Generalized T-product Tensor Bernstein Bounds

Shih Yu Chang *  Yimin Wei ‡

October 6, 2021

Abstract

Since Kilmer et al. introduced the new multiplication method between two third-order tensors around 2008 and third-order tensors with such multiplication structure are also called as T-product tensors, T-product tensors have been applied to many fields in science and engineering, such as low-rank tensor approximation, signal processing, image feature extraction, machine learning, computer vision, and the multi-view clustering problem, etc. However, there are very few works dedicated to exploring the behavior of random T-product tensors. This work considers the problem about the tail behavior of the unitarily invariant norm for the summation of random symmetric T-product tensors. Majorization and antisymmetric Kronecker product tools are main techniques utilized to establish inequalities for unitarily norms of multivariate T-product tensors. The Laplace transform method is integrated with these inequalities for unitarily norms of multivariate T-product tensors to provide us with Bernstein Bounds estimation of Ky Fan \( k \)-norm for functions of the symmetric random T-product tensors summation.

Index terms—T-product tensors, T-eigenvalues, T-singular values, Bernstein bound, Courant-Fischer theorem for T-product tensors.

1 Introduction

Since Kilmer et al. introduced the new multiplication method between two third-order tensors (T-product tensors), many new algebraic properties about such new multiplication rule between two third-order tensors are investigated recently [1, 2]. For example, the singular value decomposition (SVD) for third-order tensors via the tensor T-product is proposed in [3]. Some authors suggest a new framework by treating third-order tensors as linear operators on a space of matrices, see [4]. In [5], many useful tools of linear algebra are extended to the third-order tensors, including the T-Jordan canonical form, tensor decomposition theory, T-group inverse and T-Drazin inverse, and so on. Moreover, the authors in [6] proposed a definition of tensor functions based on the T-product of third-order F-square tensors, and Miao, Qi and Wei generalized the tensor T-function from F-square third-order tensors to rectangular tensors in [7]. These useful algebraic properties of T-product tensors have been discovered as powerful tools in many science and engineering fields: signal processing [8, 9], machine learning [10], computer vision [11, 12], image processing [13], low-rank tensor approximation [14, 15], etc.

Although T-product tensors have attracted many practical applications, all of these applications of T-product tensors assume that T-product tensors under consideration are deterministic. This assumption is

---

*Shih Yu Chang is with the Department of Applied Data Science, San Jose State University, San Jose, CA, U. S. A. (e-mail: shihyu.chang@sjsu.edu).
†This work was supported by the National Natural Science Foundation of China (No. 11771099) and Innovation Program of Shanghai Municipal Education Commission.
‡Yimin Wei is with the School of Mathematical Sciences, Shanghai Key Laboratory of Contemporary Applied Mathematics, Fudan University, Shanghai, 200433, PR China(e-mail: ymwei@fudan.edu.cn).
not practical in general scientific and engineering applications based on T-product tensors. In [17, 18], the authors have tried to establish several new tail bounds for sums of random T-product tensors. These probability bounds characterize large-deviation behavior of the extreme T-eigenvalue of the sums of random T-product tensors. The authors first apply Laplace transform method and Lieb’s concavity theorem for T-product tensors obtained from the work [17] to build several inequalities based on random T-product tensors, then utilize these inequalities to generalize the classical bounds associated with the names Chernoff, and Bernstein from the scalar to the T-product tensor setting. Tail bounds for the norm of a sum of random rectangular T-product tensors are also derived from corollaries of random symmetric T-product tensors cases. The proof mechanism is also applied to T-product tensor valued martingales and T-product tensor-based Azuma, Hoeffding and McDiarmid inequalities are also derived [18].

In this work, we will apply majorization techniques to establish new Bernstein bounds based on the summation of random symmetric T-product tensors. Compared to the previous work studied in [17, 18], we make following generalizations: (1) besides bounds related to extreme values of T-eigenvalues, we consider more general unitarily invariant norm for T-product tensors; (2) the bounds derived in [18] can only be applied to the identity map for the summation of random symmetric T-product tensors, this work can derive new bounds for any polynomial function raised by any power greater or equal than one for the summation of random symmetric T-product tensors. In order to drive these new bounds, we also establish Courant-Fischer min-max theorem for T-product tensors in Theorem 3 and majorization relation for T-singular values in Lemma 9. Our main theorem is provided below:

**Theorem 1.1 (Generalized T-product Tensor Bernstein Bound)** Consider a sequence \( \{X_j \in \mathbb{R}^{m \times m \times p}\} \) of independent, random symmetric T-product tensors with random structure defined by Definition 1. Let \( g \) be a polynomial function with degree \( n \) and nonnegative coefficients \( a_0, a_1, \ldots, a_n \) raised by power \( s \geq 1 \), i.e., \( g(x) = (a_0 + a_1 x + \cdots + a_n x^n)^s \) with \( s \geq 1 \). Suppose following condition is satisfied:

\[
g \left( \exp \left( t \sum_{j=1}^{m} X_j \right) \right) \geq \exp \left( tg \left( \sum_{j=1}^{m} X_j \right) \right) \text{ almost surely,} 
\]

where \( t > 0 \), and we also have

\[
X_j^p \leq \frac{plA^2}{2} \text{ almost surely for } p = 2, 3, 4, \cdots .
\]

Then we have following inequality:

\[
\Pr \left( \left\| g \left( \sum_{j=1}^{m} X_j \right) \right\|_{(k)} \geq \theta \right) \leq (n + 1)^{s-1} \inf_{t > 0} e^{-\theta t} k \cdot \left\{ a_0^s + \sum_{l=1}^{n} a_l^{is} \left[ 1 + mlst \Phi(m, d_1, d_2) + \frac{(mlst)^2 \sigma_1(A^2)}{2(1 - mlst)} \right] \right\}. 
\]

The rest of this paper is organized as follows. In Section 2, we review T-product tensors basic concepts and introduce a powerful scheme about antisymmetric Kronecker product for T-product tensors. In Section 3 we apply a majorization technique to prove T-product tensor norm inequalities. We then apply new derived T-product tensor norm inequalities to obtain random T-product tensor Bernstein bounds for the extreme T-eigenvalues and Ky Fan \( k \)-norm in Section 4. Finally, concluding remarks are given by Section 5.

---

1Definitions about T-eigenvalues and T-singular values associated to T-product tensors are given in Section 2.1
2 T-product Tensors

In this section, we will introduce fundamental facts about T-product tensors in Section 2.1. Several unitarily invariant norms about a T-product tensor are defined in Section 2.2. A powerful scheme about antisymmetric Kronecker product for T-product tensors will be provided by Section 2.3.

2.1 T-product Tensor Fundamental Facts

For a third order tensor $C \in \mathbb{R}^{m \times n \times p}$, we define bcirc operation to the tensor $C$ as:

$$\text{bcirc}(C) \overset{\text{def}}{=} \begin{bmatrix} C^{(1)} & C^{(p)} & C^{(p-1)} & \ldots & C^{(2)} \\ C^{(2)} & C^{(1)} & C^{(p)} & \ldots & C^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C^{(p)} & C^{(p-1)} & C^{(p-2)} & \ldots & C^{(1)} \end{bmatrix},$$

where $C^{(1)}, \ldots, C^{(p)} \in \mathbb{R}^{m \times n}$ are frontal slices of tensor $C$. The inverse operation of bcirc is denoted as $\text{bcirc}^{-1}$ with relation $\text{bcirc}^{-1}(\text{bcirc}(C)) \overset{\text{def}}{=} C$. Another operation to the tensor $C$ is unfolding, denoted as $\text{unfold}(C)$, which is defined as:

$$\text{unfold}(C) \overset{\text{def}}{=} \begin{bmatrix} C^{(1)} \\ C^{(2)} \\ \vdots \\ C^{(p)} \end{bmatrix}.$$  

The inverse operation of unfold is denoted as fold with relation $\text{fold}(\text{unfold}(C)) \overset{\text{def}}{=} C$.

The multiplication between two third order tensors, $C \in \mathbb{R}^{m \times n \times p}$ and $D \in \mathbb{R}^{n \times l \times p}$, is via T-product and this multiplication is defined as:

$$C \ast D \overset{\text{def}}{=} \text{fold}(\text{bcirc}(C) \cdot \text{unfold}(D)),$$

where $\cdot$ is the standard matrix multiplication. For given third order tensors, if we apply T-product to multiply them, we call them T-product tensors. A T-product tensor $C \in \mathbb{R}^{m \times n \times p}$ will be named as square T-product tensor if $m = n$.

For a symmetric T-product tensor $C \in \mathbb{R}^{m \times m \times p}$, we define Hermitian transpose of $C$, denoted by $C^H$, as

$$C^H = \text{bcirc}^{-1}((\text{bcirc}(C))^H).$$

And a tensor $D \in \mathbb{C}^{m \times m \times p}$ is called a Hermitian T-product tensor if $D^H = D$. Similarly, for a symmetric T-product tensor $C \in \mathbb{R}^{m \times m \times p}$, we define transpose of $C$, denoted by $C^T$, as

$$C^T = \text{bcirc}^{-1}((\text{bcirc}(C))^T).$$

And a tensor $D \in \mathbb{R}^{m \times m \times p}$ is called a symmetric T-product tensor if $D^T = D$.

The identity tensor $I_{m, m, p} \in \mathbb{R}^{m \times m \times p}$ can be defined as:

$$I_{m, m, p} = \text{bcirc}^{-1}(I_{mp}),$$

where $I_{mp}$ is the identity matrix in $\mathbb{R}^{mp \times mp}$. For a square T-product tensor, $C \in \mathbb{R}^{m \times m \times p}$, we say that $C$ is nonsingular if it has an inverse tensor $D \in \mathbb{R}^{m \times m \times p}$ such that

$$C \ast D = D \ast C = I_{m, m, p}.$$
A zero tensor, denoted as \( \mathcal{O}_{mnp} \in \mathbb{C}^{m \times n \times p} \), is a tensor that all elements inside the tensor as 0.

For any circular matrix \( C \in \mathbb{R}^{m \times m} \), it can be diagonalized with the normalized Discrete Fourier Transform (DFT) matrix, i.e., \( C = F_m^H D F_m \), where \( F_m \) is the Fourier matrix of size \( m \times m \) defined as

\[
F_m \overset{\text{def}}{=} \frac{1}{\sqrt{m}} \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & \omega & \cdots & \omega^{(m-1)} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \omega^{(m-1)} & \cdots & \omega^{(m-1)(m-1)}
\end{bmatrix},
\]

(11)

where \( \omega = \exp \left( \frac{2\pi i}{m} \right) \) with \( i^2 = -1 \). This DFT matrix can also be used to diagonalize a T-product tensor as \([2]\)

\[
\text{bcirc}(C) = (F_m^H \otimes I_m) \text{Diag} \left( C_i : i \in \{1, \cdots, m\} \right) (F_m \otimes I_m),
\]

(12)

where \( \otimes \) is the Kronecker Product and \( \text{Diag} \left( C_i : i \in \{1, \cdots, m\} \right) \in \mathbb{C}^{mp \times mp} \) is a diagonal block matrix with the \( i \)-th diagonal block as the matrix \( A_i \).

The inner product between two T-product tensors \( C \in \mathbb{C}^{m \times n \times p} \) and \( D \in \mathbb{C}^{m \times n \times p} \) is defined as:

\[
\langle C, D \rangle = \sum_{i,j,k} c_{i,j,k} \bar{d}_{i,j,k},
\]

(13)

where \( \ast \) is the complex conjugate operation.

We say that a symmetric T-product tensor \( C \in \mathbb{R}^{m \times m \times p} \) is a T-positive definite (TPD) tensor if we have

\[
\langle \mathcal{X}, C \ast \mathcal{X} \rangle > 0,
\]

(14)

holds for any non-zero T-product tensor \( \mathcal{X} \in \mathbb{R}^{m \times 1 \times p} \). Also, we said that a symmetric T-product tensor is a T-positive semidefinite (TPSD) tensor if we have

\[
\langle \mathcal{X}, C \mathcal{X} \rangle \geq 0,
\]

(15)

holds for any non-zero T-product tensor \( \mathcal{X} \in \mathbb{R}^{m \times 1 \times p} \). Given two T-product tensors \( C, D \), we use \( C \succ (\succeq) D \) if \( \langle C - D \rangle \) is a TPSD (TPD) T-product tensor.

We have the following theorem from Theorem 5 in \([19]\).

**Theorem 1** If a T-product tensor \( C \in \mathbb{R}^{m \times m \times p} \) can be diagonalized as

\[
\text{bcirc}(C) = (F_m^H \otimes I_m) \text{Diag} \left( C_i : i \in \{1, \cdots, m\} \right) (F_m \otimes I_m),
\]

(16)

where \( F \) is the DFT matrix defined by Eq. \((11)\); then \( C \) is symmetric, TPD (TPSD) if and only if all matrices \( C_i \) are Hermitian, positive definite (positive semidefinite).

Let \( C \in \mathbb{R}^{m \times m \times p} \) can be block diagonalized as Eq. \((16)\). Then, a real number \( \lambda \) is said to be a T-eigenvalue of \( C \), denoted as \( \lambda(C) \), if it is an eigenvalue of some \( C_i \) for \( i \in \{1, \cdots, m\} \). The largest and smallest T-eigenvalue of \( C \) are represented by \( \lambda_{\text{max}}(C) \) and \( \lambda_{\text{min}}(C) \), respectively. We use \( \lambda_{i,j} \) for the \( j \)-th largest T-eigenvalue of the matrix \( C_i \). We also use \( \sigma_{i,j} \), named as T-singular values, for the \( j \)-th largest singular values of the matrix \( C_i \).

We define the T-product tensor trace for a tensor \( C = (c_{ijk}) \in \mathbb{C}^{m \times m \times p} \), denoted by \( \text{Tr}(C) \), as following

\[
\text{Tr}(C) \overset{\text{def}}{=} \sum_{i=1}^{m} \sum_{k=1}^{p} c_{iik},
\]

(17)

which is the summation of all entries in \( f \)-diagonal components. Then, we have the following lemma about trace properties.
**Lemma 1** For any tensors $C, D \in \mathbb{C}^{m \times m \times p}$, we have

$$\text{Tr}(cC + dD) = c\text{Tr}(C) + d\text{Tr}(D),$$

where $c, d$ are two constants. And, the transpose operation will keep the same trace value, i.e.,

$$\text{Tr}(C) = \text{Tr}(C^T).$$

Finally, we have

$$\text{Tr}(C \star D) = \text{Tr}(D \star C).$$

**Proof:** Eqs. (18) and (19) are true from trace definition directly.

From T-product definition, the $i$-th frontal slice matrix of $D \star C$ is

$$D(i)C(1) + D(i-1)C(2) + \cdots + D(1)C(i) + D(m)C(i+1) + \cdots + D(i+1)C(m),$$

similarly, the $i$-th frontal slice matrix of $C \star D$ is

$$C(i)D(1) + C(i-1)D(2) + \cdots + C(1)D(i) + C(m)D(i+1) + \cdots + C(i+1)D(m).$$

Because the matrix trace of Eq. (21) and the matrix trace of Eq. (22) are same for each slice $i$ due to linearity and invariant under cyclic permutations of matrix trace, we have Eq. (20) by summing over all frontal matrix slices.

Below, we will define the determinant of a T-product tensor $C \in \mathbb{R}^{m \times m \times p}$, represented by $\det(C)$, as

$$\det(C) = \prod_{i=1, j=1}^{i=m, j=p} \lambda_{i,j}.$$  \hspace{1cm} (23)

We have the following theorem from Theorem 6 in [19] about symmetric T-product tensor decomposition.

