Universality in self-organized critical slope models

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The dynamics of critical slope self-organized critical models is studied, using a previous mapping into a linear interface depinning model dragged at one end. The model is solved obtaining the complete set of scaling exponents. Some results are supported by previous RG developed for constant force linear interface depinning models but others, like the linear dependency of the susceptibility with system size, are intrinsic of this model which belongs to a different universality class. The comparison of our results with numerical simulations of ricepile and vortexpile models reported in the literature reveals that, as in the constant force case, there are two universality classes corresponding to random and periodic pinning.

I. INTRODUCTION

There are many natural phenomena where the dynamics does not take place smoothly but in the form of bursts or avalanches. Avalanche dynamics has been observed in earthquakes dynamics \cite{8}, granular piles \cite{2,3}, superconductor vortex piles \cite{8}, the Barkhausen effect \cite{5}, crack propagation \cite{6}, and more. The avalanches are characterized by their size \( s \) and duration \( T \) which in general follow the power law distributions \( P(s) \sim s^{-\tau_s} \) and \( P(T) \sim T^{-\tau_T} \), respectively.

Bak, Tang, and Wiesenfeld \cite{8} introduced the notion of self-organized criticality (SOC) to explain this kind of dynamics, which seems to be common to many "different" phenomena. In their early formulation the critical state of a SOC system is an attractor of its dynamics and therefore there is no need of fine-tuning to reach the critical state. This idea was illustrated with very simple cellular automata, the sandpile models. In sandpile models an integer \cite{8,9,12} or continuous \cite{13,15} variable \( z_i \) is defined in a \( d \)-dimensional lattice. \( z_i \) can be the height of grains columns \cite{8,9,12}, the stress accumulated in certain fault \cite{13}, the vortex density \cite{15}, etc. A critical threshold condition for local relaxation (toppling) is also given, which can be a critical height \( z_i > z_c \) \cite{8,9,12}, critical slope \( m_i = z_i - z_{i+1} > m_c \) \cite{8,9,12}, or critical Laplacian \( l_i = \sum_{nn} z_i - 2d z_i > l_c \) \cite{17} threshold condition. When this condition is fulfilled the site relaxes transferring grains to its nearest neighbors (nn), otherwise certain amount of grains is added from an external field.

Critical height models have been extensively studied in the literature, either by mean field \cite{18} and field theories \cite{18}, renormalization group (RG) \cite{19,20}, and numerical simulations \cite{21,22,23}. However, the study of critical slope models has been in general limited to numerical simulations \cite{21,22,23}, which are not so extensive as in the critical height variant, while theoretical approaches are almost absent. There are many experimental situations, sandpiles and vortexpiles for instance, where the critical slope threshold condition is more appropriate, requiring a rigorous treatment which captures the general features of critical slope models. In this direction Paczuski and Boettcher \cite{22} provided an starting point. They mapped a critical slope model into a linear interface depinning (LID) model where the interface is dragged at one end, showing the existence of common behavior between these models. The importance of their observation is that LID models can be solved using continuous approaches \cite{24,25}. Their work was inconclusive because, up to our knowledge, the problem of an interface moving on a random environment and dragged at one end has not been solved. However, they observed that some scaling exponents, but not all, are identical to those measured for an interface driven uniformly.

In the present work we solve the problem of an interface moving on a random environment and dragged at one end. We start with a LID equation with the appropriate boundary conditions. We then look for the fluctuations around the average in the stationary state. In the thermodynamic limit these fluctuations satisfy an equation similar to that of an interface driven uniformly. The critical exponents are then computed using previous RG calculations for the uniformly driven interface \cite{24,25}. The dynamic and roughness scaling exponents are found identical while other exponents become different. These results show a very good agreement when compared with numerical estimates of the scaling exponents reported in the literature.

II. THE MODEL

Consider the following one-dimensional critical slope model. An integer variable \( z_i \) \((i = 1, 2, \ldots, L)\) is defined. A site where the slope \( m_i = z_i - z_{i+1} \) is greater than a threshold \( m_c \) is said to be active and relaxes according to the toppling rule

\[ z_i \to z_i - 1, \]
\[ z_{i+1} \to z_{i+1} + 1. \]
Grains are added at site \( i = 0 \) at rate \( c \) while the other boundary \( i = L − 1 \) is open. Usually \( c \) is assumed very small in such a way that a new grain is added only when no site is active, which is the usual separation of time scales assumed in numerical simulations.

