Letter

Optimization of irradiation configuration using spherical $t$-designs for laser-direct-drive inertial confinement fusion

A. Shvydky*, W. Trickey, A.V. Maximov, I.V. Igumenshchev, P.W. McKenty and V.N. Goncharov

Laboratory for Laser Energetics, University of Rochester, 250 E River Rd, Rochester, NY 14623, United States of America

E-mail: ashv@lle.rochester.edu

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Abstract

A new class of beam configurations is proposed for symmetric-direct-drive inertial confinement fusion laser systems. These configurations are based on spherical $t$-designs that are studied in spherical design theory in mathematics (Delsarte et al 1977 Geom. Dedicata 6 363). Employing $t$-design configurations offers elimination of spherical-harmonic intensity modulations for modes $\ell \leq t$. Additionally, these configurations provide fast decay of intensity nonuniformities with increasing number of beams and symmetric intensity patterns on the surface of the target. Methods developed in spherical design theory offer a convenient, systematic way of obtaining beam configurations for an arbitrary number of beams.

Keywords: inertial confinement fusion, laser irradiation, spherical design

(Some figures may appear in colour only in the online journal)

In laser-direct-drive inertial confinement fusion (ICF), a cryogenically cooled spherical shell of deuterium–tritium fuel covered with a plastic layer on the outside is irradiated by a number of laser beams. The laser irradiation ablates the outer plastic (ablator) layer, compressing the fuel to reach ignition conditions [1–3]. Because of the finite number of beams (e.g. 60 on the OMEGA Laser System [4]), the laser irradiation nonuniformity on the target surface leads to a nonuniform shell compression and reduction in the implosion performance.

The beam overlap uniformity improves with increasing the number of beams $N$. Optimization of beam port locations to minimize drive asymmetries for a given number of beams is an important consideration in designing an ICF implosion facility. There have been two basic methodologies of obtaining beam configurations presented in literature. One is based on either augmenting or composing together regular polyhedrons [4, 5]. Advantages of this method include a symmetric intensity distribution with minimized nonuniformity for a few select values of $N$. A disadvantage is the lack of a systematic extension to an arbitrary $N$. The other method is the charged-particle method, which uses a system of $N$ particles constrained to a sphere that repel each other with a Coulomb

* Author to whom any correspondence should be addressed.
force (or another distance-dependent force). The beam configurations are chosen to correspond to particle configurations that minimize the potential energy [6–8]. The advantage of the method is its simplicity in obtaining beam configurations for arbitrary \( N \). The disadvantage is a slow decay of nonuniformity with the number of beams as, e.g., \( 1/\sqrt{N} \) in [8]. Neither of the above methods offers a way of finding beam configurations that simultaneously eliminate spherical harmonic modes below a certain number, which has been recognized as an important strategy in designing the irradiation system [5].

In this Letter, we propose new beam configurations based on spherical \( t \)-designs, which are studied in the area of mathematics known as spherical designs [9]. New configurations have the advantage of combining the following properties: all of the nonuniformity modes with \( \ell \leq t \) are zero, nonuniformity amplitudes decrease strongly with the number of beams, and intensity distribution on spherical targets exhibit symmetric patterns. Computational methods developed in the field of spherical designs (see, e.g., [10, 11]) offer a systematic approach of obtaining such configurations for any number of beams [10] feasible for a direct-drive ICF facility.

The angular nonuniformity of a laser intensity distribution on a sphere, \( I(\theta, \phi) \), is characterized by the root-mean-square (rms) deviation

\[
\sigma_{rms}^2 = \frac{1}{4\pi} \left( \langle I \rangle - \langle I \rangle^2 \right) \mathrm{d}\Omega. \quad (1)
\]

