On the infinite Prandtl number limit in two-dimensional magneto-convection

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Abstract. In this paper, the infinite limit of the Prandtl number is justified for the two-dimensional incompressible magneto-convection, which describes the nonlinear interaction between the Rayleigh-Bénard convection and an externally magnetic field. Both the convergence rates and the thickness of initial layer are obtained. Moreover, based on the method of formal asymptotic expansions, an effective dynamics is constructed to simulate the motion within the initial layer.

Keywords. magneto-convection, infinite Prandtl number limit, initial layer

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1 Introduction

In this paper, we are interested in a two-dimensional Boussinesq fluid with nonlinear interaction between Rayleigh-Bénard convection and an externally magnetic field. To begin, let us consider a horizontally stratified fluid layer of characteristic height $h$, referred to as a Cartesian coordinate system with $x$-axis in the horizontal direction and $y$-axis pointing vertically upward. Assume that a fixed temperature difference, say $\theta_2 - \theta_1$, is maintained across the layer of the fluid heated from below in an externally imposed magnetic field $B_0 = \bar{B}k$. For simplicity, we also assume the periodic boundary conditions in the horizontal direction. Then, the MHD-Boussinesq approximation for incompressible viscous and resistive flows reads as follows (cf. [10]):

\begin{align}
\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p &= \mu \Delta u + g \alpha k \theta + B \cdot \nabla B, \\
\frac{\partial B}{\partial t} + u \cdot \nabla B - B \cdot \nabla u &= \nu \Delta B, \\
\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta &= \kappa \Delta \theta, \\
\nabla \cdot u &= 0, \quad \nabla \cdot B = 0
\end{align}

(1.1)

with the following initial and boundary conditions:

\begin{align}
\left\{ \begin{array}{l}
\left. u \right|_{t=0} = u_0, \quad \left. B \right|_{t=0} = B_0, \quad \left. \theta \right|_{t=0} = \theta_0, \\
\left. u \right|_{y=0,h} = 0, \quad \left. B \right|_{y=0,h} = \bar{B}k, \quad \left. \theta \right|_{y=0} = \theta_2, \quad \left. \theta \right|_{y=h} = \theta_1,
\end{array} \right.
\end{align}

(1.2)

and the periodic boundary conditions in the horizontal direction. Here, $u = (u_1, u_2)$ is the velocity, $B$ is the magnetic field, $\theta$ is the temperature and $p$ is the total pressure (incorporating

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the magnetic pressure); \( \mu \) is the kinematic viscosity, \( \nu \) is the magnetic diffusivity, \( \kappa \) is the thermal diffusivity, \( g \) is the gravitational acceleration, \( \alpha \) is the thermal expansion coefficient, and \( \mathbf{k} \) is the vertical unit vector.

This set of equations describes the nonlinear interaction between Rayleigh-Bénard convection and an externally imposed magnetic fields, which is called magneto-convection and may explain certain prominent features on the solar surface. It was shown in [2, 13] that the finite amplitude onset of steady convection became possible when the Rayleigh number is considerably below the values predicted by linear theory. Magnetic fields with sunspots are sufficiently strong to suppress convection on granular and supergranular scales (see [5, 6, 7, 21, 22, 23, 28]). However, we are far from a real understanding of the dynamic coupling between convection and magnetic fields in stars and magnetically confined high-temperature plasmas. So, it is of great importance to understand how energy transport and convection are affected by an imposed magnetic field, that is, how the Lorentz force affects the convection patterns in sunspots and magnetically confined high-temperature plasmas.

Since we aim to consider the problem of magneto-convection, it is more convenient to consider the standard and natural non-dimensional equations of (1.1). So, if using the units of the layer height \( h \) as the characteristic length scale, the thermal diffusion time \( h^2/\kappa \) as the characteristic time scale, the ratio of typical length over typical time \( \kappa/h \) as the typical velocity, the imposed field strength \( \bar{B} \) as the typical magnetic field, and the temperature on a scale where the upside is kept at 0 and the downside is kept at 1, then we obtain the following non-dimensional equations of (1.1):

\[
\begin{align*}
\frac{1}{\text{Pr}} \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \nabla p &= \Delta \mathbf{u} + \text{Ra} \mathbf{k} \theta + \text{Pm} \cdot \mathbf{Q} \mathbf{B} \cdot \nabla \mathbf{B} + \text{Pm} \cdot \mathbf{Q} \frac{\partial \mathbf{B}}{\partial y}, \\
\frac{1}{\text{Pm}} \left( \frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{u} - \frac{\partial \mathbf{u}}{\partial y} \right) &= \Delta \mathbf{B}, \\
\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta &= \Delta \theta, \\
\nabla \cdot \mathbf{u} &= 0, \quad \nabla \cdot \mathbf{B} = 0
\end{align*}
\] (1.3)

with periodic boundary conditions in the horizontal direction and

\[
\begin{align*}
\mathbf{u}|_{t=0} &= \mathbf{u}_0, \quad \mathbf{B}|_{t=0} = \mathbf{B}_0, \quad \theta|_{t=0} = \theta_0, \\
\mathbf{u}|_{y=0,1} &= 0, \quad \mathbf{B}|_{y=0,1} = 0, \quad \theta|_{y=0,1} = 1, \quad \theta|_{y=1} = 0.
\end{align*}
\] (1.4)

Here, for simplicity but without any confusion, we still use the notations \((\mathbf{u}, \mathbf{B}, \theta)\) and \((\mathbf{u}_0, \mathbf{B}_0, \theta_0)\) to denote the non-dimensional quantities and the initial data, respectively.

There are four important dimensionless parameters in (1.3): the Rayleigh number

\[
\text{Ra} = \frac{g \alpha (\theta_2 - \theta_1) h^3}{\mu \kappa},
\]

measuring the ratio of overall buoyancy force to the damping coefficients; the Chandrasekhar number

\[
\text{Q} = \frac{\bar{B}^2 h^2}{\mu \nu},
\]

measuring the ratio of Lorentz force to viscosity; the Prandtl number

\[
\text{Pr} = \frac{\mu}{\kappa},
\]

2
measuring the relative ratio of momentum diffusivity to thermal diffusivity; and the magnetic Prandtl number

\[ Pm = \frac{\nu}{\kappa}, \]

measuring the ratio of magnetic diffusivity to thermal diffusivity.

The problem of large Prandtl number (i.e., \( Pr \gg 1 \)) finds its important many applications for the fluids such as silicone oil, the earth’s melts, as well as the gases under high pressure (see, for example, [2, 7, 14, 26]). Since we have normalized the time to the thermal diffusive time scale, the large Prandtl number means that the viscous time scale of the fluid (i.e., \( h^2/\mu \)) is much shorter than the thermal diffusive time scale (i.e., \( h^2/\kappa \)). Thus, the velocity field slaved by the temperature field will settle into some “equilibrium” state due to the long-time viscosity effect (cf. [29, 30]). Formally, if the Prandtl number equal to infinity in (1.3), then the convection term can be negligible and the so-called infinite Prandtl number system reads

\[
\begin{aligned}
\nabla p^0 - \Delta u^0 &= Ra k \theta^0 + Pm \cdot QB^0 \cdot \nabla B^0 + Pm \cdot Q \frac{\partial B^0}{\partial y}, \\
\frac{1}{Pm} \left( \frac{\partial B^0}{\partial t} + u^0 \cdot \nabla B^0 - B^0 \cdot \nabla u^0 - \frac{\partial u^0}{\partial y} \right) &= \Delta B^0, \\
\frac{\partial \theta^0}{\partial t} + u^0 \cdot \nabla \theta^0 &= \Delta \theta^0, \\

\end{aligned}
\]  

(1.5)

with periodic boundary conditions in the \( x \)-direction and

\[
\begin{aligned}
B^0|_{y=0} &= B_0, & \theta^0|_{y=0} &= \theta_0, \\
u^0|_{y=0,1} &= 0, & B^0|_{y=0,1} &= 0, & \theta^0|_{y=0} &= 1, & \theta^0|_{y=1} &= 0.
\end{aligned}
\]  

(1.6)

The large Prandtl number for the incompressible fluids without magnetic effects has been studied by many authors (see, for example, [8, 9, 11, 29, 30]). Indeed, if we ignore the magnetic effects in (1.5), then it turns into

\[
\begin{aligned}
\nabla p^0 - \Delta u^0 &= Ra k \theta^0, \\
\frac{\partial \theta^0}{\partial t} + u^0 \cdot \nabla \theta^0 &= \Delta \theta^0, \\
\nabla \cdot u^0 &= 0.
\end{aligned}
\]  

(1.7)

It is easy to utilize the standard results of the Stokes equations (cf. [16, 27]) to show that there exists a global strong solution to the initial and boundary value problem of (1.7) even in the three-dimensional setting, since it readily follows from the maximum principle that the temperature \( \theta^0 \) is globally bounded. However, there is very few result about the large Prandtl number for magneto-convection. In fact, it is easy to see that system (1.5) is a coupled parabolic-elliptic system, which retains the essentially nonlinear Lorentz effect on the fluid. So, the rigorous mathematical theory (e.g., well-posedness, asymptotic behavior, etc) of (1.5) in the three-dimensional framework is full of challenge, though the stabilizing effect of magnetic field has been exploited in many works both from the physical and from the numerical point of view (see, for example, [12, 19, 20, 24]).

The global well-posedness theory of the equations for incompressible viscous fluids is classical and well-known, see, for example, [16, 23, 27] and the references cited therein. Moreover,
a similar system as that in \((1.3)\) was also studied in \([15, 18]\) and the global existence of weak solutions was proved. We also mention the interesting works \([3, 4]\), where the two-dimensional incompressible MHD equations with partial viscosities were considered. The main purpose of this paper is to justify the global-in-time asymptotic limit from the two-dimensional system \((1.3)\) to the one \((1.3)\) rigorously, as the Prandtl number \(\text{Pr} \to \infty\) (i.e., \(\text{Pr} \to \infty\)). As a by-product, the global well-posedness of strong solutions to the problem \((1.3) - (1.6)\) with large data was also proved. It is clear that the infinite limit of the Prandtl number is a singular one involving an initial layer.

Our first result is concerned with the convergence from \((u, B, \theta)\) to \((u^0, B^0, \theta^0)\) strictly away from the initial layer. Note that, since we are only interested in the infinite limit of Prandtl number, for simplicity we assume throughout the remainder of this paper that \(\text{Ra}, Q, \text{Pm} \equiv 1\).

**Theorem 1.1** Let \(\Omega \triangleq [0, L] \times [0, 1]\) and \(\text{Ra}, Q, \text{Pm} \equiv 1\). Assume that

\[
\begin{cases}
(u_0, B_0, \theta_0) \in H^2, & \nabla \cdot u_0 = 0, \quad \nabla \cdot B_0 = 0, \\
(u_0, B_0)|_{y=0,1} = 0, & \theta_0|_{y=0} = 1, \quad \theta_0|_{y=1} = 0.
\end{cases}
\]

Then for any \(0 < T < \infty\), there exists a global unique solution \((u, B, \theta)\) (resp. \((u^0, B^0, \theta^0)\)) to the problem \((1.3) - (1.4)\) (resp. the problem \((1.3) - (1.6)\)) on \(\Omega \times [0, T]\), such that

\[
\sup_{0 \leq t \leq T} \left( \| (B - B^0, \theta - \theta^0)(t) \|_{H^2}^2 + \| (\nabla (B - B^0), \nabla (\theta - \theta^0))(t) \|_{L^4}^4 \right) \leq C \varepsilon
\]

and

\[
\| (u - u^0)(t) \|_{H^2}^2 \leq \frac{C \varepsilon}{t} \quad \text{for any} \quad \varepsilon^{1-\alpha} \leq t \leq T \quad \text{with} \quad \alpha \in (0, 1),
\]

where \(\varepsilon \triangleq \text{Pr}^{-1} \in (0, 1)\) and \(C\) is a positive constant independent of \(\varepsilon\).

**Remark 1.1** It seems unsatisfactory that the quantity \(\| B \|_{H^2}\) cannot be uniformly bounded, although \(B_0 \in H^2\), and consequently, the convergence of \(B - B^0\) in \(H^2\) cannot be obtained. This is mainly due to the effects caused by the initial layer and the imposed strength \(B_k\) of magnetic field, the latter of which induces an additional term \(B_y\) in \((1.3)\).

