Planar Graphs as VPG-Graphs

Steven Chaplick$^1$ Torsten Ueckerdt$^2$

$^1$Department of Applied Mathematics, Charles University
$^2$Department of Mathematics, Karlsruhe Institute of Technology

Abstract

A graph is $B_k$-VPG when it has an intersection representation by paths in a rectangular grid with at most $k$ bends (turns). It is known that all planar graphs are $B_3$-VPG and this was conjectured to be tight. We disprove this conjecture by showing that all planar graphs are $B_2$-VPG. We also show that the 4-connected planar graphs constitute a subclass of the intersection graphs of Z-shapes (i.e., a special case of $B_2$-VPG). Additionally, we demonstrate that a $B_2$-VPG representation of a planar graph can be constructed in $O(n^{3/2})$ time. We further show that the triangle-free planar graphs are contact graphs of: L-shapes, Γ-shapes, vertical segments, and horizontal segments (i.e., a special case of contact $B_1$-VPG). From this proof we obtain a new proof that bipartite planar graphs are a subclass of 2-DIR.

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E-mail addresses: chaplick@kam.mff.cuni.cz (Steven Chaplick) torsten.ueckerdt@kit.edu (Torsten Ueckerdt)
1 Introduction

Planar graphs have a long history of being described as geometric intersection (and contact) graphs; i.e., for a planar graph $G$, each vertex can be mapped to a geometric object $O_v$ such that $(u, v)$ is an edge of $G$ if and only if $O_v$ and $O_u$ intersect. Two well-known results of this variety are that: every planar graph is an intersection graph of curves in the plane [12] (1978), and every planar graph is a contact graph of discs in the plane [21] (1936).

In this paper we consider representations of planar graphs as the intersection and contact graphs of restricted families of curves in the plane. The most general class of intersection graphs of curves in the plane is the class of string graphs. Formally, a graph $G = (V, E)$ is STRING if and only if each $v \in V$ can be associated with a curve $c_v$ in the plane such that for every pair $u, v \in V$, $(u, v) \in E$ if and only if $c_u$ and $c_v$ intersect. STRING was first considered regarding thin film RC-circuits [27].

Perhaps the most significant result describing planar graphs as intersection graphs of curves is the recent proof of Scheinerman’s conjecture that all planar graphs are segment graphs (SEG); i.e., the intersection graphs of line segments in the plane. Scheinerman conjectured this in his Ph.D. thesis (1984) [26], and it was proven in 2009 by Chalopin and Gonçalves [5]. Leading up to this result were several partial results. Bipartite planar graphs were the first subclass shown to be intersection graphs of line segments having two distinct slopes (2-DIR) [10, 4]. This was followed by triangle-free planar graphs being shown to be intersection graphs of line segments having three distinct slopes (3-DIR) [8]. It has also been proven that segment graphs include every planar graph that can be 4-colored so that no separating cycle uses all four colors [9]. Planar graphs were also shown to be representable by curves in the plane where each pair of curves intersect in at most one point (i.e., only “simple” intersections are allowed) [6] – the proof of Scheinerman’s conjecture was a strengthening of this result. The early work on this topic led West to conjecture that every planar graph is an intersection graph of line segments in four directions (4-DIR) [31].

Segment graphs have been generalized to $k$-segment graphs ($k$-SEG) where each vertex is represented by a piecewise linear curve consisting of at most $k$ segments [23]. Interestingly, a very recent result is that all planar graphs are contact 2-SEG [1]. In this context one may now consider $k$-SEG where the segments of the piecewise linear curves have a bounded number of slopes. Recently, Asinowski et al. [3] introduced the class of vertex intersection graphs of paths in a rectangular grid (VPG); equivalently, VPG is the set of intersection graphs of axis-aligned rectilinear curves in the plane (or $\bigcup_{k \geq 1} k$-SEG where each segment is either vertical or horizontal). They prove that VPG and STRING are the same graph class (this was known previously as a folklore result). Also, they focus on the subclasses which are obtained when each path in the representation has at most $k$ bends (turns) and they refer to such a subclass as $B_k$-VPG (i.e., this is $(k + 1)$-SEG with two slopes). Several relationships between existing

\footnote{In the case of contact representations, objects may only “touch” each other, but not “cross over” each other.}
graph classes and the $B_k$-VPG classes were observed. For example, every planar graph is $B_3$-VPG (this was also conjectured to be tight) and every circle graph is $B_1$-VPG. In other words, planar graphs are 4-SEG where the segments only have two distinct slopes. This result follows from the fact that every planar graph has a representation by a T-contact system \cite{11} and each T-shape can be simulated by a rectilinear curve with three bends.

In this paper we present the following results. Our main contribution is that every planar graph is $B_2$-VPG (disproving the conjecture of Asinowski et al. \cite{3}). This result consists of the following interesting components. We first demonstrate that every 4-connected planar graph is the intersection graph of Z-shapes (i.e., a special case of $B_2$-VPG). This result is extended to show that every planar graph is $B_2$-VPG (this extension involves the additional use of C-shapes – i.e., it uses the full capability of $B_2$-VPG) and that a $B_2$-VPG representation of a planar graph can be constructed in $O(n^{3/2})$ time. The secondary contribution of this paper is that every triangle-free planar graph is a contact graph of: L-shapes, Γ-shapes, vertical segments, and horizontal segments (i.e., it is a specialized contact $B_1$-VPG graph). We show how to construct such a contact representation in linear time. Moreover, if the input is bipartite then each path is a horizontal or vertical segment. In particular, we obtain a new proof that planar bipartite graphs are 2-DIR. Interestingly, the class of contact segment graphs has recently been shown to be the same as the class of contact $B_1$-VPG graphs \cite{20}.

