ON THE CHARACTERIZATION OF $p$-HARMONIC FUNCTIONS ON THE HEISENBERG GROUP BY MEAN VALUE PROPERTIES

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(Communicated by Manuel del Pino)

Abstract. We characterize $p$–harmonic functions in the Heisenberg group in terms of an asymptotic mean value property, where $1 < p < \infty$, following the scheme described in [16] for the Euclidean case. The new tool that allows us to consider the subelliptic case is a geometric lemma, Lemma 3.2 below, that relates the directions of the points of maxima and minima of a function on a small subelliptic ball with the unit horizontal gradient of that function.

1. Introduction. In this paper we study $p$–harmonic functions in the Heisenberg group in terms of an asymptotic mean value property. The corresponding characterization of the $p$-harmonic functions in terms of an asymptotic mean value property in the Euclidean case was obtained in [16]. More precisely in [16], the authors show that if $u$ is a continuous function in a domain $\Omega \subset \mathbb{R}^n$ and $p \in (1, \infty]$, then the asymptotic expansion

$$u(x) = \frac{p - 2}{2(p + n)} \left( \max_{B_\epsilon(x)} u + \min_{B_\epsilon(x)} u \right) + \frac{2 + n}{p + n} \int_{B_\epsilon(x)} u(y) \, dy + o(\epsilon^2),$$

holds as $\epsilon \to 0$ for all $x \in \Omega$ in the viscosity sense if and only if $u$ is a viscosity solution of the $p$-Laplace equation

$$\text{div} (|\nabla u(x)|^{p-2}\nabla u(x)) = 0.$$  

Here $B_\epsilon(x)$ is the Euclidean ball centered in $x$ with radius $\epsilon$.

We want to extend this characterization to functions defined on the Heisenberg group $\mathbb{H}^n$. In Section 2 we present an overview of the Heisenberg group, where the geometry and analysis are different from the Euclidean space. For this introduction

2010 Mathematics Subject Classification. Primary: 35J60, 35R03; Secondary: 35J70.

Key words and phrases. $p$-Laplacian, Heisenberg group, mean value formulas, viscosity solutions.

F. F. is supported by MURST, Italy, by University of Bologna, Italy, by EC project CG-DICE and by the ERC starting grant project 2011 EPSILON (Elliptic PDEs and Symmetry of Interfaces and Layers for Odd Nonlinearities). F. F. wishes to thank the Department of Mathematics at the University of Pittsburgh for the kind hospitality.

Q. L. and J. M. are supported by NSF award DMS-1001179.
we anticipate a few definitions from Section 2 and refer the reader to this section for full details. For \( p \in (1, \infty) \) the (subelliptic) \( p \)-Laplace operator in the Heisenberg group is

\[
\Delta_{p,\mathbb{H}^n} u = \text{div}_{\mathbb{H}^n} \left( |\nabla_{\mathbb{H}^n} u|^{p-2} \nabla_{\mathbb{H}^n} u \right).
\]

Here \( \nabla_{\mathbb{H}^n} \) and \( \text{div}_{\mathbb{H}^n} \) are respectively the intrinsic gradient and the intrinsic divergence in the Heisenberg group. From now on we will denote by \( B(P, \epsilon) \) the intrinsic ball of radius \( \epsilon \) with respect to the gauge distance, centered at the point \( P = (x, y, t) \in \mathbb{R}^{2n+1} \equiv \mathbb{H}^n \).

We point out that for \( p = 2 \), \( \Delta_{2,\mathbb{H}^n} u = \Delta_{\mathbb{H}^n} u \) is the real part of the Kohn-Laplace operator, a linear degenerate second order elliptic operator whose lowest eigenvalue is always zero. In particular, to give an idea of the structure of this linear operator corresponding to the case \( p = 2 \) and \( n = 1 \) we set

\[
\Delta_{\mathbb{H}^1} u(P) = \text{div}_{\mathbb{H}^1} \left( \frac{\nabla_{\mathbb{H}^1} u(P)}{|\nabla_{\mathbb{H}^1} u(P)|} \right),
\]

where the \( 3 \times 3 \) matrix \( M(P) \) for \( P = (x, y, t) \in \mathbb{R}^3 \equiv \mathbb{H} \) is given by

\[
M(P) = \begin{bmatrix}
1, & 0, & 2y \\
0, & 1, & -2x \\
2y, & -2x, & 4(x^2 + y^2)
\end{bmatrix}.
\]

Observe that we always have

\[
\min \left\{ \lambda: \lambda \text{ is an eigenvalue of } M(P) \right\} = 0.
\]

For \( p = 1 \) we get a subelliptic version of the mean curvature operator

\[
\Delta_{1,\mathbb{H}^n} u(P) = \text{div}_{\mathbb{H}^n} \left( \frac{\nabla_{\mathbb{H}^n} u(P)}{|\nabla_{\mathbb{H}^n} u(P)|} \right).
\]

See the monograph [9] for the intrinsic mean curvature operator in the Heisenberg setting. See also [8] and [11] for some recent results on the the flow by mean curvature in the subelliptic setting.

Our main result is the following:

**Theorem 1.1.** Let \( 1 < p < \infty \) and let \( u \) be a continuous function defined in a domain \( \Omega \subset \mathbb{H}^n \). The asymptotic expansion

\[
u(P) = \frac{\alpha}{2} \left( \min_{B(P, \epsilon)} u + \max_{B(P, \epsilon)} u \right) + \beta \int_{B(P, \epsilon)} u(x, y, t) + o(\epsilon^2), \tag{1}\]

holds as \( \epsilon \to 0 \) for every \( P \in \Omega \) in the viscosity sense if and only if

\[
\Delta_{p,\mathbb{H}^n} u = 0
\]

in \( \Omega \) in the viscosity sense, where

\[
\alpha = \frac{2(p-2)C(n)}{2(p-2)C(n) + 1}, \quad \beta = \frac{1}{2(p-2)C(n) + 1},
\]

and

\[
C(n) = \frac{1}{2(n+1)} \int_0^1 \frac{(1-s^2)^{n+1}}{(1-s^2)^{n+2}} \, dt.
\]
We remark that $\alpha + \beta = 1$ and that

$$C(n) = \frac{\Gamma \left( \frac{n+3}{2} \right)^2}{(2n + 1) \Gamma \left( \frac{n}{2} + 1 \right) \Gamma \left( \frac{n}{2} + 2 \right)}.$$ 

