Yangian, Truncated Yangian and Quantum Integrable Models

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Abstract

Based on the RTT relation for the given rational $R$-matrix the Yangian and truncated Yangian are discussed. The former can be used to generate the long-range interaction models, whereas the latter can be related to the Goryachev-Chaplygin (GC) gyrostat. The trigonometric extension of the Goryachev-Chaplygin gyrostat has also been shown.
I Introduction

The RTT relation plays the central role in setting up the connection between the completely quantum integrable systems and the quantum group symmetry [1-6]. For a given $R$-matrix satisfying the Yang-Baxter equation (YBE) variety of operator-valued transfer matrices $T(u)$ can be found to satisfy the RTT relation. On the one hand the trace $\text{tr } T(u) = \sum u^{-n} h^{(n)}$ gives the conserved quantities $h^{(n)}$ including Hamiltonian, on the other hand the commutations $[T_{ab}(u), T_{cd}(v)]$ obtained through RTT relation generate the well-known quantum group. Especially, when $R$-matrix is rational it defines Yangian [1] for some Lie algebras. From this point of view the RTT relation is an unified description of both symmetry and physical model that are closely related to each other for a considered quantum integrable system.

A solution of RTT relation [7]

$$\hat{R}(u - v)(T(u) \otimes T(v)) = (T(v) \otimes T(u))\hat{R}(u - v) \tag{1.1}$$

depends on the given form of $\hat{R}(u)$, where

$$\hat{R}(u) = R(u)P \tag{1.2}$$

and $P$ stands for the permutation.

The simplest rational solution of YBE is well-known as [8]

$$\hat{R}(u) = up + I \tag{1.3}$$

where $u$ is the spectral parameter and $I$ the identity. The RTT relation (1.1) defines the Yangian associated with $sl(n)$ algebra [1] for

$$T(u) = ||T_{ab}(u)||_{a,b=1}^{n} \tag{1.4}$$

Whereas the operators $T_{ab}$ are given by

$$T_{ab}(u) = \sum_{n=0}^{\infty} u^{-n}T_{ab}^{(n)} \tag{1.5}$$

The standard procedure to deal with the problem is as the following:
1. For the given (1.3) find commutation relations for $T_{ab}^{(n)}$ themselves on the basis of (1.1) and (1.3).

2. Re-explain such commutation relations in terms of the generators of Hopf algebra, for example, $I_\alpha$, $J_\alpha$ ($\alpha = 1, 2, 3$) introduced by Drinfeld [1].

3. Make special realization of the commutation relations, i.e. specify the particularly physical objects, satisfying these relations.

4. Obtain the Hamiltonian through such a physical realization.

In this paper we would like to carry out the procedures 1-4 to generate some quantum integrable models including long-range interaction models [9-19] and Gyrostat model that are extension of the discussed models in Refs. [20, 21].

This paper is organized as the following. In section II the general formulation of the commutation relations for $T_{ab}^{(n)}$ will be given, based on which we shall distinguish the Yangian case from the Heisenberg type of algebraic structure. In section III some long-range interaction models with internal degree of freedom will be turned out to belong to the RTT scope, in section IV the GC Gyrostat is shown to belongs to the Heisenberg type of Yangian with truncation. Finally we briefly show some more results in this respect.

II Commutation Relations of $T_{ab}^{(n)}$ for $sl(2)$

For the given $R(u)$ (1.3) and the expansion (1.3) the RTT relation gives

$$[T_{ab}^{(0)}, T_{cd}^{(n)}] = 0 \quad (a, b, c, d = 1, 2)$$

(2.1)

and

$$[T_{bc}^{(n+1)}, T_{ad}^{(m)}] - [T_{bc}^{(n)}, T_{ad}^{(m+1)}] + T_{ac}^{(n)} T_{bd}^{(m)} - T_{ac}^{(m)} T_{bd}^{(n)} = 0$$

(2.2)
From (2.2) it follows that $T^{(0)}_{ab}$ should be c-numbered matrix, thus it can always be chosen to be diagonal

$$T^{(0)} = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$  \hspace{1cm} (2.3)$$

where $\lambda$ and $\mu$ are arbitrary constants.

After calculations the relations (2.2) and (2.3) can be reduced to the following independent set of relations:

(a) \[ \begin{bmatrix} \lambda T^{(n)}_{22} + \mu T^{(n)}_{11}, T^{(1)}_{12} \end{bmatrix} = 2\lambda \mu T^{(n)}_{12}, \]
\[ \begin{bmatrix} \lambda T^{(n)}_{22} - \mu T^{(n)}_{11}, T^{(1)}_{21} \end{bmatrix} = -2\lambda \mu T^{(n)}_{21}, \] \hspace{1cm} (2.4)
\[ [T^{(n)}_{12}, T^{(1)}_{21}] = \lambda T^{(n)}_{12} - \mu T^{(n)}_{11}, \] \[ [\lambda T^{(n)}_{12} + \mu T^{(n)}_{11}, T^{(1)}_{12}] = 0 \]
for $n = 1, 2$.

(b) \[ \begin{bmatrix} \lambda T^{(2)}_{22} + \mu T^{(2)}_{11}, \lambda T^{(2)}_{22} - \mu T^{(2)}_{11} \end{bmatrix} + 2\lambda \mu (T^{(1)}_{21} T^{(2)}_{12} - T^{(2)}_{21} T^{(1)}_{12}) = 0, \]
\[ \begin{bmatrix} \lambda T^{(n)}_{22} + \mu T^{(n)}_{11}, T^{(2)}_{12} \end{bmatrix} + T^{(1)}_{12} (\lambda T^{(n)}_{22} - \mu T^{(n)}_{11}) - T^{(n)}_{12} (\lambda T^{(1)}_{22} - \mu T^{(1)}_{11}) = 0, \] \hspace{1cm} (2.5)
\[ \begin{bmatrix} \lambda T^{(n)}_{22} + \mu T^{(n)}_{11}, T^{(2)}_{21} \end{bmatrix} + T^{(n)}_{21} (\lambda T^{(1)}_{22} - \mu T^{(1)}_{11}) - T^{(1)}_{21} (\lambda T^{(n)}_{22} - \mu T^{(n)}_{11}) = 0. \]

