GRADINGS BY GROUPS ON MELIKYAN ALGEBRAS

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Abstract. In this paper we describe all gradings by abelian groups without
elements of order five on the Melikyan algebras over algebraically closed fields
of characteristic five.

1. Introduction

Let $A$ be an algebra over a field $F$, $G$ a group and Aut $A$, Aut $G$ the automor-
phism groups of $A$ and $G$, respectively. The base field $F$ will always be algebraically
closed. The field will be of characteristic five when dealing with Melikyan algebras.

Definition 1. A grading $\Gamma$ by a group $G$ on an algebra $A$, also called a $G$-grading,
is a decomposition $\Gamma : A = \bigoplus_{g \in G} A_g$ where each $A_g$ is a subspace such that
$[A_g, A_g'] \subset A_{g'g''}$ for all $g', g'' \in G$. For each $g \in G$, we call the subspace $A_g$
the homogeneous space of degree $g$. The set $\text{Supp}_\Gamma A = \{g \in G \mid A_g \neq 0\}$ is called
the support of the grading.

For a grading by a group $G$ on a simple Lie algebra $L$, it is well known that
the subgroup generated by the support is abelian [6, Lemma 2.1]. If $L$ is finite-
dimensional and the support generates $G$ we have that $G$ is finitely generated.

Definition 2. Two gradings $A = \bigoplus_{g \in G} A_g$ and $A' = \bigoplus_{h \in H} A'_h$
of an algebra $A$ are called equivalent if there exist $\Psi \in \text{Aut } A$ and $\theta \in \text{Aut } G$ such that
$\Psi(A_g) = A'_{\theta(g)}$ for all $g \in G$. If $\theta$ is the identity, we call the gradings isomorphic.

Definition 3. Let $A = \bigoplus_{g \in G} A_g$ be a grading by a group $G$ on an algebra $A$ and
$\varphi$ a group homomorphism of $G$ onto $H$. The coarsening of the $G$-grading induced
by $\varphi$ is the $H$-grading defined by $A = \bigoplus_{h \in H} A_h$ where
$$A_h = \bigoplus_{g \in G, \varphi(g) = h} A_g.$$

The task of finding all gradings on simple Lie algebras by finite groups in the
case of algebraically closed fields of characteristic zero is almost complete — see [10]
and also [1, 2, 3, 4, 5, 6, 7, 8, 9]. In the case of positive characteristic $p$, a description
of gradings on the classical simple Lie algebras, with certain exceptions, has been
obtained in [1, 2]. In the case of simple graded Cartan type Lie algebras, the
gradings by $\mathbb{Z}$ have been described in [17]. It was shown in [15] that all gradings by
groups without elements of order $p$ on the graded simple Cartan type Lie algebras,
up to isomorphism, fall into the category of what we call standard gradings (which
are coarsenings of the standard $\mathbb{Z}^k$-gradings). In [15] the gradings by arbitrary
groups on the Witt algebra $W(1;1)$ were described. The gradings on the restricted
Witt and special algebras have been announced recently by Bal'turin and Kochetov.
in [2]. This paper will deal with the gradings on the Melikyan algebras by arbitrary abelian groups with no elements of order five in the case where the base field $F$ is assumed to be algebraically closed and $p = 5$. We use the notation of [17].

Our main result is the following.

**Theorem 1.** Let $L$ be a Melikyan algebra over an algebraically closed field. Suppose $L$ is graded by a group $G$, the support generates $G$ and $G$ has no elements of order 5. Then the grading is isomorphic to a standard $G$-grading.

The correspondence between the gradings on an algebra by finite abelian groups of order coprime to the characteristic $p$ of the field and finite abelian subgroups of the automorphisms of this algebra is well known. Using the theory of algebraic groups, this extends to infinite abelian groups. Namely, a grading on an algebra $L = \bigoplus_{g \in G} L_g$ by a finitely generated abelian group without elements of order $p$ gives rise to an embedding of the dual group $\hat{G}$ into Aut $L$ using the following action:

$$\chi \ast y = \chi(g)y, \quad \text{for all } y \in L_g, \quad g \in G, \quad \chi \in \hat{G}. $$

We will denote this embedding by $\eta : \hat{G} \to \text{Aut } L$, so

$$\eta(\chi)(y) = \chi \ast y. \tag{1}$$

**Lemma 1.** Let $G$, $H$ be groups, $A$ an algebra and $\phi : G \to H$ be a group homomorphism, $\Gamma : A = \bigoplus_{g \in G} A_g$ be a $G$-grading and $\Gamma : A = \bigoplus_{h \in H} \mathfrak{A}_h$ be the $H$-grading defined by $\mathfrak{A}_h = \bigoplus_{g \in G, \ h = \phi(g)} A_g$. Then $\eta_\Gamma(H) \subset \eta_\Gamma(\hat{G})$ where the homomorphisms $\eta_\Gamma : \hat{G} \to \text{Aut } A$ and $\eta_\Gamma : H \to \text{Aut } A$ are defined by (1) with respect to the gradings $\Gamma$ and $\hat{\Gamma}$ respectively.

