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Asymptotic expansions for some integrals of quotients with degenerated divisors

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Abstract

We study asymptotic expansion as \( \nu \to 0 \) for integrals over \( \mathbb{R}^2 \) of quotients \( F(x, y) / ((x \cdot y)^2 + (\nu \Gamma(x, y))^2) \), where \( \Gamma \) is strictly positive and \( F \) decays at infinity sufficiently fast. Integrals of this kind appear in description of the four–waves interactions.

1 Introduction

Our concern is the integrals

\[
I_\nu = \int_{\mathbb{R}^d \times \mathbb{R}^d} dx \, dy \frac{F(x, y)}{(x \cdot y)^2 + (\nu \Gamma(x, y))^2}, \quad d \geq 2, \quad 0 < \nu \ll 1. \tag{1.1}
\]

Such integrals and their singular limits \( \nu \to 0 \) appear in physical works on the four-waves interaction, where the latter is suggested as a mechanism, dictating the long-time behaviour of solutions for nonlinear Hamiltonian PDEs with cubic nonlinearities and large values of the space-period. Usually the integrals \( I_\nu \) appear there in an implicit form, and become visible as a result of rigorous mathematical analysis of the objects and constructions, involved in the heuristic physical argument (see below in this section).

We denote \( (x, y) = z \in \mathbb{R}^d, \omega(z) = x \cdot y, \) and assume that \( F \) and \( \Gamma \) are \( C^2 \)-smooth real functions, satisfying

\[
|\partial^\alpha z F(z)| \leq K |z|^{-M-|\alpha|}, \quad \forall z, \forall |\alpha| \leq 2; \tag{1.2}
\]

\[
|\Gamma(z)| \geq K^{-1} |z|^{r_*}, \quad |\partial^\alpha z \Gamma(z)| \leq K |z|^{r_*-|\alpha|}, \quad \forall z, \forall |\alpha| \leq 2. \tag{1.3}
\]

Here \( r_*, M, K \) are any real constants such that

\[
M + r_* > 2d - 2, \quad M > 2d - 4, \quad K > 1. \tag{1.4}
\]
As usual we denote \( \langle z \rangle = \sqrt{|z|^2 + 1} \).

The main difficulty in the study of \( I_\nu \) comes from the vicinity of the quadric \( \Sigma = \{ \omega(z) = 0 \} \). The latter has a locus at 0 ∈ \( \mathbb{R}^{2d} \) and is smooth outside it. Firstly we will study \( I_\nu \) near 0, next – near the smooth part of the quadric, \( \Sigma \setminus \{0\} =: \Sigma_* \), and finally will combine the results obtained to get the main result of this work:

\textbf{Theorem 1.1.} As \( \nu \to 0 \), the integral \( I_\nu \) has the following asymptotic:

\[ I_\nu = \pi \nu^{-1} \int_{\Sigma_*} \frac{F(z)}{|z| \Gamma(z)} \, dz + I_\Delta. \]  

\text{(1.5)} \hspace{1cm} \text{p22}

Here \( dz \) is the volume element on \( \Sigma_* \) and \(|I_\Delta| \leq C \chi_d(\nu)\), where

\[ \chi_d(\nu) = \begin{cases} 1, & d \geq 3, \\ \max(1, \ln(\nu^{-1})), & d = 2. \end{cases} \]  

\text{(1.6)} \hspace{1cm} \text{chi_d}

The integrand in (1.5) converges absolutely, and the constant \( C \) depends on \( d, K, M \) and \( r_* \).

The integral in (1.5) may be regarded as integrating of the function \( F/\Gamma \) against a measure on \( \mathbb{R}^{2d} \), supported by \( \Sigma \), which we will denote \(|z|^{-1} \delta_{\Sigma_*} \) (here \( \delta_{\Sigma_*} \) is the delta–function of the hypersurface \( \Sigma_* \)). For any real number \( m \) let \( C_m(\mathbb{R}^{2d}) \) be the space of continuous functions on \( \mathbb{R}^{2d} \) with the finite norm \( |f|_m = \sup_z |f(z)| \langle z \rangle^m \). By (1.2) and (1.3), \( F/\Gamma \in C_{M+r_*}(\mathbb{R}^{2d}) \), where \( M + r_* > 2d - 2 \).

\textbf{Proposition 1.2.} The measure \(|z|^{-1} \delta_{\Sigma_*} \) is an atomless \( \sigma \)-finite Borel measure on \( \mathbb{R}^{2d} \). The integrating over it defines a continuous linear functional on the space \( C_m(\mathbb{R}^{2d}) \) if \( m > 2d - 2 \).

Since any function \( F \in C^\infty_0(\mathbb{R}^{2d}) \) satisfies (1.2) for every \( M \), then we have

\textbf{Corollary 1.3.} Let a \( C^2 \)-function \( \Gamma \) meets (1.3) with some \( r_* \). Then the function \( \nu/(|x|^2 + (\nu \Gamma(x,y))^2) \) converges to the measure \(|z|^{-1} \delta_{\Sigma_*} \) as \( \nu \to 0 \), in the space of distributions.

The theorem and the proposition are proved below in Sections 2-4.

In the mentioned above works from the non-linear physics, to describe the long-time behaviour of solutions for nonlinear Hamiltonian PDEs with cubic nonlinearities, physicists derived nonlinear kinetic equations, called the (four–) wave kinetic equations. The \( k \)-th component of the kinetic kernel \( K \) \((k \in \mathbb{R}^d)\) for such equation is given by an integral of the following form:

\[ K_k = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F_k(k_1, k_2, k_3) \delta_{k, k_2} \Gamma_{k_1, k_2} \, dk_1 dk_2 dk_3. \]  

\text{(1.7)} \hspace{1cm} \text{heur}

Here \( \delta_{k_1, k_2} \) is the delta-function \( \{ k + k_3 = k_1 + k_2 \} \) and \( \Gamma_{k_1, k_2} \) is the delta-function \( \{ \omega_k + \omega_{k_3} = \omega_{k_1} + \omega_{k_2} \} \), where \( \omega_k \) is the spectrum of oscillations for

\footnote{corresponding to the Riemann structure on \( \Sigma_* \), obtained by the restricting to \( \Sigma_* \) the standard Riemann stricture on \( \mathbb{R}^{2d} \).}
the linearised at zero equation. If the corresponding nonlinear PDE is the cubic NLS equation, then \( \omega_k = |k|^2 \). In this case the two delta-functions define the following algebraic set:

\[
\{ (k_1, k_2, k_3) \in \mathbb{R}^{3d} : k + k_3 = k_1 + k_2, \quad |k|^2 + |k_3|^2 = |k_1|^2 + |k_2|^2 \},
\]

see [5], p.91, and [3]. Excluding \( s_3 \) using the first relation we write the second as \(-2(k_1 - k) \cdot (k_2 - k) = 0\). Or \(-2x \cdot y = 0\), if we denote \( x = k_1 - k \), \( y = k_2 - k \). That is, \( K_k \) is given by an integral over the set \( \Sigma \subset \mathbb{R}^{2d} \) as in [1.5]. In a work in progress (see [4]) we make an attempt to derive rigorously a wave kinetic equation for NLS with added small dissipation and small random force (see [3, 4] for a discussion of this model). On this way nonlinearities of the form (1.7) appear naturally as limits for \( \nu \to 0 \) of certain integrals of the form (1.1), where, again, \( x = k_1 - k \), \( y = k_2 - k \). We strongly believe that more asymptotical expansions of integrals, similar to (1.1), will appear when more works on rigorous justification of physical methods to treat nonlinear waves will come out.

