Non-perturbative beta-function in $SU(2)$ lattice gauge fields thermodynamics

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Abstract

The new method of nonperturbative calculation of the beta function in the lattice gauge theory is proposed. The method is based on the finite size scaling hypothesis.

Monte Carlo simulations of $SU(2)$ lattice gauge theory have shown that there are rather large deviations from asymptotic scaling behaviour in the range of coupling constants accessible with today’s computing power. This raised the question of whether we are able to see continuum physics in these simulations. The answer requires the knowledge of the $\beta$-function away from the asymptotic regime where it is dominated by the two leading terms in its perturbative expansion ($g$-coupling constant)

$$\beta_f(g) = -b_0 g^3 - b_1 g^5 - \ldots,$$

where $b_0 = \frac{11}{24\pi^2}$, $b_1 = \frac{17}{96\pi^4}$.

During the last years a large effort has been put into the numerical determination of the $\beta$-function for the lattice gauge theories by measuring the deviation from the two-loop $\beta$-function (1) [1,2].

We propose a new method of determination of the nonperturbative $\beta$-function, which is based on the finite size scaling hypothesis.

We consider $SU(2)$ gauge theory at finite temperature on $N_\sigma^3 \times N_\tau$ lattices with the standard Wilson action

$$S(U) = \frac{4}{g^2} \sum_p \left( 1 - \frac{1}{2} \text{Tr} U_p \right),$$

where $U_p$ is the product of link operators around the plaquette. The number of lattice points in the space (time) direction $N_\sigma (N_\tau)$ and the lattice spacing $a$ fix the volume and temperature

$$V = (N_\sigma a)^3, \quad T = 1/(N_\tau a).$$

The $\beta$-function is defined by the expression

$$\beta_f(g) = -a \frac{dg(a)}{da}.$$
Then for lattice spacing $a$ one can obtain

$$ a = \frac{1}{\Lambda_L} \exp \left( - \int \frac{dg}{\beta_f(g)} \right) $$

(5)

where $\Lambda_L$ is the renormalization group invariant parameter. Since in Monte Carlo simulations the thermodynamic functions are calculated in the units of lattice spacing $a$, formula (5) determines the temperature dependence of these functions according (3).

In the asymptotically free (AF) regime (1) we obtain the well known formula

$$ a \Lambda_L = R(g^2) = \exp \left\{ - \frac{b_1}{2b_0} \ln(b_0g^2) - \frac{1}{2b_0g^2} \right\} $$

(6)

which is valid in the region $g^2 < 1$.

On the other hand, SU(2) gauge system at finite $N_\tau$ and undergoes the deconfinement phase transition at $g^2 \gtrsim 1$. The new nonperturbative method for the calculation of $\beta$-function is needed, which is not connected with the expansion (1).

Our approach is based on the two points: i) translation into a more conventional statistical mechanical definition of $\beta$-function and ii) the finite size scaling theory and phenomenological renormalization. As in the standard spin systems, let us make the infinitesimal transformation of the lattice spacing $a \rightarrow a' = ba = (1 + \Delta b)a$. Then

$$ -a \frac{dg}{da} = -\lim_{b \rightarrow 1} \left( a \cdot \frac{g(ba) - g(a)}{ba - a} \right) = -\lim_{b \rightarrow 1} \frac{\Delta g}{\Delta b} = -\lim_{b \rightarrow 1} \frac{dg}{db}. $$

(7)

We obtain the new definition of $\beta$-function for SU(2) lattice gauge system

$$ \beta_f(g) = -\lim_{b \rightarrow 1} \frac{dg}{db}. $$

(8)

It has been shown in the finite size scaling theory that on the finite lattice $N_3^\sigma \times N_\tau$ ($N_\tau$ fixed) the order parameter $\langle L \rangle$, the susceptibility $\chi$ and the correlation length $\xi$ can be expressed in the following form (see, for example [3])

$$ O(g^{-2}, N_\sigma) = N_\sigma^{\frac{\omega}{2}} Q_0(g^{-2}, N_\sigma). $$

(9)

Here $O$ represents $\langle L \rangle$, $\chi$ and $\xi$, $\omega = -\beta, \gamma, \nu$ is the corresponding critical index. Scaling function $Q$ has some special dependence on $g^{-2}$ and $N_\sigma$, but this is out of our consideration.

