Regression-adjusted average treatment effect estimates in stratified and sequentially randomized experiments

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Abstract
Stratified and sequentially randomized experiments are widely used in fields such as agriculture, industry, medical science, economics, and finance. Baseline explanatory covariates are often collected for each unit in the experiments. Many researchers use linear regression to analyse the experimental results for improving efficiency of treatment effect estimation by adjusting the minor imbalances of covariates in the treatment and control group. Our work proposes regression adjustment methods in stratified and sequentially randomized experiments and studies their asymptotic properties under the randomization-based inference framework. We allow both the number of strata and their sizes to be arbitrary, provided the total number of experimental units tends to infinity and each stratum has at least two treated and two control units. Under slightly stronger but interpretable conditions, we re-establish the finite population central limit theory for a stratified random sample. Based on this theorem, we prove in our main results that, with certain other mild conditions, both the stratified difference-in-means and the regression-adjusted average treatment effect estimator are consistent and asymptotically normal. The asymptotic variance of the latter is no greater and is typically lesser than that of the former when the proportion of treated units is asymptotically the same across strata or the number of stratum is bounded. The magnitude of improvement depends on the extent to which the within-strata variation of the potential outcomes can be explained by the covariates. We also provide conservative variance estimators to construct large-sample confidence intervals for the average treatment effect, which are consistent if and only if the stratum-specific treatment effect is constant. Our theoretical results are confirmed by a simulation study and we conclude with an empirical illustration.

Keywords
Blocking; Completely randomized experiments; Randomized-block design; Randomization-based inference; Regression adjustment; Stratified sampling.

1. Introduction
Randomization experiments are the ‘golden standard’ for drawing causal inference. Randomization ensures that the treatment assignment is not affected by the outcomes and by any confounders, and thus, if a clear

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difference in outcomes between treatment and control group arises, it can only be a significant treatment
effect or a random chance. The experimental results are often analysed under the Neyman–Rubin potential
outcomes model for causal inference [12, 23, 26]. Under the stable unit treatment value assumption, the
average treatment effect can be estimated without bias by the simple difference in the means of potential
outcomes in the treatment and the control group. In practice, randomization is often conducted in a strati-
fied, or blocked, mode according to some categorical stratification variables. For example, in a clinical trial
we first stratify the experimental units according to their stages of cancer, and then, we conduct completely
randomized experiments within the strata. The stratification can better balance baseline covariates and
improve the precision of treatment effect estimation when the stratification variables are correlated with
the outcomes; the reader is referred to Fisher [7], Wilk [29], Imai [13], Miratrix et al. [20], and Imbens and
Rubin [15] for more detailed discussion. Sequentially randomization is another widely used technique when
the experimenters cannot wait to conduct an experiment until all the experimental units are recruited.
For example, when a clinician studies a rare disease, he or she must randomize units sequentially [34]. In
both stratified and sequentially randomized experiments, the experimental units are divided into \( B \) strata,
blocks or groups, within which, independent completely randomized experiments are conducted. Due to
this relation, the results of stratified and sequentially randomized experiments can be analysed together
under the same randomization-based inference framework, although their motivation and aim are different.

In stratified and sequentially randomized experiments, apart from stratification variables, researchers
often observe many other baseline covariates, which may contain important information to predict the
outcomes. Even when carefully designed, the probability of existence of baseline covariates exhibiting
imbalance between treatment and control group is high [7, 22]. To find a more efficient estimator than
the simple difference-in-means, regression adjustment is a common strategy to account for the remaining
imbalances in covariates. In completely randomized experiments, Lin [18] showed that regression adjustment
can yield a consistent and asymptotically normal estimator of the average treatment effect whose asymptotic
variance is no greater than that of the difference-in-means estimator. Bloniarz et al. [4], Liu and Yang [19]
and Yue et al. [31] extended Lin’s results to a high-dimensional setting. They proposed using penalized
regression such as the lasso to perform the adjustment. [9] discussed the advantages of regression adjustment
in a paired experiment, a special case of stratified randomized experiments, and the author extended his
results to a finely stratified experiment, in which, within each stratum, there is either exactly one treated
unit or exactly one control unit [8]. The above results are obtained under a randomization-based inference
framework for a finite population [16, 17, 33]. This framework assumes the correctness of neither a fitted
regression model nor a super-population model with experimental units drawn independently and identically
from a distribution. Instead, the potential outcomes and covariates are assumed to be fixed population
quantities and the randomness comes only from the treatment assignment. The experimental units consist
of a finite population, which is embedded into a hypothetical infinite sequence of finite populations with
increasing sizes when investigating the asymptotic distribution of any sample quantity.

In what follows, we show that regression adjustment can also be leveraged for inferring the average
treatment effect in stratified or sequentially randomized experiments. Our analysis is conducted under
the randomization-based inference framework and hence enriches the literature on this aspect. Under mild
conditions, we show that the regression-adjusted average treatment effect estimator is consistent and asymp-
totically normal, which improves, or at least does not hurt, precision when compared with the stratified
difference-in-means estimator. Our main theorems require that the total number of experimental units tends
to infinity, but both the number of strata and their sizes can be arbitrary. That is, there can be many small strata or a few large strata or some combination thereof. Moreover, we provide a Neyman-type conservative variance estimator, which can be used to construct a large-sample conservative confidence interval for the average treatment effect. A caveat to our analysis, however, is that we require each stratum to have at least two treated and two control units, which rules out the paired and finely stratified randomized experiments.

2. Stratified randomized experiments

2.1. Framework and notations

Consider a stratified, or a sequentially, randomized experiment with $n$ units consisting of $K$ strata. Within the strata, independent completely randomized experiments are conducted, allowing different relative sizes of treatment and control groups. The stratum $k$ contains $n_{[k]}$ units, $n_{[k]} \geq 2$, such that $\sum_{k=1}^{K} n_{[k]} = n$, of whom $n_{[k]1}$ are sampled without replacement and receive the active treatment, while the remaining $n_{[k]0} = n_{[k]} - n_{[k]1}$ receive the control. The stratification, or blocking, variable is denoted as $B$, which takes values in $\{1, \ldots, K\}$. Let $Z_i$ be an indicator of whether or not unit $i$ is sampled and receives the treatment, and $\sum_{i:B_i=k} Z_i = n_{[k]1}$. In stratified random sampling, the probability that the indication vector $Z = (Z_1, \ldots, Z_n)$ takes a particular value $(z_1, \ldots, z_n)$ is $\prod_{i=1}^{K} n_{[k]1}! n_{[k]0}! / n_{[k]}!$, where $\sum_{i:B_i=k} z_i = n_{[k]1}$, for $k = 1, \ldots, K$. The total treated sample size is $n_{1} = \sum_{k=1}^{K} n_{[k]1}$. For each unit $i$, there are two fixed quantities called potential outcomes, $Y_i(1)$ and $Y_i(0)$, representing the outcome of this unit receiving or not receiving the treatment. Unit or individual level treatment effect, $\tau_i$, is defined as the comparison of these two potential outcomes, $\tau_i = Y_i(1) - Y_i(0)$. Since each unit is either in the treatment or in the control group, but not both, potential outcomes $Y_i(1)$ and $Y_i(0)$ cannot be observed simultaneously, and hence, $\tau_i$ is not identifiable without strong model assumptions. However, the average across all experimental units is estimable. It is often called the average treatment effect. In particular, we define the average treatment effect in stratum $B = k$ as

$$\tau_{[k]} = \frac{1}{n_{[k]}} \sum_{i:B_i=k} \tau_i = \frac{1}{n_{[k]}} \sum_{i=1}^{n} I(B_i = k) \{ Y_i(1) - Y_i(0) \},$$

where $I(\cdot)$ is an indicator function. The population average treatment effect is

$$\tau = \frac{1}{n} \sum_{i=1}^{n} \{ Y_i(1) - Y_i(0) \} = \sum_{k=1}^{K} \pi_{[k]} \tau_{[k]},$$

where $\pi_{[k]} = n_{[k]} / n$ is the proportion of stratum size. After conducting the physical experiment, we observe the outcome $Y_{i}^{\text{obs}} = Z_i Y_i(1) + (1 - Z_i) Y_i(0)$. Let $e_{[k]} = n_{[k]1} / n_{[k]}$ be the propensity score (proportion of treated units) in stratum $k$.

Apart from stratification variable $B$, each unit has a $p$-dimensional baseline covariates, $X_i = (X_{i,1}, \ldots, X_{i,p})^T$, which are measured in principle before randomization and thus are not affected by the treatment assignment. If we treat the covariates as outcomes, then $X_i(1) = X_i(0) = X_i$. The potential outcomes, covariates, and stratification variable $B$ are all fixed population quantities. The randomness comes only from the treat-
ment assignment vector \( Z \). Our goal is to infer the average treatment effect \( \tau \) using the observed data \{ (Z_i, B_i, X_i, Y_i^{obs}) \}_{i=1}^{n}.

Notations. We use the subscript ‘\([k]\)’ for stratum and the subscript ‘\(i\)’ for unit. We add a line on top of a quantity to denote its average. For example, the stratum-specific average of potential outcomes and covariates are

\[
Y_{[k]}(1) = \frac{1}{n[k]} \sum_{i:B_i=k} Y_i(1), \quad Y_{[k]}(0) = \frac{1}{n[k]} \sum_{i:B_i=k} Y_i(0), \quad X_{[k]}(1) = X_{[k]}(0) = \frac{1}{n[k]} \sum_{i:B_i=k} X_i.
\]

The population average are denoted as

\[
\bar{Y}(1) = \frac{1}{n} \sum_{i=1}^{n} Y_i(1), \quad \bar{Y}(0) = \frac{1}{n} \sum_{i=1}^{n} Y_i(0), \quad \bar{X}(1) = \bar{X}(0) = \frac{1}{n} \sum_{i=1}^{n} X_i.
\]

We add a line on top of a quantity and a subscript ‘\(1\)’ (or ‘\(0\)’) to denote its average in the treatment (or the control) group. For example,

\[
Y_{[k]}^{obs}(1) = \frac{1}{n[k]} \sum_{i=1}^{n} I(B_i = k, Z_i = 1) Y_i^{obs} = \frac{1}{n[k]} \sum_{i:B_i=k} Z_i Y_i(1), \quad X_{[k]}^{obs}(1) = \frac{1}{n[k]} \sum_{i:B_i=k} Z_i X_i,
\]

\[
Y_{[k]}^{obs}(0) = \frac{1}{n[k]} \sum_{i=1}^{n} I(B_i = k, Z_i = 0) Y_i^{obs} = \frac{1}{n[k]} \sum_{i:B_i=k} (1 - Z_i) Y_i(0), \quad X_{[k]}^{obs}(0) = \frac{1}{n[k]} \sum_{i:B_i=k} (1 - Z_i) X_i.
\]

The stratum-specific population variance of \{R_i\}_{i=1}^{n} is denoted as \( S_{[k]}^2 R \). For example, the stratum-specific population variances of \( Y_i(1) \) and \( Y_i(0) \) are

\[
S_{[k]}^2 Y(1) = \frac{1}{n[k]} - \frac{1}{n[k]} \sum_{i:B_i=k} \left( Y_i(1) - \bar{Y}(1) \right)^2, \quad S_{[k]}^2 Y(0) = \frac{1}{n[k]} - \frac{1}{n[k]} \sum_{i:B_i=k} \left( Y_i(0) - \bar{Y}(0) \right)^2.
\]

Let \( S_{[k]} RQ \) be the covariance between \( R \) and \( Q \) in stratum \( k \). For example,

\[
S_{[k]} XX(1) = S_{[k]} XX(0) = S_{[k]} XX = \frac{1}{n[k]} - \frac{1}{n[k]} \sum_{i:B_i=k} \left( X_i - \bar{X}_{[k]} \right) \left( X_i - \bar{X}_{[k]} \right)^T,
\]

\[
S_{[k]} XY(1) = \frac{1}{n[k]} - \frac{1}{n[k]} \sum_{i:B_i=k} \left( X_i - \bar{X}_{[k]} \right) \left( Y_i(1) - \bar{Y}_{[k]}(1) \right), \quad S_{[k]} XV(0) = \frac{1}{n[k]} - \frac{1}{n[k]} \sum_{i:B_i=k} \left( X_i - \bar{X}_{[k]} \right) \left( Y_i(0) - \bar{Y}_{[k]}(0) \right).
\]

Replacing \( S \) by \( s \) to denote the corresponding sample quantities. For example,

\[
s_{[k]}^2 Y(1) = \frac{1}{n[k]} - \frac{1}{n[k]} \sum_{i:B_i=k} Z_i \left( Y_i(1) - \bar{Y}_{[k]}^{obs} \right)^2, \quad s_{[k]}^2 Y(0) = \frac{1}{n[k]} - \frac{1}{n[k]} \sum_{i:B_i=k} (1 - Z_i) \left( Y_i(0) - \bar{Y}_{[k]}^{obs} \right)^2.
\]
where the subscript ‘unadj’ indicates that the estimator does not adjust imbalances of covariates. A natural unbiased estimator for the population average treatment effect in the control group can be estimated without bias by the difference of the means of observed outcomes in the treatment and the control group.

\[ \hat{\tau}[k] = \bar{Y}_{[k]}^{\text{obs}}(1) - \bar{Y}_{[k]}^{\text{obs}}(0). \]

A natural unbiased estimator for the population average treatment effect \( \tau \) is the weighted average of \( \hat{\tau}[k] \):

\[ \hat{\tau}_{\text{unadj}} = \sum_{k=1}^{K} \pi[k] \hat{\tau}[k], \]

where the subscript ‘unadj’ indicates that the estimator does not adjust imbalances of covariates. Recalling that \( \pi[k] = n[k]/n \) and the propensity score \( e[k] = n[k]/n[k] \), let

\[ S^2_{\text{unadj}} = \sum_{k=1}^{K} \pi[k] \left( \frac{S^2[k]Y(1)}{n[k]1} + \frac{S^2[k]Y(0)}{n[k]0} - S^2[k]\tau \right) = \frac{1}{n} \sum_{k=1}^{K} \pi[k] \left( \frac{S^2[k]Y(1)}{e[k]} + \frac{S^2[k]Y(0)}{1 - e[k]} - S^2[k]\tau \right). \]

We first introduce the following proposition.

**Proposition 1** [15, 20] The mean and variance of \( \hat{\tau}_{\text{unadj}} \) are \( E(\hat{\tau}_{\text{unadj}}) = \tau \) and \( \text{var}(\hat{\tau}_{\text{unadj}}) = S^2_{\text{unadj}} \).

