TOPOLOGICAL SPIN CURRENT INDUCED BY NON-COMMUTING COORDINATES - AN APPLICATION TO THE SPIN-HALL EFFECT

D. Schmeltzer
Department of Physics,
City College of the City University of New York,
New York, NY, 10031
(Dated: October 8, 2018)

A new method for computing the spin Hall conductivity for a two dimensional electron gas in the presence of the spin Orbit interaction is presented. The spin current is computed using the Many-Body wave function which is degenerated at zero momentum. The degeneracy at \( \vec{k} = 0 \) gives rise to non-commuting Cartesian coordinates. The non-commuting Cartesian coordinate are a result of an effective Aharonov-Bohm vortex at \( \vec{k} = 0 \). An explicit calculation for the Rashba model is presented. The conductivity is determined by the linear response theory which has two parts: A-a static spin Hall conductivity which is determined by the non-commuting coordinates and has the value \( \frac{e^3}{4\pi} \). B-a time dependent conductivity which renormalizes the static conductivity. The value of this renormalization depends on inelastic time scattering spin Orbit polarization energy and Zeeman energy. As a result the spin Hall conductivity vary between \( \frac{e^3}{4\pi} \) and \( \frac{e^3}{4\pi} \).

In the absence of a Zeeman field we find that the long time behavior is given by the renormalized conductivity \( \frac{e^3}{4\pi} \).

For relative small magnetic field the Zeeman field allows to probe continuously the spin Hall conductivity from the static unrenormalized value \( \frac{e^3}{4\pi} \) to the fully renormalized value \( \frac{e^3}{4\pi} \). When the Zeeman energy exceeds the Fermi energy only one Fermi Dirac band is occupied and as a result the static Hall conductivity is half the static spin Hall conductivity.

We compute the uniform magnetization without the Zeeman field and show that the spin current is covariantly conserved and satisfies effectively the continuity equation.

The effect of a time reversal scattering potential due to a single impurity in the Rashba model causes the spin Hall current to decrease with the size of the system.

I. INTRODUCTION

Two experimental groups [1, 2] have attempted to confirm the spin Hall effect without reaching conclusive results. Experimentally we observe that spins of opposite sign accumulate on opposite sides of a semiconductor in response to an external electric field. The two experimental groups [1] and [2] have reached different conclusions. The Santa Barbara group [2] suggests that the spin accumulation is extrinsic and is caused by disorder; while the Hitachi group [1] reports the experimental observation of an intrinsic spin Hall effect due to the spin Orbit interaction. Recently the Santa Barbara group [2] have done a new experiment using a two-dimensional electron gas confined in the (110) direction in AlGaAs quantum wells with an electric field applied along the crystal axis. An out of plane spin polarization is reported which might be explained by the cubic Dresselhauss model ruling out the disorder extrinsic effect [1].

The proposed theories are also controversial, a side jump disorder induced model is proposed as a mechanism for the spin Hall effect [3], while other researchers have used the spin Orbit interaction to explain the possibility for a spin Hall current [1, 4, 5, 6, 7, 8, 9, 10, 11, 12]. It seems that the definition of the spin current is also not clear [13]. The effect of static disorder introduces vertex corrections [11] causing the spin Hall conductivity to vanish [7, 11, 12, 14]. The conserved magnetic current in the presence of a orbital magnetic field has been investigated [11, 13] and a proposal to verify the spin Hall effect by applying a magnetic field gradient and measuring a charge Hall current has been suggested [15].

Microscopically, the spin Orbit interaction emerges from the non-relativistic limit of the Dirac equation [16, 17]. At low energies the Dirac Hamiltonian for spin 1/2 electrons in an electromagnetic field is projected effectively into a two component Pauli Hamiltonian. This projection replaces the U(1) gauge fields \( -e\vec{A} \) and \( -eA_0 \) by \( -e\vec{A} - \frac{g-1}{2\mu_B} \mu_B \vec{\sigma} \times \vec{E} \) and \( -eA_0 - \frac{g-1}{2\mu_B} \mu_B \vec{\sigma} \cdot \vec{B} \) where \( \vec{\sigma} \) is the Pauli matrix, \( \frac{g-1}{2\mu_B} \) is the magnetic moment, and \( \frac{g-1}{2\mu_B} \) is the Thomas precession [13]. \( \vec{E} \) and \( \vec{\sigma} \) are the electric and magnetic fields. The Rashba Hamiltonian [11, 12, 17, 20] used in Solid State physics is obtained after the replacement \( \frac{g-1}{2\mu_B} \mu_B \vec{\sigma} \times \vec{E} \rightarrow \hbar \kappa_{so}(\vec{\sigma} \times \hat{e}_3) \) where \( \hat{e}_3 \) is the unit vector perpendicular to the two dimensional plane and \( \vec{E} \) is the internal field which determines the strength of the spin Orbit momentum \( \kappa_{so} = \frac{(g-1)}{17} \mu_B |\vec{E}| \). For valence bands electrons the Rashba Hamiltonian is replaced by the Luttinger Hamiltonian [17, 13, 20].
The spin Hall conductivity has been computed in the literature \cite{4,10} without paying attention to the singularity at zero momentum.

We construct an exact ground state in the presence of the spin orbit interaction which will be used to compute the spin Hall current. As an explicit example we investigate the Rashba Hamiltonian using periodic boundary conditions with the momentum restricted to a 2-torus $T^2$. We use the spinor representation to compute the commutations relations for the coordinates in the momentum representation. These commutations emerge from the singular spinor transformation at $\vec{K} = 0$. The non-commutativity is a result of singular vector state at $\vec{K} = 0$. In the geometrical language the non-commutativity is obtained from the knowledge of the connection \cite{22}. The connection is computed in our case from the momentum derivative of the spinor-eigenstate $|\vec{K} \otimes \zeta_\alpha(\vec{K})>$, (\(\alpha = 1,2\) represents the two component spinor which is momentum dependent ). The derivative of the connection allows us to compute the geometrical curvature which is interpreted as Aharonov Bohm effect in the momentum space with a fictitious non-zero magnetic field at $\vec{K} = 0$.

The theory which emerges from the vortex at $\vec{K} = 0$ gives rise to the non-commuting Cartesian coordinates. In order to compute the spin Hall conductivity we use the linear response theory. The conductivity for a constant electric field has two contributions:

A- a static spin Hall conductivity of $|\epsilon|$ determined by by zero momentum state.

B- a time dependent conductivity which renormalizes the static conductivity which is a consequence of the gapless ground state. The time dependent corrections are given by the linear response term, $|\epsilon| \frac{1}{\tau_F} |1 - \cos(2v_{F.S.} k_s t)|$ where $v_{F.S.} = \frac{\partial E_F}{\partial K_F}$ and $\Omega_s = v_{F.S.} k_s$ are the Fermi velocity and the the spin Orbit polarization frequency.

The spin Hall conductivity is given by the sum of the two parts. We introduce the inelastic time scattering $\tau_s$ and compare it to the the Fermi Surface Spin – Orbit Polarization frequency $\Omega_s = v_{F.S.} k_s$. For times $t = \tau_s$ which obey $\Omega_s t \leq 1$ we obtain that the spin Hall conductivity is given by the static vortex contribution $\frac{|\epsilon|}{\tau_F}$. For $\Omega_s \tau_s >> 1$ we perform a time average and obtain that the spin Hall conductivity takes the value, $\frac{|\epsilon|}{\tau_F}$ in agreement with the value reported in the literature \cite{4,6,10,11,12}. According to ref.\cite{6} the spin Hall current is sensitive to the various scattering times. For time dependent electric fields our method allows for a simple separation between the static and the time dependent results.

The Zeeman interaction breaks the SU(2) symmetry and modifies the spin Hall conductivity \cite{4}. The presence of a weak Zeeman field allows the observation of a continuous variation of the spin Hall conductivity from the static value $|\epsilon|_{4\pi}$ (obtained for Zeeman energy which are larger than the spin Orbit polarization energy) to the fully renormalized conductivity $|\epsilon|_{5\pi}$ (obtained for a zero magnetic field).

For extremely large Zeeman energy which are comparable to the the Fermi energy only one of the spin bands is populated. As a result we obtain that the static conductivity is $\frac{|\epsilon|}{4\pi}$.

- The scattering effect of a non periodic time reversal invariant potential on the spin Hall current is investigated. We find that independent on the strength of the potential the spin Hall current vanishes with the size of the system. The scattering potential mixes the states. The zero momentum states remain degenerated in the presence of a time reversal invariant potential. The spectral function weight integral for the non zero momentum states diverges, as a result the eigenstate for the spin Hall current state has a vanishing overlap with the zero momentum state causing the current to vanish. Formally we find that the non-commuting Cartesian coordinates in the transformed basis (the basis which diagonalizes the scattering potential) obey the same transformation rule as the geometrical curvatures \cite{21}. The zero momentum states remain degenerated in the presence of a time reversal invariant potential. As a result the eigenstate of the scattering potential has a vanishing overlap with the zero momentum state. Therefore for a finite system the spin Hall current decreases with the size and vanishes for an infinite system. This result is not based on statistical averages and is in qualitative agreement with the transport results obtained in the literature \cite{7,11,12,13,14}.

- The plan of this paper is as following. In chapter II we introduce the Rashba Hamiltonian using the spinor representation and compute the Many-Body wave function. In chapter III we use the spinor representation to represent the Cartesian coordinates. Using this spinor representation we compute the commutations relations. This commutations relations emerge from the singular spinor transformation which diagonalizes the Rashba Hamiltonian. In chapter IV we introduce the Heisenberg equation of motion needed for computing the spin Hall current. We define the spin Hall velocity using the Heisenberg and Interaction picture. In chapter V we compute the Spin-Hall current. The calculation contains two parts: A-The static spin Hall conductivity and B-The time dependent linear response spin Hall conductivity. In chapter VI we compute the uniform magnetization induced by the electric field. We find that for $\Omega_s \tau_s \leq 1$ the uniform magnetization is finite and in the limit $\Omega_s \tau_s >> 1$ it vanishes. In chapter VII we show that the spin current is covariantly conserved and the continuity equation is satisfied to first order in the electric field. In chapter VIII we add a Zeeman interaction to the Rashba Hamiltonian and compute the spin
Hall current. In chapter IX the scattering effect of a non periodic time reversal invariant potential on the spin Hall current is investigated. We have included three appendixes: In appendix A we replace the constant electric field by a time dependent vector potential. We derive the SU(2) Berry wave function using a time dependent momentum. In appendix B the SU(2) derivation of the spin current and the covariance conserved spin current are presented. In appendix C we compute the charge Hall current in the presence of both the spin orbit interaction and the magnetic field.

II. THE MOMENTUM-SPINOR REPRESENTATION

In this chapter we will compute the Many-Body ground state wave function for the Rashba model in the absence of an external field. We will show that this ground state has a vortex at $\vec{K} = 0$. This wave function will be used to compute the spin Hall current. This method is applicable to a variety of spin orbit systems such as the cubic Dresselhaus model "n"type GaAs proposed to explain the spin Hall effect [3, 4].