**Theorem 2** Every symmetric T-product tensor $C \in \mathbb{R}^{m \times m \times p}$ can be factored as

$$C = U^T \star D \star U,$$

where $U$ is an orthogonal tensor, i.e., $U^T \star U = I_{m \times m \times p}$, and $D$ is a F-diagonal tensor, i.e., each frontal slice of $D$ is a diagonal matrix, such that diagonal entries of $(F_m \otimes I_m) \circ (D)$ $(F_m^H \otimes I_m)$ are T-eigenvalues of $C$. If $C$ is a TPD (TPSD) tensor, then all of its T-eigenvalues are positive (nonnegative).

From Theorem 2 and Lemma 1 we have the fact that

$$\text{Tr}(C) = \sum_i \lambda_i(C).$$  \hspace{1cm} (25)

If a symmetric T-product tensor $C \in \mathbb{R}^{m \times m \times p}$ can be expressed as the format shown by Eq. (16), the T-eigenvalues of $C$ with respect to the matrix $C_i$ are denoted as $\lambda_{i,k_i}$, where $1 \leq k_i \leq m$, and we assume that $\lambda_{i,1} \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,m}$ (including multiplicities). Then, $\lambda_{i,k_i}$ is the $k_i$-th largest T-eigenvalue associated to the matrix $C_i$. If we sort all T-eigenvalues of $C$ from the largest one to the smallest one, we use $\hat{k}$, a smallest integer between 1 to $m \times p$ (inclusive) associated with $p$ given positive integers $k_1, k_2, \cdots, k_p$ that satisfies

$$\lambda_{\hat{k}} = \min_{1 \leq i \leq m} \lambda_{i,k_i},$$

and we set $\hat{i}$ from $\lambda_{\hat{k}}$ as

$$\hat{i} = \arg \min_i \{ \lambda_{\hat{k}} = \lambda_{i,k_i} \}.$$  \hspace{1cm} (27)

Then, we will have the following Courant-Fischer theorem for T-product tensors.
Theorem 3 Given a symmetric T-product tensor $C \in \mathbb{R}^{m \times m \times p}$ and $p$ positive integers $k_1, k_2, \ldots, k_p$ with $1 \leq k_i \leq m$, then we have

$$\lambda_{\tilde{k}} = \max_{S \subseteq \mathbb{R}^{m \times m \times p}} \min_{A \in S} \frac{\langle X, C \ast X \rangle}{\langle X, X \rangle}$$

$$= \min_{T \in \mathbb{R}^{m \times m \times p}} \max_{A \in T} \frac{\langle X, C \ast X \rangle}{\langle X, X \rangle}$$

(28)

where $\lambda_{\tilde{k}}$ and $\tilde{i}$ are defined by Eqs. (26) and (27).

Proof:

First, we have to express $\langle X, C \ast X \rangle$ by matrices of $C_i$ and $X_i$ through the representation shown by Eq. (16). It is

$$\langle X, C \ast X \rangle = \frac{1}{p} (\text{beirc}(X), \text{beirc}(C)\text{beirc}(X))$$

$$= \frac{1}{p} \text{Tr} (\text{beirc}(X)^H \text{beirc}(C)\text{beirc}(X))$$

$$= \frac{1}{p} \text{Tr} (F_p^H \text{Diag} (x_i^H A_i x_i : i \in \{1, \ldots, p\}) F_p)$$

$$= \frac{1}{p} \text{Tr} (\text{Diag} (x_i^H A_i x_i : i \in \{1, \ldots, p\})) = \frac{1}{p} \sum_{i=1}^p x_i^H A_i x_i$$

(29)

We will just verify the first characterization of $\lambda_{\tilde{k}}$. The other is similar. Let $S_i$ be the projection of $S$ to the space with dimension $k_i$ spanned by $v_{i,1}, \ldots, v_{i,k_i}$, for every $x_i \in S_i$, we can write $x_i = \sum_{j=1}^{k_i} c_{i,j} v_{i,j}$. To show that the value $\lambda_{\tilde{k}}$ is achievable, note that

$$\langle X, C \ast X \rangle = \frac{1}{p} \sum_{i=1}^p x_i^H A_i x_i = \sum_{i=1}^{k_i} \sum_{j=1}^{k_i} \lambda_{i,j} c_{i,j}^2$$

$$= \sum_{i=1}^{k_i} \sum_{j=1}^{k_i} c_{i,j}^2 = \lambda_{\tilde{k}}$$

(30)

To verify that this is the maximum, let $T_{\tilde{i}}$ be the projection of $T$ to the space with dimension $k_{\tilde{i}}$ with dimension $n - k_{\tilde{i}} + 1$, then the intersection of $S$ and $T_{\tilde{i}}$ is not empty. We have

$$\min_{X \in S} \frac{\langle X, C \ast X \rangle}{\langle X, X \rangle} \leq \min_{X \in S \cap T_{\tilde{i}}} \frac{\langle X, C \ast X \rangle}{\langle X, X \rangle}$$

(31)

Any such $x_{\tilde{i}} \in S \cap T_{\tilde{i}}$ can be expressed as $x_{\tilde{i}} = \sum_{j=1}^{k_{\tilde{i}}} c_{\tilde{i},j} v_{\tilde{i},j}$, and any $i$ for $i \neq \tilde{i}$, we have $x_i \in S \cap T_{\tilde{i}}$.
expressed as $x_i = \sum_{j=k_i+1}^{m} c_{i,j}v_{i,j}$. Then, we have

$$\langle X, C \ast X \rangle \quad = \quad \frac{1}{p} \sum_{i=1}^{p} x_{i}^H \mathbf{A}_i x_i \quad = \quad \frac{1}{p} \sum_{i=1}^{p} x_{i}^H x_i \quad = \quad \frac{\sum_{i=1}^{p} \sum_{j=k_i+1; i \neq i}^{m} \lambda_{i,j} c_{i,j}^c c_{i,j}}{\sum_{i=1}^{p} \sum_{j=k_i; i \neq i}^{m} c_{i,j}^c c_{i,j}}$$

Therefore, for all subspaces $S$ of dimensions $\{k_1, \cdots, k_p\}$, we have $\min_{X \in S} \frac{\langle X, C \ast X \rangle}{\langle X, X \rangle} \leq \lambda_k$.

Given a symmetric T-product tensor $C$ with associated matrices $C_i$ provided by Eq. (16), next theorem is the representation of the summation of all the largest $k_i$ T-eigenvalues of $C_i$ and the summation of all the smallest $k_i$ T-eigenvalues of $C_i$.

**Theorem 4** Let $C \in \mathbb{R}^{m \times m \times p}$ be a symmetric T-product tensor with associated matrices $C_i$ provided by Eq. (16), and we sort T-eigenvalues of the matrix $C_i$ as $\lambda_{i,1} \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,k_i}$. Then, we have

$$\sum_{i=1}^{p} \max_{U_i U_i^H = I_{k_i}} \text{Tr} \left( U_i C_i U_i^H \right) = \sum_{i=1}^{p} \sum_{j=1}^{k_i} \lambda_{i,j}(C_i);$$

and

$$\sum_{i=1}^{p} \min_{U_i U_i^H = I_{k_i}} \text{Tr} \left( U_i C_i U_i^H \right) = \sum_{i=1}^{p} \sum_{j=1}^{k_i} \lambda_{i,m-j+1}(C_i),$$

where $U_i$ are $k_i \times m$ complex matrices.

**Proof:** From Theorem 1, we may assume that $C_i$ are diagonal matrices, denoted as $D_i$, since $C_i$ are symmetric T-product matrices. Therefore, we have the expression $C_i = V D_i V^H$. Then, we have

$$\text{Tr} \left( U_i D_i U_i^H \right) = \sum_{j=1}^{k_i} \sum_{l=1}^{m} u_{j,l}^* u_{j,l} \lambda_{i,l}(C_i) = \sum_{j=1}^{k_i} \sum_{l=1}^{m} p_{j,l} \lambda_{i,l}(C_i) = \left[ 1, 1, \cdots, 1 \right] \mathbf{P} \begin{bmatrix} \lambda_{i,1} \\ \lambda_{i,2} \\ \vdots \\ \lambda_{i,m} \end{bmatrix},$$

where $\mathbf{P} = (p_{j,l})$ is a $k_i \times m$ stochastic matrix. Then, we can concatenate an $(m - k_i) \times m$ matrix $Q$ to the matrix $P$ to make the following matrix $\left[ \begin{array}{c} P \\ Q \end{array} \right]$ as doubly stochastic from 2.C.1(4) from [20]. Then, Eq. (55) can be expressed as

$$\text{Tr} \left( U_i D_i U_i^H \right) = \left[ 1, 1, \cdots, 1, 0, 0, \cdots, 0 \right] \left[ \begin{array}{c} \lambda_{i,1} \\ \lambda_{i,2} \\ \vdots \\ \lambda_{i,m} \end{array} \right],$$

where $\mathbf{P} = (p_{j,l})$ is a $k_i \times m$ stochastic matrix. Then, we can concatenate an $(m - k_i) \times m$ matrix $Q$ to the matrix $P$ to make the following matrix $\left[ \begin{array}{c} P \\ Q \end{array} \right]$ as doubly stochastic from 2.C.1(4) from [20]. Then, Eq. (55) can be expressed as

$$\text{Tr} \left( U_i D_i U_i^H \right) = \left[ 1, 1, \cdots, 1, 0, 0, \cdots, 0 \right] \left[ \begin{array}{c} \lambda_{i,1} \\ \lambda_{i,2} \\ \vdots \\ \lambda_{i,m} \end{array} \right].$$
Given two lists of real numbers, \([a_1, \ldots, a_n]\) and \([b_1, \ldots, b_n]\), we use \([a_1, \ldots, a_n] \prec [b_1, \ldots, b_n]\) to represent the following relationships:

\[
\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i,
\]

holds for any \(k\) between 1 and \(n\). From Eq. (36), we have \([\lambda_{i,1}, \ldots, \lambda_{i,m}] [P^T, Q^T] \prec [\lambda_{i,1}, \ldots, \lambda_{i,m}]\) and 3.H.2.b from [20], we have

\[
\text{Tr} \left( U_i C_i U_i^H \right) \leq \sum_{j=1}^{k_i} \lambda_{i,j}(C_i);
\]

and

\[
\text{Tr} \left( U_i C_i U_i^H \right) \geq \sum_{j=1}^{k_i} \lambda_{i,m-j+1}(C_i).
\]

Finally, this theorem is proved by applying \(\sum_{i=1}^{p}\) to both sides of Eqs. (38) and (39) with respect to the index \(i\), and note that \(U_i V_i = (I_{k_i}, O)\) and \(U_i V_i = (O, I_{k_i})\), respectively. \(\square\)

### 2.2 Unitarily Invariant T-product Tensor Norms

Let us represent the T-eigenvalues of a symmetric T-product tensor \(H \in \mathbb{R}^{m \times m \times p}\) in decreasing order by the vector \(\vec{\lambda}(H) = (\lambda_1(H), \ldots, \lambda_{m \times p}(H))\), where \(m \times p\) is the total number of T-eigenvalues. We use \(\mathbb{R}_{\geq 0}(\mathbb{R}_{> 0})\) to represent a set of nonnegative (positive) real numbers. Let \(\| \cdot \|_\rho\) be a unitarily invariant tensor norm, i.e., \(\|H \ast U\|_\rho = \|U \ast H\|_\rho = \|H\|_\rho\), where \(U\) is any unitary tensor. Let \(\rho : \mathbb{R}_{\geq 0}^{m \times p} \rightarrow \mathbb{R}_{\geq 0}\) be the corresponding gauge function that satisfies Hölder’s inequality so that

\[
\|H\|_\rho = \|\|H\|_\rho = \rho(\vec{\lambda}(|H|)),
\]

where \(|H| \eqdef \sqrt{H^H \ast H}\). The bijective correspondence between symmetric gauge functions on \(\mathbb{R}_{\geq 0}^{m \times p}\) and unitarily invariant norms is due to von Neumann [21].

Several popular norms can be treated as special cases of unitarily invariant tensor norm. The first one is Ky Fan like \(k\)-norm [21] for tensors. For \(k \in \{1, 2, \ldots, m \times p\}\), the Ky Fan \(k\)-norm [21] for tensors \(H \in \mathbb{R}^{m \times m \times p}\), denoted as \(\|H\|_{(k)}\), is defined as:

\[
\|H\|_{(k)} \eqdef \sum_{i=1}^{k} \lambda_i(|H|).
\]

If \(k = 1\), the Ky Fan \(k\)-norm for tensors is the tensor operator norm, denoted as \(\|H\|\). The second one is Schatten \(p\)-norm for tensors, denoted as \(\|H\|_p\), is defined as:

\[
\|H\|_p \eqdef (\text{Tr}|H|^p)^{\frac{1}{p}},
\]

where \(p \geq 1\). If \(p = 1\), it is the trace norm.

Following inequality is the extension of Hölder inequality to gauge function \(\rho\) which will be used later to prove majorization relations.
**Lemma 2** For \( n \) nonnegative real vectors with the dimension \( r \), i.e., \( b_i = (b_{i1}, \ldots, b_{ir}) \in \mathbb{R}_{\geq 0}^r \), and \( \alpha > 0 \) with \( \sum_{i=1}^{n} \alpha_i = 1 \), we have

\[
\rho \left( \prod_{i=1}^{n} b_{i1}^{\alpha_i}, \prod_{i=1}^{n} b_{i2}^{\alpha_i}, \ldots, \prod_{i=1}^{n} b_{ir}^{\alpha_i} \right) \leq \prod_{i=1}^{n} \rho(b_i)^{\alpha_i}
\]  

(43)

**Proof:** This proof is based on mathematical induction. The base case for \( n = 2 \) has been shown by Theorem IV.1.6 from [22].

