An analytic description of the vector constrained KP hierarchy

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Abstract

In this paper we give a geometric description in terms of the Grassmann manifold of Segal and Wilson, of the reduction of the KP hierarchy known as the vector \( k \)-constrained KP hierarchy. We also show in a geometric way that these hierarchies are equivalent to Krichever’s general rational reductions of the KP hierarchy.

1 Introduction

In recent years (vector) constrained KP hierarchies have attracted considerable attention both from the mathematical as the physical community [2]-[27], [29], [31], [32]. Many interesting integrable systems like the AKNS, Yajima–Oikawa and Melnikov hierarchies appear amongst these constrained families. In the physics literature they are studied in connection with multi-matrix models.

The (vector) constrained KP hierarchies were introduced as reductions of the KP hierarchy

\[
\frac{\partial L}{\partial t_n} = [(L^n)_+ , L], \quad n \geq 1,
\]

for the first order pseudodifferential operator \( L = \partial + \sum_{j<0} \ell_j \partial^j \). This reduction consists of assuming that

\[
(L^k)_- = \sum_{j=1}^{m} q_j \partial^{-1} r_j,
\]

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such that the following conditions on the functions $q_j$ and $r_j$ hold:

$$\frac{\partial q_j}{\partial t_n} = (L^n_W)_+(q_j) \quad \text{and} \quad \frac{\partial r_j}{\partial t_n} = -(L^n_W)_-(r_j) \quad \text{for all } n \geq 1.$$ 

In this way it generalizes the well-known Gelfand-Dickey hierarchies ($(L^k)_- = 0$).

Much is known about these constrained hierarchies and many well-known features are investigated, e.g. it was shown that they possess a bi-Hamiltonian structure [9], [20], [24], [29], [32], a bilinear representation [13], [21], [22], [32] and Bäcklund-Darboux and Miura transformations [2], [4], [5], [6], [7], [10], [23]. However, until recently, the geometry remained unclear. It is well-known that one can associates to a point in an infinite Grassmannian a solution $L$ of the KP hierarchy [28], [30]. In this paper we consider the Segal-Wilson Grassmannian. Let $H$ be the Hilbert space of all square integrable functions on the circle $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$, which decomposes in a natural way as the direct sum of two infinite dimensional orthogonal closed subspaces $H_+ = \{ \sum_{n \geq 0} a_n z^n \in H \}$ and $H_- = \{ \sum_{n < 0} a_n z^n \in H \}$. The Segal-Wilson Grassmannian $Gr(H)$ consists of all closed subspaces $W \subset H$ such that the orthogonal projection on $H_-$ is a Hilbert-Schmidt operator. In this setting, the $k$-th Gelfand-Dickey hierarchy has the following simple geometrical interpretation. The KP operator $L$ belongs to the $k$-th Gelfand-Dickey hierarchy if and only if the corresponding $W \in Gr(H)$ satisfies $z^k W \subset W$. One of the authors gave in [19] (see also [18]) a simple interpretation of the constrained KP hierarchy for the case of polynomial tau-functions, viz $L$ belongs to the $m$-vector $k$-constrained KP hierarchy if and only if the corresponding $W \in Gr(H)$ has a subspace $W'$ of codimension $m$ such that $z^k(W') \subset W$. We show in this paper that the same interpretation also holds in the Segal-Wilson case. Using this geometrical interpretation, we prove in section 5 that the vector constrained KP hierarchy describes the same reduction of KP as the general rational reductions of Krichever [17] (see also [15]). Our geometrical interpretation is also useful to give solutions of these hierarchies (see e.g. [9]).

2 The KP hierarchy revisited

In this section we recall some results for the KP-hierarchy that we will need in this paper. The KP hierarchy starts with a commutative ring $R$ and a privileged derivation $\partial$ of $R$. In order to be able to take roots of differential operators in $\partial$ with coefficients form $R$, one extends this ring $R[\partial]$ to the ring $R[\partial, \partial^{-1}]$ of pseudodifferential operators with coefficients in $R$. It consists of all expressions

$$\sum_{i=-\infty}^{N} a_i \partial^i, \quad a_i \in R \quad \text{for all } i,$$

that are added in an obvious way and multiplied according to

$$\partial^i \circ a \partial^j = \sum_{k=0}^{\infty} \binom{j}{k} \partial^k(a) \partial^{i+j-k}.$$

Each operator $P = \sum p_j \partial^j$ decomposes as $P = P_+ + P_-$ with $P_+ = \sum_{j \geq 0} p_j \partial^j$ its differential operator part and $P_- = \sum_{j < 0} p_j \partial^j$ its integral operator part. We denote by $Res_0 P = p_{-1}$ the
residue of $P$. On $R[\partial, \partial^{-1}]$ we have an anti-algebra morphism called taking the adjoint. The adjoint of $P = \sum p_i \partial^i$ is given by

$$P^* = \sum_i (-\partial)^i p_i.$$ 

Further one has a set of derivations $\{\partial_n \mid n \geq 1\}$ of $R$ that commute with $\partial$. The equations of the hierarchy can be formulated in a compact way in a set of relations for a so-called Lax operator in $R[\partial, \partial^{-1}]$, i.e. an operator of the form

$$L = \partial + \sum_{j < 0} \ell_j \partial^j, \quad \ell_j \in R \quad \text{for all } j < 0. \tag{2.1}$$

These equations are

$$\partial_n(L) = \sum_{j < 0} \partial_n(\ell_j) \partial^j = [(L^n)_+, L], \quad n \geq 1. \tag{2.2}$$

