Kelvin-Froude wake patterns of a traveling pressure disturbance

Jonathan Colen$^1$ and Eugene B. Kolomeisky$^2$

$^1$Department of Physics, University of Chicago, 5720 South Ellis Ave, Chicago, Illinois 60637, USA
$^2$Department of Physics, University of Virginia, P. O. Box 400714, Charlottesville, Virginia 22904-4714, USA

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According to Kelvin, a point pressure source uniformly traveling over the surface of deep calm water leaves behind universal wake pattern confined within 39° sector and consisting of the so-called transverse and diverging wavefronts. Actual ship wakes differ in their appearance from both each other and Kelvin’s prediction. The difference can be attributed to a deviation from the point source limit and for given shape of the disturbance quantified by the Froude number $F$. For smooth pressure sources, we generalize Kelvin’s stationary phase argument to encompass finite size effects and classify resulting wake patterns. Specifically, we show that there exist two characteristic Froude numbers, $F_1$ and $F_2 > F_1$, such as the wave is only present if $F > F_1$. For $F_1 < F < F_2$, the wake consists of the transverse wavefronts confined within a sector of an angle that may be smaller than Kelvin’s. An additional 39° wake made of both the transverse and diverging wavefronts is found for $F > F_2$. We also show that if the pressure source has sharp boundary, the wake is always present and features additional interference effects. Specifically, for a constant pressure line segment source mimicking a slender ship, the wake pattern can be understood as due to two opposing effect wakes resembling Kelvin’s and originating at segment’s ends.

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I. MOTIVATION

It is impossible to overlook similarity of the wakes produced on deep water by objects as distinct in sizes, shapes and speeds as a waterfowl, a high-speed boat, or a tanker. This property has its origin in hydrodynamic similarity of the flow pattern due to a point traveling pressure source as discovered by Kelvin [1]. The conclusion is a consequence of linearity of the theory and the dispersion law of gravity waves on deep water, which in the inviscid incompressible fluid limit, and neglecting the effects of capillarity, has the form [2]

$$\omega^2(k) = gk$$

where $\omega$ is the frequency of the wave of the wave vector $k$, $g$ is the free fall acceleration and $k = |k|$. Indeed dimensional considerations imply that the flow pattern is characterized by a single length scale

$$l = \frac{v^2}{g},$$

hereafter called the Kelvin length, composed of $g$ and the speed of the source $v$. If the length is measured in Kelvin units [3], the problem of determining the wake pattern is parameter-free, i.e. all the wakes produced by point sources are geometrically similar (the proof of the statement is Eq. (1) below). The hallmark of the Kelvin wake shown in Figure 1 is its “feathered” appearance due to the so-called transverse and diverging wavefronts confined within 39° sector [1-4].

The Kelvin wake is a dispersive counterpart of the Cherenkov and Mach wakes [5] produced by light and sound, respectively. However no threshold in the form of a requirement on the speed of the source is needed for its formation. Indeed, the wake is formed if there is a wave mode whose phase velocity $\omega/k$ matches the projection of the velocity of the source onto the direction of radiation. This statement known in hydrodynamics as the condition of stationarity [3, 4] or more generally as the Mach-Cherenkov-Landau (MCL) constraint [5] applied to the dispersion law (1) acquires the form

$$\cos \varphi = \frac{\omega}{kv} = \sqrt{\frac{g}{v^2k}} = \frac{1}{\sqrt{k}l}$$

(3)

where $\varphi$ is the angle between the direction of radiation of the gravity wave and the direction of motion of the source. Since $\cos \varphi \leq 1$, the wake is present if the inequality $kl \geq 1$ holds. Clearly there always are modes satisfying this inequality and thus participating in producing the wake.

Ship wakes seen in practice are not strictly similar. Indeed, the wakes of tankers are dominated by transverse wavefronts while the wakes of high-speed boats are largely made of diverging waves. This is due to finite size of pressure sources that Kelvin’s solution neglects [6]. For fixed shape these effects are captured by the Froude number $F$ defined in terms of the ratio of the Kelvin length [2] to characteristic length scale $a$ of the source (e.g. the hull length for a slender ship)

$$F = \frac{\sqrt{\frac{F^2}{a^2}}}{a} = \frac{v}{\sqrt{ga}},$$

(4)

that also plays a central role in understanding the wave resistance [2-4].

In marine practice, applicability of the linear water wave theory [2] is limited by the effects of wave breaking, which set an upper bound to the Froude number at $F \approx 3$ that can be achieved by high-speed boats. Since the Kelvin $a = 0$ limit corresponds to infinite Froude number,
$F = \infty$, deviations from the ideal Kelvin wake seen in practice are not surprising.

The apparent wake angle is the direct signature of the wake pattern, and ship wakes narrower than Kelvin's have long been observed both in practice [7] and numerical studies [8, 9]. The interest in appearance of ship wakes has recently been reignited following a study of Rabaud and Moisy [10], who deduced the dependence of the wake angle on the Froude number $F$ from a series of airborne images of ship wakes in the Google Earth database. The angle was found to be approximately constant and close to Kelvin’s $39^\circ$ for $0.1 \lesssim F \lesssim 0.6$. For larger Froude numbers, the angle was decreasing with $F$, reaching values as small as $14^\circ$ at $F \approx 1.7$, seemingly calling Kelvin’s theory into question. Original explanation of these results [10] hinged on an assumption that an object cannot generate waves of wavelength larger than its size.

Recognizing that the angle measured by Rabaud and Moisy corresponds to the locus of the peaks of highest waves, Darmon, Benzaquin, and Raphael [11] and Ellingsen [12] provided an assumption-free analysis of the observations [10] based on the classical linear water wave theory [2, 13, 14] applied to the case of traveling isotropic Gaussian pressure source. The treatment also indicated that the entire wake was delimited by Kelvin’s $39^\circ$ angle. Benzaquin, Darmon and Raphael [15] further extended the analysis to understanding wake patterns and wave resistance due to anisotropic pressure disturbances. Similar investigation has been carried out by Moisy and Rabaud [10]. Yet their interpretation involved a conjecture that amplitude of the waves excited by a disturbance of size $a$ is small unless their wavelength is in a narrow band around $a$.

