A NADEL VANISHING THEOREM
FOR METRICS WITH MINIMAL SINGULARITIES
ON BIG LINE BUNDLES

SHIN-ICHI MATSUMURA

Abstract. The purpose of this paper is to establish a Nadel vanishing theorem for big line bundles with multiplier ideal sheaves of singular metrics admitting an analytic Zariski decomposition (such as, metrics with minimal singularities and Siu’s metrics). For this purpose, we apply the theory of harmonic integrals and generalize Enoki’s proof of Kollár’s injectivity theorem. Moreover we investigate the asymptotic behavior of the harmonic forms with respect to a family of regularized metrics.

1. Introduction

The Kodaira vanishing theorem plays an important role when we approach certain fundamental problems of algebraic geometry and the theory of several complex variables (for example, asymptotics of linear systems, extension problems of holomorphic sections, the minimal model program, and so on). By using multiplier ideal sheaves associated to singular metrics, this theorem can be generalized to the Nadel vanishing theorem, which can be seen as an analytic analogue of the Kawamata-Viehweg vanishing theorem of algebraic geometry (Kaw82, Vie82).

In this paper, we study singular metrics admitting an analytic Zariski decomposition (such as, metrics with minimal singularities and Siu’s metrics) and a Nadel vanishing theorem for them from the viewpoint of complex (differential) geometry.

**Theorem 1.1** (The Nadel vanishing theorem, Nad89, Dem82). Let $F$ be a big line bundle on a smooth projective variety $X$ and $h$ be a (singular) metric on $F$ with strictly positive curvature. Then we have

$$H^i(X, K_X \otimes F \otimes \mathcal{I}(h)) = 0 \quad \text{for any } i > 0.$$  

Here $\mathcal{I}(h)$ denotes the multiplier ideal sheaf of the (singular) metric $h$ and $K_X$ denotes the canonical bundle of $X$.

We mainly treat metrics with minimal singularities $h_{\text{min}}$ and Siu’s metrics $h_{\text{Siu}}$ (see Section 2, 3 for the definition). These metrics satisfy many important properties (for

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example they admit an analytic Zariski decomposition), thus several authors study these metrics (see [DEL00], [Dem], [Pau12], [Siu98]).

The main purpose of this paper is to establish a Nadel vanishing theorem for $h_{\min}$ and $h_{\Siu}$. When we investigate the cohomology groups with coefficients in $K_X \otimes F \otimes I(h_{\min})$ and $K_X \otimes F \otimes I(h_{\Siu})$, we encounter the following difficulties:

1. $h_{\min}$ and $h_{\Siu}$ may have non-algebraic (transcendental) singularities.
2. $h_{\min}$ and $h_{\Siu}$ do not have strictly positive curvature except the trivial case.

The proof of Theorem 1.1 heavily depends on the assumption that the curvature of $h$ is “strictly” positive. Under this assumption, we can construct solutions of the $\overline{\partial}$-equation with $L^2$-estimates, which implies Theorem 1.1 (see [Dem82]). In fact, the theorem fails without this assumption even if the curvature of $h$ is semi-positive. Nevertheless, we can expect that all higher cohomology groups with coefficients in $K_X \otimes F \otimes I(h_{\min})$ and $K_X \otimes F \otimes I(h_{\Siu})$ vanish from the special characteristics of $h_{\min}$ and $h_{\Siu}$. This is because, for a big line bundle $F$ we have already known

$$H^i(X, K_X \otimes F \otimes I(||F||)) = 0 \text{ for any } i > 0,$$

where $I(||F||)$ is the asymptotic multiplier ideal sheaf of $F$ (see [DEL00] for the precise definition). The multiplier ideal sheaves $I(h_{\min})$ and $I(h_{\Siu})$ can be seen as an analytic counterpart of $I(||F||)$. The asymptotic multiplier ideal sheaf $I(||F||)$ does not always coincide with $I(h_{\min})$, but it is conjectured that the equality $I(||F||) = I(h_{\min})$ holds if $F$ is big or more generally abundant (that is, the numerical dimension $\nu(F)$ agrees with the Kodaira dimension $\kappa(F)$). Hence it is interesting to study a Nadel vanishing theorem for $I(h_{\min})$ and $I(h_{\Siu})$. This is a natural problem, however it has been an open problem for a long time. In this paper, we give an affirmative answer for this problem, which is the main result of this paper.

**Theorem 1.2** (=Corollary 3.6). Let $F$ be a big line bundle on a smooth projective variety $X$ and $h_{\min}$ be a metric with minimal singularities on $F$. Then we have

$$H^i(X, K_X \otimes F \otimes I(h_{\min})) = 0 \text{ for any } i > 0.$$

This theorem follows from Theorem 3.1 which says that all higher cohomology groups with coefficients in $K_X \otimes F \otimes I(h)$ vanish if the singular metric $h$ is less singular than some singular metric with strictly positive curvature. From Theorem 3.1 we can also obtain the same conclusion for $h_{\Siu}$ (see Corollary 3.6).

These results are closely related to the openness conjecture of Demailly-Kollár in [DK01], which is a conjecture on singularities of plurisubharmonic (psh for short) functions. Indeed, if $F$ is a big line bundle and $h_{\min}$ has algebraic singularities, we can easily check $I(||F||) = I(h_{\min})$. However we emphasize that $h_{\min}$ does not always have algebraic singularities (see Section 4).

For the proof of the main result, we need to take a transcendental approach since we need to overcome difficulties (1), (2). The proof is based on a combination of the technique to solve the $\overline{\partial}$-equation and Enoki’s proof of Kollár’s injectivity theorem for semi-positive
line bundles (see [Eno90], [Kol86]). The strategy of the proof can be divided into four steps as follows:

In Step 1, we approximate the metric $h_{\text{min}}$ by singular metrics $\{h_\varepsilon\}_{\varepsilon > 0}$ that are smooth on a Zariski open set. Then we can represent a given cohomology class by the associated harmonic form $u_\varepsilon$ with respect to $h_\varepsilon$ on the Zariski open set.

In Step 2, by using a suitable (holomorphic) section $s$ of some positive multiple $F^m$, we show that the norm of $D''^*su_\varepsilon$ converges to zero as letting $\varepsilon$ go to zero, where $D''^*$ is the adjoint operator of $\overline{\partial}$. Unfortunately the curvature of $h_\varepsilon$ is not semi-positive. For this reason, we need to generalize Enoki’s technique for Kollár’s injectivity theorem by applying the theory of harmonic integrals.

In Step 3, we solve the $\overline{\partial}$-equation. By considering $su_\varepsilon$ instead of $u_\varepsilon$, we can construct a solution $\beta_\varepsilon$ of the $\overline{\partial}$-equation $\overline{\partial}\beta_\varepsilon = su_\varepsilon$. Moreover we can show that the $L^2$-norm $\|\beta_\varepsilon\|$ is uniformly bounded from the special characteristics of $h_{\text{min}}$.

In Step 4, we investigate the limit of $u_\varepsilon$. The above arguments yield

$$\|su_\varepsilon\|^2 = \langle \langle su_\varepsilon, \overline{\partial}\beta_\varepsilon \rangle \rangle \leq \|D''^*su_\varepsilon\| \|\beta_\varepsilon\| \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0.$$

From this convergence we prove that $u_\varepsilon$ converges to zero in a suitable sense, which completes the proof.

This paper is divided into four sections. In Section 2, we recall the fundamental results that are often used in this paper. In Section 3, we give a proof of the main result. In Section 4, we give examples that tell us that metrics with minimal singularities do not always have algebraic singularities.

