ON ALZER-KWONG’S IDENTITIES FOR BERNOUlli POLYNOMIALS

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Abstract. In this paper, we prove new identities for Bernoulli polynomials that extend Alzer and Kwong’s results. The key idea is to use the Volkenborn integral over \( \mathbb{Z}_p \) of the Bernoulli polynomials to establish recurrence relations on the integrands. Also, some known identities are obtained by our approach.

1. Introduction

Bernoulli polynomials play fundamental roles in various branches of mathematics including combinatorics, number theory, special functions and analysis, see for example [2, 4, 8, 9]. Let \( \mathbb{N} = \{1, 2, \ldots\} \) and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). The Bernoulli polynomials \( B_n(x) \) are usually defined by the generating function

\[
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi).
\]

And the Bernoulli numbers \( B_n \) can be defined by \( B_n = B_n(0) \). It is well-known that \( B_n = 0 \) for any odd \( n > 1 \). These numbers appeared for the first time in Jakob Bernoulli’s book *Ars Conjectandi*, which was published posthumously in 1713. The polynomials \( B_n(x) \) obey the relation \( B_n(x) = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} B_j \). Lehmer [5] showed that the Bernoulli polynomials satisfy the relations \( B_n(1) = (-1)^n B_n(0) \) and \( B_n(1-x) = (-1)^n B_n(x) \). It is well known that the Bernoulli polynomials have the binomial expansion \( B_n(x + y) = \sum_{j=0}^{n} \binom{n}{j} B_j(x)y^{n-j} \).

Alzer and Kwong’s [11] paper was inspired by interesting research note published by Kaneko [3] in 1995. The aim of this paper is to prove the following new identities for Bernoulli polynomials that extend Alzer and Kwong’s results (see [11]). In Section 2 we prove the following result.

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Theorem 1.1.  

(1) For $m \in \mathbb{N}$ and $\nu \in \mathbb{N}_0$ with $0 \leq \nu \leq m$, we have

$$\sum_{k=0}^{m-1} \binom{m}{k} \binom{k+m}{\nu} \binom{k+m-\nu}{m-\nu} B_k(x)$$

$$= \frac{1}{2} \sum_{j=0}^{m-1} (-1)^{j+m+1} \binom{m}{j+1} \binom{j+m}{\nu} \binom{j+m-\nu}{m-\nu} (j+m+1)x^j.$$

(2) For $m \in \mathbb{N}$ with $0 \leq \nu \leq m-1$, we have

$$\sum_{k=0}^{m-1} \binom{m}{k} \binom{k+m}{\nu} \binom{k+m-\nu}{m-\nu-1} B_{k+1}(x)$$

$$= \frac{1}{2} \sum_{j=0}^{m} (-1)^{j+m} \binom{m}{j} \binom{j+m-1}{\nu} \binom{j+m-\nu-1}{m-\nu-1} (j+m)x^j.$$

(3) For $m \in \mathbb{N}$ with $0 \leq \nu \leq m-1$ and $0 \leq \ell \leq m-\nu-1$, we have

$$\sum_{k=0}^{m-1} \binom{m}{k} \binom{k+m}{\nu} \binom{k+m-\nu}{\ell} B_{k+m-\nu-\ell}(x)$$

$$= \frac{1}{2} \sum_{j=0}^{m} (-1)^{j+m} \binom{m}{j} \binom{j+m-1}{\nu} \binom{j+m-\nu-1}{\ell} \times (j+m)x^{j+m-\nu-\ell-1}.$$

(4) For $m \in \mathbb{N}$ with $0 \leq \nu \leq m$, and $0 \leq \ell \leq m-1$ we have

$$\sum_{k=0}^{m-1} \binom{m}{k} \binom{k+m}{\nu} \binom{k+m-\nu}{\ell+m-\nu} B_{k-\ell}(x)$$

$$= \frac{1}{2} \sum_{j=0}^{m-1} (-1)^{j+m+1} \binom{m}{j+1} \binom{j+m}{\nu} (j+m+1)x^{j-\ell}.$$

