A CONE THEOREM FOR NEF CURVES

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Abstract. Following ideas of V. Batyrev, we prove an analogue of the Cone Theorem for the closed cone of nef curves: an enlargement of the cone of nef curves is the closure of the sum of a $K_X$-non-negative portion and countably many $K_X$-negative coextremal rays. An example shows that this enlargement is necessary. We also prove an analogue of the Contraction Theorem.

1. Introduction

The recent foundational work on the minimal model program in [BCHM06] and [Siu06] has resolved many important questions in birational geometry. However, the close relationship between the cone of curves, Zariski decompositions, and minimal model theory is not yet fully understood. In this paper we will analyze the cone of nef curves – that is, the set of curve classes that have non-negative intersection with every effective divisor.

The most important result concerning the cone of nef curves was formulated in [BDPP04]. Recall that an irreducible curve $C$ on a variety $X$ is called movable if it is a member of a family of curves that dominates $X$. In [BDPP04] it is shown that the cone of nef curves is the closure of the cone generated by classes of movable curves. However, one might hope to obtain more specific results for the $K_X$-negative portion of the cone of nef curves. Using the results of [BCHM06] we will prove such a result, closely mirroring the Cone Theorem for effective curves.

For a projective variety $X$, let $\overline{NE}_1(X)$ denote the closed cone of effective curves of $X$ and let $\overline{NM}_1(X)$ denote the closed cone of nef curves. Our goal is to prove the following (slightly weakened) version of a conjecture of V. Batyrev:

Theorem 1.1. Let $(X, \Delta)$ be a dlt pair. There are countably many $(K_X + \Delta)$-negative movable curves $C_i$ such that

$$\overline{NE}_1(X)_{K_X + \Delta \geq 0} + \overline{NM}_1(X) = \overline{NE}_1(X)_{K_X + \Delta \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i].$$

The rays $\mathbb{R}_{\geq 0}[C_i]$ only accumulate along hyperplanes that support both $\overline{NM}_1(X)$ and $\overline{NE}_1(X)_{K_X + \Delta \geq 0}$.

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This theorem should be seen as the analogue for $\overline{\text{NM}}_1(X)$ of the Cone Theorem for effective curves (proved by Kawamata, Kollár, Mori, Reid, Shokurov, and others – see [KM98]):

**Theorem 1.2** (Cone Theorem). Let $(X, \Delta)$ be a dlt pair. There are countably many $(K_X + \Delta)$-negative rational curves $C_i$ such that

$$\overline{\text{NE}}_1(X) = \overline{\text{NE}}_1(X)_{K_X + \Delta \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i].$$

The rays $\mathbb{R}_{\geq 0}[C_i]$ only accumulate along $\overline{\text{NE}}_1(X)_{K_X + \Delta = 0}$.

There are several important differences between the two statements. First of all, Theorem 1.1 includes the term $\overline{\text{NE}}_1(X)_{K_X + \Delta \geq 0}$ where we might expect $\overline{\text{NM}}_1(X)_{K_X + \Delta \geq 0}$ by analogy with the Cone Theorem. This enlargement is necessary: in Section 6 we construct a threefold for which

$$\overline{\text{NM}}_1(X) \neq \overline{\text{NM}}_1(X)_{K_X + \Delta \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$$

for any locally discrete countable collection of curves. This example was previously used in [Cut86] to find a divisor with no rational Zariski decomposition.

Second, Theorem 1.1 is less specific about the possible accumulation points of the rays $\mathbb{R}_{\geq 0}[C_i]$. Batyrev conjectured a stronger statement: just as in the Cone Theorem, the rays of Theorem 1.1 should not accumulate away from $\overline{\text{NE}}_1(X)_{K_X + \Delta = 0}$. This stronger version is probably true; in fact, an argument of [Ara08] derives it from the Borisov-Alexeev-Borisov conjecture concerning boundedness of Fano varieties. If we assume termination of flips, we can prove Batyrev’s conjecture when $\Delta$ is big. We will return to this question in Section 6.

Finally, we do not know whether the movable curves in Theorem 1.1 can be chosen to be rational. Again, this would follow from standard (and difficult) conjectures about Fano varieties. If we are willing to replace the $C_i$ by classes $\alpha_i$ that do not necessarily represent curves, we may choose the $\alpha_i$ to be formal numerical pullbacks of rational curves on birational models; see Remark 3.4.

We also prove an analogue of the Contraction Theorem for $(K_X + \Delta)$-negative extremal faces of $\overline{\text{NE}}_1(X)_{K_X + \Delta \geq 0} + \overline{\text{NM}}_1(X)$. The statement includes a weak condition on the face to ensure that it avoids the accumulation points of coextremal rays.

**Theorem 1.3.** Let $(X, \Delta)$ be a dlt pair. Suppose that $F$ is a $(K_X + \Delta)$-negative extremal face of $\overline{\text{NE}}_1(X)_{K_X + \Delta \geq 0} + \overline{\text{NM}}_1(X)$. Suppose furthermore that there is some pseudo-effective divisor class $\beta$ such that $\beta^+ \subset F$ but avoids $\overline{\text{NE}}_1(X)_{K_X + \Delta \geq 0}$. Then there is a birational morphism $\psi : W \to X$ and a contraction $h : W \to Z$ such that:

1. Every movable curve $C$ on $W$ with $[\psi_*C] \in F$ is contracted by $h$. 

(2) For a general pair of points in a general fiber of \( h \), there is a movable curve \( C \) through the two points with \( [\psi_* C] \in F \).

These properties determine the pair \((W,h)\) up to birational equivalence. In fact the map we construct satisfies a stronger property:

(3) There is an open set \( U \subset W \) such that the complement of \( U \) has codimension 2 in a general fiber of \( h \) and a complete curve \( C \) in \( U \) is contracted by \( h \) iff \( [\psi_* C] \in F \).

A fundamental step toward Theorem 1.1 was taken in [Bat92], where Batyrev proves this statement for threefolds. In [Ara05] and [Ara08], Araujo fixes an error in his proof and gives a very clear framework for the general case. She also shows that “most” coextremal rays can be analyzed by running the minimal model program with scaling. Work of a similar flavor has been done in [Xie05] and [Bar07]. The case where \((X,\Delta)\) is log Fano of arbitrary dimension was settled in [BCHM06]. Starting with a similar setup, we prove Theorem 1.1 by applying the results of [BCHM06]. We will prove it in an essentially equivalent form:

**Theorem 1.4.** Let \((X,\Delta)\) be a dlt pair, and let \( V \) be a closed convex cone containing \( \text{NE}_1(X)_{K_X+\Delta=0} - \{0\} \) in its interior. There are finitely many movable curves \( C_i \) such that

\[
\text{NE}_1(X)_{K_X+\Delta \geq 0} + V + \text{NE}_1(X)_{K_X+\Delta \geq 0} = \text{NE}_1(X)_{K_X+\Delta \geq 0} + V + \sum_{i=1}^N \mathbb{R}_{\geq 0}[C_i].
\]

The general strategy of the proof is as follows. Let \( D \) be a divisor such that \( D^\perp \) supports \( \text{NE}_1(X)_{K_X+\Delta \geq 0} + V + \text{NE}_1(X) \) along the \( K_X \)-negative part of the cone. Since \( D \) is positive on \( \text{NE}_1(X)_{K_X+\Delta \geq 0} \), after rescaling we can write \( D = K_X + \Delta + A \) for some ample divisor \( A \). By the results of [BCHM06], we can run the minimal model program with scaling for \( D \). After a number of divisorial contractions and flips, we obtain a birational contraction \( \phi : X \to X' \) on which \( K_{X'} + \phi_* (\Delta + A) \) is nef. Furthermore, \( X' \) has a Mori fiber space structure \( g : X' \to Z \), where \( K_{X'} + \phi_* (\Delta + A) \) vanishes on every curve contracted by \( g \). Choose a movable curve \( C \) on \( X' \) contracted by \( g \) and sufficiently general. Then the curve \( \phi^{-1}(C) \) lies on the boundary of \( \text{NE}_1(X) \) along \( D^\perp \).

