DIAMETER TWO PROPERTIES FOR SPACES OF LIPSCHITZ FUNCTIONS

RAINIS HALLER, ANDRE OSTRAK, AND MÄRT PÖLDVERE

Abstract. We solve some open problems regarding diameter two properties within the class of Banach spaces of real-valued Lipschitz functions by using the de Leeuw transform. Namely, we show that: the diameter two property, the strong diameter two property, and the symmetric strong diameter two property are all different for these spaces of Lipschitz functions; the space Lip_p(K_n) has the symmetric strong diameter two property for every n ∈ ℕ, including the case of n = 2; every local norm-one Lipschitz function is a Daugavet point.

1. Introduction

Let X be a real nontrivial Banach space. We denote the closed unit ball, the unit sphere, and the dual space of X by B_X, S_X, and X*, respectively. A slice of B_X is a set of the form

\[ S(x^*, \alpha) := \{x \in B_X : x^*(x) > 1 - \alpha\}, \]

where \( x^* \in S_{X^*} \) and \( \alpha > 0 \). If X is a dual space, then slices whose defining functional comes from (the canonical image of) the predual of X are called weak* slices.

According to the terminology in [1] and [2] (see also [3]), the Banach space X has the

- **slice diameter 2 property** (briefly, slice-D_2P) if every slice of B_X has diameter 2;
- **diameter 2 property** (briefly, D_2P) if every nonempty relatively weakly open subset of B_X has diameter 2;
- **strong diameter 2 property** (briefly, SD_2P) if every convex combination of slices of B_X has diameter 2, i.e., the diameter of \( \sum_{i=1}^{n} \lambda_i S_i \) is 2 whenever \( n \in \mathbb{N}, \lambda_1, \ldots, \lambda_n \geq 0 \) with \( \sum_{i=1}^{n} \lambda_i = 1 \), and \( S_1, \ldots, S_n \) are slices of B_X;
- **symmetric strong diameter 2 property** (briefly, SSD_2P) if, for every \( n \in \mathbb{N} \), every family \( \{S_1, \ldots, S_n\} \) of slices of B_X, and every \( \varepsilon > 0 \), there exist \( f_1 \in S_1, \ldots, f_n \in S_n \), and \( g \in B_X \) with \( \|g\| > 1 - \varepsilon \) such that \( f_i + g \in S_i \) for every \( i \in \{1, \ldots, n\} \).

If X is a dual space, then we also consider the weak* versions of these diameter two properties (w*-slice D_2P, w*-D_2P, w*-SD_2P, and w*-SSD_2P), where slices and weakly open subsets in the above definitions are replaced by weak* slices and weak* open subsets, respectively.

2020 Mathematics Subject Classification. Primary 46B04; Secondary 46B20.

Key words and phrases. Lipschitz functions spaces, diameter two properties, de Leeuw’s transform, Daugavet points.
In this paper we study diameter two properties in the space Lip$_0(M)$. Let $M$ be a pointed metric space, that is, a metric space with a fixed point 0. The space Lip$_0(M)$ is the Banach space of all Lipschitz functions $f: M \to \mathbb{R}$ with $f(0) = 0$ equipped with the norm 
\[ \|f\| = \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)} : x, y \in M, x \neq y \right\}, \]
i.e., $\|f\|$ is the Lipschitz constant of $f$. The (sub)space $\mathcal{F}(M) = \text{span}\{\delta_x : x \in M\}$ of Lip$_0(M)^*$ is called the Lipschitz-free space over $M$, where $\langle f, \delta_x \rangle = f(x)$ for every $f \in$ Lip$_0(M)$. It can be shown that, under this duality, $\mathcal{F}(M)^*$ is isometrically isomorphic to Lip$_0(M)$.

Diameter two properties of Lip$_0(M)$ have been studied in [9], [15], [7], [14], [4], and [12]. In [9], Ivakhno proved that if a metric space $M$ is unbounded or not uniformly discrete, then the space Lip$_0(M)$ has the slice-D2P. Recall that $M$ is said to be uniformly discrete if $\inf \{d(x,y) : x, y \in M, x \neq y\} > 0$. In [15], Procházka and Rueda Zoca introduced a property of metric spaces that they called the long trapezoid property (briefly, LTP; see the definition on page 12 below), and proved that Lip$_0(M)$ has the $w^*$-SD2P if and only if $M$ has the LTP. In [7], Haller et al. proved that, for every $n \in \mathbb{N}$, the space Lip$_0(K_n)$ has the $w^*$-SD2P (recall that the metric space $K_n$ is the metric subspace of the space $\ell_\infty$ of sequences with terms in \(\{0,1,\ldots,n\}\)). In [14], Ostark introduced a property of metric spaces that he called the strong long trapezoid property (briefly, SLTP; see the definition on page 8 below), and proved that Lip$_0(M)$ has the $w^*$-SD2P if and only if $M$ has the SLTP. He also gave an example of a metric space with the LTP but without the SLTP thus showing that the $w^*$-SD2P and the $w^*$-SD2P for Lip$_0(M)$ are different properties. Note that unbounded metric spaces and not uniformly discrete metric spaces as well as the spaces $K_1, K_2, \ldots$ all have the SLTP. In [5], Cascales et al. proved that if a metric space $M$ has infinitely many cluster points or $M$ is discrete but not uniformly discrete, then the space Lip$_0(M)$ has even the SSD2P. In [12], Langemets and Rueda Zoca generalised this result by proving that the same is true if $M$ is unbounded or not uniformly discrete.

**Theorem 1.1.** If the metric space $M$ is unbounded or not uniformly discrete, then the space Lip$_0(M)$ has the SSD2P.

Theorem 1.1 leaves open for which bounded but not uniformly discrete metric spaces $M$ the space Lip$_0(M)$ has the SSD2P (or SD2P or D2P or slice-D2P). In [12], Langemets and Rueda Zoca proved that the space Lip$_0(K_n)$ has the SSD2P whenever $n \in \mathbb{N}\setminus\{2\}$. In this paper (see Theorem 2.3 and Proposition 2.5 below), we show that, in fact, this is true for every $n \in \mathbb{N}$, including the case of $n = 2$.

**Theorem 1.2.** For every $n \in \mathbb{N}$, the space Lip$_0(K_n)$ has the SSD2P.

In most cases, it seems to be unknown whether for Lip$_0(M)$ the above-mentioned diameter two properties differ from each other. E.g., in [12] Introduction, the authors say that it is not known whether the slice-D2P implies the SSD2P within the class of spaces of Lipschitz functions, and it is not known whether the SSD2P and the $w^*$-SD2P coincide in general. Our Example 3.4, combined with Theorem 3.3 shows that the SD2P does not follow from $w^*$-SD2P for the spaces of Lipschitz functions. Our Example 3.6, combined with Lemma 3.4 shows that the D2P does
not follow from the \( w^*\)-SD2P for the spaces of Lipschitz functions. Therefore, we have the following.

**Theorem 1.3.** The SSD2P, the SD2P, and the D2P are three different properties for the spaces of Lipschitz functions. In fact, the \( w^*\)-SSD2P and the SD2P are different, and the \( w^*\)-SD2P and the D2P are different for the spaces of Lipschitz functions.

It remains open whether, for the spaces of Lipschitz functions, any of the above-mentioned (non-weak\*) diameter two properties coincides with its weak\* version. In fact, we don’t even know if, for the spaces of Lipschitz functions, the \( w^*\)-SSD2P implies the slice-D2P. It also remains open whether there exists a metric space \( M \) such that \( \text{Lip}_0(M) \) has the slice-D2P but not the D2P.

