Hodge decomposition for elliptic complexes over unital $C^*$-algebras

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Abstract

For a certain class of complexes of pre-Hilbert $A$-modules, we prove that their cohomology groups equipped with a canonical quotient structure are again pre-Hilbert $A$-modules and derive the Hodge decomposition for them. We call these complexes self-adjoint parametrix possessing. We show that $A$-elliptic complexes of differential operator acting on sections of finitely generated projective $A$-Hilbert bundles over compact manifolds have this property if the images of certain extensions of their Laplacians are closed.

1 Introduction

In this paper, we focus our attention to specific co-chain complexes of pre-Hilbert modules over unital $C^*$-algebras and adjointable pre-Hilbert module homomorphisms (differentials) acting between them. We would like to describe certain properties of their cohomology groups and prove the Hodge decomposition for them.

Let us recall that a co-chain complex in a category $C$ is a sequence $d^\bullet = (C^k, d_k)_{k \in \mathbb{Z}}$ such that $C^k$ are objects in the category $C$ and $d_k : C^k \to C^{k+1}$ are morphisms in $C$ which satisfy $d_{k+1}d_k = 0$, $k \in \mathbb{Z}$. For convenience, we consider complexes bounded from below, i.e., $k \geq 0$. As mentioned above, we are interested in complexes in the category of pre-Hilbert $A$-modules and pre-Hilbert $A$-module homomorphisms. By a Hodge decomposition, we mean a decomposition of the pre-Hilbert modules $C^k$ in the given complex into a direct sum of three pre-Hilbert modules, namely of the module of harmonic elements, the module of closed, and the module of co-closed elements. Notice that in particular, in order to define the harmonic elements, we have to suppose that the maps $d_k$ forming the complex are adjointable. Since the cohomology groups of a complex are quotients of the kernel of a map in the complex by the image of the preceding map, it is not surprising that the cohomology groups does not
necessarily belong to the category we started with. In the category of pre-Hilbert modules, the cohomologies need not be Hausdorff, let alone normed spaces.

We shall confine ourselves to self-adjoint endomorphisms of pre-Hilbert modules only. Let \( L : V \to V \) be such an endomorphism. We show that the existence of maps \( g, p : V \to V \) such that \( 1_V = gL + p = Lg + p, \) \( Lp = 0 \) and \( p = p^* \) is sufficient for the decomposition \( V = \ker L \oplus \operatorname{Im} L^* \) to hold (no completion involved). We call such endomorphisms self-adjoint parametrix possessing. Then we apply this result to complexes. With each complex \( d^* = (C^k, d_k)_{k \in \mathbb{N}_0} \) in the category of pre-Hilbert modules and adjointable pre-Hilbert homomorphisms, we associate the sequence of self-adjoint endomorphisms \( L_i = d_{i-1}d_{i-1}^* + d_i^*d_i : C^i \to C^i, i \in \mathbb{N}_0 \), the so-called Laplacians of the complex. When these Laplacians are self-adjoint parametrix possessing, we show that the cohomology groups of the complex are pre-Hilbert modules isomorphic to \( \ker L_i \), and moreover that \( C^i = \ker L_i \oplus \operatorname{Im} d_i^* \oplus \operatorname{Im} d_{i-1} \) (the Hodge decomposition). Let us recall that the Hodge theory, i.e., the Hodge decomposition and a description of the cohomology groups, is well known for complexes of finite dimensional vector spaces and linear maps. It is a consequence of elliptic operator theory, that the Hodge theory is valid also for elliptic complexes of differential operators acting between smooth sections of finite rank real or complex vector bundles over compact manifolds.

Here, we prove the Hodge decomposition for certain kind of complexes \( D^* = (\Gamma(F^k), D_k)_{k \in \mathbb{N}_0} \) of \( A \)-elliptic differential operators acting on the smooth sections of the \( A \)-Hilbert bundle \( F^i \) (notions introduced by the authors of \([1]\) and further worked out, e.g., in \([11]\)). We shall suppose that the operators act between smooth sections of Hilbert bundles over compact manifolds, the fibers of the bundles are finitely generated projective Hilbert \( A \)-modules, and that the following technical condition is satisfied: for each \( i \in \mathbb{N}_0 \), the image \( \operatorname{Im} (L_i)_{r_i} \) of the extension \( (L_i)_{r_i} : W^{r_i}(F^i) \to W^0(F^i) \) of the Laplacian \( L_i \) associated with \( D^* \) to the \( r_i \)-th Sobolev completion \( W^{r_i}(F^i) \) (of the space of smooth sections of the bundle \( F^i \)) is closed in \( W^0(F^i) \). Here, \( r_i \) denotes the order of the Laplacian \( L_i \) and \( W^{r_i}(F^i) \) is the \( r_i \)-th Sobolev type completion of the space of smooth sections \( \Gamma(F^i) \) of \( F^i \). We derive this application using \([5]\) which is based on results of Fomenko, Mishchenko in \([1]\). The authors of the latter article construct smoothing parametrices for extensions of \( A \)-elliptic operators to the Sobolev completions. In \([5]\), this construction is used to derive smooth parametrices for \( A \)-elliptic operators, and a generalization of these results to the case of \( A \)-elliptic complexes. In this sense, the present paper might be considered as a continuation of the work started in \([5]\).

