On The Motive of Certain Fibre Bundles and Some Applications

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December 3, 2013

Abstract

In this article we first compute the motive associated to a cellular fibration $\Gamma$ over a smooth scheme $X$ inside Voevodsky’s motivic categories. We implement this result to study the motive associated to a $G$-bundle, and additionally to study motives of varieties admitting a resolution of singularities by a tower of cellular fibrations (e.g. Schubert varieties in a twisted affine flag variety).

keywords: mixed Tate motives, cellular fibrations, principal $G$-bundles, wonderful compactification, Schubert varieties in twisted affine flag varieties

MSC(2000) 14F42, 14C25, 14D99, 20G15

Introduction

A fundamental result of V. Voevodsky states that the (higher) Chow groups of a quasi-projective variety $X$ over a perfect field $k$, of any characteristic, can be viewed as the motivic cohomologies of $X$ [34]. As a consequence of this result, A. Huber and B. Kahn established a decomposition of the pure Tate motive $M(Y)$, associated to a smooth variety $Y$, in terms of its motivic fundamental invariants (see Remark [2,3]). Regarding this we prove a motivic version of Leray-Hirsch theorem for a cellular fibration in section 2 Theorem 2.10.

Theorem 0.1. Let $X$ be a smooth irreducible variety over a perfect field $k$. Let $\pi : \Gamma \rightarrow X$ be a proper smooth locally trivial (for Zariski topology) fibration with fiber $F$. Furthermore assume that $F$ is cellular and satisfies Poincaré duality. Then one has a decomposition

$$M(\Gamma) \cong \prod_{p \geq 0} CH_p(F) \otimes M(X)(p)[2p]$$

in $DM^{eff}_{gm}(k)$.

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We implement the above result, together with the motivic version of the decomposition theorem, due to Corti and Hanamura \cite{9} and de Cataldo and Migliorini \cite{11}, to study the motive associated to a variety $X$ which admits a special kind of resolution of singularities $\bar{X}$. Namely, we assume that the resolution $\bar{X}$ can be constructed as a tower of cellular fibrations, see Theorem \ref{2.11}. In particular, one can use this approach to study the motive associated to a Schubert variety $S_\omega$ in a twisted partial affine flag variety. The Schubert varieties in a twisted partial affine flag variety were first introduced by G. Pappas and M. Rapoport \cite{26} and later received much attention due to their significance in the theory of local models for Shimura varieties, see \cite{16}, \cite{27} and \cite{31}.

Consequently at the end of Section 2 we show that

**Corollary 0.2.** The motive $M(S_\omega)$ of a minuscule Schubert variety $S_\omega$ (in a twisted affine flag variety) is mixed Tate.

Note that when char $k = 0$, one can observe that under conjectures of Grothendieck and Murre (see Remark \ref{2.12} the above statement holds for a general Schubert variety in a twisted affine flag variety.

Finally using Theorem \ref{0.1} and also the combinatorial tools provided by the theory of wonderful compactification of semi-simple algebraic groups of adjoint type (cf. \cite{12} and \cite{30}), we study motives of $G$-bundles, for a reductive group $G$ over a perfect field $k$.

Let $X$ be a smooth irreducible variety over $k$ and $G$ a $G$-bundle over $X$. One might naturally conjecture that the restriction of the motive $M(G)$ to $k^{alg}$ lies in the smallest thick tensor subcategory $\langle TDM_{gm}^{eff}(k^{alg}), M(X) \rangle$ of $DM_{gm}^{eff}(k^{alg})$, containing the category of mixed Tate motives $TDM_{gm}^{eff}(k^{alg})$ over $k^{alg}$ and the motive associated to $X$. In the following theorem we verify this expectation for some special cases.

**Theorem 0.3.** Let $X$ be a smooth irreducible variety over a perfect field $k$. The restriction of $M(G)$ to $k^{alg}$ lies in $\langle TDM_{gm}^{eff}(k^{alg}), M(X) \rangle$, in either of the following cases

a) $X$ is geometrically cellular,

b) $G$ is Zariski locally trivial over $X$ and the center $Z(G_{k^{alg}})$ of $G_{k^{alg}}$ is connected,

c) $X$ is a smooth projective curve $C$ over $k$, char $k$ does not divide the order of the fundamental group $\pi_1(G)$ and $Z(G_{k^{alg}})$ is connected.

The organization of the article is as follows. In section 1 we fix notation and conventions. In section 2 we introduce the notion of motivic (relative) cellular varieties. We show that for the motive of such a variety one has a decomposition
similar to the decomposition of the Chow motive of a relative cellular variety. Furthermore, we establish a motivic version of Leray-Hirsch theorem for cellular fibrations and then we use this theorem to study motives of varieties admitting a resolution of singularities by a tower of cellular fibrations. In Section 3 we recall some results about the geometry of wonderful compactification of a reductive group of adjoint type. Subsequently we see that the closure of a $G \times G$-orbit is (motivic) cellular. Using this result and the motivic version of Leray-Hirsch theorem we study the case when $\text{char } k = 0$ and $X$ is geometrically mixed Tate (see Proposition 3.5). We also consider the case when $\text{char } k$ is arbitrary and $X$ is geometrically cellular, proving part a) of Theorem 0.3 (see Proposition 3.7). Finally in Section 4 we discuss the case that the base scheme $X$ is not necessarily mixed Tate. We produce a nested filtration on the motive associated to $\mathcal{G}$ and consequently we prove part b) and c) of Theorem 0.3.

Acknowledgement. We are grateful to L. Barbieri Viale and B. Kahn for their helps and comments on the earlier draft of this article. We thank L. Migliorini for mentioning us an inaccuracy in the proof of Proposition 3.5 in a previous version of this article. We also thank C. V. Anghel, J. Bagalkote and J. Scholbach for editing and helpful discussions regarding this work.

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1 Notation and Conventions

Throughout this article we assume that $k$ is a perfect field. We denote by $\text{Sch}_k$ (resp. $\text{Sm}_k$) the category of schemes (resp. smooth schemes) of finite type over $k$.

For $X$ in $\text{Ob}(\text{Sch}_k)$, let $CH_i(X)$ and $CH^i(X)$ denote Fulton’s $i$-th Chow groups and let $CH_*(X) := \oplus_i CH_i(X)$ (resp. $CH^*(X) := \oplus_i CH^i(X)$).

