$N = 1$ Dualities for Exceptional Gauge Groups and Quantum Global Symmetries

Jacques Distler and Andreas Karch *

Theory Group  
Department of Physics  
University of Texas  
Austin, TX 78712, USA

Email: distler@golem.ph.utexas.edu  
Email: akarch@physics.utexas.edu

Abstract

We discuss our attempts to generalize the known examples of dualities in $N = 1$ supersymmetric gauge theories to exceptional gauge groups. We derive some dual pairs from known examples connected to exceptional groups and find an interesting phenomenon: sometimes the full global symmetry is “hidden” on the magnetic side. It is not realized as a symmetry on the fundamental fields in the Lagrangian. Rather, it emerges as a symmetry of the quantum theory. We then focus on an approach based on self-dual models. We construct duals for some very special matter content of $E_7$, $E_6$ and $F_4$. Again we find that the full global symmetry is not realized on the fundamental fields.

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1. Introduction

The past few years have seen remarkable progress in the study of supersymmetric gauge theories. Most of this progress is due to the phenomenon of duality: two seemingly different theories are shown to describe the same infrared physics. This is a symmetry of the full theories, after all the non-perturbative effects have been taken into account. Weak and strong coupling get interchanged. This allows one to obtain results in strongly coupled theories that were inaccessible before, once the duality is established.

Using these new symmetries lead to beautiful results in the last few years, especially $N = 2$ [2] and $N = 4$ [1] supersymmetric gauge theories. N. Seiberg has shown in [4,5,6] that for $N = 1$ theories a somewhat weaker version of duality still holds: for certain values of the adjustable parameters two theories different in the ultraviolet flow to the same infrared fixed point. That means for an observer only testing the low energy regime, there is no way to distinguish the two theories.

Seiberg did his original studies on $SU$ and $SO$ gauge theories with matter only in the fundamental representation. Since then many new examples have been found with more complicated matter content and different groups [15,8,9,10,18,16,17,19]. Despite all the effort that has gone in finding new dualities, we are still lacking a dual description for some of the most interesting theories: those with exceptional gauge groups. These theories appear over and over again in connection with issues like string phenomenology, dynamical supersymmetry breaking and SUSY grand unification.

In this paper, we describe our efforts to find dual descriptions for $N = 1$ gauge theories with exceptional gauge groups and an interesting phenomenon we discovered during our analysis: in some dual pairs the full global symmetry is not visible in terms of the fundamental fields but only appears as a symmetry of the effective theory in the far infrared. To use a term which has become familiar in the string context, it is a quantum symmetry, not present as a symmetry of the Lagrangian, but which appear only as a symmetry of the quantum theory.

We used two different approaches to the problem. First we tried to generalize the dualities of Pouliot and Strassler [8,9,10] between $Spin(7,8,10)$ and $SU$ groups with a symmetric tensor. These $Spin$ groups are reached when one higgses exceptional groups, so one might hope that duals for exceptional groups should look somewhat similar.

In the end, our second approach proved to be more fruitful. In [11] the special importance of self-dual models was pointed out. Ramond found [3] that $E_6$ with 6 flavors is self-dual. Similarly we argue that $E_7$ with 4 flavors is self-dual. Starting from those two theories we generate some new dual pairs by going along flat directions and perturbing the theories with mass terms. Again we find the global symmetries realized as quantum symmetries of the magnetic theory.

In Section 2 we will review the $Spin(10)$ duality of [10] which we used as a starting point for the first part of our analysis. In Section 3 we will discuss our close analysis of these $Spin \leftrightarrow SU$ dualities. We will encounter for the first time that the full global symmetry acts in a very funny way on the magnetic side: it combines fundamental fields and composite fields into multiplets of the full global symmetry.

In section 4 we will use a self-dual model obtained by Ramond [3] for $E_6$ and a new
self-dual model for $E_7$ to derive some new dual pairs with exceptional gauge groups. In particular duals are found for:

- $E_7$ with 4 fundamentals
- $E_6$ with 3 fundamentals and 3 antifundamentals
- $F_4$ with 5 fundamentals
- $F_4$ with 4 fundamentals

In Section 5 we will give our conclusions.

2. Duality in $Spin(10)$ with a Spinor

Last year Pouliot and Strassler found dualities between $Spin(7, 8, 10)$ [8,9,10] and $SU$ groups with a symmetric tensor representation. For the $Spin(7)$ and $Spin(8)$ case the situation is really strange, since these dualities map the non-chiral electric theory to a chiral magnetic theory. Since we used their $Spin(10)$ example as a starting point we’d like to review this duality. The other two examples have a very similar structure.

As usual, this duality is not proven, but the conjecture is based on some solid evidence: the global symmetries are the same, it satisfies the ’t Hooft anomaly matching conditions, and the gauge invariant operators match. Under perturbations the duals flow to new consistent dual pairs, as we will show in the next section. Under a very special perturbation (giving a vev to a 10 of $Spin(10)$ and then writing down a mass term for the spinor), they are able to connect their dual to the basic $SO$ duals of Intriligator and Seiberg [6].

