APPLICATIONS OF \( (p, q) \)-GAMMA FUNCTION TO SZÁSZ DURRMeyer OPERATORS

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Abstract. We define a \( (p, q) \) analogue of Gamma function. As an application, we propose \( (p, q) \)-Szász–Durrmeyer operators, estimate moments and establish some direct results.

1. Introduction

In the last two decades the quantum calculus is an active area of research among researchers. The quantum calculus find applications in a number of areas, including approximation theory. The relationship between approximation theory and q-calculus encouraged the mathematicians to give q-analogue of known results (see [3]). This rapid development of q-calculus has led to the discovery of new generalization of this theory. This produces some advantages like that the rate of convergence of q-operators is more flexible and better than the classical one. Since the q-calculus is based on one parameter, there is a possibility of extension of q-calculus. In this direction Sahai–Yadav [14] established some extensions to post-quantum calculus in special functions. A question arises: can we modify the operators using \( (p, q) \)-calculus such that our modified operator has better error estimation than the classical ones. For this purpose, we will define \( (p, q) \)-Szász–Durrmeyer operators. Several well-known operators may extend to \( (p, q) \)-analogue. Mursaleen et al introduced the \( (p, q) \)-analogue of the Bernstein operators in [10]. There both point-wise convergence and asymptotic formula are considered. Other important class of discrete operators has been investigated by using \( (p, q) \)-calculus. For example \( (p, q) \)-Bernstein–Stancu operators appeared in [9] \( (p, q) \) Bleimann–Butzer–Hahn and \( (p, q) \)-Szász Mirakyan operators have been studied recently in [1, 11]. Very recently, in order to obtain an approximation process in the space of \( (p, q) \)-Bernstein operators, the authors [5] defined Durrmeyer type modification of \( (p, q) \)-Bernstein operators.

\[2010 \text{ Mathematics Subject Classification: Primary } 33B15; \text{ Secondary } 41A25.\]

Key words and phrases: \( (p, q) \)-Gamma function, \( (p, q) \)-Szász–Durrmeyer operators, direct estimates, modulus of continuity.

Communicated by Gradimir Milovanović.
Motivated by all the above results we propose Durrmeyer type modification of the $(p, q)$-Szász Mirakyan operators using an integral version of $(p, q)$-Gamma function (as we know it is first in literature).

The paper is organized as follows: the next section contains some basic facts regarding $(p, q)$-calculus, we also introduce $(p, q)$-analogue of Gamma function. The construction of the announced class of operators is presented in Section 3. Section 4 deals with the quantitative type estimate with a suitable modulus of continuity. The last section is devoted to weighted Korovin type theorems and we estimate the approximation of bounded functions by announced operators with the help of a Lipschitz-type maximal function.

2. Notations and Preliminaries

Following the definitions and notations of [14]:

Set $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, the $(p, q)$-numbers are defined as

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} = \frac{p^n - q^n}{p - q}$$

for $n \in \mathbb{N}$. The $(p, q)$-factorial $[n]_{p,q}!$ of the element $n \in \mathbb{N}$ means

$$[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}, n \geq 1, [0]_{p,q}! = 1.$$

The $(p, q)$-binomial theorem is given by

$$1 \Phi_0((a, b); -; (p, q), x) = (\Phi_0((a, bx); -; (p, q)), \infty),$$

where $(a, b); -; (p, q)) = \prod_{n=0}^{\infty} (ap^n - bq^n).$ Two different $(p, q)$-expansions named $E_{p,q}$ and $e_{p,q}$ of the exponential function $x \mapsto e^x$ are given as follows:

$$e_{p,q}(x) = \sum_{n=0}^{\infty} q^{n(n-1)/2} [n]_{p,q}! x^n = 1 \Phi_0((1, 0); -; (p, q), x),$$

$$E_{p,q}(x) = \sum_{n=0}^{\infty} p^{n(n-1)/2} [n]_{p,q}! x^n = 1 \Phi_0((0, 1); -; (p, q), -x).$$

(2.1)

We know that $1 \Phi_0((1, 0); -; (p, q), x), 1 \Phi_0((0, 1); -; (p, q), x) = 1$, that is the following relation between $(p, q)$-exponential functions

$$e_{p,q}(x)E_{p,q}(-x) = 1$$

(2.2)

holds. We mention that these $(p, q)$-analogues of the classical exponential functions are valid for $0 < q < p \leq 1$. Moreover $E_{p,q}(x)$ and $e_{p,q}(x)$ tend to $e^x$ as $p \to 1^-$ and $q \to 1^-$.

