POINTWISE CONVERGENCE OF MARCINKIEWICZ-FEJĔR MEANS OF DOUBLE VILENKIN-FOURIER SERIES

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Abstract. In this paper we give a characterization of points in which Marcinkiewicz-Fejér means of double Vilenkin-Fourier series converges.

1. Introduction

Lebesgue’s [14] theorem is well known for trigonometric Fourier series: the Fejér means $\sigma_n f$ of $f$ converge to $f$ almost everywhere if $f \in L_1([0,2\pi))$ (see also Zygmund [26]). The analogous result for Walsh-Fourier series is due to Fine [3]. Later Schipp [19] showed that the maximal operator $\sigma^*$ of the Fejér means of the one-dimensional Walsh-Fourier series is of weak type (1,1), from which the a.e. convergence follows by standard arguments.

Marcinkiewicz [15] verified for two-dimensional trigonometric Fourier series that the Marcinkiewicz-Fejér means

$$\sigma_n(f) = \frac{1}{n} \sum_{j=1}^{n} S_{j,j}(f)$$

of a function $f \in L \log L([0,2\pi) \times [0,2\pi))$ converge a.e. to $f$ as $n \to \infty$, where $S_{j,j}(f)$ denotes the cubic partial sums of the Fourier series of $f$. Later Zhizhiashvili [24,25] extended this result to all $f \in L_1((0,2\pi) \times (0,2\pi))$. The analogous result for two-dimensional Walsh-Fourier series is due to Weisz [21].

In the one-dimensional case the set of convergence is characterized with the help of Lebesgue points. It is known that a.e. point $x \in [0,2\pi)$ is a Lebesgue point of $f \in L_1([0,2\pi))$ and the Fejér means of the trigonometric Fourier series of $f \in L_1((0,2\pi))$ converge to $f$ at each Lebesgue point (see Butzer and Nessel [2]). Weisz [20] introduced the notion of Walsh-Lebesgue points and proved the analogous results for Walsh-Fourier series.

For Vilenkin-Fourier series the author and Gogoladze [11] introduced the notion of Vilenkin-Lebesgue points and proved that Fejér means of the
Vilenkin-Fourier series of \( f \in L_1(G_m) \) converges to \( f \) at each Vilenkin-Lebesgue point. In this paper we will generalize these results for the Marcinkiewicz-Fejér means of double Vilenkin-Fourier series and characterize the set of convergence of these means. We introduce the Marcinkiewicz-Lebesgue points and prove that a.e. point is a Marcinkiewicz-Lebesgue point of an integrable function \( f \) and the Marcinkiewicz-Fejér means of the double Vilenkin-Fourier series of \( f \) converge to \( f \) at each Marcinkiewicz-Lebesgue point.

The problems of summability of cubical partial sums of multiple Fourier series have been investigated in ([4]-[13]).

2. Definitions and Notation

Let \( \mathbb{N}_+ \) denote the set of positive integers, \( \mathbb{N} := \mathbb{N}_+ \cup \{0\} \). Let \( m := (m_0, m_1, \ldots) \) denote a sequence of positive integers not less than 2. Denote by \( Z_{m_k} := \{0, 1, \ldots, m_k - 1\} \) the additive group of integers modulo \( m_k \). Define the group \( G_m \) as the complete direct product of the groups \( Z_{m_j} \), with the product of the discrete topologies of \( Z_{m_j} \)’s. The direct product \( \mu \) of the measures

\[
\mu_k(\{j\}) := \frac{1}{m_k} \quad (j \in Z_{m_k})
\]

is the Haar measure on \( G_m \) with \( \mu(G_m) = 1 \). If the sequence \( m \) is bounded, then \( G_m \) is called a bounded Vilenkin group. The elements of \( G_m \) can be represented by sequences \( x := (x_0, x_1, \ldots, x_j, \ldots) \), \( (x_j \in Z_{m_j}) \). The group operation + in \( G_m \) is given by \( x + y = (x_0 + y_0 \pmod{m_0}, \ldots, x_k + y_k \pmod{m_k}, \ldots) \), where \( x = (x_0, \ldots, x_k, \ldots) \) and \( y = (y_0, \ldots, y_k, \ldots) \in G_m \). The inverse of + will be denoted by −. In this paper we will consider only bounded Vilenkin group.

It is easy to give a base for the neighborhoods of \( G_m \):

\[
I_0(x) := G_m,
I_n(x) := \{ y \in G_m | y_0 = x_0, \ldots, y_{n-1} = x_{n-1} \}
\]

for \( x \in G_m, n \in \mathbb{N} \). Define \( I_n := I_n(0) \) for \( n \in \mathbb{N}_+ \). Set \( e_n := (0, \ldots, 0, 1, 0, \ldots) \in G_m \) the \( n \)th coordinate of which is 1 and the rest are zeros \( (n \in \mathbb{N}) \).

If we define the so-called generalized number system based on \( m \) in the following way: \( M_0 := 1, M_{k+1} := m_k M_k (k \in \mathbb{N}) \), then every \( n \in \mathbb{N} \) can be uniquely expressed as \( n = \sum_{j=0}^{\infty} n_j M_j \), where \( n_j \in Z_{m_j} \ (j \in \mathbb{N}_+) \) and only a finite number of \( n_j \)'s differ from zero. We use the following notation. Let \( (for \ n > 0) |n| := \max\{k \in \mathbb{N} : n_k \neq 0\} \) (that is, \( M_{|n|} \leq n < M_{|n|+1} \)).

