Bilinear Pseudo-Differential Operators with Exotic Class Symbols of Limited Smoothness

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Abstract

We consider the bilinear pseudo-differential operators with symbols in the bilinear Hörmander class $BS_{\rho,\rho}^m(\mathbb{R}^n)$ for $m \in \mathbb{R}$ and $0 \leq \rho < 1$. The aim of this paper is to discuss smoothness conditions for symbols to assure the boundedness from $h^p \times h^q$ to $h^r$ for $1 \leq r \leq 2$, $p, q \leq \infty$ satisfying $1/p + 1/q = 1/r$, where $h^\infty$ is replaced by the space $bmo$.

Keywords Bilinear pseudo-differential operators · Bilinear Hörmander symbol classes · Smoothness conditions

Mathematics Subject Classification 35S05 · 42B15 · 42B35

1 Introduction

Let $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$. The bilinear Hörmander symbol class $BS_{\rho,\delta}^m(\mathbb{R}^n)$ consists of all functions $\sigma(x, \xi, \eta) \in C^\infty((\mathbb{R}^n)^3)$ such that

$$|\partial_\alpha^\alpha \partial_\beta^\beta \partial_\gamma^\gamma \sigma(x, \xi, \eta)| \leq C_{\alpha,\beta,\gamma}(1 + |\xi| + |\eta|)^{m+\delta(|\beta|+|\gamma|)}$$

for all multi-indices $\alpha, \beta, \gamma \in \mathbb{N}_0^n = \{0, 1, 2, \ldots \}^n$. For a bounded measurable function $\sigma = \sigma(x, \xi, \eta)$ on $(\mathbb{R}^n)^3$, the bilinear pseudo-differential operator $T_\sigma$ is defined...
by
\[ T_\sigma(f, g)(x) = \frac{1}{(2\pi)^n} \int_{(\mathbb{R}^n)^2} e^{i x \cdot (\xi + \eta)} \sigma(x, \xi, \eta) \hat{f}(\xi) \hat{g}(\eta) \, d\xi \, d\eta \]
for \( f, g \in \mathcal{S}(\mathbb{R}^n) \). Regarding the boundedness of the operator \( T_\sigma \), we will use the following terminology with a slight abuse. Let \( X, Y, Z \) be function spaces. If there exists a constant \( C \) such that the estimate
\[ \| T_\sigma(f, g) \|_Z \leq C \| f \|_X \| g \|_Y \]
holds for all \( f \in \mathcal{S} \cap X \) and \( g \in \mathcal{S} \cap Y \), then we simply say that the operator \( T_\sigma \) is bounded from \( X \times Y \) to \( Z \). If \( \mathcal{A} \) is a class of symbols, we denote by \( \text{Op}(\mathcal{A}) \) the class of all bilinear operators \( T_\sigma \) such that \( \sigma \in \mathcal{A} \). If the operator \( T_\sigma \) is bounded from \( X \times Y \) to \( Z \) for all \( \sigma \in \mathcal{A} \), then we write \( \text{Op}(\mathcal{A}) \subset B(X \times Y \to Z) \). The boundedness of the bilinear pseudo-differential operators with symbols in the bilinear Hörmander classes has been studied by a lot of researches.

In the case \( \rho = 1 \) and \( \delta < 1 \), since the bilinear operator \( T_\sigma \) for \( \sigma \in BS^0_{1,1} \) can be seen as a bilinear Calderón–Zygmund operator in the sense of Grafakos–Torres [12], \( T_\sigma \) is bounded from \( L^p \times L^q \) to \( L^r \), \( 1 < p, q < \infty, 1/p + 1/q = 1/r \). See also Coifman–Meyer [7], Bényi–Torres [1], and Bényi–Maldonado–Naibo–Torres [3]. Moreover, in the case \( \rho = \delta = 1 \), Bényi–Torres [1] and Koemukka–Tomita [18] proved that the bilinear operator \( T_\sigma \) for \( \sigma \in BS^0_{1,1} \) is bounded from \( L^p_x \times L^q_y \) to \( L^r_z \), \( 1 < p, q < \infty, 1/p + 1/q = 1/r, s > 0 \), where \( L^p \) is the \( L \)-Sobolev space equipped with the norm \( \| f \|_{L^p} = \|(I - \Delta)^{s/2} f\|_{L^p} \). Hence, we might say that, in the case \( \rho = 1 \), there is no big difference between the linear and bilinear cases.

In the case \( \rho < 1 \), an interesting difference between the linear and bilinear cases appears. To see this, we shall recall the Calderón–Vaillancourt theorem [6] for linear pseudo-differential operators, which is defined by
\[ \sigma(X, f(x)) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(x, \xi) \hat{f}(\xi) \, d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n). \]
Then, if the symbol \( \sigma \) is in \( S^0_{\rho, \rho}(\mathbb{R}^n) \), \( 0 \leq \rho < 1 \), that is, \( \sigma(x, \xi) \) satisfies that
\[ |\partial_\alpha^\alpha \partial_\xi^\beta \sigma(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{\rho(|\alpha| - |\beta|)} \]
for all multi-indices \( \alpha, \beta \in \mathbb{N}_0^n \), the linear operator \( \sigma(X, D) \) is bounded on \( L^2 \). For bilinear pseudo-differential operators, it was proved by Bényi–Brenicott–Maldonado–Naibo–Torres [4] that if \( 1 \leq p, q, r < \infty \) satisfy that \( 1/p + 1/q = 1/r \), there exists a symbol \( \sigma \in BS^0_{\rho, \rho}(\mathbb{R}^n), 0 \leq \rho < 1 \), for which the corresponding bilinear operator \( T_\sigma \) is not bounded from \( L^p \times L^q \) to \( L^r \). See also Bényi–Torres [2]. In particular, this result implies that the class \( BS^0_{\rho, \rho}(\mathbb{R}^n) \) does not always provide the boundedness from \( L^2 \times L^2 \) to \( L^1 \). Therefore, we see that, when \( \rho < 1 \), the bilinear case remarkably differs from the linear one.
The result in [4] mentioned above additionally implies that we must choose the negative order \( m \), in order to have the boundedness \( \text{Op}(BS_{p,\rho}^m) \subset B(L^p \times L^q \rightarrow L') \) for \( 0 \leq \rho < 1 \). So, we shall consider the order \( m \). It was proved by Michalowski–Rule–Staubach [20] that \( \text{Op}(BS_{p,\rho}^m) \subset B(L^2 \times L^2 \rightarrow L^1) \) holds if \( m < -(1-\rho)n/2 \). Then, by Bényi–Bernicot–Maldonado–Naibo–Torres [4], this was generalized to the boundedness \( \text{Op}(BS_{p,\rho}^m) \subset B(L^p \times L^q \rightarrow L') \) for \( 1 \leq p, q, r \leq \infty \), \( 1/p + 1/q = 1/r \), and \( m < -n(1-\rho) \max\{1/2, 1/p, 1/q, 1 - 1/r\} \). For the critical case, Naibo [27] proved that \( \text{Op}(BS_{p,\rho}^{-(1-\rho)n/2}) \subset B(L^\infty \times L^\infty \rightarrow BM\Omega) \) for \( 0 < \rho < 1/2 \). After the results above, in [22,24,25], Miyachi–Tomita proved that, for \( 0 < p, q, r \leq \infty \) and \( 1/p + 1/q = 1/r \), the boundedness

\[
\text{Op}(BS_{p,\rho}^m) \subset B(H^p \times H^q \rightarrow L'),
\]

where \( H^p \) denote Hardy spaces and \( L' \) should be replaced by the space \( BM\Omega \) if \( r = \infty \), holds if and only if \( m \leq m_{\rho}(p,q) \) with

\[
m_{\rho}(p,q) = -n(1-\rho) \max\left\{ \frac{1}{2}, \frac{1}{p}, \frac{1}{q}, 1 - \frac{1}{r}, \frac{1}{r} - \frac{1}{2} \right\}.
\]

Here, it should be remarked that, in the case \( \rho = 0 \), the sharper result \( \text{Op}(BS_{0,0}^m) \subset B(h^p \times h^q \rightarrow h^r) \), \( m \leq m_0(p,q) \), holds, where \( h^p \) denote local Hardy spaces and \( h^\infty \) should be replaced by the space \( bmo \) (see [22]).

In the present paper, we are interested in the boundedness

\[
\text{Op}(BS_{p,\rho}^{-(1-\rho)n/2}) \subset B(L^p \times L^q \rightarrow L') \tag{1.1}
\]

for \( 1 \leq r \leq 2 \leq p, q \leq \infty \) satisfying that \( 1/p + 1/q = 1/r \), which is said to be a fundamental part in the bilinear case. We here remark that, in order to have (1.1), much smoothness is implicitly assumed for symbols. So, in what follows, we shall consider regularity conditions of symbols to assure the boundedness. We first recall the linear case. The smoothness condition of symbols assumed in the Calderón–Vaillancourt theorem was relaxed to, roughly speaking, the smoothness up to \( n/2 \) for each variable \( x \) and \( \xi \). For \( \rho = 0 \), see, e.g., Boulkhemair [5], Coifman–Meyer [7], Cordes [8], Miyachi [21], Muramatu [26], and Sugimoto [29]. For \( 0 < \rho < 1 \), see, e.g., Marschall [19] and Sugimoto [30]. We shall next consider the bilinear case. In the case \( \rho = 0 \), it was shown in [16,17] that the smoothness of symbols up to \( n/2 \) for each variable \( x \), \( \xi \), and \( \eta \) assures the boundedness (1.1) for \( \rho = 0 \). See also Herbert–Naibo [13,14] for the preceding results. To the best of the author’s knowledge, there is no related result for \( 0 < \rho < 1 \).

The purpose of this paper is to improve the boundedness (1.1) for \( 0 < \rho < 1 \) in two ways. Firstly, as in the case \( \rho = 0 \), we show that the sharper result \( \text{Op}(BS_{p,\rho}^{-(1-\rho)n/2}) \subset B(h^p \times h^q \rightarrow h^r) \) holds for \( 1 \leq r \leq 2 \leq p, q \leq \infty \), \( 1/p + 1/q = 1/r \), and \( 0 < \rho < 1 \), where \( h^\infty \) is replaced by the space \( bmo \). Secondly, we determine smoothness conditions of symbols for \( 0 < \rho < 1 \) to assure the boundedness (1.1).

Now, we shall state the main theorem of the present paper. To do this, we introduce a Besov type symbol class. Let \( \{\psi_k\}_{k \in \mathbb{N}_0} \) and \( \{\Psi_j\}_{j \in \mathbb{N}_0} \) be Littlewood–Paley partitions.
of unity on $\mathbb{R}^n$ and $(\mathbb{R}^n)^2$, respectively. For $j \in \mathbb{N}_0$, $k = (k_0, k_1, k_2) \in (\mathbb{N}_0)^3$, $s = (s_0, s_1, s_2) \in [0, \infty)^3$, and $\sigma = \sigma(x, \xi, \eta) \in L^\infty((\mathbb{R}^n)^3)$, write $k \cdot s = k_0s_0 + k_1s_1 + k_2s_2$, 

$$\Delta_k \sigma(x, \xi, \eta) = \psi_{k_0}(D_x)\psi_{k_1}(D_\xi)\psi_{k_2}(D_\eta)\sigma(x, \xi, \eta),$$

and

$$\sigma^\rho_j(x, \xi, \eta) = \sigma(2^{-j\rho}x, 2^{j\rho}\xi, 2^{j\rho}\eta)\Psi_j(2^{j\rho}\xi, 2^{j\rho}\eta).$$

Then, for $m \in \mathbb{R}$, we denote by $BS^m_{\rho, \rho}(s; \mathbb{R}^n)$ the set of all $\sigma \in L^\infty((\mathbb{R}^n)^3)$ such that

$$\|\sigma\|_{BS^m_{\rho, \rho}(s; \mathbb{R}^n)} = \sup_{j \in \mathbb{N}_0} \|2^{-j\rho - k\cdot s}\Delta_k(\sigma_j^\rho)\|_{L^2_{ul}(\mathbb{R}^n)^3} < \infty,$$

where $L^2_{ul}$ is the uniformly local $L^2$ space (see Sect. 2.1). By using the notations as above, the main theorem of this paper is given as follows.

**Theorem 1.1** Let $1 \leq r \leq 2 \leq p, q \leq \infty$ satisfy $1/p + 1/q = 1/r$ and let $0 \leq \rho < 1$, $m = -(1 - \rho)n/2$, and $s = (s_0, s_1, s_2) \in [0, \infty)^3$ satisfy $s_0 > n/2$ and $s_1, s_2 \geq n/2$. Then, if $\sigma \in BS^m_{\rho, \rho}(s; \mathbb{R}^n)$, the bilinear pseudo-differential operator $T_\sigma$ is bounded from $h^p(\mathbb{R}^n) \times h^q(\mathbb{R}^n)$ to $h^r(\mathbb{R}^n)$, where $h^\infty$ can be replaced by the space bmo.

**Remark 1.2** More precisely, under the assumptions in Theorem 1.1, we will prove that there exists a constant $C$ such that the estimate

$$\|T_\sigma(f, g)\|_{h^r(\mathbb{R}^n)} \leq C\|\sigma\|_{BS^m_{\rho, \rho}(s; \mathbb{R}^n)}\|f\|_{h^p(\mathbb{R}^n)}\|g\|_{h^q(\mathbb{R}^n)}$$

holds for all $f, g \in \mathcal{S}(\mathbb{R}^n)$. In this remark, we shall give a comment on the constant $C$ of (1.2). In order to prove (1.2), we utilize several types of partitions of unity on $\mathbb{R}^n$, which are constructed independently of the assumptions stated in Theorem 1.1 and, of course, the ingredients $\sigma, f,$ and $g$ of the bilinear pseudo-differential operators. So, the constant $C$ depends not only on the space dimensions $n$ and the assumptions, $p, q, \rho, s,$ but also on the choice of partitions of unity.

Now, by virtue of Theorem 1.1, we have the following boundedness related to (1.1) for the bilinear Hörmander class with limited smoothness.

**Corollary 1.3** Let $1 \leq r \leq 2 \leq p, q \leq \infty$ satisfy $1/p + 1/q = 1/r$ and let $0 \leq \rho < 1$. Then, if $\sigma \in C([n/2]+1, [n/2]+1, [n/2]+1)(\mathbb{R}^n)^3$ satisfies that

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \bar{\sigma}(x, \xi, \eta)| \leq C_{\alpha, \beta, \gamma}(1 + |\xi| + |\eta|)^{-(1 - \rho)n/2 + \rho(|\alpha| + |\beta| + |\gamma|)}$$

for all $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ with $|\alpha|, |\beta|, |\gamma| \leq [n/2] + 1$, the bilinear pseudo-differential operator $T_\sigma$ is bounded from $h^p(\mathbb{R}^n) \times h^q(\mathbb{R}^n)$ to $h^r(\mathbb{R}^n)$, where $h^\infty$ can be replaced by the space bmo.
We end this section by explaining the plan of this paper. In Sect. 2, we introduce basic notations, function spaces and their properties which will be used throughout this paper. In Sect. 3, we display two key theorems, Theorems 3.2 and 3.3, and then, we prove Theorem 1.1 and Corollary 1.3 using them. In Sect. 4, we prepare several lemmas which will be used to prove Theorems 3.2 and 3.3. After we decompose symbols into easy forms to handle in Sect. 5, we prove Theorems 3.2 and 3.3 in Sects. 6 and 7, respectively. In Sect. 8, we consider the sharpness of indices stated in Theorem 1.1. In Appendix A, we consider the existence of the decomposition used in Sect. 5. In Appendix B, we give a small remark on the critical case $s_0 = n/2$ in Theorem 1.1.

2 Preliminaries

2.1 Basic Notations

We collect notations which will be used throughout this paper. We denote by $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{N}_0$ the sets of reals, natural numbers, integers and nonnegative integers, respectively. We denote by $Q$ the $n$-dimensional unit cube $[-1/2, 1/2]^n$. A disjoint union of translations $\tau + Q$, $\tau \in \mathbb{Z}^n$, generates the Euclidean space $\mathbb{R}^n$. This implies that the integral of a function on $\mathbb{R}^n$ can be written as

$$\int_{\mathbb{R}^n} f(x) \, dx = \sum_{\tau \in \mathbb{Z}^n} \int_Q f(x + \tau) \, dx. \quad (2.1)$$

For $x \in \mathbb{R}^n$ and $R > 0$, we denote by $Q(x, R)$ the closed cube $x + [-R, R]^n$ and by $B_R = B(0, R)$ the closed ball $\{x \in \mathbb{R}^n : |x| \leq R\}$. We write the characteristic function on the set $\Omega$ as $1_\Omega$. For $1 \leq p \leq \infty$, $p'$ is the conjugate number of $p$ defined by $1/p + 1/p' = 1$. We write $[s] = \max\{n \in \mathbb{Z} : n \leq s\}$ for $s \in \mathbb{R}$.

For two nonnegative functions $A(x)$ and $B(x)$ defined on a set $X$, we write $A(x) \lesssim B(x)$ for $x \in X$ to mean that there exists a positive constant $C$ such that $A(x) \leq CB(x)$ for all $x \in X$. We often omit to mention the set $X$ when it is obviously recognized. Also $A(x) \approx B(x)$ means that $A(x) \lesssim B(x)$ and $B(x) \lesssim A(x)$.

We denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^d$ by $S(\mathbb{R}^d)$ and its dual, the space of tempered distributions, by $S'(\mathbb{R}^d)$. The Fourier transform and the inverse Fourier transform of $f \in S(\mathbb{R}^d)$ are given by

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) \, dx,$$

$$\mathcal{F}^{-1} f(x) = \check{f}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi) \, d\xi,$$

respectively. We sometimes write $\mathcal{F}[f]$ and $\mathcal{F}^{-1}[f]$ when the form of $f$ is complicated. Furthermore, we sometimes deal with the partial Fourier transform of a Schwartz function $f(x, \xi, \eta)$, $x, \xi, \eta \in \mathbb{R}^n$. We denote the partial Fourier transform with respect to the $x, \xi,$ and $\eta$ variables by $\mathcal{F}_0, \mathcal{F}_1,$ and $\mathcal{F}_2$, respectively. We also write $\mathcal{F}_{1,2} = \mathcal{F}_1 \mathcal{F}_2$. For $m \in S'(\mathbb{R}^n)$, the Fourier multiplier operator is given by

\[\mathcal{F} M \mathcal{F}^{-1} f = \mathcal{F} M \hat{f} \quad \text{with} \quad M(\xi) = \sum_{\tau \in \mathbb{Z}^n} |\xi + \tau|^m \quad (2.2)\]
We also use the notation $(m(D) f)(x) = m(D_x) f(x)$ when we indicate which variable is considered.

For $m \in \mathbb{N}$, we denote by $C((\mathbb{R}^n)^m)$ the set of all bounded and uniformly continuous functions on $(\mathbb{R}^n)^m$. For $N_i \in \mathbb{N}_0$, $i = 1, \ldots, m$, we define

$$C(N_1, \ldots, N_m)((\mathbb{R}^n)^m) = \left\{ f : \partial_{x_1}^{\alpha_1} \ldots \partial_{x_m}^{\alpha_m} f(x_1, \ldots, x_m) \in C((\mathbb{R}^n)^m) \text{ for } |\alpha_i| \leq N_i, i = 1, \ldots, m \right\}.$$

For a measurable subset $E \subset \mathbb{R}^d$, the Lebesgue space $L^p(E)$, $1 \leq p \leq \infty$, is the set of all those measurable functions $f$ on $E$ such that $\| f \|_{L^p(E)} = (\int_E |f(x)|^p \, dx)^{1/p} < \infty$ if $1 \leq p < \infty$ or $\| f \|_{L^\infty(E)} = \text{ess sup}_{x \in E} |f(x)| < \infty$ if $p = \infty$. We also use the notation $\| f \|_{L^p(E)} = \| f(x) \|_{L^p(E)}$ when we indicate the variable explicitly. The uniformly local $L^2$ space, denoted by $L^2_{ul}$, is the set of all measurable functions $f$ on $\mathbb{R}^d$ such that

$$\| f \|_{L^2_{ul}(\mathbb{R}^d)} = \sup_{v \in \mathbb{Z}^d} \left( \int_{[-1/2,1/2]^d} |f(x + v)|^2 \, dx \right)^{1/2} < \infty$$

(this notion can be found in [15, Definition 2.3]). Also, it is known that the space $L^2_{ul}$ is identical with a certain Wiener amalgam space. See, e.g., [9,31] for the definition and properties of Wiener amalgam spaces.

Let $\mathbb{K}$ be a countable set. For $1 \leq q \leq \infty$, we denote by $\ell^q = \ell^q(\mathbb{K})$ the set of all complex number sequences $\{a_k\}_{k \in \mathbb{K}}$ such that $\| a_k \|_{\ell^q} = (\sum_{k \in \mathbb{K}} |a_k|^q)^{1/q} < \infty$ if $1 \leq q < \infty$ or $\| a_k \|_{\ell^\infty} = \sup_{k \in \mathbb{K}} |a_k| < \infty$ if $q = \infty$. For the sake of simplicity, we will write $\| a_k \|_{\ell^q}$ instead of the more correct notation $\| a_k \|_{\ell^q(\mathbb{K})}$. Moreover, we use the notation $\| a_k \|_{\ell^q} = \| a_k \|_{\ell^q(\mathbb{K})}$ when we indicate the variable.

Let $X, Y, Z$ be function spaces. We denote the mixed norm by

$$\| f(x, y, z) \|_{X,Y,Z} = \| f(x, y, z) \|_{X,Y,Z}.$$  

(Here pay special attention to the order of taking norms.) We shall use these mixed norms for $X, Y, Z$ being $L^p$ or $\ell^p$.

We end this subsection by stating the Schur lemma. See, e.g., [11, Appendix A].

**Lemma 2.1** Let $\{ A_{j,k} \}_{j,k \in \mathbb{Z}}$ be a sequence of nonnegative numbers satisfying that $\| A_{j,k} \|_{\ell_1^1(\mathbb{Z})\ell^\infty(\mathbb{Z})} \leq 1$ and $\| A_{j,k} \|_{\ell_1^1(\mathbb{Z})\ell^\infty(\mathbb{Z})} \leq 1$. Then, we have

$$\sum_{j,k \in \mathbb{Z}} A_{j,k} b_j c_k \leq \| b_j \|_{\ell^2(\mathbb{Z})} \| c_k \|_{\ell^2(\mathbb{Z})}$$

for all sequences of nonnegative numbers $\{ b_j \}_{j \in \mathbb{Z}}$ and $\{ c_k \}_{k \in \mathbb{Z}}$.  

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2.2 Besov Spaces

We recall the definition of Besov spaces.

Let \( \phi \in S(\mathbb{R}^d) \) satisfy that \( \phi = 1 \) on \( \{ \xi \in \mathbb{R}^d : |\xi| \leq 1 \} \) and \( \text{supp} \phi \subset \{ \xi \in \mathbb{R}^d : |\xi| \leq 2 \} \). We set \( \phi_k = \phi(\cdot/2^k) \), \( k \in \mathbb{N}_0 \). We write \( \psi = \phi - \phi(2\cdot) \) and set \( \psi_0 = \phi \) and \( \psi_k = \psi(\cdot/2^k) \), \( k \in \mathbb{N} \). Then, \( \text{supp} \psi \subset \{ \xi \in \mathbb{R}^d : 1/2 \leq |\xi| \leq 2 \} \) and \( \sum_{k \in \mathbb{N}_0} \psi_k = 1 \). We denote the Fourier multiplier operators \( \psi_k(D) \) by \( \Delta_k \), \( k \in \mathbb{N}_0 \). We call \( \{ \psi_k \}_{k \in \mathbb{N}_0} \) a Littlewood–Paley partition of unity. For \( 1 \leq p, q \leq \infty \) and \( s \in \mathbb{R} \), the Besov space \( B^s_{p,q}(\mathbb{R}^d) \) consists of all \( f \in S'(\mathbb{R}^d) \) such that

\[
\| f \|_{B^s_{p,q}(\mathbb{R}^d)} = \left\| 2^{ks} \| \Delta_k f \|_{L^p(\mathbb{R}^d)} \right\|_{L^q(\mathbb{N}_0)} < \infty.
\]

We will sometimes write \( \Delta_k[f] \) when the form of \( f \) is complicated. It is known that Besov spaces are independent of the choice of the Littlewood–Paley partition of unity. See [32] for more properties of Besov spaces.

2.3 Local Hardy Spaces \( h^p \) and Spaces \( bmo \) and \( BMO \)

We recall the definition of the local Hardy spaces \( h^p(\mathbb{R}^n) \), \( 0 < p \leq \infty \), and the spaces \( bmo(\mathbb{R}^n) \) and \( BMO(\mathbb{R}^n) \).

Let \( \phi \in S(\mathbb{R}^n) \) be such that \( \int_{\mathbb{R}^n} \phi(x) \, dx \neq 0 \). Then, the local Hardy space \( h^p(\mathbb{R}^n) \) consists of all \( f \in S'(\mathbb{R}^n) \) such that \( \| f \|_{h^p} = \| \sup_{0 < r < 1} |\phi_r \ast f| \|_{L^p} < \infty \), where \( \phi_r(x) = t^{-n} \phi(x/t) \). It is known that \( h^p(\mathbb{R}^n) \) does not depend on the choice of the function \( \phi \), and that \( h^p(\mathbb{R}^n) = L^p(\mathbb{R}^n) \) for \( 1 < p \leq \infty \) and especially \( h^1(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n) \).