This paper is a revised version of our preprint [Mat13-A]. Ten months after we finish writing our preprint, Guan and Zhou announced that they solved the strong openness conjecture in [GZ13]. Although their celebrated result implies the main theorem, we believe that it is worth to display our techniques. This is because, our view point is quite different from that of them and our techniques seem to have some applications. In fact, we can give an interesting generalization of the injectivity theorem by applying our techniques (see [Mat13-B]).

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2. **Preliminaries**

In this section, we recall the fundamental results needed later. Refer to [Dem], [Dem-book], [Fuj12], and [Laz] for more details.

2.1. **Singular Metrics and the Nadel Vanishing Theorem.** Throughout this subsection, let $X$ be a compact complex manifold and $F$ be a line bundle on $X$. First we recall the definitions and properties of singular metrics and their multiplier ideal sheaves.
Definition 2.1. (1) A (hermitian) metric $h$ on $F$ is called a singular metric, if for a local trivialization $\theta : F|_U \cong U \times \mathbb{C}$ and a local section $\xi$ of $F$ on an open set $U \subset X$, there exists an $L^1_{\text{loc}}$-function $\varphi$ on $U$ such that $|\xi|_h = |\theta(\xi)| e^{-\varphi}$. Here $\varphi$ is called the local weight of $h$ with respect to the trivialization.

(2) A (singular) metric $h$ on $F$ is said to have algebraic (resp. analytic) singularities, if there exists an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that a local weight $\varphi$ of $h$ can be locally written as $\varphi = c_2 \log (|f_1|^2 + |f_2|^2 + \cdots + |f_k|^2) + v$, where $c \in \mathbb{Q}_{>0}$ (resp. $c \in \mathbb{R}_{>0}$), $f_1, \ldots, f_k$ are local generators of $\mathcal{I}$ and $v$ is a smooth function.

(3) The curvature current $\sqrt{-1} \Theta_h(F)$ associated to $h$ is defined by $\sqrt{-1} \Theta_h(F) = dd^c \varphi$, where $\varphi$ is a local weight of $h$.

In this paper, we simply abbreviate the singular metric (resp. the curvature current) to the metric (resp. the curvature). The Levi form $dd^c \varphi$ is taken in the sense of distributions. Note that curvatures are $(1, 1)$-currents but not always smooth $(1, 1)$-forms. The curvature $\sqrt{-1} \Theta_h(F)$ is said to be positive if $\sqrt{-1} \Theta_h(F) \geq 0$ in the sense of currents. If $\sqrt{-1} \Theta_h(F) \geq \omega$ for some hermitian form $\omega$ on $X$, then $h$ is said to have strictly positive curvature.

Definition 2.2. Let $h$ be a metric on $F$ such that $\sqrt{-1} \Theta_h(F) \geq \gamma$ for some smooth $(1, 1)$-form $\gamma$ on $X$. Then the ideal sheaf $\mathcal{I}(h)$ defined to be $\mathcal{I}(h)(U) := \{ f \in \mathcal{O}_X(U) \mid |f| e^{-\varphi} \in L^2_{\text{loc}}(U) \}$ for an open set $U \subset X$ is called the multiplier ideal sheaf associated to $h$.

The original Nadel vanishing theorem (Theorem 1.1) asserts that all higher cohomology groups with coefficients in $K_X \otimes F \otimes \mathcal{I}(h)$ vanish if $h$ has strictly positive curvature. Our motivation is that one can obtain a Nadel vanishing theorem for metrics with minimal singularities $h_{\text{min}}$. When $F$ is not an ample line bundle, the metric $h_{\text{min}}$ never has strictly positive curvature. Nevertheless, as we stated in Section 1, we can obtain a Nadel vanishing theorem for $h_{\text{min}}$ and $h_{\text{Siu}}$. Let us recall the definition of metrics with minimal singularities.

Definition 2.3. Let $h_1$ and $h_2$ be metrics on $F$ with positive curvature. The metric $h_1$ is said to be less singular than $h_2$ if $h_1 \leq C h_2$ for some positive constant $C > 0$.

When $F$ is pseudo-effective (that is, $F$ admits a metric with positive curvature), we can construct a metric $h_{\text{min}}$ on $F$ with the following properties:

- $h_{\text{min}}$ has semi-positive curvature.
- $h_{\text{min}}$ is less singular than any metric on $F$ with positive curvature.

Such a metric is uniquely determined up to equivalence of singularities (see [Dem], (6.4)}
Theorem for more details). Metrics with minimal singularities do not always have analytic singularities. Indeed, if $h_{\min}$ has analytic singularities, then $F$ admits a birational Zariski decomposition (see Section 4). However, there exist big line bundles that do not admit a birational Zariski decomposition.

2.2. Equisingular Approximations. In the proof of the main result, we apply the equisingular approximation to a given metric. In this subsection, we reformulate [DPS01, Theorem 2.3.] with our notation and give further remarks.

**Theorem 2.4.** (DPS01 Theorem 2.3.) Let $X$ be a compact Kähler manifold and $F$ be a line bundle with a metric $h$ with (semi-)positive curvature. Then there exist metrics $\{h_\varepsilon\}_{\varepsilon > 0}$ on $F$ with the following properties:

(a) $h_\varepsilon$ is smooth on $X \setminus Z_\varepsilon$, where $Z_\varepsilon$ is a subvariety on $X$.
(b) $h_{\varepsilon_2} \leq h_{\varepsilon_1} \leq h$ holds for any $0 < \varepsilon_1 < \varepsilon_2$.
(c) $I(h) = I(h_\varepsilon)$.
(d) $\sqrt{-1} \Theta_{h_\varepsilon}(F) \geq -\varepsilon \omega$.

Moreover, if the set $\{x \in X \mid \nu(\varphi, x) > 0\}$ is contained in a subvariety $Z$, then we can add the property that $Z_\varepsilon$ is contained in $Z$ for any $\varepsilon > 0$. Here $\nu(\varphi, x)$ denotes the Lelong number at $x$ of a weight $\varphi$ of $h$.

**Proof.** Fix a smooth metric $g$ on $F$. Then there exists an $L^1$-function $\varphi$ on $X$ with $h = ge^{-\varphi}$. By applying [DPS01, Theorem 2.3.] to $\varphi$, we can obtain quasi-psh functions $\varphi_\nu$ with equisingularities. For a given $\varepsilon > 0$, by taking a large $\nu = \nu(\varepsilon)$, we define $h_\varepsilon$ by $h_\varepsilon := ge^{-\varphi_\nu(\varepsilon)}$. Then the metric $h_\varepsilon$ satisfies properties (a), (b), (c), (d).

The latter conclusion follows from the proof. We will see this fact shortly, by using the notation in [DPS01]. In their proof, they locally approximate $\varphi$ by $\varphi_{\varepsilon, \nu, j}$ with logarithmic pole. By inequality (2.5) in [DPS01], the Lelong number of $\varphi_{\varepsilon, \nu, j}$ is less than or equal to that of $\varphi$. Hence $\varphi_{\varepsilon, \nu, j}$ is smooth on $X \setminus Z$ since $\varphi_{\varepsilon, \nu, j}$ has a logarithmic pole. Then $\varphi_\nu$ is obtained from Richberg’s regularization of the supremum of these functions (see around (2.5) and (2.7)). Since the supremum is continuous on $X \setminus Z$, we obtain the latter conclusion.

2.3. Theory of Harmonic Integrals. For our purpose, we recall some facts on the theory of harmonic integrals. Throughout this subsection, let $Y$ be a (not necessarily compact) complex manifold with a hermitian form $\omega$ and $E$ be a line bundle on $Y$ with a smooth metric $h$.

