Cosmological perturbations in the interacting dark sector: 
Mapping fields and fluids

Joseph P Johnson¹,* and S. Shankaranarayanan¹,†

¹Department of Physics, Indian Institute of Technology Bombay, Mumbai 400076, India

Abstract

There is no unique way to describe the dark energy-dark matter interaction, as we have little information about the nature and dynamics of the dark sector. Hence, in many of the phenomenological dark matter fluid interaction models in the literature, the interaction strength $Q_\nu$ in the dark sector is introduced by hand. Demanding that the interaction strength $Q_\nu$ in the dark sector must have a field theory description, we obtain a unique form of interaction strength. We show the equivalence between the fields and fluids for the $f(R, \chi)$ model where $f$ is an arbitrary, smooth function of $R$ and the scalar field $\chi$, which represents dark matter. Up to first order in perturbations, we show that the one-to-one mapping between the field theory description and the phenomenological fluid description of interacting dark energy and dark matter exists only for this unique form of interaction. We then classify the interacting dark energy models considered in the literature into two categories based on the field-theoretic description.

* josephpj@iitb.ac.in
† shanki@phy.iitb.ac.in
I. INTRODUCTION

Dark matter dominates the galaxy mass, and dark energy forms the majority of our Universe’s energy density [1]. However, we have little information about the properties of these two components that dominate the energy content of the Universe today [2]. The only information we have about the two components is that (i) Dark energy contributes with negative pressure to the energy budget, and (ii) Dark matter has negligible, possibly zero, pressure [1]. The above properties are based on gravitational interactions. More importantly, we do not know how they interact with each other and Baryons/Photons.

In the early Universe, due to the tight-coupling of Baryons and Photons, the baryons participate in the acoustic oscillations of the photons, and also cause Silk damping [3]. Near recombination, the baryons decouple from the photons and photons propagate freely. Solar eclipse measurements rule out dark matter interaction with Photons. Local gravity measurements rule out dark energy interactions with Baryons [4]. However, the current observations can not constraint (or rule out) the interaction strength between dark matter-dark energy. Interestingly, the dark matter-dark energy interaction provides a mechanism to alleviate the coincidence problem (see, for instance, Refs. [5]). Besides this, recently, it has been shown that the dark matter-dark energy interaction can reconcile the tensions in the Hubble constant $H_0$ [6].

Naturally, there has been a surge in constructing dark energy-dark matter models [7–30]. In all these models, phenomenologically, the interaction is proposed between the fluid terms in the dark sector. More specifically, individually, dark matter (DM) and dark energy (DE) do not satisfy the conservation equations, however, the combined sector satisfies the energy conservation equation [5], i.e.,

$$\nabla^\mu T^{\rm (DE,DM)}_{\mu\nu} = Q^{\rm (DE,DM)}_{\nu}$$  \hspace{1cm} (1)

such that

$$Q^{\rm (DE)}_{\nu} + Q^{\rm (DM)}_{\nu} = 0$$  \hspace{1cm} (2)

where $Q$ determines the interaction strength between dark matter and dark energy. Since the gravitational effects on dark matter and dark energy are opposite, even a small interaction can impact the cosmological evolution [5]. Since we have little information about the dark sector, in many of these models, the interaction strength $Q_{\nu}$ in the dark sector is put in by
However, it is unclear whether these broad classes of phenomenological models can be obtained from a field theory action. More specifically, can the above interaction strength $Q_\nu$ in the dark sector be derived systematically from a field theory action. Attempts have been made in the literature to obtain the interaction strength from the field-theoretic action [29]. The correspondence between the fluid description and field-theoretic description is established only for the background cosmology and not for the perturbations. The analysis of cosmological perturbations is essential to provide a complete understanding of these models and, more importantly, to determine if the perturbations are stable in the presence of interaction $Q_\nu$.

In this work, we show the equivalence up to first order in the perturbations of $f(R, \chi)$ model where $f$ is an arbitrary, smooth function of $R$ and the scalar field $\chi$ which represents dark matter. More specifically, under conformal transformations, we show that $f(R, \chi)$ is equivalent to a model with two coupled scalar fields. The coupling between the scalar fields which gives rise to the dark energy - dark matter interaction (1) represented by the evolution equations of the dark energy (represented by a scalar field) and dark matter (represented by a perfect fluid). We show that the interaction between the dark sectors can be rewritten in terms of the trace of the energy-momentum tensor of the dark matter fluid, and a coupling function depending on the dark energy field. We then look at several interacting dark sector models that are proposed in the literature and identify the compatible models with the field theory action proposed here.