We assume that Eq. (43) is true for \( n = m \), where \( m > 2 \). Let \( \odot \) be the component-wise product (Hadamard product) between two vectors. Then, we have

\[
\rho \left( \prod_{i=1}^{m+1} b_{i1}^{\alpha_i}, \prod_{i=1}^{m+1} b_{i2}^{\alpha_i}, \ldots, \prod_{i=1}^{m+1} b_{ir}^{\alpha_i} \right) = \rho \left( \odot_{i=1}^{m+1} b_i^{\alpha_i} \right),
\]  

(44)

where \( \odot_{i=1}^{m+1} b_i^{\alpha_i} \) is defined as \( \prod_{i=1}^{m+1} b_{i1}^{\alpha_i}, \prod_{i=1}^{m+1} b_{i2}^{\alpha_i}, \ldots, \prod_{i=1}^{m+1} b_{ir}^{\alpha_i} \) with \( b_i^{\alpha_i} \equiv (b_{i1}^{\alpha_i}, \ldots, b_{ir}^{\alpha_i}) \). Under such notations, Eq. (44) can be bounded as

\[
\rho \left( \odot_{i=1}^{m+1} b_i^{\alpha_i} \right) = \rho \left( \left( \odot_{i=1}^{m} b_i^{\alpha_i} \right)^{\sum_{j=1}^{m} \alpha_j} \odot b_{m+1}^{\alpha_m+1} \right) \leq \left[ \prod_{j=1}^{m} \rho \left( \odot_{i=1}^{m} b_i^{\alpha_i} \right) \right] \cdot \rho(b_{m+1})^{\alpha_{m+1}} \leq \prod_{i=1}^{m+1} \rho(b_i)^{\alpha_i}.
\]  

(45)

By mathematical induction, this lemma is proved. \( \square \)

### 2.3 Antisymmetric Kronecker Product for T-product Tensors

In this section, we will discuss a machinery of antisymmetric Kronecker product for T-product tensors and this scheme will be used later for log-majorization results. Let \( \mathcal{H} \) be an \( m \times p \)-dimensional Hilbert space. For each \( k \in \mathbb{N} \), let \( \mathcal{H}^{\otimes k} \) denote the \( k \)-fold Kronecker product of \( \mathcal{H} \), which is the \((m \times p)^k\)-dimensional Hilbert space with respect to the inner product defined by

\[
\langle X_1 \otimes \cdots \otimes X_k, Y_1 \otimes \cdots \otimes Y_k \rangle \overset{\text{def}}{=} \prod_{i=1}^{k} \langle X_i, Y_i \rangle.
\]  

(46)

For \( X_1, \cdots, X_k \in \mathcal{H} \), we define \( X_1 \land \cdots \land X_k \in \mathcal{H}^{\otimes k} \) by

\[
X_1 \land \cdots \land X_k \overset{\text{def}}{=} \frac{1}{\sqrt{k!}} \sum_{\sigma} (\operatorname{sgn} \sigma) X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(k)},
\]  

(47)

where \( \sigma \) runs over all permutations on \( \{1, 2, \cdots, k\} \) and \( \operatorname{sgn} \sigma = \pm 1 \) depending on \( \sigma \) is even or odd. The subspace of \( \mathcal{H}^{\otimes k} \) spanned by \( \{X_1 \land \cdots \land X_k\} \), where \( X_i \in \mathcal{H} \), is named as \( k \)-fold antisymmetric Kronecker product of \( \mathcal{H} \) and represented by \( \mathcal{H}^{\land k} \).
For each $C \in \mathbb{R}^{m \times m \times p}$ and $k \in \mathbb{N}$, the $k$-fold Kronecker product $C^\otimes k \in \mathbb{R}^{m^k \times m^k \times p^k}$ is given by

\[ C^\otimes k \star (X_1 \otimes \cdots \otimes X_k) \overset{\text{def}}{=} (C \star X_1) \otimes \cdots \otimes (C \star X_k). \]

(48)

Because $\mathcal{S}^\wedge U$ is invariant for $C^\otimes k$, the antisymmetric Kronecker product of $C^\wedge k$ of $C$ can be defined as $C^\wedge k = C^\otimes [\mathcal{S}^\wedge U]$, then we have

\[ C^\wedge k \star (X_1 \wedge \cdots \wedge X_k) = (C \star X_1) \wedge \cdots \wedge (C \star X_k). \]

(49)

We will provide the following lemmas about antisymmetric Kronecker product.

**Lemma 3** Let $A, B, C, E \in \mathbb{R}^{m^k \times m^k \times p}$ be $T$-product tensors, for any $k \in \{1, 2, \cdots, m \times p\}$, we have

1. $(A^\wedge )^T = (A^T)^\wedge k$.  
2. $(A^\wedge k) \star (B^\wedge k) = (A \star B)^\wedge k$.  
3. If $\lim_{i \to \infty} \|A_i - A\| \to 0$, then $\lim_{i \to \infty} \|A_i^\wedge k - A^\wedge k\| \to 0$.  
4. If $C \succeq O$ (zero tensor), then $C^\wedge k \succeq O$ and $(C^p)^\wedge k = (C^\wedge k)^p$ for all $p \in \mathbb{R}_{>0}$.  
5. $|A|^\wedge k = |A^\wedge k|$.  
6. If $E \succeq O$ and $E$ is invertible, $(E^z)^\wedge k = (E^\wedge k)^z$ for all $z \in \mathbb{F}$.  
7. $\|E^\wedge k\| = \prod_{i=1}^k \lambda_i(|E|)$.  

**Proof:** Items 1 and 2 are the restrictions of the associated relations $(A^H)^\otimes k = (A^\otimes k)^H$ and $(A \star B)^\otimes k = (A^\otimes k) \star (B^\otimes k)$ to $\mathcal{S}^\wedge k$. The item 3 is true since, if $\lim_{i \to \infty} \|A_i - A\| \to 0$, we have $\lim_{i \to \infty} \|A_i^\otimes k - A^\otimes k\| \to 0$ and the associated restrictions of $A_i^\otimes k, A^\otimes k$ to the antisymmetric subspace $\mathcal{S}^k$. 

For the item 4, if $C \succeq O$, then we have $C^\wedge k = ((C^{1/2})^\wedge k)^H \star ((C^{1/2})^\wedge k) \succeq O$ from items 1 and 2. If $p$ is rational, we have $(C^p)^\wedge k = (C^\wedge k)^p$ from the item 2, and the equality $(C^p)^\wedge k = (C^\wedge k)^p$ is also true for any $p > 0$ if we apply the item 3 to approximate any irrelational numbers by rational numbers.

Because we have

\[ |A|^\wedge k = \left(\sqrt{A^H A}\right)^\wedge k = \sqrt{(A^\wedge k)^H A^\wedge k} = |A^\wedge k|, \]

(50)

from items 1, 2 and 4, so the item 5 is valid. 

For item 6, if $z < 0$, item 6 is true for all $z \in \mathbb{R}$ by applying the item 4 to $E^{-1}$. Since we can apply the definition $E^z \overset{\text{def}}{=} \exp(z \ln E)$ to have

\[ C^p = E^z \iff C = \exp\left(\frac{z}{p} \ln E\right), \]

(51)

where $C \succeq O$. The general case of any $z \in \mathbb{C}$ is also true by applying the item 4 to $C = \exp(\frac{z}{p} \ln E)$.

For the item 7 proof, it is enough to prove the case that $E \succeq O$ due to the item 5. Then, from Theorem 2 there exists a set of orthogonal tensors $\{U_1, \cdots, U_r\}$ such that $|E| \star U_i = \lambda_i U_i$ for $1 \leq i \leq m \times p$. We then have

\[ |E|^\wedge k(U_{i_1} \wedge \cdots \wedge U_{i_k}) = |E|^z \star U_{i_1} \wedge \cdots \wedge |E|^z \star U_{i_k} = \left(\prod_{i=1}^k \lambda_i(|E|)\right) U_{i_1} \wedge \cdots \wedge U_{i_k}, \]

(52)

where $1 \leq i_1 < i_2 < \cdots < i_k \leq m \times p$. Hence, $\| |E|^\wedge k \| = \prod_{i=1}^k \lambda_i(|E|)$. 

\[ \square \]
3 Multivariate T-product Tensor Norm Inequalities

In this section, we will begin with the introduction of majorization techniques in Section 3.1. Then, the majorization with integral average and log-majorization with integral average will be introduced by Section 3.2 and Section 3.3. These majorization results will be used to prove T-product tensor norm inequalities in Section 3.4.

3.1 Majorization Basis

In this subsection, we will discuss majorization and several lemmas about majorization which will be used at later proofs.

Let $x = [x_1, \cdots, x_r] \in \mathbb{R}^{m \times p}$, $y = [y_1, \cdots, y_r] \in \mathbb{R}^{m \times p}$ be two vectors with following orders among entries $x_1 \geq \cdots \geq x_r$ and $y_1 \geq \cdots \geq y_r$, *weak majorization* between vectors $x, y$, represented by $x \prec_w y$, requires following relation for vectors $x, y$:

$$
\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad k \in \{1, 2, \cdots, r\}.
$$

(Majorization between vectors $x, y$, indicated by $x \prec y$, requires following relation for vectors $x, y$:

$$
\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad \text{for } 1 \leq k < r;
$$

$$
\sum_{i=1}^{m \times p} x_i = \sum_{i=1}^{m \times p} y_i, \quad \text{for } k = r.
$$

(54)

For $x, y \in \mathbb{R}^{m \times p}_{\geq 0}$ such that $x_1 \geq \cdots \geq x_r$ and $y_1 \geq \cdots \geq y_r$, *weak log majorization* between vectors $x, y$, represented by $x \prec_{w \log} y$, requires following relation for vectors $x, y$:

$$
\prod_{i=1}^{k} x_i \leq \prod_{i=1}^{k} y_i, \quad k \in \{1, 2, \cdots, r\}.
$$

(Log majorization between vectors $x, y$, represented by $x \prec_{\log} y$, requires equality for $k = r$ in Eq. (55). If $f$ is a single variable function, $f(x)$ represents a vector of $[f(x_1), \cdots, f(x_r)]$. From Lemma 1 in [23], we have

**Lemma 4** (1) For any convex function $f : [0, \infty) \to [0, \infty)$, if we have $x \prec y$, then $f(x) \prec_w f(y)$.

(2) For any convex function and non-decreasing $f : [0, \infty) \to [0, \infty)$, if we have $x \prec_{w \log} y$, then $f(x) \prec_{w \log} f(y)$.

Another lemma is from Lemma 12 in [23], we have

**Lemma 5** Let $x, y \in \mathbb{R}^{m \times p}_{\geq 0}$ such that $x_1 \geq \cdots \geq x_r$ and $y_1 \geq \cdots \geq y_r$ with $x \prec_{\log} y$. Also let $y_i = [y_{i,1}, \cdots, y_{i,r}] \in \mathbb{R}^{m \times p}_{\geq 0}$ be a sequence of vectors such that $y_{i,1} \geq \cdots \geq y_{i,r} > 0$ and $y_i \rightarrow y$ as $i \rightarrow \infty$. Then, there exists $i_0 \in \mathbb{N}$ and $x_i = [x_{i,1}, \cdots, x_{i,r}] \in \mathbb{R}^{m \times p}_{\geq 0}$ for $i \geq i_0$ such that $x_{i,1} \geq \cdots \geq x_{i,r} > 0$, $x_i \rightarrow x$ as $i \rightarrow \infty$, and

$$
x_i \prec_{\log} y_i \quad \text{for } i \geq i_0.
$$

(56)

For any function $f$ on $\mathbb{R}_{\geq 0}$, the term $f(x)$ is defined as $f(x) \overset{\text{def}}{=} (f(x_1), \cdots, f(x_r))$ with conventions $e^{-\infty} = 0$ and $\log 0 = -\infty$. 

11
3.2 Majorization with Integral Average

Let $\Omega$ be a $\sigma$-compact metric space and $\nu$ a probability measure on the Borel $\sigma$-field of $\Omega$. Let $C, D_\tau \in \mathbb{R}^{m \times m \times p}$ be symmetric T-product tensors. We further assume that tensors $C, D_\tau$ are uniformly bounded in their norm for $\tau \in \Omega$. Let $\tau \in \Omega \rightarrow D_\tau$ be a continuous function such that $\sup\{\|D_\tau\| : \tau \in \Omega\} < \infty$. For notational convenience, we define the following relation:

$$\left[\int_{\Omega} \lambda_1(D_\tau)d\nu(\tau), \ldots, \int_{\Omega} \lambda_m(D_\tau)d\nu(\tau)\right] \overset{\text{def}}{=} \int_{\Omega^{m \times p}} \tilde{\lambda}(D_\tau)d\nu^{m \times p}(\tau). \quad (57)$$

If $f$ is a single variable function, the notation $f(C)$ represents a tensor function with respect to the tensor $C$.

**Theorem 5** Let $\Omega, \nu, C, D_\tau$ be defined as the beginning part of Section 3.2 and $f : \mathbb{R} \rightarrow [0, \infty)$ be a non-decreasing convex function, we have following two equivalent statements:

$$\tilde{\lambda}(C) \overset{w}{\prec} \int_{\Omega^{m \times p}} \tilde{\lambda}(D_\tau)d\nu^{m \times p}(\tau) \iff \|f(C)\|_\rho \leq \int_{\Omega} \|f(D_\tau)\|_\rho d\nu(\tau), \quad (58)$$

where $\|\cdot\|_\rho$ is the unitarily invariant norm defined in Eq. (40).

**Proof:** We assume that the left statement of Eq. (58) is true and the function $f$ is a non-decreasing convex function. From Lemma 4, we have

$$\tilde{\lambda}(f(C)) = f(\tilde{\lambda}(C)) \overset{w}{\prec} f\left(\int_{\Omega^{m \times p}} \tilde{\lambda}(D_\tau)d\nu^{m \times p}(\tau)\right). \quad (59)$$

From the convexity of $f$, we also have

$$f\left(\int_{\Omega^{m \times p}} \tilde{\lambda}(D_\tau)d\nu^{m \times p}(\tau)\right) \leq \int_{\Omega^{m \times p}} f(\tilde{\lambda}(D_\tau))d\nu^{m \times p}(\tau) = \int_{\Omega^{m \times p}} \tilde{\lambda}(f(D_\tau))d\nu^{m \times p}(\tau). \quad (60)$$

Then, we obtain $\tilde{\lambda}(f(C)) \overset{w}{\prec} f(\int_{\Omega^{m \times p}} \tilde{\lambda}(D_\tau)d\nu^{m \times p}(\tau))$. By applying Lemma 4.4.2 in [24] to both sides of $\tilde{\lambda}(f(C)) \overset{w}{\prec} f(\int_{\Omega^{m \times p}} \tilde{\lambda}(D_\tau)d\nu^{m \times p}(\tau))$ with gauge function $\rho$, we obtain

$$\|f(C)\|_\rho \leq \rho \left(\int_{\Omega^{m \times p}} \tilde{\lambda}(f(D_\tau))d\nu^{m \times p}(\tau)\right) \leq \int_{\Omega} \rho(\tilde{\lambda}(f(D_\tau)))d\nu(\tau) = \int_{\Omega} \|f(D_\tau)\|_\rho d\nu(\tau). \quad (61)$$

Therefore, the right statement of Eq. (58) is true from the left statement.

On the other hand, if the right statement of Eq. (58) is true, we select a function $f \overset{\text{def}}{=} \max\{x + c, 0\}$, where $c$ is a positive real constant satisfying $C + cI \geq O, D_\tau + cI \geq O$ for all $\tau \in \Omega$, and tensors $C + cI, D_\tau + cI$. If the Ky Fan $k$-norm at the right statement of Eq. (58) is applied, we have

$$\sum_{i=1}^{k} (\lambda_i(C) + c) \leq \sum_{i=1}^{k} \int_{\Omega} (\lambda_i(D_\tau) + c)d\nu(\tau). \quad (62)$$

Hence, $\sum_{i=1}^{k} \lambda_i(C) \leq \sum_{i=1}^{k} \int_{\Omega} \lambda_i(D_\tau)d\nu(\tau)$, this is the left statement of Eq. (58).

Next theorem will provide a stronger version of Theorem 5 by removing weak majorization conditions.
Lemma 4, we have then,

Proof: We assume that the left statement of Eq. (63) is true and the function \( f \) is a convex function. Again, from Lemma 4 we have

\[
\tilde{\lambda}(\mathcal{C}) \prec \int_{\Omega} \tilde{\lambda}(\mathcal{D}_{\tau}) d\nu^{m \times p}(\tau) \iff \|f(\mathcal{C})\|_\rho \leq \int_{\Omega} \|f(\mathcal{D}_{\tau})\|_\rho d\nu(\tau),
\]

(63)

where \( \|\cdot\|_\rho \) is the unitarily invariant norm defined in Eq. (40).

