ABSTRACT. We study polyharmonic \((k\)-harmonic) maps between Riemannian manifolds with finite \(j\)-energies \((j = 1, \ldots, 2k - 2)\). We show that if the domain is complete and the target is the Euclidean space, then such a map is harmonic.

1. Introduction

This paper is an extension of our previous work ([24]) to polyharmonic maps. Harmonic maps play a central role in geometry; they are critical points of the energy functional\(E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 v_g\) for smooth maps \(\varphi\) of \((M, g)\) into \((N, h)\). The Euler-Lagrange equations are given by the vanishing of the tension filed \(\tau(\varphi)\). In 1983, J. Eells and L. Lemaire [6] extended the notion of harmonic map to polyharmonic map, which are, by definition, critical points of the \(k\)-energy \((k \geq 2)\)

\[
E_k(\varphi) = \frac{1}{2} \int_M |(d + \delta)^k \varphi|^2 v_g. \tag{1.1}
\]

After G.Y. Jiang [15] studied the first and second variation formulas of \(E_2\) \((k = 2)\), extensive studies in this area have been done (for instance, see [2], [4], [18], [19], [21], [25], [27], [12], [13], [14], etc.). Notice that harmonic maps are always polyharmonic by definition.

For harmonic maps, it is well known that:

If a domain manifold \((M, g)\) is complete and has non-negative Ricci curvature, and the sectional curvature of a target manifold \((N, h)\) is non-positive, then every energy finite harmonic map is a constant map (cf. [28]).

In our previous paper, we showed that...
Theorem 1.1. ([24]) Let \((M, g)\) be a complete Riemannian manifold, and the curvature of \((N, h)\) is non-positive. Then,

1. every biharmonic map \(\varphi : (M, g) \to (N, h)\) with finite energy and finite bienergy must be harmonic.
2. In the case \(\text{Vol}(M, g) = \infty\), every biharmonic map \(\varphi : (M, g) \to (N, h)\) with finite bienergy is harmonic.

Now, in this paper, we want to extend it to \(k\)-harmonic maps \((k \geq 2)\). Indeed, we will show

Theorem 1.2. (Theorems 2.4 and 3.1) Let \((M, g)\) be a complete Riemannian manifold, and \((N, h)\), the \(n\)-dimensional Euclidean space. Then,

1. every \(k\)-harmonic map \(\varphi : (M, g) \to (N, h)\) \((k \geq 2)\) with finite \(j\)-energies for all \(j = 1, 2, \cdots, 2k - 2\), must be harmonic.
2. In the case of \(\text{Vol}(M, g) = \infty\), every \(k\)-harmonic map \(\varphi : (M, g) \to (N, h)\) with finite \(j\)-energy for all \(j = 2, 4, \cdots, 2k - 2\), is harmonic.

Theorem 1.2 gives an affirmative answer to the generalized B.Y. Chen’s conjecture (cf. [4]) on \(k\)-harmonic maps \((k \geq 2)\) under the \(L^2\)-conditions.

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2. Preliminaries and statement of main theorem

In this section, we prepare materials for the first variational formula for the biharmonic maps. Let us recall the definition of a harmonic map \(\varphi : (M, g) \to (N, h)\), of a compact Riemannian manifold \((M, g)\) into another Riemannian manifold \((N, h)\), which is an extremal of the energy functional defined by

\[
E(\varphi) = \int_M e(\varphi) \, v_g,
\]

where \(e(\varphi) := \frac{1}{2} |d\varphi|^2\) is called the energy density of \(\varphi\). That is, for any variation \(\{\varphi_t\}\) of \(\varphi\) with \(\varphi_0 = \varphi\),

\[
\left. \frac{d}{dt} \right|_{t=0} E(\varphi_t) = -\int_M h(\tau(\varphi), V) v_g = 0,
\]

where \(V \in \Gamma(\varphi^{-1}TN)\) is a variation vector field along \(\varphi\) which is given by \(V(x) = \frac{d}{dt}|_{t=0} \varphi_t(x) \in T_{\varphi(x)}N, (x \in M)\), and the tension field is given
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by $\tau(\varphi) = \sum_{i=1}^{m} B(\varphi)(e_i, e_i) \in \Gamma(\varphi^{-1}TN)$, where $\{e_i\}_{i=1}^{m}$ is a locally defined frame field on $(M, g)$, and $B(\varphi)$ is the second fundamental form of $\varphi$ defined by