In this work we use the natural units where $c = 1$, $\kappa^2 = 8\pi G$, and the metric signature $(-,+,+,+)$. Greek alphabets denote the 4-dimensional space-time coordinates and Latin alphabets denote the 3-dimensional spatial coordinates. Overbarred quantities (like $\bar{\rho}(t), \bar{P}(t)$) are evaluated for the FRW background and a dot represents the derivative with respect to cosmic time $t$. Unless otherwise specified, subscript “,$\phi$” denotes derivative with respect to $\phi$, subscript “,$\chi$” denotes derivative with respect to $\chi$, and subscript $m$ denotes dark matter.
II. DARK SECTOR INTERACTION FROM A FIELD THEORY ACTION

In the field theory description of the interacting dark energy - dark matter models, the coupling between the dark sector components is represented by a coupling term, which is an arbitrary function of the dark energy scalar field. It can be shown that modified gravity models such as $f(\tilde{R}, \tilde{\chi})$ gravity can lead to such models [31]. Consider the following action in Jordan frame:

$$S_J = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{1}{2\kappa^2} f(\tilde{R}, \tilde{\chi}) - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \tilde{\chi} - V(\tilde{\chi}) \right]$$  \hspace{1cm} (3)

where $f(\tilde{R}, \tilde{\chi})$ is an arbitrary, smooth function of Ricci scalar, and scalar field $\tilde{\chi}$, and $V(\chi)$ is the self-interaction potential of the scalar field $\tilde{\chi}$. Under the conformal transformation:

$$g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}, \quad \Omega^2 = F(\tilde{R}, \tilde{\chi}) \equiv \frac{\partial f(\tilde{R}, \tilde{\chi})}{\partial \tilde{R}}$$  \hspace{1cm} (4)

and a field redefinition, the action in the Einstein frame takes the following form

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - U(\phi) - \frac{1}{2} e^{2\alpha(\phi)} g^{\mu\nu} \chi \nabla_\mu \chi - e^{4\alpha(\phi)} V(\chi) \right).$$  \hspace{1cm} (5)

where

$$U = \frac{F\tilde{R} - f}{2\kappa^2 F^2}.$$  

This action has also been considered in the context of a multi-field inflationary scenario. (See, for instance, Refs [32].) Recently, the same action is also considered in Ref. [29]. However, to our knowledge, we have not seen an explicit calculation that shows the derivation of the above action in the Einstein frame. Appendix A contains the details of the transformations in the field space to derive the above action.

From the above action (5), the field equations for $\chi$ and $\phi$, respectively, are:

$$-\nabla^\mu \nabla_\mu \chi - 2\alpha,\phi(\phi) \nabla_\mu \phi \nabla^\mu \chi + e^{2\alpha(\phi)} V_\chi(\chi) = 0$$  \hspace{1cm} (6)

$$-\nabla^\mu \nabla_\mu \phi + 4 e^{4\alpha(\phi)} \phi V(\chi) + e^{2\alpha(\phi)} \nabla^\mu \chi \nabla_\mu \chi + U,\phi(\phi) = 0$$  \hspace{1cm} (7)

where the notations such as $V_\chi, U,\phi$ denote $\partial V/\partial \chi, \partial U/\partial \phi$. The variation of action (5) with respect to the metric $g_{\mu\nu}$ gives the Einstein’s equation

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu},$$  \hspace{1cm} (8)
where the stress-tensor is given by:

\[ T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\sigma \phi \nabla_\sigma \phi - g_{\mu\nu} U(\phi) + e^{2\alpha} \nabla_\mu \chi \nabla_\nu \chi - \frac{1}{2} e^{2\alpha} g_{\mu\nu} \nabla^\sigma \chi \nabla_\sigma \chi - e^{4\alpha} g_{\mu\nu} V(\chi). \]  

(9)

In the field theoretic description, the two field equations (6), (7) and the Einstein’s equation (8) completely describe the system.

Since the dark matter and the dark energy constitute up to 95% of the energy content of the Universe today, it is good a approximation to assume that the total energy momentum tensor of the Universe is given by (9). Demanding the local conservation of the energy-momentum tensor leads to:

\[ \nabla^\mu T_{\mu\nu} = \nabla^\mu T_{\mu\nu}^{(\phi)} + \nabla^\mu T_{\mu\nu}^{(\chi)} = 0. \]  

(10)

where \( T_{\mu\nu}^{(\phi)} \) and \( T_{\mu\nu}^{(\chi)} \) refer to the stress-tensor corresponding to scalar fields \( \phi \) and \( \chi \), respectively. Due to the interaction between the two fields \( \phi \) and \( \chi \), there is no unique way to write the stress-tensor corresponding to the scalar fields, and the conservation of the energy momentum tensor of the individual components is violated. Following (1), (6) and (7), the interaction between the two scalar fields can be described as:

\[ - \nabla^\mu T_{\mu\nu}^{(\phi)} = Q_{\nu}^{(F)} = \nabla^\mu T_{\mu\nu}^{(\chi)} \]  

(11)

where

\[ T_{\mu\nu}^{(\chi)} = e^{2\alpha(\phi)} \left( \nabla_\mu \chi \nabla_\nu \chi - \frac{1}{2} g_{\mu\nu} \nabla^\sigma \chi \nabla_\sigma \chi - e^{2\alpha(\phi)} g_{\mu\nu} V(\chi) \right) \]  