Since this equations for $n = 1$ boils down to $\partial_1(\ell_j) = \partial(\ell_j)$ for all $j$, we assume from now on that $\partial = \partial_1$. Equation (2.2) has at least the trivial solution $L = \partial$ and can be seen as the compatibility equation of the linear system

$$L\psi = z\psi \quad \text{and} \quad \partial_n(\psi) = (L^n)_+(\psi) \tag{2.3}$$

One needs a context in which the actions of (2.3) make sense and that allows you to derive (2.2) from (2.3). For the trivial solution (2.3) becomes

$$\partial\psi = z\psi \quad \text{and} \quad \partial_n\psi = z^n\psi \quad \text{for all } n \geq 1.$$

Hence if one takes $\partial_n = \frac{\partial}{\partial z^n}$ then the function $\gamma(z) = \exp(\sum_{i \geq 1} t_i z^i)$ is a solution. The space $M$ of so-called oscillating functions for which we make sense of (2.3) can be seen as a collection of perturbations of this solution. It is defined as

$$M = \{(\sum_{i \leq N} a_i z^i) e^{\sum t_i z^i} \mid a_i \in R, \quad \text{for all } i\}.$$

The space $M$ becomes a $R[\partial, \partial^{-1})$-module by the natural extension of the actions

$$b\{(\sum_j a_j z^j) e^{\sum t_i z^i}\} = (\sum_j b a_j z^j) e^{\sum t_i z^i},$$

$$\partial\{(\sum_j a_j z^j) e^{\sum t_i z^i}\} = (\sum_j \partial(a_j) z^j + \sum_j a_j z^{j+1}) e^{\sum t_i z^i}.$$ 

It is even a free $R[\partial, \partial^{-1})$-module, since we have

$$(\sum p_j \partial^j) e^{\sum t_i z^i} = (\sum p_j z^j) e^{\sum t_i z^i}.$$ 

An element $\psi$ in $M$ is called an oscillating function of type $z^\ell$, if it has the form

$$\psi(z) = \{z^\ell + \sum_{j < \ell} \alpha_j z^j\} e^{\sum t_i z^i}.$$ 

The fact that $M$ is a free $R[\partial, \partial^{-1})$-module, permits you to show that each oscillating function of type $z^\ell$ that satisfies (2.3) gives you a solution of (2.2). This function is then called a wavefunction of the KP-hierarchy.
Segal and Wilson give in [30] an analytic approach to construct wavefunctions of the KP-hierarchy. They considered the Hilbert space

\[ H = \{ \sum_{n \in \mathbb{Z}} a_n z^n \mid a_n \in \mathbb{C}, \sum_{n \in \mathbb{Z}} |a_n|^2 < \infty \}, \]

with decomposition \( H = H_+ \oplus H_- \), where

\[ H_+ = \{ \sum_{n \geq 0} a_n z^n \in H \} \quad \text{and} \quad H_- = \{ \sum_{n < 0} a_n z^n \in H \} \]

and inner product \( \langle \cdot | \cdot \rangle \) given by

\[ \langle \sum_{n \in \mathbb{Z}} a_n z^n | \sum_{m \in \mathbb{Z}} b_m z^m \rangle = \sum_{n \in \mathbb{Z}} a_n b_n. \]

To this decomposition is associated the Grassmannian \( \text{Gr}(H) \) consisting of all closed subspaces \( W \) of \( H \) such that the orthogonal projection \( p_+ : W \rightarrow H_+ \) is Fredholm and the orthogonal projection \( p_- : W \rightarrow H_- \) is Hilbert-Schmidt. The connected components of \( \text{Gr}(H) \) are given by

\[ \text{Gr}^{(\ell)}(H) = \left\{ W \in \text{Gr}(H) \mid p_+: z^{\ell} W \rightarrow H_+ \text{ has index zero} \right\}. \]

On each of these components we have a natural action by multiplication of the group of commuting flows

\[ \Gamma_+ = \{ \exp(\sum_{i \geq 1} t_i z^i) \mid t_i \in \mathbb{C}, \sum |t_i| (1 + \epsilon)^i < \infty \text{ for some } \epsilon > 0 \}. \]

Now we take for \( R \) the ring of meromorphic functions on \( \Gamma_+ \) and for \( \partial_{t_n} \) the partial derivative w.r.t. \( t_n \). Then there exists for each \( W \) in \( \text{Gr}^{(-\ell)}(H) \) a wavefunction \( \psi_W \) of type \( z^\ell \) that is defined on a dense open subset of \( \Gamma_+ \) and that takes values in \( W \). Moreover, it is known that the range of \( \psi_W \) spans a dense subspace of \( W \). Hence, if we write \( \psi_W = P_W \cdot e^{\sum t_i z^i} \) with \( P_W \in R[\partial, \partial^{-1}] \), then \( L_W = P_W \partial P_W^{-1} \) is a solution of the KP-hierarchy. Each component of \( \text{Gr}(H) \) generates in this way the same set of solutions of the KP-hierarchy, so it would suffice, as is done in [30], to consider only \( \text{Gr}^{(0)}(H) \). However, it is more convenient here to consider all components.

A subsystem of the KP-hierarchy consists of all solutions \( L \) that are the \( k \)-th root of a differential operator. This gives you solutions of the KP-hierarchy that do not depend on the \( \{t_k n, \text{ with } n \geq 1\} \). Those operators satisfy the condition \( L^k = (L^k)_+ \). The set of equations corresponding to this condition is called the \( k \)-th Gelfand-Dickey hierarchy. Now it has been shown that, among the solutions coming from the Segal-Wilson Grassmannian, the ones that satisfy the \( k \)-th Gelfand-Dickey hierarchy are exactly characterized by \( z^k W \subset W \). In the next section we consider a generalization of this condition.

