Decoherence of quantum Brownian particle trapped in a penning potential in a non-commutative space

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Abstract
In this paper, the influence of non-commutativity on quantum systems interacting with the environment in the Penning trap potential is studied. The considered system is a Brownian particle expressed in non-commuting spatial coordinates coupled to a set of harmonic oscillators (environment). The equilibrium state of the total system has been evaluated using the fluctuation dissipation theorem. The effective parameters of the system are evaluated and compared with those of the system in commutative space. We found that non-commutativity effects give rise to an increase in decoherence in the system, and negative heat capacity confirms the presence of self-gravitation effects induced by non-commutativity.

1. Introduction

The twentieth century has known a remarkable evolution in the field of physics. Throughout this evolution, two main theories have attracted an increasing attention of researchers, these include the theory of general relativity and the quantum theory. Both theories have changed a lot our way of understanding and interpreting the nature. That is, several problems have then, been solved in both theories considering the space as commutative (i.e. the space-time coordinates can be measurable simultaneously according to the Heisenberg principle [1]). However, it has been shown that some theories including the string theory and quantum gravity combining with general relativity which seem to be the foundation of the unification theory could not be easily studied in a commutative space, since these theories require an infinite different (Planck’s scale). In order to solve this problem, a new theory has been introduced, here we refer to the theory of non-commutativity (NCity) which implies that the commutators relations between two or more coordinates are not more zero [2]. In addition, quantum mechanics was successfully described by the Schrödinger equation, but it is not always easy to find an exact solution to such equation. Therefore, commutative operator are generally used. However, Groenewold in 1946 demonstrated that a general systematic correspondence between quantum commutators and Poisson brackets could not hold consistently. But he found that, such systematic correspondence might exist between quantum commutators and deformation of Poisson bracket, the Moyal brackets or in general between quantum operators and classical observables [3].

Indeed, Heisenberg was the first to suggest that the notion of NCity can be extended to the coordinates as a possible way to remove the infinite quantities appearing in the fields theory and sketch the mathematical formalism of the modern quantum physics [4]. In the same idea, the properties of non-locality induced by the NCity coordinates was investigated by Snyder [5], and this allow scientists later to solve the problems of divergence in short distance of the field theory (ultra violet). In the past few decades, this notion has greatly attracted the attention of many scientists for its potential applications in the physics of condensed matter as well as in the theory of unification. The non-commutative (NC) geometry has been deeply investigated by Connes and the generalization of Bernhard Reiman’s geometry has been extended to a NC geometry, describing the space with curvature and uncertainty [6]. Moreover, it presents several applications in the physics of particles.
and plays an essential role in physics at the Planck’s scale where quantum effects of the gravity might not be negligible [7]. Although NCity embodies a puzzle piece in high-energy scenario, great interest has also been addressed to its implications in condensed matter physics though the four fundamental interactions. Thus, the studies proved that there is a trace of NCity in the hydrogen atom [8–10], quantum Hall effect [11–13], Ahoronov-Bohm effect [14, 15], graphene [16] and even in quantum information theory [17]. In the latter, several examples have been proposed and among them, we denote the phenomenon of decoherence. Quantum decoherence is considered to be a fundamental physical base for the transition from quantum to classical physics [18]. This phenomenon was first considered in 70’s and 80’s with the works of Zeh (1950) and Zureck (1991) on the emergence of classicality in the quantum framework [19, 20]. Moreover, this phenomenon presents a profound implication in very high energy universe, where important applications correspond to supplying a mechanism in which the end of inflationary phase, enters smoothly to the radiation era [20–24]. Recently, Derakhshani suggests that, quantum decoherence concept can be sensibly applied to quantum gravity theories that posit classical time parameters or matter-clock variables at a fundamental level [25]. It will be therefore, very interesting to investigate this phenomenon of decoherence considering the space as NC. Whereas, it is important to mention that, before exploring the properties of particle in quantum physics, it is often necessary to trap the particle with a convenient potential. The harmonic potential and double well potential have already been used in several works in the study of the phenomenon of decoherence [26, 27]. However, in this work, we trapped the Brownian particle in a Penning potential since its structure presents the advantage of combining the superposition of electric and magnetic field.

Recently, with the new development in the physics of quantum open systems, Dragovich et al in 2005 investigated the influence of NCity on the occurrence of the so-called decoherence effect with external magnetic field, and they found that, decoherence can be highly affected when the external control (magnetic field) is constrained by the NC parameter. In fact, in the context of quantum mechanics on non-commutative space, the parameter inducing non-commutativity admits an analogy very close to a constant magnetic field. Although the value of this non-commutative parameter is small enough, it can present an influence on open systems interacting with their environment. This requires to satisfy some conditions on the system: firstly, the redefinition of the system state basis is needed because non-commutativity appears as a distortion of the eigenstates of the system. Secondly, the decoherence time of this system must be proportional to the magnetic field and to the non-commutative parameter with the form \( \tau = \frac{1}{eB/4} \). It is observed that, for \( B = 4/e\theta \) the decoherence time is equal to zero, therefore the external control which is here the magnetic field \( B \) is inversely proportional to the non-commutative parameter [28]. In the same idea, Dias and Prata in 2009 studied decoherence in NC Brownian particle using the exact master equation, they concluded that the NC to commutative (NC-C) transition takes places before the transition from classical mechanics to ordinary quantum mechanics, by comparing the decoherence time and the NC-C time transition [29]. Moreover, Ghoroshi et al in 2013 have looked forward to see the effects of NC space on decoherence by modifying the master equation of two-dimensional harmonic oscillators under the Brownian motion, and they found that the NC leads to an increase in the rate of decoherence [30]. Furthermore, Tchoffo et al in their investigation on kinematical Brownian motion of harmonic oscillator in NC space showed that the structure of Fokkler Planck’s equation is not modified (i.e. the factorization theorem is conserved in both commutative and NC space) [31]. More recently, Santos et al have introduced the Brownian motion of a particle in two-dimensional NC space using the standard NC algebra embodied by Weyl–Moyal formalism to find that, NCity induces a non-vanishing correlation between both coordinates at different times [32]. It turns out that, several works have been intensively investigated in the field most of them using the Fockkler Planck and the Master equation approaches. However, they did not take into consideration the effects of NC property of the space on the system’s effective parameters. Thus, our main objective in this article will be based on the study of the effects of NCity on the phenomenon of decoherence of a Brownian particle in a penning trap potential. To achieve this goal, the fluctuation-dissipation theorem approach will be used to analyze whether the transition from quantum to classical world acquired in recent decades with the theory of open quantum systems is modified by the configuration of the space. To approach this new idea, we structured the paper as follows: in section 2 we modeled the Hamiltonian of Brownian particle in NC space. Section 3 is devoted to the derivation of the quantum Langevin equation, and the fluctuation-dissipation theorem is introduced thought the response and the correlation functions of the system. Finally, section 4 is devoted to the results and discussion, where we mainly discuss the effects of NCity on decoherence through effective mass, energy, heat capacity and further the entropy.
2. Model description of quantum Brownian particle and derivation of the Hamiltonian in NC space

