Effect of Accelerated Global Expansion on Bending of Light

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Abstract

In 2007 Rindler and Ishak showed that, contrary to previous claims, the value of the cosmological constant does have an effect on light deflection by a gravitating object in an expanding universe, modeled by a Schwarzschild-de Sitter spacetime. In this paper we consider light bending in the more general situation of a gravitating object in a cosmological background with varying expansion rate $H(t)$. We calculate numerically the null geodesics representing light rays deflected by a black hole in an accelerating Friedmann-Lemaître-Robertson-Walker universe, modeled by a McVittie metric. Keeping the values of the distances from the observer to the lensing object and to the source fixed, we plot the dependence of the bending angle measured by two different sets of observers in this spacetime on the rate of change of $H(t)$.

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I. INTRODUCTION

The effect of the cosmological expansion on the bending of light rays and gravitational lensing has been studied for 30 years now, at least in situations where the expansion is driven by a cosmological constant, since Islam’s 1983 paper [1] on light trajectories in Schwarzschild-de Sitter (SdS) spacetime, but the issue has received an increased amount of attention in the past 10–15 years with observations leading to the conclusion that the global rate of expansion is accelerating. Among the recent references on the subject, we refer the reader to the review paper by Ishak and Rindler [2], and especially to the comprehensive discussion by Lebedev and Lake [6], which also contain citations to the rest of the literature. In this paper we will examine light bending in a more general setting, but still representing a nonrotating gravitating object in an asymptotically homogeneous and isotropic background.

Let us introduce some relevant concepts and model spacetimes. A spatially flat, homogeneous, isotropic cosmological model is described by a Friedmann-Lemaître-Robertson-Walker (FLRW) metric of the form $d{s}^2 = dt^2 + a(t)^2 d\vec{x}^2$, where $d\vec{x}^2$ is the line element for Euclidean 3-space, in the time gauge in which $t$ is proper time along a comoving worldline, with spatial scale factor $a(t)$. The Hubble parameter $H(t) := \dot{a}/a$ is usually referred to as the “expansion rate”; an accelerating expansion rate corresponds to a situation in which $\ddot{a} \neq 0$, and is often quantified by the “deceleration parameter” $q := -\ddot{a}/\dot{a}^2 \equiv - (1 + \dot{H}/H^2)$. In the standard cosmological model, results of observations such as the redshift-luminosity relationship for supernovae are interpreted as indicating that $q < 0$. The physical reason for this acceleration is not yet well understood, but one simple possibility is the presence of a non-vanishing cosmological constant, and the results in the literature on gravitational light bending and the cosmological expansion have focused so far on this possibility. A cosmological constant $\Lambda$ corresponds to constant values $H = H_0 = \sqrt{\Lambda/3}$ and $q = -1$, and the model used to study its effects on light bending is the Schwarzschild-de Sitter spacetime, which in the common “static” coordinates takes the form of the Kottler metric [5],

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2.$$  \hspace{1cm} (1.1)

This approaches the Schwarzschild metric for a black hole of mass $m$ located at the center of the coordinate system for small $r$ (or $\Lambda = 0$), and the de Sitter metric with cosmological constant $\Lambda$, written in static coordinates, for large $r$ (or $m = 0$).

In the presence of matter fields, however, or if the accelerated expansion is due to any reason other than a cosmological constant, we do not expect $H(t)$ to be constant, and there are indeed observational indications, both from the effective equation of state for matter at cosmological scales and from discrepancies between values of $H$ calculated from data at different redshifts, that it has been varying in time (see, for example, Refs. [3, 4]). Our goal is to extend the work on gravitational light bending at cosmological scales to the case of a time-dependent expansion rate $H(t)$ and study the effects of a non-zero $\dot{H}(t)$. Our approach will be to model such a spacetime by a McVittie metric, which generalizes the SdS spacetime to non-constant $H(t)$, and examine the behavior of null geodesics in it. In Sec. [II] we review the McVittie metric, written in two convenient coordinate systems. In Sec. [III] we summarize previous results on the bending of light in the SdS spacetime. Sec. [IV] contains our results on light bending in McVittie spacetimes with varying $H(t)$, and we finish with some conclusions in Sec. [V].
II. MCVITTIE METRIC

