Research Article

Integral-Type Fractional Equations with a Proportional Riemann–Liouville Derivative

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In this paper, we present the necessary conditions where integral-type fractional equations with a proportional Riemann–Liouville derivative have a unique solution. Also, we give an example to illustrate our work.

1. Introduction

Lately, many researchers have been focusing on the study of various types of fractional problems; we refer the reader to [1–17]. The fixed point and the monotone iterative techniques can be very useful tools to prove the existence and uniqueness of a solution to this type of problems; see [1]. In this manuscript, inspired by the work of Jankowski in [1], we investigate the existence and uniqueness of a solution to the following problem:

\[ D^{\alpha, \rho}\xi(t) = g\left(t, \xi(t), \int_0^t \mathcal{F}(t, \tau)\xi(\tau)\,d\tau\right) = \mathcal{F}(t), \quad t \in J_0 = (0, a]; a > 0, \]

where \( D^{\alpha, \rho}\xi(t) \) denotes a proportional Riemann–Liouville fractional derivative for \( \rho \in [0, 1] \) and \( 0 < \alpha < 1 \). Also, \( g \in C(J \times \mathbb{R} \times \mathbb{R}), J = [0, a] \), and \( \xi(0) = t^{1-\alpha}e^{(p-1)/\rho} \xi(t)\big|_{t=0} \). Now, we remind the reader of the definition of the proportional Riemann–Liouville fractional integral and derivative.

Definition 1 (see [18]). Let \( \alpha \in \mathbb{C}; \Re(\alpha) \geq 0, 0 < \rho \leq 1, \) and \( t > 0 \).

(i) The following integral is called the proportional Riemann–Liouville fractional integral:

\[ I^{\alpha, \rho} f(t) := \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^t e^{(p-1)/\rho}(t-\tau)^{\alpha-1} f(\tau)\,d\tau. \]

(ii) The following derivative is called the proportional Riemann–Liouville fractional derivative:

\[ (D^{\alpha, \rho} f)(t) = D^{\alpha-\rho, \rho} f(t) = \frac{D^{\rho}}{\rho^\alpha \Gamma(\alpha-\rho)} \int_0^t e^{(p-1)/\rho}(t-\tau)^{\alpha-\rho-1} f(\tau)\,d\tau. \]
where $n = \lfloor \text{Re}(a) \rfloor + 1$ and $D_x^{1/\sigma} = (1 - \rho)f(t) + \rho f'(t)$.

Next, we present the following proposition.

**Proposition 1** (see [18]). If $a, \gamma \in \mathbb{C}$, where $\text{Re}(a) > 0$ and $\text{Re}(\gamma) > 0$, then for any $0 < \rho \leq 1$, we have

$$I^\sigma \left( (t^\gamma - 1) e^{(p - 1/\rho)\sigma} \right) (x) = \left( \Gamma(\gamma)/\rho^\sigma \Gamma(a + \gamma) \right) x^{\gamma + a - 1} e^{(p - 1/\rho)\gamma} x.$$

In Section 2, we prove the existence and uniqueness of a solution to problem (1) using the fixed point technique. In Section 3, we prove the existence and uniqueness of a solution to problem (1) using the monotone iterative method. In the conclusion, we present an open question.

### 2. Fixed Point Approach

First of all, let $C_{1-a}(J, \mathbb{R}) = \{ f \in C([0, a), \mathbb{R}) | f \in C(J, \mathbb{R}) \}$. Now, define the following two weighted norms:

$$\| f \|_1 = \max_{[0, a]} |f(t)|,$$

$$\| f \|_a = \max_{[0, a]} e^{-\lambda t} |f(t)|$$

for fixed $\lambda > 0$.