Paczuski and Boettcher \[22\] noted that instead of following the evolution of the column heights \( z_i \) one can follow the evolution of \( h_i(t) \), the total number of toppling events at site \( i \) up to time \( t \). The evolution rules for \( h_i \) can be easily obtained from those for \( z_i \). If we start with an initial configuration where \( z_i = 0 \) the number of grains at site \( i \) is obtained adding the number of grains received from the left neighbor \( (h_{i-1}) \) and subtracting the number of grains transferred to the right neighbor \( (h_i) \), i.e.

\[
z_i = h_{i-1} - h_i.
\] (2)

Using this expression one may compute the local slope \( m_i = z_i - z_{i+1} \), resulting

\[
m_i - m_c = h_{i+1} + h_{i-1} - 2h_i - m_c,
\] (3)

On the other hand, \( h_i \) increases by one unit every time the site \( i \) topples, i.e. every time \( m_i > m_c \), which can be written as

\[
h_i(t + 1) - h_i(t) = \Theta(m_i - m_c),
\] (4)

where \( \Theta(x) \) is the Heaviside unit step function.

The evolution rules in eqs. (3) and (4) are just the discretized evolution rules for LID models \[26\], taking \( m_i - m_c \) as the force acting on the interface. The left term of eq. (4) is a discrete time derivative while \( h_{i+1} + h_{i-1} - 2h_i \) is the discrete Laplacian in one dimension. After coarse-graining one obtains the continuum equation \[26\]

\[
\lambda \partial_t h = \Gamma \nabla^2 h - m_c + \eta(x, h).
\] (5)

Here \( \lambda \) and \( \Gamma \) are coarse-graining parameters, \( \lambda \) can be interpreted as a viscosity coefficient and \( \Gamma \) as a surface tension.

\( \eta(x, h) \) is a quenched noise which may have different origins. For instance, the critical slope may be a random variable reflecting the randomness in the local rearrangements of sand grains after each toppling. In a more realistic model \( m_c \) is thus a random variable which changes its value from site to site after each toppling event. When \( h(x, t) \) advances, i.e. the site \( x \) topplies, a new random slope is assigned. The randomness in the critical slope is thus reflected in eq. (4) through the quenched noise \( \eta(x, h) \). The quenched noise \( \eta(x, h) \) will be assumed a Gaussian noise with zero mean and noise correlator

\[
\langle \eta(x, h) \eta(x', t') \rangle = \delta^d(x - x') \Delta(t - t'),
\] (6)

where \( \Delta(h) \) is a symmetric function, i.e. \( \Delta(-h) = \Delta(h) \).

Depending on the boundary conditions for \( \Delta(h) \) one can distinguish two cases \[23\] \[24\]. For random disorder \( \Delta(h) \) goes to zero for large \( h \) while for periodic pinning forces \( \Delta(h) \) is periodic.

The problem will be completely defined after the initial and boundary conditions are specified. We are interested in the stationary solution of eq. (5) and therefore the initial condition is irrelevant. Grains will be added to the system at constant rate \( c \) at site \( i = 0 \). Hence, if we start with an initial configuration where \( z_0 = 0 \) the number of grains at site \( i = 0 \) is obtained adding the number of grains received from the external field \( (ct) \), and subtracting the number of grains transferred to the right neighbor \( (h_i) \), i.e.

\[
z_i = ct - h_i.
\] (7)

Using this expression one may compute the local slope \( m_i = z_i - z_{i+1} \), resulting

\[
m_i - m_c = h_1 - 2h_0 - m_c.
\] (8)

This evolution rule together with that in eq. (3) leads to the coarse grained equation

\[
\lambda \partial_t h = \kappa \partial_x h + ct - h - m_c + \eta(x, h),
\] (9)

which is the boundary condition at \( x = 0 \). However, in the stationary state the major contribution to the right hand side of this equation is given by the term \( ct - h \) and, therefore, this boundary condition can be approximated by the more simple condition

\[
h(0, t) = ct.
\] (10)

On the other hand, at \( x = L - 1 \) the boundary is open which implies that the site \( x = L \) will never topples, i.e.

\[
h(L, t) = 0.
\] (11)

Eq. (5) can be generalized to a \( d \)-dimensional model.