$$B(\varphi)(X, Y) = (\tilde{\nabla}d\varphi)(X, Y) = (\tilde{\nabla}_X d\varphi)(Y) = \nabla_X d\varphi(Y) - d\varphi(\nabla_X Y), \quad (2.2)$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Here, $\nabla$, and $\nabla^N$, are the Levi-Civita connections of $(M, g)$, $(N, h)$, respectively, and $\tilde{\nabla}$, and $\tilde{\nabla}^N$ are the induced ones on $\varphi^{-1}TN$, and $T^*M \otimes \varphi^{-1}TN$, respectively. By (2.1), $\varphi$ is harmonic if and only if $\tau(\varphi) = 0$.

The second variation formula is given as follows. Assume that $\varphi$ is harmonic. Then,

$$\left. \frac{d^2}{dt^2} \right|_{t=0} E(\varphi_t) = \int_M h(J(V), V) v_g, \quad (2.3)$$

where $J$ is an elliptic differential operator, called the Jacobi operator acting on $\Gamma(\varphi^{-1}TN)$ given by

$$J(V) = \overline{\Delta} V - \mathcal{R}(V), \quad (2.4)$$

where $\overline{\Delta} V = \tilde{\nabla}^T \nabla V = -\sum_{i=1}^{m}\{\nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V\}$ is the rough Laplacian and $\mathcal{R}$ is a linear operator on $\Gamma(\varphi^{-1}TN)$ given by $\mathcal{R}(V) = \sum_{i=1}^{m} R^N(V, d\varphi(e_i)) d\varphi(e_i)$, and $R^N$ is the curvature tensor of $(N, h)$ given by $R^N(U, V) = \nabla^N_U \nabla^N_V - \nabla^N_V \nabla^N_U - \nabla^N_{[U, V]}$ for $U, V \in \mathfrak{X}(N)$.

J. Eells and L. Lemaire [6] proposed polyharmonic ($k$-harmonic) maps and Jiang [15] studied the first and second variation formulas for biharmonic maps. Let us consider the bienergy functional defined by

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g, \quad (2.5)$$

where $|V|^2 = h(V, V)$, $V \in \Gamma(\varphi^{-1}TN)$. The first variation formula of the bienergy functional is given by

$$\left. \frac{d}{dt} \right|_{t=0} E_2(\varphi_t) = -\int_M h(\tau_2(\varphi), V) v_g. \quad (2.6)$$

Here,

$$\tau_2(\varphi) := J(\tau(\varphi)) = \overline{\Delta}(\tau(\varphi)) - \mathcal{R}(\tau(\varphi)), \quad (2.7)$$

which is called the bitension field of $\varphi$, and $J$ is given in (2.4).

A smooth map $\varphi$ of $(M, g)$ into $(N, h)$ is said to be biharmonic if $\tau_2(\varphi) = 0$. 
Now let us recall the definition of the $k$-energy $E_k(\varphi)\ (k \geq 2)$:

**Definition 2.1.** The $k$-energy $E_k(\varphi)\ (k \geq 2)$ is defined formally ([7]) by

$$E_k(\varphi) := \frac{1}{2} \int_M |(d + \delta)^k \varphi|^2 v_g$$

for every smooth map $\varphi \in C^\infty(M, N)$. Then, it is given ([12], p. 270, Lemma 40) by the following formula:

$$E_k(\varphi) = \begin{cases} 
\frac{1}{2} \int_M |W_{\varphi}^\ell|^2 v_g & \text{(if $k$ is even, say $2\ell$)}, \\
\frac{1}{2} \int_M |\nabla W_{\varphi}^\ell|^2 v_g & \text{(if $k$ is odd, say $2\ell + 1$)}.
\end{cases} \quad (2.9)$$

Here, $W_{\varphi}^\ell$ is given as, by definition,

$$W_{\varphi}^\ell := \overline{\Delta} \cdots \overline{\Delta} \tau(\varphi) \in \Gamma(\varphi^{-1}TN). \quad (2.10)$$

For $k = 1$, that is, $\ell = 0$, we define $W_{\varphi}^0 = \varphi$, also.

