Exact plane-symmetric non-stationary solution of the self-consistent equations of Einstein-Maxwell for a magnetoactive plasma

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Abstract
An exact plane-symmetric non-stationary solution to the Einstein-Maxwell equations for a magnetoactive plasma is obtained and studied.

1 Introduction
In Ref. [1] on the basis of an exact solution to the equations of relativistic magnetohydrodynamics (RMHD) against the plane gravitational wave (PGW) background metric a new class of relativistic essentially nonlinear phenomena arising in highly magnetized plasma under the influence of the PGW was found and called gravimagnetic shock waves (GMSW). The essence of the GMSW phenomena consists in that the highly magnetized plasma:

\[ \alpha^2 = \frac{H_{\perp}^2}{4\pi(\varepsilon_0 + p_0)} \gg 1, \]

(where \( H_{\perp} \) is the magnetic field intensity component which is perpendicular in relation to the direction of the PGW propagation, \( \varepsilon_0, p_0 \) - are the unperturbed energy density and plasma pressure without taking a magnetic field into account) anomalously highly responds even to a weak PGW at sufficiently large values of the GMSW second parameter:

\[ \Upsilon \equiv 2\beta_0\alpha^2 > 1, \]

where \( \beta_0 \) - the maximal amplitude of the PGW.

In Ref. [2] on the basis of the plasma and the PGW energy balance model it is shown that the PGW is practically completely transformed to the magnetoactive plasma acceleration (mainly in the PGW propagation direction) and to the creating of a shock wave with high densities of plasma and the magnetic field energy. In the mentioned works the isotropic plasma was considered locally, the anisotropy was formed exclusively by the magnetic field.

In the high magnetic fields as a result of magneto-bremsstrahlung local thermodynamic equilibrium (LTE) in plasma is broken. The given paper is devoted to a exact examination of this problem.
2 Ricci tensor

Let’s find solutions of the Einstein equations with a plane symmetry when the symmetry "plane" is $\Pi\{x^2, x^3\}$. The metric of the space $V_4$ of the signature $(-1, -1, -1, +1)$, supposing two spacelike vectors of Killing:

$$\xi^i = \delta^i_2; \quad i = \delta^i_3$$

(3)

and the corresponding PGW symmetry with the polarization $e_+$ can be written in the form:

$$ds^2 = \Phi - L^2 \left[ e^{2\beta}(dx^2)^2 + e^{-2\beta}(dx^3)^2 \right],$$

(4)

where

$$L = L(x^1, x^4); \quad \beta = \beta(x^1, x^4)$$

(5)

and

$$\Phi = ds^2 = g_{\alpha\beta}(x^1, x^4) \quad (\alpha, \beta = 1, 2)$$

(6)

- is the metric of the two-dimensional pseudoeuclidean surface $\Sigma$: $x^2 = \text{Const}$; $x^3 = \text{Const}$, and:

$$g_{\alpha\beta} = g_{\alpha\beta}(x^1, x^4).$$

(7)

Thus,

$$V_4 = \Sigma \times \Pi_\Sigma.$$  

(8)

As it is known (for example see Ref. [4]), the two-dimensional surface metric can be always reduced to a conform-plane form:

$$\Phi = e^{2\lambda} \left[ (dx^4)^2 - (dx^1)^2 \right].$$

(9)

Thus, with the help of the admissible transformations of the coordinates

$$x'^1 = f^1(x^1, x^4); \quad x'^4 = f^4(x^1, x^4),$$

(10)

which do not alter the metric $\Pi_\Sigma$, the metric $V_4$ can be reduced to

$$ds^2 = e^{2\lambda} \left[ (dx^4)^2 - (dx^1)^2 \right] -$$

$$- L^2 \left[ e^{2\beta}(dx^2)^2 + e^{-2\beta}(dx^3)^2 \right],$$

(11)

where

$$\lambda = \lambda(x^1, x^4).$$

Note, the metric of Eq. (11) is invariant in relation to Lorentz transformations in the plane $\Sigma$. In the retarded, $u$, and advanced, $v$, time coordinates:

$$u = \frac{1}{\sqrt{2}}(t - x) \quad v = \frac{1}{\sqrt{2}}(t + x),$$

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The metric of Eq. (11) takes the form:

\[ ds^2 = 2e^{2\beta} du dv - L^2 (e^{2\beta}(dx^2)^2 + e^{-2\beta}(dx^3)^2) . \] (12)