The key tool in the proof of Theorem 1.1 is Lemma 3.2 in Section 3. Roughly speaking, in Lemma 3.2 we prove that if $P_0 = (x_0, y_0, t_0) \in \mathbb{H}^n$ is not characteristic for the level set $\{ u = u(P_0) \}$, then the extrema of the function $u$ in the intrinsic ball of radius $\epsilon$ and center $P_0$ are attained at points $(x_\epsilon, y_\epsilon, t_\epsilon)$ and satisfying

$$\lim_{\epsilon \to 0} \left( \frac{x_\epsilon - x_0}{\epsilon}, \frac{y_\epsilon - y_0}{\epsilon}, \frac{|t_\epsilon - t_0|}{\epsilon^3} \right) = \pm \left( \frac{\nabla_{\mathbb{H}^n} u(P_0)}{\nabla_{\mathbb{H}^n} u(P_0)}, \frac{1}{2} |p(P_0)| \right),$$

where $p$ is the imaginary curvature of the level surface $\{ u = u(P_0) \}$ at $P_0$ introduced in [1] and [2] and given by

$$p(P_0) = - \frac{[X, Y](P_0)}{\nabla_{\mathbb{H}^n} u(P_0)}.$$ 

We recall now the definition of viscosity solution in the Heisenberg group taken from [4]. Observe that if $u$ is smooth then

$$-\Delta_{p,\mathbb{H}^n} u = -|\nabla_{\mathbb{H}^n} u|^{p-2} ((p - 2) \Delta_{\mathbb{H}^n} u + \Delta_{\mathbb{H}^n} u),$$

where we have used

$$\Delta_{\mathbb{H}^n} u = (D^2_{\mathbb{H}^n} u) \nabla_{\mathbb{H}^n} u \nabla_{\mathbb{H}^n} u \nabla_{\mathbb{H}^n} u \nabla_{\mathbb{H}^n} u \nabla_{\mathbb{H}^n} u.$$ 

(2)

**Definition 1.2.** Fix a value of $p \in (1, \infty)$ and consider the $p$-Laplace equation

$$-\text{div}_{\mathbb{H}^n}( |\nabla_{\mathbb{H}^n} u|^{p-2} \nabla_{\mathbb{H}^n} u) = 0.$$ 

(3)

(i) A lower semi-continuous function $u$ is a viscosity supersolution of (3) if for every $\phi \in C^2(\Omega)$ such that $u - \phi$ has a strict minimum at $P_0 \in \Omega$, and $\nabla_{\mathbb{H}^n} \phi(P_0) \neq 0$ we have

$$-(p - 2) \Delta_{\mathbb{H}^n} \phi(P_0) - \Delta_{\mathbb{H}^n} \phi(P_0) \geq 0.$$ 

(ii) A lower semi-continuous function $u$ is a viscosity subsolution of (3) if for every $\phi \in C^2(\Omega)$ such that $u - \phi$ has a strict maximum in $P_0 \in \Omega$, and $\nabla_{\mathbb{H}^n} \phi(P_0) \neq 0$, we have

$$-(p - 2) \Delta_{\mathbb{H}^n} \phi(P_0) - \Delta_{\mathbb{H}^n} \phi(P_0) \leq 0.$$ 

(iii) A continuous function $u$ is a viscosity solution of (3) if it is both a viscosity supersolution and a viscosity subsolution.

As shown in [14] for the Euclidean case and in [4] for the subelliptic case, it suffices to consider smooth test functions whose horizontal gradient does not vanish. In addition in those papers it is shown that the notions of viscosity and weak solutions agree for homogeneous equation $-\Delta_{p,\mathbb{H}^n} u = 0$.

Next we state carefully what we mean when we say that the asymptotic expansion (1) holds in the viscosity sense. Recall the familiar definition of “little o” for a real valued function $h$ defined in a neighborhood of the origin. We write

$$h(x) = o(x^2) \text{ as } x \to 0^+$$

for

$$\lim_{x \to 0^+} \frac{h(x)}{x^2} = 0.$$
Definition 1.3. Let $h$ be a real valued function defined in a neighborhood of zero. We say that
\[ h(x) \leq o(x^2) \text{ as } x \to 0^+ \]
if any of the three equivalent conditions is satisfied:

a) \( \limsup_{x \to 0^+} \frac{h(x)}{x^2} \leq 0 \),

b) there exists a nonnegative function $g(x) \geq 0$ such that
\[ h(x) + g(x) = o(x^2) \text{ as } x \to 0^+, \]

or

c) \( \lim_{x \to 0^+} \frac{h^+(x)}{x^2} \leq 0 \),

A similar definition is given for $h(x) \geq o(x^2)$ as $x \to 0^+$.

by reversing the inequalities in a) and c), requiring that $g(x) \leq 0$ in b) and replacing $h$ by $h^-$ in c).

Definition 1.4. A continuous function defined in a neighborhood of a point $P \in \mathbb{H}^n$ satisfies
\[ u(P) = \frac{\alpha}{2} \left( \min_{B(P,\epsilon)} u + \max_{B(P,\epsilon)} u \right) + \beta \int_{B(P,\epsilon)} + o(\epsilon^2), \tag{4} \]
as $\epsilon \to 0$ in viscosity sense, if

(i) for every continuous function $\phi$ defined in a neighborhood of a point $P$ such that $u - \phi$ has a strict minimum at $P$ with $u(P) = \phi(P)$ we have
\[ -\phi(P) + \frac{\alpha}{2} \left( \min_{B(P,\epsilon)} \phi + \max_{B(P,\epsilon)} \phi \right) + \beta \int_{B(P,\epsilon)} \phi \leq o(\epsilon^2), \]
as $\epsilon \to 0$, and

(ii) for every continuous function $\phi$ defined in a neighborhood of a point $P$ such that $u - \phi$ has a strict maximum at $P$ with $u(P) = \phi(P)$ then
\[ \phi(P) - \frac{\alpha}{2} \left( \min_{B(P,\epsilon)} \phi + \max_{B(P,\epsilon)} \phi \right) + \beta \int_{B(P,\epsilon)} \phi \geq o(\epsilon^2). \]
as $\epsilon \to 0$.

For the case $p = \infty$ we could consider the 1-homogeneous infinity-Laplacian (2), but there remain some technical difficulties. We certainly conjecture that the statement of Theorem 1.1 holds in this case.