Then, the definition and the first variation formula for the $k$-energy $E_k$ are given as follows:

**Definition 2.2.** ($k$-harmonic map) For each $k = 2, 3, \cdots$, and a smooth map $\varphi : (M, g) \rightarrow (N, h)$, is $k$-harmonic if

$$\frac{d}{dt} \bigg|_{t=0} E_k(\varphi_t) = 0 \quad (2.11)$$

for every smooth variation $\varphi_t : M \rightarrow N \ (-\epsilon < t < \epsilon)$ with $\varphi_0 = \varphi$.

Then, we have ([12], p.269, Theorem 39)

**Theorem 2.3.** (The first variation formula of the $k$-energy) Assume that $(N, h) = (\mathbb{R}^n, h_{\mathbb{R}^n})$ is the $n$-dimensional Euclidean space. For every $k = 2, 3, \cdots$, it holds that

$$\frac{d}{dt} \bigg|_{t=0} E_k(\varphi_t) = -\int_M \langle \tau_k(\varphi), V \rangle v_g, \quad (2.12)$$

where $V$ is a variation vector field given by $V(x) = \frac{d}{dt} \bigg|_{t=0} \varphi_t(x) \in T_{\varphi(x)}N \ (x \in M)$. The $k$-tension field $\tau_k(\varphi)$ is given by

$$\tau_k(\varphi) = J(W_{\varphi}^{k-1}) = \overline{\Delta}(W_{\varphi}^{k-1}), \quad (2.13)$$
where \( W_k^{-1} = \sum_{k-2}^{k} \tau(\varphi) \in \Gamma(\varphi^{-1}TN) \).

Thus, \( \varphi : (M, g) \to (N, h) \) is \( k \)-harmonic if and only if \( \sum_{k-2}^{k} \tau(\varphi) = 0 \) which is equivalent to \( W_k^{-1} = 0 \).

The formula (143) of the \( k \)-tension field \( \tau_k(\varphi) \) in Theorem 39 (p.269, [12]) is true only for the case that the target space \((N, h) = (\mathbb{R}^n, h_{\mathbb{R}^n})\).

Here, we denote by \( \nabla W^\ell = \nabla \varphi = d\varphi \) for \( \ell = 0 \), and \( k = 2\ell + 1 = 1 \),

\[
E_1(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 v_g.
\]

Then, we can state our main theorem.

**Theorem 2.4. (Main theorem)** Assume that the domain manifold \((M, g)\) is a complete Riemannian manifold, and the target space \((N, h)\) is the \( n \)-dimensional Euclidean space. Let \( \varphi : (M, g) \to (N, h) \) be a \( k \)-harmonic map \((k \geq 2)\). Assume that

1. \( E_j(\varphi) < \infty \) for all \( j = 2, 4, \cdots, 2k - 2 \), and
2. either
   \[
   E_j(\varphi) < \infty \text{ for all } j = 1, 3, \cdots, 2k - 3, \text{ or }
   \]
   \[
   \text{Vol}(M, g) = \infty.
   \]

Then, \( \varphi : (M, g) \to (N, h) \) is harmonic.

In the case of the \( n \)-dimensional Euclidean space \((N, h) = (\mathbb{R}^n, h_{\mathbb{R}^n})\), Theorem 2.4 and the following Theorem 3.1 are natural extensions of our previous theorem in [24] which is:

**Theorem 2.5.** Assume that \((M, g)\) is complete and the sectional curvature of \((N, h)\) is non-positive.

1. Every biharmonic map \( \varphi : (M, g) \to (N, h) \) with finite energy \( E(\varphi) < \infty \) and finite bienergy \( E_2(\varphi) < \infty \), is harmonic.
2. In the case \( \text{Vol}(M, g) = \infty \), every biharmonic map \( \varphi : (M, g) \to (N, h) \) with finite bienergy \( E_2(\varphi) < \infty \), is harmonic.

