A Principled Approach for Learning Task Similarity in Multitask Learning

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Abstract
Multitask learning aims at solving a set of related tasks simultaneously, by exploiting the shared knowledge for improving the performance on individual tasks. Hence, an important aspect of multitask learning is to understand the similarities within a set of tasks. Previous works have incorporated this similarity information explicitly (e.g., weighted loss for each task) or implicitly (e.g., adversarial loss for feature adaptation), for achieving good empirical performances. However, the theoretical motivations for adding task similarity knowledge are often missing or incomplete. In this paper, we give a different perspective from a theoretical point of view to understand this practice. We first provide an upper bound on the generalization error of multitask learning, showing the benefit of explicit and implicit task similarity knowledge. We systematically derive the bounds based on two distinct task similarity metrics: \( H \) divergence and Wasserstein distance. From these theoretical results, we revisit the Adversarial Multi-task Neural Network, proposing a new training algorithm to learn the task relation coefficients and neural network parameters iteratively. We assess our new algorithm empirically on several benchmarks, showing not only that we find interesting and robust task relations, but that the proposed approach outperforms the baselines, reaffirming the benefits of theoretical insight in algorithm design.

1 Introduction

Traditional machine learning has mainly focused on designing learning algorithms for individual problems. While significant progress has been achieved in applied and theoretical research, it still requires a large amount of labelled data in such context to obtain small generalization error. This demand is highly prohibitive in some practices, e.g., modelling users’ preferences for products [1], classifying multiple object in computer vision [2], analyzing patient data in computational healthcare [3]. In the multi-task learning (MTL) scenario, an agent learns the shared knowledge between a set of related tasks. Under different assumptions on task relations, MTL has been shown to reduce the amount of annotated examples required per task to reach acceptable performance.

Understanding the theoretical assumptions of the tasks relationship plays a core role in designing a good MTL algorithm. In fact, it determines which inductive bias should be involved in the learning procedure. Recently, there are many successful algorithms that relies on task similarity information, which assumes the \textit{Probabilistic Lipschitzness} (PL) condition [4] as the inductive bias. For instance, [1, 5, 6] minimize a weighted sum of empirical loss in which similar tasks are assigned higher weights. These approaches explicitly estimate the task similarities through a linear model. Since these approaches are estimated in the original input space, it makes it hard for handling the \textit{covariate shift} problem [7]. Therefore, lots of neural network based approaches started to explore tasks similarities implicitly: [8, 9, 10] use adversarial losses for feature adaptation, minimizing the distribution distance between the tasks for constructing a shared feature space. Then, the hypothesis for the different tasks are learned over this adapted feature space.

The implicit similarity learning approaches are inspired from the idea of Generative Adversarial Networks (GANs) [11]. However, the fundamental implications of incorporating task similarity information in MTL algorithms are not clear. The two main questions are \textit{why} should we combine explicit and implicit similarity knowledge in the MTL framework and \textit{how} can we properly do it.

Previous work either consider explicit or implicit similarity knowledge separately, or combine them heuristically in some specific applications. In contrast, the main goal of our work is to give a rigorous analysis of the benefits of task similarities and derive an algorithm which properly use this information. We start by deriving an upper bound on the generalization error of MTL under different similarity metrics (or adversarial loss). These bounds show the motivation behind the use of adversarial loss in MTL, that is to control the generalization error. Then, we derive a new procedure to update the relationship coefficients from these theoretical guarantees. This procedure allows to bridge the gap between the explicit and implicit similarities, which have been previously seen as disjoint or treated heuristically. We then derive a new algorithm to train the Adversarial Multi-task Neural Network (AMTNN) and validate it empirically on two benchmarks: digit recognition and Amazon sentiment analysis. The results show that our
method not only highlights some interesting relations, but also outperforms the previous baselines, reaffirming the benefits of theory in algorithm design.