2. Heisenberg group preliminaries. For $n \geq 1$ we denote by $\mathbb{H}^n$ the set $\mathbb{R}^{2n+1}$ endowed with the non-commutative product law given by
\[ (x_1, y_1, t_1) \ast (x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + 2(x_2 \cdot y_1 - x_1 \cdot y_2)), \]
where we have denote points in $\mathbb{R}^{2n+1}$ as $P = (x, y, t)$ with $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$ and $x \cdot y$ denote the usual inner product in $\mathbb{R}^n$. The pair $\mathbb{H}^n \equiv (\mathbb{R}^{2n+1}, \ast)$ is the Heisenberg group of order $n$. From now we will denote the group operation $\cdot$ instead of $\ast$ when there is no risk of confusion.
Given \( P = (x, y, t) \in \mathbb{H}^n \) we write \( X_i = (e_i, 0, 2y_i) \) and \( Y_i = (0, e_i, -2x_i) \) for \( i = 1, \ldots, n \), where \( \{e_i\}_{1 \leq i \leq n} \) is the canonical basis in \( \mathbb{R}^n \). We identify these vectors with the vector fields

\[
X_i = \partial_{x_i} + 2y_i \partial_t
\]

\[
Y_i = \partial_{y_i} - 2x_i \partial_t
\]

The commutator between the vector fields \( X_i \) and \( Y_j \) is 0 except when the indexes are the same \( i = j \), in which case we have

\[
[X_i, Y_i] = -4 \partial_t.
\]

The intrinsic (or horizontal) gradient of a smooth function \( u \) at the point \( P \) is given by

\[
\nabla_{\mathbb{H}^n} u(P) = \sum_{i=1}^{n} (X_i u(P) X_i(P) + Y_i u(P) Y_i(P)).
\]

The horizontal tangent space at the point \( P \) is the \( 2n \)-dimensional space \( \text{span}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\} \).

Moreover if \( W = \sum_{j=1}^{n} (\alpha_j X_j + \beta_j Y_j) \) is a smooth horizontal vector field, the intrinsic (or horizontal) divergence of \( W \) at the point \( P \) is defined as

\[
\text{div}_{\mathbb{H}^n} W(P) = \sum_{j=1}^{n} (X_j \alpha_j(P) + Y_j \beta_j(P)).
\]

We build a metric in the horizontal tangent space by declaring that the set of vectors \( \{X_1, \ldots, X_n, Y_1, \ldots, Y_n\} \) is an orthonormal basis. Thus for every pair of horizontal vectors \( U = \sum_{j=1}^{n} (\alpha_{1,j} X_j(P) + \beta_{1,j} Y_j(P)) \) and \( V = \sum_{j=1}^{n} (\alpha_{2,j} X_j(P) + \beta_{2,j} Y_j(P)) \) we have the natural inner product

\[
\langle U, V \rangle = \sum_{i=1}^{n} \alpha_{1,i} \alpha_{2,i} + \beta_{1,i} \beta_{2,i}.
\]

In particular we get the corresponding norm

\[
|U| = \sqrt{\sum_{i=1}^{n} (\alpha_{1,i}^2 + \beta_{1,i}^2)}.
\]

The norm of the intrinsic gradient of the smooth function \( u \) in \( P \) is then given by

\[
|\nabla_{\mathbb{H}^n} u(P)| = \sqrt{\sum_{i=1}^{n} (X_i u(P))^2 + (Y_i u(P))^2}
\]

If \( \nabla_{\mathbb{H}^n} u(P) = 0 \) then we say that the point \( P \) is characteristic for the surface \( \{u = u(P)\} \). For every point \( P \) that is not characteristic the intrinsic normal to the surface \( \{u = u(P)\} \) is defined, up to orientation, by the unit horizontal normal vector

\[
\nu(P) = \frac{\nabla_{\mathbb{H}^n} u(P)}{|\nabla_{\mathbb{H}^n} u(P)|}
\]

A semigroup of anisotropic dilations \( \delta_r \) is defined as follows: for every \( r > 0 \) and \( P \in \mathbb{R}^{2n+1} \equiv \mathbb{H}^n \) let

\[
\delta_r(P) = (rx, ry, r^2t).
\]
A smooth gauge that is homogeneous with respect to the anisotropic dilations \( \delta_r \) is given by

\[
\|(x, y, t)\| = \sqrt{\|x\|^2 + \|y\|^2 + t^2}.
\]

We then have \( \|\delta_r(P)\| = r\|P\| \). We use this property to define the gauge ball of radius \( r \) centered in \( 0 \) by

\[
B(0, r) = \{ P \in \mathbb{H}^n : \|P\| < r \}.
\]

The Haar measure in \( \mathbb{H}^n \) turns out to be the Lebesgue measure in \( \mathbb{R}^{2n+1} \). Moreover we have

\[
\| \delta_\lambda(B(0, 1)) \| = \lambda^{2n+2} \| B(0, 1) \|.
\]

For these properties and much more see the book [9].

The symmetrized horizontal Hessian matrix of the smooth function \( u \) at the point \( P \) is the following \( 2n \times 2n \) matrix:

\[
D_{\mathbb{H}^n}^2 u(P) = \frac{1}{2} \left( D_{\mathbb{H}^n}^2 u(P) + (D_{\mathbb{H}^n}^2 u(P))^\top \right).
\]

The \((i, j)\)-entry of \( D_{\mathbb{H}^n}^2 u \) is given by

1. \( X_i u X_j u \) for \( 1 \leq i, j, \leq n \),
2. \( X_i u Y_{j-n} u \) for \( 1 \leq i \leq n, n + 1 \leq j \leq 2n \),
3. \( X_{i-n} u Y_i u \) for \( n + 1 \leq i \leq 2n, 1 \leq j \leq n \), and
4. \( X_{i-n} u Y_{j-n} u \) for \( n + 1 \leq i \leq 2n, n + 1 \leq j \leq 2n \).

Next, we briefly review the Taylor Formula adapted to our framework. Let \( u \) be a smooth function defined in \( \Omega \), an open neighborhood of \( 0 \). Let \( \epsilon_0 \) be a positive small number such that for \( \|P\| \leq \epsilon_0 \) and for \( 0 \leq s \leq 1 \) the points \( \delta_s(P) \in \Omega \). In this way the function

\[
g(s) = u(\delta_s(P)) = u(sx, sy, s^2t)
\]

is well defined for every \( s \in [0, 1] \). By the classical Taylor’s formula centered in \( 0 \), we get

\[
g(s) = g(0) + g'(0)s + \frac{1}{2}g''(0)s^2 + o(s^2),
\]

as \( s \to 0^+ \). Computing derivatives we get

\[
g'(s) = \langle \nabla_{\mathbb{H}^n} u(\delta_s(P)), (x, y) \rangle + 2st \partial_t u(\delta_s(P)),
\]

and

\[
g''(s) = \langle D_{\mathbb{H}^n}^2 u(\delta_s(P))(x, y), (x, y) \rangle \\
+ 2st(\partial_t X_i u(\delta_s(P)) + \partial_t Y_i u(\delta_s(P))) \\
+ 2t \partial_t u(\delta_s(P)) + 4s^2 t \partial_{tt} u(\delta_s(P)).
\]

Therefore we get the expansion

\[
u(\delta_s(P)) = u(0) + \langle \nabla_{\mathbb{H}^n} u(0), (sx, sy) \rangle + 2s^2t \partial_t u(0) \\
+ \frac{1}{2} \langle D_{\mathbb{H}^n}^2 u(0)(sx, sy), (sx, sy) \rangle + o(s^2).
\]

Writing \( Q = \delta_s(P) \), noting that \( \|Q\| = s\|P\| \), that \( Q = (sx, sy, s^2t) \) and relabeling we get the horizontal Taylor expansion valid for \( P \) near zero

\[
u(P) = u(0) + \langle \nabla_{\mathbb{H}^n} u(0), (x, y) \rangle + 2t \partial_t u(0) \\
+ \frac{1}{2} \langle (D_{\mathbb{H}^n}^2 u(0)(x, y), (x, y)) + o(\|(x, y, t)\|) \rangle.
\]
3. Key tools for the proof.