3. **The iteration proposition.**

By virtue of (2.9), we have to notice the the energy conditions in (1) and (2) of Theorem 2.4:
Indeed, the condition which $E_j(\varphi) < \infty$ for all $j = 2, 4, \cdots, 2k - 2$ in (1) of Theorem 2.4 is equivalent to that
\[
\int_M |W^j_\varphi|^2 v_g < \infty \quad (j = 1, 2, \cdots, k - 1),
\] (3.1)
and the condition which $E_j(\varphi) < \infty$ for all $j = 1, 3, \cdots, 2k - 3$ in (2) of Theorem 2.4 is equivalent to that
\[
\int_M |\nabla W^j_\varphi|^2 v_g < \infty \quad (j = 0, 1, \cdots, k - 2).
\] (3.2)
Therefore, to show Theorem 2.4, we only have to prove the following theorem:

**Theorem 3.1.** Assume that the domain manifold $(M, g)$ is a complete Riemannian manifold, and the target space $(N, h)$ is the $n$-dimensional Euclidean space. Let $\varphi : (M, g) \rightarrow (N, h)$ be a $k$-harmonic map.
Assume that
\[
\begin{aligned}
(1) & \quad \int_M |W^j_\varphi|^2 v_g < \infty \text{ for all } j = 1, 2, \cdots, k - 1, \text{ and } \\
(2) & \quad \text{either } \int_M |\nabla W^j_\varphi|^2 v_g < \infty \text{ for all } j = 0, 1, \cdots, k - 2, \text{ or } \\
& \quad \text{Vol}(M, g) = \infty.
\end{aligned}
\] Then, $\varphi : (M, g) \rightarrow (N, h)$ is harmonic.

To prove Theorem 3.1 whose proof will be given in the next section, we need the following iteration proposition:

**Proposition 3.2.** (the iteration method) Let $(M, g)$ be a complete Riemannian manifold, and $(N, h)$, an arbitrary Riemannian manifold. Let $\varphi : (M, g) \rightarrow (N, h)$ be an arbitrary $C^\infty$ map satisfying that for some $j \geq 2$,
\[
W^j_\varphi = 0. \tag{3.3}
\]
If we assume the following two conditions:
\[
\begin{cases}
(1) & \quad \int_M |W^{j-1}_\varphi|^2 v_g < \infty, \text{ and } \\
(2) & \quad \text{either } \int_M |\nabla W^{j-2}_\varphi|^2 v_g < \infty \text{ or } \text{Vol}(M, g) = \infty,
\end{cases} \tag{3.4}
\]
then, we have
\[
W^{j-1}_\varphi = 0. \tag{3.5}
\]
Remark 3.3. Under the assumptions (3.2), if we have \( W_k^\phi = 0 \) for some \( k \geq 2 \), then we have automatically, \( W_1^\phi = \tau(\phi) = 0 \), i.e., \( \phi \) is harmonic.

In this section, we give a proof of Proposition 3.2 which consists of four steps.

(The first step) For a fixed point \( x_0 \in M \), and for every \( 0 < r < \infty \), we first take a cut-off \( C^\infty \) function \( \eta \) on \( M \) (for instance, see [16]) satisfying that

\[
\begin{cases}
0 \leq \eta(x) \leq 1 \quad (x \in M), \\
\eta(x) = 1 \quad (x \in B_r(x_0)), \\
\eta(x) = 0 \quad (x \notin B_{2r}(x_0)), \\
|\nabla \eta| \leq \frac{2}{r} \quad (x \in M).
\end{cases}
\]

(The second step) Notice that (3.3) is equivalent to that

\[
\Delta W_{j-1}^\phi = 0
\]

because of \( W_j^\phi = \Delta W_{j-1}^\phi \).

