LIE ALGEBRAS OF INFINITESIMAL AUTOMORPHISMS FOR THE MODEL MANIFOLDS OF GENERAL CLASSES II, III$_2$ AND IV$_2$

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ABSTRACT

We determine the Lie algebra of infinitesimal CR-automorphisms of the models of general classes II, III$_2$ and IV$_2$ through Cartan’s equivalence method.

1. INTRODUCTION

The classification of CR-manifolds up to dimension 5 has highlighted the existence of 6 non-trivial classes of CR-manifolds, which have been referred to as general classes I, II, III$_1$, III$_2$, IV$_1$ and IV$_2$ [9]. Each of these classes entails a distinguished manifold, the model, whose Lie algebra of infinitesimal CR-automorphisms is of maximal dimension. It plays a special role, as CR-manifolds belonging to the same class can be viewed as its deformations, generally by the way of Cartan connection. The aim of this paper is to determine the Lie algebra of infinitesimal CR-automorphisms of the models for general classes II, III$_2$ and IV$_2$. This is already known [1, 5] for general classes II (Engel manifolds) and IV$_2$ (2-nondegenerate, 5-dimensional CR-manifolds of constant Levi rank 1), but is unknown, to our knowledge, in the case of general class III$_2$. In our view, the main interest of this paper is to provide a unified treatment for the 3 classes through the use of Cartan’s equivalence method, in the spirit of [10]. Cartan’s equivalence method has indeed been employed recently to solve the equivalence problem for general classes II, III$_2$ and IV$_2$ [11, 12, 13]. For each of these classes, the solution to the equivalence problem for the model has been of a great help for the treatment of the general case, as a similar structure of normalizations of the group parameters occurs in both cases.

For general class II, the model is provided by Beloshapka’s cubic in $\mathbb{C}^3$, which is the CR-manifold defined by the equations:

\[ B : \]

\[
\begin{align*}
    w_1 &= \overline{w_1} + 2i z \overline{z}, \\
    w_2 &= \overline{w_2} + 2i z \overline{z} (z + \overline{z}).
\end{align*}
\]
For general class $\text{III}_2$, the model is the 5-dimensional submanifold $N \subset \mathbb{C}^4$ defined by:

$$
N: \\
w_1 = \overline{w_1} + 2i \overline{z}z, \\
w_2 = \overline{w_2} + 2i \overline{z}z(z + \overline{z}), \\
w_3 = \overline{w_3} + 2i \overline{z}(z^2 + \frac{3}{2} \overline{z} + \overline{z}^2).
$$

For general class $\text{IV}_2$, the model is provided by the tube over the future light cone, $LC \subset \mathbb{C}^3$, defined by:

$$
LC: (\text{Re} \, z_1)^2 - (\text{Re} \, z_2)^2 - (\text{Re} \, z_3)^2 = 0, \quad \text{Re} \, z_1 > 0.
$$

A Cartan connection has been constructed for CR-manifolds belonging to general class II [1, 12] and $\text{III}_2$ [13]. The equivalence problem for manifolds belonging to general class IV$_2$ has been solved either by the determination of an absolute parallelism [4, 11], or the construction of a Cartan connection [7]. We use Cartan’s equivalence method for which we refer to [10] as a standard reference.

2. CLASS II

This section is devoted to the determination of the Lie algebra of CR-automorphisms of Beloshapka’s cubic in $\mathbb{C}^3$, which is the CR-manifold defined by the equations:

$$
B: \\
w_1 = \overline{w_1} + 2i \overline{z}z, \\
w_2 = \overline{w_2} + 2i \overline{z}z(z + \overline{z}).
$$

It is the model manifold for generic 4-dimensional CR-manifolds of CR dimension 1 and real codimension 2, i.e. CR-manifolds belonging to class II, in the sense that any such manifold might be viewed as a deformation of Beloshapka’s cubic by the way of a Cartan connection [11, 12]. The main result of this section is:

**Theorem 1.** Beloshapka’s cubic,

$$
B: \\
w_1 = \overline{w_1} + 2i \overline{z}z, \\
w_2 = \overline{w_2} + 2i \overline{z}z(z + \overline{z}),
$$

has a 5-dimensional Lie algebra of CR-automorphisms. A basis for the Maurer-Cartan forms of $\text{aut}_{CR}(B)$ is provided by the 5 differential 1-forms
σ, ρ, ζ, ζ, α, which satisfy the structure equations:
\[
\begin{align*}
\sigma &= 3 \alpha \wedge \sigma + \rho \wedge \zeta + \rho \wedge \zeta, \\
\rho &= 2 \alpha \wedge \rho + i \zeta \wedge \zeta, \\
\zeta &= \alpha \wedge \zeta, \\
\overline{\zeta} &= \alpha \wedge \zeta, \\
\alpha &= 0.
\end{align*}
\]

2.1. Initial G-structure. The vectors field \( \mathcal{L}_1 \) defined by:
\[
\mathcal{L}_1 := \frac{\partial}{\partial z} + i z \frac{\partial}{\partial u_1} + i \left(2z \overline{z} + \overline{z}^2\right) \frac{\partial}{\partial u_2},
\]

and its conjugate:
\[
\overline{\mathcal{L}}_1 := \frac{\partial}{\partial \overline{z}} - i z \frac{\partial}{\partial u_1} - i \left(2z \overline{z} + z^2\right) \frac{\partial}{\partial u_2},
\]

constitute a basis of \( T_p^1 \mathcal{B} \) at each point \( p \) of \( B \). Moreover the vector fields \( \mathcal{T} \) and \( \mathcal{J} \) defined by:
\[
\mathcal{T} := i [\mathcal{L}_1, \overline{\mathcal{L}}_1],
\]

and
\[
\mathcal{J} := [\mathcal{L}_1, \mathcal{T}],
\]

complete a frame on \( B \):
\[
\{ \mathcal{T}, \mathcal{J}, \mathcal{L}, \overline{\mathcal{L}} \}.
\]

The expressions of \( \mathcal{T} \) and \( \mathcal{J} \) are:
\[
\begin{align*}
\mathcal{T} &= 2 \frac{\partial}{\partial u_1} + (4z + 4\overline{z}) \frac{\partial}{\partial u_2}, \\
\mathcal{J} &= 4 \frac{\partial}{\partial u_2}.
\end{align*}
\]

The dual coframe \( (\sigma_0, \rho_0, \zeta_0, \overline{\zeta}_0) \) is thus given by:
\[
\begin{align*}
\sigma_0 &= \frac{i}{4} z^2 dz - \frac{i}{4} z^2 d\overline{z} - \left(\frac{1}{2} z + \frac{1}{2} \overline{z}\right) du_1 + \frac{1}{4} du_2, \\
\rho_0 &= -\frac{i}{2} iz dz + \frac{i}{2} z d\overline{z} + \frac{1}{2} du_1, \\
\zeta_0 &= dz, \\
\overline{\zeta}_0 &= d\overline{z}.
\end{align*}
\]
We deduce the structure equations enjoyed by \((\sigma_0, \rho_0, \zeta_0, \bar{\zeta}_0,):\)

\[
\begin{align*}
    d\sigma_0 &= \rho_0 \wedge \zeta_0 + \rho_0 \wedge \bar{\zeta}_0, \\
    d\rho_0 &= i \zeta_0 \wedge \bar{\zeta}_0, \\
    d\zeta_0 &= 0, \\
    d\bar{\zeta}_0 &= 0.
\end{align*}
\]

(1)

As the torsion coefficients of these structure equations are constants, we have the following result:

**Lemma 1.** Beloshapka’s cubic is locally isomorphic to a Lie group whose Maurer-Cartan forms satisfy the structure equations (1).

The matrix Lie group which encodes suitably the equivalence problem for Beloshapka’s cubic (see [12]) is the 10-dimensional Lie group \(G_1\) whose elements \(g\) are of the form:

\[
g := \begin{pmatrix}
a^2 & 0 & 0 & 0 \\
a & a & 0 & 0 \\
d & b & a & 0 \\
e & b & 0 & a \end{pmatrix}.
\]

With the notations:

\[
\omega_0 := \begin{pmatrix}
\sigma_0 \\
\rho_0 \\
\zeta_0 \\
\bar{\zeta}_0
\end{pmatrix}, \quad \omega := \begin{pmatrix}
\sigma \\
\rho \\
\zeta \\
\bar{\zeta}
\end{pmatrix},
\]

we introduce the \(G_1\)-structure \(P^1\) on \(B\) constituted by the coframes \(\omega\) which satisfy the relation:

\[
\omega := g \cdot \omega_0.
\]

The proof of theorem (1) relies on successive reductions of \(P^1\) through Cartan’s equivalence method.

2.2. **Normalization of \(a\).** The structure equations for the lifted coframe \(\omega\) are related to those of the base coframe \(\omega_0\) by the relation:

\[
d\omega = dg \cdot g^{-1} \wedge \omega + g \cdot d\omega_0.
\]

(2)

The term \(dg \cdot g^{-1} \wedge \omega\) depends only on the structure equations of \(G_1\) and is expressed through its Maurer-Cartan forms. The term \(g \cdot d\omega_0\) contains the so-called torsion coefficients of the \(G_1\)-structure. We can compute it easily in terms of the forms \(\sigma, \rho, \zeta, \bar{\zeta}\), by a simple multiplication by \(g\) in the formulae (1) and a linear change of variables. The Maurer-Cartan forms
for the group $G_1$ are given by the linearly independent entries of the matrix $dg \cdot g^{-1}$, which are:

$$\alpha^1 := \frac{da}{a},$$

$$\alpha^2 := -\frac{bda}{a^3} + \frac{db}{a^3},$$

$$\alpha^3 := -\frac{cda}{a^3} - \frac{cd\bar{a}}{a^3} + \frac{dc}{a^3},$$

$$\alpha^4 := -\frac{(da\bar{a} - bc) da}{a^3} - \frac{cda}{a^3} + \frac{dd}{a^3},$$

$$\alpha^5 := -\frac{(ea\bar{a} - \bar{b}c) da}{a^3} - \frac{cda}{a^3} + \frac{de}{a^3},$$

together with their conjugates.

The first structure equation is given by:

$$d\sigma = 2\alpha^1 \wedge \sigma + \alpha^5 \wedge \sigma + \left(\frac{e}{a^2} + \frac{d}{a^5}\right) \sigma \wedge \rho - \frac{c}{a^2} \sigma \wedge \zeta - \frac{c}{a^3} \sigma \wedge \overline{\zeta} + \rho \wedge \zeta + \frac{a}{\bar{a}} \rho \wedge \overline{\zeta},$$

from which we immediately deduce that $\frac{a}{\bar{a}}$ is an essential torsion coefficient which might be normalised to 1 by setting:

$$a = \bar{a}.$$

2.3. Normalizations of $b$ and $c$. We have thus reduced the $G_1$ equivalence problem on $B$ to a $G_2$ equivalence problem, where $G_2$ is the 9 dimensional real matrix Lie group whose elements are of the form

$$g := \begin{pmatrix} a^3 & 0 & 0 & 0 \\ c & a^2 & 0 & 0 \\ d & b & a & 0 \\ e & b & 0 & a \end{pmatrix}, \quad a \in \mathbb{R}.$$ 

The Maurer-Cartan forms of $G_2$ are given by:

$$\beta^1 := \frac{da}{a},$$

$$\beta^2 := -\frac{bda}{a^3} + \frac{db}{a^3},$$

$$\beta^3 := -\frac{cda}{a^3} - \frac{cd\bar{a}}{a^3} + \frac{dc}{a^3},$$

$$\beta^4 := -\frac{(da\bar{a} - bc) da}{a^3} - \frac{cda}{a^3} + \frac{dd}{a^3},$$

$$\beta^5 := -\frac{(ea\bar{a} - \bar{b}c) da}{a^3} - \frac{cda}{a^3} + \frac{de}{a^3},$$
together with $\overline{\beta}^2$, $\overline{\beta}^3$, $\overline{\beta}^4$, $\overline{\beta}^5$. Using formula (2), we get the structure equations for the lifted coframe $(\sigma, \rho, \zeta, \overline{\zeta})$ from those of the base coframe $(\sigma_0, \rho_0, \zeta_0, \overline{\zeta}_0)$ by a matrix multiplication and a linear change of coordinates, as in the first step:

$$
\begin{align*}
\,d\sigma &= 3 \beta^1 \wedge \sigma \\
&\quad + U_\sigma^\rho \sigma \wedge \rho + U_\sigma^\zeta \sigma \wedge \zeta + U_\sigma^{\overline{\zeta}} \sigma \wedge \overline{\zeta} + \rho \wedge \zeta + \rho \wedge \overline{\zeta},
\end{align*}
$$

$$
\begin{align*}
\,d\rho &= 2 \beta^1 \wedge \rho + \beta^3 \wedge \sigma \\
&\quad + U^\rho_\sigma \sigma \wedge \rho + U^\rho_\zeta \sigma \wedge \zeta + U_\zeta^{\overline{\zeta}} \sigma \wedge \overline{\zeta} \\
&\quad + U^\rho_{\overline{\rho}} \rho \wedge \zeta + U^\rho_{\overline{\zeta}} \rho \wedge \overline{\zeta} + \iota \zeta \wedge \overline{\zeta},
\end{align*}
$$

$$
\begin{align*}
\,d\zeta &= \beta^1 \wedge \zeta + \beta^2 \wedge \rho + \beta^4 \wedge \sigma \\
&\quad + U_\sigma^\zeta \sigma \wedge \rho + U_\sigma^{\overline{\zeta}} \sigma \wedge \overline{\zeta} + U_{\overline{\sigma}}^\zeta \sigma \wedge \zeta + U_{\overline{\rho}}^\overline{\zeta} \rho \wedge \zeta \\
&\quad + U_{\overline{\rho}}^{\overline{\zeta}} \rho \wedge \overline{\zeta} + \iota^{\overline{\zeta}} \zeta \wedge \overline{\zeta}.
\end{align*}
$$

We now proceed with the absorption phase. We introduce the modified Maurer-Cartan forms:

$$
\tilde{\beta}^i = \beta^i - y_\sigma \sigma - y^i_\rho \rho - y^i_\zeta \zeta - y^i_{\overline{\zeta}} \overline{\zeta},
$$

such that the structure equations rewrite:

$$
\begin{align*}
\,d\sigma &= 3 \tilde{\beta}^1 \wedge \sigma \\
&\quad + \left( U_\sigma^\rho - 3 y^i_\rho \right) \sigma \wedge \rho + \left( U_\sigma^\zeta - 3 y^i_\zeta \right) \sigma \wedge \zeta \\
&\quad + \left( U_\sigma^{\overline{\zeta}} - 3 y^i_{\overline{\zeta}} \right) \sigma \wedge \overline{\zeta} + \rho \wedge \zeta + \rho \wedge \overline{\zeta},
\end{align*}
$$

$$
\begin{align*}
\,d\rho &= 2 \tilde{\beta}^1 \wedge \rho + \tilde{\beta}^3 \wedge \sigma \\
&\quad + \left( U^\rho_\sigma + 2 y^1_\sigma - y^2_\rho \right) \sigma \wedge \rho + \left( U^\rho_\zeta + 2 y^1_\zeta \right) \sigma \wedge \zeta \\
&\quad + \left( U^\rho_{\overline{\zeta}} - 2 y^2_{\overline{\zeta}} \right) \sigma \wedge \overline{\zeta} + \left( U^\rho_{\overline{\rho}} - 2 y^1_{\overline{\rho}} \right) \rho \wedge \zeta \\
&\quad + \left( U^\rho_{\overline{\zeta}} - 2 y^1_{\overline{\zeta}} \right) \rho \wedge \overline{\zeta} + \iota \zeta \wedge \overline{\zeta}.
\end{align*}
$$
\[ d\zeta = \tilde{\beta}_1 \wedge \zeta + \tilde{\beta}_2 \wedge \rho + \tilde{\beta}_4 \wedge \sigma \]
\[ + \left( U_{\sigma \rho}^{\zeta} + y_{\sigma}^2 - y_{\rho}^1 \right) \sigma \wedge \rho + \left( U_{\sigma \zeta}^{\zeta} + y_{\sigma}^1 - y_{\zeta}^1 \right) \sigma \wedge \zeta \]
\[ + \left( U_{\rho \zeta}^{\zeta} - y_{\rho}^4 \right) \sigma \wedge \zeta + \left( U_{\rho \zeta}^\zeta - y_{\rho}^1 \right) \rho \wedge \zeta \]
\[ + \left( U_{\rho \zeta}^\zeta - y_{\rho}^2 \right) \rho \wedge \zeta + \left( U_{\zeta \zeta}^\zeta - y_{\zeta}^1 \right) \zeta \wedge \zeta. \]

