Chaos in BEC trapped in tilted optical superlattice potential with attractive interaction

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Abstract. The chaos in Bose-Einstein condensate (BEC) in a 1-D tilted optical superlattice potential is studied numerically in this paper. Numerical analysis using the Gross- Pitaevskii (G-P) equation for the system with attractive interaction reveals that as the tilt of the optical superlattice potential is increased the chaos in the BEC increases. This makes the system more unstable and the phase-space orbit increases. The chaos increases as the secondary optical lattice potential increases.

1. Introduction
The phenomenon of chaos in non-linear systems is known to play mainly the role of destruction. But these days focus of researchers is to explore the useful and practical applications \cite{1} of the chaos. There have been intensive studies on the chaos during the past years. There has been a very wide range of different areas in the physics where application of chaos can be explored. The study of chaos in BEC has given a thrust to research in non-linear systems. Theoretically chaos phenomenon has been simply described in the framework of G-P equation \cite{2}. To explore the practical utilization of chaos, restraining chaos in BEC and to develop synchronization in BEC are the subjects of great interest these days \cite{3}. Mostly the signals which drive the dynamical system contain noise and chaos. Therefore a great attention has been devoted in developing the techniques that can synchronize the driven dynamical systems. These techniques can prove to be very useful in secure communications, biomedical systems etc. The phenomenon of chaos have been well explained by the nonlinear Schrodinger equation (NLSE) called G-P equation. Nonlinear resonances, chaos in BEC using the time varying trapping potential and also the dynamics of a weakly open BEC trapped in a double-well potentials revealing chaotic behaviour have also been investigated \cite{4}. Propagation of a bright matter wave soliton and its stabilization in a BEC has also been studied \cite{5}. The chaotic behaviour of the BEC has been widely explored in the last few years for the BEC trapped in tilted optical lattices \cite{6}. In this paper we study the chaos in 1D tilted optical superlattice potential, with attractive interaction between the atoms in a BEC which has not been studied earlier. For the \textsuperscript{7}Li, the s-wave scattering length in the spin-triplet state is negative which is the cause of attractive interatomic interactions. In earlier literature it has been predicted that BEC does not occur in such kind of systems but now there has been evidence \cite{7} of BEC in such kind of systems. By the adjustment of the wave lengths of the two laser beams, we can observe the effective change in the trajectories in the phase space.
1. Model
The model that we consider here consists of an elongated cigar-shaped BEC containing \( N \) number of \(^7\text{Li} \) atoms each having mass \( m \). The BEC we consider here is quasi-1D system whose order parameter can be described by 1D NLSE or G-P equation given as

\[
i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + \left[V_{\text{trap}}(x) + g_o \left| \psi(x,t) \right|^2 \right] \psi(x,t),
\]

where \( g_o = 4\pi \hbar^2 a / m \) is the non-linearity in the G-P equation which is basically the interatomic scattering pseudopotential. \( V_{\text{trap}}(x) \) represents the time-independent tilted optical superlattice potential which is described as

\[
v_{\text{trap}}(x) = V_{o1}\cos(k_1x) + V_{o2}\cos(k_2x) + Fx,
\]

where \( V_{o1}, V_{o2} \) are the amplitudes of the optical superlattice potential and \( k_1, k_2 \) are the wave vectors of the laser beams used to create the optical superlattice potential. The tilt of the superlattice potential is described by the term \( Fx \) in Eqn. (2) where \( F \) is the tilt-coefficient of the superlattice. As the value of \( F \) increases the superlattice potential becomes more and more tilted. For the linear part of the Hamiltonian defined as

\[
H_o = -\frac{1}{2m} \frac{\partial^2}{\partial x^2} + V_{\text{trap}}(x),
\]

the eigenstates are referred to as WS states. To simplify the problem we make use of the time-dependent variational wave function with a trivial phase as

\[
\psi(x,t) = \phi(x) e^{(-i\mu t)/\hbar},
\]

where \( \mu \) is the chemical potential and \( \phi(x) \) is the real and normalized wave function, where normalization of the wave function is described as \( \int \phi^2(x)dx = N \). By substituting above wave function in Eqn. (1), we get following equation

\[
-\frac{\hbar^2}{2m} \frac{\partial^2 \phi(x)}{\partial x^2} + V_{\text{trap}}(x)\phi(x) + g_o \phi^3(x) = \mu \phi(x).
\]

