Arithmetic area for $m$ planar Brownian paths

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Received 31 January 2012
Accepted 2 April 2012
Published 11 May 2012

Online at stacks.iop.org/JSTAT/2012/P05005
doi:10.1088/1742-5468/2012/05/P05005

Abstract. We pursue the analysis made in Desbois and Ouvry (2011 J. Stat. Mech. P05024) on the arithmetic area enclosed by $m$ closed Brownian paths. We pay particular attention to the random variable $S_{n_1,n_2,...,n_m}(m)$, which is the arithmetic area of the set of points, also called winding sectors, enclosed $n_1$ times by path 1, $n_2$ times by path 2, . . . , and $n_m$ times by path $m$. Various results are obtained in the asymptotic limit $m \to \infty$. A key observation is that, since the paths are independent, one can use in the $m$-path case the SLE information, valid in the one-path case, on the zero-winding sectors arithmetic area.

Keywords: stochastic processes (theory)

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1. Introduction

In [1], the asymptotic behaviour of the average arithmetic area enclosed by the external frontier of \( m \) independent closed Brownian planar paths, of the same length \( t \) and starting from and ending at the same point, has been obtained using a path integral approach [2, 3].

In the one-path case, the random variable of interest happens to be the arithmetic area \( S_n \) of the \( n \)-winding sectors enclosed by the path, from which the total arithmetic area \( S = \sum_n S_n \) can be computed. An \( n \)-winding sector is by definition a set of points enclosed \( n \) times by the path. Path integral techniques [3] give \( \langle S_n \rangle = \frac{t}{2\pi n^2} \) but end up being somehow less adapted for \( \langle S_0 \rangle \), the average arithmetic area of the zero-winding sectors inside the path, i.e. of the set of points enclosed an equal number of times clockwise and anti-clockwise by the path. Indeed the path integral cannot distinguish zero-winding sectors inside the path from the outside of the path, which is also zero-winding. Other techniques have to be used, in the case at hand SLE techniques [4], to get \( \langle S \rangle = \frac{\pi t}{2} \) from which \( \langle S_0 \rangle = \frac{\pi t}{30} \) can be derived. It means that \( \langle S_0 \rangle = q \langle S - S_0 \rangle \) with \( q = 1/5 \).

For \( m \) independent paths, the same path integral techniques used in the one-path case have now to focus [1] on the random variable \( S_n(m) \), the arithmetic area of the \( n \)-winding sectors enclosed by the \( m \) paths, from which the total area \( S(m) = \sum_n S_n(m) \) can, in principle, be computed. An \( n \)-winding sector is again defined as a set of points enclosed \( n \) times by the \( m \) paths as illustrated in figure 1 in the \( m = 2 \) path case. One has found that the leading asymptotic term scales like \( \ln m \), namely that \( \langle S(m) - S_0(m) \rangle \sim (\pi t/2) \ln m \) with, as already stressed, no information on \( \langle S_0(m) \rangle \) inside the \( m \) paths.

In [5], on the other hand, the asymptotic behaviour of the arithmetic area enclosed by the convex envelope of \( m \) closed paths was found to be \( \langle S(m) \rangle_{\text{convex}} \sim (\pi t/2) \ln m \), i.e. the same scaling as \( \langle S(m) - S_0(m) \rangle \). One then concluded [1] that, necessarily, \( \langle S(m) \rangle \sim (\pi t/2) \ln m \) and that \( \langle S_0(m) \rangle \) is subleading.

This particular scaling might be of interest for spatial ecology considerations where one asks about the increase in size of a natural reserve with the animal population...
Figure 1. On a square lattice two independent closed walks 1 and 2 with 38 steps each, starting from and returning to the origin. The winding sectors of each walk are labelled by their winding numbers \( \{ n_1 \} \) (for walk 1) and \( \{ n_2 \} \) (for walk 2). The winding sectors enclosed by the external frontier of the two walks \( 1 + 2 \) are labelled (i) by their joint winding numbers \( \{ n_1, n_2 \} \) and (ii) by their total winding number \( \{ n_1 + n_2 \} \equiv n \). The \( n = 0 \)-winding sectors of interest, namely inside the external frontier of \( 1 + 2 \), correspond either to \( \{ 0, 0 \} \) with at least one of the zero-winding sectors inside one of the paths, or to \( \{ n_1, n_2 \} \) with \( n_1 + n_2 = 0 \) and both \( n_1 \) and \( n_2 \neq 0 \).

(assuming of course that food is homogeneously and sufficiently available). By identifying an animal looking for food with a random walker one thereby obtains the log \( m \) scaling rather than the naive geometrical \( \sqrt{m} \) scaling.