A different explanation of narrow wake angles due to Noblesse et al. [17] implemented a classic idea of marine hydrodynamics that the wake pattern of a ship can be understood as an interference of two opposite effect Kelvin wakes originating at the bow and stern of the ship [3, 4, 18]. Assumptions underlying the three competing explanations of narrow wakes [10, 11, 15–17] have been critically assessed by He et al. [19].

These studies stimulated a series of fundamental questions regarding the gravity wake behind a ship that will be addressed in this work in a unified manner free of unnecessary assumptions:

(i) What is the physics behind the wakes more narrow than Kelvin’s? The conjectures regarding the relationship between the excited wavelength and the size of the source [10, 15] are considered to be controversial [19]. At the same time, the contention that the effect has largely a two-point interference origin [17] has not been proven.

(ii) Since there is more to the wake than apparent wake angle, how does the Froude number $F$ control the appearance of the wake, i.e. what are possible wake patterns? While there exist qualitative arguments putatively explaining the effect of the Froude number [6], quantitative understanding is lacking.

(iii) How is it possible for the wake to be bounded by Kelvin’s $39^\circ$ angle for small $F$? It is a puzzle because in this case the pattern has to be least similar to the ideal, $F = \infty$, Kelvin wake.


\[\zeta(x, y) = \frac{1}{\rho} \int \frac{d^2k}{(2\pi)^2} \frac{kp(k)e^{-ik\cdot r}}{(k\cdot v + i0)^2 - gk}, \]  

(5)

Here, \( r \) is a two-dimensional position vector, \( \rho \) is the density of water, and \( p(k) \) is a Fourier transform of the excess pressure \( \delta p(r) \) due to traveling disturbance,

\[p(k) = \int \delta p(r)e^{-ik\cdot r} d^2r.\]  

(6)

The \( +i0 \) shift in the denominator of the integrand in \( \zeta \) is required by causality [5].

Let us choose the positive \( x \) direction along the velocity vector \( v \) and measure the length in Kelvin units [2], the wave vectors in units of \( l^{-1} \), and the excess pressure \( \delta p(r) \) in units of \( \rho gl \).

Further specifying to the point source limit \( p = const = 1 \) brings the wake integral [5] to a parameter-free form

\[\zeta(x, y) = \int \frac{d^2k}{(2\pi)^2} \frac{ke^{-ik\cdot r}}{(k_x + i0)^2 - k} \]  

(7)

thus demonstrating geometric similarity of Kelvin wakes. Since the Kelvin limit continues to play central role for general \( p(k) \) in [5], we review this case first.

While the Fourier integral [7] cannot be computed in closed form, the geometry of the wake pattern can be understood by employing Kelvin’s stationary phase argument [1–5].

First, the integral [7] is dominated by the wave vectors corresponding to the pole of the integrand

\[k_x^2 = k\]  

(8)

which is the MCL condition [3] in disguise.

Second, far away from the source, at \( r \gg 1 \), the phase factor \( f = k\cdot r \) in the integrand in [7] is large and the exponential is a highly oscillatory function of \( k \). Here, contributions from various elements \( d^2k \) typically cancel each other; this is the case of destructive interference with almost zero net result. This cancellation, however, will not occur for wave vectors which both satisfy the MCL condition [3] and have a phase \( f \) which is stationary with respect to \( k \); this is the case of constructive interference.

The trace of the source divides the plane into two regions related to one another by reflection; without the loss of generality we can focus on the \( y > 0 \) half-space. Here the wake is a superposition of the waves whose wave vectors have positive components, \( k_{x,y} > 0 \). Then the phase is given by

\[f = k\cdot r = \sqrt{k} \cdot x + \sqrt{k^2 - k} \cdot y\]  

(9)

where the components \( k_{x,y} \) were expressed in terms of \( k \) according to Eq. (8). The condition of stationary phase \( df/dk = 0 \),

\[- \frac{y}{x} = \frac{\sqrt{k - 1} - 1}{2k - 1}\]  

(10)

can only be satisfied for \( x < 0 \), which is where the wake is. This relationship appeared previously [10]. The right-hand side of (10) vanishes at \( k = 1, k \rightarrow \infty \), and for \( k = 3/2 \) it reaches a maximum value of \( 1/2\sqrt{2} \). Thus Eq. (10) has two solutions for \( 0 \leq y/x < 1/2\sqrt{2} \) corresponding to the transverse and diverging wavefronts. These solutions merge at \( y/x = 1/2\sqrt{2} \), and none are found for \( y/x > 1/2\sqrt{2} \). Therefore the wake is confined (for \( y > 0 \)) by an \( \arctan(1/2\sqrt{2}) \approx 19.47^\circ \) angle which is Kelvin’s classic result.

Since the phase \( f \) is constant along the wavefront, Eqs. (9) and (10) can be solved relative to \( x \) and \( y \) to give the equation for the wavefront in parametric form:

\[x(k) = fk^{-3/2}(2k - 1),\]  

\[y(k) = -fk^{-3/2}\sqrt{k - 1}\]  

(11)

Wavewrists of the pattern [11] correspond to the choice of the phase in the form

\[f_n = -\pi \left( 2n + \frac{5}{4} \right)\]  

(12)

with \( n \) integer [20]. They are shown in Figure [1]; the \( y < 0 \) part of the wake is obtained by reflection. The wake consists of the transverse (t) wavefronts formed by elementary waves with the wave vectors in the \( 1 \leq k \leq 3/2 \) range connecting the edges of the pattern across the central line \( y = 0 \), and the diverging (d) wavefronts formed by the waves with the wave vectors in the \( k > 3/2 \) range connecting the source to the edges of the pattern [1–4]. Combined with the form of the integrand in [7], these facts imply that the transverse waves are significantly weaker than the diverging ones unless one is close to the central line \( y = 0 \). The wavelength of the pattern along the latter is \( 2\pi \) while at the wake boundaries \( y/x = \pm 1/2\sqrt{2} \) it is \( 4\pi/3 \).

Alternatively, the equation of the wavefront can be written out by substituting solutions of the equation of the stationary phase [10] into the expression for the phase [9]:

\[f(x, y) = -\frac{x^2 \left(1 + 4y^2/x^2 \pm \sqrt{1 - 8y^2/x^2}\right)^{3/2}}{4\sqrt{2}y (1 \pm \sqrt{1 - 8y^2/x^2})}.\]  

(13)

Here the upper and lower signs correspond to the diverging and transverse wavefronts, respectively. Equating Eqs. (12) and (13) then gives two implicit equations for the wavewrists of the transverse and diverging kind.

