Some remarks on the Kronheimer-Mrowka classes of algebraic surfaces

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alg-geom/9308003

1 Introduction

Recently Kronheimer and Mrowka have announced a very interesting result, which sheds new light on the Donaldson polynomials [K-M]. They find recurrence relations between the Donaldson polynomials, by finding relations between the polynomials and the minimal genus of a smooth real surface representing an homology class. To be more precise we need a definition.

For a simply connected 4-manifold $X$ with odd $b_+ \geq 3$, we denote the SU(2) polynomials on $H_0(X) \oplus H_2(X)$ by $q_k(X)$. $X$ is called simple if we have

$$q_k(X)(pt^2, -) = 4q_{k-1}(X), \quad d = 4k - \frac{3}{2}(1 + b_+).$$

For simple 4-manifolds it is convenient to label the polynomials by their degree on $H_2(X)$ i.e. we define

$$q_d(X) = \begin{cases} q_k(X)|_{H_2(X)} & \text{if } d = 4k - \frac{3}{2}(1 + b_+), \\ q_k(X)(pt, -)|_{H_2(X)} & \text{if } d = 4k - 2 - \frac{3}{2}(1 + b_+), \\ 0 & \text{otherwise} \end{cases}$$

The Donaldson series is then the formal power series

$$q(X) = \sum_d q_d(X)/d!.$$  

**Theorem 1.1.** (Kronheimer, Mrowka) For every simple 4-manifold $X$ there exist a finite number of Kronheimer-Mrowka classes $K_1, \ldots, K_p \in H^2(X)$ and non zero rational numbers $a_1, \ldots, a_p$ such that

(i) $q(X) = e^{Q/2} \sum_{i=1}^p a_i e^{K_i},$

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(ii) $K_i \equiv w_2(X) \pmod{2}$, for all $i = 1, \ldots, p$,  
(iii) if $K_i \in \{K_1, \ldots, K_p\}$ then $-K_i \in \{K_1, \ldots, K_p\}$,  
(iv) for every homologically nontrivial connected real surface $\Sigma$ with $\Sigma^2 \geq 0$ and every Kronheimer-Mrowka class $K_i$ we have  
$$2g(\Sigma) - 2 \geq \Sigma^2 + K_i \cdot \Sigma.$$  

Here $Q$ is the intersection form. The Kronheimer Mrowka classes will be abbreviated KM-classes and are called the basic classes in [K-M]. Condition (ii) is reminiscent of the Wu formula, whereas condition (iv) is similar to the the adjunction formula for the genus of a smooth algebraic curve. This suggests that for complex surfaces, the KM-classes should be closely related to the canonical divisor $K_X$. Indeed in the examples and the conjectural expression for the Donaldson series of elliptic surfaces in the announcement, the KM-classes are of type $K_i = \alpha_i K_{\min} + \sum \beta_{ij} E_j$ where $K_{\min}$ is the canonical divisor of the minimal model, $E_1, \ldots, E_l$ the $(-1)$-curves, $\alpha_i$ a rational number with $|\alpha_i| \leq 1$ and $\beta_{ij} = \pm 1$. A formulation which is a little less obvious but which generalises better, as we will see, is $K_i = C - D$ where $C$, $D$ are divisors such that $K_X = C + D$ and such that a multiple is effective. Hence in these cases the canonical class is a KM-class which is extremal from an algebraic geometric point of view.

To formulate the general relationship, we recall that the effective cone of a complex surface $\text{NE}(X)$, is the positive rational cone in $\text{NS}(X) \otimes \mathbb{Q} \subset H^2(X, \mathbb{Q})$ generated by effective divisors. Let $\overline{\text{NE}}(X)$ be its closure in the norm topology.

**Theorem 1.2.** For every KM-class $K_i$ on a simple simply connected algebraic surface $X$, there is a unique decomposition $K_X = C_i + D_i$ in $\mathbb{Z}$-divisors $C_i, D_i \in \overline{\text{NE}}(X)$ such that $K_i = C_i - D_i$. In particular $K_i = K_X$ if and only if there is a smooth hyperplane section $H$ such that $2g(H) - 2 = H^2 + K_i \cdot H$.

The results in [Kro] seem to indicate that a KM-class and a hyperplane section $H$ as in the theorem exist, at least when there is an $\omega \in H^0(K)$ such that for sufficiently large $k$, $q_k(\omega + \bar{\omega}) \neq 0$. For minimal surfaces of general type this would imply the invariance of the canonical class up to sign under orientation preserving self-diffeomorphisms.

**Corollary 1.3.** Assume in addition that $X$ is minimal and of general type, then $K_i^2 \leq K_X^2$ with equality if and only if $K_i = \pm K_X$.

