GEOMETRIC CHARACTERIZATIONS OF BIG CYCLES

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Abstract. A numerical equivalence class of $k$-cycles is said to be big if it lies in the interior of the closed cone generated by effective classes. We develop several geometric criteria that distinguish big classes from boundary classes. In particular, we construct for arbitrary cycle classes an analogue of the volume function for divisors.

1. Introduction

Let $X$ be an integral projective variety over an algebraically closed field. We will let $N_k(X)$ denote the vector space of numerical classes of $k$-cycles on $X$ with $\mathbb{R}$-coefficients. The pseudo-effective cone $\text{Eff}_k(X) \subset N_k(X)$ is defined to be the closure of the cone generated by all effective $k$-cycles. Classes that lie in the interior of the cone – known as big classes – are expected to exhibit special geometric properties. Our goal is to give geometric characterizations of big cycles similar to the well-known criteria for codimension 1 cycles and dimension 1 cycles.

An important tool for understanding big divisor classes is the volume function. The volume of a Cartier divisor $L$ is the asymptotic rate of growth of dimensions of sections of $L$. More precisely, if $X$ has dimension $n$,

$$\text{vol}(L) := \limsup_{m \to \infty} \frac{\dim H^0(X, \mathcal{O}_X(mL))}{m^n/n!}.$$ 

It turns out that the volume is an invariant of the numerical class of $L$ and satisfies many advantageous geometric properties. On a smooth variety $X$, divisors with positive volume are precisely the divisors with big numerical class.

One can interpret the volume of a divisor $L$ as an asymptotic measurement of the number of general points contained in members of $|mL|$ as $m$ increases. [DELV11] suggests studying a similar notion for arbitrary cycles. Let $N_k(X)_{\mathbb{Z}} \subset N_k(X)$ and $N_k(X)_{\mathbb{Q}} \subset N_k(X)$ denote the subsets generated by cycles with $\mathbb{Z}$-coefficients and $\mathbb{Q}$-coefficients respectively. Given a class $\alpha \in N_k(X)_{\mathbb{Z}}$, one easily verifies that there is a constant $C$ such that a cycle with class $m\alpha$ can pass through at most $Cm^{n-k}$ general points of $X$. The mobility function identifies the best possible constant $C$. (See Definition 6.13 for a more precise formulation.)

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Definition 1.1. Let \( X \) be an integral projective variety of dimension \( n \) and suppose \( \alpha \in N_k(X)_{\mathbb{Z}} \) for \( 0 \leq k < n \). The mobility of \( \alpha \) is
\[
\text{mob}(\alpha) = \limsup_{m \to \infty} \max \left\{ b \in \mathbb{Z}_{\geq 0} \mid \text{Any } b \text{ general points of } X \text{ are contained in a cycle of class } m\alpha \right\}
\]

Example 1.2. If \( X \) is a smooth variety and \( L \) is a Cartier divisor then \( \text{mob}(L) = \text{vol}(L) \) as shown in Example 6.16.

Example 1.3. Let \( \ell \) denote the class of a line on \( \mathbb{P}^3 \). The mobility of \( \ell \) is determined by an enumerative question: what is the minimal degree of a curve in \( \mathbb{P}^3 \) going through \( b \) general points?

It turns out that the answer to this question is not known (even asymptotically as the degree increases). [Per87] conjectures that the “optimal” curves are complete intersections of two divisors of equal degree, which would imply that \( \text{mob}(\ell) = 1 \). We discuss this interesting question in more depth in Section 7.1.

Example 1.4. We define the rational mobility of a class \( \alpha \in N_k(X)_{\mathbb{Z}} \) in a similar way by counting the number of general points lying on cycles in a fixed rational equivalence class inside of \( \alpha \) (see Definition 6.13). Rational mobility is interesting even for 0-cycles.

Let \( A_0(X) \) denote the set of rational equivalence classes of 0-cycles on \( X \). Recall that \( A_0(X) \) is said to be representable if the addition map \( X^{(r)} \to A_0(X)_{\text{deg}(r)} \) is surjective for some \( r > 0 \). In Section 7.2 we show for varieties over \( \mathbb{C} \) that \( A_0(X) \) is representable if and only if the rational mobility of the class of a point is the maximal possible value \((\dim X)!\).

The mobility function shares many of the important properties of the volume function for divisors. In particular mob is a homogeneous function, so the definition extends naturally to every element of \( N_k(X)_{\mathbb{Q}} \). Our first main theorem shows that bigness is characterized by positive mobility, confirming [DELV11, Conjecture 6.5].

Theorem A. Let \( X \) be an integral projective variety and suppose \( \alpha \in N_k(X)_{\mathbb{Q}} \) for \( 0 \leq k < \dim X \). Then \( \alpha \) is big if and only if \( \text{mob}(\alpha) > 0 \). In fact, mob extends to a continuous function on \( N_k(X) \).

Example 1.5. One might expect that a subvariety with “positive” normal bundle will have a big numerical class. However, [Voi10, Example 2.4] shows that even a subvariety with an ample normal bundle need not be big, indicating the need for a different approach.

Example 1.6. In contrast to the situation for divisors, it is possible for a subvariety \( V \) to have big numerical class even if no multiple of \( V \) moves in an algebraic family. For example, [FL82] constructs a surface \( S \) with ample normal bundle in a fourfold \( X \) such that no multiple of \( S \) moves in \( X \). [Pet09, Example 4.10] and a calculation of Fulger verify that \([S] \in N_2(X)\) is big.
Remark 1.7. Theorem A has analogues in the setting of other equivalence relations on cycles. The main step in the proof of Theorem A is to show that if $\text{mob}(\alpha) > 0$ then $\alpha$ is big; the proof does not use any special feature of $N_k(X)$ besides the ability to intersect against Cartier divisors. To prove the converse implication, one needs to work with an equivalence relation whose classes form a finitely generated group.

For example, suppose $X$ is an integral projective variety over $\mathbb{C}$. The statement of Theorem A holds for the subspace $N'_k(X) \subset H_{2k}(X, \mathbb{R})$ spanned by classes of cycles and for the homological analogue of the mobility function. (The other theorems below can be extended in a similar way.)

There is another natural generalization of the volume of a divisor. We can loosely interpret $\dim H^0(X, \mathcal{O}_X(L))$ as measuring “how much” the cycle $L$ can vary. For an arbitrary class $\alpha \in N_k(X)_\mathbb{Z}$, the analogous notion is the dimension of the components of the Chow variety which parametrize cycles of class $\alpha$. Theorem 5.1 shows that as $m$ increases the dimension of the components representing $m\alpha$ is bounded above by $Cm^{k+1}$ for some constant $C$.

Definition 1.8. Let $X$ be an integral projective variety and suppose $\alpha \in N_k(X)_\mathbb{Z}$ for $0 \leq k < \dim X$. The variation of $\alpha$ is

$$\text{var}(\alpha) = \limsup_{m \to \infty} \max \left\{ \frac{\dim W}{m^{k+1}/(k+1)!} \right\}$$

where $W \subset \text{Chow}(X)$ parametrizes cycles of class $m\alpha$.

It turns out that var is homogeneous and so extends naturally to classes in $N_k(X)_\mathbb{Q}$.

Example 1.9. [EH92] computes the dimension of the Chow variety of curves on $\mathbb{P}^n$. Let $\ell$ denote the class of a line in $\mathbb{P}^n$. [EH92, Theorem 3] shows that for sufficiently high degrees $d$ the maximal dimension of a component of $\text{Chow}(\mathbb{P}^n)$ with class $d\ell$ is

$$\frac{d^2 + 3d}{2} + 3(n-2)$$

so that $\text{var}(\ell) = 1$. The corresponding component of the Chow variety parametrizes planar curves of degree $d$.

The previous example is typical: components of $\text{Chow}(X)$ with large dimension tend to parametrize cycles that are as “degenerate” as possible. Thus variation usually does not accurately reflect the positivity of a class on $X$. Indeed, it is possible for non-big classes to have positive variation.

Example 1.10. Let $X$ be the blow-up of $\mathbb{P}^3$ at a closed point. Let $E$ denote the exceptional divisor and let $\alpha$ be the class of a line in $E$. For any positive integer $m$, every effective cycle of class $m\alpha$ is contained in $E$. Thus, the variation coincides with the variation of the line class on $\mathbb{P}^2$, showing that that $\text{var}(\alpha) = 1$. Note however that $\alpha$ is not big.
More generally, the pushforward of a big class on a subvariety \( V \) of \( X \) will have positive variation. Our second main theorem shows that this is essentially the only way to construct classes of positive variation. In other words, variation is a measure of positivity along subvarieties.

**Theorem B.** Let \( X \) be an integral projective variety and suppose \( \alpha \in N_k(X)_{\mathbb{Q}} \) for \( 0 \leq k < \dim X \). Then \( \text{var}(\alpha) > 0 \) if and only if there is a morphism \( f : Y \to X \) from an integral projective variety of dimension \( k + 1 \) that is generically finite onto its image and a big class \( \beta \in N_k(Y)_{\mathbb{Q}} \) such that some multiple of \( \alpha - f_*\beta \) is represented by an effective cycle.

The proof of Theorem B is modeled on a different geometric characterization of bigness for dimension 1 cycles. [BCE+02, Theorem 2.4] shows that a class \( \alpha \in N_1(X) \) is big if and only if any two points of \( X \) can be connected via a chain of effective cycles with numerical classes proportional to \( \alpha \). For cycles of higher dimension, the correct analogue is to consider chains of effective cycles that intersect in codimension 1 along “positive” subvarieties. The following definition encodes a strong version of this property.

**Definition 1.11.** Let \( X \) be an integral projective variety of dimension \( n \). Suppose that \( W \) is an integral variety and \( U \subset W \times X \) is a family of effective \( k \)-cycles. Denote the projection maps by \( p : U \to W \) and \( s : U \to X \). We say that the family is strongly big-connecting if \( s : U \to X \) is dominant and there is a big effective Cartier divisor \( B \) on \( X \) such that every \( p \)-horizontal component of \( s^*B \) is contracted by \( s \) to a subvariety of \( X \) of dimension at most \( k - 1 \).

The strongly big-connecting property forces our cycles to intersect in codimension 1 along a fixed “positive” \((k-1)\)-dimensional subvariety (controlled by the divisor \( B \)). It also implies that two general points in \( X \) can be connected by a length-two chain of members of the family. A typical example is given by fixing a \((n-k+1)\)-dimensional subspace \( V \subset H^0(X, \mathcal{O}_X(A)) \) for a very ample divisor \( A \) and taking complete intersections of \( (n-k) \) general elements of \( V \). This family is strongly big-connecting for any divisor \( A \in |V| \): the intersection of \( A \) with a general element of our family is contained in the base locus of \( V \) which has dimension \( k - 1 \).

The study of connecting chains leads to the following characterization of bigness for cycle classes. The special case when \( \alpha \) is a divisor class is one of the key steps in the proof of Theorem B.

**Theorem C.** Let \( X \) be an integral projective variety and let \( \alpha \in N_k(X) \) for \( 0 \leq k < \dim X \). Then \( \alpha \) is big if and only if there is some strongly big-connecting family of \( k \)-cycles of class \( \beta \) and a positive constant \( C > 0 \) such that \( \alpha - C\beta \) is pseudo-effective.

**Remark 1.12.** A slightly different perspective on the geometry of big cycles has been proposed by Voisin. [Voi10] conjectures that bigness can be characterized using the tangency behavior of cycles representing \( \alpha \) and shows
that this conjecture has interesting ties to the Hodge theory of complete intersections in projective space. Theorem C can be viewed as a step in this direction.

1.1. Organization. Section 2 reviews background material on cycles. Section 3 describes several geometric constructions for families of cycles. In Section 4 we bound the dimension of components of Chow$(X)$ parametrizing divisors. Section 5 introduces the variation function and proves Theorem B and Theorem C. Section 6 defines mobility and proves Theorem A. Finally, Section 7 discusses some examples of the mobility.

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2. Preliminaries

Throughout we work over a fixed algebraically closed field $K$. A variety will mean a quasiprojective scheme of finite type over $K$ (which may be reducible and non-reduced).

**Lemma 2.1.** Let $X$ be an irreducible variety. Suppose that $f : X \to Y$ is a rational map to a variety $Y$ and $g : X \to Z$ is a rational map to a variety $Z$. Let $F$ be a general fiber of $f$ over a closed point of $Y$ (so in particular $F$ is not contained in the locus where $g$ is not defined). Then $\dim(f(F)) \leq \dim(g(F)) + \dim Y$.

**Proof.** Let $U$ be an open subset of $X$ where both $f$ and $g$ are defined and let $h : U \to Y \times Z$ be the induced map. Since $F$ is a general fiber, $h(F \cap U)$ is dense in its closure in $Y \times Z$. By considering the first projection, we see that $\dim(h(U)) \leq \dim(g(F \cap U)) + \dim Y$. $\square$

We will often use the following special case of [RG71 Théorème 5.2.2].

**Theorem 2.2** ([RG71, Théorème 5.2.2]). Let $f : X \to S$ be a projective morphism of varieties such that some component of $X$ dominates $S$. There is a birational morphism $\pi : S' \to S$ such that the morphism $f' : X' \to S'$ is flat, where $X' \subset X \times_S S'$ is the closed subscheme defined by the ideal of sections whose support does not dominate $S'$.

2.1. Cycles. Suppose that $X$ is a projective variety. A $k$-cycle on $X$ is a finite formal sum $\sum a_i V_i$ where the $a_i$ are integers and each $V_i$ is an integral closed subvariety of $X$ of dimension $k$. The support of the cycle is the union of the $V_i$ (with the reduced structure). The cycle is said to be effective if each $a_i \geq 0$. For a $k$-dimensional closed subscheme $V$ of $X$, the fundamental cycle of $V$ is $\sum m_i V_i$ where the $V_i$ are the $k$-dimensional
components of the reduced scheme underlying $V$ and the $m_i$ are the lengths of the corresponding Artinian local rings $O_{V_i}$.

The group of $k$-cycles is denoted $Z_k(X)$ and the group of $k$-cycles up to rational equivalence is denoted $A_k(X)$. We will follow the conventions of [Ful84]. Chapter 19] defines a $k$-cycle on $X$ to be numerically trivial if its rational equivalence class has vanishing intersection with every weighted homogeneous degree-$k$ polynomial in Chern classes of vector bundles on $X$. Two cycles are numerically equivalent if their difference is numerically trivial. We let $N_k(X)$ denote the abelian group of numerical equivalence classes of $k$-cycles on $X$. By [Ful84, Example 19.1.4] $N_k(X)$ is a finitely generated free abelian group.

We also define
\[
N_k(X) := N_k(X) \otimes \mathbb{Q} \\
N_k(X) := N_k(X) \otimes \mathbb{R}
\]

Thus $N_k(X)$ is a finitely generated $\mathbb{R}$-vector space and there are natural injections $N_k(X) \mathbb{Z} \hookrightarrow N_k(X) \otimes \mathbb{Q} \hookrightarrow N_k(X)$. We denote the dual group of $N_k(X) \mathbb{Z}$ by $N^k(X) \mathbb{Z}$ and the dual vector spaces of $N_k(X) \mathbb{Q}$ and $N_k(X)$ by $N^k(X) \mathbb{Q}$ and $N^k(X)$ respectively.

[Ful84] defines the Chern class of a vector bundle $c_i(E)$ as an operation $A_k(X) \to A_{k-i}(X)$. It follows formally from the definition that Chern classes descend to maps $N_k(X) \to N_{k-i}(X)$.

**Convention 2.3.** When we discuss $k$-cycles on an integral projective variety $X$, we will always implicitly assume that $0 \leq k < \dim X$. This allows us to focus on the interesting range of behaviors without repeating hypotheses.

For a cycle $Z$ on $X$, we let $[Z]$ denote the numerical class of $Z$, which can be naturally thought of as an element in $N_k(X) \mathbb{Z}$, $N_k(X) \mathbb{Q}$, or $N_k(X)$. If $\alpha$ is the class of an effective cycle $Z$, we say that $\alpha$ is an effective class.

**Definition 2.4.** Let $X$ be a projective variety. The pseudo-effective cone $\overline{\text{Eff}}_k(X) \subset N_k(X)$ is the closure of the cone generated by all classes of effective $k$-cycles. The big cone is the interior of the pseudo-effective cone. The cone in $N^k(X)$ dual to the pseudo-effective cone is known as the nef cone and denoted $\text{Nef}^k(X)$.

We say that $\alpha \in N_k(X)$ is pseudo-effective (resp. big) if it lies in the pseudo-effective cone (resp. big cone), and $\beta \in N^k(X)$ is nef if it lies in the nef cone. For $\alpha, \alpha' \in N_k(X)$ we write $\alpha \preceq \alpha'$ when $\alpha' - \alpha$ is pseudo-effective.

Since $X$ is projective, $\overline{\text{Eff}}_k(X)$ is a full-dimensional salient cone. For any morphism of projective varieties $f : X \to Y$, there is a pushforward map $f_* : N_k(X) \to N_k(Y)$. It is clear that $f_*(\overline{\text{Eff}}_k(X)) \subset \overline{\text{Eff}}_k(Y)$. There is also a formal dual $f^* : N^k(Y) \to N^k(X)$ that preserves nefness.

**Convention 2.5.** Let $B$ be a Cartier divisor on an equidimensional projective variety $X$ of dimension $n$. We say that $B$ is big if for each reduced
component $X_i$ of $X$ we have that $h^0(X_i, mB|X_i) > |Cm^n|$ for some positive constant $C$. This implies that the corresponding Weil divisor has big class. With this definition bigness of a Cartier divisor is preserved by generically-finite pullback.

The following lemmas record some basic properties of pseudo-effective cycles.

**Lemma 2.6.** Let $f : X \to Y$ be a surjective morphism of projective varieties. For any effective cycle $Z$ on $Y$, there is an effective cycle $V$ on $X$ such that $f_* V = cZ$ for some $c > 0$. In particular $f_*$ takes big classes to big classes.