(c) \[ 2\lambda \mu T^{(n+1)}_{12} = \begin{bmatrix} \lambda T^{(n)}_{22} - \mu T^{(n)}_{11}, T^{(2)}_{12} \end{bmatrix} + T^{(1)}_{12} (\lambda T^{(n)}_{22} + \mu T^{(n)}_{11}) - T^{(n)}_{12} (\lambda T^{(1)}_{22} + \mu T^{(1)}_{11}), \]
\[ \lambda T^{(n+1)}_{22} - \mu T^{(n+1)}_{11} = \begin{bmatrix} T^{(n)}_{12}, T^{(2)}_{21} \end{bmatrix} + T^{(1)}_{21} T^{(n)}_{11} - T^{(n)}_{21} T^{(1)}_{11}, \] \hspace{1cm} (2.6)
\[ 2\lambda \mu T^{(n+1)}_{21} = \begin{bmatrix} T^{(2)}_{21}, \lambda T^{(n)}_{22} - \mu T^{(n)}_{11} \end{bmatrix} + T^{(1)}_{21} (\lambda T^{(n)}_{22} + \mu T^{(n)}_{11}) - T^{(n)}_{21} (\lambda T^{(1)}_{22} + \mu T^{(1)}_{11}). \]

(d) \[ [T^{(n)}_{ab}, T^{(m)}_{ab}] = 0, \]
\[ [T^{(n)}_{ab}, T^{(m)}_{cd}] = [T^{(m)}_{cd}, T^{(n)}_{cd}] \]. \hspace{1cm} (2.7)

From (2.3) one can consider only two possibilities

(i) \[ \lambda = \mu = 1, \] \hspace{1cm} (2.8)
(ii) \[ \lambda = 1, \mu = 0. \] \hspace{1cm} (2.9)
For the case (i) the set of (2.4)-(2.7) defines the Yangian associated with $sl(n)$. The Drinfeld’s Yangian is defined by [1]

\[
[I_\lambda, I_\mu] = C_{\lambda\mu\nu} I_\nu, \quad [I_\lambda, J_\mu] = C_{\lambda\mu\nu} J_\nu,
\]

(2.10)

\[
[J_\lambda, [J_\mu, I_\nu]] - [I_\lambda, [J_\mu, J_\nu]] = \hbar^2 a_{\lambda\mu\alpha\beta\gamma} \{I_\alpha, I_\beta, I_\gamma\},
\]

(2.11)

\[
[[J_\lambda, J_\mu], [I_\tau, J_\sigma]] - [[J_\tau, J_\sigma], [I_\lambda, J_\mu]]
= \hbar^2 (a_{\lambda\mu\alpha\beta\gamma} C_{\tau\sigma\nu} + a_{\tau\sigma\alpha\beta\gamma} C_{\lambda\mu\nu}) \{I_\alpha, I_\beta, I_\gamma\},
\]

(2.12)

where

\[
a_{\lambda\mu\alpha\beta\gamma} = \frac{1}{24} C_{\lambda\alpha\tau} C_{\mu\beta\sigma} C_{\nu\gamma\rho} C_{\tau\sigma\rho},
\]

\[
C_{\lambda\mu\nu} = i\epsilon_{\lambda\mu\nu},
\]

(2.13)

\[
\{x_1, x_2, x_3\} = \sum_{\pi} x_{\pi(1)} x_{\pi(2)} x_{\pi(3)}.
\]

For $sl(2)$ both of the sides of (2.11) vanish and (2.12) can be recast to

\[
[J_3, [J_+, J_-]] = \frac{\hbar^2}{4} I_3 (J_- I_+ - I_- J_+) \\
[J_\pm, [J_3, J_\pm]] = \frac{\hbar^2}{4} I_\pm (J_\pm I_3 - I_\pm J_3)
\]

(2.14)

\[
2[J_3, [J_3, J_\pm]] + [J_\pm, [J_\pm, J_\mp]] = \frac{\hbar^2}{2} I_3 (J_\pm I_3 - I_\pm J_3) + I_\pm (I_- J_+ - J_- I_+),
\]

where $I_\pm = I_1 \pm iI_2$ and $J_\pm = J_1 \pm iJ_2$, in terms of which the (2.11) read

\[
[I_3, I_\pm] = \pm I_\pm, \quad [I_+, I_-] = 2I_3,
\]

(2.15)

\[
[J_3, I_\pm] = [I_3, J_\pm] = \pm J_\pm
\]

\[
[I_\pm, J_\mp] = \pm 2J_3, \quad [I_\pm, J_\pm] = [I_3, J_3] = 0
\]

Now we set

\[
T_{12}^{(1)} = \alpha_+ I_+, \quad T_{21}^{(1)} = \alpha_- I_-, \quad T_{22}^{(1)} - T_{11}^{(1)} = 2I_3,
\]

\[
T_{12}^{(2)} = \beta_+ J_+, \quad T_{21}^{(2)} = \beta_- J_-, \quad T_{22}^{(2)} - T_{11}^{(2)} = 2\beta_3 J_3.
\]

(2.16)
Substituting (2.16) into (2.4) with \( \lambda = \mu = 1 \) (except for \( [T_{22}^{(n)} + T_{11}^{(n)}, T_{ab}^{(1)}] = 0 \)) and (2.7) with \( n, m = 1, 2 \), one finds that they are satisfied if

\[
\alpha_+ \alpha_- = 1, \quad \beta_+ \alpha_- - \alpha_+ \beta_- = \beta_3. \tag{2.17}
\]

By virtue of (2.17) substituting (2.16) into (2.5) and (2.6) where \( n = 1, 2 \) and (2.7) with \( n, m = 2, 3 \) we derive (2.14) if

\[
h^2 = 4\beta_+^{-1}\beta_-^{-1} \tag{2.18}
\]

The higher-ordered term \( T_{ab}^{(n)} \) can be generated by the set of iteration relation (2.6).

Obviously the transformation

\[
J_\alpha \longrightarrow J_\alpha + \text{const.} I_\alpha \tag{2.19}
\]

still satisfy the Yangian.

Therefore, when \( \lambda = \mu = 1 \) we derive the Yangian associated with \( sl(2) \). Without doubt it is a kind of model-independent symmetry. In order to obtain the real physical model we have to impose the physical restriction to the \( T(u) \) such that the Hamiltonian can be specified. An interesting example is the long-range interaction models.

