On the vacuum-polarization Uehling potential for a Fermi charge distribution

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Abstract. We present analytical formulas for the vacuum-polarization Uehling potential in the case where the finite size of the nucleus is modeled by a Fermi charge distribution. Using a Sommerfeld-type development, the potential is expressed in terms of multiple derivatives of a particular integral. The latter and its derivatives can be evaluated exactly in terms of Bickley-Naylor functions, whose connection to the Uehling potential was already pointed out in the pure Coulomb case, and of usual Bessel functions of the second kind. The cusp and asymptotic expressions for the Uehling potential with a Fermi charge distribution are also provided. Analytical results for the higher-order-contribution Källén-Sabry potential are given.

1 Introduction

Vacuum polarization in light atoms and ions is an important topic of Quantum Electrodynamics (QED) \cite{1,2}. In 1935, Uehling proposed a formula for the interaction potential between two point-like electric charges which contains an additional term responsible for the electric polarization of the vacuum \cite{3}. The evaluation of the vacuum-polarization potential of a point charge moving in the Coulomb field of a nucleus is a hard task. The vacuum-polarization correction for an electron in a nuclear Coulomb field can be described, up to the first order in \( \alpha \), by a correction to the Coulomb potential:

\[
\begin{align*}
\delta V(r) &= -\frac{Z e^2}{4 \pi \epsilon_0} \left[ 1 + \frac{2}{3 \pi} \frac{e^2}{4 \pi \epsilon_0 \hbar c} \int_1^\infty \sqrt{t^2 - 1} \left( \frac{1}{t^2} + \frac{1}{2t^4} \right) \right. \\
&\left. \times e^{-2mctr/\hbar} dt \right],
\end{align*}
\]

(1)

where \( Z \) is the nuclear charge, \( e \) the electron charge, \( \epsilon_0 \) the dielectric constant, \( \hbar \) the reduced Planck constant, and \( c \) the speed of light. We can write, in atomic units (\( m = \hbar = e = 1 \)) and setting \( 4 \pi \epsilon_0 = 1 \):

\[
\begin{align*}
V(r) &= -\frac{Z}{r} + \delta V(r),
\end{align*}
\]

(2)

where \( \delta V(r) \) reads

\[
\begin{align*}
\delta V(r) &= -\frac{2\alpha Z}{3 \pi r} \int_1^\infty \sqrt{t^2 - 1} \left( \frac{1}{t^2} + \frac{1}{2t^4} \right) e^{-2\alpha Z/\hbar} dt,
\end{align*}
\]

(3)

referred to as the Uehling potential \cite{3}. We keep \( \alpha \) and \( c \) in the same equation, although in atomic units (which will be used throughout the paper), one has \( \alpha = 1/c \). Formula (3) was obtained from equation (44) of Wichmann and Kröll \cite{4}, using the transformation \( t = \sqrt{y^2 + 1} \). The integral is usually evaluated numerically, but it is worth mentioning that Pyykkö et al. \cite{5,6} derived a two-parameter fitting expression:

\[
\begin{align*}
\delta V(r) &= -\frac{\alpha Z}{r} \left[ e^{-d_1 r^2} c_1 \left( \ln \left( \frac{r}{\alpha} \right) - c_2 \right) \\
&+ \frac{1 - e^{-d_1 r^2}}{c_3} e^{-2r/\alpha} d_2 \left( \frac{\alpha}{c} \right)^{0.5} + \left( \frac{\alpha}{c} \right)^{1.5} \right],
\end{align*}
\]

(4)

with \( c_1 = 2/(2\pi) \), \( c_2 = 5/6 + \gamma_E \), \( \gamma_E \) being the Euler-Mascheroni constant \cite{7}, \( c_3 = 4\sqrt{\pi} \), \( d_1 = 0.678 \) \( 10^7 \) and \( d_2 = 1.4302 \) (the formula (A1) of Ref. \cite{5} contains an error which was corrected in Ref. \cite{6} : \( \alpha/c_3 \) must be replaced by 1/\( c_3 \)). An exact expression, in terms of sine and cosine integral functions, was recently obtained by Mező \cite{8}:

\[
\begin{align*}
\delta V(r) &= -\frac{4\alpha Z}{\pi r} \int_0^1 x(1-x) \left[ \text{Shi} \left( \frac{c r}{\sqrt{x(1-x)}} \right) \\
&- \text{Chi} \left( \frac{c r}{\sqrt{x(1-x)}} \right) \right] dx,
\end{align*}
\]