Setting $\nu = 0$ in (3) and (4) of Theorem 1.1, we can derive the following corollaries:
Corollary 1.2. (1) For \( m \in \mathbb{N} \) and \( 0 \leq \ell \leq m - \nu - 1 \), we have
\[
\sum_{k=0}^{m-1} \binom{m}{k} \binom{k + m}{\ell} B_{k+m-\ell}(x) \quad \frac{1}{2} \sum_{j=0}^{m} (-1)^{j+m} \binom{m}{j} \binom{j + m - 1}{\ell} (j + m)x^{j+m-\ell-1}.
\]

(2) For \( m \in \mathbb{N} \) and \( 0 \leq \ell \leq m - 1 \) we have
\[
\sum_{k=\ell}^{m-1} \binom{m}{k} \binom{k + m}{\ell + m} B_{k-\ell}(x) \quad \frac{1}{2} \sum_{j=\ell}^{m-1} (-1)^{j+m+1} \binom{m}{j+1} (j + m + 1)x^{j-\ell}.
\]

Setting \( \ell = 0 \) in (3) of Theorem 1.1, we can derive the following identity of Alzer and Kwong [1, Theorem 1]:

Theorem 1.3 (Alzer-Kwong). Let \( m \in \mathbb{N} \) with \( 0 \leq \nu \leq m \). Then we have
\[
\sum_{k=0}^{m-1} \binom{m}{k} \binom{k + m}{\nu} B_{k+m-\nu}(x) \quad \frac{1}{2} \sum_{j=0}^{m} (-1)^{j+m} \binom{m}{j} \binom{j + m - 1}{\nu} (j + m)x^{j+m-\nu-1}.
\]

By setting \( x = 0 \) in (1), (2), (3) and (4) of Theorem 1.1, since \( 0^j = 1 \) if \( j = 0 \) and \( 0^j = 0 \) if \( j \in \mathbb{N} \), we take the following identities of of Alzer and Kwong [1, Theorem 2]:

Theorem 1.4 (Alzer-Kwong). (1) For \( 0 \leq \nu \leq m \), we have
\[
\sum_{k=0}^{m-1} \binom{m}{k} \binom{k + m}{\nu} \binom{k + m - \nu}{m - \nu} B_k = (-1)^{m+1} \frac{m(m+1)}{2} \binom{m}{\nu}.
\]

(2) For \( 0 \leq \nu \leq m - 1 \), we have
\[
\sum_{k=0}^{m-1} \binom{m}{k} \binom{k + m}{\nu} \binom{k + m - \nu}{m - 1 - \nu} B_{k+1} = (-1)^{m} \frac{m}{2} \binom{m-1}{\nu}.
\]
(3) For $0 \leq \ell \leq m - \nu - 2$, we have
\[\sum_{k=0}^{m-1} \binom{m}{k} \binom{k+m}{\nu} \binom{k+m-\nu}{\ell} B_{k+m-\nu-\ell} = 0.\]

(4) For $0 \leq \ell \leq m - 1$ and $0 \leq \nu \leq m$, we have
\[\sum_{k=\ell}^{m-1} \binom{m}{k} \binom{k+m}{\nu} \binom{k+m-\nu}{\ell+m-\nu} B_{k-\ell} = (-1)^{\ell+m+1} \frac{\ell + m + 1}{2} \binom{m}{\ell+1} \binom{\ell + m}{\nu}.\]

The following identity is due to Wu, Sun and Pan [10, Theorem 2, (6)].

**Theorem 1.5** (Wu-Sun-Pan).
\[\sum_{k=0}^{m} \binom{m}{k} B_{n+k}(x) = (-1)^{n+m} \sum_{k=0}^{n} \binom{n}{k} B_{m+k}(-x),\]
where $m$ and $n$ are positive integers.

**Theorem 1.6.** We have
\[\sum_{j=0}^{m+q} \binom{m+q}{j} (n+q+j) B_{n+q+j-1}(x) = -(-1)^{m+n} \sum_{k=0}^{n+q} \binom{n+q}{k} (m+q+k) B_{m+q+k-1}(-x),\]
where $q, m$ and $n$ are nonnegative integers and $m+n > 0$.