2 Related Work

Multi-task learning (MTL) Some survey papers [3, 12, 13] have made a broad and detailed presentation of the general MTL. More specifically related to our work, on the practical side, we note several approaches that use tasks relationship to improve empirical performances: Zhang and Yeung [14], Ciliberto et al. [15] solve a convex optimization problem in the original space or Reproducing Kernel Hilbert-Space to extract tasks relationships, Long et al. [2], Kendall et al. [16] propose probabilistic models through construction of a task covariance matrix or estimate the multitask likelihood from a deep Bayes model. On the theoretical side, Murugesan et al. [5], Murugesan and Carbonell [1], Pentina and Lampert [6] analyze the weighted sum loss algorithm and its applications in online learning, active learning and transductive learning. Moreover, Maurer et al. [17, 18] analyze generalization error of representation-based approaches, and Zhang [19] analyze the algorithmic stability in MTL.

Similarity metrics and adversarial loss The similarity metrics (or distribution distance/distribution discrepancy) is currently used in deep generative models Goodfellow et al. [11], Arjovsky et al. [20], domain adaptation Ben-David et al. [21], Ganin et al. [7], Redko et al. [22], robust learning Konstantinov and Lampert [23] and meta-learning Rakotomamonjy et al. [24]. In transfer learning, adversarial losses are widely used for feature adaptation, since the transfer procedure is much more efficient on a shared representation. In applied transfer learning, H-divergence Ganin et al. [7] and Wasserstein distance Li et al. [10] are widely used as adversarial losses. As for MTL applications, to our knowledge, Liu et al. [8], Chen and Cardie [9] and Kremer et al. [25] apply H-divergence in natural language processing for text classification and speech recognition. [26] are the first to use Wasserstein distance to estimate the similarity of linear parameters instead of the data generation distributions. As for the theoretical understanding, [27] analyzed the Minimax statistical property in the Wasserstein distance, and [28] analyzed an adaptation bound on general discrepancy.

3 Preliminaries

Considering a set of T tasks \( \{ \mathcal{D}_t \}_{t=1}^T \), in which the observations are generated by the underlying distribution \( \mathcal{D}_t \) over \( \mathcal{X} \) and the real target is determined by the underlying labelling functions \( f_t : \mathcal{X} \rightarrow \mathcal{Y} \) for \( \{(\mathcal{D}_t, f_t)\}_{t=1}^T \). Then, the goal of MTL is to find \( T \) hypothesis: \( h_1, \ldots, h_T \) over the hypothesis space \( \mathcal{H} \) to control the average expected error of all the tasks:

\[
\frac{1}{T} \sum_{i=1}^{T} R_t(h_i)
\]

Where \( R_t(h_i) \equiv R_t(h_i, f_t) = \mathbb{E}_{x \sim \mathcal{D}_t} \ell(h_i(x), f_t(x)) \) is the expected risk at task \( t \) and \( \ell \) is the loss function. Throughout the theoretical part, the loss is \( \ell(h_i(x), f_t(x)) = |h_i(x) - f_t(x)| \), which is coherent with [6, 10, 21, 7, 22]. If \( h, f \) are the binary mappings which only output \{−1, 1\}, it will recover the typical zero-one loss. \(^{1}\)

We also assume each task has \( m_t \) examples, with \( \sum_{t=1}^{T} m_t = m \) examples in total. Then for each task \( t \), we consider a minimization of weighted empirical loss for each task. That means we define a simplex \( \alpha_t \in \Delta^T = \{ \alpha_{t,i} \geq 0, \sum_{t=1}^{T} \alpha_{t,i} = 1 \} \) for the corresponding weight for task \( t \). Then the weighted empirical error w.r.t. the hypothesis \( h \) for task \( t \) can be written as:

\[
\hat{R}_t(h) = \sum_{i=1}^{T} \alpha_{t,i} \hat{R}_i(h)
\]

Where \( \hat{R}_t(h) = \frac{1}{m_t} \sum_{j=1}^{m_t} \ell(h(x_j), y_j) \) is the average empirical error for task \( t \).

4 Similarity measures

As we illustrated in the previous section, we are interested about task similarities in MTL. Therefore, the first question is how to measure the similarity between two distributions. In this section, we introduce two metrics: H-divergence [21] and Wasserstein distance [20], that are widely applied in machine learning.

