A category analogue of the density topology
non-homeomorphic with the $\mathcal{I}$-density topology

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Abstract
The paper deals with the category analogue of a density point and a density topology (with respect to a Lebesgue measure) on the real line which is different from the $\mathcal{I}$-density topology considered in Poreda et al. (Fundam Math 125:167–173, 1985; Comment Math Univ Carol 26:553–563, 1985). This topology called the intensity topology, manifests several properties analogous to that of $\mathcal{I}$-density topology, but there are also differences. The class of function which are continuous as functions from $\mathbb{R}$ equipped with an intensity topology to $\mathbb{R}$ equipped with the natural topology is included in the first class of Baire Darboux functions.

Keywords Density point · Density topology · $\mathcal{I}$-density point · $\mathcal{I}$-density topology · Intensity topology · Intensely continuous functions · Baire class

Mathematics Subject Classification 54 A 10 · 26 A 21 · 28 A 05

1 Introduction

In [8] one can find a characterization of a Lebesgue density point which uses only the $\sigma$-ideal of nullsets. This was a starting point for the definition of category density point, called the $\mathcal{I}$-density point and for the construction of $\mathcal{I}$-density topology, which was studied later in numerous papers. In this paper we shall formulate another characterization of a Lebesgue density point, again not using a Lebesgue measure but only the $\sigma$-ideal of nullsets. This characterization leads again to a definition of category density point, which is, however, not equivalent to $\mathcal{I}$-density and to a category density topology, which is different from the $\mathcal{I}$-density topology.

In the sequel $\mathcal{L}$ will denote the $\sigma$-algebra of Lebesgue measurable sets on the real line, $\mathcal{N}$—the $\sigma$-ideal of nullsets, $\mathcal{B}$—the $\sigma$-algebra of sets having the Baire property,
\( I \)—the \( \sigma \)-ideal of sets of the first category, \( \lambda \) will stand for the Lebesgue measure. We shall use also the following denotation:

\[ A' = \{ t + x : t \in A \} \text{ and } xA = \{ xt : t \in A \}. \]

Recall that \( x_0 \in \mathbb{R} \) is a density point of \( A \in \mathcal{L} \) if and only if

\[ \lim_{h \to 0^+} \frac{\lambda(A \cap [x_0 - h, x_0 + h])}{2h} = 1, \]

\( x_0 \) is a right-hand (left-hand) density point if and only if \( \lim_{h \to 0^+} \frac{\lambda(A \cap [x_0 - h, x_0])}{h} = 1 \) (\( \lim_{h \to 0^+} \frac{\lambda(A \cap [x_0 - h, x_0])}{h} = 1 \)). A point \( x_0 \) is a dispersion (right-hand dispersion, left-hand dispersion) point of \( A \in \mathcal{L} \) if and only if it is a density (right-hand density, left-hand density) point of \( A' \), or, equivalently, if the above limits are equal to 0. Observe also that \( x_0 \) is a density point of \( A \) if and only if 0 is a density point of \( A - x_0 \).

The set of all density points of \( A \) is denoted by \( \Phi(A) \). The operator \( \Phi : \mathcal{L} \to 2^\mathbb{R} \) has the following properties (see [7], Ch. 22):

1. \( \forall A \in \mathcal{L} \quad \lambda(A \Delta \Phi(A)) = 0 \),  
2. \( \forall A, B \in \mathcal{L} \quad (\lambda(A \Delta B) = 0 \implies \Phi(A) = \Phi(B)) \),
3. \( \Phi(\emptyset) = \emptyset, \Phi(\mathbb{R}) = \mathbb{R} \),
4. \( \forall A, B \in \mathcal{L} \quad \Phi(A \cap B) = \Phi(A) \cap \Phi(B) \).

From (1) it follows immediately that \( \Phi : \mathcal{L} \to \mathcal{L} \) (in fact \( \Phi : \mathcal{L} \to G_{\delta_{\sigma}} \)).

The family \( T_d = \{ A \in \mathcal{L} : A \subset \Phi(A) \} \) is a topology on \( \mathbb{R} \) (called the density topology), which is essentially stronger than a natural topology.

Since \( x_0 \) is a density (resp. dispersion) point of \( A \in \mathcal{L} \) if and only if it is simultaneously a left-hand and a right-hand density (resp. dispersion) point of \( A \), we shall deal in the sequel with the right-hand case, the remaining being obvious. The expression “in measure” will always mean with respect to the Lebesgue measure restricted to \([0, 1], [-1, 0] \) or \([-1, 1]\).

In [8] it was observed that \( x_0 \) is a right-hand density (resp. dispersion) point of \( A \in \mathcal{L} \) if and only if the sequence \( \{ \chi_{(n(A-x_0)\cap [0,1])} \}_{n \in \mathbb{N}} \) of characteristic functions converges in measure to \( \chi_{[0,1]} \) (resp. to \( \chi_{\emptyset} \)). By virtue of theorem of Riesz, \( x_0 \) is a right-hand density (resp. dispersion) point of \( A \in \mathcal{L} \) if and only if for each increasing sequence \( \{ n_m \}_{m \in \mathbb{N}} \) of positive integers there exists a subsequence \( \{ n_{mp} \}_{p \in \mathbb{N}} \) such that \( \{ \chi_{n_{mp}} \cdot (A-x_0) \cap [0,1] \}_{p \in \mathbb{N}} \) converges to \( \chi_{[0,1]} \) (resp. to \( \chi_{\emptyset} \)) almost everywhere. The last condition indeed does not use the Lebesgue measure, only the \( \sigma \)-ideal \( N \) is important.

**Definition 1** [8] A point \( x_0 \in \mathbb{R} \) is an \( \mathcal{I} \)-density point of \( A \in \mathcal{B} \) if and only if \( \{ \chi_{n(A-x_0)\cap [-1,1]} \}_{n \in \mathbb{N}} \) converges in category to \( \chi_{[-1,1]} \).

The convergence in category means that for each increasing sequence \( \{ n_m \}_{m \in \mathbb{N}} \) of positive integers there exists a subsequence \( \{ n_{mp} \}_{p \in \mathbb{N}} \) such that \( \{ \chi_{n_{mp}} \cdot (A-x_0)\cap [-1,1] \}_{p \in \mathbb{N}} \) converges to \( \chi_{[-1,1]} \) \( \mathcal{I} \)-almost everywhere (i.e. except on a set belonging to \( \mathcal{I} \)) (compare [13]).


\(\mathcal{I}\)-dispersion points, right-hand \(\mathcal{I}\)-density or \(\mathcal{I}\)-dispersion points and left-hand \(\mathcal{I}\)-density or \(\mathcal{I}\)-dispersion points are defined in an obvious way. If \(\Phi_{\mathcal{I}}(A) = \{x \in \mathbb{R} : x\) is an \(\mathcal{I}\)-density point of \(A\}\) for \(A \in \mathcal{B}\), then the operator \(\Phi_{\mathcal{I}}\) has all properties analogous to properties (1)–(4) of \(\Phi\), also \(\Phi_{\mathcal{I}} : \mathcal{B} \rightarrow \mathcal{B}\). The family \(\mathcal{T}_{\mathcal{I}} = \{A \in \mathcal{B} : A \subset \Phi_{\mathcal{I}}(A)\}\) is a topology on \(\mathbb{R}\) (called the \(\mathcal{I}\)-density topology), which is essentially stronger than a natural topology (see [8] or [3]).

\section*{2 New characterization}

Suppose that \(A\) is a Lebesgue measurable set. Put \(a_n = \lambda(A \cap [\frac{1}{n+1}, \frac{1}{n}])\) for \(n \in \mathbb{N}\) and let \(b_n = n(n+1)a_n\) be an average density of the set \(A\) on the interval \([\frac{1}{n+1}, \frac{1}{n}]\).

**Definition 2** We shall say that 0 is the right-hand \(C\)-density point of \(A\) (\(C\)-density point of \(A\)) if and only if \(\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{\infty} b_i = 1\) (\(\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{\infty} b_i = 0\)).

The definition of the left-hand \(C\)-density point (\(C\)-dispersion point) is formulated in an obvious way. 0 is the \(C\)-density (\(C\)-dispersion) point of \(A\) if and only if it is simultaneously the right-hand and the left-hand \(C\)-density (\(C\)-dispersion) point of \(A\).

We say that \(x_0\) is the right-hand (left-hand) \(C\)-density (\(C\)-dispersion) point of \(A\) if and only if 0 is the right-hand (left-hand) \(C\)-density (\(C\)-dispersion) point of \(A - x_0\).