Above equation is the stationary state G-P equation in case of zero trivial phase which describes the dynamics of BEC. Above equation can be re-written as

\[
-\frac{\partial^2 \phi(x)}{\partial x^2} + \left[\alpha_1 \cos(k_1x) + \alpha_2 \cos(k_2x) + \beta x \right] \phi(x) + \gamma \phi^3(x) = \Gamma \phi(x)
\]

where the coefficients are \( \alpha_1 = \frac{2mV_{o1}}{\hbar^2}, \alpha_2 = \frac{2mV_{o2}}{\hbar^2}, \beta = \frac{2mF}{\hbar^2}, \Gamma = \frac{2m\mu}{\hbar^2} \). To solve the above non-linear equation we substitute \( \frac{d\phi(x)}{dx} = y(x) \) and get following two coupled equations as

\[
\frac{dy(x)}{dx} = y(x).
\]
\[
\frac{dy(x)}{dx} = \left[ \alpha_1 \cos(k_1 x) + \alpha_2 \cos(k_2 x) + \beta x - \Gamma \right] \phi(x) + \gamma \phi'(x)
\]  

(7)

Now above two coupled equations can be solved simultaneously in “mathematica” with suitable initial conditions and other parameters.

3. Effect of varying the depth of optical superlattice potential and strength of tilted field on the stability of BEC

In this section we study the effect of varying the amplitude of the laser beams used to create the optical superlattice potential. By varying the amplitude of the secondary laser beam we change the height of the optical superlattice potential, which in turn affects the tunnelling of atoms between the potential wells. As the height of wells increases we should have more stability in the BEC due to decrease in tunnelling of atoms, but here the superlattice potential is tilted due to which the stability and instability of the BEC arises due to an overall effect of the tilt of lattice potential and height of the optical superlattice potential. We fix the value of \( k_1 = \frac{2 \pi}{\Lambda} \) and secondary laser beam is taken such that \( k_2 = k_1 + \pi \) with \( \Gamma = 0.01 \). We are considering the case of \( \gamma < 0 \) (means attractive interaction between the atoms in a BEC). We are fixing the value of \( \gamma = -1.0 \). In Fig. (1) we plot the phase space diagrams for a fixed value of \( \beta = 0.0001 \), \( \alpha_1 = 1.0 \), with the variation of \( \alpha_2 = 0.1, 0.7 \). As the value of the coefficient \( \alpha_2 \) is increased from 0.1 to 0.7, the motion of the system which was restricted to a small region of the phase space now extends in a large area showing a large instability in the BEC (shown in Fig. 1(b)).

![Figure 1](image1.png)

**Figure 1.** The phase orbit in the equivalent phase space of the \( \phi \) versus \( d\phi / dx \) with \( \beta = 0.0001 \), \( \phi(0) = 0.5 \), \( \phi'(0.5) = 0.0 \) (a) \( \alpha_2 = 0.1 \) (b) \( \alpha_2 = 0.7 \)

![Figure 2](image2.png)