Lemma 3.1. Let \( u \) be a smooth function. If \( \nabla_{\mathbb{H}^n} u(0) \neq 0 \), then there exists \( \epsilon_0 > 0 \) such that for every \( \epsilon \in (0, \epsilon_0) \) there exist points \( P_{\epsilon, M}, P_{\epsilon, m} \in \partial B(0, \epsilon) \) such that

\[
\max_{B(0, \epsilon)} u = u(P_{\epsilon, M})
\]

and

\[
\min_{B(0, \epsilon)} u = u(P_{\epsilon, m}).
\]

Proof. Let us consider the case of the maximum, the case of the minimum being analogous. Let us proceed by contradiction. Assume that there exist a sequence of positive numbers \( \{\epsilon_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^+ \) and a sequence of points \( \{P_j\}_{j \in \mathbb{N}} \subset B(0, \epsilon_j) \) such that \( \epsilon_j \to 0 \), as \( j \to +\infty \) and

\[
\max_{B(0, \epsilon)} u = u(P_j).
\]

Then for every \( j \in \mathbb{N} \), we have that \( \nabla u(P_j) = 0 \) because \( P_j \) is in the interior of \( B(0, \epsilon_j) \). Hence we get a contradiction with the fact that by continuity of \( \nabla u \) gives \( \nabla u(0) = 0 \), which implies \( \nabla_{\mathbb{H}^n} u(0) = 0 \).

Lemma 3.2. For small \( \epsilon > 0 \), consider points \( P_{\epsilon, M} \) and \( P_{\epsilon, m} \) in \( \partial B(0, \epsilon) \) such that

\[
\max_{B(0, \epsilon)} u = u(P_{\epsilon, M}) \quad \text{and} \quad \min_{B(0, \epsilon)} u = u(P_{\epsilon, m}).
\]

Whenever \( \nabla_{\mathbb{H}^n} u(0) \neq 0 \) we have

\[
\lim_{\epsilon \to 0} \frac{(x_{\epsilon, M}, y_{\epsilon, M})}{\epsilon} = \frac{\nabla_{\mathbb{H}^n} u(0)}{|\nabla_{\mathbb{H}^n} u(0)|}
\]

and

\[
\lim_{\epsilon \to 0} \frac{(x_{\epsilon, m}, y_{\epsilon, m})}{\epsilon} = \frac{\nabla_{\mathbb{H}^n} u(0)}{|\nabla_{\mathbb{H}^n} u(0)|},
\]

where \( P_\epsilon = (x_\epsilon, y_\epsilon, t_\epsilon) \in \mathbb{H}^n \). Moreover, we also have

\[
\lim_{\epsilon \to 0} \frac{|t_\epsilon|}{\epsilon^3} = \frac{2 |u_\epsilon(0)|}{|\nabla_{\mathbb{H}^n} u(0)|}
\]

Proof of Lemma 3.2. We consider the case of the maximum by using the method of Lagrange multipliers. There exists \( \lambda_\epsilon \in \mathbb{R} \) such that

\[
\begin{cases}
0 = 4\lambda_\epsilon \left( x_{\epsilon, j} |x_\epsilon|^2 + |y_\epsilon|^2 \right) \\
0 = 4\lambda_\epsilon \left( y_{\epsilon, j} |x_\epsilon|^2 + |y_\epsilon|^2 \right) \\
0 = 2\lambda_\epsilon |x_\epsilon|^2 + |y_\epsilon|^2 + t_\epsilon^2
\end{cases}
\]

(8)

Thus we get

\[
X_j u(P_{\epsilon, M}) = 4\lambda_\epsilon \left( x_{\epsilon, j} |x_\epsilon|^2 + |y_\epsilon|^2 \right) + 4\lambda_\epsilon y_{\epsilon, j} t_\epsilon
\]

\[
= 4\lambda_\epsilon \left( x_{\epsilon, j} (|x_\epsilon|^2 + |y_\epsilon|^2) + y_{\epsilon, j} t_\epsilon \right)
\]

and

\[
Y_j u(P_{\epsilon, M}) = 4\lambda_\epsilon \left( y_{\epsilon, j} |x_\epsilon|^2 + |y_\epsilon|^2 \right) - 4\lambda_\epsilon x_{\epsilon, j} t_\epsilon
\]

\[
= 4\lambda_\epsilon \left( y_{\epsilon, j} (|x_\epsilon|^2 + |y_\epsilon|^2) - x_{\epsilon, j} t_\epsilon \right)\]
To simplify the notation write $\rho_\epsilon = |x_\epsilon|^2 + |y_\epsilon|^2$, $x_j = x_{\epsilon,j}$, $y_j = y_{\epsilon,j}$ and $\rho_\epsilon = \rho$.

