Three-body asymptotic normalization factors for halo nuclei and their determination

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Abstract. The explicit forms of the true asymptotics of the three-body (312) bound state radial wave functions of halo nuclei $a$ both with two loosely bound external neutrons and with two charged particles, valid in different parts of the asymptotic region of the configuration space, are presented. The results for three-body asymptotic normalization factors (TBANFs) for $\alpha + n + n \to ^6\text{He}$ and $n + \alpha + \alpha \to ^9\text{Be}$ are obtained from the comparative analysis of these asymptotic forms with the specific approximate model wave functions for asymptotic regions. We use $\hbar = c = 1$ throughout the paper.

1. Introduction
In this review work, we will briefly present the true asymptotic forms derived in works [1–5] for the three-body (3+1+2) halo nucleus $a$, which is also denoted by 3+1+2 below, and the results for the three-body asymptotic normalization factors (TBANFs) for $\alpha + n + n \to ^6\text{He}$ and $n + \alpha + \alpha \to ^9\text{Be}$ are obtained from the comparative analysis of these asymptotic forms with the specific approximate model wave functions for asymptotic regions. We use $\hbar = c = 1$ throughout the paper.

2. True asymptotic forms of the three-body bound state radial wave functions for halo nuclei
Let us consider the three-body bound system ($a = (312)$) consisting of two external particles, say, particles 1 and 2, and of a core 3. Let $r$ and $\rho$ be the relative distance between the (1,2)-pair and centers of mass of third particle 3 and of the (1,2)-pair, respectively, $q$ and $p$ be the conjugate to them relative momentums as well as $r_j(q_j)$ be the relative distance (momentum) between centers of mass of the particle $j$ ($j = 1$ and 2) and of the core 3. Besides, we denote $\mu_{ij} = m_im_j/m$ and $\mu_{(ij)k} = m_km_{ij}/m$ as the reduced masses of the subsystem $(ij)$ and the $(ij)k$ system, respectively; $R = \sqrt{2\mu_{12}r^2 + 2\mu_{(12)3}\rho^2}$ as the hyperradius; $\varphi = \text{atan}(\sqrt{2\mu_{12}r}/\sqrt{2\mu_{12}r})$ as the hyperangle; $R^{(j)}(r_2, r_1) = [2\mu_{23}r_2^2 + 2\mu_{23}1(r_2 - (-1)^j\lambda_1 r_1)^2]^{1/2} = [2\mu_{13}r_1^2 + 2\mu_{(13)2}(r_1 - (-1)^j\lambda_2 r_2)^2]^{1/2}$ as a modified hyperradius with $\lambda_j = m_j/(m_j + m_3)(j = 1$ and 2); $\eta = Z_1Z_2e^2(\mu_{12}/\varepsilon_a)^{1/2}/\cos \varphi$ as the Coulomb parameter for the subsystem (12) in the three-body (312) bound system and $\varepsilon_a = m_1 + m_2 + m_3 - m$ as the binding energy for the $a$ system in the (3+2+1)-channel, where $m_{ij} = m_i + m_j$, $m = m_i + m_j + m_k$, $m_j$ and $Z_je$ are the mass and the charge of particle $j$.

The explicit forms for the true three-body asymptotics for the three-body $a = (312)$ bound state radial wave functions can be laid down by means of the following theorems:
Theorem 1. For \( r \to \infty \) and \( \rho \to \infty \) the asymptotics formula
\[
\Psi_{av}(r, \rho) \approx C_{\alpha; \nu}(\varphi) f_i(\sqrt{\varepsilon_a} R \sin^2 \varphi) f_\lambda(\sqrt{\varepsilon_a} R \cos^2 \varphi) \frac{e^{-\sqrt{\pi} \alpha}}{R^{3/2}}
\] (1)
is valid for a neutral case [1] and that
\[
\Psi_{av}(r, \rho) \approx C_{\alpha; \nu}(\varphi) f_i(\sqrt{\varepsilon_a} R \sin^2 \varphi) f_\lambda(\sqrt{\varepsilon_a} R \cos^2 \varphi)(2\sqrt{\varepsilon_a} R)^{-\eta} \frac{e^{-\sqrt{\pi} \alpha}}{R^{3/2}}
\] (2)
is valid for the bound system with two (1 and 2) the charged particles [3], where
\[
f_{L_\alpha}(x) = \sum_{n=0}^{L_\alpha} \frac{L_\alpha!}{n!(L_\alpha - n)!} \frac{1}{(2x)^n}.
\]