H-divergence Given an input space \( \mathcal{X} \) and two probability distributions \( \mathcal{D}_i \) and \( \mathcal{D}_j \) over \( \mathcal{X} \), let \( \mathcal{H} \) be a hypothesis class on \( \mathcal{X} \). We define the H-divergence of two distributions:

\[
d_H(\mathcal{D}_i, \mathcal{D}_j) = \sup_{h,h' \in \mathcal{H}} |R_i(h, h') - R_j(h, h')|
\]

And also the empirical H-divergence:

\[
h_H(\mathcal{D}_i, \mathcal{D}_j) = \sup_{h,h' \in \mathcal{H}} |\hat{R}_i(h, h') - \hat{R}_j(h, h')|
\]

Optimal transport and Wasserstein distance We assume \( \mathcal{X} \) the measurable space and denote \( \mathcal{P}(\mathcal{X}) \) the set of all probability measures over \( \mathcal{X} \). Given two probability measures \( \mathcal{D}_i \in \mathcal{P}(\mathcal{X}_1) \) and \( \mathcal{D}_j \in \mathcal{P}(\mathcal{X}_2) \). The optimal transport (or Monge-Kantorovich) problem can be defined as searching for a probabilistic coupling \( \gamma \) refined as a joint probability measure over \( \mathcal{X}_1 \times \mathcal{X}_2 \) with marginals \( \mathcal{D}_i \) and \( \mathcal{D}_j \) for all \( x, y \) which minimizes the cost of transport w.r.t. some cost function \( c \):

\[
\underset{\gamma \in \Pi(\mathcal{P}(\mathcal{X}_1), \mathcal{P}(\mathcal{X}_2))}{\arg\min} \int_{\mathcal{X}_1 \times \mathcal{X}_2} c(x, y) \rho d\gamma(x, y)
\]

s.t. \( \mathcal{P}^{X_1} \# \gamma = \mathcal{D}_i; \mathcal{P}^{X_2} \# \gamma = \mathcal{D}_j \)

Where \( \mathcal{P}^{X_i} \) is projection over \( \mathcal{X}_i \) and \# denotes the push-forward measure. This problem admits an unique solution \( \gamma_0 \) which allows to define the Wasserstein distance of order \( p \) between \( \mathcal{D}_i \) and \( \mathcal{D}_j \) for any \( p \geq 1 \):

\[
W_p(\mathcal{D}_i, \mathcal{D}_j) = \inf_{\gamma \in \Pi(\mathcal{P}(\mathcal{X}_1), \mathcal{P}(\mathcal{X}_2))} \int_{\mathcal{X}_1 \times \mathcal{X}_2} c(x, y) \rho d\gamma(x, y)
\]

where \( c : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+ \) is the cost function for transporting one unit of mass \( x \) to \( y \) and \( \Pi(\mathcal{P}(\mathcal{X}_1), \mathcal{P}(\mathcal{X}_2)) \) is a collection of all joint probability measures on \( \mathcal{X} \times \mathcal{X} \) with marginals \( \mathcal{D}_i \) and \( \mathcal{D}_j \). Throughout this paper, we mainly consider the case \( p = 1 \), i.e. Wasserstein-1 distance.

\(^{1}\)It can be extended to the multiclass problem by applying the multiclass learnability.
5 Theoretical guarantees

Based on the definitions of distribution similarity metric, we can demonstrate that the generalization error in the multi-task learning can be upper bounded by the following result:

**Theorem 1.** Let \( \mathcal{H} \) be a hypothesis family with a VC-dimension \( d \). If we have \( T \) tasks generated by the underlying distribution and labelling function \( \{(D_1, f_1), \ldots, (D_T, f_T)\} \) with observation numbers \( m_1, \ldots, m_T \). If we adopt the \( \mathcal{H} \) divergence as a similarity metric, then for any simplex \( \alpha_t \in \mathbb{R}^d_+ \), and for \( \delta \in (0, 1) \), with probability at least \( 1 - \delta \), for \( h_1, \ldots, h_T \in \mathcal{H} \), we have:

\[
\frac{1}{T} \sum_{t=1}^{T} R_t(h_t) \leq \frac{1}{T} \sum_{t=1}^{T} \hat{R}_{\alpha_t}(h_t) + C_1 \sum_{t=1}^{T} \left( \sqrt{\sum_{i=1}^{T} \frac{\alpha_{i,t}^2}{\beta_i}} \right) + C_2 \sum_{t=1}^{T} \sum_{i=1}^{T} \alpha_{i,t} \lambda_{i,t},
\]

