SINGULAR OSCILLATORY INTEGRALS ON $\mathbb{R}^n$

M. PAPADIMITRAKIS AND I. R. PARISSIS

Abstract. Let $P_{d,n}$ denote the space of all real polynomials of degree at most $d$ on $\mathbb{R}^n$. We prove a new estimate for the logarithmic measure of the sublevel set of a polynomial $P \in P_{d,n}$. Using this estimate, we prove that

$$\sup_{P \in P_{d,n}} \left| \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x)} \frac{\Omega(x/|x|)}{|x|^n} dx \right| \leq c \log d \left( \|\Omega\|_{L\log L(S^{n-1})} + 1 \right),$$

for some absolute positive constant $c$ and every function $\Omega$ with zero mean value on the unit sphere $S^{n-1}$. This improves a result of Stein from [4].

1. Introduction

We denote by $P_{d,n}$ the vector space of all real polynomials of degree at most $d$ in $\mathbb{R}^n$. Let $K$ be a $-n$ homogeneous function on $\mathbb{R}^n$, that is,

$$(1.1) \quad K(x) = \frac{\Omega(x/|x|)}{|x|^n},$$

where $\Omega$ is some function on the unit sphere $S^{n-1}$. Consider the principal value integral

$$I_n(P) = \left| \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x)} K(x) dx \right|.$$

Stein has proved in [4] that if $\Omega$ has zero mean value on the unit sphere, then

$$(1.2) \quad |I_n(P)| \leq c_d \|\Omega\|_{L^\infty(S^{n-1})},$$

for some constant $c_d$ depending on $d$. We wish to obtain sharp estimates of the form (1.2). The one dimensional analogue, namely the estimate

$$(1.3) \quad \left| \text{p.v.} \int_{\mathbb{R}} e^{iP(x)} \frac{dx}{x} \right| \leq c \log d,$$

which was proved in [3], suggests that the constant $c_d$ in (1.2) could be replaced by $c \log d$ for some absolute positive constant $c$. The fact that this is indeed the case is the content of the following theorem.

Theorem 1.1. Suppose that $K(x) = \Omega(x/|x|)/|x|^n$ where $\Omega$ has zero mean value on the unit sphere $S^{n-1}$. There exists an absolute positive constant $c$ such that

$$\sup_{P \in P_{d,n}} \left| \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x)} K(x) dx \right| \leq c \log d \left( \|\Omega\|_{L\log L(S^{n-1})} + 1 \right).$$

2000 Mathematics Subject Classification. Primary 42B20; Secondary 26D05.
Remark 1.2. Suppose that \( K(x) = \Omega(x/|x|)/|x|^n \) where the function \( \Omega \) is odd on the unit sphere. It is an immediate consequence of the one-dimensional result that

\[
\sup_{P \in \mathbb{P}_{d,n}} \left| \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x)} K(x) dx \right| \leq c \log d \| \Omega \|_{L^1(S^{n-1})}
\]

for some absolute positive constant \( c \).

The main ingredient of the proof of Theorem 1.1 is an estimate for the logarithmic measure of the sublevel set of a real polynomial in one dimension. This is a lemma of independent interest which we now state.

**Lemma 1.3** (The logarithmic measure lemma). Let \( P(x) = \sum_{k=0}^{d} b_k x^k \) be a real valued polynomial of degree at most \( d \), \( \alpha > 0 \) and \( M = \max\{|b_k| : d/2 < k \leq d\} \). If \( E = \{x \geq 1 : |P(x)| \leq \alpha\} \), then

\[
\int_E \frac{dx}{x} \leq c \min(\left( \frac{\alpha}{M} \right)^{\frac{1}{d}}, 1 + \frac{1}{d} \log^+ \frac{\alpha}{M}),
\]

where \( c \) is an absolute positive constant.

