HAMiLTON–Ostrohrads’KYj APPROACh TO
RELATiViSTiC FREE SPHERiCAL TOP DYNAMiCS

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Abstract. Dynamics of classical spinning particle in special relativity with
Pirani constraint is a typical example of the generalized Hamilton theory
recently developed by O. Krupková and discovers some characteristic features
of the latter.

Grounding

Recent developments in Ostrohrads'kyj mechanics, in particular, some subst-antial progress in understanding its Hamiltonian counterpart, may give rise to the
enrichment in the family of the generalized canonical dynamical systems which
describe certain processes in the real physical world. In this report we call upon
the Reader to follow the possibility of building yet another canonical model of the
free spinning particle motion in special relativity. One way to do that is to start
with the system of Dixon’s equations [1] in flat space-time

\[
\begin{align*}
\dot{P} &= \mathbf{0} \\
\dot{S} &= 2P \wedge u.
\end{align*}
\]

(1a) (1b)

The four-vector \( P \), the velocity four-vector \( u \), and the skew-symmetric tensor \( S \)
do not constitute a complete system of variables if one wishes to put the equations
(1) into the Hamiltonian form in the usual way. On the other hand, the system
(1) is under-determined and needs to be supplemented by some constraints.

A profound classical and quantum description of the relativistic top dynam-
ics based upon the Dirac theory of constraints was offered by A. J. Hanson and
T. Regge in [2], where they exploited the constraint \( \mathcal{P}^a \mathcal{S}^{pq} = \mathbf{0} \), sometimes re-
ferred to as Tulczyjew supplementary condition. At the same time, some relativis-
tic centre-of-mass considerations, concerning the dipole model of massive spinning

2010 Mathematics Subject Classification. 83C10, 70H40, 70H05, 70H50, 83A05, 49N45.
Key words and phrases. Higher-order mechanics, Generalized Hamilton equations, Inverse
variational problem, Invariance, Relativistic top, Classical spin.

This paper is in final form and no version of it will be submitted for publication elsewhere.
particle in relativity (see [3] and references therein), bring about the alternative supplementary condition,

\[ u_q \varepsilon^{pq} = 0, \]

sometimes named Pirani supplementary condition. In the present report I shall make an attempt to ‘hamiltonize’ the ideology of the Pirani constraint in contrast to what was already done with respect to the Tulczyjew one.

1. A brief overview of classical spinning particle settlement

In the presence of the supplementary condition (2) it is possible to re-solve with respect to \( S \) the following definition of the spin four-vector \( s \),

\[ s_p = \frac{1}{2||u||} \varepsilon_{mnqp} u_m S^n q, \]

and in this way the system of equations (1b,2) may be replaced by the following one:

\[
\begin{cases}
\dot{\mathbf{P}} = \mu_0 \frac{u}{||u||} + \frac{\dot{\mathbf{u}} \wedge u \wedge s}{||u||^2} \quad (3a) \\
\dot{s} \wedge u = 0 \\
\dot{s} \cdot u = 0.
\end{cases}
\]

The quantity \( \mu_0 = \frac{\mathbf{P} \cdot u}{||u||} \) entering in the expression (3a) may immediately be shown to constitute an integral of motion (even if we replace the right-hand side of (1a) by some force \( \mathbf{F} \), provided only that the condition \( \mathbf{F} \cdot u = 0 \) is obeyed). The equation (3b) may also be given an equivalent form of

\[ \dot{s} = \frac{\dot{\mathbf{s}} \cdot u}{||u||^2} u, \]

by means of which we deduce from (3a) that the value of the contraction \( \mathbf{P} \cdot \mathbf{s} \equiv (\mathbf{P} \cdot \mathbf{s}) - \mathbf{P} \cdot \mathbf{s} \) in fact equals \( \frac{\mu_0}{||u||} \dot{\mathbf{s}} \cdot u \), and thus, again by means of (4), the spin four-vector \( \mathbf{s} \) is constant everywhere where \( \mathbf{P} \cdot \mathbf{s} \) is null; hence in the flat space-time there is no precession due to (1a), i.e.

\[ \dot{\mathbf{s}} = 0. \]

The third-order equation of motion, obtained by substituting (3a) into (1a), coincides, within the realm of the Pirani supplementary condition, with the equation suggested by Mathisson [4] in terms of \( \mathbf{S} \).

