Homogenization of Periodic Nonlocal Partial Differential Equations

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Abstract

We study the “periodic homogenization” for a class of nonlocal partial differential equations of parabolic-type with rapidly oscillating coefficients, associated to stochastic differential equations driven by multiplicative isotropic \( \alpha \)-stable Lévy noise for \( 1 < \alpha < 2 \). Our homogenization method is probabilistic. It turns out that, under some weak regularity assumptions, the limit of the solutions satisfies a nonlocal partial differential equation with constant coefficients, associated to a symmetric \( \alpha \)-stable Lévy process.

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1 Introduction

The goal of this paper is to study the limit behavior, as \( \epsilon \to 0 \), of the solution \( u^\epsilon : \mathbb{R}^d \to \mathbb{R} \) of the following nonlocal partial differential equation (PDE) of parabolic-type with rapidly oscillating periodic and singular coefficients,

\[
\begin{aligned}
\frac{\partial u^\epsilon}{\partial t}(t, x) &= L^\alpha u^\epsilon(t, x) + \left( \frac{1}{\epsilon^{\alpha-1}} e \left( \frac{x}{\epsilon} \right) + g \left( \frac{x}{\epsilon} \right) \right) u^\epsilon(t, x), \quad t > 0, x \in \mathbb{R}^d, \\
u^\epsilon(0, x) &= u_0(x),
\end{aligned}
\]

(1.1)
where \(1 < \alpha < 2\) and the linear operator \(\mathcal{L}_\epsilon^\alpha\) is a nonlocal integro-differential operator of Lévy-type given by

\[
\mathcal{L}_\epsilon^\alpha f(x) := \int_{\mathbb{R}^d \setminus \{0\}} \left[ f \left( x + \sigma \left( \frac{x}{\epsilon}, y \right) \right) - f(x) - \sigma_i \left( \frac{x}{\epsilon}, y \right) \partial_i f(x) 1_B(y) \right] \nu^\alpha(dy)
\]
\[
+ \left[ \frac{1}{\epsilon^{\alpha-1}} b^i \left( \frac{x}{\epsilon} \right) + c^i \left( \frac{x}{\epsilon} \right) \right] \partial_i f(x), \quad x \in \mathbb{R}^d.
\]

where \(\nu^\alpha(dy) := \frac{dy}{|y|^{\alpha}}\). In this paper, we use Einstein’s convention that the repeated indices in a product will be summed automatically.

For notational simplicity, we introduce the linear operator \(A_{\sigma,\nu}^\alpha\) defined by

\[
A_{\sigma,\nu}^\alpha f(x) := \int_{\mathbb{R}^d \setminus \{0\}} \left[ f \left( x + \sigma(x, y) \right) - f(x) \right.
\]
\[
- \sigma_i(x, y) \partial_i f(x) 1_B(y) \left] \nu^\alpha(dy), \quad x \in \mathbb{R}^d.
\]

For a function \(f\) on \(\mathbb{R}^d\) (or \(F\) on \(\mathbb{R}^d \times \mathbb{R}^d\)), we denote \(f_\epsilon(x) := f \left( \frac{x}{\epsilon} \right)\) (or \(F_\epsilon(x,y) := F \left( \frac{x}{\epsilon}, y \right)\)).

Then

\[
\mathcal{L}_\epsilon^\alpha = A_{\sigma,\nu}^\alpha + \left( \frac{1}{\epsilon^{\alpha-1}} b_\epsilon + c_\epsilon \right) \cdot \nabla. \tag{1.3}
\]

The main result of this paper is the following theorem. We will prove it at the end of Section 5.

**Theorem 1.1.** Under some weak assumptions on the coefficients, the nonlocal PDE (1.1) has a unique mild solution \(u^\epsilon\) for each \(\epsilon > 0\). Moreover, for each \(t \geq 0, x \in \mathbb{R}^d\),

\[
u^\epsilon(t,x) \to u(t,x), \quad \epsilon \to 0, \quad \tag{1.4}
\]

where \(u\) satisfies the limit nonlocal PDE,

\[
\begin{aligned}
\frac{\partial u}{\partial t}(t,x) &= A^{Id_\nu,\Pi} u(t,x) + B \cdot \nabla u(t,x) + Eu(t,x), \quad t > 0, x \in \mathbb{R}^d, \\
u(0,x) &= u_0(x), \quad x \in \mathbb{R}^d,
\end{aligned}
\]

where

\[
A^{Id_\nu,\Pi} f(x) := \int_{\mathbb{R}^d \setminus \{0\}} \left[ f(x + y) - f(x) - y^i \partial_i f(x) 1_B(y) \right] \Pi(dy),
\]

and the constant coefficients \(B, E\) and measure \(\Pi\) are given by (5.4), (5.8) and (5.5) respectively. The solution is given by

\[
u(t,x) = \mathbf{E}[u_0(x + Bt + L_t)] e^{Et},
\]

where \(\{L_t\}_{t \geq 0}\) is a symmetric \(\alpha\)-stable Lévy processes with jump intensity measure \(\Pi\).

The original probabilistic approach to the homogenization of local linear second order parabolic partial differential operators is presented in [6, Chapter 3], which is based on the ergodic theorem, the Feynman-Kac formula and the functional central limit theorem. They applied a technique of removing the singular drift, which is now known as Zvonkin’s
transform, appeared originally in [38]. By now, there are lots of literature concerning the homogenization of second order local PDEs, i.e., the case of replacing the operator $\mathcal{A}^{\sigma, \nu}$ in (1.3) by a second order partial differential operator with singular coefficients. Two different scales of spatial variables involved in the coefficients have been considered in [5], by using the nonlinear Feynman-Kac formula in the context of backward stochastic differential equations (SDEs). In [25], the authors allowed the singular coefficients to be time-dependent and rapidly oscillating in time with a different scale in contrast to the spatial variable. The paper [13] dealt with the case when the second order coefficient matrix can be degenerate, using the existence of a spectral gap and Malliavin’s calculus. The work in this paper is highly motivated by [13, 24].

There are also some literature for the homogenization of nonlocal PDEs or SDEs with jumps involved. We refer the reader to [2, 28, 34] for the periodic homogenization results of some kinds of nonlocal operators involving stable-like terms or convolution type kernels. The methods used in these papers are all analytic. The probabilistic study of homogenization of periodic stable-like processes in pure jump or jump-diffusion case can be found in [10, 32]. The homogenization in random medium is slightly different from the periodic case, see [31] for related results for jump-diffusion processes in random medium.

The homogenization of a kind of one-dimensional pure jump Markov processes with the following form of generators has been investigated in the paper [14],

$$\mathcal{A}_\epsilon f(x) = \int_{\mathbb{R}} [f(x + z) - f(x) - z f'(x)] a\left(\frac{x}{\epsilon}, \frac{z}{\epsilon}\right) \frac{dz}{|z|^{\alpha+1}} + \frac{1}{\epsilon^{\alpha-1}} b\left(\frac{x}{\epsilon}\right) f'(x).$$

See [35] for a generalization for multi-dimensional case and with diffusion terms involved. But as we will see in Section 2, the change of variables allows us to write

$$\mathcal{L}_\epsilon^a f(x) := \int_{\mathbb{R} \setminus \{0\}} [f(x + z) - f(x) - z^i \partial_i f(x) 1_B(y)] h\left(\frac{x}{\epsilon}, z, \frac{1}{\epsilon^{\alpha}} b\left(\frac{x}{\epsilon}\right) + c^i\left(\frac{x}{\epsilon}\right)\right) \partial_i f(x),$$

for some function $h$. Note that the main difference is that we do not involve oscillations for $h$ in its second variable which is not periodic, while [14, 35] involve. Meanwhile, the coefficients of drift and the zeroth order term $b, c, e, g$ has two different scales.

In paper [11], the author considered the homogenization of SDEs driven by multiplicative stable processes, where the multiplicative coefficient $\sigma$ is linear in the second variable in the sense that $\sigma(x, y) = \sigma_0(x) y$, with $\sigma_0$ three-times continuously differentiable. In the present paper, we generalize his results to the general multiplicative case, and the coefficients only need to possess some Hölder or Lipschitz continuity. This will give rise to several difficulties both in analytic and probabilistic aspects. We also use the homogenization results of SDEs to study the homogenization of the nonlocal PDEs with singular coefficients involved, by utilizing Feynman-Kac formula.

We denote by $C_k^k (C_b^k)$ with integer $k \geq 0$ the space of (bounded) continuous functions possessing (bounded) derivatives of orders not greater than $k$. We shall explicitly write out the domain if necessary. Denote by $C_0(\mathbb{R}^d) := C_0^0(\mathbb{R}^d)$, it is a Banach space with the supremum norm $\|f\|_0 = \sup_{x \in \mathbb{R}^d} |f(x)|$. The space $C_b^k(\mathbb{R}^d)$ is a Banach space endowed with
the norm \[\|f\|_k = \|f\|_0 + \sum_{j=1}^{k} \|\nabla^\otimes j f\|.\] We also denote by \(C^{1-}\) the class of all Lipschitz continuous functions. For a non-integer \(\gamma > 0\), the Hölder spaces \(C^\gamma\) (\(C^\gamma_0\)) are defined as the subspaces of \(C^{[\gamma]}\) (\(C^\gamma_b\)) consisting of functions whose \([\gamma]\)-th order partial derivatives are locally Hölder continuous (uniformly Hölder continuous) with exponent \(\gamma - [\gamma]\). These two spaces \(C^\gamma\) and \(C^\gamma_0\) obviously coincide when the underlying domain is compact. The space \(C^\gamma_b(\mathbb{R}^d)\) is a Banach space endowed with the norm \(\|f\|_\gamma = \|f\|_{[\gamma]} + \|\nabla^{[\gamma]} f\|_{\gamma - [\gamma]}\), where the seminorm \([\cdot]_\gamma\) with \(0 < \gamma' < 1\) is defined as

\[\lbrack f \rbrack_{\gamma'} = \sup_{x,y \in \mathbb{R}^d, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma'}.\]

In the sequel, the torus \(\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d\) will be used frequently. Denote by \(\mathcal{D} := \mathcal{D}(\mathbb{R}_+; \mathbb{T}^d)\) the space of all \(\mathbb{T}^d\)-valued càdlàg functions on \(\mathbb{R}_+\), equipped with the Skorokhod topology. We shall always identify the periodic function on \(\mathbb{R}^d\) of period 1 with its restriction on the torus \(\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d\). This allows us to regard the space \(C^k(\mathbb{T}^d)\) (\(C^\gamma(\mathbb{T}^d)\)) as a sub-Banach space of \(C^k(\mathbb{R}^d)\) (\(C^\gamma(\mathbb{R}^d)\)).

By \(B_r\) we means the open ball in \(\mathbb{R}^d\) centering at the origin with radius \(r > 0\), we shall omit the subscript when the radius is one. The capital letter \(C\) denotes a finite positive constant whose value may vary from line to line. We also use the notation \(C(\cdots)\) to emphasize the dependence on the quantities appearing in the parentheses.

The remainder of the paper is organized as follows. In Section 2, we present some general assumptions and preliminary results. In Section 3, we study the well-posedness of the nonlocal Poisson equation and the Feller properties of the semigroup associated with \(\mathcal{L}^\alpha\). Section 4 is devoted to the strong well-posedness and exponential ergodicity of the Lévy driven SDE with generator \(\mathcal{L}^\alpha\). As a consequence, we obtained the Feynman-Kac representation for the nonlocal PDE (1.1). Lastly, Section 5 contains the homogenization results of SDEs and nonlocal PDEs, utilizing the ergodicity and the Feynman-Kac representation.