(5)

where

\[
\text{Shi}(z) = \int_0^z \frac{\sinh(t)}{t} dt
\]

(6)
is the hyperbolic sine integral and

\[
\text{Chi}(z) = \gamma_E + \ln(z) + \int_0^z \frac{\cosh(t) - 1}{t} \, dt \tag{7}
\]

the hyperbolic cosine integral. The limit of \(\delta V(r)\) for \(cr \ll 1\) was derived by Berestetski, Lifshitz and Pitaevskii in reference [9], but the calculation is rather tedious. We show that equation (5) enables one to obtain the result immediately. Noticing that only \(\text{Chi}(z)\) will contribute and that

\[
\text{Chi}(z) \approx \gamma_E + \ln(z) \quad \text{when} \quad z \ll 1, \tag{8}
\]

we get, since

\[
\int_0^1 x(1-x) \, dx = \frac{1}{6} \tag{9}
\]

and

\[
\int_0^1 x(1-x) \ln \left( \frac{cr}{\sqrt{x(1-x)}} \right) \, dx = \frac{5 + 6 \ln(cr)}{36}, \tag{10}
\]

the asymptotic form

\[
\delta V(r) \approx -\frac{2\alpha Z}{3\pi r} \left[ -\gamma_E - \frac{5}{6} + \ln \left( \frac{1}{cr} \right) \right] \quad \text{when} \quad r \ll \frac{1}{c}. \tag{11}
\]

Several methods have been proposed to calculate the integral of equation (3): for instance, Huang [10] derived series expansions which converge for all values of \(r\), whereas Fullerton and Rinker [11] have found rational approximations. Klarsfeld [12], however, expressed the Uehling potential simply in terms of Bessel functions and their integrals, thus generalizing a formula by Pauli and Rose [13] showing, that way, that on the contrary to what is mentioned in numerous textbooks on quantum mechanics, an analytical expression does exist. Such expression, which involves Bickley-Naylor functions [14–16], was rediscovered by Frolov and Wardlaw in the pure Coulomb case [17], and usual Bessel functions of the second kind. In Section 4, the asymptotic expressions for \(cr \ll 1\) and \(cr \gg 1\) are discussed and analytical results for the higher-order-contribution Källén-Sabry potential are given in Section 5.

2 Exact expansion of the potential for a Fermi density

For a charge distribution \(\rho(x)\) normalized to

\[
\int d^3x \rho(x) = 4\pi \int_0^\infty x^2 \rho(x) \, dx = Z, \tag{12}
\]

the Uehling potential \(V(r)\) may be generalized to [23]

\[
\delta V(r) = -\frac{2\alpha Z}{3\pi r} \int d^3x \rho(x) \times \int_1^\infty dt \sqrt{t^2 - 1} \left( \frac{1}{t^2} + \frac{1}{2t^4} \right) \frac{e^{-2tR}}{R}, \tag{13}
\]

with \(R = |\vec{r} - \vec{x}| = \sqrt{r^2 - 2rx \cos(\theta) + x^2}\). The integral over \(\theta\) and \(\phi\), the angles of \(\vec{x}\) in spherical coordinates, can be expressed as

\[
J(x, r) = \int_0^{2\pi} \int_0^\pi e^{-\lambda R} \frac{1}{R} d\Omega = 2\pi \int_1^\infty \frac{1}{\sqrt{t^2 - 2tx + x^2}} \, dt e^{-\lambda R} \tag{14}
\]

Using \(R\) as independent variable, we find

\[
J(x, r) = -\frac{2\pi}{\lambda x} \int_{r-x}^{r+x} e^{-\lambda R} dR = \frac{2\pi}{\lambda \sqrt{x}} \left[ e^{-\lambda(r-x)} - e^{-\lambda(r+x)} \right]. \tag{15}
\]
We may therefore express $\delta V(r)$ as

$$
\delta V(r) = -\frac{2a^2}{3r} \int_0^\infty dx \rho(x) \int_1^\infty \sqrt{t^2 - 1} \left( \frac{1}{t^3} + \frac{1}{2t^5} \right)
\times \left( e^{-2\xi |r-x|} - e^{-2\xi (r+x)} \right) dt,
$$

