Elementary solution of an infinite sequence of instances of the Hurwitz problem

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Abstract

We prove that there exists no branched cover from the torus to the sphere with degree 3h and 3 branching points in the target with local degrees (3,\ldots,3), (3,\ldots,3), (4,2,3,\ldots,3) at their preimages. The result was already established by Izmestiev, Kusner, Rote, Springborn, and Sullivan, using geometric techniques, and by Corvaja and Zannier with a more algebraic approach, whereas our proof is topological and completely elementary: besides the definitions, it only uses the fact that on the torus a simple closed curve can only be trivial (in homology, or equivalently bounding a disc, or equivalently separating) or non-trivial.

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A (topological) branched cover between surfaces is a map $f: \tilde{\Sigma} \to \Sigma$, where $\tilde{\Sigma}$ and $\Sigma$ are closed and connected 2-manifolds and $f$ is locally modeled (in a topological sense) on maps of the form $(\mathbb{C},0) \ni z \mapsto z^k \in (\mathbb{C},0)$. If $k > 1$ the point 0 in the target $\mathbb{C}$ is called a branching point, and $k$ is called the local degree at the point 0 in the source $\mathbb{C}$. There are finitely many branching points, removing which, together with their pre-images, one gets a genuine cover of some degree $d$. If there are $n$ branching points, the local degrees at the points in the pre-image of the $j$-th one form a partition $\pi_j$ of $d$ of some length $\ell_j$, and the following Riemann-Hurwitz relation holds:

$$
\chi(\tilde{\Sigma}) - (\ell_1 + \ldots + \ell_n) = d(\chi(\Sigma) - n).
$$

The very old Hurwitz problem asks whether given $\tilde{\Sigma}, \Sigma, d, n, \pi_1, \ldots, \pi_n$ satisfying this relation there exists some $f$ realizing them. (For a non-orientable $\tilde{\Sigma}$ and/or $\Sigma$ the Riemann-Hurwitz relation must actually be complemented with certain other necessary conditions, but we will not get into this here.) A
number of partial solutions of the Hurwitz problem have been obtained over the time, and we quickly mention here the fundamental \cite{4}, the survey \cite{10}, and the more recent \cite{7,8,2,9,11}.

Certain instances of the Hurwitz problem recently emerged in the work of M. Zieve \cite{12} and his team of collaborators, including in particular the case where the source surface is the torus $T^2$, the target is the sphere $S^2$, the degree is $d = 3h$, and there are $n = 3$ branching points with associated partitions $(3, \ldots, 3)$, $(3, \ldots, 3)$, $(4, 2, 3, \ldots, 3)$ of $d$. It actually turns out that this branch datum is indeed not realizable, as Zieve had conjectured, which follows from results established in \cite{6} using geometric techniques (holonomy of Euclidean structures). The same fact was also elegantly proved by Corvaja and Zannier \cite{3} with a more algebraic approach. In this note we provide yet another proof of the same result. Our approach is purely combinatorial and completely elementary: besides the definitions, it only uses the fact that on the torus a simple closed curve can only be trivial (in homology, or equivalently bounding a disc, or equivalently separating) or non-trivial.

We conclude this introduction with the formal statement of the (previously known) result established in this note:

\textbf{Theorem.} There exists no branched cover $f : T^2 \to S^2$ with degree $d = 3h$ and 3 branching points with associated partitions

$$(3, \ldots, 3), \ (3, \ldots, 3), \ (4, 2, 3, \ldots, 3).$$

1 \ Dessins d’enfant

In this section we quickly review the beautiful technique of dessins d’enfant due to Grothendieck \cite{1,5}, noting that, at the elementary level at which we exploit it, it only requires the definition of branched cover and some very basic topology.

