Abstract. We first provide some properties of the Mellin transform of nonnegative random variables, such that monotonicity, injectivity and effect of size biasing. Convergence of Mellin transforms is also entirely formalized through convergence in distribution and uniform integrability. As an application, we study a problem raised by Harkness and Shantaram (1969) who obtained, under sufficient conditions, a limit theorem for sequences of nonnegative random variables build with the iterated stationary excess operator. We reformulate this problem through the concept of multiply monotone functions and through the convergence of the families build by the continuous time version of the iterated stationary excess operator and also by size biasing. The latter allows us to show that in our context, continuous time convergence is equivalent to discrete time convergence, that the conditions of Harkness and Shantaram are actually necessary and that the only possible limits in distribution are mixture of exponential with log-normal distributions.

Convergence in distribution, Mellin transform of random variables, Log-normal, Moment indeterminate, Multiply monotone functions, Normal limit theorem, Size biasing, Stationary excess operator, Uniform integrability.

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1. Introduction and main results

The Mellin transform of a nonnegative random variable \( X \) is defined by

\[
\mathcal{M}_X(\lambda) = \mathbb{E}[X^\lambda], \quad \text{for } \lambda \text{ in some domain of definition in the complex plane,}
\]

and could be also interpreted as the moment generating function of \( \log X \). We denote \( D_X \) the domain of definition of \( \mathcal{M}_X \) restricted on the real line.

Laplace transform of nonnegative random variables and characteristic functions of real-values random variables are always well defined, respectively on the half real line and on the real line, and they entirely characterize the distribution. At the contrary, the Mellin transform could have problems of definition and for this reason formalization of its injectivity is not straightforward. Nevertheless, injectivity on the Mellin transform seems to be commonly admitted in the literature and used without a precise reference. In Chapter VI of Widder’s book [20], Theorem 6a p. 243, it is stated that if the Mellin transforms of two nonnegative random variables \( X \) and \( Y \) are well defined on some strip \( \alpha < \text{Re}(z) < \beta \), and equal there, then \( X \overset{d}{=} Y \). We improve this result by showing that the same conclusion holds if the strip is replaced by an interval.

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We will see that the only significant information one can extract from identity (2) is that $Z_\infty$ has the same integer moment as an exponential distribution multiplicatively mixed by the Log-normal distribution with parameters depending on $l$ and on the value attributed to $E[Z_\infty]$. There is then a problem of determinacy in law for $Z_\infty$ since the Log-normal distribution is well known to be moment indeterminate.

Also motivated by this problem of indeterminacy, we were concerned with the natural question: what additional information on the distribution of $Z_\infty$ can we obtain if we study the continuous scheme $Z_t = \mathcal{E}_t(X)/c_t$, $t \in [0, \infty)$ where $\mathcal{E}_t$ is the continuous time stationary-excess operator given

$$X(t) := \frac{X}{E[X]} P(X \in dx), \quad t \in D_X.$$
by
\[ P(E_t(X) > x) = \frac{t}{E[X^t]} \int_x^\infty (u-x)^{t-1} P(X > u) \, du, \quad x \geq 0 \]
(3)

Notice that the limit \( Z_\infty \) obtained by the discrete scheme has necessarily the same distribution as the one of the continuous scheme.

In section 4, we make a digression and entirely formalize convergence of the Mellin transforms of general families of random variables (in both discrete and continuous time i.e. \( t \in \mathbb{N} \) and \( t \in [0, \infty) \)) through convergence in distribution and uniform integrability. The form (3), justifies our investigation on \( t \)-monotone functions in section 5. The results obtained in both sections 4 and 5 will allow us to contribute in section 6 to the problem raised by Harkness and Shantaram [8]. We simplify their problem and solve it as follows:

1. condition (1) is not only sufficient, but also necessary for the convergence in distribution of \( Z_t \) if we require some integrability on \( Z_\infty \);
2. under (1), convergence of \( Z_t \) in both schemes \( t \in \mathbb{N} \) or \( t \in [0, \infty) \), is equivalent to 
\[ \frac{X(t)}{\rho_t} \xrightarrow{d} X_\infty \quad \text{as } t \to \infty \quad \text{and } \quad t \in \mathbb{N} \text{ or } t \in [0, \infty), \]
the normalization \( \rho_t \) being necessarily equivalent to \( t c_t \) at infinity;
3. it holds that \( Z_\infty \xrightarrow{d} \epsilon X_\infty \) where \( \epsilon \) is independent from \( X_\infty \) and is exponentially distributed;
4. the only possible distributions for the limit \( X_\infty \) is Log-normal.

In what follows, we only deal with nonnegative random variables. The notation \( T \) stands for \( \mathbb{N} \), the set of nonnegative integers, or for the interval \( [0, \infty) \).

2. DEFINITENESS, MONOTONICITY AND INJECTIVITY OF THE MELLIN TRANSFORM

The Mellin transform of a nonnegative random variable \( X \) is defined by
\[ \lambda \mapsto E[X^\lambda], \quad \lambda \in D_X = \{ x \in \mathbb{R}; E[X^x] < \infty \}. \]

We recall Hölder inequality true for every real random variables
\[ E[|UV|] \leq E[|U|^p]^{\frac{1}{p}} E[|V|^q]^{\frac{1}{q}}, \quad p, q > 0, \quad \frac{1}{p} + \frac{1}{q} = 1, \]
(4)

whenever the expectations are finite. The equality holds in (4) if and only if there exist constants \( a, b \geq 0 \), not both zero, such that \( a|U|^p = b|V|^q \). It is then clear that for a positive random variable \( X \), the standard Lyapunov inequality holds:
\[ E[X^{\lambda \mu}]^{\frac{1}{\lambda} \mu} \leq E[X^{\lambda \mu_0}]^{\frac{1}{\lambda} \mu_0} \quad \text{for } 0 < \lambda \leq \lambda_0 \quad \text{and } \quad E[X^{\mu \mu_0}]^{\frac{1}{\mu} \mu_0} \leq E[X^{\mu}]^{\frac{1}{\mu} \mu_0} \quad \text{for } 0 < \mu \leq \mu_0, \]
(5)

whenever the expectations are finite. The latter justifies that if \( D_X \) contains some \( \lambda_0 > 0 \) (respectively some \( \mu_0 < 0 \)), then \( D_X \) contains the interval \( [0, \lambda_0] \) (respectively \( [\mu_0, 0] \)). It is then seen that \( D_X \) is an interval with extremities
\[ \mu_X = \inf \{ \lambda \in \mathbb{R}; E[X^\lambda] < \infty \} \quad \text{and } \quad \lambda_X = \sup \{ \lambda \in \mathbb{R}; E[X^\lambda] < \infty \}. \]
(6)
not necessarily included. Assume $\lambda_X > 0$. By the dominated convergence theorem, we see that the Mellin transform of $X$ is often differentiable on $(0, \lambda_X)$ and by (5) that

$$
\lambda \mapsto \mathbb{E}[X^\lambda]^{1/\lambda} \text{ is nondecreasing on } [0, \lambda_X).
$$

The last fact could be also seen as a consequence of this Proposition:

**Proposition 1.** For any nonnegative random variable $X$ such that $\lambda_X > 0$, the Mellin transform $M_X$ is log-convex on $[0, \lambda_X]$. If furthermore $X$ is non-deterministic, then strict log-convexity holds.

**Proof.** Let $g(\lambda) := \log \mathbb{E}[X^\lambda]$. Trivial computations lead to

$$
g''(\lambda) = \frac{\mathbb{E}[X^\lambda (\log X)^2] \mathbb{E}[X^\lambda] - \mathbb{E}[X^\lambda \log X]^2}{\mathbb{E}[X^\lambda]^2}, \quad 0 < \lambda < \lambda_X.
$$

Taking $p = q = 2, \ U = X^{1/2} \log X$ and $V = X^{1/2}$ in (4), we deduce that $g$ is convex. It is strictly convex unless $U$ and $V$ are proportional which is equivalent to $X$ deterministic. □

Proposition 1 gives an additional information:

**Corollary 1.** Let a nonnegative random variable $X$ such that $\lambda_X > 0$. For every $\lambda \in (0, \lambda_X)$, the function $t \mapsto M_X(\lambda + t)/M_X(t)$ is nondecreasing on $[0, \lambda_X - \lambda]$. It is further increasing whenever $X$ is non-deterministic.