We denote by $\text{Sch}^{fr}_k$ (resp. $\text{Sm}^{fr}_k$) the full subcategory of $\text{Sch}_k$ (resp. $\text{Sm}_k$) consisting of those $X \in \text{Ob}(\text{Sch}_k)$ (resp. $X \in \text{Ob}(\text{Sm}_k)$) such that $CH_*(X)$ is free of finite rank over $\mathbb{Z}$.
Remark 1.1. The category $Sm_{fr}^k$ need not to be closed under fibre product, even this is not clear after passing to the coefficients in $\mathbb{Q}$, i.e. whether the full subcategory of $Sm_k$ consisting of objects $X$ with $rk_QK_0(X) < \infty$ is closed under fibre product or not. Recall that this will be implied if one assumes the Bass conjecture.

To denote the motivic categories over $k$, such as

$$DM_{gm}(k), \ DM_{gm}^{eff}(k), \ DM_{gm}^{eff}(k), \ DM_{gm}^{eff}(k) \otimes \mathbb{Q},$$

and the functors $M : Sch_k \to DM_{gm}^{eff}(k)$ and $M^c : Sch_k \to DM_{gm}^{eff}(k)$, constructed by Voevodsky, we use the same notation that was introduced by him in [35].

Note that the above constructions have been developed by Cisinski and Deglise [6]; and also Voevodsky [36], where they construct the triangulated category of motives over a general base scheme $S$.

For the definition of the geometric motives with compact support in positive characteristic we also refer to [17, Appendix B].

We simply use $A \to B \to C$ to denote a distinguished triangle

$$A \to B \to C \to A[1],$$

in either of the above categories. Moreover for any object $M$ of $DM_{gm}(k)$ we denote by $M^*$ the internal Hom-object $\text{Hom}_{DM_{gm}}(M, \mathbb{Z})$.

Definition 1.2. The thick subcategory of $DM_{gm}^{eff}(k)$, generated by $\mathbb{Z}(0)$ and the Tate object $\mathbb{Z}(1)$ is called the category of mixed Tate motives and we denote it by $TDM_{gm}^{eff}(k)$. Any object of $TDM_{gm}^{eff}(k)$ is called a mixed Tate motive. A motive $M$ is geometrically mixed Tate if it becomes mixed Tate over $k_{alg}$.

Definition 1.3. An object of $DM_{gm}(k)$ is called pure Tate if it is a (finite) direct sum of copies of $\mathbb{Z}(p)[2p]$ for $p \in \mathbb{Z}$.

CAUTION: Throughout this article we either assume that $k$ admits resolution of singularities or we consider the motivic categories after passing to coefficients in $\mathbb{Q}$.

Let us now move to the algebraic group theory side.

Let $G$ be a connected reductive linear algebraic group over $k$. Suppose that $G$ is split, fix a maximal torus $T$ and a Borel subgroup $B$ that contains $T$. 

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We denote by $G^s$ the semi-simple quotient of $G$ and by $G^{ad}$ the adjoint group of $G$.

Let $X^*(T)$ (resp. $X_*(T)$) denote the group of cocharacters (resp. characters) of $G$. Let $\Phi := \Phi(T,G)$ be the associated root system and $\Delta \subseteq \Phi(T,G)$ be a system of simple roots (i.e. a subset of $\Phi$ which form a basis for $\text{Lie}(G)$ such that any root $\beta \in \Phi$ can be represented as a sum $\beta = \sum_{\alpha \in \Delta} m_\alpha \alpha$, with $m_\alpha$ all non-negative or all non-positive integral coefficients). Let $W := W(T,G)$ and $l : W \rightarrow \mathbb{Z}_+$ denote the corresponding Weyl group and the usual length function on $W$, respectively. For any subset $I \subseteq \Delta$ we set $\Phi_I$ to be the subset of $\Phi$ spanned by $I$. Furthermore the subgroups of the Weyl group $W$ generated by the reflections associated with the elements of $\Phi_I$. Let $W^I$ denote a set of representative for $W/W_I$ with minimal length.

Let $Y$ be a variety with left $G$-action. To a $G$-bundle $\mathcal{G}$ on $X$ one associates a fibration $\mathcal{G} \times^G Y$ with fibre $Y$ over $X$, defined by the following quotient

$$\mathcal{G} \times Y / \sim,$$

where $(x, y) \sim (xg, g^{-1}y)$ for every $g \in G$.

## 2 Motive of cellular fibrations

In this section we first introduce a class of varieties which we call motivic relatively cellular varieties. Notice that this notion is slightly weaker than the geometric notion of relatively cellular introduced by Chernousov, Gille, Merkurjev [8] and also Karpenco [19].

**Definition 2.1.** A scheme $X \in \text{Ob}(\mathcal{S}ch_k)$ is called **motivic relatively cellular** (with respect to the functor $M^c(-)$) if it admits a filtration by its closed subschemes:

$$\emptyset = X_{-1} \subset X_0 \subset ... \subset X_n = X$$

together with flat equidimensional morphisms $p_i : U_i := X_i \setminus X_{i-1} \rightarrow Y_i$ of relative dimension $d_i$, such that the induced morphisms $p_i^* : M^c(Y_i)(d_i)[2d_i] \rightarrow M(U_i)$ are isomorphisms in $DM^e_{gm}(\mathbb{k})$. Here $Y_i$ is smooth proper scheme for all $1 \leq i \leq n$. Moreover we say that $X$ is cellular if $p_i$ is affine bundle and $Y_i = \text{Spec} \mathbb{k}$ for $0 \leq i \leq n$.

**Proposition 2.2.** Suppose $k$ admits resolution of singularities. Assume that $X \in \text{Ob}(\mathcal{S}ch_k)$ is equidimensional of dimension $n$, which admits a filtration as in
Then we have the following decomposition

\[ M^c(X) = \bigoplus_i M^c(Y_i)(d_i)[2d_i]. \]

Proof. We prove by induction on \( \dim X \). Consider the following distinguished triangle

\[ M^c(X_{j-1}) \to M^c(X_j) \xrightarrow{g_j} M^c(U_j) \to M^c(X_{j-1})[1]. \]

Take the closure of the graph of \( p_j : U_j \to Y_j \) in \( X_j \times Y_j \). This defines a cycle in \( CH_{\dim X_j}(X_j \times Y_j) \) and since \( Y_j \) is smooth this gives a morphism

\[ \gamma_j : M^c(Y_j)(d_j)[2d_j] \to M^c(X_j), \]

by [35, Chap. 5, Thm. 4.2.2.3] and Prop. 4.2.3, such that \( g_j \circ \gamma_j = p_j^* \). Thus the above distinguished triangle splits. Now we conclude by induction hypothesis.