## 2 Preliminaries and general assumptions

Let \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})\) be a filtered probability space endowed with a Poisson random measure \(N^\alpha\) on \((\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}_+\) with jump intensity measure \(\nu^\alpha(dy) = \frac{d\nu_{\mathfrak{L}}}{|y|^{\alpha}}\), where \(1 < \alpha < 2\). Denote by \(\tilde{N}\) the associated compensated Poisson random measure, that is, \(\tilde{N}^\alpha(dy, ds) := N^\alpha(dy, ds) - \nu^\alpha(dy)ds\). We assume that the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfies the usual conditions. Let \(L^\alpha = \{L^\alpha_t\}_{t \geq 0}\) be a \(d\)-dimensional isotropic \(\alpha\)-stable Lévy process given by

\[L^\alpha_t = \int_0^t \int_{B_r \setminus \{0\}} y \tilde{N}^\alpha(dy, ds) + \int_0^t \int_{B^c} y N^\alpha(dy, ds).\]

Given \(\epsilon > 0, x \in \mathbb{R}^d\), consider the following:

\[dX^\alpha_t = \left(\frac{1}{\epsilon^{\alpha-1}} b \left(\frac{X^\alpha_t}{\epsilon}\right) + c \left(\frac{X^\alpha_t}{\epsilon}\right)\right) dt + \sigma \left(\frac{X^\alpha_t}{\epsilon}, dL^\alpha_t\right), \quad X^\alpha_0 = x, \quad (2.1)\]
or more precisely,

\[ X^{x,\epsilon}_t = x + \int_0^t \left( \frac{1}{\epsilon^{\alpha-1}} b \left( \frac{X_s^{x,\epsilon}}{\epsilon} \right) + c \left( \frac{X_s^{x,\epsilon}}{\epsilon} \right) \right) ds + \int_0^t \int_{B(0)} \sigma \left( \frac{X_s^{x,\epsilon}}{\epsilon}, y \right) \tilde{N}^\alpha(dy, ds) + \int_0^t \int_{B(0)} \sigma \left( \frac{X_s^{x,\epsilon}}{\epsilon}, y \right) N^\alpha(dy, ds), \]

where the coefficients \( b, c, \sigma(\cdot, y) \) are periodic, for each \( y \in \mathbb{R}^d \), of periodic one in each component. The shorthand notation for the stochastic differential term in (2.1) is due to [22].

Define \( \tilde{X}^{x,\epsilon} := \frac{1}{\epsilon} X^{x,\epsilon}_{\epsilon^2 t} \). It is easy to check that

\[ d\tilde{X}^{x,\epsilon} = \left( b(\tilde{X}^{x,\epsilon}_{t^-}) + \epsilon^{\alpha-1} c(\tilde{X}^{x,\epsilon}_{t^-}) \right) dt + \frac{1}{\epsilon} \sigma \left( \tilde{X}^{x,\epsilon}_{t^-}, \epsilon d\tilde{L}^\alpha_t \right), \quad \tilde{X}^{x,\epsilon}_0 = \frac{x}{\epsilon}, \tag{2.2} \]

where \( \{\tilde{L}^\alpha_t\} := \{\frac{1}{\epsilon} L^\alpha_{\epsilon^2 t}\} \overset{d}{=} \{L^\alpha_t\} \) by virtue of the self-similarity. We shall also consider the “limit” equation, namely

\[ d\tilde{X}^x = b(\tilde{X}_{t^-}) dt + \sigma \left( \tilde{X}_{t^-}, d\tilde{L}^\alpha_t \right), \quad \tilde{X}^x_0 = x. \tag{2.3} \]

For notational simplicity, we shall allow the parameter \( \epsilon \) to be zero in \( \tilde{X}^{x,\epsilon} \) to include \( \tilde{X}^x \), i.e., \( \tilde{X}^{x,0} := \tilde{X}^x \).

In the sequel, we will regard the solutions \( \tilde{X}^{x,\epsilon}, \tilde{X}^x \) of (2.2) and (2.3) as \( \mathbb{T}^d \)-valued processes, by mapping all trajectories of the processes on \( \mathbb{R}^d \) to the torus \( \mathbb{T}^d \), via the canonical quotient map \( \pi : \mathbb{R}^d \to \mathbb{R}^d / \mathbb{Z}^d \). Then the periodicity of the coefficients implies that \( \tilde{X}^{x,\epsilon} \) and \( \tilde{X}^x \) are well-defined stochastic processes on \( \mathbb{T}^d \) (cf. [6, Section 3.3.2]).

Now we list some general assumptions for the nonlocal PDE (1.1) and the SDE (2.1). All these assumptions are assumed to hold in the sequel unless otherwise specified.

**Assumption H1.** The functions \( b, c, e, g, u_0 \) are all periodic of period 1 in each component. For every \( y \in \mathbb{R}^d \), the function \( x \to \sigma(x, y) \) is periodic of period 1 in each component.

**Assumption H2.** The functions \( b, c, e \) are of class \( C^\beta_\theta \) with exponent \( \beta \) satisfying

\[ 1 - \frac{\alpha}{2} < \beta < 1. \]

The functions \( g \) and \( u_0 \) are both continuous.

**Assumption H3.** The function \( \sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \) satisfies the following conditions.

1. **Regularity.** For every \( x \in \mathbb{R}^d \), the function \( y \to \sigma(x, y) \) is of class \( C^2 \). There exists a constant \( C > 0 \), such that for any \( x_1, x_2, y \in \mathbb{R}^d \),

\[ |\sigma(x_1, y) - \sigma(x_2, y)| \leq C|x_1 - x_2||y|. \]

2. **Oddness.** For all \( x, y \in \mathbb{R}^d \), \( \sigma(x, -y) = -\sigma(x, y) \).

3. **Bounded inverse Jacobian.** The Jacobian matrix with respect to the second variable \( \nabla_y \sigma(x, y) \) is non-degenerate for all \( x, y \in \mathbb{R}^d \), and there exists a constant \( C > 0 \) such that \( |(\nabla_y \sigma(x, y))^{-1}| \leq C \) for all \( x, y \in \mathbb{R}^d \), where \( |\cdot| \) is the operator norm on \( \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d) \).
Remark 2.1. Some comments on our assumptions will be helpful:

(1). As mentioned in the end of the introduction, $b,c,e,g,u_0$ and the function $x \to \sigma(x,y)$, for every $y \in \mathbb{R}^d$, can be regarded as functions on $\mathbb{T}^d$, and we have $b,c,e \in C^3(\mathbb{T}^d)$, $g,u_0 \in C(\mathbb{T}^d)$, under Assumptions H1 and H2.

(2). Both the oddness and the growth condition in Assumption H3 imply that $\sigma(\cdot,0) \equiv 0$.

(3). The bounded inverse Jacobian condition implies that $|\nabla_y \sigma| \geq C^{-1}$. Since by Hadamard’s inequality (see, for instance, [33]),

$$|(\nabla_y \sigma)^{-1}| \leq C \Rightarrow |\det((\nabla_y \sigma)^{-1})| \leq C^n \Leftrightarrow |\det(\nabla_y \sigma) \geq C^{-n}| \Rightarrow |\nabla_y \sigma| \geq C^{-1}.$$ 

(4). The growth condition implies that for any $\gamma > \alpha$, we have

$$\sup_{x \in \mathbb{R}^d} \int_{B \setminus \{0\}} |\sigma(x,y)|^\gamma \nu^\alpha(dy) < \infty. \quad (2.4)$$

This ensures that we can apply Itô’s formula to $f(\tilde{X}_t^x)$ (or $f(\tilde{X}_t^{x,\epsilon})$, $f(X_t^{x,\epsilon})$), for any $f \in C^\infty(\mathbb{R}^d)$ with $\gamma > \alpha$ (cf. [29, Lemma 4.2]).

(5). By virtue of the oddness condition in Assumption H3 and the symmetry of the jump intensity measure $\nu^\alpha$, for any $x \in \mathbb{R}^d$,

$$\text{P.V.} \int_{\sigma(x, \cdot)^{-1} B \setminus B} \sigma^i(x,y) \nu^\alpha(dy) = \text{P.V.} \int_{B \setminus \sigma(x, \cdot)^{-1} B} \sigma^i(x,y) \nu^\alpha(dy) = 0. \quad (2.5)$$

Consequently we can rewrite the operator $A^{\gamma,\nu^\alpha}$ in (1.2) as

$$A^{\gamma,\nu^\alpha} f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left[ f(x+z) - f(x) - z^i \partial_i f(x) \right] 1_B(z) \nu^\alpha(x,dz), \quad (2.6)$$

where the kernel $\{\nu^{\sigma,\alpha}(x, \cdot) | x \in \mathbb{R}^d\}$ is given by

$$\nu^{\sigma,\alpha}(x,A) := \int_{\mathbb{R}^d \setminus \{0\}} 1_A(\sigma(x,y)) \nu^\alpha(dy), \quad A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}). \quad (2.7)$$

Moreover, for any $\gamma > \alpha$, the growth condition in Assumption H3 implies that

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} (|z|^\gamma \wedge 1) \nu^{\sigma,\alpha}(x,dz) = \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} (|\sigma(x,y)|^\gamma \wedge 1) \nu^\alpha(dy)$$

$$\leq \sup_{x \in \mathbb{R}^d} \left( \int_{|y| \leq \phi(x)} (\phi(x)|y|^\gamma \nu^\alpha(dy)) + \int_{|y| \geq \phi(x)^{-1}} \nu^\alpha(dy) \right)$$

$$\leq \frac{1}{\gamma - \alpha} \|\phi\|_{L^\infty}^{2\gamma - \alpha} + \frac{1}{\alpha} \|\phi\|_{L^\infty}^\gamma < \infty. \quad (2.8)$$
Remark 2.2. The special case \( \sigma(x, y) = \sigma_0(x)y \) with certain \( \sigma_0 \) is considered in [11], where the author assumed the function \( \sigma_0 : \mathbb{R}^d \to \text{GL}(\mathbb{R}^d) \) is periodic and of class \( C^2 \). In our context, Assumption H3 amounts to saying that

\[
\sigma_0 : \mathbb{R}^d \to \text{GL}(\mathbb{R}^d) \text{ is periodic and Lipschitz.} \tag{2.9}
\]

Since these imply the regularity condition immediately, the bounded inverse Jacobian and growth conditions are fulfilled by continuity and periodicity, together with the observation

\[
\sup_{x \in \mathbb{R}^d} \| \sigma_0(x) \| \vee \| \sigma_0(x)^{-1} \| < \infty. \quad \text{The oddness condition is trivial in this case.} \]

We will need some regularities for the 'partial' inverse of \( \sigma \). For a function \( F : \mathbb{R}^d \times \mathbb{R}^d \ni (x, y) \to F(x, y) \in \mathbb{R} \), we say \( F \in L^\infty_2(\mathbb{R}^d; C^1_1(\mathbb{R}^d; \mathbb{R}^d)) \), if there exists a constant \( C > 0 \) such that for all \( x, y \in \mathbb{R}^d \), \( |F(x, y)| \leq C \), and for all \( x_1, x_2, y_1, y_2 \in \mathbb{R}^d \), \( |F(x_1, y_1) - F(x_2, y_2)| \leq C|x_1 - x_2| \). Then the regularity and growth conditions in Assumption H3 imply that the function \( (x, y) \to \sigma(x, y)/|y| \) is of class \( L^\infty_2(\mathbb{R}^d; C^1_1(\mathbb{R}^d; \mathbb{R}^d)) \).

Lemma 2.3. Under Assumption H3, for every \( x \in \mathbb{R}^d \), the function \( y \to \sigma(x, y) \) is a \( C^2 \)-diffeomorphism. Denote the inverse by \( \tau(x, z) := \sigma(x, \cdot)^{-1}(z) \), then for every \( z \in \mathbb{R}^d \), the function \( x \to \tau(x, z) \) is periodic of period one. Moreover, the function \( (x, z) \to \tau(x, z)/|z| \) is of class \( L^\infty_2(\mathbb{R}^d; C^1_1(\mathbb{R}^d; \mathbb{R}^d)) \).

Proof. Fix \( x \in \mathbb{R}^d \). Since the function \( y \to \sigma(x, y) \) is of class \( C^2 \), by the bounded inverse Jacobian condition in Assumption H3, together with Hadamard's global inverse function theorem (see [20, Theorem 6.2.4]), \( \sigma(x, \cdot) \) is a \( C^2 \)-diffeomorphism. The periodicity is obvious.

Now using the bounded inverse Jacobian condition, the Jacobian matrix of \( \tau(x, z) \) with respect to \( z \) satisfies \( |\nabla_z \tau(x, z)| \leq C \), for all \( x, z \in \mathbb{R}^d \). Then by the growth condition and regularity condition, the second assertion follows from the following derivation,

\[
\sup_z \frac{|\tau(x, z)|}{|z|} \leq \phi(x),
\]

\[
\sup_z \frac{|\tau(x_1, z) - \tau(x_2, z)|}{|z|} = \sup_y \frac{|\tau(x_1, \sigma(x_1, y)) - \tau(x_2, \sigma(x_1, y))|}{|\sigma(x_1, y)|}
\]

\[
= \sup_y \frac{|\tau(x_2, \sigma(x_2, y)) - \tau(x_2, \sigma(x_1, y))|}{|\sigma(x_1, y)|} \leq \|\phi\|_{L^\infty} \|\nabla_z \tau\|_{L^\infty} \sup_y \frac{|\sigma(x_2, y) - \sigma(x_1, y)|}{|y|}
\]

\[
\leq C\|\phi\|_{L^\infty} \|\nabla_z \tau\|_{L^\infty} |x_1 - x_2|.
\]

Assumption H4. \( \det( \nabla_z \tau ) \in L^\infty_2(\mathbb{R}^d; C^1_1(\mathbb{R}^d; \mathbb{R})) \).