Proof: We assume that the left statement of Eq. (63) is true and the function \( f \) is a convex function. Again, from Lemma 4 we have

\[
\tilde{\lambda}(f(A)) = f(\tilde{\lambda}(A)) \prec_w f \left( \left( \int_{\Omega} \tilde{\lambda}(\mathcal{D}_{\tau}) d\nu^{m \times p}(\tau) \right) \right) \leq \int_{\Omega} f(\tilde{\lambda}(\mathcal{D}_{\tau})) d\nu^{m \times p}(\tau),
\]

(64)

then,

\[
\|f(A)\|_\rho \leq \rho \left( \int_{\Omega} f(\tilde{\lambda}(\mathcal{D}_{\tau})) d\nu^{m \times p}(\tau) \right) \leq \int_{\Omega} \rho \left( f(\tilde{\lambda}(\mathcal{D}_{\tau})) \right) d\nu(\tau) = \int_{\Omega} \|f(\mathcal{D}_{\tau})\|_\rho d\nu(\tau).
\]

(65)

This proves the right statement of Eq. (63).

Now, we assume that the right statement of Eq. (63) is true. From Theorem 6 we already have \( \tilde{\lambda}(\mathcal{C}) \prec_w \int_{\Omega} \tilde{\lambda}(\mathcal{D}_{\tau}) d\nu^{m \times p}(\tau) \). It is enough to prove \( \sum_{i=1}^{m \times p} \lambda_i(\mathcal{C}) \geq \int_{\Omega} \sum_{i=1}^{m \times p} \lambda_i(\mathcal{D}_{\tau}) d\nu(\tau) \). We define a function \( f \overset{\text{def}}{=} \max\{c-x,0\} \), where \( c \) is a positive real constant satisfying \( \mathcal{C} \leq c\mathcal{I}, \mathcal{D}_{\tau} \leq c\mathcal{I} \) for all \( \tau \in \Omega \) and tensors \( c\mathcal{I} - \mathcal{C}, c\mathcal{I} - \mathcal{D}_{\tau} \). If the trace norm is applied, i.e., the sum of the absolute value of all eigenvalues of a symmetric T-product tensor, then the right statement of Eq. (63) becomes

\[
\sum_{i=1}^{m \times p} \lambda_i(c\mathcal{I} - \mathcal{C}) \leq \int_{\Omega} \sum_{i=1}^{m \times p} \lambda_i(c\mathcal{I} - \mathcal{D}_{\tau}) d\nu(\tau).
\]

(66)

The desired inequality \( \sum_{i=1}^{m \times p} \lambda_i(\mathcal{C}) \geq \int_{\Omega} \sum_{i=1}^{m \times p} \lambda_i(\mathcal{D}_{\tau}) d\nu(\tau) \) is established. \( \square \)

3.3 Log-Majorization with Integral Average

The purpose of this section is to consider log-majorization issues for unitarily invariant norms of TPSD T-product tensors. In this section, let \( \mathcal{C}, \mathcal{D}_{\tau} \in \mathbb{R}^{m \times m \times p} \) be TPSD T-product tensors with \( m \times p \) nonnegative T-eigenvalues by keeping notations with the same definitions as at the beginning of the Section 3.2. For notational convenience, we define the following relation for logarithm vector:

\[
\left[ \int_{\Omega} \log \lambda_1(\mathcal{D}_{\tau}) d\nu(\tau), \cdots, \int_{\Omega} \log \lambda_{m \times p}(\mathcal{D}_{\tau}) d\nu(\tau) \right] \overset{\text{def}}{=} \int_{\Omega} \log \tilde{\lambda}(\mathcal{D}_{\tau}) d\nu^{m \times p}(\tau).
\]

(67)

Theorem 7 Let \( \mathcal{C}, \mathcal{D}_{\tau} \) be TPSD T-product tensors, \( f : (0, \infty) \rightarrow [0, \infty) \) be a continuous function such that the mapping \( x \rightarrow \log f(e^x) \) is convex on \( \mathbb{R} \), and \( g : (0, \infty) \rightarrow (0, \infty) \) be a continuous function such that the mapping \( x \rightarrow g(e^x) \) is convex on \( \mathbb{R} \), then we have following three equivalent statements:

\[
\tilde{\lambda}(\mathcal{C}) \prec_w \exp \int_{\Omega} \log \tilde{\lambda}(\mathcal{D}_{\tau}) d\nu^{m \times p}(\tau);
\]

(68)
\[ \|f(C)\|_\rho \leq \exp \int_\Omega \log \|f(D_\tau)\|_\rho \, d\nu(\tau); \quad (69) \]

\[ \|g(C)\|_\rho \leq \int_\Omega \|g(D_\tau)\|_\rho \, d\nu(\tau). \quad (70) \]

**Proof:** The roadmap of this proof is to prove equivalent statements between Eq. (68) and Eq. (69) first, followed by equivalent statements between Eq. (68) and Eq. (70).

**Eq. (68) \implies Eq. (69)**

There are two cases to be discussed in this part of proof: \(C, D_\tau\) are TPD tensors, and \(C, D_\tau\) are TPSD T-product tensors. At the beginning, we consider the case that \(C, D_\tau\) are TPD tensors.

Since \(D_\tau\) are positive, we can find \(\varepsilon > 0\) such that \(D_\tau \geq \varepsilon \mathbb{I}\) for all \(\tau \in \Omega\). From Eq. (68), the convexity of \(\log f(\varepsilon^\tau)\) and Lemma 4, we have

\[ \bar{\lambda}(f(C)) = f \left( \exp \left( \log \bar{\lambda}(C) \right) \right) \prec_w f \left( \exp \int_{\Omega^{m \times p}} \bar{\lambda}(D_\tau) d\nu^{m \times p}(\tau) \right) \leq \exp \left( \int_{\Omega^{m \times p}} \log f \left( \bar{\lambda}(D_\tau) \right) d\nu^{m \times p}(\tau) \right). \quad (71) \]

Then, from Eq. (40), we obtain

\[ \|f(C)\|_\rho \leq \rho \left( \exp \left( \int_{\Omega^{m \times p}} \log f \left( \bar{\lambda}(D_\tau) \right) d\nu^{m \times p}(\tau) \right) \right). \quad (72) \]

From the function \(f\) properties, we can assume that \(f(x) > 0\) for any \(x > 0\). Then, we have following bounded and continuous maps on \(\Omega\): \(\tau \rightarrow \log f(\lambda_i(D_\tau))\) for \(i \in \{1, 2, \cdots, m \times p\}\), and \(\tau \rightarrow \log \|f(D_\tau)\|_\rho\). Because we have \(\nu(\Omega) = 1\) and \(\sigma\)-compactness of \(\Omega\), we have \(\tau_k^{(n)} \in \Omega\) and \(\alpha_k^{(n)}\) for \(k \in \{1, 2, \cdots, n\}\) and \(n \in \mathbb{N}\) with \(\sum_{k=1}^{n} \alpha_k^{(n)} = 1\) such that

\[ \int_\Omega \log f(\lambda_i(D_\tau)) d\nu(\tau) = \lim_{n \to \infty} \sum_{k=1}^{n} \alpha_k^{(n)} \log f(\lambda_i(D_{\tau_k^{(n)}})), \quad \text{for } i \in \{1, 2, \cdots, m \times p\}; \quad (73) \]

and

\[ \int_\Omega \log \|f(D_\tau)\|_\rho d\nu(\tau) = \lim_{n \to \infty} \sum_{k=1}^{n} \alpha_k^{(n)} \log \|f(D_{\tau_k^{(n)}})\|_\rho. \quad (74) \]

By taking the exponential at both sides of Eq. (73) and apply the gauge function \(\rho\), we have

\[ \rho \left( \exp \int_{\Omega^{m \times p}} \log f(\bar{\lambda}(D_\tau)) d\nu^{m \times p}(\tau) \right) = \lim_{n \to \infty} \rho \left( \prod_{k=1}^{n} f \left( \bar{\lambda}(D_{\tau_k^{(n)}}) \right)^{\alpha_k^{(n)}} \right). \quad (75) \]

Similarly, by taking the exponential at both sides of Eq. (74), we have

\[ \exp \left( \int_\Omega \log \|f(D_\tau)\|_\rho d\nu(\tau) \right) = \lim_{n \to \infty} \prod_{k=1}^{n} \|f(D_{\tau_k^{(n)}})\|_\rho^{\alpha_k^{(n)}}. \quad (76) \]
From Lemma 2, we have
\[ \rho \left( \prod_{k=1}^{n} f \left( \tilde{X} \left( D^{(n)}_{\tau_k} \right) \right) \right)^{\alpha_k^{(n)}} \leq \prod_{k=1}^{n} \rho^{\alpha_k^{(n)}} \left( f \left( \tilde{X} \left( D^{(n)}_{\tau_k} \right) \right) \right) \]
\[ = \prod_{k=1}^{n} \left\| f \left( D^{(n)}_{\tau_k} \right) \right\|^{\alpha_k^{(n)}} \right\|^{\rho} \tag{77} \]

From Eqs. (75), (76) and (77), we have
\[ \rho \left( \exp \int_{\Omega^{m \times p}} \log f \left( \tilde{X}(D_{\tau}) \right) d\nu^{m \times p}(\tau) \right) \leq \exp \int_{\Omega} \log \left\| f(D_{\tau}) \right\|^{\rho} d\nu(\tau). \tag{78} \]

Then, Eq. (69) is proved from Eqs. (72) and (78).

Next, we consider that \( C, D_{\tau} \) are TPSD T-product tensors. For any \( \delta > 0 \), we have following log-majorization relation:
\[ \prod_{i=1}^{k} (\lambda_i(C) + \epsilon_\delta) \leq \prod_{i=1}^{k} \exp \int_{\Omega} \log (\lambda_i(D_{\tau}) + \delta) d\nu(\tau), \tag{79} \]

where \( \epsilon_\delta > 0 \) and \( k \in \{1, 2, \cdots r\} \). Then, we can apply the previous case result about TPD tensors to TPD tensors \( C + \epsilon_\delta I \) and \( D_{\tau} + \delta I \), and get
\[ \|f(C) + \epsilon_\delta I\|^{\rho} \leq \exp \int_{\Omega} \log \|f(D_{\tau}) + \delta I\|^{\rho} d\nu(\tau) \tag{80} \]

As \( \delta \to 0 \), Eq. (80) will give us Eq. (69) for TPSD T-product tensors.

**Eq. (68) \iff Eq. (69)**

We consider TPD tensors at first phase by assuming that \( D_{\tau} \) are TPD T-product tensors for all \( \tau \in \Omega \). We may also assume that the tensor \( C \) is a TPD T-product tensor. Since if this is a TPSD T-product tensor, i.e., some \( \lambda_i = 0 \), we always have following inequality valid:
\[ \prod_{i=1}^{k} \lambda_i(C) \leq \prod_{i=1}^{k} \exp \int_{\Omega} \log \lambda_i(D_{\tau}) d\nu(\tau) \tag{81} \]

If we apply \( f(x) = x^p \) for \( p > 0 \) and \( \| \cdot \|^{\rho} \) as Ky Fan k-norm in Eq. (69), we have
\[ \log \sum_{i=1}^{k} \lambda_i^p(C) \leq \int_{\Omega} \log \sum_{i=1}^{k} \lambda_i^p(D_{\tau}) d\nu(\tau). \tag{82} \]

If we add \( \frac{1}{k} \log \) and multiply \( \frac{1}{p} \) at both sides of Eq. (82), we have
\[ \frac{1}{p} \log \left( \frac{1}{k} \sum_{i=1}^{k} \lambda_i^p(C) \right) \leq \frac{1}{p} \log \left( \frac{1}{k} \sum_{i=1}^{k} \lambda_i^p(D_{\tau}) \right) d\nu(\tau). \tag{83} \]

From L'Hopital's Rule, if \( p \to 0 \), we have
\[ \frac{1}{p} \log \left( \frac{1}{k} \sum_{i=1}^{k} \lambda_i^p(C) \right) \to \frac{1}{k} \sum_{i=1}^{k} \log \lambda_i(C), \tag{84} \]
and

\[
\frac{1}{p} \log \left( \frac{1}{k} \sum_{i=1}^{k} \lambda_i^p(D_\tau) \right) \rightarrow \frac{1}{k} \sum_{i=1}^{k} \log \lambda_i(D_\tau),
\]

where \( \tau \in \Omega \). Applying Eqs. (84) and (85) into Eq. (83) and taking \( p \rightarrow 0 \), we have

\[
\sum_{i=1}^{k} \lambda_i(C) \leq \int_\Omega \sum_{i=1}^{k} \log \lambda_i(D_\tau) d\nu(\tau).
\]

(86)

Therefore, Eq. (68) is true for TPD tensors.

For TPSD T-product tensors \( D_\tau \), since Eq. (69) is valid for \( D_\tau + \delta I \) for any \( \delta > 0 \), we can apply the previous case result about TPD tensors to \( D_\tau + \delta I \) and obtain

\[
\prod_{i=1}^{k} \lambda_i(C) \leq \prod_{i=1}^{k} \exp \int_\Omega \log (\lambda_i(D_\tau) + \delta) d\nu(\tau),
\]

(87)

where \( k \in \{1, 2, \cdots, r\} \). Eq. (68) is still true for TPSD T-product tensors as \( \delta \rightarrow 0 \).

Eq. (68) \( \Rightarrow \) Eq. (70)

If \( C, D_\tau \) are TPD tensors, and \( D_\tau \geq \delta I \) for all \( \tau \in \Omega \). From Eq. (68), we have

\[
\tilde{\lambda}(\log C) = \log \tilde{\lambda}(C) \preceq_w \int_{\Omega^{m \times p}} \log \tilde{\lambda}(D_\tau) d\nu^{m \times p}(\tau) = \int_{\Omega^{m \times p}} \tilde{\lambda}(\log D_\tau) d\nu^{m \times p}(\tau).
\]

(88)

If we apply Theorem 5 to \( \log C, \log D_\tau \) with function \( f(x) = g(e^x) \), where \( g \) is used in Eq. (70), Eq. (70) is implied.

If \( C, D_\tau \) are TPSD T-product tensors and any \( \delta > 0 \), we can find \( \epsilon_\delta \in (0, \delta) \) to satisfy following:

\[
\prod_{i=1}^{k} (\lambda_i(C) + \epsilon_\delta) \leq \prod_{i=1}^{k} \exp \int_\Omega \log (\lambda_i(D_\tau) + \delta) d\nu(\tau).
\]

(89)

Then, from TPD T-product tensor case, we have

\[
\|g(C + \epsilon_\delta I)\|_\rho \leq \int_\Omega \|g(D_\tau + \delta I)\|_\rho d\nu(\tau).
\]

(90)

Eq. (70) is obtained by taking \( \delta \rightarrow 0 \) in Eq. (90).

Eq. (68) \( \Leftarrow \) Eq. (70)

For \( k \in \{1, 2, \cdots, r\} \), if we apply \( g(x) = \log(\delta + x) \), where \( \delta > 0 \), and Ky Fan \( k \)-norm in Eq. (70), we have

\[
\sum_{i=1}^{k} \log (\delta + \lambda_i(C)) \leq \sum_{i=1}^{k} \int_\Omega \log (\delta + \lambda_i(D_\tau)) d\nu(\tau).
\]

(91)

Then, we have following relation as \( \delta \rightarrow 0 \):

\[
\sum_{i=1}^{k} \log \lambda_i(C) \leq \sum_{i=1}^{k} \int_\Omega \log \lambda_i(D_\tau) d\nu(\tau).
\]

(92)

Therefore, Eq. (68) can be derived from Eq. (70).

