v \mid w \rangle$ for all $v, w \in L(\Lambda), x \in \mathfrak{g}$. This form is unique up to a nonzero multiplicative scalar.

The minimal and the formal Kac-Moody group: We say that a Lie algebra $\mathfrak{l}$ acts locally nilpotent on an $\mathfrak{l}$-module $V$, if for every $v \in V$, there exists a positive integer $m \in \mathbb{N}$, such that for all $x_1, x_2, \ldots, x_m \in \mathfrak{l}$ we have $x_1 x_2 \cdots x_m v = 0$.

Call a $\mathfrak{g}$-module $V$ m-admissible, if $V$ is $\mathfrak{h}$-diagonalizable with set of weights $P(V) \subseteq P$, and $\mathfrak{g}_\alpha$ acts locally nilpotent on $V$ for all $\alpha \in \Delta_{rc}$.

Examples are the adjoint representation $(\mathfrak{g}, \text{ad})$, and the irreducible highest weight representations $(L(\Lambda), \pi_\Lambda), \Lambda \in P^+ := P \cap C[\Lambda]$.

(Note that m-admissible is slightly different from integrable, which means $V$ is $\mathfrak{h}$-diagonalizable, and $\mathfrak{g}_\alpha$ acts locally nilpotent on $V$ for all $\alpha \in \Delta_{rc}$. The weights of an integrable module can be contained in $\{ \lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}, i = 1, \ldots, n \}$. If the generalized Cartan matrix is degenerate, then this set is no lattice.)

The minimal Kac-Moody group $G = G_m = G_m(A)$ can be characterized in the following way:

• The group $G$ acts on every m-admissible representation. Two elements $g, g' \in G$ are equal if and only if for all m-admissible modules $V$, and for all $v \in V$, we have $gv = g'v$.

• (1) For every $h \in H, s \in \mathbb{F}^\times$ there exists an element $t_h(s) \in G$, such that for any m-admissible representation $(V, \pi)$ we have

$$t_h(s)v_\Lambda = s^{\lambda(h)}v_\Lambda \ , \ v_\Lambda \in V_\Lambda \ , \ \Lambda \in P(V) .$$

(2) For every $x \in \mathfrak{g}_\alpha, \alpha \in \Delta_{rc}$, there exists an element $\exp(x) \in G$, such that for any m-admissible representation $(V, \pi)$ we have

$$\exp(x)v = \exp(\pi(x))v \ , \ v \in V .$$

$G$ is generated by the elements of (1) and (2).

Call a $\mathfrak{g}$-module $V$ f-admissible, if $V$ is m-admissible, and $\mathfrak{n}^+$ acts locally nilpotent on $V$.

Examples are the representations $(L(\Lambda), \pi_\Lambda), \Lambda \in P^+ = P \cap C[\Lambda]$.

Set $\mathfrak{n}_f := \prod_{\alpha \in \Lambda_{+}} \mathfrak{g}_\alpha$ and $\mathfrak{g}_f := \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}_f$. The Lie bracket of $\mathfrak{g}$ extends in the obvious way to a Lie bracket of $\mathfrak{g}_f$. Every f-admissible $\mathfrak{g}$-module can be extended to a $\mathfrak{g}_f$-module. The Lie algebra $\mathfrak{g}_f$ should be interpreted as the Lie algebra of the formal Kac-Moody group $G_f = G_f(A)$, which can be characterized in the following way:

• The group $G_f$ acts on every f-admissible representation. Two elements $g, g' \in G_f$ are equal if and only if for all f-admissible modules $V$, and for all $v \in V$ we have $gv = g'v$.

• (3) $G_f$ contains $G$.

(4) For every $x \in \mathfrak{n}_f$ there exists an element $\exp(x) \in G_f$, such that for any f-admissible representation $(V, \pi)$ we have

$$\exp(x)v = \exp(\pi(x))v \ , \ v \in V .$$
$G_f$ is generated by $G$ and the elements of (4).

The $g$-module $g_f$ is not $f$-admissible. Nevertheless $G_f$ acts on $g_f$, extending the adjoint action of $G$ on $g$, compare Section 5.11.

Both Kac-Moody groups act faithfully on $\bigoplus_{\lambda \in P^+} L(\lambda)$. They have the following important structural properties:

1) The elements of (1) induce an embedding of the torus $H \otimes \mathbb{Z}^\times$ into $G \subseteq G_f$. Its image is denoted by $T$.

For $\alpha \in \Delta_{re}$, elements of (2) induce an embedding of $(g_\alpha, +)$ into $G \subseteq G_f$. Its image $U_\alpha$ is called the root group belonging to $\alpha$.

Let $\alpha \in \Delta_{re}^+$ and $x_\alpha \in g_\alpha, x_{-\alpha} \in g_{-\alpha}$ such that $[x_\alpha, x_{-\alpha}] = h_\alpha$. There exists an injective homomorphism of groups $\phi_\alpha : \text{SL}(2, \mathbb{F}) \rightarrow G$ with

$$\phi_\alpha \left( \begin{array}{cc} 1 & s \\ 0 & 1 \end{array} \right) := \exp(s x_\alpha), \quad \phi_\alpha \left( \begin{array}{cc} 1 & 0 \\ s & 1 \end{array} \right) := \exp(s x_{-\alpha}), \quad (s \in \mathbb{F}^\times).$$

2) Denote by $N$ the subgroup generated by $T$ and $n_\alpha := \phi_\alpha \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$, $\alpha \in \Delta_{re}$. The Weyl group $W$ can be identified with the group $N/T$ by the isomorphism $\kappa : W \rightarrow N/T$, which is given by $\kappa(\sigma_\alpha) := n_\alpha(1)T, \alpha \in \Delta_{re}$.

We denote an arbitrary element $n \in N$ with $\kappa^{-1}(nT) = \sigma \in W$ by $n_\sigma$. The set of weights $P(V)$ of an $m$-admissible $g$-module $(V, \pi)$ is $W$-invariant, and $n_\sigma V_\lambda = V_{\sigma\lambda}, \lambda \in P(V)$.

3) Let $U^\pm$ be the subgroups generated by $U_\alpha, \alpha \in \Delta_{re}^\pm$. Let $U_f := \exp(n_f)$. Then $U^\pm$ and $U_f$ are normalized by $T$. Set

$$B^\pm := T \ltimes U^\pm, \quad B_f := T \ltimes U_f.$$ 

The pairs $(B^\pm, N)$ are twinned BN-pairs of $G$ with the property $B^+ \cap B^- = B^\pm \cap N = T$. The pair $(B_f, N)$ is a BN-pair of $G_f$ with $B_f \cap N = T$. We have the following decompositions, called Bruhat and Birkhoff decompositions:

$$G = \bigcup_{\sigma \in W} B^res B^\pm, \quad G_f = \bigcup_{\sigma \in W} B^res B_f^\pm, \quad \epsilon, \delta \in \{+, -, \}. $$

4) There are also Levi decompositions of the standard parabolic subgroups. In this paper we only use the corresponding decompositions for the groups $U^\pm$ and $U_f$:

Set $\Delta_{j}^{\pm} := \Delta_{\pm} \cap \sum_{j \in J} \mathbb{Z} \alpha_j$, and $(\Delta_{j}^{\pm}) := \Delta_{\neq} \setminus \sum_{j \in J} \mathbb{Z} \alpha_j$. Similarly define $(\Delta^{\pm})^{J}_{re}$ and $(\Delta^{\pm})^{J}_{\neq}$ by replacing $\Delta^{\pm}$ by $\Delta_{re}^{\pm}$. Set $(\mathbf{n}_j)^\pm := \bigoplus_{\alpha \in \Delta_{j}^{\pm}} g_\alpha$, $(\mathbf{n}_j)^J := \prod_{\alpha \in (\Delta_{j}^{\pm})^{J}_{re}} g_\alpha$, and $(\mathbf{n}_j)^J := \prod_{\alpha \in (\Delta_{j}^{\pm})^{J}_{\neq}} g_\alpha$. We have:

$$U^\pm = U_f^J \ltimes (U^J_f)^\pm, \quad U_f = (U_f)_J \ltimes (U^J_f)^J.$$ 

Here $U^J_f$ is the group generated by the root groups $U_\alpha, \alpha \in (\Delta^{\pm})^{J}_{re}$. $(U^J_f)^\pm$ is the smallest normal subgroup of $U^\pm$, containing the root groups $U_\alpha, \alpha \in (\Delta^{\pm})^{J}_{re}$.

This group equals $\bigcap_{\sigma \in W_J} \sigma U^\pm \sigma^{-1}$. Furthermore $(U_f)_J := \exp(n_f)_J$ and
\((U_f)^J := \exp((n_f)^J)\).

The derived minimal Kac-Moody group \(G'\) is identical with the Kac-Moody group as defined in [K,P 1]. It is generated by the root groups \(U_\alpha, \alpha \in \Delta_{re}\). We have \(G = G' \times T_{\text{rest}}\), where \(T_{\text{rest}} := H_{\text{rest}} \otimes \mathbb{Z} \mathbb{F}\) is a subtorus of \(T\). The group \(G_f\) is identical with the Kac-Moody group of [Sl] for a simply connected minimal free realization.

The monoid \(\hat{G}\): The category \(\mathcal{O}\) is defined as follows: Its objects are the \(g\)-modules \(V\), which have the properties:

1. \(V\) is \(\mathfrak{h}\)-diagonalizable with finite dimensional weight spaces.
2. There exist finitely many elements \(\lambda_1, \ldots, \lambda_m \in \mathfrak{h}^*\), such that the set of weights \(P(V)\) of \(V\) is contained in the union \(\bigcup_{i=1}^m D(\lambda_i)\), where \(D(\lambda_i) := \{ \lambda \in \mathfrak{h}^* | \lambda \leq \lambda_i \}\).

The morphisms of \(\mathcal{O}\) are the morphisms of \(g\)-modules.

For \(V\) a module of \(\mathcal{O}\) and \(v \in V\), we denote by \(\text{supp}(v)\) the set of weights of the nonzero weight space components of \(v\).

A module of \(\mathcal{O}\) is \(m\)-admissible if and only if it is \(f\)-admissible, and we call such a module \(\text{admissible}\). We denote by \(\mathcal{O}_{\text{adm}}\) the full subcategory of the category \(\mathcal{O}\), whose objects are admissible modules.

There is a complete reducibility theorem. Every object of \(\mathcal{O}_{\text{adm}}\) is isomorphic to a direct sum of the admissible irreducible highest weight modules \(L(\Lambda), \Lambda \in P^+\).

The set of weights of a module of \(\mathcal{O}_{\text{adm}}\) is contained in \(X \cap P\), because we have

\[
\bigcup_{\Lambda \in P^+} P(\Lambda) = X \cap P.
\]

Let \(\Lambda \in P^+\), and \(\Theta\) be special. Because the set of weights \(P(\Lambda)\) is contained in the convex hull of \(h\Lambda \subseteq \mathfrak{h}_{\mathbb{R}}\), we find easily

\[
P(\Lambda) \cap R(\Theta) = \emptyset \text{ if and only if } \Lambda \notin R(\Theta).
\]

The monoid \(\hat{G}\) can be characterized in the following way:

- The monoid \(\hat{G}\) acts on every module of \(\mathcal{O}_{\text{adm}}\). Two elements \(\hat{g}, \hat{g}' \in \hat{G}\) are equal if and only if for all modules \(V\) of \(\mathcal{O}_{\text{adm}}\), and for all \(v \in V\), we have \(\hat{g}v = \hat{g}'v\).
- (1) \(\hat{G}\) is an extension of the minimal Kac-Moody group \(G\).
- (2) For every face \(R\) of the Tits cone there exists an element \(e(R) \in \hat{G}\), such that for every module \(V\) of \(\mathcal{O}_{\text{adm}}\) we have

\[
e(R)v_\lambda = \begin{cases} v_\lambda, & \lambda \in R \\ 0, & \lambda \in V_\lambda, \ \lambda \in P(V) \end{cases}.
\]

\(\hat{G}\) is generated by \(G\) and the elements of (2).

Note that the monoid \(\hat{G}\) acts faithfully on the sum \(\bigoplus_{\Lambda \in P^+} L(\Lambda)\).

The Chevalley involution \(*: \hat{G} \to \hat{G}\) is the involutive anti-isomorphism determined by \(\exp(x_\alpha)^* := \exp(x_\alpha^*)\), \(t^* := t\), \(e(R)^* := e(R)\), where \(x_\alpha \in \mathfrak{g}_\alpha, \ \alpha \in \Delta_{re}\).
It is compatible with any nondegenerate symmetric contravariant form \( \langle \langle \cdot | \cdot \rangle \rangle \) on any module \( V \) of \( O_{adm} \), i.e., \( \langle \langle xv | w \rangle \rangle = \langle \langle v | x^* w \rangle \rangle \), \( v, w \in V \), \( x \in \hat{G} \).

The following formulas are useful for computations in \( \hat{G} \):

- Let \( R, S \) be faces of the Tits cone, and \( n_\sigma \in N \). Then
  \[
  e(R)e(S) = e(R \cap S) \quad \text{and} \quad n_\sigma e(R)n_\sigma^{-1} = e(\sigma R) .
  \]

- An element \( g \) of \( T, N, U, U^- \), resp. \( G \) satisfies
  \[
  e(R(\Theta))g = e(R(\Theta))
  \]
  if and only if it satisfies
  \[
  g^* e(R(\Theta)) = e(R(\Theta))
  \]
  if and only if it is contained in \( T_{\Theta}, N_{\Theta}, U_{\Theta}, U^-_{\Theta} \), resp. \( G_{\Theta} \times U_{\Theta} \).

- An element \( g \) of \( T, N, U, U^- \), resp. \( G \) satisfies
  \[
  ge(R(\Theta))g^{-1} = e(R(\Theta))
  \]
  if and only if it is contained in the groups \( T, N_{\Theta \cup \Theta^+}, U_{\Theta \cup \Theta^+}, U^-_{\Theta \cup \Theta^+} \), resp. \( G_{\Theta \cup \Theta^+} \).

- In particular we have
  \[
  U e(R(\Theta)) = U_{\Theta^+}e(R(\Theta)) = e(R(\Theta)) U_{\Theta^+}, \quad \text{and} \quad \quad e(R(\Theta)) U^- = e(R(\Theta)) U^-_{\Theta^+} = U^-_{\Theta^+} e(R(\Theta)) .
  \]

The minimal Kac-Moody group \( G \) is the unit group of \( \hat{G} \). Every idempotent is \( G \)-conjugate to some idempotent \( e(R(\Theta)) \), \( \Theta \) special. We have
\[
\hat{G} = \bigcup_{\Theta \text{ special}} Ge(R(\Theta)) G .
\]

The Weyl group acts on the monoid \( (\mathcal{R}(X), \cap) \). The semidirect product \( \mathcal{R}(X) \rtimes W \) consists of the set \( \mathcal{R}(X) \times W \) with the structure of a monoid given by
\[
(R, \sigma) \cdot (S, \tau) := (R \cap \sigma S, \sigma \tau) .
\]

For \( R \in \mathcal{R}(X) \) let \( Z_W(R) := \{ \sigma \in W \mid \sigma \lambda = \lambda \text{ for all } \lambda \in R \} \) be the pointwise stabilizer of \( R \). The Weyl monoid \( \hat{W} \) is defined as the monoid \( \mathcal{R}(X) \rtimes W \) factored by the congruence relation
\[
(R, \sigma) \sim (R', \sigma') := \iff R = R' \text{ and } \sigma' \sigma^{-1} \in Z_W(R) .
\]

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We denote the congruence class of \((R,\sigma)\) by \(\varepsilon(R)\sigma\).
Assigning to \(\sigma \in \mathcal{W}\) the element \(\sigma := \sigma e(X) \in \mathcal{W}\), the Weyl group \(\mathcal{W}\) identifies with the unit group of \(\mathcal{W}\). The partition of \(\mathcal{W}\) into \(\mathcal{W}\times \mathcal{W}\)-orbits is given by
\[
\mathcal{W} = \bigcup_{\sigma \in \mathcal{W}} \mathcal{W} \varepsilon(R(\Theta)) \mathcal{W}.
\]
Assigning to \(e(R) \in \mathcal{R}(X)\) the element \(\varepsilon(R) := \varepsilon(R)1 \in \mathcal{W}\), the monoid \((\mathcal{R}(X),\cap)\) embeds into \(\mathcal{W}\). Its image are the idempotents of \(\mathcal{W}\).