The electric theory of this dual pair is $Spin(10)$ with one spinor and an arbitrary number of vectors. If $n$, the number of additional vectors, is greater or equal to 7, the theory is in its non-abelian Coulomb phase. Pouliot and Strassler established the following duality for this case:

| Gauge Group | Global Symmetries |
|-------------|-------------------|
| $Spin(10)$  | $SU(n)$          |
| $Q$ | 10 | $n$ | $-1$ | $1 - \frac{8}{n+2}$ |
| $S$ | 16 | 1 | $\frac{n}{2}$ | $1 - \frac{8}{n+2}$ |

\[ \downarrow \]

| Gauge Group | Global Symmetries |
|-------------|-------------------|
| $SU(n-5)$  | $SU(n)$          |
| $q$ | $n-5$ | $\bar{n}$ | 1 | $\frac{8}{n+2} - \frac{1}{n-5}$ |
| $q'$ | $n-5$ | 1 | $-n$ | $-1 + \frac{16}{n+2} + \frac{1}{n-5}$ |
| $s$ | $\square$ | 1 | 0 | $\frac{2}{n-5}$ |
| $Y$ | 1 | $n$ | $n-1$ | $3 - \frac{24}{n-5}$ |
| $M$ | 1 | $\square$ | $-2$ | $2 - \frac{16}{n-5}$ |

with: $W = Mqsq + \det s + Yqq'$.  

2
The gauge invariant polynomials get mapped as follows (all gauge invariants contracted with a $\delta$ are referred to as mesons whereas baryons are contracted with an $\epsilon$):

Mesons:

$$\begin{align*}
QQ & \leftrightarrow M \\
(SS)_s Q^5_a & \leftrightarrow q^{n-5} \\
(SS)_s Q & \leftrightarrow Y
\end{align*}$$

Baryons:

$$\begin{align*}
i &= 0, 1, 2 \\
Q^{10-2i} a^i W & \leftrightarrow q^{n+2i-10} s^{n-8+i} W^{2-i} q t \\
Q^9 a^{-2i} W^i (SS)_s & \leftrightarrow q^{n+2i-9} s^{n-7+i} W^{2-i}
\end{align*}$$

Even though this duality was established as a symmetry between two theories in an interacting non-abelian Coulomb phase ($N \geq 7$) one can extrapolate to the case $n = 6$. The duality maps to theories of singlets. This is similar to what one does in SUSY QCD for $N_C + 1 < N_F \leq 2N_C$. The electric theory leaves the non-abelian Coulomb phase. One can easily see this from the unitarity bound provided by the superconformal algebra. The R-charge of any operator becomes related to its scaling dimension

$$D \geq \frac{3}{2} |R|$$

the bound being saturated for chiral operators. A violation of this bound signals a breakdown of conformal symmetry. To describe the physics in this regime, one extrapolates the conjectured duality. In SUSY QCD the magnetic theory is free and one hence concludes that the electric theory confines, the composite fields being subject to the new magnetic gauge dynamics. In the present case the magnetic theory is also a theory of singlets. To maintain the notion of exchange of weak and strong coupling one should look at this case as a free magnetic phase with trivial gauge group.

As a nice application Pouliot and Strassler use this duality to reconfirm the known result [12] that Spin(10) with just one spinor dynamically breaks supersymmetry by studying the theory in the presence of mass terms.

3. More on Spin $\leftrightarrow$ SU Dualities

3.1. Derived Dualities

The three theories studied by Pouliot and Strassler [8,9,10], Spin(7) with $m$ spinors, Spin(8) with 1 spinor and $n$ vectors and Spin(10) with 1 spinor and $n$ vectors are connected to a wide variety of other very interesting gauge theories via the Higgs mechanism, including those we set out to solve, the exceptional ones:
The arrows indicate higgsing of the gauge group by giving a vev to a field transforming in the fundamental representations. For the exceptional groups this procedure is not unambiguous: one can give a fundamental a vev in different ways, resulting in different unbroken subgroups. Or put another way, a fundamental field may have more than one flat direction associated with it. Since flat directions are parametrized by gauge invariant polynomials, one can distinguish between the different possibilities by considering which gauge invariant combination of fields gets a vev. The above diagram reflects a vev to the invariant contracted with the symmetric invariants special to the exceptional groups: $d^{\alpha\beta\gamma\delta}$ for $E_7$, $d^{\alpha\beta\gamma}$ for $E_6$, and $F_4$. For $F_4$ there is another possibility by contracting with $\delta^{\alpha\beta}$, the corresponding vev breaking the group only to $Spin(9)$. For $E_6$ there is an additional invariant involving a fundamental and an antifundamental, $\delta^{\alpha}_{\beta}$, and for $E_7$ there is an additional invariant antisymmetric in two fundamentals, $f^{\alpha\beta}$. Their vevs leave unbroken, respectively, a $Spin(10)$ or a $Spin(11)$ subgroup.

In our attempt to generate a dual picture for the above diagram we tried some obvious generalizations of Pouliot’s duality. Those models had some very nice features, like matching
of ‘t Hooft anomalies and of some gauge invariant operators, but in the end they all proved to be inconsistent. The “troublemakers” were in the first place the gauge invariants symmetric in the flavor indices.