Let $f : \tilde{\Sigma} \to S^2$ be a degree-$d$ branched cover from a closed connected surface $\Sigma$ to the sphere $S^2$, branched over 3 points $p_1, p_2, p_3$ with local degrees $\pi_j = (d_{ji})_{i=1}^{\ell_j}$ over $p_j$. In $S^2$ take a simple arc $\sigma$ with vertices at $p_1$ (white) and $p_2$ (black), and we view $S^2$ as being obtained from the (closed) bigon $\tilde{B}$ of Fig. 1-left by attaching both the edges of $\tilde{B}$ to $\sigma$ so to match the vertex colors. This gives a realization of $S^2$ as the quotient of $\tilde{B}$ under the identification of its two edges. Let $\lambda : \tilde{B} \to S^2$ be the projection to the quotient. Note that the complement of $\sigma$ in $S^2$ is an open disc $B$, whose closure in $S^2$ is the whole of $S^2$, but the restriction of $\lambda$ to the interior
Now set \( D = f^{-1}(\sigma) \). Then \( D \) is a graph with white vertices of valences \( (d_1i)_{i=1}^{\ell_1} \) and black vertices of valences \( (d_2i)_{i=1}^{\ell_2} \), and \( D \) is bipartite (every edge has a white and a black end). Moreover the complement of \( D \) in \( \bar{\Sigma} \) is a union of open discs \( (R_i)_{i=1}^{\ell_3} \), where \( R_i \) is the interior of a polygon with \( 2d_{3i} \) vertices of alternating white and black color. This means that, if \( \bar{R}_i \) is the polygon of Fig. 1-right (with \( 2d_{3i} \) vertices), there exists a map \( \lambda_i : \bar{R}_i \to \bar{\Sigma} \) which restricted to the interior of \( \bar{R}_i \) is a homeomorphism with \( R_i \), and restricted to each edge is a homeomorphism with an edge of \( D \) matching the vertex colors. So \( \bar{R}_i \) can be viewed as the abstract closure of \( R_i \). The map \( \lambda_i \) may fail to be a homeomorphism between \( \bar{R}_i \) and the closure of \( R_i \) in \( \bar{\Sigma} \) if \( R_i \) is multiply incident to some vertex of \( D \) or doubly incident to some edge of \( D \). We say that \( R_i \) has embedded closure if \( \lambda_i \) is injective, hence a homeomorphism between \( \bar{R}_i \) and the closure of \( R_i \) in \( \bar{\Sigma} \).

We will way that a bipartite graph \( D \) in \( \bar{\Sigma} \) with valences \( (d_1i)_{i=1}^{\ell_1} \) at the white vertices and \( (d_2i)_{i=1}^{\ell_2} \) at the black ones, and complement consisting of polygons having \( (2d_{3i})_{i=1}^{\ell_3} \) edges, realizes the branched cover \( f : \bar{\Sigma} \to S^2 \) with 3 branching points and local degrees \( \pi_1, \pi_2, \pi_3 \) over them. This terminology is justified by the fact that \( f \) exists if and only if \( D \) does.

2 Proof of the Theorem

Suppose by contradiction that a branched cover \( f : T^2 \to S^2 \) as in the statement exists, and let \( D \) be a dessin d’enfant on \( T^2 \) realizing it, as explained in the previous section, with white and black vertices corresponding to the first two partitions, so the complementary regions are one square \( S \), some hexagons \( H \) and one octagon \( O \), shown abstractly in Fig. 2. Let \( \bar{D} \) be the graph dual to \( D \) (which is well-defined because the complement of \( D \) is a union of open discs), and let \( \Gamma \) be the set of all simple loops in \( \bar{D} \) which are

\[
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{A bigon and a polygon.}
\end{figure}
\]
The notation $V, a, b, c,$ is only needed for the base step of our induction argument.

A tube of $H$’s cannot end at an $O$ from both sides. A non-embedded $O$ gives a non-embedded $H$ simplicial (concatenations of edges), and non-trivial (non-zero in $H_1(T^2)$), or, equivalently, not bounding a disc on $T^2$, or, equivalently, not separating $T^2$). Since $T^2 \setminus \hat{D}$ is also a union of open discs, the inclusion $\hat{D} \hookrightarrow T^2$ induces a surjection $H_1(\hat{D}) \to H_1(T^2)$. Moreover $H_1(\hat{D})$ is generated by simple simplicial loops, so $\Gamma$ is non-empty. We now define $\Gamma_n$ as the set of loops in $\Gamma$ consisting of $n$ edges, and we prove by induction that $\Gamma_n = \emptyset$, thereby showing that $\Gamma = \emptyset$ and getting the desired contradiction.