Remark 2.4. In the case \( \sigma(x, y) = \sigma_0(x)y \), the Jacobian of \( \tau(x, z) \) with respect to \( z \) is \( \nabla_z \tau(x, z) \equiv \sigma_0(x)^{-1} \). Then Assumption H4 reduce to that the function \( \det(\sigma_0)^{-1} : \mathbb{R}^d \to \mathbb{R} \) is Lipschitz, which is a direct consequence of (2.9).

Now by (2.7),

\[
\nu^{\sigma, \alpha}(x, dz) = \frac{d\tau(x, z)}{dz} \frac{dz}{|\tau(x, z)|^{d+\alpha}} = |\det \nabla_z \tau(x, z)| \frac{dz}{|\tau(x, z)|^{d+\alpha}}.
\]
If we let

\[ h(x, z) = \frac{|\det \nabla_z \tau(x, z)|}{|\tau(x, z)|^{d+\alpha}}, \tag{2.10} \]

then \( \nu^{\sigma, \alpha}(x, dz) = h(x, z) \frac{dz}{|z|^{d+\alpha}}. \) Using the growth condition, we also find that for all \( x, z \in \mathbb{R}^d, \)

\[ \|\phi\|_{L^\infty}^{-1} \leq \frac{|\tau(x, z)|}{|z|} \leq \|\phi\|_{L^\infty}. \tag{2.11} \]

Combining (2.10), (2.11), Lemma 2.3 and Assumption H4, together with [9, Proposition 1.2.6], we conclude that

**Proposition 2.5.** Under Assumptions H3 and H4, \( h \in L^\infty_2(\mathbb{R}^d; C^{1-}_1(\mathbb{R}^d; \mathbb{R}^d)). \) Moreover, there exists a positive constant \( C \) such that \( h(x, z) \geq C \) for all \( x, z \in \mathbb{R}^d. \)

In particular, the kernel \( \nu^{\sigma, \alpha} \) is comparable to the jump intensity measure of an isotropic \( \alpha \)-stable process.

**Remark 2.6.** Thanks to Proposition 2.5, the general assumptions in [4, 18] are satisfied. Thus, the regularity results and heat kernel estimates therein are available in our context. Actually, these two papers only need that \( h \in L^\infty_2(\mathbb{R}^d; C^1(\mathbb{R}^d; \mathbb{R}^d)) \) for some \( 0 < \gamma < 1, \) this is the case by virtue of the natural embedding \( C^{1-} \subset C^\gamma. \) Note that [4] also needs \( \alpha + \beta \) not to be an integer, this can be fulfilled by choosing an appropriate \( \beta. \)

### 3 Nonlocal Poisson equation with zeroth-order term

As mentioned in the introduction, we will apply Zvonkin’s transform to study the homogenization of SDEs and nonlocal PDEs. Before that, we shall investigate the strong well-posedness of the SDEs presented in the previous section, and Zvonkin’s transform will also play an important role in this step (see next section). The key is to consider the following nonlocal Poisson equation with zeroth-order term,

\[ \kappa u - L^\alpha u = f, \tag{3.1} \]

where \( \kappa > 0, \) and \( L^\alpha \) is the linear integro-partial differential operator given by

\[ L^\alpha := A^{\sigma, \nu} + b \cdot \nabla, \tag{3.2} \]

which may be regarded as the infinitesimal generator of the solution process \( \tilde{X} \) of (2.3) once we prove its well-posedness in the next section.

#### 3.1 Well-posedness of nonlocal Poisson equation

We first revisit the maximum principle and the solvability of Poisson equations with zeroth-order term studied in [29]. In this subsection we always assume that Assumptions H2, H3 and H4 are in force.
Proposition 3.1. If $u \in C_b^{1+\gamma}(\mathbb{R}^d)$, $1+\gamma > \alpha$, is a solution to $\kappa u - \mathcal{L}^\alpha u = f$ with $\kappa > 0$ and $f \in C_b(\mathbb{R}^d)$, then

$$\kappa\|u\|_0 \leq \|f\|_0.$$ 

Proof. Note that the nonlocal operator $\mathcal{A}^{\sigma,\nu}$ can be rewritten in the form (2.6). For $u \in C_b^{1+\gamma}(\mathbb{R}^d)$, we have

$$|u(x+z) - u(x) - z \cdot \nabla u(x)| \leq |z| \int_0^1 |\nabla u(x+rz) - \nabla u(x)| dr \leq \frac{|\nabla u|}{1+\gamma} |z|^{1+\gamma}. \quad (3.3)$$

Then by (2.8), there exists a constant $C > 0$ such that

$$|\mathcal{L}^\alpha u(x)| \leq \int_{B\setminus\{0\}} |u(x+rz) - u(x) - z \cdot \nabla u(x)| \nu^{\sigma,\alpha}(x,dz)$$

$$+ \int_{B^c} |u(x+rz) - u(x)| \nu^{\sigma,\alpha}(x,dz) + |b(x) \cdot \nabla u(x)|$$

$$\leq 2\|u\|_{1+\gamma} \left( \int_{\mathbb{R}^d \setminus\{0\}} (|z|^{1+\gamma} \wedge 1) \nu^{\sigma,\alpha}(x,dz) + \|b\|_0 \right)$$

$$\leq C\|u\|_{1+\gamma}.$$ 

Based on this estimate, the rest of the proof is exactly the same as that of [29, Proposition 3.2], even though it is set up with $\sigma(\cdot, y) \equiv y$ there.

Now we investigate the solvability of the Poisson equation with a zeroth-order term involved. The results generalize the Schauder estimates in [29] to the anisotropic nonlocal case.

Theorem 3.2. For any $\kappa > 0$ and $f \in C_0^\beta(\mathbb{R}^d)$, the nonlocal Poisson equation (3.1) has a unique solution $u = u_\kappa \in C_b^{\alpha+\beta}(\mathbb{R}^d)$. In addition, there exists a positive constant $C = C(\kappa, \|\beta\|_\beta)$ such that

$$\|u_\kappa\|_{\alpha+\beta} \leq C(\|u_\kappa\|_0 + \|f\|_\beta). \quad (3.4)$$

Proof. The a priori estimate (3.4) is from [4, Theorem 7.1, Theorem 7.2]. We thus need to show that the equation (3.1) has a unique solution $u_\kappa \in C_b^{\alpha+\beta}(\mathbb{R}^d)$.

Now we prove the existence and uniqueness of solution in $C_b^{\alpha+\beta}(\mathbb{R}^d)$. It is shown in [29, Theorem 3.4] that when $\sigma(\cdot, y) \equiv \text{Id}_y(y) := y$, the existence and uniqueness hold in $C_b^{\alpha+\beta}(\mathbb{R}^d)$. For the general $\sigma$, we apply the method of continuity (see [12, Section 5.2]).

Define a family of linear operators by $\mathcal{L}_\theta := \theta \mathcal{A}^{\sigma,\nu} + (1-\theta) \mathcal{A}^{\text{Id}_y,\nu} + b \cdot \nabla$. We consider the family of equations:

$$\kappa u - \mathcal{L}_\theta u = f. \quad (3.5)$$

We can also rewrite the nonlocal term in $\mathcal{L}_\theta$ into the form (2.6), with the kernel given by

$$\nu_\theta := \theta \nu^{\sigma,\alpha} + (1-\theta) \nu^\alpha.$$ 

Then the a priori estimate (3.4) also holds for $u_\theta$ (cf. Remark 2.6).

As a result, the operator $\mathcal{L}_\theta$ can be considered as a bounded linear operator from the Banach space $C_b^{\alpha+\beta}(\mathbb{R}^d)$ into the Banach space $C_b^\beta(\mathbb{R}^d)$.

Note that $\mathcal{L}_0 = \mathcal{A}^{\text{Id}_y,\nu} + b \cdot \nabla$, which is the case considered in [29], and $\mathcal{L}_1 = \mathcal{L}^\alpha$. The solvability of the equation (3.1) for any $f \in C_b^\beta(\mathbb{R}^d)$ is then equivalent to the invertibility of
the operator $L_\theta$. We can see from the proof of Proposition 3.1 that $\|u_\theta\|_0 \leq C\|f\|_0$. Then together with the estimate (3.4) for $u_\theta$, we have the bound

$$\|u\|_{\alpha+\beta} \leq C\|f\|_{\beta},$$

with the constant $C$ being independent of $\theta$. Since, as discussed in [29], the operator $L_\theta = A^{1d,\nu^\alpha} + b \cdot \nabla$ maps $C^{\alpha+\beta}_b(\mathbb{R}^d)$ onto $C^{\beta}_b(\mathbb{R}^d)$, the method of continuity is applicable and the result follows.

Remark 3.3. If we take the periodicity assumption H1 into account, then we can slightly strengthen the conclusions in Theorem 3.2. That is, if $f \in C^\beta(\mathbb{T}^d)$, then the unique solution of (3.1) is of class $C^{\alpha+\beta}(\mathbb{T}^d)$.

### 3.2 Feller property

In this subsection, we will study further the operator $L^\alpha$. It turns out that it is the generator of a Feller semigroup. As a corollary, the solution of equation (3.1) can be represented in terms of a semigroup, and satisfies a finer estimate. All these results will be used in the next section.

**Theorem 3.4.** The linear operator $(L^\alpha, D(L^\alpha))$, $D(L^\alpha) = C^{\alpha+\beta}(\mathbb{T}^d)$, defined on the Banach space $(C(\mathbb{T}^d), \| \cdot \|_0)$, is closable and dissipative, its closure generates a Feller semigroup $\{P_t\}_{t \geq 0}$ on $C(\mathbb{T}^d)$.

**Proof.** Similar to (3.3), one can find that for $u \in C^{\alpha+\beta}(\mathbb{T}^d)$,

$$|u(x + \sigma(x, y)) - u(x) - \sigma(x, y) \cdot \nabla u(x)| \leq \frac{[\nabla u]_{\alpha+\beta-1}}{\alpha + \beta} |\sigma(x, y)|_{\alpha+\beta}.$$

Combining this with (2.4), a straightforward application of the dominated convergence theorem yields that $\lim_{y \to x} L^\alpha u(y) = L^\alpha u(x)$ for any $u \in C^{\alpha+\beta}(\mathbb{T}^d)$ and $x \in \mathbb{T}^d$. This amounts to saying that $L^\alpha(C^{\alpha+\beta}(\mathbb{T}^d)) \subset C(\mathbb{T}^d)$. Therefore, the operator

$$L^\alpha : C(\mathbb{T}^d) \supset C^{\alpha+\beta}(\mathbb{T}^d) \to C(\mathbb{T}^d)$$

is a densely defined unbounded operator on $C(\mathbb{T}^d)$.

Now Proposition 3.1 implies that for any $\kappa > 0$ and $u \in C^{\alpha+\beta}(\mathbb{T}^d)$, $\| (\kappa - L^\alpha) u \|_0 \geq \kappa \| u \|_0$, that is, $L^\alpha$ is dissipative. By Theorem 3.2, we have $C(\mathbb{T}^d) \subset (\kappa - L^\alpha)(C^{\alpha+\beta}(\mathbb{T}^d))$ for any $\kappa > 0$, which yields that the operator $\kappa - L^\alpha$ has dense range in $C(\mathbb{T}^d)$. In addition, $L^\alpha$ satisfies the positive maximum principle, due to the equivalent form (2.6) of $A^{\sigma,\nu^\alpha}$ and Courrèges’s theorem (see [16, Corollary 4.5.14]). Now the final assertion follows form the celebrate Hille-Yosida-Ray Theorem (see, for instance, [8, Theorem 4.2.2]).