We would like to emphasize that if \( \lambda = 1 \) and \( \mu = 0 \) in (2.3) it will give rise to the quite different algebraic structure. Under such a circumstance we have

\[
[T_{12}^{(n)}, T_{22}^{(1)}] = [T_{21}^{(n)}, T_{22}^{(1)}] = [T_{11}^{(n)}, T_{22}^{(1)}] = 0 \tag{2.20}
\]

and

\[
[T_{12}^{(1)}, T_{11}^{(n)}] = T_{12}^{(n)}
\]

\[
[T_{21}^{(1)}, T_{11}^{(n)}] = -T_{21}^{(n)} \tag{2.21}
\]

\[
[T_{12}^{(1)}, T_{21}^{(n)}] = T_{22}^{(n)}
\]

where in the left hand sides of (2.21) the relations with the interchange between \( (n) \) and (1) are also valid. Obviously, (2.21) contains a Heisenberg algebra, for example, by setting \( n = 1 \) and \( T_{22}^{(1)} = 0 \):

\[
T_{11}^{(1)} = ip, \quad T_{12}^{(1)} = e^{-q}, \quad T_{21}^{(1)} = e^{+q}, \tag{2.22}
\]
if \([p, q] = -i\). This is the reason why we call the algebra Heisenberg type.

For \(\mu = 0\) and \(\lambda = 1\) \((2.2)\) is reduced to the following independent sets of relations:

\[
[T_{22}^{(n)}, T_{11}^{(2)}] + T_{21}^{(n)} T_{12}^{(1)} - T_{21}^{(1)} T_{12}^{(n)} = 0,
\]
\[
T_{12}^{(n+1)} = [T_{12}^{(2)}, T_{11}^{(n)}] + T_{11}^{(n)} T_{12}^{(1)} - T_{11}^{(1)} T_{12}^{(n)},
\]
\[
T_{21}^{(n+1)} = [T_{11}^{(n)}, T_{21}^{(2)}] + T_{11}^{(n)} T_{21}^{(1)} - T_{11}^{(1)} T_{21}^{(n)},
\]
\[
T_{22}^{(n+1)} = [T_{12}^{(2)}, T_{21}^{(n)}] + T_{11}^{(n)} T_{21}^{(1)} - T_{11}^{(1)} T_{22}^{(n)}
\]

for \(n \geq 2\) and

\[
[T_{ab}^{(n)}, T_{ab}^{(m)}] = 0, \quad [T_{ab}^{(n)}, T_{cd}^{(m)}] = [T_{ab}^{(m)}, T_{cd}^{(n)}] = 0
\]

(2.25)

where \(a, b, c, d = 1, 2\).

In the section IV we shall return to the algebraic structure of \((2.20)-(2.23)\).

### III RTT Relation and Long-Range Interaction Models

Let us apply the Yangian approach to the long-range interaction models. Following the notation of Ref. [16] (hereafter it is denoted by BGHP) the solution of Yang-Baxter equation, \(R\)-matrix, takes the simplest form as

\[
R_{00'}(u) = u + \lambda P_{00'}
\]

(3.1)

and the RTT relation \((1.1)\) reads

\[
R_{00'}(u - v) T^0(u) T^{0'}(v) = T^{0'}(v) T^0(u) R_{00'}(u - v)
\]

(3.2)

where \(T^0(u) = T(u) \otimes 1\), \(T^{0'} = 1 \otimes T(u)\) and \(P_{00'}\) is the permutation operator exchanging the two auxiliary spaces 0 and 0'. Make the expansion \([16]\)

\[
T^0(u) = I + \sum_{a,b=1}^{p} \sum_{n=0}^{\infty} \frac{\lambda T_{ab}^{(n)}}{u^{n+1}},
\]

(3.3)

\[
P_{00'} = \sum_{a,b=1}^{p} X_{ba}^0 X_{ab}^{0'},
\]

(3.4)

\[
[X_{ab}, X_{cd}] = \delta_{bc} X_{ad} - \delta_{ad} X_{cb}.
\]

(3.5)
As was pointed above that \( \{ T_{ab}^{(n)} \} \) generate the \textit{Yangian} associated with \( sl(n) \). Substituting eqs. (3.3), (3.3) and (3.4) into eq. (3.2) one finds

\[
\sum_{a,b} \sum_{cd} X_{ab}^0 X_{dc}^\prime \sum_{n=0}^{\infty} \left\{ u^{-n-1} f_1^n - v^{-n-1} f_2^n + \sum_{m=0}^{\infty} u^{-m-1} v^{-m-1} f_3^{n,m} \right\} = 0 \tag{3.6}
\]

where

\[
f_1^n = \delta_{bc} T_{ad}^{(n)} - \delta_{da} T_{cb}^{(n)} - [T_{ab}^{(n)}, T_{cd}^{(0)}],
\]

\[
f_2^n = \delta_{bc} T_{ad}^{(n)} - \delta_{da} T_{cb}^{(n)} - [T_{ab}^{(0)}, T_{cd}^{(n)}],
\]

\[
f_3^{n,m} = \lambda (T_{ad}^{(n)} T_{cb}^{(m)} - T_{ad}^{(m)} T_{cb}^{(n)}) + [T_{ab}^{(n+1)}, T_{cd}^{(m)}] - [T_{ab}^{(n)}, T_{cd}^{(m+1)}].
\]

For any auxiliary space \( \{ X_{ab} \} \) we require \( f_1^n = f_2^n = f_3^{n,m} = 0 \). Obviously, \( f_2^n = 0 \) can be induced from \( f_1^n = 0 \). So we need only to take

\[
f_1^n = f_3^{n,m} = 0 \tag{3.7}
\]

into account.

First from \( f_3^{n,0} = 0 \) it follows

\[
\delta_{bc} T_{ad}^{(n+1)} - \delta_{da} T_{cb}^{(n+1)} = \lambda (T_{ad}^{(0)} T_{cb}^{(n)} - T_{ad}^{(n)} T_{cb}^{(0)}) + [T_{ab}^{(n)}, T_{cd}^{(1)}] \tag{3.8}
\]

which can be recast to

\[
T_{ad}^{(n+1)} = \lambda (T_{ad}^{(0)} T_{cc}^{(n)} - T_{ad}^{(n)} T_{cc}^{(0)}) + [T_{ac}^{(n)}, T_{cd}^{(1)}] \quad (a \neq d), \tag{3.9}
\]

\[
T_{aa}^{(n+1)} - T_{cc}^{(n+1)} = \lambda (T_{aa}^{(0)} T_{cc}^{(n)} - T_{aa}^{(n)} T_{cc}^{(0)}) + [T_{ac}^{(n)}, T_{ca}^{(1)}], \tag{3.10}
\]

where no summation for the repeating indices is taken. Eqs. (3.9) and (3.10) imply that \( T_{ab}^n \) can be determined by iteration for given \( T_{ab}^{(0)} \) and \( T_{ab}^{(1)} \).