To get a better understanding, where the difficulties lie and how to deal with those strange dualities, we studied the continuation of the diagram to smaller Spin groups:

\[
\begin{align*}
\text{Spin}(10) & \quad \downarrow \\
\text{Spin}(9) & \quad \text{exceptional groups} \\
\downarrow & \quad \checkmark \\
\text{Spin}(8) & \\
\downarrow & \\
\text{Spin}(7) & \quad SU \text{ with asym. tensors} \\
\downarrow & \\
\text{Spin}(6) & \quad SP \text{ groups} \\
\downarrow & \quad \checkmark \\
G_2 & \quad \text{Spin}(5) \\
\downarrow & \\
SU(3) & \quad \text{Spin}(4) \quad SU \text{ with adj. tensors} \\
\downarrow & \quad \checkmark \\
SU(2) & \quad \text{Spin}(3)
\end{align*}
\]

Here one is able to actually do calculations by starting with the Spin(10) model instead of trying to guess ones way up! We thus produced duals to the Spin groups listed above with a number of spinors corresponding to 1 spinor of Spin(10). Since these theories are connected to a variety of other interesting models, it would have been nice to generate duals for those small groups with an arbitrary number of spinors, but this proved to be very difficult, too. Our analysis nevertheless lead to a better understanding of the problems of our generalization attempts, thereby uncovering an interesting phenomenon.

In some duals models the full global symmetry is hidden on the fundamental level on the magnetic side

In addition, the self-consistency of the models obtained in this way is further evidence for the correctness of Pouliot’s duals.

First let’s consider going from Spin(10) to a smaller Spin group. To achieve this, we give a vev to \( k \) fundamental fields, breaking the electric group to Spin(10 – \( k \)). Or in terms of the gauge invariant polynomials, the meson \( M^{ij} = Q^i Q^j \) gets a vev of rank \( k \).

The matter content of the electric theory is:

- \( n – k \) vectors (\( n \) was the number of Spin(10) vectors)
• a number of spinors and conjugate spinors corresponding to the decomposition of one Spin(10) spinor

• some singlets left over from the higgsing \((k(n - k) + \frac{k(k+1)}{2})\) in number.

The global rotation symmetry is \(SU(n - k) \times SU(N_S) \times SU(N_C)\), where \(N_S\) and \(N_C\) denote the number of spinors and conjugate spinors.

Next let’s study the effect on the magnetic theory: the meson is a magnetic gauge singlet, so its vev only enters through the superpotential:

\[
W = \cdots + Mqsq + \cdots \Rightarrow \Rightarrow W = \cdots + <M>qsq + Mqsq + \cdots.
\]

Since \(<M>\) only appears with this cubic combination, the effect is simpler than in other examples, like SUSY QCD: whereas a quadratic term would give mass to some fields, this new cubic term leaves the theory almost unchanged. The only effect is that the global \(SU(n)\) symmetry is broken to a \(SU(n - k) \times SO(k)\) subgroup. The gauge group stays unchanged.

The fluctuations of the components of \(M\) that got a vev around their expectation values correspond to the remaining singlets on the electric side and one can get rid of them by writing down mass terms on both sides.

At first sight the resulting dual pairs seem to be inconsistent, since the global symmetries are not the same:

**Global Symmetries** (in addition to \(U(1)\) and \(U(1)_R\))

| gauge gr. | \(N_S + \tilde{N}_S\) | el. gl. symm. | mag. gl. symm. |
|-----------|-----------------|-----------------|-----------------|
| Spin(10)  | 1               | \(SU(n)\)       | \(SU(n)\)       |
| Spin(9)   | 1               | \(SU(n)\)       | \(SU(n)\)       |
| Spin(8)   | 1 + 1           | \(SU(n) \times U(1)\) | \(SU(n) \times SO(2)\) |
| Spin(7)   | 2               | \(SU(n) \times SU(2)\) | \(SU(n) \times SO(3)\) |
| Spin(6)   | 2 + 2           | \(SU(n) \times SU(2) \times SU(2) \times U(1)\) | \(SU(n) \times SO(4)\) |
| Spin(5)   | 4               | \(SU(n) \times SU(4)\) | \(SU(n) \times SO(5)\) |
| Spin(4)   | 4 + 4           | \(SU(n) \times SU(4) \times SU(4) \times U(1)\) | \(SU(n) \times SO(6)\) |
| Spin(3)   | 8               | \(SU(n) \times SU(8)\) | \(SU(n) \times SO(7)\) |

(Here we used \(n\) always as the number of electric fundamentals in the theory under considerations, whereas above it was the number of fundamentals in the Spin(10) we started with).

For small \(k\) some basic group isomorphisms seem to save the day, but starting from \(k = 4\) (Spin(6)), where a \(U(1)\) factor is missing on the magnetic side, the magnetic global symmetry is always only a subgroup of the full electric symmetry.

Since these theories, nonetheless, should be dual, we offer the following explanation: on the magnetic side the full symmetry is not realized linearly on the elementary fields; only part of it is. The missing generators act in a much more complicated way, exchanging fundamental fields and composite fields. Only the effective theory in the far infrared has this symmetry. Let us illustrate this in two examples chosen from the above table. In each case,
we will see how the fundamental fields combine with composite gauge invariant fields to form multiplets of the full global symmetry. The electric gauge invariants are then identified with those multiplets of invariants on the magnetic side.