A metric that can be thought of as representing a Schwarzschild black hole embedded in a FLRW spacetime was first derived in 1933 by G.C. McVittie [8]. Surprisingly, despite it being a rather old solution of the Einstein equation, its exact interpretation has been somewhat controversial, with some papers questioning the black-hole interpretation of the metric [9]; recent work [10, 11] has shown, however, that this metric does indeed model a black hole embedded in an expanding cosmology, at least when the background spacetime is spatially flat and the Hubble parameter satisfies $H(t) > 0$. We should point out though that the McVittie metric is not considered as representing a generic solution of this type. Physically, in this situation one would expect the energy density to be inhomogeneous, due to the expansion of the universe at large scales and the attraction of the black hole at small scales, which might cause the black hole to accrete matter as in a cosmological analog of the Vaidya metric [12, 13]. As Kaloper et al. argue, this is not the case; the equation relating the energy density to $H(t)$ is exactly the same as for a FLRW cosmology, while the inhomogeneity appears in the expression for the pressure [10].

The McVittie line element can be written down in the diagonal form

$$ds^2 = - \left(1 - \frac{\mu}{1 + \mu}\right)^2 dt^2 + (1 + \mu)^4 a^2(t) d\vec{X}^2,$$

where $a(t)$ is the asymptotic spatial scale factor and $m$ the mass of the central object; these coordinates might be considered as an inhomogeneous-space version of comoving coordinates, because the line element tends to that of a FLRW spacetime in comoving coordinates at large values of $|\vec{X}|$.

Several authors [10, 14, 15] have introduced a coordinate transformation $(t, \vec{X}) \mapsto (t, \vec{x})$ to “Painlevé-Gullstrand type” coordinates defined by

$$r = (1 + \mu)^2 a(t) R \quad (2.2)$$

(with $t$ and the angular coordinates unchanged), in which the line element takes the form

$$ds^2 = - \left(f(r) - H^2(t) r^2 \right) dt^2 - \frac{2H(t) r}{\sqrt{f(r)}} dr dt + \frac{dr^2}{f(r)} + r^2 d\Omega^2,$$

where $f(r) := 1 - 2m/r$ and again $H(t) = \dot{a}/a$. Thus, $r$ is an area radial coordinate, characterized by the fact that the surface area of a 2-sphere of constant $r$ is $4\pi r^2$, and the function $H(t)$ represents the asymptotic expansion rate in comoving coordinates, so we still call it the “Hubble parameter”. Note that in the ADM-type terminology for the 3+1 split of a spacetime metric, the spatial part of this metric has no time dependence and has the same form as for Schwarzschild spacetime, and the cosmological expansion terms have been moved into the shift vector, with non-vanishing component $N_r = -Hr/\sqrt{f(r)}$.

For us, it is important to note that the SdS spacetime with cosmological constant $\Lambda$ is a special case of the McVittie metric with constant $H(t) = H_0$ and $\Lambda = 3H_0^2$, as one can see by defining the time coordinate transformation

$$dt = d\bar{t} - \frac{1}{1 - \frac{2m}{r} - \frac{1}{3} \Lambda r^2} \sqrt{\frac{\Lambda r^2/3}{1 - 2m/r}} dr.$$

$$3.$$
Using $t$ as time coordinate, the McVittie metric takes the traditional Kottler form of Eq. (1.1) (see, e.g., Ref. [15]).