Since the calculation is the same for every $j = 1 \ldots n$, it suffices to solve the following system

\[
\begin{align*}
Xu &= 4\lambda_\epsilon (\xi \rho^2 + \eta t) \\
Yu &= 4\lambda_\epsilon (\eta \rho^2 - \xi t) \\
 u_t &= 2\lambda_\epsilon t \\
 \rho^4 + t^2 &= \epsilon^4,
\end{align*}
\]

where $\xi = x_j$ and $\eta = y_j$. Hence, it follows that if $u_t(P_{\epsilon,j}) \neq 0$ we have

\[
\begin{align*}
\xi \rho^2 + \eta t &= \frac{t Xu}{2u_t} \\
-\xi t + \eta \rho^2 &= \frac{t Yu}{2u_t} \\
\lambda_\epsilon &= \frac{u_t}{2t} \\
 \rho^4 + t^2 &= \epsilon^4.
\end{align*}
\]

As a consequence solving for $\xi$ and $\eta$, squaring and adding we get

\[
\rho^2 = \frac{t^2}{4\epsilon^4} \frac{|\nabla H u|^2}{u_t^2}.
\]

Thus, we obtain:

\[
|t| = 2\rho \epsilon^2 \frac{|u_t|}{|\nabla H u|}.
\]

\[\xi = \text{sgn}(t) \frac{\rho^3}{\epsilon^2} \frac{Xu}{|\nabla H u|} - \text{sgn}(t) \frac{2\rho^2}{\epsilon^2} \frac{u_t}{|\nabla H u|} \]

\[\eta = \text{sgn}(t) \frac{\rho^3}{\epsilon^2} \frac{Yu}{|\nabla H u|} + \text{sgn}(t) \frac{2\rho^2}{\epsilon^2} \frac{u_t}{|\nabla H u|}.
\]

Squaring and summing one more time and keeping in mind that

\[\left( \frac{Xu}{|\nabla H u|} \right)^2 + \left( \frac{Yu}{|\nabla H u|} \right)^2 = 1\]

we get

\[
\rho^2 = \frac{\rho^6}{\epsilon^4} + 4\rho^4 u_t^2,
\]

which implies

\[
\frac{\rho^4}{\epsilon^4} + 4\rho^2 u_t^2 = 1.
\]

We deduce that $\rho \sim \epsilon$ whenever $\epsilon \to 0$, since $u_t(0)$ is bounded. As a consequence it follows from (13) and (14) that

\[\frac{\xi}{\epsilon} \to \pm \frac{Xu}{|\nabla H u|}\]

and

\[\frac{\eta}{\epsilon} \to \pm \frac{Yu}{|\nabla H u|}.
\]

On the other hand recalling (12) and the fact that $\rho \sim \epsilon$, as $\epsilon \to 0$, we get

\[
\lim_{\epsilon \to 0} \frac{|t|}{\epsilon^3} = \frac{2|u_t(0)|}{|\nabla H u(0)|}.
\]
If there is a sequence of point \( \{P_j\}_{j \in \mathbb{N}} \) such that \( u_t(P_j) = 0 \), then either \( t_{e_j} = 0 \), or \( \lambda_{e_j} = 0 \). In the first case we get that \( \rho^4 = e^4 \), so that \( \rho = e \). Thus we get
\[
X u(P_{e_j}) = 4 \lambda_{e_j} \xi^2
\]
and
\[
Y u(P_{e_j}) = 4 \lambda_{e_j} \eta^2.
\]
As a consequence, squaring and summing once again, we get
\[
|\nabla_{\mathbb{H}^n} u(P_{e_j})|^2 = 16 \lambda_{e_j}^2 e^4 \rho^2.
\]
So that since \( |\nabla_{\mathbb{H}^n} u(0)| \neq 0 \), this implies
\[
\lambda_{e_j}^2 e^4 \rho^2 \sim \lambda_{e_j}^2 e^6 \to \frac{|\nabla_{\mathbb{H}^n} u(0)|}{16},
\]
that is
\[
|\lambda_{e_j}| \epsilon^3 \to \frac{|\nabla_{\mathbb{H}^n} u(0)|}{4}.
\]
As a consequence, as \( \epsilon_j \to 0 \) we get
\[
X u(P_{e_j}) = 4 \lambda_{e_j} \xi \epsilon^3 \sim \pm \frac{X u(0)}{|\nabla_{\mathbb{H}^n} u(0)|},
\]
and
\[
Y u(P_{e_j}) = 4 \lambda_{e_j} \eta \epsilon^3 \sim \pm \frac{Y u(0)}{|\nabla_{\mathbb{H}^n} u(0)|},
\]
that is
\[
\frac{\xi}{\epsilon} \to \pm \frac{X u(0)}{|\nabla_{\mathbb{H}^n} u(0)|},
\]
and
\[
\frac{\eta}{\epsilon} \to \pm \frac{Y u(0)}{|\nabla_{\mathbb{H}^n} u(0)|}.
\]
In the case that there exists a sequence of points \( \{P_j\}_{j \in \mathbb{N}} \) such that such that \( \lambda_{e_j} = 0 \), then we would get
\[
X u(P_{e_j}) = 0 \text{ and } Y u(P_{e_j}) = 0,
\]
getting a contradiction with the assumption that \( \nabla_{\mathbb{H}^n} u(0) \neq 0 \).

We just need to justify the sign of the limit. Using the Taylor’s formula in the Heisenberg group we get:
\[
u(P_{e,M}) = u(0) + \langle \nabla_{\mathbb{H}^n} u(0), (x_{e,M}, y_{e,M}) \rangle + \frac{1}{2} \left( (D^{e}_{\mathbb{H}^n} u(0)(x_{e,M}, y_{e,M}), (x_{e,M}, y_{e,M})) + 2 t_{e} u(0) \right) + o(\epsilon^2)
\]
Hence, dividing by \( \epsilon > 0 \) we get
\[
0 \leq \frac{u(P_{e,M}) - u(0)}{\epsilon} = \frac{(\nabla_{\mathbb{H}^n} u(0), \frac{x_{e,M}}{\epsilon}, \frac{y_{e,M}}{\epsilon})}{\epsilon} + \frac{1}{2} \left( (D^{e}_{\mathbb{H}^n} u(0)(\frac{x_{e,M}}{\epsilon}, \frac{y_{e,M}}{\epsilon}), (x_{e,M}, y_{e,M})) + 2 \frac{t_{e}}{\epsilon} \partial_t u(0) \right) + o(\epsilon).
\]
Letting \( \epsilon \to 0 \) we get
\[
0 \leq \langle \nabla_{\mathbb{H}^n} u(0), \lim_{\epsilon \to 0} (\frac{x_{e,M}}{\epsilon}, \frac{y_{e,M}}{\epsilon}) \rangle,
\]
which implies
\[
\lim_{\epsilon \to 0} (\frac{x_{e,M}}{\epsilon}, \frac{y_{e,M}}{\epsilon}) = \frac{\nabla_{\mathbb{H}^n} u(0)}{|\nabla_{\mathbb{H}^n} u(0)|}.
\]
\[\square\]
Lemma 3.3. Let \( u \) be a smooth function defined in a open subset of the Heisenberg group. Then

\[
\int_{B(P,\epsilon)} u(x, y, t) = u(P) + C(n) \Delta_{\mathbb{H}^n} u(P) \epsilon^2 + o(\epsilon^2),
\]

as \( \epsilon \to 0 \), where

\[
C(n) = \frac{1}{2(n+1)} \int_0^1 (1 - s^2)^{\frac{n+1}{2}} ds.
\]