Proof of Theorem 1.1, given below in Sections 2–4, is rather general and applies to other integrals with singular divisors. Some of these applications are discussed in Section 5.

**Notation.** By \( \chi_A \) we denote the characteristic function of a set \( A \). For an integral \( I = \int_{\mathbb{R}^{2d}} f(z) \, dz \) and a submanifold \( M \subset \mathbb{R}^{2d} \), \( \dim M = m \leq 2d \), compact or not (if \( m = 2d \), then \( M \) is an open domain in \( \mathbb{R}^{2d} \)) we write

\[
\langle I, M \rangle = \int_M f(z) \, d_M(z),
\]

where \( d_M(z) \) is the volume-element on \( M \), induced from \( \mathbb{R}^{2d} \).

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### 2 Integral over the vicinity of 0.

For \( 0 < \delta \leq 1 \) consider the domain

\[
K_\delta = \{|x| \leq \delta, |y| \leq \delta \} \subset \mathbb{R}^d \times \mathbb{R}^d,
\]

and the integral

\[
\int_{K_\delta} \frac{|F(x, y)| \, dx \, dy}{(x \cdot y)^2 + (\nu \Gamma(x, y))^2}.
\]  

So the integrand \( F_k \) depends on the parameter \( k \in \mathbb{R}^d \). This dependence should be controlled, which can be done with some extra efforts.
Obviously, everywhere in $K_\delta$, $|F(x, y)| \leq C_1$ and $\Gamma(x, y) \geq C$. So the integral is bounded by $C_1 I_\nu(\delta)$, where
\[
I_\nu(\delta) = \int_{|x| \leq \delta} \int_{|y| \leq \delta} \frac{dx \, dy}{(x \cdot y)^2 + (C\nu)^2}.
\]
We write $\bar{I}_\nu(\delta)$ as
\[
\bar{I}_\nu(\delta) = \int_{|x| \leq \delta} J_x \, dx, \quad J_x = \int_{|y| \leq \delta} \frac{dy}{(x \cdot y)^2 + (C\nu)^2}.
\]
Let us introduce in the $y$-space a coordinate system $(y_1, \ldots, y_d)$ with the first basis vector $e_1 = x/r$, where $r = |x|$. Since the volume of the layer, lying in the ball $\{|y| \leq \delta\}$ above an infinitesimal segment $[y_1, y_1 + dy_1]$ is $\leq C_d \delta^{d-1} dy_1$ and since $(x \cdot y) = ry_1$, then
\[
J_x \leq C_d r^{-2} \int_0^\delta dy_1 \frac{\delta^{d-1}}{y_1^2 + (C\nu/r)^2} = C_d \delta^{d-1} \tan^{-1}\left(\frac{r\delta}{C\nu}\right) \leq \frac{\pi}{2} C_d \delta^{d-1} \frac{\delta^{d-1}}{C\nu}.
\]
So
\[
\bar{I}_\nu(\delta) = \int_{|x| \leq \delta} J_x \, dx \leq C_d \delta^{d-1} \int_0^\delta r^{d-2} \, dr \leq C_d \delta^{2d-2} \nu^{-1}.
\]
Thus we have proved

**Lemma 2.1.** The integral (2.1) is bounded by $C\nu^{-1} \delta^{2d-2}$.

Now we pass to the global study of the integral (1.1) and begin with studying the geometry of the manifold $\Sigma_\ast$ and its vicinity in $\mathbb{R}^{2d}$.

### 3 The manifold $\Sigma_\ast$ and its vicinity.

The set $\Sigma_\ast = \Sigma \setminus (0, 0)$ is a smooth submanifold of $\mathbb{R}^{2d}$ of dimension $2d - 1$. Let $\xi \in \mathbb{R}^{2d-1}$ be a local coordinate on $\Sigma_\ast$ with the coordinate mapping $\xi \mapsto (x_\xi, y_\xi) = z_\xi \in \Sigma_\ast$. Abusing notation we write $|\xi| = |(x_\xi, y_\xi)|$. The vector $N(\xi) = (y_\xi, x_\xi)$ is a normal to $\Sigma_\ast$ at $\xi$ of length $|\xi|$, and
\[
N(\xi) \cdot (x_\xi, y_\xi) = 2 x_\xi \cdot y_\xi = 0. \quad (3.1)
\]

For any $0 \leq R_1 < R_2$ we denote
\[
S_{R_1}^{R_2} = \{z \in \mathbb{R}^{2d} : |z| = R_1\}, \quad \Sigma_{R_1}^{R_2} = \Sigma \cap S_{R_1}^{R_2},
\]
and for $t > 0$ denote by $D_t$ the dilation operator
\[
D_t : \mathbb{R}^{2d} \to \mathbb{R}^{2d}, \quad z \mapsto tz.
\]
It preserves $\Sigma_\ast$, and for any $\xi \in \Sigma_\ast$ we denote by $t\xi$ the point $D_t(x_\xi, y_\xi)$. 

\[\text{4}\]
Lemma 3.1. 1) There exists $\theta_0^* \in (0, 1]$ such that for any $0 < \theta_0 \leq \theta_0^*$ a suitable neighbourhood $\Sigma^{nbh} = \Sigma^{nbh}(\theta_0)$ of $\Sigma_*$ in $\mathbb{R}^{2d} \setminus \{0\}$ may be uniquely parametrised as

$$\Sigma^{nbh} = \{\pi(\xi, \theta) : \xi \in \Sigma_*, |\theta| < \theta_0\}, \quad (3.3)$$

where $\pi(\xi, \theta) = (x_\xi, y_\xi) + \theta N_\xi = (x_\xi, y_\xi) + \theta(y_\xi, x_\xi)$.

2) For any vector $\pi = \pi(\xi, \theta) \in \Sigma^{nbh}$ its length equals

$$|\pi| = |\xi| \sqrt{1 + \theta^2}. \quad (3.4)$$

The distance from $\pi$ to $\Sigma$ equals $|\xi| |\theta|$, and the shortest path from $\pi$ to $\Sigma$ is the segment $[\xi, \pi] = \{\pi(\xi, \theta) : 0 \leq t \leq 1\} =: S$.

