Emergent Behaviors over Signed Random Networks in Dynamical Environments

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Abstract

We study asymptotic dynamical patterns that emerge among a set of nodes that interact in a dynamically evolving signed random network. Node interactions take place at random on a sequence of deterministic signed graphs. Each node receives positive or negative recommendations from its neighbors depending on the sign of the interaction arcs, and updates its state accordingly. Positive recommendations follow the standard consensus update while two types of negative recommendations, each modeling a different type of antagonistic or malicious interaction, are considered. Nodes may weigh positive and negative recommendations differently, and random processes are introduced to model the time-varying attention that nodes pay to the positive and negative recommendations. Various conditions for almost sure convergence, divergence, and clustering of the node states are established. Some fundamental similarities and differences are established for the two notions of negative recommendations.

Keywords. Random graphs, Signed networks, Consensus dynamics, Belief clustering

1 Introduction

1.1 Motivation

The need to model, analyze and engineer large complex networks appears in a wide range of scientific disciplines, ranging from social sciences and biology to physics and engineering [1,2,3].
In many cases, these networks are composed of relatively simple agents that interact locally with their neighbors based on a very limited knowledge about the system state. Despite the simple local interactions, the resulting networks can display a rich set of emergent behaviors, including certain forms of intelligence and learning [4, 5].

Consensus problems, in which the aim is to compute a weighted average of the initial values held by a collection of nodes, play a fundamental role in the study of node dynamics over complex networks. Early work [1] focused on understanding how opinions evolve in a network of agents, and showed that a simple deterministic opinion update based on the mutual trust and the differences in belief between interacting agents could lead to global convergence of the beliefs. Consensus dynamics has since then been widely adopted for describing opinion dynamics in social networks, e.g., [5, 6, 7]. In engineering sciences, a huge amount of literature has studied these algorithms for distributed averaging, formation forming and load balancing between collaborative agents under fixed or time-varying interaction networks [8, 9, 10, 11, 12, 13, 14, 15]. Randomized consensus seeking has also been widely studied, motivated by the random nature of interactions and updates in real complex networks [16, 17, 18, 19, 20, 21, 22, 23].

Interactions in large-scale networks are not always collaborative, as nodes often take on different, or even opposing, roles. A convenient framework for modeling different roles and relationships between agents is to use signed graphs. Signed graphs were introduced in the classical work by Heider in 1940s [26] to model the structure of social networks, where a positive link represents a friendly relation between two agents, and a negative link an unfriendly one. In [27], a dynamic model based on a signed graph with positive links between nodes (representing nations) belonging to the same coalition and negative otherwise, was introduced to study the stability of world politics. In biology, sign patterns have been used to describe activator–inhibitor interactions between pairs of chemicals [28], neural networks for vision and learning [29], and gene regulatory networks [30]. In all these examples, the state updates that happen when two nodes interact depend on the sign of the arc between the nodes in the underlying graph. The understanding of the emergent dynamical behaviors of networks with agents having different roles is much more limited than for instance collaborative agents performing consensus algorithms.

It is intriguing to investigate what happens when two types of dynamics are coupled in a single network. Naturally we ask: how we should model the dynamics of positive and negative interactions, when do behaviors such as consensus, swarming and clustering emerge, and how does the structure of the sign patterns influence these behaviors? In this paper, we answer these
questions for a general model of opinion formation in dynamic signed random networks.

1.2 Contribution

We consider general randomized node interactions. A sequence of deterministic signed graphs defines a dynamical environment of the network, and then random node interactions take place under independent, but not necessarily identically distributed, random sampling of the environment. Once interaction relations have been realized, each node receives a positive recommendation consistent with the standard consensus algorithm from its positive neighbors. Nodes receive negative recommendations from the negative neighbors. Following [32, 33] we consider two models of negative recommendations. In the state-reversion recommendation, each node receives false values from its negative neighbors without necessarily knowing which of its neighbors is positive or negative [32]. In the relative-state-reversion model, nodes receive a repulsive influence from their known negative neighbors. After receiving these recommendations, each node puts a (deterministic) weight to each recommendation, and then encodes these weighted recommendations in its state update through (stochastic) attentions defined by two Bernoulli random variables.

We establish conditions for the almost sure convergence, divergence, and clustering of the node states for the considered signed random networks. Fundamental similarities and differences are established for the two models of negative recommendations. We show that strong structural balance [35] is crucial for belief clustering in the state-reversion model (which is consistent with the work of Altafini [32]), while weak structural balance is enough in the relative-state-reversion model. We also show that the deterministic weight and the stochastic attention nodes put on recommendations play a drastically different role for the state convergence and divergence of the network in the two models. The models share some consistent behaviors: if some arc independence is imposed on the random interactions, two similar no-survivor statements are established, which generalize the results for the gossiping model in [34].

1.3 Paper Organization

In Section 2 we propose the network dynamics and the node update rules. State-reversion and relative-state-reversion models are proposed, respectively, for the negative recommendations of the node updates. Section 3 presents our main results on the state-reversion model as well as the proofs. Then Section 4 moves to the relative-state-reversion model and finally some concluding
Graph Theory, Notations and Terminologies

A simple directed graph (digraph) $G = (V, E)$ consists of a finite set $V$ of nodes and an arc set $E \subseteq V \times V$, where $e = (i, j) \in E$ denotes an arc from node $i \in V$ to $j \in V$ with $(i, i) \notin E$ for all $i \in V$. We call node $j$ reachable from node $i$ if there is a directed path from $i$ to $j$. In particular, every node is supposed to be reachable from itself. A node $v$ from which every node in $V$ is reachable is called a center node (root). A digraph $G$ is strongly connected if every two nodes are mutually reachable; $G$ has a spanning tree if it has a center node; $G$ is weakly connected if a connected undirected graph can be obtained by removing all the directions of the arcs in $E$. A subgraph $G$ of $G = (V, E)$, is a graph on the same node set $V$ whose arc set is a subset of $E$. The induced graph of $V_i \subseteq V$ on $G$, denoted $G|V_i$, is the graph $(V_i, E_i)$ with $E_i = (V_i \times V_i) \cap E$. A weakly connected component of $G$ is a maximal weakly connected induced graph of $G$. If each arc $(i, j) \in E$ is associated uniquely with a sign, either ‘$+$’ or ‘$-$’, $G$ is called a signed graph and the sign of $(i, j) \in E$ is denoted as $\sigma_{ij}$. The positive and negative subgraphs containing the positive and negative arcs of $G$, are denoted as $G^+ = (V, E^+)$ and $G^- = (V, E^-)$, respectively.

Depending on the argument, $|\cdot|$ stands for the absolute value of a real number, the Euclidean norm of a vector or the cardinality of a set. The $\sigma$-algebra of a random variable is denoted as $\sigma(\cdot)$. We use $P(\cdot)$ to denote the probability and $E\{\cdot\}$ the expectation of their arguments, respectively.

2 Networks Dynamics and Node Updates

We consider a dynamic network where each user holds and updates her belief or state when interacting with other users. In this section, we present a general model specifying the network dynamics and the way users interact.

2.1 Dynamic Signed Graphs

We consider a network with a set $V = \{1, \ldots, n\}$ of $n$ users or nodes, with $n \geq 3$. Time is slotted, and at each slot $t = 0, 1, \ldots$, each user can interact with her neighbors in a simple directed graph $G_t = (V, E_t)$. The graph evolves over time in an arbitrary and deterministic manner. We assume $G_t$ is a signed graph, and we denote by $\sigma_{ij}(t)$ the sign of arc $(i, j) \in E_t$. remarks are drawn in Section 5.
The sign of arc \((i, j)\) indicates whether \(i\) is a friend \((\sigma_{ij}(t) = +)\), or an enemy \((\sigma_{ij}(t) = -)\) of node \(j\). The positive and negative subgraphs containing the positive and negative arcs of \(G_t\), are denoted by \(G^+_t = (V, \mathcal{E}^+_t)\) and \(G^-_t = (V, \mathcal{E}^-_t)\), respectively. We say that the sequence of graphs \(\{G_t\}_{t \geq 0}\) is sign consistent if the sign of any arc \((i, j)\) does not evolve over time, i.e., if for any \(s, t \geq 0\),
\[(i, j) \in \mathcal{E}_s \text{ and } (i, j) \in \mathcal{E}_t \implies \sigma_{ij}(s) = \sigma_{ij}(t).\]
We also define \(G_* = (V, \mathcal{E}_*)\) with \(\mathcal{E}_* = \bigcup_{t=0}^{\infty} \mathcal{E}_t\) as the total graph of the network. If \(\{G_t\}_{t \geq 0}\) is sign consistent, then the sign of each arc \(\mathcal{E}_*\) never changes and in that case, \(G_* = (V, \mathcal{E}_*)\) is a well-defined signed graph.

Next we introduce the notion of positive cluster in a signed directed graph (digraph), which will play an important role in the analysis of the belief dynamics.