The phase function corresponding to the diverging wavefronts (plus sign in Eq. (13)) increases in magnitude without bound as one approaches the central line
y = 0. This translates into oscillatory behavior of the water height with ever decreasing wavelength, seen as a bunching of the diverging wavefronts in Figure 1a as $y \to 0$. In practice such a behavior may be halted by a combination of effects that the present theory omits: capillarity neglected in the dispersion law (1) and non-linearity coming into play for sufficiently steep waves. Possible manifestations are discussed by Barnell and Noblesse [9] and Noblesse et al. [17]; in practice the effects are limited to a very narrow angular range near the central line $y = 0$.

III. EFFECTS OF FINITE SIZE AND SHAPE: SMOOTH BOUNDARIES

The effect of finite size and shape of pressure source on the wake pattern crucially depends on whether water surface piercing is absent or not.

A. Modified stationary phase argument

The former situation studied first may describe a hovercraft, low-flying airplane, or missile, and a boat in the planing regime [3] [4].

This class of sources can be represented by smooth spatially localized excess pressure functions $\delta p(r)$ whose Fourier transforms $\tilde{p}(k)$ are localized in the $k$-space. It has an effect of imposing an ultraviolet cutoff in the integral (5). A simple approximate way to account for this effect is to take a sharp cutoff limit consisting in strict integral (5). A simple approximate way to account for this effect has an effect of imposing an ultraviolet cutoff in the Fourier transforms

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\text{on the wake pattern crucially depends on whether water}
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\text{is hidden in the phenomenological parameter } Q
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\text{that the diverging and transverse wavefronts are found outside the}
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\text{inner wake and are bounded by Kelvin’s 39° angle. As}
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\text{Q approaches 3/2, the inner wake angle } 16 \text{ matches Kelvin’s 39°, and the}
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\text{diverging wavefronts disappear. The wakes having the}
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\text{geometry of Figure 1b have been observed both in na-}
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\text{For } 1 < Q < 3/2, \text{ Figure 1c, the wake consists entirely}
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\text{of the transverse wavefronts confined within the wedge of}
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\text{the angle } 16 \text{ smaller than Kelvin’s. As the cutoff}
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\text{parameter } Q \text{ approaches unity, the wake narrows and}
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\text{disappears for } Q < 1. \text{ The absence of the wake for } Q < 1
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\text{does not imply strict lack of the waves but indicates that}
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\text{the stationary phase approximation no longer applies,}
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\text{and that the waves are weak.}
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\text{The wakes generated by finite-size pressure sources,}
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\text{like those in Figures 1b and 1c, are hereafter called}
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\[
\text{Kelvin-Froude wakes.}
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B. General numerical procedure

All the wake integrals considered in this paper were evaluated numerically via a fast Fourier transform algorithm [21]; a sample code is available online [22]. The integration region was chosen to be a $2k_{\text{max}} \times 2k_{\text{max}}$ square centered at the origin with sides parallel to the $k_{x,y}$ axes. In all of the wake images presented below, the computation parameters $k_{\text{max}}$ and the integration step $\Delta k$ were chosen to maximize the resulting image resolution for the desired real space range $x_{\text{max}}$. It can be shown for a fast Fourier transform on an array of size $N \times N$ that the resolutions in real and the wave vector space, $\Delta x$ and $\Delta k$, respectively, obey the uncertainty relation $\Delta x \Delta k \simeq 1/N$. Here $N = k_{\text{max}}/\Delta k = x_{\text{max}}/\Delta x$, and in practice the parameter $N$ was limited by available computer memory. So at fixed $N$, one finds that $\Delta x k_{\text{max}} \simeq \Delta k x_{\text{max}} \simeq 1$. For each image, $\Delta k$ was chosen to be the smallest value compatible with the desired $x_{\text{max}}$, so that $k_{\text{max}}$ and consequently the real space image resolution could be maximized.

For viewing purposes, the water height values, $\zeta(r)$, were adjusted to be centered around zero. In order to prevent very large response at the origin from dominating the color scale and obscuring other wake features, a bound was applied such that all the values satisfying $|\zeta(r)| > \zeta_{\text{max}}$ were changed to $\pm \zeta_{\text{max}}$. The exact value of $\zeta_{\text{max}}$ was determined individually for each wake image, with the goal being to have it small enough to show all of the wake features without it being so small that slight
variations in the background became visible. Finally, the values were rescaled to fit on the range $[-1, 1]$.

### C. Sharp wave vector cutoff

To illustrate the modified stationary phase argument at work, let us consider an example where the integration domain in Eq. (7) is a $2k_{\text{max}} \times 2k_{\text{max}}$ square centered at the origin whose sides are parallel to the $x, y$ axes. Intersections of the boundary of this square with the MCL curve (3) then determine the cutoff parameter $Q$ to be given by

$$Q = \begin{cases} k_{\text{max}}^2, & \text{if } k_{\text{max}}^2 < 2 \\ \frac{1 + \sqrt{1 + 4k_{\text{max}}^2}}{2}, & \text{if } k_{\text{max}}^2 \geq 2 \end{cases}$$

(17)

One then expects no wake for $k_{\text{max}} < 1$; for $1 \leq k_{\text{max}} \leq (3/2)^{1/2}$ the wave pattern will have the geometry of Figure 1a while for $k_{\text{max}} > (3/2)^{1/2}$ the geometry will be that of Figure 1b. Moreover, the wavecrests are predicted to be given by Eqs. (11), (12), (14) and (17).

For a series of different values of $k_{\text{max}}$, the integral was computed numerically. The results for the wave displacement $\zeta(r)$, shown in Figure 2 as a series of color-coded topographical height maps, fully support phenomenological reasoning based on the modified stationary phase argument.

The wavecrests in the top row images, (a)-(e), were overlaid (for $y > 0$) with the curves given by Eqs. (11), (12), (14) and (17). As expected, the diverging wavefronts are absent in cases (a) and (b), and the wakes are narrow and weak. As one goes through the sequence (c) to (e), the diverging wavefronts grow in presence and magnitude.