3) If $z = (x, y) \in S^{R}$ is such that $\text{dist}(z, \Sigma) \leq \frac{1}{2}R\theta_0$, then $z = \pi(\xi, \theta) \in \Sigma^{nbh}$, where $|\theta| < \theta_0$ and $|\xi| \leq R \sqrt{1 + \theta^2}$.

4) If $\pi(\xi, \theta) \in \Sigma^{nbh}$, then

$$\omega(\pi(\xi, \theta)) = |\xi|^2 \theta. \quad (3.5)$$

5) If $(x, y) \in S^{R} \cap (\Sigma^{nbh}^*)^c$, then $|x \cdot y| \geq cR^2$ for some $c = c(\theta_0) > 0$.

The coordinates $\{3.3\}$ are known as the normal coordinates, and their existence follows easily from the implicit function theorem. The assertion 1) is a bit more precise than the general result since it specifies the size of the neighbourhood $\Sigma^{nbh}$.

**Proof.** 1) Fix any positive $\kappa < 1$. Then for $\theta_0^*$ small enough it is well known that the points $\pi(\xi, \theta)$ with $\xi \in \Sigma_{1-\kappa}^{1+\kappa}$ and $|\theta| < \theta_0 \leq \theta_0^*$ form a neighbourhood of $\Sigma_{1-\kappa}^{1+\kappa}$ in $\mathbb{R}^{2d}$ and parametrise it in a unique and smooth way. Besides, any point $\pi' \in \mathbb{R}^{2d}$ such that $\text{dist}(\pi', \Sigma_1) \leq \frac{1}{2}\theta_0$, may be represented as

$$\pi' = \pi(\xi', \theta'), \quad \xi' \in \Sigma_{1-\kappa}^{1+\kappa}, \quad |\theta'| < \theta_0, \quad (3.6)$$

and

$$\pi(\xi_1, \theta_1) = \pi(\xi_2, \theta_2), \quad \xi_1, \xi_2 \in \Sigma_{1-\kappa}^{1+\kappa} \Rightarrow |\theta_1|, |\theta_2| \geq 2\theta_0^*. \quad (3.7)$$

We may assume that $\theta_0^* < \frac{1}{4}\kappa$. The mapping $D_1$ sends $\Sigma_{1-\kappa}^{1+\kappa}$ to $\Sigma_{1-t_1\kappa}^{1+t_1\kappa}$ and sends $\pi(\Sigma_{1-\kappa}^{1+\kappa} \times (-\theta_0, \theta_0))$ to $\pi(\Sigma_{1-t_1\kappa}^{1+t_1\kappa} \times (-\theta_0, \theta_0))$. This implies that the set $\Sigma^{nbh}$, defined as a collection of all points $\pi(\xi, \theta)$ as in $(3.3)$, makes a neighbourhood of $\Sigma_*$. To prove that the parametrisation is unique assume that it is not. Then there exist $t_1 > t_2 > 0$, $\theta_1, \theta_2 \in (-\theta_0, \theta_0)$ and $\xi_1 \in \Sigma_1, \xi_2 \in \Sigma_2$ such that $\pi(\xi_1, \theta_1) = \pi(\xi_2, \theta_2)$. So $\pi_1 = \pi_2$, where

$$\pi_1 = \pi(t_1^{-1}\xi_1, \theta_1), \quad \pi_2 = \pi(t_1^{-1}\xi_2, \theta_2),$$

and $1 = |t_1^{-1}\xi_1| > |t_1^{-1}\xi_2|$. Let us write $|t_1^{-1}\xi_2|$ as $1 - \kappa', \kappa' > 0$. If $\kappa' < \kappa$, then $(\xi_1, \theta_1) = (\xi_2, \theta_2)$ by what was said above. If $\kappa' \geq \kappa$, then $|t_1^{-1}\xi_1 - t_1^{-1}\xi_2| \geq |t_1^{-1}\xi_1| - |t_1^{-1}\xi_2| \geq \kappa$. Since

$$|\pi_1 - t_1^{-1}\xi_1| = \theta_1, \quad |\pi_2 - t_2^{-1}\xi_2| = |\theta_2 N_{t_1^{-1}\xi_1}| \leq \theta_2,$$
then \(|\pi_1 - \pi_2| \geq \kappa - \theta_1 - \theta_2 \geq \kappa - 2\theta_0\). Decreasing \(\theta_0^*\) if needed, we achieve that \(\kappa > 2\theta_0\), so \(|\pi_1 - \pi_2| > 0\). Contradiction.

2) The first assertion holds since by (3.1) the vector \(N_\xi\) is orthogonal to \((x_\xi, y_\xi)\) and since its norm equals \(|\xi|\). The second assertion holds since the segment \(S\) is a geodesic from \(\pi\) to \(\Sigma_\ast\). Any other geodesic from \(\pi\) to \(\Sigma_\ast\) must be a segment \(S' = [\pi, \xi']\), \(\xi' \in \Sigma_\ast\), orthogonal to \(\Sigma_\ast\). It is longer than \(|\theta||\xi|\). To prove this, by scaling (i.e. by applying a delation operator), we reduce the problem to the case \(|\xi| = 1\). Now, if \(\xi' \in \Sigma_\ast^{1 - \kappa}\), then \(\pi(\xi', \theta') = \pi = \pi(\xi, \theta)\) for some real number \(\theta'\). So by (3.7), \(|\theta| \geq 2|\theta_0^*|\), which is a contradiction. While if \(\xi' \not\in \Sigma_\ast^{1 - \kappa}\), then the distance from \(\pi\) to \(\xi'\) is bigger than \(\kappa - \theta_0 > \theta_0\). Indeed, if \(|\xi'| \geq 1 + \kappa\), then the distance is bigger than \(1 + \kappa - |\pi| \geq 1 + \kappa - 1 - \theta_0 = \kappa - \theta_0\). The case \(|\xi'| \leq 1 - \kappa\) is similar.

3) If \(R = 1\), then the assertion follows from (3.6) and (3.4). If \(R \neq 1\), we apply the operator \(D_{R - 1}\) and use the result with \(R = 1\).

4) Follows immediately from (3.1).

5) If \(R = 1\), then the assertion with some \(c > 0\) follows from the compactness of \(S^1 \cap (\Sigma^{nh})^c\). If \(R \neq 1\), then again we apply \(D_{R - 1}\) and use the result with \(R = 1\).

