On the canonical forms of the multi-dimensional averaged brackets.

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Abstract

We consider here special Poisson brackets given by the “averaging” of local multi-dimensional Poisson brackets in the Whitham method. For the brackets of this kind it is natural to ask about their canonical forms, which can be obtained after transformations preserving the “physical meaning” of the field variables. We show here that the average bracket can always be written in the canonical form after a transformation of “Hydrodynamic Type” in the case of absence of annihilators of initial bracket. However, in general case the situation is more complicated. As we show here, in more general case the averaged bracket can be transformed to a “pseudo-canonical” form under some special (“physical”) requirements on the initial bracket.

1 Introduction.

We will consider here the Poisson brackets obtained by the “averaging” of local multi-dimensional Poisson brackets

\[ \{ \varphi^i(x), \varphi^j(y) \} = \sum_{l_1,\ldots,l_d} B^{ij}_{(l_1,\ldots,l_d)}(\varphi, \varphi_x, \ldots) \delta^{(l_1)}(x^1 - y^1) \cdots \delta^{(l_d)}(x^d - y^d) \equiv \]

\[ \equiv \sum_{l_1,\ldots,l_d} B^{ij}_{(l_1,\ldots,l_d)}(\varphi, \varphi_x, \ldots) \delta_{l_1x^1 \ldots l_dx^d} (x - y) , \quad (l_1, \ldots, l_d \geq 0) \] (1.1)

on the families of \( m \)-phase quasiperiodic solutions of local Hamiltonian systems

\[ \varphi^i_t = F^i(\varphi, \varphi_x, \varphi_{xx}, \ldots) \equiv F^i(\varphi, \varphi_{x^1}, \ldots, \varphi_{x^d}, \ldots) , \quad i = 1, \ldots, n \] (1.2)

which are represented in the following general form

\[ \varphi^i(x, t) = \varphi^i_{[a, \theta_0]}(x, t) = \Phi^i(k_1(a)x^1 + \ldots + k_d(a)x^d + \omega(a)t + \theta_0, a) \] (1.3)

with some smooth \( 2\pi \)-periodic in each \( \theta^a \) functions \( \Phi^i(\theta, a) \).

Thus, we assume that \( x = (x^1, \ldots, x^d) \), \( y = (y^1, \ldots, y^d) \) represent points of the Euclidean space \( \mathbb{R}^d \) and the expression (1.1) defines a skew-symmetric Hamiltonian operator on the space of smooth functions

\[ \varphi(x) = (\varphi^1(x), \ldots, \varphi^n(x)) , \]
satisfying the Jacobi identity.

The notations \( \delta^{(l)}(x - y) \) mean here the higher derivatives of the delta-function and we assume that the sum in (1.1) contains a finite number of terms.

We will call brackets (1.1) general local field-theoretic Poisson brackets in \( \mathbb{R}^d \) and assume that system (1.2) represents a Hamiltonian system generated by a local Hamiltonian functional

\[
H = \int P_H (\varphi, \varphi_x, \varphi_{xx}, \ldots) \ d^d x
\]

according to bracket (1.1).

We assume that the family (1.3) is defined with the aid of a smooth finite-parametric set \( \Lambda \) of 2\( \pi \)-periodic in each \( \theta^\alpha \) functions

\[
\Phi^i (\theta + \theta_0, a) = \Phi^i (\theta^1 + \theta_0^1, \ldots, \theta^m + \theta_0^m, a^1, \ldots, a^N)
\]

with a smooth dependence of the wave numbers \( k^i_q(a) = (k^1_q(a), \ldots, k^m_q(a)) \) and frequencies \( \omega(a) = (\omega^1(a), \ldots, \omega^m(a)) \) on the parameters \( a = (a^1, \ldots, a^N) \). All the functions \( \Phi^i(\theta, a) \) should satisfy the system

\[
\omega^\alpha \Phi^i_{\theta^\alpha} - F^i \left( \Phi, k^1_{\theta^1} \Phi_{\theta^1}, \ldots, k^d_{\theta^d} \Phi_{\theta^d}, \ldots \right) = 0
\]

The parameters \( \theta_0^\alpha \) represent the initial phase shifts of solutions (1.3) and take by definition all possible real values on the family \( \Lambda \). We assume also that the values of the parameters \( a \) do not change under the initial phase shifts. Let us denote by \( \Lambda \) the family (1.3) of the functions \( \varphi^i(x, t) \) corresponding to the family \( \Lambda \).

The procedure of averaging of a Poisson bracket is closely connected with the Whitham averaging method (39, 40, 41). For this reason we will put here additional requirements of regularity and completeness on the family \( \Lambda \) which we formulate below.

Let us say first that we will everywhere consider here the generic situation where the values \( (k_1, \ldots, k_d, \omega) \) represent independent parameters on the full family of \( m \)-phase solutions of system (1.2). Thus, we assume that the number of real parameters \( (a^1, \ldots, a^N) \) is equal to \( md + m + s, \ s \geq 0 \). In particular, the parameters \( (a^1, \ldots, a^N) \) can be locally chosen in the form \( a = (k_1, \ldots, k_d, \omega, n) \) where \( (k_1, \ldots, k_d, \omega) \) represent the wave numbers and the frequencies of the \( m \)-phase solutions and \( n = (n^1, \ldots, n^s) \) are some additional parameters (if any).

Let us consider now linear operators \( \hat{L}_{j[a, \theta_0]}^i = \hat{L}_{j[k_1, \ldots, k_d, \omega, n, \theta_0]}^i \) given by the linearization of system (1.5) on the corresponding solutions \( \Phi(\theta + \theta_0, k_1, \ldots, k_d, \omega, n) \). It’s not difficult to see that the functions \( \Phi_{\theta^\alpha}(\theta + \theta_0, k_1, \ldots, k_d, \omega, n), \ \alpha = 1, \ldots, m, \) and \( \Phi_{\omega^l}(\theta + \theta_0, k_1, \ldots, k_d, \omega, n), \ \l = 1, \ldots, s, \) represent kernel vectors of the operators \( \hat{L}_{j[k_1, \ldots, k_d, \omega, n, \theta_0]}^i \) on the space of \( 2\pi \)-periodic in each \( \theta^\alpha \) functions which depend smoothly on all the parameters \( (k_1, \ldots, k_d, \omega, n, \theta_0) \). Let us put now the following requirements on the operators \( \hat{L}_{j[k_1, \ldots, k_d, \omega, n, \theta_0]}^i \) on the family \( \Lambda \):

1) We require that the vectors \( \Phi_{\theta^\alpha}(\theta + \theta_0, k_1, \ldots, k_d, \omega, n), \ \Phi_{\omega^l}(\theta + \theta_0, k_1, \ldots, k_d, \omega, n) \) are linearly independent and represent the maximal linearly independent set among the kernel vectors of the operator \( \hat{L}_{j[k_1, \ldots, k_d, \omega, n, \theta_0]}^i \) on the space of \( 2\pi \)-periodic in each \( \theta^\alpha \) functions smoothly depending on the parameters \( (k_1, \ldots, k_d, \omega, n) \).

2) The operators \( \hat{L}_{j[k_1, \ldots, k_d, \omega, n, \theta_0]}^i \) have exactly \( m + s \) linearly independent regular left eigenvectors \( \kappa_{[k_1, \ldots, k_d, \omega, n]}^{(q)}(\theta + \theta_0), \ q = 1, \ldots, m + s, \) on the space of \( 2\pi \)-periodic in each \( \theta^\alpha \) functions, corresponding to the zero eigenvalue and depending smoothly on the parameters \( (k_1, \ldots, k_d, \omega, n) \).
Definition 1.1.

Under all the requirements formulated above we will call the corresponding family $\Lambda$ a complete regular family of $m$-phase solutions of system (1.2).

It is well known that the Whitham approach gives a description of the slowly modulated $m$-phase solutions of nonlinear PDE’s. The Whitham solutions represent asymptotic solutions of nonlinear systems with the main part having the form

$$\varphi_{(0)}(x, t, \theta) = \Phi\left(\frac{S(X, T)}{\epsilon} + \theta_{(0)}(X, T) + \theta, S_{X_1}, \ldots, S_{X^d}, S_T, n(X, T)\right) \quad (1.6)$$

where $X = \epsilon x$, $T = \epsilon t$, $\epsilon \to 0$, are the slow spatial and time variables and the function

$$S(X, T) = (S^1(X, T), \ldots, S^m(X, T))$$

represents the “modulated phase” of the solution. Thus, the main part of the Whitham solution represents an $m$-phase solution of the nonlinear system with the slow modulated parameters $a(X, T)$ and a rapidly changing phase. We have also the natural connection

$$S^\alpha_T = \omega^\alpha(X, T), \quad S^\alpha_{X_q} = k^\alpha_{qX}(X, T) \quad (1.7)$$

between the derivatives of the modulated phase and the parameters $\omega(X, T)$ and $k_q(X, T)$.

Relations (1.7) give the natural constraints

$$k^\alpha_{qT} = \omega_{X_q}^\alpha, \quad k^\alpha_{qX_p} = k^\alpha_{pX_q}$$

on the functions $\omega(X, T)$ and $k_q(X, T)$, which can be considered as the first part of the Whitham system on the parameters $a(X, T)$.

The second part of the Whitham system is defined usually by the requirement of existence of a bounded next correction to the initial approximation (1.6) and can be defined in different ways which are usually equivalent to each other (see e.g. [39, 40, 41, 25, 18, 4, 19, 20, 11, 12, 22]).

In our scheme we will define the second part of the Whitham system for a complete regular family $\Lambda$ of $m$-phase solutions of (1.2) as the orthogonality at every $X$ and $T$ of all the regular left eigen-vectors

$$\kappa^{(q)}_{p[S_{X_1}, \ldots, S_{X^d}, S_T, n(X, T)]} \left(\frac{S(X, T)}{\epsilon} + \theta_{(0)}(X, T) + \theta\right), \quad q = 1, \ldots, m + s$$

to the first $\epsilon$-discrepancy $f_1(\theta, X, T)$, obtained after the substitution of the main approximation (1.6) into the system

$$\epsilon \varphi_T = \mathcal{F}^i(\varphi, \epsilon \varphi_X, \epsilon^2 \varphi_{XX}, \ldots)$$

It is well known that the full Whitham system, defined in one of the standard ways, does not put any restrictions on the variables $\theta_{0}(X, T)$ and represents a system of PDE’s just on the parameters $a(X, T)$ (see e.g. [39, 40, 41, 25]). In particular, it is also not difficult to show that the orthogonality conditions

$$\int_0^{2\pi} \ldots \int_0^{2\pi} \kappa^{(q)}_{[S_{X_1}, \ldots, S_{X^d}, S_T, n(X, T)]i} \left(\frac{S(X, T)}{\epsilon} + \theta_{(0)}(X, T) + \theta\right) f_1^i(\theta, X, T) \frac{d^n\theta}{(2\pi)^m} = 0 \quad (1.8)$$

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defined for any complete regular family $\Lambda$, possesses the same property (see e.g. [26, 27]). In general, relations (1.8) can be written as a system of $m + s$ quasilinear equations

$$
P^{(q)}_{\alpha}(S_x, S_T, n) S^\alpha_{TT} + Q^{(q)p}_{\alpha}(S_x, S_T, n) S^\alpha_{X_T} + R^{(q)pk}_{\alpha}(S_x, S_T, n) S^\alpha_{X^p X_k} + \nonumber$$

$$\nonumber + V^{(q)}_i(S_x, S_T, n) n^i_T + W^{(q)p}_i(S_x, S_T, n) n^i_{X^p} = 0, \quad q = 1, \ldots, m + s$$

with some smooth functions $P^{(q)}_{\alpha}, Q^{(q)p}_{\alpha}, R^{(q)pk}_{\alpha}, V^{(q)}_i, W^{(q)p}_i$.

Let us say here that for the single-phase case ($m = 1$) the set of the “regular” left eigen-vectors $\kappa^{(q)}_{[k_1, \ldots, k_d, \omega, n]}(\theta + \theta_0), \quad q = 1, \ldots, s + 1$, represents usually the full set of linearly independent left eigen-vectors of the operators $\hat{L}^i_j[k_1, \ldots, k_d, \omega, n, \theta_0]$, corresponding to the zero eigen-value, for all the values of $(k_1, \ldots, k_d, \omega, n, \theta_0)$ on a complete regular family $\Lambda$. However, for the multi-phase case ($m > 1$) the situation is usually more complicated and “irregular” left eigen-vectors of $\hat{L}^i_j[k_1, \ldots, k_d, \omega, n, \theta_0]$, corresponding to the zero eigen-value, also arise for special values of parameters $(k_1, \ldots, k_d, \omega, n)$. As a result, the corrections to the main approximation (1.6) of the Whitham solution for the multi-phase case have usually rather different form in comparison with the case $m = 1$ (see e.g. [6, 7, 8]).

Let us say, however, that the regular Whitham system still plays the central role in the description of the slow-modulated $m$-phase solutions both in the cases $m = 1$ and $m > 1$. Let us give here also just some incomplete list of classical papers devoted to different questions connected with the Whitham approach: [1, 4, 5, 6, 7, 8, 9, 11, 12, 13, 18, 21, 22, 23, 25, 32, 34, 35, 39, 40, 41].