Now let us set

\[
T_{ab}^{(0)} = \sum_{i=1}^{N} T_{ab}^i,
\tag{3.11}
\]

\[
T_{ab}^{(1)} = \sum_{i=1}^{N} T_{ab}^i D_i
\tag{3.12}
\[ [I_{ab}^i, I_{cd}^j] = \delta_{ij}(\delta_{bc}I_{ad}^i - \delta_{ad}I_{cb}^i) \] (3.13)

where \( D_i \) are operators to be determined. \( \{I_{ab}^i\} \) can be understood as internal degree of freedoms, such as spins. Substituting eqs. (3.11)–(3.13) into \( f_1^1 \) we obtain

\[ \sum_i \sum_j I_{ab}^i [D_i, I_{cd}^j] = 0 \tag{3.14} \]

Further we assume

\[ \sum_i I_{ab}^i [D_i, I_{cd}^j] = 0, \text{ for any } j \tag{3.15} \]

with which the \( T_{ab}^{(2)} \) should satisfy

\[ \delta_{bc}T_{ad}^{(2)} - \delta_{ad}T_{cb}^{(2)} = \sum_{i \neq j} I_{ab}^i I_{cd}^j \left\{ \lambda \sum_{k,l} I_{kl}^i I_{lk}^j (D_j - D_i) + [D_i, D_j] \right\} \\
+ \sum_i (\delta_{bc}I_{ad}^i D_i^2 - \delta_{ad}I_{cb}^i D_i^2) \tag{3.16} \]

A sufficient solution of eq. (3.16) is

\[ T_{ab}^{(2)} = \sum_i I_{ab}^i D_i^2 \tag{3.17} \]

with

\[ [D_i, D_j] = \lambda \sum_{a,b} I_{ab}^j I_{ba}^i (D_i - D_j) \tag{3.18} \]

Thus eq. (3.12) generates long-range interaction through the eq. (3.17) and (3.18). However so far there is not simple relationship between \( D_i \) and \( I_{ab}^j \) which should satisfy eq. (3.13). It is very difficult to determine the general relationship. Fortunately, BGHP have set up the link with the help of projection. Let the permutation groups \( \Sigma_1, \Sigma_2 \) and \( \Sigma_3 \) be generated by \( K_{ij}, P_{ij} \) and the product \( P_{ij} K_{ij} \) respectively, where \( K_{ij} \) exchange the positions of particles and \( P_{ij} \) exchange the spins at position \( i \) and \( j \). The projection \( \rho \) was defined as

\[ \rho(ab) = a \text{ for } \forall a \in \Sigma_2, b \in \Sigma_1 \tag{3.19} \]
i.e. the wave function considered is symmetric. Let $I_{ab}$ be the fundamental representations, then

$$P_{ij} = \sum_{a,b} I_{iab} I_{jba}.$$  

(3.20)

Suppose that there exists

$$D_i = \rho(\hat{D}_i), \ D_i \in \Sigma_2, \ \hat{D}_i \in \Sigma_1$$  

(3.21)

and the $\hat{D}_i$ is particle-like operators, i.e.

$$K_{ij} \hat{D}_i = \hat{D}_j K_{ij}, \ K_{ij} \hat{D}_l = \hat{D}_l K_{ij} \ (l \neq i,j).$$  

(3.22)

Define

$$T_{ab}^{(m)} = \sum_i I_{iab} \rho(\hat{D}_i^m) \ (m \geq 0),$$  

(3.23)

then

(a) $[\hat{D}_j, \hat{D}_i] = \lambda \rho^{-1}(P_{ij}(D_j - D_i)) = \lambda (\hat{D}_j - \hat{D}_i) K_{ij}.$  

(3.24)

(b) $T_{ab}^{(m)}$ satisfy eq. (3.7), i.e., RTT relation eq. (3.2).

Actually $f_1^n = 0$ is easy to be checked. By using

$$[\hat{D}_i^n, \hat{D}_j^m] = \sum_{k=0}^{n-1} \hat{D}_i^k [\hat{D}_i, \hat{D}_j^m] \hat{D}_j^{n-k-1} = \lambda \sum_{k=0}^{n-1} \hat{D}_i^k (\hat{D}_i^m - \hat{D}_j^m) \hat{D}_j^{n-k-1} K_{ij},$$

we have $f_3^{n,m} = 0.$

The projection procedure is very important for it enables us to prove that eq. (3.7) is satisfied by virtue of eq. (3.21).

With the expansion eq. (3.3) and the projected long-range expansion eq. (3.23), the hamiltonian associated to $T(u)$ is obtained by the expansion of the deformed determinant $[1, 16]$

$$\det_q T(u) = \sum_{\sigma} \epsilon(\sigma) T_{1\sigma_1}(u - (p - 1)\lambda) T_{2\sigma_2}(u - (p - 2)\lambda) \cdots T_{p\sigma_p}(u).$$  

(3.25)
A calculation gives
\[
det_q T(u) = 1 + \frac{\lambda}{u} M + \frac{\lambda}{u^2} \left[ \rho \left( \sum_i \hat{D}_i - \frac{\lambda}{2} \sum_{j \neq i} K_{ij} \right) + \frac{\lambda}{2} M(M-1) \right] \\
+ \frac{\lambda}{u^3} \rho \left\{ (\sum_i \hat{D}_i - \frac{\lambda}{2} \sum_{j \neq i} K_{ij})^2 + \frac{\lambda^2}{12} \sum_{i \neq j} K_{ij} K_{jk} \right. \\
+ (M-1) \sum_i (\hat{D}_i - \frac{\lambda}{2} \sum_{j \neq i} K_{ij}) \\
+ \left. \frac{\lambda^2}{6} M(M-1)(M-2) + \frac{\lambda^2}{4} M(M-1) \right\} + \cdots .
\] (3.26)

One takes the Hamiltonian as
\[
H = \frac{1}{2} \rho \left\{ \left( \sum_i \hat{D}_i - \frac{\lambda}{2} \sum_{i \neq j} K_{ij} \right)^2 + \frac{\lambda^2}{12} \sum_{i \neq j \neq k \neq i} K_{ij} K_{jk} \right\}.
\] (3.27)

Therefore we define the Hamiltonian which have the Yangian symmetry given by eqs. (3.13), (3.18) and (3.23). The Hamiltonian (3.27) is concided with the results of Refs. [11, 12].

