Simultaneous eigenstates of the number-difference operator and a bilinear interaction Hamiltonian derived by solving a complex differential equation

Hong-yi Fan and Wei-bo Gao
1Department of Material Science and Engineering, University of Science and Technology of China, Hefei, Anhui 230026, China
2Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China

Abstract

As a continuum work of Bhaumik et al who derived the common eigenvector of the number-difference operator \( Q \equiv (a^\dagger a - b^\dagger b) \) and pair-annihilation operator \( ab \) (J. Phys. A9 (1976) 1507) we search for the simultaneous eigenvector of \( Q \) and \( (ab - a^\dagger b^\dagger) \) by setting up a complex differential equation in the bipartite entangled state representation. The differential equation is then solved in terms of the two-variable Hermite polynomials and the formal hypergeometric functions. The work is also an addendum to Mod. Phys. Lett. A 9 (1994) 1291 by Fan and Klauder, in which the common eigenkets of \( Q \) and pair creators \( a^\dagger b^\dagger \) are discussed.

1 Introduction

Coherent states [1-2] are widely used in many aspects of quantum physics. The bosonic coherent state, which describes the quantum state of laser, obeys coordinate-momentum minimum uncertainty relation and possesses non-orthonormal and over-complete properties. Generalized coherent states has been constructed and applied by theorists since 1970s, among which the coherent state \( |q, \alpha\rangle \) for charged bosons [3] is a remarkable one, when one introduces ”charge” by defining two types of quanta possessing ”charge” +1 and -1 with corresponding annihilation operators \( a \) and \( b \), so the operator \( Q = a^\dagger a - b^\dagger b \) is endowed with charge operator, \( |q, \alpha\rangle \) is constructed based on \([Q, ab] = 0, [a, a^\dagger] = [b, b^\dagger] = 1\). In the Fock space the charged coherent state is

\[
|q, \alpha\rangle = C_q \sum_{n,q=0}^{\infty} \frac{\alpha^q}{\sqrt{(n+q)!n!}} |n+q, n\rangle,
\]

\[
Q |q, \alpha\rangle = q |q, \alpha\rangle, \quad ab |q, \alpha\rangle = \alpha |q, \alpha\rangle
\]

where \( C_q \) is the normalization constant. In quantum optics theory \( |q, \alpha\rangle \) was named by Agarwal as pair-coherent state in [4], \( Q \) is the two-mode photon number-difference operator. By observing \([Q, a^\dagger b^\dagger] = 0\), in

\[\text{*Work supported by The Presidential Foundation of the Chinese Academy of Science}\]
Ref. [5] Fan and Klauder also constructed the common eigenvector of $Q$ and $a^\dagger b^\dagger$ with use of the Dirac’s $\delta$-function in contour integral form proposed by Heitler [6]-[7]. One then naturally to ask what is the simultaneous eigenstates, denoted as $|q, k\rangle$, of $Q$ and $(ab - a^\dagger b^\dagger)$? This question is full of physical meaning in quantum optics, because most nonlinear interactions in the parametric approximation reduce to a bilinear form

$$H_I = i\hbar \kappa \left( ab - a^\dagger b^\dagger \right),$$

(1)

since

$$\left[ (ab - a^\dagger b^\dagger), Q \right] = 0,$$

(2)

where $\kappa$ is related to the susceptibility in the parametric process. $H_I$ is responsible for producing two-mode squeezed states [8-11], which is not only useful for optical communication and weak signal detection, but also embodies quantum entanglement, i.e. the Einstein-Podolsky-Rosen (EPR) correlations [12] for quadrature phase amplitude are intrinsic to two-mode squeezed light, the idler-mode and the signal-mode generated from a parametric amplifier are entangled each other in a frequency domain. Solving this question is quite difficult. We recall Dirac’s guidance [13]: “When one has a particular problem to work out in quantum mechanics, one can minimize the labour by using a representation in which the representatives of the more important abstract quantities occurring in that problem are as simple as possible”. At first glance, we thought that the charged coherent state representation $|q, \alpha\rangle$ was a good candidates for tackling the problem as simple as possible, but after some tries we found that it was not. Eventually we find that the entangled state representation [14-15] is of assistance for searching for the desired common eigenvector

$$Q |q, k\rangle = q |q, k\rangle,$$

(3)

$$\left( ab - a^\dagger b^\dagger \right) |q, k\rangle = (q - k - 1) |q, k\rangle,$$