**Remark 2.7.** One cannot conclude immediately from Lemma 2.6 that the induced morphism $f_* : \text{Eff}_k(X) \to \text{Eff}_k(Y)$ is surjective, because the image of a closed cone under a linear map need not be closed. Nevertheless this statement is true; see [FL13, Corollary 3.20].

**Proof.** It suffices to prove the first statement when $Z$ is an irreducible subvariety of $Y$. Let $T$ be any integral subvariety of $X$ that is mapped surjectively onto $Z$ by $f$. By cutting $T$ down by very ample divisors, we may find a subvariety $V$ of $X$ that has the same dimension as $Z$ such that $f|_V$ surjects onto $Z$. Then $f_* V$ is a positive multiple of $Z$.

To see the final statement, fix a big effective class $\alpha \in N^k(Y)$. Choose an effective class $\beta \in N^k(X)$ such that $f_* \beta = \alpha$. Then for any big class $\gamma \in N^k(X)$, we have that $\gamma - c\beta$ is pseudo-effective for some $c > 0$; thus $f_* \gamma \geq c\alpha$. □

**Lemma 2.8.** Let $X$ be a projective variety of dimension $n$ and let $A$ be an ample Cartier divisor on $X$. Then $[A^{n-k}] \in N^k(X)$ is big.

**Proof.** Let $Z$ be a big effective $k$-cycle on $X$. Then $V = \text{Supp}(Z)$ is also big, so it suffices to prove that $j[A^{n-k}] - [V]$ is an effective class for some integer $j > 0$. Choose a sufficiently large integer $m$ such that $O_X(mA) \otimes I_V$ is globally generated. Let $V'$ be the intersection of $(n-k)$ general divisors in $|mA|$ containing $V$; then $V' \to V$ is an effective $k$-cycle. Set $j = m^{n-k}$. □

**Lemma 2.9.** Let $X$ be a projective variety.

1. If $A$ is a nef Cartier divisor then $\cdot A : N^k(X) \to N^{k-1}(X)$ takes $\text{Eff}_k(X)$ into $\text{Eff}_{k-1}(X)$.
2. If $\alpha \in N^k(X)$ is a big class and $A$ is an ample Cartier divisor then $\alpha \cdot A \in N_{k-1}(X)$ is also big.
3. Let $D$ be the support of an effective big $j$-cycle $Z$ with injection $i : D \to X$. If $\alpha \in N^k(D)$ is big (for $0 \leq k \leq j$) then $i_* \alpha \in N^k(X)$ is big.

**Proof.** By continuity and homogeneity it suffices to prove (1) when $A$ is very ample. Let $Z$ be an integral $k$-dimensional subvariety; for sufficiently general elements $H \in |A|$, the cycle underlying $H|_Z$ is an effective cycle of
class $[Z] \cdot A$, proving (1). To see (2), write $\alpha = \alpha' + cA^{n-k}$ for some pseudo-effective class $\alpha'$ and some small $c > 0$. Applying (1), it suffices to note that $A^{n-k+1}$ is a big class by Lemma 2.8. Similarly, to show (3) fix an ample Cartier divisor $A$ on $X$ and consider the class $\beta := (i^* A)^j - k$ in $N_k(D)$. Choose $m$ large enough so that $m[D] \succeq [Z]$. By the projection formula $i^*(m\beta) \succeq [A^{j-k} \cdot Z]$, so that $i^*\beta$ is big on $X$ by (2). Writing $\alpha = \alpha' + c\beta$ for a sufficiently small $c$, the claim follows from the fact that $i^*$ preserves pseudo-effectiveness.

\[ \Box \]

2.2. Analytic lemmas. We are interested in invariants constructed as asymptotic limits of functions on $N_k(X)_\mathbb{Z}$. The following lemmas will allow us to conclude several important properties of these functions directly from some easily verified conditions.

Lemma 2.10 ([Laz04] Lemma 2.2.38). Let $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a function. Suppose that for any $r, s \in \mathbb{N}$ with $f(r) > 0$ we have that $f(r + s) \geq f(s)$. Then for any $k \in \mathbb{R}_{> 0}$ the function $g : \mathbb{N} \to \mathbb{R} \cup \{\infty\}$ defined by

$$g(r) := \limsup_{m \to \infty} \frac{f(mr)}{m^k}$$

satisfies $g(cr) = c^k g(r)$ for any $c, r \in \mathbb{N}$.

Remark 2.11. Although [Laz04, Lemma 2.2.38] only explicitly address the volume function, the essential content of the proof is the more general statement above.

Lemma 2.12. Let $V$ be a finite dimensional $\mathbb{Q}$-vector space and let $C \subset V$ be a salient full-dimensional closed convex cone. Suppose that $f : V \to \mathbb{R}_{\geq 0}$ is a function satisfying

1. $f(e) > 0$ for any $e \in C^{\text{int}}$,
2. there is some constant $c > 0$ so that $f(me) = m^c f(e)$ for any $m \in \mathbb{Q}_{> 0}$ and $e \in C$, and
3. for every $v \in C$ and $e \in C^{\text{int}}$ we have $f(v + e) \geq f(v)$.

Then $f$ is locally uniformly continuous on $C^{\text{int}}$.

Proof. Endow $V$ with the Euclidean metric for some fixed basis. Let $T \subset C^{\text{int}}$ be any bounded set such that

$$\inf_{p, q \in T, q \notin C} \|p - q\| > 0.$$

We show that $f$ is uniformly continuous on $T$. Let $T$ be the cone over $T$. There is some constant $\xi > 0$ such that if $v \in T$ satisfies $\|v\| = \mu$ then the open ball $B_\xi(0)$ satisfies $B_{\xi\mu}(v) \subset C^{\text{int}}$.

Let $M = \sup_{w \in T} f(w)$; since there is some element $x \in C^{\text{int}}$ such that $T \subset x - C^{\text{int}}$, we see that $M$ is a positive real number.
Fix $\epsilon > 0$ and let $v \in T$. Note that the set $(1 - \frac{\epsilon}{M})^{1/c}v + C^{\text{int}}$ contains the open ball $B_{r_v}(v)$, where

$$r_v = \xi \left(1 - \left(1 - \frac{\epsilon}{M}\right)^{1/c}\right)\|v\|$$

Every $e \in B_{r_v}(v)$ satisfies $f(e) \geq f(v) - \epsilon$. Similarly, the set $(1 + \frac{\epsilon}{M})^{1/c}v + (-C^{\text{int}})$ contains the open ball $B_{s_v}(v)$ where

$$s_v = \xi \left(\left(1 + \frac{\epsilon}{M}\right)^{1/c} - 1\right)\|v\|$$

Every $e \in B_{s_v}(v)$ satisfies $f(e) \leq f(v) + \epsilon$.

As we vary $v \in T$, the length $\|v\|$ has a positive lower bound (since by assumption $T$ avoids a sufficiently small neighborhood of the origin). Thus, there is some $\delta > 0$ such that $\delta < \inf_{v \in T} \min\{s_v, r_v\}$. Then $|f(v') - f(v)| \leq \epsilon$ for every $v$ and $v'$ in $T$ satisfying $\|v' - v\| < \delta$, showing uniform continuity on $T$. By varying $T$, we obtain local uniform continuity on $C^{\text{int}}$. \hfill \Box

### 3. Families of cycles

Although there are several different notions of a family of cycles in the literature, the theory we will develop is somewhat insensitive to the precise choices. It will be most convenient to use a simple geometric definition.

**Definition 3.1.** Let $X$ be a projective variety. A family of $k$-cycles on $X$ consists of an integral variety $W$, a reduced closed subscheme $U \subset W \times X$, and an integer $a_i$ for each component $U_i$ of $U$, such that for each component $U_i$ of $U$ the first projection map $p : U_i \to W$ is flat dominant of relative dimension $k$. If each $a_i \geq 0$ we say that we have a family of effective cycles. We say that $\sum a_i U_i$ is the cycle underlying the family.

In this situation $p : U \to W$ will denote the first projection map and $s : U \to X$ will denote the second projection map unless otherwise specified. We will usually denote a family of $k$-cycles using the notation $p : U \to W$, with the rest of the data implicit.

For a closed point $w \in W$, the base change $w \times_W U_i$ is a subscheme of $X$ of pure dimension $k$ and thus defines a fundamental $k$-cycle $Z_i$ on $X$. The cycle-theoretic fiber of $p : U \to W$ over $w$ is defined to be the cycle $\sum a_i Z_i$ on $X$. We will also call these cycles the members of the family $p$.

**Definition 3.2.** Let $X$ be a projective variety. We say that a family of $k$-cycles $p : U \to W$ on $X$ is a rational family if every cycle-theoretic fiber lies in the same rational equivalence class.

**Remark 3.3.** Definition 3.1 has a number of deficiencies. For example, many intuitive constructions of families of cycles fail to meet the criteria: the map $\mathbb{A}^2 \times \mathbb{A}^2 \to \text{Sym}^2 \mathbb{A}^2$ is not flat over a characteristic 2 field as pointed out in [Ko96]. Since we are primarily interested in the “generic” behavior
of families of cycles, these shortcomings are not important for us. On the other hand, the geometric flexibility of Definition 3.1 will be very useful.

The following constructions show how to construct families of cycles from subsets $U \subset W \times X$.

**Construction 3.4 (Cycle version).** Let $X$ be a projective variety and let $W$ be an integral variety. Suppose that $Z = \sum a_i V_i$ is a $(k + \dim W)$-cycle on $W \times X$ such that the first projection maps each $V_i$ dominantly onto $W$. Let $W^0 \subset W$ be the (non-empty) open locus over which every projection $p : V_i \rightarrow W$ is flat and let $U \subset \text{Supp}(Z)$ denote the preimage of $W^0$. Then the map $p : U \rightarrow W^0$ defines a family of cycles where we assign the coefficient $a_i$ to the component $V_i \cap U$ of $U$.

**Construction 3.5 (Subscheme version).** Suppose that $Y$ is a reduced variety and that $X$ is a projective variety. Let $\tilde{U} \subset Y \times X$ be a closed subscheme such that the fibers of the projection $p : \tilde{U} \rightarrow Y$ are equidimensional of dimension $k$. There is a natural way to construct a finite collection of families of effective cycles associated to the subscheme $\tilde{U}$.

Consider the image $p(\tilde{U})$ (with its reduced induced structure). Let $\{\tilde{W}_j\}$ denote the irreducible components of $p(\tilde{U})$. For each there is a non-empty open subset $W_j \subset \tilde{W}_j$ such that the restriction of $p$ to each component of $p^{-1}(W_j)_{\text{red}}$ is flat. Since furthermore $p$ has equidimensional fibers, we obtain a family of effective $k$-cycles $p_j : U_j \rightarrow W_j$ where $U_j = p^{-1}(W_j)_{\text{red}}$ and we assign coefficients so that the cycle underlying the family $p_j$ coincides with the fundamental cycle of $p^{-1}(W_j)$. We can then replace $\tilde{U}$ by the closed subscheme obtained by taking the base change to $p(\tilde{U}) - \bigcup_j W_j$ and repeat. The end result is a collection of families $p_i : U_i \rightarrow W_i$ parametrizing the cycles contained in $\tilde{U}$.

If $p(\tilde{U})$ is irreducible and we are interested only in the generic behavior of the cycles in $\tilde{U}$, we can stop after the first step to obtain a single family of cycles.

### 3.1. Chow varieties and the Chow map.

Fix a projective variety $X$ and an ample divisor $H$ on $X$. For any reduced scheme $Z$ over the ground field, [Kol96, Chapter I.3] introduces a more refined definition of a family of $k$-cycles of $X$ of $H$-degree $d$ over $Z$. Kollár then constructs a semi-normal projective variety $\text{Chow}_{k,d,H}(X)$ that parametrizes families of effective $k$-cycles of $H$-degree $d$. $\text{Chow}(X)$ denotes the disjoint union over all $k$ and $d$ of $\text{Chow}_{k,d,H}(X)$ for some fixed ample divisor $H$; it does not depend on the choice of $H$.

The precise way in which $\text{Chow}(X)$ parametrizes cycles is somewhat subtle in characteristic $p$. For a discussion of the Chow functor and universal families, see [Kol96]. We will need the following properties of $\text{Chow}(X)$:

- Any family of cycles in the sense of Definition 3.1 naturally yields a family of cycles in the refined sense of [Kol96 I.3.11 Definition]
by applying \[\text{Kol96 I.3.14 Lemma}\] with the identity map (see also \[\text{Kol96 I.3.15 Corollary}\]).

- For any weakly normal integral variety \(W\) and any (refined) family of effective cycles \(p : U \to W\), there is an induced morphism \(\text{ch}_p : W \to \text{Chow}(X)\) by \[\text{Kol96 I.4.8-I.4.10}\]. (We will denote this map simply by \(\text{ch}\) when the family \(p\) is clear from the context.)

For any family of effective cycles \(p : U \to W\) the base change to the normal locus \(W^0 \subset W\) is still a family of cycles (where we assign the same coefficients). Thus there is an induced rational map \(\text{ch}_p : W \to \text{Chow}(X)\) that is a morphism on the normal locus of \(W\).

The following crucial lemma encapsulates the set-theoretical nature of the Chow functors constructed in \[\text{Kol96 Chapter I.3}\].

**Lemma 3.6.** Let \(X\) be a projective variety and let \(p : U \to W\) be a family of effective \(k\)-cycles on \(X\) over a weakly normal \(W\). A curve \(C \subset W\) is contracted by \(\text{ch} : W \to \text{Chow}(X)\) if and only if every cycle-theoretic fiber over \(C\) has the same support.

We will freely use the notation of \[\text{Kol96}\] in the verification.

**Proof.** First suppose that every cycle-theoretic fiber of \(p\) over \(C\) has the same support. Since \(\text{Chow}_{k,d,H}(X)\) is constructed by taking a semi-normalization (which is set-theoretically bijective), we may instead consider the induced map to \(\text{Chow}'_{k,d,H}(X)\). This map factors through the map \(\text{ch}\) for a projective space containing an embedding of \(X\); therefore it suffices to consider the case when \(X = \mathbb{P}\). Then the construction following \[\text{Kol96 Ch. I Eq. (3.23.1.5)}\] shows that the Cartier divisors on \((\mathbb{P}^\nu)^{k+1}\) parametrized by the image of \(C\) in \(\mathbb{H}\) must all have the same support. But this implies they are equal.

Conversely, suppose that \(C\) is contracted by \(\text{ch}\). As discussed in \[\text{Kol96 I.3.27.3}\], \(C\) is also contracted by the morphism to \(\text{Hilb}((\mathbb{P}^\nu)^{k+1})\). Again comparing with \[\text{Kol96 Ch. I Eq. (3.23.1.5)}\], we see that the support of each of the cycles parametrized by \(C\) is the same. \(\Box\)

It will often be helpful to replace a family \(p : U \to W\) by a slightly modified version.

**Lemma 3.7.** Let \(X\) be a projective variety and let \(p : U \to W\) be a family of effective cycles on \(X\). Then there is a normal projective variety \(W'\) that is birational to \(W\) and a family of cycles \(p' : U' \to W'\) such that \(\text{ch}(W') = \text{ch}(W)\).

**Proof.** Let \(\bar{W}\) be any projective closure of \(W\) and let \(\bar{U}\) be the closure of \(U\) in \(\bar{W} \times X\). Let \(\phi : W' \to \bar{W}\) be the normalization of a simultaneous flattening of the morphisms \(\bar{p} : \bar{U}_i \to \bar{W}\) for the components \(\bar{U}_i\) of \(\bar{U}\). Let \(U'\) denote the reduced subscheme of \(W' \times X\) defined by the components of \(\bar{U} \times_{\bar{W}} W'\) that dominate \(W'\). Since the components of \(U'\) are in bijection with the components of \(U\), we can assign to each component of \(U'\) the coefficient of the corresponding component of \(U\). Then \(p' : U' \to W'\) is a family of
effective $k$-cycles. Since the $p'$ and $p$ agree over an open normal subset of the base, the closure of the images under the map $\text{ch}$ agree.

**Remark 3.8.** It is also important to know whether a rational family $p : U \to W$ can be extended to a rational family over a projective closure of $W$ (although we will not need such statements below). The arguments of [Sam56, Theorem 3] show that the subset of $\text{Chow}(X)$ parametrizing cycles in a fixed rational equivalence class is a countable union of closed subvarieties. Thus we can extend families in this way when working over an uncountable algebraically closed field $K$.

### 3.2. Chow dimension of families

Let $p : U \to W$ be a family of effective $k$-cycles on a projective variety $X$. Then all the cycle-theoretic fibers of $p$ are algebraically equivalent. Indeed, for any two closed points of $W$, let $C$ be the normalization of a curve through those two points; since the base change of $U$ to $C$ is a union of flat families of effective cycles, we see that the corresponding cycle-theoretic fibers are algebraically equivalent.

**Definition 3.9.** Let $p : U \to W$ be a family of effective $k$-cycles on a projective variety $X$. We say that $p$ represents $\alpha \in N_k(X)\mathbb{Z}$ if the cycle-theoretic fibers of our family have class $\alpha$.

**Definition 3.10.** Let $X$ be a projective variety and let $p : U \to W$ be an effective family of $k$-cycles. We define the Chow dimension of $p$ to be

$$\text{chdim}_X(p) := \dim(\text{Im \ ch : } W \to \text{Chow}(X))$$

If $\alpha \in N_k(X)\mathbb{Z}$, we define

$$\text{chdim}_X(\alpha) = \max\{\text{chdim}(p) | p : U \to W \text{ represents } \alpha\}.$$  

We will usually omit the subscript when it is clear from the context.

**Remark 3.11.** When our ground field $K$ has characteristic 0, [Kol96] constructs a universal family over any component of $\text{Chow}(X)$. Using Construction 3.4 this can be turned into a family of effective cycles in the sense of Definition 3.1. Thus

$$\text{chdim}(\alpha) = \max\{\dim(Y) | Y \text{ is a component of } \text{Chow}(X) \text{ representing } \alpha\}.$$  

Even when $K$ has characteristic $p$, for any component of $\text{Chow}(X)$ [Kol96, I.4.14 Theorem] constructs a family of cycles whose chow map $\text{ch}$ is dominant, so that we still have the same interpretation.