(16)

which is the form given by Fullerton and Rinker [11]. Setting, $z = 2c(r+x)$, let us define the integral

$$
g(z) = \int_1^\infty \sqrt{t^2 - 1} \left( \frac{1}{t^3} + \frac{1}{2t^5} \right) e^{-zt} dt
$$

(17)

and consider the case of a Fermi-like distribution

$$
\rho(x) = \rho_0 f(x) = \frac{\rho_0}{1 + e^{(x-z)/a}}.
$$

(18)

The surface thickness is equal to $a = t/(4 \ln 3)$ with $t = 2.3$ fm [24] and $\xi = 2.2677 \times 10^{-5}$ $a_0$, $a_0$ being the Bohr radius. For recent reviews on the finite nuclear charge distributions, see reference [25]. Several methods have been developed to evaluate the integrals involving Fermi functions. They often rely on a particular representation of the Fermi distribution, such as the Matsubara expansion [26] or an infinite sum of contour integrals in the complex energy plane [27], etc. In order to calculate the integral

$$
\int_0^\infty dx \rho(x) x g(2c(r+x)) = \rho_0 \int_0^\infty dx f(x) x g(2c(r+x)),
$$

(19)

it is interesting to resort to the following exact development [28] (called here abusively “Sommerfeld-like” expansion as a reference to a similar expression introduced by Sommerfeld in solid-state physics [29]):

$$
\int_0^\infty f(y) H(y) dy = \int_0^\infty H(y) dy + \sum_{n=0}^\infty a^{2n+2} \left( \frac{2}{2n} - \frac{1}{2n} \right)
\times \zeta(2n+2) \frac{d^{2n+1} H}{dy^{2n+1}}(\xi) + R,
$$

(20)

where $\zeta$ represents the Riemann zeta function

$$
\zeta(n) = \sum_{n=1}^\infty \frac{1}{n^n},
$$

(21)

which is related to the Bernoulli numbers $B_n$ by

$$
\zeta(2n) = 2^{2n-1} \frac{\pi^{2n}}{(2n)!} B_n
$$

(22)

and

$$
\zeta(2) = \frac{\pi^2}{6} \quad \text{and} \quad \zeta(4) = \frac{\pi^4}{90},
$$

(23)

the residual term being equal to

$$
R = \sum_{n=1}^\infty (-1)^n e^{-n\xi/a} \int_0^\infty H(y) e^{-ny/a} dy.
$$

(24)

The normalization condition (12) gives

$$
\rho_0 = \frac{3Z}{4\pi \xi^3 N},
$$

(25)

with

$$
N = 1 + \frac{\pi^2 a^2}{\xi^2} + 6 \frac{a^3}{\xi^3} \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^3} e^{-n\xi/a}.
$$

(26)

Inspection of equation (20) shows that the problem boils down to the calculation of multiple derivatives of $g(z)$.

### 3 Exact expressions using Bickley-Naylor functions

#### 3.1 Expansion in terms of the exponential integral function

Considering the general class of functions:

$$
\chi_n(z) = \int_1^\infty \frac{1}{t^n} \left( 1 + \frac{1}{2t^2} \right) \left( 1 - \frac{1}{2t^2} \right)^{1/2} e^{-zt} dt.
$$

(27)