The space \( bmo(\mathbb{R}^n) \) consists of all locally integrable functions \( f \) on \( \mathbb{R}^n \) such that

\[
\| f \|_{bmo} = \sup_{|Q| \leq 1} \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx + \sup_{|Q| \geq 1} \frac{1}{|Q|} \int_Q |f(x)| \, dx < \infty,
\]

where \( f_Q = |Q|^{-1} \int_Q f \), and \( Q \) ranges over the cubes in \( \mathbb{R}^n \).

The space \( BMO(\mathbb{R}^n) \) consists of all locally integrable functions \( f \) on \( \mathbb{R}^n \) such that

\[
\| f \|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx < \infty,
\]

where the supremum is taken over all cubes in \( \mathbb{R}^n \).

It is known that the dual space of \( h^1(\mathbb{R}^n) \) is \( bmo(\mathbb{R}^n) \) and that the embeddings \( L^\infty(\mathbb{R}^n) \hookrightarrow bmo(\mathbb{R}^n) \hookrightarrow BMO(\mathbb{R}^n) \) hold. See Goldberg [10] for more properties.

We end this subsection by stating the following lemma. This was mentioned in the memo by Miyachi–Tomita. Hence, although this lemma is not the author’s contribution, let the author give a proof below for the reader’s convenience.

**Lemma 2.2** Let \( a \geq 0 \). Suppose that \( \varphi \in S(\mathbb{R}^n) \) and \( \psi \in S(\mathbb{R}^n) \) satisfies \( \psi(0) = 0 \). Then, the following hold for any \( f \in S(\mathbb{R}^n) \).
\[(1) \| \varphi(D/2^a) f \|_{L^\infty(\mathbb{R}^n)} \lesssim (1 + a) \| f \|_{bmo(\mathbb{R}^n)}.
\]

\[(2) \| \psi(D/2^a) f \|_{L^\infty(\mathbb{R}^n)} \lesssim \| f \|_{BMO(\mathbb{R}^n)}.
\]

Here, the implicit constants above are independent of \( a \geq 0 \).

**Proof** We first consider the assertion (1). We have

\[|\varphi(D/2^a) f(x)| = 2^{an} \left| \int_{\mathbb{R}^n} \tilde{\varphi}(2^a(x - y)) f(y) \, dy \right| \lesssim 2^{an} \int_{\mathbb{R}^n} (1 + 2^a|x - y|)^{-N} |f(y) - f_Q(x, 2^{-a})| \, dy + |f_Q(x, 2^{-a})|,\]

where \( N \) is a constant such that \( N > n \) and \( f_{\Omega} = |\Omega|^{-1} \int_{\Omega} f(y) \, dy \) for the set \( \Omega \subset \mathbb{R}^n \). The first term is estimated by a constant times \( \| f \|_{BMO} \) (see [11, Proposition 3.1.5 (ii)]). For the second term,

\[|f_Q(x, 2^{-a})| \leq \sum_{0 \leq \ell < [a]} |f_Q(x, 2^{-a+\ell}) - f_Q(x, 2^{-a+\ell+1})| + |f_Q(x, 2^{-a+([a]+1)})|,\]

Here, we have \( |f_Q(x, R) - f_Q(x, 2R)| \leq 2^n \| f \|_{BMO} \) for \( R > 0 \) (see [11, Proposition 3.1.5 (i)]) and \( |f_Q(x, 2^{-a+[a]+1})| \leq \| f \|_{bmo} \), since \(-a + ([a] + 1) > 0\). Hence, we obtain

\[|f_Q(x, 2^{-a})| \lesssim (1 + [a]) \| f \|_{BMO} + \| f \|_{bmo} \lesssim (1 + a) \| f \|_{bmo},\]

which completes the proof.

For (2), since the assumption \( \psi(0) = 0 \) means that \( \int_{\mathbb{R}^n} \psi(x) \, dx = 0 \), we have

\[\psi(D/2^a) f(x) = 2^{an} \int_{\mathbb{R}^n} \tilde{\psi}(2^a(x - y)) (f(y) - f_Q(x, 2^{-a})) \, dy.\]

Since this is estimated by a constant times \( \| f \|_{BMO} \), we complete the proof. \( \square \)

## 3 Main Theorems

### 3.1 Key Theorems

In this subsection, we display two boundedness results which immediately derive Theorem 1.1. These will be proved later in the next sections, Sects. 5, 6, and 7, by using lemmas which will be stated in Sect. 4.

To state the results, we give the definition of the Besov type symbol class.

**Definition 3.1** Let \( 0 \leq \rho < 1 \). Let \( \{ \psi_k \}_{k \in \mathbb{N}_0} \) and \( \{ \Psi_j \}_{j \in \mathbb{N}_0} \) be Littlewood–Paley partition of unity on \( \mathbb{R}^a \) and \( (\mathbb{R}^a)^2 \), respectively. For \( j \in \mathbb{N}_0, k = (k_0, k_1, k_2) \in \mathbb{N}_0^3 \),
For the critical case

3.2 Proofs of Theorem 1.1 and Corollary 1.3

Theorem 1.1 follows from the interpolation among the boundedness given in Theorems 3.2 and 3.3 and the following relation of the symbol classes: $BS_{\rho, \rho}^{m, *}(s; \mathbb{R}^n) \subset BS_{\rho, \rho}^{m, *}(s_0, s_1, s_2; \mathbb{R}^n)$ for any $\varepsilon > 0$. Corollary 1.3 is obtained by using the following lemma and Theorem 1.1. The idea contained in the argument goes back to [16, Proposition 4.7].

Lemma 3.5 Let $0 \leq \rho < 1$, $m \in \mathbb{R}$, and $s = (s_0, s_1, s_2) \in [0, \infty)^3$. Suppose that $\sigma \in C([s_0]+1, [s_1]+1, [s_2]+1)((\mathbb{R}^n)^3)$ satisfies that

$$|\partial^\alpha_x \partial^\beta_{\xi} \partial^\gamma_\eta \sigma(x, \xi, \eta)| \leq C_{\alpha, \beta, \gamma} (1 + |\xi| + |\eta|)^{m+\rho(|\alpha|+|\beta|+|\gamma|)}$$

for all $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ with $|\alpha| \leq [s_0]+1$, $|\beta| \leq [s_1]+1$, and $|\gamma| \leq [s_2]+1$. Then, $\sigma \in BS_{\rho, \rho}^{m, *}(s; \mathbb{R}^n)$. 

Note that $BS_{\rho, \rho}^{m}(s; \mathbb{R}^n) \subset BS_{\rho, \rho}^{m, *}(s; \mathbb{R}^n) \subset BS_{\rho, \rho}^{m}(s; \mathbb{R}^n)$ for $s \in [0, \infty)^3$. See Lemma 3.5 below for the first inclusion relation. Now, we have the following theorems.

Theorem 3.2 Let $0 \leq \rho < 1$, $m = -(1-\rho)n/2$, and $s = (s_0, s_1, s_2) \in [0, \infty)^3$ satisfy $s_0 > n/2$ and $s_1, s_2 \geq n/2$. Then, if $\sigma \in BS_{\rho, \rho}^{m, *}(s; \mathbb{R}^n)$, the bilinear pseudo-differential operator $T_\sigma$ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $h^1(\mathbb{R}^n)$.

Theorem 3.3 Let $0 \leq \rho < 1$, $m = -(1-\rho)n/2$, and $s = (s_0, s_1, s_2) \in [0, \infty)^3$ satisfy $s_0, s_1, s_2 \geq n/2$. Then, if $\sigma \in BS_{\rho, \rho}^{m, *}(s; \mathbb{R}^n)$, the bilinear pseudo-differential operator $T_\sigma$ is bounded from $L^2(\mathbb{R}^n) \times bmo(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and from $bmo(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

Remark 3.4 For the critical case $s_0 = n/2$ in Theorem 3.2, we can prove that $\text{Op}(BS_{\rho, \rho}^{m, *}(s; \mathbb{R}^n)) \subset B(L^2 \times L^2 \to L^1)$ for $m = -(1-\rho)n/2$ and $s_0, s_1, s_2 \geq n/2$. (See Appendix B below for the proof of this boundedness.) In order to improve the target space $L^1$ to $h^1$, we will use the small loss with respect to $s_0$.

3.2 Proofs of Theorem 1.1 and Corollary 1.3


Proof We let \( N_i = [s_i] + 1, i = 0, 1, 2 \). We first consider \( \Delta_k[\sigma_j^\rho] \) for \( j, k_0, k_1, k_2 \geq 1 \). By using the Taylor expansion with respect to the \( \eta \) variable, the \( \xi \) variable, and the \( x \) variable (in this order), together with the moment condition \( \partial^\alpha \psi(0) = \int x^\alpha \psi = 0 \), we have

\[
\Delta_k[\sigma_j^\rho](x, \xi, \eta) = 2^{(k_0 + k_1 + k_2)n} \sum_{|\alpha| = N_0} \frac{1}{\alpha!} \sum_{|\beta| = N_1} \frac{1}{\beta!} \sum_{|\gamma| = N_2} \frac{1}{\gamma!} 
\times \int_{[0,1]^3} \int_{\mathbb{R}^n} \psi(2^{k_0}x' \alpha) \psi(2^{k_1}\xi' \beta) \psi(2^{k_2}n' \gamma) 
\times \left( \prod_{i=0,1,2} N_i (1 - t_i)^{N_i - 1} \right) \frac{\partial^\alpha}{\partial_x^\alpha} \frac{\partial^\beta}{\partial_\xi^\beta} \frac{\partial^\gamma}{\partial_\eta^\gamma} (\sigma_j^\rho)(x - t_0 x', \xi - t_1 \xi', \eta - t_2 \eta') \, dT \, dX',
\]

where \( dT = dt_0 dt_1 dt_2 \) and \( dX' = dx' d\xi' d\eta' \). Here, we observe that

\[
\left| \frac{\partial^\alpha}{\partial_x^\alpha} \frac{\partial^\beta}{\partial_\xi^\beta} \frac{\partial^\gamma}{\partial_\eta^\gamma} (\sigma_j^\rho)(x, \xi, \eta) \right| \lesssim 2^{jm + j\rho(|\alpha| - |\beta| - |\gamma|)}
\]

on the support of \( \Psi_j(2^{j\rho}, 2^{j\rho} \eta) \). Then, we have

\[
\left| \frac{\partial^\alpha}{\partial_x^\alpha} \frac{\partial^\beta}{\partial_\xi^\beta} \frac{\partial^\gamma}{\partial_\eta^\gamma} (\sigma_j^\rho)(x, \xi, \eta) \right| \lesssim \sum_{\beta' \leq \beta, \gamma' \leq \gamma} 2^{-\rho(|\alpha| - |\beta'| - |\gamma'|)} 2^{jm + j\rho(|\alpha| - |\beta'| - |\gamma'|)} \approx 2^{jm}.
\]

Collecting (3.1) and (3.2), we have

\[
\left| \Delta_k[\sigma_j^\rho](x, \xi, \eta) \right| \lesssim 2^{jm} 2^{-k_0 N_0 - k_1 N_1 - k_2 N_2},
\]

and hence

\[
\| \Delta_k[\sigma_j^\rho] \|_{L^2_{ul}} \lesssim 2^{jm} 2^{-k_0 N_0 - k_1 N_1 - k_2 N_2}
\]

for \( j, k_0, k_1, k_2 \geq 1 \). For the case \( j \geq 1 \) and at least one of \( k_0, k_1, k_2 \) is zero, by avoiding the usage of the Taylor expansion for the corresponding variables, we obtain the same conclusion as above. Also, the case \( j = 0 \) is similarly obtained. Therefore, the estimate in (3.3) holds for \( j, k_0, k_1, k_2 \in \mathbb{N}_0 \). This means \( \| \sigma \|_{B^{m,\rho,s}_{p,\nu}(s;\mathbb{R}^n)} \lesssim 1 \). \( \square \)
4 Lemmas

4.1 Elementary Lemmas

In this subsection, we denote by $S$ the operator

$$S(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{(1 + |x - y|)^{n+1}} dy. \quad (4.1)$$

Note that $S$ is bounded on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. We display basic properties of the operator $S$. The proof can be found in [17, Lemmas 4.1 and 4.3].

**Lemma 4.1** Let $1 \leq p \leq \infty$. Then, the following assertions (1)-(3) hold for all nonnegative functions $f, g$ on $\mathbb{R}^n$.

1. $S(f \ast g)(x) = (S(f) \ast g)(x) = (f \ast S(g))(x)$.
2. $S(f)(x) \approx S(f)(y)$ for $x, y \in \mathbb{R}^n$ such that $|x - y| \lesssim 1$.
3. $\|S(f)(x)\|_{L^p(\mathbb{R}^n)} \approx \|S(f)(v)\|_{e_p(\mathbb{Z}^n)}$.
4. Let $\varphi$ be a function in $\mathcal{S}(\mathbb{R}^n)$ with compact support. Then, $|\varphi(D - v)f(x)|^2 \lesssim S(|\varphi(D - v)f|^2)(x)$ for any $f \in \mathcal{S}(\mathbb{R}^n)$, $v \in \mathbb{Z}^n$, and $x \in \mathbb{R}^n$.

In the following lemma, the first assertion for $R = 1$ was proved by Miyachi–Tomita [24, Lemma 3.2]. Moreover, the second assertion for $R = 1$ was proved in the unpublished memo by Miyachi–Tomita. We generalize them to the cases $R \geq 1$.

**Lemma 4.2** Let $2 \leq p \leq \infty$, $R \geq 1$, and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then, the following hold for any $f \in \mathcal{S}(\mathbb{R}^n)$.

1. $\left\| \left( \sum_{v \in \mathbb{Z}^n} \varphi \left( \frac{D - v}{R} \right) f \right)^2 \right\|_{L^p(\mathbb{R}^n)}^{1/2} \lesssim R^{n/2} \|f\|_{L^p(\mathbb{R}^n)}$.
2. $\left\| \left( \sum_{v \in \mathbb{Z}^n, \varphi(-v/R) = 0} \varphi \left( \frac{D - v}{R} \right) f \right)^2 \right\|_{L^\infty(\mathbb{R}^n)}^{1/2} \lesssim R^{n/2} \|f\|_{BMO(\mathbb{R}^n)}$.

Here, the implicit constants above are independent of $R \geq 1$.

**Proof** We consider the assertion (1). Since $\mathbb{R}^n = \bigcup_{v' \in \mathbb{Z}^n} 2\pi v' + [-\pi, \pi]^n$, we have

$$\varphi \left( \frac{D - v}{R} \right) f(x) = R^n \int_{\mathbb{R}^n} e^{i(x-y) \cdot v} \tilde{\varphi}(R(x-y)) f(y) dy$$

$$= R^n e^{ix \cdot v} \sum_{v' \in \mathbb{Z}^n} \int_{2\pi v' + [-\pi, \pi]^n} e^{-iy \cdot v} \tilde{\varphi}(R(x-y)) f(y) dy$$

$$= R^n e^{ix \cdot v} \int_{[-\pi, \pi]^n} e^{-iy \cdot v} \left\{ \sum_{v' \in \mathbb{Z}^n} \tilde{\varphi}(R(x - y - 2\pi v')) f(y + 2\pi v') \right\} dy.$$
Here, we realize that the function \( \sum_{\nu' \in \mathbb{Z}^n} \hat{\varphi}(R(x - y - 2\pi \nu')) f(y + 2\pi \nu') \) is \( 2\pi \mathbb{Z}^n \)-periodic with respect to the \( y \)-variable. Hence, we have by the Parseval identity

\[
\left\| \varphi \left( \frac{D - v}{R} \right) f(x) \right\|_{\ell_2^l}^2 = (2\pi)^n R^{2n} \int_{[-\pi,\pi]^n} \left| \sum_{\nu' \in \mathbb{Z}^n} \hat{\varphi}(R(x - y - 2\pi \nu')) f(y + 2\pi \nu') \right|^2 \, dy.
\]

By applying the Cauchy–Schwarz inequality to the sum over \( \nu' \), we have

\[
\left\| \varphi \left( \frac{D - v}{R} \right) f(x) \right\|_{\ell_2^l}^2 \lesssim R^{2n} \int_{[-\pi,\pi]^n} \sum_{\nu' \in \mathbb{Z}^n} \left| \hat{\varphi}(R(x - y - 2\pi \nu')) \right| \left| f(y + 2\pi \nu') \right|^2 \, dy = R^{2n} \left( \left| \hat{\varphi}(R \cdot) \right| * \left| f \right|^2 \right)(x),
\]

where we used that \( \sum_{\nu' \in \mathbb{Z}^n} \left| \hat{\varphi}(R(z - 2\pi \nu')) \right| \lesssim 1 \) for \( z \in \mathbb{R}^n \) and \( R \geq 1 \). Taking the \( L^{p/2} \) norm of the above, since \( 2 \leq p \leq \infty \), we have by the Young inequality

\[
\left\| \left\| \varphi \left( \frac{D - v}{R} \right) f(x) \right\|_{\ell_2^l} \right\|_{L^p_x}^2 \lesssim R^n \left\| f \right\|_{L^p}^2,
\]

which completes the proof of the assertion (1).

For the assertion (2), we have for \( \nu \in \mathbb{Z}^n \) such that \( \varphi(-\nu/R) = 0 \)

\[
\varphi \left( \frac{D - v}{R} \right) f(x) = R^n e^{ix \cdot v} \int_{\mathbb{R}^n} e^{-iy \cdot v} \hat{\varphi}(R(x - y)) \left( f(y) - c \right) \, dy
\]

for any constant \( c \in \mathbb{C} \). Repeating the same lines as above with \( f(y) - c \), we obtain

\[
\sum_{\nu \in \mathbb{Z}^n, \varphi(-\nu/R) = 0} \left\| \varphi \left( \frac{D - v}{R} \right) f(x) \right\|_{\ell_2^l}^2 \lesssim R^{2n} \left( \left| \hat{\varphi}(R \cdot) \right| * \left| f - c \right|^2 \right)(x).
\]

Choose \( c = f_{Q(x,R^{-1})} = |Q(x,R^{-1})|^{-1} \int_{Q(x,R^{-1})} f(y) \, dy \) and observe that

\[
R^n \int_{\mathbb{R}^n} \left| \hat{\varphi}(R(x - y)) \right| \left| f(y) - f_{Q(x,R^{-1})} \right|^2 \, dy \lesssim \left\| f \right\|_{BMO}^2
\]
(with the aid of [11, Proposition 3.1.5 (i) and Corollary 3.1.8]). Then, we have

\[
\sum_{\nu \in \mathbb{Z}^n: \nu(-\nu/R)=0} \left| \varphi \left( \frac{D-v}{R} \right) f(x) \right|^2 \lesssim R^n \| f \|_{BMO}^2
\]

which completes the proof of the assertion (2). \(\square\)

**Corollary 4.3** Let \( r > 0, R \geq 1, \) and \( \varphi, \phi \in \mathcal{S}(\mathbb{R}^n). \) Then,

\[
\left\| \left( \sum_{\nu \in \mathbb{Z}^n: \nu(-\nu/R)=0} \left| \varphi \left( \frac{D-v}{R} \right) \phi \left( \frac{D}{r} \right) f \right|^2 \right)^{1/2} \right\|_{L^\infty(\mathbb{R}^n)} \lesssim R^{n/2} \| f \|_{BMO(\mathbb{R}^n)},
\]

where the implicit constant above is independent of \( r > 0 \) and \( R \geq 1. \)

**Proof** The summand can be written by

\[
\varphi \left( \frac{D-v}{R} \right) \phi \left( \frac{D}{r} \right) f(x) = r^n \int_{\mathbb{R}^n} \varphi \left( \frac{D-v}{R} \right) f(x-y) \hat{\phi}(r y) \, dy.
\]

We take the \( \ell^2 \) norm restricted to the set \( \{ \nu \in \mathbb{Z}^n : \nu(-\nu/R)=0 \} \) of the above and use the Minkowski inequality for the integral. Then, by Lemma 4.2 (2), we have the desired results, since \( r^n \| \hat{\phi}(r \cdot) \|_{L^1} \approx 1 \) for \( r > 0. \) \(\square\)

### 4.2 Lemmas for Theorems 3.2 and 3.3

In this subsection, we show some lemmas for the dual form of bilinear pseudo-differential operators. The basic idea for the argument used here goes back to Boulkhemair [5]. See also [17, Proposition 5.1].

Throughout this subsection, the notation \( \| \sigma_\nu(x, \xi, \eta) \|_{L^r_{\xi} L^q_{\eta} \ell^2_{ul, x} \ell^\infty} \) will be abbreviated to \( \| \sigma_\nu \|, \) where \( L^2_{ul, x} \) is the \( L^2 \) space with respect to the \( x \) variable.

**Lemma 4.4** Let \( R_0, R_1, R_2 \geq 1 \) and \( 2 \leq p, q, r \leq \infty \) satisfy \( 1/p + 1/q + 1/r = 1. \) Let \( \Lambda, \Lambda_1, \Lambda_2 \) be subsets of \( \mathbb{Z}^n. \) Suppose that \( \kappa \in \mathcal{S}(\mathbb{R}^n) \) satisfies that \( \supp \kappa \subset [-1, 1]^n \) and that \( \{ \sigma_\nu \}, \nu = (v_1, v_2) \in (\mathbb{Z}^n)^2, \) is a sequence of bounded functions on \( \mathbb{R}^n \) satisfying that \( \| \sigma_\nu \| < \infty \) and \( \supp \mathcal{F}[\sigma_\nu] \subset B_{R_0} \times B_{R_1} \times B_{R_2} \) for \( \nu \in (\mathbb{Z}^n)^2. \) Then, the following assertions (1) and (2) hold for any \( f, g, h \in \mathcal{S}(\mathbb{R}^n), \) and the following assertion (3) holds for any \( f, g \in \mathcal{S}(\mathbb{R}^n) \) and \( \{ h_\tau \}_{\tau \in \mathbb{Z}^n} \subset \mathcal{S}(\mathbb{R}^n). \)

1. \( \sum_{\nu \in \Lambda_1 \times \Lambda_2} \left| \int_{\mathbb{R}^n} T_{\sigma_\nu}(\Box_{v_1} f, \Box_{v_2} g)(x) \, h(x) \, dx \right| \lesssim \min_{i=1, 2} \left\{ |\Lambda_i|^{1/2} \right\} (R_0 R_1 R_2)^{n/2} \| \sigma_\nu \| \| f \|_{L^p(\mathbb{R}^n)} \| \Box_{v_2} g \|_{\ell^2_{v_2}(\Lambda_2) L^q(\mathbb{R}^n)} \| h \|_{L^r(\mathbb{R}^n)}. \)
\[
\sum_{\nu \in \Lambda_1 \times \Lambda_2} \int_{\mathbb{R}^n} T_{\nu} (\Box_{v_1} f, \Box_{v_2} g)(x) \, h(x) \, dx
\]
\[\lesssim (|\Lambda_1| |\Lambda_2|)^{1/2} (R_1 R_2)^{n/2} \|\sigma_\nu\| \| f \|_{L^p(\mathbb{R}^n)} \| \Box_{v_2} g\|_{L^q_t(\mathbb{R}^n)} \| h\|_{L^r(\mathbb{R}^n)}.\]

(3)
\[
\sum_{\nu \in \Lambda} \int_{\mathbb{R}^n} T_{\nu} (\Box_{v_1} f, \Box_{v_2} g)(x) \, h_\tau(x) \, dx
\]
\[\lesssim |\Lambda|^{1/2} (R_1 R_2)^{n/2} \|\sigma_\nu\| \| f \|_{L^p(\mathbb{R}^n)} \| g\|_{L^q_t(\mathbb{R}^n)} \| h_\tau\|_{L^r_t(\mathbb{R}^n)}.
\]

Here, $\Box_{v_i}$ is the Fourier multiplier operator $\kappa(D-v_i)$, $i = 1, 2$, and the absolute values of $\Lambda$, $\Lambda_1$, and $\Lambda_2$ are the cardinality of these sets. All the implicit constants above are independent of $R_0$, $R_1$, $R_2$. In particular, $\| \Box_{v_2} g\|_{L^q_t(\mathbb{R}^n)}$ in the estimates of (1) and (2) can be replaced by $\|g\|_{L^q(\mathbb{R}^n)}$.

To prove this, we use the following lemma.

Lemma 4.5 Let $R_1, R_2 \geq 1$. Suppose that $\kappa \in \mathcal{S}(\mathbb{R}^n)$ satisfies that $\text{supp } \kappa \subset [-1, 1]^n$ and that $\{\sigma_\nu\}, \nu = (v_1, v_2) \in (\mathbb{Z}^n)^2$, is a sequence of bounded functions on $(\mathbb{R}^n)^3$ satisfying that $\|\sigma_\nu\| < \infty$ and $\text{supp } \mathcal{F}_{1,2}[\sigma_\nu](x, \cdot, \cdot) \subset B_{R_1} \times B_{R_2}$ for $\nu \in (\mathbb{Z}^n)^2$ and $x \in \mathbb{R}^n$. Then,

\[
\left| \int_{\mathbb{R}^n} T_{\nu} (\Box_{v_1} f, \Box_{v_2} g)(x) \, h_{v_1+v_2}(x) \, dx \right|
\]
\[\lesssim \|\sigma_\nu\| \sum_{v_0 \in \mathbb{Z}^n} \left\{ S \left( 1_{B_{R_1}} \ast |\Box_{v_1} f|^2 \right)(v_0) \right\}^{1/2} \left\{ S \left( 1_{B_{R_2}} \ast |\Box_{v_2} g|^2 \right)(v_0) \right\}^{1/2} \| h_{v_1+v_2}(x + v_0)\|_{L^2_t(Q)}
\]

(4.2)

for any $\nu \in (\mathbb{Z}^n)^2, x \in \mathbb{R}^n$, $f, g \in \mathcal{S}(\mathbb{R}^n)$, and $\{h_\tau\}_{\tau \in \mathbb{R}^n} \subset \mathcal{S}(\mathbb{R}^n)$. Here, the operator $S$ is as in (4.1).

