Space efficient quantum algorithms for mode, min-entropy and k-distinctness

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Abstract

We study the problem of determining if the mode of the output distribution of a quantum circuit given as a black-box is larger than a given threshold. We design a quantum algorithm for a promised version of this problem whose space complexity is logarithmic in the size of the domain of the distribution. Developing on top of that we further design an algorithm to estimate the largest probability among the outcomes of that circuit. This allows to revisit a few recently studied problems in the few-qubits scenario, namely k-distinctness and its gapped version, estimating the largest frequency in an array, and estimating the min-entropy of a distribution. In particular, our algorithm for k-distinctness on n-sized m-valued arrays requires $O(\log n + \log m)$ qubits compared to $O(poly(n))$ qubits required by all the previous algorithms, and its query complexity is optimal for $k = \Omega(n)$. We also study reductions between the above problems and derive better lower bounds for some of them. The time-complexities of our algorithms have a small overhead over their query complexities making them efficiently implementable on currently available quantum backends.

1 Introduction

A quantum circuit is always associated with a distribution, say $D$, over the observation outcomes. Given a threshold $\tau$, one may wonder if there is any outcome $x$ with probability at least $\tau$. This problem, which we denote $\text{HighDist}$, can be immediately related to some recently studied problems like that of $k$-Distinctness and Gapped k-Distinctness. The $k$-Distinctness problem [2, 3] wants to know if a list of $n$ integers have some value that appears at least $k$ times and is reducible to $\text{HighDist}$ with $\tau = \frac{k}{n}$ (assuming the ability to uniformly sample from the list). The $k = 2$ setting gives us the well studied ElementDistinctness problem [7, 2, 1] and the $k = 3$ setting has been separately studied as well [4, 9]. These problem can be generalized to the $\Delta$-Gapped $k$-Distinctness problem (introduced by Montanaro [12]) which comes with an additional promise that either some value appears at least $k$ times or every value appears at most $k - \Delta$ times for a promised gap of $\Delta$.

A variant of $\text{HighDist}$ is to compute the largest probability among all the outcomes — we call this the $P_{\text{max}}$ problem and reduce it to estimations of $F_{\infty}$ of an array and Min-Entropy of the output distribution of a quantum circuit. The $F_{\infty}$ problem [12, 8] wants to determine, or approximate, the number of times the most frequent element appears in an array, also known as the modal frequency. Montanaro related this problem to the Gapped $k$-Distinctness problem but did not provide any specific algorithm and left open its query complexity [12]. Finally, the Min-Entropy problem desires to estimate the min-entropy of a distribution $D$ and it can be approximated with an additive error provided we can estimate the largest probability in $D$ with multiplicative error. The currently known approach for this problem, in an array setting, involves reducing it to $k$-Distinctness [10].

So it appears that an efficient algorithm for $\Delta$-Gapped $k$-Distinctness can positively affect the query complexities of all the above problems. However, $\Delta$-Gapped $k$-Distinctness is not well-studied. The $k$-Distinctness problem, a special-case of $\Delta$-Gapped $k$-Distinctness with $\Delta = 1$, has been solved multiple times but the focus of all of them have been primarily to
reduce their query complexities. As a result their space requirement is significant (polynomial in the size of the list), and beyond the scope of the currently available quantum backends with a small number of qubits. One outcome of this work is a set of quantum algorithms for $k$-DISTINCTNESS, GAPPED $k$-DISTINCTNESS, $F_\infty$, and MIN-ENTROPY using $O(\log n)$ qubits. Even more interesting is the observation that the query complexity of our algorithms for $k$-DISTINCTNESS is optimal for $k = \Omega(n)$. We do this with the help of a quantum algorithm for HIGHDIST and then reducing the other problems to HIGHDIST.

When space is not a constraint, query complexity of a problem for an $n$-sized $m$-valued array is $O(n)$ since an algorithm can simply query the entire input first, store it and reuse it’s bits and pieces for further processing. However, when space is limited, querying the entire input at one go may not be feasible. This is also the scenario in a streaming setting, however, the focus of streaming algorithms is to reduce the number of passes over the input under restricted space. In contrast, our algorithms are allowed only constant many logarithmic-sized registers, and they try to optimize the number of queries and the time complexity. As a result we end up with super-linear queries for most of the problems but using $\tilde{O}(1)$ space.

The additive accuracy algorithms for HIGHDIST and $P_{\text{max}}$ are generalizations of the algorithms that we recently designed for estimating the non-linearity of a Boolean function with additive accuracy [5]. The relative accuracy algorithms that we designed can be now used to estimate the non-linearity of Boolean function with relative error.

All our algorithms work in the bounded-error setting and we shall often hide the $\log()$ factors in the complexities under $\tilde{O}()$. The time-complexities are asymptotically same as the query complexities with logarithmic overheads since the techniques rely on quantum amplitude estimating, amplification and simple classical steps.

An interesting outcome of this work is a unified study of the problems given above, each of which have received individual attention. For example, Li et al. [10] recently considered the min-entropy estimation problem of a multiset which is equivalent to computing its $F_\infty$, a problem studied just a few years ago by Montanaro [12] and Bun et al. [8]. The reductions between the problems allow us derive query complexity lower-bounds for some of these problems that either improve an existing one or is a new result for parameters that were not considered before. For the lower bounds we utilize a known lower bound given by Nayak et al. [13] for an array-based counting problem that we refer to as COUNTDECISION. We illustrate the reductions in Figure 1 (See Appendix C for the reduction algorithms.)

We now discuss our specific contributions with respect to each of the above problems along with the some existing results.
HighDist and $P_{\text{max}}$

We are not aware of any existing algorithm for the HighDist problem which is stated below.

**Problem 1 (HighDist).** We are given a $(\log(m) + a)$-qubit quantum oracle $O_D$ that generates a distribution $D : (p_x)_{x=1}^m$ upon measurement of the first $\log(m)$ qubits of

$$O_D \left| 0^{\log(m)+a} \right\rangle = \sum_{x \in \{0,1\}^{\log(m)}} \alpha_x |x\rangle |\psi_x\rangle$$

in the standard basis. We are also given a threshold $\tau \in (0,1)$ and the decision task is to identify whether there exists any $x$ such that $p_x \geq \tau$.

We generalize one of our earlier algorithms [5] to solve HighDist approximately.

**Lemma 1 (Additive-error algorithm for HighDist).** Given an oracle $O_D$ for the HighDist problem, $m$ — the domain-length of the distribution it generates and a threshold $\tau$, along with parameters $0 < \epsilon < \tau$ for additive accuracy and $\delta$ for error, there is a quantum algorithm that makes $O\left( \frac{1}{\epsilon \tau^3} \log \frac{1}{\delta} \right)$ queries to $O_D$ such that the following hold with probability at least $1 - \delta$.
1. If it returns TRUE then there exists some $x \in [m]$ such that $p_x \geq \tau - \epsilon$.
2. If it returns FALSE then for all $x \in [m]$, $p_x < \tau$.

The algorithm uses $O(\log(m) + \log(\frac{1}{\epsilon}) + a)$ qubits.

In other words, the algorithm returns TRUE if $p_x \geq \tau$ for some $x$ and returns FALSE if $p_x < \tau - \epsilon$ for all $x$; it’s behaviour is non-deterministic for the intermediate cases. It is reasonable to require that $\epsilon << \tau/2$, and in that case the query complexity can be bounded by $O\left( \frac{1}{\epsilon \tau^3} \right)$. The above algorithm can be employed to solve HighDist with a relative accuracy.

**Lemma 2 (Relative-error algorithm for HighDist).** There exists an algorithm to solve HighDist in the following sense, in which $\epsilon_r \in (0,1)$ denotes a desired relative accuracy, making $O\left( \frac{1}{\epsilon_r \tau^3} \log \frac{1}{\epsilon_r \tau^2} \right)$ queries to $O_D$ and using $O(\log(m) + \log(\frac{1}{\epsilon_r \tau^2}) + a)$ qubits.
1. If the algorithm returns TRUE then there exists some $x \in [m]$ such that $p_x \geq (1 - \epsilon_r)\tau$.
2. If the algorithm returns FALSE then for all $x \in [m]$, $p_x < \tau$.

We are able to prove a lower bound on HighDist by reducing it from CountDecision.

**Lemma 3.** For any constant $\tau \in (0,1)$, the query complexity of an additive error algorithm for HighDist satisfies $\Omega\left( \frac{1}{\tau^2} \right)$ where $\epsilon$ denotes the desired additive error.

The $P_{\text{max}}$ problem is a natural extension of HighDist.

**Problem 2 ($P_{\text{max}}$).** Given an oracle for the HighDist problem, compute $p_{\text{max}} = \max_{i \in [m]} p_i$.

We show how to approximate $p_{\text{max}}$ with absolute and relative error.