The metric of Eq. (11) coincides with the rotationally symmetric metric, see Ref. [5]. Hence, we obtain Ricci tensor nonzero components which in the coordinates \( u \) and \( v \) take the form:

\[ R_{uu} = -\frac{2}{L} (L_{uu} + L\beta_u^2 - 2\lambda_u L_u) ; \] (13)

\[ R_{vv} = -\frac{2}{L} (L_{vv} + L\beta_v^2 - 2\lambda_v L_v) ; \] (14)

\[ R_{uv} = -\frac{2}{L} (L_{uv} + L\beta_u \beta_v + L\lambda_{uv}) ; \] (15)

\[ R_2^2 + R_3^2 = -2e^{-2\lambda} \frac{(L^2)_{uv}}{L^2} ; \] (16)

\[ R_2^2 - R_3^2 = -4e^{-2\lambda} \left( \beta_{uv} + \frac{L_u \beta_v + L_v \beta_u}{L} \right). \] (17)

3 Conditions of a magnetic field freezing-in into plasma and the Einstein’s equations

Let’s study magnetoactive plasma moving in the metric (12). In Ref. [1] it is shown that the electromagnetic field of the magnetoactive plasma should satisfy freezing-in in plasma:

\[ F_{ik} v^k = 0 , \] (18)

where \( v^k \) is the vector of the plasma dynamic velocity that coincides with the vector of the electromagnetic field dynamic velocity (by Synge, Ref. [5]). Thus, the first invariant of the electromagnetic field equals to null:

\[ F_{ik} \overset{*}{F}_{ik} = 0 , \] (19)

and the second invariant is positive:

\[ \frac{1}{2} F_{ik} F^{ik} = -(H, H) \equiv H^2 > 0 , \] (20)

- where \( H_i \) is the magnetic field vector:

\[ H_i = v^k \overset{*}{F}_{ki} , \] (21)

and \( \overset{*}{F}_{ki} \) is the tensor which is dual to the Maxwell antisymmetric tensor \( F_{ki} \).

In Ref. [1] it is shown that the complete tensor of an impulse energy (EIT) of locally isotropic magnetoactive plasma takes the form:

\[ T_{ik} = (E + P)v_i n_k - P_{gik} - 2P_{H n_i n_k} , \] (22)
where

\[ P_H = \frac{H^2}{8\pi}; \quad \mathcal{E} = \varepsilon + \varepsilon_H; \quad P = p + P_H, \quad (23) \]

\( P, \mathcal{E} \) are the summary pressure and density of the magnetoactive plasma energy, and

\[ n_i = \frac{H_i}{H}; \quad (24) \]

is the single spacelike vector of the magnetic field:

\[ (n, n) = -1, \quad (25) \]

where

\[ (n, v) = 0. \quad (26) \]

Under the conditions of a plane symmetry we shall consider the plasma spreading in the direction of \( x^1 \) and the magnetic field directed along \( x^2 \). The vector potential corresponds to this field, see Ref.[1]:

\[ A_u = A_v = A_2 = 0; \quad A_3 = \psi(u, v), \quad (27) \]

where \( \psi \) is an arbitrary function of its arguments. Calculating the Maxwell tensor in relation to the potential \( (27) \) we shall find its nonzero components:

\[ F_{u3} = \psi_u; \quad F_{v3} = \psi_v. \quad (28) \]

Calculating the invariant \( (20) \) we shall obtain:

\[ H^2 = -2\frac{e^{2\beta} \psi_u \psi_v}{L^2 e^{2\lambda}}. \quad (29) \]

Thus, it should be \( \psi_u \psi_v < 0 \). Let us choose:

\[ \psi_u < 0; \quad \psi_v > 0, \quad (30) \]

the magnetic field positive direction corresponds to this choice: \( n^2 > 0 \).

