Spectral Numerical Algorithm for Solving Optimal Control Using Boubaker-Turki Operational Matrices

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Abstract. The aim of the present research is to propose a spectral method for solving optimal control problem indirectly using Boubaker - Turki polynomial functions as basis functions. To achieve this goal, explicit representation formulas for some interesting operational matrices for Boubaker - Turki polynomials functions are first derived which play an important role in dealing with the problem of optimal control. They are operational matrix of derivative, operational matrix of product. By applying the obtained operational matrices and spectral scheme, the main problem is transformed to a set of linear algebraic equations that greatly simplifies the problem. The presented method in details by solving numerical example has been investigated. A new recursive relation of the Boubaker - Turki and Chebyshev polynomials of the second kind as well as a general formula for power function as a linear combination of the Boubaker - Turki polynomial are also included in this work.

Keywords. Optimal control problems, spectral method, Boubaker-Turki polynomials, operational matrices.

1. Introduction
Polynomials can be represented in many different bases such as the power, Hermit, Chebyshev, Bernstein and Legendre basis forms [1-7]. In addition Boubaker polynomials are very important in Physical evaluation process, for example, Yücel in [8] applied Boubaker polynomials to compute the eigenvalues of fourth order Sturm-Liouville equation; The Sturm-sequences with the aide of Boubaker polynomials and their properties had been constructed by Shaikh [9]. Lots of researchers have used Boubaker polynomials for solving differential equation as well as integral equations and other problems. Zhao in [10] presented spectral type method based on Boubaker polynomials for solving differential equations: in [11] Boubaker polynomials have been utilized for finding approximate solution to system of nonlinear Volterra-Fredholm integral equations. Zhang [12] studied the solution of nonlinear mechanical system using Boubaker polynomials.

Boubaker-Turki polynomials are applied in some mathematical research. They have been utilized in a wide area of sciences and engineering, in particular Boubaker-Turki polynomials are successfully used in solving Bi-varied Head equation [13] in solving Bloch NMR flow equations [14]. See [15-17] more applications.
Some newly interesting and useful properties associated Boubaker-Turki polynomials are included in this work, then a spectral computational method are introduced for solving a class of optimal control problems. This method consists of reducing the OCP to a set of algebraic equations by using compactly supported Boubaker-Turki polynomials. The use of these polynomials is justified by their interesting properties. Several algorithms and methods have been presented to solve OCP in the literature [17-25].

2. The Boubaker Turki polynomial Functions

In this section, the modified Boubaker polynomials of Boubaker-Turki polynomials are presented. They are easily subjected to arithmetical and integral analysis. Some mew properties of Boubaker-Turki polynomials as well as their relationship with Chebyshev polynomials of the second kind are introduced throughout this section.

2.1. Historic and Definition

The first definition for Boubaker polynomials is adopted for solving heat equation

\[
\frac{\partial^2 u(x,t)}{\partial x^2} = k \frac{\partial u(x,t)}{\partial t}
\]

Defined in the following domain

\[
D = \{ -k, < x < 0 \}
\]

and have the explicit expression:

\[
B_0(t) = \sum_{k=0}^{\eta(i)} \left( \frac{(-4k)!}{(i-k)!(i-k)!} \right) \cdot (-1)^k \cdot x^{i-2k}
\]

The Boubaker-Turki polynomials were proposed through a specialized study which are the modified of Boubaker polynomial defined by the following equation

\[
B_T(t) = \sum_{k=0}^{\eta(i)} \left( \frac{(-4k)!}{(i-k)!(i-k)!} \right) \cdot (-1)^k \cdot x^{i-2k}
\]

Their graphical are shown in Fig. 1

2.2. New Operational Matrix of Derivative for Boubaker Turki Polynomials

The objective of present section is to derive and prove new formula representing the first derivative of Boubaker Turki Polynomials in terms of Boubaker Turki Polynomials themselves. A consequence an operational matrix of derivative is constructed. To illustrate the calculation procedure, the following first eight Boubaker Turki Polynomials is taken

\[
B_T_0(t) = 1
\]
\[
B_T_1(t) = 2t
\]
\[
B_T_2(t) = 4t^2 + 2
\]
\[
B_T_3(t) = 8t^3 + 2t
\]
\[
B_T_4(t) = 16t^4 - 2
\]
\[
B_T_5(t) = 32t^5 - 8t^3 - 6t
\]
\[
B_T_6(t) = 64t^6 - 32t^4 - 12t^2 + 2
\]
\[
B_T_7(t) = 128t^7 - 96t^5 - 16t^3 + 10t
\]
\[
B_T_8(t) = 256t^8 - 256t^6 + 32t^4 - 2
\]