In comparison to the known models we list the expressions for \( \hat{D}_i \) satisfying eq. (3.24)

1. \( \hat{D}_i = p_i + \frac{\lambda}{2} \sum_{i \neq j} [\text{sgn}(x_i - x_j) + 1] K_{ij}, \ \lambda = 2i l, \)

\[
H = \frac{1}{2} \sum_i p_i^2 + \frac{1}{2} \sum_{i \neq j} l(l - P_{ij}) \delta(x_i - x_j) .
\] (3.28)

2. \( \hat{D}_i = p_i + \sum_{i \neq j} \left[ i \cot a(x_i - x_j) + 1 \right] K_{ij}, \ \lambda = 2i l, \)

\[
H = \frac{1}{2} \sum_i p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{l(l - aP_{ij})}{\sin^2 a(x_i - x_j)} .
\] (3.29)

This is the famous Calogero-Sutherland model with an internal degree of freedom.

3. \( \hat{D}_i = p_i + il \sum_{i \neq j} \left[ \coth a(x_i - x_j) P_{ij}^+ + \tanh a(x_i - x_j) P_{ij}^- + 1 \right] K_{ij}, \ \lambda = 2i l, \)

\[
H = \frac{1}{2} \sum_i p_i^2 + \frac{1}{2} \sum_{i \neq j} l(l - aP_{ij}) \left( \frac{P_{ij}^+}{\sinh^2 a(x_i - x_j)} - \frac{P_{ij}^-}{\cosh^2 a(x_i - x_j)} \right) .
\] (3.30)

where \( P_{ij}^\pm = \frac{1 \pm \sigma_i \sigma_j}{2}, \ \sigma \) is quantum number with \( \sigma^2 = 1. \) At the case of \( P_{ij} = 1, \) eq. (3.30) was first introduced by Calogero etc. [17] and studied recently by Sutherland and Römer in Ref. [18].

An alternative description of transfer matrix was given by BGHP. Define
\[
\tilde{D}_i = \hat{D}_i - \lambda \sum_{i < j} K_{ij} ,
\] (3.31)
then
\[
[D_i, D_j] = 0 , \quad (3.32)
\]
\[
[K_{ij}, D_k] = 0 \quad (k \neq i, j) , \quad (3.33)
\]
\[
K_{ij} D_i - D_j K_{ij} = \lambda . \quad (3.34)
\]

It was proved that
\[
\bar{T}_i(u) = 1 + \lambda \frac{P_{0i}}{u - D_i} , \quad \bar{T}(u) = \prod_i \bar{T}_i(u) \quad \text{and} \quad \rho(\bar{T}(u)) \quad (3.35)
\]
all satisfy the RTT relation.

The deformed determinant of \( \bar{T}(u) \) was defined by
\[
\det_q \bar{T}(u) = \frac{\Delta_M(u + \lambda)}{\Delta_M(u)} , \quad \Delta_M(u) = \prod_{i=1}^M (u - D_i) . \quad (3.36)
\]

It was proved that
\[
\rho(\det_q \bar{T}(u)) = \det_q (T(u)) . \quad (3.37)
\]

We can give another kind \( \hat{D}_i \) satisfies eqs.(3.32), and (3.33) etc.
\[
[D_i, D_j] = 2 \beta P_{ij} (\hat{D}_i - \hat{D}_j) K_{ij} \quad (3.38)
\]

and
\[
D_i = \hat{D}_i - \beta \sum_{j<i} P^+_{ij} K_{ij} \quad (3.39)
\]

The Hamiltonian can be obtained from eq.(3.36) or from the following equation
\[
H = \frac{1}{2} \rho \left\{ \sum_i \bar{D}_i^2 + \frac{\beta^2}{6} \sum_{i \neq j \neq k \neq i} P^+_{ijk} K_{ij} K_{jk} \right\} . \quad (3.40)
\]

where \( P^+_{ijk} = P^+_{ij} P^+_{jk} \). Now we have two sufficient solutions of \( \hat{D}_{ij} \):

(1) \[ \hat{D}_i = p_i + \sum_{i \neq j} l[i \cot a(x_i - x_j) + 1] K_{ij} P^+_{ij} , \quad \lambda = 2l, \]
\[
H = \frac{1}{2} \sum_i p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{l(l - aP_{ij})}{\sin^2 a(x_i - x_j)} P^+_{ij} . \quad (3.41)
\]

Eq. (3.41) is the generalization of the spin chain model (3.29) considered by BGHP [16].
\[
\begin{align*}
(2) \quad \hat{D}_i &= p_i + \frac{1}{2} \sum_{i \neq j} [\text{sgn}(x_i - x_j) + 1] K_{ij} P_{ij}^+ , \quad \lambda = 2il , \\
H &= \frac{1}{2} \sum_i p_i^2 + \frac{1}{2} \sum_{i \neq j} l(l - P_{ij}) \delta(x_i - x_j) P_{ij}^+ . \\
\end{align*}
\]

On condition that \( P_{ij} = \pm 1 \), eq. (3.42) was first pointed out by Yan [19] through Bethe Ansatz, he also found the Y-operator defined by Yang for his model

\[
Y^\alpha_{ij} = \frac{1}{ik_{ij}(ik_{ij} - c)} [ik_{ij} - cP^+_{ij}] [-ik_{ij}P^\alpha_{ij} + cP^+_{ij}] 
\]

where \( c = l(l \pm 1) \), \( P^\alpha_{ij} \) is the permutation, and \( Y \) satisfies \[8\]

\[
Y^\alpha_{jk} Y^\beta_{ik} Y^\gamma_{ij} = Y^\beta_{jk} Y^\gamma_{ik} Y^\alpha_{ij}
\]

In conclusion of this section we have shown the consistence between Yangian symmetry and the integrability of Polychronakos for long-range interaction models and given the interpretation of model (3.30) and Yan model (3.42) from the point of view of YB system.

**IV Truncated Yangian and GC Gyrostat**

In this section we shall show that by making the truncation of the expansion (1.3) at \( n = 4 \) (i.e. \( T^{(4)}_{ab} = 0 \)) and set \( \mu = 0 \) in (2.3) the set of commutation relation (2.20)-(2.25) is equivalent to the Goryachev-Chaplygin(GC) gyrostat.

