$\mu$-$\tau$ symmetry in Zee-Babu model

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Abstract

We study the Zee-Babu two-loop neutrino mass generation model and look for a possible flavor symmetry behind the tri-bimaximal neutrino mixing. We find that there probably exists the $\mu$-$\tau$ symmetry in the case of the normal neutrino mass hierarchy, whereas there may not be in the inverted hierarchy case. We also propose a specific model based on a Froggatt-Nielsen-like $Z_5$ symmetry to naturally accomplish the $\mu$-$\tau$ symmetry on the neutrino mass matrix for the normal hierarchy case.

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I. INTRODUCTION

Neutrino oscillation experiments have almost completely established that neutrinos have tiny masses and mix with each other through the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) leptonic mixing matrix \([1]\). From the latest global analysis of three-neutrino mixing \([2]\), one currently has the following best fit values with \(1\sigma\) errors:

\[
\Delta m^2_{21} = (7.59 \pm 0.20) \times 10^{-5} \text{eV}^2, \\
\Delta m^2_{31} = \begin{cases} 
(-2.36 \pm 0.11) \times 10^{-3} \text{eV}^2 & \text{for inverted hierarchy} \\
(+2.46 \pm 0.12) \times 10^{-3} \text{eV}^2 & \text{for normal hierarchy}
\end{cases}, \\
\sin^2 \theta_{12} = 0.319 \pm 0.016, \quad \sin^2 \theta_{23} = 0.462^{+0.082}_{-0.050}, \quad \sin^2 \theta_{13} = 0.0095^{+0.013}_{-0.007}.
\]

The data indicate the existence of, at least, two massive neutrinos with a very suggestive neutrino mixing matrix, that is, the tri-bimaximal (TB) mixing matrix \([3]\):

\[
V_{TB} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & \sqrt{2} & 0 \\ -1 & \sqrt{2} & \sqrt{3} \\ 1 & -\sqrt{2} & \sqrt{3} \end{pmatrix}.
\]

However, the standard model (SM) neither includes neutrino mass terms nor provides us with any explanation for the TB mixing. Clearly, we need new physics beyond the SM. In fact, many extensions of the SM have been proposed so far. For instance, in the type-I \([4]\), type-II \([5]\) and type-III \([6]\) seesaw mechanisms, the SM is extended by introducing extra heavy fermions or scalars to generate neutrino masses suppressed by the mass scale of the heavy particles, while in Ref. \([7]\) tiny neutrino masses come from the dimension-five Weinberg operators. These scenarios have been extensively studied with some flavor symmetries to explain the TB mixing \([8\text{-}12]\).

Yet another possibility of leading to tiny neutrino masses is to use radiative corrections. It was first pointed out by Zee in Ref. \([13]\) in which new scalars are added in the Higgs sector with neutrino masses induced at the one-loop level. After that, a two-loop scenario called Zee-Babu model \([14]\) was proposed\(^1\). In these kinds of scenarios, discussions about the neutrino phenomena can be much different form those of the tree level scenarios.

\(^1\) Other types of multi-loop scenarios have also been studied in Ref. \([15]\).
because the induced neutrino mass matrix elements are the products of the anti-symmetric Yukawa coupling and charged lepton mass. Hence, it is non-trivial whether a flavor symmetry can play an important role in the radiative scenarios. Since there is a claim that the original Zee model may not be able to reproduce current neutrino oscillation data [16], we focus on the Zee-Babu two-loop model in this Letter. We re-analyze the model and try to explain the TB pattern of neutrino mixings in terms of the \( \mu-\tau \) symmetry which is the prime candidate of a flavor symmetry in the tree level scenarios. Note that other phenomenological studies have been discussed in Refs. [17, 18].

This Letter is organized as follows. In Section II, we summarize the Zee-Babu model and show some definitions of parameters. In Section III, we investigate the model along with the \( \mu-\tau \) symmetry. We propose a specific flavor model in Section IV. Finally, we conclude our discussions in Section V.