**Lemma 4 (Approximating $p_{\text{max}}$ with additive error).** Given an oracle as required for the HighDist problem, additive accuracy $\epsilon \in (0,1)$ and error $\delta$, there is a quantum algorithm that makes $O\left( \frac{1}{\epsilon \tau^3} \log \frac{1}{\delta} \log \frac{1}{\epsilon} \right)$ queries to the oracle and outputs an estimate $\tilde{p}_{\text{max}}$ such that $|p_{\text{max}} - \tilde{p}_{\text{max}}| \leq \epsilon$ with probability $1 - \delta$. The algorithm uses $O(\log(m) + \log(\frac{1}{\epsilon}) + a)$ qubits.

The min-entropy of a distribution $D = (p_i)_{i=1}^m$ is defined as $\max_{i \in [m]} \log(1/p_i)$ and the MIN-ENTROPY problem is to estimate this value; clearly, estimating it with an additive accuracy is equivalent to estimating $\max_{i=1}^m p_i$ with relative accuracy.
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Table 1 Results for the k-Distinctness problem

| k-Distinctness | Prior upper bound | Our upper bound |
|----------------|-------------------|-----------------|
| \( k = \omega(1) \) | \( O(r + (\frac{k}{2})^{k/2}/\sqrt{r}) \) queries, \( O(r(\log(m) + \log(n))) \) space for \( r \geq k \) | \( \tilde{O}(n^{3/2}/\sqrt{k}) \) queries, \( O(\log(m) + \log(n)) \) space |
| \( k = O(1) \) and \( k \geq 4 \) | \( O(n^{k/2}) \) queries, \( O(r(\log(m) + \log(n))) \) space for \( r \geq k \) | \( \tilde{O}(n^{3/2}/\sqrt{k}) \) queries, \( O(\log(m) + \log(n)) \) space |

**Lemma 5** (Approximating \( p_{\max} \) with relative error). Given an oracle as required for the HighDist problem, relative accuracy \( \epsilon \in (0, 1) \) and error \( \delta \), there is a quantum algorithm that makes \( \tilde{O}(\frac{n^{3/2}}{\epsilon}) \) queries to the oracle and outputs an estimate \( \hat{p}_{\max} \) such that with probability \( 1 - \delta \), it holds that \((1 - \epsilon)^2 \hat{p}_{\max} \leq p_{\max} \leq \hat{p}_{\max} \). The algorithm uses \( \log(\frac{n}{\epsilon}) + 4 \) qubits.

Li et al. used the polynomial method to show that any quantum algorithm using an oracle \( O_S \) to access an \( m \)-valued \( n \)-sized array requires \( \Omega(\frac{n}{\epsilon^2 m}) \) queries to estimate \( p_{\max} \) with additive error \( \epsilon \) \[1\] Proposition 9.1 \[4\]. Instead, we reduce from the F\(_{\infty}\) problem and obtain a tighter lower bound of \( \Omega(\frac{1}{\epsilon}) \) queries.

**Lemma 6.** The query complexity to estimate \( p_{\max} \) with additive accuracy \( \epsilon \) is \( \Omega(\frac{1}{\epsilon}) \).

Upper bounds for \( k \)-Distinctness

Our results on the \( k \)-Distinctness problem are summarised in Table 1.

**Problem 3** \( (k \)-Distinctness). Given an oracle to an \( n \)-sized \( m \)-valued array \( A \), decide if there exists \( k \) distinct indices such that the values of \( A \) at those indices are identical.

Here by an \( m \)-valued array we mean an array whose entries are from \( \{0, \ldots, m - 1\} \). The \( k = 2 \) version is the element distinctness problem which was first solved by Buhrman et al. \[7\]; their algorithm makes \( O(n^{3/4} \log(n)) \) queries (with roughly the same time complexity), and stores \( O(\sqrt{m} \log(n)) \) indices using \( O(\sqrt{m} \log m) \) qubits. A better algorithm was later proposed by Ambainis \[2\] using a quantum walk on a Johnson graph whose nodes represent \( r \)-sized subsets of \( [n] \), for some suitable parameter \( r \geq k \). He used the same technique to design an algorithm for \( k \)-Distinctness as well that uses \( \tilde{O}(r) \) qubits and \( O(r + (n/r)^{k/2}/\sqrt{r}) \) queries (with roughly the same time complexity). Later Belovs designed a learning-graph for the \( k \)-Distinctness problem, but only for constant \( k \), and obtained a tighter bound of \( O(n^{\frac{3}{2} - \frac{2k}{2k+1}}) \). It is not clear whether the bound holds for non-constant \( k \), and it is often tricky to construct efficiently implementable algorithms base on the dual-adversary solutions obtained from the learning graphs.

Thus it appears that even though efficient algorithms may exist for small values of \( k \), the situation is not very pleasant for large \( k \), especially \( k = \Omega(n) \) — the learning graph idea may not work (even if the corresponding algorithm could be implemented in a time-efficient manner) and the quantum walk algorithm uses \( \Omega(k) \) space. Our algorithm addresses this

\[1\] The proof of the proposition reveals that \( \Omega(\sqrt{m}) \) query is a lower bound to distinguish whether \( p_{\max} = \frac{1}{m} \) or \( p_{\max} \geq \frac{1}{m} + \frac{1}{m} \). Using \( \epsilon = \epsilon'/m \) gives us the bound to estimate \( p_{\max} \) with accuracy \( \epsilon \).
Table 2 Results for the $\Delta$-Gapped $k$-Distinctness problem. ($\Delta$ denotes additive error.)

| Prior upper bound | Our upper bound | Prior lower bound | Our lower bound |
|-------------------|-----------------|-------------------|-----------------|
| None              | $\tilde{O}(\frac{n^{3/2}}{\Delta \sqrt{k}})$ queries, $\tilde{O}(1)$ space | None              | $\Omega(\frac{n}{\Delta})$ |

The Gapped $k$-Distinctness problem

The Gapped $k$-Distinctness problem was introduced by Montanaro [12] Sec 2.3 as a generalization of the $k$-Distinctness problem to solve the $F_\infty$ problem; we slightly change the problem to consider an additive gap to suit the results of this paper.

Problem 4 ($\Delta$-Gapped $k$-Distinctness). This is the same as the $k$-Distinctness problem along with a promise that either there exists a set of $k$ distinct indices with identical values or no value appears more than $k - \Delta$ times.

Montanaro observed that this problem can be reduced to $F_\infty$ estimation and vice-versa with a $\log(n)$ overhead for binary search; however, he left open an algorithm or the query complexity of this problem. We are able to design a constant space algorithm by reducing it to our HighDist problem. Our results are summarised in Table 2.

Lemma 8. There is a quantum algorithm to solve the $\Delta$-Gapped $k$-Distinctness problem that makes $\tilde{O}(\frac{n^{3/2}}{\Delta \sqrt{k}})$ queries and uses $O(\log(m) + \log(n))$ qubits.

A reduction from CountDecision can be used to derive a lower-bound of $\Omega(n/\Delta)$ on the query complexity of Gapped $k$-Distinctness.

Lemma 9. Any quantum algorithm that solves the $\Delta$-Gapped $k$-Distinctness problem with $k = \frac{n}{t}$ for any constant $t$ uses $\Omega(n/\Delta)$ queries to the given array.
Table 3: Algorithms for $F_\infty$ ($\epsilon < n$ denotes additive error)

| Approach | Query and space complexity | Nature of error |
|----------|---------------------------|-----------------|
| binary search with $k$-distinctness [12] | $O(n \log(n))$ queries, $O(n)$ space | exact |
| $k$-distinctness with $k = \lceil \frac{16 \log(n)}{\epsilon^2} \rceil$ [10] | $O(n)$ queries, $O(n)$ space | $\epsilon$ additive error |
| quantum maximum finding over frequency table [12, Sec 3.3] | $O(n^{3/2})$ queries, $O(\log(m) + \log^2(n))$ space | exact |
| reducing to $p_{\text{max}}$ (binary search with HIGHDIST) [this] | $O((n/\epsilon)^{3/2} \log(n/\epsilon))$ queries, $O(\log(m) + \log(\frac{n}{\epsilon}))$ | $\epsilon$ additive error, set $\epsilon = 0.99$ for exact |

Table 4: Lower bound for $F_\infty$

| Approach | Query complexity | Error |
|----------|-----------------|-------|
| reducing from element-distinctness and $k$-distinctness [12] | $\Omega(n^{3/4}, 1/\epsilon)$ | relative error $\epsilon \in (0, 1)$ |
| reduction from $k$-distinctness [8] | $\Omega(n^{1/2 - \frac{1}{2^k}})$ | relative error $\epsilon \leq \frac{1}{2}$ |
| reduction from $k$-distinctness [11] | $\Omega(n^{1/2 - \frac{1}{2^k}})$ | relative error $\epsilon \leq \frac{1}{2}$ |
| polynomial method [10] | $\Omega(\frac{n}{\epsilon^{2/3}})$ | additive error $\epsilon \in [n]$ |
| reduction from COUNTDECISION [this] | $\Omega(\frac{n}{\epsilon^2})$ | additive error $\epsilon \in [n]$ |

Upper and lower bounds for $F_\infty$

Problem 5 ($F_\infty$). Given an oracle to query an $n$-sized array $A$ with values in $\{1, \ldots, m\}$, compute the frequency of the most frequent element, also known as the modal frequency.