For \(J \subseteq I\) denote by \(\mathcal{W}^J\) the minimal coset representatives of \(\mathcal{W}/\mathcal{W}^J\), and denote by \(J\mathcal{W}\) the minimal coset representatives of \(\mathcal{W}^J\mathcal{W}\). It is easy to see that there are the following uniquely determined normal forms of an element \(\hat{\sigma} \in \mathcal{W}\):
\[
\hat{\sigma} = \sigma_1\varepsilon(R(\Theta))\sigma_2 \quad \text{with} \quad \Theta \text{ special}, \quad \sigma_1 \in \mathcal{W}^{\Theta,\Theta^\perp}, \quad \sigma_2 \in \Theta \mathcal{W}.
\]
\[
\hat{\sigma} = \tau_1\varepsilon(R(\Theta))\tau_2 \quad \text{with} \quad \Theta \text{ special}, \quad \tau_1 \in \mathcal{W}^\Theta, \quad \tau_2 \in \Theta^\perp\Theta \mathcal{W}.
\]

Let \(J \subseteq I\). We call the submonoid \(\mathcal{W}^J\), which is generated by \(\mathcal{W}^J\) and the elements \(\varepsilon(R(\Theta))\), \(\Theta \subseteq J\) special, a parabolic submonoid. Denote by \(\mathcal{J}^\infty\) the union of all connected components of nonfinite type of \(J\). We have
\[
\mathcal{W}^J = \bigcup_{\theta \in \mathcal{R}(\mathcal{X})} \mathcal{W}^J \varepsilon(R) = \bigcup_{\Xi \subseteq \mathcal{J}^\infty} \mathcal{W}^J \varepsilon(R(\Xi)) \mathcal{W}^J.
\]
We get an abelian submonoid of \(\mathcal{G}\) by \(\hat{T} := \bigcup_{R \in \mathcal{R}(X)} T e(R)\). We get a submonoid of \(\mathcal{G}\) by \(\hat{N} := \bigcup_{R \in \mathcal{R}(X)} N e(R)\). Define a congruence relation on \(\hat{N}\) as follows:
\[
\hat{n} \sim \hat{n}' \iff \hat{n}T = \hat{n}'T \iff \hat{n}T = \hat{n}'T \iff \hat{n} \in \hat{n}'T.
\]
The Weyl monoid \(\mathcal{W}\) is isomorphic to the monoid \(\hat{N}/T\), an isomorphism \(\kappa : \mathcal{W} \to \hat{N}/T\) given by \(\kappa(\varepsilon(R)) = n e(R)T\).
\(\mathcal{G}\) has Bruhat and Birkhoff decompositions:
\[
\mathcal{G} = \bigcup_{\hat{n} \in \hat{N}} U^\epsilon \hat{n} U^\delta = \bigcup_{\hat{\sigma} \in \mathcal{W}} B^\epsilon \hat{\sigma} B^\delta, \quad \epsilon, \delta \in \{+, -, \}.
\]

The coordinate ring of \(\mathcal{G}\), and the algebra of strongly regular functions: For a module \(V\) of \(\mathcal{O}\_adm\), \(v, w \in V\), and \(\langle \langle v \mid xw \rangle \rangle\) a nondegenerate symmetric contravariant bilinear form on \(V\), call the function \(\hat{f}_{vw} : \mathcal{G} \to \mathbb{F}\) defined by \(\hat{f}_{vw}(x) := \langle \langle v \mid xw \rangle \rangle\), \(x \in \mathcal{G}\), a matrix coefficient of \(\mathcal{G}\). The set of all such matrix coefficients \(\mathbb{F}[\mathcal{G}]\) is an algebra. It is an integral domain, and admits a Peter-Weyl theorem: Equip \(\mathbb{F}[\mathcal{G}]\) with an action \(\pi \times \mathcal{G}\), and an involutive automorphism \(\ast\) by:
\[
\pi(g,h) f(x) := f(g^* x h), \quad f^*(x) := f(x^*), \quad g, x, h \in \mathcal{G}, \quad f \in \mathbb{F}[\mathcal{G}].
\]

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For every $\Lambda \in P^+$ fix a nondegenerate symmetric contravariant bilinear form on $L(\Lambda)$. The map $\bigoplus_{\Lambda \in P^+} L(\Lambda) \otimes L(\Lambda) \rightarrow F[\hat{G}]$ induced by $v \otimes w \mapsto f_{vw}$ is an isomorphism of $\hat{G} \times G$-modules. It identifies the direct sum of the switch maps of the factors with the involution.

The monoids $\hat{T}$, $\hat{N}$, $\hat{G}$ are the Zariski closures of $T$, $N$, $G$, and $G$ is the Zariski open dense unit group of $\hat{G}$.

The algebra of strongly regular functions $F[G]$ is obtained by restricting the functions of $F[\hat{G}]$ onto $G$. The restriction map is an isomorphism from $F[\hat{G}]$ to $F[G]$.

Restricting the functions of $F[G]$ onto $G'$, resp. $T_{rest}$ gives the algebras $F[G']$, resp. $F[T_{rest}]$, the first identical with the algebra of strongly regular functions as defined in [K,P 2], the second the classical coordinate ring of the torus $T_{rest}$. $F[G]$ is isomorphic to $F[G'] \otimes F[T_{rest}]$, by the comorphism dual to the multiplication map $G' \times T_{rest} \rightarrow G$.

**Substructures:** For $\emptyset \neq J \subseteq I$ the submatrix $A_J := (a_{ij})_{i,j \in J}$ of $A$ is a generalized Cartan matrix. There exist saturated sublattices $H(A_J) \subseteq H$, $P(A_J) \subseteq P$ with $(h_{ij})_{i,j \in J} \subseteq H(A_J)$, $(a_{ij})_{i,j \in J} \subseteq P(A_J)$, giving a simply connected minimal free realization of $A_J$. We have $P = P(A_J) \oplus H(A_J)^\perp$, and the projections of $\Lambda_J, j \in J$, to $P(\hat{A}_J)$ are a system of fundamental dominant weights.

The corresponding Kac-Moody algebra $g(A_J)$ embeds in $g$. If we identify $g(A_J)$ with its image, then the set of roots $\Delta(A_J)$ identifies with $\Delta_J := \Delta \cap \sum_{j \in J} \mathbb{Z} a_{ij}$, and the Weyl group $W(A_J)$ identifies with the parabolic subgroup $W_J$.

The face lattice of the Tits cone of $A_J$ embeds onto a sublattice of the face lattice of the Tits cone of $A$, and the Weyl monoid $W(A_J)$ identifies with the parabolic submonoid $W_J$.

The minimal and formal Kac-Moody groups $G(A_J), G_f(A_J)$, the monoid $\hat{G}(A_J)$ embed in $G, G_f, \hat{G}$ in the obvious way.

The images of these embedding depend on the choice of the sublattice $H(A_J)$, only $H_J := \mathbb{Z} \text{-span} \{ h_j \mid j \in J \}$ is uniquely determined by $A_J$. Denote by $G_I$ the submonoid of $\hat{G}$, which is generated by $G'$ and the elements $e(R), R$ a face of $X$. The images of $g(A_J)', G(A_J)', G_f(A_J)',$ and $G(\hat{A}_J)'$ are independent of this choice, and denoted by $g_J, G_J, (G_f)_J, \text{ and } \hat{G}_J$.

An admissible irreducible highest weight module $L(\Lambda)$ of $g$, equipped with a nondegenerate contravariant symmetric bilinear form, decomposes as a $g(A_J)$-module into an orthogonal direct sum of admissible irreducible highest weight modules of $g(A_J)$, which are $\mathbf{h}$-invariant. In particular $L(\Lambda) := U(\mathbf{n}_J) \otimes L(\Lambda)_{\Lambda}$ is an admissible irreducible highest weight module of $g(A_J)$, its highest weight given by the projection of $\Lambda$ to $P(A_J)$, $(\Lambda \in P^+)$.

The coordinate rings $F[\hat{G}(A_J)'], F[G(\hat{A}_J)']$ identify with the restrictions of $F[\hat{G}], F[G]$ to $\hat{G}_J, G_J$. (A similar statement for $F[\hat{G}(\hat{A}_J)], F[G(A_J)]$ is not
valid.)
To simplify the notation of many formulas, it is useful to set $g_\emptyset := \{0\}$, and $G_\emptyset := \hat{G}_\emptyset := (G_f)_\emptyset := \{1\}$.

For $M \subseteq \hat{G}$ and $J \subseteq I$ set $M_J := M \cap \hat{G}_J$, and similarly for $M \subseteq G_f$ set $M_J := M \cap (G_f)_J$.

2 An easy algebraic geometric setting

In this section we develop an easy algebraic geometric setting, which is useful to determine the $\mathbb{F}$-valued points of the algebra of strongly regular functions. This section is a complement to [M 1], Section 3, but it can be read independently. The definition of a morphism in [M 1], Section 3, is more restrictive, because it has to preserve an extra structure, which we do not need here. We will consider certain nonempty sets $A$ equipped with point separating algebras of functions $\mathbb{F}[A]$, which we call coordinate rings. The closed sets of the Zariski topology on such a set $A$ are given by the zero sets of the functions of $\mathbb{F}[A]$. Note that $A$ is irreducible if and only if $\mathbb{F}[A]$ is an integral domain. A morphism of sets with coordinate rings $(A, \mathbb{F}[A])$ and $(B, \mathbb{F}[B])$ consists of a map $\phi : A \to B$, whose comorphism $\phi^* : \mathbb{F}[B] \to \mathbb{F}[A]$ exists. In particular a morphism is Zariski continuous.

If $(B, \mathbb{F}[B])$ is a set with coordinate ring, and $A$ is a nonempty subset of $B$, then we get a coordinate ring on $A$ by restricting the functions of $\mathbb{F}[B]$ to $A$. If $(A, \mathbb{F}[A])$ is a set with coordinate ring and $f \in \mathbb{F}[A] \setminus \{0\}$, the principal open set $D_A(f) := \{ a \in A \mid f(a) \neq 0 \}$ is equipped with a coordinate ring by identifying the localization $\mathbb{F}[A]_f$ in the obvious way with an algebra of functions on $D_A(f)$. If $A$ is irreducible, then also $D_A(f)$ is irreducible.

If $(A, \mathbb{F}[A])$ and $(B, \mathbb{F}[B])$ are sets with coordinate rings, then $A \times B$ is equipped with a coordinate ring by identifying the tensor product $\mathbb{F}[A] \otimes \mathbb{F}[B]$ in the obvious way with an algebra of functions on $A \times B$. If $A$ and $B$ are irreducible, then also $A \times B$ is irreducible.

To construct the sets with coordinate rings, which we will use later, fix nondegenerate symmetric contravariant forms $\langle \langle \mid \rangle \rangle$ on all modules $L(\Lambda)$, $\Lambda \in P^+$, and extend to a form on $\bigoplus_{\Lambda \in P^+} L(\Lambda)$, also denoted by $\langle \langle \mid \rangle \rangle$, by requiring $L(\Lambda)$ and $L(\Lambda')$ to be orthogonal for $\Lambda \neq \Lambda'$. For $v, w \in L(\Lambda)$, $\Lambda \in P^+$, define a linear function

$$f_{vw} : \text{End} \left( \bigoplus_{\Lambda \in P^+} L(\Lambda) \right) \to \mathbb{F}$$
by \( f_{vw}(\phi) := \langle v | \phi w \rangle \), where \( \phi \in \text{End} \left( \bigoplus_{\Lambda \in P^+} L(\Lambda) \right) \).

We equip the subalgebra

\[
gr-\text{Adj} := \left\{ \phi \in \text{End} \left( \bigoplus_{\Lambda \in P^+} L(\Lambda) \right) \bigg| \begin{array}{c}
\phi \text{ exists and} \\
\phi(L(\Lambda)) \subseteq L(\Lambda), \Lambda \in P^+.
\end{array} \right\}
\]

doing the algebra of endomorphisms with the coordinate ring \( F[gr-\text{Adj}] \), which is generated by the linear subspace

\[
\text{span} \left\{ f_{vw} \big|_{gr-\text{Adj}} \big| v, w \in L(\Lambda), \Lambda \in P^+ \right\}
\]

of the linear dual of \( gr-\text{Adj} \). This coordinate ring is isomorphic to the symmetric algebra in this subspace.

The multiplication map is no morphism, but the left and right multiplications with elements of \( gr-\text{Adj} \), and also the adjoint map \( * : gr-\text{Adj} \to gr-\text{Adj} \) are morphisms.

For a set \( M \subseteq gr-\text{Adj} \) we denote by \( \overline{M} \) its Zariski closure in \( gr-\text{Adj} \). If \( M \) is a \( (\ast\text{-invariant}) \) submonoid of \( gr-\text{Adj} \), then also \( \overline{M} \) is a \( (\ast\text{-invariant}) \) submonoid.

The left and right translations with elements of \( \overline{M} \) (and the map \( * : \overline{M} \to \overline{M} \)) are morphisms.

A function \( f \in F[M] \) induces a function on \( \text{Specm} F[M] \), assigning \( x \in \text{Specm} F[M] \) the value \( x(f) \). In this way, \( \text{Specm} F[M] \) is equipped with a coordinate ring isomorphic to \( F[M] \). Its Zariski topology coincides with the relative topology induced by the topology of the spectrum of \( F[M] \).

To investigate \( \text{Specm} F[M] \), we introduce two new monoids \( \overline{M}, \overline{M}^{\ast} \) with coordinate rings.

Equip the subalgebra

\[
gr-\text{End} := \left\{ \phi \in \text{End} \left( \bigoplus_{\Lambda \in P^+} L(\Lambda) \right) \bigg| \begin{array}{c}
\phi(L(\Lambda)) \subseteq L(\Lambda), \Lambda \in P^+.
\end{array} \right\}
\]

of the algebra of endomorphisms with the coordinate ring \( F[gr-\text{End}] \) generated by the linear subspace

\[
\text{span} \left\{ f_{vw} \big|_{gr-\text{End}} \big| v, w \in L(\Lambda), \Lambda \in P^+ \right\}
\]

of the linear dual of \( gr-\text{End} \), which is isomorphic to \( \bigoplus_{\Lambda \in P^+} L(\Lambda) \otimes L(\Lambda) \). This coordinate ring is isomorphic to the symmetric algebra in this subspace. The restriction of \( F[gr-\text{End}] \) to \( gr-\text{Adj} \) coincides with the coordinate ring of \( gr-\text{Adj} \).

The right multiplication with an element \( \phi \in gr-\text{End} \) is a morphism. We can guarantee the left multiplication \( l_{\phi} \) to be a morphism, only if we restrict to \( \phi \in gr-\text{Adj} \), because then we have

\[
f_{vw} \big|_{gr-\text{End}} \circ l_{\phi} = f_{\phi v w} \big|_{gr-\text{End}} \quad \text{for all} \quad v, w \in L(\Lambda), \Lambda \in P^+.
\]

For a set \( M \subseteq gr-\text{End} \) we denote by \( \overline{M} \) its Zariski closure in \( gr-\text{End} \). If \( M \) is a submonoid of \( gr-\text{End} \), then \( \overline{M} \) is not necessarily a submonoid of \( gr-\text{End} \). But we have:
Proposition 2.1 If $M$ is a submonoid of gr-Adj, then $\overline{M}$ is a submonoid of gr-End.

Also the following proposition is easy to prove:

Proposition 2.2 Let $M$ be a submonoid of gr-Adj. For $\phi \in \overline{M}$ and $\psi \in \overline{M}$ we get a homomorphism of algebras $\pi(\phi, \psi) : \mathbb{F}[M] \to \mathbb{F}[M]$ by

$$\pi(\phi, \psi)(f_{vw}|_M) := f_{\phi v \psi w}|_M , \quad v, w \in L(\Lambda) , \; \Lambda \in P^+ .$$

$\pi$ is an action of $\overline{M} \times \overline{M}$ on $\mathbb{F}[M]$.