2 Preliminaries

A grid path (a path in the plane square grid) consists of horizontal and vertical segments that appear alternatingly along the path. Every horizontal segment has a left endpoint and a right endpoint, and every vertical segment an upper endpoint and a lower endpoint in the obvious meaning. A path is a $k$-bend path if it has $k$ bends, i.e., $k$ points that are the endpoints of a horizontal and a vertical segment. Equivalently, $k$-bend paths are those with precisely $k + 1$ segments.

A $B_k$-VPG representation of a graph $G$ is a set of grid paths (one for each vertex) with at most $k$ bends such that two paths intersect if and only if the corresponding vertices are adjacent in $G$. For every vertex $v$ we denote the corresponding grid path in a given $B_k$-VPG representation by $v$. Consequently a $B_k$-VPG representation of a graph $G$ is denoted by $G$. A graph is called $B_k$-VPG if it has a $B_k$-VPG representation.

3 Planar Graphs are $B_2$-VPG

In this section we show that every planar graph $G$ has a $B_2$-VPG representation. We fix any plane embedding of $G$ and assume without loss of generality that $G$ is a maximally planar graph, i.e., all faces are triangular. To achieve this we
may put a dummy vertex into each face of $G$ and triangulate it. In a $B_2$-VPG representation of this graph the paths corresponding to dummy vertices may be removed to obtain a $B_2$-VPG representation of $G$.

Our construction of the $B_2$-VPG representation of the maximally planar graph $G$ relies on two well-known concepts. Using the separation tree $T_G$ of $G$, we show in Section 3.1 how to divide $G$ into its 4-connected maximally planar subgraphs. Each such subgraph, if we remove one outer edge, has a rectangular dual, i.e., a contact representation with axis-aligned rectangles. In Section 3.2 we show how to construct a $B_2$-VPG representation from a rectangular dual. In particular we will convert each rectangle to a Z-shaped path by choosing “part” of the top of it, the complementary “part” of the bottom of it and connecting them via a vertical segment. In Section 3.3 we put the obtained representations of all 4-connected maximally planar subgraphs of $G$ together to obtain a $B_2$-VPG representation of our graph. The same method has been used to prove that every planar graph is a $B_4$-EPG graph, where EPG stands for emphedge-intersecting paths in the grid [18].

3.1 Separation Tree

A triangle $\Delta$ in a graph is a triple of pairwise adjacent vertices. We say that a triangle is separating when its removal disconnects the graph. Also, in a maximally planar graph $G$ a triangle $\Delta$ is said to be non-empty when at least one vertex of $G$ lies inside the bounded region inscribed by $\Delta$. Notice that every separating triangle is non-empty. In fact, each non-empty triangle is either the outer triangle or separating.

We say that a triangle $\Delta_1$ is contained in a triangle $\Delta_2$, denoted by $\Delta_1 \subseteq \Delta_2$, if the bounded region enclosed by $\Delta_1$ is strictly contained in the one enclosed by $\Delta_2$. For example, the outer triangle contains every triangle in the graph (except itself), and no triangle in $G$ is contained in an inner facial triangle.

**Definition 1 ([28])** The separation tree of $G$ is the rooted tree $T_G$ whose vertices are the non-empty triangles in $G$, with $\Delta$ being a descendant of $\Delta'$ if and only if $\Delta$ is contained in $\Delta'$.

The separation tree has been introduced by Sun and Sarrafzadeh [28]. The root of $T_G$ is the outer triangle. For every non-empty triangle $\Delta$ we define $H_\Delta$ to be the unique 4-connected maximally planar subgraph of $G$ that contains $\Delta$ and at least one vertex of $G$ that lies inside $\Delta$. Equivalently, $H_\Delta$ is the union of $\Delta$ and all triangles contained in $\Delta$ but not contained in any triangle that itself is contained in $\Delta$; i.e., $H_\Delta = \Delta \cup (\bigcup_{\Delta \subseteq \Delta'} \Delta)$.

**Theorem 1 ([28])** The separation tree of $G$ and all subgraphs $H_\Delta$ can be computed in $O(n^{3/2})$.

3.2 Rectangular Duals

Throughout this section let $H$ be a triangulation of the 4-gon, i.e., $H$ is a plane graph with quadrangular outer face and solely triangular inner faces. Such
graphs are also known as irreducible triangulations of the 4-gon. We denote the outer vertices by \( T, R, B, L \) in this clockwise order around the outer face.

**Definition 2** A rectangular dual of \( H \) is a set of \( |V(H)| \) non-overlapping axis-aligned rectangles in the plane (one for each vertex) such that every edge of \( H \) corresponds to a non-trivial overlap of the boundaries of the corresponding rectangles.

The rectangle corresponding to a vertex \( v \) is denoted by \( R(v) \). In every rectangular dual the rectangles \( R(T), R(B), R(L) \) and \( R(R) \) that correspond to the outer vertices of \( H \) inscribe a rectangular hole that contains all the remaining rectangles. We assume without loss of generality that \( R(T), R(B), R(L) \) and \( R(R) \) are laid out as in Fig. 1(a), i.e., the bottom side of \( R(T) \) forms the top side of the hole, the left side of \( R(R) \) forms the right side of the hole, and so on.