It is easily seen from \((1.10)\) that there is an initial layer between \(u\) and \(u^0\), whose thickness is almost of the value \(\varepsilon\). Motivated by this fact, to capture the effective dynamics of the initial layer, we adopt the so-called two-time scale approach (cf. \([14, 15, 17]\)) by introducing the following fast time scale

\[
\tau = \text{Pr} \cdot t = \frac{t}{\varepsilon} \quad \text{with} \quad \varepsilon = \frac{1}{\text{Pr}}
\]

and the formal asymptotic expansions

\[
\begin{align*}
u &= u^{(0)}(t, \tau) + \varepsilon u^{(1)}(t, \tau) + \text{h.o.t.}, \\
B(t, x) &= B^{(0)}(t, \tau) + \varepsilon B^{(1)}(t, \tau) + \text{h.o.t.}, \\
\theta(t, x) &= \theta^{(0)}(t, \tau) + \varepsilon \theta^{(1)}(t, \tau) + \text{h.o.t.},
\end{align*}
\]

where “h.o.t.” represents the higher-order terms in \(\varepsilon\). Moreover, to ensure the validity of the formal asymptotic expansion for large values of the fast variable \(\tau\), we also impose the customary sublinear growth condition

\[
\lim_{\tau \to \infty} \frac{(u^{(1)}, B^{(1)}, \theta^{(1)})(t, \tau)}{\tau} = 0.
\]
Inserting the formal asymptotic expansion (1.11) into (1.3) and noting that
\[ D_t = \frac{\partial}{\partial t} + \frac{1}{\varepsilon} \frac{\partial}{\partial \tau}, \]
we obtain after collecting the leading-order terms that \((Ra, Q, Pm \equiv 1)\)
\[
\begin{aligned}
    \frac{\partial u^{(0)}}{\partial \tau} + \nabla p^{(0)} &= \Delta u^{(0)} + k\theta^{(0)} + B^{(0)} \cdot \nabla B^{(0)} + \frac{\partial B^{(0)}}{\partial y}, \\
    \frac{\partial B^{(0)}}{\partial \tau} &= 0, \quad \frac{\partial \theta^{(0)}}{\partial \tau} = 0, \quad \nabla \cdot u^{(0)} = 0, \quad \nabla \cdot B^{(0)} = 0.
\end{aligned}
\]
(1.13)

Let \(A\) be the Stokes operator defined as
\[ Au = f, \]
(1.14)
if and only if \(u\) satisfies
\[ \nabla p - \Delta u = f, \quad \nabla \cdot u = 0, \quad u|_{y=0,1} = 0, \]
and the periodic conditions in the \(x\)-direction.

Then it follows from (1.13) that
\[ B^{(0)}(t, \tau) = B^{(0)}(t), \quad \theta^{(0)}(t, \tau) = \theta^{(0)}(t), \]
(1.15)
and
\[
\begin{aligned}
    u^{(0)}(t, \tau) &= e^{-\tau A} u^{(0)}(t, 0) + A^{-1} \mathbb{P} \left( k\theta^{(0)} + B^{(0)} \cdot \nabla B^{(0)} + \frac{\partial B^{(0)}}{\partial y} \right)(t) \\
    &= e^{-\tau A} A^{-1} \mathbb{P} \left( k\theta^{(0)} + B^{(0)} \cdot \nabla B^{(0)} + \frac{\partial B^{(0)}}{\partial y} \right)(t),
\end{aligned}
\]
(1.16)
where \(\mathbb{P}\) is the Leray-Hopf projector and \(A = -\mathbb{P}\Delta\) (see, for example, [27]).

The next-order dynamics is governed by
\[
\begin{aligned}
    \frac{\partial u^{(1)}}{\partial \tau} + Au^{(1)} &= -\frac{\partial u^{(0)}}{\partial t} - \mathbb{P} \left( u^{(0)} \cdot \nabla u^{(0)} \right) \\
    &\quad + \mathbb{P} \left( k\theta^{(1)} + B^{(0)} \cdot \nabla B^{(1)} + B^{(1)} \cdot \nabla B^{(0)} + \frac{\partial B^{(1)}}{\partial y} \right), \\
    \frac{\partial B^{(1)}}{\partial \tau} &= \Delta B^{(0)} - u^{(0)} \cdot \nabla B^{(0)} + B^{(0)} \cdot \nabla u^{(0)} + \frac{\partial u^{(0)}}{\partial y} - \frac{\partial B^{(0)}}{\partial t}, \\
    \frac{\partial \theta^{(1)}}{\partial \tau} &= \Delta \theta^{(0)} - u^{(0)} \cdot \nabla \theta^{(0)} - \frac{\partial \theta^{(0)}}{\partial t}.
\end{aligned}
\]
(1.17)

In view of the sublinear growth condition (1.12), we have
\[
\begin{aligned}
    0 &= \Delta B^{(0)} - u^{(0)} \cdot \nabla B^{(0)} + B^{(0)} \cdot \nabla u^{(0)} + \frac{\partial u^{(0)}}{\partial y} - \frac{\partial B^{(0)}}{\partial t}, \\
    0 &= \Delta \theta^{(0)} - u^{(0)} \cdot \nabla \theta^{(0)} - \frac{\partial \theta^{(0)}}{\partial t},
\end{aligned}
\]
(1.18)
which is the limit model of infinite Prandtl number.

The equation (1.17) for \( u^{(1)} \) is dissipative. However, similarly to that in \cite{29}, there are three terms of \( u^{(0)} \) in (1.16): one term slaved by the leading-order terms of temperature and Lorentz force, and another two terms exponentially decaying in time (initial layer type). This means that no more dynamics on \( u^{(0)} \) is necessary except the ones in (1.16). Moreover, by modifying the initial layer terms in such a way so that the initial data are fixed, we can propose the following effective dynamics within the initial layer \((\tau = t/\varepsilon)\):

\[
\begin{cases}
u^{(0)} = e^{-\tau A}u_0 - e^{-\tau A}A^{-1}P \left( k\theta_0 + B_0 \cdot \nabla B_0 + \frac{\partial B_0}{\partial y} \right) \\
+ A^{-1}P \left( k\theta^{(0)} + B^{(0)} \cdot \nabla B^{(0)} + \frac{\partial B^{(0)}}{\partial y} \right), \\
\frac{\partial B^{(0)}}{\partial t} + u^{(0)} \cdot \nabla B^{(0)} - B^{(0)} \cdot \nabla u^{(0)} - \frac{\partial u^{(0)}}{\partial y} = \Delta B^{(0)}, \quad \nabla \cdot B^{(0)} = 0, \\
\frac{\partial \theta^{(0)}}{\partial t} + u^{(0)} \cdot \nabla \theta^{(0)} = \Delta \theta^{(0)},
\end{cases}
\]

which is completed with periodic boundary conditions in the \( x \)-direction and

\[
\begin{cases}
B^{(0)}|_{t=0} = B_0, \quad \theta^{(0)}|_{t=0} = \theta_0, \\
B^{(0)}|_{y=0,1} = 0, \quad \theta^{(0)}|_{y=0} = 1, \quad \theta^{(0)}|_{y=1} = 0.
\end{cases}
\]

The solutions of (1.19)–(1.20) will be compared with the ones of the problems (1.3)–(1.4) and (1.5)–(1.6) with \( Ra, \bar{Q}, \bar{P}m \equiv 1 \), respectively.

With the help of the effective dynamics (1.19)–(1.20), we can prove the following main theorem which is concerned with the behavior of the initial layer.

**Theorem 1.2** Let the conditions of Theorem 1.1 be in force. For any fixed \( 0 < T < \infty \), assume that \((u, B, \theta), (u^0, B^0, \theta^0)\) and \((u^{(0)}, B^{(0)}, \theta^{(0)})\) are the solutions of the problems (1.3)–(1.4), (1.5)–(1.6) and (1.19)–(1.20) on \( \Omega \times [0, T] \), respectively. Then, in addition to (1.9) and (1.10), there exists a positive constant \( C, \) independent of \( \varepsilon, \) such that for any \( t \in [0, T], \)

\[
\| B - B^0 \|_{L^2} + \| \theta - \theta^0 \|_{L^2} \leq C\varepsilon, \tag{1.21}
\]

\[
\| u - u^0 - e^{-\tau A}u_0 + e^{-\tau A}A^{-1}P (k\theta_0 + B_0 \cdot \nabla B_0 + \partial y B_0) \|_{L^2} \leq C\varepsilon \tag{1.22}
\]

and

\[
\| \nabla (u - u^0 - e^{-\tau A}u_0 + e^{-\tau A}A^{-1}P (k\theta_0 + B_0 \cdot \nabla B_0 + \partial y B_0)) \|_{H^1} \leq C\varepsilon^{1/2}. \tag{1.23}
\]

**Remark 1.2** It is easily seen from (1.22) and (1.23) that the motion within the initial layer can be modelled by the initial-layer correction function \( e^{-\tau A}u_0 - e^{-\tau A}A^{-1}P (k\theta_0 + B_0 \cdot \nabla B_0 + \partial y B_0) \) in the sense of uniform convergence. Indeed, the first term \( u_0 \) is the initial data of \( u \) and the second one \( A^{-1}P (k\theta_0 + B_0 \cdot \nabla B_0 + \partial y B_0) \) is the initial data of \( u^0 \).

**Remark 1.3** The \( L^2 \)-convergence rates of order \( \varepsilon \) in (1.21) and (1.22) are optimal, which can be justified via the systematic asymptotic expansion with the small parameter \( \varepsilon = 1/\text{Pr} \).

The proofs of Theorems 1.1 and 1.2 are based on the global (uniform) estimates of \((u, B, \theta), (u^0, B^0, \theta^0)\) and \((u^{(0)}, B^{(0)}, \theta^{(0)})\), which will be achieved by making a full use of the estimates of
the Stokes equations in a manner similar to that used for the standard incompressible Navier-Stokes/MHD equations (cf. [16, 25, 27]). It is worth pointing out that due to the presence of initial layer corrections in (1.19)–(1.20) is close to the one of the infinite Prandtl number dynamics (1.5). It is natural to show that as \( \varepsilon \to 0 \), the solution of the effective dynamics (1.19)–(1.20) is close to the solution of the velocity cannot be obtained directly. To circumvent this difficulty, we observe that the leading-order term of temperature and Lorentz force (i.e., \( A^{-1}\mathbf{k}\theta(0) + \mathbf{B}(0) \cdot \nabla \mathbf{B}(0) + \mathbf{B}_y(0) \)) plays an important role and its \( t \)-derivative acts as a correction term between \( \mathbf{u} \) and \( \mathbf{u}_0 \) in some sense (see (3.40)–(3.43)). The optimal convergence rates in (1.21) and (1.22) also need some careful analysis, based on the elementary energy methods and the application of the Poincaré’s inequality (see sections 3.2 and 3.3).

2 Proof of Theorem 1.1

The global existence of classical solutions \((\mathbf{u}, \mathbf{B}, \theta)\) to the problem (1.3)–(1.4) with smooth data can be easily proved via the standard Faedo-Galerkin method and the global a priori estimates. The global solutions \((\mathbf{u}^0, \mathbf{B}^0, \theta^0)\) of (1.5)–(1.6) can be obtained as the vanishing \( \varepsilon \)-limit of \((\mathbf{u}, \mathbf{B}, \theta)\). Thus, for any given \( 0 < T < \infty \), we assume that \((\mathbf{u}, \mathbf{B}, \theta)\) and \((\mathbf{u}^0, \mathbf{B}^0, \theta^0)\) are smooth solutions of (1.3)–(1.4) and (1.5)–(1.6) on \( \Omega \times [0, T] \), respectively. To prove Theorem 1.1, it suffices to derive some global (uniform-in-\( \varepsilon \)) estimates of \((\mathbf{u}, \mathbf{B}, \theta)\) and \((\mathbf{u}^0, \mathbf{B}^0, \theta^0)\).