In [11], Jung and Rueda Zoca studied Daugavet points in \( \text{Lip}_0(M) \) in connection with locality properties of Lipschitz functions. They posed and addressed the question of whether every local norm-one \( f \) in \( \text{Lip}_0(M) \) is a Daugavet point. We answer that question with the following theorem.

**Theorem 1.4 (cf. [11, Proposition 3.4 and Theorem 3.6]).** Let \( M \) be a pointed metric space. If \( f \) is local, then \( f \) is a Daugavet point.

In general, one faces difficulties when dealing with the dual space \( \text{Lip}_0(M) \) due to the lack of a useful characterisation of the space. Our results will heavily rely on the following observation.

Set
\[
\Gamma_1 = \{(x, y) \in \widetilde{M} : x \in A\} \quad \text{and} \quad \Gamma_2 = \{(x, y) \in \widetilde{M} : y \in A\}.
\]

Given a metric space \( M \), a point \( x \) in \( M \), and \( r \geq 0 \), we denote by \( B(x, r) \) the open ball in \( M \) centred at \( x \) of radius \( r \). For \( x, y \in M \) with \( x \neq y \), we denote by \( m_{x,y} \) the norm-one element \( \delta_x - \delta_y \) in \( F(M) \).

### 2. The SSD2P for spaces of Lipschitz functions

We start this section by giving two sufficient conditions for the space \( \text{Lip}_0(M) \) to have the SSD2P. The first one is a consequence of identifying the dual space of \( \text{Lip}_0(M) \) via the de Leeuw’s transform. From this, we derive the second one, which involves only conditions on the metric of the space \( M \) and which we will then use to prove Theorems [11] and [12].
Lemma 2.1. Let $M$ be a pointed metric space and let $\tilde{M}$ be as in (1.1). Suppose that, whenever $\delta > 0$, $n \in \mathbb{N}$, $h_1, \ldots, h_n \in \text{Lip}_0(M)$ with $\|h_i\| \leq 1 - \delta$ for every $i \in \{1, \ldots, n\}$, and $\mu \in ba(\tilde{M})$ with only non-negative values, there exist a subset $A$ of $M$ and functions $f_1, \ldots, f_n, g \in \text{Lip}_0(M)$ satisfying

- $\mu(\Gamma_{1,A}) < \delta$ and $\mu(\Gamma_{2,A}) < \delta$;
- $f_i|_{M \setminus A} = h_i|_{M \setminus A}$ for every $i \in \{1, \ldots, n\}$;
- $g|_{M \setminus A} = 0$ and $|g| \geq 1 - \delta$;
- $\|f_i \pm g\| \leq 1$ for every $i \in \{1, \ldots, n\}$.

Then the space $\text{Lip}_0(M)$ has the SSD2P.

Proof. Let $n \in \mathbb{N}$, let $F_1, \ldots, F_n \in \text{Lip}_0(M)^*$, and let $\varepsilon > 0$. It suffices to find $f_i \in S(F_i, \varepsilon)$, $i = 1, \ldots, n$, and $g \in \text{Lip}_0(M)$ with $\|g\| > 1 - \varepsilon$ such that $f_i \pm g \in S(F_i, \varepsilon)$ for every $i \in \{1, \ldots, n\}$.

For every $i \in \{1, \ldots, n\}$, let $\mu_i \in ba(\tilde{M})$ with $|\mu_i|(\tilde{M}) = 1$ satisfy (1.2) with $F$ and $\mu$ replaced by $F_i$ and $\mu_i$, respectively. Define $\mu = |\mu_1| + \cdots + |\mu_n|$. Fix a real number $\delta > 0$ satisfying $8\delta \leq \varepsilon$. For every $i \in \{1, \ldots, n\}$, pick a function $h_i \in S(F_i, 2\delta)$ with $\|h_i\| \leq 1 - \delta$.

Let a subset $A$ of $M$ and functions $f_1, \ldots, f_n, g \in \text{Lip}_0(M)$ satisfy the conditions in the lemma. Setting $\Gamma_A = \Gamma_{1,A} \cup \Gamma_{2,A}$, one has $\mu(\Gamma_A) < 2\delta$, hence, whenever $i \in \{1, \ldots, n\}$,

$$|F_i(g)| = \left| \int_{\tilde{M}} \tilde{g} \, d\mu_i \right| = \left| \int_{\Gamma_A} \tilde{g} \, d\mu_i \right| \leq |\mu_i|(\Gamma_A) < 2\delta$$

and (observing that $\tilde{f}_i|_{\tilde{M} \setminus \Gamma_A} = \tilde{h}_i|_{\tilde{M} \setminus \Gamma_A}$)

$$F_i(f_i) = \int_{\tilde{M}} \tilde{f}_i \, d\mu_i = \int_{\tilde{M}} \tilde{h}_i \, d\mu_i + \int_{\Gamma_A} (\tilde{f}_i - \tilde{h}_i) \, d\mu_i$$

$$= F_i(h_i) + \int_{\Gamma_A} (\tilde{f}_i - \tilde{h}_i) \, d\mu_i$$

$$> 1 - 2\delta - 2|\mu_i|(\Gamma_A) > 1 - 6\delta,$$

and thus

$$F_i(f_i \pm g) \geq F_i(f_i) - |F_i(g)| > 1 - 6\delta - 2\delta \geq 1 - \varepsilon.$$  

One can prove Theorems 1.1 and 1.2 by directly applying Lemma 2.1. However, we prefer to first prove (and then use) a further sufficient condition for $\text{Lip}_0(M)$ to have the SSD2P, which involves only conditions on the metric of the space $M$ and is therefore easy to handle.

Definition 2.2 (cf. [13] Definition 1.3). We say that a metric space $M$ has the sequential strong long trapezoid property (briefly, seq-SLTP) if, for every $\varepsilon > 0$, there exist pairwise disjoint subsets $A_1, A_2, \ldots$ of $M$ such that, for every $m \in \mathbb{N}$, there are $u_m, v_m \in A_m$ with $u_m \neq v_m$ satisfying, for all $x, y \in M \setminus A_m$,

$$d(x, y) + d(u_m, v_m) \leq d(x, u_m) + d(y, v_m),$$

and, for all $x, y, z, w \in M \setminus A_m$,

$$d(x, y) + d(z, w) + 2d(u_m, v_m) \leq d(x, u_m) + d(y, u_m) + d(z, v_m) + d(w, v_m).$$


Theorem 2.3 (cf. [13] the proof of Theorem 2.1, (ii)⇒(i)). Let $M$ be a pointed metric space. If $M$ has the seq-SLTP, then $\text{Lip}_0(M)$ has the SSDP.

Remark. We do not know whether the converse of Theorem 2.3 holds. However, the seq-SLTP is strictly stronger than the SLTP (see Example 2.4 below).

Proof of Theorem 2.3. Assume that $M$ has the seq-SLTP. Let $\delta > 0$, $n \in \mathbb{N}$, $h_1, \ldots, h_n \in \text{Lip}_0(M)$ with $\|h_i\| \leq 1 - \delta$ for every $i \in \{1, \ldots, n\}$, and let $\mu \in \text{ba}(\widehat{M})$ with only non-negative values where $\widehat{M}$ is as in (1.1). By Lemma 2.1 it suffices to find a subset $A$ of $M$ and functions $f_1, \ldots, f_n, g \in \text{Lip}_0(M)$ satisfying the conditions of that lemma.