The main point is to show that finitely many curves will suffice. In fact, all we need to know is that we obtain only finitely many different Mori fiber spaces as we vary \( D \). The theorems of [BCHM06] give us the needed finiteness so long as we work within a finite-dimensional space of real Weil divisors. In order to apply this theorem, we work in a small neighborhood of \( D \) first, and then extend to a global result using compactness.

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2. Preliminaries

We first fix notation. Suppose $X$ is a normal projective variety. I will let $N^1(X) = NS(X) \otimes \mathbb{R}$ denote the Néron-Severi space of $\mathbb{R}$-Cartier divisors up to numerical equivalence, and $N_1(X)$ the dual space of curves up to numerical equivalence. As we run the minimal model program we will need to be careful to distinguish between divisors (or curves) and their numerical equivalence classes. If $D$ is a Cartier divisor, I will denote the numerical class of $D$ by $[D] \in N^1(X)$, and similarly for curves.

I have already introduced the cone of effective curves $NE_1(X)$ – that is, the closure of the cone generated by classes of irreducible curves. I will use $NE_1(X)$ to denote the cone of pseudo-effective divisors. We define the cone of nef curves $NM_1(X) \subset N_1(X)$ as the cone dual to $NE_1(X)$ under the intersection product. Finally, given a cone $\sigma$ and an element $K$ of the dual vector space, $\sigma K \geq 0$ denotes the intersection of $\sigma$ with the closed half-space on which $K$ is non-negative, and $\sigma K = 0$ denotes the intersection $\sigma \cap K^\perp$.

The standard definitions and theorems of the minimal model program may be found in [KM98]. However, in our definition of a pair $(X, \Delta)$ we allow $K_X + \Delta$ to be $\mathbb{R}$-Cartier, not just $\mathbb{Q}$-Cartier. This distinction makes little difference in the proofs.

We will also use one other non-standard piece of terminology. Suppose that $(X, \Delta)$ is a $\mathbb{Q}$-factorial klt pair, and $\phi : X \to X'$ is a composition of $K_X + \Delta$ flips and divisorial contractions. We let $\phi_i : X \to X_i$ denote the result after $i$ steps, and $R_i$ the extremal ray on $X_i$ that defines the $(i + 1)$ step. Given a divisor $D$ on $X$, we say that $\phi$ is non-negative with respect to $D$ if $\phi_i^* D \cdot R_i \leq 0$ for every $i$.

We will use the following lemma many times, often without explicit mention:

**Lemma 2.1** ([Laz04], Example 9.2.29). Let $(X, \Delta)$ be a klt pair, and suppose that $H$ is an ample $\mathbb{R}$-Cartier divisor. There is some effective $H'$ that is $\mathbb{R}$-linearly equivalent to $H$ such that $(X, \Delta + H')$ is klt.

Suppose that an irreducible curve $C \subset X$ is a member of a family of curves dominating $X$. I will say that $C$ is a movable curve. Every movable curve $C$ is nef, but not conversely. Recall the following fundamental result of [BDPP04]. (Although the proof in [BDPP04] is only for smooth varieties, the theorem holds for singular varieties as well; see [Laz04].)

**Theorem 2.2** ([BDPP04], Theorem 1.5). Let $X$ be an irreducible normal projective variety. The nef cone of curves of $X$ is the closure of the cone generated by classes of movable curves.

We apply this in the form of the following lemma, which we will use to analyze non-$\mathbb{Q}$-factorial varieties.

**Lemma 2.3.** Let $\pi : Y \to X$ be a birational morphism of normal projective varieties. Then $\pi_*NE_1(Y) = NE_1(X)$ and $\pi_*NM_1(Y) = NM_1(X)$. 

Proof. Note that $\pi$ must be surjective. The pushforward of the class of a curve on $Y$ is of course in $\text{NE}_1(X)$. Conversely, every curve on $X$ is the image of some curve on $Y$. This gives the first equality. Similarly, since $\pi_*$ on curves is dual to $\pi^*$ on Cartier divisors, the pushforward of a nef curve on $Y$ is nef on $X$. Conversely, suppose that $C$ is movable on $X$, so that it is a member of a dominant family of curves. Composing with $\pi^{-1}$, we see that the strict transform of $C$ is movable on $Y$. By Theorem 2.2 this gives the second equality. □

3. Running the Minimal Model Program

As explained in the introduction, we will use the minimal model program to show that certain rays in $N_1(X)$ are generated by movable curves. In this section we will extract the application of the minimal model program in the form of Proposition 3.3.

Lemma 3.1. Suppose that $(X, \Delta)$ is a klt pair, and that $\{H_j\}_{j=1}^m$ is a finite collection of ample divisors. Consider the set of divisors

$$\mathcal{H} = \left\{ \sum_{j=1}^m c_j H_j \left| 0 \leq c_j \leq 1 \forall j \right. \right\}.$$

There are finitely many ample divisors $\{W_j\}_{j=1}^m$ such that every element of $\mathcal{H}$ is $\mathbb{R}$-linearly equivalent to a linear combination $\sum_{j=1}^m a_j W_j$ and

$$\left( X, \Delta + \sum_{j=1}^m a_j W_j \right)$$

is a klt pair.

Proof. If $(X, \Delta)$ is a klt pair and $H$ is an ample divisor, then Lemma 2.1 guarantees the existence of an effective divisor $W$, $\mathbb{R}$-linearly equivalent to $H$, such that $(X, \Delta + W)$ is a klt pair. Using this fact inductively, we find effective $W_j$, $\mathbb{R}$-linearly equivalent to $H_j$, such that $(X, \Delta + \sum W_j)$ is klt. Finally, recall that if $(X, \Delta + D)$ is any klt pair, then so is $(X, \Delta + D')$ for every effective $D' \leq D$. Thus any sum of the $W_j$ with smaller coefficients still yields a klt pair. □

In order to limit ourselves to finitely many curves in Theorem 1.3, we need to use finiteness of ample models as proved in [BCHM06]. The particular result we need is the following:

Theorem 3.2 ([BCHM06], Corollaries 1.1.5 and 1.3.2). Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial klt pair, and let $A$ be an ample divisor on $X$. Suppose $V$ is a finite dimensional subspace of the space of real Weil divisors. Define

$$\mathcal{E}_A = \{ \Gamma \in V \mid \Gamma \geq 0 \text{ and } K_X + \Delta + A + \Gamma \text{ is klt and pseudo-effective } \}.$$
For every $\Gamma \in E_A$, we can run the minimal model program for $D := K_X + \Delta + A + \Gamma$. If $D$ lies on the boundary of the pseudo-effective cone, this will result in a birational contraction $\phi_i : X \to X_i$ with a Mori fiber space structure $g_i : X_i \to Z_i$, where $K_{X_i} + \phi_{i*}(\Delta + A + \Gamma)$ vanishes on every curve contracted by $g_i$.

Furthermore, there will be only finitely many such $g_i : X_i \to Z_i$ realized by all the $\Gamma$ in $E_A$.