We can also consider the analogous construction for rational families.

**Definition 3.12.** Let $X$ be a projective variety. For $\alpha \in N_k(X)\mathbb{Z}$, we define

$$\text{rchdim}(\alpha) = \max\{\text{chdim}(p) | p : U \to W \text{ is a rational family representing } \alpha\}.$$  

Similarly, for $\tau \in A_k(X)$ we define

$$\text{rchdim}(\tau) = \max\{\text{chdim}(p) | p : U \to W \text{ is a rational family of class } \tau\}.$$
3.3. Geometry of families.

**Definition 3.13.** Let $X$ be a projective variety and let $p : U \to W$ be a family of effective cycles on $X$.

- We say that $p$ is a reduced family if every coefficient $a_i$ is 1. Any family of effective cycles yields a reduced family by simply changing the coefficients.
- We say that $p$ is an irreducible family if $U$ only has one component.

For any component $U_i$ of $U$, we have an associated irreducible family $p_i : U_i \to W$ (with coefficient $a_i$).

**Lemma 3.14.** Let $X$ be a projective variety and let $p : U \to W$ be an effective family of $k$-cycles. Let $p' : U' \to W$ denote the reduced family for $p$. Then $\text{chdim}(p') = \text{chdim}(p)$.

*Proof.* Note that $p : U \to W$ and $p' : U' \to W$ agree set-theoretically. The statement then follows immediately from Lemma 3.6. $\square$

**Lemma 3.15.** Let $X$ be a projective variety and let $p : U \to W$ be a family of effective $k$-cycles on $X$. For each component $U_i$ of $U$, let $p_i := p|_{U_i} : U_i \to W$ denote the irreducible family induced by $U_i$. Then $\text{chdim}(p) \leq \sum_i \text{chdim}(p_i)$.

*Proof.* By Lemma 3.6 the map $\text{ch}_p : W \to \text{Chow}(X)$ factors rationally through the map $\prod_i \text{ch}_{p_i} : W \to \prod_i \text{Chow}(X)$. $\square$

We will also need several geometric constructions.

**Construction 3.16** (Flat pullback families). Let $g : Y \to X$ be a flat morphism of projective varieties of relative dimension $d$. Suppose that $p : U \to W$ is a family of effective $k$-cycles on $X$ with underlying cycle $V$. The flat pullback cycle $g^*V$ on $W \times Y$ is effective and has relative dimension $(d + k)$ over $W$. We define the flat pullback family $g^*p : U' \to W^0$ of effective $(d + k)$-cycles on $Y$ over an open subset $W^0 \subset W$ by applying Construction 3.4 to $g^*V$.

**Construction 3.17** (Pushforward families). Let $f : X \to Y$ be a morphism of projective varieties. Suppose that $p : U \to W$ is a family of effective $k$-cycles on $X$ with underlying cycle $V$. Consider the cycle pushforward $f_*V$ on $W \times Y$ and assume $f_*V \neq 0$. Construction 3.4 yields a family of $k$-cycles $f_*p : \tilde{U} \to W^0$ over an open subset of $W$. We call $f_*p$ the pushforward family. Note that this operation is compatible with the pushforward on cycle-theoretic fibers over $W^0$ by [Kol96 1.3.2 Proposition].

**Construction 3.18** (Restriction families). Let $X$ be a projective variety and let $p : U \to W$ be a family of effective $k$-cycles on $X$. Let $W' \subset W$ be an integral subvariety. For each component $U_i$ of $U$, the restriction $U_i \times_W W'$ is flat over $W'$ of relative dimension $k$. Consider the cycle $V$ on $W' \times X$ defined as the sum $V = \sum_i a_i V_i$ where $V_i$ is the fundamental cycle of $U_i$ restricted to $W'$. We define the restriction of the family $p$ to $W'$ over an
open subset $W^0 \subset W'$ by applying Construction 3.4 to $V$. Note that this operation leaves the cycle-theoretic fibers unchanged over $W^0$. Note also that if $W' \subset W$ is open, then we may take $W^0 = W'$ and the family $p$ is simply the base-change to $W^0$.

**Construction 3.19** (Family sum). Let $X$ be a projective variety and let $p : U \rightarrow W$ and $q : S \rightarrow T$ be two families of effective $k$-cycles on $X$. We construct the family sum of $p$ and $q$ over an open subset of $W \times T$ as follows.

Let $V_p$ and $V_q$ denote the underlying cycles for $p$ and $q$ on $W \times X$ and $T \times X$ respectively. The family sum of $p$ and $q$ is the family defined by applying Construction 3.4 to the sum of the flat pullbacks of $V_p$ and $V_q$ to $W \times T \times X$.

**Construction 3.20** (Strict transform families). Let $X$ be an integral projective variety and let $p : U \rightarrow W$ be a family of effective $k$-cycles on $X$. Suppose that $\phi : X \rightarrow Y$ is a birational map. We define the strict transform family of effective $k$-cycles on $Y$ as follows.

First, modify $U$ by removing all irreducible components whose image in $X$ is contained in the locus where $\phi$ is not an isomorphism. Then define the cycle $U'$ on $W \times Y$ by taking the strict transform of the remaining components of $U$. We define the strict transform family by applying Construction 3.4 to $U'$ over $W$.

**Construction 3.21** (Intersecting against divisors). Let $X$ be a projective variety and let $p : U \rightarrow W$ be a family of effective $k$-cycles on $X$. Let $D$ be an effective Cartier divisor on $X$. If every cycle in our family has a component contained in $\text{Supp}(D)$, we say that the intersection family of $p$ and $D$ is empty.

Otherwise, let $s : U \rightarrow X$ denote the projection map. By assumption the effective Cartier divisor $s^*D$ does not contain any component of $U$, so we may take a cycle-theoretic intersection of $s^*D$ with the cycle underlying the family $p$ to obtain a $(k - 1 + \dim W)$-cycle $V$ on $W \times \text{Supp}(D)$. We then apply Construction 3.4 to obtain a family of cycles on $\text{Supp}(D)$ over an open subset of $W$. We can also consider the intersection as a family of cycles on $X$ by pushing forward and we denote this family by $p \cdot D$.

Finally, suppose that we have a linear series $|L|$. We define the intersection of $|L|$ with a family $p : U \rightarrow W$ as follows. Consider the flat pullback family $q : U' \rightarrow W^0$ on $\mathbb{P}(|L|) \times X$. Then intersect the family $q$ against the pullback of the universal divisor on $\mathbb{P}(|L|) \times X$ to obtain a family of cycles on $\mathbb{P}(|L|) \times X$. The underlying cycle has dimension $k - 1 + \dim W + \dim(\mathbb{P}(|L|))$; by using Construction 3.4 we can convert this cycle to a family of effective $(k - 1)$-cycles on $X$ over an open subset of $W \times \mathbb{P}(|L|)$.

We conclude this section with a brief analysis of how these constructions affect the Chow dimension.

**Lemma 3.22.** Let $f : X \rightarrow Y$ be a morphism of projective varieties. Let $p : U \rightarrow W$ be a family of effective $k$-cycles on $X$ such that for every
component \( U_i \) of \( U \) the image \( s(U_i) \) is not contracted to a variety of smaller dimension by \( f \). Then \( \text{chdim}(p) = \text{chdim}(f_\ast p) \).

**Proof.** Let \( T \) be an integral curve through a general point of \( W \) that is not contracted by \( ch \) and set \( S = p^{-1}(T) \). Then \( \dim(s(S)) = \dim(f(s(S))) \). Thus the cycle-theoretic fibers parametrized by \( T \) do not pushforward to the same cycle on \( Y \). We conclude by Lemma 3.6 that \( T \) is not contracted by \( ch_{f_\ast p} \).

**Lemma 3.23.** Let \( X \) be a projective variety and let \( p : U \to W \) and \( q : S \to T \) be two families of effective \( k \)-cycles on \( X \). Then \( \text{chdim}(p + q) = \text{chdim}(p) + \text{chdim}(q) \).

**Proof.** A curve through a general point of \((W \times T)^0\) is contracted by \( ch_{p+q} \) if and only if its projection to \( W \) and to \( T \) are contracted by \( ch_p \) and \( ch_q \) respectively. We conclude by Lemma 3.6. \( \square \)

## 4. Families of divisors

In this section we analyze the Chow dimension of families of effective divisors. The goal is to find bounds on the Chow dimension that depend only on the numerical class of the divisor. The key result is Proposition 4.9, which is the base case of inductive arguments used in the following sections.

### 4.1. A Lefschetz hyperplane-type theorem for Weil divisors

Let \( X \) be an integral projective variety. Fix a closed point \( p \) in the smooth locus of \( X \); then \( X \) admits an Albanese mapping \( \text{alb} : X \to \text{Alb}(X) \) sending \( p \) to 0 characterized by the properties:

- the image of \( \text{alb} \) generates \( \text{Alb}(X) \), and
- the map \( \text{alb} \) is universal among rational maps from \( X \) to abelian varieties that map \( p \) to 0.

We will let \( P(X) \) denote the abelian variety dual to \( \text{Alb}(X) \). Note that \( P(X) \) is a birational invariant of \( X \).

Suppose that \( A \) is a very ample Cartier divisor on \( X \). A general \( H \in |A| \) is an integral variety by the Bertini theorems. After choosing a basepoint \( p \in H \) that is also in the smooth locus of \( X \), by the universal property of the Albanese we obtain an induced morphism \( \text{Alb}(H) \to \text{Alb}(X) \). We let \( r : P(X) \to P(H) \) denote the dual map. Note that the restriction map is unchanged if we replace the inclusion \( i : H \to X \) by a birationally equivalent map (that is compatible with the basepoint).

When \( X \) is normal, [BGS11] shows that the \( K \)-points of \( P(X) \) parametrize the rational equivalence classes of Weil divisors on \( X \) that are algebraically equivalent to 0. More precisely, let \( X_{\text{sm}} \subset X \) be the open smooth locus of \( X \) and fix a closed point \( p \in X_{\text{sm}} \) (corresponding to the base point of the Albanese map). Define the functor \( P_X^0 \) which assigns to a reduced \( K \)-variety \( T \) the collection of invertible sheaves \( \mathcal{L} \) on \( X_{\text{sm}} \times T \) (up to isomorphism) such that \( \mathcal{L}|_{\{p\} \times T} \) is trivial and \( \mathcal{L}|_{X_{\text{sm}} \times \{t\}} \) is algebraically equivalent to zero for
every $K$-point $t$ of $T$. [BGS11, Proposition 3.2] shows that $P(X)$ represents the functor $P_X^0$ on the category of reduced $K$-varieties.

4.1.1. Relationship with Pic$^0$. For an integral projective variety $X$, let $X^\nu$ denote the normalization of $X$ and let $X^\nu_{sm} \subset X^\nu$ denote the smooth locus. Let Pic$^0(X)$ denote the connected component of the identity of Pic$(X)$. Then Pic$^0(X)_{red}$ represents the functor Pic$^0_X$ on the category of reduced $K$-varieties.

We define a morphism $q : \text{Pic}^0(X)_{red} \to P(X)$ via the following natural transformation on the corresponding functors. Let $T$ be a reduced variety. For an invertible sheaf $L$ on $X \times T$ representing an element of Pic$^0_X(T)$, we pull-back $L$ to $X^\nu \times T$ and restrict to $X^\nu_{sm} \times T$. After twisting by the pullback of a line bundle on $T$ to ensure triviality over the basepoint, we obtain an element of $P_X^0(T)$. Since this natural transformation is injective on the set-theoretic level, $q$ is a monomorphism in the category of reduced $K$-varieties; in particular it is set-theoretically injective.

Theorem 4.1. Let $X$ be an integral projective variety. Let $A$ be a very ample Cartier divisor on $X$ and let $H$ be a general element of $|A|$. Then the kernel of the restriction map $r : P(X) \to P(H)$ is supported on a finite number of points of $P(X)$. In particular, $r$ is finite flat onto its image.

Proof. Let $X^\nu$ denote the normalization of $X$ and let $X^\nu_{sm}$ denote the smooth locus of $X^\nu$. We will first need a special case of a result of [Ben12] (obtained by applying [Ben12, Lemma 6] when $U$ is the smooth locus of $X^\nu$).

Lemma 4.2 ([Ben12], Lemma 6). Let $X^\nu$ be a projective integral normal variety over an algebraically closed field. Then there exists a birational map $\phi : Y \to X^\nu$ from a projective normal variety $Y$ that is an isomorphism over the smooth locus of $X^\nu$ such that every rational equivalence class of divisors corresponding to a $K$-point of $P(Y)$ is Cartier as a class on $Y$.

Let $Y_{sm} \subset Y$ be the smooth locus of $Y$ and let $\psi : Y \to X$ denote the composition $\nu \circ \phi$. Suppose $T$ is a normal reduced variety and $L$ is an invertible sheaf on $Y_{sm} \times T$ parametrizing algebraically-trivial line bundles. $L$ extends to a reflexive rank one sheaf $D$ on the (normal) variety $Y \times T$. Furthermore, [BGS11, Lemme 2.6] shows that $D$ is an invertible sheaf on $Y \times T$. Thus we see that on the category of normal reduced $K$-varieties the defining functors for Pic$^0(Y)_{red}$ and $P(Y)$ are naturally isomorphic, so that $r : \text{Pic}^0(Y)_{red} \to P(Y)$ induces an isomorphism of normalizations. Since both Pic$^0(Y)_{red}$ and $P(Y)$ are smooth, $r$ is in fact an isomorphism.

Let $H$ be a general element of $|A|$ on $X$ so that $H$ is integral. We may also suppose that $H$ does not contain any exceptional center for the birational morphism $\psi : Y \to X$, so that the pullback $H' := \psi^*H$ is integral as well.
Consider the diagram

\[
\begin{array}{ccc}
\text{Pic}^0(Y)_{\text{red}} & \longrightarrow & P(Y) \\
\downarrow r & & \downarrow r \\
\text{Pic}^0(H')_{\text{red}} & \longrightarrow & P(H')
\end{array}
\]

This diagram commutes; on the level of functors, both maps \( \text{Pic}^0(Y)_{\text{red}} \to P(H') \) are defined by the pull-back of invertible sheaves. The morphism on the top is an isomorphism and the morphism on the bottom is set-theoretically injective. Since \( H' \) is big and basepoint free, the kernel of the morphism on the left is supported on a finite number of points by the following theorem. (In the statement \( \text{Pic}^\tau \) denotes the torsion components of \( \text{Pic} \).)

**Theorem 4.3** ([BdJ13], Theorem 2.8). Let \( Y \) be a normal integral projective variety of dimension \( \geq 2 \) over a field \( K \) and let \( D \) be a big semiample divisor on \( X \). Then the restriction map \( \text{Pic}^\tau(Y) \to \text{Pic}^\tau(D) \) is

- injective if \( \text{char}(K) = 0 \).
- has finite and \( p^\infty \)-torsion kernel if \( \text{char}(K) = p > 0 \).

By identifying \( P(X) = P(Y) \) and \( P(H') = P(H) \), we obtain the desired statement.

### 4.2. Restrictions to very ample divisors

Let \( X \) be an integral projective variety and let \( p : U \to W \) be a family of effective divisors on \( X \). Fix a general closed point \( w \in W \). This data yields a rational map \( v_{p,w} : W \dashrightarrow P(X) \) as follows. Let \( \nu : X' \to X \) be a normalization of \( X \). Let \( p' : U' \to W^0 \) denote the strict transform family of \( p \) to \( X' \). After shrinking \( W^0 \) we may suppose that it is smooth. Let \( w \in W^0 \) be a closed point and let \( L \) denote the corresponding divisor. The effective cycle underlying the family \( p' \) defines a reflexive rank 1 sheaf \( \mathcal{D} \) on the normal variety \( W^0 \times X' \). The restriction \( \mathcal{D} \) to the smooth locus \( W^0 \times X'_{\text{sm}} \) is invertible. Furthermore, the divisor \( L \) restricts to define an invertible sheaf \( \mathcal{L} \) on \( X'_{\text{sm}} \). There is an induced morphism \( v : W^0 \to P(X') = P(X) \) given by twisting \( \mathcal{D} \) by the pull-back of \( \mathcal{L}^{-1} \) and using the functorial definition of \( P(X') \).

**Lemma 4.4.** Let \( X \) be an integral projective variety and let \( p : U \to W \) be a family of effective divisors on \( X \). Fix a general closed point \( w \) in \( W \) and let \( F \) be a component (with the reduced structure) of a general fiber of \( v_{p,w} : W \dashrightarrow P(X) \). Then \( \text{chdim}(p) = \dim(\overline{v_{p,w}(W)}) + \text{chdim}(p|_F) \).

**Proof.** Let \( p' \) denote the strict transform family on \( X' \) as above. Let \( W^0 \subseteq W \) denote the open locus where the maps \( \text{ch}_p, \text{ch}_{p'} \), and \( v_{p,w} \) are all defined. Note that the only divisors removed by the construction of \( p' \) are contained in \( \nu \)-exceptional centers and are thus fixed components of \( p \). By Lemma 3.6 a curve \( C \subseteq W^0 \) through a general point is contracted by \( \text{ch}_p \) if and only if it is contracted by \( \text{ch}_{p'} \). But if \( C \) is contracted by \( \text{ch}_{p'} \) it must also be contracted by \( v_{p,w} \), and the lemma follows. \( \square \)
We next prove two lemmas in preparation for Proposition 4.7.

**Lemma 4.5.** Let $\phi : Y \to X$ be a birational map of integral projective varieties. Suppose that $p : U \to W$ is a family of effective divisors on $X$ and that some cycle in our family intersects the locus where $\phi$ is an isomorphism.

1. The strict transform family $p' : U' \to W'$ on $Y$ has $\text{chdim}(p') = \text{chdim}(p)$.
2. Suppose that $D$ is an effective Cartier divisor on $X$ that does not contain any $\phi$-exceptional center or any component of $s(U)$. Then $\text{chdim}_D(p \cdot D) = \text{chdim}_{\phi^* D}(p' \cdot \phi^* D)$.