By the seq-SLTP, there exist subsets $A_1, A_2, \ldots$ of $M$ and points $u_m, v_m \in A_m$, $m = 1, 2, \ldots$, as in Definition 2.2 with $\varepsilon = \delta$. Since the sets $A_1, A_2, \ldots$ are pairwise disjoint, there exists an $m \in \mathbb{N}$ such that $\mu(\Gamma_{1,A_m}) < \delta$ and $\mu(\Gamma_{2,A_m}) < \delta$. Let $A = A_m$, $u = u_m$, and $v = v_m$. In order to define the suitable functions $f_1, \ldots, f_n, g$, we follow the idea of [13] proof of Theorem 2.1, (ii)⇒(i)].

Setting
\[
    r_0 = \frac{1}{2} \inf_{x, y \in M \setminus A} (d(x, u) + d(y, u) - (1 - \delta)d(x, y)),
\]
and
\[
    s_0 = \frac{1}{2} \inf_{z, w \in M \setminus A} (d(z, v) + d(w, v) - (1 - \delta)d(z, w)),
\]
we have $r_0 + s_0 \geq (1 - \delta)d(u, v)$. Thus, there exist $r, s \geq 0$ with $r \leq r_0$ and $s \leq s_0$ such that
\[
    r + s = (1 - \delta)d(u, v).
\]
We may assume that $r > 0$. Define a function $g : M \to \mathbb{R}$ by
\[
    g(x) = \begin{cases} 
        r - d(x, u) & \text{if } x \in B(u, r); \\
        -s + d(x, v) & \text{if } x \in B(v, s); \\
        0 & \text{otherwise.}
    \end{cases}
\]
Observe that $|g| \leq 1$ (here we use that, whenever $x \in B(u, r)$ and $y \in B(v, s)$, one has $g(y) \leq 0 \leq g(x)$, and thus $|g(x) - g(y)| = g(x) - g(y)$). One also has $|g| \geq 1 - \delta$, because
\[
    |g(u) - g(v)| = g(u) - g(v) = r + s = (1 - \delta)d(u, v).
\]
Set $L = (M \setminus A) \cup B$, where $B = B(u, r) \cup B(v, s)$. Observe that $B \cap (M \setminus A) = \emptyset$. Fix $i \in \{1, \ldots, n\}$. We first define $f_i$ on the set $L$. Let $f_i|_{M \setminus A} = h_i|_{M \setminus A}$. We next show that there is a $c_i \in \mathbb{R}$ such that, by defining $f_i|_B = c_i$, one has $\|f_i \pm g\|_{\text{Lip}_0(L)} \leq 1$ and $\|f_i \pm g\|_{\text{lip}_0(L)} \leq 1$.

Set
\[
    \tilde{a}_i = \sup_{x \in M \setminus A} (h_i(x) - d(x, u)), \quad \hat{a}_i = \inf_{x \in M \setminus A} (h_i(x) + d(x, u)),
\]
\[
    \tilde{b}_i = \sup_{x \in M \setminus A} (h_i(x) - d(x, v)), \quad \hat{b}_i = \inf_{x \in M \setminus A} (h_i(x) + d(x, v)).
\]
Whenever $x, y \in M \setminus A$, since $\|h_i\| \leq 1 - \delta$, one has
\[
    h_i(x) + d(x, u) - (h_i(y) - d(y, u)) \geq d(x, u) + d(y, u) - (1 - \delta)d(x, y) \geq 2r,
\]
and, by \((2.1)\),
\[
  h_i(x) + d(x, u) - (h_i(y) - d(y, v)) \geq d(x, u) + d(y, v) - (1 - \delta)d(x, y)
\]
\[
  \geq (1 - \delta)d(u, v) > r + s.
\]
Thus, \(\hat{a}_i - r \geq \tilde{a}_i + r\) and \(\hat{a}_i - r \geq \tilde{b}_i + s\). Similarly, one observes that \(\hat{b}_i - s \geq \tilde{b}_i + s\) and \(\widehat{b}_i - s \geq \tilde{a}_i + r\). It follows that there exists a \(c_i \in [\hat{a}_i + r, \hat{a}_i - r] \cap [\tilde{b}_i + s, \tilde{b}_i - s]\). This \(c_i\) does the job. By setting
\[
  f_i(y) = \sup_{x \in L} (f_i(x) + |g(x)| - d(x, y)) \quad \text{for every } y \in M \setminus L,
\]
one has \(\|f_i \pm g\|_{\lip_p(M)} \leq 1\). For details, we refer the reader to the corresponding parts of [13] proof of Theorem 2.1, (ii)\(\Rightarrow\)(i), where the argumentation reads nearly word-for-word when replacing \(N\) by \(M \setminus A\).

Theorems 1.1 and 1.2 are immediate corollaries of Theorem 2.3 teamed with the following Propositions 2.4 and 2.5, respectively.

**Proposition 2.4.** An unbounded or not uniformly discrete metric space has the seq-SLTP.

**Proposition 2.5.** For every \(n \in \mathbb{N}\), the metric space \(K_n\) has the seq-SLTP.

In the proof of Proposition 2.4 we make use of the following lemma.

**Lemma 2.6.** Let \(\varepsilon > 0, p \in M, 0 \leq s < r\), and \(u, v \in B(p, r) \setminus B(p, s)\) be such that
\[
  4s \leq \varepsilon d(u, v)
\]
and, for every \(x \in M \setminus B(p, r)\), one has
\[
  2d(u, v) \leq \varepsilon \min_{x \in L} \{d(x, u), d(x, v)\}.
\]
Then, for all \(x, y, z, w \in M \setminus A\), where \(A = B(p, r) \setminus B(p, s)\), one has
\[
  (1 - \varepsilon)(d(x, y) + d(u, v)) \leq d(x, u) + d(y, v)
\]
and
\[
  (1 - \varepsilon)(d(x, y) + d(z, w) + 2d(u, v)) \leq d(x, u) + d(y, u) + d(z, v) + d(w, v).
\]

**Proof.** Let \(x, y, z, w \in M \setminus A\). First suppose that \(x, y, z, w \in B(p, s)\). For this case, observe that, whenever \(a, b, c, d \in B(p, s)\), one has
\[
  \varepsilon d(u, v) \geq 4s \geq d(a, b) + d(c, d)
\]
and thus
\[
  d(a, u) + d(b, v) \geq (1 - \varepsilon)d(u, v) + \varepsilon d(u, v) - d(a, b)
\]
\[
  \geq (1 - \varepsilon)(d(u, v) + d(c, d)).
\]
Taking, in the above, \(a = c = x\) and \(b = d = y\), one obtains (2.3). Taking, respectively, \(a = c = x, b = z,\) and \(d = y\), and \(a = y, b = d = w,\) and \(c = z\), one obtains
\[
  d(x, u) + d(z, v) \geq (1 - \varepsilon)(d(u, v) + d(x, y))
\]
and
\[
  d(y, u) + d(w, v) \geq (1 - \varepsilon)(d(u, v) + d(z, w)),
\]
and (2.4) follows.
Now suppose that (at least) one of the points \( x \) and \( y \), say \( x \), is not in \( B(0, r) \). In this case
\[
d(x, u) + d(y, v) \geq (1 - \varepsilon)d(x, u) + 2d(u, v) + d(y, v)
\]
\[
\geq (1 - \varepsilon)(d(x, u) + d(y, v) + d(u, v))
\]
\[
\geq (1 - \varepsilon)(d(x, y) + d(u, v)).
\]
Finally, if (at least) one of the points \( x, y, z, w \), say, \( x \), is not in \( B(0, r) \), then
\[
d(x, u) + d(y, u) + d(z, v) + d(w, v)
\]
\[
\geq (1 - \varepsilon)(d(x, u) + d(y, u) + d(z, v) + d(w, v)) + 2d(u, v)
\]
\[
\geq (1 - \varepsilon)(d(x, y) + d(z, w) + 2d(u, v)),
\]
and the proof is complete. \( \Box \)