2.1 Global \( \varepsilon \)-independent estimates of \((\mathbf{u}, \mathbf{B}, \theta)\)

The purpose of this subsection is to derive the global uniform estimates of \((\mathbf{u}, \mathbf{B}, \theta)\). For simplicity, throughout this paper we use the same letter \( C \) to denote the \( \varepsilon \)-independent constant.

**Proposition 2.1** Let \((\mathbf{u}, \mathbf{B}, \theta)\) be a smooth solution of (1.3)–(1.4) on \( \Omega \times [0, T] \). Then there exists a positive constant \( C \), independent of \( \varepsilon \), such that for any \( p \geq 2 \),

\[
\sup_{0 \leq t \leq T} \| (\mathbf{B}, \theta)(t) \|_{H^1 \cap W^{1,p}} + \int_0^T \left( \| (\mathbf{B}, \theta) \|_{H^2}^2 + \| (\mathbf{B}, \theta_t) \|_{L^2}^2 \right) dt + \sup_{0 \leq t \leq T} \| \mathbf{u}(t) \|_{H^2} + \int_0^T \left( \| \mathbf{u} \|_{H^3}^2 + \varepsilon \| \mathbf{u}_t \|_{H^1}^2 \right) dt \leq C(p).
\]

**Proof.** The proofs are split into three steps.

**Step I. the \( L^2 \)-estimates**

First, it is easily deduced from (1.3) and the maximum principle that

\[
\sup_{0 \leq t \leq T} \| \theta(t) \|_{L^\infty} \leq C.
\]

Let \( \Theta \triangleq \theta - (1 - y) \). Then, it holds that \( \Theta|_{y=0,1} = 0 \) and

\[
\Theta_t + \mathbf{u} \cdot \nabla \Theta = \Delta \Theta + u_2
\]
So, multiplying \((1.3)_1\), \((1.3)_2\) and \((2.4)\) by \(u\), \(B\) and \(\Theta\) in \(L^2\) respectively, integrating by parts, and using the Gronwall’s inequality, we get that

\[
\sup_{0 \leq t \leq T} (\varepsilon \|u\|_{L^2}^2 + \|B\|_{L^2}^2 + \|\Theta\|_{L^2}^2) (t) + \int_0^T \| (\nabla u, \nabla B, \nabla \Theta) \|_{L^2}^2 dt \leq C. \tag{2.4}
\]

To prove the \(\varepsilon\)-independent estimate of \(\|u\|_{L^2}\), we first multiply \((1.3)_2\) by \(\|B\|^2 B\) and integrate by parts to get

\[
\frac{d}{dt} \|B\|^4_{L^4} + \|\nabla B\|^2_{L^2} + \|\nabla |B|^2\|^2_{L^2} \leq C \|\nabla u\|_{L^2} \left( \|B\|^2_{L^4} + \|B\|_{L^4} \|B\|^2_{L^4} \right) 
\leq C \left( 1 + \|\nabla u\|_{L^2}^2 \right) \left( 1 + \|B\|_{L^4}^4 \right) + \frac{1}{2} \|\nabla |B|^2\|^2_{L^2},
\tag{2.5}
\]

where we have used the Cauchy-Schwarz’s inequality and the following Sobolev’s inequality:

\[
\|f\|_{L^2}^2 \leq C \|f\|_{L^2}^2 + C \|f\|_{L^2} \|\nabla f\|_{L^2} \tag{2.6}
\]

for any \(f \in \{ f \in H^1 : f \) is \(x\)-periodic and \(f = 0\) on \(y = 0, 1\)\).

Thus, using \((2.4)\) and the Gronwall’s inequality, we infer from \((2.5)\) that

\[
\sup_{0 \leq t \leq T} \|B(t)\|_{L^4}^4 + \int_0^T \left( \|\nabla B\|^2_{L^2} + \|\nabla |B|^2\|^2_{L^2} \right) dt \leq C. \tag{2.7}
\]

Now, multiplying \((1.3)_1\) by \(u\) in \(L^2\) again and integrating by parts, we find

\[
\varepsilon \frac{d}{dt} \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq C \left( \|u\|_{L^2} \|\theta\|_{L^2} + \|\nabla u\|_{L^2} \|B\|_{L^4}^4 + \|\nabla u\|_{L^2} \|B\|_{L^2} \right).
\]

Thus, thanks to \((2.4)\), \((2.7)\) and the Poincaré’s inequality:

\[
\|f\|_{L^2} \leq C \|\nabla f\|_{L^2}
\]

for any \(f \in \{ f \in H^1 : f \) is \(x\)-periodic and \(f = 0\) on \(y = 0, 1\)\), we have

\[
\varepsilon \frac{d}{dt} \|u\|_{L^2}^2 + \|u\|_{L^2}^2 \leq C,
\]

and consequently,

\[
\frac{d}{dt} \left( e^{t/\varepsilon} \|u\|_{L^2}^2 \right) \leq C e^{-1} e^{t/\varepsilon},
\]

which, integrated in time, shows that \(\|u(t)\|_{L^2}\) is uniformly bounded in \(\varepsilon\) for all \(t \in [0, T]\).

To summarize, we have proved that

\[
\sup_{0 \leq t \leq T} \left( \|u(t)\|_{L^2}^2 + \|B(t)\|_{L^4}^4 + \|\theta(t)\|_{L^\infty} \right) 
+ \int_0^T \left( \|\nabla u, \nabla B, \nabla \Theta\|_{L^2}^2 + \|B\| \|\nabla B\|_{L^2}^2 \right) dt \leq C. \tag{2.8}
\]

**Step II. the \(H^1\)-estimates**

Note that

\[
\| (\Delta u, \Delta B, \Delta \Theta) \|_{L^2} = \| (\nabla^2 u, \nabla^2 B, \nabla^2 \Theta) \|_{L^2}
\]
and
\[ \| \nabla f \|^2_{L^4} \leq C \| \nabla f \|^2_{L^2} + C \| \nabla f \|_{L^2} \| \nabla^2 f \|_{L^2}, \]  
(2.9)
where \( f \in \{ u, B, \Theta \} \). Thus, multiplying (1.3), (1.6) and (2.3) by \(-\Delta u, -\Delta B\) and \(-\Delta \Theta\) in \( L^2 \) respectively, and integrating by parts, we infer from (2.6), (2.8), (2.10) and the Cauchy-Schwarz’s inequality that
\[
\frac{d}{dt} (\varepsilon \| \nabla u \|^2_{L^2} + \| \nabla B \|^2_{L^2} + \| \nabla \Theta \|^2_{L^2}) + \|(\nabla^2 u, \nabla^2 B, \nabla^2 \Theta)\|^2_{L^2} \\
\leq C \left( 1 + \| (\nabla u, \nabla B) \|^2_{L^2} \right) \left( 1 + \varepsilon \| \nabla u \|^2_{L^2} + \| \nabla B \|^2_{L^2} + \| \nabla \Theta \|^2_{L^2} \right),
\]  
(2.10)
and hence, it follows from (2.8), (2.10) and the Gronwall’s inequality that
\[
\sup_{0 \leq t \leq T} (\varepsilon \| \nabla u \|^2_{L^2} + \| \nabla B \|^2_{L^2} + \| \nabla \Theta \|^2_{L^2}) (t) + \int_0^T \|(\nabla^2 u, \nabla^2 B, \nabla^2 \Theta)\|^2_{L^2} dt \leq C,
\]  
(2.11)
which, together with (1.3) and (2.3), also gives
\[
\int_0^T \left( \varepsilon^2 \| u_t \|^2_{L^2} + \| (B_t, \Theta_t) \|^2_{L^2} \right) dt \leq C.
\]  
(2.12)
To prove the \( \varepsilon \)-independent estimate of \( \| \nabla u \|_{L^2} \), we multiply (1.3) by \( u_t \) in \( L^2 \) and integrate by parts to get that
\[
\frac{d}{dt} \| \nabla u \|^2_{L^2} + \varepsilon \| u_t \|^2_{L^2} = \langle k \theta + B \cdot \nabla B + B_y, u_t \rangle - \varepsilon \langle u \cdot \nabla u, u_t \rangle \\
\triangleq \frac{d}{dt} \langle k \theta + B \cdot \nabla B + B_y, u \rangle + I,
\]  
(2.13)
where \( \langle \cdot, \cdot \rangle \) denotes the standard \( L^2 \)-inner product and
\[ I \triangleq -\langle k \theta_t + B_t \cdot \nabla B + B \cdot \nabla B_t + B_{ty}, u \rangle - \varepsilon \langle u \cdot \nabla u, u_t \rangle. \]

On one hand, by (2.8) we have
\[
\langle k \theta + B \cdot \nabla B + B_y, u \rangle = \langle k \theta, u \rangle - \langle B \cdot \nabla u, B \rangle - \langle B, u_y \rangle \leq \frac{1}{2} \| \nabla u \|^2_{L^2} + C.
\]
On the other hand, based upon integration by parts, we have
\[
I = -\langle k \theta_t, u \rangle + \langle B_t \cdot \nabla u, B \rangle + \langle B \cdot \nabla u, B_t \rangle + \langle B_t, u_y \rangle - \varepsilon \langle u \cdot \nabla u, u_t \rangle \\
\leq C \left( 1 + \| (u, B) \|^2_{H^2} \right) \| \nabla u \|^2_{L^2} + C \left( \| \theta_t \|^2_{L^2} + \| B_t \|^2_{L^2} + \varepsilon^2 \| u_t \|^2_{L^2} \right),
\]
where have used the Sobolev embedding inequality:
\[
\| f \|_{L^\infty} \leq C \| f \|_{H^2} \quad \text{for any} \quad f \in H^2.
\]  
(2.14)
Thus, inserting the above estimates into (2.13), using (2.8), (2.11), (2.12) and the Gronwall’s inequality, we obtain
\[
\sup_{0 \leq t \leq T} \| \nabla u(t) \|^2_{L^2} + \varepsilon \int_0^T \| u_t \|^2_{L^2} dt \leq C.
\]  
(2.15)

**Step III.** The higher regularities.
As aforementioned, it is difficult to obtain the uniform $H^2$-estimate of $B$. Indeed, instead of this, we have the $W^{1,p}$-estimate of $B$ for any $p > 2$, which particularly indicates that $B$ is uniformly bounded in $\varepsilon$. To do this, differentiating $u_{3,2}$ with respect to $y$, multiplying the resulting equation by $p|B_y|^{p-2}B_y$, and integrating by parts, we deduce that (noting that $(B_y + u)_y = 0$ on $y = 0, 1$)

$$\frac{d}{dt}\|B_y\|_{L^p}^p + \int \left( |B_y|^{p-2} |B_{yy}|^2 + |B_y|^{p-4} \left( |(B_y)^2|_y^2 \right) \right) dx dy$$

$$+ \int \left( |B_y|^{p-2} |B_{xy}|^2 + |B_y|^{p-4} \left( |(B_y)^2|_x^2 \right) \right) dx dy$$

$$\leq C(p) \int \left( |u|^2 |\nabla B|^2 + |B|^2 |\nabla u|^2 + |u|^4 \right) |B_y|^{p-2} dx dy$$

$$\leq C(p) \left( \|u, B\|_2^2 \| (\nabla u, x) \|_{H^1}^2 + \| u\|^2_{L^2} \| B_y \|_{L^p}^{p-2} \right) \leq C(p) \left( \| B \|_{H^1}^{2} + \| u \|^2_{H^1} \right) (1 + \| B_y \|^p_{L^p}) ,$$

where we have used (2.11), (2.15), and the Sobolev embedding inequality $H^1 \hookrightarrow L^q$ for any $q \geq 1$. So, using (2.11) and the Gronwall’s inequality, we have from (2.16) that

$$\sup_{0 \leq t \leq T} \|B_y(t)\|_{L^p} \leq C(p) \quad \forall \ p \geq 2,$$

and similarly,

$$\sup_{0 \leq t \leq T} \|B_y(t)\|_{L^p} \leq C(p) \quad \forall \ p \geq 2,$$

so that, it follows from the Sobolev’s embedding inequality $W^{1,p} \hookrightarrow L^\infty$ for $p > 2$ that

$$\|B(t)\|_{L^\infty} + \|B(t)\|_{W^{1,p}} \leq C(p) \quad \forall \ t \in [0, T]. \quad (2.18)$$