Observe that 0 is the right-hand \(C\)-density point of \(A\) if and only if it is the right-hand \(C\)-dispersion point of \(A\).

**Theorem 3** 0 is the right-hand \(C\)-dispersion point of \(A\) if and only if 0 is the right-hand \(C\)-dispersion point of \(A\).

**Proof** “⇒” Take \(\varepsilon > 0\). There exists \(n_0 \in \mathbb{N}\) such that for \(n \geq n_0\) we have \(\frac{1}{2n-1} \cdot \sum_{i=1}^{2n-1} b_i < \frac{\varepsilon}{4}\). Hence also \(\frac{1}{2n-1} \cdot \sum_{i=n}^{2n-1} b_i < \frac{\varepsilon}{4}\). Since \(2n \leq 4 \cdot \frac{n(n+1)}{2n-1} \leq 4 \cdot \frac{(i+1)}{2n-1}\) for \(i \geq n\), then

\[
\frac{\lambda(A \cap [\frac{1}{n+\frac{1}{2n}}, \frac{1}{n}])}{\frac{1}{n}} = 2n \cdot \sum_{i=n}^{2n-1} a_i \leq \frac{4}{2n-1} \sum_{i=n}^{2n-1} i(i+1)a_i = \frac{4}{2n-1} \sum_{i=n}^{2n-1} i \cdot \frac{\varepsilon}{4} = \frac{n \cdot \varepsilon}{4} = \frac{\varepsilon}{4} \quad \text{for} \quad n \geq n_0.
\]

This implies \(\lim_{n \to \infty} \frac{\lambda(A \cap [0, \frac{1}{n}])}{\frac{1}{n}} = 0\) and 0 is the right-hand dispersion point of \(A\).

“⇐” Take \(\varepsilon > 0\). Fix a positive integer \(k > \frac{3}{\varepsilon} + 1\). Then we have \(\frac{n}{nk-1} < \frac{\varepsilon}{3}\) for each \(n \in \mathbb{N}\). Since \(\lim_{n \to \infty} \frac{\lambda(A \cap [\frac{1}{n}, \frac{1}{nk}])}{\frac{1}{n}} = 0\), then also \(\lim_{n \to \infty} \frac{\lambda(A \cap [\frac{1}{nk}, \frac{1}{n}])}{\frac{1}{n-nk}} = \lim_{n \to \infty} \frac{\lambda(A \cap [\frac{1}{nk}, \frac{1}{nk}])}{\frac{1}{nk-k}}\).

\(\lambda(A \cap [\frac{1}{nk}, \frac{1}{nk}]) = 0\). There exists \(n_1 \in \mathbb{N}\) such that \(\frac{nk}{k-1} \lambda(A \cap [\frac{1}{nk}, \frac{1}{nk}]) < \frac{\varepsilon}{3k}\) for \(n \geq n_1\). Since \(\frac{(i+1)}{2n-1} \leq \frac{nk}{k-1}\), then \(\frac{1}{nk} \sum_{i=n}^{nk-1} i(i+1)a_i = nk \lambda(A \cap [\frac{1}{nk}, \frac{1}{nk}]) < \frac{\varepsilon}{3k} \cdot (k-1) < \frac{\varepsilon}{3}\).

Let \(n_2 \in \mathbb{N}\) be such that \(\frac{k}{nk-1} < \frac{\varepsilon}{3}\) for \(n \geq n_2\). Then for \(n \geq n_0\), 

\(\lambda(A \cap [\frac{1}{nk}, \frac{1}{nk}]) = 0\) and \(nk - 1 < m \leq (n+1)k - 1\) we have \(\frac{1}{m} \sum_{i=nk}^{m} b_i = \frac{1}{m} \sum_{i=nk}^{m} b_i = \frac{1}{m} \sum_{i=nk}^{m} \sum_{i=nk}^{m} b_i \leq \frac{1}{m} \sum_{i=nk}^{m} b_i + \frac{1}{m} \sum_{i=1}^{m} b_i < \frac{\varepsilon}{3} + \frac{1}{nk} \sum_{i=1}^{m} b_i \leq \frac{\varepsilon}{3} + \frac{1}{nk} \sum_{i=1}^{m} b_i < \frac{\varepsilon}{3} + \frac{2}{3} \varepsilon = \varepsilon\). Hence \(\frac{1}{m} \sum_{i=1}^{m} b_i < \varepsilon\) for \(m \geq n_0k - 1\), which means that 0 is the right-hand \(C\)-dispersion point of \(A\).
Remark 4 Obviously similar theorem holds also for the left-hand C-dispersion point of \( A \). Also \( 0 \) is the right-hand (or left-hand) C-density point of \( A \) if and only if it is the right-hand (or left-hand) density point of \( A \).

3 Intensity topology

Now we shall formulate the definition of C-dispersion (and C-density) point without using the Lebesgue measure. By virtue of Theorem 3 it will be another characterization of the dispersion (and density) point. This will enable us to introduce the category analogue of the density and dispersion point.

Put \( A_n = [n(n + 1) \cdot (A - \frac{1}{n+1})] \cap [0, 1] \). Then \( b_n = \lambda(A_n) = \int_0^1 \chi_{A_n}(t) dt \), so

\[
\frac{1}{n} \sum_{i=1}^n b_i = \frac{1}{n} \int_0^1 \sum_{i=1}^n \chi_{A_i}(t) dt.
\]

If we denote \( f_n = \frac{1}{n} \sum_{i=1}^n \chi_{A_i} \) for \( n \in \mathbb{N} \), then we have \( 0 \leq f_n(t) \leq 1 \) for each \( n \in \mathbb{N} \) and \( t \in [0, 1] \).

So \( 0 \) is a right hand C-dispersion point (dispersion point) of \( A \) if and only if \( \int_0^1 f_n \to 0 \) as \( n \to \infty \), which means \( f_n \to 0 \) in \( L^1 \) norm. Since the sequence \( \{f_n\}_{n \in \mathbb{N}} \) is commonly bounded this is equivalent to \( f_n \to 0 \) in measure (compare [2], p. 245, Th. 5.36 and p. 246 Th. 5.37).

Definition 5 We say that \( 0 \) is the right-hand rarefaction (intensity) point of \( A \in \mathcal{B} \) if and only if the sequence \( \{f_n\}_{n \in \mathbb{N}} \) defined above converges to \( 0 = \chi_\emptyset \) (converges to \( \chi_{[0,1]} \)) in category.

The left-hand rarefaction (intensity) point is defined in the obvious way. \( 0 \) is rarefaction (intensity) point of \( A \in \mathcal{B} \) if and only if it is simultaneously the right-hand and the left-hand rarefaction (intensity) point of \( A \). A point \( x_0 \) is the right-hand rarefaction (intensity) point of \( A \) if and only if \( 0 \) is the right-hand rarefaction (intensity) point of \( A - x_0 \).

In the sequel we shall use the following denotation:

\[
A_n(x) = \left[ n(n + 1)(A - x) - \frac{1}{n + 1} \right] \cap [0, 1].
\]

If \( x = 0 \) we shall write \( A_n \) rather than \( A_n(0) \).

Let \( \Phi_i(A) = \{x \in \mathbb{R} : x \text{ is the intensity point of } A\} \) for \( A \in \mathcal{B} \).

Theorem 6 \( \Phi_i \) is a lower density operator, i.e.

1. \( \forall A \in \mathcal{B} \quad \Phi_i(A) \triangle A \in \mathcal{T} \),
2. \( \forall A, B \in \mathcal{B} \quad (A \triangle B \in \mathcal{T} \implies \Phi_i(A) = \Phi_i(B)) \),
3. \( \Phi_i(\emptyset) = \emptyset, \Phi_i(\mathbb{R}) = \mathbb{R} \),
4. \( \forall A, B \in \mathcal{B} \quad \Phi_i(A \cap B) = \Phi_i(A) \cap \Phi_i(B) \).

Proof Ad 1. Observe that if \( A = G \triangle P \), where \( G \) is open and \( P \in \mathcal{T} \), then \( G \subset \Phi_i(A) \subset \bar{G} \) and \( \bar{G} \setminus G \) is nowhere dense.

Proofs of 2, 3 and 4 are standard. \( \square \)
Theorem 7 The family $T_I = \{ A \in B : A \subset \Phi(I(A)) \}$ is a topology essentially stronger than a natural topology on the real line.