Corollary 2.3. Keep the notation and the assumptions of the above proposition. Assume that each \( Y_i \) belongs to \( \text{Ob}(\text{Sm}^{fr}_k) \), then \( X \in \text{Ob}(\text{Sch}^{fr}_k) \).

Proof. Apply the functor \( \text{Hom}(\mathbb{Z}(i)[2i], -) \) to the decomposition

\[ M^c(X) = \bigoplus_i M^c(Y_i)(d_i)[2d_i], \]

which we obtained in the above proposition. The above corollary now follows from [25, Prop. 19.18].

Remark 2.4. Note that one can define a variant of the definition 2.1 with respect to the functor \( M(-) \). In this case one has to replace \( p_i^* \) by \( p_i \). Furthermore it is not necessary to assume that \( p_i \)'s are flat. With this definition it is not hard to see that a variant of the Proposition 2.2 holds after imposing some additional condition. Namely, to apply Gysin triangle (see proof of Proposition 2.2) we have to assume that all \( X_i \)'s appearing in the filtration of \( X \) are smooth. Note that in this case we don’t need to assume \( k \) admits resolution of singularities. The proof goes similar to the proof of Proposition 2.2.

Remark 2.5. Assume that \( X \) is a motivic relatively cellular scheme, such that \( Y_i \) is pure Tate for every \( 1 \leq i \leq n \). Then using noetherian induction and Gysin triangle one can show that \( X \) is pure Tate.

Let \( Ab \) be the category of abelian groups. Let us recall that there is a fully faithful tensor triangulated functor

\[ i : D^b_f(Ab) \to D\text{M}^{eff}_{gm}(k), \]

where \( D^b_f(Ab) \) is the full subcategory of the bounded derived category \( D^b(Ab) \), consisting of objects with finitely generated cohomology groups, see [17, Prop. 4.5].
Proposition 2.6. Assume that $k$ admits resolution of singularities. For a cellular variety $X \in \text{Ob}(\text{Sch}_k)$, there is a canonical isomorphism

$$\prod_{p \geq 0} CH_p(X) \otimes \mathbb{Z}(p)[2p] \to M^c(X),$$

which is functorial both with respect to proper or flat morphisms.

Proof. C.f. [18] Prop. 3.4. \qed

Proposition 2.7. Let $X \in \text{Ob}(\text{Sm}_k)$ and assume further that it is equidimensional. Then there is a natural isomorphism in $\text{DM}^\text{eff}_{gm}(k)$:

$$\prod_{p \geq 0} CH^p(X)^\vee \otimes \mathbb{Z}(p)[2p] \to M(X),$$

where $CH^p(X)^\vee$ denotes the dual $\mathbb{Z}$-module.

Proof. C.f. [18] Cor. 3.5. \qed

Remark 2.8. More generally in [17] Prop. 4.10 Huber and Kahn show that for a smooth variety $X$ over $k$, if the associated motive $M(X)$ is pure Tate then there is a natural decomposition $M(X) \cong \bigoplus_p c_p(X)(p)[2p]$, in terms of the corresponding motivic fundamental invariants $c_p(X)$.

The most well-known class of cellular varieties consists of generalized flag varieties. Let us state the following easy consequence, obtained by applying the above proposition to this particular case.

Corollary 2.9. Let $G$ be a split reductive group over a perfect field $k$, and let $P$ be a parabolic subgroup of $G$ which is conjugate with a standard parabolic subgroup $P_I$. Then there is an isomorphism

$$M(G/P) \cong \prod_{w \in W^I} \mathbb{Z}(l(w))[2l(w)],$$

in particular $G/P$ is pure Tate.

Proof. The decomposition $G = \coprod_{w \in W^I} BwP$ induces a cell decomposition $G/P \cong G/P_I = \coprod_{w \in W^I} X_w$, where $X_w \cong \mathbb{A}^{l(w)}$. The cycles $[X_w]$ form a set of generators for the free module $CH_*^c(G/P)$. Thus we may conclude by properness of $G/P$ and Proposition 2.6. \qed

We now want to compute the motive associated to a fiber bundle. Recall that the naive version of Leray-Hirsch theorem does not even hold for the Chow functor. One way to tackle the problem in the algebraic set-up is to impose some stronger conditions on the fiber. For instance one has to assume that the
fiber admits cell decomposition and satisfies Poincaré duality (i.e. the degree map $CH_0(F) \to \mathbb{Z}$ is an isomorphism and the intersection pairings $CH_p(F) \otimes CH^p(F) \to CH_0(F)$ are perfect pairings).

Let $f : \Gamma \to X$ be a smooth proper morphism that is locally trivial for Zariski topology, with fiber $F$ which satisfies the above conditions. Let also $\zeta_1, \ldots, \zeta_m$ be homogeneous elements of $CH^*(X)$ whose restriction to any fiber form a basis of its Chow group over $\mathbb{Z}$. Then the Leray-Hirsch theorem for Chow groups states that the homomorphism

$$\varphi : \bigoplus_{i=1}^m CH_*(X) \to CH_*(\Gamma), \quad \varphi(\oplus \alpha_i) = \Sigma \zeta_i \cap f^* \alpha_i$$

is an isomorphism. When $X$ is non-singular, it means that $\zeta_i$ form a free basis for $CH^*(\Gamma)$ as a $CH^*(X)$-module. For the proof we refer to [7, appendix C].

Let us now state the motivic version of the Leray-Hirsch theorem.

**Theorem 2.10.** Let $X$ be a smooth irreducible variety over $k$. Let $\pi : \Gamma \to X$ be a proper smooth locally trivial (for Zariski topology) fibration with fiber $F$. Furthermore assume that $F$ is cellular and satisfies Poincaré duality. Then one has an isomorphism

$$M(\Gamma) \cong \bigoplus_{p \geq 0} CH_p(F) \otimes M(X)(p)[2p]$$

in $DM_{gir}(k)$.