**Proof.** Theorem 5.1.1 [19] p. 194 says that convexity of $x \mapsto g(x) = \log M_X(x)$ yields that its slopes are nondecreasing:

$$
\frac{g(y) - g(x)}{y - x} \leq \frac{g(z) - g(x)}{z - x} \leq \frac{g(z) - g(y)}{z - y}, \quad 0 \leq x < y < z < \lambda_X.
$$

Then,

$$
g(\lambda + s) - g(s) \leq g(\lambda + t) - g(t), \quad \text{for } 0 \leq s < t \text{ and } \lambda + t < \lambda_X, \tag{7}
$$

and the first assertion is proved. For the strict monotonicity, notice that equality holds in (7) only in case where the function $r \mapsto g(\lambda + r) - g(r)$ is not injective. Because of the differentiability of $g$, the latter reads $g'(\lambda + r) = g'(r)$ for some value of $r$. The latter is possible only if $g'$ is not injective, that is $g''(x) = 0$ for some value of $x$ and the second statement in Proposition 1 allows to conclude. □

Our result in Corollary 1 is the same than the one stated in Lemma 3.1 in [8] when $\lambda$ and $t$ are positive integers. Corollary 1 is also proved in [11], where the author also adapts the arguments of Lemma 3.1 in [8]. However, we found that the argument of continuity used in [11], appealing to a result of Kingman [10], does not fit his context. We clarify the approach of [11] with this second proof:

**Second proof of Corollary 1.** We first show that the sequence $u_n = M_X(\lambda u)$ satisfies

$$
\frac{u_{n+m}}{u_n} \leq \frac{u_{n+m+1}}{u_{n+1}}, \quad \text{for every } n, m \in \mathbb{N}, \ u > 0. \tag{8}
$$

Schwarz inequality (4), with $p = q = 2$, gives $\mathbb{E}[X^{d(n+1)}]^2 \leq \mathbb{E}[(X^{u(n+2)}] \mathbb{E}[X^{u(n+2)}]$. Then

$$
\frac{u_{n+1}}{u_n} \leq \frac{u_{n+2}}{u_{n+1}}, \quad \text{for every } n \in \mathbb{N},
$$
which is also equivalent to (8) from which we deduce that for each $u > 0$ and $m \in \mathbb{N}$, the sequence
\[ n \mapsto \frac{E[X^{(n+m)}]}{E[X^m]} \] is nondecreasing. \hfill (9)

Now, take $\lambda > 0$ and $t > s > 0$ with $s, t$ rationals of the form $s = p/q$ and $t = k/l$ so that $pl < qk$. Applying (9) with $u = \frac{\lambda}{ql}$, we obtain the inequality
\[ \frac{E[X^{\lambda(s+1)}]}{E[X^\lambda s]} \leq \frac{E[X^{\lambda(t+1)}]}{E[X^\lambda t]}, \] for all $\lambda > 0$ and all real numbers $s, t$ s.t. $0 < s < t$.

The proof is finished by replacing the couple $(s, t)$ by $(\frac{s}{\lambda}, \frac{t}{\lambda})$. Strict monotonicity is obtained as in the end of the first proof.

We found that in the literature, many papers invoke the injectivity of the Mellin transform without a precise reference. For instance, an informal discussion in exercise 1.13 in [5] appeals to Chapter VI in Widder’s book [20] where we found Theorem 6a p. 243 stating the following:

*If the Mellin transforms of two nonnegative random variables $X$ and $Y$ are well defined on some strip $\alpha < Re(z) < \beta$, and equal there, then $X \overset{d}{=} Y$.*

Widder’s theorem could be improved by the following Lemma:

**Lemma 1.** Let $X$ and $Y$ two nonnegative random variables such that their Mellin transforms are well defined on some interval $(\alpha, \beta) \subset \mathbb{R}$ and equal there, then $X \overset{d}{=} Y$.

The proof of this Lemma is a technique borrowed from [1] and based on a Blaschke’s theorem that allows to identify holomorphic functions given their restriction along suitable sequences:

**Theorem 1** (Blaschke, Corollary p. 312 in Rudin [14]). *If $f$ is holomorphic and bounded on the open unit disc $D$, if $z_1, z_2, z_3, \cdots$ are the zeros of $f$ in $D$ and if $\sum_{k=1}^{\infty} (1 - |z_k|) = \infty$, then $f(z) = 0$ for all $z \in D$.*

Using the one-to-one mapping of the strip $S = \{z \in \mathbb{C}, \ 0 < Re(z) < 1\}$ onto the open unit disc
\[ z \mapsto \theta(z) = \frac{e^{i\pi z} - i}{e^{i\pi z} + i}, \]
one can easily rephrase Blaschke’s theorem for function defined on the strip $S$:

**Corollary 2.** Two holomorphic functions on the strip $S$ are identical if their difference is bounded and if they coincide along a sequence $\alpha_1, \alpha_2, \alpha_3, \cdots$ in $S$, such that the series $\sum_{k=1}^{\infty} (1 - \frac{e^{i\pi \alpha_k - i}}{e^{i\pi \alpha_k + i}})$ diverge. For instance, the series diverge for the sequence $\alpha_k = \frac{1}{k}$, $k \geq 1$.

**Proof of Lemma 1.** It is enough to take $(\alpha, \beta) = (0, 1)$, to notice that both $\mathcal{M}_X$ and $\mathcal{M}_Y$ extend holomorphically on the strip $S$ and to conclude with Corollary 2 since $\mathcal{M}_X$ and $\mathcal{M}_Y$ coincide along the sequence $\alpha_k = k^{-1}$, $k \geq 2$ which is contained in $(0, 1)$.
3. **Mellin Transform and Size Biased Laws**

Let $X$ denote a non-deterministic nonnegative random variable. For $t \in \mathcal{D}_X$, the size biased law of order $t$ is denoted $X(t)$ and is a version of the weighted law

$$
\mathbb{P}(X(t) \in dx) = \frac{x^t}{\mathbb{E}[X^t]} \mathbb{P}(X \in dx), \quad x \geq 0.
$$

(10)

Chebychev’s association inequality says that

$$
\mathbb{E}[f(X)g(X)] \geq \mathbb{E}[f(X)] \mathbb{E}[g(X)],
$$

whenever the expectations are well defined and $f, g$ are both nondecreasing or nonincreasing real-valued functions. Taking $f(u) = u^t$, $g(u) = 1_{u>x}$, we see that $X(t) \geq_X t$ for $t \geq 0$, i.e.

$$
\mathbb{P}(X(t) > x) = \frac{\mathbb{E}[X^t 1_{X>x}]}{\mathbb{E}[X^t]} \geq \mathbb{P}(X > x), \quad \forall t, x \geq 0.
$$

(11)

Notice that the last stochastic inequality also justifies Corollary 1, since the Mellin transform could be computed as

$$
\mathbb{E}[X^\lambda] = \lambda \int_0^\infty x^{\lambda-1} \mathbb{P}(X > x) \, dx, \quad \lambda \in (0, \lambda_X),
$$

(12)

whenever the extremity $\lambda_X$ given by (6) is positive.

We list some elementary properties for the size biased law of a r.v. $X$:

(P0) For every $c > 0$ and $t \in \mathcal{D}_X$, we have

$$
X(0) = X \quad \text{and} \quad (cX)_t = cX(t).
$$

(P1) For every $\lambda, t$ such that $t, t+\lambda \in \mathcal{D}_X$ and measurable bounded function $g$, we have

$$
\mathbb{E}[X^\lambda(t)] = \frac{\mathbb{E}[X^{t+\lambda}]}{\mathbb{E}[X^t]}, \quad \mathbb{E}[g(X)_t] = \frac{\mathbb{E}[X^tg(X)]}{\mathbb{E}[X^t]}.
$$

(P2) For every $s, t \in \mathcal{D}_X$ such that $t+s \in \mathcal{D}_X$, we have

$$
(X(s))_t \overset{d}{=} X(s+t) \overset{d}{=} (X(t))_s.
$$

(P3) For every $s, t$ such that $st \in \mathcal{D}_X$, we have

$$
(X^s)_t \overset{d}{=} (X(st))_s.
$$

(P4) For every independent random variables $X, Y$ and $t \in \mathcal{D}_X \cap \mathcal{D}_Y$, we have

$$
(XY)_t \overset{d}{=} X(t)Y(t) \quad \text{(assuming that $X(t)$ and $Y(t)$ are independent)}.
$$

4. **Convergence of Sequences and Families of Mellin Transforms**

This section contains results dealing with sequences or families of Mellin transforms. As we did for the injectivity in Lemma 1, we felt it was important to also clarify the notion of convergence via Mellin transform. Next Theorem 2 and Proposition 3, will be crucial for handling section 6 below.