Figure 1: (a) A rectangular dual; and (b) its transversal structure.

Rectangular duals have been considered several times independently in the literature \[30, 24, 22, 29, 25\]. In particular, the following theorem is well-known.

**Theorem 2** A triangulation of a 4-gon admits a rectangular dual if and only if it is 4-connected, i.e., contains no non-empty triangle.

We define here transversal structures as introduced by Fusy \[14\], which were independently considered by He \[17\] under the name regular edge labelings. For a nice overview about regular edge labelings and their relations to geometric structures we refer to the introductory article by D. Eppstein \[13\].

**Definition 3 (Fusy \[14\])** A transversal structure of a triangulation \( H \) with outer vertices \( T, L, B, R \) is a coloring and orientation of the inner edges of \( H \) with colors red and blue such that:

(i) All edges at \( T \) are incoming and blue, all edges at \( B \) are outgoing and blue, all edges at \( R \) are incoming and red, all edges at \( L \) are outgoing and red.

(ii) Around each inner vertex \( v \) the edges appear in the following clockwise cyclic order: One or more incoming red edges, one or more outgoing blue edges, one or more outgoing red edges, one or more incoming blue edges.
We denote a transversal structure by \((E_r, E_b)\), where \(E_r\) and \(E_b\) is the set of red and blue edges, respectively.

We obtain a transversal structure from any rectangular dual of \(H\) as follows. If the right side of a rectangle \(R(u)\) has a non-trivial overlap with the left side of some rectangle \(R(v)\), then we color the edge \(\{u, v\}\) in \(H\) red and orient it from \(u\) to \(v\). Similarly, if the topside of \(R(u)\) overlaps with the bottom side of \(R(v)\) then \(\{u, v\}\) is colored blue and oriented from \(u\) to \(v\). Fig. 1(b) depicts the transversal structure obtained from the rectangular dual in Fig. 1(a). It is known that every transversal structure of \(H\) arises from a rectangular dual of \(H\) in this way.

**Theorem 3 (Kant & He [19])** Every transversal structure maps to a rectangular dual.

If we identify combinatorially equivalent rectangular duals, i.e., those in which any two rectangles touch with the same sides in both duals, then Theorem 3 actually states that rectangular duals and transversal structures are in bijection. Transversal structures (and hence combinatorially equivalent rectangular duals) can be endowed with a distributive lattice structure [15]. For our purposes, we describe the minimal transversal structure of \(H\); i.e., the minimum element in the distributive lattice of all transversal structures of \(H\).

**Lemma 1 (Fusy [15])** Consider four vertices \(v, w, x, y\) in the minimal transversal structure \((E_r, E_b)\), such that \(v \rightarrow w \in E_b\), \(x \rightarrow y \in E_b\), \(v \rightarrow x \in E_r\), \(w \rightarrow y \in E_r\). Then we have neither \(x \rightarrow w \in E_b\) nor \(v \rightarrow y \in E_r\).

Moreover, the minimal transversal structure can be computed in linear time.

![Figure 2: Two configurations that do not appear in the minimal transversal structure.](image)
Let us call a rectangular dual non-degenerate if the top sides of two rectangles lie on the same horizontal line only if there is a rectangle whose bottom side overlaps with both of them. It is not difficult to see that there always exists a non-degenerate minimal rectangular dual.

Given a rectangular dual and any inner vertex \( v \) we consider the rightmost rectangle overlapping the top side of \( R(v) \). We denote the corresponding vertex of \( H \) by \( v^* \). In other words, \( (v, v^*) \) is the outgoing blue edge at \( v \) whose clockwise next edge is red (and outgoing). Similarly, \( R(v^*) \) is the bottommost rectangle overlapping the right side of \( R(v) \), i.e., \( (v, v^*) \) is the outgoing red edge at \( v \) whose clockwise next edge is blue (and incoming). Moreover, \( R(*) \) is the leftmost (topmost) rectangle overlapping the bottom side (left side) of \( R(v) \), which means that \( (*, v) \) is the incoming blue (red) edge at \( v \) whose clockwise next edge is red (blue). Note that if the transversal structure is minimal then every inner edge of \( H \) can be written as \( (v, v^*), (v, v^*), (**, v), (**, v) \) or \( (**, v) \) for some inner vertex \( v \).

From \( H \) and its fixed transversal structure \( (E_r, E_b) \) we define a new graph \( H^* \), called the split graph, and its transversal structure \( (E^*_r, E^*_b) \) as follows.

- The outer vertices of \( H \) and \( H^* \) are the same.
- For every inner vertex \( v \) of \( H \) there are two vertices \( v_1 \) and \( v_2 \) in \( H^* \).
  - There is a red edge \( v_1 \rightarrow v_2 \) in \( E^*_r \).
  - There is a red edge \( v_2 \rightarrow v_1 \) in \( E^*_r \) for every edge \( v \rightarrow w \in E_r \).
  - There are blue edges \( v_1 \rightarrow w_1 \) and \( v_1 \rightarrow w_2 \) in \( E^*_b \) for every edge \( v \rightarrow w \in E_b \).
  - There is a blue edge \( v_2 \rightarrow (v^*)_2 \) in \( E^*_b \).