Let us recall the notion of martingale problem (see [8, Section 4.3]). First recall that $\mathcal{D} = \mathcal{D}(\mathbb{R}_+; \mathbb{T}^d)$ is the space of all $\mathbb{T}^d$-valued càdlàg functions on $\mathbb{R}_+$, equipped with the Skorokhod topology. Let $w_t(\omega) = \omega(t), \omega \in \mathcal{D}$, be the coordinate process on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$, and $\{\mathcal{F}_t^w\}_{t \geq 0} := \sigma(w_s : 0 \leq s \leq t)$ be the canonical filtration. Given a probability measure $\nu$
on $\mathbb{T}^d$, we say that a probability measure $P^\nu$ on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$ is a solution of the martingale problem for $(\mathcal{L}^\alpha, \nu)$, if $P^\nu \circ \omega_0^{-1} = \nu$ and the process
\[
M^f(t) := f(w_t) - f(w_0) - \int_0^t \mathcal{L}^\alpha f(w_s) \, ds
\]
is a $(\mathcal{D}, \mathcal{B}(\mathcal{D}), \{\mathcal{F}_t^\nu\}_{t \geq 0}, P^\nu)$-martingale, for any $f \in D(\mathcal{L}^\alpha) = C^{\alpha+\beta}(\mathbb{T}^d)$. We denote by $\delta_x$ the Dirac measure, or equivalently, the Dirac function as distribution, focusing on $x \in \mathbb{R}^d$.

**Theorem 3.5.** For every $x \in \mathbb{T}^d$, the martingale problem for $(\mathcal{L}^\alpha, \delta_x)$ has a unique solution $P^x$. Moreover, the coordinate process $\{w_t\}_{t \geq 0}$ is a Feller process with generator being the closure of $(\mathcal{L}^\alpha, C^{\alpha+\beta}(\mathbb{T}^d))$, and has a jointly continuous transition probability density $p(t; x, y)$, i.e., $P^x(w_t \in A) = \int_A p(t; x, y) \, dy$, $A \in \mathcal{B}(\mathbb{T}^d)$, which satisfies the following estimates
\[
C_1^{-1} \sum_{j \in \mathbb{Z}^d} \left( \frac{t}{|x - y + j|^{d+\alpha}} \wedge t^{-\frac{d}{\alpha}} \right) \leq p(t; x, y) \leq C_1 \sum_{j \in \mathbb{Z}^d} \left( \frac{t}{|x - y + j|^{d+\alpha}} \wedge t^{-\frac{d}{\alpha}} \right),
\]
where $C_1 > 1, C_2 > 0$ are two constants.

**Proof.** The existence of solution of the martingale problem is in [23, Proposition 3]. Taking Theorem 3.4 into account, the uniqueness and the Feller property follow from [8, Theorem 4.4.1]. The existence of transition density and the two estimates can be found in [18, Theorem 1.4].

**Remark 3.6.** (1). Combining Theorem 3.4 and Theorem 3.5, we see that the Feller semigroup $\{P_t\}_{t \geq 0}$ generated by the closure of $\mathcal{L}^\alpha$ has the representation
\[
P_t f(x) = \int_{\mathbb{T}^d} f(y) p(t; x, y) \, dy, \quad f \in C(\mathbb{T}^d),
\]
and the following gradient estimate holds
\[
|\nabla P_t f(x)| \leq C_2 \|f\|_0 t^{-\frac{1}{2}} \int_{\mathbb{T}^d} \sum_{j \in \mathbb{Z}^d} \left( \frac{t}{|y - j|^{d+\alpha}} \wedge t^{-\frac{d}{\alpha}} \right) \, dy
\]
\[
= C_2 \|f\|_0 t^{-\frac{1}{2}} \int_{\mathbb{R}^d} \left( \frac{t}{|y|^{d+\alpha}} \wedge t^{-\frac{d}{\alpha}} \right) \, dy
\]
\[
\leq C_2 \left( 1 + \frac{1}{\alpha} \right) \|f\|_0 t^{-\frac{1}{\alpha}}.
\]

(2). Denote the formal generator of $\tilde{X}^{x, \epsilon}$ by $\tilde{\mathcal{L}}^\alpha_{\epsilon}$, i.e.,
\[
\tilde{\mathcal{L}}^\alpha_{\epsilon} f(x) := \int_{\mathbb{R}^d \setminus \{0\}} \left[ f \left( x + \frac{1}{\epsilon} \sigma(x, \epsilon y) \right) - f(x) - \frac{1}{\epsilon} \sigma^i(x, \epsilon y) \partial_i f(x) \mathbf{1}_B(y) \right] \nu^\alpha(dy)
\]
\[
+ \left[ \mathcal{Y}(x) + \epsilon^{\alpha-1} c^i(x) \right] \partial_i f(x), \quad x \in \mathbb{R}^d.
\]
Then Theorem 3.4 and 3.5 still hold true with $\tilde{\mathcal{L}}^\alpha_{\epsilon}$ in place of $\mathcal{L}^\alpha$. \qed
Then (3.11) follows by choosing $\kappa$ a version of $L$ posedness by Zvonkin’s transform. Thus we consider the following operator, which is a “flat” version of $L^\alpha$:

$$L^{\alpha,\beta} f(x) = \int_{B \setminus \{0\}} \left[ f(x + \sigma(x,y)) - f(x) - \sigma^i(x,y) \partial_i f(x) \right] \nu^\alpha(dy) - b^i(x) \partial_i f(x).$$

(3.9)

We have the following regularity result for $L^{\alpha,\beta}$:

**Corollary 3.8.** There exists a constant $\kappa_* > 0$ such that for any $\kappa > \kappa_*$ and $f \in C^\beta(\mathbb{T}^d)$, there exists a unique solution $u = u^\beta_\kappa \in C^{\alpha+\beta}(\mathbb{T}^d)$ to the equation

$$\kappa u - L^{\alpha,\beta} u = f.$$  

(3.10)

In addition, there exists a constant $C > 0$ independent of $u, f, b, \kappa$, such that for any $\kappa > \kappa_*$,

$$(\kappa - \kappa_*) \|u^\beta_\kappa\|_0 + \kappa^{\alpha+\beta-1} \|\nabla u^\beta_\kappa\|_0 + [\nabla u^\beta_\kappa]_{\alpha+\beta-1} \leq C \|f\|_\beta.$$  

(3.11)

**Proof.** To obtain the a priori estimate (3.11), we rewrite the equation (3.10) in the form

$$\kappa u - L^\alpha u = f - \int_{B^c} [u(x + \sigma(x,y)) - u(x)] \nu^\alpha(dy).$$

The estimate (3.8) implies that

$$\kappa \|u^\beta_\kappa\|_0 + \kappa^{\alpha+\beta-1} \|\nabla u^\beta_\kappa\|_0 + [\nabla u^\beta_\kappa]_{\alpha+\beta-1} \leq C(\|f\|_\beta + 2\nu^\alpha(B^c)) \|u\|_0).$$

Then (3.11) follows by choosing $\kappa_* = 2C\nu^\alpha(B^c)$.

Now define a family of operators by

$$L^\alpha_0 = L^{\alpha,\beta} + \theta \int_{B^c} [u(x + \sigma(x,y)) - u(x)] \nu^\alpha(dy).$$

Then $L^\alpha_1 = L^\alpha, L^\alpha_0 = L^{\alpha,\beta}$. The well-posedness of equation (3.10) follows from the method of continuity and the a priori estimate (3.11), just as in the proof of Theorem 3.2. \qed
4 SDEs with multiplicative stable Lévy noise

The goal of this section is to study the strong well-posedness of SDEs (2.2) and (2.3), as well as the ergodic properties of the solution processes $\tilde{X}^{x,\epsilon}$ for each $\epsilon > 0$. As corollaries, we also obtain the Feynman-Kac formula and the well-posedness of nonlocal Poisson equation without zeroth-order term, which will be used to study homogenization in the next two sections.

4.1 Strong well-posedness of SDEs

We only consider the strong well-posedness for SDE (2.3) since (2.2) has the same form. As we have seen in Theorem 3.5, the existence and uniqueness hold for the martingale problem for $(\mathcal{L}^\alpha, \delta_x)$. Meanwhile, it is known that the martingale solution for $(\mathcal{L}^\alpha, \delta_x)$ is equivalent to the weak solution of SDE (2.3), see [21, Theorem 2.3, Corollary 2.5]. Thus, the existence and uniqueness of weak solution hold for SDE (2.3).

Moreover, utilizing the fact shown in [3, Theorem 1.2] that the weak existence and pathwise uniqueness for SDE (2.3) imply strong existence, we only need to prove the pathwise uniqueness. The key is to reduce the SDE (2.3), whose coefficients have low regularity, to an SDE with Lipschitz coefficients by using Zvonkin’s transform.

For $\kappa > \kappa_*$, let $\hat{b}_\kappa \in C^{\alpha+\beta}(\mathbb{T}^d)$ be the solution of

$$\kappa \hat{b}_\kappa - \mathcal{L}^{\alpha,\beta} \hat{b}_\kappa = b,$$

where $\mathcal{L}^{\alpha,\beta}$ is the operator in (3.9). The existence and uniqueness of solution $\hat{b}_\kappa$ is ensured by Corollary 3.8. Define a map $\Phi_\kappa : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\Phi_\kappa(x) = x + \hat{b}_\kappa(x).$$

Then $\Phi_\kappa$ is of class $C^{\alpha+\beta}$. Moreover, we have

**Lemma 4.1.** For $\kappa > 0$ large enough, the map $\Phi_\kappa : \mathbb{R}^d \to \mathbb{R}^d$ is a $C^1$-diffeomorphism and its inverse $\Phi_\kappa^{-1}$ is also of class $C^{\alpha+\beta}$.

**Proof.** By the estimate in Corollary 3.8, we have

$$\kappa^{\alpha+\beta-1} \frac{\alpha}{\alpha+\beta-1} \| \nabla \hat{b}_\kappa \|_0 \leq C \| b \|_\beta, \quad \kappa > \kappa_*.$$  

Now by choosing $\kappa > \kappa_* \vee (2C \| b \|_\beta)^{\alpha / (\alpha+\beta-1)}$, we get that $\| \nabla \hat{b}_\kappa \|_0 \leq \frac{1}{2}$. Thus

$$\frac{1}{2} |x_1 - x_2| \leq |\Phi_\kappa(x_1) - \Phi_\kappa(x_2)| \leq \frac{3}{2} |x_1 - x_2|,$$

i.e., $\Phi_\kappa$ is bi-Lipschitz. In particular, $\Phi_\kappa$ is a $C^1$-diffeomorphism. Moreover,

$$\nabla (\Phi_\kappa^{-1}) = \text{Inv} \circ \nabla \Phi_\kappa \circ \Phi_\kappa^{-1},$$

where the matrix inverse map $\text{Inv} : \text{GL}(\mathbb{R}^d) \to \text{GL}(\mathbb{R}^d)$ is of class $C^\infty$. We arrive at the second conclusion of the lemma by applying [9, Proposition 1.2.7]. \qed
To solve SDE (2.3), by a standard interlacing technique (cf. [1, Section 6.5] or [15, Theorem IV. 9.1]), it suffices to solve the following SDE with no jumps greater than one:

\[ \tilde{X}_t^{x,b} = x + \int_0^t b(\tilde{X}_s^{x,b})ds + \int_0^t \int_B \sigma(\tilde{X}_s^{x,b}, y)\tilde{N}_\alpha(dy, ds). \]

Now fix \( \kappa > 0 \) large enough such that the conclusions in Lemma 4.1 hold. We introduce Zvonkin’s transform

\[ \tilde{X}_t^* = \Phi_\kappa(\tilde{X}_t^{x,b}). \]

Then by applying Itô’s formula, we have

\[ \tilde{X}_t^* = \Phi_\kappa(x) + \int_0^t b^*(\tilde{X}_{s-}^*)ds + \int_0^t \int_B \sigma^*(\tilde{X}_{s-}^*, y)\tilde{N}_\alpha(dy, ds), \quad (4.1) \]

where

\[ b^*(x) = \kappa\hat{b}_\kappa(\Phi_\kappa^{-1}(x)), \]

\[ \sigma^*(x, y) = \hat{b}_\kappa(\Phi_\kappa^{-1}(x) + \sigma(\Phi_\kappa^{-1}(x), y)) - \hat{b}_\kappa(\Phi_\kappa^{-1}(x)) + \sigma(\Phi_\kappa^{-1}(x), y). \]

**Theorem 4.2.** For each \( x \in \mathbb{R}^d \), there is a unique strong solution \( \tilde{X}^x = \{\tilde{X}_t^x\}_{t \geq 0} \) to SDE (2.3).