Li et al. [10] studied this problem in the context of min-entropy of an array. They reduced the problem of MIN-ENTROPY estimation (of an $m$-valued array with additive error $\epsilon \in (0, 1)$) to that of $k$-DISTINCTNESS with $k = \lceil \frac{16 \log(m)}{\epsilon^2} \rceil$. Min-entropy, or its estimate, could be used to calculate $F_\infty$ but they did not proceed further and made the remark that “Existing quantum algorithms for the $k$-distinctness problem . . . do not behave well for super-constant $k$s.” Indeed, it is possible to run the quantum-walk based algorithm for $k$-DISTINCTNESS [2] and thereby solve $F_\infty$ estimation; this turns out to be not very effective with $O(n)$ query complexity and $O(n)$ space complexity. (See Appendix [8] for a rough analysis.)

Our technique reduces the $F_\infty$ problem to that of HIGHDIST and yields a $\tilde{O}(1)$-space algorithm to estimate the modal frequency with additive error. Montanaro proposed two methods to accurately compute the modal frequency, one of which closely matches the complexities of our proposed algorithm but our approach has a lower query complexity when $\epsilon = poly(n)$. The results are summarised in Table [3]

Lemma 10. There are quantum algorithms to estimate $F_\infty$ with additive accuracy $\epsilon$ using $\tilde{O}\left(\left(\frac{n}{\epsilon}\right)^{3/2} \log \frac{n}{\epsilon}\right)$ queries and $O(\log(m) + \log(\frac{n}{\epsilon}))$ qubits and to estimate with relative accuracy $\epsilon$ using $\tilde{O}\left(\frac{n}{\epsilon^{2/3}}\right)$ queries and $O(\log(\frac{n}{\epsilon^2}) + \log(n))$ qubits.

There has been few previous attempts at providing lower bounds on the query complexity of $F_\infty$, but for the case where relative accuracy is used rather than additive accuracy. We are aware of only prior attempt for the additive accuracy case in which the polynomial method was used by Li et al. [10]. We are able to obtain a better bound by reducing from the COUNTDECISION problem. All these lower bounds are summarised in Table [4].
Lemma 11. The $F_\infty$ problem with additive accuracy $\epsilon$ has query complexity $\Omega(\frac{n}{\epsilon})$.

2 Background: Additive accuracy algorithms for HighDist and $P_{\text{max}}$

For both of these problems we are given a quantum black-box $O_D$ such that

$$O_D \left| \psi^{\log(m)} \right\rangle \left| \psi^p \right\rangle = \sum_{x=0}^{m-1} \alpha_x \left| x \right\rangle \left| \xi_x \right\rangle$$

in which $\{ \left| \xi_x \right\rangle : x \in [m] \}$ are normalized states. Let $p_x = |\alpha_x|^2$ denote the probability of observing the first $\log(n)$ qubits in the standard-basis $\left| x \right\rangle$. The objective of HighDist is to determine whether there exists any $x$ such that $p_x \geq \tau$ for any specified threshold $\tau \in (0, 1)$ and the task of $P_{\text{max}}$ is to compute $\max_x p_x$.

In a recent work we designed algorithms for specific versions of these problems in the context of Boolean functions and for additive error [5]. In this work we have generalized them to be usable for arbitrary problems and also with relative error. We have inducted the complete generalized algorithms, with a few improvements as well, and their detailed analysis in Appendix A.

Imagine two scenarios, one in which $p_{\text{max}} > \tau$ and another in which $p_{\text{max}} < \tau$ but $p_{\text{max}}$ being very close to $\tau$ in both of the scenarios. It is clear that differentiating between these two scenarios would require significant effort. We therefore design algorithms for promise versions of these problems.

2.1 Choice of oracle

Bravyi et al. [6] worked on designing quantum algorithms to analyse probability distributions induced by multisets. They considered an oracle, say $O_S$, to query an $n$-sized multiset, say $S$, in which an element can take one of $m$ values. Hence, the probabilities in the distribution of elements in those multisets are always multiples of $\frac{1}{n}$. They further proved that the query complexity of an algorithm in this oracle model is same as the sample complexity when sampled from the said distribution in a classical scenario. Li and Wu [10] too used the same type of oracles for estimating entropies of a multiset.

We consider a general oracle in which the probabilities can be any real number, and are encoded in the amplitudes of the superposition generated by an oracle. We show below how to implement an oracle of our type for any distribution $D$, denoted $O_D$, using $O_S$.

$$\left| \psi^{\log(n)} \right\rangle \left| \psi^{\log(m)} \right\rangle \xrightarrow{H^{\log(n)}} \frac{1}{\sqrt{n}} \sum_{i \in [n]} \left| i \right\rangle \left| \psi^{\log(m)} \right\rangle \xrightarrow{O_S} \frac{1}{\sqrt{m}} \sum_{i \in [n]} \left| i \right\rangle \left| S_i \right\rangle = \sum_{j \in [m]} \alpha_j \left| \xi_j \right\rangle \left| j \right\rangle \quad (1)$$

It should be noted that one call to $O_D$ invokes $O_S$ only once. Here the $\left| \xi_j \right\rangle$ states are normalized, and the probability of observing the second register is $|\alpha_j|^2$. Hence, ignoring the first register gives us the desired output of $O_D \left| \psi^{\log(m)} \right\rangle$ in the second register.

We use $O_D$ for HighDist and $P_{\text{max}}$, and $O_S$ for the other array-based problems, namely, $F_\infty$ and variants of element distinctness.

2.2 Algorithm for Promise–HighDist (additive error)

We first study a promise version of HighDist with additive error, which we refer to as Promise–HighDist. The task here is to identify whether there exists any $x$ such that $p_x \geq \tau$ or if for all $x$, $p_x < \tau - \epsilon$ under the promise that only one of the cases is true.
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For this task we generalize QBoundFMax from our earlier work on estimating non-linearity [5], Algorithm 3, except that we use $\epsilon/2$ instead of $\epsilon$. The repurposing of that algorithm follows from two observations. First, QBoundFMax identified whether there exists any basis state whose probability, upon observing the output of a Deutsch-Jozsa circuit, is larger than a threshold in a promised setting. However, no specific property of that circuit was being used and the Deutsch-Jozsa circuit could be replaced with any other circuit. Secondly, amplitude estimation can be used to estimate $|\alpha_x|^2$ (with bounded error) in $\sum_x \alpha_x |x\rangle \langle x|$ for any $x \in [n]$ by designing a sub-circuit on only the first $\log(n)$ qubits to identify “good” states (this sub-circuit was referred to as EQ in QBoundFMax). To prove Lemma 1 we refer to the complexity analysis and circuit diagram of QBoundFMax (see [5] Lemma 4.2). For the number of qubits used and other details refer to Appendix A.1.

2.3 Algorithm for $P_{\text{max}}$ (additive error)

We can estimate $p_{\text{max}}$ by performing binary search over all integer multiples of $1/2^k$ to find the largest $t$ such that $p_{\text{max}} \geq t/2^k$. Since $\sum_x p_x = 1$, $p_{\text{max}} \geq 1/n$, thus there is no need to search among thresholds less than $1/n$. An algorithm for Promise-HighDist, with additive accuracy set to some $\epsilon$, is used to decide whether to search in the right half or the left half. It suffices to choose $k$ and $\epsilon$ such that $1/2^k \leq \epsilon/2$ and $\epsilon \leq \epsilon/2$ and repeatedly call the Promise-HighDist algorithm with accuracy $\epsilon$. Suppose $1/2^k$ is the threshold passed to the Promise-HighDist algorithm at some point. Then, if the algorithm returns TRUE then $p_{\text{max}} \geq 1/2^k - \epsilon$, and so we continue to search towards the right of the current threshold; on the other hand if the algorithm returns FALSE then $p_{\text{max}} < 1/2^k$, so we search towards its left. At the end some $t$ is obtained such that $p_{\text{max}} \in [t/2^k - \epsilon, t/2^k + \epsilon)$, an interval of length at most $\epsilon$. This is the idea behind the IntervalSearch algorithm from our earlier work on non-linearity estimation [5] Algorithm 1. Once such a $t$ is obtained, $t/2^k$ can be output as an estimate of $p_{\text{max}}$ which is at most $\epsilon$ away from the actual value. Lemma 2 follows from Lemma 1 and the observation that $k$ binary searches have to be performed. We have included the complete algorithm with a detailed analysis in Appendix A.2.

