Modeling of stresses in inorganic composite plates under non-uniform high temperature heating

Yu I Dimitrienko, E S Egoleva, D O Yakovlev and S V Sborschikov
Bauman Moscow State Technical University, Moscow, 105005, Russia
E-mail: dimit@bmstu.ru

Abstract. An asymptotic theory of thin composite plates with multi-stage high-temperature phase transformations is proposed. The theory is based on an asymptotic analysis of 3-dimensional equations of the mechanics of composite materials, taking into account phase transformations. Phase transformations are described by a system of kinetic equations, which is solved in conjunction with the equations of internal heat and mass transfer. The so-called local problems for plate theory are obtained and the averaged equations of plate theory with phase transformations are derived. An example of a composite based on an aluminum-chromo-phosphate matrix under uneven heating is considered. A numerical-analytical solution showed that although the heating is uneven due to the peculiarities of phase transformations, the stress-strain state changes significantly in time and is variable across the plate thickness. The developed theory allows us to calculate the stress distribution in the plate with high accuracy.

1. Introduction
Composite materials based on inorganic matrices, for example, aluminum phosphate, aluminum chromium phosphate, magnesium phosphate, and ceramic fibers based on SiO2, Al2O3, ZrO2, and others, are promising materials for the electrical, engineering, and aerospace industries [1-3]. These materials are able to withstand high temperatures, for a long time up to 500 °C, and for a short time, more than 1000 °C, while maintaining a sufficiently high mechanical and electromagnetic properties. In contrast to polymeric materials, in which mechanical properties are degraded upon heating, inorganic matrix composites (IMC) under certain conditions, temperature hardening is possible, and only then degradation. The reason for such unusual properties of IMC is the complex chain of phase transformations that occur in the matrix and fibers when heated to high temperatures. There are currently very few mathematical models of the mechanical behavior of IMC taking into account phase transformations. Apparently, the first such model was proposed in [3,4]. The aim of this work is to use this model of thermomechanical behavior of IMC to calculate stresses in thin plates under non-uniform high-temperature heating.

To calculate the stresses in thin plates, an asymptotic analysis of 3-dimensional equations of the theory of elasticity [5-15] is used, which provides high accuracy in calculating the fields of stresses, strains and displacements, and is not internally contradictory, unlike many traditional plate theories.

2. Statement of the three-dimensional problem of the linear theory of thermoelasticity for a thin plate with phase transformations
Consider a thin plate of tissue IMC, in which multi-stage high-temperature phase transformations (MHPT) occur in the matrix and fibers upon heating. We introduce a small parameter \( \chi \), where \( h \) is the thickness and \( L \) is the plate length. We denote the dimensionless coordinates: \( \hat{x}_k = \gamma x_k / L \), \( k = 1, 2, 3 \), where \( \hat{x}_k \) are the ordinary rectangular Cartesian coordinates, such that the axis \( O\hat{\gamma}_3 \) is normal to the upper
and lower planes of the plate, and the axis $O_1$, $O_2$ belong to the median plane of the plate. We write in a dimensionless form the non-stationary problem of heat conduction, the three-dimensional problem of the linear theory of thermoelasticity for a plate with MHPT, and also the system of kinematic equations for phase concentrations in the matrix and fibers

$$\rho(\varphi)C(\varphi)\frac{\partial \theta}{\partial t} = -\chi^2 \frac{\partial q_1}{\partial x_1}, \quad q_i = -\lambda_i(\varphi)\frac{\partial \theta}{\partial x_i}, \quad (1)$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad \sigma_{ij} = A_{ijkl}(\varphi)(\varepsilon_{ij} - \varepsilon_{ij}^T), \quad \varepsilon_{ij}^T = \alpha_{ij}(\varphi)(\theta - \theta_0). \quad (2)$$

$$\sum_{s=1}^{s} \Sigma_{t}: \sigma_{ij} = -\chi^2 \rho_{st} p_{ij} \delta_{ij}, \quad \theta = \theta_{st}, \quad (3)$$

$$\Sigma_{r}: q_i n_j = 0; \quad u_i = u_{st}, \quad (4)$$

$$\rho_{sw} \frac{\partial \varphi_{sw}}{\partial t} = \sum_{s=1}^{s} J_{sw} \varphi_{sw}, \quad w = \{a, b\}, \quad \rho_{sw} \frac{\partial \varphi_{sw}}{\partial t} = J_{sw}^0 \varphi_{hw}(\theta). \quad (5)$$

$$\sum_{s=1}^{s} \varphi_{sw} = \varphi_a, \quad \varphi_a + \varphi_b = \varphi_f, \quad \varphi_a + \varphi_b + \varphi_f + \varphi_h = 1, \quad (6)$$

$$t = 0: \quad \varphi_{sw} = \varphi_{sw}^0, \quad \varphi_h = \varphi_f, \quad \theta = \theta^0. \quad (7)$$

