Characterization of Cyclic and Separating Vectors and Application to an Inverse Problem in Modular Theory
II. Semifinite Factors

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Abstract

This paper generalizes the results obtained in an earlier paper ([Bol]) for finite factors to infinite but still semifinite factors. First we give a characterization of cyclic and separating vectors for infinite semifinite factors in terms of operators associated with this vector and being affiliated with the factor. Further we show how this operator generates the modular objects of the given cyclic and separating vector generalizing an idea of Kadison and Ringrose. With the help of these results we can show that the second simple class of solutions for the inverse problem constructed in [Bol] never exists in infinite semifinite factors. Finally we give a classification of the solutions of the inverse problem in the case of modular operators having pure point spectrum completely analogous to the finite case.

1 The Inverse Problem in Modular Theory

Let $M_0$ be a von Neumann algebra on a separable Hilbert space $H_0$ with a cyclic and separating vector $u_0$. Then modular theory shows the existence of a modular operator $\Delta_0$ and a modular conjugation $J_0$ (the modular objects $(\Delta_0, J_0)$) belonging to the vector $u_0$. In this paper we examine the inverse problem of constructing algebras $M$ having the same cyclic and separating vector and modular objects as $M_0$:

**The Inverse Problem**

Let $(\Delta_0, J_0)$ be the modular objects for the von Neumann algebra $M_0$ with cyclic and separating vector $u_0$. Characterize all von Neumann algebras $M$ isomorphic to $M_0$ with the following properties:

1. $u_0$ is also cyclic and separating for $M$,
2. $(\Delta_0, J_0)$ are the modular objects for $(M, u_0)$. 
Let $NF_{M_0}(\Delta_0, J_0, u_0)$ denote all solutions $M$ of the inverse problem.

In [Bol] the following theorems were shown:

**Theorem 1.1.** Let $(M_0, H_0)$ be a finite von Neumann factor. Let further $u \in H_0$. Then there is exactly one operator $T_u \eta M_0$ associated with the vector $u$, s.t. $u = T_u u_{tr}$ where $u_{tr} \in H_0$ is a cyclic trace vector. This operator has the following properties:

1. $\text{tr}(T_u T_u^*) = \text{tr}(T_u^* T_u) < \infty$.
2. $u$ is cyclic, iff $T_u$ is injective.
3. $u$ is separating, iff $T_u$ has dense range.
4. $u$ is cyclic and separating iff $T_u$ is injective and has dense range, i.e. iff $T_u$ is invertible.

**Theorem 1.2.** Let $T \eta M_0$. Then there is a vector $u \in H_0$ s.t $T = T_u$ in the sense of Theorem [Bol] iff $\text{tr}TT^* = \text{tr}T^*T < \infty$.

In the second section of this paper these theorems were generalized to infinite semifinite factors. For this purpose we first consider a special case of such factors, a matrix algebra of finite factors, where the trace vector is replaced by a sequence of vectors, constructed from the trace vectors of the constituting factors.

With the help of this result we show the analogue of the following result, also obtained in [Bol] for finite factors:

**Theorem 1.3.** Let $M_0$ be a finite von Neumann factor with cyclic and separating vector $u_0 \in M_0$ and cyclic trace vector $u_{tr} \in H_0$. Let further $T_{u_0} \eta M_0$ be the invertible operator corresponding to $u_0$ and $T_{u_0} = HV = (T_{u_0} T_{u_0}^*)^{1/2} V$ the polar decomposition of $T_{u_0}$. Then we can calculate the modular objects $(\Delta_0, J_0)$ of $(M_0, u_0)$ as follows:

$$J_0 = JV^* JV = VJ^* V,$$

where $J$ is the conjugation corresponding to $u_{tr}$, and

$$\Delta_0 = J_0 H_0^{-1} J_0 H_0,$$

where $H_0 = H^2 = T_{u_0} T_{u_0}^*$.

Then we will be in exactly the same situation as in the finite case, and can examine the inverse problem as in [Bol]. In contrast to that case the second simple class of solutions will never exists in this case (s. §4), but the classification of the solutions in the pure point spectrum case will be the same.

Notice that in this paper all Hilbert spaces are separable, i.e. the von Neumann algebras are countably decomposable.
2 Characterization of Vectors by Affiliated Operators

In this section we consider infinite but still semifinite factors, i.e. \( \mathcal{M}_0 \) is of type \( I_\infty \) or \( II_\infty \). In this case we have no trace vectors left. But nevertheless we can make a similar construction as in the finite case by considering the infinite factor as an infinite matrix of finite factors and using the results presented in [Bo].

As a model for such a matrix of finite factors we examine now the semifinite factor \( \mathcal{R} = \mathcal{T} \otimes L(\mathcal{H}_\infty) \otimes \mathbb{C} \) on \( \mathcal{K} = \mathcal{H} \otimes \mathcal{H}_\infty \otimes \mathcal{H}_\infty \), where \( (\mathcal{T}, \mathcal{H}) \) is a finite factor possessing a cyclic and separating vector and \( \mathcal{H}_\infty \) is an infinite dimensional separable Hilbert space which we can identify with \( l_2(\mathbb{N}) \). Now \( \mathcal{R} \) is an (infinite) type \( I \) (II) factor, if \( \mathcal{T} \) is type \( I \) (II). Further, since \( (\mathcal{T}, \mathcal{H}) \) is finite and possesses a cyclic and separating vector, it possesses a cyclic trace vector \( u_{\text{tr}} \in \mathcal{H} \) (cf. [KR86, Th. 8.2.8, Lem. 7.2.8]).

In the following we consider the elements of \( \mathcal{K} \) as infinite dimensional matrices \( u = (u^k_{ij})_{i,j \in \mathbb{N}} \) with entries \( u^k_i \in \mathcal{H} \) s.t. \( \sum_{i,k} \|u^k_i\|^2 < \infty \), where the lower index corresponds to the second component of the tensor product and the upper to the third, resp. Then we can write the elements of \( \mathcal{R} \) as matrices \( T = (T_{li})_{l,i \in \mathbb{N}} \) with entries \( T_{li} \in \mathcal{T} \), where

\[
Tu = \left( \sum_i T_{li} u^k_i \right)_l^k.
\]

Then the commutant \( \mathcal{R}' \) of \( \mathcal{R} \) is \( \mathcal{T}' \otimes \mathbb{C} \otimes L(\mathcal{H}_\infty) \), where we can write an element in \( \mathcal{R}' \) as \( T' = (T'^{lk})_{l,k \in \mathbb{N}} \) with entries \( T'^{lk} \in \mathcal{T}' \), where

\[
T'u = \left( \sum_k T'^{lk} u^k_i \right)_l^i.
\]

For the proofs in this section the following subalgebras of \( \mathcal{R} \) and \( \mathcal{R}' \) are important:

\[
\mathcal{R}_0 := \{ M = (M_{ij})_{i,j} \in \mathcal{R} | M_{ij} \neq 0 \text{ for only finitely many } i,j \in \mathbb{N} \}
\]

\[
\mathcal{R}'_0 := \{ M' = (M'^{ij})_{i,j} \in \mathcal{R}' | M'^{ij} \neq 0 \text{ for only finitely many } i,j \in \mathbb{N} \}.
\]