Analogously to the proof of (2.17), one also gets that for any $t \in [0, T]$,

$$\|\Theta(t)\|_{W^{1,p}} + \|\theta(t)\|_{W^{1,p}} \leq C(p) \quad \forall \ p > 2. \quad (2.19)$$

The estimate of $\|u\|_{H^2}$ needs more works. For this purpose, we first differentiate $u_{3,1}$ with respect to $t$, multiply the resulting equation by $u_t$ in $L^2$, and integrate by parts to deduce that

$$\varepsilon \frac{d}{dt} \|u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 = - \varepsilon \langle u_t \cdot \nabla u, u_t \rangle + \langle k \theta_t, u_t \rangle + \langle B_{yt}, u_t \rangle$$

$$+ \langle B \cdot \nabla B_t, u_t \rangle + \langle B_t \cdot \nabla B, u_t \rangle \triangleq II.$$

Using (2.14), (2.18), and the Poincaré’s inequality, we obtain after integrating by parts that

$$II = \varepsilon \langle u_t \cdot \nabla u_t, u_t \rangle + \langle k \theta_t, u_t \rangle - \langle B_t, u_{yt} \rangle - \langle B \cdot \nabla u_t, B_t \rangle - \langle B_t \cdot \nabla u_t, B \rangle$$

$$\leq \frac{1}{2} \|\nabla u_t\|_{L^2}^2 + C \varepsilon \|u_t\|_{L^2}^2 + C \| (B_t, \theta_t) \|_{L^2}^2,$$

so that

$$\varepsilon \frac{d}{dt} \|u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \leq C \varepsilon \|u_t\|_{L^2}^2 + C \| (B_t, \theta_t) \|_{L^2}^2. \quad (2.20)$$

In a similar manner, we also have

$$\frac{1}{2} \frac{d}{dt} \| (B_t, \theta_t) \|_{L^2}^2 + \| (\nabla B_t, \nabla \theta_t) \|_{L^2}^2 = \langle B \cdot \nabla u_t, B_t \rangle + \langle B_t \cdot \nabla u_t, B \rangle$$

$$- \langle u_t \cdot \nabla B, B_t \rangle = \langle u_t \cdot \nabla \theta, \theta_t \rangle \triangleq III,$$
where we can utilize (2.2), (2.14), (2.18) and the Poincaré inequality to get that
\[
\begin{align*}
III = & \langle B \cdot \nabla u_t, B_t \rangle - \langle u_t \cdot \nabla B_t, u_t \rangle + \langle u_{yt}, B_t \rangle + \langle u_t \cdot \nabla B_t, B \rangle + \langle u_t \cdot \nabla \theta_t, \theta \rangle \\
\leq & \frac{1}{2} \| \nabla B_t \nabla \theta_t \|^2_{L^2} + C (1 + \| u \|^2_{H^2}) \| (B_t, \theta_t) \|^2_{L^2} + C \| \nabla u_t \|^2_{L^2},
\end{align*}
\]
and consequently,
\[
\begin{align*}
\frac{d}{dt} \| (B_t, \theta_t) \|^2_{L^2} + \| (\nabla B_t, \nabla \theta_t) \|^2_{L^2} & \leq C (1 + \| u \|^2_{H^2}) \| (B_t, \theta_t) \|^2_{L^2} + C \| \nabla u_t \|^2_{L^2}. \tag{2.21}
\end{align*}
\]
As a result of (2.20) and (2.21), we find
\[
\begin{align*}
\frac{d}{dt} (\varepsilon \| u_t \|^2_{L^2} + \| (B_t, \theta_t) \|^2_{L^2}) & + \| (\nabla u_t, \nabla B_t, \nabla \theta_t) \|^2_{L^2} \\
& \leq C (1 + \| u \|^2_{H^2}) (\varepsilon \| u_t \|^2_{L^2} + \| (B_t, \theta_t) \|^2_{L^2}). \tag{2.22}
\end{align*}
\]
To eliminate the effect of initial layer, multiplying (2.22) by \( \varepsilon \) and integrating it over \((0, T)\), we deduce from (2.12), (2.15) and the Gronwall’s inequality that
\[
\begin{align*}
\sup_{0 \leq t \leq T} \left[ t (\varepsilon \| u_t(t) \|^2_{L^2} + \| (B_t, \theta_t(t)) \|^2_{L^2}) \right] + \int_0^T t \| (\nabla u_t, \nabla B_t, \nabla \theta_t) \|^2_{L^2} dt \\
& \leq C \int_0^T (\varepsilon \| u_t \|^2_{L^2} + \| (B_t, \theta_t) \|^2_{L^2}) dt \leq C. \tag{2.23}
\end{align*}
\]
Since \( u_0 \in H^2 \) implies that \( \varepsilon \| u_t \|^2_{L^2} \in L^2 \), after multiplying (2.22) by \( \varepsilon \) and integrating it over \((0, T)\), we arrive at
\[
\begin{align*}
\sup_{0 \leq t \leq T} (\varepsilon^2 \| u_t \|^2_{L^2} + \varepsilon \| (B_t, \theta_t) \|^2_{L^2}) + \varepsilon \int_0^T \| (\nabla u_t, \nabla B_t, \nabla \theta_t) \|^2_{L^2} dt & \leq C. \tag{2.24}
\end{align*}
\]
In view of (2.11), (2.18) and (2.24), we can make use of (2.6), (2.9) and the standard estimates of the Stokes equations to get from (1.3) that
\[
\begin{align*}
\| \nabla^2 u \|_{L^2} & \leq C \varepsilon (\| u \|_{L^2} + \| u \|_{L^4} \| \nabla u \|_{L^4}) + C (\| \theta \|_{L^2} + \| B \|_{L^2} \| \nabla B \|_{L^2} + \| \nabla B \|_{L^2}) \\
& \leq C + C \| \nabla^2 u \|_{L^2}^{1/2} \leq \frac{1}{2} \| \nabla^2 u \|_{L^2} + C,
\end{align*}
\]
which immediately results in
\[
\begin{align*}
\sup_{0 \leq t \leq T} \| \nabla^2 u(t) \|_{L^2} & \leq C, \tag{2.25}
\end{align*}
\]
and moreover,
\[
\begin{align*}
\int_0^T \| u \|^2_{H^3} dt & \leq C + C \int_0^T (\varepsilon^2 \| u_t \|^2_{H^1} + \| (u, B) \|^2_{L^4} \| (u, B) \|^2_{H^2}) dt \\
& + C \int_0^T (\| (\nabla u, \nabla B) \|^2_{L^4} + \| (B, \theta) \|^2_{H^2}) dt \tag{2.26}
\end{align*}
\]
\[
\begin{align*}
& \leq C.
\end{align*}
\]
Now, collecting (2.8), (2.11), (2.12), (2.15), (2.18), (2.19) and (2.23)–(2.26) together leads to the desired estimates stated in (2.1). The proof of Proposition 2.1 is therefore complete. □
2.2 Global estimates of $(u^0, B^0, \theta^0)$

This subsection is devoted to the global estimates of the solutions to the problem (1.5)–(1.6), which can be achieved via the standard estimates of the Stokes equations.

**Proposition 2.2** Let $(u^0, B^0, \theta^0)$ be a smooth solution of (1.5)–(1.6) on $\Omega \times [0, T]$. Then there exists a positive constant $C$, such that

$$
\sup_{0 \leq t \leq T} \left( \| (u^0, B^0, \theta^0) \|_{H^2}^2 + \| (B_t^0, \theta_t^0) \|_{L^2}^2 \right) + \int_0^T \| (u_t^0, B_t^0, \theta_t^0) \|_{H^1}^2 dt \leq C. \tag{2.27}
$$

**Proof.** First, one easily gets from (1.5) that

$$
\sup_{0 \leq t \leq T} \left( \| (B^0, \theta^0) (t) \|_{L^2}^2 + \| \theta^0 (t) \|_{L^\infty} \right) + \int_0^T \| (\nabla u^0, \nabla B^0, \nabla \theta^0) \|_{L^2}^2 dt \leq C. \tag{2.28}
$$

Next, similarly to the proof of (2.27), by (2.28) we have

$$
\sup_{0 \leq t \leq T} \| B^0 (t) \|_{L^2}^4 + \int_0^T \left( \| B^0 \| \nabla B^0 \|_{L^2}^2 + \| \nabla B^0 \|_{L^2}^2 \right) dt \leq C. \tag{2.29}
$$

Thus, it is easily seen from (1.5) and (2.29) that

$$
\| \nabla u^0 (t) \|_{L^2} \leq C \left( \| B^0 (t) \|_{L^2}^2 + \| (B^0, \theta^0) (t) \|_{L^2} \right) \leq C, \quad \forall t \in [0, T], \tag{2.30}
$$

and moreover, it follows from the $H^2$-estimate of the Stokes equations that

$$
\int_0^T \| \nabla^2 u^0 \|_{L^2}^2 dt \leq C \int_0^T \left( \| B^0 \| \nabla B^0 \|_{L^2}^2 + \| \nabla B^0 \|_{L^2}^2 + \| \theta^0 \|_{L^2}^2 \right) dt \leq C. \tag{2.31}
$$

With the help of (2.28)–(2.31), we can obtain in a manner similar to the derivations of (2.11) and (2.12) that

$$
\sup_{0 \leq t \leq T} \| (\nabla B^0, \nabla \theta^0) (t) \|_{L^2}^2 + \int_0^T \left( \| (\nabla^2 B^0, \nabla^2 \theta^0) \|_{L^2}^2 + \| (B_t^0, \theta_t^0) \|_{L^2}^2 \right) dt \leq C. \tag{2.32}
$$

Finally, differentiating (1.5)1, (1.5)2 and (1.5)3 with respect to $t$ and multiplying the resulting equations by $u_t$, $B_t$ and $\theta_t$ in $L^2$ respectively, and integrating by parts, we have

$$
\frac{d}{dt} \left( \| B_t^0 \|_{L^2}^2 + \| \theta_t^0 \|_{L^2}^2 \right) + \| (\nabla u_t^0, \nabla B_t^0, \nabla \theta_t^0) \|_{L^2}^2 \\
\leq C \left( 1 + \| (u^0, B^0, \theta^0) \|_{H^2} \right) \left( \| B_t^0 \|_{L^2}^2 + \| \theta_t^0 \|_{L^2}^2 \right), \tag{2.33}
$$

where we have used (2.4), (2.28)–(2.32) and the Poincaré’s inequality to get that

$$
\langle k \theta_t^0, u_t^0 \rangle + \langle \partial_y B_t^0, u_t^0 \rangle + \langle \partial_y u_t^0, B_t^0 \rangle \leq \| u_t^0 \|_{L^2} \| \theta_t^0 \|_{L^2} + \| \nabla u_t^0 \|_{L^2} \| B_t^0 \|_{L^2} + \| \nabla u_t^0 \|_{L^2} \| \nabla \theta_t^0 \|_{L^2} + \| \nabla u_t^0 \|_{L^2} \| \nabla \theta_t^0 \|_{L^2} \leq \frac{1}{8} \| \nabla u_t^0 \|_{L^2}^2 + C \left( \| B_t^0 \|_{L^2}^2 + \| \theta_t^0 \|_{L^2}^2 \right),
$$

and

$$
\langle (u^0 \cdot \nabla B^0)_t, B_t^0 \rangle - \langle (u^0 \cdot \nabla \theta^0)_t, \theta_t^0 \rangle = -\langle u_t^0 \cdot \nabla B^0, B_t^0 \rangle - \langle u_t^0 \cdot \nabla \theta^0, \theta_t^0 \rangle \leq \| u_t^0 \|_{L^4} \| \nabla B^0 \|_{L^4} + \| \nabla \theta^0 \|_{L^4} \left( \| B_t^0 \|_{L^2} + \| \theta_t^0 \|_{L^2} \right) \leq \frac{1}{8} \| \nabla u_t^0 \|_{L^2}^2 + C \left( \| \nabla B^0 \|_{H^1}^2 + \| \nabla \theta^0 \|_{H^1}^2 \right) \left( \| B_t^0 \|_{L^2}^2 + \| \theta_t^0 \|_{L^2}^2 \right),
$$

as desired.
and
\[ (B^0 \cdot \nabla B^0_t, u^0_t) + (B^0 \cdot \nabla u^0_t, B^0_t) = (B^0 \cdot \nabla B^0_t, u^0_t) + (B^0_t \cdot \nabla u^0, B^0_t) \]
\[ \leq \|B^0_t\|_{L^4} \|\nabla B^0\|_{L^4} \|u^0_t\|_{L^2} + \|B^0_t\|_{L^4} \|\nabla u^0\|_{L^4} \|B^0_t\|_{L^2} \]
\[ \leq C \left( \|B^0_t\|_{L^2} + \|B^0\|_{L^4}^{1/2} \|\nabla B^0_t\|_{L^2}^{1/2} \right) \left( \|\nabla B^0\|_{L^2} + \|\nabla B^0\|_{L^2} \|\nabla B^0\|_{L^2} \right) \|u^0_t\|_{L^2} \]
\[ + C \left( \|B^0_t\|_{L^2} + \|B^0\|_{L^4}^{1/2} \|\nabla B^0_t\|_{L^2}^{1/2} \right) \left( \|\nabla u^0\|_{L^2} + \|\nabla u^0\|_{L^2} \|\nabla u^0\|_{L^2} \right) \|B^0_t\|_{L^2} \]
\[ \leq \frac{1}{8} \|\nabla u^0, \nabla B^0_t\|_{L^2}^2 + C \left( 1 + \|\nabla^2 u^0, \nabla^2 B^0\|_{L^2} \right) \|B^0_t\|_{L^2}^2. \]