**Proof.** (1) Recall that the strict transform family is constructed by throwing away any components of $U$ with image contained in a $\phi$-exceptional center and then taking the strict transform of a general element of the remaining components of $U$. In particular, two general cycles in the family $p$ have the same support if and only if the corresponding cycles in $p'$ have the same support. We conclude by Lemma 3.6.

(2) Note that over an open subset of $W$ the pushforward family of $p' \cdot \phi^* D$ to $X$ only differs from $p \cdot D$ by a constant cycle (namely, the intersection of $D$ with the divisor removed in constructing $p'$). Furthermore pushing forward a family of effective divisors over a generically finite morphism does not change the Chow dimension: since the only components of the family that can be contracted are those representing a constant cycle, this follows from Lemma 3.22. Combining these two facts yields the statement. □

**Lemma 4.6.** Let $X$ be a normal integral projective variety and let $p : U \to W$ be a rational family of effective divisors on $X$ representing the class $\alpha$. Suppose that $\pi : X \to Y$ is a birational morphism and $A$ is a very ample Cartier divisor on $Y$ such that $\alpha - [\pi^* A]$ is not pseudo-effective. Then for a general element $H \in |A|$ we have $\text{chdim}_X(p) = \text{chdim}_{\pi^* H}(p \cdot \pi^* H)$.

**Proof.** Let $D$ be the reflexive rank one sheaf on $X$ defined by the cycle-theoretic fibers of $p$. Let $T \subset \mathbb{P}(H^0(X, D))$ denote the closure of the locus defined by the cycle-theoretic fibers of $p$. Over an open subset of $T$ we obtain a family of divisors and we replace $p$ by this family; note that this change does not affect the Chow dimension $\dim T$.

By [Tak07, Proposition 1.1-1.4] there is a birational morphism $\phi : X' \to X$ from a normal projective variety $X'$ and a line bundle $\mathcal{L}$ on $X'$ such that $\phi_* \mathcal{L} = D$ and there is an equality $H^0(X', \mathcal{L}) = H^0(X, D)$. Thus our (modified) family $p$ yields a new family $p'$ on $X'$ defined over an open subset of the same subvariety $T$ of $\mathbb{P}(H^0(X, \mathcal{L}))$. Note that $p'$ pushes forward to $p$ (over an open subset of $T$). In particular, if $\beta$ is the class of $p'$ then $\beta - [\phi^* \pi^* A]$ is still not pseudo-effective. Furthermore Lemma 4.5 (2) shows that the Chow dimension of $p' \cdot \phi^* \pi^* H$ coincides with $p \cdot \pi^* H$. In sum, after a birational modification we may suppose that $D$ is a line bundle.

A general element $H \in |A|$ does not contain any $\pi$-exceptional center. We may also assume that $H$ does not contain the $\pi$-image of any component of
Thus the pullback $\pi^* H$ is an integral projective variety such that the intersection family $p \cdot \pi^* H$ is defined. Consider the exact sequence

$$0 \rightarrow \mathcal{O}_X(-\pi^* A) \otimes \mathcal{D} \rightarrow \mathcal{D} \rightarrow \mathcal{O}_{\pi^* H} \otimes \mathcal{D} \rightarrow 0.$$ 

By assumption $H^0(X, \mathcal{O}_X(-\pi^* A) \otimes \mathcal{D}) = 0$ since the corresponding numerical class is not pseudo-effective. Thus there is an injection $H^0(X, \mathcal{D}) \rightarrow H^0(\pi^* H, \mathcal{D}|_{\pi^* H})$.

Let $\psi : H' \rightarrow \pi^* H$ be a normalization. There is an injective morphism $H^0(\pi^* H, \mathcal{D}|_{\pi^* H}) \rightarrow H^0(H', \psi^*(\mathcal{D}|_{\pi^* H}))$. Precomposing with the injection above, we can identify $T$ as a subvariety of $\mathbb{P}(H^0(H', \psi^*(\mathcal{D}|_{\pi^* H})))$. The corresponding family of divisors on $H'$ has Chow dimension equal to $\dim T$ and its pushforward to $\pi^* H$ agrees with $p \cdot \pi^* H$ over an open subset of $T$. Since pushing forward this family does not change the Chow dimension by Lemma 3.22 we have proved the result. 

**Proposition 4.7.** Let $X$ be an integral projective variety and let $p : U \rightarrow W$ be a family of effective divisors on $X$ representing $\alpha$. Suppose that $A$ is a very ample Cartier divisor on $X$ such that $\alpha - [A]$ is not pseudo-effective. Then for a general element $H \in |A|$ we have

$$\chdim_X(p) = \chdim_H(p \cdot H).$$

**Proof.** We return to the setting of the proof of Theorem 4.1 keeping the notation there. Let $p' : U' \rightarrow W^0$ denote the strict transform family of $p$ to $Y$. Let $\beta$ denote the class of $p'$ and note that $\beta - [\psi^* A]$ is not pseudo-effective.

Let $H \in |A|$ be a general element so that $H' = \psi^* H$ is an integral projective variety and $p' \cdot H'$ is defined. By shrinking $W^0$ we may suppose that $p' \cdot H'$ is defined over $W^0$. Fix a general closed point $w$ of $W^0$ yielding a map $v_{p',w} : W^0 \rightarrow P(X)$. We first show that the map $v_{p',H',w} : W^0 \rightarrow P(H')$ coincides over an open subset of $W^0$ with the composition $r \circ v_{p',w}$.

This is a consequence of the key property of $Y$: every rational equivalence class of divisors parametrized by $P(Y)$ is Cartier on $Y$. Thus if $H'$ is the normalization of $H'$ the pullbacks of the rational classes parametrized by $v_{p',w}(W^0) \subset P(Y)$ to $H'$ are the same as the rational classes on $H'$ given in the construction of $v_{p',H',w}$.

Lemma 4.3 shows that

$$\chdim(p' \cdot H') = \dim(v_{p',H',w}(W)) + \chdim(p' \cdot H'|_F)$$

where $F$ is a component of a general fiber of $v_{p',H',w}$. Since $r$ is finite flat onto its image by Theorem 4.1 the argument above shows that

$$\dim(v_{p',H',w}(W)) = \dim(v_{p',w}(W)).$$

Similarly, since $r$ is finite flat onto its image a component $F$ of a general fiber of $v_{p',H',w}$ can be identified with a component of a general fiber of $v_{p',w}$. Lemma 4.4 shows that $\chdim(p' \cdot H'|_F) = \chdim(p'|_F)$. We conclude that $\chdim_H(p' \cdot H') = \chdim_Y(p')$. 


By Lemma 4.5 (1) this implies $\text{chdim}_{H'}(p' \cdot H') = \text{chdim}_X(p)$. By Lemma 4.5 (2) the Chow dimension of $p' \cdot H'$ agrees with the Chow dimension of $p \cdot H$, giving the desired statement. □

**Corollary 4.8.** Let $X$ be an integral projective variety of dimension $n$ and let $\alpha \in N_{n-1}(X)$. Suppose that $A$ is a very ample Cartier divisor on $X$ such that $\alpha - [A]$ is not pseudo-effective. Let $H \in |A|$ be a general element. Then

$$\text{chdim}_X(\alpha) \leq \text{chdim}_H(\alpha \cdot H).$$

**Proof.** Let $p : U \to W$ be a family of effective divisors representing $\alpha$ with maximal Chow dimension. Apply Proposition 4.7 to $p$. □

### 4.3. Dimensions of families of divisors.

We next develop bounds several dimension estimates that govern the behavior of families of divisors. The goal is to construct bounds that depend only on intersection numbers.

**Proposition 4.9.**

(1) Let $X$ be an equidimensional projective variety of dimension $n$ and let $\alpha \in N_{n-1}(X) \subset Z$. Let $A$ be a Cartier divisor that is the pullback of a very ample divisor by a generically finite map. Then

$$\text{chdim}(\alpha) < \left( \frac{\alpha \cdot A^{n-1} + n}{n} \right).$$

(2) Let $X$ be an integral projective variety of dimension $n$ and let $\alpha \in N_{n-1}(X) \subset Z$. Let $H$ be a very ample divisor such that $\alpha - [H]$ is not pseudo-effective and let $A$ be a Cartier divisor that is the pullback of a very ample divisor by a generically finite map. Then

$$\text{chdim}(\alpha) < \left( \frac{\alpha \cdot A^{n-2} \cdot H + n - 1}{n - 1} \right).$$

**Proof.** (1) Let $\pi : X \to \mathbb{P}^n$ be a generically finite morphism defined by a general subspace of $|A|$. Suppose that $p : U \to W$ is a family of effective divisors representing $\alpha$. Note that any component of $s(U)$ that is contracted by $\pi$ must map to a fixed divisor in $X$. By Lemma 3.22 we see that $\text{chdim}(p) = \text{chdim}(\pi_* p)$. But $\pi_* p$ is a family of divisors on $\mathbb{P}^n$ of degree $\alpha \cdot A^{n-1}$, yielding the result.

(2) Replace $H$ by a general element in its linear series. By Corollary 4.8 we have $\text{chdim}_X(\alpha) \leq \text{chdim}_H(\alpha \cdot H)$ and we conclude by (1). □

For later reference we restate these bounds in a simpler (but weaker) form.

**Corollary 4.10.** Let $X$ be an integral projective variety of dimension $n$ and let $p : U \to W$ be a family of effective $(n-1)$-cycles on $X$ representing $\alpha \in N_k(X) \subset Z$.

(1) Suppose that $A$ is a very ample Cartier divisor on $X$ and $s$ is a positive integer such that $\alpha \cdot A^{n-1} \leq sA^n$. Then

$$\text{chdim}(p) < (n + 1)s^{n-1} \alpha \cdot A^{n-1}.$$
(2) Let $A$ and $H$ be very ample divisors and let $s$ be a positive integer such that $\alpha - [H]$ is not pseudo-effective and $\alpha \cdot A^{n-2} \cdot H \leq sA^{n-1} \cdot H$. Then

$$\text{chdim}(p) < ns^{n-2} \alpha \cdot A^{n-2} \cdot H.$$ 

**Proof.** An easy inductive argument using difference functions shows that for a positive integer $d$ and non-negative integer $n$ we have

$$\binom{d+n}{n} \leq (n+1)d^n.$$ 

Finally, we prove a similar statement for sections of line bundles.

**Lemma 4.11.** Let $X$ be an equidimensional projective variety of dimension $n$ and let $A$ be a very ample Cartier divisor on $X$. Then

$$h^0(X, \mathcal{O}_X(A)) \leq (n+1)A^n.$$ 

**Proof.** Computing the cohomology of the exact sequence

$$0 \to \mathcal{O}_X(-A) \to \mathcal{O}_X \to \mathcal{O}_A \to 0$$

shows by induction on dimension that $h^0(X, \mathcal{O}_X) \leq A^n$. Computing the cohomology of the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(A) \to \mathcal{O}_A(A) \to 0$$

gives the desired statement by induction on dimension. \qed

5. **The Variation Function**

The variation of a class $\alpha \in N_k(X)_{\mathbb{Z}}$ measures the rate of growth of the dimensions of components of Chow($X$) that represent $m\alpha$ as $m$ increases. The main theorem in this section is Theorem 5.20 which shows that variation is in some sense a measure of bigness along subvarieties of $X$.

5.1. **Dimensions of families of cycles.** Before defining the variation, we need to find bounds for the dimension of components of Chow($X$). The following theorem incorporates a suggestion of Voisin who pointed out that the coefficient could be improved by considering a generically finite map to projective space.

**Theorem 5.1.** Let $X$ be an equidimensional projective variety of dimension $n$ and let $\alpha \in N_k(X)_{\mathbb{Z}}$. Suppose that $A$ is a very ample divisor on $X$ and set $d = \alpha \cdot A^k$. Then we have

$$\text{chdim}(\alpha) < (n-k) \binom{d+k+1}{k+1}.$$
Proof. Let \( p : U \to W \) be a family of effective cycles representing \( \alpha \). Since the desired upper bound is superadditive in \( d \), by Lemma 3.15 we may prove the bound for each irreducible component of \( U \) separately. By Lemma 3.14 we may furthermore suppose that \( p \) is a reduced family.

Since \( p \) is an irreducible family, the image \( s(U) \) is irreducible, and \( \text{chdim}(p) \) is the same whether we consider it as a family of cycles on \( X \) or on \( s(U) \) (with its reduced induced structure) by Lemma 3.6. In sum, we have reduced to the case where \( X \) is integral and \( p \) is a irreducible reduced family such that the map \( s : U \to X \) is dominant.

Let \( \pi : X \to \mathbb{P}^n \) be a generically finite morphism defined by a generic subspace of \(|A|\). Lemma 3.22 shows that \( \text{chdim}(p) = \text{chdim}(\pi_*p) \). But \( \pi_*p \) is a family of effective cycles on \( \mathbb{P}^n \) of degree \( d \), so it suffices to consider this case.

Suppose that \( q : R \to T \) is a dominant irreducible family of effective \( k \)-cycles on \( \mathbb{P}^n \) with degree \( d \). We prove the desired bound on \( \text{chdim}(q) \) by induction on the codimension \( n - k \). The base case \( n - k = 1 \) is clear.

Let \( f : Y \to \mathbb{P}^n \) be the blow-up of a general closed point and \( g : Y \to \mathbb{P}^{n-1} \) the resolution of the projection away from the point. Since \( q \) is a dominant irreducible family on \( \mathbb{P}^n \), Lemma 3.22 shows that \( \text{chdim}(q) = \text{chdim}(q') \) where \( q' \) is the strict transform family on \( Y \). Let \( F \) be a component of a general fiber of \( \text{ch}_g^*q' : T \to \text{Chow}(\mathbb{P}^{n-1}) \). Note that the cycles parametrized by \( F \) all map to the same \( k \)-dimensional subvariety \( V \subset \mathbb{P}^{n-1} \). Then the restriction \( q'|_F \) of \( q' \) to (an open subset of) \( F \) parametrizes a family of \( k \)-cycles on the variety \( g^{-1}(V) \) of pure dimension \( k + 1 \). Altogether

\[
\text{chdim}(q') \leq \text{chdim}(g_*q') + \text{chdim}(q'|_F) \text{ by Lemma 2.1}
\]

\[
< (n - k - 1) \left( \begin{array}{c} d + k + 1 \\ k + 1 \end{array} \right) + \text{chdim}(q'|_F) \text{ by induction}
\]

\[
< (n - k - 1) \left( \begin{array}{c} d + k + 1 \\ k + 1 \end{array} \right) + \left( \begin{array}{c} d + k + 1 \\ k + 1 \end{array} \right) \text{ by Proposition 4.9}
\]

Theorem 5.1 shows that for any class \( \alpha \in N_k(X)_{\mathbb{Z}} \), there is some positive constant \( C \) such that \( \text{chdim}(m\alpha) < Cm^{k+1} \). Furthermore, this order of growth can always be achieved by some class on \( X \) as in the following example.

**Example 5.2.** Let \( H_1, \ldots, H_{n-k} \) be general very ample divisors on \( X \) and set \( \alpha = H_1 \cdot \ldots \cdot H_{n-k} \). Let \( V \) denote the scheme-theoretic intersection \( H_1 \cap \ldots \cap H_{n-k-1} \). The linear series \( |m(H_{n-k}|_V) \) defines a rational family of \( k \)-cycles representing \( m\alpha \). Thus

\[
\text{rchdim}(m\alpha) \geq (H_1 \cdot \ldots \cdot H_{n-k-1} : H_{n-k}^{k+1}) \frac{m^{k+1}}{(k+1)!} + O(m^k)
\]
5.2. Definitions. Theorem 5.1 and Example 5.2 suggest that one should compare the growth rate of chdim($m\alpha$) against $m^{k+1}$. The choice of the coefficient $(k + 1)!$ is justified by Example 5.7.

**Definition 5.3.** Let $X$ be an integral projective variety. For any $\alpha \in N_k(X)_\mathbb{Z}$, we define the variation of $\alpha$ to be

$$\text{var}(\alpha) := \limsup_{m \to \infty} \frac{\text{chdim}(m\alpha)}{m^{k+1}/(k + 1)!}.$$ 

We define the rational variation of $\alpha$ in the analogous way using rchdim($\alpha$) in place of chdim($\alpha$).

**Remark 5.4.** [Nol97, Corollary 1.6] shows that there are infinitely many components of Hilb($\mathbb{P}^3$) parametrizing subschemes whose underlying cycle is a double line. Furthermore the dimensions of these components is unbounded. Thus there is no analogue of Theorem 5.1 for components of the Hilbert scheme with a fixed underlying cycle class.

Any attempt to formulate an analogue of the variation using the Hilbert scheme will need to either consider only special components of Hilb($X$) or account for all the terms in the Hilbert polynomial.

**Example 5.5.** Let $X$ be a normal integral projective variety of dimension $n$ and suppose that $X$ admits a resolution $\phi : Y \to X$ such that the kernel of $\phi_* : N_{n-1}(Y) \to N_{n-1}(X)$ is spanned by $\phi$-exceptional divisors. (For example $X$ could be smooth or normal $\mathbb{Q}$-factorial over $\mathbb{C}$.) Then for any Cartier divisor $D$ on $X$ we have $\text{ratvar}([D]) = \text{var}([D]) = \text{vol}(D)$.