It is possible to write [10]

$$
\chi_2(z) = f_a(z) E_1(z) + f_b(z) e^{-z},
$$

(28)

where $f_a$ and $f_b$ are entire functions of $z$ and

$$
E_1(z) = \int_1^\infty \frac{e^{-zt}}{t} dt
$$

(29)

is the exponential integral. We have $g(z) = \chi_2(z)$. The functions $f_a$ and $f_b$ can be expanded in power series

$$
f_a(z) = \sum_{k=0}^\infty C_k z^{2k+1}
$$

(30)

and

$$
f_b(z) = \sum_{k=0}^\infty D_k z^{2k+1},
$$

(31)

where the coefficients $C_k$ and $D_k$ have been obtained by McKee [30] following Glauber et al. [31]. Then $d^m g(z)/dz^n$ is simple to obtain using Leibniz formula for the derivative of a product. It involves multiple derivative of the
exponential integral function of equation (29). One has:
\[
\frac{d^n}{dx^n}E_1(z) = (-1)^n e^{-z} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{k!}{z^{k+1}}
\]
\[
= (-1)^n e^{-z} \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-1-k)!} \frac{1}{z^{k+1}}
\]
\[
= \left(-\frac{1}{z}\right)^n \Gamma(n,z),
\]
where
\[
\Gamma(n,z) = \int_z^{\infty} t^n e^{-t} dt
\]

is the incomplete Gamma function. Roesel calculated the function \(g(z)\) using an expansion in terms of Chebyshev polynomials which represent rapidly converging series [32,33]. In contrast to rational approximation, such an approach leads to an expansion where the accuracy is only determined by the number of terms taken into account in the series.

### 3.2 Expansion in terms of Bickley-Naylor functions

The above formulas are definitely useful, but in the following we show that it is possible to derive an exact formulation of \(g(z)\) in terms of Bickley-Naylor functions, formulation that may be of interest for obtaining exact expressions of the Uehling potential in case of a Fermi charge distribution. Making the change of variables \(t \rightarrow \cosh(u)\) in equation (17), we get
\[
g(z) = \int_0^1 e^{-z \cosh(u)}
\times \left(\frac{1}{\cosh(u)} - \frac{1}{2 \cosh^3(u)} - \frac{1}{2 \cosh^4(u)}\right) du
\]
\[
= -Ki_1(z) + \frac{1}{2} Ki_3(z) + \frac{1}{2} Ki_5(z),
\]
where \(Ki_n(z)\) is the Bickley-Naylor function defined as
\[
Ki_n(z) = \int_0^\infty e^{-z \cosh(t)} \cosh^n(t) dt,
\]
with \(Ki_0(z) = K_0(z)\) the modified Bessel function of zeroth order. The Bickley-Naylor functions satisfy the following differentiation and integration rules
\[
\frac{d}{dz} Ki_{n+1}(z) = -Ki_n(z)
\]
and
\[
Ki_{n+1}(z) = \int_z^{\infty} Ki_n(y) dy.
\]
They also obey the following recursion relation
\[
(n-1)Ki_n(z) = (n-2)Ki_{n-2}(z) + z [Ki_{n-3}(z) - Ki_{n-1}(z)]
\]
and follow the asymptotic form [7]
\[
Ki_n(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z} \left[1 + \frac{1}{(n-1)!} \right]
\times \sum_{m=1}^{\infty} \frac{(-1)^m (2m-1)!}{z^m 2^{2m-1}(m-1)!}
\times \sum_{k=0}^{m} \frac{(2k)![(n+m-k-1)!]}{8^k (k!)^2 (m-k)!},
\]
when \(r \to \infty\). It is worth mentioning that Hem Prabha and Yadav [34] proposed polynomial expressions for Bickley-Naylor functions up to \(n = 7\). Using equation (38), equation (34) can be put in the form
\[
g(z) = \left(\frac{7}{16} + \frac{z^2}{48}\right) K_0(z) - \left(\frac{9}{16} + \frac{z^2}{48}\right) Ki_1(z)
\]
\[
- \left(\frac{19z}{48} + \frac{z^3}{48}\right) Ki_2(z)
\]
and we therefore have to calculate
\[
\delta V(r) = -\frac{2a^2}{3r} \int_0^\infty dx x \rho(x)
\times \left[\left(\frac{7}{16} + \frac{c^2(r+x)^2}{12}\right) K_0(2c(r+x))
- \left(\frac{9}{16} + \frac{c^2(r+x)^2}{12}\right) Ki_1(2c(r+x))
- \left(\frac{19c(r+x)}{24} + \frac{c^2(r+x)^3}{48}\right) Ki_2(2c(r+x))\right].
\]

We consider here the terms involving the argument \(2c(r+x)\) in the exponential in equation (16), but the formalism can be applied in the same way to the part for which the argument of the exponential is \(2c|r-x|\). If we keep the first expression of \(g(z)\) given in equation (34), we have to consider three Bickley-Naylor functions, namely \(Ki_1, Ki_3\) and \(Ki_5\). The expression in equation (40) is simpler than equation (34), since it involves only Bickley functions \(Ki_1, Ki_2\) and the usual Bessel function \(K_0\).