**Proof.** Take a set of homogeneous elements $\{\zeta_{i,p}\}_{i,p}$ of $CH^*(\Gamma)$ such that for any $p$ the restrictions of $\{\zeta_{i,p}\}_i$ to any fiber $\Gamma_x \cong F$ form a basis for $CH^p(\Gamma_x)$. Notice that since $X$ is irreducible, it is enough that the restrictions of the $\zeta_{i,p}$’s generate $CH_*(\Gamma_x)$ for the fiber over a particular $x$.

By [25] Theorem 14.16 and Theorem 19.1, for each $i$, $\zeta_{i,p}$ defines a morphism $M(\Gamma) \to Z(p)[2p]$. Summing up all these morphisms and taking dual, by Poincaré duality we get the following morphism

$$\varphi : M(\Gamma) \to \bigoplus_p CH_p(F) \otimes Z(p)[2p].$$

Composing $M(\Delta) : M(\Gamma) \to M(\Gamma \times \Gamma) \cong M(\Gamma) \otimes M(\Gamma)$, which is induced by the diagonal map $\Delta : \Gamma \times \Gamma \to \Gamma$, with $M(\pi) \otimes \varphi$, we obtain

$$M(\Gamma) \to \bigoplus_p CH^p(F) \otimes M(X)(p)[2p].$$

Now take a covering $\{U_i\}$ of $X$ that trivializes $\Gamma$. The restriction of this global morphism to $U_j$ is induced by the restriction of $\zeta_{i,p}$s to $U_j$. The same holds over intersections, i.e. these morphisms fit together when we pass to $U_j \cap U_k$. Thus by
Mayer-Vietoris triangle (see [35, Chapter 5, (4.1.1)]) we may reduce to the case that \( \Gamma \) is a trivial fibration \( X \times_k F \). This precisely follows from Künneth formula [35, Prop. 4.1.7] and Proposition 2.7 above.

Let us now state the first application which we propose in this article of the above theorem. Let \( \tilde{X} \) be a variety over a perfect field \( k \). Suppose that \( \tilde{X} \) sits in a tower

\[
\begin{array}{c}
\tilde{X}_n := \tilde{X} \\
\uparrow \\
\tilde{X}_{n-1} \\
\vdots \\
\downarrow \\
\tilde{X}_0
\end{array}
\]

where \( \tilde{X}_i \rightarrow \tilde{X}_{i-1} \) is a proper smooth locally trivial fibration with fibre \( F_i \). Suppose in addition that \( F_i \) is cellular and satisfies Poincaré duality. We call such a variety a tower variety over \( \tilde{X}_0 \).

**Theorem 2.11.** Let \( f : \tilde{X} \rightarrow X \) be a surjective semismall morphism. Further assume that \( \tilde{X} \) is a tower variety over a smooth proper scheme \( \tilde{X}_0 \). Then the motive \( M(X) \) is a summand of

\[
\bigotimes_{i=0}^{n-1} \left( \prod_{p \geq 0} CH_p(F_i) \otimes \mathbb{Z}(p)[2p] \right) \otimes M(\tilde{X}_0)
\]

in \( DM_{gm}(k) \otimes \mathbb{Q} \).

**Proof.** This follows from the motivic version of the decomposition theorem [11, Thm. 2.3.8], Theorem 2.10 and the embedding theorem [35, Chap. 5, Prop. 2.1.4].

**Remark 2.12.** Assuming conjectures of Grothendieck and Murre, see [9, Paragraph 3.6], Corti and Hanamura prove that the decomposition theorem holds in the category of relative Chow motives with rational coefficients \( CHM_{S, \mathbb{Q}} \). In this case one may drop the semismallness from the hypotheses of the above corollary.

Let \( G \) be a connected reductive algebraic group over the local field \( K := k((z)) \) of Laurent series with algebraically closed residue field \( k \) and ring of integers \( \mathcal{O}_K = k[[z]] \).
Recall that to a connected reductive group $G$ over a local field $K$, Bruhat and Tits associate a building $B(G)$. Moreover any maximal split torus $S$ defines an apartment $A := A(G, S)$ which is called the reduced apartment of $G$ associated with $S$. For any $F \in A(G, S)$ let $P_F$ denote the corresponding parahoric group scheme (see [4]). Let $\mathcal{F}_{\ell P_F}$ denote the twisted affine flag variety associated to $G$ and $F$. For an element $\omega$ of the Iwahori-Weyl group $\tilde{W}$, let $S_\omega$ denote the associated Schubert variety in $\mathcal{F}_{\ell P_F}$, see [26, Sec. 8].

**Corollary 2.13.** Assume that $S_\omega$ is minuscule (or assume that $\text{char } k = 0$ and the conjectures of Grothendieck and Murre hold, see remark 2.12). Then $M(S_\omega)$ is mixed Tate.

**Proof.** The source of the Bott-Samelson-Demazure resolution $m : \Sigma \to S_\omega$, constructed in [31, Cor. 3.5], is an iterated extension of homogeneous varieties [31, Rmk 2.9]. Therefore by Theorem 2.10 the motive $M(\Sigma)$ is pure Tate. Note that generalized flag varieties satisfy Poincaré duality (see [21]). Now the above corollary follows from Theorem 2.11 (and Remark 2.12).

**3 Motive of a $G$-bundle**

In this section we study motives of $G$-bundles over a base scheme which is (geometrically) cellular, or more generally, over a base scheme which is (geometrically) mixed Tate. Let us first recall the following result of A. Huber and B. Kahn.

**Proposition 3.1.** An object $M \in \mathcal{D}M_{gm}^{eff}(k)$ is geometrically mixed Tate if and only if there is a finite separable extension $E$ of $k$ such that the restriction of $M$ to $\mathcal{D}M_{gm}^{eff}(E)$ is mixed Tate.