Proof Exactly the same as in [8] for $T_I$. \( \square \)

We shall show that the notion of intensity point is not equivalent to that of $I$-density point. However, the topology $T_I$ shares most of the properties (but not all) with the topology $T_{\mathcal{I}}$.

Theorem 8 There exists a set $A \subset [0, 1]$ such that $0$ is a right-hand rarefaction point of $A$ but not the right-hand $I$-dispersion point of $A$.

Proof Let $\{ \phi_n \}_{n \in \mathbb{N}}$ be a standard example of a sequence of functions defined on $[0, 1]$ which converges in measure and diverges everywhere on $[0, 1]$, i.e. $\phi_1 = \chi([0, 1])$, $\phi_2 = \chi([0, \frac{1}{4}, 1])$, $\phi_3 = \chi([\frac{1}{4}, \frac{1}{2}, 1])$, $\ldots$, $\phi_{2k} = \chi([0, \frac{1}{2k}, 1])$, $\ldots$, $\phi_{2k+1} = \chi([1-\frac{1}{2k}, 1])$. Put $E_n = \{ x \in (0, 1) : \phi_n(x) = 1 \}$.

Let $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n = \frac{1}{n(n+1)} \cdot E_n + \frac{1}{n+1}$. Observe that for each $k \in \mathbb{N}$ we have $\sum_{n=1}^{2k+1} \chi_{(n(n+1), 1)}(A - \frac{1}{n+1} \cdot 0, 1) = \sum_{n=1}^{2k+1} \chi E_n (x) = 1$ for each $x \in (0, 1]$. Hence $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{(n(n+1), 1)}(A - \frac{1}{n+1} \cdot 0, 1) (x) = 0$ for each $x \in (0, 1]$ and $0$ is a right-hand rarefaction point of $A$.

At the same time the set $A \cap \left[ \frac{1}{n+1}, \frac{1}{n} \right]$ includes a non-empty open set for each $n \in \mathbb{N}$, from which it follows easily that $0$ is not a right-hand $I$-dispersion point of $A$. \( \square \)

Theorem 9 There exists a set $A \subset [0, 1]$ such that $0$ is the right-hand $I$-dispersion point of $A$ but not the right-hand rarefaction point of $A$.

Proof Put $C_1 = \left[ \frac{1}{6}, 1 \right] \setminus \left( \frac{1}{4}, \frac{1}{3} \right)$. Let $I_1^1$ and $I_1^2$ be closed components of $C_1$ numbered from the left to the right. Suppose that we have defined $C_1, C_2, \ldots, C_{n-1}$ such that $C_{n-1} \subset \left[ \frac{1}{6n-1}, \frac{1}{6n-2} \right]$ and $I_1^1, \ldots, I_{2n}^1$ is the sequence of closed components of $C_{n-1}$ numbered from the left to the right. We shall define $C_n \subset \left[ \frac{1}{6n}, \frac{1}{6n-1} \right]$ in the following way: let $J_i^i$ for $i = 1, \ldots, 2n-1$ be an open interval of the form $\left( \frac{1}{m+1}, \frac{1}{m} \right)$, where $m$ is a positive integer such that the center of $\frac{1}{6} \cdot I_{n-1}^i$ belongs to $\left[ \frac{1}{m+1}, \frac{1}{m} \right]$. Put $C_n = \frac{1}{6} C_{n-1} \setminus \bigcup_{i=1}^{2n} J_i^i$. Then $C_n$ is the union of $2^n$ closed intervals $I_1^n, \ldots, I_{2n}^n$.

At last put $A = \bigcup_{n=1}^{\infty} C_n$. Observe that $B = \bigcap_{n=1}^{\infty} 6^n A \cap \left[ \frac{1}{6}, 1 \right]$ is a perfect nowhere dense subset of $[\frac{1}{6}, 1]$. Observe also that the interval $\left[ \frac{1}{6n}, \frac{1}{6n-1} \right]$ is the union of $5 \cdot 6^n-1$ intervals of the form $\left[ \frac{1}{m+1}, \frac{1}{m} \right]$, namely $\left[ \frac{1}{6n}, \frac{1}{6n-1} \right] = \bigcup_{m=6n+1}^{6^n} \left[ \frac{1}{m+1}, \frac{1}{m} \right]$ while the set $C_n$ is the union of $5 \cdot 6^n-1 - (6^n-1 + 2 \cdot 6^n-2 + \ldots + 2n-2 \cdot 6 + 2n-1)$ among these intervals. Since $A_k = (k(k+1) - 1 \cdot \frac{1}{6n-1}) \cap \{0, 1\} = \{k(k+1)(C_n - \frac{1}{k+1}) \} \cap \{0, 1\}$ for $k \in \{6^n-1, \ldots, 6^n - 1\}$, we have $\frac{1}{6n-1} \cdot \sum_{k=6^n-1}^{6^n-1} \chi_{A_k} (t) > \frac{7}{10}$ for each $n \in \mathbb{N}$ and $t \in \{0, 1\}$ and $0$ is not a right-hand rarefaction point of $A$. Indeed observe that $\frac{1}{5} \cdot \frac{1}{6n-1} \cdot (6^n-1 + 2 \cdot 6^n-2 + \ldots + 2n-2 \cdot 6 + 2n-1) = \frac{1}{5} (1 + \frac{2}{6} + \frac{2}{6^2} + \ldots + \frac{2n-1}{6^n-1}) < \frac{3}{10}$.

To show that $0$ is a right-hand $I$-dispersion point of $A$ take an increasing sequence $\{ n_m \}_{m \in \mathbb{N}}$ of positive integers. For each $m \in \mathbb{N}$ there exists $k_m \in \mathbb{N}$
such that $\frac{1}{n_m} \in [\frac{1}{6^m}, \frac{1}{6^{m-1}})$. Since the sequence $\frac{6^{km}}{n_m}$ is bounded ($1 \leq \frac{6^{km}}{n_m} < 6$), there exists a convergent subsequence $\left\{\frac{6^{km}}{n_{m_p}}\right\}$. Put $c = \lim_{p \to \infty} \frac{6^{km}}{n_{m_p}}$. We have
\[
\limsup_{p \to \infty}(n_{m_p} \cdot A) \cap [0, 1] = (c \cdot B) \cap [0, 1].
\]
Since this is a perfect nowhere dense set, the proof is finished. \hfill $\Box$

Now we shall present some basic properties of the topology $T_i$. The proofs go exactly in the same way as for the topology $T_1$, so we shall give only references at the end of the following theorem.

**Theorem 10** The topology $T_i$ on $\mathbb{R}$ has the following properties:

1. a subset $A$ of $\mathbb{R}$ is closed and discrete with respect to $T_i$ if and only if $A \in \mathcal{I}$,
2. a subset $A$ of $\mathbb{R}$ is $T_i$-nowhere dense if and only if it is $T_i$—of the first category if and only if $A \in \mathcal{I}$,
3. $T_i$ is neither first countable, nor separable, nor has the Lindelöf property,
4. $T_i$ is not regular,
5. a subset $A$ of $\mathbb{R}$ is compact with respect to $T_i$ if and only if $A$ is finite,
6. the family of $T_i$—Borel sets coincides with the family of sets having the Baire property (in the natural topology), each set which is a $T_i$—Borel set is the union of $T_i$-open and $T_i$-closed set,
7. if $A \subset \mathbb{R}$, then $\text{Int}_i(A) = A \cap \Phi_i(B)$, where $B \subset A$ is a Baire kernel of $A$.

For the proofs of 1, 3, 5 see [3], p. 37, proof of 2, 6 see [9], proof of [4] see [14] and of 7 see [4].

The following property of $T_i$ needs a separate proof.

**Theorem 11** Each interval $[a, b] \subset \mathbb{R}$ is connected in $T_i$.

**Proof** Suppose that $[a, b] = A \cup B$, where $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$ and $A, B \in T_i$ (it means that $A, B$ are open in the subspace topology $T_i|_{[a,b]}$). We have $A \subset \Phi_i(A)$ and $B \subset \Phi_i(B)$, where at the endpoints of $[a, b]$ one sided intensity is considered.

At the same time $A = G_1 \triangle P_1$, $B = G_2 \triangle P_2$, where $G_1, G_2$ are regular open and $P_1, P_2$ are of the first category (all in the natural topology). Let $(c, d) \subset [a, b]$ be a component of $G_1$ (in the natural topology). We shall prove that $(c, d) \subset A$. First observe that $P_1 \cap (c, d) = \emptyset$. Indeed, if there exists $x_0 \in P_1 \cap (c, d)$, then $x_0 \in B$, so $x_0 \not\in \Phi_i(B)$. It is impossible because $A$ is residual in some neighbourhood of $x_0$ (in the natural topology) and $x_0 \not\in \Phi_i(A)$. A point $d$ is a left-hand point of intensity of $A$, so it cannot belong to $B$ and similarly for a point $c$. Obviously if $(c, d)$ is a component of $G_2$, then $(c, d) \subset B$.

