CYLINDRICITY OF COMPLETE EUCLIDEAN SUBMANIFOLDS WITH RELATIVE NULLITY

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Abstract. We show that a complete Euclidean submanifold with minimal index of relative nullity \( \nu_0 > 0 \) and Ricci curvature with a certain controlled decay must be a \( \nu_0 \)-cylinder. This is an extension of the classical Hartman cylindricity theorem.

1. Introduction

The simplest examples of isometric immersions \( f : M^n \rightarrow \mathbb{R}^m \) such that the index of relative nullity is positive everywhere are the \( s \)-cylinders. The isometric immersion \( f \) is said to be an \( s \)-cylinder if there exists a Riemannian manifold \( N^{n-s} \) such that \( M^n, \mathbb{R}^m \) and \( f \) have factorizations

\[
M^n = \mathbb{R}^s \times N^{n-s}, \quad \mathbb{R}^m = \mathbb{R}^s \times \mathbb{R}^{m-s} \quad \text{and} \quad f = I \times h,
\]

where \( h : N^{n-s} \rightarrow \mathbb{R}^{m-s} \) is an isometric immersion and \( I : \mathbb{R}^s \rightarrow \mathbb{R}^s \) is the identity map. Clearly, in this case the minimal index of relative nullity \( \nu_0 \) of \( f \) is precisely \( s \), as long as that of \( h \) is zero.

The classical Hartman theorem states that these are the only possible complete examples with nonnegative Ricci curvature.

Theorem 1 (Maltz [1]). Let \( M^n \) be a complete manifold with nonnegative Ricci curvature and let \( f : M^n \rightarrow \mathbb{R}^m \) be an isometric immersion with minimal index of relative nullity \( \nu_0 > 0 \). Then \( f \) is a \( \nu_0 \)-cylinder.

The main purpose of this article is to extend the above result to submanifolds with Ricci curvature having a certain controlled decay.

Theorem 2. Let \( M^n \) be a complete manifold with

\[
\text{Ric} \geq - \left( \text{Hess} \psi + \frac{d\psi \otimes d\psi}{n-1} \right)
\]

for some function \( \psi \) bounded from above on \( M^n \) and let \( f : M^n \rightarrow \mathbb{R}^m \) be an isometric immersion with minimal index of relative nullity \( \nu_0 > 0 \). Then \( f \) is a \( \nu_0 \)-cylinder.

Note that we recover Theorem 1 from the above by simply taking \( \psi \) to be constant.

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Remarks 1. (i) In Wylie [2], such a Riemannian manifold satisfying (1.1) was said to be CD(0,1) with respect to the potential function $\psi$.

(ii) We actually prove a version of Theorem 2 that is more general in two ways. The first is that we can weaken the upper bound on $\psi$ assumption to an integral condition along geodesics, the so-called bounded energy distortion. Secondly the function $\psi$ can be replaced with a vector field $X$. We delay discussing this result until Section 4.

2. Preliminaries

The main step in the proof of Theorem 2 is Lemma 1 below (see Maltz [1]).

Lemma 1. Suppose $M^n = \mathbb{R} \times N^{n-1}$ is the Riemannian product of $\mathbb{R}$ and a connected Riemannian manifold $N^{n-1}$, and suppose $f : M^n \to \mathbb{R}^m$ is an isometric immersion mapping a geodesic of the form $\mathbb{R} \times \{q\}$ onto a straight line in $\mathbb{R}^m$. Then $f$ is a 1-cylinder.

Our result also relies on the fundamental fact that the leaves of the minimum relative nullity distribution of a complete submanifold of $\mathbb{R}^m$ are also complete (cf. Dajczer [3]).

Lemma 2. Let $M^n$ be a complete Riemannian manifold and let $f : M^n \to \mathbb{R}^m$ be an isometric immersion with $\nu > 0$ everywhere. Then, the leaves of the relative nullity distribution are complete on the open subset where $\nu = \nu_0$ is minimal.

Theorem 1 follows easily from Lemmas 1 and 2 above together with the Cheeger-Gromoll splitting theorem. Indeed, under the assumptions of Theorem 1, Lemma 2 yields that $M^n$ contains $\nu_0$ linearly independent lines through each point where the index of relative nullity is minimal. By the splitting theorem of Cheeger-Gromoll, $M^n$ is isometric to a Riemannian product $\mathbb{R}^{\nu_0} \times N^{n-\nu_0}$, and Theorem 1 then follows inductively from Lemma 1.