Their graphical are shown in Fig. 1
Figure 1. Graph of the Boubaker Turki polynomial of order 0, 1, 2, 3, 4, 5, 6, 7 and 8

Differentiating both sides with respect to $t$ and get

\begin{align*}
\ddot{B}T_0(t) &= 0 \\
\ddot{B}T_1(t) &= 2BT_0(t) \\
\ddot{B}T_2(t) &= 4BT_1(t) \\
\ddot{B}T_3(t) &= 6BT_2(t) - 10BT_0(t) \\
\ddot{B}T_4(t) &= 8BT_3(t) - 8BT_1(t) \\
\ddot{B}T_5(t) &= 10BT_4(t) - 6BT_2(t) + 26BT_0(t) \\
\ddot{B}T_6(t) &= 12BT_5(t) + 4BT_3(t) + 28BT_1(t) \\
\ddot{B}T_7(t) &= 14BT_6(t) + 2BT_4(t) + 30BT_2(t) - 82BT_0(t) \\
\ddot{B}T_8(t) &= 16BT_7(t) + 32BT_5(t) - 80BT_3(t)
\end{align*}

Rewrite the above equation as

\begin{equation}
\ddot{B}T(t) = M(\dot{B}T)(t)
\end{equation}

where

\((\dot{B}T)(t) = [\dot{B}T_0(t), \dot{B}T_1(t), ..., \dot{B}T_8(t)]\), \((\ddot{B}T)(t) = [\ddot{B}T_0(t), \ddot{B}T_1(t), ..., \ddot{B}T_8(t)]\) and the matrix M is 8×8 and namely operational matrix of derivative.

\[
M = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
-10 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\
0 & -8 & 0 & 8 & 0 & 0 & 0 & 0 \\
26 & 0 & -6 & 0 & 10 & 0 & 0 & 0 \\
0 & 28 & 0 & -4 & 0 & 12 & 0 & 0 \\
-82 & 0 & 30 & 0 & -2 & 0 & 14 & 0 \\
0 & -80 & 0 & 32 & 0 & 0 & 0 & 16
\end{pmatrix}
\]

Note that the derivative of BTPs can be written in the compact form

\[
\dot{B}T(t) = (\dot{B}T)_i(t) = \sum_{k=0}^{i-1} b_{i,k} (BT)_k(t)
\]

where

\[
b_{i,k} = \begin{cases} 
2i, & k > 1 \\
(-1)^{i-k}(i+1-k)(i+2-k), & k = 1 \\
(i-1, k-1) + (i-1, k+1) - (i-2, k), & k > 1 \\
i < k, & i > k
\end{cases}
\]
2.3. The Product Operational Matrix of Boubaker Turki Polynomials

It is necessary to compute the product of $BT(t)$ and $(BT)^T(t)$, that is the product for the Boubaker Turki polynomials basis. Assume

$$P(t) = BT(t)(BT)^T(t)$$

To illustrate the calculation we choose $m = 4$

$$BT(t)(BT)^T(t) = \begin{pmatrix} BT_0 & BT_1 & BT_0 & BT_3 \\ BT_1 & BT_2 & BT_1 & BT_3 \\ BT_2 & BT_3 & BT_2 & BT_3 \\ BT_3 & BT_4 & BT_3 & BT_3 \end{pmatrix}$$

Expanding each product by Boubaker Turki polynomials basis, yields

$$\begin{align*}
BT_0^2 &= BT_0^2 \\
BT_1 &= BT_0 + 2BT_2 \\
BT_2 &= BT_0 + 4BT_3 \\
BT_3 &= BT_0 + 6BT_4
\end{align*}$$

In general

$$BT_i^2 = BT_0^2 - 2BT_0 + 4 \sum_{k=1}^{i-1} BT_2k$$

2.4. The Initial and Final Conditions of Boubaker Turki Polynomials

One of the important properties of Boubaker Turki Polynomials is given in the following Lemma.

**Lemma 1**

$$BT_i(0) = \begin{cases} 0, & i \text{ odd} \\ (-1)^{i/2}, & i \text{ even} \end{cases}$$

$$BT_i(1) = 2 + 4(i-1), i = 1, 2, \ldots$$

where $BT_0(1) = 1$

2.5. The Relation Between Boubaker Turki Polynomials and Chebyshev Polynomials of the Second Kind

The Boubaker-Turki polynomials can be written in terms of Chebyshev polynomials of the second kind as given through the following lemmas.