**Definition 1.** Let \(G\) be a signed digraph with positive subgraph \(G^+\). A subset \(V_*\) of the set of nodes \(V\) is a positive cluster if \(V_*\) constitutes a weakly connected component of \(G^+\). A positive cluster partition of \(G\) is a partition of \(V\) into \(V = \bigcup_{i=1}^{T_p} V_i\) for some \(T_p \geq 1\), where for all \(i = 1, \ldots, T_p\), \(V_i\) is a positive cluster.

Note that negative arcs may exist between the nodes of a positive cluster. Therefore, \(G\) admitting a positive-cluster partition is a generalization of the classical definition of weakly structural balance for which negative links are strictly forbidden [36]. From the above definition, it is clear that for any signed graph \(G\), there is a unique positive cluster partition \(V = \bigcup_{i=1}^{T_p} V_i\) of \(G\), where \(T_p\) is the number of positive clusters covering the entire set \(V\) of nodes.

2.2 Random Interactions

At time \(t\), node \(i\) may only interact with her neighboring nodes in \(G_t\). We present a general model on the random node interactions at a given time \(t\). This model includes the classical Erdős-Rényi random graph [24], gossiping models where a single pair of nodes is chosen at random for interaction [17], as well as where all nodes interact with their neighbors at a given time [18]. At time \(t\), some pairs of nodes are randomly selected for interaction. We denote by \(E_t \subset \mathcal{E}_t\) the random subset of arcs corresponding to interacting node pairs at time \(t\). To be precise, \(E_t\) is sampled from the distribution \(\mu_t\) defined over the set \(\Omega_t\) of all subsets of arcs in \(\mathcal{E}_t\). We assume that \(E_0, E_1, \ldots\) form a sequence of independent sets of arcs. Formally, we introduce the probability space \((\Theta, \mathcal{F}, P)\) obtained by taking the product of the probability spaces \((\Omega_t, \mathcal{S}_t, \mu_t)\),
Figure 1: A signed network and its three positive clusters. The positive arcs are solid, and the negative arcs are dashed. Note that negative arcs are allowed within positive clusters.

where $S_t$ is the discrete $\sigma$-algebra on $\Omega$: $\Theta = \prod_{t \geq 0} \Omega_t$, $\mathcal{F}$ is the product of $\sigma$-algebras $S_t$, $t \geq 0$, and $P$ is the product probability measure of $\mu_t$, $t \geq 0$. We denote by $G_t = (\mathcal{V}, E_t)$ the random subgraph of $\mathcal{G}_t$ corresponding to the random set $E_t$ of arcs. The disjoint sets $E_t^+$ and $E_t^-$ denote the positive and negative arc set of $E_t$, respectively. Finally, we split the random set of nodes interacting with node $i$ at time $t$ depending on the sign of the corresponding arc: for node $i$, the set of positive neighbors is defined as $N_i^+(t) := \{ j : (j, i) \in E_t^+ \}$, whereas similarly, the set of negative neighbors is $N_i^-(t) := \{ j : (j, i) \in E_t^- \}$.

2.3 Node updates

Next we explain how nodes update their states. Each node $i$ holds a state $s_i(t) \in \mathbb{R}$ at $t = 0, 1, \ldots$. To update her state at time $t$, node $i$ considers recommendations received from her positive and negative neighbors:

(i) The positive recommendation node $i$ receives at time $t$ is $h_i^+(t) := -\sum_{j \in N_i^+(t)} (s_i(t) - s_j(t))$.

(ii) The negative recommendations node $i$ receives at time $t$ are modeled by two different maps:

- The state-reversion recommendation $h_i^-(t) := -\sum_{j \in N_i^-(t)} (s_i(t) + s_j(t))$;
- The relative-state-reversion recommendation $h_i^-(t) := \sum_{j \in N_i^-(t)} (s_i(t) - s_j(t))$. 


In the above expressions, we use the convention that summing over empty sets yields a recommendation equal to zero, e.g., when node $i$ has no positive neighbors, then $h^+_i(t) = 0$.

**Remark 1.** The two definitions of negative recommendations have different physical interpretations and make different assumptions on the knowledge that nodes possess about their neighbor relationships. The state-reversion model can be interpreted as a situation where negative nodes provide false values of their states by flipping the true sign \[32\]. However, the receiving node does not necessarily know which of its neighbors are positive or negative. In the relative-state-reversion model, on the other hand, nodes must know if a specific neighbor is positive or negative to implement the state update that causes the repulsive influence from its negative neighbors \[33\].

Now let $\{B_t\}_{t \geq 0}$ and $\{D_t\}_{t \geq 0}$ be two sequences of independent Bernoulli random variables. We assume that $\{B_t\}_{t \geq 0}$, $\{D_t\}_{t \geq 0}$, and $\{G_t\}_{t \geq 0}$ define independent processes. For any $t \geq 0$, define $b_t = E\{B_t\}$ and $d_t = E\{D_t\}$. The processes $\{B_t\}_{t \geq 0}$ and $\{D_t\}_{t \geq 0}$ represent how much attention node $i$ pays to the positive and negative recommendations, respectively.

Node $i$ updates her state as follows:

$$s_i(t + 1) = s_i(t) + \alpha B_t h^+_i(t) + \beta D_t h^-_i(t),$$

where $\alpha, \beta > 0$ are two positive constants marking the weight each node put on the positive and negative recommendations, respectively. Depending on the definition of $h^-_i(t)$, we call the corresponding model the state-reversion model and the relative-state-reversion model, respectively.

Let $s(t) = (s_1(t) \ldots s_n(t))^T$ be the random vector representing the network state at time $t$. The main objective of this paper is to analyze the behavior of the stochastic process $\{s(t)\}_{t \geq 0}$. In the following, we denote by $P$ the probability measure capturing all random components driving the evolution of $s(t)$.

### 3 The State-Reversion Model

In this section, we study the system dynamics under the state-reversion model. We provide conditions for convergence and divergence. The results are stated in the following subsection, and the remaining of the section is devoted to their proofs.
3.1 Main Results

We begin by stating two natural assumptions on the way nodes are selected for updates, and on the graph dynamics. In the first assumption, we impose that at time $t$, any arc is selected in $E_t$ with positive probability. The second assumption states that the unions of the graphs $\mathcal{G}_t$ over time-windows of fixed duration are strongly connected.

**A1.** There is a constant $p_* \in (0, 1)$ such that for all $t \geq 0$ and $i, j \in \mathcal{V}$, $P((i, j) \in E_t) \geq p_*$ if $(i, j) \in \mathcal{E}_t$.

**A2.** There is an integer $K \geq 1$ such that the union graph $\mathcal{G}([t, t+K-1]) = (\mathcal{V}, \bigcup_{\tau \in [t, t+K-1]} \mathcal{E}_\tau)$ is strongly connected for all $t \geq 0$.

The following theorem provides conditions under which the system dynamics converges almost surely. Surprisingly, these conditions are mild: we just require that the sum of the updating parameters $\alpha$ and $\beta$ is small enough, and that node updates occur with constant probabilities, i.e., $E\{B_t\}$ and $E\{D_t\}$ do not evolve over time. In particular, the state of each node converges almost surely even if the signs of arcs change over time.

**Theorem 1.** Assume that A1 and A2 hold, and that $\alpha, \beta > 0$ are such that $\alpha + \beta \leq 1/(n-1)$. Further assume that for any $t \geq 0$, $b_t \equiv b$ and $d_t \equiv d$ for some $b, d \in (0, 1)$. Then under the state-reversion model, we have, for all $i \in \mathcal{V}$ and all initial states $s(0)$, $P(\lim_{t \to \infty} s_i(t) exists) = 1$.

In the above theorem, we say that $\lim_{t \to \infty} s_i(t)$ exists if $s_i(t)$ converges to a finite limit as $t$ tends to infinity. Characterizing the limiting states is in general challenging. There are however scenarios where this can be done, which require the notion of structural balance.

**Definition 2.** Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a signed digraph. $\mathcal{G}$ is strongly balanced if we can divide $\mathcal{V}$ into two disjoint nonempty subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ where negative arcs exist only between these two subsets.

To predict the limiting system behavior, we make the following assumption.

**A3.** $\{\mathcal{G}_t\}_{t \geq 0}$ is sign consistent admitting a total graph $\mathcal{G}_*$.

**Theorem 2.** Assume that A1, A2 and A3 hold, and that $\alpha, \beta > 0$ are such that $\alpha + \beta \leq 1/(n-1)$. We suppose $\mathcal{G}_*$ contains at least one negative arc and every negative arc in $\mathcal{G}_*$ appears infinitely often in $\{\mathcal{G}_t\}_{t \geq 0}$. Further assume that for any $t \geq 0$, $b_t \equiv b$ and $d_t \equiv d$ for some $b, d \in (0, 1)$. Then under the state-reversion model, we have, for any initial state $s(0)$:
(i) If \( G_* \) is strongly balanced, then there is a random variable \( y_* \), with \( y_* \leq \| s(0) \|_1 \) almost surely, such that \( \mathbb{P}( \lim_{t \to \infty} s_i(t) = y_* , \forall i \in V_1 ; \lim_{t \to \infty} s_i(t) = -y_* , \forall i \in V_2 ) = 1 \);

(ii) If \( G_* \) is not strongly balanced, then \( \mathbb{P}( \lim_{t \to \infty} s_i(t) = 0 , \forall i \in V ) = 1 \).