(4)

because it possesses well-behaved features. Thus we shall briefly review the properties of entangled state $|\xi\rangle$ in Sec. 2. Then in Sec. 3 we shall make full use of $|\xi\rangle$ to set up a complex differential equation for the overlap $\langle \xi | q, k \rangle$. In Sec. 4 we employ the two-variable Hermite polynomials and the hypergeometric function to solve the differential equation. The Gauss’ contiguous relation of hypergeometric function is essential for us to derive the common eigenvector of $Q$ and $(ab - a^\dagger b^\dagger)$.

2 Brief review of the bipartite entangled state representations

In [14] and [15] we have constructed the bipartite entangled state

$$\exp \left[ -\frac{|\xi|^2}{2} + \xi a^\dagger + \xi^* b^\dagger - a^\dagger b^\dagger \right] |0\rangle \equiv |\xi\rangle, \quad \xi = |\xi|e^{i\phi},$$

(5)

$|\xi\rangle$ satisfy the eigenvator equations

$$(a + b^\dagger) |\xi\rangle = \xi |\xi\rangle, \quad (a^\dagger + b) |\xi\rangle = \xi^* |\xi\rangle.$$

(6)

or

$$\frac{1}{2} (X_1 + X_2) |\xi\rangle = \frac{1}{\sqrt{2}} \xi_1 |\xi\rangle, \quad (P_1 - P_2) |\xi\rangle = \sqrt{2} \xi_2 |\xi\rangle,$$

(7)

i.e. $|\xi\rangle$ is just the simultaneous eigenvector of two-particles’ center-of-mass position $X_1 - X_2$ and the relative momentum $P_1 - P_2$ in Fock space, we name it the EPR eigenstate since EPR were the first who used the commutative property $[X_1 - X_2, P_1 + P_2] = 0$ to challege the incompleteness of quantum mechanics [11].
According to Dirac’s representation theory [13]: “To set up a representation in a general way, we take a complete set of bra vectors, i.e. a set such that any bra can be expressed linearly in terms of them,” we use the normal ordering form of the two-mode vacuum state projector
\[ |00\rangle \langle 00| = e^{-a^\dagger a - b^\dagger b}, \] (8)
and the technique of integration within an ordered product (IWOP) of operators [16]-[17], we can prove
\[ \int \frac{d^2 \xi}{\pi} |\xi\rangle \langle \xi| = 1. \] (9)

3 Complex differential equation for the new state \(|q, k\rangle\)

By noticing (5) we see
\[ a |\xi\rangle = (\xi - b^\dagger) |\xi\rangle, \quad b |\xi\rangle = (\xi^* - a^\dagger) |\xi\rangle, \] (10)
\[ \langle \xi| a^\dagger = \langle \xi| (\xi^* - b), \quad \langle \xi| b^\dagger = \langle \xi| (\xi - a), \] so
\[ \langle \xi| (a^\dagger b - a b^\dagger) = \langle \xi| [a(b - (\xi^* - b)b)] = \langle \xi| [b(\xi - (\xi^* - b))] + 1]. \] (11)
Then using
\[ a^\dagger |\xi\rangle = \left( \frac{\partial}{\partial \xi} + \frac{\xi^*}{2} \right) |\xi\rangle, \quad b^\dagger |\xi\rangle = \left( \frac{\partial}{\partial \xi^*} + \frac{\xi}{2} \right) |\xi\rangle, \] (12)
we re-write (11) as
\[ \langle \xi| (a^\dagger b - a b^\dagger) = \left[ \left( \frac{\partial}{\partial \xi} + \frac{\xi^*}{2} \right) + \left( \frac{\partial}{\partial \xi^*} + \frac{\xi}{2} \right) \right] \xi^* - |\xi|^2 + 1 \] (13).

