Checkerboard Julia Sets for Rational Maps

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In this paper, we consider the family of rational maps

\[ F_{\lambda}(z) = z^n + \frac{\lambda}{z^d}, \]

where \( n \geq 2, \ d \geq 1, \) and \( \lambda \in \mathbb{C}. \) We consider the case where \( \lambda \) lies in the main cardioid of one of the \( n-1 \) principal Mandelbrot sets in these families. We show that the Julia sets of these maps are always homeomorphic. However, two such maps \( F_{\lambda} \) and \( F_{\mu} \) are conjugate on these Julia sets only if the parameters at the centers of the given cardioids satisfy \( \mu = \nu^{j(d+1)} \lambda \) or \( \mu = \nu^{j(d+1)} \lambda \) where \( j \in \mathbb{Z} \) and \( \nu \) is an \((n-1)\)th root of unity. We define a dynamical invariant, which we call the minimal rotation number. It determines which of these maps are conjugate on their Julia sets, and we obtain an exact count of the number of distinct conjugacy classes of maps drawn from these main cardioids.

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Introduction

In recent years there have been many papers dealing with the family of rational maps of the Riemann sphere, \( \overline{\mathbb{C}} \), given by

\[ F_{\lambda}(z) = z^n + \frac{\lambda}{z^d}, \]

where \( n \geq 2, \ d \geq 1, \) and \( \lambda \in \mathbb{C} \) [Devaney, 2007]. For many parameter values, the Julia sets for these maps are Sierpiński curves, i.e., planar sets that are homeomorphic to the well-known Sierpiński carpet fractal.
One distinguishing property of Sierpiński curve Julia sets is that the Fatou set consists of infinitely many open disks, each bounded by a simple closed curve, but no two of these bounding curves intersect.

There are many different ways in which these Sierpiński curves arise as Julia sets in these families. For example, the Julia set is a Sierpiński curve if

1. the critical orbits enter the immediate basin of attraction of $\infty$ after two or more iterations [Devaney et al., 2005];
2. the parameter lies in the main cardioid of a “buried” baby Mandelbrot set [Devaney & Look, 2005]; or
3. the parameter lies on a buried point in a Cantor necklace in the parameter plane [Devaney, 2004].

The parameter planes for these maps in the cases where $n = d = 3$ and $n = d = 4$ are shown in Figure 1. The red disks not centered at the origin are regions where the first case above occurs. These disks are called Sierpiński holes.

Many small copies of the Mandelbrot set are visible in Figure 1. The ones that touch the external red region are not “buried,” so their main cardioids do not contain Sierpiński curve Julia sets. Only the ones that do not meet this boundary contain parameters from case 2.

Finally, numerous Cantor necklaces, i.e., sets homeomorphic to the Cantor middle-thirds set with the removed open intervals replaced by open disks, appear in these figures. The buried points in the Cantor set portion of the necklace are the parameters for which case 3 occurs.

![Fig. 1. Two parameter planes: $n = d = 3$ (left) and $n = d = 4$ (right). The colors are given by the standard escape time coloring algorithm applied to the free critical points (see the comments that follow Symmetry Lemma 2).](image-url)

The dynamical behavior on Sierpiński curve Julia sets drawn from non-symmetrically located Sierpiński holes is never the same [Devaney & Pilgrim, 2009]. That is, only symmetrically located Sierpiński holes contain parameters for which the corresponding maps have conjugate dynamics. While it is known that two such maps are not conjugate on their Julia sets, there is no known dynamical invariant that explains this lack of conjugacy.

In this paper, we describe the topology of and dynamics on a very different type of Julia set, the “checkerboard” Julia sets, that arise in these families. For most values of $n$ and $d$ considered in this paper,
the \( \lambda \)-parameter plane contains \( n-1 \) “principal Mandelbrot sets” [Devaney & Look, 2006], and we consider the Julia sets for parameters that lie within the main cardioids of these sets. For such parameters \( \lambda \), the maps \( F_\lambda \) have two distinct types of Fatou components (see Figure 2). Since \( \infty \) is a superattracting fixed point, the immediate basin of \( \infty \) and its preimages lie in the Fatou set. These components are the escaping Fatou components. The Fatou set also contains a collection of components corresponding to other finite attracting periodic orbits and their preimages. These components are the non-escaping Fatou components. As we shall show, none of the boundaries of the escaping Fatou components intersect. Likewise, the boundaries of the non-escaping Fatou components do not intersect. However, each such boundary intersects infinitely many boundaries of the escaping Fatou components, and each boundary of an escaping Fatou component intersects infinitely many non-escaping boundaries. Hence, the topology of these Julia sets is very different from the topology of Sierpiński curve Julia sets. We use the word “checkerboard” to describe this pattern of Fatou components.

In Figure 2, we display the Julia set for the map \( F_{0.18}(z) = z^4 + 0.18/z^3 \). The red regions are the preimages of the attracting basin of \( \infty \), and the black regions are the preimages of the basins of the finite attracting cycles. The boundary of each red region touches infinitely many boundaries of the black regions, but it does not touch the boundary of any other red region. Similarly, the boundary of each black region touches infinitely many boundaries of the red regions, but it does not touch the boundary of any other black region.