The stratum-specific variance of \( \tau_i \), \( S^2[k]\tau \), is generally not estimable because we cannot observe \( \tau_i \). This term equals zero if and only if the stratum-specific treatment effect is constant, that is, \( \tau_i = c \) for all \( i \) such that \( B_i = k \), where \( c \) is a constant. The variances \( S^2[k]Y(1) \) and \( S^2[k]Y(0) \) can be estimated consistently by the sample variances \( s^2[k]Y(1) \) and \( s^2[k]Y(0) \), respectively. Therefore, the variance of the stratified difference-in-means estimator, \( S^2_{\text{unadj}} \), can be estimated consistently if and only if \( \tau_i \) is constant within each stratum. It does not require \( \tau_i \) to be constant between strata, which is different from that in completely randomized experiments. In general, the variance \( S^2_{\text{unadj}} \) can be estimated by a Neyman-type conservative estimator

\[ S^2_{\text{unadj}} = \sum_{k=1}^{K} \pi[k] \left( \frac{s^2[k]Y(1)}{n[k]1} + \frac{s^2[k]Y(0)}{n[k]0} \right) = \frac{1}{n} \sum_{k=1}^{K} \pi[k] \left( \frac{s^2[k]Y(1)}{e[k]} + \frac{s^2[k]Y(0)}{1 - e[k]} \right). \]

To conduct statistical inference of \( \tau \) based on \( \hat{\tau}_{\text{unadj}} \), we need to study the sampling distribution, or asymptotic distribution, of \( \hat{\tau}_{\text{unadj}} \) induced by the stratified random sampling. Finite population central
limit theory plays an essential role for randomization-based asymptotic inference [17, 25, 27]. In what follows, we re-establish the same asymptotic normality result as it in Bickel and Freedman [3], which is similar to the classical Lindeberg–Feller central limit theory for independent random variables. To obtain this result, we propose a much more interpretable condition to replace the Lindeberg–Feller condition, although it is slightly stronger than the latter.

3. Asymptotic normality of stratified difference-in-means estimator

3.1. Finite population central limit theory

The group of treated and control units can be seen as a stratified random sample from the population \( \Pi_{1N} = \{ Y_i(1) : i = 1, \ldots, n \} \) and \( \Pi_{0N} = \{ Y_i(0) : i = 1, \ldots, n \} \), respectively. The finite population central limit theory for a stratified random sample \( \{ Y_i(1) : Z_i = 1; \, i = 1, \ldots, n \} \) depends crucially on the maximum weighted squared distance of \( Y_i(1) \) from its stratum-specific average \( \bar{Y}_{[k]}(1) \):

\[
m_{1n} = \max_{k=1, \ldots, K} \max_{i:B_i=k} \left( \frac{n_{[k]}^{[1]}}{n_{[k]}^{[1]}} \right)^2 \frac{\{ Y_{i}(1) - \bar{Y}_{[k]}(1) \}^2}{\sum_{k=1}^{K} \pi_{[k]} S_{[k]}^2 Y(1) n_{[k]}^{[0]} / n_{[k]}^{[1]}}.
\]

Let \( \hat{Y}(1) = \sum_{k=1}^{K} \pi_{[k]} \hat{Y}_{[k]}^{obs} \) be the weighted sample mean. Denote the weighted population and sample variance as

\[
S_1^2 = \sum_{k=1}^{K} \pi_{[k]}^2 S_{[k]}^2 Y(1) \left( \frac{1}{n_{[k]}^{[1]}} - \frac{1}{n_{[k]}^{[1]}} \right) = \frac{1}{n} \sum_{k=1}^{K} \pi_{[k]} S_{[k]}^2 Y(1) n_{[k]}^{[0]} / n_{[k]}^{[1]},
\]

\[
s_1^2 = \sum_{k=1}^{K} \pi_{[k]}^2 \bar{Y}_{[k]}^{obs} (1) \left( \frac{1}{n_{[k]}^{[1]}} - \frac{1}{n_{[k]}^{[1]}} \right) = \frac{1}{n} \sum_{k=1}^{K} \pi_{[k]} \bar{Y}_{[k]}^{obs} (1) n_{[k]}^{[0]} / n_{[k]}^{[1]},
\]

respectively. We have the following results.

**Proposition 2** [6] For stratum \( k \), the mean and variance of the sample mean \( \hat{Y}_{[k]}^{obs} \) are

\[
E(\hat{Y}_{[k]}^{obs}) = \bar{Y}_{[k]}(1), \quad \text{var}(\hat{Y}_{[k]}^{obs}) = S_{[k]}^2 Y(1) \left( \frac{1}{n_{[k]}^{[1]}} - \frac{1}{n_{[k]}^{[1]}} \right).
\]

Furthermore, the mean and variance of the weighted sample mean \( \hat{Y}(1) \) are

\[
E \left( \hat{Y}(1) \right) = \bar{Y}(1), \quad \text{var} \left( \hat{Y}(1) \right) = S_1^2.
\]

**Theorem 1** As \( n \to \infty \), if \( m_{1n}/n \to 0 \), then \( \{ \hat{Y}(1) - \bar{Y}(1) \} / S_1 \) converges in distribution to a Gaussian random variable with mean zero and variance one. Furthermore, if for each stratum \( k \), \( 2 \leq n_{[k]}^{[1]} \leq n_{[k]}^{[0]} - 2 \), then, \( s_1^2 / S_1^2 \) converges to one in probability.

Theorem 1 holds for any proportion of stratum size \( \pi_{[k]} = n_{[k]}^{[1]} / n \) such that \( 0 \leq \pi_{[k]} \leq 1 \) and \( \sum_{k=1}^{K} \pi_{[k]} = 1 \), that is, the number of strata and their sizes can be arbitrary as \( n \to \infty \). When there is only one stratum,
K = 1, the condition \( m_{1n}/n \to 0 \) is reduced to that in Li and Ding [17], which is proposed to obtain the finite population central limit theory for a simple random sample. Theorem 1 substantially generalizes the result, from simple random sampling to stratified random sampling. It also suggests a strategy to construct a large-sample confidence interval for the population average, \( \bar{Y}(1) \), based on the Normal approximation.

### 3.2. Asymptotic normality of stratified difference-in-means estimator

We apply Theorem 1 to derive the asymptotic normality of stratified difference-in-means estimator \( \hat{\tau}_{\text{unadj}} \) for inferring the average treatment effect \( \tau \). To present the theoretical results, we propose the following conditions.

**Condition 1** The propensity score (proportion of treated units), \( e_{[k]} \), tends to a constant \( p_k \) not depending on \( n \), such that \( 0 < \min_{k=1,\ldots,K} p_k \leq \max_{k=1,\ldots,K} p_k < 1 \), as \( n \to \infty \).

**Condition 2** As \( n \to \infty \), the maximum stratum-specific squared distances divided by \( n \) tend to zero, that is,

\[
\frac{1}{n} \max_{k=1,\ldots,K} \max_{i:B_i=k} \left( Y_i(1) - \bar{Y}[k](1) \right)^2 \to 0, \quad \frac{1}{n} \max_{k=1,\ldots,K} \max_{i:B_i=k} \left( Y_i(0) - \bar{Y}[k](0) \right)^2 \to 0.
\]

**Condition 3** The weighted variances \( \sum_{k=1}^{K} \pi[k] S^2_{[k]Y(1)}/e_{[k]} \sum_{k=1}^{K} \pi[k] S^2_{[k]Y(0)}/(1-e_{[k]}) \) and \( \sum_{k=1}^{K} \pi[k] S^2_{[k]\tau} \) tend to finite limits (positive for their combination \( nS^2_{\text{unadj}} \)), as \( n \to \infty \).

**Remark 1** For Condition 1, the propensity score \( e_{[k]} \) is usually the same for all strata in stratified or sequentially randomized experiments, that is, \( e_{[k]} = p \) \( (k = 1, \ldots, K) \). In this case, the stratification is beneficial if based on the covariates that are strongly correlated with the outcomes. Otherwise, the stratification may hurt precision if there is no between-strata variation, but the penalty is small and vanishes asymptotically. We refer the reader to Neyman [24] and Miratrix et al. [20] for more detailed discussion. Condition 2 is slightly stronger, but simpler and more interpretable, than the Lindeberg–Feller condition proposed by Bickel and Freedman [3]; see the arguments in Li and Ding [17]. It is much weaker than the fourth moment condition proposed by Fogarty [9] and Fogarty [8]. In Condition 3, the term \( S^2_{[k]Y(1)}/e_{[k]} + S^2_{[k]Y(0)}/(1-e_{[k]}) - S^2_{[k]\tau} \) is the variance of \( \sqrt{n[k]}(\hat{\tau}_{[k]} - \tau_{[k]}) \). Condition 3 implies that their weighted average tends to a finite and positive limit. It also implies that the variance \( S^2_{\text{unadj}} \) is of order \( 1/n \).

**Theorem 2** If Conditions 1–3 hold, then \( (\hat{\tau}_{\text{unadj}} - \tau)/S_{\text{unadj}} \) converges in distribution to a Gaussian random variable with mean zero and variance one. Furthermore, if for each stratum \( k, 2 \leq n_{[k]} \leq n_{[k]} - 2 \), then the estimator \( nS^2_{\text{unadj}} \) converges in probability to the limit of

\[
\sum_{k=1}^{K} \pi[k] \left( \frac{S^2_{[k]Y(1)}}{e_{[k]}} + \frac{S^2_{[k]Y(0)}}{1-e_{[k]}} \right),
\]

which is greater than or equal to the limit of \( nS^2_{\text{unadj}} \), and the difference is the limit of \( \sum_{k=1}^{K} \pi[k] S^2_{[k]\tau} \).
Remark 2 Theorem 2 is very general in two aspects. First, the asymptotic normality holds for arbitrary number of strata and their sizes, provided the total number of experimental units $n$ tends to infinity. It also means this result includes the paired and finely stratified randomized experiments as special cases. Second, we do not require the proportion of treated units or the propensity score $e_{[k]}$ to be asymptotically the same across strata. A caveat to our analysis, however, is that when estimating the asymptotic variance, we require each stratum to have at least two treated and two control units, which rules out the paired and finely stratified randomized experiments. This is because $s^2_{[k]Y(1)}$ or $s^2_{[k]Y(0)}$ is not well-defined when there is exactly one treated or one control unit. Suitable but still conservative variance estimators can be found in Imai [13], Fogarty [9], and Fogarty [8].

Theorem 2 provides a large-sample Normal approximation of $\hat{\tau}_{unadj}$, which can produce an asymptotically conservative confidence interval for the average treatment effect $\tau$: $[\hat{\tau}_{unadj} - q_{\alpha/2}s_{unadj}, \hat{\tau}_{unadj} + q_{\alpha/2}s_{unadj}]$, where $\alpha$ is the significance level and $q_{\alpha/2}$ is the upper $\alpha/2$ quantile of a standard normal distribution. For comparison, in the special cases of paired and finely stratified randomized experiments, Fogarty [9] and Fogarty [8] have established the asymptotic normality of $\hat{\tau}_{unadj}$, but under much stronger conditions, the fourth moment conditions on the potential outcomes. Our theorem substantially generalizes their results.

4. Covariates adjustment

4.1. Covariates adjustment in general

Covariates sometimes are predictive for the potential outcomes and we may improve the efficiency of treatment effect estimation by adjusting their imbalances between treatment and control group. In completely randomized experiments, Lin [18] proposed to regress the observed outcome on the treatment indicator, covariates and their interactions, and he showed that the ordinary least squares estimator for the coefficient of the treatment indicator is consistent and asymptotically normal with asymptotic variance no larger than that of the simple difference-in-means estimator. This procedure is equivalent to run two regressions, regressing $Y_{i}\text{obs}$ on $X_i$ in the treatment and control group separately, and estimate the average treatment effect by the difference in the means of fitted values of these two regressions. We perform a similar regression analysis in stratified randomized experiments. Precisely, we compute the weighted least squares estimator based on the treated units that

$$\hat{\beta}(1) = \arg\min_{\beta} \sum_{k=1}^{K} \frac{n_{[k]} \pi_{[k]}}{n_{[k]} - 1} n_{[k]} \sum_{i:B_i=k} Z_i \left\{ Y_i(1) - \bar{Y}_{[k]} [1 - \left( X_i - \bar{X}_{[k]} \right)^T \hat{\beta} \right\}^2 .$$

Similarly, we can obtain $\hat{\beta}(0)$ based on the control units. The stratum-specific regression-adjusted average treatment effect estimator is

$$\hat{\tau}_{[k], \text{ols}} = \left\{ \bar{Y}_{[k]} [1] - \left( \bar{X}_{[k]} [1] - \bar{X}_k \right)^T \hat{\beta}(1) \right\} - \left\{ \bar{Y}_{[k]} [0] - \left( \bar{X}_{[k]} [0] - \bar{X}_k \right)^T \hat{\beta}(0) \right\} ,$$

where $\left( \bar{X}_{[k]} [1] - \bar{X}_k \right)^T \hat{\beta}(1)$ and $\left( \bar{X}_{[k]} [0] - \bar{X}_k \right)^T \hat{\beta}(0)$ adjust imbalances of covariate means between treatment and control group in stratum $k$. When $n_{[k]}$ is large, we can use stratum-specific adjusted coefficients
\( \hat{\beta}_k(1) \) and \( \hat{\beta}_k(0) \) instead of common adjusted coefficients \( \hat{\beta}(1) \) and \( \hat{\beta}(0) \) for all strata, which will be discussed later. We define the regression-adjusted average treatment effect estimator as the weighted average:

\[
\hat{\tau}_{\text{ols}} = \sum_{k=1}^{K} \pi[k] \hat{\tau}[k]_{\text{ols}}.
\]

To study the asymptotic behavior of \( \hat{\tau}_{\text{ols}} \), we project the potential outcomes onto the space spanned by the linear combination of covariates with stratum-specific intercepts: for unit \( i \) in stratum \( k \) (\( B_i = k \)),

\[
Y_i(1) = Y[k](1) + (X_i - X[k])^T \beta(1) + \varepsilon_i(1), \quad Y_i(0) = Y[k](0) + (X_i - X[k])^T \beta(0) + \varepsilon_i(0), \quad (1)
\]

where \( \beta(1) \) and \( \beta(0) \) are the projection coefficients, that is,

\[
\beta(1) = \arg\min_{\beta} \sum_{k=1}^{K} \frac{\pi[k]}{n[k]} - 1 \sum_{i:B_i=k} \left\{ Y_i(1) - \bar{Y}[k](1) - (X_i - \bar{X}[k])^T \beta \right\}^2,
\]

and similar definition on \( \beta(0) \). All terms in equation (1) are fixed population quantities, which is different from assuming a linear regression model where \( \varepsilon_i(1) \) and \( \varepsilon_i(0) \) are independently and identically distributed random errors. The following Theorem 3 shows that the asymptotic variance of \( \hat{\tau}_{\text{ols}} \) is

\[
S_{\text{ols}}^2 = \sum_{k=1}^{K} \pi[k]^2 \left( \frac{S[k]^{2}(1)}{n[k]1} + \frac{S[k]^{2}(0)}{n[k]0} - \frac{S[k]^{2}(1)-\varepsilon(0)}{n[k]} \right) = \frac{1}{n} \sum_{k=1}^{K} \pi[k] \left( \frac{S[k]^{2}(1)}{e[k]} + \frac{S[k]^{2}(0)}{1 - e[k]} - S[k]^{2}(1)-\varepsilon(0) \right)
\]

To obtain an appropriate variance estimator, we observe that the first variance term \( S[k]^{2}(1) \) can be estimated by the sample quantity based on residuals:

\[
S[k]^{2}(1) = \frac{1}{n[k]1-1} \sum_{i:B_i=k} \left( Y_i(1) - \bar{Y}[k]1 - (X_i - \bar{X}[k])^T \hat{\beta}(1) \right)^2.
\]

Similarly, we can estimate \( S[k]^{2}(0) \) by \( S[k]^{2}(0) \). The third variance term \( S[k]^{2}(1)-\varepsilon(0) \) is generally not estimable. We can obtain a Neyman-type conservative variance estimator for \( \hat{\tau}_{\text{ols}} \):

\[
S_{\text{obs}}^2 = \sum_{k=1}^{K} \pi[k]^2 \left( \frac{S[k]^{2}(1)}{n[k]1} + \frac{S[k]^{2}(0)}{n[k]0} \right) = \frac{1}{n} \sum_{k=1}^{K} \pi[k] \left( \frac{S[k]^{2}(1)}{e[k]} + \frac{S[k]^{2}(0)}{1 - e[k]} \right).
\]

We now provide conditions under which covariates adjustment improves or at least does not hurt the precision of average treatment effect estimation in stratified and sequentially randomized experiments.