(12)

\[ T_{\mu\nu}^{(\phi)} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\sigma \phi \nabla_\sigma \phi - g_{\mu\nu} U(\phi) \]  

(13)

\[ Q_{\nu}^{(F)} = \nabla^\mu T_{\mu\nu}^{(\chi)} = -e^{2\alpha(\phi)\alpha,\phi}(\phi) \nabla_\nu \phi \left[ \nabla^\sigma \chi \nabla_\sigma \chi + 4e^{2\alpha(\phi)} V(\chi) \right] \]  

(14)

It is important to note that starting from (3), we can obtain interaction strength \( Q_{\nu}^{(F)} \) in terms of \( \phi \) and \( V(\chi) \). We can equally rewrite \( Q_{\nu}^{(F)} \) in-terms of \( U(\phi) \). While this field theory description may be considered a fundamental description of the system, the fluid description turns out to be more useful to analyze the cosmological observations. In that regard, the most common description of the interacting dark sector is in terms of dark matter fluid.

A. Fluid description of the interacting dark sector

In the fluid description, it is often convenient to consider the dark matter to be fluid. For this purpose, we replace the dark matter scalar field and related quantities by the
corresponding energy density $\rho_m$, pressure $p_m$ of the dark matter fluid [29]:

$$p_m = -\frac{1}{2}e^{2\alpha}\left[g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\chi + e^{2\alpha}V(\chi)\right], \quad \rho_m = -\frac{1}{2}e^{2\alpha}\left[g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\chi - e^{2\alpha}V(\chi)\right]. \quad (15)$$

The four velocity $u_\mu$ of the dark matter fluid is given by

$$u_\mu = -\left[-g^{\alpha\beta}\nabla_\alpha\chi\nabla_\beta\chi\right]^{-\frac{1}{2}}\nabla_\mu\chi \quad (16)$$

In this description, the Einstein’s equation can be rewritten in terms of dark energy scalar field and dark matter fluid:

$$G_{\mu\nu} = \kappa^2 \left[\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\sigma\phi\nabla_\sigma\phi - g_{\mu\nu}V(\phi) + p_m g_{\mu\nu} + (\rho_m + p_m)u_\mu u_\nu\right], \quad (17)$$

where the energy-momentum tensor for the dark matter fluid is given by

$$T^{(m)\mu}_\nu = p_m g_{\mu\nu} + (\rho_m + p_m)u_\mu u_\nu, \quad (18)$$

and the interaction term can be rewritten as

$$Q^{(F)}_\nu = \nabla_\mu T^{(m)\mu}_\nu = -e^{2\alpha(\phi)}\alpha(\phi)\nabla_\nu\phi \left[\nabla_\sigma\chi\nabla_\sigma\chi + 4e^{2\alpha(\phi)}V(\chi)\right] = -\alpha(\phi)\nabla_\nu\phi(\rho_m - 3p_m) \quad (19)$$

Identifying $T^{(m)} = T^{(m)\mu}_\mu = -(\rho_m - 3p_m)$, we get

$$Q^{(F)}_\nu = T^{(m)}\nabla_\nu\alpha(\phi) \quad (20)$$

Thus, we see that in the fluid description of interacting dark matter, the interaction term is proportional to trace of the energy-momentum tensor of the dark matter and the coupling $\alpha$. It is important to note that starting from the Jordan frame action (3), the form of the interaction term $Q^{(F)}_\nu$ is uniquely written in terms of dark energy scalar field and dark matter fluid.

This has to be contrasted with the dark matter interaction fluid models in the literature Refs. [7–30]) where $Q_\nu$ can take any form. In the next section we show that a one-to-one correspondence between the fields and the fluids is only true if the interaction term is given by $Q^{(F)}_\nu$ in Eq. (20).
III. COSMOLOGICAL EVOLUTION WITH DARK ENERGY - DARK MATTER INTERACTION

To study the cosmological evolution with interacting dark sector, we consider the spatially flat FRW metric with first order scalar perturbations in synchronous gauge [3]:

\[
g_{00} = -1, \quad g_{0i} = 0, \quad g_{ij} = a^2 \left[(1 + A)\delta_{ij} + \frac{\partial^2 B}{\partial x^i \partial x^j}\right]. \tag{21}
\]

where \(a \equiv a(t)\) is the scale factor with Hubble parameter given by \(H = \dot{a}/a\) and \(A \equiv A(t, x, y, z)\) and \(B \equiv B(t, x, y, z)\) are scalar perturbations. At the linear order, the scalar, vector and tensor perturbations decouple, and can be treated separately. Since the scalar perturbations couple to the energy density (\(\delta \rho\)) and pressure (\(\delta P\)) leading to the growing inhomogeneities, we only consider scalar perturbations.

