Periodic Quasi-Exactly Solvable Models

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Abstract

Various quasi-exact solvability conditions, involving the parameters of the periodic associated Lamé potential, are shown to emerge naturally in the quantum Hamilton-Jacobi approach. It is found that, the intrinsic nonlinearity of the Riccati type quantum Hamilton-Jacobi equation is primarily responsible for the surprisingly large number of allowed solvability conditions in the associated Lamé case. We also study the singularity structure of the quantum momentum function, which yields the band edge eigenvalues and eigenfunctions.
I. INTRODUCTION

Quasi-exactly solvable (QES) Hamiltonians, interconnecting a diverse array of physical problems, have been the subject of extensive study in recent times [1,2,3,4,5,6]. These systems, containing a finite number of exactly obtainable eigenstates, have been linked with classical electrostatic problems, as also to the finite dimensional irreducible representations of certain algebras. Some of these studies employ group theoretical methods, others are based on the symmetry of the relevant differential equations [2,3,4]. The key to the existence of the finite number of identifiable states is the quasi-exact solvability condition, relating certain potential parameters of these dynamical systems. An interesting feature, distinguishing these QES systems from known exactly solvable cases, is the presence of complex zeros of the wave functions. Hence, quantum Hamilton-Jacobi (QHJ) formalism, being naturally formulated in the complex domain, is ideally suited for studying the QES problems [5]. Although polynomial potentials have been studied rather exhaustively, QES periodic potentials have not received significant attention in the literature.

The associated Lamé potential (ALP)

\[ V(x) = a(a + 1)m \text{sn}^2(x, m) + b(b + 1)m \frac{\text{cn}^2(x, m)}{\text{dn}^2(x, m)}, \]  

(1)

is an interesting example of a periodic potential, which is exactly solvable, when \( a = b \) and shows QES property, when \( a \neq b \). Here, \( \text{sn}(x, m), \text{cn}(x, m) \) and \( \text{dn}(x, m) \) are the doubly periodic elliptic functions with modulus parameter \( m \) [7]. The ALP has a periodic lattice of period \( K(m) \) with the basis composed of two different atoms which are alternately placed. It possesses a surprisingly large variety of QES solvability conditions depending on the nature of the potential parameters \( a \) and \( b \).

For \( a, b \) being unequal integers, with \( a > b > 0 \), there are \( a \) bound bands followed by a continuum band. If \( a - b \) is odd (even) integer, it has \( b \) doubly degenerate band edges of period \( 2K(4K) \), which can not be obtained analytically. For \( a, b \) having half-integral values (with \( a > b \)), there are infinite number of bands with band edge wave functions having period \( 2K(4K) \), if \( a - b \) is odd (even). Of these infinite number of bands, \( a - b \) bands have band edges, which are non-degenerate with period \( 2K(4K) \) and \( b + \frac{1}{2} \) doubly degenerate states of period \( 2K(4K) \) which can be obtained analytically. For \( a \) being an integer and \( b \) being a half integer or vice versa, one can obtain some exact analytical results for mid-band states [8, 9, 10].
It is quite natural to enquire about the origin of this rich QES structure in the associated Lamé potential \[11,12\]. This paper is devoted to the study of the same through the quantum Hamilton-Jacobi approach. The quasi-exact solvability conditions, involving the parameters of the periodic associated Lamé potential, are shown to emerge naturally. The intrinsic nonlinearity of the Riccati type QHJ equation is responsible for these allowed solvability conditions in the associated Lamé case. We also study the singularity structure of the quantum momentum function, which yields the band edge eigenvalues and eigenfunctions.

In our earlier studies, we had looked at non-periodic ES, QES and ES periodic potentials through the QHJ formalism which was initiated by Leacock and Padgett \[13,14\]. We were successful in obtaining the quasi exact solvability condition \[5\] for QES models and in obtaining the eigenvalues and eigenfunctions for both the ES \[15\] and QES models \[5\]. Within the same formalism, we have also proposed a new method to solve ES periodic potentials \[16\] and obtain the band edge eigenvalues and eigenfunctions. The present study makes use of the QHJ formalism as devolved in our earlier papers \[15,5\] and \[16\]. We refer the interested readers to them for all the details.