Another possibility would be to express \(g(z)\) in terms of functions \(K_0, K_1\) and \(K_i\) [32]:
\[
g(z) = \left(\frac{21 + z^2 + 48}{48}\right) K_0(z) - \left(\frac{19z^2 + z^4}{48}\right) K_1(z)
\]
\[
- \left(\frac{27 - 18z^2 - z^4}{48}\right) Ki_1(z).
\]
In equation (41), we need to consider the six following functions:

\[
\begin{align*}
H_1(x) &= xK_0(2c(r + x)), \\
H_2(x) &= x[2c(r + x)]^2 K_0(2c(r + x)), \\
H_3(x) &= xK_1(2c(r + x)), \\
H_4(x) &= x[2c(r + x)]^2 K_1(2c(r + x)), \\
H_5(x) &= x[2c(r + x)] K_2(2c(r + x)), \\
H_6(x) &= x[2c(r + x)]^3 K_2(2c(r + x)),
\end{align*}
\]

and then using, after expanding the function ln(x) in power series, the expression

\[
\int_0^\infty \frac{y^k}{1 + e^{(y-\xi)/a}} dy = \frac{\xi^k}{k+1} + \sum_{n=0}^{k} \frac{(2n+1)!}{\xi^{k+1}} \cdot \frac{[\xi^{k+1}]}{a^{2n+2}} \sum_{n=0}^{k-2n-1} \xi^{2n+2} + R, \quad \text{(49)}
\]

with

\[
R = \sum_{n=1}^{\infty} (-1)^n \frac{e^{-n\xi/a}}{n^{k+1}}, \quad \text{(50)}
\]

which is simpler than the expression published in references [32,35]:

\[
\int_0^\infty \frac{y^k}{1 + e^{(y-\xi)/a}} dy = 4\pi a^{k+3} \left\{ \left( \frac{\xi}{a} \right)^{k+3} \frac{1}{k+3} - (k+2)! \left( \frac{\xi}{a} \right)^{k+2} (k+2)! \right. \\
+ 2\pi \left( \frac{\xi}{a} \right)^{k+1} \sum_{p=0}^{\lfloor (k+1)/2 \rfloor} (2p+1-1) \left( \frac{\pi a}{\xi} \right)^{2p+1} \\
\left. \times \frac{|B_{2p+2}|}{(k+1-2p)!(2p+1)!} \right\}, \quad \text{(51)}
\]

involving Euler’s polylogarithm

\[
Li_n(z) = \sum_{q=1}^{\infty} \frac{z^q}{q^n} \quad \text{(52)}
\]

and Bernoulli number \( B_p \). \([X]\) is the integer part of \( X \).

### 3.3 Calculation using power expansion

One possibility to obtain an exact expression for the integral in equation (20) consists in expanding all the functions \( H_i \), \( i = 1, 6 \) in power series using the exact expression

\[
Ki_n(z) = 2^{n-2} \sum_{k=0}^{n-1} (-z/2)^k \frac{(n-k)}{k!(n-k-1)!} \left[ \frac{\Gamma \left( \frac{n-k}{2}, 0 \right)}{2} \right]^2 \\
+ (-z)^n \sum_{k=0}^{\infty} \frac{(z/2)^{2k}(2k)!}{(k!)^2(n+2k)!!} \\
\times \Phi(k+1) - \Phi(2k+1) + \Phi(2k+n+1) \\
- \gamma_E - \ln \left( \frac{z}{2} \right), \quad \text{(47)}
\]

where

\[
\Phi(k+1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}, \quad \text{(48)}
\]

and the issue boils down to the calculation of

\[
\frac{d^n}{dz^n} H_i(x), \quad i = 1, 6. \quad \text{(44)}
\]