### 3 An extension of the condition \( z^k W \subset W \)

In this section we consider, for each \( k \) and \( m \) in \( \mathbb{N} = \{0, 1, 2, \ldots \} \), \( k \neq 0 \) subspaces \( W \) in \( \text{Gr}(H) \) that possess the \textbf{\( m \)-Vector \( k \)-Constrained (\( mVkC \))-condition}:

\[ \text{There is a subspace } W' \text{ of } W \text{ of codimension } m \text{ such that } z^k(W') \subset W. \]
This is a natural generalization of the condition that describes inside $Gr(H)$ the solutions of the $k$-th Gelfand-Dickey hierarchy. We will show here in a geometric way how you can associate to each $W$, satisfying the $mVkC$-condition, $2m$ functions $\{q_j | 1 \leq j \leq m\}$ and $\{r_j | 1 \leq j \leq m\}$ for which the following equations hold:

$$\partial_n(q_j) = (L^k_n)_+(q_j) \quad \text{for all} \quad n \geq 1,$$  \hspace{1cm} (3.2)

$$\partial_n(r_j) = -(L^k_n)_+(r_j) \quad \text{for all} \quad n \geq 1. \hspace{1cm} (3.3)$$

Here $A^*$ denotes the adjoint of $A$ in $R[\partial, \partial^{-1}]$. Moreover $L_W$ satisfies

$$L^k_W = (L^k_W)_+ + \sum_{j=1}^{m} q_j \partial^{-1} r_j. \hspace{1cm} (3.4)$$

At the same time we will give links with the paper of Zhang [31].

Take any $W$ in $Gr^{(-\ell)}$ that satisfies the $mVkC$-condition. It is no restriction to assume that the $m$ occurring in (3.1) is optimal, i.e. there is an orthonormal basis $\{u_1, \ldots, u_m\}$ of the orthocomplement of $W'$ in $W$ such that

$$(\Span\{z^k u_1, \ldots, z^k u_m\}) \cap W = \{0\}.$$ 

Since multiplication with $z$ is unitary, the vectors $\{z^k(u_1), \ldots, z^k(u_m)\}$ are an orthonormal basis of the orthocomplement of $W$ in $z^k W + W$. To the space $W$ we associate the subspaces $W_j = W \oplus \mathbb{C} z^k u_j, 1 \leq j \leq m$.

Clearly the $W_j$ all belong to $Gr^{(-\ell+1)}$ and hence, they have wavefunctions $\psi_{W_j}$ of type $z^{\ell-1}$ i.e.

$$\psi_{W_j} = \psi_{W_j}(t, z) = \{z^{\ell-1} + \sum_{s \geq 1} a_{js}(t) z^{\ell-1-s} \} e^{\sum_{i} t_i z^i}. \hspace{1cm} (3.5)$$

Recall that $\psi_{W_j}(t, z)$ is well-defined for all $t$ belonging to the open dense subset

$$\Gamma_{W_j}^+ = \{\gamma(z) = \exp(\sum_{i} t_i z^i) \in \Gamma_+ | \gamma^{-1} W_j \text{ is transverse to } z^{\ell-1} H_+\}.$$ 

On $\Gamma_{W_j}^+$ we consider the function

$$s_j(t) = < \psi_{W_j}(t, z) | z^k u_j >. \hspace{1cm} (3.6)$$

Since the vectors $\{\psi_{W_j}(t, z) | t \in \Gamma_{W_j}^+\}$ are lying dense in $W_j$ and $m$ was assumed to be optimal, the functions $\{s_j\}$ do not vanish. Hence, on a dense open subset of $\Gamma_+$, there is defined the function

$$\varphi_j = \frac{1}{s_j} \psi_{W_j} := r_j \psi_{W_j}. \hspace{1cm} (3.7)$$

It takes values in $W_j$ and has moreover the following useful property

$$\varphi_j(t) - z^k u_j \in W, \hspace{1cm} (3.8)$$
Recall that \( j \), we have that for all \( W \)

By construction, there holds

This equation is part of the system of differential equations for the \( W \). In [31], similar functions \( \{ \varphi_j \} \) are introduced, only not using the geometry, but as solutions of a certain system of differential equations. In particular, we can dispose of the condition (a) in the Proposition of [31]. Thus we have obtained \( m \) functions \( \{ r_j \} \).

To define the \( \{ q_j \} \) we consider

\[
 z^k \psi_W - (L^k_W)_+(\psi_W) = (L^k_W)_-(\psi_W) = \{ \sum_{s \geq 0} b_s(t) z^{\ell-1-s} \} e^\sum t_i z^i. \tag{3.9}
\]

For each \( j, 1 \leq j \leq m \), we have a function \( q_j \) on \( \Gamma^W_+ \).

\[
 q_j(t) = \langle z^k \psi_W(t, z) - (L^k_W)_+ \psi_W(t, z) \mid z^k u_j \rangle \\
 = \langle z^k \psi_W(t, z) \mid z^k u_j \rangle \\
 = \langle \psi_W(t, z) \mid u_j \rangle.
\]

Because \( m \) is optimal, the functions \( \{ q_j \} \) are non-zero on an open dense subset of \( \Gamma_+ \). Since \( u_j \) does not depend on \( t \) and since \( \frac{\partial}{\partial n} \psi_W = (L^k_W)_+(\psi_W) \), we get directly for \( q_j \)

\[
 \frac{\partial q_j}{\partial n} = \sum_{n} \langle \psi_W \mid u_j \rangle = \langle (L^k_W)_+(\psi_W) \mid u_j \rangle = (L^k_W)_+(\psi_W) - (L^k_W)_+ q_j. \tag{3.10}
\]

Thus the equations (3.2) for the derivatives of the \( \{ q_j \} \) are clear. Those for the \( \{ r_j \} \) require more work.