**Condition 4** As \( n \to \infty \), for each covariate \( j \) (\( j = 1, \ldots, p \)),

\[
\frac{1}{n} \max_{k=1,\ldots,K} \max_{i:B_i=k} \left\{ X_{ij} - (\bar{X}[k])_j \right\}^2 \to 0.
\]
Condition 5 The weighted variances $\sum_{k=1}^{K} \pi[k] \cdot S_{k}[\varepsilon(1)]/e[k]$, $\sum_{k=1}^{K} \pi[k] S_{k}[\varepsilon(0)]/(1-e[k])$ and $\sum_{k=1}^{K} \pi[k] S_{k}[\varepsilon(1)-\varepsilon(0)]$ tend to finite limits (positive for their combination $nS_{\text{ols}}^2$), as $n \rightarrow \infty$.

Condition 6 The weighted covariance matrix $\sum_{k=1}^{K} \pi[k] S_{k}[XX]$ converges to a finite, invertible matrix; the weighted covariances $\sum_{k=1}^{K} \pi[k] S_{k}[XY(1)]$, $\sum_{k=1}^{K} \pi[k] S_{k}[XY(0)]$, and the weighted absolute covariances $\sum_{k=1}^{K} \pi[k] |S_{k}[XX]|$, $\sum_{k=1}^{K} \pi[k] |S_{k}[XY(1)]|$, $\sum_{k=1}^{K} \pi[k] |S_{k}[XY(0)]|$ converge to finite limits, as $n \rightarrow \infty$.

Theorem 3 If Conditions 1–6 hold, and assume that the propensity score is asymptotically the same across stratum ($p_k = p$), and that $2 \leq n[k] \leq n[k] - 2$, for $k = 1, \ldots, K$, then, as $n \rightarrow \infty$, $(\hat{\tau}_{\text{ols}} - \tau)/S_{\text{ols}}$ converges in distribution to a Gaussian random variable with mean zero and variance one. The difference between the asymptotic variance of $\sqrt{n}\hat{\tau}_{\text{ols}}$ and $\sqrt{n}\hat{\tau}_{\text{unadj}}$ is the limit of $-\sum_{k=1}^{K} \pi[k] \Delta_k^2$, where

$$\Delta_k^2 = \frac{1}{e[k}(1-e[k]) \begin{pmatrix} \beta^k \end{pmatrix}^T S_{k}[XX] \begin{pmatrix} \beta^k \end{pmatrix}, \quad \beta^k = (1-e[k])\beta(1)+e[k]\beta(0).$$

Furthermore, the estimator $nS_{\text{ols}}^2$ converges in probability to the limit of

$$\sum_{k=1}^{K} \pi[k] \left( S_{k}[\varepsilon(1)]/e[k] + S_{k}[\varepsilon(0)]/(1-e[k]) \right),$$

which is greater than or equal to the limit of $nS_{\text{ols}}^2$, and is no greater than the limit of $nS_{\text{unadj}}^2$. The differences, $n(s_{\text{ols}}^2 - S_{\text{ols}}^2)$ and $n(s_{\text{unadj}}^2 - S_{\text{unadj}}^2)$, converge in probability to the limits of

$$\sum_{k=1}^{K} \pi[k] S_{k}[\varepsilon(1)-\varepsilon(0)] \quad \text{and} \quad -\frac{1}{p}\beta(1)^T \left( \sum_{k=1}^{K} \pi[k] S_{k}[XX] \right) \beta(1) - \frac{1}{1-p}\beta(0)^T \left( \sum_{k=1}^{K} \pi[k] S_{k}[XX] \right) \beta(0),$$

respectively.

Remark 3 As pointed out by Cochran [6], stratified random sampling does not always result in a smaller variance for the population mean than that given by simple random sampling. However, if intelligently used, for example, the sampling proportion is the same in all strata, stratified random sampling gives a variance no larger than that of the simple random sampling. In Theorem 3, both the number of strata and their sizes can be arbitrary and the propensity score $e[k]$ can vary slightly across strata. Our main requirement is that $e[k]$ ($k = 1, \ldots, K$) tends to a common limit $p$ ($0 < p < 1$) for all strata, as $n$ tends to infinity. In this theorem, even for the valid of asymptotic normality, we require that each stratum has at least two treated and two control units, otherwise, the adjusted coefficients $\hat{\beta}(1)$ and $\hat{\beta}(0)$ are not well-defined.

Theorem 3 implies that the regression-adjusted average treatment effect estimator $\hat{\tau}_{\text{ols}}$ is consistent and asymptotically normal with asymptotic variance no more than that of the stratified difference-in-means estimator $\hat{\tau}_{\text{unadj}}$. Thus, regression or covariates adjustment improves or does not hurt precision, at least asymptotically. The improvement depends on whether the covariates are predictive to the potential outcomes in such a way that $\Delta_k^2 > 0$ for some stratum $k$. This theorem also provides a conservative variance
estimator, which is consistent if and only if the stratum-specific treatment effect is constant asymptoti-
cally, or equivalently, the limit of \( \sum_{k=1}^{K} \pi[k] S(k) (1) - (1 - \pi[k]) S(k) (0) \) is zero. Moreover, the variance estimator \( s^2_{ols} \) is
asymptotically no worse than the unadjusted variance estimator \( s^2_{unadj} \), and thus, we can use it to con-
struct a large-sample confidence interval for the average treatment effect that is at least as good as the one
constructed by \( s^2_{unadj} \).

4.2. Covariates adjustment for a few large strata

In some stratified or sequentially randomized experiments, the number of strata \( K \) is small, each with a
large stratum size \( n[k] \). For example, in a clinical trial with many males and females, experimental units are
stratified according to their gender, resulting in two large strata. The stratified or sequentially randomized
experiment is hence the independent combination of a few completely randomized experiments, each with a
large sample size. It should be more efficient to use stratum-specific adjusted coefficients \( \hat{\beta}[k](1) \) and \( \hat{\beta}[k](0) \)
instead of common ones for all strata. More precisely, within stratum \( k \), we regress the outcomes \( Y_{obs}^{i} \)
(\( B_i = k \)) on covariates in the treatment and control group separately, and obtain \( \hat{\beta}[k](1) \) that
\[
\hat{\beta}[k](1) = \arg \min_\beta \sum_{i:B_i=k} Z_i \left\{ Y_i(1) - \bar{Y}^{obs}[k] - \left( X_i - \bar{X}[k] \right)^T \beta \right\}^2,
\]
and similar definition on \( \hat{\beta}[k](0) \). The above procedure is equivalent to include the interaction of stratum
indicator and covariates in the regression, hopefully resulting in smaller variance of approximation errors,
and thus, decreasing the variance of average treatment effect estimator. The stratum-specific average
treatment effect estimator is
\[
\hat{\tau}[k],\text{ols\_int} = \left\{ \bar{Y}^{obs}[k] - \left( \bar{X}[k] - \bar{X}[k] \right)^T \hat{\beta}[k](1) \right\} - \left\{ \bar{Y}^{obs}[k] - \left( \bar{X}[k] - \bar{X}[k] \right)^T \hat{\beta}[k](0) \right\}.
\]
The regression with interaction-adjusted average treatment effect estimator is
\[
\hat{\tau}_\text{int} = \sum_{k=1}^{K} \pi[k] \hat{\tau}[k],\text{ols\_int}.
\]
The asymptotic behavior of \( \hat{\tau}_\text{int} \) depends on the projection of the potential outcomes onto the space
spanned by the linear combination of covariates, stratum indicator, and their interactions, or equivalently,
the projection of the potential outcomes onto the linear space of covariates with stratum-specific intercepts
and projection coefficients. That is, for \( i \) such that \( B_i = k \),
\[
Y_i(1) = \bar{Y}[k](1) + (X_i - \bar{X}[k])^T \beta[k](1) + \eta_i(1), \quad Y_i(0) = \bar{Y}[k](0) + (X_i - \bar{X}[k])^T \beta[k](0) + \eta_i(0),
\]
where the projection coefficient \( \beta[k](1) \) is
\[
\beta[k](1) = \arg \min_\beta \frac{1}{n[k]-1} \sum_{i:B_i=k} \left\{ Y_i(1) - \bar{Y}[k](1) - \left( X_i - \bar{X}[k] \right)^T \beta \right\}^2.
\]
and similar definition on $\beta_k(0)$. Again, all quantities in equation (2) are not random.

The asymptotic variance of $\hat{\tau}_{ols,\text{int}}$ is

$$S_{ols,\text{int}}^2 = \sum_{k=1}^{K} \pi[k] \left( \frac{S[k]\eta(1)}{n[k]} + \frac{S[k]\eta(0)}{n[k]} - \frac{S[k]\eta(1) - \eta(0)}{n[k]} \right) = \frac{1}{n} \sum_{k=1}^{K} \pi[k] \left( \frac{S[k]\eta(1)}{e[k]} + \frac{S[k]\eta(0)}{1 - e[k]} - S[k]\eta(1) - \eta(0) \right).$$

The stratum-specific variance of $\eta_k(1)$, $S[k]\eta(1)$, can be estimated by the residual sum of squares divided by its degree of freedom:

$$s[k]\eta(1) = \frac{1}{n[k] - p - 1} \sum_{i:B_i=k} \left( Y_i(1) - \bar{Y}_{k[1]}^{ols} - \left( X_i - \bar{X}_{k[1]}^{ols} \right)^T \hat{\beta}_k(1) \right)^2.$$

Following Blomarz et al. [4], we adjust the residuals’ degree of freedom. Similarly, we can define $s[k]\eta(0)$ to estimate $S[k]\eta(0)$. As discussed before, the stratum-specific variance of $\eta_k(1) - \eta(0)$, $S[k]\eta(1) - \eta(0)$, cannot be estimated in general. However, we can conservatively estimate the variance of $\hat{\tau}_{ols,\text{int}}$ by

$$S_{obs,\text{int}}^2 = \sum_{k=1}^{K} \pi[k] \left( \frac{s[k]\eta(1)}{n[k]} + \frac{s[k]\eta(0)}{n[k]} \right) = \frac{1}{n} \sum_{k=1}^{K} \pi[k] \left( \frac{s[k]\eta(1)}{e[k]} + \frac{s[k]\eta(0)}{1 - e[k]} \right).$$

We now present the condition and asymptotic result as follows.

**Condition 7** (a) There exists a constant $K_{\text{max}}$ not depending on $n$, such that $K \leq K_{\text{max}}$ for every $n$. The stratum size $n[k] \to \infty$ as $n \to \infty$ for $k = 1, \ldots, K$. (b) The stratum-specific covariance matrix $S[k]XX$ converges to a finite, invertible matrix. The stratum-specific variances $S[k]Y(1)$, $S[k]Y(0)$, $S[k]r$, $S[k]\eta(1)$, $S[k]\eta(0)$, $S[k]\eta(1) - \eta(0)$, and covariances $S[k]XY(1)$, $S[k]XY(0)$ converge to finite limits (positive for the variances).

**Remark 4** By Conditions 1 and 7, $n[k]_1/n[k] \to p_k \in (0, 1)$ and $n[k] \to \infty$, thus, $n[k]_1 \to \infty$, which implies that $n[k]_1 \geq 2$ holds asymptotically.

**Theorem 4** If Conditions 1, 2, 4 and 7 hold, then as $n \to \infty$, $(\hat{\tau}_{ols,\text{int}} - \tau)/S_{obs,\text{int}}$ converges in distribution to a Gaussian random variable with mean zero and variance one. The difference between the asymptotic variance of $\sqrt{n}\hat{\tau}_{ols,\text{int}}$ and $\sqrt{n}\hat{\tau}_{unadj}$ tends to the limit of $-\sum_{k=1}^{K} \pi[k]\hat{\Delta}_k^2 \leq 0$, where

$$\hat{\Delta}_k^2 = \frac{1}{e[k](1 - e[k])} \left( \hat{\beta}_k \right)^T S[k]XX \left( \hat{\beta}_k \right), \quad \hat{\beta}_k = (1 - e[k])\beta_k(1) + e[k]\beta_k(0).$$

Furthermore, the estimator $nS_{obs,\text{int}}^2$ converges in probability to the limit of

$$\sum_{k=1}^{K} \pi[k] \left( \frac{s[k]\eta(1)}{e[k]} + \frac{s[k]\eta(0)}{1 - e[k]} \right).$$
which is greater than or equal to the limit of \(nS_{\text{obs, int}}^2\), and is no greater than the limit of \(ns^2_{\text{unadj}}\). The differences, \(n(s_{\text{obs, int}}^2 - S_{\text{obs, int}}^2)\) and \(n(s_{\text{obs, int}}^2 - s_{\text{unadj}}^2)\), converge in probability to the limits of

\[
\sum_{k=1}^{K} \pi[k]s^2[k]\eta(1) - \eta(0) \quad \text{and} \quad - \sum_{k=1}^{K} \frac{\pi[k]}{e[k]} \beta[k](1)^T S[k]XX \beta[k](1) - \sum_{k=1}^{K} \frac{\pi[k]}{1 - e[k]} \beta[k](0)^T S[k]XX \beta[k](0),
\]

respectively.

**Remark 5** When the projection of potential outcomes is homogeneous across strata, \(\beta[k](1) = \beta(1)\) and \(\beta[k](0) = \beta(0)\), for \(k = 1, \ldots, K\), \(\hat{\tau}_{\text{obs, int}}\) and \(\hat{\tau}_{\text{ols}}\) are asymptotically equivalent. However, when the projection is heterogeneous, \(\hat{\tau}_{\text{obs, int}}\) is more efficient in general than \(\hat{\tau}_{\text{ols}}\); see the following Corollary 1.

Theorem 4 states that, compared with the stratified difference-in-means estimator \(\hat{\tau}_{\text{unadj}}\), the regression with interaction-adjusted estimator \(\hat{\tau}_{\text{obs, int}}\) improves or does not hurt precision when there are only a few large strata. The asymptotic variance can be conservatively estimated by the weighted residual sum of squares, \(s_{\text{obs, int}}^2\), which performs no worse than the unadjusted estimator \(s_{\text{unadj}}^2\). Moreover, this theorem does not require asymptotically the same propensity score across strata as assumed in Theorem 3.