Using Leibniz formula for the multiple derivative of a product, we obtain, for \( H_6(x) \):

\[
\frac{d^n}{dz^n} \left\{ x[2c(r + x)] K_2(2c(r + x)) \right\} \\
= \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dz^k} K_2(2c(r + x)) \frac{d^{n-k}}{dz^{n-k}} \left\{ x[2c(r + x)]^3 \right\}, \quad \text{(45)}
\]

where for \( k \geq 0 \):

\[
\frac{d^{n+k}}{dz^{n+k}} Ki_n(z) = (-1)^n \frac{d^k}{dz^k} K_0(z) = (-1)^{n+k} K_k(z). \quad \text{(46)}
\]
that, integrating $I(p, q)$ by parts, we obtain the recurrence relation

$$I(p, q) = \frac{1}{p+1} \left[ \gamma^{p+1} K_i(q) - \delta^{p+1} K_i(q) \right] - I(p+1, q-1),$$  

(55)

which can be initialized by

$$\begin{align*}
I(0, 0) &= \int_0^\delta y e^{-\alpha n y} K_i(y) dy, \\
I(1, 0) &= \int_0^\delta y e^{-\alpha n y} K_0(y) dy.
\end{align*}$$  

(56)

The integrals involved in the residual term $R$ (see Eq. (50)) can be calculated in a similar manner, being expressed through quantities of the kind

$$L_n(p, q) = \int_0^\delta y^p e^{-\alpha n y} K_i(y) dy,$$

(57)

with $\alpha$ strictly positive. Integral (53) corresponds to the case $\alpha=0$. Integrating the right-hand side of equation (57) by parts, we find:

$$L_n(p, q) = \frac{1}{\alpha n} \left[ e^{-\alpha n \gamma} K_i(\gamma) - e^{-\alpha n \delta} K_i(\delta) \right] + pL_n(p-1, q) - L_n(p, q-1),$$

which can be initialized by

$$\begin{align*}
L_n(0, 0) &= \int_0^\delta e^{-\alpha n y} K_0(y) dy, \\
L_n(1, 0) &= \int_0^\delta y e^{-\alpha n y} K_0(y) dy, \\
L_n(0, 1) &= \int_0^\delta e^{-\alpha n y} K_1(y) dy.
\end{align*}$$

(59)

Using the expression (see Ref. [7], 9.6.13 p. 375):

$$K_0(z) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left( \frac{z^2}{4} \right)^k \left[ \psi(k+1) - \ln \left( \frac{1}{2} \right) \right],$$

(60)

where

$$\psi(k+1) = -\gamma_E + \Phi(k+1),$$

(61)

as well as equation (47) for $n=1$, we see that the evaluation of the integrals in equation (59) reduces to the calculation of simple integrals of the kind

$$\int_0^\delta y^k dy = \frac{\delta^{k+1} - \gamma^{k+1}}{k+1}. $$

(62)

In addition, the last integrals defined in equation (59) involve terms of the kind

$$\int_0^\delta y^k e^{-\alpha n y} dy$$

$$= \frac{1}{(\alpha n)^{p+1}} \left[ \Gamma(k+1, \gamma \alpha n) - \Gamma(k+1, \delta \alpha n) \right].$$

(63)

The approach presented above applies for all the functions $H_1$ to $H_6$ given in equation (43), yielding an analytical expression for $\delta V(r)$. The quantity $R$ is much smaller (a few orders of magnitude) than the main terms of the summation in equation (20), and the series in equation (20) requires 10 terms for a convergence of 1%.