**Proof.** By the above argument, we only need to prove the pathwise uniqueness for SDE (4.1). First of all, we have, for any \( x, x_1, x_2, y \in \mathbb{R}^d \),

\[ |b^*(x_1) - b^*(x_2)| \leq C(||\hat{b}_\kappa||_1, ||\Phi_\kappa^{-1}||_1)|x_1 - x_2|, \quad (4.2) \]

Note that for \( \gamma \in (0, 1) \), \( f \in \mathcal{C}^{1+\gamma}_b(\mathbb{R}^d) \), \( x, u, v \in \mathbb{R}^d \), there exists a constant \( C > 0 \) such that

\[ |f(u + x) - f(u) - f(v + x) - f(v)| \leq C||f||_{1+\gamma}|u - v||x|^\gamma, \]

the proof can be found in [4, Theorem 5.1.(c)]. Then for any \( x_1, x_2 \),

\[ |\sigma^*(x_1, y) - \sigma^*(x_2, y)| \]

\[ \leq |\hat{b}_\kappa(\Phi_\kappa^{-1}(x_1) + \sigma(\Phi_\kappa^{-1}(x_1), y)) - \hat{b}_\kappa(\Phi_\kappa^{-1}(x_2) + \sigma(\Phi_\kappa^{-1}(x_2), y))| \\
- |\hat{b}_\kappa(\Phi_\kappa^{-1}(x_2) + \sigma(\Phi_\kappa^{-1}(x_2), y)) - \hat{b}_\kappa(\Phi_\kappa^{-1}(x_1)) + \sigma(\Phi_\kappa^{-1}(x_1), y))| \\
+ |\hat{b}_\kappa(\Phi_\kappa^{-1}(x_2) + \sigma(\Phi_\kappa^{-1}(x_1), y)) - \hat{b}_\kappa(\Phi_\kappa^{-1}(x_2)) + \sigma(\Phi_\kappa^{-1}(x_2), y))| \\
+ |\sigma(\Phi_\kappa^{-1}(x_1), y) - \sigma(\Phi_\kappa^{-1}(x_2), y)| \quad (4.3) \]

\[ \leq C||\hat{b}_\kappa||_{1+\beta}||\Phi_\kappa^{-1}(x_1) - \Phi_\kappa^{-1}(x_2)||^\alpha \sigma(\Phi_\kappa^{-1}(x_1), y)|^\kappa \]

\[ + (||\nabla \hat{b}_\kappa||_0 + 1)\sigma(\Phi_\kappa^{-1}(x_1), y) - \sigma(\Phi_\kappa^{-1}(x_2), y)| \]

\[ \leq C \left(||\hat{b}_\kappa||_{1+\beta}, ||\Phi_\kappa^{-1}||_1, ||\phi||_{L^\infty}\right)|x_1 - x_2|(|y|^\alpha + |y|). \]

where we have used the regularity condition for \( \sigma \) in Assumption H3, and \( \phi \) is the positive bounded function in the growth condition in that assumption. Noting that \( 2(\alpha + \beta - 1) > \alpha \) by Assumption H2, we arrive at

\[ \int_B |\sigma^*(x_1, y) - \sigma^*(x_2, y)|^2 \nu_\alpha(dy) \leq C \left(||\hat{b}_\kappa||_{1+\beta}, ||\Phi_\kappa^{-1}||_1, ||\phi||_{L^\infty}\right)|x_1 - x_2|^2. \quad (4.4) \]

The pathwise uniqueness of SDE (4.1) follows from (4.2), (4.4) and the classical result [15, Theorem 4.9.1]. The proof is complete. □

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Corollary 4.3. The solution process $\tilde{X}^x$ is a Feller process with generator being the closure of $(\mathcal{L}^\alpha, \mathcal{C}^{\alpha+\beta}(\mathbb{T}^d))$. In particular, $\tilde{X}^x$ is a strong Markov process.

Proof. By applying Itô’s formula, it is easy to see that for any $f \in D(\mathcal{L}^\alpha) = \mathcal{C}^{\alpha+\beta}(\mathbb{T}^d)$, the following process is a $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$-martingale

$$\tilde{M}^f(t) := f(\tilde{X}^x_t) - f(\tilde{X}^x_0) - \int_0^t \mathcal{L}^\alpha f(\tilde{X}^x_s)ds.$$  

It is easy to see that $\tilde{X}^x$ has càdlàg paths almost surely. Let $\mathbb{P}_{\tilde{X}^x} := \mathbb{P} \circ \tilde{X}^x$ be the pushforward probability measure of $\tilde{X}^x$ on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$, then $\mathbb{P}_{\tilde{X}^x}$ is a solution of martingale problem for $(\mathcal{L}^\alpha, \delta_x)$. By Theorem 3.5, we find that $\mathbb{P}_{\tilde{X}^x} = \mathbb{P}_x$, the Feller property follows. The strong Markov property follows from [30, Theorem III.3.1].

Remark 4.4. The Feller semigroup $\{P_t\}_{t \geq 0}$ in Theorem 3.4 is the semigroup associated with the solution process $\tilde{X}^x$, that is,

$$P_t f(x) = E(f(\tilde{X}^x_t)), \quad f \in \mathcal{C}(\mathbb{T}^d).$$

As a consequence of the Feller property, we can obtain the well-posedness of the parabolic nonlocal PDE and the corresponding Feynman-Kac representation. See [26] for the classical version for second order PDE.

Theorem 4.5. The parabolic nonlocal PDE

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = \mathcal{L}^\alpha u(t, x) + g(x)u(t, x), & t > 0, x \in \mathbb{R}^d, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

admits a unique mild solution in the sense that $\int_0^t u(s)ds \in D(\mathcal{L}^\alpha)$ for all $t \geq 0$ and

$$u(t) = u_0 + \mathcal{L}^\alpha \int_0^t u(s)ds + g \int_0^t u(s)ds.$$

Moreover, the unique solution has the following Feynman-Kac representation

$$u(t, x) = E\left[u_0(\tilde{X}^x_t) \exp\left(\int_0^t g(\tilde{X}^x_s)ds\right)\right].$$

Proof. Choose $G > 0$ large enough such that $\|g\|_0 < G$. Define

$$P^g_t f(x) = E\left[f(\tilde{X}^x_t) \exp\left(\int_0^t g(\tilde{X}^x_s)ds - Gt\right)\right], \quad f \in \mathcal{C}(\mathbb{T}^d).$$

Then by an argument similar to that used in [1, Section 6.7.2], one can show that $\{P^g_t\}_{t \geq 0}$ is a Feller semigroup with generator being the closure of $(\mathcal{L}^\alpha + g - G, \mathcal{C}^{\alpha+\beta}(\mathbb{T}^d))$. This yields that $\{e^{Gt}P^g_t\}_{t \geq 0}$ is a $C_0$-semigroup on $\mathcal{C}(\mathbb{T}^d)$ with generator being the closure of $(\mathcal{L}^\alpha + g, \mathcal{C}^{\alpha+\beta}(\mathbb{T}^d))$. Note that $u_0 \in \mathcal{C}(\mathbb{T}^d)$. Now applying the classic result [7, Proposition II.6.4] in the theory of $C_0$-semigroups, we conclude that the parabolic nonlocal PDE admits a unique mild solution, which can be given by the orbit map $u(t) = e^{Gt}P^g_t u_0$. The desired conclusions follow immediately.
4.2 Ergodicity

Now we deal with the ergodicity of SDEs. To this end, we need the following scaling assumption for the coefficient $\sigma$.

**Assumption H5.** $\frac{1}{\epsilon}\sigma(x,\epsilon y) \to \sigma(x, y)$ as $\epsilon \to 0$, uniformly in $x, y \in \mathbb{R}^d$.

**Remark 4.6.** In the case $\sigma(x, y) = \sigma_0(x)y$, this assumption holds automatically. Moreover, this assumption implies that $\frac{1}{\epsilon}\sigma(x, y) - \sigma(x, \epsilon y) \to 0$ as $\epsilon \to 0$, uniformly in $x, y$.

By the above discussion, for every $\epsilon > 0$, SDE (2.2) also admits a unique strong solution $\tilde{X}^{x,\epsilon}_{\cdot \epsilon}$ which is a $\mathbb{T}^d$-valued Feller process. Denote by $\{P_t^\epsilon\}_{t \geq 0}$ the transition probability density of $\tilde{X}^{x,\epsilon}_{\cdot \epsilon}$, by $\{P_t\}_{t \geq 0}$ the associated Feller semigroup.

**Proposition 4.7.** For each $\epsilon \geq 0$, the process $\tilde{X}^{x,\epsilon}_{\cdot \epsilon}$ possesses a unique invariant distribution $\mu_\epsilon$ on $\mathbb{T}^d$. Moreover, there exist positive constants $C$ and $\rho$ such that for any periodic bounded Borel function $f$ on $\mathbb{R}^n$ (i.e., $f$ is Borel bounded on $\mathbb{T}^d$),

$$\sup_{x \in \mathbb{T}^d} \left| P_t^\epsilon f(x) - \int_{\mathbb{T}^d} f(y)\mu_\epsilon(dy) \right| \leq C\|f\|_0 e^{-\rho t}$$

for every $t \geq 0$.

**Proof.** Thanks to the Doeblin-type result presented in [6, Theorem 3.3.1], it is enough to ensure that the transition probability density $P_t^\epsilon(t; x, y)$ is bounded from below by a positive constant (cf. [11, Proposition 1]), which follows immediately from the density estimates in Theorem 3.5. We also need an elementary lemma. The proof is very similar to (3.3) and shall be omitted.

**Lemma 4.8.** Let $0 < \gamma \leq 1$ and $f \in C^{1+\gamma}_b(\mathbb{R}^d)$. For any $x, u, v \in \mathbb{R}^d$, it holds that

$$|f(x + u) - f(x + v) - (u - v) \cdot \nabla f(x)| \leq \frac{\|\nabla f\|_\gamma}{1 + \gamma} |u - v|^{1+\gamma}.$$

Denote by $\mu = \mu_0$ the invariant probability measure for the limit process $\tilde{X}_t^\gamma$ in (2.3). Then we can prove the following lemma, as in [13, Lemma 2.4].

**Lemma 4.9.** As $\epsilon \to 0$, we have $\mu_\epsilon \to \mu$.

**Proof.** It suffices to prove that $P_t^\epsilon f \to P_t f$ in $C(\mathbb{T}^d)$ as $\epsilon \to 0$ for any $f \in C(\mathbb{T}^d)$ and $t \geq 0$. By Theorem 3.5 and Remark 3.6.(2), we know that $C^{\alpha+\beta}(\mathbb{T}^d)$ is a core for $L^\alpha$ and each $L^\alpha_\epsilon$,
\( \epsilon > 0 \). Fix an arbitrary \( f \in C^{\alpha+\beta}(\mathbb{T}^d) \),
\[
|\hat{L}_\epsilon^\alpha f(x) - \mathcal{L}^\alpha(x)| \leq \epsilon^{-1} |c(x) \cdot \nabla f(x)| + \int_{B^c} |f \left( x + \frac{1}{\epsilon} \sigma(x, \epsilon y) \right) - f(x + \sigma(x, y))| \, \nu^\alpha(dy) \\
+ \int_{B \setminus \{0\}} \left| f \left( x + \frac{1}{\epsilon} \sigma(x, \epsilon y) \right) - f(x + \sigma(x, y)) \right| \, \nu^\alpha(dy) \\
- \left( \frac{1}{\epsilon} \sigma(x, \epsilon y) - \sigma(x, y) \right) \cdot \nabla f(x) \, \nu^\alpha(dy) \\
\leq \epsilon^{-1} \|c\| \|f\|_1 + \int_{B^c} |f \left( x + \frac{1}{\epsilon} \sigma(x, \epsilon y) \right) - f(x + \sigma(x, y))| \, \nu^\alpha(dy) \\
+ \frac{\|f\|_{\alpha+\beta}}{\alpha + \beta} \int_{B \setminus \{0\}} \left| \frac{1}{\epsilon} \sigma(x, \epsilon y) - \sigma(x, y) \right|^{\alpha+\beta} \, \nu^\alpha(dy),
\]
it converges to zero as \( \epsilon \to 0 \), uniformly in \( x \), by the dominated convergence and Assumption H5. Using the Trotter-Kato approximation theorem (see [7, Theorem III.4.8]), \( P^\epsilon_t f \to P_t f \) in \( C(\mathbb{T}^d) \) as \( \epsilon \to 0 \) for all \( f \in C(\mathbb{T}^d) \), uniformly for \( t \) in compact intervals.