3.1.1. Spin(5) with 4 Spinors

Consider the following derived dual pair:

|       | Spin(5) | SU(n) | SU(4) | U(1) | U(1)_R |
|-------|---------|-------|-------|------|--------|
| Q     | 5       | n     | 1     | -1   | 1 - \frac{3}{n+2} |
| S     | 4       | 1     | 4     | \frac{n}{2} | 1 - \frac{3}{n+2} |

\[\downarrow\]

|       | SU(n) | SU(n) | SO(5) = Sp(2) | U(1) | U(1)_R |
|-------|-------|-------|---------------|------|--------|
| q     | n     | \bar{n} | 1             | 1   | \frac{3}{n+2} - \frac{1}{n} |
| q'    | n     | 1     | 1             | -n  | -1 + \frac{6}{n+2} + \frac{1}{n} |
| s     | 1     | 1     | 0             | \frac{2}{n} |
| t     | \bar{n} | 1     | 5             | 0   | 1 - \frac{4}{n} |
| Y     | 1     | n     | n - 1         | 3 - \frac{9}{n+2} |
| M     | 1     | \bar{n} | 1             | -2  | 2 - \frac{6}{n+2} |
| N     | 1     | 1     | 5             | n   | 2 - \frac{6}{n+2} |

with: \( W = Mqsq + \det s + Yqqq + Ntqt + tst. \)

We find the following mapping of gauge invariant operators:

**Baryons:**

\[ Q_a^{5-2i}W^i \leftrightarrow q^{n+2i-5}s^{n-3+i}W^{2-i}q' \]
\[ (SS)_a Q_a^{3} \leftrightarrow q^{n-3}t^3 \]
\[ (SS)_a Q_a^{4} \leftrightarrow \{q^{n-4}s^{n-2}W^2 + q^{n-4}t^4\} \]
\[ (SS)_a W^2 \leftrightarrow \{q^n s^n + W^2 N\} \]

\[ 6 = 1 + 5 \]

**Mesons:**

\[ QQ \leftrightarrow M \]
\[ SS_a \leftrightarrow \{N + q^n\} \]

\[ 6 = 1 + 5 \]

\[ (SS)_s Q_a^{2} \leftrightarrow q^{n-2}t^2 \]
\[ (SS)_a Q \leftrightarrow \{Y + tq^{n-1}\} \]

\[ 6 = 1 + 5 \]

The operators transforming under the asymmetric tensor representation (the 6) of the global \( SU(4) \) decompose as a 5 + 1 under the \( Sp(2) \) symmetry that is visible on the fundamental fields on the magnetic side. Among the mesons, the 1s are fundamental fields in the lagrangian, whereas the 5s are composite fields. Among the baryons, both the 1s and 5s are composite, but they have a quite different structure.
Another interesting thing is to observe how the operators transforming like a symmetric
tensor (as a 10) under \( SU(4) \) are realized on the magnetic side (for example \((SS)_aQ^2\)). As
mentioned before, these were the most problematic in trying to generalize to more interesting
dual pairs. Here the system avoids the trouble in a rather tricky way: while on the electric
side the 10 is a symmetric combination of two 4s, on the magnetic side it is an antisymmetric
combination of two 5s, which can easily be realized in terms of a magnetic baryon.

3.1.2. Spin(3) with 8 Spinors

To see that the above construction is not special to the \( Spin(5) \) case we’ll demonstrate that
it also works in this derived duality. Again, the full \( SU(8) \) multiplet corresponding to a given
electric invariant is realized through combining elementary fields and composite invariants
transforming under an \( SO(7) \) subgroup.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
& Spin(3) & SU(n) & SU(8) & U(1) & U(1)_R \\
\hline
Q & 3 & n & 1 & -1 & 1 - \frac{1}{n+2} \\
S & 2 & 1 & 8 & \frac{n}{2} & 1 - \frac{1}{n+2} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
& SU(n+2) & SU(n) & SO(7) & U(1) & U(1)_R \\
\hline
q & n+2 & \tilde{n} & 1 & 1 & 0 \\
qt & n+2 & 1 & 1 & -n & -1 + \frac{3}{n+2} \\
s & 1 & 1 & 0 & \frac{2}{n+2} \\
t & n+2 & 1 & 7 & 0 & 1 - \frac{1}{n+2} \\
Y & 1 & n & 1 & n-1 & 3 - \frac{3}{n+2} \\
M & 1 & \blacksquare & 1 & -2 & 2 - \frac{3}{n+2} \\
N & 1 & 1 & 7 & n & 2 - \frac{2}{n+2} \\
\hline
\end{array}
\]

with: \( W = Mqsq + \det s + Yqqt + Ntq + tst \).

Mesons:

\[
\begin{align*}
QQ & \leftrightarrow M \\
SS_a & \leftrightarrow \{N + q^at^2\} \\
28 & = 7 + 21 \\
(SS)_sQ & \leftrightarrow Y + q^{n-1}t^3 \\
36 & = 1 + 35
\end{align*}
\]

Again the symmetric combination, the 36, on the electric side is matched with an anti-
symmetric combination of three 7s on the magnetic side.

In the same way, the other examples can be resolved. The obvious part of the magnetic
group for \( Spin(4) \) and \( Spin(6) \) turns out to be the diagonal subgroup of the full symmetry.