Lemma 1.3 should be compared to the following variation of a classical result of Vinogradov which can be found in [6]:

**Lemma 1.4.** Let \( P(x) = \sum_{k=0}^{d} b_k x^k \) be a real valued polynomial of degree at most \( d \), \( \alpha > 0 \) and \( M_r = \max\{|b_k| : r \leq k \leq d\} \). Let \( 1 < R \). Then

\[
|\{x \in [1, R] : |P(x)| \leq \alpha\}| \leq c R^{1 - \frac{r}{d}} \frac{\alpha^{\frac{r}{d}}}{M_r^{\frac{r}{d}}},
\]

where \( c \) is an absolute positive constant.

The estimates above depend on the length of the interval \([1, R]\) in all cases but the one where \( r = d \). The dependence on \( R \) is sharp as can be seen by a scaling argument.

When \( r = d \) we get

\[
(1.4) \quad |\{x \in [1, R] : |P(x)| \leq \alpha\}| \leq c \frac{\alpha^{\frac{r}{d}}}{|b_d|^{\frac{1}{d}}},
\]

The last inequality corresponds to the following more general result about sublevel sets which was proved in [1]:

**Lemma 1.5.** Let \( \phi \) be a \( C^k \) function on the interval \([1, R]\) for some \( k \geq 1 \) and \( R > 1 \). Suppose that \( |\phi^{(k)}(x)| \geq M \) on \([1, R]\). Then

\[
|\{x \in [1, R] : |\phi(x)| \leq \alpha\}| \leq c k \frac{\alpha^{\frac{k}{d}}}{M^{\frac{r}{d}}},
\]

where \( c \) is an absolute positive constant.

Observe that inequality (1.4) can be deduced by Lemma 1.5 by taking \( k = d \) derivatives of the phase function \( \phi(x) = P(x) \).

In case \( n = 1 \) the "linear" part \( (\frac{d}{2})^{\frac{1}{d}} \) of the estimate of \( \int_E \frac{1}{x} dx \) in Lemma 1.3 is enough for the proof of Theorem 1.1. In fact, the author in [3] used Lemma 1.4 in some appropriate way to prove the above "linear" estimate of Lemma 1.3.
In case $n > 1$ the “logarithmic” part of the estimate of $\int_E \frac{1}{x} dx$ is essential in the proof of Theorem 1.1 as can easily be seen by examining the argument therein.

The structure of the rest of this work is as follows. In section 2 we state some preliminary results. In section 3 we present the proof of Lemma 1.3 and section 3 contains the proof of Theorem 1.1. Finally in section 4 we give a proof of Theorem 1.1 in case $n = 1$ which uses (the "linear" estimate in) Lemma 1.3 and not Lemma 1.4 and which is thus simpler than the proof appearing in [3].

**Notation.** We will use the letter $c$ to denote an absolute positive constant which might change even in the same line of text.

## 2. Preliminary Results

As is usually the case when one deals with oscillatory integrals, a key Lemma is the classical van der Corput Lemma.

**Lemma 2.1** (van der Corput). Let $\phi : [a, b] \to \mathbb{R}$ be a $C^1$ function and suppose that $|\phi'(t)| \geq 1$ for all $t \in [a, b]$ and $\phi'$ changes monotonicity $N$ times in $[a, b]$. Then, for every $\lambda \in \mathbb{R}$,

$$\left| \int_a^b e^{i\lambda \phi(x)} dx \right| \leq \frac{cN}{|\lambda|}$$

where $c$ is an absolute constant independent of $a, b$ and $\phi$.

The proof of Lemma 2.1 is a simple integration by parts.

We will also need a precise estimate for the Lebesgue measure of the sublevel set of a polynomial on $\mathbb{R}^n$.

**Theorem 2.2** (Carbery, Wright). Suppose that $K \subset \mathbb{R}^n$ is a convex body of volume 1 and $P \in P_{d,n}$. Let $1 \leq q \leq \infty$. Then,

$$|\{x \in K : |P(x)| \leq \alpha\}| \leq c \min\{qd, n\} \alpha^d \|P\|_{L^q(K)}^{-1}.$$ 

This is a consequence of a more general Theorem of Carbery and Wright and can be found in [2].