The scalar fields \(\phi\) and \(\chi\), dark matter fluid energy density (\(\rho_m\)), dark matter fluid pressure (\(p_m\)) and the interaction strength (\(Q_\nu\)) can be split into background and perturbed parts as:

\[
\phi = \bar{\phi} + \delta \phi, \quad \chi = \bar{\chi} + \delta \chi, \quad \rho_m = \bar{\rho}_m + \delta \rho_m, \quad p_m = \bar{p}_m + \delta p_m, \quad Q_\nu = \bar{Q}_\nu + \delta Q_\nu \tag{22}
\]

Usually in the literature, in the fluid description, the dark matter is assumed to be pressureless dust, i.e. \(\bar{p}_m = \delta \rho_m = 0\). In this work, we do not make this assumption for the dark matter fluid, i.e., \(\bar{p}_m \neq 0\) and \(\delta p_m \neq 0\). Although, all our calculations are valid in the special case of pressureless dust.

Components of dark matter fluid four velocity can be written as:

\[
u_\mu = \bar{\nu}_\mu + \delta \nu_\mu, \quad \nu_0 = -1, \quad \delta \nu_0 = 0, \quad \nu_i = 0, \quad \delta \nu_i = \frac{\partial \delta u^s}{\partial x^i}, \quad \delta u^s = -\frac{\delta \chi}{\chi} \tag{23}
\]

In the following subsections, we present the evolution equations for the background and the first-order perturbations.
A. Correspondence between fields and fluids in the FRW background

In the fluid description, the Friedmann equations for the interacting dark sector are given by [5]:

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{\kappa^2}{3} \left( \rho_m + \frac{\dot{\phi}^2}{2} + V(\phi) \right),
\]

\[
2 \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 = -\frac{\kappa^2}{3} \left( \rho_m + \frac{\dot{\phi}^2}{2} - V(\phi) \right). \tag{24}
\]

From Eq. (1), the conservation equations for the dark energy field and dark matter fluid in the FRW background are given by:

\[
\ddot{\phi} + 3H\dot{\phi} + V(\phi)\dot{\phi} = \overline{Q},
\]

\[
\overline{\rho}_m + 3H(\overline{\rho}_m + \overline{p}_m) = -\overline{Q}. \tag{25}
\]

In the phenomenological description of the dark matter fluid interaction, there is no unique form of \( \overline{Q} \). Several authors have considered many different forms of \( \overline{Q} \) in the literature (See, for instance, Refs. [7–30]). However, as discussed in Sec. (II A), starting from the Jordan frame action (3), the interaction term \( Q^{(F)}_\nu \) in Eq. (20) is uniquely written in terms of dark energy scalar field and dark matter fluid. In this case, the background interaction term is given by

\[
\overline{Q}^{(F)} = -\alpha_{\phi}(\phi)(\dot{\phi}e^{2\alpha}(\phi) - 3\overline{p}_m). \tag{26}
\]

We now show that the above equations are consistent with the field theory description only for this form of interaction term \( \overline{Q}^{(F)} \). Using the definition of \( \rho_m \) and \( p_m \) in Eq. (15), the evolution equations for the scalar field \( \phi \) and \( \chi \) are given by:

\[
\ddot{\chi} + 3H\dot{\chi} + e^{2\alpha}V_\chi(\chi) + 2\alpha_{\phi}(\phi)\dot{\phi}\chi = 0,
\]

\[
\ddot{\phi} + 3H\dot{\phi} + U_\phi(\phi) + 4e^{4\alpha}\alpha_{\phi}(\phi)V(\chi) - e^{2\alpha}\alpha_{\phi}(\phi)\chi^2 = 0. \tag{27}
\]

The background interaction term in the field theory picture also can be obtained by the direct substitution of the variables:

\[
\overline{Q}^{(F)} = \alpha_{\phi}(\phi)\dot{\phi}e^{2\alpha}(\phi) \left[ \chi^2 - 4e^{2\alpha}V(\chi) \right]. \tag{28}
\]

Similarly, the Friedmann equations in the field theory description can be obtained by substituting \( \overline{\rho}_m \) and \( \overline{p}_m \) with corresponding field theory variables. From the above analysis,
it is clear that there is a one-to-one correspondence between the fluids and fields only for interaction term $Q^{(F)}$. For any other form of the interaction term, the correspondence may not exist. In Sec. (IV), we classify various models used in the literature based on this correspondence.