The proof of our Theorem 2 uses the same ideas above, taking advantage of a recent warped product version of the splitting theorem by Wylie [2]. According to this latter result, estimate (1.1) is sufficient to split a complete Riemannian manifold $M^n$ that admits a line into a warped product $\mathbb{R} \times \rho N^{n-1}$ over $\mathbb{R}$. But since this splitting comes from a line of relative nullity, our goal is to show that the warping function $\rho$ must be constant, and thus $\mathbb{R} \times \rho N^{n-1}$ is actually a Riemannian product, so that Lemma 1 can be applied to conclude the proof. To do this we need to collect geometric information on the behavior of a warped product as above along the line $\mathbb{R}$. For later use, we carry out this study within the broader class of twisted products $M^n = \mathbb{R} \times \rho N^{n-1}$ over $\mathbb{R}$, where $(N, h)$ is a Riemannian manifold, $\rho : M^n \to \mathbb{R}_+$ the twisting function, and $M^n$ is endowed with the metric $g = dr^2 + \rho^2 h$. If $\rho$ is a function of $r$ only, then we have a warped product over $\mathbb{R}$. The following lemma describes how vector fields vary along $\mathbb{R}$.

Lemma 3. Let $M^n = \mathbb{R} \times \rho N^{n-1}$ be a twisted product over $\mathbb{R}$. Then

\begin{equation}
\nabla_{\partial_r} \partial_r = 0
\end{equation}

and

\begin{equation}
\nabla_{\partial_r} X = \nabla_X \partial_r = \frac{1}{\rho} \frac{\partial \rho}{\partial r} X
\end{equation}

for all $X \in \mathfrak{X}(N)$. 

Proof. Let us write \( \rho_r = \rho(r, \cdot) \) and denote by \( N_{\rho_r} \) the Riemannian manifold \( N \) endowed with the conformal metric rescaled by \( \rho_r^2 \). It is straightforward to check that \( \nabla \) given by (2.1), (2.2) and
\[
\nabla_X Y = \nabla_X^{N_{\rho_r}} Y - \langle X, Y \rangle \frac{1}{\rho} \frac{\partial \rho}{\partial r} \frac{\partial}{\partial r}
\]
for all \( X, Y \in \mathfrak{X}(N) \) defines a compatible symmetric connection on \( TM \), hence it coincides with the Levi-Civita connection of \( M^n \).

Next, we use Lemma 3 to compute the sectional curvatures along planes containing \( \partial_r \).

Lemma 4. Let \( M^n = \mathbb{R} \times_\rho N^{n-1} \) be a twisted product over \( \mathbb{R} \). Then
\[
(2.3) \quad K(\partial_r, X) = -\frac{1}{\rho} \frac{\partial^2 \rho}{\partial r^2}
\]
for all unit vector \( X \in T_x N \) and all \( x \in N^{n-1} \).

Proof. Differentiating \( \langle X, X \rangle = \rho^2 \) twice with respect to \( r \) gives
\[
(\nabla_{\partial_r} \nabla_{\partial_r} X, X) + \| \nabla_{\partial_r} X \|^2 = \rho \frac{\partial^2 \rho}{\partial r^2} + \left( \frac{\partial \rho}{\partial r} \right)^2.
\]
Using (2.1) and (2.2), we conclude that
\[
\langle R(\partial_r, X) \partial_r, X \rangle = \rho \frac{\partial^2 \rho}{\partial r^2},
\]
from which the result follows.

We are now in a position to state and prove our main lemma, in which by a line of nullity of a Riemannian manifold \( M^n \) we mean a curve \( \gamma : \mathbb{R} \to M^n \) such that \( \gamma'(t) \in \Gamma(\gamma(t)) \) for all \( t \in \mathbb{R} \), where
\[
\Gamma(x) = \{ X \in T_x M : R(X, Y) = 0 \text{ for all } Y \in T_x M \}
\]
is the nullity subspace at \( x \in M^n \).

Lemma 5. Let \( M^n = \mathbb{R} \times_\rho N^{n-1} \) be a twisted product over \( \mathbb{R} \). If \( \mathbb{R} \times \{ q \} \) is a line of nullity of \( M^n \) for some \( q \in N^{n-1} \), then \( \rho_r = \rho_0 \) does not depend on \( r \), and hence \( M^n \) is actually the Riemannian product \( \mathbb{R} \times N^{n-1}_{\rho_0} \).

Proof. It follows from (2.3) that
\[
\frac{\partial^2 \rho}{\partial r^2} \equiv 0,
\]
but since the twisting function \( \rho \) is positive on the whole real line it must be constant.