We recall that $\mathbb{T} = \mathbb{N}$ or $[0, \infty)$. In what follows,
(1) a property \((P_t)\) is said to be \textit{true for} \(t\) \textit{big enough}, if there exists \(t_0 \in T\) such that \((P_t)\) is true for \(t \geq t_0\);
(2) \((X_t)_{t \in T}\) denotes a family of nonnegative random variables indexed by the time \(t \in T\);
(3) by a subsequence of \((X_t)_{t \in T}\), we mean a collection of random variables \((X_{t(n)})_{n \in \mathbb{N}}\) obtained through a nondecreasing function \(t : \mathbb{N} \to T\) such that \(t(n) \to \infty\) as \(n \to \infty\);
(4) we always assume that for \(t\) big enough,
\[
\lambda_{X_t} = \sup \{ \lambda \in \mathbb{R}, \ E[X_t^\lambda] < \infty \} > 0;
\]
(5) for \(\lambda \geq 0\), we define informally
\[
m(\lambda) := \lim \inf_{t \in T} E[X_t^\lambda] \quad \text{and} \quad M(\lambda) := \lim \sup_{t \in T} E[X_t^\lambda].
\]

We also recall some basic ingredients related to convergence in distribution.

\textbf{Definition 1} (Billingsley [4]). Let a sequence \((X_n)_{n \in \mathbb{N}}\) of real-valued random variables.

(i) \((X_n)_{n \in \mathbb{N}}\) is called tight if
\[
\sup_{n \in \mathbb{N}} \mathbb{P}(|X_n| > x) \to 0 \quad \text{as} \quad x \to \infty.
\]

(ii) \((X_n)_{n \in \mathbb{N}}\) is called uniformly integrable if
\[
\sup_{n \in \mathbb{N}} E[|X_n| 1_{X_n > x}] \to 0 \quad \text{as} \quad x \to \infty.
\]

(iii) \(X_n\) converges in distribution to \(X_\infty\) if \(E[f(X_n)] \to E[f(X_\infty)]\), as \(n \to \infty\), for every continuous bounded (or continuous compactly supported) real function \(f\).

We are going to study convergence in distribution for families of nonnegative random variables.

\textbf{Definition 2.} Let family \((X_t)_{t \in T}\) a family of nonnegative random variables.

(i) We say that the family \((X_t)_{t \in T}\) ultimately tight if
\[
\limsup_{t \in T} \mathbb{P}(X_t > x) \to 0, \quad \text{as} \quad x \to \infty.
\]

(ii) We say that the family \((X_t)_{t \in T}\) is \(\lambda\)-uniformly integrable, and we denote \((X_t)_{t \in T}\) is \(\lambda - UI\), if \(\lambda \in (0, \lambda_{X_t})\) for \(t\) big enough and
\[
\limsup_{t \in T} E[X_t^\lambda 1_{X_t > x}] \to 0, \quad \text{as} \quad x \to \infty.
\]

(iii) We say that the family \((X_t)_{t \in [0, \infty)}\) converge in distribution, if every subsequence \((X_{t(n)})_{n \in \mathbb{N}}\) converges in distribution in the usual sense (iii) of the preceding Definition.

\textbf{Remark 1.} We can notice that:

a) \(\lambda - \text{uniform integrability of} \ (X_t)_{t \geq 0}\) is equivalent to \(1 - \text{uniform integrability of} \ (X_t^\lambda)_{t \geq 0}\).

b) If \(T = \mathbb{N}\), then Definitions 1 and 2 are equivalent \(0 \lambda_{X_t} \in (0, \infty)\) for every \(t \in \mathbb{N}\). In general, this is untrue if \(T = [0, \infty)\).
c) If $T = [0, \infty)$, then $(X_t)_{t \geq 0}$ is ultimately tight (respectively $\lambda - UI$) if and only if there exists some positive $t_0$, big enough, such that $(X_t)_{t \geq t_0}$ is tight (respectively $(X_t^\lambda)_{t \geq t_0}$ is uniformly integrable) in the same sense that Billingsley gave in Definition 1.

We start this section with the following result that clarifies the link between ultimate tightness and uniform integrability:

**Proposition 2.** Let $\lambda_0 > 0$ and $(X_t)_{t \in T}$ a family of nonnegative random variables. Recall the function $m(.)$ and $M(.)$ are given by (13).

1) If $(X_t)_{t \in T}$ is $\lambda_0$-uniformly integrable, then $M(\lambda_0) < \infty$.

2) If $M(\lambda_0) < \infty$, then $(X_t)_{t \in T}$ is ultimately tight and also $\lambda$-uniformly integrable for every $\lambda \in (0, \lambda_0)$.

3) Assume $m(\lambda_0) > 0$ and $M(\lambda_0 + \epsilon) < \infty$, for some $\epsilon > 0$. Then $\lambda_0$-uniform integrability of $(X_t)_{t \in T}$ is equivalent to its ultimate tightness.

**Proof.** 1) Write

$$M(\lambda_0) \leq x^\lambda + \limsup_{t \in T} E[X_t^\lambda 1_{X_t > x}],$$

for $x$ big enough and deduce that $M(\lambda_0) < \infty$.

2) Hölder and Markov inequalities give

$$\limsup_{t \in T} E[X_t^\lambda 1_{X_t > x}] \leq M(\lambda_0) \frac{\lambda}{\lambda_0} \limsup_{t \in T} P(X_t > x) \frac{\lambda_0 - \lambda}{\lambda_0} \leq \frac{M(\lambda_0)}{x^{\lambda_0 - \lambda}}, \quad 0 < \lambda < \lambda_0; \quad (14)$$

Ultimate tightness and $\lambda$-uniform integrability are then immediate.

3) Inequality (11) gives

$$E[X_t^\lambda] P(X_t > x) \leq E[X_t^\lambda 1_{X_t > x}], \quad \text{for all } x > 0 \text{ and } t \in T.$$

Using again inequality (14), with the couple $(\lambda, \lambda_0)$ replaced by $(\lambda_0, \lambda_0 + \epsilon)$, obtain

$$m(\lambda_0) \limsup_{t \in T} P(X_t > x) \leq \limsup_{t \in T} E[X_t^\lambda 1_{X_t > x}] \leq M(\lambda_0 + \epsilon) \frac{\lambda_0}{\lambda_0 + \epsilon} \limsup_{t \in T} P(X_t > x) \frac{\epsilon}{\lambda_0 + \epsilon}. \quad \square$$

Next Theorem rephrases and improves some results borrowed from the monograph of Billingsley [4]:

**Theorem 2.** Let $(X_t)_{t \in T}$ a family of nonnegative random variables such that $\lambda_0 \in (0, \lambda_X)$, for some $\lambda_0 > 0$ and $t$ big enough.

1) Let $X_\infty$ a nonnegative random variable. The following assertions are equivalent, as $t \to \infty$:

   (i) $X_t \xrightarrow{d} X_\infty$ and $(X_t)_{t \in T}$ is $\lambda_0$-uniformly integrable;

   (ii) $X_t \xrightarrow{d} X_\infty$, $\lambda_0 \in D_{X_\infty}$ and $E[X_t^\lambda_0] \to E[X_\infty^\lambda_0]$;

   (iii) $\lambda_0 \in D_{X_\infty}$ and for every $\lambda \in [0, \lambda_0]$, $E[X_t^\lambda] \to E[X_\infty^\lambda]$.

2) Let $\lambda_1 \in (0, \lambda_0)$ and assume that $E[X_t^\lambda]$ converges as $t \to \infty$ to a well defined function $f(\lambda)$, $\lambda \in [\lambda_1, \lambda_0]$. Then (iii) holds and $f$ is well defined on $[0, \lambda_0]$ by $f(\lambda) = E[X_\infty^\lambda]$. 

Proof. The proof is conducted by reasoning on subsequences.

1) (i) $\implies$ (iii): it is a direct application of Theorem 25.12 p. 338 in [4], using Remark 1 and Proposition 2.