For $E$-valued $(p,q)$-forms $u$, $v$, the point-wise inner product $\langle u, v \rangle_{h, \omega}$ can be defined. Further, the (global) inner product $\langle \langle u, v \rangle \rangle_{h, \omega}$ can also be defined by

$$\langle \langle u, v \rangle \rangle_{h, \omega} := \int_Y \langle u, v \rangle_{h, \omega} \omega^n.$$  

Then the $L^2$-space of the $E$-valued $(p,q)$-forms is defined as follows:

$L_{\omega}^{p,q}(Y, E)_{h, \omega} := \{u \mid u$ is an $E$-valued $(p,q)$-form with $\|u\|_{h, \omega} < \infty\}$.  

The connection $D_h$ on $E$ is determined by the holomorphic structure of $E$ and the hermitian metric $h$, which is called the Chern connection. The Chern connection $D_h$ can be written as $D_h = D_h' + D_h''$ with the $(1,0)$-connection $D_h'$ and the $(0,1)$-connection $D_h''$. (The $(0,1)$-connection $D_h''$ agrees with $\overline{\partial}$.) The connections $D_h'$ and $D_h''$ can be seen as a closed and densely defined operator on $L_{(2)}^{p,q}(Y,E)_{h,\omega}$. If $\omega$ is a complete metric on $Y$, then the formal adjoints $D_h'^*$ and $D_h''^*$ coincide with the Hilbert space adjoints (see [Dem-book, (3.2) Theorem in Chapter 8]).

**Proposition 2.5.** Let $\omega$ be a complete Kähler metric on $Y$ of dimension $n$. Assume that $\sqrt{-1}\Theta_h(E) \geq -C\omega$ for some positive constant $C > 0$. Then for every $u \in L_{(2)}^{n;i}(Y,E)_{h,\omega}$ with $u \in \text{Dom}D_h'^* \cap \text{Dom}\overline{\partial}$, the following equality holds:

$$\|D_h'^*u\|_{h,\omega}^2 + \|\overline{\partial}u\|_{h,\omega}^2 = \|D_h''^*u\|_{h,\omega}^2 + \langle \sqrt{-1}\Theta_h(E)\Lambda_\omega u, u \rangle_{h,\omega}.$$  

Here $\Lambda_\omega$ denotes the adjoint operator of the wedge product $\omega \wedge \cdot$.

**Proof.** This proposition can be obtained from Nakano’s identity and the density lemma. Since $\omega$ is a Kähler form, we have

$$\Delta'' = \Delta'' + [\sqrt{-1}\Theta_h(E), \Lambda_\omega]$$

by Nakano’s identity. Here $\Delta'$ (resp. $\Delta''$) is the Laplacian operator defined by $\Delta' := D_h'D_h'^* + D_h''D_h''$ (resp. $\Delta'' := D_h''D_h'^* + D_h''D_h''$) and $[\cdot, \cdot]$ is the graded Lie bracket. Therefore if $u$ is smooth and compactly supported, the equality in the proposition holds. For a given $u$, there exist smooth and compactly supported $E$-valued $(n,i)$-forms $\{u_k\}_{k=1}^\infty$ such that $u_k \to u$, $D_h'^*u_k \to D_h'^*u$ and $\overline{\partial}u_k \to \overline{\partial}u$ in $L_{(2)}^{n;i}(Y,E)_{h,\omega}$ as $k \to \infty$ since $\omega$ is a complete metric (see [Dem-book, Chapter 8]). By the assumption of $\sqrt{-1}\Theta(E) \geq -C\omega$, the second term of the right hand side

$$\langle \sqrt{-1}\Theta_h(E)\Lambda_\omega u_k, u_k \rangle_{h,\omega} = \int_Y \langle \sqrt{-1}\Theta_h(E)\Lambda_\omega u_k, u_k \rangle_{h,\omega} \omega^n.$$  

is bounded below. Therefore we obtain the conclusion by Lebesgue’s convergence theorem. \hfill $\Box$

### 2.4. Topics Related to the Main Problem.

It is worth mentioning that the main result of this paper is related to the openness conjecture of Demailly-Kollár on singularities of psh functions ([DK01, 5.3 Remark]).

When $h_{\text{min}}$ is a metric with minimal singularities on a big line bundle $F$, we have

$$\mathcal{I}_+(h_{\text{min}}) \subset \mathcal{I}(\|F\|) \subset \mathcal{I}(h_{\text{min}}),$$

where $\mathcal{I}_+(h_{\text{min}})$ is the upper semi-continuous regularization of the multiplier ideal sheaf of $h_{\text{min}}$ (see [DEL00]). Further all higher cohomology groups with coefficients in $K_X \otimes F \otimes \mathcal{I}_+(h_{\text{min}})$ vanish. Indeed, it is easy to check that there exists a metric $h$ on $F$ such that the curvature of $h$ is strictly positive and $\mathcal{I}(h) = \mathcal{I}_+(h_{\text{min}})$. However the main problem of
this paper has been an open problem even if $F$ is big since it is more difficult to handle $\mathcal{I}(h_{\min})$.

It is an interesting question to ask whether or not $\mathcal{I}_+(\varphi)$ agrees with $\mathcal{I}(\varphi)$ for a psh function $\varphi$, which was first posed in [DEL00]. It is a natural question, but it has been an open problem. It is known that $\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi)$ holds when $\varphi$ has analytic singularities, but unfortunately $h_{\min}$ does not always have analytic singularities (see Section 4). This question is also related to the openness conjecture of Demailly-Kollár. Recently, Berndtsson announced that this conjecture was affirmatively solved in [Ber13]. However the openness conjecture does not, at least directly, imply that $\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi)$ holds. See [Kim10, Proposition 3.2] for relations between this question and the openness conjecture.

3. Main results

In this section, we prove Theorem 3.1 and Corollary 3.6 which lead to Theorem 1.2.

3.1. Proof of the Main Theorem. In this subsection, we give a proof of Theorem 3.1.

**Theorem 3.1.** Let $F$ be a big line bundle on a smooth projective variety $X$ and $h$ be a (singular) metric on $F$ with (semi-)positive curvature. Assume that there exist a (non-zero) section $s$ of some positive multiple $F^m$ of $F$ and a (singular) metric $h_{\text{big}}$ on $F$ satisfying the following conditions:

1. The metric $h_{\text{big}}$ has strictly positive curvature.
2. The metric $h$ is less singular than $h_{\text{big}}$.
3. The point-wise norm $|s|_{h_{\text{big}}}$ of $s$ with respect to $h_{\text{big}}$ is bounded on $X$.

Then we have

$$H^i(X, K_X \otimes F \otimes \mathcal{I}(h)) = 0 \quad \text{for any } i > 0.$$  

**Proof of Theorem 3.1.** The idea of the proof is as follows: We observe the case where $h$ is smooth on $X$ (or some Zariski open set of $X$). Take an arbitrary cohomology class $\{u\} \in H^i(X, K_X \otimes F \otimes \mathcal{I}(h))$ represented by the $F$-valued $(n, i)$-form $u$ such that the norm $\|u\|_h$ is bounded and $u$ is harmonic. If the norm $\|u\|_{h_{\text{big}}}$ is bounded, we can construct a solution $\gamma \in L^{n, i-1}_2(X, F)_h$ of the $\overline{\partial}$-equation $\overline{\partial}\gamma = u$ by assumptions (1), (2), which completes the proof. However we can never expect that $\|u\|_{h_{\text{big}}}$ is bounded since $h_{\text{big}}$ is more singular than $h$. Now we consider the $\overline{\partial}$-equation $\overline{\partial}\beta = su$ instead of $\overline{\partial}\gamma = u$. Then the norm $\|su\|_{h_{\text{big}}}$ is bounded since the singularities of $h_{\text{big}}$ are canceled by $s$ thanks to assumption (3). Further the metric $h_{\text{big}}^m h$ has strictly positive curvature. Therefore we can obtain a solution $\beta \in L^{n, i-1}_2(X, F^{m+1})_{h^{m+1}}$ of the $\overline{\partial}$-equation $\overline{\partial}\beta = su$. On the other hand, $su$ can be shown to be harmonic since $u$ is harmonic and the curvature of $h$ is semi-positive. It follows from Enoki's technique for the injectivity theorem. From these arguments, we can conclude that $u$ is zero.