It is obvious by the $(p, q)$-derivative formula $D_{p,q}f(x) = \frac{f(qx) - f(px)}{(p-q)x}, x \neq 0$ that

$$D_{p,q}E_{p,q}(x) = E_{p,q}(qx),$$

$$D_{p,q}E_{p,q}(ax) = aE_{p,q}(aqx).$$

(2.3)
Proposition 2.1. [13] The formula of \((p, q)\)-integration by part is given by
\[
\int_a^b f(px)D_{p,q}g(x)\,d_{p,q}x = f(b)g(b) - f(a)q(a) - \int_0^a g(qx)D_{p,q}f(x)\,d_{p,q}x.
\]

Definition 2.1. For any \(n \in \mathbb{N}\), we define a \((p, q)\)-Gamma function by
\[
\Gamma_{p,q}(n) = \int_0^\infty p^{(n-1)(n-2)/2}x^{n-1}E_{p,q}(-qx)\,d_{p,q}x.
\]

Lemma 2.1. For any \(n \in \mathbb{N}\), we have \(\Gamma_{p,q}(n+1) = [n]_{p,q}!\).

Proof. From (2.1) we have \(E_{p,q}(0) = 1\) and from (2.2) we have
\[
E_{p,q}(\infty) = \lim_{x \to \infty} E_{p,q}(x) = \lim_{x \to \infty} e_{p,q}(1,0; -; (p,q), -x)
\]
\[
= \lim_{x \to \infty} ((p,0); -; (p,q))_\infty = 0.
\]

Also from (2.3) we can write
\[
\Gamma_{p,q}(n+1) = \int_0^\infty p^{n(n-1)/2}x^nE_{p,q}(-qx)\,d_{p,q}x
\]
\[
= -\int_0^\infty p^{n(n-1)/2}x^nD_{p,q}E_{p,q}(-x)\,d_{p,q}x.
\]

By Proposition 2.1 using \((p, q)\)-integration by parts for \(f(x) = x^n\) and \(g(x) = E_{p,q}(-x)\), we have
\[
\Gamma_{p,q}(n+1) = [n]_{p,q}p^n\int_0^\infty p^{n(n-1)/2}x^{n-1}E_{p,q}(-qx)\,d_{p,q}x
\]
\[
= [n]_{p,q}\int_0^\infty p^{n(n-1)/2}x^{n-1}E_{p,q}(-qx)\,d_{p,q}x = [n]_{p,q}\Gamma_{p,q}(n).
\]

Thus, we have
\[
\Gamma_{p,q}(n+1) = [n]_{p,q}\Gamma_{p,q}(n) = [n]_{p,q}[n-1]_{p,q}\Gamma_{p,q}(n-1) = [n]_{p,q}!.
\]

Also from (2.3) we can write
\[
\Gamma_{p,q}(n+1) = [n]_{p,q}p^n\int_0^\infty p^{n(n-1)/2}x^{n-1}E_{p,q}(-qx)\,d_{p,q}x
\]
\[
= [n]_{p,q}\int_0^\infty p^{n(n-1)/2}x^{n-1}E_{p,q}(-qx)\,d_{p,q}x = [n]_{p,q}\Gamma_{p,q}(n).
\]

An alternate form of \((p, q)\)-Gamma function without integral expression for \(n\) nonnegative integer, is given in [12] by
\[
\Gamma_{p,q}(n+1) = \frac{(p \odot q)^n}{(p - q)^n} = [n]_{p,q}!, \quad 0 < q < p.
\]

3. \((p, q)\)-Szász–Durrmeyer Operators and Moments

In order to introduce a \((p, q)\) Durrmeyer variant for Szasz–Mirakjan operators, we present a construction due to Acar [1]. The \((p, q)\)-analogue of Szász operators for \(x \in [0, \infty)\) and \(0 < q < p \leq 1\) defined by in the following way
\[
S_{n,p,q}(f; x) = \sum_{k=0}^n s_{n,k}^{p,q}(x)f\left(\frac{[k]_{p,q}}{q^{k-n}[n]_{p,q}}\right).
\]
where

\[ s_{n,p,q}^k(x) = \frac{1}{E_{p,q}([n]_{p,q}x)} q^{k(k-1)/2} ([n]_{p,q}x)^k. \]

In case \( p = 1 \), we get the \( q \)-Szász operators \([2]\). If \( p = q = 1 \), we get at once the well known Szász operators.