This type of construction will be encountered over and over again in what follows. But
already here we get a taste of what it means: at no level of our analysis is it possible to
generalize the obtained theory to a higher number of spinors. Each of the subtle matchings
depends crucially on the number of flavors. The theories get more and more complicated the further we move away from our starting point. This behaviour is different from that encountered in SUSY QCD and the other well known dualities, where perturbations drive us from one dual in a series of equally fundamental ones to another one from the same series. The duality we started with is a rather special case, not a part in a chain. Going along flat directions produces new dual pairs because it has to, but these still reflect the structure of the original dual, leading to complicated matter content and hidden global symmetries. Duals for similar groups, like higher Spin groups, Spin with more matter or exceptional groups have, in general, a similarly complicated structure and it seems to be more a coincidence that in some special cases the magnetic theory simplifies to the examples from above. Still, they can be regarded as a hint of how complicated this structure may look.

3.2. Two Simple Duals

In a similar way we can derive a dual for SU(3) with n flavors in terms of an SU gauge group with a symmetric tensor by following the G2 branch in the above diagram obtained by giving a vev to the spinor of Spin(7). It is interesting since it is an example of a theory having two dual descriptions in terms of a simple gauge group. A number of theories with more than one dual are known [18,17], but usually at least one of the possible dual descriptions involves a product gauge group.

The electric theory is the well-known SUSY QCD with $N_C = 3$, the first dual is the original one constructed by Seiberg, the second dual is the one derived from Pouliot’s theory via G2. This derivation involves the same calculations as in the previous examples. Starting from the dual of [8] (Spin(7) with $n$ spinors dual to $SU(n - 4)$ with a symmetric tensor and $n$ antifundamentals) one gives vev to a spinor of Spin(7) and then to a fundamental of the resulting $G_2$. The corresponding magnetic fields are fundamental gauge singlets multiplying cubic superpotential terms. Only the global symmetry is broken, the gauge group and matter content of the magnetic theory stay the same.

| $SU(3)$ | $SU(n)_L$ | $SU(n)_R$ | $U(1)$ | $U(1)_R$ |
|---------|-----------|-----------|--------|----------|
| 3       | $n$       | 1         | 1      | $1 - \frac{2}{n}$ |
| 3       | 1         | $n$       | $-1$   | $1 - \frac{2}{n}$ |

Seiberg:

| $SU(n - 3)$ | $SU(n)_L$ | $SU(n)_R$ | $U(1)$ | $U(1)_R$ |
|-------------|-----------|-----------|--------|----------|
| $q$         | $n - 3$   | $\bar{n}$ | $1$    | $-1 + \frac{3}{n}$ |
| $\bar{q}$   | $\bar{n} - 3$ | $\bar{n}$ | $1 - \frac{3}{n}$ | $\frac{3}{n}$ |
| $M$         | 1         | $n$       | $n$    | 0        | $2 - \frac{6}{n}$ |

with: $W = Mq\bar{q}$.

Derived from Pouliot:
Again, the second dual has only parts of the global symmetry visible on the fundamental fields. As in the examples we considered before different fundamental gauge singlets combine with composite fields into multiplets of the full global symmetry. We get the following mapping of gauge invariant operators in the two dual descriptions:

Note that while in the original Seiberg dual all mesons become fundamental fields which are singlets under the magnetic gauge group, in the alternative dual description, the meson multiplet in magnetic theory consists partly of fundamental and partly of composite fields.

4. New Dualities

Recently a dual description for $E_6$ with 6 fundamentals has been found [3]. The author shows that this theory is actually self-dual. Self-dual models deserve special attention since it was argued in [11] that they are closely related to the existence of exactly marginal operators and hence to the existence of fixed lines rather than fixed points. One can get a condition for this to occur by studying $\beta$ functions. The requirements that the $\beta$ functions for all the couplings involved vanish, have to be linear dependent to allow a line of solutions. With the analysis of [11] it is easy to see that this is associated with an exactly marginal operator quadratic in the mesons appearing in the superpotential. If we are dealing with only one type of field and have only one candidate operator to become a fundamental magnetic gauge singlet field, this happens if the R-charge of the this meson becomes 1. For $E_6$ this candidate operator is the 3-index symmetric composite field and its R-charge becomes 1 for $N_F = 6$.

Similarly, we will argue that $E_7$ with 4 flavors is self-dual. By following flat directions of those two theories and perturbing them with mass terms we will construct some more new duals. As evidence that we really constructed new dual theories, we take that the ’t Hooft anomaly matching conditions are satisfied and the fact that we can consistently flow from one to the other. The fact that, perturbing a conjectured dual pair, we obtain a self-consistent new dual pair is a highly non-trivial check of the validity of the dual we started with.
It is much more difficult to check that the gauge invariant operators match for the exceptional groups than for the classical groups. Even though the basic invariant tensors are well known, one can build higher invariant tensors out of the fundamental ones. There exist several relations between contracted products of invariant tensors. The real task is to find the independent ones. This problem is not yet solved for the exceptional groups other than $G_2$.