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**Theorem 1.** Let $0 < a < 1, 0 < \rho \leq 1$, and $g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{X} \in C(J \times J, \mathbb{R})$. Let $b_1 = (\rho - 1/\rho)$. Also, assume the following two hypotheses:

1. There exist nonnegative constants $H, V$, and $W$ such that $|\mathcal{X}(t, s)| < H$ and

$$g(t, u_1, u_2) - g(t, v_1, v_2) \leq V|u_1 - v_1| + W|v_2 - u_2|.$$  

2. $b \equiv (a^\sigma(\Gamma(2a)/\rho^\sigma)) [V + (HWa/2a)] < 1$, for $a \in (0, 1/2)$.

Then, initial value problem (1) has a unique solution.

**Proof.** First, let $S\xi(t) = t^{1-a} e^{(p - 1/\rho)\xi} + (1/\rho^\sigma(\alpha)) \int_0^t e^{(p - 1/\rho)\xi}(t - \tau)^{a-1} \mathcal{X}(\tau) \, d\tau$. Note that if $S$ has a unique fixed point and that is $S\xi(t) = \xi(t)$, then initial value problem (1) has a unique solution, i. e., it will be enough to show that $S$ is a contraction map. So, let $\xi, \gamma \in C_{1-a}(J, \mathbb{R})$; we have two cases:

Case 1: $a \in (0, 1/2)$.

$$\| S\xi - S\gamma \|_1 = \frac{1}{\rho^\sigma(\alpha)} \max_{t \in J} t^{1-a} \int_0^t e^{(p - 1/\rho)\xi}(t - \tau)^{a-1} \mathcal{X}(\tau) \, d\tau$$

$$\leq \frac{1}{\rho^\sigma(\alpha)} \max_{t \in J} t^{1-a} \int_0^t e^{(p - 1/\rho)\xi}(t - \tau)^{a-1} \left[ V|\xi(\tau) - \gamma(\tau)| + W\int_0^\tau |\mathcal{X}(t, s)\gamma(s) - \gamma(s)| \, ds \right] \, d\tau$$

$$\leq \frac{1}{\rho^\sigma(\alpha)} \| \xi - \gamma \|_1 \max_{t \in J} t^{1-a} \int_0^t e^{(p - 1/\rho)\xi}(t - \tau)^{a-1} \left[ V\tau^{a-1} + HW\int_0^\tau \tau^{a-1} \, ds \right] \, d\tau$$

$$= \frac{1}{\rho^\sigma(\alpha)} \| \xi - \gamma \|_1 \max_{t \in J} t^{1-a} e^{\rho t} \int_0^t e^{(p - 1/\rho)\xi}(t - \tau)^{a-1} \left[ V\tau^{a-1} + HWe^{\rho \tau a} \right] \, d\tau$$

$$= \frac{1}{\rho^\sigma(\alpha)} \| \xi - \gamma \|_1 \max_{t \in J} t^{1-a} e^{\rho t} \left[ I^\sigma \left( V\tau^{a-1} e^{\rho t} \right) + I^\sigma \left( \frac{HWa}{2a} e^{\rho t} \right) \right]$$

$$= \frac{\Gamma(a) (a)^{\sigma}}{\Gamma(a) (2a)^{\sigma}} \left[ V + HWa/2a \right] \| \xi - \gamma \|_1$$

$$= b \cdot \| \xi - \gamma \|_1.$$

Hence, $S$ is a contraction map. Therefore, $S$ has a unique fixed point as desired.

Case 2: $a \in ((1/2), 1)$; in this case, we use $\| \cdot \|_1$ with the positive constant $\lambda > 0$ such that

$$\sqrt{\lambda - \beta} > b_1 = \frac{e^{\beta \lambda} (V\alpha + HWa) \Gamma(2a - 1) \sqrt{\alpha^{a-1}}}{\alpha \rho^\sigma \Gamma(a) \sqrt{\Gamma(2(2a - 1))}}.$$  

(7)
It is not difficult to see the following:

(1) \[ \int_0^t e^{a(t-\beta)r} \, dr \leq \left( e^{a(t-\beta)/20} \right)^{(2-\lambda)} \]

Also, recall the Schwarz inequality for integrals:

\[ \int_0^t |f(r)|^2 \, dr \leq \left( \int_0^t f(r) \, dr \right)^2 \]

As an application to Theorem 1, consider the following lemma is a consequence of Theorem 1.