## II. ZEE-BABU MODEL

In addition to the SM particles, the Zee-Babu model contains two \( SU(2)_L \) singlet new scalars: a singly charged scalar \( h^\pm \) and doubly charged scalar \( k^{\pm\pm} \). Accordingly, new interactions appear and terms relevant to our study are

\[
\mathcal{L}_{ZB} = F_{ab}(L_a^T C L_b h^+) + Y_{ab}(\ell_{Ra}^T C \ell_{Rb} k^{++}) - \mu h^+ k^{-} + h.c.,
\]

where \( C \) is the charge conjugation matrix, \( L_{a=\epsilon,\mu,\tau} \) stand for the left-handed \( SU(2)_L \) doublet leptons and \( \ell_{Ra} \) are the right-handed singlet charged leptons in the diagonal basis of the charged lepton mass matrix. \( F_{ab} \) and \( Y_{ab} \) are \( 3 \times 3 \) complex Yukawa matrices, parametrized as

\[
F_{ab} = \begin{pmatrix}
0 & f_{e\mu} & f_{e\tau} \\
-f_{e\mu} & 0 & f_{\mu\tau} \\
-f_{e\tau} & -f_{\mu\tau} & 0
\end{pmatrix}, \quad Y_{ab} = \begin{pmatrix}
y_{ee} & y_{e\mu} & y_{e\tau} \\
y_{e\mu} & y_{\mu\mu} & y_{\mu\tau} \\
y_{e\tau} & y_{\mu\tau} & y_{\tau\tau}
\end{pmatrix}.
\]

The Majorana neutrino mass term is induced at the two-loop level as depicted in Fig. 1 with the mass matrix given by

\[
\mathcal{M}_{ab} = 8\mu(f_{ac} m_c y_{ca}^* m_d f_{db}) I,
\]

(3)
where $m_a$ indicate the charged lepton masses and

$$ I \simeq \frac{1}{(16\pi^2)^2} \frac{1}{M_h^2} \int_0^1 dx \int_0^{1-x} dy \frac{-(1-y)}{x + [(M_k/M_h)^2 - 1] y + y^2} \log \frac{y(1-y)}{x + (M_k/M_h)^2 y} \tag{6} $$

is the two-loop integral function with the masses of the new scalars, $M_k$ and $M_h$. Note that Eq. (6) is simplified by neglecting the charged lepton masses [17]. The elements of the neutrino mass matrix in Eq. (5) are written as

$$ M_{11} = 8 \mu_f^2 \mu_\tau (\tilde{f}_{e\tau} \omega_{\tau\tau} - 2 \tilde{f}_{e\mu} \tilde{f}_{e\tau} \omega_{\mu\tau} - \tilde{f}_{e\mu}^2 \omega_{\mu\mu}) I, $$
$$ M_{22} = 8 \mu_f^2 \mu_\tau (\omega_{\tau\tau} + 2 \tilde{f}_{e\mu} \omega_{e\tau} - \tilde{f}_{e\mu}^2 \omega_{ee}) I, $$
$$ M_{33} = 8 \mu_f^2 \mu_\tau (\omega_{\mu\mu} - 2 \tilde{f}_{e\tau} \omega_{ee} - \tilde{f}_{e\mu}^2 \omega_{ee}) I, $$
$$ M_{12} = 8 \mu_f^2 \mu_\tau (\tilde{f}_{e\tau} \omega_{\tau\tau} - \tilde{f}_{e\mu} \omega_{\mu\tau} + \tilde{f}_{e\mu}^2 \omega_{ee}) I = M_{21}, $$
$$ M_{13} = 8 \mu_f^2 \mu_\tau (\tilde{f}_{e\tau} \omega_{\mu\tau} + \tilde{f}_{e\mu} \omega_{\mu\tau} + \tilde{f}_{e\mu}^2 \omega_{ee}) I = M_{31}, $$
$$ M_{23} = 8 \mu_f^2 \mu_\tau (\omega_{\tau\tau} + \tilde{f}_{e\tau} \omega_{e\tau} - \tilde{f}_{e\mu} \omega_{e\mu} - \tilde{f}_{e\mu} \tilde{f}_{e\tau} \omega_{ee}) I = M_{32}, $$

with the following redefinitions of parameters:

$$ \tilde{f}_{e\mu} \equiv \frac{f_{e\mu}}{f_{e\tau}}, \quad \tilde{f}_{e\tau} \equiv \frac{f_{e\tau}}{f_{e\mu}}, \quad \omega_{ab} \equiv m_a y_{ab}^* m_b. \tag{8} $$

It is clear that the mass matrix in Eq. (8) always has a zero-eigenvalue because of the vanishing determinant of $F_{ab}$. Although all three active neutrinos should have non-zero masses if we take into account the higher order loop contributions, we ignore these contributions in this Letter.