First we derive an expression for \( (L^k_W)_-(\psi_W) \). Thereo we consider

\[
 \Phi(t) = z^k \psi_W - (L^k_W)_+(\psi_W) - \sum_{j=1}^m q_j \varphi_j. \tag{3.11}
\]

Since \( \varphi_j \) takes values in \( W \), the function \( (L^k_W)_+(\psi_W) \) does so in the space \( W \) and \( z^k \psi_W \) in \( z^k W \). Hence we have that \( \Phi(t) \) belongs to \( W + z^k W \) for all relevant \( t \). By construction we have that for all \( j, 1 \leq j \leq m, \Phi(t) \) is orthogonal to \( z^k u_j \), hence \( \Phi(t) \) even belongs to \( W \). From the form of the \( \varphi_j \), we see that on an open dense set of \( \Gamma_+ \) one has

\[
 \Phi(t) = \{ \sum_{s \geq 0} c_s z^{\ell-1-s} \} e^\sum t_i z^i.
\]

By construction, there holds

\[
 W \cap (z^\ell H_+)^\bot \gamma(z) = \{ 0 \},
\]

so that we arrive at

\[
 z^k \psi_W - (L^k_W)_+(\psi_W) = \sum_{j=1}^m q_j \varphi_j. \tag{3.12}
\]

This equation is part of the system of differential equations for the \( \varphi_j \) as used in [Z].

Recall that \( \varphi_j \) has the form

\[
 \varphi_j = \{ r_j z^{\ell-1} + \text{ lower order terms in } z \} e^\sum t_i z^i.
\]
Hence,
\[ \frac{\partial \varphi_j}{\partial x} = \frac{\partial \varphi_j}{\partial t_1} = \{r_j z^\ell + \text{lower order terms}\} e^{\sum t_i z^i}. \]

On the other hand we know that \( \varphi_j(t) - z^k u_j \) belongs to \( W \) for all \( t \). Thus also \( \frac{\partial \varphi_j}{\partial x}(t) \) belongs to \( W \). In \( W \) we have that
\[ \frac{\partial \varphi_j}{\partial x} - r_j \psi_W = \{\sum_{s \geq 0} \alpha_s z^{\ell-1-s} e^{\sum t_i z^i} \in (z^\ell H_+)^{1-\gamma} \}
\]
and this has to be zero. By definition we have \( \varphi_j = r_j \psi_W \) and differentiation w.r.t. \( x \) gives
\[ \psi_W = \frac{1}{r_j} \partial(r_j \psi_W) = (r_j^{-1} \partial r_j)(\psi_W). \quad (3.13) \]

Consequently, we have for \( \phi_j \)
\[ \varphi_j = r_j \psi_W = r_j(r_j^{-1} \partial r_j) \psi_W = \partial^{-1} r_j \psi_W. \]

Now we substitute this in equation (3.12) and obtain
\[ (L^k_W)(\psi_W) = \{\sum_{j=1}^m q_j \partial^{-1} r_j\} \psi_W. \quad (3.14) \]

Since the pseudodifferential operators act freely on wavefunctions, we see that \( L_W \) and the functions \( \{q_j\} \) and \( \{r_j\} \) are exactly connected by equation (3.14)
\[ (L^k_W) = \sum_{j=1}^m q_j \partial^{-1} r_j. \]

What remains to be shown, is the differential equation (3.3) for the \( r_j \). As \( \varphi_j(t) - z^k u_j \) belongs to \( W \), it follows that for all \( n \geq 1 \), \( \frac{\partial \varphi_j}{\partial t_n}(t) \) lies in \( W \). Recall that
\[ \varphi_j = \{r_j z^{\ell-1} + \text{lower order terms in } z\} e^{\sum t_i z^i}. \]

Then we have
\[ \frac{\partial \varphi_j}{\partial t_n} = \{r_j z^{n+\ell-1} + \text{lower order terms}\} e^{\sum t_i z^i} = \{r_j \partial^{n-1} \psi_W + \{\sum_{s \geq 0} \alpha_s z^{n+\ell-1-s} \} e^{\sum t_i z^i} = A_{nj}(\psi_W) + \{\sum_{s \geq 0} \beta_s z^{\ell-1-s} \} e^{\sum t_i z^i}, \]

with \( A_{nj} \) a uniquely determined differential operator in \( \partial \) of order \( n - 1 \) and with leading coefficient \( r_j \). Since both \( \frac{\partial \varphi_j}{\partial t_n} \) as \( A_{nj}(\psi_W) \) are lying in \( W \), we get
\[ \frac{\partial \varphi_j}{\partial t_n} - A_{nj}(\psi_W) = 0 = W \cap (z^\ell H_+)^{1-\gamma}(z). \]

On the other hand we know that \( \varphi_j = \partial^{-1} r_j \psi_W \) and this leads to
\[ A_{nj}(\psi_W) = \partial^{-1} \frac{\partial r_j}{\partial t_n} \psi_W + \partial^{-1} r_j(L^n_W)(\psi_W). \quad (3.15) \]
This gives you an expression for $A_{nj}$ in $L_W$ and $r_j$

$$A_{nj} = \partial^{-1}(\frac{\partial r_j}{\partial t_n} + r_j(L^n_W)^+).$$

By taking the residue in $\partial$ of the operators in this equation, we see that

$$\text{Res}_\partial(A_{nj}) = 0 = \frac{\partial r_j}{\partial t_n} + \text{Res}_\partial(\partial^{-1}r_j(L^n_W)^+) = \frac{\partial r_j}{\partial t_n} + (L^n_W)^+(r_j).$$

The last equality is a direct consequence of the following property of residues of pseudodifferential operators.