Dimensionless quantities are indicated here: $t$ - time, $u_j$ - components of the displacement vector, $\theta$ - temperature, $\rho$ - density, $c$ - heat capacity of the composite, $\sigma_{ij}, \varepsilon_{ij}$ - components of the tensors of stresses and deformations, $A_{ijkl}$ - components of the tensor of elastic moduli, $\alpha_{ij}$ - components of the tensor of thermal conductivity, $q_i$ - components of the heat flux vector, $u_{st}$ - given components displacement vectors at the ends $\Sigma_{t}$ of the plate, $p_{ij}, \theta_{st}$ - specified pressures and temperature on the outer surfaces $\Sigma_{st}$ of the plate, $\varphi_{sw}$ - volumetric phase concentrations in the matrix, $w$ - index of the initial component in the matrix, $s$ - number of the phase of each component of the matrix ($s = 1, ..., N$), $\varphi_h$ - is the concentration of the initial (amorphous) phase of the fiber, $\varphi_f$ - is the crystalline phase in the fibers, $\varphi_a$ - is the porosity, $\rho_{sw}, \rho_h$ - is the phase density. The matrix $J_{sw}$ of multistage phase transformations rates has a size $N \times (N-1)$ and is characterized by a ribbon structure (2 elements in each row, one with “+”, the other with “-”), for example, for the case $N = 4$

$$J_{sw} = \begin{pmatrix} -J_{1w}^0 f_{1w}(\theta) & 0 & 0 \\ J_{1w}^0 f_{1w}(\theta)(1 - \Gamma_1) & -J_{2w}^0 f_{2w}(\theta) & 0 \\ 0 & J_{2w}^0 f_{2w}(\theta) & -J_{3w}^0 f_{3w}(\theta) \\ 0 & 0 & J_{3w}^0 f_{3w}(\theta)(1 - \Gamma_1) \end{pmatrix}. \quad (8)$$

The functions $f_{sw}(\theta)$ and $f_{hw}(\theta)$ obey the law of Arrhenius: $f_{sw}(\theta) = \exp(-E_{sw}/\theta)$, $f_{hw}(\theta) = \exp(-E_{hw}/\theta)$, where $E_{sw}$ and $E_{hw}$ are the dimensionless activation energies of phase transformations. All plate characteristics: $A_{ijkl}(\varphi), \lambda_{st}(\varphi), \alpha_{st}(\varphi), \rho(\varphi)C(\varphi)$ - depend on phase concentrations $\varphi = (\varphi_{sw}, \varphi_h, \varphi_f), \ w = \{a, b\}$. 


3. Asymptotic expansions

We introduce the local dimensionless coordinate \( \xi = x_2 / \chi \), \(-0.5 < \xi < 0.5\). The solution to problem (1) - (7) will be sought in the form of asymptotic expansions in terms of a small parameter \( \chi \)

\[
\varphi = \varphi^{(0)}(x_1, \xi) + \chi \varphi^{(1)}(x_1, \xi) + \chi^2 \varphi^{(2)}(x_1, \xi) + \chi^3 \ldots \\
\theta = \theta^{(0)}(x_1, \xi) + \chi \theta^{(1)}(x_1, \xi) + \chi^2 \theta^{(2)}(x_1, \xi) + \chi^3 \ldots \\
u_k = u_k^{(0)}(x_1, \xi) + \chi u_k^{(1)}(x_1, \xi) + \chi^2 u_k^{(2)}(x_1, \xi) + \chi^3 u_k^{(3)}(x_1, \xi) + \ldots \\
\sigma_{ij} = \sigma_{ij}^{(0)} + \chi \sigma_{ij}^{(1)} + \chi^2 \sigma_{ij}^{(2)} + \ldots
\]

