A Representation of Stable Banach Spaces

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Abstract

We show that any separable stable Banach space can be represented as a group of isometries on a separable reflexive Banach space, which extends a result of S. Guerre and M. Levy. As a consequence, we can then represent homeomorphically its space of types.

1 Introduction

Stable Banach spaces have been introduced for the first time by J.L. Krivine and B. Maurey [K.M]. In this famous paper, a rather simple condition on the norm is given to ensure, for any \( \varepsilon > 0 \) and any infinite dimensional subspace \( Y \) of a given Banach space \( X \), the existence of a further subspace \( Z \) of \( Y \) that is \((1 + \varepsilon)\) isomorphic to \( l_p \), for some \( 1 \leq p < \infty \); i.e. there exists an isomorphism \( T : l_p \rightarrow Z \) so that \( \|T\|\|T^{-1}\| \leq (1 + \varepsilon) \). This generalized the result of D. Aldous which was established when \( X = L_1 \) and \( 1 \leq p \leq 2 \). Subsequently, this class has been intensively studied by various authors. We refer the reader to the book by S. Guerre [G] for an extensive survey on stability in Banach spaces. In this note, we are interested in getting a representation for separable stable Banach spaces. In [G.L] S. Guerre and M. Levy characterized the real number \( p \) so that a subspace \( E \) of \( L_1 \) contains \( l_p \), as being the upper bound of the set of all reals \( q \) so that \( E \) embeds into \( L_q \). Actually, since this number \( p(E) \) is also the upper bound of the set of all reals \( q \) so that \( E \) is of type \( q \)-Rademacher, and B. Maurey and G. Pisier [M.P] have proved that \( l_{p(E)} \) is finitely representable in \( E \), S. Guerre and M. Levy prove that for any infinite dimensional closed subspace \( E \) of \( L_1 \), if \( l_p \) is finitely representable in \( E \), then there exist a \( q \leq p \) so that \( E \) contains a subspace isomorphic to \( l_q \). To do so, S. Guerre and M. Levy used a representation of the norm of subspaces of \( L_1 \) and then a representation of the types on these subspaces, (and on their ultraproducts) as an inner product on some Hilbert space. Our results were inspired from this proof but they don’t generalize to give a similar result for a general stable Banach space. To see this one can take for example the \( l_p \)-sum of \( l_2^n \) which is stable, has \( l_2^n \) finitely represented, but there exist no \( q < p \) such that this space contains an isomorphic copy of \( l_q \). However we do obtain some generalizations of the techniques in [G.L].
In Theorem 1, we obtain that any stable separable Banach space $X$ can be represented homeomorphically as a closed subgroup $G$, of the group of all isometries on a reflexive separable Banach space $R$, when $G$ is endowed with the strong operator topology. This is used to represent the types on $X$ in the following way: there exists a homeomorphism $T$ from $F_X$, the Alexandroff compactification of $F_X$, the space of types on $X$, endowed with topology of pointwise convergence on $X$, to $\tilde{G}$, closure for the weak operator topology of $G$ so that $T_{\sigma \star \tau} = T_{\tau \star \sigma} = T_{\sigma} \circ T_{\tau}$ for all types $\sigma$ and $\tau$ on $X$. This will be Theorem 4. We also obtain that $T$ is actually a homeomorphism between $(F_X, s)$ and $(\tilde{G}, \tau_s)$ where $s$ is the topology of uniform convergence on bounded subsets of $X$ and $\tau_s$ is the topology of pointwise convergence on the reflexive Banach space $R$. This will be Theorem 7.

It would be interesting to investigate if there is a similar representation of the norm of some non stable Banach spaces, in particular when they don’t contain $l_p$ (Tsirelson’s space, Schlumprecht’s space or Gower-Maurey’s space), and on what kind of spaces do they have such a representation, if there is any.

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## 2 Definitions and Notations:

Throughout this paper the unit ball $\{x \in X : \|x\| \leq 1\}$ of a Banach space $X$ will be denoted by $BaX$. Let $X$ be a separable Banach space. A type $\tau$ on $X$ is a function $\tau : X \rightarrow [0, \infty)$ which is realized by a sequence $(x_n) \subset X$, meaning that

$$\tau(x) = \lim_{n \to \infty} \|x + x_n\|.$$ 

The sequence $(x_n)$ is then called type determining. Note that since the space $X$ was taken to be separable the original definition in [K.M] and this one are the same for anytime $x_n$ is a bounded sequence in $X$ there exists a subsequence $y_n$ that satisfies $(\ast)$. Indeed given $x_n$ bounded in $X$ and $D$ countable dense in $X$ choose $y_n$ a subsequence of $x_n$ so that $\lim_{n \to \infty} \|x_n + d\|$ exists for all $d \in D$, by a diagonal argument. Then $y_n$ generates a unique type $\tau$.