Let us fix any \(0 < \theta_0 \leq \theta_0^*\), and consider the manifold \(\Sigma^{nh} = \Sigma^{nh}(\theta_0)\). Below we provide it with some additional structures and during the corresponding constructions decrease \(\theta_0^*\), if needed. Consider the set \(\Sigma^1\). It equals

\[
\Sigma^1 = \{(x, y) : x \cdot y = 0, x^2 + y^2 = 1\}.
\]

Since the differentials of the two relations, defining \(\Sigma^1\), are independent on \(\Sigma^1\), then this set is a smooth compact submanifold of \(\mathbb{R}^{2d}\) of codimension 2. Let us cover it by some finite system of charts \(N_1, \ldots, N_N, N_j = \{\eta^1 = (\eta^1_j, \ldots, \eta^{2d - 2}_j)\}\). Denote by \(m(d\eta)\) the volume element on \(\Sigma^1\), induced from \(\mathbb{R}^{2d}\), and denote the coordinate maps as \(N_j \ni \eta \rightarrow (x_\eta, y_\eta) \in \Sigma^1\). We will write points of \(\Sigma^1\) both as \(x\) and \((x_\eta, y_\eta)\).

The mapping

\[
\Sigma^1 \times \mathbb{R}^+ \rightarrow \Sigma_\ast, \quad ((x_\eta, y_\eta), t) \rightarrow D_t(x_\eta, y_\eta),
\]

is 1-1 and is a local diffeomorphism; so this is a global diffeomorphism. Accordingly, we can cover \(\Sigma_\ast\) by the \(\tilde{n}\) charts \(N_j \times \mathbb{R}^+\), with the coordinate maps

\[
(\eta^j, t) \mapsto D_t(x_{\eta^j}, y_{\eta^j}), \quad \eta^j \in N_j, \quad t > 0,
\]

and can apply Lemma 3.1, taking \((\eta, t)\) for the coordinates \(x\). In these coordinates the volume element on \(\Sigma^1\) is \(t^{2d - 2}m(d\eta)\). Since \(\partial / \partial t \in T_{\eta, t}\Sigma_\ast\) is a vector of unit length, perpendicular to \(\Sigma^1\), then the volume element on \(\Sigma_\ast\) is

\[
d_{\Sigma^1} = t^{2d - 2}m(d\eta)\, dt. \tag{3.8}
\]

\[\text{vol_on_}\]

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4as \(\partial / \partial t \perp S^1\) and \(S^1 \supset \Sigma^1\).
The coordinates \((\eta, t, \theta)\) with \(\eta \in \mathcal{N}_j, t > 0, |\theta| < \theta_0\), where \(1 \leq j \leq \bar{n}\), make coordinate systems on the open set \(\Sigma_{nbh}^n\). Since the vectors \(\partial/\partial t\) and \(t^{-1}\partial/\partial \theta\) form an orthonormal base of the orthogonal complement in \(\mathbb{R}^d\) to \(T_{(\eta, t, 0)} \Sigma^t\), then in \(\Sigma_{nbh}^n\) the volume element \(dx \, dy\) may be written as

\[
dx \, dy = t^{2d-1} \mu(\eta, t, \theta) m(d\eta) \, dt \, d\theta, \quad \text{where} \quad \mu(\eta, t, 0) = 1.
\]

For \(r > 0\) the transformation \(D_r\) multiplies the form in the l.h.s. by \(r^{2d}\), preserves \(d\eta\) and \(d\theta\), and multiplies \(dt\) by \(r\). Hence, \(\mu\) does not depend on \(t\), and we have got

\[\text{Lemma 3.2.} \quad \text{The coordinates} \quad (\eta^j, t, \theta), \quad \text{where} \quad \eta^j \in \mathcal{N}_j, \quad t > 0, \quad |\theta| < \theta_0, \quad (3.10)\]

and \(1 \leq j \leq \bar{n}\), define on \(\Sigma_{nbh}^n\) coordinate systems, jointly covering \(\Sigma_{nbh}^n\). In these coordinates the dilations \(D_r\), \(r > 0\), read as

\[D_r : (\eta, t, \theta) \mapsto (\eta, rt, \theta),\]

and the volume element has the form \((3.9)\), where \(\mu\) does not depend on \(t\).

Besides, since at a point \(z = (x, y) = \pi(\xi, \theta) \in \Sigma_{nbh}^n\) we have \((\partial/\partial \theta) = \nabla_z \cdot (y, x)\), then in view of \((1.2), (1.3)\)

\[
\left| \frac{\partial^k}{\partial \theta^k} F(\eta, t, \theta) \right| \leq K'(1 + t)^{-M}, \quad \left| \frac{\partial^k}{\partial \theta^k} \Gamma(\eta, t, \theta) \right| \leq K'(1 + t)^{r_*} \quad (3.11)\]

for \(0 \leq k \leq 2\) and for all \((\eta, t, \theta)\).

For \(0 \leq R_1 < R_2\) we denote

\[
(\Sigma_{nbh}^n)_{R_2}^{R_1} = \pi \left( \Sigma_{R_2}^{R_1} \times (-\theta_0, \theta_0) \right).
\]

In a chart \((3.10)\) this domain is \(\{(\eta^j, t, \theta) : \eta^j \in \mathcal{N}_j, \quad R_1 < t < R_2, \quad |\theta| < \theta_0\}\).

4 Global study of the integral \((1.1)\)

To study \(I_{\nu}\) and the integrals \((I_{\nu}, (\Sigma_{nbh}^n)_{R_1}^{R_2})\), \(0 \leq R_1 < R_2\), we pass to the coordinates \((\eta, t, \theta)\) and write the former using \((3.9)\). First we will examine in details the integrals over \(\mu(\eta, \theta) \, d\theta\), and next integrate them over \(t^{2d-1}m(d\eta) \, dt\).

\[\text{Since the vector} \partial/\partial t \text{is perpendicular to} \Sigma^t \text{and lies in} \ T_{(\eta, t, 0)} \Sigma^t, \quad \partial/\partial \theta \text{is proportional to the vector} \ N_{(\eta, t, 0)}, \text{normal to} \Sigma^t \text{at} \ (\eta, t, 0).\]
4.1 Desintegration of $I_\nu$ and integrating over $\mu \, d\theta$.