To verify this, note that since chdim is preserved by passing to strict transform families of divisors we have

$$\text{chdim}(m[D]) = \max \left\{ \text{chdim}(\beta) \mid \beta \text{ is a class on } Y \text{ with } \phi_* \beta = m[D] \right\}.$$ 

Let $L$ denote any divisor in the class $\beta$ attaining this maximum value. We may write $L \equiv m\phi^*D + E$ where $E$ is some $\phi$-exceptional divisor. Since increasing the coefficients in $E$ can only increase chdim, we may assume that $E$ is effective. But then

$$h^0(Y, \mathcal{O}_Y(L)) = h^0(Y, \mathcal{O}_Y(L - E))$$

by the negativity of contraction lemma (see for example [Nak04, III.5.7 Proposition]). Thus

$$h^0(X, \mathcal{O}_X(mD)) - 1 \leq \text{rchdim}(m[D]) \leq \text{chdim}(m[D])$$

$$\leq \dim \text{Pic}^0(Y) + \max_{D' \equiv m\phi^*D} h^0(Y, \mathcal{O}_Y(D')) - 1.$$ 

While the rightmost term may be greater than $h^0(X, \mathcal{O}_X(mD)) - 1$, the difference is bounded by a polynomial of degree $n - 1$ in $m$ (see the proof of [Laz04, Proposition 2.2.43]). Thus $\text{ratvar}([D])$ and $\text{var}([D])$ agree with the volume.
Example 5.6. Suppose that $X$ is an integral projective variety of dimension $n$. There is an isomorphism $\deg : N_0(X)_\mathbb{Z} \to \mathbb{Z}$. For $\alpha \in N_0(X)_\mathbb{Z}$ of positive degree $\text{chdim}(\alpha) = n \deg(\alpha)$ so that

$$\text{var}(\alpha) = n \deg(\alpha)$$

The behavior of $\text{ratvar}(\alpha)$ is somewhat more subtle. For simplicity suppose that $\alpha$ is the positive generator of $N_0(X)_\mathbb{Z}$. A result of [Ro˘ı72] shows that there are non-negative integers $d(X), j(X)$ such that for sufficiently large $m$

$$\text{rchdim}(m\alpha) \geq m(n - d(X)) - j(X).$$

This gives a lower bound on $\text{ratvar}$. However, since $\text{ratvar}$ calculates the maximal variation (and not the “minimum variation” as in [Ro˘ı72]), it may happen that $\text{ratvar}(\alpha) > n - d(X)$. At the very least we know that $\text{ratvar}(\alpha)$ is always positive. In fact, by considering families of points lying on a fixed curve $C \subset X$ we see that $\text{ratvar}(\alpha) \geq 1$.

Example 5.7. Let $h \in N_k(\mathbb{P}^n)$ denote the class of a $k$-dimensional hyperplane. By analogy with the case of curves, [EH92] predicts that for sufficiently large $d$ the maximal dimension of a component of Chow($\mathbb{P}^n$) representing $dh$ parametrizes degree $d$ hypersurfaces in a $(k + 1)$-dimensional hyperplane. The expected value of $\text{chdim}(dh)$ is thus

$$\left(\frac{d + k + 1}{k + 1}\right) - 1 + (k + 2)(n - k - 1)$$

which would yield $\text{var}(h) = 1$. Note that this conjecture would yield a better bound in Theorem 5.1.

Remark 5.8. The components of Chow($X$) of maximal dimension tend to parametrize degenerate subvarieties. To measure the bigness of a class, one should instead only consider the dimensions of components of Chow($X$) that are “general” in some sense. This intuition is captured by the mobility function defined in Section 6; however, it would be interesting to see a formulation using the Chow variety directly. For curves on $\mathbb{P}^3$, [Per87] conjectures that calculating the dimensions of components of Chow($X$) parametrizing “general” curves of degree $d$ – in the sense that the corresponding cycles are not contained in any hypersurface of degree $< d^{1/2}$ – will yield the value of the mobility function.

5.3. Basic properties. We next verify some of the basic properties of the variation.

Lemma 5.9. Let $X$ be an integral projective variety and let $\alpha \in N_k(X)_\mathbb{Z}$. Then for any positive integer $c$ we have $\text{var}(c\alpha) = c^{k+1}\text{var}(\alpha)$ (and similarly for $\text{ratvar}$).

Proof. If $\text{chdim}(\alpha) > 0$ then $\alpha$ is represented by an effective cycle $Z$. Thus $\text{chdim}(\alpha + \beta) \geq \text{chdim}(\beta)$ for any class $\beta$: if $p$ is a family of effective cycles of class $\beta$ then we can add the constant cycle $Z$ to $p$ (using the family sum
construction) to obtain a family representing $\alpha + \beta$ with the same Chow dimension. We conclude by Lemma 2.10.

Lemma 5.9 allows us to extend the definition of variation to any $\mathbb{Q}$-class by homogeneity. Thus we obtain a function

$$\text{var} : N_k(X)_\mathbb{Q} \to \mathbb{R}_{\geq 0}.$$ 

**Lemma 5.10.** Let $X$ be an integral projective variety. Suppose that $\alpha, \beta \in N_k(X)_\mathbb{Q}$ are classes such that some positive multiple of each is represented by an effective cycle. Then $\text{var}(\alpha + \beta) \geq \text{var}(\alpha) + \text{var}(\beta)$ (and similarly for $\text{ratvar}$).

*Proof.* Note that we may check the inequality after rescaling $\alpha$ and $\beta$ by the same positive integer $c$. Thus we may suppose that every multiple of $\alpha$ and $\beta$ is represented by an effective cycle.

Suppose that $p : U \to W$ is a family representing $m\alpha$ and $q : S \to T$ is a family representing $m\beta$. Then the family sum $p + q$ represents $m(\alpha + \beta)$. Lemma 3.23 shows that $\text{chdim}(p + q) = \text{chdim}(p) + \text{chdim}(q)$ and the desired inequality follows. 

By Example 5.2 we find:

**Corollary 5.11.** Let $X$ be an integral projective variety and let $\alpha \in N_k(X)_\mathbb{Q}$ be a big class. Then $\text{var}(\alpha) > 0$.

As a consequence, we see that var is a continuous function on the big cone.

**Theorem 5.12.** Let $X$ be an integral projective variety. The function $\text{var} : N_k(X)_\mathbb{Q} \to \mathbb{R}_{\geq 0}$ is locally uniformly continuous on the interior of $\text{Eff}_k(X)_\mathbb{Q}$ (and similarly for $\text{ratvar}$).

*Proof.* var verifies conditions (1)-(3) of Lemma 2.12.

The behavior of the variation along the pseudo-effective boundary is more subtle. Probably the most one can hope for is:

**Question 5.13.** Let $X$ be an integral projective variety. Is the function $\text{var} : \text{Eff}_k(X)_\mathbb{Q} \to \mathbb{R}_{\geq 0}$ upper semi-continuous?

By analogy with the volume, the variation should satisfy some form of concavity on the big cone. The following conjecture is a weak statement in this direction. It is easy to show that the conjecture would imply the upper semi-continuity of var as a function on $\text{Eff}_k(X)_\mathbb{Q}$.

**Conjecture 5.14.** Let $X$ be an integral projective variety. Suppose that $\alpha, \beta, \gamma \in N_k(X)_{\mathbb{Z}}$ are classes with $\alpha$ pseudo-effective and $\beta$ and $\gamma$ big. Then

$$\text{chdim}(\alpha + \beta) - \text{chdim}(\alpha) \leq \text{chdim}(\alpha + \beta + \gamma) - \text{chdim}(\alpha + \gamma).$$
Finally we note that variation behaves well with respect to inclusions of subvarieties.

**Lemma 5.15.** Let $X$ be an integral projective variety and $i : W \to X$ an integral closed subvariety. For any class $\beta \in N_k(W)_{\mathbb{Q}}$ we have $\text{var}(\beta) \leq \text{var}(i_*\beta)$ (and similarly for $\text{ratvar}$).

**Proof.** Let $p$ be a family of effective cycles on $W$ and consider the push-forward family $q$ on $X$. Recall that for a general cycle-theoretic fiber $Z$ of $p$ the corresponding cycle in the push-forward family is just $i_*Z$; thus Lemma 3.6 shows that $\text{chdim}(p) = \text{chdim}(q)$ and the result follows. □

### 5.4. Variation, connecting chains, and bigness.

Example 1.10 shows that a class may have positive variation even when it is not big. This class is constructed by pushing forward a big class on a subvariety. In this section we show that every class with positive variation arises in this way.

The main step in the proof is to develop a criterion for bigness of a class using connecting chains of cycles. This criterion is modeled on [BCE+02, Theorem 2.4] which describes big curve classes via connecting chains. The correct analogue in higher dimensions should require that the cycles in our chain intersect “positively” in some sense. The next theorem shows that such a statement holds under a very strong positivity condition.

**Definition 5.16.** Let $X$ be an integral projective variety of dimension $n$ and let $p : U \to W$ be a family of effective $k$-cycles. We say that $p$ is strongly big-connecting if $s : U \to X$ is dominant and there is a big effective divisor $B$ on $X$ such that every $p$-horizontal component of $s^*B$ is contracted to a subvariety of $X$ of dimension at most $k-1$.

The following lemma is in preparation for Theorem 5.18.

**Lemma 5.17.** Let $p : U \to W$ be a flat map of projective varieties of relative dimension $k$ with $W$ integral. Let $A$ be a big effective Cartier divisor on $U$. There is a big effective $k$-cycle $Z$ on $U$ such that every component of $\text{Supp}(Z)$ is contained either in a fiber of $p$ or in a $p$-horizontal component of $A$.

**Proof.** Set $n = \dim U$. The proof is by induction on the codimension $n-k$. For the base case $n-k = 1$ we can simply take $Z = A$.

For the general case, note that by Lemma 2.9 (3) there is a big effective cycle $V$ on $U$ contained in $\text{Supp}(A)$. Let $D$ denote the support of the $p$-vertical components of $A$ and let $V'$ denote the part of $V$ whose support is not contained in any $p$-horizontal component of $A$.

Choose a very ample divisor $H$ on $W$ sufficiently positive so that $p^*H - D$ is linearly equivalent to an effective divisor. For $H$ general, and hence integral, we can apply the induction hypothesis to $p : p^*H \to H$ and $A|_{p^*H}$ to obtain a big effective $k$-cycle $Z'$ on $p^*H$ satisfying the support condition. In particular, for an ample divisor $\tilde{H}$ on $U$ there is some $c$ sufficiently large so that $c[Z'] \geq p^*H \cdot \tilde{H}^{n-k-1}$. This shows that some multiple of $[Z']$ will
also dominate any effective cycle supported in $D$, so for some $c'$ we have $c'[Z'] \geq [V']$. Set $Z = c'Z' + (V - V')$.

**Theorem 5.18.** Let $X$ be an integral projective variety. A class $\alpha \in N_k(X)$ is big if and only if there is some strongly big-connecting family of effective $k$-cycles $p : U \to W$ with class $\beta$ and a positive constant $c$ such that $c\alpha - \beta$ is pseudo-effective.

**Proof.** We first prove the forward implication. It suffices to construct an example of a strongly big-connecting family of effective $k$-cycles on $X$. Fix a very ample divisor $A$ on $X$ and an $(n-k+1)$-dimensional subspace $V \subset |A|$. Consider the family of effective $k$-cycles $p : U \to \mathbb{P}(V')$ defined by taking intersections of $(n-k)$ general elements of $V$. Choose an element $A \in |V|$. A general cycle in our family $p$ only intersects $A$ along the base locus of $V$; thus any $p$-horizontal component of $A$ must be mapped under $s$ to the base-locus of $V$ which has dimension $k-1$. So $p$ is a strongly big-connecting family for the divisor $A$.

Conversely, it suffices to show that a strongly big-connecting family has big class $\beta$. We may restrict our family $p$ to (an open subset of) a general complete intersection of very ample divisors on $W$ to ensure that $\dim U = \dim X$ without changing the strongly big-connecting property. We may then modify $p$ as in Lemma 3.7 to make $U$ and $W$ projective without changing the strongly big-connecting property.

Then $s^*A$ is a big effective Cartier divisor on $U$. Lemma 5.17 shows that there is a big effective $k$-cycle $Z$ on $U$ whose support is contained in fibers of $p$ and in $p$-horizontal components of $s^*A$. The former components are dominated by the cycle-theoretic fibers of $p$; the latter push forward to 0. Thus there is some constant $d$ such that $d\beta - s_*[Z]$ is a pseudo-effective class. Since $s_*[Z]$ is a big class on $X$ by Lemma 2.6 we find that $\beta$ is also a big class. 

**Example 5.19.** Suppose that $X$ is a smooth variety and $D$ is an irreducible divisor. This proposition is similar to the fact that $D$ is big if $D|_D$ is ample (see [Voi10, Lemma 2.3]).

**Theorem 5.20.** Let $X$ be an integral projective variety and let $\alpha \in N_k(X)_{\mathbb{Q}}$. Then the following conditions are equivalent:

1. $\text{ratvar}(\alpha) > 0$.
2. $\text{var}(\alpha) > 0$.
3. There is an integral $(k+1)$-dimensional projective variety $Y$, a big class $\beta \in N_k(Y)_{\mathbb{Q}}$, and a morphism $f : Y \to X$ that is generically finite onto its image such that some multiple of $\alpha - f_*\beta$ is represented by an effective cycle.

**Remark 5.21.** The proof shows that Theorem 5.20 is also true if the morphism $f$ in (3) is instead required to be a closed immersion.

**Proof.** The case when $k = 0$ is explained in Example 5.6 so we may assume $k \geq 1$. 


(1) \implies (2) is obvious.

We next show (2) \implies (3). Suppose that \( \text{var}(\alpha) > 0 \). We may rescale \( \alpha \) so that \( \alpha \in N_k(X)_\mathbb{Z} \) and every positive multiple of \( \alpha \) is represented by an effective cycle. Fix a very ample Cartier divisor \( A \), and choose some positive integer \( m \) sufficiently large so that

\[
\text{chdim}(m\alpha) > (n - k + 1) \left( \frac{m\alpha \cdot A^k + k}{k} \right).
\]

Let \( p : U \to W \) denote a family of effective \( k \)-cycles that has maximal Chow dimension among all the families representing \( m\alpha \). Denote the projection map to \( X \) by \( s : U \to X \). By replacing \( A \) by a linearly equivalent divisor, we may suppose that \( A \) does not contain any component of \( s(U) \).

Let \( q : R \to W^0 \) denote the intersection family of \( p \) with \( A \) as in Construction 3.21 (where \( W^0 \) is an appropriately chosen open set of \( W \)). The family \( q \) has class \( \beta := m\alpha \cdot A \in N_{k-1}(X) \). By Theorem 5.1, we have

\[
\text{chdim}(\beta) \leq (n - k + 1) \left( \frac{m\alpha \cdot A^k + k}{k} \right) < \text{chdim}(p).
\]

Thus there is a curve \( T \subset W^0 \) through a general point of \( W \) that is contracted by \( \text{ch}_q \) but not by \( \text{ch}_p \). Let \( q_T : R_T \to T \) denote the restriction of the family to (an open subset of) \( T \). Using Lemma 3.7 we may extend the family \( q_T \) to a projective closure of \( T \); from now on we let \( q_T \) denote this family over a projective base.

By Lemma 3.6 there is some irreducible component \( R' \) of \( R_T \) whose \( s \)-image in \( X \) has dimension \( k + 1 \). Set \( Y = s(R') \) with the reduced induced structure and \( f : Y \to X \) the corresponding closed immersion. By construction every \( q_T \)-horizontal component of \( s^*A \) on \( R' \) has \( s \)-image of dimension at most \( k - 1 \). Thus \( q_T' : R' \to T \) defines a strongly big-connecting family of divisors on \( Y \) with respect to \( A \). If we set \( \beta' \) to be the class on \( Y \) of the family \( q_T' \), then \( \beta' \) is a big class on \( Y \) by Theorem 5.18. Furthermore \( m\alpha - f_*\beta' \) is the class of an effective cycle. Setting \( \beta = \frac{1}{m}\beta' \) finishes the second implication.

To show (3) \implies (1), suppose that there is a generically finite morphism \( f : Y \to X \) from an integral \((k + 1)\)-dimensional projective variety \( Y \) and a big class \( \beta \in N_k(Y)_\mathbb{Q} \) so that some multiple of \( \alpha - f_*\beta \) is represented by an effective cycle. Let \( V = f(Y) \) with the induced reduced structure, so we have morphisms \( f' : Y \to V \) and \( i : V \subset X \). By Lemma 2.6 \( f'_*\beta \) is big on \( V \) so that \( \text{ratvar}(f'_*\beta) > 0 \). By Lemma 5.15 \( \text{ratvar}(i_*f'_*\beta) > 0 \). Thus \( \text{ratvar}(\alpha) > 0 \) as well.

\[\square\]

6. The Mobility Function

As suggested by [DELV11], we will define the mobility of a class \( \alpha \in N_k(X)_\mathbb{Z} \) to be the asymptotic growth rate of the number of general points contained in cycles representing multiples of \( \alpha \). We prove that big classes are
precisely those with positive mobility, confirming \cite{DELV11} Conjecture 6.5. Recall that by Convention \ref{Conv:dim} we only consider $k$-cycles for $0 \leq k < \dim X$.

6.1. **Mobility count.** The mobility count of a family of effective cycles can be thought of informally as a count of how many general points of $X$ are contained in members of the family. Although we are mainly interested in families of cycles, it will be helpful to set up a more general framework.

**Definition 6.1.** Let $X$ be an integral projective variety and let $W$ be a reduced variety. Suppose that $U \subset W \times X$ is a subscheme and let $p : U \to W$ and $s : U \to X$ denote the projection maps. The mobility count $mc(p)$ of the morphism $p$ is the maximum non-negative integer $b$ such that the map

$$U \times_W U \times_W \ldots \times_W U \times X \times \ldots \times X$$

is dominant, where we have $b$ terms in the product on each side. (If the map is dominant for every positive integer $b$, we set $mc(p) = \infty$.)

For $\alpha \in N_k(X)_\mathbb{Z}$, the mobility count of $\alpha$, denoted $mc(\alpha)$, is defined to be the largest mobility count of any family of effective cycles representing $\alpha$. We define the rational mobility count $rmc(\alpha)$ in the analogous way by restricting our attention to rational families.

**Example 6.2.** Let $X$ be an integral projective variety and let $p : U \to W$ be a family of effective $k$-cycles on $X$. Then $mc(p) \leq (\dim W)/(\dim X - k)$. Indeed, if the map of Definition 6.1 is dominant then dimension considerations show that $mc(p)k + \dim W \geq mc(p) \dim X$.

**Example 6.3.** Let $X$ be an integral projective variety and let $A$ be a very ample divisor such that $h^i(X, \mathcal{O}_X(A)) = 0$ for every $i > 0$. For any positive integer $s$, one can construct by induction a collection of $s$ distinct reduced closed points $P_s \subset X$ with

$$h^1(X, I_{P_s}(A)) = \max\{0, s - h^0(X, \mathcal{O}_X(A))\}.$$

Furthermore this is the maximal value of $h^1(X, I_P(A))$ for any collection of $s$ distinct closed points $P$.