We get the following absorption equations:

\[
3 y_{\rho}^1 = U_{\sigma \rho}^{\sigma}, \quad 3 y_{\zeta}^1 = U_{\sigma \zeta}^{\sigma}, \quad 3 y_{\zeta}^1 = U_{\sigma \zeta}^{\sigma}, \\
-2 y_{\sigma}^1 + y_{\rho}^3 = U_{\sigma \rho}^{\rho}, \quad y_{\zeta}^3 = U_{\sigma \zeta}^{\rho}, \quad y_{\zeta}^3 = U_{\sigma \zeta}^{\rho}, \\
2 y_{\zeta}^1 = U_{\rho \zeta}^{\rho}, \quad 2 y_{\zeta}^1 = U_{\rho \zeta}^{\rho}, \quad -y_{\rho}^2 + y_{\zeta}^4 = U_{\rho \zeta}^{\zeta}, \\
y_{\rho}^4 = U_{\rho \zeta}^{\zeta}, \quad y_{\zeta}^4 = U_{\rho \zeta}^{\zeta}, \quad -y_{\rho}^1 + y_{\zeta}^2 = U_{\rho \zeta}^{\zeta}. \\
y_{\zeta}^2 = U_{\rho \zeta}^{\zeta}, \quad y_{\zeta}^1 = U_{\rho \zeta}^{\zeta}, \quad U_{\zeta \zeta}^{\zeta} = \frac{1}{2} U_{\rho \zeta}^{\zeta} = \frac{1}{3} U_{\sigma \zeta}^{\zeta}. \\
\]

Eliminating \( y_{\zeta}^1 \) among the previous equations leads to:

\[
U_{\zeta \zeta}^{\zeta} = \frac{1}{2} U_{\rho \zeta}^{\rho} = \frac{1}{3} U_{\sigma \zeta}^{\sigma}. \\
\]

that is:

\[
\frac{ib}{a^2} = \frac{1}{2} \left( \frac{c}{a^3} - \frac{ib}{a^2} \right) = -\frac{1}{3} \frac{c}{a^3}, \\
\]

from which we easily deduce that

\[
b = c = 0.\]

2.4. **Normalizations of \( d \) and \( e \).** We have thus reduced the group \( G_2 \) to a new group \( G_3 \), whose elements are of the form

\[
g := \begin{pmatrix}
a^3 & 0 & 0 & 0 \\
0 & a^2 & 0 & 0 \\
d & 0 & a & 0 \\
e & 0 & 0 & a
\end{pmatrix}. \\
\]
The Maurer Cartan forms of $G_3$ are:

\[ \gamma^1 := \frac{da}{a}, \]
\[ \gamma^2 := -\frac{dda}{a^4} + \frac{dd}{a^3}, \]
\[ \gamma^3 := -\frac{eda}{a^4} + \frac{de}{a^3}. \]

The third loop of Cartan’s method is straightforward. We get the following structure equations:

\[ d\sigma = 3\gamma^1 \wedge \sigma + \frac{d+e}{a^4} \sigma \wedge \rho + \rho \wedge \zeta + \rho \wedge \bar{\zeta}, \]
\[ d\rho = 2\gamma^1 \wedge \rho + i \frac{e}{a^3} \sigma \wedge \zeta - i \frac{d}{a^3} \sigma \wedge \bar{\zeta} + i \zeta \wedge \bar{\zeta}, \]
\[ d\zeta = \gamma^1 \wedge \zeta + \gamma^2 \wedge \sigma + \frac{d(d+e)}{a^6} \sigma \wedge \rho + \frac{d}{a^3} \rho \wedge \zeta + \frac{d}{a^3} \rho \wedge \bar{\zeta}, \]
\[ d\bar{\zeta} = \gamma^1 \wedge \bar{\zeta} + \gamma^3 \wedge \sigma + \frac{e(d+e)}{a^6} \sigma \wedge \rho + \frac{e}{a^3} \rho \wedge \zeta + \frac{e}{a^3} \rho \wedge \bar{\zeta}, \]

from which we deduce that we can perform the normalizations:

\[ e = d = 0. \]

With the 1-dimensional group $G_4$ whose elements $g$ are of the form:

\[ g := \begin{pmatrix} a^3 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \]

and whose Maurer-Cartan form is given by

\[ \alpha := \frac{da}{a}, \]

we get the following structure equations:

\[ d\sigma = 3\alpha \wedge \sigma + \rho \wedge \zeta + \rho \wedge \bar{\zeta}, \]
\[ d\rho = 2\alpha \wedge \rho + i \zeta \wedge \bar{\zeta}, \]
\[ d\zeta = \alpha \wedge \zeta, \]
\[ d\bar{\zeta} = \alpha \wedge \bar{\zeta}. \]

No more normalizations are allowed at this stage. We thus just perform a prolongation by adjoining the form $\alpha$ to the structure equations, whose exterior derivative is given by:

\[ d\alpha = 0. \]
This completes the proof of Theorem 1.

3. CLASS III$_2$

This section is devoted to the determination of the Lie algebra of CR-automorphisms of the model manifold of class III$_2$ which is defined by the equations:

\[ N : \]
\[
\begin{align*}
  w_1 &= \overline{w_1} + 2i \overline{z}, \\
  w_2 &= \overline{w_2} + 2i \overline{z}(z + \overline{z}), \\
  w_3 &= \overline{w_3} + 2i \overline{z}(z^2 + \frac{3}{2} z \overline{z} + \overline{z}^2).
\end{align*}
\]

It is the model manifold for CR-manifolds belonging to class III$_2$, in the sense that any such manifold might be viewed as a deformation of $N$ by the way of a Cartan connection (13). The main result of this section is the following:

**Theorem 2.** The model of the class III$_2$:

\[ N : \]
\[
\begin{align*}
  w_1 &= \overline{w_1} + 2i \overline{z}, \\
  w_2 &= \overline{w_2} + 2i \overline{z}(z + \overline{z}), \\
  w_3 &= \overline{w_3} + 2i \overline{z}(z^2 + \frac{3}{2} z \overline{z} + \overline{z}^2),
\end{align*}
\]

has a 6-dimensional Lie algebra of CR-automorphisms. A basis for the Maurer-Cartan forms of $\text{aut}_{\text{CR}}(N)$ is provided by the 6 differential 1-forms $\tau, \sigma, \rho, \zeta, \overline{\zeta}, \alpha$, which satisfy the structure equations:

\[
\begin{align*}
  d\tau &= 4 \alpha \wedge \tau + \sigma \wedge \zeta + \sigma \wedge \overline{\zeta}, \\
  d\sigma &= 3 \alpha \wedge \sigma + \rho \wedge \zeta + \rho \wedge \overline{\zeta}, \\
  d\rho &= 2 \alpha \wedge \rho + i \zeta \wedge \overline{\zeta}, \\
  d\zeta &= \alpha \wedge \zeta, \\
  d\overline{\zeta} &= \alpha \wedge \overline{\zeta}, \\
  d\alpha &= 0.
\end{align*}
\]

3.1. **Initial G-structure.** The vector fields :

\[
\mathcal{L} := \frac{\partial}{\partial z} + i \overline{z} \frac{\partial}{\partial u_1} + i(2z \overline{z} + \overline{z}^2) \frac{\partial}{\partial u_2} + i(3z^2 \overline{z} + 3z \overline{z}^2 + \overline{z}^3) \frac{\partial}{\partial u_3},
\]

with its conjugate:

\[
\overline{\mathcal{L}} := \frac{\partial}{\partial \overline{z}} - iz \frac{\partial}{\partial u_1} - i(2z \overline{z} + \overline{z}^2) \frac{\partial}{\partial u_2} - i(3z^2 \overline{z} + 3z \overline{z}^2 + z^3) \frac{\partial}{\partial u_3},
\]
constitute a basis of $T^1_p \text{N}$ and of $T^0_p \text{N}$ at each point $p$ of $\text{N}$. Moreover the vector fields $\mathcal{T}$, $\mathcal{I}$ and $\mathcal{R}$ defined by:

$$
\mathcal{T} := i [\mathcal{L}, \mathcal{L}_1],
$$

$$
\mathcal{I} := [\mathcal{L}_1, \mathcal{I}],
$$

and

$$
\mathcal{R} := [\mathcal{L}_1, \mathcal{R}],
$$

complete a frame on $\text{N}$:

$$\{\mathcal{R}, \mathcal{I}, \mathcal{T}, \mathcal{L}, \mathcal{L}_1\}.$$ 

The expressions of $\mathcal{T}$, $\mathcal{I}$ and $\mathcal{R}$ are:

$$
\mathcal{T} := 2 \frac{\partial}{\partial u_1} + (4z + 4\overline{z}) \frac{\partial}{\partial u_2} + (6z^2 + 12z\overline{z} + 6\overline{z}^2) \frac{\partial}{\partial u_3},
$$

$$
\mathcal{I} := 4 \frac{\partial}{\partial u_2} + (12z + 12\overline{z}) \frac{\partial}{\partial u_3},
$$

$$
\mathcal{R} := 12 \frac{\partial}{\partial u_3}.
$$

The dual coframe $\{\tau_0, \sigma_0, \rho_0, \overline{\zeta}_0, \zeta_0\}$ is thus given by:

$$
\tau_0 = -\frac{i}{12} \overline{z}^3 dz + \frac{i}{12} z^3 d\overline{z} + \left( \frac{1}{4} z^2 + \frac{1}{2} z\overline{z} + \frac{1}{4} \overline{z}^2 \right) du_1 - \left( \frac{1}{4} z + \frac{1}{4} \overline{z} \right) du_2 + \frac{1}{12} du_3,
$$

$$
\sigma_0 = \frac{i}{4} z^2 dz - \frac{i}{4} z^2 d\overline{z} - \left( \frac{1}{2} z + \frac{1}{2} \overline{z} \right) du_1 + \frac{1}{4} du_2,
$$

$$
\rho_0 = -\frac{i}{2} \overline{z} dz + \frac{i}{2} z d\overline{z} + \frac{1}{2} du_1,
$$

$$
\zeta_0 = dz,
$$

$$
\overline{\zeta}_0 = d\overline{z}.
$$

We deduce the structure equations enjoyed by the base coframe $\{\tau_0, \sigma_0, \rho_0, \overline{\zeta}_0, \zeta_0\}$:

$$
d\tau_0 = \sigma_0 \land \zeta_0 + \sigma_0 \land \overline{\zeta}_0,
$$

$$
d\sigma_0 = \rho_0 \land \zeta_0 + \rho_0 \land \overline{\zeta}_0,
$$

$$
d\rho_0 = i \zeta_0 \land \overline{\zeta}_0,
$$

$$
d\zeta_0 = 0,
$$

$$
d\overline{\zeta}_0 = 0.\tag{3}$$
As the torsion coefficients of these structure equations are constants, we have the following result:

**Lemma 2.** The model of the class \( \text{III}_2 \) is locally isomorphic to a Lie group whose Maurer-Cartan forms satisfy the structure equations (3).

The matrix Lie group which encodes suitably the equivalence problem for the model of class \( \text{III}_2 \) (see [13]) is the 18-dimensional Lie group \( G_1 \) whose elements \( g \) are of the form:

\[
g := \begin{pmatrix}
a^3a & 0 & 0 & 0 & 0 \\
f & a^2a & 0 & 0 & 0 \\
g & c & a & 0 & 0 \\
h & d & b & a & 0 \\
k & e & b & 0 & a
\end{pmatrix}.
\]

With the notations:

\[
\omega_0 := \begin{pmatrix}
\tau_0 \\
\sigma_0 \\
\rho_0 \\
\zeta_0 \\
\zeta_0
\end{pmatrix}, \quad \omega := \begin{pmatrix}
\tau \\
\sigma \\
\rho \\
\zeta \\
\zeta
\end{pmatrix},
\]

we introduce the \( G_1 \)-structure \( P^1 \) on \( N \) constituted by the coframes \( \omega \) which satisfy the relation:

\[
\omega := g \cdot \omega_0.
\]

As in the case of Beloshapka’s cubic, the proof of theorem (2) relies on successive reductions of \( P^1 \) through Cartan’s equivalence method.
3.2. **Normalization of** a. The Maurer-Cartan forms of $G_1$ are given by:

\[
\begin{align*}
\alpha_1 &= \frac{da}{a}, \\
\alpha_2 &= -\frac{bda}{a^2a} - \frac{d\overline{a}}{a}, \\
\alpha_3 &= -\frac{cda}{a^2a} + \frac{dc}{a^2a}, \\
\alpha_4 &= -\frac{(da - bc) d\overline{a}}{a^2a^2} - \frac{cd\overline{b}}{a^2a^2} + \frac{dd}{a^2a^3}, \\
\alpha_5 &= -\frac{(ea - bc) d\overline{a}}{a^2a^2} - \frac{cd\overline{b}}{a^2a^2} + \frac{de}{a^2a^3}, \\
\alpha_6 &= -2\frac{fda}{a^4} - \frac{f\overline{da}}{a^3a^2} + \frac{df}{a a^3}, \\
\alpha_7 &= -\frac{(ga^2 - cf) d\overline{a}}{a^2a^6} - \frac{(ga^2 - cf) d\overline{a}}{a^3a^5} - \frac{f\overline{d}c}{a^6a^2} + \frac{dg}{a^3a^3}, \\
\alpha_8 &= -\frac{(ha^3 - dfa - bga^2 + bcf) d\overline{a}}{a^7a^4} - \frac{(ga^2 - cf) d\overline{b}}{a^6a^3} - \frac{f\overline{d}d}{a^6a^2} + \frac{dh}{a^3a^3}, \\
\alpha_9 &= -\frac{(ka^3a^2 - efa - bga^2 + bcf) d\overline{a}}{a^6a^4} - \frac{(ga^2 - cf) d\overline{b}}{a^6a^3} - \frac{f\overline{d}ef}{a^6a^2} + \frac{dk}{a^3a^3},
\end{align*}
\]

together with their conjugates.

The first structure equation is given by:

\[
d\tau = 3 \alpha_1 \wedge \tau + \overline{\alpha_1} \wedge \tau + T_{\tau \sigma} \tau \wedge \sigma + T_{\tau \rho} \tau \wedge \rho + T_{\tau \zeta} \tau \wedge \zeta
\]

\[
+ T_{\tau \zeta} \tau \wedge \zeta + T_{\sigma \rho} \sigma \wedge \rho + \sigma \wedge \zeta - \frac{a}{a} \sigma \wedge \zeta,
\]

from which we immediately deduce that $\frac{a}{a}$ is an essential torsion coefficient which shall be normalized to 1 by setting:

\[
a = \overline{a}.
\]

We thus have reduced the $G_1$ equivalence problem to a $G_2$ equivalence problem, where $G_2$ is the 10 dimensional real matrix Lie group whose elements are of the form

\[
g = \begin{pmatrix}
a^4 & 0 & 0 & 0 & 0 \\
f & a^3 & 0 & 0 & 0 \\
g & c & a^2 & 0 & 0 \\
h & d & b & a & 0 \\
k & e & b & 0 & a
\end{pmatrix},
\]
3.3. Normalizations of $f$, $b$ and $c$. The Maurer-Cartan forms of $G_2$ are
given by:

\[
\beta^1 := \frac{da}{a},
\]

\[
\beta^2 := -\frac{bd}{a^3} + \frac{db}{a^2},
\]

\[
\beta^3 := -2 \frac{cd}{a^4} + \frac{dc}{a^3},
\]

\[
\beta^4 := -\frac{(da^2 - bc)}{a^6} da - \frac{cdb}{a^5} + \frac{dd}{a^3},
\]

\[
\beta^5 := -\frac{(ea^2 - b^2 c)}{a^6} da - \frac{cd}{a^5} + \frac{de}{a^3},
\]

\[
\beta^6 := -3 \frac{f da}{a^5} + \frac{df}{a^4},
\]

\[
\beta^7 := -2 \frac{(ga^3 - cf)}{a^8} da - \frac{f dc}{a^7} + \frac{dg}{a^4},
\]

\[
\beta^8 := -\frac{(ha^5 - dfa^2 - bg a^3 + bcf)}{a^{10}} da - \frac{(ga^3 - cf)}{a^9} db - \frac{f dd}{a^7} + \frac{dh}{a^4},
\]

\[
\beta^9 := -\frac{(ka^5 - efa^2 - Bga^3 + B^2 cf)}{a^{10}} da - \frac{(ga^3 - cf)}{a^9} dB - \frac{f de}{a^5} + \frac{dk}{a^4},
\]

together with $\overline{\beta}^i$, $i = 2 \ldots 9$.