In the literature on light deflection in SdS spacetimes authors use the Kottler, diagonal form of the metric, because of its greater simplicity. If $H(t)$ is not constant, however, a coordinate transformation of the type (2.4) does not yield a diagonal, “generalized Kottler” metric with “time-varying $\Lambda$”, and we will use instead the form (2.3). We will call (2.3) the (spatially) “static” form of the McVittie metric, and the corresponding observers “static”, as opposed to the “comoving” ones of the line element (2.1), in which the spatial metric does not have a static form even when $H(t)$ is constant.

III. NULL GEODESICS AND LIGHT BENDING IN SDS SPACETIMES

In this section we will review some of the definitions and results in the literature on null geodesics in SdS spacetime (see, for example, the papers by Ishak & Rindler [2] and Lebedev & Lake [6]). In view of the fact that we will later generalize those results, however, we will replace $\Lambda$ by $3H_0^2$, where $H_0$ is the constant value of $H(t)$, use the line element (2.3), and add some remarks motivated by the more general situation. The whole McVittie line element is now time-independent, and we can use the static and rotational Killing vector fields $\xi = \partial/\partial t$ and $\eta = \partial/\partial \phi$, to define two conserved quantities along a geodesic $x^\mu(\lambda)$ with tangent vector $K^\mu := dx^\mu/d\lambda$, $e := -g_{\alpha\beta} \xi^\alpha K^\beta = [f(r) - H_0^2 r^2] K^t + \frac{H_0 r}{\sqrt{f(r)}} K^r$, $\ell := g_{\alpha\beta} \eta^\alpha K^\beta = r^2 K^\phi$, (3.1) for a geodesic in the equatorial coordinate plane. In other words, the geodesic satisfies

$$K^\phi = \frac{\ell}{r^2}, \quad K^t = \frac{e}{f(r) - H_0^2 r^2} - \frac{H_0 r}{\sqrt{f(r)} [f(r) - H_0^2 r^2]} K^r. \quad (3.2)$$

Using these equations, the conditions for a null geodesic with $g_{\mu\nu} K^\mu K^\nu = 0$ can be reduced to a single differential equation, which in terms of the conserved quantities becomes

$$\frac{1}{\ell^2} \left( \frac{dr}{d\lambda} \right)^2 + \frac{1}{r^2} (f(r) - H_0^2 r^2) = \frac{1}{b^2}, \quad (3.3)$$

where we have defined $b := \ell/e$, the impact parameter. Note that, as expected, if $H_0 \to 0$ this is the same equation one gets for a null geodesic in Schwarzschild spacetime (see, for example, Ref. [16] or most of the references on light bending in SdS spacetime).

The shape of the light trajectory, in the sense of the relationship between the coordinates $r$ and $\phi$, can be obtained using the expression for $\ell$ in (3.3), and one finds that

$$\frac{d\phi}{dr} = \pm \frac{1}{r^2} \left[ \frac{1}{b^2} + H_0^2 - \left( 1 - \frac{2m}{r} \right) \right]^{-1/2}. \quad (3.4)$$

Notice that, since these equations do not involve $t$ or the geometry of a constant-$t$ hypersurface, we would have obtained the same relationship between $r$ and $\phi$ using the form (1.1) for the metric. Eq. (3.4) encodes some aspects of the effect a cosmological constant has on the formal description of light bending near a nonrotating gravitating mass, but one should
be careful when discussing any measurable effects on observations. A controversy in this sense arose around the interpretation of (3.4), because although the equation does depend on $\Lambda$, it is of the same form as the one obtained in Schwarzschild spacetime if one replaces the impact parameter $b$ in the latter by an “effective impact parameter” $B$ defined by

$$\frac{1}{B^2} := \frac{1}{b^2} + H_0^2,$$

and one might conclude that the cosmological constant does not affect observations because it is simply “absorbed” into the impact parameter, which is not directly measurable. This aspect has been extensively discussed starting with the original paper by Islam [1]. However, as pointed out in the more recent literature, the relationship between $r$ and $\phi$ only tells us part of the story on how the deflection angle depends on $\Lambda$, for various reasons.