Next theorem will extend Theorem 7 to non-weak version.
Theorem 8 Let $C, D_\tau$ be TPSD T-product tensors with $\int_\Omega \left\| D_\tau^{-p} \right\| d\nu(\tau) < \infty$ for any $p > 0$, $f : (0, \infty) \to (0, \infty)$ be a continuous function such that the mapping $x \to \log f(e^x)$ is convex on $\mathbb{R}$, and $g : (0, \infty) \to (0, \infty)$ be a continuous function such that the mapping $x \to g(e^x)$ is convex on $\mathbb{R}$, then we have following three equivalent statements:

$$\tilde{\lambda}(C) \prec_{\log} \exp \int_{\Omega^{m \times p}} \log \tilde{\lambda}(D_\tau) d\nu^{m \times p}(\tau);$$

$$\| f(C) \|_{\rho} \leq \exp \int_\Omega \log \| f(D_\tau) \|_{\rho} d\nu(\tau);$$

$$\| g(C) \|_{\rho} \leq \int_\Omega \| g(D_\tau) \|_{\rho} d\nu(\tau).$$

Proof:

The proof plan is similar to the proof in Theorem 7. We prove the equivalence between Eq. (93) and Eq. (94) first, then prove the equivalence between Eq. (93) and Eq. (95).

**Eq. (93) $\implies$ Eq. (94)**

First, we assume that $C, D_\tau$ are TPD T-product tensors with $D_\tau \geq \delta \mathcal{I}$ for all $\tau \in \Omega$. The corresponding part of the proof in Theorem 7 about TPSD tensors $C, D_\tau$ can be applied here.

For case that $C, D_\tau$ are TPSD T-product tensors, we have

$$\prod_{i=1}^k \lambda_i(C) \leq \prod_{i=1}^k \exp \int_\Omega \log (\lambda_i(D_\tau) + \delta_n) d\nu(\tau),$$

where $\delta_n > 0$ and $\delta_n \to 0$. Because $\int_\Omega \log \left( \tilde{\lambda}(D_\tau) + \delta_n \right) d\nu^{m \times p}(\tau) \to \int_\Omega \log \tilde{\lambda}(D_\tau) d\nu^{m \times p}(\tau)$ as $n \to \infty$, from Lemma 5 we can find $a^{(n)}$ with $n \geq n_0$ such that $a^{(n)}_1 \geq \cdots \geq a^{(n)}_r > 0$, $a^{(n)} \to \tilde{\lambda}(C)$ and $a^{(n)} \prec_{\log} \exp \int_\Omega \log \tilde{\lambda}(D_\tau + \delta_n \mathcal{I}) d\nu^{m \times p}(\tau)$.

Selecting $C^{(n)}$ with $\tilde{\lambda}(C^{(n)}) = a^{(n)}$ and applying TPD tensors case to $C^{(n)}$ and $D_\tau + \delta_n \mathcal{I}$, we obtain

$$\| f(C^{(n)}) \|_{\rho} \leq \exp \int_\Omega \log \| f(D_\tau + \delta_n \mathcal{I}) \|_{\rho} d\nu(\tau)$$

where $n \geq n_0$.

There are two situations for the function $f$ near 0: $f(0^+) < \infty$ and $f(0^+) = \infty$. For the case with $f(0^+) < \infty$, we have

$$\| f(C^{(n)}) \|_{\rho} = \rho(f(a^{(n)})) \to \rho(f(\tilde{\lambda}(C))) = \| f(C) \|_{\rho},$$

and

$$\| f(D_\tau + \delta_n \mathcal{I}) \|_{\rho} \to \| f(D_\tau) \|_{\rho},$$

where $\tau \in \Omega$ and $n \to \infty$. From Fatou–Lebesgue theorem, we then have

$$\limsup_{n \to \infty} \int_\Omega \log \| f(D_\tau + \delta_n \mathcal{I}) \|_{\rho} d\nu(\tau) \leq \int_\Omega \log \| f(D_\tau) \|_{\rho},$$

$$\leq \int_\Omega \log \| f(D_\tau) \|_{\rho},$$

17
By taking $n \to \infty$ in Eq. (97) and using Eqs. (98), (99), (100), we have Eq. (94) for case that $f(0^+) < \infty$.

For the case with $f(0^+) = \infty$, we assume that $\int_{\Omega} \log \|f(D_\tau)\|_\rho \, d\nu(\tau) < \infty$ (since the inequality in Eq. (94) is always true for $\int_{\Omega} \log \|f(D_\tau)\|_\rho \, d\nu(\tau) = \infty$). Since $f$ is decreasing on $(0, \epsilon)$ for some $\epsilon > 0$. We claim that the following relation is valid: there are two constants $a, b > 0$ such that
\[
a \leq \|f(D_\tau + \delta_n I)\|_\rho \leq \|f(D_\tau)\|_\rho + b,\tag{101}
\]
for all $\tau \in \Omega$ and $n \geq n_0$. If Eq. (101) is valid and $\int_{\Omega} \log \|f(D_\tau)\|_\rho \, d\nu(\tau) < \infty$, from Lebesgue’s dominated convergence theorem, we also have Eq. (94) for case that $f(0^+) = \infty$ by taking $n \to \infty$ in Eq. (97).

Below, we will prove the claim stated by Eq. (101). By the uniform boundedness of tensors $D_\tau$, there is a constant $\kappa > 0$ such that
\[
0 < D_\tau + \delta_n I \leq \kappa I,\tag{102}
\]
where $\tau \in \Omega$ and $n \geq n_0$. We may assume that $D_\tau$ is TPD tensors because $\|f(D_\tau)\|_\rho = \infty$, i.e., Eq. (101) being true automatically, when $D_\tau$ is TPSD T-product tensors. From Theorem 2, we have
\[
f(D_\tau + \delta_n I) = \sum_{i, \text{s.t. } \lambda_i(D_\tau) + \delta_n < \epsilon} f(\lambda_i(D_\tau) + \delta_n) U_i \ast U_i^H + \sum_{j, \text{s.t. } \lambda_j(D_\tau) + \delta_n \geq \epsilon} f(\lambda_j(D_\tau) + \delta_n) U_j \ast U_j^H \
\leq \sum_{i, \text{s.t. } \lambda_i(D_\tau) + \delta_n < \epsilon} f(\lambda_i(D_\tau)) U_i \ast U_i^H + \sum_{j, \text{s.t. } \lambda_j(D_\tau) + \delta_n \geq \epsilon} f(\lambda_j(D_\tau) + \delta_n) U_j \ast U_j^H \
\leq f(D_\tau) + \sum_{j, \text{s.t. } \lambda_j(D_\tau) + \delta_n \geq \epsilon} f(\lambda_j(D_\tau) + \delta_n) U_j \ast U_j^H.\tag{103}
\]
Therefore, the claim in Eq. (101) follows by the triangle inequality for $\|\cdot\|_\rho$ and $f(\lambda_j(D_\tau) + \delta_n) < \infty$ for $\lambda_j(D_\tau) + \delta_n \geq \epsilon$.

Eq. (93) $\iff$ Eq. (94)

The weak majorization relation
\[
\prod_{i=1}^{k} \lambda_i(C) \leq \prod_{i=1}^{k} \exp \int_{\Omega} \log \lambda_i(D_\tau) \, d\nu(\tau),\tag{104}
\]
is valid for $k < m \times p$ from Eq. (68) $\implies$ Eq. (69) in Theorem 7. We wish to prove that Eq. (104) becomes equal for $k = m \times p$. It is equivalent to prove that
\[
\log \det(C) \geq \int_{\Omega} \log \det(D_\tau) \, d\nu(\tau),\tag{105}
\]
where $\det(\cdot)$ is defined by Eq. (23). We can assume that $\int_{\Omega} \log \det(D_\tau) \, d\nu(\tau) \geq -\infty$ since Eq. (105) is true for $\int_{\Omega} \log \det(D_\tau) \, d\nu(\tau) = -\infty$. Then, $D_\tau$ are TPD tensors.

If we scale tensors $C, D_\tau$ as $aC, aD_\tau$ by some $a > 0$, we can assume $D_\tau \leq I$ and $\lambda_i(D_\tau) \leq 1$ for all $\tau \in \Omega$ and $i \in \{1, 2, \cdots, m \times p\}$. Then for any $p > 0$, we have
\[
\frac{1}{m \times p} \left\|D_\tau^{-\theta}\right\|_1 \leq \lambda_{\tau}^{-\theta}(D_\tau) \leq (\det(D_\tau))^{-\theta},\tag{106}
\]
and

\[ \frac{1}{\varrho} \log \left( \frac{\|D_\tau^{-\varrho}\|_{m \times p}}{m \times p} \right) \leq - \log \det(D_\tau). \]  

(107)

If we use tensor trace norm, represented by \(\|\cdot\|_1\), as unitarily invariant tensor norm and \(f(x) = x^{-\varrho}\) for any \(\varrho > 0\) in Eq. (94), we obtain

\[ \log \|C^{-\varrho}\|_1 \leq \int_{\Omega} \log \|D_\tau^{-\varrho}\|_{1} d\nu(\tau). \]  

(108)

By adding \(\frac{1}{m \times p}\) and multiplying \(\frac{1}{\varrho}\) for both sides of Eq. (108), we have

\[ \frac{1}{\varrho} \log \left( \frac{\|C^{-\varrho}\|_{m \times p}}{m \times p} \right) \leq \int_{\Omega} \frac{1}{\varrho} \log \left( \frac{\|D_\tau^{-\varrho}\|_{m \times p}}{m \times p} \right) d\nu(\tau) \]  

(109)

Similar to Eqs. (84) and (85), we have following two relations as \(\varrho \to 0\):

\[ \frac{1}{\varrho} \log \left( \frac{\|C^{-\varrho}\|_{m \times p}}{m \times p} \right) \to \frac{-1}{m \times p} \log \det(C), \]  

(110)

and

\[ \frac{1}{\varrho} \log \left( \frac{\|D_\tau^{-\varrho}\|_{m \times p}}{m \times p} \right) \to \frac{-1}{m \times p} \log \det(D_\tau). \]  

(111)

From Eq. (107) and Lebesgue’s dominated convergence theorem, we have

\[ \lim_{\varrho \to 0} \int_{\Omega} \frac{1}{\varrho} \log \left( \frac{\|D_\tau^{-\varrho}\|_{m \times p}}{m \times p} \right) d\nu(\tau) = \frac{-1}{m \times p} \int_{\Omega} \log \det(D_\tau) d\nu(\tau) \]  

(112)

Finally, we have Eq. (105) from Eqs. (109) and (112).

Eq. (94) \(\Rightarrow\) Eq. (95)

First, we assume that \(C, D_\tau\) are TPD tensors and \(D_\tau \geq \delta I\) for \(\tau \in \Omega\). From Eq. (93), we can apply Theorem 6 to \(\log C, \log D_\tau\) and \(f(x) = g(e^x)\) to obtain Eq. (95).

For \(C, D_\tau\) are TPSD T-product tensors, we can choose \(C^{(n)}\) and corresponding \(D_\tau^{(n)}\) for \(n \geq n_0\) given \(\delta_n \to 0\) with \(\delta_n > 0\) as the proof in Eq. (93) \(\Rightarrow\) Eq. (94). Since tensors \(C^{(n)}, D_\tau + \delta_n I\) are TPD T-product tensors, we then have

\[ \|g(C^{(n)})\|_{\rho} \leq \int \|g(D_\tau + \delta_n I)\|_{\rho} d\nu(\tau). \]  

(113)

If \(g(0^+) < \infty\), Eq. (95) is obtained from Eq. (113) by taking \(n \to \infty\). On the other hand, if \(g(0^+) = \infty\), we can apply the argument similar to the portion about \(f(0^+) = \infty\) in the proof for Eq. (93) \(\Rightarrow\) Eq. (94) to get \(a, b > 0\) such that

\[ a \leq \|g(D_\tau + \delta_n I)\|_{\rho} \leq \|g(D_\tau)\|_{\rho} + b, \]  

(114)
for all $\tau \in \Omega$ and $n \geq n_0$. Since the case that $\int_{\Omega} \|g(D_\tau)\|_\rho d\nu(\tau) = \infty$ will have Eq. (95), we only consider the case that $\int_{\Omega} \|g(D_\tau)\|_\rho d\nu(\tau) < \infty$. Then, we have Eq. (95) from Eqs. (113), (114) and Lebesgue’s dominated convergence theorem.

**Eq. (93) ⇐ Eq. (95)**

The weak majorization relation

$$
\sum_{i=1}^{k} \log \lambda_i(C) \leq \sum_{i=1}^{k} \int_{\Omega} \log \lambda_i(D_\tau) d\nu(\tau)
$$

is true from the implication from Eq. (68) to Eq. (70) in Theorem 7. We have to show that this relation becomes identity for $k = m \times p$. If we apply $\|\cdot\|_\rho = \|\cdot\|_1$ and $g(x) = x^{-\theta}$ for any $\theta > 0$ in Eq. (95), we have

$$
\frac{1}{\theta} \log \left( \frac{\|C^{-\theta}\|_1}{m \times p} \right) \leq \frac{1}{\theta} \log \left( \int_{\Omega} \frac{\|D^{-\theta}_\tau\|_1}{m \times p} d\nu(\tau) \right).
$$

Then, we will get

$$
- \log \det(C) = \lim_{\theta \to 0} \frac{1}{\theta} \log \left( \frac{\|C^{-\theta}\|_1}{m \times p} \right) \leq \lim_{\theta \to 0} \frac{1}{\theta} \log \left( \int_{\Omega} \frac{\|D^{-\theta}_\tau\|_1}{m \times p} d\nu(\tau) \right) = 1 \frac{\log \det(D_\tau) d\nu(\tau)}{m \times p},
$$

which will prove the identity for Eq. (115) when $k = m \times p$. The equality in $= 1$ will be proved by the following Lemma 6.

**Lemma 6** Let $D_\tau$ be TPSD T-product tensors with $\int_{\Omega} \|D^{-p}_\tau\|_\rho d\nu(\tau) < \infty$ for any $p > 0$, then we have

$$
\lim_{p \to 0} \left( \frac{1}{p} \log \int_{\Omega} \frac{\|D^{-p}_\tau\|_1}{m \times p} d\nu(\tau) \right) = - \frac{1}{m \times p} \int_{\Omega} \log \det(D_\tau) d\nu(\tau).
$$

**Proof:** Because $\int_{\Omega} \|D^{-p}_\tau\|_\rho d\nu(\tau) < \infty$, we have that $D_\tau$ are TPD tensors for $\tau$ almost everywhere in $\Omega$. Then, we have

$$
\lim_{p \to 0} \left( \frac{1}{p} \log \int_{\Omega} \frac{\|D^{-p}_\tau\|_1}{m \times p} d\nu(\tau) \right) = \lim_{p \to 0} \frac{\int_{\Omega} \frac{\log \lambda_i(D_\tau)}{m \times p} d\nu(\tau)}{\int_{\Omega} \frac{\|D^{-p}_\tau\|_1}{m \times p} d\nu(\tau)} = \frac{-1}{m \times p} \int_{\Omega} \log \lambda_i(D_\tau) d\nu(\tau) = 2 \frac{-1}{m \times p} \int_{\Omega} \log \det(D_\tau) d\nu(\tau),
$$

where $= 1$ is from L’Hopital’s rule, and $= 2$ is obtained from det definition. 