One of the most elegant ways of constructing the Whitham system was suggested by B.A. Dubrovin and S.P. Novikov, who introduced the concept of the Hamiltonian structure of Hydrodynamic Type. In general, the Dubrovin-Novikov bracket in $\mathbb{R}^d$ can be written in the following local form

$$
\varphi^j_i = \hat{j}^{ij} \frac{\delta H}{\delta \varphi^j(x)}
$$

where $\hat{j}^{ij}$ is the Hamiltonian operator

$$
\hat{j}^{ij} = \sum_{l_1, \ldots, l_d} B^{ij}_{l_1, \ldots, l_d}(\varphi, \varphi_x, \ldots) \left( \frac{d}{dx^1} \right)^{l_1} \cdots \left( \frac{d}{dx^d} \right)^{l_d},
$$

defined by the Poisson bracket (1.1), and $H$ is the Hamiltonian functional having the form (1.4).

The Hamiltonian theory of the Whitham equations was started by B.A. Dubrovin and S.P. Novikov, who introduced the concept of the Hamiltonian structure of Hydrodynamic Type. In general, the Dubrovin-Novikov bracket in $\mathbb{R}^d$ can be written in the following local form

$$
\{U^\nu(x), U^\mu(y)\} = g^{\nu \mu l}(U(x)) \delta_{X^l}(X - Y) + b^{\nu \mu l}_\lambda(U(x)) U^\lambda_{X^l} \delta(X - Y)
$$

(summation over repeated indices).

\footnote{The definition of $f_1$ in [26, 27] differs by a phase shift from that used here which is included there also in the corresponding orthogonality conditions.}
The general theory of the brackets (1.9) is rather nontrivial. Rather deep results on the classification of brackets (1.9) were obtained in [10, 30, 31] where the full description of brackets (1.9), satisfying special non-degeneracy conditions, was presented. However, there are many interesting examples where a nontrivial structure of a system is defined by a non-generic bracket (1.9) (see e.g. [16, 17]). In general, we can say that the full theory of the brackets (1.9) represents an important branch of the theory of the Poisson brackets and is still waiting for its final completion.

A special class of the Dubrovin-Novikov brackets (1.9) is given by the one-dimensional brackets of Hydrodynamic Type. The brackets (1.9) have in this case the following general form

\[
\{U^\nu(X), U^\mu(Y)\} = g^{\nu\mu}(U(X)) \delta'(X - Y) + b^\nu_{\lambda\mu}(U(X)) U^\lambda_X \delta(X - Y) , \quad \nu, \mu = 1, \ldots, N
\]

and are closely connected with Differential Geometry. Thus, it can be proved ([9, 10, 11, 12]) that the expression (1.10) with non-degenerate tensor \(g^{\nu\mu}(U)\) defines a Poisson bracket on the space of fields \(U(X)\) if and only if the tensor \(g^{\nu\mu}(U)\) defines a flat pseudo-Riemannian metric with upper indices on the space of \(U\) while the values \(\Gamma^\nu_{\mu\gamma} = -g_{\mu\lambda}b^\lambda_{\nu\gamma}\) represent the corresponding Christoffel symbols \((g_{\nu\tau}(U)g^{\tau\mu}(U) = \delta^\mu_{\nu})\).

As a consequence, we can claim in fact that every Poisson bracket (1.10) with non-degenerate tensor \(g^{\nu\mu}(U)\) can be locally written in the constant form

\[
\{n^\nu(X), n^\mu(Y)\} = \epsilon^\nu \delta^{\nu\mu} \delta'(X - Y) , \quad \epsilon^\nu = \pm 1
\]

after the transition to the flat coordinates \(n^\nu = n^\nu(U)\) of the metric \(g_{\nu\mu}(U)\).

It’s not difficult to see also, that the functionals

\[
N^\nu = \int_{-\infty}^{+\infty} n^\nu(X) \, dX
\]

represent then the annihilators of the bracket (1.10) while the functional

\[
P = \int_{-\infty}^{+\infty} \frac{1}{2} \sum_{\nu=1}^{N} \epsilon^\nu (n^\nu)^2(X) \, dX
\]

gives the momentum operator for the bracket (1.10) ([9, 10, 11, 12]).

The statement, formulated above, plays in fact the role of an analog of the Darboux Theorem for the brackets (1.10) with non-degenerate tensor \(g^{\nu\mu}(U)\). Following B.A. Dubrovin and S.P. Novikov, we will call the form (1.11) of the bracket (1.10) the Canonical Form of a non-degenerate one-dimensional Poisson bracket of Hydrodynamic type. Let us note here also, that the theory of the brackets (1.10) with degenerate tensor \(g_{\nu\mu}(U)\) can be also formulated in a nice Differential Geometric form which we will not consider here in detail ([3]).

The theory of the Poisson brackets of Hydrodynamic Type gives the basement for the theory of integrability of multi-component one-dimensional Hydrodynamic Type systems

\[
U_T^\nu = V_{\nu\mu}^\nu(U) U^\mu_X , \quad \nu = 1, \ldots, N
\]

Thus, according to conjecture of S.P. Novikov, every diagonalizable system (1.12) which is Hamiltonian with respect to some bracket (1.10) with the Hamiltonian of Hydrodynamic Type

\[
H = \int_{-\infty}^{+\infty} h(U) \, dX
\]
can be integrated.

The conjecture of S.P. Novikov was proved by S.P. Tsarev \((37, 38)\) who suggested a method for solving of diagonal Hamiltonian systems

\[
U^\nu_T = V^\nu(U) U^\nu_X, \quad \nu = 1, \ldots, N
\]  \((1.13)\)

The method of Tsarev can be applied in fact to a wider class of systems \((1.13)\) which were called by S.P. Tsarev semi-Hamiltonian. In particular, the class of the semi-Hamiltonian systems contains the diagonal systems, Hamiltonian with respect to the weakly nonlocal Poisson brackets of Hydrodynamic Type - the Mokhov-Ferapontov bracket \((29)\) and more general Ferapontov brackets \((14, 15)\), which appeared as generalizations of the brackets of B.A. Dubrovin and S.P. Novikov. The diagonal semi-Hamiltonian systems represent the widest class of integrable one-dimensional systems of Hydrodynamic Type.

B.A. Dubrovin and S.P. Novikov suggested also a method of averaging of local field-theoretic Hamiltonian structures for the case of one spatial dimension.

The Dubrovin-Novikov procedure is based on the existence of \(N\) local integrals of system \((1.2)\)

\[
I^\nu = \int P^\nu(\varphi, \varphi_x, \ldots) dx
\]

which commute with the Hamiltonian \(H\) and with each other

\[
\{I^\nu, H\} = 0, \quad \{I^\nu, I^\mu\} = 0
\]  \((1.14)\)

according to the bracket \((1.1)\) \((d = 1)\). It is supposed also that the set of parameters \(a\) on the family \(\Lambda\) can be chosen in the form \((a^1, \ldots, a^N) = (U^1, \ldots, U^N)\), where

\[
U^\nu = \langle P^\nu \rangle \equiv \int_0^{2\pi} \ldots \int_0^{2\pi} P^\nu(\Phi, k^\alpha \phi^\alpha, \ldots) \frac{d^m \theta}{(2\pi)^m}
\]

represent the values of the densities \(P^\nu(\varphi, \varphi_x, \ldots)\) on \(\Lambda\), averaged over the angle (phase) variables.

We can write for the time evolution of the densities \(P^\nu(\varphi, \varphi_x, \ldots)\) according to system \((1.2)\):

\[
P^\nu_t(\varphi, \varphi_x, \ldots) \equiv Q^\nu_x(\varphi, \varphi_x, \ldots),
\]

where \(Q^\nu(\varphi, \varphi_x, \ldots)\) are some smooth functions of \(\varphi\) and its spatial derivatives. It is convenient to write also the Whitham system as a system of conservation laws

\[
\langle P^\nu \rangle_T = \langle Q^\nu \rangle_X, \quad \nu = 1, \ldots, N
\]  \((1.15)\)

using the functions \(P^\nu(\varphi, \varphi_x, \ldots)\) and \(Q^\nu(\varphi, \varphi_x, \ldots)\).

The procedure of construction of the Dubrovin-Novikov bracket for system \((1.15)\) can be described in the following way:

Let us calculate the pairwise Poisson brackets of the densities \(P^\nu(x), P^\mu(y)\), which can be represented in the form:

\[
\{P^\nu(x), P^\mu(y)\} = \sum_{k \geq 0} A^\nu_\mu(k)(x-y)
\]

which some smooth functions \(A^\nu_\mu(k)(\varphi, \varphi_x, \ldots)\).
According to conditions (1.14) we can write the relations

\[ A_{0}^{\mu\nu}(\varphi, \varphi_{x}, \ldots) \equiv \partial_{x}Q^{\mu\nu}(\varphi, \varphi_{x}, \ldots) \]

for some functions \( Q^{\mu\nu}(\varphi, \varphi_{x}, \ldots) \).

Let us put now \( U^{\mu} = \langle P^{\mu} \rangle \) and define the Poisson bracket

\[
\{U^{\mu}(X), U^{\nu}(Y)\} = \langle A_{1}^{\mu\nu}(U)\rangle \delta'(X - Y) + \frac{\partial Q^{\mu\nu}}{\partial U^\gamma} U^\gamma_X \delta(X - Y) \quad (1.16)
\]
on the space of functions \( U(X) \).

System (1.15) can be defined now as a Hamiltonian system with respect to the bracket (1.16) with the Hamiltonian functional

\[ H_{av} = \int_{-\infty}^{+\infty} \langle P_{H}(U(X)) \rangle \, dX \]

Let us say that the complete justification of the Dubrovin-Novikov procedure represents in fact a nontrivial question. Let us give here the reference on paper [26] where some review of this question and the most detailed consideration of the justification problem were presented. In particular, we can state that the Dubrovin-Novikov procedure is well justified for a complete regular family \( \Lambda \) having certain regular Hamiltonian properties [26].

In the case of several spatial dimensions \((d > 1)\) the procedure of bracket averaging should be actually modified, which is connected mostly with a special role of the variables \( S(X) \) revealed in this situation. Let us formulate here the corresponding procedure and the conditions of its applicability according to the scheme proposed in [27] [28].

Let us consider a complete regular family \( \Lambda \) of \( m \)-phase solutions of system (1.2) parametrized by the \( m(d + 1) + s \) parameters \((k_{1}, \ldots, k_{d}, \omega, n)\) and \( m \) initial phase shifts \( \theta_{0} \).

**Definition 1.2.**

We will call the complete regular family \( \Lambda \) a complete Hamiltonian family of \( m \)-phase solutions of (1.2) if it satisfies the following requirements:

1) The bracket (1.1) has at every point \((k_{1}, \ldots, k_{d}, \omega, n, \theta_{0})\) of \( \Lambda \) the same number \( s' \) of “annihilators” defined by linearly independent solutions \( v^{(k)}_{[a, \theta_{0}]}(x) \) of the equation

\[
\sum_{l_{1}, \ldots, l_{d}} B^{ij}_{(l_{1}, \ldots, l_{d})}(\varphi_{[a, \theta_{0}]}, \varphi_{[a, \theta_{0}]}x_{i}, \ldots) v^{(k)}_{[a, \theta_{0}], l_{1}, \ldots, l_{d} x^{i}}(x) = 0 , \quad (1.17)
\]
such that all the functions \( v^{(k)}_{[a, \theta_{0}], i}(x) \) can be represented in the form

\[
v^{(k)}_{[a, \theta_{0}], i}(x) = \hat{v}^{(k)}_{[a, \theta_{0}], i}(k_{1} x^{1} + \ldots + k_{d} x^{d}) \quad (1.18)
\]

for some smooth \( 2\pi \)-periodic in each \( \theta_{i} \) functions \( \hat{v}^{(k)}_{[a, \theta_{0}], i}(\theta) \).

2) For the derivatives \( \varphi_{\omega_{\alpha}}, \varphi_{n_{i}} \) of the functions \( \varphi_{[a, \theta_{0}]}(x) = \varphi_{[k_{1}, \ldots, k_{d}, \omega, n, \theta_{0}]}(x) \) we have the relations

\[
\text{rank } \begin{pmatrix} (\varphi_{\omega_{\alpha}} \cdot v^{(k)}) \\ (\varphi_{n_{i}} \cdot v^{(k)}) \end{pmatrix} = s' \]
\[ (\varphi_{\omega_\alpha} \cdot v^{(k)}) \equiv \lim_{K \to \infty} \frac{1}{(2K)^d} \int_{-K}^{K} \int_{-K}^{K} \varphi_{\omega_{\alpha}}^{i}(x) v^{(k)}_i(x) \, d^d x \]

\[ (\varphi_{n'} \cdot v^{(k)}) \equiv \lim_{K \to \infty} \frac{1}{(2K)^d} \int_{-K}^{K} \int_{-K}^{K} \varphi_{n'}^{i}(x) v^{(k)}_i(x) \, d^d x \]

represent the convolutions of the variation derivatives of annihilators with the tangent vectors \( \varphi_{\omega_{\alpha}} \), \( \varphi_{n'} \).