The Rashba model in the momentum eigenstate-spinor basis is given by, $\hat{h} = \hat{h}_0 + \hat{h}(ext)$ where $\hat{h}_0$ is the Rashba Hamiltonian.

$$\hat{h}_0 = \frac{1}{2m}[\vec{p} - \hbar k_{so}(\vec{\sigma} \times \vec{\epsilon}_3)]^2$$

The external potential is given by, $\hat{h}(ext) = -eE^{(ext)} \cdot R^{(1)}$, where $e = -|e|$ is the electron charge, $E^{(ext)}$ is the applied electric field at $t \geq 0$ and $R^{(1)}$ is the Cartesian coordinate in the $i = 1$ direction. A careful investigation of the two component Pauli Hamiltonian [10] suggest that the Rashba Hamiltonian in an external electric field has an additional external potential. This is seen when an external field is applied to the Pauli Hamiltonian which is a function of the total electric field, $\vec{E} = \vec{E}_{(int)} + E^{(ext)}$. We substitute in the Pauli Hamiltonian the total electric field, $\vec{E} = \vec{E}_{(int)} + E^{(ext)}$ and find $\frac{\partial\hat{h}}{\partial\vec{E}_{(int)}}(\vec{\sigma} \times (\vec{E}_{(int)} + E^{(ext)})) \rightarrow \hbar k_{so}(\vec{\sigma} \times \vec{\epsilon}_3) + (\hbar \frac{\partial}{\partial\vec{E}_{(int)}})(\vec{\sigma} \times E^{(ext)})(E_{(int)}$ is the absolute value of the internal electric field). As a result the external potential will contain an additional term, $\delta\hat{h}(ext) = -(-\frac{\hbar k_{so}}{E_{(int)}})\sigma(\vec{E}^{(ext)}_2)$. This potential is controlled by the magnetic charge and therefore in agreement with the literature will be ignored. The fluctuations of the spin Orbit are best described by the $SU(2)$ Aharonov - Casher spin Orbit Hamiltonian [24] (see appendix B).

The Rashba Hamiltonian in eq.1 is given in the momentum $\vec{p} = \hbar\vec{K}$ and coordinate $\vec{R} = i\vec{\partial}/\partial\vec{K}$ representation. We diagonalize this Hamiltonian using a spinor representation. The spinor which diagonalizes the Hamiltonian is singular at $\vec{K} = 0$ and therefore gives rise to non commuting coordinates. This methodology has been used for the Quantum Hall case where the magnetic Bloch functions have zero’s in the magnetic Brillouin zone which gives rise to vortexes and to a quantized Hall effect [24] (see eqs.2,13,3,7,3.8 and 3.9). For a solid with a periodic potential the plane wave functions is replaced by the Bloch functions $u_{n,K}(\vec{q})$. The transformation from the plane wave representation to the Bloch representation modifies the coordinate representation from $\vec{R} = i\vec{\partial}/\partial\vec{K}$ to $\vec{\rho} = i\vec{\partial}/\partial\vec{K} + \vec{A}_{n,m}(\vec{K})$ where $\vec{A}_{n,m}(\vec{K}) = \int \frac{d^3q}{(2\pi)^3} u_{n,K}(\vec{q}) \frac{\vec{\partial}}{\partial\vec{q}} n_{m,K}(\vec{q})$. Zak has shown [25] that for a degenerated point in the Brillouin zone the Berry phase $\vec{A}_{n,K}(\vec{K})$ gives rise to a non zero band curvature, $\Omega(\vec{K}, n) = \frac{\pi}{\hbar^2} \times \vec{A}_{n,m}(\vec{K})$ giving rise to non-commuting coordinates, $[r^{(1)}_1, r^{(2)}_2] = i\Omega(K, n)$. We will show that the eigensfunctions which diagonalizes the Rashba Hamiltonian have a singular point in the momentum space and therefore the methodology used in ref. [25] applies to our case.

The Rashba Hamiltonian has two eigenvectors, $|K \otimes \zeta_\alpha(K) \rangle$, $\alpha = 1, 2$ which replace the free particle-spinor $|K \otimes S \rangle = |K \times S \rangle$ for $S \geq \frac{1}{2} |K > |S < |1, 0 \rangle$, $\frac{1}{2} |S = 2 \rangle$. Due to the periodic boundary conditions the vectors $K^{(1)}$ and $K^{(2)}$ are restricted to a two dimensional Torus, this ensures that for a finite system the state $\vec{K} = 0$ is included in the Many-Body wave function. The eigenvector spinor $|K \otimes \zeta_\alpha(K) \rangle$ is obtained with the help of the SU(2) transformation used to diagonalize the Rashba Hamiltonian. We find, $|K \otimes \zeta_\alpha(K) \rangle = \frac{1}{\sqrt{2}} U(\varphi(\vec{K}), \vartheta = \frac{\pi}{2}) |K \otimes S \rangle$ with the SU(2) transformation which is given by, $U(\varphi(\vec{K}), \vartheta = \frac{\pi}{2}) = \left( e^{(i/2)} \varphi(\vec{K}) \sigma^{(3)} \right) \left( e^{(-i/2)} \varphi(\vec{K}) \sigma^{(2)} \right)$ and $\varphi(\vec{K}) = \arctan(\frac{\vec{K}^{(1,2)}}{K^{(3)}})$ is the azimuth angle in the plane, $\vec{K} = (K^{(1)}, K^{(2)})$. The Rashba Hamiltonian in the diagonal representation $\frac{h^2\vec{K}^{(1,2)}_2}{2m} - \sigma^{(3)} \hbar^2 k_{so}|\vec{K}| + \epsilon_{so}$ has two eigenvalues, $\epsilon_{\sigma(\alpha)}(K) = \frac{h^2\vec{K}^{(1,2)}_2}{2m} - \sigma(\alpha) \hbar^2 k_{so}|\vec{K}| + \epsilon_{so} \equiv \epsilon(|\vec{K}|, \sigma(\alpha))$, where $\sigma(\alpha = 1) = 1$, $\sigma(\alpha = 2) = -1$ and $\epsilon_{so} = \frac{\hbar^2 k_{so}^2}{2m}$ is the spin orbit single particle energy. At $\vec{K} = 0$ the eigenvalues are degenerated and the spinor $|K \otimes \zeta_\alpha(K) \rangle$ is Singular (multivalued). The degeneracy at $\vec{K} = 0$ gives rise to a situation where the eigenvalue function which is a function of the momentum in the Brillouin zone returns to its original value after a $4\pi$ rotation in the plane. Therefore the eigenvalue function has the topology of a connected sum of two tori $T_g = 2$.  

In appendix B the connected sum of two tori $T_g = 2$.  

In appendix B the connected sum of two tori $T_g = 2$.  

In appendix B the connected sum of two tori $T_g = 2$.
Using the eigen-spinor basis we obtain the diagonal form of the Rashba Hamiltonian,

\[ \hat{h}_0 = \int \frac{d^2 K}{(2\pi)^2} \sum_{\alpha=1}^{2} \epsilon_{\sigma(\alpha)}(\mathbf{K}) |K \otimes \zeta_{\alpha}(K) > < \zeta_{\alpha}(K) \otimes K| \]  

(2)

It is important to stress that the macroscopic spin current is determined by the ground state Many-Body wave function computed in the absence of the external field.

For the Rashba model the wave function is determined by the products of the SU(2) transformations in the momentum space which act on the state \( |G> = \prod_{K,S} |K \otimes S> \). As a result we find that in the transformed frame the Pauli spin \( \sigma^{(3)} \) is aligned in the plane (the angle in the plane is determined by the value of the two component momenta). The Many-Body ground state \( |F.S.> \) for the Rashba model is determined by the product of the SU(2) rotations.

\[ |F.S.> = \prod_{K,\alpha} |K \otimes \zeta_{\alpha}(K) > \equiv \prod_{K,S} U(\varphi(\mathbf{K})), \vartheta = \pi/2 |K \otimes S> \]  

(3)

The wave function \( |F.S.> \) is defined in terms of the two Fermi surface momenta \( K_{F} \) and \( \mathbf{K}_{F}^\perp \). We introduce the notation \( |F.S.> \equiv |K_{F}, K_{F}^\perp> \) to emphasize the fact that the occupation function is restricted by two maximal momenta \( K_{F}^\perp \) and \( K_{F}^\perp \).

III. THE CARTESIAN COORDINATES IN THE SPINOR REPRESENTATION

The basis \( |K \otimes \zeta_{\alpha}(K) > \) which builds the ground state wave function \( |F.S.> \) is used to find the Cartesian coordinates \( r^{(i)}, i = 1, 2 \) representation,

\[ r^{(i)} = \int \frac{d^2 K}{(2\pi)^2} \int \frac{d^2 P}{(2\pi)^2} \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} r_{\alpha,\beta}^{(i)}(K,P) |K \otimes \zeta_{\alpha}(K) > < \zeta_{\beta}(P) \otimes P| \]  

(4)

with the matrix elements,

\[ r_{\alpha,\beta}^{(i)}(K,P) = I_{\alpha,\beta} \otimes \frac{id}{dK^{(i)}} \delta(\mathbf{K} - \mathbf{P}) - \frac{1}{2} \sigma^{(1)}_{\alpha,\beta} \otimes \delta(\mathbf{K} - \mathbf{P}) \frac{d\varphi(\mathbf{K})}{dK^{(i)}} \]  

and momentum-spinor representation;

\[ r^{(i)} = I \otimes \frac{id}{dK^{(i)}} - \frac{1}{2} \sigma^{(1)} \otimes \frac{d\varphi(\mathbf{K})}{dK^{(i)}} \]  

where \( I \) is the identity operator and \( \sigma^{(1)} \) is the Pauli matrix. The momentum derivative \( \frac{id}{dK^{(i)}} \) represents the Cartesian coordinate \( R^{(i)} \) for spinless particles and \( \frac{d\varphi(\mathbf{K})}{dK^{(i)}} \) is the spinor connection which is a result of the SU(2) transformation which rotate the spin frame.

The representation of the Cartesian coordinate in the momentum spinor representation given in eq.4 allows to compute the commutator for the Cartesian coordinates. In order to identify the singularity and to compute the two dimensional commutator it is advantageous to work in the complex plane, \( Z = K^{(1)} + iK^{(2)}, \overline{Z} = K^{(1)} - iK^{(2)} \). We perform a change of variables from the momentum plane \( \mathbf{K} = (K^{(1)}, K^{(2)}) \) to the complex plane \( Z \) and \( \overline{Z} \). We find the commutator,

\[ [r^{(1)}, r^{(2)}]dK^{(1)}dK^{(2)} = \frac{\sigma^{(1)}}{4} \left[ \frac{\partial}{\partial Z}(\frac{1}{Z})dZd\overline{Z} - \frac{\partial}{\partial \overline{Z}}(\frac{1}{Z})d\overline{Z}dZ \right] \]  

(5)

The commutator in eq.5 is zero for \( \mathbf{K} \neq 0 \). The right hand side of eq.5 represents the two dimensional delta function in the complex plane, \( \delta^{(2)}(\mathbf{K}) = \frac{1}{\pi} \frac{\partial}{\partial Z} \left( \frac{1}{Z} \right) = \frac{1}{\pi} \frac{\partial}{\partial \overline{Z}} \left( \frac{1}{\overline{Z}} \right) \) which is a result of the multivalued phase \( \varphi(\mathbf{K}) = 0 \). At \( \mathbf{K} = 0 \) the eigenvalues are degenerated and the spinor \( |K \otimes \zeta_{\alpha}(K) > \) is Singular. The singularity in the plane at \( \mathbf{K} = 0 \) represent a vortex [21] (see section 5.2 "Closed Forms and Exact Forms", page 156) and gives rise to non-commuting coordinates. The origin of this result follows from the fact that the energy spectrum forms a double Torus structure in the momentum space. The central result of this section is the result given by the commutator in eq.5. This result will be used to compute the spin Hall conductivity in the next chapters.