A Banach space $X$ is called stable if whenever $(x_n)$ and $(y_n)$ are type determining then

$$\lim_{n \to \infty} \lim_{m \to \infty} \|x_n + x_m\| = \lim_{m \to \infty} \lim_{n \to \infty} \|x_n + x_m\|. $$

2
Let $\mathcal{F}_X$ be the set of all types on $X$. Several topologies can be defined on $\mathcal{F}_X$.

The weak topology $w$: this topology is induced by the metric

$$d_w(\tau, \sigma) = \sum_{k=1}^{\infty} 2^{-k} \frac{|\tau(r_k) - \sigma(r_k)|}{\|r_k\|}$$

where $\{r_k\}$ is a dense subset of $X$. Note that $\tau_n$ converges to $\tau$ for the metric $d_w$ if and only if $\tau_n$ converges to $\tau$ pointwise on $X$. Also, if $X$ is separable, the space $\mathcal{F}_X$ is locally compact and separable [K.M].

The strong topology: which is defined by

$$d_s(\tau, \sigma) = \sum_{k=1}^{\infty} 2^{-k} \sup_{\|x\| \leq k} |\tau(x) - \sigma(x)|.$$ 

If $X$ is stable separable then $(\mathcal{F}_X, d_s)$ is separable, see [O] or [R].

The uniform topology: which is defined by

$$d_u(\tau, \sigma) = \sup_{x \in X} |\tau(x) - \sigma(x)|.$$ 

If $X$ is uniformly stable (see [C]) then $(\mathcal{F}_X, d_u)$ is separable, however $\mathcal{F}_{L_1}$ [H], the space of types on $L_1$, as well as $\mathcal{F}_{Ts}$ [O] the space of types on Tsirelson’s space, are separable.

A degenerate type $\tau_x$ is a map from $X$ to $[0, \infty)$ defined by $\tau_x(y) = \|x + y\|$. Note that the space of all degenerate types is naturally homeomorphic to $X$ when endowed with the pointwise convergence topology. Usually the space of types on $X$ is viewed as the closure of the set $\{\tau_x : x \in X\}$ in the product topology of $R^X$. In case $X$ is separable, for all type $\tau$ on $X$, there exists a sequence $(x_n)$ in $X$ so that $\|x_n + y\|$ converges to $\tau(y)$ for all $y \in X$.

For $\sigma$ and $\tau$ two types on a stable separable Banach space $X$ the convolution is defined by:

$$\sigma \ast \tau(x) = \lim_{n \to \infty} \lim_{m \to \infty} \|x + x_n + y_m\|$$

where $(x_n)$ determines $\sigma$ and $(y_m)$ determines $\tau$. We would like to point out that the hypothesis of stability of $X$ is essential here, and that is the reason why these results don’t extend to general Banach spaces.

Under the stability hypothesis, the bracket of $\sigma$ and $\tau$ is defined by:

$$[\sigma, \tau] = \sigma \ast \tau(0)$$
and the norm of $\sigma$ is defined by:

$$\|\sigma\| = \sigma(0).$$

Let us now recall some facts about Lions-Peetre interpolation spaces. We refer the reader to [L.P] or [B] for an exhaustive study of these spaces:

An ordered pair $(A_0, A_1)$ of Banach spaces is an interpolation pair if these Banach spaces are given as subspaces of a common Hausdorff topological linear space $Z$ so that the embeddings of $A_0$ and $A_1$ in $Z$ are continuous. In particular if $A_0$ embeds in $A_1$ then one can take $A_1$ as the common space.

Given an interpolation pair $(A_0, A_1)$, $0 < \theta < 1$ and $1 \leq p \leq \infty$, the interpolation space $[A_0, A_1]_{\theta,p}$ is the space of all $z \in A_0 + A_1$ which admit a representation $z = x_1(t) + x_2(t)$, $t \in \mathbb{R}$ so that $e^{\theta t} x_1(t) \in L_p(A_0)$ and $e^{-(1-\theta)t} x_2(t) \in L_p(A_1)$. The norm on this space is defined by

$$\|z\|_{\theta,p} = \inf \left\{ \|e^{\theta t} x_1(t)\|^{1-\theta}_p \|e^{-(1-\theta)t} x_2(t)\|^\theta_p \right. \}$$

over all possible $z = x_1(t)+x_2(t)$ $t \in \mathbb{R}$

This norm satisfies the following important inequality which we will refer to as the interpolation inequality: For all $x \in A_0 \cap A_1$ we have

$$\|x\|_{\theta,p} \leq C(\theta,p) \|x\|_{A_0}^{1-\theta} \|x\|_{A_1}^\theta$$

(1)

where $C(\theta,p)$ is an absolute constant depending only on $\theta$ and $p$. This follows from the fact that $z = \psi(t)z + (1-\psi(t))z$ where $\psi(t)$ is so that $0 \leq \psi(t) \leq 1$, $e^{\theta t} \psi(t) \in L_p(\mathbb{R})$ and $e^{-(1-\theta)t} \psi(t) \in L_p(\mathbb{R})$.