Let us notice that for deriving results of this paper, we were motivated by the mathematics connected with Quantum theory, in particular, by results of Kostant in \([4]\) and the work of Habermann on the so-called symplectic Dirac operator. For it, see Habermann, Habermann \([3]\). There are also generalization of the classical Hodge theory (for elliptic operators) in directions different from the one we present here. See, e.g., Smale et al. \([10]\) and the reference there. For more \( K \)-theoretically and/or analytically oriented works, see Pavlov \([7]\), Schick \([9]\), Shubin \([8]\), Troitsky \([12]\), and Troitsky and Frank \([13]\).
In the second section, we set the terminology and the notation and derive some simple properties of projections, complementability, and pre-Hilbert module structures on quotients. Then we prove that a self-adjoint parametrix possessing endomorphism \( L \) of a pre-Hilbert \( A \)-module \( V \) admits a decomposition of the form \( V = \ker L \oplus \im L \) (Theorem 3). In the third section, we derive the Hodge decomposition for self-adjoint parametrix possessing complexes (Theorem 5) and give a characterization of their cohomology groups (Corollary 7). In the fourth section, the definitions of an \( A \)-Hilbert bundle and an \( A \)-elliptic complexes are recalled. At the end, Theorem 8 on the Hodge theory for the class of \( A \)-elliptic complexes mentioned above is proved.

**Preamble:** All manifolds and bundle structures (total spaces, base spaces, and bundle projections) are smooth. Base spaces of all bundles are finite dimensional. Further, if an index of an labeled object exceeds its allowed range, we consider it to be zero.

# 2 Parametrix possessing endomorphisms of pre-Hilbert \( A \)-modules

Let \( A \) be a unital \( C^* \)-algebra. We denote the involution, the norm in \( A \), the partial ordering on the double cone of hermitian elements in \( A \), and the unit by \( ^* \), \( | |_A \), \( \leq \), and \( 1 \), respectively.

Let us recall that a pre-Hilbert \( A \)-module is firstly, a complex vector space \( U \) on which \( A \) acts. For definiteness, we consider that \( A \) acts from the left, and denote the action by a dot. Secondly, \( U \) has to be equipped with a map \(( \cdot, \cdot )_U : U \times U \to A \) such that for all \( a \in A \) and \( u, v \in U \), the following properties hold

1) \((a.u, v)_U = a^*(u, v)_U \)
2) \((u, v)_U = (v, u)^*_U \)
3) \((u, u)_U \geq 0 \)
4) \((u, u)_U = 0 \) if and only if \( u = 0 \).

We call such a map \(( \cdot, \cdot )_U : U \times U \to A \) an \( A \)-product. For a pre-Hilbert \( A \)-module \((U, ( \cdot, \cdot )_U)\), one defines a norm \(| |_U : U \to \mathbb{R}_0^+ \) (induced by \(( \cdot, \cdot )_U \)) by the formula \( U \ni u \mapsto |u|_U = \sqrt{|(u, u)_U|_A} \in \mathbb{R}_0^+ \). A homomorphism \( L \) between pre-Hilbert \( A \)-modules \( U, V \) has to be \( A \)-linear, i.e., \( L(a.u) = a.L(u) \) for each \( a \in A \) and \( u \in U \), and continuous with respect to the norms \(| |_U \) and \(| |_V \).

An adjoint of a pre-Hilbert \( A \)-module homomorphism \( L : U \to V \) is a map from \( V \) to \( U \) denoted by \( L^* \) such that for each \( u \in U \) and \( v \in V \), the identity \(( Lu, v)_V = (u, L^* v)_U \) holds. If the adjoint exists, it is unique and a pre-Hilbert \( A \)-module homomorphism. (See, e.g., Lance [6].) We denote the set of pre-Hilbert \( A \)-module homomorphisms from \( U \) to \( V \) by \( \text{Hom}_A(U, V) \). If \( U = V \), \( \text{End}_A(U) \) denotes \( \text{Hom}_A(U, V) \). Quite often, in the literature a homomorphism
\( L : U \rightarrow V \) of pre-Hilbert or Hilbert \( A \)-modules is supposed to be adjointable. For technical reasons, we don’t follow this convention. Let us recall, that a pre-Hilbert \( A \)-module \((U, (\cdot, \cdot)_U)\) is called a Hilbert \( A \)-module if it is complete with respect to \( \| \cdot \|_U \).

Elements \( u, v \in U \) are called orthogonal if \((u, v)_U = 0\). For any pre-Hilbert \( A \)-submodule \( U \) of \( V \), we denote by \( U^\perp \) the orthogonal complement of \( U \) defined by \( U^\perp = \{ v \in V \mid (v, u)_V = 0 \text{ for all } u \in U \} \). We call \( U \) orthogonally complementable if there exists a pre-Hilbert \( A \)-submodule \( U' \subseteq V \) such that \( U \oplus U' = V \). Let us notice that if we write a direct sum of pre-Hilbert \( A \)-submodules, we suppose that the elements belonging to different summands are mutually orthogonal. It is immediate to see that for any pre-Hilbert \( A \)-submodules \( V \subseteq W \) of a pre-Hilbert \( A \)-module \( U \), the operation of taking the orthogonal complement changes the inclusion sign, i.e.,

\[
V^\perp \supseteq W^\perp
\]

(1)

### 2.1 Complementability, quotients and parametrix possessing maps

Let us start with the following simple observation. For any pre-Hilbert \( A \)-module \( V \), an element \( p \) of \( \text{End}_A(V) \) is called a projection if \( p^2 = p \). Let \( p \) be a projection and let us denote by \( U \) the \( A \)-submodule \( \text{Im} \ p \). For each \( z \in U \), there exists \( x \in V \) such that \( px = z \). Thus, \( p^2x = pz \) which in turn implies \( pz = px = z \), i.e., if \( p \) is a projection onto an \( A \)-submodule \( U \), then if restricted to \( U, p \) is the identity on \( U \). If \( V = U \oplus U' \) and \( p(x_U + x_{U'}) = x_U \), where \( x_U \in U \) and \( x_{U'} \in U' \), we call \( p \) a projection onto \( U \) along \( U' \).