Take a closed component $[c_1, d_1] \subset [a, b]$ of $A$. Obviously $c_1 > a$ or $d_1 < b$.

Suppose that the second case holds. Consider the interval $I_1 = [c_1, d_1 + (d_1 - c_1)] = [c_1, 2d_1 - c_1]$. Observe that there exists a point $x_1 \in B \cap (d_1, 2d_1 - c_1)$. Since $B \subset \Phi_i(B)$, in each neighbourhood of $x_1$ one can find a closed component of $B$. In particular there exists a closed component $[c_2, d_2]$ of $B$ such that $c_2 \in (d_1, \min(b, 2d_1 - c_1))$. Similarly as above we can find a point $x_2 \in A \cap (2c_2 - d_2, c_2)$ and a closed component $[c_3, d_3]$ of $A$ such that $d_3 \in (2c_2 - d_2, c_2)$. Continuing in this way we obtain a sequence of disjoint intervals $\{(c_n, d_n)\}_{n \in \mathbb{N}}$ such that $\bigcup_{n=1}^{\infty} [c_{2n-1}, d_{2n-1}] \subset A$. 

\[\checkmark\] Springer
\[ \bigcup_{n=1}^{\infty} [c_{2n}, d_{2n}] \subset B, \quad d_n - c_n \to 0 \quad \text{and there exists } x_0 \in (a, b) \text{ such that } c_n \to x_0 \quad \text{and } d_n \to x_0. \]

Observe also that the sequence \( \{c_{2n-1}\}_{n \in \mathbb{N}} \) is increasing and \( \{c_{2n}\}_{n \in \mathbb{N}} \) is decreasing. From the construction it follows that \( d_n - c_n \geq \text{dist}(x_0, [c_n, d_n]) \). Consider the sequence \( \{(c_{2n}, d_{2n})\}_{n \in \mathbb{N}} \). Let \( k_n \) and \( m_n \) for \( n \in \mathbb{N} \) be positive integers such that

\[ \bigcup_{i=k_n}^{m_n} \left[ x_0 + \frac{1}{i+1}, x_0 + \frac{1}{i} \right] \subset [c_{2n}, d_{2n}] \subset B. \]

Both sequences \( \{k_n\}_{n \in \mathbb{N}}, \{m_n\}_{n \in \mathbb{N}} \) are increasing and tend to infinity. Moreover \( \liminf_{n \to \infty} \frac{m_n}{k_n} \geq 2 \). Then \( \limsup_{n \to \infty} \frac{1}{m_n} \sum_{i=1}^{m_n} \chi_{i(i+1)/2} \left( B - x_0 - \frac{1}{i+1} \right)(x) \geq \frac{1}{2} \) for each \( x \in [0, 1] \) and the same holds for each subsequence \( \{m_{n_p}\}_{p \in \mathbb{N}} \) of \( \{m_n\}_{n \in \mathbb{N}} \). Hence \( x_0 \) is not a right-hand rarefaction point of \( B \). Similarly we see that \( x_0 \) is not a left-hand rarefaction point of \( A \). Then \( x_0 \in (a, b) \) is neither intensity point of \( A \), nor of \( B \)—a contradiction.

**Corollary 12** The family of all \( T_i \)-connected subsets of \( \mathbb{R} \) coincides with the family of intervals of all kinds (open, closed, half-open, bounded and unbounded).

### 4 Intense continuity

To introduce the notion of intense continuity (right- or left intense continuity) we shall need the notion of interior intensity point of an arbitrary set \( A \subset \mathbb{R} \). Recall that the Baire kernel of \( A \subset \mathbb{R} \) is a set \( B \subset A \) having the Baire property such that for each set \( C \) having the Baire property if \( C \subset A \), then \( C \setminus B \in \mathcal{I} \). For each set \( A \subset \mathbb{R} \) there exists the Baire kernel of \( A \) (compare [11] or [6], §11.IV, Cor. 1, p. 90).

**Definition 13** We shall say that a function \( f : \mathbb{R} \to \mathbb{R} \) is intensely continuous at a point \( x_0 \in \mathbb{R} \) if and only if \( x_0 \) is an intensity point of a Baire kernel of \( f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon)) \), in another words \( x_0 \) is an interior intensity point of \( f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon)) \) for each \( \varepsilon > 0 \). The right- or left intense continuity of \( f \) at \( x_0 \) is defined in an obvious way.

**Definition 14** We shall say that a function \( f : \mathbb{R} \to \mathbb{R} \) is intensely continuous if for each set \( G \) open in the natural topology \( (T_{\text{nat}}) \) \( f^{-1}(G) \in \mathcal{I} \), i.e. \( f \) is continuous as a function from \( (\mathbb{R}, T_i) \) to \( (\mathbb{R}, T_{\text{nat}}) \). The right- or left intense continuity as well as intense continuity \( \mathcal{I} \)-almost everywhere is defined in an obvious way.

**Theorem 15** A function \( f : \mathbb{R} \to \mathbb{R} \) is intensely continuous (right intensely continuous) if and only if is intensely continuous (right intensely continuous) at each point \( x \in \mathbb{R} \).

**Proof** We shall prove first that if for each point \( x \in A \) there exists a set \( B_x \subset A \) having the Baire property such that \( x \in \Phi_i(B_x) \), then \( A \) has the Baire property. Suppose that it is not the case. Then there exists an interval \( (a, b) \subset \mathbb{R} \) such that for
each \((c, d) \subset (a, b)\) the set \(A \cap (c, d)\) is of the second category but not residual in \((c, d)\). If \(B\) is an arbitrary Baire kernel of \(A \cap (a, b)\), then \(B\) is of the first category, so for each \(x \in A \cap (a, b)\) a point \(x\) is not an intensity point of \(B\), a contradiction. The rest of the proof is routine. The proof in the case of right intense continuity is analogue.

To prove basic properties of intensely continuous functions we shall need some lemmas. In the sequel \(\{(a_i, b_i)\}_{i \in \mathbb{N}}\) will denote the basis of the natural topology on \((0, 1)\).

**Lemma 16** If \(0\) is a right-hand intensity point of a regular open set \(A\), then for each \(m \in \mathbb{N}\) there exists a positive integer \(n_m\) such that for each \(i \in \{1, 2, \ldots, m\}\) there exists an open interval \((c_i^m, d_i^m) \subset (a_i, b_i)\) such that

\[
\frac{1}{n_m} \sum_{j=1}^{n_m} \chi_{A_j}(t) \geq \frac{m - 1}{m} \quad \text{for each } t \in \bigcup_{i=1}^{m} (c_i^m, d_i^m).
\]

The sequence \(\{n_m\}_{m \in \mathbb{N}}\) can be chosen to be increasing. Here \(A_j = j(j + 1)(A - \frac{1}{j+1}) \cap (0, 1)\), as before.

**Proof** This is obvious for \(m = 1\). Suppose that the thesis holds for some \(m \in \mathbb{N}\).

Let \(\{r_p\}_{p \in \mathbb{N}}\) be an arbitrary increasing sequence of natural numbers such that \(r_1 > n_m \geq m\). There exists a subsequence \(\{r_{p_k}\}_{k \in \mathbb{N}}\) such that \(\lim_{k \to \infty} \frac{1}{r_{p_k}} \sum_{j=1}^{r_{p_k}} \chi_{A_j}(t) = 1\) \(\mathcal{I}\)-almost everywhere on \((0, 1)\). Hence for each \(i \in \{1, 2, \ldots, m, m + 1\}\) there exists \(t_i \in (a_i, b_i)\) and \(p_k\) such that for each \(p \geq p_k\) we have \(\frac{1}{r_p} \sum_{j=1}^{r_p} \chi_{A_j}(t_i) \geq \frac{m}{m+1}\).

Let \(p(m + 1) = \max\{p_{k_1}, p_{k_2}, \ldots, p_{k_n}, p_{m+1}\}\) and \(n_{m+1} = r_{p(m+1)}\). Then for each \(i \in \{1, 2, \ldots, m, m + 1\}\) we have \(\frac{1}{n_{m+1}} \sum_{j=1}^{n_{m+1}} \chi_{A_j}(t_i) \geq \frac{m}{m+1}\) and obviously \(n_{m+1} \geq n_m\).