**Proof.** Let $\chi \in \hat{H}$. For $y \in A_g$ we have $\eta_\Gamma(\chi)(y) = \chi(\phi(g))y$ since $A_g \subset \mathfrak{A}_{\phi(g)}$. Let $\zeta : G \to F^\times$ be the map defined by $\zeta(g) = \chi(\phi(g))$ for all $g \in G$. Then

$$\zeta(g_1g_2) = \chi(\phi(g_1g_2)) = \chi(\phi(g_1)\phi(g_2)) = \chi(\phi(g_1))\chi(\phi(g_2)) = \zeta(g_1)\zeta(g_2)$$

for all $g_1, g_2 \in G$. Hence $\zeta \in \hat{G}$. Furthermore, for all $y \in A_g$ we have

$$\eta_\Gamma(\chi)(y) = \chi(\phi(g))y = \zeta(g)y = \eta_\Gamma(\zeta)(y).$$

Hence $\eta_\Gamma(\chi) \in \eta_\Gamma(\hat{G})$. $\square$

**Lemma 2.** Let $G$, $H$ be abelian groups without elements of order $p$, $A$ an algebra $\Gamma : A = \bigoplus_{g \in G} A_g$ be a $G$-grading and $\hat{\Gamma} : A = \bigoplus_{h \in H} \mathfrak{A}_h$ be an $H$-grading such that the groups are generated by their support respectively. If $\eta_\Gamma(H) \subset \eta_\Gamma(G)$ then $\hat{\Gamma}$ is a coarsening of the $G$-grading where $\eta_\Gamma(H)$ and $\eta_\Gamma(G)$ are defined by (1).

**Proof.** The eigenspaces of $\eta_\Gamma(G)$ and $\eta_\Gamma(H)$ are $A_g$ and $\mathfrak{A}_h$ respectively for all $g \in \text{Supp}_\Gamma A$ and $h \in \text{Supp}_\Gamma A$. Since $\eta_\Gamma(H) \subset \eta_\Gamma(G)$ we have that for any $g \in \text{Supp}_\Gamma A$ the eigenspace $A_g$ of $\eta_\Gamma(G)$ is contained in some eigenspace $\mathfrak{A}_h$ of $\eta_\Gamma(H)$ for some $h \in \text{Supp}_\Gamma A$ where $h$ depends on $g$. Let $\phi : \text{Supp}_\Gamma A \to \text{Supp}_\Gamma A$ be the map defined by $\phi(g) = h$ for $g \in \text{Supp}_\Gamma A$ where $h \in \text{Supp}_\Gamma A$ and $A_g \subset \mathfrak{A}_h$. The map $\phi$ extends to a homomorphism of $G$ onto $H$ since $A_gA_{g'} \subset A_{g+g'}$ and $\mathfrak{A}_h\mathfrak{A}_{h'} \subset \mathfrak{A}_{h+h'}$ by the property of gradings and the groups are generated by their supports respectively. Then $\hat{\Gamma}$ is a coarsening of $\Gamma$. $\square$
If $L$ is finite-dimensional, then $\text{Aut } L$ is an algebraic group, and the image $\eta(\hat{G})$ belongs to the class of algebraic groups called quasi-tori. Recall that a quasi-torus is an algebraic group that is abelian and consists of semisimple elements. Conversely, given a quasi-torus $Q$ in $\text{Aut } L$, we obtain the eigenspace decomposition of $L$ with respect to $Q$, which is a grading by the group of characters of $Q$, $G = \hat{X}(Q)$.

In this paper, $L$ is a Melikyan algebra $M(2; \underline{n})$, where $\underline{n} = (n_1, n_2)$ is a pair of positive integers — see the definitions in the next section. Unless it is stated otherwise, $m$ is a positive integer and $\underline{n} = (n_1, \ldots, n_m)$ is an $m$-tuple of positive integers. We denote by $a$ and $b$ some $m$-tuples of non-negative integers and by $i, j, k, l$ some integers.

2. Melikyan Algebras and Their Standard Gradings

In this section we introduce some basic definitions, closely following [17, Chapter 2]. We start by defining the commutative algebras $O(m; \underline{n})$ and the Witt algebras $W(m; \underline{n})$ which we will use to define the Melikyan algebras when $m = 2$.

Definition 4. Let $O(m; \underline{n})$ be the commutative algebra

$$O(m; \underline{n}) := \left\{ \sum_{0 \leq \alpha \leq \tau(\underline{n})} \alpha(a)x^{(\alpha)} \mid \alpha(a) \in F \right\}$$

er over a field of characteristic $p$, where $\tau(\underline{n}) = (p^{n_1} - 1, \ldots, p^{n_m} - 1)$, with multiplication

$$x^{(a)}x^{(b)} = \left(\frac{a + b}{a}\right)x^{(a+b)}$$

where $\left(\frac{a + b}{a}\right) = \frac{1}{\prod_{i=1}^{m} \left(\frac{a_i + b_i}{a_i}\right)}$.