B. Correspondence between fields and fluids in first order perturbations

In the fluid description, the first order scalar perturbations, in synchronous gauge, satisfy the following equations:

\[
\dot{A} = \kappa^2 \left[ \left( \bar{p}_m + \bar{p}_m \right) \delta u^s - \bar{\phi} \delta \phi \right] \tag{29}
\]

\[
\dot{B} + 3H \dot{B} - \frac{A}{a^2} = 0 \tag{30}
\]

\[
\frac{3}{2} \ddot{A} + \nabla^2 \left[ \frac{1}{2} \dot{B} + H \dot{B} \right] + 3H \dot{A} = \frac{\kappa^2}{2} \left[ -\delta \rho_m - 3\delta p_m - 4\dot{\phi} \delta \phi + 2V_\phi(\bar{\phi}) \delta \phi \right] \tag{31}
\]

\[
-\frac{1}{2} \ddot{A} + \frac{1}{2} \alpha^2 \nabla^2 A - 3H \dot{A} - \frac{1}{2} H \nabla^2 \dot{B} = \frac{\kappa^2}{2} \left[ -\delta \rho_m + \delta p_m - 2V_\phi(\bar{\phi}) \delta \phi \right] \tag{32}
\]

From Eq. (1), the conservation equations for the dark energy field and dark matter fluid in the first order perturbations are given by:

\[
\delta \dot{\rho}_m + 3H (\delta p_m + \delta \rho_m) + (\bar{p}_m + \bar{p}_m) \left[ \frac{\nabla^2 \delta u^s}{a^2} + \frac{3}{2} \dot{A} + \frac{\nabla^2 \dot{B}}{2} \right] = -\delta Q \tag{33}
\]

\[
\ddot{\phi} \left( \delta \phi - \frac{\nabla^2 \delta \phi}{a^2} + V_{\phi\phi}(\bar{\phi}) \delta \phi \right) + \dot{\phi} \left( \ddot{\phi} + 6H \dot{\phi} + V_\phi(\bar{\phi}) \right) + \frac{\phi^2}{2} \left( \nabla^2 \dot{B} + 3\dot{A} \right) = \delta Q . \tag{34}
\]

The above equations are generic equations for the coupled dark matter fluid and dark energy field with arbitrary interaction term $\delta Q$. As mentioned earlier, there is no unique form of $\delta Q$ in the phenomenological description of the dark matter fluid interaction. Several authors have considered many different forms of $\delta Q$ in the literature (See, for instance, Refs. [7–30]). However, as discussed in Sec. (II A), starting from the Jordan frame action (3), the interaction term $Q^{(F)}$ in Eq. (20) is uniquely written in terms of dark energy scalar field and dark matter fluid. In this case, the perturbed interaction term is given by

\[
\delta Q^{(F)} = -(\delta \rho_m - 3\delta p_m) \alpha_{\phi}(\bar{\phi}) \dot{\phi} - (\bar{p}_m - 3\bar{p}_m) \left[ \alpha_{\phi\phi}(\bar{\phi}) \ddot{\phi} + \alpha_{\phi}(\bar{\phi}) \delta \phi \right] \tag{35}
\]

Like in the previous subsection, we now show that the above equations are consistent with the field theory description only for this form of interaction $Q^{(F)}$. Substituting $\rho_m, p_m, \delta \rho_m,$
and $\delta p_m$ from Eq. (15), the perturbed equations of motion for $\phi$ and $\chi$, respectively, are:

\[
\ddot{\delta \chi} - \nabla^2 \delta \chi + \frac{1}{a^2} \left( \nabla^2 \dot{B} + 3 \dot{A} \right) + 3H \dot{\delta \chi} + 2\alpha_{\phi,\phi}(\phi) \left( \ddot{\phi} \delta \chi + \dddot{\phi} \right) + 2\delta \dot{\phi} \left[ \dddot{\phi} \delta \chi \alpha_{\phi,\phi}(\phi) + e^{2\alpha_{\phi,\phi}(\phi)} V_{\chi}(\chi) \right] = 0 \tag{36}
\]

\[
\ddot{\delta \phi} - \nabla^2 \delta \phi + \frac{1}{a^2} \left( \nabla^2 \dot{B} + 3 \dot{A} \right) + U_{\phi,\phi}(\phi) \delta \phi + \frac{1}{2} \left( \nabla^2 \dot{B} + 3 \dot{A} \right) + 2\alpha_{\phi,\phi}(\phi) \left[ 2e^{2\alpha_{\phi,\phi}(\phi)} V_{\chi}(\chi) \delta \chi - \dddot{\chi} \right] + 2e^{2\alpha_{\phi,\phi}(\phi)} \delta \phi \left[ 8e^{2\alpha} V(\chi) - \dddot{\chi} \right] + e^{2\alpha_{\phi,\phi}(\phi)} \delta \phi \left[ 4e^{2\alpha} V(\chi) - \dddot{\chi} \right] = 0 \tag{37}
\]

The above perturbed field equations are identical to the equations obtained from Eqs. (6, 7), respectively. The perturbed interaction term in the field theory picture also can be obtained by the direct substitution of the variables:

\[
\delta Q^{(F)} = 2e^{2\alpha_{\phi,\phi}(\phi)} \dddot{\chi} \delta \chi + 2e^{2\alpha_{\phi,\phi}(\phi)} \dddot{\phi} \delta \phi \left[ \dddot{X} - 4V(\chi) \right] \tag{38}
\]

\[
+ 2e^{2\alpha_{\phi,\phi}(\phi)} \dddot{\phi} \delta \phi \left[ \dddot{X} - 8e^{2\alpha} V(\chi) \right] + e^{2\alpha_{\phi,\phi}(\phi)} \delta \phi \left[ \dddot{X} - 4e^{2\alpha} V(\chi) \right] \tag{39}
\]