3 Relative accuracy algorithms for HighDist and $P_{\text{max}}$

The algorithm for Promise-HighDist with additive error can be immediately used to solve Promise-HighDist with relative error $\epsilon_r$ by setting $\epsilon = \epsilon_r \tau$. This proves Lemma 3.

The algorithm for $P_{\text{max}}$ with relative accuracy, denoted $\epsilon_r$, follows a similar idea as that of its additive accuracy version (see Section 2.3), except that it searches among the thresholds $1, (1 - \epsilon_r), (1 - \epsilon_r)^2, \ldots, (1 - \epsilon_r)^{k-1}$ in which $k$ is chosen to be the smallest integer for which $(1 - \epsilon_r)^{k-1} \leq 1/m$. Further, it calls the above algorithm for Promise-HighDist with relative error $\epsilon_r$. At the end of the binary search among the $k$ thresholds, we obtain some $t$ such that $p_{\text{max}} \in [(1 - \epsilon_r)(1 - \epsilon_r)^{t+1}, (1 - \epsilon_r)^{t}]$. Clearly, if we output $(1 - \epsilon_r)^t$ as the estimate $\tilde{p}_{\text{max}}$, then $p_{\text{max}} \leq \tilde{p}_{\text{max}}$ and $\max p_{\text{max}} \geq (1 - \epsilon_r)^2 \tilde{p}_{\text{max}}$ as required.

The algorithm queries $O_D$ only during the calls to the Promise-HighDist algorithm. At most $\log(k)$ such calls are made, and the query complexity of each call is $\tilde{O}(\frac{1}{\epsilon_r \tau^{3/2}})$ in which $\tau$ denotes the threshold passed to the algorithm. Observe that $\tau \geq (1 - \epsilon_r)^{k-1}$, and with this we can bound the total query complexity by $\tilde{O}(\log(k) \frac{1}{\epsilon_r} \left(\frac{1}{(1 - \epsilon_r)^{3/2}}\right)^{3/2})$. By our choice of $k$, $(1 - \epsilon_r)^{k-2} > \frac{1}{m}$, so $(1 - \epsilon_r)^{k-1} > \frac{1}{m^2}$ and thus we get the total query complexity as $\tilde{O}(\frac{m^{3/2}}{\epsilon_r})$ (we also use the observation that $1 - \epsilon_r$ can be lower bounded by a constant for
all practical purposes). This proves Lemma 3. The number of qubits for both the above algorithms are that required by the algorithm for Promise-HIGHDIST with additive error.

4 Application of HighDist for \( k \)-Distinctness

Montanaro hinted at a possible algorithm for the promise problem \( \Delta \)-GAPPED \( k \)-DISTINCTNESS by reducing it to the \( F_\infty \) problem.\(^2\) The idea is to estimate the modal frequency of an array \( A \) up to an additive accuracy \( \Delta/2 \) and then use this estimate to decide if there is some element of \( A \) with frequency at least \( k \). The query complexity would be same as that of \( F_\infty \).

Here we show a reduction from \( \Delta \)-GAPPED \( k \)-DISTINCTNESS to a promise version of HIGHDIST which allows us to shave off a \( \log \left( \frac{4}{\Delta} \right) \) factor from the above complexity. For \( \Delta \)-GAPPED \( k \)-DISTINCTNESS we are given an oracle \( O_S \) to access the elements of \( A \). First use \( O_S \) to implement an oracle \( O_D \) for the distribution \( D = (p_i)_{i=1}^m \) induced by the frequencies of the values in \( A \). Then call the algorithm for Promise-HIGHDIST with threshold \( k/n \) and additive accuracy \( \Delta/n \). Now observe that if there exists some \( i \in \{1, \ldots, m\} \) whose frequency is at least \( k \), then \( p_i \geq \frac{k}{2n} \) and the Promise-HIGHDIST algorithm will return TRUE. On the other hand, if the frequency of every element is less than \( k - \Delta \), then for all \( i \), \( p_i < \frac{k}{2n} - \frac{\Delta}{n} \); the Promise-HIGHDIST algorithm will return FALSE. The query complexity of this algorithm is \( \tilde{O} \left( \frac{1}{\sqrt{n}} \frac{1}{\sqrt{\epsilon/n}} \right) \) which proves Lemma 8. The space complexity is the same as that of solving Promise-HIGHDIST problem.

As for Lemma 7 it is easy to see that \( k \)-distinctness is equivalent to \( \Delta \)-GAPPED \( k \)-DISTINCTNESS with \( \Delta = 1 \) and so the above algorithm can be used.

5 Application of \( P_{\text{max}} \) for \( F_\infty \)

To compute the modal frequency of an array \( A \), given an oracle \( O_S \) to it, we first use \( O_S \) to implement \( O_D \) whose amplitudes contain the distribution \( D_A \) induced by the values of \( A \): \( D_A = (p_i)_{i=1}^m \) where \( p_i = \frac{1}{n} |\{i \in [n]: A[i] = x\}| \). Then we can use the algorithms for \( P_{\text{max}} \) for \( O_D \). The estimate obtained from that algorithm has to rescaled by multiplying it by \( n \) to obtain an estimate of the largest frequency of \( A \). If we call the additive accuracy algorithm for \( P_{\text{max}} \) with accuracy set to \( \epsilon/n \), then we get an estimate of \( F_\infty \) with additive error \( \epsilon \). No such scaling of the error is required if we call the relative accuracy algorithm for \( P_{\text{max}} \) to obtain an estimate of \( F_\infty \) with relative error. Thus Lemma 10 is proved.

6 Lower bounds

For our lower bounds we reduce from the COUNTDECISION problem that takes as input a binary string of length \( n \) and decides if the number of ones in \( X \), denoted \( |X| \), is \( l_1 \) or \( l_2 > l_1 \), given a promise that one of the two cases is true. We use the following lower bound on the query complexity of this problem that was proved by Nayak and Wu [13].

\[ \Delta \text{ COUNTDECISION lower bound [13]. Let } l_1, l_2 \in [n] \text{ be two integers such that } l_2 > l_1, \text{ and } X \text{ be an } n \text{-bit binary string. Further, let } l_2 - l_1 = 2\Delta \text{ for some integer } \Delta. \text{ Then any quantum algorithm takes } \Omega \left( \frac{\sqrt{n}}{\Delta} + \frac{1}{\sqrt{l_2 - \Delta}(n - (l_2 - \Delta))} \right) \text{ queries to solve the COUNTDECISION problem.} \]

---

\(^2\) His reduction was to a relative-gap version of GAPPED \( k \)-DISTINCTNESS; however, the same idea works for the additive-gap version that we consider in this paper.
Proof of Lemma 9 (lower bound for Δ-Gapped k-Distinctness). Consider the following reduction from CountDecision to Δ-Gapped k-Distinctness. Let \( X \in \{0, 1\}^n \) be a given input string with the promise that either \(|X| = \frac{n}{7} - 2d\) or \(|X| = \frac{n}{7}\) for some \( t = O(1) \) and some \( 1 \leq d \leq \frac{n}{2} \). Theorem 12 states \( \Omega(n/2) \) queries are required to decide the value of |X|.

Now, construct an array \( A^X \) of size \( n \) such that \( A^X[i] = i \) if \( X_i = 0 \) and \( A^X[i] = n + 1 \) if \( X_i = 1 \). For any \( X \) such that \(|X| = \frac{n}{7} - 2d\), there are \( \frac{n}{7} - 2d \) many distinct indices of \( A \) that have the same element \( n + 1 \). Similarly, for any \( X \) such that \(|X| = \frac{n}{7}\), there are \( \frac{n}{7} \) many distinct indices that have the same element. Therefore, any algorithm that decides the Δ-Gapped k-Distinctness problem with \( k = \frac{n}{7} \) and \( \Delta = 2d \) also decides if \(|X| = \frac{n}{7} - 2d\) or \(|X| = \frac{n}{7}\), and using Theorem 12 we get that the quantum query complexity of the Δ-Gapped k-Distinctness problem is \( \Omega(n/d) = \Omega(n/2) \).

Proof of Lemma 11 (lower bound for \( F_{\infty} \)). Consider the array \( A^X \) that was constructed in proof of Lemma 9. To decide whether \( A^X \) has at least \( \frac{n}{7} \) identical elements or no element appears more than \( \frac{n}{7} - 2\Delta \) times, it suffices to run any algorithm for \( F_{\infty} \) on \( A^X \) with additive accuracy \( \Delta/2 \). In the former case, the algorithm will return an estimate that is at least \( \frac{n}{7} - \Delta/2 \), and in the latter case, the algorithm will return an estimate that is at most \( \frac{n}{7} - 3\Delta \). Clearly, both the cases can be easily distinguished and so we obtain the desired lower bound for \( F_{\infty} \) on \( n \)-sized arrays with \( \Delta/2 \) additive accuracy.