The bottom row images, (f)-(j), have (for $y > 0$) only the Kelvin $-y/x = 1/2 \sqrt{2}$ and Froude (15) rays overlaid, and they clearly demonstrate well-developed diverging wavefronts and narrowing of the inner wake according to Eqs. (16) and (17) with increase of $k_{\text{max}}$. The last three images, (h)-(j), also show that if the nearly zero wave displacement is formed between the Froude ray (15) and the part of the wake containing both the diverging and transverse wavefronts, as $k_{\text{max}}$ increases, this sliver widens due to narrowing of the inner wake. The appearance of this peculiar region is an artifact reflecting our inability to properly display rapidly oscillating diverging wavefronts, the property discussed following Eq. (13). We then conclude that as $k_{\text{max}} \rightarrow \infty$, the wave pattern approaches Kelvin’s ideal limit sketched in Figure 1a.

Since $k_{\text{max}}$ has a meaning of an inverse spatial scale of the pressure source, these conclusions can be recast in terms of the Froude number $F \equiv \sqrt{k_{\text{max}}}$ as follows: there exist two critical Froude numbers, $F_1 = 1$ and $F_2 = (3/2)^{1/4}$, such as no wake is found for $F < F_1$, for $F_1 \leq F \leq F_2$ the wake has the geometry of Figure 1a, while for $F > F_2$ the wake has the geometry of Figure 1b.

A different implementation of the sharp ultraviolet cutoff in the integral (7) only changes the critical values $F_{1,2}$ without affecting the gross picture that there are two possibilities for the wake patterns controlled by the Froude number and encapsulated in Figures 1a or 1b. The underlying physics is a suppression of the short-wavelength modes by finite size effects which (coming from large to small Froude numbers) progressively decrease the presence of the diverging wavefronts until they disappear at $F = F_2$ followed by gradual exclusion of the transverse wavefronts until the wake is gone at $F = F_1$.

### D. Generalization to smooth pressure sources $p(k)$

While the wakes of sources sharply localized in the wave vector space can be understood by generalization of the argument that explains the ideal Kelvin wake pattern, they mimic realistic disturbances poorly: their direct space counterparts $\delta p(r)$ feature non-physical slowly decaying oscillatory behavior as one moves away from the pressure center. Yet, we contend that the reasoning developed to understand these cases largely carries over to smooth localized pressure disturbances $p(k)$ whose real space counterparts $\delta p(r)$ are free of artifacts.

Indeed, the origin of the critical Froude numbers $F_{1,2}$ can be traced back to the existence of the two critical values of the cutoff parameter (14), $Q_1 = 1$ (appearance of the wake in the form of the transverse wavefronts), and $Q_2 = 3/2$ (onset of the diverging wavefront generation). However, for smooth $p(k)$ the cutoff parameter $Q$ is no longer sharply defined because contribution of large wave vector modes into the wake pattern is only suppressed rather than excluded. The well-localized nature of the pressure function $p(k)$ still allows us to define a characteristic cutoff parameter $Q$. The instants when the latter takes on the threshold values of $Q_1 = 1$ and $Q_2 = 3/2$ define the characteristic Froude numbers $F_1$ and $F_2$, respectively. Thus the transitions at $F = F_{1,2}$ become smeared. Moreover, observation of the narrow wake pattern like in Figure 1b may become problematic as the tail of the function $p(k)$ admits high wave vector modes, causing the wake to appear wider.

In order to test these predictions, we focus on smooth pressure distributions characterized by a single length scale $a \equiv 1/F^2$ (see Eq. (4)). This limitation is not crucial and is only made for the sake of simplicity. Then the response to the moving pressure source is given by a generalization of Eq. (7):

$$\zeta(r) = \int \frac{d^2k}{(2\pi)^2} \frac{kp(k/F^2)e^{ik\cdot r}}{(k_p^2 + a^2)^2}$$

(18)

For isotropic disturbance $p(k/F^2) \equiv p(k/F^2)$ the cutoff parameter $Q_{14}$ then can be estimated as

$$Q \simeq F^2$$

(19)

As a result the opening angle of the inner wake (10) for
$F \gg 1$ behaves as

$$2\theta \simeq \frac{1}{F}$$  \hspace{1cm} (20)

Curiously, the $1/F$ asymptotic behavior was predicted for the angle of largest waves due to isotropic Gaussian pressure source \[10, 11\].

For strongly anisotropic disturbance $p(k/F^2) \equiv p(k_x/F^2)$ the $x$-components of the wave vectors are effectively suppressed at $k_x \simeq F^2$; the MCL condition \[3\] then implies that

$$Q \simeq F^4$$  \hspace{1cm} (21)

In this case the opening angle of the inner wake \[16\] for $F \gg 1$ behaves as

$$2\theta \simeq \frac{1}{F^2}$$  \hspace{1cm} (22)

Curiously, the $1/F^2$ asymptotic dependence was predicted for the angle of largest waves by Noblesse \textit{et al.} \[17\] and by Moisy and Rabaud \[16\] for two different model anisotropic pressure disturbances. The integral \[18\] is evaluated numerically for two particular choices of isotropic and strongly anisotropic pressure disturbances to which the predictions \[19\]-\[22\] apply. To probe the effects of the pressure disturbance rather than that of the cutoff we made sure that $F^2 \ll k_{\text{max}}$ for isotropic case, $p(k/F^2) \equiv p(k_x/F^2)$, and $F^4 \ll k_{\text{max}}$ for strongly anisotropic one, $p(k/F^2) \equiv p(k_x/F^2)$.

1. \textit{Isotropic Gaussian pressure source}

As an example of an isotropic pressure source we look at the Gaussian disturbance

$$p \left( \frac{k}{F^2} \right) = \exp \left( - \frac{k^2}{F^4} \right).$$  \hspace{1cm} (23)

While this case has been studied previously \[11, 12\], our focus is different as we test quantitative reasoning regarding evolution of the wake pattern with the Froude number. In the regime of small Froude numbers we also find a disagreement with Refs.\[11\]. Our convention \[23\] gives a Froude number which is $\sqrt{2}/\pi$ times larger than that employed in Refs.\[11, 12\].

