Research Article

$(p, q)$-Growth of Meromorphic Functions and the Newton-Padé Approximant

Mohammed Harfaoui, Loubna Lakhmaili, and Abdellah Mourassil

University Hassan II Mohammedia, Laboratory of Mathematics, Cryptography, Mechanical and Numerical Analysis, F. S. T., BP 146, Mohammedia 20650, Morocco

Correspondence should be addressed to Mohammed Harfaoui; mharfaoui04@yahoo.fr

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In this paper, we have considered the generalized growth $(p, q)$-order and $(p, q)$-type in terms of coefficient of the development $p_n$ given in the $(n, n)$-th Newton-Padé approximant of meromorphic function. We use these results to study the relationship between the degree of convergence in capacity of interpolating functions and information on the degree of convergence of best rational approximation on a compact of $C$ (in the supremum norm). We will also show that the order of meromorphic functions puts an upper bound on the degree of convergence.

1. Introduction

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a nonconstant entire function and $M(f, r) = \max_{|z|=r} |f(z)|$.

It is well known that the function $r \mapsto \log(M(f, r))$ is an indefinitely increasing convex function of $\log(r)$. To estimate the growth of $f$ precisely, Boas (see [1]) has introduced the concept of order, defined by the number $\rho(0 \leq \rho \leq +\infty)$:

$$\rho = \limsup_{r \to +\infty} \frac{\log \log(M(f, r))}{\log(r)}.$$ (1)

It is known that the order and type of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ are given, respectively, by

$$\rho_f = \limsup_{n \to +\infty} \frac{n \ln n}{\ln |a_n|} \quad \left(\sigma_f = \frac{1}{\rho_f} \limsup_{n \to +\infty} n |a_n|^{1/n}\right).$$ (2)

The concept of type has been introduced to determine the relative growth of two functions of the same nonzero finite order. An entire function, of order $\rho$, $0 < \rho < +\infty$, is said to be of type $\sigma$, $0 \leq \sigma \leq +\infty$, if

$$\sigma = \limsup_{r \to +\infty} \frac{\log(M(f, r))}{r^{\rho}}.$$ (3)

If $f$ is an entire function of infinite or zero order, the definition of type is not valid and the growth of such function cannot be precisely measured by the above concept. Bajpai et al. (see [2]) have introduced the concept of index-pair of an entire function. Thus, for $p > q \geq 1$, they have defined the number

$$\rho(p, q) = \limsup_{r \to +\infty} \frac{\log(p)(M(f, r))}{\log(q)(r)},$$ (4)

where $b = 0$ if $p > q$ and $b = 1$ if $p = q$, where $\log^{(0)}(x) = x$, and $\log^{(p)}(x) = \log(\log^{(p-1)}(x))$, for $p \geq 1$.

The function $f$ is said to be of index-pair $(p, q)$ if $\rho(p - 1, q - 1)$ is a nonzero finite number. The number $\rho(p, q)$ is called the $(p, q)$-order of $f$.

Bajpai et al. have also defined the concept of the $(p, q)$-type $\sigma(p, q)$, for $b < \rho(p, q) < +\infty$, by

$$\sigma(p, q) = \limsup_{r \to +\infty} \frac{\log^{(p-1)}((M(f, r)))}{\log^{(q-1)}(r)^p(pq)}.$$ (5)

In their works, the authors established the relationship of $(p, q)$-growth of $f$ with respect to the coefficients $a_n$ in the Maclaurin series of $f$. 
We also have many results in terms of polynomial approximation in the classical case. Let $K$ be a compact subset of the complex plane $C$ of positive logarithmic capacity, and $f$ be a complex function defined and bounded on $K$. For $k \in \mathbb{N}$, put

$$E_n(K, f) = \| f - T_n \|_K,$$

where the norm $\| \cdot \|_K$ is the maximum on $K$ and $T_n$ is the $n$th Chebyshev polynomial of the best approximation to $f$ on $K$.

Bernstein showed (see [3], p. 14), for $K = [-1, 1]$, that there exists a constant $\eta > 0$ such that

$$\rho_n \bigg( \frac{E_n(K, f)}{n^{1/\rho}} \bigg)$$

is finite, and if only if $f$ is the restriction to $K$ of an entire function of order $\rho$ and some finite type.