Using formula (2), we get the structure equations for the lifted coframe
$(\tau, \sigma, \rho, \varsigma, \overline{\varsigma})$ from those of the base coframe
$(\tau_0, \sigma_0, \rho_0, \varsigma_0, \overline{\varsigma}_0)$ by a matrix
multiplication and a linear change of coordinates, as in the first step:

\[
d\tau = 4 \beta^1 \wedge \tau
\]

\[
+ U^\tau_{\tau \sigma} \tau \wedge \sigma + U^\tau_{\tau \rho} \tau \wedge \rho + U^\tau_{\tau \varsigma} \tau \wedge \varsigma + U^\tau_{\tau \overline{\varsigma}} \tau \wedge \overline{\varsigma}
\]

\[
+ U^\tau_{\sigma \rho} \sigma \wedge \rho + \sigma \wedge \varsigma + \sigma \wedge \overline{\varsigma},
\]

\[
d\sigma = 3 \beta^1 \wedge \sigma + \beta^6 \wedge \tau
\]

\[
+ U^\sigma_{\tau \sigma} \tau \wedge \sigma + U^\sigma_{\tau \rho} \tau \wedge \rho + U^\sigma_{\tau \varsigma} \tau \wedge \varsigma
\]

\[
+ U^\sigma_{\tau \overline{\varsigma}} \tau \wedge \overline{\varsigma} + U^\sigma_{\sigma \rho} \sigma \wedge \rho + U^\sigma_{\sigma \varsigma} \sigma \wedge \varsigma
\]

\[
+ U^\sigma_{\sigma \overline{\varsigma}} \sigma \wedge \overline{\varsigma} + \rho \wedge \varsigma + \rho \wedge \overline{\varsigma},
\]
\[ d\rho = 2\beta^1 \wedge \rho + \beta^3 \wedge \sigma + \beta^7 \wedge \tau \]
\[ + U^{\sigma}_{\tau\sigma} \tau \wedge \sigma + U^{\rho}_{\tau\rho} \tau \wedge \rho + U^{\rho}_{\tau\zeta} \tau \wedge \zeta + U^{\rho}_{\tau\zeta} \rho \wedge \zeta + U^{\sigma}_{\sigma\rho} \sigma \wedge \rho \]
\[ + U^{\rho}_{\sigma\zeta} \sigma \wedge \zeta + U^{\rho}_{\rho\zeta} \rho \wedge \zeta + U^{\rho}_{\zeta\rho} \zeta \wedge \rho + i \zeta \wedge \zeta, \]

\[ d\zeta = \beta^1 \wedge \zeta + \beta^2 \wedge \rho + \beta^4 \wedge \sigma + \beta^8 \wedge \tau \]
\[ + U^{\zeta}_{\tau\sigma} \tau \wedge \sigma + U^{\zeta}_{\tau\rho} \tau \wedge \rho + U^{\zeta}_{\tau\zeta} \tau \wedge \zeta + U^{\zeta}_{\tau\zeta} \tau \wedge \zeta \]
\[ + U^{\zeta}_{\sigma\rho} \sigma \wedge \rho + U^{\zeta}_{\sigma\zeta} \sigma \wedge \zeta + U^{\zeta}_{\rho\zeta} \rho \wedge \zeta \]
\[ + U^{\zeta}_{\rho\zeta} \rho \wedge \zeta + U^{\zeta}_{\rho\zeta} \rho \wedge \zeta + U^{\zeta}_{\zeta\rho} \zeta \wedge \rho. \]

We now proceed with the absorption phase. We introduce the modified Maurer-Cartan forms:

\[ \tilde{\beta}^i = \beta^i - y^i_\tau \tau - y^i_\sigma \sigma - y^i_\rho \rho - y^i_\zeta \zeta - y^i_\zeta \zeta. \]

The structure equations rewrite:

\[ d\tau = 4 \tilde{\beta}^1 \wedge \tau \]
\[ + (U^{\tau}_{\tau\sigma} - 4 y^1_{\tau}) \tau \wedge \sigma + (U^{\tau}_{\tau\rho} - 4 y^1_{\rho}) \tau \wedge \rho \]
\[ + (U^{\tau}_{\tau\zeta} - 4 y^1_{\zeta}) \tau \wedge \zeta + (U^{\tau}_{\tau\zeta} - 4 y^1_{\zeta}) \tau \wedge \zeta \]
\[ + U^{\tau}_{\sigma\rho} \sigma \wedge \rho + \sigma \wedge \zeta + \sigma \wedge \zeta, \]

\[ d\sigma = 3 \tilde{\beta}^1 \wedge \sigma + \tilde{\beta}^6 \wedge \tau \]
\[ + (U^{\sigma}_{\tau\sigma} + 3 y^1_{\tau} - y^6_{\sigma}) \tau \wedge \sigma + (U^{\sigma}_{\tau\rho} - y^6_{\rho}) \tau \wedge \rho \]
\[ + (U^{\sigma}_{\tau\zeta} - y^6_{\zeta}) \tau \wedge \zeta + (U^{\sigma}_{\tau\zeta} - y^6_{\zeta}) \tau \wedge \zeta \]
\[ + (U^{\sigma}_{\sigma\rho} - 3 y^1_{\sigma}) \sigma \wedge \rho + (U^{\sigma}_{\sigma\zeta} - 3 y^1_{\zeta}) \sigma \wedge \zeta \]
\[ + (U^{\sigma}_{\zeta\rho} - 3 y^1_{\zeta}) \sigma \wedge \zeta + \rho \wedge \zeta + \rho \wedge \zeta, \]
\[
\begin{align*}
\, d\rho = & 2\tilde{\beta}^1 \wedge \rho + \tilde{\beta}^3 \wedge \sigma + \tilde{\beta}^7 \wedge \tau \\
+ & \left( U_{\tau\sigma}^\rho + y_2^3 - y_7^7 \right) \tau \wedge \sigma + \left( U_{\tau\rho}^\sigma + 2 y_7^1 - y_7^7 \right) \tau \wedge \rho \\
+ & \left( U_{\tau\zeta}^\rho - y_7^7 \right) \tau \wedge \zeta + \left( U_{\tau\zeta}^\sigma - y_7^7 \right) \rho \wedge \zeta \\
+ & \left( U_{\sigma\rho}^\rho + 2 y_7^1 - y_7^7 \right) \sigma \wedge \rho + \left( U_{\sigma\zeta}^\rho - y_7^7 \right) \sigma \wedge \zeta \\
+ & \left( U_{\sigma\zeta}^\rho - y_7^7 \right) \sigma \wedge \zeta + \left( U_{\rho\zeta}^\rho - 2 y_7^1 \right) \rho \wedge \zeta \\
+ & \left( U_{\rho\zeta}^\rho - 2 y_7^1 \right) \rho \wedge \zeta + i \zeta \wedge \zeta ,
\end{align*}
\]

\[
\begin{align*}
\, d\zeta = & \tilde{\beta}^1 \wedge \zeta + \tilde{\beta}^2 \wedge \rho + \tilde{\beta}^4 \wedge \sigma + \tilde{\beta}^8 \wedge \tau \\
+ & \left( U_{\zeta\sigma}^\rho + y_2^4 - y_8^8 \right) \tau \wedge \sigma + \left( U_{\zeta\rho}^\sigma + y_2^4 - y_8^8 \right) \tau \wedge \rho \\
+ & \left( U_{\zeta\zeta}^\rho + y_2^4 - y_8^8 \right) \tau \wedge \zeta + \left( U_{\zeta\zeta}^\sigma - y_8^8 \right) \tau \wedge \zeta \\
+ & \left( U_{\sigma\rho}^\rho + y_2^4 - y_8^8 \right) \sigma \wedge \rho + \left( U_{\sigma\zeta}^\rho + y_2^4 - y_8^8 \right) \sigma \wedge \zeta \\
+ & \left( U_{\sigma\zeta}^\rho + y_2^4 - y_8^8 \right) \sigma \wedge \zeta + \left( U_{\rho\zeta}^\rho + y_2^4 - y_8^8 \right) \rho \wedge \zeta \\
+ & \left( U_{\rho\zeta}^\rho - y_8^8 \right) \rho \wedge \zeta + \left( U_{\zeta\zeta}^\rho - y_2^4 \right) \zeta \wedge \zeta .
\end{align*}
\]

We get the following absorption equations:

\[
\begin{align*}
4 y_7^1 &= U_{\tau\sigma}^\tau, & 4 y_7^1 &= U_{\tau\rho}^\tau, & 4 y_7^1 &= U_{\tau\zeta}^\tau, \\
4 y_7^1 &= U_{\tau\zeta}^\sigma, & -3 y_7^1 + y_6^6 &= U_{\tau\sigma}^\sigma, & y_6^6 &= U_{\tau\sigma}^\tau, & 3 y_7^1 &= U_{\tau\rho}^\sigma, \\
y_7^6 &= U_{\tau\zeta}^\zeta, & y_7^6 &= U_{\tau\zeta}^\tau, & -2 y_7^1 + y_7^7 &= U_{\tau\rho}^\rho, & y_7^7 &= U_{\tau\rho}^\tau, & y_7^7 &= U_{\tau\zeta}^\tau, \\
y_7^3 &= U_{\rho\zeta}^\sigma, & 3 y_7^1 &= U_{\rho\zeta}^\tau, & -2 y_7^1 + y_3^3 &= U_{\rho\rho}^\rho, & y_3^3 &= U_{\rho\rho}^\tau, & y_3^3 &= U_{\rho\zeta}^\tau, \\
2 y_7^1 &= U_{\rho\zeta}^\rho, & 2 y_7^1 &= U_{\rho\zeta}^\rho, & -2 y_7^1 + y_2^2 &= U_{\rho\psi}^\psi, & y_2^2 &= U_{\rho\psi}^\rho, & -y_7^1 + y_8^8 &= U_{\zeta\sigma}^\rho, \\
-2 y_7^1 + y_8^8 &= U_{\zeta\rho}^\rho, & -y_7^1 + y_8^8 &= U_{\zeta\rho}^\zeta, & y_8^8 &= U_{\zeta\sigma}^\zeta, \\
-2 y_7^1 + y_4^4 &= U_{\zeta\zeta}^\rho, & -y_7^1 + y_4^4 &= U_{\zeta\zeta}^\zeta, & y_4^4 &= U_{\zeta\zeta}^\zeta, \\
-4 y_7^1 &= U_{\rho\zeta}^\zeta, & y_7^2 &= U_{\rho\zeta}^\zeta, & y_7^2 &= U_{\zeta\zeta}^\zeta.
\end{align*}
\]
Eliminating $y_\zeta$ among the previous equations leads to:

\[ U_\zeta = \frac{1}{2} U_\rho = \frac{1}{3} U_\sigma = \frac{1}{4} U_\tau, \]

that is:

\[ \frac{ib}{a^2} = \frac{1}{2} \left( \frac{c}{a^3} - \frac{ib}{a^2} \right) = -\frac{1}{3} \left( \frac{c}{a^3} + \frac{f}{a^4} \right) = -\frac{1}{4} \frac{f}{a^4}, \]

from which we easily deduce that

\[ b = c = f = 0. \]

We have thus reduced the group $G_2$ to a new group $G_3$, whose elements are of the form

\[ g := \begin{pmatrix} a^4 & 0 & 0 & 0 & 0 \\ 0 & a^3 & 0 & 0 & 0 \\ g & 0 & a^2 & 0 & 0 \\ h & d & 0 & a & 0 \\ k & e & 0 & 0 & a \end{pmatrix}. \]

3.4. **Normalization of $g$, $d$ and $e$.** The Maurer Cartan forms of $G_3$ are:

\[ \gamma^1 := \frac{da}{a}, \]
\[ \gamma^2 := -\frac{dd}{a^4} + \frac{dd}{a^3}, \]
\[ \gamma^3 := -\frac{ed}{a^4} + \frac{de}{a^3}, \]
\[ \gamma^4 := -2 \frac{gda}{a^5} + \frac{dg}{a^4}, \]
\[ \gamma^5 := -\frac{hda}{a^5} + \frac{dh}{a^4}, \]
\[ \gamma^6 := -\frac{kda}{a^5} + \frac{dk}{a^4}. \]

We get the following structure equations:

\[ d\tau = 4 \gamma^1 \wedge \tau + V_{\tau\sigma}^\tau \tau \wedge \sigma + \sigma \wedge \zeta + \sigma \wedge \bar{\zeta}, \]
\[ d\sigma = 3 \gamma^1 \wedge \sigma + V_{\tau\rho}^\sigma \tau \wedge \rho + V_{\tau\zeta}^\sigma \tau \wedge \zeta + V_{\tau\bar{\zeta}}^\sigma \tau \wedge \bar{\zeta} + V_{\sigma\rho}^\tau \sigma \wedge \rho \wedge \zeta + \rho \wedge \bar{\zeta}, \]
\[ d\rho = 2 \gamma^1 \wedge \rho + \gamma^4 \wedge \tau + V_{\tau\sigma}^\rho \tau \wedge \sigma + V_{\tau\zeta}^\rho \tau \wedge \zeta + V_{\tau\bar{\zeta}}^\rho \tau \wedge \bar{\zeta} + V_{\sigma\rho}^\tau \sigma \wedge \zeta + V_{\sigma\rho}^\bar{\zeta} \sigma \wedge \bar{\zeta} + i \zeta \wedge \bar{\zeta}, \]
\[ d\zeta = \gamma^1 \wedge \zeta + \gamma^2 \wedge \sigma + \gamma^5 \wedge \tau \\
+ V_\tau^\zeta \tau \wedge \sigma + V_\tau^\zeta \tau \wedge \rho + V_\tau^\zeta \tau \wedge \zeta + V_\tau^\zeta \tau \wedge \zeta \\
+ V_\sigma^\zeta \sigma \wedge \rho + V_\sigma^\zeta \sigma \wedge \zeta + V_\sigma^\zeta \sigma \wedge \zeta + V_\rho^\zeta \rho \wedge \zeta \\
\]

and

\[ d\zeta = \gamma^1 \wedge \zeta + \gamma^3 \wedge \sigma + \gamma^6 \wedge \tau \\
+ V_\tau^\zeta \tau \wedge \sigma + V_\tau^\zeta \tau \wedge \rho + V_\tau^\zeta \tau \wedge \zeta + V_\tau^\zeta \tau \wedge \zeta \\
+ V_\sigma^\zeta \sigma \wedge \rho + V_\sigma^\zeta \sigma \wedge \zeta + V_\sigma^\zeta \sigma \wedge \zeta + V_\rho^\zeta \rho \wedge \zeta . \]

From these equations, we immediately see that \( V_\tau^\sigma, V_\tau^\zeta \) and \( V_\rho^\zeta \) are essential torsion coefficients. As we have:

\[ V_\tau^\sigma = -\frac{g}{a^4}, \quad V_\tau^\zeta = \frac{d}{a^3}, \quad V_\rho^\zeta = \frac{e}{a^3}, \]

we obtain the new normalizations:

\[ d = e = g = 0. \]

