Estimation the components of deflection of the vertical for a point in Baghdad City – Iraq

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Abstract. In the age of the satellite geodesy, the GNSS is widely used in development of geodetic networks and that give much application, for that reason the study is to estimate deflection of the vertical of a point in Baghdad City in Iraq, the estimated depends on the orthometric height, which computed through the levelling and the ellipsoidal height, which computed from the GPS observations. Due to the lack of survey data and the difficulty of taking terrestrial measurements as a result of the security situation the authors used only 5 points for the national network in Baghdad. The deflection of the vertical component ξ in the north-south direction calculated as ξ=2.5"±0.001" and the prime vertical component η in the east-west direction calculated as η=5.23" ±1.5". The results were obtained showing that it can calculate the deflection of the vertical at sufficient accuracy.

Keywords: GNSS, Leveling, Estimation, Deflection of the vertical.

1. Introduction
The deflection of the vertical ε is the angle between the direction of gravity at a certain point and the direction of the normal on the ellipsoidal at the same point. The surveyors need to compute the deflection of the vertical and applied it on the observations such as slope distance and azimuths to reduction onto the ellipsoid, and it has many applications in civil engineering that require knowledge of the geoid, such as the direction of slope for highway construction, water flow direction for sewers, drains, and dams. The deflection of the vertical ε has two components: meridional component ξ in the north-south direction (it has a positive sign in the north direction) and the prime vertical component η in the east-west direction (it has a positive sign in the east direction), so that the elevation is rising if the two components are positive and direction of the gravity in the south and west directions of the ellipsoidal normal at the same point[1, 2].

The deflection can be computed in classic geodesy via astronomical observations or gravitational measurements[3]. The first method used the astronomical coordinates (Φ,Λ) together with the geodetic coordinates (φ,λ) to compute the deflection of the vertical ε, this method is expensive and needs professional surveyors and knowing the theory and practice of (astronomy, triangulation, …) and knowing using terrestrial instruments (high precise theodolites, Electronics Distance Measurement EDMs). The Gravimetric methods depend on gravity anomalies that computed by Stokes’ formula. In the era of satellites navigation, the GNSS has the availability to measure the ellipsoidal height, and the orthometric heights can be acquired through the spirit leveling, this opens the door for researches to investigate the availability of determining the deflection of the vertical from this composition of GNSS ellipsoidal height and orthometric heights through leveling observations.

This study aims to estimate the deflection of the vertical components at a point Baghdad city using GPS and leveling, the data used are produced by many previous works and from the general authority of survey-Iraq.
2. Materials and methods

The relationship between the deflection and the geoid height are presented by Heiskanen and Moritz 1967, through the following formula:

\[ dN = -\varepsilon ds \]  

Or

\[ \varepsilon = -\frac{dN}{ds} \]  

The relationship between the deflection of the vertical \( \varepsilon \) and its two components: meridional component \( \xi \) and the prime vertical component \( \eta \) and the geodetic azimuth \( \alpha \), through the following formula:

\[ \varepsilon = \xi \cos \alpha + \eta \sin \alpha \]  

Merge equation 2 and 3, the result will be:

\[ -\frac{dN}{ds} = \xi \cos \alpha + \eta \sin \alpha \]  

Figure 1. Spherical definition of deflection components [4].

Figure 2. Relationship between deflection of the vertical and geoid height [5].
Referring to figure 2, for short distances the differential quantities $dN, ds$ in formula 4 can be substituted by different quantities $\Delta N$ geoid undulation and $\Delta s$ spheroid distance, which can be obtained from geodetic measurements.

$$-\frac{\Delta N}{\Delta s} \approx \xi \cos \alpha + \eta \sin \alpha$$

(5)

The geoid undulation for any two points (P, Q) that close to each other on the earth surface can be computed using the following formula:

$$N_p = h_p - H_p$$

$$N_Q = h_Q - H_Q$$

(6)

(7)

The geoid height difference in equation 5 can be derived by subtracting equation 6 from 7.