Now we define the following sequence of vectors in \( \mathcal{K} \), which is the analogue to the trace vector:

\[
v_k := (\delta_i^j \delta_{ik} u_{\text{tr}})_i^j.
\]

Further we define

\[
D_0 := \text{lin}\{ M v_k | M \in \mathcal{R}, k \in \mathbb{N} \} \subset \mathcal{K}
\]

and

\[
D'_0 := \text{lin}\{ M' v_k | M' \in \mathcal{R}', k \in \mathbb{N} \} \subset \mathcal{K}.
\]
Now the trace \( \text{tr} \) of \( \mathcal{R} \), which is a n.s.f. tracial weight, is \( \text{tr} = \text{tr}_T \otimes \text{tr}_{L(\mathcal{H}_\infty)} \), where \( \text{tr}_T \) is the trace on \( \mathcal{T} \) and \( \text{tr}_{L(\mathcal{H}_\infty)} \) the standard trace on \( \mathcal{H}_\infty \). It can be written with the help of the vectors \( (v_k) \):

\[
\text{tr}(M) = \sum_k \text{tr}_T(M_{kk}) = \sum_k \langle M_{kk}u_\text{tr}|u_\text{tr} \rangle = \sum_k \int \lambda d \| E^M_\lambda v_k \|^2 \quad \forall M = (M_{ij}) \in \mathcal{R}^+,
\]

where \( M = \int \lambda d E^M_\lambda \) is the spectral measure of \( M \). As in [Bol] we can continue the trace to all the positive closed operators \( A \) affiliated with \( \mathcal{R} \) by

\[
\text{tr}(A) := \sum_k \int \lambda d \| E^A_\lambda v_k \|^2,
\]

(2.1)

where \( E^A_\lambda \) is the spectral measure of \( A \).

Now we can associate an operator \( T_{ij} \) with every component \( u_{ij} \) of a vector \( u = (u_{ij}) \in \mathcal{H} \), s.t. \( u_\text{tr} \in D(T_{ij}) \) and \( T_{ij}u_\text{tr} = u_{ij} \) (cf. [Bol]). These operators give rise to a linear operator \( \tilde{T}u \) defined by

\[
\tilde{T}u : D(\tilde{T}u) := D_0' \subset \mathcal{K} \rightarrow \mathcal{K} \\
M' v_k \mapsto \tilde{T}uM' v_k := (T_{ik}M'_{jk}u_\text{tr})_i.
\]

(2.2)

Now we can prove

**Lemma 2.1.** Let \( \tilde{T}u \) defined by (2.2). Then \( \tilde{T}u \) is densely defined and closable. Let \( T_u \) be its closure. Then \( D_0' \subset D(T^*_u) \), \( T_u \) is affiliated with \( \mathcal{R} \), and

\[
\sum_k T_u v_k = u,
\]

where the convergence is absolute.

**Proof.** 1. First we must show that \( \tilde{T}u \) is well defined. Observe first that

\[
\sum_{i,j} \| T_{ik} \delta^j_k u_\text{tr} \|^2 = \sum_i \| T_{ik}u_\text{tr} \|^2 = \sum_i \| u_{ik} \|^2 < \infty,
\]

i.e. \( \tilde{T}u v_k \in \mathcal{K} \) for every \( k \in \mathbb{N} \). Let now \( M' \in \mathcal{R}' \) be arbitrary. Then
$M^* \tilde{T}_u v_k \in K$ and

$$\infty > \left\| M^* \tilde{T}_u v_k \right\|^2 = \sum_{i,j} \left\| M'_{jk} T_{ik} u_{tr} \right\|^2 = \sum_{i,j} \left\| T_{ik} M'_{jk} u_{tr} \right\|^2 \quad \text{(cf. [Bol, Prop.2.1.])}$$

(2.3)

for every $k \in \mathbb{N}$, hence $\tilde{T}_u$ is well defined.

2. Now we show that $\mathcal{D}(\tilde{T}_u)$ is dense in $K$. First the elements with only finitely many entries not 0 are dense in $K$. Further every such element is a linear combination of elements of the type $(u^j \delta_{ik})^i_j$, again all but a finite number equal 0. Since $u_{tr} \in \mathcal{H}$ is cyclic for $\tilde{T}$, we can approximate these elements by elements of the form $(M'_{jk} \delta_{ik} u_{tr})^i_j =: M' v_k$ with $M' = (M'_{jk} \delta_{ik})^i_j \in \mathcal{R}'_0 \subset \mathcal{R}'$, hence $\mathcal{D}(\tilde{T}_u) = \mathcal{D}_0'$ is dense in $K$.

3. In this step we want to show that $\tilde{T}_u$ is closable. Let $x = M' v_k \in \mathcal{D}(\tilde{T}_u)$, $y = N' v_j \in \mathcal{D}(S) := \mathcal{D}_0' (k, j \in \mathbb{N})$, where $S := (T_{il}^*)_{i,l}$ is defined analogously to $\tilde{T}_u$, hence it is a densely defined operator, too (All $T_{il}^*$ are closed operators affiliated with $\mathcal{T}$ and $u_{tr} \in \mathcal{D}(T_{il}^*)$, cf. [Bol, Prop. 2.1.], and $\left\| T_{il}^* u_{tr} \right\|^2 = \left\| T_{il} u_{tr} \right\|^2$).

Now

$$\langle \tilde{T}_u x | y \rangle = \langle \tilde{T}_u M' v_k | N' v_j \rangle = \sum_{i,l} \langle T_{ik} M'_{lk} u_{tr} | N'_{lj} \delta_{ij} u_{tr} \rangle = \sum_{i,l} \langle M'_{lk} u_{tr} | T_{il}^* N'_{lj} u_{tr} \rangle = \sum_{i,l} \langle \delta_{ik} M'_{lk} u_{tr} | T_{lj}^* N'_{lj} u_{tr} \rangle = \langle x | S y \rangle.$$  

This shows $y \in \mathcal{D}(\tilde{T}_u^*)$, $\tilde{T}_u^* y = S y$, and $S \subset (\tilde{T}_u)^*$, hence $\tilde{T}_u$ is closable. This shows also, that $\mathcal{D}_0' \subset \mathcal{D}((\tilde{T}_u)^*) = \mathcal{D}(\tilde{T}_u^*)$.

4. To show that $T_u$ is affiliated with $\mathcal{R}$, let $U' = (U'^{ij})_{i,j \in \mathbb{N}} \in \mathcal{R}'$ be a
unitary. Then $U' D_0' = D_0'$. Let now $x = M' v_k \in D_0' = D(\tilde{T}_u)$. Then

$$U' \tilde{T}_u x = U' (T_i k M'_{jk} u_{tr})_i^j$$
$$= (\sum_j U'_{lj} T_i k M'_{jk} u_{tr})_i^j$$
$$= (\sum_j T_i k U'_{lj} M'_{jk} u_{tr})_i^j \quad (\text{cf. [Bol, Prop.2.1.]})$$
$$= (T_i k \sum_j U'_{lj} M'_{jk} u_{tr})_i^j$$
$$= \tilde{T}_u U' x$$

This shows $U' \tilde{T}_u = \tilde{T}_u U'$ for every unitary $U' \in \mathcal{R}'$, hence, since $D_0'$ is a core for $T_u$,

$$U' T_u = T_u U' \quad \forall U' \in \mathcal{U}(\mathcal{R}')$$.