Note that $V$ is not a subspace of $N^1(X)$; we need to have a finite dimensional space of actual divisors, not numerical classes. However, we would like to have a finiteness statement for certain open subsets $U$ in $N^1(X)$. So, we apply Lemma 3.1 to find finitely many divisors such that every element of $U$ has a klt representative in the space spanned by these divisors. In this way we obtain:

**Proposition 3.3.** Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial klt pair. Suppose that $B$ is a big effective $\mathbb{R}$-divisor such that $(X, \Delta + B)$ is klt. Then there is some neighborhood $U \subset N^1(X)$ of $[K_X + \Delta + B]$ and a finite set of movable curves $\{C_i\}$ so that:

- Every class $[\alpha] \in U$ that lies on the pseudo-effective boundary satisfies $[\alpha] \cdot C_i = 0$ for some $C_i$.

**Proof.** First we apply Lemma 3.1 to restrict our attention to a finite set of Weil divisors.

Since $B$ is big, it is numerically equivalent to $H + E$ for some ample $\mathbb{R}$-divisor $H$ and effective $\mathbb{R}$-divisor $E$. For sufficiently small $\tau > 0$, the pair $(X, \Delta + B + \tau E)$ is still klt, and then so is the pair $(X, \Delta + \frac{1}{1+\tau}(B + \tau E))$. Let $A$ be some sufficiently small ample divisor so that $\frac{\tau}{1+\tau}H - A$ is ample. If we replace $A$ by some $\mathbb{R}$-linearly equivalent divisor, we can ensure that

$$(X, \Delta + A + \frac{1}{1+\tau}(B + \tau E))$$

is klt.

Let $\{H_j\}_{j=1}^m$ be a finite set of ample divisors such that the convex hull of the $[H_j]$ contains an open set around $[\frac{\tau}{1+\tau}H - A]$ in $N^1(X)$. Choose an open neighborhood $U'$ of $[B - A]$ that is contained in the convex hull of the classes $[B - \frac{\tau}{1+\tau}H + H_j]$. We may apply Lemma 3.1 to $(X, \Delta + A + \frac{1}{1+\tau}(B + \tau E))$ and the $H_j$ to obtain a finite set of ample divisors $W_j$. Let $V$ be the vector space of $\mathbb{R}$-Weil divisors spanned by the irreducible components of $\frac{1}{1+\tau}(B + \tau E)$ and of the $W_j$. Thus, $V$ is a finite dimensional vector space of Weil divisors so that every class in $U'$ has an effective representative $\Gamma \in V$ with $(X, \Delta + A + \Gamma)$ klt.

Now we apply Theorem 3.2 with $V$ and $A$ as chosen. According to the first part of the theorem, every class in $U'$ has a representative $\Gamma$ so that, setting $D := K_X + \Delta + A + \Gamma$, we can run the $D$-minimal model program with scaling.
If $D$ is on the pseudo-effective boundary, we obtain a birational contraction $\phi_i : X_i \to X_i$ with a Mori fibration $g_i : X_i \to Z_i$, where $K_{X_i} + \phi_i^* (\Delta + A + \Gamma)$ vanishes on every curve contracted by $g_i$. Pick a curve $B_i$ in a general fiber of $g_i$ that avoids the (codimension at least 2) locus where $\phi_i^{-1} : X_i \to X$ is not an isomorphism. If we choose $B_i$ sufficiently general, then it belongs to a family of curves dominating $X_i$. Define $C_i$ to be the image of $B_i$ under $\phi_i^{-1}$. Of course $C_i$ is also a movable curve, and since $\phi$ is an isomorphism on a neighborhood of $C_i$ we have $D \cdot C_i = 0$.

According to the second part of Theorem 3.2, as we vary $\Gamma$ we obtain only finitely many Mori fibrations $g_i : X_i \to Z_i$. Applying this construction to each fibration in turn yields a finite set of curves $\{C_i\}$. If we let $U$ be the open neighborhood of $B$ given by $U' + [A]$, then this finite set of curves satisfies the conclusion. □

**Remark 3.4.** I am unable to show that the curves in Theorem 1.4 can be chosen to be rational curves. In the proof of Proposition 3.3 we needed to choose curves that avoided certain codimension 2 subvarieties. The obstacle to finding rational curves with this property comes from the singularities of the ambient variety. In fact, finding suitable rational curves is essentially equivalent to finding rational curves contained in the smooth locus. It is conjectured that one can do this for any $\mathbb{Q}$-Fano variety with klt singularities, but it seems difficult to prove.

However, if we are willing to replace the $C_i$ by classes $\alpha_i$ that do not necessarily represent curves, we can say something more. Since a Mori fibration has relative Picard number 1 and the general fiber is a log Fano variety, the class of any curve on the fiber is proportional to the class of a rational curve. Thus we may choose $\alpha_i$ to be the numerical pullback of the class of a nef rational curve. See [Ara08] for more details on numerical pullbacks.

Using compactness we can extend Proposition 3.3 to a global result.

**Corollary 3.5.** Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial klt pair. Suppose that $S \subset NE^1(X)$ is a set of divisor classes satisfying

1. $S$ is closed.
2. For each element $\beta \in S$, there is some big effective divisor $B$ such that $(X, \Delta + B)$ is klt and the class $[K_X + \Delta + B] = c\beta$ for some $c > 0$.

Then there is a finite set of movable curves $\{C_i\}$ so that every class $\alpha \in S$ that lies on the pseudo-effective boundary satisfies $\alpha \cdot C_i = 0$ for some $C_i$.

**Proof.** Let $R$ be the cone over $S$. Suppose that $\gamma \in R$. Since some positive multiple of $\gamma$ is in $S$, we have $[K_X + \Delta + B] = c\gamma$ for some $c > 0$ and for some big effective divisor $B$ with $(X, \Delta + B)$ klt. Apply Proposition 3.3 to $B$ to find an open neighborhood $U$ of $[K_X + \Delta + B]$ and finitely many movable curves $C_i$ such that every $\alpha \in U$ on the pseudo-effective boundary satisfies $\alpha \cdot C_i = 0$ for some $C_i$. If we rescale $U$ by $1/c$ we find a neighborhood $U_{\gamma}$ of $\gamma$.
such that every $\alpha \in U_\gamma$ on the pseudo-effective boundary satisfies $\alpha \cdot C_i = 0$ for some $C_i$.

Now fix some compact slice $Q$ of the cone $\overline{NE_1}(X)$. Since $S$ is closed, $R \cap Q$ is compact. To each point $\gamma \in R \cap Q$ we have associated finitely many curves $C_i$ and an open set $U_\gamma$ containing $\gamma$. The $U_\gamma$ define an open cover of $R \cap Q$, so by compactness there is a finite subcover. The finite set of corresponding curves satisfies the conclusion. $\square$

4. The Cone of Nef Curves

We now analyze the cone of nef curves. Suppose that $(X, \Delta)$ is a klt pair.

We start by identifying the rays in $\overline{NM_1}(X)$ that we are interested in:

**Definition 4.1.** A coextremal ray $R \geq 0 [\alpha] \subset \overline{N_1}(X)$ is a $(K_X + \Delta)$-negative ray of $\overline{NM_1}(X)$ that is extremal for $\overline{NE_1}(X)_{K_X+\Delta \geq 0} + \overline{NM_1}(X)$. That is, it must satisfy:

1. $\alpha \in \overline{NM_1}(X)$ has $(K_X + \Delta) \cdot \alpha < 0$.
2. If $\beta_1, \beta_2$ are classes in $\overline{NE_1}(X)_{K_X+\Delta \geq 0} + \overline{NM_1}(X)$ with
   $$\beta_1 + \beta_2 \in \mathbb{R}_{\geq 0} \alpha$$
   then $\beta_1, \beta_2 \in \mathbb{R}_{\geq 0} \alpha$.