Proof We simply denote by $I$ the left hand side of (4.2). Observe that

\[
(2\pi)^{2n} \, T_{\nu} (\Box_{v_1} f, \Box_{v_2} g)(x)
\]
\[= \int_{(\mathbb{R}^n)^2} \mathcal{F}_{1,2}[\sigma_\nu](x, y - x, z - x) \, 1_{B_{R_1}}(x - y) \, \Box_{v_1} f(y) \, 1_{B_{R_2}}(x - z) \, \Box_{v_2} g(z) \, dydz.
\]
Then, the Cauchy–Schwarz inequalities and the Plancherel theorem give

\[
|T_{\sigma_{\nu}}(\square_{v_1}f, \square_{v_2}g)(x)|^2 \\
\lesssim \|\sigma_{\nu}(x, \xi, \eta)\|_{L_{\xi}^{\infty}L_{\eta}^{2}}^2 \left(1_{B_{R_1}} \ast |\square_{v_1}f|^2\right)(x) \left(1_{B_{R_2}} \ast |\square_{v_2}g|^2\right)(x)
\]

for any \(\nu \in (\mathbb{Z}^n)^2\) and \(x \in \mathbb{R}^n\). From this, it holds that

\[
I \lesssim \int_{\mathbb{R}^n} \left\|\sigma_{\nu}(x, \xi, \eta)\right\|_{L_{\xi}^{2}L_{\eta}^{2}} |h_{v_1+v_2}(x)| \\
\times \left\{\left(1_{B_{R_1}} \ast |\square_{v_1}f|^2\right)(x)\right\}^{1/2} \left\{\left(1_{B_{R_2}} \ast |\square_{v_2}g|^2\right)(x)\right\}^{1/2} dx.
\]

(4.3)

We separate the integral by using (2.1). Then, the inequality (4.3) coincides with

\[
I \lesssim \sum_{\nu_0 \in \mathbb{Z}^n} \int_{Q} \left\|\sigma_{\nu}(x + \nu_0, \xi, \eta)\right\|_{L_{\xi}^{2}L_{\eta}^{2}} |h_{v_1+v_2}(x + \nu_0)| \\
\times \left\{\left(1_{B_{R_1}} \ast |\square_{v_1}f|^2\right)(x + \nu_0)\right\}^{1/2} \left\{\left(1_{B_{R_2}} \ast |\square_{v_2}g|^2\right)(x + \nu_0)\right\}^{1/2} dx.
\]

Since we have by Lemma 4.1 (4), (1), and (2)

\[
\left(1_{B_{R_1}} \ast |\square_{v_1}f|^2\right)(x + \nu_0) \lesssim S\left(1_{B_{R_1}} \ast |\square_{v_1}f|^2\right)(\nu_0)
\]

for \(x \in Q\), we obtain

\[
I \lesssim \sum_{\nu_0 \in \mathbb{Z}^n} \left\{S\left(1_{B_{R_1}} \ast |\square_{v_1}f|^2\right)(\nu_0)\right\}^{1/2} \left\{S\left(1_{B_{R_2}} \ast |\square_{v_2}g|^2\right)(\nu_0)\right\}^{1/2} \\
\times \int_{Q} \left\|\sigma_{\nu}(x + \nu_0, \xi, \eta)\right\|_{L_{\xi}^{2}L_{\eta}^{2}} |h_{v_1+v_2}(x + \nu_0)| dx
\]

\[
\leq \sum_{\nu_0 \in \mathbb{Z}^n} \left\{S\left(1_{B_{R_1}} \ast |\square_{v_1}f|^2\right)(\nu_0)\right\}^{1/2} \left\{S\left(1_{B_{R_2}} \ast |\square_{v_2}g|^2\right)(\nu_0)\right\}^{1/2} \\
\times \left\|\sigma_{\nu}(x + \nu_0, \xi, \eta)\right\|_{L_{\xi}^{2}L_{\eta}^{2}(Q)} \left\|h_{v_1+v_2}(x + \nu_0)\right\|_{L_{\xi}^{2}(Q)}.
\]

Taking the supremum over \(\nu_0\) and \(\nu\) of the factor containing \(\sigma_{\nu}\), we complete the proof.

\[\square\]

Now, we shall prove Lemma 4.4.

**Proof of Lemma 4.4 (1)** We may assume that \(|\Lambda_1| < \infty\) or \(|\Lambda_2| < \infty\). We simply denote by \(I\) the left hand side of the assertion (1). We first observe that

\[
\mathcal{F}\left[T_{\sigma_{\nu}}(\square_{v_1}f, \square_{v_2}g)\right](\xi) \\
= \frac{1}{(2\pi)^{2n}} \int_{(\mathbb{R}^n)^2} \mathcal{F}[\sigma_{\nu}](\xi - (\xi + \eta), \xi, \eta) \kappa(\xi - v_1) \hat{f}(\xi) \kappa(\eta - v_2) \hat{g}(\eta) \, d\xi \, d\eta.
\]
Here, since supp $F_0[\sigma_v](\cdot, \xi, \eta) \subset B_{R_0}$ and supp $\kappa(\cdot - v_i) \subset v_i + [-1, 1]^n$, we see that
\[
\text{supp } \mathcal{F} \left[ T_{\sigma_v}(\square_{v_1} f, \square_{v_2} g) \right] \subset \{ \xi \in \mathbb{R}^n : |\xi - (v_1 + v_2)| \lesssim R_0 \}.
\]
We take a function $\varphi \in C(\mathbb{R}^n)$ satisfying that $\varphi = 1$ on $\{ \xi \in \mathbb{R}^n : |\xi| \lesssim 1 \}$. Then,
\[
I = \sum_{v \in \Lambda_1 \times \Lambda_2} \left| \int_{\mathbb{R}^n} T_{\sigma_v}(\square_{v_1} f, \square_{v_2} g)(x) \varphi \left( \frac{D + v_1 + v_2}{R_0} \right) h(x) \, dx \right|.
\]
By the use of Lemma 4.5,
\[
I \lesssim \| \sigma_v \| \sum_{v_0 \in \mathbb{Z}^n} \sum_{v \in \Lambda_1 \times \Lambda_2} \left\| \varphi \left( \frac{D + v_1 + v_2}{R_0} \right) h(x + v_0) \right\|_{L^p_2(Q)} \times \left\{ S\left( 1_{B_{R_1}} * |\square_{v_1} f|^2 \right)(v_0) \right\}^{1/2} \left\{ S\left( 1_{B_{R_2}} * |\square_{v_2} g|^2 \right)(v_0) \right\}^{1/2}. \tag{4.4}
\]
In what follows, we simply write each summand by
\[
F(v_1, v_0) = \left\{ S\left( 1_{B_{R_1}} * |\square_{v_1} f|^2 \right)(v_0) \right\}^{1/2},
\]
\[
G(v_2, v_0) = \left\{ S\left( 1_{B_{R_2}} * |\square_{v_2} g|^2 \right)(v_0) \right\}^{1/2},
\]
\[
H(v_3, v_0) = \left\| \varphi \left( \frac{D + v_3}{R_0} \right) h(x + v_0) \right\|_{L^p_2(Q)}.
\]
Then the inequality (4.4) is rewritten as
\[
I \lesssim \| \sigma_v \| \, II \tag{4.5}
\]
with
\[
II = \sum_{v_0 \in \mathbb{Z}^n} \sum_{v_1 \in \Lambda_1} \sum_{v_2 \in \Lambda_2} F(v_1, v_0) G(v_2, v_0) H(v_1 + v_2, v_0).
\]
Let us estimate $II$. We first consider the case $|\Lambda_1| \leq |\Lambda_2|$. We apply the Cauchy–Schwarz inequality firstly to the sum over $v_2$ and secondly to the sum over $v_1$, and thirdly apply the Hölder inequality with $1/p + 1/q + 1/r = 1$ to the sum over $v_0$. Then,
\[
II \leq \sum_{v_0 \in \mathbb{Z}^n} \sum_{v_1 \in \Lambda_1} F(v_1, v_0) \| G(v_2, v_0) \|_{L^2_{p_1}(\Lambda_2)} \| H(v_3, v_0) \|_{L^2_{p_3}(\mathbb{Z}^n)}
\]
\[
\leq |\Lambda_1|^{1/2} \sum_{v_0 \in \mathbb{Z}^n} \| F(v_1, v_0) \|_{L^2_{p_1}(\Lambda_1)} \| G(v_2, v_0) \|_{L^2_{p_2}(\Lambda_2)} \| H(v_3, v_0) \|_{L^2_{p_3}}
\]
\[
\leq |\Lambda_1|^{1/2} \| F(v_1, v_0) \|_{L^2_{p_1}} \| G(v_2, v_0) \|_{L^2_{p_2}} \| H(v_3, v_0) \|_{L^2_{p_3}}.
\]
For the opposite case $|\Lambda_2| \leq |\Lambda_1|$, we switch the order to use the Cauchy–Schwarz inequalities with respect to $\nu_1$ and $\nu_2$. Thus, we obtain

$$\| F(\nu_1, \nu_0) \|_{\ell_{\nu_1}^2 \ell_{\nu_0}^p} \cong \left\| S(1_{B_{R_1}} \ast \Box_{\nu_1} f \|_{\ell_{\nu_1}^2}) \right\|_{L^{p/2}}^{1/2}.$$  \hspace{1cm} (4.6)

We shall estimate the norms of $F$, $G$, and $H$. Firstly, for the norm of $F$, we have by Lemma 4.1 (3)

$$\| F(\nu_1, \nu_0) \|_{\ell_{\nu_1}^2 \ell_{\nu_0}^p} \cong \left\| S(1_{B_{R_1}} \ast \Box_{\nu_1} f \|_{\ell_{\nu_1}^2}) \right\|_{L^{p/2}}^{1/2}.$$  \hspace{1cm} (4.7)

Since the operator $S$, defined in (4.1), is bounded on $L^{p/2}(\mathbb{R}_n)$, $p \geq 2$, the right hand side of (4.7) is estimated by a constant times

$$\left\| S(1_{B_{R_1}} \ast \Box_{\nu_1} f \|_{\ell_{\nu_1}^2}) \right\|_{L^{p/2}} \lesssim \| f \|_{L^p}.$$  \hspace{1cm} (4.8)

Since $\Box_{\nu_1} = \kappa(D - \nu_1)$ with $\kappa \in \mathcal{S}(\mathbb{R}_n)$, we have $\| \Box_{\nu_1} f \|_{\ell_{\nu_1}^2} \lesssim \| f \|_{L^p}$ by Lemma 4.2 (1) with $R = 1$. Therefore, we obtain

$$\| F(\nu_1, \nu_0) \|_{\ell_{\nu_1}^2 \ell_{\nu_0}^p} \lesssim R_1^{n/2} \| f \|_{L^p}.$$  \hspace{1cm} (4.9)

For the norm of $G$, repeating the same line as for $F$, we have by (4.7) and (4.8)

$$\| G(\nu_2, \nu_0) \|_{\ell_{\nu_2}^2(\Lambda_2) \ell_{\nu_0}^q} \lesssim R_2^{n/2} \| \Box_{\nu_2} g \|_{\ell_{\nu_2}^2(\Lambda_2)}.$$  \hspace{1cm} (4.10)

Lastly, we consider the norm of $H$. Since $L^r(Q) \hookrightarrow L^2(Q)$ for $2 \leq r \leq \infty$, we have by Lemma 4.2(1)
\[ \| H(v_3, v_0) \|_{\ell^2_t \ell^r_v} = \left\| \varphi \left( \frac{D + v_3}{R_0} \right) h(x + v_0) \right\|_{\ell^2_t L^r_v(Q) \ell^r_v} \leq \left\| \varphi \left( \frac{D + v_3}{R_0} \right) h(x) \right\|_{\ell^2_t L^r_v(\mathbb{R}^d)} \lesssim R_0^{n/2} \| h \|_{L^r}. \quad (4.11) \]

Thus, collecting (4.5), (4.6), (4.9), (4.10), and (4.11), we obtain the desired estimate. Moreover, by virtue of Lemma 4.2 (1), \( \| \square v_2 g \|_{\ell^2_t (\Lambda_2) L^q} \) can be replaced by \( \| g \|_{L^q} \). \( \square \)

**Proof of Lemma 4.4 (2)** We may assume \( |\Lambda_1|, |\Lambda_2| < \infty \). We simply denote by \( I \) the left hand side of the inequality of the assertion (2). Since \( h \) is independent of \( v_1 + v_2 \), we have by Lemma 4.5

\[ I \lesssim \| \sigma_v \| \sum_{v_0 \in \mathbb{Z}^d} \| h(x + v_0) \|_{L^2(Q)} \]
\[ \times \sum_{v \in \Lambda_1 \times \Lambda_2} \left\{ S \left( 1_{B_{R_1} \ast |\square v_1 f|^2} \right)(v_0) \right\}^{1/2} \left\{ S \left( 1_{B_{R_2} \ast |\square v_2 g|^2} \right)(v_0) \right\}^{1/2}. \]

We use the Cauchy–Schwarz inequality for the sums over \( v_1 \) and \( v_2 \) and then use the Hölder inequality for the sum over \( v_0 \) with \( 1/p + 1/q + 1/r = 1 \). Then,

\[ I \lesssim (|\Lambda_1| |\Lambda_2|)^{1/2} \| \sigma_v \| \| h(x + v_0) \|_{L^2_t(Q) \ell^r_v} \]
\[ \times \left\{ S \left( 1_{B_{R_1} \ast |\square v_1 f|^2} \right)(v_0) \right\}^{1/2} \left\{ S \left( 1_{B_{R_2} \ast |\square v_2 g|^2} \right)(v_0) \right\}^{1/2} \]
\[ \| \sigma_v \| \sum_{v_0 \in \mathbb{Z}^d} \sum_{\tau \in \Lambda} \sum_{v_1 + v_2 = \tau} \| h_\tau(x + v_0) \|_{L^2_t(Q)} \]
\[ \times \left\{ S \left( 1_{B_{R_1} \ast |\square v_1 f|^2} \right)(v_0) \right\}^{1/2} \left\{ S \left( 1_{B_{R_2} \ast |\square v_2 g|^2} \right)(v_0) \right\}^{1/2}. \]

Here, from (4.9) and (4.10), the factors of \( f \) and \( g \) are bounded by a constant times \( R_1^{n/2} \| f \|_{L^p} \) and \( R_2^{n/2} \| \square v_2 g \|_{\ell^2_t (\Lambda_2) L^q} \), respectively. From the embedding \( L^r(Q) \hookrightarrow L^2(Q) \) for \( 2 \leq r < \infty \), \( \| h(x + v_0) \|_{L^2_t(Q) \ell^r_v} \leq \| h \|_{L^r} \). Hence, we obtain the desired estimate. Also, by Lemma 4.2 (1), \( \| \square v_2 g \|_{\ell^2_t (\Lambda_2) L^q} \) can be replaced by \( \| g \|_{L^q} \). \( \square \)

**Proof of Lemma 4.4 (3)** We may assume \( |\Lambda| < \infty \). We denote by \( I \) the left hand side of the assertion (3). By Lemma 4.5,

\[ I \lesssim \| \sigma_v \| \sum_{v_0 \in \mathbb{Z}^d} \sum_{\tau \in \Lambda} \sum_{v_1 + v_2 = \tau} \| h_\tau(x + v_0) \|_{L^2_t(Q)} \]
\[ \times \left\{ S \left( 1_{B_{R_1} \ast |\square v_1 f|^2} \right)(v_0) \right\}^{1/2} \left\{ S \left( 1_{B_{R_2} \ast |\square v_2 g|^2} \right)(v_0) \right\}^{1/2}. \]

Firstly, we use the Cauchy–Schwarz inequality for the sum over \( v_1 \), since \( v_2 = \tau - v_1 \). Secondly, we use the Cauchy–Schwarz inequality for the sum over \( \tau \) and thirdly use the Hölder inequality for the sum over \( v_0 \) with \( 1/p + 1/q + 1/r = 1 \). Then,
\[ I \lesssim |\Lambda|^{1/2} \| \sigma \| \| h_\tau (x + v_0) \|_{L^2(Q)} \ell^2_2(\Lambda) e_{v_0}^c \times \left\| \left\{ S \left( 1_{B_{R_1}} \ast \Box_{v_1} f \right) (v_0) \right\}^{1/2} \ell^p_{v_1} \ell^q_{v_0} \right\|^{1/2} \ell^2_2 \ell^q_{v_0} \].

By (4.9), the factors of \( f \) and \( g \) are bounded by a constant times \( R_1^{n/2} \| f \|_{L^p} \) and \( R_2^{n/2} \| g \|_{L^q} \), respectively. For the factor of \( h_\tau \), since \( L^r(Q) \hookrightarrow L^2(Q) \) for \( 2 \leq r \leq \infty \), \( \| h_\tau (x + v_0) \|_{L^2(Q)} \ell^2_2(\Lambda) e_{v_0}^c \leq \| h_\tau \|_{L^2(Q)} \ell^2_2(\Lambda) L^r(\mathbb{R}^n) \). Hence, we obtain the desired estimate.

\[ \square \]

5 Decomposition of Symbols

In this section, we decompose symbols of the bilinear operator \( T_\sigma \) by Littlewood–Paley partitions and the following lemma given by Sugimoto [29, Lemma 2.2.1]. An explicit proof can be found in [17, Lemma 4.4].

**Lemma 5.1** There exist functions \( \kappa \in S(\mathbb{R}^n) \) and \( \chi \in S(\mathbb{R}^n) \) satisfying that
\[ \text{supp } \kappa \subset [-1, 1]^n, \text{supp } \hat{\chi} \subset B_1, |\chi| \geq c > 0 \text{ on } [-1, 1]^n, \text{and} \]
\[ \sum_{\nu \in \mathbb{Z}^n} \kappa(\xi - \nu) \chi(\xi - \nu) = 1, \xi \in \mathbb{R}^n. \]

As a first step of this section, we decompose symbols by a Littlewood–Paley partition of unity \( \{ \Psi_j \}_{j \in \mathbb{N}_0} \) on \( \mathbb{R}^n \) as follows:
\[ \sigma(x, \xi, \eta) = \sum_{j \in \mathbb{N}_0} \sigma(x, \xi, \eta) \Psi_j(\xi, \eta) = \sum_{j \in \mathbb{N}_0} \sigma_j(x, \xi, \eta) \]
with
\[ \sigma_j(x, \xi, \eta) = \sigma(x, \xi, \eta) \Psi_j(\xi, \eta). \tag{5.1} \]

Here, we observe the following two identities for \( \Psi_j \). Firstly, we let a function \( \phi \in S(\mathbb{R}^n) \) satisfy that \( \phi = 1 \) on \( [\xi \in \mathbb{R}^n : |\xi| \leq 2] \) and supp \( \phi \subset [\xi \in \mathbb{R}^n : |\xi| \leq 4] \), and write \( \phi_j = \phi(\cdot/2^j) \). Then, we have for \( j \geq 0 \)
\[ \Psi_j(\xi, \eta) = \Psi_j(\xi, \eta) \phi_j(\xi) \phi_j(\eta). \]

Secondly, there exist functions \( \phi', \psi', \psi'' \in S(\mathbb{R}^n) \) satisfying that for \( j \geq 0 \)
\[ \Psi_j(\xi, \eta) = \Psi_j(\xi, \eta) \phi_j'(\xi) \psi_j'(\eta) + \Psi_j(\xi, \eta) \phi_j'(\xi) \psi_j''(\eta) + \Psi_j(\xi, \eta) \psi_j''(\xi) \psi_j''(\eta), \]
where \( \phi_j' = \phi'(\cdot/2^j), \psi_j' = \psi'(\cdot/2^j), \) and \( \psi_j'' = \psi''(\cdot/2^j), \) and that
\[ \text{supp } \phi' \subset \left\{ \xi \in \mathbb{R}^n : |\xi| \leq 2^{-5} \right\}, \quad (5.2) \]

\[ \text{supp } \psi' \subset \left\{ \xi \in \mathbb{R}^n : 2^{-4} \leq |\xi| \leq 2^2 \right\}, \quad (5.3) \]

\[ \text{supp } \psi'' \subset \left\{ \xi \in \mathbb{R}^n : 2^{-6} \leq |\xi| \leq 2^2 \right\}. \quad (5.4) \]

See Appendix A for the existence of such functions. (By this, the annulus of \( \text{supp } \Psi_j \) is decomposed into the three parts: \(|\xi| \ll |\eta|, |\xi| \gg |\eta|, |\xi| \approx |\eta|\).) Then, we have

\[
\sigma(x, \xi, \eta) = \sum_{j \leq 1} \sigma_j(x, \xi, \eta) \phi_j(\xi) \phi_j(\eta) + \sum_{j \gg 1} \sigma_j(x, \xi, \eta) \phi'_j(\xi) \psi'_j(\eta) 
+ \sum_{j \gg 1} \sigma_j(x, \xi, \eta) \psi'_j(\xi) \phi'_j(\eta) + \sum_{j \gg 1} \sigma_j(x, \xi, \eta) \psi''_j(\xi) \psi''_j(\eta),
\]

where the implicit constant of the sum over \( j \) depends only on \( \rho \) and dimensions.

As a second step, we rewrite the dual form of \( T_\sigma(f, g) \). By using the decomposition of symbols just above, the dual form of \( T_\sigma(f, g) \) can be written as

\[
\int_{\mathbb{R}^n} T_\sigma(f, g)(x) h(x) \, dx = \sum_{j \leq 1} \int_{\mathbb{R}^n} T_{\sigma_j}(\phi'_j(D)f, \phi'_j(D)g)(x) h(x) \, dx
+ \sum_{j \gg 1} \int_{\mathbb{R}^n} T_{\sigma_j}(\phi'_j(D)f, \psi'_j(D)g)(x) h(x) \, dx
+ \sum_{j \gg 1} \int_{\mathbb{R}^n} T_{\sigma_j}(\psi'_j(D)f, \phi'_j(D)g)(x) h(x) \, dx
+ \sum_{j \gg 1} \int_{\mathbb{R}^n} T_{\sigma_j}(\psi''_j(D)f, \psi''_j(D)g)(x) h(x) \, dx =: I_0 + I_1 + I_2 + I_3. \quad (5.5)
\]

In what follows, we denote these forms by \( I_i, i = 0, 1, 2, 3 \).