(2) \[ t^{1-a} \int_0^t (t-r)^{2(a-1)} r^{2(a-1)} \, dr = (\Gamma (2a - 1) / \sqrt{2(2a - 1)}) \]

Hypothesis \( H_1 \)

(1) \[ L(t) = L, \; t \in J \]

(2) The function \( L \) is nonconstant on \( J \)

\[ \frac{a^a}{\rho^a T(2a)} \max_{t \in J} |L(t)| < 1 \quad \text{only if} \quad \alpha \in \left( 0, \frac{1}{2} \right) \]

The following lemma is a consequence of Theorem 1.

Lemma 1. If \( \alpha \in (0, 1) \), \( L \in C(J, \mathbb{R}) \), \( z \in C_{1-a}(J, \mathbb{R}) \), and hypothesis \( (H_1) \) holds, then problem (11) has a unique solution.

We would like to bring to the reader’s attention that, in [1], in the hypothesis \( \rho \) should be as follows: \( \rho = (T^2/2q)[K + (WLT/2q)] \) which he used to prove the case where \( q \in (0, 1/2) \). This way, his result will be stronger or he can just change the last equality to the inequality.
3. Monotone Iterative Method

First of all, we start by introducing the following hypothesis.

**Hypothesis 2 (H₂)**

(1) \( L(t) = L \), \( t \in J \) or
(2) The function \( L \) is nonconstant, and if \( L(t) \) is negative, then there exists \( T \) which is nondecreasing, where \( -L(t) \leq T(t) \) on \( J \) and for every \( x \in J \), we have

\[
\frac{e^{\beta x}}{\rho^{-1 \alpha} \Gamma(\alpha)} \int_0^x (a - \tau)^{\alpha - 1} e^{\beta (a - \tau)} T(\tau) d\tau < 1. \tag{13}
\]

Now, for our purpose, we prove the following useful lemma.

**Lemma 2.** Let \( \alpha \in (0,1) \) and \( L \in C(J, [0, \infty)) \) or \( L \in C(J, (-\infty, 0]) \). Also, denote by \( \beta = (\rho - 1/\rho) \). Assume that \( q \in C_{1-a} (J, \mathbb{R}) \) is a solution to the following problem:

\[
D^{\alpha \rho} q(t) \leq - L(t) q(t), \quad t \in J_0, \quad \overline{q}(0) < 0.
\]  

If \( (H₂) \) holds, then \( q(t) \leq 0 \) for all \( t \in J \).

**Proof.** Assume that our lemma is false, that is, there exist \( x, y \in [0, a) \) such that \( q(x) = 0 \), \( q(y) > 0 \), and \( q(t) \leq 0 \) for \( t \in (0, x] \); \( q(t) > 0 \) for \( t \in (x, y] \). Let \( x_0 \) be the first maximal point of \( q \) on \([x, y]\).

Case 1: assume that \( L(t) \geq 0 \) for all \( t \in J \). Thus, \( D^{\alpha \rho} q(t) \leq 0 \) for \( t \in [x, y] \). Hence,

\[
\int_x^y D^{\alpha \rho} q(t) dt \leq 0. \tag{15}
\]

Therefore, \( B \equiv L^{1-a} q(x_0) - L^{1-a} q(y) \leq 0 \), but

\[
B = -\frac{1}{\rho^{-1 \alpha} \Gamma(1 - \alpha)} \int_0^{x_0} e^{\beta (x_0 - \tau)} (x_0 - \tau)^{-\alpha} q(\tau) d\tau - \int_0^x e^{\beta (x - \tau)} (x - \tau)^{-\alpha} q(\tau) d\tau
\]

\[
= \frac{1}{\rho^{-1 \alpha} \Gamma(1 - \alpha)} \left\{ \int_0^{x_0} e^{\beta (x_0 - \tau)} (x_0 - \tau)^{-\alpha} q(\tau) d\tau - \int_0^x e^{\beta (x - \tau)} (x - \tau)^{-\alpha} q(\tau) d\tau \right\}
\]

\[
> \frac{1}{\rho^{-1 \alpha} \Gamma(1 - \alpha)} \int_x^{x_0} e^{\beta (x_0 - \tau)} (x_0 - \tau)^{-\alpha} q(\tau) d\tau > 0,
\]

which leads us to a contradiction given the fact that \( B \leq 0 \).