It is convenient to introduce here also the families \( \Lambda_{k_1, \ldots, k_d} \) representing the functions \( \varphi_{[k_1, \ldots, k_d, \omega, n, \theta_0]} \) with the fixed parameters \( (k_1, \ldots, k_d) \). Following [28], we will give here the following definition:

**Definition 1.3.**

We say that a complete Hamiltonian family \( \Lambda \) is equipped with a minimal set of commuting integrals if there exist \( m + s \) functionals \( I^\gamma \), \( \gamma = 1, \ldots, m + s \), having the form

\[ I^\gamma = \int P^\gamma (\varphi, \varphi_x, \varphi_{xx}, \ldots) \, d^d x \] (1.19)

such that:

1) The functionals \( I^\gamma \) commute with the Hamiltonian functional (1.4) and with each other according to the bracket (1.7):

\[ \{ I^\gamma, H \} = 0 , \quad \{ I^\gamma, I^\rho \} = 0 , \] (1.20)

2) The values \( U^\gamma \):

\[ U^\gamma = \lim_{K \to \infty} \frac{1}{(2K)^d} \int_{-K}^{K} \int_{-K}^{K} P^\gamma (\varphi[a, \theta_0], \varphi[a, \theta_0]x, \ldots) \, d^d x \]

of the functionals \( I^\gamma \) on \( \Lambda \) represent independent parameters on every family \( \Lambda_{k_1, \ldots, k_d} \), such that the total set of parameters on \( \Lambda \) can be represented in the form \( (k_1, \ldots, k_d, U^1, \ldots, U^{m+s}, \theta_0) \);

3) The Hamiltonian flows, generated by the functionals \( I^\gamma \), leave invariant the family \( \Lambda \) and the values of all the parameters \( (k_1, \ldots, k_d, U) \) of the functions \( \varphi_{[k_1, \ldots, k_d, U, \theta_0]}(x) \) and generate the linear time evolution of the phase shifts \( \theta_0 \) with constant frequencies \( \omega^\gamma = (\omega^1, \ldots, \omega^{m+s}) \), such that

\[ \text{rk} \| \omega^\gamma (k_1, \ldots, k_d, U) \| = m \] (1.21)

everywhere on \( \Lambda \);

4) At every point \( (k_1, \ldots, k_d, U, \theta_0) \) of \( \Lambda \) the linear space, generated by the variation derivatives \( \delta I^\gamma / \delta \varphi^i(x) \), contains the variation derivatives \( v^{(k)}_{[k_1, \ldots, k_d, U, \theta_0]}(x) \) of all the annihilators of bracket (1.7) introduced above. In other words, at every point \( (k_1, \ldots, k_d, U, \theta_0) \) we can write for a complete set \( \{ v^{(k)}_{[k_1, \ldots, k_d, U, \theta_0]}(x) \} \) of linearly independent quasiperiodic solutions of (1.17) the relations:

\[ v^{(k)}_{[k_1, \ldots, k_d, U, \theta_0]}(x) = \sum_{\gamma=1}^{m+s} \gamma^k_\gamma (k_1, \ldots, k_d, U) \left. \frac{\delta I^\gamma}{\delta \varphi^i(x)} \right|_\Lambda \]
with some functions \( \gamma^k_{\gamma}(k_1, \ldots, k_d, U) \) on \( \Lambda. \)

It should be noted here that the definition given above implies in fact that the number of the additional parameters \((n^1, \ldots, n^s)\) on \( \Lambda \) is equal to the number of annihilators of the bracket (1.1). So, in this scheme the additional parameters \((n^1, \ldots, n^s)\) are directly connected with the annihilators of the Poisson bracket.

Like in the one-dimensional case, we can write the following relations for the time evolution of the densities \( P^\gamma(\varphi, \varphi_x, \ldots) \):

\[
P^\gamma_t(\varphi, \varphi_x, \ldots) = Q^\gamma_1(\varphi, \varphi_x, \ldots) + \ldots + Q^\gamma_d(\varphi, \varphi_x, \ldots)
\]

Let us consider now the modulation equations for a complete Hamiltonian family \( \Lambda \) equipped with a minimal set of commuting integrals \( \{I^1, \ldots, I^{m+s}\} \). It is convenient to choose now the parameters of the slowly modulated solutions of (1.2) in the form

\[
(S(X, T), U(X, T)) = (S^1(X, T), \ldots, S^m(X, T), U^1(X, T), \ldots, U^{m+s}(X, T))
\]
such that the parameters \( k_q(X, T) \) are defined by the relations \( k_q = S_{X_q} \ (X = \epsilon x, \ T = \epsilon t) \).

The regular Whitham system can be written now in the following form

\[
S^\alpha_T = \omega^\alpha (S^1, \ldots, S^d, U), \quad \alpha = 1, \ldots, m, \tag{1.22}
\]

\[
U^\gamma_T = \{Q^\gamma\}_1 \ldots + \{Q^\gamma_d\}_X, \quad \gamma = 1, \ldots, m + s,
\]

which is equivalent to the system defined by (1.7)-(1.8) (28).

The procedure of averaging of the Poisson bracket (1.1) represents a modification of the Dubrovin-Novikov procedure and can be formulated in the following way (27 28):

Like in the one-dimensional case, let us calculate the pairwise Poisson brackets of the densities \( P^\gamma(x), P^\rho(y) \), which can be represented now in the form

\[
\{P^\gamma(x), P^\rho(y)\} = \sum_{l_1, \ldots, l_d} A^\rho_{l_1 \ldots l_d}(\varphi, \varphi_x, \ldots) \delta^{(l_1)}(x^1 - y^1) \ldots \delta^{(l_d)}(x^d - y^d)
\]

\((l_1, \ldots, l_d \geq 0)\).

In the same way, we can write here the relations

\[
A^\rho_{0 \ldots 0}(\varphi, \varphi_x, \ldots) \equiv \partial_x^1 Q^\gamma_1(\varphi, \varphi_x, \ldots) + \ldots + \partial_x^d Q^\gamma_d(\varphi, \varphi_x, \ldots)
\]

for some functions \((Q^\gamma_1(\varphi, \varphi_x, \ldots), \ldots, Q^\gamma_d(\varphi, \varphi_x, \ldots))\).

We define the averaged Poisson bracket \( \{\ldots, \ldots\}^{AV} \) on the space of fields \((S(X), U(X))\) by the following equalities:

\[
\{S^\alpha(X), S^\beta(Y)\}^{AV} = 0, \quad \alpha, \beta = 1, \ldots, m,
\]

\[
\{S^\alpha(X), U^\gamma(Y)\}^{AV} = \omega^\alpha (S^1, \ldots, S^d, U(X)) \delta(X - Y),
\]

\[
\{U^\gamma(X), U^\rho(Y)\}^{AV} = \{A^\rho_{0 \ldots 0}\}(S^1, \ldots, S^d, U(X)) \delta(X - Y) + \ldots + \{A^\rho_{0 \ldots 0}\}(S^1, \ldots, S^d, U(X)) \delta(X - Y) + \]

\[
+ [\{Q^\gamma p\}(S^1, \ldots, S^d, U(X))]_{X_p} \delta(X - Y), \quad \gamma, \rho = 1, \ldots, m + s.
\]
The detailed consideration and justification of the above procedure for a complete Hamiltonian family $\Lambda$ equipped with a minimal set of commuting integrals can be found in [28]. Let us say, that the same procedure was considered also under some other requirements on the family of $m$-phase solutions of (1.2) in [27]. Let us formulate here also a theorem claiming the invariance of the procedure of the bracket averaging.

**Theorem 1.1 (28).**

Let a family $\Lambda$ represent a complete Hamiltonian family of $m$-phase solutions of system (1.2) equipped with a minimal set of commuting integrals $\{I^1, \ldots, I^{m+s}\}$. Let the set $\{I^1, \ldots, I^{m+s}\}$ represent another minimal set of commuting integrals for the family $\Lambda$, satisfying all the requirements of Definition 1.3.

Then the Poisson brackets (1.23), obtained with the aid of the sets $\{I^1, \ldots, I^{m+s}\}$ and $\{I^1, \ldots, I^{m+s}\}$, coincide with each other.

In other words, under the requirements of Theorem 1.1 we can claim, that the expressions (1.23), obtained with the aid of the sets $\{I^1, \ldots, I^{m+s}\}$ and $\{I^1, \ldots, I^{m+s}\}$, transform into each other under the coordinate transformation

$$(S^1(X), \ldots, S^m(X), U^1(X), \ldots, U^{m+s}(X)) \rightarrow (S^1(X), \ldots, S^m(X), U^n(X), \ldots, U^{m+s}(X))$$

where $(U^1, \ldots, U^{m+s})$ and $(U^n, \ldots, U^{m+s})$ are the parameters on the family $\Lambda$, corresponding to the sets $\{I^1, \ldots, I^{m+s}\}$ and $\{I^1, \ldots, I^{m+s}\}$, respectively.

The main purpose of this article is to study the canonical forms of the brackets (1.23) so we could have an analog of the Darboux Theorem for the averaged Poisson brackets in the multi-dimensional case.

Let us say, however, that we will be interested here just in the special coordinate transformations, preserving the “physical” meaning of the fields $(S^1(X), \ldots, S^m(X))$ and $(U^1(X), \ldots, U^{m+s}(X))$. Thus, we will always keep here the variables $(S^1(X), \ldots, S^m(X))$, representing the “phase” functions, as the first part of canonical variables for the bracket (1.23). So, the transformations we consider here will have in fact the form (1.24) written above. Besides that, we will always assume here that the variables $(U^1(X), \ldots, U^{m+s}(X))$ represent some densities of “Hydrodynamic Type”, which means in fact that transformations $U(X) \rightarrow U'(X)$ should have the “Hydrodynamic” form

$$U^{\gamma}(X) = U^{\gamma}(S_X, \ldots, S_{X^d}, U(X))$$

As we will see in Chapter 2, any bracket (1.23) with the additional condition (1.21) can be transformed to the standard canonical form by a transformation (1.24) - (1.25) in the special case $s = 0$. This case corresponds in fact to the absence of annihilators of the bracket (1.1) on the space of the quasiperiodic functions and allows always the construction of the second part of canonical variables $(Q^1(X), \ldots, Q^m(X))$, conjugated to the variables $(S^1(X), \ldots, S^m(X))$.

In Chapter 3 we consider more complicated case of the presence of additional parameters $(n^1, \ldots, n^s)$ connected with the presence of annihilators of the bracket (1.1). As we will see, the
situation is more complicated in this case. We suggest here a generalization of the canonical form for the bracket (1.23) which represents the sum of the standard (“action - angle”) part and an independent Poisson bracket for some additional variables \((\bar{N}^1(X), \ldots, \bar{N}^s(X))\). As we will show, the averaged bracket (1.23) can be transformed into the “pseudocanonical” form by a coordinate transformation (1.24) - (1.25) under some additional (“physical”) requirements on the initial bracket (1.1). We have to say, however, that an abstract Poisson bracket (1.23) can not in general be written in the pseudo-canonical form after a coordinate transformation (1.24) - (1.25) which is demonstrated by a special example at the end of Chapter 3.

2 The Canonical form of the averaged bracket.

First, let us introduce here special coordinates for the bracket (1.23) which will give a basis for it’s further consideration. Everywhere below we will assume that the bracket (1.23) represents the averaging of the bracket (1.1) on a complete Hamiltonian family of \(m\)-phase solutions of system (1.2) equipped with a minimal set of commuting integrals.

Let us consider an \((md + m + s)\)-dimensional manifold with coordinates \((k_1, \ldots, k_d, U)\) and the \((m + s)\)-dimensional submanifolds given by the relations \((k_1, \ldots, k_d) = \text{const}\.\) Consider the vector fields

\[
\bar{\xi}_{(\alpha)} = (\omega^{\alpha 1}(k_1, \ldots, k_d, U), \ldots, \omega^{\alpha m+s}(k_1, \ldots, k_d, U))^t
\]

on the submanifolds \((k_1, \ldots, k_d) = \text{const}\).

Using the Jacobi identities

\[
\{\{U^\gamma(X), S^\alpha(Y)\}, S^\beta(Z)\} - \{\{U^\gamma(X), S^\beta(Z)\}, S^\alpha(Y)\} \equiv 0
\]

define the relations

\[
[\bar{\xi}_{(\alpha)}, \bar{\xi}_{(\beta)}] \equiv 0, \quad \alpha, \beta = 1, \ldots, m
\]

for the commutators of the vector fields \(\bar{\xi}_{(\alpha)}\) on the submanifolds \((k_1, \ldots, k_d) = \text{const}\).

According to relations (1.21) we can state also that the set of vector fields \(\{\bar{\xi}_{(\alpha)}\}\) is linearly independent at every point.