**Proof.** c.f. [17, Prop. 5.3].

**Definition 3.2.** Let $X \in \text{Ob}(\text{Sch}_k)$. We say that $X$ is mixed Tate if the associated motive $M_c(X)$ is an object of the subcategory of mixed Tate motives $\mathcal{T}DM_{gm}^{eff}(k)$. Let $\{X_i\}_{i=1}^n$ be the set of irreducible components of $X$. We call $X$ a configuration of mixed Tate varieties if

i) $X_i$ is mixed Tate for $1 \leq i \leq n$, and

ii) the union of the elements of any arbitrary subset of $\{X_{ij} := X_i \cap X_j\}_{i \neq j}$ is a configuration of mixed Tate varieties or is empty.

**Lemma 3.3.** The motive of a configuration of mixed Tate varieties is mixed Tate.
Proof. We prove by induction on $r$, the dimension of the mixed Tate configuration. The statement is obvious for $r = 0$. Suppose that the lemma holds for all mixed Tate configurations of dimension $r < m$. Let $X = X_1 \cup \cdots \cup X_n$ be a configuration of mixed Tate varieties of dimension $m$, here $X_i$ denote an irreducible component of $X$. For inclusion $\bigcup_{i \neq j} X_{ij} \subset \bigcup_{i=1}^{n} X_i$, we have the following induced localization distinguished triangle

$$M^c(\bigcup_{i \neq j} X_{ij}) \to M^c(X_1 \cup \cdots \cup X_n) \to M^c(\bigcup_{i=1}^{n} X_i \setminus \bigcup_{i \neq j} X_{ij}) \to M^c(\bigcup_{i \neq j} X_{ij})[1].$$

By the induction assumption, $M^c(\bigcup_{i \neq j} X_{ij})$ is mixed Tate. On the other hand we have:

$$M^c(\bigcup_{i=1}^{n} (X_i \setminus \bigcup_{i \neq j} X_{ij})) = \bigoplus_{i=1}^{n} M^c(X_i \setminus \bigcup_{i \neq j} X_{ij}).$$

It only remains to show that for every $i$, $M^c(X_i \setminus \bigcup_{i \neq j} X_{ij})$ is mixed Tate. To see this, for a given $i$, consider the following distinguished triangle

$$M^c(\bigcup_{j \neq i} X_{ij}) \to M^c(X_i) \to M^c(X_i \setminus \bigcup_{j \neq i} X_{ij}) \to M^c(\bigcup_{j \neq i} X_{ij})[1].$$

Notice that $M^c(\bigcup_{i \neq j} X_{ij})$ is mixed Tate by induction hypothesis.

In the sequel, we use the theory of wonderful compactification of semi-simple algebraic groups of adjoint type to relate motive of a $G$-bundle to the motives associated to certain cellular fiber bundles (see section 2).

Wonderful Compactification
In [10] De Concini and Procesi have introduced the wonderful compactification of a symmetric space. In particular their method produces a smooth canonic compactification $\overline{G}$ of an algebraic group $G$ of adjoint type. Note that in [10] they study the case that the group $G$ is defined over $\mathbb{C}$. Nevertheless most of the theory carries over for any algebraically closed field of arbitrary characteristic. However there are some subtleties in positive characteristic which we mention later.

As a feature of this compactification there is a natural $G \times G$-action on $\overline{G}$, and the arrangement of the orbits can be explained by the associated weight polytope. Let us briefly recall some facts about the construction of $\overline{G}$ and the geometry of its $G \times G$-orbits and their closure.

Let $\rho_\lambda : G \to GL(V_\lambda)$ be an irreducible faithful representation of $G$ with strictly dominant highest weight $\lambda$. One defines the compactification $X_\lambda$ of $G$ as the closure $\overline{P(\rho_\lambda(G))}$ (in $\overline{P(End(V_\lambda))}$) of the projectivization $\overline{P(\rho_\lambda(G))}$ of $\rho_\lambda(G)$.

It is verified in [12] that when $G$ is of adjoint type, $X_\lambda$ is smooth and independent of the choice of the highest weight. This compactification is called wonderful
compactification. In this section we denote the wonderful compactification of $G$ by $\overline{G}$.

The following proposition explains the geometry of the wonderful compactification and the closures of its $G \times G$-orbits. Furthermore this provides an effective method to compute their cohomologies. Before stating this proposition let us fix some notation. Consider the correspondence between polytopes and fans, which associates to a polytope its normal fan. Let $\mathcal{P}_C$ denote the polytope associated to the fan of Weyl chamber and its faces.

**Proposition 3.4.** Keep the above notation, we have the following statements:

a) There is a one-to-one correspondence between the $G \times G$-orbits of $\overline{G}$ and the orbits of the action of the Weyl group on the faces of the polytope $\mathcal{P}_C$, which preserves the incidence relation among orbits (i.e. for two faces $\mathcal{F}_1, \mathcal{F}_2$ of the polytope $\mathcal{P}_C$, if $\mathcal{F}_1 \subseteq \mathcal{F}_2$ then the orbit corresponding to the face $\mathcal{F}_1$ is contained in the closure of the orbit corresponding to $\mathcal{F}_2$).

b) Let $I \subset \Delta$ and $\mathcal{F} = \mathcal{F}_I$ the associated face of $\mathcal{P}_C$. Let $D_{\mathcal{F}}$ denote the closure of the orbit corresponding to the face $\mathcal{F}$. Then $D_{\mathcal{F}} = \sqcup_{\alpha \in W \times W} C_{\mathcal{F}, \alpha}$, such that for each $\alpha := (u, v)$ there is a bijective morphism

$$A^{n_{\mathcal{F}, \alpha}} \to C_{\mathcal{F}, \alpha},$$

where $n_{\mathcal{F}, \alpha} = l(w_0) - l(u) + |I \cap I_u| + l(v)$ and $w_0$ denotes the longest element of the Weyl group. In particular when char $k = 0$ (resp. char $> 0$) $D_{\mathcal{F}}$ is cellular (resp. motivic cellular).

c) $\overline{G} \setminus G$ is a normal crossing divisor, and its irreducible components form a mixed Tate configuration.

**Proof.** For the proof of a) we refer to [33, Prop. 8]. The existence of the bijective morphism in part b) is the main result of Renner in [30]. The fact that $D_{\mathcal{F}}$ is cellular in characteristic zero follows from Zariski main theorem. In positive characteristic this follows from the fact that any universal topological homeomorphism induces isomorphism of the associated h-sheaves, see [37, Prop. 3.2.5]. Finally c) follows from a), b) and Remark 2.5.