Let $p : U \to W$ be the family of hyperplanes in $|A|$. Then $rmc(p) = h^0(X, \mathcal{O}_X(A)) - 1$, since this is the largest number of points for which we are guaranteed to have $h^0(X, I_p(A)) > 0$.

**Lemma 6.4.** Let $X$ be an integral projective variety. Let $W$ be a reduced variety and let $p : U \to W$ denote a closed subscheme of $W \times X$. Suppose that $T$ is another reduced variety and $q : S \to T$ is a closed subscheme of $T \times X$ such that every fiber of $p$ over a closed point of $W$ is contained in a fiber of $q$ over some closed point of $T$ (as subsets of $X$). Then $mc(p) \leq mc(q)$.

**Proof.** The conditions imply that for any $b > 0$, the $s^b$-image of any fiber of $p^b : U^{\times b} \to W$ over a closed point of $W$ is set-theoretically contained in the image of a fiber of $q^b : S^{\times b} \to T$ over a closed point of $T$ (as subsets of $X^{\times b}$). The statement follows. \hfill $\square$
Proposition 6.5. Let $X$ be an integral projective variety.

1. Let $W$ be an integral variety and let $U \subset W \times X$ be a closed subscheme such that $p : U \to W$ is flat. For an open subvariety $W^0 \subset W$ let $p^0 : U^0 \to W^0$ be the base change to $W^0$. Then $mc(p) = mc(p^0)$.

2. Let $p : U \to W$ be a family of effective cycles on $X$. For an open subvariety $W_0 \subset W$ let $p_0 : U_0 \to W_0$ be the restriction family. Then $mc(p) = mc(p_0)$.

3. Let $W$ be a normal integral variety and let $U \subset W \times X$ be a closed subscheme such that:
   - Every fiber of the first projection map $p : U \to W$ has the same dimension.
   - Every component of $U$ dominates $W$ under $p$.

   Let $W^0 \subset W$ be an open subset and $p_0 : U_0 \to W^0$ be the preimage of $W^0$. Then $mc(p) = mc(p_0)$.

Remark 6.6. Proposition 6.5 indicates that the mobility count is insensitive to the choice of definition of a family of effective cycles.

Proof. (1) The map $p^b : U^b \times W^b \to W$ is proper flat, so that every component of $U^b \times W^b$ dominates $W$. Then $(U^0)^b \times W^b$ is dense in $U^b \times W^b$ for any $b$. Thus $mc(p^b) = mc(p)$.

(2) Let $\{U_i\}$ denote the irreducible components of $U$. Every irreducible component of $U^b \times W^b$ is contained in a product of the $U_i$ over $W$. Since each $p|_{U_i} : U_i \to W$ is flat, we can apply the same argument as in (1).

(3) The inequality $mc(p) \geq mc(p^0)$ is clear. To show the converse inequality, we may suppose that $U$ is reduced. We may also shrink $W^0$ and assume that $p^0$ is flat.

Let $p' : U' \to W'$ be a flattening of $p$ via the birational morphism $\phi : W' \to W$. We may ensure that $\phi$ is an isomorphism over $W^0$. Choose a closed point $w \in W$ and let $T \subset W'$ be the set-theoretic preimage. Since $W$ is normal $T$ is connected.

Choose a closed point $w' \in T$. By construction the fiber $U'_{w'}$ is set theoretically contained in $U_w$ (as subsets of $X$). Since they have the same dimension, $U'_{w'}$ is a union of components of $U_w$. Since $p'$ is flat over $T$ and $T$ is connected, in fact $U'_{w'}$ and $U_w$ have the same number of components and thus are set-theoretically equal. Applying Lemma 6.4 and part (1) we see that $mc(p) \leq mc(p') = mc(p^0)$. \qed

We can now describe how the mobility count changes under certain geometric constructions of families of cycles.

Lemma 6.7. Let $X$ be an integral projective variety and let $p : U \to W$ be a family of effective $k$-cycles. Suppose that $U$ has a component $U_i$ whose image in $X$ is contained in a proper subvariety. Then $mc(p) = mc(p')$ where $p'$ is the family defined by removing $U_i$ from $U$. 

Proof. This is immediate from the definition. \qed

Lemma 6.8. Let $\psi : X \dashrightarrow Y$ be a birational morphism of integral projective varieties. Let $p : U \to W$ be a family of effective $k$-cycles on $X$ and let $p'$ denote the strict transform family on $Y$. Then $mc(p) = mc(p')$.

Proof. By Lemma 6.7 we may assume that every component of $U$ dominates $X$. Using Proposition 6.5 (2), we may replace $p$ by the restricted family $p^0 : U^0 \to W^0$, where $W^0$ is the locus of definition of the strict transform family $p' : U' \to W^0$. The statement is then clear using the fact that the morphisms $(U^0) \times_W b \to X^b$ and $(U')^0 \times_W b \to Y^b$ are birationally equivalent for every $b$. \qed

Lemma 6.9. Let $X$ be an integral projective variety. Suppose that $W$ and $T$ are reduced varieties and that $p_1 : U \to W$ and $p_2 : S \to T$ are closed subschemes of $W \times X$ and $T \times X$ respectively. Let $q : V \to W \times T$ denote the subscheme

$$U \times T \cup W \times S \subset W \times T \times X.$$ 

Then $mc(q) = mc(p_1) + mc(p_2)$.

In particular, if $p_1$ and $p_2$ are families of effective $k$-cycles, then the mobility count of the family sum is the sum of the mobility counts.

Proof. Set $b_1 = mc(p_1)$ and $b_2 = mc(p_2)$. There is a dominant projection map

$$\left( U^{xWb_1} \times T \right)^{W \times T} \left( W \times S^{xTb_2} \right) \to X^{(b_1+b_2)}.$$ 

Since the domain is naturally a subscheme of $V^{xW \times T b_1 + b_2}$, we obtain $mc(q) \geq mc(p_1) + mc(p_2)$.

Conversely, any irreducible component of $V^{xW \times T c}$ is (up to reordering the terms) a subscheme of

$$\left( U^{xWc_1} \times T \right)^{W \times T} \left( W \times S^{xTc_2} \right)$$

for some non-negative integers $c_1$ and $c_2$ with $c = c_1 + c_2$ where the map to $X^b$ is component-wise. This yields the reverse inequality.

To extend the lemma to the family sum, first replace $p_1$ and $p_2$ by their restrictions to the normal locus of $W$ and $T$ respectively; this does not change the mobility count by Proposition 6.5 (2). Then by Proposition 6.5 (3) the mobility count of the family sum is the same as the mobility count of the subscheme $U \times T \cup W \times S$ as defined in Construction 3.19. \qed

Corollary 6.10. Let $X$ be an integral projective variety and let $p : U \to W$ be a family of effective $k$-cycles. Let $p_i : U_i \to W$ denote the irreducible components of $U$. Then $mc(p) \leq \sum_i mc(p_i)$.

Proof. Let $q : S \to T$ denote the family sum of the $p_i$. By Lemma 6.4 we have $mc(p) \leq mc(q)$; by Lemma 6.9 $mc(q) = \sum_i mc(p_i)$. \qed
6.2. The mobility function. As indicated by [DELV11], we have:

**Proposition 6.11.** Let \( X \) be an integral projective variety of dimension \( n \) and let \( \alpha \in N_k(X) \). Fix a very ample divisor \( A \) and choose a positive constant \( c < 1 \) so that \( h^0(X, mA) \geq \lceil cm^n \rceil \) for every positive integer \( m \). Then any family \( p: U \to W \) representing \( \alpha \) has

\[
mc(p) \leq (n + 1)2^n \left( \frac{2(k + 1)}{c} \right)^{\frac{n+k}{n-k}} \frac{1}{n-k} (\alpha \cdot A^k)^{\frac{n}{n-k}} A^n.
\]

In particular, there is some constant \( C \) so that \( mc(m\alpha) \leq Cm^{n} \).

We will develop a similar bound that does not depend on the constant \( c \) in Theorem 6.24.

**Proof.** By Lemma 4.11 the support \( Z \) of any effective cycle representing \( \alpha \) satisfies

\[
h^0(X, I_Z(dA)) \geq \lceil cd^n \rceil - (k + 1) d^k (\alpha \cdot A^k)
\]

for any positive integer \( d \). Thus an effective cycle representing \( \alpha \) is set-theoretically contained in an element of \( |[d]A| \) as soon as \( d \) is sufficiently large to make the right hand side greater than 1, and in particular, for

\[
d = \left( \frac{2(k + 1)}{c} \right)^{\frac{n}{n-k}} \frac{1}{n-k} (\alpha \cdot A^k)^{\frac{n}{n-k}}.
\]

Let \( q: \tilde{U} \to \mathbb{P}(|[d]A|) \) denote the family of divisors defined by the linear series. By Lemma 6.4 we have \( mc(p) \leq mc(q) \). Since \( c < 1 \), \( d \geq 1 \) so that \( |d| < 2d \). Applying Lemma 4.11 again, Example 6.3 indicates that

\[
mc(p) \leq h^0(X, [d]A) - 1 < (n + 1)2^n \left( \frac{2(k + 1)}{c} \right)^{\frac{n}{n-k}} \frac{1}{n-k} (\alpha \cdot A^k)^{\frac{n}{n-k}} A^n.
\]

Furthermore, the growth rate of \( Cm^{\frac{n}{n-k}} \) is always achieved by some big class as demonstrated by the next example.

**Example 6.12.** Let \( X \) be an integral projective variety and let \( A \) be a very ample divisor such that \( h^i(X, \mathcal{O}_X(mA)) = 0 \) for every \( i > 0 \), \( m > 0 \) and \( h^0(X, \mathcal{O}_X(mA)) > 0 \). Choose a positive constant \( c \) such that we have \( h^0(X, \mathcal{O}_X(mA)) > cm^n \) for every positive integer \( m \).

Let \( P \subset X \) be any collection of \( k \) reduced closed points on \( X \). Example 6.3 shows that the ideal sheaf \( I_P \) is \( m \)-regular as soon as \( c(m - n)^n > k \). In particular, for large \( m \) we can find a complete intersection of \( n \) elements of \( |mA| \) that has dimension 0 and contains any \( [c(m - n)^n] \) closed points of \( X \). Setting \( a = A^{n-k} \), it is then clear that

\[
rmc(m^{n-k}a) \geq [c(m - n)^n].
\]

Proposition 6.11 and Example 6.12 indicate that we should make the following definition.
Definition 6.13. Let $X$ be an integral projective variety and let $\alpha \in N_k(X)$. The mobility of $\alpha$ is

$$\text{mob}_X(\alpha) = \limsup_{m \to \infty} \frac{\text{mc}(m\alpha)}{m^{n/k}/n!}.$$ 

We will omit the subscript $X$ when the ambient variety is clear from the context. We define the rational mobility $\text{ratmob}(\alpha)$ in an analogous way using $\text{rmc}$.

The coefficient $n!$ is justified by Section 7.1. We verify in Example 6.16 that the mobility agrees with the volume function for Cartier divisors on a smooth integral projective variety.

Example 6.14. Let $X$ be an integral projective variety of dimension $n$ and let $\alpha \in N_0(X)$ be the class of a point. Then $\text{mc}(m\alpha) = m$, so that the mobility of the point class is $n!$. Rational mobility is more interesting; we will analyze it in more detail in Section 7.2.

6.3. Basic properties. We now turn to the basic properties of the mobility function.

Lemma 6.15. Let $X$ be an integral projective variety and let $\alpha \in N_k(X)$. Fix a positive integer $a$. Then $\text{mob}(a\alpha) = a^{n/k} \text{mob}(\alpha)$ (and similarly for $\text{ratmob}$).

Proof. If $\text{mc}(\alpha) > 0$ then $\alpha$ is represented by a family of effective cycles. Thus $\text{mc}(\alpha + \beta) \geq \text{mc}(\beta)$ for any class $\beta$ by the additivity of mobility count under family sums as in Lemma 6.9. Then apply Lemma 2.10. \qed

Lemma 6.15 allows us to extend the definition of mobility to any $\mathbb{Q}$-class by homogeneity, obtaining a function $\text{mob} : N_k(X) \to \mathbb{R}_{\geq 0}$.

Example 6.16. Let $X$ be a normal integral projective variety of dimension $n$ and suppose that $X$ admits a resolution $\phi : Y \to X$ such that the kernel of $\phi_* : N_{n-1}(Y) \to N_{n-1}(X)$ is spanned by $\phi$-exceptional divisors. (For example $X$ could be smooth or normal $\mathbb{Q}$-factorial over $\mathbb{C}$.) For any Cartier divisor $L$ on $X$ we have

$$\text{ratmob}(\lfloor L \rfloor) = \text{mob}(\lfloor L \rfloor) = \text{vol}(L).$$

To prove this, note first that by Example 6.2 we have

$$\text{rmc}(m\lfloor L \rfloor) \leq \text{mc}(m\lfloor L \rfloor) \leq \text{chdim}(m\lfloor L \rfloor)$$

so that by Example 5.5 $\text{vol}(L) = \text{var}(\lfloor L \rfloor) \geq \text{mob}(\lfloor L \rfloor) \geq \text{ratmob}(\lfloor L \rfloor)$.

We first show the converse inequality when $L$ is ample. After rescaling, we may suppose that $L$ is very ample and that $h^i(X, \mathcal{O}_X(mL)) = 0$ for every $i > 0, m > 0$. Example 6.3 indicates that $h^0(X, \mathcal{O}_X(mL)) - 1 \leq \text{rmc}(mL)$, showing that $\text{ratmob}(L) = \text{vol}(L)$ in this case. By homogeneity, we also obtain equality for any ample $\mathbb{Q}$-divisor $A$. 
Suppose now that $L$ is big. Let $\phi : Y \to X$ be an $\epsilon$-Fujita approximation for $L$ (constructed by [Tak07] in arbitrary characteristic), so that there is an ample $\mathbb{Q}$-divisor $A$ and an effective $\mathbb{Q}$-divisor $E$ on $Y$ with $\phi^* L \equiv A + E$ and $\text{vol}(A) > \text{vol}(L) - \epsilon$. Then
\[
\text{vol}(L) - \epsilon < \text{vol}(A) = \text{ratmob}([A]) \\
\leq \text{ratmob}(f_*[A]) \leq \text{ratmob}([L]).
\]
Since $\epsilon > 0$ is arbitrary, we obtain the desired equalities.

**Lemma 6.17.** Let $X$ be an integral projective variety. Suppose that $\alpha, \beta \in N_k(X)_{\mathbb{Q}}$ are classes such that some positive multiple of each is represented by an effective cycle. Then $\text{mob}(\alpha + \beta) \geq \text{mob}(\alpha) + \text{mob}(\beta)$ (and similarly for $\text{ratmob}$).

**Proof.** We may verify the inequality after rescaling $\alpha$ and $\beta$ by the same positive constant $c$. Thus we may suppose that every multiple of each is represented by an effective class. Using the additivity of mobility counts under family sums as in Lemma 6.9, we see that
\[
\text{mc}(m(\alpha + \beta)) \geq \text{mc}(m\alpha) + \text{mc}(m\beta)
\]
and the conclusion follows. \qed

By Example 6.12 we find:

**Corollary 6.18.** Let $X$ be an integral projective variety and let $\alpha \in N_k(X)_{\mathbb{Q}}$ be a big class. Then $\text{mob}(\alpha) > 0$ (and similarly for $\text{ratmob}$).

**Theorem 6.19.** Let $X$ be an integral projective variety. The function $\text{mob} : N_k(X)_{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$ is locally uniformly continuous on the interior of $\text{Eff}_k(X)_{\mathbb{Q}}$ (and similarly for $\text{ratmob}$).

Theorem 6.28 extends this result to prove that $\text{mob}$ is continuous on all of $N_k(X)$.

**Proof.** The conditions (1)-(3) in Lemma 2.12 are verified by Lemma 6.15, Lemma 6.17 and Corollary 6.18. \qed

The mobility should also have good concavity properties. Here is a strong conjecture in this direction:

**Conjecture 6.20.** Let $X$ be an integral projective variety. Then $\text{mob}$ is a log-concave function on $\text{Eff}_k(X)$: for any classes $\alpha, \beta \in \text{Eff}_k(X)$ we have
\[
\text{mob}(\alpha + \beta)^{n-k} \geq \text{mob}(\alpha)^{n-k} + \text{mob}(\beta)^{n-k}
\]

6.4. **Mobility and bigness.** We now show that big cycles are precisely those with positive mobility:

**Theorem 6.21.** Let $X$ be an integral projective variety and let $\alpha \in N_k(X)_{\mathbb{Q}}$. The following statements are equivalent:

1. $\alpha$ is big.
(2) \( \text{ratmob}(\alpha) > 0 \).
(3) \( \text{mob}(\alpha) > 0 \).

The implication (1) \( \implies \) (2) follows from Corollary 6.18 and (2) \( \implies \) (3) is obvious. The implication (3) \( \implies \) (1) is a consequence of the more precise statement in Corollary 6.25.

Example 6.22. Let \( X \) be a normal integral projective variety with \( \mathbb{Q} \)-factorial singularities over \( \mathbb{C} \). By Example 6.16, Theorem 6.21 is equivalent to the usual characterization of big divisors using the volume function.

Example 6.23. Let \( \alpha \) be a curve class on a complex variety \( X \). [BCE+02, Theorem 2.4] shows that if two general points of \( X \) can be connected by an effective curve whose class is proportional to \( \alpha \), then \( \alpha \) is big. Theorem 6.21 is a somewhat weaker statement in this situation.