The QHJ formalism revolves round the logarithmic derivative of the wave function \(p\),

\[
p = -\frac{i\hbar}{dx} \ln \psi \tag{2}
\]

known as the quantum momentum function (QMF), which satisfies a Riccati type equation

\[
p^2 - i\hbar \frac{dp}{dx} = 2M(E - V(x)). \tag{3}
\]

We have found that, the knowledge of the singularity structure of the QMF and the residue at these singular points are the only information needed to obtain the QES condition and the solutions. In all our earlier studies, it was assumed that, the point at infinity is an isolated singularity, which turned out to be true in all the cases. With this same assumptions on \(p\), we proceed to obtain the QES condition and the forms of the band edge wave functions for the general ALP in the next section. In this paper we concentrate on the cases where, \(a\) and \(b\) are either integers or half integers. In section III, we analyze the situation where both \(a\), \(b\) are integers taking, the values 2 and 1, respectively. The case when \(a\) and \(b\) are both half integers, with values 7/2 and 1/2 respectively, is analyzed in section IV.
II. QES CONDITION AND THE FORMS OF THE WAVE FUNCTIONS

The QHJ equation for the ALP, putting $\hbar = 2M = 1$, is given by

$$p^2 - ip' = \left( E - a(a + 1)m \sin^2(x) - b(b + 1)m \frac{\csc^2(x)}{\text{dn}^2(x)} \right).$$  \hspace{1cm} (4)

Note that with the transformation $b \to -b - 1$ or $a \to -a - 1$, the potential does not change. Hence, for our analysis, without loss of generality, we take $a, b$ positive, with $a > b$. Defining $p \equiv -iq$ and substituting it in (4), one obtains

$$q^2 + \frac{dq}{dx} = a(a + 1)m \sin^2(x) + b(b + 1)m \frac{\csc^2(x)}{\text{dn}^2(x)} - E.$$  \hspace{1cm} (5)

Changing the variable to $t \equiv \text{sn} (x)$, and writing $q = \sqrt{(1 - t^2)(1 - mt^2)} \phi$, with $\phi = \chi + \frac{1}{2} \left( \frac{mt}{1 - mt^2} + \frac{t}{1 - t^2} \right)$, Equation (5), gets transformed into

$$\chi^2 + \frac{d\chi}{dt} + \frac{m^2t^2 + 2m(1 - 2b(b + 1))}{4(1 - mt^2)^2} + \frac{2 + t^2}{4(1 - t^2)^2} + \frac{2E - mt^2(1 - 2a(a + 1))}{2(1 - t^2)(1 - mt^2)} = 0.$$  \hspace{1cm} (7)

For all our later calculations, we shall treat $\chi$ as the QMF and (7) as the QHJ equation.

**Singularity Structure :** From (7), we see that $\chi$ has fixed poles at $t = \pm 1$ and $t = \pm \frac{1}{\sqrt{m}}$. In addition to the fixed poles, the QMF has finite number of moving poles and no other singular points in the complex plane. Hence, one can write $\chi$ as a sum of the singular and analytical parts as follows

$$\chi = \frac{b_1}{t - 1} + \frac{b'_1}{t + 1} + \frac{d_1}{t - \frac{1}{\sqrt{m}}} + \frac{d'_1}{t + \frac{1}{\sqrt{m}}} + \frac{P'_n}{P_n} + Q(t),$$  \hspace{1cm} (8)

where $b_1, b'_1, d_1$ and $d'_1$ are the residues at $t = \pm 1$ and $\pm \frac{1}{\sqrt{m}}$ respectively, which need to be calculated. $P_n$ is an $n^{th}$ degree polynomial with $\frac{P'_n}{P_n} = \sum_{k=1}^{n} \frac{1}{r_k}$ being the summation of terms coming from the $n$ moving poles with residue one. The function $Q(t)$ is analytic and bounded at infinity. Hence, from Liouville’s theorem it is a constant, say $C$. The residues at the fixed poles can be calculated by taking the Laurent expansion around each individual pole and substituting them in (7). Comparing the coefficients of different powers of $t$, one gets two values of residues at each pole owing to the quadratic nature of the QHJ equation. Thus the two values of the residues, $b_1, b'_1$ at $t = \pm 1$ are

$$b_1 = \frac{3}{4}, \frac{1}{4} \text{ and } b'_1 = \frac{3}{4}, \frac{1}{4}.$$  \hspace{1cm} (9)
At \( t = \pm \frac{1}{\chi_m} \), one has

\[
d_1 = \frac{3}{4} + \frac{b}{2} \cdot \frac{1}{4} - \frac{b}{2} \quad \text{and} \quad d'_1 = \frac{3}{4} + \frac{b}{2} \cdot \frac{1}{4} - \frac{b}{2}.
\]

(10)

Since, there is no way of ruling out one of the two values of these residues, we need to consider both the values. We demand \( b_1 = b'_1 \) and \( d_1 = d'_1 \), a condition coming from the parity constraint \( \chi(t) = -\chi(t) \).