Using (3.9) we write the integral $\langle I_\nu, (\Sigma^{nbh})_{R_1}^R \rangle$ as

$$\int_{\Sigma^1} m(d\eta) \int_{R_1}^{R_2} dt \, t^{2d-1} \int_{-\theta_0}^{\theta_0} d\theta \, \frac{F(\eta, t, \theta) \mu(\eta, \theta)}{(x \cdot y)^2 + (\nu \Gamma(\eta, t, \theta))^2} = \int_{\Sigma^1} m(d\eta) \int_{R_1}^{R_2} dt \, t^{2d-1} J_\nu(\eta, t),$$

where by (3.5)

$$J_\nu(\eta, t) = \int_{-\theta_0}^{\theta_0} d\theta \, F(\eta, t, \theta) \mu(\eta, \theta) \frac{h(\theta)}{\theta^2 h^{-2}(\theta) + (\nu \Gamma(\eta, t, \theta))^2}.$$ 

To study $J_\nu(\eta, t) =: J_\nu$ we write $\Gamma = \Gamma(\eta, t, \theta)$ as

$$\Gamma(\eta, t, \theta) = h_{\eta,t}(\theta) \Gamma(\eta, t, 0).$$

The function $h(\theta) := h_{\eta,t}(\theta)$ is $C^2$-smooth, and in view of (4.11) and (1.3) it satisfies

$$|h(\theta)| \geq C^{-1}_0, \quad \left| \frac{\partial^k}{\partial \theta^k} h(\theta) \right| \leq C_k \quad \forall \eta, t, \theta, \ 0 \leq k \leq 2.$$ 

Denoting

$$\epsilon = \nu t^{-2} \Gamma(\eta, t, 0),$$

we write $J_\nu$ as

$$J_\nu = t^{-4} \int_{-\theta_0}^{\theta_0} F(\eta, t, \theta) \mu(\eta, \theta) h^{-2}(\theta) \, d\theta \frac{1}{\theta^2 h^{-2}(\theta) + \epsilon^2}.$$ 

Since $h(0) = 1$, then in view of (4.2) the mapping

$$f = f_{\eta,t} : [-\theta_0, \theta_0] \ni \theta \mapsto \theta = \theta/h(\theta)$$

is a $C^2$-diffeomorphism on its image such that $f(0) = 0, f'(0) = 1$ and the $C^2$-norms of $f$ and $f^{-1}$ are bounded by a constant, independent from $\eta, t$ (to achieve that, if needed, we decrease $\theta_0^*$. Denote

$$\theta_0^* = f(\theta_0), \quad \theta_0^- = -f(-\theta_0), \quad \theta_0 = \min(\theta_0^*, \theta_0^-).$$

Then $2^{-1} \theta_0 \leq \theta_0^* \leq 2\theta_0$ if $\theta_0^*$ is small, and

$$J_\nu = t^{-4} \int_{\theta_0^-}^{\theta_0^*} F(\eta, t, \theta) \mu(\eta, \theta) h^{-2}(\theta) (f^{-1}(\theta))' \, d\theta \frac{1}{\theta^2 + \epsilon^2}.$$ 

Denote the nominator of the integrand as $\Phi(\eta, t, \tilde{\theta})$. This is a $C^2$-smooth function, and by (3.11) and (4.2) it satisfies

$$\left| \frac{\partial^k}{\partial \theta^k} \Phi \right| \leq C(1 + t)^{-M} \quad \text{for} \quad 0 \leq k \leq 2.$$ 

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Moreover, since $h(0) = 1$ and $(f^{-1}(0))' = f'(0) = 1$, then in view of \(4.9\) we have that
\[
\Phi(\eta, t, 0) = F(\eta, t, 0). \tag{4.3}
\]
Consider the interval $\Delta_{\eta,t} = f^{-1}_{\eta,t}(-\bar{\theta}_0, \bar{\theta}_0)$. Then
\[
(-\theta_0/2, \theta_0/2) \subset \Delta_{\eta,t} \subset (-\theta_0, \theta_0)
\]
for all $\eta$ and $t$. Now we modify the neighbourhood $\Sigma^{nbh}(\theta_0)$ to
\[
\Sigma^{nbhm} = \Sigma^{nbh}(\bar{\theta}_0) = \{ \pi(\eta, t, \theta) : \eta \in \Sigma^1, \ t > 0, \ \theta \in \Delta_{\eta,t} \}.
\]
Then
\[
\Sigma^{nbhm}(\bar{\theta}_0) \subset \Sigma^{nbh}(\theta_0) \subset \Sigma^{nbhm}(\theta_0). \tag{4.4}
\]

The modified analogy $J^{m}_{\nu}$ of the integral $J_{\nu}$ has the same form as $J_{\nu}$, but the domain of integrating becomes not $(-\theta_0, \theta_0)$, but $\Delta_{\eta,t}$. Then
\[
J^{m}_{\nu} = t^{-4} \int_{-\bar{\theta}_0}^{\bar{\theta}_0} \Phi(\eta, t, \theta) d\bar{\theta}.
\]

To estimate $J^{m}_{\nu}$, consider first the integral $J^{0m}_{\nu}$, obtained from $J^{m}_{\nu}$ by freezing $\Phi$ at $\bar{\theta} = 0$:
\[
J^{0m}_{\nu} = t^{-4} \int_{-\bar{\theta}_0}^{\bar{\theta}_0} \Phi(\eta, t, 0) d\bar{\theta} = 2t^{-4}F(\eta, t, 0)\varepsilon^{-1}\tan^{-1}\frac{\bar{\theta}_0}{\varepsilon}.
\]

(we use \(4.3\)). From here
\[
|J^{0m}_{\nu}| \leq \pi\varepsilon^{-1}t^{-4}|F(\eta, t, 0)|. \tag{4.5}
\]
As $0 < \frac{\varepsilon}{2} - \tan^{-1}\frac{1}{\varepsilon} < \bar{\varepsilon}$ for $0 < \varepsilon \leq \frac{1}{2}$, then also
\[
0 < \pi\nu^{-1}t^{-2}(F/\Gamma) \big|_{\theta=0} - J^{0m}_{\nu} < \frac{2}{\bar{\theta}_0}t^{-4}F(\eta, t, 0), \tag{4.6}
\]
if
\[
\nu t^{-2}\Gamma(\eta, t, 0) \leq \frac{1}{2}\bar{\theta}_0. \tag{4.7}
\]

Now we estimate the difference between $J^{m}_{\nu}$ and $J^{0m}_{\nu}$. We have:
\[
J^{m}_{\nu} - J^{0m}_{\nu} = t^{-4} \int_{-\bar{\theta}_0}^{\bar{\theta}_0} \Phi(\eta, t, \bar{\theta}) - \Phi(\eta, t, 0) \frac{\bar{\theta}^2 d\bar{\theta}}{\bar{\theta}^2 + \varepsilon^2}.
\]
Since each $C^k$-norm of $\Phi$, $k \leq 2$, is bounded by $C(1 + t)^{-M}$, then
\[
\Phi(\eta, t, \bar{\theta}) - \Phi(\eta, t, 0) = A(\eta, t)\bar{\theta} + B(\eta, t, \bar{\theta})\bar{\theta}^2,
\]
where $|A|, |B| \leq C(1 + t)^{-M}$. From here
\[
|J^{m}_{\nu} - J^{0m}_{\nu}| \leq C(1 + t)^{-M}t^{-4} \int_{0}^{\bar{\theta}_0} \frac{\bar{\theta}^2 d\bar{\theta}}{\bar{\theta}^2 + \varepsilon^2} \leq C_1(1 + t)^{-M}t^{-4}\bar{\theta}_0.
\]
Denote
\[
J_{\nu}(\eta, t) = \pi t^{-2}(FT^{-1})(\eta, t, 0). \tag{4.8}
\]
Then, jointly with \(4.6\), the last estimate tell us that
\[
|J^{m}_{\nu} - \nu^{-1}J_{\nu}(\eta, t)| \leq C(1 + t)^{-M}t^{-4}\bar{\theta}_0^{-1} \quad \text{if} \quad \nu \leq \frac{1}{2} \tag{4.9}
\]
4.2 Proof of Theorem 1.1