![Fig. 2. The image on the left is the checkerboard Julia set for \( F_{0.18}(z) = z^4 + 0.18/z^3 \). The image on the right is a magnification of one-seventh of the Julia set. In both images, the points that are colored black remain bounded under iteration while points that are not colored black have orbits that escape to infinity. The orbits that escape fastest are colored red, followed by those colored orange, and so on.](image_url)

The external red region in the left-hand image of Figure 2 is the immediate basin of attraction of \( \infty \). We denote this basin by \( B_\lambda \). The central red region that contains the pole at the origin is mapped to \( B_\lambda \). We let \( T_\lambda \) denote this Fatou component. All other red regions are also eventually mapped to \( B_\lambda \).

Note that there are \( n + d \) (= 7 in this example) large black regions that touch \( B_\lambda \) and \( T_\lambda \) at unique points. These Fatou components are (eventually) periodic. We call them the connecting (Fatou) components since they are the only Fatou components that extend from \( B_\lambda \) to \( T_\lambda \). Each of these connecting components
seems to be separated by another red region that touches exactly two boundaries of the adjacent connecting components. On one side of these red regions, we see \( d - 1 \) (\( = 2 \) in this example) smaller black components. On the other side, we see \( n - 1 \) (\( = 3 \) in this example) smaller black components. Each such black component connects to a pair of red regions. If we were to magnify this image, we would see that this pattern repeats itself at any scale.

In Section 2, we make this construction precise. In particular, we give an algorithm for describing the topological structure of these Julia sets. This algorithm also describes the dynamics on these Julia sets via symbolic dynamics, and it provides a proof of:

**Theorem 1.** Let \( F_\lambda(z) = z^n + \lambda/z^d \) with \( n \geq 2 \) and \( d \geq 1 \). Any two Julia sets that correspond to parameters in the main cardioids of the principal Mandelbrot sets in the parameter plane for these maps are homeomorphic.

Theorem 1 says that checkerboard Julia sets are analogous to Sierpiński curve Julia sets [Whyburn, 1958] because all checkerboard Julia sets with the same \( n \) and \( d \) are homeomorphic.

As in the Sierpiński case [Devaney & Pilgrim, 2009], only certain symmetrically located cardioids give rise to conjugacies on their respective Julia sets. However, unlike the Sierpiński case, we can define a dynamical invariant for checkboard Julia sets. We call it the minimal rotation number, and we prove that it is a conjugacy invariant for checkerboard Julia sets.

**Theorem 2.** Two maps drawn from different main cardioids of principal Mandelbrot sets are topologically conjugate on their Julia sets if and only if their minimal rotation numbers are equal. In particular, two such maps restricted to their Julia sets are topologically conjugate only if the parameters are symmetric either under the rotation \( z \mapsto \nu^{j(d+1)}z \) or under the map \( z \mapsto \nu^{j(d+1)}\bar{z} \), where \( j \in \mathbb{Z} \) and \( \nu^{n-1} = 1 \).

Theorem 2 leads to an exact count of the number of main cardioids that have non-conjugate dynamics.

**Theorem 3.** Let \( g \) be the greatest common divisor of \( n - 1 \) and \( n + d \). If \( g \) is even, then there are exactly \( 1 + g/2 \) distinct conjugacy classes among the maps drawn from the main cardioids of the principal Mandelbrot sets. If \( g \) is odd, then the number of conjugacy classes is \( (g + 1)/2 \).

1. **Preliminaries**

Consider the family of maps on the Riemann sphere, \( \overline{\mathbb{C}} \),

\[
F_\lambda(z) = z^n + \frac{\lambda}{z^d}
\]

where \( n \geq 2, d \geq 1, \) and \( \lambda \in \mathbb{C} \). The point at infinity is superattracting of order \( n \). As above, we denote the immediate basin of \( \infty \) by \( B_\lambda \). Also, \( 0 \) is a pole of order \( d \), so there is a neighborhood of \( 0 \) that is mapped into \( B_\lambda \). If this neighborhood is disjoint from \( B_\lambda \), we use the term “trap door” for the preimage of \( B_\lambda \) that contains 0. We denote the trap door by \( T_\lambda \).

The map \( F_\lambda(z) \) has \( n + d \) “free” critical points. They satisfy the equation

\[
z^{n+d} = \frac{d\lambda}{n}
\]
Hence, they are equally spaced on the circle of radius
\[ n + d \sqrt{\frac{d|\lambda|}{n}} \]
centered at the origin. There are also \( n + d \) prepoles. They satisfy the equation \( z^{n+d} = -\lambda \).

For a given \( n \) and \( d \), let \( \lambda_0 \) be the positive parameter
\[ \lambda_0 = \left( \frac{d}{n} + 1 \right) \frac{n + d}{n} \left( \frac{d}{n} \right)^{\frac{d + 1}{n - 1}}. \]
The map \( F_{\lambda_0}(z) = z^n + \lambda_0 / z^d \) has a real, superattracting fixed point.