**Lemma 1:** Let \( V \) be a pre-Hilbert \( A \)-module and \( U \) be an orthogonally complementable pre-Hilbert \( A \)-submodule of \( V \). If \( V = U \oplus U' \), then \( U' = U^\perp \) and the projection \( p \) onto \( U \) along \( U^\perp \) is self-adjoint. Conversely, if \( p \) is a self-adjoint projection in \( V \), then \( U = \text{Im} \ p \) is an orthogonally complementable pre-Hilbert \( A \)-submodule and \( 1 - p \) is a projection onto \( U^\perp \) along \( U \).

**Proof.** For \( x \in U^\perp \), there are uniquely determined \( x_U \in U \) and \( x_{U'} \in U' \) for which \( x = x_U + x_{U'} \). Let us compute \((x_U, x_U)_V = (x - x_U, x_U)_V = (x, x_{U'})_V - (x_{U'}, x_{U'})_V = (x, x_U)_V = 0 \) since \( x \perp U \). Thus, \( x_U = 0 \) proving \( U^\perp \subseteq U' \). The opposite inclusion follows from the definition of the orthogonal complement immediately. Further, for any \( x \in V \) and \( y = y_U + y_U \in V, y_U \in U, y_U \in U' \), let us write \((px, y)_V = (x_U, y_U + y_{U'})_V = (x_U, y_U)_V = (x, y_U)_V = (x, y_U)_V = (x, py)_V \), i.e., \( p \) is self-adjoint.

For the other statement, set \( U = p(V) \) and \( U' = (1 - p)(V) \). We have \( V = U + U' \). For \( x \in U \cap U' \), we get \((x, x)_V = (px, (1 - p)x)_V = (px, x)_V - (px, px)_V = (x, x)_V = (x, x)_V = 0 \), i.e., \( x = 0 \) and thus, the sum is direct. It is immediate to see that \( 1 - p \) is self-adjoint and a projection. To prove that \( 1 - p \) projects onto \( U^\perp \), let us consider an element \( y \in U^\perp \). We may compute \((py, py)_V = (y, p^2y)_V = (y, py)_V = (y, py)_V = 0 \). Thus, \( py = 0 \) and therefore \( y = (1 - p)y \). Obviously, \( 1 - p \) annihilates the elements from \( U \). Thus, \( 1 - p \) is a self-adjoint projection onto \( U^\perp \) along \( U \). \( \square \)
For a normed space \((Y, \| \cdot \|_Y)\) and its closed normed subspace \(X\), one usually considers the quotient space \(Y/X\) equipped with the norm \(||\cdot||_q : Y/X \to \mathbb{R}\) defined by

\[||y||_q = \inf \{ \|y - x\|_Y, x \in X\},\]

where \(y \in Y\) and \([y]\) denotes the equivalence class of \(y\). We call \(||\cdot||_q\) the quotient norm. It is immediate to see that if \(Y\) is a Banach space, the quotient is a Banach space as well. Now, we focus our attention to quotients of pre-Hilbert \(A\)-modules. When we speak of a quotient \(V/U\) of a pre-Hilbert module \(V\) and its orthogonally complementable submodule \(U\), we think of \(V/U\) as of an \(A\)-module equipped with the following \(A\)-product \(\langle \cdot, \cdot \rangle_{V/U}\). Let \(p\) be the projection onto \(U^\perp\) along \(U\). We set \(\langle [u], [v] \rangle_{V/U} = (p(u), p(v))_V\), \(u, v \in V\). The map \(\langle \cdot, \cdot \rangle_{V/U}\) is easily seen to be correctly defined. Further, it is evident that it maps into the set of non-negative elements of \(A\). Suppose that \(\langle [u], [u] \rangle_{V/U} = 0\) for an element \(u \in V\). Then \(p(u), p(u)\|_V = 0\) and consequently, \(p(u) = 0\). Thus \(u \in U\), i.e., \([u] = 0\), proving the positive definiteness of \(\langle \cdot, \cdot \rangle_{V/U}\) (properties 3 and 4 in the definition of the \(A\)-product). Summing up, in the case of an orthogonally complementable pre-Hilbert \(A\)-submodule \(U\) of a pre-Hilbert \(A\)-module \(V\), we obtain a canonical pre-Hilbert \(A\)-module structure \((V/U, \langle \cdot, \cdot \rangle_{V/U})\).

**Lemma 2:** Let \(U\) be an orthogonally complementable pre-Hilbert \(A\)-submodule of a pre-Hilbert \(A\)-module \((V, \langle \cdot, \cdot \rangle_V)\). Then

1) \(V/U\) and \(U^\perp\) are isomorphic pre-Hilbert \(A\)-modules and

2) the quotient norm \(||\cdot||_q\) coincides with the norm induced by \(\langle \cdot, \cdot \rangle_{V/U}\).

**Proof.** Let \(p\) be the projection onto \(U^\perp\) along \(U\) and \(p' = 1 - p\) be the projection onto \(U\) along \(U^\perp\) (Lemma 1). For any \(v \in V\), we have

\[||v||_q = \inf_{u \in U} |v - u|^2_V = \inf_{u \in U} (v - u, v - u)_V|_A = \inf_{u \in U} (p'v + pv - u, p'v + pv - u)_V|_A = \inf_{u \in U} (p'v - u, p'v + pv - u)_V + (pv, pv)_V|_A = \inf_{u \in U} (p'v - u, p'v - u)_V + (pv, pv)_V|_A = ||pv_v = ||v||_U^\perp_A,\]

where in the second last step, we used the fact that \(|a + b|_A \geq |a|_A\) which holds for any non-negative \(a, b \in A\). This proves the second item.

