Characteristic hypersurfaces
in a relativistic superfluid theory

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Abstract

By discussing the Cauchy problem, we determine the covariant equation of the characteristic hypersurfaces in a relativistic superfluid theory.

1 Introduction

Phenomenological approaches to relativistic superfluidity had been initially proposed by Rothen [1], Dixon [2] and Israel [3, 4]. Then, Khalatnikov and Lebedev [5, 6] and Carter [7, 8] have developed general superfluid theories which are in fact equivalent as they have later shown [9, 10]. This is a relativistic generalization of the non-dissipative version of the Landau theory of superfluidity.

In the framework of this superfluid theory, Carter and Langlois [11] have studied the velocity of propagation of the sound following a method due to Hadamard. The story of infinitesimal disturbances, giving the sound speed, is described by an hypersurface for which the field variables are continuous but their derivatives are discontinuous across this one. The problem is to find the equation of these characteristic hypersurfaces. However, Carter and Langlois [11] do not solve the general case but the low-temperature limit for a relativistic superfluid with phonon-like excitation spectrum whose they have derived the Lagrangian. Likewise, Vlasov [12, 13] analyses the shock waves in this relativistic superfluid theory.

The purpose of the present work is to give the covariant equation determining the characteristic hypersurfaces in this relativistic superfluid theory. Our method is based on the fact that the Cauchy problem of the field equations has no unique solution on a characteristic hypersurface. To do this, we consider the covariant field equations of the

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superfluid in a spacetime with a background metric $g_{\mu\nu}$ in a general coordinate system $(x^\mu)$. We attempt to solve the initial value problem on the hypersurface $x^1 = 0$ by finding the Cauchy data and by expressing merely the derivatives with respect to $x^1$ in terms of the Cauchy data, method used in hydrodynamics for instance by Lichnerowicz [14]. If it is not possible to determine uniquely the derivatives with respect to $x^1$ then $x^1 = 0$ is a characteristic hypersurface. Also, discontinuities of the normal derivatives of the field variables can occur across $x^1 = 0$. In another coordinate system $(y^\mu)$, the equation $x^1 = 0$ will be $f(y^\mu) = 0$ and by this way we will find a covariant equation for the function $f$.

The plan of the work is as follows. In section 2, we recall the field equations of the relativistic superfluid theory. We discuss the Cauchy problem in section 3. The determination of the covariant equation of the characteristic hypersurfaces is carried out in section 4. We add in section 5 some concluding remarks.

2 Field equations of the relativistic superfluid

The relativistic superfluid theory of Carter is built on a particle number $n^\rho$ and an entropy current $s^\rho$ which are conserved. In a general coordinate system, these conservation laws can be written

\[ \nabla_\sigma n^\rho = 0 \]  
\[ \nabla_\rho s^\rho = 0 \]

where $\nabla_\sigma$ denotes the covariant derivative associated with the metric $g_{\mu\nu}$. A master function $\Lambda$ of $n^\rho$ and $s^\rho$ defines the chemical potential $\mu_\sigma$ and the temperature $\theta_\sigma$ by the differential relation $d\Lambda = \mu_\rho dn^\rho + \theta_\rho ds^\rho$. The convective variational principle of Carter yields the equation

\[ s^\sigma (\nabla_\sigma \theta_\rho - \nabla_\rho \theta_\sigma) = 0 \]  

and a similar equation for $\mu_\sigma$ but the assumption of superfluidity requires the compatible condition

\[ \nabla_\sigma \mu_\rho - \nabla_\rho \mu_\sigma = 0 \]

For our purpose, it is more convenient to introduce a Lagrangian $L$ having the expression $L = \Lambda - \mu_\rho n^\rho$ which is considered as a function of $s^\rho$ and $\mu_\sigma$ defining $\theta_\sigma$ and $n^\rho$ by the differential relation $dL = \theta_\rho ds^\rho - n^\rho d\mu_\rho$. The Lagrangian $L$ is in fact a function of the following scalars

\[ s^2 = -s^\rho s_\rho \quad , \quad \mu^2 = -\mu^\rho \mu_\rho \quad \text{and} \quad y^2 = -s^\rho \mu_\rho \]

defined for a signature $(-+++)$ of the metric. In consequence, we can express $n^\rho$ and $\theta^\rho$ in terms of $\mu_\sigma$ and $s^\rho$ in the following form

\[ n^\rho = B \mu^\rho - A s^\rho \quad \text{and} \quad \theta^\rho = C s^\rho + A \mu^\rho \]
where the functions $A$, $B$ and $C$ of $s^2$, $\mu^2$ and $y^2$ are the partial derivatives of $L$

$$B = 2\frac{\partial L}{\partial \mu^2}, \quad C = -2\frac{\partial L}{\partial s^2} \quad \text{and} \quad A = -\frac{\partial L}{\partial y^2}. \quad (7)$$