Due to this proposition we get an action of $\overline{M} \times \overline{M}$ on $\text{Specm} \mathbb{F}[M]$ from the right by

$$\alpha \circ \pi(\phi, \psi) , \quad \text{where} \; \alpha \in \text{Specm} \mathbb{F}[M], \; \phi \in \overline{M}, \; \psi \in \overline{M} .$$

This is an action by morphisms of $\text{Specm} \mathbb{F}[M]$. But in general the map $(\overline{M} \times \overline{M}) \times \text{Specm} \mathbb{F}[M] \to \text{Specm} \mathbb{F}[M]$ is no morphism. In general also the right translations of this map are no morphisms.

The next Proposition, which is easy to prove, describes the part of $\text{Specm} \mathbb{F}[M]$ obtained by applying $\overline{M} \times \overline{M}$ to the evaluation map of $\mathbb{F}[M]$ in the unit of $M$.

Proposition 2.3 Let $M$ be a submonoid of gr-Adj.

1) For $\phi \in \overline{M}$ and $\psi \in \overline{M}$ we get a point $\alpha(\phi, \psi) \in \text{Specm} \mathbb{F}[M]$ by

$$\alpha(\phi, \psi)(f_{vw}|_M) := (\phi v | \psi w) \quad \text{for all} \; \; v, w \in L(\Lambda) , \; \Lambda \in P^+ .$$

If we equip $\overline{M} \times \overline{M}$ with the right action on itself, then the map $\alpha : \overline{M} \times \overline{M} \to \text{Specm} \mathbb{F}[M]$ is equivariant. Furthermore we have

$$\alpha(\phi, z \psi) = \alpha(z^* \phi, \psi) \quad \text{for all} \; \; \phi \in \overline{M}, \; \psi \in \overline{M}, \; \text{and} \; \; z \in \overline{M} .$$

2) For every $\phi \in \overline{M}$ the map $\alpha(\phi, \cdot) : \overline{M} \to \text{Specm} \mathbb{F}[M]$, which assigns $\psi$ the point $\alpha(\phi, \psi)$, is a morphism. The map $\alpha(1, \cdot)$ is injective.

For every $\psi \in \overline{M}$ the map $\alpha(\cdot, \psi) : \overline{M} \to \text{Specm} \mathbb{F}[M]$ which assigns $\phi$ the point $\alpha(\phi, \psi)$, is a morphism. The map $\alpha(\cdot, 1)$ is injective.

Remarks:

1) We often say $\overline{M} \times \overline{M}$ maps to $\mathbb{F}[M]$, without mentioning the map $\alpha$. In general this map is no morphism.

2) The maps $\{1\} \times \overline{M} \to \text{Specm} \mathbb{F}[M]$, and $\overline{M} \times \{1\} \to \text{Specm} \mathbb{F}[M]$ are injective. But if $M$ is nontrivial, then the map $\overline{M} \times \overline{M} \to \text{Specm} \mathbb{F}[M]$ is not injective:

Let $\sim$ be the equivalence relation on $\overline{M} \times \overline{M}$ generated by

$$(z^* x, y) \sim (x, z y) , \quad z \in \overline{M} .$$
Similarly, the set \( B \) or irreducible subgroup of \( \overline{M^*} \).

3) The map \( * : M \to M^* \) is an isomorphism. Its commorphism \( * : \mathbb{F}[M^*] \to \mathbb{F}[M] \) is given by

\[
(f_{vw}|_{M^*})^* = f_{wv}|_M \quad \text{for all} \quad v, w \in L(\Lambda), \Lambda \in P^+.
\]

We omit to state the easy compatibility conditions with the maps \( \alpha, \pi, \) which correspond to \( \mathbb{F}[M], \) and \( \mathbb{F}[M^*] \).

The properties stated in the second part of the last proposition are sufficient to guarantee the irreducibility of certain orbits, belonging to the action of the product of a irreducible subgroup of \( \overline{M^*} \) and a irreducible subgroup of \( \overline{M} \).

**Theorem 2.4** Let \( M \) be a submonoid of \( \text{gr-Adj} \), and \( x \in \overline{M} \). Let \( D_1 \) be an irreducible subgroup of \( \overline{M^*} \), let \( D_2 \) be an irreducible subgroup of \( \overline{M} \). Then the \( D_1 \times D_2 \)-orbit of the element \( \alpha(1, x) \in \text{Specm} \mathbb{F}[M] \) is irreducible.

**Proof:** Let \( Or \) be the \( D_1 \times D_2 \)-orbit of \( \alpha(1, x) \), i.e., \( Or = \alpha(D_1, xD_2) = \alpha(x^*D_1, D_2) \). Let \( A_1 \) and \( A_2 \) be closed subsets of \( \text{Specm} \mathbb{F}[M] \), such that \( Or \subseteq A_1 \cup A_2 \). We have to show \( Or \subseteq A_1 \) or \( Or \subseteq A_2 \).

Let \( d_1 \in D_1 \). Due to the last proposition the map \( \gamma_{d_1} : D_2 \to \text{Specm} \mathbb{F}[M] \) defined by \( \gamma_{d_1}(d_2) := \alpha(x^*d_1, d_2), d_2 \in D_2, is a morphism. Similarly, for \( d_2 \in D_2 \), the map \( \delta_{d_2} : D_1 \to \text{Specm} \mathbb{F}[M] \) defined by \( \delta_{d_2}(d_1) := \alpha(d_1, xd_2), d_1 \in D_1, is a morphism.

Let \( d_1 \in D_1 \). Because of \( \gamma_{d_1}(D_2) \subseteq Or \), we have \( \gamma_{d_1}^{-1}(A_1) \cup \gamma_{d_1}^{-1}(A_2) = D_2 \). Furthermore \( \gamma_{d_1}^{-1}(A_1) \) and \( \gamma_{d_1}^{-1}(A_2) \) are closed. Because of the irreducibility of \( D_2 \) we get \( \gamma_{d_1}^{-1}(A_1) = D_2 \) or \( \gamma_{d_1}^{-1}(A_2) = D_2 \).

Therefore the sets

\[
B_1 := \{ d_1 \in D_1 | \gamma_{d_1}^{-1}(A_1) = D_2 \}, \quad B_2 := \{ d_1 \in D_1 | \gamma_{d_1}^{-1}(A_2) = D_2 \}
\]

satisfy \( B_1 \cup B_2 = D_1 \).

Note that for \( d_1 \in D_1 \) and \( d_2 \in D_2 \) we have \( \gamma_{d_1}(d_2) = \delta_{d_2}(d_1) \). The set \( B_1 \) is closed, because of

\[
B_1 = \{ d_1 \in D_1 | \gamma_{d_1}(d_2) \in A_1 \quad \text{for all} \quad d_2 \in D_2 \} = \bigcap_{d_2 \in D_2} \delta_{d_2}^{-1}(A_1)_{\text{closed}}.
\]

Similarly, the set \( B_2 \) is closed. Because of the irreducibility of \( D_1 \) we get \( B_1 = D_1 \) or \( B_2 = D_1 \), which is equivalent to \( Or \subseteq A_1 \) or \( Or \subseteq A_2 \).

\( \square \)
We have $gr-Adj = gr-End$. Due to Proposition 2.3 we get a map $\alpha : gr-End \times gr-End \to Specm \mathbb{F}[gr-Adj]$. The following technical proposition, which follows immediately from the definition of the Zariski closures in $gr-End$, is very useful for determining such closures:

**Proposition 2.5** Let $M$ be a submonoid of $gr-Adj$. We have:

$$\overline{M} = \{ \phi \in gr-End \mid \alpha(1, \phi) \text{ factors to a homomorphism } \mathbb{F}[M] \to \mathbb{F} \} .$$

$$\overline{M}^\circ = \{ \phi \in gr-End \mid \alpha(\phi, 1) \text{ factors to a homomorphism } \mathbb{F}[M] \to \mathbb{F} \} .$$

The formal Kac-Moody group $G_f$ acts faithfully on $\bigoplus_{\Lambda \in P^+} L(\Lambda)$. We identify $G_f$ with the corresponding subgroup of $gr-End$. Under this identification the minimal Kac-Moody group $G \subseteq G_f$ is identified with a subgroup of $gr-Adj$, invariant under taking the adjoint.

The results of this section can be applied to the $\mathbb{F}$-valued points of the algebra of strongly regular functions $\mathbb{F}[G]$, because the restriction of the coordinate ring $\mathbb{F}[gr-Adj]$ to $G$ coincides with $\mathbb{F}[G]$.

Also the monoid $\hat{G}$ can be identified with a submonoid of $gr-Adj$. In [M 1], Theorem 5.14, we showed $G = \hat{G}$. In this paper we determine $G$. We show the bijectivity of the map $\tilde{\alpha} : G \times G \to Specm \mathbb{F}[G]$. To prove its surjectivity, we use an induction over $|J|$, $J \subseteq I$, describing $Specm \mathbb{F}[G_J]$. To prepare this proof, in the next two sections we first determine $G$-valued points of some other coordinate rings.

### 3 The $\mathbb{F}$-valued points of $\mathbb{F}[T_J]$, $\mathbb{F}[T]$, $\mathbb{F}[T_{rest}]$ ($J \subseteq I$)

Recall that the torus $T$ of the Kac-Moody group can be described by the following isomorphism of groups:

$$H \otimes_{\mathbb{Z}} \mathbb{F}^\times \rightarrow \sum_{i=1}^{2n-l} h_i \otimes s_i \rightarrow \prod_{i=1}^{2n-l} t_{h_i}(s_i)$$

The group algebra $\mathbb{F}[P]$ of the lattice $P$ can be identified with the classical coordinate ring on $T$, identifying $\sum c_\lambda e_\lambda \in \mathbb{F}[P]$ with the function on $T$, which is defined by

$$\left( \sum_{\lambda} c_\lambda e_\lambda \right) \left( \prod_{i=1}^{2n-l} t_{h_i}(s_i) \right) : = \sum_{\lambda} c_\lambda \prod_{i=1}^{2n-l} (s_i)^{\lambda(h_i)} , \quad (s_i \in \mathbb{F}^\times) .$$

We have similar descriptions for the tori $T_J := \{ \prod_{j \in J} t_{h_j}(s_j) \mid s_j \in \mathbb{F}^\times \}$, $T_{rest}$, and its classical coordinate rings, replacing the lattices $H, P$ by $H_J, P_J := \mathbb{Z}$-span \{ $\Lambda_j \mid j \in J$ \} or $H_{rest}, P_{rest} := \mathbb{Z}$-span \{ $\Lambda_i \mid i = n + 1, \ldots, 2n - l$ \},

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In \[\mathbb{M}_1\], Proposition 5.1, we determined the coordinate rings \( F[T_J], F[T], \) and \( F[T_{\text{rest}}] \). They are in general only subalgebras of the classical coordinate rings of these tori:

**Theorem 3.1** Let \( J \subseteq I \). Let \( p_J : P \to P_J \) be the projection defined by \( p_J(\lambda) := \sum_{j \in J} \lambda(h_j)\Lambda_j \). We have:
1) \( F[T_J] = F[p_J(X \cap P)] \).
2) \( F[T] = F[X \cap P] \).
3) \( F[T_{\text{rest}}] = F[P_{\text{rest}}] \).

In \[\mathbb{M}_1\], Theorem 5.2, we described the relative closures of \( T_J, T \) and \( T_{\text{rest}} \) in \( \text{gr-Adj} \). In the same way, but replacing \( \text{gr-Adj} \) by \( \text{gr-End} \), we can determine the closures of \( T_J, T \) and \( T_{\text{rest}} \). The proof of \[\mathbb{M}_1\], Theorem 5.2, also shows, that \( \{1\} \times \hat{T}_J, \{1\} \times \hat{T}, \) resp. \( \{1\} \times T_{\text{rest}} \) map bijectively to \( \text{Specm} F[T_J], \text{Specm} F[T], \) resp. \( \text{Specm} F[T_{\text{rest}}] \). Therefore we get:

**Theorem 3.2** Let \( J \subseteq I \).
1) We have \( T_J = \hat{T}_J \), and \( \{1\} \times T_J \) maps bijectively to \( \text{Specm} F[T_J] \).
2) We have \( T = \hat{T} \), and \( \{1\} \times T \) maps bijectively to \( \text{Specm} F[T] \).
3) \( T_{\text{rest}} \) is closed, and \( \{1\} \times T_{\text{rest}} \) maps bijectively to \( \text{Specm} F[T_{\text{rest}}] \).

4 The \( F \)-valued points of \( F[U_J] \) and \( F[U^J] \) (\( J \subseteq I \))

The coordinate ring of \( U \) has been described by Kac and Peterson in \[K,P_2\], Lemma 4.3. From this Lemma follows immediately part 1) of the next theorem. Part 2) has been shown in \[\mathbb{M}_1\], Theorem 5.6.

**Theorem 4.1** Let \( J \subseteq I \).
1) Let \( \Lambda \in F_{I,J} \cap P^+ \), and \( v_\Lambda \in L(\Lambda)_\Lambda \setminus \{0\} \). Then:
   \( F[U_J] \) is a symmetric algebra in \( \{ f_{v_\Lambda^y_n} |_{U_J} \mid y \in n_J \} \).
2) Let \( N \in F_J \cap P^+ \), and \( v_N \in L(N)_N \setminus \{0\} \). Then:
   \( F[U^J] \) is as algebra generated by \( \{ f_{v_N^y_w} |_{U^J} \mid y \in (n^J)^- \} \).

Using these descriptions of the coordinate rings, it is possible to determine its \( F \)-valued points:

**Theorem 4.2** Let \( J \subseteq I \).
1) We have \( U_J = (U_J)_J \), and \( \{1\} \times U_J \) maps bijectively to \( \text{Specm} F[U_J] \).
2) We have \( U^J = (U^J)_J \), and \( \{1\} \times U^J \) maps bijectively to \( \text{Specm} F[U^J] \).

**Proof of 1)**: The case \( J = \emptyset \) is trivial. Let \( J \) be nonempty. We first show \( (U_J)_J \subseteq U_J \). The coordinate ring \( F[\text{gr-End}] \) is a symmetric algebra in the linear span of the functions \( f_{vw} |_{\text{gr-End}}, v, w \in L(\Lambda), \Lambda \in P^+ \), which is isomorphic
to $\bigoplus_{\Lambda \in P^+} L(\Lambda) \otimes L(\Lambda)$. We get a representation of the Lie algebra $(\mathfrak{n}_f)_J$, by assigning $x \in (\mathfrak{n}_f)_J$ the derivation $\delta_x : \mathbb{F}[\text{gr-End}] \to \mathbb{F}[\text{gr-End}]$, which is defined by

$$\delta_x(f_{vw}) := f_{vwx}, \quad v, w \in L(\Lambda), \quad \Lambda \in P^+.$$ 

The derivation $\delta_x$ is locally nilpotent, because $x \in (\mathfrak{n}_f)_J$ acts locally nilpotent on $L(\Lambda), \quad \Lambda \in P^+$. The homomorphism of algebras $\exp(\delta_x)$ satisfies

$$\alpha(1,1) \circ \exp(\delta_x) = \alpha(1, \exp(x)).$$

Let $I(U_J)$ be the vanishing ideal of $U_J$ in $\mathbb{F}[\text{gr-End}]$. Due to the last identity and Proposition 2.5, it is sufficient to show $\delta_x(I(U_J)) \subseteq I(U_J)$ for all $x \in (\mathfrak{n}_f)_J$. Let $f \in I(U_J)$. Let $x \in \mathfrak{g}_n, \alpha \in (\Delta_J)_e$. Therefore also $\delta_x(f) \in I(U_J)$ for all $x \in \mathfrak{n}_J$.

For an element $v \in L(\Lambda), \quad \Lambda \in P^+$, there exist only finitely many roots $\alpha \in \Delta^+$ with $\mathfrak{g}_\alpha v \neq 0$. Therefore for an element $x \in (\mathfrak{n}_f)_J$ there exists an element $\tilde{x} \in \mathfrak{n}_J$, depending on $f$, such that $\delta_x(f) = \delta_{\tilde{x}}(f) \in I(U_J)$.

We have $(U_J)_J \subseteq \bigcap U_J$, and $(1) \times \bigcup U_J$ maps injectively to $\text{Specm} \mathbb{F}[U_J]$. To prove 1), it remains to show that $(1) \times (U_J)_J$ maps surjectively to $\text{Specm} \mathbb{F}[U_J]$.