Since \(A\) is regular open and all \(A_n\)’s are regular open too, for each \(i \in \{1, 2, \ldots, m, m + 1\}\) there exists a neighbourhood \((c_i^{m+1}, d_i^{m+1}) \subset (a_i, b_i)\) of \(t_i\) such that \(\frac{1}{n_{m+1}} \sum_{j=1}^{n_{m+1}} \chi_{A_j}(t) \geq \frac{m}{m+1}\) for each \(t \in \bigcup_{i=1}^{m+1} (c_i^{m+1}, d_i^{m+1})\).

Indeed, let \(C_{m+1, i} = \{j : 1 \leq j \leq n_{m+1} \land t_i \in A_j\}\) for \(i \in \{1, 2, \ldots, m + 1\}\). If \(j \in C_{m+1, i}\), then there exists \(\varepsilon_{i, j}^{m+1} > 0\) such that \((t_i - \varepsilon_{i, j}^{m+1}, t_i + \varepsilon_{i, j}^{m+1}) \subset (a_i, b_i) \cap A_j\). Put \(\varepsilon_{i}^{m+1} = \min\{\varepsilon_{i, j} : j \in C_{m+1, i}\}\). We have \((c_i^{m+1}, d_i^{m+1}) = (t_i - \varepsilon_{i}^{m+1}, t_i + \varepsilon_{i}^{m+1}) \subset (a_i, b_i) \cap A_j\) for each \(i \in \{1, 2, \ldots, m, m + 1\}\) and \(j \in C_{m+1, i}\), so \(\frac{1}{n_{m+1}} \sum_{j=1}^{n_{m+1}} \chi_{A_j}(t) \geq \frac{m}{m+1}\) for each \(t \in \bigcup_{i=1}^{m+1} (c_i^{m+1}, d_i^{m+1})\).

Hence there exists an increasing sequence \(\{n_m\}_{m \in \mathbb{N}}\) of positive integers fulfilling all requirements.

**Lemma 17** If \(0\) is a right-hand intensity point of a regular open set \(A\) if \(\{n_m\}_{m \in \mathbb{N}}\) is a sequence of positive integers and \(\{(c_i^m, d_i^m)\}_{m \in \mathbb{N}, i \in \{1, 2, \ldots, m\}}\) — a sequence of intervals constructed in the previous lemma, then for each \(m \in \mathbb{N}\) there exist intervals \((c_i^m, f_i^m) \subset (c_i^m, d_i^m)\) for \(i = \{1, 2, \ldots, m\}\) and a neighbourhood \(O_m\) of \(0\) such that
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\[
\frac{1}{n_m} \sum_{j=1}^{n_m} \chi_{A_j}(x)(t) \geq \frac{m - 1}{m}
\]
for each \( x \in O_m \) and each \( t \in \bigcup_{i=1}^{m} (e_i^m, f_i^m) \).

**Proof** Fix \( m \in \mathbb{N} \). Put \( C_{m,i} = \{ j : 1 \leq j \leq n_m \land (c_i^m, d_i^m) \subset A_i \} \) for \( i \in \{1, 2, \ldots, m\} \). Observe that \( A_j(x) = (j(j+1)(A - \frac{1}{j+1}) - j(j+1)x) \cap (0, 1) \), so if \( \varepsilon_{i,j}^m < \frac{1}{3(j+1)}(d_i^m - c_i^m) \) for \( i = 1, 2, \ldots, m \) and \( j \in C_{m,i} \), then \( (\varepsilon_i^m, f_i^m) = (\frac{2}{3}c_i^m + \frac{1}{3}d_i^m, \frac{1}{3}c_i^m + \frac{2}{3}d_i^m) \subset A_j(x) \) for \( x \in (-\varepsilon_{i,j}^m, \varepsilon_{i,j}^m) \). Put now \( \varepsilon_i^m = \min\{\varepsilon_{i,j}^m : j \in C_{m,i}\} \) and \( \gamma_i^m = \min\{\varepsilon_{i,j}^m : i \in \{1, 2, \ldots, m\}\} \). Then for \( x \in (-\varepsilon_m, \varepsilon_m) = O_m \) we have \( \frac{1}{n_m} \sum_{j=1}^{n_m} \chi_{A_j}(x)(t) \geq \frac{m - 1}{m} \) for \( t \in \bigcup_{i=1}^{m} (e_i^m, f_i^m) \). \( \square \)

Since the intensity of a set at points different from zero is defined with the use of the translation, we obtain immediately:

**Corollary 18** If \( x_0 \in \mathbb{R} \) is a right-hand intensity point of a regular open set \( A \), then for each \( m \in \mathbb{N} \) there exists a natural number \( n_m(x_0) \geq m \) such that there exists a neighbourhood \( O_m(x_0) \) of \( x_0 \) and for each \( i = 1, 2, \ldots, m \) there exists an open interval \( (e_i^m, f_i^m) \subset (a_i, b_i) \) such that

\[
\frac{1}{n_m(x_0)} \sum_{j=1}^{n_m(x_0)} \chi_{A_j}(x)(t) \geq \frac{m - 1}{m}
\]
for each \( m \in \mathbb{N} \), each \( x \in O_m(x_0) \) and each \( t \in \bigcup_{i=1}^{m} (e_i^m, f_i^m) \).

**Theorem 19** If a function \( f : \mathbb{R} \to \mathbb{R} \) is right intensely continuous at each point, then \( f \) is of the first Baire class.

**Proof** Suppose that this is not the case. Then there exists a perfect set \( P \subset \mathbb{R} \) and two real numbers \( a, b \) (\( a < b \)) such that the sets \( T_1 = \{ x \in P : f(x) < a \} \) and \( T_2 = \{ x \in P : f(x) > b \} \) are both dense in \( P \) in the natural topology (see for example [10] or [6], p. 395). Let \( C_i < T_i \) be a countable set, dense in \( T_i \) (thus also dense in \( P \)) for \( i = 1, 2 \). Consider first the set \( C_1 = \{ x_1, x_2, \ldots, x_k, \ldots \} \). For each \( m, k \in \mathbb{N} \) let \( O_{m,k} \) be the neighbourhood of \( x_k \) such that

\[
\frac{1}{n_m(x_k)} \sum_{j=1}^{n_m(x_k)} \chi_{A_j}(x)(t) \geq \frac{m - 1}{m}
\]
for each \( x \in O_{m,k} \) and each \( t \in \bigcup_{i=1}^{m} (e_i^m, f_i^m) \),

where \( \{n_m(x_k)\}_{m \in \mathbb{N}} \), a sequence associated with \( x_k \) is described in the above corollary and \( \{(e_i^m, f_i^m)\}_{m \in \mathbb{N}, i \in \{1, 2, \ldots, m\}} \) also depends on \( x_k \).

Put \( O_m = P \cap \bigcup_{k=1}^{\infty} O_{m,k} \) and \( O_a = P \cap \bigcap_{m=1}^{\infty} O_m \). The set \( O_a \) is residual in \( P \), since each \( O_m \) is open and dense in \( P \).
From the assumption it follows that each point of \( T_1 \) is a right intensity point of \( T_1 \). We shall show that if \( x \in O_a \cap P \) and \( G_a \) is a regular open part of \( T_1 \), then \( x \) is not a rarefaction point of \( G_a \). Indeed, for each \( m \in \mathbb{N} \) there exists \( k_m \in \mathbb{N} \) such that \( x \in \bigcap_{m=1}^{\infty} O_{m,k_m} \).

Let \( n_m = n_m(x_{km}) \) be a number associated with \( m \in \mathbb{N} \) and \( x_{km} \in C_1 \). We can (and, choosing a subsequence if necessary, shall) suppose that \( \{n_m\}_{m \in \mathbb{N}} \) is an increasing sequence. For each \( m \in \mathbb{N} \) we have

\[
\frac{1}{n_m} \sum_{j=1}^{n_m} \chi_{G_a,j(x)}(t) \geq \frac{m-1}{m} \quad \text{for each } t \in \bigcup_{i=1}^{m} (e_{i,k_m}^{m}, f_{i,k_m}^{m}),
\]

where

\[
G_a,j(x) = j \cdot (j + 1) \left( (G_a - x) - \frac{1}{j+1} \right) \cap (0, 1),
\]

as usual. So

\[
\limsup_{m \to \infty} \frac{1}{n_m} \sum_{j=1}^{n_m} \chi_{G_a,j(x)}(t) = 1 \quad \text{for } t \in \limsup_{m \to \infty} \bigcup_{i=1}^{m} (e_{i,k_m}^{m}, f_{i,k_m}^{m})
\]

and the last set is residual in \((0, 1)\). The same argument works also for each subsequence \( \{n_m\}_{p \in \mathbb{N}} \) of \( \{n_m\}_{m \in \mathbb{N}} \). That means that \( x \) is not a right-hand rarefaction point of \( G_a \). Since \( G_a \triangle T_1 \) is the set of the first category, \( x \) is not a right-hand rarefaction point of \( T_1 \). Similarly, considering the set \( C_2 \subset T_2 \) one can prove the existence of the set \( O_b \) residual in \( P \) such that if \( x \in O_b \cap T \), then \( x \) is not a right-hand rarefaction point of \( T_2 \).