The reduced group \( G_4 \) is of the form:

\[ g := \begin{pmatrix}
  a^4 & 0 & 0 & 0 & 0 \\
  0 & a^3 & 0 & 0 & 0 \\
  0 & 0 & a^2 & 0 & 0 \\
  h & 0 & 0 & a & 0 \\
  k & 0 & 0 & 0 & a
\end{pmatrix}. \]

Its Maurer-Cartan forms are given by:

\[ \delta^1 := \frac{da}{a}, \]

\[ \delta^2 := -\frac{hda}{a^5} + \frac{dh}{a^4}, \]

\[ \delta^3 := -\frac{kda}{a^5} + \frac{dk}{a^4}. \]

The structure equations are easily computed as:
\[ d\tau = 4 \delta^1 \wedge \tau + \frac{h + k}{a^4} \tau \wedge \sigma + \sigma \wedge \zeta + \sigma \wedge \overline{\zeta}, \]
\[ d\sigma = 3 \delta^1 \wedge \sigma + \frac{h + k}{a^4} \tau \wedge \rho + \rho \wedge \zeta + \rho \wedge \overline{\zeta}, \]
\[ d\rho = 2 \delta^1 \wedge \rho + i \frac{k}{a^4} \tau \wedge \zeta - i \frac{h}{a^4} \tau \wedge \overline{\zeta} + i \zeta \wedge \overline{\zeta}, \]
\[ d\zeta = \delta^1 \wedge \zeta + \delta^2 \wedge \tau + \frac{h (h + k)}{a^8} \tau \wedge \sigma + \frac{h}{a^4} \sigma \wedge \zeta + \frac{h}{a^4} \sigma \wedge \overline{\zeta}, \]
\[ d\overline{\zeta} = \delta^1 \wedge \overline{\zeta} + \delta^3 \wedge \tau + \frac{k (h + k)}{a^8} \tau \wedge \sigma + \frac{k}{a^4} \sigma \wedge \zeta + \frac{k}{a^4} \sigma \wedge \overline{\zeta}. \]

We deduce from these equations that we can perform the normalization:
\[ h = k = 0. \]

With the 1-dimensional group \( G_5 \) of the form:
\[
g := \begin{pmatrix}
a^4 & 0 & 0 & 0 & 0 \\
0 & a^3 & 0 & 0 & 0 \\
0 & 0 & a^2 & 0 & 0 \\
0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & a
\end{pmatrix},
\]
whose Maurer-Cartan form is given by
\[ \alpha := \frac{da}{a^4}, \]
we get the following structure equations:
\[ d\tau = 4 \alpha \wedge \tau + \sigma \wedge \zeta + \sigma \wedge \overline{\zeta}, \]
\[ d\sigma = 3 \alpha \wedge \sigma + \rho \wedge \zeta + \rho \wedge \overline{\zeta}, \]
\[ d\rho = 2 \alpha \wedge \rho + i \zeta \wedge \overline{\zeta}, \]
\[ d\zeta = \alpha \wedge \zeta, \]
\[ d\overline{\zeta} = \alpha \wedge \overline{\zeta}. \]

No more normalizations are allowed at this stage. We thus just perform a prolongation by adjoining the form \( \alpha \) to the structure equations, whose exterior derivative is given by:
\[ d\alpha = 0. \]

This completes the proof of Theorem 2.
4. CLASS IV₂

Class IV₂ is constituted by the 5-dimensional real hypersurfaces \( M^5 \subset \mathbb{C}^3 \) which are of CR-dimension 2, whose Levi form is of constant rank 1 and which are 2-nondegenerate, i.e. their Freeman forms are non-zero. The most symmetric manifold of this class is the tube over the future light cone, which is defined by the equation:

\[
\text{LC} : \quad (\text{Re} \, z_1)^2 - (\text{Re} \, z_2)^2 - (\text{Re} \, z_3)^2 = 0, \quad \text{Re} \, z_1 > 0.
\]

This section is devoted to the determination of the Lie algebra \( \text{aut}_{\text{CR}}(\text{LC}) \) of infinitesimal CR-automorphisms of \( \text{LC} \). This has been done before by Kaup and Zaitsev [5]. We prove the following result:

**Theorem 3.** The tube over the future light cone:

\[
\text{LC} : \quad (\text{Re} \, z_1)^2 - (\text{Re} \, z_2)^2 - (\text{Re} \, z_3)^2 = 0, \quad \text{Re} \, z_1 > 0.
\]

has a 10-dimensional Lie algebra of CR-automorphisms. A basis for the Maurer-Cartan forms of \( \text{aut}_{\text{CR}}(\text{LC}) \) is provided by the 10 differential 1-forms \( \rho, \kappa, \zeta, \overline{\kappa}, \overline{\zeta}, \pi^1, \pi^2, \overline{\pi^1}, \overline{\pi^2}, \Lambda \), which satisfy the Maurer-Cartan equations:

\[
\begin{align*}
    d\rho &= \pi^1 \wedge \rho + \overline{\pi^1} \wedge \rho + i \kappa \wedge \overline{\kappa}, \\
    d\kappa &= \pi^1 \wedge \kappa + \pi^2 \wedge \rho + \zeta \wedge \overline{\kappa}, \\
    d\zeta &= i \pi^2 \wedge \kappa + \overline{\pi^1} \wedge \zeta - \pi^1 \wedge \overline{\zeta}, \\
    d\overline{\kappa} &= \pi^1 \wedge \kappa + \pi^2 \wedge \rho - \kappa \wedge \overline{\zeta}, \\
    d\overline{\zeta} &= -i \pi^2 \wedge \kappa + \overline{\pi^1} \wedge \zeta - \pi^1 \wedge \overline{\zeta}, \\
    d\pi^1 &= \Lambda \wedge \rho + i \kappa \wedge \overline{\pi^2} + \zeta \wedge \overline{\zeta}, \\
    d\pi^2 &= \Lambda \wedge \kappa + \zeta \wedge \overline{\pi^2} + \pi^2 \wedge \overline{\pi^1}, \\
    d\overline{\pi^1} &= \Lambda \wedge \rho - i \kappa \wedge \pi^2 - \zeta \wedge \overline{\zeta}, \\
    d\overline{\pi^2} &= \Lambda \wedge \pi^1 + \overline{\zeta} \wedge \pi^2 - \pi^1 \wedge \overline{\pi^2}, \\
    d\Lambda &= -\pi^1 \wedge \Lambda + i \pi^2 \wedge \overline{\pi^2} - \pi^1 \wedge \overline{\Lambda}.
\end{align*}
\]

4.1. **Geometric set-up.** In order to motivate our subsequent notations, it is convenient to introduce some general results on CR-manifolds belonging to class IV₂, for which we refer to [11] for a proof.

Let \( M \subset \mathbb{C}^3 \) be a smooth hypersurface locally represented as a graph over the 5-dimensional real hyperplane \( \mathbb{C}z_1 \times \mathbb{C}z_2 \times \mathbb{R}v \):

\[
u = F(z_1, z_2, \overline{z_1}, \overline{z_2}, v),
\]
where $F$ is a local smooth function depending on 5 arguments. We assume that $M$ is a CR-submanifold of CR dimension 2 which is 2-nondegenerate and whose Levi form is of constant rank 1. The two vector fields $\mathcal{L}_1$ and $\mathcal{L}_2$ defined by:

$$\mathcal{L}_j = \frac{\partial}{\partial z_j} + A^j \frac{\partial}{\partial v}, \quad A^j := -i \frac{F_{z_j}}{1 + i F_v}, \quad j = 1, 2,$$

constitute a basis of $T_p^{1,0}M$ at each point $p$ of $M$ and thus provide an identification of $T_p^{1,0}M$ with $\mathbb{C}^2$ at each point. Moreover, the real 1-form $\sigma$ defined by:

$$\sigma := dv - A^1 dz_1 - A^2 dz_2 - \overline{A^1 d\overline{z}_1} - \overline{A^2 d\overline{z}_2},$$

satisfies

$$\{\sigma = 0\} = T^{1,0}M \oplus T^{0,1}M,$$

and thus provides an identification of the projection

$$\mathbb{C} \otimes T_pM \rightarrow \mathbb{C} \otimes T_pM / (T_p^{1,0}M \oplus T_p^{0,1}M)$$

with the map $\sigma_p: \mathbb{C} \otimes T_pM \rightarrow \mathbb{C}$. With these two identifications, the Levi form $LF$ can be viewed at each point $p$ as a skew hermitian form on $\mathbb{C}^2$ represented by the matrix:

$$LF = \left( \begin{array}{cc} \sigma_p (i [\mathcal{L}_1, \overline{\mathcal{L}_1}]) & \sigma_p (i [\mathcal{L}_2, \overline{\mathcal{L}_1}]) \\ \sigma_p (i [\mathcal{L}_1, \overline{\mathcal{L}_2}]) & \sigma_p (i [\mathcal{L}_2, \overline{\mathcal{L}_2}]) \end{array} \right).$$

The fact that $LF$ is supposed to be of constant rank 1 ensures the existence of a certain function $k$ such that the vector field $\mathcal{K} := k \mathcal{L}_1 + \mathcal{L}_2$

lies in the kernel of $LF$. Here are the expressions of $\mathcal{K}$ and $k$ in terms of the graphing function $F$:

$$\mathcal{K} = k \partial_{z_1} + \partial_{z_2} - \frac{i}{1 + i F_v} (k F_{z_1} + F_{z_2}) \partial_v,$$

$$k = - \frac{F_{z_2, \overline{z}_1} + F_{z_2, \overline{v}} F_{\overline{z}_1}^2 \overline{F}_{\overline{z}_1} - i F_{\overline{z}_1} F_{\overline{z}_2, v} - F_{\overline{z}_2} F_v F_{z_2, z_1} + i F_{z_2} \overline{F}_{\overline{z}_1} F_{\overline{z}_2} F_v - F_{z_2} F_v F_{\overline{z}_1} \overline{F}_{\overline{z}_2}}{F_{\overline{z}_1} + F_{\overline{z}_2} \overline{F}_{\overline{z}_1}^2 - i F_{\overline{z}_1} F_{\overline{z}_2, v} - F_{\overline{z}_2} F_v F_{z_2, z_1} + i F_{z_2} \overline{F}_{\overline{z}_1} F_{\overline{z}_2} F_v - F_{z_2} F_v F_{\overline{z}_1} \overline{F}_{\overline{z}_2}}.$$

From the above construction, the four vector fields $\mathcal{L}_1$, $\mathcal{K}$, $\overline{\mathcal{L}_1}$, $\overline{\mathcal{K}}$ constitute a basis of $T_p^{1,0}M \oplus T_p^{0,1}M$ at each point $p$ of $M$. It turns out that the vector field $\mathcal{I}$ defined by:

$$\mathcal{I} := i [\mathcal{L}_1, \overline{\mathcal{L}_1}]$$

is linearly independent from $\mathcal{L}_1$, $\mathcal{K}$, $\overline{\mathcal{L}_1}$, $\overline{\mathcal{K}}$. 
It is well known (see [3, 8]) that the tube over the future light cone is locally biholomorphic to the graphed hypersurface:

\[ u = \frac{z_1 \overline{z}_1 + \frac{1}{2} z_1^2 \overline{z}_2 + \frac{1}{2} \overline{z}_1^2 z_2}{1 - z_2 \overline{z}_2}. \]

The five vector fields \( \mathcal{L}_1, \mathcal{K}, \mathcal{L}_1, \mathcal{K} \) and \( \mathcal{T} \), which constitute a local frame on \( \text{LC} \), have thus the following expressions:

\[ \mathcal{L}_1 := \frac{\partial}{\partial z_1} - \frac{i \overline{z}_1 + z_1 \overline{z}_2}{1 - z_2 \overline{z}_2} \frac{\partial}{\partial v}, \]

\[ \mathcal{K} := -\frac{\overline{z}_1 + z_1 \overline{z}_2}{1 - z_2 \overline{z}_2} \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \frac{i z_1^2 + 2 z_1 \overline{z}_1 \overline{z}_2 + z_1^2 \overline{z}_2^2}{2 (1 - z_2 \overline{z}_2)^2} \frac{\partial}{\partial v}, \]

and

\[ \mathcal{T} := -\frac{2}{1 - z_2 \overline{z}_2} \frac{\partial}{\partial v}. \]

Moreover the function \( k \) is given by

\[ k := -\frac{\overline{z}_1 + z_1 \overline{z}_2}{1 - z_2 \overline{z}_2}. \]

Let \((\rho_0, \kappa_0, \zeta_0, \overline{\kappa}_0, \overline{\zeta}_0)\) be the dual coframe of \((\mathcal{T}, \mathcal{L}_1, \mathcal{K}, \mathcal{L}_1, \mathcal{K})\). We have:

\[
\rho_0 = -\frac{i}{2} \left( \frac{\overline{z}_1 + z_1 \overline{z}_2}{1 - z_2 \overline{z}_2} \right) dz_1 - \frac{i \overline{z}_1^2 + 2 z_1 \overline{z}_1 \overline{z}_2 + z_1^2 \overline{z}_2^2}{4 (1 - z_2 \overline{z}_2)} dz_2 + \frac{i}{2} \overline{z}_1 (z_1 + \overline{z}_1 \overline{z}_2) d\overline{z}_1
\]

\[
+ \frac{i z_1^2 + 2 z_1 \overline{z}_1 \overline{z}_2 + z_1^2 \overline{z}_2^2}{4 (1 - z_2 \overline{z}_2)} dz_2 + \frac{1}{2} (-1 + z_2 \overline{z}_2) dv, \]

\[ \kappa_0 = dz_1 + \frac{\overline{z}_1 + z_1 \overline{z}_2}{1 - z_2 \overline{z}_2} dz_2, \]

\[ \zeta_0 = dz_2, \]

\[ \overline{\kappa}_0 = d\overline{z}_1 + \frac{z_1 + \overline{z}_1 \overline{z}_2}{1 - z_2 \overline{z}_2} d\overline{z}_2, \]

\[ \overline{\zeta}_0 = d\overline{z}_2. \]

A direct computation gives the structure equations enjoyed by the coframe \((\rho_0, \kappa_0, \zeta_0, \overline{\kappa}_0, \overline{\zeta}_0)\):
The matrix Lie group which encodes the equivalence problem for LC is the 10 dimensional Lie group \( G_1 \) whose elements are of the form:

\[
g := \begin{pmatrix}
    c & 0 & 0 & 0 & 0 \\
    b & c & 0 & 0 & 0 \\
    d & e & f & 0 & 0 \\
    \bar{b} & 0 & 0 & \bar{e} & 0 \\
    \bar{d} & 0 & 0 & \bar{f} & 0
\end{pmatrix},
\]

where \( c \) and \( f \) are non-zero complex numbers whereas \( b, d \) and \( e \) are arbitrary complex numbers (see [11, 9]). We introduce the 5 new one-forms \( \rho, \kappa, \zeta, \bar{\kappa}, \bar{\zeta} \) by the relation:

\[
\begin{pmatrix}
\rho \\
\kappa \\
\zeta \\
\bar{\kappa} \\
\bar{\zeta}
\end{pmatrix} := g \cdot
\begin{pmatrix}
\rho_0 \\
\kappa_0 \\
\zeta_0 \\
\bar{\kappa}_0 \\
\bar{\zeta}_0
\end{pmatrix},
\]

which we abbreviate as:

\[
\omega := g \cdot \omega_0.
\]

The coframes \( \omega \) define a \( G_1 \) structure \( P^1 \) on LC. The rest of this section is devoted to reduce \( P^1 \) to an absolute parallelism on LC through Cartan equivalence method.
4.2. Normalization of \( f \). The Maurer Cartan forms of \( G_1 \) are the following:

\[
\begin{align*}
\alpha^1 & := \frac{dc}{c}, \\
\alpha^2 & := \frac{db}{2c} - \frac{b \, dc}{c^2}, \\
\alpha^3 & := \frac{dd}{2c} - \frac{b \, de}{c^2} + \frac{(-dc + eb) \, df}{c^2f}, \\
\alpha^4 & := \frac{de}{c} - \frac{ef}{cf}, \\
\alpha^5 & := \frac{df}{f}.
\end{align*}
\]

The structure equations read as:

\[
\begin{align*}
d\rho &= \alpha^1 \wedge \rho + \bar{\alpha}^3 \wedge \rho \\
&+ T^p_{\rho \kappa} \rho \wedge \kappa + T^p_{\rho \zeta} \rho \wedge \zeta + T^p_{\rho \bar{\kappa}} \rho \wedge \bar{\kappa} + i \kappa \wedge \bar{\kappa}, \\
d\kappa &= \alpha^1 \wedge \kappa + \alpha^2 \wedge \rho \\
&+ T^\kappa_{\rho \kappa} \rho \wedge \kappa + T^\kappa_{\rho \zeta} \rho \wedge \zeta + T^\kappa_{\rho \bar{\kappa}} \rho \wedge \bar{\kappa} \\
&+ T^\kappa_{\rho \bar{\kappa}} \rho \wedge \bar{\kappa} + T^\kappa_{\rho \kappa} \kappa \wedge \zeta + T^\kappa_{\rho \kappa} \kappa \wedge \bar{\kappa} + T^\kappa_{\rho \kappa} \kappa \wedge \bar{\kappa}, \\
d\zeta &= \alpha^3 \wedge \rho + \alpha^4 \wedge \kappa + \alpha^5 \wedge \zeta \\
&+ T^\zeta_{\rho \kappa} \rho \wedge \kappa + T^\zeta_{\rho \zeta} \rho \wedge \zeta + T^\zeta_{\rho \bar{\kappa}} \rho \wedge \bar{\kappa} \\
&+ T^\zeta_{\rho \bar{\kappa}} \rho \wedge \bar{\kappa} + T^\zeta_{\rho \kappa} \kappa \wedge \zeta + T^\zeta_{\rho \kappa} \kappa \wedge \bar{\kappa} + T^\zeta_{\rho \kappa} \kappa \wedge \bar{\kappa},
\end{align*}
\]

where the expressions of the torsion coefficients \( T^\bullet_{\bullet \bullet} \) are given in the appendix.