For $1 \leq i \leq m$, let $\epsilon_i := (0, \ldots, 0, 1, 0 \ldots, 0)$, where the 1 is at the $i$-th position, and $x_i := x^{(\epsilon_i)}$.

There are standard derivations on $O(m; \underline{n})$ defined by $\partial_i(x^{(\alpha)}) = x^{(\alpha - \epsilon_i)}$ for $1 \leq i \leq m$.

Definition 5. Let $W(m; \underline{n})$ be the Lie algebra

$$W(m; \underline{n}) := \left\{ \sum_{1 \leq i \leq m} f_i \partial_i \mid f_i \in O(m; \underline{n}) \right\}$$

with the commutator defined by

$$[f \partial_i, g \partial_j] = f(\partial_i g)\partial_j - g(\partial_j f)\partial_i, \quad f, g \in O(m; \underline{n}).$$

The Lie algebras $W(m; \underline{n})$ are called Witt algebras. $W(m; \underline{n})$ is a subalgebra of $\text{Der } O(m; \underline{n})$, the Lie algebra of derivations of $O(m; \underline{n})$.

From now on the base field $F$ is algebraically closed and its characteristic is 5. We set $\hat{W}(2; \underline{n}) = O(2; \underline{n})\hat{\partial}_1 + O(2; \underline{n})\hat{\partial}_2$. We define the map $\text{div} : \hat{W}(2; \underline{n}) \to O(2; \underline{n})$ by

$$\text{div}(f_1 \partial_1 + f_2 \partial_2) := \partial_1(f_1) + \partial_2(f_2)$$
for all \( f_1, f_2 \in O(2; \mathfrak{n}) \). Also set
\[
f_1 \partial_1 + f_2 \partial_2 := f_1 \tilde{\partial}_1 + f_2 \tilde{\partial}_2
\]
for all \( f_1, f_2 \in O(2; \mathfrak{n}) \).

**Definition 6.** Let \( M(2; \mathfrak{n}) := O(2; \mathfrak{n}) \oplus W(2; \mathfrak{n}) \oplus \tilde{W}(2; \mathfrak{n}) \) be the algebra whose multiplication is defined by the following equations. For all \( D \in W(2; \mathfrak{n}), E \in \tilde{W}(2; \mathfrak{n}), f_1, f_2, g_1, g_2 \in O(2; \mathfrak{n}) \) we set
\[
[D, E] := [\widehat{D}, E] + 2 \text{div}(D) \widehat{E},
\]
\[
[D, f] := D(f) - 2 \text{div}(D)f,
\]
\[
[f, E] := fE
\]
\[
[f_1, f_2] := 2(f_1 \partial_1(f_2) - f_2 \partial_1(f_1))\tilde{\partial}_2 + 2(f_2 \partial_2(f_1) - f_1 \partial_2(f_2))\tilde{\partial}_1.
\]
\[
[f_1 \tilde{\partial}_1 + f_2 \tilde{\partial}_2, g_1 \tilde{\partial}_1 + g_2 \tilde{\partial}_2] := f_1 g_2 - f_2 g_1.
\]
We call \( M(2; \mathfrak{n}) \) the Melikyan algebra.

The algebras \( O(m; \mathfrak{n}), W(m; \mathfrak{n}), M(2; \mathfrak{n}) \) defined above have well known canonical \( \mathbb{Z} \)-gradings.

**Definition 7.** Let \( A = O(m; \mathfrak{n}), W(m; \mathfrak{n}), M(2; \mathfrak{n}) \). The canonical \( \mathbb{Z} \)-grading of \( A \),
\[
A = \bigoplus_{i \in \mathbb{Z}} = \{ y \in A | \text{deg}_A(y) = i \},
\]
is defined by declaring their degrees, \( \text{deg}_O, \text{deg}_W \) and \( \text{deg}_M \), respectively, as follows:
\[
\text{deg}_O(x^{(a)}) := a_1 + \cdots + a_m,
\]
\[
\text{deg}_W(x^{(a)} \partial_1) := a_1 + \cdots + a_m - 1,
\]
\[
\text{deg}_M(x^{(a)} \partial_1) := 3 \text{deg}_W(x^{(a)} \partial_1),
\]
\[
\text{deg}_M(x^{(a)} \tilde{\partial}_1) := 3 \text{deg}_W(x^{(a)} \partial_1) + 2,
\]
\[
\text{deg}_M(x^{(a)}) := 3 \text{deg}_O(x^{(a)}) - 2,
\]
for \( 0 \leq a \leq \tau(\mathfrak{n}) \). The canonical filtration of \( A \), is defined by declaring \( A_{(i)} = \bigoplus_{j \geq i} A_j \).

Note that \( W(2; \mathfrak{n}) = \bigoplus_{i \in \mathbb{Z}} M_{3i} \).