**Lemma 3.1** In the ring $R(\partial, \partial^{-1})$ of pseudodifferential operators with coefficients in $R$, we have for each $f$ in $R$ and $P = \sum_{j \leq N} p_j \partial^j$ in $R(\partial, \partial^{-1})$

$$\text{Res}_\partial(\partial^{-1}fP) = (P^*)_+(f),$$

where $(P^*)_+ = \sum_{0 \leq j \leq N} (-\partial)^j p_j$ is the differential operator part of the adjoint of $P$.

**Proof.** First we recall that $\text{Res}_\partial$ behaves as follows w.r.t. to taking the adjoint $P^* = \sum_{j \leq N} (-\partial)^j p_j$ of $P$

$$\text{Res}_\partial(P^*) = -\text{Res}_\partial P.$$

This is easily reduced to operators of the form $a\partial^n, n \in \mathbb{Z}$. Next one notices that it suffices to prove the equality in the lemma for differential operators. The left hand side for such a $P$ transforms as

$$\text{Res}_\partial(\partial^{-1}fP) = -\text{Res}_\partial(P^*f(-\partial)^{-1}) = \text{Res}_\partial(P^*f\partial^{-1}).$$

As $P^*f$ is a differential operator with constant term $P^*(f)$, this gives the proof of the lemma. \(\square\)

So we have shown that each $r_j$ satisfies the equation (3.3):

$$\frac{\partial r_j}{\partial t_n} = -(L^n_W)^+(r_j).$$

and we can conclude that $L_W$, the $\{q_j\}$ and the $\{r_j\}$ form a solution of the $m$ vector $k$-constrained KP-hierarchy.

### 4 The main theorem

In this subsection we will prove the converse of the result from the foregoing subsection and thus come to the main theorem. So we start with a $W$ in $G_{r^r(-\ell)}$ and functions $\{q_j\}$ and $\{r_j\}$, all defined on a dense open subset of $\Gamma_+$, such that the equations (3.2), (3.3) and (3.4) are satisfied. We will show that such a $W$ fulfills the $mVkC$-condition from section 3. Recall that there is a unique pseudodifferential operator $P_W$ such that $\psi_W = P_W(e^{\sum t_i z^i})$. It has the form

$$P_W = \partial^\ell + \sum_{j < \ell} p_j \partial^j = \{1 + \sum_{s<0} p_{s+s} \partial^s\} \partial^\ell.$$  \hspace{0.5cm} (4.1)
It is not difficult to see that the fact that $\psi_W$ is a wavefunction is equivalent to $P_W$ satisfying the Sato-Wilson equations
\begin{equation}
\frac{\partial P_W}{\partial r_n} P_W^{-1} = -(P_W \partial^n P_W^{-1})_-, \tag{4.2}
\end{equation}
where $P_-$ denotes the integral operator part $\sum_{i < 0} p_i \partial^i$ of the element $P = \sum p_j \partial^i$ in $R[\partial, \partial^{-1}]$. Next we consider for each $j, 1 \leq j \leq m$, the operators $Q_j$ and $R_j$ defined by
\begin{equation}
Q_j := q_j \partial q_j^{-1} P_W \quad \text{and} \quad R_j = r_j^{-1} \partial^{-1} r_j P_W. \tag{4.3}
\end{equation}
We want to show that the $Q_j$ and the $R_j$ also satisfy the Sato-Wilson equations. To do so, we need some properties of the ring $R[\partial, \partial^{-1}]$ of pseudodifferential operators with coefficients from $R$. We resume them in a lemma

**Lemma 4.1** If $f$ belongs to $R$ and $Q$ to $R[\partial, \partial^{-1}]$, then the following identities hold
\begin{enumerate}[(a)]
\item $(Qf)_- = Q_- f$,
\item $(fQ)_- = fQ_- $,
\item $\text{Res}_\partial(Qf) = \text{Res}_\partial(fQ) = f \text{Res}_\partial(Q)$,
\item $(\partial Q)_- = \partial Q_- - \text{Res}_\partial(Q)$,
\item $(Q\partial)_- = Q_- \partial - \text{Res}_\partial(Q)$,
\item $(Q^{-1})_- = Q_- \partial^{-1} + \text{Res}_\partial(Q\partial^{-1}) \partial^{-1}$,
\item $(\partial^{-1} Q)_- = \partial^{-1} Q_- + \partial^{-1} \text{Res}_\partial(Q^* \partial^{-1})$.
\end{enumerate}

Since the proof of this lemma consists of straightforward calculations, we leave this to the reader. Now we can show

**Proposition 4.1** The operators $Q_j$ and $R_j, 1 \leq j \leq m$, satisfy the Sato-Wilson equations.

**Proof.** If we denote $\frac{\partial}{\partial r_n}$ by $\partial_n$, then we get for $Q_j = q_j \partial q_j^{-1} P_W$ that
\begin{align*}
\partial_n(Q_j) Q_j^{-1} &= \partial_n(q_j \partial q_j^{-1}) q_j^{-1} P_W^{-1} q_j \partial q_j^{-1} + q_j \partial q_j^{-1} \partial_n(P_W) P_W^{-1} q_j \partial q_j^{-1} q_j^{-1} \\
&= -q_j \partial q_j^{-1}(L^n_W) q_j \partial q_j^{-1} q_j^{-1} + \partial_n(q_j \partial q_j^{-1}) q_j \partial q_j^{-1} q_j^{-1}.
\end{align*}