Let $\Lambda \in F_I \cap P^+$, and $v_\Lambda \in L(\Lambda)_\Lambda \setminus \{0\}$. Due to the description of $\mathbb{F}[U_J]$ of the last theorem, it is sufficient to show, that for every linear map

$$l : \{ f_{v_\Lambda y v_\Lambda} | y \in \mathfrak{n}_J \} \to \mathbb{F}$$

there exists an element $\phi \in (U_J)_J$ with $l(f_{v_\Lambda y v_\Lambda}) = \alpha(1,\phi)(f_{v_\Lambda y v_\Lambda})$ for all $y \in \mathfrak{n}_J$.

Choose $(\_ | \_)$-dual bases of $\mathfrak{n}_J^+, \mathfrak{n}_J$, adopted to the root space decomposition:

$$x_{\alpha i} \in \mathfrak{g}_\alpha, \quad \alpha \in \Delta^+_J, \quad i = 1, \ldots, m_\alpha, \quad y_{\beta i} \in \mathfrak{g}_{-\beta}, \quad \beta \in \Delta^+_J, \quad i = 1, \ldots, m_\beta$$

such that $\langle x_{\alpha i} | y_{\beta j} \rangle = \delta_{\alpha \beta} \delta_{ij}$.

We have $(\Lambda | \beta) > 0$ for all $\beta \in \Delta^+_J$. Define recursively elements $b_{\beta j} \in \mathbb{F}$, $j = 1, \ldots, m_\beta, \quad \beta \in \Delta^+_J$, as follows:

$$b_{\beta 1} := \frac{1}{(\Lambda | \beta) \langle v_\Lambda | v_\Lambda \rangle} l(f_{v_\Lambda y_{\beta 1} v_\Lambda}) \quad \text{for} \quad \text{ht} \beta = 1.$$ 

Let $\beta \in \Delta^+_J$ with $\text{ht} \beta > 1$, and let $b_{\alpha i}$ be defined for all $i = 1, \ldots, m_\alpha, \alpha \in \Delta^+_J$ with $\text{ht} \alpha < \text{ht} \beta$. Set:

$$b_{\beta j} := \frac{1}{(\Lambda | \beta) \langle v_\Lambda | v_\Lambda \rangle} l(f_{v_\Lambda y_{\beta j} v_\Lambda}) - \langle v_\Lambda | \exp(\sum_{\substack{\alpha \in \Lambda \setminus \Lambda^J \beta \in \Delta^+_J \text{ht} \alpha < \text{ht} \beta}} b_{\alpha i} x_{\alpha i} y_{\beta j} v_\Lambda) \rangle.$$

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Next we show that we can take \( \phi = \exp(\sum_{\alpha,i} b_{\alpha i} x_{\alpha i}) \). Let \( \beta \in \Delta^+_J \). By expanding the exponential function we find for \( j = 1, \ldots, m_\beta \):

\[
\langle v_\Lambda | \exp(\sum_{\alpha,i} b_{\alpha i} x_{\alpha i}) y_{\beta j} v_\Lambda \rangle = \langle v_\Lambda | \exp(\sum_{\alpha,i} b_{\alpha i} x_{\alpha i}) y_{\beta j} v_\Lambda \rangle + \langle v_\Lambda | \exp(\sum_{\alpha,i} b_{\alpha i} x_{\alpha i}) y_{\beta j} v_\Lambda \rangle .
\]

Denote by \( \nu : h \to h^* \) the linear isomorphism induced by the invariant bilinear form \( \langle \cdot | \cdot \rangle \). Using \([K, P 2]\), Lemma 4.2, we find that the second summand on the right side equals \( b_{\beta j}(\Lambda | \beta) \langle v_\Lambda | v_\Lambda \rangle \). After inserting the definition of \( b_{\beta j} \), the right side equals \( l(f_{v_\Lambda y_{\beta j} v_\Lambda} | U_J) \).

To prepare the proof of 2), we first show two propositions:

**Proposition 4.3** Let \( \Lambda \in P^+ \) and \( J \subseteq I \). Then \((U_f)^J\) fixes the points of \( U(n_J)(\Lambda) \).

**Proof:** The case \( J = \emptyset \) is obvious, let \( J \neq \emptyset \). If we fix an element \( v \in U(n_J)(\Lambda) \), then for every element \( \tilde{x} \in (n_J)^J \), there exists an element \( x \in n_J \), such that \( \exp(\tilde{x}) v = \exp(x) v \). Therefore it is sufficient to show, that every element \( x \in n_J \) acts trivially on \( U(n_J)(\Lambda) \).

Let \( v_\Lambda \in L(\Lambda) \setminus \{0\} \). We show by induction over \( n \in \mathbb{N}_0 \):

\[
\begin{align*}
  n = 0 : & \quad x v_\Lambda = 0 \quad \text{for all } x \in n_J , \\
  n \in \mathbb{N} : & \quad x y_1 \cdots y_n v_\Lambda = 0 \quad \text{for all } x \in n_J , y_1, \ldots, y_n \in n_J^- .
\end{align*}
\]

Clearly the statement for \( n = 0 \) is valid. The induction step from \( n \) to \( n + 1 \) follows from the equation:

\[
x y_1 y_2 \cdots y_{n+1} v_\Lambda = [x, y_1] y_2 \cdots y_{n+1} v_\Lambda + y_1 (x y_2 \cdots y_{n+1} v_\Lambda) ,
\]

together with \([n_J, n_J^-] \subseteq n_J^\perp) .

\[ \square \]

**Proposition 4.4** Let \( J \subseteq I \), and let \( \Lambda \in F_{I \setminus J} \cap P^+ , N \in F_J \cap P^+ \). The comorphism \( m^* : \mathbb{F}[U] \to \mathbb{F}[U_J] \otimes \mathbb{F}[U^J] \), dual to the multiplication map \( m : U_J \times U^J \to U \), is an isomorphism of algebras. Furthermore we have:

\[
\begin{align*}
m^*(f_{v_{\Lambda} y_{\Lambda}} | U) = f_{v_{\Lambda} y_{\Lambda}} | U_J \otimes 1 , & \quad y \in n_J^- . \\
m^*(f_{v_{\Lambda} y_{\Lambda}} | U) = 1 \otimes f_{v_{\Lambda} y_{\Lambda}} | U^J , & \quad y \in (n_J^-)^\perp .
\end{align*}
\]

**Proof:** Due to \([K, P 2]\), Lemma 4.2, \( \mathbb{F}[U] \) is a Hopf algebra. Therefore for \( f \in \mathbb{F}[U] \) there exist functions \( f_i , g_i \in \mathbb{F}[U] \), \( i = 1, \ldots, m \), such that \( f(u_1 u_2) = \sum_{i=1}^m f_i(u_1) g_i(u_2) \) for all \( u_1 , u_2 \in U \). Restricting to \( u_1 \in U_J , u_2 \in U^J \) we get \( f \circ m = \sum_{i=1}^m f_i | U_J \otimes g_i | U^J \in \mathbb{F}[U_J] \otimes \mathbb{F}[U^J] \).

\( m^* \) is injective, because \( m \) is surjective. To show the surjectivity of \( m^* \), due to
Theorem 4.1. It is sufficient to show the equations (1) and (2). Due to the last proposition we find for all \( y \in n_j^- \), and for all \( u_1 \in U_J, u_2 \in U^J \subseteq (U_f)^J \):

\[
m^* (f_{v_\lambda y_\lambda} | v)(u_1, u_2) = \langle \langle v_\lambda | u_1 u_2 y_\lambda \rangle \rangle = \langle \langle v_\lambda | u_1 y_\lambda \rangle \rangle = (f_{v_\lambda y_\lambda} | v_J \otimes 1)(u_1, u_2).
\]

For \( j \in J \) the \( \alpha_j \)-string of \( P(N) \) through the highest weight \( N \), due to \( N = \sigma J N \). Therefore \( G_J \) is contained in the stabilizer of \( v_N \), and we find for \( y \in (n)^J^- \), and for all \( u_1 \in U_J, u_2 \in U^J \):

\[
m^* (f_{v_N y_N} | v)(u_1, u_2) = \langle \langle (u_1)^* v_N | u_2 y_N \rangle \rangle = \langle \langle v_N | u_2 y_N \rangle \rangle = (1 \otimes f_{v_N y_N} | v_J)(u_1, u_2).
\]

\[\square\]

Proof of Theorem 4.2: The case \( J = I \) is trivial. Let \( J \neq I \). We first show \((U_f)^J \subseteq U^J \): Let \( \phi \in (U_f)^J \). We have \((U_f)^J \subseteq U_f = \overline{U} \subseteq G \). Denote by \( \alpha(1, \phi) : F[G] \rightarrow F \) the homomorphism of algebras corresponding to \( \phi \). Denote by \( I(U^J) \) the vanishing ideal of \( U^J \) in \( F[G] \). Because of Proposition 2.3 it is sufficient to show that \( I(U^J) \) is contained in the kernel of \( \alpha(1, \phi) \).

Let \( f \in I(U^J) \). Due to the Peter and Weyl theorem \( f \) is of the form \( f = \sum_{v_i m_i} f_{v_i m_i} | G \). Because \( \phi \) is of the form \( \phi = \exp(\prod_{\alpha} x_{\alpha}) \), \( x_{\alpha} \in (n)^J_{\alpha} \), \( \alpha \in (J^+) \), it is sufficient to show:

\[
0 = \sum_i \langle \langle v_i | w_i \rangle \rangle, \quad (3)
\]

\[
0 = \sum_i \langle \langle v_i | x_1 \cdots x_k w_i \rangle \rangle \quad \text{for all } x_1, \ldots, x_k \in n^J, k \in \mathbb{N}. \quad (4)
\]

Equation (3) follows because of \( 1 \in U^J \). It is sufficient to show equation (4) for a system of generators of \( n^J \).

Define recursively a multibracket for elements of \( g \):

\[
[x] := x, \quad x \in g, \\
[x_{k+1}, x_k, \ldots, x_1] := [x_{k+1}, [x_k, \ldots, x_1]], \quad x_1, \ldots, x_k \in g, k \in \mathbb{N}.
\]

By an easy induction, a multibracket of the form \([y, x_1, \ldots, x_k]\) is a linear combination of multibrackets \([x_{i_1}, \ldots, x_{i_k}, y]\), where \((i_1, \ldots, i_k)\) is a permutation of \((1, \ldots, k)\).

\(n^+\) is generated by \( g_{\alpha}, \alpha \in \Delta^+_r = (\Delta_J^+)_{re} \cup (\Delta^+_J)_{re} \). We have \( n = n_j \oplus n^J \) and \([n_j, n^J] \subseteq n^J \). All multibrackets with all elements in \( g_{\alpha}, \alpha \in (\Delta^+_J)_{re} \) are in \( n_j \), all multibrackets with an element in \( g_{\beta}, \beta \in (\Delta_J^+)_{re} \) are in \( n^J \). Therefore \( n^J \) is generated by all multibrackets, which contain an element in \( g_{\beta}, \beta \in (\Delta_J^+)_{re} \).

By the remark of above it is sufficient to consider the multibrackets:

\([x_{\gamma_m}, \cdots, x_{\gamma_1}, x_{\gamma_0}]\)
with \( x_{\gamma_i} \in \mathfrak{g}_{\gamma_i}, \ i = 0, \ldots, m, \) and \( \gamma_1, \ldots, \gamma_m \in \Delta^+_r; \ \gamma_0 \in (\Delta^+_J)_r, \ m \in \mathbb{N}. \) \( U^J \) is normal in \( U. \) Therefore for \( t_0, \ldots, t_m \in \mathbb{F} \) we have

\[
u(t_m, \ldots, t_1, t_0) := \exp(t_m x_{\gamma_m}) \cdots \exp(t_1 x_{\gamma_1}) \exp(t_0 x_{\gamma_0}) \exp(-t_1 x_{\gamma_1}) \cdots \exp(-t_m x_{\gamma_m}) \in U^J.
\]

We consider a product \( p \) of \( k \) factors of such expressions in different variables. To simplify our notation, we only write down a factor in the middle of \( p: \)

\[
p = \cdots u(t_m, \ldots, t_1, t_0) \cdots \text{ where } \ldots, t_m, \ldots, t_1, t_0, \ldots \in \mathbb{F}.
\]

Because of \( p \in U^J \) we get

\[
0 = f(p) = \sum_i \langle \langle v_i | \cdots u(t_m, \ldots, t_1, t_0) \cdots w_i \rangle \rangle.
\]

Because the root vectors belonging to real roots act locally nilpotent, the right side is polynomial in \( \cdots t_0, t_1, \ldots, t_m \cdots \). Because of \(|\mathbb{F}| = \infty \) the coefficients of the monomials are zero. In particular for the monomial \( \cdots t_m \cdots t_1 t_0 \cdots \) we find

\[
\sum_i \langle \langle v_i | \cdots [x_{\gamma_m}, \ldots, x_{\gamma_1}, x_{\gamma_0}] \cdots w_i \rangle \rangle = 0.
\]

Now we show \((U_f)^J \supseteq \overline{U^J} \): Let \( u \in \overline{U^J} \). Denote by \( \alpha \) the map \( \overline{U^-} \times \overline{U} \to \text{Spec} \mathbb{F} [U] \), and by \( \tilde{\alpha} \) the maps \( \overline{U^-} \times \overline{U} \to \text{Spec} \mathbb{F} \{U_j\}, \overline{U^-} \times \overline{U} \to \text{Spec} \mathbb{F} \{U'_j\} \). Using the last proposition and its notation, and 1), there exists an element \( \tilde{u} \in U_f \) such that

\[
\alpha(1, \tilde{u}) = (\tilde{\alpha}(1, 1) \otimes \tilde{\alpha}(1, u)) \circ m^*.
\]

Write \( \tilde{u} \) in the form \( u_j u'_j \) with \( u_j \in (U_f)_J, u'_j \in (U_f)_J^J \). Choose elements \( \Lambda \in F_J \cap P^+ \) and \( v_{\Lambda} \in L(\Lambda)_{\Lambda} \setminus \{0\} \), and apply the left and right sides of the last equation to the elements \( f_{v_{\Lambda} y v_{\Lambda}} | u, y \in n_J \). Using equation (5) of the last proposition we get

\[
\langle \langle v_{\Lambda} | u_j u'_j y v_{\Lambda} \rangle \rangle = \langle \langle v_{\Lambda} | y v_{\Lambda} \rangle \rangle \text{ for all } y \in n_J^J.
\]

Due to Proposition 4.3 the left side is equal to \( \langle \langle v_{\Lambda} | u_j y v_{\Lambda} \rangle \rangle \). Using Theorem 4.1, 1), and 1) we conclude \( u_j = 1 \).

Insert \( \tilde{u} = u_j \) in equation (6). Choose elements \( N \in F_J \cap P^+ \) and \( v_N \in L(N)_{N} \setminus \{0\} \), and apply the left and right sides of (6) to the elements \( f_{v_N y v_N} | u, y \in (n_J^J)^- \). Using equation (5) of the last proposition we find

\[
\langle \langle v_N | u'_j y v_N \rangle \rangle = \langle \langle v_N | u y v_N \rangle \rangle \text{ for all } y \in (n_J^J)^-.
\]

We have \( 1, u'_j \in (U_f)_J \subseteq \overline{U^J} \). Using Theorem 4.1, 2) we conclude \( \tilde{\alpha}(1, u'_j) = \tilde{\alpha}(1, u) \), from which follows \( u = u'_j \in (U_f)_J \).

\[\square\]
5 The $\mathbb{F}$-valued points of $\mathbb{F}[G]$ and the Birkhoff decomposition

Let $J \subseteq I$. Define the following subsets of $gr$-$End$:

$$
\hat{G}_f := \hat{G}U_f = \bigcup_{\hat{n} \in \hat{N}} U^\pm \hat{n}U_f ,
$$

$$
(\hat{G}_f)_J := \hat{G}_J(U_f)_J = \bigcup_{\hat{n} \in \hat{N}_J} U^+_J \hat{n}(U_f)_J .
$$

We call the unions on the right the Bruhat and Birkhoff coverings. During our investigation of $\text{Specm} \mathbb{F}[G]$, we will find, that $\hat{G}_f$ is a monoid, and the Bruhat and Birkhoff coverings are really decompositions. Similar things hold for $(\hat{G}_f)_J$.