Proof. Without loss of generality we set \( P = 0 \). We average the Taylor expansion (7) to get

\[
\int_{B(0,\epsilon)} u(x, y, t) = u(0) + \int_{B(0,\epsilon)} \langle \nabla_{\mathbb{H}^n} u(0), (x, y) \rangle + \frac{1}{2} \left( \int_{B(0,\epsilon)} \langle D^2_{\mathbb{H}^n} u(0)(x, y), (x, y) \rangle + \int_{B(0,\epsilon)} 2i \partial_t u(0) \right) + o(\| (x, y, t) \|^2)
\]

\[
= u(0) + \frac{1}{2} \int_{B(0,\epsilon)} \langle D^2_{\mathbb{H}^n} u(0)(x, y), (x, y) \rangle + \int_{B(0,\epsilon)} o(\| (x, y, t) \|^2)
\]

\[
= u(0) + C(n) \Delta_{\mathbb{H}^n} u(P) \epsilon^2 + o(\epsilon^2).
\]

Indeed we have that the linear terms vanish:

\[
\int_{B(0,\epsilon)} t \partial_t u(0) dx dy dt = 0,
\]

\[
\int_{B(0,\epsilon)} \langle \nabla_{\mathbb{H}^n} u(0), (x, y) \rangle dx dy dt = \int_{-\epsilon^2}^{\epsilon^2} \int_{S(\epsilon, t)} \langle \nabla_{\mathbb{H}^n} u(0), (x, y) \rangle dx dy dt = 0,
\]

where we have set \( S(\epsilon, t) = \{ (x, y) : |x|^2 + |y|^2 < \sqrt{\epsilon^4 - t^2} \} \). Proceeding analogously for the second order terms we get

\[
\int_{B(0,\epsilon)} \langle D^2_{\mathbb{H}^n} u(0)(x, y), (x, y) \rangle dx dy dt = \int_{-\epsilon^2}^{\epsilon^2} \int_{S(\epsilon, t)} \langle D^2_{\mathbb{H}^n} u(0)(x, y), (x, y) \rangle dx dy dt
\]

\[
= \sum_{i=1}^{(n+1)/2} \left( X_i^2 u(0) + Y_i^2 u(0) \right) \int_{-\epsilon^2}^{\epsilon^2} \int_{S(\epsilon, t)} (x_i^2 + y_i^2) dx dy dt
\]

\[
= \frac{1}{(n+1)} \int_0^1 (1 - s^2)^{\frac{n+1}{2}} ds |B(0, \epsilon)| \Delta_{\mathbb{H}^n} u(0) \epsilon^2
\]
Let $P$. We denote by $\omega(n)$ the Euclidean surface area of the unit sphere $\partial B(0,1)$ in $\mathbb{R}^n$, and have used the formula $|B(0,\epsilon)| = \epsilon^{n+2} \frac{\omega(n)}{n} \int_0^1 (1-t^2)^{\frac{n+1}{2}} dt$ for the Lebesgue measure in $\mathbb{R}^{n+1}$ of the gauge ball of radius $\epsilon$.

4. Proof of Theorem 1.1. The first step in the proof of Theorem 1.1 is the following expansion valid for smooth functions.

Let $P \in \Omega$ be a point and $\phi$ be a $C^2$-function defined in a neighborhood of $P$. We denote by $P_{e,M} \in \overline{B(P,\epsilon)}$ and $P_{e,m} \in \overline{B(P,\epsilon)}$ the points of maxima and minima

$\phi(P_{e,M}) = \max_{\overline{B(P,\epsilon)}} \phi$, and $\phi(P_{e,m}) = \min_{\overline{B(P,\epsilon)}} \phi$.

Lemma 4.1. Let $p \in (1, +\infty)$ and $\phi$ be a $C^2$-function in a domain $\Omega \subset \mathbb{H}^n$. Let $C(n), \alpha, \beta$ be given in the statement of Theorem 1.1. Consider the vectors

$$(h_{e,M}, l_{e,M}) = \left( \frac{x_{e,M} - x}{\epsilon}, \frac{y_{e,M} - y}{\epsilon} \right) \quad \text{and} \quad (h_{e,m}, l_{e,m}) = \left( \frac{x_{e,m} - x}{\epsilon}, \frac{y_{e,m} - y}{\epsilon} \right)$$

For $p \geq 2$ the following expansions hold near every $P \in \Omega$:

$$\beta C(n)\epsilon^2 \left[ \Delta_{\mathbb{H}^n} \phi(P) + (p-2) \langle D_{\mathbb{H}^n}^2 \phi(P) \rangle (h_{e,M}, l_{e,M}, (h_{e,M}, l_{e,M})) \right] \geq$$

$$\beta \int_{B(P,\epsilon)} \phi(x,y,t) + \frac{\alpha}{2} \left( \min_{\overline{B(P,\epsilon)}} \phi + \max_{\overline{B(P,\epsilon)}} \phi \right) dt - \phi(P) + o(\epsilon^2),$$

as $\epsilon \to 0$

and

$$\beta C(n)\epsilon^2 \left[ \Delta_{\mathbb{H}^n} \phi(P) + (p-2) \langle D_{\mathbb{H}^n}^2 \phi(P) \rangle (h_{e,m}, l_{e,m}, (h_{e,m}, l_{e,m})) \right] \leq$$

$$\beta \int_{B(P,\epsilon)} \phi(x,y,t) + \frac{\alpha}{2} \left( \min_{\overline{B(P,\epsilon)}} \phi + \max_{\overline{B(P,\epsilon)}} \phi \right) dt - \phi(P) + o(\epsilon^2),$$

as $\epsilon \to 0$.
While for $1 < p < 2$ we have the expansions:

$$
\beta C(n)\epsilon^2 \left[ \Delta_{\mathbb{H}^n} \phi(P) + (p-2)\langle D^n_{\mathbb{H}^n} \phi(P)(h_{\epsilon,M}, t_{\epsilon,M}), (h_{\epsilon,M}, t_{\epsilon,M}) \rangle \right] \leq 
$$

$$
\beta \int_{B(P,\epsilon)} \phi(x,y,t) + \frac{\alpha}{2} \left( \min_{B(P,\epsilon)} \phi + \max_{B(P,\epsilon)} \phi \right) - \phi(P) + o(\epsilon^2),
$$

$\epsilon \to 0$, and

$$
\beta C(n)\epsilon^2 \left[ \Delta_{\mathbb{H}^n} \phi(P) + (p-2)\langle D^n_{\mathbb{H}^n} \phi(P)(h_{\epsilon,m}, t_{\epsilon,m}), (h_{\epsilon,m}, t_{\epsilon,m}) \rangle \right] \geq 
$$

$$
\beta \int_{B(P,\epsilon)} \phi(x,y,t) + \frac{\alpha}{2} \left( \min_{B(P,\epsilon)} \phi + \max_{B(P,\epsilon)} \phi \right) - \phi(P) + o(\epsilon^2),
$$

$\epsilon \to 0$.