Unfortunately $h$ may not smooth even on Zariski open sets of $X$. In Step 1, we approximate the metric $h$ by metrics $\{h_\varepsilon\}_{\varepsilon > 0}$ that are smooth on a Zariski open set. Then we must investigate the asymptotic behavior of the harmonic forms that represent a given
cohomology class since the positivity of the curvature of $h_\varepsilon$ can be lost. In this step, we fix the notation to apply the theory of harmonic integrals. At the end of this step, we give the sketch of the proof.

**Step 1 (The equisingular approximation of $h_\varepsilon$)**

We can take a Kähler form $\omega$ on $X$ such that $\sqrt{-1}\Theta_{h_{\text{big}}}(F) \geq \omega$ since the curvature of $h_{\text{big}}$ is strictly positive. For the proof we apply the theory of harmonic integrals, but $h$ may not be smooth. For this reason, we need to approximate $h$ by metrics $\{h_\varepsilon\}_{\varepsilon > 0}$ that are smooth on a Zariski open set. By Theorem 2.4, we can obtain metrics $\{h_\varepsilon\}_{\varepsilon > 0}$ on $F$ with the following properties:

(a) $h_\varepsilon$ is smooth on $X \setminus Z_\varepsilon$, where $Z_\varepsilon$ is a subvariety on $X$.
(b) $h_{\varepsilon_2} \leq h_{\varepsilon_1} \leq h$ holds for any $0 < \varepsilon_1 < \varepsilon_2$.
(c) $\mathcal{I}(h) = \mathcal{I}(h_\varepsilon)$.
(d) $\sqrt{-1}\Theta_{h_\varepsilon}(F) \geq -\varepsilon \omega$.

By assumptions (2), (3), the point-wise norm $|s|_{h,m}$ is also bounded on $X$, thus the set $\{x \in X \mid \nu(\varphi, x) > 0\}$ is contained in the subvariety $Z := \{z \in X \mid s(x) = 0\}$. Here $\nu(\varphi, x)$ denotes the Lelong number at $x \in X$ of a weight $\varphi$ of $h$. This implies that $h_\varepsilon$ is smooth on $X \setminus Z$ (see the latter conclusion of Theorem 2.4). Hence we can add a stronger property than property (a), namely

(e) $h_\varepsilon$ is smooth on $Y := X \setminus Z$, where $Z$ is a subvariety independent of $\varepsilon$.

Now we construct a complete Kähler metric on $Y$ with suitable potential function. Take a quasi-psh function $\psi$ on $X$ such that $\psi$ is smooth on $Y$ and $\psi$ has a logarithmic pole along $Z$. Since quasi-psh functions are upper semi-continuous, we may assume $\psi \leq -e$. Then we define the function $\varphi$ and the $(1,1)$-form $\tilde{\omega}$ on $Y$ by

$$\varphi := \frac{1}{\log(-\psi)} \quad \text{and} \quad \tilde{\omega} := \ell \omega + dd^c \varphi,$$

where $\ell$ is a positive number. By taking a sufficiently large $\ell > 0$, we can easily see that the $(1,1)$-form $\tilde{\omega}$ satisfies the following properties:

(A) $\tilde{\omega}$ is a complete Kähler metric on $Y$.
(B) $\varphi$ is bounded on $X$.
(C) $\tilde{\omega} \geq \omega$.

Indeed, properties (B), (C) are obvious. Property (A) follows from straightforward computations (see [Fuj12, Lemma 3.1]).

In the proof, we consider the harmonic forms on $Y$ with respect to $h_\varepsilon$ and $\tilde{\omega}$. Let $L^n_{(2)}(Y, F)_{h_\varepsilon, \tilde{\omega}}$ be the space of the square integrable $F$-valued $(n,i)$-forms $\alpha$ with respect to the inner product defined by

$$||\alpha||_{h_\varepsilon, \tilde{\omega}}^2 := \int_Y |\alpha|^2_{h_\varepsilon, \tilde{\omega}} \tilde{\omega}^n.$$

Then we have the orthogonal decomposition

$$L^n_{(2)}(Y, F)_{h_\varepsilon, \tilde{\omega}} = \text{Im} \tilde{\mathcal{F}} \oplus \mathcal{H}^{n,i}(F)_{h_\varepsilon, \tilde{\omega}} \oplus \text{Im} \mathcal{D}^{n,*}_{h_\varepsilon}.$$
Here the operators $D_h^{\ast}, D_h^{\prime\prime}$ are the Hilbert space adjoints of $D_h$ and $\partial$. (Since $\tilde{\omega}$ is complete, these coincide with the closed extensions of the formal adjoints in the sense of distributions.) The space $\mathcal{H}^{n,i}(F)_{h,\tilde{\omega}}$ denotes the space of the harmonic forms with respect to $h$ and $\tilde{\omega}$, namely

$$\mathcal{H}^{n,i}(F)_{h,\tilde{\omega}} = \{ \alpha \mid \alpha \text{ is an } F\text{-valued } (n,i)\text{-form such that } \partial \alpha = D_h^{\prime\prime} \alpha = 0 \}.$$  

Harmonic forms are smooth by the regularization theorem for elliptic operators. These facts may be known to specialists. The precise proof can be found in [Fuj12, Claim 1].

From property (C) we have the inequality $|\beta|_\omega^2 \tilde{\omega}^n \leq |\beta|_\omega^2 \omega^n$ for any $(n,i)$-form $\beta$. From this inequality and property (b) of $h$, we obtain

$$\|\alpha\|_{h,\tilde{\omega}} \leq \|\alpha\|_{h,\omega} \leq \|\alpha\|_{h,\omega}$$

for any $F$-valued $(n,i)$-form $\alpha$. This inequality is often used in the proof.

Take an arbitrary cohomology class $\{u\} \in H^i(X,K_X \otimes F \otimes I(h))$ represented by an $F$-valued $(n,i)$-form $u$ with $\|u\|_{h,\omega} < \infty$. Our goal is to show that the cohomology class $\{u\}$ is actually zero. By inequality (3.1), we know $u \in L^{n,i}(Y,F)_{h,\tilde{\omega}}$ for any $\varepsilon > 0$. By the orthogonal decomposition above, there exist $u_\varepsilon \in \mathcal{H}^{n,i}(F)_{h,\tilde{\omega}}$ and $v_\varepsilon \in L^{n,i-1}(Y,F)_{h,\tilde{\omega}}$ such that

$$u = u_\varepsilon + \partial v_\varepsilon.$$  

Notice that the component of $\text{Im}D_h^{\prime\prime}$ is zero since $u$ is $\partial$-closed.