□

20
Lemma 7 Let \( m \in \mathbb{N} \) and \( (\mathcal{L}_k)_{k=1}^m \) be a finite sequence of bounded T-product tensors with dimensions \( \mathcal{L}_k \in \mathbb{R}^{m \times m \times p} \), then we have
\[
\lim_{n \to \infty} \left( \prod_{k=1}^m \exp\left( \frac{\mathcal{L}_k}{n} \right) \right)^n = \exp\left( \sum_{k=1}^m \mathcal{L}_k \right) \tag{120}
\]

Proof:
We will prove the case for \( m = 2 \), and the general value of \( m \) can be obtained by mathematical induction. Let \( \mathcal{L}_1, \mathcal{L}_2 \) be bounded tensors act on some Hilbert space. Define \( \mathcal{C} = \exp((\mathcal{L}_1 + \mathcal{L}_2)/n) \), and \( \mathcal{D} = \exp(\mathcal{L}_1/n) \ast \exp(\mathcal{L}_2/n) \). Note we have following estimates for the norm of tensors \( \mathcal{C}, \mathcal{D} \):
\[
\|\mathcal{C}\|, \|\mathcal{D}\| \leq \exp\left( \frac{\|\mathcal{L}_1\| + \|\mathcal{L}_2\|}{n} \right) = \left[ \exp\left( \|\mathcal{L}_1\| + \|\mathcal{L}_2\| \right) \right]^{1/n}. \tag{121}
\]
From the Cauchy-Product formula, the tensor \( \mathcal{D} \) can be expressed as:
\[
\mathcal{D} = \exp(\mathcal{L}_1/n) \ast \exp(\mathcal{L}_2/n) = \sum_{i=0}^{\infty} \frac{(\mathcal{L}_1/n)^i}{i!} \ast \sum_{j=0}^{\infty} \frac{(\mathcal{L}_2/n)^j}{j!} = \sum_{l=0}^{\infty} n^{-l} \sum_{i=0}^{l} \frac{\mathcal{L}_1^i}{i!} \ast \frac{\mathcal{L}_2^{l-i}}{(l-i)!}. \tag{122}
\]
then we can bound the norm of \( \mathcal{C} - \mathcal{D} \) as
\[
\|\mathcal{C} - \mathcal{D}\| = \left\| \sum_{i=0}^{\infty} \frac{([\mathcal{L}_1 + \mathcal{L}_2]/n)^i}{i!} - \sum_{i=0}^{\infty} n^{-l} \sum_{i=0}^{l} \frac{\mathcal{L}_1^i}{i!} \ast \frac{\mathcal{L}_2^{l-i}}{(l-i)!} \right\|
\leq \frac{1}{k^2} \left[ \exp(\|\mathcal{L}_1\| + \|\mathcal{L}_2\|) + \sum_{l=2}^{\infty} n^{-l} \sum_{i=0}^{l} \frac{\|\mathcal{L}_1\|^i}{i!} \cdot \frac{\|\mathcal{L}_2\|^{l-i}}{(l-i)!} \right]
\leq \frac{1}{k^2} \left[ \exp(\|\mathcal{L}_1\| + \|\mathcal{L}_2\|) + \sum_{l=2}^{\infty} n^{-l} (\|\mathcal{L}_1\| + \|\mathcal{L}_2\|)^l \right]
\leq \frac{2\exp(\|\mathcal{L}_1\| + \|\mathcal{L}_2\|)}{n^2}. \tag{123}
\]
For the difference between the higher power of \( \mathcal{C} \) and \( \mathcal{D} \), we can bound them as
\[
\|\mathcal{C}^n - \mathcal{D}^n\| = \left\| \sum_{l=0}^{n-1} \mathcal{C}^l (\mathcal{C} - \mathcal{D}) \mathcal{C}^{n-l-1} \right\|
\leq 1 \exp(\|\mathcal{L}_1\| + \|\mathcal{L}_2\|) \cdot n \cdot \|\mathcal{L}_1 - \mathcal{L}_2\|. \tag{124}
\]
where the inequality \( \leq 1 \) uses the following fact

\[
\|C\|^l \|D\|^{n-l-1} \leq \exp (\|L_1\| + \|L_2\|) \leq \exp (\|L_1\| + \|L_2\|),
\]

based on Eq. (121). By combining with Eq. (123), we have the following bound

\[
\|C^n - D^n\| \leq \frac{2\exp (2\|L_1\| + 2\|L_2\|)}{n}.
\]

Then this lemma is proved when \( n \) goes to infinity. \( \square \)

Below, new multivariate norm inequalities for T-product tensors are provided according to previous majorization theorems.

**Theorem 9** Let \( C_i \in \mathbb{R}^{m \times m \times p} \) be TPD tensors, where \( 1 \leq i \leq n \), \( \| \cdot \| \) be a unitarily invariant norm with corresponding gauge function \( \rho \). For any continuous function \( f : (0, \infty) \to [0, \infty) \) such that \( x \to \log f(e^x) \) is convex on \( \mathbb{R} \), we have

\[
\left\| f \left( \exp \left( \sum_{i=1}^{n} \log C_i \right) \right) \right\|_{\rho} \leq \exp \int_{-\infty}^{\infty} \log \left\| f \left( \prod_{i=1}^{n} C_i^{1+it} \right) \right\|_{\rho} \beta_0(t) \, dt,
\]

where \( \beta_0(t) = \frac{n}{2(\cosh(\pi t) + 1)} \).

For any continuous function \( g(0, \infty) \to [0, \infty) \) such that \( x \to g(e^x) \) is convex on \( \mathbb{R} \), we have

\[
\left\| g \left( \exp \left( \sum_{i=1}^{n} \log C_i \right) \right) \right\|_{\rho} \leq \int_{-\infty}^{\infty} \left\| g \left( \prod_{i=1}^{n} C_i^{1+it} \right) \right\|_{\rho} \beta_0(t) \, dt.
\]

**Proof:** From Hirschman interpolation theorem [25] and \( \theta \in [0, 1] \), we have

\[
\log |h(\theta)| \leq \int_{-\infty}^{\infty} \log |h(t)|^{1-\theta} \beta_{1-\theta}(t) \, dt + \int_{-\infty}^{\infty} \log |h(1+it)|^{\theta} \beta_\theta(t) \, dt,
\]

where \( h(z) \) be uniformly bounded on \( S \equiv \{ z \in \mathbb{C} : 0 \leq \Re(z) \leq 1 \} \) and holomorphic on \( S \). The term \( d\beta_\theta(t) \) is defined as:

\[
\beta_\theta(t) \overset{\text{def}}{=} \frac{\sin(\pi \theta)}{2\theta(\cos(\pi t) + \cos(\pi \theta))}.
\]

Let \( H(z) \) be a uniformly bounded holomorphic function with values in \( \mathbb{C}^{m \times m \times p} \). Fix some \( \theta \in [0, 1] \) and let \( U, V \in \mathbb{C}^{m \times m \times p} \) be normalized tensors such that \( \langle U, H(\theta) \star V \rangle = \|H(\theta)\| \). If we define \( h(z) \) as \( h(z) \overset{\text{def}}{=} \langle U, H(z) \star V \rangle \), we have following bound: \( |h(z)| \leq \|H(z)\| \) for all \( z \in S \). From Hirschman interpolation theorem, we then have following interpolation theorem for tensor-valued function:

\[
\log \|H(\theta)\| \leq \int_{-\infty}^{\infty} \log \|H(it)\|^{1-\theta} \beta_{1-\theta}(t) \, dt + \int_{-\infty}^{\infty} \log \|H(1+it)\|^{\theta} \beta_\theta(t) \, dt.
\]

Let \( H(z) = \prod_{i=1}^{n} C_i^z \). Then the first term in the R.H.S. of Eq. (131) is zero since \( H(it) \) is a product of unitary tensors. Then we have

\[
\log \left\| \prod_{i=1}^{n} C_i^\theta \right\|^{\frac{1}{n}} \leq \int_{-\infty}^{\infty} \log \left\| \prod_{i=1}^{n} C_i^{1+it} \right\| \beta_\theta(t) \, dt.
\]
From Lemma 3 we have following relations:

$$\left| \prod_{i=1}^{n} \left( \wedge^k C_i \right)^{\theta} \right|^{\frac{1}{\theta}} = \wedge^k \left| \prod_{i=1}^{n} C_i^{\theta} \right|^{\frac{1}{\theta}},$$

and

$$\left| \prod_{i=1}^{n} \left( \wedge^k C_i \right)^{1+\beta t} \right| = \wedge^k \left| \prod_{i=1}^{n} C_i^{1+\beta t} \right|.$$  \hfill (134)

If Eq. (132) is applied to $\wedge^k C_i$ for $1 \leq k \leq r$, we have following log-majorization relation from Eqs. (133) and (134):

$$\log \left| \prod_{i=1}^{n} C_i^{\theta} \right|^{\frac{1}{\theta}} < \int_{-\infty}^{\infty} \log \left| \prod_{i=1}^{n} C_i^{1+\beta t} \right|^{\frac{1}{\theta}} \beta(t) dt.$$  \hfill (135)

Moreover, we have the equality condition in Eq. (135) for $k = r$ due to following identities:

$$\det \left| \prod_{i=1}^{n} C_i^{\theta} \right|^{\frac{1}{\theta}} = \det \left| \prod_{i=1}^{n} C_i^{1+\beta t} \right| = \prod_{i=1}^{n} \det C_i.$$  \hfill (136)

At this stage, we are ready to apply Theorem 8 for the log-majorization provided by Eq. (135) to get following facts:

$$\left\| f \left( \prod_{i=1}^{n} C_i^{\theta} \right)^{\frac{1}{\theta}} \right\|_{\rho} \leq \exp \int_{-\infty}^{\infty} \left\| f \left( \prod_{i=1}^{n} C_i^{1+\beta t} \right) \right\|_{\rho} \beta(t) dt,$$  \hfill (137)

and

$$\left\| g \left( \prod_{i=1}^{n} C_i^{\theta} \right)^{\frac{1}{\theta}} \right\|_{\rho} \leq \int_{-\infty}^{\infty} \left\| g \left( \prod_{i=1}^{n} C_i^{1+\beta t} \right) \right\|_{\rho} \beta(t) dt.$$  \hfill (138)

From Lie product formula for tensors given by Lemma 7 we have

$$\left| \prod_{i=1}^{n} C_i^{\theta} \right|^{\frac{1}{\theta}} \to \exp \left( \sum_{i=1}^{n} \log C_i \right).$$  \hfill (139)

By setting $\theta \to 0$ in Eqs. (137), (138) and using Lie product formula given by Eq. (139), we will get Eqs. (127) and (128).

\section{Applications of T-product Tensor Norm Inequalities}

The purpose of this section is to apply new derived T-product tensor norm inequalities to obtain random symmetric T-product tensor Bernstein bounds. In Section 4.1, Ky Fan $k$-norm inequalities for T-product tensors will be provided and such Ky Fan $k$-norm inequalities will be utilized to establish T-product tensor Bernstein bounds in Section 4.2 and Section 4.3.
4.1 Ky Fan $k$-norm Tail Bounds

We will present several lemmas required to prove Ky Fan $k$-norm tail bounds.

**Lemma 8** Given a symmetric T-product tensor $C \in \mathbb{R}^{m \times m \times p}$ which can be expressed as the format shown by Eq. (16), the T-eigenvalues of $C$ with respect to the matrix $C_i$ are denoted as $\lambda_{i,k_i}$, where $1 \leq k_i \leq m$, and we assume that $\lambda_{i,1} \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,m}$ (including multiplicities). We have following relation about T-eigenvalues summation representation:

\[
\sum_{i=1}^{p} \max_{U_iU_i^H=I_{k_i}} \text{Tr} U_iC_iU_i^H = \sum_{i=1}^{p} \sum_{j=1}^{k_i} \lambda_{i,j}(C_i), \quad (140)
\]

and

\[
\sum_{i=1}^{p} \min_{U_iU_i^H=I_{k_i}} \text{Tr} U_iC_iU_i^H = \sum_{i=1}^{p} \sum_{j=1}^{k_i} \lambda_{i,m-j+1}(C_i). \quad (141)
\]

**Proof:** From Theorem 1, each matrix $C_i$ associated to $C$ based on the format shown by Eq. (16) is Hermitian, then the matrix $C_i$ can be diagonalized as $D_i$ by the unitary matrix $V_i$. Without loss of generality, we may assume that $C_i$ are diagonal matrices. Then, we have

\[
\text{Tr} U_iD_iU_i^H = \sum_{j=1}^{k_i} \sum_{l=1}^{m} u_{i,j,l}^* u_{i,j,l} \lambda_{i,l} = \sum_{j=1}^{k_i} \sum_{l=1}^{m} p_{i,j,l} \lambda_{i,l} = \left[ 1, \cdots, 1 \right] P_i \left[ \lambda_{i,1}, \cdots, \lambda_{i,m} \right], \quad (142)
\]

where the superscript $*$ is the operation of a complex conjugate, and $P_i = (p_{i,j,l})$ is a $k_i \times m$ stochastic matrix. From the fact provided by 2.C.1 in [20], there exists an $(m-k_i) \times m$ matrix $Q_i$ such that $\left[ P_i \atop Q_i \right]$ is a doubly stochastic matrix. Then, Eq. (142) can be expressed as

\[
\text{Tr} U_iD_iU_i^H = \left[ 1, \cdots, 1, 0, \cdots, 0 \right] \left[ \begin{array}{c} \lambda_{i,1} \\ \vdots \\ \lambda_{i,m} \end{array} \right] = \left[ 1, \cdots, 1 \right] P_i \left[ \lambda_{i,1}, \cdots, \lambda_{i,m} \right], \quad (143)
\]

Because we have

\[
[\lambda_{i,1}, \cdots, \lambda_{i,m}] \left[ P_i^T Q_i^T \right] \preceq [\lambda_{i,1}, \cdots, \lambda_{i,m}], \quad (144)
\]

then, we can apply the fact 3.H.2 about majorization in [20] to get

\[
\text{Tr} U_iD_iU_i^H \leq \sum_{j=1}^{k_i} \lambda_{i,j}(C_i), \quad (145)
\]

and

\[
\text{Tr} U_iD_iU_i^H \geq \sum_{j=1}^{k_i} \lambda_{i,m-j+1}(C_i), \quad (146)
\]

24
Given two symmetric $T$-product tensors $C, D \in \mathbb{R}^{m \times m \times p}$. We have the following majorization relation about $T$-singular values:

\[
\sigma(C + D) \prec_w \sigma(C) + \sigma(D).
\]

**Proof:** Since we have

\[
\sum_{i=1}^{p} \sum_{j=1}^{k_i} \sigma_{i,j}(C + D) = \max_{U_i, U_i^H = 1} \Re \left( \sum_{i=1}^{p} \text{Tr} U_i (C_i + D_i) U_i^H \right)
\]

\[
\leq \max_{U_i, U_i^H = 1} \Re \left( \sum_{i=1}^{p} \text{Tr} U_i C_i U_i^H \right) + \max_{U_i, U_i^H = 1} \Re \left( \sum_{i=1}^{p} \text{Tr} U_i D_i U_i^H \right)
\]

\[
= \sum_{i=1}^{p} \sum_{j=1}^{k_i} \sigma_{i,j}(C) + \sum_{i=1}^{p} \sum_{j=1}^{k_i} \sigma_{i,j}(D)
\]

where $\Re$ is the operation to take the real part, and the equalities $= _1$ and $= _2$ come from Lemma 8. □

We are ready to introduce the following two lemmas about Ky Fan $k$-norm inequalities for the product of tensors (Lemma 10) and the summation of tensors (Lemma 11).

**Lemma 10** Let $C_i \in \mathbb{R}^{m \times m \times p}$ be symmetric $T$-product tensors and let $p_i$ be positive real numbers satisfying $\sum_{i=1}^{m} \frac{1}{p_i} = 1$. Then, we have

\[
\left\| \prod_{i=1}^{m} C_i \right\|_{s, (k)} \leq \prod_{i=1}^{m} \left( \| C_i \|_{s, p_i} \right)^{1/p_i} \leq \sum_{i=1}^{m} \left\| C_i \right\|_{s, p_i, (k)}
\]

where $s \geq 1$ and $k \in \{1, 2, \cdots, m \times p\}$.