Then taking the relation of the velocity vector normalization into consideration the conditions of freezing-in \( (18) \) for the nonzero components of this vector give:

\[ v_v = e^\lambda \sqrt{-\frac{\psi_v}{\psi_u}}; \quad v_u = e^\lambda \sqrt{-\frac{\psi_u}{\psi_v}}. \quad (31) \]

Taking these relations and Eq.\( (22) \) into account let us write Einstein’s nontrivial equations out:

\[ L_{uu} + L\beta_u^2 - 2\lambda_u L_u = \varepsilon + p + \frac{H^2}{4\pi}; \quad (32) \]
\[ L_{uv} + L_\beta^2 - 2\lambda_v L_v = \kappa \frac{L e^{2\lambda} \psi_u}{4\psi_u} \left( \varepsilon + p + \frac{H^2}{4\pi} \right); \]  
(33)

\[ L_{uv} + L_\beta u \beta_v + L_{\lambda uv} = -\frac{\kappa}{2} L e^{2\lambda} p; \]  
(34)

\[ (L^2)_{uv} = \frac{\kappa}{2} L^2 e^{2\lambda} (\varepsilon - p); \]  
(35)

\[ \beta_{uv} + L_u \beta_v + L_v \beta_u = -\frac{\kappa}{16\pi} e^{2\lambda} H^2. \]  
(36)

The Eqs. (32) simultaneously with the definition \( H^2 \) (29) and the local equation of the plasma state:
\[ p = p(\varepsilon) \]  
(37)
represent a complete equations set in relation to the five unknowns of the scalar functions: \( \lambda, \beta, L, \psi, \varepsilon \). Note due to the last equation of the Eqs. (36):
\[ \beta = \text{Const} \rightarrow H^2 = 0. \]  
(38)

It is this property that calls in question the correctness of the energy balance model used in Refs. [1] and [2] in order to take the GMSW influence back on the gravitational wave metric into account.

If we suppose all functions depending only on one variable \( t \) in Eqs. (32) - (36), we shall obtain a homogeneous anisotropic universe model with a magnetic field. If we suppose all functions depending only on the variable \( x \), we shall obtain a static model of the plane anisotropic stratum. In vacuum the system (32) - (36) supposes also a retarded solution (all functions depend only on the variable \( u \)), or an advanced solution (all functions depend only on the variable \( v \)), called plane gravitational waves. In these cases from the system (32) - (36) one nontrivial equation for three metric functions is left. It gives an opportunity to choose, for example:
\[ \lambda(u) = 0, \]  
(39)

and to let the function \( \beta(u) \) which is the PGW amplitude be arbitrary. Then for the PGW background factor, i.e. for \( L(u) \), we get the equation (see Ref. [6])
\[ L_{uu} + \beta_u^2 L = 0. \]  
(40)

## 4 Static solution

Let us assume there is no gravitational wave
\[ \psi = \psi(x), \]  
(41)
, then we get \( \psi_u = -\psi_v \), and according to the Eq. (31) \( v_u = v_v = v = 0 \), i.e., plasma is at rest. For a static metric the first two equations of the Eqs. (32) (33) coincide, the independent Einstein’s equations take the form:

\[
L'' + L\beta'^2 - 2\lambda' L' = -\frac{\kappa}{2} L e^{2\lambda} \left( \varepsilon + p + \frac{H^2}{4\pi} \right) ;
\]

(42)

\[
L'' + L\beta'^2 + \lambda'' L = \kappa L e^{2\lambda} p ;
\]

(43)

\[
(L^2)'' = -4\kappa L^2 e^{2\lambda}(\varepsilon - p) ;
\]

(44)

\[
\beta'' + 2\beta' \frac{L'}{L} = \frac{\kappa}{16\pi} e^{2\lambda} H^2 .
\]

(45)

From the four equations Eqs. (42) - (45), two equations Eqs. (44) and Eqs. (45) are the definitions of \( \varepsilon \) and \( H^2 \). Thus, for the three metric functions \( \lambda \), \( \beta \) and \( L \) there are only two equations, it gives an opportunity to impose an additional condition on these functions, defining a class of solutions.