Since a perfect isolated quantum system does not exist, all realistic quantum systems are influenced by their surrounding. Therefore, several systems studied in quantum physics display dissipative dynamics where energy and other conserved quantities are exchanged between the system and its environment [33]. Although the ultimate sources of dissipation and irreversibility in physics are open problem, there is consensus that dissipation may occur due to incomplete knowledge on a system which can be described in coarse-grained form [21]. The study of those systems can be modelled in classical mechanics by going beyond the Hamiltonian formalism for instance by linking the equation of motion of free system to the friction and noise terms [34]. However, in quantum mechanics it is not really true due to the fact that correlations must be established between the two parts of the system, implying a deep consequence to the dynamics of the systems. The theory of decoherence is precisely the study of destruction of interference due to the interaction between a system and its environment [35, 36]. In recent years, the study of decoherence was still a rather new subject, but nowadays physicists are struggling to prove that decoherence might be useful for some physical experiments. It is known as the most important effect that make impossible the realization of quantum computers. Since due to decoherence, qubits which is the units of quantum information are extremely fragile and their ability to stay in superposition and entangled state is severely jeopardized [37, 38]. However, wide works have been investigated to study this phenomenon, but most of them consider the space as commutative. Therefore, we ask ourselves what could be the change in this phenomenon if we consider our system as Brownian particle trapped in a penning potential but in NC space.

2.1. Formalism of non-commutativity

In a general case, the NC phase-space is realized when the coordinates and momentum operators satisfy the following relations:

\[ [q_i, q_j] = i\theta_{ij} [p_i, p_j] = \tilde{\theta}_{ij}, [q_i, p_j] = i\hbar\delta_{ij}, i, j = 1, 2, \]

(1)

taking into account the effective Plank constant \( \hbar_{\text{eff}} = \hbar \left( 1 + \frac{\theta_{ij}}{4\pi^2} \right) \), where the quantities \( \theta_{ij}, \tilde{\theta}_{ij} \) are matrix elements which do not depend on the position and momentum, with \( (\theta_{ij})^\dagger = -\theta_{ji}, (\tilde{\theta}_{ij})^\dagger = \tilde{\theta}_{ji} \) [39, 40]. This setup is consistent with the usual commutative space-time quantum mechanics only if \( \theta \ll 4\hbar^2 \). Due to the fact that the NCity of phase space is purely geometrical property, its physical effects are independent of composition of the particle. Therefore, NCity between momenta arises naturally as a consequence of NC coordinates, since momenta are define to be partial derivatives of the action with respect to the NC coordinates [41]. We shall however consider the simple version which refer to the NCity of space only which is introduced by the basic formula of NC algebra.

\[ [q_i, q_j] = i\theta_{ij} [p_i, p_j] = 0; [q_i, p_j] = i\hbar\delta_{ij}, i, j = 1, 2, \]

(2)

this is firstly due to the fact that a direct description stemming from string theory where the momentum NCity is absent [42]. Secondly, they are some interesting connections with landau problem and quantum hall effect. Finally, for the particular case of Brownian particle interacting with a bath such as our model, the application of fluctuation dissipation theorem is encumbering and the deformation of momentum commutation relation would make this task even harder.

We will refer to \( \theta_{ij} = \theta_{\epsilon_{ij}} \) as the NCity parameter, where \( \theta \) is a real constant and \( \epsilon_{ij} \) the antisymmetric tensor. From these equations we observe that the NCity parameter has the dimension of length squared. At the fundamental level, \( \sqrt{\theta} \) is a candidate for the Planck’s length and show how the NCity is connected to quantum gravity [43].

2.2. Model description

In this work, we consider a single quantum particle interacting with a large reservoir, alternatively called bath or environment. Whenever the system is driven out equilibrium by external perturbation, the coupling between the system and the bath makes the system relaxing back to equilibrium. In most of the cases, the system has only a few degree of freedom, whereas the bath needs to have infinite degree of freedom in order to generate truly irreversible dynamics. This might be the electromagnetic field into which the atom can radiate its energy, or the crystal lattice that gets distortion by an electron moving along. The model described above forms the standard pattern of quantum Brownian motion (QBM), and is motivated by possible observation of macroscopic effects in quantum system (such as dissipation in tunneling).

In this section, we describe the theory of Brownian particle moving with a certain velocity in the XY-plane in the Penning trap potential and a constant homogeneous magnetic field \( \mathbf{B} \) applied along the Z-axis. This model
can be taken as the basic reality of macroscopic description of a quantum particle linearly coupled to a passive heat bath. The Brownian particle trapped in the penning potential is a charged particle (electron or ion devoted to the spin–orbit coupling) and the trapping mechanism combines an electrostatic quadrupole potential [44],

\[ V(q, z) = \frac{U_0}{2d^2} \left[ -\frac{1}{2}q^2 + z^2 \right], \tag{3} \]

where \( q^2 = x^2 + y^2 \), the parameters \( U_0 > 0 \) and \( d \) are respectively the electric potential and the trap dimension. The Hamiltonian of such open quantum system is written as:

\[ H = H_{\text{part}} + H_{\text{bath}} + H_{\text{int}}, \tag{4} \]

\[ H_{\text{int}} = -q \sum_{n=1}^{N} \mu_n \omega_n^2 (\varphi_n^{(n)})^2 + \frac{q^2}{2} \sum_{n=1}^{N} \mu_n \omega_n^2, \tag{5} \]

\[ H_{\text{bath}} = \sum_{n=1}^{N} \left( \frac{\sigma_n^{(n)}}{2\mu_n} + \frac{\mu_n \omega_n^2}{2} (\varphi_n^{(n)})^2 \right), \tag{6} \]