The monoid $\overline{G}$ contains $G$, as well as the closures $\overline{T} = \overline{T}$ and $\overline{U} = U_f$. Therefore it also contains $\hat{G}_f$, and due to Proposition 2.3 we have:

$$
\hat{G}_f \times \hat{G}_f \text{ maps to } \text{Specm} \mathbb{F}[G] . \quad (6)
$$

Similarly we get:

$$
(\hat{G}_f)_J \times (\hat{G}_f)_J \text{ maps to } \text{Specm} \mathbb{F}[G_J] . \quad (7)
$$

Our first aim is to show the surjectivity of (6). The key step of the proof is an induction over $|J|$, showing the surjectivity of (7). The next two theorems prepare the induction step. They relate $\text{Specm} \mathbb{F}[G_J]$ to $\text{Specm} \mathbb{F}[G_{J \setminus \{j\}}], j \in J$.

For $\Lambda \in P^+$ choose a nonzero element $v_\Lambda \in L(\Lambda)_\Lambda$ and define

$$
\theta_\Lambda := \frac{f_{v_\Lambda v_\Lambda}|_G}{\langle v_\Lambda | v_\Lambda \rangle} \in \mathbb{F}[G] . \quad (8)
$$

The function $\theta_\Lambda$ is independent of the chosen element $v_\Lambda$, and the chosen non-degenerate contravariant symmetric bilinear form on $L(\Lambda)$. Set $\theta_i := \theta_{\Lambda_i}, i = 1, \ldots, n$.

Kac and Peterson showed in [K,P 2] by checking on the dense principal open set $U^-TU^+$ of $G$:

$$
\theta_\Lambda \theta_{\Lambda'} = \theta_{\Lambda + \Lambda'} \quad \text{for all } \Lambda, \Lambda' \in P^+ . \quad (9)
$$

The next theorem gives a covering of $\text{Specm} \mathbb{F}[G_J]$ by principal open sets, which are built with these functions. A variant of this covering for the full spectrum of $\mathbb{F}[G]$ has been given by Kashiwara in [Kas], Proposition 6.3.1.

Denote the action of $G_J \times G_J$ on $\mathbb{F}[G_J]$, which is induced by the action $\pi$ of $G \times G$ on $\mathbb{F}[G]$, also by $\pi$. Write $\theta_\Lambda$ instead of $\theta_\Lambda|_{G_J}$ for short.
Theorem 5.1  Let \( \emptyset \neq J \subseteq I \). We have

\[
\bigcup_{g,h \in G_J} \bigcup_{j \in J} D_{\text{Specm } \mathbb{F}[G_J]}(\pi(g,h)\theta_j)
\]

\[
= \begin{cases} 
\text{Specm } \mathbb{F}[G_J] & \text{ if } J \text{ is not special} \\
(\text{Specm } \mathbb{F}[G_J]) \setminus \{ \alpha(1,e(R(J))) \} & \text{ if } J \text{ is special} 
\end{cases}
\]

Proof: Suppose there exists an element \( \alpha \in \text{Specm } \mathbb{F}[G_J] \) not contained in the union on the left. Then for all \( g,h \in G_J \) and \( j \in J \) we have \( \alpha(\pi(g,h)\theta_j) = 0 \). Here \( \alpha \circ \pi(g,h) \) is a homomorphism of algebras. Because of the multiplicative property \( \hat{\mathfrak{m}} \) we find \( \alpha(\pi(g,h)\theta_{\Lambda}) = 0 \) for all \( g,h \in G_J \) and \( \Lambda \in P_J^+ \setminus \{0\} \), where \( P_J^+ := P_J \cap P^+ = \mathbb{N}_0\text{-span} \{ \Lambda_j \mid j \in J \} \).

Now \( L_J(\Lambda) := U_{\text{specm }} \mathbb{F}(\Lambda) \) is an irreducible \( G_J \)-module, \( \Lambda \in P_J^+ \). Since \( G_JL(\Lambda) \) spans \( L_J(\Lambda) \), we find \( \alpha(f_vw)_{|G_J} = 0 \) for all \( v,w \in L_J(\Lambda) \), \( \Lambda \in P_J^+ \setminus \{0\} \). Due to Theorem 5.12 of \( \text{[M1]} \) this is impossible if \( J \) is not special, and \( \alpha = \alpha(1,e(R(J))) \) if \( J \) is special.

The principal open subsets, which have been used in the covering of \( \text{Specm } \mathbb{F}[G_J] \), can be obtained by

\[
D_{\text{Specm } \mathbb{F}[G_J]}(\pi(g,h)\theta_j) = (\text{Specm } \mathbb{F}[D_{G_J}(\pi(g,h)\theta_j)])|_{\mathbb{F}[G_J]}.
\]

The next theorem gives a product decomposition of the principal open sets \( D_{G_J}(\theta_j), j \in J \). (Set \( L = J \setminus \{j\} \) and \( \Lambda = \Lambda_j, j \in J \). It can be proved in the same way as Theorem 5.11 a), and b) in \( \text{[M1]} \). A decomposition of the coordinate rings analogous to the second part of b) has been given in \( \text{Kas} \), Lemma 5.3.4 and 5.3.5.

Theorem 5.2  Let \( L \subsetneq J \subseteq I \). Set \( (U_J^+)^L := \bigcap_{\sigma \in W_u} \sigma U_J^+ \sigma^{-1} \). For \( \Lambda \in P_J \cap F_L \) we have:

a) \( D_{G_J}(\theta_{\Lambda}) = (U_J^-)^L G_L T_{J \setminus L} (U_J^+)^L \).

b) The multiplication map

\[
m : (U_J^-)^L \times G_L \times T_{J \setminus L} \times (U_J^+)^L \to D_{G_J}(\theta_{\Lambda})
\]

is bijective, and its comorphism

\[
m^* : \mathbb{F}[D_{G_J}(\theta_{\Lambda})] \to \mathbb{F}[(U_J^-)^L] \otimes \mathbb{F}[G_L] \otimes \mathbb{F}[T_{J \setminus L}] \otimes \mathbb{F}[(U_J^+)^L]
\]

exists, and is an isomorphism of algebras.

Theorem 5.3  Let \( J \subseteq I \). Then \( (\widehat{\Gamma}_J) \times (\widehat{\Gamma}_J) \) maps surjectively to \( \text{Specm } \mathbb{F}[G_J] \).

Proof: We show the surjectivity by induction over \( |J| \). The case \( J = \emptyset \) is trivial. For \( J = \{j\} \) the map \( \{1\} \times G_{\langle j \rangle} \to \text{Specm } \mathbb{F}[G_{\langle j \rangle}] \) is already surjective,
because \((G_J, \mathbb{F}[G_J])\) can be identified with \((\text{SL}(2, \mathbb{F}), \mathbb{F}[\text{SL}(2, \mathbb{F})])\).

Now the step of the induction from \(|J| \leq m\) to \(|J| = m + 1\), \((1 < m < |I|)\):

Let \(j \in J\). Due to the induction assumption we have a surjective map
\[
\tilde{\alpha} : (\hat{G}_f)_{J \setminus \{j\}} \times (\hat{G}_f)_{J \setminus \{j\}} \rightarrow \text{Specm} \mathbb{F}[G_{J \setminus \{j\}}].
\]

Because \(\mathbb{F}[P_{(j)}]\) is the classical coordinate ring of the torus, we get a bijective map
\[
\tilde{\alpha} : \{1\} \times T_{(j)} \rightarrow \text{Specm} \mathbb{F}[P_{(j)}].
\]

Due to Theorem 4.2, Proposition 2.3, and Remark 3) after Proposition 2.3, we have bijective maps
\[
\tilde{\alpha} : \{1\} \times ((U_f)_{J \setminus \{j\}}) \rightarrow \text{Specm} \mathbb{F}[[U_{(j)}]_{J \setminus \{j\}}],
\]
\[
\tilde{\alpha} : ((U_f)_{J \setminus \{j\}}) \times \{1\} \rightarrow \text{Specm} \mathbb{F}[[U_{\setminus J\setminus \{j\}}].
\]

Due to the last theorem an element \(\beta \in \text{Specm} \mathbb{F}[D_G(J_I)]\) can be written in the form
\[
\beta = (\tilde{\alpha}(u, 1) \otimes \tilde{\alpha}(x, y) \otimes \tilde{\alpha}(1, t) \otimes \tilde{\alpha}(1, \tilde{u})) \circ m^0
\]
with \(u, \tilde{u} \in ((U_f)_{J \setminus \{j\}}, t \in T_{(j)}\), and \(x, y \in (\hat{G}_f)_{J \setminus \{j\}}\).

Let \(N \in P^+\). Choose \((\langle | \rangle)-dual\) bases of \(L(N)_\lambda\), by choosing \((\langle | \rangle)-dual\) bases
\[
(a_{\lambda i})_{i=1, \ldots, m_\lambda}, \quad (c_{\lambda i})_{i=1, \ldots, m_\lambda}
\]
of \(L(N)_\lambda\) for every \(\lambda \in P(N)\). Let \(v \in L(N)_\lambda\), \(w \in L(N)_\mu\). By applying \(m^0\) to \(f_{vw\lvert G_J}\) we find
\[
m^0(f_{vw\lvert G_J})
= \sum_{\langle a_{\lambda i} | a_{\mu j} \rangle \geq \lambda \geq \mu} f_{e_{\lambda i} a_{\mu j} \lvert G_{J \setminus \{j\}}} \otimes f_{e_{\mu j} \lvert G_{J \setminus \{j\}}} \otimes \delta_{\lambda, \mu} \delta_{j, 0} \times \mathbb{F}[[U_{(j)}]_{J \setminus \{j\}}.
\]

This sum is finite, due to \(P(N) \subseteq N - Q_0^+\). By applying \(\beta\) to \(f_{vw\lvert G_J}\) we get
\[
\beta(f_{vw\lvert G_J})
= \sum_{\langle a_{\lambda i} | a_{\mu j} \rangle \geq \lambda \geq \mu} \langle x_{\lambda i} | gta_{\mu j} \rangle \langle yta_{\mu j} | x_{\lambda i} \rangle
= \langle x_{uv} | yta_{\tilde{u}} \rangle.
\]

In particular for \(v \in U(n^+_J) L(N)_N \cap L(N)_\lambda\), \(w \in U(n^-_J) L(N)_N \cap L(N)_\mu\) we have
\[
\beta(f_{vw\lvert G_J}) = \langle x_{uv} | yta_{\tilde{u}} \rangle = \alpha(xu, yta_{\tilde{u}})(f_{vw\lvert G_J}).
\]
Here α denotes the map \((\widehat{G_f})_J \times (\widehat{G_f})_J \to \text{Spec}m\mathbb{F}[G_f]\) due to [5]. Because \(\mathbb{F}[G_f]\) is already spanned by the functions \(f_{vw}|_{G_f}, v, w \in \text{U}(\mathfrak{n}_f^{-1})L(N)N, N \in P_f^+ = P_f \cap P^+\), we conclude \(\beta|_{\mathbb{F}[G_f]} = \alpha(xu, yt\tilde{u}) \in \text{Spec} \mathbb{F}[G_f]\).

It is easy to check that for \(g, h \in G_f\) we have

\[
\left(\text{Spec} \mathbb{F}[D_{G_f}((\pi(g, h)\theta_f))]|_{\mathbb{F}[G_f]}\right) = \left(\text{Spec} \mathbb{F}[D_{G_f}((\theta_f))]|_{\mathbb{F}[G_f]} \circ \pi(g^{-1}, h^{-1})\right).
\]

Because of Theorem 5.1, and because of the equation \(\alpha(a, b) \circ \pi(c, d) = \alpha(ac, bd)\), we get

\[
\bigcup_{g, h \in G_f} \bigcup_{j \in J} \left\{ \alpha(xug^{-1}, y\tilde{u}h^{-1}) \mid x, y \in (\widehat{G_f})_{\pi(j)} , u, \tilde{u} \in ((U_f)_j)^{\pi(j)}, t \in T_f(j) \right\}
\]

\[
= \begin{cases} 
\text{Spec} \mathbb{F}[G_f] & \text{if } J \text{ is not special} \, , \\
\text{Spec} \mathbb{F}[G_f] \setminus \{ \alpha(1, e(R(J))) \} & \text{if } J \text{ is special} \, .
\end{cases}
\]

Using the Bruhat covering of \((\widehat{G_f})_J\), the Bruhat decompositions of \((G_f)_J\) and \(\widehat{G_f}_J\), we find

\[
(\widehat{G_f})_J \subseteq (\widehat{G_f})_J(G_f)_J = U_f\tilde{N}_f(U_f)_J(G_f)_J = \frac{U_f\tilde{N}_fU_fN_f(U_f)_J}{G_f}\, .
\]

From this follows \((\widehat{G_f})_J \backslash T_f(J)((U_f)_J)^{\pi(j)}G_f \subseteq (\widehat{G_f})_J\) for all \(j \in J\). Furthermore, if \(J\) is special, then \(e(R(J)) \in (\widehat{G_f})_J\). Therefore the map \(\alpha : (\widehat{G_f})_J \times (\widehat{G_f})_J \to \text{Spec}m\mathbb{F}[G_f]\) is surjective.

\[\square\]

**Corollary 5.4** \(\widehat{G_f} \times \widehat{G_f}\) maps surjectively to \(\text{Spec}m\mathbb{F}[G]\).

**Proof:** Let \(m : G_f \times T_{rest} \to G\) be the multiplication map, and \(m^* : \mathbb{F}[G] \to \mathbb{F}[G_f] \otimes \mathbb{F}[T_{rest}]\) its comorphism. Due to the bijectivity of \(m^*\), and due the last theorem and Theorem 3.2, 3), the elements of \(\text{Spec} \mathbb{F}[G]\) are given by

\[
(\hat{\alpha}(x, y) \otimes \hat{\alpha}(1, t)) \circ m^* \, ,
\]

where \(x, y \in (\widehat{G_f})_J\) and \(t \in T_{rest}\). Applying this expression to the matrix coefficients \(f_{vw}|_{\mathcal{G}}, v \in L(\Lambda), w \in L(\Lambda)^\mu, \lambda, \mu \in P(\Lambda), \Lambda \in P^+,\) we find

\[
(\hat{\alpha}(x, y) \otimes \hat{\alpha}(1, t)) \circ (m^*(f_{vw}|_{\mathcal{G}})) = (\hat{\alpha}(x, y) \otimes \hat{\alpha}(1, t))(f_{vw}|_{\mathcal{G}_\lambda} \otimes e^\mu|_{T_{rest}}) = (\langle xv \mid yw \rangle) e^\mu(t) = \alpha(x, yt)(f_{vw}|_{\mathcal{G}}) \, .
\]

Here α denotes the map \(\widehat{G_f} \times \widehat{G_f} \to \text{Spec} \mathbb{F}[G]\) due to [5]. Therefore \(\text{Spec} \mathbb{F}[G]\) = \(\left\{ \alpha(x, y) \mid x \in (\widehat{G_f})_J, y \in (\widehat{G_f})_J \cap T_{rest} \right\}\), in particular α is surjective.

\[\square\]

The next theorem gives one of the main results of this paper: A description of the \(\widehat{G_f} \times \widehat{G_f}\)-set \(\text{Spec} \mathbb{F}[G]\).

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Theorem 5.5
1) We have $\hat{G} = \hat{\hat{G}}$ and $\hat{G} = \hat{G_f}$. In particular $\hat{G}_f$ is a monoid.
2) The map $\hat{G} \times G \rightarrow \text{Spec} F[G]$ is a $G \times G$-equivariant bijection.

Proof: The first equation of 1) has been shown in [M 1], Theorem 5.14. To prepare the proof of the rest of the theorem, we first show the following statements a) - d):

a) We show $B^- \subseteq \hat{B}^-$. For $\Lambda \in P^+$ and $\lambda \in P(\Lambda)$ define

$$L(\Lambda)_{\lambda} := \bigoplus_{\mu \in P(\Lambda) \cap (\lambda - Q_0^+)} L(\Lambda)_{\mu}.$$ 

Since $L(\Lambda)_{\lambda}$ is $B^-$-invariant, we have

$$\langle \langle w | B^- v \rangle \rangle = 0 \quad \text{for all} \quad v \in L(\Lambda)_{\lambda}, \ w \in (L(\Lambda)_{\lambda})^\perp.$$ 

These equations are also valid, if $B^-$ is replaced by its Zariski closure $\overline{B}^-$, and due to the orthogonality of the weight spaces we have $(L(\Lambda)_{\lambda})^\perp \perp = L(\Lambda)_{\lambda}$. Therefore $L(\Lambda)_{\lambda}$ is also $\overline{B}^-$-invariant.