By the Lefschetz (1,1) theorem [G-H, p. 163], the algebraicity of the KM-classes is equivalent to the $K_i$ being of type $(1,1)$, In fact this is what
we will prove, using that the Donaldson polynomials are of pure Hodge type as in [Br1]. Since we assume that $p_g > 0$ this shows that the lattice spanned by the $K_i$ is a proper sublattice. Moreover since the $K_i$ are defined by the differentiable structure, they are contained in the fixed lattice of the variation of Hodge structures defined by a family of complex structures on $X$. In favourable circumstances this should force the KM-classes to be in $H^2(X, \mathbb{Z}) \cap [K_{\min}, E_1, \ldots, E_l]$.

Another implication of the theorem is that the KM-classes are trivial on $H^{0,2}(X) \subset H^2(X, \mathbb{C})$. Thus we get the following corollary.

**Corollary 1.4.** If $X$ is a simple and simply connected surface, then for all $\omega \in H^0(K_X)$ we have

$$q(\omega + \bar{\omega}) = q_0 e^{\int \omega \wedge \bar{\omega}}$$

where $q_0$ is the Donaldson polynomial of degree 0.

It would be rather interesting to understand this formula from an algebraic geometric point of view, possibly clarifying the role of the simpleness condition. Combining the corollary with O'Grady’s non vanishing result, we see that $q_0 \neq 0$ if $X$ is of general type, $p_g$ is odd and $|K_{\min}|$, the linear system of the canonical class of the minimal model, contains a reduced curve [O'G, th. 2.4], [Li2, appendix], [Mor, th. 1].

In a similar vein, since the Neron-Severi group has a non degenerate intersection form, the $K_i$ are determined by their intersection products with divisors. Thus we get

**Corollary 1.5.** For a simple simply connected surface, the Donaldson series $q(X)$ is determined by $q(X)|_{NS(X)}$.

The corollary says that by knowing the algebraic part of all Donaldson polynomials we can reconstruct the transcendental part as well, i.e. in the simple case, the polynomials defined by Jun-Li [Li1] contain as much information as the full polynomials. Moreover since $\text{NE} \cap K_X - \text{NE}$ is a bounded subset of $\text{NS}(X, \mathbb{Q})$, and the $K_i$ are integral, at least in principle, we get an effective bound on the number of Kronheimer Mrowka classes, hence on the number of polynomials one has to compute in order to reconstruct the Donaldson series.

### 2 Proof of theorem 1.2 and corollary 1.3

We have to prove that the KM-classes are of type $(1, 1)$. Accepting this, the rest of the statement of theorem 1.2 and corollary 1.3 is a consequence of
property (ii), (iii), and (iv) of the KM-classes.

By property (ii) we can write \( K_i = K_X - 2D \) for some \( \mathbb{Z} \)-divisor \( D \). For every very ample line bundle \( \mathcal{O}(H) \), we choose a smooth connected hyperplane section \( H \). Then we get

\[
2g(H) - 2 = H^2 + H \cdot K_X \geq H^2 + H \cdot K_i = H^2 + H \cdot K_X - 2H \cdot D
\]
i.e. \( D \cdot H \geq 0 \). Thus by the duality of the closure of the effective cone \( \overline{\text{NE}}(X) \) and the nef cone, we conclude that \( D \in \overline{\text{NE}}(X) \) [Wil, prop. 2.3].

We also have \( C = (K_X + K_i)/2 \in \overline{\text{NE}} \) by property (iii). Rewriting we get \( K_X = C + D \) and \( K_i = C - D \) as claimed. Note that nothing is gained by applying the inequality to other smooth connected divisors \( C \) with \( C^2 \geq 0 \), since such divisors are nef. Finally note that \( H \cdot K_i = 2g(H) - 2 - H^2 = H \cdot K_X \) if and only if \( D = 0 \), since \( \overline{\text{NE}} \cap H^\perp = \{0\} \). This proves theorem 1.2 up to algebraicity.

Now assume temporarily that \( X \) is minimal and of general type. Write \( K_i^2 = K_X^2 + 4(D^2 - K_X \cdot D) \). Since \( K_X \) is nef, we get \( K_i^2 \leq K_X^2 \) if \( D^2 \leq 0 \), with equality iff \( D^2 = K \cdot D = 0 \). The latter is equivalent to \( D = 0 \) by the Hodge index theorem. Interchanging \( K_i \) and \( -K_i \) if necessary, we are left with the case \( C^2 > 0 \), \( D^2 > 0 \). Since \( C, D \in \overline{\text{NE}} \), we have \( C \cdot D > 0 \) by the numerical connectedness of \( K_X \) [BPV, prop VII.6.1] (strictly speaking we need that \( C \) and \( D \) are effective, but the proof of [loc. cit] carries over without change). Thus \( K_X^2 > D^2 \). Then again by the Hodge index theorem we get \( (K \cdot D)^2 \geq K^2D^2 > (D^2)^2 \), so \( K_i^2 < K^2 \). This proves the corollary.