For the isotropic Gaussian pressure source \[23\] and a series of different values of $F$ the integral \[18\] was computed numerically. The results displayed in Figure 3 largely support the reasoning regarding the existence of two characteristic Froude numbers $F_{1,2}$ and two types of the wake patterns having the geometries of Figures 1b and 1c. Specifically, the results are reasonably well-described in terms of the cutoff parameter $Q \approx 1.4F^2$ (in agreement with the scaling prediction \[19\]) and characteristic Froude numbers $F_1 \approx 0.84$ and $F_2 \approx 1.04$.

The wakes in the first row images, (a)-(e), only feature transverse wavefronts which are bounded within a wedge marked by two black-dotted rays. Their bounding angle $2\theta$ is smaller than Kelvin’s; the Kelvin ray $-y/x = 1/2\sqrt{2}$ is also shown for comparison. The bounding wedge was drawn by fitting lines through the points of maximal $|\partial \zeta/\partial y|$ as demonstrated in the second from the top row of images, (a’)-(e’), that display topographical map of $\partial \zeta/\partial y$. As the Froude number increases tending...
FIG. 3. Evolution of the wake pattern generated by isotropic Gaussian pressure source, Eqs. (18) and (23), for a series of different values of the Froude number $F$. The legend is the same as that of Figure 2. The bounding rays for the top row were drawn via a procedure based on the topographical map of $\partial \zeta / \partial y$, the second from the top row, and explained in the main text. The source point was set as the intersection of these bounding rays with the $y = 0$ line, and was found to be at some $x = X_{\text{int}} > 0$ for $F < F_2$. The cutoff parameter employed in the third from the top row for the wavecrests (11) and in the fourth from the top row for the Froude ray (15) is $Q = 1.4F^2$.

to $F = F_2$, the wake widens approaching the Kelvin 39° limit. This would be an illustration of the geometry of Figure 1c, if not for the fact that the apex of the wake at $x = X_{\text{int}}$ is found to be ahead of the source, i.e. $X_{\text{int}} > 0$, a feature commonly observed in marine practice [3]. As the Froude number increases toward $F = F_2$, the apex of the wake $X_{\text{int}}$ approaches the disturbance center $x = 0$. Numerically determined values of the wake angle $2\theta$ and the apex position $X_{\text{int}}$ along with corresponding Froude numbers $F$ are quoted in images (a')-(e'). It is in this range of low Froude numbers that we find ourselves in disagreement with conclusions of Refs. [11, 12] who found that the wake is always delimited by Kelvin’s 39° angle.

The wavecrests of the third row images, (f)-(j), were overlaid with the curves given by Eqs. (11), (12), and (14) with $Q = 1.4F^2$. As the Froude number increases from 1.2 to 1.6, the diverging wavefronts develop and grow in presence.

The bottom row images, (k)-(o), only have the Kelvin $-y/x = 1/2\sqrt{2}$ and Froude (15) rays overlaid, and they clearly demonstrate well-developed diverging wavefronts and narrowing of the inner wake with increase of $F$ in accordance with Eq. (16).

The wakes in the $F = 1.25 - 2.5$ range of the Froude numbers can be visually compared with those reported in Refs. [11, 12], and we find ourselves in agreement on their appearance and evolution with the Froude number.

2. Strongly anisotropic Gaussian pressure source

As an example of a strongly anisotropic pressure source we look at the Gaussian disturbance

$$p \left( \frac{k}{F^2} \right) = \exp \left( -\frac{k_x^2}{F^4} \right).$$

(24)
The results of numerical evaluation of the integral \((18)\) shown in Figure 4 for a series of Froude numbers largely support the reasoning regarding the existence of two characteristic Froude numbers \(F_1, 2\) and two types of the wake patterns encapsulated in Figures 3 and 4. Specifically, numerical results are reasonably well-described in terms of the cutoff parameter \(Q \approx 3.1F^4\) (in agreement with the scaling prediction \((21)\) and characteristic Froude numbers \(F_1 \approx 0.75\) and \(F_2 \approx 0.83\).

The wakes in the first row images, (a)-(e), only feature transverse wavefronts bounded by the Kelvin ray \(-y/x = 1/2\sqrt{2}\) This would be an illustration of the geometry of Figure 3 if not for the fact that the wake is wider (a possibility already anticipated); the wake angle appears to be exactly Kelvin’s 39°. In contrast to the low-F regime of isotropic case, Figure 3 (a-e), the apex of the wake coincides with the center of the source. It is curious that low-F wakes of isotropic pressure source, Figure 3, are more narrow than low-F wakes of strongly anisotropic one, Figure 4.

The wavecrests of the first and second row images, (a)-(j), were overlaid with the curves given by Eqs. (11), (12), and (14) with \(Q = 3.1F^4\). As the Froude number increases from 0.9 to 0.98, the diverging wavefronts develop and grow in presence.

The bottom row images, (k)-(o), only have the Kelvin \(-y/x = 1/2\sqrt{2}\) and Froude \((15)\) rays overlaid, and they clearly demonstrate well-developed diverging wavefronts and narrowing of the inner wake with increase of \(F\) in accordance with Eq. (18).

**IV. EFFECTS OF FINITE SIZE AND SHAPE: SHARP BOUNDARIES**

Due to water piercing, pressure disturbances describing actual ships correspond to functions \(\delta p(r)\) that are finite within a compact spatial domain and zero otherwise. Their Fourier transforms \(p(k)\) entering Eq. (6) are slowly decaying oscillating functions; they do not suppress large wave vector modes, thus implying that the wake is always present. Oscillatory behavior exhibited by \(p(k)\), a source to a host of additional interference effects, has to be treated on the same footing as that of the oscillating exponentials in the integral \((5)\).