(iii) $\implies$ (ii) is treated as follows: Since $M(\lambda_0) < \infty$, then by Proposition 2, the family $(X_t)_{t \in T}$ is ultimately tight. Lemma 1 insures that any subsequence $(X_{t(n)})_{n \in \mathbb{N}}$, if converging in distribution as $n \to \infty$, necessarily converge to the law of $X_\infty$. Corollary in [4] p.337 allows to conclude that $X_t \xrightarrow{d} X_\infty$ as $t \to \infty$.

(ii) $\implies$ (i): we use the following representation valid for any nonnegative random variables $Z$ such that $E[Z^4] < \infty$:

$$E[Z^4 \mathbb{1}_{Z \leq x}] = x^4 P(Z \leq x) - \lambda \int_0^x u^{\lambda - 1} P(Z \leq u) \, du, \quad x \geq 0.$$  

Choose $x_0$ a continuity point of $u \mapsto P(X_\infty \leq u)$. By the dominated convergence theorem, we have

$$\lim_{t \to \infty} E[X_t^4 \mathbb{1}_{X_t \leq x_0}] = \lim_{t \to \infty} \left( x^4 P(X_t \leq x) - \lambda \int_0^{x_0} u^{\lambda - 1} P(X_t \leq u) \, du \right) = x^4 P(X_\infty \leq x_0) - \lambda \int_0^{x_0} u^{\lambda - 1} P(X_\infty \leq u) \, du = E[X_\infty^4 \mathbb{1}_{X_\infty \leq x_0}]$$

Since $\lim_{t \to \infty} E[X_t] = E[X_\infty]$, we also have $\lim_{t \to \infty} E[X_t^4 \mathbb{1}_{X_t > x_0}] = E[X_\infty^4 \mathbb{1}_{X_\infty > x_0}]$, that is, for every $\epsilon > 0$, there exists $t_0 \in T$ such $|E[X_t^4 \mathbb{1}_{X_t > x_0}] - E[X_\infty^4 \mathbb{1}_{X_\infty > x_0}]| < \epsilon$. Now choose $\epsilon > 0$, then $x_0$ big enough so that $E[X_\infty^4 \mathbb{1}_{X_\infty > x_0}] < \epsilon$. We deduce that

$$E[X_t^4 \mathbb{1}_{X_t > x}] \leq E[X_t^4 \mathbb{1}_{X_t > x_0}] < E[X_\infty^4 \mathbb{1}_{X_\infty > x_0}] + \epsilon < 2\epsilon, \quad t \geq t_0, \quad x \geq x_0.$$  

2) We adapt a part of the proof of Corollary 1.6 p. 5 given in Schilling and al. [15] in the context of convergence of sequences on completely monotone functions. Helly’s selection theorem allows a shortcut since there exists a subsequence $(X_{t(n)})_{n \in \mathbb{N}}$ satisfying $X_{t(n)} \xrightarrow{d} X_\infty$, as $n \to \infty$.

Fix $\lambda \in [\lambda_1, \lambda_0]$. For every function $h : [0, \infty) \to [0, 1]$, compactly supported, we find, by Fatou Lemma, that

$$E[h(X_\infty) X_\infty^\lambda] = \lim_{s \to \infty} E[h(X_{t(n)}) X_{t(n)}^\lambda] \leq \lim_{s \to \infty} E[X_{t(n)}^\lambda] = f(\lambda).$$  

Monotone convergence theorem gives a first inequality

$$E[X_\infty^\lambda] = \sup_h E[h(X_\infty) X_\infty^\lambda] \leq f(\lambda).$$  

Now, fix $\epsilon > 0$, choose a continuity point $x$ of the distribution function of $X_\infty$, then apply the fact that $X_{t(n)} \xrightarrow{d} X_\infty$, identity (12) and the dominated convergence theorem, in order to get that for $n$ big enough,

$$E[X_{t(n)}^\lambda \mathbb{1}_{X_{t(n)} \leq x}] - E[X_\infty^\lambda \mathbb{1}_{X_\infty \leq x}] = \lambda \int_0^x u^{\lambda - 1} \left( P(u < X_{t(n)} \leq x) - P(u < X_\infty \leq x) \right) \, du \leq x^\lambda \epsilon.$$  

Since $(X_{t(n)})_{n \in \mathbb{N}}$ is $\lambda - UI$, then $E[X_{t(n)}^\lambda \mathbb{1}_{X_{t(n)} > x}] < \epsilon$ for all $x$, $n$ big enough. Finally get for all $\epsilon > 0$,

$$E[X_{t(n)}^\lambda] - E[X_\infty^\lambda] \leq E[X_{t(n)}^\lambda \mathbb{1}_{X_{t(n)} > x}] + E[X_{t(n)}^\lambda \mathbb{1}_{X_{t(n)} \leq x}] - E[X_\infty^\lambda \mathbb{1}_{X_\infty \leq x}] < \epsilon(1 + x^\lambda).$$
The latter proves the second inequality \( f(\lambda) = \lim_{t \to \infty} \mathbb{E}[X_{t(n)}^\lambda] \leq \mathbb{E}[X_{\infty}^\lambda] \). All in one, we have that

\[
f(\lambda) = \mathbb{E}[X_{\infty}^\lambda], \quad \text{for every } \lambda \in [\lambda_1, \lambda_0].
\]

As in point 1) above, notice that the family \((X_t)_{t \in \mathbb{T}}\) is ultimately tight, and by Lemma 1, each subsequence of it, if converging in distribution, necessarily converge to the distribution of \(X_{\infty}\). Use again the Corollary in [4] p. 337 in order to have \(X_t \xrightarrow{d} X_{\infty} \) as \( t \to \infty \). To conclude, use \(\lambda\)-uniform integrability of \((X_t)_{t \in \mathbb{T}}\) and then implication \((i) \implies (ii)\) in point 1) above.

Now, consider two families of nonnegative random variables \((U_t)_{t \in \mathbb{T}}\) and \((V_t)_{t \in \mathbb{T}}\) such that \(U_t \xrightarrow{d} U_{\infty} , \ V_t \xrightarrow{d} V_{\infty}\) and \(U_t\) and \(V_t\) independent for every \(t \in \mathbb{T}\). It then is trivial that \(U_t V_t \xrightarrow{d} U_{\infty} V_{\infty}\). As a consequence of Theorem 2, we deduce a kind of converse:

**Corollary 3.** Let \((U_t)_{t \in \mathbb{T}}\), \((V_t)_{t \in \mathbb{T}}\) and \((W_t)_{t \in \mathbb{T}}\) three families of nonnegative random variables such that \(U_t\) and \(V_t\) are independent for each \(t \in \mathbb{T}\) and such that

(i) the factorizations in law \(W_t \xrightarrow{d} U_t V_t\) holds;

(ii) the convergences in distribution \(W_t \xrightarrow{d} U_{\infty}\) and \(V_t \xrightarrow{d} V_{\infty}\) hold as \( t \to \infty\);

(iii) there exists \(\lambda_0 > 0\) such that \((W_t)_{t \in \mathbb{T}}\) is \(\lambda_0 - UI\) or such that \(\lim_{t \to \infty} \mathbb{E}[W_t^\lambda] < \infty\).

Then, \(U_t \xrightarrow{d} U_{\infty}\), where the distribution of the random variable \(U_{\infty}\) is well defined by its Mellin transform given by \(\mathbb{E}[U_{\infty}^\lambda] = \mathbb{E}[W_{\infty}^\lambda] / \mathbb{E}[V_{\infty}^\lambda], \) for every \(\lambda \in [0, \lambda_0]\).