Now we combine Proposition 4.7 and Lemma 4.9 to get the following ergodic theorem.

**Theorem 4.10.** Let \( f \) be a bounded Borel function on \( \mathbb{T}^d \). Then for any \( t > 0 \),
\[
\int_0^t \left| f \left( \frac{X^{x,\epsilon}_s}{\epsilon} \right) - \int_{\mathbb{T}^d} f(x) \mu(dx) \right| \, ds \to 0
\]
in probability, as \( \epsilon \to 0 \).

**Proof.** For \( \epsilon > 0, 0 \leq s < t \), let \( \bar{f} \) be a bounded measurable function on \( \mathbb{T}^d \) satisfying \( \int_{\mathbb{T}^d} \bar{f}(x) \mu(dx) = 0 \). Then by Proposition 4.7,
\[
\mathbb{E} \left[ \left\| \bar{f}(\frac{X^{x,\epsilon}_t}{\epsilon}) \right\| \frac{X^{x,\epsilon}_t}{\epsilon} \right] = \int_{\mathbb{T}^d} |\bar{f}(y)| \left| p^\epsilon(t-s, \frac{X^{x,\epsilon}_s}{\epsilon}, y)dy - \mu(dx) \right| \leq C \|\bar{f}\|_0 e^{-\rho(t-s)}.
\]
The rest of the proof is similar to that of [24, Proposition 2.4] by applying Lemma 4.9.

For every \( \gamma > 0 \), denote by \( C^{\gamma}_\mu(\mathbb{T}^d) \) the class of all \( f \in C^\gamma(\mathbb{T}^d) \) which are centered with respect to the invariant measure \( \mu \) in the sense that \( \int_{\mathbb{T}^d} f(x) \mu(dx) = 0 \). It is easy to check that \( C^{\gamma}_\mu(\mathbb{T}^d) \) is closed, and hence a sub-Banach space of \( C^\gamma(\mathbb{T}^d) \) under the norm \( \| \cdot \|_\gamma \).

Thanks to Theorem 3.2 and Proposition 4.7, we can use the Fredholm alternative to obtain the solvability of the following Poisson equation without zeroth-order term in the smaller space \( C^{\alpha+\beta}_\mu(\mathbb{T}^d) \),
\[
\mathcal{L}^\alpha u + f = 0,
\]
for \( f \in C^{\beta}_\mu(\mathbb{T}^d) \). Before that, we need some lemmas.

**Lemma 4.11.** The restrictions \( \{ P_t^\mu := P_t|_{C^{\gamma}_\mu(\mathbb{T}^d)} \}_{t \geq 0} \) form a \( C_0 \)-semigroup on the Banach space \( (C^{\gamma}_\mu(\mathbb{T}^d), \| \cdot \|_0) \), with generator given by \( \mathcal{L}^\alpha f := \mathcal{L}^\alpha f, D(\mathcal{L}^\mu) := C^{\alpha+\beta}_\mu(\mathbb{T}^d) \).
Theorem 4.13. For any $f \in C_\mu(\mathbb{T}^d)$ and $t \geq 0$, we have
\[
\int_{\mathbb{T}^d} P_t f(x) \mu(dx) = \int_{\mathbb{T}^d} f(x) \mu(dx) = 0.
\]
That is, $C_\mu(\mathbb{T}^d)$ is $\{P_t\}_{t \geq 0}$-invariant, in the sense that $P_t(C_\mu(\mathbb{T}^d)) \subset C_\mu(\mathbb{T}^d)$ for all $t \geq 0$. The lemma then follows from the corollary in [7, Subsection II.2.3].

Lemma 4.12. If $f \in C_\mu^\beta(\mathbb{T}^d)$, then the unique solution $u_\kappa$ of (3.1) is of class $C_{\mu}^{\alpha+\beta}(\mathbb{T}^d)$, for any $\kappa > 0$.

Proof. Since $f$ is centered with respect to $\mu$, by Proposition 4.7 we have
\[
\|P_t f\|_0 \leq C \|f\|_0 e^{-\rho t}.
\]
Note the fact that $\mu$ is invariant with respect to $\{P_t\}_{t \geq 0}$. Then combining (4.7) and the representation (3.7), a straightforward application of Fibini’s theorem implies that
\[
\int_{\mathbb{T}^d} u_\kappa(x) \mu(dx) = \int_{\mathbb{T}^d} \int_0^\infty e^{-\kappa t} P_t f(x) dt \mu(dx) = \int_0^\infty e^{-\kappa t} \left( \int_{\mathbb{T}^d} P_t f(x) \mu(dx) \right) dt = 0.
\]
That is, $u_\kappa$ is also centered with respect to $\mu$. \qed 

The following theorem will solve the well-posedness of equation (4.6), which is more general than the results in [11, Proposition 3]. We formulate it as follows, referring to [27, Theorem 1] for the classical version for second order partial differential operators.

Theorem 4.13. For any $f \in C_\mu^\beta(\mathbb{T}^d)$, there exists a unique solution in $C_\mu^{\alpha+\beta}(\mathbb{T}^d)$ to the equation (4.6), which satisfies the estimate
\[
\|u\|_{\alpha+\beta} \leq C(\|u\|_0 + \|f\|_\beta),
\]
where $C = C(\|b\|_\beta)$ is a positive constant. Moreover, the unique solution admits the representation
\[
u(x) = \int_0^\infty P_t f(x) dt.
\]
\( C_{\mu}^{\alpha+\beta}(\mathbb{T}^d) \subset C_{\mu}^\beta(\mathbb{T}^d) \) (see, for instance, [12, Lemma 6.36]), the resolvent \( \mathcal{R}_{\kappa_0} := (\kappa_0 - \mathcal{L}^\alpha)^{-1} \) is compact from \( C_{\mu}^\beta(\mathbb{T}^d) \) to \( C_{\mu}^\beta(\mathbb{T}^d) \). Consider then the equation

\[
\begin{align*}
    u - \kappa_0 \mathcal{R}_{\kappa_0} u &= \mathcal{R}_{\kappa_0} f, \quad f \in C_{\mu}^\beta(\mathbb{T}^d),
    \end{align*}
\]

Then the Fredholm alternative (see [12, Section 5.3]) implies that the equation (4.10) always has a unique solution \( u \in C_{\mu}^\beta(\mathbb{T}^d) \) provided the homogeneous equation \( u - \kappa_0 \mathcal{R}_{\kappa_0} u = 0 \) has only the trivial solution \( u = 0 \).

To rephrase these statements in terms of the Poisson equation (4.6), we observe first that since \( \mathcal{R}_{\kappa_0} \) maps \( C_{\mu}^\beta(\mathbb{T}^d) \) onto \( C_{\mu}^{\alpha+\beta}(\mathbb{T}^d) \), any solution \( u \in C_{\mu}^\beta(\mathbb{T}^d) \) of (4.10) must also belong to \( C_{\mu}^{\alpha+\beta}(\mathbb{T}^d) \). Hence, operating on (4.10) with \( \kappa_0 - \mathcal{L}^\alpha \) we obtain

\[
    -\mathcal{L}^\alpha u = (\kappa_0 - \mathcal{L}^\alpha)(u - \kappa_0 \mathcal{R}_{\kappa_0} u) = f.
\]

Thus, the solutions of (4.10) are in one-to-one correspondence with the solutions of the Poisson equation (4.6). Consequently, (4.10) has a unique solution in \( C_{\mu}^{\alpha+\beta}(\mathbb{T}^d) \) if we can show that the homogeneous equation \( \mathcal{L}^\alpha u = 0 \) has only the zero solution, while the latter follows from the representation (4.9).

\begin{remark}
4.14. The assumption that \( f \) is centered with respect to \( \mu \) in Theorem 4.13 is necessary. To see this informally, let’s recall the Riesz-Schauder theory for compact operators (cf. [37, Theorem X.5.3]). The equation (4.10) admits a solution \( u \in C(\mathbb{T}^d) \) if and only if \( \mathcal{R}_{\kappa} f \in \text{Ker}(I^* - \kappa \mathcal{R}_{\kappa}^*)^\perp \), where the superscript \(*\) denotes the adjoint of operators. This is equivalent to say that the equation (4.6) admits a solution \( u \in C(\mathbb{T}^d) \) if and only if \( f \in \text{Ker}(L^{\alpha,*})^\perp \). On the other hand, we have \( \mu \in \text{Ker}(L^{\alpha,*}) \) since \( \mu \) is the invariant measure with respect to \( \{P_t\}_{t \geq 0} \). Thus a necessary condition for the existence of (4.6) is \( \langle \mu, f \rangle = 0 \), regarding \( \mu \) as an element in the dual space of \( C(\mathbb{T}^d) \).
\end{remark}

5 Homogenization results

5.1 Homogenization of SDEs

The aim of this subsection is to show the homogenization result of the solutions \( X_{x,\epsilon} \) of SDEs (2.1). It is quite natural to get rid of the drift term involving \( \frac{1}{\epsilon} \) in (2.1). For this purpose, we again use Zvonkin’s transform,

\[
    \hat{X}_{t,\epsilon} := X_{t,\epsilon} + \epsilon \left( \hat{b} \left( \frac{X_{t,\epsilon}}{\epsilon} \right) - \hat{b} \left( \frac{x}{\epsilon} \right) \right), \tag{5.1}
\]

where \( \hat{b} \) is the solution of the Poisson equation

\[
    \mathcal{L}^\alpha \hat{b} + b = 0, \tag{5.2}
\]

with the linear operator \( \mathcal{L}^\alpha \) given by (3.2). Note that the transform here is slightly different from that used in Section 4. Due to Theorem 4.13, \( \hat{b} \in C_{\mu}^{\alpha+\beta}(\mathbb{T}^d) \) is uniquely determined under the following assumption.
Assumption H6. The functions \( b \) and \( e \) satisfy the centering condition,
\[
\int_{\mathbb{R}^d} b(x) \mu(dx) = 0, \quad \int_{\mathbb{R}^d} e(x) \mu(dx) = 0.
\]

Note that this Assumption is quite natural in the homogenization problems and the reader can also find it in [6, 13, 24]. We will let Assumptions H5 and H6 hold true in this and next subsection. Now we are in a position to study the homogenization of SDEs with multiplicative stable noise.

**Theorem 5.1.** In the sense of weak convergence on the space \( \mathcal{D} \), we have that,
\[
X^{x,\epsilon}_t \Rightarrow X^x, \quad \text{where } X_t^x := x + Bt + L_t,
\]
as \( \epsilon \to 0 \). The homogenized coefficient \( B \) is given by
\[
B = \int_{\mathbb{T}^d} (I + \nabla \hat{b}) c(x) \mu(dx),
\]
and \( \{L_t\}_{t \geq 0} \) is a symmetric \( \alpha \)-stable Lévy processes with jump intensity measure
\[
\Pi(A) = \int_{\mathbb{R}^d \setminus \{0\}} \int_{\mathbb{T}^d} 1_A(\sigma(x,y)) \mu(dx) \nu^\alpha(dy), \quad A \in \mathcal{B}((\mathbb{R}^d \setminus \{0\})).
\]

**Proof.** Since \( \hat{b} \) is bounded, the theorem will follow if we prove that \( \hat{X}^{x,\epsilon} \Rightarrow X^x \), as \( \epsilon \to 0 \) (cf. [36, Theorem 2.7.(iv)]).