For $n = 1$ we prove the slightly stronger fact (needed below) that every region has embedded closure, namely, that its closure in $T^2$ is homeomorphic to its abstract closure. Taking into account the symmetries (including a color switch) this may fail to happen only if some edge $a$ in Fig. 2 is glued to $b$ or $c$ of the same region (if two vertices of a region are glued together then two edges also are, since the vertices have valence 3). The case $b = a$ implies $V$ has valence 1, so it is impossible. If $c = a$ in $H$ we have the situation of Fig. 3-left, and each of the neighboring regions already has 3 vertices of one color, so it cannot be $S$. If it is an $H$, it also has a gluing of type $c = a$. Iterating, we have a tube of $H$’s as in Fig. 3-centre that at some point must hit $O$ from both sides, which is impossible because the terminal region already contains 5 vertices of each color. If $c = a$ in $O$ then we have Fig. 3-right, so a neighboring region also has non-embedded closure, which was already excluded.

Let us now assume that $n \geq 2$ and $\Gamma_m = \emptyset$ for all $m < n$. By contradiction, take $\gamma \in \Gamma_n$. From now on in our figures we will use for $\gamma$
Figure 4: The ways $\gamma$ can cross a region. Note that the vertex colors may be switched.

Figure 5: Impossible configurations.

a thicker line than that used for $D$. We first note that $\gamma$ cannot enter a region through an edge and leave it from an adjacent edge (otherwise we could reduce its length), so the only ways $\gamma$ can cross a region are those shown in Fig. 4. Therefore $\gamma$ is described by a word in the letters $S, H_d, H_\ell, H_r, O_d, O_\ell, O_L, O_r, O_R$, from which we omit the $H_d$'s for simplicity. The vertex coloring implies that the total number of $S, H_\ell, H_r, O_d, O_L, O_R$ in $\gamma$ is even.

We now prove that any subword $H_rH_\ell, H_rH_\ell$, $SH_\ell$, or $SH_\ell$ is impossible in $\gamma$, as shown in Fig. 5 (here the thick dashed line gives a new $\gamma$ contradicting the minimality of the original one). This already implies the former of the following claims:

1. No $\gamma \in \Gamma_n$ can contain $S$ but not $O$;
2. There exists $\gamma \in \Gamma_n$ consisting of $H$’s only.

To establish the latter, we suppose $O \in \gamma \in \Gamma_n$ and list all the possible cases up to symmetry (which includes switching colors and/or reversing the direction of $\gamma$):
Figure 6: \( \gamma = O_d H_r (H_l H_r)^p \Rightarrow \exists \gamma' \ldots \) (left); \( \gamma = O_r (H_l H_r)^p \Rightarrow \exists \gamma' \ldots \) (centre for \( p > 0 \) and right for \( p = 0 \)). On the right, as in many figures below, we decorate some edges to indicate that they are glued in pairs.

For each of these cases we show in Figg. 6 to 9 a modification of \( \gamma \) which gives a new loop \( \gamma' \) isotopic to \( \gamma \) (and hence in \( \Gamma \)), and not longer than \( \gamma \). When \( \gamma' \) is shorter than \( \gamma \) we have a contradiction to the minimality of \( \gamma \), so the case is impossible. To conclude we must show that \( \gamma' \) does not contain \( O \) in the cases where it is as long as \( \gamma \). To do this, suppose that \( \gamma' \) contains \( O \), and construct two loops \( \gamma_1, \gamma_2 \) by applying one of the three moves of Fig. 10. Note that whatever move applies, \( \gamma \) is the homological sum of \( \gamma_1 \) and \( \gamma_2 \), so at least one of them is non-trivial. If one of the moves of Fig. 10-left/centre applies, the total length of \( \gamma_1 \) and \( \gamma_2 \) is 1 plus the length of \( \gamma \), but we know that there is no length-1 loop at all (trivial or not), so both \( \gamma_1 \) and \( \gamma_2 \) are shorter than \( \gamma \), a contradiction. If only the move of Fig. 10-right applies then we are either in Fig. 7-right, or Fig. 8-left or Fig. 8-right and \( O \) is the region where \( \gamma' \) makes a left turn; in this case the total length of \( \gamma_1 \) and \( \gamma_2 \) is 2 plus the length of \( \gamma \), but \( \gamma_1 \) and \( \gamma_2 \) both have length at least 3, so they are both shorter than \( \gamma \), and again we have a contradiction.

Our next claim is the following:

(3) There exists \( \gamma \in \Gamma_n \) described by a word \( (H_l H_r)^p \) with \( p \leq 1 \).