**Lemma 3.** Let \( \Gamma_M : M(2; \mathfrak{n}) = \bigoplus_{(a, a_2) \in \mathbb{Z}^2} M_{(a_1, a_2)} \) where
\[
M_{(a_1, a_2)} := \text{Span}\{x^{(a+\varepsilon_i) \partial_1}| 1 \leq i \leq 2\}
\]
\[
M_{(3a_1, 3a_2)+\mathbb{1}} := \text{Span}\{x^{(a+\varepsilon_i) \tilde{\partial}_1}| 1 \leq i \leq 2\}
\]
\[
M_{(3a_1, 3a_2)-\mathbb{1}} := \text{Span}\{x^{(a)}\}.
\]
The decomposition above is \( \mathbb{Z}^2 \)-grading on \( M(2; \mathfrak{n}) \).
Remark 1. The support of the \( \mathbb{Z}^2 \)-grading \( \Gamma_M \) does not generate \( \mathbb{Z}^2 \). The support generates the subgroup \( G = \langle (3i + j, j) \mid i, j \in \mathbb{Z} \rangle \) which is isomorphic to \( \mathbb{Z}^2 \). Hence we can define a \( \mathbb{Z}^2 \)-grading for which the support generates \( \mathbb{Z}^2 \). Let \( \phi_M : \mathbb{Z}^2 \to \mathbb{Z}^2 \) defined by \( \phi_M((1,0)) = (3,0) \) and \( \phi_M((0,1)) = (1,1) \). If we set \( L_a = M_{\phi_M((a))} \) for \( a \in \mathbb{Z}^2 \) then \( \Gamma_M : M(2;\mathbb{N}) = \bigoplus_{a \in \mathbb{Z}^2} L_a \) is a \( \mathbb{Z}^2 \)-grading since \( \phi_M(\mathbb{Z}^2) = G \). Also since \( L_{(-1,0)} = M_{(-3,0)} = \text{Span}\{\partial_t\} \) and \( L_{(0,-1)} = M_{(-1,-1)} = F \) we have that the support of the \( \Gamma_M \) grading generates \( \mathbb{Z}^2 \).

Note that the grading in Lemma 3 is a coarsening of the \( \Gamma_M \) grading. By Lemma 4 we have that \( \eta_{\Gamma_M}(\mathbb{Z}^2) \subset \eta_{\Gamma_M}(\mathbb{Z}^2) \). We will mainly work with the grading \( \Gamma_M \) and get results for \( \Gamma_M \). We will show that \( \eta_{\Gamma_M}(\mathbb{Z}^2) \) is a maximal abelian subgroup of \( \text{Aut} M(2;\mathbb{N}) \) which implies that \( \eta_{\Gamma_M}(\mathbb{Z}^2) = \eta_{\Gamma_M}(\mathbb{Z}^2) \).

Definition 8. We call the \( \mathbb{Z}^2 \)-grading \( \Gamma_M \) in Remark 4 the standard \( \mathbb{Z}^2 \)-grading on \( M(2;\mathbb{N}) \). Let \( \text{deg}_{\Gamma_M}(y) \) and \( \text{deg}(y) \) be the degrees of \( y \) with respect to the \( \mathbb{Z}^2 \)-gradings \( \Gamma_M \) and \( \Gamma_M \) respectively.

Remark 2. The canonical \( \mathbb{Z} \)-grading is a coarsening of the \( \mathbb{Z}^2 \)-grading \( \Gamma_M \) from Lemma 5 and hence a coarsening of the standard \( \mathbb{Z}^2 \)-grading \( \Gamma_M \). Explicitly,

\[
M_i = \bigoplus_{a_1 + a_2 = i} M_{(a_1, a_2)}.
\]

Definition 9. Let \( G \) be an abelian group and \( \varphi : \mathbb{Z}^2 \to G \) a homomorphism. The decomposition \( M(2;\mathbb{N}) = \bigoplus_{g \in G} M_g \), given by

\[
M_g = \text{Span}\{y \in M(2;\mathbb{N}) \mid \varphi(\text{deg}_{\Gamma_M}(y)) = g\},
\]

is a \( G \)-grading on \( M(2;\mathbb{N}) \). We call such decomposition a standard \( G \)-grading induced by \( \varphi \) on \( M(2;\mathbb{N}) \). We will refer to a standard \( G \)-grading induced by \( \varphi \) as a standard \( G \)-grading when \( \varphi \) is not specified.

The grading \( \Gamma_M \) on \( M(2;\mathbb{N}) \) gives rise to a quasi-torus \( \eta_{\Gamma_M}(\mathbb{Z}^2) \). We will show later that \( \eta_{\Gamma_M}(\mathbb{Z}^2) \) is actually a maximal torus. Let \( t^a := t_1^{a_1} t_2^{a_2} \) for all \( \underline{a} = (t_1, t_2) \in (F^\times)^2 \) and \( \alpha(\underline{a}) := t_1 t_2 \). We define \( \lambda : (F^\times)^2 \to \text{Aut} M(2;\mathbb{N}) \) where

\[
\begin{align*}
\lambda(t^a)x(a)\partial_i & := t^{3a-3x(a)}(a)\partial_i, \\
\lambda(t^a)x(a)\partial_i & := t^{3a-3x(a)}\alpha(\underline{a})(x(a))\partial_i, \\
\lambda(t^a)x(a) & := t^{3a}\alpha(\underline{a})^{-1}x(a).
\end{align*}
\]

For any element \( y \) in \( M_{(a_1, a_2)} \) of the grading \( \Gamma_M \) we have \( \lambda(t(y)) = \underline{t}^a y \) which is the same as saying \( \lambda(t)(y) = \underline{t}^{\text{deg}(y)} y \).