where \( \beta_i = \frac{m_i}{m} \), \( C_1 = 2 \sqrt{2(d \log(\frac{4m}{2m}) + \log(\frac{16e}{\delta}))} \) and \( C_2 = 2 \min_{i,j} \sqrt{2d \log(2m_{i,j}) + \log(\frac{16e}{\delta})} \), with \( m_{i,j} = \min\{m_i, m_j\} \) and \( \lambda_{i,j} = \inf_{h \in \mathcal{H}} \{R_t(h) + R_j(h)\} \) (joint expected minimal error w.r.t. \( \mathcal{H} \)).

The proof is provided in the supplementary material. Theorem 1 illustrates the upper bound of the generalization error in the MTL framework and can be decomposed by the following terms:

1. The empirical loss and empirical distribution similarities control the weights (or task relation coefficient) \( \alpha_{1:t, t} \). For instance, for a given task \( t \), if task \( i \) has a small empirical distance \( d(D_i, D_t) \) and hypothesis \( h_i \) has a small empirical loss \( \hat{R}_i(h_i) \) on task \( i \), it means that task \( i \) is very similar to \( t \). Hence, more information should be borrowed from task \( i \) when learning \( t \) and the corresponding coefficient \( \alpha_{i,t} \) should have high values.

2. Simultaneously, the coefficient regularization term prevents the relation coefficients locating only on the \( \alpha_{i,t} \), in which it will completely recover the independent MTL framework. Then the coefficient regularization term proposed a trade-off between learning the single task and sharing information from the others tasks.

3. The complexity and optimal terms depend on the setting hypothesis family \( \mathcal{H} \). Given a fixed hypothesis family such as neural network, the complexity is constant. As for the optimal expected loss, throughout this paper we assume \( \lambda_{i,t} \) is much smaller than the empirical term, which indicates that hypothesis family \( \mathcal{H} \) can learn the multiple tasks with a small expected risk. This is a natural setting in the MTL problem since we want the predefined hypothesis family to learn well for all the tasks. While a high expected risk means such a hypothesis set can not perform well, which contradicts our assumption.

In Theorem 1 we have derived a bound based on the \( \mathcal{H} \) divergence and applied in the classification problem. Then we proposed another bound based on the Wasserstein distance, which can be applied in the classification and regression problem.

**Theorem 2.** Let \( \mathcal{H} \) be a hypothesis family from \( \mathcal{X} \) to \([0, 1]\), with pseudo-dimension \( d \) and each member \( h \in \mathcal{H} \) is \( K \) Lipschitz. If we have \( T \) tasks generated by the underlying distribution and labelling function \( \{(D_1, f_1), \ldots, (D_T, f_T)\} \) with observation numbers \( m_1, \ldots, m_T \). If we adopt Wasserstein-1 distance as a similarity metric with cost function \( c(x, y) = ||x - y||_2 \), then for any simplex \( \alpha_t \in \mathbb{R}^d_+ \), and for \( \delta \in (0, 1) \), with probability at least \( 1 - \delta \), for \( h_1, \ldots, h_T \in \mathcal{H} \), we have:

\[
\frac{1}{T} \sum_{t=1}^{T} R_t(h_t) \leq \frac{1}{T} \sum_{t=1}^{T} \hat{R}_{\alpha_t}(h_t) + C_1 \sum_{t=1}^{T} \left( \sqrt{\sum_{j=1}^{T} \frac{\alpha_{j,t}^2}{\beta_j}} \right) + C_2 \sum_{t=1}^{T} \sum_{i=1}^{T} \alpha_{i,t} \lambda_{i,t},
\]