To be more specific we consider a \( d \)-dimensional hypercube of linear size \( L \) where grains are added at face \( x|_i = 0 \) and the boundary \( x|_i = L \) is open. The equation of motion of \( h(x, t) \) will be given by eq. (5), where \( \nabla^2 \) will be now the \( d \)-dimensional Laplacian, with the boundary conditions

\[
h|_{x|i=0} = ct, \quad h|_{x|i=L} = 0.
\] (12)

Periodic boundary conditions are assumed in the other directions. Eqs. (5) and (12) completely defines the LID model dragged at one end. For \( c \rightarrow 0 \) we recover the equation of motion proposed by Paczuski and Boettcher \[22\], which describes the dynamics of critical slope models in the limit of separation of time scales and no local dissipation.
III. STATIONARY STATE

Eq. (6) is similar to the equation of motion of LID models. However, in this case the interface is dragged at one end instead of being driven by a constant force. The drag at one end makes the problem asymmetric, which leads to a gradient in $h(x, t)$. It is thus easier to look for an expansion around this gradient. With this idea in mind we look for a solution of the form

$$h(x, t) = h_0(x_{||}, t) + y(x, t),$$

where $h_0(x, t)$ will be determined imposing the constraint $y|_{x_{||}=0} = y|_{x_{||}=L} = 0$ (symmetric boundary conditions for $y(x, t)$) and introducing a constant force term in the equation for $y(x, t)$. These requirements lead to the following problem for $h_0(x, t)$

$$\Gamma \frac{\partial^2 h_0}{\partial x_{||}^2} = F - m_c,$$  \hspace{1cm} (14)

with the boundary conditions in eq. (13). The solution of this problem is given by

$$h_0(x, t) = \frac{F - m_c}{2\Gamma} x_{||}^2 - \left( ct + \frac{F - m_c L^2}{2\Gamma} \right) \frac{x_{||}}{L} + ct.$$  \hspace{1cm} (15)

Then, substituting eq. (13) in eq. (14), with $h_0(x, t)$ given by eq. (15), and taking the limits $x t \gg (F - m_c) L^2 / 2\Gamma$ and $x_{||} \ll L$ we obtain the following equation for $y(x, t)$

$$\lambda \partial_t y = \Gamma \nabla^2 y + F - \lambda c + \eta(x, ct + y).$$  \hspace{1cm} (16)

This equation describe the fluctuations of the interface profile $h(x, t)$ around $h_0(x_{||}, t)$, away from the boundaries and for very long times. The constant force $F$, introduced in eq. (14), will be determined self-consistently using the constraint $y(x_{||}) = 0$. Within this approximations the anisotropy introduced by the boundary conditions, such that the resulting equation for $y(x, t)$ is isotropic. The influence of the anisotropy will be considered in the next subsection to compute the avalanche exponents.

Eq. (16) is identical to the one obtained for the fluctuations of an elastic interface driven uniformly by a constant force $F$ [23,24]. However, in this case $F$ is a fixed parameter while the interface velocity $v$ ($c$ in eq. (16)) is obtained self-consistently from the equation of motion. A depinning transition takes place at certain critical force $F_c$ determined by the disorder. For $F < F_c$ the interface is pinned after certain finite time while for $F > F_c$ it moves with finite average velocity $v \sim (F - F_c)\beta$. On the contrary, in our model the interface velocity $c$ is the fixed parameter, while $F$ is determined self-consistently from the equation of motion. Since $c > 0$ the system will always be above the depinning transition, i.e. $F > F_c$. Moreover, to obtain an average interface velocity $c$ we should have $c \sim (F - F_c)\beta$ and therefore

$$F = F_c + \text{const.} c^{1/\beta}.$$  \hspace{1cm} (17)

To reach the critical state $F = F_c$ we must then fine-tune $c$ to zero. In other words, the critical state will be obtained in the limit of separation of time scales $c \to 0$, which is usually satisfied in numerical simulations. According to eq. (15) adjusting the constant force $F$ we are just adjusting the curvature of the interface profile $h(x, t)$ along the $x_{||}$ direction. Hence, the system self-organizes into a stationary state where the curvature of the interface balances the pinning forces. This conclusion, which was already conjectured by Paczuski and Boettcher, was obtaining here by solving the LID model dragged at one end.