From the set (??), having carried out identical transformations, we get a consequence:

\[
L'' + L\beta'^2 + L\lambda'' = -\frac{1}{2L} [L^2 \nu' - \frac{1}{2}(L^2)']' ,
\]

(46)

introducing a new variable:

\[
\nu = \beta - \lambda .
\]

(47)

we write the consequence in the form

\[
L'' + L\beta'^2 + L\lambda'' = -\frac{1}{2L} [L^2 \nu' - \frac{1}{2}(L^2)']' .
\]

(48)

With the help of the Eq. (46), the relations (43) - (45) can be reduced to a more convenient form:

\[
[L^2 \nu' - \frac{1}{2}(L^2)']' = -2\kappa L^2 e^{2\lambda} p ;
\]

(49)

\[
(L^2)'' = -4\kappa L^2 e^{2\lambda}(\varepsilon - p) ;
\]

(50)

\[
(\beta' L^2)' = -\frac{\kappa}{16\pi} L^2 e^{2\lambda} H^2 .
\]

(51)

If we accept the barotropic equation

\[
p = k\varepsilon ; \quad (0 < k < 1) .
\]

(52)

as a local equation of the plasma state;

From the Eqs. (43) and (50) we shall obtain an algebraic consequence:

\[
[(1 - k) L^2 \nu' - \frac{1 + 3k}{2}(L^2)']' = 0 ,
\]

(53)
from which we get the first integral:

\[ L^2 \nu' = C_1 + \frac{1 + 3k}{2(1 - k)} (L^2)' , \quad (54) \]

where \( C_1 = \text{Const.} \). Due to Eq. (54), the following relations are valid, for example:

\[ \left[ L^2 \nu' - \frac{1}{2} (L^2)' \right]' = \frac{2k}{1 - k} (L^2)' ; \]

\[ \left[ L^2 \nu' + \frac{1}{2} (L^2)' \right]' = \frac{1 + k}{1 - k} (L^2)' . \]

With the help of them the definitions of the plasma energy density (50) and the magnetic field (51) can be written in a more symmetric form:

\[ \kappa \varepsilon = -\frac{2e^{-2\lambda}}{1 + 3k} \frac{(L^2 \nu')'}{L^2} ; \quad (55) \]

\[ \frac{\kappa H^2}{8\pi} = -\frac{2e^{-2\lambda}(L^2 \beta')'}{L^2} . \quad (56) \]

Due to the nonnegativity of the functions \( \varepsilon \) and \( H^2 \) the solution of the Einstein equations should satisfy the conditions:

\[ (L^2 \nu')' \leq 0 ; \quad (L^2 \beta')' \leq 0 . \quad (57) \]

The use of the integral (54) allows us to write the first of the conditions (57) in a more compact form:

\[ (L^2)'' \leq 0 . \quad (58) \]

Thus we obtain the following local value of the parameter \( \alpha^2(x) \) introduced in Ref. 11:

\[ \alpha^2 = \frac{H^2}{4\pi(\varepsilon + p)} = \frac{1 + 3k}{1 + k} \frac{(L^2 \beta')'}{(L^2 \nu')'} . \quad (59) \]

In the case of the barotropic equation of state the Einstein’s set of equations is reduced to two independent nonlinear differential equations for the three metric functions, one of which, Eq. (54), is of the first order and the second one, Eq. (54), is of the second order. It is possible to impose one additional condition on the three functions which does not contradict Eq. (57). Choosing in the integral (54):

\[ C_1 = 0 , \quad (60) \]

we get a private solution:

\[ \nu = \frac{1 + 3k}{1 - k} \ln L . \quad (61) \]
Being free in choosing the additional condition for the metric functions, we can assume, for example, in Eq. (59):

\[ \alpha^2 = \alpha_0^2 = \text{Const}. \]  