**Proof.** Notice that \(\lambda_0 \in D_{W_t} = D_{U_t} \cap D_{V_t}\) for \(t\) big enough and that both conditions in (iii) are equivalent by Theorem 2. Choose \(v_0 > 0\) a continuity point of \(x \mapsto \mathbb{P}(V_{\infty} > x)\) such that \(\mathbb{P}(V_{\infty} > v_0) > 1/2\) and notice also that for every \(t \in \mathbb{T}\) and \(x \geq 0\),

\[
\mathbb{E}[W_t^\lambda \mathbb{1}_{W_t > x}] \geq \mathbb{E}[(U_t V_t)^\lambda \mathbb{1}_{U_t > \frac{v_0}{\mathbb{V}_{\infty}^\lambda}, V_t > v_0}] = \mathbb{E}[U_t^\lambda \mathbb{1}_{U_t > \frac{v_0}{\mathbb{V}_{\infty}^\lambda}}] \mathbb{E}[V_t^\lambda \mathbb{1}_{V_t > v_0}]
\]

\[
\geq \mathbb{E}[U_t^\lambda \mathbb{1}_{U_t > \frac{v_0}{\mathbb{V}_{\infty}^\lambda}}] v_0 \mathbb{P}(V_t > v_0).
\]

Then use the fact that there exists \(t_0 \in \mathbb{T}\) such that \(\mathbb{P}(V_t > v_0) > \mathbb{P}(V_{\infty} > v_0) - 1/4 > 1/4\) for \(t \geq t_0\) and then

\[
\mathbb{E}[W_t^\lambda \mathbb{1}_{W_t > x}] \geq \frac{v_0}{4} \mathbb{E}[U_t^\lambda \mathbb{1}_{U_t > \frac{v_0}{\mathbb{V}_{\infty}^\lambda}}], \quad t \geq t_0, \ x \geq 0.
\]

The latter yields that the family \((U_t)_{t \in \mathbb{T}}\) is \(\lambda_0 - UI\), then apply Theorem 2. \(\square\)

**Next proposition studies the convergence of biased laws and improves Theorem 2.3 in [2]:**

**Proposition 3.** Let \((X_t)_{t \in \mathbb{T}}\) a family of nonnegative random variables such that \(X_t\) converges in distribution to a non-null random \(X_{\infty}\). Suppose \(0 < \lambda_0 < \min(\lambda_X, \lambda_{X_{\infty}})\) for \(t\) big enough and \(\lim_{t \to \infty} \mathbb{E}[X_t^\lambda] \to \mathbb{E}[X_{\infty}^\lambda]\). Then, we have the convergence of the size biased distributions of \((X_t)_{t \in \mathbb{T}}\):

\[
(X_t)_{\lambda} \xrightarrow{d} (X_{\infty})_{\lambda}, \quad as \ t \to \infty, \quad for \ every \ \lambda \in [0, \lambda_0].
\]
Proof. a) We start by proving (15) for $\lambda = \lambda_0$. By assumption, we have $0 < E[X^\lambda_0] < \infty$. Convergence in distribution of $X_t$ to $X_\infty$ is equivalent to $E[g(X_t)] \to E[g(X_\infty)]$ for every continuous, compactly supported function $g$, as $t \to \infty$. The function $h(x) = |x|^\lambda_0 g(x)$ is also a continuous, compactly supported function. By property (P1), we also have

$$E[h(X_t)] = E[X^\lambda_0] \to E[h(X_\infty)] = E[X^\lambda_0] E[g((X_\infty)_{\lambda_0})].$$

The limit (15) for $\lambda = \lambda_0$ follows by simplification in both sides of the last limit.

b) By Proposition 2, notice that $(X_t)_{t \in \mathbb{T}}$ is $\lambda - UI$ for every $\lambda \in (0, \lambda_0)$. Deduce by Theorem 2 that $\lim_{t \to \infty} E[X^\lambda_t] \to E[X^\lambda_\infty]$ for and reproduce step a) for $\lambda \in [0, \lambda_0)$.

\[\Box\]

Remark 2. By Theorem 2, finiteness of the quantity $M(\lambda_0)$ given by (13), or $\lambda_0$,-uniform integrability of $(X_t)_{t \in \mathbb{T}}$ is sufficient to insure that $\lambda_0 \in D_{X_\infty}$.

5. $t$-MONOTONE DENSITY FUNCTIONS

Let $x_+$ denotes $\max\{0, x\}$, $x \in \mathbb{R}$. The following definition extends the one of Schilling et al. [15] given for $t$ nonnegative integer.

Definition 3. Let $t \in (0, \infty)$. A function $f : (0, \infty) \to [0, \infty)$ is $t$-monotone if it is represented by

$$f(x) = c + \int_{(0,\infty)} (u-x)^{t-1} \nu(du), \quad x > 0$$

for some $c \geq 0$ and some measure $\nu$ on $(0, \infty)$.

Remark 3. When $t = 1$, representation (16) holds if and only if $f$ is nonincreasing and right-continuous. When $t = n$ is an integer greater than or equal to 2, representation (16) holds if and only if $f$ is $n-2$ times differentiable, $(-1)^j f^{(j)}(x) \geq 0$ for all $j = 0, 1, \cdots, n-2$ and $x > 0$, and $(-1)^{n-2} f^{(n-2)}(x)$ is nonincreasing and convex. Furthermore, by Theorem 1.11 p.8, [15]), the couple $(c, \nu)$ in (16) uniquely determines $f$.

A random variable $b_{a,b}$ is said to have the beta distribution with parameter $(a, b)$, $a, b > 0$, if it has the density function

$$\frac{1}{\beta(a, b)} x^{a-1} (1-x)^{b-1}, \quad x \in (0, 1) \quad \text{with} \quad \beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$ 

A random variable $g_a$ is said to have the Gamma distribution with parameter $a > 0$, if it has the density function $\Gamma(a)^{-1} x^{a-1} e^{-x}$, $x \in (0, \infty)$. It is well known that

$$g_a \overset{d}{=} b_{a,b} g_{a+b} \quad \text{and} \quad b_{a,b+c} \overset{d}{=} b_{a,b} b_{a+b,c}, \quad \text{for all } a, b, c > 0,$$

where in the first (respectively second) identity $b_{a,b}$ and $g_{a+b}$ (respectively $b_{a,b}$ and $b_{a+b,c}$) are assumed to be independent. Biasing on Beta and Gamma variables is nicely expressed by

$$(b_{a,b})(t) \overset{d}{=} b_{a+t,b}, \quad \text{and} \quad (g_a)(t) \overset{d}{=} g_{a+t}, \quad \text{for all } t > 0.$$

In the sequel, we denote $b_t$ the random variable defined by $b_0 = 1$ and $b_{1,t}$ if $t > 0$, that is

$b_t$ has the density function $t(1-x)^{t-1}$, $x \in (0, 1), t > 0.$
Also, $e = g_1$ denotes a random variable with standard exponential distribution. It is clear that $b_t d - e^{-\frac{x}{t}}$ and that $tb_t d - e$, as $t \to \infty$.

We propose the following characterization for $t$-monotone densities.

**Proposition 4.** Let $t \in (0, \infty)$.

1) The density function $f : (0, \infty) \to [0, \infty)$ of a positive random variable $Z$, is $t$-monotone, if and only if there exists a positive random variable $Y_t$ such that $f$ is represented

$$f(x) = t \int_{(0, \infty)} \left(1 - \frac{x}{u} \right)^{t-1} \frac{\mathbb{P}(Y_t \in du)}{u}, \quad x > 0.$$  

This equivalent to the factorization in law $Z d - b_t Y_t$, where $b_t$ has the beta distribution as in (19) and is independent from $Y_t$.

2) If $f$ is $t$-monotone, then it is also $s$-monotone for every $s \in (0, t)$.

3) Furthermore, the $\nu$-measure associated to $f$ through (16) is finite if and only if there exists a positive random variable $X$ such that $Y_t$ has the same distribution as the size biased random variable $X(t)$ given by (10) i.e.

$$Z d - b_t X(t).$$  

**Remark 4.** Bernstein characterization for completely monotone functions says that a function $f : (0, \infty) \to \mathbb{R}$ is $n$-monotone for every $n \in \mathbb{N}$, if and only if it is represented as the Laplace transform of a (unique) Radon measure $\nu$ on $[0, \infty)$:

$$f(x) = \int_{(0, \infty)} e^{-xu} \nu(du), \quad \lambda > 0 \text{ (respectively } x \geq 0);$$  

When $f$ is a density function associated to a positive random variable $Z$, the latter is equivalent to $Z d - eY$ where $Y$ is positive and independent from the exponentially distributed random variable $e$ and also to $f(x) = \mathbb{E}[e^{-x/Y}/Y], \ x > 0$. In this case, Bernstein characterization for $f$ could be reinterpreted as follows:

- use the Beta-Gamma algebra (17) in order to write

$$Z d - e d - b_t g_t Y, \quad \mathbb{P}(Z > x) = \mathbb{E} \left[ \left(1 - \frac{x}{g_t Y} \right)^t \right], \text{ for every } t x > 0;$$  

- use the fact that $(1 - \frac{x}{t})^t \to e^{-x}$ uniformly in $x > 0$, as $t \to \infty$, and that $\frac{g_t}{t} d - 1$, then rephrase (23) as:

$$\mathbb{P}(Z > x) = \lim_{n \to \infty} \mathbb{E} \left[ \left(1 - \frac{t}{g_t} \frac{x}{t Y} \right)^n \right] = \mathbb{E}[e^{-x/Y}] = \mathbb{P}(e Y > x), \text{ for every } x > 0.$$  

This clarifies the discussion made right after Proposition 2.2 in [9].