An operator $T$ is said to be bounded from the interpolation pair $(A_0, A_1)$ to the interpolation pair $(B_0, B_1)$ if $T$ is defined on the sum $A_0 + A_1$ and is a bounded operator from $A_0$ to $B_0$ and from $A_1$ to $B_1$.

Let us now recall the important interpolation Theorem (see [L.P] or [B]):

**Interpolation Theorem:** Let $T$ be a bounded operator from the interpolation pair $A_0, A_1$ to the interpolation pair $(B_0, B_1)$. Then, for any $0 < \theta < 1$ and $1 \leq p \leq \infty$ we have that $T$ is a bounded operator from $[A_0, A_1]_{\theta,p}$ to $[B_0, B_1]_{\theta,p}$; furthermore,

$$\|T\|_{\theta,p} \leq \|T\|_{0}^{1-\theta} \|T\|_1^\theta$$

where $\|T\|_i$ $i = 0, 1$ is the norm of $T$ as an operator from $A_i$ to $B_i$.

and the density theorem: [B]

**Density Theorem:** If $0 < \theta < 1$ and $1 \leq p \leq \infty$ then $A_0 \cap A_1$ is dense in $[A_0, A_1]_{\theta,p}$.
3 Main result:

**Theorem .1** Let $X$ be a separable stable Banach space. There exists a representation $T$ from $X$ onto a closed subgroup $G$ of the group $U$ of all the isometries on a separable reflexive space $R$, where $U$ is endowed with the strong operator topology, so that $T_{x+y} = T_x \circ T_y$ and $(T_x)^{-1} = T_{-x}$.

Here the word representation means a group isomorphism that is bicontinuous when $X$ is endowed with the strong operator topology and $U$ is endowed with the strong operator topology.

To prove this theorem, we first need to establish the following proposition:

**Proposition .2** Let $X$ be a separable stable Banach space, and define an operator $\Phi : \mathcal{M}(X) \to C^b(X)$ from the space of Radon measures on $X$ to the space of bounded continuous functions on $X$ by:

$$\Phi(\mu)(y) = \int_X e^{-\|x-y\|} \mu(dx)$$

for all $\mu \in \mathcal{M}(X)$ and $y \in X$. Then the operator $\Phi$ is weakly compact.

We first establish the following lemma:

**Lemma .3** Let $F_X$ be the Alexandroff compactification of $F_X$, the space of all types on $X$; and let $f : F_X \times F_X \to \mathbb{R}$ defined by $\hat{f}(\sigma, \tau) = \exp(-[\sigma, -\tau])$ if both $\sigma$ and $\tau$ are in $F_X$, and $f(\sigma, \tau) = 0$ otherwise. Then $\hat{f}$ is separately continuous.

**Proof:** Let $f : X \times X \to \mathbb{R}$ defined by: $f(x, y) = e^{-\|x-y\|}$ for $x, y$ in $X$. It is clear that $f$ is then separately continuous and satisfies the inequalities

$$f(x, y) \leq e^{-\|x\|} e^{-\|y\|}$$

and

$$f(x, y) \leq e^\|x\| e^{-\|y\|}.$$

By density of $X$ in $F_X$, we have that $\hat{f} : F_X \times F_X \to \mathbb{R}$ defined by $\hat{f}(\sigma, \tau) = \exp(-[\sigma, -\tau])$ is separately continuous, since the bracket $[..]$ is, and satisfies the inequalities:

$$\hat{f}(\sigma, \tau) \leq \exp(-\|\sigma\|) \exp \|\tau\|$$

and

$$\hat{f}(\sigma, \tau) \leq \exp \|\sigma\| \exp (-\|\tau\|).$$

Using the last inequalities one can then extend $\hat{f}$ to $F_X \times F_Y$ by setting $\hat{f}(\sigma_\infty, \tau) = 0$ if $\sigma_\infty$ is the ”point at infinity”, $\hat{f}(\sigma, \tau_\infty) = 0$ if $\tau_\infty$ is the ”point at infinity” and $\hat{f}(\sigma_\infty, \tau_\infty) = \lim_{\tau \to \tau_\infty} \hat{f}(\sigma_\infty, \tau) = 0$. \[\square\]
We are now ready to prove Proposition 2. These ideas were already contained in [K.M] and [R], however we include the proof here for the sake of completeness.