Then, we have

\[
0 = \int_M \langle \eta^2 W_{j-1}^\phi, \Delta W_{j-1}^\phi \rangle v_g \\
= \int_M \sum_{i=1}^m \langle \nabla e_i (\eta^2 W_{j-1}^\phi), \nabla e_i W_{j-1}^\phi \rangle v_g \\
= \int_M \eta^2 \sum_{i=1}^m |\nabla e_i W_{j-1}^\phi|^2 v_g + 2 \int_M \sum_{i=1}^m \eta e_i(\eta) \langle W_{j-1}^\phi, \nabla e_i W_{j-1}^\phi \rangle v_g.
\]

By moving the second term in the last equality of (3.8) to the left hand side, we have

\[
\int_M \eta^2 \sum_{i=1}^m |\nabla e_i W_{j-1}^\phi|^2 = -2 \int_M \sum_{i=1}^m \langle \eta \nabla e_i W_{j-1}^\phi, e_i(\eta) W_{j-1}^\phi \rangle v_g = -2 \int_M \sum_{i=1}^m \langle S_i, T_i \rangle v_g,
\]

where we put \( S_i := \eta \nabla e_i W_{j-1}^\phi \), and \( T_i := e_i(\eta) W_{j-1}^\phi \) \((i = 1 \ldots, m)\).

Now let recall the following inequality:

\[
\pm 2 \langle S_i, T_i \rangle \leq \epsilon |S_i|^2 + \frac{1}{\epsilon} |T_i|^2
\]
for all positive $\epsilon > 0$ because of the inequality $0 \leq |\sqrt{\epsilon} S_i \pm \frac{1}{\sqrt{\epsilon}} T_i|^2$. Therefore, for (3.10), we obtain

$$-2 \int_M \sum_{i=1}^m \langle S_i, T_i \rangle v_g \leq \epsilon \int_M \sum_{i=1}^m |S_i|^2 v_g + \frac{1}{\epsilon} \int_M \sum_{i=1}^m |T_i|^2 v_g. \tag{3.11}$$

If we put $\epsilon = \frac{1}{2}$, we obtain, by (3.9) and (3.11),

$$\int_M \eta^2 \sum_{i=1}^m |\nabla e_i W^j - 1\phi|^2 v_g \leq \frac{1}{2} \int_M \sum_{i=1}^m \eta^2 |\nabla e_i W^j - 1\phi|^2 v_g$$

$$+ 2 \int_M \sum_{i=1}^m e_i(\eta)^2 |W^j - 1\phi|^2 v_g. \tag{3.12}$$

Thus, by (3.12) and (3.6), we obtain

$$\int_M \eta^2 \sum_{i=1}^m |\nabla e_i W^j - 1\phi|^2 v_g \leq 4 \int_M |\nabla \eta|^2 |W^j - 1\phi|^2 v_g$$

$$\leq \frac{16}{\tau^2} \int_M |W^j - 1\phi|^2 v_g. \tag{3.13}$$

(The third step) By definition of $\eta$ in the first step, (3.13) turns out that

$$\int_{B_r(x_0)} |\nabla W^j - 1\phi|^2 v_g \leq \frac{16}{\tau^2} \int_M |W^j - 1\phi|^2 v_g. \tag{3.14}$$

Here, recall our assumption that $(M, g)$ is complete and non-compact, and (1) $\int_M |W^j - 1\phi|^2 v_g < \infty$. When we tend $r \rightarrow \infty$, the right hand side of (3.12) goes to zero, and the left hand side of (3.12) goes to $\int_M |\nabla W^j - 1\phi|^2 v_g$. Thus, we obtain

$$0 \leq \int_M |\nabla W^j - 1\phi|^2 v_g \leq 0,$$

which implies that

$$\nabla W^j - 1\phi = 0 \tag{3.15}$$

everywhere on $M$.

(The fourth step) (a) In the case that $\int_M |\nabla W^j - 2\phi|^2 v_g < \infty$, let us define a smooth 1-form $\alpha$ on $M$ by

$$\alpha(X) := \langle W^j - 1\phi, \nabla X W^j - 2\phi \rangle \quad (X \in \mathfrak{X}(M)). \tag{3.16}$$