We shall consider the following form:

\[
I := \int_{\mathbb{R}^n} T_{\sigma_j}(F, G)(x) h_j(x) \, dx,
\]

where \( \sigma_j \) is as in (5.1). By changes of variables, we have

\[
I = \frac{2^{-j\rho n}}{(2\pi)^{2n}} \int_{(\mathbb{R}^n)^3} e^{ix \cdot (\xi + \eta)} \sigma_j(2^{-j\rho}x, 2^{j\rho} \xi, 2^{j\rho} \eta) \widehat{F}(2^{-j\rho} \cdot)(\xi) \widehat{G}(2^{-j\rho} \cdot)(\eta) \widehat{h}(2^{-j\rho}x) \, dX
= 2^{-j\rho n} \int_{\mathbb{R}^n} T_{\sigma_j}(F_j, G_j)(x) h_j(x) \, dx,
\]
where we simply wrote $dX = dxd\xi d\eta$,

$$
s_j^\rho = s_j(2^{-j\rho}, 2^{i\rho}, 2^{j\rho}),
F_j = F(2^{-j\rho}), \quad G_j = G(2^{-j\rho}), \quad \text{and} \quad h_j = h(2^{-j\rho}), \quad (5.6)
$$

with $s_j$ as in (5.1). We next decompose the symbol $s_j^\rho$. We use the product type operator $\Delta_k$ defined in Definition 3.1 and then use Lemma 5.1 to have

$$
\sigma_j^\rho(x, \xi, \eta) = \sum_{k=(k_0, k_1, k_2) \in (\mathbb{N}_0)^3} \Delta_k[\sigma_j^\rho](x, \xi, \eta)
= \sum_{k \in (\mathbb{N}_0)^3} \sum_{(v_1, v_2) \in (\mathbb{Z}^n)^2} \Delta_k[\sigma_j^\rho](x, \xi, \eta) \chi(\xi - v_1) \chi(\eta - v_2) \kappa(\xi - v_1) \kappa(\eta - v_2).
$$

Then, by writing as

$$
\sigma_{j, k, v}^\rho(x, \xi, \eta) = \Delta_k[\sigma_j^\rho](x, \xi, \eta) \chi(\xi - v_1) \chi(\eta - v_2) \quad (5.7)
$$

for $k = (k_0, k_1, k_2) \in (\mathbb{N}_0)^3$ and $v = (v_1, v_2) \in (\mathbb{Z}^n)^2$, we can see that

$$
T_{\sigma_j^\rho}(F_j, G_j)(x) = \sum_{k \in (\mathbb{N}_0)^3} \sum_{v \in (\mathbb{Z}^n)^2} T_{\sigma_j^\rho, k, v}(\kappa(D - v_1)F_j, \kappa(D - v_2)G_j)(x).
$$

By denoting the Fourier multiplier operator $\kappa(D - v_1)$ by $\Box_{v_i}$, $i = 1, 2$, we have

$$
I = \sum_{k \in (\mathbb{N}_0)^3} \sum_{v \in (\mathbb{Z}^n)^2} 2^{-j\rho n} \int_{\mathbb{R}^n} \phi_{j, k, v}(\Box_{v_1} F_j, \Box_{v_2} G_j)(x) \, h_j(x) \, dx. \quad (5.8)
$$

Now, we rewrite the form $I_1$ defined in (5.5). For $I_0$, substituting $F = \phi_j(D)f$ and $G = \phi_j(D)g$ into (5.8), then we have

$$
I_0 = \sum_{j \leq 1} \sum_{k \in (\mathbb{N}_0)^3} \sum_{v \in (\mathbb{Z}^n)^2} 2^{-j\rho n} \int_{\mathbb{R}^n} T_{\sigma_j^\rho, k, v}(\Box_{v_1} f_j, \Box_{v_2} g_j)(x) \, h_j(x) \, dx, \quad (5.9)
$$

where $\sigma_{j, k, v}$ is as in (5.7) with (5.6) and (5.1),

$$
f_j = \phi_j(D)f(2^{-j\rho}). \quad g_j = \phi_j(D)g(2^{-j\rho}). \quad \text{and} \quad h_j = h(2^{-j\rho}).
$$

Also, since $\text{supp } \phi \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 4 \}$, $f_j$ and $g_j$ satisfy that

$$
\text{supp } \hat{f}_j \subset \left\{ \xi \in \mathbb{R}^n : |\xi| \leq 2^{j(1-\rho)+2} \right\},
\text{supp } \hat{g}_j \subset \left\{ \eta \in \mathbb{R}^n : |\eta| \leq 2^{j(1-\rho)+2} \right\}. \quad (5.10)
$$
Substituting \((F, G) = (\phi_j(D)f, \psi_j(D)g), (\psi_j(D)f, \phi_j(D)g),\) and \((\psi''_j(D)f, \psi''_j(D)g)\) into (5.8), the corresponding forms \(I_1, I_2,\) and \(I_3\) are respectively given as follows. For \(I_1\), we have

\[
I_1 = \sum_{j \gg 1} \sum_{k \in (\mathbb{N}_0)^3} \sum_{v \in (\mathbb{Z}^n)^2} 2^{-j\rho n} \int_{\mathbb{R}^n} T_{\sigma_j, k, v} (\Box_{v_1} f_j^{(1)}, \Box_{v_2} g_j^{(1)})(x) \, h_j(x) \, dx,
\]

(5.11)

where

\[
f_j^{(1)} = \phi_j(D)f(2^{-j\rho} \cdot), \quad g_j^{(1)} = \psi_j(D)g(2^{-j\rho} \cdot), \quad \text{and} \quad h_j = h(2^{-j\rho} \cdot).
\]

(5.12)

Also, \(f_j^{(1)}\) and \(g_j^{(1)}\) satisfy from (5.2) and (5.3) that

\[
\text{supp} \, \hat{f}_j^{(1)} \subset \left\{ \xi \in \mathbb{R}^n : \left| \xi \right| \leq 2^{j(1-\rho) - 5} \right\},
\]

\[
\text{supp} \, \hat{g}_j^{(1)} \subset \left\{ \eta \in \mathbb{R}^n : \left| \eta \right| \leq 2^{j(1-\rho) - 4} \leq 2^{j(1-\rho) + 2} \right\}.
\]

(5.13)

Moreover, in this case, it should be remarked that

\[
2^{j(1-\rho) - 5} \leq |\xi + \eta| \leq 2^{j(1-\rho) + 3}, \quad \text{if} \quad (\xi, \eta) \in \text{supp} \, \hat{f}_j^{(1)} \times \text{supp} \, \hat{g}_j^{(1)}.
\]

(5.14)

For \(I_2\), we have

\[
I_2 = \sum_{j \gg 1} \sum_{k \in (\mathbb{N}_0)^3} \sum_{v \in (\mathbb{Z}^n)^2} 2^{-j\rho n} \int_{\mathbb{R}^n} T_{\sigma_j, k, v} (\Box_{v_1} f_j^{(2)}, \Box_{v_2} g_j^{(2)})(x) \, h_j(x) \, dx,
\]

(5.15)

where

\[
f_j^{(2)} = \psi_j(D)f(2^{-j\rho} \cdot), \quad g_j^{(2)} = \phi_j(D)g(2^{-j\rho} \cdot), \quad \text{and} \quad h_j = h(2^{-j\rho} \cdot).
\]

(5.16)

Also, \(f_j^{(2)}\) and \(g_j^{(2)}\) satisfy from (5.3) and (5.2) that

\[
\text{supp} \, \hat{f}_j^{(2)} \subset \left\{ \xi \in \mathbb{R}^n : 2^{j(1-\rho) - 4} \leq |\xi| \leq 2^{j(1-\rho) + 2} \right\},
\]

\[
\text{supp} \, \hat{g}_j^{(2)} \subset \left\{ \eta \in \mathbb{R}^n : \left| \eta \right| \leq 2^{j(1-\rho) - 5} \right\}.
\]

(5.17)
Again, in this case, it should be remarked that
\[ 2^{j(1-\rho)-5} \leq |\xi + \eta| \leq 2^{j(1-\rho)+3}, \quad \text{if} \quad (\xi, \eta) \in \text{supp} \hat{f}_j^{(2)} \times \text{supp} \hat{g}_j^{(2)}. \]
\[ (5.18) \]
For \( I_3 \), we have
\[ I_3 = \sum_{j \gg 1} \sum_{k \in (\mathbb{N}_0)^3} \sum_{v \in (2^j)^3} 2^{-j \rho n} \int_{\mathbb{R}^n} T_{\sigma_j^{(3)}, k, v} \left( \square_{v_1} f_j^{(3)}, \square_{v_2} g_j^{(3)} \right)(x) h_j(x) \, dx, \]
\[ (5.19) \]
where
\[ f_j^{(3)} = \psi_j''(D) f(2^{-j \rho} \cdot), \quad g_j^{(3)} = \psi_j''(D) g(2^{-j \rho} \cdot), \quad \text{and} \quad h_j = h(2^{-j \rho} \cdot). \]
\[ (5.20) \]
Also, \( f_j^{(3)} \) and \( g_j^{(3)} \) satisfy from \( (5.4) \) that
\[ \text{supp} \hat{f}_j^{(3)}, \text{supp} \hat{g}_j^{(3)} \subset \left\{ \xi \in \mathbb{R}^n : 2^{j(1-\rho)-6} \leq |\xi| \leq 2^{j(1-\rho)+2} \right\}. \]
\[ (5.21) \]
We remark that a fact like \( (5.14) \) and \( (5.18) \) does not hold in this case.

In the forthcoming sections, we will estimate the dual forms in \( (5.9), (5.11), (5.15), \) and \( (5.19) \) to show the boundedness in Theorems 3.2 and 3.3.

**Remark 5.2** The method of decomposing symbols by using a Littlewood–Paley partition of unity on \( \mathbb{R}^n \times \mathbb{R}^n \) and a frequency-uniform partition of unity on \( \mathbb{R}^n \) can be found in, e.g., Miyachi–Tomita [24, Sect. 3]. Moreover, the idea of decomposing symbols by using the functions \( \kappa \) and \( \chi \) in Lemma 5.1 comes from Sugimoto [29]. Changing of variables as \( \zeta \mapsto 2^{\pm j \rho} \zeta \) is a usual method when we consider the boundedness for the case \( 0 < \rho < 1 \). See, e.g., [28, Chapter VII, Sect. 2.5], where the \( L^2 \)-boundedness for linear pseudo-differential operators with symbols in the class \( S^0_{\rho, \rho}, 0 \leq \rho < 1 \), is shown.

### 6 Proof of Theorem 3.2

In this section, we prove Theorem 3.2. To this end, we show that the absolute values of \( I_i \), given in \( (5.9), (5.11), (5.15), \) and \( (5.19) \), are bounded by a constant times
\[ \| \sigma \|_{B_{\rho, \rho}^{m, s}(s; \mathbb{R}^n)} \| f \|_{L^2} \| g \|_{L^2} \| h \|_{\text{bmo}} \]
with \( m = -(1 - \rho)n/2 \) and \( s = (n/2 + \varepsilon, n/2, n/2) \) for any \( \varepsilon > 0 \). Then, these complete the proof of Theorem 3.2 by recalling the expression in \( (5.5) \). In what follows, we will first consider \( I_1 \). However, we omit the proof for \( I_2 \) because of symmetry.
between $I_2$ and $I_1$. After that, we consider $I_3$ and finally $I_0$. The basic idea contained in the proof goes back to Miyachi–Tomita [24].

Before the proof, we shall give two remarks. In order to obtain the desired boundedness, we will apply Lemma 4.4 to the dual forms $I_i$. This means that this lemma will be used under the setting $\sigma_\nu = \sigma_{j,k,\nu}^\rho$. Recall from (5.7) and Lemma 5.1 that

$$
\sigma_{j,k,\nu}^\rho(x, \xi, \eta) = \Delta_k[\sigma_{j}^\rho](x, \xi, \eta) \chi(\xi - v_1) \chi(\eta - v_2)
$$

with $\chi \in \mathcal{S}(\mathbb{R}^n)$ satisfying that $\text{supp} \hat{\chi} \subset B_1$ and $|\chi| \geq c > 0$ on $[-1, 1]^n$.

Firstly, let us investigate the support of the Fourier transform of $\sigma_{j,\nu}$. Since $\text{supp} \hat{\chi} \subset B_1$,

$$
\text{supp } \mathcal{F}[\sigma_{j,k,\nu}^\rho] \subset B_{2^{k_0+1}} \times B_{2^{k_1+2}} \times B_{2^{k_2+2}}
$$

(6.1)

holds. Hence, we will use Lemma 4.4 with $R_0 = 2^{k_0+1}$ and $R_i = 2^{k_i+2}$, $i = 1, 2$.

Secondly, we have

$$
\|\sigma_{j,k,\nu}^\rho(x, \xi, \eta)\|_{L^2_\xi L^2_\eta L^\infty_{\mu}} \lesssim \|\Delta_k[\sigma_{j}^\rho]\|_{L^2_\mu([\mathbb{R}^n]^3)}.
$$

(6.2)

To see (6.2), we separate the integrals of $L^2_\xi L^2_\eta$ by using (2.1). Then, we have

$$
\|\sigma_{j,k,\nu}^\rho(x + v_0, \xi, \eta)\|_{L^2_\xi L^2_\eta L^\infty_{\mu}}^2 = \sum_{\mu_1, \mu_2 \in \mathbb{Z}^n} \int_{Q^3} |\Delta_k[\sigma_{j}^\rho](x + v_0, \xi + \mu_1, \eta + \mu_2) \chi(\xi + \mu_1 - v_1) \chi(\eta + \mu_2 - v_2)|^2 dX
$$

$$
\lesssim \sum_{\mu_1, \mu_2 \in \mathbb{Z}^n} \prod_{i = 1, 2} (1 + |\mu_i - v_i|)^{-(n+1)} \int_{Q^3} |\Delta_k[\sigma_{j}^\rho](x + v_0, \xi + \mu_1, \eta + \mu_2)|^2 dX
$$

for $v_0 \in \mathbb{Z}^n$ and $\nu = (v_1, v_2) \in (\mathbb{Z}^n)^2$, where $dX = dx d\xi d\eta$. Thus, it holds that

$$
\|\sigma_{j,k,\nu}^\rho(x + v_0, \xi, \eta)\|_{L^2_\xi L^2_\eta L^\infty_{\mu}} \lesssim \|\Delta_k[\sigma_{j}^\rho]\|_{L^2_\mu}
$$

for $v_0, v_1, v_2 \in \mathbb{Z}^n$, which yields the inequality $\lesssim$ of (6.2). The opposite inequality $\gtrsim$ of (6.2) can be proved in a similar way by using the fact $|\chi| \geq c > 0$ on $[-1, 1]^n$.

Therefore, when we use Lemma 4.4 under the situation $\sigma_\nu = \sigma_{j,k,\nu}^\rho$, the equivalence (6.2) allows us to replace $\|\sigma_{j,k,\nu}^\rho\| = \|\sigma_{j,k,\nu}^\rho(x, \xi, \eta)\|_{L^2_\xi L^2_\eta L^\infty_{\mu}}$ by $\|\Delta_k[\sigma_{j}^\rho]\|_{L^2_\mu}$.

### 6.1 Estimate for $I_1$

In this subsection, we consider the dual form $I_1$ in (5.11). We decompose the factor of $f$ by a Littlewood–Paley partition $\{\psi_\ell\}$ on $\mathbb{R}^n$ as

$$
\Box_{v_1} f_j^{(1)}(x) = \sum_{\ell \in \mathbb{N}_0} \Delta_\ell \left[ \Box_{v_1} f_j^{(1)} \right] = \sum_{\ell \in \mathbb{N}_0} \Box_{v_1} \Delta_\ell f_j^{(1)}.
$$
Then, $I_1$ can be expressed by

$$I_1 = \sum_{j \gg 1} \sum_{k \in (\mathbb{N}_0)^3} \sum_{\ell \in \mathbb{N}_0} \sum_{v \in \mathbb{Z}^n} I^{(1)}_{j,k,\ell,v}$$

with

$$I^{(1)}_{j,k,\ell,v} = 2^{-j \rho n} \int_{\mathbb{R}^n} T_{\sigma_j,\ell,v} (\Box_{v_1} \Delta_{\ell} f_j^{(1)}, \Box_{v_2} g_j^{(1)}) (x) h_j (x) \, dx.$$  \hspace{1cm} (6.3)

The sums over $\ell$ and $v$ are restricted by the factors $\Box_{v_1} \Delta_{\ell} f_j^{(1)}$ and $\Box_{v_2} g_j^{(1)}$. Firstly, recalling the notation $\Box_{v_i} = \kappa (D - v_i)$, $i = 1, 2$, with $\kappa \in \mathcal{S}(\mathbb{R}^n)$ such that $\text{supp} \kappa \subset [-1, 1]^n$, we have by (5.13)

$$v_1 \in \Lambda_{1,\ell} = \left\{ v_1 \in \mathbb{Z}^n : |v_1| \lesssim 2^\ell \right\}, \quad v_2 \in \Lambda_{2, j} = \left\{ v_2 \in \mathbb{Z}^n : |v_2| \lesssim 2^{j(1-\rho)} \right\}. \hspace{1cm} (6.4)$$

Secondly, from the factor $\Delta_{\ell} f_j^{(1)}$, we see that $2^{\ell-1} \leq 2^{j(1-\rho)-5}$, which implies that

$$\ell \leq j (1 - \rho) - 4 \leq j (1 - \rho). \hspace{1cm} (6.5)$$

Note that the set \{ $\ell \in \mathbb{N}_0 : \ell \leq j (1 - \rho) - 4$ \} is not empty, since $j \gg 1$.

Hence,

$$I_1 = \sum_{j \gg 1} \sum_{k \in (\mathbb{N}_0)^3} \sum_{\ell \in \mathbb{N}_0 : \ell \leq j (1 - \rho)} \sum_{v \in \Lambda_{1,\ell} \times \Lambda_{2, j}} I^{(1)}_{j,k,\ell,v}.$$ 

For this expression, we further separate the sum over $j$ as

$$I_1 = \sum_{k_0 \in \mathbb{N}_0} \left\{ \sum_{j \gg 1 : j (1 - \rho) \leq k_0 + L} + \sum_{j \gg 1 : j (1 - \rho) > k_0 + L} \right\} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{\ell \in \mathbb{N}_0 : \ell \leq j (1 - \rho)} \sum_{v \in \Lambda_{1,\ell} \times \Lambda_{2, j}} I^{(1)}_{j,k,\ell,v}$$

$$=: I^{(1,1)} + I^{(1,2)} \hspace{1cm} (6.6)$$

for some sufficiently large constant $L > 0$, where $I^{(1)}_{j,k,\ell,v}$ is as in (6.3).

### 6.1.1 Estimate of $I^{(1,1)}$ in (6.6)

Firstly, we observe that

$$\mathcal{F} \left[ T_{\sigma_j,\ell,v} (\Box_{v_1} \Delta_{\ell} f_j^{(1)}, \Box_{v_2} g_j^{(1)}) \right] (\xi)$$

$$= \frac{1}{(2\pi)^{2n}} \int_{(\mathbb{R}^n)^2} \mathcal{F}_0 [\sigma_j^{\rho} \Box_{v_1} \Delta_{\ell} f_j^{(1)}(\xi) \Box_{v_2} g_j^{(1)}(\eta)] \, d\xi \, d\eta,$$
Combining this with the fact \( \text{supp} \mathcal{F}_0[\sigma_{j,k,v}^\rho](\cdot, \xi, \eta) \subset B_{2^k0+1} \) from (6.1), we have

\[
\text{supp} \mathcal{F} \left[ T_{\sigma_{j,k,v}^\rho} (\square v_1 \Delta_{\ell} f_j^{(1)}, \square v_2 g_j^{(1)}) \right] \\
\subset \left\{ \xi \in \mathbb{R}^n : |\xi - (\xi + \eta)| \leq 2^{k+1}, \xi \in \text{supp} \square v_1 \Delta_{\ell} f_j^{(1)}, \eta \in \text{supp} \square v_2 g_j^{(1)} \right\}.
\]

(6.7)

Here, since we are considering the sum over \( j \) such that \( j(1-\rho) \leq k_0 + L \), we have \(|\xi|, |\eta| \lesssim 2^{k_0}\) for \( \xi \in \text{supp} \square v_1 \Delta_{\ell} f_j^{(1)} \) and \( \eta \in \text{supp} \square v_2 g_j^{(1)} \). Hence, by (6.7)

\[
\text{supp} \mathcal{F} \left[ T_{\sigma_{j,k,v}^\rho} (\square v_1 \Delta_{\ell} f_j^{(1)}, \square v_2 g_j^{(1)}) \right] \subset \{ \xi \in \mathbb{R}^n : |\xi| \lesssim 2^{k_0} \}.
\]

We take a function \( \varphi \in S(\mathbb{R}^n) \) such that \( \varphi = 1 \) on \( \{ \xi \in \mathbb{R}^n : |\xi| \lesssim 1 \} \). Then, \( I^{(1,1)} \) can be rewritten as

\[
I^{(1,1)} = \sum_{k_0 \in \mathbb{N}_0} \sum_{j : j(1-\rho) \leq k_0 + L} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{\ell : \ell \leq j(1-\rho)} \sum_{v \in \Lambda_{1,\ell} \times \Lambda_{2,j}}
\times 2^{-j\rho n} \int_{\mathbb{R}^n} T_{\sigma_{j,k,v}^\rho} (\square v_1 \Delta_{\ell} f_j^{(1)}, \square v_2 g_j^{(1)}) (x) \varphi(D/2^{k_0}) [h_j] (x) \, dx.
\]

In what follows, we shall estimate \( I^{(1,1)} \) using this last formula. Again, recall that \( \square v_i = \kappa(D - v_i), i = 1, 2, \) with \( \kappa \in S(\mathbb{R}^n) \) such that \( \text{supp} \kappa \subset [-1, 1]^n \). Then, since \( \min(|\Lambda_{1,\ell}|, |\Lambda_{2,j}|) \lesssim 2^{n} \), we have by Lemma 4.4 (1) with \( p = q = 2 \) and \( r = \infty \)

\[
|I^{(1,1)}| \lesssim \sum_{k_0 \in \mathbb{N}_0} \sum_{j : j(1-\rho) \leq k_0 + L} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{\ell : \ell \leq j(1-\rho)}
\times 2^{-j\rho n} 2^{(n/2) 2^{k_0+1} k_0/2} \| \Delta_k [\sigma_{j}^\rho] \|_{L^2} \| \Delta_{\ell} f_j^{(1)} \|_{L^2} \| g_j^{(1)} \|_{L^2} \| \varphi(D/2^{k_0}) [h_j] \|_{L^\infty}.
\]

Here, recall the notations of \( f_j^{(1)}, g_j^{(1)} \), and \( h_j \) from (5.12). Then, we have

\[
\| \Delta_{\ell} f_j^{(1)} \|_{L^2} = \| \Delta_{\ell} [\psi_j(D)f(2^{-j\rho})] \|_{L^2} \lesssim 2^{j\rho n/2} \| \Delta_{\ell+j\rho} f \|_{L^2},
\]

\[
\| g_j^{(1)} \|_{L^2} = 2^{j\rho n/2} \| \psi_j(D)g \|_{L^2},
\]

(6.8)

where, \( \Delta_{\ell+j\rho} = \psi_{\ell}(D/2^{j\rho}) \) for \( \ell \geq 0 \). Moreover, since \( j(1-\rho) \leq k_0 + L \), by Lemma 2.2 (1), we have for any \( \varepsilon > 0 \)

\[
\| \varphi(D/2^{k_0}) [h_j] \|_{L^\infty} = \| \varphi(D/2^{k_0+j\rho}) h(2^{-j\rho}) \|_{L^\infty}
\lesssim (1 + k_0 + j\rho) \| h \|_{bmo} \leq C \varepsilon 2^{k_0} \| h \|_{bmo}.
\]

(6.9)

(The role of the condition \( j(1-\rho) \leq k_0 + L \) is finished by obtaining this estimate.)
Hence, by denoting the Fourier multiplier operator $\psi_j(D)$ by $\Delta'_j$, we have

$$
\left| f^{(1,1)} \right| \lesssim \sum_{k_0 \in \mathbb{N}_0} \sum_{j \gg 1} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{\ell : \ell \leq j(1-\rho)} 2^{\ell n/2} 2^{k_0(n/2+\varepsilon)} 2^{(k_1+k_2)n/2} \| \Delta_k [\sigma_{\Delta'}^{\rho}] \|_{L^2_{\mu}} \| \Delta_{\ell+j\rho} f \|_{L^2} \| \Delta'_{j} g \|_{L^2} \| h \|_{bmo}
$$

$$
\leq \| \sigma \|_{B_{s_\rho}^{m_{\alpha} \sigma} (\mathbb{R}^n)} \| h \|_{bmo} \sum_{j \gg 1} \sum_{\ell : \ell \leq j(1-\rho)} 2^{-j(1-\rho)n/2} 2^{\ell n/2} \| \Delta_{\ell+j\rho} f \|_{L^2} \| \Delta'_{j} g \|_{L^2}
$$

$$
= \| \sigma \|_{B_{s_\rho}^{m_{\alpha} \sigma} (\mathbb{R}^n)} \| h \|_{bmo} II ,
$$

(6.10)

where $m = -(1-\rho)n/2$, $s = (n/2 + \varepsilon, n/2, n/2)$ for any $\varepsilon > 0$, and

$$
II := \sum_{j \gg 1} \sum_{0 \leq \ell \leq j(1-\rho)} 2^{-j(1-\rho)n/2} 2^{\ell n/2} \| \Delta_{\ell+j\rho} f \|_{L^2} \| \Delta'_{j} g \|_{L^2}.
$$

(6.11)

In what follows, we shall show that

$$
II \lesssim \| f \|_{L^2} \| g \|_{L^2}.
$$

(6.12)

To this end, we divide $II$ into the two parts $\ell = 0$ and $\ell \geq 1$. We write

$$
II = II_{\ell=0} + II_{\ell \geq 1}
$$

(6.13)

with

$$
II_{\ell=0} = \sum_{j \gg 1} 2^{-j(1-\rho)n/2} \| \psi_0 (D/2^{j\rho}) f \|_{L^2} \| \Delta'_{j} g \|_{L^2},
$$

$$
II_{\ell \geq 1} = \sum_{j \gg 1} \sum_{1 \leq \ell \leq j(1-\rho)} 2^{-j(1-\rho)n/2} 2^{\ell n/2} \| \psi_\ell (D/2^{j\rho}) f \|_{L^2} \| \Delta'_{j} g \|_{L^2}.
$$

The sum for $\ell = 0$ is estimated as

$$
II_{\ell=0} \lesssim \| f \|_{L^2} \| g \|_{L^2} \sum_{j \gg 1} 2^{-j(1-\rho)n/2} \approx \| f \|_{L^2} \| g \|_{L^2},
$$

(6.14)

since $0 \leq \rho < 1$. For the sum in the case $\ell \geq 1$, we take a function $\psi_\ell \in \mathcal{S}(\mathbb{R}^n)$ such that $\psi_\ell = 1$ on $\{ \xi \in \mathbb{R}^n : 1/4 \leq |\xi| \leq 4 \}$ and supp $\psi_\ell \subset \{ \xi \in \mathbb{R}^n : 1/8 \leq |\xi| \leq 8 \}$. Then, we realize that $\psi_\ell (\cdot/2^{\ell+1[j\rho]}) = 1$ on supp $\psi_\ell (\cdot/2^{j\rho})$, since $[j\rho] \leq j\rho \leq [j\rho] + 1$. Hence, by writing the operator $\Delta_{\ell+j\rho}^{\dagger} = \psi_\ell (D/2^{\ell+1[j\rho]})$, we have

$$
\| \psi_\ell (D/2^{j\rho}) f \|_{L^2} = \| \Delta_{\ell+j\rho}^{\dagger} \psi_\ell (D/2^{j\rho}) f \|_{L^2} \lesssim \| \Delta_{\ell+j\rho}^{\dagger} f \|_{L^2}.
$$
Therefore, we see that
\[
II_{\ell \geq 1} \lesssim \sum_{j \gg 1} \sum_{1 \leq \ell' \leq j} 2^{-j(1-\rho)n/2} 2^{\ell' n/2} \| \Delta_{\ell' j}^{\ell} f \|_{L^2} \| \Delta_{j}^{\ell} g \|_{L^2}.
\]

By using the fact \(2^{j\rho} \approx 2^{[j\rho]}\) and the translation as \(\ell + [j\rho] \mapsto \ell'\), we have
\[
II_{\ell \geq 1} \lesssim \sum_{j \gg 1} \sum_{1 \leq \ell' \leq j} 2^{-j n/2} 2^{\ell' n/2} \| \Delta_{\ell'}^{\ell} f \|_{L^2} \| \Delta_{j}^{\ell} g \|_{L^2}.
\]

Here, we verify that
\[
\sup_{j \in \mathbb{N}} \left\{ \sum_{\ell' \in \mathbb{N}} 1_{\ell' \leq j} 2^{-j n/2} 2^{\ell' n/2} \right\} \approx 1 \quad \text{and} \quad \sup_{\ell' \in \mathbb{N}} \left\{ \sum_{j \in \mathbb{N}} 1_{\ell' \leq j} 2^{\ell' n/2} 2^{-j n/2} \right\} \approx 1.
\]

Then, we obtain from Lemma 2.1
\[
II_{\ell \geq 1} \lesssim \| \Delta_{\ell'}^{\ell} f \|_{L^2} \ell_{\rho}^2(\mathbb{N}) \| \Delta_{j}^{\ell} g \|_{L^2} \ell_{\rho}^2(\mathbb{N}) \lesssim \| f \|_{L^2} \| g \|_{L^2}. \tag{6.15}
\]

Therefore, from (6.13), (6.14) and (6.15), we obtain (6.12).