**Proof of Proposition 2.4.** Let \( M \) be a pointed metric space, and let \( \varepsilon > 0 \).

First assume that \( M \) is unbounded. Letting \( r_0 = 1 \), we can inductively define points \( u_m, v_m \in M \setminus B(0, r_{m-1}) \) with \( u_m \neq v_m \) and real numbers \( r_m > 0 \), \( m = 1, 2, \ldots \), satisfying, for every \( m \in \mathbb{N} \), the inequalities \( 4r_m - 1 < \varepsilon d(u_m, v_m) \), \( r_m > r_{m-1} \), and
\[
2d(u_m, v_m) \leq \varepsilon \min \{d(x, u_m), d(x, v_m)\} \quad \text{for every } x \in M \setminus B(0, r_m).
\]
Now the sets \( A_m := B(0, r_m) \setminus B(0, r_{m-1}) \), \( m = 1, 2, \ldots \), are pairwise disjoint. For every \( m \in \mathbb{N} \), Lemma 2.6 with \( p = 0 \), \( r = r_m \), \( s = r_{m-1} \), \( u = u_m \), and \( v = v_m \) implies that, for all \( x, y, z, w \in M \setminus A_m \), the inequalities (2.1) and (2.2) hold.

Assume now that \( M \) is not uniformly discrete. We first consider the case when \( M \) has a limit point \( p \). Starting with \( r_1 = 1 \), we can inductively define points \( u_m, v_m \in B(p, r_m) \setminus \{p\} \) with \( u_m \neq v_m \) and real numbers \( r_m > 0 \), \( m = 1, 2, \ldots \), satisfying, for every \( m \in \mathbb{N} \), the condition (2.5), \( r_{m+1} < r_m \), and \( 4r_{m+1} - 1 < \varepsilon d(u_m, v_m) \). Now the sets \( A_m := B(p, r_m) \setminus B(p, r_{m+1}) \), \( m = 1, 2, \ldots \), are pairwise disjoint. For every \( m \in \mathbb{N} \), Lemma 2.6 with \( r = r_m \), \( s = r_{m+1} \), \( u = u_m \), and \( v = v_m \) implies that, for all \( x, y, z, w \in M \setminus A_m \), the inequalities (2.1) and (2.2) hold.

Finally, consider the case when \( M \) has no limit points. Since \( M \) is not uniformly discrete, there exist points \( u_m, v_m \in M \) with \( u_m \neq v_m \), \( m = 1, 2, \ldots \), such that \( d(u_m, v_m) \to 0 \). Since \( M \) has no limit points, we may assume, after passing to subsequences if necessary, that there is an \( r > 0 \) such that \( d(u_m, u_n) \geq 2r \) whenever \( m, n \in \mathbb{N} \) with \( m \neq n \). Furthermore, we may assume that \( v_m \in B(u_m, r) \) and \( 4d(u_m, v_m) \leq \varepsilon r \) for every \( m \in \mathbb{N} \). Now the sets \( A_m := B(u_m, r) \), \( m = 1, 2, \ldots \), are pairwise disjoint. For every \( m \in \mathbb{N} \) and every \( x \in M \setminus B(u_m, r) \), one has
\[
\varepsilon d(x, u_m) \geq \varepsilon r > 2d(u_m, v_m)
\]
and
\[
\varepsilon d(x, v_m) \geq \varepsilon (d(x, u_m) - d(u_m, v_m)) > \varepsilon (r - \frac{1}{4}r) = \frac{3}{4} \varepsilon r \geq 2d(u_m, v_m),
\]
thus Lemma 2.6 with \( p = u = u_m \), \( s = 0 \), and \( v = v_m \) implies that, for all \( x, y, z, w \in M \setminus A_m \), the inequalities (2.1) and (2.2) hold. \( \Box \)

**Proof of Proposition 2.4.** Let \( n \in \mathbb{N} \). For every \( m \in \mathbb{N} \), define
\[
A_m = \{(x_j)_{j=1}^\infty \in K_n : \max x_j = n \text{ and } x_j < n \text{ for every } j \notin \{2m - 1, 2m\}\},
\]
sets $u = n \epsilon_{2m-1} + (n-1) \epsilon_{2m}$, and $v = (n-1) \epsilon_{2m-1} + n \epsilon_{2m}$. Note that the sets $A_1, A_2, \ldots$ are pairwise disjoint. Fix an $m \in \mathbb{N}$. Clearly, $u_m, v_m \in A_m$ and $d(u_m, v_m) = 1$. We show that, for all $x, y \in K_n \setminus A_m$,
\[ d(x, y) + d(u_m, v_m) \leq d(x, u_m) + d(y, v_m), \]
and, for all $x, y, z, w \in K_n \setminus A_m$,
\[ d(x, y) + d(z, w) + 2d(u_m, v_m) \leq d(x, u_m) + d(y, u_m) + d(z, v_m) + d(w, v_m). \]

Fix $x = (x_j)_{j=1}^\infty, y = (y_j)_{j=1}^\infty \in K_n \setminus A_m$. It suffices to show that the following inequalities hold:
\[ d(x, y) + d(u_m, v_m) \leq d(x, u_m) + d(y, v_m), \]
\[ d(x, y) + d(u_m, v_m) \leq d(x, u_m) + d(y, u_m), \]
\[ d(x, y) + d(u_m, v_m) \leq d(x, v_m) + d(y, v_m). \]

To this end, let $j \in \mathbb{N}$ be such that $d(x, y) = |x_j - y_j|$. Without loss of generality, we assume that $x_j \geq y_j$. Notice that, if $j \notin \{2m - 1, 2m\}$, then $x_j \geq d(x, y)$, and therefore $d(x, u_m) \geq d(x, y)$ and $d(x, v_m) \geq d(x, y)$. If $j \in \{2m - 1, 2m\}$, then $y_j \leq n - 1 - d(x, y)$ because $x_j \leq n - 1$, and hence $d(y, u_m) \geq d(x, y)$ and $d(y, v_m) \geq d(x, y)$. Since $d(u_m, v_m) = 1$, the desired inequalities hold.

Recall [13] Definition 1.3 that a metric space $M$ has the strong long trapezoid property (briefly, SLTP) if, for every $\varepsilon > 0$ and every finite subset $N$ of $M$, there exist elements $u, v \in M$ with $u \neq v$ satisfying, for all $x, y \in N$,
\[ (1 - \varepsilon)(d(x, y) + d(u, v)) \leq d(x, u) + d(y, v), \]
and, for all $x, y, z, w \in N$,
\[ (1 - \varepsilon)(d(x, y) + d(z, w) + 2d(u, v)) \leq d(x, u) + d(y, u) + d(z, v) + d(w, v). \]

We end this section by giving an example of a metric space $M$ with the SLTP but without the seq-SLTP. In fact, this $M$ does not even have the seq-LTP (see Definition 3.2 below). By [14] Theorem 2.1, the space $\text{Lip}_0(M)$ has the $w^*$-SSD2P. It remains unknown whether $\text{Lip}_0(M)$ has the SSD2P.

**Example 2.7.** Let $M = \{a_k, b_k, c_k : k \in \mathbb{N}\}$ be the metric space where, for every $k \in \mathbb{N}$,
\[ d(a_k, c_k) = 2, \]
and, for all $k, l \in \mathbb{N}$ with $k < l$,
\[ d(a_k, b_l) = d(b_k, b_l) = d(c_k, b_l) = 2, \]
and the distance between two different elements is 1 in all other cases.