**Remark 1.7.** Substituting $q = 1$ and $x = 0$ into Theorem 1.6 we can derive the extension of Kaneko’s [3] given by Momiyama [6]. It was proved by using a $p$-adic integral over $\mathbb{Z}_p$. The Kaneko identity is stated as follows
\[\sum_{j=0}^{n+1} \binom{n+1}{j} \tilde{B}_{n+j} = 0,\]
where $\tilde{B}_n = (n+1)B_n$. 
Theorem 1.8. We have
\[ \sum_{j=0}^{m} \binom{m}{j} \binom{n+j}{\nu} B_{n+j-\nu}(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{m+k}{\nu} B_{m+k-\nu}(x+1), \]
where \( \nu, m \) and \( n \) are nonnegative integers and \( m+n > 0 \).

Sun [9, Theorem 1.2, (1.15)] derived the next identity on Bernoulli polynomials (see also [2, Theorem 5.1]). This identity can be verified by our approach.

Theorem 1.9 (Sun). We have
\[ (-1)^{m} \sum_{j=0}^{m} \binom{m}{j} x^{m-j} B_{n+j}(y) = (-1)^{n} \sum_{k=0}^{n} \binom{n}{k} x^{n-k} B_{m+k}(z), \]
where \( x + y + z = 1 \).

2. Proofs of Theorem 1.1, 1.5 and 1.6

Throughout this section \( \mathbb{Z}_p, \mathbb{Q}_p \) and \( \mathbb{C}_p \) will, respectively, denote the ring of \( p \)-adic integers, the field of \( p \)-adic rational numbers and the completion of algebraic closure of \( \mathbb{Q}_p \).

For the fundamental properties of \( p \)-adic integrals and \( p \)-adic distributions, which are given briefly below, we may refer the references [4, 7, 8] and the references cited therein.

The Volkenborn integral of a function \( f : \mathbb{Z}_p \rightarrow \mathbb{C}_p \) is defined by
\[ \int_{\mathbb{Z}_p} f(t) dt = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{j=0}^{p^N-1} f(j) \]
and that this limit exists if \( f \) is uniformly (or strictly) differentiable on \( \mathbb{Z}_p \). A function \( f : X \rightarrow \mathbb{C}_p \) is uniformly differentiable on \( X \subset \mathbb{C}_p \) (assumed not to have isolated points), denoted by \( f \in UD(\mathbb{Z}_p) \), if at all points \( a \in X \)
\[ \lim_{(x,y) \rightarrow (a,a)} \frac{f(x) - f(y)}{x - y} \]
exists, the limit being restricted to \( x, y \in X, x \neq y \) (see [7, p. 218]).

Let \( S \subset \mathbb{C}_p \) be an arbitrary subset closed under \( x \rightarrow x + t \) for \( t \in \mathbb{Z}_p \) and \( x \in S \). That is, \( S \) could be \( \mathbb{C}_p \setminus \mathbb{Z}_p, \mathbb{Q}_p \setminus \mathbb{Z}_p \) or \( \mathbb{Z}_p \). Suppose \( f : S \rightarrow \mathbb{C}_p \) is strictly differentiable on \( S \), so that for fixed \( x \in S \) the function \( t \rightarrow f(x + t) \) is uniformly differentiable on \( \mathbb{Z}_p \).
For $f \in UD(\mathbb{Z}_p)$, the Volkenborn integral

$$F(x) = \int_{\mathbb{Z}_p} f(x+t)dt, \quad (x \in S)$$

is given then satisfied the equation

$$F(x+1) - F(x) = f'(x)$$

(see, e.g., [7, p. 265] and [8, Proposition 55.5(ii)]). From (2.4), we can be written as

$$F(x + q) - F(x + q - 1) = f'(x + q - 1),$$

where $x \in S$ and $q \in \mathbb{N}$. It is easily checked that

$$F(x + n) - F(x) = \sum_{i=0}^{n-1} f'(x + i)$$

with $x \in S$ and $n \in \mathbb{N}$.

In order to prove Theorem 1.1, 1.5 and 1.6, we need the following lemmas. It should be noted that the following lemma was obtained by Schikhof in [8, Proposition 55.7].

**Lemma 2.1.** Let $f \in UD(\mathbb{Z}_p)$. Then we have the functional equation

$$\int_{\mathbb{Z}_p} f(-t)dt = \int_{\mathbb{Z}_p} f(t+1)dt = \int_{\mathbb{Z}_p} f(t)dt + f'(0).$$

In particular, if $f$ is an odd function, then

$$\int_{\mathbb{Z}_p} f(t)dt = -\frac{1}{2}f'(0).$$

It is worth nothing that the Witt’s formula for $B_n(x)$ is indeed efficient in deriving recurrence relations for the ordinary Bernoulli polynomials in an elementary way.