## 4 Asymptotic form of $g(z)$

The asymptotic expression of $g(z)$ for small values of $r$ can be determined following the method described in reference [9] for the Uehling potential of a point-like nucleus. We first split the integral in two parts:

$$\int_1^\infty \sqrt{t^2 - 1} \left( 1 + \frac{1}{2t^2} \right) e^{-2c tr} dt$$

$$= \int_1^\infty \sqrt{t^2 - 1} e^{-2c tr} dt + \int_1^\infty \sqrt{t^2 - 1} 2t^2 e^{-2c tr} dt$$

$$= I_1 + I_2,$$

(64)

and choose $t_1 (\frac{1}{c r} \gg t_1 \gg 1)$, such that

$$I_1 = \int_1^{t_1} \sqrt{t^2 - 1} e^{-2c tr} dt + \int_{t_1}^\infty \sqrt{t^2 - 1} 2t^2 e^{-2c tr} dt$$

$$= I_1 + I_2.$$  

(65)

$I_1$ can be estimated setting $r = 0$ and the change of variable $u^2 = t^2 - 1$ yields

$$J_1 = \int_0^{\sqrt{t_1^2 - 1}} \frac{u^2}{(u^2 + 1)^2} du$$

$$= \frac{1}{2} \left( -\sqrt{t_1^2 - 1} + \arctan \left( \frac{1}{\sqrt{t_1^2 - 1}} \right) \right).$$

(66)

For $t_1 \to \infty$, we get $J_1 \to \pi/4$. The other integrals are easy to evaluate. In $J_2$, we can neglect 1 in the square root, which yields, after two successive integrations by parts

$$J_2 = \int_{t_1}^\infty \frac{1}{t^2} e^{-2c tr} dt$$

$$= \frac{1}{t_1} e^{-2c t_1 r} + 2c r \ln(t_1) e^{-2c t_1 r}$$

$$- (2c r)^2 \int_{t_1}^\infty \ln(t) e^{-2c tr} dt.$$  

(67)

The first two terms tend to zero when $t_1 \to \infty$ and the last term is

$$\int_{t_1}^\infty \ln(t) e^{-2c tr} dt = \int_{2c r}^\infty \left[ \ln(u) - \ln(2c r) \right] e^{-u} du,$$  

(68)
yielding, for \( r \) close to zero
\[
\int_0^\infty \ln(t) e^{-2crt} dt = \frac{1}{2cr} \left[ \ln \left( \frac{1}{2cr} \right) - \gamma_E \right]
\]
and therefore, for \( r \to 0 \), we have
\[
J_2 \approx -2cr \ln \left( \frac{1}{2cr} \right).
\] (70)

For \( I_2 \) we can set directly \( r = 0 \):
\[
I_2 = \int_1^\infty \sqrt{t^2 - 1} \frac{dt}{2t^5} = \frac{\pi}{32}
\] (71)
and then
\[
I_1 + I_2 = \frac{\pi}{4} + \frac{\pi}{32} - 2cr \ln \left( \frac{1}{2cr} \right),
\] leading to the asymptotic form:
\[
g(z) \approx \frac{9\pi}{32} - 2cr \ln \left( \frac{1}{2cr} \right).
\] (73)

This result can be obtained by integrating the quantity \(-3\pi r^2 V(r)/(2\alpha Z)\) in expression (11) with respect to variable \((2cr)\), noticing that
\[
g(0) = \int_1^\infty \sqrt{t^2 - 1} \left( 2t^2 + 1 \right) dt = \frac{9\pi}{32}.
\] (74)

For large values of \( r \) we find the same asymptotic form
\[
\delta V(r) \approx -\frac{Z}{r} \left( 1 + \frac{\alpha}{4\sqrt{\pi}(cr)^{3/2}} e^{-2cr} \right),
\] (75)
which is the same as for the Uehling potential in the pure Coulomb case.

5 Higher-order contribution: the Källén-Sabry potential

The procedure presented for the Uehling potential can be used for the calculation of fourth-order QED corrections in \( \alpha^2(Z\alpha) \) (the corresponding Feynman diagrams are the two-loop diagram and three diagrams with an additional photon line within a single electron-positron loop) using the Källén-Sabry potential [36–38]:
\[
V_{KS}(r) = \frac{\alpha^2(Z\alpha)}{\pi^2 r} \int_0^\infty dx e^{x r} (x) \times \left[ L_0(2c|r - x|) - L_0(2c(r + x)) \right],
\] (76)
where
\[
L_0(x) = -\int_x^\infty L_1(u) du,
\] (77)
with
\[
L_1(u) = \int_1^\infty \left\{ \left( \frac{2}{3t^5} - \frac{8}{3t} \right) f(t) \right. \\
\left. + \left( \frac{2}{3t^4} + \frac{4}{3t^2} \right) \sqrt{t^2 - 1} \ln \left[ 8t(t^2 - 1) \right] \right\} + \frac{\sqrt{t^2 - 1}}{44} \left( \frac{2}{9t^6} + \frac{5}{4t^5} + \frac{2}{3t^3} - \frac{44}{9t} \right)
\]
\[
\times \ln \left[ \sqrt{t^2 - 1} + t \right] e^{-ut} dt
\] (78)
and
\[
f(t) = \int_1^\infty \left[ \frac{(3x^2 - 1) \ln \left[ \sqrt{x^2 - 1} + x \right]}{x(x^2 - 1)} - \ln \left[ \frac{8x(x^2 - 1)}{\sqrt{x^2 - 1}} \right] \right] dx.
\] (79)