To prove that the KM-classes are of type (1,1), we need a slight generalization of [Br1, prop. 3.1]. For simplicity we restrict ourselves to the SU(2) case and a statement about Hodge types, but the proof can easily be modified to show that all SO(3) polynomials \( q_{L,k}(X) \) with \( L \in \text{NS}(X) \) come form an algebraic cycle (cf. [loc. cit.]). Consider the Donaldson polynomial \( q_d \) as an element of \( S^dH^2(X) \) endowed with its natural Hodge structure.

**Lemma 2.1.** For every \( d \geq 0 \), the Donaldson polynomials \( q_d \) are pure of type \((d,d)\).

**Proof.** We temporarily index the polynomials on \( H_2(X) \) by \( k = c_2 \). By [Br1, prop. 3.1] the lemma is true for \( q_k \) with odd \( k \gg 0 \) and evaluated on 2 dimensional classes. Here sufficiently large means that the moduli space \( \mathcal{M}_k(X) \) of stable bundles on \( X \) for generic polarisation \( H \) is generically smooth and reduced of the proper dimension, and that the lower moduli spaces \( \mathcal{M}_{k'}(X) \) for \( k' < k \) have sufficiently low dimension, so that we can
apply Morgan’s comparison result \[\text{Mor}\]. For the general case we use stabilization.

By \[\text{M-O, th. 2.1.1}\], for every \(k\) we can find an \(l_0\), and \(\epsilon_1, \ldots, \epsilon_l\) sufficiently small such that for a generic polarisation on the \(l \geq l_0\)-fold blow-up of type \(H - \sum \epsilon_i E_i\), the moduli space \(\mathcal{M}_{k+l}(\hat{X}(x_1, \ldots, x_l))\) is generically smooth of the proper dimension and the lower moduli spaces \(\mathcal{M}_{k'}(\hat{X})\) with \(k' < k\), have sufficiently low dimension. Thus \(q_{k+l}(\hat{X}(x_1, \ldots, x_l))\) is pure of type \((d + 4l, d + 4l)\).

Now by the blow-up formula \[\text{F-M, th. 4.3.1}\], we have

\[
q_k(X) = (-\frac{1}{2})^{l} q_{k+l}(\hat{X}(x_1, \ldots, x_l))(E_1^4, \ldots, E_l^4, -).
\]

Clearly the exceptional divisors \(E_i\) are pure of type \((1, 1)\), hence \(q_k(X)\) is pure of type \((d, d)\). Finally

\[
q_k(X)(pt, -) = -\frac{1}{2} q_{k+1}(\hat{X}(x))(E^6, -),
\]

by \[\text{F-M, below cor. 4.3.2}\] (without proof) or \[\text{Br2}\], and so we can use the same argument as above.

\[\Box\]

The rest of the proof is now straightforward. Let

\[
C(X) = e^{-Q/2} q(X) = \sum a_i e^{K_i}.
\]

Then \(C(X) = \sum C_d(X)\), where \(C_d(X)\) is a homogeneous polynomial of pure type \((d, d)\), since both \(\exp(-Q/2)\) and \(q(X)\) is a sum of homogeneous polynomials of pure type. Write \(K_i = \alpha_i + \beta_i + \bar{\alpha}_i\), with \(\alpha_i \in H^{0,2}, \beta_i \in H^{1,1}\). Let \(z \in H^{1,1}\) be variable and denote by \(\frac{\partial}{\partial z}\) the directional derivative. Then we have

\[
\frac{\partial^n C(X)}{\partial z^n} = \sum_{d=0}^{\infty} \frac{\partial^n C_d}{\partial z^n} = \sum a_i \langle \beta_i, z \rangle^n e^{K_i}.
\]

Clearly \(\frac{\partial^n C_d}{\partial z^n}\) has pure type \((d - n, d - n)\). Thus, upon restriction to \(H^{0,2}(X)\) only the constant term contributes, and we get

\[
\frac{\partial^n C(X)}{\partial z^n} \bigg|_{H^{0,2}} = \text{constant} = \sum a_i \langle \beta_i, z \rangle^n e^{\bar{\alpha}_i}.
\]

We conclude that

\[
\sum_{\alpha_i \neq 0} a_i \langle \beta_i, z \rangle^n e^{\bar{\alpha}_i} = 0.
\]

Since \(z\) and \(n\) are arbitrary we find that the sum must be empty, i.e. all KM-classes \(K_i\) are of type \((1, 1)\).

\[\Box\]
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