One reason is that, as pointed out by Ishak & Rindler [2], from (3.4) alone and without knowing the spatial metric, one can only obtain the value that an angle would have in flat space. Consider for example the angle $\theta$ in Fig. 1. This is not the full bending angle, but as the following argument illustrates it is relevant for the calculation of the latter. In Schwarzschild spacetime one can imagine placing the source and observer at infinity and calculate a bending angle that does not refer to specific locations for them and depends only on the impact parameter $b$, in addition to the lens mass $m$. Because of the curved geometry of the constant-$t$ surfaces, this is not an option in SdS spacetime [2]. The simplest alternative is to set up the coordinates so that the point of closest approach is at $\phi = \pi/2$ and use as measure of total bending the sum $\alpha + \theta$ of the angles the light ray makes with the half-lines $\phi = \pi$ and $\phi = 0$ at the respective intersection points, labeled $S$ and $O$ in Fig. 1. Because the spacetime is static and spherically symmetric, in this situation the radial positions of $S$ and $O$ are equal, $R_{SL} = R_{L}$, and so are the angles, $\alpha = \theta$. The “Euclidean” value $\theta_E = \tan^{-1}(rd\phi/dr)$ of half of the total bending angle can then be easily calculated; neglecting quadratic terms in $m$, one finds the well-known result (see, e.g., Ref. [2])

$$\theta_E \approx \frac{2m}{B}.$$

However, a more physically meaningful, covariant value for the angle between two lines takes into account the actual metric on a constant-$t$ spatial hypersurface in SdS spacetime. As first derived by Ishak & Rindler [2], this measurable value $\theta_M$ of the angle $\theta$, to leading order in $\Lambda$, is given by an expression with a different dependence on $H_0$,

$$\theta_M = \frac{2m}{b} - \frac{\Lambda b^3}{12m} = \frac{2m}{b} - \frac{H_0^2 b^3}{4m}.$$

Secondly, when discussing observational consequences of $H_0$ for light bending we need to specify exactly what question we are asking. For example, if we said that Eq. (3.6) implies that the Euclidean bending angle $\theta_E$ depends on $H_0$ through the effective parameter $B$, we would be assuming that the comparison is made between situations in which the impact parameter $b$ is held constant. However, because $b$ is not directly measurable, it may be more meaningful to specify in some other way which situations we are comparing when we change the value of $H_0$. We take the point of view that, for each value of $H_0$, the geodesic we use to find the angle $\theta$ needs to be uniquely determined by a fixed set of values for parameters that can in principle be measured by an observer. As we will see in the next section, our choice will lead to a different conclusion for the dependence of $\theta_E$ on $H_0$. 

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FIG. 1: Definition of the main angles and static-coordinate distances used in the calculation of
the light bending angle. The lensing object \( L \) is located a distance \( R_L \) and the light source at a
distance \( R_s \) from the observer \( O \). The full bending angle is \( \alpha + \theta \) but only \( \theta \) is measurable by the
observer. In SdS spacetime the point of closest approach is at \( \phi = \pi/2 \) and the figure is symmetric
about the vertical axis; in general, this is no longer the case in McVittie spacetime.

The third thing to keep in mind is that values of angles are observer-dependent. Thus,
the angle \( \theta_M \) in (3.7) is the one that would be measured by a “static observer”, one along
whose worldline \( r \) and the angular coordinates are constant, but one may want to know what
angle would be measured by a “comoving observer”, one along whose worldline \( R \) rather
than \( r \) is constant. This point is discussed in detail by Lebedev & Lake [6], who give a
general expression for the angle measured by an observer with 4-velocity \( U^\mu \), as calculated
from the dot product between the projections orthogonal to \( U^\mu \) of the vector \( K^\mu \) tangent to
the deflected null geodesic and a radial null vector \( W^\mu \) (see Eq. 80 in Ref. [6]),

\[
\cos \theta_M = \frac{K \cdot W}{(U \cdot K)(U \cdot W)} + 1.
\]