The family \( F_{\lambda} \) has symmetries in both the dynamical plane and the parameter plane. The proofs of the following three symmetry lemmas are straightforward and are left to the reader.

**Symmetry Lemma 1.** The map \( F_{\lambda} \) is conjugate to \( F_{\lambda}^{\infty} \) by the conjugacy \( z \mapsto \bar{z} \).

This first symmetry implies that important subsets of the the parameter plane are symmetric under complex conjugation.

**Symmetry Lemma 2.** If \( \omega \) is a \((n + d)\)th root of unity, then
\[ F_{\lambda}(\omega z) = \omega^n F_{\lambda}(z). \]

This second symmetry implies that the Julia set of \( F_{\lambda} \) is symmetric under the map \( z \mapsto \omega z \). Similarly, \( B_{\lambda} \) and \( T_{\lambda} \) possess this \((n + d)\)-fold symmetry.

Moreover, since the free critical points are arranged symmetrically with respect to \( z \mapsto \omega z \), all of the free critical orbits behave symmetrically with respect to this rotation. However, it is not necessarily true that all of these critical orbits behave in the same manner. For example, consider the map \( F_{0.18}(z) = z^4 + 0.18 / z^3 \) (see Figure 2). The orbit of the free critical point on the positive real axis is asymptotic to a fixed point, but the other six free critical orbits are asymptotic to one of two attracting period-three orbits whose basins are the other six connecting Fatou components. This symmetry implies that all of these basins are arranged symmetrically.

The most important consequence of Symmetry Lemma 2 is the fact that the orbits of all of the free critical points can be determined from the orbit of any one of them (see Figure 3). So the one-dimensional \( \lambda \)-plane is a natural parameter plane for these maps.

**Symmetry Lemma 3.** Suppose that \( \eta \) is an \((n + d)(n - 1)\)st root of unity. Let \( \nu = \eta^{n+d} \) and \( \omega = \eta^{n-1} \). Then
\[ F_{\nu \lambda}^k(\eta z) = \eta^{nk} F_{\lambda}^k(z) \]
for \( k = 1, 2, 3, \ldots \).

Note that \( \nu \) is an \((n - 1)\)st root of unity and \( \omega \) is an \((n + d)\)th root of unity. Symmetry Lemma 3 is proved by induction on \( k \).

This symmetry allows us to determine the orbit diagram of \( F_{\nu \lambda} \) from the orbit diagram of \( \lambda \). In particular, if \( c_\lambda \) is a critical point for \( F_{\lambda} \), then \( \eta c_\lambda \) is a critical point for \( F_{\nu \lambda} \). We denote this critical point by \( c_{\nu \lambda} \). From Symmetry Lemma 3, we have
\[ F_{\nu \lambda}^k(c_{\nu \lambda}) = F_{\nu \lambda}^k(\eta c_\lambda) = \eta^{nk} F_{\lambda}^k(c_\lambda). \]
Fig. 3. The orbit diagram for the critical points of $F_{\lambda_0}(z) = z^{13} + \lambda_0/z^7$. The critical point on the positive real axis is a fixed point, and it is labeled with the number 0. The orbits of the remaining critical points are determined from the orbit of the fixed point using Symmetry Lemma 2.

Therefore, the orbits of the critical points of $F_\lambda$ and $F_{\nu\lambda}$ behave symmetrically with respect to rotation by some power of $\eta$ (see Figure 4).

Fig. 4. The orbit diagram for the critical points of $F_{\lambda_1}(z) = z^{13} + \lambda_1/z^7$ where $\lambda_1 = \nu\lambda_0$.

Symmetry Lemma 3 also implies that the full basin of $\infty$ for $F_\lambda$ is homeomorphic to the full basin of $F_{\nu\lambda}$ under the rotation $z \mapsto \eta z$. Since this basin is completely invariant, the Julia set of $F_{\nu\lambda}$ is the rotation of the Julia set of $F_\lambda$ under the rotation $z \mapsto \eta z$. This observation implies Theorem 1.

As we shall see, the dynamics of $F_\lambda$ and $F_{\nu\lambda}$ are not necessarily conjugate. For example, if $\lambda$ lies in the main cardioid of the right-hand principal Mandelbrot set in the $n = d = 3$ case, the map $F_\lambda$ has a pair of attracting fixed points (see Figure 1). In contrast, if $\lambda$ is an element of the the main cardioid of the left-hand principal Mandelbrot set, the map $F_\lambda$ has an attracting cycle of period two.
The map $F_{\nu\lambda}$ is conjugate to $\omega F_\lambda$ by the rotation $z \mapsto \eta z$ because

$$F_{\nu\lambda}(\eta z) = (\eta z)^n + \frac{\nu \lambda}{(\eta z)^d} = \eta^n z^n + \frac{\nu \lambda}{\eta^d z^d} = \eta^n \left( z^n + \frac{\lambda}{z^d} \right) = \eta \omega \left( z^n + \frac{\lambda}{z^d} \right).$$

More generally, the map $F_{\nu j\lambda}$ is conjugate to the map $\omega^j F_\lambda$ via the rotation $z \mapsto \eta^j z$.