**Corollary 1** Under Conditions 1–7, if \(p_k = p\), then the following holds (in probability for the second one),

\[
ns_{\text{obs, int}}^2 \leq ns_{\text{ols}}^2 \leq ns_{\text{unadj}}^2, \quad ns_{\text{obs, int}}^2 \leq ns_{\text{ols}}^2 \leq ns_{\text{unadj}}^2.
\]

Corollary 1 shows that the regression with interaction-adjusted estimator \(\hat{\tau}_{\text{obs, int}}\) performs better (at least no worse) than the regression-adjusted estimator \(\hat{\tau}_{\text{ols}}\), when there are a few large strata. Furthermore, the asymptotic variance estimator \(s_{\text{obs, int}}^2\) performs at least as good as \(s_{\text{ols}}^2\).

5. Simulation study

In this section, we conduct a simulation study to evaluate the finite-sample performance of regression adjustment in stratified and sequentially randomized experiments. Our data generating process is similar to that of Bloniarz et al. [4]. Specifically, we generate the potential outcomes from the following nonlinear models:

\[
Y_i(1) = X_i^T \beta_{11} + \exp(X_i^T \beta_{12}) + \varepsilon_i(1), \quad i = 1, \ldots, n,
\]

\[
Y_i(0) = X_i^T \beta_{01} + \exp(X_i^T \beta_{02}) + \varepsilon_i(0), \quad i = 1, \ldots, n,
\]

where \(\varepsilon_i(1)\) and \(\varepsilon_i(0)\) are generated from Gaussian distribution with mean zero and variance \(\sigma^2\) such that the signal-to-noise ratio equals ten. The \(X_i\) is a 10-dimensional covariates vector generated from a multivariate normal distribution with mean zero and covariance matrix \(\Sigma: \Sigma_{jj} = 1\) and \(\Sigma_{jl} = \rho^{j-l}, j \neq l, j, l = 1, \ldots, 10\), where \(\rho\) controls the correlation between covariates and it takes values of 0 and 0.5. The number of strata, stratum size, and coefficients \(\beta_{11}, \beta_{12}, \beta_{01}, \beta_{02}\) are generated in three scenarios.

1. **Scenario 1.** There are many small strata. We set the number of strata \(K = 25, 50, 100\) and the stratum size equals ten. The elements of coefficients \(\beta_{11}, \beta_{12}, \beta_{01}, \beta_{02}\) are generated as follows: \(\beta_{11j} \sim t_3, \beta_{12j} \sim t_5, \beta_{01j} \sim t_5, \beta_{02j} \sim t_3\).
\( \beta_{12j} \sim 0.1 \cdot t_3, \; \beta_{01j} \sim \beta_{11j} + t_3, \; \beta_{02j} \sim \beta_{12j} + 0.1 \cdot t_3, \) for \( j = 1, \ldots, 10, \) where \( t_3 \) denote the \( t \) distribution with three degrees of freedom.

(2) Scenario 2. There are two large homogeneous strata, \( K = 2, \) and we consider three stratum size, \( n_{[k]} = 100, 200, 500. \) The coefficients \( \beta_{11}, \beta_{12}, \beta_{01}, \beta_{02} \) are generated in the same way as in Scenario 1.

(3) Scenario 3. There are two large heterogeneous strata with the same stratum size as in Scenario 2. The difference is that the coefficients \( \beta_{11}, \beta_{12}, \beta_{01}, \beta_{02} \) are generated separately and independently for different stratum.

Both the covariates and potential outcomes are generated once, and then kept fixed. Thereafter, we simulate a stratified randomized experiment for 10000 times, assigning \( n_{[k]} = n_{[k]} / 2 \) units to the treatment and the remainders to the control. In Scenario 1, since the stratum size is small, we estimate and infer the average treatment effect using only the stratified difference-in-means estimator \( \hat{\tau}_{\text{unadj}} \) and the regression-adjusted estimator \( \hat{\tau}_{\text{ols}} \), while in Scenarios 2 and 3, we add the regression with interaction-adjusted estimator \( \hat{\tau}_{\text{ols, int}} \). We compare their performance in terms of absolute bias, standard deviation, root mean squared error, empirical coverage probability and mean confidence interval length of 95% confidence interval constructed as \( [\hat{\tau} - 1.96s, \hat{\tau} + 1.96s] \), where \( \hat{\tau} \) is the average treatment effect estimator and \( s \) is the estimated standard deviation.

The results are shown in Fig. 1 and Tables 1 – 2. The main findings are summarized as follows. In all scenarios, the bias of each method is substantially smaller, more than 10 times, than its standard deviation. This is because \( \hat{\tau}_{\text{unadj}} \) is unbiased and both \( \hat{\tau}_{\text{ols}} \) and \( \hat{\tau}_{\text{ols, int}} \) are asymptotically unbiased. In the first two scenarios, \( \hat{\tau}_{\text{ols}} \) performs the best, reducing the standard deviation and root mean squared error of \( \hat{\tau}_{\text{unadj}} \) by 32% – 43%, although it introduces a small and negligible amount of finite-sample bias. The estimator \( \hat{\tau}_{\text{ols}} \) performs slightly better than \( \hat{\tau}_{\text{ols, int}} \) when the potential outcomes are generated homogeneously across strata as in Scenario 2. In Scenario 3 where there are two large heterogeneous strata, \( \hat{\tau}_{\text{ols}} \) can still reduce the standard deviation and root mean squared error of \( \hat{\tau}_{\text{unadj}} \) by 3% – 10%, although it is not as significant as in the first two scenarios. In this scenario, \( \hat{\tau}_{\text{ols, int}} \) performs the best, which further reduces the standard deviation and root mean squared error of \( \hat{\tau}_{\text{ols}} \) by 17% – 33%. Furthermore, for all scenarios and methods, the variance estimators are conservative, resulting in conservative confidence intervals with empirical coverage probabilities much higher than the confidence level. However, when compared with the unadjusted estimator \( \hat{\tau}_{\text{unadj}} \), \( \hat{\tau}_{\text{ols}} \) substantially reduces, 4% – 19%, the mean confidence interval length in all scenarios. The \( \hat{\tau}_{\text{ols, int}} \) performs similarly to \( \hat{\tau}_{\text{ols}} \) in Scenario 2 and performs better, 5% – 10%, than \( \hat{\tau}_{\text{ols}} \) in Scenario 3.

Overall, the regression-adjusted estimator \( \hat{\tau}_{\text{ols}} \) always improve the precision of average treatment effect estimation, when compared with the unadjusted estimator \( \hat{\tau}_{\text{unadj}} \). Only when there are a few large heterogeneous strata, regression with interaction-adjusted estimator \( \hat{\tau}_{\text{ols, int}} \) is preferable. Moreover, the variance estimators are all conservative. These findings agree with our theoretical results.

### 6. Analysis of iron deficiency and schooling attainment data

We applied regression adjustment methods to analyse iron deficiency and schooling attainment data from a stratified randomized trial conducted by Chong et al. [5]. The trial aimed to study the effect of iron deficiency on educational attainment and cognitive ability.

We provide here only a brief description of the experimental setup and refer the reader to Chong et al. [5] for more details. The experiment was carried out in a rural secondary school in the Cajamara district of Peru.
Figure 1: Box plot of average treatment effect estimators minus the true average treatment effect, \( \hat{\tau} - \tau \). In each sub-slot of each scenario, the box plot corresponds to the methods, from left to right, \( \hat{\tau}_{\text{unadj}} \), \( \hat{\tau}_{\text{ols}} \), and \( \hat{\tau}_{\text{ols,int}} \). In Scenario 1, we do not compute and present \( \hat{\tau}_{\text{ols,int}} \) because the stratum size is too small.
Table 1: Comparison results of average treatment effect estimates in Scenario 1

| ρ   | K | methods | Bias (×1000) | SD (×100) | √MSE (×100) | CP (%) | CI length (×100) |
|-----|---|---------|--------------|-----------|-------------|--------|-----------------|
| 0   | 25 | τ_{unadj} | 2(6)         | 114(7)    | 114(23)     | 99.0   | 232             |
|     |    | τ_{ols}   | 14(2)        | 68(5)     | 68(8)       | 99.6   | 190             |
| 50  |    | τ_{unadj} | 2(3)         | 89(6)     | 89(14)      | 98.8   | 200             |
|     |    | τ_{ols}   | 4(1)         | 52(3)     | 52(5)       | 99.5   | 164             |
| 100 |    | τ_{unadj} | 0(2)         | 66(4)     | 66(8)       | 98.8   | 173             |
|     |    | τ_{ols}   | 1(1)         | 38(3)     | 38(3)       | 99.7   | 142             |
| 0.5 | 25 | τ_{unadj} | 1(4)         | 99(7)     | 99(18)      | 98.9   | 216             |
|     |    | τ_{ols}   | 13(1)        | 57(4)     | 57(6)       | 99.6   | 174             |
| 50  |    | τ_{unadj} | 1(3)         | 77(5)     | 77(10)      | 98.8   | 187             |
|     |    | τ_{ols}   | 3(1)         | 46(3)     | 46(4)       | 99.5   | 154             |
| 100 |    | τ_{unadj} | 1(2)         | 59(4)     | 59(6)       | 98.8   | 164             |
|     |    | τ_{ols}   | 1(1)         | 34(2)     | 34(2)       | 99.6   | 135             |

SD, standard deviation; √MSE, root mean squared error; CP, coverage probability; CI length, mean confidence interval length. The numbers in parentheses are the corresponding standard errors estimated by using the bootstrap with 500 replications.

between October and December in 2009. The experimental units consist of 215 students who were stratified by the number of years of secondary school completed. There were five strata in total and the strata size ranges from 30 to 58. Within each stratum, students were randomly, with approximate equal probability, exposed to one of the following three videos: an educational video in which a well-known soccer player explained the importance of iron for maximizing energy and encouraged iron supplementation; a similar educational video in which the soccer player was replaced by a doctor who encouraged iron supplements for overall health; and a ‘placebo’ video unrelated to iron in which a dentist encouraged good oral hygiene. As suggested by Chong et al. [5], we grouped the first two videos related to iron as treatment and the placebo video as control. Chong et al. [5] examined the effect of treatment on a variety of outcomes, among them the most important ones are: number of iron supplement pills taken between October and December in 2009; average grade point of the last two quarter; and cognitive ability measured by the average score on five Wii games, Big Brain Academy: Wii Degree, testing the ability of identification, memorization, analysis, computation, and visualization, respectively. There were a few baseline covariates, and among them we select five related ones as illustrations to perform covariates adjustment. The selected covariates are gender, age, distance to school, first quarter grades, and year of mother’s education.

We estimate and construct 95% confidence intervals for the average treatment effect of the educational video about the importance of iron on the outcomes mentioned above, using the simple stratified difference-in-mean estimator \( \hat{\tau}_{unadj} \), regression-adjusted estimator \( \hat{\tau}_{ols} \), and regression with interaction-adjusted estimator \( \hat{\tau}_{ols,int} \). Figure 2 shows the results. For the first outcome, all three methods produce confidence intervals not containing zero, meaning that the iron educational video has a significant effect on the number of iron supplement pills taken. This conclusion agrees with that of Chong et al. [5]. However, for the other two outcomes related to schooling attainment, the average treatment effects are not significant. This is not in conflict with the findings in [5], because we infer here the average treatment effect on all experimental
Table 2: Comparison results of average treatment effect estimates in Scenario 2 and 3

| ρ  | n_[k] | methods          | Bias (×1000) | SD (×100) | √MSE (×100) | CP (%) | CI length (×100) |
|----|-------|------------------|--------------|-----------|-------------|--------|------------------|
|    |       |                  | τ_unadj      | 4(6)      | 116(8)      | 116(23) | 99.1             | 234    |
|    |       |                  | τ_ols        | 26(3)     | 78(5)       | 78(11)  | 99.3             | 196    |
|    |       |                  | τ_ols_int    | 67(3)     | 83(5)       | 83(12)  | 99.2             | 203    |
|    | 200   |                  | τ_unadj      | 4(4)      | 99(7)       | 99(17)  | 98.8             | 213    |
|    |       |                  | τ_ols        | 4(2)      | 61(4)       | 61(6)   | 99.5             | 177    |
|    |       |                  | τ_ols_int    | 1(2)      | 63(4)       | 63(7)   | 99.4             | 180    |
|    | 500   |                  | τ_unadj      | 1(2)      | 66(4)       | 66(8)   | 98.8             | 173    |
|    |       |                  | τ_ols        | 1(1)      | 38(3)       | 38(3)   | 99.6             | 142    |
|    |       |                  | τ_ols_int    | 0(1)      | 38(2)       | 38(3)   | 99.5             | 143    |
|    | 0.5   |                  | τ_unadj      | 5(6)      | 110(7)      | 110(21) | 98.9             | 226    |
|    |       |                  | τ_ols        | 22(2)     | 70(5)       | 70(8)   | 99.3             | 185    |
|    |       |                  | τ_ols_int    | 59(3)     | 74(5)       | 74(9)   | 99.2             | 191    |
|    | 200   |                  | τ_unadj      | 5(4)      | 91(6)       | 91(14)  | 98.8             | 204    |
|    |       |                  | τ_ols        | 3(1)      | 56(4)       | 56(5)   | 99.5             | 170    |
|    |       |                  | τ_ols_int    | 1(2)      | 58(4)       | 58(6)   | 99.4             | 173    |
|    | 500   |                  | τ_unadj      | 0(2)      | 59(4)       | 59(6)   | 98.9             | 164    |
|    |       |                  | τ_ols        | 1(1)      | 34(2)       | 34(2)   | 99.6             | 135    |
|    |       |                  | τ_ols_int    | 0(1)      | 34(2)       | 34(2)   | 99.6             | 135    |