(62)

Assuming \( \alpha_0^2 \neq 0 \), from Eq. (59) we get one more first integral:

\[ L^2 \nu' = \frac{2(1 + 3k)}{\alpha_0^2(1 + k)} L^2 \beta' + C_2, \]  

(63)

where \( C_2 = \text{Const}. \)

Choosing here again:

\[ C_2 = 0, \]  

(64)

we have:

\[ \beta = \alpha_0^2 \frac{1 + k}{2(1 - k)} \ln L. \]  

(65)

Let us write the integrals (61) and (65) in the form:

\[ \nu = q_1 \ln L, \]

where

\[ q_1 = q_1(k) = \frac{1 + 3k}{1 - k}, \]

\[ \beta = q_2 \ln L, \]

where

\[ q_2 = q_2(k, \alpha_0) = \alpha_0^2 \frac{1 + k}{2(1 - k)}. \]

Substituting these integrals in the left Eq. (46) we obtain the equation closed in relation to the function \( L \):

\[ L''L = q_3 L'^2, \]  

(66)

where

\[ q_3 = q_3(k, \alpha_0) = \frac{1 - 3q_1 + 2q_2 - 2q_2^2}{1 + 2q_2 - q_1}. \]

Solving it we get:

\[ L = \left( \frac{1}{\mu_1(v)u(1 + q_3) + \mu_2(v)} \right)^{1 + q_3} \]  

(67)

where \( \mu_1(v), \mu_2(v) \) – are arbitrary functions. Let us define their form from the static condition of solution:

\[ \psi = \psi(x) = \psi(v - u). \]
whence it follows:

\[ \mu_1(v) = -\frac{A}{\sqrt{2(1 + q_3)}} = \text{Const}; \quad (68) \]

\[ \mu_2(v) = \frac{A}{\sqrt{2}} v + B; \quad (69) \]

Coming from the variable \( v \) to the variable \( x \), according to Eq.(2), expression (67) takes the form:

\[ L = (Ax + B)q_4 \quad (70) \]

where

\[ q_4 = q_4(k, \alpha_0) = \frac{1}{1 + q_3}, \quad (71) \]

Supposing \( B = 1 \) we obtain a condition on the function \( L \):

\[ L(0) = 1 \quad (72) \]

And the solution (70) takes the form:

\[ L = (Ax + 1)q_4 \quad (73) \]

Substituting the obtained solution into Eqs. (47), (56) and (65) we have

\[ \lambda = q_4(q_2 - q_1) \ln(Ax + 1) \quad (74) \]

\[ \varepsilon = \varepsilon_0(Ax + 1)^{q_5}, \quad (75) \]

where

\[ \varepsilon_0 = \frac{2A^2q_1q_4(1 - 2q_4)}{\kappa(1 + 3k)}, \]

\[ q_5 = q_5(k, \alpha_0) = -2q_4(q_2 - q_1) - 2. \]

\[ H^2 = H_0^2(Ax + 1)^{q_5} \quad (76) \]

where

\[ H_0^2 = \frac{16\pi A^2q_2q_4(1 - 2q_4)}{\kappa}. \]

\[ \nu = q_1q_4 \ln(Ax + 1) \quad (77) \]

\[ \beta = q_2q_4 \ln(Ax + 1) \quad (78) \]

Thus the metric (11) is

\[ ds^2 = (Ax^1 + 1)^2q_4(q_2 - q_1) \left[ (dx^1)^2 - (dx^1)^2 \right] - \\
- \left[ (Ax^1 + 1)^2q_4(1 + q_2)(dx^2)^2 + \\
+ (Ax^1 + 1)^2q_4(1 - q_2)(dx^3)^2 \right], \quad (79) \]
Constant $A$ can be easily derived from Eq. (76):

$$A^2 = \frac{H_0^2 \kappa}{16\pi q_2 q_4 (1 - 2q_4)}$$  \hfill (80)