**A. Finite length segment constant pressure source: two-point interference argument**

Below we do not attempt to discuss all the possibilities but focus on the case of a boundary with two sharp corners. Specifically, we will be employing the function

\[ p \left( \frac{k}{F^2} \right) = \frac{2F^2}{k_x} \sin \left( \frac{k_x}{2F^2} \right) = \frac{2}{k_x a} \sin \left( k_x \frac{a}{2} \right) \]

which corresponds to a constant pressure line segment of length \(a \approx 1/F^2\) parallel to the \(x\)-axis and centered at the origin. The function \((25)\) mimics the geometry of a slender ship. Substituting \((25)\) into Eq. (18) we find

\[ \zeta(r) = \int \frac{2F^2 k d^2 k \sin(k_x/2F^2) e^{ikx}}{(2\pi)^2k_x^2 (k_x + i0)^2 - k} \]

\[ = \int \frac{kd^2 k e^{ikx(a/2) + k_y y}}{(2\pi)^2 k_x a (k_x + i0)^2 - k - k} \]

\[ \approx \frac{2}{k_x a} \sin \left( k_x \frac{a}{2} \right) \]

\[ \approx \frac{2}{k_x a} \sin \left( k_x \frac{a}{2} \right) \]

\[ \approx \frac{2}{k_x a} \sin \left( k_x \frac{a}{2} \right) \]

The second representation lends support to interpretation of the outcome as a superposition of the two opposing effect wakes originating at the segment’s end \(x = -a/2\) and front \(x = a/2\). This is largely in agreement with the classic idea of marine hydrodynamics that the wake pattern of a ship can be understood as an interference of two opposite effect bow and stern Kelvin wakes \((18)\). The difference is that the two wakes implicit in Eq. (26) are not exactly Kelvin’s and the function \((25)\) is not a pressure dipole in real space. This distinction is important because the function \(p(k/F^2)\) in Eq. (18) is the property of the pressure source at rest. Thus if a vessel is symmetric about its center, so is the pressure function; a pressure dipole is incompatible with central symmetry.

In the stationary phase approximation, the wakes originating at the segment’s edges, \(x = \mp a/2\) are described by the phase functions \(f(x \pm a/2, y)\), respectively, of the form \((13)\) found in the Kelvin case. For \(x < -a/2\) and \(|y/(x + a/2)| < 1/2\sqrt{2}\) the two wakes interfere. This wedge-shaped region of the two-point interference is our main interest. The rest of the pattern appears due to the front of the segment \(x = a/2\) generating its own Kelvin-like wake pattern.

The locus of the points of largest waves is given by the condition of constructive interference

\[ f \left( x + \frac{a}{2} \right) - f \left( x - \frac{a}{2} \right) = \pi (2n + 1), \quad n = 0, 1, 2, \ldots \]  

(27)

Likewise, the locus of the points of smallest waves is given by similar expression with \(2n + 1\) replaced by \(2n\). The relationship \((27)\) encompasses two different equations corresponding to the two sign choices in the phase function \((13)\); with upper signs we have the case of the diverging (dd) wavefronts interfering while with lower signs chosen, it is the transverse (tt) wavefronts that interfere. The two-letter abbreviations, dd or tt, are hereafter refer to the two-point interference nature of the effects.

Far away from the source, \(|x| \gg a\), Eq. (27) simplifies to

\[ \frac{\partial f}{\partial x} = \frac{\pi (2n + 1)}{a} \equiv \pi (2n + 1) F^2. \]

Curiously, evaluation of the derivative \(\partial f/\partial x\) with two different phase functions \((13)\) leads to the same result

\[ -\frac{y}{x} = \frac{\sqrt{k_n - 1}}{2k_n - 1}, \quad k_n = \pi^2 (2n + 1)^2 F^4 \]  

(29)

with the provision that the tt interference takes place if \(1 \leq k_n \leq 3/2\) while the dd interference corresponds
FIG. 4. Evolution of the wake pattern generated by strongly anisotropic Gaussian pressure source, Eqs. (18) and (24), for a series of different values of the Froude number \( F \). The legend is the same as that of Figures 2 and 3. The cutoff parameter employed in the first and second rows for the wavecrests, Eqs. (11) and (12), and in the third row for the Froude ray (15) is \( Q = 3.1F^4 \).

to the \( k_n > 3/2 \) case. Eq. (29) is a discrete counterpart of the equation of stationary phase (10); the role of the magnitude of the wave vector \( k \) is now played by the discrete set of \( k_n \) hereafter called the wave numbers.

The relationship (29) was already given [17] where it was obtained by assuming that the wake pattern is a result of interference of the two opposing Kelvin wakes originating at the bow and stern of a ship mimicked by a pressure dipole. In our analysis the physical picture of the two out of phase interfering wakes arises naturally in the course of treatment of integral (26); the latter also contains the information about magnitude of the effects. However as far as the geometry of the wake is concerned, the idea of Ref. [17] is vindicated. Hereafter our analysis complements that of Ref. [17]; numerical demonstration of the effects is where our results largely lie.

If the wave numbers \( k_n \) (29) are subject to the MCL condition [1], one finds a set of the \( x \)-components of the wave vectors, \( k_{x,n} = \pi(2n + 1)F^2 = \pi(2n + 1)/a \geq \pi/a \). This largely justifies the conjectures [10, 16] as consequences of the established theory [13, 14] applied to the pressure source (25).

According to Eq. (29), the constructive \( tt \) interference takes place along the central line \( y = 0 \) provided \( k_n = 1 \) which is satisfied for a discrete set of the Froude numbers

\[
F_n = \frac{1}{\sqrt{\pi(2n + 1)}} \approx 0.564, 0.326, 0.252, 0.213, \ldots \quad (30)
\]

Likewise, the condition of the destructive \( tt \) interference is

\[
F_n = \frac{1}{\sqrt{2\pi n}} \approx 0.399, 0.282, 0.230, 0.199, \ldots \quad (31)
\]

These two sets are familiar from analysis of a one-dimensional toy model that only allows for the transverse wavefronts [3]. This model is recovered along the central line where \( f(x, 0) = x \), Eq. (13). The ”unfavorable” set (30) leading to large waves is known to correspond to the maxima of the wave resistance versus Froude number dependence [3]. On the contrary, for the ”favorable” set (31) the waves are small, and the wave resistance exhibits minima [3].

Evolution of the wake pattern can be understood by following the ”flow” of the set of the wave numbers \( k_n \) (29) along the curve of the stationary phase (10) with decrease of the Froude number as shown in Figure 5. If \( \pi^2 F^4 > 3/2 \) or \( F > 0.624 \) all the wave numbers belong to the descending, \( k > 3/2 \), part of the curve and represent the \( dd \) interference effects. An example is shown in Figure 5a. Opaque grayscale is used for the curve of the
stationary phase \(\theta\), and the wave numbers \(k_n\) visible within the \(1 \leq k \leq 128\) range are shown as solid red circles. The numbers next to the circles are the subscripts in \(k_n\) labeling the fringes. The entire interference pattern is confined within a wedge of an angle given by Eq. (16) with

\[ Q = k_0 = \pi^2 F^4 \] (32)

This is the main interference fringe.