**Proof of Proposition 21.** 1) The density function $f$ is of the form (16) if and only if $c = 0$ and

$$f(x) = \int_{(0, \infty)} (1 - \frac{x}{u})^{t-1} u^{t-1} \nu(du), \quad x > 0,$$  

some measure \( \nu \) on \((0, \infty)\) such that
\[
1 = \int_0^\infty f(x)dx = \int_{(0, \infty)} t^{-1} u^t \nu(du) \quad \text{so that} \quad t^{-1} u^t \nu(dx) \text{ is a probability measure associated to some random variable, say } Y_t. \]
The second assertion is due to the fact that the density of the independent product of a non negative random variables \( U \) and \( V \) such that
\[
E_3 \quad \nu \text{ is a probability measure.} \]

2) It is enough to use the Beta-algebra (17): \( b_t \overset{d}{=} b_{s,t+} \).

3) \( E[Y_t^{-t}] = t^{-1} \nu(0, \infty) < \infty \) is equivalent to the identity \( Y_t \overset{d}{=} X(t) \) where \( X \) has the distribution \( \nu(0, \infty)^{-1} \nu(x) \).

\[ \square \]

6. Application: Refinement of the results of Harkness and Shantaram [8]

Recall that \( T = N \) or \([0, \infty)\). In what follows \( X \) is a nonnegative random variable, non identically null, such that \([0, \infty) \subset D_X\) i.e. \( X \) has moments of all positive orders.

We are willing to obtain a limit theorem for the family obtained by size biasing the distribution of \( X \). Identity (21) suggests to introduce a family of random variables \( (E_t(X))_{t \in T} \), such that each \( E_t(X) \) has its distribution defined through an operator \( E_t \) stemming from the identity
\[
E_t(X) :\overset{d}{=} \mathcal{B}_t(X(t), \mathcal{B}_t \text{ independent from } X(t)). \tag{24}
\]

By property (P2) and identity (17) for Beta distributions, notice that the family \( (E_t)_{t \in T} \) forms a semi-group of commuting operators:
\[
E_t(E_s(X)) \overset{d}{=} E_t(E_s(X)) \overset{d}{=} E(t+s)(X), \quad s, t \in T.
\]

By simple computations, we obtain that identity (24) is equivalent to one of the following expressions for the Mellin transform or for the distribution function of \( E_t(X) \):
\[
\frac{E[E_t(X)^\lambda]}{\Gamma(\lambda + 1)} = \frac{E[X^{\lambda+t}]}{\Gamma(\lambda + t + 1)} = \frac{E[X^{\lambda+t}]}{\Gamma(\lambda + t + 1)} E[X(t)], \quad \lambda \geq 0,
\]
\[
\mathbb{P}(E_t(X) > x) = E \left[ \left( 1 - \frac{x}{X(t)} \right)_+ \right] = \frac{E[(X-x)_+]}{E[X]} = \frac{t}{[X]} \int_x^\infty (u-x)^{t-1} \mathbb{P}(X > u) \, du, \quad x \geq 0,
\]
so that
\[
\mathbb{P}(E_1(X) \leq x) = \frac{1}{E[X]} \int_0^x \mathbb{P}(X > u) \, du, \quad x \geq 0.
\]

It is then clear that the operator \( E_1 \) corresponds to the stationary excess operator studied by Harkness and Shantaram [8] and also in [16, 11, 12, 18]. It is also seen that the operator \( E_n \) corresponds the \( n \)-th iterate by the composition of \( E_1 \):
\[
E_{n+1} = E_1 \circ E_n, \quad n \in N \setminus \{0\}.
\]

Harkness and Shantaram [8] solved the discrete time problem \((T = \mathbb{N})\) of:

- finding a deterministic normalization speed \( c_t \), \( t \in \mathbb{N} \), and sufficient conditions such that
\[
Z_t := \frac{E_t(X)}{c_t} \overset{d}{\rightarrow} Z_\infty \quad \text{as } t \in \mathbb{N} \text{ and } t \rightarrow \infty. \tag{25}
\]
describing the set of possible distributions for $Z_\infty$.

It is natural to study what kind of additional information we can recover from the continuous time problem, i.e. convergence (25) in case $T = [0, \infty)$ instead of $T = \mathbb{N}$, and to find the necessary and sufficient conditions such that

$$Z_t = \frac{1}{ct} b_t X(t) \xrightarrow{d} Z_\infty \quad \text{when } t \in T \text{ and } t \to \infty. \quad (26)$$

A direction for solving this problem is given by (19): we have that $t b_t \xrightarrow{d} c$ as $t \to \infty$. Take in Corollary 3

$$(U_t, V_t, W_t) = (t b_t, X(t), Z_t), \quad \text{with } \rho_t = t c_t$$

and assume $E[Z_\infty^\lambda]$ is finite for some $\lambda_0 \in T \setminus \{0\}$. Under the last assumption, it could be noticed that problem (26) is equivalent to finding necessary and sufficient conditions on the deterministic and positive normalization speed $\rho_t$, such that

$$X_t := \frac{X(t)}{\rho_t} \xrightarrow{d} X_\infty \quad \text{when } t \in T \text{ and } t \to \infty, \quad (27)$$

and such that $E[X_\infty^\lambda]$ is finite for some $\lambda_0 \in T \setminus \{0\}$. The random variable $Z_\infty$ in (26) is then linked to $X_\infty$ by

$$Z_\infty \overset{d}{=} e X_\infty, \quad \text{where } e \text{ is exponentially distributed, independent from } X_\infty. \quad (28)$$

Theorems 3 and 4 below, improve the discrete time problem in (26, case $T = \mathbb{N}$) studied by Harkness and Shantaram [8], by giving a sharper answer through the continuous time problem in (26, 27, case $T = [0, \infty)$).

**Theorem 3** (A normal limit theorem). Let $(X_t)_{t \in T}$ the family given by (27).

1) Assertions (i)-(ii)-(iii) are equivalent as $t \to \infty$:

(i) $X_t$ converges in distribution to a non-null and nonnegative random variable $X_\infty$ and

$$\lim_{t \to \infty} E[X_t^\lambda] = E[X_\infty^\lambda] < \infty, \quad \text{for some } \lambda_0 \in T \setminus \{0\};$$

(ii) $X_t$ converges in distribution to a non-null and nonnegative random variable $X_\infty$ and

$$\limsup_{t \to \infty} \frac{\rho_t + s \rho_t}{\rho_t} < \infty, \quad \text{for some } s \in T \setminus \{0\};$$

(iii) there exists a non-null and nonnegative random variable $X_\infty$ such that $[0, \infty) \in \mathcal{D}_{X_\infty}$ and

$$E[X_t^\lambda] \to E[X_\infty^\lambda], \quad \text{for all } \lambda \in [0, \infty).$$

2) In this case, necessarily, $\rho_t \xrightarrow{t \to \infty} E[X_\infty] E[X_t^{t+1}] / E[X_t^t]$ and there exists $c \geq 0$ such that

$$l(s) := \lim_{t \to \infty} \frac{\rho_t + s \rho_t}{\rho_t} = e^{cs}, \quad \text{for every } s \in T. \quad (29)$$

3) Assume one of the equivalent assertions in 1) and let $c$ given by (29). Choose $a := \log E[X_\infty]$ and let $N$ a random variable normally distribution with mean $a - \frac{c}{2}$ and variance $c$ (it is understood that $N = a$ when $c = 0$). Then the following holds:
(i) If $T = \mathbb{N}$, then the law of the random variable $X_\infty$ is not determined by its integer moments, we only have
\[
E[X_\infty^\lambda] = E[e^{\lambda N}] = e^{(a - \frac{1}{2})\lambda + \frac{1}{2}\lambda^2}, \quad \text{for all } \lambda \in \mathbb{N};
\]
(ii) if $T = [0, \infty)$, then $\log X_\infty \overset{d}{=} N$, i.e. (30) holds for all $\lambda \in [0, \infty)$.
(iii) We have the identity in law
\[
(X_\infty)_{(s)} \overset{d}{=} e^{cs} X_\infty, \quad \text{for every } s \in T.
\]