A. Scaling exponents

The fluctuations around the average profile are characterized by the pair correlation function. At the critical state the pair correlation function is expected to obey the scaling law [23,24]

$$\langle |y(x, t) - y(0, 0)|^2 \rangle \sim |x|^{2z} g(t/|x|^\zeta),$$

where $g(x)$ is an scaling function and $z$ and $\zeta$ are the dynamic and roughness exponents, respectively. These exponents are obtained through a RG analysis of eq. (16) [23,24]. The upper critical dimension is found to be $d_c = 4$ and below $d_c$ the scaling exponents are given by

$$z = 2 - \frac{2}{9}(4 - d), \quad \zeta = \frac{4 - d}{3},$$  \hspace{1cm} (19)

for a random distribution of pinning forces [23,24] and

$$z = 2 - \frac{4 - d}{3}, \quad \zeta = 0,$$  \hspace{1cm} (20)

for periodic distribution of pinning [24]. The exponents $z$ and $\zeta$ are thus identical to those obtained for the constant force case.

However, other scaling exponents result different because of the anisotropy introduced by the boundary conditions. In the case of an interface driven by a constant force the average interface profile above the depinning transition is given by $\langle y(x_{||}, t) \rangle$ and, therefore, there is no preference direction in space. On the contrary, in the case of an interface dragged at one end the average interface profile is given by $h_0(x_{||}, t)$, which is clearly non-uniform along the $x_{||}$ direction as one can see from eq. (13).

To analyze the influence of the anisotropy introduced by the boundary conditions let us analyze the dynamics of active sites. Let $\rho_a(x_{||}, x_{\perp}, t)$ the average density of active sites at site $(x_{||}, x_{\perp})$ and time $t$, given was a site active at $x_{||} = 0$. Here $x_{||}$ as above denotes the position along the preferential direction and $x_{\perp}$ is a $d - 1$
dimensional vector denoting the position in a plane perpendicular to \( x_{\|} \). Since the flow of grains takes place, in average, along the positive \( x_{\|} \) direction then the average flux of particles through the plain \( x_{\|} = L \) is given by

\[
J(L) = \int dt d^{d-1}x_{\perp} \rho_a(L, x_{\perp}, t).
\]  

(21)

On the other hand, at the critical state \( c \to 0 \) the average density of active sites should satisfy the scaling law

\[
\rho_a(x_{\|}, x_{\perp}, t) = t^{-d/z} f \left( \frac{x_{\|}}{t^{1/z}}, \frac{x_{\perp}}{t^{1/z}} \cdot \frac{t}{L^z} \right),
\]

(22)

where \( \eta \) is another scaling exponent and \( f \) is an scaling function. Then, substituting this expression in eq. (21) it results that

\[
J(L) \sim L^{(1+\eta)z-1}.
\]

(23)

Now, if the system is in a stationary state for each grain we put at \( x_{\|} = 0 \) one grain should go out at \( x_{\|} = L \) and, therefore, \( J(L) = 1 \). This stationary condition will be satisfied only if

\[
(1 + \eta)z = 1,
\]

(24)

in eq. (23). A more familiar scaling relation is obtained if one computes the mean avalanche size, which is given by

\[
\langle s \rangle = \int dt dx_{\|} d^{d-1}x_{\perp} \rho_a(x_{\|}, x_{\perp}, t).
\]

(25)

Substituting the scaling law for \( \rho_a \) in eq. (22) in this expression it results that

\[
\langle s \rangle \sim L^{(1+\eta)z} \sim L,
\]

(26)

which is the usual scaling dependency of the mean avalanche size with system size in critical slope models. This derivation using such simple scaling arguments is reported here for the first time.