We would like to stress the following points regarding the above results: First, there is no unique form of $\delta Q$, in the phenomenological description of the dark matter fluid interaction. However, demanding a one-to-one correspondence between the field and fluid picture leads to a unique interaction term $Q^{(F)}_{\nu}$. Second, we see that apart from the convenience of relating the variable to cosmological observables, evolution equations in the fluid description are simpler than those in the fluid theory description, making it easier for the numerical analysis of the model. Third, while the form of the interaction term is unique, it still contains unknown functions like $\alpha(\phi)$, $\chi$ and $V(\chi)$. In the next section, we now use this correspondence to clarify the phenomenological dark matter fluid interaction models in the literature [7–30].

### IV. INTERACTING DARK ENERGY MODELS IN THE LITERATURE

Since we have little information about the nature and dynamics of the dark sector, there is no unique way of describing the interaction between dark energy and dark matter. Till now, the interaction strength $Q_{\nu}$ is described by phenomenological models, with model parameters constrained by cosmological observations [5]. In many of the models, the interaction strength $Q_{\nu}$ in the dark sector is constructed using the energy densities of the dark energy and dark
matter, and other dynamic quantities appearing in the model. However, it is not clear whether the models can be written from a field-theoretic action.

In this work, starting from the Jordan frame action (3), we showed that the interaction term $Q^F_{\nu}$ is unique. We showed that this interaction provides a one-to-one mapping between the field and fluid description of the dark matter sector. Armed with this, in this section, we classify the interacting dark energy models considered in the literature into two categories based on the field-theoretic description. The table below identifies the models that can (or can not) be described by the field theory approach considered in this work. The list is not exhaustive but gives a good representation of the various models discussed in the literature.

| Interacting DE-DM model | DE-DM Interaction $\nabla^{\mu}T^{\text{(DE,DM)}}_{\mu\nu} = Q^{\text{(DE,DM)}}_{\nu}$ | Is $Q^{\nu} \propto Q^{(F)}_{\nu}$? |
|------------------------|-----------------------------------------------|-------------------------------|
| Amendola - 1999 [7]    | $\dot{\rho}_m + 3H\rho_m = -C\rho_m\dot{\phi}$ | Yes                          |
| Amendola - 1999 [8]    | $\dot{\rho}_m + 3H\rho_m = -C\rho_m\dot{\phi}$ | Yes                          |
| Billyard & Coley -1999 [9] | $\dot{\phi}(\dot{\phi} + 3H\dot{\phi} + kV) = \frac{(4-3\gamma)}{2\sqrt{\omega+\frac{2}{3}}} \dot{\phi}_\mu$ | Yes |
| Authors            | Equation                                                                 | 1-to-1 Correspondence |
|--------------------|--------------------------------------------------------------------------|-----------------------|
| Avelino & Silva - 2012 [21] | \( \dot{\rho}_m + 3H\rho_m = \alpha H\rho_w \)                           | No                    |
| Pan et al. - 2012 [22]  | \( \dot{\rho}_m + 3H\rho_m = 3\lambda_m H\rho_m + 3\lambda_d H\rho_d \) | No                    |
| Salvatelli et al. - 2013 [23]  | \( \dot{\rho}_{dm} + 3H\rho_{dm} = \xi H\rho_{de} \)                      | No                    |
| Amendola et al. - 2014 [24]  | \( \dot{\rho}_\alpha + 3H\rho_\alpha = -\kappa \sum_i C_{i\alpha} \dot{\phi}_i \rho_\alpha \) | Yes                   |
| Marra - 2015 [25]  | \( \dot{\rho}_m + 3H\rho_m = \nu \rho_m \dot{\phi} / M_{Pl} \)          | No                    |
| Bernardi & Landim - 2016 [26]  | \( \dot{\rho}_m + 3H\rho_m = Q (\rho_\phi + \rho_m) \dot{\phi} \)       | No                    |
| Pan & Sharov - 2016 [27]  | \( \dot{\rho}_{dm} + 3H\rho_{dm} = 3\lambda_m H\rho_{dm} + 3\lambda_d H\rho_d \) | No                    |
| Bruck & Mifsud - 2017 [28]  | \( \nabla^\mu T_{DM}^{\mu\nu} = Q \nabla_\nu \phi \)                      | Yes                   |
| Gonzalez & Trodden - 2018 [29]  | \( \dot{\rho}_\chi + 3H\rho_\chi = \alpha' \dot{\phi} \rho_\chi \)     | Yes                   |
| Barros et al. - 2018 [30]  | \( \dot{\rho}_c + 3H\rho_c = -\kappa \dot{\phi} \rho_c \)               | Yes                   |

V. CONCLUSIONS AND DISCUSSIONS

In this work, we have constructed the dark energy - dark matter interaction from a field theory action. This action is obtained from the \( f(\tilde{R}, \tilde{\chi}) \) action using a conformal transformation and redefinition of the scalar fields. While the total energy-momentum tensor is conserved due to interaction, the energy-momentum of the individual components in the dark sector is not satisfied. This lead to an unique interaction term \( Q_{\nu}^{(F)} \).