IV. THE HEISENBERG EQUATION OF MOTION FOR NON-COMMUTING COORDINATES

The singular transformation gives rise to the following commutation relations, \( [K^{(i)}, K^{(j)}] = 0 \), \( [r^{(i)}, K^{(j)}] = i\delta_{i,j} \) and \( [r^{(i)}, r^{(j)}] = (1 - \delta_{i,j}) \frac{-1}{2} \sigma^{(1)}(2\pi)^2(\mathbf{K}) \) replaces the commutator, \( [R^{(i)}, R^{(j)}] = 0 \) (\( \delta_{i,j} \) represents the Kronecker delta).
function). Using the Heisenberg equation of motion we find that the time derivative of the momentum 
\( K^{(1)} \) and the coordinate \( r^{(2)} \) are linear functions of the electric field 
\( E_{1}^{\text{ext}}(t) \); \( i\hbar \frac{dK^{(1)}(t)}{dt} = -eE_{1}^{\text{ext}} \) and, \( V_{2,H}(\vec{K}, t) = \frac{d\ell^{(2)}(t)}{dt} = \frac{1}{2r} \left\{ [r^{(2)}, \hat{h}_{0}] - eE_{1}^{\text{ext}}[r^{(2)}, r^{(1)}] \right\} \) is the velocity in the Heisenberg representation. Due to the non-commutativity of the coordinates, we find that the velocity operator in the \( i = 2 \) direction depends on the electric field in the \( i = 1 \) direction. The time dependent velocity \( V_{i,H}(\vec{K}, t) \) (in the Heisenberg picture) is given in terms of the Schrödinger Velocity \( V_{i}(\vec{K}) \). The velocity and operator in the Schrödinger picture is given by,

\[
V_{i,H}(\vec{K}, t = 0) = V_{i}(\vec{K}) = \left( \frac{d\ell^{i}(\vec{K}, t)}{dt} \right)_{t=0} = \left( \frac{1}{\hbar} \right) \left\{ [r^{(i)}, \hat{h}_{0}] - eE_{1}^{\text{ext}}[r^{(i)}, r^{(1)}] \right\}_{t=0} \tag{6}
\]

We will use the first quantized form to define the single particle operators in the momentum representation. For any single particle operator in the Schrödinger representation \( O(\vec{K}) \) we define the single particle operator in the Heisenberg \( O(\vec{K}, t) \) and interaction picture \( O_{I}(\vec{K}, t) \). We will apply this formulation to the velocity and spin current.

The time dependent velocity \( V_{i,H}(\vec{K}, t) \) in the Heisenberg picture is given in terms of the Schrödinger Velocity \( V_{i}(\vec{K}) \) defined in eq.6. We find, \( V_{i,H}(\vec{K}, t) = e^{\hat{h}_{0}(\vec{K})t}V_{i}(\vec{K})e^{\hat{h}_{0}(\vec{K})t} = V_{i,I}(\vec{K}, t) + (\frac{i}{\hbar}) \int_{0}^{t} dt_{1}[V_{i,I}(\vec{K}, t_{1}), -e\ell^{(1)}(\vec{K}, t_{1})]E_{1}^{\text{ext}}(t_{1}) + ... \) where \( V_{i,I}(\vec{K}, t) = e^{\hat{h}_{0}(\vec{K})t}V_{i}(\vec{K})e^{\hat{h}_{0}(\vec{K})t} \) is the velocity in the interaction picture.

The spin-velocity is defined according to the Noether’s theorem [31] (see appendix B). We will introduce the spin velocity in the Schrödinger picture,

\[
\ell^{(A)}_{i}(\vec{K}) = \frac{1}{2} \{ V_{i}(\vec{K}), \hat{\sigma}^{(A)} \}_{+} \tag{7}
\]

\( \{ , \} _{+} \) stands for the symmetric product and \( \hat{\sigma}^{(A)} = U^{\dagger}(\vec{K})\sigma^{(A)}U(\vec{K}) \) represents the transformed Pauli matrix with \( A = 1, 2, 3 \). (This definition is equivalent to the symmetric projection of the \( 2 \times 2 \) velocity matrix \( V_{i}(\vec{K}) \) into the spin space and represents the velocity of the particle with a given spin polarization.)

The spin-velocity in the \( i = 2 \) direction and polarization \( A = 3 \) takes in the Heisenberg representation the form,

\[
\ell^{(3)}_{i,H}(\vec{K}, t) = \ell^{(3)}_{i,I}(\vec{K}, t) + (\frac{-i}{\hbar}) \int_{0}^{t} dt_{1}[\ell^{(3)}_{i,I}(\vec{K}, t_{1}), -e\ell^{(1)}_{i}(\vec{K}, t_{1})]E_{1}^{\text{ext}}(t_{1}) + ... \tag{8}
\]

where \( \ell^{(3)}_{2,I}(\vec{K}, t) \) and \( r^{(1)}_{i}(\vec{K}, t) \) are defined according to the Interaction picture,

\[
\ell^{(3)}_{2,I}(\vec{K}, t) = e^{\hat{h}_{0}(\vec{K})t}\ell^{(3)}_{2,I}(\vec{K})e^{\hat{h}_{0}(\vec{K})t} \equiv \ell^{(3,0)}_{2,I}(\vec{K}, t) + \ell^{(3,\text{ext})}_{2,I}(\vec{K}, t) \tag{9}
\]

Where \( \ell^{(3,0)}_{2,I}(\vec{K}, t) = e^{\hat{h}_{0}(\vec{K})t}\ell^{(3,0)}_{2,I}(\vec{K})e^{\hat{h}_{0}(\vec{K})t} \) represents the homogeneous spin velocity in the Schrödinger picture \( \ell^{(3)}_{2,I}(\vec{K}) = \frac{1}{2} \left\{ \hat{h}_{0}(\vec{K})t, \ell^{(3,0)}_{2,I}(\vec{K}) \right\}_{+} \). The second term in eq.9 represent the static spin-velocity generated by the non-commuting Cartesian. This term is generated by external electric field \( E_{1}^{\text{ext}} \) at \( t = 0 \) (the second term in eq.6). We find that the external spin velocity \( \ell^{(3,\text{ext})}_{2,I}(\vec{K}) \) in the Interaction picture is the same as the Schrödinger picture \( \ell^{(3,\text{ext})}_{2,I}(\vec{K}) = \ell^{(3,\text{ext})}_{2,I}(\vec{K}, t) \) where \( \ell^{(3,\text{ext})}_{2,I}(\vec{K}) = \frac{1}{2} \left\{ \hat{h}_{0}(\vec{K})t, -eE_{1}^{\text{ext}}[r^{(2)}, r^{(1)}] \right\}_{+} \).

The linear response result in our case is given by eq.8.

The first term will give rise to the static linear response part , \( \ell^{(3,\text{ext})}_{2,I}(\vec{K}) = \ell^{(3,\text{ext})}_{2,I}(\vec{K}, t) = \frac{1}{2} \left\{ \hat{h}_{0}(\vec{K})t, -eE_{1}^{\text{ext}}[r^{(2)}, r^{(1)}] \right\}_{+} \). This part is due to the non-commuting coordinates. The second term in eq.8 (the term which is linear in the external electric field \( E_{1}^{\text{ext}} \)) will give rise to the time dependent linear response when we substitute in the commutator of eq.8 the homogeneous spin velocity operator \( \ell^{(3,0)}_{2,I}(\vec{K}, t) \).

We will compute the expectation values using the second quantized form. We introduce the two component Spinors in the momentum representation \( \Psi^{\dagger}(\vec{K}) \) and \( \Psi(\vec{K}) \) which act on the Many-Body ground state.\( |F.S.> \). We introduce the Heisenberg operator in the second quantized form , \( \hat{O}_{H} = \int \frac{d^{2}K}{(2\pi)^{2}} \Psi^{\dagger}(\vec{K})\hat{O}_{H}(\vec{K}, t)\Psi(\vec{K}) \). The second quantized form of the spin current \( j^{(3)}_{2,H} \) in the Heisenberg picture is obtained from the Heisenberg representation of the single particle operators. The spin current is obtained in the limit \( q \to 0 \) by taking the expectation value of the spin current operator with respect the ground state \( |F.S.> \).
\[ J_2^{(3)} = < F.S. | J_2^{(3)} | F.S. > \]
\[ = \lim_{\hbar \to 0} \frac{\hbar}{2} \int \frac{d^2 K}{(2\pi)^2} < F.S. | \Psi^\dagger(\tilde{K}) (t_2^{(3)}(\tilde{K}, t)) e^{i\tilde{r}_H(\tilde{K}, t)} \Psi(\tilde{K}) | F.S. > \]
\[ = J_2^{(3, ext., static)} + \delta J_2^{(3, time-dependent)} \]

(10)

\( \tilde{r}_H(\tilde{K}, t) \) is the coordinate operator in the Heisenberg picture and the exponential acts as a shift operator in the momentum space, \( e^{i\tilde{r}_H(\tilde{K})} \Psi(\tilde{K}) = \Psi(\tilde{K} - \tilde{q}) \).

The first term \( J_2^{(3, ext., static)} \) represents the static linear response current due to the vortex at \( \tilde{K} = 0 \). The second term \( \delta J_2^{(3, time-dependent)} \) is the time dependent linear response. This term gives rise to renormalization effects of the static current. In the presence of a gap this renormalization effect is negligible. (This is the situation for the integer Quantum Hall where the conductivity is given by the static part.)

V. THE SPIN-HALL CURRENT

In order to compute the spin Hall current we will use the ground state wave function \( |F.S. > \) characterized by the Fermi - Dirac occupation function \( f_{F.D.} [\epsilon(\tilde{K}), \sigma(\alpha)] - E_F \) which at \( T = 0 \) is given by the step function, \( \theta(\epsilon(\tilde{K}), \sigma(\alpha)) - E_F \). The spin Hall current has two parts: the static linear response given in the first term in eq.8 and the time dependent linear response term given by the second term in eq.8.