See Fig. 3(b) for an example of a split graph and its rectangular dual. It is straightforward to check that \( (E^*_r, E^*_b) \) is indeed a transversal structure, namely that for every \( v \in V(H) \) incoming and outgoing red and blue edges appear around \( v_1 \) and \( v_2 \) in accordance with Definition 3. We refer to Fig. 3(b) for an illustration of this fact. Note that defining \( R(v) := R(v_1) \cup R(v_2) \) for every vertex \( v \) we obtain the transversal structure we started with.

### 3.3 VPG-representation

We want to construct a \( B_2 \)-VPG representation for every maximally planar graph \( G \). To this end we split \( G \) into its 4-connected maximally planar subgraphs. The outer face \( \Delta \) of such a subgraph \( H_\Delta \) is either the outer face of \( G \) or an inner face of \( H_{\Delta'} \), where \( \Delta' \) is the father of \( \Delta \) in the separation tree. We start by representing the outer face of \( G \) as depicted in Fig. 4. The highlighted area in the figure is called the frame for \( H_\Delta \). Formally, the frame for \( H_\Delta \) is a rectangular area such that either: the paths corresponding to two vertices of \( \Delta \) pass through it vertically and the path for the third vertex passes through it horizontally, or the paths corresponding to two vertices of \( \Delta \) pass through it horizontally and third passes through it vertically. When defining the \( B_2 \)-VPG
representation of any $H_\Delta$ we assume that we have already constructed the paths for the vertices in $\Delta$ and that there is a frame for $H_\Delta$.

We now describe how to obtain a $B_2$-VPG representation of a 4-connected maximally planar graph $H_\Delta$ given a frame $F$ for it. Our construction is based on a non-degenerate minimal rectangular dual and its split graph. Let $u$ and $w$ be the two vertices of $\Delta$ whose paths do not intersect inside $F$ and denote the third vertex in $\Delta$ by $v$. Then we consider the graph $H$ obtained from $H_\Delta$ by removing the edge \{u, w\}. Notice that $H$ is a 4-connected triangulation of a 4-gon and we assume without loss of generality that $u = L$, $v = T$, and $w = R$. Consider the minimal transversal structure, a corresponding non-degenerate minimal rectangular dual of $H$, and its split graph $H^*$ together with the transversal structure $(E^*_r, E^*_b)$. By rotating and stretching it appropriately we place the non-degenerate rectangular dual of $H^*$ inside the frame $F$, such that the right side of $L$, the bottom side of $T$ and the left side of $R$ is contained in $u$, $v$ and $w$, respectively.

We define the 2-bend path $B$ for the vertex $B$ to be a C-shape path that is contained in $F$ and whose horizontal segments intersect $u$ and $v$, the upper one being contained in the top side of $R(B)$. See Fig. 4 for an illustration.

We define a 2-bend path $v$ for every inner vertex $v$ of $H$ as follows. First, let $v$ be the union of the top side and right side of $R(v_1)$ and the bottom side
of $R(v_2)$. Now consider the vertex $v$. We extend the left horizontal end of $v$ to the right side of $R(v_1)$. In case $v = L$ we do not extend the left end of $v$. Similarly we extend the right horizontal end of $v$ horizontally to the right side of $R((v_1))$, unless $v = R$. See Fig. 5(a) for an illustration.

Figure 5: [a] The path $v$ based on the rectangles $R(v_1)$ and $R(v_2)$ in the rectangular dual of the split graph. Note: the wide edges indicate the border between split rectangles. [b] The Z-shapes arising from the split graph in Fig. 3(b).

Lemma 2 The above construction gives a $B_2$-representation of $H$.

Proof: Clearly every path defined above has at most two bends. So it remains to prove that the paths $u$ and $v$ intersect if and only if $\{u, v\}$ is an edge in $G$. Evidently, all outer edges $\{T, L\}, \{L, B\}, \{B, R\}$, and $\{T, R\}$ are realized, i.e., the corresponding paths intersect. Moreover, $T \cap B = \emptyset = L \cap R$ which means that no unwanted edge is created.

Now consider a blue edge $u \rightarrow v \in E_b$. By definition of the split graph and its transversal structure $(E^*_r, E^*_b)$ we have an edge $u_1 \rightarrow v_2$ in $E^*_b$, i.e., the top side of $R(u_1)$ and the bottom side of $R(v_2)$ overlap. In particular $u \cap v \neq \emptyset$, since $u$ and $v$ contains the top side of $R(u_1)$ and the bottom side of $R(v_2)$, respectively.

Next consider a red edge of $G$. Since the underlying rectangular dual is minimal, it does not contain the configuration in the right of Fig. 2. Thus, every red edge can be written as $(v, v_*)$ or $(v, v)$ for some inner vertex $v$. By
definition the right end of \( v \) lies on the right side of \( R((v)_{1}) \) (or \( R \) in case \( v = R \)) and the left end of \( v \) lies on the right side of \( R((v)_{1}) \) (or \( L \) in case \( v = L \)). Hence both edges are properly represented by intersecting paths.