For example

$$ \begin{cases} 
\langle L \rangle = N_\sigma^{-\frac{\omega}{2}} Q_{\langle L \rangle}(g^{-2}, N_\sigma), \\
\xi = N_\sigma Q_{\xi}(g^{-2}, N_\sigma). 
\end{cases} $$

(10)

The existence of the scaling function $Q$ allows to develop a procedure to renormalize the coupling constant $g^{-2}$ by the use of two different lattice sizes $N_\sigma$ and $N'_\sigma$. Let us fix the physical size $L = N_\sigma a$ and make a scale transformation

$$ \begin{cases} 
a \rightarrow a' = ba, \\
N_\sigma \rightarrow N'_\sigma = N_\sigma/b. 
\end{cases} $$

(11)
Then the phenomenological renormalization is defined by the equation

\[ Q(g^{-2}, N_\sigma) = Q((g')^{-2}, N_\sigma/b) \]  

(12)

It expresses that the scaling function \( Q \) remains unchanged if the lattice size is rescaled by a factor \( b \) and the inverse coupling \( g^{-2} \) is shifted to \( (g')^{-2} \) simultaneously. Taking the derivative with respect to the scale parameter \( b \) of the both sides of (12) and using (8) it is easy to obtain the expression

\[ a \frac{dg^{-2}}{da} = \frac{\partial \ln Q / \partial \ln N_\sigma}{\partial \ln Q / \partial g^{-2}}. \]  

(13)

The approximation of the derivative with respect to \( N_\sigma \) by the finite difference yields the final formula for the \( \beta \)-function

\[ a \frac{dg^{-2}}{da} = \frac{\ln \left[ \frac{Q(N'_\sigma)}{Q(N_\sigma)} \right]}{\ln \left( \frac{N'_\sigma}{N_\sigma} \right)} - \frac{1}{2} \left[ \frac{dQ(N_\sigma)}{dg^{-2}} \frac{dQ(N'_\sigma)}{dg^{-2}} \right] \left[ \frac{Q(N_\sigma)}{Q(N'_\sigma)} \right]^{-1/2}. \]  

(14)

So far we have considered the scaling function \( Q \). It is very interesting to apply the same analysis to the correlation length \( \xi \). Using (10) we obtain instead of (12)

\[ \frac{\xi(g, N_\sigma)}{N_\sigma} = \frac{\xi(g', N'_\sigma)}{N'_\sigma} \]  

(15)

In the case of the large enough lattice size \( (N_\sigma \to \infty) \) the dependence of the correlation length \( \xi \) on \( N_\sigma \) become negligible and (15) yields

\[ \xi(g) = b \xi(g'). \]  

(16)

This is the renormalization group equation for the bulk system, which is wellknown in the standard spin theory. From this by the manipulations mentioned above one can obtain the expression for the \( \beta \)-function

\[ \beta(f)(g) = \left( \frac{d \ln \xi(g)}{dg} \right)^{-1}. \]  

(17)

Substituting (17) into (5) we have an extremely simple formula for the lattice spacing

\[ a\Lambda_L = \frac{1}{\xi(g)}. \]  

(18)

The first attempt to calculate the correlation length \( \xi(g) \) has been made in [4], but with rather pure statistics and on rather small lattices \( (N_\sigma = 18, N_\tau = 3, 4, 5) \). We can see that the new method of calculation of the nonperturbative \( \beta \)-function for SU(2) lattice gauge thermodynamics only needs the correct calculation of the correlation length \( \xi \) in the wide coupling constant interval.

On the other hand, the best studied quantities in MC lattice calculations of \( SU(N) \) gauge theories are the string tension \( \sqrt{\sigma} \) and the deconfinement transition temperature...
The MC calculations give the values of the critical coupling $\beta_{c}^{MC}$ of the deconfinement transition and dimensionless string tension $(\sqrt{\sigma})_{MC}$. The values $\beta_{c}^{MC}$ were found for the finite lattices and the extrapolation to spatially infinite volume ("thermodynamical limit") $N_{\sigma} \to \infty$ has been done (see Ref. [5] and references therein). For the $SU(2)$ gauge theory the MC values of the critical couplings $\beta_{c}^{MC}$ are presented in Table 1 for different $N_{\tau}$. One observes a rather strong dependence of $T_{c}/\Lambda_{AF}^{L}$ on $N_{\tau}$. This means that the perturbative AF relation (6) does not work even on the largest available lattices. This fact is known as an absence of the asymptotic scaling.