Theorem 2 states that strong structural balance is crucial to ensure convergence to nontrivial clustering states, which is consistent with the result of [32] derived for fixed graphs under continuous-time node updates. Instead of a spectral analysis as in [32], we study the asymptotic behavior of each sample path. From the above theorem, we know that under the strong structural balance condition, the states of nodes in the same positive cluster converge to the same limit, and that the limits of two nodes in different positive clusters are exactly opposite. Using similar arguments as in [32], the value of \( y_* \) can be described as the limit of a random consensus process with the help of a gauge transformation.

Next we are interested in determining whether the states could diverge depending on the values of the updating parameters \( \alpha \) and \( \beta \). We show that by increasing \( \beta \), i.e., the strength of the negative recommendations, one may observe such divergence. To this aim, we make the following assumptions.

A4. There is an integer \( K \geq 1 \) such that the union graph \( G^+([t, t+K]) = (V, \bigcup_{\tau \in [t, t+K-1]} E^+_{\tau}) \) is strongly connected for all \( t \geq 0 \).

A5. There is an integer \( K \geq 1 \) such that the union graph \( G^-([t, t+K]) = (V, \bigcup_{\tau \in [t, t+K-1]} E^-_{\tau}) \) is strongly connected for all \( t \geq 0 \).

A6. The events \( \{(i, j) \in G_t\} , i, j \in V , t = 0, 1, \ldots \) are independent and there is a constant \( p^* \in (0, 1) \) such that for all \( t \geq 0 \) and \( i, j \in V \), \( \mathbb{P}( (i, j) \in G_t ) \leq p^* \) if \( (i, j) \in E_t \).

Theorem 3. Assume that A1, A4, A5 and A6 hold, and that for any \( t \geq 0 \), \( b_t \equiv b \) and \( d_t \equiv d \) for some \( b, d \in (0, 1) \). Fix \( \alpha \in [0, (2n)^{-1}] \). Then under the state-reversion model, there is \( \beta_* > 0 \) such that whenever \( \beta > \beta_* \), we have \( \mathbb{P}( \lim_{t \to \infty} \max_{i \in V} |s_i(t)| = \infty ) = 1 \) for almost all initial states \( s(0) \) (under the standard Lebesgue measure).

Theorem 3 shows that under appropriate conditions, \( \max_{i \in V} |s_i(t)| \) diverges almost surely if the negative updating parameter \( \beta \) is sufficiently large. Actually, one may even prove that when \( \max_{i \in V} |s_i(t)| \) grows large when \( t \to \infty \), the state of any node diverges. This result is referred to as the no-survivor property, and is formally stated in the following proposition.

Proposition 1. Assume that A1, A2 and A6 hold, and that for any \( t \geq 0 \), \( b_t \equiv b \) and \( d_t \equiv d \)
for some \( b, d \in (0, 1) \). Fix the initial state \( s(0) \). Then under the state-reversion model, we have

\[
P \left( \limsup_{t \to \infty} |s_i(t)| = \infty, i \in V \middle| \limsup_{t \to \infty} \max_{i \in V} |s_i(t)| = \infty \right) = 1.
\]

In all above results, it can be seen from their proofs that extensions to time-varying \( \{b_t \} \geq 0 \) and \( \{d_t \} \geq 0 \) are straightforward under mild assumptions. The resulting expressions are however more involved. We omit those discussions here to simplify the presentation.

### 3.2 Supporting Lemmas

Before proving the presented results, we first provide a few lemmas that will prove instrumental. For any \( t \geq 0 \), we define\( M(t) = \max_{i \in V} |s_i(t)| \).

**Lemma 1.** Suppose \( \alpha + \beta \leq 1/(n - 1) \). Then \( M(t + 1) \leq M(t) \).

**Proof.** Define \( Y(t) = \alpha B_t |N_i^+(t)| + \beta D_t |N_i^-(t)| \). Observe that \( |N_i^+(t)| + |N_i^-(t)| \leq n - 1 \), and hence \( Y(t) \in [0, 1] \) as long as \( \alpha + \beta \leq 1/(n - 1) \). Now for any \( i \in V \),

\[
|s_i(t + 1)| \leq \left( |1 - Y(t)| + Y(t) \right) \max_{j \in V} |s_j(t)| = \max_{j \in V} |s_j(t)|,
\]

which completes the proof. \( \blacksquare \)

**Lemma 2.** Assume that \( \alpha + \beta \leq 1/(n - 1) \). Let \( i \in V \) and assume that \( |s_i(t)| \leq \zeta_0 M(t) \) for some \( 0 < \zeta_0 < 1 \). Then

\[
|s_i(t + k)| \leq \left( 1 - (1 - \zeta_0) \gamma_*^k \right) M(t), \quad k = 0, 1, \ldots
\]

where \( \gamma_* = 1 - (\alpha + \beta)(n - 1) \).

**Proof.** Let \( Y(t) \) be as defined in Lemma 1. We have:

\[
|s_i(t + 1)| \leq \left( 1 - Y(t) \right) |s_i(t)| + Y(t) M(t)
\]

\[
\leq \left( 1 - Y(t) \right) \zeta_0 M(t) + Y(t) M(t)
\]

\[
\leq \left( 1 - (\alpha + \beta)(n - 1) \right) \zeta_0 M(t) + (\alpha + \beta)(n - 1) M(t)
\]

\[
= (1 - (1 - \zeta_0) \gamma_*^k) M(t).
\]

The lemma is then obtained by applying a simple induction argument. \( \blacksquare \)
Lemma 3. Assume that $\alpha + \beta \leq 1/(n-1)$. Let $i \in V$ and assume that $|s_i(t)| \leq \zeta_0 M(t)$ for some $0 < \zeta_0 < 1$. Let $(i, j) \in E_t$. Then conditioned on $B_t = 1$ if $(i, j) \in E_t^+$, $D_t = 1$ if $(i, j) \in E_t^-$, we have

$$|s_j(t + 1)| \leq (1 - (1 - \zeta_0) \min\{\alpha, \beta\}) M(t).$$

Proof. Based on the update rule it can be easily seen that:

$$|s_j(t + 1)| \leq \min\{\alpha, \beta\}|s_i(t)| + (1 - \min\{\alpha, \beta\}) M(t).$$

Plugging in $|s_i(t)| \leq \zeta_0 M(t)$, one gets the desired result. □

Note that if the conditions in Lemmas 2 and 3 are replaced by $|s_i(t)| < \zeta_0 M(t)$, then we have the same conclusions but with strict inequalities. Moreover, in view of Lemma 1, the following limit is well defined: $M_\ast = \lim_{t \rightarrow \infty} M(t)$.

Lemma 4. Assume that A1 and A2 hold, $\alpha, \beta > 0$, and $\alpha + \beta \leq 1/(n-1)$. Further assume that for any $t \geq 0$, $b_t \equiv b$ and $d_t \equiv d$ for some $b, d \in (0,1)$. Then for any initial state $s(0)$, we have $\mathbb{P}(\lim_{t \rightarrow \infty} |s_i(t)| = M_\ast, \forall i \in V) = 1$.

Proof. We prove this lemma using sample path arguments by contradiction. Let us assume that:

H1. There exist $i_0 \in V$ and $\delta, q_\ast \in (0,1)$ such that $\mathbb{P}(\liminf_{t \rightarrow \infty} |s_{i_0}(t)| < \delta M_\ast) \geq q_\ast$.

Let $\epsilon > 0$. Define

$T(\epsilon) := \inf_{k \geq 0} \{ M(t) \leq (1 + \epsilon) M_\ast, \forall t \geq k \}$

and

$T^* := \inf_{t \geq T} \{ s_{i_0}(t) < \delta M_\ast \}.$

Note that $T(\epsilon)$ is a stopping time, and the monotonicity of $M(t)$ guarantees that $T$ is bounded almost surely [25]. Moreover, $T^*$ is also a stopping time, and it is bounded with probability at least $q_\ast$ in view of H1. Next, we use Lemmas 2 and 3 to get a contradiction. Applying Lemma 2 conditioned on $\{T^* < \infty\}$, we have that for all $k = 0, 1, \ldots$:

$$|s_{i_0}(T^* + k)| < (1 - (1 - \delta) \gamma^k) M_\ast (1 + \epsilon). \quad (2)$$

Now consider the time interval $[T^*, T^* + K - 1]$. The independence of $\{B_t\}_{t \geq 0}$, $\{D_t\}_{t \geq 0}$, and $\{G_t\}_{t \geq 0}$ guarantee that $(G_{T^*}, B_{T^*}, D_{T^*}), (G_{T^*+1}, B_{T^*+1}, D_{T^*+1}), \ldots$ are independent random variables, and they are independent of $\mathcal{F}_{T^* - 1}$ (cf. Theorem 4.1.3 in [25]). From their definitions we also know that $(B_{T^*}, D_{T^*})$, $(B_{T^*+1}, D_{T^*+1})$, $\ldots$ are i.i.d. with the same distribution as