Take now a point \( x_0 \in O_a \cap O_b \) and \( \varepsilon \) such that \( 0 < \varepsilon < \frac{b-a}{3} \). Put \( T = f^{-1}(f(x_0) - \varepsilon, f(x_0) + \varepsilon) \). A point \( x_0 \) is a right-hand intensity point of \( T \) and is a right-hand rarefaction point neither of \( T_1 \) nor of \( T_2 \). Hence \( T \cap T_1 \neq \emptyset \) and \( T \cap T_2 \neq \emptyset \). Let \( x' \in T \cap T_1 \) and \( x'' \in T \cap T_2 \). We have \( |f(x') - f(x_0)| < \varepsilon \) and \( |f(x'') - f(x_0)| < \varepsilon \). Recall that \( f(x') < a \) and \( f(x'') > b \), so \( f(x'') - f(x') > b - a > 3\varepsilon \), but simultaneously \( |f(x'') - f(x')| < 2\varepsilon \), a contradiction.

\[ \square \]

**Theorem 20** If \( f : \mathbb{R} \to \mathbb{R} \) is intensely continuous at each point, then \( f \) is of the first Baire class and has the Darboux property.

**Proof** The first part follows from the previous theorem. The second is an immediate consequence of the following result of Z. Zahorski: if \( f : \mathbb{R} \to \mathbb{R} \) is Baire one and for each \( a \in \mathbb{R} \) the sets \( \{ x : f(x) > a \} \) and \( \{ x : f(x) < a \} \) are bilaterally dense in itself, then \( f \) has the Darboux property ([16], see also [1], Th. 1.1 (8)). \[ \square \]

**Theorem 21** A function \( f : \mathbb{R} \to \mathbb{R} \) has the Baire property if and only if it is intensely continuous \( I \)-almost everywhere.

**Proof** Suppose that \( f \) has the Baire property. Then there exists a residual set \( E \subset \mathbb{R} \) such that the restriction \( f|_E \) is continuous. Then \( f \) is intensely continuous at each point of \( E \), so it is intensely continuous \( I \)-almost everywhere.
Suppose now that \( f \) is intensely continuous \( \mathcal{I} \)-almost everywhere. Consider the set \( A = \{ x \in \mathbb{R} : f(x) < a \} \) for \( a \in \mathbb{R} \). Let \( E \) be the set of points where \( f \) is intensely continuous. Since \( A \setminus E \) is of the first category, it is enough to show that \( E \cap A \) has the Baire property. If \( x \in E \cap A \), then there exists a \( \mathcal{T}_i \)-neighbourhood \( A(x) \) of \( x \) such that \( A(x) \subset A \). Then \( A(x) \cap E \) is also a \( \mathcal{T}_i \)-neighbourhood of \( x \) and \( E \cap A = \bigcup_{x \in E \cap A} \{ A(x) \cap E \} \), so \( E \cap A \in \mathcal{T}_i \) and obviously it has the Baire property. \( \square \)

5 Restrictional intense continuity

Until now we have considered “topological” intense continuity of a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) at a point \( x_0 \in \mathbb{R} \). One can also consider a kind of “restrictional” or “path” intense continuity defined in the following way (compare [9] or [12]).

**Definition 22** We say that a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is restrictively intensely continuous at a point \( x_0 \) if and only if there exists a \( \mathcal{T}_i \)-neighbourhood \( U \) of \( x_0 \) such that \( f(x_0) = \lim_{x \rightarrow x_0} f(x) \).

According to [5] if a topology \( \mathcal{T} \) on the real line is invariant with respect to translations, then the topological and restrictional continuities (both with respect to \( \mathcal{T} \)) are equivalent if and only if the following condition (called (\( J_2 \)) in [12]), pp. 28–30) holds:

\( (J_2) \) For any descending sequence \( \{ U_n \}_{n \in \mathbb{N}} \) of right-hand \( \mathcal{T} \)-neighbourhoods of 0 there exists a decreasing sequence \( \{ h_n \}_{n \in \mathbb{N}} \) tending to 0 such that \( 0 \cup \bigcup_{n=1}^{\infty} (U_n \cap [h_{n+1}, h_n)) \) is also a right-hand \( \mathcal{T} \)-neighbourhood of 0 and the analogous condition for the left-hand case (with necessary changes) holds.

For the topology \( \mathcal{T}_i \) the above condition for the right-hand case can be formulated in the following more suitable way:

For each ascending sequence \( \{ A_n \}_{n \in \mathbb{N}} \) of sets having the Baire property such that 0 is a right-hand rarefaction point of each \( A_n \) there exists a decreasing sequence \( \{ h_n \}_{n \in \mathbb{N}} \) of positive numbers tending to 0 such that 0 is a right-hand rarefaction point of \( \bigcup_{n=1}^{\infty} (A_n \cap [h_{n+1}, h_n)) \). The formulation for the left-hand case is analogous.

Observe that restrictional continuity for the above mentioned topologies always implies the topological continuity.

It is well known ([1], Th. 5.6, p. 23 or [12], Ex. 14.1, p. 27) that a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is approximately continuous at a point \( x_0 \) if and only if it is restrictively approximately continuous at that point. In [9] it was proved that for \( \mathcal{I} \)-approximately continuous the situation is different, namely the condition (\( J_2 \)) does not hold. Now we shall prove that for intensely continuous functions restrictional and topological continuities at a point also are different because (\( J_2 \)) fails for \( \mathcal{T}_i \).

**Lemma 23** There exists a double sequence \( \{ E_{n,m} \}_{n,m \in \mathbb{N}} \) of subsets of the interval \( (0, 1) \) having the Baire property such that \( E_{n,m} \subset E_{n,m+1} \) for each \( n,m \in \mathbb{N} \), \( \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{E_{n,m}}(x) = 0 \) for each \( x \in (0, 1) \) and each \( m \in \mathbb{N} \) and for each increasing sequence \( \{ k_m \}_{m \in \mathbb{N} \cup \{0\}} \) of positive integers (where \( k_0 = 0 \)) if \( F_n = E_{n,m} \) for \( k_{m-1} \leq n < k_m \), then \( \lim \sup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{F_i}(x) = 1 \) on a set residual in \( (0, 1) \).
**Proof** For \((a, b) \subset (0, 1)\) let \(\{B_n(a, b)\}_{n \in \mathbb{N}}\) be a descending sequence of intervals defined in the following way: \(B_n(a, b) = (a, a + \frac{i}{n}(b - a))\) for \(n \in \mathbb{N}\). It is not difficult to see that \(\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{B_n(a, b)}(x) = 0\) everywhere on \((0, 1)\). Put \(E_{n, 1} = B_n(0, 1), E_{n, 2} = B_n(0, \frac{1}{2}) \cup B_n(0, \frac{1}{2}) \cup B_n(\frac{1}{2}, 1) = E_{n, 1} \cup B_n(0, \frac{1}{2}) \cup B_n(\frac{1}{2}, 1)\) for \(n \in \mathbb{N}\) and generally \(E_{n, m+1} = E_{n, m} \cup \bigcup_{i=1}^{m+1} B_n\left(i \frac{1}{m+1} \cdot i \frac{1}{m+1}\right)\) for \(n \in \mathbb{N}\). Immediately, we have \(\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{E_{i, m}}(x) = 0\) for each \(x \in (0, 1)\) and each \(m \in \mathbb{N}\).

If \(\{k_m\}_{m \in \mathbb{N} \cup \{0\}} (k_0 = 0)\) is an increasing sequence of positive integers, we choose a subsequence \(\{k_{m_p}\}_{p \in \mathbb{N}}\) (where \(k_{m_0} = k_0 = 0\)) such that \(k_{m_p} \geq p \cdot k_{m_p-1}\) for \(p \in \mathbb{N}\).