In the parameter planes, there are numerous subsets that are homeomorphic to the Mandelbrot set [Devaney, 2006] (see the black regions in Figure 1). We abuse terminology and refer to these subsets as Mandelbrot sets as well.

Note that, in the left parameter plane in Figure 1, there are two large Mandelbrot sets along the real axis. Similarly, in the right parameter plane in the same figure, there are three large Mandelbrot sets symmetrically located with respect to the rotation $z \mapsto \nu z$ where $\nu = e^{2\pi i/3}$. With the exception of the $n = d = 2$ and the $d = 1$ cases, the parameter plane for the family $F_\lambda$ contains $n - 1$ symmetrically located Mandelbrot sets [Devaney, 2006]. We call these sets the principal Mandelbrot sets for the family $F_\lambda$. In [Devaney, 2006], the existence of these sets was proved for the case $n = d > 2$. However, the same proof works if $d \neq 1$ and $n \neq d$. In this paper, we describe the structure of and dynamics on the Julia sets for parameters that lie in the main cardioids of these principal Mandelbrot sets. Consequently, we denote these main cardioids using the following conventions:

1. Let the main cardioid whose center is $\lambda_0$ be denoted $\mathcal{M}_0$.
2. The remaining $n - 2$ main cardioids are denoted $\mathcal{M}_j$ where the ordering is in the counterclockwise direction.

If $n = d = 2$, there does not exist a principal Mandelbrot set in the parameter plane. In this case, the “tail” of the Mandelbrot set, i.e., the parameter corresponding to $c = -2$ in the Mandelbrot set for $z^2 + c$, extends to the origin, where the map is just $F_0(z) = z^2$. So, we do not have a complete Mandelbrot set. Nonetheless, there is a main cardioid $\mathcal{M}_0$ in which each parameter has four connecting Fatou components (see the left-hand parameter plane in Figure 5).

If $d = 1$, there are no principal Mandelbrot sets in the parameter plane. However, there are $n - 1$ distinct cardioid-shaped regions arranged symmetrically around the origin (see the right-hand parameter plane in Figure 5). We denote these regions by $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_{n-2}$. Parameters drawn from these regions have $n + 1$ connecting Fatou components.

For the maps that we study, the boundary of each of the Fatou components is a simple closed curve.

**Proposition 1.** Suppose $\lambda$ lies in some $\mathcal{M}_j$. Then each of the Fatou components of $F_\lambda$ is bounded by a simple closed curve. Consequently, the Julia sets of these maps are compact, connected sets.

**Proof:** Since the set of Fatou components consists of $B_\lambda$ and all of its preimages together with the connecting Fatou components and all of their preimages, it suffices to show that the boundary of $B_\lambda$ and the boundaries of the connecting Fatou components are simple closed curves. By the symmetry lemmas, in fact, we need only prove this for one of the connecting Fatou components.

First consider $\partial B_\lambda$. Since all of the critical orbits tend to attracting cycles, $F_\lambda$ is hyperbolic on its Julia set. Consequently, $J(F_\lambda)$ is locally connected. In particular, $\partial B_\lambda$ is a locally connected set (see Lemma 19.3 in [Milnor, 2006]). Thus we need only show that the set $\overline{\mathbb{C}} - \overline{B_\lambda}$ is connected.
We argue by contradiction. Suppose that $C - B_\lambda$ is disconnected. Let $W_0$ denote its component that contains the trap door. The second symmetry lemma implies that $W_0$ is symmetric under $z \mapsto \omega z$. Also, $T_\lambda$ is contained in $W_0$ since, if not, there would be a critical point in $\partial T_\lambda \cap \partial B_\lambda$, which contradicts the assumption that $F_\lambda$ is hyperbolic on $J(F_\lambda)$.

At least one other component of $C - B_\lambda$, say $W_1$, is mapped over $W_0$. If not, $\partial W_0$ would be backward invariant, which cannot happen. Another application of the second symmetry implies that $W_j = \omega^j W_1$ is also mapped onto $W_0$ for $j = 1, \ldots, n + d - 1$. We have $n + d$ distinct preimages of $W_0$.

However, we claim that there are points in $W_0$ that are also mapped into $W_0$. To see why, recall that $\partial T_\lambda$ is mapped over the entire boundary of $B_\lambda$ by $F_\lambda$ and that $\partial T_\lambda$ lies in $W_0$. Thus there is a point $z_0$ in $\partial T_\lambda$ that is mapped into $\partial W_0$. Then there is a neighborhood of $z_0$ in $W_0$ mapped to a neighborhood of $F_\lambda(z_0)$, and hence there are points in this neighborhood that are mapped inside $W_0$. We arrive at a contradiction since we have found points in $W_0$ that have more than $n + d$ preimages.

For the case of the connecting Fatou components, consider a parameter drawn from the main cardioid $M_0$. As mentioned above, we always have an attracting fixed point for such a parameter. Since the pole does not lie in this basin, it follows that the immediate basin of this fixed point is simply connected. By hyperbolicity, the boundary of this basin is again locally connected. Hence, all internal rays extending from the fixed point to the boundary of this basin land at a single point.