By (2) and the fact that subwords \( H_l H_l \) or \( H_r H_r \) are impossible in \( \gamma \in \Gamma_n \), we have a \( \gamma \in \Gamma_n \) described by a word \( (H_l H_r)^p \). Now suppose \( p \geq 2 \), consider a portion of \( \gamma \) described by \( H_r H_l H_r H_l \) as in Fig. 11-left and try to construct
Figure 7: $\gamma = O_R H_r (H_l H_r)^p$ impossible (left); $\gamma = O_R H_l (H_r H_l)^p \Rightarrow \exists \gamma'$ ... (right).

Figure 8: $\gamma = SO_d (H_r H_l)^p \Rightarrow \exists \gamma'$ ... (left); $\gamma = H_r SO_r (H_r H_l)^p$ impossible (centre); $\gamma = H_l SO_r (H_l H_r)^p \Rightarrow \exists \gamma'$ ... (right).
Figure 9: $SO_R \in \gamma$ impossible: if in $\gamma$ there are $m$ copies of $H$ outside the word $SO_R$, we treat separately the cases $m \geq 2$ (left), $m = 1$ (centre), and $m = 0$ (right). In the last case, if there are not even $H_d$’s between $S$ and $O_R$, the absurd comes from the fact that a bigon is created.

Figure 10: If $O$ appears in $\gamma$ and immediately to the right of $\gamma$, we can construct two loops $\gamma_{1,2}$ of which $\gamma$ is the homological sum.
the two loops \( \gamma_\ell \) and \( \gamma_r \) as in Fig. 11-centre by repeated application of the moves in Fig. 11-right. If one of \( \gamma_\ell \) or \( \gamma_r \) exists it belongs to \( \Gamma_n \) and it is described by \( (H_\ell H_r)^{p-1} \), so we can conclude recursively. The construction of \( \gamma_\ell \) or \( \gamma_r \) may fail only if when we apply an elementary move as in Fig. 11-right to \( \alpha \in \Gamma_n \) the region \( B \) is...

- the square \( S \); this would contradict (1), so it is impossible;
- already in \( \alpha \); but then \( B \) is not one of \( A_1, A_2, A_3 \) because all regions are embedded, and it easily follows that \( \alpha \) is homologous to the sum of two shorter loops, which is absurd because at least one of them would be non-trivial;
- the octagon \( O \); this is indeed possible, but it cannot happen both to the left and to the right, otherwise we would get a simplicial loop in \( \widehat{D} \) intersecting \( \gamma \) transversely at one point, whence non-trivial, and shorter than \( \gamma \) (actually, already at least by 1 shorter than the portion of \( \gamma \) described by \( H_r H_\ell H_r H_\ell \)).

We now include again the \( H_d \)'s in the notation for the word describing a loop. It follows from (3) that there exists \( \gamma \in \Gamma_n \) of shape \( H_d^q \) or \( H_\ell H_d^q H_r H_d^t \).

To conclude the proof we set \( \gamma_\ell = \gamma_r = \gamma \) and we apply to \( \gamma_\ell \) and \( \gamma_r \) as long as possible the following moves (that we describe for \( \gamma_\ell \) only):

Figure 11: Reducing the number of turns.
Figure 12: Evolution of $\gamma_{\ell}$. The second move is performed so that $O$ is not included in the new loop.

- If $O$ is not incident to the left margin of $\gamma_{\ell}$ we entirely push $\gamma_{\ell}$ to its left, as in Fig. 12-left;

- If $\gamma$ has shape $H_{\ell}H_{d}^{q}H_{r}H_{d}^{r}$ and $O$ is incident to the left margin of $\gamma_{\ell}$ but not to $H_{\ell}$, we note that $O$ is not incident to either $H_{d}^{r}H_{\ell}$ or to $H_{d}^{q}H_{r}$, and we partially push $\gamma_{\ell}$ to its left so not to include $O$, as in Fig. 12-right (this is the case where $O$ is not incident to $H_{d}^{q}H_{\ell}$).

Note that by construction the new $\gamma_{\ell}$ does not contain $O$, so it also does not contain $S$ by (1), hence it has the same shape $H_{d}^{q}$ or $H_{\ell}H_{d}^{q}H_{r}H_{d}^{r}$ as the old $\gamma_{\ell}$. Therefore at any time $\gamma_{\ell}$ and $\gamma_{r}$ have the same shape as the original $\gamma$. We stop applying the moves when one of the following situations is reached:

(a) The left margin of $\gamma_{\ell}$ and the right margin of $\gamma_{r}$ overlap;

(b) $\gamma_{\ell}$ and $\gamma_{r}$ have shape $H_{d}^{q}$ and $O$ is incident to the left margin of $\gamma_{\ell}$ and to the right margin of $\gamma_{r}$;

(c) $\gamma_{\ell}$ and $\gamma_{r}$ have shape $H_{\ell}H_{d}^{q}H_{r}H_{d}^{r}$ and $O$ is incident to the left margin of $\gamma_{\ell}$ in $H_{\ell}$ and to the right margin of $\gamma_{r}$ in $H_{r}$.