Lemma 6 follows from Lemma 11 since the \( F_{\infty} \) problem can be trivially reduced to the \( P_{\max} \) problem by scaling the accuracy parameter.

Proof of Lemma 3 (lower bound for HighDist). We reduce an instance of CountDecision on a \( n \)-bit string \( X \) with \( l_1 = \frac{n}{7} \) and \( l_2 = \frac{n}{7} + \varepsilon n \) to Promise-HighDist. Observe that an oracle to \( X \) can be used to implement an oracle \( O_D \) to the distribution \( D \) induced by the frequencies of \( 0 \) and \( 1 \) in \( X \), and each call to \( O_D \) corresponds to one call to \( X \) (see Section 2.1). By Theorem 12, the query complexity to decide the above CountDecision instance is \( \Omega(n/2) \). If \(|X| = \frac{n}{7}\) then \( \Pr_D[0] = \Pr_D[1] = \frac{1}{2} \), and if \(|X| = \frac{n}{7} + \varepsilon n\), then \( \Pr_D[1] = \frac{1}{2} + \varepsilon \).

Thus, the output of a Promise-HighDist algorithm with \( \tau = \frac{1}{2} + \varepsilon \) and additive accuracy \( \varepsilon = \varepsilon \) can be used to decide our CountDecision instance. This proves a lower bound of \( \Omega(n/7) \) for Promise-HighDist, and also of its generalized version HighDist.

7 Conclusions

We studied several problems in the low-space complexity regime but reducing them to one another and to two very generic problems which we call as HighDist and \( P_{\max} \). Our algorithm for HighDist can roughly determine if the probability of any outcome of a given quantum circuit is larger a specified threshold and that for \( P_{\max} \) uses the earlier algorithm to approximate the largest probability among all its outcomes. These algorithms are then used to solve the k-distinctness of an \( n \)-sized array with \( O(n^{3/2}) \) query complexity, and estimate the \( F_{\infty} \) of an \( n \)-sized array using \( O((n/\varepsilon)^{3/2} \log(n/\varepsilon)) \) queries, requiring only \( O(\log n) \) qubits in both scenarios. Our algorithms are designed using amplitude amplification and estimation and their time-complexities have a small overhead over their query complexities.

It would be interesting to study the time-space tradeoffs of the above problems and design quantum algorithms in the streaming setting in which, apart from the number of qubits, the objective is to also minimize the number of passes over an input.
Algorithm for HighDist and $P_{\text{max}}$

In this section we present the algorithm for HighDist and for $P_{\text{max}}$ with additive accuracy that generalize $\text{BundFM}ax$ and $\text{IntervalSearch}$ from an earlier work [5]. Minor changes have been introduced during the generalization.

A.1 Algorithm for HighDist problem with additive accuracy

Our space-efficient algorithm for Promise-HighDist requires a few subroutines which we borrow from our earlier work on estimating non-linearity [5].
EQ$_m$: Given two computational basis states $|x\rangle$ and $|y\rangle$ each of $k$ qubits, EQ$_m$ checks if the $m$-sized prefix of $x$ and that of $y$ are equal. Mathematically, $\text{EQ}_m(x) \mid y\rangle = (-1)^c \mid x\rangle \mid y\rangle$ where $c = 1$ if $x_i = y_i$ for all $i \in [m]$, and $c = 0$ otherwise.

HD$_q$: When the target qubit is $|0^q\rangle$, and with a $q$-bit string $y$ in the control register, HD computes the absolute difference of $y_{int}$ from $2^q-1$ and outputs it as a string where $y_{int}$ is the integer corresponding to the string $y$. It can be represented as $\text{HD}_q \mid y\rangle \mid b\rangle = |b \oplus y\rangle |y\rangle$ where $y, b \in \{0, 1\}^q$ and $\tilde{y}$ is the bit string corresponding to the integer $2^q-1 - y_{int}$.

Even though the operator HD requires two registers, the second register will always be in the state $|0^q\rangle$ and shall be reused by uncomputing (using HD$^\dagger$) after the CMP gate. For all practical purposes, this operator can be treated as the mapping $|y\rangle \mapsto |\tilde{y}\rangle$.

**Algorithm 1 Algorithm HighDist_Algo**

**Require:** Oracle $O_D$, oracle size $r = \log(m) + a$, size of the distribution $m$, threshold $\tau$, accuracy $\epsilon$ and error $\delta$.

**1:** Set $\tau' = \tau - \frac{\epsilon}{2}$ and $q = \lceil \log\left(\frac{1}{\delta}\right) \rceil + 4$.

**2:** Set $\tau_1 = \left\lfloor \frac{2^q}{\tau'} \sin^{-1}\left(\frac{\tau}{\sqrt{\tau^2}}\right) \right\rfloor$.

**3:** Initialize 5 registers $R_1R_2R_3R_4R_5$ as $|0^r\rangle |0^q\rangle |0\rangle |\tau_1\rangle |0\rangle$. Fourth register is on $q$ qubits.

**Stage 1:** Apply $O_D$ on $R_1$.

**Stage 2:** Apply quantum amplitude estimation (AmpEst) on $O_D$ with $R_2$ as the input register, $R_3$ as the precision register and $R_1$ is used to determine the “good state”. AmpEst is called with error at most $\delta/2$ and additive accuracy $\frac{1}{2\tau}$.

**Stage 3:** Use HD$_q$ on $R_3$ and $R_4$ individually.

**Stage 4:** Use CMP on $R_3$ and $R_4 = |\tau_1\rangle$ as input registers and $R_5$ as output register.

**Amplification stage:** Apply Amplitude Amplification (AA) $O\left(\frac{1}{\tau}\right)$ times on $R_5$ with error at most $\delta/2$ and measure $R_5$ as $\text{out}$.

**if** $\text{out} = |1\rangle$ **then**

**return** TRUE

**else**

**return** FALSE

**end if**

Using these operators as subroutines, we now present HighDist_Algo (described in Algorithm [1]) that solves the Promise-HighDIST problem, with additive accuracy. The operation of HighDist_Algo can be explained in stages. For convenience, let us call the set $G = \{z : z \in [m], p_z \geq \tau\}$ as the ‘good’ set and its elements as the ‘good’ states. In the first stage, we initialize the registers $R_1R_2R_3R_4R_5$ in the state $|0^r\rangle |0^q\rangle |0^q\rangle |\tau_1\rangle |0\rangle$. We then apply the oracle $O_D$ on $R_1$ to obtain the state of $R_1$ as $\sum_{x \in [m]} a_x |x\rangle |\xi_2\rangle$. Let $x, \xi_2$ denote the state $|x\rangle |\xi_2\rangle$ and let $p_z = |a_z|^2$.

In stage two, we apply amplitude estimation collectively on the registers $R_1$, $R_2$ and $R_3$ in a way that for every basis state $|z\rangle$ in the first log$(m)$ qubits of $R_1$, a string $a_z$ is output on $R_3$ such that $\sin^2\left(\frac{\tau}{2\sqrt{\tau^2}}\right) = \tilde{p}_z (\text{say}) \in \left[p_2 - \frac{\epsilon}{2}, p_2 + \frac{\epsilon}{2}\right]$. This leaves the circuit in a state of the form $|\psi_2\rangle = \sum_{x \in [m]} a_x |x, \xi_2\rangle |\phi\rangle |a_x\rangle |\tau_1\rangle |0\rangle$ where $|\phi\rangle = \sum_{y \in [m]} a_y |y, \xi_9\rangle$. The algorithm for Promise-HighDIST with additive error is presented in Algorithm [1] and is same as the QBoundFMax algorithm in our earlier work [5, Algorithm 3] with a minor change in parameters.
Stage three is essentially about filtering out the good states. We use the subroutines HD₃ and CMP to perform the filtering and marking all the good states |z⟩ by flipping the state of R₅ to |1⟩ for such states. So, the state in the circuit after stage three is |ψ₃⟩ = ∑ₓ∈[n] αₓ |x, ξₓ⟩ |φ⟩ |aₓ⟩ |τ₁⟩ |˜pₓ ≥ τ₁⟩. Notice that the probability of measuring R₅ as |1⟩ in |ψ₃⟩ is either 0 or is lower bounded by τ due to the promise.

Finally at stage four, we use the amplitude amplification to amplify the probability of obtaining the state |1⟩ in R₅. Now for any x that is marked, we have ˜pₓ ≥ τ₁. If the probability of observing |1⟩ in R₅ is non-zero, then since we have a lower bound on that probability, we have an upper bound on the number of amplifications needed to observe the state |1⟩ in R₅ with high probability.

> **Lemma 1 (Additive-error algorithm for HIGHDIST).** Given an oracle Oₐ for the HIGHDIST problem, m — the domain-length of the distribution it generates and a threshold τ, along with parameters 0 < ε < τ for additive accuracy and δ for error, there is a quantum algorithm that makes O(ετ − ε² log 1/δ) queries to Oₐ such that the following hold with probability at least 1 − δ.