Theorem 2. For a neutral case the asymptotic formula
\[
\Psi_{av}(r, \rho) \approx C_{\alpha; \nu}(\varphi) g_i(\sqrt{\varepsilon_a} R \sin^2 \varphi) f_\lambda(\sqrt{\varepsilon_a} R \cos^2 \varphi) \frac{e^{-\sqrt{\pi} \alpha}}{R^{3/2}}
\] (3)
is valid for \( r \to \infty \) and \( \rho \to 0 \) (\( \rho \neq 0 \), \( \varphi \to 0 \)) and that
\[
\Psi_{av}(r, \rho) \approx C_{\alpha; \nu}(\varphi) f_i(\sqrt{\varepsilon_a} R \sin^2 \varphi) g_\lambda(\sqrt{\varepsilon_a} R \cos^2 \varphi) \frac{e^{-\sqrt{\pi} \alpha}}{R^{3/2}}
\] (4)
is valid for \( \rho \to \infty \) and \( r \to 0 \) (\( r \neq 0 \), \( \varphi \to \pi/2 \)) [5], where
\[
g_{L}(x) = \frac{1}{2} [f_{L}(x) + (-1)^{L+1} e^{2x} f_{L}(-x)] \approx x^{L+1} / (2L + 1)!! \ll 1 \text{ at } x \to 0.
\] (5)

In Eqs. (1)-(4), the TBANF \( C_{av}(\varphi) \) is determined by [1, 6]
\[
C_{\alpha; \nu}(\varphi) = (-1)^{l_1 + \lambda + 1} \frac{(m \sqrt{\varepsilon_a})^{3/2}}{2 \pi^{3/2}} W_{\alpha; \nu}(i \sqrt{\varepsilon_a} \cos \varphi, i \sqrt{\varepsilon_a} \sin \varphi),
\] (6)
where \( \nu = \{ \lambda ; L \} \) and \( W_{\alpha; \nu}(i \sqrt{\varepsilon_a} \cos \varphi, i \sqrt{\varepsilon_a} \sin \varphi) \) is the on-shell partial vertex function (OSWF) for the virtual decay \( a \to 3 + 1 + 2 \), i.e., values of the partial vertex functions \( W_{\alpha; \nu}(q, p) \) when all the particles 3, 1 and 2 are on-shell \( (\varepsilon = q^2 / 2\mu_1 + p^2 / 2\mu_2) = -\varepsilon_a \).

Theorem 3. For \( r_1 \to \infty \) and \( r_2 \to \infty \) the asymptotic formula
\[
\Psi_{av_1}(r_1, r_2) \approx \{C_{av_1}(r_1, r_2) f_i(q_1^{(1)}(r_2)) f_i(q_1^{(1)}(r_1)) \exp[-\sqrt{\varepsilon_a} R^{(1)}(r_2, r_1)] / [R^{(1)}(r_2, r_1)]^{3/2}
\]
\[-C_{av_2}(r_1, r_2) f_i(q_2^{(2)}(r_2)) f_i(q_1^{(2)}(r_1)) \exp[-\sqrt{\varepsilon_a} R^{(2)}(r_2, r_1)] / [R^{(2)}(r_2, r_1)]^{3/2} / r_2 r_1
\] (7)
is valid [2], where \( q_1^{(1)} = i 2 \mu_{(23)} \sqrt{\varepsilon_a} r_1 - (-1)^{l_1} \lambda_1 r_1 / R^{(1)} \) and \( q_2^{(2)} = i 2 \mu_{(13)} 2 \sqrt{\varepsilon_a} r_2 - (-1)^{l_2} \lambda_1 r_1 / R^{(2)} \). In (7), the TBANFs \( C_{av_1}(r_1, r_2) \) is determined by
\[
C_{av_1}(r_1, r_2) = N(-1)^{l_1 + l_2 + L} \sum_{l'_1 l'_2 L'} \xi^{(j)}_{l_1 l'_1 l'_2 L'} \sqrt{l_1 l'_1 l'_2 l_2} W_{\alpha; \nu}(q_1^{(1)}(r_1), q_2^{(2)}(r_2)) \left( \begin{array}{ccc} l_2 & l'_2 & L' \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} l_1 & l'_1 & L' \\ 0 & 0 & 0 \end{array} \right)
\]
\[
\times \left\{ \begin{array}{c} l_1 \\ l'_1 \\ l'_2 \\ l_2 \\ L' \end{array} \right\}, \quad N = (2\pi)^{-5/2} m_3 m_2 m_3 / (m^{1/2} \varepsilon_a^{1/4}).
\] (8)
where $W_{a_1; n_1}(q_1^{(j)}, q_2^{(j)})$ is the OSVF for the virtual decay $a \rightarrow 3+2+1$ but written for the variables $q_1$ and $q_2$ at $\varepsilon = -\varepsilon_a$, $\xi_{L}^{(1)} = 1$, $\xi_{L}^{(2)} = (-1)^{L'}$, the cycle and figured brackets are the $(3j)$- and $(6j)$- symbols, respectively, $\nu_l = \{l_1 l_2 L S_{12} S\}$, $l_j$ is the relative orbital angular momentum of particle $j$ and a core 3, $s_{12}$ is a total spin of the (1,2)- pair and $L_0 = 2L_a + 1$. One notes that when $m_3 \rightarrow \infty$ (a heavy particle 3 for which $m_j / m_3 << 1$), the asymptotics formula for $\Psi_{a; \nu}(r_1, r_2)$ passes to the same form for $\Psi_{a; \nu}^{(a)}(r, \rho)$ at $r \rightarrow \infty$ and $\rho \rightarrow \infty$ given by (1) since for the limit at $m_3 \rightarrow \infty$ $R^{(j)} = R$ and $C_{a; \nu}^{(j)} = C_{a; \nu}$ [2]. As is seen from Eqs.(1)–(4), (7) and (8), the true asymptotic formulas obtained for the three-body bound state radial wave functions $\Psi_{a; \nu}(r, \rho)$ and $\Psi_{a; \nu}(r_1, r_2)$ in the different asymptotic regions are expressed through the unknown TBANFs $C_{a; \nu}(\varphi)$ and $C_{a; \nu}(r_1/r_2)$ for $3+2+1 \rightarrow a$.