(3.8)

While we will not go into the details of how this is used to derive explicit expressions for the
angles measured by different observers, let us summarize some results we can compare with
our numerical ones for McVittie spacetime in the next section. From (3.8) one can derive a
relationship between the Euclidean angle \( \theta_E \) and the angle \( \theta_M \) measured by a static observer
in SdS spacetime [6],

\[
\tan \theta_M = \sqrt{f(r) - H_0^2 r^2} \tan \theta_E.
\]

(3.9)

On the other hand, from Eq. (3.3) for null geodesics one obtains the well-known first-order
lens equation, valid for the general case of a source at a distance \( y \) from the lens-observer
axis (see, for example, Refs. [2, 6]). The result, using the notation \( R_s = R_{sl} + R_L \), is

\[
y = R_s \theta_E - \frac{4m R_{sl}}{R_L \theta_E}.
\]

(3.10)

When the source is on the axis, \( y = 0 \), if we solve (3.10) for the Euclidean angle we get

\[
\theta_E = \sqrt{\frac{4m R_{sl}}{R_L R_s}};
\]

(3.11)

this \( \theta_E \) can in turn be plugged into (3.9) and, using the small-angle approximation, for the
static-observer measurable angle we get

\[
\theta_M = \sqrt{\left(1 - 2m \frac{R_s}{R_L} - H_0^2 R_L^2\right) \frac{4m R_{sl}}{R_L R_s}}.
\]

(3.12)
An expression for the comoving-observer measurable angle, which is physically related to the static-observer one by an appropriate aberration factor, was obtained in Refs. \[6\], and can be written as

$$\theta_M = \sqrt{(1 + D_L H_0 + D_S^2 H_0^2) \frac{4m D_{SL}}{D_S D_L}}, \quad (3.13)$$

where $D_L$ and $D_S$ are measurable angular-diameter distances to the lens and source, and $D_{SL} = D_S^2 - D_L$. For later purposes notice that: (i) when $H = 0$ a McVittie metric reduces to the Schwarzschild metric of the same mass, in which comoving and static observers coincide, so the two corresponding values of $\theta_M$ are equal, consistently with what (3.12) and (3.13) tell us about the leading-order behavior in $m$; and (ii) while the comoving-observer angle in (3.13) increases with $H_0$, the static-observer one in (3.12) decreases.

For more comments on formulating physically relevant questions about the effect of $\Lambda$ on the bending angle in SdS spacetime, we refer to the work of Ishak and Rindler [2] or the more recent papers by Lebedev and Lake [6] and by Hammad [7].

**IV. NULL GEODESICS IN MCVITTIE SPACETIME**

Let us now consider light bending in a McVittie spacetime with non-constant expansion rate $H(t)$. In the coordinates of the “static” line element (2.3), a light ray in this spacetime obeys the null condition

$$\left[ f(r) - r^2 H(t)^2 \right] (K^t)^2 + \frac{2r H(t)}{\sqrt{f(r)}} (K^r K^t) - \frac{1}{f(r)} (K^r)^2 - \frac{\ell^2}{r^2} = 0, \quad (4.1)$$

where as before the angular momentum $\ell = r^2 K^\phi$ is conserved. Unlike in the earlier SdS case, however, there is no second conserved quantity $e$, because the spacetime has no timelike Killing vector field in general. Therefore, to find the deflected light ray and determine its tangent vector at the observer’s location we must actually solve at least one component of the null geodesic equation in addition to (4.1), as opposed to just using first integrals.