Here \( \langle I \rangle = \int I(\theta, \phi) \mathrm{d}\Omega/4\pi \). For a system of \( N \) beams that have identical intensity profiles azimuthally symmetric around each beam axis with each beam axis passing through the center of the sphere, \( \sigma_{rms} \) can be written as [5, 8, 12]

\[
\sigma_{rms}^2 = \sum_{\ell=1}^{\infty} \sigma_{\ell}^2, \quad \sigma_{\ell}^2 = (2\ell + 1) \left( \frac{a_{\ell}}{a_0} \right)^2 G_\ell^2, \quad (2)
\]

where \( \sigma_{\ell} \) is rms nonuniformity of mode \( \ell \), \( a_{\ell} = \int_0^{2\pi} I_\ell(\theta) P_\ell(\cos\theta) \sin\theta \mathrm{d}\theta \) are coefficients in the Legendre polynomial expansion of the single-beam absorbed intensity distribution \( I_\ell(\theta) \). For the absorbed intensity, we adopt a model proposed in [7],

\[
I_\ell(\theta) = I_0 \left[ 1 - (1 - n) \cos^\eta \theta \right] \exp\left[ -\left( \sin\theta/c \right)^n \cos\theta \right], \quad (3)
\]

where angle \( \theta \) is measured from the beam axis, \( I_0 \) is the incident intensity on the beam axis, \( \eta \) is the absorption fraction of the incident intensity, and \( c \) and \( n \) are parameters corresponding to the 1/e radius and order of the super-Gaussian incident-intensity profile. In equation (2) \( G_\ell \) is the geometrical factor [5, 8] which can be written as

\[
G_\ell^2 = \frac{4\pi}{(2\ell + 1)N^2} \sum_{m=-\ell}^{\ell} \left[ \sum_{p=1}^{N} Y_{\ell m}(\theta_p, \phi_p) \right]^2, \quad (4)
\]

where \( Y_{\ell m}(\theta, \phi) \) are spherical harmonic functions of degree \( \ell \) and order \( m \); \( \theta_p \) and \( \phi_p \) are the polar and azimuthal angles of the \( p \)th beam central axis, \( p = 1, \ldots, N \). One can see that rms nonuniformity \( \sigma_{\ell} \) is a product of the geometrical factor \( G_\ell \), which depends only on the beam configuration, and the single-beam factor \( \delta \sqrt{1 + \Omega} / a_0 \), which depends only on the single-beam absorption profile on the target. Therefore, if \( G_\ell = 0 \) then \( \sigma_{\ell} = 0 \) for any single-beam absorption profile \( I_0(\theta) \).

Spherical \( t \)-design is a set of points \( p = 1, \ldots, N \) on a unit sphere with coordinates \( (x_p, y_p, z_p) = (\sin\theta_p \cos\phi_p, \sin\theta_p \sin\phi_p, \cos\theta_p) \) that satisfy

\[
\frac{1}{N} \sum_{p=1}^{N} x_p^a y_p^b z_p^c = \frac{1}{4\pi} \int x^a y^b z^c \mathrm{d}\Omega, \quad (5)
\]

where \( x = \sin\theta \cos\phi \), \( y = \sin\theta \sin\phi \), \( z = \cos\theta \). Powers \( a, b, c \) are integers that satisfy \( a + b + c = k, k = 1, 2, \ldots, t \). Any set of points that satisfies equation (5) for \( k \leq t \) has the same mass, center of mass, inertial tensor, \ldots, \( t \)-th moment as a unit sphere with uniform surface mass distribution. The higher the order \( t \), the more moments of a sphere are reproduced by a given set of points, making it in this sense a better discrete approximation of a sphere.

One can show that any polynomial \( x^a y^b z^c \) with degree \( a + b + c \leq t \) on a unit sphere \( x^2 + y^2 + z^2 = 1 \) is a linear combination of either \( x^a y^b \), \( a + b < t \), or \( x^a y^b z, a + b < t - 1 \), polynomials (since \( z^2 = 1 - x^2 - y^2 \)). The total number of linearly independent polynomials \( x^a y^b z, a + b < t \) is \( (t + 1)(t + 2)/2 \) [the number of \( x^a y^b \) polynomials with \( a + b < t \) plus \( (t + 1)/2 \) [the number of \( x^a y^b z \) polynomials with \( a + b < t - 1 \)] which is equal to \( (t + 1)^2 \). Since for the zero degree polynomial, \( a = b = c = 0 \), equation (5) is satisfied identically, there are, therefore, total of \( (t + 1)^2 \) \(-\) independent equations in the system of equations equation (5). The number of unknown variables in equation (5) is the number of angles \( \theta_p \) and \( \phi_p \), \( p = 1, \ldots, N \), which is \( 2N \) minus 3, where 3 accounts for rotational symmetry. One should expect to find a solution of equation (5) when the number of unknowns, \( 2N - 3 \), is about or larger than the number of equations, \( (t + 1)^2 - 1 \), which for large \( N \) leads to

\[
N \gtrsim t^2/2. \quad (6)
\]