Using (2.28), (2.31), (2.32) and the Gronwall’s inequality, we deduce from (2.33) that
\[ \sup_{0 \leq t \leq T} \left( \|B^0_t\|_{L^2}^2 + \|\theta^0_t\|_{L^2}^2 \right) + \int_0^T \|\nabla u^0_t, \nabla B^0_t, \nabla \theta^0_t\|_{L^2}^2 \, dt \leq C. \] (2.34)
Thus, it follows from (1.3), (2.32), (2.34) and the Sobolev embedding inequality that
\[ \|B^0, \theta^0\|(t)_{H^2} \leq C + C \left( \|B^0\|_{L^4} + \|\nabla B^0\|_{L^2} + \|\nabla u^0\|_{H^1} \right) \]
\[ + C \left( \|\nabla u^0\|_{L^2} + \|B^0\|_{H^1} + \|\nabla u^0\|_{L^2} \right) \] (2.35)
and similarly, it follows from (1.3) and the standard estimates of the Stokes equations that
\[ \|u^0(t)\|_{H^2} \leq C + C \left( \|\nabla B^0\|_{L^2} + \|\nabla B^0\|_{H^2} \right) \]
\[ \leq C + C \|B^0\|_{H^2}^{1/2} \] (2.36)
As a result, we conclude from (2.35), (2.36) and the Cauchy-Schwarz’s inequality that
\[ \sup_{0 \leq t \leq T} \|(u^0, B^0, \theta^0)(t)\|_{H^2} \leq C. \]
This, together with (2.28), (2.32) and (2.34), finishes the proof of Proposition 2.2.

2.3 Convergence from \((u, B, \theta)\) to \((u^0, B^0, \theta^0)\)

With the help of the global (uniform) estimates stated in Propositions 2.1 and 2.2, we are now ready to prove Theorem 1.1. First of all, the global existence of strong solutions to the problem (1.3)–(1.4) is an immediate consequence of the global estimates in Proposition 2.1 and the global solutions of (1.5)–(1.6) can be obtained as the vanishing \(\varepsilon\)-limit of \((u, B, \theta)\). So, it only remains to prove the convergence rates. To do this, we define
\[ \bar{u} = u - u^0, \quad \bar{B} = B - B^0, \quad \bar{\theta} = \theta - \theta^0. \]
Then it is easily derived from (1.3)–(1.4) and (1.5)–(1.6) that
\[ \varepsilon \left( \frac{\partial \bar{u}}{\partial t} + u \cdot \nabla \bar{u} \right) + \nabla \bar{q} - \Delta \bar{u} = k \bar{\theta} + B \cdot \nabla B + B^0 \cdot \nabla B + \frac{\partial B}{\partial y} \]
\[ - \varepsilon \left( u^0_t + u^0 \cdot \nabla u^0 + \bar{u} \cdot \nabla u^0 \right), \] (2.37)
\[ \frac{\partial \vec{B}}{\partial t} + \mathbf{u} \cdot \nabla \vec{B} - \Delta \vec{B} = -\mathbf{u} \cdot \nabla \mathbf{B}^{0} + \vec{B} \cdot \nabla \mathbf{u} + \mathbf{B}^{0} \cdot \nabla \mathbf{u} + \frac{\partial \mathbf{u}}{\partial y}, \tag{2.38} \]

and
\[ \frac{\partial \bar{\theta}}{\partial t} + \mathbf{u} \cdot \nabla \bar{\theta} - \Delta \bar{\theta} = -\mathbf{u} \cdot \nabla \theta^{0}, \tag{2.39} \]

with \( \text{div} \bar{\mathbf{u}} = \text{div} \vec{B} = 0 \) and the vanishing initial-boundary conditions:
\[ (\bar{\mathbf{B}}, \bar{\theta})|_{t=0} = 0 \quad \text{and} \quad (\bar{\mathbf{u}}, \bar{\mathbf{B}}, \bar{\theta})|_{y=0,1} = 0. \]

First, multiplying (2.37) by \( \bar{\mathbf{u}} \) in \( L^{2} \) and integrating by parts, we have from Propositions 2.1 and 2.2 that
\[ \varepsilon \frac{d}{dt} \| \bar{\mathbf{u}} \|_{L^{2}}^{2} + \| \nabla \bar{\mathbf{u}} \|_{L^{2}}^{2} \leq C \left( \| \bar{\theta} \|_{L^{2}}^{2} + \| \mathbf{B} \|_{L^{2}}^{2} \right) + \left( \| \mathbf{u} \|_{L^{2}}^{2} + \| \theta^{0} \|_{L^{2}}^{2} \right) \]
\[ \leq C \varepsilon + C \left( \| \bar{\mathbf{u}} \|_{L^{2}}^{2} + \| \bar{\mathbf{B}} \|_{L^{2}}^{2} + \| \bar{\theta} \|_{L^{2}}^{2} \right), \]

and analogously,
\[ \frac{d}{dt} \left( \| \bar{\mathbf{B}} \|_{L^{2}}^{2} + \| \bar{\theta} \|_{L^{2}}^{2} \right) + \| \nabla \bar{\mathbf{B}} \|_{L^{2}}^{2} + \| \nabla \bar{\theta} \|_{L^{2}}^{2} \leq C \left( \| \bar{\mathbf{u}} \|_{L^{2}}^{2} + \| \bar{\mathbf{B}} \|_{L^{2}}^{2} + \| \bar{\theta} \|_{L^{2}}^{2} \right). \]

Owing to the Poincaré’s inequality, it holds that \( \| \bar{\mathbf{u}} \|_{L^{2}} \leq C \| \nabla \bar{\mathbf{u}} \|_{L^{2}} \). Thus,
\[ \frac{d}{dt} \left( \varepsilon \| \bar{\mathbf{u}} \|_{L^{2}}^{2} + \| \bar{\mathbf{B}} \|_{L^{2}}^{2} + \| \bar{\theta} \|_{L^{2}}^{2} \right) + \| \nabla \bar{\mathbf{u}} \|_{L^{2}}^{2} + \| \nabla \bar{\mathbf{B}} \|_{L^{2}}^{2} + \| \nabla \bar{\theta} \|_{L^{2}}^{2} \]
\[ \leq C \left( \varepsilon \| \bar{\mathbf{u}} \|_{L^{2}}^{2} + \| \bar{\mathbf{B}} \|_{L^{2}}^{2} + \| \bar{\theta} \|_{L^{2}}^{2} \right) + C \varepsilon, \]

so that
\[ \sup_{0 \leq t \leq T} \left( \varepsilon \| \bar{\mathbf{u}} \|_{L^{2}}^{2} + \| \bar{\mathbf{B}} \|_{L^{2}}^{2} + \| \bar{\theta} \|_{L^{2}}^{2} \right) (t) + \int_{0}^{T} \| \nabla \bar{\mathbf{u}}, \nabla \bar{\mathbf{B}}, \nabla \bar{\theta} \|_{L^{2}}^{2} dt \leq C \varepsilon. \tag{2.40} \]

Multiplying (2.38) and (2.39) by \( \bar{\mathbf{B}} \) and \( \bar{\theta} \) in \( L^{2} \) respectively, integrating by parts, using Propositions 2.1, 2.2 and the Poincaré’s inequality, we deduce
\[ \frac{d}{dt} \left( \| \nabla \bar{\mathbf{B}} \|_{L^{2}}^{2} + \| \nabla \bar{\theta} \|_{L^{2}}^{2} \right) + \| \bar{\mathbf{B}} \|_{L^{2}}^{2} + \| \bar{\theta} \|_{L^{2}}^{2} \]
\[ \leq C \| \mathbf{u} \|_{H^{2}} \left( \| \nabla \bar{\mathbf{B}} \|_{L^{2}} + \| \nabla \bar{\theta} \|_{L^{2}} \right) + \left( 1 + \| \mathbf{B}^{0} \|_{H^{2}} + \| \theta^{0} \|_{H^{2}} \right) \| \nabla \bar{\mathbf{u}} \|_{L^{2}}^{2} \]
\[ \leq C \left( \| \nabla \bar{\mathbf{B}} \|_{L^{2}}^{2} + \| \nabla \bar{\theta} \|_{L^{2}}^{2} + \| \nabla \bar{\mathbf{u}} \|_{L^{2}}^{2} \right), \]

and hence, by (2.40) we have
\[ \sup_{0 \leq t \leq T} \left( \| \nabla \bar{\mathbf{B}} \|_{L^{2}}^{2} + \| \nabla \bar{\theta} \|_{L^{2}}^{2} \right) (t) + \int_{0}^{T} \left( \| \bar{\mathbf{B}} \|_{L^{2}}^{2} + \| \bar{\theta} \|_{L^{2}}^{2} \right) dt \leq C \varepsilon, \tag{2.41} \]

which, together with (2.38) and (2.39), also yields
\[ \int_{0}^{T} \left( \| \nabla \bar{\mathbf{B}} \|_{L^{2}}^{2} + \| \nabla \bar{\theta} \|_{L^{2}}^{2} \right) dt \leq C \varepsilon. \tag{2.42} \]
Applying $\nabla$ to both sides of (2.38), multiplying it by $|\nabla \tilde{B}|^2 \nabla \tilde{B}$, and integrating by parts, we deduce in a manner similar to the derivation of (2.17) that

$$\frac{d}{dt} ||\nabla \tilde{B}||^4_{L^4} + ||\nabla \tilde{B}||^2 ||\nabla^2 \tilde{B}||^2_{L^2} \leq C \int (|u|^2 |\nabla \tilde{B}|^2 + |\bar{u}|^2 |\nabla \tilde{B}|^2 + |\tilde{B}|^2 |\nabla \bar{u}|^2 + |\bar{B}|^2 |\nabla \bar{u}|^2 + |\bar{u}_y|^2) \, |\nabla \tilde{B}|^2 \, dx \, dy$$

(2.43)

where we have used (2.41), Propositions 2.1–2.2, the Sobolev embedding inequality, the Poincaré’s inequality, and the following simple fact (noting that $\bar{u}$ is defined in the form: $\bar{u} = u - u_0 \in L^\infty(0, T; H^2)$)

$$||\bar{u}_y||^4_{L^4} \leq ||\bar{u}_y||^4_{L^4} + ||\bar{u}_y||^2_{L^2} ||\nabla \bar{u}_y||^2_{L^2} \leq C \|
abla \bar{u}||^2_{L^2}.$$  

Thus, it follows from (2.40), (2.43) and the Gronwall’s inequality that

$$\sup_{0 \leq t \leq T} \|
abla \tilde{B}(t)||^4_{L^4} \leq C \varepsilon.$$  

(2.44)

In the exactly same way, we also have

$$\sup_{0 \leq t \leq T} \|
abla \tilde{\theta}(t)||^4_{L^4} \leq C \varepsilon.$$  

(2.45)

Combining (2.40), (2.41), (2.44), (2.45) and the Sobolev embedding inequality, we see that

$$(B, \theta) \to (\tilde{B}, \tilde{\theta}) \text{ in } C(\Omega \times [0, T]),$$  

(2.46)

which particularly implies that there is no initial-layer between $(B, \theta)$ and $(\tilde{B}, \tilde{\theta})$ in the sense of uniform convergence.