The question is whether two (or more) rays land at the same point on the boundary. If this were the case, then portions of the Julia set would protrude in towards the attracting fixed point in the region between these rays. In fact, we would necessarily have infinitely many such protruding regions. Let $U$ be the closure of the basin of this fixed point together with all of the protruding regions (and the Fatou components that they surround). So $U$ is a closed disk. By symmetry, none of the protruding regions or the basin can surround the origin, so there are no poles in $U$. Therefore none of the protruding regions can be mapped outside of $U$. But these regions contain points in the Julia set, and neighborhoods of these points must eventually be mapped over the entire Julia set and, in particular, outside $U$. This gives a contradiction and shows that there cannot be any such protruding regions. Hence each internal ray lands
at a unique point and so the boundary of this Fatou component is also a simple closed curve. By symmetry, the same holds for the other connecting Fatou components.

Thus the Julia set is the complement in the Riemann sphere of infinitely many open, disjoint disks, and so the Julia set is compact and connected.

\[\square\]

2. Checkerboard Julia Sets

In this section we present an algorithm for constructing the Julia sets for parameters in the main cardioids of the principal Mandelbrot sets. This algorithm gives an alternate proof of Theorem 1.

First, consider \(M_0\) and let \([a, b]\) denote the interval of intersection of \(M_0\) with the real axis. By considering the graph of \(F_\lambda | \mathbb{R}\) for \(\lambda \in (a, b)\), we see that each such map has an attracting fixed point that is real and positive. Hence, for each \(\lambda \in M_0\), the map \(F_\lambda\) also has an attracting fixed point. We denote this fixed point by \(p_0^\lambda\) and its immediate basin of attraction by \(C_\lambda^0\). As shown earlier, \(\partial C_\lambda^0\) is a simple closed curve. Furthermore, if \(\lambda \in (a, b)\), the graph of \(F_\lambda | \mathbb{R}\) shows that \(C_\lambda^0\) extends from \(\partial B_\lambda\) to \(\partial T_\lambda\). Consequently, \(C_\lambda^0\) extends from \(\partial B_\lambda\) to \(\partial T_\lambda\) for all values of \(\lambda \in M_0\). If \(\lambda \in (a, b)\), the intersection \(\partial B_\lambda \cap \partial C_\lambda^d\) contains a repelling fixed point \(q_\lambda^0\) that is real and positive. Using the fact that \(\partial C_\lambda^d\) is invariant and the fact that the map is conjugate to \(z \mapsto z^n\) on \(\partial B_\lambda\), it follows that \(q_\lambda^0\) is the only point in the intersection. Similarly, the intersection \(\partial T_\lambda \cap \partial C_\lambda^d\) is also just one point that is real and positive. We denote it by \(u_\lambda^0\). Note that \(F_\lambda(u_\lambda^0) = q_\lambda^0\).

From the second symmetry, we obtain \(n + d - 1\) other Fatou components that are symmetrically located around the origin. We denote these Fatou components by \(C_\lambda^j\) with \(j = 1, \ldots, n + d - 1\). These are ordered in the counterclockwise direction. Recall that these are called the connecting (Fatou) components since each of these components extends from \(T_\lambda\) to \(B_\lambda\). Some of these Fatou components are immediate basins of attracting cycles, and others are eventually periodic components. The exact configuration of these components is determined by Symmetry Lemma 2 with \(\omega = \exp(2\pi i/(n + d))\). For example, since \(F_\lambda(\omega z) = \omega^n F_\lambda(z)\), we have \(F_\lambda(C_\lambda^1) = C_\lambda^n\), \(F_\lambda(C_\lambda^2) = C_\lambda^{2n}\), and so forth. In particular, if \(n = d = 3\), both \(C_\lambda^0\) and \(C_\lambda^3\) are fixed basins, \(C_\lambda^1\) and \(C_\lambda^2\) are mapped to \(C_\lambda^3\), and \(C_\lambda^3\) and \(C_\lambda^4\) are mapped to \(C_\lambda^0\).

Let \(p_\lambda^j = \omega^j p_\lambda^0\), \(q_\lambda^j = \omega^j q_\lambda^0\), and \(u_\lambda^j = \omega^j u_\lambda^0\). Then both \(q_\lambda^j\) and \(u_\lambda^j\) lie on \(\partial C_\lambda^j\). Also, \(p_\lambda^j\) lies in the interior of \(C_\lambda^j\) and is either periodic or preperiodic (see Figure 6).

For \(M_1, M_2, \ldots, M_{n-2}\), we have a similar structure due to the \((n - 1)\)-fold symmetry in the parameter plane. More precisely, if \(\nu\) is an \((n - 1)\)th root of unity, the orbits of the critical points of \(F_\lambda\) and \(F_{\nu \lambda}\) behave symmetrically with respect to multiplication by some (fractional) power of \(\nu\), as was shown immediately following Symmetry Lemma 3. Consequently, the configuration of the basins for \(F_{\nu \lambda}\) is similar to that of \(F_\lambda\).