Lemma 4. \( \lambda \) is a homomorphism of algebraic groups.

Proof. We start by showing that for \( \underline{a} \in (F^\times)^2 \) we have \( \lambda(t^a) \in \text{Aut} M(2;\mathbb{N}) \). Lemma 3 gives us that \( \text{deg}([y, z]) = \text{deg}(y) + \text{deg}(z) \) when \( y, z \) are homogeneous elements. For homogeneous \( y, z \) we have

\[
\lambda(t)([y, z]) = \underline{t}^{\text{deg}([y,z])} [y, z] = \underline{t}^{\text{deg}(y) + \text{deg}(z)} [y, z] = \underline{t}^{\text{deg}(y)} \underline{t}^{\text{deg}(z)} [y, z].
\]

Hence \( \lambda(t) \in \text{Aut} M(2;\mathbb{N}) \).
Now we show that $\lambda$ is a homomorphism. Let $s, t \in (F^\times)^2$ and $y$ be a homogeneous element. Then

$$\lambda(sy) = (sx)^{\deg(y)} = s^{\deg(y)}x^{\deg(y)} = s^{\deg(y)}\lambda(x)(y)$$

which shows that $\lambda$ is a homomorphism.

It is obvious that $\lambda$ is a rational map and it is a homomorphism. $\square$

Let $T_M := \lambda((F^\times)^2)$. The kernel of $\lambda$ is $\{(t_1, t_2) \in (F^\times)^2 \mid t_1^3 = t_2^3 = 1, t_1t_2 = 1\}$. Since the kernel is finite and $\lambda$ is a regular homomorphism we have that $T_M$ is a torus.

**Lemma 5.** The torus $T_M$ is $\eta_{\hat{\Gamma}M}(\widehat{\mathbb{Z}^2})$.

**Proof.** First we show that $\eta_{\hat{\Gamma}M}(\widehat{\mathbb{Z}^2}) \subset T_M$. Let $\chi \in \widehat{\mathbb{Z}^2}$ and $\chi((1,0)) = t_1 \in F^\times$ and $\chi((0,1)) = t_2 \in F^\times$. For $y \in M_{(a_1, a_2)}$ we have

$$\eta_{\hat{\Gamma}M}(\chi)(y) = \chi((a_1, a_2))y = \chi((a_1, 0))\chi((0, a_2))y = \chi((0, 1))^{a_1}y = (t_1, t_2)^{\deg(y)}y = \lambda((t_1, t_2))(y).$$

Hence $\eta_{\hat{\Gamma}M}(\chi) \in T_M$ and we have $\eta_{\hat{\Gamma}M}(\widehat{\mathbb{Z}^2}) \subset T_M$.

Now we show that $T_M \subset \eta_{\hat{\Gamma}M}(\widehat{\mathbb{Z}^2})$. For $\chi : \mathbb{Z}^2 \rightarrow F^\times$ let $\chi : \mathbb{Z}^2 \rightarrow F^\times$ be the element of $\widehat{\mathbb{Z}^2}$ defined by $\chi(a) = t^a$ for any $a \in \mathbb{Z}^2$. For $y \in M_{a}$, $a \in \mathbb{Z}^2$ we have

$$\lambda(\chi)(y) = \chi(y) = \eta_{\hat{\Gamma}M}(\chi)(y).$$

Hence $\lambda(\chi) \in \eta_{\hat{\Gamma}M}(\widehat{\mathbb{Z}^2})$ and we have $T_M \subset \eta_{\hat{\Gamma}M}(\widehat{\mathbb{Z}^2})$. $\square$

The following proposition shows that if we want to know more about the quasitorus $\eta(\hat{G})$ up to conjugation by an automorphism of $M(2; n)$ then we should look at the normalizer of a maximal torus in $\Aut M(2; n)$. This follows from [10] Corollary 3.28.

**Proposition 1.** A quasi-torus of an algebraic group belongs to the normalizer of a maximal torus. $\square$

In Section 3, we will show that $T_M$ is a maximal torus of $\Aut M(2; n)$. This leads us to look at the normalizer of the restriction of $T_M$ on $W(2; n)$ in $\Aut W(2; n)$. Using that the automorphisms of $W(2; n)$ can extend to $\Aut M(2; n)$ (– see [12]) we can then extend the information of the normalizer in $\Aut W(2; n)$ to get the normalizer of $T_M$ in $\Aut M(2; n)$.

The goal of Section 3 is to show that if $G$ has no elements of order five then $\eta(\hat{G})$ is always contained in a maximal torus.