One note that the residue of the three-body partial amplitude $M_{a}(q, p)$ for the 1 + 2 + 3 scattering at the pole $\varepsilon = \varepsilon_a$ is expressed in terms of the OSVF $W_{a; \nu}(q, p)$ for the virtual processes $a \leftrightarrow 3 + 2 + 1$. Therefore, the OSWF and TBANF are fundamental nuclear characteristics for the three-particle (312) bound system and carry an information on the dynamics of strong two-particle interactions. Consequently, knowledge of the TBANF (or the OSVF) allows to get the information both on the three-body cluster structure of halo nuclei and on types of two-particle (cluster-cluster, cluster-nucleon and nucleon-nucleon) interactions as well as on the mechanism of the two nucleon transfer reactions (for example, the $\alpha^6\text{He}, \alpha^6\text{He}$ and $\alpha^6\text{Li}, \alpha^6\text{Li}$($E^* = 3.532$ MeV,$0^+$) reactions). Besides, a value of OSVF (or TBANF) enables one to calculate reliable values of the three-body bound state radial wave function in the asymptotic regions, which in turn are very important for reliable quantitative verification of behaviour of different approximate model three-body bound state wave functions in the asymptotic region of the configuration space. Therefore, systematic collection of data about TBANFs for different three-body halo nuclei must be extremely encouraged nowadays. For this end, at the present time a comparative analysis of the approximate model wave functions with the obtained asymptotic expressions [1–5] is usually done to extract the TBANF values.

3. Comparative analysis for the approximate model three-body bound state model wave functions for the $^6\text{He}$ and $^9\text{Be}$ nuclei