In the present work we will give a more detailed analysis of the random variable \( S_n(m) \). In particular, we will pay attention to the random variable \( S_{n_1,n_2,...,n_m}(m) \), the arithmetic area of the set of points enclosed \( n_1 \) times by path 1, \( n_2 \) times by path 2, . . . , and \( n_m \) times by path \( m \), with \( \sum_i n_i = n \). Happily enough, this variable can be tackled by path integral techniques analogous to the one used in \([1,3]\) provided that at least one of the \( n_i \neq 0 \). Note, when \( \sum_i n_i = 0 \), that \( S_{n_1,n_2,...,n_m}(m) \) contains a part of the inside
zero-winding sector area $S_0(m)$. To get a hand on the other part $S_0,0,\ldots,0(m)$, with at least one of the $n_i = 0$-winding sectors inside the corresponding path $i$ (otherwise one would be trivially outside the external frontier of the $m$ paths), one key observation is that since the paths are independent one can use in the $m$-path case the SLE information $\langle S_0 \rangle = q \langle S - S_0 \rangle$ valid in the one-path case.

From these considerations will follow $\langle S(m) \rangle$ and $\langle S_0(m) \rangle$. We will show in particular that, when $m \to \infty$, the subleading $\langle S_0(m) \rangle$ remains finite. Also some information on the asymptotics of $\langle S_{n_1,n_2,0,\ldots,0}(m) \rangle$ when both $n_1$ and $n_2 \to \infty$ will be obtained. Finally $\langle S_0(2) \rangle$ and the overlap between two paths, $\langle 2S(1) - S(2) \rangle$, will be considered. This quantity might have some interest in polymer physics where polymers are modelled by Brownian paths.

In view of unifying notations between [1] and [3], we always denote the $n$-winding sector arithmetic area and total arithmetic area for $m$ paths by $S_n(m)$ and $S(m)$, which means that, from now on, $S_n(1)$ and $S(1)$ stand for $S_n$ and $S$, the $n$-winding sector arithmetic area and total arithmetic area in the one-path case.

2. Winding sectors

2.1. Winding angle and propagator

As stated in section 1, the arithmetic area of the winding sectors enclosed by planar Brownian paths can be obtained from their winding properties. Consider a path of length $t$, starting from and ending at $\vec{r}$ and the angle $\theta$ wound by the path around the origin $O$. The average of the random variable $e^{i\alpha \theta}$ over the set of such paths is

$$\langle e^{i\alpha \theta} \rangle = \frac{G_{\alpha}(\vec{r},\vec{r})}{G_0(\vec{r},\vec{r})}$$

where

$$G_{\alpha}(\vec{r},\vec{r}) = \int_{\vec{r}(0) = \vec{r}}^{\vec{r}(t) = \vec{r}} \mathcal{D}\vec{r}(\tau) e^{-1/2 \int_0^t \vec{r}^2(\tau) d\tau + i\alpha \int_0^t \theta(\tau) d\tau}$$

is the quantum propagator of a charged particle coupled to a vortex at location $O$. By symmetry, it depends only on $r$:

$$G_{\alpha}(\vec{r},\vec{r}) = \frac{1}{2\pi t} e^{-r^2/t} \sum_{k=-\infty}^{+\infty} I_{|k-\alpha|} \left( \frac{r^2}{t} \right)$$

where $I_{|k-\alpha|}$ is a modified Bessel function and

$$G_0(\vec{r},\vec{r}) = \frac{1}{2\pi t}.$$

Additional symmetry and periodicity considerations lead to $G_{\alpha} = G_{\alpha+1} = G_{1-\alpha}$ so that $\alpha$ can be restricted to $0 \leq \alpha \leq 1$. We also set $r^2/t \equiv x$ so that

$$\langle e^{i\alpha \theta} \rangle = e^{-x} \sum_{k=-\infty}^{+\infty} I_{|k-\alpha|}(x) \equiv G_{\alpha}(x)$$

doi:10.1088/1742-5468/2012/05/P05005
with \( G_0(x) = 1 \). Clearly areas are proportional to \( t \), so we can, without loss of generality, set \( t = 1 \).