**Lemma 3.1.** \([1]\) For \( x \in x \in [0, \infty) \), \( 0 < q < p \leq 1 \), we have

1. \( S_{n,p,q}(1; x) = 1 \),
2. \( S_{n,p,q}(t; x) = qx \),
3. \( S_{n,p,q}(t^2; x) = pqx^2 + \frac{q^2}{[n]_{p,q}}x. \)

The Szász operators defined by (3.1) are discrete operators. The integral modification of these operators was proposed in \([7]\). Different variants and \( q \)-analogues in \([3]\) and \([6]\). As an application of the \((p,q)\)-Gamma function, we introduce below the Durrmeyer type \((p,q)\)-variant of the Szász operators as

**Definition 3.1.** The \((p,q)\)-analogue of Szász–Durrmeyer operator for \( x \in [0, \infty) \) and \( 0 < q < p \leq 1 \) is defined by

\[
\tilde{S}_{n,p,q}(f; x) = [n]_{p,q} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \int_0^{\infty} p^{k(k-1)/2} \left( \frac{[n]_{p,q}t^k}{[k]_{p,q}} \right) E_{p,q}(-q[n]_{p,q}t)f(q^{1-k}p^kt) \, dp_q t,
\]

where \( s_{n,k}^{p,q}(x) \) is defined in (3.1).

It may be remarked here that for \( p = q = 1 \) these operators reduce to the Szász–Durrmeyer operators.

**Lemma 3.2.** For \( x \in x \in [0, \infty) \), \( 0 < q < p \leq 1 \), we have

1. \( \tilde{S}_{n,p,q}(1; x) = 1 \),
2. \( \tilde{S}_{n,p,q}(t; x) = \frac{x}{[n]_{p,q}} + px \),
3. \( \tilde{S}_{n,p,q}(t^2; x) = \frac{p^2}{q} x^2 + \frac{[2]_p^2}{[n]_{p,q}} x + \frac{[2]_q^2}{p[n]_{p,q}}. \)

**Proof.** Using Definition 2.1, Lemmas 2.1 and 3.1, we have

\[
\tilde{S}_{n,p,q}(1; x) = [n]_{p,q} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \int_0^{\infty} p^{k(k-1)/2} \left( \frac{[n]_{p,q}t^k}{[k]_{p,q}} \right) E_{p,q}(-q[n]_{p,q}t) \, dp_q t
\]

\[ = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{\Gamma_{p,q}(k + 1)}{[k]_{p,q}} = 1 \]

and next using \([k+1]_{p,q} = q^k + p[k]_{p,q}\), we have

\[
\tilde{S}_{n,p,q}(t; x) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \int_0^{\infty} p^{k(k-1)/2} q^{1-k} \left( \frac{[n]_{p,q}t^k}{[k]_{p,q}} \right) E_{p,q}(-q[n]_{p,q}t) \, dp_q t
\]

\[ = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) q^{1-k} \frac{k_{p,q}(k + 2)}{[k]_{p,q}[n]_{p,q}} = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) q^{1-k} \frac{k_{p,q}}{[n]_{p,q}} \]

\[ = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) q^{1-k} \frac{(q^k + p[k]_{p,q})}{[n]_{p,q}}. \]
\[ f(x) = 4 \int_0^y \int_0^z [2f(x + u + v) - f(x + 2(u + v))] \, du \, dv \]
By simple computation, it is observed that

(i) \[ \|f_h - f\|_{C_B} \leq \omega_2(f, h). \]

(ii) If \( f \) is continuous, then \( f'_h, f''_h \in C_B \) and

\[ \|f'_h\|_{C_B} \leq \frac{5}{h} \omega(f, h), \quad \|f''_h\|_{C_B} \leq \frac{9}{h^2} L \omega_2(f, h), \]

where the first and second order modulus of continuity are respectively defined by

\[ \omega(f, \delta) = \sup_{x, u, v \geq 0 \atop |u - v| \leq \delta} |f(x + u) - f(x + v)|, \]

\[ \omega_2(f, \delta) = \sup_{x, u, v \geq 0 \atop |u - v| \leq \delta} |f(x + 2u) - 2f(x + u + v) + f(x + 2v)|, \quad \delta \geq 0. \]