This result has been generalized by Reddy (see [4, 5]) as follows:

$$\lim_{n \to \infty} \frac{E_n(K, f)}{n^{1/\rho}} = (\rho \eta)^{2^\rho},$$

if and only if $f$ is the restriction to $K$ of an entire function of order $\rho$ and type $\sigma$ for $K = [-1, 1]$.

In the same way Winiarski (see [6]) generalized this result to a compact $K$ of the complex plane $C$ of positive logarithmic capacity, denoted by $c = \text{cap}(K)$ as follows:

If $K$ is a compact subset of the complex plane $C$, of positive logarithmic capacity, then

$$\lim_{n \to \infty} n^{1/\rho} \sqrt[n]{E_n(K, f)} = c^\rho \rho \sigma,$$

if and only if $f$ is the restriction to $K$ of an entire function of order $\rho$ ($0 < \rho < +\infty$) and type $\sigma$.

Recall that the capacity of $[-1, 1]$ is $\text{cap}([-1, 1]) = 1/2$, and the capacity of a unit disc is $\text{cap}(D(0, 1)) = 1$.

The authors considered, respectively, the Taylor development of $f$ with respect to the sequence $(z_n)_{n}$ and the development of $f$ with respect to the sequence $(\omega_n)_{n}$ defined by

$$\omega_n(z) = \prod_{j=1}^{n+1} (z - \eta_{n_j}), \quad n = 1, 2, \ldots,$$

where $\eta = (\eta_0, \eta_1, \ldots, \eta_m)$ is the $n$th extremal points system of $K$ (see [6], p. 260).

We remark that the above results suggest that the rate at which the sequence $(\sqrt[n]{E_n(K, f)})_n$ tends to zero depends on the growth of the entire function (order and type).

Harfouf (see [7–9]) obtained a result of generalized order and type in terms of approximation in $L^p$-norm for a compact of $C^n$.

Remark 1. If we let $r_n$ range over the polynomials of degree $n$ instead of over the rational functions, we get the class of entire functions.

We need the following notations and lemma which will be used in the sequel (see [2]):

\begin{align}
(N) \text{For } p \in \mathbb{Z}, \text{ put} & \\
\log^{[p]}(x) &= \log (\log^{[p-1]}(x)) ; \\
\log^{[0]}(x) &= x ; A_{[p]} &= \prod_{k=1}^{p} \log^{[k]}(x) , \\
\exp^{[p]}(x) &= \exp (\exp^{[p-1]}(x)) ; \\
\exp^{[0]}(x) &= x , \\
E_{[p]}(x) &= \prod_{k=0}^{p} \exp^{k}(x) .
\end{align}

We will also show that the order of meromorphic functions puts an upper bound on the degree of convergence.

A relation between the degree of convergence (in capacity) of Padé approximants and the degree of best rational is derived for functions in Goncar’s class $\mathcal{B}_0$ (see [11]), where $\mathcal{B}_0$ is the class of functions $f$ such that on some compact circular disk $\Delta_0$ (depending on $f$) we have

$$\lim_{n \to \infty} \inf_{r_n \in \Delta_0} \sup_{z \in \Delta_0} (f - r_n)(z)^{1/n} = 0 ,$$

where $r_n$ ranges over the rational functions of type $n$ with poles off $\Delta_0$. 

2. Auxiliary Results: The Newton-Padé Approximants

First, we recall some definitions and notations which will be used later.

Definition 1. If $\Delta$ is a compact subset of $C$, we define its logarithmic capacity (transfinite diameter) by

$$\text{cap}(\Delta) = \lim_{n \to \infty} \left( \inf_{P_n} \| P_n \|_{\Delta} \right)^{1/n} ,$$

where $P_n$ ranges over all polynomials of degree $n$ with leading coefficient 1 and $\| P_n \|_{\Delta} = \sup_{z \in \Delta} | P_n(z) |$.

Let $\Delta$ be a compact subset of the complex plane $C$ such that $\text{cap}(\Delta) > 0$, and $f$ is a complex function defined and bounded on $\Delta$. For $n \in \mathbb{N}$, put (error of best rational approximation)

$$e_n(\Delta, f) = \inf_{r_n \in \Delta} \| f - r_n \|_{\Delta} .$$

We will denote by $\mathcal{R}$, the class of functions $f$, such that on some compact circular disk $\Delta$ (depending on $f$) we have

$$\lim_{n \to \infty} \sqrt[n]{e_n(\Delta, f)} = 0 ,$$

where $r_n$ ranges over the rational functions of type $n$ ($r_n = P_n/Q_n$) with poles off $\Delta$.