It is easy to check that \(\Phi([v]) = pv\) is a well defined \(A\)-module homomorphism of \(V/U\) into \(U^\perp\). Consider also the map \(\Psi : U^\perp \to V/U\) defined by \(\Psi(u) = [u], u \in U^\perp\). Both of the maps are continuous with respect to the norm topology on \(U^\perp\) (inherited from \((V, \| \cdot \|_V)\)) and the quotient topology on \(V/U\). Because the topology induced by \(\| \cdot \|_q\) coincides with the quotient topology, and \(||\cdot||_q\) coincides with \(||\cdot||_{V/U}\) (due to the first paragraph of this proof), we conclude that both \(\Phi\) and \(\Psi\) are continuous with respect to the norms \(||\cdot||_{U^\perp}\) and \(||\cdot||_{V/U}\), i.e., they are homomorphisms of the corresponding pre-Hilbert \(A\)-modules. Here, \(||\cdot||_{U^\perp}\) denotes the restriction of \(||\cdot||_V\) to \(U^\perp\). Further, for any \(u \in U^\perp\), we have
\( \Phi(\Psi(u)) = \Phi([u]) = pu = u \) since \( p \) projects onto \( U^\perp \). For each \( [v] \in V/U \), we may write \( \Psi(\Phi([v])) = \Psi(pv) = [pv] \). Because the difference of \( v \) and \( pv \) lies in \( U \), we get \( \Psi \circ \Phi = 1_{|V/U} \). □

**Remark 1:** As a consequence of Lemma 2, for a pre-Hilbert \( A \)-module \( V \) and an orthogonally complementable pre-Hilbert \( A \)-submodule \( U \) of \( V \) if \( (V/U, \| \cdot \|_q) \) is a Banach space, \( (V/U, (,)_V/U) \) is a Hilbert \( A \)-module. Further if moreover, \( V \) is Hilbert \( A \)-module, then \( (V/U, (,)_V/U) \) is a Hilbert \( A \)-module as well.

Now, we shall focus our attention to relations between orthogonal complementability of images of pre-Hilbert \( A \)-module endomorphisms and the property described in the next definition.

**Definition 1:** Let \( L \) be an endomorphism of a pre-Hilbert \( A \)-module \( V \). We call \( L \) parametrix possessing if there exists pre-Hilbert \( A \)-module endomorphisms \( p, g : V \to V \) such that

\[
1_{|V} = gL + p \quad 1_{|V} = Lg + p \quad Lp = 0
\]

where \( 1_{|V} \) denotes the identity on \( V \). We call a parametrix possessing map \( L \) self-adjoint parametrix possessing if \( L \) and \( p \) are self-adjoint.

**Remark 2:** The first two equations in Definition 1 are called parametrix equations. Notice that there exist pre-Hilbert \( A \)-module endomorphisms which are not parametrix possessing and also such for which, \( g \) and \( p \) are not uniquely determined. The name parametrix is borrowed from the theory of elliptic PDEs.

**Theorem 3:** Let \( L : V \to V \) be a self-adjoint parametrix possessing endomorphism of a pre-Hilbert \( A \)-module \( V \) with the corresponding maps denoted by \( g \) and \( p \). Then

1) \( p \) is a projection onto \( \text{Ker} L \) and
2) \( V = \text{Ker} L \oplus \text{Im} L \).

**Proof.**

1) Composing the first parametrix equation from the right by \( p \) and using the third equation from the definition of a parametrix possessing endomorphism, we get that \( p^2 = p \), i.e., \( p \) is an idempotent. Restricting \( 1_{|V} = gL+p \) to \( \text{Ker} L \), we get \( 1_{|\text{Ker} L} = p_{|\text{Ker} L} \) which implies that \( \text{Im} p \supseteq \text{Ker} L \). Further, \( Lp = 0 \) forces \( \text{Im} p \subseteq \text{Ker} L \). Thus, \( \text{Im} p = \text{Ker} L \) and consequently, \( p \) is a projection onto \( \text{Ker} L \).

2) Since \( p \) is self-adjoint, we may use Lemma 1 to conclude that \( V = \text{Ker} L \oplus (\text{Ker} L)^\perp \). It is sufficient to prove the equality

\[
\text{Im} L = (\text{Ker} L)^\perp
\]

First, we prove that \( \text{Im} L \subseteq (\text{Ker} L)^\perp \). Let \( y = Lx \) for an element \( x \in V \). For any \( z \in \text{Ker} L \), we may write \( (y, z) = (Lx, z) = (x, L^* z) = (x, Lz) = 0 \).
Thus, \( y \perp \ker L \). Now, we prove that \((\ker L)^\perp \subseteq \text{Im } L\). Let \( x \in (\ker L)^\perp \). Using the second parametrix equation, we obtain \( Lgx = (1-p)g = x \) since \( 1-p \) projects onto \((\ker L)^\perp\) (Lemma 1). Therefore \( x = Lgx \in \text{Im } L\).

Summing up, \( \text{Im } L = (\ker L)^\perp \), and the equation \( V = \ker L \oplus \text{Im } L \) follows.

\[ \square \]

**Remark 3:** Let us notice that in particular, any self-adjoint parametrix possessing endomorphism has closed image.

### 3 Cohomology and Hodge decomposition

In this section, we focus our attention to co-chain complexes \( d^\bullet = (C^k, d_k)_{k \in \mathbb{N}_0} \) of pre-Hilbert \( A \)-modules and adjointable pre-Hilbert \( A \)-module homomorphisms, i.e., for each \( k \in \mathbb{N}_0 \), \( d_k : C^k \to C^{k+1} \) is an adjointable pre-Hilbert \( A \)-module homomorphism which satisfies \( d_{k+1}d_k = 0 \). We will transfer Theorem 3 to the situation of co-chain complexes. Let us consider the sequence of Laplacians \( L_k = d_k^*d_k + d_{k-1}d_{k-1}^* \), \( k \in \mathbb{N}_0 \), associated with \( d^\bullet \).