Now we apply successively the identities from Lemma 4.1 to the first operator of the right-hand side
\begin{align*}
q_j \partial q_j^{-1}(L^n_W) q_j \partial q_j^{-1} q_j^{-1} &= q_j \partial q_j^{-1}(L^n_W q_j) \partial q_j^{-1} q_j^{-1} = \\
q_j \partial q_j^{-1} L^n_W q_j \partial q_j^{-1} q_j^{-1} &= q_j \partial \text{Res}_\partial(q_j^{-1} L^n_W q_j \partial q_j^{-1}) \partial q_j^{-1} q_j^{-1} = \\
q_j \partial \text{Res}_\partial(q_j^{-1} L^n_W q_j \partial q_j^{-1}) \partial q_j^{-1} q_j^{-1} &= (q_j \partial q_j^{-1} L^n_W q_j \partial q_j^{-1}) q_j^{-1} = \\
q_j^{-1} \text{Res}_\partial(L^n_W q_j \partial q_j^{-1}) q_j^{-1} &= q_j^{-1} \text{Res}_\partial(L^n_W q_j \partial q_j^{-1}) q_j^{-1} = \\
(q_j \partial q_j^{-1} L^n_W q_j \partial q_j^{-1}) q_j^{-1} &= q_j \partial q_j^{-1} \text{Res}_\partial(L^n_W q_j \partial q_j^{-1}) q_j^{-1}.
\end{align*}

By applying Lemma 3.1 to these last two residues we get
\begin{equation}
(q_j \partial q_j^{-1} L^n_W q_j \partial q_j^{-1})_+ + (L^m_W)_+(q_j) q_j^{-1} - q_j \partial q_j^{-1}(L^n_W)_+(q_j) \partial q_j^{-1} q_j^{-1}.
\end{equation}
On the other hand
\[ \partial_n(q_j \partial q_j^{-1}) q_j \partial^{-1} q_j^{-1} = \partial_n(q_j) q_j^{-1} - q_j \partial q_j^{-2} \partial_n(q_j) q_j \partial^{-1} q_j^{-1}. \]

Thus we see that, if \( \partial_n(q_j) = (L^n_W)_+ (q_j) \), the operator \( Q_j \) satisfies the Sato-Wilson equation
\[ \partial_n(Q_j) Q_j^{-1} = -(Q_j \partial^n Q_j^{-1})_. \] (4.4)

For \( R_j \), we proceed in a similar fashion
\[ \partial_n(R_j) R_j^{-1} = -r_j^{-1} \partial^{-1} r_j (L^n_W)_- r_j \partial r_j + \partial_n(r_j^{-1} \partial^{-1} r_j) r_j^{-1} \partial r_j \]
\[ = -r_j^{-1} \partial^{-1} (r_j L^n_W r_j^{-1})_\partial r_j + \partial_n(r_j) r_j^{-1} + r_j^{-1} \partial^{-1} (\partial_n(r_j) r_j^{-1})_\partial r_j. \]

Now we successively apply Lemma [4.3] (g) and (e) and (4.2) to the first term of the right hand side of this equation
\[ -r_j^{-1} \partial^{-1} (r_j L^n_W r_j^{-1})_\partial r_j = -r_j^{-1} \{ (\partial^{-1} r_j L^n_W r_j^{-1})_- - \partial^{-1} \text{Res}(r_j^{-1} (L^n_W)^* r_j \partial^{-1}) \} \partial r_j \]
\[ = -r_j^{-1} (\partial^{-1} r_j L^n_W r_j^{-1})_\partial r_j + r_j^{-1} \partial^{-1} r_j L^n_W r_j^{-1} \partial r_j \]
\[ = -r_j^{-1} (\partial^{-1} r_j L^n_W r_j^{-1})_\partial r_j - r_j^{-1} (L^n_W)^* (r_j) + r_j^{-1} \partial^{-1} r_j L^n_W r_j^{-1} \partial r_j. \]

Since \( \partial_n(t_j) = - (L^n_W)^* (r_j) \), we see that the last two terms cancel \( \partial_n(r_j^{-1} \partial r_j) r_j^{-1} \partial r_j \) and thus we have obtained the Sato-Wilson equation for \( R_j \)
\[ \partial_n(R_j) R_j = -(R_j \partial^n R_j^{-1})_. \] (4.5)

This concludes the proof of proposition [4.4].