Finally we need to argue that no two paths that correspond to non-adjacent vertices of \( G \) intersect. Therefore consider the parts of \( v \) that lie outside \( R(v) \). The left extension of \( v \) passes through \( R((v)_{2}) \). This could be along the top side of \( R((v)_{2}) \), which is by definition of the split-graph strictly contained in the bottom side of some \( R(w_{2}) \). Similarly, the right extension of \( v \) passes through \( R((v)_{1}) \) and this could be along the bottom side of this rectangle, which is strictly contained in some \( R(w_{1}) \). In other words all left extensions are contained in \( \bigcup_{v \in V} R(v_{2}) \) and all right extension are contained in \( \bigcup_{v \in V} R(v_{1}) \).

Thus a left extension may intersect a right extension only if these pass through \( R(v_{2}) \) and \( R(v_{1}) \) corresponding to the same vertex \( v \), respectively. Since the underlying rectangular dual is non-degenerate the two extensions lie on distinct \( y \)-coordinates and hence are disjoint. \( \square \)

Slightly changing the paths corresponding to outer vertices we can easily transform them into Z-shapes and make \( L \) and \( R \) intersect. Thus we obtain the following corollary.

**Corollary 1** Every 4-connected planar graph has a \( B_{2} \)-representation where every path has a Z-shape and no two paths cross. \( \square \)

We have shown so far how to define a \( B_{2} \)-VPG representation of \( H_{\Delta} \) given a frame for \( H_{\Delta} \). It remains to identify a frame for each \( \Delta' \subset \Delta \) that is a son of \( \Delta \) in the separation tree. We modify the representation for this purpose.

Consider a horizontal line \( \ell \) that supports horizontal sides of some rectangles different from \( R(T) \). We partition the paths that have a horizontal segment on \( \ell \) into two sets: \( A \) contains all paths whose vertical segment lies above \( \ell \) and \( B \) all paths whose vertical segment lies below \( \ell \). Next we extend the vertical segments of all paths in \( B \) by some small amount, keeping all lower horizontal segments unchanged. The extension is chosen small enough so that no unwanted intersections are created. See Fig. 6 for an illustration. Since the underlying rectangular dual is minimal, it does not contain the configuration in the left of Fig. 2. It follows that all vertical segments of paths in \( A \) lie to the left of the vertical segments of paths in \( B \). Thus, if \( v \in A \) and \( w \in B \) were touching before, then they are crossing after this operation.

![Figure 6: Extending the vertical segments of all paths in B.](image)

Next we identify a frame for every inner face \( \Delta' \) of \( H \). In case \( \Delta' \) is a non-empty triangle of \( G \) this will be the frame for \( H_{\Delta'} \).
Lemma 3 One can find in $H_\Delta$ a frame for every inner face of $H_\Delta$, such that each frame is contained in $F$ and all frames are pairwise disjoint.

Proof: First consider the triangle $\{L, B, R\}$, which is an inner face of $H_\Delta$ but not after the removal of the edge $\{L, R\}$. We define the frame for $\{L, B, R\}$ as illustrated in Fig. 1 to partly contain the lower horizontal segment of $B$ and the vertical segments of $L$ and $R$.

Now consider any inner face $f$ of $H_\Delta$ different from $\{L, B, R\}$ and let $u, v, w$ be the vertices of $f$ appearing in this clockwise order. Then $f$ is an inner face of $H$ corresponding to the three mutually touching rectangles $R(u), R(v)$ and $R(w)$ in the rectangular dual. Thus there is a point $p_f$ where those three rectangles meet; two rectangles having a corner at $p_f$. Without loss of generality let $R(v)$ be the rectangle that does not have corner at $p_f$. We distinguish the four cases according to which side of $R(v)$ contains $p_f$. See Fig. 7 for an illustration.

If the top side of $R(v)$ contains $p_f$, then consider the point $p$ where $R(u_1), R(u_2)$ and $R(v_1)$ meet. By definition $p$ is the lower bend of $u$ and the right horizontal end of $w$. Moreover, the upper horizontal segment of $v$ lies immediately above $p$, crossing $u$. Now, the frame for $f$ is defined around $p$ as illustrated in Fig. 7 a).

If the bottom side of $R(v)$ contains $p_f$, then consider the point $p$ where $R(u_1), R(u_2)$ and $R(v_2)$ meet. Now right above $p$ lies the upper bend of $u$ and the left horizontal end of $w$, while $v$ goes horizontally through $p$. The frame for $f$ is then defined as illustrated in Fig. 7 b).

If the right side of $R(v)$ contains $p_f$, let $p$ be the common point of $R(u_1), R(w_1)$ and $R(w_2)$, i.e., $p$ is the lower bend of $u$. The upper horizontal segment of $w$ lies right above $p$ and ends on the vertical segment of $v$. The frame for $f$ is then defined as illustrated in Fig. 7 c).

Finally, if the left side of $R(v)$ contains $p_f$, let $p$ be the common point of $R(u_2), R(w_1)$ and $R(w_2)$, i.e., right above $p$ lies the upper bend of $w$. The lower horizontal segment of $u$ runs through $p$ and ends on the vertical segment of $v$. The frame for $f$ is then defined as illustrated in Fig. 7 d).

Clearly, each frame is contained in the frame for $H_\Delta$. Moreover, each frame contains one bend or lies very close to one. Given the bend one can find the corresponding $p_f$ to the left if it is a lower bend, and to the bottom-right if it is an upper bend. It follows that frames and bends are in bijection and hence that all frames are pairwise disjoint. □
We end this section with its main theorem. It is not difficult to see that this theorem follows from Theorem 1, and Lemmas 2 and 3.