**Definition 2:** Let \( d^\bullet = (C^k, d_k)_{k \in \mathbb{N}_0} \) be a co-chain complex of pre-Hilbert \( A \)-modules and adjointable pre-Hilbert \( A \)-module homomorphisms. We call \( d^\bullet \) a parametrix possessing complex if for each \( k \in \mathbb{N}_0 \), the associated Laplacian \( L_k \) is a parametrix possessing endomorphism of \( C^k \). We call \( d^\bullet \) a self-adjoint parametrix possessing complex if the operators \( L_k \) are self-adjoint parametrix possessing pre-Hilbert \( A \)-module endomorphisms for all \( k \in \mathbb{N}_0 \).

Obviously, a parametrix possessing complex is self-adjoint parametrix possessing if and only if the projection \( p_k \) is self-adjoint, \( k \in \mathbb{N}_0 \). Notice that (in concordance with the preamble), \( L_0 = d_0^*d_0 \).

**Lemma 4:** Let \( d^\bullet = (C^k, d_k)_{k \in \mathbb{N}_0} \) be a co-chain complex of pre-Hilbert \( A \)-modules and adjointable pre-Hilbert \( A \)-module homomorphisms. Then

\[ \ker L_k = \ker d_k \cap \ker d_{k-1}^* \]

**Proof.** The inclusion \( \ker L_k \supseteq \ker d_k \cap \ker d_{k-1}^* \) follows from the definition of the Laplacian \( L_k \). To prove the opposite one, let \( x \in \ker L_k \). We may write \( 0 = (x, L_kx)_{C^k} = (x, d_k^*d_kx + d_{k-1}d_{k-1}^*x)_{C^k} = (d_kx, d_kx)_{C^{k+1}} + (d_{k-1}x, d_{k-1}^*x)_{C^{k-1}} \). Thus, \( d_kx = d_{k-1}^*x = 0 \) due to the positive definiteness of the \( A \)-products on \( C^{k+1} \) and \( C^{k-1} \), respectively. \( \square \)

**Theorem 5:** Let \( d^\bullet = (C^k, d_k)_{k \in \mathbb{N}_0} \) be a self-adjoint parametrix possessing complex. Then for any \( k \in \mathbb{N}_0 \), we have the decomposition

\[ C^k = \ker L_k \oplus \text{Im } d_{k-1} \oplus \text{Im } d_k^* \]

**Proof.** Because \( d^\bullet \) is a parametrix possessing complex, there exist maps \( g_k \) and \( p_k \) satisfying the parametrix equations for \( L_k \) and the identity \( L_kp_k = 0 \), \( k \in \mathbb{N}_0 \).
1) Due to Lemma 4, we have \( \ker L_k \subseteq \ker d_{k-1}^* \). Therefore using the formulas (1) and (2), we get \( (\ker d_{k-1}^*)^\perp \subseteq (\ker L_k)^\perp = \text{Im} L_k \). Further, due to Lemma 4 again, we have \( \ker L_k \subseteq \ker d_k \). In the same way as above, we get \( (\ker d_k)^\perp \subseteq (\ker L_k)^\perp = \text{Im} L_k \). Summing up, \( (\ker d_{k-1}^*)^\perp + (\ker d_k)^\perp \subseteq \text{Im} L_k \).

2) The inclusion \( \text{Im} d_{k-1} \subseteq (\ker d_{k-1}^*)^\perp \) holds since for any \( x \in C^{k-1} \) and \( y \in \ker d_{k-1}^* \), we have \( \langle d_{k-1} x, y \rangle \in C^k = \langle x, d_{k-1}^* y \rangle_{C^{k-1}} = 0 \). Similarly, \( \text{Im} d_k \subseteq (\ker d_k)^\perp \). Combining this with the result of item 1 of this proof, we get \( \text{Im} d_{k-1} + \text{Im} d_k^* \subseteq (\ker d_{k-1}^*)^\perp + (\ker d_k)^\perp \subseteq \text{Im} L_k \). Now, we show that the sum \( \text{Im} d_k^* + \text{Im} d_{k-1} \) is direct. Let \( y = d_k^* x = d_{k-1} z \) for elements \( x \in C^{k+1} \) and \( z \in C^{k-1} \). We have \( \langle y, y \rangle \in C_k = \langle d_k^* x, d_{k-1} z \rangle_{C^{k+1}} = 0 \), and consequently, \( y = 0 \). Summing up, \( \text{Im} d_k^* \oplus \text{Im} d_{k-1} \subseteq \text{Im} L_k \).

3) It is easy to prove that \( \text{Im} L_k \subseteq \text{Im} d_k^* \oplus \text{Im} d_{k-1} \). Indeed, for any \( y \in \text{Im} L_k \), there exists \( \in C^k \) such that \( y = L_k x = d_k^* d_k x + d_{k-1} d_{k-1}^* x = d_k^* (d_k x) + d_{k-1} (d_{k-1}^* x) \in \text{Im} d_k^* + \text{Im} d_{k-1} \). This together with item 2 proves that \( \text{Im} L_k = \text{Im} d_k^* \oplus \text{Im} d_{k-1} \).

4) Because \( L_k \) is a self-adjoint parametrix possessing pre-Hilbert \( A \)-module endomorphism of \( C^k \), we have due to Theorem 3, the equality \( C^k = \text{Im} L_k \oplus \ker L_k \). Substituting for \( \text{Im} L_k \) from item 3 of this proof, we get the decomposition.

\[ \square \]

**Remark 4:** During the proof of the previous theorem, we obtained for \( (\text{a self-adjoint parametrix-possessing complex}) d^* \), the decomposition

\[ \text{Im} L_k = \text{Im} d_k^* \oplus \text{Im} d_{k-1} \]

Notice that if \( d^* = (C^k, d_k)_{k \in \mathbb{N}_0} \) is a co-chain complex, then its adjoint \((C^k, d_k^*)_{k \in \mathbb{N}_0}\) is a chain complex as one easily sees from \( d_k^* d_{k+1} = (d_{k+1}^* d_k)^* \).