The strategy of the proof is as follows: In Step 2, we show

$$\|D_h^{\prime\prime} su_\varepsilon\|_{h^{m+1},\tilde{\omega}} \to 0 \quad \text{as } \varepsilon \to 0.$$  

This step can be regarded as a generalization of Enoki’s proof of Kollár’s injectivity theorem. In Step 3, we construct a solution $\beta_\varepsilon$ of the $\partial$-equation $\partial \beta_\varepsilon = su_\varepsilon$ such that the norm $\|\beta_\varepsilon\|_{h^{m+1},\tilde{\omega}}$ is uniformly bounded. By these steps, we know

$$\|su_\varepsilon\|^2_{h^{m+1},\tilde{\omega}} = \langle su_\varepsilon, \partial \beta_\varepsilon \rangle_{h^{m+1},\tilde{\omega}} \leq \|D_h^{\prime\prime} su_\varepsilon\|_{h^{m+1},\tilde{\omega}} \|\beta_\varepsilon\|_{h^{m+1},\tilde{\omega}} \to 0 \quad \text{as } \varepsilon \to 0.$$

In Step 4, we show that $u_\varepsilon$ converges to zero in a suitable sense. This completes the proof.

**Step 2 (A generalization of Enoki’s proof of the injectivity theorem)**  
The aim of this step is to prove the following claim.

**Claim 3.2.** The norm $\|D_h^{\prime\prime} su_\varepsilon\|_{h^{m+1},\tilde{\omega}}$ converges to zero as letting $\varepsilon$ go to zero.

**Proof of Claim 3.2.** From the definition of $u_\varepsilon$ and inequality (3.1), we have

$$\|u_\varepsilon\|_{h,\tilde{\omega}} \leq \|u\|_{h,\omega} \leq \|u\|_{h,\omega}.$$  

In the proof, these inequalities play an important role. By applying Proposition 2.3 to $u_\varepsilon$, we obtain

$$0 = \langle \sqrt{-1}\Theta_{h}\Lambda_{\tilde{\omega}}u_\varepsilon, u_\varepsilon \rangle_{h,\tilde{\omega}} + \|D_h^{\prime\prime} u_\varepsilon\|^2_{h,\tilde{\omega}}.$$
Note that the left hand side is zero since $u_\varepsilon$ is harmonic. Let $A_\varepsilon$ be the first term and $B_\varepsilon$ the second term of the right hand side of equality (3.3). From now on, we will show that the first term $A_\varepsilon$ and the second term $B_\varepsilon$ converge to zero. For simplicity, we put

$$ g_\varepsilon := \langle \sqrt{-1} \Theta_{h^m_\varepsilon}(F) \Lambda_{\overline{\omega}} u_\varepsilon, u_\varepsilon \rangle_{h^m_\varepsilon, \overline{\omega}}. $$

Then we can easily see that there exists a positive constant $C$ such that

$$ g_\varepsilon \geq -\varepsilon C |u_\varepsilon|_{h^m_\varepsilon, \overline{\omega}}^2. $$

Indeed, let $\lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq \cdots \leq \lambda_n^\varepsilon$ be the eigenvalues of $\sqrt{-1} \Theta_{h^m_\varepsilon}(F)$ with respect to $\overline{\omega}$. Then for any point $y \in \mathcal{Y}$, there exists a local coordinates $(z_1, z_2, \ldots, z_n)$ centered at $y$ such that

$$ \sqrt{-1} \Theta_{h^m_\varepsilon}(F) = \sum_{j=1}^n \lambda_j^\varepsilon dz_j \wedge d\overline{z}_j \quad \text{and} \quad \overline{\omega} = \sum_{j=1}^n dz_j \wedge d\overline{z}_j \quad \text{at } y. $$

When we locally write $u_\varepsilon$ as $u_\varepsilon = \sum_{|K|=i} f_K^\varepsilon dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_K$, by straightforward computations we have

$$ g_\varepsilon = \sum_{|K|=i} \left( \sum_{j \in K} \lambda_j^\varepsilon \right) |f_K^\varepsilon|_{h^m_\varepsilon}^2. $$

On the other hand, from property (C) of $\overline{\omega}$ and property (d) of $h^m$, we have $\sqrt{-1} \Theta_{h^m}(F) \geq -\varepsilon \omega \geq -\varepsilon \overline{\omega}$, which implies that $\lambda_j^\varepsilon \geq -\varepsilon$. Therefore we obtain inequality (3.4).

From inequalities (3.2), (3.4) and equality (3.3), we obtain

$$ 0 \geq A_\varepsilon = \int_{\mathcal{Y}} g_\varepsilon \overline{\omega}^n $$

$$ \geq -\varepsilon C \int_{\mathcal{Y}} |u_\varepsilon|_{h^m_\varepsilon, \overline{\omega}}^2 \overline{\omega}^n $$

$$ \geq -\varepsilon C |u|^2_{h^m_\varepsilon, \omega}. $$

Hence $A_\varepsilon$ converges to zero. By equality (3.3), $B_\varepsilon$ also converges to zero.

We will apply Proposition 2.5 to $su_\varepsilon$. We first see that the norm $|su_\varepsilon|_{h_{m+1}^\varepsilon, \overline{\omega}}$ is bounded. By assumptions (2), (3), the point-wise norm $|s|_{h^m}$ with respect to $h^m$ is bounded. Further, we have $|s|_{h_m^\varepsilon} \leq |s|_{h^m}$ from property (b) of $h^m$, thus we obtain

$$ |su_\varepsilon|_{h_{m+1}^\varepsilon, \overline{\omega}} \leq \sup_X |s|_{h^m} |u_\varepsilon|_{h^m_\varepsilon, \overline{\omega}} \leq \sup_X |s|_{h^m} |u|_{h^m_\varepsilon, \omega} < \infty. $$

By applying Proposition 2.5 to $su_\varepsilon$, we obtain

$$ \|D_{h_{m+1}^\varepsilon}^*su_\varepsilon\|_{h_{m+1}^\varepsilon, \overline{\omega}}^2 = \langle \sqrt{-1} \Theta_{h_{m+1}^\varepsilon}(F_{m+1}) \Lambda_{\overline{\omega}} su_\varepsilon, su_\varepsilon \rangle_{h_{m+1}^\varepsilon, \overline{\omega}} + \|D_{h_{m+1}^\varepsilon}^*su_\varepsilon\|_{h_{m+1}^\varepsilon, \overline{\omega}}^2. $$

Here we used $\overline{\mathcal{D}}su_\varepsilon = 0$. First we see that the second term of the right hand side converges to zero. Since $s$ is a holomorphic $(0, 0)$-form, we can check the equality $D_{h_{m+1}^\varepsilon}^*su_\varepsilon = sD_{h^m_\varepsilon}^*u_\varepsilon$, which yields

$$ \|D_{h_{m+1}^\varepsilon}^*su_\varepsilon\|_{h_{m+1}^\varepsilon, \overline{\omega}}^2 \leq \sup_X |s|_{h^m}^2 \int_{\mathcal{Y}} |D_{h^m_\varepsilon}^*u_\varepsilon|_{h^m_\varepsilon, \overline{\omega}}^2 \overline{\omega}^n \leq \sup_X |s|_{h^m}^2 B_\varepsilon. $$
Since $B_{\varepsilon}$ converges to zero, the norm $\|D^*_{h_{m+1}} s u_{\varepsilon}\|_{h_{m+1}, \omega}$ also converges to zero.