**Theorem 4.1.** Let \( q \in (0, 1) \) and \( p \in (q, 1) \) The operator \( \tilde{S}_{n, p, q} \) maps space \( C_B \) into \( C_B \) and \( \|\tilde{S}_{n, p, q}(f)\|_{C_B} \leq \|f\|_{C_B}. \)

**Proof.** Let \( q \in (0, 1) \) and \( p \in (q, 1) \). From Lemma 3.2 we have

\[ |\tilde{S}_{n, p, q}(f, x)| \leq \sum_{k=0}^{\infty} \int_0^\infty p^{(k-1)/2} \left( \frac{[n]_{p, q}}{[k]_{p, q}} \right)^k E_{p, q}(-q[n]_{p, q}t) f(q^{1-k} p^k t) dt. \]

We are going to study the degree of approximation in terms of first and second order modulus of continuity.

**Theorem 4.2.** Let \( q \in (0, 1) \) and \( p \in (q, 1) \). If \( f \in C_B[0, \infty) \), then

\[ |\tilde{S}_{n, p, q}(f, x) - f(x)| \leq 5\omega \left( f, \frac{1}{\sqrt{[n]_{p, q}}} \right) \left( \frac{q}{\sqrt{[n]_{p, q}}} + \sqrt{[n]_{p, q}(p - 1)x} \right) + 9\omega^2 \left( f, \frac{1}{\sqrt{[n]_{p, q}}} \right) \left[ 2 + \frac{(p^3 - 2pq + q)[n]_{p, q}x^2}{q} + \frac{[2]_{p, q}^2 q^2}{p[n]_{p, q}} \right]. \]

**Proof.** For \( x \geq 0 \) and \( n \in \mathbb{N} \) and using the Steklov mean \( f_h \) defined by (4.1), we can write

\[ |\tilde{S}_{n, p, q}(f, x) - f(x)| \leq \tilde{S}_{n, p, q}(|f - f_h|, x) + \tilde{S}_{n, p, q}(f_h - f_h(x), x) + |f_h(x) - f(x)|. \]

First by Theorem 4.1 and property (i) of Steklov mean we have

\[ \tilde{S}_{n, p, q}(|f - f_h|, x) \leq \|\tilde{S}_{n, p, q}(f - f_h)\|_{C_B} \leq \|f - f_h\|_{C_B} \leq \omega_2(f, h). \]
By Lemma 3.2, we have

\[|\tilde{S}_{n,p,q}(f_h - f_h(x), x)| \leq |f'_h(x)|\tilde{S}_{n,p,q}(t - x, x) + \frac{1}{2}\|f''\|_{C_n}\tilde{S}_{n,p,q}((t - x)^2, x).\]

By Lemma 3.2, we have

\[|\tilde{S}_{n,p,q}(f_h - f_h(x), x)| \leq \frac{5}{h}\omega(f, h)\left(\frac{q}{[n]_{p,q}} + (p - 1)x\right)
 + \frac{9}{2n^2}\omega_2(f, h)\tilde{S}_{n,p,q}((t - x)^2, x),\]

where \(\tilde{S}_{n,p,q}((t - x)^2, x)\) is given by (3.2). For \(x \geq 0\), \(h > 0\) and choosing \(h = \sqrt{1/[n]_{p,q}}\), we get the desired result. \(\square\)

REMARK 4.1. For \(q \in (0, 1)\) and \(p \in (q, 1]\) it is seen that \(\lim_{n \to \infty}[n]_{p,q} = 1/(q - p)\). In order to consider the convergence of \((p, q)\)-Szász–Durrmeyer operators, we assume \(p = (p_n)\) and \(q = (q_n)\) such that \(0 < q_n < p_n \leq 1\) and for \(n\) sufficiently large \(p_n \to 1, q_n \to 1, p_n^a \to a, q_n^a \to b\), so that \([n]_{p_n,q_n} \to \infty\). Such a sequence can always be constructed for example, we can take \(p_n = 1 - 1/2n\) and \(q_n = 1 - 1/n\). Clearly \(\lim_{n \to \infty} p_n^a = e^{-1/2}, \lim_{n \to \infty} q_n^a = e^{-1}\) and \(\lim_{n \to \infty}[n]_{p_n,q_n} = \infty\).