**Proof.** We can assume without any restriction that $P = 0$ ($x = 0$, $y = 0$, and $t = 0$) just moving $P$ to the origin by a left translation of the group. The Taylor formula in the Heisenberg group gives

$$
\phi(P_{\epsilon,M}) = \phi(0) + \langle \nabla_{\mathbb{H}^n} \phi(0), (x_{\epsilon,M}, y_{\epsilon,M}) \rangle + \frac{1}{2} \langle D^n_{\mathbb{H}^n} \phi(0)(x_{\epsilon,M}, y_{\epsilon,M}), (x_{\epsilon,M}, y_{\epsilon,M}) \rangle + 2\epsilon \partial_t \phi(0) + o(\epsilon^2).
$$

and

$$
\phi(-P_{\epsilon,M}) = \phi(0) - \langle \nabla_{\mathbb{H}^n} \phi(0), (x_{\epsilon,M}, y_{\epsilon,M}) \rangle + \frac{1}{2} \langle D^n_{\mathbb{H}^n} \phi(0)(x_{\epsilon,M}, y_{\epsilon,M}), (x_{\epsilon,M}, y_{\epsilon,M}) \rangle - 2\epsilon \partial_t \phi(0) + o(\epsilon^2).
$$

Adding the last two inequalities we get

$$
\phi(P_{\epsilon,M}) + \phi(-P_{\epsilon,M}) = 2\phi(0) + \langle D^n_{\mathbb{H}^n} \phi(0)(x_{\epsilon,M}, y_{\epsilon,M}), (x_{\epsilon,M}, y_{\epsilon,M}) \rangle + o(\epsilon^2).
$$

Using the definition of $P_{\epsilon,M}$ if follows that

$$
\max_{B(0,\epsilon)} \phi + \min_{B(0,\epsilon)} \phi \leq \max_{B(0,\epsilon)} \phi + \phi(-P_{\epsilon,M})
$$

$$
= \phi(0) + \langle D^n_{\mathbb{H}^n} \phi(0)(x_{\epsilon,M}, y_{\epsilon,M}), (x_{\epsilon,M}, y_{\epsilon,M}) \rangle + o(\epsilon^2),
$$

(18)

which implies the inequality

$$
\phi(0) + \frac{1}{2} \langle D^n_{\mathbb{H}^n} \phi(0)(x_{\epsilon,M}, y_{\epsilon,M}), (x_{\epsilon,M}, y_{\epsilon,M}) \rangle \geq \frac{1}{2} \left( \max_{B(0,\epsilon)} \phi + \min_{B(0,\epsilon)} \phi \right) + o(\epsilon^2).
$$

(19)

Multiplying this relation by $\alpha \geq 0$, the expansion in Lemma 3.3 by $\beta$, adding and using the fact that $\alpha + \beta = 1$ we obtain

$$
\phi(0) + C(n) \beta \Delta_{\mathbb{H}^n} \phi(0) \epsilon^2 + \frac{\alpha}{2} \langle D^n_{\mathbb{H}^n} \phi(0)(x_{\epsilon,M}, y_{\epsilon,M}), (x_{\epsilon,M}, y_{\epsilon,M}) \rangle 
$$

$$
\geq \beta \int_{B(0,\epsilon)} \phi(x,y,t) + \frac{\alpha}{2} \left( \min_{B(0,\epsilon)} \phi + \max_{B(0,\epsilon)} \phi \right) + o(\epsilon^2),
$$

as we wanted to show in the case $\alpha \geq 0$.

We determine $\alpha$ and $\beta$ in such a way that

$$
\frac{\alpha}{2C(n)\beta} = p - 2,
$$

for some $p$. The choice $p = 2$ gives $\alpha = 0$ and $\beta = 1$, whereas $p > 2$ gives $\alpha > 0$ and $\beta < 1$.
Thus together the requirement \( \alpha + \beta = 1 \) we get

\[
\frac{1 - \beta}{2C(n)\beta} = p - 2,
\]
giving

\[
\alpha = \frac{2(p - 2)C(n)}{2(p - 2)C(n) + 1} \quad \text{and} \quad \beta = \frac{1}{2(p - 2)C(n) + 1}.
\]

We can now write

\[
\frac{\epsilon^2C(n)}{2(p - 2)C(n) + 1} \left( \Delta_{\mathbb{H}^n} \phi(0) + (p - 2)\left( D_{\mathbb{H}^n}^{2\epsilon} \phi(0) \left( \frac{x_{\epsilon,M}}{\epsilon}, \frac{y_{\epsilon,M}}{\epsilon} \right), \left( \frac{x_{\epsilon,M}}{\epsilon}, \frac{y_{\epsilon,M}}{\epsilon} \right) \right) \right) + o(\epsilon^2)
\]

\[
\geq \left( \frac{1}{2(p - 2)C(n) + 1} \int_{B(0,\epsilon)} \phi + \frac{(p - 2)C(n)}{2(p - 2)C(n) + 1} \left[ \min_{B(0,\epsilon)} \phi + \max_{B(0,\epsilon)} \phi \right] - \phi(0) \right).
\]

This computation works for \( \alpha \geq 0 \), that is for every \( p \geq 2 \).

When \( \alpha < 0 \) the procedure is the same but the sign of the inequality is reversed, that is

\[
\frac{\epsilon^2C(n)}{2(p - 2)C(n) + 1} \left( \Delta_{\mathbb{H}^n} \phi(0) + (p - 2)\left( D_{\mathbb{H}^n}^{2\epsilon} \phi(0) \left( \frac{x_{\epsilon,M}}{\epsilon}, \frac{y_{\epsilon,M}}{\epsilon} \right), \left( \frac{x_{\epsilon,M}}{\epsilon}, \frac{y_{\epsilon,M}}{\epsilon} \right) \right) \right) + o(\epsilon^2)
\]

\[
\leq \left( \frac{1}{2(p - 2)C(n) + 1} \int_{B(0,\epsilon)} \phi + \frac{(p - 2)C(n)}{2(p - 2)C(n) + 1} \left[ \min_{B(0,\epsilon)} \phi + \max_{B(0,\epsilon)} \phi \right] - \phi(0) \right).
\]

and \( p \in (1, 2) \).