Remark 1. If we let $r_n$ range over the polynomials of degree $n$ instead of over the rational functions, we get the class of entire functions.
Let the function $f_{n,m}$ be the Newton–Padé approximant of the function $f$ with respect to the sequence $(z_n)_{n=1}^\infty$. In the sequel, we will consider the sequences of Newton–Padé approximants $(f_{n,m})$ with $m$ fixed and with $n$ tending to infinity. It will be useful to simplify the notations. Denote

$$ f_n = f_{n,m} = \frac{p_n}{q_n}, $$

where

$$ p_n(z) = \sum_{i=0}^n p_i z^i, $$

$$ q_n(z) = (z - z_{n,1}), \ldots, (z - z_{n,m}), $$

and

$$ \log^{[p]} = \log^{[1]} \cdot \log^{[2]} \cdot \ldots \cdot \log^{[p-1]} $$

The following results describe the asymptotic behavior of the approximants $f_{n,m}$.

**Lemma 1.** (see [2]).

With the above notations we have the following results:

$$ E_{-[p]}(x) = \frac{x}{\Lambda_{[p-1]}(x)}, $$

$$ \Lambda_{-[p]}(x) = \frac{x}{E_{[p-1]}(x)}, $$

$$ \frac{d}{dx} \exp^{[p]}(x) = \frac{E_{[p]}(x)}{x} = \frac{1}{\Lambda_{[p-1]}(x)}, $$

$$ \frac{d}{dx} \log^{[p]}(x) = \frac{E_{-[p]}(x)}{x} = \frac{1}{\Lambda_{[p-1]}(x)} $$

(16)

$$ E_{-[p]}^{(1)}(x) = \begin{cases} 
  x, & \text{if } p = 0, \\
  \log^{[p-1]} \left[ \log(x) - \log^{[2]}(x) + o(\log^{[3]}(x)) \right], & \text{if } p = 1, 2, \ldots,
\end{cases} $$

$$ \lim_{x \to +\infty} E_{[p-2]}(x) = \begin{cases} 
  e, & \text{if } p = 2, \\
  1, & \text{if } p \geq 3,
\end{cases} $$

$$ \lim_{x \to +\infty} \left[ \exp^{[p]}(E_{[p-2]}(x)) \right]^{1/x} = \begin{cases} 
  e, & \text{if } p = 2, \\
  1, & \text{if } p \geq 3.
\end{cases} $$

where $z_{n,1}, \ldots, z_{n,m}$ are the poles of the approximant $f_n$. Then, the polynomials $p_n$ and $q_n$ have no common divisors of degree higher than zero. Assume that

$$ |z_{n,1}| \leq \cdots \leq |z_{n,m}|. $$

(20)

**3. The $(p,q)$-Growth of Meromorphic Functions**

In our work we assume that $p > q \geq 3$.

Let $M_m(C)$ be the class of meromorphic functions whose number of poles is not greater than $m$. The main result of this paper is as follows:

**Lemma 2.** Let $(z_n)_{n=1}^\infty$ be a bounded sequence of complex numbers and let $f$ be a function meromorphic in $C$, holomorphic in a neighbourhood of the set $z_n : 1 \leq n < \infty$. Suppose that $f$ has exactly $m$ poles in $C$, counted with their multiplicities. Then,

(1) For almost every $n$ there exists the approximant $f_n$

(2) The poles of $f_n$ tend to the poles of $f$ when $n$ tends to infinity

(3) $\limsup_{n \to \infty} f_n(z) = f(z)$ in $C$, except for the poles of $f$

(4) $f$ can be extended to a function of the class $M_m(C)$

This lemma is a slight modification of the staff theorem, so we omit the proof.
Theorem 1. Let \((z_n)_{n=1}^{\infty}\) be a bounded sequence of complex numbers. Let \(\omega\) be a domain containing the set \(z_n: 1 \leq n < \infty\). Assume that there exists a limit point of the sequence \((z_n)\) in \(\omega\). Let \(f\) be a function meromorphic in \(\omega\) and holomorphic at each point of \(z_n\) for \(1 \leq n < \infty\). Assume that for almost every \(n\), there exists the \((n,m)\)-th Newton-Padé approximant \(f_n\), with respect to the sequence \((z_n)_{n=1}^{\infty}\) and that for some positive numbers \(\mu\) and \(\nu\)

\[
\limsup_{n \to \infty} \frac{\log^{|p-2|}(n)}{\log^{|p-2|}(-(1/n)\log|p_m|)} 
\leq \nu,
\]