Next, we introduce on \( G_m \) an orthonormal system which is called the Vilenkin system. At first define the complex valued functions \( r_k(x) : G_m \rightarrow \mathbb{C} \), the generalized Rademacher functions in this way

\[
r_k(x) := \exp \left( \frac{2\pi i x_k}{m_k} \right) \quad (i^2 = -1, \ x \in G_m, \ k \in \mathbb{N}).
\]
It is known that
\[ \sum_{i=0}^{m_n - 1} r_i^n (x) = \begin{cases} 0, & \text{if } x_n \neq 0 \\ m_n, & \text{if } x_n = 0 \end{cases} \]

Now define the Vilenkin system \( \psi := (\psi_n : n \in \mathbb{N}) \) on \( G_m \) as follows:
\[ \psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}). \]

Specifically, we call this system the Walsh-Paley one if \( m \equiv 2 \).

The Vilenkin system is orthonormal and complete in \( L^1(G_m) \) \([1]\).

We consider the double system \( \{ \psi_n(x) \times \psi_m(y) : n, m \in \mathbb{N} \} \) on \( G_m \times G_m \).

The notation \( a \lesssim b \) in the whole paper stands for \( a \leq cb \), where \( c \) is an absolute constant.

The rectangular partial sums of the double Vilenkin-Fourier series are defined as follows:
\[ S_{M,N}(x,y; f) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \hat{f}(i,j) \psi_i(x) \psi_j(y), \]
where the number
\[ \hat{f}(i,j) = \int_{G_m \times G_m} f(x,y) \psi_i(x) \psi_j(y) \, d\mu(x,y) \]
is said to be the \( (i,j) \)th Vilenkin-Fourier coefficient of the function \( f \).

The norm (or quasinorm) of the space \( L_p(G_m \times G_m) \) is defined by
\[ \| f \|_p := \left( \int_{G_m \times G_m} |f(x,y)|^p \, d\mu(x,y) \right)^{1/p} \quad (0 < p < +\infty). \]

The space weak-\( L_p(G_m \times G_m) \) consists of all measurable functions \( f \) for which
\[ \| f \|_{\text{weak-}L_p(G_m \times G_m)} := \sup_{\lambda > 0} \lambda \mu(|f| > \lambda)^{1/p} < +\infty. \]

The \( \sigma \)-algebra generated by the dyadic 2-dimensional \( I_k \times I_k \) cube of measure \( M_{k-1}^2 \times M_{k-1}^2 \) will be denoted by \( F_k \) \((k \in \mathbb{N})\). Denote by \( f = (f^{(n)}, n \in \mathbb{N}) \) one-parameter martingales with respect to \((F_n, n \in \mathbb{N})\) (for details see, e. g. \([22]\)). The maximal function of a martingale \( f \) is defined by
\[ f^* = \sup_{n \in \mathbb{N}} |f^{(n)}|. \]

In case \( f \in L_1(G_m \times G_m) \), the maximal function can also be given by
\[ f^*(x,y) = \sup_{n \in \mathbb{N}} \frac{1}{\mu(I_n(x) \times I_n(y))} \left| \int_{I_n(x) \times I_n(y)} f(t,u) \, d\mu(t,u) \right|. \]
\((x, y) \in G_m \times G_m\).

For \(0 < p < \infty\) the dyadic martingale Hardy space \(H_p(G \times G)\) consists of all martingales for which

\[
\|f\|_{H_p} := \|f^*\|_p < \infty.
\]

If \(f \in L_1(G_m \times G_m)\) then it is easy to show that the sequence \((S_{M_n, M_n}(f) : n \in \mathbb{N})\) is a martingale. If \(f\) is a martingale, that is \(f = (f^{(0)}, f^{(1)}, \ldots)\) then the Vilenkin-Fourier coefficients must be defined in a little bit different way:

\[
\hat{f}(i, j) = \lim_{k \to \infty} \int_{G_m \times G_m} f^{(k)}(x, y) \psi_i(x) \psi_j(y) \, d\mu(x, y).
\]

The Vilenkin-Fourier coefficients of \(f \in L_1(G_m \times G_m)\) are the same as the ones of the martingale \((S_{M_n, M_n}(f) : n \in \mathbb{N})\) obtained from \(f\).

For \(n = 1, 2, \ldots\) and a martingale \(f\) the Marcinkiewicz-Fejér means of order \(n\) of the 2-dimensional Vilenkin-Fourier series of the function \(f\) is given by

\[
\sigma_n(x, y; f) = \frac{1}{n} \sum_{j=0}^{n-1} S_{j,j}(x, y; f).
\]

If

\[
K_n(x, y) := \frac{1}{n} \sum_{k=0}^{n-1} D_k(x) D_k(y)
\]

denotes the 2-dimensional Marcinkiewicz-Fejér kernel of order \(n\) then

\[
(2) \quad \sigma_n(x, y; f) = \int_{G_m \times G_m} f(t, u) K_n(x - t, y - u) \, d\mu(t, u).
\]

A bounded measurable function \(a\) is a p-atom, if there exists a generalized square \(I \times J \in F_n\), such that

a) \(\int_{I \times J} ad\mu = 0\);

b) \(\|a\|_\infty \leq \mu(I \times J)^{-1/p}\);

c) \(\text{supp } a \subset I \times J\).

An operator \(T\) which maps the set of martingales into the collection of measurable functions will be called p-quasi-local if there exist a constant \(C_p > 0\) such that for every p-atom \(a\)

\[
\int_{G_m \times G_m \setminus (I \times J)} |Ta|^p \leq C_p < \infty,
\]

where \(I \times J\) is the support of the atom.
3. MARCINKIEWICZ-LEBESGUE POINTS

In the one-dimensional case a point $x \in (-\infty, \infty)$ is called a Lebesgue point of a function $f$ if

$$
\lim_{h \to 0} \frac{1}{h} \int_{0}^{h} |f(x + t) - f(x)| \, dt = 0.
$$

It is known that a.e. point $x \in [0, 2\pi)$ is a Lebesgue point of a function $f \in L^1([0, 2\pi))$ and that the Fejér means of the trigonometric Fourier series of $f \in L^1([0, 2\pi))$ converge to $f$ at each Lebesgue point (see Butzer and Nessel [2]). Feichtinger and Weisz [18] extended these results to two-dimensional trigonometric Fourier series, to arbitrary summability methods and to all $f \in L(\log L)^+([0, 2\pi]^2)$.