To estimate $I_\nu$ we write it as $I_\nu = \langle I_\nu, (\Sigma^{\text{nbh}})_{0}^{C^{-1}\sqrt{\nu}} \rangle + \langle I_\nu, (\Sigma^{\text{nbh}})_{C^{-1}\sqrt{\nu}}^{\infty} \rangle$. We will show that the first term is small by Lemma 2.1 and that the second may be approximately calculated using the representation (4.1) and the estimate (4.9). Doing that we have to distinguish the cases $r_* \leq 2$ and $r_* > 2$.

Let $r_* \leq 2$. Then by (1.3) assumption (4.7) holds if $t \geq C^{-1}\sqrt{\nu}$ (where $C$ depends on $\theta_0$). Integrating $J_{\nu}^{m}(\eta, t)$ and $\nu^{-1} J_{\nu}(\eta, t)$ with respect to the measure $t^{2d-1}m(d\eta)\,dt$ and using (4.9), (4.1) and (1.4) we get that

$$\langle I_\nu, (\Sigma^{\text{nbh}})_{C^{-1}\sqrt{\nu}}^{\infty} \rangle \leq C \int_{\Sigma_{1}} m(d\eta) \int_{C^{-1}\sqrt{\nu}}^{\infty} dt \, t^{2d-1} \, J_{\nu}(\eta, t)$$

for the quantity $\chi_{d}(\nu)$ see (1.6). In view of (4.8) and Lemma 2.1 with $\delta = 2C^{-1}\sqrt{\nu}$,

$$\langle I_\nu, (\Sigma^{\text{nbh}})_{0}^{C^{-1}\sqrt{\nu}} \rangle - \nu^{-1} \int_{\Sigma_{1}} m(d\eta) \int_{0}^{C^{-1}\sqrt{\nu}} dt \, t^{2d-1} \, J_{\nu}(\eta, t) \leq C \nu^{-1} d^{-1} + C \nu^{-1} \int_{0}^{C^{-1}\sqrt{\nu}} t^{2d-1} \, dt \leq C_1$$

as $d \geq 2$. Next, by (4.8) and (1.4)

$$\int_{\Sigma_{1}} m(d\eta) \int_{0}^{\infty} dt \, t^{2d-1} |J_{\nu}(\eta, t)| \leq C \int_{0}^{\infty} t^{2d-1-2} (1 + t)^{-M-r_*} \leq C_1,$$

and by (3.8)

$$\int_{\Sigma_{1}} m(d\eta) \int_{0}^{\infty} dt \, t^{2d-1} J_{\nu} = \pi \int_{\Sigma_{1}} m(d\eta) \int_{0}^{\infty} dt \, t^{2d-3}(F/\Gamma) |_{\theta=0}$$

$$= \pi \int_{\Sigma_{1}} |z|^{-1} (F/\Gamma)(z) d_{\Sigma_{1}} z.$$

This gives us asymptotic description as $\nu \to 0$ of the integral (1.1), calculated over the vicinity $\Sigma^{\text{nbh}}$ of $\Sigma_*$. It remains to estimate the integral over the complement to $\Sigma^{\text{nbh}}$. But this is easy: by (4.4),

$$\langle I_\nu, \mathbb{R}^{2d} \setminus \Sigma^{\text{nbh}} \rangle \leq \langle I_\nu, \{ |(x, y)| \leq 2\nu \} \rangle$$

$$+ C_d \int_{0}^{\infty} dr \, r^{2d-1} \int_{S^{1} \setminus \Sigma^{\text{nbh}}(\theta_0/2)} F(x, y) \, dS$$

$$= \pi \int_{\nu}^{\infty} \theta^{2d-3} (F/\Gamma) |_{\theta=0} \, dS \, s$$

$$+ C_d \int_{0}^{\infty} dr \, r^{2d-1} \int_{S^{1} \setminus \Sigma^{\text{nbh}}(\theta_0/2)} (x \cdot y)^{2} + (\nu \Gamma(x, y)^2)^{2}$$

By item 5) of Lemma 3.1 the divisor of the integrand is $\geq C^{-2} r^4$. Due to this and (1.2), the second term in the r.h.s. is bounded by

$$C \int_{0}^{\infty} (1 + r)^{-M} r^{2d-5} \, dr \leq C_1 \chi_{d}(\nu).$$
This estimate and Lemma 2.1 with \( \delta = 2 \nu \) imply that
\[
|\langle I_{\nu}, R^2 \setminus \Sigma^{n \text{th}} \rangle| \leq C\chi_d(\nu). \tag{4.14} \]

Now relations (4.10), (4.11), (4.13), (4.14) imply (1.5), while (3.8) and (4.12) imply that the integral in (1.5) converges absolutely.

Let \( r_\ast > 2 \). Then condition (4.6) holds if
\[
C_\beta \nu^{-\beta} \geq t \geq C^{-1} \sqrt{\nu}, \quad \beta = \frac{1}{r_\ast - 2}.
\]

Accordingly, the term in the l.h.s. of (4.10) should be split in two. The first corresponds to the integrating from \( C^{-1} \sqrt{\nu} \) to \( C_\beta \nu^{-\beta} \) and estimates exactly as before. The second is
\[
|\langle I_{\nu}, (\Sigma^{n \text{th}})_{C_\beta \nu^{-\beta}} \rangle - \nu^{-1} \int_{\Sigma} m(d\eta) \int_{C_\beta \nu^{-\beta}} dt t^{2d-3} f(\eta, t)|. \tag{4.15} \]

To bound it we estimate the norm of the difference of the two integrals via the sum of their norms. In view of (4.5) and (4.8) both of them are bounded by
\[
C \nu^{-1} \int_{\Sigma} m(d\eta) \int_{C_\beta \nu^{-\beta}} dt t^{2d-3} (F/\Gamma)(\eta, t, 0).
\]

So
\[
(4.15) \leq C_\beta \nu^{-1+\beta(M+r_\ast+2-2d)} \leq C_\beta
\]
since \( M > 2d - 4 \).

Adding this relation to (4.10), applied to the integrating from \( C^{-1} \sqrt{\nu} \) to \( C_\beta \nu^{-\beta} \), and – as before – using this jointly with (4.11), (4.13), (4.14), we again get (1.5) (while the absolute convergence of the integral still follows from (4.12)).