Furthermore, we can embed the terms associated with the electron mass into $\omega_{\tau\tau}, \omega_{\mu\tau}$ and $\omega_{\mu\mu}$ terms, such that

$$ \omega'_{\tau\tau} \equiv \omega_{\tau\tau} - 2 \tilde{f}_{e\mu} \omega_{e\tau} + \tilde{f}_{e\mu}^2 \omega_{ee}, $$
\[ \omega_{\mu \tau}^i \equiv \omega_{\mu \tau} + \tilde{f}_{\mu \tau} \omega_{\tau \tau} - \tilde{f}_{e \mu} \omega_{e \mu} - \tilde{f}_{e \tau} \omega_{e \tau}, \quad (9) \]
\[ \omega_{\mu \mu}^i \equiv \omega_{\mu \mu} + 2 \tilde{f}_{e \tau} \omega_{e \mu} + \tilde{f}_{e \tau} ^2 \omega_{e \tau}. \]

Then, we obtain the following simplified mass matrix

\[ M_{ab} = 8 \mu \tilde{f}_{\mu \tau}^2 \omega_{\mu \mu}^i I M_{ab} \quad (10) \]

with

\[
M_{ab} = \begin{pmatrix}
-\tilde{f}_{e \tau}^2 \tilde{\omega}_{\tau \tau} - 2 \tilde{f}_{e \mu} \tilde{f}_{e \tau} \tilde{\omega}_{\mu \tau} - \tilde{f}_{e \mu}^2 & -\tilde{f}_{e \tau} \tilde{\omega}_{\tau \tau} & \tilde{\omega}_{\mu \tau} \\
* & -\tilde{\omega}_{\tau \tau} & \tilde{\omega}_{\mu \tau} \\
* & * & -1
\end{pmatrix}, \quad (11)
\]

where \( \tilde{\omega}_{\mu \tau} \) and \( \tilde{\omega}_{\tau \tau} \) are defined as

\[ \tilde{\omega}_{\mu \tau} = \frac{\omega_{\mu \tau}^i}{\omega_{\mu \mu}^i}, \quad \tilde{\omega}_{\tau \tau} = \frac{\omega_{\tau \tau}^i}{\omega_{\mu \mu}^i}. \quad (12) \]

As partially discussed in Ref. \[17\], we can represent \( \tilde{f}_{e \mu}, \tilde{f}_{e \tau}, \tilde{\omega}_{\mu \tau} \) and \( \tilde{\omega}_{\tau \tau} \) in terms of the neutrino mass ratios, mixing angles and CP violating phases in the PMNS matrix, parametrized by

\[ U_{PMNS} = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{pmatrix} \begin{pmatrix}
c_{13} & s_{13} e^{-i \delta} & 0 \\
0 & 1 & 0 \\
-s_{13} e^{i \delta} & 0 & c_{13}
\end{pmatrix} \begin{pmatrix}
c_{12} & s_{12} & 0 \\
-s_{12} & c_{12} & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i \gamma/2} & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad (13)
\]

where \( \delta \) and \( \gamma \) are the Dirac and Majorana CP phase, respectively, and \( s_{ij} (c_{ij}) = \sin \theta_{ij} (\cos \theta_{ij}) \geq 0 \). Since we consider the diagonal basis of the charged leptons, the Majorana mass matrix is diagonalized by the PMNS matrix, such that \( U_{PMNS}^T M U_{PMNS} = \text{diag}(m_1, m_2, m_3) \). In the case of the normal mass hierarchy, the four parameters are described as