**Theorem 4**  
Every planar graph is $B_2$-VPG. Moreover, a $B_2$-VPG representation can be found in $O(n^{3/2})$, where $n$ denotes the number of vertices in the graph.

**Proof:** Given a maximally planar graph $G$ with a fixed embedding, we find the separation tree of $G$ in $O(n^{3/2})$ and all 4-connected maximally planar subgraphs $H_\Delta$ of $G$ (Theorem 1). We define a $B_2$-VPG representation of the outer triangle $\Delta$ of $G$ as explained in Section 3.3 and identify the frame for $H_\Delta$ (Fig. 4). Then we traverse the separation tree starting with the root and consider for each non-empty triangle $\Delta$ the frame $F$ for the corresponding graph $H_\Delta$. If $u$ and $w$ are the vertices of $\Delta$ whose paths $u$ and $w$ do not intersect within $F$, we consider the graph $H = H_\Delta \setminus \{u, w\}$. We find the minimal transversal structure of $H$ in $O(|V(H)|)$ (Lemma 1) and build the split graph $H'$ as described in Section 3.2. We then construct a $B_2$-VPG representation of $H$ within the frame $F$ as described in Section 3.3 and identify frames for each non-empty triangle $\Delta'$ that is an inner face of $H_\Delta$. The construction of the split graph and the $B_2$-VPG representation can be easily done in $O(|V(H)|)$. Hence the overall running time is dominated by the time needed to find the separation tree, i.e., a $B_2$-VPG representation can be constructed in $O(|V(G)|^{3/2})$. \[\square\]

## 4 Triangle-Free Planar Graphs are $B_1$-VPG

In this section we prove that every triangle-free planar graph is $B_1$-VPG with a very particular $B_1$-VPG representation. Namely, every vertex is represented by either a 0-bend path or a 1-bend path whose vertical segment is attached to the left end of its horizontal segment. This means that we use only two out of the four possible shapes of a grid path with exactly one bend. Moreover, whenever two paths intersect, it is at an endpoint of exactly one of these paths; i.e., no two paths cross. We call a 1-bend path an $L$ if the left endpoint of the horizontal segment is the lower endpoint of its vertical segment, and a $\Gamma$ if the left endpoint of the horizontal segment is the upper endpoint of its vertical segment. A VPG representation in which each path that has a bend is an $L$ or a $\Gamma$, and in which no two paths cross, is called a contact-L-$\Gamma$ representation.

We say that two contact-L-$\Gamma$ representations of the same graph $G$ are equivalent if the underlying combinatorics is the same. That means that paths corresponding to the same vertex have the same type (either $L$, $\Gamma$, horizontal or vertical segment), the inherited embedding of $G$ is the same, and that the fashion in which two paths touch is the same, e.g., the right endpoint of $u$ is contained in the vertical segment of $v$ in both representations. However, it is convenient in our proofs to deal with actual contact-L-$\Gamma$ representations instead of equivalence classes of contact-L-$\Gamma$ representations. Therefore we need the following lemma.
Lemma 4 Let $G$ be a plane graph and $v$ be a vertex of $G$. Let $u$ and $w$ be two paths in $G$ that touch $v$ at the same segment but from different sides. Then there exists a contact-L-$\Gamma$ representation of $G$ that is equivalent to $G$ in which the touching points of $u$ and $w$ with $v$ come in the reversed order along $v$.

Proof: We obtain the required representation from $G$ with a simple operation, called slicing. Assume without loss of generality that the segment $s_v$ of $v$ that is touched by $u$ and $w$ is vertical, i.e., the horizontal segments $s_u$ of $u$ and $s_w$ of $w$ touch $s_v$. Assume further without loss of generality that $s_u \cap s_v$ lies above $s_w \cap s_v$ and that $s_u$ lies to the left and $s_w$ to the right of $s_v$, respectively. Consider any 2-bend grid path $P$ containing $s_u$ and $s_w$ and extend its left and right endpoints to the left and to the right to infinity, respectively. Then $P$ divides the plane into two unbounded regions. We denote the lower region by $A$ and consider $s_u$ to be contained in $A$, and the upper region by $B$ and consider $s_w$ to be contained in $B$. Now we increase the $y$-coordinates of every point in $B$ by some amount large enough that $s_w \cap s_v$ lies above $s_u \cap s_v$. All vertical segments that cross $P$, including $s_v$ and maybe the vertical segments of $u$ and $w$ are extended so that the corresponding paths are connected again.

The slicing operation is illustrated in Fig. 8. Figuratively speaking, we cut the plane along $P$ and pull the two pieces apart until $s_u$ and $s_w$ change the order along $s_v$, while paths that cross $P$ are stretched instead of cut. \[\square\]

The main result of this section is the following.

Theorem 5 Every triangle-free planar graph has a contact-L-$\Gamma$ representation.