(21)

Then,

(1) The order of \(f\) is not greater than \(\nu\).
(2) If \(\rho(f) = \nu\) then the type of \(f\) is not greater than \(\mu\).
(3) If

\[
\limsup_{n \to \infty} \frac{\log^{|p-2|}(n)}{\log^{|p-2|}(-(1/n)\log|p_m|)} = \nu,
\]

and if

\[
\limsup_{n \to \infty} |z_{n,m}|^{1/n} \leq 1,
\]

(23)

then \(\rho(f) = \mu\) and \(\sigma(f) = \nu\).

Proof. Let \(z \in C/D_\theta\), suppose that there exists a sequence \((n_i)\) and a neighbourhood \(U\) of the point \(z\) such that for every \(f\) the function \(f_n\) has no poles in \(U\). Then, it can be shown that \(\lim_{n \to \infty} f_n(z) = f(z)\). So, we have shown that \(\lim_{n \to \infty} f_n(z) = f(z)\) in \(C/D_\theta\) except for at most \(n\) points. We can choose a number \(R_0\) such that for every point

\[
z \in \left(C \setminus \overline{D_\theta}\right) B(0, R_0),
\]

(24)

Then

\[
M_f(R) \leq M_f(R_\theta) \leq \|f_n\| C(0, R_\theta)
\]

+ \[
\sum_{n=n_0+1}^{\infty} \|f_n - f_{n-1}\| C(0, R_\theta).
\]

(25)

According to (25), we have

\[
M_f(R) \leq A_1(R_\theta)^n + \sum_{n=n_0+1}^{\infty} |p_m| \theta^{-2mn}(R_\theta + s)^{m+n}.
\]

(26)

Let \(K\) be an arbitrary number greater than \(\mu\). Then, it follows from (21) that there exists a number \(n_1 \geq n_0\) such that

\[
|p_m| \leq \exp\left(-n \left(\frac{\exp[q-2]\log[q-2](n)}{\mu}\right)^{1/\nu}\right), \quad \text{for } n \geq n_1,
\]

(27)

if \(R\) is large enough.

\[
M_f(R) \leq A_2 R^n + \left(\theta^{-\mu} \cdot R\right)^n
\]

\[
\sum_{n=n_0+1}^{\infty} \exp\left(-n \left(\frac{\exp[q-2]\log[q-2](n)}{\mu}\right)^{1/\nu}\right) \left(\theta^{-3n} \cdot R\right)^n,
\]

(28)

where \(A_2\) depends only on \(\theta\).

Let \(n_\theta\) be the smallest integer greater than \(\exp[|\theta-1| (2\theta^{-3\mu} \cdot R)^\mu]\). Then, \(n_\theta\) is greater than \(n_1\), if \(R\) is large enough, and the sum \(\sum_{n=n_0+1}^{\infty} |p_m| (\theta^{-3n} \cdot R)^n\) is smaller than 1. Consequently,

\[
M_f(R) \leq A_2 R^n + \left(\theta^{-\mu} \cdot R\right)^n
\]

\[
\cdot \left(n_\theta \cdot \max \left\{\exp\left(-n \left(\frac{\exp[q-2]\log[q-2](n)}{\mu}\right)^{1/\nu}\right) \right\} \right.
\]

\[
\cdot \left(\theta^{-3n} \cdot R\right)^n + 1\right)
\]

(29)

when \(R\) is large enough. From (29), we get

\[
M_f(R) \leq A_2 R^n + \left(\theta^{-\mu} \cdot R\right)^n
\]

\[
\cdot \left(n_\theta \cdot \left(R \exp\left(-\exp[q-2](\log(R) - \epsilon)\right)\right)^n\right),
\]

(30)

where \(A_2\) depends only on \(\theta, \mu, \) and \(K\). Therefore, we can show that using the general formula of a \((\rho, q)\) type

\[
\left(R \exp\left(-\exp[q-2](\log(R) - \epsilon)\right)\right)^n \leq \exp[|\theta-1| (K \log[q-1] (R)^\mu - \epsilon)],
\]

(31)

which implies that the order of \(f\) is not greater than \(\mu\) and if \(\rho(f) = \mu\), then the type of \(f\) does not exceed \(K\), consequently not greater than \(\nu\). This proves 1 and 2.