**Lemma 2.2** (Witt’s formula for $B_n(x)$). For any $n \in \mathbb{N}_0$, we have

$$\int_{\mathbb{Z}_p} (x+t)^n dt = B_n(x).$$

It is known that the Witt’s formula for $B_n(x)$ is a power tool in the study of the Bernoulli numbers, Bernoulli polynomials and its generalization, $p$-adic analytic number theory, etc. (see [4, 7]).
2.1. **Proof of Theorem 1.1.** Note that

\[
\left( \frac{d}{dt} \right)^\nu t^m = \begin{cases} 
\nu! \binom{m}{\nu} t^{m-\nu} & \text{if } m \geq \nu, \\
0 & \text{otherwise.}
\end{cases}
\]

For \( x \in S \), we define

\[ R(t; x) := (x + t)^m(x + t - 1)^m \]
on \( \mathbb{Z}_p \). Thus, by the binomial expansion, we find

\[ R(t; x) = \sum_{k=0}^{m} (-1)^{k+m} \binom{m}{k} (x + t)^{k+m} \]
and

\[ R(t + 1; x) = \sum_{k=0}^{m} \binom{m}{k} (x + t)^{k+m}. \]

Let \( D_t \) be a differentiation operator. Then we set

\[ R^{(\nu)}(t; x) = D_t^\nu R(t; x) = \left( \frac{\partial}{\partial t} \right)^\nu R(t; x). \]

It is clear from the definition that

\[ R^{(\nu)}(t + 1; x) - R^{(\nu)}(t; x) = \nu! \sum_{k=0}^{m} \binom{m}{k} \binom{k + m}{\nu} (x + t)^{k+m-\nu} \left[ 1 - (-1)^{k+m} \right]. \]

Since \( 1 - (-1)^{k+m} = 0 \) if \( k \) and \( m \) have same parity, we have

\[ R^{(\nu)}(t + 1; x) - R^{(\nu)}(t; x) = \begin{cases} 
2\nu! \sum_{k=0}^{m-1} \binom{m}{k} \binom{k+m}{\nu} (x + t)^{k+m-\nu} & \text{if } k + m \text{ odd}, \\
0 & \text{if } k + m \text{ even.}
\end{cases} \]

Similarly, from

\[ \binom{j + m}{\nu}(j + m - \nu) = \binom{j + m - 1}{\nu}(j + m), \]
we have

\[(2.9)\]
\[R^{(\nu+1)}(0; x) = R^{(\nu+1)}(t; x)\bigg|_{t=0}\]
\[= \nu! \sum_{j=0}^{m} (-1)^{j+m} \binom{m}{j} \binom{j + m}{\nu} (j + m - \nu)(x + t)^{j+m-\nu-1}\bigg|_{t=0}\]
\[= \nu! \sum_{j=0}^{m} (-1)^{j+m} \binom{m}{j} \binom{j + m - 1}{\nu} (j + m)x^{j+m-\nu-1}.
\]

(1) Let \(k + m\) be an odd integer. Applying \(D_t^{m-\nu}\) to both side of equation \[(2.7)\], we see that

\[(2.10)\]
\[D_t^{m-\nu}\left[R^{(\nu)}(t+1; x) - R^{(\nu)}(t; x)\right]\]
\[= 2\nu!(m - \nu)! \sum_{k=0}^{m-1} \binom{m}{k} \binom{k + m - 1}{\nu} (k + m - \nu)(x + t)^k.\]

On the other hand, recalling \[(2.9)\], we get

\[(2.11)\]
\[R^{(\nu+1)}(t; x) = \nu! \sum_{j=1}^{m} (-1)^{j+m} \binom{m}{j} \binom{j + m}{\nu} (j + m - \nu)(x + t)^{j+m-\nu-1}\]
\[+ (-1)^{m}\nu! \binom{m}{\nu} (m - \nu)(x + t)^{m-\nu-1}.\]