where \( \beta_i = \frac{m_i}{m} \), \( C_1 = 2 \sqrt{2(d \log(\frac{4m}{2m}) + \log(\frac{16e}{\delta}))} \) and \( C_2 = 2 \sum_{t=1}^{T} \sum_{i=1}^{T} \alpha_{i,t} \lambda_{i,t} \cdot W_1(D_t, D_i) + \sum_{t=1}^{T} \sum_{i=1}^{T} \alpha_{i,t} \lambda_{i,t} \cdot W_1(D_t, D_i) \cdot \gamma_{i,t} \), where \( \gamma_{i,t} = \frac{\mu t m_i^{-1/s} + \mu m_i^{-1/s} + 2K \sum_{t=1}^{T} \sum_{i=1}^{T} \alpha_{i,t} \lambda_{i,t} \cdot W_1(D_t, D_i) \cdot \gamma_{i,t} \cdot \left( 1 \frac{1}{m_i} + \frac{1}{m_i} \right) \) and \( s \) and \( \mu \) are some specified constants.

The proof w.r.t. the Wasserstein-1 distance is analogue to the proof in the \( \mathcal{H} \) divergence but with different assumptions, which is also provided in the supplementary material.

**Remark.** The upper bound of the generalization error shows some intuitions that we should not only minimize the weighted empirical loss, but also minimize the empirical distribution distance between each task. Moreover, in the neural network based MTL approaches, these conclusions proposed a theoretical support for understanding the role of adversarial losses, which exactly minimize the “distance” between two distributions.

6 Adversarial Multi-task Neural Network

From the generalization error upper bound in the MTL framework, we developed a new training algorithm in the Adversarial Multi-task Neural Network (AMTNN). It consists in multiple iterative training steps: we optimize the parameters in the neural network, giving a fixed relation coefficient \( \alpha_1, \ldots, \alpha_T \); we estimate the relation coefficient, given fixed weights in the neural network.

Moreover, we have three kinds of parameters in AMTNN: \( \theta^1, \theta^2 \) and \( \theta^3 \), corresponding the parameter for feature extractor, adversarial loss (distribution similarity) and task loss.

To simplify the problem, we assume that each task has the same number of observations, i.e \( \beta_i = \frac{1}{T} \) and that the regularization coefficient will be the \( l_2 \) norm of \( \alpha_t \).

\(^*\)This bound can be extended to any Wasserstein \( p > 1 \) distance with restricting the hypothesis satisfies \( K \) Hölder condition.
6.1 Neural network Parameter updating

Given a fixed $\alpha_1, \ldots, \alpha_T$, according to the theoretical bound, we want to minimize the weighted empirical error $\frac{1}{T} \sum_{t=1}^{T} \hat{R}_{\alpha_t}(\theta^f_t, \theta^h_t)$ and the empirical distribution ‘distance’ $d(D_t, D_i)$ with $t, i = 1, \ldots, T$. Inspired by [7], the minimization of the distribution ‘distance’ is equivalent to maximization the adversarial loss $\hat{E}_{t,i}(\theta^f_t, \theta^d_{t,i})$. Overall, we have the following loss function with the trade-off coefficient $\rho$:

$$
\min_{\theta^f_t, \theta^h_t} \max_{\alpha_t} \sum_{i=1}^{T} \hat{R}_{\alpha_t}(\theta^f_t, \theta^h_t) + \rho \sum_{i,t=1}^{T} \alpha_t \hat{E}_{t,i}(\theta^f_t, \theta^d_{t,i})
$$

We will give some discussions about loss (1):

- For a given task $t$, the sum loss can be expressed as:
  $$\frac{1}{T} \sum_{t=1}^{T} \alpha_{t,i} \sum_{x \in D_i} f((x, y), \theta^f_t, \theta^h_t),$$
  with $f$ the cross entropy loss, meaning the empirical loss is a weighted sum of all the task loss, determined by task relation coefficient $\alpha_t$. This part is coherent with [5], in which they did not provide a theoretical understanding.