We can claim then that on every submanifold \((k_1, \ldots, k_d) = \text{const}\) there exists a locally invertible change of coordinates

\[
(U^1, \ldots, U^{m+s}) \rightarrow \left(\hat{Q}_1(k_1, \ldots, k_d, U), \ldots, \hat{Q}_m(k_1, \ldots, k_d, U), \hat{N}^1(k_1, \ldots, k_d, U), \ldots, \hat{N}^s(k_1, \ldots, k_d, U)\right),
\]

depending smoothly on the parameters \((k_1, \ldots, k_d)\), which leads to the following coordinate representation

\[
\bar{\xi}_{(1)} = (1, 0, \ldots, 0)^t, \quad \ldots, \quad \bar{\xi}_{(m)} = (0, \ldots, 0, 1, 0, \ldots, 0)^t
\]

of the vector fields \(\bar{\xi}_{(\alpha)}\) on these submanifolds.

It is not difficult to see then that for the functionals

\[
\hat{Q}_\alpha(X) = \hat{Q}_\alpha(S_{X^1}, \ldots, S_{X^d}, U(X)), \quad \hat{N}^l(X) = \hat{N}^l(S_{X^1}, \ldots, S_{X^d}, U(X))
\]
we get immediately the following relations

\[ \left\{ S^\alpha(X), \tilde{Q}_\beta(Y) \right\} = \delta^\alpha_\beta \delta(X - Y) , \quad \left\{ S^\alpha(X), \tilde{N}^l(Y) \right\} = 0 \]

The pairwise Poisson brackets of the functionals \( \tilde{Q}_\alpha(X), \tilde{N}^l(X) \) have a local translationally invariant form which we can write in general as

\[ \left\{ \tilde{Q}_\alpha(X), \tilde{Q}_\beta(Y) \right\} = J_{\alpha\beta}(X, Y) , \quad \left\{ \tilde{Q}_\alpha(X), \tilde{N}^l(Y) \right\} = J^l_{\alpha}(X, Y) \]

\[ \left\{ \tilde{N}^l(X), \tilde{N}^q(Y) \right\} = J^{lq}(X, Y) \]

Using now the Jacobi identities

\[ \left\{ \left\{ \tilde{Q}_\alpha(X), \tilde{Q}_\beta(Y) \right\}, S^\gamma(Z) \right\} + c.p. \equiv 0 , \quad \left\{ \left\{ \tilde{Q}_\alpha(X), \tilde{N}^l(Y) \right\}, S^\gamma(Z) \right\} + c.p. \equiv 0 \]

we obtain also the following relations

\[ \frac{\delta J_{\alpha\beta}(X, Y)}{\delta \tilde{Q}_\gamma(Z)} \equiv 0 , \quad \frac{\delta J^l_{\alpha}(X, Y)}{\delta \tilde{Q}_\gamma(Z)} \equiv 0 , \quad \frac{\delta J^{lq}(X, Y)}{\delta \tilde{Q}_\gamma(Z)} \equiv 0 , \quad \gamma = 1, \ldots, m \]

for the distributions \( J_{\alpha\beta}(X, Y), J^l_{\alpha}(X, Y), J^{lq}(X, Y) \).

Finally, we can write the Poisson bracket (1.23) in coordinates \( S(X), \tilde{Q}_\alpha(X), \tilde{N}^l(X) \) in the following general form

\[ \left\{ S^\alpha(X), S^\beta(Y) \right\} = 0 , \quad \left\{ S^\alpha(X), \tilde{Q}_\beta(Y) \right\} = \delta^\alpha_\beta \delta(X - Y) , \quad \left\{ S^\alpha(X), \tilde{N}^l(Y) \right\} = 0 , \]

\[ \left\{ \tilde{Q}_\alpha(X), \tilde{Q}_\beta(Y) \right\} = \Omega^p_{\alpha\beta}(S_X, \tilde{N}) \delta_{xp} X - Y + \Gamma^{pq}_{\alpha\beta\gamma}(S_X, \tilde{N}) S^r_{xp} S_q \delta(X - Y) + \Pi^p_{\alpha\beta\gamma}(S_X, \tilde{N}) \tilde{N}^r_{xp} \delta(X - Y) \]

\( (\Gamma^{pq}_{\alpha\beta\gamma} \equiv \Gamma^{qp}_{\alpha\beta\gamma}) \),

\[ \left\{ \tilde{Q}_\alpha(X), \tilde{N}^l(Y) \right\} = A^l_{\alpha}(S_X, \tilde{N}) \delta_{xp} X - Y + B^{lp}_{\alpha\gamma}(S_X, \tilde{N}) S^r_{xp} \delta(X - Y) + C^{lp}_{\alpha r}(S_X, \tilde{N}) \tilde{N}^r_{xp} \delta(X - Y) \]

\( (B^{lp}_{\alpha\gamma} \equiv B^{lp}_{\alpha\gamma}) \),

\[ \left\{ \tilde{N}^l(X), \tilde{N}^k(Y) \right\} = g^{lk}_{xp}(S_X, \tilde{N}) \delta_{xp} X - Y + b^{l}_{k\gamma}(S_X, \tilde{N}) \tilde{N}^r_{xp} \delta(X - Y) + M^{lp}_{\gamma}(S_X, \tilde{N}) S^r_{xp} \delta(X - Y) \]

\( (M^{lp}_{\gamma} \equiv M^{lp}_{\gamma}) \).
Easy to see that, according to their definition, the variables $\tilde{Q}_\alpha(X)$ and $\tilde{N}^l(X)$ are defined modulo the transformations

$$
\tilde{Q}_\alpha(X) \rightarrow \tilde{Q}_\alpha(X) + \int_a \left( S_X, \tilde{N}(X) \right), \quad \tilde{N}^l(X) \rightarrow \tilde{N}^l(X) + \left( S_X, \tilde{N}(X) \right)
$$

where

$$
\det \left| \frac{\partial \tilde{N}^l}{\partial \tilde{N}^k} \right| \neq 0
$$

It is not difficult to see also that the values $\{\tilde{N}^l(X), \tilde{N}^k(Y)\}$ define a Poisson bracket on the space of fields $\tilde{N}(X)$ at any fixed values of $(S^1(X), \ldots, S^m(X))$.

We will consider in this chapter an important case when the number of annihilators of the bracket \((\ref{1.23})\) and the number of additional parameters $(n^1, \ldots, n^s)$ on $\Lambda$ are equal to zero. As we will see, the investigation of the canonical form of the bracket \((\ref{1.23})\) represents in this case a special interest.

Let us write down the averaged bracket \((\ref{1.23})\) in coordinates $(S^\alpha(X), \tilde{Q}_\alpha(X))$, such that we will have

$$
\{S^\alpha(X), S^\beta(Y)\} = 0, \quad \{S^\alpha(X), \tilde{Q}_\beta(Y)\} = \delta_\beta^\alpha \delta(X - Y),
$$

$$
\{\tilde{Q}_\alpha(X), \tilde{Q}_\beta(Y)\} = J_{\alpha\beta} \left[ S \left( X, Y \right) = \Omega^{\mu}_{\alpha\beta} (S_X) \delta_{X^\mu}(X - Y) + \Gamma^{pq}_{\alpha\beta\gamma}(S_X) S^\gamma_{X^pX^q} \delta(X - Y) \right.
$$

The Jacobi identities

$$
\left\{ \left\{ \tilde{Q}_\alpha(X), \tilde{Q}_\beta(Y) \right\}, \tilde{Q}_\gamma(Z) \right\} + c.p. \equiv 0
$$

give now the following relations

$$
\frac{\delta J_{\alpha\beta}[S](X, Y)}{\delta S^\gamma(Z)} + \frac{\delta J_{\beta\gamma}[S](Y, Z)}{\delta S^\alpha(X)} + \frac{\delta J_{\gamma\alpha}[S](Z, X)}{\delta S^\beta(Y)} \equiv 0
$$

for the functionals $J_{\alpha\beta}[S](X, Y)$, which mean the closeness of the two-form

$$
\int J_{\alpha\beta}[S](X, Y) \delta S^\alpha(X) \wedge \delta S^\beta(Y) \, d^d X \, d^d Y
$$

on the space of fields $(S^1(X), \ldots, S^m(X))$.

According to the terminology of S.P. Novikov (\[33\]), the brackets of the form \((\ref{2.1})\) represent "variationally admissible" Poisson brackets, connected with the Lagrangian representation for the corresponding Hamiltonian systems. As was shown in \[33\], the variationally admissible Poisson brackets lead in general to a nontrivial Lagrangian representation of the Hamiltonian systems where the Lagrangian represents in fact a closed 1-form on the functional space. As can be also shown, the variationally admissible Hamiltonian structures have in general rather nontrivial topological invariants connected with the topology of the functional space \((\ref{33})\).

Let us say that in general case we can admit that the variables $S^\alpha(X)$ represent "unobservable" quantities, such that only their spatial and time derivatives can appear in all "physically measurable" values. As a corollary, we can admit also, that only the spatial and time derivatives of the functions
but not the functions \(S(X)\) themselves, are in general uniquely defined for solutions of the corresponding Hamiltonian systems. Certainly, the most important class of solutions of this kind is represented by solutions containing \((d-1)\)-dimensional singularities (“vertices”) in \(X\)-space, where the functions \(S(X)\) are not defined, while the increments of the functions \(S^\alpha(X)\) along any closed 1-dimensional contour surrounding the vertex are different from zero.

According to the circumstance mentioned above we will separately consider here the values having immediate “physical” meaning. As examples of the variables of this kind we can mention here the values \(U^\gamma(X), \tilde{Q}_\alpha(X)\) or the derivatives \(S^\alpha_X\).

Let us formulate now the theorem about the canonical form of the bracket (2.1).

**Theorem 2.1.**

For every bracket (2.1) there exists locally a change of coordinates

\[
Q_\alpha(X) = \tilde{Q}_\alpha(X) + \tilde{q}_\alpha(S_{X^1}, \ldots, S_{X^d})
\]

which transforms bracket (2.1) into the form

\[
\{S^\alpha(X), S^\beta(Y)\} = 0, \quad \{S^\alpha(X), \tilde{Q}_\beta(Y)\} = \delta_\alpha^\beta \delta(X - Y), \quad \{Q_\alpha(X), Q_\beta(Y)\} = 0 \quad (2.5)
\]

According to Theorem 2.1 we can claim that for every bracket (2.1) there exist canonical variables \((Q_1(X), \ldots, Q_m(X))\) conjugated to the variables \((S^1(X), \ldots, S^m(X))\), which can be chosen among the “physically observable” fields.

It can be easily seen that Theorem 2.1 permits us to state also the following theorem about the bracket (1.23):

**Theorem 2.1′.**

Let the relations (1.23) represent a Poisson bracket on the space of \(2m\) fields

\[
(S^1(X), \ldots, S^m(X), U^1(X), \ldots, U^m(X))
\]

satisfying the conditions

\[
\text{rk } ||\omega^\alpha_{\gamma} (k_1, \ldots, k_d, U) || = m
\]

Then there exists locally an invertible change of coordinates

\[
(S^1(X), \ldots, S^m(X), U^1(X), \ldots, U^m(X)) \rightarrow (S^1(X), \ldots, S^m(X), Q_1(X), \ldots, Q_m(X)) \quad (2.6)
\]

where

\[
Q_\alpha(X) = Q_\alpha(S_{X^1}, \ldots, S_{X^d}, U(X)),
\]

such that bracket (1.23) has in the coordinates \((S(X), Q(X))\) the non-degenerate canonical form:

\[
\{S^\alpha(X), S^\beta(Y)\} = 0, \quad \{S^\alpha(X), \tilde{Q}_\beta(Y)\} = \delta_\alpha^\beta \delta(X - Y), \quad \{Q_\alpha(X), Q_\beta(Y)\} = 0
\]

As we said already, Theorem 2.1′ corresponds to the special case, when the number of annihilators of the bracket (1.1) and the number of the additional parameters \(n^1, \ldots, n^s\) on \(\Lambda\) are equal to zero.

Theorem 2.1′ was first formulated in [27] with a brief idea of the proof. We will give in this chapter a detailed proof of Theorem 2.1 which will imply also Theorem 2.1′ as a corollary.
It seems that the most compact proof of Theorem 2.1 can be given by combining of the geometric and pure computational methods. For the proof we will need to prove first two following lemmas:

**Lemma 2.1.**

Any divergence-free vector field \( \xi^r(X) \) having the form
\[
\xi^r(X) = F^{qp,r}_\alpha (S_X) S^\alpha_{XqXp}
\]
\( (F^{qp,r}_\alpha \equiv F^{pq,r}_\alpha ) \), can be locally represented in the form
\[
\xi^r(X) = \sum_{s \neq r} \left[ f^{sr}(S_X) \right]_{X^s},
\]
where \( f^{sr}(S_X) \equiv -f^{rs}(S_X) \).