**Proof:** Since we have

\[
\left\| \prod_{i=1}^{m} C_i \right\|_{s, (k)} = \sum_{j=1}^{k} \lambda_j \left( \prod_{i=1}^{m} C_i \right)^{s} = \sum_{j=1}^{k} \lambda_j^s \left( \prod_{i=1}^{m} C_i \right) = \prod_{j=1}^{k} \sigma_j^s \left( \prod_{i=1}^{m} C_i \right),
\]

where we have orders for eigenvalues as $\lambda_1 \geq \lambda_2 \geq \cdots$, and singular values as $\sigma_1 \geq \sigma_2 \geq \cdots$.

From Lemma 3 we have

\[
\left\| \prod_{i=1}^{m} C_i \right\|_{\wedge k} = \prod_{j=1}^{k} \sigma_j \left( \prod_{i=1}^{m} C_i \right).
\]

Apply Theorem H.1. in [20] to each matrix at block diagonal of $\text{bcirc}(C_i)$ by Eq. (16), we will have

\[
\sum_{j=1}^{k} \sigma_j^s \left( \prod_{i=1}^{m} C_i \right) \leq \sum_{j=1}^{k} \left( \prod_{i=1}^{m} \sigma_j^s(C_i) \right).
\]
Then, we can apply Hölder’s inequality to Eq. (152) and obtain

$$\sum_{j=1}^{k} \left( \prod_{i=1}^{m} \sigma_{j}^{s}(C_{i}) \right) \leq \prod_{i=1}^{m} \left( \sum_{j=1}^{k} \sigma_{j}^{sp_{i}}(C_{i}) \right)^{\frac{1}{p_{i}}} = \prod_{i=1}^{m} \left( \sum_{j=1}^{k} \lambda_{j}^{s}(|C_{i}|^{sp_{i}}) \right)^{\frac{1}{p_{i}}} = \prod_{i=1}^{m} \left( \|\|C_{i}|^{sp_{i}}\|_{(k)} \right)^{\frac{1}{p_{i}}}$$ (153)

The second inequality in Eq. (149) is obtained by applying Young’s inequality to numbers \(\|C_{i}|^{sp_{i}}\|_{(k)}\) for \(1 \leq i \leq m\). This completes the proof.

**Lemma 11** Let \(C_{i} \in \mathbb{R}^{m \times m \times p}\) be symmetric T-product tensors, then we have

$$\left\| \sum_{i=1}^{m} C_{i}^{s} \right\|_{(k)} \leq m^{s-1} \sum_{i=1}^{m} \|\|C_{i}|^{s}\|_{(k)}$$ (154)

where \(s \geq 1\) and \(k \in \{1, 2, \cdots, m \times p\}\).

**Proof:** Since we have

$$\left\| \sum_{i=1}^{m} C_{i}^{s} \right\|_{(k)}^{k} = \sum_{j=1}^{k} \lambda_{j}^{s} \left( \sum_{i=1}^{m} C_{i}^{s} \right) = \sum_{j=1}^{k} \sum_{i=1}^{m} \sigma_{j}^{s}(C_{i}) \sum_{i=1}^{m} C_{i} \right) = \sum_{j=1}^{k} \sigma_{j}^{s} \left( \sum_{i=1}^{m} C_{i} \right)$$ (155)

where we have orders for eigenvalues as \(\lambda_{1} \geq \lambda_{2} \geq \cdots\), and singular values as \(\sigma_{1} \geq \sigma_{2} \geq \cdots\).

From Lemma 9 we have

$$\sum_{j=1}^{k} \sigma_{j} \left( \sum_{i=1}^{m} C_{i} \right) \leq \sum_{j=1}^{k} \left( \sum_{i=1}^{m} \sigma_{j}(C_{i}) \right)$$ (156)

where \(k \in \{1, 2, \cdots, m \times p\}\). Then, we have

$$\sum_{j=1}^{k} \sigma_{j}^{s} \left( \sum_{i=1}^{m} C_{i} \right) \leq \sum_{j=1}^{k} \left( \sum_{i=1}^{m} \sigma_{j}(C_{i}) \right)^{s} \leq m^{s-1} \sum_{j=1}^{k} \left( \sum_{i=1}^{m} \sigma_{j}(C_{i}) \right) = m^{s-1} \sum_{j=1}^{k} \left( \sum_{i=1}^{m} \sigma_{j}(C_{i}) \right)^{s} = m^{s-1} \sum_{i=1}^{m} \|\|C_{i}|^{s}\|_{(k)}$$ (157)

Now, we are ready to present our main theorem about Ky Fan k-norm probability bound for a function of tensors summation.

**Theorem 10** Consider a sequence \(\{X_{j} \in \mathbb{R}^{m \times m \times p}\}\) of independent, random, symmetric T-product tensors. Let \(g\) be a polynomial function with degree \(n\) and nonnegative coefficients \(a_{0}, a_{1}, \cdots, a_{n}\) raised by power \(s \geq 1\), i.e., \(g(x) = (a_{0} + a_{1}x + \cdots + a_{n}x^{n})^{s}\). Suppose following condition is satisfied:

$$g \left( \exp \left( t \sum_{j=1}^{m} X_{j} \right) \right) \geq \exp \left( t \sum_{j=1}^{m} (X_{j}) \right) \text{ almost surely},$$ (158)

26
where \( t > 0 \). Then, we have

\[
\Pr \left( \left\| g \left( \sum_{j=1}^{m} X_j \right) \right\|_{(k)} \geq \theta \right) \leq (n + 1)^{s-1} \inf_{t,p_j} \exp (-\theta t) \cdot \left( ka_0^s + \sum_{l=1}^{m} \sum_{j=1}^{m} a_l^s \mathbb{E} \| \exp \left( p_j l s X_j \right) \|_{(k)} \right), \quad (159)
\]

where \( \sum_{j=1}^{m} \frac{1}{p_j} = 1 \) and \( p_j > 0 \).

**Proof:** Let \( t > 0 \) be a parameter to be chosen later. Then

\[
\Pr \left( \left\| g \left( \sum_{j=1}^{m} X_j \right) \right\|_{(k)} \geq \theta \right) = \Pr \left( \left\| \exp \left( t g \left( \sum_{j=1}^{m} X_j \right) \right) \right\|_{(k)} \geq \exp (\theta t) \right)
\]

\[
\leq_1 \exp (-\theta t) \mathbb{E} \left( \left\| \exp \left( t g \left( \sum_{j=1}^{m} X_j \right) \right) \right\|_{(k)} \right)
\]

\[
\leq_2 \exp (-\theta t) \mathbb{E} \left( \left\| g \left( \exp \left( t \sum_{j=1}^{m} X_j \right) \right) \right\|_{(k)} \right) \quad (160)
\]

where \( \leq_1 \) uses Markov’s inequality, \( \leq_2 \) requires conditions provided by Eq. (158).

We can further bound the expectation term in Eq. (159) as

\[
\mathbb{E} \left( \left\| g \left( \exp \left( t \sum_{j=1}^{m} X_j \right) \right) \right\|_{(k)} \right) \leq_3 \mathbb{E} \int_{-\infty}^{\infty} \left\| g \left( \prod_{j=1}^{m} e^{(1+\tau) t X_j} \right) \right\|_{(k)} \beta_0(\tau) d\tau
\]

\[
\leq_4 (n + 1)^{s-1} \left( ka_0^s + \sum_{l=1}^{m} \sum_{j=1}^{m} a_l^s \mathbb{E} \left\| e^{(1+\tau) t X_j} \right\|_{(k)} d\tau \right), \quad (161)
\]

where \( \leq_3 \) from Eq. (128) in Theorem 9 \( \leq_4 \) is obtained from function \( g \) definition and Lemma 11. Again, the expectation term in Eq. (161) can be further bounded by Lemma 10 as

\[
\mathbb{E} \int_{-\infty}^{\infty} \left\| \prod_{j=1}^{m} e^{(1+\tau) t X_j} \right\|_{(k)}^{ls} \beta_0(\tau) d\tau \leq \mathbb{E} \int_{-\infty}^{\infty} \sum_{j=1}^{m} \left\| e^{t X_j} \right\|_{(k)}^{ls} \frac{\beta_0(\tau) d\tau}{p_j} \leq m \sum_{j=1}^{m} \mathbb{E} \left\| e^{p_j l s X_j} \right\|_{(k)} \frac{\beta_0(\tau) d\tau}{p_j} \quad (162)
\]

Note that the final equality is obtained due to that the integrand is independent of the variable \( \tau \) and \( \int_{-\infty}^{\infty} \beta_0(\tau) d\tau = 1 \).
Finally, this theorem is established from Eqs. (160), (161), and (162).

Remarks: The condition provided by Eq. (158) can be achieved by normalizing tensors $X_j$ through scaling.

### 4.2 T-product Tensor Bernstein Bound

In this section, we will present a tensor Bernstein bound for the maximum and the minimum T-eigenvalue for summation of random symmetric T-product tensors. We will provide the following definition to define a random structure for the T-product tensor $X \in \mathbb{R}^{m \times n \times p}$.

**Definition 1** Random structure for random symmetric T-product tensor $X \in \mathbb{R}^{m \times n \times p}$

1. There are $p$ Hermitian matrices with size $m \times m$, denoted as $X_1, X_2, \ldots, X_p$, obtained from Eq. (16). The entries for the matrix $X_i$ are denoted by $(x_{i,j,k})$, where $x_{i,j,k}$ is a complex number.
2. For each $X_i$, the random variables $x_{i,j,k}$, $\Re x_{i,j,k}$ for $j < k$, and $\Im x_{i,j,k}$ for $j < k$, are independent.
3. For each $X_i$, the random variables $x_{i,j,k}$ follow Gaussian distribution with zero mean and variance as $\frac{1}{m}$.
4. For each $X_i$, the random variables $\Re x_{i,j,k}$ for $j < k$, and $\Im x_{i,j,k}$ for $j < k$, follow Gaussian distribution with zero mean and variance as $\frac{1}{2m}$.

Following lemma is about the expectation of the largest T-eigenvalue of symmetric T-product tensor $\exp(\gamma X)$, where $\gamma$ is a real number.

**Lemma 12** Given a random symmetric T-product tensor $X \in \mathbb{R}^{m \times n \times p}$ satisfying Definition 1 and any real number $\gamma$, we have

$$\mathbb{E} \lambda_1 (\exp(\gamma X)) \leq \frac{3mc_1 c_2}{2} \int_{-\infty}^{\infty} \exp \left( \frac{1}{2} \gamma y + c_2 m y - 2 \right) dy \equiv \Psi(m, \gamma, c_1, c_2) \quad (163)$$

where $\lambda_1$ is the largest T-eigenvalue, and $c_1, c_2$ are constants related to the bound of cumulative distribution function of the largest eigenvalue of the random Hermitian matrix $X$.

**Proof:** From random structure of discussed random symmetric T-product tensor $X \in \mathbb{R}^{m \times n \times p}$, all random Hermitian matrix $X_i$ have same probability density distributions. The maximum T-eigenvalue of $X$ will be equal to the maximum eigenvalue of $X_i$, and we use $X$ to represent any random Hermitian matrix $X_i$ since they share same distribution.

From Eq. (2) in [26], given a $m \times m$ random Hermitian matrix $X$, we have

$$\Pr (\lambda_1(X) \leq y) \leq 1 - c_1 \exp(-c_2 m(y - 2)\frac{3}{2}) \quad (164)$$

where $c_1, c_2$ are constants related to the bound of cumulative distribution function of the largest eigenvalue of any random Hermitian matrix $X$. Then, we have

$$\mathbb{E} \lambda_1 (\exp(\gamma X)) = \mathbb{E} \exp(\gamma \lambda_1(X)) = \int_{-\infty}^{\infty} \exp(\gamma y) d(\Pr (\lambda_1(X) \leq y))$$

$$\leq \int_{-\infty}^{\infty} \exp(\gamma y) d \left\{ 1 - c_1 \exp \left[ c_2 m(y - 2)\frac{3}{2} \right] \right\}$$

$$= \frac{3mc_1 c_2}{2} \int_{-\infty}^{\infty} \exp \left( \frac{1}{2} \gamma y + c_2 m y - 2 \right) dy, \quad (165)$$

where $=1$ comes from the spectral mapping theorem.

We are ready to present our theorem about the maximum and the minimum of T-eigenvalue for the summation of random symmetric T-product tensors.
Theorem 11 (T-product Tensor Bernstein Bound for T-eigenvalue) Consider a sequence \( \{X_j \in \mathbb{R}^{m \times m \times p}\} \) of independent, random, symmetric T-product tensors with random structure defined by Definition 1. Then we have following inequalities: given \( \theta_1 > 0 \), we have

\[
\Pr \left( \lambda_{\max} \left( \sum_{j=1}^{m} X_j \right) \geq \theta_1 \right) \leq \inf_{t>0} \left[ \frac{\exp(-\theta_1 t)}{m} \sum_{j=1}^{m} \Psi(m, mt, c_1, c_2) \right],
\]

(166)

and, given \( \theta_2 < 0 \), we have

\[
\Pr \left( \lambda_{\min} \left( \sum_{j=1}^{m} X_j \right) \leq \theta_2 \right) \leq \inf_{t>0} \left[ \frac{\exp(\theta_2 t)}{m} \sum_{j=1}^{m} \Psi(m, -mt, c_1, c_2) \right].
\]

(167)

The \( \Psi \) function is defined by Eq. (163).

Proof: Since we have

\[
\Pr \left( \lambda_{\max} \left( \sum_{j=1}^{m} X_j \right) \geq \theta_1 \right) =_1 \Pr \left( \sigma_{\max} \left( \sum_{j=1}^{m} X_j \right) \geq \theta_1 \right)
\]

\[
\leq_2 \inf_{t, p_j} \exp(-\theta_1 t) \left( \sum_{j=1}^{m} \frac{\mathbb{E} \sigma_{\max} (\exp(p_j t X_j))}{p_j} \right)
\]

\[
\leq_3 \inf_{t, p_j} \exp(-\theta_1 t) \left( \sum_{j=1}^{m} \frac{\Psi(m, p_j t, c_1, c_2)}{p_j} \right)
\]

\[
\leq_4 \inf_{t>0} \left[ \frac{\exp(-\theta_1 t)}{m} \sum_{j=1}^{m} \Psi(m, mt, c_1, c_2) \right],
\]

(168)

where \( =_1 \) comes from that maximum singular value equals to the maximum absolute value of an T-eigenvalue and the maximum and the minimum of T-eigenvalue has same distribution due to the symmetry of random structure given by Definition 1; the inequality \( \leq_2 \) comes from Theorem 10 when \( g \) is the identity function; the equality \( \leq_3 \) comes from Lemma 12 and \( \sigma_{\max} (\exp(p_j t X_j)) = \lambda_{\max} (\exp(p_j t X_j)) \) due to TPD of \( \exp(p_j t X_j) \); the inequality \( \leq_4 \) is obtained by selecting \( p_j = m \). Therefore, we have Eq. (166).
For the minimum T-eigenvalue, we also have

$$\Pr \left( \lambda_{\min} \left( \sum_{j=1}^{m} x_j \right) \leq \theta_2 \right) = \Pr \left( \lambda_{\max} \left( \sum_{j=1}^{m} -x_j \right) \geq -\theta_2 \right)$$

$$= \Pr \left( \sigma_{\max} \left( \sum_{j=1}^{m} -x_j \right) \geq -\theta_2 \right)$$

$$\leq 3 \inf_{t,p} \exp(\theta_2 t) \left( \sum_{j=1}^{m} \frac{\mathbb{E} \sigma_{\max}(\exp(-p_j t x_j))}{p_j} \right)$$

$$\leq 4 \inf_{t,p} \exp(\theta_2 t) \left( \sum_{j=1}^{m} \frac{\Psi(m, -p_j t, c_1, c_2)}{p_j} \right)$$

$$\leq 5 \inf_{t>0} \left[ \frac{\exp(\theta_2 t)}{m} \sum_{j=1}^{m} \Psi(m, -mt, c_1, c_2) \right],$$

(169)

where $=1$ comes from Theorem 3; $=2$ is true since the maximum singular value equals to the maximum absolute value of a T-eigenvalue and the maximum and the minimum of T-eigenvalue has same distribution due to the symmetry of random structure given by Definition 1; the inequality $\leq 3$ comes from Theorem 10 again when $g$ is an identity map; the equality $\leq 4$ comes from Lemma 12 and $\sigma_{\max}(\exp(p_j t x_j)) = \lambda_{\max}(\exp(p_j t x_j))$ due to TPD of $\exp(p_j t x_j)$; the inequality $\leq 5$ is obtained by selecting $p_j = m$. Hence, we have Eq. (167).