Weisz introduced the one-dimensional Walsh-Lebesgue point in [20]: $x \in G_2$ is a Walsh-Lebesgue point of $f \in L^1(G_2)$, if

$$
\lim_{A \to \infty} \sum_{n=0}^{n} 2^k \int_{I_n(e_k)} |f(x + t) - f(x)| \, dt = 0.
$$

He proved that a.e. point $x \in G_2$ is a Walsh-Lebesgue point of an integrable function $f$. Moreover, the Fejér means of the Walsh-Fourier series of $f \in L^1(G_2)$ converge to $f$ at each Walsh-Lebesgue point. The higher dimensional extension of this result can be found in [23, 11].

In [11] it is characterized the set of convergence of Vilenkin-Fejér means. We introduced the operator

$$
W_A f(x) := \sum_{s=0}^{A-1} \sum_{r=1}^{m_s-1} \int_{I_A(x-r_se_s)} |f(t) - f(x)| \, d\mu(t).
$$

A point $x \in G_m$ is a Vilenkin-Lebesgue point of $f \in L^1(G_m)$, if

$$
\lim_{A \to \infty} W_A f(x) = 0.
$$

The following are proved in [11].

**Theorem GG ([11]).** Let $f \in L^1(G_m)$, where $G_m$ is a bounded Vilenkin group. Then

$$
\lim_{n \to \infty} \sigma_n f(x) = f(x)
$$

for all Vilenkin-Lebesgue points of $f$.

**Corollary GG1 ([11]).** Let $f \in L^1(G_m)$, where $G_m$ is a bounded Vilenkin group. Then

$$
\lim_{A \to \infty} W_A f(x) = 0 \text{ for a.e. } x \in G_m,
$$

thus a.e. point is a Vilenkin-Lebesgue point of $f$. 
**Corollary GG2 ([11]).** Let $f \in L_1(G_m)$, where $G_m$ is a bounded Vilenkin group. Then

$$\sigma_n(x; f) \rightarrow f(x) \text{ for a.e. } x \in G_m.$$  

For two-dimensional Walsh-Fourier series Weisz [21] has proved that for all $f \in L_1(G \times G)$ the Marcinkiewicz-Féjér means $\sigma_n f$ converge a.e. to $f$ as $n \to \infty$.

In [13] it is characterized the set of convergence of Marcinkiewicz-Féjér means of two-dimensional Walsh-Fourier series.

For two-dimensional Vilenkin-Fourier series Gát [4] has proved that for all $f \in L_1(G_m \times G_m)$ the Marcinkiewicz-Féjér means $\sigma_n f$ converge a.e. to $f$ as $n \to \infty$.

In this paper we will characterize the set of convergence of Marcinkiewicz-Féjér means with respect to bounded Vilenkin system. We introduce the Marcinkiewicz-Lebesgue points and prove that a.e. point is a Marcinkiewicz-Lebesgue point of an integrable function $f$ and the Marcinkiewicz-Féjér means of the two-dimensional Vilenkin-Fourier series of $f$ converge to $f$ at each Marcinkiewicz-Lebesgue point.

Set

$$W_j (x, y; f) := M_j^{-1} \sum_{q=0}^{j-1} \sum_{k=1}^{j-1} \sum_{u_q=1}^{m_q-1} M_q M_k^2$$

$$\times \int_{I_k \times I_k(u_q e_q)} |f(x-t, y-u) - f(x, y)| r_{k+1,j-1} (t, u) \, d\mu(t, u)$$

$$+ M_j^{-1} \sum_{q=0}^{j-1} \sum_{k=1}^{j-1} \sum_{t_q=1}^{m_q-1} M_q M_k^2$$

$$\times \int_{I_k(t_q e_q) \times I_k} |f(x-t, y-u) - f(x, y)| r_{k+1,j-1} (t, u) \, d\mu(t, u)$$

$$+ \sum_{s=0}^{j} \sum_{i=1}^{j} M_s M_i \sum_{u_s=1}^{m_s-1}$$

$$\times \int_{I_i \times I_i(u_s e_s)} |f(x-t, y-u) - f(x, y)| \, d\mu(t, u)$$

$$+ \sum_{s=0}^{j} \sum_{i=1}^{j} M_s M_i \sum_{t_s=1}^{m_s-1}$$

$$\times \int_{I_i(t_s e_s) \times I_i} |f(x-t, y-u) - f(x, y)| \, d\mu(t, u),$$
where
\[ r_{i,n}(x,y) := \prod_{l=i}^{n} \left( \sum_{s=0}^{m_l-1} \psi_{M_l}^s(x+y) \right). \]

By (1) it is easy to show that
\[ r_{i,n}(x,y) = \begin{cases} m_i m_{i+1} \cdots m_n, x_j + y_j \pmod{m_j} = 0, j = i, i+1, \ldots, n \\ 0, \text{otherwise} \end{cases} \]

A point \((x,y) \in G_m \times G_m\) is a Marcinkiewicz-Lebesgue point (for bounded Vilenkin group) of \(f \in L_1(G_m \times G_m)\), if
\[ \lim_{n \to \infty} W_n(x,y;f) = 0. \]