**Proof.**

From the conditions
\[
\sum_r \left( F^{qp,r}_\alpha (S_X) S^\alpha_{XqXp} \right)_{X^r} \equiv 0
\]
we can get, in particular, the following relations
\[
F^{qq,q}_\alpha (S_X) \equiv 0, \quad F^{qq,r}_\alpha (S_X) \equiv -2 F^{qr,q}_\alpha (S_X), \quad q \neq r
\]
(2.8)

\[
\frac{\partial F^{qq,r}_\alpha}{\partial S^\beta_{Xq}} \equiv -2 \frac{\partial F^{qr,q}_\alpha}{\partial S^\alpha_{Xq}} \equiv \frac{\partial F^{qq,r}_\alpha}{\partial S^\alpha_{Xq}}, \quad q \neq r
\]
(2.9)

(no summation).

\[
\frac{\partial F^{qq,r}_\alpha}{\partial S^\beta_{Xq}} \equiv - \frac{\partial F^{rr,q}_\alpha}{\partial S^\alpha_{Xq}}, \quad q \neq r
\]
(2.10)

(no summation).

Form relations (2.9) and (2.10) we can conclude that locally there exist functions \( g^{qr}(S_X) \), satisfying the relations
\[
F^{qq,r}_\alpha (S_X) \equiv \frac{\partial g^{qr}}{\partial S^\alpha_{Xq}}, \quad g^{qr}(S_X) \equiv -g^{rq}(S_X)
\]
(2.11)

We easily get then also from (2.8) the relations
\[
F^{qr,q}_\alpha (S_X) \equiv -\frac{1}{2} \frac{\partial g^{qr}}{\partial S^\alpha_{Xq}}, \quad q \neq r
\]
(2.12)

Let us consider now the vector field
\[
\tilde{\xi}^r(X) = \xi^r(X) - \sum_{q \neq r} \left[ g^{qr}(S_X) \right]_{X^q} = \xi^r(X) - \sum_{q \neq r} \frac{\partial g^{qr}}{\partial S^\alpha_{Xq}} S^\alpha_{XqXs}
\]

Using relations (2.11) and (2.12) we conclude now that the field \( \tilde{\xi}^r(X) \) represents a divergence-free vector field having the form
\[
\tilde{\xi}^r(X) = \sum \tilde{F}^{qp,r}_\alpha (S_X) S^\alpha_{XqXp}
\]

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where
\[ \tilde{F}_{\alpha}^{qp,r} \equiv \tilde{F}_{\alpha}^{qr,q} \equiv 0, \quad (\tilde{F}_{\alpha}^{qp,r} \equiv \tilde{F}_{\alpha}^{pq,r}). \]

From the relations
\[ \sum_{q\neq p \neq r} \left( \tilde{F}_{\alpha}^{qp,r} (S_X)_X S_{X^q X^r}^{\alpha} \right)_{X^q} \equiv 0 \]
we now get the relations
\[ \tilde{F}_{\alpha}^{qp,r} + \tilde{F}_{\alpha}^{rp,q} + \tilde{F}_{\alpha}^{qr,p} \equiv 0 \quad (2.13) \]
and
\[ \frac{\partial \tilde{F}_{\alpha}^{qp,r}}{\partial S_{\beta X^q}} + \frac{\partial \tilde{F}_{\alpha}^{qs,p}}{\partial S_{\beta X^r}} + \frac{\partial \tilde{F}_{\beta}^{r s , p}}{\partial S_{\alpha X^p}} + \frac{\partial \tilde{F}_{\beta}^{r s , q}}{\partial S_{\alpha X^p}} \equiv 0 \quad (2.14) \]

In particular, for \( s = r \) we have:
\[ \frac{\partial \tilde{F}_{\alpha}^{qp,r}}{\partial S_{\beta X^r}} \equiv 0 \quad (2.15) \]
(no summation).

Applying the operator \( \partial^2 / \partial S_{\gamma X^q} \partial S_{\delta X^p} \) to the relations \( (2.13) \) we easily get the relations:
\[ \frac{\partial^2 \tilde{F}_{\alpha}^{qp,r}}{\partial S_{\beta X^q} \partial S_{\gamma X^p}} \equiv 0 \]
(no summation).

Using also relations \( (2.15) \) we then get locally the following representation for \( \tilde{F}_{\alpha}^{qp,r} \):
\[ \tilde{F}_{\alpha}^{qp,r} = a_{\alpha}^{qp,r} \left( S_{X^1}, \ldots, S_{X^q}, \ldots, S_{X^r}, \ldots, S_{X^d} \right) + a_{\alpha}^{pq,r} \left( S_{X^1}, \ldots, S_{X^p}, \ldots, S_{X^r}, \ldots, S_{X^d} \right) \]
where the hat over the variable means the absence of this variable among the arguments of a function.

For the functions \( a_{\alpha}^{qp,r} \) we get now the relations:
\[ a_{\alpha}^{qp,r} \left( S_{X^1}, \ldots, S_{X^q}, \ldots, S_{X^r}, \ldots, S_{X^d} \right) + a_{\alpha}^{pq,r} \left( S_{X^1}, \ldots, S_{X^p}, \ldots, S_{X^r}, \ldots, S_{X^d} \right) + \]
\[ + a_{\alpha}^{rp,q} \left( S_{X^1}, \ldots, S_{X^q}, \ldots, S_{X^r}, \ldots, S_{X^d} \right) + a_{\alpha}^{pr,q} \left( S_{X^1}, \ldots, S_{X^p}, \ldots, S_{X^r}, \ldots, S_{X^d} \right) + \]
\[ + a_{\alpha}^{qr,p} \left( S_{X^1}, \ldots, S_{X^q}, \ldots, S_{X^r}, \ldots, S_{X^d} \right) + a_{\alpha}^{q p, r} \left( S_{X^1}, \ldots, S_{X^p}, \ldots, S_{X^r}, \ldots, S_{X^d} \right) \equiv 0 \]

Applying the operators \( \partial / \partial S_{\gamma X^q} \) to the relations above, we easily get now the following relations:
\[ a_{\alpha}^{pq,r} \left( S_{X^1}, \ldots, S_{X^p}, \ldots, S_{X^r}, \ldots, S_{X^d} \right) = - a_{\alpha}^{rq,p} \left( S_{X^1}, \ldots, S_{X^p}, \ldots, S_{X^r}, \ldots, S_{X^d} \right) + \]
\[ + a_{\alpha}^{rp,q} \left( S_{X^1}, \ldots, S_{X^r}, \ldots, S_{X^p}, \ldots, S_{X^d} \right) \quad (2.16) \]

Putting now \( s = q \) in the relations \( (2.14) \) we have:
\[ \frac{\partial \tilde{F}_{\alpha}^{qp,r}}{\partial S_{\beta X^q}} + \frac{\partial \tilde{F}_{\beta}^{r q,p}}{\partial S_{\alpha X^q}} \equiv 0 \quad (\text{no summation}), \]

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which gives the following relations
\[
\frac{\partial a_{\alpha}^{pq,r}}{\partial S_{Xq}^\alpha} + \frac{\partial a_{\beta}^{r,q,p}}{\partial S_{Xq}^\beta} \equiv 0 \quad \text{(no summation)}
\]
for the functions \(a_{\alpha}^{pq,r}\). From the relations (2.16) we get then
\[
\frac{\partial a_{\alpha}^{pq,r}}{\partial S_{Xq}^\beta} - \frac{\partial a_{\alpha}^{pq,r}}{\partial S_{Xq}^\alpha} \equiv 0 \quad \text{(no summation)},
\]
which gives locally
\[
a_{\alpha}^{pq,r} \left( S_{X^1}, \ldots, \hat{S}_{X^p}, \ldots, \hat{S}_{X^r}, \ldots, S_{X^d} \right) \equiv \frac{\partial h^{p,r}(S_{X^1}, \ldots, \hat{S}_{X^p}, \ldots, \hat{S}_{X^r}, \ldots, S_{X^d})}{\partial S_{Xq}^\alpha}
\]
for some functions \(h^{p,r}(S_{X^1}, \ldots, \hat{S}_{X^p}, \ldots, \hat{S}_{X^r}, \ldots, S_{X^d})\), \(h^{p,r} \equiv -h^{r,p}\).
We get then
\[
\sum_p \left[ 2 h^{p,r} \right]_{X^p} = \sum_p 2 \frac{\partial h^{p,r}}{\partial S_{Xq}^\alpha} S_{X^q,X^p}^\alpha = 2 a_{\alpha}^{pq,r} S_{X^q,X^p}^\alpha = \tilde{F}^{pq,r} S_{X^q,X^p}^\alpha = \tilde{\xi}^r(X)
\]
Finally, we obtain
\[
\xi^r(X) \equiv \sum_{p \neq r} \left[ g^{p,r}(S_X) \right]_{X^p} + \sum_{p \neq r} \left[ 2 h^{p,r}(S_X) \right]_{X^p}
\]
which gives the proof of the lemma.

Lemma 2.1 is proved.

Lemma 2.2.
Let a closed 1-form on the space of functions \(S(X) = (S^1(X), \ldots, S^m(X))\) have the form
\[
q = \int q_\alpha(X) \delta S^\alpha(X) \, d^dX \equiv \int M^{qp}_{\alpha\gamma}(S_X) S_{X^q,X^p}^\gamma \delta S^\alpha(X) \, d^dX \quad (2.17)
\]
\((M^{qp}_{\alpha\gamma} \equiv M^{pq}_{\alpha\gamma})\).
Then the functions \(q_\alpha(X) \equiv M^{qp}_{\alpha\gamma}(S_X) S_{X^q,X^p}^\gamma\) can be locally represented as
\[
q_\alpha(X) \equiv \frac{\delta}{\delta S^\alpha(X)} \int h(S_W) \, d^dW
\]
with some smooth function \(h(S_W)\).

Proof.
Using the standard “homotopy operator” (see e.g. [36]), based on the mapping:
\[
\hat{F} : [0,1] \times \{S(X)\} \rightarrow \{S(X)\}, \quad (\lambda, S(X)) \rightarrow \lambda S(X), \quad \lambda \in [0,1],
\]
we can claim that the coefficients \(q_\alpha(X)\) can be written in the form
\[
M^{qp}_{\alpha\gamma}(S_X) S_{X^q,X^p}^\gamma \equiv \frac{\delta}{\delta S^\alpha(X)} \int S^p(W) \hat{M}^{qp}_{\alpha\gamma}(S_W) S_{W^q,W^p}^\gamma \, d^dW
\]
where
\[
\bar{M}^{qp}_{\rho\gamma}(S_W) \equiv \int_0^1 M^{qp}_{\rho\gamma}(\lambda S_W) \, d\lambda
\]

To get a representation of the required form let us first note that the 1-form (2.17) is evidently invariant under the transformations
\[
S^\rho(X) \rightarrow S^\rho(X) + C^\rho, \quad C^\rho = \text{const (2.18)}
\]

As a corollary, we can claim that the values \( M^{qp}_{\rho\gamma}(S_X) S^q_X S^p_X \) can be also represented in the form
\[
M^{qp}_{\rho\gamma}(S_X) S^q_X S^p_X \equiv \delta \int (S^\rho(W) + C^\rho) \bar{M}^{qp}_{\rho\gamma}(S_W) S^q_W S^p_W dW
\]

for arbitrary values of \( C^\rho \).

We can claim then, that all the functionals
\[
M(\rho) = \int \bar{M}^{qp}_{\rho\gamma}(S_W) S^q_W S^p_W \, dW
\]

have identically zero variation derivatives on the space of functions \((S^1(X), \ldots, S^m(X))\).

According to the classical theorem (see e.g. [36], Chpt. 4, Thm. 4.7), any density \( \sigma(\rho)(W) = \bar{M}^{qp}_{\rho\gamma}(S_W) S^q_W S^p_W \)

can then be represented as the full divergence of the vector field \( v(\rho)(W) \):
\[
\bar{M}^{qp}_{\rho\gamma}(S_W) S^q_W S^p_W \equiv \left[ v^r(\rho)(W) \right]_W,
\]

where the values \( v^r(\rho)(W) \) have in general the form
\[
v^r(\rho)(W) \equiv S^\mu(W) V^{rqp}_{(\rho)\mu\gamma}(S_W) S^q_W S^p_W + u^r(\rho)(S_W)
\]

The vector fields \( \xi_{(\rho\mu)} \) given by the components
\[
\xi^r_{(\rho\mu)}(W) = V^{rqp}_{(\rho)\mu\gamma}(S_W) S^q_W S^p_W
\]

represent divergence-free vector fields, so we can write according to Lemma 2.1:
\[
V^{rqp}_{(\rho)\mu\gamma}(S_W) S^q_W S^p_W \equiv \sum_{s \neq r} \left[ f^{sr}_{(\rho\mu)}(S_W) \right]_W
\]

for some functions \( f^{sr}_{(\rho\mu)}(S_W) \).