5. In the last step we calculate

$$\sum_k T_u v_k = \sum_k (T_i k \delta_{jk} u_{tr})_i^j$$
$$= (T_{ij} u_{tr})_i^j = u.$$

Now we can give the following definition:

**Definition 2.1.** For every vector $u = (u_i^j) \in \mathcal{K}$ we denote by $T_u = (T_{ij})$ an operator affiliated with $\mathcal{R}$ s.t. $u_{tr} \in D(T_{ij})$ for all $i, j \in \mathbb{N}$, $T_{ij} u_{tr} = u_i^j$, and $\sum_k T_u v_k = u$, which exists according to Lemma 2.1.

The next proposition shows some useful properties of the operators occurring in Definition 2.1.

**Proposition 2.2.** Let $T \in \mathcal{T}$, $v_k \in D(T)$ $(k \in \mathbb{N})$, $\sum_k ||Tv_k||^2 < \infty$. Then

1. $\mathcal{R} v_k \subset D(T)$, $\mathcal{R} v_k \subset D(T^*)$, and $\mathcal{R} v_k \subset D((T^* T)^{1/2})$ for all $k \in \mathbb{N}$.
2. $D_0$ is a core for $T$, $T^*$, and $(T^* T)^{1/2}$.
3. $\mathcal{R}' v_k \subset D(T)$, $\mathcal{R}' v_k \subset D(T^*)$, and $\mathcal{R}' v_k \subset D((T^* T)^{1/2})$ for all $k \in \mathbb{N}$.
4. $D'_0$ is a core for $T$, $T^*$, and $(T^* T)^{1/2}$.
Proof. 1. Let $T = VH$ the polar decomposition of $T$, and $E_\lambda$ the spectral resolution of $H$. Then

$$\int \lambda^2 d\|E_\lambda U v_k\|^2 \leq \sum_l \int \lambda^2 d\|E_\lambda U v_l\|^2$$

$$= \sum_l \int \lambda^2 d\|U^* E_\lambda v_l\|^2 \quad (\text{s. (2.1)})$$

$$= \sum_l \int \lambda^2 d\|E_\lambda v_l\|^2$$

$$= \sum_l \|H v_l\|^2 = \sum_l \|T v_l\|^2 < \infty$$

for every unitary $U \in \mathcal{R}$ and every $k \in \mathbb{N}$, i.e. $\mathcal{R}v_k \subset \mathcal{D}(H) = \mathcal{D}(T)$ for every $k \in \mathbb{N}$. Now $T^* = HV^*$, and, since $V^* \in \mathcal{R}$, also $\mathcal{R}v_k \subset \mathcal{D}(T^*)$.

2. 1) shows that $D_0 \subset \mathcal{D}(T)$, further $D_0$ is dense in $\mathcal{K}$. Now $D_0$ is invariant under the unitary group $e^{itH}$, i.e. $D_0$ is a core for $H$ and also for $T$. The assertion for $T^*$ follows analogous.

3. This follows from 1) and [Bol, Prop. 2.1].

4. Now for every $M = (M_{ij})_{ij} \in \mathcal{R}$ there exists exactly one $M' = (M'^{ij}) \in \mathcal{R}'$ s.t. $M'^{ij} u_{tr} = M_{ji} u_{tr}$ ($M'^{ij} := J M_{ji} J$, where $J$ is the conjugation w.r.t. $u_{tr}$). Now define

$$M'_{(k,l)} := E'_{k} M' E'_{l},$$

where $E'_{(k)} := (\delta_{ik} \delta^{ij})^{ij}$. Then

$$M v_k = \sum_l M'_{(k,l)} v_l$$

(2.4)

and

$$\sum_l \|T M'_{(k,l)} v_l\|^2 = \sum_l \|M'_{(k,l)} T v_l\|^2$$

$$\leq \sum_l \|M'\|^2 \|T v_l\|^2$$

$$\leq \|M\|^2 \sum_l \|T v_l\|^2 < \infty.$$ 

Hence $\sum_l T M'_{(k,l)} v_l$ converges and therefore $\sum_l M'_{(k,l)} v_l$ converges in the graph norm of $T$ to $M v_k$, i.e. also $D'_0$ is a core, since $D_0$ is it.

Lemma 2.3. Let $T \in \mathcal{R}$ be as in Proposition 2.2. Then there are $T'_{ij} \in \mathcal{R}$ with $u_{tr} \in \mathcal{D}(T_{ij})$ s.t. $T = (T_{ij})_{i,j \in \mathbb{N}}$ and $T = T_u$ with $u := \sum_k T v_k$ in the sense of Definition 2.2.
Proof. Set $E_k := [R' \mathcal{v}_k] \in \mathcal{R} \ (k \in \mathbb{N})$. Then matrix calculation shows that $(E_k)$ is a family of orthogonal, equivalent, finite projections, s.t.

$$E_k \mathcal{R} E_k = (\delta_{ki} \delta_{ij} T)_{i,j \in \mathbb{N}}$$

and $\sum_k E_k = \operatorname{Id}$. Now $E_{ij} : M' \mathcal{v}_j \mapsto M' \mathcal{v}_i$ defines a selfadjoint system of matrix units $(E_{ij})$ s.t. $E_{kk} = E_k$ for every $k \in \mathbb{N}$. Now define operators $S_{ij} : \mathcal{D}(S_{ij}) := D_0' \subset \mathcal{K} \to \mathcal{K}$

$$\sum_k M_k' v_k \mapsto \sum_k E_{ki} T E_{jk} M_k' v_k$$

and

$$\tilde{S}_{ji} : \mathcal{D}(\tilde{S}_{ji}) := D_0' \subset \mathcal{K} \to \mathcal{K}$$

$$\sum_k M_k' v_k \mapsto \sum_k E_{kj} T^* E_{ik} M_k' v_k.$$
hence \( \tilde{T}_{ij} \) is affiliated with \( \mathcal{T} \otimes \mathbb{C} \otimes \mathbb{C} = \{E_{mn}|m, n \in \mathbb{N}\} \cap \mathcal{R} \). Now set \( T_{ij} := V^* \tilde{T}_{ij} V \), where

\[
V : \mathcal{H} \to \mathcal{K} \\
v \mapsto (\delta_{ij} v)_i^j
\]

is the canonical partial isometry from \( \mathcal{H} \) to \( \mathcal{K} = \mathcal{H} \otimes \mathcal{H}_\infty \otimes \mathcal{H}_\infty \).