Arguing with the inequality coming from the minimum in the case \( p \geq 2 \) we get

\[
\phi(0) + \frac{1}{2} \left( D_{\mathbb{H}^n}^{2\epsilon} \phi(0)(x_{\epsilon,m}, y_{\epsilon,m}), (x_{\epsilon,m}, y_{\epsilon,m}) \right) + o(\epsilon^2) \leq \frac{1}{2} \left( \min_{B(0,\epsilon)} \phi + \max_{B(0,\epsilon)} \phi \right)
\]

and

\[
\frac{\epsilon^2C(n)}{2(p - 2)C(n) + 1} \left( \Delta_{\mathbb{H}^n} \phi(0) + (p - 2)\left( D_{\mathbb{H}^n}^{2\epsilon} \phi(0) \left( \frac{x_{\epsilon,M}}{\epsilon}, \frac{y_{\epsilon,M}}{\epsilon} \right), \left( \frac{x_{\epsilon,M}}{\epsilon}, \frac{y_{\epsilon,M}}{\epsilon} \right) \right) \right) + o(\epsilon^2)
\]

\[
\leq \left( \frac{1}{2(p - 2)C(n) + 1} \int_{B(0,\epsilon)} \phi + \frac{(p - 2)C(n)}{2(p - 2)C(n) + 1} \left[ \min_{B(0,\epsilon)} \phi + \max_{B(0,\epsilon)} \phi \right] - \phi(0) \right),
\]

while in the case \( 1 < p < 2 \) we end up getting

\[
\frac{\epsilon^2C(n)}{2(p - 2)C(n) + 1} \left( \Delta_{\mathbb{H}^n} \phi(0) + (p - 2)\left( D_{\mathbb{H}^n}^{2\epsilon} \phi(0) \left( \frac{x_{\epsilon,M}}{\epsilon}, \frac{y_{\epsilon,M}}{\epsilon} \right), \left( \frac{x_{\epsilon,M}}{\epsilon}, \frac{y_{\epsilon,M}}{\epsilon} \right) \right) \right) + o(\epsilon^2)
\]

\[
\geq \left( \frac{1}{2(p - 2)C(n) + 1} \int_{B(0,\epsilon)} \phi + \frac{(p - 2)C(n)}{2(p - 2)C(n) + 1} \left[ \min_{B(0,\epsilon)} \phi + \max_{B(0,\epsilon)} \phi \right] - \phi(0) \right).
\]

(20)

\[ \square \]

**Proof of Theorem 1.1.** Suppose that \( u \) satisfies the asymptotic expansion in the viscosity sense as in Definition 1.4. Let \( \phi \) be a smooth function such that \( u - \phi \)
Definition 1.4, proof of condition (ii). Which is condition (i) in the Definition 1.4. An analogous computation gives the
that is $u$, where the authors consider a very similar problem. Write
During the preparation of this manuscript, we learned about the paper Remark 1.

Their asymptotic mean value property is

$u(P) = \frac{\alpha}{2} \left( \min_{\frac{B(P, \epsilon)}{B(P, \epsilon)}} u + \max_{\frac{B(P, \epsilon)}{B(P, \epsilon)}} u \right) + \beta \int_{\frac{B(P, \epsilon)}{B(P, \epsilon)}} \Psi(P^{-1} \ast Q) u(Q) dQ + o(\epsilon^2), \quad (21)$

where the kernel $\Psi$ is given by

$\Psi(x, y, t) = \frac{x^2 + y^2}{(x^2 + y^2)^2 + t^2}.$
Note that our formula (1) is simpler than (21) without the presence of this kernel. But, of course, both asymptotic expansions are indeed equivalent.

REFERENCES

[1] N. Arcozzi and F. Ferrari, Metric normal and distance function in the Heisenberg group, Math. Z., 256 (2007), 661–684.
[2] N. Arcozzi and F. Ferrari, The Hessian of the distance from a surface in the Heisenberg group, Ann. Acad. Sci. Fenn. Math., 33 (2008), 35–63.
[3] N. Arcozzi, F. Ferrari and F. Montefalcone, CC-distance and metric normal of smooth hypersurfaces in sub-Riemannian two-step Carnot groups, preprint, arXiv:0910.5648v1.
[4] T. Bieske, Equivalence of weak and viscosity solutions to the $p$-Laplace equation in the Heisenberg group, Ann. Acad. Sci. Fenn. Math., 31 (2006), 363–379.
[5] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni, Stratified Lie groups and potential theory for their sub-Laplacians, Springer Monographs in Mathematics, Springer, Berlin, 2007.
[6] J.-M. Bony, Principe du maximum et inégalité de Harnack pour les opérateurs elliptiques dégénérés, in 1969 Séminaire de Théorie du Potentiel, dirigé par M. Brelot, G. Choquet et J. Deny: 1967/1968, Exp. 10, Secrétariat Mathématique, Paris, 1969, 20 pp.
[7] J.-M. Bony, Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés, Ann. Inst. Fourier (Grenoble), 19 (1969), 277–304.
[8] L. Capogna and G. Citti, Generalized mean curvature flow in Carnot groups, Comm. Partial Differential Equations, 34 (2009), 937–956.
[9] L. Capogna, D. Danielli, S. D. Pauls and J. T. Tyson, An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem, Progress in Mathematics, 259, Birkhäuser Verlag, Basel, 2007.
[10] B. Franchi, R. Serapioni and F. Serra Cassano, Rectifiability and perimeter in the Heisenberg group, Math. Ann., 321 (2001), 479–531.
[11] F. Ferrari, Q. Liu and J. J. Manfredi, On the horizontal mean curvature Flow for axisymmetric surfaces in the Heisenberg group, to appear in Commun. Contemp. Math.
[12] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
[13] C. Gutiérrez and E. Lanconelli, Classical viscosity and average solutions for PDE’s with nonnegative characteristic form, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., 15 (2004), 17–28.
[14] P. Juutinen, P. Lindqvist and J. J. Manfredi, On the equivalence of viscosity solutions and weak solutions for a quasi-linear elliptic equation, SIAM J. Math. Anal., 33 (2001), 699–717.
[15] H. Liu and X. Yang, Asymptotic mean value formula for sub-$p$-harmonic functions on the Heisenberg group, J. Funct. Anal., 264 (2013), 2177–2196.
[16] J. J. Manfredi, M. Parviainen and J. D. Rossi, An asymptotic mean value characterization for $p$-harmonic functions, Proc. Amer. Math. Soc., 138 (2010), 881–889.
[17] Y. Peres, O. Schramm, S. Sheffield and D. Wilson, Tug-of-war and the infinity Laplacian, J. Amer. Math. Soc., 22 (2009), 167–210.
[18] C. Pucci and G. Talenti, Elliptic (second-order) partial differential equations with measurable coefficients and approximating integral equations, Advances in Math., 19 (1976), 48–105.

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