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(B_0, D_0), and Assumption A2 guarantees that $\mathcal{G}([T^*, T^* + K - 1]) = (\mathcal{V}, \bigcup_{t \in [T^*, T^* + K - 1]} \mathcal{E}_t)$ is strongly connected. Therefore, there exists a node $i_1 \neq i_0$ and $\tau_1 \leq K$ such that $(i_0, i_1) \in \mathcal{E}_{T^* + \tau_1}$ (note that $i_1$ and $\tau_1$ are random variables, but they are independent with $\mathcal{F}_{T^* - 1}$ since $T^*$ is a stopping time). Hence we can apply Lemma 3 and conclude that

$$|s_{i_1}(T^* + \tau_1)| < (1 - (1 - \delta)\gamma^2 \min\{\alpha, \beta\})M_\epsilon(1 + \epsilon)$$

with a probability at least $p \min\{b, d\}$. Again by Lemma 2 we have that for all $k = K, K + 1, \ldots,$

$$|s_{i_1}(T^* + k)| < (1 - (1 - \delta)\gamma^k \min\{\alpha, \beta\})M_\epsilon(1 + \epsilon).$$

We can repeat the same argument over time intervals $[T^* + K, T^* + 2K - 1], \ldots, [T^* + (n - 2)K, T^* + (n - 1)K - 1]$, and find $i_2, \ldots, i_{n-1}$ such that $\mathcal{V} = \{i_0, \ldots, i_{n-1}\}$ and bound the absolute values of their states. Finally, we get:

$$P(M(T^* + (n - 1)K) < \lfloor 1 - \gamma^{(n-1)K} \min\{\alpha, \beta\} \rfloor - 2 \times (1 - \delta)M_\epsilon(1 + \epsilon) | T^* < \infty) \geq (p_\epsilon \min\{b, d\})^{n-1}. \quad (3)$$

Now select $\epsilon$ sufficiently small so that $\theta := (1 - (1 - \delta)\gamma^{(n-1)K} \min\{\alpha, \beta\})^{n-2}(1 + \epsilon) < 1$. Using the monotonicity of $M(t)$ established in Lemma 1 we deduce from (3):

$$P(M_\epsilon < \theta M_\epsilon | T^* < \infty) \geq (p_\epsilon \min\{b_\epsilon, d_\epsilon\})^{n-1},$$

which is impossible and hence, H1 is not true. We have proved that:

$$P\left(\liminf_{t \to \infty} |s_i(t)| = M_\epsilon, \forall i \in \mathcal{V}\right) = 1.$$

The claim then follows easily from Lemma 1

Lemma 5. Let $\alpha < (2n)^{-1}$ and $\beta > 16n^{n+1}$. Then $M(t + 1) \geq (2n)^{-1}M(t)$. 

Proof. Let us first assume that $D_t = 0$. Let $i \in \mathcal{V}$ such that $|s_i(t)| = M(t)$. Then with $\alpha < (2n)^{-1}$, we have

$$M(t + 1) \geq |s_i(t + 1)| \geq \left|1 - \alpha B_i |N_i^+(t)|\right| \cdot |s_i(t)| - \alpha B_i |N_i^+(t)| \cdot M(t) \geq \left|1 - 2\alpha B_i |N_i^+(t)|\right| \cdot M(t) \geq \frac{1}{n}M(t) \geq (2n)^{-1}M(t).$$
Now assume that $D_1 = 1$. We first prove the following claim.

**Claim.** Consider $i_1$ such that $s_{i_1}(t) \in [(1-Z_2)M(t),(1-Z_1)M(t)]$ with $0 \leq Z_1 < Z_2 < nZ_2 < 1/4$ and $\beta Z_2 \geq 2$. Then $\mathcal{H}_1 \cup \mathcal{H}_2$ is a sure event, where

$$\mathcal{H}_1 = \{ M(t+1) \geq M(t)/4 \}$$

and

$$\mathcal{H}_2 = \{ \exists i_2 : s_{i_2}(t) \in (- (1-Z_2)M(t), -(1-nZ_2)M(t)) \}.$$

To prove this claim, we distinguish two cases:

(i) Assume that there exists $j_* \in \mathcal{V}$ such that $j_* \in N_{i_1}^-(t)$ and $s_{j_*}(t) \in [-(1-nZ_2)M(t), M(t)]$.

Then $s_{i_1}(t) + s_{j_*}(t) \geq (n-1)Z_2M(t) \geq 0$ and $s_{i_1}(t) + s_{j}(t) \geq -Z_2M(t)$ for all $j \in \mathcal{V} \setminus \{i_1, j_*\}$. Thus, taking out the term $s_{i_1}(t) + s_{j_*}(t)$ in $h_{j_*}^-(t)$ from (1), some simple algebra leads to

$$M(t+1) \geq |s_{j_*}(t+1)|$$

$$\geq \beta |s_{i_1}(t) + s_{j_*}(t)| - M(t) - \alpha(n-1)M(t)$$

$$- \beta(n-2)Z_2M(t)$$

$$\geq |\beta Z_2 - 1 - (n-1)(2n)^{-1}| \cdot M(t)$$

$$\geq \frac{1}{2} M(t). \quad (4)$$

(ii) Assume that $s_{j}(t) \in [-M(t), -(1-Z_2)(M(t))]$ for all $j \in N_{i_1}^-(t)$. Then $s_{i_1}(t) + s_{j}(t) \leq -Z_1M(t) \leq 0$ for all $j \in N_{i_1}^-(t)$, which implies that $h_{i_1}^-(t) \geq 0$. Observing that $s_{i_1}(t) \geq 0$, we obtain

$$M(t+1) \geq |s_{i_1}(t+1)|$$

$$\geq |s_{i_1}(t)| - \alpha(n-1)M(t)$$

$$\geq \left| 1 - Z_2 - (n-1)(2n)^{-1} \right| \cdot M(t)$$

$$\geq \frac{1}{4} M(t). \quad (5)$$

From (4) and (5), we deduce that if $\mathcal{H}_2$ does not hold, then $\mathcal{H}_1$ is true, which proves the claim.

Finally, we complete the proof of the lemma using the claim we just established. Take $\epsilon = 8^{-1}n^{-n-1}$ and $\beta = 16n^{n+1}$. We proceeds in steps.

(1) Let $m_1 \in \mathcal{V}$ with $|s_{m_1}(t)| = M(t)$. Without loss of generality, by symmetry we can assume that $s_{m_1}(t) = M(t)$. Applying the claim with $Z_1 = 0$ and $Z_2 = \epsilon$, we deduce that either the
lemma holds or there is another node \( m_2 \in V \) such that \( s_{m_2}(t) \in ((1 - n)M(t), (1 - n\epsilon)M(t)) \).

(2) If in the first step, we could not conclude that the lemma holds, we can apply the claim to \( m_2 \) (observe that the claim we established is also valid when all states \( s_i(t) \) are replaced by \(-s_i(t))\). We then obtain that either the lemma holds, or there is a node \( m_3 \) such that \( s_{m_3}(t) \in ((1 - n^2\epsilon)M(t), (1 - n\epsilon)M(t)) \).

The argument can be repeated for \( m_3, \ldots \) applying the claim adapting the value of \( \epsilon \) and \( \beta \). Since the number of nodes is bounded, the above repeated procedure necessarily ends, so the lemma holds.

\[ \blacksquare \]

### 3.3 Proofs of the Main Results

#### Proof of Theorem 1

From Lemma 4, we know that for any \( i \in V \), one of the following events happens almost surely: \( \{ \lim_{t \to \infty} s_i(t) = M \ast \}; \{ \lim_{t \to \infty} s_i(t) = -M \ast \}; \{ \liminf_{t \to \infty} s_i(t) = -M \ast \) and \( \limsup_{t \to \infty} s_i(t) = M \ast \}. \) Therefore, we just need to rule out the last case. We actually prove that: \( \mathbb{P}(M \ast > 0, \liminf_{t \to \infty} s_i(t) = -M \ast, \limsup_{t \to \infty} s_i(t) = M \ast, \lim_{t \to \infty} |s_i(t)| = M \ast) = 0. \)

Let \( \epsilon > 0 \) and define \( T_1(\epsilon) := \inf \{ k : M(k) \leq M \ast (1 + \epsilon) \} \). Using a similar recursive argument as that used in the proof of Lemma 2, we get: for all \( k = 0, 1, \ldots \) and \( t \geq T_1 \),

\[ s_i(t + k) \leq \gamma_i^k s_i(t) + (1 - \gamma_i^k)M \ast (1 + \epsilon). \]  

Let \( M \ast > 0 \). Assume that \( \liminf_{t \to \infty} s_i(t) = -M \ast \). Then for the given \( \epsilon \), we can find an infinite sequence \( T_j(\epsilon) < t_1 < t_2 < \ldots \) such that \( s_i(t_m) \leq -M \ast (1 - \epsilon) \). Now, if \( \limsup_{t \to \infty} s_i(t) = M \ast \), for any \( t_m \), we can find \( \bar{t}_m > t_m \) with \( s_i(\bar{t}_m) \geq M \ast (1 - \epsilon) \). Then based on (6), there must be \( \bar{t}_m \in [t_m, \bar{t}_m] \) such that \( s_i(\bar{t}_m) \in [ -\gamma_i M \ast (1 - \epsilon) + (1 - \gamma_i)M_\ast (1 + \epsilon), -\gamma_i^2 M \ast (1 - \epsilon) + (1 - \gamma_i^2)M_\ast (1 + \epsilon)] \). We deduce that \( |s_i(\bar{t}_m)| < M \ast (1 + \max\{|1 - 2\gamma_i|, |1 - 2\gamma_i^2|\})/2, \) \( m = 1, 2, \ldots \), when \( \epsilon < (1 - \max\{|1 - 2\gamma_i|, |1 - 2\gamma_i^2|\})/2. \) This contradicts \( \lim_{t \to \infty} |s_i(t)| = M \ast \) since by our assumption we have \( 0 < \gamma_i < 1 \).

\[ \blacksquare \]

#### Proof of Theorem 2

In view of Theorem 1, at least one of the following sets is non-empty:

\[ V^+_1 := \{ i \in V : \lim_{t \to \infty} s_i(t) = -M \ast \}, \]
and

$$V^*_2 := \{ i \in V : \lim_{t \to \infty} s_i(t) = M_* \}.$$ 

Without loss of generality, we assume $V^*_1 \neq \emptyset$ and $P(M_*) > 0$.