Now let us fix the parametrization of the world line of the particle by means of choosing the coordinate time as the parameter along the trajectory. We introduce the space vs. time splitting of the variables with the help of the following notations:

\[ u = (1, v); \quad \mathbf{P} = (\mathbf{P}_0, \mathbf{P}); \quad \mathbf{s} = (s_0, \mathbf{s}), \]
by which the formulae (3a) and (3c) take the shape (please notice $v^2 = v \cdot v = v_\alpha v^\alpha = -\sum_{a=1}^3 v_\alpha v_a$, although all constructions bear the same appearance independent of the signature of the (pseudo-)Euclidean metrics)

$$
\begin{cases}
P = \frac{\mu_0}{\sqrt{1 + v^2}} v + \frac{1}{(1 + v^2)^{3/2}} (v' \times s - s_0 \cdot v' \times v) \\
s_0 + s \cdot v = 0.
\end{cases}
$$

(6)

(7)

2. Hamiltonian dynamics of free relativistic top

We shall follow the approach of [5] and describe the Hamiltonian dynamics by means of the kernel of the Lepagean differential two-form

$$
- dH \wedge dt + dp_a \wedge dx^a + dp'_a \wedge dv^a
$$

(8)

with the Hamilton function

$$
H = p \cdot v - \frac{M_0 \sqrt{1 + v^2}}{(s_0^2 + s^2)^{3/2}}.
$$

(9)

One observation consists in that it is possible to define such functions $p$ and $p'$ of the variables $v$ and $v'$, that in (8) all one-contact terms of the second order cancel out, and the expression (8) becomes

$$
- \frac{\partial p_a}{\partial v^b} \omega^a \wedge \omega^b - \frac{\partial p'_a}{\partial v^b} \omega^a \wedge \omega^b - \frac{\partial p_a}{\partial v^b} \omega^a \wedge dt - \frac{\partial p'_a}{\partial v^b} \omega^a \wedge dv^b,
$$

with $\omega$ and $\omega'$ denoting the contact forms of the first and of the second order resp. The functions $p$ and $p'$ constitute the generalized Legendre transformation, and the Lagrangian counterpart of dynamics is described by the Euler-Poisson expression

$$
- \frac{d}{dt} p = -(v' \cdot \partial_v + v'' \cdot \partial_{v'}) p. \tag{10}
$$

We can suggest the following expression of the Legendre transformation which I believe points at the adequate way to hamiltonize the dynamics governed by the system of equations (1a & 3a),

$$
\begin{cases}
p = \frac{M_0}{(s_0^2 + s^2)^{3/2}} v + \frac{v' \times (s - s_0 v)}{\sqrt{1 + v^2} \left[ (s - s_0 v)^2 + (s \times v)^2 \right]^{3/2}} \\
p' = \frac{\xi \times (s - s_0 v)}{3 \left( (s_0^2 + s^2) \left[ (s - s_0 v)^2 + (s \times v)^2 \right]^{1/2} \right)}
\end{cases}
$$

(11a)

(11b)

where

$$
\xi_a = \frac{1}{s_0} \frac{(s_0 + s \cdot v) s_a - (s_0^2 + s^2) v_a}{(s - s_0 v)^2 - (s_0 - s_0 v_0)^2 + (s \times v)^2},
$$

(12)

and $M_0$ is some constant number.

Looking closer at the expression (10) with $p$ given by (11a) convinces that the Lagrange system, defined by (10), carries along the primary semispray constraint
(if stick to the terminology of [5])

$$\frac{M_0}{s_0 - s^2} \left[ \frac{s \cdot v'}{\sqrt{1 + v^2}} - \frac{(s_0 + s \cdot v)(v \cdot v')}{(1 + v^2)^{3/2}} \right] = 0. \quad (13)$$

The expression included within square brackets in (13) presents an exact total derivative, so we obtain the first integral of motion,

$$\frac{s_0 + s \cdot v}{\sqrt{1 + v^2}}, \quad (14)$$

that clearly generalizes the genuine constraint (7) which in turn—we recall—is nothing else but the rudiment of the Pirani supplementary condition (2).