where \( \varphi^2 \) is the potential vector expressed in terms of the position vector of the particle \( \varphi = \frac{1}{2} \hat{B} \wedge \vec{r} \) and related to the uniform magnetic field \( B \) along the Z-axis, \( \vec{r} = \vec{q} + \hat{z} \), \( e \) is the charge of the particle, \( m \) the mass of the particle, \( c \) the celerity of the light and \( V(\vec{r}) \) the penning potential. Using the potential vector in the symmetric gauge, we rewrite the above Hamiltonian of the system in the following form:

\[
\begin{align*}
H_{||} &= \frac{p_{||}^2}{2m} + \frac{1}{2} m \Omega^2 q^2 + \sum_{n=1}^{N} \left( \frac{\pi_n^{(n)}}{2\mu_n} + \frac{\mu_n \omega_n^2}{2} (\varphi_n^{(n)})^2 - q^2 \right), \\
H_L &= \frac{p_L^2}{2m} + \frac{1}{2} m \omega_L^2 Z^2,
\end{align*}
\]  

where \( \omega_l = \frac{4\hbar}{mc} \) is the angular (or cyclotron) frequency, \( \omega_2 = \sqrt{\frac{2e}{md}} \) is the axial frequency and \( \Omega = \sqrt{\frac{2}{m} - \frac{\omega_2^2}{2}} \) the radial cyclotron frequency. Considering the Boop’s shifts operators defined by the following relations:

\[ Q_{l} = q_{l} + \frac{\theta}{2\hbar} \varepsilon_{l} \rho_{l} ; P_{l} = p_{l} ; \varphi_{l}^{(n)} = \varphi_{l}^{(n)} + \frac{\theta}{2\hbar} \varepsilon_{l} \phi_{l}^{(n)} ; \Pi_{l}^{(n)} = \Pi_{l}^{(n)} ; \]

the Hamiltonian can then, be transformed as follows:

\[
\begin{align*}
H_{||} &= \frac{p_{||}^2}{2m} + \frac{1}{2} M N Q_{l}^2 - \frac{1}{2} M N \frac{\theta}{h} \varepsilon_{l} \phi_{l}^{(n)} P_{l} + \sum_{n=1}^{N} \left( \frac{\Pi_{l}^{(n)}(\phi_{l}^{(n)})^2}{2m_n} + \frac{1}{2} m_n \Omega_n^2 (\varphi_{l}^{(n)})^2 - \frac{m_n \omega_{n}^2}{2} \varepsilon_{l} \phi_{l}^{(n)} \Pi_{l}^{(n)} \right) \\
&+ \sum_{n=1}^{N} \frac{m_n \omega_{n}^2}{2} \left[ -2 \phi_{l}^{(n)} Q_{l} + \frac{\theta}{\hbar} \varepsilon_{l} \Pi_{l}^{(n)} Q_{l} + \frac{\theta}{\hbar} \varepsilon_{l} \phi_{l}^{(n)} P_{l} - \frac{\theta}{2\hbar} \varepsilon_{l} \phi_{l}^{(n)} P_{l} \right] \Pi_{l}^{(n)} P_{l} + Q_{l}^2 - \frac{\theta}{\hbar} \varepsilon_{l} \phi_{l}^{(n)} P_{l}, \\
H_L &= \frac{p_{L}^2}{2m} + \frac{1}{2} m \omega_{L}^2 Z^2,
\end{align*}
\]  

with \( \lambda = \frac{m \omega_{l}^2 \theta}{2\hbar} ; M = \left[ \frac{m}{1 + \frac{\lambda}{\omega_{l}^2}} \right] ; \Lambda = \sqrt{1 + \left( \frac{\lambda}{\omega_{l}^2} \right)^2} \); and \( m_n = \frac{\mu_n}{1 + \left( \frac{\lambda}{\omega_{n}^2} \right)^2} \); \( \lambda_n = \frac{\mu_n \omega_{n}^2 \theta}{2\hbar} ; \Omega_n = \omega_n \sqrt{1 + \left( \frac{\lambda_n}{\omega_{n}^2} \right)^2} \).

The quantities \( M \) and \( \Lambda \) define respectively the NC mass and NC frequency of the quantum particle while \( m_n \) and \( \Omega_n \) refer to the NC mass and NC frequency of \( n \)th oscillator of the bath. In addition, \( P \) and \( Q \) are respectively the coordinate and the momentum operators for the particle in NC space while, \( \Pi_{l}^{(n)} \) and \( \phi_{l}^{(n)} \) are the corresponding quantities for the \( n \)th oscillator of the bath, while \( \lambda \) and \( \lambda_n \) are the parameters which depend of NC parameter \( \theta \). Equation (8) above represent the new Hamiltonian of the system in the NC space. The aim of our investigation here is to show the effects of NCity of the space on decoherence of the system when coupled to a bath of oscillator in the Penning potential. The transversal part of Hamiltonian of the system introduced by the geometry of Penning potential is not influenced by the NC space due to the fact that the NCity can only be expressed in at least two-dimensional space. Having the Hamiltonian of our particle in NC space, we can now determine the quantum Langevin equation (QLE) in NC space which reflects obviously the equation of motion of Brownian particle containing a stochastic term representing the interaction with the bath.
3. Quantum Langevin equation in non-commutative space

The QLE offers a certain pure representation of the quantum Brownian motion. It is the Heisenberg equation of motion in terms of coordinate operator \(Q\) of quantum particle coupled to a heat bath. Before deriving the QLE of our system in NC space, we will move from canonical equations of Hamilton to obtain the equation of motion of the particle and the bath. The corresponding differential equations are found in the following relations:

\[
Q_i = -\frac{\hbar^2 \theta}{2\hbar} \epsilon_{ij} Q_j + N^2 Q_i - \frac{1}{M} \sum_{n=1}^{N} \frac{m_n \Omega_n}{2} \left( \frac{\theta}{\hbar} \epsilon_{ij} m_n \phi_j^{(n)} + 2Q_i \right) - \frac{\theta}{\hbar} \epsilon_{ij} M Q_j = 0, \tag{9}
\]

\[
\phi_j^{(n)}(t) + \Omega_n^2 \phi_j^{(n)}(t) - \frac{1}{2} \sum_{n=1}^{N} m_n \Omega_n^2 \frac{\theta}{\hbar} \epsilon_{ij} \phi_j^{(n)}(t) + \Omega_n^2 Q_i - M \Omega_n^2 \frac{\theta}{\hbar} \epsilon_{ij} Q_j = 0, \tag{10}
\]

from where the homogeneous part corresponding to (10) is given by:

\[
\phi_j^{(n)}(t) + \Omega_n^2 \phi_j^{(n)}(t) - \frac{1}{2} \sum_{n=1}^{N} m_n \Omega_n^2 \frac{\theta}{\hbar} \epsilon_{ij} \phi_j^{(n)}(t) = 0 \tag{11}
\]