Probably it is instructive to illustrate this problem with the solved $G_2$ example, following the lines of [7]. $G_2$ has the symmetric invariant $\delta^{\alpha\beta}$, the 7-index totally antisymmetric $\epsilon$ tensor and an 3-index antisymmetric $f^{\alpha\beta\gamma}$. Several relations between contracted products of those can be deduced from Fierz identities of $Spin(7)$, since $G_2$ is a subgroup of $Spin(7)$, for example

$$f^{\alpha\beta\gamma} f^{\alpha\delta\epsilon} + f^{\alpha\delta\gamma} f^{\alpha\beta\epsilon} = 2\delta^{\beta\delta}\delta^{\gamma\epsilon} - \delta^{\gamma\delta}\delta^{\beta\epsilon} - \delta^{\beta\gamma}\delta^{\delta\epsilon}$$

Only very few relations like this are known for the larger exceptional groups. For $G_2$ the full set of those relations tells us that the only independent higher order invariant is

$$f^{\alpha\beta\gamma\delta} = \epsilon^{\alpha\beta\gamma\delta\epsilon\nu\rho} f_{\epsilon\nu\rho}.$$

The full set of gauge invariant operators for $G_2$ is hence a 2-index symmetric meson and 3-,4- and 7-index totally antisymmetric higher composites.

4.1. $E_6$ with 6 Flavors

In [3] Ramond found that $E_6$ with 6 flavors is self-dual. The electric and magnetic fields transform under the global $SU(6) \times U(1)_R$ symmetry as follows:

|       | $E_6$ | $SU(6)$ | $U(1)_R$ |
|-------|-------|---------|-----------|
| $Q$   | 27    | 6       | $\frac{1}{3}$ |

|       | $E_6$ | $SU(6)$ | $U(1)_R$ |
|-------|-------|---------|-----------|
| $g$   | 27    | 6       | $\frac{1}{3}$ |
| $\bar{Z}$ | 1     | 1       | 1         |

with: $W = Zq^3$.

As evidence, he showed that the ’t Hooft anomaly matching conditions are satisfied and constructed a matching of some of the gauge invariant operators (as mentioned above, finding all independent gauge invariant operators of $E_6$ is still an unsolved problem). The three flavor symmetric invariant becomes a fundamental field in the magnetic theory. The $N_F = 6$ case is exactly the case, where this field gets R-charge 1. Since this is the only invariant appearing as a fundamental gauge singlet in the magnetic theory, this indicates a possible self-duality, as mentioned before.

Ramond found that there exist at least one independent higher invariant, the sixth order composite invariant transforming like

$$(27, \Box)^6 \sim (1, \bar{\Box}).$$
This is mapped to the corresponding magnetic invariant, the flavor indices raised with 2 $\epsilon$ tensors. The latter construction is only possible with 6 flavors.

4.2. $E_7$ with 4 Flavors

As in the $E_6$ case from [3], the dual theory of $E_7$ with 4 flavors is again $E_7$ with 4 flavors. The theory has a global $SU(4) \times U(1)_R$ symmetry, the electric fields transforming as

| $E_7$ | $SU(4)$ | $U(1)_R$ |
|-------|---------|---------|
| Q     | 56      | $\frac{1}{4}$ |

and the magnetic fields as

| $E_7$ | $SU(4)$ | $U(1)_R$ |
|-------|---------|---------|
| q     | 56      | $\frac{1}{4}$ |
| M     | 1       | $\frac{1}{4}$ |

In addition the magnetic theory has a superpotential $W = M q^4$.

The global symmetries satisfy the ’t Hooft anomaly matching conditions. $E_7$ has as invariant tensors an 2-index antisymmetric tensor $f^{\alpha\beta}$ and a 4-index symmetric invariant $d^{\alpha\beta\gamma\delta}$ [7]. While the gauge invariant polynomial associated with $d^{\alpha\beta\gamma\delta}$ gets mapped to the gauge singlet $M$ in the magnetic theory, the antisymmetric invariant gets mapped to the corresponding invariant of the magnetic theory:

$$Q^i_\alpha Q^j_\beta f^{\alpha\beta} \leftrightarrow \epsilon^{ijkl} q_{k,\alpha} q_{l,\beta} f^{\alpha\beta}$$

(with $i, j, k, l = 1, \ldots, 4$ being flavor indices). $M$ is again the only operator mapped to an elementary gauge singlet, and has R-charge 1, which we already met as an indication of self-duality.

For some of the following duals to be consistent there must exist a 6th order invariant transforming like:

$$(56, \square)^6 \sim (1, \Box)$$

Under the duality, this would be mapped to the corresponding magnetic invariant, the flavor indices this time raised with 3 $\epsilon$ tensors.

This matching again is rather special to the 4 flavor case. As with all of the exceptional group duals, one cannot dial the number of colors in the magnetic theory to compensate for changing the number of flavors in the electric theory.
4.3. Derived Dualities

4.3.1. Going Along Flat Directions

An interesting thing to do is to go along the flat directions of this $E_7$ theory and the $E_6$ theory of [3] and study the resulting dual pairs.

As a first step we study the flat directions associated with those gauge invariant polynomials appearing as fundamental fields on the magnetic side. (the 4-index symmetric tensor $M$ in $E_7$ and the 3-index symmetric tensor $Z$ in $E_6$)

The electric theory gets higgsed to:

$E_7 \rightarrow E_6 \rightarrow F_4 \rightarrow Spin(8) \rightarrow \ldots$

and:

$E_6 \rightarrow F_4 \rightarrow Spin(8) \rightarrow \ldots$

and some additional singlet fields.