Note that if some graph $G$ admits a contact-L-$\Gamma$ representation then so does every subgraph $H$ of $G$. Indeed every edge $(u,v)$ in $E(G) \setminus E(H)$ corresponds to a contact point of $u$ and $v$ in the representation $G$. Moreover, this contact point is an endpoint of one of the two paths. If we shorten this path a little bit, and do this for every edge that is in $G$ but not in $H$, then we obtain a contact-L-$\Gamma$ representation of $H$. Thus we assume for the remainder of the section without loss of generality that $G$ is a maximally triangle-free planar graph, i.e., $G$ is 2-connected and every face of $G$ is a quadrangle or a pentagon. Moreover, we can assume by adding one vertex (if necessary) that the outer face of $G$ is a quadrangle.
Consider a contact-L-Γ representation $C$ of a cycle $C$ on four vertices $v_1, v_2, v_3, v_4$ and assume without loss of generality that any two paths in $C$ touch at most once. Then $v_1 \cup v_2 \cup v_3 \cup v_4$ inscribes a simple rectilinear polygon $P$. We call the parts of $C$ that do not lie in the interior of $P$ the outside of $C$. See Fig. 9 for an example.

![Figure 9: A contact-L-Γ representation of a 4-cycle. Its outside is highlighted.](image)

We prove the following stronger version of Theorem 5.

**Theorem 6** Let $G$ be a maximally triangle-free planar graph with a fixed plane embedding and a quadrangular outer face $C_{\text{out}}$. Let $C_{\text{out}}$ be any contact-L-Γ representation of $C_{\text{out}}$. Then there is a contact-L-Γ representation of $G$ with the same underlying embedding in which the outside of the induced representation of $C_{\text{out}}$ is equivalent to that in $C_{\text{out}}$.

**Proof:** We do induction on the number of vertices in $G$, distinguishing the following three cases.

**Case 1:** $G$ has a separating 4-cycle $C$. Let $V_C$ be the set of vertices interior to $C$ and $G_1$ be the graph $G - V_C$. Note that $G_1$ is maximally triangle-free and with outer face $C_{\text{out}}$. Hence by induction we find a contact-L-Γ representation $G_1$ of $G_1$ such that $C_{\text{out}}$ is represented with an equivalent outside as in $C_{\text{out}}$. Since the representation $G_1$ respects the embedding of $G_1$, the interior of $C$ is empty. We again apply induction to $G_2 = G[V \cup V_C]$ with respect to the representation $C$ induced by $G_1$ and obtain a contact-L-Γ representation $G_2$. Since the outside of the representation of $C$ in $G_2$ is equivalent to that in $G_1$, we can put together $G_1$ and $G_2$ and obtain a contact-L-Γ representation $G$ of $G$ that satisfies our requirements.

**Case 2:** $G$ has a facial 4-cycle $C = \{v_1, v_2, v_3, v_4\}$. Let $v_1$ and $v_3$ be two opposite vertices on $C$ that have distance (counted by the number of edges) at least 4 in $G - \{v_2, v_4\}$. Since $G$ is triangle-free and planar, such vertices exist and we can moreover assume without loss of generality that $v_1$ is not an outer vertex. Let $\tilde{G}$ be the graph resulting from $G$ by merging $v_1$ and $v_3$, and denoting the new vertex by $\tilde{v}$. Note that $\tilde{G}$ is a maximally triangle-free planar graph that inherits a plane embedding from $G$. Moreover $\tilde{G}$ has outer cycle $C_{\text{out}}$ where possibly $v_3$ is replaced by $\tilde{v}$. By induction we find a contact-L-Γ representation $\tilde{G}$ of $\tilde{G}$. Next we split the path $\tilde{v}$ in $\tilde{G}$ into two, one for $v_1$ and one for $v_3$, which will result in a contact-L-Γ representation $G$ of $G$. See Fig. 10 for an example.
Consider the circular ordering of contacts when tracing around $\tilde{v}$ in $\tilde{G}$. The paths $v_2$ and $v_4$ split the circular ordering into two consecutive blocks, that is, subsets of contacts corresponding to neighbors of $v_1$ and one corresponding to neighbors of $v_3$ in $G$. (There are no common neighbors of $v_1$ and $v_3$ apart from $v_2$ and $v_4$, because $v_1$ and $v_3$ are at distance at least 4 in $G - \{v_2, v_4\}$.) Now define $v_3$ to be the sub-path of $\tilde{v}$ defined by the block of neighbors of $v_3$. Moreover define $v_1$ in the same way w.r.t. the neighbors of $v_1$, except that $v_1$ is translated by some small amount “towards its block”. Finally, every path $u$ corresponding to a neighbor $u$ of $v_1$ different from $v_2$ and $v_4$ is shortened or extended so that it touches $v_1$. The procedure for Case 2 is illustrated in Fig. 10.

It is important to note that, even if an outer edge is involved in the above construction, the outsides of $C_{\text{out}}$ in $G$ is equivalent to that in $\tilde{G}$.

Case 3: Neither Case 1 nor Case 2 applies and there is an edge $(u, v)$ in $G$ with interior vertices $u$ and $v$. We contract the edge $(u, v)$ and denote by $\tilde{v}$ the new vertex in the resulting graph $\tilde{G}$. Since neither Case 1 nor Case 2 applies, $u$ and $v$ are at distance 4 in $G - (u, v)$ and thus $\tilde{G}$ is maximally triangle-free. Moreover $\tilde{G}$ has outer cycle $C_{\text{out}}$ and inherits its plane embedding from $G$. By induction we find a contact-L-Γ representation $\tilde{G}$, in which we want to split $\tilde{v}$ into two paths $v$ and $u$, such that the result is a contact-L-Γ representation $G$ of $G$.