This equation has been obtained in several works including the work of [28] when studying the exact master equation of NC Brownian motion, and the solution of such equation can be found [28] under the form below:

\[
\phi_j^{(n)}(t) = \lambda_j^{(n)}(t) \phi_j^{(0)}(0) + \rho_j^{(n)}(t) \frac{\Pi_j^{(n)}(0)}{m_n}, \tag{12}
\]

subject to the following initial conditions:

\[
\begin{cases}
\phi_j^{(n)}(t = 0) = \phi_j^{(0)}(0), \\
\phi_i^{(n)}(t = 0) = \frac{\Pi_i^{(n)}(0)}{m_n},
\end{cases} \tag{13}
\]

where the matrix elements \(\lambda_j^{(n)}(t)\) and \(\rho_j^{(n)}(t)\) are given by:

\[
\lambda_j^{(n)}(t) = \frac{1}{2\Omega_n} \sum_{m=\pm} \left[ \delta_{ij} (\Omega_n - \sigma \lambda_n) \cos(\Omega_n t + \sigma \lambda_n t) + \sigma \epsilon_{ij} (\Omega_n - \sigma \lambda_n) [\sin(\Omega_n t + \sigma \lambda_n t)], \right] \tag{14}
\]

and

\[
\rho_j^{(n)}(t) = \frac{1}{2\Omega_n} \sum_{m=\pm} \left[ \delta_{ij} \sin(\Omega_n t + \sigma \lambda_n t) - \sigma \epsilon_{ij} \cos(\Omega_n t + \sigma \lambda_n t) \right]. \tag{15}
\]

The particular solution is written under the following form:

\[
\phi_j^{(n)}(t) = \nu_1(t) \lambda_j^{(n)}(t) + \nu_2(t) \rho_j^{(n)}(t), \tag{16}
\]

where \(\nu_1(t)\) and \(\nu_2(t)\) are obtained through the following initial conditions,

\[
\begin{cases}
\lambda_j^{(n)}(t = 0) = \delta_{ij} \nu_2^{(n)}(t = 0) = 0;
\\
\lambda_j^{(n)}(t = 0) = 0; \nu_j^{(n)}(t = 0) = \delta_{ij};
\end{cases} \tag{17}
\]

thus, the final solution of (10) is expressed as:

\[
\phi_j^{(n)}(t) = \lambda_j^{(n)}(t) \phi_j^{(0)}(0) + \rho_j^{(n)}(t) \frac{\Pi_j^{(n)}(0)}{m_n} - \Omega_n^2 \int_0^t ds \rho_j(t - s) \left( \delta_{ij} - \frac{\theta}{2\hbar} \epsilon_{ij} \frac{d}{ds} Q(s) \right). \tag{18}
\]

Considering (9) giving the canonical equation of Hamilton in terms of coordinates, we eliminate the bath degrees of freedom by inserting \(\phi_j^{(n)}(t)\) and \(\phi_j^{(n)}(t)\) into the homogeneous part of (10). For \(\lambda_j^{(n)}(t) = -\Omega_n^2 \rho_j^{(n)}(t)\), \(\rho_j^{(n)}(t) = \lambda_j^{(n)}(t) - 2\lambda_n \epsilon_{ij} \rho_j(t)\) and \(\lambda_n = \frac{m_n - \theta}{2\hbar}\), we then, obtain the dissipative dynamics of the reduced system expressing the so called QLE of Brownian particle in NC space as follows:

\[
Q_i + N^2 Q_i - \frac{\alpha^2 \theta}{2\hbar} \epsilon_{ij} Q_j - \frac{1}{M} \Omega_n^2 \int_0^t ds \left( \delta_{ij} \frac{d}{ds} Q_j - \frac{\theta}{2\hbar} \epsilon_{ij} \frac{d}{ds} Q(s) \right) = \frac{f(t)}{M}. \tag{19}
\]
with \( \alpha = (M \dot{X}^2 + m_n \Omega_n^2 \dot{x}_j^2) \), \( \rho_j(t - s) = \frac{4}{\Omega_n} \rho_j(t - s) \) and
\[
f(t) = \sum_{n=1}^{N} m_n \Omega_n^2 \left\{ \chi_j^{(n)}(t) + \frac{m_n \theta}{\hbar} \Omega_n^2 \rho_j^{(n)}(t) \right\} \phi_j^{(n)}(0) + \left\{ 1 + \frac{2 \lambda_n^2}{\Omega_n^2} \right\} \rho_j^{(n)}(t) - \frac{\lambda_n}{\Omega_n^2} \chi_j^{(n)}(t) + \left\{ \Pi_j^{(n)}(0) \right\} \frac{m_n}{\hbar} \right\}.
\]

(20)

The QLE is a stochastic differential equation describing the time evolution of the subset of the degree of freedom, and is interesting in the discussion of autocorrelation introduced by two temporal regimes in the evolution of quantum system. \( f(t) \) describes the properties of the heat bath and is independent from external force. Physically, we can understand this by saying that, it includes a kind of competition between the noise and dissipation. Indeed, the thermal noise injects energy in the particle and simultaneously, dissipation absorbs energy (via the friction force \( f(t) \)). Moreover, noise is responsible for decoherence and entropy generation [23]. If there was not dissipation, the energy of the particle would be in perpetual growth and reciprocally, if there was not noise the particle will finish dissipating all its energy. The behavior of a particle is thus, a relation between both quantities.