(i). Applying the same sample-path analysis as in the proof of Theorem 1, one can easily show that the arcs among nodes in $V^*_1$ are necessarily positive since each negative link appears for infinite time slots. Now the total graph $G_*$ is strongly balanced with nonempty $V_1$ and $V_2$, and hence $V^*_1$ is for example included in $V_1$, which in turns implies that $V^*_2 \neq \emptyset$. Again there are only positive arcs among nodes of $V^*_2$. We simply deduce that $\{V_1, V_2\} = \{V^*_1, V^*_2\}$.

(ii). Since $P(M_* > 0) > 0$, we have $V^*_1 \cap V^*_2 = \emptyset$. Again arcs between nodes in the same set from $V^*_i, i = 1, 2$ are necessarily positive. However there is at least one negative link in $G_*$ by our assumption, which can only be an arc between $V^*_1$ and $V^*_2$. Thus both $V^*_1$ and $V^*_2$ are nonempty, which implies that $G_*$ must be strongly balanced. This contradicts our standing assumption and the proof is complete. ■

Proof of Theorem 3

Let $\beta > 16^n^{n+1}$ so the conditions of Lemma 5 hold. Let us fix $t \geq 0$ and assume that $|s_{i_0}(t)| = M(t)$ for some $i_0 \in V$. By symmetry, we can also assume without loss of generality that $s_{i_0}(t) = M(t)$. Let $i_* \in V \setminus \{i_0\}$. Under Assumptions A4 and A6, we prove the following claim.

Claim. There is an integer $N_0 \geq 1$ and $q_0 > 0$ such that

$$P\left( s_{i_0}(t + N_0K) = M(t), s_{i_*}(t + N_0K) \geq M(t)/2 \right) \geq q_0.$$ 

In view of the connectivity condition A4 and the arc independence condition A6, the event $\{s_{i_*}(t + N_0K - 1) \geq M(t)/2\}$ given $s_{i_0}(t) = M(t)$ can be easily constructed by selecting a proper sequence of positive arcs for time slots $t, t + 1, \ldots, t + N_0K - 1$, and by imposing that $B_{\tau} = 1, D_{\tau} = 0, \tau = t, t + 1, \ldots, t + N_0K - 1$. Here $N_0$ and $q_0$ depend on $\alpha, b_*, d_*, p_*, p^*, n$ but do not rely on $\beta$. The analysis follows arguments to analyze basic consensus algorithms, and we omit the details.

In addition, in view of Assumption A5, we can select a node $i_* \neq i_0$ satisfying $(i_*, i_0) \in$
\[ \bigcup_{\tau \in [t + N_0K; t + (N_0+1)K-1]} \mathcal{E}_\tau^- \]. It then follows that

\[
\begin{align*}
\mathbb{P}
\left[
|s_{i_0}(t + (N_0 + 1)K)| \geq \left(\frac{3}{2} \beta - 1 - \frac{n - 1}{2n}\right) M(t)
\right]
\geq \mathbb{P}
\left[
 s_{i_0}(t + N_0K) = M(t), s_{i_0}(t + N_0K) \geq M(t)/2
\right]
\times \mathbb{P}
\left[
 \exists \tau \in [t + N_0K, t + (N_0 + 1)K - 1] \text{ s.t. } (i_*, i_0) \in E^-_\tau
\right]
\times \mathbb{P}
\left[
 D_\tau = 1
\right]
\times \mathbb{P}
\left[
 B_m = D_m = 0, m \neq \tau \in [t + N_0K, t + (N_0 + 1)K - 1]
\right)
\geq \vartheta_0,
\end{align*}
\]

where \( \vartheta_0 = q_0 \rho_* d((1 - d)(1 - b))^{K-1} \). This implies

\[
\mathbb{P}(M(t + N_0K) \geq 3(\beta - 1) M(t)/2) \geq \vartheta_0. \tag{7}
\]

Now assume that \( M(0) > 0 \) so that \( U(m) = \log(M(mN_0K)) \) for \( m \geq 0 \) is well defined. Note that from Lemma 5 and (7), we have:

\[
\mathbb{E} \{ U(m + 1) - U(m) \} \geq -N_0K \log(2n) + \vartheta_0 \log (3(\beta - 1)/2).
\]

For \( \beta \) large enough, the r.h.s. in the above inequality is strictly positive. We can then easily conclude, using classical arguments in random walks that the process \( U(m) \) has a strictly positive drift, from which it can be deduced that \( \mathbb{P}(\liminf_{m \to \infty} M(mN_0K) = \infty) = 1 \) (for \( \beta \) large enough). Using Lemma 5 one can easily conclude the desired theorem. \( \blacksquare \)

**Proof of Proposition 1**

Assume that for some \( q_* > 0 \) we have \( \mathbb{P}(\limsup_{t \to \infty} \max_{i \in \mathcal{V}} |s_i(t)| = \infty) \geq q_* \). There must be a node \( i_0 \) satisfying \( \mathbb{P}(\limsup_{t \to \infty} |s_{i_0}(t)| = \infty) \geq q_*/n \). Let \( C_0 > 0 \), and define \( T^*_1 := \inf_t \{ |s_{i_0}(t)| \geq C_0 \} \). \( T^*_1 \) is a stopping time. Let \( Y_0 > 0 \) be an integer. We can further recursively define \( T^*_2, \ldots, T^*_m, \ldots \)

\[
T^*_{m+1} := \inf_{t \geq T^*_m + Y_0} \{ |s_{i_0}(t)| \geq C_0 \}.
\]

Based on Theorem 4.1.3 in [20], each \( T^*_m \) is a stopping time for all \( m \geq 0 \) and \((G_{T^*_1}, B_{T^*_1}, D_{T^*_1}), \ldots, (G_{T^*_1 + Y_0-1}, B_{T^*_1 + Y_0-1}, D_{T^*_1 + Y_0-1}); (G_{T^*_2}, B_{T^*_2}, D_{T^*_2}), \ldots, (G_{T^*_2 + Y_0-1}, B_{T^*_2 + Y_0-1}, D_{T^*_2 + Y_0-1}); \ldots\)

are independent random variables that are also independent of \( \mathcal{F}_{T^*_1-1} \). In addition, we have \( \mathbb{P}(T^*_m < \infty, m = 1, 2, \ldots) \geq q_*/n \). Under Assumption A5, \( \mathcal{G}([T^*_1, T^*_1 + K - 1]) \) being strongly connected is a sure event. As a result, there exists another node \( i_1 \in \mathcal{V} \setminus i_0 \) and \( \tau_0 \in [T^*_1, T^*_1 + K - 1] \)}
such that \((i_0, i_1) \in \mathcal{E}_{\tau_0}\). Assume that \(s_{i_0}(\tau_0) = s_{i_0}(T_1^* )\). We treat two cases: \(\sigma_{i_0 i_1} = -\) and \(\sigma_{i_0 i_1} = +\).

(i) \(\sigma_{i_0 i_1} = -\).

- If \(\beta = 1\), then \(|\beta s_{i_0}(\tau_0) + (1 - \beta)s_{i_1}(\tau_0)| = |\beta s_{i_0}(\tau_0)| = |s_{i_0}(T_1^* )| \geq C_0;\)
- If \(\beta \neq 1\) and \(|s_{i_1}(\tau_0)| < \beta C_0/(2|1 - \beta|), \) then \(|\beta s_{i_0}(\tau_0) + (1 - \beta)s_{i_1}(\tau_0)| \geq \beta C_0 - (1 - \beta)|s_{i_1}(\tau_0)| \geq \beta C_0/2.\)

(ii) \(\sigma_{i_0 i_1} = +\).