Let us rewrite \( G_\alpha(x) \) in a more suitable form: one has

\[
G_\alpha(x) = e^{-x} \sum_{k=0}^{+\infty} I_{k+\alpha}(x) + \{ \alpha \to 1 - \alpha \}.
\]

Observing that \( (d/dx) (e^{-x} \sum_{k=0}^{+\infty} I_{k+\alpha}(x)) = \frac{1}{2} e^{-x} (I_{\alpha-1}(x) - I_\alpha(x)) \), we get

\[
\frac{dG_\alpha(x)}{dx} = \frac{1}{2} e^{-x} (I_{\alpha-1}(x) - I_\alpha(x) + I_\alpha(x) - I_{1-\alpha}(x)) = \frac{\sin(\alpha \pi)}{\pi} e^{-x} (K_\alpha(x) + K_{1-\alpha}(x)).
\]

Using the integral representation [6] of \( K_\alpha(x) \):

\[
K_\alpha(x) = \frac{1}{2} \int_{-\infty}^{\infty} du e^{-x \cosh u} \cosh(\alpha u)
\]

we deduce

\[
G_\alpha(x) = \frac{\sin(\alpha \pi)}{\pi} \int_{-\infty}^{\infty} du \frac{1 - e^{-x(1+\cosh u)}}{1 + \cosh u} \cosh \left( \frac{u}{2} \left( \alpha - \frac{1}{2} \right) u \right).
\]

Clearly

\[
\lim_{x \to \infty} G_\alpha(x) = \frac{\sin(\alpha \pi)}{\pi} \int_{-\infty}^{\infty} du \frac{1}{1 + \cosh u} \cosh \left( \frac{u}{2} \left( \alpha - \frac{1}{2} \right) u \right) = 1
\]

so that

\[
1 - G_\alpha(x) = \frac{\sin(\alpha \pi)}{\pi} \int_{-\infty}^{\infty} du \frac{e^{-x(1+\cosh u)}}{1 + \cosh u} \cosh \left( \frac{u}{2} \left( \alpha - \frac{1}{2} \right) u \right).
\]

Equation (10) will be extensively used in the following.

2.2. Average area \( \langle S_n(m) \rangle \) of \( n \)-winding sectors

Let us first consider the average arithmetic area \( \langle S_n(m) \rangle \) of the \( n \)-winding sectors labelled by their winding number \( n \). For \( m \) Brownian paths of same length unity, starting from and ending at the same point, one has [1]

\[
Z_\alpha(m) \equiv \pi \int_{0}^{\infty} dx \left( 1 - (G_\alpha(x))^m \right) = \sum_{n \neq 0} \langle S_n(m) \rangle \left( 1 - e^{i2\pi \alpha n} \right).
\]

Rewriting \( G_\alpha(x) = 1 - (1 - G_\alpha(x)) \) leads, for \( n \neq 0 \), to

\[
\langle S_n(m) \rangle = - \int_{0}^{1} d\alpha \ Z_\alpha(m) \cos(2\pi \alpha n) = \sum_{j=1}^{m} (-1)^{j+1} \binom{m}{j}
\]

\[
\times \left( -\pi \right)^{j} \int_{0}^{1} d\alpha \int_{0}^{\infty} dx \left( 1 - G_\alpha(x) \right)^j \cos(2\pi \alpha n)
\]

\[
\text{doi:10.1088/1742-5468/2012/05/P05005}
\]
where \( \binom{m}{j} \) is the binomial coefficient. Thus

\[
\langle S_n(m) \rangle = \sum_{j=1}^{m} (-1)^{j+1} \binom{m}{j} I_{n,j}
\]

with

\[
I_{n,j} = -\pi \int_0^1 d\alpha \int_0^\infty dx \, (1 - G_\alpha(x))^j \cos(2\pi \alpha n).
\]

Using (10) one rewrites \( I_{n,j} \) as

\[
I_{n,j} = -\pi \prod_{i=1}^j \int_{-\infty}^{\infty} \frac{du_i}{2\pi \cosh u_i/2} \frac{\Phi_{n,j}}{j + \sum_{i=1}^j \cosh u_i}
\]

where the \( \Phi_{n,j} \) s follow from the integration over \( \alpha \)

- \( j \) odd = 2k + 1

\[
\Phi_{n,j} = \frac{(-1)^{k+1}}{2^{2k+1}} 2\pi \cosh \left( \frac{\sum_{i=1}^j u_i}{2} \right) \sum_{N=n-k}^{n+k+1} (-1)^{N+k-n} \binom{2k+1}{N+k-n}
\]

\[
\times \frac{2N-1}{(\pi(2N-1))^2 + (\sum_{i=1}^j u_i)^2}
\]

- \( j \) even = 2k

\[
\Phi_{n,j} = \frac{(-1)^k}{2^{2k}} \left( \sum_{i=1}^j u_i \right) \sinh \left( \frac{\sum_{i=1}^j u_i}{2} \right) \sum_{N=n-k}^{n+k} (-1)^{N+k-n} \binom{2k}{N+k-n}
\]

\[
\times \frac{1}{(2\pi N)^2 + (\sum_{i=1}^j u_i)^2}
\]