**Lemma 2**

$$BT_i(t) = U_i(t) + 3U_{i-2}(t) \quad i = 2, 3, \ldots$$

where $BT_0(t) = U_0(t), BT_1(t) = U_1(t)$

**Proof**

The mathematical induction principle is used to proof Eq. 9

First Eq. 9 is true when $i = 2$ by direct calculation, since

$$BT_2(t) = U_2(t) + 3U_0(t) = 4t^2 + 2.$$

Assume that Eq. 9 is true for a particular positive integer $i = k$, this means that

$$BT_k(t) = U_k(t) + 3U_{k-2}(t)$$

Now, to see that Eq. 9 is true for $i = k + 1$, using the following identity

$$BT_{k+1}(t) = 2x(BT_k(t) - BT_{k-1}(t))$$

$$= 2x(U_k(t) + 3U_{k-2}(t) - U_{k-1}(t) - 3U_{k-3}(t))$$

$$= 2xU_k(t) + 3U_{k-1}(t) + 3(2xU_{k-2}(t) - U_{k-3}(t))$$

This means that

$$BT_{k+1}(t) = U_k(t) + 3U_{k-2}(t)$$ which is the required results.

**Lemma 3**
\[ \frac{d \text{B}_1(t)}{dt} = \frac{d u_1(t)}{dt} + 3 \frac{d u_{i-2}(t)}{dt} \]

where \( \frac{d u_i(t)}{dt} = \sum_{k=0}^{i-1} 2(k + 1) u_k(t) \), \( i = 1, 2, \ldots \), \( i + k = \text{odd} \)

**Proof**: by differentiating both sides of (10), the result can be reached.

### 2.6. Explicit Expression for Powers of \( t \) In terms of Boubaker Turki Polynomials

A general formula for \( (1, t, t^2, \ldots, t^n) \) can be presented as linear combination of the Boubaker-Turki polynomials.

One can observe that the powers \( (1, t, t^2, \ldots, t^n) \) with Boubaker-Turki polynomials are described as follows

\[ \begin{bmatrix} 1 \\ t^1 \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \\ t^7 \\ t^8 \end{bmatrix} = \frac{1}{2^n} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \text{B}_0(t) \\ \text{B}_1(t) \\ \text{B}_2(t) \\ \text{B}_3(t) \\ \text{B}_4(t) \\ \text{B}_5(t) \\ \text{B}_6(t) \\ \text{B}_7(t) \\ \text{B}_8(t) \end{bmatrix} \]

or

\[ \begin{bmatrix} 1 \\ t^1 \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \\ t^7 \\ t^8 \end{bmatrix} = \frac{1}{2^n} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \text{B}_0(t) \\ \text{B}_1(t) \\ \text{B}_2(t) \\ \text{B}_3(t) \\ \text{B}_4(t) \\ \text{B}_5(t) \\ \text{B}_6(t) \\ \text{B}_7(t) \\ \text{B}_8(t) \end{bmatrix} \]

The compact form which relates the powers \( (1, t, t^2, \ldots, t^n) \) and Boubaker-Turki polynomials can be written as

\[ k_{ij} = \frac{1}{2^n} \begin{cases} 1, & i = j \\ -2 k_{i-1,j+1}, & i = 1 \\ k_{i-1,j-1} + k_{i-1,j+1}, & i > j \end{cases} \]
3. The Design Spectral Operation Matrix Boubaker-Turki Approach for Solving Optimal Control Problem

New spectral parameterization associated Boubaker-Turki polynomials is proposed in this section for solving approximately a class of optimal control problem.

Consider the performance index

\[ J = \int F(x(t), u(t)) \, dt \]

with dynamical equation

\[ \dot{x}(t) = f(x(t), u(t)) \]

where \( x(t) \) is the state variables and \( u(t) \) is the control variables at time \( t \), and the boundary conditions are defined as \( x(0) = x_0 \), \( x(1) = x_1 \).