With the help of (2.41) and Propositions 2.1–2.2, it is easy to derive the convergence of $\bar{u}$ strictly away from the initial layer. Indeed, if rewriting (2.37) in the form:

$$\nabla q - \Delta \bar{u} = -\varepsilon (u_t \cdot u + u \cdot \nabla u) + k \tilde{\theta} + B \cdot \nabla \tilde{B} + \tilde{B} \cdot \nabla B^0 + \tilde{B}_y,$$

then we can utilize the estimates of the Stokes equations to infer from (2.40), (2.41) and Proposition 2.1 that

$$||\bar{u}||^2_{H^2} \leq \varepsilon^2 (||u_t||^2_{L^2} + ||u||^4_{H^2}) + ||(B, \theta)||^2_{H^1} \leq C \varepsilon + C \varepsilon^2 ||u||^2_{L^2},$$

which, combined with (2.23), yields

$$t ||\bar{u}(t)||^2_{H^2} \leq C \varepsilon + C \varepsilon^2 t ||u_t(t)||^2_{L^2} \leq C \varepsilon \quad \text{for } \forall t \in [0, T].$$  

(2.47)

As a result, it follows from (2.37) and the Sobolev embedding inequality that

$$||\bar{u}(t)||^2_{C(\overline{\Omega})} \leq C ||\bar{u}(t)||^2_{H^2} \leq \frac{C \varepsilon}{t} \to 0, \quad \text{if } t \geq \varepsilon^{1-\alpha} \quad \text{with } \forall \alpha \in (0, 1),$$  

(2.48)

which indicates that as $\varepsilon \to 0$, $u$ converges to $u^0$ in $H^2$ strictly away from the initial layer, whose width is of the value $O(\varepsilon^{1-\alpha})$ with any $\alpha \in (0, 1)$.

Collecting (2.40), (2.42) and (2.43), (2.48) together leads to the convergence results stated in (1.9) and (1.10). The proof of Theorem 1.1 is therefore complete.  \[ \square \]
Proposition 3.1

Let \( u \) and consequently, there exists a positive constant \( C \) after integrating by parts that

\[
\|u\|_{L^2} \leq C\|\nabla u\|_{L^2} \leq C\|Au\|_{L^2},
\]

and the semigroup \( e^{-tA} \) satisfies

\[
\|e^{-tA}u\|_{L^2} \leq Ce^{-t}\|u\|_{L^2}.
\]

Moreover, if \( u = A^{-1}f \) is a solution of the problem (1.14), then it follows from (3.1) that

\[
\|\nabla u\|_{L^2}^2 = \langle Au, u \rangle = \langle f, u \rangle \leq \|f\|_{H^{-1}}\|u\|_{H^1} \leq C\|f\|_{H^{-1}}\|\nabla u\|_{L^2}
\]

and consequently,

\[
\|A^{-1}f\|_{L^2} = \|u\|_{L^2} \leq C\|\nabla u\|_{L^2} \leq C\|f\|_{H^{-1}}.
\]

With the help of (3.1)–(3.4) and the estimates of Stokes equations, we can prove that

Proposition 3.1 Let \((u^{(0)}, B^{(0)}, \theta^{(0)})\) be a smooth solution of (1.19)–(1.20) on \( \Omega \times [0, T] \). Then there exists a positive constant \( C \), such that for any \( p > 2 \),

\[
\sup_{0 \leq t \leq T} \left( \|u^{(0)}(t)\|_{H^2} + \|(B^{(0)}, \theta^{(0)})(t)\|_{H^1 \cap W^{1,p}} \right)
\]

\[
+ \int_0^T \left( \|(B^{(0)}, \theta^{(0)})(t)\|_{H^2}^2 + \|(B^{(0)}, \theta^{(0)})\|_{L^2}^2 \right) dt \leq C(p).
\]

Proof. For completeness, we sketch the proofs. First, it readily follows from (1.19) and the maximum principle that

\[
\|\theta^{(0)}(t)\|_{L^\infty} \leq C, \quad \forall t \in [0, T].
\]

Next, operating \( A \) to both sides of (1.19), multiplying it by \( u^{(0)} \) in \( L^2 \) and integrating by parts, we have from (3.1), (3.2) and the Poincaré’s and Cauchy-Schwarz’s inequalities that

\[
\|
\nabla u^{(0)}\|_{L^2}^2 \leq C \|e^{-\tau A}u_0\|_{H^1}^2 + C \|e^{-\tau A}(k\theta_0 + B_0 \cdot \nabla B_0 + \theta B_0)\|_{L^2}^2
\]

\[
+ C \left( \|\theta^{(0)}\|_{L^2}^2 + \|B^{(0)}\|_{L^2}^2 \right) + \langle B^{(0)} \cdot \nabla B^{(0)}, u^{(0)} \rangle
\]

\[
\leq C \left( 1 + \|\theta^{(0)}\|_{L^2}^2 + \|B^{(0)}\|_{L^2}^2 \right) + \langle B^{(0)} \cdot \nabla B^{(0)}, u^{(0)} \rangle.
\]

Similarly, multiplying (1.19) and (1.19) by \( B^{(0)} \) and \( \theta^{(0)} - (1 - y) \) in \( L^2 \) respectively, we obtain after integrating by parts that

\[
\frac{d}{dt} \left( \|B^{(0)}\|_{L^2}^2 + \|\theta^{(0)}\|_{L^2}^2 \right) + \|
\nabla B^{(0)}\|_{L^2}^2 + \|
\nabla \theta^{(0)}\|_{L^2}^2
\]

\[
\leq \frac{1}{2} \|
\nabla u^{(0)}\|_{L^2}^2 + C \left( 1 + \|B^{(0)}\|_{L^2}^2 \right) + \langle B^{(0)} \cdot \nabla u^{(0)}, B^{(0)} \rangle.
\]

3 Proof of Theorem [1.2]

In this section, we aim to prove Theorem [1.2] by comparing the solutions \((u^{(0)}, B^{(0)}, \theta^{(0)})\) of the problem (1.19)–(1.20) with the ones of the problems (1.3)–(1.6) and (1.3)–(1.4) successively.

3.1 Global estimates of \((u^{(0)}, B^{(0)}, \theta^{(0)})\)

This subsection is devoted to the global estimates of \((u^{(0)}, B^{(0)}, \theta^{(0)})\). To this end, we first recall some known facts of the Stokes operator \( A \). It is well known that (cf. [16, 27]) the operator \( A \) is coercive and satisfies

\[
\|\nabla u\|_{L^2} \leq C\|\nabla u\|_{L^2} \leq C\|Au\|_{L^2},
\]

and the semigroup \( e^{-tA} \) satisfies

\[
\|e^{-tA}u\|_{L^2} \leq Ce^{-t}\|u\|_{L^2}.
\]
Thus, combining (3.7) with (3.8) and integrating by parts, we infer from the Gronwall's inequality that

$$\sup_{0 \leq t \leq T} \left( \| B^{(0)} \|_{L^2}^2 + \| \theta^{(0)} \|_{L^2}^2 \right) (t) + \int_0^T \| (\nabla u^{(0)}, \nabla B^{(0)}, \nabla \theta^{(0)}) \|_{L^2}^2 \, dt \leq C. \quad (3.9)$$

In a manner similar to the derivation of (2.7), by (3.9) we have

$$\sup_{0 \leq t \leq T} \| B^{(0)}(t) \|_{L^4}^4 + \int_0^T \left( \| B^{(0)} \|_{L^2}^2 \| \nabla B^{(0)} \|_{L^2}^2 + \| \nabla B^{(0)} \|_{L^2}^2 \right) \, dt \leq C, \quad (3.10)$$

and hence, it is easily obtained from (3.7) that

$$\| u^{(0)}(t) \|_{H^1}^2 \leq C + C \| B^{(0)}(t) \|_{L^4}^4 \leq C, \quad \forall \ t \in [0, T]. \quad (3.11)$$

Thanks to (3.2) and (3.3), we have

$$\| A u^{(0)} \|_{L^2} \leq \| A e^{-\tau A} u_0 \|_{L^2} + \| e^{-\tau A} P (k \theta_0 + B_0 \cdot \nabla B_0 + \partial_y B_0) \|_{L^2} + \| P \left( k \theta^{(0)} + B^{(0)} \cdot \nabla B^{(0)} + \partial_y B^{(0)} \right) \|_{L^2} \leq C \left( 1 + \| B^{(0)} \|_{L^2}^2 + \| \nabla B^{(0)} \|_{L^2}^2 \right), \quad (3.12)$$

so that, by (3.9) and (3.10) we deduce

$$\int_0^T \| \nabla^2 u^{(0)} \|_{L^2}^2 \, dt \leq C + C \int_0^T \left( \| B^{(0)} \|_{L^2}^2 \| \nabla B^{(0)} \|_{L^2}^2 + \| \nabla B^{(0)} \|_{L^2}^2 \right) \, dt \leq C, \quad (3.13)$$

since it holds that $\| \nabla^2 u^{(0)} \|_{L^2} = \| A u^{(0)} \|_{L^2}$ (see, for example, [16, 27]).

Similarly to the proofs of (2.11), (2.12) and (2.17), by (3.9) – (3.13) we can show that

$$\sup_{0 \leq t \leq T} \| (\nabla B^{(0)}, \nabla \theta^{(0)})(t) \|_{L^2 \cap L^p}^2 + \int_0^T \left( \| (B^{(0)}, \theta^{(0)}) \|_{H^2}^2 + \| (B_t^{(0)}, \theta_t^{(0)}) \|_{L^2}^2 \right) \, dt \leq C, \quad (3.14)$$

which, combined with (3.12), yields

$$\| u^{(0)}(t) \|_{H^2} \leq C + C \left( 1 + \| B^{(0)}(t) \|_{L^\infty} \right) \| B^{(0)}(t) \|_{H^1} \leq C, \quad \forall \ t \in [0, T]. \quad (3.15)$$

Therefore, collecting (3.9) – (3.15) together finishes the proof of Proposition 3.1 \hfill \Box

### 3.2 Convergence from $(u^{(0)}, B^{(0)}, \theta^{(0)})$ to $(u^0, B^0, \theta^0)$

In this subsection, we verify that as $\varepsilon \to 0$, the solution of the effective dynamics (1.19)–(1.20) is close to the one of the infinite Prandtl number model dynamics (1.5)–(1.6), based on the global (uniform) estimates stated in Propositions 2.2 and 3.1. Indeed, it is the infinite Prandtl number dynamics if the initial-layer corrections in (1.19) are neglected.