**Corollary 2.3.** Let $P$ be a real homogeneous polynomial of degree $k$ on $\mathbb{R}^n$. Then

$$\int_{S^{n-1}} \frac{\|P\|_{L^\infty(S^{n-1})}^{\frac{1}{k}}}{|P(x')|^{\frac{1}{k}}} d\sigma_{n-1}(x') \leq c.$$ 

**Proof of Corollary 2.3.** Let $B = B(0, \rho)$ be the ball of volume 1 on $\mathbb{R}^n$. For $\epsilon < \frac{1}{k}$ and some $\lambda > 0$ to be defined later, we have

$$\int_B |P(x)|^{-\epsilon} dx = \int_0^\infty \{x \in B : |P(x)|^{-\epsilon} \geq \alpha\} d\alpha$$

$$\leq \lambda + \int_\lambda^\infty \{x \in B : |P(x)| < \alpha^{-\frac{1}{k}}\} d\alpha$$

$$\leq \lambda + cn \|P\|_{L^\infty(B)} \frac{\lambda^{\frac{1}{k\epsilon} + 1}}{\kappa \epsilon - 1},$$

using Theorem 2.2. Optimizing in $\lambda$ we get

$$\int_B |P(x)|^{-\epsilon} dx \leq \left( cn \frac{k\epsilon}{1 - k\epsilon} \right)^{k\epsilon} \|P\|_{L^\infty(B)}^{-\epsilon}.$$
Using polar coordinates and setting $\epsilon = \frac{1}{2R} < \frac{1}{R}$, we then get
\[
\|P\|_{L^\infty(S^{n-1})} \int_{S^{n-1}} |P(x')|^\frac{1}{n-1} \, d\sigma_{n-1}(x') \leq \frac{c n^\frac{2}{n}}{\rho^n} = \frac{c n^\frac{2}{n}}{\Gamma(\frac{n}{2} + 1)} \leq \frac{c n^\frac{2}{n} (e\pi)^\frac{2}{n}}{(\frac{n}{2} + 1) \frac{n}{2}} \leq c,
\]
which completes the proof. \qed

3. The Logarithmic Measure Lemma

The proof of Lemma 1.3 is motivated by an argument of Vinogradov from [6], used to estimate the Lebesgue measure of the sublevel set of a polynomial in a bounded interval. We fix a polynomial $P(x) = \sum_{k=0}^d b_k x^k$ and look at the set $E = \{ x \geq 1 : |P(x)| \leq \alpha \}$. Note that by replacing $\alpha$ with $\alpha M$ in the statement of the lemma, it is enough to consider the case $M = 1$. Since $E$ is a closed set we can find points $x_0, x_1, \ldots, x_d \in E$ such that $x_0 < x_1 < \cdots < x_d$ and
\[
\frac{1}{d} \int_E \frac{dx}{x} = \int_{E \cap [x_j, x_{j+1}]} \frac{dx}{x} \leq \log \frac{x_{j+1}}{x_j}, \quad 0 \leq j \leq d - 1.
\]
We set $\mu = \int_E \frac{dx}{x}$ and $t = e\# > 1$ and we have that $x_{j+1} \geq tx_j$, $0 \leq j \leq d - 1$. The Lagrange interpolation formula is
\[
P(x) = \sum_{j=0}^d P(x_j) \frac{(x-x_0) \cdots (x-x_j) \cdots (x-x_d)}{(x_j-x_0) \cdots (x_j-x_j) \cdots (x_j-x_d)}, \quad x \in \mathbb{R},
\]
where $\hat{u}$ means that $u$ is omitted. Thus,
\[
b_k = \sum_{j=0}^d P(x_j) (-1)^{d-k} \frac{\sigma_{d-k}(x_0, \ldots, \hat{x}_j, \ldots, x_d)}{(x_j-x_0) \cdots (x_j-x_j) \cdots (x_j-x_d)},
\]
where $\sigma_l$ is the $l$-th elementary symmetric function of its variables. Therefore
\[
|b_k| \leq \alpha \sum_{j=0}^d \frac{\sigma_{d-k}(x_0, \ldots, \hat{x}_j, \ldots, x_d)}{|x_j-x_0| \cdots |x_j-x_j| \cdots |x_j-x_d|} \leq \alpha \sum_{j=0}^d \frac{\sigma_k(\frac{1}{x_0}, \ldots, \frac{1}{x_j}, \ldots, \frac{1}{x_d})}{(\frac{1}{x_0} - 1) \cdots (\frac{1}{x_{j-1}} - 1) (1 - \frac{1}{x_{j+1}}) \cdots (1 - \frac{1}{x_d})} \leq \alpha \sum_{j=0}^d \frac{\sigma_k(1, \ldots, \frac{1}{t^j}, \ldots, \frac{1}{t^d})}{(t^j - 1) \cdots (t^j - 1) (1 - \frac{1}{t^{j+1}}) \cdots (1 - \frac{1}{t^{d+1}})}.
\]
It is easy to see that there exists precisely one $j$, $0 \leq j \leq \frac{d-1}{2} < d$, for which
\[
t^{j-1} < \frac{2t^d}{t^{d+1} + 1} \leq t^j.
\]
It is exactly for this $j$ that $(t^j - 1) \cdots (t^j - 1) (1 - \frac{1}{t^j}) \cdots (1 - \frac{1}{t^{d+1}})$ takes its minimum value as $j$ runs from 0 to $d$. On the other hand we have
\[
\sum_{j=0}^d \sigma_k(1, \ldots, \frac{1}{t^j}, \ldots, \frac{1}{t^d}) = (d+1-k)\sigma_k(1, \ldots, \frac{1}{t^d})
\]
and, hence