Since \(m - \nu > m - \nu - 1\), the second term of right-hand side of \[(2.11)\] gives the identity

\[D_t^{m-\nu}\left((-1)^{m}\nu! \binom{m}{\nu} (m - \nu)(x + t)^{m-\nu-1}\right) = 0.\]

Thus, from \[(2.11)\], we have

\[(2.12)\]
\[D_t^{m-\nu}\left[R^{(\nu+1)}(t; x)\right] = \nu!(m - \nu)! \sum_{j=0}^{m-1} (-1)^{j+m+1} \binom{m}{j+1} \binom{j + m + 1}{\nu} \]
\[\times \binom{j + m - \nu}{m - \nu} (j + m - \nu + 1)(x + t)^j.\]
Moreover,

\[
D^{m-\nu}_t \left[ R^{(\nu+1)}(0; x) \right] = \nu!(m - \nu)! \sum_{j=0}^{m-1} (-1)^{j+m+1} \binom{m}{j+1} \left( \binom{j + m + 1}{\nu} \right) (j + m - \nu) (x+t)^j.
\]

In particular,

\[
\int_{\mathbb{Z}} D^{m-\nu}_t \left[ R^{(\nu)}(t + 1; x) - R^{(\nu)}(t; x) \right] dt = D^{m-\nu}_t \left[ R^{(\nu+1)}(0; x) \right]
\]

by using Lemma 2.1. On expanding (2.14) by (2.10) and (2.13), we obtain

\[
\sum_{k=0}^{m-1} \binom{m}{k} \left( \binom{k + m}{\nu} \right) \left( \binom{k + m - \nu}{m - \nu} \right) \int_{\mathbb{Z}} (x+t)^k dt
\]

\[
= \frac{1}{2} \sum_{j=0}^{m-1} (-1)^{j+m+1} \binom{m}{j+1} \left( \binom{j + m + 1}{\nu} \right) \left( \binom{j + m - \nu}{m - \nu} \right) (j + m + 1) (x+t)^j,
\]

since

\[
\left( \binom{j + m + 1}{\nu} \right) (j + m - \nu + 1) = \left( \binom{j + m}{\nu} \right) (j + m + 1).
\]

Therefore, Part (1) follows from Lemma 2.2.

(2) From (2.7) and (2.9), we find

\[
D^{m-\nu}_t \left( R^{(\nu)}(t + 1; x) - R^{(\nu)}(t; x) \right)
\]

\[
= 2\nu!(m - \nu - 1)! \sum_{k=0}^{m-1} \binom{m}{k} \left( \binom{k + m}{\nu} \right) \left( \binom{k + m - \nu}{m - \nu - 1} \right) (x+t)^{k+1}
\]

and

(2.16)

\[
D^{m-\nu}_t \left( R^{(\nu+1)}(t; x) \right) = \nu!(m - \nu - 1)! \sum_{j=0}^{m} (-1)^{j+m} \binom{m}{j} \left( \binom{j + m - 1}{\nu} \right) \left( \binom{j + m - \nu - 1}{m - \nu - 1} \right) (j + m)(x+t)^j.
\]
Combining Lemma 2.1 with (2.15) and (2.16), we get
\[
\sum_{\substack{k=0 \\
    k+m \text{ odd}}}
\binom{m}{k}\binom{k+m}{\nu}(k+m-\nu) \int_{\mathbb{Z}_p} (x+t)^{k+1} dt \\
= \frac{1}{2} \sum_{j=0}^{m} (-1)^{j+m} \binom{m}{j}\binom{j+m-1}{\nu}\binom{j+m}{\nu} (j+m)x^j.
\]
So we have Part (2) by Lemma 2.2.