We first show that the space $M$ has the SLTP. To this end, let $N$ be a finite subset of $M$. Then there exists a $K \in \mathbb{N}$ such that $N \subseteq \{a_k, b_k, c_k : k < K\}$. Let $u = b_K$ and $v = b_{K+1}$. Then, for every $x \in N$, one has $d(x, u) = 2$ and $d(x, v) = 2$, and therefore, for all $x, y \in N$,
\[ d(x, y) + d(u, v) \leq 4 = d(x, u) + d(y, v), \]
and, for all $x, y, z, w \in N$,
\[ d(x, y) + d(z, w) + 2d(u, v) \leq 8 = d(x, u) + d(y, u) + d(z, v) + d(w, v). \]
Now suppose for contradiction that $M$ has the seq-SLTP. Then there exist pairwise disjoint subsets $A_1$, $A_2$, and $A_3$ of $M$ such that, for every $m \in \{1, 2, 3\}$, there are $u_m, v_m \in A_m$ with $u_m \neq v_m$ such that the inequality \ref{2.1} holds for all $x, y \in M \setminus A_m$. Let $K \in \mathbb{N}$ be such that $u_1, v_1, u_2, v_2, u_3, v_3 \in \{a_k, b_k, c_k : k < K\}$. For every $m \in \{1, 2, 3\}$, one has

\[(1 - \varepsilon)(d(a_K, c_K) + d(u_m, v_m)) \geq 3(1 - \varepsilon) > 2 = d(a_K, u_m) + d(c_K, v_m),\]

which implies $a_K \in A_m$ or $c_K \in A_m$. It follows that $A_1, A_2$, and $A_3$ are not pairwise disjoint, a contradiction.

3. The SSD2P, SD2P, and D2P are three different properties for spaces of Lipschitz functions

In this section, we give an example of a metric space $M$ such that the corresponding space $\operatorname{Lip}_0(M)$ has the SD2P but fails the SSD2P, and of a metric space $M$ such that the corresponding space $\operatorname{Lip}_0(M)$ has the D2P but fails the SD2P, thus showing that the SSD2P, SD2P, and D2P are three different properties for the spaces of Lipschitz functions. This answers an implicit question in \cite[Introduction]{12}. For these two examples, we first give sufficient conditions for the space of Lipschitz functions to have the SD2P, and the D2P, as we did for the SSD2P in Section 2. We start with an analogue of Lemma \ref{2.1} for the SD2P.

**Lemma 3.1.** Let $M$ be a pointed metric space and let $\tilde{M}$ be as in \ref{1.1}. Suppose that, whenever $\delta > 0$, $n \in \mathbb{N}$, $h_1, \ldots, h_n \in \operatorname{Lip}_0(M)$ with $\|h_i\| \leq 1 - \delta$ for every $i \in \{1, \ldots, n\}$, and $\mu \in ba(\tilde{M})$ with only non-negative values, there exist a subset $A$ of $M$, elements $u, v \in A$ with $u \neq v$, and functions $f_1, \ldots, f_n \in B_{\operatorname{Lip}_0(M)}$ satisfying

- $\mu(\Gamma_{1, A}) < \delta$ and $\mu(\Gamma_{2, A}) < \delta$;
- $f_i|_{M \setminus A} = h_i|_{M \setminus A}$ for every $i \in \{1, \ldots, n\}$;
- $f_i(u) - f_i(v) \geq (1 - \delta)d(u, v)$ for every $i \in \{1, \ldots, n\}$.

Then the space $\operatorname{Lip}_0(M)$ has the SD2P.

**Proof.** Let $n \in \mathbb{N}$, let $F_1, \ldots, F_n \in S_{\operatorname{Lip}_0(M)}$, and let $\varepsilon > 0$. By \cite[Corollary 2.2]{3} and \cite[Proposition 2.2]{8}, it suffices to find $u, v \in M$ such that $\|F_i + m_{u,v}\| \geq 2 - \varepsilon$ for every $i \in \{1, \ldots, n\}$.

For every $i \in \{1, \ldots, n\}$, let $\mu_i \in ba(\tilde{M})$ with $|\mu_i|(\tilde{M}) = 1$ satisfy \ref{1.2} with $F$ and $\mu$ replaced by $F_i$ and $\mu_i$, respectively. Define $\mu = |\mu_1| + \cdots + |\mu_n|$. Fix a real number $\delta > 0$ satisfying $7\delta \leq \varepsilon$. For every $i \in \{1, \ldots, n\}$, pick a function $h_i \in S(F_i, 2\delta)$ with $\|h_i\| \leq 1 - \delta$. Let a subset $A$ of $M$, elements $u, v \in A$ with $u \neq v$, and functions $f_1, \ldots, f_n \in \operatorname{Lip}_0(M)$ satisfy the conditions in the lemma. Setting $\Gamma_A = \Gamma_{1, A} \cup \Gamma_{2, A}$, one has $\mu(\Gamma_A) < 2\delta$, hence, whenever $i \in \{1, \ldots, n\}$, (observing that $\tilde{f}_i|_{M \setminus \Gamma_A} = \tilde{h}_i|_{M \setminus \Gamma_A}$)

\[
F_i(f_i) = \int_{\tilde{M}} \tilde{f}_i d\mu_i = \int_{\tilde{M}} \tilde{h}_i d\mu_i + \int_{\Gamma_A} (\tilde{f}_i - \tilde{h}_i) d\mu_i
\]

\[
= F_i(h_i) + \int_{\Gamma_A} (\tilde{f}_i - \tilde{h}_i) d\mu_i
\]

\[
> 1 - 2\delta - 2|\mu_i|(\Gamma_A) > 1 - 6\delta,
\]
and thus
\[(F_i + m_{u,v})(f_i) = F_i(f_i) + \frac{f_i(u) - f_i(v)}{d(u, v)} > 1 - 6\delta + 1 - \delta \geq 2 - \varepsilon.\]

\[\square\]

We next prove (and then use) a further sufficient condition for \(\text{Lip}_0(M)\) to have the SD2P—an analogue of Theorem 2.3—which involves only conditions on the metric of the space \(M\) and is therefore easy to handle.

**Definition 3.2** (cf. [13, Theorem 3.1, (3)]). We say that a metric space \(M\) has the *sequential long trapezoid property* (briefly, seq-LTP) if, for every \(\varepsilon > 0\), there exist pairwise disjoint subsets \(A_1, A_2, \ldots\) of \(M\) such that, for every \(m \in \mathbb{N}\), there are \(u_m, v_m \in A_m\) with \(u_m \neq v_m\) satisfying, for all \(x, y \in M\),

\[(1 - \varepsilon)(d(x, y) + d(u_m, v_m)) \leq d(x, u_m) + d(y, v_m).\]

Clearly every metric space with the seq-SLTP has the seq-LTP as the conditions (2.1) and (3.1) are the same.

**Theorem 3.3** (cf. [13, Theorem 3.1, (3)\(\Rightarrow\)1)]). Let \(M\) be a pointed metric space. If \(M\) has the seq-LTP, then \(\text{Lip}_0(M)\) has the SD2P.

Remark. We do not know whether the converse of Theorem 3.3 holds. Note that the seq-LTP is strictly stronger than the LTP (see Example 2.7 above).