This proposition has some important consequences. Since the \( \{r_j\} \) and the \( \{q_j\} \) are non-zero on a dense open subset of \( \Gamma_+ \), we define on such a subset of \( \Gamma_+ \) oscillating functions \( \psi Q_j \) and \( \psi R_j \) of type \( z^{\ell+1} \) resp. \( z^{\ell-1} \) by
\[ \psi Q_j = q_j \partial q_j^{-1} \cdot \psi_W \quad \text{and} \quad \psi R_j = r_j^{-1} \partial^{-1} r_j \cdot \psi_W. \] (4.6)

Consider the following subspaces in \( Gr (H) \)
\[ W_{Q_j} = \text{Span} \{ \psi Q_j(t,z) \} \quad \text{and} \quad W_{R_j} = \text{Span} \{ \psi R_j(t,z) \}. \]

Then we can conclude from proposition [4.3]

**Corollary 4.1** The functions \( \psi Q_j \) and \( \psi R_j \) are the wavefunctions of the planes \( W_{Q_j} \) and \( W_{R_j} \). Moreover we have the following codimension 1 inclusions:
\[ W_{Q_j} \subset W \quad \text{and} \quad W \subset W_{R_j}. \]

**Proof.** From the Sato-Wilson equations one deduces directly that for all \( n \geq 1 \),
\[ \partial_n \psi Q_j = (Q_j \partial^n Q_j^{-1})_+ \psi Q_j \quad \text{and} \quad \partial_n \psi R_j = (R_j \partial^n R_j^{-1})_+ \psi R_j. \]

This shows the first part of the claim. The inclusions between the different spaces follows from the relations
\[ \psi Q_j = (q_j \partial q_j^{-1}) (\psi_W) \quad \text{and} \quad \psi W = (r_j \partial r_j^{-1}) \psi R_j. \]
the fact that the values of a wavefunction corresponding to an element of $\text{Gr}(H)$ are lying dense in that space. Since for a suitable $\gamma$ in $\Gamma_+$ the orthogonal projections of $\gamma^{-1}W_{R_j}$ on $z^\ell H_+$ resp. $\gamma^{-1}W$ on $z^{\ell+1}H_+$ have a one dimensional kernel, one obtains the codimension one result. This concludes the proof of the corollary.

Now we can formulate the main results of this paper.

**Theorem 4.1** Let $W$ be a plane in $\text{Gr}(H)$ and let $L_W$ be the corresponding solution of the KP-hierarchy. Then for $m, k \in \mathbb{N}$, $k \neq 0$, the following 2 conditions are equivalent

(a) The space $W$ satisfies the $mVkC$-condition.

(b) There exist functions $\{q_j | 1 \leq j \leq m\}$ and $\{r_j | 1 \leq j \leq m\}$ defined on an open dense subset of $\Gamma_+$ such that the following conditions are fulfilled:

(i) $\partial_n(q_j) = (L^\ell_W)_{+}(q_j)$ for all $n \geq 1$,

(ii) $\partial_n(r_j) = -(L^\ell_W)_{+}(r_j)$ for all $n \geq 1$,

(iii) $L^k_W = (L^k_W)_{+} + \sum_{j=1}^{m} q_j \partial^{-1}r_j$.

**Proof.** In section 2 it has been shown that (a) implies (b). So we assume from now on (b). The relation (b) (iii) leads to

\[
L^k_W(\psi_W) = z^k\psi_W = (L^k_W)_{+}(\psi_W) + \sum_{j=1}^{m} q_j \partial^{-1}r_j \psi_W = (L^k_W)_{+}(\psi_W) + \sum_{j=1}^{m} q_j r_j \partial^{-1}r_j \psi_W = (L^k_W)_{+}(\psi_W) + \sum_{j \neq 0}^{m} q_j r_j \psi_{R_j}.
\]

Thus we see with the usual density argument that

\[
z^kW \subset W + \sum_j W_{R_j} = \sum_j W_{R_j} = \tilde{W}.
\]

Since each $W$ has codimension one in $W_{R_j}$, we see that the codimension of $W$ in $\tilde{W}$ is $\leq m$. Let $W_1$ be the orthocomplement of $W$ in $\tilde{W}$ and $p_1 : H \to W_1$ the orthogonal projection on $W_1$. Inside $W$ we consider

\[W^1 = \{w \in W | p_1(z^k w) = 0\}.
\]

Since $\dim(W_1) \leq m$, we see that $W^1$ is a subspace of $W$ of codimension $\leq m$ and by construction $z^kW^1 \subset W$. This completes the proof of the theorem.

5 General rational reductions of the KP hierarchy

We are now going to connect the vector constrained KP hierarchy to reductions of the KP hierarchy introduced by Krichever [17]. For that purpose we assume that $W$ is a plane in $\text{Gr}(H)$ that satisfies the $mVkC$-condition, where we choose $m$ to be as minimal as possible for that plane. Let $L_W = P_W \partial P^{-1}_W$, with $P_W$ of the form (4.1), be the corresponding
solution of the KP hierarchy and let $W^1 \subset W$ be the subspace of codimension $M$ such that $W_1 = z^k W^1 \subset W$. Notice first that $W_1$ is a subspace of $W$ and $z^k W$ of codimension $k + m$ and $m$, respectively. Hence there exist differential operators $L_1$ and $L_2$ of order $k + m$ and $m$, respectively, such that

$$L_1 \psi_W = \psi_{W_1}, \quad L_2 z^k \psi_W = \psi_{W_1}$$

(5.1)

and that $\psi_{W_1}$ is again a wavefunction. From (5.1) one immediately deduces that

$$L_2^k = L_2^{-1} L_1.$$  

(5.2)

We first prove the following lemma.