Next, we show that a spherical \( t \)-design eliminates nonuniformity modes with numbers \( \ell \leq t \). As it is well known, each spherical harmonic \( Y_{\ell m} \) is a homogeneous polynomial of degree \( \ell \) restricted to a unit sphere,

\[
Y_{\ell m}(\theta, \phi) = \sum_{a+b+c=\ell} C_{\ell m}^{a,b,c} x^a y^b z^c, \quad (7)
\]

where \( C_{\ell m}^{a,b,c} \) are standard coefficients in Cartesian representation of the spherical harmonics. For a set of points with angles \( (\theta_p, \phi_p) \) that constitute a \( t \)-design, using equations (5) and (7), one finds

\[
\frac{1}{N} \sum_{p=1}^{N} Y_{\ell m}(\theta_p, \phi_p) = \frac{1}{4\pi} \int Y_{\ell m}(\theta, \phi) \mathrm{d}\Omega = \frac{1}{2\sqrt{\pi}} \int Y_{\ell m}(\theta_p, \phi_p) Y_{00}^{\ell m}(\theta, \phi) \mathrm{d}\Omega = \frac{1}{2\sqrt{\pi}} \delta_{\ell 0} \delta_{m 0} = 0, 0 < \ell \leq t. \quad (8)
\]
Comparing equation (8) to equation (4) one can see that $G^2_t = 0$ and therefore $\sigma_t = 0$ for $0 < t \leq t$.

The beam configuration of the OMEGA Laser System ($\Omega_{60}$) [4] is practically a 60-beam 9-design (a spherical t-design with $t = 9$). It has all $\ell < 9$ exactly equal to zero, except for mode 6, which is close to zero. $\Omega_{60}$ is not a regular truncated icosahedron, ‘soccer ball’, which is only a 5-design, but rather a ‘stretched soccer ball’ whose hexagonal faces have unequal sides A and B with the stretch factor $A/B = 1.2$ (see figure 1(a)). The ratio $A/B = 1.21$ is quoted in [5] to eliminate mode $\ell = 6$ in the stretched soccer ball configuration. Values of the stretch factor in both $\Omega_{60}$ and [5] are within 0.5% from the exact value $A/B = 1.205285 \ldots$ that can be calculated following the method from [13], where it was recognized that the stretched soccer ball with the optimal stretch factor is a better approximation for a sphere than the regular soccer ball.

Remarkably, a 10-design exists for a set of 60 points [14], which we will call $T_{60}$. $T_{60}$ configuration eliminates $\ell$ modes up to and including mode 10 and is shown in figure 1(b) along with the $M_{60}$ configuration (figure 1(c)) obtained in [6] using the charged-particle method. Although $T_{60}$ has $[3,3]^+$ symmetry group and is the union of 5 snub tetrahedrons, [14] there are no visually discernible symmetries in $T_{60}$ point locations (much like in $M_{60}$) in comparison to aesthetically pleasing $\Omega_{60}$. Figures 1(d)–(f) show the intensity distributions produced by 60 laser beams from $\Omega_{60}$, $T_{60}$, and $M_{60}$ configurations, respectively. As an example, for figure 1, the beams were chosen to have profiles given by equation (3) with parameters $n = 4.19$ and $c = 0.635$, which correspond to profiles produced by SG5-650 phase plates [15] on 450 µm-diameter targets on the OMEGA Laser System and assuming 100% absorption ($\eta = 1$).

$$I_\ell(\theta) = I_0 \exp\left(-\left(\frac{\sin \theta}{0.635}\right)^{4.19}\right) \cdot \max(\cos \theta, 0).$$

Similarly to $\Omega_{60}$, $T_{60}$ produces a symmetric illumination pattern (compare figures 1(d) and (e)), while $M_{60}$ produces less regular illumination pattern (see figure 1(f)). Out of the three configurations, the ten-design configuration produces the lowest intensity variation amplitude.