\[
\tilde{f}_{e \mu} = \frac{s_{12} s_{23}}{c_{12} c_{13}} - \frac{s_{13}}{c_{13}} c_{23} e^{i \delta}, \\
\tilde{f}_{e \tau} = \frac{s_{12} c_{23}}{c_{12} c_{13}} + \frac{s_{13}}{c_{13}} c_{23} e^{i \delta}, \\
\tilde{\omega}_{\mu \tau} = -\frac{2 c_{13}^2 c_{23}^2}{c_{13}^2 c_{23}^2 + r_{2/3}(s_{12} s_{13} c_{23} - e^{-i \delta} + c_{12} s_{23})^2 e^{-i \gamma}} \\
\tilde{\omega}_{\tau \tau} = \frac{2 s_{13}^2}{c_{13}^2 c_{23} + r_{2/3}(s_{12} s_{13} s_{23} - e^{-i \delta} - c_{12} c_{23})^2 e^{-i \gamma}}.
\quad (14)
\]
with \( r_{2/3} = m_2/m_3 \), while for the inverted one

\[
\tilde{f}_{\mu\tau} = \frac{r_{2/1}}{r_{2/3}} \frac{c_{12} s_{13}}{s_{13}} e^{-i\delta},
\]

\[
\tilde{f}_{\tau\tau} = -s_{23} \frac{c_{12}}{s_{13}} e^{i\delta},
\]

\[
\tilde{\omega}_{\mu\tau} = -\frac{r_{2/1}(s_{12} s_{13} c_{23} e^{-i\delta} + c_{12} s_{23})(s_{12} s_{13} s_{23} e^{-i\delta} - c_{12} c_{23})e^{-i\gamma}}{r_{2/3}(s_{12} s_{13} c_{23} e^{-i\delta} + c_{12} c_{23})(s_{12} s_{13} s_{23} e^{-i\delta} + s_{12} s_{23})} (c_{12} s_{13} c_{23} e^{-i\delta} - s_{12} s_{23})(c_{12} s_{13} s_{23} e^{-i\delta} - s_{12} s_{23})^2,
\]

\[
\tilde{\omega}_{\tau\tau} = \frac{r_{2/1}(s_{12} s_{13} s_{23} e^{-i\delta} - c_{12} c_{23})^2 e^{-i\gamma}}{r_{2/3}(s_{12} s_{13} c_{23} e^{-i\delta} + c_{12} c_{23})^2 e^{-i\gamma} + (c_{12} s_{13} c_{23} e^{-i\delta} + s_{12} c_{23})^2} (c_{12} s_{13} c_{23} e^{-i\delta} - s_{12} s_{23})(c_{12} s_{13} s_{23} e^{-i\delta} - s_{12} s_{23})^2.
\]

with \( r_{2/1} = m_2/m_1 \). From the first two equations in Eq. (14), one can see that \( \tilde{f}_{\mu\mu} \) will be close to \( \tilde{f}_{\tau\tau} \) in the limit of \( \theta_{13} \to 0 \) and \( \theta_{23} \to \pi/4 \). This fact turns out to be one of the origins of the \( \mu-\tau \) symmetry as shown in the next section. On the other hands, \( \tilde{f}_{\mu\mu} \) and \( \tilde{f}_{\tau\tau} \) in Eq. (15) always have an opposite sign. This indicates that the inverted case cannot be consistent with the \( \mu-\tau \) symmetry.\(^2\)

**III. \( \mu-\tau \) symmetric limit and deviation**

In this section, we investigate the Zee-Babu model by considering the \( \mu-\tau \) symmetric type of the matrix in Eq. (11) as follows

\[
M^{\mu\tau} = \begin{pmatrix}
A & -B & B \\
-B & C & D \\
B & D & C
\end{pmatrix},
\]