We henceforth consider that the field variables of the relativistic superfluid theory are $\mu_{\sigma}$ and $s^\rho$. From (3) and (5) we derive the following equations

$$\mu^\rho \partial_\rho B + B \nabla_\rho \mu^\rho - s^\rho \partial_\rho A = 0, \quad (8)$$

$$s^\rho s^\sigma \partial_\sigma C + C s^\rho \nabla_\sigma s^\rho + \mu^\rho s^\sigma \partial_\sigma A + s^2 g^{\rho\sigma} \partial_\sigma s^2 + y^2 g^{\rho\sigma} \partial_\sigma A = 0 \quad (9)$$

where we have used (2) and (4). The derivatives of $A$, $B$ and $C$ appearing in (8) and (9) have the expressions

$$\partial_\sigma A = \frac{\partial A}{\partial s^2} \partial_\sigma s^2 + \frac{\partial A}{\partial \mu^2} \partial_\sigma \mu^2 + \frac{\partial A}{\partial y^2} \partial_\sigma y^2. \quad (10)$$

We notice the identities

$$\frac{\partial A}{\partial \mu^2} = -\frac{1}{2} \frac{\partial B}{\partial y^2}, \quad \frac{\partial A}{\partial s^2} = \frac{1}{2} \frac{\partial C}{\partial y^2} \quad \text{and} \quad \frac{\partial B}{\partial s^2} = -\frac{\partial C}{\partial \mu^2}. \quad (11)$$

For a Lagrangian $L$ function of $s^2$, $\mu^2$ and $y^2$, the field equations are (2), (4), (8) and (9) with definitions (7) and (10).

3  Cauchy problem

We now discuss the Cauchy problem on the hypersurface $x^1 = 0$ for the field equations of the previous section, the field variables being $\mu_{\sigma}$ and $s^\rho$. The Cauchy data are $s^\rho$ and $\mu_{\sigma}$ on $x^1 = 0$. The problem is to determine $\partial_1 \mu_{\sigma}$ and $\partial_1 s^\rho$ on $x^1 = 0$ in terms of $s^\rho$ and $\mu_{\sigma}$ on $x^1 = 0$ and their derivatives with respect to $x^a (a = 0, 2, 3)$. Hereafter, we call $dC$ all quantities calculable in terms of the Cauchy data.

We immediately remark by virtue respectively of (2) and (4) that

$$\partial_1 \mu_a = dC \quad (a = 0, 2, 3) \quad \text{and} \quad \partial_1 s^1 = dC. \quad (11)$$

According to (3), we thus have

$$\partial_1 \mu^2 = -2 \mu \partial_1 \mu_1 + 2C \quad \text{and} \quad \partial_1 s^2 = -2 s_a \partial_1 s^a + 2C. \quad (12)$$

We insert (11) and (12) in equations (8) and (9) for $\rho = a$ and we obtain four linear equations determining $\partial_1 \mu_1$ and $\partial_1 s^a$ in terms of the Cauchy data

$$\begin{pmatrix} a & a_a \\ c_b & d_{ab} \end{pmatrix} \begin{pmatrix} \partial_1 \mu_1 \\ \partial_1 s^a \end{pmatrix} = \begin{pmatrix} dC \\ dC_b \end{pmatrix} \quad (a, b = 0, 2, 3). \quad (13)$$
The hypersurface \( x^1 = 0 \) is characteristic if the determinant of the matrix in (13) vanishes. Unfortunately, it is not easy to find the covariant equation of the characteristic hypersurfaces in this manner.

Instead of making use of (13), we are going to show that equations (16) and (17) determine \( \partial_1 \mu_1, \partial_1 s^2 \) and \( \partial_1 y^2 \) in terms of the Cauchy data

\[
\begin{pmatrix}
  m_{11} & m_{12} & m_{13} \\
  m_{21} & m_{22} & m_{23} \\
  m_{31} & m_{32} & m_{33}
\end{pmatrix}
\begin{pmatrix}
  \partial_1 \mu_1 \\
  \partial_1 s^2 \\
  \partial_1 y^2
\end{pmatrix}
= \begin{pmatrix}
  dC \\
  dC \\
  dC
\end{pmatrix}. \tag{14}
\]

Furthermore, we will get

\[
s^1 \partial_1 s^a = dC \tag{15}
\]

when \( \partial_1 \mu_1, \partial_1 s^2 \) and \( \partial_1 y^2 \) are known from equations (14).