- If \(\alpha = 1\), then \(|\alpha s_{i_0}(\tau_0) + (1 - \alpha)s_{i_1}(\tau_0)| = C_0.\)
- If \(\alpha \neq 1\) and \(|s_{i_1}(\tau_0)| < \alpha C_0/(2|1 - \alpha|), \) then \(|\alpha s_{i_0}(\tau_0) + (1 - \alpha)s_{i_1}(\tau_0)| \geq \alpha C_0/2.\)

Now \(s_{i_1}(\tau_0 + 1) = -\beta s_{i_0}(\tau_0) + (1 - \beta)s_{i_1}(\tau_0) \) when \(i_0\) is the unique node in \(N_{i_1}^{-}(\tau_0)\) and \(D_{\tau_0} = 1\).

Also observe that \(s_{i_1}(\tau_0 + 1) = \alpha s_{i_0}(\tau_0) + (1 - \alpha)s_{i_1}(\tau_0) \) when \(i_0\) is the unique node in \(N_{i_1}^{+}(\tau_0)\) and \(B_{\tau_0} = 1.\)

Independence ensures that \((B_{T_1^*}, D_{T_1^*}), \ldots, (B_{T_{m+1}^*} + Y_{t_0-1}, D_{T_{m+1}^*} + Y_{t_0-1})\) have the same distribution as \((B_0, D_0).\)

We can therefore simply bound the probabilities of the above events and establish

\[
\mathbb{P}(\exists i_1 \in \mathcal{V} \setminus \{i_0\} : |s_{i_1}(T_1^* + K)| \geq \phi C_0) \geq \chi_0,
\]

where \(\chi_0 = ((1-b)(1-d))^{2K-1} \min\{b, d\} p_s(1-p^*)^{n-2} \) and \(\phi = \min\{[\alpha/(2|1 - \alpha|), \alpha/2, [\beta/(2|1 - \beta|), \beta/2, 1]\}

(we use \([\cdot]\) to indicate that the corresponding term is taken into account in the min only if it is well defined). Repeating the analysis on \(T_2^*\) we obtain

\[
\mathbb{P}(\exists i_m \in \mathcal{V} \setminus \{i_0\} : |s_{i_m}(T_m^* + K)| \geq \phi C_0) \geq \chi_0.
\]

Since we have a finite number of nodes, independence allows us to invoke the second Borel-Cantelli Lemma (cf. Theorem 2.3.6 in [25]) and conclude that

\[
\mathbb{P}(\exists \text{ (deterministic) } i_1 \in \mathcal{V} \setminus \{i_0\} : \limsup_{t \to \infty} |s_{i_1}(t)| \geq \phi C_0 |T_m^* < \infty, m = 1, \ldots) = 1. \tag{8}
\]

Note that \(C_0\) can be chosen arbitrarily, and hence \([8]\) implies that there exists \(i_1 \in \mathcal{V} \setminus \{i_0\}\) such that

\[
\mathbb{P}(\limsup_{t \to \infty} |s_{i_1}(t)| = \infty | \limsup_{t \to \infty} \max_{i \in \mathcal{V}} |s_i(t)| = \infty) = 1. \tag{9}
\]

We can apply the same argument recursively, to show that \([9]\) holds for any node \(i_1\) in the network. □
4 The Relative-State-Reversion Model

In this section, we investigate the system dynamics under the relative-state-reversion model, and provide, as for the state-reversion model, conditions for convergence and divergence of the node states.

4.1 Main Results

The following theorem provides general conditions for convergence and divergence.

**Theorem 4.** Assume that for any \( t \geq 0 \), \( \mathcal{G}_t \equiv \mathcal{G} \) for some digraph \( \mathcal{G} \), and that each positive cluster of \( \mathcal{G} \) admits a spanning tree in \( \mathcal{G}^+ \). Further assume that A1 holds and that \( \alpha \in (0, (n - 1)^{-1}) \). Under the relative-state-reversion model, we have:

(i) If \( \sum_{t=0}^{\infty} d_t < \infty \), then \( P(\lim_{t \to \infty} s_i(t) \text{ exits}) = 1 \) for all node \( i \in \mathcal{V} \) and all initial states \( s(0) \);

(ii) If \( \sum_{t=0}^{\infty} d_t = \infty \), then there is an infinite number of initial states \( s(0) \) such that, as long as \( \beta > 0 \) and \( \mathcal{G} \) has at least two positive clusters,

\[
P(\lim_{t \to \infty} \max_{i,j \in \mathcal{V}} |s_i(t) - s_j(t)| = \infty) = 1. \tag{10}
\]

The first part of the above theorem indicates that when the environment is frozen, and when positive clusters are properly connected, then irrespective of the mean of the positive attentions \( \{b_t\}_{t=0}^{\infty} \), the system states converge if the attention each node puts in her negative neighbors decays sufficiently fast over time. The second part of the theorem states that when this attention does not decay, divergence is typically observed. We can easily build examples showing that essentially, the conditions in Theorem 4 cannot be relaxed.

Next, we provide a sufficient condition for weak consensus (meaning that the distances among the node states converge to zero almost surely). It is based on the following assumption.

**A7.** There is an integer \( K \geq 1 \) such that the union graph \( \mathcal{G}^+(\lceil t, t+K \rceil) = (\mathcal{V}, \bigcup_{\tau \in \lceil t, t+K-1 \rceil} \mathcal{E}_\tau^+) \) has a spanning tree for all \( t \geq 0 \).

**Theorem 5.** Assume that A1 and A7 hold and that \( \alpha \in (0, (n - 1)^{-1}) \). Denote \( K_0 = (2n-3)K \) and \( \rho_* = \min\{\alpha, 1 - (n-1)\alpha\} \). Define \( X_m = \frac{\rho_*^{-1}}{2} \prod_{t=mK_0}^{(m+1)K_0-1} (b_t(1 - d_t)) \), and \( Y_m = (1 + 2\beta(n-1))^{K_0} \prod_{t=mK_0}^{(m+1)K_0-1} (1 - d_t) \). Then under the relative-state-reversion model, if \( 0 \leq X_m - Y_m \leq 1 \) for all \( m \geq 0 \) and \( \sum_{m=0}^{\infty} (X_m - Y_m) = \infty \), we have \( P(\limsup_{t \to \infty} \max_{i,j \in \mathcal{V}} |s_i(t) - s_j(t)| = 0) = 1 \) for all initial states.
A direct consequence of Theorem 5 is that if \( b_t \equiv b \) and \( d_t \equiv d \) with \( b, d \in (0, 1) \) and \( \beta > 0 \), there exists \( d_* > 0 \) such that whenever \( d < d_* \), weak consensus is achieved almost surely. Observe that weak consensus does not necessarily guarantee the convergence of the state of each node. In fact, simple examples can be constructed with arbitrarily small \( \beta \) such that under the relative-state-reversion model, the state of each node grows arbitrarily large while weak consensus still holds. This contrasts the result for the state-reversion model: the condition \( \alpha + \beta < (n - 1)^{-1} \) of Theorem 1 prevents the state of individual nodes to diverge.

Next we provide conditions under which the maximal gap between the states of two nodes grows large almost surely, and establish a no-survivor property.

**A8.** There is an integer \( K \geq 1 \) such that the union graph \( G^-([t, t + K]) = (V, \bigcup_{t \in [t, t+K-1]} E^-_t) \) is weakly connected for all \( t \geq 0 \).

**Theorem 6.** Assume that \( A1, A6, \) and \( A8 \) hold and that \( \alpha \in [0, (2(n - 1))^{-1}] \). Let \( b_t \equiv b \) and \( d_t \equiv d \) for some constants \( b, d \in (0, 1) \). Let \( \beta > 0 \) and fix \( d \). Then under the relative-state-reversion, there is \( b_* > 0 \) such that whenever \( b < b_* \), we have \( \mathbb{P}(\lim_{t \to \infty} \max_{i,j \in V} |s_i(t) - s_j(t)| = \infty) = 1 \) for almost all initial states (under the standard Lebesgue measure).

**Proposition 2.** Assume that \( A1, A4 \) and \( A6 \) hold. Let \( b_t \equiv b \) and \( d_t \equiv d \) for some constants \( b, d \in (0, 1) \). Let \( \alpha, \beta > 0 \). Fix the initial value \( s(0) \). Then under the relative-state-reversion model, we have \( \mathbb{P}(\limsup_{t \to \infty} |s_i(t) - s_j(t)| = \infty, i \neq j \in V) = 1 \).

Finally, we investigate the clustering of states of nodes within each positive cluster.

**A9.** Assume that \( A3 \) holds and let \( V = \bigcup_{i=1}^{T_p} V_i \) be a positive-cluster partition of the total graph \( G_* \). There is an integer \( K \geq 1 \) such that the union graph \( G^+([t, t + K])|_{V_i} = (V_i \cup \bigcup_{t \in [t, t+K-1]} E^+_t|_{V_i}) \) has a spanning tree for all \( t \geq 0 \).