$$\Delta N_{PQ} = N_p - N_Q = h_p - H_p - (h_Q - H_Q) = (h_p - h_Q) - (H_p - H_Q) = \Delta h_{PQ} - \Delta H_{PQ}$$

(8)

Equation 8 is the geoid height difference, and can be substituted in equation 5, so we obtained;

$$-\frac{\Delta h_{PQ} - \Delta H_{PQ}}{\Delta s_{PQ}} \approx \xi \cos \alpha_{PQ} + \eta \sin \alpha_{PQ}$$

(9)

The terms in equation 9 are; the value of $\Delta h_{PQ}$ is the orthometric height, which computed through the precise leveling. The value of $\Delta H_{PQ}$ is the ellipsoidal height, which computed from the GPS observations. The values $\Delta s_{PQ}$ and $\alpha_{PQ}$ are the geodetic distance and azimuth respectively, are computed from geodetic coordinates. The only unknowns terms are the components of the deflection of the vertical $\xi$ and $\eta$. We can see from equation 9 that the values of $\xi$ and $\eta$ can be computed for any point such as P, we need another point to calculate the differences in $\Delta h$, $\Delta H$, and $\Delta s$.

The errors in the deflection can be computed by considering that the variables $\Delta h, \Delta H$ are not correlated, and apply the theory of error propagation on left-hand terms of equation 9.

$$\sigma_\xi^2 = (\frac{\partial f}{\partial x_1})^2 \sigma_{x_1}^2 + (\frac{\partial f}{\partial x_2})^2 \sigma_{x_2}^2 + \cdots + (\frac{\partial f}{\partial x_n})^2 \sigma_{x_n}^2$$

[6]

$$\Delta h_{PQ} - \Delta H_{PQ}$$

(10)

$$\Delta s_{PQ}$$

(11)

$$\sigma_\epsilon^2 = \frac{1}{\Delta s^2} (\sigma_{\Delta h}^2 + \sigma_{\Delta H}^2) + \left(\frac{\Delta h - \Delta H}{\Delta s^2}\right)^2 \sigma^2_s$$

(12)

According to (Tse and Iz, 2006), equation 11 can be generalized as follow:

$$\sigma_\epsilon^2 = \frac{1}{\Delta s^2} (\sigma_{\Delta h}^2 + \sigma_{\Delta H}^2)$$

(13)

We can see from equation (12) that the standard deviation of the deflection of the vertical is inversely proportional to the distance and direct proportion to the difference in GPS and leveling heights.

Table 1 shows how the errors in the GPS and leveling measurements and the distance can affect the accuracy of the deflection.

| $S$ (km) | $\sigma_\epsilon$ | $\sigma_{\Delta h} = \pm 1$mm | $\sigma_{\Delta H} = \pm 1$mm |
|---------|-------------------|-----------------------------|-----------------------------|
| 1 km    | $\pm 0.29''$      | $\pm 0.53''$                | $\pm 1.46''$                |
| 2 km    | $\pm 0.15''$      | $\pm 0.73''$                | $\pm 2.31''$                |
| 5 km    | $\pm 0.06''$      | $\pm 0.21''$                | $\pm 1.51''$                |
| 10 km   | $\pm 0.03''$      | $\pm 0.15''$                | $\pm 0.46''$                |
| 20 km   | $\pm 0.01''$      | $\pm 0.07''$                | $\pm 0.23''$                |
3. **Study Area:**
The study area is 5 national network points in Baghdad city and the desired point (P) located at the Campus of University of Technology – Iraq. Table 2. contains the necessary data for calculation of the deflection such as orthometric and ellipsoidal height differences between the network point and point P.