Recall that \(\partial B_\lambda\) and \(\partial T_\lambda\) are simple closed curves. Since there are no critical points in \(\partial B_\lambda \cap \partial T_\lambda\), these curves do not intersect. Let \(A_\lambda\) denote the closed annulus bounded by \(\partial B_\lambda\) and \(\partial T_\lambda\). Let \(I^j_\lambda\) denote the closed set in \(A_\lambda\) that is contained in the region located between the open disks \(C_\lambda^j\) and \(C_\lambda^{j + 1}\). Note that the intersection of \(I^j_\lambda\) and \(I^{j + 1}_\lambda\) is the pair of points \(q^{j + 1}_\lambda\) and \(u^{j + 1}_\lambda\). Thus there are four points on the boundary of each \(I^j_\lambda\) that also lie on the boundary of another such set: a pair of points lies in \(I^j_\lambda \cap I^{j + 1}_\lambda\) and another pair in \(I^j_\lambda \cap I^{j - 1}_\lambda\). We call the points \(q^j_\lambda\) the outer junction points and the points \(u^j_\lambda\) the inner
junction points (see Figure 6).

**Proposition 2.** $F_\lambda$ maps each $I^j_\lambda$ univalently (except at the junction points) over the region that is the complement of the three sets $B_\lambda$, $F_\lambda(C^j_\lambda)$, and $F_\lambda(C^{j+1}_\lambda)$.

**Proof:** Since $F_\lambda$ is conjugate to $z \mapsto z^n$ on $\partial B_\lambda$, the portion of $\partial B_\lambda$ that meets $I^j_\lambda$, i.e., the arc in $\partial B_\lambda$ connecting $q^j_\lambda$ to $q^{j+1}_\lambda$, is mapped to an arc in $\partial B_\lambda$ that passes through exactly $n+1$ outer junction points. Similarly, the portion of $\partial T_\lambda$ that meets $I^j_\lambda$ is mapped to the complementary arc in $\partial B_\lambda$. These two arcs meet at a pair of outer junction points in $\partial B_\lambda$. Also, the portion of the boundary of $I^j_\lambda$ that meets $\partial C^j_\lambda$ is mapped one-to-one onto the boundary of $F_\lambda(C^j_\lambda)$ except at the junction points. The junction points are both mapped to the same point. Similarly the other boundary of $I^j_\lambda$ that lies in $\partial C^{j+1}_\lambda$ is mapped onto $\partial F_\lambda(C^{j+1}_\lambda)$. Therefore, the boundary of $I^j_\lambda$ is mapped to the boundary of the three sets $B_\lambda$, $F_\lambda(C^j_\lambda)$, and $F_\lambda(C^{j+1}_\lambda)$. Since there are no critical points in $I^j_\lambda$, the result follows.

We call the two arcs in $I^j_\lambda$ that lie in the boundaries of $C^j_\lambda$ and $C^{j+1}_\lambda$ the internal boundary components of $I^j_\lambda$. By Proposition 2, there must be a preimage of $T_\lambda$ in each $I^j_\lambda$. Moreover, the boundary of this preimage must meet each internal boundary component of $I^j_\lambda$ in exactly one point, namely the preimage of the inner junction points lying in the portions of the boundary of $F_\lambda(C^j_\lambda)$ and $F_\lambda(C^{j+1}_\lambda)$ that lie in $I^j_\lambda$. Thus the preimage of $T_\lambda$ in each $I^j_\lambda$ is an open region whose boundary meets exactly one point in each of the boundaries of the connecting Fatou components that are adjacent to $I^j_\lambda$ (see Figure 7).

The preimage of $T_\lambda$ separates $I^j_\lambda$ into two pieces: an external piece that abuts $\partial B_\lambda$ and an internal piece that abuts $\partial T_\lambda$. The external piece is mapped by $F_\lambda$ over the portion of $A_\lambda$ that stretches from $F_\lambda(C^j_\lambda)$ to $F_\lambda(C^{j+1}_\lambda)$ in the counterclockwise direction. Since $F_\lambda(\omega z) = \omega^n F_\lambda(z)$, this region is mapped over exactly $n$ of the $I^j_\lambda$ and $n-1$ of the $C^j_\lambda$. Similarly, the internal piece is mapped over exactly $d$ of the $I^j_\lambda$ and $d-1$ of the $C^j_\lambda$. So each of $I^j_\lambda$ can be further subdivided as shown in Figure 7. The portion of $I^j_\lambda$ lying outside the preimage of $T_\lambda$ has $n-1$ preimages of the connecting components, and the internal portion has $d-1$ such preimages. Between each preimage including $C^j_\lambda$ and $C^{j+1}_\lambda$, there is a region that is mapped univalently.
onto one of the $I_k^\lambda$’s. Hence there is a preimage of each of the sets just constructed in each of these smaller regions (see Figure 7).

Fig. 7. The regions $I_k^\lambda$ if $n = 3$ and $d = 2$.

Continuing in this fashion, we always find the same picture in each region bounded by $k^{th}$ and earlier preimages of $T_\lambda$ and $k^{th}$ and earlier preimages of the $C_j^\lambda$’s. It is a central $(k + 1)^{st}$ preimage of $T_\lambda$ flanked by $n - 1$ $(k + 1)^{st}$ preimages of the connecting components on one side and $d - 1$ other $(k + 1)^{st}$ preimages on the other side.