- The adversarial loss $\hat{E}_{t,i}(\theta^f_t, \theta^d_{t,i})$: since it is symmetric metric and we only need to compute $\hat{E}_{t,i}$ for $t < i$. Motivated by [7], for a pair of observed unlabeled tasks $(D_t, D_i)$, the neural network will output a score $[0, 1]$ for discriminating which distribution it comes. Supposing the output function is $g_{t,i}(x, (\theta^f_t, \theta^d_{t,i})) = g_{t,i}(x)$, the adversarial loss will be the followings under different distance metrics:

  - $\mathcal{H}$ divergence, $\hat{E}_{t,i} = \left( \sum_{x \in D_i} \log(g_{t,i}(x)) + \sum_{x \in D_t} \log(1 - g_{t,i}(x)) \right)$

  - Wasserstein-1 distance. Since the primal form of Wasserstein-1 distance has a high computational complexity, then we adopted the same strategy with [20], by estimating the empirical Kantorovich-Rubinstein duality. We have $W_1(D_t, D_i) = \frac{1}{\kappa} \sup ||f|| \leq \kappa \left( E_{x \in D_i} g_{t,i}(x) - E_{x \in D_t} g_{t,i}(x) \right)$. Combining the conclusion in Theorem 2, we can derive $\hat{E}_{t,i} = \left( E_{x \in D_i} g_{t,i}(x) - E_{x \in D_t} g_{t,i}(x) \right)$.

6.2 Relation coefficient updating

The second step after updating neural network parameter, we need to re-estimate the coefficient $\alpha_1, \ldots, \alpha_T$ when giving fixed $\theta^f_t, \theta^h_t, \theta^d_t$. According to the theoretical guarantees, we need to solve the following convex constraint optimization problem:

$$
\min_{\alpha_1, \ldots, \alpha_T} \sum_{t=1}^{T} \hat{R}_{\alpha_t}(\theta^f_t, \theta^h_t) + \kappa_1 \sum_{t=1}^{T} \left\| \alpha_t \right\|_2
$$

$$
+ \kappa_2 \sum_{i,t=1}^{T} \alpha_{t,i} \hat{d}_{t,i}(\theta^f_t, \theta^d_{t,i})
$$

s.t \left\| \alpha_t \right\|_1 = 1, \quad \alpha_{t,i} \geq 0 \quad \forall t, i
$$

Figure 1: AMTNN: The general framework of Adversarial Multi-task Neural Network.

**Algorithm 1 AMTNN Updating algorithm**

Require: Samples from different tasks $\{D_t\}_{t=1}^{T}$, initial coefficient $\{\alpha_t\}_{t=1}^{T}$ and learning rate $\eta$

Ensure: Neural network parameters $\theta^f, \theta^h, \theta^d$ and relationship coefficient $\alpha_1, \ldots, \alpha_T$

1: for mini-batch of samples $\{(x^t_i, y^t_i)\}$ from $\{D_t\}_{t=1}^{T}$ do
2:  For the distribution pair $(t, i)$ with $(t < i)$, computing the adversarial loss $\hat{E}_{t,i}(\theta^f_t, \theta^d_{t,i})$.
3:  For each task $t$, define the empirical loss matrix $\hat{R}_{t,i} = \sum_{(x^t_i, y^t_i) \in D_t} \ell((x^t_i, y^t_i), \theta^f_t, \theta^h_t)$ computing the label loss:
4:  Updating $\theta^f_t, \theta^h_t, \theta^d_t = \theta^f_t - \eta \frac{\partial \hat{R}_{\alpha_t}}{\partial \theta^f_t}$, and $\theta^f_t = \theta^f_t - \eta \frac{\partial \hat{E}_{t,i}}{\partial \theta^d_{t,i}}$.
5:  Updating $\theta^d_{t,i} (t < i)$: $\theta^d_{t,i} = \theta^d_{t,i} + \eta \left( \alpha_{t,i} + \kappa_1 \frac{\partial \hat{E}_{t,i}}{\partial \theta^d_{t,i}} \right)$.
6:  end for
7: Re-estimating $\{\alpha_t\}_{t=1}^{T}$ by optimizing (2).

Where $\kappa_1$ and $\kappa_2$ are hyper-parameters and $\hat{d}$ is the estimated distribution ‘distance’, having different forms with different similarity metrics:

- $\mathcal{H}$ divergence. According to [6, 21, 7], the distribution ‘distance’ is proportional to accuracy of the discriminator $\theta^d$, i.e we applied $g_{t,i}(x)$ to predict $x$ coming from distribution $t$ or $i$. The prediction accuracy reflects the difficulty to distinguish two distributions. Hence, we set $\hat{d}_{t,i}$ is the accuracy of the discriminator $g_{t,i}(x)$;

- Wasserstein-1 distance. According to [20], they approximated $\hat{d}_{t,i} = -\hat{E}_{t,i}$.