We also need a “dual” notion for divisors.

**Definition 4.2.** A bounding divisor $D$ is any non-zero $\mathbb{R}$-Cartier divisor $D$ satisfying the following properties:

1. $D \cdot \alpha \geq 0$ for every class $\alpha$ in $\overline{NE_1}(X)_{K_X+\Delta \geq 0} + \overline{NM_1}(X)$.
2. $D^\perp$ contains some coextremal ray.

The zero divisor is not considered to be a bounding divisor.

We will often need a more restrictive notion. Suppose that $V \subset N_1(X)$ is any subset. If a bounding divisor satisfies $D \cdot \alpha \geq 0$ for every $\alpha \in V$, I will call it a $V$-bounding divisor. Note that as $V$ gets larger, the set of $V$-bounding divisors gets smaller.

Bounding divisors support the cone $\overline{NE_1}(X)_{K_X+\Delta \geq 0} + \overline{NM_1}(X)$ along a face that includes some coextremal ray. In particular, every bounding divisor is on the pseudo-effective boundary.

**Lemma 4.3.** Let $(X, \Delta)$ be a klt pair such that $K_X + \Delta$ is not pseudo-effective. Let $V$ be a closed convex cone containing $\overline{NE_1}(X)_{K_X+\Delta = 0} - \{0\}$ in its interior. Then every $V$-bounding divisor $D$ can be written as

$$D = \delta_D(K_X + \Delta) + A_D$$

for some ample $A_D$ and some $\delta_D > 0$.

**Proof.** Following [Ara08], we suppose the lemma fails and derive a contradiction. That is, suppose there is some $V$-bounding divisor $D$ such that the interior of the cone

$$\sigma = \mathbb{R}_{\geq 0}[D] + \mathbb{R}_{\geq 0}[-K_X - \Delta]$$


never intersects the ample cone. Then there is a curve class $\alpha$ for which the cone $\sigma$ is contained in $\alpha \leq 0$, but the ample cone is contained in $\alpha > 0$. By Kleiman’s criterion $\alpha$ is in the closed cone of effective curves; in particular $\alpha \in \overline{NE}_1(X)_{K_X + \Delta \geq 0}$.

Because $D$ is a $V$-bounding divisor and $V$ is not contained in any hyperplane, $D$ must be positive on the interior of $V$. So $D$ is positive on $\overline{NE}_1(X)_{K_X + \Delta = 0} - \{0\}$, and thus also positive on all of $\overline{NE}_1(X)_{K_X + \Delta \geq 0} - \{0\}$. In particular we must have $D \cdot \alpha > 0$, contradicting $\sigma \subset \alpha \leq 0$. □

Now we apply the results of the previous section.

**Proposition 4.4.** Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial klt pair. Let $V$ be a closed convex cone containing $\overline{NE}_1(X)_{K_X + \Delta = 0} - \{0\}$ in its interior. There is a finite set of movable curves $\{C_i\}$ such that for any $V$-bounding divisor $D$, there is some $C_i$ for which $D \cdot C_i = 0$.

**Proof.** The statement is vacuous when the set of $V$-bounding divisors is empty (for example, when $K_X + \Delta$ is pseudo-effective or when $V$ is the entire space), so we assume otherwise.

We apply Corollary 3.5 to the set of $V$-bounding divisors. The first hypothesis holds since $V$-bounding divisors are precisely the pseudo-effective divisors $D$ satisfying the closed condition

$$D \cdot \alpha \geq 0$$

for every class $\alpha \in \overline{NE}_1(X)_{K_X + \Delta \geq 0} + V + \overline{NM}_1(X)$.

We verify the second hypothesis by using Lemma 4.3. Each $V$-bounding divisor $D$ satisfies

$$\frac{1}{\delta_D} D = K_X + \Delta + \frac{1}{\delta_D} A_D$$

for some ample $A_D$ and some $\delta_D > 0$. By replacing $\frac{1}{\delta_D} A$ by some $\mathbb{R}$-linearly equivalent effective divisor $A'$, we may ensure that $(X, \Delta + A')$ is a klt pair. Thus the second hypothesis holds as well. Since every $V$-bounding divisor is on the pseudo-effective boundary, an application of Corollary 3.5 finishes the proof. □

**Corollary 4.5.** Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial klt pair, and let $V$ be a closed convex cone containing $\overline{NE}_1(X)_{K_X + \Delta = 0} - \{0\}$ in its interior. There are finitely many movable curves $C_i$ such that

$$\overline{NE}_1(X)_{K_X + \Delta \geq 0} + V + \overline{NM}_1(X) = \overline{NE}_1(X)_{K_X + \Delta \geq 0} + V + \sum_{i=1}^{N} \mathbb{R}_{\geq 0}[C_i].$$

**Proof.** Note that as $V$ increases, the statement of the theorem becomes easier to prove. Therefore, we may assume by shrinking $V$ that $\overline{NE}_1(X) + V$ does not contain any 1-dimensional subspace of $N_1(X)$. In particular, this implies that there is some ample divisor $A$ that is positive on $V - \{0\}$.

Let $\{C_i\}$ be the set of curves found in Proposition 4.4. Suppose that there were some curve class $\alpha \in \overline{NM}_1(X)$ not contained in $\overline{NE}_1(X)_{K_X + \Delta \geq 0} + V + \overline{NM}_1(X) = \overline{NE}_1(X)_{K_X + \Delta \geq 0} + V + \sum_{i=1}^{N} \mathbb{R}_{\geq 0}[C_i]$. □
\[ V + \sum R_{\geq 0}[C_i]. \] In particular, there is some divisor \( B \) that is positive on \( \overline{NE}_1(X)_{K_X+\Delta \geq 0} + V + \sum R_{\geq 0}[C_i] \), but for which \( B \cdot \alpha < 0 \).

Let \( A \) be an ample divisor positive on \( V - \{0\} \), and consider \( A + \tau B \), where \( \tau > 0 \) is the maximum over all \( t \) such that \( A + tB \) is pseudoeffective. Then \( A + \tau B \) is a \( V \)-bounding divisor, but \( (A + \tau B) \cdot C_i > 0 \) for every \( C_i \), a contradiction.

So we have proven Theorem 1.4 for \( \mathbb{Q} \)-factorial klt pairs \((X, \Delta)\). It only remains to reduce the problem for general dlt pairs \((X, \Delta)\) to this specific case. The following lemma from \[KM98\] shows that dlt pairs can be approximated by klt pairs on any projective variety. (Although \[KM98\] assumes that \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier, the proof works equally well for \( \mathbb{R} \)-Cartier divisors.)

**Lemma 4.6** ([KM98], Proposition 2.43). Suppose that \((X, \Delta)\) is a dlt pair, and that \( A \) is an ample Cartier divisor on \( X \). Let \( \Delta_1 \) be an effective \( \mathbb{Q} \)-Weil divisor such that \( \text{Supp}(\Delta) = \text{Supp}(\Delta_1) \). Then there is some rational \( c > 0 \) and effective \( \mathbb{Q} \)-divisor \( D \), numerically equivalent to \( \Delta_1 + cA \), such that \((X, \Delta + \tau D - \tau \Delta_1)\) is klt for any sufficiently small \( \tau > 0 \).

We can now prove Theorem 1.4.

**Proof (of Theorem 1.4):** I’ll assume that \( K_X + \Delta \) is not pseudo-effective, as otherwise the statement is vacuous.