While the field theory description helps us to obtain the interaction from the action principle, the fluid description turns out to be more useful for analyzing cosmological observations. In that regard, the most common description of the interacting dark sector is in terms of dark matter fluid. However, in the phenomenological description of the dark matter fluid interaction, there is no unique form of \( Q_{\nu} \). In many of the models in the literature, the interaction strength \( Q_{\nu} \) in the dark sector is introduced by hand. We have systematically shown that the one-to-one correspondence between the fluids and fields is possible only if the interaction term is given by \( Q_{\nu}^{(F)} \). In this specific case, the equations in the field theory description can be obtained from the fluid equations by a simple substitution of the

\[ \frac{\partial}{\partial \chi} T_{DM}^{\mu\nu} = \frac{D}{\partial \chi} T_{DM}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \nabla_\mu \left( \frac{D}{\partial \chi} T_{DM}^{\mu\nu} \nabla_\nu \phi \right) \]

if \( D = 0 \)

\[ Q = \frac{C}{\partial \chi} T_{DM}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \nabla_\mu \left[ \frac{D}{\partial \chi} T_{DM}^{\mu\nu} \nabla_\nu \phi \right] \]

\[ \nabla^\mu T_{DM}^{\mu\nu} = Q \nabla_\nu \phi \]

1 Violates causality condition \( (D(\phi) > 0) \) for the disformal transformations [33]
variables.

We classified the interacting dark energy models considered in the literature into two categories based on the field-theoretic description. While many of the models have a field-theoretic description, many of the dark matter fluid interaction models do not have a field-theoretic description used in this work. The field-theoretic description used in this work is the simplest possible. It may be possible that by considering a generalized action, like Horndeski Lagrangian, some of these models may have a field-theoretic description [34]. This needs further investigation.

While the form of the interaction term \((Q_\nu^{(F)})\) is unique, it still contains unknown functions like \(\alpha(\phi), \chi\) and \(V(\chi)\). The immediate question that arises is whether one can use some other tools to constrain further the suitable dark matter-dark energy model from the observations. This is currently under investigation.

VI. ACKNOWLEDGEMENTS

We thank T. Padmanabhan for fruitful discussions and for drawing us into the dark sector dynamics! JPJ thanks M. Trodden for clarifications on Ref. [29]. JPJ is supported by CSIR Senior Research Fellowship, India. The work is partially supported by IRCC Seed Grant, IIT Bombay.

Appendix A: Field theoretic formulation of the dark energy - dark matter interaction

Consider the following Jordan frame action:

\[
S_J = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{1}{2\kappa^2} f(\tilde{R}, \tilde{\chi}) - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\chi} \tilde{\nabla}_\nu \tilde{\chi} - V(\tilde{\chi}) \right]
\]  

where \(f(\tilde{R}, \tilde{\chi})\) is arbitrary, smooth function of Ricci scalar \((\tilde{R})\) defined in the 4-D metric \(\tilde{g}_{\mu\nu}\), and the scalar field \(\tilde{\chi}\). Under conformal transformation and redefining the scalar fields, one can bring it to the Einstein frame with two interacting scalar fields [32].

To keep calculations tractable, we assume the form of \(f(\tilde{R}, \tilde{\chi})\) as

\[
f(\tilde{R}, \tilde{\chi}) = h(\tilde{\chi}) f(\tilde{R})
\]  

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The above action can be rewritten as:

\[
S = \int d^4x \sqrt{-\hat{g}} \left[ h(\tilde{\chi}) \left( \frac{F\tilde{R}}{2\kappa^2} - \tilde{U} \right) - \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \tilde{\chi} \partial_\nu \tilde{\chi} - V(\tilde{\chi}) \right]
\]  
(A3)

where

\[
F = \frac{\partial f}{\partial \tilde{R}} \quad \text{and} \quad \tilde{U} = \frac{F\tilde{R} - f}{2\kappa^2}
\]

Under the following conformal transformation

\[
\hat{g}_{\mu\nu} = F g_{\mu\nu}
\]  
(A4)

the above action (A3) becomes

\[
S = \int d^4x \sqrt{-\hat{g}} \left[ h(\tilde{\chi}) \left( \frac{\tilde{R}}{2\kappa^2} - h(\tilde{\chi})\tilde{U} + h(\tilde{\chi})\sqrt{\frac{3}{2\kappa^2}} \Box \psi - \frac{h(\tilde{\chi})}{2} \hat{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi 
\right.
\]

\[
- e^{-\sqrt{\frac{2\kappa^2}{3}}} \hat{g}^{\mu\nu} \partial_\mu \tilde{\chi} \partial_\nu \tilde{\chi} - \tilde{V}(\tilde{\chi}) \right]\n\]  
(A5)

where

\[
\psi = \sqrt{\frac{3}{2\kappa^2}} \ln F, \quad \tilde{U} = \frac{\tilde{U}}{F^2}, \quad \tilde{V} = \frac{V}{F^2}.
\]  
(A6)