Observe that if \(x \in D_p = \bigcup_{i=1}^{m_p} \left(\frac{i-1}{m_p}, \frac{i-1}{m_p} + \frac{1}{k_{m_p-1}} \cdot \frac{1}{m_p}\right)\), then \(x \in F_n = E_{n, m_p}\) for each \(n\) such that \(k_{m_p-1} \leq n < k_{m_p}\). Hence \(\frac{1}{k_{m_p-1}} \sum_{n=1}^{k_{m_p-1}} \chi_{F_n}(x) \geq \frac{1}{k_{m_p-1}} \sum_{n=k_{m_p-1}}^{k_{m_p}-1} \chi_{F_n}(x) = (k_{m_p} - k_{m_p-1}) \cdot \frac{1}{k_{m_p-1}} \geq \frac{k_{m_p} - k_{m_p-1}}{k_{m_p}} = (1 - \frac{1}{p})\) for each \(x \in D_p\).

So, if \(x \in \lim_{p \to \infty} D_p\), then \(\lim_{p \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{F_n}(x) = 1\). Observe that \(\lim_{p \to \infty} D_p\) is residual in \((0, 1)\). In fact \(\lim_{p \to \infty} \frac{1}{k_{m_p-1}} \sum_{n=1}^{k_{m_p-1}} \chi_{F_n}(x) = 1\) for \(x \in \lim_{p \to \infty} D_p\) and the same result holds for each subsequence \(\{k_{m_p}\}_{p \in \mathbb{N}}\) of \(\{k_{m_p}\}_{p \in \mathbb{N}}\).

**Theorem 24** There exists an increasing sequence \(\{A_m\}_{m \in \mathbb{N}}\) of subsets of \((0, 1)\) having the Baire property such that \(0\) is a right-hand rarefaction point of \(A_m\) for each \(m \in \mathbb{N}\) and for each increasing sequence \(\{k_m\}_{m \in \mathbb{N}}\) of positive integers \(0\) is not a right-hand rarefaction point of the set \(\bigcup_{m=1}^{\infty} (A_m \cap (\frac{1}{k_m}, 1))\).

**Proof** Put \(A_m = \bigcup_{n=1}^{\infty} \left\{E_{n, m} + \frac{1}{n+1} \cdot \frac{1}{k_m}\right\}\) for each \(m \in \mathbb{N}\), where \(\{E_{n, m}\}_{n, m \in \mathbb{N}}\) is a double sequence from Lemma 23.

Then \(E_{n, m} = n(n+1)(A_m - \frac{1}{n+1}) \cap (0, 1)\) for \(n, m \in \mathbb{N}\) and \(0\) is a right-hand rarefaction point of each \(A_m\), so the result follows immediately from the above lemma.

**Corollary 25** The topology \(\mathcal{T}_i\) does not fulfil the condition \((J_2)\).

**Proof** Let \(\{A_m\}_{m \in \mathbb{N}}\) be a sequence of sets from Theorem 11 and let \(\{h_m\}_{m \in \mathbb{N} \cup \{0\}}\) be an arbitrary decreasing sequence convergent to \(0\). If \(\{k_m\}_{m \in \mathbb{N} \cup \{0\}}\) is an increasing sequence of positive integers such that \(\frac{1}{k_m} \leq h_m\) for each \(m \in \mathbb{N}\), then

\[
\bigcup_{m=1}^{\infty} \left( A_m \cap \left(\frac{1}{k_m}, \frac{1}{k_{m-1}}\right) \right) \setminus \{h_m : m \in \mathbb{N}\} \subset \bigcup (A_m \cap (h_m, h_{m-1}))
\]

and the conclusion follows immediately from the above theorem. □
6 More properties of the intense topology

It is well known that if $A \subset [0, 1]$ is a regular open set such that 0 is a right-hand $\mathcal{I}$-density point of $A$, then there exists a right interval set $E \subset A$ such that 0 is also a right-hand $\mathcal{I}$-density point of $E$ (see, for example [3], Lemma 2.2.4, p. 25). We shall show that for intensity the situation is different.

Recall that a right interval set (open or closed) at $a \in \mathbb{R}$ is a set of the form $\bigcup_{n=1}^{\infty}(a_n, b_n)$ (or $\bigcup_{n=1}^{\infty}[a_n, b_n]$) such that $b_{n+1} < a_n < b_n$ for $n \in \mathbb{N}$ and $a = \lim_{n \to \infty} a_n$. A left interval set at $a$ is defined in the same way. The set $E$ is an interval set if it is the union of a right interval set and a left interval set at the same point.

**Theorem 26** There exists a sequence $\{A_n\}_{n \in \mathbb{N}}$ of regular open subsets of $[0, 1]$ such that $\lim_{j \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{A_j}(x) = 1$ $\mathcal{I}$-a.e. and for each sequence $\{B_n\}_{n \in \mathbb{N}}$ of open sets such that $B_n$ consists of finite number of components of $A_n$ there exists an increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ of positive integers such that for each subsequence $\{n_{k_p}\}_{p \in \mathbb{N}}$

$$\lim_{p \to \infty} \frac{1}{n_{k_p}} \sum_{j=1}^{n_{k_p}} \chi_{B_j}(x) \leq \frac{1}{2}$$

on a residual $G_\delta$ subset of $[0, 1]$.

**Proof** Each positive integer $n$ can be represented uniquely in a form $n = 2^k + l$, where $k \in \mathbb{N} \cup \{0\}$ and $l \in \{0, 1, \ldots, 2^k - 1\}$. Let $q \in (0, 1)$.

Put $E_n = (0, 1) \setminus \bigcup_{m=0}^{\infty}[q^{2^{k-m+l+1}}, q^{2^{k-m+l}}]$ for $n = 2^k + l$ and $A_n = \bigcup_{i=0}^{k}(E_n + i \cdot q^{-k+1}) \cap \bigcup_{i=0}^{k} \bigcup_{j=1}^{l} (E_n + i \chi_{A_n}(x) \geq l$ for all $x \in (0, 1)$ except on the set $\bigcup_{i=0}^{k} \bigcup_{j=1}^{l} \chi_{A_n}(x) = 1$ except on a set $\mathbb{Q} \cap (0, 1)$, which means that the convergence is $\mathcal{I}$-a.e. which is stronger than the convergence in category. We shall show that the subsequence $n_k = 2^k$ is a required subsequence. Indeed, let $B_n \subset A_n$ for each $n \in \mathbb{N}$ be a set consisting of a finite number of components of $A_n$. Observe that for each $k \in \mathbb{N}$ there exists a positive number $\varepsilon_k$ such that $B_n \cap \bigcup_{i=0}^{k} \left( \frac{i}{k+1}, \frac{i}{k+1} + \varepsilon_k \right) = \emptyset$ for each $n \in \{2^k, \ldots, 2^{k+1} - 1\}$.

Hence $\frac{1}{2^k} \sum_{n=2^k}^{2^{k+1}-1} \chi_{B_n}(x) = 0$ for $x \in \bigcup_{i=0}^{k} \left( \frac{i}{k+1}, \frac{i}{k+1} + \varepsilon_k \right) = D_k$. From this it follows immediately that if $D = \limsup_{k \to \infty} D_k$, then $\liminf_{k \to \infty} \frac{1}{2^k} \sum_{n=1}^{2^{k+1}-1} \chi_{B_n}(x) \leq \frac{1}{2}$. Obviously $D$ is dense $G_\delta$ set in $(0, 1)$, so it is residual in $(0, 1)$.

The same holds for each subsequence of $\{n_k\}_{k \in \mathbb{N}}$, which ends the proof. □

**Theorem 27** There exists a set $A \subset [0, 1]$ such that 0 is a right-hand intensity point of $A$ but it is not a right-hand intensity point of any interval set included in $A$.
\textbf{Proof} Let \( \{A_n\}_{n \in \mathbb{N}} \) be a sequence of sets constructed in the above lemma. The set \( A = \bigcup_{n=1}^{\infty} \left( \frac{1}{n(n+1)} A_n + \frac{1}{n+1} \right) \) fulfills all requirements. 

\textbf{Theorem 28} Topological spaces \((\mathbb{R}, \mathcal{T}_1)\) and \((\mathbb{R}, \mathcal{T}_2)\) are not homeomorphic.