It has been proposed in Ref. [2] that a deviation from the asymptotic scaling can be described by a universal non-perturbative (NP) beta function, i.e. $\beta_{f}(g)$ is the same for all lattice observables and it does not depend on the lattice size if $N_{\sigma}, N_{\tau}$ are not too small. The following ansatz was suggested [2]:

$$ a\Lambda_{L}^{NP} = \lambda(g^2) R(g^2), \quad (19) $$

where $R(g^2)$ is given by Eq. (6) and $\lambda(g^2)$ is thought to describe a deviation from the perturbative behaviour. The equation (6) has been expected at $g \to 0$ so that an additional constraint, $\lambda(0) = 1$, has been assumed. The values of $T_{c}/\Lambda_{L}^{NP}$ can be calculated then as

$$ T_{c}/\Lambda_{L}^{NP} = \frac{1}{N_{\tau} \lambda(g_{c}^2) R(g_{c}^2)} \quad (20) $$

A simple formula for the function $\lambda(g^2)$ was suggested [2]:

$$ \lambda(g^2) = \exp\left(\frac{c_{3}g^{6}}{2\Lambda_{0}^{2}}\right) \quad (21) $$

Parameter $c_{3}$ in (21) and a new one, $T_{c}^{*}/\Lambda_{L}^{NP} = \text{const}$, were considered as free parameters and determined from fitting the MC values of $T_{c}/\Lambda_{L}^{NP}$ at different $N_{\tau}$ to the constant value $T_{c}^{*}/\Lambda_{L}^{NP}$. This procedure gives:

$$ T_{c}^{*}/\Lambda_{L}^{NP} = 21.45(14), \quad c_{3} = 5.529(63) \cdot 10^{-4} \quad (22) $$

In comparison to $T_{c}/\Lambda_{L}^{AF}$ the much weaker $N_{\tau}$ dependent values of $T_{c}/\Lambda_{L}^{NP}$ have been obtained, which become now close to the constant value $T_{c}^{*}/\Lambda_{L}^{NP}$ (22).

In spite of the phenomenological success of the above procedure of [2] the crucial question regarding the existence of the universal NP beta function, with does not depend on the lattice size, is not solved and remains just a postulate. To answer this question we reanalyze the same MC data using the different strategy as in Ref. [2]. A principal difference of our analysis is that we do not assume the existence of the universal beta function and take into account the finite size effects of the lattice.

Usually finite size scaling (FSS) in the vicinity of a finite-temperature phase transition is discussed for lattice SU(N) gauge models, without trying to make contact with the continuum limit, i.e. the scaling properties are studied on lattices of $N_{\tau} \times N_{\sigma}^{3}$ with fixed $N_{\tau}$ and varying $N_{\sigma}$, and the model is viewed as a 3-dimensional spin system. In the continuum limit the FSS properties of these non-abelian models should, of course, be discussed in terms of the physical volume $V = L^{3}$ and the temperature $T$ in the vicinity of the deconfinement transition temperature $T_{c}$. We will study here how the
scaling behaviour of the continuum theory emerges from the lattice free energy on arbitrary lattices, i.e. when varying \( N_\sigma \) and \( N_\tau \).

On a lattice \( N_\tau \times N_\sigma^3 \) the length scale \( L \) and the temperature \( T \) are given in units of the lattice spacing \( a \), therefore it is advantageous to replace the length scale \( L \) by the dimensionless combination

\[
LT = \frac{N_\sigma}{N_\tau}
\]

(23)

Using this ratio the singular part of the free energy density is described by a universal finite-size scaling function \([3,5]\)

\[
f_s(t, h, N_\sigma, N_\tau) = \left(\frac{N_\sigma}{N_\tau}\right)^{-3} Q_{fs}\left(t\left(\frac{N_\sigma}{N_\tau}\right)^{1/\nu}, h\left(\frac{N_\sigma}{N_\tau}\right)^{\frac{2+\gamma}{\nu}}\right),
\]

(24)

where \( \beta, \gamma, \nu \) are the critical indexes of the theory, the scaling function \( Q_{fs} \) depends on the reduced temperature \( t = (T - T_c)/T_c \) and the external field strength \( h \).