Set
\[ (3) V_n(x,y;f) := \sum_{q=0}^{n-1} \sum_{k=q}^{n-1} \sum_{t_q=1}^{m_q-1} M_q M_k \]
\[ \times \int_{I_k(t_q e_q) \times I_k} f(x-t, y-u) \mathbb{I}_{\{t_r+u_r \pmod{m_r} = 0, r=k+1, \ldots, n-1\}}(t,u) \, d\mu(t,u) \]
\[ + \sum_{q=0}^{n-1} \sum_{k=q}^{n-1} \sum_{u_q=1}^{m_k} M_q M_k \]
\[ \times \int_{I_k \times I_k(u_q e_q)} f(x-t, y-u) \mathbb{I}_{\{t_r+u_r \pmod{m_r} = 0, r=k+1, \ldots, n-1\}}(t,u) \, d\mu(t,u) \]
\[ + \sum_{s=0}^{n} \sum_{i=s}^{n} M_s M_i \sum_{u_i=1}^{m_i-1} \]
\[ \times \int_{I_n \times I_i(u_s e_s)} f(x-t, y-u) \, d\mu(t,u) \]
\[ + \sum_{s=0}^{n} \sum_{i=s}^{n} M_s M_i \sum_{t_i=1}^{m_i-1} \]
\[ \times \int_{I_i(t_i e_s) \times I_n} f(x-t, y-u) \, d\mu(t,u) \]
\[ := \sum_{j=1}^{4} V_n^{(j)}(x,y;f), \]
where \(\mathbb{I}_E\) is characteristic function of the set \(E\).

It is easy to see that \(W_n f (x,y) \to 0\) as \(n \to \infty\) if and only if
\[ \lim_{n \to \infty} V_n (|f - f(x,y)|)(x,y) = 0. \]
Let
\[ Vf := \sup_n |V_n f|, \quad V^{(i)} f := \sup_n |V_n^{(i)} f|, \quad i = 1, 2, 3, 4. \]

4. Main Results

In this paper we prove that the following are true

**Theorem 1.** Let \( f \in L_1 (G_m \times G_m) \). Then
\[ \lim_{n \to \infty} \sigma_n (x, y; f) = f (x, y) \]
for all Marcinkiewicz-Lebesgue points of \( f \).

**Theorem 2.** Let \( p > 1/2 \). Then
\[ \| Vf \|_p \leq c_p \| f \|_p \quad (f \in H_p (G_m \times G_m)) \]
and
\[ \sup_{\lambda} \nu \{ Vf > \lambda \} \leq c \| f \|_1. \]

It is easy to show that \( \lim_{n \to \infty} W_n (x, y; f) = 0 \) for every Vilenkin polynomials and \((x, y) \in G_m \times G_m \). Since the Vilenkin polynomials are dense in \( L_1 (G_m \times G_m) \), Theorem 2 and the usual density argument (see Marcinkiewicz and Zygmund [16]) imply

**Corollary 1.** Let \( f \in L_1 (G_m \times G_m) \). Then
\[ \lim_{n \to \infty} W_n (x, y; f) = 0 \quad a. e. \quad (x, y) \in G_m \times G_m, \]
thus a. e. points is a Marcinkiewicz-Lebesgue point of \( f \).

**Corollary 2.** (Gat [14]) Let \( f \in L_1 (G_m \times G_m) \). Then
\[ \lim_{n \to \infty} \sigma_n (x, y; f) = f (x, y) \quad a. e. \quad (x, y) \in G_m \times G_m. \]

5. Auxiliary Propositions

**Theorem W.** (Weisz [22]) Suppose that the operator \( T \) is \( \sigma \)-sublinear and \( p \)-quasi-local for each \( 0 < p_0 < p \leq 1 \). If \( T \) is bounded from \( L_\infty (G_m \times G_m) \) to \( L_\infty (G_m \times G_m) \), then
\[ \| T f \|_p \leq c_p \| f \|_p \quad (f \in H_p (G_m \times G_m)) \]
for every \( 0 < p_0 < p < \infty \). In particular for \( f \in L_1 (G_m \times G_m) \), holds
\[ \| T f \|_{\text{weak}_1 (G_m \times G_m)} \leq c \| f \|_1. \]

**Lemma 1.** We have
\[
M_{AK} (x, y) = \sum_{k=0}^{A-1} r_{k+1,A-1} (x, y) M_k \sum_{q=0}^{m_k-1} \left( \sum_{s=0}^{r_1-1} \psi_{M_k}^q (x) \right) \left( \sum_{s=0}^{r_1-1} \psi_{M_k}^s (y) \right) \\
\times D_{M_k} (x) D_{M_k} (y)
\]
\[ + \sum_{k=0}^{A-1} r_{k+1,A-1} (x, y) \sum_{r=1}^{m_{k-1}} \left( \sum_{q=0}^{r-1} \psi_{M_k}^q (x) \right) \psi_{M_k}^r (y) D_{M_k} (x) M_k K_{M_k} (y) \]

\[ + \sum_{k=0}^{A-1} r_{k+1,A-1} (x, y) \sum_{r=1}^{m_{k-1}} \left( \sum_{s=0}^{r-1} \psi_{M_k}^s (y) \right) \psi_{M_k}^r (x) D_{M_k} (y) M_k K_{M_k} (x) \]

\[ + r_{1,A-1} (x + y) . \]