It is possible to compute \( I_{n,1} \) exactly [4]:

\[
I_{n,1} = \frac{\pi}{2} \int_{-\infty}^{\infty} \frac{du}{1 + \cosh u} \left( \frac{2N-1}{(\pi(2N-1))^2 + u^2} - \frac{2N+1}{(\pi(2N+1))^2 + u^2} \right) = \frac{1}{2\pi n^2}.
\]

On the other hand, when \( n \to \infty \)

- \( j \) odd

\[
I_{n,j} \sim \frac{(-1)^{(j-1)/2} j!}{(2\pi)^j} n^{j+1} c_j
\]

\[
c_j = \int_0^\infty dx \, e^{-jx} (K_0(x))^j
\]

- \( j \) even

\[
I_{n,j} \sim \frac{(-1)^{(j+1)/2} j (j+1)!}{(2\pi)^{j+1}} d_j
\]

\[
d_j = \int_0^\infty dx \, e^{-jx} (K_0(x))^{j-1} \int_0^\infty du \, e^{-x \cosh u} u \tanh \frac{u}{2}.
\]
We finally obtain
\[ \langle S_n(1) \rangle = I_{n,1} = \frac{1}{2\pi n^2} \]

\[ \langle S_n(2) \rangle = \frac{2}{2\pi n^2} - I_{n,2} = n \cdot \frac{2}{2\pi n^2} - \frac{3}{2\pi^3 n^4} d_2 + O\left(\frac{1}{n^6}\right) \]

and in the general case

\[ \langle S_n(m) \rangle = \frac{m}{2\pi n^2} - \left(\begin{array}{c} m \\ 2 \end{array}\right) I_{n,2} + \left(\begin{array}{c} m \\ 3 \end{array}\right) I_{n,3} - \cdots \]

\[ = n \cdot \frac{m}{2\pi n^2} - \frac{3}{4\pi^3 n^4} \left(2 \left(\begin{array}{c} m \\ 2 \end{array}\right) d_2 + \left(\begin{array}{c} m \\ 3 \end{array}\right) c_3\right) + O\left(\frac{1}{n^6}\right) \]

with the convention that \( \left(\begin{array}{c} m \\ j \end{array}\right) = 0 \) if \( j > m, \) \( d_2 \simeq 2.84 \) and \( c_3 \simeq 5.73. \) Clearly, and as discussed in section 1, we have no information so far on the zero-winding sector inside the \( m \) paths. To make some progress on this issue, we have to turn to \( \langle S_{n_1,n_2,...,n_m}(m) \rangle, \)

the average arithmetic area of the sectors enclosed \( n_1 \) times by path 1, \( n_2 \) times by path 2, \ldots, and \( n_m \) times by path \( m. \)

### 2.3. Average area \( \langle S_{n_1,n_2,...,n_m}(m) \rangle \)

Winding sectors can as well be labelled by the set \( \{n_1, n_2, \ldots, n_m\} \) of the individual winding numbers \( n_i \) enclosed by each path \( i. \) In line with section 2.2, one has

\[
Z_{\alpha_1,\alpha_2,...,\alpha_m}(m) \equiv \pi \int_0^\infty dx \left(1 - \prod_{i=1}^m G_{\alpha_i}(x)\right) = \sum' \langle S_{n_1,n_2,...,n_m}(m) \rangle (1 - e^{2\pi i (\sum_{i=1}^m \alpha_i n_i)}).
\]

(23)

where in \( \sum' \) the set \( n_1 = n_2 = \cdots = n_m = 0 \) is excluded from the sum. So

\[
\langle S_{n_1,n_2,...,n_m}(m) \rangle = -\int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_n Z_{\alpha_1,\alpha_2,...,\alpha_m}(m) \cos \left(2\pi \sum_{i=1}^m \alpha_i n_i\right)
\]

\[ = -\int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_n Z_{\alpha_1,\alpha_2,...,\alpha_m}(m) \prod_{i=1}^m \cos(2\pi \alpha_i n_i)
\]

(24)

where we have used \( \int_0^1 d\alpha \sin(\pi \alpha) \sin(2\pi n \alpha) \cosh((\alpha - \frac{1}{2})u) = 0. \) Now \( \langle S_{n_1,n_2,...,n_m}(m) \rangle \)

is invariant by permutation on the \( n_i, \) so one can focus without loss of generality on \( \langle S_{n_1,n_2,...,n_m}(m) \rangle, 1 \leq j \leq m, \) with \( n_1, \ldots, n_j \neq 0. \)