**Behaviour at infinity:** Equation (8) gives the behaviour of \( \chi \) in the entire complex plane. Hence, for large \( t \),

\[
\chi \sim \frac{2b_1 + 2d_1 + n}{t},
\]

(11)

where the restriction \( b_1 = b'_1 \) and \( d_1 = d'_1 \) has been applied. This should match with the leading behaviour of \( \chi \) obtainable from (7). Note that, the assumption on the singularity structure on \( \chi \) is equivalent to the point at infinity being an isolated singularity. Since \( \chi \) has at most an isolated singular point at infinity, one can expand \( \chi \) in Laurent series around the point at infinity as,

\[
\chi(t) = \lambda_0 + \frac{\lambda_1}{t} + \frac{\lambda_2}{t^2} + \cdots.
\]

(12)

Substituting (12) in (7) and comparing various powers of \( t \), one gets \( \lambda_0 = 0 \), thus making \( Q(t) \) in (8) equal to zero. Further one obtains

\[
\lambda_1 = a + 1, \quad -a.
\]

(13)

Since both the equations (11) and (13) give the leading behavior of \( \chi \) at infinity, both should be equal. Thus

\[
2b_1 + 2d_1 + n = \lambda_1.
\]

(14)

Taking various combinations of \( b_1 \) and \( d_1 \) from (9) and (10), substituting them in (14) one obtains the QES condition for each combination as given in table 1 for \( \lambda = a + 1 \). Thus one sees that all the allowed combinations of residues give one of the forms of QES condition [8], where \( n = 0, 1, 2 \cdots \). Note that, the other value of \( \lambda \), i.e., \( -a \), when substituted instead of \( a+1 \) in (14), gives the QES condition for negative values of \( a, b \) i.e., for \( a \to -a-1, b \to -b-1 \) in \( b_1, d_1 \).

**Forms of wave function:** From (2), one can write \( \psi \) in terms of \( p \) as

\[
\psi(x) = \exp \left( \int ipdx \right).
\]

(15)
Changing the variable to \( t \) and writing \( p \) in terms of \( \chi \), one gets

\[
\psi(x) = \exp \left( \int \left( \chi + \frac{1}{2} \left( \frac{mt}{1-mt^2} + \frac{t}{1-t^2} \right) \right) dt \right). \tag{16}
\]

Substituting \( \chi \) from (8) in the above equation gives the wave function in terms of the residue \( b_1, d_1 \) and the polynomial \( P_n \):

\[
\psi(t) = \exp \left( \int \left( \frac{(1 - 4b_1)t}{2(1-t^2)} + \frac{(1 - 4d_1)mt}{2(1-mt^2)} + \frac{P'_n}{P_n} \right) dt \right). \tag{17}
\]

In terms of the original variable \( x \) the wave function takes the form,

\[
\psi(x) = (\text{cn } x)^\alpha (\text{dn } x)^\beta P_n(\text{sn } x) \tag{18}
\]

where \( \alpha = \frac{4b_1 - 1}{2}, \ \beta = \frac{4d_1 - 1}{2} \). Hence, for each set of \( b_1, d_1 \) one gets a wave function given by (18). The degree \( n \) of this polynomial, which is obtained from (14) as,

\[
n = a + 1 - 2b_1 - 2d_1 \tag{19}
\]

is in terms of either \( a + b \) or \( a - b \), as evident from table 1. The forms of the wave function can be found and are given in table 1, for the two different cases, when \( a + b \) and \( a - b \) are odd and even separately.

**Case 1**, Both \( a + b, a - b \) are even: We introduce \( N = \frac{a+b}{2} \) and \( M = \frac{a-b}{2} \), where \( M \) and \( N \) are integers, and obtain the forms of the wave functions in table 2, in terms of \( M \) and \( N \), for the four sets of combinations of \( b_1 \) and \( d_1 \) in table 1.