Operating (13) on the state \(|q, k\rangle\), and using (4) we obtain a complex differential equation
\[ \langle \xi| (a^\dagger b - a b^\dagger) |q, k\rangle = \left( \xi^* + \frac{\partial}{\partial \xi^*} + 1 \right) \langle \xi| q, k \rangle = (q - k - 1) \langle \xi| q, k \rangle \] (14)

On the other hand, from (5) we have
\[ (a^\dagger a - b^\dagger b) |\xi\rangle = (\xi a^\dagger - b^\dagger b^*) |\xi\rangle \] (15)
\[ = |\xi| \left( e^{-i\phi} a^\dagger - e^{i\phi} b^\dagger \right) \exp \left[ -\frac{|\xi|^2}{2} + |\xi| \left( e^{-i\phi} a^\dagger + e^{i\phi} b^\dagger \right) - a^\dagger b^\dagger \right] |00\rangle \]
\[ = -i \frac{\partial}{\partial \phi} |\xi\rangle, \]
which together with (3) leads to another differential equation
\[ \langle \xi| (a^\dagger a - b^\dagger b) |q, k\rangle = i \frac{\partial}{\partial \phi} \langle \xi| q, k \rangle = q |q, k\rangle. \] (16)

In the next section we want to solve (14) and (16).
4 Solution to Eq. (16)

After many tries we find the solution of (14) and (16) is

\[ \langle \xi \mid q, k \rangle = C(k) e^{-|\xi|^2/2} A, \]  

(17)

where \( C(k) \) is the normalization constant, which keeps undisturbed in the following calculations, so we neglect it in the derivation process, and

\[ A \equiv \sum_{n=0}^{\infty} \frac{1}{n!} H_{n+q,n} (\xi^*, \xi) \ 2F_1(-n, k/2 + 1; q + 1; 2) \]  

(18)

\( 2F_1 \) is the hypergeometric function defined as

\[ 2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n z^n}{(\gamma)_n n!} = \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha) \Gamma(n+\beta) z^n}{\Gamma(n+\gamma) n!}, \]  

(19)

the symbol \((\alpha)_n\) means

\[ (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \alpha (\alpha+1) (\alpha+2) \cdots (\alpha+n-1), \]  

(20)

and the two-variable Hermite polynomials is defined as [18]

\[ H_{m,n} (\xi, \xi^*) = \sum_{l=0}^{\min(m,n)} \frac{m! n!}{l!(m-l)!(n-l)!} (-1)^l \xi^{m-l} \xi^{*-l} = H_{n,m} (\xi^*, \xi), \]  

(21)

whose generating function is

\[ \sum_{m,n=0}^{\infty} \frac{t^m t^n}{m! n!} H_{m,n} (\xi, \xi^*) = \exp \left(-tt' + t\xi + t'\xi^* \right). \]  

(22)

Note that the convergent condition for the hypergeometric function defined in (19) is \(|z| < 1, \gamma \neq 0, -1, -2, \cdots\), so \( 2F_1(-n, k/2 + 1; q + 1; 2) \) is a formal power series.

Now we prove (14): Firstly, we notice

\[ \xi \frac{\partial}{\partial \xi} \langle \xi \mid q, k \rangle = \xi \left( -\frac{\xi^*}{2} e^{-|\xi|^2/2} A + e^{-|\xi|^2/2} \frac{\partial}{\partial \xi} A \right), \]  

(23)

and

\[ \xi^* \frac{\partial}{\partial \xi^*} \langle \xi \mid q, k \rangle = \xi^* \left( -\frac{\xi}{2} e^{-|\xi|^2/2} A + e^{-|\xi|^2/2} \frac{\partial}{\partial \xi^*} A \right), \]  

(24)

it then follows

\[ \left( \xi \frac{\partial}{\partial \xi} + \xi^* \frac{\partial}{\partial \xi^*} \right) \langle \xi \mid q, k \rangle = -|\xi|^2 \langle \xi \mid q, k \rangle + e^{-|\xi|^2/2} \left( \xi \frac{\partial}{\partial \xi} + \xi^* \frac{\partial}{\partial \xi^*} \right) A \]  

(25)

so

\[ \left( \xi \frac{\partial}{\partial \xi} + \xi^* \frac{\partial}{\partial \xi^*} + 1 \right) \langle \xi \mid q, k \rangle = e^{-|\xi|^2/2} \left( \xi \frac{\partial}{\partial \xi} + \xi^* \frac{\partial}{\partial \xi^*} + 1 - |\xi|^2 \right) \sum_{n=0}^{\infty} \frac{1}{n!} H_{n+q,n} (\xi^*, \xi) \ 2F_1(-n, k/2 + 1; q + 1; 2). \]  