One would have to prove that the Hamiltonian dynamics defined by (8,9,11) really has some connection with the classical spinning particle dynamics given by (1a,3a,3c, and 5). This connection clears up in two steps. First, prove the following algebraic identity:

$$\frac{(s - s_0 v)^2 + (s \times v)^2}{(s_0^2 + s^2)(1 + v^2)} = 1 - \frac{(s_0 + s \cdot v)^2}{(s_0^2 + s^2)(1 + v^2)}, \quad (15)$$

Then, multiply (11a) by the constant of motion \([s - s_0 v, \dot{v}, s]\) and compare with (6) to conclude that there must exist a link-up between the constants \(\mu_0\) and \(M_0\):

$$\mu_0 = M_0 \left[ 1 - \frac{(s_0 + s \cdot v)^2}{(s_0^2 + s^2)(1 + v^2)} \right]^{3/2}. \quad (16)$$

We may summarize the results of the preceding calculations in a couple of statements:

- As far as Pirani supplementary condition is recognized, the phase space of the free classical spinning particle may be augmented in the way that the dynamics allows a generalized Hamiltonian description with the Hamilton function

$$H = -\frac{M_0}{(s_0^2 + s^2)^{3/2}} \frac{[\dot{v}', v, s]}{\sqrt{1 + v^2}} - \frac{[\dot{v}', v, s]}{[(s - s_0 v)^2 + (s \times v)^2]^{3/2}}; \quad (17)$$

- The mass \(\mu_0\) of the ‘hamiltonized’ particle depends upon its spin according to the expression (16); it is worthwhile to mention at this place that the Hamiltonian description of [2] demanded an arbitrary dependence of the particle’s mass on its spin;

- Any dynamical subsystem, obtained by prescribing a fixed value to the integral of motion (14), never is Hamiltonian by itself; in particular, we could not have obtained a variational description of the spinning particle motion if the constant of motion (14) had been frozen by means of the equation (7) or, equivalently, by the demand that \(\mu_0\) and \(M_0\) take the same value in (16);
The Legendre transformation, given by (11), is not globally defined in an intrinsic sense, as may be seen from (12); nevertheless, the Hamilton function is defined quite nicely via the expression (17).

Guessing the form of the Legendre transformation (11) is equivalent to solving the Poincaré-invariant inverse problem of calculus of variations in order 3. That was treated in [6] and the corresponding Euler-Poisson expression (10) found. But I did not know the appropriate expression for the Legendre transformation until 1995 when a set of Lagrange functions corresponding to (10) was discovered [7].

References

[1] W. G. Dixon, Dynamics of extended bodies in general relativity I. Momentum and angular momentum, Proc. Royal Soc. London, Ser. A. 314 (1970), 499–527.
[2] A. J. Hanson and T. Regge, The relativistic spherical top, Ann. Phys. 87 (1974), no. 2, 498–566.
[3] C. Møller, The theory of relativity, Springer, Berlin–Heidelberg, 1972.
[4] M. Mathisson, Neue Mechanik materieller Systeme, Acta Phys. Polon. 6 (1937), no. 3, 163–200.
[5] Olga Krupková, The geometry of ordinary variational equations, Lecture Notes in Mathematics, vol. 1678, Springer, 1997.
[6] R. Ya. Matsyuk, Poincaré-invariant equations of motion in Lagrangian mechanics with higher derivatives, Ph.D. Thesis, Institute for Applied Problems in Mechanics and Mathematics, Academy of Science of Ukraine, Lviv, 1984 (Russian).
[7] R. Ya. Matsyuk, Spin dependence of classical test particle mass in the third-order relativistic mechanics, 14th International Conference on General Relativity and Gravitation (Florence, August 6), Abstracts of Contributed Papers, pp. A.120.
Spin Dependence of Classical Test Particle Mass in the Third-Order Relativistic Mechanics

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From time to time there arise some higher-order equations of motion in the classical mechanics of particles. Perhaps the best known one is the Lorentz-Dirac equation of the radiating electron. Unfortunately, in the real physical dimension of four one may never derive a third-order equation of motion with Lorentz symmetry from a variational principle.¹

The Lorentz-Dirac equation, however, may be put down into a more general framing of the Dixon system of equations

\[ \frac{1}{2} \dot{S} = p \land u; \quad \dot{p} = \mathcal{F} \]

by introducing the momentum \( p = \frac{m_0 u}{|u|} + \frac{\dot{S} \cdot u}{u} \) and the spin tensor \( \frac{1}{2} S = \frac{4}{9} S_2 \frac{u \land u}{u^2} \).