We also assume $\hat{d}_{t,t} = 0$ since the discriminator cannot distinguish two identical distributions. Moreover, the expected loss $\alpha_t \lambda_{t,i}$ is omitted since we assume that $\lambda_{t,i}$ is much smaller.
than the empirical term. Then, we only use the empirical parts to re-estimate the relationship coefficient.

As it is mentioned in the theoretical part, the \( L_2 \) norm regularization aims at preventing all the relation coefficient to be concentrated on the current task \( \alpha_{t,t} \). The theoretical bound proposed an elegant interpretation for training AMTNN, which is shown in the Algorithm 1.

### 6.3 Training algorithm

The general framework of the neural network is shown in Figure 1. We proposed a complete iteration step on how to update the neural network parameters and relation coefficients in Algorithm 1. When updating the feature extraction parameter \( \theta^f \), we applied gradient reversal [7] in the training procedure. We also add the gradient penalty [29] for improving the Lipschitz property when training the Wasserstein distance based adversarial loss.

### 7 Experiments

We evaluate the modified AMTNN method on two benchmarks, that is digits datasets and the Amazon sentiment dataset. We also consider the following approaches, as baselines to make comparisons:

- **MTL_uni:** the vanilla MTL framework where \( \frac{1}{T} \sum_{t=1}^{T} \hat{R}_t(\theta^f, \theta^h) \) is minimized;
- **MTL_weighted:** minimizing \( \frac{1}{T} \sum_{t=1}^{T} \hat{R}_t(\theta^f, \theta^h), \) computation of \( \alpha_t \) depending on \( \hat{R}_t(\cdot, \cdot) \), similarly to Murugesan et al. [5].
- **MTL_disH and MTL_disW:** from [8]. We apply the same kind of loss function but with two different adversarial losses (\( \mathcal{H} \) divergence and Wasserstein distance) and a general neural network without special part for the NLP;
- **AMTNN_H and AMTNN_W:** proposed approaches with two different adversarial losses, \( \mathcal{H} \) divergence and Wasserstein distance respectively.

The details on the experiments (e.g., dataset descriptions, neural network training configurations, hyper-parameter settings) are provided in the supplementary material.

#### 7.1 Digit recognition

We first evaluate our algorithm on three benchmark datasets of digit recognition, which are datasets, MNIST, MNIST-M, and SVHN. The MTL setting is to jointly learn a system to recognize the digits from the three datasets, which can differ significantly. In order to show the effectiveness of MTL, only a small portion of the original dataset is used for training (i.e., 3K, 5K and 8K for each task).

We use the LeNet-5 architecture, and define the feature extractor \( \theta^f \) as the two convolutional layers of the network, followed by multiple blocks of two fully connected layers as label prediction parameter \( \theta^h \) and discriminator parameter \( \theta^d \). Five repetitions are conducted for each approach, and the average test accuracy (%) reported in Table 1. We also show the estimated coefficient \( \{\alpha_t\}_{t=1} \) of AMTNN_H and AMTNN_W, in Fig. 2.

![Figure 2: Estimated task relation coefficients matrix from the two proposed algorithms, with training set of 8K instances.](image)

**Discussion**

Reported results show that the proposed approaches outperform all the baselines in the task average and also in most single tasks. Particularly for the AMTNN_W, it outperforms the baselines with 1.0% - 2.9% in the test accuracy. The reason can be that the Wasserstein-1 distance is more efficient for measuring the high dimensional distribution, which has been verified theoretically [22]. Moreover, the \( \mathcal{H} \) divergence-based approach (AMTNN_H) outperforms the baselines without significant increment (< 0.3%). The reason may be that the VC-dimension with \( \mathcal{H} \) divergence is not good metric for measuring high dimensional complex dataset, coherently with Li et al. [10].