**Proposition 3.5.** Assume that char $k = 0$. Let $G$ be a connected reductive group over $k$ with connected center $Z(G)$. Let $\mathcal{G}$ be a $G$-bundle over an irreducible variety $X \in \operatorname{Ob}(\text{Sm}_k)$. Suppose that $\mathcal{G}$ is locally trivial for Zariski topology and $X$ is geometrically mixed Tate, then $M(\mathcal{G})$ is also geometrically mixed Tate.

**Proof.** We may assume that the base field $k$ is algebraically closed. Let us first assume that $G$ is a semisimple group of adjoint type. Then $G$ admits a
wonderful compactification $\overline{G}$ which is smooth. By construction, there is $(G \times G)$-action on $\overline{G}$. Consider the $G$-fibration $\overline{G} := G \times G \to X$ (here $G$ acts on $\overline{G}$ via the embedding $G \hookrightarrow G \times e \subseteq G \times_k G$). Consider the following generalized Gysin distinguished triangle

$$M(\overline{G}) \to M(G) \to M_c(\overline{G} \setminus G)^*(n)[2n] \to M(\overline{G})[1],$$

corresponding to the open immersion $G \hookrightarrow \overline{G}$ (see [33 page 197]), where $n := \dim G + \dim X$.

By Proposition [3.4] $\overline{G}$ admits a cell decomposition and therefore by Theorem [2.10] $M_c(\overline{G})$ is mixed Tate. Hence it suffices to show that $M_c(\overline{G} \setminus G)$ is mixed Tate.

Let’s now look at the geometry of the closures of $(G \times G)$-orbits. As it is mentioned in Proposition [3.4 a), these orbit closures could be indexed by a subset of faces of Weyl chamber, in such a way that the incidence relation between faces gets preserved. Note that by Proposition [3.4 b) the closure of these orbits also admit a cell decomposition. Thus by Theorem [2.10] the irreducible components of $\overline{G} \setminus G$ form a mixed Tate configuration. Now Lemma [3.3] implies that $M_c(\overline{G} \setminus G)$ is mixed Tate.

Now let us assume that $G$ is a reductive algebraic group. Then $Z := Z(G)$ is a split torus. Let $G'$ denote the $G^{ad}$-bundle associated with $G$. As we have shown above the motive $M(G')$ is mixed Tate. Notice that any torus bundle is locally trivial for Zariski topology by Hilbert’s Theorem 90. Take a toric compactification $\overline{Z}$ of $Z$ and embed $G$ into $Z := G \times \overline{Z}$, which is a toric fibration over $G'$. The irreducible components of the complement of $G$ in $Z$ are toric fibrations over $G'$. Since fibers are toric (and hence cellular) and $M(G')$ is mixed Tate, by Theorem [2.10] we may argue that these irreducible components form a mixed Tate configuration and hence we conclude as above.

In the above proposition, the assumption that the $G$-bundle $G$ is locally trivial for Zariski topology may look restrictive, nevertheless as we will see below this assumption is not necessary when $X$ is geometrically cellular. Before proving this let us state the following lemma.

**Lemma 3.6.** Let $G$ be a connected reductive group over $k$, then the motive associated to $G$ is geometrically mixed Tate. Furthermore if $G$ is a split reductive group then $M(G)$ is mixed Tate.

**Proof.** For the first statement we may assume that $k$ is algebraically closed. Let $T$ be a maximal split torus in $G$ of rank $r$. We view $G$ as a $T$-bundle over $G/T$ under the projection $\pi : G \to G/T$, and $\mathbb{P}^r$ as a compactification of $T$. Let $\overline{T} := G \times T \mathbb{P}^r$ be the associated projective bundle over $G/T$. By projective bundle formula

$$M(\overline{T}) = M(\mathbb{P}^r) \otimes M(G/T),$$
see [25, Thm. 15.12]. On the other hand $B = T \ltimes U$, where $B$ is a Borel subgroup of $G$ containing $T$ and $U$ is the unipotent part of $B$. Notice that, as a variety, $U$ is isomorphic to an affine space over $k$. Since the fibration $G \to G/B$ is the composition of $G \to G/T$ and $U$-fibration $G/T \to G/B$, we deduce by Corollary 2.9 that $M(T)$ is pure Tate. As in the last paragraph of the proof of Proposition 3.5 one can embed $G$ into $T$ over $G/T$ and verify that the irreducible components of its complement form a mixed Tate configuration. The second part of the lemma is similar, only since $G$ is split one doesn’t need to pass to an algebraic closure. 

**Proposition 3.7.** Let $G$ be a connected reductive group over $k$. Let $G$ be a $G$-bundle over an irreducible variety $X \in \mathcal{Ob}(Sm_k)$. Suppose in addition that $X$ is geometrically cellular. Then $M(G)$ is geometrically mixed Tate.

**Proof.** We may assume that $k$ is algebraically closed. Let

$$\emptyset = X_{-1} \subset X_0 \subset \ldots \subset X_n = X$$

be a cell decomposition for $X$, where $U_i := X_i \setminus X_{i-1}$ is isomorphic to $k^{d_i}$. We prove by induction on $n$. Consider the following Gysin distinguished triangle:

$$M(G|_{U_n}) \to M(G) \to M(G|_{X_{n-1}}).$$

By Raghunathan’s Theorem [29] (this theorem is in fact a generalized version of the well-known conjecture of Serre on triviality of vector bundles over an affine space), the restriction of $G$ to $U_n$ is trivial. Therefore $M(G|_{U_n})$ is mixed Tate by Lemma 3.6 and Künneth theorem [35, Prop. 4.1.7]. On the other hand $M(G|_{X_{n-1}})$ is mixed Tate by induction hypothesis. 

### 4 Filtration on the motive of a G-bundle

Recall that in section 3 we studied the motive associated to a $G$-bundle over a base scheme $X$ whose motive $M(X)$ is geometrically mixed Tate. In the sequel, we produce a nested filtration (in terms of incidence relations between faces of a convex body) on the motive of a $G$-bundle over a more general base scheme.

Let us first recall the following filtration of the motive of a torus bundle in $DM_{gm}^{eff}(k)$, constructed by A. Huber and B. Kahn [17]. In fact, this filtration is constructed as an application of the theory of slice filtration. They in particular use this filtration to study motive of a split reductive group, see [17, Lem. 9.1]. Let us briefly recall their construction.