5. Direct Estimates

Let us denote by \(H_{zz}[0, \infty)\) the set of all functions \(f\) defined on the positive real axis satisfying the condition \(|f(x)| \leq M_f(1 + x^2)\), where \(M_f\) is an absolute constant depending on \(f\). By \(C_{zz}[0, \infty)\), we mean the subspace of all continuous functions belonging to \(H_{zz}[0, \infty)\). Also, let \(C^*_z[0, \infty)\) denote the subspace of all functions \(f \in C_{zz}[0, \infty)\), for which \(\lim_{|x| \to \infty} \frac{f(x)}{1 + x^2}\) is finite. The class \(C^*_z[0, \infty)\) is endowed with the norm

\[\|f\|_{zz} = \sup_{x \in [0, \infty)} \frac{|f(x)|}{1 + x^2}.\]

We discuss below the weighted approximation theorem, where the approximation formula is valid for the positive real axis (see [4]).

THEOREM 5.1. Let \(p = p_n\) and \(q = q_n\) satisfies \(0 < q_n < p_n \leq 1\) and for \(n\) sufficiently large \(p_n \to 1, q_n \to 1\) and \(q_n^a \to a\) and \(p_n^a \to b\). For each \(f \in C^*_z[0, \infty)\), we have \(\lim_{n \to \infty} \|\tilde{S}_{n,p_n,q_n}(f) - f\|_{zz} = 0\).

PROOF. Using the Korovkin’s theorem, it is sufficient to verify the following three conditions

\[(5.1) \quad \lim_{n \to \infty} \|\tilde{S}_{n,p_n,q_n}(t^\nu, x) - x^\nu\|_{zz} = 0, \quad \nu = 0, 1, 2.\]

Since \(\tilde{S}_{n,p_n,q_n}(1, x) = 1\) the first condition of (5.1) is fulfilled for \(\nu = 0\).

For \(n \in \mathbb{N}\), we can write,

\[\|\tilde{S}_{n,p_n,q_n}(t, x) - x\|_{zz} \leq \frac{q_n}{[n]_{p_n,q_n}} + (p_n - 1) \sup_{x \in [0, \infty)} \frac{x}{1 + x^2}\]
\( \| \tilde{S}_{n,p,q}(t^2, x) - x^2 \|_{L^2} \leq \left( \frac{p^3}{q_n} - 1 \right) \sup_{x \in [0, \infty)} \frac{x^2}{1 + x^2} + \frac{[2]^2_{p,q}}{[n]_{p,q}} \sup_{x \in [0, \infty)} \frac{x}{1 + x^2} + \frac{[2]_{p,q}^2}{[p][n]_{p,q}} \sup_{x \in [0, \infty)} \frac{1}{1 + x^2}, \)

which implies that for \( v = 1, 2 \) we have \( \lim_{n \to \infty} \| \tilde{S}_{n,p,q}(t^v, x) - x^v \|_{L^2} = 0. \) \( \square \)

We give the following theorem to approximate all functions in \( C_{\infty}(0, \infty) \).

**Theorem 5.2.** Let \( p = p_n \) and \( q = q_n \) satisfies \( 0 < q_n < p_n \leq 1 \) and for \( n \) sufficiently large \( p_n \to 1 \), \( q_n \to 1 \) and \( q_n^\alpha \to \alpha \) and \( p_n^\alpha \to b \). For each \( f \in C_{\infty}(0, \infty) \) and \( \alpha > 0 \), we have

\[
\lim_{n \to \infty} \sup_{x \in (0, \infty)} \frac{\| \tilde{S}_{n,p,q}(f, x) - f(x) \|}{(1 + x^2)^{1+\alpha}} = 0.
\]

**Proof.** For any fixed \( x_0 > 0 \),

\[
\sup_{x \in [0, \infty)} \frac{|\tilde{S}_{n,p,q}(f, x) - f(x)|}{(1 + x^2)^{1+\alpha}}
\]

\[
= \sup_{x \in (0, x_0)} \frac{|\tilde{S}_{n,p,q}(f, x) - f(x)|}{(1 + x^2)^{1+\alpha}} + \sup_{x \in [x_0, \infty)} \frac{|\tilde{S}_{n,p,q}(f, x) - f(x)|}{(1 + x^2)^{1+\alpha}}
\]

\[
\leq \| \tilde{S}_{n,p,q}(f) - f \|_{C(0, \infty)} + \sup_{x \geq x_0} \frac{|\tilde{S}_{n,p,q}(1 + t^2, x)|}{(1 + x^2)^{1+\alpha}} + \sup_{x \geq x_0} \frac{|f(x)|}{(1 + x^2)^{1+\alpha}}.
\]