The algorithm steps are summarized as follows

Step 1: Approximate state variable \( x(t) \) as first trial solution using Boubaker-Turki polynomials

\[ x(t) = \sum_{i=0}^{n} a_i B(t) = B^T A(t) \]

where the vectors \( A \) and \( B \) are defined as

\[ A = [a_0, a_1, a_2] \quad \text{and} \quad B = [B_0, B_1, B_2] \]

Step 2: Use Eq. 13 to obtain

\[ a_0 = \frac{1}{2} (x_1 + x_0 - 2a_2), \quad a_1 = \frac{1}{2} (x_1 - x_0) \]

Step 3: Rewrite Eq. 14 as follows

\[ x_1(t) = a_2 B(t) + \frac{1}{2} [(x_1 - x_0) B(t) + (x_1 + x_2 - 2a_2)B_0(t)] \]

Step 4: Calculate the control variable, which is defined in Eq. 12

\[ u(t) = f(x(t), \dot{x}(t)) \quad \text{or} \quad u(t) = f(A^T B(t), A^T D B(t)) \]

where the matrix \( D \) is the operation matrix of derivative.

Step 5: Obtain the value of \( J \) as a function of \( a_2 \) by determining

\[ J = \int F(A^T B(t), A^T D B(t)) \, dt \]

Step 6: Apply the second iteration approximation for the state variable to be

\[ x_2(t) = x_1(t) + \sum_{i=1}^{n} a_i B_i(t) \]

Step 7: Use the same procedure in order to obtain the unknown coefficients \( a_1, a_2, a_3 \).

In general the \( n \)th iteration approximation for the state variable is

\[ x_n(t) = x_{n-1}(t) + \sum_{i=n-1}^{n} a_i B_i(t) \]

4. Numerical Examples

The efficiency of the suggested method is illustrated by considering a section in which some test examples are solved. A comparison with results of the exact solution is made to allow the validation of our proposed algorithm.

Application test 1

Consider the non-quadratic optimal control problem

Minimize \[ J = \int (-0.5u^2(t) + x(t)) \, dt \quad , \quad 0 \leq t \leq 1 \]

Together with the constraints

\[ u(t) = \dot{x}(t) + x(t) \quad , \quad x(0) = 0 \quad , \quad x(1) = 0.1997882004 \]

By following the steps presented in the previous section, one can get the first approximations for the state variable and the corresponding control variable as below

\[ x_1(t) = -0.1023 B(t) + 0.2046 B^T(t) + 0.3044 B^T(t) \]

\[ u_1(t) = -0.1023 B^T(t) + 0.0998 B^T(t) + 0.6084 B^T(t) \]

The approximate value for \( J \) corresponding to the first trial solution \( x_1(t) \) and \( u_1(t) \) is \( J_1 = 0.0840 \).

The obtained approximate result for the state variable and the exact solution \( x(t) = 1 - 0.5e^{-t} + 0.3591e^{-t} \) are plotted in Fig. 2. The obtained approximate result for the control variable and the exact
solution \( u(t) = 1 - e^{t-1} \) are also plotted in Fig. 2. Note that in the second trail approximation for \( x_2(t) \) and \( u_2(t) \), the obtained approximate value is \( J_2 = 0.08404 \).

**Figure 2.** The solution in the first and second trail approximations together with the exact one for example 1

Application test 2

Consider the quadratic optimal control problem

Minimize \( J = \int_0^1 (u^2(t) + x^2(t)) dt \), \( 0 \leq t \leq 1 \)

Together with the constraints

\[
\dot{x}(t) = u(t) \\
x(0) = 0, \quad x(1) = 0.5
\]

By following the steps presented in the previous section, one can get the first approximations for the state variable and the corresponding control variable as below

\[
x_1(t) = 0.0284BT_2(t) + 0.1932BT_1(t) - 0.0568BT_0(t) \\
u_1(t) = 0.1136BT_2(t) + 0.3864BT_0(t)
\]

The approximate value for \( J \) corresponding to the first trial solution \( x_1(t) \) and \( u_1(t) \) is \( J_1 = 0.3284 \).

The obtained approximate result for the state variable and the exact solution \( x(t) = 0.21273(e^t - e^{-t}) \) are plotted in Fig. 3. The obtained approximate result for the control variable and the exact solution \( u(t) = 0.21273(e^t + e^{-t}) \) are also plotted in Fig. 3. Note that in the second trail approximation for \( x_2(t) \) and \( u_2(t) \), the obtained approximate value is \( J_2 = 0.3283 \).

**Figure 3.** The solution in the first and second trail approximations together with the exact one for example 2
5. Conclusion
The explicit formulas of the derivatives operational matrix as well as the formula of the operational matrix of multiplication of two Boubaker-Turki polynomials are presented in this work. These operational matrices are utilized in the development of an approximate technique to solve optimal control problem. This technique converts the optimal control problem into a system of linear algebraic equations. Also, the recurrence relation which combining Boubaker-Turki polynomials and Chebyshev polynomials of the second kind is devoted through this work.

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