The Goryachev-Chaplygin(GC) top is completely integrable in sense of Liouville as shown in the literature [20, 21]. In the quantum case the GC top is extended to the quantum GC gyrostat whose Hamiltonian takes the form [20, 21]:

\[
H = \frac{1}{2}(J^2 + 3J_3^2) - bx_1 + pJ_3
\]

where \( J^2 = J_1^2 + J_2^2 + J_3^2 \) and \( b \) are parameters. The quantities appearing in eq. (4.1) satisfy the following commutation relations:

\[
\begin{align*}
[J_i, J_j] &= -i\epsilon_{ijk}J_k , \\
[J_i, x_j] &= -i\epsilon_{ijk}x_k , \\
\end{align*}
\]
\[
[x_i, x_j] = 0 , \tag{4.2}
\]
\[
[p, q] = -i ,
\]
\[
[p, J_i] = [p, x_i] = [q, x_i] = [q, J_i] = 0
\]

where \(i, j, k\) take over 1, 2 and 3. The momentum \(p\) and coordinate \(q\) of mass center commute with all the locally dynamic variables. The Hamiltonian eq. (4.1) commutes with another integral of motion:
\[
G = (2J_3 + p)(J^2 - J_3^2) + 2b[x_3, J_1] \tag{4.3}
\]
where \([A, B]_+ \equiv AB + BA\). In addition to the commutation relations eq. (4.2) there are two constraints [21]:
\[
\sum_{i=1}^{3} J_i x_i = 0 , \tag{4.4}
\]
\[
\sum_{i=1}^{3} x_i^2 = 1 . \tag{4.5}
\]

Eq. (4.4) plays important roles in proving the quantum integrability of the model.

Sklyanin has pointed out that the GC gyrostat can be systematically investigated in the framework of RTT relation [21]. It is interesting to re-derive the GC Gyrostat on the basis of the detail analysis of Yangian and explore some points not touched in [21], for example, the derivation of more general solution of the transfer matrix \(T(u)\) in terms of \(R\)-matrix formalism.

In this section we shall mainly focus on the two points. The first is to show that the GC gyrostat represents a new type of Yangian which can be called “truncated Yangian”. The second goal is to look for more general form of the transfer matrix \(T(u)\) on the basis of \(R\)-matrix formalism.

For solving (2.20)-(2.25) with \(T^{(n)}_{ab} = 0\) for \(n \geq 4\) we first put [21]
\[
T^{(1)}_{11} = \alpha p , \quad T^{(1)}_{22} = 0 , \quad T^{(1)}_{12} = \beta e^{q} x_{+} , \quad T^{(1)}_{21} = \gamma e^{-q} , \tag{4.6}
\]
and

\[ T^{(2)}_{11} = f_1 J^2 + f_2 J_3^2 + f_3 p J_3 + f_4 x_+ + f_5, \]
\[ T^{(3)}_{11} = (p + g_3 J_3)(g_1 J^2 + g_2 J_3^2 + g_5) + g_4 [J_-, x_3]_+ \]  \hspace{1cm} (4.7)

where \( f_1, \ldots, f_5, g_1, \ldots, g_4, \alpha, \beta, \gamma \) and \( \tau \) all parameters to be determined and \( J_{\pm} = J_1 \pm i J_2, x_{\pm} = x_1 \pm i x_2 \) obey the commutation relations shown by eq. (1.2). Obviously not all the parameters are independent. \( \lambda \) can be normalized to be one. Substituting eqs. (4.6) and (4.7) together with eq. (4.4) into eq. (2.20)-(2.25) after lengthy calculations by hand we find

\[ \tau = -i \alpha^{-1} \lambda, \quad f_1 = -\frac{1}{4} \lambda, \quad f_2 = -\frac{3}{4} \lambda, \quad f_3 = \alpha, \quad f_5 = -\frac{1}{16} \lambda, \]
\[ g_1 = -g_2 = -\frac{1}{4} \alpha, \quad g_3 = -\lambda \alpha^{-1}, \quad g_4 = \frac{1}{4} f_4, \quad g_5 = -\frac{1}{16} \alpha. \]

Denoting \( f_4 = f \) the solution \( T^{(n)}_{ab} \) reads:

\[ T^{(2)}_{11} = -\frac{1}{4} \lambda (J^2 + 3 J_3^2 + \frac{1}{4}) + \alpha p J_3 + f x_-, \quad T^{(2)}_{22} = \lambda^{-1} \gamma \beta x_+, \]
\[ T^{(2)}_{12} = -\lambda^{-1} \beta e^{\tau q} \left( -\frac{\lambda}{4} [J_+, x_3]_+ + x_+ (\lambda J_3 - \alpha p) \right), \quad T^{(2)}_{21} = \gamma e^{-\tau q} J_3, \]
\[ T^{(3)}_{11} = -\frac{1}{4} \alpha (p - \lambda \alpha^{-1} J_3)(J^2 - J_3^2 + \frac{1}{4}) + \frac{1}{4} f [J_-, x_3]_+, \]
\[ T^{(3)}_{22} = \frac{1}{4} \lambda^{-1} \beta \gamma [J_+, x_3]_+, \]
\[ T^{(3)}_{12} = -\lambda^2 \beta e^{\tau q} \left\{ f x_3^2 - \frac{1}{4} \alpha [J_+, x_3]_+ (p - \lambda \alpha^{-1} J_3) \right\}, \]
\[ T^{(3)}_{21} = -\frac{1}{4} \gamma e^{-\tau q} (J^2 - J_3^2 + \frac{1}{4}). \]  \hspace{1cm} (4.8)

Therefore the truncated Yangian can be viewed as a mapping of the algebra given by eq. (4.2).