Let $T$ be a split torus of rank $r$ and let $T$ be a $T$-bundle over a scheme $X \in \mathcal{Ob}(Sm_k)$. Let $\Xi := Hom(\mathbb{G}_m, T)$ denote the cocharacter group. Then one has the following diagram of distinguished triangles in $DM_{gm}^{eff}(k)$
where \( \lambda_p(X, T) := M(X)(p)[p] \otimes \Lambda^p(\Omega) \) for \( 0 \leq p \leq r \). Note that \( M(T) \cong \nu_X^{p=0} M(T) \) and \( \nu_X^{p=r+1} M(T) = 0 \).

For more details on the construction of relative slice filters \( \nu_X^{p} M(T) \) see [17, Sec. 8].

Now, following the approach we introduced in section 3, we want to explain how to construct a filtration on the motive of a \( G \)-bundle.

Let \( G \) be a \( G \)-bundle over \( X \), where \( G \) is a linear algebraic group. Let \( \overline{G} \) be a compactification of \( G \). Suppose that the irreducible components of \( D := \overline{G} \setminus G \) form a mixed Tate configuration \( D = \bigcup_{i=1}^m D_i \), such that \( D^J := \cap_{i \in J} D_i \) is either irreducible or empty for any \( J \subseteq \{1, \ldots, m\} \). We assume that there exist a polytope whose faces correspond to those subsets \( J \) of \( \{1, \ldots, m\} \) such that \( D^J \) is non-empty (with face relation \( F_2 \) is a face of \( F_1 \) if we have the inclusion \( J_2 \subseteq J_1 \) of the corresponding sets). Let \( P \) be the dual of this polytope. For each face \( F \) of \( P \), we denote by \( D_F \) the associated subvariety of \( D \), regarding the above correspondence. Furthermore we set \( D_P := \overline{G} \). For each \( 1 \leq r \leq m \), let \( Q_r \) be the set consisting of all faces in \( P \) of codimension \( r \). Let \( \partial F \) denote the boundary of \( F \), i.e. the set \( \{ F \cap F' | F' \in Q_1 \} \setminus \{ F \} \).

Let \( G \) denote the compactification \( G \times \mathbb{G} \) of \( G \) and let \( D_F \) be the associated \( D_F \)-fibration over \( X \). Furthermore set \( D_P := \overline{G} \). We may now form the following nested filtration on \( M^c(G) \) by distinguished triangles, indexed by codimension \( r \) and faces \( F \in Q_r \)

\[
M^c(\bigcup_{F \in Q_1} D_F) = \overline{G} \setminus G \to M^c(D_P = \overline{G}) \to M^c(D_P \setminus \bigcup_{F \in Q_1} D_F = G)
\]

\[
\vdots
\]

\[
M^c(\bigcup_{F \in Q_{r+1}} D_F) \to M^c(\bigcup_{F \in Q_r} D_F) \to \bigoplus_{F \in Q_r} M^c(D_F \setminus \bigcup_{F \in \partial F} D_F),
\]

\[
\vdots
\]

and for each \( F \in Q_r \) the triangle

\[
M^c(\bigcup_{F \in \partial F} D_F) \to M^c(D_F) \to M^c(D_F \setminus \bigcup_{F \in \partial F} D_F),
\]

where \( \lambda_p(X, T) := M(X)(p)[p] \otimes \Lambda^p(\Omega) \) for \( 0 \leq p \leq r \). Note that \( M(T) \cong \nu_X^{p=0} M(T) \) and \( \nu_X^{p=r+1} M(T) = 0 \).

For more details on the construction of relative slice filters \( \nu_X^{p} M(T) \) see [17, Sec. 8].

Now, following the approach we introduced in section 3, we want to explain how to construct a filtration on the motive of a \( G \)-bundle.

Let \( G \) be a \( G \)-bundle over \( X \), where \( G \) is a linear algebraic group. Let \( \overline{G} \) be a compactification of \( G \). Suppose that the irreducible components of \( D := \overline{G} \setminus G \) form a mixed Tate configuration \( D = \bigcup_{i=1}^m D_i \), such that \( D^J := \cap_{i \in J} D_i \) is either irreducible or empty for any \( J \subseteq \{1, \ldots, m\} \). We assume that there exist a polytope whose faces correspond to those subsets \( J \) of \( \{1, \ldots, m\} \) such that \( D^J \) is non-empty (with face relation \( F_2 \) is a face of \( F_1 \) if we have the inclusion \( J_2 \subseteq J_1 \) of the corresponding sets). Let \( P \) be the dual of this polytope. For each face \( F \) of \( P \), we denote by \( D_F \) the associated subvariety of \( D \), regarding the above correspondence. Furthermore we set \( D_P := \overline{G} \). For each \( 1 \leq r \leq m \), let \( Q_r \) be the set consisting of all faces in \( P \) of codimension \( r \). Let \( \partial F \) denote the boundary of \( F \), i.e. the set \( \{ F \cap F' | F' \in Q_1 \} \setminus \{ F \} \).

Let \( G \) denote the compactification \( G \times \mathbb{G} \) of \( G \) and let \( D_F \) be the associated \( D_F \)-fibration over \( X \). Furthermore set \( D_P := \overline{G} \). We may now form the following nested filtration on \( M^c(G) \) by distinguished triangles, indexed by codimension \( r \) and faces \( F \in Q_r \)

\[
M^c(\bigcup_{F \in Q_1} D_F) = \overline{G} \setminus G \to M^c(D_P = \overline{G}) \to M^c(D_P \setminus \bigcup_{F \in Q_1} D_F = G)
\]

\[
\vdots
\]

\[
M^c(\bigcup_{F \in Q_{r+1}} D_F) \to M^c(\bigcup_{F \in Q_r} D_F) \to \bigoplus_{F \in Q_r} M^c(D_F \setminus \bigcup_{F \in \partial F} D_F),
\]

\[
\vdots
\]

and for each \( F \in Q_r \) the triangle

\[
M^c(\bigcup_{F \in \partial F} D_F) \to M^c(D_F) \to M^c(D_F \setminus \bigcup_{F \in \partial F} D_F),
\]
is the first line of a nested filtration obtained by replacing $P$ by $F$.