To begin, noticing that

$$u^0 = A^{-1} P \left( k \theta^0 + B^0 \cdot \nabla B^0 + \frac{\partial B^0}{\partial y} \right),$$

...
we infer from (3.15) and (1.19) that
\[
\begin{align*}
\|u^* - u^0\|_{H^1} &\leq Ce^{-\tau} \left( \|u_0\|_{H^2} + \|B_0\|_{H^2} + \|\theta_0\|_{H^2} \right) \\
&\quad + C \left( \|\theta^*\|_{L^2} + \|B^0\|_{L^\infty} + \|B^*\|_{L^2} \right) \\
&\quad + C_1 |(\theta^*, \theta^*)|_{L^2} + C_2 e^{-\tau} \|B^*, \theta^*\|_{L^2},
\end{align*}
\] (3.19)

which, combined with (3.19) and the Cauchy-Schwarz’s inequality, yields
\[
\frac{d}{dt} \|B^*, \theta^*\|_{L^2}^2 + \|\nabla B^*, \nabla \theta^*\|_{L^2}^2 \leq C_1 \|B^*, \theta^*\|_{L^2}^2 + C_2 e^{-\tau} \|B^*, \theta^*\|_{L^2}. \quad (3.20)
\]

Keeping in mind that \(\tau = t/\varepsilon\) and that \((B^*, \theta^*)|_{t=0} = 0\), we deduce from (3.20) that
\[
\|B^*, \theta^*(t)\|_{L^2} \leq \frac{C_2 \varepsilon}{2} e^{C_1 t/\varepsilon} \quad \forall \ t \in [0, T],
\] (3.21)

which, inserted into (3.19) and combined with (3.16), shows that for any \(0 \leq t \leq T\),
\[
\|u^0 - u^0 - e^{-\tau A} u_0 + e^{-\tau A} A^{-1} \mathbb{P} (k\theta_0 + B_0 \cdot \nabla B_0 + B_0 y) \|_{H^1} \leq C \varepsilon. \quad (3.22)
\]

To prove the \(H^2\)-convergence, we first utilize (3.19), (3.21), Propositions 2.2 and 3.1 to deduce from (3.17) and (3.18) in a manner similar to the proof of (2.41) that
\[
\frac{d}{dt} \|\nabla B^*, \nabla \theta^*\|_{L^2}^2 \leq C \|\nabla B^*, \nabla \theta^*\|_{L^2}^2 + C \|u^*\|_{H^1}^2 \\
\leq C \|\nabla B^*, \nabla \theta^*\|_{L^2}^2 + C \varepsilon^2 + C e^{-2\tau},
\]
from which we find
\[ \| (\nabla B^*, \nabla \theta^*) \|_{L_2} \leq C \varepsilon. \] (3.23)
and thus, it follows directly from (3.16) that
\[ \| u^{(0)} - u^0 - e^{-rA} u_0 + e^{-rA} A^{-1} P (k \theta_0 + B_0 \cdot \nabla B_0 + B_0 y) \|_{H^2}^2 \]
\[ \leq \| A^{-1} P (k \theta^* + B^* \cdot \nabla B^{(0)} + B^0 \cdot \nabla B^* + B_y^*) \|_{H^2}^2 \]
\[ \leq C (\| \theta^* \|_{L^2} + \| B^* \|_{H^1}) \leq C \varepsilon. \] (3.24)

In short, collecting (3.21)–(3.24) together, we arrive at

**Theorem 3.1** For any given \( T > 0 \), assume that \((u^0, B^0, \theta^0)\) and \((u^{(0)}, B^{(0)}, \theta^{(0)})\) are the solutions of (1.5)–(1.6) and (1.19)–(1.20) on \( \Omega \times [0, T] \), respectively. Then, there exists a positive constant \( C \), independent of \( \varepsilon \), such that for any \( t \in [0, T] \),
\[ \| u^{(0)} - u^0 - e^{-rA} u_0 + e^{-rA} A^{-1} P (k \theta_0 + B_0 \cdot \nabla B_0 + B_0 y) \|_{H^1} \leq C \varepsilon, \] (3.25)
\[ \| u^{(0)} - u^0 - e^{-rA} u_0 + e^{-rA} A^{-1} P (k \theta_0 + B_0 \cdot \nabla B_0 + B_0 y) \|_{H^2} \leq C \varepsilon^{1/2}, \] (3.26)
and
\[ \| (B^{(0)} - B^0, \theta^{(0)} - \theta^0) \|_{L^2} \leq C \varepsilon, \quad \| \nabla (B^{(0)} - B^0, \theta^{(0)} - \theta^0) \|_{L^2} \leq C \varepsilon^{1/2}. \] (3.27)

### 3.3 Convergence from \((u, B, \theta)\) to \((u^{(0)}, B^{(0)}, \theta^{(0)})\)

This subsection aims to justify the limit from \((u, B, \theta)\) to \((u^{(0)}, B^{(0)}, \theta^{(0)})\), which is more complicated than the previous ones. To do this, let
\[ \tilde{u} = u - u^{(0)}, \quad \tilde{B} = B - B^{(0)}, \quad \tilde{\theta} = \theta - \theta^{(0)}. \]

Then it is easily derived from (1.23) and (1.19) that
\[ \varepsilon \left( \frac{\partial \tilde{u}}{\partial t} + u \cdot \nabla \tilde{u} + \tilde{u} \cdot \nabla u^{(0)} \right) + \nabla p = \Delta \tilde{u} + k \tilde{\theta} + B \cdot \nabla \tilde{B} + \tilde{B} \cdot \nabla B^{(0)} + \frac{\partial \tilde{B}}{\partial y} + f, \] (3.28)
where \( f \) is defined as follows:
\[ f \triangleq - \varepsilon \frac{\partial u^{(0)}}{\partial t} - \varepsilon u^{(0)} \cdot \nabla u^{(0)} + \Delta u^{(0)} + k \theta^{(0)} + B^{(0)} \cdot \nabla B^{(0)} + \frac{\partial B^{(0)}}{\partial y} \]
\[ = Ae^{-rA} u_0 - e^{-rA} P \left( k \theta_0 + B_0 \cdot \nabla B_0 + \frac{\partial B_0}{\partial y} \right) \]
\[ - \varepsilon A^{-1} P \frac{\partial}{\partial t} \left( k \theta^{(0)} + B^{(0)} \cdot \nabla B^{(0)} + \frac{\partial B^{(0)}}{\partial y} \right) \]
\[ - \varepsilon u^{(0)} \cdot \nabla u^{(0)} - A u^{(0)} + k \theta^{(0)} + B^{(0)} \cdot \nabla B^{(0)} + \frac{\partial B^{(0)}}{\partial y} + \nabla p^{(0)} \]
\[ = - \varepsilon A^{-1} P \frac{\partial}{\partial t} \left( k \theta^{(0)} + B^{(0)} \cdot \nabla B^{(0)} + \frac{\partial B^{(0)}}{\partial y} \right) - \varepsilon u^{(0)} \cdot \nabla u^{(0)} + \nabla p^{(0)}. \] (3.29)
In the following, we prove the convergence of \( \tilde{u} \). First, multiplying (3.28) by \( \tilde{u} \) in \( L^2 \) and integrating by parts, we deduce

\[
\frac{\varepsilon}{2} \frac{d}{dt} \| \tilde{u} \|^2_{L^2} + \| \nabla \tilde{u} \|^2_{L^2} = -\varepsilon \langle u \cdot \nabla u(0), \tilde{u} \rangle + \langle k\tilde{\theta} + B \cdot \nabla B + \tilde{B} \cdot \nabla B(0) + \tilde{B}_y, \tilde{u} \rangle \\
- \varepsilon \langle A^{-1}P \frac{\partial}{\partial t}(k\tilde{\theta}(0) + B(0) \cdot \nabla B(0) + B_y(0)), \tilde{u} \rangle \triangleq \sum_{i=1}^{3} I_i,
\]

where we have used the facts that \( \operatorname{div} \tilde{u} = 0 \), \( \tilde{u}|_{y=0,1} = 0 \) and \( u = \tilde{u} + u(0) \). Using Propositions 2.1, 3.1 and the Poincaré’s inequality, we have

\[
I_1 \leq C\varepsilon \| u \|_{L^\infty} \| \nabla u(0) \|_{L^2} \| \tilde{u} \|_{L^2} \leq C\varepsilon \| \nabla \tilde{u} \|_{L^2} \leq \frac{1}{4} \| \nabla \tilde{u} \|^2_{L^2} + C\varepsilon^2,
\]

and similarly,

\[
I_2 \leq C \left( \| \tilde{\theta} \|_{L^2} \| \tilde{u} \|_{L^2} + \| (B, B(0)) \|_{L^\infty} \| \tilde{B} \|_{L^2} \| \nabla \tilde{u} \|_{L^2} + \| \tilde{B} \|_{L^2} \| \nabla \tilde{u} \|_{L^2} \right) \\
\leq \frac{1}{4} \| \nabla \tilde{u} \|^2_{L^2} + C \left( \| \tilde{B} \|^2_{L^2} + \| \tilde{\theta} \|^2_{L^2} \right).
\]

In view of (3.3) and Proposition 3.1 we obtain after integrating by parts that

\[
I_3 \leq C\varepsilon \left( \| \tilde{\theta}(0) \|_{H^{-1}} \| \tilde{u} \|_{L^2} + \| B(0) \|_{L^\infty} \| B^0 \|_{H^{-1}} \| \nabla \tilde{u} \|_{L^2} + \| B^0 \|_{H^{-1}} \| \tilde{B} \|_{L^2} \| \nabla \tilde{u} \|_{L^2} \right) \\
\leq \frac{1}{4} \| \nabla \tilde{u} \|^2_{L^2} + C\varepsilon^2 \left( \| \tilde{\theta}(0) \|^2_{H^{-1}} + \| B^0 \|^2_{H^{-1}} \right) \\
\leq \frac{1}{4} \| \nabla \tilde{u} \|^2_{L^2} + C\varepsilon^2,
\]

where we have used (1.19) and Proposition 3.1 to get that

\[
\| (B^0, \tilde{\theta}(0)) \|_{H^{-1}} \leq \| (B(0), \tilde{\theta}(0)) \|_{H^1} + \| u(0) \|_{L^\infty} \| (B(0), \tilde{\theta}(0)) \|_{L^2} + \| u(0) \|_{L^2} \leq C.
\]

Thus, substituting the estimates of \( I_i \) \( (i = 1, 2, 3) \) into (3.30), we arrive at

\[
\frac{\varepsilon}{2} \frac{d}{dt} \| \tilde{u} \|^2_{L^2} + \| \nabla \tilde{u} \|^2_{L^2} \leq C \left( \| \tilde{B} \|^2_{L^2} + \| \tilde{\theta} \|^2_{L^2} \right) + C\varepsilon^2.
\]

Clearly, we need to deal with \( \| (\tilde{B}, \tilde{\theta}) \|^2_{L^2} \). Indeed, by (1.3) and (1.19) we have

\[
\begin{align*}
\frac{\partial \tilde{B}}{\partial t} + u \cdot \nabla \tilde{B} - \Delta \tilde{B} & = -\tilde{u} \cdot \nabla B(0) + \tilde{B} \cdot \nabla u + B(0) \cdot \nabla \tilde{u} + \frac{\partial \tilde{u}}{\partial y}, \\
\frac{\partial \tilde{\theta}}{\partial t} + u \cdot \nabla \tilde{\theta} - \Delta \tilde{\theta} & = -\tilde{u} \cdot \nabla \theta(0),
\end{align*}
\]

from which we easily get that

\[
\frac{d}{dt} \| (\tilde{B}, \tilde{\theta}) \|^2_{L^2} + \| (\nabla \tilde{B}, \nabla \tilde{\theta}) \|^2_{L^2} \leq C \left( 1 + \| (u, B(0), \theta(0)) \|_{L^\infty}^2 \right) \| (\tilde{u}, \tilde{B}) \|^2_{L^2} \\
\leq C \| (\tilde{u}, \tilde{B}) \|^2_{L^2},
\]

20
where we have used Propositions 2.1 and 3.1. Thus, for any $0 \leq t \leq T$ one has
\begin{align}
\|(\bar{B}, \bar{\theta})(t)\|^2_{L^2} + \int_0^t \|(\nabla \bar{B}, \nabla \bar{\theta})\|^2_{L^2}ds &\leq C \int_0^t \|\bar{u}\|^2_{L^2}ds. \tag{3.33}
\end{align}

In view of (3.31), (3.33) and the Poincaré’s inequality, we have
\begin{align*}
\frac{d}{dt} \left( e^{t/\varepsilon} \|\bar{u}\|^2_{L^2} \right) &\leq e^{-1} e^{t/\varepsilon} \int_0^t \|\bar{u}\|^2_{L^2}ds + C \varepsilon e^{t/\varepsilon},
\end{align*}
which, integrated in time, yields (noting that $\|\bar{u}\|_{t=0} = 0$)
\begin{align*}
\|\bar{u}(t)\|^2_{L^2} &\leq C \int_0^t \|\bar{u}\|^2_{L^2}ds + C \varepsilon ^2,
\end{align*}
and hence, by Gronwall’s inequality we see that
\begin{align}
\|\bar{u}(t)\|^2_{L^2} &\leq C \varepsilon ^2, \quad \forall t \in [0, T]. \tag{3.34}
\end{align}
This, combined with (3.31) and (3.33), also leads to
\begin{align}
\sup_{0 \leq t \leq T} \|\bar{u}(t)\|^2_{L^2} + \int_0^T \|(\nabla \bar{u}, \nabla \bar{B}, \nabla \bar{\theta})\|^2_{L^2}dt &\leq C \varepsilon ^2. \tag{3.35}
\end{align}