1. If it returns TRUE then there exists some x ∈ [m] such that pₓ ≥ τ − ε.
2. If it returns FALSE then for all x ∈ [m], pₓ < τ.

*The algorithm uses O(log(n) + log(1/ε) + a) qubits.*

**Proof.** Before we provide the correctness of the algorithm we introduce a few propositions that will be useful in proving the correctness of the algorithm.

> **Proposition 13.** For any two angles θ₁, θ₂ ∈ [0, π], sin θ₁ ≤ sin θ₂ ⇔ ∣π/2 − θ₁∣ ≥ ∣π/2 − θ₂∣.

**Proof.** The proof of this proposition follows from the proof of Proposition 4.1 of [5].

> **Proposition 14.** τ′ and τ₁ satisfy 0 ≤ τ′ − 2π/2ₐ ≤ sin²(πτ₁/2ₐ).

**Proof.** The proof of this proposition follows from the proof of Proposition 4.3 of [5].

We now show the correctness of the algorithm. The state of the circuit just before the amplitude estimation is

|ψ₁⟩ = ∑ₓ∈[m] αₓ |x, ξₓ⟩ |0′⟩ |0₀⟩ |τ₁⟩ |0⟩.

After the amplitude estimation, we obtain a state of the form

|ψ₂⟩ = ∑ₓ∈[m] αₓ |x, ξₓ⟩ |φ⟩ (βₓ₁ |aₓ⟩ + βₓ₂ |Eₓ⟩) |τ₁⟩ |0⟩,

where |φ⟩ = ∑ₓ∈[m] √Pₓ |x, ξₓ⟩ and aₓ is a q-bit string such that sin²(π/2ₐ) = ˜pₓ (say) is a 1/2ₐ-approximate estimate of pₓ. It is known that in the case of estimating the amplitude of a basis state |x⟩ in some given state |η⟩ = ∑ₓ γₓ |y⟩ using amplitude estimation, the final state just prior to the final measurement is of the form |η⟩ (βₓ |aₓ⟩ + βₓ |E⟩) where aₓ is such that sin²(π/2ₐ) approximates |γₓ|² up to q bits of accuracy and any basis state in |E⟩ corresponds to a wrong estimate. In our case, we have βₓ₁ in |ψ₂⟩ such that |βₓ₁|² = (1 − 2π/2ₐ). For the rest of the analysis we assume that the error probability is negligible. Now, we consider the following two cases:
Case(i): Let $z$ be a string such that $z \in [m]$ and $p_z \geq \tau$. Then after the amplitude estimation in stage two, the state $|a_z\rangle$ in $R_3$ corresponding to the basis state $|z\rangle$ in $R_1$ would be such that if $\sin^2\left(\frac{a_z \pi}{2q}\right) = \tilde{p}_z$ (say) then $\tilde{p}_z \geq \tau - \frac{\epsilon}{8}$. This gives us that

$$\sin^2\left(\frac{a_z \pi}{2q}\right) \geq \tau - \frac{\epsilon}{8} = \tau'$$

or, $\sin\left(\frac{a_z \pi}{2q}\right) \geq \sqrt{\tau'} \geq \sin\left(\frac{\tau_1 \pi}{2q}\right)$

The last inequality follows from the fact that $\tau_1$ is an integer in $[0, 2^{q-1}]$ and $\tau_1 = \left\lceil \frac{2\tau}{\pi} \sin^{-1}\left(\sqrt{\tau'}\right) \right\rceil$. Now, on applying $\text{HD}_q$ on $R_3$ and $R_4$, we obtain $|\tilde{a}_z\rangle$ and $|\tilde{\tau}_1\rangle$ respectively in $R_3$ and $R_4$ such that $\tilde{a}_z = |2^{q-1} - a_z| \text{ and } \tilde{\tau}_1 = |2^{q-1} - \tau_1|$. Using Proposition 13 we have $|\tilde{\tau}_1\rangle$ is at most $|\tau_1\rangle$ and flips the state in $R_5$ to $|1\rangle$ if $\tilde{a}_z \leq \tilde{\tau}_1$ and leaves the state in $R_5$ as $|0\rangle$ if $\tilde{a}_z > \tilde{\tau}_1$. Since $\tilde{a}_z \leq \tilde{\tau}_1$, the state in $R_5$ is such that the probability of measuring $|1\rangle$ in $R_5$ is non-zero. As $p_z \geq \tau$, we have that the probability of measuring $|1\rangle$ in $R_5$ is at least $\tau$. So, if one obtains $|0\rangle$ on measurement, then there does not exist any $z \in [m]$ such that $p_z \geq \tau$, i.e., all $z \in [m]$ is such that $p_z < \tau$.

Case(ii): Let for all $x \in [m]$, $p_x < \tau - \epsilon$. Then after stage two, the state $|a_x\rangle$ in $R_3$ for any arbitrary $|x\rangle$ in $R_1$ will be such that $\tilde{p}_x$ (say) $= \sin^2\left(\frac{a_x \pi}{2q}\right) < \tau - \frac{\epsilon}{8}$. From Proposition 14 we have $\tau' < \frac{\epsilon}{8} \sin^2\left(\frac{\pi a_x}{2q}\right)$. Since we have $2^q \geq \frac{16 \epsilon}{\pi^2}$, we get that $\frac{\epsilon}{8} > \frac{\pi a_x}{2q}$. Using this, we have $\tau - \frac{\epsilon}{8} < \tau' < \frac{\epsilon}{8} \sin^2\left(\frac{\pi a_x}{2q}\right)$. Since, $\tilde{p}_x < \tau - \frac{\epsilon}{8}$, we gave $\tilde{p}_x = \sin^2\left(\frac{a_x \pi}{2q}\right) < \sin^2\left(\frac{\pi a_x}{2q}\right)$. Now on applying $\text{HD}_q$ on $R_3$ and $R_4$, we obtain $|\tilde{a}_x\rangle$ and $|\tilde{\tau}_1\rangle$ respectively in $R_3$ and $R_4$ such that $\tilde{a}_x = |2^{q-1} - a_x| \text{ and } \tilde{\tau}_1 = |2^{q-1} - \tau_1|$. Using Proposition 13 on the fact that $\tilde{p}_x < \sin^2\left(\frac{\pi a_x}{2q}\right)$ we get $\tilde{a}_x > \tilde{\tau}_1$. Then on using $\text{CMP}$ on $R_3$ and $R_4$, since $\tilde{a}_x > \tilde{\tau}_1$, the state in $R_5$ remains $|0\rangle$. It is not hard to see that this is true for all $x \in [m]$ since $p_x < \tau - \epsilon$ for all $x \in [m]$. Hence, if $p_x < \tau - \epsilon$ for all $x \in [m]$, the state in $R_5$ is $|0\rangle$. In other words, the probability of measuring $|1\rangle$ in $R_5$ is 0. This implies that if $|1\rangle$ was measured in $R_5$ then there exists some $x \in [m]$ such that $p_x \geq \tau - \epsilon$.

Now, we present the query complexity of the algorithm. It is obvious that the number of calls made by amplitude estimation with accuracy $\frac{1}{2\sqrt{q}}$ and error $\frac{\epsilon}{8}$ is $O(2^q \log \left(\frac{1}{\epsilon}\right)) = O\left(\frac{1}{\epsilon} \log \frac{1}{\epsilon}\right)$. The subroutines $\text{HD}_q$ and $\text{CMP}$ make no queries to the oracle. Now, after stage three, if there was no $x \in [m]$ such that $p_x \geq \tau$, then the probability of obtaining $|1\rangle$ on measuring $R_5$ is 0 which remains unchanged on amplification. On the other hand if there was some $x \in [m]$ such that $p_x \geq \tau$, then the probability of $|1\rangle$ on measuring $R_5$ is lower bounded by $\tau$. So, it suffices to perform $O\left(\frac{1}{\sqrt{q}}\right)$ many iterations of fixed point amplitude amplification to obtain the probability of observing $|1\rangle$ in $R_0$ arbitrarily close to 1. Hence, the total number of queries made by $\text{HighDist}_\text{Algo}$ is $O\left(\frac{1}{\sqrt{\epsilon q}}\right)$.

Now, as for the error analysis in $\text{HighDist}_\text{Algo}$, note that $\text{HD}_q$ and $\text{CMP}$ are exact algorithms. So, the error in $\text{HighDist}_\text{Algo}$ is only due to the amplitude estimation and the amplitude amplification. Since, the amplitude estimation and amplitude amplification are called with at most $\frac{\delta}{2}$ error, using union bounds, it is straightforward that the total error is at most $\delta$.