Let us calculate the Casimirs through

\[
\text{det}_q T(u) = T_{11}(u)T_{22}(u-1) - T_{12}(u)T_{21}(u-1) \\
= \sum_{n,l=0,m=1}^{\infty} u^{-n-m-\binom{l+m-1}{2}} (T^{(n)}_{11}T^{(m)}_{22} - T^{(n)}_{12}T^{(m)}_{21}) \\
\equiv \sum_{i=0}^{\infty} u^{-i} C_i
\]  \hspace{1cm} (4.9)
it gives

\[ C_0 = 0, \quad C_1 = \lambda T^{(1)}_{22}, \]

\[ C_j = \lambda T^{(j)}_{22} + \lambda \sum_{m+l=j, m, l \neq 0} \frac{(l+m-1)!}{(m-1)!!} T^{(m)}_{22} \]

\[ + \sum_{m+n+l=j, m, n \neq 0} \frac{(l+m-1)!}{(m-1)!!} \left( T^{(n)}_{11} T^{(m)}_{22} - T^{(n)}_{12} T^{(m)}_{21} \right) (j \geq 1). \quad (4.10) \]

Substituting the derived \( T(u) \) into eq. (4.10) we obtain

\[ \det_q T(u) = u^{-3} (u-1)^{-3} \lambda^{-1} f \left( u^2 - u + \frac{3}{16} \right) (x_+ x_- + x^2_3) . \quad (4.11) \]

So only \( \sum_{i=1}^3 x_i^2 \) is a Casimir operator, i.e. \( C_m = 0 \) for \( m < 4 \), \( C_4 = \lambda^{-1} f \sum_{i=1}^3 x_i^2 \) that commute with \( J_i \). The constraint eq. (4.5) does not play role in solving the RTT relation.

By choosing a proper representation eq. (4.3) can be taken that is automatically satisfied by virtue of \( [\det_q T(u), T_{ab}(v)] = 0 \). Notice that \( \det_q T(u) \) has zero’s at \( u = \frac{1}{4} \) and \( u = \frac{3}{4} \) where the inverse of \( T(u) \) can not be defined.

It is worth noting that the commutation relations shown by eq. (4.2) are invariant subject to a transformation:

\[ J'_a = \sum_b A_{ab} J_b, \quad x'_a = \sum_b A_{ab} x_b, \quad (a, b = 1, 2) \quad (4.12) \]

where

\[ A_{11} = \epsilon A_{22}, \quad A_{12} = -\epsilon A_{21}, \]

\[ A_{11}^2 + A_{21}^2 = 1, \quad \epsilon = \pm 1 = \det A. \quad (4.13) \]

This transformation is useful to transform the Hamiltonian in preserving the RTT relation.

Let us turn to the conserved quantities for GC gyrostat.

By taking the trace of the transfer matrix one obtains

\[ \text{tr} T(u) = \lambda + u^{-1} \alpha \beta + u^{-2} \left\{ -\frac{1}{4} \lambda (J^2 + 3 J_3^2 + \frac{1}{4}) + \alpha \beta + \frac{1}{4} \right\} + \frac{u^{-3}}{4} \left\{ -\left( \alpha \beta - \lambda J_3 \right) (J^2 - J_3^2 + \frac{1}{4}) + f [J_- x_3]^+ + \lambda^{-1} \beta \gamma [J_+, x_3]^+ \right\} . \quad (4.14) \]
Hence we have the Hamiltonian $H_p$ and another conserved quantities $G_p$:

$$H_p = \frac{1}{2} \left\{ \frac{1}{4} \lambda (J^2 + 3J_3^2) - fx_- - \lambda^{-1} \beta \gamma x_+ - \alpha p J_3 \right\} + \frac{1}{16} \lambda, \quad (4.15)$$

$$G_p = \frac{1}{4} \left\{ \left( \alpha p - \lambda J_3 \right) \left( J^2 - J_3^2 + \frac{1}{4} \right) + f [J_-, x_3]_+ + \lambda^{-1} \beta \gamma [J_+, x_3]_+ \right\}. \quad (4.16)$$

If one requires

$$4 \lambda^{-1} \beta \gamma f = b^2, \quad (4.17)$$

then there is rotational invariance about the $x_3$ axis. By virtue of eqs. (4.8) and (4.12) we find that eqs. (4.15) and (4.16) are reduced to

$$H_p = \frac{1}{2} \left\{ \frac{1}{4} \lambda (J'^2 + 3J_3'^2) - bx'_1 - \alpha \epsilon J'_3 \right\} + \text{const.} \quad (4.18)$$

$$G_p = \frac{1}{4} \left\{ \left( \alpha p - \epsilon \lambda J'_3 \right) \left( J'^2 - J_3'^2 + \frac{1}{4} \right) + b [x_3, J'_1]_+ \right\} \quad (4.19)$$

which are exactly those given by Sklyanin [21]. Obviously, the parameters $\lambda$ and $\tau$ are trivial in the solution of $T(u)$ and $\beta$, $\gamma$ can be viewed as the consequence of $T(u)$ subject to a similar transformation. Therefore only parameters $f$ and $\alpha$ are essential in determining the form of $T(u)$ though $f$ does not appear in the Hamiltonian. The simplest realization of quantum GC top is a quantum mechanical system on sphere.

Taking the constraints eqs. (4.4) and (4.5) into account the simplest realization of $J_i$ and $x_i$ can be made through

$$x_i = n_i(t), \quad (4.20)$$

$$J_i = -i \epsilon_{ijk} n_j(t) \dot{n}_k(t) \quad (i, j, k = 1, 2, 3) \quad (4.21)$$

that give $J^2 = -\hat{n} \cdot \hat{n}$.

In conclusion we have shown that the truncated Yangian gives rise to quantum integrable system with finite number of conserved quantities and make the system work in higher than (1+1) dimensions.

The trigonometric extension of the truncated Yangian will be the “truncated” affine quantum algebra which is the generalization of Drinfeld’s statement discussed in Ref. [3]. It gives $q$-deformed GC gyrostat. In the following section, we will show it.
V Further Development

Along the same lines discussed in the section III and IV we are able to develop the idea: for a given $R$-matrix find the corresponding $T(u)$-matrix.

(1) The first extension should be made for the $R$-matrix given by the standard 6-vertex form whose rational limit leads to (1.3). The trigonometric $R$-matrix takes the form:

$$\hat{R}_T(x) = \begin{bmatrix} a(x) & w & b(x) \\ w & b(x) & w \\ b(x) & w & a(x) \end{bmatrix}$$ (5.1)

$$\hat{R}_T(xy^{-1})(T(x) \otimes T(y)) = (T(y) \otimes T(x))\hat{R}_T(xy^{-1})$$ (5.2)

where $a(x) = qx - q^{-1}x^{-1}, b(x) = x - x^{-1}, w = q - q^{-1}, x = e^u$ and $q = e^\gamma$. Following Drinfeld [3] the transfer matrix $T(u)$ obeying (5.2) for $\hat{R} = \hat{R}_T$ should have the expansion form

$$T(x) = \sum_{n=-\infty}^{+\infty} x^n T^{(n)}$$ (5.3)

We want to find the truncated (affine) solution $T^{(u)}(|n| \leq 3)$, i.e., which possesses the form $T(x) = \sum_{n=-3}^{3} x^n T^{(n)}$ which takes the solution (4.6)-(4.8) as a rational limit. We shall see that this extension will give rise to noncommutative geometry rather than those discussed in Ref. [21].