Using the RG estimates for the exponents \( z \) and \( \zeta \) in eqs. (14) or (24) and the scaling relation \( \langle s \rangle \sim L \) we can compute the avalanche scaling exponents. Let \( s \) be the avalanche size and \( T \) its duration, which are distributed according to \( P(s) \) and \( P(T) \), respectively. Just at the critical state one expect that these distributions satisfy the power law behavior \( P(s) \sim s^{-\tau_s} \) and \( P(T) \sim T^{-\tau_t} \), where \( \tau_s \) and \( \tau_t \) are the avalanche distribution exponents. However, for a system of finite size a characteristic avalanche size \( s_c \sim L^D \) and duration \( T_c \sim L^2 \) will appear, where \( D \) is the avalanche fractal dimension. Near the critical state the distributions of avalanche size and duration will thus satisfy the scaling laws

\[
P(s) \sim s^{-\tau_s} f(s/L^D), \quad P(T) \sim T^{-\tau_t} g(T/L^2),
\]

(27)

where \( f(x) \) and \( g(x) \) are some cutoff functions with the asymptotic behaviors \( f(x), g(x) \sim 1 \) for \( x \ll 1 \) and \( f(x), g(x) \ll 1 \) for \( x \gg 1 \).

The exponents \( \tau_s, \tau_t, D \) and \( z \) are not all independent. Since \( s \sim T^{z/D} \) then the condition \( \int ds P(s) = \int dTP(T) \) implies

\[
(\tau_s - 1)D = (\tau_t - 1)z.
\]

(28)

Another scaling relation can be obtained taking into account that \( \langle s \rangle \sim L \) (see eq. (25), resulting

\[
(2 - \tau_s)D = 1.
\]

(29)

Then, from eqs. (28) and (29) it follows that

\[
\tau_s = 2 - \frac{1}{D}, \quad \tau_t = 1 + \frac{D - 1}{z}.
\]

(30)

Finally there is a scaling relation which relates the avalanche dimension exponent \( D \) with the roughness exponent \( \zeta \). Below the upper critical dimension the avalanches are compact objects \([27]\) and therefore \( s \sim \Delta h r^d \), where \( \Delta h \) is the characteristic fluctuation of the interface width during the avalanche and \( r \) its characteristic linear extent in the \( d \)-dimensional substrate. Then since \( \Delta h \sim r^\zeta \) and \( s \sim r^D \) one obtains \([27]\)

\[
D = d + \zeta.
\]

(31)

Above the upper critical dimension the avalanches are no more compact and \( D = d_c = 4 \) \([27]\).

IV. COMPARISON WITH EXPERIMENTS AND NUMERICAL SIMULATIONS

Using the values of \( z \) and \( \zeta \) obtained from the RG analysis in eqs. (14) or (24) and the scaling relations in eqs. (25) and (21) we can determine all the avalanche exponents. The results in one and two dimensions are shown in table I and II for random and periodic pinning, respectively. Some numerical estimates for critical slopes model are also shown for comparison.

In \( d = 1 \) we count with numerical simulations of rice pile models, which are critical slope sandpile models with certain randomness in the toppling rule. For instance Frette \([11]\) considered a rice pile model where the slope threshold \( m_c \) is selected at random after each toppling. A modified version of this model was later used by Christensen et al \([16]\). In a somehow different model Nunes-Amaral and Lauritsen \([12]\) considered a rice pile model where the toppling rule is stochastic in certain range of slopes \( m_c1 < m < m_c2 \) while it is deterministic above \( m_c2 \). All these rice pile models give the same avalanche exponents and should belong to the random disorder universality class. In table I we display the more accurate numerical estimates reported by Nunes-Amaral and Lauritsen.

In \( d = 2 \) we count with a critical slope model introduced by Bassler and Paczuski \([13]\) to describe the avalanche dynamics in superconducting vortexpiles. In
their model \( z \) is the density of vortex at site \( i \). The evolution rules are not exactly those described above but a critical slope condition was used. They also considered quenched random pinning forces and therefore it should belong to the random disorder universality class.