Figure 2 shows the $\ell$-mode spectra, $\sigma_{\ell}$ versus $\ell$, corresponding to configurations in figure 1. One can see dominant mode 10 and zero modes $\ell < 10$ for $\Omega_{60}$, $M_{60}$ has significant amplitude in a broad range of modes from $\ell = 2$ to $\ell = 14$. $T_{60}$ has modes $\ell = (11$ to 15) with amplitudes that are similar to that of $M_{60}$ but, as expected, has zero modes $\ell < 11$ (see figure 2(b)).

Future direct-drive ICF laser systems will look to improve uniformity, which means increasing $t$ in $t$-design configurations using the least number of beams $N$. Finding $t$-designs with the smallest $N$ for a given $t$ happens to be one of the main problems in the spherical design theory. In view of equation (6), it was conjectured that the spherical $t$-designs exist for any $t$ and require at least number of points $N \approx \frac{t^2}{2}$, see [14], where $t$-designs are listed for every $N \leq 100$. The authors’ website in [14] also contains point coordinates for $t$-designs when $N = 12m$, $m = 1, 2, 3 \ldots$ with the number of points up to $N = 240$ and $t = 21$. Below, we will use these designs as an example. It is worth noting that numerical
methods have been developed that make it possible to calculate \( t \)-designs with \( t \leq 1000 \) and corresponding \( N \leq 10^6 \) (see, e.g. \cite{10}).

Figure 3 shows the intensity distribution for four \( N = 12m \) spherical \( t \)-designs with \( t = 11, 13, 16, 21 \) and \( N = 72, 96, 144, 240 \), which we call \( T72, T96, T144, T240 \) respectively. Similar to \( T60 \), the \( N = 12m \) configurations with higher number of beams have intrinsic symmetry groups and show symmetric intensity distribution patterns. Note a dramatic reduction of the nonuniformity amplitude with \( t \) (see colorbar ranges in figure 3).

Figure 4 shows \( \ell \)-mode spectra \( \sigma_\ell \) for the five \( N = 12m \) \( t \)-designs, four designs from figure 3, and \( T60 \) from figure 1(c). One can see zero amplitude of modes \( \ell \leq t \) and a sharp decrease of the \( \sigma_\ell \) for large \( \ell \) for all \( t \)-designs.

As one can see from figure 3, the intensity distribution nonuniformity of the \( t \)-design configurations decays strongly with increase in \( t \) and \( N \). Figure 5(a) shows that for \( N = 12m \) \( t \)-design configurations with \( N = 60, 72, 84, 96, 108, 120, 144, 156, 180, 204, 216, 240 \) and corresponding \( t = 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21 \), the total \( \sigma_{\text{rms}} \) decays with \( N \) approximately as \( N^{-4.9} \). Such a sharp dependence of the \( \sigma_{\text{rms}} \) on \( N \) can be understood as follows. One can numerically verify that for \( \ell \) in the range from 10 to 25 the Legendre coefficients \( a_\ell/a_0 \) which strongly varies with \( \ell \). So, \( \sigma_{\text{rms}} = \sqrt{\sum_{\ell=1}^{\infty} (2\ell+1)(a_\ell/a_0)^2 G^2_\ell} \sim \sqrt{t \cdot N^{-9.8}} = t^{-0.93} \sim N^{-4.65} \) (where we used \( N \approx \frac{1}{2} t^2 \)), which is close to the \( \sigma_{\text{rms}} \sim N^{-4.9} \) dependence in figure 5(a). As another example, consider a Gaussian beam profile \( I_0 = I_0 \exp(-\sin^2(\theta/0.25)^2 \cdot \max(\cos \theta, 0)) \) used with the above \( t \)-designs. The corresponding Legendre coefficients \( a_\ell/a_0 \) can be approximated as \( \exp(-0.03 \cdot \ell^{1.8}) \) for \( \ell \) in the range from 1 to 40, so that \( \sigma_{\text{rms}} \sim \sqrt{t} \exp(-0.03 \cdot \ell^{1.8}) \sim N^{1/4} \exp(-0.03 \cdot (2N)^{0.9}) \), i.e. \( \sigma_{\text{rms}} \) decays with \( N \) nearly exponentially. From the above consideration one can see that the fast decay of \( \sigma_{\text{rms}} \) with increasing number of beams \( N \) for \( t \)-design configurations is the result of \( G_\ell = 0 \) for modes \( \ell \leq t \) (which increases with \( N \) as \( t \sim \sqrt{2N} \)) and a sharp decay in single-beam Legendre coefficients \( a_\ell \), hence \( \sigma_\ell \), with \( \ell \) (see \( T96 \) (red) and \( T240 \) (black) lines in figure 6).