For positive integers \( n \) and for \( 0 \leq k < n \), define \( \epsilon_{n,k} \) inductively by setting \( \epsilon_{n,n-1} = 1 \) and

\[
\epsilon_{n,k} = \frac{n-k-1}{n-k+1} \epsilon_{n-1,k} - \frac{n-k}{n-k+1} \epsilon_{n-1,k}.
\]

For positive integers \( n \) and for \( 0 \leq k < n \), define \( \tau_{n,k} \) inductively by setting \( \tau_{n,n-1} = 1 \) and

\[
\tau_{n,k} = \min \left\{ \frac{n-k-1}{n-1} \tau_{n-1,k}, \frac{n-k-1}{n-k+1} \tau_{n-1,\tau_{n-1,k} \tau_{n-1,k} \tau_{n-1,k}} \right\}.
\]

It is easy to verify that \( 0 < \tau_{n,k} \leq \epsilon_{n,k} \leq \frac{1}{n-k} \) and that the last inequality is strict as soon as \( n-k > 1 \).

Theorem 6.24. Let \( X \) be an integral projective variety and let \( \alpha \in \bar{N}_k(X) \). Let \( A \) be a very ample divisor and let \( s \) be a positive integer such that \( \alpha \cdot A^k \leq sA^n \). Then

(1) \[
\text{mc}(\alpha) \leq 2^{kn+3n}(k+1)s^{\frac{n}{n-k}}A^n.
\]

(2) Suppose furthermore that \( \alpha - [A]^{n-k} \) is not pseudo-effective. Then

\[
\text{mc}(\alpha) \leq 2^{kn+3n}(k+1)s^{\frac{n}{n-k}}\epsilon_{n,k}A^n.
\]

(3) Suppose that \( t \) is a positive integer such that \( t \leq s \) and \( \alpha - t[A]^{n-k} \) is not pseudo-effective. Then

\[
\text{mc}(\alpha) \leq 2^{kn+3n}(k+1)s^{\frac{n}{n-k}}\tau_{n,k}A^n.
\]

Proof. We prove (1) by induction on the dimension \( n \) of \( X \). The induction step may reduce the codimension \( n-k \) of our cycle class by at most 1. Thus, for the base case it suffices to consider when \( k = 0 \) or when \( n \) is arbitrary and \( n-k = 1 \). These cases are proved by Example 6.14 and Corollary 4.10 (1) respectively.
Let \( p : U \to W \) be a family of effective \( k \)-cycles representing \( \alpha \). By Proposition 6.5 we may modify \( p \) by Lemma 3.7 to assume \( W \) is projective without changing the mobility count of \( p \). Choose a general divisor \( H \) in the very ample linear series \( |s^{n-\pi}A| \) on \( X \) such that \( H \) is integral and does not contain the image of any component of \( U \). We can associate several families of subschemes to \( p \) and to \( H \).

- Consider the base change \( U \times_X H \). We can view this as a subscheme \( U_H \subset W \times H \) with projection map \( \pi : U_H \to W \).
- We can intersect the family \( p \) with the divisor \( H \) to obtain a family of effective \((k-1)\) cycles \( q : S \to T \) on \( H \) as in Construction 3.21. Note that \( T \) is an open subset of \( W \); we may shrink \( T \) so that it is normal. By Lemma 3.7 we may extend \( q \) over a projective closure of \( T \) which we continue to denote by \( q : S \to T \).
- Let \( V \subset U_H \) be the reduced closed subset consisting of points whose local fiber dimension for \( \pi \) attains the maximal possible value \( k \). The map \( \pi|_V : V \to W \) has equidimensional fibers of dimension \( k \). Thus we can associate to \( \pi|_V \) a collection of families \( p_i : V_i \to W_i \) as in Construction 3.5. We will think of these as families of effective \( k \)-cycles on \( H \).

It will be useful to combine \( q \) and the \( p_i \) as follows. Let \( \tilde{W} \) denote the disjoint union of the irreducible varieties \( T \times W_i \) as we vary over all \( i \). This yields the subscheme \( S \times (\cup_i W_i) \cup T \times (\cup_i V_i) \) of \( \tilde{W} \times X \). We denote the first projection map by \( \tilde{p} \).

Using the universal property, one can see that
\[
(U \times_X H)^{\times W} \cong (U^{\times W})^{\times H}.
\]

Since the base change of a surjective map is surjective, we see that \( \text{mc}_H(\pi) \geq \text{mc}_X(p) \). Furthermore, by Krull’s principal ideal theorem every component of a fiber of \( \pi \) over a closed point of \( W \) has dimension \( k \) or \( k-1 \). In particular, any member of the family \( \pi \) is set theoretically contained in a member of the family \( \tilde{p} \). Applying Lemmas 5.9 and 6.4 we obtain
\[
\text{mc}_X(p) \leq \text{mc}_H(\pi) \leq \text{mc}_H(\tilde{p}) = \text{mc}_H(q) + \sup_i \text{mc}_H(p_i).
\]

We will use induction to bound the two terms on the right, giving us our overall bound for \( \text{mc}_X(p) \).

The family \( q \) of effective \((k-1)\)-cycles on \( H \) has class \( \alpha \cdot H \). Note that
\[(\alpha \cdot H) \cdot A_{\pi}^{k-1} \leq sA_{\pi}^{n-1} \cdot \] By induction on the dimension of the ambient variety,
\[
\text{mc}_H(q) \leq 2^{(k-1)(n-1)+3(n-1)}k^3s^{\frac{n-1}{\pi}}(A_{\pi}^{n-1}) \\
\leq 2^{(k-1)(n-1)+3(n-1)}k^3s^{\frac{n-1}{\pi}}(2^{n-\pi}A^n) \\
\leq 2^{(k-1)(n-1)+3(n-1)+1}(k+1)s^{\frac{n}{\pi}}A^n.
\]
Next consider a family $p_i$ of effective $k$-cycles on $H$. Let $\alpha_i$ denote the corresponding class. Let $j : H \to X$ be the inclusion; by construction, it is clear that $\alpha - j_*\alpha_i$ is the class of an effective cycle. In particular

$$\alpha_i \cdot A|_H^k \leq \alpha \cdot A^k \leq \lceil s^{n-k-1} \rceil A|_H^{n-1}. $$

By induction on the dimension of the ambient variety,

$$mc_H(p_i) \leq 2^{(n-1)k+3(n-1)}(k+1)\lfloor s^{n-k-1} \rfloor \lceil n \rceil (A|_H^{n-1})$$

$$\leq 2^{(n-1)k+3(n-1)}(k+1)2^{n-k-1} s^{n-k} (2s^{n-k} A^n)$$

$$\leq 2^{(n-1)k+3(n-1)+k+2}(k+1)s^{n-k} A^n.$$  

By adding these contributions, we see that

$$mc_X(p) \leq 2^{kn+3n}(k+1)s^{n-k} A^n.$$  

(2) is proved in a similar way. The argument is by induction on the codimension $n-k$ of $\alpha$. The base case – when $n$ is arbitrary and $n-k = 1$ – is a consequence of Corollary 4.10 (2) (applied with $H = A$).

Let $p : U \to W$ be a family of effective $k$-cycles representing $\alpha$. Set $c := 1 - \epsilon_{n,k}$. Let $H$ be an integral element of $[s^c A]$ that does not contain the image of any component of $U$. We construct the families $q : S \to T$ and $p_i : V_i \to W_i$ just as in (1). The same argument shows that

$$mc_X(p) \leq mc_H(q) + \sup_i mc_H(p_i).$$

The family $q$ of effective $(k-1)$-cycles on $H$ has class $\alpha \cdot H$. Note that

$$(\alpha \cdot H) \cdot A|_H^{k-1} \leq sA|_H^{n-1}. $$

By (1), we have

$$mc_H(q) \leq 2^{(k-1)(n-1)+3(n-1)}ks^{n-1-k} (A|_H^{n-1})$$

$$\leq 2^{(k-1)(n-1)+3(n-1)}ks^{n-1-k} (2s^c A^n)$$

$$\leq 2^{(k-1)(n-1)+3(n-1)+1}(k+1)s^{n-k} \epsilon_{n,k} A^n.$$  

Next consider the family $p_i$ of effective $k$-cycles on $H$. Let $\alpha_i$ denote the class of the family $p_i$ on $H$. Let $j : H \to X$ be the inclusion; by construction, it is clear that $\alpha - j_*\alpha_i$ is the class of an effective cycle. In particular

$$\alpha_i \cdot A|_H^k \leq \alpha \cdot A^k \leq \lceil s^{1-c} \rceil A|_H^{n-1}. $$

Note furthermore that $\alpha_i - [A|_H]^{n-1-k}$ is not pseudo-effective; otherwise it would push forward to a pseudo-effective class on $X$, contradicting the fact that $\alpha - [A]^{n-k}$ is not pseudo-effective. By induction on the codimension of the cycle,

$$mc_H(p_i) \leq 2^{(n-1)+3(n-1)}(k+1)\lceil s^{1-c} \rceil \lceil n \rceil - \epsilon_{n-1,k} (A|_H^{n-1})$$

$$\leq 2^{(n-1)+3(n-1)}(k+1)2^{n-k-1} s^{(1-c)(n-1)} \lceil 1-c \rceil \epsilon_{n-1,k} (2s^c A^n)$$

$$\leq 2^{(n-1)+3(n-1)+k+2}(k+1)s^{n-k} \epsilon_{n,k} A^n.$$  

Also, we have that  
\[
\text{class of the family } p = \alpha \text{ it is clear that }
\]
Adding the two contributions proves the statement as before.

The proof of (3) is also very similar. The argument is by induction on the codimension \( n - k \) of \( \alpha \). The base case – when \( n \) is arbitrary and \( n - k = 1 \) – is a consequence of Corollary 4.10 (2) (applied with \( H = tA \)).

Let \( p : U \to W \) be a family of effective \( k \)-cycles representing \( \alpha \). Set \( c := \frac{1}{n-k} - \tau_{n,k} \). Let \( H \) be an integral element of \( \left[ [s^c t^{\tau_{n,k}}] A \right] \) that does not contain the image of any component of \( U \). We construct the families \( q : S \to T \) and \( p_i : V_i \to W_i \) just as in (1). The same argument shows that  
\[
mc_X(p) \leq mc_H(q) + \sup_i mc_H(p_i).
\]

The family \( q \) of effective \((k - 1)\)-cycles on \( H \) has class \( \alpha \cdot H \). Note that  
\[
(\alpha \cdot H) \cdot A^{k-1}_H \leq sA^{n-1}_H.
\]
By (1), we have  
\[
mc_H(q) \leq 2^{(k-1)(n-1)+3(n-1)}kS^{\frac{n-1}{n-k}}(A^{n-1}_H)
\]
\[
\leq 2^{(k-1)(n-1)+3(n-1)}kS^{\frac{n-1}{n-k}}(2s^c t^{\tau_{n,k}}A^n)
\]
\[
\leq 2^{(k-1)(n-1)+3(n-1)+1}(k+1)S^{\frac{n-1}{n-k}} - \tau_{n,k} t^{\tau_{n,k}}A^n.
\]

Next consider the family \( p_i \) of effective \( k \)-cycles on \( H \). Let \( \alpha_i \) denote the class of the family \( p_i \) on \( H \). Let \( j : H \to X \) be the inclusion; by construction, it is clear that \( \alpha - j_*\alpha_i \) is the class of an effective cycle. In particular  
\[
\alpha_i \cdot A^k_H \leq \alpha \cdot A^k \leq \left[ s^{1-c} t^{-\tau_{n,k}} \right] A^{n-1}_H.
\]
Also, we have that  
\[
\alpha_i \cdot \left[ t^{1-\tau_{n,k} - c} \right] A^k_H \leq \left[ s^{1-c} t^{-\tau_{n,k}} \right] A^{n-1}_H.
\]
is not pseudo-effective, since the difference between \( \alpha - t[A]^{n-k} \) and the push forward of this class to \( X \) is pseudo-effective. Finally, note that  
\[
\left[ s^{1-c} t^{-\tau_{n,k}} \right] \geq \left[ t^{1-\tau_{n,k} - c} \right]
\]
so that we may apply (3) inductively to the family \( p_i \) with the constants \( s' = \left[ s^{1-c} t^{-\tau_{n,k}} \right] \) and \( t' = \left[ t^{1-\tau_{n,k} - c} \right] \).

There are two cases to consider. First suppose that \( t^{1-\tau_{n,k} - c} \geq 1 \). Then by induction on the codimension of the cycle,  
\[
mc_H(p_i) \leq 2^{(k-1)+3(n-1)}(k+1)\left[ s^{1-c} t^{-\tau_{n,k}} \right] \frac{\frac{n-1}{n-k}}{\tau_{n-1,k}} \frac{\frac{n-1}{n-k}}{\tau_{n-1,k}} \frac{\frac{n-1}{n-k}}{\tau_{n-1,k}}
\]
\[
\leq 2^{(k-1)+3(n-1)}(k+1)2^{\frac{n-1}{n-k}} \frac{\frac{n-1}{n-k}}{\tau_{n-1,k}} \frac{\frac{n-1}{n-k}}{\tau_{n-1,k}} \frac{\frac{n-1}{n-k}}{\tau_{n-1,k}}
\]
\[
\leq 2^{(k-1)+3(n-1)+k+2}(k+1)S^{\frac{n-1}{n-k}} - \tau_{n,k} t^{\tau_{n,k}}A^n,
\]

or for \( t^{1-\tau_{n,k} - c} < 1 \).
Since \( \tau_{n-1,k} \geq \frac{n-1}{n-k-1} \tau_{n,k} \), the part of the exponents in parentheses is negative for \( s \) and positive for \( t \). By assumption \( s \geq t \), so
\[
mc_H(p_i) \leq 2^{kn+3(n-1)+2(k+1)}s^{\frac{n}{n-k}}\tau_{n,k} A^n
\]
Next suppose that \( 1-\tau_{n,k} s^{-c} < 1 \). Then by (2) we find
\[
mc_H(p_i) \leq 2^{k(n-1)+3(n-1)}(k+1)\left[s^{1-c}t^{-\tau_{n,k}}\right]^{\frac{n-1}{n-k-1}} - \epsilon_{n-1,k} (A_H^{n-1})
\]
\[
\leq 2^{k(n-1)+3(n-1)}(k+1)\left[s^{1-c}t^{-\tau_{n-1,k}}(A_H^{n-1})\right]
\]
\[
\leq 2^{kn+3(n-1)+2(k+1)}s^{\frac{n}{n-k}}\tau_{n,k} A^n.
\]
This upper bound for the two cases is the same; by adding it to the upper bound for \( mc_H(q) \) we obtain the desired upper bound for \( mc(p) \).

We can apply Theorem \[6.24\] (2) to any class \( \alpha \in \partial \text{Eff}_k(X) \cap N_k(X)_\mathbb{Q} \) to obtain the following corollary.

**Corollary 6.25.** Let \( X \) be an integral projective variety and suppose that \( \alpha \in N_k(X)_\mathbb{Q} \) is not big. Let \( A \) be a very ample divisor and let \( s \) be a positive integer such that \( \alpha \cdot A^k \leq sA^n \). Then
\[
mc(\alpha) \leq 2^{kn+3n}(k+1)s^{\frac{n}{n-k}} - \epsilon_{n,k} A^n.
\]

**Remark 6.26.** The exponent \( \frac{n}{n-k} - \epsilon_{n,k} \) in Corollary \[6.25\] is not optimal in general. For example, \[BCE^*02\], Theorem 2.4 shows that for a curve class \( \alpha \) that is not big there is a positive constant \( C \) such that \( mc(\alpha C m) \leq C m \).

**Example 6.27.** Let \( f : X \to Z \) be a surjective morphism from a smooth integral projective variety of dimension \( n \) to a smooth integral projective variety of dimension \( k \) for some \( 1 < k < n \). Fix ample divisors \( A \) on \( X \) and \( H \) on \( Z \) and define \( \alpha = [A]_{n-k-1} \cdot [f^*H] \). \( \alpha \) is not big since \( \alpha \cdot [f^*H]^k = 0 \). By taking the complete intersection of \( (n-k-1) \) elements of \( H^0(X, \mathcal{O}_X([m \frac{n}{(n-k)(k+1)}] A)) \) with an element of \( H^0(X, \mathcal{O}_X([m \frac{n}{(n-k)(k+1)}] f^*H)) \) we see
\[
mc(m \alpha) \geq C m \frac{n^k}{(n-k)(k+1)}
\]
for some positive constant \( C \). Rewriting
\[
\frac{n^k}{(n-k)(k+1)} = \frac{n}{n-k} - \frac{n}{(n-k)(k+1)}
\]
shows that the optimal value of \( \epsilon_{n,k} \) is at most \( \frac{n}{(n-k)(k+1)} \).

**6.5. Continuity of mobility.** Theorem \[6.24\] also allows us to prove the continuity of the mobility function.

**Theorem 6.28.** Let \( X \) be an integral projective variety. Then the mobility function \( \text{mob} : N_k(X)_\mathbb{Q} \to \mathbb{R} \) can be extended to a continuous function on \( N_k(X) \).
Proof. Note that mob can be extended to a continuous function on the interior of $\mathbb{P}^k_{\mathbb{Q}}(X)$ by Theorem 6.19. Furthermore mob is identically 0 on every element in $N_k(X)_{\mathbb{Q}}$ not contained in $\mathbb{P}^k_{\mathbb{Q}}(X)$. Thus it suffices to show that mob approaches 0 for classes approaching the boundary of $\mathbb{P}^k_{\mathbb{Q}}(X)$.

Let $\alpha$ be a point on the boundary of $\mathbb{P}^k_{\mathbb{Q}}(X)$. Fix $\mu > 0$; we show that there exists a neighborhood $U$ of $\alpha$ such that mob approaches 0 for classes approaching the boundary of $\mathbb{P}^k_{\mathbb{Q}}(X)$.

Choose $\delta$ sufficiently small so that
$$n!2^{k\beta+3n+1}(k+1)s^\frac{n}{n-k}A^n\delta^{\tau_{n,k}} < \mu.$$ Let $U$ be a sufficiently small neighborhood of $\alpha$ so that:
- $\beta \cdot A^k \leq sA^n$ for every $\beta \in U$, and
- $\beta - \delta s[A]^{n-k}$ is not pseudo-effective for every $\beta \in U$.