Then, we have:

$$\text{div}(\alpha) = -|W^j - 1\phi|^2. \tag{3.17}$$
Because we have
\[
\text{div}(\alpha) = \sum_{i=1}^{m}(\nabla_{e_i}\alpha)(e_i)
\]
\[
= \sum_{i=1}^{m}\left\{e_i(\alpha(e_i)) - \alpha(\nabla_{e_i}e_i)\right\}
\]
\[
= \sum_{i=1}^{m}\left\{e_i\left(\langle W_{\varphi}^{j-1}, \nabla_{e_i}W_{\varphi}^{j-2}\rangle\right) - \langle W_{\varphi}^{j-1}, \nabla_{\nabla_{e_i}e_i}W_{\varphi}^{j-2}\rangle\right\}
\]
\[
= \sum_{i=1}^{m}\left\{\langle \nabla_{e_i}W_{\varphi}^{j-1}, \nabla_{e_i}W_{\varphi}^{j-2}\rangle + \langle W_{\varphi}^{j-1}, \nabla_{e_i}e_iW_{\varphi}^{j-2}\rangle
\]
\[
- \langle W_{\varphi}^{j-1}, \nabla_{\nabla_{e_i}e_i}W_{\varphi}^{j-2}\rangle\right\}
\]
\[
= \langle W_{\varphi}^{j-1}, -\overline{\Delta}W_{\varphi}^{j-2}\rangle \quad \text{(because of (3.15) and definition of } \overline{\Delta})
\]
\[
= -|W_{\varphi}^{j-1}|^2, \quad (3.18)
\]
which is (3.17).

Furthermore, we have
\[
\int_{M} |\alpha| v_g < \infty. \quad (3.19)
\]

Because we have, by definition of \(\alpha\) in (3.16),
\[
\int_{M} |\alpha| v_g = \int_{M} |\langle W_{\varphi}^{j-1}, \nabla W_{\varphi}^{j-2}\rangle| v_g
\]
\[
\leq \left(\int_{M} |W_{\varphi}^{j-1}|^2 v_g \right)^{\frac{1}{2}} \left(\int_{M} |\nabla W_{\varphi}^{j-2}|^2 v_g \right)^{\frac{1}{2}}
\]
\[
< \infty \quad (3.20)
\]
because of our assumptions \(\int_{M} |W_{\varphi}^{j-1}|^2 v_g < \infty\) and \(\int_{M} |\nabla W_{\varphi}^{j-2}|^2 v_g < \infty\). Thus, we can apply Gaffney’s theorem to this \(\alpha\) (cf. [10], and Theorem 4.1 in Appendix in [24]). We obtain
\[
0 = \int_{M} \text{div}(\alpha) v_g = -\int_{M} |W_{\varphi}^{j-1}|^2 v_g, \quad (3.21)
\]
which implies that \(W_{\varphi}^{j-1} = 0\).

(b) In the case that \(\text{Vol}(M, g) = \infty\), we first notice that \(|W_{\varphi}^{j-1}|^2\) is constant on \(M\), say \(C_0\). Because for every \(X \in \mathfrak{X}(M)\), we have
\[
X |W_{\varphi}^{j-1}|^2 = 2 \langle \nabla_X W_{\varphi}^{j-1}, W_{\varphi}^{j-1}\rangle = 0 \quad (3.22)
\]
due to (3.15). Then, due to the assumption (1) of Proposition 3.2, and the above, we obtain

$$\infty > \int_M |W^j_\varphi - 1|^2 v_g = C_0 \int_M v_g = C_0 \text{Vol}(M, g).$$

(3.23)

By our assumption that \(\text{Vol}(M, g) = \infty\), (3.23) implies that \(C_0 = 0\). We obtain \(W^j_\varphi - 1 \equiv 0\). We obtain Proposition 3.2.

Proof of Theorem 3.1.  We apply Proposition 3.2 to our map \(\varphi : (M, g) \to (N, h)\), then the iteration procedure works well since \(\varphi\) is \(k\)-harmonic, i.e., \(W^k_\varphi = 0\). Then, we have \(W^{k-1}_\varphi = 0\), and then we have \(W^{k-2}_\varphi = 0\), etc. Finally, we obtain \(\tau(\varphi) = W^1_\varphi = 0\). Thus, \(\varphi : (M, g) \to (N, h)\) is harmonic. We obtain Theorem 3.1.

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