As for the coefficients \( \alpha_t \), the proposed algorithms appear robust at estimating these task relationships, with almost identical values under different similarity metrics. Moreover, in contrast to previous approaches, we obtain a non-symmetric matrix with a better interpretability. For instance, when learning for the MNIST dataset, only information from MNIST_M is used, which is reasonable since these two tasks have the same digits configurations with different background, while SVHN is different in most ways (i.e., digits taken from street view house numbers). However, when learning MNIST_M, information from SVHN is beneficial because it provides some information on the background, which is absent from MNIST but similar to MNIST_M. Therefore, the information of both tasks are involved in training for MNIST_M.

In order to show of role of weighted sum, we use t-SNE to visualise in Fig. 3 the embedded space of the MNIST task from the training data. Information from SVHN is not relevant for learning MNIST as \( \alpha_{1,2} = 0 \) (see Fig. 2), such that SVHN data is arbitrarily distributed in the embedded space without influence on the final result. At the same time, information from MNIST_M is used for training on MNIST task (\( \alpha_{1,3} = 0.28 \)), which can be seen by a slight overlap in the embedded space. From that perspective, the role of weighted loss, which helps us to achieve some reasonable modifications of the decision boundary, is trained by the relevant and current tasks jointly. For small scale task (typically the MTL scenario), during the test procedure, the agent predicts the labels by borrowing its neighbors (relevant tasks) information. This is coherent with the Probabilistic Lipschitzness condition [4].
### Tables:

**Table 1:** Average test accuracy (in %) of MTL algorithms on the digits datasets.

| Approach | 3K | 5K | 8K |
|----------|----|----|----|
| MNIST    | 97.23 | 77.12 | 81.13 |
| MNIST_M  | 91.43 | 74.07 | 85.04 |
| MNIST_M  | 81 | 83.05 | 71.19 |
| SVHN     | 71.33 | 73.81 | 85.92 |
| Average. | 81.13 | 74.07 | 85.04 |
| MTL_uni  | 93.23 | 76.85 | 81.13 |
| MRL_weighted | 89.09 | 73.69 | 77.13 |
| MTL_disH | 91.92 | 82.68 | 80.45 |
| MTL_disW | 95.47 | 83.48 | 82.86 |
| AMTNN_H  | 97.47 | 76.28 | 76.06 |
| AMTNN_W  | 97.67 | 82.5 | 85.36 |

**Table 2:** Average test accuracy (in %) of MTL algorithms in the sentiment dataset.

| Approach | 1000 | 1600 |
|----------|------|------|
| Book     | 81.31 | 81.35 |
| DVDs     | 78.44 | 80.14 |
| Kitchen  | 80.91 | 80.72 |
| Elec     | 85.31 | 82.86 |
| Average. | 83.28 | 83.92 |
| MTL_uni  | 88.95 | 88.54 |
| MRL_weighted | 81.20 | 79.81 |
| MTL_disH | 87.34 | 87.80 |
| MTL_disW | 84.82 | 87.26 |
| AMTNN_H  | 85.53 | 85.82 |
| AMTNN_W  | 86.94 | 85.63 |

### Figure 3:

Figure 3: tSNE in the feature space of task MNIST in AMTNN_W for 8K. Red: MNIST dataset; Blue: MNIST_M dataset; Green: SVHN data set.

### Discussions

We found the proposed approaches outperform all the baselines in the task average and also in the most tasks. Meanwhile, we observed that the role of adversarial loss (MTL_disH, MTL_disW, AMTNN_H and AMTNN_W) is not that significant (gains < 0.25%), compared to the results on the digits datasets. The possible reason is that we applied algorithm on the pre-processed feature instead of the original feature, making the discriminator $\theta^d$ less powerful in the feature adaptation. By the contrary, adding the weighted loss can improve performance by 0.4% ~ 0.9%, enhancing the importance of role of the explicit similarity, which is also coherent with Murugesan et al. [5].

### 8 Conclusion

In this paper, we propose a principle approach for using the task similarity information in the MTL framework. We first derive an upper bound of the generalization error in the MTL. Then, according to the theoretical results, we design a new training algorithm on the Adversarial Multi-task Neural Network (AMTNN). Finally, the empirical results on the benchmarks are showing that the proposed algorithm outperforms the baselines, reaffirming the benefits of theoretical insight in the algorithm design.

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