Next we consider \( y = N_\sigma/N_\tau \) fixed, varying \( N_\sigma \) and therefore \( N_\tau \) accordingly as is needed to reach the continuum limit. Rescaling \( N_\sigma \) and \( N_\tau \) by a factor \( b \) leads to a phenomenological renormalization \( g'(g, b, y) \) by the following identity for a scaling function \( Q \)

\[
Q\left(t(g, N_\tau) \cdot \left(\frac{N_\sigma}{N_\tau}\right)^{1/\nu}\right) = Q\left(t(g', N_\tau/b) \cdot \left(\frac{bN_\sigma}{bN_\tau}\right)^{1/\nu}\right).
\]

(25)

It follows from (25)

\[
t(g, N_\tau) = t(g', N_\tau/b).
\]

(26)

In general the reduced temperature \( t = (T - T_c)/T_c \) is a complicated function of the coupling \( \beta = 2N/g^2 \), which in the vicinity of the critical temperature \( T_c \) can be approximated by \([5]\)

\[
t = (\beta - \beta_c) \frac{1}{4Nb_0} \left[1 - \frac{2Nb_1}{b_0 \beta_c^{-1}}\right]
\]

(27)

This approximation reproduces the correct reduced temperature in the continuum limit, which is easy verified by using (4). Taking the derivative with respect to the scale parameter \( b \) of the both sides of (26) and using (8) and (27) it is easy to obtain the expression for the beta function:

\[
\beta_f(g) = -B_0(N_\tau)g^3 - B_1(N_\tau)g^5,
\]

(28)

where

\[
\begin{cases}
B_0(N_\tau) = \frac{1}{4N} \left(1 - \frac{2Nb_1}{b_0 \beta_c^{-1}}\right) \frac{d\beta_c}{d \ln N_\tau} \\
B_1(N_\tau) = B_0(N_\tau) \frac{b_1}{b_0}
\end{cases}
\]

(29)

Then the equation (4) leads to

\[
a \Lambda_L = \exp\left(-\frac{1}{2B_0 g^2}\right) (B_0 g^2)^{-B_1/2B_0^2}.
\]

(30)

Using (3) and (30) one can obtain the critical temperature \( T_c \). The only problem is the calculation of the derivative \( d\beta_c/d \ln N_\tau \) in expression (29). For the \( SU(2) \) gauge
theory the calculation has been made by fitting the MC data for the critical couplings $\beta_{c}^{MC}$ with a Spline interpolation and numerical differentiation of this curve. The result of the calculation is represented in Table 1. In comparison to $T_c/\Lambda_{AF}^L$ the much weaker dependence on $N_\tau$ of the critical temperature $T_c/\Lambda_L$ is observed.

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Table 1: MC data for $\beta_c$ are taken from [5]. The values of $T_c/\Lambda_{AF}^L$ are calculated from (6). Our results for $T_c/\Lambda_L$ are obtained from (30).

| $N_\tau$ | $\beta_c = \frac{4}{g_c^2}$ | $T_c/\Lambda_{AF}^L$ | $\frac{d\beta_c}{dN_\tau}$ | $T_c/\Lambda_L$ |
|----------|-----------------|---------------------|-----------------|-----------------|
| 2        | 1.880           | 29.7                | —               | —               |
| 3        | 2.177           | 41.4                | 0.158           | 25.22           |
| 4        | 2.299           | 42.1                | 0.086           | 25.46           |
| 5        | 2.373           | 40.6                | 0.063           | 25.38           |
| 6        | 2.427           | 38.7                | 0.045           | 24.13           |
| 8        | 2.512           | 36.0                | 0.040           | 24.24           |
| 16       | 2.739           | 32.0                | 0.017           | —               |

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