4.3 Proof of Proposition 1.2

For any \( R > r > 0 \) let us denote by \( B^R \) and \( B^R_r \) the ball \( \{|z| \leq R\} \) and the spherical layer \( \{r \leq |z| \leq R\} \), and consider the measure \( \mu^R = \chi_{B^R}(z)|z|^{-1}\delta_{\Sigma_*} \).

This is a well defined Borel measure on the manifold \( \Sigma_* \) and on the space \( \mathbb{R}^{2d} \), supported by \( B^R \). By the Riesz theorem the family of measures \( \{\mu^R, 0 < r \leq R\} \) is weakly compact in the space of measures on \( B^R \), and it is easy to see that for any bounded continuos function \( f(z) \) the curve \( \varepsilon \to \int f(z) d\mu^R(dz) \) satisfies the Cauchy condition as \( \varepsilon \to 0 \). So the measures \( \mu^R \) weakly converge to a limit as \( \varepsilon \to 0 \). This is the restriction of the measure \( |z|^{-1}\delta_{\Sigma_*} \) to \( B^R \), and its further restriction to \( B^R_r \) equals \( \mu^R_r \). So \( |z|^{-1}\delta_{\Sigma_*} \) is a \( \sigma \)-additive Borel measure on \( \mathbb{R}^{2d} \), and it has no atoms outside the origin. Let us abbreviate \( |z|^{-1}\delta_{\Sigma_*} = \mu \). By (3.8), for any \( 0 < r \leq R \) and \( \varepsilon \leq r \),
\[
\int_{B^R_r} d\mu = \int_{B^R_r} d\mu^R_r = \int_r^R r^{2d-2} t^{-1} dt = \frac{1}{2d-2}(R^{2d-2} - r^{2d-2}).
\]
From here and the weak convergence of the measures $\mu^R$ to $\mu_{|B^R}$ we get that $\mu\{ |z| < \rho \} \leq \rho^{2d-2}/(2d-2)$ if $\rho < R$, so $\mu$ has no atom in the origin and is atomless. Next, for any function $f \in C_m(\mathbb{R}^{2d})$ we have

$$
\int f(z) \mu(dz) \leq |f|_m \sum_{R=0}^{\infty} \int_{B^R_{R+1}} (z)^{-m} \mu(dz)
$$

$$
\leq C_1 |f|_m \sum_{R=0}^{\infty} \langle R \rangle^{-m} \frac{(R+1)^{2d-2} - R^{2d-2}}{2d-2}
$$

$$
\leq C_2 |f|_m \sum_{R=0}^{\infty} \langle R \rangle^{-m} = C_3 |f|_m,
$$

if $m > 2d-2$. This proves the proposition.

5 Other integrals

The geometrical approach to treat integrals (1.1), developed above, applies to various modifications of these integrals. Below we briefly discuss three more examples.

5.1 Integrals (1.1) with $d = 1$

The restriction $d \geq 2$ was imposed in the previous sections since in the one-dimensional case some integrals, involved in the construction, strongly diverge at the locus of the quadric $\Sigma$. This problem disappears if the function $F$ vanishes near the origin. The quadric $\Sigma' = \{ xy = 0 \} \subset \mathbb{R}^2$ is one dimensional, has a singularity at the origin, and its smooth part $\Sigma'^* = \Sigma' \setminus 0$ has four connected components. Consider one of them: $C_1 = \{ (x,y) : y = 0, x > 0 \}$. Now the coordinate $\xi$ is a point in $\mathbb{R}^+$ with $(x_\xi, y_\xi) = (\xi, 0)$ and with the normal $N(\xi) = (0, \xi)$, the set $\Sigma_1 \cap C_1$ is the single point $(1,0)$ and the coordinate $(\eta, t, \theta)$ in the vicinity of $C_1$ degenerates to $(t, \theta)$, $t > 0$, $|\theta| < \theta_0$, with the coordinate-map $(t, \theta) \mapsto (t, t\theta)$. The relations (3.8) and (3.9) are now obvious, and the integral (2.1) vanishes if $\delta > 0$ is sufficiently small. Interpreting $z = (x,y)$ as a complex number, we write the assertion of Theorem 1.1 as

$$
|J' - \pi \nu^{-1} \int_{\Sigma'} \frac{F(z)}{|z|^2} \Gamma(z) \, dz| \leq C,
$$

where the integral is a contour integral in the complex plane.
5.2 Integrals of quotients with divisors, linear in $\omega$.

Consider
\[ I'_{v} = \int_{\mathbb{R}^{2d}} \frac{F(x, y) \, dx \, dy}{x \cdot y + i \Gamma(x, y)}. \]  
(5.1)

Now there is no need to separate the integral over the vicinity of the origin, and we just split $I'_{v}$ to an integral over $\Sigma_{nbh}^{m}$ and over its complement.

To calculate $\langle I'_{v}, \Sigma_{nbh}^{m} \rangle$ we observe that an analogy of $J_{\nu}^{m}$ is the integral
\[ \int_{-\theta_{0}}^{\theta_{0}} \int_{\mathbb{R}^{2d}} F(\eta, t, 0) \, d\bar{\theta} = t^{-2} F(\eta, t, 0) \ln \frac{\theta_{0}^{+} + i \nu t^{-2} \Gamma(\eta, t, 0)}{-\theta_{0} + i \nu t^{-2} \Gamma(\eta, t, 0)}, \]
which equals $\pi t^{-2} F(\eta, t, 0) + O(\nu)$. So $\langle I'_{v}, \Sigma_{nbh}^{m} \rangle = \pi \int_{\Sigma_{*}} F(z) \, dz + O(\nu)$.

The integral over the complement to $\Sigma_{nbh}^{m}$ is
\[ \int_{\mathbb{R}^{2d} \setminus \Sigma_{nbh}^{m}} \frac{F(x, y) \, dx \, dy}{x \cdot y + i \Gamma(x, y)} = \int_{\mathbb{R}^{2d} \setminus \Sigma_{nbh}^{m}} \frac{F(x, y) \, dx \, dy}{x \cdot y} + o(1) \]
as $\nu \to 0$ (the integral in the r.h.s. is regular). In difference with (1.1) the last integral is of the same order as the integral over $\Sigma_{nbh}^{m}$. So we have that
\[ I'_{v} = \pi \int_{\Sigma_{*}} \frac{F(z)}{|z|} \, dz + \int_{\mathbb{R}^{2d} \setminus \Sigma_{nbh}^{m}} \frac{F(x, y) \, dx \, dy}{x \cdot y} + o(1), \]
in agreement with the estimate (6.4), applied to (5.1).