However, this construction does not give the entire Julia set of $F_\lambda$. Indeed, the portion of the Julia set produced thus far contains only preimages of the boundaries of $B_\lambda$ and the $C_j^\lambda$. None of these preimages contain any periodic points; the only periodic points here lie in $\partial B_\lambda$ and $\partial C_j^\lambda$. So there must be more to the Julia set.

To complete the construction of the Julia set, note that each closed region $I_k^\lambda$ is almost mapped univalently over the union of all of the $I_k^\lambda$’s. The map is univalent except at the four junction points. One pair of junction points is mapped to an outer junction point in the image, and the other pair is mapped to a different outer junction point. We can use symbolic dynamics to identify each point in the Julia set. Let $\Sigma$ denote the set of sequences $(s_0, s_1, s_2, \ldots)$ where each $s_j$ is one of the integers $0, 1, \ldots, n + d - 1$. We identify each point in $J(F_\lambda)$ with a point in $\Sigma$ by assigning to each $z \in J(F_\lambda)$ its itinerary $S(z) = (s_0, s_1, s_2, \ldots)$ where $s_k = j$ if $F_k^\lambda(z) \in I_k^\lambda$. However, infinitely many points are assigned to a pair of sequences. The points $q_j^\lambda$ and $u_j^\lambda$ each have a pair of sequences attached to them since these points reside in two of the $I_k^\lambda$’s. For example, $S(q_0^\lambda) = (0) = (n + d - 1)$ and $S(u_0^\lambda) = (0, n + d - 1) = (n + d - 1, 0)$. Similarly, any point that is eventually mapped onto a $q_j^\lambda$ or a $u_j^\lambda$ also has a pair of itineraries, e.g., the itineraries $(s_0, \ldots, s_k, 0, n + d - 1)$ and $(s_0, \ldots, s_k, n + d - 1, 0)$ correspond to the same points.

We let $\Sigma'$ denote the sequence space with the above identifications and endow $\Sigma'$ with the quotient topology. Since each $I_j^\lambda$ is mapped univalently (except at the junction points) over the union of the $I_k^\lambda$ and the Julia set is contained in this union, standard arguments then show that the Julia set is homeomorphic to $\Sigma'$. The subsets $\Sigma_j'$ of $\Sigma'$ consisting of all sequences that start with the digit $j$ correspond to points in
$I_A^* \cap J(F_\lambda)$, and they are homeomorphic to $\Sigma'_k$. It is important to note that the dynamics on these sets are not the same even though they are homeomorphic. We have described the topological structure of each $I_A^* \cap J(F_\lambda)$, and this description implies Theorem 1 (see [Çilingir et al., 2010] for a similar argument).

3. Dynamical Invariants

In this section we prove Theorems 2 and 3. Let $\nu = \exp(2\pi i/(n-1))$. We show that two maps drawn from the main cardioids of different principal Mandelbrot sets are conjugate on their Julia sets if and only if the cardioids are located symmetrically under either the maps $z \mapsto \nu^{j(d+1)}z$ or $z \mapsto \nu^{j(d+1)}\overline{z}$ for some integer $j$ (see Proposition 4).

We first observe that it suffices to prove this result for the special maps whose parameter is the center of these main cardioids. The set of critical points is invariant under the map, so the critical points are either periodic or preperiodic. The following proposition follows from the work of Mañé, Sad, and Sullivan [1983].

**Proposition 3.** Suppose $\lambda$ lies at the center of an $M_j$ and $\mu \in M_j$. Then $F_\lambda$ and $F_\mu$ are quasiconformally conjugate on their Julia sets.

**Remark.** It is not true that $F_\lambda$ and $F_\mu$ are globally conjugate since $F_\lambda$ has a superattracting cycle while the attracting cycle for $F_\mu$ need not be superattracting.

By Proposition 3, we need only consider parameters that lie at the centers of the $M_j$. So for the remainder of this section, we assume that $\lambda$ and $\mu$ are centers. Then $\mu = \nu^j \lambda$ for some $j \in \mathbb{Z}$. The proof of one direction of Theorem 2 is straightforward:

**Proposition 4.** If $\mu = \nu^{j(d+1)}\lambda$ or $\mu = \nu^{j(d+1)}\overline{\lambda}$ for some integer $j$, then $F_\mu$ is conjugate to $F_\lambda$.

**Proof:** Let $\mu = \nu^{j(d+1)}\lambda$, then

$$F_\mu(\nu^jz) = \nu^{jn}z^n + \frac{\lambda \nu^{j(d+1)}}{\nu^{jd+1}} = \nu^j \left( z^n + \frac{\lambda}{z^d} \right) = \nu^j F_\lambda(z).$$

So $F_\lambda$ is conjugate to $F_\mu$ via the linear map $z \mapsto \nu^j z$.

By Symmetry Lemma 1, $F_\lambda$ and $F_\overline{\lambda}$ have conjugate dynamics. So if $\mu = \nu^{j(d+1)}\overline{\lambda}$, then $F_\mu$ is conjugate to $F_\overline{\lambda}$ and hence also to $F_\lambda$.