Using relations (2.19) - (2.20) we now easily get the relations
\[
\left[ S^\mu(W) V^{rqp}_{(\rho)\mu\gamma}(S_W) S^q_W S^p_W \right]_W \equiv \left[ S^\mu(W) \sum_{s \neq r} \left[ f^{sr}_{(\rho\mu)}(S_W) \right]_W \right]_W \equiv S^\mu_W \sum_{s \neq r} \left[ f^{sr}_{(\rho\mu)}(S_W) \right]_W
\]
and
\[ S^\rho(W) \tilde{M}_{\rho\gamma}^p (S_W) S_{WqWp}^\gamma \equiv S^\rho(W) S_{W^s}^\mu \sum_{s \neq r} [f^s_{(\rho)} (S_W)]_{W^s} + S^\rho(W) [u^r_{(\rho)} (S_W)]_{W^r} \]

Finally, using integration by parts, we can claim that the functional
\[ H \equiv \int S^\rho(W) \tilde{M}_{\rho\gamma}^p (S_W) S_{WqWp}^\gamma \, d^4W \]
can be represented in the form
\[ H \equiv - \int \left[ S^\rho_{W^s} S_{W^r}^\mu \sum_{s \neq r} f^s_{(\rho)} (S_W) + S^\rho_{W^r} u^r_{(\rho)} (S_W) \right] d^4W \]

which gives the proof of the lemma.

Lemma 2.2 is proved.

Proof of Theorem 2.1.

Using the homotopy operator approach for the closed 2-form (2.3) we obtain the relations
\[ J_{\alpha\beta}[S] (X, Y) = \frac{\delta}{\delta S^\beta(Y)} q_\alpha(X) - \frac{\delta}{\delta S^\alpha(X)} q_\beta(Y) \]
where
\[ q_\alpha(X) \equiv \int_0^1 d\lambda \int 2\lambda J_{\alpha \rho} [\lambda S] (X, W) S^\rho(W) \, d^4W \equiv \int_0^1 d\lambda \lambda \Omega_{\alpha \rho}^\beta (\lambda S_X) + 2 S_{X^p X^q}^\gamma S^\rho(X) \int_0^1 d\lambda \lambda \Gamma_{\alpha \rho \gamma}^{pq} (\lambda S_X) \]

We can see now that the coordinate change
\[ \tilde{Q}_\alpha(X) \rightarrow \tilde{Q}_\alpha(X) + q_\alpha(X) \quad (2.21) \]
gives the required form for the bracket (2.1). However, we can see also that the transformation (2.21) does not have the required form (2.4). To get a transformation of the form (2.4) let us note again that the 2-form (2.3) is evidently invariant under the transformations (2.18). As a consequence, we easily get then that any transformation (2.18), applied to the set \( \{q_\alpha(X)\} \), gives a set of functions \( \{q'_\alpha(X)\} \) having the property that the change
\[ \tilde{Q}_\alpha(X) \rightarrow \tilde{Q}_\alpha(X) + q'_\alpha(X) \]
transforms the bracket (2.1) to the canonical form (2.5). It’s not difficult to see, that this circumstance means in fact that all the functions
\[ \omega_{(\rho)}(X) = (\omega_{(\rho)1}(X), \ldots, \omega_{(\rho)m}(X)) \]
defined as
\[ \omega_{(\rho)\alpha}(X) \equiv 2 S_{X^p X^q}^\gamma \int_0^1 d\lambda \lambda \Gamma_{\alpha \rho \gamma}^{pq} (\lambda S_X) \]

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represent coefficients of closed 1-forms on the space \((S^1(X), \ldots, S^m(X))\):

\[
\frac{\delta \omega_{(\rho)\alpha}(X)}{\delta S^\beta(Y)} \equiv \frac{\delta \omega_{(\rho)\beta}(Y)}{\delta S^\alpha(X)}, \quad \rho = 1, \ldots, m
\]

Using Lemma 2.2 we can claim then that the functions \(\omega_{(\rho)}(X)\) can be locally represented in the form

\[
\omega_{(\rho)\alpha}(X) \equiv \frac{\delta}{\delta S^\alpha(X)} \int h_{(\rho)}(S_W) \, d^dW
\]

for some functions \(h_{(\rho)}(S_W)\).

Let us now

\[
\bar{q}_\alpha(X) = q_\alpha(X) - \frac{\delta}{\delta S^\alpha(X)} \int S^\rho(W) \, h_{(\rho)}(S_W) \, d^dW
\]

and define

\[
Q_\alpha(X) = \bar{Q}_\alpha(X) + \bar{q}_\alpha(X) \quad (2.22)
\]

It can be seen now that the coordinate change (2.22) has the necessary form (2.4). Besides that, the change (2.22) transforms the bracket (2.1) to the canonical form like the transformation (2.21).

Theorem 2.1 is proved.

As a corollary of Theorems 2.1 - 2.1’ we can claim that any system (1.22) with \(s = 0\) can be written locally in the Lagrangian form

\[
\delta \int \left[ Q_\alpha(X) S^\alpha_T - \langle P_H \rangle(S_{X^1}, \ldots, S_{X^d}, Q(X)) \right] \, d^dX \, dT = 0
\]

after the transition to the variables \((S^1(X), \ldots, S^m(X), Q_1(X), \ldots, Q_m(X))\).

In the non-degenerate case, when the values \(Q_\alpha(X)\) can be expressed in terms of \((S_T, S_{X^1}, \ldots, S_{X^d})\) from the first part of system (1.22), we can write system (1.22) in the “standard” Lagrangian form

\[
\delta \int \left[ \sum_{\alpha=1}^m (S^\alpha_T)^2 - \langle P_H \rangle(S_T, S_{X^1}, \ldots, S_{X^d}) \right] \, d^dX \, dT = 0
\]

It is not difficult to see, that we have to require the non-degeneracy conditions

\[
\det \left| \frac{\partial \omega^\alpha}{\partial Q_\beta} \right|_{S_X} = \det \left| \frac{\partial^2 \langle P_H \rangle}{\partial Q_\alpha \partial Q_\beta} \right|_{S_X} \neq 0
\]

in this case.

At the end of the chapter let us discuss the group of canonical transformations for the bracket (2.5) having the “physical” form

\[
Q_\alpha'(X) = Q_\alpha(X) + q_\alpha(S_{X^1}, \ldots, S_{X^d}) \quad (2.23)
\]

It is easy to see that the transformation (2.23) represents a canonical transformation for the bracket (2.5) if and only if we have the identities

\[
\frac{\delta}{\delta S^\beta(Y)} q_\alpha(S_X) - \frac{\delta}{\delta S^\alpha(X)} q_\beta(S_Y) \equiv 0
\]
It is easy to check also that the identities above are equivalent to the following relations

\[ \frac{\partial q_\alpha}{\partial S^\beta_{Xq}} \equiv - \frac{\partial q_\beta}{\partial S^\alpha_{Xq}} , \quad \frac{\partial^2 q_\beta}{\partial S^\alpha_{Xq} \partial S^\gamma_{Xp}} + \frac{\partial^2 q_\beta}{\partial S^\alpha_{Xq} \partial S^\gamma_{Xq}} \equiv 0 \quad (2.24) \]

From (2.24) we easily get also the following relations

\[ \frac{\partial q_\alpha}{\partial S^\alpha_{Xq}} \equiv 0 , \quad \frac{\partial^2 q_\beta}{\partial S^\alpha_{Xq} \partial S^\alpha_{Xp}} \equiv 0 , \quad \frac{\partial^2 q_\beta}{\partial S^\alpha_{Xq} \partial S^\gamma_{Xq}} \equiv 0 \quad \text{(no summation).} \]

It’s not difficult to see that the group of the canonical transformations (2.23) represents in fact a finite-dimensional linear space with the basis elements, which can be described in the following way:

Consider all possible pairs of sets \((M_1, M_2)\):

\[ M_1 = (\alpha_1, \ldots, \alpha_{l+1}) \quad , \quad \alpha_j \in \{1, \ldots, m\} \quad , \quad \alpha_1 < \alpha_2 < \ldots < \alpha_{l+1} \]

\[ M_2 = (q_1, \ldots, q_l) \quad , \quad q_j \in \{1, \ldots, d\} \quad , \quad q_1 < q_2 < \ldots < q_l \]

for all possible \(l \geq 0\).

Consider the functions \(q_{(M_1, M_2)}\) having the form

\[ q_\alpha(M_1, M_2) = \begin{cases} 0 , & \alpha \notin M_1 \\ \Delta_\alpha^{(M_1, M_2)} , & \alpha \in M_1 \end{cases} \]

where

\[ \Delta_\alpha^{(M_1, M_2)} \equiv (-1)^{l-1} \det \begin{bmatrix} S_{Xq_1}^{\alpha_1} & S_{Xq_2}^{\alpha_1} & \cdots & S_{Xq_l}^{\alpha_1} \\
\hat{S}_{Xq_1}^{\alpha_j} & \hat{S}_{Xq_2}^{\alpha_j} & \cdots & \hat{S}_{Xq_l}^{\alpha_j} \\
\cdots & \cdots & \cdots & \cdots \\
S_{Xq_1}^{\alpha_{l+1}} & S_{Xq_2}^{\alpha_{l+1}} & \cdots & S_{Xq_l}^{\alpha_{l+1}} \end{bmatrix} , \quad \alpha = \alpha_j \in M_1 , \quad l \geq 1 \]

(the hats mean that the corresponding row is absent in the matrix).

Let us also put by definition

\[ \Delta_\alpha^{(M_1, M_2)} \equiv 1 \]

for \(l = 0\), \(M_1 = \{\alpha\}\), \(M_2 = \emptyset\).

The functions \(\{q_{(M_1, M_2)}\}\) can be considered now as the basis of the linear space, representing the group of the canonical transformations (2.23).

3 On more complicated (pseudo-)Canonical forms.

In this chapter we will consider brackets (1.23) in the case of presence of additional parameters \((n^1, \ldots, n^s)\), connected with the presence of annihilators of the initial bracket (1.1). As we said in Introduction, we will assume here that the bracket (1.23) is obtained by the averaging of the bracket (1.1) on a complete Hamiltonian family \(\Lambda\) of \(m\)-phase solutions of system (1.2), equipped with a minimal set of commuting integrals, which implies, in particular, that the number of the
parameters \((n_1, \ldots, n^s)\) is exactly equal to the number of annihilators of the bracket (1.1) on the space of quasiperiodic functions.

As we will see below, the canonical form of the bracket (1.23) should be understood in this case in more general sense and represents in fact the separation of the “standard” canonical variables \((S^1(X), \ldots, S^m(X), Q_1(X), \ldots, Q_m(X))\) and some special variables \((\tilde{N}^1(X), \ldots, \tilde{N}^s(X))\) with their own Poisson bracket.

As in the previous chapter, we consider here the transformations of the “physical” variables \((U^1(X), \ldots, U^{m+s}(X))\) having the form

\[
U^\gamma(X) = U^\gamma(S_{X^1}, \ldots, S_{X^d}, U(X)), \quad \gamma = 1, \ldots, m + s,
\]

which can be called the transformations of “Hydrodynamic Type”. As above, the variables \((S^1(X), \ldots, S^m(X))\) will be always considered here as the first part of canonical variables for every bracket (1.23).

Let us consider now a special class of the Poisson brackets (1.1) having some special “physical” property.

**Definition 3.1.**

Let us say that the bracket (1.1) has annihilators of the physical form if all the independent annihilators of (1.1) on the space of quasiperiodic functions can be represented in the form:

\[
C^l = \int c^l(\varphi, \varphi_x, \ldots) \, dx, \quad l = 1, \ldots, s
\]

with some smooth functions \((\varphi, \varphi_x, \ldots)\).

In particular, for a complete Hamiltonian family \(\Lambda\) of \(m\)-phase solutions of system (1.2) the Definition 3.1 requires that the functions

\[
v^{(l)}_{[a, \theta_0]}(x) = \left(v^{(l)}_{[a, \theta_0]1}(x), \ldots, v^{(l)}_{[a, \theta_0]n}(x)\right), \quad v^{(l)}_{[a, \theta_0]i}(x) = \left. \frac{\delta C^l}{\delta \varphi^i(x)} \right|_{\Lambda_{k_1 \ldots k_d}}, \quad l = 1, \ldots, s
\]

represent the full basis of solutions of system (1.17), having the form (1.18), everywhere on \(\Lambda\) in accordance with the Definition 1.2.

Let us come back now to the variables

\[
(S^1(X), \ldots, S^m(X), \tilde{Q}_1(X), \ldots, \tilde{Q}_m(X), \tilde{N}^1(X), \ldots, \tilde{N}^s(X))
\]

for the bracket (1.23), introduced in the previous chapter. We can formulate here the following lemma:

**Lemma 3.1.**

Let the bracket (1.1) have annihilators of the physical form and the family \(\Lambda\) represent a complete Hamiltonian family of \(m\)-phase solutions of system (1.2) equipped with a minimal set of commuting integrals. Let the bracket (1.23) represent the averaging of the bracket (1.1) on the family \(\Lambda\).
Then the variables
\[
\left( \tilde{Q}_1(X), \ldots, \tilde{Q}_m(X), \tilde{N}^1(X), \ldots, \tilde{N}^s(X) \right)
\]
can be chosen in the form
\[
\left( \tilde{Q}_1(X), \ldots, \tilde{Q}_m(X), N^1(X), \ldots, N^s(X) \right)
\]
where
\[
N^l \equiv \langle c^l \rangle \equiv \int_0^{2\pi} \cdots \int_0^{2\pi} c^l(\Phi, k^1_{\alpha_1} \Phi_{g^1}, \ldots, k^d_{\alpha_d} \Phi_{g^d}, \ldots) \frac{d^m \theta}{(2\pi)^m}
\]
represent the averaged densities of annihilators.