Suppose now that $\beta \in U \cap N_k(X)_{\mathbb{Q}}$ and that $m$ is any positive integer such that $m\beta \in N_k(X)_{\mathbb{Z}}$. Then:
- $m\beta \cdot A^k \leq smA^n$ and
- $m\beta - \lceil \delta ms \rceil [A]^{n-k}$ is not pseudo-effective.

Theorem 6.24 shows that
$$mc(m\beta) \leq 2^{k\beta+3n}(k+1)(m\beta)^{\frac{n}{n-k}}s^\frac{n}{n-k}A^n\delta^{\tau_{n,k}} A^n.$$ When $m$ is sufficiently large, $\lceil \delta ms \rceil \leq 2\delta ms$, so we obtain for such $m$
$$mc(m\beta) \leq 2^{k\beta+3n+1}(k+1)m^\frac{n}{n-k}s^\frac{n}{n-k}A^n\delta^{\tau_{n,k}} A^n < \frac{\mu}{n!m^\frac{n}{n-k}}$$ showing that mob($\beta$) < $\mu$ as desired. □

6.6. Alternative definitions of mobility. The mobility seems difficult to calculate explicitly. In this section we discuss two alternatives which may be easier to compute. However, they seem to be less flexible from a theoretical perspective.

6.6.1. Smooth mobility.

Definition 6.29. Let $X$ be an integral projective variety and let $p : U \to W$ be a family of effective cycles on $X$. We say that $p$ is a mostly smooth family if every component of a general cycle-theoretic fiber of $p$ is smooth.

To obtain a good theory of mostly smooth families, it is important that mostly smoothness is compatible with the geometric constructions outlined in Section 3. In particular, mostly smoothness is compatible with
- closure of families,
- family sums,
- intersections against general very ample divisors, and
- pushforwards from subvarieties.
Definition 6.30. Let $X$ be an integral projective variety and let $\alpha \in N_k(X)$. Define $\text{mc}_{sm}(\alpha)$ to be the maximum mobility count of any mostly smooth family of effective cycles representing a class $\beta$ such that $\alpha - \beta$ is an effective class. Define

$$\text{mob}_{sm}(\alpha) = \limsup_{m \to \infty} \frac{\text{mc}_{sm}(m\alpha)}{m^n/n^{k}/n!}.$$ 

Since mostly smoothness is compatible with family sums, just as for the mobility function one verifies that $\text{mob}_{sm}$ is homogenous and for $\alpha, \beta \in N_k(X) \mathbb{Q}$ we have $\text{mob}_{sm}(\alpha + \beta) \geq \text{mob}_{sm}(\alpha) + \text{mob}_{sm}(\beta)$. Furthermore, if $X$ is a smooth integral variety, the Bertini theorems imply that by taking complete intersections of very ample divisors we obtain mostly smooth families. Thus $\text{mob}_{sm}$ is positive on any big class on a smooth variety.

By Lemma 2.12, $\text{mob}_{sm}$ extends to a continuous function on the interior of $\text{Eff}_k(X)$. Since $\text{mob}_{sm} \leq \text{mob}$, Theorem 6.28 then implies:

Theorem 6.31. Let $X$ be an integral projective variety. Then $\text{mob}_{sm}$ extends to a homogeneous continuous function on $N_k(X)$. If $X$ is smooth, then $\text{mob}_{sm}$ is positive on the big cone.

6.6.2. ACM mobility. Let $X$ be an integral projective variety with a fixed embedding into some projective space $\mathbb{P}^r$. $X$ is said to be arithmetically Cohen-Macaulay for this embedding (or simply ACM) if its homogeneous coordinate ring is Cohen-Macaulay.

Definition 6.32. Let $X$ be an integral projective variety with a fixed embedding into projective space and let $p : U \to W$ be a family of effective $k$-cycles on $X$. We say that $p$ is a mostly ACM family if for a general cycle-theoretic fiber $Z$ of $p$ there is a finite collection of positive integers $a_j$ and of ACM subschemes $Y_j$ (with respect to the fixed embedding) of pure dimension $k$ with fundamental cycles $Z_j$ such that $Z = \sum a_j Z_j$.

Note that mostly ACMness is compatible with
- closure of families,
- family sums,
- intersections against multiples of the hyperplane class on the ambient projective space, and
- pushforwards from subvarieties (with the compatible embedding).

Definition 6.33. Let $X$ be an integral projective variety with a fixed embedding into projective space and let $\alpha \in N_k(X)$. Define $\text{mc}_{ACM}(\alpha)$ to be the maximum mobility count of any mostly ACM family of effective cycles representing a class $\beta$ such that $\alpha - \beta$ is an effective class. Define

$$\text{mob}_{ACM}(\alpha) = \limsup_{m \to \infty} \frac{\text{mc}_{ACM}(m\alpha)}{m^n/n^{k}/n!}.$$ 

If $X$ is ACM with respect to its fixed embedding in $\mathbb{P}^r$, then families of complete intersections of $X$ with hyperplanes in $\mathbb{P}^r$ yield mostly ACM
families of effective cycles. Thus on an ACM variety $X$ any big class has positive $\text{mob}_{\text{ACM}}$. Since $\text{mob}_{\text{ACM}}$ is compatible with family sums, arguing as before for $\text{mob}_{\text{sm}}$ we see that

**Theorem 6.34.** Let $X$ be a integral projective variety with a fixed embedding in projective space. Then $\text{mob}_{\text{ACM}}$ extends to a homogeneous continuous function on $N_k(X)$. If $X$ is ACM, then $\text{mob}_{\text{ACM}}$ is positive on the big cone.

7. Examples of mobility

The mobility seems difficult to calculate explicitly. By analogy with the volume, one wonders whether the mobility is related to intersection numbers for “sufficiently positive” classes (just as the volume of an ample divisor is a self-intersection product). In particular, we ask:

**Question 7.1.** Let $X$ be an integral projective variety and let $H$ be an ample Cartier divisor. For $0 < k < n$, is

$$\text{mob}(H^{n-k}) = \text{vol}(H)?$$

More generally, if $L$ is a big Cartier divisor, and $\alpha = \langle L^{n-k} \rangle$ for some $0 < k < n$, where $\langle - \rangle$ denotes the positive product of [BDP13], is

$$\text{mob}(\alpha) = \text{vol}(L)?$$

An affirmative answer would imply that the “optimal cycles” with respect to the mobility count for $H^{n-k}$ are complete intersections of $(n-k)$ general elements of $|\mathcal{O}_X|$.

**Remark 7.2.** Note that the statements in Question 7.1 do not hold for point classes: for an ample divisor $H$ we have $\text{vol}(H) = H^n$ but $\text{mob}(H^n) = n!H^n$.

In the remainder of this section we discuss two examples in detail. We will work over the base field $\mathbb{C}$ to cohere with the cited references.

7.1. Curves on $\mathbb{P}^3$. Let $\ell$ denote the class of a line on $\mathbb{P}^3$ over $\mathbb{C}$. The mobility of $\ell$ is determined by the following enumerative question: what is the minimal degree of a curve in $\mathbb{P}^3$ going through $b$ very general points? The answer is unknown (even in the asymptotic sense).

[Per87] conjectures that the “optimal” curves are the complete intersections of two hypersurfaces of degree $d$. Indeed, among all curves not contained in a hypersurface of degree $(d-1)$, [GP78] shows that these complete intersections have the largest possible arithmetic genus, and thus conjecturally the corresponding Hilbert scheme has the largest possible dimension.

Complete intersections of two hypersurfaces of degree $d$ have degree $d^2$ and pass through $\approx \frac{1}{6}d^3$ general points. Letting $d$ go to infinity, we find the lower bound

$$1 \leq \text{mob}(\ell)$$

and conjecturally equality holds.
Theorem 7.3. Let \( \ell \) be the class of a line on \( \mathbb{P}^3 \). Then:

1. \( 1 \leq \text{mob}(\ell) < 3.54 \).
2. \( 1 \leq \text{mob}_{sm}(\ell) \leq 3 \). (\cite{Per87}, Proposition 6.29)

7.1.1. Mobility. To prove the first bound on the mobility, we simply repeat the argument of Theorem 6.24 with more careful constructions of families and better estimates.

Fix a degree \( d \). Let \( s = \lceil \sqrt{\frac{9-\sqrt{69}}{2}} d \rceil \) and let \( S \) be a Noether-Lefschetz general hypersurface of degree \( s \). Then every curve on \( S \) is the restriction of a hypersurface on \( \mathbb{P}^3 \). In particular, \( \text{Pic}(S) \cong \mathbb{Z} \), and if \( H \) denotes the hyperplane class on \( \mathbb{P}^3 \) then the mobility count of \( cH|S \) is

\[
\left( \frac{c+3}{3} \right) - \left( \frac{c-s+3}{3} \right)
\]

(where we use the convention that the rightmost term is 0 when \( c < s \)). Let \( p : U \to W \) be a family of degree \( d \) curves on \( \mathbb{P}^3 \). Consider the base change \( p' : U \times_{\mathbb{P}^3} S \to W \). Every component of a fiber of \( p' \) has dimension 1 or 0. We can stratify \( W \) by locally closed subsets \( W_i \) based on the degree \( d' \) of the components of the fibers of dimension 1; the components of dimension 0 then have degree \( (d-d')s \). Applying Construction 3.5 to construct families of cycles as in the proof of Theorem 6.24, Lemma 6.4 implies that

\[
\text{mc}(d\ell) \leq \max_{0 \leq d' \leq d} \text{mc}_{S}((d-d')\ell \cdot S) + \text{mc}_{S} \left( \left\lfloor \frac{d'}{s} \right\rfloor H|S \right)
\leq \max_{0 \leq d' \leq d} (d-d')s + \left( \left\lfloor \frac{d'}{s} \right\rfloor + 3 \right) - \left( \left\lfloor \frac{d'}{s} \right\rfloor - s + 3 \right).
\]

A straightforward computation shows that for \( \left\lfloor d'/s \right\rfloor \geq s \) the maximum value is achieved when \( d' = d \) and for \( \left\lfloor d'/s \right\rfloor < s \) the maximum value is achieved when \( d' = 0 \). In either case, the asymptotic value of the computation above is

\[
\text{mc}(d\ell) \leq \sqrt{\frac{9-\sqrt{69}}{2}} d^{3/2} + O(d)
\]

yielding the desired bound.

7.1.2. Smooth mobility. The bound \( \text{mob}_{sm}(\ell) \leq 3 \) is proved by \cite{Per87}, Proposition 6.29]. The argument below is more-or-less the same as that of Perrin. Suppose that \( p : U \to W \) is a mostly smooth family of degree \( d \) curves on \( \mathbb{P}^3 \). We show that

\[
\text{mc}(p) \leq \frac{1}{2} d^{3/2} + O(d)
\]

which immediately yields \( \text{mob}_{sm}(\ell) \leq 3 \). By Lemmas 6.9 and 6.5, it suffices to prove this bound when \( p \) is generically irreducible and reduced. By replacing \( p \) by the flattening of a Stein factorization, we may suppose that a
general fiber is an integral smooth curve (since we are working over a field of characteristic 0).

Set \( s = \lfloor \sqrt{d} \rfloor \). First suppose that a general cycle-theoretic fiber of \( p \) is contained in a surface of degree \( \leq s \). Then Lemma 6.4 shows that \( \text{mc}(p) \leq \text{mc}(sH) \approx \frac{1}{2}d^{3/2} \).

Otherwise, a general cycle-theoretic fiber \( C \) of \( p \) is a smooth curve not contained on a surface of degree \( \leq s \). Let \( g \) denote the genus of \( C \). The exact sequence

\[
0 \to T_C \to T_{P^3}|C \to N_{C/P^3} \to 0
\]

shows that \( h^0(C, N_{C/P^3}) \leq h^0(C, T_{P^3}|C) - \chi(T_C) \). An elementary argument shows that \( T_{P^3}|C \) admits a filtration by line bundles of positive degree; in particular \( h^0(C, T_{P^3}|C) \leq \deg(T_{P^3}|C) = 4d \). Thus

\[
h^0(C, N_{C/P^3}) \leq 4d + g - 1.
\]

[GP78, Théorème 3.1] shows that curves not lying on a surface of degree \( \leq s \) have arithmetic genus at most \( s^3 - 2s^2 + 1 \). Thus,

\[
h^0(C, N_{C/P^3}) \leq d^{3/2} + 2d.
\]

This gives an upper bound on the dimension of the corresponding component of the Hilbert scheme, hence also the corresponding component of the Chow scheme. Since general points impose at least 2 conditions on curves, we see that in this case

\[
\text{mc}(p) \leq \frac{1}{2}d^{3/2} + d.
\]

7.2. **Rational mobility of points.** In this section we relate rational mobility with the theory of rational equivalence of 0-cycles. In order to cohere with the cited references, we work only with normal integral varieties \( X \) over \( \mathbb{C} \) (although the results easily extend to a more general setting). Recall that \( A_0(X) \) denotes the group of rational-equivalence classes of 0-cycles on \( X \). We will denote the \( r \)-th symmetric power of \( X \) by \( X^{(r)} \); by [Kol96, I.3.22 Exercise] this is the component of Chow(\( X \)) parametrizing 0-cycles of degree \( r \).

**Remark 7.4.** The universal family of 0-cycles of degree \( r \) (in the sense of Definition 3.1) is not \( u : X^{(r)} \to X^{(r)} \) but a flattening of this map. However, note that the rational mobility computations are the same whether we work with \( u \) or a flattening by Lemma 6.5. For simplicity we will work with \( u \) and \( X^{(r)} \) despite the slight incongruity with Definition 3.1.

We start by recalling the results of [Roï72] concerning \( A_0(X) \). Consider the map \( \gamma_{m,n} : X^{(m)} \times X^{(n)} \to A_0(X) \) sending \((p, q) \mapsto p - q\). [Roï72, Lemma 1] shows that the fibers of \( \gamma_{m,n} \) are countable unions of closed subvarieties.

A subset \( V \subset A_0(X) \) is said to be irreducible closed if it is the \( \gamma_{m,n} \)-image of an irreducible closed subset \( Y \) of \( X^{(m)} \times X^{(n)} \) for some \( m \) and \( n \). The dimension of such a subset \( V \) is defined to be the dimension of \( Y \) minus the
minimal dimension of a component of a fiber of $\gamma_{m,n}$ [Roî72] Lemma 9 shows that the dimension is independent of the choice of $Y$, $m$, and $n$.

**Lemma 7.5.** Let $V, W \subset A_0(X)$ be irreducible closed subsets with $V \subsetneq W$. Then $\dim(V) < \dim(W)$.

**Proof.** Let $Z \subset X^{(m)} \times X^{(n)}$ be an irreducible closed subset whose $\gamma_{m,n}$-image is $W$. Let $Y \subset Z$ denote the preimage of $V$; [Roî72] Lemma 5] shows that $Y$ is a countable union of closed subsets. By [Roî72] Lemma 6] some component $Y' \subset Y$ dominates $V$. But then $\dim(Y') < \dim(Z)$, proving the statement. \hfill \qed

We are mainly interested in when $A_0(X)$ is an irreducible closed set. This is equivalent to the following notion:

**Definition 7.6.** $A_0(X)$ is said to be representable if there is a positive integer $r$ such that the addition map $a_r : X^{(r)} \to A_0(X)_{\deg r}$ is surjective.

We now relate these notions to the rational mobility of 0-cycles on $X$.

**Example 7.7.** Let $X$ be an integral projective variety of dimension $n$ and let $\alpha$ be the class of a point in $N_0(X)$. Let $A$ be a very ample divisor on $X$; for sufficiently large $m$ we have $h^0(X, \mathcal{O}_X(mA)) \approx \frac{1}{m!} A^n$. By taking complete intersections of $n$ elements of $|mA|$, we see that $\text{ratmob}(\alpha) \geq 1$.

**Proposition 7.8.** Let $X$ be a normal integral projective variety and let $\alpha$ denote the class of a point in $N_0(X)$. Then the following are equivalent:

1. $A_0(X)$ is representable.
2. $\text{ratmob}(\alpha) = n!$.
3. $\text{ratmob}(\alpha) > n!/2$.

**Proof.** (1) $\implies$ (2). Suppose that $A_0(X)$ is representable. There is some positive integer $r$ such that the addition map $X^{(r)} \to A_0(X)_{\deg r}$ is surjective. Fix $m > 0$ and choose some class $r \in A_0(X)_{\deg (m+r)}$. For any effective 0-cycle $Z$ of degree $m$, there is an effective 0-cycle $T_Z$ of degree $r$ such that $T_Z + Z \subset r$. As $Z \in X^{(m)}$ varies, the effective cycles $Z + T_Z$ are rationally equivalent, showing that $\text{rmc}((m+r)\alpha) \geq m$ and $\text{ratmob}(\alpha) = n!$.

(3) $\implies$ (1). Suppose that $A_0(X)$ is not representable. Fix a closed point $p_0 \in X$. Note that non-representability implies that $a_m(X^{(m)}) + p_0 \subseteq a_{m+1}(X^{(m+1)})$ for every $m$: if we had equality for some $m$, we would also have equality for every larger $m$ and the map $a_m$ would be surjective. By Lemma 7.5 $\dim(a_m(X^{(m)}))$ strictly increases in $m$.

Suppose that $\text{rmc}(m\alpha) = b$. This implies that there is some rational equivalence class $\tau$ of degree $m$ so that for any $p \in X^{(b)}$, there is an element $q \in X^{(m-b)}$ such that $p + q \in \tau$. In particular, the subset $\tau - X^{(b)} \subset A_0(X)_{\deg m-b}$ is contained in $a_{m-b}(X^{(m-b)})$. By Lemma 7.5 $\dim(a_b(X^{(b)})) \leq \dim(a_{m-b}(X^{(m-b)}))$. But since these dimensions are strictly increasing in $m$ we must have $m \geq 2b$. Thus we see that $\text{ratmob}(\alpha) \leq n!/2$, proving the statement. \hfill \qed
Example 7.9. Let $X$ be a smooth surface and let $\alpha$ be the class of a point. By combining Example 7.7 with Proposition 7.8 we see that there are two possibilities:

- $A_0(X)$ is representable and $\text{ratmob}(\alpha) = 2$.
- $A_0(X)$ is not representable and $\text{ratmob}(\alpha) = 1$.

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