Rewriting \( G_{\alpha_i}(x) = 1 - (1 - G_{\alpha_i}(x)) \) in (23), (24) leads us to consider when \( n_i \neq 0: \)

\[
\int_0^1 d\alpha_i (1 - G_{\alpha_i}(x)) \cos(2\pi \alpha_i n_i) = \int_{-\infty}^\infty du_i e^{-x(1+\cosh u_i)} P(u_i, n_i)
\]

(25)

with

\[
P(u_i, n_i) = \frac{u_i^2 + (1 - 4n_i^2)^{\pi^2}}{(u_i^2 + (\pi(2n_i + 1))^2)(u_i^2 + (\pi(2n_i - 1))^2)} \sim_{i-n} \frac{1}{4\pi^2 n_i^3}.
\]

(26)
It follows that
\[
\langle S_{n_1,\ldots,n_j,0,\ldots,0}(m) \rangle = \pi(-1)^j \int_0^\infty dx \left( \prod_{i=1}^{j} \int_{-\infty}^{\infty} du_i e^{-x(1+\cosh u_i)} P(u_i, n_i) \right) (1 - f(x))^{m-j}
\]
(27)
where
\[
f(x) = \int_0^1 d\alpha \left( 1 - G_\alpha(x) \right) = \int_{-\infty}^{\infty} du \frac{e^{-x(1+\cosh u)}}{u^2 + \pi^2}.
\]
(28)

For example, when \(m = 1, 2\) one gets
\[
\langle S_{n_1}(1) \rangle = -\pi \int_0^\infty dx \int_{-\infty}^\infty du e^{-x(1+\cosh u)} P(u, n_1) = \frac{1}{2\pi n_1^2}
\]
\[
\langle S_{n_1,n_2}(2) \rangle = \pi \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{du_1 du_2}{2 + \cosh u_1 + \cosh u_2} P(u_1, n_1) P(u_2, n_2)
\]
\[
\langle S_{n_1,0}(2) \rangle = \frac{1}{2\pi n_1^2} + \pi \int_{-\infty}^\infty \int_{-\infty}^{\infty} \frac{du_1 du_2}{2 + \cosh u_1 + \cosh u_2} P(u_1, n_1) P(u_2, 0).
\]

When \(n_i \to \infty\) one obtains
\[
\langle S_{n_1,\ldots,n_j,0,\ldots,0}(m) \rangle \sim \frac{1}{2^j \pi^{2j-1}} \frac{1}{n_1^2 \cdots n_j^2} c_{j,m}
\]
where the constant
\[
c_{j,m} = \int_0^\infty dx e^{-jx} (K_0(x))^j (1 - f(x))^{m-j}
\]
has to be evaluated numerically with the exception of
\[
c_{2,2} = \frac{3}{2} \zeta(2) - 1 + \frac{3}{2} \zeta(2) \sum_{k=1}^\infty \frac{1}{k} \left(1 - \frac{1}{2k}\right)^2 - \sum_{k=1}^\infty \prod_{i=1}^k \left(1 - \frac{1}{2k+1}\right)^2.
\]

3. **Arithmetic area enclosed by \(m\) Brownian paths**

We are now in a position to compute \(\langle S(m) \rangle\), the average arithmetic area enclosed by the \(m\) paths. Obviously
\[
\langle S(m) \rangle = \sum'_n \langle S_{n_1,\ldots,n_m}(m) \rangle + \langle S_{0,\ldots,0}(m) \rangle
\]
(29)
where \(\langle S_{0,\ldots,0}(m) \rangle\) is the area of the finite \(\{n_1 = 0, n_2 = 0, \ldots, n_m = 0\}\) winding sectors inside at least one of the paths\(^2\). From (23), we get
\[
\sum'_n \langle S_{n_1,\ldots,n_m}(m) \rangle = \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_m Z_{\alpha_1,\ldots,\alpha_m}(m)
\]
(30)

\(^2\) Remember that \(\langle S_0(m) \rangle = \langle S_{0,\ldots,0}(m) \rangle + \sum'_n \langle S_{n_1,\ldots,n_m}(m) \rangle\), with \(\sum n_i = 0\) and at least one of the \(n_i \neq 0\).
so that, using (28),
\[
\sum' \langle S_{n_1,\ldots,n_m}(m) \rangle = \pi \int_0^\infty dx \,(1-(1-f(x))^m).
\] (31)