**Case 2** Both \( a + b, a - b \) odd: Introducing \( N' = \frac{a+b}{2} \) and \( M' = \frac{a-b}{2} \), where \( M' \) and \( N' \) are integers, we obtain the wave functions, in terms of \( M' \) and \( N' \), for the four sets of combinations of \( b_1 \) and \( d_1 \) in table 1.

From the forms of the wave functions in tables 2 and 3, one observes that the number of linearly independent solutions is different for the two cases. The unknown polynomial in the wave function can be obtained by substituting \( \chi \) from (8) in the QHJ equation (7), which gives

\[
P''_n(t) + 4P_n(t) \left( \frac{b_1 t}{t^2 - 1} + \frac{md_1 t}{mt^2 - 1} \right) + G(t)P_n(t) = 0 \tag{20}
\]
where
\[ G(t) = \frac{t^2(4b_1^2 - 2b_1 + \frac{1}{4}) - 2b_1 + \frac{1}{2}}{(t^2 - 1)^2} + \frac{m^2t^2(4d_1^2 - 2d_1 + \frac{1}{4}) - 2md_1 + m(\frac{1-2b(b+1)}{2})}{(mt^2 - 1)^2} + \frac{2E + (16b_1d_1 - 1 - 2a(a + 1))mt^2}{2(1 - t^2)(1 - mt^2)}. \]

The above differential equation is equivalent to a system of \( n \) linear equations for the coefficients of the different powers of \( t \) in \( P_n(t) \). The energy eigenvalues are obtained by setting the corresponding determinant equal to zero. In the next section we obtain the band edge wave functions for the associated Lamé potential, when \( a = 2 \) and \( b = 1 \).

### III. ALP WITH \( a, b \) INTEGERS:

For this case, we consider the associated Lamé potential with \( a = 2, b = 1 \). For the purpose of comparison with literature, we work with the supersymmetric potential
\[ V_-(x) = 6m \text{sn}^2(x) + 2m \frac{\text{cn}^2 x}{\text{dn}^2 x} - 4m \tag{21} \]

This potential is same as (1) with \( a = 2 \) and \( b = 1 \), except that a constant has been added to make the lowest energy equal to zero. The QHJ equation in terms of \( \chi \) is
\[ \chi^2 + \chi' + \frac{m^2t^2 - 6m}{4(1 - mt^2)^2} + \frac{2 + t^2}{4(1 - t^2)^2} + \frac{2E + 8m - 13mt^2}{2(1 - t^2)(1 - mt^2)} = 0. \tag{22} \]

Apart from \( n \) moving poles, \( \chi \) has poles at \( t = \pm 1 \) and \( t = \pm 1/\sqrt{m} \). As in the previous section one can write \( \chi \), with the parity constraint as
\[ \chi = \frac{2b_1t}{t^2 - 1} + \frac{2md_1t}{t^2 - \frac{1}{m}} + \frac{P'_n(t)}{P_n(t)}. \tag{23} \]

This gives the form of \( \chi \) in the entire complex plane, where \( P_n \) is yet to be determined. Note that for this potential the combination \( a + b \) and \( a - b \) are both odd i.e, 3 and 1 respectively. Hence, we use table 3 to obtain all the information regarding the residues at the fixed poles, number of moving poles of \( \chi \), number of linearly independent solutions and their form, for each set, by taking the values of \( a = 2, b = 1, M' = 0 \) and \( N' = 1 \). The unknown polynomial in the wave function can be obtained from (20), where \( G(t) \) for this potential satisfies,
\[ G(t) = \frac{t^2(4b_1^2 - 2b_1 + \frac{1}{4}) - 2b_1 + \frac{1}{2}}{(t^2 - 1)^2} + \frac{m^2t^2(4d_1^2 - 2d_1 + \frac{1}{4}) - 2md_1 - \frac{3m}{2}}{(mt^2 - 1)^2} + \frac{2E + 8m + (16b_1d_1 - 13)mt^2}{2(1 - t^2)(1 - mt^2)}. \tag{24} \]
Using (20) and (24), one gets the explicit expressions for the eigenfunctions and the eigenvalues as given in table 4. From the table, we see that the first set of residues gives \( n = -1 \), which will not be considered as \( n \) cannot be negative. Thus this particular case of Lamé potential has five band edge solutions, which can be obtained analytically out of an infinite number of possible states.