For the proof of the claim, it remains to show that the first term of the right hand side of equation (3.5) converges to zero. For this purpose, we investigate $A_{\varepsilon}$ in details. By the definition of $A_{\varepsilon}$, we have

$$A_{\varepsilon} = \int_{\{g_{\varepsilon} \geq 0\}} g_{\varepsilon} \tilde{\omega}^n + \int_{\{g_{\varepsilon} \leq 0\}} g_{\varepsilon} \tilde{\omega}^n.$$ 

Let $A_{\varepsilon}^+$ be the first term and $A_{\varepsilon}^-$ be the second term of the right hand side. Then $A_{\varepsilon}^+$ and $A_{\varepsilon}^-$ converge to zero. Indeed, by simple computations and inequalities (3.2), (3.4), we have

$$0 \geq A_{\varepsilon}^- \geq -\varepsilon C \int_{\{g_{\varepsilon} \leq 0\}} |u_{\varepsilon}|^2 |h_{\varepsilon, \omega}| \tilde{\omega}^n$$ 

$$\geq -\varepsilon C \int_Y |u_{\varepsilon}|^2 |h_{\varepsilon, \omega}| \tilde{\omega}^n$$ 

$$\geq -\varepsilon C \|u\|_{h_{\varepsilon, \omega}}^2.$$ 

It implies that $A_{\varepsilon}^+$ and $A_{\varepsilon}^-$ go to zero since $A_{\varepsilon} = A_{\varepsilon}^+ + A_{\varepsilon}^-$ goes to zero. Now we have

$$\langle \sqrt{-1} \Theta h_{m+1} F_{m+1} \Lambda s u_{\varepsilon}, s u_{\varepsilon} \rangle_{h_{m+1}, \omega} = (m + 1) \left| s \right|_{h_{m+1}^p}^2 \langle \sqrt{-1} \Theta h_{\varepsilon} (F) \Lambda s u_{\varepsilon}, u_{\varepsilon} \rangle_{h_{\varepsilon, \omega}}$$

$$= (m + 1) \left| s \right|_{h_{m}^p}^2 g_{\varepsilon}. $$

Therefore we obtain

$$\langle \left\langle \sqrt{-1} \Theta h_{m+1} (F_{m+1}) \Lambda s u_{\varepsilon}, s u_{\varepsilon} \right\rangle_{h_{m+1}, \omega} \rangle_{h_{m}, \omega} = (m + 1) \left\{ \int_{\{g_{\varepsilon} \geq 0\}} |s|_{h_{m}^p}^2 g_{\varepsilon} \tilde{\omega}^n + \int_{\{g_{\varepsilon} \leq 0\}} |s|_{h_{m}^p}^2 g_{\varepsilon} \tilde{\omega}^n \right\}.$$ 

Let $C_{\varepsilon}^+$ be the first term and $C_{\varepsilon}^-$ be the second term of the right hand side. Then we have

$$0 \leq C_{\varepsilon}^+ \leq (m + 1) \sup_X |s|_{h_{m}^p}^2 \int_{\{g_{\varepsilon} \geq 0\}} g_{\varepsilon} \tilde{\omega}^n$$ 

$$\leq (m + 1) \sup_X |s|_{h_{m}^p}^2 A_{\varepsilon}^+.$$ 

On the other hand, we have

$$0 \geq C_{\varepsilon}^- \geq (m + 1) \sup_X |s|_{h_{m}^p}^2 \int_{\{g_{\varepsilon} \leq 0\}} g_{\varepsilon} \tilde{\omega}^n$$

$$\geq -\varepsilon C (m + 1) \sup_X |s|_{h_{m}^p}^2 \int_{\{g_{\varepsilon} \leq 0\}} |u_{\varepsilon}|_{h_{\varepsilon, \omega}}^2 \tilde{\omega}^n$$

$$\geq -\varepsilon C (m + 1) \sup_X |s|_{h_{m}^p}^2 \int_Y |u_{\varepsilon}|_{h_{\varepsilon, \omega}}^2 \tilde{\omega}^n$$

$$\geq -\varepsilon C (m + 1) \sup_X |s|_{h_{m}}^2 \|u\|_{h_{\varepsilon, \omega}}^2.$$
By these inequalities, we know that $C_\varepsilon^+$ and $C_\varepsilon^-$ converge to zero. Thus we obtain the conclusion of Claim 3.2 from equality (3.5). □

Step 3 (Solutions of the $\overline{\partial}$-equation with uniformly bounded $L^2$-norms)

A positive multiple $s^k$ of the section $s$ also satisfies assumption (3), thus we may assume that $m$ is greater than $\ell$. Under this additional assumption, we prove the following claim.

Claim 3.3. For every $\varepsilon > 0$, there exists an $F^m$-valued $(n,i-1)$-form $\beta_\varepsilon$ such that

$$\overline{\partial}\beta_\varepsilon = su_\varepsilon$$

and $\|\beta_\varepsilon\|_{h^m_{\tilde{\omega}},\tilde{\omega}}$ is uniformly bounded.

Proof of Claim 3.3. To construct solutions of the $\overline{\partial}$-equation $\overline{\partial}\beta_\varepsilon = su_\varepsilon$, we apply [Dem82, Théorème 4.1]. For this purpose, we consider the new metric $H_\varepsilon$ on $F^m_{\tilde{\omega},\tilde{\omega}}$ defined by $H_\varepsilon := h_{big}^m h_\varepsilon e^{-\varphi}$. First we compute the curvature of $H_\varepsilon$. We have

$$\sqrt{-1}\Theta_{H_\varepsilon}(F^{m+1}) = m\sqrt{-1}\Theta_{h_{big}}(F) + \sqrt{-1}\Theta_{h_\varepsilon}(F) + dd^c\varphi \geq m\omega - \varepsilon\omega + dd^c\varphi.$$

By the definition of $\tilde{\omega}$ and the additional assumption of $m > \ell$, we obtain $\sqrt{-1}\Theta_{H_\varepsilon}(F^{m+1}) \geq \tilde{\omega}$. Now we consider the norm of $su_\varepsilon$ with respect to $H_\varepsilon$. By assumption (3), the pointwise norm $|s|_{h_{big}^m}$ is bounded on $X$. Further, since $\varphi$ is bounded on $X$, there exist positive constants $C_1$ and $C_2$ such that $C_1 \leq e^{-\varphi} \leq C_2$ on $X$. Then by inequality (3.2) we obtain

$$\|su_\varepsilon\|_{H_\varepsilon,\tilde{\omega}} \leq C_2 \sup_X |s|_{h_{big}^m} \|u_\varepsilon\|_{h_\varepsilon,\tilde{\omega}} \leq C_2 \sup_X |s|_{h_{big}^m} \|u\|_{h,\omega}.$$

In particular, the norm $\|su_\varepsilon\|_{H_\varepsilon,\tilde{\omega}}$ is bounded. Moreover, the right hand side does not depend on $\varepsilon$. By applying [Dem82, Théorème 4.1], we can find a solution $\beta_\varepsilon$ of the $\overline{\partial}$-equation $\overline{\partial}\beta_\varepsilon = su_\varepsilon$ with

$$\|\beta_\varepsilon\|_{H_\varepsilon,\tilde{\omega}}^2 \leq \frac{1}{\ell} \|su_\varepsilon\|_{H_\varepsilon,\tilde{\omega}}^2.$$

Since $h$ is less singular than $h_{big}$, there exists a positive constant $C_3 > 0$ such that $h_\varepsilon \leq h \leq C_3 h_{big}$. Then we can easily see

$$C_3^{-m} C_1 \|\beta_\varepsilon\|_{h^m_{\tilde{\omega},\tilde{\omega}}} \leq \|\beta_\varepsilon\|_{H_\varepsilon,\tilde{\omega}}.$$

Since the right hand side can be estimated by the constant independent of $\varepsilon$, the norm $\|\beta_\varepsilon\|_{h^m_{\tilde{\omega},\tilde{\omega}}}$ is uniformly bounded. This completes the proof. □

Step 4 (The limit of harmonic forms)

In this step, we investigate the limit of $u_\varepsilon$ and complete the proof of Theorem 3.1. First we prove the following claim.