Now \( u_{tr} \in \mathcal{D}(T_{ij}) \) since \( V u_{tr} = v_1 \in \mathcal{D}(T_{ij}) \), and with \( u_i^j := T_{ij} u_{tr} \)

\[
\sum_{i,j} \left\| u_i^j \right\|^2 = \sum_{i,j} \left\| T_{ij} u_{tr} \right\|^2 = \sum_{i,j} \left\| V^* \tilde{T}_{ij} V u_{tr} \right\|^2 = \sum_{i,j} \left\| E_{1i} T E_{j1} v_1 \right\|^2 = \sum_{i,j} \left\| E_i^j T v_j \right\|^2 = \sum_{j} \left\| T v_j \right\|^2 < \infty
\]

s.t. \( u := \sum_k T v_k = (u_i^j)_i^j = (T_{ij} u_{tr})_i^j \in \mathcal{K} \). This means that we can construct the operator \( T_u = (T_{ij})_{ij} \) according to Lemma 2.7. Now \( T_u \) and \( T \) coincide on the core \( D_0' \), and hence they are equal.

**Corollary 2.4.** The operator \( T_u \) defined in Definition 2.1 is unique.

**Corollary 2.5.** Let \( T_u \) be the operator defined in Definition 2.1. Then \( \mathcal{R} v_k \in \mathcal{D}(T_u) \) for every \( k \in \mathbb{N} \) and

\[
T_u M v_k = (\sum_l T_{il} M_{lk} \delta^j_{li} u_{tr})^j_i.
\]

**Proof.** Proposition 2.2 shows that \( \mathcal{R} v_k \in \mathcal{D}(T_u) \) for every \( k \in \mathbb{N} \) and \( M v_k = \sum_{l} M_{(k,l)}^l v_l \) (cf. (2.4)). Now

\[
T_u M v_k = T_u \sum_{l} M_{(k,l)}^l v_l = \sum_{l} T_u M_{(k,l)}^l v_l = \sum_{l} (T_{il} M_{lk}^l \delta^j_{ij} u_{tr})^j_i = \sum_{l} (T_{il} M_{lk}^l \delta^j_{lj} u_{tr})^j_i.
\]
Now we can formulate the following lemma:

**Lemma 2.6.** Let $T_u$ be the operator defined in Definition 2.1. Then:

1. $\text{tr}(T_u^* T_u) = \text{tr}(T_u T_u^*) < \infty$.
2. $u$ is cyclic, iff $T_u$ is injective.
3. $u$ is separating, iff $T_u$ has dense range.
4. $u$ is cyclic and separating iff $T_u$ is injective and has dense range, i.e. iff $T_u$ is invertible.

For the proof we need:

**Proposition 2.7.** Let $T$ be a (finite) von Neumann algebra with cyclic trace vector $u_{tr}$. Let further $S, T \in \mathcal{D}(\mathcal{S}) \cap \mathcal{D}(\mathcal{T})$ and $M, N \in \mathcal{T}$. Then

$$\langle MT_{u_{tr}}|NS_{u_{tr}} \rangle = \langle S^* N^* u_{tr}|T^* M^* u_{tr} \rangle. \quad (2.5)$$

**Proof.** Let $(E_n)$ and $(F_n)$ be bounding sequences for $T$ and $S$, resp. (cf. [KR83, Lem. 5.6.14]). Then:

$$\langle MT_{u_{tr}}|NS_{u_{tr}} \rangle = \lim_{n \to \infty} \langle MTE_n u_{tr}|NSF_n u_{tr} \rangle = \lim_{n \to \infty} \langle (SF_n)^* N^* u_{tr}|(TE_n)^* M^* u_{tr} \rangle\left|_{(S,T) \in \mathcal{D}(S) \cap \mathcal{D}(T)}\right.$$  

since $N^* u_{tr} \in \mathcal{D}(S^*)$ and $M^* u_{tr} \in \mathcal{D}(T^*)$ (cf. [Bo, Prop2.1]).

**Proof of Lemma 2.6.** 1. Since $v_k \in \mathcal{D}(T_{u_{tr}}) = \mathcal{D}(H)$ for all $k \in \mathbb{N}$, where $T_u = VH$ is the polar decomposition of $T_u$, we can write the trace, defined in (2.1), as follows ($E_\lambda$ is the spectral measure of $H$):

$$\text{tr}(T_u^* T_u) = \text{tr}(H^2) = \sum_k \lambda^2 d \|E_\lambda v_k\|^2 = \sum_k \|H v_k\|^2 = \sum_k \|T_u v_k\|^2.$$

Since the $[R v_k]$ are mutually orthogonal, we have

$$\text{tr}(T_u^* T_u) = \sum_k \langle T_u v_k | T_u v_k \rangle = \sum_{k,j} \langle T_u v_k | T_u v_j \rangle = \left\| \sum_k T_u v_k \right\|^2 = \|u\|^2 < \infty.$$
Further

\[ \text{tr}(T_u^* T_u) = \sum_j \langle T_u^* v_j | T_u^* v_j \rangle \]

\[ = \sum_j \sum_{k,i} \langle T_{jk}^* \delta^{ji} u_{tr} | T_{jk}^* \delta^{ji} u_{tr} \rangle \]

\[ = \sum_{k,i} \langle T_{ik}^* u_{tr} | T_{ik}^* u_{tr} \rangle \]

\[ = \sum_k \sum_i \langle T_{ik}^* u_{tr} | T_{ik}^* u_{tr} \rangle \]

\[ = \sum_k \langle T_u u_k | T_u u_k \rangle \]

\[ = \text{tr}(T_u^* T_u). \]

2. Let \( u \) be cyclic. Then there are \( M^{(n)} = (M_{ik}^{(n)}) \in \mathcal{R} \) with

\[ \lim_{n \to \infty} M^{(n)} u = v \]

for every \( v = (S_{ij} u_{tr})_i \in \mathcal{K} \), where \( S = (S_{ij}) \in \mathcal{R}_0 \). This means, using Proposition 2.7 and Corollary 2.5,

\[ 0 \leq \sum_{i,j} \left\| \sum_k M^{(n)}_{ik} T_{kj} u_{tr} - S_{ij} u_{tr} \right\|^2 \]

\[ = \sum_{i,j} \left( \left\| \sum_k M^{(n)}_{ik} T_{kj} u_{tr} \right\|^2 - 2 \sum_k \text{Re} \langle M^{(n)}_{ik} T_{kj} u_{tr} | S_{ij} u_{tr} \rangle + \| S_{ij} u_{tr} \|^2 \right) \]

\[ = \sum_{i,j} \left( \left\| \sum_k T_{kj}^* (M^{(n)}_{ik})^* u_{tr} \right\|^2 - 2 \sum_k \text{Re} \langle T_{kj}^* (M^{(n)}_{ik})^* u_{tr} | S_{ij}^* u_{tr} \rangle + \| S_{ij}^* u_{tr} \|^2 \right) \]

\[ = \sum_{i,j} \left\| \sum_k T_{kj}^* (M^{(n)}_{ik})^* u_{tr} - S_{ij}^* u_{tr} \right\|^2, \]

i.e., since \((T_{kj}^*)_{i,k} \subset T_u^* \) and \( \mathcal{R}_0 u_{tr} \) is dense in \( \mathcal{K} \), \( T_u^* \) has dense range, i.e. \( T_u \) is injective.