\[
|b_k| \leq \alpha \frac{(d + 1 - k)\sigma_k(1, \ldots, \frac{1}{t^d})}{(t-1) \cdots (t-1)(1-\frac{1}{t}) \cdots (1-\frac{1}{t^d})}.
\]

(3.2)

\[
\leq \frac{\alpha (d + 1 - k)^{d+1}}{1 \cdot t \cdots t^k} \frac{1}{(t-1) \cdots (t-1)(1-\frac{1}{t}) \cdots (1-\frac{1}{t^d})}.
\]

From (3.1) we easily see that \(t^j < 2\) and, since \(\log(1-\frac{x}{x})\) is increasing in the interval \((1, 2)\), we find

\[
\log(t-1) + \cdots + \log(t^j - 1)
\]

\[
= \frac{1}{t-1} \left( \log \left( \frac{t-1}{t} \right) + \cdots + \log \left( \frac{t^j - 1}{t^j - 1} \right) \right)
\]

\[
\geq \frac{1}{t-1} \int_1^{t^j} \log \left( \frac{x}{1-x} \right) dx = \frac{1}{t-1} \int_0^{1-\frac{1}{t^d}} \log x \cdot \frac{1}{1-x} dx.
\]

Similarly, since \(\log(1-x)\) is decreasing in the interval \((0, 1)\) we get

\[
\log \left( \frac{1}{1 - \frac{1}{t^{d-j}}} \right) + \cdots + \log \left( \frac{1}{1 - \frac{1}{t^j}} \right)
\]

\[
= \frac{1}{t-1} \left( \log \left( \frac{1}{1 - \frac{1}{t^{d-j}}} \right) - \frac{1}{t^{d-j}} \log \left( \frac{1}{1 - \frac{1}{t^{d-j}}} \right) + \log \left( \frac{1}{1 - \frac{1}{t}} \right) \right)
\]

\[
\geq \frac{1}{t-1} \int_{1/\sqrt{t^d}}^1 \log \left( \frac{1-x}{x} \right) dx = \frac{1}{t-1} \int_0^{1-\frac{1}{\sqrt{t^d}}} \log x \cdot \frac{1}{1-x} dx.
\]