3. **The Automorphism Groups of Melikyan Algebras**

The automorphism group of $M(2; n)$ respects the canonical filtration on $M(2; n)$ (– see proof of [13] Theorem 4.7). Also [12] says that any automorphism of $W(2; n)$ can be extended to an automorphism of $M(2; n)$.

We start by looking at a maximal torus of $\Aut W(2; n)$. Let

$$T_W := \{ \psi \in \Aut W(2; n) \mid \psi(x^{(a)}\partial_k) = t_1^{a_1}t_2^{a_2}t_k^{-1}x^{(a)}\partial_k, t_j \in F^\times \}.$$ 

According to [17], p. 371, $T_W$ is indeed a maximal torus of $W(2; n)$.

Let $\Aut W M(2; n) = \{ \Psi \in \Aut M(2; n) \mid \Psi(W(2; n)) = W(2; n) \}$ and $\pi : \Aut W M(2; n) \rightarrow \Aut W(2; n)$ is the respective restriction map on
\( \text{Aut}_W M(2; \underline{n}) \). Since \( T_M = n_{\overline{M}}(\overline{2^2}) \) with respect to the \( \mathbb{Z}^2 \)-grading \( T_M \) on \( M(2; \underline{n}) \) and \( W(2; \underline{n}) \) is a graded subspace of this grading we have \( T_M \subset \text{Aut}_W M(2; \underline{n}) \).

**Lemma 6.** The restriction of \( T_M \) to \( W(2; \underline{n}) \) is \( T_W \).

**Proof.** We start by showing \( T_W \subset \pi(T_M) \). For any \( \psi \in T_W \) we have a pair \((s_1, s_2) \in (F^\times)^2\) such that \( \psi(x^{(a)} \partial_i) = s_1^{a_1} s_2^{a_2} s_i^{-1} x^{(a)} \partial_i \). For any element \( u \in F^\times \) there is at least one element \( v \) such that \( v^3 = u \) because \( F \) is algebraically closed. Hence there exist \( t_1 \) and \( t_2 \) in \( F^\times \) such that \( t_1^3 = s_1 \) and \( t_2^3 = s_2 \). Computing \( \lambda(t) \) on \( x^{(a)} \partial_i \) we get
\[
\lambda(t)(x^{(a)} \partial_i) = t_1^{3a_1} t_2^{3a_2} t_i^{-1} x^{(a)} \partial_i = s_1^{a_1} s_2^{a_2} s_i^{-1} x^{(a)} \partial_i.
\]
This shows that \( \psi = \pi(\lambda((t_1, t_2))) \in \pi(T_M) \) and we have \( T_W \subset \pi(T_M) \).

The kernel of \( \pi \) on \( T_M \) is \( \{ \lambda(t) \in T_M \mid t_1^3 = t_2^3 = 1 \} \).

**Lemma 7.** [12, Lemma 5] If \( \Theta \in \text{Aut}_W M(2; \underline{n}) \) is such that \( \pi(\Theta) = \text{Id}_W \) then for \( y \in M_i, i \in \mathbb{Z} \), there exists a \( \beta \) such that \( \Theta(y) = \beta^i y \) where \( \beta^3 = 1 \).

We now fix \( \beta \) to be a primitive third root of unity and set \( \Theta := \lambda(\beta^2, \beta^2) \). Note that \( \Theta \in T_M \).

**Corollary 1.** Let \( \Psi \) and \( \Phi \) be elements of \( \text{Aut}_W M(2; \underline{n}) \). If \( \pi(\Psi) = \pi(\Phi) \) then there exists an \( l \) such that \( 0 \leq l \leq 2 \) and \( \Psi = \Phi \Theta^l \).

**Proof.** If \( \pi(\Psi) = \pi(\Phi) \) then \( \pi(\Phi^{-1} \Psi) = \text{Id}_W \). By Lemma 7 we have \( \Phi^{-1} \Psi = \Theta^l \) for some \( 0 \leq l \leq 2 \).

**Corollary 2.** If \( \Psi \in \text{Aut}_W M(2; \underline{n}) \) is such that \( \pi(\Psi) \in T_W \) then \( \Psi \in T_M \).

**Proof.** Lemma 6 shows that there exists \( \Phi \in T_M \) such that \( \pi(\Phi) = \pi(\Psi) \) and Corollary 1 says that \( \Psi = \Phi \Theta^l \) for some \( 1 \leq l \leq 2 \). Hence \( \Psi \in T_M \).

In order to describe the normalizers in \( \text{Aut} W(2; \underline{n}) \) and \( \text{Aut} M(2; \underline{n}) \) we introduce the automorphism \( \nu \) of \( O(2; \underline{n}) \) that induces an automorphism \( \sigma \) of \( W(2; \underline{n}) \) and finally we extend \( \sigma \) to \( \text{Aut} M(2; \underline{n}) \). For \( \underline{n} = (n_1, n_2) \) we define \( \overline{n} := (a_2, a_1) \) for \( a = (a_1, a_2) \in \mathbb{Z}^2 \). Let \( n_1 = n_2 \). The linear maps \( \nu \) and \( \sigma \) of \( O(2; \underline{n}) \) and \( W(2; \underline{n}) \) respectively, defined by \( \nu(x^{(a)}) := x^{\overline{n}} \) and \( \sigma(D) := D \nu^{-1} \) for \( x^{(a)} \in O(2; \underline{n}) \), and all \( D \in W(2; \underline{n}) \).