Let $b \in B^-$. Choose a pair of $\langle \langle \ | \rangle \rangle$-dual bases of $L(\Lambda)$, by choosing $\langle \langle \ | \rangle \rangle$-dual bases $(a^A_{\lambda i})_{i=1,\ldots,m_{\lambda}}, (c^A_{\lambda i})_{i=1,\ldots,m_{\lambda}}$ of $L(\Lambda)$ for every $\lambda \in P(\Lambda)$. For a fixed weight $\mu \in P(\Lambda)$ we have

$$\langle \langle a^A_{\mu i} | b c^A_{\mu' i'} \rangle \rangle \neq 0$$

at most for the finitely many weights $\mu' \in P(\Lambda)$ with $\mu' \geq \mu$. Therefore we get a well defined linear map $\psi_\Lambda$ by

$$\psi_\Lambda v := \sum_{\mu i} \sum_{\mu' i'} a^A_{\mu i} \langle \langle a^A_{\mu i} | b c^A_{\mu' i'} \rangle \rangle \langle \langle c^A_{\mu i} | v \rangle \rangle, \ v \in L(\Lambda).$$

These maps $\psi_\Lambda$, $\Lambda \in P^+$, define an element $\psi$ of $\text{gr-End}$. It is easy to check, that $\psi$ is the adjoint of $b$. Therefore $b \in \overline{B}^- \cap \text{gr-Adj} = \overline{B}^-.$

b) Let $\psi \in \overline{G}$, $b \in \overline{B}$ with $\alpha(1, \psi) = \alpha(b, 1)$. We show

$$b, \psi \in \hat{G} \quad \text{and} \quad \psi^* = b.$$

Due to the second part of Proposition 5.1, the homomorphism $\alpha(b, 1) = \alpha(1, \psi)$ factors to a homomorphism $F[B^-] \rightarrow F$. Due to the first part of Proposition 5.1 and part a) of above, we find

$$\psi \in \overline{B}^- \subseteq \overline{B}^- \subseteq \overline{G} = \hat{G} \subseteq \text{gr-Adj}.$$
By checking on the matrix coefficients, using the non-degeneracy of the contravariant bilinear forms, we find $\psi^* = b$.

c) Let $g, h \in G_f$ and $x, y, \tilde{x}, \tilde{y} \in \widehat{G}$. By checking on the matrix coefficients we find:

$$\alpha(xg, yh) = \alpha(\tilde{x}, \tilde{y}) \iff \alpha(x, y) = \alpha(\tilde{x}g^{-1}, \tilde{y}h^{-1}) .$$

d) We show $\text{Specm} \, F[G] = \alpha(U_f, \tilde{N}U_f)$: Due to the last corollary, every element of $\text{Specm} \, F[G]$ is of the form $\alpha(\psi, \tilde{\psi})$ with $\psi, \tilde{\psi} \in \widehat{G_f}$. Due to the Birkhoff covering of $\widehat{G_f}$, we can write $\psi, \tilde{\psi}$ in the form

$$\psi = u_- n_\sigma e(R) u_+ \quad \psi = \tilde{u} - \tilde{n}_\tilde{\sigma} e(\tilde{R}) \tilde{u}_+ \quad \text{with} \quad u_-, \tilde{u}_- \in U^-, \quad u_+, \tilde{u}_+ \in U_f, \quad n_\sigma, \tilde{n}_{\tilde{\sigma}} \in N .$$

From this follows:

$$\alpha(\psi, \tilde{\psi}) \sim (u_+, e(R)n_\sigma^* u_- \tilde{u} - \tilde{n}_{\tilde{\sigma}} e(\tilde{R}) \tilde{u}_+) .$$

Due to the Birkhoff decomposition of $\widehat{G}$, we can write $e(R)n_\sigma^* u_- \tilde{u} - \tilde{n}_{\tilde{\sigma}} e(\tilde{R})$ in the form $u'_- \tilde{n}'u'_+$. We get

$$\alpha(\psi, \tilde{\psi}) \sim ((u'_-)^* u_+, \tilde{n}'u'_+ \tilde{u}_+) .$$

Therefore $\alpha(\psi, \tilde{\psi}) = \alpha((u'_-)^* u_+, \tilde{n}'u'_+ \tilde{u}_+)$.

Now we can prove the theorem:

To 1) To show $\overline{G} = \widehat{G_f}$, it is sufficient to show $\overline{G} \subseteq \widehat{G_f}$. Let $\phi \in \overline{G}$. Then due to d) there exist elements $u, \tilde{u} \in U_f, e(R)n_\sigma \in \tilde{N}$ such that

$$\alpha(1, \phi) = \alpha(u, e(R)n_\sigma \tilde{u}) .$$

Using c) this is equivalent to

$$\alpha(1, \phi(n_\sigma \tilde{u})^{-1}) = \alpha(e(R)u, 1) .$$

The monoid $\overline{B}$ contains $\overline{T}$ and $U_f$. Therefore $e(R)u \in \overline{T}U_f \subseteq \overline{B}$, and due to b) we get $\phi \in \overline{Gn_\sigma \tilde{u}} \subseteq \widehat{G_f}$.

To 2) Due to the last corollary, we only have to show the injectivity in 2). Due to the proof of d), we may start with elements $u, \tilde{u} \in U_f, g, \tilde{g} \in NU_f$ such that

$$\alpha(e(R)u, g) = \alpha(e(\tilde{R})\tilde{u}, \tilde{g}) .$$

Using c) this equation is equivalent to

$$\alpha(e(R)u \tilde{u}^{-1}, 1) = \alpha(1, e(\tilde{R})\tilde{g}g^{-1}) .$$
Due to b) we find $e(R)u\tilde{u}^{-1} \in \hat{G}$ and $(e(R)u\tilde{u}^{-1})^* = e(\tilde{R})\tilde{g}^{-1}$. From this follows
\[
(e(R)u, g) = ((e(R)u\tilde{u}^{-1})\tilde{u}, g) \sim (\tilde{u}, (e(R)u\tilde{u}^{-1})^* g) = (\tilde{u}, e(\tilde{R})\tilde{g}) \sim (e(\tilde{R})\tilde{u}, \tilde{g}) .
\]

The group $U_f \times U_f$ acts on $\text{Specm} \mathbb{F}[G]$. The corresponding partition into orbits is described by the Birkhoff decomposition in the following theorem:

**Theorem 5.6**

1) There are the following Bruhat- and Birkhoff decompositions of $\hat{G}_f$:

\[
\hat{G}_f = \bigcup_{n \in \hat{N}} U^\pm nU_f .
\]

2) There is the following Birkhoff decomposition of $\text{Specm} \mathbb{F}[G]$:

\[
\text{Specm} \mathbb{F}[G] = \bigcup_{n \in \hat{N}} \alpha(U_f, nU_f) .
\]

**Proof:** Due to the Bruhat and Birkhoff coverings of $\hat{G}_f$ and part d) of the proof of the last theorem, we only have to show that these unions are disjoint.

a) First we do this for 2). Suppose there exist elements $u_1, u_2, \tilde{u}_1, \tilde{u}_2 \in U_f$ such that

\[
\alpha(u_1, n_\sigma e(R)u_2) = \alpha(\tilde{u}_1, n_\tilde{\sigma} e(\tilde{R})\tilde{u}_2) .
\]

Using part c) of the proof of the last theorem, we find for all $v, w \in L(\Lambda)$, $\Lambda \in P^+$:

\[
\langle \langle v | n_\sigma e(R)w \rangle \rangle = \langle \langle \tilde{u}_1(u_1)^{-1}v | \tilde{n}_\tilde{\sigma} e(\tilde{R})\tilde{u}_2(u_2)^{-1}w \rangle \rangle .
\] (10)

Fix an element $\Lambda \in P^+$ with $P(\Lambda) \cap X \setminus R \neq \emptyset$. Fix elements $\mu \in P(\Lambda) \cap X \setminus R$ and $w_\mu \in L(\Lambda)_\mu \setminus \{0\}$. By inserting $w = w_\mu$ in the last equation we find

\[
0 = \langle \langle \tilde{v} | e(\tilde{R})\tilde{u}_2(u_2)^{-1}w_\mu \rangle \rangle \quad \text{for all} \quad \tilde{v} \in L(\Lambda) .
\]

Because of $\tilde{u}_2(u_2)^{-1} \in U_f$ we find $\mu \in X \setminus \tilde{R}$.

Since $\bigcup_{\Lambda \in P^+} P(\Lambda) = X \cap P$ this shows $(X \setminus R) \cap P \subseteq (X \setminus \tilde{R}) \cap P$, from which follows $R \supseteq \tilde{R}$. We may interchange the variables with and without $\tilde{\cdot}$, and get $R = \tilde{R}$.

Fix an element $\Lambda \in P^+$ with $P(\Lambda) \cap R \neq \emptyset$. Fix elements $\mu \in R \cap P(\Lambda)$ and $w_\mu \in L(\Lambda)_\mu \setminus \{0\}$. Because of $\tilde{u}_2(u_2)^{-1} \in U_f$ we have

\[
\tilde{\sigma}_\mu \in \text{supp}(\tilde{n}_\tilde{\sigma} e(\tilde{R})\tilde{u}_2(u_2)^{-1}w_\mu) .
\]
Choose a maximal weight $m$ of $\text{supp}(\tilde{n}_\sigma e(R)\tilde{u}_2(u_2)^{-1}w_\mu)$ with $m \geq \tilde{\sigma} \mu$. By inserting $w = w_\mu$ in equation (10), using $\tilde{u}_1(u_1)^{-1} \in U_f$, we find
\[
\langle \langle v | n_\sigma w_\mu \rangle \rangle = \langle \langle v | \tilde{n}_\sigma e(R)\tilde{u}_2(u_2)^{-1}w_\mu \rangle \rangle \quad \text{for all } v \in L(\Lambda)_m \quad (11)
\]
Because $\langle \langle \cdot | \cdot \rangle \rangle$ is nondegenerate on $L(\Lambda)_m$, there exists an element $v_m$, such that the right side is nonzero. Therefore $\sigma \mu = m \geq \tilde{\sigma} \mu$. Interchanging the variables with and without $\tilde{}$ gives $\tilde{\sigma} \mu = \sigma \mu = m$.
Inserting $m = \tilde{\sigma} \mu$ in (11), and using $\tilde{u}_2(u_2)^{-1} \in U_f$, we find
\[
\langle \langle v | n_\sigma w_\mu \rangle \rangle = \langle \langle v | \tilde{n}_\sigma w_\mu \rangle \rangle \quad \text{for all } v \in L(\Lambda)_m = L(\Lambda)_{\sigma \mu} = L(\Lambda)_{\tilde{\sigma} \mu} .
\]
Because of the orthogonality of the weight spaces, this equation is also valid for all $v \in L(\Lambda)$.
This shows
\[
\langle \langle v | n_\sigma e(R)w \rangle \rangle = \langle \langle v | \tilde{n}_\sigma e(R)w \rangle \rangle \quad \text{for all } v, w \in L(\Lambda), \Lambda \in P^+ .
\]
Therefore we get $n_\sigma e(R) = \tilde{n}_\sigma e(R) = \tilde{n}_\sigma e(\tilde{R})$.

b) The Birkhoff decomposition of $\tilde{G}_f$ follows from 2), by using the embedding of $\tilde{G}_f$ in $\text{Specm} \mathbb{F}[G]$. Suppose there exist elements $u_1, \tilde{u}_1 \in U^-$, $u_2, \tilde{u}_2 \in U_f$ such that
\[
(\tilde{u}_1)^* n_\sigma e(R)u_2 = (\tilde{u}_1)^* \tilde{n}_\sigma e(\tilde{R})\tilde{u}_2 .
\]
From this equation follows equation (10), and as in part a) of the proof we can deduce $R = \tilde{R}$. Also similar as in part a), but now using a minimal weight $m$ of $\text{supp}(\tilde{n}_\sigma e(R)\tilde{u}_2(u_2)^{-1}w_\mu)$ with $m \leq \sigma \mu$, we find $n_\sigma e(R) = \tilde{n}_\sigma e(R)$.

\[
\Box
\]

6   The stratification of the spectrum of $\mathbb{F}$-valued points of $\mathbb{F}[G]$ in $G_f \times G_f$-orbits

In this section, we show that the $G_f \times G_f$-orbits of $\text{Specm} \mathbb{F}[G]$ are locally closed, irreducible, and in one to one correspondence with the finitely many special subsets of $I$. The closure relation is given by the inverse inclusion of the special sets. We give a countable covering of each orbit by big cells. We show that there exist stratified transversal slices to the orbits at any of their points.
To cut short our notation, we denote by $x \diamond y$ the image of $(x, y) \in \tilde{G}_f \times \tilde{G}_f$ under the surjective map $\alpha : \tilde{G}_f \times \tilde{G}_f \to \text{Specm} \mathbb{F}[G]$. We make use of this map as a parametrization of $\text{Specm} \mathbb{F}[G]$.
Recall that we have $x \diamond z = z^* x \diamond y$, $x, y, z \in \tilde{G}_f$. Recall that $\tilde{G}_f \times \tilde{G}_f$ acts on $\text{Specm} \mathbb{F}[G]$ by morphisms from the right, i.e.,
\[
(x \diamond y)(\tilde{x}, \tilde{y}) = x \tilde{x} \diamond y \tilde{y} , \quad x, y, \tilde{x}, \tilde{y} \in \tilde{G}_f .
\]
The Chevalley involution of \( F[G] \) induces an involutive morphism \( * \) on \( \text{Specm} F[G] \), which we also call Chevalley involution. It is given by the switch map:

\[
(x \diamond y)^* = y \diamond x, \quad x, y \in \widehat{G}_f.
\]

In this section we also denote by \( f_{vw} \), the function on \( \text{Specm} F[G] \), induced by the matrix coefficient \( f_{vw} \mid G \in F[G] \). It is given by

\[
f_{vw}(x \diamond y) = \langle\langle xv | yw\rangle\rangle, \quad x, y \in \widehat{G}_f.
\]

For a set \( M \subseteq \text{Specm} F[G] \) denote by \( \overline{M} \) its Zariski closure. Note, that for a subset \( A \subseteq \widehat{G} \) we have \( 1 \diamond A \subseteq 1 \diamond A \). Similarly, for a subset \( A \subseteq \widehat{G}_f \), we have \( 1 \diamond A \subseteq 1 \diamond A \). These formulas are useful to determine closures in \( \text{Specm} F[G] \).

**Theorem 6.1**

1) The partition of \( \text{Specm} F[G] \) in \( G_f \times G_f \)-orbits is given by

\[
\text{Specm} F[G] = \bigcup_{\Xi \text{ special}} G_f \diamond e(R(\Xi))G_f.
\]

2) Let \( \Theta \) be special. The orbit \( G_f \diamond e(R(\Theta))G_f \) is locally closed and irreducible. Its closure is given by

\[
\bigcup_{\Xi \supseteq \Theta} G_f \diamond e(R(\Xi))G_f.
\]

**Proof:**
a) We first decompose the \( G_f \times G_f \)-orbit \( G_f \diamond e(R(\Theta))G_f \) in a union of \( U_f \times U_f \)-orbits, i.e., we show

\[
G_f \diamond e(R(\Theta))G_f = \bigcup_{\sigma \in W^e(R(\Theta))W} U_f \diamond (\sigma T)U_f.
\]

By using the Birkhoff decomposition of \( G_f \), and Proposition 2.14, Theorem 2.15 b) of \([M 1]\), compare also the section preliminaries, we get

\[
G_f \diamond e(R(\Theta))G_f = U_f \diamond e(\sigma R(\Theta))G_f = \bigcup_{\sigma \in W} U_f \diamond e(\sigma R(\Theta))G_f.
\]

Because \( W_{\Theta,\Theta^+} \) is the stabilizer of the face \( R(\Theta) \) as a whole, we may restrict the last union to the minimal coset representatives \( W_{\Theta,\Theta^+} \) of \( W/W_{\Theta,\Theta^+} \), characterized by \( W_{\Theta,\Theta^+} = \{ \sigma \in W \mid \sigma \alpha_i \in \Delta^+_\Theta \text{ for all } i \in \Theta \cup \Theta^+ \} \). Next we insert in \( U_f \diamond e(\sigma R(\Theta))G_f \) the Birkhoff decomposition \( G_f = (\sigma U_f \sigma^{-1})NU_f \).