The number of qubits used in the algorithm is $2 \cdot r + 2 \cdot q + 1 = O(\log(m) + \log\left(\frac{1}{\epsilon}\right) + a)$ as has been initialized in Line 3 of the algorithm.
A.2 Algorithm for $P_{\text{max}}$ problem with additive accuracy

We now describe a quantum algorithm to estimate $\max_{x \in [n]} p_x = |\alpha_x|^2$ with an additive accuracy given a quantum black-box $O_D$ with the following behaviour.

$$O_D \left| 0^{\log(m) + a} \right> = \sum_{x \in \{0, 1\}^{\log(m)}} \alpha_x |x \rangle |\psi_x\rangle$$

The black-box generates the distribution $D = (p_x)_{x=1}^m$ when its first $\log(m)$ qubits are measured in the standard basis.

Lemma 4 (Approximating $p_{\text{max}}$ with additive error). Given an oracle as required for the HIGHDIST problem, additive accuracy $\epsilon \in (0, 1)$ and error $\delta$, there is a quantum algorithm that makes $O(\frac{1}{\epsilon^2} \log^3 \frac{1}{\epsilon} \log \frac{1}{\delta})$ queries to the oracle and outputs an estimate $\hat{p}_{\text{max}}$ such that $|p_{\text{max}} - \hat{p}_{\text{max}}| \leq \epsilon$ with probability $1 - \delta$. The algorithm uses $O(\log(m) + \log(\frac{1}{\epsilon}) + a)$ qubits.

We design an algorithm namely IntervalSearch to prove the lemma. The algorithm originally appeared in [5]. The idea behind Algorithm IntervalSearch is quite simple. The algorithm essentially combines the HIGHDIST_Algo with the classical binary search. Recall that given any threshold $\tau$, accuracy $\epsilon$ and error $\delta$, if HIGHDIST_Algo outputs TRUE then $p_{\text{max}} \geq \tau - 2\epsilon$ else if the output is FALSE then $p_{\text{max}} < \tau$. The IntervalSearch algorithm is as presented in Algorithm 2.

Algorithm 2 Algorithm IntervalSearch to find out an $\epsilon$-length interval containing $\max_{i \in [n]} p_i$

**Require:** Distribution oracle $O_D$, size of the oracle $r = \log(m) + a$, size of the distribution $m$, accuracy $\epsilon$ and probability of error $\delta$

Set $k = \lceil \log_2 \frac{\epsilon}{\delta} \rceil + 1$ \hspace{1cm} \triangleright k$ is the smallest integer s.t. $\frac{1}{2^k} \leq \frac{\epsilon}{\delta}$; thus, $\frac{\epsilon}{2^k} < \frac{1}{2^k} \leq \frac{\epsilon}{2}$

Set gap $g = \frac{\epsilon}{2}$

Set boundaries $\text{lower} = \frac{1}{n}$, $\text{upper} = 1$ and threshold $\tau = \frac{1}{2}$

for $i = 1 \ldots k$

if $\tau \leq \epsilon$ then

Update $\text{upper} = \epsilon$

Break

endif

if HIGHDIST_Algo $(r, m, \tau, g, \frac{\epsilon}{2})$ \rightarrow TRUE then

Update $\text{lower} = \tau - g$, $\tau = \tau + \frac{1}{2^{i+1}}$; $\text{upper}$ is unchanged

else

Update $\text{upper} = \tau$, $\tau = \tau - \frac{1}{2^{i+1}}$; $\text{lower}$ is unchanged

dendif

end for

return $[\text{lower}, \text{upper}]$

**Proof.** Notice that the interval $[\text{lower}, \text{upper}]$ at the start of the $i^{th}$ iteration is such that the size of the interval is either $\frac{1}{2^i}$ or $\frac{1}{2^i - 1}$. The algorithm essentially attempts to find a $\tau$ which is a multiple of $\frac{1}{2^i}$ in such a way that at $k - 1^{st}$ iteration, $\tau$ is (almost) the center of an interval $J$ of size $\frac{1}{2^k}$ and $p_{\text{max}} \in J$. It is clear that after the $k^{th}$ iteration the algorithm returns an interval of the form $[\tau - g, \tau + g]$ for $t \in \{1, 2, \cdots 2^k - 1\}$ and the length of the returned interval is at most $\frac{1}{2^k} + g \leq \frac{\epsilon}{2} + g \leq \epsilon$ as desired. The correctness of the algorithm then follows from the correctness of HIGHDIST_Algo. IntervalSearch makes $k = O(\log(\frac{1}{\epsilon}))$ invocations of the HIGHDIST_Algo.
Figure 2 Illustration of the IntervalSearch algorithm

Since the accuracy parameter of each invocation of HighDist_Algo in IntervalSearch is $\frac{\epsilon}{4}$ and the error parameter is $\frac{\delta}{4}$, from Lemma 1 we get that the query complexity of each invocation of HighDist_Algo is $O \left( \frac{1}{\sqrt{\epsilon}} \log \log(\epsilon) \log \left( \frac{1}{\delta} \right) \right)$ where $\tau_i$ denotes the threshold at iteration $i$. Hence, we get the total query complexity of IntervalSearch as $\sum_{i=1}^{k} O \left( \frac{1}{\sqrt{\epsilon}} \log \log(\epsilon) \log \left( \frac{1}{\delta} \right) \right) = O \left( \frac{1}{\sqrt{\epsilon}} \log \log(\epsilon) \log \left( \frac{1}{\delta} \right) \right)$. The last equality uses the fact that $\tau_i \geq \epsilon$ for any $i \in [k]$. Now, since each time HighDist_Algo is invoked with the error parameter $\frac{\delta}{4}$, using union bounds we can say that the IntervalSearch algorithm returns an errored output with probability at most $\delta$. ◀

A.3 Algorithm for $P_{\text{max}}$ with relative accuracy

Now, we present an algorithm to approximate $p_{\text{max}}$ with relative error.

Lemma (Approximating $p_{\text{max}}$ with relative error). Given an oracle as required for the HighDist problem, relative accuracy $\epsilon \in (0, 1)$ and error $\delta$, there is a quantum algorithm that makes $\tilde{O}(m^{3/2})$ queries to the oracle and outputs an estimate $\tilde{p}_{\text{max}}$ such that with probability $1 - \delta$, it holds that $(1 - \epsilon)\tilde{p}_{\text{max}} \leq p_{\text{max}} < \tilde{p}_{\text{max}}$. The algorithm uses $\log \left( \frac{m}{\epsilon} \right) + a$ qubits.

To solve the relative version of the $P_{\text{max}}$ problem, we introduce a relative version of IntervalSearch which we call IntervalSearchRel. Similar to the IntervalSearch algorithm, IntervalSearchRel also combines the HighDist_Algo with a classical binary search. But here the binary search is over the powers of $(1 - \epsilon^{'})$ where $\epsilon^{'} = (1 - \sqrt{1 - \epsilon})$ rather than on intervals of length $\frac{1}{2^i}$. The algorithm is as in Algorithm 3.

Proof. First observe that for any relative accuracy $\epsilon'$ and a threshold $\tau$, deciding the HighDist problem with relative accuracy $\epsilon'$ is equivalent to deciding the additive HighDist problem with additive accuracy $\epsilon' \tau$. So, HighDist_Algo($r, m, \tau, (\epsilon' \tau), \frac{\delta}{4}$) even though is in additive terms, essentially solves the HighDist problem with relative accuracy $\epsilon'$. Next, in IntervalSearchRel, at the end of $i^{th}$ iteration, any interval $[\text{lower}, \text{upper})$ not on the extremes is of the form $(1 - \epsilon')^{\frac{(2^i - 1) + 1}{2^i}}, (1 - \epsilon')^{\frac{2^i}{2^i + 1}}$ where $t \in \{1, 2, \ldots, 2^i - 2\}$. The left and
the right extreme intervals are of the form \( \left[ \frac{1}{m}, (1 - \epsilon')^{(1 - \frac{\epsilon}{3})^2k} \right] \) and \( \left[ (1 - \epsilon')^{(\frac{\epsilon}{3} + 1)}, 1 \right] \) respectively. In contrast to \text{IntervalSearch} by the end of \( k - 1 \)th iteration \text{IntervalSearchRel} tries to find a \( \tau \) which is a power of \( (1 - \epsilon')^{\frac{1}{2t + 1}} \) such that \( \tau \) lies strictly inside an interval \( J = \left( (1 - \epsilon')^{(t + 2)}, (1 - \epsilon')^{t} \right) \) where \( J \) contains \( p_{\max} \). At the end of \( k \)th iteration, the interval \( I = [lower, upper] \) is of the form \( \left[ (1 - \epsilon')^{(t + 1)}, (1 - \epsilon')^{t} \right) \) where \( t \in \{1, 2, \ldots, 2^k - 2\} \) and \( p_{\max} \in I \) if \( I \) is not in the extremes. The left and the right extreme intervals are of the form \( \left[ \frac{1}{m}, (1 - \epsilon')^{2^{k - 1}} \right] \) and \( \left[ (1 - \epsilon')^{2}, 1 \right] \) respectively. The algorithm then returns \( \hat{p}_{\max} = upper \). Now, since we know that \( p_{\max} \in I \), we have that \( p_{\max} < upper = \hat{p}_{\max} \) and \( p_{\max} \geq lower \geq (1 - \epsilon')^2 \hat{p}_{\max} = (1 - \epsilon)\hat{p}_{\max} \). So, we have \( (1 - \epsilon)\hat{p}_{\max} \leq p_{\max} < \hat{p}_{\max} \) as required.