A. The static linear response current \( J_2^{(3, ext., static)} \).

The formula for the static spin current in the \( i = 2 \) direction is given by the term, \( J_2^{(3, ext., static)} \). The expectation value with respect to the ground state \( |F.S. > \) at zero temperature gives,

\[ J_2^{(3, ext., static)} = \frac{\hbar}{2} \int \frac{d^2 K}{(2\pi)^2} < F.S. | \Psi^\dagger(\tilde{K}) (t_2^{(3, ext.)}(\tilde{K})) \Psi(\tilde{K}) | F.S. > \]
\[ = \frac{1}{(2\pi)^2} \sum_{\alpha=1}^{2} \int \int -\frac{e}{2\hbar} E_1^{(ext)}(\alpha)[r^{(1)}, r^{(2)}] \theta(\epsilon(\tilde{K}), \sigma(\alpha)) - E_F | dK^{(1)} dK^{(2)} \]
\[ = \frac{1}{(2\pi)^2} \sum_{\alpha=1}^{2} \int \int \frac{e}{2\hbar} E_1^{(ext)}(\alpha)[\epsilon(\tilde{K}), \sigma(\alpha)) - E_F] \frac{(\sigma^{(1)} \alpha)}{4} \left[ \partial Z(\frac{1}{Z}) dZ d\bar{Z} - \partial Z(\frac{1}{Z}) d\bar{Z} dZ \right] \]
\[ = \frac{e}{2} E_1^{(ext)} \frac{\pi}{2(2\pi)^2} \sum_{\alpha=1}^{2} \frac{1}{2\pi i} \oint \frac{dZ}{Z} - \frac{1}{2\pi i} \oint \frac{d\bar{Z}}{\bar{Z}} = \frac{-e}{4\pi} E_1^{(ext)} \]

(11)

The second row in eq.11 is a function of the commutator \( [r^{(1)}, r^{(2)}] \). The third row in eq.11 is obtained after we replace the commutator \( [r^{(1)}, r^{(2)}] \) with its complex representation given in eq.5. In the third row we recognize the two dimensional delta function \( \delta^{(2)}(\tilde{K}) = \frac{1}{\pi} \frac{\partial}{\partial \tilde{K}^2}(\frac{1}{Z}) = \frac{1}{\pi} \frac{\partial}{\partial \tilde{K}^2}(\frac{1}{Z}) \) which enable us to replace a two dimensional integral with a closed line integral in agreement with the Stokes theorem. The theorem allows us to replace the two dimensional integral over the Fermi surface with a closed line integral at the boundary of on the Fermi surface (the two Fermi surfaces, for spin up and spin down). As a result we find that the static Spin – Hall current in eq.11 is replaced by a Fermi – Surface (the line integral in the fourth row) integral gives the exact value of \( \sigma_2^{(1)} \sigma_2^{(2)} = \frac{e}{2\pi} \) for the static Spin – Hall conductivity. The origin of this exact result, is caused by the current which is carried by the vortex at \( \tilde{K} = 0 \). The Fermi Dirac occupation function at \( \tilde{K} = 0 \) takes the value of one for spin up and spin down, therefore the contribution from the two spin polarization in eq.11 are equal.

B. The linear response time dependent linear response spin Hall current \( \delta J_2^{(3, time-dependent)} \).

The linear response time dependent spin Hall current is given by the second term in eq.9. We use the interaction picture for the spin velocity , Cartesian coordinate and the Pauli matrix .

\[ r_2^{(1)}(\tilde{K}, t) = e^{i\int_{t_1}^{t} \dot{\tilde{r}}_I(\tilde{K}, t) dt} e^{i\int_{t_1}^{t} \dot{\tilde{r}}_I(\tilde{K}, t) dt} \]
\[ = e^{i\int_{t_1}^{t} \dot{\tilde{r}}_I(\tilde{K}, t) dt} e^{i\int_{t_1}^{t} \dot{\tilde{r}}_I(\tilde{K}, t) dt} \]

\[ Tr[r_2^{(1)}(\tilde{K}, t), (-\epsilon)r_2^{(1)}(\tilde{K}, t)] = Tr[\frac{\hbar}{m} K^{(2)} \dot{\tilde{r}}_I^{(3)}(\tilde{K}, t) \frac{e}{2} \dot{\tilde{r}}_I^{(3)}(\tilde{K}, t) \frac{d\varphi}{dK^{(1)}}] \]
\[ = -\frac{\hbar}{2} e^{i\int_{t_1}^{t} \dot{\tilde{r}}_I(\tilde{K}, t) dt} e^{i\int_{t_1}^{t} \dot{\tilde{r}}_I(\tilde{K}, t) dt} \frac{d\varphi}{dK^{(1)}} \]

(12)
The result in equation 14 can be considered for different limits of inelastic scattering (the inelastic scattering time 

\[ \delta J_2^{(3, \text{time-dependent})} = \frac{e}{8\pi} \int_0^\infty \left[ \frac{\hbar^2 K^2}{2m} + \epsilon_{so} - \frac{\hbar^2 k_{so}}{m} |\vec{K} - E_F| - F.D. \right] \cdot \int_0^t \left[ E_1^{\text{ext}}(t_1) - e(2\Omega_o(t_1) - E_F) \right] \left[ \frac{2\hbar k_{so}}{m} |t - t_1| \right] \] 

At zero temperature the difference between the two Fermi-Dirac step functions for spin up and spin down is equal to the Fermi-Surface Spin-Orbit Polarization energy \( \hbar \Omega_o \) multiplied by the delta function \( \delta[(\frac{\hbar^2 K^2}{2m} + \epsilon_{so} - E_F)] \). The polarization frequency \( \Omega_o = v_F \cdot k_{so} \) is defined in terms of the spin Orbit momentum \( k_{so} = \frac{(\hbar^2 K^2 - \hbar^2 k_{so})}{m} |\vec{K} - E_F| \) and the Fermi velocity \( v_F = 2(\frac{E_F - \epsilon_{so}}{\hbar K_F}) \). The momentum integration is replaced by the energy integration \( \int_0^\infty \left[ \frac{\hbar^2 K^2}{2m} + \epsilon_{so} - \frac{\hbar^2 k_{so}}{m} |\vec{K} - E_F| - F.D. \right] \) restricted by the delta function \( \delta[(\frac{\hbar^2 K^2}{2m} + \epsilon_{so} - E_F)] \). As a result we find that the linear response time dependent current for a time dependent electric field is given by \( \delta J_2^{(3, \text{time-dependent})} = \frac{e}{8\pi} \int_0^t \left[ E_1^{\text{ext}}(t_1) - e(2\Omega_o(t_1) - E_F) \right] \left[ \frac{2\hbar k_{so}}{m} |t - t_1| \right] \) (see eq.14).

VI. THE MAGNETIZATION

The application of an electric field in the \( i = 1 \) direction generates an uniform magnetization with an in plane spin polarization \( M^{(A=2)} \) in the \( i = 2 \) direction which is time dependent and vanishes in the long time limit. The uniform magnetization is given by

\[ M^{(A)}(t) = \frac{\hbar}{2} \int \frac{d^2 K}{(2\pi)^2} < F.S. | \Psi^1(\vec{K}) \sigma^{(A)}_H(\vec{K}, t) \Psi(\vec{K}) | F.S. > \]

As a result of the \( SU(2) \) transformation the spin operators are transformed; \( \sigma^{(1)}(\vec{K}) = \sigma^{(3)} \cos \varphi(\vec{K}) - \sigma^{(2)} \sin \varphi(\vec{K}) \), \( \sigma^{(2)}(\vec{K}) = \sigma^{(2)} \cos \varphi(\vec{K}) - \sigma^{(3)} \sin \varphi(\vec{K}) \) and \( \sigma^{(3)} = -\sigma^{(1)} \). The Pauli matrices \( A = 1, 2, 3 \) in the Heisenberg representation are represented in terms of the Pauli matrices and the coordinate \( t^{(1)}_i \) in the Interaction picture.

\[ \sigma^{(A)}_H(\vec{K}, t) = \sigma^{(A)}(\vec{K}, t) - i \frac{\hbar}{2} \int_0^t dt_1 [\sigma^{(A)}_I(\vec{K}, t), -e \epsilon^{(1)}_v(\vec{K}, t_1)] E_1^{\text{ext}}(t_1) \ldots \]

Using the explicit form of the Pauli matrices and the \( SU(2) \) angular \( \varphi(\vec{K}) \) dependence, \( \sin \varphi(\vec{K}) = \frac{K^{(2)}}{|\vec{K}|} \) and \( \cos \varphi(\vec{K}) = \frac{K^{(1)}}{|\vec{K}|} \) we find that the only non zero term is \( M^{(2)} \). This is the magnetization with the \( A = 2 \) polarization.
(this is the only term which is invariant under the transformation $\vec{K} \rightarrow -\vec{K}$ and has a finite trace over the Pauli matrices). We have,

$$M^{(2)}(t) = \frac{\hbar}{2} \int \frac{d^2 K}{(2\pi)^2} <F.S.|\Psi(\vec{K})|\cos\varphi(\vec{K})|\sigma_1^{(2)}(\vec{K}, t) - \frac{i}{\hbar} \int dt [\sigma_1(\vec{K}, t), -i\epsilon^{(1)}(\vec{K}, t_1)]E_{1}^{ext}(t_1)|\Psi(\vec{K})|F.S.>$$

The commutator in the last equation is proportional to $\frac{d\varphi(\vec{K})}{dK^{(2)}} = \frac{-K^{(2)}}{|K|}$ therefore only the product with the term proportional to $\cos\varphi(\vec{K}) = \frac{K^{(2)}}{|K|}$ will be nonzero. As a result we obtain,

$$M^{(2)}(t) = \frac{\hbar}{2} \left(\frac{-i}{\hbar}\frac{-e}{2}\int \frac{d^2 K}{(2\pi)^2} \right)\left\langle F.S. | \Psi(\vec{K}) | \cos \varphi(\vec{K}) \right| \frac{d\varphi(\vec{K})}{dK^{(1)}} \int dt_1 [\sigma_1^{(2)}(\vec{K}, t), -i\epsilon^{(1)}(\vec{K}, t_1)]E_{1}^{ext}(t_1) |\Psi(\vec{K})|F.S.>$$

For a constant electric field we perform the time integral and find the time dependent magnetization.

$$M^{(2)}(t) = \left(\frac{\hbar}{2}\right) \left(\frac{1}{4\pi}\right)k_{so} \left| -\frac{eE_{1}^{ext}t}{\hbar} \right| \left| \sin(2\Omega_o t) \right|$$

where $\Omega_o$ is the the polarization frequency $\Omega_o = v_{F.S.} k_{so}$ defined in the previous chapter. We observe that for inelastic scattering times $\tau_s$ which obey $\Omega_o \tau_s \leq 1$ the magnetization is finite, $M^{(2)} \approx \left(\frac{\hbar}{2}\right) \left(\frac{1}{4\pi}\right)k_{so} \left| -\frac{eE_{1}^{ext}t}{\hbar} \right|$. In the opposite limit we perform the time average and find that the uniform magnetization vanishes.