As the Froude number decreases, the wave numbers \(k_n\) flow leftward along the curve of the stationary phase \(\theta\). As a result, the confining angle of the wake given by Eqs. (16) and (32) increases approaching its maximal \(39^\circ\) value as \(F \rightarrow (3/2\pi^2)^{1/4} \approx 0.624\) from above. At a slightly smaller \(F\) the wave number \(k_0\) finds itself on the ascending, \(k < 3/2\), part of the curve \(\theta\). It is now shown as solid orange circle and it corresponds to the tt interference effect. At the same time the remaining wave numbers, \(k_n, n \geq 1\), still belong to the \(k > 3/2\) part of the curve \(\theta\). The confining angle of the pattern now decreases with decrease of \(F\) as evident from Figure 5.

As the Froude number further decreases, \(k_0\) moves down while the remaining \(k_n\)’s move up the curve \(\theta\). The confining angle of the interference pattern decreases and continues to be given by Eqs. (16) and (32) until \(k_0\) and \(k_1\) reach the same height. At this point, \(F = 0.568\) (Figure 5), the interference pattern is confined within fairly narrow wedge because \(F = 0.568\) is very close to the \(F_0 = 0.564\), Eq. (30), threshold.

Upon further decrease of the Froude number, the entire interference pattern is confined within a wedge of an angle given by Eq. (16) with

\[ Q = k_1 = 9\pi^2 F^4 \] (33)

It contains the single \(k_0\) tt interference fringe and an infinite set of the \(n \geq 1\), dd interference fringes. As the Froude number further decreases toward \(F = F_0 = 0.564\), the \(k_0\) tt fringe approaches the central line \(y = 0\), the wedge, confining the pattern according to Eqs. (16) and (33) widens, and at \(F = 0.564\), depicted in Figure 5, the tt fringe is about to become extinct.

At a Froude number slightly smaller than 0.564, the situation shown in Figure 5 occurs. Here only the \(n \geq 1\) dd interference fringes are present. Apart from the substitution \(n \rightarrow n + 1\), this looks like a counterpart to the case depicted in Figure 5a, indicative of a cyclic change in the interference pattern. However this is only a half-cycle as there also are ”destructive” \((2n + 1 \rightarrow 2n)\) wave numbers sandwiched between two existing neighboring ”constructive” \(k_n\)’s (29). To complete the cycle the Froude number has to further decrease until the tt fringe corresponding to the wave number \(k_1\) is extinct.

B. Numerical visibility requirements

Numerical evaluation of the integral (26) was carried over a \(2k_{max} \times 2k_{max}\) domain with \(k_{max} = 128 \gg 1\). In order to properly understand numerical results we observe that the two wakes generated by the segment’s front and end have the geometry of Figure 1b. They feature inner wakes of the opening angle (16).

\[ 2\theta \approx \frac{1}{\sqrt{Q}} \approx \frac{1}{\sqrt{k_{max}}} = \frac{1}{8\sqrt[4]{2}} \] (34)

This is a manifestation of strict elimination of the diverging \(k > k_{max}\) wave modes from contributing into the wake pattern. The same reasoning applies to the set of the wave numbers \(k_n\) (29) participating in the constrictive interference. Specifically, the presence of finite \(k_{max}\) limits numerical visibility of the effects to those satisfying the condition \(k_n < k_{max}\) thus setting an upper limit to the number of interference fringes. Equating \(k_n\) and \(k_{max}\) one arrives at the set of characteristic ”constructive” Froude numbers

\[ F_c^{(n)} = \frac{k_{max}^{1/4}}{\sqrt{\pi(2n + 1)}} = 1.898, 1.096, 0.849, 0.717, ... \] (35)

Similarly, in the ”destructive” \((2n + 1 \rightarrow 2n)\) case one finds

\[ F_d^{(n)} = \frac{k_{max}^{1/4}}{\sqrt{2\pi n}} = 1.342, 0.949, 0.775, 0.671... \] (36)
The meaning of these characteristic values is that for given Froude number \( F \) only the interference fringes whose order \( n \) (and type) satisfies the inequalities \( F < F_{c_d}^{(n)} \) are visible. Specifically, for \( k_{\text{max}} = 128 \) the two-point interference effects are only visible if \( F < F_{c}^{(0)} = 1.898 \). Explicit examples of what fringes are visible for given \( F \) are also shown in Figures 5 (a)-(e), where the horizontal range was deliberately chosen to cover the \( 1 \leq k \leq 128 \) interval. We note that there also are destructive fringes (not shown in Figure 5) sandwiched between two neighboring constructive ones.

C. Numerical confirmation

For a series of Froude numbers the integral (20) was computed choosing the square \( 2k_{\text{max}} \times 2k_{\text{max}} \) integration domain with \( k_{\text{max}} = 128 \). A series of wake patterns corresponding to Froude numbers \( F = 1.00 - 0.399 \) is shown in Figure 6. The wake source, the pressure segment, is shown as a bold interval of length \( a = 1/F^2 \) centered at the origin, the wakes are overlaid with the Kelvin ray \( y/(x + a/2) = 1/2\sqrt{2} \) (blue dotted line) originating at the segment’s end \( x = -a/2 \). The loci of the points of constructive interference given by Eqs. (27) and (13) for all visible \( n \), are indicated by red dotted lines if they represent the dd interference effects, \( k_n > 3/2 \), or orange dotted lines if the effect has the nt nature, \( 1 \leq k_n \leq 3/2 \). While for larger \( F \) these are nearly identical to the straight line rays (29), use of the more accurate expressions (27) and (13) makes a difference as \( F \) gets smaller.

First we observe that the dd interference effects are invisible in the maps of \( \zeta(\mathbf{r}) \). The reason is that \( k_{\text{max}} = 128 \) is sufficiently large so that our inability to properly display rapidly oscillating functions comes into play before the wave vector cutoff effect. That is why the inner wakes appear to be significantly wider than the prediction \( 2\theta = 1/8\sqrt{2} \). We have already faced this artifact in the problem of point source visible in Figures 2 (i, j) as a sliver of nearly zero water displacement sandwiched between the Froude ray (15) and the part of the wake containing both the diverging and transverse wavefronts. It is apparent from Figure 6 that the loci of the points of constructive interference may be masked by this artifact.