**Proof of Lemma 7.** Since (see (7))

\[ D_{j+r M_A} (x) = \left( \sum_{q=0}^{r-1} \psi_{M_A}^q (x) \right) D_{M_A} (x) + \psi_{M_A}^r (x) D_j (x) \]

\[ j = 0, 1, ..., M_A - 1, r = 1, 2, ..., m_{A-1} - 1, \]

we can write

\[ M_A K_{M_A} (x, y) = \sum_{j=0}^{M_A-1} D_j (x) D_j (y) \]

\[ = M_{A-1} K_{M_{A-1}} (x, y) + \sum_{r=1}^{m_{A-1}-1} \sum_{j=0}^{M_{A-1}-1} D_{j+r M_{A-1}} (x) D_{j+r M_{A-1}} (y) \]

\[ = M_{A-1} K_{M_{A-1}} (x, y) + \sum_{r=1}^{m_{A-1}-1} \sum_{j=0}^{M_{A-1}-1} \left( \sum_{q=0}^{r-1} \psi_{M_{A-1}}^q (x) \right) D_{M_{A-1}} (x) \]

\[ \times \left( \sum_{q=0}^{r-1} \psi_{M_{A-1}}^q (y) \right) D_{M_{A-1}} (y) \]

\[ + \sum_{r=1}^{m_{A-1}-1} \sum_{j=0}^{M_{A-1}-1} \psi_{M_{A-1}}^r (x) \psi_{M_{A-1}}^r (y) D_j (x) D_j (y) \]

\[ + \sum_{r=1}^{m_{A-1}-1} \sum_{j=0}^{M_{A-1}-1} \left( \sum_{q=0}^{r-1} \psi_{M_{A-1}}^q (x) \right) D_{M_{A-1}} (x) \]

\[ \times \psi_{M_{A-1}}^r (y) D_j (y) \]

\[ + \sum_{r=1}^{m_{A-1}-1} \sum_{j=0}^{M_{A-1}-1} \left( \sum_{s=0}^{r-1} \psi_{M_{A-1}}^s (y) \right) D_{M_{A-1}} (y) \]

\[ \times \psi_{M_{A-1}}^r (x) D_j (x) \]

\[ = \left( \sum_{r=0}^{m_{A-1}-1} \psi_{M_{A-1}}^r (x + y) \right) M_{A-1} K_{M_{A-1}} (x, y) \]
Then from (4) and (5) we can write

\[ \frac{m_{A-1}}{x} \sum_{r=1}^{r-1} \left( \sum_{q=0}^{\psi_{MA-1}^q(x)} \left( \sum_{s=0}^{\psi_{MA-1}^s(y)} D_{MA-1}(x) D_{MA-1}(y) \right) \right) \]

\[ + \frac{m_{A-1}}{x} \sum_{r=1}^{r-1} \left( \sum_{q=0}^{\psi_{MA-1}^q(x)} \psi_{MA-1}^s(y) D_{MA-1}(x) M_{A-1} K_{MA-1}(y) \right) \]

\[ + \frac{m_{A-1}}{x} \sum_{r=1}^{r-1} \left( \sum_{s=0}^{\psi_{MA-1}^s(y)} \psi_{MA-1}^r(x) D_{MA-1}(y) M_{A-1} K_{MA-1}(x) \right). \]

Iterating this equality we obtain the proof of Lemma 1 \[ \square \]

By results in [17] we have

\[ |K_{MA}(x)| \lesssim \sum_{s=0}^{A} M_{s} \sum_{s=1}^{mA-1} D_{M_{A}}(x - x_{s}e_{s}) \quad (4) \]

and

\[ n |K_{n}(x)| \lesssim \sum_{j=0}^{A} M_{j} |K_{M_{j}}(x)|, M_{A} \leq n < M_{A+1}. \quad (5) \]

Then from (4) and (5) we can write

\[ \frac{n}{x} |K_{n}(x)| \lesssim \sum_{s=0}^{A} \sum_{j=0}^{j} M_{s} \sum_{s=1}^{mA-1} D_{M_{j}}(x - x_{s}e_{s}) \]

\[ \lesssim \sum_{s=0}^{A} M_{s} \sum_{j=s}^{A} \sum_{s=1}^{mA-1} D_{M_{j}}(x - x_{s}e_{s}) \]

\[ \lesssim \sum_{s=0}^{A} M_{s} \sum_{j=s}^{A} \sum_{s=1}^{mA-1} D_{M_{j}}(x - x_{s}e_{s}). \]

\[ \text{Lemma 2. Let } M_{A} \leq n < M_{A+1}. \text{ Then we have} \]

\[ n |K_{n}(x,y)| \lesssim \sum_{j=0}^{A} \sum_{q=0}^{j-1} \sum_{k=0}^{j-1} r_{k+1,j-1}(x,y) \]

\[ \times M_{q} D_{M_{k}}(x) \sum_{y_{q}=1}^{mA-1} D_{M_{k}}(y - y_{q}e_{q}) \]

\[ + \sum_{j=0}^{A} \sum_{q=0}^{j-1} \sum_{k=0}^{j-1} r_{k+1,j-1}(x,y) \]

\[ \times M_{q} D_{M_{k}}(y) \sum_{x_{q}=1}^{mA-1} D_{M_{k}}(x - x_{q}e_{q}) \]

\[ + \sum_{j=0}^{A} D_{M_{j}}(x) \sum_{s=0}^{j} M_{s} \sum_{i=s}^{j} m_{i-1} D_{M_{i}}(y - y_{s}e_{s}) \]
\[
+ \sum_{j=0}^{A} D_{M_j}(y) \sum_{s=0}^{j} \sum_{i=s}^{j} \sum_{x_s=1}^{m_{s-1}} D_{M_i}(x - x_s e_s).
\]

proof of Lemma 2. It is proved in [9] that
\[
n |K_{n}(x, y)| \lesssim \sum_{j=0}^{A} M_j \left| K_{M_j}(x, y) \right|
\]
\[
+ \sum_{j=0}^{A} D_{M_j}(x) \max_{1 \leq n \leq n^{(j)}} n |K_{n}(y)|
\]
\[
+ \sum_{j=0}^{A} D_{M_j}(y) \max_{1 \leq n \leq n^{(j)}} n |K_{n}(x)|,
\]

where \(n^{(j)} := \sum_{k=0}^{j} n_k M_k\).