Finally, collecting (6.10) and (6.12), we obtain
\[
\left| I^{(1,1)} \right| \lesssim \| \sigma \|_{B_{S_{d,v}(s;\mathbb{R}^n)}} \| f \|_{L^2} \| g \|_{L^2} \| h \|_{hmo},
\]
where \(m = -(1 - \rho)n/2\) and \(s = (n/2 + \epsilon, n/2, n/2)\) for any \(\epsilon > 0\).

### 6.1.2 Estimate for \(I^{(1,2)}\) in (6.6)

In this subsection, since we are considering the sum over \(j\) such that \(j(1-\rho) > k_0 + L\) with some large \(L > 0\), we have by (6.7)
\[
\text{supp} \mathcal{F} \left[ T_{\sigma_{f,k,v}}^{(1)} \left( \Box_{v_1} \Delta_{\ell} f_j^{(1)}, \Box_{v_2} g_j^{(1)} \right) \right] \\
\subset \left\{ \xi \in \mathbb{R}^n : |\xi - (\xi + \eta)| \leq 2^{j(1-\rho)+1-L}, \eta \in \text{supp} \Box_{v_1} \Delta_{\ell} f_j^{(1)}, \right\}.
\]

Here, we recall from (5.14) that if \(\xi \in \text{supp} \hat{f}_j^{(1)}\) and \(\eta \in \text{supp} \hat{g}_j^{(1)}\), then \(2^{j(1-\rho)-5} \leq |\xi + \eta| \leq 2^{j(1-\rho)+3}\) holds. Then, we see that
\[
\text{supp} \mathcal{F} \left[ T_{\sigma_{f,k,v}}^{(1)} \left( \Box_{v_1} \Delta_{\ell} f_j^{(1)}, \Box_{v_2} g_j^{(1)} \right) \right] \\
\subset \left\{ \xi \in \mathbb{R}^n : 2^{j(1-\rho)-6} \leq |\xi| \leq 2^{j(1-\rho)+4} \right\}. \tag{6.16}
\]
We take a function \( \psi^\frac{1}{2} \in S(\mathbb{R}^n) \) satisfying that \( \psi^\frac{1}{2} = 1 \) on \( \{ \zeta \in \mathbb{R}^n : 2^{-6} \leq |\zeta| \leq 2^4 \} \) and \( \text{supp} \, \psi^\frac{1}{2} \subset \{ \zeta \in \mathbb{R}^n : 2^{-7} \leq |\zeta| \leq 2^5 \} \). Then, \( I^{(1,2)} \) can be rewritten as

\[
I^{(1,2)} = \sum_{k_0 \in \mathbb{N}_0} \sum_{j: j(1-\rho) > k_0 + L} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{\ell: \ell \leq j(1-\rho)} \sum_{v \in \Lambda_1, \ell \times \Lambda_2, j} \times 2^{-jn} \int_{\mathbb{R}^n} T_{\sigma^{\rho}, k, v} (\square_{v_1} \Delta_\ell f_j^{(1)}, \square_{v_2} g_j^{(1)})(x) \psi^\frac{1}{2}(D/2j^{(1-\rho)})|h_j|(x) \, dx.
\]

(The role of the condition \( j(1-\rho) > k_0 + L \) is finished by obtaining this expression.)

Now, we shall estimate the newly given \( I^{(1,2)} \). Since \( \min(|\Lambda_1, \ell|, |\Lambda_2, j|) \lesssim 2^\ell n \), we have by using Lemma 4.4 (1) with \( p = q = 2 \) and \( r = \infty \)

\[
|I^{(1,2)}| \lesssim \sum_{k_0 \in \mathbb{N}_0} \sum_{j \gg 1} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{\ell: \ell \leq j(1-\rho)} \times 2^{-jn} 2^{\ell n/2} 2^{k_0 + k_1 + k_2} n/2 \|\Delta_k[\sigma^\rho]\|_{L^2_{ul}} \|\Delta_\ell f_j^{(1)}\|_{L^2} \|g_j^{(1)}\|_{L^2} \|\psi^\frac{1}{2}(D/2j^{(1-\rho)})\|_{L^\infty} |h_j|.
\]

Here, for the factors of \( f \) and \( g \), we have the estimate (6.8). For the factor of \( h \), since \( \psi^\frac{1}{2}(0) = 0 \), we have by Lemma 2.2 (2)

\[
\|\psi^\frac{1}{2}(D/2j^{(1-\rho)})\|_{L^\infty} = \|\psi^\frac{1}{2}(D/2j)h(2^{-j}\rho)\|_{L^\infty} \lesssim \|h\|_{BMO}.
\]

Therefore, by writing as \( \Delta_j' = \psi_j'(D) \), we have

\[
|I^{(1,2)}| \lesssim \sum_{k_0 \in \mathbb{N}_0} \sum_{j \gg 1} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{\ell: \ell \leq j(1-\rho)} \times 2^{\ell n/2} 2^{k_0 + k_1 + k_2} n/2 \|\Delta_k[\sigma^\rho]\|_{L^2_{ul}} \|\Delta_\ell f_j^{(1)}\|_{L^2} \|g_j^{(1)}\|_{L^2} \|\psi\|_{BMO} \sum_{j \gg 1} \sum_{\ell \leq j(1-\rho)} 2^{-j(1-\rho)n/2} 2^{\ell n/2} \|\Delta_{\ell+j}\rho f\|_{L^2} \|\Delta_j' g\|_{L^2},
\]

where \( m = -(1-\rho)n/2 \) and \( s = (n/2, n/2, n/2) \). Since the sums over \( j \) and \( \ell \) are exactly identical with \( II \) defined in (6.11), we see from (6.12) that

\[
|I^{(1,2)}| \lesssim \|\sigma\|_{B_{sp_{m, s}}(\mathbb{R}^n)} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{BMO},
\]

where \( m = -(1-\rho)n/2 \) and \( s = (n/2, n/2, n/2) \).

### 6.2 Estimate for \( I_2 \)

In this subsection, we consider the dual form \( I_2 \) given in (5.15). However, by comparing (5.16) with (5.12), (5.17) with (5.13), and (5.18) with (5.14), we realize that \( I_2 \) and \( I_1 \) are in symmetrical positions. Moreover, in this section, we are considering the
boundedness on $L^2 \times L^2$. Therefore, following the same lines as in Sect. 6.1, we obtain

$$|I_2| \lesssim \| \sigma \|_{B^m_{2,2}(\mathbb{R}; \mathbb{R}^n)} \| f \|_{L^2} \| g \|_{L^2} \| h \|_{bmo}$$

with $m = -(1 - \rho)n/2$ and $s = (n/2 + \varepsilon, n/2, n/2)$ for any $\varepsilon > 0$.

### 6.3 Estimate for $I_3$

In this subsection, we consider the dual form $I_3$ given in (5.19). As in the previous subsections, we put

$$I^{(3)}_{j,k,v} = 2^{-jn} \int_{\mathbb{R}^n} T_{\sigma_{j,k,v}} (\square_{\nu_1} f^{(3)}_j, \square_{\nu_2} g^{(3)}_j) (x) h_j (x) \, dx$$

and also remark that

$$\text{supp } F \left[ T_{\sigma_{j,k,v}} (\square_{\nu_1} f^{(3)}_j, \square_{\nu_2} g^{(3)}_j) \right] \subset \left\{ \xi \in \mathbb{R}^n : |\xi - (\xi + \eta)| \leq 2^{k_0+1}, \xi \in \square_{\nu_1} f^{(3)}_j, \eta \in \square_{\nu_2} g^{(3)}_j \right\}. \quad (6.17)$$

Now, we separate the sum of $I_3$ into three parts with slight changes in the way of summing over $v$ as follows. For some sufficiently large constants $N > 0$ and $C > 0$,

$$I_3 = \sum_{k_0 \in \mathbb{N}_0} \sum_{j \gg 1: j(1-\rho) \leq k_0 + N} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{v \in \mathbb{Z}^n} I^{(3)}_{j,k,v}$$

$$+ \sum_{k_0 \in \mathbb{N}_0} \sum_{j \gg 1: j(1-\rho) > k_0 + N} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{v_1, v_2 \in \mathbb{Z}^n: |\tau| \leq C2^{k_0}} \sum_{v_1 + v_2 = \tau} I^{(3)}_{j,k,v}$$

$$+ \sum_{k_0 \in \mathbb{N}_0} \sum_{j \gg 1: j(1-\rho) > k_0 + N} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{v_1, v_2 \in \mathbb{Z}^n: |\tau| > C2^{k_0}} \sum_{v_1 + v_2 = \tau} I^{(3)}_{j,k,v}$$

$$=: I^{(3,1)} + I^{(3,2)} + I^{(3,3)}. \quad (6.18)$$

#### 6.3.1 Estimate for $I^{(3,1)}$ in (6.18)

Due to the factors $\square_{\nu_1} f^{(3)}_j$ and $\square_{\nu_2} g^{(3)}_j$ in $I^{(3)}_{j,k,v}$ and (5.21), we have the following restrictions for the sums over $\nu_1$ and $\nu_2$:

$$\nu_1, \nu_2 \in \Lambda_j = \left\{ v \in \mathbb{Z}^n : |v| \lesssim 2^{j(1-\rho)} \right\}.$$
We next observe that $|\xi|, |\eta| \lesssim 2^{k_0}$ hold for $\xi \in \{v_1 f_j^{(3)}\}$ and $\eta \in \{v_2 g_j^{(3)}\}$, since $j(1-\rho) \leq k_0 + N$. Then, we see from (6.17) that

$$\supp \mathcal{F} \left[ T_{\sigma_{\rho,j,k,v}}(\{v_1 f_j^{(3)}\}, \{v_2 g_j^{(3)}\}) \right] \subset \{\xi \in \mathbb{R}^n : |\xi| \lesssim 2^{k_0}\}.$$  

We take a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\varphi = 1$ on $\{\xi \in \mathbb{R}^n : |\xi| \lesssim 1\}$. Then, $I^{(3,1)}$ can be expressed as

$$I^{(3,1)} = \sum_{k_0 \in \mathbb{N}_0} \sum_{j: j(1-\rho) \leq k_0 + N} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{v \in \Lambda_j \times \Lambda_j} 2^{-j\rho n} \int_{\mathbb{R}^n} T_{\sigma_{\rho,j,k,v}}(\{v_1 f_j^{(3)}\}, \{v_2 g_j^{(3)}\})(x) \varphi(D/2^{k_0})[h_j](x) \, dx.$$  

By Lemma 4.4 (1) with $p = q = 2, r = \infty$, and the fact $|\Lambda_j| \lesssim 2^{j(1-\rho)n}$, we have

$$|I^{(3,1)}| \lesssim \sum_{k_0 \in \mathbb{N}_0} \sum_{j: j(1-\rho) \leq k_0 + N} \sum_{k_1, k_2 \in \mathbb{N}_0} 2^{-j\rho n} 2^{j(1-\rho)n/2} 2^{k_0(k_1+k_2)n/2} \|\Delta_k[\sigma_{\rho,j}^0]\|_{L^2_{ul}} \|f_j^{(3)}\|_{L^2} \|g_j^{(3)}\|_{L^2} \|\varphi(D/2^{k_0})[h_j]\|_{L^\infty}.$$  

Here, $\|f_j^{(3)}\|_{L^2} = 2^{jn/2}\|\psi_j''(D)f\|_{L^2}$ and $\|g_j^{(3)}\|_{L^2} = 2^{jn/2}\|\psi_j''(D)g\|_{L^2}$ hold from (5.20). Also, since $j(1-\rho) \leq k_0 + N$, $\|\varphi(D/2^{k_0})[h_j]\|_{L^\infty} \leq C_\varepsilon 2^{k_0\varepsilon} \|h\|_{bmo}$ holds for $\varepsilon > 0$ from (6.9). (Again, the role of the condition $j(1-\rho) \leq k_0 + N$ is finished by obtaining this estimate.) Hence, we obtain from the Cauchy–Schwarz inequality

$$|I^{(3,1)}| \lesssim \sum_{k_0 \in \mathbb{N}_0} \sum_{j > 1} \sum_{k_1, k_2 \in \mathbb{N}_0} 2^{j(1-\rho)n/2} 2^{k_0(n+\varepsilon)/2} \|\Delta_k[\sigma_{\rho,j}^0]\|_{L^2_{ul}} \|\psi_j''(D)f\|_{L^2} \|\psi_j''(D)g\|_{L^2} \|h\|_{bmo} \sum_{j > 1} \|\psi_j''(D)f\|_{L^2} \|\psi_j''(D)g\|_{L^2} \|h\|_{bmo},$$  

$$\lesssim \|\sigma_{BS_{\rho,0}}(x;\mathbb{R}^n)\|_b \|h\|_{bmo} \sum_{j > 1} \|\psi_j''(D)f\|_{L^2} \|\psi_j''(D)g\|_{L^2} \|h\|_{bmo},$$

where $m = -(1-\rho)n/2$ and $s = (n/2 + \varepsilon, n/2, n/2)$ for any $\varepsilon > 0$.

6.3.2 Estimate for $I^{(3,2)}$ in (6.18)

We here write $\Lambda_{k_0} = \{\tau \in \mathbb{Z}^n : |\tau| \leq C 2^{k_0}\}$.

Observe that $|\xi + \eta| \lesssim 2^{k_0}$ holds for $(\xi, \eta) \in \sup \{v_1 f_j^{(3)}\} \times \sup \{v_2 g_j^{(3)}\}$, since $|\xi - v_1| \lesssim 1$ and $|\eta - v_2| \lesssim 1$ for this $(\xi, \eta)$ and $v_1 + v_2 = \tau \in \Lambda_{k_0}$ in the sum of $I^{(3,2)}$. Then, we see from (6.17) that

$$\supp \mathcal{F} \left[ T_{\sigma_{\rho,j,k,v}}(\{v_1 f_j^{(3)}\}, \{v_2 g_j^{(3)}\}) \right] \subset \{\xi \in \mathbb{R}^n : |\xi| \lesssim 2^{k_0}\}.$$
Taking a function $\varphi \in S(\mathbb{R}^n)$ such that $\varphi = 1$ on $\{\xi \in \mathbb{R}^n : |\xi| \lesssim 1\}$, we have

$$I^{(3,2)} = \sum_{k_0 \in \mathbb{N}_0} \sum_{j:(1-\rho)>k_0+N} \sum_{k_1,k_2 \in \mathbb{N}_0} \sum_{r \in \Lambda_{k_0}} v_1 \cdot v_1 = \tau \times 2^{-j\rho n} \int_{\mathbb{R}^n} T_{\sigma_j,\varphi}^{\rho}(\square v_1 f^{(3)}_j, \square v_2 g^{(3)}_j)(x) \varphi(D/2^{k_0})[h_f](x) \, dz.$$}

Now, we shall estimate this $I^{(3,2)}$. By Lemma 4.4 (3) with $p = q = 2$ and $r = \infty$,

$$|I^{(3,2)}| \lesssim \sum_{k_0 \in \mathbb{N}_0} \sum_{j:(1-\rho)>k_0+N} \sum_{k_1,k_2 \in \mathbb{N}_0} \times 2^{-j\rho n} 2^{k_0n/2} 2^{(k_1+k_2)n/2} \|\Delta_k[\sigma_j^\rho]\|_{L^2_{ul}} \|f^{(3)}_j\|_{L^2} \|g^{(3)}_j\|_{L^2} \|\varphi(D/2^{k_0})[h_f]\|_{\ell^2(\Lambda_{k_0})L^\infty}.$$}

Here, recalling (5.20), we have $\|f^{(3)}_j\|_{L^2} \lesssim 2^{j\rho n/2} \|f\|_{L^2}$ and $\|g^{(3)}_j\|_{L^2} \lesssim 2^{j\rho n/2} \|g\|_{L^2}$. Moreover, since $\varphi(D/2^{k_0})[h_f]$ is independent of $\tau$, we have by Lemma 2.2 (1)

$$\|\varphi(D/2^{k_0})[h_f]\|_{\ell^2(\Lambda_{k_0})L^\infty} = |\Lambda_{k_0}|^{1/2} \|\varphi(D/2^{k_0})[h_f]\|_{L^\infty} \lesssim 2^{k_0n/2}(1 + k_0 + j\rho) \|h\|_{bmo} \leq 2^{k_0n/2}(1 + j) \|h\|_{bmo},$$

where we used the condition $j(1-\rho) > k_0 + N$ in the last inequality. From these,

$$|I^{(3,2)}| \lesssim \sum_{k_0 \in \mathbb{N}_0} \sum_{j:(1-\rho)>k_0+N} \sum_{k_1,k_2 \in \mathbb{N}_0} \times (1 + j) 2^{k_0n/2} 2^{(k_0+k_1+k_2)n/2} \|\Delta_k[\sigma_j^\rho]\|_{L^2_{ul}} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{bmo} \times \sup_{k_0 \in \mathbb{N}_0} \sum_{j \in \mathbb{N}_0} \sum_{k_1,k_2 \in \mathbb{N}_0} 2^{(1-\rho)n/2} 2^{(k_0+k_1+k_2)n/2} \|\Delta_k[\sigma_j^\rho]\|_{L^2_{ul}} \times 2^{k_0n/2} \sum_{j:(1-\rho)>k_0+N} (1 + j) 2^{-j(1-\rho)n/2} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{bmo}.$$}

Here, we have for $0 < \varepsilon < n/2$

$$\sum_{j:(1-\rho)>k_0+N} (1 + j) 2^{-j(1-\rho)n/2} \leq C_{\varepsilon,\rho} 2^{-k_0(n/2-\varepsilon)}, \quad k_0 \in \mathbb{N}_0,$n/2 and $s = (n/2 + \varepsilon, n/2, n/2)$ for any $\varepsilon > 0$. 

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6.3.3 Estimate for $I^{(3,3)}$ in (6.18)

The sum over $\tau$ in $I^{(3,3)}$ is further restricted to

$$\tau \in \Lambda_{j,k_0} = \{ \tau \in \mathbb{Z}^n : C2^{k_0} < |\tau| \lesssim 2^{j(1-\rho)} \},$$

by the factors $\Box_{v_1} f_j^{(3)}$ and $\Box_{v_2} g_j^{(3)}$. Here, note that the set $\Lambda_{j,k_0}$ is not empty, since $j(1-\rho) > k_0 + N$ with a sufficiently large $N$. Moreover, we have by (6.17)

$$\text{supp} \mathcal{F} [ T_{\sigma_j,\nu}^{\rho} (\Box_{v_1} f_j^{(3)}, \Box_{v_2} g_j^{(3)}) ] \subset \{ \xi \in \mathbb{R}^n : |\xi - \tau| \lesssim 2^{k_0} \},$$

since $|\xi - v_1| \lesssim 1$ and $|\eta - v_2| \lesssim 1$ hold for $\xi \in \text{supp} \Box_{v_1} f_j^{(3)}$ and $\eta \in \text{supp} \Box_{v_2} g_j^{(3)}$, where $v_1 + v_2 = \tau$. Hence, taking a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\varphi = 1$ on $\{ \xi \in \mathbb{R}^n : |\xi| \lesssim 1 \}$, we are able to rewrite $I^{(3,3)}$ as

$$I^{(3,3)} = \sum_{k_0 \in \mathbb{N}_0} \sum_{j : j(1-\rho) > k_0 + N} \sum_{k_1,k_2 \in \mathbb{N}_0} \sum_{\tau \in \Lambda_{j,k_0} : v_1 + v_2 = \tau} \sum_{j} T^{\rho}_{\sigma_j,\nu} (\Box_{v_1} f_j^{(3)}, \Box_{v_2} g_j^{(3)}) (x) \varphi \left( \frac{D + \tau}{2^{k_0}} \right) |h_j| dx.$$

Now, we shall estimate this $I^{(3,3)}$. By Lemma 4.4 (3) with $p = q = 2$ and $r = \infty$,

$$|I^{(3,3)}| \lesssim \sum_{k_0 \in \mathbb{N}_0} \sum_{j : j(1-\rho) > k_0 + N} \sum_{k_1,k_2 \in \mathbb{N}_0} \sum_{\tau \in \Lambda_{j,k_0} : v_1 + v_2 = \tau} 2^{-j\rho n} 2^{j(1-\rho)n/2} 2^{(k_1+k_2)n/2} \| \Lambda_k \sigma_j \|_{L^2} \| f_j^{(3)} \|_{L^2} \| g_j^{(3)} \|_{L^2} \| \varphi \left( \frac{D + \tau}{2^{k_0}} \right) |h_j| \|_{L^\infty}.$$

Here, from (5.20), $\| f_j^{(3)} \|_{L^2} = 2^{j\rho n/2} \| \psi_j^{(1)} (D) f \|_{L^2}$ and $\| g_j^{(3)} \|_{L^2} = 2^{j\rho n/2} \| \psi_j^{(1)} (D) g \|_{L^2}$. Moreover, since $\varphi(\tau/2^{k_0}) = 0$ for $\tau \in \Lambda_{j,k_0}$, we see from Lemma 4.2 (2) that

$$\| \varphi \left( \frac{D + \tau}{2^{k_0}} \right) |h_j| \|_{L^\infty} \lesssim 2^{k_0 n/2} \| h(2^{-j\rho}) \|_{BMO} = 2^{k_0 n/2} \| h \|_{BMO}.$$

Here, we recall that $BMO$ is scaling invariant, that is, $\| f(\cdot \lambda) \|_{BMO} = \| f \|_{BMO}$ for $\lambda > 0$, although the space $bmo$ is not so in general. See, e.g., [11, Proposition 3.1.2
In this section we prove Theorem 3.3. However, we only give a precise proof for the
former boundedness from $L^2(\mathbb{R}^n) \times bmo(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, since the latter boundedness
from $bmo(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ follows by symmetry.

Now, to this end, we will estimate the dual forms $I_i, i = 0, 1, 2, 3$, given in (5.9),
(5.11), (5.15), and (5.19), by a constant times

$$\|\sigma\|_{B^m_{p,\rho}(s;\mathbb{R}^n)} \|f\|_{L^2} \|g\|_{bmo} \|h\|_{L^2}.$$
for \(m = -(1 - \rho)n/2\) and \(s = (n/2, n/2, n/2)\). Then, these complete the proof of the boundedness from \(L^2(\mathbb{R}^n) \times bmo(\mathbb{R}^n)\) to \(L^2(\mathbb{R}^n)\) in Theorem 3.3. The basic idea of the proof here again goes back to [24].

We will again use Lemma 4.4 under the setting \(\sigma_v = \sigma_{j,k,v}^\rho\). Hence, for the same reasons as stated in Sect. 6, we will use the lemma with \(R_0 = 2^{k_0+1}\), \(R_i = 2^{k_i+2}\), \(i = 1, 2\), and the equivalence (6.2). Now, we shall start the proof of Theorem 3.3.