On the magnetic side the field that gets a vev is a fundamental gauge singlet. It only affects the dynamics through the superpotential. It acts in a very simple way: the gauge group stays unchanged, no fields become massive, only the global symmetry gets broken to the subgroup leaving $< M > q^4$ ($< Z > q^3$) in the $E_7$ ($E_6$) theory invariant. This is exactly the same mechanism that was at work in the theories studied in Section 3.

After giving mass to the singlet fields that remain after higgsing, one gets the following dual pairs:

$F_4$ with 5 fund. and $E_6$:

This dual is obtained by giving a vev to the 3-index invariant of $E_6$ ($Z$), breaking the electric theory to $F_4$: 

\[
\begin{array}{|c|c|c|c|}
\hline
& F_4 & SU(5) & U(1)_R \\
\hline
Q & 26 & 5 & \frac{1}{5} \\
\hline
\end{array}
\]

\[
\downarrow
\]

\[
\begin{array}{|c|c|c|c|}
\hline
& E_6 & SU(5) & U(1)_R \\
\hline
q & 27 & 5 & \frac{1}{5} \\
\hline
t & 27 & 1 & \frac{1}{2} \\
\hline
Z & 1 & & \frac{6}{5} \\
\hline
X & 1 & & \frac{6}{5} \\
\hline
\end{array}
\]

with: $W = t^3 + X t q^2 + Z q^3$.

The magnetic gauge singlets $Z$ and $X$ correspond to the electric gauge singlets built with the gauge invariant tensors $d^{\alpha \beta \gamma}$ and $\delta^{\alpha \beta}$. In addition, there are 6th order magnetic composite invariants $q^6$, $t q^5$, $t^2 q^4$ and the remaining 3rd order $t^2 q$.

After raising flavor indices, these should correspond to electric singlets transforming like:

\[
\left( \begin{array}{llll}
\vdots & \vdots & \vdots & \vdots \\
\end{array} \right)
\]

We are hence led to conjecture, that these correspond to higher order invariants of $F_4$. 

13
Spin(8) with 4/4/4 and $E_6$:

From the above dual we can go down another step to Spin(8) by giving a vev to a component of the 3-index invariant of $F_4$ (Z):

|         | $Spin(8)$ | $SU(4)_V$ | $SU(4)_S$ | $SU(4)_C$ | $U(1)_R$ |
|---------|-----------|------------|------------|------------|-----------|
| $Q$     | 8$_V$     | 4          | 1          | 1          | 1         |
| $S$     | 8$_S$     | 1          | 4          | 1          | 1         |
| $C$     | 8$_C$     | 1          | 1          | 4          | 1         |

\[
\uparrow
\]

|         | $E_6$ | $SU(4)_{Diag.}$ | $U(1)_R$ |
|---------|-------|-----------------|----------|
| $q$     | 27    | 4               | 1/6      |
| $t$     | 27    | 1               | 1/3      |
| $v$     | 27    | 1               | 1/3      |
| $Z$     | 1     | 1               | 1/2      |
| $X$     | 1     | 1               | 1        |
| $Y$     | 1     | 1               | 1        |

with: $W = t^3 + v^3 + Xtq^2 + Yvq^2 + Zq^3$.

Here only the diagonal subgroup of the global symmetry is visible on the fundamental fields. The gauge singlets combine with composite fields into multiplets of the full symmetry.

| el. invariant | $SU(4)^3$ | mag. invariant | $SU(4)_{Diag.}$ |
|---------------|-----------|---------------|-----------------|
| $QSC$         | $\times \times \times \times$ | $Z + tq^5 + vq^5 + tvq$ | $\mathbb{1} + 2 \cdot 1$ |
| $QQ$          | $\times \times 1 \times 1$ | $X$           | $\mathbb{1}$ |
| $SS$          | $1 \times \times \times \times$ | $Y$           | $\mathbb{1}$ |
| $CC$          | $1 \times \times \times \times$ | $q^6$         | $\mathbb{1}$ |

$E_6$ with 3/3 and $E_7$

This dual is obtained by starting with the $E_7$ theory from above and giving a vev to the 4-index invariant (M):

|         | $E_6$ | $SU(3)_L$ | $SU(3)_R$ | $U(1)_R$ |
|---------|-------|-----------|-----------|----------|
| $Q$     | 27    | 3         | 1         | 1/6      |
| $Q$     | 27    | 1         | 3         | 1/6      |

\[
\uparrow
\]

|         | $E_7$ | $SU(3)_{Diag.}$ | $U(1)_R$ |
|---------|-------|-----------------|----------|
| $q$     | 56    | 3               | 1/6      |
| $q$     | 56    | 1               | 1/6      |
| $M$     | 1     | $\mathbb{1}$   | 1/3      |
| $Z$     | 1     | $\mathbb{1}$   | 1        |
| $X$     | 1     | $\mathbb{1}$   | 1/3      |
| $Y$     | 1     | $\mathbb{1}$   | 1/3      |

with: $W = Mq^4 + Zq^3t + Xq^2t^2 + Yq^2$.
Again only the diagonal subgroup of the flavor symmetry is manifest in the Lagrangian.

| el. invariant | $SU(3) \times SU(3)$ | mag. invariant | $SU(3)_{Diag.}$ |
|---------------|-----------------------|---------------|-----------------|
| $QQ$          | $\square \times \square$ | $X + tq$      | $\square + \square$ |
| $Q^2\bar{Q}^2$ | $\square \times \square$ | $M + \cdots$ | $\square + \cdots$ |
| $Q^3$         | $\square \times 1$    | $Z$           | $\square$        |
| $\bar{Q}^3$  | $1 \times \square$    | $q^3t^3$      | $\square$        |

In addition the magnetic invariants $Y$ and $qt^3$ seem to correspond to an 5th order composite $E_6$ invariant involving fundamentals and antifundamentals, totally antisymmetric under both flavor symmetries.