We let

\[
A = \frac{t^d - 1}{t^d + 1}, \quad B = t^j - 1, \quad \Gamma = 1 - \frac{1}{t^{d-j}},
\]

and, obviously, \(0 < A, B, \Gamma < 1\). From (3.3) and (3.4) we have

\[
\log(t-1) + \cdots + \log(t^j - 1) + \log \left( \frac{1}{1 - \frac{1}{t^{d-j}}} \right) + \cdots + \log \left( \frac{1}{1 - \frac{1}{t}} \right)
\]

\[
\geq \frac{t}{t-1} \int_0^{t^j - 1} \log \left( \frac{x}{1-x} \right) dx + \frac{1}{t-1} \int_0^{1-\frac{1}{\sqrt{t^d}}} \log \left( \frac{x}{1-x} \right) dx
\]

\[
= \frac{t}{t-1} \int_0^{B} \log \left( \frac{x}{1-x} \right) dx + \frac{1}{t-1} \int_0^{1/\sqrt{t^d}} \log \left( \frac{x}{1-x} \right) dx
\]

\[
= -\frac{t}{t-1} B \log \frac{1}{B} - \frac{1}{t-1} \Gamma \log \frac{1}{\Gamma} - O \left( \frac{t}{t-1} B \right) - O \left( \frac{1}{t-1} \Gamma \right).
\]

From (3.1) we get \(B, \Gamma \leq \frac{t^d + 1}{t^d + 1} \) and, since \(\frac{t^{d+1} - 1}{t^d + 1} \) is decreasing in \(t \in (1, +\infty)\), we find

\[
\frac{t}{t-1} B \leq \frac{t + 1}{t - 1} \frac{t^{d+1} - 1}{t^{d+1} + 1} \leq d + 1
\]

and, similarly,

\[
\frac{1}{t-1} \Gamma \leq \frac{t + 1}{t - 1} \frac{t^{d+1} - 1}{t^{d+1} + 1} \leq d + 1.
\]
Therefore
\[
\log(t - 1) + \cdots + \log(t^j - 1) + \log \left( 1 - \frac{1}{td-j} \right) + \cdots + \log \left( 1 - \frac{1}{t} \right)
\geq - \frac{t}{t-1}B \log \frac{1}{B} - \frac{1}{t-1} \Gamma \log \frac{1}{\Gamma} - cd
\geq - \frac{2}{t-1}A \log \frac{1}{A} - \frac{1}{t-1} \left( B \log \frac{1}{B} + \Gamma \log \frac{1}{\Gamma} - 2A \log \frac{1}{A} \right) - cd.
\]

Now
\[
B \log \frac{1}{B} + \Gamma \log \frac{1}{\Gamma} - 2A \log \frac{1}{A} = \frac{B + \Gamma}{A} - 2 \log \frac{1}{A} + cA.
\]

Using (3.1)
\[
\frac{B + \Gamma}{A} - 1 \leq \frac{2(t-1)}{td+1+1}
\]
and we conclude that
\[
\frac{1}{t-1} \left( B \log \frac{1}{B} + \Gamma \log \frac{1}{\Gamma} - 2A \log \frac{1}{A} \right) \leq \frac{2}{td+1+1} A \log \frac{1}{A} + \frac{c}{t-1} A
\leq c + c \frac{t+1}{t-1} \frac{t^d-1}{td+1} \leq cd.
\]

Therefore
\[
\log(t - 1) + \cdots + \log(t^j - 1) + \log \left( 1 - \frac{1}{td-j} \right) + \cdots + \log \left( 1 - \frac{1}{t} \right)
\geq - \frac{2}{t-1}A \log \frac{1}{A} - cd
\]
and, finally, (3.2) implies that for some \(k > \frac{d}{2}\)
\[
1 \leq \frac{c_o \alpha}{t^{2kd-1} \left( \frac{A}{A} \right)^{\frac{2d}{t^d+1}}} ,
\]
where \(c_o\) is an absolute positive constant.

**case 1:** \(c_o \alpha^{\frac{d}{2}} < \frac{1}{2}\). Then, since \(\frac{2d}{t^d+1} A \leq d\), we get
\[
A^d \leq A^{\frac{2d}{t^d+1}} \leq c_o^d \alpha
\]
which implies
\[
\frac{t^d-1}{td+1} = A \leq c_o \alpha^{\frac{d}{2}}
\]
and, finally,
\[
\mu \leq e^\mu - 1 = t^d - 1 \leq 4c_o \alpha^{\frac{d}{2}}.
\]