**Theorem 6:** Let \( d^* = (C^k, d_k)_{k \in \mathbb{N}_0} \) be a self-adjoint parametrix possessing complex. Then for any \( k \in \mathbb{N}_0 \),

\[ \ker d_k = \ker L_k \oplus \text{Im} d_{k-1} \]

\[ \ker d_k^* = \ker L_{k+1} \oplus \text{Im} d_{k+1} \]

**Proof.** Because of Theorem 5, we know that the sums in both rows are direct.

The inclusion \( \ker L_k \oplus \text{Im} d_{k-1} \subseteq \ker d_k \) is an immediate consequence of the definition of a co-chain complex and of Lemma 4. To prove the opposite inclusion, let us consider an element \( y \in \ker d_k \). Due to Theorem 5, there exist elements \( y_1 \in \ker L_k, y_2 \in \text{Im} d_{k-1} \) and \( y_3 \in \text{Im} d_k^* \) such that \( y = y_1 + y_2 + y_3 \). It is sufficient to prove that \( y_3 = 0 \). Let \( z_3 \in C^{k+1} \) be such that \( y_3 = d_k^* z_3 \). We have \( 0 = (d_k y, z_3) = (d_k y_1 + d_k y_2 + d_k y_3, z_3) = (d_k y_3, z_3) = (y_3, d_k z_3) = (y_3, z_3) \) which implies \( y_3 = 0 \), and the first relation follows.
The inclusion \( \text{Ker} \, L_{k+1} \oplus \text{Im} \, d_{k+1}^* \subseteq \text{Ker} \, d_k^* \) follows from Lemma 4 and the second part of Remark 4. To prove the inclusion \( \text{Ker} \, d_k^* \subseteq \text{Ker} \, L_{k+1} \oplus \text{Im} \, d_{k+1}^* \), we proceed similarly as in the previous paragraph. For \( y \in \text{Ker} \, d_k^* \), we have \( y_1 \in \text{Ker} \, L_{k+1} \), \( y_2 \in \text{Im} \, d_k \), and \( y_3 \in \text{Im} \, d_{k+1}^* \) such that \( y = y_1 + y_2 + y_3 \) (Theorem 5). Let us consider an element \( z_2 \in C^k \) for which \( y_2 = d_k z_2 \). We have \( 0 = (d_k^* y_1 + d_k^* y_2 + d_k^* y_3, z_2) = (d_k^* y_2, z_2) = (y_2, y_2) \). Thus, \( y_2 = 0 \), and the second relation follows. \( \square \)

For any complex \( d^* = (C^k, d_k)_{k \in \mathbb{N}_0} \), we consider the cohomology groups

\[
H^i(d^*, A) = \frac{\text{Ker} \, (d_i : C^i \to C^{i+1})}{\text{Im} \, (d_{i-1} : C^{i-1} \to C^i)}, \quad i \in \mathbb{N}_0.
\]

Notice that in general, the \( A \)-module \( Z^i(d^*, A) = \text{Im} \, (d_{i-1} : C^{i-1} \to C^i) \) of co-boundaries needs not be orthogonally complementable or even not closed in the appropriate pre-Hilbert \( A \)-module. Thus, the cohomology group needn’t be a Hausdorff space with respect to the quotient topology. Nevertheless, in the case of a self-adjoint parametrix possessing complex, we derive

**Corollary 7**: Let \( d^* = (C^k, d_k)_{k \in \mathbb{N}_0} \) be a self-adjoint parametrix possessing complex. Then for each \( i \), the cohomology group \( H^i(d^*, A) \) is a pre-Hilbert \( A \)-module. If \( d^* \) is a self-adjoint parametrix possessing complex of Hilbert \( A \)-modules, then for each \( i \), the cohomology group \( H^i(d^*, A) \) is a Hilbert \( A \)-module.

**Proof.** Because of Theorem 6, \( U = \text{Im} \, d_{i-1} \) is orthogonally complementable in \( V = \text{Ker} \, d_i \). Thus using Lemma 2, the cohomology group \( H^i(d^*, A) = \text{Ker} \, d_i / \text{Im} \, d_{i-1} \) equipped with the canonical \( A \)-product \((,)_{V/U}\) is a pre-Hilbert \( A \)-module. The second statement follows from Remark 1. \( \square \)

**Remark 5**: Notice that moreover due to Theorem 6, we have

\[
H^i(d^*, A) \cong \text{Ker} \, L_i
\]

as pre-Hilbert \( A \)-modules.

### 4 Application to differential operators

Let \( M \) be a finite dimensional manifold and \( p : F \to M \) be a Banach bundle over \( M \). We call \( p : F \to M \) an \( A \)-Hilbert bundle if there exists a Hilbert \( A \)-module \((S,(,)_S)\) and a bundle atlas \( \mathcal{A} \) of \( p \) compatible with the bundle atlas of \( p \) considered as the Banach bundle only, such that

1) \((S,|_S)\) is the typical fiber of the Banach bundle \( p \)

2) for each \( m \in M \), the fiber \( F_m = p^{-1}(m) \) is equipped with a Hilbert \( A \)-product, denoted by \((,)_m\)

3) for each \( m \in M \) and each chart \((\phi_U, U) \in \mathcal{A}, U \ni m \), the map \( \phi_U|_{F_m} : (F_m,(,)_m) \to (S,(,)_S) \) is a Hilbert \( A \)-module isomorphism and

4) the transition maps of the charts in \( \mathcal{A} \) are maps into \( \text{Aut}_A(S) \).
Let us recall that for two bundle charts \( \phi_U : p^{-1}(U) \rightarrow U \times S \) and \( \phi_V : p^{-1}(V) \rightarrow V \times S \), their transition map \( \phi_{UV} : U \cap V \rightarrow \text{Aut}_A(S) \) is defined by the formula 
\[ (\phi_U \circ \phi_V^{-1})(m, v) = (m, \phi_{UV}(m)v) \]
for each \( m \in U \cap V \) and \( v \in S \). A homomorphism of \( A \)-Hilbert bundles \( p_1 : F_1 \rightarrow M \) and \( p_2 : F_2 \rightarrow M \) is a map \( F : F_1 \rightarrow F_2 \) between the total spaces of \( p_1 \) and \( p_2 \) such that \( p_2 \circ F = p_1 \), and in each fiber, \( F \) is a Hilbert \( A \)-module homomorphism, i.e., for each \( m \in M \), 
\[ F|_{U_1 \cap (m)} : (F_1)_m \rightarrow (F_2)_m \]
is such a map. An \( A \)-Hilbert bundle is called finitely generated projective if and only if its typical fiber, the Hilbert \( A \)-module \( (S, (,)_S) \), is a finitely generated projective Hilbert \( A \)-module.