Let \( A \) be some ample Cartier divisor on \( X \), and \( \Delta_1 \) some effective \( \mathbb{Q} \)-Weil divisor with the same support as \( \Delta \). Choose \( c \) and \( D \) as in Lemma 4.6. Suppose that we choose \( \tau \) small enough so that

\[ \overline{NE}_1(X)_{K_X+\Delta+\tau D-\tau \Delta_1=0} \subset V \]

Then the result for \((\Delta, V)\) follows from the result for \((\Delta + \tau D - \tau \Delta_1, V)\). Thus we may assume that \((X, \Delta)\) is klt.

Suppose that \((X, \Delta)\) is not \( \mathbb{Q} \)-factorial. By Corollary 1.4.3 of \[BCHM06\] we may find a log terminal model \((Y, \Gamma)\) over \((X, \Delta)\). That is, \((Y, \Gamma)\) is a \( \mathbb{Q} \)-factorial klt pair, and there is a small birational map \( \pi: Y \to X \) such that \( \pi_*\Gamma = \Delta \) and \( K_Y + \Gamma = \pi^*(K_X + \Delta) \). Since the map \( \pi_*: N_1(Y) \to N_1(X) \) is linear, \( \pi_*^{-1}V \) is still closed and convex. Furthermore, \( \pi_*^{-1}V \) contains \( \overline{NE}_1(Y)_{K_Y+\Gamma=0} \) in its interior, since \( \pi_* \) on curves is dual to \( \pi^* \) on Cartier divisors.

Apply Corollary 1.5 to \((Y, \Gamma)\) and \( \pi_*^{-1}V \) to obtain the equality of cones

\[ \overline{NE}_1(Y)_{K_Y+\Gamma \geq 0 + \pi_*^{-1}V + \overline{NM}_1(Y)} = \overline{NE}_1(Y)_{K_Y+\Gamma \geq 0 + \pi_*^{-1}V + \sum_{i=1}^N R_{\geq 0}[C_i]} \]

If we take the image of these cones under the map \( \pi_* \) and apply Lemma 2.3, we obtain

\[ \overline{NE}_1(X)_{K_X+\Delta \geq 0} + V + \overline{NM}_1(X) = \overline{NE}_1(X)_{K_X+\Delta \geq 0} + V + \sum_{i=1}^N R_{\geq 0}[\pi_*C_i]. \]
The pushforward of a movable curve is again movable. Thus the coextremal rays are still spanned by curves belonging to covering families, giving the result of Theorem 1.4 for $(X, \Delta)$.

\[ \text{□} \]

Proof (of Theorem 1.1): Let \( \{V_j\} \) be a countable set of nested closed convex cones containing \( \overline{NE_1(X)_{K_X+\Delta=0}} - \{0\} \) in their interiors such that

\[ \bigcap_j V_j = \overline{NE_1(X)_{K_X+\Delta=0}}. \]

Let \( \mathcal{A}_j \) be the finite set of curves found by applying Theorem 1.4 to \((X, \Delta)\) and \(V_j\). By tossing out redundant curves, we may ensure that each curve in \( \mathcal{A}_j \) generates a coextremal ray. We define the countable set of curves \( \mathcal{A} = \bigcup_j \mathcal{A}_j \).

We first show the equality of cones. Suppose that there is some curve class \( \alpha \in \overline{NM_1(X)} \) such that

\[ \alpha \notin \overline{NE_1(X)_{K_X+\Delta=0}} + \sum_A \mathbb{R}_{\geq 0}[C_i]. \]

Since this cone is closed and convex, there is a convex open neighborhood \( U \) of the cone which also does not contain \( \alpha \). For sufficiently high \( j \), we have \( V_j \subset U \), so

\[ \alpha \notin \overline{NE_1(X)_{K_X+\Delta=0}} + V_j + \sum_A \mathbb{R}_{\geq 0}[C_i]. \]

But this contradicts Theorem 1.4. This proves the non-trivial containment of Theorem 1.1.

We also need to verify the accumulation condition for the rays generated by curves in \( \mathcal{A} \). Suppose that \( \alpha \) is a point on the \((K_X + \Delta)\)-negative portion of the boundary of \( \overline{NE_1(X)_{K_X+\Delta=0}} + \overline{NM_1(X)} \), and that \( \alpha \) does not lie on a hyperplane supporting both \( \overline{NM_1(X)} \) and \( \overline{NE_1(X)_{K_X+\Delta=0}} \). For a sufficiently small open neighborhood \( U \) of \( \alpha \) the points of \( \overline{U} \) still do not lie on such a hyperplane. We may also assume that \( \overline{U} \) is disjoint from \( \overline{NE_1(X)_{K_X+\Delta=0}} \). We define

\[ \mathcal{P} := \overline{U} \cap \partial \left( \overline{NE_1(X)_{K_X+\Delta=0}} + \overline{NM_1(X)} \right). \]

Fix a compact slice of \( \overline{NE_1(X)} \), and let \( \mathcal{D} \) denote the bounding divisors on this slice that have vanishing intersection with some element of \( \mathcal{P} \). By construction \( \mathcal{D} \) is positive on \( \overline{NE_1(X)_{K_X+\Delta=0}} - \{0\} \). By passing to a compact slice it is easy to see that \( \mathcal{D} \) is also positive on \( V_j - \{0\} \) for a sufficiently large \( j \). In other words, every element of \( \mathcal{P} \) is on the boundary of \( \overline{NE_1(X)_{K_X+\Delta=0}} + V_j + \overline{NM_1(X)} \). By Theorem 1.4 there are only finitely many coextremal rays that lie on this cone, and thus only finitely many coextremal rays through \( U \). So \( \alpha \) can not be an accumulation point. \( \text{□} \)
5. Proof of the Contraction Theorem

Our first step will be to prove a version of Theorem 1.3 for divisors \( K_X + \Delta \) with \( \Delta \) big.

**Theorem 5.1.** Let \((X, \Delta)\) be a klt pair, with \( \Delta \) big. Suppose that \( D := K_X + \Delta \) lies on the pseudo-effective boundary. Then there is a birational morphism \( \psi : W \to X \) and a contraction \( h : W \to Z \) such that:

1. Every movable curve \( C \) on \( W \) with \( \psi^* D \cdot C = 0 \) is contracted by \( h \).
2. For a general pair of points in a general fiber of \( h \), there is a movable curve \( C \) through the two points with \( \psi^* D \cdot C = 0 \).

These properties determine the pair \((W, h)\) up to birational equivalence. In fact the map we construct satisfies a stronger property:

3. There is an open set \( U \subset W \) such that the complement of \( U \) has codimension 2 in a general fiber of \( h \) and a complete curve \( C \) in \( U \) is contracted by \( h \) iff \( \psi^* D \cdot C = 0 \) and \( [C] \in \overline{NM}_1(W) \).

The proof goes as follows. Since \( X \) is not assumed to be \( \mathbb{Q} \)-factorial, we must first pass to a log terminal model \((Y, \Gamma)\) for \((X, \Delta)\). We still know that \( \Gamma \) is big, so we can run the \((K_Y + \Gamma)\)-minimal model program. We obtain a birational contraction \( \phi : Y \dashrightarrow Y' \) such that \( K_{Y'} + \phi_* \Gamma \) is nef, and an application of the usual Contraction Theorem on \( Y' \) gives us the desired map on an open subset of \( Y \).

There is one subtlety; in order to prove the stronger Property (3), we will need to improve the properties of the contraction \( g' : Y' \to Z \). This is accomplished by the following lemma.