(26)
Then we use the property
\[
\frac{\partial}{\partial \xi} H_{m,n} (\xi^*, \xi) = n H_{m,n-1} (\xi^*, \xi), \quad \frac{\partial}{\partial \xi} H_{m,n} (\xi^*, \xi) = m H_{m-1,n} (\xi^*, \xi) \tag{27}
\]
and
\[
H_{m+1,n} + n H_{m,n-1} = \xi^* H_{m,n}, \quad H_{m,n+1} + m H_{m-1,n} = \xi H_{m,n}, \tag{28}
\]
which can be derived from (21) and (23), as well as
\[
|\xi|^2 H_{m,n} = \xi^* (H_{m+1,n} + n H_{m,n-1}) = H_{m+1,n+1} + nm H_{m-1,n-1} + (m + n + 1) H_{m,n}, \tag{29}
\]
we have
\[
\left( \xi \frac{\partial}{\partial \xi} + \xi^* \frac{\partial}{\partial \xi^*} + 1 \right) (\xi | q, k) = e^{-|\xi|^2/2} \sum_{n=0}^{\infty} \frac{1}{n!} [\xi n H_{n+q,n-1} + \xi^* (q + n) H_{n+q-1,n}] F_1(-n, \frac{k}{2} + 1; q + 1; 2)
\]
\[
+ H_{n+q+1,n+1} - n(n + q) H_{n+q-1,n-1} - (q + 2n) H_{n+q,n} \right) F_1(-n, \frac{k}{2} + 1; q + 1; 2)
\]
\[
= e^{-|\xi|^2/2} \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ n [H_{n+q,n} + (n + q) H_{n+q-1,n-1}] \right. + (q + n)[H_{n+q,n} + n H_{n+q-1,n}] \right.
\]
\[
- n(n + q) H_{n+q-1,n-1} - (q + 2n) H_{n+q,n} \right) F_1(-n, \frac{k}{2} + 1; q + 1; 2)
\]
\[
= e^{-|\xi|^2/2} \sum_{n=0}^{\infty} \frac{1}{n!} H_{n+q,n}[(q + n + 1) 2 F_1(-n - 1, \frac{k}{2} + 1; q + 1; 2)
\]
\[
- n 2 F_1(-n - 1, \frac{k}{2} + 1; q + 1; 2). \tag{30}
\]

Then using two Gauss’ contiguous relations about the hypergeometric function [19-20]
\[
2 F_1(\alpha, \beta; \gamma; \varepsilon) = 2 F_1(\beta, \alpha; \gamma; \varepsilon), \tag{31}
\]
and
\[
[\gamma - 2 \beta + (\beta - \alpha) \varepsilon] 2 F_1(\alpha, \beta; \gamma; \varepsilon) + \beta(1 - \varepsilon) 2 F_1(\alpha, \beta + 1; \gamma; \varepsilon)
\]
\[
- (\gamma - \beta) 2 F_1(\alpha, \beta - 1; \gamma; \varepsilon) = 0, \tag{32}
\]
and letting \( \alpha = \frac{k}{2} + 1, \beta = -n, \gamma = q + 1, \varepsilon = 2 \), we have
\[
(q - k - 1) 2 F_1\left(\frac{k}{2} + 1, -n; q + 1; 2\right) + n 2 F_1\left(\frac{k}{2} + 1, -n + 1; q + 1; 2\right)
\]
\[
- (q + n + 1) 2 F_1\left(\frac{k}{2} + 1, -n - 1; q + 1; 2\right) = 0 \tag{33}
\]
Substituting (30) into the right-hand side of (30) we finally obtain (recovering \( C(k) \))
\[
\left( \xi \frac{\partial}{\partial \xi} + \xi^* \frac{\partial}{\partial \xi^*} + 1 \right) (\xi | q, k) = (q - k - 1) C(k) e^{-|\xi|^2/2} \sum_{n=0}^{\infty} \frac{1}{n!} H_{n+q,n} (\xi^*, \xi) 2 F_1(-n, \frac{k}{2} + 1; q + 1; 2)
\]
\[
= (q - k - 1) (\xi | q, k), \tag{34}
\]
thus we have proved the solution of (14). On the other hand, from (21) we know
\[
H_{n+q,n} (\xi^*, \xi) = e^{-i q \xi} H_{n+q,n} (|\xi|, |\xi|), \tag{35}
\]
so the solution (14) automatically satisfied with (16). The solution seems new.
5 The simultaneous eigenstate of $Q$ and $(ab - a^\dagger b^\dagger)$