In this section, we present the result of the analysis of the asymptotic behaviour of the approximate three-body $(ann)$ and $(nna)$ radial wave functions for the ground state of $^6\text{He}$ and $^9\text{Be}$ nuclei obtained in [2,7–9] and [10], respectively, as well as the results of the TBANF values for $\alpha + n + n \rightarrow ^6\text{He}$ and $n + \alpha + \alpha \rightarrow ^9\text{Be}$ obtained in Refs. [2, 5] and in Ref. [3], respectively. The approximated radial wave functions $\Psi_{a; \nu}(r, \rho)$ were derived within the multichannel stochastic variational approach for the $^6\text{He}(ann)$ nucleus [9] and the $^9\text{Be}(nna)$ one [10], which is based on two dimensional gaussian basic expansion. At this, in [9] the Sack-Biedeharn-Breit for $\alpha\text{N}$ interaction and the hard-core Reid for the NN potential (M1-model; $\varepsilon_{He}=0.3048$ MeV,$\varepsilon_{He}^{(exp)}=0.975$ MeV) as well as the Kukulin et al potential for $\alpha\text{N}$-interaction and the hard-core Reid for the $NN$ potential (M2-model; $\varepsilon_{He}=0.1848$ MeV,$\varepsilon_{He}^{(exp)}=0.975$ MeV) were used, whereas, in [10] the Ali-Bodmer $\alpha\alpha$ potential and the Kukulin et al $\alpha\alpha$ potential (M3-model; $\varepsilon_{Be}=1.76$ MeV,$\varepsilon_{Be}^{(exp)}=1.56$ MeV) as well as the Buck-Friedrich-Wheatley $\alpha\alpha$ potential and the Kukulin et al $\alpha n$ potential (M4-Model; $\varepsilon_{Be}=2.86$ MeV,$\varepsilon_{Be}^{(exp)}=1.56$ MeV) were applied. In [2, 7, 8], the radial wave function $\Psi_{a; \nu}(r_1, r_2)$ were obtained within the Lagrange-mesh approach by using the Kanada et al for $\alpha\text{N}$-interaction and the Minnesota NN potential.

In [1, 3] and [5], the asymptotic behaviour of the approximate three-body $(ann)$ and $(nna)$ bound state radial wave functions of [9, 10] has been compared with the asymptotic formulas given by Eqs. (1)–(4) and information has been obtained about the values of the TBANFs for the $\alpha + n + n \rightarrow ^6\text{He}$ and $n + \alpha + \alpha \rightarrow ^9\text{Be}$ as a function of the hyperangle $\varphi$. To this end, for
Figure 1. Ratio of the radial wave functions $\Psi_{H_{c;v}}(r, \rho)$ for M1-model (or $\Psi_{H_{c;v1}}(r_1, r_2)$) to the asymptotic expression $\Psi_{H_{c;v}}^{(as)}(r, \rho)$ (or $\Psi_{H_{c;v1}}^{(as)}(r_1, r_2)$) in (a) (or (c), where $r_{31} = r_1$ and $r_{23} = r_2$) obtained in [5]/[2]). In (b), the result of a comparison of the numerical radial wave function of [9] for the M1-model (solid line) with the asymptotic wave function (4) (dashed line) calculated in [5] for the $^6$He($nn$) nucleus at $\varphi=5.0^o$.

For each fixed value of $\varphi$ the ratio

$$R_{MLS}(\varphi) = \frac{\Psi_{MLS}(r, \rho)}{\Psi_{as}(r, \rho)} C_{a\nu}(\varphi)$$

(9)

was considered. The similar way of testing for the asymptotic behaviour of the approximate three-body ($ann$) radial wave functions $\Psi_{H_{c;v1}}(r_1, r_2)$ was applied in [2] and information has been obtained about the values of the TBANFs $C_{a\nu}(r_1/r_2)$ ($j=1$ and 2).

The analysis performed in Refs. [5] and [3] showed that the approximate wave functions $\Psi_{a\nu}(r, \rho)$ derived in Refs. [9] and [10] for the $^6$He($nn$) and the $^9$Be($n\alpha\alpha$) nuclei, respectively,
have a correct asymptotic behaviour in the fairly narrow range of the asymptotical regions. As an example, this is illustrated in Fig. 1 a and b only for the $^6\text{He}(\alpha nn)$ nucleus. But, as is illustrated in Fig. 1c, the approximated radial three-body $^6\text{He}(\alpha nn)$ wave functions $\Psi_{\alpha,\nu_1}(r_1, r_2)$ for the $^6\text{He}$ [2, 7, 8] have a correct asymptotic behaviour in the fairly wide range of the asymptotical regions.

Figure 2. The TBANF values for $\alpha + n + n \rightarrow ^6\text{He}$ (a, c and d) and $n + \alpha + \alpha \rightarrow ^9\text{Be}$ (b) for different values of the quantum numbers. A detail caption is given in the text.