The relevant components of the geodesic equation $dK^\mu / d\lambda + \Gamma^\mu_{\alpha\beta} K^\alpha K^\beta = 0$, using the explicit form of the connection coefficients $\Gamma^\mu_{\alpha\beta}$ for the McVittie metric (2.3), are

$$\frac{dK^t}{d\lambda} + \left( \frac{r H(t) [2r H(t)^2 - f'(r)]}{2 \sqrt{f(r)}} \right) (K^t)^2 + \frac{f'(r) - 2r H(t)^2}{f(r)} (K^r K^t)$$

$$+ \frac{H(t)}{f(r)^{3/2}} (K^r)^2 + \frac{H(t) \ell^2}{\sqrt{f(r)} r^2} = 0, \quad (4.2)$$

$$\frac{dK^r}{d\lambda} + \frac{2r H(t)^2 - f'(r)}{2 f} (K^t)^2 + \frac{r H(t)}{\sqrt{f(r)}} (f'(r) - 2r H(t)^2) K^t K^r + \frac{\ell^2}{r^3} (r^2 H^2(t) - f(r))$$

$$+ \frac{1}{2} \left( (f(r) - r^2 H^2(t))(f'(r) - 2r H^2(t)) - 2r \sqrt{f(r)} H'(t) \right) (K^t)^2 = 0, \quad (4.3)$$

and our goal is to solve Eqs. (4.2) and (4.3) for $K^\mu = d x^\mu / d\lambda$, with initial values at the source satisfying the condition (4.1). We do not know how to solve those equations analytically, so we integrate them numerically instead, using the Runge-Kutta method.
Before describing our numerical simulations, we need to discuss how we will determine the effect of the time dependence of \( H(t) \) on the light bending angle. We will be comparing with each other McVittie metrics with the same mass \( m \) and value of the Hubble parameter \( H(t) \) at the time \( t_0 \) when the null geodesic leaves the source, but different \( \dot{H}(t) \). Treating \( H \) as a slowly varying function over the relevant times, we will parameterize it simply by giving the values of \( H_0 = H(t_0) \) and \( A = \dot{H}(t_0) \), or

\[
\dot{H}(t) = \dot{H}_0 + A(t - t_0) .
\]

(4.4)

Because McVittie metrics with different values of \( A \) are to be thought as different spacetime manifolds, the key point when comparing them is to formulate a criterion for determining which null geodesic in each spacetime is to be used for the comparison. One possibility, mentioned when we discussed SdS metrics, might have been to use geodesics with the same impact parameter \( b \). We take instead the point of view that a meaningful criterion is one which allows someone who observes light bending from some source-lens pair to determine which null geodesic corresponds to a deflected light ray from that source in each McVittie model (values for \( H_0 \) and \( A \)), just using quantities that are in principle measurable. They will then be able to compare the angle \( \theta_s \) they measure with the theoretically predicted angle for their type of observer (for light bending on cosmological scales, presumably a comoving observer) in each model spacetime, and determine which ones fit their measured \( \theta_m \).

For simplicity, in this paper we will assume that, in addition to the angle \( \theta_m \), the data observers have access to are the distances to the source and lens (which to leading order in \( m \) we identify with the values of \( R_s \) and \( R_L \)), as well as the information that the source and lens are aligned, and look for how \( \theta_m \) varies with \( A \) when \( m, H_0, R_s \) and \( R_L \) are kept constant. In a McVittie spacetime, fixing a set of values for those parameters determines uniquely a pair of null geodesics, one on each side of the lens, and for definiteness we will choose the clockwise one, as in Fig. 1. Considering \( R_s \) and \( R_L \) as directly measurable is overly simplistic, and considering only cases in which the source, lensing object and observer are aligned limits the generality of the results. Removing those limitations is left for future work, but we should also point out that aligned configurations are the physically most relevant ones; although light bending certainly occurs in more general settings, it is much more likely to be noticed observationally when all objects are nearly aligned, or \( y = 0 \) in Eq. (3.10).