Introducing one more conformal transformation:

\[
g_{\mu\nu} = h(\tilde{\chi})\hat{g}_{\mu\nu}
\]  
(A7)

the above action can be rewritten as:

\[
S = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \left[ \frac{1}{2he\sqrt{\frac{2\kappa^2}{3}}\psi} + \frac{3h^2}{4\kappa^2 h^2} \right] g^{\mu\nu} \partial_\mu \tilde{\chi} \partial_\nu \tilde{\chi}
\]

\[
- \frac{1}{2} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - \sqrt{\frac{3}{2\kappa^2}} h \hat{g}^{\mu\nu} \partial_\mu \tilde{\chi} \partial_\nu \psi - \hat{W} \right],
\]  
(A8)

where

\[
\hat{W} = \frac{FR - f}{\kappa h F^2} + \frac{V}{h^2 F^2}
\]  
(A9)

The above action in the Einstein frame neatly separates into Ricci scalar and the scalar fields. However, the scalar fields are not in canonical form. Since the metric \( g_{\mu\nu} \) appears in all the kinetic part of the scalar fields, the field space line-element can be written as:

\[
d\ell^2 \left[ \frac{1}{he\sqrt{\frac{2\kappa^2}{3}}\psi} + \frac{3h^2}{2\kappa^2 h^2} \right] d\chi^2 + 2\sqrt{\frac{3}{2\kappa^2}} h \frac{h}{h} d\tilde{\chi} d\psi + d\psi^2
\]  
(A10)
It has to be noted that it is impossible to bring the above line element to Euclidean form by redefinition of the fields. Thus, the field-space line-element can be written in many different ways, leading to different interaction between the two scalar fields. We list two cases below:

1. One of the simplest option is to redefine the fields as [31]:

\[
\sqrt{\frac{3}{2\kappa^2}} \ln h + \psi = \phi, \quad \tilde{\chi} = \chi
\]  

(A11)

Then the field space line element (A10) reduces to:

\[
d\ell^2 = \frac{1}{he^{\frac{2\kappa^2}{3}\psi}} d\tilde{\chi}^2 + \left[ d\sqrt{\frac{3}{2\kappa^2} \ln h + \psi} \right]^2 = e^{-\sqrt{\frac{2\kappa^2}{3}}\phi} d\chi^2 + d\phi^2
\]  

(A12)

Under this field redefinition, the Einstein frame action (A8) is given by:

\[
S_E = \int d^4x \sqrt{-g} \left[ R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - U(\phi) - \frac{1}{2} e^{-\sqrt{\frac{2\kappa^2}{3}}\phi} g^{\mu\nu} \nabla_\mu \chi \nabla_\nu \chi - e^{2\alpha(\phi)} V(\chi) \right]
\]  

(A13)

2. Let us now consider the following redefinition of the fields:

\[
e^{2\alpha(\phi)} \left( \frac{\partial \chi}{\partial \psi} \right)^2 + \left( \frac{\partial \phi}{\partial \psi} \right)^2 = 1
\]

\[
e^{2\alpha(\phi)} \frac{\partial \chi}{\partial \psi} \frac{\partial \chi}{\partial \psi} + \frac{\partial \phi}{\partial \psi} \frac{\partial \phi}{\partial \psi} = \sqrt{\frac{3}{2\kappa^2} \frac{h}{\tilde{\chi}^2}}
\]

\[
e^{2\alpha(\phi)} \left( \frac{\partial \chi}{\partial \chi} \right)^2 + \left( \frac{\partial \phi}{\partial \chi} \right)^2 = \frac{1}{he^{\frac{2\kappa^2}{3}\psi}} + \frac{3}{2\kappa^2} \frac{h^2}{\tilde{\chi}^2}
\]  

(A14)

where \( \chi \equiv \chi(\tilde{\chi}, \psi) \), \( \phi \equiv \phi(\tilde{\chi}, \psi) \), and \( \alpha(\phi) \) is an arbitrary function of \( \phi \). Under this redefinition, the field space line-element (A10) reduces to:

\[
ds^2 = e^{2\alpha(\phi)} d\chi^2 + d\phi^2
\]  

(A15)

Thus, the Einstein frame action takes the form

\[
S_E = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - U(\phi) - \frac{1}{2} e^{2\alpha(\phi)} g^{\mu\nu} \nabla_\mu \chi \nabla_\nu \chi - e^{4\alpha(\phi)} V(\chi) \right)
\]  

(A16)

and is identical to the action (5) in Sec. (II). This action describes two interacting scalar fields with an arbitrary coupling represented by the function \( \alpha(\phi) \).
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