\textbf{Proof} The families of \( \mathcal{T}_1 \) - and \( \mathcal{T}_2 \) - connected sets are equal. Both consist of all intervals. Suppose that \( h : (\mathbb{R}, \mathcal{T}_1) \to (\mathbb{R}, \mathcal{T}_2) \) is a homeomorphism. Then it is not difficult to see that if \([a, b] \subset \mathbb{R}\) is an arbitrary interval, then \( h([a, b]) \) is also a closed interval and since \( h \) is one-to-one, it must be a homeomorphism \( h : (\mathbb{R}, \mathcal{T}_{\text{nat}}) \to (\mathbb{R}, \mathcal{T}_{\text{nat}}) \). Suppose that \( h \) is increasing. In the remaining case the proof is similar.

Let \( A \subset \mathbb{R} \) be a \( \mathcal{T}_1 \)-open set such that \( 0 \) is a right-hand intensity point of \( A \), but it is not a right-hand intensity point of any interval set included in the regular part \( G \) of \( A \). Such a set exists by virtue of Theorem 27. Then \( h(A) \in \mathcal{T}_2 \) and \( h(0) \) is a right-hand \( \mathcal{T}_2 \)-density point of \( h(G) \). Let \( B \subset h(G) \) be an interval set such that \( h(0) \) is a right-hand \( \mathcal{T}_2 \)-density point of \( B \). Then \( 0 \) should be a right-hand intensity point of \( h^{-1}(B) \). But \( h^{-1}(B) \) is an interval set included in \( G \)—a contradiction.

\textbf{Lemma 29} There exists a sequence \( \{E_n\}_{n \in \mathbb{N}} \) of subsets of \( (0, \frac{1}{2}) \) having the Baire property such that \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{E_i}(x) = 0 \) except on a set of the first category but a sequence \( \left\{ \frac{1}{n} \sum_{i=1}^{n} \chi_{F_i} \right\}_{n \in \mathbb{N}}, \) where \( F_n = \frac{2n+1}{2n} E_n \) for \( n \in \mathbb{N}, \) does not converge to 0 in category.

\textbf{Proof} (Compare the proof of Theorem 4 in [15]) Let \( D_k = \bigcup_{i=1}^{2^{k-1}} (\frac{i}{2^k} - \varepsilon_k, \frac{i}{2^k}) \) for \( k \in \mathbb{N}, \) where \( \varepsilon_k \) is a positive number small enough to assure that if \( E_n = \frac{2n+1}{2n} D_k \) for \( 2^{k-1} \leq n < 2^k - 1, \) then \( \{E_n\}_{2^{k-1} \leq n < 2^k - 1} \) is a family of pairwise disjoint sets. To prove the existence of positive number \( \varepsilon_k \) fulfilling the above requirement it suffices to show that if \( i, j \in \{1, 2, \ldots, 2^{k-1}\} \) and \( m, n \in \{2^{k-1}, \ldots, 2^k - 1\} \), then the equality \( \frac{i}{2^k} \cdot \frac{2n}{2n+1} = \frac{j}{2^k} \cdot \frac{2m}{2m+1} \) is possible only when \( i = j \) and \( m = n \). Indeed, suppose that \( \frac{i}{2^k} \cdot \frac{2n}{2n+1} = \frac{j}{2^k} \cdot \frac{2m}{2m+1} \) and \( i \neq j \). Then \( i \cdot 2n \cdot (2m + 1) = j \cdot 2m \cdot (2n + 1) \), so \( 2nm(i - j) = mj - ni \). Observe that \( |2n| \cdot |j - i| \geq 2 \cdot 2^k \cdot 2^{k-1} \cdot |j - i| \geq 2^{k-1} \) and \( |mj - ni| \leq (2^k - 1) \cdot 2^k - 1 - 2^{k-1} - 2^k - 1 = 2^{k-1} - 2^k \) (since \( m \leq 2^k - 1, j \leq 2^k - 1, n \geq 2^{k-1} \) and \( i \geq 1 \)) for each \( i, j, m \) and \( n \) from their domains, a contradiction. Hence \( i = j \) and also \( m = n \). So in the set \( B_k = \{ \frac{i}{2^k} \cdot \frac{2n}{2n+1} : i \in \{1, 2, \ldots, 2^{k-1}\}; n \in \{2^{k-1}, \ldots, 2^k - 1\} \} \cup \{0, \frac{1}{2}\} \) all points are different and each positive \( \varepsilon_k \) less than the smallest distance between points of \( B_k \) does the job.

Observe now that for each \( x \in (0, 1) \) we have \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{E_i}(x) = 0 \). Simultaneously \( F_n = \frac{2n+1}{2n} E_n = D_k \) for \( 2^{k-1} \leq n < 2^k - 1 \). The set \( D = \lim \sup_{k \to \infty} D_k \) is a \( G_\delta \) set dense in \((0, 1/2)\).

If \( x \in D \), then for infinitely many \( k \)'s \( x \in \bigcap_{n=2^{k-1}}^{2^k-1} F_n = D_k \) and then \( \lim \sup_{k \to \infty} \frac{1}{2^k} \sum_{i=1}^{2^k-1} \chi_{F_i}(x) \geq \frac{1}{2} \). For each subsequence \( \{2^{kp}\}_{p \in \mathbb{N}} \) of the sequence \( \{\frac{1}{2} - 1\}_{k \in \mathbb{N}} \) we have also \( \lim \sup_{p \to \infty} \frac{1}{2^{kp}} \sum_{i=1}^{2^{kp}-1} \chi_{F_i}(x) \geq \frac{1}{2} \) for \( x \in \hat{D} = \lim \sup_{p \to \infty} D_{kp} \) and the set \( \hat{D} \) is also a \( G_\delta \) set dense in \((0, 1/2)\).

\textbf{Theorem 30} There exists a set \( A \subset (0, 1) \) such that \( 0 \) is a right-hand rarefaction point of \( A \) but not a right-hand rarefaction point of \( 1/2 A \).
Proof} Put $A_n = \frac{1}{n+1} \cdot E_n + \frac{1}{n+1} \cdot \mathbb{N}$ for $n \in \mathbb{N}$, where \{E_n\}_{n \in \mathbb{N}} is a sequence of sets from the lemma 5. Then $A_n \subseteq \left( \frac{1}{n+1}, \frac{1}{n+1} + \frac{1}{2} \frac{1}{n(n+1)} \right) = \left( \frac{1}{n+1}, \frac{2n+1}{2n(n+1)} \right)$, which means that $A_n$ is included in the left half of the interval $(\frac{1}{n+1}, \frac{1}{n})$. Let $A = \bigcup_{n=1}^{\infty} A_n$. We have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{i}(-j+1) \cap (0,1) \cdot (A - \frac{1}{j+1}) = 0$$

$\mathcal{I}$-a.e., because $i(i+1) \left( A - \frac{1}{j+1} \right) \cap (0,1) = E_i$ for $i \in \mathbb{N}$. This means that 0 is a right-hand rarefaction point of $A$.

Consider now $\frac{1}{2} A = \frac{1}{2} \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \left( \frac{1}{2} A \right)$. Observe that $\frac{1}{2} A_n$ is included in $(\frac{1}{2(n+1)}, \frac{1}{2(n+1)} + \frac{1}{2})$ (and not necessarily in $(\frac{1}{2(n+1)}, \frac{1}{2n+1} + \frac{1}{2})$, because $\frac{1}{2n+1} < \frac{2n+1}{4n(n+1)}$.

Further $2(n+1)(2n+1) \left( \frac{1}{2} A_n - \frac{1}{2(n+1)} \right) = \frac{1}{2} \cdot 2 \cdot (n+1)(2n+1)(A_n - \frac{1}{n+1}) = \frac{(n+1)(2n+1)}{n(n+1)} \cdot E_n = \frac{2n+1}{2n} E_n \subset (0, \frac{2n+1}{2n})$. Simultaneously $2(n+1)(2n+1) \left( \frac{1}{2} A - \frac{1}{2(n+1)} \right) \cap (0, \frac{2n+1}{2n}) = (2n+1)(2n+1) \left( \frac{1}{2} A - \frac{1}{2(n+1)} \right) = 2F_n$. Since \{ $\frac{1}{2} \sum_{i=1}^{n} \chi_{2F_i}$ \}_{n \in \mathbb{N}} does not converge to 0 in category, the sequence \{ $\frac{1}{n} \sum_{i=1}^{n} \chi_{2F_i \cap [0,1]}$ \}_{n \in \mathbb{N}} has the same property, so 0 is not a right-hand rarefaction point of $\frac{1}{2} A$.

\[\square\]

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