Case (a) with $\gamma_{\ell}$ and $\gamma_{r}$ of shape $H_{d}^{q}$ is impossible, because the left margin of $\gamma_{\ell}$ and the right margin of $\gamma_{r}$ would close up like a zip, leaving no space for $S$ and $O$, see Fig. 13-left. We postpone the treatment of case (a) with $\gamma_{\ell}$ and $\gamma_{r}$ of shape $H_{\ell}H_{d}^{q}H_{r}H_{d}^{r}$, to face the easier cases (b) and (c). For (b), we have the situation of Fig. 13-right, where in the direction given by the
arrow we must have a strip of identical hexagons that can never close up. Case (c), excluding (a), is trivial: the region that should be $O$ cannot close up with fewer than 10 vertices, see Fig. 14 top/left.

In case (a) for the shape $H_\ell H_d H_r H_t$, the left margin of $\gamma_\ell$ can overlap with the right margin of $\gamma_r$ only along a segment as in Fig. 14 top/right —this segment has type $H_r H_d^2$ in $\gamma_\ell$ and $H_d^2 H_\ell$ in $\gamma_r$, in particular it uses $H_r$ from $\gamma_\ell$ and $H_\ell$ from $\gamma_r$, so there is only one. Therefore the rest of $\gamma_\ell$ and $\gamma_r$ delimit an $x \times y$ rhombic area $R$ as in Fig. 14 bottom/left (with $x \times y = 3 \times 4$ in the figure), that must contain $S$ and $O$. Note that the $H$’s incident to $\partial R$ are pairwise distinct: for the initial $\gamma$ the left margin cannot be incident to the right margin, otherwise a move as in Fig. 10 left/centre would contradict its minimality, and during the construction of $\gamma_\ell$ and $\gamma_r$ only new $H$’s are added. If $O$ is not incident to one of the four sides of $R$ we can modify $\gamma_\ell$ or $\gamma_r$ as suggested already in Fig. 14 bottom/left. This modification changes the shape of $\gamma_\ell$ or $\gamma_r$, but:

- The modified loop is still minimal and does not contain $O$, so it does not contain $S$;
- The area $R$ into which $O$ and $S$ are forced to lie remains a rhombus,
- The $H$’s incident to $\partial R$ are pairwise distinct (otherwise $R$ closes up leaving no space for $O$ or $S$).

We can iterate this modification, shrinking $R$ until $O$ is incident to all the four sides of $\partial R$. If $R$ is $1 \times 1$ of course there is space in $R$ only for an $H$. If $R$ is $1 \times y$ or $x \times 1$ with $x, y \geq 2$, the fact that the $H$’s incident to $\partial R$ are distinct implies that the vertices of $\partial R$ are distinct, so a region incident to all the four sides or $\partial R$ must have at least 10 vertices. If $R$ is $x \times y$ with $x, y \geq 2$ then $O$ contains some of the germs of regions $O^{(s)}$ in Fig. 14 bottom/right so as to touch all the $r/t/\ell/b$ sides of $\partial R$. An easy analysis shows that any identification between two vertices of the $O^{(s)}$’s would force two $H$’s incident to $\partial R$ to coincide, so it is impossible. This implies that any...
Figure 14: Conclusion for the shape $H, H_d^2, H, H_d^t$. 
$O^{(*)}$ actually contained in $O$ contributes to the number of vertices of $O$ with as many vertices as one sees in Fig. [14] bottom/right, namely 3 for $O^{(r)}$, $O^{(t)}$, $O^{(ℓ)}$, $O^{(b)}$, then 4 for $O^{(tℓ)}$, $O^{(br)}$, and finally 5 for $O^{(tr)}$, $O^{(bℓ)}$. Therefore, a region can touch all of $r/t/ℓ/b$ with a total of no more that 8 vertices only if it includes $O^{(tℓ)}$ and $O^{(br)}$, but then the vertex colors again imply that the number of vertices is at least 10. This gives the final contradiction and concludes the proof.

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