We now proceed with the absorption step of Cartan’s method. We introduce the modified Maurer-Cartan forms \( \tilde{\alpha}^i \), which are related to the 1-forms \( \alpha^i \) by the relations:

\[
\tilde{\alpha}^i := \alpha^i - x^i \rho - x^i \kappa - x^i \zeta - x^i \bar{\kappa} - x^i \bar{\zeta},
\]

where \( x^1, x^2, x^3, x^4 \) and \( x^5 \) are arbitrary complex-valued functions. The previously written structure equations take the new form:

\[
\begin{align*}
d\rho &= \tilde{\alpha}^1 \wedge \rho + \bar{\tilde{\alpha}}^3 \wedge \rho \\
&+ \left(T^p_{\rho \kappa} - x^1_{\rho} - x^1_{\kappa}\right) \rho \wedge \kappa + \left(T^p_{\rho \zeta} - x^1_{\rho} - \bar{x}^1_{\kappa}\right) \rho \wedge \zeta \\
&+ \left(T^p_{\rho \bar{\kappa}} - x^1_{\rho} - \bar{x}^1_{\kappa}\right) \rho \wedge \bar{\kappa} + \left(T^p_{\rho \zeta} - x^1_{\rho} - \bar{x}^1_{\kappa}\right) \rho \wedge \bar{\zeta} \\
&+ i \kappa \wedge \bar{\kappa},
\end{align*}
\]
\[ d\kappa = \tilde{\alpha}^1 \wedge \kappa + \tilde{\alpha}^2 \wedge \rho + \frac{1}{(1-z^2z_2)^e} c \]

\[ + \left( T^\kappa_{\rho \kappa} - x^2_\kappa + x^1_\rho \right) \rho \wedge \kappa + \left( T^\kappa_{\rho \kappa} - x^2_\kappa \right) \rho \wedge \zeta \]

\[ + \left( T^\kappa_{\rho \kappa} - x^3_\kappa \right) \rho \wedge \bar{\rho} + \left( T^\kappa_{\rho \kappa} - x^3_\zeta \right) \rho \wedge \bar{\zeta} \]

\[ + \left( T^\kappa_{\kappa \kappa} + x^4_\zeta \right) \kappa \wedge \zeta + \left( T^\kappa_{\kappa \kappa} - x^4_\kappa \right) \kappa \wedge \bar{\kappa} \]

\[ + T^\kappa_{\zeta \kappa} \zeta \wedge \bar{\kappa} + \left( T^1_{\kappa \kappa} - x^4_\kappa \right) \kappa \wedge \zeta, \]

\[ d\zeta = \tilde{\alpha}^3 \wedge \rho + \tilde{\alpha}^4 \wedge \kappa + \tilde{\alpha}^5 \wedge \zeta \]

\[ + \left( T^\zeta_{\rho \kappa} - x^3_\kappa + x^4_\rho \right) \rho \wedge \kappa + \left( T^\zeta_{\rho \zeta} - x^3_\zeta + x^5_\rho \right) \rho \wedge \zeta \]

\[ + \left( T^\zeta_{\rho \zeta} - x^3_\zeta \right) \rho \wedge \bar{\rho} + \left( T^\zeta_{\rho \zeta} - x^3_\zeta \right) \rho \wedge \bar{\zeta} \]

\[ + \left( T^\zeta_{\kappa \zeta} - x^4_\kappa \right) \kappa \wedge \bar{\kappa} + \left( T^\zeta_{\kappa \zeta} - x^4_\zeta \right) \zeta \wedge \bar{\kappa} \]

\[ + \left( x^5_\kappa - x^4_\zeta \right) \kappa \wedge \zeta - x^4_\kappa \kappa \wedge \bar{\kappa} \]

\[ + \left( x^5_\zeta - x^4_\zeta \right) \zeta \wedge \zeta - x^5_\zeta \zeta \wedge \bar{\zeta}. \]

We then choose \( x^1, x^2, x^3, x^4 \) and \( x^5 \) in a way that eliminates as many torsion coefficients as possible. We easily see that the only coefficient which can not be absorbed is the one in front of \( \zeta \wedge \bar{\kappa} \) in \( d\kappa \), because it does not depend on the \( x^i \)'s. We choose the normalization

\[ T^\kappa_{\zeta \kappa} = 1, \]

which yields to:

\[ f = - \frac{c}{\bar{\rho} \left( 1 - z^2z_2 \right)} \]

We notice that the absorbed structure equations take the form:

\[ d\rho = \tilde{\alpha}^1 \wedge \rho + \tilde{\alpha}^1 \wedge \rho + i \kappa \wedge \bar{\kappa}, \]

\[ d\kappa = \tilde{\alpha}^1 \wedge \kappa + \tilde{\alpha}^2 \wedge \rho + \zeta \wedge \bar{\kappa}, \]

\[ d\zeta = \tilde{\alpha}^3 \wedge \rho + \tilde{\alpha}^4 \wedge \kappa + \tilde{\alpha}^5 \wedge \zeta. \]

The normalization of \( f \) gives the new relation:

\[
\begin{pmatrix}
\rho \\
\kappa \\
\zeta \\
\bar{\kappa} \\
\bar{\zeta}
\end{pmatrix} = \begin{pmatrix}
c \bar{\zeta} & 0 & 0 & 0 & 0 \\
b & c & 0 & 0 & 0 \\
d & e & \frac{1}{(1+z^2z_2)^e} & 0 & 0 \\
b & 0 & 0 & \bar{c} & 0 \\
0 & 0 & \bar{d} & \bar{e} & \frac{1}{(1+z^2z_2)^e}
\end{pmatrix}
\begin{pmatrix}
\rho_0 \\
\kappa_0 \\
\zeta_0 \\
\bar{\kappa}_0 \\
\bar{\zeta}_0
\end{pmatrix}.
\]
We thus introduce the new one-form
\[ \hat{\zeta}_0 = -\frac{1}{1 - z_2 \zeta_0} \cdot \zeta_0, \]
such that the previous equation rewrites:
\[
\begin{pmatrix}
\rho \\
\kappa \\
\zeta \\
\overline{\zeta}
\end{pmatrix}
= \begin{pmatrix}
c \overline{c} & 0 & 0 & 0 \\
b & c & 0 & 0 \\
d & e & \xi & 0 \\
b & 0 & \overline{c} & 0
\end{pmatrix}
\begin{pmatrix}
\rho_0 \\
\kappa_0 \\
\hat{\zeta}_0 \\
\overline{\zeta}_0
\end{pmatrix}.
\]

We have reduced the $G_1$ equivalence problem to a $G_2$ equivalence problem, where $G_2$ is the 8 dimensional real matrix Lie group whose elements are of the form
\[
g = \begin{pmatrix}
c \overline{c} & 0 & 0 & 0 \\
b & c & 0 & 0 \\
d & e & \xi & 0 \\
b & 0 & \overline{c} & 0 \\
0 & 0 & \overline{d} & \overline{\xi}
\end{pmatrix}.
\]

We determine the new structure equations enjoyed by the base coframe $(\rho_0, \kappa_0, \hat{\zeta}_0, \overline{\zeta}_0)$. We get:
\[
d \rho_0 = -\overline{z_2} \rho_0 \wedge \hat{\zeta}_0 - z_2 \rho_0 \wedge \overline{\zeta}_0 + i \kappa_0 \wedge \overline{\kappa}_0,
\]
\[
d \kappa_0 = -\overline{z_2} \kappa_0 \wedge \hat{\zeta}_0 + \hat{\zeta}_0 \wedge \overline{\kappa}_0,
\]
\[
d \hat{\zeta}_0 = z_2 \hat{\zeta}_0 \wedge \overline{\zeta}_0.
\]

4.3. **Normalization of** b. The Maurer forms of the $G_2$ are given by the independent entries of the matrix $dg \cdot g^{-1}$. We have:
\[
dg \cdot g^{-1} = \begin{pmatrix}
\beta^1 + \overline{\beta^1} & 0 & 0 & 0 & 0 \\
\beta^2 & \beta^1 & 0 & 0 & 0 \\
\beta^3 & \beta^1 - \overline{\beta^1} & 0 & 0 & 0 \\
\overline{\beta^2} & 0 & 0 & \overline{\beta^1} & 0 \\
\overline{\beta^3} & 0 & 0 & \overline{\beta^1} & -\beta^1 + \overline{\beta^1}
\end{pmatrix},
\]
where the forms $\beta^1, \beta^2, \beta^3$ and $\beta^4$ are defined by

\[
\begin{align*}
\beta^1 &:= \frac{dc}{c}, \\
\beta^2 &:= \frac{db}{cc} - \frac{b dc}{c^2 c}, \\
\beta^3 &:= \left(-\frac{dc + eb}{c^3 c^3} \right) \frac{dc}{c^3 c} - \left(-\frac{dc + eb}{c^2 c^2} \right) \frac{dd}{cc} - \frac{b dc}{c^2 c}, \\
\beta^4 &:= -\frac{edc}{c^2} + \frac{edc}{cc} + \frac{de}{c}.
\end{align*}
\]

Using formula (2), we get the structure equations for the lifted coframe $(\rho, \kappa, \zeta, \overline{\kappa}, \overline{\zeta})$ from those of the base coframe $(\rho_0, \kappa_0, \hat{\zeta}_0, \overline{\kappa}_0, \overline{\zeta}_0)$:

\[
\begin{align*}
d\rho &= \beta^1 \wedge \rho + \beta^3 \wedge \rho \\
&\quad + U^\rho_{\rho \kappa} \rho \wedge \kappa + U^\rho_{\rho \zeta} \rho \wedge \zeta + U^\rho_{\rho \overline{\kappa}} \rho \wedge \overline{\kappa} \\
&\quad + U^\rho_{\rho \overline{\zeta}} \rho \wedge \overline{\zeta} + i \kappa \wedge \overline{\kappa},
\end{align*}
\]

\[
\begin{align*}
d\kappa &= \beta^1 \wedge \kappa + \beta^2 \wedge \rho \\
&\quad + U^\kappa_{\rho \kappa} \rho \wedge \kappa + U^\kappa_{\rho \zeta} \rho \wedge \zeta + U^\kappa_{\rho \overline{\kappa}} \rho \wedge \overline{\kappa} \\
&\quad + U^\kappa_{\rho \overline{\zeta}} \rho \wedge \overline{\zeta} + \kappa \wedge \overline{\kappa} + \zeta \wedge \overline{\zeta},
\end{align*}
\]

\[
\begin{align*}
d\zeta &= \beta^3 \wedge \rho + \beta^4 \wedge \kappa + \beta^1 \wedge \zeta - \beta^1 \wedge \zeta \\
&\quad + U^\zeta_{\rho \kappa} \rho \wedge \kappa + U^\zeta_{\rho \zeta} \rho \wedge \zeta + U^\zeta_{\rho \overline{\kappa}} \rho \wedge \overline{\kappa} \\
&\quad + U^\zeta_{\rho \overline{\zeta}} \rho \wedge \overline{\zeta} + \zeta \wedge \overline{\kappa} + \zeta \wedge \overline{\zeta} + \kappa \wedge \overline{\zeta}.
\end{align*}
\]

We introduce the modified Maurer-Cartan forms $\tilde{\beta}^i$ which differ from the $\beta^i$ by a linear combination of the 1-forms $\rho, \kappa, \zeta, \overline{\kappa}, \overline{\zeta}$, i.e. that is:

\[
\tilde{\beta}^i = \beta^i - y^j_\rho \rho - y^j_\kappa \kappa - y^j_\zeta \zeta - y^j_\overline{\kappa} \overline{\kappa} - y^j_\overline{\zeta} \overline{\zeta}.
\]

The structure equations rewrite:

\[
\begin{align*}
d\rho &= \tilde{\beta}^1 \wedge \rho + \beta^1 \wedge \rho \\
&\quad + \left(U^\rho_{\rho \kappa} - y^1_\kappa - \overline{y}^1_\rho \right) \rho \wedge \kappa + \left(U^\rho_{\rho \zeta} - y^1_\zeta - \overline{y}^1_\rho \right) \rho \wedge \zeta \\
&\quad + \left(U^\rho_{\rho \overline{\kappa}} - y^1_\overline{\kappa} - \overline{y}^1_\rho \right) \rho \wedge \overline{\kappa} + \left(U^\rho_{\rho \overline{\zeta}} - y^1_\overline{\zeta} - \overline{y}^1_\rho \right) \rho \wedge \overline{\zeta} + i \kappa \wedge \overline{\kappa},
\end{align*}
\]
\[ d\kappa = \beta^1 \wedge \kappa + \beta^2 \wedge \rho \]
\[ + (U^\kappa_{\rho\kappa} + y^1_\rho - y^2_\kappa) \rho \wedge \kappa + (U^\kappa_{\rho\kappa} - y^2_\kappa) \rho \wedge \zeta + (U^\kappa_{\rho\kappa} - y^2_\kappa) \rho \wedge \bar{\kappa} + (U^\kappa_{\rho\kappa} - y^2_\kappa) \rho \wedge \bar{\zeta} + (U^\kappa_{\rho\kappa} - y^2_\kappa) \rho \wedge \bar{\bar{\kappa}} + (U^\kappa_{\rho\kappa} - y^2_\kappa) \rho \wedge \bar{\bar{\zeta}} + \rho \wedge \zeta + \rho \wedge \bar{\zeta}, \]
\[ d\zeta = \beta^3 \wedge \rho + \beta^4 \wedge \kappa + \beta^1 \wedge \zeta - \beta^1 \wedge \zeta \]
\[ + (U^\zeta_{\rho\kappa} - y^3_\kappa + y^4_\rho) \rho \wedge \kappa + (U^\zeta_{\rho\kappa} - y^3_\kappa + y^4_\rho - y^1_\rho) \rho \wedge \zeta + (U^\zeta_{\rho\kappa} - y^3_\kappa + y^4_\rho - y^1_\rho) \rho \wedge \bar{\kappa} + (U^\zeta_{\rho\kappa} - y^3_\kappa + y^4_\rho - y^1_\rho) \rho \wedge \bar{\zeta} + (U^\zeta_{\rho\kappa} - y^3_\kappa + y^4_\rho - y^1_\rho) \rho \wedge \bar{\bar{\kappa}} + (U^\zeta_{\rho\kappa} - y^3_\kappa + y^4_\rho - y^1_\rho) \rho \wedge \bar{\bar{\zeta}}. \]