Substituting expansions (9) into problem (1) - (7) and grouping the terms by degrees \( \chi \), we obtain a recurrent sequence of local problems of thermoelasticity with allowance for multistage phase transformations. The problem in the zeroth approximation has the form:

\[
\rho \frac{\partial \varphi^{(0)}}{\partial t} = \sum_{i=1}^{N-1} \int_{svw} \varphi^{(0)}(x_0, \xi) w = (a, b), \quad s = 1, 4, \quad \rho_h \frac{\partial \varphi^{(0)}}{\partial t} = J_h^{(0)} \varphi^{(0)} h \left( \theta^{(0)} \right),
\]

\[
\rho C \left( \varphi^{(0)} \right) \frac{\partial \theta^{(0)}}{\partial t} = \left( \lambda^{(0)}_{33} \theta^{(0)} \right)_{,3}.
\]

\[
\sigma_{33,3}^{(0)} = 0, \quad \sigma_{33}^{(0)} = 2 A_{33} K_3 \left( \varphi^{(0)} \right) \varepsilon_{K3}^{(0)}, \quad \sigma_{33}^{(0)} = A_{33} \left( \varphi^{(0)} \right) \left( \varepsilon_{12}^{(0)} - \varepsilon_{12}^{(0)T} \right) + A_{3333} \left( \varphi^{(0)} \right) \left( \varepsilon_{33}^{(0)} - \varepsilon_{33}^{(0)T} \right),
\]

\[
\varepsilon^{(0)}_{ij} = \frac{1}{2} \left( u_{i,j}^{(0)} + u_{j,i}^{(0)} \right), \quad \varepsilon_{i3}^{(0)} = \frac{1}{2} \left( u_{i,3}^{(0)} + u_{3,j}^{(0)} \right), \quad \varepsilon_{33}^{(0)} = u_{3,3}^{(0)},
\]

\[
\Sigma_{33} : \theta^{(0)} = \theta_{33}, \quad \sigma_{33}^{(0)} = 0; \quad \int_{-0.5}^{0.5} u_{3,i}^{(0)} d \xi = 0;
\]

\[
t = 0: \quad \varphi_{33}^{(0)} = \varphi_{33,3}^{(0)} = 0, \quad \varphi_{i3}^{(0)} = \varphi_{i3}^{(0)} = 0, \quad \theta^{(0)} = \theta^{(0)}, \quad \varphi_{33}^{(0)} = \varphi_{33}^{(0)} = 0, \quad \varphi_{i3}^{(0)} = \varphi_{i3}^{(0)} = 0.
\]

Hereinafter, the notation for the derivatives is introduced: \( f^{(0)} = f, \quad \frac{\partial f}{\partial \xi} = f, \quad \frac{\partial f}{\partial x_i} = f, \quad \frac{\partial f}{\partial x_j} = f, \quad \frac{\partial f}{\partial x_i} = f \).

The problems for higher approximations at \( m = 1, 2, \ldots \) are as follows:

\[
\rho_m \frac{\partial \varphi^{(m)}}{\partial t} = \sum_{i=1}^{N-1} \int_{svw} \varphi^{(m)}(x_0, \xi) w = (a, b), \quad s = 1, N, \quad \rho_h \frac{\partial \varphi^{(m)}}{\partial t} = \sum_{m=0}^{m} J_h^{(m)} \varphi^{(m)} h \left( \theta^{(m)} \right),
\]

\[
\rho C^{(m)} \frac{\partial \theta^{(m)}}{\partial t} = \left( \lambda^{(m)}_{33} \theta^{(m)} \right)_{,3},
\]

\[
\sigma_{33,3}^{(m)} = 0, \quad \sigma_{33}^{(m)} = 2 A_{33} K_3 \left( \varphi^{(m)} \right) \varepsilon_{K3}^{(0)}, \quad \sigma_{33}^{(m)} = A_{33} \left( \varphi^{(m)} \right) \left( \varepsilon_{12}^{(m)} - \varepsilon_{12}^{(m)T} \right) + A_{3333} \left( \varphi^{(m)} \right) \left( \varepsilon_{33}^{(m)} - \varepsilon_{33}^{(m)T} \right),
\]