**Proof of Proposition 2** Let \( \hat{\Phi} \) be defined on \( \mathcal{M}(F_X) \) the space of Radon measures on \( F_X \), with values in \( C(F_X) \), the space of continuous functions on \( F_X \), by

\[
\hat{\Phi}(\mu)(\tau) = \int_{F_X} \hat{f}(\sigma, \tau) \mu(d\sigma).
\]

Then the operator \( \Phi(\mu) \) is just the restriction of \( \hat{\Phi}(\mu) \) to the degenerate types. On the other hand, \( \hat{f}(\sigma, \tau) \) is a separately continuous map on a product of two compact metrizable spaces, and thus is of the first Baire class on the product \( F_X \times F_X \) [K.M]. It then follows that \( g \) is Borel on \( F_X \times F_X \), and by Fubini’s theorem, if \( \nu \in C(F_X)^* \) then \( \hat{\Phi}^*(\nu)(\sigma) = \int_{F_X} \exp(-[\sigma, -\tau]) \nu(d\tau) \) for \( \sigma \in F_X \). Using Lebesgue dominated convergence theorem, we get that \( \hat{\Phi}^*(\nu) \) is actually in \( C(F_X)^* \) i.e \( \hat{\Phi}^*: C(F_X)^* \rightarrow C(F_X) \), an equivalent way of saying that \( \Phi^* \) is a weakly compact operator. Now let \( i \) be the canonical embedding of \( \mathcal{M}(X) \), the space of Radon measures on \( X \), in \( \mathcal{M}(F_X) = C(F_X)^* \); and let \( r \) be the operator from \( C(F_X) \) to the space of bounded continuous functions \( C_b(X) \) on \( X \). Then \( \Phi = r \circ \hat{\Phi} \circ i \) and this proves the proposition. \( \blacksquare \)

**Proof of the Main Result:** Let \( \Phi : \mathcal{M}(X) \rightarrow C_b(X) \) be the operator defined in the previous proposition. By a result of W.J. Davis T. Figiel W.B. Johnson and A. Pelczynski [D.F.J.P] the weakly compact operator \( \Phi \) factors through a reflexive space \( R \). More precisely, B. Beauzamy proved in [3] that if a Banach space \( A_0 \) embeds in a Banach space \( A_1 \), then the interpolation spaces \([A_0, A_1]_{\theta,p}\) for any \( 0 < \theta < 1 \) and \( 1 < p < \infty \) are reflexive if and only if the injection of \( A_0 \) in \( A_1 \) is weakly compact. Let \( A_0 = \Phi(\mathcal{M}(X)) \) endowed with the norm defined by the gauge of the closed bounded convex symmetric set \( \Phi(Ba\mathcal{M}(X)) \), and \( A_1 = C_b(X) \) with the uniform norm. Then \( A_0 \) is a Banach space whose unit ball is \( \Phi(Ba\mathcal{M}(X)) \), is isometric to \( \mathcal{M}(X)/\text{Ker}\Phi \), and the embedding of \( A_0 \) in \( A_1 \) is weakly compact. The Banach spaces \([A_0, A_1]_{\theta,p}\) for all \( 0 < \theta < 1 \) and \( 1 < p < \infty \) are then reflexive and \( \Phi \) factors through these spaces since \( \phi = j \circ i \circ \Phi \) where \( i : A_0 \rightarrow [A_0, A_1]_{\theta,p} \) and \( j : [A_0, A_1]_{\theta,p} \rightarrow A_1 \) are the canonical injections.

For each \( z \in X \), we define a translation operator on \( C_b(X) \) by \( T_zf(x) = f(x-z) \) for all \( x \in X \). By duality we can also define an operator on \( \mathcal{M}(X) \), which we still denote \( T_z \), by

\[
<T_z\mu, f> = <\mu, T_{-z}f> \quad \text{for all } \mu \in \mathcal{M}(X) \text{ and } f \in C_b(X).
\]
We now verify that $T_z$ commutes with $\Phi$, i.e. $T_z(\Phi(\mu)) = \Phi(T_z(\mu))$ for all $\mu \in \mathcal{M}(X)$. It suffices for that to see that for all $x \in X$ and $y \in X$ we have

\[
T_z(\Phi(\delta_x))(y) = \Phi(\delta_x)(y - z) = e^{-\|x-(y-z)\|} = \Phi(\delta_{x+z})(y) = \Phi(T_z(\delta_x))(y)
\]

The last equality is easily checked since for all $f \in C_b(X)$ we have that

\[
\langle T_z \delta_x, f \rangle = \langle \delta_x, T_z f \rangle = \langle \delta_x, f + z \rangle = \langle \delta_{x+z}, f \rangle.
\]

Now the operator $T_z$ is an isometry of $C_b(X)$, and when restricted to $A_0 = \Phi(M(X))$ is an operator of norm $\|T_z\|_0 \leq 1$. Using the Interpolation Theorem stated in Section 2, $T_z$ is also an operator on $[A_0, A_1]_{\theta,p}$ and its norm satisfies

\[
\|T_z\|_{\theta,p} \leq \|T_z\|^{1-\theta}_0 \|T_z\|^\theta_1 \leq 1.
\]

The same argument applied to the operator $T_{-z}$, inverse operator of $T_z$, yields $\|T_{-z}\|_{\theta,p} \leq 1$. In other words, $T_z$ is an isometry of the reflexive space $R = [A_0, A_1]_{\theta,p}$.