Claim 3.4. The norm $\|su_\varepsilon\|_{h^m_{\tilde{\omega},\tilde{\omega}}}$ converges to zero as $\varepsilon \to \infty$. 
Proof of Claim 3.4. Take \( \beta_\varepsilon \in L^{n,i+1}(F^{m+1})_{h^{m+1,\omega}} \) satisfying the properties in Claim 3.3. Then straightforward computations yield
\[
\|su_\varepsilon\|^2_{h^{m+1,\omega}} = \langle su_\varepsilon, \beta_\varepsilon \rangle_{h^{m+1,\omega}}
\]
\[
= \langle D_{h^{m+1}} su_\varepsilon, \beta_\varepsilon \rangle_{h^{m+1,\omega}}
\]
\[
\leq \|D_{h^{m+1}} su_\varepsilon\|_{h^{m+1,\omega}} \|\beta_\varepsilon\|_{h^{m+1,\omega}}.
\]

The norm of \( \beta_\varepsilon \) is uniformly bounded by Claim 3.3. On the other hand, the norm \( \|D_{h^{m+1}} su_\varepsilon\|_{h^{m+1,\omega}} \) converges to zero by Claim 3.2. Therefore the norm \( \|su_\varepsilon\|_{h^{m,\omega}} \) also converges to zero.

From now on, we fix a small positive number \( \varepsilon_0 > 0 \). Then for any positive number \( \varepsilon \) with \( 0 < \varepsilon < \varepsilon_0 \), by property (b) of \( h_\varepsilon \), we obtain
\[
\|u_\varepsilon\|_{h^{0,\omega}} \leq \|u_\varepsilon\|_{h^{\varepsilon,\omega}} \leq \|u\|_{h,\omega}.
\]

These inequalities say that the norms of \( \{u_\varepsilon\}_{\varepsilon > 0} \) with respect to \( h_{\varepsilon_0} \) are uniformly bounded. Therefore there exists a subsequence of \( \{u_\varepsilon\}_{\varepsilon > 0} \) that converges to \( \alpha \in L^{n,i}(Y,F)_{h_{\varepsilon_0}\omega} \) with respect to the weak \( L^2 \)-topology. For simplicity, we denote this subsequence by the same notation \( \{u_\varepsilon\}_{\varepsilon > 0} \). Then we prove the following claim.

Claim 3.5. The weak limit \( \alpha \) of \( \{u_\varepsilon\}_{\varepsilon > 0} \) in \( L^{n,i}(Y,F)_{h_{\varepsilon_0}\omega} \) is zero.

Proof of Claim 3.5. For every positive number \( \delta > 0 \), we define the subset \( A_\delta \) of \( Y \) by \( A_\delta := \{x \in Y \mid |s|_{h_{\varepsilon_0}}^2 > \delta \} \). Since a weight of \( h_{\varepsilon_0} \) is upper semi-continuous, \( |s|_{h_{\varepsilon_0}}^2 \) is lower semi-continuous. Hence \( A_\delta \) is an open set of \( Y \). We estimate the norm of \( u_\varepsilon \) on \( A_\delta \). By easy computations, we have
\[
\|su_\varepsilon\|^2_{h^{m+1,\omega}} \geq \|su_\varepsilon\|^2_{h^{0,\omega}} \geq \int_{A_\delta} |s|_{h_{\varepsilon_0}}^2 \|u_\varepsilon\|^2_{h_{\varepsilon_0}\omega} \omega^n \geq \delta \int_{A_\delta} \|u_\varepsilon\|^2_{h_{\varepsilon_0}\omega} \omega^n \geq 0
\]
for any \( \delta > 0 \).

Note that \( u_\varepsilon|_{A_\delta} \) converges to \( \alpha|_{A_\delta} \) with respect to the weak \( L^2 \) topology in \( L^{n,i}(A_\delta,F)_{h_{\varepsilon_0}\omega} \), where \( u_\varepsilon|_{A_\delta} \) (resp. \( \alpha|_{A_\delta} \)) is the restriction of \( u_\varepsilon \) (resp. \( \alpha \)) to \( A_\delta \). Indeed, for every \( \gamma \in L^{n,i}(A_\delta,F)_{h_{\varepsilon_0}\omega} \), the inner product \( \langle u_\varepsilon|_{A_\delta}, \gamma \rangle_{A_\delta} = \langle u_\varepsilon, \gamma \rangle_Y \) converges to \( \langle \alpha, \tilde{\gamma} \rangle_Y = \langle \alpha, \gamma \rangle_{A_\delta} \), where \( \tilde{\gamma} \) is the zero extension of \( \gamma \) to \( Y \). Since \( u_\varepsilon|_{A_\delta} \) converges to \( \alpha|_{A_\delta} \), we obtain
\[
\|\alpha|_{A_\delta}\|_{h_{\varepsilon_0},\omega} \leq \liminf_{\varepsilon \to 0} \|u_\varepsilon|_{A_\delta}\|_{h_{\varepsilon_0},\omega} = 0.
\]
This is because the norm of the weak limit can be estimated by the limit inferior of norms of a sequence. Therefore we know that \( \alpha|_{A_\delta} = 0 \) for any \( \delta > 0 \). By the definition of \( A_\delta \), the union of \( \{A_\delta\}_{\delta > 0} \) is equal to \( Y = X \setminus Z \). Hence the weak limit \( \alpha \) is zero on \( Y \).
By using Claim 3.5, we complete the proof of Theorem 3.1. By the definition of $u_\varepsilon$, we have
\[ u = u_\varepsilon + \overline{\partial}v_\varepsilon. \]
Claim 3.5 implies that $\overline{\partial}v_\varepsilon$ converges to $u$ with respect to the weak $L^2$-topology. Then we can easily see that $u$ is a $\overline{\partial}$-exact form (that is, $u \in \text{Im} \overline{\partial}$). This is because the subspace $\text{Im} \overline{\partial}$ is closed in $L_{(2)}^{n,i}(Y,F)_{h_\varepsilon,\tilde{\omega}}$ with respect to the weak $L^2$-topology. Indeed, for every $\gamma_1 + D''\gamma_2 \in \mathcal{H}^{n,i}(F)_{h_\varepsilon,\tilde{\omega}} \oplus \text{Im}D''$, we have
\[ \langle \langle u, \gamma_1 + D''\gamma_2 \rangle \rangle = \lim_{\varepsilon \to 0} \langle \langle \overline{\partial}v_\varepsilon, \gamma_1 + D''\gamma_2 \rangle \rangle = 0. \]
Therefore we know $u \in \text{Im} \overline{\partial}$.

To summarize, we proved that $u$ is a $\overline{\partial}$-exact form in $L_{(2)}^{n,i}(Y,F)_{h_\varepsilon,\tilde{\omega}}$, which implies that the cohomology class $\{u\}$ of $u$ is zero in $H^i(X, K_X \otimes F \otimes \mathcal{I}(h_\varepsilon))$. By property (c), we obtain the conclusion of Theorem 3.1. □

3.2. Proof of Corollary. In this subsection, we prove that $h_{\text{min}}$ and $h_{\text{Siu}}$ satisfy the assumptions of Theorem 3.1.

First we recall the definition of Siu’s metrics $h_{\text{Siu}}$ (which was first introduced and plays an central role in [Siu98]). Let $F$ be a line bundle whose Kodaira dimension is non-negative (that is, some positive multiple of $F$ admits sections). For every positive integer $m > 0$, let $h_m$ be the singular metric on $F$ induced by a basis of $H^0(X,F^m)$, namely $h_m$ is (locally) defined by
\[ -\log h_m := \frac{1}{2m} \log \left( \sum_{j=1}^{N_m} |s_j^{(m)}|^2 \right), \]
where $\{s_j\}_{j=1}^{N_m}$ is a basis of $H^0(X,F^m)$. Then with a choice of positive numbers $\{\varepsilon_m\}_{m\geq 1}$, Siu’s metric $h_{\text{Siu}}$ is defined by
\[ -\log h_{\text{Siu}} = \log \sum_{m\geq 1} \varepsilon_m \frac{1}{h_m}. \]
This metric $h_{\text{Siu}}$ and the multiplier ideal sheaf $\mathcal{I}(h_{\text{Siu}})$ depend on the choice of $\{\varepsilon_k\}_{k=1}^\infty$ (see [Kim]), but $h_{\text{Siu}}$ always admits an analytic Zariski decomposition. Thus we can expect a Nadel vanishing theorem for $\mathcal{I}(h_{\text{Siu}})$.