Let us suppose that \( M = \text{compact} \), equipped with a Riemannian metric \( g \) and let us choose a volume element \( \text{vol}_g \) of \( (M, g) \). The (positive definite) Laplace-Beltrami operator will be denoted by \( \triangle \). For each \( A \)-Hilbert bundle \( p : F \rightarrow M \) over \( M \) and each \( t \in \mathbb{Z} \), Fomenko and Mishchenko in \([1]\) define a certain Hilbert \( A \)-module, the so-called Sobolev completion \( W^t(F) \) of the space of smooth sections \( \Gamma(F) \) of \( F \). Let us sketch their construction briefly. Obviously, the space \( \Gamma(F) \) of smooth sections of \( F \) carries a left \( A \)-module structure given by \( (a.s)(m) = a.(s(m)) \), \( a \in A \), \( s \in \Gamma(F) \) and \( m \in M \). One defines an \( A \)-product by the formula

\[ (s, s')_t = \int_{m \in M} (s, s')_m |\text{vol}_g|_m, \ s' \in \Gamma(F). \]

Setting

\[ (s, s')_t = \int_{m \in M} (s, (1 + \triangle)^t s')_m |\text{vol}_g|_m, \ s' \in \Gamma(F), \]

we obtain further pre-Hilbert \( A \)-modules \( (\Gamma(F), (,)_t) \). Obviously, \( (,)_0 = (,)_0 \).

For definiteness, we consider (the appropriate manifold version of) the Bochner integral of Banach space valued functions. The Sobolev completion \( W^t(F) \) is defined as the completion of \( \Gamma(F) \) with respect to the norm \( ||.||_t \) induced by the \( A \)-product \( (,)_t \). We keep denoting the Hilbert \( A \)-products by \( (,)_t \) also if we consider their extensions to \( W^t(F) \). See Fomenko, Mishchenko \([1]\) or Solovyov, Troitsky \([1]\) for details on this construction if necessary.

Our reference for the statements in the upcoming paragraph is Solovyov, Troitsky \([1]\). For a definition of an \( A \)-differential operators we refer to Solovyov, Troitsky \([1]\), pp. 79 and 80. We omit the prefix \( A \) and call these operators differential operators only. For any differential operator \( D : \Gamma(F_1) \rightarrow \Gamma(F_2) \), we have the order \( \text{ord}(D) \in \mathbb{Z} \) of \( D \), the adjoint \( D^* : \Gamma(F_2) \rightarrow \Gamma(F_1) \) (Theorem 2.1.37 in \([1]\)) and for each \( t \in \mathbb{Z} \), the (continuous) extension \( D_t : W^t(F_1) \rightarrow W^{t-\text{ord}(D)}(F_2) \) of \( D \) at our disposal \([1]\), pp. 89, Theorem 2.1.60). Let us denote by \( \pi : T^*M \rightarrow M \) the cotangent bundle and let \( \pi' \) be the restriction of \( \pi \) to \( T^*M' = T^*M \setminus \{(m, 0) \in T^*M | m \in M \} \). For a differential operator \( D \), one defines the notion of its symbol \( \sigma(D) : \pi^*(F_1) \rightarrow F_2 \). See \([1]\) pp. 79 and 80. If \( T^*M \) is considered with the trivial \( A \)-Hilbert bundle structure, i.e., \( a \alpha_m = \alpha_m \) for each \( a \in A \) and \( \alpha_m \in T^*_m M, m \in M \), the symbol \( \sigma(D) : \pi^*(F_1) \rightarrow F_2 \) is an adjointable \( A \)-Hilbert bundle homomorphism. The restriction of the symbol \( \sigma = \sigma(D) \) of \( D \) to \( \pi'^*(F_1) \) will be denoted by \( \sigma' \).
Let \((p_k : \mathcal{F}_k \rightarrow M)_{k \in \mathbb{N}_0}\) be a sequence of finitely generated projective \(A\)-Hilbert bundles over \(M\) and \(D^\bullet = (\Gamma(\mathcal{F}_k), D_k)_{k \in \mathbb{N}_0}\) be a complex of differential operators in \(\mathcal{F}^\bullet\), i.e., \(D_k : \Gamma(\mathcal{F}_k) \rightarrow \Gamma(\mathcal{F}_{k+1})\) is a differential operator and \(D_{k+1}D_k = 0, k \in \mathbb{N}_0\). Let us set \(\sigma_k = \sigma(D_k)\) for the symbol of \(D_k\). The symbol sequence \(\sigma^\bullet = (\pi^*F^k, \sigma_k)_{k \in \mathbb{N}_0}\) is a complex in the category of \(A\)-Hilbert bundles.

**Definition 3:** A complex \(D^\bullet = (\Gamma(\mathcal{F}_k), D_k)_{k \in \mathbb{N}_0}\) of differential operators in \(A\)-Hilbert bundles is called \(A\)-elliptic if its restricted symbol sequence \(\sigma^{\bullet'} = (\sigma_k', \pi'^*(F^k))_{k \in \mathbb{N}_0}\) is an exact sequence in the category of \(A\)-Hilbert bundles.