**VII. THE COVARIANTLY CONSERVED SPIN CURRENT**

Following Noether’s theorem [30, 31] (see Appendix B) we compute the spin currents. We apply this formulation to the Rashba model and find from eq.B9 that the spin current is covariantly conserved. The three spin polarizations currents obey in the long wave limit $q \rightarrow 0$ the following continuity equation : $\partial_t J_0^{(1)} + iq \cdot \vec{J}^{(1)}(q) = 0$, $\partial_t J_0^{(2)} + iq \cdot \vec{J}^{(2)}(q) = 0$. Only the $A = 3$ component violates the continuity equation

$$\partial_t J_0^{(3)}(q) = 2k_{so}(J_1^{(1)}(q) + J_2^{(2)}(q))$$

$$\frac{-q \rightarrow 0}{2k_{so}(\frac{\hbar}{2})} < F.S. | \int \frac{d^2 K}{(2\pi)^2} \frac{1}{2} \left[ V_{1,H}(\vec{K}, t), \sigma_{H}^{(1)}(\vec{K}, t) \right] + \frac{1}{2} \left[ V_{2,H}(\vec{K}, t), \sigma_{H}^{(2)}(\vec{K}, t) \right] |\Psi(\vec{K})|F.S.>$$

$$= \left(\frac{\hbar}{2}\right) \int \frac{d^2 K}{(2\pi)^2} < F.S. | \Psi^{+}(\vec{K}) \frac{1}{2} \left[ \sigma_{H}^{(3)}(\vec{K}, t), (\vec{V}_{H}(\vec{K}, t) \times \frac{\vec{K}}{|K|}) \right] + \left(\frac{\hbar}{2}\right) \left[ \sigma_{H}^{(3)}(\vec{K}, t), \frac{\vec{K}}{|K|} \right] |\Psi(\vec{K})|F.S.>$$

This equation has been obtained from the Heisenberg representation of the spin velocity and the $SU(2)$ transformed Pauli matrices,$\sigma_1^{(1)} = \sigma_3^{(1)} \cos \varphi(\vec{K}) - \sigma_2^{(1)} \sin \varphi(\vec{K}), \sigma_1^{(2)} = \sigma_2^{(2)} \cos \varphi(\vec{K}) - \sigma_3^{(2)} \sin \varphi(\vec{K})$. The right hand side of equation 21 is evaluated using the Heisenberg representation of the operators. The velocity and spin operators are a function of the external electric field, see eqs.8 and 17. To first order in the electric field we observed that the right hand side term vanishes. (The first term on the right hand side of the equation vanishes as a result of the momentum integration, and the second term vanishes as a result of the expectation value with respect the Pauli matrices.) Therefore we conclude that the continuity equation is effectively satisfied.
VIII. THE SPIN HALL EFFECT IN THE PRESENCE OF A ZEEMAN INTERACTION

From the analysis in chapters IV and V we learn that the static spin current is determined by the state \( \vec{K} = 0 \). Since the Fermi-Dirac step function for the state \( \vec{K} = 0 \) at zero temperature is not affected by a Zeeman field (if this energy is much less than the Fermi energy) the static spin-Hall current will be same as for the zero magnetic field (see eq.11). The effect of the magnetic field will be to renormalize the static current.

We will present first the modification caused by the Zeeman field to the Rashba model. The Zeeman energy \( b_3 = \frac{1}{2} g \mu_B B \) (\( B \) is the effective magnetic field) changes the the single particle eigenvalues from \( \epsilon_{\alpha}(K) = \frac{\hbar^2(K)^2}{2m} - \sigma(\alpha) \cdot \vec{h} \cdot \vec{K} + \epsilon_{so} \) to \( \epsilon_{\alpha}(\vec{K}) = \frac{\hbar^2(K)^2}{2m} - \sigma(\alpha) \Delta(K) + \epsilon_{so} \) where \( \Delta(K) = \sqrt{b_3^2 + \left( \frac{\hbar^2 k_{so} |\vec{K}|}{m} \right)^2} \). The \( SU(2) \) transformation is replaced by, \( U(\varphi(\vec{K}), \theta(\vec{K})) = e^{(-i/2)\varphi(\vec{K})\sigma(3)} e^{(-i/2)\theta(\vec{K})\sigma(2)} \) with the polar angle \( \vartheta = \frac{\pi}{2} \) (zero magnetic field) replaced by \( \tan \theta(\vec{K}) = \frac{\hbar^2 k_{so} |\vec{K}|}{m b_3} \) (finite magnetic field).

Following the steps described in equations 3 – 6 we find that the polar angle \( \vartheta(\vec{K}) \) modifies the transformed the Cartesian coordinates \( i = 1,2 \).

\[
r^{(i)}(\vec{K}) = R^{(i)}(\vec{K}) = -\frac{1}{2} \left[ \sigma(1) \sin \vartheta(\vec{K}) \frac{d\varphi(\vec{K})}{d\vec{K}(0)} - \sigma(2) \frac{d\theta(\vec{K})}{d\vec{K}(0)} - \sigma(3) \cos \vartheta(\vec{K}) \frac{d\varphi(\vec{K})}{d\vec{K}(0)} \right]
\]

and the commutation relations are transformed,

\[
\left[ r^{(1)}, r^{(2)} \right] = \frac{i}{2} \left[ -\sigma(1) \sin \vartheta(\vec{K}) + \sigma(3) \cos \vartheta(\vec{K}) \right] 2\pi \delta^2(K)
\]

where the transformed Pauli matrix is given by, \( \sigma(3) \to U^\dagger \sigma(3) U = \sigma(3) \cos \vartheta(\vec{K}) - \sigma(1) \sin \vartheta(\vec{K}) \) and the delta function \( \delta(2)(\vec{K}) = \frac{\delta_{(x)}(\vec{K})}{\sqrt{2}} = \frac{\delta_{(x)}(\vec{K})}{\sqrt{2}} \).

At zero temperature the ground state is described by the two Fermi–Dirac for spin up and spin down step functions, \( \theta(\epsilon(\vec{K}), \sigma(\alpha) - E_F) \). We will consider two cases:

a) The magnetic Zeeman energy is much less than the Fermi energy therefore the state \( \vec{K} = 0 \) with spin up or spin down have equal occupation, \( \theta(\epsilon_{so} + b_3 - E_F) = \theta(\epsilon_{so} - b_3 - E_F) = 1 \). For this case the static spin Hall current is identical to the case without the Zeeman field. Following the method used in section V we find that the spin Hall current is given by two contributions, the static one (see eq.11) and the the time dependent contribution (see eq.14). The sum of the two parts is given by an expression similar to the one given in eq.15.

\[
J^{(3)}_2 = \frac{-e}{4\pi} E^{(ext)}_1 + \frac{e}{8\pi} \left( \frac{\hbar \Omega^2}{\hbar \Omega^2 + b_3^2} \right) [1 - \cos(2(\Omega^2 + b_3^2)\frac{t}{\hbar})] E^{(ext)}_1
\]

The static spin Hall conductivity is determined by the degenerate state at \( \vec{K} = 0 \) and is given by the first term in eq.24 which is identical with the first term in eq.15. The second term in eq.24 represents the time dependent linear response part. Here we observe that the Zeeman magnetic interaction controls the renormalization effects. The Zeeman interaction modifies the value of the commutator given in eq.12, which is replaced by the term \( \sin^2 \vartheta(\vec{K}) = \left( \frac{\hbar \Omega^2}{\hbar \Omega^2 + b_3^2} \right) \). When the Zeeman energies \( b_3 \) is larger than polarization energy \( \hbar \Omega \), the linear response term can be ignored and the spin Hall current is determined by the static solution, \( J^{(3)}_2 \approx \frac{|e|}{4\pi} E^{(ext)}_1 \). For Zeeman energies which are comparable or less than the polarization energy we find in the long time limit, \( (\Omega^2 + b_3^2)^2 \gg \tau_\alpha \gg 1 \) that the spin Hall current is given by, \( \frac{|e|}{8\pi} \left( 1 - \frac{1}{2} \left( \frac{\hbar \Omega^2}{\hbar \Omega^2 + b_3^2} \right) \right) \).

Therefore in this case the magnetic field can be used to vary continuously the spin Hall conductivity from the static conductivity \( \frac{|e|}{4\pi} \) for \( \frac{b_3}{\hbar \Omega} \gg 1 \) (for large magnetic fields comparable to the polarization energy) to the fully renormalized spin Hall conductivity \( \frac{|e|}{8\pi} \) for \( b_3 = 0 \) (zero magnetic).

b)–When the Zeeman interaction is comparable or larger than the Fermi energy only one of the Dirac step function contributes. For this case we find a spin Hall conductivity is determined by the static (without normalization by the time dependent linear response term). In this case only one of the Dirac step functions contribute to the conductivity \( \frac{|e|}{8\pi} \).
IX. THE SPIN-HALL EFFECT IN THE PRESENCE OF A NON-PERIODIC SCATTERRING POTENTIAL

We will investigate the effect of a non-periodic time reversal invariant potential $V(\vec{r})$ on the Rashba hamiltonian in the absence of inelastic and scattering and Zeeman field. We will see that the spin Hall current depends on the ratio $\lambda_{so}/L$ where $L$ is the size of the system and $\lambda_{so}$ is the spin orbit length defined by the inverse of the spin orbit momentum $\lambda_{so} = \frac{2\hbar}{k_{so}}$ where $k_{so} = \frac{\mu_B}{\hbar} |\vec{H}|$.

The hamiltonian, $\hat{h} = \hbar \hat{H} + V(\vec{r})$ is represented with the help of the the Rashba eigenstates basis,

$$\hat{h} = \int \frac{d^2K}{(2\pi)^2} \sum_{\alpha \beta} \frac{1}{2} (\vec{K} \cdot \vec{r})^{(\alpha,\beta)} |\vec{K} \cdot \zeta_\alpha(K) > < \zeta_\beta(K) \cdot \vec{K}| + \int \frac{d^2K}{(2\pi)^2} \sum_{\alpha \beta} \frac{2}{2} \sum_{\beta \lambda} \hat{V}_{\alpha,\beta}(\vec{K} - \vec{P}) |\vec{K} \cdot \zeta_\alpha(K) > < \zeta_\beta(P) \cdot \vec{P}|$$

(25)

Where the matrix elements of the scattering potential are given by the product of the Fourier component in the momentum representation $V(\vec{K} - \vec{P})$ with the $SU(2)$ matrix elements, $U(\varphi(\vec{K}), \vartheta = \varphi) = \left( e^{-i/2} \varphi(\vec{K}) \varphi \right)$ $(e^{-i/2} \varphi(\vec{K}) \varphi)$. We define the transformed potential, $\hat{V}_{\alpha,\beta}(\vec{K} - \vec{P}) = \sum_{\alpha \lambda} U(\varphi(\vec{K}), \vartheta = \varphi)_{\alpha,\lambda} V(\vec{K} - \vec{P}) U(\varphi(\vec{P}), \vartheta = \varphi)_{\lambda,\beta}$. The eigenstates $|\Phi^{(\alpha)}(K) >$ of the hamiltonian, $\hat{h} = \hbar \hat{H} + V(\vec{r})$ replaces the eigenstate $|\vec{K} \cdot \zeta_\alpha(K) >$ given in eq.2.

$$|\Phi^{(\alpha)}(K) > = \frac{1}{\sqrt{N^{(\alpha)}(K)}} |\vec{K} \cdot \zeta_\alpha(K) > + (1 - \delta_{R,0}) \sum_{\lambda} \delta_{\lambda,1} \delta_{\lambda,2} + \delta_{\lambda,2} \delta_{\lambda,1} A^{(\alpha)}_K(P, \lambda) |\vec{K} \cdot \zeta_\lambda(K) > + \sum_{P \neq K} \sum_{\lambda} A^{(\alpha)}_K(P, \lambda) |P \cdot \zeta_\lambda(P) >$$

(26)