Proof. Using properties (P1) and (P2), we start by noticing the following identity valid for every $t, s, \mu \in T$ and $x \geq 0$:
\[
E[X_t^{s+\mu} \mathbb{1}_{X_t>x}] = E\left[\left. (X_{(t)})^\lambda \mathbb{1}_{X_{(t)}>xp_t} \right| \rho_t \right] = \frac{E[X_t^{s+\mu} \mathbb{1}_{X>xp_t}]}{\rho_t} E[X_t^\lambda] = \frac{E[X_t^{s+\mu} \mathbb{1}_{X>t}]}{\rho_t} E[X_t^\lambda] E[X_{(t+s)}^{\mu} \mathbb{1}_{X_{(t+s)}>xp_t}].
\]
In particular, for every $t, s, \mu \in T$ and $x, y \geq 0$, we have
\[
E[X_t^{s+\mu} \mathbb{1}_{X_t>x}] = E[X_t^s] \left(\frac{\rho_{t+s}}{\rho_t}\right)^\mu E[X_{t+s}^{\mu} \mathbb{1}_{X_{t+s}>x\rho_{t+s}^\mu}] = E[X_{t+s}^s] \mathbb{P}(X_{t+s} > x) \left(\frac{\rho_{t+s}}{\rho_t}\right)^\mu E[X_{t+s}^{\mu} \mathbb{1}_{X_{t+s}>xp_{t+s}}] = E[X_{t+s}^s] \mathbb{P}(X_{t+s} \leq y) \left(\frac{\rho_{t+s}}{\rho_t}\right)^\mu E[X_{t+s}^{\mu} \mathbb{1}_{X_{t+s}>yp_{t+s}}].
\]
If $X_t$ converges in distribution to $X_\infty$ and if $z$ is a continuity point of $u \mapsto \mathbb{P}(X_\infty \leq Ku)$, then for every $\epsilon \in (0, 1)$, there exists $t_\epsilon \in T$ such that
\[
|\mathbb{P}(X_\infty \leq Kz) - \mathbb{P}(X_{t_\epsilon} \leq z)| = |\mathbb{P}(X_{t_\epsilon} > z) - \mathbb{P}(X_\infty > Kz)| < \epsilon, \quad \text{for all } t \geq t_\epsilon.
\]
1) (iii) $\Rightarrow$ (ii) is easy, since by (32) with $x = 0$, we have
\[
E[X_{\infty}^{s+\mu}] = \lim_{t \to \infty} E[X_t^{s+\mu}] = E[X_\infty^s] E[X_\infty^\mu] \lim_{t \to \infty} \left(\frac{\rho_{t+s}}{\rho_t}\right)^\mu \text{ for every } s, \mu \in T \setminus \{0\}.
\]
(ii) $\Rightarrow$ (iii) Here $s \in T \setminus \{0\}$ is fixed and we proceed through two steps:

**Step 1:** we know that there exits $K > 0$, $t_s \in T$ such that
\[
\rho_t/\rho_{t+s} \geq K, \quad \text{for } t \geq t_s.
\]
The last inequality combined with identities (33) and (34) give that for every $t \geq t_s$, $x, y > 0$,
\[
E[X_{t}^s \mathbb{1}_{X_{t}>x}] \leq E[X_{t}^s] \mathbb{P}(X_{t+s} > Kx) \quad \text{and} \quad y^s \geq E[X_{t}^s] \mathbb{P}(X_{t+s} \leq Ky).
\]
Now we choose $\epsilon = 1/4$ in (35) with a continuity point $z = y_0$ such that $\mathbb{P}(X_\infty \leq Ky_0) > 1/2$. We get by the second inequality in (38) that
\[
y_0^s \geq \mathbb{E}[X_{t}^s \mathbb{1}_{X_{t}\leq y_0}] = E[X_{t}^s] \left(\mathbb{P}(X_\infty \leq Ky_0) - \frac{1}{4}\right) > \frac{1}{4} E[X_{t}^s], \quad t \geq t_{y_0}.
\]
We deduce that $C(y_0) := \sup_{t \geq t_0} E[X_{t}^s] < \infty$. The first inequality in (38) gives
\[
E[X_{t}^s \mathbb{1}_{X_{t}>x}] \leq C(y_0) \mathbb{P}(X_{t+s} > xK), \quad t \geq t_{y_0}, \ x > 0.
\]
The next step is to choose $\epsilon$ arbitrary small in (35) with a continuity point $z = x_0$, big enough, so that $\mathbb{P}(X_\infty > Kx_0) < \epsilon$ in order to have for $t \geq \max(t_s, t_{x_0}, t_{y_0})$ and $x \geq x_0$, that

$$
\mathbb{E}[X_t^s \mathbb{I}_{X_t > x}] \leq \mathbb{E}[X_t^s \mathbb{I}_{X_t > x_0}] \leq C(y_0) \mathbb{P}(X_{t+s} > xK) \leq C(y_0) \mathbb{P}(X_\infty > x_0K) + \epsilon \leq (1 + C(y_0))\epsilon.
$$

The latter justifies that $(X_t)_{t \in \mathbb{T}}$ is $s$-uniformly integrable.

**Step 2:** By (32) and (37), we have for every $t \geq t_s$,

$$
\mathbb{E}[X_t^s \mathbb{I}_{X_t > x}] = \mathbb{E}[X_t^s] \left( \frac{\rho_{t+s}}{\rho_t} \right)^s \mathbb{E}[X_{t+s}^s \mathbb{I}_{X_{t+s} > x}] \leq \mathbb{E}[X_t^s] \left( \frac{\rho_{t+s}}{\rho_t} \right)^s \mathbb{E}[X_{t+s}^s \mathbb{I}_{X_{t+s} > xK}].
$$

In step 1, we gained that the family $M(s) = \limsup_{t \to \mathbb{T}} \mathbb{E}[X_t^s] < \infty$ that $(X_t)_{t \in \mathbb{T}}$ is $s - UI$. The last inequality shows that $(X_t)_{t \in \mathbb{T}}$ is also $2s - UI$. Repeating the procedure, we obtain that $(X_t)_{t \in \mathbb{T}}$ is also $ms - UI$ for every positive integer $m$ and then, by Proposition 2, $(X_t)_{t \in \mathbb{T}}$ is also $\lambda - UI$ for every positive number $\lambda > 0$. Then we apply Theorem 2.

$(iii) \implies (i)$ is an immediate application of point 1) in Theorem 2.

$(i) \implies (iii)$: Identity (32) shows that

$$
\mathbb{E}[(X_t)^\mu_{(\lambda_0)}] = \left( \frac{\rho_{t+s}}{\rho_t} \right)^s \mathbb{E}[X_{t+s}^\mu \mathbb{I}_{X_{t+s} > x}] \cdot \mathbb{E}[X_{t+s}^\mu \mathbb{I}_{X_{t+s} > x}], \quad \text{for every } t, \mu \in \mathbb{T}.
$$

By Lemma 1, we deduce that

$$
X_{t+\lambda_0} \overset{d}{=} \frac{\rho_t}{\rho_{t+\lambda_0}} (X_t)_{(\lambda_0)}.
$$

By Proposition 3, we have that $(X_t)_{(\lambda_0)}$ converges in distribution and that the triplet $(U_t, V_t, W_t) = (X_{t+\lambda_0}, (X_t)_{(\lambda_0)}, \rho_{t+\lambda_0}/\rho_t)$ satisfies Corollary 3. We obtain that $\rho_{t+\lambda_0}/\rho_t$ converges as $t \to \infty$ and then we use $(ii) \implies (iii)$.