We get the following absorption equations:
\[ y^1_\kappa + y^1_\rho = U^\rho_{\rho\kappa}, \]
\[ y^1_\zeta + y^1_\rho = U^\rho_{\rho\kappa}, \]
\[ y^1_\kappa + y^1_\rho = U^\rho_{\rho\kappa}, \]
\[ y^1_\zeta + y^1_\rho = U^\rho_{\rho\kappa}, \]
\[ y^1_\zeta + y^1_\rho = U^\rho_{\rho\kappa}, \]
\[ y^1_\zeta + y^1_\rho = U^\rho_{\rho\kappa}, \]
\[ y^1_\zeta + y^1_\rho = U^\rho_{\rho\kappa}, \]
\[ y^1_\zeta + y^1_\rho = U^\rho_{\rho\kappa}, \]
\[ y^1_\zeta + y^1_\rho = U^\rho_{\rho\kappa}, \]
\[ y^1_\zeta + y^1_\rho = U^\rho_{\rho\kappa}. \]

Eliminating the \( y^*_\kappa \) among these equations leads to the following relations between the torsion coefficients:
\[ U^\rho_{\rho\kappa} = \bar{U}^\rho_{\rho\kappa}, \]
\[ U^\rho_{\rho\kappa} = \bar{U}^\rho_{\rho\kappa}, \]
\[ U^\rho_{\rho\kappa} = \bar{U}^\rho_{\rho\kappa}, \]
\[ U^\rho_{\rho\kappa} = \bar{U}^\rho_{\rho\kappa}, \]
\[ U^\rho_{\rho\kappa} = \bar{U}^\rho_{\rho\kappa}, \]
\[ 2U^\rho_{\rho\kappa} = U^\rho_{\rho\kappa} + U^\rho_{\rho\kappa}. \]

We verify easily that the first four equations do not depend on the group coefficients and are already satisfied. However, the last one does depend on the group coefficients. It gives us the normalization of \( \mathbf{b} \) as it rewrites:
\[ b = -i \bar{\mathbf{c}}. \]
The absorbed structure equations rewrite:

\[
d\rho = \beta_1^1 \wedge \rho + \beta_1^1 \wedge \rho + i \kappa \wedge \bar{\kappa},
\]

\[
d\kappa = \beta_1^1 \wedge \kappa + \beta_2^1 \wedge \rho + \zeta \wedge \bar{\kappa},
\]

\[
d\zeta = \beta_3^1 \wedge \rho + \beta_4^1 \wedge \kappa + \beta_1^1 \wedge \zeta - \beta_1^1 \wedge \zeta + \left( U^\zeta_{\zeta \bar{\kappa}} + U^\rho_{\rho \bar{\kappa}} - 2 U^\kappa_{\kappa \bar{\kappa}} \right) \zeta \wedge \bar{\kappa}.
\]

4.4. **Normalization of** \(d\). We have thus reduced the group \(G_2\) to a new group \(G_3\), whose elements are of the form

\[
g = \begin{pmatrix}
  c \bar{c} & 0 & 0 & 0 & 0 \\
  -i e \bar{c} & c & 0 & 0 & 0 \\
  d & e & \zeta & 0 & 0 \\
  i \bar{c} & 0 & \bar{\zeta} & 0 & 0 \\
  \bar{d} & 0 & 0 & \bar{\zeta} & \bar{\zeta}
\end{pmatrix}.
\]

It is a six-dimensional real Lie group. We compute its Maurer Cartan forms with the usual formula

\[
dg \cdot g^{-1} = \begin{pmatrix}
  \gamma^1 + \bar{\gamma}^1 & 0 & 0 & 0 & 0 \\
  \gamma^2 & \gamma^1 & 0 & 0 & 0 \\
  \gamma^3 & i \gamma^2 & \gamma^1 - \bar{\gamma}^1 & 0 & 0 \\
  \bar{\gamma}^2 & 0 & 0 & \bar{\gamma}^1 & 0 \\
  -\gamma^3 & 0 & 0 & -i \gamma^2 & -\gamma^1 + \bar{\gamma}^1
\end{pmatrix},
\]

where

\[
\gamma^1 := \frac{dc}{c},
\]

\[
\gamma^2 := i e \frac{dc}{c^2} - i \frac{e \bar{c} \bar{c}}{c \bar{c}} - \frac{de}{c},
\]

and

\[
\gamma^3 := \left( \frac{dc + i e \bar{c}}{c^2 \bar{c}} \right) \left( \frac{d \bar{c}}{\bar{c}} - \frac{dc}{c} \right) + \frac{dd}{c \bar{c}} + i \frac{ede}{c^2}.
\]

As the normalization of \(b\) does not depend on the base variables, the third loop of Cartan’s method is straightforward. We get the following structure equations:

\[
d\rho = \gamma^1 \wedge \rho + \bar{\gamma}^1 \wedge \rho + V_{\rho \kappa}^\rho \wedge \kappa + V_{\rho \zeta}^\rho \wedge \zeta + V_{\rho \bar{\kappa}}^\rho \wedge \bar{\kappa} + V_{\rho \zeta}^\rho \wedge \zeta + i \kappa \wedge \bar{\kappa},
\]
\[ d\kappa = \gamma^3 \wedge \kappa + \gamma^2 \wedge \rho \]
\[ + V^\kappa_{\rho\kappa} \rho \wedge \kappa + V^{\kappa}_{\rho\rho} \rho \wedge \zeta + V^{\kappa}_{\rho\rho} \rho \wedge \kappa \]
\[ + V^\kappa_{\rho\kappa} \rho \wedge \zeta + V^{\kappa}_{\rho\rho} \kappa \wedge \zeta + V^{\kappa}_{\rho\rho} \kappa \wedge \kappa + \zeta \wedge \kappa , \]

\[ d\zeta = \gamma^3 \wedge \rho + i \gamma^2 \wedge \kappa + \gamma^1 \wedge \zeta - \bar{\gamma}^1 \wedge \zeta \]
\[ + V^\zeta_{\rho\kappa} \rho \wedge \kappa + V^{\zeta}_{\rho\rho} \rho \wedge \zeta + V^{\zeta}_{\rho\rho} \rho \wedge \zeta + V^{\zeta}_{\rho\rho} \rho \wedge \zeta \]
\[ + V^{\zeta}_{\rho\rho} \kappa \wedge \zeta + V^{\zeta}_{\rho\rho} \kappa \wedge \kappa + V^{\zeta}_{\rho\rho} \zeta \wedge \zeta + V^{\zeta}_{\rho\rho} \zeta \wedge \zeta + V^{\zeta}_{\rho\rho} \zeta \wedge \zeta , \]

We now start the absorption step. We introduce:
\[ \widetilde{\gamma}^i := \gamma^i - z_i^\rho \rho - z_i^\kappa \kappa - z_i^\zeta \zeta - z_i^\kappa \kappa - z_i^\zeta \zeta. \]

The structure equations are modified accordingly:
\[ d\rho = \tilde{\gamma}^1 \wedge \rho + \bar{\gamma}^1 \wedge \rho \]
\[ + \left( V^\rho_{\rho\kappa} - z_1^\kappa - z_1^\kappa \right) \rho \wedge \kappa + \left( V^\rho_{\rho\rho} - z_1^\kappa - z_1^\kappa \right) \rho \wedge \zeta \]
\[ + \left( V^\rho_{\rho\rho} - z_1^\kappa - z_1^\kappa \right) \rho \wedge \kappa + \left( V^\rho_{\rho\rho} - z_1^\kappa - z_1^\kappa \right) \rho \wedge \zeta , \]

\[ d\kappa = \gamma^1 \wedge \kappa + \tilde{\gamma}^2 \wedge \kappa \]
\[ + \left( V^\kappa_{\rho\kappa} - z^2_\kappa + z^2_\rho \right) \rho \wedge \kappa + \left( V^\kappa_{\rho\rho} - z^2_\kappa \right) \rho \wedge \zeta \]
\[ + \left( V^\kappa_{\rho\rho} - z^2_\rho \right) \rho \wedge \kappa + \left( V^\kappa_{\rho\rho} - z^2_\kappa - z_1^\kappa \right) \kappa \wedge \zeta \]
\[ + \left( V^\kappa_{\rho\rho} - z_1^\kappa \right) \kappa \wedge \kappa + \zeta \wedge \kappa - z_1^\kappa \kappa \wedge \zeta , \]

and
\[ d\zeta = \tilde{\gamma}^3 \wedge \rho + i \tilde{\gamma}^2 \wedge \kappa + \tilde{\gamma}^1 \wedge \zeta - \bar{\gamma}^1 \wedge \zeta \]
\[ + \left( V^\zeta_{\rho\kappa} - z^3_\kappa + i z^2_\rho \right) \rho \wedge \kappa + \left( V^\zeta_{\rho\rho} + z^1_\kappa - z^3_\kappa \right) \rho \wedge \zeta \]
\[ + \left( V^\zeta_{\rho\rho} + z^1_\kappa - z^3_\kappa \right) \rho \wedge \zeta + \left( V^\zeta_{\rho\rho} - z^2_\rho \right) \rho \wedge \zeta \]
\[ + \left( V^\zeta_{\rho\rho} - i z^2_\rho \right) \kappa \wedge \zeta + \left( V^\zeta_{\rho\rho} + z^1_\kappa + z^2_\rho \right) \zeta \wedge \zeta . \]

We thus want to solve the system of linear equations:
This is easily done as:

\[
\begin{align*}
  z_1^\kappa + \overline{z_1^\kappa} &= V_\rho^\rho, \\
  \overline{z_1^\zeta} + z_1^\zeta &= V_\rho^\rho, \\
  z_1^\zeta &= V_\rho^\zeta, \\
  z_1^\kappa &= 0, \\
  -z_1^\rho + \overline{z_1^\rho} + z_3^\zeta &= V_\rho^\zeta, \\
  z_3^\kappa &= V_\rho^\rho, \\
  z_1^\kappa - \overline{z_1^\kappa} &= V_\zeta^\zeta, \\
  z_1^\zeta - \overline{z_1^\zeta} &= 0,
\end{align*}
\]

\[
\begin{align*}
  z_2^\kappa &= V_\rho^\kappa, \\
  z_2^\zeta &= V_\rho^\zeta, \\
  z_2^\zeta &= V_\rho^\zeta, \\
  z_3^\kappa &= V_\rho^\kappa, \\
  z_3^\zeta &= V_\rho^\zeta, \\
  z_3^\zeta &= V_\rho^\zeta,
\end{align*}
\]
where \( z^1_\rho \) and \( z^2_\rho \) may be chosen freely. Eliminating the \( z^* \) we get the following additional conditions on the \( V^* \):

\[
\begin{align*}
V^\rho_{\rho\kappa} &= V^\rho_{\rho\kappa}, \\
V^\rho_{\rho\zeta} &= V^\rho_{\rho\zeta}, \\
V^\rho_{\rho\zeta} &= V^\kappa_{\kappa\zeta}, \\
i V^\rho_{\rho\zeta} &= V^\zeta_{\kappa\zeta}, \\
V^\rho_{\rho\zeta} &= -V^\zeta_{\zeta\zeta}, \\
2V^\rho_{\kappa\zeta} &= V^\rho_{\rho\kappa} + V^\zeta_{\zeta\zeta},
\end{align*}
\]

(6)

and

\[
\begin{align*}
i V^\rho_{\rho\kappa} &= V^\zeta_{\kappa\kappa}, \\
V^\rho_{\rho\zeta} + V^\zeta_{\kappa\kappa} &= i V^\kappa_{\rho\kappa}.
\end{align*}
\]

(7)

We easily verify that the equations (6) are indeed satisfied. However the remaining two equations are not and they provide two essential torsion coefficients, namely \( i V^\kappa_{\rho\kappa} - V^\zeta_{\kappa\kappa} \) and \( V^\rho_{\rho\zeta} + V^\zeta_{\kappa\kappa} - i V^\kappa_{\rho\kappa} \), which will give us at least one new normalization of the group coefficients. Indeed we have

\[
i V^\kappa_{\rho\kappa} - V^\zeta_{\kappa\kappa} = -2i \frac{d}{c\bar{c}} + \frac{e^2}{c^2}.
\]

Setting this expression to 0, we get the normalization of the parameter \( d \):

\[
d = -i \frac{e^2}{2c}.
\]

4.5. Prologation of the \( G_4 \) structure. We have reduced the previous \( G_3 \)-structure to a \( G_4 \)-structure, where \( G_4 \) is the four dimensional matrix Lie group whose elements are of the form:

\[
\begin{pmatrix}
c\bar{c} & 0 & 0 & 0 \\
-i e\bar{c} & c & 0 & 0 \\
-\frac{i}{2} \bar{e}c & e & \bar{c} & 0 \\
i \bar{e}c & 0 & 0 & \bar{c}
\end{pmatrix}.
\]

The basis for the Maurer-Cartan forms of \( G_4 \) is provided by the four forms

\[
\delta^1 := \frac{dc}{c}, \quad \delta^2 := i \frac{e dc}{c^2} - i \frac{e d\bar{c}}{c\bar{c}} - i \frac{de}{c}, \quad \bar{\delta}^1, \quad \bar{\delta}^2.
\]
Now we just substitute the parameter $d$ by its normalization in the structure equations at the third step. We have to take into account the fact that $dd$ is modified accordingly. Indeed we have:

$$dd = -\frac{ie\tau}{c} - \frac{i}{2} e^2 \tau \left( \frac{d\tau}{c} - \frac{dc}{c} \right).$$

The forms $\gamma^1$ and $\gamma^2$ are not modified as they do not involve terms in $dd$, but this is not the case for $\gamma^3$ which is transformed into:

$$\gamma^3 = \frac{dd}{cc} + \frac{i}{c^2} e - \frac{d}{cc} dc - \frac{i}{c^2} e^2 dc + \frac{d}{cc} d\tau + \frac{i}{c^2} e^2 d\tau = 0.$$

The expressions of $d\rho$ and $d\kappa$ are thus unchanged from the expressions given by the structure equations at the third step, except the fact that we shall replace $d$ by $-\frac{i}{2} e^2 \tau + i \frac{1}{2} H$ in the expression of each torsion coefficient $V_\bullet^{\bullet}$, which we rename $W_\bullet^{\bullet}$, and the fact that the forms $\gamma^1$ and $\gamma^1$ shall be replaced by the forms $\delta^1$ and $\delta^2$, that is:

$$d\rho = \delta^1 \wedge \kappa + \delta^2 \wedge \rho + W_\rho^\rho \rho \wedge \kappa + W_\rho^\kappa \rho \wedge \kappa + W_\rho^\rho \rho \wedge \kappa + W_\rho^\kappa \rho \wedge \kappa + i \kappa \wedge \kappa,$$

and

$$d\kappa = \delta^1 \wedge \kappa + \delta^2 \wedge \rho + W_\rho^\rho \rho \wedge \kappa + W_\rho^\kappa \rho \wedge \kappa + W_\rho^\rho \rho \wedge \kappa + W_\rho^\kappa \rho \wedge \kappa + \kappa \wedge \kappa + \zeta \wedge \zeta.$$

The expression of $d\zeta$ is obtained in the same way, setting $\gamma_3$ to zero, and renaming $V_\nu^{\nu}$ the coefficients $V_\nu^{\nu}$ in which one performs the substitution $d = -\frac{i}{2} e^2 \tau$:

$$d\zeta = i \delta_1 \wedge \zeta + \delta_1 \wedge \zeta - \delta_1 \wedge \zeta + W_\rho^\rho \rho \wedge \kappa + W_\rho^\kappa \rho \wedge \zeta + W_\rho^\rho \rho \wedge \kappa + W_\rho^\kappa \rho \wedge \kappa + W_\rho^\rho \rho \wedge \kappa + W_\rho^\kappa \rho \wedge \kappa + \kappa \wedge \zeta + \zeta \wedge \zeta.$$