The results for the TBANF values ($C_{^6\text{He},\nu}(\varphi)$ and $C_{^9\text{Be},\nu}(\varphi)$) recommended in [5] and [3] for the $\alpha + n + n \rightarrow ^6\text{He}$ and $n + \alpha + \alpha \rightarrow ^9\text{Be}$, respectively, are displayed in Figs. 2a and 2b as a function of the hyperangle $\varphi$. In Fig. 2a, the solid and dashed lines are related to $\alpha + n + n \rightarrow ^6\text{He}$ for the potentials of the M1-and M2-models, respectively. In Fig. 2b, the solid and dashed curves correspond to $n + \alpha + \alpha \rightarrow ^9\text{Be}$ obtained for the potentials of the M3-and M4-models, respectively. There the upper and the lower curves present, respectively, the maximum and the minimum values of $C_{^9\text{Be},\nu}(\varphi)$. The TBANF values were obtained by matching the approximated radial wave functions with the asymptotical expressions given (1)–(4) for the main components of the quantum numbers $(\lambda, l, L, S)$. As is seen from this figures
the TBANF values are sensitive to the forms of the nuclear $\alpha N$ and $\alpha\alpha$ potentials used. The results for the TBANFs $C^{(j)}_{\text{He}_4}(r_1/r_2)$ recommended in [2] are displayed in Figs. 2c and d as a function of the ratio $r = r_1/r_2$. In Figs. 2c and 2d, ranges of obtained functions $C^{(j)}_{\text{Li}_4}(r_1/r_2)$ (delimited by dashed lines) and recommended value (solid lines) as a function of the coordinate ratio $r_1/r_2$ for $L = S = 0$ and $l_1 = l_2 = 0$, 1 and 2 (curves in Fig. 2c) as well as for $L = S = 1$ and $l_1 = l_2 = 1$ and 2 (curves in Fig. 2d).

4. Conclusion
The true asymptotic forms are presented for the three-body (312) bound state wave function of halo nuclei $a$ with two valence neutrons $(1=2=n)$ and with two charged particles $(1=2=\alpha)$ in respect to the Jacobi coordinates and that with two valence neutrons in the relative coordinates of the valence neutrons. These asymptotic expressions contain the unknown TBANF for $3 + 1 + 2 \rightarrow a$ being a new fundamental characteristic three-body (312) nucleus $a$ and relating to the OSVFs for the virtual decay $a \rightarrow 3 + 1 + 2$ given by (6).

The TBANF values for $\alpha + n + n \rightarrow {}^6\text{He}$ and $\alpha + \alpha + n \rightarrow {}^9\text{Be}$ recommended in [2, 3, 5] are presented and their sensitivity in the forms of the adopted two-body ($\alpha n$ and $\alpha\alpha$) nuclear potentials is also demonstrated. On the other hand, the TBANFs (OSVFs) for $\alpha + n + n \rightarrow {}^6\text{He}$ and $\alpha + \alpha + n \rightarrow {}^9\text{Be}$ can be calculated within the Faddeev’s equation in a correct manner.

The TBANFs (OSVFs) are in principle observable quantities. For instance, the TBANFs (or OSVFs) for $\alpha + n + n \rightarrow {}^6\text{He}$ can be obtained from the exchanged scattering $\alpha(^6\text{He}, \alpha)^6\text{He}$. It would be interesting to compare the present results with experiment. It is possible to get additional information about the form of the two-body potential and to verify an accuracy of the different approximated three-body wave functions as a source of reliable information on the TBANFs. Therefore, systematic collection of information about the TBANF values for different three-body halo nuclei is now encouraged in the future.

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References
[1] Blokhintsev L.D., Ubaydullaeva M.K., Yarmukhamedov R. 1999 Phys. At. Nucl. 62 1289
[2] Yarmukhamedov R., Baye D., Leclercq-Willain C. 2002 Nucl. Phys. A 705 335.
[3] Blokhintsev L.D., Ubaydullaeva M.K. and Yarmukhamedov R., 2005 Phys. At. Nucl. 68 1372.
[4] Yarmukhamedov R. and Ubaydullaeva M.K. 2008 Uzbek Math. J. 4 15.
[5] Yarmukhamedov R. and Ubaydullaeva M.K. 2009 Int. J. Mod. Phys. 18 1561.
[6] Mukhamedzhano A.M., Ubaydullaeva M.K., and Yarmukhamedov R. 1993 Theor. and Math. Phys. 94 315.
[7] Baye D., Kruglanski M., Vincke M. 1994 Nucl. Phys. A 573 431.
[8] Baye D. 1997 Nucl. Phys. A 627 305.
[9] Kukulin V.I., Pomerantsev V.N., Razikov Kh.D., Voronchev V.T., Ryzhikh G.G. 1995 Nucl. Phys. A 586 151.
[10] Voronchev V.T., Kukulin V.I., Pomerantsev V.N., and Ryzhikh G.G. 1995 Few-body System. 18 191.