By applying Itô’s formula, and note that \( \hat{b} \in \mathcal{C}^{a+\beta}(\mathbb{T}^d) \) is the solution of Poisson equation (5.2),
\[
\hat{X}^{x,\epsilon}_t = x + \int_0^t (I + \nabla \hat{b}) c \left( \frac{X^{x,\epsilon}_s}{\epsilon} \right) ds - \int_0^t \frac{1}{\epsilon} \alpha - 1 A^{\alpha,\nu^\alpha} \hat{b} \left( \frac{X^{x,\epsilon}_s}{\epsilon} \right) ds + \int_0^t \epsilon A^{\alpha,\nu^\alpha} \hat{b}_t \left( \frac{X^{x,\epsilon}_s}{\epsilon} \right) ds
\]
\[
+ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \epsilon \left[ \hat{b}_t \left( X^{x,\epsilon}_s + \sigma (X^{x,\epsilon}_s, y) \right) - \hat{b}_t \left( X^{x,\epsilon}_s \right) \right] \tilde{N}^\alpha(dy,ds)
\]
\[
+ \int_0^t \int_{\mathbb{B} \setminus \{0\}} \sigma (X^{x,\epsilon}_s, y) \tilde{N}^\alpha(dy,ds) + \int_0^t \int_{\mathbb{B} \setminus \{0\}} \sigma (X^{x,\epsilon}_s, y) N^\alpha(dy,ds)
\]
\[
= x + \Lambda^\epsilon_1 (c)_t - \Lambda^\epsilon_2 \hat{b} (A^{\alpha,\nu^\alpha})_t + \Lambda^\epsilon_3 \left( \hat{b}, A^{\alpha,\nu^\alpha} \right)_t + \Lambda^\epsilon_4 (\hat{b}, \tilde{N}^\alpha)_t + \Lambda^\epsilon_5 (\sigma, \tilde{N}^\alpha)_t + \Lambda^\epsilon_6 (\sigma, N^\alpha)_t.
\]

For the last three stochastic integral terms, we figure out the characteristics of them as semimartingales (cf. [17, Proposition IX.5.3]). Choose the truncation function \( h_1(x) = x 1_B(x) \). Denote by \( \Xi^\epsilon(s, y) := \epsilon [\hat{b}_t (X^{x,\epsilon}_s + \sigma (X^{x,\epsilon}_s, y)) - \hat{b}_t (X^{x,\epsilon}_s)] \). Note that \( \Xi^\epsilon(\cdot, 0) \equiv 0 \) by virtue of \( \sigma(\cdot, 0) \equiv 0 \) as mentioned in Remark 2.1 (2). Then the characteristics of \( \Lambda^\epsilon_4 (\hat{b}, \tilde{N}^\alpha) \) associated with \( h_1 \) is given by
\[
\begin{align*}
B^\epsilon_4 (t) &= - \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \Xi^\epsilon(s, y) 1_B(c(\Xi^\epsilon(s, y))) \nu^\alpha(dy) ds,
C^\epsilon_4 &\equiv 0,
\nu^\epsilon_4 (A \times [0, t]) &= \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} 1_A(\Xi^\epsilon(s, y)) \nu^\alpha(dy) ds, \quad A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).
\end{align*}
\]
The characteristics of \( \Lambda_5^\epsilon(\sigma, \tilde{N}^\alpha) + \Lambda_6^\epsilon(\sigma, N^\alpha) \) is given by

\[
\begin{aligned}
B_{5+6}^\epsilon(t) &= \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \sigma_\epsilon \left( X_{s-}^{x,y} \right) [1_B (\sigma_\epsilon (X_{s-}^{x,y})) - 1_B(y)] \nu^\alpha(dy)ds, \\
C_{5+6}^\epsilon &\equiv 0, \\
\nu_{5+6}^\epsilon(A \times [0,t]) &= \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} 1_A (\sigma_\epsilon (X_{s-}^{x,y})) \nu^\alpha(dy)ds, \quad A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).
\end{aligned}
\]

By the same argument as in (2.5), we have \( B_{5+6}^\epsilon \equiv 0 \).

Then the theorem is a consequence of the functional central limit theorem in [17, Theorem VIII.2.17] and the following lemma.

**Lemma 5.2.** For any \( t \in \mathbb{R}_+ \), and any bounded continuous function \( f : \mathbb{R}^d \to \mathbb{R} \) which vanishes in a neighbourhood of the origin, the following convergences hold in probability \( \mathbb{P} \) when \( \epsilon \to 0 \):

(i) \( \sup_{0 \leq s \leq t} |\Lambda_1^\epsilon(c)_s - Bs| \to 0; \)

(ii) \( \sup_{0 \leq s \leq t} \left| \Lambda_3^\epsilon \left( \hat{b}, A^{\sigma,\nu^\alpha} \right)_s - \Lambda_2^\epsilon (\hat{b}, A^{\sigma,\nu^\alpha})_s \right| \to 0; \)

(iii) \( \sup_{0 \leq s \leq t} |B_4^\epsilon(s)| \to 0; \)

(iv) \( \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} f(x) \nu_4^\epsilon(dx,ds) \to 0; \)

(v) \( \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} f(x) \nu_{5+6}^\epsilon(dx,ds) \to t \int_{\mathbb{R}^d \setminus \{0\}} f(x) \Pi(dx); \)

where \( B \) and \( \Pi \) are defined in (5.4) and (5.5), respectively.

**Proof.** (i). By Theorem 4.10, the convergence in probability of the first integral is immediate,

\[
\sup_{0 \leq s \leq t} |\Lambda_1^\epsilon(c)_s - Bs| \leq \int_0^t \left| (I + \nabla \hat{b})c \frac{X_{s-}^{x,y}}{\epsilon} - B \right| ds \to 0, \quad \epsilon \to 0.
\]

(ii). Note that \( \nu^\alpha(\epsilon A) = \epsilon^{-\alpha} \nu^\alpha(A), \quad A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}) \). By the oddness condition in Assumption H3,

\[
\frac{1}{\epsilon^{\alpha-1}} A^{\sigma,\nu^\alpha} \hat{b} \left( \frac{x}{\epsilon} \right) = \int_{\mathbb{R}^d \setminus \{0\}} \epsilon \left[ \frac{\hat{b} (x/\epsilon + \sigma (x/\epsilon,y/\epsilon)) - \hat{b} (x/\epsilon)}{\epsilon} \right] dy - \sigma^i (x/\epsilon,y/\epsilon) \partial_i \hat{b} \left( \frac{x}{\epsilon} \right) 1_B(y) \nu^\alpha(dy).
\]
Then we have
\[
\left| \epsilon A^\sigma_{\nu,\nu} \hat{b}_\epsilon(x) - \frac{1}{\epsilon^{a-1}} A^\sigma_{\nu,\nu} \hat{b} \left( \frac{x}{\epsilon} \right) \right| \\
= \int_{B \setminus \{0\}} \left| \epsilon \left[ \hat{b} \left( \frac{x}{\epsilon} + \frac{1}{\epsilon} \sigma \left( \frac{x}{\epsilon}, y \right) \right) - \hat{b} \left( \frac{x}{\epsilon} + \sigma \left( \frac{x}{\epsilon}, y \right) \right) \right. \right. \\
- \left. \left. \left( \frac{1}{\epsilon} \sigma^{i} \left( \frac{x}{\epsilon}, y \right) - \sigma^{i} \left( \frac{x}{\epsilon}, y \right) \right) \partial_i \hat{b} \left( \frac{x}{\epsilon} \right) \right] \nu^\alpha(dy) \right. \\
+ \int_{B^c} \left| \epsilon \left[ \hat{b} \left( \frac{x}{\epsilon} + \frac{1}{\epsilon} \sigma \left( \frac{x}{\epsilon}, y \right) \right) - \hat{b} \left( \frac{x}{\epsilon} \right) \right] \nu^\alpha(dy) \right. \\
=: I_1^\epsilon + I_2^\epsilon,
\]
where $I_1^\epsilon$ and $I_2^\epsilon$ are the two integral terms in the second equality. By the boundness of $\hat{b}$ and the bounded convergence theorem, $I_2^\epsilon$ go to zero as $\epsilon \to 0$. It follows from Lemma 4.8, Assumption H5 and the dominated convergence that
\[
I_1^\epsilon \leq \|\hat{b}\|_{a+\beta} \int_{B \setminus \{0\}} \epsilon \left| \frac{1}{\epsilon} \sigma \left( \frac{x}{\epsilon}, y \right) - \sigma \left( \frac{x}{\epsilon}, y \right) \right|^{a+\beta} \nu^\alpha(dy) \to 0.
\]
Thus,
\[
\sup_{0 \leq s \leq t} \left| \Lambda_3^\epsilon(\hat{b}, A^\sigma_{u,\nu})_s - \Lambda_2^\epsilon(\hat{b}, A^\sigma_{u,\nu})_s \right| \\
\leq \int_0^t \left| \epsilon A^\sigma_{\nu,\nu} \hat{b}_\epsilon \left( X_{s,\epsilon}^x \right) - \frac{1}{\epsilon^{a-1}} A^\sigma_{\nu,\nu} \hat{b} \left( \frac{X_{s,\epsilon}^x}{\epsilon} \right) \right| ds \\
\to 0, \quad \epsilon \to 0.
\]

(iii) and (iv). Since the stochastic integral of $\Xi^\epsilon$ with respect to $\hat{N}^\alpha$ is definable, the third characteristic of $\Lambda_4^\epsilon$ satisfies that $\int_0^t \int_{\mathbb{R}^d \setminus \{0\}} (|x|^2 \wedge 1) \nu_4^\epsilon(dx, ds) < \infty$ for each $\epsilon > 0$ and $t \in \mathbb{R}_+$ (cf. [17, Proposition II.2.9]). By the hypothesis, there exist $\rho > 0$ and $M > 0$ such that $|f| \leq M$ on $B^c_\rho$ and $f = 0$ on $B_\rho$. Then for any $t \in \mathbb{R}_+$,
\[
\int_0^t \int_{\mathbb{R}^d \setminus \{0\}} f(x) \nu_4^\epsilon(dx, ds) \leq M \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} 1_{B^c_\rho}(x) \nu_4^\epsilon(dx, ds),
\]
which goes to zero almost surely as $\epsilon \to 0$ by the boundness of $\hat{b}$ and the dominated convergence theorem, and (iv) follows.

For $B_4^\epsilon$, we have the estimate
\[
\sup_{0 \leq s \leq t} |B_4^\epsilon(s)| \leq \left[ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} |x|^2 \nu_4^\epsilon(dx, ds) \right]^{\frac{1}{2}} \left[ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} 1_{B^c}(x) \nu_4^\epsilon(dx, ds) \right]^{\frac{1}{2}} \\
=: \sqrt{J_1} \cdot \sqrt{J_2}.
\]
By (iv) and a usual approximation procedure, $J_2^\epsilon$ goes to zero surely as $\epsilon \to 0$. For $J_1^\epsilon$, 

\[ J_1^\epsilon = \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \left| \epsilon \left[ b_\epsilon \left( X_{s,\epsilon}^x + \sigma_\epsilon \left( X_{s,\epsilon}^x, y \right) \right) - \hat{b}_\epsilon \left( X_{s,\epsilon}^x \right) \right] \right|^2 \nu^\alpha(dy)ds \]

\[ = \int_0^t \left( \int_{B_\epsilon} + \int_{B_\epsilon \setminus \{0\}} \right) \cdots |^2 \nu^\alpha(dy)ds \]

\[ \leq 4t \| \hat{b} \|_2^2 \lambda \left( \mathbb{S}^{d-1} \right) \epsilon^{2-\alpha} + \| \hat{b} \|_1^2 \int_{B_\epsilon \setminus \{0\}} \int_0^t \left| \sigma \left( X_{s,\epsilon}^x, y \right) \right|^2 d\nu^\alpha(dy). \]

By the growth condition in Assumption H3,

\[ \int_{B_\epsilon \setminus \{0\}} \int_0^t \left| \sigma \left( \frac{X_{s,\epsilon}^x}{\epsilon}, y \right) \right|^2 d\nu^\alpha(dy) \leq t \lambda \left( \mathbb{S}^{d-1} \right) \epsilon^{2-\alpha} \int_0^t \left| \phi \left( \frac{X_{s,\epsilon}^x}{\epsilon} \right) \right|^2 ds. \]

Then (iii) follows from these estimates and Theorem 4.10.