Scenario 3

| ρ  | n_[k] | methods          | Bias (×1000) | SD (×100) | √MSE (×100) | CP (%) | CI length (×100) |
|----|-------|------------------|--------------|-----------|-------------|--------|------------------|
|    | 0     |                  | τ_unadj      | 8(7)      | 124(8)      | 124(28) | 99.1             | 244    |
|    |       |                  | τ_ols        | 44(6)     | 121(8)      | 121(26) | 98.6             | 231    |
|    |       |                  | τ_ols_int    | 58(4)     | 91(6)       | 91(15)  | 99.3             | 213    |
|    | 200   |                  | τ_unadj      | 1(6)      | 117(8)      | 117(22) | 98.9             | 231    |
|    |       |                  | τ_ols        | 7(5)      | 112(8)      | 112(20) | 98.6             | 221    |
|    |       |                  | τ_ols_int    | 7(4)      | 92(6)       | 92(14)  | 99.1             | 208    |
|    | 500   |                  | τ_unadj      | 1(2)      | 63(4)       | 63(7)   | 99.1             | 172    |
|    |       |                  | τ_ols        | 0(2)      | 58(4)       | 58(6)   | 99.1             | 166    |
|    |       |                  | τ_ols_int    | 1(1)      | 43(3)       | 43(3)   | 99.5             | 150    |
|    | 0.5   |                  | τ_unadj      | 7(6)      | 117(8)      | 117(24) | 99.3             | 240    |
|    |       |                  | τ_ols        | 34(6)     | 113(8)      | 113(23) | 98.8             | 227    |
|    |       |                  | τ_ols_int    | 48(4)     | 97(6)       | 97(16)  | 99.1             | 215    |
|    | 200   |                  | τ_unadj      | 1(4)      | 102(7)      | 102(17) | 99.1             | 217    |
|    |       |                  | τ_ols        | 7(4)      | 96(6)       | 96(15)  | 98.8             | 207    |
|    |       |                  | τ_ols_int    | 7(3)      | 83(5)       | 83(11)  | 99.1             | 197    |
|    | 500   |                  | τ_unadj      | 0(1)      | 50(3)       | 50(4)   | 99.2             | 157    |
|    |       |                  | τ_ols        | 1(1)      | 45(3)       | 45(3)   | 99.3             | 149    |
|    |       |                  | τ_ols_int    | 2(0)      | 33(2)       | 33(2)   | 99.6             | 134    |

SD, standard deviation; √MSE, root mean squared error; CP, coverage probability; CI length, mean confidence interval length. The numbers in parentheses are the corresponding standard errors estimated by using the bootstrap with 500 replications.
Figure 2: Average treatment effect estimates and 95% confidence intervals for three outcomes: number of iron pills taken, average grade score, and average score of Wii games. The circle dots are average treatment effect estimators and the bars are 95% confidence intervals, with the lengths shown on top. The stratified difference in means estimator $\hat{\tau}_{unadj}$, regression-adjusted estimator $\hat{\tau}_{ols}$, and regression with interaction-adjusted estimator $\hat{\tau}_{ols\text{ int}}$ are abbreviated as ‘unadj’, ‘ols’, and ‘ols\text{ int}’, respectively.

units, while Chong et al. [5] studied the effect only on students who are anaemic. Our results confirm the existing knowledge that students who are non-anaemic receive no benefit from the additional iron taken, and further show that it can cancel out the effect on students who are anaemic. Therefore, iron supplements should be considered only for students who are anaemic. Furthermore, compared with $\hat{\tau}_{unadj}$, both $\hat{\tau}_{ols}$ and $\hat{\tau}_{ols\text{ int}}$ produce much shorter confidence intervals, 6% − 32% shorter, which is in accordance with our theoretical results. The $\hat{\tau}_{ols}$ performs the best in this empirical study, whose mean confidence interval lengths are 1% − 8% shorter than that of $\hat{\tau}_{ols\text{ int}}$. This may be because there is one stratum with a very small treated group size, size of ten, resulting in large finite-sample bias and variance for $\hat{\tau}_{ols\text{ int}}$.

7. Discussion

We study the theoretical advantage of regression adjustment in stratified and sequentially randomized experiments. We assume that the number of covariates is fixed, not depending on the sample size. In practice, however, the number of covariates can converge to infinity at a rate of the sample size. In other words, the covariates are high-dimensional. For example, in a clinical trial, each patient’s demographic and genetic information may be recorded. Performing high-dimensional regression adjustment and investigating its asymptotic properties in stratified and sequentially randomized experiments are still challenging. The most difficult technical aspect is to establish an appropriate concentration inequality for the mean of a stratified random sample, extending the Massart concentration inequality for the mean of a simple random sample used in completely randomized experiments [4, 19, 31].

This paper focuses on using covariates adjustment to infer the average treatment effect for a binary treatment in stratified and sequentially randomized experiments. It would be interesting to extend the results to more complicated settings, such as covariates adjustment for multiple value treatment [10], for binary outcomes using logistic regression [11, 21, 32], and covariate adjustment in experiments when there is
noncompliance [1, 2, 14]. Moreover, we proposed Neyman-type conservative variance estimators to construct large-sample conservative confidence intervals for the average treatment effect. It would be also interesting to explore other variance estimators under the randomized-based inference framework for stratified and sequentially randomized experiments, such as the Huber–White robust variance estimator for linear models.

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A. Proof of main theorems

A.1. Proof of Theorem 1

Proof 1 Our proof relies on the following finite population central limit theorem proposed by Bickel and Freedman [3], who assumed a slightly weaker Lindeberg–Feller condition. Let \( v_i^2 \) be the “variance weight”:

\[
   v_i^2 = \pi[k]^2 S[k]Y(1) \left( \frac{1}{n[k]} - \frac{1}{n[k]} \right) / S[1] = \pi[k]^2 S[k]Y(1) n[k] n[0] / (n[k] n[k] S[1]) = \text{var} \left( \pi[k] \bar{Y}_{\text{obs}}[k] / S[1] \right),
\]

and let \( \rho_i \) be the “the effective sample size”: \( \rho_i = n[k] n[0] / n[k] \).

Condition 8 (Lindeberg–Feller condition) For any \( \epsilon > 0 \), suppose that

\[
   \sum_{k=1}^{K} \frac{1}{n[k] - 1} \sum_{j: \epsilon v_i \{ Y_i(1) - \bar{Y}_{k}(1) \} / S[k]Y(1) > \epsilon} v_i^2 \{ Y_i(1) - \bar{Y}_{k}(1) \}^2 / S[k]Y(1) \to 0.
\]

Theorem 5 (Bickel and Freedman, 1984) If Condition 8 holds, then \( \hat{Y}(1) - \bar{Y}(1) \) converges in distribution to a Gaussian random variable with mean zero and variance one. Furthermore, if for each stratum \( k \), \( 2 \leq n[k] \leq n[k] - 1 \), then \( s^2 / S^2 \) converges to one in probability.
What we have to show is that the condition $m_{1n}/n \to 0$ implies the Lindeberg–Feller condition. If

$$\lim_{n \to \infty} \max_{k=1, \ldots, K} \max_{i: B_i = k} \frac{v_i^2 \{ Y_i(1) - \bar{Y}_k[1](1) \}^2}{S_{k}^2 Y(1) \rho_i} = 0,$$

then for any $\epsilon > 0$, there exists $N_\epsilon$ such that when $n \geq N_\epsilon$, $v_i^2 \{ Y_i(1) - \bar{Y}_k[1](1) \}^2 \leq \epsilon^2 S_{k}^2 Y(1) \rho_i$ for all $i$ such that $B_i = k$ and $k = 1, \ldots, K$. Thus, the second summation in the Lindeberg–Feller condition is zero, and hence, this condition holds. Recall that $\pi_k = n_k/n$. By definite and simple calculation, we have

$$\frac{v_i^2 \{ Y_i(1) - \bar{Y}_k[1](1) \}^2}{S_{k}^2 Y(1) \rho_i} = \frac{\pi_k^2 S_{k}^2 Y(1) n_k[0]/(n_k[1] n_k[0]) \{ Y_i(1) - \bar{Y}_k[1](1) \}^2}{n \sum_{i: B_i = k} \frac{\{ Y_i(1) - \bar{Y}_k[1](1) \}^2}{n_k[1] n_k[0]/n_k[0]} \pi_k^2 S_{k}^2 Y(1) n_k[0]/(n_k[1] n_k[0])} \leq \frac{\max_{i: B_i = k} \{ Y_i(1) - \bar{Y}_k[1](1) \}^2}{n} \left( \frac{n_k[1]}{n_k[0]} \right)^2 \leq m_{1n}/n.$$

Therefore, the condition $m_{1n}/n \to 0$ as $n \to \infty$ implies that the condition (3) holds, which further implies that the Lindeberg–Feller condition holds.

A.2. Proof of Theorem 2

Proof 2 We write $\hat{\tau}_k - \tau_k$ as the average of a stratified random sample:

$$\hat{\tau}_k - \tau_k = \sum_{i: B_i = k} \left\{ \frac{Z_i Y_i(1)}{n_k[1]} - \frac{(1 - Z_i) Y_i(0)}{n_k[0]} \right\} - \left\{ \bar{Y}_k[1](1) - \bar{Y}_k[0](0) \right\}$$

$$= \sum_{i: B_i = k} \left[ \frac{Z_i \{ Y_i(1) - \bar{Y}_k[1](1) \}}{n_k[1]} - \frac{(1 - Z_i) \{ Y_i(0) - \bar{Y}_k[0](0) \}}{n_k[0]} \right]$$

$$= \frac{1}{n_k[1]} \sum_{i: B_i = k} Z_i \left[ \frac{Y_i(1) - \bar{Y}_k[1](1)}{n_k[1]} + \frac{n_k[1]}{n_k[0]} \{ Y_i(0) - \bar{Y}_k[0](0) \} \right],$$

where the second equality is because $\sum_{i: B_i = k} Z_i/n_k[1] = \sum_{i: B_i = k} (1 - Z_i)/n_k[0] = 1$ and the last equality is due to $\sum_{i: B_i = k} \{ Y_i(0) - \bar{Y}_k[0](0) \} = 0$. Let $a_i = Y_i(1) - \bar{Y}_k[1](1) + (n_k[1]/n_k[0]) \{ Y_i(0) - \bar{Y}_k[0](0) \}$, then $\hat{\tau}_k - \tau_k$ is the stratum-specific sample mean of a stratified random sample drawn from the population $\Pi_a = \{ a_i : i = 1, \ldots, n \}$. Thus, $\tau_{\text{unadj}} - \tau = \sum_{k=1}^K \pi_k(\hat{\tau}_k - \tau_k)$ is the weighted sample mean. We can apply Theorem 1 to a stratified random sample drawn from $\Pi_a$ with the stratum-specific population mean.
\[ \bar{a}_i = \frac{\sum_{i:B_i=k} a_i}{n[k]} \] is 0. It is easy to show that the stratum-specific variance of \( a_i \) (in stratum \( k \)) is

\[ S_{[k]a}^2 = \frac{1}{n[k] - 1} \sum_{i:B_i=k} (a_i - \bar{a}_i)^2 \]

\[ = \frac{1}{n[k] - 1} \sum_{i:B_i=k} \left[ Y_i(1) - \bar{Y}_k(1) + \frac{n[k]_1}{n[k]_0} \left\{ Y_i(0) - \bar{Y}_k(0) \right\} \right]^2 \]

\[ = S_{[k]Y(1)}^2 + \frac{n[k]_1^2}{n[k]_0} S_{[k]Y(0)}^2 + \frac{n[k]_1}{n[k]_0} \times \left( \frac{2}{n[k] - 1} \right) \sum_{i:B_i=k} \left\{ Y_i(1) - \bar{Y}_k(1) \right\} \left\{ Y_i(0) - \bar{Y}_k(0) \right\} \]

The above covariance term can be further expressed as

\[ -\frac{2}{n[k] - 1} \sum_{i:B_i=k} \left\{ Y_i(1) - \bar{Y}_k(1) \right\} \left\{ Y_i(0) - \bar{Y}_k(0) \right\} \]

\[ = \frac{1}{n[k] - 1} \sum_{i:B_i=k} \left\{ Y_i(1) - \bar{Y}_k(1) \right\} \left\{ Y_i(0) - \bar{Y}_k(0) \right\} \]

\[ - \frac{1}{n[k] - 1} \sum_{i:B_i=k} \left\{ Y_i(0) - \bar{Y}_k(0) \right\} \]

\[ = \frac{1}{n[k] - 1} \sum_{i:B_i=k} \left( \tau_i - \tau_k \right)^2 - S_{[k]Y(1)}^2 - S_{[k]Y(0)}^2 \]

\[ = S_{[k] \tau}^2 - S_{[k]Y(1)}^2 - S_{[k]Y(0)}^2 \]

Since \( n[k]_1 + n[k]_0 = n[k] \), then

\[ S_{[k]a}^2 = S_{[k]Y(1)}^2 + \frac{n[k]_1^2}{n[k]_0} S_{[k]Y(0)}^2 + \frac{n[k]_1}{n[k]_0} S_{[k]Y(1)}^2 + \frac{n[k]_1}{n[k]_0} S_{[k]Y(0)}^2 - \frac{n[k]_1}{n[k]_0} S_{[k] \tau}^2 \]

\[ = \frac{n[k]}{n[k]_0} S_{[k]Y(1)}^2 + \frac{n[k]_1}{n[k]_0} S_{[k]Y(0)}^2 - \frac{n[k]_1}{n[k]_0} S_{[k] \tau}^2. \]

Since the randomization is independent across strata, then, the variance of \( \hat{\tau}_{\text{unadj}} - \tau \) is

\[ \sum_{k=1}^{K} \pi[k] S_{[k] \tau}^2 = \sum_{k=1}^{K} \pi[k] S_{[k]a}^2 = \sum_{k=1}^{K} \pi[k] \left( \frac{S_{[k]Y(1)}^2}{n[k]_1} + \frac{S_{[k]Y(0)}^2}{n[k]_0} - \frac{S_{[k] \tau}^2}{n[k]} \right) = \sigma_{\text{unadj}}^2. \]

Moreover,

\[ \sum_{k=1}^{K} \pi[k] S_{[k]a}^2 = \sum_{k=1}^{K} \pi[k] \left( \frac{S_{[k]Y(1)}^2}{n[k]_1} + \frac{S_{[k]Y(0)}^2}{n[k]_0} - S_{[k] \tau}^2 \right) = nS_{\text{unadj}}^2 \]
has a finite and positive limit according to Condition 3. Together with Condition 1, 
\(e_{[k]} \rightarrow p_k \in (0, 1)\), the condition for asymptotic normality of a stratified random sample drawn from \(\Pi_n\) is equivalent to

\[
\frac{1}{n} \max_{k=1, \ldots, K} \max_{i : B_i = k} (a_i - \bar{a}_{[k]})^2 \rightarrow 0,
\]

which is implied by Condition 1 and 2.