The new eigenstate $|\Phi^{(\alpha)}(K) >$ is given in terms of the amplitude $A^{(\alpha)}_K(P, \lambda)$ which represents the overlap between the eigenstate $|\Phi^{(\alpha)}(K) >$ and the spinor, $|P \cdot \zeta_\lambda(P) >$. The amplitude $|\Phi^{(\alpha)}(K) >$ is computed with the help of the matrix elements of the scattering potential given in eq.26. The function $N^{(\alpha)}(K)$ represents the normalization factor defined by the orthonormality condition, $< \Phi^{(\alpha)}(K) | \Phi^{(\beta)}(P) > = \delta_{\alpha,\beta} \delta_{R,0} \delta_{P,0}$. The non-periodic potential is time reversal invariant therefore the zero momentum eigenstate $|\Phi^{(\alpha)}(K = 0) >$ remains double degenerated! In order to deal with the state $K = 0$ we replace the integrals over the momenta with a discrete sum and replace the two dimensional delta function $\delta^2(K = 0)$ with the Kronecker delta function $\delta_{K,0}$. We will use the new basis given in eq.26 to represent the coordinates (similar to eq.4). We find the matrix elements of the Cartesian commutator in the new basis (the basis due to the scattering potential).

$$< \Phi^{(\alpha)}(K) | r^{(1)}, r^{(2)} | \Phi^{(\beta)}(P) > = -i \frac{1}{2} \sum_{\alpha,\beta} \delta_{R,0} \delta_{P,0} |N^{(\alpha)}(0) N^{(\beta)}(0)|^{-\frac{1}{2}}$$

(27)

Using the result given in eq.27 we compute the spin Hall current following the steps given in eq.11. The static current is determined by the normalization function $N^{(\alpha)}(\vec{K} = 0)$ given by eq.26.

$$J^{(3,ext,static)}_2 = \frac{-e}{2} E_{1}^{(ext)} \sum_{\alpha} \frac{1}{2} \sum_{K} \delta_{K,0} \frac{1}{N^{(\alpha)}(K) 2\pi} = \frac{1}{2} \sum_{\alpha} \frac{1}{N^{(\alpha)}(K) 2\pi} \frac{1}{4\pi} E_{1}^{(ext)}$$

(28)

The normalization factor $N^{(\alpha)}(\vec{K} = 0)$ is a function of the amplitude $A^{(\alpha)}_{K = 0}(P, \lambda)$ (the projection of the eigenstate $|\Phi^{(\alpha)}(K = 0) >$ on the spinor $|P \cdot \zeta_\lambda(P) >$). From eq.26 we determine he normalization factor,

$$\frac{1}{2} \sum_{\alpha} \frac{1}{N^{(\alpha)}(K^2 = 0)} = \frac{1}{2} \sum_{\alpha} \left[ 1 + \sum_{\lambda} \int \frac{d^2P}{(2\pi)^2} |A^{(\alpha)}_{K = 0}(P, \lambda)|^2 \right]^{-\frac{1}{2}}$$

(29)

The amplitude $A^{(\alpha)}_{K = 0}(P, \lambda)$ is a function of the matrix elements $V(|\vec{q}|)$ and is obtained within perturbation theory , $|A^{(\alpha)}_{K = 0}(P, \alpha)|^2 = \frac{V(\vec{K} = 0 - \vec{P})}{(\frac{\hbar}{m})^2 |k_{SO}| P + \frac{1}{2} P^2}$. This allows to compute the current in eq.28. We substitute the amplitude
\[ A^{(\alpha)}_{K=0}(P, \lambda) \] into the normalization function \( N^{(\alpha)}(\vec{K} = 0) \) and obtain from eq.28 the current;

\[
J_{2}^{(3, \text{ext.-static})} = [1 + \frac{m/\hbar^2}{(\hbar^2/2m)k_{so}^2} \int_{\kappa_{so}}^{\infty} \frac{(V(q))^2}{q(1 + \frac{q^2}{2})^2} dq]^{-1} \left[ -e \right] E_{1}^{(\text{ext})} \]  

(30)

In eq.30 we have introduced the dimensionless momentum \( q \) (the momentum has been normalized by the spin orbit momentum \( k_{so} \)). The last integral is evaluated with the help of the dimensionless infrared cutoff \( \frac{k_{so}}{\Lambda} = \frac{L}{\Lambda} \). where \( L \) is the size of the two dimensional system, \( \lambda_{so} \) is the spin orbit wavelength and \( \Lambda \) is an arbitrary ultraviolet cutoff (which does not affect the integration). The momentum integration can be performed when the matrix elements \( (V(q))^2 \) are replaced by a momentum independent potential. This introduces the scattering strength exponent \( \gamma_{sc} \)

\[ \gamma_{sc} \equiv \frac{(V(q))^2 m/\hbar^2}{k_{so}^2/\Lambda^2}. \]

We observe that for an infinite systems the integral diverges causing the spin Hall current to decreases with the size of the system \( L \),

\[
J_{2}^{(3, \text{ext.-static})} = [1 + \gamma_{sc} \ln \left( \frac{L}{\lambda_{so}} \right)]^{-1} \left[ -e \right] E_{1}^{(\text{ext})} \]

Next we add the time dependent part spin Hall current. This is done using equations 10 – 15. In the absence of inelastic scattering \( \Omega_{\alpha}r_{s} \rightarrow \infty \) we find following the result in eq.15 that the total spin Hall current is given by,

\[
J_{2}^{(3)} = [1 + \gamma_{sc} \ln \left( \frac{L}{\lambda_{so}} \right)]^{-1} \left[ -e \right] E_{1}^{(\text{ext})} \]

(31)

We observe that for an infinite systems the integral diverges causing the spin Hall current to decreases with the size of the system \( L \). The spin Hall current vanishes for an infinite system since the scattering causes the weight of the zero momentum state to spread over to all other states. Since the integrated weight for the non zero states diverges in two dimensions we find that the projection of the eigenstate with the \( K = 0 \) state vanishes. This result has been obtained using general arguments, the only assumption is that the absolute value of the square of the scattering potential is momentum independent, such a situation is realized for a single impurity potential. For many impurities we observe that as for the single impurity if we replace the square of the scattering potential by an ensemble average over the impurity configuration, \( (V(q))^2 \) \( \propto \langle (V(q))^2 \rangle_{\text{configuration-average}} = V_{sc}^2 \) and obtain the scattering exponent \( \gamma_{sc} \equiv \frac{V_{sc}^2 m/\hbar^2}{k_{so}^2/\Lambda^2}. \) For many impurities the fluctuations of the scattering potential will cause the current to vanishes faster than our power law prediction.

X. CONCLUSION

A new formulation for the spin Hall current based on exact ground state of the Rashba Hamiltonian has been presented. This formulation shows that the degeneracy of the wave function at \( \vec{K} = 0 \) gives rise to non-commuting coordinates. Due to the fact that the spin Orbit wave function has no gap the spin Hall conductivity is composed from two contribution, a static part caused by the non-commuting coordinates and a time dependent linear response part. When the Zeeman interaction is larger than the polarization energy the spin Hall conductivity is determined by the static part. The uniform magnetization is computed and we show that the spin current which is covariantly conserved is effectively conserved to first order in the electric field.

Using this new formulation we find the exact value for the spin Hall conductivity which vary from \( \frac{1}{15} \) to \( \frac{1}{11} \).

An exact derivation for the vanishing the spin Hall current caused by a scattering potentials has been presented. The spin Hall current vanishes for an infinite system since the scattering causes the weight of the zero momentum state to spread over to all other states. Since the integrated weight for the non zero states diverges in two dimensions we find that the projection of the eigenstate with the \( K = 0 \) state is zero. This result has been obtained using general arguments, the only assumption is that the absolute value of the square of the scattering potential is momentum independent, such a situation is realized for a single impurity potential.

APPENDIX A: THE SU(2) BERRY PHASE

The effect of the static electric field is equivalent to a time dependent vector potential. As a result obtain a time dependent momentum vector, \( K^{(1)}(t) = K^{(1)}(0) + \frac{e}{\hbar} E_{1}^{(\text{ext})} t, K^{(2)}(t) = K^{(2)}(0) \). Therefore according to the adiabatic approximation we replace the spinors \( |K \otimes \zeta_{\alpha}(K) > \) by the time dependent state, \( |K(t) \otimes \zeta_{\alpha}(K(t)) > \). According to Berry [27, 28] one finds instead of the spinor \( |K(t) \otimes \zeta_{\alpha}(K(t)) > = \exp \left[ \frac{i}{\hbar} \int_{0}^{t} \epsilon_{\sigma(\alpha)}(K(t')) dt' \right] |K(0) \otimes \zeta_{\alpha}(K(0)) > \) the SU(2) time dependent solution \( |\Psi_{\alpha}(K, t) > \) is given by,
\[ |\Psi_\alpha(K,t) > = T \exp[-i \int_0^t \Omega(K(t')) dt'] \exp[-i \hbar \int_0^t \epsilon_{\sigma(\alpha)}(K(t')) dt'] |K(0) \otimes \zeta_\alpha(K(0)) > \] (A1)

We observe that the Berry phase has been replaced by a time ordered matrix, \( T \exp[-i \int_0^t \Omega(K(t')) dt'] \) with the \( SU(2) \) matrix elements, \( \Omega_{\alpha\alpha}(K(t)) = \langle K(t) \otimes \zeta_\alpha(K(t)) | \frac{d}{dt} |K(t) \otimes \zeta_\alpha(K(t)) > \) and non-diagonal matrix elements, \( \Omega_{\alpha\beta}(K(t)) = \epsilon^{(1)}_{\sigma(\alpha)}(K(t')) - \epsilon^{(1)}_{\sigma(\beta)}(K(t')) \) \[ \times \int_0^t \Omega(K(t')) dt' \langle K(t) \otimes \zeta_\alpha(K(t)) | \frac{d}{dt} |K(t) \otimes \zeta_\beta(K(t)) > \). The \( SU(2) \) spinor \( |\Psi_\alpha(K,t) > \) is used to construct the time dependent representation for the coordinate and the velocity operator.

\[ r^{(i)}(t) = \int \frac{d^2 K}{(2\pi)^2} \int \frac{d^2 P}{(2\pi)^2} \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} |K \otimes \zeta_\alpha(K) > \langle \zeta_\beta(P) \otimes P| \]

\[ T(e^{i \int_0^t \Omega(K(t')) dt'} \epsilon^{(1)}_{\alpha}(K(t')) \int_0^t \epsilon^{(1)}_{\alpha}(K(t')) \int_0^t \epsilon^{(1)}_{\beta}(K(t')) dt' \]

\[ \frac{dr^{(2)}(K(t))}{dt} = \frac{1}{i\hbar} [r^{(2)}(K(t)), \h_0(K(t))] + \frac{1}{i\hbar} [r^{(2)}(K(t)), \Omega(K(t))] \]

\[ = \frac{1}{i\hbar} [r^{(2)}(K(t)), \h_0(K(t))] + \frac{1}{i\hbar} (-\epsilon) E^{(ext)}_1 [r^{(2)}(K(t)), r^{(1)}(K(t))] \] (A3)

The first term in eq.A2 represents the velocity in the absence of the external field. The second term describes the effect of the external field proportional to the commutator of the Cartesian coordinates. This can be seen in the following way, the matrix \( \Omega(K(t)) \) is equal to the product between the time derivative of the momentum and a new matrix \( \Omega(\frac{d}{dK}) \). For this matrix we replace the momentum derivative by the time derivative momentum derivative. We find, \( \Omega(K(t)) = \frac{dK^{(2)}}{dt} \Omega(\frac{d}{dK^{(2)}}) \). The momentum time derivative is proportional to the electric field and the matrix \( \frac{d}{dK^{(2)}} \) is identified with the Cartesian coordinate, \([r^{(2)}, \Omega(\frac{d}{dK^{(2)}})] \equiv [r^{(2)}, r^{(1)}] \). As a result of the \( SU(2) \) adiabatic evolution, the velocity operator given in eq.A2 is equivalent to the the Heisenberg equation of motion given in eq.6 (with the same commutation rules as given in eq.4).