**Figure 2.** The phase orbit in the equivalent phase space of the \( \phi \) versus \( d\phi / dx \) with \( \beta = 0.1 \), \( \phi(0) = 0.5 \), \( \phi'(0.5) = 0.0 \) (a) \( \alpha_2 = 0.1 \) (b) \( \alpha_2 = 0.7 \)
Similarly in the Fig. (2) we show the same plots of phase space diagrams as in Fig. (1) but for $\beta = 0.1$. Here the phase space orbit extends almost up to 6 times when the secondary lattice depth is increased. In all the cases we have taken the initial conditions as $\phi(0) = 0.5 , \phi'(0.5) = 0.0$. Corresponding to the above phase space plots, we show in the Fig. (3) and Fig. (4) the spatial evolution of the solution of the coupled Eqn. (7). We observe that as the depth of optical superlattice increases (by increasing the coefficient $\alpha_2$) with a fixed value of the coefficient $\beta$ and $\alpha_1 = 1.0$ the amplitude of the wave function increases. This increase is less with the lower value of external field. As the strength of the external field is increased the effect of varying the superlattice depth is more pronounced and the wave function increases abruptly for $\alpha_2 = 0.7$ with $\beta = 0.1$. We observe that the motion is chaotic because on changing the initial conditions slightly, trajectories in the phase space are highly changed. Now we study the effect of changing the initial conditions on the stability of BEC. In the Fig. (5) we show that as the coefficient of tilt is increased from 0.0001 to 0.1 with the initial conditions as $\phi(0) = -2.0 , \phi(0.5) = 0.0$ the phase space orbit which appears to be a little periodic in

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure3a.png}
\hspace{0.1cm}
\includegraphics[width=0.4\textwidth]{figure3b.png}
\caption{The spatial evolution of the solution $\phi(x)$ with $\beta = 0.0001$, $\phi(0) = 0.5 , \phi'(0.5) = 0.0$ (a) $\alpha_2 = 0.1$ (b) $\alpha_2 = 0.7$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure4a.png}
\hspace{0.1cm}
\includegraphics[width=0.4\textwidth]{figure4b.png}
\caption{The spatial evolution of the solution $\phi(x)$ with $\beta = 0.1$, $\phi(0) = 0.5 , \phi'(0.5) = 0.0$ (a) $\alpha_2 = 0.1$ (b) $\alpha_2 = 0.7$}
\end{figure}
Figure 5. The phase orbit in the equivalent phase space of the $\phi$ versus $d\phi/dx$ with $\alpha_2 = 0.1$, $\phi(0) = -2.0$, $\phi'(0.5) = 0.0$. (a) $\beta = 0.0001$ (b) $\beta = 0.1$

Figure 6. The spatial evolution of the solution $\phi(x)$ with $\alpha_2 = 0.1$, $\phi(0) = -2.0$, $\phi'(0.5) = 0.0$. (a) $\beta = 0.0001$ (b) $\beta = 0.1$

Fig. 5(a) with its wave function almost stable (shown in Fig. 6(a)) becomes extremely chaotic on increasing the value of $\beta$ as shown in Fig. 5(b). The large increase in wave function corresponding to Fig. 5(b) is shown in Fig. 6(b). The effect of changing the initial conditions is clearly seen in Fig. 5(a), 5(b) though the other parameters are same as in Fig. 1(a) and 2(a) respectively. It is observed that the system’s trajectory remains almost deterministic for any initial condition, but the instability can also be suppressed to an extent by varying the initial conditions of the system. Also as the value of $\beta$ is increased the trajectory of the system occupies the more region in phase-space plots which shows that the chaos and instability is increasing in the BEC, but still the trajectory in the phase space remains confined to a particular area showing that the system behaves as chaotic attractor. For higher values of $\beta$ and $\alpha_2$ the BEC becomes highly unstable.

4. Conclusions
In this paper we have investigated the chaos in a BEC system trapped in 1D tilted optical superlattice potential. We have shown that even the slightest tilt of the lattice potential leads to great chaos in the system and makes the BEC highly unstable with the abrupt increase in the wave function of the system. We can control this instability of the BEC by adjusting the tilt of the superlattice potential and the ratio of wave lengths of the two laser beams used to create the superlattice. The increase in the amplitude of the superlattice potential further enhances the chaos in the system. The initial conditions of the system also have a deep impact on the chaos in BEC. There are some initial conditions of the system which can make the system’s orbit periodic instead of chaotic as in Fig. 5(a). Therefore we can
control the instability and chaos in the BEC by adjusting the coefficients $\beta, \alpha_2$ and ratio of wave vectors of the laser beams used.

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