3.1. Response function of the system

The response function contains valuable information about the thermodynamic behavior of the system [46]. Generally, for a single degree of freedom, linear response theory yields for the change of the expectation value of an operator \( G \) due to the action of a classical force \( F(t) \) coupled to the conjugate dynamical operator \( F \) is defined as:
\[
\chi_{FG}(t) = \frac{i}{\hbar} \left\{ [F(t), G] \right\} \Theta(t),
\]

(21)

where the Heaviside step function \( \Theta(t) \) is introduced to make causality manifest. Let us now consider our particular case, the response function \( \chi_{qq} \) for our system which may now be obtained by taking the commutator of the equation of motion (equation (19)) with \( Q(0) \). Since the commutator with the noise term in (20) vanishes, we found that the response function obeys the following differential equation:
\[
\ddot{\chi}_{qq} + M \dot{\chi}_{qq} + \frac{\alpha^2 \theta}{2 \hbar} \varepsilon_j \chi_{qq} \delta_j \gamma + \frac{\Omega_n^2 \theta}{M \hbar} \varepsilon_j \chi_{qq} = 0,
\]

(22)

with the initial conditions \( \chi_{qq}(0) = \frac{i}{\hbar} \left\{ \left[ \frac{1}{2} (Q(0), Q(0)) \right] \right\} = 0 \) and \( \chi_{qq}(0) = \frac{i}{M \hbar} \left\{ \left[ P(0), Q(0) \right] \right\} = \frac{1}{M} \) and
\[
\chi_{qq}(\xi) = \frac{\xi \chi_{qq}(0) + \chi_{qq}(0) + 3 \chi_{qq}(0)}{\xi^2 + \lambda^2 - \frac{\theta}{2 \hbar} \varepsilon_j \xi \alpha^2 + \frac{\Omega_n^2 \theta}{2 M \hbar} \varepsilon_j}.
\]

(23)

By taking into account the initial conditions and for \( \gamma = \left[ -\frac{\theta}{2 \hbar} \varepsilon_j \alpha^2 + \frac{\Omega_n^2 \theta}{2 M \hbar} \varepsilon_j \right] \), we get
\[
\chi_{qq}(\xi) = \frac{1}{M \zeta} \left( \frac{\zeta}{\zeta + \xi} \right )^\frac{1}{2},
\]

(24)

The reaction of the system is contained in the response function \( \chi_{qq}(t) \) with the so-called dissipation part. Therefore, the fluctuation-dissipation theorem will allow us to make a connection between the damped motion as given by the response function and the fluctuations described by correlation functions. We discuss in detail the equilibrium correlation functions of our damped system while paying special attention to the low temperature properties.

3.2. Fluctuation-dissipation theorem and correlation function of the system

Historically, there have been some different approaches to the problems of open quantum systems, master equation and Fokker-Planck’s equation in both commutative and NC space. However, in this paper, we introduce the fluctuation-dissipation theorem in NC space. This method will allow us to find the thermodynamics properties of the system. The response of the system from external perturbation is accompanied by the absorption and emission of energy. This follows because under influence of the external perturbation, the system changes. The difference between the absorbed energy and the emitted one is the dissipative energy. This dissipated energy is closely related to the susceptibility of the system which the imaginary part is therefore related to actual energy dissipated by the system.
In fact, dissipation is an ubiquitous phenomenon in real physical systems. Its nature is made clear by considering the damped system, a paradigm for dissipative systems in the classical as well as in the quantum regime, after starting at a non-equilibrium position. Looking closely, one will notice that even in equilibrium oscillator, coordinates fluctuate. This effect is related to the Brownian motion of a free particle. Damping and fluctuation are both caused by the coupling of the harmonic oscillator to other degrees of freedom. A particle’s motion is damped because of collisions with molecules of air in the trap during the oscillations. The identical origin of both effects manifest itself in the fluctuation-dissipation theorem (F-D theorem). The Fourier transform of the response function $\chi_{qq}(t)$ will be denoted by $\chi_{qq}(\omega) = \int_0^\infty \Theta(t') \frac{1}{M} e^{-i\omega t'} \sin(\omega t') e^{i\omega t'} dt'$, where after integration we get:

$$\chi_{qq}(\omega) = \frac{(\omega^2 - \omega^2)}{M[(\omega^2 - \omega^2) + (\omega \gamma)^2]} + i \frac{(\omega \gamma)}{M[(\omega^2 - \omega^2) + (\omega \gamma)^2]},$$

with $\omega$ the frequency and $\gamma$ a function of the coupling factor $\eta$ between the Brownian particle and its environment.

Due to the fact that, the dissipation is induced by the imaginary part of this function, we will then focus on it to derive the correlation. Knowing that the F-D theorem was derived within linear response theory, therefore an exact relation for the damped harmonic oscillator is derived. Following [18] we may thus use the fluctuation-dissipation theorem to determine the equilibrium correlations functions for this system. For this reason, it is sufficient to calculate the position autocorrelation function as follows:

$$C_{qq}(t) = \langle Q(t) Q(0) \rangle,$$

where $Q(t)$ defines the position of the Brownian particle at time $t$ and $Q(0)$ at initial time. By the F-D theorem, (26) can be rewritten as:

$$C_{qq}(t) = \frac{\hbar}{M\pi} \int_{-\infty}^{+\infty} d\omega \chi(\omega) \frac{e^{-i\omega t}}{1 - e^{-\beta \omega}},$$

Let us consider $\frac{1}{1 - e^{-\beta \omega}} = \frac{1}{2} + \frac{1}{2} \cot \left( \frac{\hbar \beta \omega}{2} \right)$ therefore, we can decompose the correlation function into its symmetric and antisymmetric part given by $C_{qq}(t) = S_{qq}(t) + iA_{qq}(t)$ with

$$S_{qq}(t) = \frac{\hbar}{2M\pi} \int_{-\infty}^{+\infty} d\omega \frac{\omega \gamma}{M[(\omega^2 - \omega^2) + (\omega \gamma)^2]} \cos(\omega t) \cot \left( \frac{\hbar \beta \omega}{2} \right),$$

and its antisymmetric part:

$$A_{qq}(t) = \frac{\hbar}{2M\pi} \int_{-\infty}^{+\infty} d\omega \frac{\omega \gamma}{M[(\omega^2 - \omega^2) + (\omega \gamma)^2]} \sin(\omega t).$$

Here, it can be observed that the antisymmetric part vanishes at $t = 0$ thus, we will focus on the symmetric part which has a real physical meaning. This symmetric part will then, help in obtaining the autocorrelation function at the initial time. Thus, $C_{qq}(t) = \langle Q(0) Q(0) \rangle = \langle Q^2 \rangle$ which implies that,

$$\langle Q^2 \rangle = \frac{\hbar}{M\pi} \int_{-\infty}^{+\infty} d\omega \frac{\omega \gamma}{M[(\omega^2 - \omega^2) + (\omega \gamma)^2]} \cot \left( \frac{\hbar \beta \omega}{2} \right),$$

and

$$\langle P^2 \rangle = \frac{\hbar M}{\pi} \int_{-\infty}^{+\infty} d\omega \frac{\omega \gamma}{M[(\omega^2 - \omega^2) + (\omega \gamma)^2]} \omega^2 \cot \left( \frac{\hbar \beta \omega}{2} \right).$$

The presence of the environment means that, the system is communicating with the external world. However, the attention is focused on the subsystem under study. In this work, we are interested in the case where the total system is in Gibbs states at temperature $T$, and due to the effects of the environment, this temperature change to an effective one and the mass to the effective mass. Therefore, we will be interested on deriving these parameters in the next section.