Note that this filtration is particularly interesting when $\mathcal{D}_F$ is a cellular fibration. In this situation we may apply Theorem 2.10 to compute $M^c(\mathcal{D}_F)$. Let us consider the following two cases.

**Example 4.1.** Let $T$ be a split torus of rank $r$ as above and $\mathcal{T}$ be a $T$-bundle over $X$. Consider the projective space $\mathbb{P}^r$ as a toric compactification of $T$ corresponding to the standard $r$-simplex $\Delta^r$. So we have $\mathcal{P} = \Delta^r$. Note that in this case for each face $\mathcal{F} \in \Delta^r$, $\mathcal{D}_F$ is in fact a projective bundle over $X$. Hence one may use the projective bundle formula [25, Thm. 15.11] to compute $M^c(\mathcal{D}_F)$. In particular when $M^c(X)$ is mixed Tate, using the above filtration, one may prove recursively that $M^c(T)$ is mixed Tate.

**Example 4.2.** Let $G$ be a semi-simple group of adjoint type and $\overline{G}$ its wonderful compactification. In this case the polytope $\mathcal{P}$ coincide with the one in Proposition 3.4. Note that for each face $\mathcal{F}$ of $\mathcal{P}$, $\mathcal{D}_F$ admits a cell decomposition, see Proposition 3.4. Let us mention that for a regular compactification of a reductive group $G$, each closed orbit $D_F$ corresponding to a vertex $\mathcal{F}$ is isomorphic to product of flag varieties $G/B \times G/B$. In particular $\mathcal{D}_F$ is a cellular fibration (see [3] for details).

Recall that we phrased part a) of Theorem 0.3 in Proposition 3.7. Below we prove the remaining parts of this theorem.

**Proof. of part b) of Theorem 0.3** Since $G$ is reductive, $Z(G) = Z(G)^{\circ}$ is a torus. Thus it suffices to prove the statement for the associated $G^{ad}$-bundle $\mathcal{G}'$, see Example 4.1. Note that the statement for $\mathcal{G}'$ follows from filtration (4.12) and Theorem 2.10 (see Example 4.2).

**Remark 4.3.** In practice it might happen that the motive of the base scheme $X$ is far from being mixed Tate. One may already expect this from the case of 1-motives. Assume that $X = C$ is a curve. Recall that the motive $M(C)$ decomposes in $DM^c_{\text{eff}}(k) \otimes \mathbb{Q}$ as follows

$$M(C) = M_0(C) \oplus M_1(C) \oplus M_2(C),$$

where $M_i(C) := \text{TotLiAlb}^i(C)[i]$. For the definition of $\text{LiAlb}^i(C)$ and detailed explanation of the theory we refer to section 3.12 of [1].

**Proof. of part c) of Theorem 0.3** Let $\mathcal{G}_s$ denote the associated $G^s$-bundle. Fix a closed point $p$ of $C$ and set $\mathcal{C} := C \setminus \{p\}$. Since $\text{char} \ k$ does not divide the order of the fundamental group $\pi_1(G)$, by the well-known theorem of Drinfeld
and Simpson [13], we may take a finite extension $k'$ of $k$ which simultaneously trivializes the restriction of $G_s$ over $\hat{C}$ and the fiber over $p$. Therefore we obtain the following Gysin distinguished triangle

$$M(G^s) \otimes M(\hat{C}_k') \to M(G_{s,k'}) \to M(G^s \times k')(n)[2n].$$

Since the split reductive group $G$ has a connected center, the group $Z = Z(G)$ is a split torus. Thus we may apply either the filtration in example 4.1 or the relative slice filtration (4.1) to the torus bundle $G \to G_s$. For instance from the latter filtration we get the following

i) A filtration \(\{\varphi_i : M_i \to M_{i-1}\}_{i \in \mathbb{N}}\) where \(M_i := \nu_{G_{s,k'}}^i M(G_{k'})\).

ii) The following sort of distinguished triangles

\[M(\hat{C}_k') \otimes M(G^s) \to M(G_{s,k'}) \to M(G^s) \otimes M(k')(n)[2n]\]

\[M_{i+1} \to M_i \to M(G_{s,k'})(i)[i] \otimes F_i,\]

where \(F_i\) is the \(i\)-th wedge power of the group \(\Xi := Hom(G_m, Z)\).

Note that \(M_0 = M(G_{k'})\) and \(M_i = 0\) for \(i > \text{rk } Z\).

At the end of this section, it may look worthy to state the corresponding fact in the K-ring \(K_0(\text{Var}_k)\) (as well as the K-ring \(K_0(\text{DM}_{gm}^\text{eff}(k))\)). Recall that for a fibration \(X \to Y\) with fiber \(F\), which is locally trivial for the Zariski topology, one has \([X] = [Y].[F]\) (here \([\cdot]\) denotes the corresponding class in \(K_0(\text{Var}_k)\), see Gillet and Soulé [15, Prop. 3.2.2.5]. As we show in the following proposition, when \(k\) is algebraically closed, one can see that a similar fact (under certain assumption on the characteristic of \(k\)) holds for \(G\)-bundles over \(C\) (note however that they might not be locally trivial for Zariski topology).

**Proposition 4.4.** Let \(G\) be a reductive group over \(k\). Assume that char \(k\) does not divide the order of the fundamental group \(\pi_1(G)\). For a \(G\)-bundle \(G\) over a relative curve \(C_S\) the class \([G] = [G \times_S C_S]\) in the Grothendieck ring of varieties \(K_0(\text{Var}_S)\) lies in the kernel of the natural morphism \(K_0(\text{Var}_S) \to K_0(\text{Var}_{S'})\) induced by an étale morphism \(S' \to S\). In particular when \(k\) is algebraically closed and \(S' = \text{Spec } k\) then \([G] = [G].[C]\).

**Proof.** According to Hilbert’s Theorem 90 torus bundles are locally trivial for Zariski topology, thus by the result of Gillet and Soulé [15], which we mentioned above, we can reduce to the case that \(G = G^s\) is semi-simple. Then the above proposition follows from the theorem of Drinfeld-Simpson, in a similar way as we discussed in the proof of part c) of the Theorem [13].
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