4.3 Generalized T-product Tensor Bernstein Bound

In this section, we will present a generalized tensor Bernstein bound for Ky Fan $k$-norm, and we will begin with a lemma to bound exponential of a random T-product tensor.

**Lemma 13** Suppose that $X \in \mathbb{R}^{m \times m \times p}$ is a random symmetric T-product tensor that satisfies

$$X^p \preceq \frac{p! A^2}{2} \quad \text{almost surely for } p = 2, 3, 4, \ldots,$$

(170)

where $A$ is a fixed TPD tensor. Then, we have

$$e^{tx} \leq \mathcal{I} + tx + \frac{t^2 A^2}{2(1-t)} \quad \text{almost surely},$$

(171)

where $0 < t < 1$.

**Proof:** From Taylor series of the tensor exponential expansion, we have

$$e^{tx} = \mathcal{I} + tx + \sum_{p=2}^{\infty} \frac{t^p (X^p)}{p!} \leq \mathcal{I} + tx + \sum_{p=2}^{\infty} \frac{t^p A^2}{2} = \mathcal{I} + tx + \frac{t^2 A^2}{2(1-t)}.$$
Lemma 14  Given a random symmetric T-product tensor $X \in \mathbb{R}^{m \times m \times p}$ satisfying Definition 7 we have

$$\mathbb{E} \sigma_1 (X) \leq \int_{-2}^{\infty} d_1 \exp(-d_2 mz^{3/2}) \, dz \overset{\text{def}}{=} \Phi(m, d_1, d_2). \quad (173)$$

where $\sigma_1$ is the largest T-singular value, and $d_1, d_2$ are constants related to the upper bound of the largest eigenvalue of the random Hermitian matrix $X$.

Proof: From random structure of discussed random symmetric T-product tensor $X \in \mathbb{R}^{m \times m \times p}$, all random Hermitian matrix $X_i$ have same probability density distributions. The maximum T-singular value of $X$ will be equal to the maximum singular value of $X_i$, and we use $X$ to represent any random Hermitian matrix $X_i$ since they share the same distribution.

From Eq. (8) in [26], given a $m \times m$ random Hermitian matrix $X$, we have

$$\Pr (\sigma_1 (X) > y) \leq d_1 \exp(-d_2 m(y - 2)^{3/2}) \quad (174)$$

where $d_1, d_2$ are constants related to the upper bound of the largest (or smallest) eigenvalue of any random Hermitian matrix $X$. Then, we have

$$\mathbb{E} \sigma_1 (X) = \int_0^{\infty} \Pr (\sigma_1 (X) > y) \, dy$$

$$\leq \int_0^{\infty} d_1 \exp \left[ -d_2 m(y - 2)^{3/2} \right] \, dy$$

$$= \int_{-2}^{\infty} d_1 \exp(-d_2 mz^{3/2}) \, dz \quad (175)$$

Following lemma is about Ky Fan $k$-norm bound for the exponential of a random T-product tensor with subexponential constraints.

Lemma 15  Given a symmetric random T-product tensor $X \in \mathbb{R}^{m \times m \times p}$ with random structure defined by Definition 7 and $X^p \preceq \frac{p! A^2}{2}$ almost surely for $p = 2, 3, 4, \cdots$, where $A$ is a TPD T-product tensor. Then, we have following bound about the expectation value of Ky Fan $k$-norm for the random T-product tensor $\exp(\theta X)$

$$\mathbb{E} \left\| \exp(\theta X) \right\|_k \leq k \left[ 1 + \theta \Phi(m, d_1, d_2) + \frac{\theta^2}{2(1 - \theta)} \sigma_1 (A^2) \right]. \quad (177)$$

Proof: From Lemma 13 we have

$$\mathbb{E} \left\| \exp(\theta X) \right\|_k = \sum_{l=1}^{k} \mathbb{E} \sigma_l (\exp(\theta X))$$

$$\leq \sum_{l=1}^{k} \mathbb{E} \sigma_l \left( I + \theta X + \frac{\theta^2 A^2}{2(1 - \theta)} \right) \leq k \mathbb{E} \sigma_1 \left( I + \theta X + \frac{\theta^2 A^2}{2(1 - \theta)} \right) \quad (178)$$

where $\sigma_l(\cdot)$ is the $l$-th largest T-singular value.
From Lemma\textsuperscript{[9]} we have $\sigma_1(A + B) \leq \sigma_1(A) + \sigma_1(B)$ for two symmetric T-product tensors $A$ and $B$. Then, we can bound $\mathbb{E}\sigma_1 \left( I + \theta X + \frac{\theta^2 A^2}{2(1 - \theta)} \right)$ as

$$
\mathbb{E}\sigma_1 \left( I + \theta X + \frac{\theta^2 A^2}{2(1 - \theta)} \right) \leq 1 + \theta \mathbb{E}\sigma_1(X) + \frac{\theta^2}{2(1 - \theta)} \sigma_1(A^2)
$$

$$
\leq 1 + \theta \mathbb{E}\Phi(m, d_1, d_2) + \frac{\theta^2}{2(1 - \theta)} \sigma_1(A^2)
$$

(179)

where we use $\Phi(m, d_1, d_2)$ from Lemma\textsuperscript{[14]} to bound $\mathbb{E}\sigma_1(X)$ in the last inequality. This Lemma is proved by multiplying $k$ at Eq. (179).

We are ready to present our main theorem about the generalized T-product tensor Bernstein bound.

**Theorem 1.1 (Generalized T-product Tensor Bernstein Bound)** Consider a sequence $\{X_j \in \mathbb{R}^{m \times m \times p}\}$ of independent, random symmetric T-product tensors with random structure defined by Definition 7. Let $g$ be a polynomial function with degree $n$ and nonnegative coefficients $a_0, a_1, \ldots, a_n$ raised by power $s \geq 1$, i.e., $g(x) = (a_0 + a_1 x + \cdots + a_n x^n)^s$ with $s \geq 1$. Suppose following condition is satisfied:

$$
g \left( \exp \left( t \sum_{j=1}^{m} X_j \right) \right) \geq \exp \left( t g \left( \sum_{j=1}^{m} X_j \right) \right) \text{ almost surely},
$$

(1)

where $t > 0$, and we also have

$$
X_j^p \leq \frac{p! A^2}{2} \text{ almost surely for } p = 2, 3, 4, \ldots.
$$

(2)

Then we have following inequality:

$$
\Pr \left( \left\| g \left( \sum_{j=1}^{m} X_j \right) \right\|_{(k)} \geq \theta \right) \leq (n + 1)^{s-1} \inf_{t > 0} e^{-\theta t} \cdot \left\{ a_0^s + \sum_{l=1}^{n} a_l^s \left[ 1 + mls^t \Phi(m, d_1, d_2) + \left( \frac{mls^t}{2(1 - mls)} \right)^2 \sigma_1(A^2) \right] \right\}.
$$

(3)

**Proof:** Since we have

$$
\Pr \left( \left\| g \left( \sum_{j=1}^{m} X_j \right) \right\|_{(k)} \geq \theta \right) \leq_1 (n + 1)^{s-1} \inf_{t, p_j} e^{-\theta t} \left\{ k a_0^s + \sum_{l=1}^{n} \sum_{j=1}^{m} a_l^s \mathbb{E} \left\| \exp \left( p_j l X_j \right) \right\|_{(k)} \right\}
$$

$$
\leq_2 (n + 1)^{s-1} \inf_{t, p_j} e^{-\theta t} \left\{ k a_0^s + \sum_{l=1}^{n} \sum_{j=1}^{m} a_l^s \left[ 1 + p_j l X_j \Phi(m, d_1, d_2) + \left( \frac{p_j l X_j}{2(1 - p_j l X_j)} \right)^2 \sigma_1(A^2) \right] \right\}
$$

$$
\leq_3 (n + 1)^{s-1} \inf_{t > 0} e^{-\theta t} \left\{ a_0^s + \sum_{l=1}^{n} a_l^s \left[ 1 + mls^t \Phi(m, d_1, d_2) + \left( \frac{mls^t}{2(1 - mls)} \right)^2 \sigma_1(A^2) \right] \right\}
$$

(180)

where the inequality $\leq_1$ comes from Theorem\textsuperscript{[10]} the inequality $\leq_2$ comes from Lemma\textsuperscript{[15]} the inequality $\leq_3$ is obtained by setting $p_j = m$. \hfill \Box

32
5 Conclusions

This work extend previous work in [18] by making following generalizations via majorization techniques: (1) besides bounds related to extreme values of T-eigenvalues, this works considers more general unitarily invariant norm for T-product tensors; (2) this work derives new bounds for any polynomial function raised by any power greater or equal than one for the summation of random symmetric T-product tensors. We also establish the Courant-Fischer min-max theorem for T-product tensors and majorization relation for T-singular values which are by-products of our procedure to prove the generalized random T-product Bernstein bounds. Possible future work about this research is to consider tail bounds behaviors for the summation of random symmetric T-product tensors equipped with other random structures different from random structure provided by Definition [1].

References

[1] M. E. Kilmer, C. D. Martin, and L. Perrone, “A third-order generalization of the matrix svd as a product of third-order tensors,” Tufts University, Department of Computer Science, Tech. Rep. TR-2008-4, 2008.

[2] M. E. Kilmer, K. Braman, N. Hao, and R. C. Hoover, “Third-order tensors as operators on matrices: A theoretical and computational framework with applications in imaging,” SIAM Journal on Matrix Analysis and Applications, vol. 34, no. 1, pp. 148–172, 2013.

[3] M. E. Kilmer and C. D. Martin, “Factorization strategies for third-order tensors,” Linear Algebra and its Applications, vol. 435, no. 3, pp. 641–658, 2011.

[4] K. Braman, “Third-order tensors as linear operators on a space of matrices,” Linear Algebra and its Applications, vol. 433, no. 7, pp. 1241–1253, 2010.

[5] Y. Miao, L. Qi, and Y. Wei, “T-jordan canonical form and t-drazin inverse based on the t-product,” Communications on Applied Mathematics and Computation, vol. 3, no. 2, pp. 201–220, 2021.

[6] K. Lund, “The tensor t-function: A definition for functions of third-order tensors,” Numerical Linear Algebra with Applications, vol. 27, no. 3, p. e2288, 2020.

[7] Y. Miao, L. Qi, and Y. Wei, “Generalized tensor function via the tensor singular value decomposition based on the t-product,” Linear Algebra and its Applications, vol. 590, pp. 258–303, 2020.

[8] Z. Zhang and S. Aeron, “Exact tensor completion using t-svd,” IEEE Transactions on Signal Processing, vol. 65, no. 6, pp. 1511–1526, 2016.

[9] O. Semerci, N. Hao, M. E. Kilmer, and E. L. Miller, “Tensor-based formulation and nuclear norm regularization for multienergy computed tomography,” IEEE Transactions on Image Processing, vol. 23, no. 4, pp. 1678–1693, 2014.

[10] B. Settles, M. Craven, and S. Ray, “Multiple-instance active learning,” Advances in neural information processing systems, vol. 20, pp. 1289–1296, 2007.

[11] Z. Zhang, G. Ely, S. Aeron, N. Hao, and M. Kilmer, “Novel methods for multilinear data completion and de-noising based on tensor-svd,” in Proceedings of the IEEE conference on computer vision and pattern recognition, 2014, pp. 3842–3849.
[12] C. D. Martin, R. Shafer, and B. LaRue, “An order-p tensor factorization with applications in imaging,” *SIAM Journal on Scientific Computing*, vol. 35, no. 1, pp. A474–A490, 2013.

[13] N. Khalil, A. Sarhan, and M. A. Alshewimy, “An efficient color/grayscale image encryption scheme based on hybrid chaotic maps,” *Optics & Laser Technology*, vol. 143, p. 107326, 2021.

[14] Y. Xu, R. Hao, W. Yin, and Z. Su, “Parallel matrix factorization for low-rank tensor completion,” *Inverse Problems and Imaging*, vol. 9, no. 2, pp. 601–624, Dec. 2013.

[15] P. Zhou, C. Lu, Z. Lin, and C. Zhang, “Tensor factorization for low-rank tensor completion,” *IEEE Transactions on Image Processing*, vol. 27, no. 3, pp. 1152–1163, 2017.

[16] L. Qi and G. Yu, “T-singular values and t-sketching for third order tensors,” 2021.

[17] S. Y. Chang, “T product tensors part i: Inequalities,” *arXiv preprint arXiv:2107.06285*, 2021.

[18] ———, “T product tensors part ii: Tail bounds for sums of random t product tensors,” *arXiv preprint arXiv:2107.06224*, 2021.

[19] M.-M. Zheng, Z.-H. Huang, and Y. Wang, “T-positive semidefiniteness of third-order symmetric tensors and t-semidefinite programming,” *Computational Optimization and Applications*, vol. 78, no. 1, pp. 239–272, 2021.

[20] A. W. Marshall, I. Olkin, and B. C. Arnold, *Inequalities: theory of majorization and its applications*. Springer, 2011, vol. 143.

[21] K. Fan and A. J. Hoffman, “Some metric inequalities in the space of matrices,” *Proceedings of the American Mathematical Society*, vol. 6, no. 1, pp. 111–116, 1955.

[22] R. Bhatia, *Matrix analysis*. Springer Science & Business Media, 2013, vol. 169.

[23] F. Hiai, R. König, and M. Tomamichel, “Generalized log-majorization and multivariate trace inequalities,” in *Annales Henri Poincaré*, vol. 18, no. 7. Springer, 2017, pp. 2499–2521.

[24] F. Hiai, “Matrix analysis: matrix monotone functions, matrix means, and majorization,” *Interdisciplinary Information Sciences*, vol. 16, no. 2, pp. 139–248, 2010.

[25] D. Sutter, M. Berta, and M. Tomamichel, “Multivariate trace inequalities,” *Communications in Mathematical Physics*, vol. 352, no. 1, pp. 37–58, 2017.

[26] G. Aubrun, “A sharp small deviation inequality for the largest eigenvalue of a random matrix,” in *Séminaire de Probabilités XXXVIII*. Springer, 2005, pp. 320–337.