**Proof of Theorem 3.3**. Assume that \(M\) has the seq-LTP. Let \(\delta > 0\), let \(n \in \mathbb{N}\), let \(h_1, \ldots, h_n \in \text{Lip}_0(M)\) with \(\|h_i\| \leq 1 - \delta\) for every \(i \in \{1, \ldots, n\}\), and let \(\mu \in \text{ba}(\hat{M})\) with only non-negative values where \(\hat{M}\) is as in (1.1). By Lemma 3.4, it suffices to find a subset \(A\) of \(M\), elements \(u, v \in A\) with \(u \neq v\), and functions \(f_1, \ldots, f_n \in \text{Lip}_0(M)\) satisfying the conditions of that lemma.

By the seq-LTP, there exist subsets \(A_1, A_2, \ldots\) of \(M\) and points \(u_m, v_m \in A_m\), \(m = 1, 2, \ldots\), as in Definition 3.2 with \(\varepsilon = \delta\). Since the sets \(A_1, A_2, \ldots\) are pairwise disjoint, there exists an \(m \in \mathbb{N}\) such that \(\mu(\Gamma_{1,A_m}) < \delta\) and \(\mu(\Gamma_{2,A_m}) < \delta\). Let \(A = A_m\), \(u = u_m\), and \(v = v_m\).

Fix \(i \in \{1, \ldots, n\}\). Define the function \(f_i\) by \(f_i|_{M\setminus A} = h_i|_{M\setminus A}\),

\[f_i(u) = \inf_{x \in M\setminus A} (f_i(x) + d(x, u)),\]

and

\[f_i(y) = \sup_{x \in \{u\} \cup M\setminus A} (f_i(x) - d(x, y)) \quad \text{for every } y \in A\setminus \{u\}.\]

Since \(\|f_i\| \leq 1\), it remains to show that \(f_i(u) - f_i(v) \geq (1 - \delta)d(u, v)\). If \(f_i(v) = f_i(u) - d(u, v)\), then the inequality holds. Suppose now that this is not the case. Then

\[f_i(u) - f_i(v) = \inf_{x,y \in M\setminus A} (f_i(x) + d(x, u) - f_i(y) + d(y, v))\]

\[\geq \inf_{x,y \in M\setminus A} \left( -(1 - \delta)d(x, y) + d(x, u) + d(y, v) \right)\]

\[\geq (1 - \delta)d(u, v).\]

\[\square\]
The following example from [13, Example 3.1] was the first known example of a metric space $M$ which has the LTP but not the SLTP or, equivalently, for which the corresponding Lipschitz function space $\text{Lip}_0(M)$ has the $w^*$-SD2P but not the $w^*$-SSD2P.

We further show that $M$ has the seq-LTP. Therefore, $\text{Lip}_0(M)$ has the SD2P but not the $w^*$-SSD2P. To our knowledge, this is the first known example of such a Lipschitz function space, showing that the properties SD2P and ($w^*$-)SSD2P are really different for the class of Lipschitz function spaces.

**Example 3.4.** Let $M = \{a_1, a_2, b_1, b_2\} \cup \{u_m, v_m : m \in \mathbb{N}\}$ be the metric space where, for all $i, j \in \{1, 2\}$ and all $m \in \mathbb{N}$,

$$d(a_i, b_j) = d(a_i, u_m) = d(b_j, v_m) = d(u_m, v_m) = 1,$$

and the distance between two different elements is 2 in all other cases.

We show that $M$ has the seq-LTP. To this end, it suffices to show that, whenever $m \in \mathbb{N}$, one has, for all $x, y \in M \setminus A_m$,

$$d(x, y) + d(u_m, v_m) \leq d(x, u_m) + d(y, v_m),$$

where $A_m = \{u_m, v_m\}$. Fix an $m \in \mathbb{N}$ and let $x, y \notin A_m$. If $d(x, u_m) + d(y, v_m) \leq 3$, then the desired inequality holds because $d(u_m, v_m) = 1$. If $d(x, u_m) + d(y, v_m) = 2$, then $x \in \{a_1, a_2\}$ and $y \in \{b_1, b_2\}$, and therefore $d(x, y) = 1$; thus the desired inequality holds.

Our second aim in this section is to give an example of a metric space $M$ such the corresponding space $\text{Lip}_0(M)$ has the D2P but fails the SD2P. For this example, we need a sufficient condition for the space $\text{Lip}_0(M)$ to have the D2P. Here we do not have a condition which involves only the metric of the underlying space $M$ as we did with Theorems 2.3 and 3.3 for the SSD2P and the SD2P, respectively. However, the following analogue of Lemmas 2.4 and 3.4 for the D2P is suitable for our purposes.

**Lemma 3.5.** Let $M$ be a pointed metric space and let $\tilde{\Gamma}$ be as in (1.1). Suppose that, whenever $\delta > 0$, $h \in \text{Lip}_0(M)$ with $\|h\| \leq 1 - \delta$, and $\mu \in ba(M)$ with only non-negative values, there exist a subset $A$ of $M$, elements $u, v \in A$ with $u \neq v$, and functions $f, g \in B_{\text{Lip}_0(M)}$ satisfying

- $\mu(\Gamma_{1, A}) < \delta$ and $\mu(\Gamma_{2, A}) < \delta$;
- $f|_{M \setminus A} = g|_{M \setminus A} = h|_{M \setminus A}$;
- $f(u) - f(v) \geq (1 - \delta)d(u, v)$ and $g(u) - g(v) \leq -(1 - \delta)d(u, v)$.

Then the space $\text{Lip}_0(M)$ has the D2P.

**Proof of Lemma 3.5.** Let $n \in \mathbb{N}$, let $F_1, \ldots, F_n \in S_{\text{Lip}_0(M)}$, let $\varepsilon > 0$, and let $\phi \in B_{\text{Lip}_0(M)}$. It suffices to find $f, g \in \text{Lip}_0(M)$ with $\|f\| \leq 1$ and $\|g\| \leq 1$ such that $|F_i(f - \phi)| < \varepsilon$ and $|F_i(g - \phi)| < \varepsilon$ for every $i \in \{1, n\}$, and $\|f - g\| > 2 - \varepsilon$.

For every $i \in \{1, n\}$, let $\mu_i \in ba(M)$ with $|\mu_i| = 1$ satisfy (1.2) with $F$ and $\mu$ replaced by $F_i$ and $\mu_i$, respectively. Define $\mu = |\mu_1| + \cdots + |\mu_n|$. Fix a real number $\delta > 0$ satisfying $5\delta \leq \varepsilon$. Let $h = (1 - \delta)\phi$.

Let a subset $A$ of $M$, elements $u, v \in A$ with $u \neq v$, and functions $f, g \in \text{Lip}_0(M)$ satisfy the conditions in the lemma. Setting $\Gamma_A = \Gamma_{1, A} \cup \Gamma_{2, A}$, one has $\mu(\Gamma_A) < 2\delta$, and
hence, for every \( i \in \{1, \ldots, n\} \) (observing that \( \tilde{f}|_{\tilde{\Omega}} = \tilde{h}|_{\tilde{\Omega}} \)),

\[
|F_i(f - \phi)| \leq |F_i(f - h)| + \delta = \left| \int_{\Gamma} (\tilde{f} - \tilde{h}) \, d\mu_i \right| + \delta = \left| \int_{\Gamma} (\tilde{f} - \tilde{h}) \, d\mu_i \right| + \delta \\
\leq 2|\mu_i|(\Gamma_A) + \delta < 5\delta \leq \varepsilon.
\]

Similarly, \( |F_i(g - \phi)| < \varepsilon \) for every \( i \in \{1, \ldots, n\} \). It remains to observe that

\[
\|f - g\| \geq \frac{(f-g)(u) - (f-g)(v)}{d(u,v)} = \frac{f(u) - f(v) - g(u) + g(v)}{d(u,v)} \\
\geq \frac{2(1 - \delta)}{d(u,v)} = 2(1 - \delta) > 2 - \varepsilon.
\]

\[\square\]

We now introduce a space \( M \) for which the corresponding Lipschitz function space \( \text{Lip}_0(M) \) has the D2P but not the \((w^*)\text{-SD2P}\). To our knowledge this is the first such example in the class of Lipschitz function spaces.