7.1 Estimate for \(I_1\)

In this subsection, we consider the dual form \(I_1\) given in (5.11). As was done in the previous subsections, we divide the sum over \(j\) as follows. For some sufficiently large constant \(L > 0\),

\[
I_1 = \sum_{k_0 \in \mathbb{N}_0} \left\{ \sum_{j > 1: (1-\rho) \leq k_0 + L} + \sum_{j > 1: (1-\rho) > k_0 + L} \right\} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{v \in (\mathbb{Z}^n)^2} I_{j,k,v}^{(1)}
\]

\[
=: I^{(1,1)} + I^{(1,2)}
\]

with

\[
I_{j,k,v}^{(1)} = 2^{-j\rho n} \int_{\mathbb{R}^n} T_{\sigma_{j,k,v}^\rho} (\square_{v_1} f_j^{(1)}, \square_{v_2} g_j^{(1)}) (x) h_j(x) \, dx.
\]

7.1.1 Estimate for \(I^{(1,1)}\) in (7.1)

The sum over \(v\) of \(I^{(1,1)}\) is restricted to

\[
v_1, v_2 \in \Lambda_j = \left\{ v \in \mathbb{Z}^n : |v| \leq 2^{j(1-\rho)} \right\}
\]

in view of the factors \(\square_{v_1} f_j^{(1)}\) and \(\square_{v_2} g_j^{(1)}\) and (5.13). Hence, \(I^{(1,1)}\) is given by

\[
I^{(1,1)} = \sum_{k_0 \in \mathbb{N}_0} \sum_{j: (1-\rho) \leq k_0 + L} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{v \in \Lambda_j \times \Lambda_j} I_{j,k,v}^{(1)}
\]

By Lemma 4.4 (2) with \(p = r = 2\) and \(q = \infty\), we have

\[
\left| I^{(1,1)} \right| \lesssim \sum_{k_0 \in \mathbb{N}_0} \sum_{j: (1-\rho) \leq k_0 + L} \sum_{k_1, k_2 \in \mathbb{N}_0} \times 2^{-j\rho n} 2^{j(1-\rho)n/2} 2^{j(1-\rho)n/2} 2^{(k_1+k_2)n/2} \| \Delta_k [\sigma_{j,v}^\rho] \|_{L^2_n} \| f_j^{(1)} \|_{L^2} \| g_j^{(1)} \|_{L^\infty} \| h_j \|_{L^2}.
\]

Here, \(\| f_j^{(1)} \|_{L^2} \lesssim 2^{j\rho n/2} \| f \|_{L^2}\) and \(\| h_j \|_{L^2} = 2^{j\rho n/2} \| h \|_{L^2}\) hold. Moreover, it holds from Lemma 2.2 (2) that

\[
\| g_j^{(1)} \|_{L^\infty} = \| \psi_j^{(D)} g(2^{-j\rho}. \|_{L^\infty} \lesssim \| g \|_{BMO}.
\]
Hence, since \( \sum j : j(1 - \rho) \leq k_0 + L \ 2^{j(1 - \rho)n/2} \lesssim 2^{k_0 n/2} \), we have

\[
\left| I^{(1,1)} \right| \lesssim \|f\|_{L^2} \|g\|_{BMO} \|h\|_{L^2} \\
\times \sum_{k_0 \in \mathbb{N}_0} \sum_{j : j(1 - \rho) \leq k_0 + L} 2^{j(1 - \rho)n/2} \\
\sup_{j \in \mathbb{N}_0} \left\{ \sum_{k_1, k_2 \in \mathbb{N}_0} 2^{j(1 - \rho)n/2} 2^{(k_1 + k_2)n/2} \|\Delta_k[\sigma_j^\rho]\|_{L^2_{\text{al}}} \right\} \\
\lesssim \|\sigma\|_{B_{\rho}^s_{\rho,\rho}(\mathbb{R};\mathbb{R}^n)} \|f\|_{L^2} \|g\|_{BMO} \|h\|_{L^2},
\]

where \( m = -(1 - \rho)n/2 \) and \( s = (n/2, n/2, n/2) \).

### 7.1.2 Estimate for \( I^{(1,2)} \) in (7.1)

We follow the same ideas as in Sect. 6.1.

We use a Littlewood–Paley partition on \( \mathbb{R}^n \) to decompose \( \Box v_j f_j^{(1)} \) as \( \sum_{\ell} \Box v_j \Delta_\ell f_j^{(1)} \). Then, the sum over \( \nu \) is restricted to \( \nu_1 \in \Lambda_{1,\ell} = \{ \nu_1 \in \mathbb{Z}^n : |\nu_1| \leq 2^\ell \} \) and \( \nu_2 \in \Lambda_{2,\ell} = \{ \nu_2 \in \mathbb{Z}^n : |\nu_2| \leq 2^{j(1 - \rho)} \} \) (see (6.4)), and the sum over \( \ell \) is restricted to \( \ell \leq j(1 - \rho) \) (see (6.5)). Moreover, recall (6.16) and take a function \( \psi^{\frac{\kappa}{2}} \in S(\mathbb{R}^n) \) such that \( \psi^{\frac{\kappa}{2}} = 1 \) on \( \{ \xi \in \mathbb{R}^n : 2^{-6} \leq |\xi| \leq 2^4 \} \) and supp \( \psi^{\frac{\kappa}{2}} \subset \{ \xi \in \mathbb{R}^n : 2^{-7} \leq |\xi| \leq 2^5 \} \). Then, \( I^{(1,2)} \) can be expressed as

\[
I^{(1,2)} = \sum_{k_0 \in \mathbb{N}_0} \sum_{j : j(1 - \rho) > k_0 + L} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{\ell : \ell \leq j(1 - \rho)} \sum_{\nu \in \Lambda_{1,\ell} \times \Lambda_{2,\ell}} \\
\times 2^{-j\rho n} \int_{\mathbb{R}^n} T_{\sigma_j^\rho,\nu,\nu} \Box v_1 \Delta_\ell f_j^{(1)}, \Box v_2 g_j^{(1)}(x) \psi^{\frac{\kappa}{2}}(D/2^{j(1 - \rho)})[h_j](x) \, dx.
\]

We shall estimate this new \( I^{(1,2)} \). We use Lemma 4.4 (1) with \( p = r = 2 \) and \( q = \infty \) to the sum over \( \nu \), and use the fact \( \min(|\Lambda_{1,\ell}|, |\Lambda_{2,\ell}|) \lesssim 2^{\ell n} \) to have

\[
\left| I^{(1,2)} \right| \lesssim \sum_{k_0 \in \mathbb{N}_0} \sum_{j(1 - \rho) > k_0 + L} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{\ell : \ell \leq j(1 - \rho)} 2^{-j\rho n/2} 2^{(k_0 + k_1 + k_2)n/2} \\
\times \|\Delta_k[\sigma_j^\rho]\|_{L^2_{\text{al}}} \|\Delta_\ell f_j^{(1)}\|_{L^2} \|g_j^{(1)}\|_{L^\infty} \|\psi^{\frac{\kappa}{2}}(D/2^{j(1 - \rho)})[h_j]\|_{L^2}.
\]

Here, \( \|\Delta_\ell f_j^{(1)}\|_{L^2} \lesssim 2^{j\rho n/2}\|\Delta_\ell f\|_{L^2} \) and \( \|g_j^{(1)}\|_{L^\infty} \lesssim \|g\|_{BMO} \) hold (see (6.8) and (7.2), respectively). Also, \( \|\psi^{\frac{\kappa}{2}}(D/2^{j(1 - \rho)})[h_j]\|_{L^2} = 2^{j\rho n/2}\|\psi^{\frac{\kappa}{2}}(D/2^{j})h\|_{L^2} \).
7.2 Estimate for $l_2$

As in the previous sections, we divide the sum of $I_2$ defined in (5.15) as follows. For some sufficiently large constants $M > 0$ and $C > 0$,

$$I_2 = \sum_{j > 1} \sum_{k \in (\mathbb{N}_0)^3} \sum_{v_1 \in \mathbb{Z}^n} I_{j,k,v}^{(2)}$$

$$+ \sum_{k_0 \in \mathbb{N}_0} \sum_{j > 1} \sum_{k_1,k_2 \in \mathbb{N}_0} \sum_{v_1 \in \mathbb{Z}^n} I_{j,k,v}^{(2)}$$

$$+ \sum_{k_0 \in \mathbb{N}_0} \sum_{j > 1} \sum_{k_1,k_2 \in \mathbb{N}_0} \sum_{v_1 \in \mathbb{Z}^n} I_{j,k,v}^{(2)}$$

$$= I^{(2,1)} + I^{(2,2)} + I^{(2,3)}$$

with

$$I_{j,k,v}^{(2)} = 2^{-j\rho n} \int_{\mathbb{R}^n} T_{\sigma_f^{(2)}} \left( \Box_{v_1} f_j^{(2)}, \Box_{v_2} g_j^{(2)} \right) (x) h_j(x) \, dx.$$ 

7.2.1 Estimate for $I^{(2,1)}$ in (7.3)

We write $\Lambda_2 = \{v_2 \in \mathbb{Z}^n : |v_2| \leq C \}$ and observe that $\min(|\mathbb{Z}^n|, |\Lambda_2|) \lesssim 1$. Then, by Lemma 4.4 (1) with $p = r = 2$ and $q = \infty$

$$\left| I^{(2,1)} \right| \lesssim \sum_{j > 1} \sum_{k \in (\mathbb{N}_0)^3} 2^{-j\rho n} 2^{(k_0+k_1+k_2)n/2} \| \Delta_k [\sigma_f^\rho] \|_{L^2_w} \| f_j^{(2)} \|_{L^2} \| g_j^{(2)} \|_{L^\infty} \| h_j \|_{L^2}.$$ 

Here, we have $\| f_j^{(2)} \|_{L^2} \lesssim 2^{j\rho n/2} \| f \|_{L^2}$ and $\| h_j \|_{L^2} = 2^{j\rho n/2} \| h \|_{L^2}$ by (5.16), and

$$\| g_j^{(2)} \|_{L^\infty} = \| \phi_j^{(2)} (D) g(2^{-j\rho} \cdot) \|_{L^\infty} \lesssim (1 + j) \| g \|_{BMO}.$$
by Lemma 2.2 (1). Hence, recalling the assumption $0 \leq \rho < 1$, we obtain

$$
|I^{(2,1)}| \lesssim \|\sigma\|_{BS^{m,s}_{\rho,\rho}(\mathbb{R}^n)} \|f\|_{L^2} \|g\|_{BMO} \|h\|_{L^2} \sum_{j \gg 1} (1 + j) 2^{-j(1-\rho)n/2}
$$

$$
\lesssim \|\sigma\|_{BS^{m,s}_{\rho,\rho}(\mathbb{R}^n)} \|f\|_{L^2} \|g\|_{BMO} \|h\|_{L^2},
$$

where $m = -(1-\rho)n/2$ and $s = (n/2, n/2, n/2)$.

### 7.2.2 Estimate for $I^{(2,2)}$ in (7.3)

The sum over $\nu$ is restricted to

$$
v_1 \in \left\{ v_1 \in \mathbb{Z}^n : |v_1| \lesssim 2^{j(1-\rho)} \right\} \quad \text{and} \quad v_2 \in \left\{ v_2 \in \mathbb{Z}^n : C < |v_2| \lesssim 2^{j(1-\rho)} \right\}.
$$

Denote by $\Lambda_{2,j}$ the set to which $v_2$ belongs. Then, we have by Lemma 4.4 (2) with $p = r = 2$ and $q = \infty$

$$
|I^{(2,2)}| \lesssim \sum_{k_0 \in \mathbb{N}_0} \sum_{j : j(1-\rho) \leq k_0 + M} \sum_{k_1, k_2 \in \mathbb{N}_0} 2^{-j\rho n} 2^{j(1-\rho)n/2} 2^{j(1-\rho)n/2} 2^{(k_1 + k_2)n/2}
$$

$$
\times \|\Delta_k [\sigma_j]\|_{L^2_d} \|f_j^{(2)}\|_{L^2} \|\square_{v_2} g_j^{(2)}\|_{\ell^2_{v_2}(\Lambda_{2,j})L^\infty} \|h_j\|_{L^2}.
$$

Here, we have $\|f_j^{(2)}\|_{L^2} \lesssim 2^{j\rho n/2} \|f\|_{L^2}$ and $\|h_j\|_{L^2} = 2^{j\rho n/2} \|h\|_{L^2}$. Moreover, we have

$$
\|\square_{v_2} g_j^{(2)}\|_{\ell^2_{v_2}(\Lambda_{2,j})L^\infty} \lesssim \|g\|_{BMO}. \quad (7.4)
$$

To see this, we observe that

$$
\square_{v_2} g_j^{(2)}(x) = \kappa(D - v_2) \left[ \phi_j'(D) g(2^{-j\rho \cdot}) \right](x)
$$

$$
= \kappa(D - v_2) \phi'(D/2^{j(1-\rho)}) [g(2^{-j\rho \cdot})](x)
$$

and also that $\kappa(-v_2) = 0$ for $v_2 \in \Lambda_{2,j}$, since supp $\kappa \subset [-1, 1]^n$. Then, by Corollary 4.3 and the scaling invariance of the space $BMO$, we have

$$
\|\square_{v_2} g_j^{(2)}\|_{\ell^2_{v_2}(\Lambda_{2,j})L^\infty} \lesssim \|g(2^{-j\rho \cdot})\|_{BMO} = \|g\|_{BMO}.
$$
Therefore, since \( \sum j: j(1 - \rho) \leq k_0 + M 2^{j(1 - \rho)n/2} \lesssim 2^{k_0 n/2} \), we obtain

\[
\left| I^{(2,2)} \right| \lesssim \| f \|_{L^2} \| g \|_{BMO} \| h \|_{L^2} 
\times \sum_{k_0 \in \mathbb{N}_0} \sum_{j: j(1 - \rho) \leq k_0 + M} 2^{j(1 - \rho)n/2} 
\sup_{j \in \mathbb{N}_0} \left\{ \sum_{k_1, k_2 \in \mathbb{N}_0} 2^{j(1 - \rho)n/2} 2^{(k_1 + k_2)n/2} \| \Delta_k [\sigma_j^{\rho}] \|_{L^2} \right\}
\lesssim \| \sigma \|_{BS^{m,s}_{\rho,\rho} (\mathbb{R}^n)} \| f \|_{L^2} \| g \|_{BMO} \| h \|_{L^2},
\]

where \( m = -(1 - \rho)n/2 \) and \( s = (n/2, n/2, n/2) \).

### 7.2.3 Estimate for \( I^{(2,3)} \) in (7.3)

The sum over \( \nu \) in \( I^{(2,3)} \) is restricted to

\[
\nu_1 \in \Lambda_{1,j} = \{ \nu_1 \in \mathbb{Z}^n : |\nu_1| \lesssim 2^{j(1 - \rho)} \}, \\
\nu_2 \in \Lambda_{2,j} = \{ \nu_2 \in \mathbb{Z}^n : C < |\nu_2| \lesssim 2^{j(1 - \rho)} \}.
\]

Moreover, in the sum over \( j \) such that \( j(1 - \rho) > k_0 + M \), we have

\[
\text{supp } \mathcal{F} \left[ T_{\sigma_j^{\rho}, \nu} \left( \Box \nu_1 f^{(2)}_j, \Box \nu_2 g^{(2)}_j \right) \right] 
\subset \{ \xi \in \mathbb{R}^n : |\xi - (\xi + \eta)| \lesssim 2^{k_0 + 1}, \xi \in \text{supp } \Box \nu_1 f^{(2)}_j, \eta \in \text{supp } \Box \nu_2 g^{(2)}_j \}
\subset \{ \xi \in \mathbb{R}^n : 2^{j(1 - \rho) - 6} \lesssim |\xi| \lesssim 2^{j(1 - \rho) + 4} \},
\]

recalling from (5.18) that \( 2^{j(1 - \rho) - 5} \lesssim |\xi + \eta| \lesssim 2^{j(1 - \rho) + 3} \) holds for \( \xi \in \text{supp } f^{(2)}_j \) and \( \eta \in \text{supp } g^{(2)}_j \). Hence, by taking a function \( \psi^\check{\cdot} \in \mathcal{S}(\mathbb{R}^n) \) such that \( \psi^\check{\cdot} = 1 \) on \( \{ \xi \in \mathbb{R}^n : 2^{-6} \leq |\xi| \leq 2^4 \} \) and \( \text{supp } \psi^\check{\cdot} \subset \{ \xi \in \mathbb{R}^n : 2^{-7} \leq |\xi| \leq 2^5 \} \), we have

\[
I^{(2,3)} = \sum_{k_0 \in \mathbb{N}_0} \sum_{j: j(1 - \rho) > k_0 + M} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{\nu \in \Lambda_{1,j} \times \Lambda_{2,j}} 
\times 2^{-j \rho n} \int_{\mathbb{R}^n} T_{\sigma_j^{\rho}, \nu} \left( \Box \nu_1 f^{(2)}_j, \Box \nu_2 g^{(2)}_j \right)(x) \psi^\check{\cdot} (D/2^{j(1 - \rho)}) [h_j](x) \, dx.
\]
Now, we shall estimate $I^{(2,3)}$ using the above formula. Using Lemma 4.4 with $p = r = 2$ and $q = \infty$ and recalling the estimate (7.4), we have

$$
|I^{(2,3)}| \lesssim \sum_{k_0 \in \mathbb{N}_0} \sum_{j \gg 1} \sum_{k_1, k_2 \in \mathbb{N}_0} 2^{-j \rho n} 2^{j(1-\rho)n/2} 2^{(k_0 + k_1 + k_2)n/2} \times \| \Delta_k [\sigma_j^p]\|_{L^2_{ul}} \| f_j^{(2)} \|_{L^2} \| \Box_v g_j^{(2)} \|_{L^2(A_{2,j})} \| \psi_\delta(D/2^{j(1-\rho)})[h_j] \|_{L^2} \\
\lesssim \| \sigma \|_{B_{p,p} \epsilon \mathbb{R}^n} \| g \|_{BMO} \sum_{j \gg 1} \| \psi_j(D) f \|_{L^2} \| \psi_j(D/2^j) h \|_{L^2} \\
\lesssim \| \sigma \|_{B_{p,p} \epsilon \mathbb{R}^n} \| f \|_{L^2} \| g \|_{BMO} \| h \|_{L^2},
$$

where $m = -(1 - \rho)n/2$ and $s = (n/2, n/2, n/2)$.

### 7.3 Estimate for $I_3$

As in Sect. 6.3, we split the sum into two parts and slightly change the way to sum over $\nu$. For some sufficiently large $N > 0$,

$$
I_3 = \sum_{\nu \in \mathbb{Z}^n} I^{(3)}_{\nu} \\
= \sum_{k_0 \in \mathbb{N}_0} \sum_{j \gg 1} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{\nu \in \mathbb{Z}^n} I^{(3)}_{j, k, \nu} \\
+ \sum_{k_0 \in \mathbb{N}_0} \sum_{j \gg 1} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{\nu \in \mathbb{Z}^n} \sum_{\nu_1 + \nu_2 = \tau} I^{(3)}_{j, k, \nu}
$$

(7.5)

with

$$
I^{(3)}_{j, k, \nu} = 2^{-j \rho n} \int_{\mathbb{R}^n} T_{\sigma_j^p} (\Box_{\nu_1} f_j^{(3)} \Box_{\nu_2} g_j^{(3)})(x) h_j(x) \, dx.
$$

#### 7.3.1 Estimate of $I^{(3,1)}$ in (7.5)

The sum over $\nu$ is restricted to $\nu_1, \nu_2 \in \{ \nu \in \mathbb{Z}^n : |\nu| \lesssim 2^j(1-\rho) \}$. Then, by Lemma 4.4 (2) with $p = r = 2$ and $q = \infty$, we have

$$
|I^{(3,1)}| \lesssim \sum_{k_0 \in \mathbb{N}_0} \sum_{j : j(1-\rho) \leq k_0 + N} \sum_{k_1, k_2 \in \mathbb{N}_0} 2^{-j \rho n} 2^{j(1-\rho)n/2} 2^{k_1} 2^{(k_1 + k_2)n/2} \| \Delta_k [\sigma_j^p]\|_{L^2_{ul}} \| f_j^{(3)} \|_{L^2} \| g_j^{(3)} \|_{L^\infty} \| h_j \|_{L^2}.
$$
Hence, since $\|g_j^{(3)}\|_{L^\infty} \lesssim \|g\|_{BMO}$ from Lemma 2.2 (2), we obtain

$$\left| I^{(3,1)} \right| \lesssim \|f\|_{L^2} \|g\|_{BMO} \|h\|_{L^2} \times \sum_{k_0 \in \mathbb{N}_0} \sum_{j: j(1-\rho) \leq k_0 + N} 2^{j(1-\rho)n/2} \sup_{j \in \mathbb{N}_0} \left\{ \sum_{k_1, k_2 \in \mathbb{N}_0} 2^{j(1-\rho)n/2} 2^{(k_1 + k_2)n/2} \|\Delta_k [\sigma_j^\rho]\|_{L^2} \right\} \lesssim \|\sigma\|_{BS^{m,\ast}_{\rho,\rho}(s; \mathbb{R}^n)} \|f\|_{L^2} \|g\|_{BMO} \|h\|_{L^2},$$

where $m = -(1 - \rho)n/2$ and $s = (n/2, n/2, n/2)$. 

### 7.3.2 Estimate of $I^{(3,2)}$ in (7.5)

Let $\{\psi_\ell\}_{\ell \in \mathbb{N}_0}$ be the Littlewood–Paley partition of unity on $\mathbb{R}^n$. Then, it holds that

$$\sum_{\ell \in \mathbb{N}_0} \psi_\ell(\zeta/2^{k_0}) = 1, \quad \zeta \in \mathbb{R}^n, \quad k_0 \in \mathbb{N}_0.$$

By using this, we further decompose $I^{(3,2)}$ as follows:

$$I^{(3,2)} = \sum_{k_0 \in \mathbb{N}_0} \sum_{j: j(1-\rho) > k_0 + N} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{\ell \in \mathbb{N}_0} \sum_{\tau \in \mathbb{Z}^n} \sum_{v_1 + v_2 = \tau} I^{(3)}_{j, k, \ell, v}(3),$$

with

$$I^{(3)}_{j, k, \ell, v} = 2^{-j\rho n} \int_{\mathbb{R}^n} T_{(j, k, \ell, v)} \big(\Box_v f_j^{(3)}, \Box_v g_j^{(3)}\big)(x) \psi_\ell(D/2^{k_0})[h_j](x) \, dx.$$

As in the previous sections, observe that

$$\text{supp} \mathcal{F} \left[ T_{(j, k, \ell, v)} \big(\Box_v f_j^{(3)}, \Box_v g_j^{(3)}\big) \right] \subseteq \left\{ \zeta \in \mathbb{R}^n : |\zeta - \tau| \lesssim 2^{k_0} \right\}$$

and take a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\varphi = 1$ on $\{ \zeta \in \mathbb{R}^n : |\zeta| \lesssim 1 \}$. Then, $I^{(3)}_{j, k, \ell, v}$ can be rewritten as

$$I^{(3)}_{j, k, \ell, v} = 2^{-j\rho n} \int_{\mathbb{R}^n} T_{(j, k, \ell, v)} \big(\Box_v f_j^{(3)}, \Box_v g_j^{(3)}\big)(x) \varphi \left( \frac{D + \tau}{2^{k_0}} \right) \psi_\ell(D/2^{k_0})[h_j](x) \, dx.$$

(7.6)
We next investigate restrictions of $\ell$ and $\tau$ by considering the multipliers acting on $h_j$. Observe that $|\nu_1|, |\nu_2| \lesssim 2^{j(1-\rho)}$ with $\nu_2 = \tau - \nu_1$. Then, since $j(1-\rho) > k_0 + N$,

$$\text{supp } \phi \left( \frac{\cdot + \tau}{2^{k_0}} \right) \subset \left\{ \zeta \in \mathbb{R}^n : |\zeta + \tau| \lesssim 2^{k_0} \right\} \quad (7.7)$$

$$\subset \left\{ \zeta \in \mathbb{R}^n : |\zeta| \lesssim 2^{j(1-\rho)} \right\}. \quad (7.8)$$

Furthermore,

$$\text{supp } \psi_0(\cdot/2^{k_0}) \subset \left\{ \zeta \in \mathbb{R}^n : |\zeta| \leq 2^{k_0+1} \right\}, \quad \text{if } \ell = 0,$n

$$\text{supp } \psi_\ell(\cdot/2^{k_0}) \subset \left\{ \zeta \in \mathbb{R}^n : 2^{k_0+\ell-1} \leq |\zeta| \leq 2^{k_0+\ell+1} \right\}, \quad \text{if } \ell \geq 1. \quad (7.9)$$

From (7.8) and (7.9), the sum over $\ell$ is restricted to

$$\ell \in \Omega_{j,k_0} = \{ \ell \in \mathbb{N}_0 : \ell \leq j(1-\rho) - k_0 + C \}$$

with a suitable constant $C > 0$ depending on dimensions. From (7.7) and (7.9), the sum over $\tau$ is restricted to

$$\tau \in \Lambda_{k_0,\ell} = \left\{ \tau \in \mathbb{Z}^n : |\tau| \lesssim 2^{k_0+\ell} \right\}.$$

Therefore, with $I_{j,k,\ell,\nu}^{(3)}$ newly given in (7.6), $I^{(3,2)}$ can be rewritten by

$$I^{(3,2)} = \sum_{k_0 \in \mathbb{N}_0} \sum_{j : j(1-\rho) > k_0 + N} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{\ell \in \Omega_{j,k_0}} \sum_{\tau \in \Lambda_{k_0,\ell}} \sum_{\nu_1, \nu_2} I_{j,k,\ell,\nu}^{(3)}.$$

We shall estimate this new $I^{(3,2)}$. By Lemma 4.4 (3) with $p = r = 2$ and $q = \infty$

$$\left| I^{(3,2)} \right| \lesssim \sum_{k_0 \in \mathbb{N}_0} \sum_{j : j(1-\rho) > k_0 + N} \sum_{k_1, k_2 \in \mathbb{N}_0} \sum_{\ell \in \Omega_{j,k_0}} \sum_{\tau \in \Lambda_{k_0,\ell}} 2^{-j \rho n} 2^{(k_0+\ell)n/2} 2^{(k_1+k_2)n/2} \times \| \Delta_k \left[ \sigma_j^{\rho} \right] \|_{L^2_{ul}} \| f_j^{(3)} \|_{L^2} \| g_j^{(3)} \|_{L^\infty} \left\| \phi \left( \frac{D + \tau}{2^{k_0}} \right) \psi_\ell(D/2^{k_0})[h_j] \right\|_{\ell^2(\Lambda_{k_0,\ell}) L^2}.$$