Let now \( T_u \) be injective and \( M' = (M'^{ij}) \in \mathcal{R}' \) with \( M' u = 0 \). Now

\[ M' u = (\sum_j M'^{ij} T_{kj} u_{tr})_{i,k} = 0, \]

and

\[ 0 = \left\| M' u \right\|^2 = \sum_{i,k} \left\| \sum_j M'^{ij} T_{kj} u_{tr} \right\| \]

\[ = \sum_{i,k} \left\| \sum_j T_{kj} M'^{ij} u_{tr} \right\| = \left\| T_u v \right\|. \]
where $v := (M'_{ij}u_{tr})^j_i = \sum_k M'v_k \in \mathcal{D}(T_u)$ ($T_u$ is closed), hence $T_u v = 0$, and, since $T_u$ is injective, $v^j_i = M'_{ij}u_{tr} = 0$ for all $i, j \in \mathbb{N}$. Because $u_{tr}$ is cyclic for $T$ hence separating for $T'$, $M'_{ij} = 0$ for all $i, j \in \mathbb{N}$, s.t. $M' = 0$.

3. Let $u$ be separating. This means that $u$ is cyclic for $\mathcal{R}'$. Then there are $M(n) = (M^j_i(n)) \in \mathcal{R}'$ and

$$\lim_{n \to \infty} M(n) u = v$$

for every $v = (S_{ij}u_{tr})^j_i \in \mathcal{K}$, where $(S_{ij}) \in \mathcal{R}_0' \ (i, j \in \mathbb{N})$. This means

$$0 \leftarrow \sum_{i,j} \left\| \sum_k M^j_k(n) T_{ik} u_{tr} - S_{ij} u_{tr} \right\|^2 = \sum_{i,j} \left\| \sum_k T_{ik} M^j_k(n) u_{tr} - S_{ij} u_{tr} \right\|^2.$$

Since $\mathcal{R}_0' u_{tr}$ is dense in $\mathcal{K}$ we have proven that $T_u$ has dense range.

For the converse read the argument backwards.

4. This follows from 2. and 3.

Remark 2.1. Also here, as in the finite case, the finite trace condition of Lemma 2.6 is not only necessary but also sufficient for an operator being the operator associated with a vector in the sense of Definition 2.1. Suppose that $\text{tr}(T^*T) < \infty$ with $T \in \mathcal{R}$. Then

$$\infty > \text{tr}(T^*T) = \text{tr}(H^2) = \sum_k \int \lambda^2 d \|E_{\lambda} v_k\|$$

hence

$$\int \lambda^2 d \|E_{\lambda} v_k\| < \infty \ \forall k \in \mathbb{N},$$

i.e. $v_k \in \mathcal{D}(H) = \mathcal{D}(T)$, and

$$\sum_k \|Tv_k\|^2 = \sum_k \|Hv_k\|^2 = \sum_k \int \lambda^2 d \|E_{\lambda} v_k\| < \infty.$$

This shows that the assumptions of Lemma 2.3 are fulfilled.

Corollary 2.8. $\mathcal{R}$ possesses a cyclic and separating vector $u_0 \in \mathcal{K}$.

Proof. Set $T := (\delta_{ij}j^{-2}I)_{i,j}$ or $u_0 := \sum_j j^{-2}v_j$. Then $T$ fulfills the conditions of Lemma 2.3 and is invertible, s.t. from Lemma 2.6 follows that $u_0$ is cyclic and separating. □
In the last step of this subsection we show that the model we have just treated is really representative for the general situation, in the sense that all infinite type $I$ or type $II$ factors can be considered as a matrix algebra of finite type $I$ or type $II$ factors, resp. This is shown by the next Lemma 2.9.

**Lemma 2.9.** Every infinite but semifinite von Neumann factor $(\mathcal{M}_0, \mathcal{H}_0)$ with cyclic and separating vector $u_0 \in \mathcal{H}_0$ is unitarily equivalent to $\mathcal{T} \otimes L(\mathcal{H}_\infty) \otimes \mathbb{C} =: \mathcal{R}, \mathcal{H} \otimes \mathcal{H}_\infty \otimes \mathcal{H}_\infty =: \mathcal{K}$, where $\mathcal{T}$ is a finite von Neumann factor acting on the Hilbert space $\mathcal{H}$ with cyclic and separating vector and $\mathcal{H}_\infty$ is a separable infinite dimensional Hilbert space.

**Proof.** Since $\mathcal{M}_0$ is infinite but semifinite there is a countable orthogonal family of finite equivalent projections $(E_n)_{n \in \mathbb{N}}$ in $\mathcal{M}_0$, s.t. $\sum E_n = \text{Id}$. Now there is a selfadjoint system of matrix units $(E_{ab})_{a,b \in \mathbb{N}}$ with $E_{aa} = E_a$ (cf. [KR86, 6.6.4]). This shows that $\mathcal{M}_0$ is isomorphic to $\tilde{\mathcal{T}} \otimes L(\mathcal{H}_\infty)$ where $\tilde{\mathcal{T}} := \{E_{ab}\} \cap \mathcal{M}_0$ and $\tilde{\mathcal{T}}$ is isomorphic to every $E_n \mathcal{M}_0 E_n$ ($n \in \mathbb{N}$). Since the projections $E_n$ are finite also $\tilde{\mathcal{T}}$ is a finite factor.

Since $\mathcal{M}_0$ possesses the separating vector $u_0$ we can represent the algebras $E_n \mathcal{M}_0 E_n$ by the GNS representation for the faithful state $\omega_n$ induced by the separating vector $E_n u_0$ on a Hilbert space $\mathcal{H}_n$, s.t. the vector $u_n \in \mathcal{H}_n$ implementing the state $\omega_n$ is a cyclic and separating vector for $E_n \mathcal{M}_0 E_n$. Since all the $E_n \mathcal{M}_0 E_n$ are isomorphic and they possess in this representation a cyclic and separating vector, they are all unitarily equivalent. This means that we can choose as $\mathcal{T}$ one of the $E_n \mathcal{M}_0 E_n$ acting on the representation space $\mathcal{H}_n$.

Since the factor $(\mathcal{T} \otimes L(\mathcal{H}_\infty) \otimes \mathbb{C} =: \mathcal{R}, \mathcal{H} \otimes \mathcal{H}_\infty \otimes \mathcal{H}_\infty =: \mathcal{K})$ possesses a cyclic and separating vector if $(\mathcal{T}, \mathcal{H})$ does (see Corollary 2.8) and it is isomorphic to $\mathcal{M}_0$ it is unitarily equivalent to $\mathcal{M}_0$. \hfill \Box

The results of this section (and the analogues in [Bo]) can be subsumed in the next two theorems:

**Theorem 2.10.** Let $(\mathcal{M}_0, \mathcal{H}_0)$ be a semifinite von Neumann factor. Let further $u \in \mathcal{H}_0$. Then there is exactly one operator $T_u \eta \mathcal{M}_0$ associated with the vector $u$ in the sense of [Bo, Def 2.1.] in the finite case and in the sense of Definition 2.1 in the infinite case, resp., having the following properties:

1. $\text{tr}(T_u T_u^*) = \text{tr}(T_u^* T_u) < \infty$.
2. $u$ is cyclic, iff $T_u$ is injective.
3. $u$ is separating, iff $T_u$ has dense range.
4. $u$ is cyclic and separating iff $T_u$ is injective and has dense range, i.e. iff $T_u$ is invertible.