More recently Cruz, Mulet and Atshuler \[23\] have made simulations of the model introduced by Bassler and Paczusky but considering both random and periodic pinning forces. In the case of random pinning their numerical estimates reproduced, within the numerical error, those reported by Bassler and Paczuski \[19\]. However, in the case of periodic pinning they obtained very different exponents (see table \[4\]) suggesting that there are two universality class. Within our analysis the existence of these two universality classes is clear, corresponding with random and periodic distributions of pinning forces.

In all cases we observe a very good agreement between the numerical estimates and our predictions. This agreement is more surprising because the dimensions considered are far from the upper critical dimension \( d_c = 4 \). In table \[4\] we also display the scaling exponents \( D \) and \( z \) obtained from numerical simulations of the LID model driven at constant force, which are expected to be identical to those for the LID model driven at one end. As one can see the agreement is quite good, event better than with the RG estimates.

Recently Tadić and Nowak \[25\] has observed that the scaling exponents of the random-field Ising model (RFIM) in the low disorder regime are very close to those of the ricepile model (see table \[4\]). In particular they considered a diluted two-dimensional Ising model with weak random fields and with an anisotropic initial condition. In the low disorder regime a single domain wall separating two regions with different spin orientations is observed. In this regime the Barkhausen avalanches are attributed to the fluctuations in the domain wall. The agreement with the exponents for the ricepile model suggests that the domain walls in this regime may be also described by the equation of motion \[9\] while the anisotropy was introduced through the initial conditions. On the contrary, on the high disorder regime a multi-domain structure is obtained, resulting in different avalanche exponents.

In this analysis we have not included a comparison with some available experimental results for sandpiles \[2\], ricepiles \[3\] and vortexpiles \[4\] because it is very difficult to compare our results with experimental measurements. In most of the experiments the measured magnitude is the number of grains (vortices) leaving (entering) the system which will be denoted by \( s_0 \). In general \( s_0 \) can be also described by the scaling law \( P(s_0) \sim s_0^{-\tau_0} f(s_0/L^{D_0}) \). However, in average, for one grain (vortex) entering the system one grain (vortex) goes out and therefore \( s_0 = 1 \). Then if one computes \( s_0 \) using the scaling law for \( P(s_0) \) one obtains \( s_0 \sim L^{(2-\tau_0)D_0} \sim 1 \), which immediately gives \( \tau_0 = 2 \). This value is near the one reported for real sandpiles, ricepiles and vortex piles. On the contrary, in numerical simulations and in the model analyzed in this work the avalanche size \( s \) is given by the number of toppling events which is not necessarily proportional to the flow of grains (vortices), the magnitude measured in experiments. In fact, in the case of the avalanche size \( s \), the conservation law yields \( \tau_s = 2 - 1/D < 2 \) (see eq. \[24\]). Hence, we conclude that \( \tau_0 \neq \tau_s \).

V. CONCLUSIONS

We conclude that ricepile models, the vortexpile model by Bassler and Paczuski, and the RFIM in the low disorder regime belong to the same universality class, that of critical slope sandpile models. The prototype model for this universality class is the LID model dragged at one end, as it was already conjectured by Paczuski and Boettcher \[22\]. Our analytical treatment has demonstrated that their guess was correct. For the first time, we have solved the LID model dragged at one end, obtaining the complete set of scaling exponents. Some of these results were supported by previous RG calculations developed for constant force LID but others, like the linear dependence of the susceptibility with system size, are intrinsic to this model. Moreover, as in the constant force case, we found two different universality classes corresponding with random and periodic pinning forces. Our predictions were found in very good agreement with numerical simulations of different critical slope models.

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d | Model | $\tau$ | $\tau_t$ | $D$ | $z$ | Ref.
--- | --- | --- | --- | --- | ---
1 | RP | 1.53(5) | 1.84(5) | 2.20(5) | 1.40(5) | [12]
2 | RFIM | 1.58 | 1.89 | 2.23 | 1.45 | [29]
3 | LID | 2.25 | 1.42(3) | 1.42(3) | 1.42(3) | [26]
4 | RG | $\frac{3}{2}$ | 1.5 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}$ | 1.75 | $\frac{2}{2}