The dd interference effects are revealed in topographical maps of the magnitude of the wake \( \zeta(\mathbf{r}) \). This happens because rapid oscillations in \( \zeta \) now average to a positive constant. This creates a smooth background for the interference effects to be clearly visible. The maps of \( \zeta \) are shown in grayscale in the second and fourth rows of Figure 6 right below corresponding maps of \( \zeta \). They all feature a very narrow innermost wake, the finite \( k_{\text{max}} \) effect, whose opening angle is indeed given by \( 2\theta = 1/2\sqrt{8} \). Evolution of the \( \zeta \) pattern with the Froude number is in full correspondence with the analysis given; constructive interference fringes are dark while the destructive ones are light. As a visual aid to the fringe count, the Froude numbers for images d’-g’ in Figure 6 were chosen to coincide with those in Figure 5 (a, b, d, and e).

As \( F \) decreases, existing dark dd fringes move in the direction of larger angles toward Kelvin’s and get progressively wider. The new narrow fringes become visible at the threshold Froude numbers satisfying the condition \( k_n = k_{\text{max}} \), and their overall number increases. The maps of \( \zeta \) also reveal “echoes” which represent magnified patterns of \( \zeta \) in the vicinity of the inner wake boundaries. These artifacts of finite \( k_{\text{max}} \) make it easier to discern the fringes, especially when their number is large, the case of \( F \) small.

At \( F = 0.580 \) the \( n = 0 \) fringe ends up on the transverse part of the curve of the stationary phase, Figure 5, and we do not see tt interference effects in \( \zeta \). At the same time, the evolution of the original wake patterns \( \zeta(\mathbf{r}) \) with decrease of \( F \) reveals that the height of the transverse waves steadily increases in magnitude in the \( F = 1.000 - 0.564 \) range. The right boundary of this interval corresponds to the first unfavorable Froude number \( F_0 \) (30); the \( n = 0 \) tt fringe which is now along the central line \( y = 0 \) is about to disappear. We checked that the transverse waves are indeed largest at \( F = F_0 \) but this maximum is flat and the effect is weak. This is already visible from the F(30) (c), \( F = 0.564 \) (f), and \( F = 0.500 \) (g) images of Figure 6 which all feature transverse waves of comparable magnitude. On the other hand, the \( F = 0.399 \) (h) image of Figure 6, corresponding to the first favorable Froude number \( F_1 \) (31), illustrates a strong destructive interference effect where transverse wavefronts are practically invisible.

Despite confirmation of the two-point interference picture of the wake patterns, the images of Figure 6 do not yet make it obvious to the eye that we are dealing with two interfering wakes. In order to see that this is the case, we need to look at even smaller Froude numbers where the pressure source is sufficiently long and the effects of the interval ends are spatially well-separated. This is demonstrated in Figure 7 where we show topographical maps \( \zeta(\mathbf{r}) \) of the wakes whose Froude numbers were chosen from the sets (30), images b, f, and j, and (31), images d, h, and l, of special Froude numbers. These are supplemented by generic Froude numbers, images a, e, i, c, g, and k, sandwiched between the special ones.

Like in Figure 6 the dd interference effects can be “unmasked” by looking at the maps of \( \zeta(\mathbf{r}) \). These are not shown as they do not contain anything new compared to what was already demonstrated in Figure 6. Therefore, the images in Figure 7 are only overlaid with the Kelvin ray \( y/(x + a/2) = 1/2\sqrt{2} \) originating at segment’s end to aid visual separation of the effects of the interval edges.

Moreover, the tt interference effects are more pronounced than those in Figure 6. Specifically, the waves are largest at unfavorable Froude numbers, \( F = 0.326, 0.252 \) and 0.213 (second column of images), Eq. (30), and smallest at favorable ones, \( F = 0.282, 0.230 \) and 0.199 (fourth column of images), Eq. (31). Cyclic
FIG. 6. (Color online) Evolution of the two-point interference effects in the wake pattern generated by the uniform pressure segment source of length $a = 1/F^2$, Eq.(26), for a series of moderately large Froude numbers $F$. The first and third rows display topographical map of $\zeta(r)$ while its magnitude $|\zeta(r)|$ (grayscale) is presented in the second and fourth rows followed by grayscale bars. The source is shown as a bold segment centered at the origin, and each of the wakes is overlaid with the Kelvin ray $-y/(x + a/2) = 1/2\sqrt{2}$ (blue dotted line). The loci of the points of constructive interference given by Eqs.(27) and (13) for all visible $n$ are indicated by red dotted lines if they represent the dd interference effects, $k_n > 3/2$, or orange dotted lines if the effect has the tt nature, $1 \leq k_n \leq 3/2$.

changes in the appearance of the wake patterns occurring with decrease of $F$ are also obvious to the eye.

It is now certain that each wake pattern is a result of superposition of the two opposite effect wakes originating at the interval edges: the water at the front of the segment, $x = a/2$, is pulled up while at its end, $x = -a/2$, it is pushed down. The part of the wake outside the $|y/(x + a/2)| = 1/2\sqrt{2}$ wedge is solely due to the Kelvin-like wake originating at the segment’s front and resembling $k_{max} = 64$ wake of Figure 2.

V. DISCUSSION

We now discuss Figure 1a of Ref.[10], an image of a wake due to a cargo ship characterized by the Froude
number $F \approx 0.15$. It resembles the images shown in Figure 7 with two sources, at the bow and stern, determining overall wake pattern. The apparent wake angle that the authors associate with this Froude number corresponds to Kelvin-like wake originating at the bow of the ship. It is then no surprise that this angle is the same as Kelvin’s despite the fact that the Froude number is very small. There also is little evidence of waves in the part of the wake where one would expect to see the two-point interference effects: Figure 1a of Ref. [10] is very similar to the last column of images in Figure 7. This is also not surprising as $F \approx 0.15$ is very close to the favorable Froude number $F_7$ given by Eq. (31) when the destructive interference effects drastically diminish the waves along the central line $y = 0$.

### VI. ACKNOWLEDGEMENTS

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