**Lemma 5.1** Let $L = P \partial^k P^{-1}$ be a pseudodifferential operator of order $k$ and let $L_1$ and $L_2$ be differential operators of order $k + m$ and $m$, respectively, such that $L = L_2^{-1} L_1$. Then one has the following identities:

$$L_1(L_2^{-1} L_1)^{i/k} = (L_1 L_2^{-1})^{i/k} L_1, \quad L_2(L_2^{-1} L_1)^{i/k} = (L_1 L_2^{-1})^{i/k} L_2.$$  

Proof. Since $L_1 P = L_2 P \partial^k$, one can find a pseudodifferential operator $Q$ of the same order as $P$ such that $L_1 = Q \partial^{k+m} P^{-1}$, $L_2 = Q \partial^m P^{-1}$ and thus $L_1 L_2^{-1} = Q \partial^k Q^{-1}$. Since also $L_2^{-1} L_1 = P \partial^k P^{-1}$, one finds that their $k$-th roots satisfy

$$(L_2^{-1} L_1)^{1/k} = P \partial P^{-1}, \quad (L_1 L_2^{-1})^{1/k} = Q \partial Q^{-1}. $$

Using this, one easily verifies the identities of the Lemma. \qed

Since both $\psi_W$ and $\psi_{W_1}$ are wavefunctions that are connected by equations (5.1), we find, using (5.2) and Lemma 5.1, that

$$L_W = (L_2^{-1} L_1)^{1/k} \quad \text{and} \quad L_{W_1} = L_1 (L_2^{-1} L_1)^{1/k} L_1^{-1} = (L_1 L_2^{-1})^{1/k}. $$

(5.3)

Hence

$$\partial_i \psi_{W_1} = ((L_1 L_2^{-1})^{i/k})_+ \psi_{W_1} = ((L_1 L_2^{-1})^{i/k})_+ L_1 \psi_W$$

and on the other hand is also equal to

$$\partial_i (L_1 \psi_W) = \partial_i (L_1) \psi_W + L_1 ((L_2^{-1} L_1)^{i/k})_+ \psi_W.$$  

From which one deduces that

$$\partial_i L_1 = ((L_1 L_2^{-1})^{i/k})_+ L_1 - L_1 ((L_2^{-1} L_1)^{i/k})_+. $$

(5.4)

In a similar way one obtains from the other identity of (5.1) that

$$\partial_i L_2 = ((L_1 L_2^{-1})^{i/k})_+ L_2 - L_2 ((L_2^{-1} L_1)^{i/k})_+. $$

(5.5)

Notice that in this way we have exactly obtained Krichever’s general rational reductions of the KP hierarchy [17]. Krichever considers KP pseudodifferential operators $L$ of the form (2.4), such that $L^k = L_2^{-1} L_1$, where $L_1$ and $L_2$ are coprime differential operators of order $k + m$ and $m$, respectively. It can be shown that the equations (5.4) and (5.5) for $L_1$ and $L_2$ are equivalent to the KP Lax equations for $L$. It is not difficult to see that our operators must be coprime, since we have chosen our $m$ to be minimal. We will now prove that the converse also holds, i.e., that the following theorem holds.
Theorem 5.1 Let \( W \) be a plane in \( Gr(H) \) and let \( L_W \) be the corresponding solution of the KP-hierarchy. Then for \( m, k \in \mathbb{N}, k \neq 0 \), the following 2 conditions are equivalent

(a) The space \( W \) satisfies the \( mVkC \)-condition, with \( m \) as minimal as possible.

(b) There exist coprime differential operators \( L_1 \) and \( L_2 \) of order \( k + m \) and \( m \), respectively, such that the following conditions are fulfilled:

\begin{align*}
(i) & \quad L_W^k = L_2^{-1}L_1 \\
(ii) & \quad \partial_i L_1 = ((L_1L_2^{-1})^{i/k})_+ L_1 - L_1((L_2^{-1}L_1)^{i/k})_+ \\
(iii) & \quad \partial_i L_2 = ((L_1L_2^{-1})^{i/k})_+ L_2 - L_2((L_2^{-1}L_1)^{i/k})_+.
\end{align*}

Proof. We have already shown that (a) implies (b). So we assume from now on (b). Let \( \psi_1 \) be the oscillating function \( L_1\psi_W \), then by using Lemma 5.1:

\[
(L_1L_2^{-1})^{1/k}\psi_1 = (L_1L_2^{-1})^{1/k}L_1\psi_W = L_1(L_2^{-1}L_1)^{1/k}\psi_W = zL_1\psi_W = z\psi_1.
\]

Now consider

\[
\partial_i \psi_1 = \partial_i(L_1)\psi_W + L_1\partial_i\psi_W = (((L_1L_2^{-1})^{i/k})_+ L_1 - L_1((L_2^{-1}L_1)^{i/k})_+)\psi_W = ((L_1L_2^{-1})^{i/k})_+ L_1\psi_W = ((L_1L_2^{-1})^{i/k})_+ \psi_1.
\]

Hence \( \psi_1 \) is again a wavefunction of the KP hierarchy. If we let \( W_1 \) be the closure of the span of the \( \psi_1(t, z) \) then \( \psi_W = \psi_1 \). Since \( z^k\psi_W \) is also a wavefunction,

\[
L_2z^k\psi_W = \psi_W.
\]

Thus we see with the usual density argument that

\[
\begin{align*}
W_1 & \subset z^kW \text{ of codimension } m \\
W_1 & \subset W \text{ of codimension } k + m
\end{align*}
\]

\[\tag{5.6}\]

Hence \( W_1 = z^{-k}W_1 \) is a subset of \( W \) of codimension \( m \) such that \( z^kW_1 \subset W \). Since our differential operators are coprime, one cannot find lower order operators \( M_1 \) and \( M_2 \) such that \( L_W = M_2^{-1}M_1 \). Hence there is no smaller subspace \( W_1 \) and no smaller \( m \) such that (5.6) is satisfied. \( \square \)

As a consequence of this, we obtain that in the Segal-Wilson setting, the vector constrained KP hierarchy and Krichever’s general rational reduction define the same reduction of the KP hierarchy.

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