After tedious calculations we find that the algebra for truncated $T^{(n)}_{ab} (|n| \leq 3)$ in (13) is equivalent to the algebra $A^q$ formed by [22] $[P,Q] = -i$, $(P, Q$ commute with $\hat{J}_\pm, \hat{x}_\pm, \hat{J}_3$ and $\hat{x}_3$) and

$$[[\hat{J}_\pm, \hat{x}_\pm] = [[\hat{J}_3, \hat{x}_3] = 0,$

$$[[\hat{J}_\pm, \hat{J}_3] = \pm \hat{J}_3, [\hat{J}_+, \hat{J}_-] = -g[\hat{J}_3]_q,$$

$$[[\hat{x}_\pm, \hat{J}_3] = \pm \hat{x}_\pm,$$

$$q^{\delta} \hat{J}_\pm \hat{x}_3 = \hat{x}_3 \hat{J}_\pm \pm \tau_\pm K^{\pm \delta} \hat{x}_\pm,$$

$$[[\hat{x}_+, \hat{x}_-] = 0, \hat{x}_+ \hat{x}_3 = q^{\delta} \hat{x}_3 \hat{x}_+, q^{-1} \hat{J}_\pm \hat{x}_3 = q^{\delta} \hat{x}_3 \hat{x}_+ \pm \tau_\mp^{-1} g K^{\mp \delta} \hat{x}_3.$$

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with \( \delta_+ + \delta_- = \pm 1 \), and the constraint condition

\[
g[\hat{J}_3]q\hat{x}_3 + \tau_- K^{-\delta_-} \hat{J}_+ \hat{x}_- + \tau_+ K^{\delta_+} \hat{J}_- \hat{x}_+ = 0 \tag{5.5}
\]

when \( q \to 1 \), \( \mathcal{A}^q \) returns to (4.2), namely (5.4) is nothing but the trigonometric extension of (4.2).

The expression of \( T^{(n)}_{ab}(n \leq 3) \) in terms of \( \{ \hat{J}_\pm, \hat{x}_\pm, \hat{J}_3, \hat{x}_3 \} \) is explicitly given in Ref. [22]. Also the Hamiltonian \( H \) and another conserved quantity \( G \) have the forms:

\[
H = 2i\lambda_3 \{ \alpha\alpha^{-1}(q - 1)wg^{-1} \cos[\eta(P + \xi\hat{J}_3)] \hat{J}_+ \hat{J}_- \\
+ 2\sin(\xi\eta\hat{J}_3)(\sin[\eta(P + \xi\hat{J}_3) - \alpha^{-1}\alpha_2q^{1 \over 2} \sin[\eta(P + \xi\hat{J}_3 + {1 \over 2})]]) \\
+ (\alpha\alpha^{-1}(q + 1) + 2) \cos(\xi\eta) \} + D_+,
\tag{5.6}
\]

\[
G = -2i\lambda_3 \left\{ \alpha\alpha^{-1}(q - 1)wg^{-1}\hat{J}_+ \hat{J}_- + 2(\alpha\alpha^{-1}q^{1 \over 2} \cos[\xi(\hat{J}_3 + {1 \over 2})]) \\
+ \cos(\xi\eta\hat{J}_3) \sin[\eta(P + \xi\hat{J}_3)] + D_+ \right\}
\tag{5.7}
\]

where \( q = e^{i\xi\eta} \) and \( D_\pm \) are given by

\[
D_\pm = \epsilon_\pm \left\{ \lambda_3^{-1}\alpha_2\delta_+^{(1)} (\pm \hat{J}_\pm \hat{x}_3 K^{-1} + (1 - q^{-1})^{-1} q^{-2+\delta_+} \tau_+ K^{\delta_+} (q^{-1} - 1) \hat{x}_+ ) \\
+ \lambda_+^{(1)} (\hat{J}_- \hat{x}_3 + (1 - q^{-1})^{-1} q^{-\delta_+} \tau_- K^{1+\delta_+} (1 \mp q^{-1}\alpha^{-1}\alpha_2) \hat{x}_- ) \right\}
\tag{5.8}
\]

for \( \delta_+ + \delta_- = 1 \), and

\[
D_\pm = \epsilon_\pm \left\{ \lambda_3^{-1}\alpha_2\delta_+^{(1)} (\hat{J}_\pm \hat{x}_3 K + (1 - q)^{-1} q^{2+\delta_+} \tau_+ K^{\delta_+} (q^{-1} - 1) \hat{x}_+ ) \\
+ \lambda_+^{(1)} (\mp \hat{J}_+ \hat{x}_3 + (1 - q)^{-1} q^{-\delta_+} \tau_- K^{1+\delta_+} (\alpha\alpha^{-1}q \mp 1) \hat{x}_- ) \right\}
\tag{5.9}
\]

for \( \delta_+ + \delta_- = -1 \), with \( \epsilon_+ = i, \epsilon_- = 1 \).

It is noted that

\[
\alpha = \pm q^{-1/2}\alpha_2
\tag{5.10}
\]

in (5.8) and (5.3), respectively.
We would like to remark that for the given standard 6-vertex $R$-matrix (5.1) we find a set of solution for truncated RTT relation \([22]\). The solution can be realized through the algebra (5.4) and corresponding conserved quantities (5.6) and (5.7). Eq. (5.4) naturally yields the non-commutative geometry. Since the considered system is axially symmetric so that the coordinates on \((x_1, x_2)\) plane are commute with each other, whereas the third coordinates \(x_3\) does not commute with them.

(2) For the Yangian case \((\lambda = \mu = 1\) in (2.3)) more possibility of realizations of the transfer matrix \(T_{ab}^{(n)}\) in (1.3) can be found.

The interesting examples are those expressed in terms of the second quantization forms. They allow to obtain some physical Hamiltonian from Yang-Baxter systems expressed by annihilation and creation operators. For instance we may ask whether the Hamiltonian of 1-dim. Hubbard model can be obtained from the RTT relation. The answer is yes. The calculation will appear elsewhere.

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