Recall \cite{Haller16} Theorem 3.1, (3)] that a metric space \( M \) has the long trapezoid property (briefly, LTP) if, for every \( \varepsilon > 0 \) and every finite subset \( N \) of \( M \), there exist elements \( u, v \in M \) with \( u \neq v \) satisfying, for all \( x, y \in N \),

\[(1 - \varepsilon)(d(x, y) + d(u, v)) \leq d(x, u) + d(y, v).
\]

**Example 3.6.** Let \( M = \{a_i, u_m^i, v_m^i : i \in \{1, 2, 3\}, m \in \mathbb{N}\} \) be the metric space where, for all \( i, j \in \{1, 2, 3\} \) with \( i \neq j \) and all \( m \in \mathbb{N} \),

\[
d(a_i, u_m^i) = d(a_i, v_m^j) = 1,
\]

and, for all \( j \in \{1, 2, 3\} \) and all \( m \in \mathbb{N} \),

\[
d(u_m^i, v_m^j) = 1,
\]

and the distance between two different elements is 2 in all other cases.

We first show that the space \( \text{Lip}_0(M) \) has the D2P. We make use of Lemma 3.5

Let \( \delta > 0 \), let \( h \in \text{Lip}_0(M) \) with \( \|h\| \leq 1 \), and let \( \mu \in \text{ba}(M) \) with only non-negative values. We may assume that \( h(a_1) \leq h(a_2) \leq h(a_3) \). Set \( L = \inf h(M) \).

If \( h(a_2) \leq L + 1 \), then let \( k = 3 \) and \( c = L \); otherwise, let \( k = 1 \) and \( c = L + 1 \). Choose an \( m \in \mathbb{N} \) so that \( \mu(\Gamma_{1,A}) < \delta \) and \( \mu(\Gamma_{2,A}) < \delta \) where \( A = \{u_m^k, v_m^k\} \). Let \( u = u_m^k \) and \( v = v_m^k \), and define \( f, g : M \to \mathbb{R} \) by

\[
f(x) = \begin{cases} h(x) & \text{if } x \in M \setminus A; \\ c + 1 & \text{if } x = u; \\ c & \text{if } x = v, \end{cases}
\]

and

\[
g(x) = \begin{cases} c & \text{if } x = u; \\ c + 1 & \text{if } x = v. \end{cases}
\]

Then

\[
f(u) - f(v) = g(v) - g(u) = 1 = d(u,v).
\]

It is straightforward to verify that \( \|f\| = 1 \) and \( \|g\| = 1 \). By Lemma 3.5 \( \text{Lip}_0(M) \) has the D2P.

We now show that the space \( \text{Lip}_0(M) \) does not have the \((w^*)\text{-SD2P}\). It suffices to show that space \( M \) does not have the LTP. Let \( N = \{a_1, a_2, a_3\} \) and let \( \varepsilon < 1/3 \). Whenever \( u, v \in M \) with \( u \neq v \), there exist \( x, y \in N \) with \( x \neq y \) such that \( d(x, u) \leq 1 \) and \( d(y, v) \leq 1 \). Since \( d(x, y) = 2 \), one has

\[
(1 - \varepsilon)(d(u, v) + d(x, y)) \geq 3(1 - \varepsilon) > 2 \geq d(x, u) + d(y, v).
\]
4. LOCAL NORM-ONE LIPSCHITZ FUNCTION IS A DAUGAVET POINT

Let $M$ be a pointed metric space. In this section, we show that certain norm-one elements $f$ of $\text{Lip}_0(M)$ are Daugavet points, i.e., given a slice $S$ of the unit ball of $\text{Lip}_0(M)$ and an $\varepsilon > 0$, there exists a $g \in S$ with $\|f - g\| > 2 - \varepsilon$.

**Definition 4.1** (see [11] Definition 2.5). A function $f \in \text{Lip}_0(M)$ is said to be local if, for every $\varepsilon > 0$, there are $u, v \in M$ with $u \neq v$ such that $d(u, v) < \varepsilon$ and $f(m_{u,v}) > \|f\| - \varepsilon$.

The question of whether every local $f$ in the unit sphere of $\text{Lip}_0(M)$ is a Daugavet point was posed and addressed in [11]; there it was shown that consequence of the following result.

Let, as in (1.1), and let $\mu \in \text{ba}(\widetilde{M})$ with $|\mu|_1(\widetilde{M}) = 1$ satisfy (1.2). It suffices to show that there is a subset $A$ of $M$ with $|\mu(\Gamma_{1,A})| < \delta$ and $|\mu(\Gamma_{2,A})| < \delta$ such that, for some $n \in \mathbb{N}$, there exists a function $f \in B_{\text{Lip}_0(M)}$ such that $f|_{M \setminus A} = h|_{M \setminus A}$ and $f(u_n) - f(v_n) \geq (1 - \delta)d(u_n, v_n)$. Indeed, suppose that such $A$, $n$, and $f$ have been found. Then, setting $\Gamma_A = \Gamma_{1,A} \cup \Gamma_{2,A}$, one has $|\mu|_A(\Gamma_A) < 2\delta$, and thus (observing...
that \( \tilde{f}_{|\tilde{M} \setminus \Gamma_A} = \tilde{h}_{|\tilde{M} \setminus \Gamma_A} \)

\[
F(f) = \int_{\tilde{M}} \tilde{f} \, d\mu = \int_{\tilde{M}} \tilde{h} \, d\mu + \int_{\Gamma_A} (\tilde{f} - \tilde{h}) \, d\mu \\
= F(h) + \int_{\Gamma_A} (\tilde{f} - \tilde{h}) \, d\mu \\
> 1 - 2\delta - 2|\mu|(\Gamma_A) \geq 1 - 6\delta,
\]

and, therefore,

\[
\|F + m_{u_n,v_n}\| \geq F(f) + \frac{f(u_n) - f(v_n)}{d(u_n,v_n)} > 2 - 7\delta \geq 2 - \varepsilon.
\]

It remains to find the \( A, n, \) and \( f \) as above. To this end, choose a real number \( \theta \in (0, 1) \) satisfying \( \frac{\theta}{\theta - \theta'} < \frac{3}{2} \). Without loss of generality, one may assume that one of the following (mutually exclusive) conditions holds:

1. no subsequence of the sequence \( \{u_n\}_{n=1}^\infty \) converges;
2. there is a \( u \in M \) such that \( u_n = u \) for every \( n \in \mathbb{N} \);
3. there is a \( u \in M \) with \( u_n \neq u \) and \( v_n \neq u \) for every \( n \in \mathbb{N} \) such that \( u_n \to u \).

(1). In this case, by passing to a subsequence, one may assume that there is an \( r > 0 \) such that the open balls \( A_n := B(u_n, r), \) \( n = 1, 2, \ldots, \) are pairwise disjoint. It follows that there is an \( N \in \mathbb{N} \) such that, for every \( n \geq N, \) one has \( |\mu|(\Gamma_{1,A_n}) < \delta \) and \( |\mu|(\Gamma_{2,A_n}) < \delta \). Pick an \( n \geq N \) so that \( d(u_n, v_n) < \theta r \). One may assume that \( 0 \notin A_n \). Let \( A = A_n \) and define the function \( f \) by \( f_{|M \setminus A} = \tilde{h}_{|M \setminus A}, \) \( f(u_n) = h(u_n) \), \( f(v_n) = h(u_n) - (1 - \delta)d(u_n, v_n) \), and by extending the definition norm-preservingly to the whole \( M \).