Then from Lemma 1 and estimation (5) we can write
\[
n |K_{n}(x, y)| \lesssim \sum_{j=0}^{A} \sum_{k=0}^{j-1} r_{k+1,j-1}(x, y) D_{M_k}(x) M_k \left| K_{M_k}(y) \right|
\]
\[
+ \sum_{j=0}^{A} \sum_{k=0}^{j-1} r_{k+1,j-1}(x, y) D_{M_k}(y) M_k \left| K_{M_k}(x) \right|
\]
\[
+ \sum_{j=0}^{A} D_{M_j}(x) \max_{1 \leq n \leq n^{(j)}} n |K_{n}(y)|
\]
\[
+ \sum_{j=0}^{A} D_{M_j}(y) \max_{1 \leq n \leq n^{(j)}} n |K_{n}(x)|
\]
\[
\lesssim \sum_{j=0}^{A} \sum_{k=0}^{j-1} r_{k+1,j-1}(x, y) D_{M_k}(x)
\]
\[
\times \sum_{q=0}^{k} \sum_{y_q=1}^{m_{q-1}} D_{M_k}(y - y_q e_q)
\]
\[
+ \sum_{j=0}^{A} \sum_{k=0}^{j-1} r_{k+1,j-1}(x, y) D_{M_k}(y)
\]
\[
\times \sum_{q=0}^{k} \sum_{x_q=1}^{m_{q-1}} D_{M_k}(x - x_q e_q)
\]
\[
+ \sum_{j=0}^{A} D_{M_j}(x) \sum_{s=0}^{j} \sum_{i=s}^{j} \sum_{x_s=1}^{m_{s-1}} D_{M_i}(y - y_s e_s)
\]
\[
+ \sum_{j=0}^{A} D_{M_j}(y) \sum_{s=0}^{j} \sum_{i=s}^{j} \sum_{y_s=1}^{m_{s-1}} D_{M_i}(y - y_s e_s)
\]
Proof of Theorem 1. We can write

\[ |\sigma_n (x, y; f) - f (x, y)| \]

\[ \leq \int_{G_m \times G_m} |f (x - t, y - u) - f (x, y)| |K_n (t, u)| d\mu (t, u) \]

\[ \leq \frac{c}{n} \sum_{j=0}^{A} \sum_{q=0}^{j-1} \sum_{k=q}^{j-1} \sum_{u_q=1}^{m_q-1} \int_{G_m \times G_m} |f (x - t, y - u) - f (x, y)| \]

\[ \times r_{k+1, j-1} (t, u) M_q D_{M_k} (t) D_{M_k} (u - u_q e_q) d\mu (t, u) \]

\[ + \frac{c}{n} \sum_{j=0}^{A} \sum_{s=0}^{j} \sum_{i=s}^{j} \sum_{s_{i}=1}^{m_{s}-1} \int_{G_m \times G_m} |f (x - t, y - u) - f (x, y)| \]

\[ \times r_{k+1, j-1} (t, u) M_q D_{M_k} (t - t_q e_q) D_{M_k} (u) d\mu (t, u) \]

\[ + \frac{1}{n} \sum_{j=0}^{A} \sum_{s=0}^{j} \sum_{i=s}^{j} \sum_{s_{i}=1}^{m_{s}-1} M_s \int_{G_m \times G_m} |f (x - t, y - u) - f (x, y)| \]

\[ \times D_{M_j} (t) D_{M_i} (u - u_s e_s) d\mu (t, u) \]

6. Proofs of Main Results

Proof of Theorem 1. We can write

\[ |\sigma_n (x, y; f) - f (x, y)| \]

\[ \leq \int |f (x - t, y - u) - f (x, y)| |K_n (t, u)| d\mu (t, u) \]

\[ \leq \frac{c}{n} \sum_{j=0}^{A} \sum_{q=0}^{j-1} \sum_{k=q}^{j-1} \sum_{u_q=1}^{m_q-1} \int_{G_m \times G_m} |f (x - t, y - u) - f (x, y)| \]

\[ \times r_{k+1, j-1} (t, u) M_q D_{M_k} (t) D_{M_k} (u - u_q e_q) d\mu (t, u) \]

\[ + \frac{c}{n} \sum_{j=0}^{A} \sum_{s=0}^{j} \sum_{i=s}^{j} \sum_{s_{i}=1}^{m_{s}-1} \int_{G_m \times G_m} |f (x - t, y - u) - f (x, y)| \]

\[ \times r_{k+1, j-1} (t, u) M_q D_{M_k} (t - t_q e_q) D_{M_k} (u) d\mu (t, u) \]

\[ + \frac{1}{n} \sum_{j=0}^{A} \sum_{s=0}^{j} \sum_{i=s}^{j} \sum_{s_{i}=1}^{m_{s}-1} M_s \int_{G_m \times G_m} |f (x - t, y - u) - f (x, y)| \]

\[ \times D_{M_j} (t) D_{M_i} (u - u_s e_s) d\mu (t, u) \]

\[ = \frac{c}{n} \sum_{j=0}^{A} \sum_{q=0}^{j-1} \sum_{k=q}^{j-1} \sum_{u_q=1}^{m_q-1} \int_{I_k \times I_k (u_q e_q)} |f (x - t, y - u) - f (x, y)| \]