Now, assume that the conditions (22) and (23) are satisfied. Then, of course, \(f\) can be extended to a function of the class \(M_m(C)\). Then, we can write \(f = \varphi/Q\), where \(\varphi\) is an entire function and \(Q\) is a polynomial of the form

\[
Q(z) = (t - \xi_1), \ldots, (t - \xi_k),
\]

(32)

where \(k\) is the number of poles of \(f\). Then, of course, the order of \(\varphi\) is equal to the order of \(f\) and the type of \(\varphi\) is equal to the type of \(f\).

Assume that either the order of \(f\) is smaller than \(\mu\) or the type of \(f\) is smaller than \(\nu\). Then, there exist a number \(K < \nu\), such that
By the Poisson–Jensen formula, we have
\[ |\varphi(z)| \leq \exp^{[p-1]}(K \log^{[q-1]}(z)), \tag{33} \]
when |z| is large enough. Using the Cauchy formula we get from (17) and (32),
\[ p_m = \frac{1}{2\pi i} \int_{C(\alpha r)} \frac{\varphi(z)Q_n(z)}{\log^{[q+1]}(z)} \, dz, \tag{34} \]
for \( r > s \). Using (17), (18), and (33), we obtain the estimation
\[ |p_m| \leq r \cdot 2m^{[p-2]} \frac{p+1}{K} \log^{[q+1]}(r), \tag{35} \]
when \( r \) is large enough. Put \( r = \exp^{[p-1]}(\log^{[p-2]}(n)/K) \). Then, for almost every \( n \), the estimation (35) is true. Hence, we derive
\[ |p_m| \leq \exp\left(-n \exp^{[p-2]}(\frac{\log^{[p-2]}(n)}{K})^{1/n}\right), \tag{36} \]
and this contradicts the assumed equality (22). We have proved 3.6.

\[ \square \]

4. Best Rational Approximation in Terms of \((p, q)\)-Growth

The aim of this section is to give a generalisation of the following theorems (see [11]).

**Theorem 2.** Let \( f \) be a meromorphic function of order at most \( \rho \), \( 0 < \rho < \infty \). Then,
\[ n^{1/2} \epsilon_{n}(\Delta, f) \leq \frac{1}{m}, \forall \alpha \tag{37} \]

**Remark 2.** A function \( f \) is entire of order at most \( \rho \), \( 0 < \rho < \infty \), if and only if
\[ n^{1/2} \epsilon_{n}(\Delta, f) \leq \frac{1}{m}, \forall \alpha \tag{38} \]
where \( n \) are replaced by polynomials.

**Theorem 3.** Let \( f \) be a meromorphic function of order \( \leq \rho(p, q) \), \( 0 \leq \rho(p, q) \leq \infty \). Then,
\[ \limsup_{n \to \infty} \frac{\log^{[p-2]}(n)^{\alpha}}{\log^{[q-1]}(-1/n) \log^{(\epsilon_{n}(\Delta, f))}} \leq 1. \tag{39} \]