IV. ALP WITH \( a, b \) HALF INTEGERS :

The potential studied here is the supersymmetric associated Lamé potential, with \( a = \frac{7}{2}, b = \frac{1}{2} \):

\[
V_\pm = \frac{63}{4} m \sin^2 x + \frac{3}{4} m \frac{\csc^2 x}{\csc^2 x} - 2 - \frac{29}{4} m + \delta_9
\]  

where \( \delta_9 = \sqrt{4 - 4m + 25m^2} \). In terms of \( \chi(t) \), the QHJ equation takes the form,

\[
\chi^2 + \chi' + \frac{2 + t^2}{4(t^2 - 1)^2} + \frac{m^2t^2 - m}{4(mt^2 - 1)^2} + \frac{4E + 8 + 29m - 4\delta_9 - 65mt^2}{4(t^2 - 1)(mt^2 - 1)} = 0.
\]

Note that, for this case, \( a + b = 4 \) and \( a - b = 3 \), these are even and odd respectively. Hence, for such cases, one needs to use sets 1 and 3 from table 2 and sets 2 and 4 from table 3 in order to get the four groups of the eigenfunctions. The solutions for this potential are given in table 5. We see that there is a degeneracy in the band edge energy eigenvalue, \( 14 - 7m + \delta_9 \). All the solutions agree with the known results [8].

V. CONCLUSIONS

In conclusion, in this study, we have demonstrated the applicability of QHJ formalism to QES periodic potentials. We have been successful in obtaining the quasi-exact solvability conditions and band edge solutions for cases when both \( a \) and \( b \) are integers or half-integers. The origin of the large number of solvability conditions for the ALP case, comes out naturally in QHJ approach. Interestingly, the singularity structure of the QMF for the QES periodic potentials is similar to that of the ES periodic potentials. This structure is completely different from those of the polynomial potentials. The case where one of the parameter \( a \) and \( b \) is an integer, the other being a half-integer, requires further careful study and will be reported elsewhere.

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TABLE I: The quasi exact solvability condition from the four permitted combinations of $b_1$ and $d_1$ for the general associated Lamé potential.

| set | $b_1$ | $d_1$ | $2b_1 + 2d_1 + n = \lambda_1$ | QES condition |
|-----|-------|-------|-------------------------------|---------------|
| 1   | $3/4$ | $\frac{3}{4} + \frac{b}{2}$ | $2 + b + n = a$ | $b-a = -n -2$ |
| 2   | $3/4$ | $\frac{1}{4} - \frac{b}{2}$ | $1 - b + n = a$ | $a + b + 1 = n + 2$ |
| 3   | $1/4$ | $\frac{3}{4} + \frac{b}{2}$ | $1 + b + n = a$ | $b-a = -n -1$ |
| 4   | $1/4$ | $\frac{1}{4} - \frac{b}{2}$ | $- b + n = a$ | $a + b = n$ |

TABLE II: The form of the wave functions for the four sets of residue combinations when $a + b$ and $a - b$ are even and equal to $2N$ and $2M$ respectively.

| set | $b_1$ | $d_1$ | $n = \lambda_1 - 2b_1 - 2d_1$ | $n(M, N)$ | wave function $\psi(x)$ | LI solutions |
|-----|-------|-------|-------------------------------|------------|--------------------------|--------------|
| 1   | $3/4$ | $\frac{3}{4} + \frac{b}{2}$ | $a - b - 2$ | $2M - 2$ | $cn x (dn x)^{1+b} P_{2M-2} (sn x)$ | $M$ |
| 2   | $3/4$ | $\frac{1}{4} - \frac{b}{2}$ | $a + b - 1$ | $2N - 1$ | $\frac{cn x}{(dn x)^b} P_{2N-1} (sn x)$ | $N$ |
| 3   | $1/4$ | $\frac{3}{4} + \frac{b}{2}$ | $a - b - 1$ | $2M - 1$ | $(dn x)^{b+1} P_{2M-1} (sn x)$ | $M$ |
| 4   | $1/4$ | $\frac{1}{4} - \frac{b}{2}$ | $a + b$ | $2N$ | $P_{2N} (sn x) (dn x)^b$ | $N + 1$ |

TABLE III: The form of the wave functions for the four sets of residue combinations when $a + b$ and $a - b$ are odd and equal to $2N' + 1$ and $2M' + 1$ respectively.