4. Main results and discussion

This study set out to investigate the effects of NCity on a Brownian particle in a Penning potential. The following results will focus on the effective parameters where the comparison between the commutative and the NC case is presented.

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4.1. Effective temperature and effective mass of Brownian particle in NC space

Considering that the whole system is in equilibrium, in particular it has been shown in [45] that even at zero temperature, we may have a non-zero effective temperature \( \tilde{T} \). Thus, the reduced density matrix will not be a pure state and the particle cannot be stable in its ground state. The density operator at thermal equilibrium is defined by

\[
\hat{\rho} = \sum_i \rho_i \frac{\hat{Z}}{Z},
\]

where \( \hat{Z} \) and \( Z \) are respectively the effective mass and the partition function playing the role of a normalization constant to ensure that \( \text{tr} \sigma = 1 \), and \( \tilde{T} \) defines the effective temperature. At high frequency, the last term of (32) may be neglected, therefore we will be working in this condition. From the above equations, it is clear that the system is decoherent. In these conditions, we fall on an effective Hamiltonian which has the same form as that found in the work of [26], but with the difference that our parameters contain NC effects. So, (30) and (31) can be rewritten as follows:

\[
\langle Q^2 \rangle = \text{tr} \sigma Q^2 = \frac{\hbar}{2\lambda} \cot \frac{\hbar \lambda}{2 k_b \tilde{T}},
\]

and

\[
\langle P^2 \rangle = \text{tr} \sigma P^2 = \frac{\hbar \lambda}{2 k_b \tilde{T}} \cot \frac{\hbar \lambda}{2 k_b \tilde{T}}.
\]

Taking the product and the quotient of \( \langle Q^2 \rangle \) and \( \langle P^2 \rangle \), we obtain respectively the effective temperature and effective mass given by:

\[
k_b \tilde{T} = \frac{\hbar \lambda}{2} \frac{1}{\text{arctanh} \left( \frac{\hbar}{2\sqrt{\langle Q^2 \rangle}} \right)},
\]

and

\[
\tilde{M} = \frac{\langle P^2 \rangle}{\lambda^2 \langle Q^2 \rangle}.
\]

The effective temperature \( \tilde{T} \) and effective mass \( \tilde{M} \) differ from the real temperature \( T \) and mass \( M \) of the particle respectively, due to the coupling between the latter and the environment. In figure 1, we have plotted their behaviors versus real temperature for different values of the coupling factor for both the commutative and NC cases.

Figure 1. These figures depict simultaneously the impact of NC effects and that of the environment on our system. In these figures, the blue dash curves represent the effective temperature (left panel) and mass (right panel) when the NC effects are neglected \( (\theta = 0) \), while the solid red curves represent the same parameters but now with NC effects \( (\theta \neq 0) \), where the dimension of the NC parameter is length square.
It is observed that, when the real temperature increases, the effective temperature increases while the effective mass decreases in the both cases. This implies that the system exchanges the energy with its environment, thus we conclude that the particle abrupt the effective thermal effects of the bath and which is less important for NC space and for high real temperature too. Moreover, according to the difference found on the figures it is interesting to mention that the NCity acts in the system by reducing effective thermal effect for a Brownian particle coupled to a bath, implying that the NCity of the space might be used to control the thermal effects. This can be more important when the coupling factor between the particle and the bath become higher.

4.2. Effective energy of the particle in non-commutative space

When the Brownian particle interacts with its environment, there are some phenomena which emerge such as fluctuation and dissipation. Therefore, an effective energy appears in the system. We will be focused in this subsection on the investigation of the effects of NCity in this effective energy. For this reason, let us reconsider our Hamiltonian given by (7), therefore the eigenvectors of \( H_P \) and \( H_\perp \) will be those of the total Hamiltonian \( H \) with eigenvalues \( E = E_P + E_\perp \). After simplification, we get:

\[
E = \hbar \lambda (n_x + n_y + 1) - \kappa (n_x - n_y) + \hbar \omega_z \left( n_z + \frac{1}{2} \right).
\]

(37)

where \( n_x, n_y, \) and \( n_z \) are non-negative integers representing the principal quantum number, \( \kappa = \frac{\lambda'}{\lambda} \) and \( \lambda' = \frac{M/v_0}{2\hbar} \). Then, the partition function is found by \( Z = \sum_{n=0}^{N} e^{-\beta E} \), where \( \beta = \frac{1}{kT} \). This allows us to evaluate the effective energy of a quantum Brownian particle in Penning potential as follows: \( \tilde{E} = \frac{-\partial \log Z}{\partial \beta} \), where after some transformations we obtain the following expression:

\[
\tilde{E} = \hbar \lambda \left[ \frac{\sinh (\hbar \beta \lambda) - \kappa \sinh (\hbar \beta \lambda \kappa)}{\cosh (\hbar \beta \lambda) - \cosh (\hbar \beta \lambda \kappa)} \right] + \frac{\hbar \omega_z}{2} \cot \left( \frac{\hbar \beta \omega_z}{2} \right).
\]

(38)

Figure 2 depicts the evolution of this energy with real temperature for different coupling constant considering both the NC and commutative cases. The overall effect of NCity over the energy of the system is analyzed and a noticeable effect is found when the coupling factor \( \eta \) between the particle and its bath increase. The energy increases with the real temperature, however when the coupling factor increases, the curve of NC energy merge with that of commutative case as the real temperature increases. This represents the fact that, when the coupling between the particle and the bath is strong, the whole system takes a large dimension and then, the effects of quantum gravity manifest in NC space are not more detectable.