3. Proof

As previously discussed, Lemma 1 is at the core of the proof of Theorem 2 whereas Lemma 5 is the principle behind its use.
Proof. We can assume that $\nu_0 = 1$, since the general case follows easily by induction on $\nu_0$. Take a point $p \in M^n$ where $\nu = 1$. It follows from Lemma 2 that $M^n$ contains a line $l$ through $p$. By the warped product version of the splitting theorem of Cheeger-Gromoll due to Wylie [2], the Riemannian manifold $M^n$ is isometric to a warped product $\mathbb{R} \times \rho N^{n-1}$ over $\mathbb{R}$, the line $l$ corresponding to $\mathbb{R} \times \{q\}$ for some $q \in N^{n-1}$. Since $l$ is a leaf of the relative nullity foliation, we have in particular that $\mathbb{R} \times \{q\}$ is a line of nullity of $\mathbb{R} \times \rho N^{n-1}$, and thus, by Lemma 5, $\rho_r = \rho_0$ does not depend on $r$ and $\mathbb{R} \times \rho N^{n-1}$ is actually the Riemannian product $\mathbb{R} \times N^{n-1}$. Hence, we may consider $f: \mathbb{R} \times N^{n-1} \rho_0 \to \mathbb{R}^m$, and as $f$ maps $\mathbb{R} \times \{q\}$ onto a straight line in $\mathbb{R}^m$, the result then follows from Lemma 1. □

4. Generalization

In this section we explain how the result above also has a version for non-gradient potential fields. Curvature inequality (1.1) has a natural extension to vector fields $X$ and can be regarded as the special case where $X = \nabla \psi$. Our result in the gradient case assumes boundness of the potential function $\psi$. While there is no potential function for a non-gradient field, we can still make sense of bounds by integrating $X$ along geodesics. Let $X$ be a vector field on a Riemannian manifold $M^n$. Let $\gamma: (a, b) \to M^n$ be a geodesic that is parametrized by arc-length. Define

$$\psi_\gamma(t) = \int_a^t \langle \gamma'(s), X(\gamma(s)) \rangle \, ds,$$

which is a real valued function on the interval $(a, b)$ with the property that $\psi'_\gamma(t) = \langle \gamma'(t), X(\gamma(t)) \rangle$. When $X = \nabla \psi$ is a gradient field then $\psi_\gamma(t) = \psi(\gamma(t)) - \psi(\gamma(a))$, in the non-gradient case we think of $\psi_\gamma$ as being the anti-derivative of $X$ along the geodesic $\gamma$. We now recall the notion of ‘bounded energy distortion’, introduced by Wylie [2].

**Definition 1.** Let $M^n$ be a non-compact complete Riemannian manifold and $X \in \mathfrak{X}(M)$ a vector field. Then we say $X$ has bounded energy distortion if, for every point $x \in M^n$,

$$\limsup_{r \to \infty} \inf_{l(\gamma) = r} \left\{ \int_0^r e^{-\frac{2\psi_\gamma(\gamma(s))}{n-1}} \, ds \right\} = \infty,$$

where the infimum is taken over all minimizing unit speed geodesics $\gamma$ with $\gamma(0) = x$.

In general, $\psi_\gamma$ depends on the parametrization of $\gamma$ only up to an additive constant, so the notion of bounded energy distortion does not depend on the parametrization of the geodesic. Also note that if a vector field $X$ has the property that $\psi_\gamma$ is bounded for all unit speed minimizing geodesics then it has bounded energy distortion. However, even in the gradient case, bounded energy distortion is a weaker condition than $\psi$ bounded above.

Our most general cylindricity theorem is the following.

**Theorem 3.** Let $(M^n, g)$ be a complete manifold with

$$\text{Ric} \geq -\left( \frac{1}{2} L_X g + \frac{X^2 \otimes X^2}{n-1} \right)$$

(4.1)
for some vector field $X$ with bounded energy distortion and let $f : M^n \to \mathbb{R}^m$ be an isometric immersion with minimal index of relative nullity $\nu_0 > 0$. Then $f$ is a $\nu_0$-cylinder.

In particular, when $X = \nabla \psi$, we conclude that Theorem 2 still holds under the weaker condition that $\psi$ has bounded energy distortion rather than being bounded from above.

By Wylie [2], inequality (4.1) allows to split $M^n$ as a twisted product $\mathbb{R} \times_{\rho} N^{n-1}$ over $\mathbb{R}$, provided there is a line. But since Lemma 5 actually holds for twisted products, the proof of Theorem 3 then follows by the same arguments as in Section 3.

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