According to classical conventions, we denote the Laplacians \(L_k\) with a complex \(D^\bullet\) of differential operators by \(\triangle_k\). Let \(r_k\) denote the order of \(\triangle_k\).

**Remark 6:**

1) If \(D : \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{F})\) is a single differential operator, then we consider it as the complex
\[
0 \rightarrow \Gamma(\mathcal{E}) \xrightarrow{D} \Gamma(\mathcal{F}) \rightarrow 0.
\]
In this case, the definition of an \(A\)-elliptic complex of differential operators coincides with the (classical) definition of an \(A\)-elliptic operator given in Solovyov, Troitsky [11].

2) If \(D^\bullet\) is an \(A\)-elliptic complex of differential operators, then for each \(i \in \mathbb{N}_0\), the Laplacian \(\triangle_i\) is an \(A\)-elliptic operator. For it, see Corollary 10 in Krýsl [5].

Let us state the following

**Theorem 8:** Let \(A\) be a unital \(C^*\)-algebra and \(D^\bullet = (\Gamma(\mathcal{F}_k), D_k)_{k \in \mathbb{N}_0}\) be an \(A\)-elliptic complex in finitely generated projective \(A\)-Hilbert bundles over a compact manifold \(M\). Let for each \(k \in \mathbb{N}_0\), the image of the extension \((\triangle_k)_{r_k}\) of \(\triangle_k\) be closed in \(W^0(\mathcal{F}_k)\). Then for any \(i \in \mathbb{N}_0\),

1) \(H^i(D^\bullet, A)\) is a finitely generated projective \(A\)-module

2) \(\Gamma(\mathcal{F}^i) = \text{Ker} \triangle_i \oplus \text{Im} D_i \oplus \text{Im} D_{i-1}^*\)

3) \(\text{Ker} D_i = \text{Ker} \triangle_i \oplus \text{Im} D_i^*\)

4) \(\text{Ker} D_i^* = \text{Ker} \triangle_{i+1} \oplus \text{Im} D_i\)

**Proof.** In the proof of Theorem 8 in [5], a projection \(P\) is constructed which satisfies the parametrix equations for a self-adjoint \(A\)-elliptic operator \(K : \Gamma(\mathcal{F}) \rightarrow \Gamma(\mathcal{F})\) of order \(r\) provided \(\text{Im} K_r\) is closed in \(W^0(\mathcal{F})\). (Recall that \(K_r\) denotes the extension of \(K\) to \(W^r(\mathcal{F})\).) Let us sketch this construction briefly. For \(K_r : W^r(\mathcal{F}) \rightarrow W^0(\mathcal{F})\), consider its adjoint \((K_r)^* : W^0(\mathcal{F}) \rightarrow W^r(\mathcal{F})\) and the projection \(p_{\text{Ker} K^*_r}\) from the space \(W^0(\mathcal{F})\) onto the kernel \(\text{Ker} (K_r)^*\). Let us denote the restriction of \(p_{\text{Ker} K^*_r}\) to \(\Gamma(\mathcal{F})\) by \(P\). Besides \(P\), a further map \(G : \Gamma(\mathcal{F}) \rightarrow \Gamma(\mathcal{F})\) is constructed in the mentioned proof, which satisfies \(1_{|\Gamma(\mathcal{F})} = \)

\(\text{Ker} \triangle_i \oplus \text{Im} D_i \oplus \text{Im} D_{i-1}^*\)

\(\text{Ker} D_i = \text{Ker} \triangle_i \oplus \text{Im} D_i^*\)

\(\text{Ker} D_i^* = \text{Ker} \triangle_{i+1} \oplus \text{Im} D_i\)
$GK + P = KG + P$ and $KP = 0$. (See Theorem 8 in [5].) In particular, $K$ is a parametrix possessing endomorphism of the pre-Hilbert $A$-module $(\Gamma(F), (,)_F)$.

Now, let us prove that the operator $K$ is also self-adjoint parametrix possessing. Due to the mentioned closed image assumption on the extension $K_r$ of $K$, $\text{Ker}(K_r)^*$ is orthogonally complementable in $W^0(F)$ according to Lance [6], Theorem 3.2, pp. 22. Thus, the projection $p_{\text{Ker}K^*}$ is self-adjoint according to Lemma 1. Restricting $p_{\text{Ker}K^*}$ to $\Gamma(F)$ does not change the property of being idempotent. Moreover, the restriction keep being self-adjoint, because the $A$-product $(,)_r$ in $\Gamma(F)$ coincides with the $A$-product $(,)_0$ in $W^0(F)$ when restricted to $\Gamma(F) \times \Gamma(F)$. Thus, $K$ is not only parametrix possessing, but it is also self-adjoint parametrix possessing.

Now, let us pass to the statement we are proving. Let $i \in \mathbb{N}_0$. Using item 2 of Remark 6, $\Delta_i$ is $A$-elliptic. Thus, we may use the conclusion of the previous paragraph for $K = \Delta_i$ obtaining that $\Delta_i$ is self-adjoint parametrix possessing, and consequently, that $D^\bullet$ is a self-adjoint parametrix possessing complex. We may utilize the theorems derived in this paper. Namely, using Theorems 5 and 6 and Corollary 7, one derives the items 2, 3 and 4. According to Theorem 11 in [5], $H^i(D^\bullet, A)$ is a finitely generated $A$-module and a Banach space with respect to the quotient norm $| |$. Thus due to Remark 1, $H^i(D^\bullet, A)$ equipped with the canonical quotient structure, is a finitely generated Hilbert $A$-module. Using Theorem 5.9 in Frank, Larson [2], the cohomology $H^i(D^\bullet, A)$ is projective due to the unitality of $A$. □

Remark 7: Let us notice that the decomposition in item 2 of the previous theorem is meant with respect to the $A$-product $(,)_F$. Also the adjoints are considered with respect to this $A$-product.

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