| from   | To    | Azimuth (α) ° ′ ″ | Distance (S) (m) | ΔH (m) | Δh (m) |
|--------|-------|--------------------|------------------|--------|--------|
| 20108  | 20081 | 24 18 29.1531      | 9425.9108        | 23.966 | 24.23  |
| 20081  | 20073 | 133 43 00.3987     | 9624.5400        | 27.666 | 27.73  |
| 20073  | 20080 | 234 15 50.2667     | 17231.7500       | 7.98   | 7.23   |
| 20080  | 20098 | 243 02 26.6666     | 5860.3696        | 53.571 | 52.93  |
| 20098  | P     | 46 26 19.3554      | 35060.1669       | 1.286  | 2.33   |

4. **Results and discussion**
The calculations of the deflection components depend on equation (9). For statistical model equation (9) can be rewritten as:

\[
\frac{1}{\Delta s_{pq}} \Delta h_{pq} - \frac{1}{\Delta s_{pq}} \Delta H_{pq} + \xi \cos \alpha_{pq} + \eta \sin \alpha_{pq} + \frac{\Delta h_{pq} - \Delta H_{pq}}{\Delta s_{pq}} = 0
\]

We can see that equation (14) include the two unknowns components of the deflection of the vertical, which can be estimated using the least squares observation method, and include observations of the height difference of the orthometric and ellipsoidal height, which can be adjusted using least-squares condition method. This type of equation can be adjusted using the general least squares method [8, 9].

\[
B.V + AX + W = 0
\]

Or in matrix form

\[
\begin{bmatrix}
N & A \\
A^T & 0
\end{bmatrix}
\begin{bmatrix}
K \\
X
\end{bmatrix}
+
\begin{bmatrix}
W \\
0
\end{bmatrix}
= 0
\]

\[
K = -N^{-1}A^T W
\]

Where

\[
N = BP^{-1}B^T
\]

\[
X = -[A^T N^{-1} A]^{-1} A^T N^{-1} W
\]

\[
K = -N^{-1}(AX + W)
\]

Where the order of B matrix is 5 × 10, while the order of A matrix is 5 × 2. The V matrix is the correction matrix and X represents the estimated parameters.

The authors assumed that the observations of orthometric and ellipsoidal height differences were uncorrelated such that the weight matrix:

\[
P = \begin{bmatrix}
\frac{1}{\sigma_{h}^2} & 0 \\
0 & \frac{1}{\sigma_{H}^2}
\end{bmatrix}
\]

After applying the general least squares, the estimated parameters of the components of the deflection of the vertical were 2.5”, 5.23”.

To compute the standard error of unit weight:
\[ \sigma_o = \sqrt{\frac{V^T PV}{r - u}} \]  \hspace{1cm} (22)

Where

\[ V^T PV = -K^T W \]  \hspace{1cm} (23)

\( r \) = number of conditions
\( u \) = number of unknown parameters

The standard error of unit weight was 0.82 [10].

To compute the weight coefficient matrix of the most probable value:

\[ Q_{XX} = [A^T N^{-1} A]^{-1} \]  \hspace{1cm} (24)

And finally the standard error of the parameters:

\[ \sigma_i = \sigma_o \sqrt{q_{ii}} \]  \hspace{1cm} (25)

And it was 0.001” and 1.5”

Thus the final results of the components of the deflection of the vertical were \( \xi = 2.5'' \pm 0.001 '' \) and \( \eta = 5.23'' \pm 1.5'' \)

5. Conclusions

This study is investigating the feasibility of computing the deflection of the vertical depending on the orthometric height, which computed through the leveling and the ellipsoidal height, which computed from the GPS observations.

The deflection of the vertical component \( \xi \) in the north-south direction calculated as \( \xi = 2.5'' \pm 0.001'' \) and the prime vertical component \( \eta \) in the east-west direction calculated as \( \eta = 5.23'' \pm 1.5'' \). The general least squares method was used to compute the estimated components because the derived equation to compute the deflection of the vertical have two unknowns components of the deflection of the vertical, which can be estimated using the least squares observation method, and include observations of the height difference of the orthometric and ellipsoidal height, which can be adjusted using least-squares condition method. Iraq has little data for terrestrial gravity so the authors did not make any comparison with other methods.

6. References

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