\[ \times r_{k+1, j-1} (t, u) d\mu (t, u) \]

\[ + \frac{c}{n} \sum_{j=0}^{A} \sum_{s=0}^{j} \sum_{i=s}^{j} \sum_{s_{i}=1}^{m_{s}-1} \int_{I_k (t_q e_q) \times I_k} |f (x - t, y - u) - f (x, y)| \]

\[ \times r_{k+1, j-1} (t, u) d\mu (t, u) \]
\[\frac{c}{n} \sum_{j=0}^{A} \sum_{s=0}^{j} \sum_{i=s}^{j} \sum_{u=1}^{m_i-1} M_q M_i M_j \]
\[\times \int_{I_j \times I_s(e_s)} |f(x - t, y - u) - f(x, y)| \, d\mu(t, u)\]
\[ + \frac{c}{n} \sum_{j=0}^{A} \sum_{s=0}^{j} \sum_{i=s}^{j} \sum_{t_s=1}^{m_i-1} M_q M_i M_j \]
\[\times \int_{I_i(t_s e_s) \times I_j} |f(x - t, y - u) - f(x, y)| \, d\mu(t, u)\]
\[\leq \frac{c}{n} \sum_{j=0}^{A} \sum_{s=0}^{j} \sum_{t_s=1}^{m_i-1} M_j W_j(x, y; f)\]

which tends to 0 as \(n \to \infty\). This completes the proof of Theorem 1. □

\textbf{Proof of Theorem 2.} Since

\[Vf \leq \sum_{j=1}^{4} V^{(j)} f,\]

by Theorem W, the proof of Theorem 2 will be complete if we show that the operators \(V^{(i)}, i = 1, 2, 3, 4\) are \(p\)-quasi-local for each \(1/2 < p \leq 1\) and bounded from \(L_\infty(G_m \times G_m)\) to \(L_\infty(G_m \times G_m)\).

It follows from \(3\) that

\[\|Vf\|_\infty \leq c \|f\|_\infty \sup_n \sum_{q=0}^{n} \sum_{k=q}^{m} M_q M_k \leq c \|f\|_\infty.\]

Let \(a\) be an arbitrary atom with support \(I_N(z', z'') = I_N(z') \times I_N(z'').\)

It is easy to see that \(V_n(a) = 0\) if \(n < N\). Therefore we can suppose that \(n \geq N\). We may assume that \(z' = z'' = 0\). Hence

\[\text{supp}(a) \subset I_N \times I_N.\]

\textbf{Step 1: Integrating over} \(\overline{I}_N \times \overline{I}_N\). If \(q \geq N\) then \(y - u \notin I_N\) by \(3\). Hence

\[a(x - t, y - u) = 0.\]

Consequently, we can write

\[V_n^{(1)}(x, y; a) := \sum_{q=0}^{N-1} \sum_{k=q}^{N-1} \frac{M_k M_q}{M_k M_n} \sum_{t_q=1}^{m_q-1}\]
\[\int_{I_k(t_q e_q) \times I_k} a(x - t, y - u) \mathbb{I}_{\{t_r + u_r (\text{mod}_{m_r}) = 0, r = k+1, \ldots, n-1\}}(t, u) \, d\mu(t, u).\]
Then from (7) \( a (x - t, y - u) \neq 0 \) implies that
\[
t = (0, ..., 0, t_q, 0, ..., 0, x_k, ..., x_{N-1}, t, \ldots)\]
\[
x = (0, ..., 0, t_q, 0, ..., 0, x_k, ..., x_{N-1}, \ldots)\]
\[
u = (0, ..., 0, y_k, m_{k+1} - x_{k+1}, \ldots, m_{N-1} - x_{N-1}, u, \ldots)\]
\[
y = (0, ..., 0, y_k, m_{k+1} - x_{k+1}, \ldots, m_{N-1} - x_{N-1}, y, \ldots)\].

Hence
\[
\left| V_n^{(1)} (x, y; a) \right| \leq \frac{M_N^{2/p}}{M_N^2} \sum_{q=0}^{N-1} \sum_{k=q}^{N-1} M_k M_q \mathbb{I}_{I_k(t_q \sim q) (x)} \times \mathbb{I}_{I_{N} (0, ..., 0, y_k, m_{k+1} - x_{k+1}, ..., m_{N-1} - x_{N-1}) (y)}
\]
and
\[
\int_{\mathcal{I}_N \times \mathcal{I}_N} \left( V^{(1)} (x, y; a) \right)^p d\mu (x, y) \leq c_p M_N^{2/p} \sum_{q=0}^{N-1} \sum_{k=q}^{N-1} M_k^p M_q^p \frac{1}{M_k M_N} \leq c_p < \infty \quad (1/2 < p \leq 1)
\]

Analogously, we can prove that
\[
\int_{\mathcal{I}_N \times \mathcal{I}_N} \left( V^{(2)} (x, y; a) \right)^p d\mu (x, y) \leq c_p < \infty \quad (1/2 < p \leq 1)
\]

Since \( x + t \notin I_N \) we obtain,
\[
V_n^{(3)} (x, y; a) = 0.
\]

Analogously, we can prove that
\[
V_n^{(4)} (x, y; a) = 0.
\]