**Corollary 3.6** (=Theorem 1.2). Let $F$ be a big line bundle on a smooth projective variety $X$ and let $h$ be either a metric with minimal singularities $h_{\text{min}}$ or Siu’s metric $h_{\text{Siu}}$ on $F$. Then we have
\[ H^i(X, K_X \otimes F \otimes \mathcal{I}(h)) = 0 \quad \text{for any } i > 0. \]

**Proof.** It is sufficient to prove that the metrics $h_{\text{min}}$ and $h_{\text{Siu}}$ satisfy the assumptions in Theorem 3.1. Since $F$ is big, some positive multiple $F^m$ of $F$ can be written as $F^m = A \otimes E$, where $A$ is a very ample line bundle and $E$ is an effective line bundle. Take a (non-zero)
section $s_A$ (resp. $s_E$) of $A$ (resp. $E$). Let $h_A$ be the metric induced by a basis of $H^0(X,A)$. Then $h_A$ is smooth and has strictly positive curvature since $A$ is very ample. Further let $h_E$ be the singular metric defined by the section $s_E$. Now we consider the section $s$ of $F^m$ and the singular metric $h_{big}$ on $F$ defined by

$$s := s_A \otimes s_E \quad \text{and} \quad h_{big} := (h_A h_E)^{1/m}.$$ 

Since the curvature of $h_{big}$ is equal to $(\sqrt{-1}\Theta h_A(A) + [\text{div} s_E])/m$, the metric $h_A$ has strictly positive curvature. Further the point-wise norm $|s|_{h_{big}}$ is bounded. This is because, the norm $|s|_{h_{big}}$ is equal to $|s_A|_{h_A}$ and $h_A$ is a smooth metric. It remains to confirm assumption (2) in Theorem 3.1. When $h$ is a metric with minimal singularities, assumption (2) is obvious by the definition of $h_{min}$. We consider the case where $h$ is Siu’s metric $h_{Siu}$. By the construction of $h_{Siu}$, the metric $h_{Siu}$ is less singular than $h_{big}$. When $\{t_j\}_{j=1}^N$ is a basis of $H^0(X,A)$, sections $\{s_E \otimes t_j\}_{j=1}^N$ become a part of a basis of $H^0(X,F^m)$. Therefore, by the construction of $h_{min}$ and $h_{big}$, the metric $h_{min}$ is less singular than $h_{big}$. □

4. Appendix

In this section, we show that metrics with minimal singularities do not always have analytic singularities even if line bundles are big. The content of this section is known to specialists, but we give the references and proof for the reader’s convenience. Throughout this section, let $X$ be a compact Kähler manifold and $D$ be a divisor on $X$. We denote by $O_X(D)$, the line bundle (the invertible sheaf) defined by $D$.

**Proposition 4.1.** Let $h_{min}$ be a metric with minimal singularities on $O_X(D)$. Assume that $h_{min}$ has analytic singularities. Then $O_X(D)$ admits a birational Zariski decomposition. That is, there exist a modification $\pi : \tilde{X} \to X$, a nef $\mathbb{R}$-divisor $P$ and an $\mathbb{R}$-effective divisor $N$ on $\tilde{X}$ with the following properties:

- $\pi^* D = P + N$
- For any positive integer $k > 0$, the map

$$H^0(X, O_{\tilde{X}}(\lceil kP \rceil)) \to H^0(X, O_{\tilde{X}}(kD))$$

induced by the section $e_k$ is an isomorphism, where $e_k$ is the natural section of $\lceil kN \rceil$. Here $[G]$ (resp. $[G]$) denotes the divisor defined by the round-downs (resp. the round-ups) of the coefficients of an $\mathbb{R}$-divisor $G$.

**Proof.** Since $h_{min}$ has analytic singularities, we can take an ideal sheaf $\mathcal{I} \subset O_X$ such that a local weight $\varphi_{min}$ can be written as

$$\varphi_{min} = \frac{c}{2} \log \left( |f_1|^2 + |f_2|^2 + \cdots + |f_k|^2 \right) + v,$$

where $f_i$ are local generators of $\mathcal{I}$ (see Definition 2.1). By taking a resolution of $\mathcal{I}$, we obtain a modification $\pi : \tilde{X} \to X$ with

$$\sqrt{-1}\Theta_{\pi^* h_{min}}(\pi^* D) = \beta + [N],$$
where $\beta$ is a smooth semi-positive $(1,1)$-form and $N$ is an effective $\mathbb{R}$-divisor. On the other hand, the pull-back $\pi^*h_{\min}$ of $h_{\min}$ is a metric with minimal singularities on $\mathcal{O}_X(\pi^*D)$. Indeed, we fix a smooth metric $h_0$ on $\mathcal{O}_X(D)$ and take an arbitrary singular metric $g$ on $\mathcal{O}_X(\pi^*D)$ with positive curvature. Then there exists an $L^1$-function $\varphi_g$ on $\tilde{X}$ such that $g = \pi^*h_0e^{-\varphi_g}$ since $\tilde{X}$ is also a compact Kähler manifold. Now we define the function $\psi_g$ on $X$ by

$$\psi_g(x) := \sup_{y \in \pi^{-1}(x)} \varphi_g(y) \text{ for any } x \in X.$$ 

Then the curvature of $h_0e^{-\psi_g}$ is also positive. Hence $h_{\min} \leq Ch_0e^{\psi_g}$ for some positive constant $C > 0$ by the definition of $h_{\min}$. It yields $\pi^*h_{\min} \leq C\pi^*h_0e^{-\varphi_g} = Cg$. Therefore $\pi^*h_{\min}$ is a metric with minimal singularities.

Put $P := \pi^*D - N$. Then the first Chern class of $P$ contains the semi-positive form $\beta$ (in particular $P$ is nef). Finally we show that the map in Proposition 4.1 is isomorphic. For an arbitrary section $s \in H^0(X, \mathcal{O}_X(kD))$, the metric $\pi^*h_{k\min}$ is less singular than the metric $h_s$ induced by the section $s$, since $\pi^*h_{k\min}$ is a metric with minimal singularities $\mathcal{O}_{\tilde{X}}(k\pi^*D)$. In particular, the Lelong number of a weight of $k\pi^*h_{\min}$ is less than or equal to that of $h_s$, thus we obtain $kN \leq \text{div}s$. Since $\text{div}s$ is a $\mathbb{Z}$-divisor, we have $\lceil kN \rceil \leq \text{div}s$. It implies that $s/e_k$ is a (holomorphic) section. This completes the proof. □

In general, it is difficult to compute metrics with minimal singularities. Proposition 4.1 says that metrics with minimal singularities on line bundles admitting no birational Zariski decomposition never have analytic singularities. However there are at least two known examples which tell us that line bundles do not admit a birational Zariski decomposition, even if the line bundles are big. (see [Nak, IV, §2.6, Example 6.4], [Les12, Theorem 1.1].)

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KAGOSHIMA UNIVERSITY, 1-21-35 Koorimoto, KAGOSHIMA 890-0065, JAPAN.

E-mail address: shinichi@sci.kagoshima-u.ac.jp, mshinichi0@gmail.com