which can be diagonalized by the PMNS matrix in Eq. (13) with \( \theta_{23} = \pi/4 \) and \( \theta_{13} = 0 \), where \( A, B, C \) and \( D \) are complex values in general. Note that for the matrix in Eq. (11) there are only two possible \( \mu-\tau \) symmetric limits: (i) \( \tilde{\omega}_{\mu\tau} = \tilde{\omega}_{\tau\tau} = 1 \) (\( \omega'_{\mu\mu} = \omega'_{\mu\tau} = \omega'_{\tau\tau} \)) and (ii) \( \tilde{\omega}_{\mu\tau} = 1 \) and \( \tilde{f}_{\tau\tau} = \tilde{f}_{\mu\mu} \) (\( \omega'_{\mu\mu} = \omega'_{\tau\tau} \) and \( f_{\mu\mu} = f_{\tau\tau} \)). However, the former condition results in \( m_1 = m_3 = 0 \) or \( m_2 = m_3 = 0 \), which must be largely broken in order to fit the experimental data. Thus, we will focus on only the latter one.

\(^2\) There is actually a special case in which \( f_{\mu\tau} = 0 \). However, once we force the neutrino mass matrix to be \( \mu-\tau \) symmetric, the theory suffers from the dangerous lepton flavor violating processes, such as \( \tau \to \mu\gamma \).
A. Normal mass hierarchy

In the $\mu$-$\tau$ symmetric limit, the matrix $M_{ab}$ in Eq. (11) becomes

$$M_{ab} = \begin{pmatrix}
-2\tilde{f}_{e\mu}^2 (1 + \tilde{\omega}_{\mu\tau}) & -\tilde{f}_{e\mu} (1 + \tilde{\omega}_{\mu\tau}) & \tilde{f}_{e\mu} (1 + \tilde{\omega}_{\mu\tau}) \\
* & -1 & \tilde{\omega}_{\mu\tau} \\
* & * & -1
\end{pmatrix}$$

and three mixing angles are given by

$$\theta_{23} = \frac{\pi}{4}, \quad \theta_{13} = 0, \quad \tan 2\theta_{12} = \frac{2\sqrt{2}\tilde{f}_{e\mu}}{1 - 2\tilde{f}_{e\mu}^2}. \quad (18)$$

The three eigenvalues are found to be

$$\lambda_1 = \left| (2\tilde{f}_{e\mu}^2 c_{12}^2 - 2\sqrt{2}\tilde{f}_{e\mu} s_{12} c_{12} + s_{12}^2)(\tilde{\omega}_{\mu\tau} + 1) \right|,$$

$$\lambda_2 = \left| (2\tilde{f}_{e\mu}^2 s_{12}^2 + 2\sqrt{2}\tilde{f}_{e\mu} s_{12} c_{12} + c_{12}^2)(\tilde{\omega}_{\mu\tau} + 1) \right|,$$

$$\lambda_3 = |\tilde{\omega}_{\mu\tau} - 1|,$$

where either $\lambda_1$ or $\lambda_2$ always vanishes. Hence, this limit is only consistent with the normal mass hierarchy case. For example, the exact TB mixing is obtained from $\tilde{f}_{e\mu} = 1/2^3$, while the central value of the mass ratio, which is $m_2/m_3 \simeq 0.176$, corresponds to $\tilde{\omega}_{\mu\tau} \simeq -1.27$. Moreover, in order to fit all central values in Eq. (1), we need to deviate from the $\mu$-$\tau$ symmetric limit and it can be realized with the following data set:

$$\tilde{f}_{e\mu} \simeq 0.47 - 0.07e^{i(0 - 2\pi)}, \quad \tilde{f}_{e\tau} \simeq 0.51 + 0.07e^{i(0 - 2\pi)},$$

$$\tilde{\omega}_{\tau\tau} \simeq 0.85 + (0.00 - 0.06)e^{i(0 - 2\pi)},$$

$$\tilde{\omega}_{\mu\tau} \simeq -0.95 + (0.19 - 0.23)e^{i(0 - 2\pi)}, \quad (20)$$

where we have varied $\delta$ and $\gamma$ from 0 to $2\pi$ based on Eq. (14). Although the $\mu$-$\tau$ conditions are no longer exact, they remain as good approximations, i.e., $\tilde{\omega}_{\tau\tau} \simeq 1$ and $\tilde{f}_{e\tau} \simeq \tilde{f}_{e\mu}$. Therefore, we conclude that there probably exists the $\mu$-$\tau$ symmetry behind the TB pattern of neutrino mixings in the case of the normal mass hierarchy.