Next, multiplying (3.32) by $\bar{B}_t$ and $\bar{\theta}_t$ in $L^2$ respectively, integrating by parts, using Propositions 2.1 and 3.1, and the Poincaré’s inequality, we obtain
\begin{align*}
\frac{d}{dt} \left( \|\nabla \bar{B}\|^2 + \|\nabla \bar{\theta}\|^2 \right) + \left( \|\bar{B}_t\|_{L^2}^2 + \|\bar{\theta}_t\|_{L^2}^2 \right) &\leq C \|u\|_{L^\infty} \left( \|\nabla \bar{B}\|^2 + \|\nabla \bar{\theta}\|^2 \right) + C \|\nabla \bar{u}\|^2_{L^2} + C \|\bar{B}(0)\|_{L^\infty} \|\nabla \bar{u}\|^2_{L^2} \\
&\quad + C \left( \|\nabla \bar{B}(0)\|^2_{L^2} + \|\nabla \bar{\theta}(0)\|^2_{L^2} \right) \|\bar{u}\|^2_{L^2} + C \|\nabla \bar{u}\|^2_{L^2} \|\bar{B}\|^2_{L^4} \\
&\leq C \left( \|\nabla \bar{B}\|^2_{L^2} + \|\nabla \bar{\theta}\|^2_{L^2} + \|\nabla \bar{u}\|^2_{L^2} \right),
\end{align*}
which, combined with (3.35), results in
\begin{align}
\sup_{0 \leq t \leq T} \left( \|\nabla \bar{B}\|^2_{L^2} + \|\nabla \bar{\theta}\|^2 \right) + \int_0^T \left( \|\bar{B}_t\|^2_{L^2} + \|\bar{\theta}_t\|^2_{L^2} \right) dt \leq C \varepsilon ^2, \tag{3.36}
\end{align}
and moreover, it follows from (3.32) that
\begin{align}
\int_0^T \left( \|\bar{B}\|^2_{H^2} + \|\bar{\theta}\|^2_{H^2} \right) dt \leq C \varepsilon ^2. \tag{3.37}
\end{align}

Similarly, multiplying (3.28) by $\bar{u}_t$ in $L^2$ and integrating by parts, we obtain
\begin{align*}
\frac{d}{dt} \|\nabla \bar{u}\|^2_{L^2} + \varepsilon \|\bar{u}_t\|^2_{L^2} &\leq C \left( \varepsilon \|(u, u(0))\|_{H^1} \|\bar{u}\|_{H^1} + \varepsilon \|u(0)\|^2_{H^2} + \|\bar{\theta}\|_{L^2} + \|\bar{B}\|_{H^1} \right) \|\bar{u}_t\|_{L^2} \\
&\quad + C \left( \|B\|_{L^\infty} \|\bar{B}\|_{H^1} + \|\nabla B(0)\|_{L^4} \|\bar{B}\|_{H^1} \right) \|\bar{u}_t\|_{L^2} \\
&\quad + C \varepsilon \left( \|\bar{B}(0)\|^2_{L^2} + \|\bar{\theta}(0)\|^2_{L^2} + |A^{-1} \bar{B}(0) + B(0) - B_y(0)|^2 \right) \|\bar{u}_t\|_{L^2} \\
&\leq C \varepsilon \left( 1 + \|\bar{\theta}(0)\|^2_{H^{-1}} + \|B(0)\|_{L^\infty} \|\bar{B}(0)\|^2_{L^2} + \|\bar{B}(0)\|^2_{L^2} \right) \|\bar{u}_t\|_{L^2} \\
&\leq \frac{\varepsilon}{2} \|\bar{u}_t\|^2_{L^2} + C \varepsilon \left( 1 + \|\bar{B}(0)\|^2_{L^2} \right),
\end{align*}
and...
where Propositions 2.1, 3.1, 3.3 and (3.34)–(3.36) were used. Hence, by (3.37) we find
\[ \sup_{0 \leq t \leq T} \| \nabla \tilde{u}(t) \|_{L^2}^2 + \varepsilon \int_0^T \| \tilde{u}_t(t) \|_{L^2}^2 dt \leq C \varepsilon + C \varepsilon \int_0^T \| B_t^{(0)} \|_{L^2}^2 dt \leq C \varepsilon. \]  

(3.38)

Finally, it remains to derive the $H^2$-convergence of $\tilde{u}$, which is more subtle and needs more work. To do this, let
\[ v \triangleq u - u^{(0)} + A^{-1}p \left( k\theta^{(0)} + B^{(0)} \cdot \nabla B^{(0)} + B_y^{(0)} \right). \]

It is easy to deduce from (1.3)
\[ 1 \leq \frac{\int_{\Omega} \nabla v \cdot \nabla |u| \, dx}{\int_{\Omega} |\nabla v|^2 \, dx} \leq \frac{1}{\varepsilon} \leq \frac{1}{\varepsilon} \parallel u \parallel_{H^1}^2. \]

Moreover, by direct calculations we have
\[
\begin{align*}
\varepsilon v_t + Av &= p \left( k\theta + B \cdot \nabla B + B_y - \varepsilon u \cdot \nabla u \right).
\end{align*}
\]

(3.39)

and hence,
\[ v_{|t=0} = \lim_{t \to 0} v_t = p(u_0 \cdot \nabla u_0) \in H^1. \]  

(3.40)

Now, differentiating (3.39) with respect to $t$ and multiplying it by $v_t$ in $L^2$, we deduce after integrating by parts that
\[
\frac{\varepsilon}{2} \frac{d}{dt} \| v_t \|_{L^2}^2 + \| \nabla v_t \|_{L^2}^2 = \langle (k\theta + B \cdot \nabla B + B_y - \varepsilon u \cdot \nabla u)_t, v_t \rangle \triangleq I,
\]  

(3.41)

where the terms on the right-hand side can be estimated as follows, using Proposition 2.1 and the Poincaré’s inequality.
\[
\begin{align*}
|I| &\leq |\langle k\theta_t, v_t \rangle| + |\langle B_t \cdot \nabla v_t, B \rangle| + |\langle B \cdot \nabla v_t, B_t \rangle| \\
&+ \varepsilon |\langle B_t, v_{ty} \rangle| + \varepsilon |\langle u_t \cdot \nabla v_t, u \rangle| + \varepsilon |\langle u \cdot \nabla v_t, u_t \rangle| \\
&\leq \frac{1}{2} \| \nabla v_t \|_{L^2}^2 + C \left( \| (B_t, \theta_t) \|_{L^2}^2 + \varepsilon^2 \| u_t \|_{L^2}^2 \right)
\end{align*}
\]

Thus, by virtue of (3.40) and Proposition 2.1 we get that
\[ \varepsilon \sup_{0 \leq t \leq T} \| v_t \|_{L^2}^2 + \int_0^T \| \nabla v_t \|_{L^2}^2 dt \leq C. \]  

(3.42)

Using (2.24), Proposition 3.1 and (3.4), we infer from (3.42) that
\[
\begin{align*}
\varepsilon \| \tilde{u}_t \|_{L^2}^2 &\leq C \| v_t \|_{L^2}^2 + \varepsilon A^{-1}p(kt^{(0)} + B^{(0)} \cdot \nabla B^{(0)} + B_y^{(0)})_t \|_{L^2}^2 \\
&\leq C + C \varepsilon \left( \| \theta_t^{(0)} \|_{H^{-1}} + \| B^{(0)} \|_{L^\infty} \| B_t^{(0)} \|_{L^2} + \| B_t^{(0)} \|_{L^2} \right) \\
&\leq C + C \varepsilon \left( 1 + \| B_t^{(0)} \|_{L^2} \right).
\end{align*}
\]  

(3.43)
Similarly to the derivation of (2.21), by Proposition 3.1 we have
\[
\frac{1}{2} \frac{d}{dt} \| B_t(0) \|_{L^2}^2 + \| \nabla B_t(0) \|_{L^2}^2 = \langle B(0) \cdot \nabla u(0), B_t(0) \rangle + \langle B_t(0) \cdot \nabla u(0), B_t(0) \rangle \\
+ \langle u_t(0), B_t(0) \rangle - \langle u_t(0), \nabla B(0), B_t(0) \rangle \\
= - \langle B(0) \cdot \nabla B_t(0), u(0) \rangle - \langle B_t(0) \cdot \nabla B_t(0), u(0) \rangle \\
- \langle u_t(0), B_t(0) \rangle + \langle u_t(0), \nabla B(0), B(0) \rangle \\
\leq \frac{1}{2} \| \nabla B_t(0) \|_{L^2}^2 + C \left( \| u_t(0) \|_{L^2}^2 + \| B_t(0) \|_{L^2}^2 \right) \\
\leq \frac{1}{2} \| \nabla B_t(0) \|_{L^2}^2 + C \left( \| \tilde{u}_t \|_{L^2}^2 + \| u_t \|_{L^2}^2 + \| B_t(0) \|_{L^2}^2 \right).
\]

Inserting (3.43) into (3.44) and using (2.15), by Gronwall’s inequality we obtain
\[
\varepsilon \sup_{0 \leq t \leq T} \| B_t(0)(t) \|_{L^2}^2 + \varepsilon \int_0^T \| \nabla B_t(0) \|_{L^2}^2 dt \leq C + C \varepsilon \int_0^T \| u_t \|_{L^2}^2 dt \leq C,
\]
which, together with (3.43), immediately gives
\[
\varepsilon \sup_{0 \leq t \leq T} \| \tilde{u}(t) \|_{L^2}^2 \leq C.
\]

Now, using Propositions 2.1, 3.1, (3.34)–(3.38), (3.45), (3.46) and the estimates of the Stokes equations, we deduce from (3.28) and (3.29) that for any \( 0 \leq t \leq T \),
\[
\| \nabla^2 \tilde{u} \|_{L^2}^2 \leq C \left( \| \tilde{\theta} \|_{L^2}^2 + \| B \|_{L^2}^2 + \| \nabla B(0) \|_{L^2}^2 \right) \\
+ C \varepsilon^2 \left( \| \tilde{u}_t \|_{L^2}^2 + \| (u, u(0)) \|_{H^2}^2 \| \tilde{u} \|_{H^1}^2 + \| u(0) \|_{H^2}^2 \right) \\
+ C \varepsilon^2 \| A^{-1}P(k\theta(0) + B(0) \cdot \nabla B_0 + B_0(0)) \|_{L^2}^2 \\
\leq C \varepsilon^2 + C \varepsilon^2 \left( 1 + \| \tilde{u}_t \|_{L^2}^2 + \| \tilde{u} \|_{H^1}^2 + \| \theta(0) \|_{H^1}^2 + \| B_t(0) \|_{L^2}^2 \right) \\
\leq C \varepsilon.
\]

Thus, collecting (3.34)–(3.36), (3.38) and (3.47) together, we conclude that

**Theorem 3.2** Let \((u, B, \theta)\) and \((u(0), B(0), \theta(0))\) be the solutions of (1.3)–(1.4) and (1.19)–(1.20) on \( \Omega \times [0, T) \) with \( 0 < T < \infty \), respectively. Then, there exists a positive constant \( C \), independent of \( \varepsilon \), such that for any \( t \in [0, T) \),
\[
\| u - u(0) \|_{L^2} + \| (B - B(0), \theta - \theta(0)) \|_{H^1} \leq C \varepsilon
\]
and
\[
\| \nabla (u - u(0)) \|_{L^2} + \| \nabla^2 (u - u(0)) \|_{L^2} \leq C \varepsilon^{1/2}.
\]

**Proof of Theorem 1.2** Now, combining Theorems 3.1 and 3.2 we readily obtain the desired convergence results stated in (1.21)–(1.23) of Theorem 1.2. \( \square \)

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