In order to find \( \langle S_{0,\ldots,0}(m) \rangle \) we first consider
\[
\sum_{n_1 \neq 0} \langle S_{n_1,0,\ldots,0}(m) \rangle = \pi \int_0^\infty dx \, f(x)(1-f(x))^{m-1}
\]
and, the sum
\[
A_1 \equiv \sum_{n_1 \neq 0} (\langle S_{n_1,0,\ldots,0}(m) \rangle + \langle S_{0,n_1,0,\ldots,0}(m) \rangle + \cdots + \langle S_{0,\ldots,0,n_1}(m) \rangle)
\]
\[
= \left( \frac{m}{1} \right) \pi \int_0^\infty dx \, f(x)(1-f(x))^{m-1}
\]
where one has taken into account permutation invariance. Similarly one can consider
\[
A_2 \equiv \sum_{n_1,n_2 \neq 0} (\langle S_{n_1,n_2,0,\ldots,0}(m) \rangle + \cdots + \langle S_{0,\ldots,0,n_1,0,\ldots,0,n_2,\ldots,0}(m) \rangle + \cdots + \langle S_{0,\ldots,0,n_1,n_2}(m) \rangle)
\]
\[
= \left( \frac{m}{2} \right) \pi \int_0^\infty dx \, f(x)^2(1-f(x))^{m-2}
\]
and in general
\[
A_m \equiv \sum_{n_1,n_2,\ldots,n_m \neq 0} \langle S_{n_1,n_2,\ldots,n_m}(m) \rangle = \left( \frac{m}{m} \right) \pi \int_0^\infty dx \, f(x)^m(1-f(x))^{m-m}
\]
with, obviously,
\[
\sum_{i=1}^m A_i = \pi \int_0^\infty dx \,(1-(1-f(x))^m) = \sum' \langle S_{n_1,n_2,\ldots,n_m}(m) \rangle.
\] (32)

We are interested in \( \langle S_{0,\ldots,0}(m) \rangle \): in the case of one closed path one knows from SLE [4] that
\[
q = \frac{\langle S_0(1) \rangle}{\sum_{n \neq 0} \langle S_n(1) \rangle} = \frac{1}{5}
\] (33)

This means that for any point inside the path, with winding number \( n \), \( q \) is the ratio of the probability to have a zero-winding to the probability to have a \( n \neq 0 \)-winding.

\( A_1 \) counts the points with only one non-zero winding number. It follows that the corresponding contribution to \( \langle S_{0,\ldots,0}(m) \rangle \) is necessarily \( q \, A_1 \). Similarly, \( A_2 \) counts the points with only two non-zero winding numbers. Since the \( m \) paths are independent, it follows that the corresponding contribution to \( \langle S_{0,\ldots,0}(m) \rangle \) is \( q^2 A_2 \). This line of reasoning generalizes to \( A_k \): the contribution to \( \langle S_{0,\ldots,0}(m) \rangle \) is \( q^k A_k \). Finally
\[
\langle S_{0,\ldots,0}(m) \rangle = \sum_{i=1}^m q^i A_i = \Phi_0(m) - \Phi_q(m)
\] (34)
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\[
\Phi_q(m) = \pi \int_0^\infty dx (1 - (1 - (1 - q) f(x))^m).
\]  

(35)

Clearly \( \sum_{i=1}^m A_i \) in (32) coincides with \( \Phi_0(m) \). It follows that the average arithmetic area enclosed by the \( m \) paths is

\[
\langle S(m) \rangle = \sum_{i=1}^m (1 + q^i) A_i = 2\Phi_0(m) - \Phi_q(m).
\]  

(36)

When \( m \to \infty \) (see equation (A.5) in the appendix) one obtains

\[
\langle S(m) \rangle = \frac{\pi}{2} \ln m - \frac{\pi}{4} \ln \ln m + \frac{\pi}{2} \left( \ln \sqrt{\frac{25}{4\pi^2}} + C \right) + o\left( \frac{1}{\sqrt{\ln m}} \right)
\]  

(37)

where \( C \) is the Euler constant. In figure 2, numerical simulations for \( \langle S(m) \rangle \) show that the agreement with equation (37) is quite correct, even for values of \( m \) which are not large.

Moreover, the asymptotic [1] of \( \langle S(m) - S_0(m) \rangle \) when \( m \to \infty \) is known to be

\[
\langle S(m) - S_0(m) \rangle = \frac{\pi}{2} \ln m - \frac{\pi}{4} \ln \ln m + \frac{\pi}{2} \left( -\ln \sqrt{4\pi} + C \right) + o\left( \frac{1}{\sqrt{\ln m}} \right).
\]  

(38)
Comparing equations (37) and (38), one deduces that the subleading zero-winding sector average arithmetic area:

$$\lim_{m \to \infty} \langle S_0(m) \rangle = \frac{\pi}{2} \ln \frac{5}{\pi}$$

remains finite in the $m \to \infty$ limit.