By using Proposition 2.14 of \([M 1]\), compare the section preliminaries, we get

\[
\bigcup_{\sigma \in W^{\Theta,\Theta^+}, \tau \in W} U_f \diamond \sigma U_f^{-1} \sigma^{-1} e(\sigma R(\Theta))\tau B_f = \bigcup_{\sigma \in W^{\Theta,\Theta^+}, \tau \in W} U_f \diamond e(\sigma R(\Theta))\tau B_f.
\]
Since \( \bigcup_{\sigma \in W \cap \Theta^+} \tau \in W \) \( \varepsilon \left( \sigma R(\Theta) \right) \varepsilon = W \varepsilon \left( R(\Theta) \right) W \), we have shown the equality in (14). The disjointness of the union follows from the Birkhoff decomposition of \( \text{Specm} \mathbb{F} \left[ G \right] \).

Taking into account the partition \( \tilde{W} = \bigcup_{\Theta \in sp} W \varepsilon \left( R(\Theta) \right) W \), part 1) of the theorem follows from (14) the Birkhoff decomposition of \( \text{Specm} \mathbb{F} \left[ G \right] \).

b) Next we show that the union \( \bigcup_{\Xi \in sp., \Xi \supseteq \Theta} G_f \circ e(\Xi) G_f \) is closed, i.e., we show that it is the common zero set of the functions \( f_{vw} \), \( v, w \in L(\Lambda) \), \( \Lambda \in P^+ \setminus \overline{F_\Theta} \).

Due to part 1), every element of \( \text{Specm} \mathbb{F} \left[ G \right] \) is of the form \( g \circ e(\Xi) h \) with \( g, h \in G_f, \Xi \) special. The equations

\[
0 = f_{vw} (g \circ e(\Xi) h) = \langle gv | e(\Xi)hw \rangle \quad \text{for all } v, w \in L(\Lambda), \Lambda \in P^+ \setminus \overline{F_\Theta}
\]

are equivalent to

\[
e(\Xi)L(\Lambda) = \{0\} \quad \text{for all } \Lambda \in P^+ \setminus \overline{F_\Theta} = P^+ \setminus R(\Theta).
\]

Recall that for an element \( \Lambda \in P^+ \) we have \( P(\Lambda) \cap R(\Xi) = \emptyset \) if and only if \( \Lambda \notin R(\Xi) \). Therefore these equations are equivalent to \( \Lambda \in P^+ \setminus \overline{F_\Theta} \) for all \( \Lambda \in P^+ \setminus \overline{F_\Theta} \). This is equivalent to \( P^+ \setminus \overline{F_\Theta} \subseteq P^+ \setminus \overline{F_\Theta} \), which in turn is equivalent to \( \Theta \subseteq \Xi \).

c) The closure of the \( G_f \times G_f \)-orbit \( G_f \circ e(\Theta) G_f \) is a union of \( G_f \times G_f \)-orbits.

Due to b) it is contained in \( \bigcup_{\Xi \in sp., \Xi \supseteq \Theta} G_f \circ e(\Xi) G_f \). To show equality, it is sufficient to show, that the closure contains the elements \( 1 \circ e(\Xi), \Xi \supseteq \Theta, \Xi \) special.

Because left multiplications with elements of \( \widehat{G} \) are Zariski continuous on \( \widehat{G} \), we find

\[
e(\Theta)\widehat{G} = e(\Theta)\overline{\widehat{G}} \subseteq e(\Theta)\widehat{G}.
\]

Therefore we get

\[
1 \circ e(\Theta)\widehat{G} \subseteq 1 \circ e(\Theta)G_f \subseteq G_f \circ e(\Theta)G_f.
\]

Now \( e(\Theta)\widehat{G} \) contains for every \( \Xi \supseteq \Theta, \Xi \) special, the element \( e(\Theta)e(\Xi) = e(\Theta) \cap R(\Xi) = e(\Xi) \).

d) If \( \Theta \) is the biggest special set with respect to the inclusion, then the orbit \( G_f \circ e(\Theta)G_f \) is closed due to c).

If \( \Theta \) is not the biggest special set, then due to c) we find

\[
G_f \circ e(\Theta)G_f = \overline{G_f \circ e(\Theta)G_f} \cup \bigcup_{\Xi \in sp., \Xi \supseteq \Theta} G_f \circ e(\Xi)G_f.
\]

There are only finitely many special sets. Therefore \( G_f \circ e(\Theta)G_f \) is locally closed.

e) The algebra of strongly regular functions \( \mathbb{F} \left[ G \right] \) is an integral domain. Therefore every subset of \( \overline{G} = \widehat{G}_f \), which contains \( G \), is irreducible. In particular \( G_f \).
is irreducible. From Theorem 2.4 follows, that the $G_f \times G_f$-orbits of Specm $F[G]$ are irreducible.

Our next aim is to define and describe big cells of every $G_f \times G_f$-orbit of Specm $F[G]$. As a preparation we first determine certain principal open sets of Specm $F[G]$.

Recall that $\theta_\Lambda$ denotes the function defined by

$$\theta_\Lambda := \frac{1}{\langle\langle v_\Lambda | n_\sigma e(R)u \rangle \langle\langle v_\Lambda | v_\Lambda \rangle \rangle} \langle\langle v_\Lambda | n_\sigma e(R)u \rangle \langle\langle v_\Lambda | v_\Lambda \rangle \rangle$$

$$\langle\langle v_\Lambda | v_\Lambda \rangle \rangle$$

where $v_\Lambda \in L(\Lambda) \setminus \{0\}$, $\Lambda \in P^+$. Recall the multiplicative property $\theta_\Lambda \theta_{\Lambda'} = \theta_{\Lambda + \Lambda'}$, $\Lambda, \Lambda' \in P^+$.

**Theorem 6.2** Let $\Theta$ be special. The principal open subset of Specm $F[G]$ associated with $\theta_\Lambda$, $\Lambda \in F_\Theta \cap P$, is given by

$$\bigcup_{\sigma \in \hat{W}_\Theta} U_f \circ (\sigma T)U_f$$

This set, as well as its coordinate ring as a principal open set, is independent of the chosen element $\Lambda \in F_\Theta \cap P$.

We denote this principal open set by $D(\Theta)$, and its coordinate ring as a principal open set by $F[D(\Theta)]$.

**Proof:** a) We first show that the principal open set of $\theta_\Lambda$ is given by (15): Due to the Birkhoff decomposition, every element of Specm $F[G]$ can be written in the form $u \circ n_\sigma e(R)\bar{u}$, with $u, \bar{u} \in U_f$, $n_\sigma \in N$, and $R$ a face of $X$. Let $v_\Lambda \in L(\Lambda) \setminus \{0\}$. We find

$$0 \neq \theta_\Lambda(u \circ n_\sigma e(R)\bar{u}) = \frac{\langle\langle uv_\Lambda | n_\sigma e(R)\bar{u}v_\Lambda \rangle \langle\langle v_\Lambda | v_\Lambda \rangle \rangle}{\langle\langle v_\Lambda | v_\Lambda \rangle \rangle} = \frac{\langle\langle v_\Lambda | n_\sigma e(R)v_\Lambda \rangle \rangle}{\langle\langle v_\Lambda | v_\Lambda \rangle \rangle}$$

if and only if $\Lambda \in R$ and $\sigma \Lambda = \Lambda$. Because $\Lambda$ is an interior point of the face $R(\Theta)$, the first condition is equivalent to $R(\Theta) \subseteq R$. The second condition is equivalent to $\sigma \in W_\Theta$.

Because of

$$\hat{W}_\Theta = \bigcup_{\text{a face of } X, S \supseteq R(\Theta)} W_\Theta \in (S)$$

we get $\sigma \in (R) \in \hat{W}_\Theta$. We also find that for every $\sigma \in \hat{W}_\Theta$ we have $U_f \circ \sigma TU_f \subseteq D(\theta_\Lambda)$.

b) Clearly (15) does not depend on $\Lambda \in F_\Theta \cap P$. To show that the coordinate rings of the principal open sets are independent of $\Lambda \in F_\Theta \cap P$, we only have to show, that for any $N, N' \in F_\Theta \cap P$ there exists a function $f$ of the coordinate ring of Specm $F[G]$, and an integer $n \in \mathbb{N}$, such that

$$f_{(\theta_N)\Lambda} |_{D(\Theta)} = \frac{1}{\theta_{N'}} |_{D(\Theta)}$$

(16)
Because $F_\Theta$ is open in the linear span of $F_\Theta$, there exists an integer $n \in \mathbb{N}$, such that $N - \frac{1}{n}N' \in F_\Theta$. Since $F_\Theta$ is a cone, we find $nN - N' \in F_\Theta \cap P$. The function $f = \theta_{nN - N'}$ satisfies equation (16).

Let $\Theta$ be special. We call the set $BC(\Theta) := \bigcup f \cdot e(R(\Theta))TU_f$, as well as every translate $U_fg \cdot e(R(\Theta))TU_fh$, where $g,h \in G_f$, a big cell of the orbit $G_f \cdot e(R(\Theta))G_f$. This name is justified by the following three theorems.

**Theorem 6.3** Let $\Theta$ be special. The big cell $BC(\Theta)$ is principal open in the closure of $G_f \cdot e(R(\Theta))G_f$, i.e.,

$$BC(\Theta) = \overline{G_f \cdot e(R(\Theta))G_f \cap D(\Theta)}.$$  (17)

The big cell $BC(\Theta)$ is dense in the closure of $G_f \cdot e(R(\Theta))G_f$.

**Proof:** a) We first show formula (17). Due to the Birkhoff decomposition of $\text{Spec} \mathbb{F}[G]$, and the formulas (13), (14), and (15), we only have to show

$$\{ \varepsilon (R(\Theta)) \} = \bigcup_{\Xi \in \text{sp.}, \Xi \supseteq \Theta} W_\varepsilon (R(\Xi)) W \cap \widehat{W}_\Theta .$$

Because of $\widehat{W}_\Theta = \bigcup_{\Xi \in \text{sp.}, \Xi \supseteq \Theta} W_\Theta \varepsilon (R(\Xi)) W_\Theta$, the intersection on the right equals $W_\Theta \varepsilon (R(\Theta)) W_\Theta$. Since $W_\Theta$ is the pointwise stabilizer of $R(\Theta)$, its elements fix $\varepsilon (R(\Theta))$. Therefore this intersection contains only the element $\varepsilon (R(\Theta))$.

b) Due to Theorem 6.1 the orbit $G_f \cdot e(R(\Theta))G_f$ is irreducible. Therefore also its closure is irreducible. Because the big cell is open in the closure, and nonempty, it is dense. 

Let $\Theta$ be special. We equip the big cell $BC(\Theta)$ with its coordinate ring $\mathbb{F} [BC(\Theta)]$ as a principal open set in the closure of $G_f \cdot e(R(\Theta))G_f$.

Set $T^\Theta := T_{I \setminus T \text{rest}}$. Set $P^\Theta := \mathbb{Z} - \text{span} \{ A_i \mid i = 1, \ldots, 2n - l, i \notin \Theta \}$, and identify the group algebra $\mathbb{F} [P^\Theta]$ with the classical coordinate ring of the torus $T^\Theta$.

Note that due to Theorem 4.2 the coordinate ring $\mathbb{F} [U_f^\Theta]$ is isomorphic to $\mathbb{F} [U^\Theta]$ by the restriction map.

**Theorem 6.4** Let $\Theta$ be special. We get an isomorphism

$$m : U_f^\Theta \times T^\Theta \times U_f^\Theta \rightarrow BC(\Theta)$$

by $m(u,t,\tilde{u}) := u \cdot e(R(\Theta))t \tilde{u}$, where $u, \tilde{u} \in U_f^\Theta$ and $t \in T^\Theta$.

**Proof:** a) First we show, that $m$ is surjective: Due to [M 1], Proposition 2.13, compare the section preliminaries, we have $e(R(\Theta))T_{\Theta} = e(R(\Theta))$. Because of $T = T_{\Theta} T^\Theta$ we get $e(R(\Theta))T = e(R(\Theta))T^\Theta$.

Due to the same proposition we also have $e(R(\Theta))U_{\Theta} = e(R(\Theta))$. Left and
right multiplications with elements of $\overline{G} = \widehat{G}$ are Zariski continuous on $\overline{G} = \widehat{G}$. Therefore we get by using Theorem 4.2:

$$e(R(\Theta))(U_f)_\Theta = e(R(\Theta))(U_\Theta) \subseteq e(R(\Theta))(U_\Theta) = \{ e(R(\Theta)) \}.$$  

Because of $U_f = (U_f)_\Theta \times (U_f)^\Theta$ we find $e(R(\Theta))U_f = e(R(\Theta))(U_f)^\Theta$.

Because of these formulas, and because $\widehat{T}$ is abelian, we get:

$$U_f \circ e(R(\Theta))TU_f = U_f \circ e(R(\Theta))(U_f)^\Theta = e(R(\Theta))U_f \circ T^\Theta(U_f)^\Theta$$

$$= (U_f)^\Theta \circ e(R(\Theta))T^\Theta(U_f)^\Theta.$$

b) Next we show that the comorphism $m^* : \mathbb{F}[BC(\Theta)] \to \mathbb{F}[U_f^\Theta] \otimes \mathbb{F}[P^\Theta] \otimes \mathbb{F}[U_N^\Theta]$ is well defined and surjective: For $\Lambda \in P^+$ choose $\langle \langle | \rangle \rangle$-dual bases of $L(\Lambda)$, by choosing $\langle \langle | \rangle \rangle$-dual bases

$$\left( \begin{array}{c} a_{\lambda i} \end{array} \right)_{i=1,..,m} \quad , \quad \left( \begin{array}{c} b_{\lambda i} \end{array} \right)_{i=1,..,m}$$

of $L(\Lambda)_\lambda$ for every $\lambda \in P(\Lambda)$. Fix elements $N \in F_\Theta \cap P$ and $v_N \in L(N)_N \setminus \{0\}$. For $v, w \in L(\Lambda)$, and $u, \bar{u} \in U_f^\Theta$, $t \in T^\Theta$ we get

$$\frac{f_{vw}}{(\theta_N)^k}(u \circ e(R(\Theta))t\bar{u}) = \sum_{\lambda \in P(\Lambda) \cap R(\Theta)} \langle \langle uv | a_{\lambda i} \rangle \rangle \cdot e_{\lambda - kN}(t) \langle \langle b_{\lambda i} | \bar{u}w \rangle \rangle.$$  

This sum has only finitely many nonzero summands, because $supp(uw)$ and $supp(\bar{u}w)$ are finite. Denote by $p^\Theta : P \to P^\Theta$ the projection corresponding to the decomposition $P = P_\Theta \oplus P^\Theta$. Due to the last formula we have

$$\frac{f_{vw}}{(\theta_N)^k}|_{BC(\Theta)} \circ m = \sum_{\lambda \in P(\Lambda) \cap R(\Theta)} f_{a_{\lambda i}v}|U^\Theta \otimes e_{p^\Theta(\lambda) - kN} \otimes f_{b_{\lambda i}w}|U^\Theta.$$  

There are only finitely many nonzero summands of this sum. The function $f_{a_{\lambda i}v}|U^\Theta$ is nonzero at most if $v \neq 0$ and if $\lambda$ is bigger than a weight of $supp(v)$, which is only possible for finitely many weights $\lambda$ in $P(\Lambda)$. Similar things hold for $f_{b_{\lambda i}w}|U^\Theta$.