Conversely, it may be cut off from (A1) by means of the constraint \( \dot{S} \cdot u + \frac{2e^2}{3} \left( \dot{u} - \frac{(u \land u) u}{|u|^2} \right) = 0 \) and defining the force in (A1) by \( \mathcal{F} = -\frac{2}{3} e^2 \kappa u \), where \( \kappa \) stands for the curvature of the particle’s world line.

These considerations suggest that there may still exist a third-order equation of motion with the Lorentz symmetry, derivable from a variational principle if only some spin variable were retained in it. In course of developing some previous work² we offer here the following form of the spinning particle equations of motion

\[ \frac{\mu_0}{|\sigma|^3} \left[ \frac{(u \cdot \dot{u})}{|u|^3} u - \frac{\dot{u}}{|u|} \right] - \frac{\dot{u} \land u \land \sigma}{|\sigma \land u|^3} + 3 \frac{\dot{u} \land u \land \sigma}{|\sigma \land u|^3} (\sigma \land \dot{u}) \cdot (\sigma \land u) = \frac{|u|}{|\sigma \land u|^3} \mathcal{F}, \]

\[ \dot{\sigma} \land u = 0 \]

where we have set \( \mathcal{F}_\alpha = \frac{1}{2} R_{\alpha \beta \gamma} \sigma^\beta \gamma S_{\gamma \delta} \).

The system of equations (A2, A3) may be put into the Dixon form (A1) by introducing the spin tensor \( \frac{1}{2} S = \frac{u \land \sigma}{|u|} \) and imposing the constraint

\[ (\sigma \cdot u) = 0. \]

Conversely, if we introduce the spin four-vector \( \sigma \), then the Pirani auxiliary condition \( S \cdot u = 0 \) will cut off the system of equations (A2–A4) from the Dixon system (A1). This way the Dixon system (A1) may be thought of as a “covering equation” both to the Lorentz-Dirac equation and to the equation (A2). But the latter admits a lagrangian. In the flat Lorentz space-time the lagrangian for the equation (A2) reads

\[ L = \frac{\mu_0}{|\sigma|^3} |u| + \frac{L_{(\alpha)}}{|\sigma|^2 |\sigma \land u|}, \quad \text{with} \quad L_{(\alpha)} = \frac{\dot{u} \land u \land \sigma \land \epsilon_{(\alpha)}}{(u \cdot \sigma - \sigma \cdot u)^2 - (\sigma \land u)^2} \left( \sigma^2 u_{\alpha} + (\sigma \cdot u) \sigma_{\alpha} \right), \]

where \( \epsilon_{(\alpha)} \) denotes the \( \alpha \)-component of the Lorentz frame. Each \( L_{(\alpha)} \) will do because each differs from the others by some total derivative. This holds irrespective of the constraint (A4).

From the variational point of view it is even more natural to consider the equation (A2) also in the region outside the manifold defined by the constraint (A4). If we assume that (A1) together with the Pirani auxiliary condition governs the motion of a spinning particle with the constant mass \( m_0 \), then it follows that the equation (A2) admits (at least in the flat space-time) a spectrum of the particles with the variable mass \( m = \mu_0 \left[ 1 - \frac{(\sigma \cdot u)^2}{\sigma^2 u^2} \right] \). This quantity \( m \) is a constant of motion if only \( \dot{\sigma} = 0 \) and \( \mathcal{F} = 0 \). The shortcoming of our approach is that we freeze the spin variable in the variational procedure. We shall try to overcome this threshold in future work. On the other hand, it is common that the spin four-vector \( \sigma \) does not change along the world line of the particle in the flat space-time.