Let now $G$ be the group of all the isometries $T_z$, for $z \in X$, endowed with the strong operator topology $s$. Suppose that $y_n$ converges in norm to $y$ as $n \to \infty$ and $r = \phi(\mu)$ is an arbitrary element of $A_0$. Since $T_y(\Phi(\mu)) = \Phi(T_y(\mu))$, we have that $T_{y_n}r - T_y(r) \in A_0$ and

\[
\|T_{y_n}r - T_y(r)\|_{A_0} \leq 2\|r\|_{A_0} \quad (2)
\]

On the other hand, by definition of the norm in $A_1$,

\[
\|T_{y_n}r - T_y(r)\|_{A_1} = \sup_{z \in X} |r(z - y_n) - r(z - y)| = \sup_{z \in X} \{\phi(\mu + \nu)(z - y_n) - \Phi(\mu + \nu)(z - y) : \nu \in \text{Ker} \Phi\} \leq \|\mu + \nu\| \|y_n - y\| \quad \text{for all} (\nu \in \text{Ker} \Phi)
\]

But since $A_0$ is isometric to $\mathcal{M}(X)/\text{Ker} \Phi$, we have $\|\Phi(\mu)\|_{A_0} = \inf\{\|\mu + \nu\|, \Phi(\nu) = 0\}$, and so

\[
\|T_{y_n}r - T_y(r)\|_{A_1} \leq \|r\|_{A_0} \|y_n - y\| \quad (3)
\]
Combining (2), (3) and the interpolation inequality of section 2, we get
\[ \|T_{y_n} r - T_y (r)\|_{\theta, p} \leq C(\theta, p) 2^{1-\theta} \|r\|_{A_0} \|y_n - y\|_{A_1}^\theta. \] (4)

Since the group \( G \) is equi-continuous and \( A_0 \) is dense in \([A_0, A_1]_{\theta, p}\), this inequality shows that for any \( r \in [A_0, A_1]_{\theta, p} \) we have \( T_{y_n} r \to T_y r \) as \( n \to \infty \) by Ascoli’s theorem.

We now show that \( G \) is sequentially closed. Suppose indeed that \( T_{y_n} \) converges strongly to an operator \( S \). For any \( r \in R \) the sequence \( T_{y_n} r \) is Cauchy, in particular when \( r = \Phi(\theta_0) \). Using the embedding of \( R \) in \( C_b(X) \) we get:

\[ \|T_{y_n} r - T_{y_m} r\|_R \geq C^{-1} \|T_{y_n} r - T_{y_m} r\|_\infty \]
\[ = C^{-1} \|\exp (-\|y_n - y_m\|) - \exp (-\|y_n - y_m\|)\|_\infty \]
\[ \geq |\exp (-\|y_n - y_m\|) - 1| \]

Thus \( y_n \) is Cauchy in \( X \); so letting \( y = \lim_{n \to \infty} y_n \) and using the previous arguments we get that \( T_{y_n} r \to T_y r \) for any \( r \in R \), and since also \( T_{y_n} r \to S \), we have that \( T_y = S \) i.e \( G \) is sequentially closed.

We now show that the reflexive space \( R = [A_0, A_1]_{\theta, p} \) is separable provided \( 0 < \theta < 1 \) and \( 1 < p < \infty \). Recall that \( A_1 = C_b(X) \) while \( A_0 = \Phi(M(X)) = r \circ \Phi \circ i(M(X)) \) where \( i \) is the canonical embedding of \( M(X) \), the space of Radon measures on \( X \), in \( M(F_X) \) and \( r \) the restriction to the types realized in \( X \) defined from \( C(F_X) \) to the space of bounded continuous functions \( C_b(X) \) on \( X \). Then \( i(A_0) \) is separable for the \( \mathcal{C}(F_X) \) norm, and therefore \( A_0 \) is separable for the \( C_b(X) \) norm, and let then \( a_n \) be a dense sequence in \( A_0 \) for the \( A_1 \) norm. Let \( r \in BaR \) and pick \( r_1 \in BaA_0 \) so that \( \|r - r_1\|_{\theta, p} < \varepsilon/2 \). Since \( A_0 \) embeds densely in \([A_0, A_1]_{\theta, p}\), there exists \( m \) so that \( \|r_1 - \frac{a_n}{\|a_n\|_{A_1}}\|_{\theta, p} < \varepsilon/2C(\theta, p)^{-1}2^{\theta-1} \). Using the interpolation inequality of section 2, we get that \( \|r_1 - \frac{a_n}{\|a_n\|_{A_1}}\|_{\theta, p} < \varepsilon/2 \). Therefore \( R \) is separable.