5.3 Integrals, coming from the three-waves interaction

The three-waves interacting systems lead to integrals, similar to (1.7), where $\mathbb{R}^{3d}$ is replaced by $\mathbb{R}^{2d}$ and the $\delta$-factor is replaced by $\delta_{k_{1}k_{2}} \delta(\omega_{k_{1}k_{2}})$, which gives rise to the algebraic set
\[ \{(k_{1}, k_{2}) : k_{1} + k_{2} = k, \ |k_{1}|^{2} + |k_{2}|^{2} = |k|^{2}\} \]
(see [5], Section 6). I.e., $k_{2} = k - k_{1}$, where $k_{1} \in \{r \in \mathbb{R}^{d} : |r - \frac{1}{2} k|^{2} = \frac{1}{4} |k|^{2}\}$. Accordingly, in the variable $z = r - \frac{1}{2} k$ some constructions from the study of the three-waves interaction lead to the integrals
\[ I'_{v} = \int_{\mathbb{R}^{d}} \frac{F(z)}{\omega(z)^{2} + (\nu \Gamma(z))^{2}} \, d\Sigma \geq 2, \]
with $\omega(z) = |z|^{2} - \frac{1}{4} |k|^{2}$. Now the quadric $\Sigma = \{\omega = 0\}$ is a sphere, i.e. a smooth compact manifold. Denoting by $\eta$ a local coordinate on $\Sigma$ with a coordinate mapping $\eta \mapsto z(\eta) \in \Sigma$ and the volume form $m(d\eta)$ we see that, similar to Section [6], the local coordinate in the vicinity $\Sigma_{nbh}^{m}$ of $\Sigma$ is $(\eta, \theta)$, $|\theta| < \theta_{0}$, with the coordinate mapping $(\eta, \theta) \mapsto z(\eta)(1 + \theta)$ and the volume form $(\frac{1}{2} |k|)^{-1} \mu(m(d\eta)d\theta, \mu(\eta, 0) \equiv 0$. The proof in Sections [2][4] simplifies and leads to the asymptotic
\[ |I'_{v} - \pi \nu^{-1} \int_{\Sigma} (F(z)/\Gamma(z)) \, dz| \leq \text{Const}, \]
valid for $C^{2}$-functions $F$ and $\Gamma$, satisfying some mild restriction.
6 Appendix

Let \( \varphi(x) \) and \( g(x) \) be smooth functions on \( \mathbb{R}^n \) and \( \varphi \) has a compact support. Consider the integral

\[
I(\lambda) = \int_{\mathbb{R}^n} \varphi(x) e^{i\lambda g(x)} \, dx, \quad \lambda \geq 1.
\]

Assume that \( g(x) \) has a unique critical point \( x_0 \), which is non-degenerate. Then, by the stationary phase method,

\[
I(\lambda) = \left( \frac{2\pi}{\lambda} \right)^{n/2} \det g_{xx}(x_0)^{-1/2} \varphi(x_0) e^{i\lambda g(x_0)} + (i\pi/4) \text{sgn}   g_{xx}(x_0) + R\lambda^{-n/2-1}
\]  

for \( \lambda \geq 1 \), where \( R \) depends on \( \|\varphi\|_{C^2}, \|g\|_{C^3} \), the measure of the support of \( \varphi \) and on \( \sup_{x \in \text{supp} \varphi} |x - x_0|/|\nabla g(x)| =: C^\#(g) \). See Section 7.7 of [2] and Section 5 of [1].

If the functions \( \varphi \) and \( g \) are not \( C^\infty \)-smooth, but \( \varphi \in C^2_0(\mathbb{R}^n) \) and \( g \in C^3(\mathbb{R}^n) \), then, approximating \( \varphi \) and \( g \) by smooth functions and applying the result above we get from (6.1) that

\[
|I(\lambda)| \leq C' \lambda^{-n/2} \quad \forall \lambda \geq 1,
\]

with \( C' \) depending on \( \|\varphi\|_{C^2}, \|g\|_{C^3} \) and \( C^\#(g) \).

Now let \( f(x) \in C^2_0(\mathbb{R}^d) \) and \( g(x) \in C^3(\mathbb{R}^d) \) be such that

\[
|f| \leq C, \quad \text{meas(\text{supp} f)} \leq C.
\]

Let an \( x_0 \) be the unique critical point of \( g(x) \) and

\[
C^{-1} \leq |\det \text{Hess} g(x_0)| \leq C.
\]

Consider the integral

\[
\mathcal{I}(\nu) = \int_{\mathbb{R}^d} \frac{f(x)}{\Gamma + i\nu^{-1}g(x)} \, dx = \nu \int_{\mathbb{R}^d} \frac{f(x)}{\nu \Gamma + ig(x)} \, dx, \quad 0 < \nu \leq 1,
\]

where \( \Gamma \) is a positive constant. Let us write it as

\[
\mathcal{I}(\nu) = \int_{\mathbb{R}^d} \int_{-\infty}^{0} f(x) e^{i(\Gamma + i\nu^{-1}g(x))} \, dt \, dx =: \int_{\mathbb{R}^d} \int_{-\infty}^{0} F_\nu(t, x) \, dt \, dx = \mathcal{I}_1 + \mathcal{I}_2,
\]

where

\[
\mathcal{I}_1 = \int_{\mathbb{R}^d} \int_{-\nu}^{0} F_\nu(t, x) \, dt \, dx, \quad \mathcal{I}_2 = \int_{\mathbb{R}^d} \int_{-\infty}^{-\nu} F_\nu(t, x) \, dt \, dx.
\]

Clearly, \( |\mathcal{I}_1| \leq C^2 \nu \). To estimate \( \mathcal{I}_2 \) consider the internal integral

\[
J(t) = e^{\Gamma} \int_{\mathbb{R}^d} f(x) e^{-i\nu^{-1}|t|g(x)} \, dx,
\]
and apply to it the stationary phase method with \( \lambda = \nu^{-1}|t| \geq 1 \). By (6.2) and (6.3), \(|J(t)|\) is bounded by \( e^{t\Gamma} K_1(f, g)(\nu^{-1}|t|)^{-d/2} \). So

\[
|I_2| = \int_{-\infty}^{\nu} J(t) \, dt \leq K_1(f, g) \nu^{d/2} \int_{-\infty}^{\nu} |t|^{-d/2} e^{t\Gamma} \, dt \leq K_2(f, g) \nu^{d/2} \nu^{-d/2+1} \chi_d(\nu)
\]

since \( \Gamma > 0 \), where \( \chi_d(\nu) \) is defined in (1.6).

Thus

\[
|I(\nu)| \leq |I_1| + |I_2| \leq K(f, g) \nu \chi_d(\nu) .
\]

The constant \( K(f, g) \) depends on \( C, \|f\|_{C^2}, \|g\|_{C^3} \) and \( C^\#(g) \).

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