3 Although $\tilde{f}_{e\mu} = -1$ also implies the exact TB mixing, it leads to a vanishing $\lambda_2$ at the same time.
TABLE I: A particle content and charge assignments

|     | L | \ell_R | H | h^+ | k^{++} | \phi |
|-----|---|--------|---|------|---------|------|
| SU(2)_L | 2 | 1 | 2 | 1 | 1 |     |
| U(1)_Y | -1 | -2 | 1 | 2 | 4 | 0   |
| Z_5 | (2,0,1) | (2,0,1) | 0 | 0 | 0 | -1 |

B. Inverted mass hierarchy

As mentioned in the previous subsection, the \( \mu-\tau \) condition has to be largely broken in the case of the inverted mass hierarchy. For instance, the central values in Eq. (1) can be obtained from

\[
\tilde{f}_{e\mu} \simeq 7.49e^{i(0 - 2\pi)}, \quad \tilde{f}_{e\tau} \simeq -6.94e^{i(0 - 2\pi)},
\]

\[
\tilde{\omega}_{\tau\tau} \simeq 2.00 + (0.00 - 1.64)e^{i(0 - 2\pi)}, \quad \tilde{\omega}_{\mu\tau} \simeq 1.52 + (0.00 - 0.78)e^{i(0 - 2\pi)},
\]

(21)

where we have varied \( \delta \) and \( \gamma \) from 0 to \( 2\pi \) based on Eq. (15). However, contrary to the normal hierarchy case, it is difficult to find out possible remnants of the \( \mu-\tau \) symmetry from Eq. (21), i.e., \( \tilde{f}_{e\mu} \neq \tilde{f}_{e\tau} \) and \( \tilde{\omega}_{\tau\tau} \neq 1 \). This suggests that, in the inverted hierarchy case, the TB mixing may just be an accidental result due to the suitable parameter tunings.

IV. FROGGATT-NIELSEN-LIKE \( Z_5 \) MODEL

In the previous section, we have obtained the conditions: \( \tilde{f}_{e\tau} = \tilde{f}_{e\mu} \) and \( \tilde{\omega}_{\tau\tau} = 1 \) to derive the \( \mu-\tau \) symmetric matrix. The former condition is easy to achieve by imposing a permutation symmetry, whereas the latter one may not be because \( \tilde{\omega}_{\tau\tau} \) includes not only Yukawa couplings but also the charged lepton masses. For instance, if we ignore the electron mass, the condition becomes \( m_\tau^2 y_{e\tau}^* = m_\mu^2 y_{e\mu}^* \), and it requires a hierarchy between Yukawa couplings rather than a permutation relation. To naturally realize the \( \mu-\tau \) conditions for the normal mass hierarchy case, we adopt the scheme of the Froggatt-Nielsen mechanism \[19\] and show a specific model based on an \( Z_5 \) symmetry. The charge
$\mu \rightarrow e \gamma$  Br. $< 1.2 \times 10^{-11}$
$\mu \rightarrow e^+ e^- e^-$  Br. $< 1.0 \times 10^{-12}$
$\tau \rightarrow \mu^+ \mu^- \mu^-$  Br. $< 3.2 \times 10^{-8}$

| TABLE II: Lepton flavor violating processes and their experimental bounds used in our calculation. |
|-----------------------------------------------|

assignments of the particles under the symmetries are summarized in Table I. In this model, we introduce a gauge singlet scalar $\phi$ with the charge $-1$ under $Z_5$ and consider the higher dimensional operators. Because of the $Z_5$ symmetry, at the leading order, the Yukawa matrices $F_{ab}$ and $Y_{ab}$ in Eq. (3) turn out to be