Next, we prove the asymptotical conservativeness of the variance estimator \(s^2_{\text{unadj}}\). The rescaled treated units

\[
\left\{ \left( \frac{n_{[k]}}{n_{[k]0}} \right)^{1/2} Y_i(1) : Z_i = 1; \ i = 1, \ldots, n \right\}
\]

can be seen as a stratified random sample drawn from the rescaled population

\[
\left\{ \left( \frac{n_{[k]}}{n_{[k]0}} \right)^{1/2} Y_i(1) : i = 1, \ldots, n \right\},
\]

then, apply the second statement of Theorem 1, we have, the following variance ratio

\[
\left( \sum_{k=1}^{K} \pi^2_{[k]} s^2_{[k]} Y(1)/n_{[k]1} \right) / \left( \sum_{k=1}^{K} \pi^2_{[k]} S^2_{[k]} Y(1)/n_{[k]1} \right)
\]

tends to one in probability. Similarly, apply Theorem 1 to the rescaled control units

\[
\left\{ \left( \frac{n_{[k]}}{n_{[k]0}} \right)^{1/2} Y_i(0) : Z_i = 0; \ i = 1, \ldots, n \right\},
\]

we have, the variance ratio

\[
\left( \sum_{k=1}^{K} \pi^2_{[k]} s^2_{[k]} Y(0)/n_{[k]0} \right) / \left( \sum_{k=1}^{K} \pi^2_{[k]} S^2_{[k]} Y(0)/n_{[k]0} \right)
\]

tends to one in probability. Therefore, the ratio

\[
\sum_{k=1}^{K} \pi^2_{[k]} \left( \frac{s^2_{[k]} Y(1)}{n_{[k]1}} + \frac{s^2_{[k]} Y(0)}{n_{[k]0}} \right) / \sum_{k=1}^{K} \pi^2_{[k]} \left( \frac{S^2_{[k]} Y(1)}{n_{[k]1}} + \frac{S^2_{[k]} Y(0)}{n_{[k]0}} \right)
\]

tends to one in probability. Under Condition 3, the variance \(S^2_{\text{unadj}}\) is of order \(1/n\) because

\[
S^2_{\text{unadj}} = \sum_{k=1}^{K} \pi_{[k]} \frac{n_{[k]}}{n} \left( \frac{s^2_{[k]} Y(1)}{n_{[k]1}} + \frac{s^2_{[k]} Y(0)}{n_{[k]0}} - \frac{S^2_{[k]} \tau}{n_{[k]}} \right) = \frac{1}{n} \sum_{k=1}^{K} \pi_{[k]} \left( \frac{s^2_{[k]} Y(1)}{e_{[k]}} + \frac{s^2_{[k]} Y(0)}{1 - e_{[k]}} - S^2_{[k]} \tau \right).
\]
Therefore, \( ns^{2}_{\text{unadj}} \) converges in probability to the limit of

\[
\sum_{k=1}^{K} \pi_{[k]} \left( \frac{S^{2}_{[k]}Y(1)}{e_{[k]}} + \frac{S^{2}_{[k]}Y(0)}{1 - e_{[k]}} \right),
\]

and the difference \( n \left( s^{2}_{\text{unadj}} - S^{2}_{\text{unadj}} \right) \) tends to the limit of \(- \sum_{k=1}^{K} \pi_{[k]}S^{2}_{[k]} \tau \leq 0 \).

A.3. Some useful lemmas

In order to obtain the asymptotic normality of \( \hat{\tau}_{\text{obs}} \), we require the following lemmas regarding the consistency of regression-adjusted coefficients \( \hat{\beta}(1), \hat{\beta}(0) \) and the asymptotic normality of stratified random samples from the covariates.

Let \( \Pi_{b} = \{ b_{i} : i = 1, \ldots, n \} \) and \( \Pi_{d} = \{ d_{i} : i = 1, \ldots, n \} \) be two series of fixed population quantities satisfying the following conditions:

**Condition 9** Let \( \bar{b}_{[k]1} \) and \( \bar{b}_{[k]1}^{\text{obs}} \) be the sample means of treated units in stratum \( k \). As \( n \to \infty \),

(a) the maximum squared distances satisfy

\[
\frac{1}{n} \max_{k=1,\ldots,K} \max_{i:B_{i}=k} (b_{i} - \bar{b}_{[k]})^{2} \to 0,
\]

(b) the weighted variances \( \sum_{k=1}^{K} \pi_{[k]}S^{2}_{[k]b} \) and \( \sum_{k=1}^{K} \pi_{[k]}S^{2}_{[k]d} \), and the weighted covariance \( \sum_{k=1}^{K} \pi_{[k]}S_{[k]bd} \), converge to finite limits (positive for variances).

In our notations, \( S_{[k]bb} = S^{2}_{[k]b} \) and \( S_{[k]dd} = S^{2}_{[k]d} \). Let \( s_{[k]bd} \) be the sample covariance of \( b \) and \( d \) in stratum \( k \):

\[
s_{[k]bd} = \frac{1}{n_{[k]1} - 1} \sum_{i:B_{i}=k} Z_{i}(b_{i} - \bar{b}_{[k]1})(d_{i} - \bar{d}_{[k]1}).
\]

In what follows, \( b_{i} \) and \( d_{i} \) can be \( Y_{i}(1), Y_{i}(0) \) and \( X_{ij} (j = 1, \ldots, p) \).

**Lemma 1** Under Conditions 1 and 9, if \( 2 \leq n_{[k]1} \leq n_{[k]} - 2 \) \( (k = 1, \ldots, K) \), then, \( \sum_{k=1}^{K} \pi_{[k]}S_{[k]bd} - \sum_{k=1}^{K} \pi_{[k]}S_{[k]bd} \) converges to zero in probability.

**Lemma 2** Under Conditions 1–6, if \( 2 \leq n_{[k]1} \leq n_{[k]} - 2 \) \( (k = 1, \ldots, K) \), we have \( \hat{\beta}(1) - \beta(1) \) and \( \hat{\beta}(0) - \beta(0) \) converge to zero in probability.

The proofs of Lemma 1 and Lemma 2 are left to the last section of the appendix. Let

\[
S^{2}_{Xj} = \frac{K}{n} \sum_{k=1}^{K} \frac{1}{n_{[k]} - 1} \sum_{i:B_{i}=k} \left( X_{ij} - (\bar{X}_{[k]})_{j} \right)^{2},
\]

which is of order \( 1/n \) according to Conditions 1 and 6. Applying the finite population central limit theory for stratified random sampling (Theorem 1) to each covariate \( X_{ij} (j = 1, \ldots, p) \), we can obtain the following lemma.

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**Lemma 3** Under Conditions 1–6, for each covariate $X_{ij}$ ($j = 1, \ldots, p$), $\sum_{k=1}^{K} \pi[k] (\bar{X}_{[k]} - \bar{X})_j / S_{Xj}$ converges in distribution to a Gaussian random variable with mean zero and variance one.

**A.4. Proof of Theorem 3**

**Proof 3** We first prove the asymptotic normality of $\hat{\tau}_{\text{ols}}$. Recall that, the projection of potential outcomes is

$$Y_i(1) = \tilde{Y}_{[k]}(1) + (X_i - \bar{X}_{[k]})^T \beta(1) + \varepsilon_i(1), \quad Y_i(0) = \tilde{Y}_{[k]}(0) + (X_i - \bar{X}_{[k]})^T \beta(0) + \varepsilon_i(0), \quad \text{for} \quad B_i = k. \quad (4)$$

It is easy to verify that $\tilde{\varepsilon}_{[k]}(1) = \tilde{\varepsilon}_{[k]}(0) = 0$. If we define $h_1 = \hat{\beta}(1) - \beta(1), \quad h_0 = \hat{\beta}(0) - \beta(0)$, by substitution, we have

$$\hat{\tau}_{[k],\text{ols}} = \left\{ \sum_{k=1}^{K} \pi[k] (Y_{[k]}(1) - \bar{X}_{[k]})^T \hat{\beta}(1) - \left\{ \sum_{k=1}^{K} \pi[k] (Y_{[k]}(0) - \bar{X}_{[k]})^T \hat{\beta}(0) \right\} \right\} = \left\{ \sum_{k=1}^{K} \pi[k] \left\{ \sum_{k=1}^{K} \pi[k] S[k]XX \right\}^{-1} \right\} \left\{ \sum_{k=1}^{K} \pi[k] S[k]XY(1) \right\}.$$  

Thus,

$$\hat{\tau}_{\text{ols}} - \tau = \frac{\sum_{k=1}^{K} \pi[k] (Y_{[k]}(1) - \bar{X}_{[k]})(X_i - \bar{X}_{[k]})^T \beta(1)}{\sum_{k=1}^{K} \pi[k] S[k]XX} - \frac{\sum_{k=1}^{K} \pi[k] (Y_{[k]}(0) - \bar{X}_{[k]})(X_i - \bar{X}_{[k]})^T \beta(0)}{\sum_{k=1}^{K} \pi[k] S[k]XY(1)}.$$  

We analyze the two terms on the right hand side separately, showing that the first term is asymptotically normal and the second term converges to zero in probability.

The first term is the stratified difference-in-means estimator divided by its standard deviation $S_{\text{ols}}$ for the potential outcomes $\varepsilon_i(1)$ and $\varepsilon_i(0)$, whose population average treatment effect is $\sum_{k=1}^{K} \pi[k] \{\tilde{\varepsilon}_{[k]}(1) - \tilde{\varepsilon}_{[k]}(0)\} = 0$. By the definition of projection coefficient $\beta(1) = (\beta_1(1), \ldots, \beta_p(1))^T$, we have

$$\beta(1) = \arg \min_{\beta} \sum_{k=1}^{K} \frac{\pi[k]}{n[k] - 1} \sum_{i:B_i=k} \left\{ Y_i(1) - \tilde{Y}_{[k]}(1) - (X_i - \bar{X}_{[k]})^T \beta(1) \right\}^2 = \left( \sum_{k=1}^{K} \pi[k] S[k]XX \right)^{-1} \left( \sum_{k=1}^{K} \pi[k] S[k]XY(1) \right),$$

which has a finite limit (a $p$-dimensional vector) as $n \to \infty$ according to Condition 6. Thus, the maximum squared distance for $\varepsilon_i(1)$ satisfies

$$\max_{k=1,\ldots,K} \max_{i:B_i=k} \{\varepsilon_i(1) - \tilde{\varepsilon}_{[k]}(1)\}^2 \leq 2 \max_{k=1,\ldots,K} \max_{i:B_i=k} \{Y_i(1) - \tilde{Y}_{[k]}(1)\}^2 + 2p^2 \max_{j=1,\ldots,p} \max_{k=1,\ldots,K} \max_{i:B_i=k} \{X_{ij} - (\bar{X}_{[k]})\}^2 \max_{j=1,\ldots,p} |\beta_j(1)|^2.$$  

Therefore, $\max_{k=1,\ldots,K} \max_{i:B_i=k} \{\varepsilon_i(1) - \tilde{\varepsilon}_{[k]}(1)\}^2 / n$ tends to zero due to Conditions 2 and 4 (on the maximum squared distance of $Y_i(1)$ and $X_{ij}$). Similarly, $\max_{k=1,\ldots,K} \max_{i:B_i=k} \{\varepsilon_i(0) - \tilde{\varepsilon}_{[k]}(0)\}^2 / n$ tends to zero. Therefore, the maximum squared distance condition for the potential outcomes $\varepsilon_i(1)$ and $\varepsilon_i(0)$ holds. Applying the finite population central limit theorem for stratified random sampling (Theorem 2) to $\varepsilon_i(1)$ and
\( \varepsilon_i(0) \), we obtain the asymptotic normality of the first term, i.e., \( \sum_{k=1}^{K} \pi[k] \left\{ \varepsilon_{[k]1} - \varepsilon_{[k]0} \right\} / S_{\text{ols}} \) converges in distribution to a Gaussian random variable with mean zero and variance one.

For the second term in (5), Lemma 2 implies that both \( h_1 \) and \( h_0 \) converge to zero in probability and Lemma 3 implies that each coordinate \( \sum_{k=1}^{K} \pi[k] \left\{ X_{[k]}^{\text{ols}} - X_{[k]} \right\} / S_{Xj} \ (j = 1, \ldots, p) \) converges in distribution to a Gaussian random variable with mean zero and variance one. By Condition 5, the variance \( S_{\text{ols}}^2 \) is of order \( 1/n \) (Similar to the argument for \( S_{\text{unadj}}^2 \)). Since both \( S_{Xj}^2 \) and \( S_{\text{ols}}^2 \) are of order \( 1/n \), the second term converges to zero in probability.

Second, we compare the asymptotic variance of \( \sqrt{n} \tilde{\tau}_{\text{ols}} \) and \( \sqrt{n} \tilde{\tau}_{\text{unadj}} \). For stratum \( k \),

\[
S_{[k]}^2 \tau(1) = \frac{1}{n[k]-1} \sum_{i:B_i=k} \left\{ Y_i(1) - \bar{Y}_{[k]}(1) \right\}^2
\]

\[
= \frac{1}{n[k]-1} \sum_{i:B_i=k} \left\{ (X_i - \bar{X}_{[k]})^T \beta(1) \right\}^2 + \frac{1}{n[k]-1} \sum_{i:B_i=k} \varepsilon_i(1)^2 + \frac{2}{n[k]-1} \sum_{i:B_i=k} \varepsilon_i(1) (X_i - \bar{X}_{[k]})^T \beta(1)
\]

\[
= \beta(1)^T S_{[k]XX} \beta(1) + S_{[k]\varepsilon(1)}^2 + 2 S_{[k]X\varepsilon(1)}^T \beta(1).
\]

Similarly,

\[
S_{[k]}^2 \tau(0) = \beta(0)^T S_{[k]XX} \beta(0) + S_{[k]\varepsilon(0)}^2 + 2 S_{[k]X\varepsilon(0)}^T \beta(0),
\]

\[
S_{[k]}^2 \tau = \{ \beta(1) - \beta(0) \}^T S_{[k]XX} \{ \beta(1) - \beta(0) \} + S_{[k]\varepsilon(1-\varepsilon(0)}^2 + 2 \left( S_{[k]X\varepsilon(1)} - S_{[k]X\varepsilon(0)} \right)^T \{ \beta(1) - \beta(0) \}.
\]

Thus, simple calculus gives

\[
\left( \frac{S_{[k]}^2 \tau(1)}{n[k]} + \frac{S_{[k]}^2 \tau(0)}{n[k]} - \frac{S_{[k]}^2 \tau}{n[k]} \right) - \left( \frac{S_{[k]}^2 \varepsilon(1)}{n[k]} + \frac{S_{[k]}^2 \varepsilon(0)}{n[k]} - \frac{S_{[k]}^2 \varepsilon(1-\varepsilon(0)}{n[k]} \right)
\]

\[
= \frac{1}{n[k]} e_{[k]}(1 - e_{[k]}) \left( \beta^k \right)^T S_{[k]XX} \left( \beta^k \right) + 2 \frac{1}{n[k]} e_{[k]} \left( S_{[k]X\varepsilon(1)}^T \beta^k + \frac{2}{n[k]} \frac{1}{1 - e_{[k]}} S_{[k]X\varepsilon(0)} \beta^k 
\]

\[
= \frac{1}{n[k]} \Delta_k^2 + \frac{1}{n[k]} e_{[k]} \left( S_{[k]X\varepsilon(1)}^T \beta^k + \frac{2}{n[k]} \frac{1}{1 - e_{[k]}} S_{[k]X\varepsilon(0)} \beta^k 
\]

where \( e_{[k]} \to p, \beta^k = (1 - e_{[k]}) \beta(1) + e_{[k]} \beta(0) \to \lim_{n \to \infty} (1 - p) \beta(1) + p \beta_0 \), and

\[
\Delta_k = \frac{1}{\{e_{[k]}(1 - e_{[k]})\}} \left( \beta^k \right)^T S_{[k]XX} \left( \beta^k \right).
\]
Therefore, the difference between the asymptotic variance of $\sqrt{n} \hat{\tau}_{\text{unadj}}$ and $\sqrt{n} \hat{\tau}_{\text{ols}}$ is

\[
ns_{\text{unadj}}^2 - ns_{\text{ols}}^2 = \sum_{k=1}^{K} \frac{\pi[k]}{n[k]} \Delta_k^2 + \sum_{k=1}^{K} \frac{\pi[k]}{n[k]} e[k] S[k] X(1) \beta^k + \sum_{k=1}^{K} \frac{\pi[k]}{n[k]} \frac{1}{1 - e[k]} S[k] X(0) \beta^k.
\]