**Proof.** The finite case is just Theorem 1.1. In the infinite case the existence and the asserted properties follow from Lemma 2.9 and Lemma 2.6 infinite case, the uniqueness from Corollary 2.4. \hfill \Box
Theorem 2.11. Let $T_{\eta}M_0$. Then there is a vector $u \in H_0$ s.t $T = Tu$ iff $\text{tr}(TT^*) = \text{tr}(T^*T) < \infty$.

Proof of Theorem 2.11. Again the finite case is just Theorem 1.2. In the infinite case the necessity of the trace condition follows from Theorem 2.10 and the sufficiency from Remark 2.1, resp.

3 Generation of Modular Objects

In this section we show how the modular objects of a cyclic and separating vector $u_0 \in H$ for a semifinite von Neumann factor $(M_0, H_0)$ are related to the operator $T_{u_0}$ constructed in the last section. As in §2 we consider as a model for the infinite but semifinite factor the factor $T \otimes L(H_\infty) \otimes \mathbb{C} =: \mathcal{R}$, where $T$ is a finite factor with cyclic trace vector $u_{\text{tr}} \in H$.

If $u_0 \in \mathcal{R}$ is a cyclic and separating vector for $\mathcal{R}$, according to Lemma 2.6, there is an invertible operator $T_{u_0} \eta \mathcal{R}$, s.t. $u_0 = \sum_k T_{u_0} v_k$, where $v_k = (\delta_i^j \delta_i^k u_{\text{tr}})_{ij}$.

Using this operator we can formulate the following analogue to Theorem 1.3:

Theorem 3.1. Use the notations from above. Let further $T_{u_0} = HV = (H_{ij})_{ij}(V_{ij})_{ij}$ be the polar decomposition of $T_{u_0}$. With the conjugation $\tilde{J}$ defined as

$$\tilde{J}(M_{ij} u_{\text{tr}})_i^j := (M_i^* u_{\text{tr}})_i^j := (JM_{ij} u_{\text{tr}})_i^j \quad \forall M = (M_{ij})_{ij} \in \mathcal{R},$$

where $J$ is the conjugation corresponding to the trace vector $u_{\text{tr}}$, we can calculate the modular objects $(\Delta_0, J_0)$ of $(M_0, u_0)$ as follows:

$$J_0 = \tilde{J}V^* J\tilde{J} = VJ^*,$$

and

$$\Delta_0 = J_0H_0^{-1}J_0H_0,$$

where $H_0 = H^2 = T_{u_0}T_{u_0}^*$. 

Proof. 1. First we observe that $\tilde{J}R\tilde{J} \in \mathcal{R}'$ for every $R \in \mathcal{R}$. For let $R = (R_{ij}) \in \mathcal{R}$ and $v = (u_i^j)_i^j = (M_{ij} u_{\text{tr}})_i^j \in \mathcal{K}$, $(M_{ij}) \in \mathcal{R}_0$, then

$$\tilde{J}R\tilde{J}v = \tilde{J}R(JM_{ij} u_{\text{tr}})_i^j = \tilde{J}(\sum_i R_{ki}JM_{ji} u_{\text{tr}})_k^i = (J \sum_i R_{ji}JM_{ki} u_{\text{tr}})_k^i = (JR_{ji}J^i_k)(M_{ki} u_{\text{tr}})_k^i := R' \in \mathcal{R}'$$

$$= R'v.$$
Further
\[ \tilde{J} v = \tilde{J}(J M_{ij} u_{\text{tr}})_{ij} = (M_{ij} u_{\text{tr}})_{ij} = v, \]
s.t. \( \tilde{J} \) is an (algebraic) conjugation for \( \mathcal{R} \).

2. Let now \( T_{u_0} \) be bounded (⇒ all the \( T_{ij} \) and \( H_{ij} \), resp. are bounded). Then we show that the Tomita operator \( S \) defined by
\[ S A u_0 = A^* u_0 \quad \forall A \in \mathcal{R} \]
can be written as
\[ S = H^{-1} V \tilde{J} V^* H. \] (3.1)
For this let \( A = (A_{ij})_{ij} \in \mathcal{R} \) and \( u_0 = (\sum_k H_{jk} V_{kl} u_{\text{tr}})_l^j \). Then
\[ A u_0 = (\sum_{j,k} A_{ij} H_{jk} V_{kl} u_{\text{tr}})_i^l \]
and
\[ A^* u_0 = (\sum_{j,k} A_{ji}^* H_{jk} V_{kl} u_{\text{tr}})_i^l. \]
Now
\[ (H^{-1} V \tilde{J} V^* H) A u_0 = H^{-1} V \tilde{J} (\sum_{i,j,k,m} V_{mn}^* H_{mi} A_{ij} H_{jk} V_{kl} u_{\text{tr}})_n^l \]
\[ = H^{-1} V (\sum_{i,j,k,m} J V_{ml}^* H_{mi} A_{ij} H_{jk} V_{kn} u_{\text{tr}})_n^l \]
\[ = H^{-1} V (\sum_{i,j,k,m} V_{km}^* H_{kj} A_{ij}^* H_{im} V_{ml} u_{\text{tr}})_n^l \]
\[ = H^{-1} (\sum_{i,j,m} H_{nj} A_{ij}^* H_{im} V_{ml} u_{\text{tr}})_n^l \]
\[ = (\sum_{i,m} A_{in}^* H_{im} V_{ml} u_{\text{tr}})_n^l = A^* u_0, \]
which proves (3.1). Now \( S^* = H V \tilde{J} V^* H^{-1} \) and
\[ \Delta_0 = S^* S \]
\[ = H V \tilde{J} V^* H^{-1} H^{-1} V \tilde{J} V^* H \]
\[ = V \tilde{J} V^* H^{-2} V \tilde{J} V^* H^2 \]
\[ = J_0 H_0^{-1} J_0 H_0. \]
Further
\[ J_0 \Delta_0^{1/2} = H^{-1} J_0 H = S, \]
and all the assertions are proven in the bounded case.
3. In the last step we approximate the (unbounded) operator $T_{u_0}$ by bounded operators $T_n$ in exactly the same way as in the proof of Theorem 3.1. in [Bol] and show the assertions like there also in the unbounded case.