$$F_{ab} = \begin{pmatrix} 0 & f_{e\mu} \lambda^2 & f_{e\tau} \lambda^2 \\ -f_{e\mu} \lambda^2 & 0 & f_{\mu\tau} \lambda \\ -f_{e\tau} \lambda^2 & f_{\mu\tau} \lambda & 0 \end{pmatrix}, \quad Y_{ab} = \begin{pmatrix} y_{ee} \lambda & y_{e\mu} \lambda^2 & y_{e\tau} \lambda^2 \\ * & y_{\mu\mu} & y_{\mu\tau} \lambda \\ * & * & y_{\tau\tau} \lambda^2 \end{pmatrix}, \quad (22)$$

where $\lambda = \frac{\langle \phi \rangle}{\Lambda}$ is the suppression factor of the higher dimensional operators with the typical energy scale of the $Z_5$ symmetry, $\Lambda$. Note that due to the symmetry, the charged lepton mass matrix is diagonal up to the leading order. We remark that to simplify our discussion, we have assumed that the terms like $LHLH\phi^{\pm \mu}$ are strongly suppressed by an extremely-high energy scale. It is easy to see that if we assume

$$\lambda = \frac{m_\mu}{m_\tau} \simeq 0.06, \quad f_{e\tau} = f_{e\mu}, \quad y_{\mu\mu} = y_{\tau\tau} \quad (23)$$

and $m_e = 0$, we obtain the $\mu-\tau$ symmetric Majorana neutrino mass matrix, given by

$$M_{ab} = 8 \mu_f^2 \lambda^2 \omega_{\mu\mu} \begin{pmatrix} -2 \tilde{f}_{e\mu}^2 \lambda^2 (1 + \tilde{\omega}_{\mu\tau}) & -\tilde{f}_{e\mu} \lambda (1 + \tilde{\omega}_{\mu\tau}) & \tilde{f}_{e\mu} \lambda (1 + \tilde{\omega}_{\mu\tau}) \\ * & -1 & \tilde{\omega}_{\mu\tau} \\ * & * & -1 \end{pmatrix} I, \quad (24)$$

where $\tilde{\omega}_{\mu\tau} = y_{\mu\tau}^* / y_{\mu\mu}^*$ and $\omega_{\mu\mu} = \frac{m_\mu^2}{y_{\mu\mu}^*}$. 

In the rest of this section, we discuss several experimental constraints including some non-standard lepton flavor violating processes. (See Table II) As discussed in Sec. III, the realistic neutrino mixings requires

$$f_{e\mu} \simeq f_{e\tau} \simeq \frac{f_{\mu\tau}}{2\lambda}. \quad (25)$$
In this case, the strongest constraint on the $f_{ab}$ couplings comes from the $\mu \to e\gamma$ process. Combined with the neutrino mixing data, we get the lower bounds on $f_{\mu\tau}$ and $y_{\mu\mu}$:

\[ f_{\mu\tau} \lambda > 0.008, \]  
\[ y_{\mu\mu} > 0.13, \]  
and the allowed range for the singly charged scalar mass:

\[ 10^2 \text{ GeV} < M_h < 10^4 \text{ GeV} \]  

as discussed in Ref. [17]. Moreover, since it may be natural to take all $y_{ab}$ to be the same order, we can estimate the branching ratios of $\ell_a^- \rightarrow \ell_b^+ \ell_c^- \ell_c^-$ mediated by the doubly charged scalar. The stringent constraint comes form either $\tau \to 3\mu$ or $\mu \to 3e$, given by

\[ \frac{|y_{ab}|^2}{M_k^2} < 10^{-7} \text{ GeV}^{-2}. \]  

It is clear that these processes can be accessible in the near future with the TeV scale doubly charged scalar if $y_{ab} \simeq \mathcal{O}(0.1)$.

V. SUMMARY

We have investigated the Zee-Babu model and tried to find out a possible flavor symmetry behind the TB neutrino mixing matrix. We have found that there probably exists the $\mu$-$\tau$ symmetry in the normal neutrino mass hierarchy case, but the TB mixing may be accidental in the case of the inverted one. We have also attempted to derive the $\mu$-$\tau$ symmetric neutrino mass matrix with a Froggatt-Nielsen-like $Z_5$ symmetry and estimated several constraints coming from lepton flavor violating processes.

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