The fast decay of \( \sigma_{\text{rms}} \) with \( N \) for \( t \)-design configurations can be contrasted with much slower \( \sigma_{\text{rms}} \sim N^{-1.1} \) decay for charged particle configurations with the same number of beams and beam profile given by equation (3) as for the \( t \)-design configurations (see figure 5(a)). As illustrated in figure 6(a), low \( \ell \) modes in charged particle configurations are not identically zero and dominate the contribution to the total \( \sigma_{\text{rms}} \) even though their geometrical factors look negligible (on a linear plot) compared to those of the modes with highest \( G_\ell \)'s (\( \ell \sim 18 \) for \( M96 \) and \( \ell \sim 29 \) for \( M240 \)) (see figure 6(b)). Also, \( \sigma_\ell \) of low \( \ell \) modes decrease only by a factor of few between charged particle configurations with 96 beams (\( M96 \)) and 240 beams (\( M240 \)) (compare blue and green lines in figure 6(a)), which explains why the total \( \sigma_{\text{rms}} \) decays slowly with \( N \) for the charged particle configurations (figure 5(a)). Previous studies \cite{7, 8} also reported slow decay of \( \sigma_{\text{rms}} \) with \( N \) for charged particle configurations. As a note, the above
Figure 5. (a) Total rms nonuniformity $\sigma_{rms}$ as a function of the number of beams $N$ for the $N = 12m$ t-design configurations (black) and charged particle configurations (red). (b) Single-beam Legendre coefficients $a_\ell/a_0$ as a function of mode number $\ell$.

The contrast between charged particle and t-design configurations is similar to that between basic methods of numerical integration (such as trapezoidal or Simpson) and the Gaussian quadratures, [16] which show much faster convergence with the number of integration points compared to the basic methods. The above similarity is not circumstantial—the spherical t-designs are a particular case of more general spherical quadratures [17].

In conclusion, we have proposed a new approach to systematically obtain beam configurations for laser-direct-drive ICF systems that are based on t-designs from the area of mathematics known as spherical designs. The t-design beam configurations offer the following advantages: (a) they eliminate all nonuniformity modes $\ell \leq t$, where $t$ increases with the number of beams as $\sim \sqrt{2N}$; (b) the rms nonuniformity drops rapidly with the number of beams as a high power of $N$ or close to exponentially with $N$; and (c) t-designs with intrinsic symmetries show symmetric intensity-distribution patterns (although, it may have only aesthetic benefits). We envision that future laser-direct-directive ICF facilities will use t-design beam configurations with a number of beams that will be determined by the nonuniformity requirements. As a final note, spherical t-designs can be used, more generally, in applications where a uniformity of an action on a sphere applied at discrete points is required, e.g. a uniformity of pressure applied with a system of identical actuators.

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ORCID iDs

A. Shvydky https://orcid.org/0000-0002-8015-0425
W. Trickey https://orcid.org/0000-0001-7013-4395
A.V. Maximov https://orcid.org/0000-0002-5742-3034
I.V. Igumenshchev https://orcid.org/0000-0001-5495-8503
P.W. McKenty https://orcid.org/0000-0002-8048-1726
V.N. Goncharov https://orcid.org/0000-0003-2730-1666

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