Let us now proceed with the absorption phase. We make the two substitutions:

$$\delta^1 := \tilde{\delta}^1 + w_1^1 \rho + w_1^1 \kappa + w_1^1 \zeta + w_1^1 \kappa \wedge \kappa + w_1^1 \kappa \wedge \zeta,$$

$$\delta^2 := \tilde{\delta}^2 + w_2^2 \rho + w_2^2 \kappa + w_2^2 \zeta + w_2^2 \kappa \wedge \kappa + w_2^2 \kappa \wedge \zeta.$$
in the previous equations. We get:

\[
    d\rho = \delta^1 \wedge \rho + \delta^1 \wedge \rho \\
    + \left( W^\rho_{\rho \kappa} - w^1_\kappa - \bar{w}^1_\kappa \right) \rho \wedge \kappa \\
    + \left( W^\rho_{\rho \zeta} - \bar{w}^1_\zeta - w^1_\zeta \right) \rho \wedge \zeta,
\]

\[
    d\kappa = \delta^1 \wedge \kappa + \delta^2 \wedge \rho \\
    + \left( W^\kappa_{\rho \kappa} - w^2_\kappa + w^1_\rho \right) \rho \wedge \kappa \\
    + \left( W^\kappa_{\rho \zeta} - \bar{w}^2_\zeta \right) \rho \wedge \zeta \\
    + \left( W^\kappa_{\kappa \zeta} - w^1_\zeta \right) \kappa \wedge \zeta \\
    + \left( W^\kappa_{\kappa \kappa} - w^1_\kappa \right) \kappa \wedge \kappa \\
    + \left( W^\kappa_{\kappa \kappa} - \bar{w}^1_\kappa \right) \kappa \wedge \zeta,
\]

and

\[
    d\zeta = i \delta^2 \wedge \kappa + \delta^1 \wedge \zeta - \delta^1 \wedge \zeta \\
    + \left( W^\zeta_{\rho \kappa} + i w^2_\rho \right) \rho \wedge \kappa \\
    + \left( W^\zeta_{\rho \zeta} + w^1_\rho - \bar{w}^1_\rho \right) \rho \wedge \zeta \\
    + W^\zeta_{\rho \zeta} \rho \wedge \zeta \\
    + \left( W^\zeta_{\kappa \zeta} - i w^2_\kappa \right) \kappa \wedge \zeta \\
    + \left( W^\zeta_{\kappa \kappa} - w^1_\kappa \right) \kappa \wedge \zeta.
\]

From the last equation, we immediately see that \( W^\zeta_{\rho \zeta} \) and \( W^\zeta_{\rho \kappa} \) are two new essential torsion coefficients. We find the remaining ones by solving the set of equations:

\[
\begin{align*}
    w^1_\kappa + \bar{w}^1_\kappa &= W^\rho_{\rho \kappa}, \\
    w^1_\zeta + \bar{w}^1_\zeta &= W^\rho_{\rho \zeta}, \\
    w^1_\zeta &= W^\rho_{\rho \zeta}, \\
    w^2_\kappa - w^1_\rho &= W^\kappa_{\rho \kappa}, \\
    w^2_\zeta &= W^\kappa_{\rho \zeta}, \\
    w^1_\zeta &= W^\kappa_{\rho \zeta}, \\
    w^1_\zeta - \bar{w}^1_\zeta - i w^2_\kappa &= -W^\zeta_{\kappa \zeta}, \\
    i w^2_\kappa &= W^\zeta_{\kappa \zeta}, \\
    -w^1_\rho + \bar{w}^1_\rho &= W^\zeta_{\rho \kappa}, \\
    w^1_\kappa - \bar{w}^1_\kappa - i w^2_\kappa &= -W^\zeta_{\kappa \zeta}, \\
    i w^2_\kappa &= W^\zeta_{\kappa \zeta}, \\
    w^1_\kappa - \bar{w}^1_\kappa &= W^\zeta_{\zeta \kappa}, \\
    i w^2_\kappa &= W^\zeta_{\kappa \kappa}, \\
    w^1_\kappa - \bar{w}^1_\kappa &= W^\zeta_{\zeta \zeta},
\end{align*}
\]
which lead easily as before to:

\[
\begin{align*}
    w_1^\kappa &= \frac{\mathcal{W}^\kappa_{\rho\kappa}}{\mathcal{W}^\kappa_{\rho\kappa}}, \\
    w_1^\zeta &= \frac{\mathcal{W}^\zeta_{\kappa\zeta}}{\mathcal{W}^\zeta_{\kappa\zeta}}, \\
    w_1^\zeta &= 0, \\
    w_2^\kappa &= \frac{\mathcal{W}^\kappa_{\rho\kappa}}{\mathcal{W}^\kappa_{\rho\kappa}}, \\
    w_2^\zeta &= \frac{\mathcal{W}^\zeta_{\kappa\zeta}}{\mathcal{W}^\zeta_{\kappa\zeta}}, \\
    w_2^\zeta &= \frac{\mathcal{W}^\zeta_{\rho\zeta} + w_1^\kappa}{\mathcal{W}^\zeta_{\rho\zeta}}, \\
    w_2^\rho &= \frac{\mathcal{W}^\rho_{\kappa\rho}}{\mathcal{W}^\rho_{\kappa\rho}}, \\
    -w_1^\rho + w_1^\rho &= \frac{\mathcal{W}^\rho_{\kappa\rho}}{\mathcal{W}^\rho_{\kappa\rho}}.
\end{align*}
\]

(8)

Eliminating the \(w_i^i\) from (8), we get one additional condition on the \(\mathcal{W}_i^i\) which has not yet been checked, namely that \(\mathcal{W}^\zeta_{\rho\zeta}\) shall be purely imaginary.

The computation of \(\mathcal{W}^\zeta_{\rho\zeta}\), \(\mathcal{W}^\zeta_{\kappa\rho}\) and \(\mathcal{W}^\zeta_{\rho\zeta}\) gives:

\[
\begin{align*}
    \mathcal{W}^\zeta_{\rho\zeta} &= i \frac{\mathcal{E} \mathcal{E}^*}{\mathcal{E}^*} - \frac{i}{2} \frac{\mathcal{E}^2 \mathcal{E}^*}{\mathcal{E}^*} z_2 - \frac{i}{2} \frac{\mathcal{E}^2 \mathcal{E}^*}{\mathcal{E}^*} z_2, \\
    \mathcal{W}^\zeta_{\rho\kappa} &= 0,
\end{align*}
\]

and

\[
\begin{align*}
    \mathcal{W}^\zeta_{\rho\zeta} &= 0,
\end{align*}
\]

which indicates that no further normalizations are allowed at this stage and that we must perform a prolongation of the problem. Let us introduce the modified Maurer Cartan forms of the group \(G_4\), namely:

\[
\begin{align*}
    \tilde{\delta}_1^i &= \delta_1^i - w_1^i \rho - w_1^i \kappa - w_1^i \zeta - w_1^i \xi, \\
    \tilde{\delta}_2^i &= \delta_2^i - w_2^i \rho - w_2^i \kappa - w_2^i \zeta - w_2^i \xi,
\end{align*}
\]

where \(w_1^i, w_1^i, w_1^i, w_1^i, \ k = 1, 2\), are the solutions of the system of equations (8) corresponding to \(w_1^i + w_1^i = 0\), that is :

\[
\begin{align*}
    \tilde{\delta}_1^i &= \delta_1^i + \frac{1}{2} V_{\rho\kappa}^i \rho - \frac{V_{\rho\kappa}^i \kappa}{\mathcal{W}^\kappa_{\rho\kappa}} - \frac{V_{\rho\kappa}^i \zeta}{\mathcal{W}^\zeta_{\rho\kappa}} - \frac{V_{\rho\kappa}^i \xi}{\mathcal{W}^\xi_{\rho\kappa}}, \\
    \tilde{\delta}_2^i &= \delta_2^i - V_{\rho\kappa}^i \rho - \left( \frac{V_{\rho\kappa}^i \kappa}{\mathcal{W}^\kappa_{\rho\kappa}} - \frac{1}{2} V_{\rho\kappa}^i \right) \kappa - \frac{V_{\rho\kappa}^i \zeta}{\mathcal{W}^\zeta_{\rho\kappa}} - \frac{V_{\rho\kappa}^i \xi}{\mathcal{W}^\xi_{\rho\kappa}}.
\end{align*}
\]

We also introduce the modified Maurer Cartan forms which correspond to solutions of the system (8) when \(\text{Re}(w_1^i)\) is not necessarily set to zero,
namely:

\[
\begin{align*}
\pi^1 &:= \delta^1 - \Re(w^1_\rho)\rho, \\
\pi^2 &:= \delta^2 - \Re(w^1_\rho)\kappa.
\end{align*}
\]

(10)

Let \( P^9 \) be the nine dimensional \( G_4 \)-structure constituted by the set of all coframes of the form \((\rho, \kappa, \zeta, \pi, \delta)\) on \( M^5 \). The initial coframe \((\rho_0, \kappa_0, \zeta_0, \pi_0, \delta_0)\) gives a natural trivialisation \( P^9 \to M^5 \times G_4 \) which allows us to consider any differential form on \( M^5 \) or \( G_4 \) as a differential form on \( P^9 \). If \( \omega \) is a differential form on \( M^5 \) for example, we just consider \( p^* \left( pr^*_1(\omega) \right) \), where \( pr_1 \) is the projection on the first component \( M^5 \times G_4 \to M^5 \). We still denote this form by \( \omega \) in the sequel. Following [10], we introduce the two coframes \((\rho, \kappa, \zeta, \pi, \delta)\) and \((\rho, \kappa, \zeta, \pi, \delta)\) on \( P^9 \). Setting \( t := -\Re(w^1_\rho) \), they relate to each other by the relation:

\[
\begin{pmatrix}
\rho \\
\kappa \\
\zeta \\
\pi^1 \\
\pi^2 \\
\pi^1 \\
\pi^2
\end{pmatrix} = \begin{pmatrix}
\rho \\
\kappa \\
\zeta \\
\pi^1 \\
\pi^2 \\
\pi^1 \\
\pi^2
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
t & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
t & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
t & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},
\]

where \( g_t \) is defined by

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
t & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
t & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
t & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.
\]

The set \( \{g_t, t \in \mathbb{R}\} \) defines a one-dimensional Lie group, whose Maurer Cartan form is given by \( dt \), which we rename \( \Lambda \) in the sequel. We now start the reduction step in the equivalence problem on \( P^9 \). From the definition of \( \pi^1 \) and \( \pi^2 \) as the solutions of the absorption equations [8], the five first
structure equations read as

\[\begin{align*}
    d\rho &= \pi^1 \wedge \rho + \overline{\pi^1} \wedge \rho + i \kappa \wedge \overline{\kappa}, \\
    d\kappa &= \pi^1 \wedge \kappa + \pi^2 \wedge \rho + \zeta \wedge \overline{\kappa}, \\
    d\zeta &= i \pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \overline{\pi^1} \wedge \zeta, \\
    d\overline{\rho} &= \pi^1 \wedge \overline{\rho} + \overline{\pi^1} \wedge \overline{\rho} + \kappa \wedge \overline{\kappa}, \\
    d\overline{\kappa} &= -i \pi^2 \wedge \overline{\rho} + \pi^1 \wedge \overline{\kappa} - \pi^1 \wedge \overline{\kappa}.
\end{align*}\]

We could obtain the expressions of \(d\pi^1\) and \(d\pi^2\) by taking the exterior derivative of the previous five equations. But for now, as we have explicit expressions of \(\pi^1\) and \(\pi^2\) given by formulae (9) and (10), we can perform an actual computation:

\[
    \begin{align*}
    d\pi^1 &= dt \wedge \rho \\
            &+ X_{\rho\kappa}^1 \rho \wedge \kappa + X_{\rho\zeta}^1 \rho \wedge \zeta + X_{\rho\overline{\kappa}}^1 \rho \wedge \overline{\kappa} + X_{\rho\overline{\zeta}}^1 \rho \wedge \overline{\zeta} \\
            &+ X_{\rho\pi^1}^1 \rho \wedge \pi^1 + X_{\rho\pi^2}^1 \rho \wedge \pi^2 + X_{\rho\overline{\pi}^1}^1 \rho \wedge \overline{\pi^1} \\
            &+ X_{\rho\overline{\pi}^2}^1 \rho \wedge \overline{\pi^2} + i \kappa \wedge \overline{\pi^2} + \zeta \wedge \overline{\pi^2} + \kappa \wedge \overline{\pi^1} + \kappa \wedge \overline{\pi^2} + \zeta \wedge \overline{\pi^1},
    \end{align*}
\]

and

\[
    \begin{align*}
    d\pi^2 &= dt \wedge \kappa \\
            &+ X_{\rho\kappa}^2 \rho \wedge \kappa + X_{\rho\zeta}^2 \kappa \wedge \zeta + X_{\rho\overline{\kappa}}^2 \kappa \wedge \overline{\kappa} + X_{\rho\overline{\zeta}}^2 \kappa \wedge \overline{\zeta} \\
            &+ X_{\kappa\pi^1}^2 \kappa \wedge \pi^1 + X_{\kappa\pi^2}^2 \kappa \wedge \pi^2 + X_{\kappa\overline{\pi}^1}^2 \kappa \wedge \overline{\pi^1} \\
            &+ X_{\kappa\overline{\pi}^2}^2 \kappa \wedge \overline{\pi^2} + \zeta \wedge \overline{\pi^2} + \pi^2 \wedge \overline{\pi^1} + \pi^2 \wedge \overline{\pi^1}.
    \end{align*}
\]

From these equations, we see that the absorption is straightforward and that there remain no nonconstant essential torsion term. Indeed if we define the absorbed form \(\Lambda\) by:

\[
    \Lambda = dt - X_{\rho\kappa}^2 \rho - X_{\rho\kappa}^1 \kappa - \sum_{\nu=\zeta,\pi^1,\ldots,\pi^2} X_{\rho\nu}^1 \nu,
\]

the previous two equations become:

\[
    \begin{align*}
    d\pi^1 &= \Lambda \wedge \rho + i \kappa \wedge \overline{\pi^2} + \zeta \wedge \overline{\zeta}, \\
    d\pi^2 &= \Lambda \wedge \kappa + \zeta \wedge \overline{\pi^2} + \pi^2 \wedge \overline{\pi^1}.
    \end{align*}
\]

A straightforward computation gives the expression of \(d\Lambda\):

\[
    d\Lambda = -\pi^1 \wedge \Lambda + i \pi^2 \wedge \overline{\pi^2} - \pi^1 \wedge \overline{\Lambda}.
\]
Let us summarize the results that we have obtained so far: The ten 1-forms \( \rho, \kappa, \zeta, \pi, \zeta^1, \pi^2, \zeta^1, \pi^2, \Lambda \) satisfies the structure equations given by (4). This completes the proof of Theorem 3.