**Lemma 5.2.** Let \((Y', \Gamma')\) be a \( \mathbb{Q} \)-factorial klt pair such that \( \Gamma' \) is big. Suppose that \( g' : Y' \to Z \) is a \((K_{Y'} + \Gamma')\)-trivial fibration. We can construct a birational contraction over \( Z \), \( \phi' : Y' \dashrightarrow Y^+ \) (with morphism \( g^+ : Y^+ \to Z \)), and a Zariski-closed subset \( N^+ \subset Y^+ \) of codimension at least 2 such that every curve contracted by \( g^+ \) that does not lie in \( N^+ \) is nef.

**Proof.** Since \( \Gamma' \) is big and \( g' \) is \((K_{Y'} + \Gamma')\)-trivial, \( g' \) is a log Fano fibration. Thus, there are only finitely many birational contractions over \( Z \) (see [BCHM06, Corollary 1.3.1]). If we let \( N' \) be the union of the (finitely many) closed subsets where these birational maps are not isomorphisms, then any curve \( C \) contracted by \( g' \) and avoiding \( N' \) is nef. We’ll call \( N' \) the “bad” locus of \( g' : Y' \to Z \).

Let \( \{E_i\} \) denote the divisorial components of \( N' \). Through any point \( x \in E_i \) there is some curve \( C \) that is not nef. If we pick \( x \) to be very general in \( E_i \), then \( C \) deforms to cover \( E_i \). The only way that \( C \) could not be nef is if \( E_i \cdot C < 0 \). Thus, every component \( E_i \) is covered by curves with \( E_i \cdot C < 0 \).

Fix some component \( E \), and choose \( \epsilon > 0 \) such that \((Y', \Gamma' + \epsilon E)\) is still klt. Note that \( K_{Y'} + \Gamma' + \epsilon E \) is numerically equivalent over \( Z \) to \( \epsilon E \). Thus when we run the relative \((K_{Y'} + \Gamma' + \epsilon E)\)-minimal model program, we must contract the component \( E \). By performing this process inductively, we...
eventually find a variety $Y^+$ such that the “bad” locus $N^+$ has codimension at least 2.

\[\]

Proof (of Theorem 5.1): Let $(Y, \Gamma)$ be a log terminal model for $(X, \Delta)$ with small contraction $\pi : Y \to X$. Since $\pi_*\Gamma = \Delta$, $\Gamma$ is also big. Thus, we can run the $(K_Y + \Gamma)$-minimal model program with scaling to obtain a birational map $\phi : Y \dasharrow Y'$ such that $K_{Y'} + \phi_*\Gamma$ is nef. By applying the basepoint freeness theorem to $K_{Y'} + \phi_*\Gamma$ we obtain a contraction morphism $g' : Y' \to Z$. Applying Lemma 5.2 we obtain a birational contraction $\phi' : Y' \to Y^+$ and $g^+ : Y^+ \to Z$. There is a codimension 2 locus $N^+ \subset Y^+$ such that every curve contracted by $g^+$ but not contained in $N^+$ is nef. We denote the composition $\phi' \circ \phi : Y \to Y^+$ by $\Phi$. We define $W$ by taking a resolution of the map $\Phi$, so that we have maps $s : W \to Y$, $s^+ : W \to Y^+$. We let $\psi = \pi \circ s : W \to X$ and $h = g^+ \circ s^+ : W \to Z$.

Note that the map $\Phi$ is $(K_Y + \Gamma)$-non-positive, since $\phi$ is $(K_Y + \Gamma)$-negative and $\phi'$ is $(K_Y + \Gamma)$-trivial. Thus, for some effective divisor $E$ we have

$$s^*(K_Y + \Gamma) = s^{++}(K_{Y^+} + \Phi_*\Gamma) + E$$

(see [KM98], Lemma 3.38). As a consequence, if $C$ is a movable curve on $W$ with $\psi^*D \cdot C = 0$, then we also have $s^{++}(K_{Y^+} + \Phi_*\Gamma) \cdot C = 0$. In particular, $C$ is contracted by $h$, showing Property (1).

Let $U_Y$ be the open subset of $Y$ on which $\Phi$ is an isomorphism. Since $\Phi^{-1}$ does not contract a divisor, the complement of $\Phi(U_Y)$ has codimension at least 2. So by shrinking $U_Y$ we can remove all fibers of $g^+$ on which the complement has codimension 1. By shrinking $U_Y$ further we can remove all the reducible fibers of $g^+$. Finally, we remove the subset $\Phi^{-1}(N^+ \cap \Phi(U_Y))$; note that $\Phi(U_Y)$ still has codimension 2 in a general fiber. We define the open subset $U \subset W$ by taking $s^{-1}(U_Y)$.

Since $s^+(U)$ has codimension 2 in a general fiber of $g^+$, we can connect two general points in a general fiber of $g^+$ by a movable curve $C^+ \subset s^+(U)$: we just intersect the fiber with the appropriate number of very ample divisors that contain the two points. Since $\Phi$ is an isomorphism on $U$, the movable curve $\Phi^{-1}C^+$ satisfies $\psi^*D \cdot \Phi^{-1}C^+ = (K_{Y^+} + \Phi_*\Gamma) \cdot C^+ = 0$. This shows Property (2).

In fact, suppose that $C \subset U$ is a complete curve contracted by $h$. By construction $s^+(C)$ is nef, so by Lemma 2.29 $C$ is also nef. Furthermore, since $s$ and $s^+$ are isomorphisms on a neighborhood of $C$, we must have $\psi^*D \cdot C = s^{++}(K_{Y^+} + \Phi_*\Gamma) \cdot C = 0$. Conversely, any curve $C$ on $W$ with $s^{++}(K_{Y^+} + \Phi_*\Gamma) \cdot C = 0$ must be contracted by $h$, showing Property (3).

Finally, we must show the uniqueness of $h : W \to Z$ up to birational equivalence. So, suppose that $h' : W' \to Z'$ is another map satisfying Properties (1) and (2). Let $\tilde{W}$ be a common resolution. The maps $\tilde{W} \to Z$ and $\tilde{W} \to Z'$ still satisfy Properties (1) and (2), and so they must coincide on an open subset.
When we rephrase Theorem 5.1 in terms of faces of $NE_1(X)_{K_X+\Delta \geq 0} + NM_1(X)$, we obtain Theorem 1.3. The main difficulty is to show that most $(K_X + \Delta)$-negative extremal faces of the cone admit a divisor $D$ supporting the cone precisely along that face.

**Proof (of Theorem 1.3):** Our first goal is to find a pseudo-effective divisor $D$ such that $D \perp$ supports $NE_1(X)_{K_X+\Delta \geq 0} + NM_1(X)$ exactly along $F$. This will follow from our assumption that there is some pseudo-effective divisor class $\beta$ such that $\beta \perp$ contains $F$ but doesn’t intersect $NE_1(X)_{K_X+\Delta \geq 0}$.

We let $F_\beta$ denote the face $F_\beta := \beta \cap (NE_1(X)_{K_X+\Delta \geq 0} + NM_1(X))$.

Note that $F$ is a subface of $F_\beta$. For a sufficiently small closed convex cone $V$ containing $NE_1(X)_{K_X+\Delta \geq 0} - \{0\}$ in its interior, $\beta$ is still positive on $(NE_1(X)_{K_X+\Delta \geq 0} + V) - \{0\}$.

This means that $F_\beta$ is also a $(K_X + \Delta)$-negative face of the larger cone $NE_1(X)_{K_X+\Delta \geq 0} + NM_1(X)$ exactly along that subface. In particular this is true for the subface $F$. Furthermore, $D$ must be pseudo-effective since it is non-negative on $NM_1(X)$. This finishes the construction of $D$.