By Lemma 3.2 and well known Korovkin theorem, the first term of the above inequality tends to zero for sufficiently large \( n \). By Lemma 3.2 for any fixed \( x_0 > 0 \) it is easily seen that \( \sup_{x \geq x_0} \frac{|\tilde{S}_{n,p,q}(1 + t^2, x)|}{(1 + x^2)^{1+\alpha}} \) tends to zero as \( n \to \infty \). We can choose \( x_0 > 0 \) so large that the last part of above inequality can be made small enough. This completes the proof of the theorem. \( \square \)

Now we establish some point-wise estimates of the rate of convergence of \( (p, q) \)-Szász–Durrmeyer operators. First, we give the relationship between the local smoothness of \( f \) and local approximation. A function \( f \in C(0, \infty) \) is said to satisfy Lipschitz condition \( \text{Lip}_\alpha \) on \( D \), \( \alpha \in (0, 1] \), \( D \subset [0, \infty) \) if

\[
|f(t) - f(x)| \leq M_f |t - x|^{\alpha}, \quad t \in [0, \infty) \text{ and } x \in D,
\]

where \( M_f \) is a constant depending only \( \alpha \) and \( f \).

**Theorem 5.3.** Let \( f \in \text{Lip}_\alpha \) on \( D \), \( D \subset [0, \infty) \) and \( \alpha \in (0, 1] \). We have

\[
|\tilde{S}_{n,p,q}(f, x) - f(x)| \leq \left( \frac{p^3 - 2pq + q}{q} x^2 + \frac{[2]^2_{p,q} - 2q}{[n]_{p,q}} + \frac{[2]_{p,q}^2}{p[n]_{p,q}^2} \right)^{\alpha/2} + 2d^\alpha(x; D)
\]

where \( d(x; D) \) represents the distance between \( x \) and \( D \).
Then, with Hölder’s inequality with $p$ we have

\begin{equation}
|S_{n,p,q}(f,x) - f(x)| \leq |S_{n,p,q}(f(t) - f(x_0)|, x) + |f(x_0) - f(x)| \\
\leq M_f S_{n,p,q}(|t-x_0|^\alpha, x) + M_f |x_0 - x|^\alpha.
\end{equation}

Using (5.3), (5.4) and (3.2), we get desired result. □

\begin{equation}
ω_α(S_{n,p,q}(|t-x|^\alpha, x)) \leq (\tilde{S}_{n,p,q}(|t-x|^\alpha, x))^\frac{2}{p} + |x-x|^\alpha.
\end{equation}

Using (5.3), (5.4) and (3.2), we get desired result.

Now, we give local direct estimate for $(p,q)$-Szász–Durrmeyer operators using the Lipschitz-type maximal function of order $α$ introduced by Lenze [8] as

\begin{equation}
\omega_α(f,x) = \sup_{t \neq x, t \in [0, \infty)} \frac{|f(t) - f(x)|}{|t-x|^\alpha}, \quad x \in [0, \infty) \text{ and } α \in (0, 1].
\end{equation}

**Theorem 5.4.** Let $f \in \text{Lip}_α$ on $D$ and $f \in C_D[0, \infty)$. Then for all $x \in [0, \infty)$, we have

\begin{equation}
|S_{n,p,q}(f,x) - f(x)| \leq \omega_α(f,x) \left( (p^3 - 2pq + q)x^2 + \frac{[2]_p^2 - 2q}{[n]_p^2} + \frac{[2]_p^2q^2}{p[n]_p^2} \right)^\frac{2}{p}.
\end{equation}

**Proof.** From (5.5) we have

\begin{align*}
|f(t) - f(x)| & \leq \omega_α(f,x)|t-x|^\alpha, \\
|S_{n,p,q}(f,x) - f(x)| & \leq S_{n,p,q}(|f(t) - f(x)|, x) \\
& \leq \omega_α(f,x) S_{n,p,q}(|t-x|^\alpha, x).
\end{align*}

Applying Hölder’s inequality with $p := \frac{2}{α}$ and $\frac{1}{p} := 1 - \frac{1}{p}$, we have

\begin{equation}
|S_{n,p,q}(f,x) - f(x)| \leq \omega_α(f,x) S_{n,p,q}(|t-x|^2, x)^\frac{2}{p}.
\end{equation}

Using (3.2), we have our assertion. □

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