APPENDIX A. TORSION COEFFICIENTS FOR THE \( G \)-STRUCTURES ON \( B \)

A.1. Coefficients \( U_{\sigma \rho} \).

\[
U_{\sigma \rho} = \frac{e}{a^3} + \frac{d}{a^3},
\]

\[
U_{\sigma \zeta} = \frac{-c}{a^3},
\]

\[
U_{\sigma \zeta} = \frac{-c}{a^3},
\]

\[
U_{\rho \sigma \rho} = \frac{ce}{a^6} + \frac{cd}{a^6} - \frac{ib}{a^6} + \frac{id}{a^6},
\]

\[
U_{\rho \sigma \zeta} = \frac{ie}{a^6} - \frac{i\overline{\omega}}{a^6} - \frac{c^2}{a^6},
\]

\[
U_{\rho \sigma \zeta} = \frac{ibc}{a^6} - \frac{c^2}{a^6} - \frac{id}{a^6},
\]

\[
U_{\rho \zeta \zeta} = \frac{c}{a^3} + \frac{i\overline{\omega}}{a^3},
\]

\[
U_{\rho \zeta \zeta} = \frac{c}{a^3} - \frac{i\overline{\omega}}{a^3},
\]

\[
U_{\zeta \sigma \rho} = \frac{d^2}{a^9} + \frac{i\overline{\omega}db}{a^9} - \frac{i\overline{\omega}b^2}{a^9} + \frac{de}{a^9},
\]

\[
U_{\zeta \sigma \zeta} = \frac{ib}{a^9} - \frac{i\overline{\omega}cb}{a^9} - \frac{cd}{a^9},
\]

\[
U_{\zeta \sigma \zeta} = -\frac{i\overline{\omega}d}{a^9} + \frac{i\overline{\omega}b^2 c}{a^9},
\]

\[
U_{\zeta \rho \zeta} = \frac{d}{a^3} + \frac{i\overline{\omega}b}{a^3},
\]

\[
U_{\zeta \rho \zeta} = \frac{d}{a^3} - \frac{i\overline{\omega}b^2}{a^3},
\]

\[
U_{\zeta \zeta} = \frac{i\overline{\omega}}{a^2}.
\]
APPENDIX B. TORSION COEFFICIENTS FOR THE \( G \)-STRUCTURES ON \( N \)

B.1. **Coefficients \( U^{\tau}_{\tau \sigma} \):**

\[
U^{\tau}_{\tau \sigma} = \frac{h}{a^4} - \frac{bg}{a^6} - \frac{bg}{a^6} + \frac{k}{a^6}.
\]

\[
U^{\tau}_{\tau \rho} = \frac{bf}{a^6} + \frac{bf}{a^6}.
\]

\[
U^{\tau}_{\tau \zeta} = -\frac{f}{a^4},
\]

\[
U^{\tau}_{\tau \zeta} = -\frac{f}{a^4},
\]

\[
U^{\tau}_{\sigma \rho} = -\frac{b}{a^2} - \frac{b}{a^2},
\]

\[
U^{\sigma}_{\tau \sigma} = \frac{ge}{a^4} - \frac{hc}{a^7} - \frac{fc}{a^7} + \frac{gd}{a^7} + \frac{dk}{a^7} + \frac{fh}{a^7} - \frac{fbg}{a^{10}} - \frac{fbg}{a^{10}}.
\]

\[
U^{\sigma}_{\tau \rho} = \frac{bf^2}{a^{10}} + \frac{bf^2}{a^{10}} - \frac{fe}{a^7} - \frac{fe}{a^7} - \frac{k}{a^4} + \frac{h}{a^4},
\]

\[
U^{\sigma}_{\tau \zeta} = \frac{g}{a^4} + \frac{cf}{a^7} - \frac{f^2}{a^7},
\]

\[
U^{\sigma}_{\zeta \zeta} = \frac{g}{a^4} + \frac{cf}{a^7} - \frac{f^2}{a^7},
\]

\[
U^{\sigma}_{\sigma \rho} = \frac{e}{a^4} + \frac{d}{a^3} - \frac{bf}{a^6} - \frac{bf}{a^6},
\]

\[
U^{\sigma}_{\sigma \zeta} = -\frac{c}{a^4} + \frac{f}{a^4},
\]

\[
U^{\sigma}_{\zeta \zeta} = -\frac{c}{a^4} + \frac{f}{a^4},
\]

\[
U^{\rho}_{\tau \sigma} = -\frac{i e b g}{a^9} - \frac{i b c h}{a^9} + \frac{i b c g}{a^9} + \frac{ibck}{a^{10}} + \frac{egc}{a^{10}} + \frac{dgc}{a^{10}} + \frac{dk}{a^7} + \frac{i c^2 h}{a^{10}} - \frac{c^2 k}{a^{10}} - \frac{g^2 b}{a^{10}} + \frac{gk}{a^8} + \frac{gh}{a^8} + \frac{g^2 \bar{b}}{a^{10}},
\]

\[
U^{\rho}_{\tau \rho} = -\frac{cd e f}{a^{10}} + \frac{f b g}{a^{10}} - \frac{f b g}{a^{10}} + \frac{c e f}{a^{10}} + \frac{h c}{a^7} + \frac{k c}{a^7} - \frac{ibk}{a^6} + \frac{ib e f}{a^9} - \frac{i d b f}{a^9} + \frac{i d b f}{a^9} - \frac{i d b h}{a^6},
\]

\[
U^{\rho}_{\tau \zeta} = \frac{i b c f}{a^9} - \frac{i e f}{a^7} - \frac{i b c g}{a^7} + \frac{c^2 f}{a^7} + \frac{gc}{a^7} + \frac{ik}{a^4} - \frac{g h}{a^8},
\]

\[
U^{\rho}_{\zeta \zeta} = -\frac{i b c f}{a^9} + \frac{i d f}{a^7} + \frac{i b c g}{a^7} + \frac{c^2 f}{a^7} - \frac{gc}{a^7} - \frac{g h}{a^8} - \frac{g h}{a^8} - \frac{ih}{a^8},
\]
\[ U^\rho_{\sigma\rho} = \frac{ce}{a^6} + \frac{cd - gb}{a^6} - \frac{gb}{a^6} - \frac{i be}{a^6} + \frac{i d b}{a^6}, \]
\[ U^\rho_{\sigma\zeta} = \frac{ie}{a^3} - \frac{i b c}{a^6} + \frac{c^2}{a^6} + \frac{g}{a^6}, \]
\[ U^\rho_{\zeta\zeta} = \frac{ibc}{a^6} - \frac{c^2}{a^6} - \frac{id}{a^6} + \frac{g}{a^6}, \]
\[ U^\rho_{\varphi\zeta} = \frac{c}{a^3} + \frac{i b}{a^2}, \]
\[ U^\rho_{\varphi\zeta} = \frac{c}{a^3} - \frac{i b}{a^2}, \]
\[ U^\zeta_{\tau\sigma} = \frac{\frac{i e b^2 g}{a^{11}} - \frac{i d k b}{a^9} + \frac{i e h b + i b^2 c k}{a^{11}} + \frac{h k + d^2 g - i b c h b}{a^{10}}}{a^{10}}. \]
APPENDIX C. TORSION COEFFICIENTS FOR THE $G$-STRUCTURES ON LC

C.1. Coefficients $T^\kappa$. 

\[ T^\rho_\kappa = i \frac{b}{cc} - \frac{e}{cf} \frac{1}{1 - z_2}, \]
\[ T^\rho_\kappa = \frac{1}{f} \frac{z_2}{1 - z_2}, \]
\[ T^\rho_\kappa = -i \frac{b}{cc} + \frac{e}{cf} - \frac{1}{1 - z_2}, \]
\[ T^\rho_\kappa = \frac{1}{f} \frac{z_2}{1 - z_2}, \]

\[ T^{\kappa}_{\rho\kappa} = \frac{eb}{cc^2f} \frac{1}{1 - z_2} + \frac{d}{ccf} \frac{z_2}{1 - z_2} + i \frac{b^2}{cc^2f} - \frac{eb}{cc^2f} \frac{1}{1 - z_2}, \]
\[ T^{\kappa}_{\rho\kappa} = -\frac{b}{cc^2f} \frac{1}{1 - z_2}, \]

\[ T^{\kappa}_{\kappa\kappa} = \frac{b}{ccf} \frac{z_2}{1 - z_2}, \]
\[ T^{\kappa}_{\kappa\kappa} = \frac{1}{f} \frac{z_2}{1 - z_2}, \]
\[ T^{\kappa}_{\kappa\kappa} = \frac{e}{ccf} \frac{1}{1 - z_2} + i \frac{b}{cc}, \]
\[ T^{\kappa}_{\kappa\kappa} = -\frac{c}{ccf} \frac{1}{1 - z_2}, \]
\[ T^{\kappa}_{\kappa\kappa} = \frac{e^2b}{cc^2f} \frac{1}{1 - z_2} + i \frac{d}{cc^2f}, \]
\[ T^{\kappa}_{\kappa\kappa} = -\frac{eb}{cc^2f} \frac{1}{1 - z_2} - \frac{eb}{cc^2f} \frac{z_2}{1 - z_2} + \frac{d}{ccf} \frac{z_2}{1 - z_2}, \]
\[ T^{\kappa}_{\kappa\kappa} = \frac{ed}{cc^2f} \frac{1}{1 - z_2} - \frac{eb}{cc^2f} \frac{1}{1 - z_2} - i \frac{db}{ccf} \frac{z_2}{1 - z_2}, \]
\[ T^{\kappa}_{\kappa\kappa} = \frac{d}{ccf} \frac{z_2}{1 - z_2}, \]
\[ T^{\kappa}_{\kappa\kappa} = \frac{e}{ccf} \frac{z_2}{1 - z_2}. \]
\[ T_{\zeta\kappa} = \frac{e^2}{c^2} \frac{1}{1 - z_2 \bar{z}_2} + i \frac{d}{c\bar{d}}, \]
\[ T_{\zeta\zeta} = -\frac{e}{c} \frac{1}{1 - z_2 \bar{z}_2}. \]

C.2. Coefficients \( U_{\zeta\zeta} \).

\[ U_{\rho\kappa} = \frac{B}{c\bar{c}} + \frac{e}{c^2} \frac{\bar{c}}{z_2}, \]
\[ U_{\rho\zeta} = -\frac{c}{c^2} \frac{z_2}{z_2}, \]
\[ U_{\rho\rho} = -\frac{b}{c^2} \frac{z_2}{z_2}, \]
\[ U_{\rho\zeta} = \frac{e}{c^2} \frac{z_2}{z_2}, \]
\[ U_{\rho\kappa} = -\frac{d}{c^2} \frac{z_2}{z_2}, \]
\[ U_{\rho\zeta} = -\frac{e}{c^2} \frac{z_2}{z_2}, \]
\[ U_{\kappa\kappa} = -\frac{e}{c} + i \frac{b}{c\bar{c}} \]
\[ U_{\kappa\zeta} = \frac{e}{c^2} \frac{z_2}{z_2}, \]
\[ U_{\kappa\kappa} = -\frac{d}{c} \frac{z_2}{z_2} + \frac{e}{c^2} \frac{z_2}{z_2}, \]
\[ U_{\kappa\zeta} = -\frac{e}{c^2} \frac{z_2}{z_2}, \]
\[ U_{\kappa\xi} = \frac{e\bar{c}}{c^2} z_2 - \frac{e^2}{c^2} + i \frac{d}{c\bar{c}}, \]

\[ U_{\kappa\zeta} = \frac{e}{c} z_2, \]

\[ U_{\zeta\xi} = \frac{\bar{c}e}{c^2} z_2 + \frac{e}{c}, \]

\[ U_{\zeta\zeta} = \frac{c}{c} z_2. \]

C.3. Coefficients \( V_{\bullet\bullet} \).

\[ V_{\rho\kappa}\rho = -\frac{\bar{e}}{c} + \frac{e\bar{c}}{c^2} \frac{z_2}{c^2}, \]

\[ V_{\rho\kappa}\zeta = -\frac{\bar{e}}{c} \frac{z_2}{c}, \]

\[ V_{\rho\zeta}\rho = -\frac{e}{c} + \frac{\bar{e}c}{c^2} \frac{z_2}{c^2}, \]

\[ V_{\rho\zeta}\zeta = -\frac{c}{c} \frac{z_2}{c^2}, \]

\[ V_{\rho\kappa}\zeta = -\frac{d}{c^2} \frac{z_2}{c^2} - i \frac{e^2\bar{c}}{c^3} \frac{z_2}{c^3}, \]

\[ V_{\rho\kappa}\kappa = i \frac{\bar{e}}{c}, \]

\[ V_{\rho\zeta}\kappa = -\frac{d}{c\bar{c}} - i \frac{e\bar{c}}{c^2} \frac{z_2}{c^2}, \]

\[ V_{\rho\zeta}\zeta = i \frac{e}{c} \frac{z_2}{c^2}, \]

\[ V_{\kappa\kappa}\kappa = 0, \]

\[ V_{\kappa\kappa}\zeta = -\frac{d}{c^2} \frac{z_2}{c^2} - i \frac{e\bar{c}}{c^3} \frac{z_2}{c^3} - i \frac{e^2\bar{c}}{c^2} \frac{z_2}{c^2}, \]

\[ V_{\kappa\zeta}\kappa = -\frac{d}{c\bar{c}} z_2 + i \frac{e\bar{c}}{c^2} \frac{z_2}{c^2} - i \frac{c\bar{e}^2}{c^3} \frac{z_2}{c^3} - i \frac{d}{c^2} \frac{z_2}{c^2}, \]

\[ V_{\kappa\zeta}\zeta = 2 \frac{d\bar{e}}{c^3} \frac{z_2}{c^3} + i \frac{c\bar{e}^2}{c^4} \frac{z_2}{c^4} - 2 \frac{ed}{c^2} \frac{z_2}{c^2} - i \frac{e^3}{c^3}, \]

\[ V_{\kappa\kappa}\kappa = -2 \frac{d}{c^2} \frac{z_2}{c^2} - i \frac{e^2}{c\bar{c}} \frac{z_2}{c\bar{c}}. \]
\[ V_{\kappa\kappa}^\zeta = -\frac{e\zeta}{c^2} z_2, \]
\[ V_{\kappa\pi}^\zeta = \frac{e\pi}{c^2} z_2 - \frac{e^2}{c^2} + i \frac{d}{c\zeta}, \]
\[ V_{\pi\zeta}^\kappa = -\frac{e\zeta}{c^2} z_2, \]
\[ V_{\pi\pi}^\zeta = -\frac{\pi\pi}{c^2} z_2 + \frac{e}{c}, \]
\[ V_{\zeta\zeta}^\zeta = \frac{\zeta\zeta}{c^2} z_2. \]

C.4. Coefficients \( X_{\bullet\bullet}^\bullet \).

\[ X_{\rho\kappa}^1 = -\frac{1}{2} \frac{te}{c} \bar{z}_2 - \frac{3}{8} i \frac{e^2 e}{c^3} \bar{z}_2 + \frac{1}{2} \frac{te}{c} + \frac{1}{8} i \frac{e^2 e}{c^2} \bar{z}_2 \bar{z}_2 + \frac{1}{8} i \frac{e^2 e}{c^3} \bar{z}_2 + \frac{1}{4} i \frac{e^2 e}{c^2}, \]
\[ X_{\rho\zeta}^1 = -\frac{1}{4} i \frac{e^2}{c^2} + \frac{1}{2} i \frac{e}{c^2} \bar{z}_2 - \frac{1}{4} i \frac{e^2 e}{c^4} \bar{z}_2, \]
\[ X_{\rho\kappa}^1 = \overline{X_{\rho\kappa}^1}, \]
\[ X_{\rho\zeta}^1 = \overline{X_{\rho\zeta}^1}, \]
\[ X_{\rho\pi^1} = -t, \]
\[ X_{\rho\pi^2}^1 = \frac{1}{2} \frac{e}{c} + \frac{1}{2} \frac{e^2 e}{c^2}, \]
\[ X_{\rho\pi^1}^1 = -t, \]
\[ X_{\rho\pi^2}^1 = \overline{X_{\rho\pi^2}^1}, \]
\[ X_{\rho\pi^2}^2 = -\frac{1}{4} \frac{e^2 e}{c^4} \bar{z}_2 + \frac{1}{4} \frac{e^2 e}{c^4} \bar{z}_2 + \frac{1}{8} \frac{e^2 e}{c^2} \bar{z}_2 + \frac{1}{16} \frac{e^2 e}{c^4} \bar{z}_2 + \frac{1}{16} \frac{e^2 e}{c^6} \bar{z}_2 - t^2, \]
\[ X_{\kappa\nu}^2 = X_{\rho\nu}^1 \quad \text{for} \quad \nu = \zeta, \pi^1, \ldots, \pi^2. \]
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