Now we hope to obtain $|q,k\rangle$ from the information of $\langle \xi | q,k \rangle$. Using the completeness relation (9) of $|\xi\rangle$ and (17)-(18) we can have

$$|q,k\rangle = \int \frac{d^2\xi}{\pi} |\xi\rangle \langle \xi | q,k \rangle$$

$$= C(k) \int \frac{d^2\xi}{\pi} |\xi\rangle e^{-|\xi|^2/2} \sum_{n=0}^{\infty} \frac{1}{n!} H_{n+q,n} (\xi^*, \xi) \ _2F_1(-n, \frac{k}{2} + 1; q + 1; 2).$$

Then using (22) we expand $|\xi\rangle$ in (5),

$$|\xi\rangle = e^{-|\xi|^2/2} \sum_{l,j=0}^{\infty} \frac{1}{\sqrt{l!j!}} H_{l,j} (\xi, \xi^*) |l,j\rangle,$$

where $|l,j\rangle$ is the two-mode Fock state. Substituting (37) into (36) and using the integration formula

$$\int \frac{d^2\xi}{\pi} e^{-|\xi|^2} H_{l,j} (\xi, \xi^*) H_{n,m}^* (\xi, \xi^*) = \delta_{l,m} \delta_{j,n} n!,$$

we have

$$|q,k\rangle = C(k) \sum_{n=0}^{\infty} \frac{1}{n!} \ _2F_1(-n, \frac{k}{2} + 1; q + 1; 2)$$

$$\times \int \frac{d^2\xi}{\pi} e^{-|\xi|^2} \sum_{l,j=0}^{\infty} \frac{1}{\sqrt{l!j!}} H_{l,j} (\xi, \xi^*) H_{n+q,n}^* (\xi, \xi^*) |l,j\rangle$$

$$= C(k) \sum_{n=0}^{\infty} \frac{\sqrt{n+q}}{\sqrt{n!}} \ _2F_1(-n, \frac{k}{2} + 1; q + 1; 2) |n+q,n\rangle$$

$$= C(k) a^q \sum_{n=0}^{\infty} \frac{a^\dagger b^\dagger^n}{n!} \ _2F_1(-n, \frac{k}{2} + 1; q + 1; 2) |0,0\rangle.$$

As we have known that the convergent condition for the hypergeometric function defined in (19) is $|z| < 1$, $\gamma \neq 0, -1, -2, \cdots$, so $|q,k\rangle$ seems not normalized as a finite number, but as a divergent one. To see this more clearly, using (17)-(18), (9) and (38), formally we have

$$\langle q,k | q,k \rangle = \int \frac{d^2\xi}{\pi} \langle q,k | \xi \rangle \langle \xi | q,k \rangle$$

$$= |C(k)|^2 \int \frac{d^2\xi}{\pi} \sum_{n=0}^{\infty} \frac{1}{n!} H_{n+q,n} (\xi^*, \xi) \ _2F_1(-n, \frac{k}{2} + 1; q + 1; 2)$$

$$\times \sum_{n'=0}^{\infty} \frac{1}{n'!} H_{n'+q,n'} (\xi, \xi^*) \ _2F_1(-n', \frac{k}{2} + 1; q + 1; 2)$$

$$= |C(k)|^2 \sum_{n=0}^{\infty} \frac{(n+q)!}{n!} \ _2F_1(-n, \frac{k}{2} + 1; q + 1; 2)^2.$$
Therefore, the common eigenvector of $Q$ and $(ab - a^\dagger b^\dagger)$, like the common eigenvector of $Q$ and $a^\dagger b^\dagger$, is normalized as a singular function, so its applications are greatly limited. However, the exploration for formal solution of $|q, k\rangle$ has its own sense in mathematical physics.