(v). It follows from Theorem 4.10 that,

\[ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} f(y) \nu^{\delta + \beta}(dy, ds) = \int_{\mathbb{R}^d \setminus \{0\}} \int_0^t f \left( \sigma \left( \frac{X_{s,\epsilon}^x}{\epsilon}, y \right) \right) d\nu^\alpha(dy) \]

\[ \to t \int_{\mathbb{R}^d \setminus \{0\}} \int_{\mathbb{T}^d} f(\sigma(x, y)) \mu(dx) \nu^\alpha(dy) \]

\[ = t \int_{\mathbb{R}^d \setminus \{0\}} f(y) \Pi(dy), \quad \epsilon \to 0, \]

where the convergence is in probability.

\[ \square \]

### 5.2 Homogenization of linear nonlocal PDEs

Define

\[ Y_t^\epsilon := \int_0^t \left( \frac{1}{\epsilon^{\alpha-1}} e \left( \frac{X_{s,\epsilon}^x}{\epsilon} \right) + g \left( \frac{X_{s,\epsilon}^x}{\epsilon} \right) \right) ds. \]  \hfill (5.6)

Thanks to Theorem 4.5, the nonlocal PDE (1.1) has a unique mild solution, which is given by the Feynman-Kac formula,

\[ u^\epsilon (t, x) = E \left[ u_0(X_t^x) \exp(Y_t^\epsilon) \right]. \]  \hfill (5.7)

Similar to $\hat{X}_{t}^{x,\epsilon}$, we define

\[ \hat{Y}_t^\epsilon := Y_t^\epsilon + \epsilon \left( \hat{e} \left( \frac{Y_t^\epsilon}{\epsilon} \right) - \hat{e} \left( \frac{x}{\epsilon} \right) \right). \]

Here $\hat{e} \in \mathcal{C}_0^{\alpha+\beta}(\mathbb{T}^d)$, thanks to Theorem 4.13 and Assumption H6, is the unique solution of the Poisson equation

\[ \mathcal{L}^\alpha \hat{e} + e = 0, \]

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with $\mathcal{L}^\alpha$ given by (3.2). Again using Itô’s formula,

$$
\dot{Y}_t^\epsilon = \int_0^t (g + \nabla \hat{c}) \left( \frac{X^{x,\epsilon}_{s-}}{\epsilon} \right) ds - \int_0^t \frac{1}{\epsilon^{\alpha-1}} A^{\sigma,\nu\alpha} \hat{e} \left( \frac{X^{x,\epsilon}_{s-}}{\epsilon} \right) ds + \int_0^t \epsilon A^{\sigma,\nu\alpha} \hat{e}_\epsilon (X^{x,\epsilon}_{s-}) ds 
$$

$$
+ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \epsilon |\hat{e}_\epsilon (X^{x,\epsilon}_{s-}) + \sigma (X^{x,\epsilon}_{s-}, y)| - \hat{e}_\epsilon (X^{x,\epsilon}_{s-})| \tilde{N}^\alpha(dy, ds) 
$$

$$
=: \Lambda_1^\epsilon (c, \sigma) \epsilon - \Lambda_2^\epsilon (\hat{e}, A^{\sigma,\nu\alpha}) t + \Lambda_3^\epsilon (\hat{e}, A^{\sigma,\nu\alpha}) t + \Lambda_4^\epsilon (\hat{e}, \tilde{N}^\alpha)_t. 
$$

Then in the same way as the proof of Theorem 5.1, we have the convergence of $Y^\epsilon$.

**Lemma 5.3.** In the sense of weak convergence on the space $\mathcal{D}$, both $Y^\epsilon$ and $\dot{Y}^\epsilon$ converge in distribution to a deterministic path $y(t) = Et$ as $\epsilon \to 0$, where the homogenized coefficient $E$ is given by

$$
E := \int_{\mathbb{R}^d} (g + \nabla \hat{c})(x) \mu(dx). 
$$

(5.8)

Now we are in the position to prove the main result of this section. Since $\hat{b}$ and $\hat{v}$ are bounded on $\mathbb{R}^d$, $u^\epsilon$ has the same limit behavior with

$$
\dot{u}_\epsilon(t, x) := \mathbb{E}[u_0(X^{x,\epsilon}_t) \exp(\hat{Y}^\epsilon_t)] 
$$

(5.9)
as $\epsilon \to 0$.

**Proof of Theorem 1.1.** We only need to show $\dot{u}_\epsilon(t, x) \to u(t, x)$, $\epsilon \to 0$ for any $t \geq 0, x \in \mathbb{R}^d$. For the convenience of notation, we shall write $\Lambda_1^\epsilon (c, g)_t$, $\Lambda_2^\epsilon (\hat{e}, A^{\sigma,\nu\alpha})_t$, $\Lambda_3^\epsilon (\hat{e}, A^{\sigma,\nu\alpha})_t$, $\Lambda_4^\epsilon (\hat{e}, \tilde{N}^\alpha)_t$ as $\Lambda_1^\epsilon (t)$, $\Lambda_2^\epsilon (t)$, $\Lambda_3^\epsilon (t)$, $\Lambda_4^\epsilon (t)$, respectively. We fix a $t \in \mathbb{R}_+$.

Firstly, we prove the uniform integrability of the set $\{e^{\Lambda_4^\epsilon (t)} | 0 < \epsilon \leq 1 \}$ for each $t \in \mathbb{R}_+$. This follows by proving that it is uniformly bounded in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Denoting the integrand in $\Lambda_4^\epsilon (\hat{e}, \tilde{N}^\alpha)$ by

$$
\Gamma^\epsilon(s, y) := \epsilon [\hat{e}_\epsilon (X^{x,\epsilon}_{s-}) + \sigma (X^{x,\epsilon}_{s-}, y)] - \hat{e}_\epsilon (X^{x,\epsilon}_{s-})]. 
$$

Then by Itô’s formula,

$$
e^{2\Lambda_4^\epsilon (t)} = 1 - \int_0^t \int_{\mathbb{R}^d} 2e^{2\Lambda_4^\epsilon (s-)} \Gamma^\epsilon(s, y) \nu^\alpha(dy)ds 
$$

$$
+ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} e^{2\Lambda_4^\epsilon (s-)} \left( e^{2\Gamma^\epsilon(s, y)} - 1 \right) \tilde{N}^\alpha(dy, ds) 
$$

$$
+ \int_0^t \int_{\mathbb{R}^d} e^{2\Lambda_4^\epsilon (s-)} \left( e^{2\Gamma^\epsilon(s, y)} - 1 \right) \tilde{N}^\alpha(dy, ds) 
$$

$$
+ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} e^{2\Lambda_4^\epsilon (s-)} \left[ e^{2\Gamma^\epsilon(s, y)} - 1 - 2\Gamma^\epsilon(s, y) \right] \nu^\alpha(dy)ds. 
$$

Since $\hat{e}$ is bounded, $\Gamma^\epsilon$ has a uniform bound for all $\epsilon > 0$. Then there exists a large constant $C > 0$ such that for each $\epsilon > 0$ and $t \in \mathbb{R}_+$,

$$
\mathbb{E} \int_0^t \int_{\mathbb{R}^d} \left| e^{2\Lambda_4^\epsilon (s-)} \left( e^{2\Gamma^\epsilon(s, y)} - 1 - 2\Gamma^\epsilon(s, y) \right) \right|^2 \nu^\alpha(dy)ds \leq C \nu^\alpha(B^c) \mathbb{E} \int_0^t e^{2\Lambda_4^\epsilon (s-)} ds < \infty. 
$$
Thus, combining these, there exists $\theta \in (0, 1)$ such that

$$E e^{2\Lambda'_t(t)} = 1 + E \int_0^t \int_{B\setminus\{0\}} e^{2\Lambda'_t(s-)} \left[ e^{2\Gamma(s,y)} - 1 - 2\Gamma(s,y) \right] \nu^\alpha(dy)ds$$

$$\leq 1 + E \int_0^t e^{2\Lambda'_t(s-)} \int_{B\setminus\{0\}} 2e^{2\theta\Gamma(s,y)} |\Gamma(s,y)|^2 \nu^\alpha(dy)ds$$

$$\leq 1 + C(\|\hat{e}\|_0)E \int_0^t e^{2\Lambda'_t(s-)} \int_{B\setminus\{0\}} |\Gamma(s,y)|^2 \nu^\alpha(dy)ds.$$

As shown in the proof of part (iii) and (iv) in Lemma 5.2,

$$\int_{B\setminus\{0\}} |\Gamma(s,y)|^2 \nu^\alpha(dy) \leq e^2 \|\hat{e}\|^2 \int_{B\setminus\{0\}} |\sigma_s(X_{s-}, y)|^2 \nu^\alpha(dy) \leq \frac{\lambda(S^{-1})}{2 - \alpha} e^2 \|\hat{e}\|^2 \|\psi\|_{L^\infty}.$$ 

Thus,

$$E e^{2\Lambda'_t(t)} \leq 1 + e^2 C(\alpha, \lambda(S^{-1}), \|\hat{e}\|_1, \|\psi\|_{L^\infty}) \int_0^t E e^{2\Lambda'_t(s-)}ds.$$

By Grönwall’s inequality, the uniform boundness of $\{e^{\Lambda'_t(t)}\}_{0 < \epsilon \leq 1}$ in $L^2(\Omega, P)$ follows.

Secondly, the proof of Lemma 5.2 shows that the set $\{\Lambda'_3(t) - \Lambda'_2(t)\}_{0 < \epsilon \leq 1}$ is bounded. The set $\{\Lambda'_1(t)\}_{0 < \epsilon \leq 1}$ is bounded by virtue of the boundness of $c, g$ and $\hat{e}$. Also since $u_0$ is periodic and continuous, $\{u_0(X_t)\}_{0 < \epsilon \leq 1}$ is bounded. Thus, the set $\{u_0(X_t^{x,\epsilon}) \exp(\hat{Y}_t^\epsilon)\}_{0 < \epsilon \leq 1}$ is uniformly integrable.

Finally, we pass to the limit. It is easy to see that $e^{\hat{Y}_t^\epsilon} \rightarrow e^{y(t)}$ in probability as $\epsilon \rightarrow 0$ (by, for instance, [36, Theorem 2.7.(iii)]). Then for any subsequence $\{\epsilon_n\} \rightarrow 0$, there exists a subsequence $\{\epsilon_{n_k}\} \rightarrow 0$ such that $e^{\hat{Y}_{t^{n_k}}^\epsilon} \rightarrow e^{y(t)}$ almost uniformly (cf. [19, Lemma 4.2]). That is, for any $\rho > 0$, there exists a set $N \in \mathcal{F}$ with $P(N) \leq \rho$, such that

$$\left\| e^{\hat{Y}_{t^{n_k}}^\epsilon} - e^{y(t)} \right\|_{L^\infty(\mathcal{N}_c, P)} \rightarrow 0, \quad k \rightarrow \infty. \tag{5.10}$$

By the boundness of $u_0$, we know the set $\{u_0(X_t^{x,\epsilon})[\exp(\hat{Y}_t^\epsilon) - \exp(y(t))]\}_{0 < \epsilon \leq 1}$ is also uniformly integrable. Then for any $\delta > 0$, there exist $\rho_0 > 0$ and $N_0 \in \mathcal{F}$ with $P(N_0) \leq \rho_0$, such that

$$E \left| u_0(X_t^\epsilon) \left( e^{\hat{Y}_t^\epsilon} - e^{y(t)} \right) 1_{N_0} \right| < \delta. \tag{5.11}$$

Now along the sequence $\{\epsilon_{n_k}\}$, we combining (5.10) with (5.11) to get

$$E \left| u_0(X_t^{\epsilon_{n_k}}) \left( e^{\hat{Y}_{t^{n_k}}^\epsilon} - e^{y(t)} \right) \right| \leq E \left| \cdots 1_{N_0} \right| + E \left| \cdots 1_{N_0^c} \right|$$

$$\leq \delta + \|u_0\|_{L^\infty(\mathcal{N}_c, P)} \left\| e^{\hat{Y}_{t^{n_k}}^\epsilon} - e^{y(t)} \right\|_{L^\infty(\mathcal{N}_c, P)}$$

$$\leq 2\delta.$$

To summarize these together, for any subsequence $\{\epsilon_n\} \rightarrow 0$, there exists a subsequence $\{\epsilon_{n_k}\} \rightarrow 0$ such that

$$E \left| u_0(X_t^{\epsilon_{n_k}}) \left( e^{\hat{Y}_{t^{n_k}}^\epsilon} - e^{y(t)} \right) \right| \rightarrow 0, \quad k \rightarrow \infty,$$
which implies that the convergence holds on the whole line $0 < \epsilon \leq 1$. On the other hand, by Theorem 5.1, we know that $E|u_0(X^\epsilon_t) - u_0(X_t)| \to 0$ as $\epsilon \to 0$. The result (1.4) follows immediately.

Remark 5.4. We close this section by some comments for the proof of Theorem 1.1. In [24], the author applied Girsanov's transform to get rid of the stochastic integral term involved in $\hat{Y}^\epsilon$, since this term may not possess the uniformly integrability. While in our case, since the stochastic integral term in $Y^\epsilon_t$ has an infinitesimal integrand $\Gamma^\epsilon(s, y)$, the uniform integrability of $\{\exp(\hat{Y}^\epsilon_t)|0 < \epsilon \leq 1\}$ is easier to treat.

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