(3) From (2.7) and (2.9), we known that
\[
D_{t}^{\ell}(R^{(\nu)}(t + 1; x) - R^{(\nu)}(t; x)) \\
= 2\nu! \sum_{\substack{k=0 \\
    k+m \text{ odd}}}
\binom{m}{k}\binom{k+m}{\nu}(k+m-\nu) \int_{\mathbb{Z}_p} (x+t)^{k+m-\nu-\ell} dt
\]
and
\[
D_{t}^{\ell}(R^{(\nu+1)}(t; x)) = \nu! \sum_{j=0}^{m} (-1)^{j+m} \binom{m}{j}\binom{j+m-1}{\nu}\binom{j+m}{\nu} \binom{j+m-\nu-1}{\ell} \times (j+m-\nu)(x+t)^{j+m-\nu-\ell-1}.
\]

Combining Lemma 2.1 with (2.17), (2.18) and using (2.8), we get
\[
\sum_{\substack{k=0 \\
    k+m \text{ odd}}}
\binom{m}{k}\binom{k+m}{\nu}(k+m-\nu) \int_{\mathbb{Z}_p} (x+t)^{k+m-\nu-\ell} dt \\
= \frac{1}{2} \sum_{j=0}^{m} (-1)^{j+m} \binom{m}{j}\binom{j+m-1}{\nu}\binom{j+m}{\nu} \binom{j+m-\nu-1}{\ell} x^{j+m-\nu-\ell-1},
\]
which implies Part (3) by using Lemma 2.2.

(4) By (2.7) and (2.11), we have
\[
D_{t}^{\ell+m-\nu}(R^{(\nu)}(t + 1; x) - R^{(\nu)}(t; x)) \\
= 2\nu!(\ell + m - \nu)! \sum_{\substack{k=0 \\
    k+m \text{ odd}}}
\binom{m}{k}\binom{k+m}{\nu}(k+m-\nu) \binom{k+m-\nu}{\ell} (x+t)^{k-\ell}
\]
and
\[(2.20)\]
\[D^\nu \ell+m-n \left(R^{(\nu+1)}(t; x)\right) = \nu! (\ell + m - \nu)! \sum_{j=\ell}^{m-1} (-1)^{j+\nu+1} \binom{m}{j+1} \binom{\nu}{j+1} \times (m + j - \nu + 1) (x + t)^{j-\ell}.\]

Combining Lemma 2.1 with (2.19) and (2.20), we get
\[\sum_{k=0}^{m-1} \binom{m}{k} \binom{k + m - \nu}{\ell + m - \nu} \int_{\mathbb{Z}_p} (x + t)^{k-\ell} dt = \frac{1}{2} \sum_{j=\ell}^{m-1} (-1)^{j+\nu+1} \binom{m}{j+1} \binom{\nu}{j+1} (j + m + 1) (j + m + 1) (x + t)^{j-\ell},\]

the right hand side following from
\[\binom{j + m + 1}{\nu} (m + j - \nu + 1) = \binom{j + m}{\nu} (j + m + 1).\]

Hence, Part (4) is obtained by Lemma 2.2.

2.2. Proof of Theorem 1.5. Let \(m\) and \(n\) be positive integers. For \(x \in S\), we define
\[G(t; x) := (-1)^m (x + t)^m (x + t - 1)^n\]
on \(\mathbb{Z}_p\). By the binomial expansion, the formula \(G(-t; x)\) and \(G(t+1; x)\) can be rewritten as
\[G(-t; x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-x + t)^{m+k}\]
and
\[G(t + 1; x) = (-1)^m \sum_{k=0}^{m} \binom{m}{k} (x + t)^{n+k}.\]

Applying Lemma 2.1 to the above two equations, we have
\[(-1)^m \sum_{k=0}^{m} \binom{m}{k} \int_{\mathbb{Z}_p} (x + t)^{n+k} dt = (-1)^n \sum_{k=0}^{n} \binom{n}{k} \int_{\mathbb{Z}_p} (-x + t)^{m+k} dt.\]