Let \( \mathcal{U} \) be the group of all isometries of \( R \). If \( (h_n)_n \) is a dense subset of \( R \), then
\[ d(S, T) = \sum_{n=1}^{\infty} 2^{-n} \|Sh_n - Th_n\|_R \] for \( S, T \in \mathcal{L}(R) \) is a distance that is uniformly equivalent on \( \mathcal{U} \) to the uniform structure of pointwise convergence on \( R \), since \( \mathcal{U} \) is equicontinuous. Therefore \( G \) is closed. Finally to see that \( T^{-1} \) is continuous, it suffices to repeat the same argument used to show that \( G \) is closed. So \( X \) is homeomorphic to a group of isometries on the reflexive space \( R \).

\[ \text{Remark} \]
If $S$ is the symmetrization operator defined on $C_b(X)$ by $Sf(x) = f(-x)$ for $x \in X$ then $S$ is an isometry of $R$ so that $ST_x = T_{-x}S$.

Indeed $S$ commutes with $\Phi$ and with the same argument used for the $T'_x$’s we can define $S$ as an isometry of $R$.

Furthermore, if $f \in R$ and $z \in X$ then $T_{-x}Sf(z) = T_{-x}f(-z) = f(-z + x) = f(-(z - x)) = ST_x f(z)$.

Let us now consider the space $\mathcal{F}_X$ of all types on a separable stable Banach space $X$. Denote by $\tilde{G}$ the closure of the group $G$ for the weak operator topology. Recall that this topology $\tau_{R \times R^*}$ is generated by the family of semi-norms $p_{r,r^*} (T) = | <Tr,r^*> |$ where $r \in R$ and $r^* \in R^*$. Our goal here is to represent $\mathcal{F}_X$ in a similar way as it is done by S. Guerre and M. Levy in [G.L].

**Theorem .4** Let $X$ be a separable stable Banach space. There exists a homeomorphism $T$ from $(\mathcal{F}_X, w)$ onto $(\tilde{G}, \tau_{R \times R^*})$ so that:

- If $\sigma_\infty$ is the "infinite type" then $T(\sigma_\infty) = 0$ and reciprocally, if $T \sigma = 0$ then $\sigma$ is the infinite type.

- If $S$ is the symmetrization operator then $S \circ T_{-\sigma} = T_{\sigma} \circ S$.

- If $\sigma$ and $\tau$ are two types on $X$ then $T_{\sigma^* \tau} = T_{\tau^* \sigma} = T_{\sigma} \circ T_{\tau}$.

**Proof:** Let $\sigma \in \mathcal{F}_X$ be a type defined by a sequence $(x_n)$ and an ultrafilter $\mathcal{V}$. Since $R$ is a separable reflexive space, the unit ball of $(\mathcal{L}(R), \tau_{R \times R^*})$ is compact metrizable, and so is $(\tilde{G}, \tau_{R \times R^*})$. Let $T_\sigma$ be the operator so that $<T_\sigma r, r^*> = \lim_n, \mathcal{V} <T_{x_n} r, r^*>$. Let $r = \Phi(\delta_y)$ and $r^* = j^*(\delta_z)$ where $j$ is the natural embedding of $[\Phi(M(X)), C_b(X)]_{\theta,p}$ in $C_b(X)$. The operator $T_\sigma$ satisfies:

$$<T_\sigma \Phi(\delta_y), j^*(\delta_z)> = <j T_\sigma \Phi(\delta_y), \delta_z>$$

$$= \lim_{n, \mathcal{V}} <j T_{x_n} \Phi(\delta_y), \delta_z>$$

$$= \lim_{n, \mathcal{V}} e^{-\|x_n + y - z\|}$$

$$= e^{-\sigma(y - z)}$$

(In case $\sigma_\infty$ is the infinite type, we have by what precedes that each $<T_{\sigma_m} \Phi(\delta_y), j^*(\delta_z)> = e^{-\sigma_m(y - z)}$ where $\sigma_m \rightarrow \sigma$.)

So to show that the operator $T_\sigma$ depends only on $\sigma$, that $T$ is continuous, and $T_{\sigma_\infty} = 0$ it suffices to show the following:
Lemma 5 Under the same hypothesis as above, the subset \( A = \{ \Phi(\delta_y) \mid y \in X \} \) (respectively \( A^* = \{ j^*(\delta_z) \mid z \in X \} \)) is total in \( R \) (respectively in \( R^* \)).

Proof: It is well known that \( \{ \delta_y, y \in X \} \) is weak-* total in \( BaM(X) \). By weak compactness of \( \Phi \), the set \( W = \Phi(BaM(X)) \) is weakly compact in \( C_b(X) \) so \( A = \{ \Phi(\delta_y) \mid y \in X \} \) is weakly total in \( W \), which is convex, so in fact \( A = \{ \Phi(\delta_y) \mid y \in X \} \) is total for the \( C_b(X) \) norm, in \( W \).