Here, $\| g_j^{(3)} \|_{L^\infty} \lesssim \| g \|_{BM} \otimes \phi$ hold from Lemma 2.2 (2). Moreover, we have by Lemma 4.2 (1) and changes of variable

$$\left\| \phi \left( \frac{D + \tau}{2^{k_0}} \right) \psi_\ell(D/2^{k_0})[h_j] \right\|_{\ell^2(\mathbb{Z}^n) L^2} \lesssim 2^{k_0n/2} \| \psi_\ell(D/2^{k_0})[h_j] \|_{L^2}$$

$$= 2^{k_0n/2} 2^{j \rho n/2} \| \psi_\ell(D/2^{k_0+j\rho}) h \|_{L^2}.$$
Hence, by denoting the operators $\psi''_j(D)$ by $\Delta''_j$ and $\psi_\ell(D/2^{k_0+j})$ by $\Delta_{\ell+k_0+j}$,

$$
|I^{(3,2)}| \lesssim \sum_{k_0 \in \mathbb{N}_0} \sum_{j: j(1-\rho)>k_0+N} \sum_{k_1,k_2 \in \mathbb{N}_0} \sum_{\ell \in \Omega_{j,k_0}} 2^{(k_0+\ell)n/2} 2^{(k_0+k_1+k_2)n/2} \\
\times \|\Delta k[\sigma_j^\rho]\|_{L^2_{ul}} \|\Delta''_j f\|_{L^2} \|g\|_{BMO} \|\Delta_{\ell+k_0+j} h\|_{L^2} \\
\lesssim \|g\|_{BMO} \sum_{k_0 \in \mathbb{N}_0} \sup_{j \in \mathbb{N}_0} \left\{ \sum_{k_1,k_2 \in \mathbb{N}_0} 2^{j(1-\rho)n/2} 2^{(k_0+k_1+k_2)n/2} \|\Delta k[\sigma_j^\rho]\|_{L^2_{ul}} \right\} \\
\times \sum_{j: j(1-\rho)>k_0+N} \sum_{\ell \in \Omega_{j,k_0}} 2^{(k_0+\ell)n/2} 2^{-j(1-\rho)n/2} \|\Delta''_j f\|_{L^2} \|\Delta_{\ell+k_0+j} h\|_{L^2} \\
\lesssim \|\sigma\|_{\mathcal{B}S_{m,s}^{m,s}(\mathbb{R};\mathbb{R}^n)} \|g\|_{BMO} \sup_{k_0 \in \mathbb{N}_0} \mathcal{M}_{k_0} 
$$

(7.10)

with $m = -(1-\rho)n/2$, $s = (n/2, n/2, n/2)$, and

$$
\mathcal{M}_{k_0} = \sum_{j: j(1-\rho)>k_0+N} \sum_{\ell \in \Omega_{j,k_0}} 2^{(k_0+\ell)n/2} 2^{-j(1-\rho)n/2} \|\Delta''_j f\|_{L^2} \|\Delta_{\ell+k_0+j} h\|_{L^2}.
$$

We shall estimate $\mathcal{M}_{k_0}$ and prove that this is bounded by a constant times $\|f\|_{L^2} \|h\|_{L^2}$ for all $k_0 \in \mathbb{N}_0$. The way to estimate $\mathcal{M}_{k_0}$ is almost the same as was done to have (6.12). We divide $\mathcal{M}_{k_0}$ into the two parts $\ell = 0$ and $\ell \geq 1$. That is,

$$
\mathcal{M}_{k_0} = \mathcal{M}_{k_0}^{\ell=0} + \mathcal{M}_{k_0}^{\ell \geq 1}
$$

(7.11)

with

$$
\mathcal{M}_{k_0}^{\ell=0} = \sum_{j: j(1-\rho)>k_0+N} 2^{k_0n/2} 2^{-j(1-\rho)n/2} \|\Delta''_j f\|_{L^2} \|\psi_0(D/2^{k_0+j})h\|_{L^2},
$$

$$
\mathcal{M}_{k_0}^{\ell \geq 1} = \sum_{j: j(1-\rho)>k_0+N} \sum_{\ell \in \mathbb{N}_0 \cap \Omega_{j,k_0}} 2^{(k_0+\ell)n/2} 2^{-j(1-\rho)n/2} \|\Delta''_j f\|_{L^2} \|\psi_\ell(D/2^{k_0+j})h\|_{L^2}.
$$

For the first sum $\mathcal{M}_{k_0}^{\ell=0}$, we have

$$
\mathcal{M}_{k_0}^{\ell=0} \lesssim \|f\|_{L^2} \|h\|_{L^2} \sum_{j: j(1-\rho)>k_0+N} 2^{k_0n/2} 2^{-j(1-\rho)n/2} \lesssim \|f\|_{L^2} \|h\|_{L^2}.
$$

(7.12)

We next consider the second sum $\mathcal{M}_{k_0}^{\ell \geq 1}$. We take a function $\psi^\dagger \in \mathcal{S}(\mathbb{R}^n)$ such that

$$
\psi^\dagger = 1 \text{ on } \{ \xi \in \mathbb{R}^n : 1/4 \leq |\xi| \leq 4 \} \text{ and } \supp \psi^\dagger \subset \{ \xi \in \mathbb{R}^n : 1/8 \leq |\xi| \leq 8 \}.
$$

Then, by writing as $\Delta^\dagger_{\ell+k_0+j} = \psi^\dagger(D/2^{\ell+k_0+j})$, we have

$$
\|\psi_\ell(D/2^{k_0+j})h\|_{L^2} = \|\Delta^\dagger_{\ell+k_0+j} \psi_\ell(D/2^{k_0+j})h\|_{L^2} \lesssim \|\Delta^\dagger_{\ell+k_0+j} h\|_{L^2}.
$$
From this and the fact $2^{j\rho} \approx 2^{[j\rho]}$, it holds that

$$
\|f\|_{L^2} \lesssim \sum_{j \geq 1} 2^{-jn/2} \|\Delta_j f\|_{L^2} \|\Delta_j h\|_{L^2}.
$$

Changing the sum of $\ell$ as $\ell \to \ell'$, we have by Lemma 2.1

$$
\|f\|_{L^2} \lesssim \sum_{j \geq 1} 2^{\ell' n/2} 2^{-jn/2} \|\Delta_j f\|_{L^2} \|\Delta_j h\|_{L^2} \lesssim \|f\|_{L^2} \|h\|_{L^2}.
$$

(7.13)

Lastly, collecting (7.10), (7.11), (7.12), and (7.13), we obtain that

$$
|I_{(3,2)}| \lesssim \|\sigma\|_{BS_{m,\rho,s}^*(\mathbb{R}^n)} \|f\|_{L^2} \|g\|_{BMO} \|h\|_{L^2},
$$

where $m = -(1 - \rho)n/2$ and $s = (n/2, n/2, n/2)$.

7.4 Estimate for $I_0$

In this section, we consider $I_0$ in (5.9). Considering $\Box_v f_j$ and $\Box_v g_j$, we see from (5.10) that $v_1, v_2 \in \mathbb{Z}^n$ satisfy that $|v_1|, |v_2| \lesssim 1$, since $j \lesssim 1$. Hence, we have by Lemma 4.4 (2) with $p = r = 2$, $r = \infty$ and Lemma 2.2 (1),

$$
|I_0| \lesssim \sum_{j \lesssim 1} \sum_{k \in (N_0)^3} \sum_{|v_1|, |v_2| \lesssim 1} 2^{-j\rho n} \left| \int_{\mathbb{R}^n} T_{\sigma, j, k, v} (\Box_v f_j, \Box_v g_j)(x) h_j(x) \, dx \right|
\lesssim \sum_{j \lesssim 1} \sum_{k \in (N_0)^3} 2^{-j\rho n} 2^{(k_1 + k_2)n/2} \|\Delta_k [\sigma_j^\rho]\|_{L^2} \|f_j\|_{L^2} \|g_j\|_{L^\infty} \|h_j\|_{L^2}
\lesssim \|\sigma\|_{BS_{m,\rho,s}^*(\mathbb{R}^n)} \|f\|_{L^2} \|g\|_{BMO} \|h\|_{L^2},
$$

where $m = -(1 - \rho)n/2$ and $s = (0, n/2, n/2)$.

8 Sharpness of Theorem 1.1

In this section, we consider the sharpness of the conditions of the order $m$ and the smoothness $s = (s_0, s_1, s_2)$ stated in Theorem 1.1.

8.1 Sharpness of the Order $m$

In this subsection, we show the following.
Proposition 8.1 Let \(0 \leq \rho < 1, m \in \mathbb{R}, s = (s_0, s_1, s_2) \in [0, \infty)^3, \) and \(1 \leq r \leq 2 \leq p, q \leq \infty\) satisfy \(1/p + 1/q = 1/r.\) If all bilinear pseudo-differential operators with symbols in \(BS^m_{\rho, \rho}(s; \mathbb{R}^n)\) are bounded from \(L^p \times L^q \) to \(L^r,\) then \(m \leq -(1 - \rho)n/2.\)

This is immediately obtained by the fact that \(BS^m_{\rho, \rho}(\mathbb{R}^n) \subset BS^m_{\rho, \rho}(s; \mathbb{R}^n), s \in [0, \infty)^3,\) from Lemma 3.5 and the following theorem proved by Miyachi–Tomita [22, Theorem A.2].

Theorem 8.2 Let \(0 \leq \rho < 1, m \in \mathbb{R},\) and \(1 \leq r \leq 2 \leq p, q \leq \infty\) satisfy \(1/p + 1/q = 1/r.\) If all bilinear pseudo-differential operators with symbols in \(BS^m_{\rho, \rho}(\mathbb{R}^n)\) are bounded from \(L^p \times L^q \) to \(L^r,\) then \(m \leq -(1 - \rho)n/2.\)

8.2 Sharpness of the Smoothness \(s_1\) and \(s_2\)

In this subsection, we show the following. The idea of the proof comes from Miyachi–Tomita [23, Sect. 7].

Proposition 8.3 Let \(0 \leq \rho < 1, m \in \mathbb{R}, s = (s_0, s_1, s_2) \in (0, \infty)^3,\) and \(1 \leq p, q, r \leq \infty\) satisfy \(1/p + 1/q = 1/r.\) Suppose that the inequality

\[
\|T_\sigma(f, g)\|_{L^r} \lesssim \|\sigma\|_{BS^m_{\rho, \rho}(s; \mathbb{R}^n)} \|f\|_{L^p} \|g\|_{L^q}
\]

holds for all \(\sigma \in BS^m_{\rho, \rho}(s; \mathbb{R}^n)\) and \(f, g \in \mathcal{S}(\mathbb{R}^n).\) Then, \(s_1, s_2 \geq n/2.\)

We will use the following fact. See, e.g., [30, Proposition 1.1 (i)].

Lemma 8.4 Let \(1 \leq p, q \leq \infty\) and \(s > 0.\) Then, we have

\[
\|f(\lambda \cdot)|_{B^s_{p,q}(\mathbb{R}^n)} \lesssim \lambda^{-s/p} \max\{1, \lambda^s\} \|f\|_{B^s_{p,q}(\mathbb{R}^n)}, \quad \lambda > 0.
\]

Proof of Proposition 8.3 It suffices to prove \(s_1 \geq n/2.\) Let \(u, v \in \mathcal{S}(\mathbb{R}^n)\) satisfy that

\[
\text{supp } \hat{u} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 1\}, \quad \text{supp } \hat{v} \subset \{\eta \in \mathbb{R}^n : 9/10 \leq |\eta| \leq 11/10\}, \quad \hat{v} = 1 \quad \text{on} \quad \{\eta \in \mathbb{R}^n : 19/20 \leq |\eta| \leq 21/20\},
\]

and put for \(\varepsilon > 0\)

\[
\sigma(x, \xi, \eta) = \sigma(\xi, \eta) = \hat{u}(\xi/\varepsilon) \hat{v}(\eta), \quad \hat{f}(\xi) = \varepsilon^{n/p-n} \hat{u}(\xi/\varepsilon), \quad \hat{g}(\xi) = \varepsilon^{n/q-n} \hat{u}(\eta - e_1)/\varepsilon)
\]

where \(e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n.\) Then,

\[
\|T_\sigma(f, g)\|_{L^r} \approx 1, \quad \|f\|_{L^p} \approx 1, \quad \text{and} \quad \|g\|_{L^q} \approx 1
\]

for a sufficiently small \(\varepsilon > 0.\) In fact, since \(\hat{v} = 1\) on the support of \(\hat{u}((\cdot - e_1)/\varepsilon)\) by choosing a suitably small \(\varepsilon > 0,\) we have

\[
T_\sigma(f, g)(x) = \varepsilon^{n/p}(u * u)(\varepsilon x) \varepsilon^{n/q} e^{ix_1} u(\varepsilon x),
\]
where \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). Hence, \( \|T_\sigma(f, g)\|_{L^r} \approx 1 \), since \( 1/p + 1/q = 1/r \). The second and third equivalences are obvious.

Next, we let a function \( \Psi_0 \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \) satisfy that for a sufficiently small \( \delta > 0 \)

\[
\text{supp } \Psi_0 \subset \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n : |(\xi, \eta)| \leq 2^{1/2+\delta}\},
\]

\[
\Psi_0 = 1 \quad \text{on } \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n : |(\xi, \eta)| \leq 2^{1/2-\delta}\}.
\]

We put \( \Psi = \Psi_0(2\cdot,2\cdot) \) and \( \Psi_j = \Psi(\cdot/2^j,\cdot/2^j) , j \in \mathbb{N} \). Then, we have

\[
\text{supp } \Psi \subset \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n : 2^{-1/2-\delta} \leq |(\xi, \eta)| \leq 2^{1/2+\delta}\},
\]

\[
\Psi = 1 \quad \text{on } \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n : 2^{-1/2+\delta} \leq |(\xi, \eta)| \leq 2^{1/2-\delta}\},
\]

and \( \sum_{j \in \mathbb{N}_0} \Psi_j = 1 \) on \( \mathbb{R}^n \times \mathbb{R}^n \). This \( \{\Psi_j\}_{j \in \mathbb{N}_0} \) is a Littlewood–Paley partition of unity on \( \mathbb{R}^n \times \mathbb{R}^n \). Now, observe that \( \text{supp } \sigma \subset \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n : |(\xi, \eta)| \leq 2^{1/2-\delta}\} \), choosing \( \varepsilon > 0 \) and \( \delta > 0 \) suitably. Then, we see that

\[
\sigma_j^\rho(\xi, \eta) = \sigma(2^j \rho \xi, 2^j \rho \eta) \Psi_j(2^j \rho \xi, 2^j \rho \eta) = \begin{cases} 
\sigma, & \text{if } j = 0, \\
0, & \text{if } j \neq 0.
\end{cases}
\]

Moreover, since the symbol \( \sigma \) is independent of \( x \), we have

\[
\Delta_k[\sigma_j^\rho](x, \xi, \eta) = \begin{cases} 
\psi_{k_1}(D_\xi)\psi_{k_2}(D_\eta)[\sigma_j^\rho](\xi, \eta), & \text{if } k_0 = 0, \\
0, & \text{if } k_0 \neq 0.
\end{cases}
\]

These two facts mean that

\[
\|\sigma\|_{B_{p',q}(\mathbb{R}^n)} = \sum_{k_1,k_2 \in \mathbb{N}_0} 2^{s_1k_1+s_2k_2} \|\psi_{k_1}(D_\xi)\psi_{k_2}(D_\eta)\sigma(\xi, \eta)\|_{L^2_{ul}(\mathbb{R}^n)^2}. 
\]

Therefore, from the embedding \( L^2 \hookrightarrow L^2_{ul} \) and Lemma 8.4, we obtain

\[
\|\sigma\|_{B_{p',q}(\mathbb{R}^n)} \leq \sum_{k_1,k_2 \in \mathbb{N}_0} 2^{s_1k_1+s_2k_2} \|\psi_{k_1}(D_\xi)\psi_{k_2}(D_\eta)\sigma(\xi, \eta)\|_{L^2(\mathbb{R}^n)^2},
\]

\[
\frac{\|\sigma\|_{B_{p',q}(\mathbb{R}^n)}}{\|\sigma\|_{B_{p',q}(\mathbb{R}^n)}} \approx \varepsilon^{n/2-s_1}
\]

for \( \varepsilon > 0 \) sufficiently small.

Test (8.2) to the assumption (8.1). Then, by (8.3) and (8.4), we have \( \varepsilon^{n/2-s_1} \gtrsim 1 \) for \( \varepsilon > 0 \) sufficiently small. This means that \( s_1 \geq n/2 \).

\[\square\]

### 8.3 Sharpness of the Smoothness \( s_0 \)

In this subsection, we show the following.

\[\Box\] Birkhäuser
Let assertion (1) was stated in [30, Theorem 1.4]. See also [32, Theorem 2.8.2 (i) and Remark 1 in Section 2.8.2]. The two variables version of the assertion (2) was mentioned in [30, Proposition 1.1]. Following the same arguments, one can show the assertion (2) for three variables. Therefore, we omit these proofs.

**Proposition 8.5** Let $0 \leq \rho < 1$, $m = -(1 - \rho)n/2$, $s = (s_0, s_1, s_2) \in (0, \infty)^3$, and $1 \leq p, q, r \leq \infty$ satisfy $1/p + 1/q = 1/r$. Suppose that the inequality

$$
\| T_\sigma (f, g) \|_{L^r} \lesssim \| \sigma \|_{BS^{m,}^0 (s; \mathbb{R}^n)} \| f \|_{L^p} \| g \|_{L^q}
$$

(8.5)

holds for all $\sigma \in BS^{m,}^0 (s; \mathbb{R}^n)$ and $f, g \in S(\mathbb{R}^n)$. Then, $s_0 \geq n/2$.

To prove this, we employ a strategy by Miyachi–Tomita [22, Appendix A]. Define

$$
\| \sigma \|_{BS^{m,}^0 (s; \mathbb{R}^n)} = \sup_{j \in \mathbb{N}_0, k \in (\mathbb{N}_0)^3} 2^{j m + k s} \| \Delta_k [\sigma^p_f] \|_{L^\infty((\mathbb{R}^n)^3)},
$$

with the same notations as in Definition 3.1. Then, we have the following.

**Lemma 8.6** Let $0 < \rho < 1$, $m \in \mathbb{R}$, $s = (s_0, s_1, s_2) \in (0, \infty)^3$, and $1 \leq p, q, r \leq \infty$ satisfy $1/p + 1/q = 1/r$. Suppose that the inequality

$$
\| T_\zeta (f, g) \|_{L^r} \lesssim \| \zeta \|_{BS^{m,}^0 (s; \mathbb{R}^n)} \| f \|_{L^p} \| g \|_{L^q}
$$

(8.6)

holds for all $\zeta \in BS^{m,}^0 (s; \mathbb{R}^n)$ and $f, g \in S(\mathbb{R}^n)$. Then, the inequality

$$
\| T_\sigma (f, g) \|_{L^r} \lesssim \| \sigma \|_{BS^{m,}^0 (s; \mathbb{R}^n)} \| f \|_{L^p} \| g \|_{L^q}
$$

holds for all $\sigma \in BS^{m,}^0 (s; \mathbb{R}^n)$ with $m' < m/(1 - \rho)$ and $f, g \in S(\mathbb{R}^n)$.

To prove this, we will use the following lemma given by Sugimoto [30].

**Lemma 8.7** Let $\{\psi_k\}_{k \in \mathbb{N}_0}$ be a Littlewood–Paley partition of unity on $\mathbb{R}^n$.

1. Let $(s_1, s_2) \in (0, \infty)^2$. Then,

$$
\sup_{k_1, k_2 \in \mathbb{N}_0} 2^{s_1 k_1 + s_2 k_2} \| \psi_{k_1} (D_\xi) \psi_{k_2} (D_\eta) [f_1 f_2] (\xi, \eta) \|_{L^\infty((\mathbb{R}^n)^2)} \lesssim \prod_{i=1, 2} \sup_{k_1, k_2 \in \mathbb{N}_0} 2^{s_1 k_1 + s_2 k_2} \| \psi_{k_1} (D_\xi) \psi_{k_2} (D_\eta) [f_i] (\xi, \eta) \|_{L^\infty((\mathbb{R}^n)^2)}.
$$

2. Let $s = (s_0, s_1, s_2) \in (0, \infty)^3$. Then,

$$
\sup_{k \in (\mathbb{N}_0)^3} 2^{k s} \| \Delta_k [f (\lambda_0 \cdot, \lambda_1 \cdot, \lambda_2 \cdot)] \|_{L^\infty((\mathbb{R}^n)^3)} \lesssim \max \{1, \lambda_0 \} \max \{1, \lambda_1 \} \max \{1, \lambda_2 \} \sup_{k \in (\mathbb{N}_0)^3} 2^{k s} \| \Delta_k f \|_{L^\infty((\mathbb{R}^n)^3)}
$$

for $\lambda_0, \lambda_1, \lambda_2 \in (0, \infty)$. 

The assertion (1) was stated in [30, Theorem 1.4]. See also [32, Theorem 2.8.2 (i) and Remark 1 in Section 2.8.2]. The two variables version of the assertion (2) was mentioned in [30, Proposition 1.1]. Following the same arguments, one can show the assertion (2) for three variables. Therefore, we omit these proofs.
Proof of Lemma 8.6 Assume \( \sigma \in BS_{0,0}^{m',\dagger}(s;\mathbb{R}^n) \) with \( m' < m/(1-\rho) \). Let \( \{\Psi_\ell\}_{\ell \in \mathbb{N}_0} \) be a Littlewood–Paley partition of unity on \( (\mathbb{R}^n)^2 \). Then,

\[
\sigma(x, \xi, \eta) = \sum_{\ell \in \mathbb{N}_0} \sigma(x, \xi, \eta) \Psi_\ell(\xi, \eta) = \sum_{\ell \in \mathbb{N}_0} \sigma(x, \xi, \eta)
\]

(8.7)

with \( \sigma_\ell(x, \xi, \eta) = \sigma(x, \xi, \eta) \Psi_\ell(\xi, \eta) \). For simplicity, we write \( \varrho = 1 - \rho \) for \( 0 < \rho < 1 \). We have by changes of variables

\[
T_\sigma(f, g)(x) = \frac{1}{(2\pi)^n} \int_{(\mathbb{R}^n)^2} e^{i2^{-\varrho}x \cdot (\xi + \eta)} \sigma(x, 2^{-\varrho} \xi, 2^{-\varrho} \eta) \widehat{f}(2^{-\varrho} \cdot)(\xi) \widehat{g}(2^{-\varrho} \cdot)(\eta) \, d\xi \, d\eta = T_\varrho(f_\ell, g_\ell)(2^{-\varrho} x)
\]

(8.8)

with

\[
\varrho_\ell = \sigma_\ell(2^{-\varrho} \cdot, 2^{-\varrho} \cdot, 2^{-\varrho} \cdot), \quad f_\ell = f(2^{-\varrho} \cdot), \quad \text{and} \quad g_\ell = g(2^{-\varrho} \cdot).
\]

Then, \( \varrho_\ell \in BS_{0,0}^{m',\dagger}(s;\mathbb{R}^n) \), and more precisely, the inequality

\[
\| \varrho_\ell \|_{BS_{0,0}^{m',\dagger}(s;\mathbb{R}^n)} \lesssim 2^{\ell(m' - m/(1-\rho))} \| \sigma \|_{BS_{0,0}^{m',\dagger}(s;\mathbb{R}^n)}
\]

(8.9)

holds for any \( \ell \in \mathbb{N}_0 \). We shall prove (8.9). Since

\[
\text{supp } \varrho_\ell(x, \cdot, \cdot) \subset \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n : 2^{\ell/(1-\rho)-1} \leq |(\xi, \eta)| \leq 2^{\ell/(1-\rho)+1}\}
\]

for \( x \in \mathbb{R}^n \), recalling the notation of \( (\varrho_\ell)_j^\rho \):

\[
(\varrho_\ell)_j^\rho(x, \xi, \eta) = \varrho_\ell(2^{-j\varrho} x, 2^{j\varrho} \xi, 2^{j\varrho} \eta) \Psi_j(2^{j\varrho} \xi, 2^{j\varrho} \eta),
\]

we see that \( j \) must be in the set

\[
\Omega_\ell = \left\{ j \in \mathbb{N}_0 : \max \left\{ 0, \frac{\ell}{1-\rho} - 2 \right\} \leq j \leq \frac{\ell}{1-\rho} + 2 \right\}
\]

(otherwise, \( (\varrho_\ell)_j^\rho \) vanishes). Then, we have

\[
\| \varrho_\ell \|_{BS_{0,0}^{m',\dagger}(s;\mathbb{R}^n)} \approx 2^{-\ell m/(1-\rho)} \sup_{j \in \Omega_\ell} \sup_{k \in \mathbb{N}_0^3} 2^{k \cdot s} \left\| \Delta_k \left[ (\varrho_\ell)_j^\rho \right] \right\|_{L^\infty}
\]

(8.10)
for $\ell \in \mathbb{N}_0$. Here, since
\[
\Delta_k \left[ (\zeta)_{\ell j}^\rho \right] = \psi_k(D_\xi)\psi_k(D_\eta) \left[ \psi_0(D_x) \left[ \zeta_{\ell j}(2^{-j\rho}x, 2^{j\rho}\xi, 2^{j\rho}\eta) \right] \right] \times \Psi_j(2^{j\rho}\xi, 2^{j\rho}\eta),
\]
it holds from Lemma 8.7 (1) that
\[
\sup_{k \in (\mathbb{N}_0)^3} 2^k s \left\| \Delta_k \left[ (\zeta)_{\ell j}^\rho \right] \right\|_{L_\infty} \lesssim \sup_{k \in (\mathbb{N}_0)^3} 2^k s \left\| \Delta_k \left[ (\zeta_{\ell j})_{\ell}^\rho \right] \right\|_{L_\infty} \lesssim 1
\] (8.11)
for $j \in N_0$, where we used the fact that
\[
\sup_{k_1, k_2 \in \mathbb{N}_0} 2^{s_1 k_1 + s_2 k_2} \left\| \psi_{k_1}(D_\xi)\psi_{k_2}(D_\eta) \left[ \psi_0(D_x) \left[ \zeta_{\ell j}(2^{-j\rho}x, 2^{j\rho}\xi, 2^{j\rho}\eta) \right] \right] \right\|_{L_\infty} \lesssim 1
\]
holds with the implicit constant independent of $j \in \mathbb{N}_0$. Moreover, since
\[
\zeta_{\ell j}(2^{-j\rho}, 2^{j\rho}, 2^{j\rho}) = \sigma_{\ell j}(2^{(\rho_0-j)\rho}, 2^{-(\rho_0-j)\rho}, 2^{-(\rho_0-j)\rho}),
\]
and $2^{\rho_0-j\rho} \approx 1$ for $j \in \Omega_\ell$, we have by Lemma 8.7 (2)
\[
\sup_{k \in (\mathbb{N}_0)^3} 2^k s \left\| \Delta_k \left[ \zeta_{\ell j}(2^{-j\rho}, 2^{j\rho}, 2^{j\rho}) \right] \right\|_{L_\infty} \lesssim \sup_{k \in (\mathbb{N}_0)^3} 2^k s \left\| \Delta_k \left[ \sigma_{\ell j} \right] \right\|_{L_\infty} \lesssim 1
\] (8.12)
for $j \in \Omega_\ell$. Collecting (8.10), (8.11), and (8.12), we obtain
\[
\| \zeta_{\ell j} \|_{B^{m_\rho,3}_{\rho,\beta}(S; \mathbb{R}^n)} \lesssim 2^{-\ell m/(1-\rho)} \sup_{k \in (\mathbb{N}_0)^3} 2^k s \left\| \Delta_k \left[ \sigma_{\ell j} \right] \right\|_{L_\infty} \lesssim 2^{-\ell m/(1-\rho)} 2^{\ell m'} \| \sigma \|_{B^{m_\rho,4}_{\rho,0}(S; \mathbb{R}^n)},
\]
since $\sigma_{\ell j}^\rho = \sigma_{\ell j}^0 = \sigma_{\ell j}$ if $\rho = 0$. This is the desired inequality (8.9).