Finally, as another illustration of the path integral formalism, consider the average overlap $\langle 2S(1) - S(2) \rangle$ of the arithmetic areas of two paths and $\langle S_0(2) \rangle$, the average zero-winding sector arithmetic area of two paths. One has

$$\langle S(1) \rangle = \pi \int_0^\infty dx (1 + q) f(x) = \frac{\pi}{5}$$

$$\langle S(2) \rangle = \pi \int_0^\infty dx (2(1 + q)f(x) + f(x)^2((1 - q)^2 - 2))$$

so that

$$\langle 2S(1) - S(2) \rangle = \pi(2 - (1 - q)^2) \int_0^\infty dx f(x)^2$$

$$= \frac{34\pi}{25} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{du_1 du_2}{(2 + \cosh u_1 + \cosh u_2)((\pi^2 + u_1^2)(\pi^2 + u_2^2)).}$$

Numerically

$$\frac{\langle 2S(1) - S(2) \rangle}{\langle S(1) \rangle} \approx 0.286$$

is close to what one would obtain if the paths attached in $O$ were two circles of radius $R$: the overlap in units of $\pi R^2$ would then be

$$\frac{1}{2} - \frac{2}{\pi^2} \approx 0.297.$$

Also, as far as $\langle S_0(2) \rangle$ is concerned, we rewrite

$$\langle S_0(2) \rangle = \langle S(2) \rangle - \sum_{n \neq 0} \langle S_n(2) \rangle = 2\langle S(1) \rangle - \langle 2S(1) - S(2) \rangle - \sum_{n \neq 0} \langle S_n(2) \rangle.$$

From section 2.2 one has

$$\sum_{n \neq 0} \langle S_n(2) \rangle = \frac{\pi}{3} - \pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} du_1 du_2 \frac{\tanh(u_1/2) + \tanh(u_2/2)}{(2 + \cosh u_1 + \cosh u_2)(u_1 + u_2)((2\pi)^2 + (u_1 + u_2)^2))}$$

so that, using $\langle S_0(1) \rangle = \frac{\pi}{30}$ and equation (40), one finds $\langle 2S_0(1) - S_0(2) \rangle$ to be

$$\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{du_1 du_2}{(2 + \cosh u_1 + \cosh u_2)}$$

$$\times \left( \frac{34/25}{(\pi^2 + u_1^2)(\pi^2 + u_2^2)} - \frac{\tanh(u_1/2) + \tanh(u_2/2)}{(u_1 + u_2)((2\pi)^2 + (u_1 + u_2)^2)} \right).$$

If the paths were not overlapping, then one would necessarily have $\langle 2S_0(1) - S_0(2) \rangle = 0$. The nonvanishing result (41) clearly indicates that the two paths do, on average, overlap as already seen in (40). Note that the zero-winding sector ‘overlap’ $\langle 2S_0(1) - S_0(2) \rangle$ is different in nature from the arithmetic area overlap $\langle 2S(1) - S(2) \rangle$: the latter is a purely geometric overlap, whereas the former is more subtle since the superposition of the two paths destroys some original zero-winding sectors in each of the paths and creates new zero-winding sectors for the two paths.
4. Conclusion

Path integral techniques have been extensively used to tackle the issue of $n$-winding sector arithmetic area of $m$ Brownian paths. Some information stemming from SLE techniques, valid only in the one-path case, has also proved useful in the $m$-path case, merely because the paths are independent.

Equations (37) and (39) are the main results of this paper. In particular, the subleading zero-winding arithmetic area has been shown to remain finite in the asymptotic limit. Numerical simulations have nicely confirmed these asymptotic results. The overlap between two paths is also computed numerically. Applications to polymer physics will be studied in a forthcoming publication. It is indeed well known that polymers can be modelled in some cases by Brownian paths, in particular, in the temperature regime where attractive van der Waals forces between monomers are exactly cancelled by excluded-volume effects. This takes place also at high polymer density which is the situation we consider in the large $m$ limit when we estimate the area occupied by the $m$ Brownian paths [7].

Acknowledgment

One of us (SO) would like to thank S N Majumdar for stimulating conversations.