When performing the simulations, we start each light ray from the coordinate location \( r = R_{SL} = R_S - R_L, \phi = \pi \) at \( t = t_0 \), and aim it in some tentative direction \( \alpha \). The geodesic equations (4.2) and (4.3) are then integrated numerically until the light ray reaches \( \phi = 0 \), where the value of \( r \) is checked against the chosen \( R_L \). If we find that \( r(\phi = 0) \) is smaller (greater) than the desired value, the simulation is repeated using a larger (smaller) \( \alpha \), and the procedure is iterated until the final \( r \) equals \( R_L \), at which point the components of \( K^\mu \) are recorded. Notice that when \( H(t) \) is time-dependent the spatial projection of a deflected null geodesic is not symmetric about the point of closest approach, and in a sketch of the trajectory of the light ray similar to the SdS one in Fig. 1 even with \( R_{SL} = R_L \), the angles \( \alpha \) and \( \theta \) will not be equal and the point of closest approach will not be at \( \phi = \pi/2 \).

Once the components of \( K^\mu \) at the location of the observer are known, we use (3.8) to calculate the angle \( \theta_m \) measured by an observer with 4-velocity \( U^\mu \). For static and comoving observers the static-coordinate components of the 4-velocities are

\[
U^\mu_{\text{stat}} = \left( \frac{1}{\sqrt{f(r) - H_0^2 r^2}}, 0, 0, 0 \right) , \quad U^\mu_{\text{comov}} = \left( \frac{r [1 + \sqrt{f(r)}]}{r [1 + \sqrt{f(r)}] - 2m H(t) r}, 0, 0 \right) .
\]

(4.5)
FIG. 2: A plot of the Euclidean angle $\theta_E$ for an observer at fixed distances $R_s = 2$ Gpc from the source and $R_L = 1$ Gpc from the lensing object (in static coordinates) vs. the acceleration parameter $A$, for various values of $H_0$. For $A = 0$ the bending angle is independent of $H_0$, to a good approximation, and its value agrees with (3.11).

respectively, and finding a null radial vector is simple; we choose

$$W^\mu = \left(1, \sqrt{f(r)} \left[\sqrt{f(r)} + Hr\right], 0, 0\right).$$

(4.6)

To specify the metric we choose a single value $m = 10^{14} M_{\text{sun}}$ for the mass of the lensing object, considered to be representative of situations in which a light ray traveling over cosmological distances is deflected by a galaxy cluster. For $H_0$ we choose three values in the range from 0 to 70 km/s/Mpc (from no expansion to approximately the currently accepted value). To come up with a realistic range of values for $A$ we use the fact that $q = \frac{1}{2}(1 + 3w)$, where $w$ is the cosmological equation of state parameter, and we take values consistent with current estimates to be approximately in the range from 1.05 to 1.25 [3]. Based on this and the relationship $\dot{H} = -(1 + q)H_0^2$, we choose the range from $-1.0 \times 10^{-9}$ to $+1.0 \times 10^{-9}$ km/s/Mpc/yr for the values for $A$. To specify the null geodesic, for the coordinate values of the source-lens and lens-observer distances we choose $R_{SL} = R_L = 1$ Gpc, considered as representative of typical values in a cosmological lensing situation, with a tolerance of 0.1 Mpc in the vale of $r$ for the intersection of the geodesic with $\phi = 0$.

The results of our simulations are shown in Figs. 2-4. For comparison with what we said in Sec. III about SdS spacetime, Fig. 2 shows the Euclidean angle $\theta_e = \tan^{-1}(r K^\phi/K^r)$ as a function of $A$, for various values of $H_0$. For $A = 0$ we recover the SdS situation and the value of $\theta_e$ agrees with what one finds from (3.11). In particular, as we see from the plot and as can also be concluded more generally from (3.6), when we use our criterion for identifying light rays in different spacetimes, based on fixing the values of $R_s$ and $R_L$, the Euclidean angle in SdS spacetime does not depend on $H_0$, contrary to what one would conclude by fixing $b$; when $A \neq 0$, however, $\theta_e$ does depend on $H_0$.  