4 The Second Simple Class of Solutions of the Inverse Problem

In this section we want to use the results of the last two sections to examine the second simple classes of solutions of the inverse problem introduced by Wollenberg in [Wolb] for type $I$ factors, and considered in [Bol] also for type $II_1$ factors. For the construction of this class it is crucial that the inverse $\Delta_0^{-1}$ of the modular operator is again a modular operator. To this scope there was shown the following

Lemma 4.1. Let $\Delta_0 = J_0 H_0^{-1} J_0 H_0$ be the decomposition of the modular operator $\Delta_0$, where $J_0 = JV^* J V = VJV^*$ and $T_{u_0} = H_0^{1/2} V$ is the operator corresponding to $u_0$ (cf. Theorem 3.1). Then $\Delta_0^{-1} = J_0 H_0 J_0 H_0^{-1}$ and the following is equivalent:

1. $(\Delta_0^{-1}, J_0)$ are the modular objects w.r.t. a cyclic and separating vector $u_1 \in \mathcal{H}_0$.

2. $\text{tr}(H_0^{-1}) < \infty$. (4.1)

This lemma can be proven with the same techniques as in [Bol] also for the infinite case taking into account Theorem 2.10, Theorem 2.11, and Theorem 3.1.

Now we must examine, whether or not the second condition in Lemma 4.1 is fulfilled:

Lemma 4.2. For type $I_\infty$ and type $II_\infty$ factors the condition (4.1) is never true.

Proof. Let $\mathcal{M}_0$ now be a type $I_\infty$ or $II_\infty$ factor and $T_{u_0} = H_0^{1/2} V$ the operator corresponding to the cyclic and separating vector $u_0$. Let further $E_\lambda \in \mathcal{M}_0$ the spectral resolution of $H_0$. Then we can define a positive measure $\mu_{tr}$ on the $\sigma$-algebra of Borel sets in $\mathbb{R}$, s.t.

\[
\text{tr}(H_0) = \int \lambda d\mu_{tr}(\lambda),
\]

where

\[
\mu_{tr}(B) := \text{tr} E(B)
\]

for all Borel sets $B$. Now $c := \text{tr}(H_0) < \infty$. Assume w.l.o.g. $c = 1$. Then

\[
1 = \int_0^1 \lambda d\mu_{tr}(\lambda) \geq \int_0^1 \lambda d\mu_{tr}(\lambda) + \int_{(1,\infty)} d\mu_{tr}(\lambda),
\]

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i.e.

\[ \int_{(1,\infty)} d\mu_{tr}(\lambda) < \infty. \]

Since \( \mathcal{M}_0 \) is infinite \( \infty = \text{tr}(\text{Id}) = \mu_{tr}(\mathbb{R}) \), i.e.

\[ \infty = \int_{\lambda} d\mu_{tr}(\lambda) = \int_{[0,1]} d\mu_{tr}(\lambda) + \int_{(1,\infty)} d\mu_{tr}(\lambda), \]

hence

\[ \int_{[0,1]} d\mu_{tr}(\lambda) = \infty. \]

Suppose now that also \( \text{tr}(\Pi_0^{-1}) < \infty \), then

\[ \infty > \int_{\lambda} \lambda^{-1}d\mu_{tr}(\lambda) \geq \int_{[0,1]} d\mu_{tr}(\lambda) + \int_{(1,\infty)} \lambda^{-1}d\mu_{tr}(\lambda), \]

which is a contradiction. \( \square \)

Hence the last lemma shows that for infinite semifinite factors the second class of solutions of the inverse problem can never be constructed. This result was yet obtained by Wollenberg in [Wolb] for the type \( I_\infty \) case.

5 The Classification of Solutions in the Pure Point Spectrum Case

In this section we want to show the modifications of classification results obtained in [Bo1]. The definition of the equivalence relation does not use any special properties of the finite factors, and can just be repeated here:

**Definition 5.1.** Two semifinite von Neumann factors \( \mathcal{M}, \mathcal{N} \in NF_{\mathcal{M}_0}(\Delta_0, J_0, u_0) \) are called equivalent, \( \mathcal{M} \sim \mathcal{N} \), if \( \mathcal{M} \in NF_{\mathcal{N}}^1(\Delta_0, J_0, u_0) \), i.e. if there exists a unitary operator \( U \) on \( \mathcal{H}_0 \), s.t. \( \mathcal{M} = U\mathcal{N}U^* \), \( U \) commutes with \( \Delta_0 \) and \( J_0 \) and \( U^*u_0 = \pm u_0 \) (For the definition of the class \( NF_{\mathcal{N}}^1(\Delta_0, J_0, u_0) \) see [Bo1]).

Also the next lemmas can be formulated and proved in exactly the same way as in the finite case. Assume in the following that \( \mathcal{H}_0 \) has pure point spectrum, i.e. \( \mathcal{H}_0 = \sum_{k \in K} \mu_k E_k \) where the \( \mu_k \) (\( k \in K \)) are the eigenvalues of \( \mathcal{H}_0 \) and \( E_k \in \mathcal{M}_0 \) are the corresponding (orthogonal) eigenprojections with
\( m_k := \text{tr} E_k =: D_{\mathcal{M}_0}(E_k) \) their von Neumann dimension. Then we have for \( \Delta_0 \) the following decomposition
\[
\Delta_0 = H_0 J_0 H_0^{-1} J_0 \\
= \sum_{k,l \in K} \mu_k \mu_l^{-1} E_k J_0 E_l J_0 \\
= \sum_{j \in J} \lambda_j F_j, \tag{5.1}
\]
where the \( \lambda_j \) \((j \in J)\) are the eigenvalues of \( \Delta_0 \) and \( F_j \) are the corresponding eigenprojections. Now

**Lemma 5.1.** With the notations introduced above we can compute the spectrum of \( \Delta_0 \) in the following way:
\[
\{ \lambda_j | j \in J \} = \{ \mu_k \mu_l^{-1} | k, l \in K \} \quad \forall j \in J \tag{5.2}
\]
and
\[
n_j = \sum_{\mu_k \mu_l^{-1} = \lambda_j} m_k m_l \quad \forall j \in J \text{ if } \mathcal{M}_0 \text{ is type } I, \tag{5.3a}
\]
\[
n_j = \infty \quad \forall j \in J \text{ if } \mathcal{M}_0 \text{ is type } II, \tag{5.3b}
\]
where \( n_j := D_{L(H_0)}(F_j) \) with \( D_{L(H_0)}(F_j) \) the dimension function in the type \( I_\infty \) factor \( L(H_0) \), which corresponds to the normalized Hilbert space dimension.

**Lemma 5.2.** If there are two solutions of the inverse problem \( \mathcal{M}_1, \mathcal{M}_2 \) s.t. the corresponding selfadjoint operators \( H_1 \) and \( H_2 \) have the same eigenvalues modulo a positive constant \( c > 0 \) and same (von Neumann) multiplicities, then \( \mathcal{M}_1 \sim \mathcal{M}_2 \).

**Lemma 5.3.** If there are two equivalent solutions \( \mathcal{M}_1, \mathcal{M}_2 \) of the inverse problem with the corresponding positive operators \( H_1 \) and \( H_2 \), resp., (having pure point spectrum) then \( H_1 \) and \( H_2 \) have the same eigenvalues (up to a positive constant) and von Neumann multiplicities, i.e. they are unitarily equivalent in \( \mathcal{M}_0 \).