For the metric (79) let us obtain the following nonzero components of the Riemann’s tensor:

$$R_{2323} = -q_4^2 (q_2^2 - 1) A^2 (Ax^1 + 1) q_5 + 4q_4,$$  \hfill (81)

$$R_{1434} = q_4^2 (q_2 - q_1) (q_2 - 1) A^2 (Ax^1 + 1) -2 - 2q_4(q_2 - 1),$$  \hfill (82)

$$R_{1313} = q_4 (q_2 - 1) A^2 (q_4(2q_2 - q_1 - 1) +$$

$$+1)(Ax^1 + 1) -2 q_4(q_2 - 1) - 2,$$  \hfill (83)

$$R_{1212} = q_4 (1 + q_2) A^2 (q_4 + q_4 q_1 -$$

$$-1)(Ax^1 + 1) q_4(q_2 + 1) - 2,$$  \hfill (84)

$$R_{2424} = -q_4^2 (q_2 - q_1) A^2 (1+$$

$$+q_2)(Ax^1 + 1) q_4(q_2 + 1) - 2,$$  \hfill (85)

$$R_{1414} = q_4 (q_2 - q_1) A^2 (Ax^1 + 1) q_5$$  \hfill (86)

In the limit $\alpha_0 \to \infty$, from the mentioned function it follows:

$$q_2 = \infty \quad q_3 = \infty \quad q_4 = 0 \quad q_5 = 0$$  \hfill (87)

And the metric (79) is reduced to:

$$ds^2 = (Ax^1 + 1)^{-2} \left[ (dx^4)^2 - (dx^1)^2 \right] -$$

$$- \left[ (Ax^1 + 1)^{-2}(dx^2)^2 + (Ax^1 + 1)^2(dx^3)^2 \right]$$  \hfill (88)

If $\alpha_0 \to 0$ we obtain

$$q_2 = 0$$

$$q_3 = \frac{5k + 1}{2k} \quad q_4 = \frac{2k}{7k + 1}$$

$$q_5 = -\frac{2(13k^2 - 4k - 1)}{7k^2 - 6k - 1}$$  \hfill (89)

Then the metric (79) will take the form

$$ds^2 = \left( Ax^1 + 1 \right)^{4k(1 + 3k)} \frac{4k(1 + 3k)}{(7k + 1)(k - 1)} \left[ (dx^4)^2 - (dx^1)^2 \right] -$$

$$- (Ax^1 + 1)^{4k} \frac{4k}{7k + 1} \left[ (dx^2)^2 + (dx^3)^2 \right],$$  \hfill (90)
Let us study some special cases. If $k = 0$ the metric (79) is transformed to

$$ds^2 = (dx^4)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

(91)

Supposing $k = 1/3$, we have

$$ds^2 = (Ax^1 + 1) \frac{6}{5} \left[(dx^4)^2 - (dx^1)^2\right] -$$

$$- (Ax^1 + 1) \frac{2}{5} \left[(dx^2)^2 + (dx^3)^2\right],$$

(92)

For $k = 1$ we get

$$ds^2 = (Ax^1 + 1)^\infty \left[(dx^4)^2 - (dx^1)^2\right] -$$

$$- (Ax^1 + 1)^1 \left[(dx^2)^2 + (dx^3)^2\right],$$

(93)

References

[1] Ignat’ev Yu.G. *Gravitation & Cosmology*, Vol.1, (1995), No 4, 287;

[2] Ignat’ev Yu.G., *Gravitation & Cosmology*, Vol.2., (1996), No 4, 213;

[3] Ignat’ev Yu.G. *Phys. Letters*, A, (1997);

[4] Norden .P. *Differential Giometry,* ( Moscow, 1948)[in Russian];

[5] Synge J.L., *Relativity: The General Theory*, Nort-Holland Publishing Company, Amsterdam, 1963;

[6] Misner C.W., Torn K.S., Wheeler J.A., *Gravitation*, W.H.Freeman and Company, San Francisco, 1973;