2) The first claim stems from $\mathbb{E}[X_\infty] = \lim_{t \to \infty} \mathbb{E}[X_{t}] / \rho_t$. For the second claim, notice by Corollary 1, that the function $t \mapsto \rho_t$ is asymptotically increasing, so that $l(s) \geq 1$ for every $s \in \mathbb{T}$. By (36), we recover that

$$
l(s)^\mu = \lim_{t \to \infty} \left( \frac{\rho_{t+s}}{\rho_t} \right)^s \frac{\mathbb{E}[X_{t+s}^\mu]}{\mathbb{E}[X_{\infty}^\mu] \mathbb{E}[X_{t}^\mu]}, \quad \text{for every } s, \mu \in \mathbb{T} \setminus \{0\}.
$$

From the symmetry in (39), it is seen that $l(s)^\mu = l(\mu)^s$ for every $s, \mu \in \mathbb{T}$. Taking $c = \log l(1) \geq 0$, we get representation (29). The latter could be also deduced from Lemma 1 in [1].

3) By Proposition 3, $(X_t)_{(s)} \overset{d}{\to} (X_\infty)_{(s)}$ for all $s \in \mathbb{T}$, as $t \to \infty$, and by properties (P0) and (P3), we obtain

$$
(X_t)_{(s)} \overset{d}{=} \frac{\rho_{t+s}}{\rho_t} X_{(t+s)} \overset{d}{\to} e^{c}\lambda X_\infty \overset{d}{=} (X_\infty)_{(s)}.
$$

The latter gives that the Mellin transform $\lambda \mapsto \mathcal{M}_{X_\infty}(\lambda) = \mathbb{E}[X_\infty^\lambda]$ is a solution of the functional equation:

$$
h(1) = \mathbb{E}[X_\infty] = e^a, \quad h(s + \mu) = e^{c\mu} h(s) \cdot h(\mu), \quad \text{for every } s, \mu \in \mathbb{T},
$$

and this could be also from identity (39). The function $h_0(\lambda) = e^{c\lambda(\lambda-1)}$ solves the equation without the initial condition. Any solution of the form $h = h_0 k$, has necessary $k(s + \mu) = k(s) k(\mu)$ for every $s, \mu \in \mathbb{T}$, so that $k(\lambda) = k(1)^\lambda$. Due the initial condition, necessarily $k(1) = e^a$. \qed
If $T = \mathbb{N}$, then identity (31) true for every $s \in \mathbb{N}$ is equivalent to the same identity with $s = 1$:

$$(X_\infty)_1 \overset{d}{=} e^c X_\infty.$$  \hfill (41)

We stress that identity (41) does not allow to recover the log-normal distribution of $X_\infty$ which is not moment determinate. This situation was studied by many authors, [2, 3, 6, 16, 11, 12, 18] for instance and all these works were motivated by finding the set or possible limit for the discrete problem (25). We also stress that Harkness and Shantaram [8] only showed, and in case $T = \mathbb{N}$, that condition (iii) in our Theorem 3, implies (i) and (ii) without specifying the distribution of $Z_\infty$ which we know equal in distribution to $e X_\infty$. Theorem 3 distinguishes between the situation below shows that the converse is true and that actually in both discrete and continuous time problems, the only possible limits of normalized biased families are the log-normal distributions. Equivalently, the only possible limits of normalized families obtained by the stationary excess operator are the mixture of the exponential and the log-normal distribution.

**Theorem 4** (The normal limit theorem improved). *The following statements are equivalent:*

(i) convergence (26, case $T = \mathbb{N}$) holds and $D_{Z_\infty}$ contains some value $\lambda_0 \in (0, \infty)$;

(ii) convergence (26, case $T = [0, \infty)$) holds and $D_{Z_\infty}$ contains some value $\lambda_0 \in (0, \infty)$;

(iii) convergence (27, case $T = \mathbb{N}$) holds and $D_{X_\infty}$ contains some value $\lambda_0 \in (0, \infty)$;

(iv) convergence (27, case $T = [0, \infty)$) holds and $D_{X_\infty}$ contains some value $\lambda_0 \in (0, \infty)$.

*In all cases, $D_{Z_\infty}$ and $D_{X_\infty}$ necessarily contain $[0, \infty)$ and convergence (29) holds with some $c \geq 0$. We also have $Z_\infty \overset{d}{=} e X_\infty$ where $e$ and $X_\infty$ are independent and have respectively the standard exponential distribution and the log-normal distribution, i.e., if for every choice of $\alpha = \mathbb{E}[X_\infty]$, the random variable $\log X_\infty$ has the normal distribution with mean equal to $\log \alpha - \frac{c}{2}$ and variance equal to $\frac{c^2}{4}$. It is understood that $X_\infty = \alpha$ if $c = 0$. Furthermore we have the identity in law

$$Z_\infty \overset{d}{=} e^{-c s} b_s (Z_\infty)_s, \quad \text{for every } s \geq 0$$

where $b_s$ is assumed to be independent from $(Z_\infty)_s$.*

**Proof.** By the discussion before Theorem 3, we know that $(i) \iff (iii)$ and that $(ii) \iff (iv)$. $(iv) \implies (iii)$ being trivial, it remains to show $(iii) \implies (iv)$. By Theorem 3, it is enough to show that

$$\limsup_{n \in \mathbb{N}, n \to \infty} \frac{\rho_{n+1}}{\rho_n} < \infty \quad \text{is equivalent to} \quad \limsup_{t \in [0, \infty), t \to \infty} \frac{\rho_{t+s}}{\rho_t} < \infty, \quad \text{for all } s > 0.$$  \hfill (42)

By Theorem 3, we also know that

$$\rho_{t \to \infty} r(t) = \mathbb{E}[X_\infty] \mathbb{E}[X(t)]$$

and Corollary 1 says that the function $t \mapsto r(t)$ is nondecreasing. Let $[x]$ the integer of the real number $x$. We have $[t] \leq t \leq t + s \leq [t] + [s] + 2$ for every $t$, $s > 0$ and then $r(t + s)/r(t) \leq
\[ r([t] + [s] + 2)/r([t]). \] It is then immediate that
\[
\limsup_{t \in [0, \infty), \ t \to \infty} \frac{\rho_{t+s}}{\rho_t} = \limsup_{t \in [0, \infty), \ t \to \infty} \frac{r(t + s)}{r(t)} \leq \limsup_{t \in [0, \infty), \ t \to \infty} \frac{r([t] + [s] + 2)}{r([t])} = \limsup_{n \in \mathbb{N}, \ n \to \infty} \frac{r(n + [s] + 2)}{r(n)} \leq \limsup_{n \in \mathbb{N}, \ n \to \infty} \left( \frac{r(n + 1)}{r(n)} \right)^{[s]+2} = \limsup_{n \in \mathbb{N}, \ n \to \infty} \left( \frac{\rho_{n+1}}{\rho_n} \right)^{[s]+2} < \infty.
\]

On the other hand,
\[
\limsup_{t \in [0, \infty), \ t \to \infty} \frac{\rho_{t+1}}{\rho_t} \geq \limsup_{n \in \mathbb{N}, \ n \to \infty} \frac{\rho_{n+1}}{\rho_n},
\]
which shows (42). Last identity in the theorem stems from property (P4), identities (28), (18) and then Beta-Gamma algebra identities (17):
\[ (Z_\infty)_s \overset{d}{=} (e X_\infty)_s \overset{d}{=} g_{s+1} (X_\infty)_s \]
which yields
\[ e^{-cs} b_s (Z_\infty)_s \overset{d}{=} e X_\infty \overset{d}{=} Z_\infty. \]
Example 1. If $X$ is a random variable such that $g'_{X}$ is a concave function, then (29) is satisfied. For instance, assume $\log X$ is an infinite divisible random variable such that its Lévy exponent $g_{X} = \log M_{X}$ has the form

$$g_{X}(\lambda) = d\lambda + \frac{\sigma^{2}}{2}\lambda^{2} + \int_{(0,\infty)}(e^{-\lambda x} - 1 + \lambda x 1_{x\leq 1})\pi(dx), \quad \lambda \geq 0,$$

(44)

with $d \in \mathbb{R}$, $\sigma \geq 0$ and the Lévy measure $\pi$ satisfy the $\int_{(0,\infty)}(x^{2} \wedge 1)\pi(dx) < \infty$. It is easy to check that $g'_{X}$ is concave. Furthermore, we have

$$\Delta_{1}\Delta_{s}g(t) = \sigma^{2}s + \int_{(0,\infty)}e^{-tx}(1 - e^{-x})(1 - e^{-sx})\pi(dx), \quad t, s > 0,$$

and by (43), $X$ satisfies (29) with $c = \sigma^{2}$.

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