At iteration \( i \), as the accuracy parameter and the error parameter or \text{HighDist_Alg} in \text{IntervalSearchRel} are \( \epsilon_i \tau_i \) and \( \frac{\delta}{k} \), \text{HighDist_Alg} makes \( O\left( \frac{1}{(\epsilon_i \tau_i)^{\frac{1}{2t+1}}} \log \left( \frac{1}{\epsilon_i \tau_i} \right) \log \left( \frac{k}{\delta} \right) \right) \) queries to the oracle. So, algorithm \text{IntervalSearchRel} makes \( \sum_{i=1}^{k} O\left( \frac{1}{(\epsilon_i \tau_i)^{\frac{1}{2t+1}}} \log \left( \frac{1}{\epsilon_i \tau_i} \right) \log \left( \frac{k}{\delta} \right) \right) = O\left( \frac{m^{\frac{1}{2t+1}}}{\epsilon} \left( \log \log \left( \frac{\alpha^i}{\epsilon} \right) - \log \log (1 - \epsilon) + \log (\frac{1}{\delta}) \right) \right) = O(\frac{m^{\frac{1}{2t+1}}}{\epsilon} \log \log (\frac{\alpha^i}{\epsilon})) \) queries to the oracle. The error analysis simply follows from the union bound of errors at each iteration.

**B. Complexity analysis if \( p_{\max} \) estimation proposed by Li et al [10]**

It is well known that the current best known algorithm for solving \( k \)-distinctness problem for any general \( k \) is the quantum walk based algorithm due to Ambainis[2] which has a query complexity of \( O(n^{k/k+1}) \). Here, we show that using that quantum walk based algorithm, the query complexity of \( p_{\max} \) estimation algorithm proposed in [10] Algorithm 7, which we call \text{LiWuAlgo}, with \( \epsilon \) relative error is in fact \( O(n) \). Theorem 7.1 of [10] states that the quantum query complexity of approximating \( \max_{i \in [n]} p_i \) within a multiplicative error \( 0 < \epsilon < 1 \) with success probability at least \( \Omega(1) \) using \text{LiWuAlgo} is the query complexity of \( \frac{16\log(n)}{\epsilon^3} \)-distinctness problem.
So we have the complexity of $\frac{16\log(n)}{\epsilon^2}$-distinctness as $n^{\frac{16\log(n)}{\epsilon^2}}$. Now,

$$\frac{16\log(n)}{16\log(n) + \epsilon^2} = \frac{\log(n)}{\log(n) + (\epsilon^2/16)}$$

Since we have $0 < \epsilon \leq 1$, $\frac{\epsilon^2}{16} \leq \frac{1}{16}$. This gives us that $\frac{\log(n)}{\log(n) + (\epsilon^2/16)} \geq \frac{\log(n)}{\log(n) + (1/16)} = 1 - \frac{1}{16\log(n) + 1}$.

So, we have

$$n^{\frac{16\log(n)}{\epsilon^2}} \geq n^{1 - \frac{1}{16\log(n) + 1}} \geq n^{1 - \frac{1}{16\log(n)}} = \frac{n}{n^{16\log(n)}} = \frac{n}{n^{\frac{16}{16}}} \geq n/2.$$ 

The second last equality is due to the fact that $n^{\frac{1}{16\log(n)}} = \epsilon$. So for any relative error $\epsilon$, the algorithm makes $O(n)$ queries to the oracle.

## C Reductions between problems

In this section, we describe all the reductions between various problems encountered in this draft.

\textbf{HighDist} $\rightarrow_R$ \textbf{P}_{\text{max}}: Given \textsc{HighDist} $(O_D, \tau, \epsilon)$, solve $\text{P}_{\text{max}} (O_D, \epsilon/3)$ and return $\text{TRUE}$ if $\hat{p}_{\text{max}} \geq \tau - \frac{\epsilon}{2}$ else return $\text{FALSE}$.

$\text{P}_{\text{max}} \rightarrow_R \text{HighDist}$: Given $\text{P}_{\text{max}} (O_D, \epsilon)$ search for the largest integer $t \in \{1, 2, \ldots, 2^k\}$ such that $\text{HighDist} (O_D, \frac{1}{2^t}, \frac{1}{2^t})$ returns $\text{TRUE}$ where $k = \lceil \log(\frac{1}{2^t}) + 1 \rceil$ and return the interval $[\frac{1}{2^t} - \epsilon, \frac{1}{2^t}]$ if $t \neq 2^k$ and return $[1 - \frac{1}{2^t}, 1]$ if $t = 2^k$. The search is performed using binary search which imparts an additional log factor overhead to the complexity of solving $\text{P}_{\text{max}}$. See Section 2.3.

\textbf{Δ-Gapped $k$-Distinctness} $\rightarrow_R$ \textbf{HighDist}: Given $\text{Δ-Gapped $k$-Distinctness} (O_A, Δ)$, solve $\text{HighDist} (O_A, \frac{Δ}{2^n})$ and return as $\text{HighDist} (O_A, \frac{Δ}{n})$ returns. See Section 4.

\textbf{Δ-Gapped $k$-Distinctness} $\rightarrow_R$ \textbf{F}_∞: Given $\text{Δ-Gapped $k$-Distinctness} (O_A, Δ)$, solve $\text{F}_∞ (O_A, \frac{Δ}{2^n})$ and return $\text{TRUE}$ if $\hat{f}_{\text{inf}} \geq k - \frac{1}{2}$ else return $\text{FALSE}$.  

\textbf{Δ-Gapped $k$-Distinctness} $\rightarrow_R$ \textbf{k-distinctness}: Given $\text{Δ-Gapped $k$-Distinctness} (O_A, Δ)$, solve $\text{k-distinctness} (O_A)$ and return as $\text{k-distinctness} (O_A)$ returns.

\textbf{k-distinctness} $\rightarrow_R$ \textbf{Δ-Gapped $k$-Distinctness} : Given $\text{k-distinctness} (O_A)$, solve $\text{Δ-Gapped $k$-Distinctness} (O_A, 1)$ and return as $\text{Δ-Gapped $k$-Distinctness} (O_A, 1)$ returns.

$\text{F}_∞ \rightarrow_R \text{P}_{\text{max}}$: Given $\text{F}_∞ (O_A, \epsilon)$, solve $\text{P}_{\text{max}} (O_A, \epsilon)$ and return as $\text{P}_{\text{max}} (O_A, \epsilon)$ returns. See Section 5.

$\text{F}_∞ \rightarrow_R \text{k-distinctness}$: Given $\text{F}_∞ (O_A, \epsilon)$, return the largest $k$ such that $\text{k-distinctness} (O_A)$ returns $\text{TRUE}$. The search is performed using a binary search which imparts an additional log factor overhead to the complexity of solving $\text{F}_∞$.

$\text{k-distinctness} \rightarrow_R \text{F}_∞$: Given $\text{k-distinctness} (O_A)$, solve $\text{F}_∞ (O_A, 1/3)$ and return $\text{TRUE}$ if $\hat{f}_{\text{inf}} \geq k - \frac{1}{2}$ else return $\text{FALSE}$.
**CountDecision → \(_R^\Delta\)-Gapped \(k\)-Distinctness**: Given CountDecision \((O_X, l, \Delta)\), solve \(_\Delta\)-GAPPED \(k\)-DISTINCTNESS \((O_X, n/3\Delta)\) and return as \(_\Delta\)-GAPPED \(k\)-DISTINCTNESS \((O_X, n/3\Delta)\) returns. See proof of Lemma 9.

**CountDecision → \(_R^k\)-distinctness**: Given CountDecision \((O_X, l, \Delta)\), solve \(k\)-distinctness \((O_X)\) with \(k = l\) and return as \(k\)-distinctness \((O_X)\) returns.

**CountDecision → \(_R^F\infty\)**: Given CountDecision \((O_X, l, \Delta)\), solve \(F_{\infty} ((O_X, \Delta/3)\) and return \(TRUE\) if \(\tilde{f}_\infty \geq l - \frac{\Delta}{2}\) else return \(FALSE\). See proof of Lemma 11.

**CountDecision → \(_R^\text{HighDist}\)**: Given CountDecision \((O_X, l, \Delta)\), solve HighDist \((O_X, l/n, \Delta/3n)\) and return as HighDist \((O_X, l/n, \Delta/3n)\) returns. See proof of Lemma 3 in Section 6.