| set | $b_1$ | $d_1$ | $n = \lambda_1 - 2b_1 - 2d_1$ | $n(M', N')$ | wave function $\psi(x)$ | LI solutions |
|-----|-------|-------|-------------------------------|--------------|--------------------------|--------------|
| 1   | $3/4$ | $\frac{3}{4} + \frac{b}{2}$ | $a - b - 2$ | $2M' - 1$ | $cn x (dn x)^{1+b} P_{2M'-1} (sn x)$ | $M'$ |
| 2   | $3/4$ | $\frac{1}{4} - \frac{b}{2}$ | $a + b - 1$ | $2N'$ | $\frac{cn x}{(dn x)^b} P_{2N'} (sn x)$ | $N' + 1$ |
| 3   | $1/4$ | $\frac{3}{4} + \frac{b}{2}$ | $a - b - 1$ | $2M'$ | $(dn x)^{b+1} P_{2M'} (sn x)$ | $M' + 1$ |
| 4   | $1/4$ | $\frac{1}{4} - \frac{b}{2}$ | $a + b$ | $2N' + 1$ | $P_{2N'+1} (sn x) (dn x)^b$ | $N' + 1$ |
TABLE IV: For the Associated Lamé potential $V_-(x) = 6m \text{sn}^2(x) + 2m \frac{cn(x)^2}{dn(x)} - 4m$, with $a = 2$ and $b = 1$ the residues, the value of $n$, number of linear independent solutions, the band edge eigenfunctions and eigenvalues are as follows. Here $a + b = 3$ and $a - b = 1$ which give $N' = 1$ and $M' = 0$.

| set | $b_1$ | $d_1$ | $n$ | LI solutions | eigenfunction $\psi(x)$ | eigenvalues |
|-----|-------|-------|-----|--------------|--------------------------|-------------|
| 1   | 3/4   | 5/4   | -1  | -            | -                        | -           |
| 2   | 3/4   | -1/4  | 2   | 2            | $\frac{cn x}{dn x}(3m \text{sn}^2 x - 2 \pm \sqrt{4 - 3m})$ | $5 - 3m \pm 2\sqrt{4 - 3m}$ |
| 3   | 1/4   | 5/4   | 0   | 1            | $dn^2 x$                 | 0           |
| 4   | 1/4   | -1/4  | 2   | 2            | $\frac{cn x}{dn x}(3m \text{sn}^2 x - 2 - m \pm \sqrt{4 - 5m + m^2})$ | $5 - 2m \pm 2\sqrt{m^2 - 5m + 4}$ |

TABLE V: For the Associated Lamé potential $V_-(x) = \frac{63}{4}m \text{sn}^2 x + \frac{3}{4}m \frac{cn^2 x}{dn^2 x} - 2 - \frac{29}{4}m + \delta_9$ with $a = 7/2$ and $b = 1/2$ the residues, the value of $n$, number of linear independent solutions, the band edge eigenfunctions and eigenvalues are as follows. Here $a + b = 4$ and $a - b = 3$ which give $N = 2$ and $M' = 1$.

| set | $b_1$ | $d_1$ | $n$ | LI solutions | eigenfunction $\psi(x)$ | eigenvalues |
|-----|-------|-------|-----|--------------|--------------------------|-------------|
| 1   | 3/4   | 1     | 1   | 1            | $cn x(dn x)^{3/2} \text{sn} x$ | $\delta_9 - m + 2$ |
| 2   | 3/4   | 0     | 3   | 2            | $cn x(dn x)^{3/2} \text{sn} x$ | $\delta_9 - m + 2$ |
|     |       |       |     |              | $cn x(dn x)^{-1/2} \text{sn} x(1 - 2\text{sn}^2 x)$ | $14 - 7m + \delta_9$ |
| 3   | 1/4   | 1     | 2   | 2            | $(dn x)^{3/2}(12m \text{sn}^2 x - 5m - 2 - \delta_9)$ | 0           |
|     |       |       |     |              | $(dn x)^{3/2}(12m \text{sn}^2 x - 5m - 2 + \delta_9)$ | $2\delta_9$ |
| 4   | 1/4   | 0     | 4   | 3            | $(dn x)^{3/2}(12m \text{sn}^2 x - 5m - 2 - \delta_9)$ | 0           |
|     |       |       |     |              | $(dn x)^{3/2}(12m \text{sn}^2 x - 5m - 2 + \delta_9)$ | $2\delta_9$ |
|     |       |       |     |              | $1 - 8\text{sn}^2 x cn^2 x$ | $14 - 7m + \delta_9$ |