**case 2:** \(c_o \alpha^{\frac{d}{2}} \geq \frac{1}{2}, \ t^d < 2\). Then
\[
1 < e^\mu = t^d < 4c_o \alpha^{\frac{d}{2}}
\]
which shows that
\[
\mu < \log^+ (4c_o) + \frac{\log^+ \alpha}{d}.
\]
**case 3:** $c_0 \alpha^+ \geq \frac{1}{2}$, $t^d \geq 2$. Then $A \geq \frac{1}{2}$ and $\frac{2A}{t-1} \leq \frac{t+1}{t-1} A \leq d$ and, hence,

$$
\frac{1}{3^d t^{\frac{k(k-1)}{2}}} \leq c_0^\alpha.
$$

We conclude that

$$
\mu \leq \frac{2d^2}{k(k-1)} \left( \log^+ (3c_0) + \frac{\log^+ \alpha}{d} \right) \leq \epsilon \left( 1 + \frac{\log^+ \alpha}{d} \right)
$$

since $k > \frac{d}{2}$.  

Let $\Omega$ be a function with zero mean value on the unit sphere $S^{n-1}$ belonging to the class $L \log L(S^{n-1})$, that is

$$
\|\Omega\|_{L \log L(S^{n-1})} = \int_{S^{n-1}} |\Omega(x')|(1 + \log^+ |\Omega(x')|) d\sigma_{n-1}(x') < \infty.
$$

Set $K(x) = \Omega(x/|x|)/|x|^n$ and let $P \in \mathcal{P}_{d,n}$. We will show the theorem for $d = 2^m$, for some $m \geq 0$. The general case is then an immediate consequence.

We set

$$
C_d = \sup_{0 < \epsilon < R} \sup_{P \in \mathcal{P}_{d,n}} \left| \int_{|x| \leq R} e^{iP(x)} K(x) dx \right|,
$$

where $C_d$ is a constant depending on $d$, $\Omega$ and $n$. For $0 < \epsilon < R$ and $P \in \mathcal{P}_{d,n}$ we write,

$$
I_{\epsilon,R}(P) = \int_{|x| \leq R} e^{iP(x)} K(x) dx = \int_{S^{n-1}} \int_{\epsilon}^{R} e^{iP(rx')} \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x').
$$

For $x' \in S^{n-1}$, we have that $P(rx') = \sum_{j=1}^{d} P_j(x') e^{j}$ where $P_j$ is a homogeneous polynomial of degree $j$. Observe that we can omit the constant term, without loss of generality. Set also $m_j = \|P_j\|_{L^\infty(S^{n-1})}$. Since $\epsilon$ and $R$ are arbitrary positive numbers, by a dilation in $r$ we can assume that $\max_{\frac{d}{2} < j \leq d} m_j = 1$ and, in particular, that $m_{j_0} = 1$ for some $\frac{d}{2} < j_0 \leq d$. We also write $Q(x) = \sum_{j=1}^{d} P_j(x)$. We split the integral in two parts as follows

$$
|I_{\epsilon,R}(P)| \leq \left| \int_{S^{n-1}} \int_{\epsilon}^{1} e^{iP(rx')} \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x') \right| + \left| \int_{S^{n-1}} \int_{1}^{R} e^{iP(rx')} \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x') \right| = I_1 + I_2.
$$

For $I_1$ we have that

$$
I_1 \leq \int_{S^{n-1}} \int_{0}^{1} \left| e^{iP(rx')} - e^{iQ(rx')} \right| \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x') + \int_{S^{n-1}} \int_{1}^{1} \left| e^{iQ(rx')} \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x') \right| \leq \sum_{\frac{d}{2} < j \leq d} m_j \|\Omega\|_{L^1(S^{n-1})} + C_4 \epsilon \leq \epsilon \|\Omega\|_{L^1(S^{n-1})} + C_4 \epsilon.
$$
For $I_2$ we write
\[
I_2 \leq \int_{S^{n-1}} \left| \int_{r \in [1,R]} \frac{e^{iP(x')}}{r} \right| \Omega(x') |d\sigma_{n-1}(x') \right| \int_{r \in [1,R]: |\frac{\partial P(x')}{\partial r}| > d} \frac{dr}{r} |\Omega(x')| |d\sigma_{n-1}(x')| \right.
\]
\[
+ \int_{S^{n-1}} \left| \int_{r \in [1,R]: |\frac{\partial P(x')}{\partial r}| \leq d} \frac{dr}{r} |\Omega(x')| |d\sigma_{n-1}(x')| \right.
\]