Case 2: assume that \( L(t) \leq 0 \) for all \( t \in J \), and consider \( T \) to be nondecreasing on \( J \). Now, if we apply \( L^{\alpha \rho} \) on problem \( (14) \), we obtain

\[
q(t) - \overline{q}(0) e^{\beta t^{1-a}} \rho^{-1 \alpha} \Gamma(\alpha) \leq - L^{\alpha \rho} [L(t) q(t)], \quad t \in [x, x_0]. \tag{17}
\]

Notice that \( \overline{q}(0) e^{\beta t^{1-a}} \rho^{-1 \alpha} \Gamma(\alpha) \leq 0 \) which is due to the fact that \( \overline{q}(0) \leq 0 \). Thus,
Assume that Theorem 2.

Proof. Using Lemmas 1 and 2, the proof is similar to the proof of Theorem 2 in [1].

Now, we present the following example.

\[
\mathcal{F}(t) = \frac{\rho^a \beta^1}{\Gamma(2 - \alpha)} + A(t) \left[ e^{\beta t} - 1 - \xi(t) \right]^3 + \frac{\beta}{e^{\beta t} - 1} B(t) \int_0^t [\sin(xt)]^4 \xi(xt) dx.
\]
Now, let $x_0(t) = 0$ and $y_0(t) = t e^{\beta t}$; first, note that $x_0(t)$ is a lower solution of problem (22). Next, we show that $y_0(t)$ is an upper solution of problem (22):

\[
\mathcal{F} y_0(t) = \frac{\rho \alpha}{\Gamma(2-\alpha)} - \alpha(t) + \frac{\beta}{\epsilon^{\beta} - 1} B(t) \int_0^t [\sin(\tau)]^4 e^{\beta \tau} d\tau \\
\leq \frac{\rho \alpha}{\Gamma(2-\alpha)} - \alpha(t) + \frac{\beta}{\epsilon^{\beta} - 1} B(t) \int_0^t e^{\beta \tau} d\tau \\
= \frac{\rho \alpha}{\Gamma(2-\alpha)} - \alpha(t) + \frac{\beta}{\epsilon^{\beta} - 1} B(t) \left[ \frac{e^{\beta t} - 1}{\beta} \right] < \frac{\rho \alpha}{\Gamma(2-\alpha)} - \alpha(t) + \frac{\beta}{\epsilon^{\beta} - 1} B(t) \left[ \frac{e^{\beta t} - 1}{\beta} \right] \leq \frac{\rho \alpha}{\Gamma(2-\alpha)} - \alpha(t) + \frac{\beta}{\epsilon^{\beta} - 1} B(t) \left[ \frac{e^{\beta t} - 1}{\beta} \right] \\
= D^\alpha_0 y_0(t).
\]

Thus, $y_0(t)$ is an upper solution of problem (22). Now, it is not difficult to see that all the hypotheses of Theorem 2 are satisfied. Therefore, problem (22) has solutions in $[x_0, y_0]$ if $\alpha \in (1/2, 1)$, and for $\alpha \in (0, 1/2)$, we need to assume that $(1/\rho \alpha \Gamma(2\alpha))max_{t\in[0,1]}|\alpha(t)| < 1$.

4. Conclusion

In closing, note that the results of Jankowski [1] are a special case of our work which is by taking $\rho = 1$. Also, we would like to bring to the reader attention the following open question.

What are the necessary and sufficient conditions for problem (1) to have a unique solution if $\rho$ is not constant , but it is a function of $t$ say $g(t)$, so that the problem involves $D^\alpha_0 g(t)$?

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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