Proof.

Let the set \( \{I^1, \ldots, I^{m+s}\} \) represent a minimal set of commuting integrals (1.19) for the family \( \Lambda \). Let us assume without lost of generality that we have
\[
\text{rk } ||\omega^{\alpha\gamma}(k_1, \ldots, k_d, U)|| = m , \quad \gamma = 1, \ldots, m
\]
for the corresponding frequencies \( \omega^\gamma \).

It’s not difficult to see that the variation derivatives
\[
\left( \frac{\delta I^1}{\delta \varphi^i(X)}, \ldots, \frac{\delta I^m}{\delta \varphi^i(X)}, \frac{\delta C^1}{\delta \varphi^i(X)}, \ldots, \frac{\delta C^s}{\delta \varphi^i(X)} \right)
\]
represent in this case a linearly independent system on \( \Lambda \), so the values
\[
U^\gamma = \langle P^\gamma \rangle , \quad \gamma = 1, \ldots, m , \quad N^l = \langle c^l \rangle , \quad l = 1, \ldots, s ,
\]
give a set of independent parameters on every family \( \Lambda_{k_1, \ldots, k_d} \). We can easily see then that the functionals
\[
(I^1, \ldots, I^m, C^1, \ldots, C^s)
\]
represent a minimal set of commuting integrals for the family \( \Lambda \), satisfying all the requirements of Definition 1.3.

Consider now the bracket (1.23) in the coordinates
\[
(S^1(X), \ldots, S^m(X), U^1(X), \ldots, U^m(X), N^1(X), \ldots, N^s(X))
\]
Using the Jacobi identities
\[
\{ \{U^\gamma(X), S^\alpha(Y)\} , S^\beta(Z) \} - \{ \{U^\gamma(X), S^\beta(Z)\} , S^\alpha(Y) \} \equiv 0 , \quad \gamma = 1, \ldots, m ,
\]
we easily get that the vector fields
\[
\tilde{\xi}_{(\alpha)} = (\omega^\alpha(k_1, \ldots, k_d, U, N), \ldots, \omega^{\alpha m}(k_1, \ldots, k_d, U, N))^t
\]
represent commuting vector fields, tangent to the submanifolds \( (k_1, \ldots, k_d, N) = \text{const} \).

Since the set \( \{\tilde{\xi}_{(\alpha)}\} \) is linearly independent we can claim again that we can choose the variables
\[
\left( \tilde{Q}_1(k_1, \ldots, k_d, U, N), \ldots, \tilde{Q}_m(k_1, \ldots, k_d, U, N) \right)
\]
on each submanifold, such that the vectors $\vec{\xi}_{(\alpha)}$ get the coordinate representation:

$$\vec{\xi}_{(\alpha)} = (0, \ldots, 1, \ldots, 0)$$

Easy to see that we get now the required coordinate system using the variables $(N^1(X), \ldots, N^s(X))$ and

$$\tilde{Q}_\alpha(X) = \tilde{Q}_\alpha(S_{X^1}, \ldots, S_{X^d}, U(X), N(X)), \quad \alpha = 1, \ldots, m$$

Lemma 3.1 is proved.

**Lemma 3.2.**

Let the bracket (1.7) have annihilators of the physical form and the family $\Lambda$ represent a complete Hamiltonian family of $m$-phase solutions of system (1.2) equipped with a minimal set of commuting integrals. Let the bracket (1.23) represent the averaging of the bracket (1.7) on the family $\Lambda$.

Consider the variables on the space $U(X)$ introduced in Lemma 3.1.

Then the bracket (1.23) has in the variables

$$\left( S^1(X), \ldots, S^m(X), \tilde{Q}_1(X), \ldots, \tilde{Q}_m(X), N^1(X), \ldots, N^s(X) \right)$$

the form

$$\{S^\alpha(X), S^\beta(Y)\} = 0, \quad \{S^\alpha(X), \tilde{Q}_\beta(Y)\} = \delta^\alpha_\beta \delta(X - Y), \quad \{S^\alpha(X), N^l(Y)\} = 0$$

$$\{\tilde{Q}_\alpha(X), \tilde{Q}_\beta(Y)\} = \Omega^{p}_{\alpha\beta}(S_X, N) \delta_{X^p}(X - Y) +$$

$$+ \Gamma^{pq}_{\alpha\beta\gamma}(S_X, N) S^\gamma_{X^p X^q} \delta(X - Y) + \Pi^{p}_{\alpha\beta\gamma}(S_X, N) N^\gamma_{X^p} \delta(X - Y)$$

($\Gamma^{pq}_{\alpha\beta\gamma} \equiv \Gamma^{qp}_{\alpha\beta\gamma}$),

$$\{\tilde{Q}_\alpha(X), N^l(Y)\} = A^{lp}_{\alpha}(S_X, N) \delta_{X^p}(X - Y) \tag{3.1}$$

$$\{N^l(X), N^k(Y)\} = g^{lk}_{\gamma}(S_X, N) \delta_{X^\gamma}(X - Y)$$

**Proof.**

What we have actually to prove is the absence of the “$\delta$ - terms” in the last two expressions of (3.1).

To prove this fact let us first note that, according to the definition of annihilators, the corresponding terms are absent in the Poisson brackets of the densities $P^\gamma(\varphi, \varphi_x, \ldots)$ and $C^l(\varphi, \varphi_x, \ldots)$ with $C^k(\varphi, \varphi_y, \ldots)$:

$$\{P^\gamma(x), C^k(y)\} = \sum_{l_1, \ldots, l_d} G^{lk}_{l_1 \ldots l_d}(\varphi, \varphi_x, \ldots) \delta^{(l_1)}(x^1 - y^1) \ldots \delta^{(l_d)}(x^d - y^d)$$

$$\{C^l(x), C^k(y)\} = \sum_{l_1, \ldots, l_d} W^{lk}_{l_1 \ldots l_d}(\varphi, \varphi_x, \ldots) \delta^{(l_1)}(x^1 - y^1) \ldots \delta^{(l_d)}(x^d - y^d)$$

($l_1, \ldots, l_d \geq 0$, $(l_1, \ldots, l_d) \neq (0, \ldots, 0)$).
According to the averaging procedure we can claim then the absence of the terms, containing \( \delta (X - Y) \), in the Poisson brackets \( \{ U^\gamma (X), N^k (Y) \} \) and \( \{ N^l (X), N^k (Y) \} \) for the bracket (1.23). Easy to see then, that the same property is valid also for the brackets \( \{ \tilde{Q}_o (X), N^k (Y) \} \) and \( \{ N^l (X), N^k (Y) \} \) after the transition to the variables \( (\tilde{Q}_1 (X), \ldots, \tilde{Q}_m (X), N^1 (X), \ldots, N^s (X)) \).

Let us note also here that from the form of the Poisson brackets for the functionals \( N^k (Y) \) we can also conclude that all the functionals
\[
n^l = \int N^l (X) \, d^d X
\]
represent annihilators of the physical form for the averaged bracket (1.23).

It’s not difficult to check that just from the skew-symmetry of the bracket (3.1) we get the relations
\[
g^{lk,p} = g^{kl,p} \ , \quad [g^{kl,p}]_{X^p} = 0
\]
From the second relation above we then easily get the relations
\[
\frac{\partial g^{kl,p}}{\partial N^\tau} = 0 \ , \quad \frac{\partial g^{kl,p}}{\partial S^\alpha_{X^\gamma}} + \frac{\partial g^{kl,q}}{\partial S^\alpha_{X^\gamma}} = 0
\]
for the functions \( g^{kl,p} (S_{X^1}, \ldots, S_{X^d}, N) \).

So, we can put now \( g^{kl,p} = g^{kl,p} (S_{X^1}, \ldots, S_{X^d}) \) for all the functions \( g^{kl,p} \). It can be also seen, that the second part of the relations above implies the relations
\[
\frac{\partial^2 g^{kl,p}}{\partial S^\gamma_{X^\delta} \partial S^\beta_{X^\epsilon}} = - \frac{\partial^2 g^{kl,p}}{\partial S^\gamma_{X^\delta} \partial S^\beta_{X^\epsilon}}
\]
Like in Chapter 2, we can actually claim here that all the functions
\[
g^{kl} = g^{ik} = (g^{kl,1}, \ldots, g^{kl,d})
\]
belong to a finite-dimensional linear space with the basis elements, which can be described in the following way:

Consider all possible pairs of sets \( (\mathcal{N}_1, \mathcal{N}_2) \):
\[
\mathcal{N}_1 = (\alpha_1, \ldots, \alpha_l) \ , \quad \alpha_j \in \{1, \ldots, m\} \ , \quad \alpha_1 < \alpha_2 < \ldots < \alpha_l
\]
\[
\mathcal{N}_2 = (q_1, \ldots, q_{l+1}) \ , \quad q_j \in \{1, \ldots, d\} \ , \quad q_1 < q_2 < \ldots < q_{l+1}
\]
for all possible \( l \geq 0 \).

Consider the functions \( g^{p} (\mathcal{N}_1, \mathcal{N}_2) \) having the form
\[
g^{p} (\mathcal{N}_1, \mathcal{N}_2) = \begin{cases} 0 & , \quad p \notin \mathcal{N}_2 \\ \Delta^p_{(\mathcal{N}_1, \mathcal{N}_2)} & , \quad p \in \mathcal{N}_2 \end{cases}
\]
where
\[
\Delta^p_{(\mathcal{N}_1, \mathcal{N}_2)} \equiv (-1)^{j-1} \det \begin{vmatrix} S^\alpha_{X^\gamma_1} & \ldots & \tilde{S}^\alpha_{X^\gamma_1} & \ldots & S^\alpha_{X^{p_{l+1}}} \\ S^\alpha_{X^\gamma_1} & \ldots & \tilde{S}^\alpha_{X^\gamma_1} & \ldots & S^\alpha_{X^{p_{l+1}}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ S^\alpha_{X^\gamma_1} & \ldots & \tilde{S}^\alpha_{X^\gamma_1} & \ldots & S^\alpha_{X^{p_{l+1}}} \end{vmatrix}, \quad p = p_j \in \mathcal{N}_2 , \quad l \geq 1
\]
We also put by definition

$$\tilde{\Delta}_{(\mathcal{N}_1,\mathcal{N}_2)}^p \equiv 1$$

for \( l = 0 \), \( \mathcal{N}_1 = \emptyset \), \( \mathcal{N}_2 = \{ p \} \).

So, we can write here

$$g^{kl} \in \text{Span} \{ g_{(\mathcal{N}_1,\mathcal{N}_2)} \}$$

for all \( k, l = 1, \ldots, d \).

**Definition 3.2.**

Let the bracket (1.1) have annihilators of the physical form and the family \( \Lambda \) represent a complete Hamiltonian family of \( m \)-phase solutions of system (1.2) equipped with a minimal set of commuting integrals. Let the bracket (1.23) represent the averaging of the bracket (1.1) on the family \( \Lambda \).

1) We say that the bracket (1.23) has a non-degenerate annihilator part if we have

$$\det |g^{lk,p}| \neq 0$$

at least for one \( p \) in coordinates \((S^1(X), \ldots, S^m(X), \tilde{Q}_1(X), \ldots, \tilde{Q}_m(X), N^1(X), \ldots, N^s(X))\).

2) We say that the bracket (1.23) has a simple annihilator part if we have

$$g^{lk,p} = \text{const}$$

(all \( p \)) in coordinates \((S^1(X), \ldots, S^m(X), \tilde{Q}_1(X), \ldots, \tilde{Q}_m(X), N^1(X), \ldots, N^s(X))\).

Let us formulate now the Theorem related to the canonical form of the brackets (1.23), which are obtained by the averaging of the brackets (1.1) having the special property, formulated above.

**Theorem 3.1.**

Let the bracket (1.1) have annihilators of the physical form and the family \( \Lambda \) represent a complete Hamiltonian family of \( m \)-phase solutions of system (1.2) equipped with a minimal set of commuting integrals. Let the bracket (1.23) represent the averaging of the bracket (1.1) on the family \( \Lambda \).

Let the bracket (1.23) have a simple non-degenerate annihilator part.

Then there exists locally a smooth change of coordinates

$$(U^1, \ldots, U^{m+s}) \rightarrow (Q_1, \ldots, Q_m, N^1, \ldots, N^s)$$

(3.2)

$$Q_\alpha = Q_\alpha (S_{X1}, \ldots, S_{Xd}, U), \quad \tilde{N}^l = \tilde{N}^l (S_{X1}, \ldots, S_{Xd}, U),$$

such that we have for the Poisson brackets of the functionals \( S(X), \ Q(X), \ \tilde{N}(X) \):

$$\{S^\alpha(X), S^\beta(Y)\} = 0, \quad \{S^\alpha(X), Q_\beta(Y)\} = \delta_\alpha^\beta \delta(X - Y), \quad \{S^\alpha(X), \tilde{N}^l(Y)\} = 0$$

$$\{Q_\alpha(X), Q_\beta(Y)\} = 0, \quad \{Q_\alpha(X), \tilde{N}^l(Y)\} = 0$$

(3.3)

$$\{\tilde{N}^l(X), \tilde{N}^k(Y)\} = g^{lk,p} \delta_{X^p}(X - Y)$$

($$g^{lk,p} = \text{const}$$).