In a similar fashion, one should get an $E_6$ dual of $Spin(10)$ and $E_7$ duals of $F_4$ and $Spin(8)$.

### 4.3.2. Mass Perturbations

Even though the two duals we started with do not allow mass terms, most of the new pairs we derived in the previous subsection do. While this time the effect on the electric side is really simple (the mass term just removes one field from the low energy effective action), the magnetic case gets rather complicated. The analysis for the $F_4$ case is sketched in the Appendix.

### 4.4. Summary and Interpretation

In this section we started with two self-dual models: $E_6$ with 6 flavors found in [3] and $E_7$ with 4 flavors. From the first one we got dual descriptions for $F_4$ with 5 flavors and $Spin(8)$ with 4 vectors, 4 spinors and 4 conjugate spinors in terms of a magnetic $E_6$ gauge group with 6 flavors via the Higgs mechanism. From the second one we got a dual for $E_6$ with 3 fundamentals and 3 antifundamentals in terms of a $E_7$ gauge theory with 4 flavors. Finally we studied a mass term in the $F_4$ theory with 5 flavors and obtained a dual for $F_4$ with 4 flavors with a magnetic $Spin(9)$ group.

We consider the fact that the anomaly matching conditions are satisfied, the gauge invariant operators match and that perturbations drive us to new consistent theories as enough evidence, that these conjectured duals are in fact correct. Nevertheless we encounter the same features as in our discussion of the dual pairs derived from $Spin(10)$: the duals become more and more complicated the further we go from our starting pair, all the constructions depend crucially on the number of flavors, only a subgroup of the global symmetry group is classically-realized in the magnetic theory. It is far from obvious how to generalize to higher number of flavors.

### 5. Conclusions

We gave a detailed discussion about some new duals derived from known dual pairs. We found that often only a subgroup of the global symmetry is linearly-realized on the fundamental fields of the magnetic theory. The fundamental fields combine with composite fields
into multiplets of the full symmetry. We studied self-dual models and applied the method of [11] to find candidates for self-dual models. With this we presented some duals for very special matter content of $E_6$ and $E_7$ theories, some of which are new. In the same spirit one would expect $E_8$ with 2 adjoints to be self-dual, but we haven’t checked this conjecture yet.

While we were finishing this work we received the very interesting paper [21] where the authors also found these accidental symmetries. Our results overlap with their work.

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Appendix: Mass Terms for $F_4$

We start from the $F_4$ theory with 5 fundamentals. With the additional mass term for a fundamental the magnetic superpotential becomes

$$W = t^3 + X t q^2 + Z q^3 + m X.$$  

The equation of motion for $X$ reads:

$$< t q^2 > + m = 0.$$  

Hence the operator $t q^2$ gets a vev, breaking the magnetic $E_6$ to $Spin(9)$. After integrating out the heavy fields we arrive at the following dual pair:

|  | $F_4$ | $SU(4)$ | $U(1)_R$ |
|---|---|---|---|
| $Q$ | 26 | 4 | $\frac{1}{2}$ |


|  | $Spin(9)$ | $SU(4)$ | $U(1)_R$ |
|---|---|---|---|
| $s$ | 16 | 4 | $\frac{1}{2}$ |
| $q$ | 9 | 4 | $\frac{3}{2}$ |
| $Z$ | 1 | 4 | $\frac{3}{2}$ |
| $X$ | 1 | 4 | $\frac{1}{2}$ |
| $N$ | 1 | 4 | $\frac{1}{2}$ |

with: $W = Z s^2 q + N s^2 + M q^2 + q^2 s^2$.

The magnetic gauge singlets $X$ and $Z$ are again easily identified as the symmetric electric invariants built out of three and two fields. The appearance of $N$ is somewhat mysterious. It has the right R-charge to correspond to an electric gauge invariant built out of 6 fields, but its transformation under flavor rotations seems to be rather strange. But since the exact form of the higher invariants of $F_4$ is not known we can not rule out this possibility.

Applying the same procedure once again one can arrive at a duality between $F_4$ with 3 flavors and $G_2$ with 4 flavors. This can be easily retraced to a duality between $E_6$ and $G_2$, both with 4 flavors. The latter is known to be confining [7]. As we are unable to recover the exact magnetic superpotential, we cannot extract any useful information. But it is nice to see that again the ’t Hooft matching conditions are satisfied and the gauge invariant operators seem to work out, too.

Similarly one should be able to obtain a $Spin$ dual for $E_6$ with two fundamentals and two antifundamentals by adding a mass term.