**Lemma 8.** For \( n_1 = n_2 \), the maps \( \nu \) and \( \sigma \) are automorphisms of \( O(2; \underline{n}) \) and \( W(2; \underline{n}) \) respectively.

**Proof.** It follows easily from [17, Theorem 6.3.2] that \( \nu \) is a continuous automorphism of \( O(2; \underline{n}) \) (which are in \( \text{Aut} O(2; \underline{n}) \) as the name implies). It follows from [17, Theorem 7.3.2] that conjugating an element \( D \) of \( W(2; \underline{n}) \) by a continuous automorphism \( \psi \) of \( O(2; \underline{n}) \), \( D \mapsto \psi \circ D \circ \psi^{-1} \) is an automorphism of \( W(2; \underline{n}) \) and hence \( \sigma \in \text{Aut} W(2; \underline{n}) \).

**Lemma 9.** [15] The normalizer of \( T_W \) in \( \text{Aut} W(2; \underline{n}) \) is \( T_W \) if \( n_1 \neq n_2 \) and \( T_W \langle \sigma \rangle_2 \) if \( n_1 = n_2 \).

The following follows from the first paragraph on p.3920 of [12].
Proposition 2. For every automorphism $\psi$ of $W(2; n)$ there exists a $\psi_M$ of $M(2; n)$ which respects $W(2; n)$ and whose restriction to $W(2; n)$ is $\psi$. □

By Proposition 2 there exists a $\sigma_M \in \text{Aut} M(2; n)$ which respects $W(2; n)$ and whose restriction is $\sigma$. We fix this $\sigma_M$.

Now we can prove that $T_M$ is a maximal torus in $\text{Aut} M(2; n)$. Let $U_M = \langle \sigma_M \rangle$ when $n_1 = n_2$ and identity otherwise.

Proposition 3. The normalizer of $T_M$ in $\text{Aut} M$ is $T_M U_M$.

Proof. In [12] p. 3921 it is stated that we can decompose $\Psi \in \text{Aut} M$ as the product of $\Psi = \Phi \Omega$ where $\Phi$, $\Omega \in \text{Aut} M(2; n)$ are such that for all $i \in \mathbb{Z}$ we have

$$\Phi(y) = y + M_{(i+1)} \quad \text{for } y \in M_i,$$

$$\Omega(M_i) = M_i.$$ Let $\Psi \in N_{\text{Aut} M(2; n)}(T_M)$. Then we will show that $\Phi = \text{Id}_M$.

Let $y \in M_i$ be a nonzero eigenvector of $T_M$. Since $\Psi(y) \in M_{(i+1)}$ by (2) and $0 \neq \Psi(y) \in M_k$ we have $k \geq i$. Hence $\Phi(y) = y$ for all $i \in \mathbb{Z}$. Now we use the decomposition of $\Psi = \Phi \Omega$. We have $\Omega(y) = w \in M_i$ since $\Omega(M_i) = M_i$ for all $i$. (2)

$$\Psi(y) = \Phi(\Omega(y)) = \Phi(w) \in w + M_{(i+1)}.$$ Since $w \neq 0$ the calculations above show that $\Psi(y) \in M_{(i)}$ and $\Psi(y) \notin M_{(i+1)}$.

The intersection of $M_k$ and $M_{(i)} = \bigoplus_{j \geq i} M_j$ is zero if $k < i$. Since $\Psi(y) \in M_{(i)}$ by (2) and $0 \neq \Psi(y) \in M_k$ we have $k \geq i$. Since $\Psi(y) \notin M_{(i+1)}$ by (2) and $\psi(y) \in M_k$ we have $k \leq i$. Hence $k = i$. We have shown that $\Psi(M_i) = M_i$ for all $i$.

Hence $\Phi = \text{Id}_M$. Since $W(2; n) = \bigoplus_{i \in \mathbb{Z}} M_i$, we have $\Psi(W(2; n)) = W(2; n)$. We conclude that if $\Psi \in N_{\text{Aut} M(2; n)}(T_M)$ then $\Psi$ preserves the standard $\mathbb{Z}$-grading of $M(2; n)$ and that $\pi(\Psi) \in N_{\text{Aut} W(2; n)}(T_W)$ since $\pi(T_M) = T_W$ (Lemma 6).

According to Lemma 6, $N_{\text{Aut} W(2; n)}(T_W) = T_W$ when $n_1 \neq n_2$ and $T_W(\sigma)$ if $n_1 = n_2$. By Corollary 2 the set of automorphisms of $M(2; n)$ which when restricted to $W(2; n)$ are in $T_M$ is $\text{Aut} W(2; n)$. When $n_1 = n_2$, Corollary 4 says that if $\Psi \in \text{Aut} W(2; n)$ and $\pi(\Psi) = \rho \sigma$, where $\rho \in T_W$ then there exists $z \in T_M$ such that $\pi(z) = \rho$ and $\Psi = z\sigma M \Theta$ for $0 \leq l \leq 2$. The automorphism $\Theta$ is in $T_M$ since $\Theta = \lambda(\beta^2, \beta^2)$. Hence, $N_{\text{Aut} M(2; n)}(T_M) \subset T_M U_M$.