The only difference to the finite case is shown by the next

**Lemma 5.4.** Let \((\mu_k, m_k)_{k \in K}\) be a sequence of pairs of positive reals \( \mu_k > 0 \) and \( m_k > 0 \), s.t.
\[
m_k \in \mathbb{N} \text{ if } \mathcal{M}_0 \text{ is type } I_\infty, \tag{5.4a}
\]
\[
m_k \in \mathbb{R}_{>0} \text{ if } \mathcal{M}_0 \text{ is type } II_\infty, \tag{5.4b}
\]
and
\[ \sum_{k \in K} m_k = \infty \]  \hspace{1cm} (5.4c)

and
\[ \sum_{k \in K} m_k \mu_k = 1 \]  \hspace{1cm} (5.4d)

and the relations (5.2) and (5.3) are fulfilled. Then there exists a solution \( M = U M_0 U^* \in NF_{M_0}(\Delta_0, J_0, u_0) \), s.t. \( U^* \Delta_0 U = H J_0 H^{-1} J_0 \) and \( H \) has the eigenvalues and multiplicities \( (\mu_k, m_k)_{k \in K} \) (cf. [Wolb, prop.4.1]).

For the proof we need the following auxiliary results:

**Proposition 5.5.** If \( (m_k) \) is countable family of positive reals with \( \sum m_k = \infty \), then there exists in a type II\( \infty \) von Neumann factor \( M \) a family of pairwise orthogonal projections \( (E_k) \), s.t.

\[ D(E_k) = m_k \text{ for every } k \]

Proof. We construct the \( E_k \) inductively: Since the range of \( D_M \) is all of \( \mathbb{R}_{\leq 0} \) (cf. [KR86, 8.4.4]) there is a projection in \( M \), s.t.

\[ D(M(E_k)) = m_k \text{ for } 1 \leq k < N \].

Setting \( F_N := \text{Id} - \sum_{k=1}^{N} E_k \) the restricted algebra \( F_N M F_N \) is again a type II factor, finite, if \( F_N \) is finite, and infinite, if \( F_N \) is infinite (cf. [KR86, Ex. 6.9.16]) with the dimension function

\[ D_N(F_n E F_N) := D(M_0(F_n E F_N)/D(F_n)) \text{ } \forall F_n E F_N \in F_N M F_N, \]

if \( F_N \) is finite, and \( D_N = D_{M_0} \) else, where

\[ D_{M_0}(F_N) = D_{M_0}(\text{Id} - \sum_{k=1}^{N} E_k) = 1 - \sum_{k=1}^{N} D_{M_0}(E_k) \geq m_N. \]

With the same argument as above there is again a projection \( E_N \in F_N M F_N \subset M \), s.t. \( D_N(E_N) = D(F_N)^{-1} m_N \leq 1 \), if \( F_N \) is finite, and \( D_N(E_N) = m_N \) else. In both cases \( D_{M_0}(E_N) \) and \( E_N < F_N \perp E_k \) (1 \leq k < N).

Now the proof of Lemma 5.4 is the same as in [Bo].

**Remark 5.1.** (5.4) show that in the infinite case we have always an infinite set of eigenvalues with 0 as cummulation point, i.e. \( K = \mathbb{N} \) and 0 is in the spectrum \( \sigma(H) \) of \( H \).

Now we can summarize the lemmas of this section in the following

**Theorem 5.6.** Let \( M_0 \) be a semifinite von Neumann factor with cyclic and separating vector \( u_0 \) and \( T u_0 = H_0^{-1/2} V \) the operator corresponding to \( u_0 \). If \( H_0 \) has pure point spectrum, also \( \Delta_0 \) has it. In this case let \( (\lambda_j) \) (j \in J) be the eigenvalues of \( \Delta_0 \). Then
1. Two solutions $\mathcal{M}_1, \mathcal{M}_2 \in NF_{M_0}(\Delta_0, J_0, u_0)$ of the inverse problem with corresponding invertible operators $H_i M_0$ ($i = 1, 2$) having pure point spectrum are equivalent iff $H_1$ and $H_2$ have the same eigenvalues and (von Neumann) multiplicities.

2. A positive invertible operator $H_\eta M_0$ with pure point spectrum gives rise to a solution of the inverse problem iff its eigenvalues and multiplicities satisfy (5.2), (5.3), and (5.4).

3. When the corresponding operators $H$ has pure point spectrum the equivalence classes of $\sim$ are completely classified by the spectrum of the corresponding operators, i.e. by sequences of pairs of positive reals $(\mu_k, m_k)$ satisfying (5.2), (5.3), and (5.4).

**Example 5.1.** Here we want to give some examples to illustrate Theorem 5.6.

1. In [Wolb] you can find some examples for the type $I$ case.

2. Let

$$(\ldots, 10^{-3}, 10^{-2}, 10^{-1}, 1, 10^2, 10^3, \ldots)$$

be the eigenvalues of a modular operator for a type $II_\infty$ factor. Then

$$( (c_1 \cdot 1, 1), (c_1 \cdot 10^{-1}, 1), (c_1 \cdot 10^{-2}, 1), (c_1 \cdot 10^{-3}, 1), \ldots)$$

and

$$( (c_2 \cdot 1, 1), (c_2 \cdot 10^{-1}, 1), (c_2 \cdot 10^{-3}, 1), (c_2 \cdot 10^{-5}, 1), \ldots)$$

characterize two different classes of solutions of the inverse problem, i.e. they both satisfy (5.2), (5.3), and (5.4), where $c_i$ ($i = 1, 2$) are appropriate chosen constants. This shows that in this case there are more than the simple classes of solutions of the inverse problem.

3. Let $(\mu_k, m_k)_{k \in \mathbb{N}}$ characterize a class of solutions of the inverse problem in the type $II_\infty$ case, where $m_l \neq m_k$ for at least one pair $k, l \in \mathbb{N}$, then for every finite permutation $\sigma$ of $\mathbb{N}$ interchanging $k$ and $l$ also $(c\mu_k, m_{\sigma(k)})$ characterize another class of solutions of the inverse problem ($c > 0$ a norming constant) which is really a new one.

4. Let again $(\mu_k, m_k)_{k \in \mathbb{N}}$ be a solution of the inverse problem in the type $II_\infty$ case, and let $k, l \in \mathbb{N}$ be a pair of indices and $\epsilon > 0$. Then we get another class by adding $\epsilon$ to $m_k$ and subtracting it from $m_l$ where again we have really a new class.

**Remark 5.2.** 1. Example 5.1.3 and Example 5.1.4 shows that in the type $II_\infty$ case, when $H_0$ has pure point spectrum, we can always construct a second class of solutions, different from the simple class discussed in §4, i.e. $NF_{M_0} \neq NF_{M_0}^1$, in contrast to the type $I$ case, where for modular operators with generic spectrum we have $NF_{M_0} = NF_{M_0}^1$ (cf. [Wolb]).
2. Unfortunately the classification result presented here applies only to operators with pure point spectrum. Whereas in general there are also operators with more complicated spectrum (cf. [Bol, Remark 4.1]), for type $I$ factors this is no restriction, since all operators generating modular operators are trace class operators, hence have pure point spectrum.

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