**APPENDIX B: GAUGE INVARIANCE IN THE MOMENTUM SPACE-THE COVARIANTLY CONSERVED SPIN CURRENT**

In this appendix we will consider the \( U(1) \times SU(2) \) gauge invariance of Aharon-Casher (A-C) 23 model for a periodic lattice in the momentum space. The A-C effect is viewed as a moving magnetic moment which is equivalent to a moving magnetic current, due to the Lorentz transformation an induced electric-charge appears in the laboratory frame, which interacts with an electrostatic potential.

\[ H = \frac{\hbar^2}{2m} [-i\partial_r - \frac{e}{\hbar c} \vec{A}(r) - \frac{g - 1}{2\hbar c} (\vec{B} \times \vec{E}(r))]^2 - \frac{g}{2} \mu_B \vec{B} \cdot (\vec{\nabla} \times \vec{A}) \] (B1)

where \( \vec{A}(r) \) is the \( U(1) \) electromagnetic vector potential, \( A_E(r) \equiv (\vec{\sigma} \times \vec{E}(r)) \) is the \( SU(2) \) “electrostatic vector potential” (\( \vec{\sigma} \) is the Pauli matrix and \( \vec{E}(r) \) is the electrostatic field), \( \mu_B \) is the Bohr magneton, \( g \approx 2 \) is the gyro magnetic factor with \( g \rightarrow (g - 1)/2 \) being the Thomas precession and “e” is the electrostatic charge.

For a periodic lattice we use the quasi-momentum representation with momentum integration restricted to the first Brillouin zone. The single particle energy and the two component spinor obey the symmetry in the momentum space (for simplicity we replace the Bloch wave function by the free particle representation), \( \epsilon \equiv \epsilon(K) = \epsilon(K + \vec{G}) \), \( \Psi^+(K) = \Psi^+(K + \vec{G}) \), \( \Psi(K) = \Psi(K + \vec{G}) \) where \( \vec{G} \) is the reciprocal Lattice vector. In order to describe the two dimensional Rashba model we take the z component of the electric field to be constant in space. We introduce the static \( SU(2) \) Rashba term,
\[ \sum_{A=1,2} \tilde{\omega}_A(K) \sigma^{(A)} = \kappa_{so}(\hat{\sigma} \times \hat{e}_3) \]

Only two components are nonzero: \( \omega_{(1,A=2)}(K) \sigma^{(A=2)} = -\kappa_{so} \sigma^{(2)} \), \( \omega_{(1,A=1)}(K) \sigma^{(A=1)} = \kappa_{so} \sigma^{(1)} \) and \( \omega_{(1,A=1)}(K) \sigma^{(A=1)} = \omega_{(2,A=2)}(K) \sigma^{(A=2)} = 0 \). The fluctuation of the electric field gives rise to the SU(2) gauge field, \( \tilde{A}(K) = \sum_{A=1,2} \tilde{A}_A(K) \sigma^{(A)} \). In the absence of a magnetic field we have \( \sum_{A=1,2,3} \omega_{0,A}(K) \sigma^{(A)} = 0 \) and the fluctuating part is given by \( \sum_{A=1,2,3} \Lambda_{0,A}(K) \sigma^{(A)} \). In the momentum representation the model given in eq.B1 the external gauge field and without the static magnetic field takes form: \( S = S_0 + S_{ext} \), \( S_0 \) is the free electron action in the presence of the Rashba term and \( S_{ext} \) is the electromagnetic \( U_{em}(1) \times SU(2)_\text{spin} \) action.

\[ S_0 = \int dt \int \frac{d^2 K}{(2\pi)^2} \left( \Psi^+(K,t)(-i\hbar \partial_t - E_F)\Psi(K,t) - \Psi^+(\tilde{K},t)[\epsilon(K) - \sum_{A=1,2} \tilde{\omega}_A(K) \sigma^{(A)}] + \sum_{A=1,2,3} \omega_{0,A}(K) \sigma^{(A)} | \Psi(K,t) \right) \]

(B2)

In momentum space the external vector potential \( \tilde{A}(q,t) \) and the scalar potential \( A_0(q,t) \) generate trough the A-C term a vector SU(2) gauge potential \( \tilde{A}(q) = \sum_{A=1,2} \tilde{A}_A(q) \sigma^{(A)} \) and a static SU(2) gauge potential \( \sum_{A=1,2,3} \Lambda_{0,A}(q) \sigma^{(A)} \). The external action \[29\] is given by: \( S_{ext} = S_{ext}^{(para)} + S_{ext}^{(dia)} \).

\[ S^{(para)}_{ext} = \int dt \int \frac{d^2 q}{(2\pi)^2} \int \frac{d^2 K}{(2\pi)^2} \frac{\hbar^2}{2m} \left( \Psi^+(K,t) \left( \frac{\hbar}{m} (K - \frac{1}{2} \tilde{q}) - \sum_{A} \tilde{\omega}_A \sigma^{(A)} \right) + \sum_{A=1,2,3} \bar{\Lambda}_A(-\tilde{q}) \sigma^{(A)} \right) \]

\[ + e A_0(-\tilde{q}) + \sum_{A=1,2,3} \Lambda_{0,A}(-\tilde{q}) \sigma^{(A)} ] e^{iq \cdot \tilde{R}(K)} \Psi(K) \]

(B3)

\[ S^{(dia)}_{ext} = \int dt \int \frac{d^2 q}{(2\pi)^2} \int \frac{d^2 K}{(2\pi)^2} \frac{\hbar^2}{2m} \Psi^+(K,t) [\epsilon \tilde{A}(q) + \sum_{A=1,2} \bar{\Lambda}_A(q) \sigma^{A} ] e^{i\tilde{q} \cdot \tilde{R}(K)} \Psi(K) \]

(B4)

We observe that the action is \( U(1) \times SU(2) \) gauge invariant in momentum space. As a result the coordinate is SU(2) transformed, \( \tilde{R}(K) \rightarrow R(K) \). Using Noether’s theorem \[30\] we obtain that the Spin-current is determined by the derivative with respect to the SU(2) gauge potential \( \Lambda_{\mu,A}(q) \).

\[ j_0^{(A)}(q) = \frac{\hbar}{2} \frac{\partial S}{\partial A_{0,\mu = (-\tilde{q})}} = \frac{\hbar}{2} \int \frac{d^2 K}{(2\pi)^2} \Psi^+(K) \sigma^{(A)} e^{i\tilde{q} \cdot \tilde{R}(K)} \Psi(K) \]

(B5)

\[ j_i^{(A)}(q) = \frac{\hbar}{2} \frac{\partial S}{\partial A_{\mu i,\mu = (-\tilde{q})}} = \frac{\hbar}{2} \int \frac{d^2 K}{(2\pi)^2} \Psi^+(K) \left[ \frac{1}{2} \left( \frac{\hbar}{m} (K - \frac{1}{2} \tilde{q} - e \tilde{A}(q)) \sigma_i^{(A)} \right) + e^{i \tilde{q} \cdot \tilde{R}(K)} \Psi(K) \right] \]

(B6)

where \( \sigma^{(A)} \) are the Pauli matrices.

In order to study the electromagnetic response for an electric field without an orbital magnetic effect \( \tilde{q} \times \tilde{A}(q) = 0 \) we take the vector potential to be zero \( \tilde{A}(q) = 0 \) and represent the electric field by a scalar potential \( A_{(ext)}(q,t) \) which satisfies the condition \( E_{(ext)}(q,t) = i \hbar A_{(ext)}(q,t) \). An alternative choice is to replace the vector potential by a time dependent vector potential. As a result we have to use a time dependent momentum \( K^{(1)}(t) = K^{(1)}(0) + \frac{\hbar}{m} E_{(ext)}^2(t) \) this approach has been used in the appendix A. From eq.B6 we obtain the spin velocity is given by the symmetric product between the spin operator and the velocity operator. Using the SU(2) transformation which diagonalizes the Rashba Hamiltonian introduced in chapter III with the velocity operator \( V_i(K) = (\frac{dt}{dt}(\Phi(K,t)))_{|t=0} \) gives us according to eq.B6 the spin velocity operator \( \delta_i^{(A)}(K) = \frac{1}{2} \{ V_i(K), \sigma_i^{(A)} \}_+ \) where \( \{ , \}_+ \) stands for the symmetric product and \( \delta^{(A)} = U^{(1)}(K) \sigma^{(A)} U^{(1)}(K) \) represents the transformed Pauli matrix with \( A = 1, 2, 3 \). The transformed spin current operator in the Heisenberg picture is given by:

\[ j_{i,H}^{(A)}(q,t) = \frac{\hbar}{2} \int \frac{d^2 K}{(2\pi)^2} \Psi^+(K) \left[ \frac{1}{2} \{ V_i,H(K), \sigma_i^{(A)}(K,t) \}_+ e^{i \tilde{q} \cdot \tilde{R}(K,t)} - \frac{1}{2} \bar{q}_i \{ e^{i \tilde{q} \cdot \tilde{R}(K,t)}, \sigma_i^{(A)}(K,t) \}_+ \right] \Psi(K) \]
where \( \hat{\sigma}^{(A)} = U^{+}(K)\sigma^{(A)}U(K) \) with the Heisenberg representation of the matrix elements, \( V_{i,H}(\vec{K},t) \), \( r_{i}^{(A)}(\vec{K},t) \) and \( \hat{\sigma}^{(A)}(\vec{K},t) \). The spin currents are computed by taking the expectation value with respect the Many-Body state, \( \left| F.S. > = |K_{p}, K_{p}^{\perp} >, \right. \)

\[
J_{i}^{(A)}(\vec{q},t) = < F.S.|J_{i}^{(A)}(\vec{q},t)|F.S. >
\]  

(B8)

Using Noether’s theorem \[30, 31\], we obtain that the spin currents are only covariantly conserved.

\[
\frac{\partial}{q}J_{0}^{(1)}(\vec{q},t) + i\vec{q} \cdot \vec{J}^{(1)}(\vec{q},t) = 2e_{ABC}[\omega_{0,B} + \Lambda_{0,B}(\vec{q},t)]J_{0}^{(C)}(\vec{q},t) + (\vec{\omega}_{B} + \vec{\Lambda}_{B}(\vec{q})) \cdot \vec{J}^{(C)}(\vec{q},t)
\]  

(B9)

where \( e_{ABC} \) is the antisymmetric tensor which takes the values, 1, 0, -1. We apply this equation to the Rashba model. The spin current components in the limit \( q \to 0 \) obey the following continuity equation for the three polarizations:

\[
\frac{\partial}{q}J_{0}^{(1)} + i\vec{q} \cdot \vec{J}^{(1)}(\vec{q}) = 0, \quad \frac{\partial}{q}J_{0}^{(2)} + i\vec{q} \cdot \vec{J}^{(2)}(\vec{q}) = 0.
\]

Only the A = 3 component violates the continuity equation

\[
\frac{\partial}{q}J_{0}^{(3)} + i\vec{q} \cdot \vec{J}^{(3)}(\vec{q}) = 2e_{3BC}\vec{\omega}_{B}(\vec{K}) \cdot \vec{J}^{(C)}(\vec{q}) = 2k_{so}(J_{1}^{(1)}(q) + J_{2}^{(2)}(q))
\]

The first term on the right hand side of the equation vanishes as a result of the momentum integration. The second term vanishes as a result of the expectation value with respect the Pauli matrices. The external electric field induces a first order contribution to the spin velocity which vanishes after the momentum integration is performed. Therefore we conclude that the continuity equation is satisfied effectively.