\[
\varepsilon^{(m)}_{ij} = \frac{1}{2} \left( u_{i,j}^{(m)} + u_{j,i}^{(m)} \right), \quad \varepsilon_{i3}^{(m)} = \frac{1}{2} \left( u_{i,3}^{(m)} + u_{3,j}^{(m)} \right), \quad \varepsilon_{33}^{(m)} = u_{3,3}^{(m)},
\]

\[
\Sigma_{33} : \theta^{(m)} = \theta_{33}^{(m)}, \quad \sigma^{(m)} = -p \delta_{33} \delta_{33}, \quad \varepsilon_{33}^{(m)} = 0, \quad \sigma_{ij}^{(m)} = \sigma_{ij}^{(m)} = 0, \quad \theta^{(m)} = 0.
\]
After the introduction of the functions, the equilibrium equations (2) take the form:

\[ h_t^{(0)} + \chi h_t^{(1)} + \chi^2 h_t^{(2)} + \ldots = 0 \]  

(13)

4. The solution of local problems

The problem of non-stationary thermal conductivity and the task of calculating the concentrations in system (10) are solved separately from the problem of mechanical equilibrium, and they are solved numerically.

As a result, we find the distribution of temperatures \( \theta^{(0)}(\xi, x, t) \) and phase concentrations \( \varphi_w^{(0)}, \varphi_h^{(0)} \). The solution of the equilibrium equations in system (10) has the form

\[
\begin{align*}
\sigma_{13}^{(0)} &= 0, \\
u_1^{(1)} &= -\xi u_2^{(1)}, \\
u_3^{(1)} &= -\epsilon_{KL} U_{3, KL} + U_3^T, \\
U_{iKL}(\xi) &= \langle Z_{iKL} >_\xi, \\
U_1^T(\xi) &= \langle Z_{iKL} >_\xi 
\end{align*}
\]

(14)

where \( Z_{iKL} = A_{i3,j3} A_{j3,k,li} \).

Solving local problems of the theory of elasticity in system (12), we find recurrence formulas for stresses of higher approximations

\[
\sigma_{13}^{(m)} = -\sigma_{13}^{(m-1)}(\xi), \quad \sigma_{13}^{(m+1)} = -\left\{ \sigma_{13}^{(m)}(\xi) \right\}_{\xi} - \left( p_\theta + \Delta p (\xi + 0.5) ) \right\} \delta_1 \delta_m
\]

(15)

as well as expressions for constants

\[
\sigma_{13}^{(1)} = \xi A_{iKL} \text{KL} - A_{iKL} e_{i}^{(1)}, \\
\epsilon_{KL}^{(0)} = \xi \eta_{KL}, \\
\eta_{KL} = -u_{3, KL}^{(0)}
\]

(16)

For \( m = 1,2,3 \) stress and strain, we calculate explicitly through the derivatives of displacements \( u^{(0)}_3 \) and \( u^{(0)}_1 \)

\[
\sigma_{13}^{(1)} = \xi A_{iKL} \text{KL} - A_{iKL} e_{i}^{(1)}, \\
\epsilon_{KL}^{(0)} = \xi \eta_{KL}, \\
\eta_{KL} = -u_{3, KL}^{(0)}
\]

(16)

5. Averaged equations of plates with multi-phase transformations

We denote the forces \( N_{ij} \), transverse forces \( Q_i \) and moments \( M_{ij} \) in the plate

\[
N_{ij} = -\sigma_{ij}^{(0)} + \chi < \sigma_{ij}^{(0)} > + \ldots, \\
M_{ij} = \chi < \xi \sigma_{ij}^{(0)} > + \chi^2 < \xi^2 \sigma_{ij}^{(0)} > + \ldots
\]

(17)

Averaging the equilibrium equations in system (12) with allowance for (17), we obtain the moment equations and equilibrium equations for thin plates [15]

\[
N_{ij} = 0, \quad Q_i = \chi^2 \Delta p, \quad M_{ij} - Q_i = 0.
\]

(18)

The averaged constitutive relations for plates with phase transformations have the following form

\[
N_{ij} = \bar{A}_{iKL} \left( \sigma_{ij}^{(0)} \right) e_{ij}^{(0)} + B_{iKL} \left( \sigma_{ij}^{(0)} \right) \eta_{KL} - N_{ij}^T, \\
M_{ij} = B_{iKL} \left( \sigma_{ij}^{(0)} \right) e_{ij}^{(0)} + \bar{D}_{iKL} \left( \sigma_{ij}^{(0)} \right) \eta_{KL} - M_{ij}^T
\]