To better study and understand the phenomenon of decoherence, we will introduce the entropy to measure the loss of information and the specific heat capacity for the dissipated energy in our system.

Figure 2. The figure above depict simultaneously the impact of NCity effect and that of the environment on our system. In this figure, the blue dash curve represents the effective energy when the NCity effects are neglected (\( \theta = 0 \)), while the solid red curve represents the same parameter but now with NCity effects (\( \theta \neq 0 \)).
4.3. Entropy and specific heat capacity of the system

Due to the interaction of the particle with its environment, there is an exchange of information between this particle and the bath therefore, the particle can lose information and the system become decoherent. In this subsection, we study the effects of NCity on entropy and specific heat capacity of the system in order to compare with that of the commutative case and determine which of space structure of the system is more coherent.

Equation (39) gives the expression of the entropy of our system:

\[
S = \hbar \beta \Lambda \left[ \sinh \left( \frac{\hbar \beta \Lambda}{2} \right) - \sinh \left( \hbar \beta \Lambda \kappa \right) \right] + \frac{\hbar \beta \omega^2}{2} \cot h \left( \frac{\hbar \beta \omega^2}{2} \right) - \log \left[ \cosh \left( \frac{\hbar \beta \Lambda}{2} \right) - \cosh \left( \hbar \beta \Lambda \kappa \right) \right].
\] (39)

Figure 3 depicts the evolution of this entropy with real temperature for different coupling constant considering both the NC and commutative cases. It is observed that the entropy of the system increases faster in the NC case than in the commutative one as the real temperature increases. Thus, we can point out that the NCity increases the rate of decoherence in the system and this effect becomes more important for weak coupling. It turns out that, when the particle exchanges energy with its environment in the strong coupling, the size of the system becomes larger than the Planck scale thus, the effects of NCity tend to vanish in the system. Analogically, we have calculated the specific heat capacity of the system as given by:

\[
C = - K_b \left\{ \frac{(\hbar \beta \Lambda)^2 \left[ \cosh (\hbar \beta \Lambda) \cosh (\hbar \beta \Lambda \kappa) - 1 \right]}{\cosh (\hbar \beta \Lambda) - \cosh (\hbar \beta \Lambda \kappa)^2} - \frac{2(\hbar \beta \Lambda)^2 \kappa \sinh (\hbar \beta \Lambda) \sinh (\hbar \beta \Lambda \kappa)}{[\cosh (\hbar \beta \Lambda) - \cosh (\hbar \beta \Lambda \kappa)]^2} \right\}
+ \frac{\hbar^2 \beta \omega^2}{4} \frac{1}{\sinh^2 \left( \frac{\hbar \beta \omega}{2} \right)}.
\] (40)

Figure 4 depicts its evolution with real temperature for different coupling constant considering both the NC and commutative cases. Similarly as in the case of the entropy, we compare the specific heat capacity of the system. It is observed that, its value decreases and gets negative. This confirms the presence of self-gravitation effects in the system at the Planck scale [47]. The opposite behavior of specific heat capacity compared to the entropy shows the fluctuations and dissipation of the energy in the system. Moreover, comparing the NC specific heat capacity with the commutative one, it turns out that, for high temperature, the NCity curve merges with that of commutative. Implying that, the effects of quantum gravity are not noticeable when the real temperature becomes sufficiently high.
5. Concluding remarks

In this work, we studied decoherence of Brownian particle trapped in a Penning potential and coupled to a set of harmonic oscillators in NC space. The aim was to find the effects of NCity on the phenomenon of decoherence in the system. That is, the NCity in the space metric have been introduced by replacing the coordinates with their NC counterpart in order to derive the quantum Langevin equation in NC space. Thereafter, with the help of fluctuation-dissipation theorem, the correlation function has been found. This has allowed us to find the effective parameters of the system such as the effective temperature and effective mass. Similarly, the energy, the entropy and heat capacity have also been evaluated. We found that the NCity of the space contributes valuably to increase the rate of entropy and therefore increase the rate of decoherence in the system, while its effects rather give rise to a negative value of specific heat capacity and confirm the presence of self-gravitation effects in the system. We can therefore conclude that, the idea of considering the system in NC space seems more realistic.

Appendix A

With the help ordinary quantum mechanics, the Hamiltonian of the particle is written in the x-y plane under the form below:

$$H_0 = \frac{p_x^2}{2M} + \frac{1}{2} m\Omega^2 x^2 + \frac{p_y^2}{2m} + \frac{1}{2} m\Omega^2 y^2$$  \hspace{1cm} (A.1)

the non-commuting coordinates are expressed in term of the following commutating coordinates: $X = x - \frac{\theta}{2\hbar} P_y$; $P_x = p_x$; $Y = y + \frac{\theta}{2\hbar} P_x$; $P_Y = p_y$ the Hamiltonian in terms of these new coordinates is given by:

$$H_j = \frac{p_x^2}{2M} + \frac{p_y^2}{2m} + \frac{M^2 X^2}{2} + \frac{M^2 Y^2}{2} - \frac{M^2 \theta}{4\hbar} (XP_Y - YP_X)$$  \hspace{1cm} (A.2)

where the parameters $M$ and $\Lambda$ were being defined when we obtained (8), and the ladder operators were introduced through the following equations:

$$a_+ = \sqrt{\frac{M^2 \Lambda}{2\hbar}} (X + i \frac{P_X}{M\Lambda}), \quad a_+_+ = \sqrt{\frac{M\Lambda}{2\hbar}} (X - i \frac{P_X}{M\Lambda})$$  \hspace{1cm} (A.3)

$$X = \frac{\hbar}{2M \Lambda} (a_X + a_+^+) \quad P_X = \frac{1}{i} \sqrt{\frac{M\Lambda}{2}} (a_X - a_+^+)$$  \hspace{1cm} (A.4)