Therefore, we have by (8.7), (8.8), (8.6), and (8.9)
\[
\| T_\sigma(f, g) \|_{L'} \leq \sum_{\ell \in \mathbb{N}_0} \| T_{\sigma_\ell}(f, g) \|_{L'} \leq \sum_{\ell \in \mathbb{N}_0} 2^{\ell m/(1-\rho)} \| T_{\zeta_{\ell j}}(f, g) \|_{L'} \lesssim \sum_{\ell \in \mathbb{N}_0} 2^{\ell m/(1-\rho)} \| \zeta_{\ell j} \|_{B^{m_\rho,4}_{\rho,0}(S; \mathbb{R}^n)} \| f \|_{L^p} \| g \|_{L^q} \lesssim \sum_{\ell \in \mathbb{N}_0} 2^{(\ell m'-m/(1-\rho))} \| \sigma \|_{B^{m_\rho,4}_{\rho,0}(S; \mathbb{R}^n)} \| f \|_{L^p} \| g \|_{L^q} \approx \| \sigma \|_{B^{m_\rho,4}_{\rho,0}(S; \mathbb{R}^n)} \| f \|_{L^p} \| g \|_{L^q},
\]
since $1/p + 1/q = 1/r$ and $m' < m/(1-\rho)$. This completes the proof. \qed
We also use the following lemma proved by Wainger [33, Theorem 10] and Miyachi–Tomita [22, Lemma 6.1].

**Lemma 8.8** Let $1 \leq p \leq \infty$, $0 < a < 1$, $0 < b < n$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. For $t > 0$, put

$$f_{a,b,t}(x) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} e^{-t|k|} |k|^{-b} e^{i|k|a} e^{i\xi \cdot x} \varphi(x).$$

Then, if $b > n(1 - a/2 - 1/p + a/p)$, we have $\sup_{t > 0} \|f_{a,b,t}\|_{L^p(\mathbb{R}^n)} < \infty$.

By using this lemma, we shall show the following. See also [17, Proposition 7.3].

**Lemma 8.9** Let $m \geq -n$, $s = (s_0, s_1, s_2) \in [0, \infty)^3$, and $1 \leq p, q, r \leq \infty$. Suppose that

$$\|T_\sigma(f, g)\|_{L^r} \lesssim \|\sigma\|_{B_{s_0,0}^{m,\dagger}(s; \mathbb{R}^n)} \|f\|_{L^p} \|g\|_{L^q}$$

(8.13)

holds for all $\sigma \in B_{s_0,0}^{m,\dagger}(s; \mathbb{R}^n)$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then, $s_0 \geq m + n$.

**Proof** In this proof, we will use nonnegative functions $\varphi, \tilde{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ satisfying that $\text{supp} \varphi \subset [-1/4, 1/4]^n$, $\tilde{\varphi} = 1$ on $[-1/4, 1/4]^n$, and $\text{supp} \tilde{\varphi} \subset [-1/2, 1/2]^n$. Define

$$\sigma_{a_1,a_2}(x, \xi, \eta) = \varphi(x)e^{-ix \cdot (\xi + \eta)}$$

$$\times \sum_{k, \ell \in \mathbb{Z}^n \setminus \{0\}} (1 + |k| + |\ell|)^{m-s_0} e^{-i|k|a_1} e^{-i|\ell|a_2} \varphi(\xi - k) \varphi(\eta - \ell),$$

$$f_{a_1,b_1,t}(x) = \sum_{v \in \mathbb{Z}^n \setminus \{0\}} e^{-t|v|} |v|^{-b_1} e^{i|v|a_1} e^{i\xi \cdot x} \mathcal{F}^{-1}\tilde{\varphi}(x),$$

$$g_{a_2,b_2,t}(x) = \sum_{\mu \in \mathbb{Z}^n \setminus \{0\}} e^{-t|\mu|} |\mu|^{-b_2} e^{i|\mu|a_2} e^{i\mu \cdot x} \mathcal{F}^{-1}\tilde{\varphi}(x),$$

where $t > 0$, $0 < a_1, a_2 < 1$, $b_1 = n(1 - a_1/2 - 1/p + a_1/p) + \varepsilon_1$, and $b_2 = n(1 - a_2/2 - 1/q + a_2/q) + \varepsilon_2$, with $\varepsilon_1, \varepsilon_2 > 0$. Note that $f_{a_1,b_1,t}, g_{a_2,b_2,t} \in \mathcal{S}(\mathbb{R}^n)$ due to the exponential decay factors.

For these functions, the following hold:

$$\|\sigma_{a_1,a_2}\|_{B_{s_0,0}^{m,\dagger}(s; \mathbb{R}^n)} \lesssim 1,$$

$$\sup_{t > 0} \|f_{a_1,b_1,t}\|_{L^p(\mathbb{R}^n)} \lesssim 1,$$

$$\sup_{t > 0} \|g_{a_2,b_2,t}\|_{L^q(\mathbb{R}^n)} \lesssim 1.$$  (8.14)

The second and third inequalities follow from Lemma 8.8. We shall consider the first inequality. We write $N_i = [s_i] + 1$, $i = 0, 1, 2$ and recall the notation $\sigma^{0}_{j}(x, \xi, \eta) = \sigma(x, \xi, \eta)\Psi_j(\xi, \eta)$ for $\rho = 0$. Then, observing that

$$\left|\partial^\alpha_\xi \partial^\beta_\eta \partial^\gamma_\eta \sigma_{a_1,a_2}(x, \xi, \eta)\right| \lesssim (1 + |\xi| + |\eta|)^{|\alpha|+m-s_0},$$
and that $1 + |\xi| + |\eta| \approx 2^j$ on the support of $\Psi_j$, we realize that

$$
\left| \partial_x^a \partial_\xi^b \partial_\eta^c (\sigma_{a_1,a_2})_j^0 (x, \xi, \eta) \right| \lesssim 2^{j(|a|+m-s_0)}.
$$

Hence, applying the Taylor expansion (see (3.1)) to $\Delta_k [(\sigma_{a_1,a_2})^0_j]$, we have

$$
\left\| \Delta_k \left[ (\sigma_{a_1,a_2})^0_j \right] \right\|_{L^\infty} \lesssim 2^{-k_0 N_0 - k_1 N_1 - k_2 N_2} 2^{j(N_0 + m - s_0)} \quad \text{(8.15)}
$$

for $j, k_0, k_1, k_2 \geq 1$. For the case $j \geq 1$ and at least one of $k_0, k_1, k_2$ is zero, by avoiding the usage of the Taylor expansion for the corresponding variables, we obtain the same conclusion as in (8.15). The case $j = 0$ is similarly obtained. Therefore, the estimate in (8.15) holds for $j, k_0, k_1, k_2 \in \mathbb{N}_0$. On the other hand, in the derivation of (3.1), by avoiding the Taylor expansion with respect to the $x$-variable, we have

$$
\Delta_k \sigma (x, \xi, \eta) = 2^{n(k_0+k_1+k_2)} \sum_{|\beta| = N_1} 1 \sum_{|\gamma| = N_2} 1 \int_{(\mathbb{R}^n)^3} \psi (2^{k_0} x') \psi (2^{k_1} \xi') (x' - \xi')^\theta \psi (2^{k_2} \eta') (\eta')^\gamma
\times \int_{[0,1]^2} \left( \prod_{i=1,2} N_i (1-t_i) N_i^{-1} \right) (\partial_x^\beta \partial_\xi^\gamma \sigma)(x - x', \xi - t_1 \xi', \eta - t_2 \eta') \, dT \, dX',
$$

where $dT = dt_1 dt_2$ and $dX' = dx' d\xi' d\eta'$. Then, we have by the same lines as above

$$
\left\| \Delta_k \left[ (\sigma_{a_1,a_2})^0_j \right] \right\|_{L^\infty} \lesssim 2^{-k_1 N_1 - k_2 N_2} 2^{j(m - s_0)} \quad \text{(8.16)}
$$

for $j \geq 0$ and $k \in \mathbb{N}_0^3$. Take $0 < \theta < 1$ such that $N_0 \theta = s_0$. By (8.15) and (8.16)

$$
\left\| \Delta_k \left[ (\sigma_{a_1,a_2})^0_j \right] \right\|_{L^\infty} = \left\| \Delta_k \left[ (\sigma_{a_1,a_2})^0_j \right] \right\|_{L^\infty}^{\theta} \left\| \Delta_k \left[ (\sigma_{a_1,a_2})^0_j \right] \right\|_{L^\infty}^{1-\theta}
\lesssim 2^{-k_0 s_0} 2^{-k_1 N_1 - k_2 N_2} 2^{j m}.
$$

Therefore, we obtain the first inequality in (8.14).

We next investigate the left hand side of (8.13). Observe that supp $\varphi(\cdot - k) \cap$ supp $\tilde{\varphi}(\cdot - k') = \emptyset$ if $k \neq k'$, and $\tilde{\varphi} = 1$ on supp $\varphi$. Then,

$$
T_{\sigma_{a_1,a_2}} (f_{a_1,b_1,t}, g_{a_2,b_2,t})(x)
= \frac{\|\varphi\|_{L^1}^2}{(2\pi)^{2n}} \varphi(x) \sum_{k,\ell} e^{-t(|k|+|\ell|)} (1 + |k| + |\ell|)^{m-s_0} |k|^{-b_1} |\ell|^{-b_2}.
$$
Taking the $L^r$ norm of both sides, we have
\[
\|T_{\sigma}, a_2 (f_{a_1, b_1, t}, g_{a_2, b_2, t})\|_{L^r} \gtrsim \sum_{k \in \mathbb{Z} \setminus \{0\}} \sum_{\ell \leq |k|} e^{-2t|k|} (1 + |k|)^{m - s_0 - b_1 - b_2 - n}.
\]

Collecting (8.13), (8.14), and (8.17), we obtain
\[
\sum_{k \in \mathbb{Z} \setminus \{0\}} e^{-2t|k|} (1 + |k|)^{m - s_0 - b_1 - b_2 + n} \lesssim 1
\]
with the implicit constant independent of $t$. Thus, we have by the Fatou lemma
\[
\sum_{k \in \mathbb{Z} \setminus \{0\}} (1 + |k|)^{m - s_0 - b_1 - b_2 + n} \lesssim 1.
\]
This yields $m - s_0 - b_1 - b_2 + n < -n$, which is identical with
\[
s_0 > m + 2n - n \left(1 - \frac{a_1}{2} - \frac{1}{p} + \frac{a_1}{p}\right) - n \left(1 - \frac{a_2}{2} - \frac{1}{q} + \frac{a_2}{q}\right) - \varepsilon_1 - \varepsilon_2.
\]
Since $0 < a_i < 1$ and $\varepsilon_i > 0$, $i = 1, 2$, are arbitrary, if we take the limits as $a_i \to 1$ and $\varepsilon_i \to 0$, we obtain the condition $s_0 \geq m + n$, which gives the desired result.  \hfill \Box

**Proof of Proposition 8.5** We first observe that $BS_{\rho_\varepsilon, \rho}^{m, \Delta} (s_\varepsilon; \mathbb{R}^n) \subset BS_{\rho_\varepsilon, \rho}^m (s; \mathbb{R}^n)$ holds with $s_\varepsilon = (s_0 + \varepsilon, s_1 + \varepsilon, s_2 + \varepsilon)$ for any $\varepsilon > 0$. Now, we first consider the case $0 < \rho < 1$. In this case, we have by (8.5) and the inclusion relation above
\[
\|T_\sigma (f, g)\|_{L^r} \lesssim \|\sigma\|_{BS_{\rho_\varepsilon, \rho}^{m, \Delta} (s_\varepsilon; \mathbb{R}^n)} \|f\|_{L^p} \|g\|_{L^q},
\]
and then, by Lemma 8.6
\[
\|T_\sigma (f, g)\|_{L^r} \lesssim \|\sigma\|_{BS_{s_0, \rho}^{m, \Delta} (s, \mathbb{R}^n)} \|f\|_{L^p} \|g\|_{L^q}
\]
for any $\delta > 0$. Thus, we conclude by Lemma 8.9 that $s_0 + \varepsilon \geq n/2 - \delta$. Since $\varepsilon > 0$ and $\delta > 0$ are both arbitrary, if we take the limits as $\varepsilon \to 0$ and $\delta \to 0$, we obtain $s_0 \geq n/2$, which is the desired result for the case $0 < \rho < 1$. The case $\rho = 0$ is similarly proved by the embedding above, Lemma 8.9, and a limit argument.  \hfill \Box

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Appendix A: Existence of Decomposition

In this appendix, we determine functions used to decompose symbols in Sect. 5. Let \( \phi \in \mathcal{S}(\mathbb{R}^n) \) satisfy that \( \text{supp} \phi \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2 \} \) and \( \phi = 1 \) for \( |\xi| \leq 1 \). For \( k \in \mathbb{Z} \), we write \( \phi_k = \phi(\cdot/2^k) \). Then, we see that \( \text{supp} \phi_k \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2^{k+1} \} \) and \( \text{supp} (1 - \phi_k) \subset \{ \xi \in \mathbb{R}^n : |\xi| \geq 2^k \} \) for \( k \in \mathbb{Z} \), since \( \phi_k = 1 \) for \( |\xi| \leq 2^k \).

Let \( \{ \Psi_j \}_{j \in \mathbb{N}_0} \) be a Littlewood–Paley partition on \((\mathbb{R}^n)^2\). Then, for \( j \geq 1 \), \( \Psi_j \) can be expressed into the following form:

\[
\Psi_j(\xi, \eta) = \Psi_j(\xi, \eta)\phi_{j+1}(\xi)\phi_{j+1}(\eta)
× \{ \phi_{j-6}(\xi) + (1 - \phi_{j-6}(\xi)) \} \{ \phi_{j-6}(\eta) + (1 - \phi_{j-6}(\eta)) \}.
\]

Since \( \phi_j \phi_{j'} = \phi_j \) if \( j > j' \) and \( \phi_{j-6}(\xi)\phi_{j-6}(\eta) \) vanishes on \( \text{supp} \Psi_j \), \( j \geq 1 \), we have

\[
\Psi_j(\xi, \eta) = \Psi_j(\xi, \eta)\phi_{j-6}(\xi)\phi_{j+1}(\eta)(1 - \phi_{j-6}(\eta))
+ \Psi_j(\xi, \eta)\phi_{j+1}(\xi)(1 - \phi_{j-6}(\xi))\phi_{j-6}(\eta)
+ \Psi_j(\xi, \eta)\phi_{j+1}(\xi)(1 - \phi_{j-6}(\xi))\phi_{j+1}(\eta)(1 - \phi_{j-6}(\eta)).
\]

We further decompose the first factor above as follows:

\[
\Psi_j(\xi, \eta)\phi_{j-6}(\xi)\phi_{j+1}(\eta)(1 - \phi_{j-6}(\eta)) \{ \phi_{j-4}(\eta) + (1 - \phi_{j-4}(\eta)) \}.
\]

Then, this is equal to \( \Psi_j(\xi, \eta)\phi_{j-6}(\xi)\phi_{j+1}(\eta)(1 - \phi_{j-4}(\eta)) \), since \( \phi_{j-6}(\xi)\phi_{j-4}(\eta) \) vanishes on \( \text{supp} \Psi_j \), \( j \geq 1 \), and \( (1 - \phi_j)(1 - \phi_{j'}) = (1 - \phi_j) \) if \( j > j' \). The second factor can be expressed similarly because of symmetry. Therefore, we have

\[
\Psi_j(\xi, \eta) = \Psi_j(\xi, \eta)\phi_{j-6}(\xi)\phi_{j+1}(\eta)(1 - \phi_{j-4}(\eta))
+ \Psi_j(\xi, \eta)\phi_{j+1}(\xi)(1 - \phi_{j-4}(\xi))\phi_{j-6}(\eta)
+ \Psi_j(\xi, \eta)\phi_{j+1}(\xi)(1 - \phi_{j-6}(\xi))\phi_{j+1}(\eta)(1 - \phi_{j-6}(\eta))
=: \Psi_j(\xi, \eta)\psi_j'(\xi)\psi_j'(\eta) + \Psi_j(\xi, \eta)\psi_j''(\xi)\psi_j''(\eta)
\]

with \( \phi_j' = \phi_j(\cdot/2^j), \psi_j' = \psi_j(\cdot/2^j) \), and \( \psi_j'' = \psi_j''(\cdot/2^j) \), and then we realize that

\[
\text{supp} \phi' \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2^{-5} \},
\]
\[
\text{supp} \psi' \subset \{ \xi \in \mathbb{R}^n : 2^{-4} \leq |\xi| \leq 2^2 \},
\]
\[
\text{supp} \psi'' \subset \{ \xi \in \mathbb{R}^n : 2^{-6} \leq |\xi| \leq 2^2 \}.
\]

Hence, we obtain the information (5.2), (5.3), and (5.4) given in Sect. 5.
Appendix B: Boundedness from $L^2 \times L^2$ to $L^1$

In this appendix, we shall prove the following boundedness stated in Remark 3.4.

**Theorem B.1** Let $0 \leq \rho < 1$, $m = -(1 - \rho)n/2$, and $s = (s_0, s_1, s_2) \in [0, \infty)^3$ satisfy $s_0, s_1, s_2 \geq n/2$. Then, if $\sigma \in BS^n_{m,0}(s; \mathbb{R}^n)$, the bilinear pseudo-differential operator $T_\sigma$ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

The proof is much simpler than that of the boundedness to $h^1$ done in Sect. 6.

**Proof** As in Sect. 5 and Appendix A, we decompose a Littlewood–Paley partition $\{\Psi_j\}_{j \in \mathbb{N}_0}$ on $(\mathbb{R}^n)^2$ into the following form: For $j \geq 1$, 
\[
\Psi_j(\xi, \eta) = \Psi_j(\xi, \eta)\phi'_j(\xi)\psi'_j(\eta) + \Psi_j(\xi, \eta)\psi'_j(\xi)\phi'_j(\eta),
\]
where $\phi'_j = \phi'(\cdot/2^j)$, $\psi'_j = \psi'(\cdot/2^j)$, $\text{supp } \phi' \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2^{-2}\}$, and $\text{supp } \psi' \subset \{\xi \in \mathbb{R}^n : 2^{-3} \leq |\xi| \leq 2^2\}$. Then, repeating the same lines as in Sect. 5, the dual form of $T_\sigma(f, g)$ can be expressed by the sum of the forms $I_0$, $I_1$, and $I_2$ as follows. The form $I_0$ is the same as in (5.9). The form $I_1$ is the following:
\[
I_1 = \sum_{j \geq 1} \sum_{k \in \mathbb{N}_0^3} \sum_{\nu \in \{\pm\}^2} 2^{-j\rho n} \int_{\mathbb{R}^n} T_{\sigma, j, k, \nu} (\square_{v_1} f'_j, \square_{v_2} g'_j)(x) h_j(x) \, dx,
\]
where $f'_j = \phi'_j(D) f(2^{-j\rho} \cdot)$, $g'_j = \psi'_j(D) g(2^{-j\rho} \cdot)$, and $h_j = h(2^{-j\rho} \cdot)$. Also, we have
\[
\text{supp } \hat{f}'_j \subset \left\{ \xi \in \mathbb{R}^n : |\xi| \leq 2^{j(1-\rho)-2} \right\}, \quad \text{supp } \hat{g}'_j \subset \left\{ \eta \in \mathbb{R}^n : 2^{j(1-\rho)-3} \leq |\eta| \leq 2^{j(1-\rho)+2} \right\}.
\]

The form $I_2$ is in a symmetrical position with $I_1$, and thus we omit stating it.

We shall consider the three forms above. However, we only consider $I_1$, since the proof for $I_0$ is exactly the same as in Sect. 6.4 and the proof for $I_2$ is similar to that for $I_1$ because of symmetry. We take a Littlewood–Paley partition $\{\psi_{\ell}\}$ on $\mathbb{R}^n$ and decompose the factor of $f$ as
\[
I_1 = \sum_{j \geq 1} \sum_{k \in \mathbb{N}_0^3} \sum_{\ell \in \mathbb{N}_0} \sum_{\nu \in \{\pm\}^2} 2^{-j\rho n} \int_{\mathbb{R}^n} T_{\sigma, j, k, \ell, \nu} (\square_{v_1} \Delta_{\ell} f'_j, \square_{v_2} g'_j)(x) h_j(x) \, dx.
\]

Then, the sums over $\nu$ and $\ell$ are restricted to $v_1 \in \{v_1 \in \mathbb{Z}^n : |v_1| \lesssim 2^\ell\}$, $v_2 \in \{v_2 \in \mathbb{Z}^n : |v_2| \lesssim 2^{j(1-\rho)}\}$, and $\ell \leq j(1 - \rho)$ (see Sect. 6.1). Applying Lemma 4.4 (1) with $p = q = 2$ and $r = \infty$ to the restricted sums, we have
\[
|I_1| \lesssim \sum_{j, k, \ell : \ell \leq j(1-\rho)} 2^{\ell n/2} 2^{2(k_0 \ell + k_2) n/2} \|\Delta_k [\sigma_j^\rho]\|_{L^2} \|\Delta_{\ell+j \rho} f\|_{L^2} \|\psi'_j(D) g\|_{L^2} \|h\|_{L^\infty}
\]

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where we used the calculation as in (6.8) to the factors of $f$ and $g$. Since we are not dividing the sum over $j$, the right hand side above is simply bounded by

$$\|\sigma \|BS_{\rho,\rho}^m(s;\mathbb{R}^n)\|L^\infty \sum_{j \gg 1} \sum_{\ell: \ell \leq j(1-\rho)} 2^{\ell n/2} 2^{-j(1-\rho)n/2} \|\Delta_{\ell+j} f \|L^2 \|\psi_j(D)g\|L^2,$$

where $m = -(1-\rho)n/2$ and $s = (n/2, n/2, n/2)$. The sums over $j$ and $\ell$ are bounded by a constant times $\|f\|L^2 \|g\|L^2$, recalling the proof of (6.12). Hence, we obtain

$$|I_1| \lesssim \|\sigma \|BS_{\rho,\rho}^m(s;\mathbb{R}^n)\|f\|L^2 \|g\|L^2 \|h\|L^\infty,$$

where $m = -(1-\rho)n/2$ and $s = (n/2, n/2, n/2)$. This completes the proof. \hfill \Box

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