By density of \( W \) in \( BaR \), if \( \epsilon > 0 \) and \( \psi \in BaR \) there exists \( \eta \in W \) so that \( \| \psi - \eta \|_R < \epsilon / 2 \); and by totality in norm of \( A \) in \( W \) there exist \( (a_i)_{i=1}^n \) reals and \( (y_i)_{i=1}^n \) in \( X \) so that \( \| \eta - \sum_{i=1}^n a_i \Phi(\delta_{y_i}) \|_{C_b(X)} < \epsilon / 2(C(\theta,p))^{-1} \). The totality of \( A = \{ \Phi(\delta_y) \mid y \in X \} \) follows then from the interpolation inequality of Section 2, for

\[
\| \eta - \sum_{i=1}^n a_i \Phi(\delta_{y_i}) \|_{\theta,p} \leq C(\theta,p) \| \eta - \sum_{i=1}^n a_i \Phi(\delta_{y_i}) \|_{\frac{1}{\theta}C_{\theta,\varphi}(\theta_{\theta,\varphi})} \times \| \eta - \sum_{i=1}^n a_i \Phi(\delta_{y_i}) \|_{C_b(X)}. 
\]

Now for the set \( A^* = \{ j^*(\delta_z) \mid z \in X \} \), it is weak-* total in \( BaR^* \). Since \( R \) is reflexive, \( A \) is weakly total in \( BaR^* \), which is convex so \( A^* \) is in fact total in norm. This proves the lemma.

Now to see that \( T^{-1} \) is continuous, it suffices to observe that if \( T_{\sigma_n} \rightarrow T_\sigma \) for the weak operator topology then for all \( y \in X \) we have that

\[
<T_{\sigma_n} \Phi(\delta_y), j^*(\delta_0)> = e^{-\sigma_n(y)} \rightarrow e^{-\sigma(y)}.
\]

Part 2 of the Theorem follows from the identities:

\[
<ST_{-\sigma} r, r^* > = \lim_{n,v} <ST^{-x_n}r, r^* > = \lim_{n,v} <T^{-x_n}r, S^*r^* > = \lim_{n,v} <T^{-x_n}S^2r, S^*r^* > = \lim_{n,v} <ST^{-x_n}Sr, S^*r^* > = \lim_{n,v} <T^{-x_n}Sr, S^*S^*r^* > = \lim_{n,v} <T^{-x_n}Sr, r^* > = <T_\sigma Sr, r^* >
\]

Part 3 of the Theorem follows from the following lemma and the fact that \( T_\sigma T_\tau = T_{\sigma+\tau} \) in our case. We will prove the later fact after Lemma 6.
Lemma 6 Let \( G \) be an abelian group of isometries on a reflexive Banach space \( R \). Let \( S \) and \( T \) two elements of the closure \( \tilde{G} \) for the weak operator topology. Then \( ST = TS \).

Proof: For all \( r \in R \) and \( r^* \in R^* \) we easily have:

\[
< ST r, r^* > = < Tr, S^* r^* >
\]

\[
= \lim_{\alpha, \beta} < T_{\alpha} r, S^* r^* >
\]

\[
= \lim_{\alpha, \beta} < ST_{\alpha} r, r^* >
\]

\[
= \lim_{\alpha, \beta} \lim_{\gamma} < S_{\beta} T_{\alpha} r, r^* >
\]

\[
= \lim_{\alpha, \beta} < T_{\alpha} S_{\beta} r, r^* >
\]

\[
= \lim_{\alpha, \beta} \lim_{\gamma} < S_{\beta} T_{\alpha} r, r^* >
\]

\[
= \lim_{\alpha, \beta} < T_{\alpha} S_{\beta} r, r^* >
\]

\[
= \lim_{\beta, \gamma} < TS_{\beta} r, r^* >
\]

\[
= \lim_{\beta, \gamma} < S_{\beta} r, T^* r^* >
\]

\[
= \lim_{\beta, \gamma} < TS r, T^* r^* >
\]

This proves Lemma 6. To see that \( T_{\sigma} T_{\tau} = T_{\sigma \tau} \) let \( r = \Phi(\delta_y) \) and \( r^* = j^*(\delta_z) \) where \( j \) is the natural embedding of \( [\Phi(M(X)), C_b(X)]_{\theta, \rho} \) in \( C_b(X) \). Let also \( x_n \) and \( y_m \) be determining sequences for the types \( \sigma \) and \( \tau \) respectively. We then have:

\[
< T_{\sigma} T_{\tau} r, r^* > = < T_{\sigma} r, (T_{\sigma})^* r^* >
\]

\[
= \lim_{m, \beta} < T_{y_m} r, (T_{\sigma})^* r^* >
\]

\[
= \lim_{m, \beta} < T_{\sigma} T_{y_m} r, r^* >
\]

\[
= \lim_{m, \beta} \lim_{n, \alpha} < T_{\alpha} T_{x_n} y_m r, r^* >
\]

\[
= \lim_{m, \beta} < T_{x_n + y_m} r, r^* >
\]

\[
= \lim_{n, \beta} < j T_{x_n + y_m} \Phi(\delta_y), \delta_z >
\]
\[
\lim \lim_{n, k, m, v} \exp -\|x_n + y_m + y - z\| \\
= \exp -\sigma * \tau(y - z) \\
= <T_{\sigma * \tau} \Phi(\delta_y), j^* (\delta_z)>
\]

The fact mentioned above follows then from Lemma 5.