We have always equation (8) which gives

\[
\mu^1 \partial_1 B + B g^{11} \partial_1 \mu_1 - s^1 \partial_1 A = dC. \tag{16}
\]

We add equation (9) for \( \rho = 1 \) in the form

\[
(s^1)^2 \partial_1 C + s^1 \mu^1 \partial_1 A + s^2 g^{11} \partial_1 C + \frac{1}{2} C g^{11} \partial_1 s^2 + y^2 g^{11} \partial_1 A = dC. \tag{17}
\]

taking into account (14). By contracting equation (9) by \( \mu_\sigma \), we obtain thereby

\[
- y^2 s^\sigma \partial_\sigma C - C s^\sigma \partial_\sigma y^2 - C s^\sigma s^\rho \nabla_\sigma \mu_\rho - \mu^2 s^\sigma \partial_\sigma A + s^2 \mu^\rho \partial_\rho C + \frac{1}{2} C \mu^\rho \partial_\rho s^2 + y^2 \mu^1 \partial_1 A = 0 \tag{18}
\]

where we have used the identity

\[
s^\rho \mu_\rho \nabla_\sigma s^\sigma = - s^\sigma \partial_\sigma y^2 - s^\sigma s^\rho \nabla_\sigma \mu_\rho.
\]

We now write down equation (13) in the form

\[
- y^2 s^1 \partial_1 C - C s^1 \partial_1 y^2 - C (s^1)^2 \partial_1 \mu_1 - \mu^2 s^1 \partial_1 A + s^2 \mu^1 \partial_1 C + \frac{1}{2} C \mu^1 \partial_1 s^2 + y^2 \mu^1 \partial_1 A = dC. \quad (19)
\]

In consequence, the three equations (16), (17) and (19) give a system for determining \( \partial_1 \mu_1, \partial_1 s^2 \) and \( \partial_1 y^2 \) in terms of the Cauchy data. We obtain explicitly the following system

\[
\begin{align*}
-2(\mu^1)^2 \frac{\partial B}{\partial \mu^2} + 2 s^1 \mu^1 \frac{\partial A}{\partial \mu^2} + B g^{11} & \partial_1 \mu_1 \\
+ \left[ \mu^1 \frac{\partial B}{\partial s^2} - s^1 \frac{\partial A}{\partial s^2} \right] \partial_1 s^2 & + \left[ \mu^1 \frac{\partial B}{\partial y^2} - s^1 \frac{\partial A}{\partial y^2} \right] \partial_1 y^2 = dC, \tag{20}
\end{align*}
\]

\[
\begin{align*}
-2 \mu^1 (s^1)^2 \frac{\partial C}{\partial \mu^2} - 2(\mu^1)^2 s^1 \frac{\partial A}{\partial \mu^2} & - 2 \mu^1 g^{11} \frac{\partial C}{\partial \mu^2} s^2 - 2 \mu^1 g^{11} \frac{\partial A}{\partial \mu^2} y^2 & \partial_1 \mu_1 \\
+ \left[ (s^1)^2 \frac{\partial C}{\partial s^2} + s^1 \mu^1 \frac{\partial A}{\partial s^2} + g^{11} \frac{\partial C}{\partial s^2} s^2 + g^{11} \frac{\partial A}{\partial s^2} y^2 + \frac{1}{2} C g^{11} \right] \partial_1 s^2 \\
+ \left[ (s^1)^2 \frac{\partial C}{\partial y^2} + s^1 \mu^1 \frac{\partial A}{\partial y^2} + g^{11} \frac{\partial C}{\partial y^2} s^2 + g^{11} \frac{\partial A}{\partial y^2} y^2 \right] \partial_1 y^2 & = dC. \tag{21}
\end{align*}
\]
\[
\begin{aligned}
2\mu_1 s_1^1 \frac{\partial C}{\partial \mu_2^2} y^2 + 2\mu_1 s_1^1 \frac{\partial A}{\partial \mu_2^2} \mu^2 - 2(\mu_1^1)^2 \frac{\partial C}{\partial \mu_2^2} s^2 - 2(\mu_1^1)^2 \frac{\partial A}{\partial \mu_2^2} y^2 - C(s^1)^2 \bigg] \partial_1 \mu_1 \\
+ \left[-s_1^1 \frac{\partial C}{\partial s_2^2} y^2 - s_1^1 \frac{\partial A}{\partial s_2^2} \mu^2 + \mu_1^1 \frac{\partial C}{\partial s_2^2} s^2 + \mu_1^1 \frac{\partial A}{\partial s_2^2} y^2 + \frac{1}{2} C \mu_1^1 \right] \partial_1 s_2 \\
+ \left[-s_1^1 \frac{\partial C}{\partial y^2} y^2 - s_1^1 \frac{\partial A}{\partial y^2} \mu^2 + \mu_1^1 \frac{\partial C}{\partial y^2} s^2 + \mu_1^1 \frac{\partial A}{\partial y^2} y^2 - C s^1 \right] \partial_1 y^2 = dC
\end{aligned}
\]

which corresponds to system (14). Moreover, from equation (9) for \( \rho = a \) we find (15) when the equations (20), (21) and (22) are satisfied.