Then \( \|f\| \leq 1 \) because, whenever \( x \in M \setminus A \), one has (taking into account that \( d(u_n, v_n) \leq \frac{\delta}{2}d(x, v_n) \) by Lemma 4.3)

\[
|f(x) - f(v_n)| = |h(x) - h(u_n) + (1 - \delta)d(u_n, v_n)|
\leq (1 - \delta)d(x, v_n) + (1 - \delta)d(u_n, v_n)
\leq (1 - \delta)(d(x, v_n) + 2d(u_n, v_n) \leq d(x, v_n).
\]

(2). Pick an \( n \in \mathbb{N} \) and \( r, s > 0 \) so that \( d(v_n, u) < \theta r \), \( s < \theta d(v_n, u) \), and \( |\mu|(\Gamma_{1,A}) < \delta \) and \( |\mu|(\Gamma_{2,A}) < \delta \) where \( A = B(u, r) \setminus B(u, s) \). One may assume that \( 0 \notin B(u, r) \setminus \{u\} \). Letting \( A \) be as above, define the function \( f \) by \( f_{|M \setminus A} = \tilde{h}_{|M \setminus A}, \) \( f(v_n) = h(u) - (1 - \delta)d(u, v_n) \), and by extending the definition norm-preservingly to the whole \( M \). One has \( \|f\| \leq 1 \). In fact, if \( x \in B(u, s) \), then (taking into account that \( d(x, u) \leq \frac{\delta}{2}d(x, v_n) \) by Lemma 4.3)

\[
|f(x) - f(v_n)| = |h(x) - h(u) + (1 - \delta)d(u, v_n)|
\leq (1 - \delta)d(x, u) + (1 - \delta)d(u, v_n)
\leq (1 - \delta)(2d(x, u) + d(x, v_n)) \leq d(x, v_n);
\]

if \( x \in M \setminus B(u, r) \), then, keeping in mind that \( u = u_n \), the desired inequality \( |f(x) - f(v_n)| \leq d(x, v_n) \) is obtained as in the case (1).

(3). Pick an \( n \in \mathbb{N} \) and \( r, s > 0 \) so that

\[
\max\{d(u, u_n), d(u, v_n)\} < \theta r, \quad s < \theta \min\{d(u, u_n), d(u, v_n)\},
\]

and \( |\mu|(\Gamma_{1,A}) < \delta \) and \( |\mu|(\Gamma_{2,A}) < \delta \) where \( A = B(u, r) \setminus B(u, s) \). Letting \( A \) be as above, define the function \( f \) by \( f_{|M \setminus A} = \tilde{h}_{|M \setminus A}, \) \( f(u_n) = h(u) + (1 - \delta)d(u, u_n), \)
Recall that a metric space $M$ is local if for every $\varepsilon > 0$ and for every Lipschitz function $f : M \to \mathbb{R}$ there are two distinct points $u, v \in M$ such that $d(u, v) < \varepsilon$ and $f(m_{u,v}) > \|f\| - \varepsilon$. The following result is an immediate consequence of the previous theorem.

**Corollary 4.4** (see [6, Proposition 3.4 and Theorem 3.5], cf. [10, Theorem 3.1]). Let $M$ be a pointed metric space. If $M$ is local, then $\text{Lip}_0(M)$ has the Daugavet property.

**Remark.** The converse statement of this result also holds (see [10] Proposition 2.3 and the remark following its proof) and [6, Theorem 3.5]). On the other hand, the converse statement of our Theorem 1.4 does not hold since there exist uniformly discrete metric spaces $M$ for which $\text{Lip}_0(M)$ has Daugavet points. For example, let $M$ be an infinite pointed metric space where the distance between two different elements is 1 if one of the elements is the fixed point 0, and the distance between two different elements is 2 in all other cases. Then the norm-one element $f \in \text{Lip}_0(M)$, given by $f(x) = 1$ for every $x \in M \setminus \{0\}$, is a Daugavet point.

**Acknowledgment**

The authors wish to express their thanks to Triinu Veerorg for her active interest in the publication of this paper.

This work was supported by the Estonian Research Council grant (PRG1901). The research of Andre Ostrak was supported by the University of Tartu ASTRA Project PER ASPERA, financed by the European Regional Development Fund.

**References**

1. Trond A. Abrahamsen, Vegard Lima, and Olav Nygaard, *Remarks on diameter 2 properties*, J. Convex Anal. 20 (2013), no. 2, 439–452. MR 3098474
2. Trond A. Abrahamsen, Olav Nygaard, and Märt Pöldvere, *New applications of extremely regular function spaces*, Pacific J. Math. 301 (2019), no. 2, 385–394. MR 4023351
3. Julio Becerra Guerrero, Ginés López-Pérez, and Abraham Rueda Zoca, *Octahedral norms and convex combination of slices in Banach spaces*, J. Funct. Anal. 266 (2014), no. 4, 2424–2435. MR 3150166
4. Bernardo Cascales, Rafael Chichlana, Luis C. García-Lirola, Miguel Martín, and Abraham Rueda Zoca, *On strongly norm attaining Lipschitz maps*, J. Funct. Anal. 277 (2019), no. 6, 1677–1717. MR 3985517
5. Luis García-Lirola, Antonín Prochážka, and Abraham Rueda Zoca, *A characterisation of the Daugavet property in spaces of Lipschitz functions*, J. Math. Anal. Appl. 464 (2018), no. 1, 473–492. MR 3794100
6. Rainis Haller, Johann Langemets, Vegard Lima, and Rihhard Nadel, *Symmetric strong diameter two property*, Mediterr. J. Math. 16 (2019), no. 2, Paper No. 35, 17. MR 3917942
7. Rainis Haller, Johann Langemets, and Märt Pöldvere, *On duality of diameter 2 properties*, J. Convex Anal. 22 (2015), no. 2, 465–483. MR 3346197
8. Yevgen Ivakhno, *Big slice property in the spaces of Lipschitz functions*, Visn. Khark. Univ., Ser. Mat. Prykl. Mat. Mekh. 749 (2006), no. 56, 109–118 (English).
9. Yevgen Ivakhno, Vladimir Kadets, and Dirk Werner, *The Daugavet property for spaces of Lipschitz functions*, Math. Scand. 101 (2007), no. 2, 261–279. MR 2379289
11. Mingu Jung and Abraham Rueda Zoca, *Daugavet points and Δ-points in Lipschitz-free spaces*, Studia Math. (2022), 19.

12. Johann Langemets and Abraham Rueda Zoca, *Octahedral norms in duals and biduals of Lipschitz-free spaces*, J. Funct. Anal. 279 (2020), no. 3, 108557, 17. MR 4093788

13. Andre Ostrak, *Characterisation of the weak-star symmetric strong diameter 2 property in Lipschitz spaces*, J. Math. Anal. Appl. 483 (2020), no. 2, 123630, 10. MR 4026495

14. ______, *On the duality of the symmetric strong diameter 2 property in Lipschitz spaces*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 115 (2021), no. 2, Paper No. 78, 10. MR 4233633

15. Antonín Procházka and Abraham Rueda Zoca, *A characterisation of octahedrality in Lipschitz-free spaces*, Ann. Inst. Fourier (Grenoble) 68 (2018), no. 2, 569–588. MR 3803112

16. Nik Weaver, *Lipschitz algebras*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018, Second edition of [MR 1832645]. MR 3792558