It is enough to show that as $n \rightarrow \infty$,

\[
\sum_{k=1}^{K} \frac{\pi[k]}{n[k]} e[k] S[k] X(1) \beta^k \rightarrow 0, \quad \sum_{k=1}^{K} \frac{\pi[k]}{1 - e[k]} S[k] X(0) \beta^k \rightarrow 0.
\]  

(9)

By the definition of projection (or the definition of projection coefficient $\beta(1)$), the projection error $\varepsilon_i(1)$ is orthogonal to the covariates $X_i$ in the following sense:

\[
\sum_{k=1}^{K} \pi[k] \frac{1}{n[k]} \sum_{i: B_i = k} (X_i - \bar{X}[k]) \varepsilon_i(1) = \sum_{k=1}^{K} \pi[k] S[k] X(1) = 0.
\]

On the other hand, $\varepsilon_i(1) = Y_i(1) - \bar{Y}[k](1) - (X_i - \bar{X}[k]) \beta(1)$, then

\[
S[k] X(1) = S[k] XY(1) - S[k] XX \beta(1).
\]

Therefore, if let $\beta_0 = \lim_{n \rightarrow \infty} \beta^k = \lim_{n \rightarrow \infty} (1 - p) \beta(1) + p \beta(0)$, then

\[
\sum_{k=1}^{K} \frac{\pi[k]}{e[k]} S[k] X(1) \beta^k = \sum_{k=1}^{K} \frac{\pi[k]}{e[k]} S[k] X(1) \beta^k - \sum_{k=1}^{K} \pi[k] S[k] X(1) \beta_0 / p
\]

\[
= \sum_{k=1}^{K} \pi[k] S[k] X(1) \left( \frac{1}{e[k]} \beta^k - \beta_0 / p \right)
\]

\[
\leq \sum_{k=1}^{K} \pi[k] S[k] XY(1) \left| \left( \frac{1}{e[k]} \beta^k - \beta_0 / p \right) \right| + |\beta(1)| \sum_{k=1}^{K} \pi[k] S[k] XX \left| \left( \frac{1}{e[k]} \beta^k - \beta_0 / p \right) \right|,
\]

which tends to zero because $\beta^k/e[k] - \beta_0 / p$ tends to zero, for all $k = 1, \ldots, K$, and the weighted absolute covariances $\sum_{k=1}^{K} \pi[k] S[k] XY(1)$, $\sum_{k=1}^{K} \pi[k] S[k] XX$ tend to finite limits (Condition 6). Similarly, $\sum_{k=1}^{K} \pi[k] S[k] X(0) \beta^k/(1 - e[k])$ tends to zero.

Finally, we prove the conservativeness of the variance estimator $s_{\text{ols}}^2$. By definition and projection of
potential outcomes as in (4), we have

\[
S_{[k] \varepsilon(1)}^2 = \frac{1}{n_{[k]} - 1} \sum_{i: B_i = k} Z_i \left\{ Y_i(1) - \bar{Y}_{[k]}^{\text{obs}} - \left( X_i - \bar{X}_{[k]}^{\text{obs}} \right)^T \hat{\beta}(1) \right\}^2
\]

\[
= \frac{1}{n_{[k]} - 1} \sum_{i: B_i = k} Z_i \left\{ (X_i - \bar{X}_{[k]}^{\text{obs}})^T \beta(1) - \hat{\beta}(1) + \varepsilon_i(1) - \delta_{[k]1} \right\}^2
\]

\[
= \left( \beta(1) - \hat{\beta}(1) \right)^T S_{[k]XX(1)} \left( \beta(1) - \hat{\beta}(1) \right) + S_{[k] \varepsilon(1)}^2 + \left( \beta(1) - \hat{\beta}(1) \right)^T S_{[k] \varepsilon(1)}.
\]

Thus,

\[
n \sum_{k=1}^{K} \pi[k] S_{[k] \varepsilon(1)}^2 = n \sum_{k=1}^{K} \pi[k] S_{[k] \varepsilon(1)}^2
\]

\[
= \sum_{k=1}^{K} \pi[k] e[k] \left( \frac{S_{[k] \varepsilon(1)}^2}{e[k]} + \frac{S_{[k] \varepsilon(0)}^2}{1 - e[k]} \right)
\]

\[
= \left( \beta(1) - \hat{\beta}(1) \right)^T \sum_{k=1}^{K} \pi[k] \frac{1}{e[k]} S_{[k]XX(1)} \left( \beta(1) - \hat{\beta}(1) \right) + \left( \sum_{k=1}^{K} \pi[k] \frac{S_{[k] \varepsilon(1)}^2}{e[k]} - \sum_{k=1}^{K} \pi[k] \frac{S_{[k] \varepsilon(0)}^2}{1 - e[k]} \right) + \left( \beta(1) - \hat{\beta}(1) \right)^T \sum_{k=1}^{K} \pi[k] \frac{1}{e[k]} S_{[k] \varepsilon(1)}.
\]

Therefore,

\[
n s_{\text{ols}}^2 - n \sum_{k=1}^{K} \pi[k] \left( \frac{S_{[k] \varepsilon(1)}^2}{e[k]} + \frac{S_{[k] \varepsilon(0)}^2}{1 - e[k]} \right)
\]

\[
= \sum_{k=1}^{K} \pi[k] \left( \frac{S_{[k] \varepsilon(1)}^2}{e[k]} + \frac{S_{[k] \varepsilon(0)}^2}{1 - e[k]} \right) - \sum_{k=1}^{K} \pi[k] \left( \frac{S_{[k] \varepsilon(1)}^2}{e[k]} + \frac{S_{[k] \varepsilon(0)}^2}{1 - e[k]} \right)
\]

\[
= \left( \beta(1) - \hat{\beta}(1) \right)^T \sum_{k=1}^{K} \pi[k] \frac{1}{e[k]} S_{[k]XX(1)} \left( \beta(1) - \hat{\beta}(1) \right) + \left( \beta(0) - \hat{\beta}(0) \right)^T \sum_{k=1}^{K} \pi[k] \frac{1}{1 - e[k]} S_{[k]XX(0)} \left( \beta(0) - \hat{\beta}(0) \right)
\]

\[
+ \left\{ \sum_{k=1}^{K} \pi[k] \left( \frac{S_{[k] \varepsilon(1)}^2}{e[k]} + \frac{S_{[k] \varepsilon(0)}^2}{1 - e[k]} \right) - \sum_{k=1}^{K} \pi[k] \left( \frac{S_{[k] \varepsilon(1)}^2}{e[k]} + \frac{S_{[k] \varepsilon(0)}^2}{1 - e[k]} \right) \right\}
\]

\[
+ \left( \beta(1) - \hat{\beta}(1) \right)^T \sum_{k=1}^{K} \pi[k] \frac{1}{e[k]} S_{[k] \varepsilon(1)} + \left( \beta(0) - \hat{\beta}(0) \right)^T \sum_{k=1}^{K} \pi[k] \frac{1}{1 - e[k]} S_{[k] \varepsilon(0)}.
\]

(10)
Applying Lemma 1 to \( b_i = X_{ij} \) and \( d_i = X_{ij'} \), we can obtain the element-wise convergence of \( \sum_{k=1}^{K} \pi[k]S[k]XX(1) \) to \( \sum_{k=1}^{K} \pi[k]S[k]XX(1) \) in probability. Since \( e[k] \to p \), we have

\[
\sum_{k=1}^{K} \pi[k] \frac{1}{e[k]} S[k]XX(1) - \frac{1}{p} \sum_{k=1}^{K} \pi[k]S[k]XX(1)
\]

tends to zero in probability. Lemma 2 implies \( \beta(1) - \hat{\beta}(1) \) converges to zero in probability element-wise. Therefore, the first two terms in the summation, (10), tends to zero in probability. Theorem 2 implies that the third term in the summation, (11), tends to zero in probability. Similarly, for the fourth term, (12), applying Lemma 1 to \( b_i = X_{ij} \) and \( d_i = \epsilon_i(1) \) or \( \epsilon_i(0) \), together with Lemma 2, it tends to zero in probability. Therefore, \( nS^2_{ols} \) converges in probability to the limit of

\[
\sum_{k=1}^{K} \pi[k] \left( \frac{S^2[k]\epsilon(1)}{e[k]} + \frac{S^2[k]\epsilon(0)}{1 - e[k]} \right)
\]

Since

\[
nS^2_{ols} = \sum_{k=1}^{K} \pi[k] \left( \frac{S^2[k]\epsilon(1)}{e[k]} + \frac{S^2[k]\epsilon(0)}{1 - e[k]} - S^2[k]\epsilon(1) - \epsilon(0) \right),
\]

then, the difference \( nS^2_{ols} - nS^2_{ols} \) converges in probability to the limit of \( \sum_{k=1}^{K} \pi[k]S^2[k]\epsilon(1) - \epsilon(0) \geq 0 \).

By Theorem 2, \( nS^2_{unadj} \) converges in probability to the limit of

\[
\sum_{k=1}^{K} \pi[k] \left( \frac{S^2[k]Y(1)}{e[k]} + \frac{S^2[k]Y(0)}{1 - e[k]} \right)
\]

thus, the difference \( nS^2_{unadj} - nS^2_{ols} \) converges in probability to the limit of

\[
\sum_{k=1}^{K} \pi[k] \left( \frac{S^2[k]Y(1)}{e[k]} + \frac{S^2[k]Y(0)}{1 - e[k]} \right) - \sum_{k=1}^{K} \pi[k] \left( \frac{S^2[k]\epsilon(1)}{e[k]} + \frac{S^2[k]\epsilon(0)}{1 - e[k]} \right).
\]

We have shown in (6) and (7) that

\[
S^2[k]Y(1) = \beta(1)^T S[k]XX \beta(1) + S^2[k]\epsilon(1) + 2S^T[k]X\epsilon(1)\beta(1),
\]

\[
S^2[k]Y(0) = \beta(0)^T S[k]XX \beta(0) + S^2[k]\epsilon(0) + 2S^T[k]X\epsilon(0)\beta(0).
\]
Thus,
\[
\sum_{k=1}^{K} \pi[k] \left( \frac{S^2[k]Y(1)}{e[k]} + \frac{S^2[k]Y(0)}{1 - e[k]} \right) - \sum_{k=1}^{K} \pi[k] \left( \frac{S^2[k]\varepsilon(1)}{e[k]} + \frac{S^2[k]\varepsilon(0)}{1 - e[k]} \right) = \beta(1)^T \sum_{k=1}^{K} \pi[k] \frac{1}{e[k]} S[k]XX\beta(1) + \beta(0)^T \sum_{k=1}^{K} \pi[k] \frac{1}{1 - e[k]} S[k]XX\beta(0) + 2 \sum_{k=1}^{K} \pi[k] \frac{1}{e[k]} S[k]\varepsilon(1)X(1)\beta(1) + 2 \sum_{k=1}^{K} \pi[k] \frac{1}{1 - e[k]} S[k]\varepsilon(0)X(0)\beta(0).
\]

Similar to (9), we can show that
\[
\sum_{k=1}^{K} \pi[k] \frac{1}{e[k]} S[k]XX(1)\beta(1) \to 0, \quad \sum_{k=1}^{K} \pi[k] \frac{1}{1 - e[k]} S[k]\varepsilon(0)X(0)\beta(0) \to 0,
\]
and
\[
\beta(1)^T \sum_{k=1}^{K} \pi[k] \frac{1}{e[k]} S[k]XX(1) \to \lim_{n \to \infty} \frac{1}{p} \beta(1)^T \left( \sum_{k=1}^{K} \pi[k] S[k]XX \right) \beta(1),
\]
\[
\beta(0)^T \sum_{k=1}^{K} \pi[k] \frac{1}{1 - e[k]} S[k]XX(0) \to \lim_{n \to \infty} \frac{1}{1 - p} \beta(0)^T \left( \sum_{k=1}^{K} \pi[k] S[k]XX \right) \beta(0).
\]

Therefore, the difference \(n_s^2_{\text{unadj}} - n_s^2_{\text{obs}}\) converges in probability to the limit of
\[
\frac{1}{p} \beta(1)^T \left( \sum_{k=1}^{K} \pi[k] S[k]XX \right) \beta(1) + \frac{1}{1 - p} \beta(0)^T \left( \sum_{k=1}^{K} \pi[k] S[k]XX \right) \beta(0).
\]

A.5. Proof of Theorem 4

**Proof 4** We use the following asymptotic normality result for regression-adjusted average treatment effect estimator in completely randomized experiments. See Example 9 in Li and Ding [17] and also Theorem 1 and Corollary 1.1 in Lin [18] who assumed the stronger fourth moment condition. Let
\[
S^2[k,\text{obs} \text{int}] = \frac{S^2[k]\eta(1)}{n[k]} + \frac{S^2[k]\eta(0)}{n[k]} - \frac{S^2[k]\eta(1) - \eta(0)}{n[k]}, \quad S^2[k,\text{obs} \text{int}] = \frac{s^2[k]\eta(1)}{n[k]} + \frac{s^2[k]\eta(0)}{n[k]}, \quad S^2[k,\text{unadj}] = \frac{s^2[k]Y(1)}{n[k]} + \frac{s^2[k]Y(0)}{n[k]}.
\]

**Proposition 3** (Li and Ding [17]) If Conditions 1, 2, 4 and 7 hold, then for stratum \(k\), \((\hat{\tau}[k,\text{obs} \text{int}] - \tau[k])/S[k,\text{obs} \text{int}]\) converges in distribution to a Gaussian random variable with mean zero and variance one as \(n[k] \to \infty\). The difference between the asymptotic variance of \(\sqrt{n[k]}(\hat{\tau}[k,\text{obs} \text{int}] - \tau[k])\) and \(\sqrt{n[k]}(\hat{\tau}[k] - \tau[k])\)
tends to the limit of $-\hat{\Delta}_k^2$, where

$$\hat{\Delta}_k^2 = \frac{1}{e_\lfloor k\rfloor (1 - e_\lfloor k\rfloor)} \left( \tilde{\beta}^k \right)^T S_{\lfloor k\rfloor} XX \left( \tilde{\beta}^k \right), \quad \tilde{\beta}^k = (1 - e_\lfloor k\rfloor) \beta_k(1) + e_\lfloor k\rfloor \beta_k(0).$$

Furthermore, the estimator $n_{\lfloor k\rfloor} S^2_{\lfloor k\rfloor, \text{adj}}$ converges in probability to the limit of

$$\frac{S^2_{\lfloor k\rfloor \eta(1)}}{e_\lfloor k\rfloor} + \frac{S^2_{\lfloor k\rfloor \eta(0)}}{1 - e_\lfloor k\rfloor},$$

which is greater than or equal to the limit of $n_{\lfloor k\rfloor} S^2_{\lfloor k\rfloor, \text{obs, adj}}$, and the difference is the limit of $S^2_{\lfloor k\rfloor \eta(1)-\eta(0)}$. The difference $n_{\lfloor k\rfloor} (s^2_{\lfloor k\rfloor, \text{obs, int}} - s^2_{\lfloor k\rfloor, \text{adj}})$ converges in probability to the limit of

$$-\frac{1}{e_\lfloor k\rfloor} \beta_k(1)^T S_{\lfloor k\rfloor} XX \beta_k(1) - \frac{1}{1 - e_\lfloor k\rfloor} \beta_k(0)^T S_{\lfloor k\rfloor} XX \beta_k(0).$$

Since $(\hat{\tau}_{\lfloor k\rfloor, \text{obs}} - \tau_{\lfloor k\rfloor})$ are independent across strata, and the number of strata $K$ is bounded, Theorem 4 holds immediately.