¹R. Ya. Matsyuk. Poincaré-invariant equations of motion in lagrangian mechanics with higher derivatives. Thesis, Institute for Applied Problems in Mechanics and Mathematics, Academy of Science, Ukraine. Lviv, 1984 (in Russian).
²R. Ya. Matsyuk. In: Abstracts of Contributed Papers. 11th International Conference on General Relativity and Gravitation. Stockholm, July 6–12, 1986. Vol. II, p. 648
³R. Ya. Matsyuk. Dokl. Akad. Nauk SSSR, 285 (1985), No. 2, 327–330, English translation: Soviet Phys. Dokl., 30 1985, No. 11, 923–925, MR0820861(87d:70029).
LAGRANGIAN APPROACH TO SPINNING OR RADIATING PARTICLE HIGHER-ORDER EQUATIONS OF MOTION IN SPECIAL RELATIVITY

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We recognize it worthwhile to distinguish among the higher-order equations of motion those admitting a variational reformulation. Considering Dixon’s form of Papapetrou equations either for free particle,

\begin{equation}
\frac{\mathbf{p}}{m} = 0, \quad \frac{\mathbf{S}}{m} = 2 \mathbf{p} \wedge \mathbf{u},
\end{equation}

or introducing an external force by means of \( p' = ku' u \), and imposing within the restriction \( \|u\| = 1 \) a supplementary condition \( \mathbf{S}' + ku' = 0 \), one regains the Lorentz-Dirac equation, \( m\mathbf{u}' = k(u'' + u' u) \), in the second case with constant mass \( m = pu \), reversing thus the considerations of A. O. Barut. The Lorentz-Dirac equation, however, cannot be reformulated in terms of a variational principle.* In the presence of another supplementary condition, \( Su = 0 \), the equations (B1) can be given the form

\begin{equation}
\frac{m}{u} \mathbf{u}' = Su'' \quad \text{(M. Mathisson, 1937)}, \quad \frac{\mathbf{S}}{u} \wedge \frac{\mathbf{u}}{u} = 0, \quad (u^2 = 1).
\end{equation}

In terms of the star operator \( * \), the spin four-vector \( \mathbf{s} = \frac{\mathbf{u} \wedge \mathbf{S}}{2\|u\|} \), and complying with the obvious constraint \( su = 0 \), equation (B2) can be rewritten as

\begin{equation}
\frac{m}{u^2} \mathbf{u}' = * (u'' \wedge \mathbf{u} \wedge \mathbf{s}), \quad (u^2 = 1, \quad s' = 0).
\end{equation}

Renormalizing the mass by means of

\begin{equation}
m = m_0 \left( 1 - \frac{(su)^2}{s^2 u^2} \right)^{3/2},
\end{equation}

and taking \( m_0 \) to be an arbitrary but fixed constant, we state here that the following third-order equation of motion defining the same set of world lines as (B3),

\begin{equation}
m_0 \frac{u_2 u'}{u^2} - \frac{(u') u}{u^2} = \frac{*(u'' \wedge \mathbf{u} \wedge \mathbf{s})}{\|\mathbf{u} \wedge \mathbf{s}\|^3} - 3* (u' \wedge \mathbf{u} \wedge \mathbf{s}) \frac{(s \wedge \mathbf{u}) \cdot (s \wedge u)}{\|\mathbf{u} \wedge \mathbf{s}\|^5},
\end{equation}

is the Euler-Poisson equation of a parameter-invariant variational problem. Proceeding further in eliminating the spin variables from (B3) results in the fourth-order equation

\begin{equation}
u''' = \frac{p'^2}{s^2} u', \quad (u^2 = 1),
\end{equation}

which under the assumption \( p^2 > 0 \) was suggested from the higher-order Lagrangian point of view by F. Riewe without indicating any direct relationship with Papapetrou equations.

REFERENCES. A. O. Barut. In: “Quantum Opt. Exp. Gravity, and Meas. Theory”, Proc. NATO Adv. Study Inst., Bad Windsheim, 1981, N. York–London, 1983, 155–168. M. Mathisson. Acta Phys. Polon., 6 (1937), fasc. 3, 163–200. *R. Ya. Matsyuk. Dokl. Akad. Nauk SSSR, 282 (1985), No. 4, 841–844, English translation: Soviet Phys. Dokl., 30 (1985), No. 6, 458460, MR0802859 (87d:70028). T. F. Riewe. Il Nuovo Cim. B, 8 (1972), No. 1, 271–277.