Figure 4. This figure shows simultaneously the impact of NCity effect and that of the environment on our system. In this figure, the blue dash curves represent the capacity of the system when the NCity effects are neglected ($\theta = 0$), while the solid red curve represent the same parameter but now with NCity effects ($\theta \neq 0$).
\begin{align}
  a_y &= \sqrt{\frac{\mathcal{M} \Lambda}{2\hbar}} \left( Y + i \frac{P_y}{\mathcal{M} \Lambda} \right) \quad a_y^+ = \sqrt{\frac{\mathcal{M} \Lambda}{2\hbar}} \left( Y - i \frac{P_y}{\mathcal{M} \Lambda} \right) \\
  Y &= \frac{\hbar}{2\mathcal{M}\Lambda} \left( a_y + a_y^+ \right); \quad P_y = \frac{1}{i} \sqrt{\frac{\hbar \mathcal{M} \Lambda}{2}} (a_y - a_y^+) \tag{A.6}
\end{align}

following the Schwinger representation for the angular momentum,

\begin{align}
  J_h &= (a_x^+ a_y + a_y^+ a_x) \tag{A.7} \\
  J_z &= \frac{1}{2i} (a_x^+ a_y - a_y^+ a_x) \tag{A.8} \\
  J_y &= \frac{1}{2} (a_x^+ a_y - a_y^+ a_x) \tag{A.9}
\end{align}

and considering the following transformation in which the angular momentum part takes the diagonal form of \( J_3 \)

\begin{align}
  \begin{pmatrix}
    a_x \\
    a_y
  \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
    1 & -i \\
    i & 1
  \end{pmatrix} \begin{pmatrix}
    a' \\
    a'
  \end{pmatrix} \tag{A.10}
\end{align}

and the Hamiltonian in NC space becomes

\begin{align}
  H_{J_3} &= \hbar \Lambda (a_x'^+ a_x' + a_y'^+ a_y') + 1 - \frac{\mathcal{M} \Lambda^2 \hbar}{2h}(a_x'^+ a_x' - a_y'^+ a_y') \tag{A.11}
\end{align}

if \( N_x = a_x'^+ a_x' \) and \( N_y = a_y'^+ a_y' \) are the numbers operators of the harmonic oscillator in the x-y direction, we may write the Hamiltonian in NC space under the following form:

\begin{align}
  H_{J_3} &= \hbar \Lambda (N_x + N_y + 1) - \frac{\mathcal{M} \Lambda^2 \hbar}{2h}(N_x - N_y) \tag{A.12}
\end{align}

the eigenvalue of Hamiltonian where we added the transversal part of the Hamiltonian is therefore given by:

\begin{align}
  H_{J_3} &= \hbar \Lambda (n_x + n_y + 1) - \frac{\mathcal{M} \Lambda^2 \hbar}{2h}(n_x - n_y) + \hbar \omega_0 \left( n_z + \frac{1}{2} \right) \tag{A.13}
\end{align}

for \( \lambda' = \frac{\mathcal{M} \Lambda \hbar}{2h} \) we obtain the value of energy found in (37).

**Appendix B**

The partition function of this system where the parallel and transversal part are consider is given by:

\begin{align}
  Z &= Z_{||} Z_{\perp} \tag{B.1}
\end{align}

from the relation (37), we have the partition function under the following form

\begin{align}
  Z &= \sum_{n_x=n_y=0}^{\infty} e^{-\hbar \Lambda \beta \left( 1 + n_x (1 - \kappa) + n_y (1 + \kappa) \right)} \sum_{n_z=0}^{\infty} e^{-\hbar \omega_0 n_z} \tag{B.2}
\end{align}

after some transformations we obtain the partition function given by:

\begin{align}
  Z &= \frac{1}{2} \frac{\cos \left( \hbar \beta \Lambda \right)}{\cos \left( \hbar \beta \Lambda \kappa \right)} \cdot \frac{1}{2 \sin \left( \frac{\hbar \beta \omega_0}{2} \right)} \tag{B.3}
\end{align}

where \( \kappa = \frac{\lambda'}{\Lambda} \).

In commutative limit, \( \beta \) tends to zero, thus \( \kappa \) tends to zero, \( \Lambda = \Omega \) and \( M = m \). We obtain the following expressions respectively for the effective temperature, effective mass and the partition function:

\begin{align}
  k_B \tilde{T} &= \frac{\hbar \Omega}{2} \frac{1}{\text{argth} \left( \frac{\hbar \beta}{\sqrt{4 \left( \Omega^2 \langle p^2 \rangle \right)}} \right)}, \tag{B.4} \\
  \tilde{m} &= \sqrt{\frac{\langle p^2 \rangle}{\Omega^2 \langle Q^2 \rangle}}, \tag{B.5}
\end{align}
and

\[ Z = \frac{1}{2} \cot h \left( \frac{\hbar \Omega}{2} \right) - \frac{1}{2} \sin h \left( \frac{\hbar \omega_z}{2} \right) \]  

where \( \langle Q^2 \rangle \) and \( \langle P^2 \rangle \) are defined as follows:

\[ \langle Q^2 \rangle = \text{tr} \hat{\sigma} Q^2 = \frac{\hbar}{2m\Omega} \cot h \left( \frac{\hbar \Omega}{2k_B T} \right) \]  

and

\[ \langle P^2 \rangle = \text{tr} \hat{\sigma} P^2 = \frac{\hbar \omega_z}{2} \cot h \left( \frac{\hbar \Omega}{2k_B T} \right) \]  

From the value of effective energy obtained in (38) and considering the commutative limit \( \theta = 0 \) we have commutative effective Energy given by:

\[ \frac{E}{k_B} = \hbar \Omega \cot h \left( \frac{\hbar \Omega}{2} \right) + \frac{\hbar \omega_z}{2} \cot h \left( \frac{\hbar \omega_z}{2} \right). \]  

this effective energy is in accordance with the result found in [26], where the transversal part related to the \( Z \)-axis is added due to the configuration of Penning trap potential.

Similarly, the Entropy of the system is also found in commutative space and is given by:

\[ \frac{S}{k_B} = \hbar \Omega \cot h \left( \frac{\hbar \Omega}{2} \right) + \frac{\hbar \omega_z}{2} \cot h \left( \frac{\hbar \omega_z}{2} \right) - \ln \left( 2 \sin h \left( \frac{\hbar \Omega}{2} \right) \right) - \ln \left( 2 \sin h \left( \frac{\hbar \omega_z}{2} \right) \right). \]  

and Heat capacity in commutative space is express as:

\[ C = -k_B \left( \frac{\hbar \Omega}{2} \right)^2 \left[ \frac{1}{\sin h \left( \frac{\hbar \Omega}{2} \right)} \right]^2 + \left( \frac{\hbar \omega_z}{2} \right)^2 \left[ \frac{1}{\sin h \left( \frac{\hbar \omega_z}{2} \right)} \right]^2. \]  

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