### 4 Covariant equation of the characteristic hypersurfaces

When the condition \( \det m_{ij} = 0 \) is verified it is not possible to determine uniquely \( \partial_1 \mu_1, \partial_1 s_1 \) and \( \partial_1 y^2 \) in terms of the Cauchy data. This condition defines the characteristic hypersurfaces.

In a coordinate system \((y^\mu)\), for instance the Minkowskian coordinates, the equation of an hypersurface is \( f(y^\mu) = 0 \). If we perform the change of coordinates

\[
x^1 = f(y^\mu) \quad \text{and} \quad x^\alpha = y^\alpha \quad (a = 0, 2, 3)
\]

then the equation \( f(y^\mu) = 0 \) becomes \( x^1 = 0 \). We can then apply the results of the previous section in the coordinates \((x^\mu)\). It is crucial to notice that

\[
s^1 = s^\alpha \partial_\alpha f \quad , \quad \mu^1 = \mu^\alpha \partial_\alpha f \quad \text{and} \quad g^{11} = g^{\alpha\beta} \partial_\alpha f \partial_\beta f
\]

where here \( s^\alpha \), \( \mu^\alpha \) and \( g^{\alpha\beta} \) are the components in the coordinates \((y^\mu)\). Furthermore, \( s^2 \), \( \mu^2 \) and \( y^2 \) are scalars which can be written in the coordinates \((y^\mu)\).

Now matrix \( m_{ij} \) depends only on \( s^1, \mu^1, g^{11} \) and the scalars \( s^2, \mu^2 \) and \( y^2 \) taking into account the expressions of the functions \( A, B \) and \( C \), therefore the condition \( \det m_{ij} = 0 \) gives a covariant equation for the function \( f \)

\[
\mathcal{F}(s^\alpha \partial_\alpha f, \mu^\alpha \partial_\alpha f, g^{\alpha\beta} \partial_\alpha f \partial_\beta f, s^2, \mu^2, y^2) = 0
\]

which is the desired equation of the characteristic hypersurfaces. According to (15) we must add the another possibility

\[
s^\alpha \partial_\alpha f = 0 .
\]

The speed of the sound \( u \) with respect to the superfluid \( \mu^\rho \) is given by

\[
u^2 = \frac{(\mu^\rho \partial_\rho f)^2}{(\mu^2 g^{\sigma\rho} + \mu^\rho \mu^\sigma) \partial_\rho f \partial_\sigma f}
\]

in units for which \( c = 1 \).
5 Conclusion

Of course, it is difficult to exploit equation (25) determining the characteristic hypersurfaces. For obvious reasons of causality, we must have $u < 1$ and so these characteristic hypersurfaces must be timelike, i.e.

$$g^{\alpha\beta} \partial_\alpha f \partial_\beta f > 0 .$$

By solving (25), we must find condition (27) for all solutions to the field equations corresponding to a given Lagrangian $L$. This restricts the generality of the function $L$.

We can apply our formalism in the case of the Lagrangian found by Carter and Langlois [11] which has the expression

$$L = L_0(\mu^2) - \frac{3k}{4} \left[(c_P^2 - 1)\frac{y^4}{\mu^2} + s^2\right]^{2/3}$$

(28)

where $k$ is a physical constant and $c_P$ the velocity of sound for the fluid with one constituent defined by $L_0$ such that $c_P < 1$. We assume that the relative velocity $v$ between $\mu^\rho$ and $s^\rho$ defined by

$$1 - v^2 = \frac{s^2 \mu^2}{y^4}$$

is smaller than $c_P$. The temperature $T$ is given by

$$T = \frac{k}{y^2} \left[(c_P^2 - 1)\frac{y^4}{\mu^2} + s^2\right]^{2/3} .$$

With expression (28), we can evaluate the functions $A$, $B$ and $C$ and their derivatives that we express as functions of $T$, $\mu$ and $v$. We take the low-temperature limit, $T \to 0$, keeping $\mu$ and $v$ fixed. Hence the condition det $m_{ij} = 0$ reduces to two separate conditions

$$\lim_{T \to 0} m_{11} = 0 ,$$

(29)

$$\lim_{T \to 0} T(m_{22}m_{33} - m_{23}m_{32}) = 0 .$$

(30)

Condition (29) gives immediately $u_1 = c_P$. Condition (30) is complicated and we are not enabled to obtain a simple formula in order to compare with the one of Carter and Langlois [11]. For the case in which $v = 0$, we find $u_{II} = c_P/\sqrt{3}$ as expected.
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