In particular, $m^*$ is well defined. From this formula we find for $v \in L(N)$ and $\Lambda \in \overline{F}_\Theta \cap P$:

$$m^*(\frac{f_{vw}}{\theta_N}) = f_{v,N,v}|U^\Theta \otimes 1 \otimes 1,$$

$$m^*(\frac{\theta^k}{(\theta_N)^k}) = 1 \otimes e_{A - kN} \otimes 1,$$

$$m^*(\frac{f_{uw}}{\theta_N}) = 1 \otimes 1 \otimes f_{vw}|U^\Theta.$$  

It is easy to see, that $(\overline{F}_\Theta \cap P) - N_0 N = P^\Theta$. Taking into account Theorem 4.1, 2), we have found elements of $\mathbb{F}[BC(\Theta)]$, which are mapped onto a system of
generators of $F[U^\Theta_f] \otimes F[P^\Theta] \otimes F[U^\Theta_f]$. Therefore $m^*$ is surjective.

c) $m^*$ is injective, because $m$ is surjective. To show the injectivity of $m$, let $u_1, u_2, \tilde{u}_1, \tilde{u}_2 \in U^\Theta_f$, $t_1, t_2 \in T^\Theta$ such that $m(u_1, t_1, \tilde{u}_1) = m(u_2, t_2, \tilde{u}_2)$. Then for all $f \in F[U^\Theta]$ we have

$$f(u_1) = (m^*)^{-1}(f \otimes 1 \otimes 1)(m(u_1, t_1, \tilde{u}_1)) = f(u_2).$$

Therefore we find $u_1 = u_2$. In a similar way, we get $t_1 = t_2$, and $\tilde{u}_1 = \tilde{u}_2$. □

In the next theorem we give countable coverings of every $G_f \times G_f$-orbit of $\text{Specm} F[G]$ by big cells.

Recall that $W_J$ denotes the set of minimal coset representatives of $W/W_J$, and $JW$ denotes the set of minimal coset representatives of $W_J/W$, $J \subseteq I$.

**Theorem 6.5** Let $\Theta$ be special. We have

$$G_f \circ e(R(\Theta))G_f = \bigcup_{\sigma \in \Theta_W, \tau \in \Theta_W} U_f \sigma \circ e(R(\Theta))TU_f \tau$$

$$= \bigcup_{\sigma \in \Theta_W, \tau \in \Theta_W} U_f \sigma \circ e(R(\Theta))TU_f \tau.$$

**Proof:** It is sufficient to prove the second covering of the theorem, the first follows by applying the Chevalley involution of $\text{Specm} F[G]$. Obviously the sets $U_f \sigma \circ e(R(\Theta))TU_f \tau$ are contained in the orbit $G_f \circ e(R(\Theta))G_f$. Therefore it is sufficient to show that the orbit $G_f \circ e(R(\Theta))G_f$ is contained in the second union.

By writing the elements $W \in (R(\Theta))W$ in normal form, and inserting in (14), we get

$$G_f \circ e(R(\Theta))G_f = \bigcup_{\sigma \in \Theta_W \cup \Theta^+, \tau \in \Theta_W} U_f \circ \sigma e(R(\Theta))T \tau U_f.$$

To transform the expression $U_f \circ \sigma e(R(\Theta))T \tau U_f$, we use the following decomposition of $U_f$, associated to an element $w \in W$, proved in [3], Section 5.5:

Set $\Delta^+_w := \{ \alpha \in \Delta^+ \mid w\alpha \in \Delta^- \}$ and $(\Delta^+)^w := \{ \alpha \in \Delta^+ \mid w\alpha \in \Delta^+ \}$. Then $\Delta^+_w$ consists of finitely many positive real roots, and we have

$$U_f = U_w U^w_f \text{ where } U_w := \exp( \bigoplus_{\alpha \in \Delta^+_w} g_{\alpha}) \text{, } U^w_f := \exp( \prod_{\alpha \in (\Delta^+)^w} g_{\alpha}).$$

Using this decomposition, and Proposition 2.14 of [4], compare the section preliminaries, we find

$$U_f \circ \sigma e(R(\Theta))T \tau U_f = U_f \circ \sigma e(R(\Theta))T \tau U_f \tau^{-1} \tau U_f \tau^{-1} \tau \subseteq U_f \subseteq U^w_f.$$

$$\subseteq U_f \circ U^w_f \circ e(R(\Theta))TU_f \tau = (\sigma U^w_f \circ e^{-1})^* U_f \circ e(R(\Theta))TU_f \tau.$$
Because of $\sigma \in W^{\Theta, \Theta^+} = \{ \sigma \in W \mid \sigma \alpha_i \in \Delta_i^+ \text{ for all } i \in \Theta \cup \Theta^+ \}$, we have

$$\sigma U_{\Theta^+} \sigma^{-1} \subseteq U^-,$$

and the last expression equals $U_f \circ \sigma e(R(\Theta)) TU_f \tau$. By doing similar transformations, we get

$$U_f \circ \sigma e(R(\Theta)) TU_f \tau = \sigma^{-1} U_f \circ e(R(\Theta)) TU_f \tau$$

$$= (\sigma^{-1} U_{\Theta^+} \sigma)(\sigma^{-1} U_f \sigma^{-1}) \sigma^{-1} \circ e(R(\Theta)) TU_f \tau \subseteq U_f \sigma^{-1} \circ U e(R(\Theta)) TU_f \tau$$

$$= U_f \sigma^{-1} \circ e(R(\Theta)) TU_{\Theta} \sigma U_f \tau = U_f \sigma^{-1} \circ e(R(\Theta)) TU_f \tau.$$  

Because of $(W^{\Theta, \Theta^+})^{-1} = \Theta \cup \Theta^+ W$ we have shown, that the orbit is contained in the second union.

Our last aim is to show, that there exist stratified transversal slices to the $G_f \times G_f$-orbits of $\text{Specm} \mathbb{F}[G]$ at any of their points.

Because $G_f \times G_f$ acts by isomorphisms on $\text{Specm} \mathbb{F}[G]$, it is sufficient to find stratified transversal slices at the points $1 \circ e(R(\Theta)) \in G_f \circ e(R(\Theta)) G_f$, $\Theta$ special.

As transversal slice at $1 \circ e(R(\Theta))$, we will use the closure $G_\Theta := 1 \circ G_\Theta$, equipped with its coordinate ring as a closed subset of $\text{Specm} \mathbb{F}[G]$. We have the following description:

**Theorem 6.6** Let $\Theta$ be special. The restriction map $\mathbb{F}[G] \to \mathbb{F}[G_\Theta]$ induces a closed embedding $\text{Specm} \mathbb{F}[G_\Theta] \to \text{Specm} \mathbb{F}[G]$ with image

$$\overline{G}_\Theta = (\tilde{G}_f)_\Theta \circ (\tilde{G}_f)_\Theta = \bigcup_{\Xi \subseteq \Theta, \Xi \text{ special}} (G_f)_\Theta \circ e(R(\Xi))(G_f)_\Theta$$

$$= \bigcup_{\sigma \in \tilde{W}_\Theta} (U_f)_\Theta \circ (\sigma T_\Theta)(U_f)_\Theta.$$

**Proof:** It is not difficult to check, that the map $\text{Specm} \mathbb{F}[G_\Theta] \to \text{Specm} \mathbb{F}[G]$ is a closed embedding with image $\overline{G}_\Theta$. Due to Theorem 6.6 $(\tilde{G}_f)_\Theta \times (\tilde{G}_f)_\Theta$ maps surjectively to $\text{Specm} \mathbb{F}[G_\Theta]$. To show the first equation of the theorem, we have to show, that the concatenation of the maps

$$(\tilde{G}_f)_\Theta \times (\tilde{G}_f)_\Theta \xrightarrow{\alpha} \text{Specm} \mathbb{F}[G_\Theta] \to \text{Specm} \mathbb{F}[G]$$

coincides with the restricted map $\circ : (\tilde{G}_f)_\Theta \times (\tilde{G}_f)_\Theta \to \text{Specm} \mathbb{F}[G]$.

Let $x, y \in (\tilde{G}_f)_\Theta$. Let $v, w \in L(\Lambda), \Lambda \in P^+$. Choose a decomposition $L(\Lambda) = \bigoplus_{j \in J} V_j$ of $L(\Lambda)$ in an orthogonal direct sum of irreducible highest weight $(\mathfrak{g}_\Theta + \mathfrak{h})$-modules. Write $v, w$ as sums $v = \sum_{j \in J} v_j, w = \sum_{j \in J} w_j$ with $v_j, w_j \in V_j$. Then:

$$\alpha(x, y)(f_{vw} | G_\Theta) = \alpha(x, y)(\sum_{j \in J} f_{v_j w_j} | G_\Theta) = \sum_{j \in J} \langle (x v_j | y w_j) \rangle$$

$$= \langle (x v | y w) \rangle = (x \circ y)(f_{vw})$$

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The other equations of the theorem follow from
\[(\tilde{G}_f)_\Theta \circ (\tilde{G}_f)_\Theta = U\Theta \cdot \tilde{N}_\Theta (U_f)_\Theta \circ (\tilde{G}_f)_\Theta = (U_f)_\Theta \circ (U\Theta \cdot \tilde{N}_\Theta (U_f)_\Theta )^*(\tilde{G}_f)_\Theta \]
\[= (U_f)_\Theta \circ (\tilde{G}_f)_\Theta = (U_f)_\Theta \circ U\Theta \cdot \tilde{N}_\Theta (U_f)_\Theta = (U\Theta)^* (U_f)_\Theta \circ \tilde{N}_\Theta (U_f)_\Theta \]
\[= (U_f)_\Theta \circ \tilde{N}_\Theta (U_f)_\Theta \subseteq \bigcup_{\Xi \subseteq \Theta, \Xi \text{ special}} (G_f)_\Theta \circ e(R(\Xi))(G_f)_\Theta \subseteq (\tilde{G}_f)_\Theta \circ (\tilde{G}_f)_\Theta . \]

The unions in the equations of the theorem are disjoint, because the unions \(\bigcup_{\Xi \subseteq \Theta, \Xi \text{ special}} G_f \circ e(R(\Xi))G_f, \bigcup_{\Theta \in \tilde{V}} U_f \circ \tilde{w}TU_f\) are disjoint.

As open neighborhood of \(1 \circ e(R(\Theta))\), we take the principal open set \(D(\Theta)\).

**Theorem 6.7** Let \(\Theta\) be special.

1) We have \(G_f \circ e(R(\Theta))G_f \cap D(\Theta) = BC(\Theta)\). The big cell \(BC(\Theta)\) is closed in the principal open set \(D(\Theta)\). Its coordinate ring coincides with the coordinate ring as a closed subset of the principal open set \(D(\Theta)\).

2) We have \(1 \circ e(R(\Theta)) \in \Theta \subseteq D(\Theta)\). The coordinate ring of \(\Theta\) coincides with the coordinate ring as a closed subset of the principal open set \(D(\Theta)\).

3) a) We get an isomorphism
\[\Psi : \Theta \times BC(\Theta) \rightarrow D(\Theta)\]
by \(\Psi(x \circ y, u \circ e(R(\Theta))t\tilde{u}) := xu \circ yt\tilde{u}\), where \(x, y \in (\tilde{G}_f)_\Theta\), \(u, \tilde{u} \in (U_f)_\Theta\), and \(t \in T\Theta\).

b) Inserting \(1 \circ e(R(\Theta))\) in the first entry (resp. second entry) of \(\Psi\) induces the identity map on \(BC(\Theta)\) (resp. \(\Theta\)).

c) The partition of \(\text{Specm} F[G]\) into \(G_f \times G_f\)-orbits induces partitions of \(\Theta\) and \(D(\Theta)\). \(\Psi\) preserves the orbits, i.e.,
\[\Psi(\Theta \cap G_f \circ e(R(\Xi))G_f, BC(\Theta)) = D(\Theta) \cap G_f \circ e(R(\Xi))G_f, \Xi \text{ special} . \]

**Proof:** 1) The first part of the theorem follows immediately from the definition of the big cell and its coordinate ring.

2) From the last theorem and the description of \(D(\Theta)\) given in Theorem 3.2 follows \(1 \circ e(R(\Theta)) \in \Theta \subseteq D(\Theta)\).

The statement of part 2) about the coordinate rings can be seen as follows: \(D(\Theta)\) is the principal open set of \(\theta_\Lambda\) for an element \(\Lambda \in F_\Theta \cap P\). Now \(G_\Theta\) stabilizes every point of the highest weight space \(L(\Lambda)_\Lambda\), because for \(j \in \Theta\) the \(\alpha_j\)-string of \(P(\Lambda)\) through \(\Lambda\) consists only of \(\Lambda\). Therefore \(\theta_\Lambda\) takes the value 1 on \(1 \circ G_\Theta\), and also on the closure \(1 \circ G_\Theta = G_\Theta\).

3) Because of Theorem 3.4, the statement of 3 a) is equivalent to the following statement, which we will prove: We get a bijective map
\[\tilde{\Psi} : (U_f)_\Theta \times G_\Theta \times T^{T^\Theta} \times (U_f)_\Theta \rightarrow D(\Theta)\]
by \( \Psi(u, x \circ y, t, \tilde{u}) := xu \circ yt \tilde{u} = (x \circ y)(u, t \tilde{u}) \), where \( x, y \in (\hat{G}_f)_\Theta \), \( u, \tilde{u} \in (U_f)^\Theta \), \( t \in T^\Theta \). Its comorphism exists and is an isomorphism of algebras.

From the the descriptions of \( D(\Theta) \) and \( G_\Theta \) given in Theorem 4.2 and in the last theorem follows, that the image of the map \( \tilde{\Psi} \) is \( D(\Theta) \).

Next we show that the comorphism \( \tilde{\Psi}^* \) exists and is surjective: Choose an element \( A \in F_\Theta \cap F_f \). It is easy to check, that the multiplication maps \( D_G^{\prime}(\theta_A) \times T_{\text{rest}} \to D_G(\theta_A) \), and \( T_{\text{rest}} \times T_{\text{rest}} \to T^\Theta \) are bijective. Furthermore their comorphisms are isomorphisms \( F[D_G(\theta_A)] \to F[D_G^{\prime}(\theta_A)] \otimes F[P_{\text{rest}}] \), and \( F[P^\Theta] \to F[P_{\text{rest}}] \otimes F[P_{\text{rest}}] \). Also the comorphism of the bijective map \( \ast : U^\Theta \to (U^\Theta)^- \) is an isomorphism \( F[\hat{G}_f^\Theta] \to F[U^\Theta] \). Taking into account Theorem 5.2 for \( J = I \) and \( L = \Theta \), we find that the map

\[
\check{m} : U^\Theta \times G_\Theta \times T^\Theta \times U^\Theta \to D_G(\theta_A)
\]

given by \( \check{m}(u, g, t, \tilde{u}) := u^*gt \tilde{u} \) is bijective, its comorphism exists, and is an isomorphism of algebras.

Now identify \( G \) with \( 1 \circ G \). Then \( D_G(\theta_A) \) is contained in \( D(\Theta) \). Due to Theorem 4.2 the set \( U^\Theta \) is dense in \( (U_f)^\Theta \). The coordinate ring \( F[\hat{G}_f^\Theta] \) is isomorphic to \( F[U^\Theta] \) by the restriction map. Similar things hold for \( G_\Theta, G_\Theta^{\prime} \) and their coordinate rings.

For a coordinate ring \( F[B] \) and a nonempty subset \( A \subseteq B \) denote by \( \text{res}^B_A \) the restriction map \( F[B] \to F[A] \). It is easy to check, that the surjective map

\[
\left( \text{res}^{U^\Theta}_U \otimes \text{res}^{\hat{G}_f^\Theta}_{G_\Theta} \otimes \text{res}^{T^\Theta}_T \otimes \text{res}^{(U_f)^\Theta}_U \right)^{-1} \circ \check{m}^* \circ \text{res}^{D(\Theta)}_{D_G(\theta_A)}
\]

is the comorphism of \( \check{\Psi} \). (Make use of \( \check{\Psi}(u, g, t, \tilde{u}) = \check{m}(u, g, t, \tilde{u}) \) for \( u, \tilde{u} \in U^\Theta \), \( g \in G_\Theta \), and \( t \in T^\Theta \).)

Because the maps \( \check{\Psi} \) and \( \check{\Psi}^* \) are surjective, they are also injective. This is shown in the same way as the injectivity of the maps \( m \) and \( m^* \) in the proof of Theorem 5.2.

3 b) follows immediately from the definition of \( \Psi \). 3 c) can be checked easily by using the definition and the bijectivity of \( \Psi \), and the last theorem.

\[ \square \]

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