Conversely let $\Psi \in \text{Aut} W(2; n)$ such that $\pi(\Psi) \in N_{\text{Aut} W(2; n)}(T_W)$. Then

$$\pi(\Psi \lambda(\beta) \Psi^{-1}) = \pi(\Psi) \pi(\lambda(\beta)) \pi(\Psi)^{-1} \in T_W.$$ By Corollary 2 we have that $\Psi(\lambda(\beta) \Psi^{-1} \in T_M$ and hence it follows that $\Psi \in N_{\text{Aut} M(2; n)}(T_M)$. Since for $n_1 = n_2$ we have $\pi(\sigma_M) = \sigma \in N_{\text{Aut} W(2; n)}(T_W)$ and it follows that $\sigma_M \in N_{\text{Aut} M(2; n)}(T_M)$. We have shown that $T_M U_M \subset N_{\text{Aut} M(2; n)}(T_M)$.

□

Corollary 3. The centralizer of $T_M$ in $\text{Aut} M(2; n)$ is $T_M$. Moreover, $T_M$ is a maximal torus and $T_M = \eta_{\mathbb{Z}^2}(\mathbb{Z}^2)$.

Proof. The centralizer of $T_M$ is contained in the normalizer of $T_M$. By Proposition 3 the normalizer of $T_M$ is in $\text{Aut} W(2; n)$. This implies that if $\Psi \in \text{Aut} W(2; n)$ and $\Psi$ is in the centralizer of $T_M$ then $\pi(\Psi)$ must be in the centralizer of $\pi(T_M) = T_W$ (Lemma 6). Since $T_W$ is a maximal torus and (for $n_1 = n_2$) $\sigma \notin T_W$ we have
that $\sigma$ is not in the centralizer of $T_W$. Hence $\sigma_M$ is not in the centralizer of $T_M$. We have shown that the centralizer of $T_M$ in $\text{Aut} M(2; \underline{n})$ is $T_M$ and that $T_M$ is a maximal torus.

By Lemma 1 we have $T_M = \eta_{T_M}(\hat{Z}^2) \subset \eta_{M(2; \underline{n})}(\hat{Z}^2)$. Since $\eta_{T_M}(\hat{Z}^2)$ is abelian and contains $T_M$ it must be in the centralizer of $T_M$ which is $T_M$. □

**Proposition 4.** Let $Q$ be a quasi-torus in $\text{Aut} M(2; \underline{n})$. There is an automorphism $\Psi \in \text{Aut} M(2; \underline{n})$ such that $\Psi Q \Psi^{-1} \subset T_M$.

**Proof.** By Proposition 1 $Q$ is inside the normalizer of a maximal torus. Up to conjugation we can assume $Q \subset \mathcal{N}_{\text{Aut} M(2; \underline{n})}(T_M)$. Then $Q$ must preserve $W(2; \underline{n})$ since $\mathcal{N}_{\text{Aut} M(2; \underline{n})}(T_M) = T_M U_M$ (Proposition 3). Let $Q' = \pi(Q)$. It follows that $Q' \subset \mathcal{N}_{\text{Aut} W(2; \underline{n})}(T_W)$. In [15] it is shown that there exists a $\psi \in \text{Aut} W(2; \underline{n})$ such that $\psi Q' \psi^{-1} \subset T_W$. Proposition 2 says that there is $\Psi \in \text{Aut} M(2; \underline{n})$ such that $\pi(\Psi) = \psi$. Hence $\pi(\Psi Q \Psi^{-1}) = \psi Q' \psi^{-1} \subset T_W$. Since $\pi(\Psi Q \Psi^{-1}) \subset T_W$, Corollary 2 gives us that $\Psi Q \Psi^{-1} \subset T_M$. □

We can now prove Theorem 1.

**Proof.** Let $L = M(2; \underline{n})$. Suppose $\Gamma : L = \bigoplus_{g \in G} L_g$ is a $G$-grading where $G$ is a group without elements of order five. Without loss of generality, we assume that the support of the grading generates $G$. Let $\eta_T : \hat{G} \to \text{Aut} L$ be the corresponding embedding and $Q = \eta_T(\hat{G})$. Then by Proposition 4 there is a $\Psi \in \text{Aut} M(2; \underline{n})$ such that $\Psi Q \Psi^{-1} \subset T_M$. Recall that by Corollary 3 that $T_M = \eta_{T_M}(\hat{Z}^2)$. It follows from Lemma 2 that $L = \bigoplus_{g \in G} L'_g$, where $L'_g = \Psi(L_g)$, is a coarsening of the standard $Z^2$-grading $\Gamma_M$ which is isomorphic to the original grading. □

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