As in the previous case we trace the contour of $\tilde{v}$ and see two disjoint blocks, each consisting of those contacts that correspond to neighbors of $u$ and $v$ in $G$, respectively. We denote the block corresponding to $u$ and $v$ by $B_u$ and $B_v$, respectively. Without loss of generality assume that $B_u \cup B_v$ is the entire contour of $\tilde{v}$. We distinguish the following four sub-cases. By symmetry we assume that $\tilde{v}$ is not a Γ-shape and denote its vertical segment (if existent) by $s$.

In Case 3a either $s$ is completely covered by one block, say $B_u$, or $\tilde{v}$ is only a horizontal segment and $B_u$ is the block that contains the left endpoint of it. We define $u$ and $v$ to be the sub-paths of $\tilde{v}$ that are covered by $B_u$ and $B_v$, respectively. We shift $v$ a little bit up or down and attach a short vertical segment to its left endpoint so as to touch $u$. The construction is illustrated in Fig. 11.
In Case 3b the left side of $s$ is completely covered by one block, say $B_u$. We define $u$ to be the sub-path of $\tilde{v}$ that is covered by $B_u$. If $B_v$ is contained in $s$, we define $v$ to be a very short horizontal segment touching the right side of $s$ immediately below the $B_v$. Otherwise we define $v$ to be the sub-path of the horizontal segment of $\tilde{v}$ that is covered by $B_v$ and shift $v$ a little bit up. Note that each path that touches the right side of $s$ is only a horizontal segment. We shorten the left endpoint of each such path that corresponds to a neighbor of $v$ a little bit and attach a vertical segment to it that touches $v$ from above. This can be done so that no two such paths intersect. Moreover, every vertical segment touching $\tilde{v}$ and corresponding to $B_v$ is shortened or extended a bit so as to touch $v$. See the left of Fig. 12 for an illustration.

In Case 3c either the horizontal segment of $\tilde{v}$ is completely covered by one block, say again $B_u$, or $\tilde{v}$ is only a vertical segment and $B_u$ is the block that contains the lower endpoint of it. Note that since Case 3b does not apply, $B_v$ partially covers the left side of $s$. By Lemma 4 we can assume that no point of $s$ is covered on the left by $B_u$ and on the right by $B_v$. We define $u$ and $v$ to be the sub-paths of $\tilde{v}$ that are covered by $B_u$ and $B_v$, respectively, and shift $v$ a little bit to the left. Again we shorten or extend each path that corresponds to a neighbor of $v$ so that it touches $v$. See the middle of Fig. 12 for an illustration.

In the remaining case, Case 3d, both blocks $B_u$ and $B_v$ appear on both sides of the vertical and horizontal segment of $\tilde{v}$. Let $B_u$ be the block that contains
the upper end of $\tilde{v}$. Consider paths that touch the horizontal segment of $\tilde{v}$ on the upper side and within the block $B_u$. By Lemma 4, may can assume that the horizontal segment of each such path lies above the block $B_u$. We define $u$ and $v$ to be the sub-paths of $\tilde{v}$ that are covered by $B_u$ and $B_v$, respectively. We shift the horizontal segment of $u$ up to the upper endpoint of $v$ and move $u$ a little bit to the left so that $v$ touches $u$ from below. Moreover, we shorten or extend every path corresponding to a neighbor of $u$ so that it touches $u$. This completes Case 3.

Finally, if neither of Case 1, Case 2 and Case 3 applies, then $G$ consists only of the outer cycle $C_{out}$, for which a Contact-L-Γ representation $C_{out}$ is given by assumption. This concludes the proof. □

Theorem 5 can be easily transferred into a linear-time algorithm to find a contact-L-Γ representation of a triangle-free planar graph. Note that such an algorithm should first construct the combinatorics of the representation, since slicing operation would have to be done in $O(1)$. The computation of the actual coordinates of each path can be easily carried out afterwards in linear time. Moreover the constructed representation can be placed into the $n \times n$ grid, since every path requires only one horizontal and one vertical grid line. Here $n$ denotes the number of vertices in $G$.

5 Future Work and Open Problems

We have disproved the conjecture of Asinowski et al. [2] that $B_3$-VPG is the simplest $B_k$-VPG graph class containing planar graphs. Specifically, we have demonstrated that every planar graph is $B_2$-VPG and that 4-connected planar graphs are the intersection graphs of Z-shapes (i.e., a special subclass of $B_2$-VPG). We have also shown that these representations can be produced from a planar graph in $O(n^{3/2})$ time. We have additionally shown that every triangle-free planar graph is a contact graph of: L-shapes, Γ-shapes, vertical segments, and horizontal segments (i.e., it is a specialized contact $B_1$-VPG graph). Furthermore, we demonstrated how to construct such a contact representation in linear time. As an further consequence, we obtain a new proof that planar bipartite graphs are 2-DIR.

Interestingly, there is no known planar graph which does not have an intersection representation of L-shapes; i.e., even this very restricted form of $B_1$-VPG is still a good candidate to contain all planar graphs. Further to this, a colleague of ours has observed (via computer search) that all planar graphs on at most ten vertices are intersection graphs of L-shapes [16]. Similarly, all small triangle-free planar graphs seem to be contact graphs of L-shapes. These observations lead to the following two conjectures.

**Conjecture 1** Every planar graph is the intersection graph of L-shapes.

**Conjecture 2** Every triangle-free planar graph is the contact graph of L-shapes.
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