**Theorem 7.** Assume that \( A1, A3 \) and \( A9 \) hold and let \( V = \bigcup_{i=1}^{T_p} V_i \) be a positive-cluster partition of \( G_* \). Let \( \alpha \in (0, (n - 1)^{-1}) \). Define \( J(m) = \prod_{t=m}^{(m+1)K_0-1} b_t \) and \( W(m) = \sum_{t=m}^{(m+1)K_0-1} d_t \) with \( K_0 = (2n-3)K \). Further assume that \( \sum_{m=0}^{\infty} J(m) = \infty \), \( \sum_{t=0}^{\infty} d_t < \infty \), and \( \lim_{m \to \infty} W(m)/J(m) = 0 \). Then under the relative-state-reversion model, for any initial state \( s(0) \), there are \( T_p \) real-valued random variables, \( w^*_1, \ldots, w^*_p \), corresponding to each of the positive clusters, such that \( \mathbb{P}(\lim_{t \to \infty} s_i(t) = w^*_j, i \in V_j, j = 1, \ldots, T_p) = 1 \).
4.2 Supporting Lemmas

We list three martingale convergence lemmas (see e.g. [31]), and a result that will be instrumental in the analysis of the system convergence under the relative-state-reversion model.

**Lemma 6.** Let \( \{v_t\}_{t \geq 0} \) be a sequence of non-negative random variables with \( \mathbb{E}\{v_0\} < \infty \). Assume that for any \( t \geq 0 \),

\[
\mathbb{E}\{v_{t+1}|v_0, \ldots, v_t\} \leq (1 + \xi_t)v_t + \theta_t,
\]

where \( \{\xi_t\}_{t \geq 0} \) and \( \{\theta_t\}_{t \geq 0} \) are two (deterministic) sequences of non-negative numbers satisfying \( \sum_{t=0}^{\infty} \xi_t < \infty \) and \( \sum_{t=0}^{\infty} \theta_t < \infty \). Then \( \lim_{t \to \infty} v_t = v \) a.s. for some random variable \( v \geq 0 \).

**Lemma 7.** Let \( \{v_t\}_{t \geq 0} \) be a sequence of non-negative random variables with \( \mathbb{E}\{v_0\} < \infty \). Assume that for any \( t \geq 0 \),

\[
\mathbb{E}\{v_{t+1}|v_0, \ldots, v_t\} \leq (1 - \xi_t)v_t + \theta_t,
\]

where \( \{\xi_t\}_{t \geq 0} \) and \( \{\theta_t\}_{t \geq 0} \) are two (deterministic) sequences of non-negative numbers satisfying \( \forall t \geq 0, 0 \leq \xi_t \leq 1, \sum_{t=0}^{\infty} \xi_t = \infty, \sum_{t=0}^{\infty} \theta_t < \infty, \) and \( \lim_{t \to \infty} \theta_t/\xi_t = 0. \) Then \( \lim_{t \to \infty} v_t = 0 \) a.s..

**Lemma 8.** (Robbins-Siegmund) Let \( \{v_t\}_{t \geq 0}, \{\xi_t\}_{t \geq 0}, \{\theta_t\}_{t \geq 0} \) be sequences of non-negative random variables. Assume that for any \( t \geq 0 \),

\[
\mathbb{E}\{v_{t+1}|\mathcal{F}_t\} \leq (1 + \xi_t)v_t + \theta_t,
\]

where \( \mathcal{F}_t = \sigma(v_0, \ldots, v_t; \xi_0, \ldots, \xi_t; \theta_0, \ldots, \theta_t) \). Suppose \( \sum_{t=0}^{\infty} \xi_t < \infty \) and \( \sum_{t=0}^{\infty} \theta_t < \infty \) almost surely. Then \( \lim_{t \to \infty} v_t = 0 \) a.s. for some random variable \( v \geq 0 \).

**Lemma 9.** Define \( h(t) := \min_{i \in V} s_i(t), H(t) := \max_{i \in V} s_i(t), \) and \( \mathcal{H}(t) := H(t) - h(t). \) Assume that \( \alpha \in [0, (n-1)^{-1}] \) and that \( \sum_{t=0}^{\infty} d_t < \infty \). Then under the relative-state-reversion model, for all initial states, \( h(t), H(t), \mathcal{H}(t) \) converges almost surely as \( t \) grows large.

**Proof.** We first prove the convergence of \( \mathcal{H}(t). \) On can easily see that when \( \alpha \in [0, (n-1)^{-1}] \) and \( D_t = 0, H(t+1) \leq H(t), h(t+1) \geq h(t), \) and thus \( \mathcal{H}(t+1) \leq \mathcal{H}(t). \) When \( D_t = 1, \mathcal{H}(t+1) \leq (2\beta(n-1)+1)\mathcal{H}(t). \) We deduce that:

\[
\mathbb{E}\{\mathcal{H}(t+1)|\mathcal{H}(t)\} \leq (1 + 2\beta(n-1)d_t)\mathcal{H}(t), \tag{11}
\]

which, in view of Lemma 8, implies that \( \mathcal{H}(t) \to \mathcal{H}_* \) almost surely for some \( \mathcal{H}_* > 0. \) Similarly, for \( H(t), \) we have \( \mathbb{E}\{H(t+1)|H(t)\} \leq H(t) + \zeta(t), \) where \( \zeta(t) := (1 + 2\beta(n-1)d_t)H(t). \) Since \( \mathcal{H}(t) \)}
converges a.s. and \( \sum d_t < \infty \), we deduce that \( \sum \zeta(t) < \infty \) a.s.. Now, the first Borel-Cantelli Lemma (Theorem 2.3.1, [25]) and the fact that \( H(t) \geq h(t) \) ensure that \( P(\inf_{t \geq 0} H(t) > -\infty) = 1 \). In other words, \( H(t) \) is almost surely lower bounded. Hence, we can still invoke Lemma 8 to conclude that \( H(t) \) converges almost surely as \( t \) grows large. The convergence of \( h(t) \) follows from a symmetric argument. 

4.3 Proofs of Main Results

We now prove the various results stated previously.

Proof of Theorem 4

(i) We investigate two cases: \( \sum_{t=0}^{\infty} b_t < \infty \), and \( \sum_{t=0}^{\infty} b_t = \infty \).

a). Assume that \( \sum_{t=0}^{\infty} b_t < \infty \). Since we also have \( \sum_{t=0}^{\infty} d_t < \infty \), the first Borel-Cantelli Lemma guarantees that almost surely, each node revises its state for only a finite number of slots. The desired claim follows obviously.

b). Assume that \( \sum_{t=0}^{\infty} b_t = \infty \). With \( \sum_{t=0}^{\infty} d_t < \infty \), from the first Borel-Cantelli Lemma, \( K(*) := \inf\{k \geq 0 : D_t = 0, \forall t \geq k\} \) is almost surely bounded. We note that \( K(*) \) is not a stopping time for \( \{D_t\}_{t \geq 0} \), but a stopping time for \( \{B_t\}_{t \geq 0} \) by independence of \( \{B_t\}_{t \geq 0} \) and \( \{D_t\}_{t \geq 0} \). Hence, the second Borel-Cantelli lemma ensures that

\[
K_{m+1} := \inf\{t \geq K_m : B_t = 1\}, \quad m = 0, 1, \ldots
\]

and \( K_0 := \inf\{t \geq K(*) : B_t = 1\} \) are stopping times for \( \{B_t\}_{t \geq 0} \). Now in view of the independence of \( \{G_t\}_{t \geq 0} \), \( \{B_t\}_{t \geq 0} \), and \( \{D_t\}_{t \geq 0} \), we know that \( \{G_{K_m}\}_{m \geq 0} \) is an independent process and each \( G_{K_m} \) satisfies \( P((i, j) \in E_{K_m}) \geq p_* \) for all \( (i, j) \in \mathcal{G} \) under Assumption A1.

Let \( \mathcal{V}^{\dagger} \) be a positive cluster of \( \mathcal{G} \). By assumption, \( \mathcal{V}^{\dagger} \) has a spanning tree. Since \( \alpha < 1/(n-1) \), the above discussion shows that at times \( K_m, m = 0, 1, \ldots \), the considered relative-state-reversion model defines a standard consensus dynamics on independent random graphs where each arc exists with probability at least \( p_* \) for any fixed time slot. It has become clear from existing works on randomized consensus dynamics (as follows from connectivity-independent graphs in [37], combining Theorem 1 in [9] and Theorem 3 in [21] for the i.i.d. case with \( \mu_t \equiv \mu \), or various other implicit results in the literature [22]), that the connectivity of \( \mathcal{V}^{\dagger} \) ensures that
\( P(\lim_{m \to \infty} \mathcal{H}^t(K_m) = 0) = 1 \), where \( \mathcal{H}^t(t) = \max_{i \in V^t} s_i(t) - \min_{i \in V^t} s_i(t) \). The monotonicity of \( \mathcal{H}^t(t) \) for \( t \geq K \) further ensures that \( P(\lim_{t \to \infty} \mathcal{H}^t(t) = 0) = 1 \). Finally, applying the analysis of Lemma 9 restricted to \( V^t \), we show that \( P(\lim_{t \to \infty} \max_{i \in V^t} s_i(t) = H^t_{\ast}) = 1 \) and that \( P(\lim_{t \to \infty} \min_{i \in V^t} s_i(t) = h^t_{\ast}) = 1 \) for some \( H^t_{\ast} \) and \( h^t_{\ast} \). Therefore, \( H^t_{\ast} = h^t_{\ast} \) almost surely, which implies \( \lim_{t \to \infty} s_i(t) = H^t_{\ast} = h^t_{\ast} \) almost surely for all \( i \in V^t \). This completes the proof.