An exact expression of \( f(t) \) is given in Appendix A. The method proposed in the present paper for the Uehling potential can be applied to the Källén-Sabry potential using the fit proposed by Indelicato [38]:
\[
L_1(u) = \left( a + b \sqrt{u} + cu + du^{3/2} + eu^2 + fu^{5/2} \right) \frac{e^{-u}}{u^{7/2}}
\] (80)
for \( u > 3 \). \( L_0(x) \) is then obtained by direct integration of the latter expression, the integration constant being fixed assuming that \( L_0 \to 0 \) when \( r \to \infty \). For \( u \leq 3 \), the form, inspired from Blomqvist [39], is
\[
L_1(u) = uh_2(u)[\ln(u)]^2 + uh_1(u) \ln(u) + h_0(u).
\] (81)
The coefficients \( a, b, c, d, e \) and \( f \) are given in Appendix A of reference [38] and functions \( h_0(u) \), \( h_1(u) \) and \( h_2(u) \), also derived by Indelicato, in Appendix B of the latter article (there are two small typos in Appendix B of Ref. [38]: \( g_1 \) and \( g_2 \) should be replaced by \( h_1 \) and \( h_2 \) respectively).

6 Conclusion

We proposed a closed formula for the Uehling potential in case of a Fermi charge distribution. The result combines a Sommerfeld-type expansion of the involved integral together with Bickley-Naylor functions and Bessel functions of the second kind. The Fermi distribution is widely used in QED computations in order to overcome the assumption of a point-like nucleus. The obtained expression is an extension of the result obtained by Frolov and Wardlaw in the pure Coulomb case and enables one to avoid numerical integration and analytical fitting formulas. The relations given here may also serve as guides for the derivation of rational approximations. We do not pretend that the formulas presented in this work are likely
to bring any significant improvement in numerical accuracy or speed. The most efficient method to compute the Uehling potential for any nuclear charge distribution (and in particular for the Fermi or Woods-Saxon distribution) is probably the rational approximation published by Fullerton and Rinker, which provides nine-digit accuracy with a low numerical cost. Thus, the most natural solution is to use this rational approximation, and to evaluate the integral by usual numerical integration methods. We would be happy if the mathematical expressions, relations and properties discussed in the present article could help to bring new ideas in the field. In the future, we plan to investigate fourth-order QED corrections in $\alpha^2(Z\alpha)$ using the Källén-Sabry potential [36–38], still in the case of a Fermi nuclear charge distribution.

Appendix A: Analytical expression for the function $f(t)$ involved in the Källén-Sabry potential

The function $f(t)$, defined in equation (79), is equal to

$$f(t) = \frac{2\pi^2}{3} - \ln(\eta) \ln \left[ \left( \frac{\eta^4 - 1}{\eta^2} \right) \eta^2 \right] + \operatorname{Li}_2 \left( -\frac{1}{\eta^2} \right) - 2\mathcal{R} \left[ \operatorname{Li}_2 (\eta^2) \right], \quad (A.1)$$

where $\eta = t + \sqrt{t^2 - 1}$, $\operatorname{Li}_2(z)$ is the dilogarithm function defined in equation (52) and $\mathcal{R}$ the real part. The integral form of the dilogarithm function is

$$\operatorname{Li}_2(z) = -\int_0^1 \frac{\ln(1 - zt)}{t} dt. \quad (A.2)$$

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