**Proof.**
Let us consider the bracket \((1.23)\) in the coordinates
\[
\left( S^1(X), \ldots, S^m(X), \tilde{Q}_1(X), \ldots, \tilde{Q}_m(X), N^1(X), \ldots, N^s(X) \right)
\]
where it has the form \((3.1)\).

Let us assume here without loss of generality that we have
\[\det \left| g_{lk} \right| \neq 0\]
for the pairwise Poisson brackets of the densities of annihilators.

Let us consider now the Jacobi identities
\[
\{ \{ \tilde{Q}_\alpha(X), N^l(Y) \}, N^k(Z) \} - \{ \{ \tilde{Q}_\alpha(X), N^k(Z) \}, N^l(Y) \} - \{ \tilde{Q}_\alpha(X), \{ N^l(Y), N^k(Z) \} \} \equiv 0
\]
It’s not difficult to check that the identities above are equivalent to the following set of relations
\[\frac{\partial A_{\alpha}^{l,p}}{\partial N^r} g^{r,k,q} - \frac{\partial A_{\alpha}^{k,q}}{\partial N^r} g^{r,l,p} = 0 \quad (3.4)\]
Let us put now in \((3.4)\): \(q = p = 1\). We get then
\[\frac{\partial A_{\alpha}^{l,1}}{\partial N^r} g^{r,k,1} = \frac{\partial A_{\alpha}^{k,1}}{\partial N^r} g^{r,l,1}\]
Using the inverse tensor \(g_{lk}^1\) we can write the relation above in the equivalent form
\[\frac{\partial}{\partial N^l} \left( A_{\alpha}^{r,1} g_{rl}^1 \right) = \frac{\partial}{\partial N^k} \left( A_{\alpha}^{r,1} g_{rl}^1 \right)\]
which implies
\[A_{\alpha}^{r,1} g_{rl}^1 \equiv \frac{\partial f_\alpha}{\partial N^k}\]
or
\[A_{\alpha}^{l,1} \equiv \frac{\partial f_\alpha}{\partial N^k} g^{kl,1}\]
for some functions \(f_\alpha(S_{X^1}, \ldots, S_{X^d}, N)\).

Let us just put now \(q = 1\) in relations \((3.4)\). We immediately get then
\[\frac{\partial A_{\alpha}^{l,p}}{\partial N^j} = \frac{\partial A_{\alpha}^{k,1}}{\partial N^r} g^{r,p} g_{k,l}^1 \equiv \frac{\partial}{\partial N^r} \left( A_{\alpha}^{k,1} g^{r,p} g_{k,l}^1 \right) = \frac{\partial}{\partial N^r} \frac{\partial}{\partial N^j} \left( f_\alpha g^{r,p} \right), \quad \forall p\]
Thus, we can put
\[A_{\alpha}^{l,p} \equiv \frac{\partial f_\alpha}{\partial N^k} g^{kl,p} + \gamma_{\alpha}^{l,p} (S_{X^1}, \ldots, S_{X^d}) \quad (p \geq 2)\]
for some functions \(\gamma_{\alpha}^{l,p} (S_{X^1}, \ldots, S_{X^d})\).

Let us put now
\[Q'_\alpha(X) = \tilde{Q}_\alpha(X) - f_\alpha (S_{X^1}, \ldots, S_{X^d}, N)\]

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Easy to see that we have then:

\[ \{Q'_\alpha(X), N^l(Y)\} = \sum_{p=2}^s \gamma^{l,p}_{\alpha} (S_{X^1}, \ldots, S_{X^d}) \delta_{X^p} (X - Y), \]

\[ \{S^\alpha(X), Q'_\beta(Y)\} = \delta_{\beta}^\alpha \delta(X - Y) \]

for the new coordinates \(Q'_\alpha(X)\).

Consider now the Jacobi identities

\[ \{Q'_\alpha(X), \{Q'_\beta(Y), N^l(Z)\}\} - \{Q'_\beta(Y), \{Q'_\alpha(X), N^l(Z)\}\} \equiv \\equiv \{\{Q'_\alpha(X), Q'_\beta(Y)\}, N^l(Z)\} \quad (3.5) \]

It’s not difficult to check that from the identities (3.5) and the conditions

\[ \gamma_{\alpha}^{l,1} \equiv 0, \quad \det ||g^{kl,1}|| \neq 0 \]

we immediately get the relations

\[ \frac{\delta}{\delta N^k(W)} \{Q'_\alpha(X), Q'_\beta(Y)\} \equiv 0 \]

which means

\[ \{Q'_\alpha(X), Q'_\beta(Y)\} = \{Q'_\alpha(X), Q'_\beta(Y)\}[S] \]

Applying now Theorem 2.1 to the set of the variables \((S^1(X), \ldots, S^m(X), Q'_1(X), \ldots, Q'_m(X))\) we can define the variables

\[ Q_\alpha(X) = Q'_\alpha(X) + q'_\alpha (S_{X^1}, \ldots, S_{X^d}) \]

satisfying the relations

\[ \{S^\alpha(X), Q_\beta(Y)\} = \delta_{\beta}^\alpha \delta(X - Y), \quad \{Q_\alpha(X), Q_\beta(Y)\} = 0, \]

\[ \{Q_\alpha(X), N^l(Y)\} = \sum_{p=2}^s \gamma^{l,p}_{\alpha} (S_{X^1}, \ldots, S_{X^d}) \delta_{X^p} (X - Y) \]

Another corollary of the identities (3.5) is given by the relations

\[ \frac{\partial \gamma^{l,p}_{\alpha}}{\partial S_{X^p}^\beta} \equiv - \frac{\partial \gamma^{l,p}_{\beta}}{\partial S_{X^p}^\alpha}, \quad \frac{\partial \gamma^{l,p}_{\alpha}}{\partial S_{X^q}^\alpha} \equiv - \frac{\partial \gamma^{l,q}_{\alpha}}{\partial S_{X^p}^\beta} \quad (3.6) \]

\((p = 2, \ldots, s, \ q = 1, \ldots, s)\), and

\[ \left[ \frac{\partial \gamma^{l,p}_{\alpha}}{\partial S_{X^q}^\beta} \right]_{X^q} \equiv 0 \]

Relations (3.6) give, in particular, the relations

\[ \frac{\partial \gamma^{l,p}_{\alpha}}{\partial S_{X^q}^\alpha} \equiv \frac{\partial \gamma^{l,q}_{\alpha}}{\partial S_{X^p}^\alpha} \]

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which implies the relations
\[ \gamma_{\alpha}^{l,p} \equiv \frac{\partial}{\partial S_{X_p}^\alpha} g^l(S_{X^2}, \ldots, S_{X^d}), \quad (p \geq 2) \]
for some functions \( g^l(S_{X^2}, \ldots, S_{X^d}) \).

Let us put now \( \bar{N}_l(X) = N_l(X) - g^l(S_{X^2}, \ldots, S_{X^d}) \).

Using again the relations (3.6) it’s not difficult to show then that we obtain
\[ \{ Q_{\alpha}(X), \bar{N}_l(Y) \} = 0 \]
for the variables \( \bar{N}_l(Y) \).

Finally, we get the variables \( (S(X), Q(X), \bar{N}(X)) \) satisfying all the relations (3.3).

Theorem 3.1 is proved.

At the end, let us say that in general the separation of the variables \( (S(X), Q(X)) \) and \( N(X) \) into two independent brackets by a transformation of the form (3.2) is impossible for the bracket (1.23). As an example, let us consider the bracket (1.23) which has in the variables \( (S(X), \tilde{Q}(X), \tilde{N}_1(X), \tilde{N}_2(X)) \) the following form
\[ \begin{align*}
\{ S(X), \tilde{Q}(X), \tilde{N}_1(X), \tilde{N}_2(X) \} &= 0 \\
\{ S(X), S(Y) \} &= 0, \quad \{ S(X), \tilde{Q}(Y) \} = \delta(X - Y), \quad \{ \tilde{Q}(X), \tilde{Q}(Y) \} = 0 \\
\{ S(X), \tilde{N}_1(Y) \} &= \{ S(X), \tilde{N}_2(Y) \} = 0 \\
\{ \tilde{Q}(X), \tilde{N}_1(Y) \} &= \tilde{N}_2(X) \delta_{X}(X - Y), \quad \{ \tilde{Q}(X), \tilde{N}_2(Y) \} = 0 \\
\begin{pmatrix} \tilde{N}_1(X) \\ \tilde{N}_2(X) \end{pmatrix}, \begin{pmatrix} \tilde{N}_1(Y) \\ \tilde{N}_2(Y) \end{pmatrix} \end{align*} \]
\[ (X = (X^1, X^2)) \]

It can be checked by direct calculation that the bracket (3.7) is skew-symmetric and satisfies the Jacobi identity. However, it’s not difficult to check that no transformation
\[ \begin{align*}
\begin{pmatrix} \tilde{Q}(X), \tilde{N}_1(X), \tilde{N}_2(X) \end{pmatrix} &\rightarrow \begin{pmatrix} Q(X), N_1^l(X), N_2^l(X) \end{pmatrix} \\
\end{align*} \]
where
\[ Q(X) = \tilde{Q}(X) + \tilde{q} \begin{pmatrix} S_{X^1}, S_{X^2}, \tilde{N}_1(X), \tilde{N}_2(X) \end{pmatrix}, \quad N_l^l(X) = N_l \begin{pmatrix} S_{X^1}, S_{X^2}, \tilde{N}_1(X), \tilde{N}_2(X) \end{pmatrix} \]
can give the relations
\[ \{ Q(X), N_1^l(Y) \} = \{ Q(X), N_2^l(Y) \} = 0 \] (3.8)
Indeed, according to the Jacobi identities
\[
\{ \{ N^l(X), N^k(Y) \}, Q(Z) \} + \{ \{ Q(Z), N^l(X) \}, N^k(Y) \} + \\
+ \{ \{ N^k(Y), Q(Z) \}, N^l(X) \} \equiv 0 ,
\]
the fulfillment of the conditions (3.8) would imply the relations
\[
\frac{\delta}{\delta S(Z)} \{ N^l(X), N^k(Y) \} \equiv 0 ,
\]
which means the independence of the brackets \( \{ N^l(X), N^k(Y) \} \) on the variables \([S(Z)]\).

On the other hand, the Poisson brackets \( \{ N^l(X), N^k(Y) \} \) can be represented in the form
\[
\left\{ \left( \begin{array}{c} N^1(X) \\ N^2(X) \end{array} \right), \left( \begin{array}{c} N^1(Y) \\ N^2(Y) \end{array} \right) \right\} = \\
= \left( \begin{array}{cc} \partial N^1/\partial S^1 & \partial N^1/\partial S^2 \\ \partial N^2/\partial S^1 & \partial N^2/\partial S^2 \end{array} \right) \times \left[ \left( \begin{array}{cc} S^2_1 & 1 \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} \partial N^1/\partial S^1 & \partial N^1/\partial S^2 \\ \partial N^2/\partial S^1 & \partial N^2/\partial S^2 \end{array} \right) \right] \delta_X(X-Y) - \\
- \left( \begin{array}{cc} 0 & 0 \\ S^2_1 & 1 \end{array} \right) \left( \begin{array}{cc} \partial N^1/\partial S^1 & \partial N^1/\partial S^2 \\ \partial N^2/\partial S^1 & \partial N^2/\partial S^2 \end{array} \right) \delta_X(X-Y) + \\
+ \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} \partial N^1/\partial S^1 & \partial N^1/\partial S^2 \\ \partial N^2/\partial S^1 & \partial N^2/\partial S^2 \end{array} \right)_X \delta_X(X-Y) - \\
- \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} \partial N^1/\partial S^1 & \partial N^1/\partial S^2 \\ \partial N^2/\partial S^1 & \partial N^2/\partial S^2 \end{array} \right)_Y \delta_X(X-Y)
\]

We can see then, that the absence of the terms containing the derivatives \( S^1_{X^1} \) and \( S^1_{X^2} \) in the brackets \( \{ N^l(X), N^k(Y) \} \) requires, in particular, the relations
\[
\frac{\partial^2 N^l}{\partial N^1 \partial S^1} = 0 , \quad \frac{\partial^2 N^l}{\partial N^1 \partial S^2} = 0 , \quad l = 1, 2 ,
\]
which means in fact
\[
N^l \left( S^1_{X^1}, S^1_{X^2}, \tilde{N}^1, \tilde{N}^2 \right) \equiv N^l \left( \tilde{N}^1, \tilde{N}^2 \right) + N'' \left( S^1_{X^1}, S^1_{X^2}, \tilde{N}^2 \right)
\]
(3.9)

On the other hand, it’s not difficult to see that the transformations (3.9) can not transform the metric
\[
g^{ik,2}(S_X) = - \left( \begin{array}{c} S^1_X \\ 0 \\ 0 \end{array} \right)
\]
into a form, independent on \( S^1_X \). So, we get now our statement.

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