In summary, as a continuum work of Bhaumik et al [3] we have set a complex differential equation in the entangled state representation for deriving the new the common eigenstate $|q, k\rangle$ of parametric interaction Hamiltonian and number-difference operator, the complex differential equation has been solved in terms of hypergeometric function and the two-variable Hermite polynomials and the solution seems new. Thus this paper embodies a new use of hypergeometric functions and is of theoretical mathematical physics meaning. This work is also an addendum to Ref. [5], in which the common eigenkets of $Q$ and two-mode creators $a^\dagger b^\dagger$ are discussed.

References

[1] Klauder J R and Skargerstam B S, *Coherent States*, World Scientific, Singapore, 1985; Klauder J R and Sudarshan E C G, 1968 *Fundamentals of Quantum Optics*, W A Benjamin, New York

[2] Glauber R J, 1963 Phys. Rev. 130 2529; 1963 131 2766

[3] Bhaumik D et al 1976 J. Phys. A9 1507

[4] G. S. Agarwal, J. Opt. Soc. Am. B, 5 (1988) 1940

[5] Fan Hong-yi and J. R. Klauder, Mod. Phys. Lett. A 9 (1994) 1291

[6] Heitler W, *The Quantum Theory of Radiation*, 3rd ed, London: Oxford Claredon Press, 1954; A. Vourdas and R. F. Bishop, Phys. Rev. A 53 (1996) R1205; J. Phys. A 31 (1998) 8563

[7] Hong-yi Fan, Zu-wei Liu, and Tu-nan Ruan, Commun. Theor. Phys. 3 (1984) 175; Hong-yi Fan and A. Wünsche, Ann. Phys. (LPG) 1, (1992) 181

[8] For a very recent review, see Dodonov V V, 2002 J. Opt. B: Quantum Semiclass. Opt. 4 R1-R33

[9] See e.g., D’Ariano G M, Rassetti M G, Katriel J and Solomon A I 1989 *Squeezed and Nonclassical Light* ed P Tombesi and E R Pike (New York: Plenum) p 301; Bužek V 1990 J. Mod. Opt. 37 303; .

[10] See e.g., Walls D F and Milburn G J, *Quantum Optics*, Springer-Verlag, Berlin, 1994; Wolfgang P Schleich, 2001 *Quantum Optics in Phase Space*, Wiley-Vch, Berlin; Scully M O and Zubairy M S, 1997 Cambridge University Press

[11] See e.g., Loudon R and Knight P L, 1987 J. Mod. Opt. 34 709; Mandel L and Wolf E 1995, *Optical Coherence and Quantum Optics* (Cambridge press, London)

[12] Einstein A, Podolsky B, and Rosen N, 1935 Phys. Rev. 47 777

[13] Dirac P A M, *The Principle of Quantum Mechanics*, (fourth edition), Oxford University Press(1958)

[14] Fan Hong-yi and Klauder J R 1994 Phys. Rev. A 49 704; Fan Hongyi and Ye Xiong, Phys. Rev. A 51 (1995) 3343; Fan Hongyi and Fan Yue, 1996 Phys. Rev. A 54 958

[15] Hong-yi Fan, 2004 Inter. J. Mod. Phys. 18 1387
[16] Hong-yi Fan, 2003 J Opt B, Quantum Semiclass. Opt. 5 R147; Fan Hong-yi, Zaidi H R and Klauder J R, 1987 Phys. Rev. D35 1831; Fan Hong-yi and Klauder J R, 1988 J. Phys. A21 L725; Fan Hong-yi and Zaidi H R, 1988 Phys. Rev. A39 2985; Fan Hong-yi and Vanderlinde, 1988 J. Phys. Rev. A39 2987

[17] Wünsche A 1999 J Opt B, Quantum Semiclass. Opt. 1 R11-21

[18] A. Erdélyi 1953 Higher Transcendental Functions, The Bateman Manuscript Project, McGraw Hill,

[19] I. S. Gradshteyn and I. M. Ryzhik, Table of integrals, series and products, Academic press, New York, 1980, page 844

[20] W. Magnus, et al, Formulas and Theorems for the Special Functions of Mathematical Physics, 3rd ed. Springer, 1966 Berlin