The proof now follows directly from Lemma 2.2.
2.3. **Proof of Theorem 1.6**. Let \( m, n \) and \( q \) be nonnegative integers with \( m + n > 0 \). For \( x \in S \), we define 
\[
H(t; x) := (x+t)^{m+q}(x+t-1)^{n+q} + (-1)^{m+n}(-x+t)^{n+q}(-x+t-1)^{m+q}
\]
on \( \mathbb{Z}_p \). By the binomial expansion, the formula \( H(-t; x) \) and \( H(t+1; x) \) can be rewritten as 
\[
H(-t; x) = (-1)^{m+n}(-x+t)^{m+q}(-x+t+1)^{n+q} + (x+t)^{n+q}(x+t+1)^{m+q}
\]
and 
\[
H(t+1; x) = (x+t+1)^{m+q}(x+t)^{n+q} + (-1)^{m+n}(-x+t+1)^{n+q}(-x+t)^{m+q}.
\]
Since \( H(-t; x) = H(t+1; x) \), we have 
\[
-H'(t; x) = H'(t+1; x),
\]
where \( H' = \frac{\partial H}{\partial t} \). Hence, by Lemma 2.1, we obtain 
\[
- \int_{\mathbb{Z}_p} H'(t+1; x) dt = \int_{\mathbb{Z}_p} H'(-t; x) dt = \int_{\mathbb{Z}_p} H'(t+1; x) dt.
\]
By the above equation, we have
\[
(2.21) \quad \int_{\mathbb{Z}_p} H'(t+1; x) dt = 0.
\]
In particular, 
\[
(2.22) \quad H'(t+1; x) = \sum_{j=0}^{m+q} \binom{m+q}{j} (n+q+j)(x+t)^{n+q+j-1}
\]
\[
+ (-1)^{m+n} \sum_{k=0}^{n+q} \binom{n+q}{k} (m+q+k)(-x+t)^{m+q+k-1}.
\]
From equations (2.21) and (2.22), we get
\[
0 = \int_{\mathbb{Z}_p} H'(t+1; x) dt
\]
\[
= \sum_{j=0}^{m+q} \binom{m+q}{j} (n+q+j) \int_{\mathbb{Z}_p} (x+t)^{n+q+j-1} dt
\]
\[
+ (-1)^{m+n} \sum_{k=0}^{n+q} \binom{n+q}{k} (m+q+k) \int_{\mathbb{Z}_p} (-x+t)^{m+q+k-1} dt.
\]
Thus the result follows from Lemma 2.2.
2.4. **Proof of Theorem 1.8.** We begin by proving some binomial coefficient identities. We start with the identity

\[
0 = (x + t)^n(x + t + 1)^m - (x + t + 1)^m(x + t + 1 - 1)^n
\]

\[
= \sum_{j=0}^{m} \binom{m}{j} (x + t)^{n+j} - \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k}(x + t + 1)^{m+k},
\]

or equivalently,

\[
(2.23) \quad \sum_{j=0}^{m} \binom{m}{j} (x + t)^{n+j} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k}(x + t + 1)^{m+k}.
\]

Differentiating both sides of the identity (2.23) with respect to \(t\), \(\nu\) times, leads to following relation

\[
(2.24) \quad \sum_{j=0}^{m} \binom{m}{j} \binom{n+j}{\nu} (x+t)^{n+j-\nu} = \sum_{k=0}^{n} (-1)^{n-k}(x + t + 1)^{m+k-\nu}.
\]

Applying Lemma 2.1 to (2.24), we have

\[
\sum_{j=0}^{m} \binom{m}{j} \binom{n+j}{\nu} \int_{\mathbb{Z}_p} (x + t)^{n+j-\nu} dt = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{m+k}{\nu} \int_{\mathbb{Z}_p} (x + t + 1)^{m+k-\nu} dt.
\]

From Lemma 2.2 this gives the formula of our theorem.

2.5. **Proof of Theorem 1.9.** Let \(x + y + z = 1\). For \(x \in S\), we define

\[
L(t; x) := (-1)^m(y - 1 + t)^n(y + x - 1 + t)^m
\]

on \(\mathbb{Z}_p\). By the binomial expansion, we have

\[
(2.25) \quad L(t + 1; x) = (-1)^m \sum_{j=0}^{m} \binom{m}{j} x^{m-j}(t + y)^{n+j},
\]

and since \(x + y + z = 1\),

\[
L(-t; x) = (-1)^n(t + x + z)^n(t + z)^m
\]

\[
(2.26) \quad = (-1)^n \sum_{k=0}^{n} \binom{n}{k} x^{n-k}(t + z)^{m+k}.
\]
Applying Lemma 2.1 to (2.25) and (2.26), we have

\[(2.27)\]

\[(-1)^m \sum_{j=0}^{m} \binom{m}{j} x^{m-j} \int_{\mathbb{Z}_p} (y+t)^{n+j} dt = (-1)^n \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \int_{\mathbb{Z}_p} (z+t)^{m+k} dt.\]

Therefore, the result follows from Lemma 2.2.

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