Since $D$ is positive on $NE_1(X)_{K_X+\Delta = 0}$, Lemma 4.3 shows that (after rescaling) $D = K_X + \Delta + A$ for some ample $A$. Thus, we can apply Theorem 5.1 to $D$ to obtain $h : W \to Z$. Properties (1) and (2) and the uniqueness up to birational equivalence follow immediately. To show Property (3), we just need to note that for any irreducible curve $C$ on $W$ not contracted by $\psi$, $[\psi_*C] \in F$ iff $\psi^*D \cdot C = 0$ and $[C] \in NM_1(W)$.

**6. Comparisons with the Cone Theorem**

**Cutkosky’s Example.** In this section we give an example of a threefold for which the stronger statement

\[ (* ) \quad NM_1(X) = NM_1(X)_{K_X+\Delta \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i] \]

does not hold for any locally discrete countable collection of curves. Although this statement seems appealing, it is much less natural from the viewpoint of the minimal model program. The problem is that the bounding divisors are no longer of the form $K_X + \Delta + A$ for an ample $A$, but $K_X + \Delta + B$ for a big $B$. In general we cannot say anything about the singularities of such divisors, and so we can not apply Proposition 3.3.

**Remark 6.1.** If we dualize (*), we obtain a statement about the structure of the pseudo-effective cone of divisors: let $\sigma(X)$ denote the cone in $N^1(X)$ generated by $[K_X+\Delta]$ and $NE_1(X)$. Suppose that $U \subset \sigma(X)$ is a relatively
A CONE THEOREM FOR NEF CURVES

open subcone such that \([K_X + \Delta] \not\in U\). (*) would imply that \(\overline{NE}^1(X) \cap U\) is cut out in \(U\) by finitely many hyperplanes \(C_i\), each defined by a curve. Theorem 1.4 guarantees this local structure along the boundary of the nef cone of divisors, so it only needs to be verified along the boundary of the pseudo-effective cone of divisors. Thus (*) is closely related to the existence of Zariski decompositions.

The statement (*) is probably true for surfaces. It is shown in [BKS04] that for a surface the portion of the pseudo-effective boundary that is not nef is locally polyhedral. However, it may be possible for extremal rays to accumulate where the pseudo-effective cone and nef cone first coincide. More generally, the results of [Nak04] and [Bou04] under some circumstances give a similar structure theorem for the pseudo-effective cone of divisors.

Example 6.2. We construct a threefold for which (*) does not hold. In particular, we’ll construct a smooth variety \(X\) such that \(-K_X\) is big, but \(\overline{NM}^1(X)\) is circular along certain portions of its boundary. This example is due to Cutkosky, who uses it to find a divisor with no rational Zariski decomposition (see [Cut86]).

Let \(Y\) be an abelian surface with Picard number at least 3. As \(Y\) is abelian, the nef cone of divisors and the pseudo-effective cone of divisors coincide. This cone is circular in \(N^1(Y)\): it consists of all the curve classes with non-negative self intersection and non-negative intersection with some ample divisor.

Choose a divisor \(L\) on \(Y\) such that \(-L\) is ample, and define \(X\) to be the \(\mathbb{P}^1\)-bundle \(X := \mathbb{P}_Y(\mathcal{O} \oplus \mathcal{O}(L))\) with projection \(\pi : X \to Y\). Let \(S\) denote the zero section of \(\pi\), that is, the section such that \(S|_S\) is the linear equivalence class of \(L\). Every divisor on \(X\) can be written as \(aS + \pi^*D\) for some integer \(a\) and some divisor \(D\) on \(Y\). Using adjunction we find \(K_X = -2S + \pi^*L\).

By general theory, a divisor of the form \(S + \pi^*D\) is pseudo-effective on \(X\) iff there is a pseudo-effective divisor on \(Y\) in the cone generated by \(D\) and \(D + L\). Since \(-L\) is ample, this amounts to requiring that \(D\) be pseudo-effective. Thus the pseudo-effective cone \(\overline{NE}^1(X)\) is generated by \(S\) and \(\pi^*\overline{NE}^1(Y)\). Note that \(-K_X\) is big.

Dualizing \(\overline{NE}^1(X)\), we see that \(\overline{NM}^1(X)\) is circular along some portions of its boundary. Since \(-K_X\) is big, all of \(\overline{NM}^1(X)\) is \(K_X\)-negative, so (*) fails for \(X\).

It is helpful to see why Theorem 1.4 holds on \(X\). From the viewpoint of the minimal model program \(X\) is very simple; there is only one Mori fibration \(\pi : X \to Y\). This simplicity is reflected in the structure of the cones: the only coextremal ray is generated by a fiber of \(\pi\). \(\square\)

Accumulation on \(K_X^1\). Finally, we consider whether coextremal rays accumulate only along \(\overline{NE}^1(X)_{K_X + \Delta = 0}\) as conjectured by Batyrev. This would imply that for some countable collection of curves \(C_i\)

\[\overline{NE}^1(X)_{K_X + \Delta \geq 0} + \overline{NM}^1(X) = \overline{NE}^1(X)_{K_X + \Delta \geq 0} + \sum\mathbb{R}_{\geq 0}[C_i].\]
In [Ara08], Araujo shows this stronger statement for terminal threefolds. She first finds movable curves generating coextremal rays by running the minimal model program. Using boundedness of terminal threefolds of Picard number 1, she bounds the degrees of all of these curves with respect to a fixed polarization. So in fact, for any ample divisor $A$ there are only finitely many $(K_X + \Delta + A)$-negative coextremal rays.

The situation in higher dimensions is expected to be similar. The following conjecture is due to Alexeev [Ale94], and A. Borisov and L. Borisov [Bor96].

**Conjecture 6.3** (Borisov-Alexeev-Borisov). For any $\epsilon > 0$, the family of $\mathbb{Q}$-Fano varieties of a given dimension with log discrepancy greater than $\epsilon$ is bounded.

Since running the minimal model program can only increase log discrepancies, this conjecture combined with Araujo’s argument should settle the question.

Unfortunately our methods are not strong enough to prove Batyrev’s conjecture. However, if we assume termination of flips, we can prove the case when $\Delta$ is big.

**Conjecture 6.4** (Termination of Flips). Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial klt pair. Then there is no infinite sequence of $(K_X + \Delta)$ flips.

**Theorem 6.5.** Let $(X, \Delta)$ be a dlt pair with $\Delta$ big. Assume termination of flips. Then there are finitely many $(K_X + \Delta)$-negative movable curves $C_i$ such that

$$\overline{NE}_1(X)_{K_X + \Delta \geq 0} + \overline{NM}_1(X) = \overline{NE}_1(X)_{K_X + \Delta \geq 0} + \sum R_{\geq 0}[C_i].$$

**Remark 6.6.** For the usual Cone Theorem, the case when $\Delta$ is big implies the statement in general. However, Theorem 6.5 does not imply the full strength of Batyrev’s conjecture. The problem is the presence of the term $\overline{NE}_1(X)_{K_X + \Delta \geq 0}$. If we add a small ample divisor $\epsilon H$ to $\Delta$, we are actually changing the shape of this cone, so that the limiting behavior as $\epsilon$ vanishes is not very precise.

The proof of Theorem 6.5 uses the same arguments as before. Given a bounding divisor $D$, we find a birational contraction $\phi : X \dasharrow X'$ and a Mori fibration $g : X' \to Z$ that is $\phi_*D$-trivial. We then use termination of flips to prove that we need only finitely many fibrations.