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FIG. 3: A plot of the angle $\theta_M$ measured by a static observer at fixed distances $R_s = 2$ Gpc from the source and $R_L = 1$ Gpc from the lensing object (in static coordinates) vs. the acceleration parameter $A$, for various values of $H_0$. For each value of $A$, the arrival angle decreases with $H_0$.

FIG. 4: A plot of the angle $\theta_M$ measured by a comoving observer at fixed distances $R_s = 2$ Gpc from the source and $R_L = 1$ Gpc from the lensing object (in static coordinates) vs. the acceleration parameter $A$, for various values of $H_0$. For each value of $A$, the arrival angle increases with $H_0$. 
Figs. 3 and 4 show the more physically meaningful actual (curved-geometry) angle $\theta_M$ measured by static and comoving observers, respectively, as functions of $A$ for various values of $H_0$. It can be checked that for $A = 0$, in SdS spacetime, the values of the static-observer angle $\theta_M$ agree with (3.12). It is worth noting that, while in Fig. 3 the angles decrease with $H_0$, in Fig. 4 they increase with $H_0$; this is consistent with what Eqs. (3.12) and (3.13), respectively, tell us for the $A = 0$ case.

V. CONCLUSIONS

In this paper we examined the bending of null geodesics in spatially flat McVittie metrics, exact solutions to the Einstein equation which have been shown to represent nonrotating black holes embedded in FLRW background spacetimes. We used a slowly varying Hubble parameter $H(t) = H_0 + A(t - t_0)$, and the null geodesics were found numerically for situations in which the source and lens were aligned as seen by the observer. Simulations were run with fixed values for the mass of the central object and distances from the observer to the source and lens, in static coordinates, while we used three different values for $H_0$ and in each case we looked at how the angle of arrival of null geodesics at the observer’s location varied with $A$; in the $A = 0$ case, our results are in agreement with earlier perturbative calculations for light bending in Schwarzschild-de Sitter spacetimes [2, 6].

Each simulation gave us three different values for the angle of arrival. The Euclidean angle (calculated using an auxiliary, fictitious flat spatial metric and therefore not physically measurable, but nevertheless useful for comparison with previous work and as a check on the results) is independent of $H_0$ if there is no acceleration, $A = 0$, but interestingly our simulations show a small dependence on $H_0$ if one considers cases with $A \neq 0$. The two other types of angles are the ones that would be measured by static and comoving observers, respectively, and both show a dependence on $H_0$ as well as on $A$.

Our approach can be improved in various ways, two of which are the fact that we fixed the values of the source and lens coordinates $R_S$ and $R_L$ as if they were directly measurable, and the fact that we considered only cases in which the source, lensing object and observer are aligned. Obtaining results beyond the latter limitation is essentially straightforward, although one will have to extend our criterion for identifying geodesics in different spacetimes to a more general setting. To replace the criterion based on values of $R$ by a more realistic one based on redshifts, one would have to use the relationship between redshifts and distances, which depends on the cosmological expansion history. Related to this is the fact that, although the simulations themselves and the calculations of the arrival angles are non-perturbative, fixing values of $R$ is equivalent to fixing those of actual spatial distances only to leading order in $m$; and using the same values of $H$ in different spacetimes at the time the geodesic leaves the source is equivalent to using the same values of $H$ when the geodesics arrive at the observer’s location also only to leading order.

Unfortunately, the expansion of the universe is such that the current value of $\dot{H}$ is small, and the effects we described would seem to be dwarfed by measurement uncertainties and departures of actual galaxy clusters from the spherically symmetric objects used here to model them. We view this work as a first step in quantifying the effect of $\dot{H}$ on light bending and lensing in a useful way, and it will have to be considerably extended before these extra bending effects can be meaningfully related to measurements.
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