Let us now turn to comparing the strong topology on types and the strong operator topology \(\tau_s\) on the representation group \(G\). Let us recall that Y. Raynaud proved in \[R\] that the topology \(s\) of uniform convergence on bounded subsets of \(X\) on the space of types \(F_X\) is separable provided the space \(X\) is stable separable.

**Theorem 7** Let \(X\) be a separable stable Banach space and let \(T\) be the representation of \(F_X\). If \((\sigma_n)\) is a sequence of types in \(F_X\) then \((\sigma_n)\) converges for the topology \(s\) to a type \(\sigma\) if and only if \(T_{\sigma_n}\) converges strongly to \(T_{\sigma}\).

**Proof:** Suppose that \(\sigma_n \rightarrow \sigma\) for the topology \(s\). This means that for any positive real \(M\) we have that \(d_M(\sigma_n, \sigma) = \sup_{\|x\| \leq M} |\sigma_n(x) - \sigma(x)| \rightarrow 0\) as \(n \rightarrow \infty\). Consider now the contractions \(T_{\sigma_n}\) and \(T_{\sigma}\). Since we have seen before that the family \(\Phi(\delta_x)\) was total in \(X\), it suffices to show that \(T_{\sigma_n} \Phi(\delta_x) = T_{\sigma_n} T_x \Phi(\delta_0) = T_x T_{\sigma_n} \Phi(\delta_0)\) converges in \(R\), i.e. if \(U_x = \Phi(\delta_x)\) then \(T_{\sigma_n} U_0\) converges to \(T_{\sigma} U_0\) in \(R\). Now each \(T_{\sigma_n}\) and \(T_{\sigma}\) are weak limits of elements in \(\Phi(M(X))\), which is convex, so its weak closure is the same as its closure for the norm defined by its gauge. Therefore \(\|T_{\sigma_n} U_0 - T_{\sigma} U_0\|_{A_0} \leq 2\). On the other hand,

\[
\|T_{\sigma_n} U_0 - T_{\sigma} U_0\|_{C_{\mu}(X)} = \sup_{y \in X} |(T_{\sigma_n} U_0 - T_{\sigma} U_0)(y)| \\
= \sup_{y \in X} |<j(T_{\sigma_n} U_0 - T_{\sigma} U_0), \delta_y>| \\
= \sup_{y \in X} |<T_{\sigma_n} U_0 - T_{\sigma} U_0, j^*(\delta_y)>| \\
= \sup_{y \in X} |e^{-\sigma_n(-y)} - e^{-\sigma(-y)}|
\]

But the last quantity tends to zero as \(n\) tends to infinity since on the bounded sets of \(X\) that is the convergence in the \(s\) sense, while outside the bounded sets, this is a property of the exponential function. We then conclude the proof of one direction using the interpolation inequality

\[
\|T_{\sigma_n} U_0 - T_{\sigma} U_0\|_R \leq C(\theta, p)2^{1-\theta}\|T_{\sigma_n} U_0 - T_{\sigma} U_0\|^\theta_{C_{\mu}(X)}.
\]
For the converse, if $T_{\sigma_n} \longrightarrow T_\sigma$ in $R$ as $n \longrightarrow \infty$, then $j(T_{\sigma_n}U_0) \longrightarrow j(T_\sigma U_0)$ in $C_b(X)$. But for that norm we clearly have:

$$
\|j(T_{\sigma_n}U_0) \longrightarrow j(T_\sigma U_0)\|_{C_b(X)} \geq |<T_{\sigma_n}U_0 - T_\sigma U_0, \delta_{-x}>|
$$

$$
= |e^{-\sigma_n(x)} - e^{-\sigma(x)}|
$$

$$
= |e^{-\sigma(x)}[e^{-\sigma_n(x)} + \sigma(x) - 1]|
$$

$$
\geq e^{-M - \|\sigma\|} |e^{-\sigma_n(x)} + \sigma(x) - 1|
$$

if we suppose that $\|x\| \leq M$. Thus $d_M(\sigma_n, \sigma) = \sup_{\|x\| \leq M} |\sigma_n(x) - \sigma(x)| \longrightarrow 0$ as $n \longrightarrow \infty$.

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