**Proof.** By the Poisson–Jensen formula, we have
\[ \log \left| \frac{Q_{n} - P_{n}}{z^{2n+1}} \right| = \frac{1}{2\pi} \int_{0}^{2\pi} \log \left| \frac{Q_{n} - P_{n}}{R^{e^{i\theta}}} \right| \, d\theta, \]
\[ R^{2} - R^{2} + 2\pi R \cos(\theta - \phi) \]
\[ + \sum_{b | z - b_{r}} \log \left| \frac{R^{2} - b_{r}z}{R(z - a_{r})} \right| \]
\[ - \sum_{b | z - b_{r}} \log \left| \frac{R^{2} - a_{r}z}{R(z - a_{r})} \right|, \tag{40} \]
where \( z = re^{i\theta} \) and \( a_{r} \) and \( b_{r} \) are the zeros and poles, respectively, of \( fQ_{n} - P_{n} \). Let \( Q_{n}(z) = \prod \{z - z_{n}\} \). Since \( P_{n} \) is the \( n \)th Taylor polynomial to \( fQ_{n} \) and hence majorized by a constant times \( |Q_{n}| \) near the origin, we have on \( |\omega| = R : |P_{n}(\omega) \leq \prod \{1 + |z_{n}|\} |(R \cos)^{\alpha} \) by the Walsh–Bernstein lemma. With the usual notation of the Nevanlinna theory,
\[ m(R) = \frac{1}{2\pi} \int_{0}^{2\pi} \log^{+} |f(R^{e^{i\theta}})| \, d\theta, \tag{41} \]
\[ N(R) = \log \prod_{b | R} \left| \frac{R}{b_{r}} \right|, T(R) = N(R) + m(R), \]
by replacing the integrand with \( \log^{+} |f| + \log(\log^{+} R) \prod \{1 + |z_{n}|\} - \log R^{2n-1} \) and integrating, using the fact that the Poisson kernel had integral 1 and is bounded for \( r < R \), we get
\[ \log \left| \frac{Q_{n} - P_{n}}{z^{2n+1}} \right| \leq \log r^{2n-1} - \log R^{2n-1} + \log(\log^{+} R) \prod \{1 + |z_{n}|\} \]
\[ + \log \left( \log^{+} R \prod \{1 + |z_{n}|\} \right) \]
\[ + \sum_{b | R} \log \left( \frac{2R}{b_{r} + 2} \right) \]
\[ + \sum_{b | R} \log \left( \frac{2R}{e} \right), \tag{42} \]
if \( |z - b_{r}| > e \). Now, if \( f \) is of order \( \leq \rho \), we have by definition that \( T(R) \leq R^{1/\alpha} \) for any \( \alpha < \rho^{-1} \) for sufficiently large \( R \), and we get
\[ \log \left| \frac{Q_{n} - P_{n}}{z^{2n+1}} \right| \leq \log R^{2n-1} \log \mathrm{const} \log \mathrm{const} R^{\alpha} \prod \{1 + |z_{n}|\} \log R^{2n-1} \]
\[ + \sum_{b | R} \log 2R - \sum_{b | R} \log e. \tag{43} \]
We take \( R = \exp^{\alpha-1} \log^{[p-2]}(n)^{\alpha} \) for so small \( r \), and the two sums will disappear, and then subtract \( \log (Q_{n}) \) to get
\[
\log(f - (P_n/Q_n))(z) \leq \log(r^{2n+1}2^n/\exp^{q-1}(\log^{p-2}(n)^{a^{x+1}} + n\log \text{const} - \log \delta))\quad \text{except when } |Q_n(z)| \leq \delta.
\]

Exponentiating, we get
\[
\epsilon_n(\Delta, f) = \left| \left( f - \frac{P_n}{Q_n} \right)(z) \right|.
\]

Then,
\[
-\frac{1}{n}\log(\epsilon_n(\Delta, f)) \geq \log\left(r^{(1/n)+2}\text{const} + \exp^{q-2}(\log^{p-2}(n)^{a^{x+1}})\right)^{1/(1/n)},
\]
for \(n\) assez grand
\[
-\frac{1}{n}\log(\epsilon_n(\Delta, f)) \geq \exp^{q-2}(\log^{p-2}(n)^{a^{x+1}}),
\]
\[
\frac{\log(\log^{p-2}(n)^{a^{x+1}})}{\log^{q-1}((-1/n)\log(\epsilon_n(\Delta, f)))} \leq 1.
\]

Hence, the theorem is proved. \(\Box\)

**Theorem 4.** Let \(f\) be a meromorphic function of finite order \(\rho(p, q)\), \(0 < \rho(p, q) < \infty\), and finite type. Then,
\[
\limsup_{n \to \infty} \log^{[q-2]}\left(\frac{\log^{[p-2]}(n)}{((-1/n)\log(\epsilon_n(\Delta, f)))}\right) \leq c,
\]
where \(c\) is a constant.

**Proof.** For the proof we use exactly the same step of Theorem 3. \(\Box\)

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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