**A.6. Proof of Corollary 1**

**Proof 5** We have shown in the proof of Theorem 3 that $nS^2_{\text{obs}} \leq nS^2_{\text{adj}}$. Now we show that $nS^2_{\text{obs, int}} \leq nS^2_{\text{obs}}$. According to the projections in (4) and (2) in the main text, we have for $i$ such that $B_i = k$,

$$Y_i(1) = \bar{Y}_{\lfloor k\rfloor}(1) + (X_i - \bar{X}_{\lfloor k\rfloor})^T \beta(1) + \varepsilon_i(1), \quad Y_i(1) = \bar{Y}_{\lfloor k\rfloor}(1) + (X_i - \bar{X}_{\lfloor k\rfloor})^T \beta_k(1) + \eta_i(1).$$

Taking difference gives

$$\varepsilon_i(1) = (X_i - \bar{X}_{\lfloor k\rfloor})^T (\beta_k(1) - \beta(1)) + \eta_i(1).$$

Thus,

$$S^2_{\lfloor k\rfloor \varepsilon(1)} = \frac{1}{n_{\lfloor k\rfloor} - 1} \sum_{i:B_i=k} \left\{ \varepsilon_i(1) - \varepsilon_{\lfloor k\rfloor}(1) \right\}^2$$

$$= \frac{1}{n_{\lfloor k\rfloor} - 1} \sum_{i:B_i=k} \left\{ (X_i - \bar{X}_{\lfloor k\rfloor})^T (\beta_k(1) - \beta(1)) \right\}^2 + \frac{1}{n_{\lfloor k\rfloor} - 1} \sum_{i:B_i=k} \eta_i(1)^2$$

$$+ \frac{2}{n_{\lfloor k\rfloor} - 1} \sum_{i:B_i=k} \eta_i(1) (X_i - \bar{X}_{\lfloor k\rfloor})^T (\beta_k(1) - \beta(1))$$

$$= \frac{1}{n_{\lfloor k\rfloor} - 1} \sum_{i:B_i=k} \left\{ (X_i - \bar{X}_{\lfloor k\rfloor})^T (\beta_k(1) - \beta(1)) \right\}^2 + S^2_{\lfloor k\rfloor \eta(1)}$$

$$= \left\{ \beta_k(1) - \beta(1) \right\}^T S_{\lfloor k\rfloor} XX \left\{ \beta_k(1) - \beta(1) \right\} + S^2_{\lfloor k\rfloor \eta(1)}, \quad (13)$$
where the last but second equality is due to the property of projection that the covariates \( X_i - \bar{X}_k \) and the projection error \( \eta_i(1) \) are orthogonal in stratum \( k \), i.e., \( \sum_{i:B_i=k} \eta_i(1) (X_i - \bar{X}_k) = 0 \). Similarly,

\[
S_{[k]e(0)}^2 = (\beta_{[k]}(0) - \beta(0))^T S_{[k]XX} (\beta_{[k]}(0) - \beta(0)) + S_{[k]r(0)}^2, \tag{14}
\]

\[
S_{[k]e(1)-e(0)}^2 = \{\beta_{[k]}(1) - \beta(1) - \beta_{[k]}(0) + \beta(0)\}^T S_{[k]XX} \{\beta_{[k]}(1) - \beta(1) - \beta_{[k]}(0) + \beta(0)\} + S_{[k]r(1)-r(0)}^2. \tag{15}
\]

Combining (13) – (15), we have

\[
nS_{\text{ols}}^2 - nS_{\text{ols, int}}^2 = \sum_{k=1}^{K} \pi[k] \frac{1}{e[k]} \{\beta_{[k]}(1) - \beta(1)\}^T S_{[k]XX} \{\beta_{[k]}(1) - \beta(1)\}
+ \sum_{k=1}^{K} \pi[k] \frac{1}{1-e[k]} \{\beta_{[k]}(0) - \beta(0)\}^T S_{[k]XX} \{\beta_{[k]}(0) - \beta(0)\}
- \sum_{k=1}^{K} \pi[k] \{\beta_{[k]}(1) - \beta(1) - \beta_{[k]}(0) + \beta(0)\}^T S_{[k]XX} \{\beta_{[k]}(1) - \beta(1) - \beta_{[k]}(0) + \beta(0)\}
= \sum_{k=1}^{K} \pi[k] \frac{1}{e[k](1-e[k])} \gamma_k S_{[k]XX} \gamma_k \geq 0,
\]

where

\[
\gamma_k = (1-e[k]) \{\beta_{[k]}(1) - \beta(1)\} + e[k] \{\beta_{[k]}(0) - \beta(0)\}.
\]

Next, we show that the inequality \( nS_{\text{ols}}^2 \leq nS_{\text{ols, int}}^2 \leq nS_{\text{unadj}}^2 \) holds in probability. We have shown in Theorem 3 that \( nS_{\text{ols}}^2 - nS_{\text{unadj}}^2 \) converges in probability to a limit no larger than zero, and \( nS_{\text{ols}}^2 \) converges in probability to the limit of \( \sum_{k=1}^{K} \pi[k] \{S_{[k]e(1)}^2/p + S_{[k]e(0)}^2/(1-p)\} \) (in this corollary we assume the propensity score is asymptotically the same across strata, that is, \( p_k = p \)). Theorem 4 shows that \( nS_{\text{ols, int}}^2 \) converges in probability to the limit of \( \sum_{k=1}^{K} \pi[k] \{S_{[k]r(1)}^2/p + S_{[k]r(0)}^2/(1-p)\} \). Therefore, it is enough to show that the limit of

\[
\sum_{k=1}^{K} \pi[k] \{S_{[k]e(1)}^2/p + S_{[k]e(0)}^2/(1-p)\} - \sum_{k=1}^{K} \pi[k] \{S_{[k]r(1)}^2/p + S_{[k]r(0)}^2/(1-p)\}
\]
is greater than or equal to zero. Combing (13) and (14), we have

\[
K \sum_{k=1}^{K} \pi[k] \left\{ S^2_{k|c} / p + S^2_{k|v} / (1-p) \right\} - K \sum_{k=1}^{K} \pi[k] \left\{ S^2_{k|v} / p + S^2_{k|v} / (1-p) \right\}
\]

\[
= \frac{1}{p} \sum_{k=1}^{K} \pi[k] \left\{ \beta[k](1) - \beta(1) \right\}^T S_{k|XX} \left\{ \beta[k](1) - \beta(1) \right\}
\]

\[
+ \frac{1}{1-p} \sum_{k=1}^{K} \pi[k] \left\{ \beta[k](0) - \beta(0) \right\}^T S_{k|XX} \left\{ \beta[k](0) - \beta(0) \right\},
\]

which is greater than or equal to zero because the covariance matrix \( S_{k|XX} \) is positive-definite.

B. Proof of lemmas

B.1. Proof of lemma 1

**Proof** Applying standard sampling theory (using Theorem 2.1 and Theorem 2.3 in Cochran [6]), it is

\[
E \left( \sum_{k=1}^{K} \pi[k] s_{k|bd} - \sum_{k=1}^{K} \pi[k] S_{k|bd} \right) = 0.
\]

Using Markov inequality, it is enough for Lemma 1 to show that the variance \( \text{var}(\sum_{k=1}^{K} \pi[k] s_{k|bd}) \) tends to zero. Since completely randomized experiment is conducted independently across strata, we have

\[
\text{var}(\sum_{k=1}^{K} \pi[k] s_{k|bd}) = \sum_{k=1}^{K} \pi[k]^2 \text{var}(s_{k|bd}).
\]

Next, we bound the stratum-specific variance \( \text{var}(s_{k|bd}) \). Since

\[
s_{k|bd} = \frac{1}{n[k] - 1} \sum_{i:B_i = k} Z_i(b_i - \bar{b}_{k|1})(d_i - \bar{d}_{k|1})
\]

\[
= \frac{1}{n[k] - 1} \sum_{i:B_i = k} Z_i(b_i - \bar{b}_{k})(d_i - \bar{d}_{k}) - \frac{n[k] - 1}{n[k] - 1} \left( \bar{b}_{k|1} - \bar{b}_{k} \right) \left( \bar{d}_{k|1} - \bar{d}_{k} \right),
\]

then

\[
\text{var}(s_{k|bd}) \leq 2 \text{var} \left( \frac{1}{n[k] - 1} \sum_{i:B_i = k} Z_i(b_i - \bar{b}_{k})(d_i - \bar{d}_{k}) \right) + 2 \text{var} \left( \frac{n[k] - 1}{n[k] - 1} \left( \bar{b}_{k|1} - \bar{b}_{k} \right) \left( \bar{d}_{k|1} - \bar{d}_{k} \right) \right),
\]

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The first term is bounded as follows:

\[
\text{var} \left( \frac{1}{n[k]_1 - 1} \sum_{i:B_i = k} Z_i (b_i - \bar{b}_[k]) (d_i - \bar{d}_[k]) \right) \\
= \frac{n^2[k]_1}{(n[k]_1 - 1)^2} \text{var} \left( \frac{1}{n[k]_1} \sum_{i:B_i = k} Z_i (b_i - \bar{b}_[k]) (d_i - \bar{d}_[k]) \right) \\
\leq \frac{n^2[k]_1}{(n[k]_1 - 1)^2} \frac{1}{n[k]_1 - 1} \sum_{i:B_i = k} \left\{ (b_i - \bar{b}_[k]) (d_i - \bar{d}_[k]) \right\}^2 \left( \frac{1}{n[k]_1} - \frac{1}{n[k]} \right) \\
\leq \max_{i:B_i = k} (b_i - \bar{b}_[k])^2 \frac{n^2[k]_1}{(n[k]_1 - 1)^2} \frac{1}{n[k]_1 - 1} \sum_{i:B_i = k} (d_i - \bar{d}_[k])^2 \left( \frac{1}{n[k]_1} - \frac{1}{n[k]} \right) \\
\leq 4 \max_{i:B_i = k} (b_i - \bar{b}_[k])^2 \frac{n[k]_0}{n[k] n[k]_1} S^2_{id}. \quad (16)
\]

where the last inequality is due to $n^2[k]_1/(n[k]_1 - 1)^2 \leq 4 \ n[k]_1 \geq 2$.

To bound the second term, we first observe that by Cauchy-Schwarz inequality

\[
(b_{[k]_1} - \bar{b}_[k])^2 = \left( \frac{1}{n[k]_1} \sum_{i:B_i = k} Z_i (b_i - \bar{b}_[k]) \right)^2 \leq \frac{1}{n[k]_1} \sum_{i:B_i = k} (b_i - \bar{b}_[k])^2 \leq \max_{i:B_i = k} (b_i - \bar{b}_[k])^2.
\]

Thus, the second term is bounded as follows:

\[
\text{var} \left( \frac{n[k]_1}{n[k]_1 - 1} (\bar{b}_{[k]} - \bar{b}_[k]) (\bar{d}_{[k]} - \bar{d}_[k]) \right) \\
\leq \frac{n^2[k]_1}{(n[k]_1 - 1)^2} E \left( (\bar{b}_{[k]} - \bar{b}_[k])^2 (\bar{d}_{[k]} - \bar{d}_[k])^2 \right) \\
\leq \max_{i:B_i = k} (b_i - \bar{b}_[k])^2 \frac{n^2[k]_1}{(n[k]_1 - 1)^2} E \left( \frac{1}{n[k]_1} \sum_{i:B_i = k} Z_i (d_i - \bar{d}_[k]) \right)^2 \\
\leq \max_{i:B_i = k} (b_i - \bar{b}_[k])^2 \frac{n^2[k]_1}{(n[k]_1 - 1)^2} S^2_{id} \left( \frac{1}{n[k]_1} - \frac{1}{n[k]} \right) \\
\leq 4 \max_{i:B_i = k} (b_i - \bar{b}_[k])^2 \frac{n[k]_0}{n[k] n[k]_1} S^2_{id}. \quad (17)
\]

Combining (16) and (17), we have

\[
\text{var}(s_{[k]id}) \leq 16 \max_{i:B_i = k} (b_{ij} - \bar{b}_[k])^2 \frac{n[k]_0}{n[k] n[k]_1} S^2_{id}.
\]
Therefore,

\[
\text{var}\left(\sum_{k=1}^{K} \pi[k] s[k] \bar{b}d\right) \leq 16 \sum_{k=1}^{K} \pi[k] \max_{i: B_i = k} (b_i - \bar{b}[k])^2 \frac{n[k]0}{n[k]} S_{id}^2
\]

\[
\leq 16 \frac{1}{n} \max_{k=1, \ldots, K} \max_{i: B_i = k} (b_i - \bar{b}[k])^2 \sum_{k=1}^{K} \pi[k] \frac{n[k]0}{n[k]} S_{id}^2,
\]

which converges to zero by Condition 1 and Condition 9.

### B.2. Proof of lemma 2

By definition,

\[
\hat{\beta}(1) = \arg\min_{\beta} \sum_{k=1}^{K} \frac{\pi[k]}{n[k]} \max_{i: B_i = k} Z_i \left\{Y_i(1) - \bar{Y}_{k}[1] - \left(X_i - \bar{X}_{k}[1]\right)^T \beta\right\}^2.
\]

\[
= \left(\sum_{k=1}^{K} \pi[k] s[k]XX(1)\right)^{-1} \left(\sum_{k=1}^{K} \pi[k] s[k]XY(1)\right),
\]

where \(s[k]XX(1)\) is the sample covariance matrix of the covariate for the treated units in stratum \(k\), and \(s[k]XY(1)\) is the sample covariance between \(X_i\) and \(Y_i(1)\) for the treated units in stratum \(k\). Applying Lemma 1 to \(b_i = X_{ij}\) and \(d_i = X_{ij}'\), we have \(\sum_{k=1}^{K} \pi[k] s[k]XX(1)\) and \(\sum_{k=1}^{K} \pi[k] s[k]XY(1)\) converge, element-wise, to zero in probability. Moreover, by definite of \(\beta(1)\), we have

\[
\beta(1) = \left(\sum_{k=1}^{K} \pi[k] S[k]XX\right)^{-1} \left(\sum_{k=1}^{K} \pi[k] S[k]XY(1)\right).
\]

Therefore, \(\hat{\beta}(1) - \beta(1)\) converges to zero in probability. Similarly, \(\hat{\beta}(0) - \beta(0)\) converges to zero in probability.