(ii) Let \( V = \bigcup_{i=1}^{T_p} V_i \) be a positive-cluster partition of \( G \) with \( T_p \geq 2 \). Let \( \epsilon > 0 \). Set \( s_i(t) = j\epsilon \) for all \( i \in V_j \), \( j = 1, \ldots, T_p \). Applying the second Borel-Cantelli lemma, the divergence condition (10) can be simply established by investigating the evolution of \( \mathcal{H}(t) \) under the assumption that \( \sum_{t=0}^{\infty} d_t = \infty \).

**Proof of Theorem 5**

The proof relies on Lemma 7, cf., 22 for the analysis of randomized consensus.

Consider \( 2n - 3 \) intervals \([mK, (m + 1)K - 1], m = 0, \ldots, 2(n - 2)\). With Assumption A7, there is a center node \( v_m \in V \) in each of \( G([mK, (m + 1)K - 1]) \). As a result, we can find \( n - 1 \) center nodes (repetitions are allowed) out of the \( v_m \)’s and denote them as \( v_{m_1}, \ldots, v_{m_{n-1}} \), that satisfy either \( s_{v_{m_j}}(0) \leq (h(0) + H(0))/2 \), or \( s_{v_{m_j}}(0) > (h(0) + H(0))/2 \), for all \( j = 1, \ldots, n - 1 \). Without loss of generality, we consider the first case only.

Assume that \( D_t = 0 \) for \( t = 0, \ldots, 2(n - 2)K - 1 \). The following facts can be established using a similar method as that used to prove Lemma 2. Let \( i \in V \).

**F1.** If \( s_i(t) \leq \zeta_0 h(0) + (1 - \zeta_0)H(0) \) for some \( \zeta_0 \in (0, 1) \), then \( s_i(t+1) \leq \lambda_s \zeta_0 h(0) + (1 - \lambda_s \zeta_0)H(0) \), where \( \lambda_s = 1 - \alpha(n - 1) \).

**F2.** If \( s_i(t) \leq \zeta_0 h(0) + (1 - \zeta_0)H(0) \) for some \( \zeta_0 \in (0, 1) \) and \( (i, j) \in G_t \), then \( s_j(t+1) \leq \alpha \zeta_0 h(0) + (1 - \alpha \zeta_0)H(0) \).

By recursively applying F1 and F2, and exploiting the properties of \( v_{m_1}, \ldots, v_{m_{n-1}} \), we obtain

\[
P\left( s_i(K_0) \leq \frac{\rho_s K_0}{2} h(0) + (1 - \frac{\rho_s K_0}{2})H(0), \ i \in V \right) \\
\geq p_s^{n-1} \prod_{t=0}^{K_0-1} \left( b_t(1 - d_t) \right).
\]

This implies

\[
P\left( \mathcal{H}(K_0) \leq (1 - \frac{\rho_s K_0}{2})H(0) \right) \geq p_s^{n-1} \prod_{t=0}^{K_0-1} \left( b_t(1 - d_t) \right).
\]
On the other hand, from the definition of the algorithm we know that

$$P(H(t+1) \leq (1 + 2\beta(n-1))H(0)) = 1$$  \hspace{1cm} (13)

and

$$P(H(K_0) > H(0)) \leq 1 - \prod_{t=0}^{K_0-1} (1 - d_t).$$  \hspace{1cm} (14)

Since $\{B_t\}_{t \geq 0}$, $\{D_t\}_{t \geq 0}$, and $\{G_t\}_{t \geq 0}$ define independent processes, we conclude from (12), (13), and (14) that

$$E\{H((m + 1)K_0) | H(mK_0)\} \leq (1 - X_m + Y_m)H(mK_0).$$

The claim follows directly from Lemma 7 and (13).  

**Proof of Theorem 6**

One can easily see that:

$$P(H(t+1) \geq (1 - 2(n-1)\alpha)H(t)) = 1,$$

and

$$P(H(t+1) < H(t)) \leq b.$$

Define $L_0 := \inf\{t \in \mathbb{Z} : (1 + \beta)^t \geq 2(n-1)\}$. Consider time intervals $[mK, (m + 1)K - 1]$ for $m = 0, 1, \ldots, (n^2 - n)(L_0 - 1)$. Denote $K_{L_0} = K((n^2 - n)(L_0 - 1) + 1)$. Under Assumption A8 and based on the fact that there are at most $n(n-1)$ arcs, there are two nodes $i_*, j_* \in V$ and $L_0$ instants $0 \leq \tau_1 < \tau_2 < \cdots < \tau_{L_0} < K_{L_0}$ such that $(i_*, j_*) \in G_{\tau_k}^{-}$ and $|s_{i_*}(\tau_k) - s_{j_*}(\tau_k)| \geq H(\tau_k)/(n-1)$ for all $\tau_k$. As a result, we have

$$P(H(K_{L_0}) \geq |s_{i_*}(K_{L_0}) - s_{j_*}(K_{L_0})| \geq H(0)(1 + \beta)^{L_0} \cdot (n-1)^{-1}) \geq (dp_* (1 - p^*)^{n-2})^{L_0} (1 - b)^{K_{L_0}}.$$

The previous inequality is obtained by considering the events where $D_{\tau_k} = 1$ and $i_* = N_{j_*}^{-}(\tau_k)$ for all $\tau_k$ and $B_t = 0$ for all $t \in [0, K_{L_0} - 1]$.

Then the desired conclusion is obtained by applying the same argument as that used at the end of the proof of Theorem 3. This completes the proof.

**Proof of Proposition 2**

The result follows from a similar sample-path analysis using the second Borel-Cantelli lemma as in the proof of Proposition 1. Hence we omit the details.
Proof of Theorem 7

Let us focus on a given positive cluster $V^\dagger$ of $G$. Again, we use the following notations $H^\dagger(t) = \max_{i \in V^\dagger} s_i(t)$, $h^\dagger(t) = \min_{i \in V^\dagger} s_i(t)$, $H^\dagger(t) = H^\dagger(t) - h^\dagger(t)$. Applying the analysis of Lemma 6 on $V^\dagger$ we know that $H^\dagger(t)$, $H^\dagger(t)$, and $h^\dagger(t)$ converge to finite limits almost surely if $\sum_{t \geq 0} d_t < \infty$.

Applying the analysis of Theorem 5 on $V^\dagger$, we get

$$
E\{H^\dagger((m + 1)K_0)\mid H^\dagger(mK_0)\} \leq (1 - X_m) H^\dagger(mK_0) + (1 + 2\beta) \sum_{t = mK_0}^{(m+1)K_0-1} d_t H(t),
$$

where $X_m$ is defined in Theorem 5. From (11) we know that $E(H(t)) \leq H_0 \prod_{t=0}^\infty (1 + 2\beta(n-1)d_t)$ for all $t \geq 0$. Taking the expectation on both sides in (15), we obtain:

$$
E\{H^\dagger((m + 1)K_0)\} \leq (1 - X_m) E\{H^\dagger(mK_0)\} + \left[(1 + 2\beta)H_0 \prod_{t=0}^\infty (1 + 2\beta(n-1)d_t)\right] W(m).
$$

Moreover, since $\prod_{t=0}^\infty (1 - d_t) < \infty$, $\sum_{m=0}^\infty J(m) = \infty$ implies $\sum_{m=0}^\infty X_m = \infty$. In view of Lemma 7 we have $\lim_{m \to \infty} E\{H^\dagger((m + 1)K_0)\} = 0$ if $\lim_{m \to \infty} W(m)/J(m) = 0$. Invoking Fatou’s lemma (e.g., Theorem 1.6.5, [25]), we further conclude that $E\{\liminf_{m \to \infty} H^\dagger((m + 1)K_0)\} \leq \lim_{t \to \infty} E\{H^\dagger((m + 1)K_0)\} = 0$. Hence $E\{\lim_{m \to \infty} H^\dagger((m + 1)K_0)\} = 0$ since $H^\dagger((m+1)K_0)$ converges almost surely if $\sum_{t \geq 0} d_t < \infty$. Therefore, we have $P(\lim_{m \to \infty} H^\dagger((m + 1)K_0) = 0) = 1$, and $P(\lim_{t \to \infty} H^\dagger(t) = 0) = 1$. This means that $H^\dagger(t)$ and $h^\dagger(t)$ converge to the same limit, which must be the limit of the each node state in $V^\dagger$, which completes the proof.

\section{Conclusions}

Inspired by various examples from social, biological and engineering networks, the emerging behaviors of node states evolving over signed random networks in a dynamical environment were studied. Each node received positive and negative recommendations from its neighbors determined by the sign of the interaction arcs. After receiving recommendations, each node put a deterministic weight and a random attention on each of the recommendations and then updated its state. Various conditions were derived the almost sure convergence, divergence, and clustering for both state-reversion model and relative-state-reversion model. The results showed explicitly in general positive arcs contribute to convergence, negative arcs contribute to divergence, while
the structure of the sign patterns contribute to clustering. Some interesting future directions include the co-evolution of the signs of the interaction links along with the node states, as well as the optimal placement of negative links with the aim of breaking the effect of positive updates as much as possible.

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