Appendix

In this appendix we derive the $m \to \infty$ asymptotic limit of $\Phi_q(m)$ defined in (35):

$$\Phi_q(m) = \pi \int_0^\infty dx (1 - (1 - (1 - q)f(x))^m)$$

with $f(x)$ given in (28):

$$f(x) = \int_{-\infty}^{\infty} du \frac{e^{-x(1+\cosh u)}}{\pi^2 + u^2}.$$ 

Setting $x = y \ln m / 2$, $\Phi_q(m)$ becomes

$$\Phi_q(m) = \frac{\pi \ln m}{2} \int_0^\infty dy \left( 1 - \left(1 - (1 - q)f\left(\frac{y \ln m}{2}\right)\right)^m \right) 
\approx \frac{\pi \ln m}{2} \int_0^\infty dy \left( 1 - e^{-m(1-q)f(y \ln m / 2)} \right). \tag{A.1}$$

Using $f(x) \simeq (e^{-2x/\pi^2})\sqrt{2\pi/x}$ when $x \to \infty$, the integrand in (A.1):

$$1 - e^{-m(1-q)f(y \ln m / 2)} \approx 1 - e^{-(1-q)(m^{1-y}/\pi^2)}\sqrt{4\pi/y \ln m}$$

behaves, in the $m \to \infty$ limit, like

$$0 \quad \text{if } y > 1$$
$$1 \quad \text{if } y < 1$$

and so, at leading order, $\Phi_q(m) = \pi \ln m / 2$. 

doi:10.1088/1742-5468/2012/05/P05005

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Focusing now on the subleading correction at order $1/\sqrt{\ln m}$, namely

$$\frac{\pi \ln m}{2} \left( \int_0^\infty dy \left( 1 - e^{-\left((1-q)(m^{1-y}/\pi^2)\sqrt{4\pi/\ln m}\right)} - \int_0^1 dy e^{-\left((1-q)(m^{1-y}/\pi^2)\sqrt{4\pi/\ln m}\right)} \right) \right) \tag{A.2}$$

let us first consider the $y > 1$ integration. One has to compute

$$a \approx \frac{\pi \ln m}{2} \int_1^\infty dy (1-q) \frac{m^{1-y}}{\pi^2} \sqrt{4\pi/\ln m} \approx \frac{\pi \ln m}{2} \int_1^\infty dy (1-q) \frac{m^{1-y}}{\pi^2} \sqrt{4\pi/\ln m}$$

where one has used that, because of the $m^{1-y}$ factor, $y$ is peaked to 1 when $m \to \infty$. One obtains

$$a = \frac{1 - q}{\sqrt{\pi \ln m}} + o\left( \frac{1}{\sqrt{\ln m}} \right). \tag{A.3}$$

Considering next the $y < 1$ integration one has to compute

$$b \approx -\frac{\pi \ln m}{2} \int_0^1 dy e^{-\left((1-q)(m^{1-y}/\pi^2)\sqrt{4\pi/\ln m}\right)} \approx -\frac{\pi \ln m}{2} \int_0^1 dy e^{-\left((1-q)(m^{1-y}/\pi^2)\sqrt{4\pi/\ln m}\right)} \approx -\frac{\pi \ln m}{2} \int_0^1 dy e^{-a'm^{1-y}} \quad \text{with} \quad a' = \frac{1 - q}{\pi^2} \sqrt{4\pi/\ln m}.$$ Setting $a'm^{1-y} = z$ with $-dy \ln m = dz/z$ leads to

$$b \approx -\frac{\pi}{2} \int_{a'}^{a'm} e^{-z} \frac{dz}{z} = -\frac{\pi}{2} \left( [\ln z e^{-z}]_{a'}^{a'm} + \int_{a'}^{a'm} \ln z e^{-z} dz \right).$$

At order $1/\sqrt{\ln m}$, one has $[\ln z e^{-z}]_{a'}^{a'm} \approx -\ln a' (1 - a')$ and $\int_{a'}^{a'm} \ln z e^{-z} dz \approx \int_0^{a'm} \ln z e^{-z} dz - \int_0^{a'} \ln z e^{-z} dz \approx C - \int_0^{a'} \ln z (1 - z) dz$, where $C$ is the Euler constant. The last remaining integral is straightforward and finally

$$b \approx \frac{\pi}{2} (\ln a' + C - a'). \tag{A.4}$$

Noticing that $a'(\pi/2) = (1 - q)/\sqrt{\pi \ln m} \approx a$, we are left with $a + b \approx (\pi/2)(\ln a' + C)$ so that

$$\Phi_q(m) = \frac{\pi}{2} \ln m - \frac{\pi}{4} \ln \ln m + \frac{\pi}{2} \left( \ln(1 - q) + \ln \sqrt{\frac{4}{\pi^3} + C} \right) + o\left( \frac{1}{\sqrt{\ln m}} \right). \tag{A.5}$$

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