Our first lemma is an analogue of Lemma 4.3 for an arbitrary bounding divisor.

**Lemma 6.7.** Let $(X, \Delta)$ be a klt pair such that $K_X + \Delta$ is not pseudo-effective. Every bounding divisor $D$ can be written

$$D = \delta_D(K_X + \Delta) + N_D$$

for some nef $N_D$ and some $\delta_D \geq 0$. 
Proof. Just as before, we suppose the lemma fails and derive a contradiction. That is, suppose there is some bounding divisor $D$ such that the cone
$$\sigma = \mathbb{R}_{\geq 0}[D] + \mathbb{R}_{\geq 0}[-K_X - \Delta]$$
never intersects the nef cone. Then there is a curve class $\alpha$ for which the cone $\sigma$ is contained in $\alpha_{<0}$, but the nef cone is contained in $\alpha_{>0}$. By Kleiman's criterion $\alpha$ is in the closed cone of effective curves; in particular $\alpha \in NE_1(X)_{K_X + \Delta_{\geq 0}}$. Because $D$ is a bounding divisor, we should have $D \cdot \alpha \geq 0$. This contradicts $\sigma \subset \alpha_{<0}$. \qed

Given a bounding divisor $D$, we want to find curves that have vanishing intersection with $D$ by running the minimal model program. During the process, we must ensure that $D$ is still a bounding divisor, or at least that it still intersects some coextremal ray. This issue is handled by the next lemma.

Lemma 6.8. Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial klt pair. Suppose $D$ is a bounding divisor, and $\phi : X \to X'$ is a composition of $(K_X + \Delta)$ flips and divisorial contractions that are each non-positive with respect to $D$. Then there is a class $\alpha$ on the boundary of $\mathbb{NE}_1(X')$ such that $\phi_* D \cdot \alpha = 0$ and $(K_{X'} + \phi_* \Delta) \cdot \alpha < 0$.

Proof. Let $Y$ be a smooth variety resolving the rational map $\phi : X \to X'$. We denote the corresponding maps by $g : Y \to X$ and $g' : Y \to X'$. Choose a class $\beta$ on a $D$-trivial coextremal ray. By Lemma 2 of [1], there is some class $\gamma \in \mathbb{NE}_1(Y)$ with $g_* \gamma = \beta$. Define $\alpha = g'_* \gamma$.

By [2], Lemma 3.38, there is an effective divisor $E$ such that $g^*(K_X + \Delta) = g'^*(K_{X'} + \phi_* \Delta) + E$.

Since $\gamma \cdot E \geq 0$, we have
$$(K_{X'} + \phi_* \Delta) \cdot \alpha = g'^*(K_{X'} + \phi_* \Delta) \cdot \gamma \leq g^*(K_X + \Delta) \cdot \gamma \leq (K_X + \Delta) \cdot \beta < 0$$

Similarly, since each step of $\phi$ is $D$ non-positive, there is some effective divisor $E'$ such that
$$g^* D = g'^* \phi_* D + E'.$$

The same argument shows that $\phi_* D \cdot \alpha \leq 0$. However, since $\phi_* D$ is pseudoeffective, we must have $\phi_* D \cdot \alpha = 0$. \qed

The next lemma is used to show the finiteness of coextremal rays. In fact, the only reason we must assume that $\Delta$ is big is so that we may apply this lemma.

Lemma 6.9. Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial klt pair, with $\Delta$ big. Assume termination of flips. Then there are only finitely many models that can be obtained by a sequence of $K_X + \Delta$ flips and divisorial contractions.
Proof. By the Cone Theorem, if $\Delta$ is big, then there are only finitely many $(K_X + \Delta)$-negative extremal rays.

We construct a tree in the following way. The bottom node represents the variety $X$. For each $(K_X + \Delta)$-flip or divisorial contraction $\phi : X \rightarrow X'$, we add a node representing $X'$, and connect it to the node for $X$ by an edge representing $\phi$. After we have completed the (finitely many) additions of edges around the node $X$, we perform the same process on each model $X'$ and adjoint divisor $K_{X'} + \phi_*\Delta$. We continue until we have exhausted all possible models $X'$ that can be obtained by a sequence of flips or divisorial contractions.

Note that for any birational map $\phi$ the divisor $\phi_*\Delta$ is still big. This implies that every model $X'$ only has finitely many $(K_{X'} + \phi_*\Delta)$-negative extremal rays. In other words, our tree is locally finite at each node. By König’s Lemma, if there were infinitely many nodes, there would have to be an infinite branch of the tree, contradicting termination of flips. □

Proof (of Theorem 6.5): We first consider the case when $(X, \Delta)$ is a $\mathbb{Q}$-factorial klt pair. Since $\Delta$ is big, we can write $\Delta = A + B$ for some ample divisor $A$ and some effective divisor $B$. Since $A$ is ample, it is $\mathbb{R}$-linearly equivalent to some ample effective divisor $A'$ such that $(X, A' + B)$ is klt. We define $\Gamma := \frac{1}{2}A' + B$; of course $(X, \Gamma)$ is still klt.

Our next goal is to show that for any bounding divisor $D$, there is a $(K_X + \Gamma)$-minimal model program $\phi : X \rightarrow X'$ and a Mori fibration $g : X' \rightarrow Z$ such that $\phi_*D$ is trivial on the fibers. By Lemma 6.7, we can write $D = \delta_D(K_X + \Delta) + N_D$ for some nef $N_D$ and some $\delta_D \geq 0$. We separate into two cases.

First suppose that $\delta_D > 0$. By rescaling $D$ we can write

$$D = K_X + \Gamma + \left(\frac{1}{2}A + N_D\right).$$

Note that $\frac{1}{2}A + N_D$ is ample; after replacing by a suitable $\mathbb{R}$-equivalent divisor, we may run the $D$-minimal model program. The result is a map $\phi : X \rightarrow X'$ and a Mori fibration $g : X' \rightarrow Z$ such that $\phi_*D$ is trivial on the fibers.

Now suppose that $\delta_D = 0$, so that $D = N_D$ is nef. Suppose that $D$ has vanishing intersection with an extremal ray that corresponds to a flip or divisorial contraction. If $\psi : X \rightarrow X_1$ is this operation, then $\psi_*D$ is still nef. We can repeat this process inductively; by termination of flips, there is some birational contraction $\phi : X \rightarrow X'$ such that $\phi_*D$ does not have vanishing intersection with any extremal ray corresponding to a flip or divisorial contraction. Since $D$ is a bounding divisor, Lemma 6.5 shows that $\phi_*D$ still has vanishing intersection with some $(K_{X'} + \phi_*\Delta)$-negative class $\alpha \in \overline{NM}_1(X')$. Since $\phi_*D$ is nef, this means that it must also have vanishing intersection with a $(K_X + \Delta)$-negative extremal ray. The contraction of this extremal ray is a $\phi_*D$-trivial Mori fibration $g : X' \rightarrow Z$. 
So, to each bounding divisor $D$ we have associated a rational map $\phi : X \dasharrow X'$ found by running the $(K_X + \Gamma)$-minimal model program. Lemma 6.9 shows there can only by finitely many such models $X'$. Furthermore, since $\phi_*\Gamma$ is big for any of these maps, there can by only finitely many Mori fibrations on each $X'$. By choosing a sufficiently general curve $C$ in a fiber of each Mori fibration, we find a finite set of movable curves $\{C_i\}$ such that for any bounding divisor $D$, there is some $C_i$ for which $D \cdot C_i = 0$. By an easy cone argument, this finishes the $\mathbb{Q}$-factorial klt case. The reduction of the general case to this situation is essentially the same as before. □

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