Since $\{ r \in [1,R]: |\frac{\partial P(x')}{\partial r}| > d \}$ consists of at most $O(d)$ intervals where $|\frac{\partial P(x')}{\partial r}|$ is monotonic, a simple corollary to van der Corput’s lemma for the first derivative [5, corollary on p. 334] gives the bound
\[
\int_{S^{n-1}} \left| \int_{r \in [1,R]: |\frac{\partial P(x')}{\partial r}| > d} \frac{e^{iP(x')}}{r} \right| \Omega(x') |d\sigma_{n-1}(x') \leq c \left| \frac{\Omega}{L^1(S^{n-1})} \right|
\]

On the other hand, the logarithmic measure lemma implies that
\[
\int_{S^{n-1}} \left| \int_{r \in [1,R]: |\frac{\partial P(x')}{\partial r}| \leq d} \frac{dr}{r} |\Omega(x')| |d\sigma_{n-1}(x')| \right.
\]
\[
\leq c \left| \frac{\Omega}{L^1(S^{n-1})} \right| + \frac{c}{d} \int_{S^{n-1}} \log \max_{\frac{4}{d} < \epsilon \leq d} \frac{1}{\epsilon^d} \left| j |P_j(x')| \right| |\Omega(x')| |d\sigma_{n-1}(x')|.
\]

Combining the estimates we get
\[
C_d \leq c \left| \frac{\Omega}{L^1(S^{n-1})} \right| + C_d \frac{c}{d} \int_{S^{n-1}} \log \left| \frac{P_j(x')}{\Omega(x')} \right| |\Omega(x')| |d\sigma_{n-1}(x')|.
\]

and, from Young’s inequality,
\[
C_d \leq c \left| \frac{\Omega}{L^1(S^{n-1})} \right| + C_d \frac{c}{d} \int_{S^{n-1}} \log \left| \frac{P_j(x')}{\Omega(x')} \right| |\Omega(x')| |d\sigma_{n-1}(x')| + c \int_{S^{n-1}} \left| \frac{P_j(x')}{\Omega(x')} \right| |\Omega(x')| |d\sigma_{n-1}(x')|.
\]

Now, using corollary 2.3 we get
\[
C_d \leq C_d \frac{c}{d} + c \left( \left| \frac{\Omega}{L^1(S^{n-1})} \right| + 1 \right).
\]

Since $d = 2^m$, this means that
\[
C_d \leq C_d \frac{c}{d} + c \left( \left| \frac{\Omega}{L^1(S^{n-1})} \right| + 1 \right).
\]

Using induction on $m$ we get that $C_d \leq C_1 + cm \left( \left| \frac{\Omega}{L^1(S^{n-1})} \right| + 1 \right)$. Observe that $C_1$ corresponds to some polynomial $P(x) = b_1 x_1 + \cdots + b_n x_n$. We write
\[
\left| \int_{x \in [a,b] \times \mathcal{L}^1} e^{iP(x')} K(x) dx \right| = \left| \int_{S^{n-1}} \int_{r \in [1,R]} e^{iP(x')} \Omega(x') |d\sigma_{n-1}(x')| \right|.
\]

Using the simple estimate
\[
\left| \int_{r \in [1,R]} e^{iP(x')} \Omega(x') |d\sigma_{n-1}(x')| \right| \leq c + c \log \left| \frac{b}{a} \right|
\]
we get
\[ \left| \int_{|x|<R} e^{iP(x)} K(x) \, dx \right| \leq c \|\Omega\|_{L^1(S^{n-1})} + 
+ c \int_{S^{n-1}} \log \frac{\|P\|_{L^\infty(S^{n-1})}}{|P(x')|^{\frac{1}{2}}} |\Omega(x')| \, d\sigma_{n-1}(x'). \]

Hence, \( C_1 \leq c \|\Omega\|_{L^1(S^{n-1})} + c \) and \( C_2 = c m \|\Omega\|_{L^{\log L}(S^{n-1})} + 1 \).