(19)

where membrane, mixed, flexural and also shear modules of the plate stiffnesses are indicated

\[
\bar{A}_{iKL} \left( \sigma_{ij}^{(0)} \right) = A_{iKL} \left( \sigma_{ij}^{(0)} \right) >, \quad A_{iKL}^{(0)} = A_{iKL} - A_{i3} A_{3, KL}^{-1} A_{3, KL}, \\
B_{iKL} \left( \sigma_{ij}^{(0)} \right) = \chi < \xi A_{iKL}^{(0)} >, \quad \bar{D}_{iKL} \left( \sigma_{ij}^{(0)} \right) = \chi^2 < \xi^2 A_{iKL}^{(0)} >, \\
N_{ij}^T = A_{i3,ij} e_{ij}^{(1)} >, \quad M_{ij}^T = \chi < A_{3,ij} U_{3,ij}^T > + \chi < A_{3,ij} e_{ij}^{(1)} >
\]

(20)
6. Numerical calculation results for a thin plate with uneven heating and bending

Figures 1 - 3 show the results of numerical modeling of the solution to the problem of deflection of a multilayer plate when exposed to a uniformly distributed pressure \( p^+ = 0.1 \text{ MPa} \) and a uniform temperature field \( \theta_\text{up} \) on the upper surface \((\xi = 0.5)\). The left end of the plate is pinched, the right one is freely supported, the lower surface \((\xi = -0.5)\) is thermally insulated, the plate thickness is 0.025 m, length 1 m.

**Figure 1.** Temperature change from warm-up time at \( \xi = 0.5; 0.25; 0; -0.5 \).

**Figure 2.** Distribution of the deflection along the length of the plate at different time moments.

Figure 1 shows the temperature change over time at different distances from the heating surface. As shown in Figure 1, a constant temperature is set on the upper surface, and the plate layer located at a distance of 0.25 from the upper surface \((\xi = 0.25)\) warms up faster than the layer in the middle of the plate \((\xi = 0)\) and the lower surface \((\xi = -0.5)\). At the beginning of heating, only the first phase was present in the composite material matrix; phases 4 and 5 remained at the end of heating. With an increase in the heating time, the concentration of the “amorphous” phase of the fiber decreases and the concentration of the “crystalline” phase of the fiber increases at a distance of 0.25 from the upper surface of the plate.

The maximum temperature of 1800 K is on the upper surface of the plate, and in time \( t_1 \) the middle surface of the plate has time to warm up to a temperature of about 700 K, while the lower surface warms up only to 400 K. By the time \( t_1 \) at the point \( \xi = 0.3 \), the elastic modulus of the binder is much higher than on the upper surface of the plate, this shows that there is a phase ratio of the binder in a certain temperature range at which the elastic properties will be greater than at the maximum temperature. This is a feature of the
proposed model of phase transformations of the binder, which is characterized by an increase in the elastic properties of the binder in the temperature range 800–1100 K [6].

Figures 2 and 3 show the changes in shear stresses and deflection over the plate thickness in the cross section \( x_1 = 0.75 \) at time instants \( t_1 < t_2 < t_3 < t_4 \). Plate deflection increases nonlinearly over time.

![Figure 3. Distribution of the in shear stress \( \sigma_{13} \) along the length of the plate at different time moments in the cross section \( x_1 = 0.75 \).](image)

7. Conclusions
A modified version of the asymptotic theory of the calculation of thin elastic multilayer plates from high-temperature composites with multi-stage phase transformations is developed, taking into account the finite values of the shear characteristics of the composites.

A numerical-analytical solution to the problem of the stress-strain state of a thin-walled plate made of a fabric composite material with high-temperature phase transformations under non-uniform non-stationary heating and uniform pressure is obtained. The calculations performed to establish the effect of phase transformations on the kinetics of changes in the stress-strain state of the plate for different time instants, in particular, the appearance of internal zones of hardening of the material after preliminary heating to the beginning of the intermediate stages of phase transformations, and, as a result, an increase in the level of bending stresses in these zones.

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