Combining (3) and (8-11) we obtain
\[
\int_{\mathcal{I}_N \times \mathcal{I}_N} (V (x, y; a))^p d\mu (x, y) \leq c_p \quad (1/2 < p \leq 1).
\]
Step 2: Integrating over $\overline{T_N} \times I_N$. Since $V_n^{(1)}(x, y; a) = 0$ for $q \geq N$ we have

$$V_n^{(1)}(x, y; a) = \sum_{q=0}^{N-1} \sum_{k=q}^{N-1} \frac{M_k M_q}{m_k} \sum_{t_q=1}^{m_q-1} \int_{I_k(t_q e_q) \times I_k} a(x - t, y - u) \mathbb{1}_{\{t_r + u_1 \text{ (mod e_r)} = 0, r = k + 1, \ldots, n - 1\}}(t, u) \, d\mu(t, u)$$

\[+ \sum_{q=0}^{N-1} \sum_{k=q}^{N} \frac{M_k M_q}{m_k} \sum_{t=1}^{m_q-1} \int_{I_k(t_q e_q) \times I_k} a(x - t, y - u) \mathbb{1}_{\{t_r + u_1 \text{ (mod e_r)} = 0, r = k + 1, \ldots, n - 1\}}(t, u) \, d\mu(t, u)\]

$$= V_n^{(1,1)}(x, y; a) + V_n^{(1,2)}(x, y; a).$$

From (7) $V_n^{(1,1)}(x, y; a) \neq 0$ implies that

$$u = (0, \ldots, 0, u_N, \ldots)$$

$$y = (0, \ldots, 0, y_N, \ldots).$$

$$t = (0, \ldots, 0, t_q, 0, \ldots, 0, t_k, 0, \ldots, 0, t_N, \ldots)$$

$$x = (0, \ldots, 0, t_q, 0, \ldots, 0, t_k, 0, \ldots, 0, x_N, \ldots)$$

Consequently,

$$\left|V_n^{(1,1)}(x, y; a)\right| \lesssim \frac{M_N^{2/p}}{M_N^2} \sum_{q=0}^{N-1} \sum_{k=q}^{N-1} \frac{M_k M_q}{m_k} \sum_{t_q=1}^{m_q-1} \sum_{t_k=1}^{m_k-1} \mathbb{1}_{I_N(e_q t_q + e_k t_k)}(x) \mathbb{1}_{I_N(y)}$$

and

$$\int_{\overline{T_N} \times I_N} \left(\sup_n \left|V_n^{(1,1)}(x, y; a)\right|\right)^p \leq c_p \frac{M_N^{2/p}}{M_N^2} \sum_{q=0}^{N-1} \sum_{k=q}^{N-1} M_k^p M_q^p \frac{1}{M_N^2} \leq c_p.$$

From (7) $V_n^{(1,2)}(x, y; a) \neq 0$ implies that

$$t = (0, \ldots, 0, t_q, 0, \ldots, 0, t_k, \ldots, t_{n-1}, t_n, \ldots)$$

$$x = (0, \ldots, 0, t_q, 0, \ldots, 0, x_N, \ldots)$$

$$u = (0, \ldots, 0, u_k, \alpha_k+1, \ldots, \alpha_{n-1}, u_n, \ldots)$$

$$y = (0, \ldots, 0, y_N, \ldots),$$

where

$$\alpha_j := \begin{cases} m_j - t_j, t_j \neq 0 \\ 0, t_j = 0, j = k + 1, \ldots, n - 1. \end{cases}$$
Thus

\[
\left| V_n^{(1,2)} (x, y; a) \right| \lesssim \frac{M_N^{2/p} \sum_{q=0}^{N-1} \sum_{k=q}^{N-1} M_k M_q \sum_{t_q=1}^{m_q-1} \mathbb{I}_{I_N(e_q t_q)} (x) \mathbb{I}_{I_N} (y)}{M_n} \leq \frac{c M_n^{2/p} \sum_{q=0}^{N-1} M_q \mathbb{I}_{I_N(e_q t_q)} (x) \mathbb{I}_{I_N} (y)}{M_n}
\]

and \( (n \geq N) \)

\[
(15) \int_{T_n \times I_N} \left( \sup_n \left| V_n^{(1,2)} (x, y; a) \right| \right)^p d\mu (x, y) \leq \frac{c_p M_N^2}{M_n^{2/p} M_n^p} \sum_{q=0}^{N-1} M_q^p \leq c_p < \infty.
\]

Combining (13)-(15) we conclude that

\[
(16) \int_{T_n \times I_N} \left( \left| V_n^{(1)} (x, y; a) \right| \right)^p d\mu (x, y) \leq c_p < \infty.
\]

Let \( q < N \). Then it is easy to show that \( y-u \notin I_N \) and consequently,

\[
a (x-t, y-u) = 0,
\]

\[
V_n^{(2)} (x, y; a) = 0.
\]

Let \( q \geq N \). Then \( x-t \notin I_N \) and

\[
V_n^{(2)} (x, y; a) = 0.
\]

Hence

\[
(17) \quad V^{(2)} (x, y; a) = 0.
\]

Analogously, we can prove that

\[
(18) \quad V^{(4)} (x, y; a) = 0.
\]

The estimation of \( V_n^{(3)} (x, y; a) \) is analogous to the estimation of \( V_n^{(1)} (x, y; a) \) and we can prove that

\[
(19) \int_{T_n \times I_N} \left( \left| V_n^{(3)} (x, y; a) \right| \right)^p d\mu (x, y) \leq c_p < \infty \ (1/2 < p \leq 1).
\]

Combining (16)-(19) we conclude that

\[
(20) \int_{T_n \times I_N} \left( \left| V (x, y; a) \right| \right)^p d\mu (x, y) \leq c_p < \infty \ (1/2 < p \leq 1).
\]
Step 3: Integrating over $I_N \times T_N$. This case is analogous to the step 2 and we obtain that

$$\int_{I_N \times T_N} \left( \| V(x, y; a) \| \right)^p d\mu(x, y) \leq c_p < \infty \quad (1/2 < p \leq 1).$$

Combining (12), (20) and (21) we complete the proof of Theorem 2. □

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