The case of general \( d \) is now trivial. If \( 2^{m-1} < d \leq 2^m \) then
\[ C_d \leq C_2 m \leq c m \|\Omega\|_{L^{\log L}(S^{n-1})} + 1 \]
\[ \leq c \log d \|\Omega\|_{L^{\log L}(S^{n-1})} + 1. \]

4. THE ONE DIMENSIONAL CASE REVISITED

We will attempt to give a short proof of the one dimensional analogue of theorem 1.1. This is a slight simplification of the proof in [3], with the aid of the logarithmic measure lemma.

So, fix a real polynomial \( P(x) = b_0 + b_1 x + \cdots + b_d x^d \) and consider the quantity
\[ C_d = \sup_{0<|x|<R} \left| \int_{|x|<R} e^{iP(x)} \frac{dx}{x} \right|. \]

By the same considerations as in the \( n \)-dimensional case, we can assume that \( P \) has no constant term and that it can be decomposed in the form
\[ P(x) = \sum_{0<j \leq \frac{d}{2}} b_j x^j + \sum_{\frac{d}{2}<j \leq d} b_j x^j = Q(x) + R(x), \]
where \( \max_{\frac{d}{2}<j \leq d} |b_j| = 1 \). As a result
\[ \left| \int_{|x|<R} e^{iP(x)} \frac{dx}{x} \right| \leq C_\frac{d}{2} + \int_{0<|x|<1} \frac{|R(x)|}{x} \, dx + \int_{1<|x|<R} \frac{e^{iP(x)} \, dx}{x} \]
\[ \leq C_\frac{d}{2} + c + I. \]

We split \( I \) as follows
\[ I \leq \int_{\{x\in[1,R]:|P'(x)|>d\}} \frac{e^{iP(x)} \, dx}{x} + \int_{\{x\geq 1:|P'(x)|\leq d\}} \frac{dx}{x}. \]

Now, using Proposition 2.1 for the first summand in the above estimate and the logarithmic measure lemma to estimate the second summand, we get that \( I \leq c \).

But this means that \( C_d \leq C_\frac{d}{2} + c \) which completes the proof by considering first the case \( d = 2^m \) for some \( m \), as in the \( n \)-dimensional case.
References

1. G. I. Arkhipov, A. A. Karacuba, and V. N. Čubarikov, *Trigonometric integrals*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 5, 971–1003, 1197. MR MR552548 (81f:10050) 2

2. A. Carbery and J. Wright, *Distributional and $L^q$ norm inequalities for polynomials over convex bodies in $\mathbb{R}^n$*, Math. Res. Lett. **8** (2001), no. 3, 233–248. MR MR1839474 (2002h:26033) 3

3. I. R. Parissis, *A sharp bound for the Stein-Wainger oscillatory integral*, Proc. Amer. Math. Soc. **136** (2008), no. 3, 963–972 (electronic). MR MR2361870 1, 2, 3, 9

4. E. M. Stein, *Oscillatory integrals in Fourier analysis*, Beijing lectures in harmonic analysis (Beijing, 1984), Ann. of Math. Stud., vol. 112, Princeton Univ. Press, Princeton, NJ, 1986, pp. 307–355. MR MR864375 (88g:42022) 1

5. Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. MR MR1232192 (95c:42002) 8

6. I. M. Vinogradov, *Selected works*, Springer-Verlag, Berlin, 1985, With a biography by K. K. Mardzhanishvili, Translated from the Russian by Naidu Psv [P. S. V. Naidu], Translation edited by Yu. A. Bakhturin. MR MR807530 (87a:01042) 2, 4

E-mail address: papadim@math.uoc.gr

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CRETE, KNOSSOS AVENUE 71409, IRAKLIO–CRETE, GREECE

E-mail address: ioannis.parissis@gmail.com

INSTITUTIONEN FÖR MATematik, KUNGliga TEkniska Högskolan, SE 100 44, STOCKHOLM, SWEDEN.