**APPENDIX C: THE SPIN AND CHARGE HALL EFFECT IN THE PRESENCE OF A ZEEMAN INTERACTION**

In this appendix we will consider the charge Hall effect in the static linear response approximation. We will consider the current for the following cases: No orbital magnetic contribution, magnetic orbital current without spin Orbit and at the end we compute the charge Hall in the presence of the spin Orbit interaction and a weak magnetic field.

In the absence of the orbital motion the charge Hall conductivity is given by:

\[
\sigma_{Hall}^{C} = \frac{e^{2}}{2h}[f_{F.D.}(\epsilon_{so} + b_{3} - E_{F}) - f_{F.D.}(\epsilon_{so} - b_{3} - E_{F})]
\]  

(C1)

This result follows directly from equation 5 once we replace the Pauli matrix \( \sigma^{(3)} \) with the identity matrix I. The conductivity is zero in the absence of the Zeeman interaction and vanishes at zero temperature.

The orbital effects are investigated for a periodic potential \( W(\vec{q}) \) with the Bloch eigenfunctions \( u_{n,K}(\vec{q}) \). \( n \) is the band index and \( \vec{q} \) is the coordinate in the unit cell. The coordinate \( \vec{R} \) is a function of \( \vec{q}, \vec{R} = \vec{q} + \vec{r} = \vec{q} + i\frac{\partial}{\partial \vec{K}} \).

The curvature is determined by the band connection \( \tilde{A}_{n,m}(\vec{K}) = i\int \frac{d^{2}q}{(2\pi)^{2}} u_{m,K}^{*}(\vec{q}) \frac{\partial}{\partial \vec{K}} u_{n,K}(\vec{q}) \) in the one band approximation the commutator is given by \( [r^{(1)}, r^{(2)}] = i\Omega(\vec{K}, n) \) with the band curvature, \( \Omega(\vec{K}, n) = \frac{\vec{q}}{2\pi} \times \tilde{A}_{n,n}(\vec{K}) \).

We apply this formalism to electrons in a constant magnetic field \( b \) perpendicular to the two dimensional plane. (For a strong magnetic field which is such that the magnetic flux per unit cell is a rational multiple \( \frac{\Phi}{l} \) of the flux unit \( \Phi_{0} \) we obtain a magnetic unit cell with \( l \) cites and a reduced Brillouin zone. The complex eigenfunction \( u_{n,K}(\vec{q}) \) has zero’s in the Brillouin zone which is the origin for the U(1) Berry phase connections.) In order to study orbital effects we replace the Rashba Hamiltonian equation by \( h_{0}(\vec{K}, \vec{r}) = \frac{\vec{q}}{2m} \cdot (\vec{K} - \frac{\vec{q}}{2m} \times \vec{r}) - k_{so}(\vec{M} \times \vec{r})^{2} \). We define the kinetic crystal momentum \( \vec{\kappa} \) which replaces the momentum \( \vec{K} \) by \( \vec{K} = \frac{\vec{q}}{2m} \times \vec{r} \). The kinetic crystal momentum components do not commute in a magnetic field, \( [\kappa_{1}, \kappa_{2}] = -i\frac{e}{c} b \).

In the absence of the spin – orbit interaction the Bloch function in a magnetic field \[32\] gives rise to a U(1) Berry phase connection \( \tilde{A}_{n,n}(\vec{K}) = i\int \frac{d^{2}q}{(2\pi)^{2}} u_{n,K}^{*}(\vec{q}) \frac{\partial}{\partial \vec{K}} u_{n,K}(\vec{q}) \) and a Berry curvature \( \Omega(\vec{K}, n) = \frac{\vec{q}}{2m} \times \tilde{A}_{n,n}(\vec{K}) \).
the one band approximation we find \([r^{(1)}, r^{(2)}] = i\Omega(\vec{K}, n)\). We introduce the \(4 \times 4\) Symplectic matrix to describe the commutators of the Cartesian coordinates and kinetic momentum, \(J_{i,j} = [\xi_i, \xi_j]\) where \(\vec{\xi} = (r_1, r_2, \kappa_1, \kappa_2) = (\xi_1, \xi_2, \xi_3, \xi_4)\). The Heisenberg equation of motion are given by,

\[
(i\hbar) \frac{d\xi_i(\vec{K})}{dt} = \sum_j J_{i,j} \frac{\partial h_0(\vec{K}, \vec{r}(\vec{K}))}{\partial \xi_j(\vec{K})}
\]

We obtain the same equation of motion as the one given in ref. [34] (see eqs.1, 2). Using the equation of motion we compute the charge Hall current in the \(i = 2\) direction,

\[
J^{(i)}_{2,\text{Hall}} = \frac{e}{l^2} \sum_{\vec{K},n} \text{Tr}[\frac{d\vec{r}(\vec{K})}{dt} F_{F,D}(E_n(\vec{K}), E_F)]
\]

"Tr" stands for the trace over the spin degrees of freedom. \(F_{F,D}(\vec{K}, E_F) = \delta(\sigma, \uparrow)f_{F,D}^\uparrow(E_n(\vec{K}), E_F) + \delta(\sigma, \downarrow)f_{F,D}^\downarrow(E_n(\vec{K}), E_F)\) is the Fermi-Dirac function and \(E_n(\vec{K})\) are the Bloch eigenvalues as a function of the magnetic band index \(n\). We find that the conductivity is given by,

\[
\frac{\epsilon^2}{\hbar} \int d^2K \sum_n \Omega(\vec{K}, \vec{n})[f_{F,D}(E_n(\vec{K}) + b_3 - E_F) + f_{F,D}(E_n(\vec{K}) - b_3 - E_F)]
\]

This result is in agreement with the result in ref. [34] (see eq.3). For strong magnetic fields the commutator \([r^{(1)}, r^{(2)}] = i\Omega(\vec{K})\) describes the vorticity for the Hall wave function [34]. We find that the conductance is determined by the integral over the curvature and is given by the first Chern number [24] (see eqs.3.9 and 4.8).

Next we consider the combined effect of a spin Orbit interaction and a weak magnetic field. For each Bloch band with magnetic eigenfunctions \(u_{n,K}(\vec{q})\) we compute the following SU(2) matrix elements, \((\epsilon(\vec{\kappa}))_{n,n} + (\vec{\sigma} \cdot \vec{R})_{n,n}\) where \((\vec{R})_{n,n} = ((-\frac{\hbar k_F}{m}K_2, \frac{\hbar k_F}{m}K_1, \frac{1}{2} g\mu_B (B + b)))_{n,n}\) is a fictitious magnetic field for the band \(n\) and \((\epsilon(\vec{\kappa}))_{n,n} \equiv E_n(\kappa)\) is the eigenvalue of the Bloch state and \(b_3 = \frac{1}{2} g\mu_B (B + b)\) is the total Zeeman field. We introduce an SU(2) transformation to rotate the Pauli matrix \(\vec{\sigma}\) in the direction of the fictitious magnetic field \((\vec{R})_{n,n}\) where \((\vec{K})_{n,n} = \vec{K} - \frac{\epsilon}{2\hbar c} \vec{b} \times (\vec{r})_{n,n} = \vec{K} - \frac{\epsilon}{2\hbar c} \vec{b} \times \vec{A}_{n,n}(\vec{K})\). For weak magnetic field we find that the curvature for the band \(n\) is given by, \([r^{(1)}, r^{(2)}]_{n,n} = i \int \Omega(\vec{K}, \vec{n}) + \frac{1}{2} \left[\sigma^3 \cos(\kappa_{n,n}) + \sigma^1 \sin(\kappa_{n,n})\right] 2\pi \delta^2((\kappa), n)\). The first term is the Berry curvature of the magnetic Bloch function (generated by the connection) \(\vec{A}_{n,n}(\vec{K}) = \frac{i}{\pi} \frac{d^2}{r_{\vec{q}}} u_{n,K}(\vec{q}) \frac{\partial}{\partial \vec{K}} u_{n,K}(\vec{q})\). The second part is the curvature due to the SU(2) spin-orbit interaction. This part is obtained by replacing \(\vec{K} \rightarrow (\vec{R})_{n,n} = \vec{K} - \frac{\epsilon}{2\hbar c} \vec{b} \times (\vec{r})_{n,n} = \vec{K} - \frac{\epsilon}{2\hbar c} \vec{b} \times \vec{A}_{n,n}(\vec{K})\). As a result we find that the Hall conductivity has two parts, a band contribution and a spin Orbit part;

\[
\frac{\epsilon^2}{\hbar} \int d^2K \sum_n \Omega(\vec{K}, \vec{n})[f_{F,D}(E_n(\vec{K}) + \epsilon_{so} + b_3 - E_F) + f_{F,D}(E_n(\vec{K}) + \epsilon_{so} - b_3 - E_F)]
\]

\[
+ \frac{\epsilon^2}{\hbar} \sum_n \left[f_{F,D}(E_n(0) + \epsilon_{so} + b_3 - E_F) - f_{F,D}(E_n(0) + \epsilon_{so} - b_3 - E_F)\right]
\]

In the present case the spin orbit contribution to the current (the second term) is determined by the magnetic cyclotron eigenvalues \(E_n(0) \approx \frac{\epsilon_{so}}{\hbar^2}(n + \frac{1}{2}), n = 0, 1, 2, 3, \ldots\). The Spin Orbit contribution to the current decreases with the increase of the magnetic field faster than in the case where orbital effects are absent.

[1] J.Wunderlich, B.Kaestner, J.Sinova, and T.Jungwirth, Phys.Rev.Lett. 94, 047204 (2005).
[2] Y.K.Kato, R.C.Myers, A.C.Gossard and D.D.Awshalm, Science 306, 1910 (2004).
[3] V.Sih, R.C.Myers, Y.K.Kato, W.H.Lau, A.C.Gossard and D.D.Awshalm, Nature Physics 1, 31-35 (2005).
[4] B.Andrei Bernevig and Shou-Cheng Zhang, cond-mat/0412550
[5] J.E.Hirsch, Phys.Rev.Lett. 83, 1834 (1999).
[6] J.Sinova et al., Phys.Rev.Lett. 92, 126603 (2004).
[7] E.G.Mishchenko, A.V.Shytov, and B.I.Halperin, Phys.Rev.Lett.93, 226602 (2004).