From Proposition 4, we know that all centers whose parameters are of the form $\nu^k \lambda$ or $\nu^k \overline{\lambda}$ where $k = j(d + 1) \mod (n - 1)$ have conjugate dynamics. That is, any two main cardioids that are located symmetrically with respect to either rotation by $\nu^{d+1}$ or complex conjugation have conjugate dynamics. Note that $\nu^{d+1} = \nu^{n-1} \nu^{d+1} = \nu^{n+d}$, so we can say that any two cardioids that are located symmetrically with respect to either rotation by $\nu^{n+d}$ or complex conjugation have conjugate dynamics.

Using basic facts about the greatest common divisor of two numbers, we can restate this relationship among the centers with conjugate dynamics in terms of the greatest common divisor $g$ of $d + 1$ and $n - 1$. In fact, all centers whose parameters are of the form $\nu^k \lambda$ or $\nu^k \overline{\lambda}$ where $k$ is an integer multiple of $g$ have conjugate dynamics.
Now we show that these symmetrically located centers are the only centers with conjugate dynamics. First we define the minimum rotation number for parameters in the $M_j$. For each such parameter we have $n + d$ connecting components $C^j_{\lambda}$ with $j$ defined mod $n + d$, and the $C^j_{\lambda}$ are ordered in the counterclockwise direction as $j$ increases. Each of these connecting components is mapped two-to-one onto another such component since each contains a unique critical point (see Figure 6).

Suppose $F_\lambda(C^j_{\lambda}) = C^k_{\lambda}$. We define the rotation number $\rho_j$ of $C^j_{\lambda}$ to be the value of $k - j$ that is closest to 0 for any $k$ mod $n + d$. Note that it is possible for $k$ to be negative (see Figure 8). For example, if $F_\lambda(C^0_{\lambda}) = C^{n+d-1}_{\lambda}$, then the rotation number of $C^0_{\lambda}$ would be $-1$ since $C^{n+d-1}_{\lambda} = C^{-1}_{\lambda}$. We say that $C^j_{\lambda}$ is rotated through $k - j$ components if $\rho_j = k$. We then define the minimum rotation number $\rho(\lambda)$ for $F_\lambda$ to be the minimum value of $|\rho_j|$ over all $j$. For example, if $F_\lambda$ has an attracting fixed point in some $C^j_{\lambda}$, $\rho(\lambda) = 0$. If there is no such attracting fixed point, then $\rho(\lambda) > 0$.

![Fig. 8. A $C^j_{\lambda}$ with $\rho_j = -1$.](image)

**Proposition 5.** Let $\lambda$ and $\mu$ be centers of the $M_j$. Then $F_\lambda$ is conjugate to $F_\mu$ if and only if $\rho(\lambda) = \rho(\mu)$. Equivalently, $\mu = v^k \lambda$ or $\mu = v^k \overline{\lambda}$ where $k = j(d + 1)$ mod $(n - 1)$ for some integer $j$. The conjugating map is either a rotation about the origin or a rotation followed by complex conjugation.

**Proof:** First suppose that two such centers $\lambda$ and $\mu$ have different minimum rotation numbers. Then $F_\lambda$ and $F_\mu$ cannot be conjugate on their Julia sets. To see why, recall that the connecting components each touch $\partial B_\lambda$ at exactly one point. Now $\partial B_\lambda$ must be sent to itself by any conjugacy between $F_\lambda$ and $F_\mu$ since this set is the only invariant subset of the Julia set that touches the boundaries of all of the connecting components. Thus the ordering of the $C^j_{\lambda}$ is either preserved or reversed by the conjugacy, i.e., either the conjugacy rotates the connecting components in one direction or the other, or the conjugacy first applies complex conjugation followed by some rotation. In either case, if the minimum rotation numbers of $F_\lambda$ and $F_\mu$ are different, then such a conjugacy cannot exist.

To prove the converse, we consider the set of critical points, $c^0_{\lambda}, \ldots, c^{n+d-1}_{\lambda}$, of $F_\lambda$ where $\lambda$ is the center of an $M_j$. The point $c^j_{\lambda}$ is the unique critical point that lies in the connecting component $C^j_{\lambda}$. We note that this set of critical points is invariant under $F_\lambda$. 
Suppose $F_\lambda$ and $F_\mu$ have the same minimum rotation number $m$. By assumption, there is at least one critical point $c_\lambda^i$ for $F_\lambda$ for which either $F_\lambda(c_\lambda^i) = c_\lambda^{i+m} = \omega^m c_\lambda^i$ or $F_\lambda(c_\lambda^i) = c_\lambda^{i-m} = \omega^{-m} c_\lambda^i$. (Recall that $\omega^{n+d} = 1$.) There is also a critical point $c_\mu^i$ for $F_\mu$ for which either $F_\mu(c_\mu^i) = c_\mu^{i+m} = \omega^m c_\mu^i$ or $F_\mu(c_\mu^i) = c_\mu^{i-m} = \omega^{-m} c_\mu^i$.

We consider the first case for $\lambda$ and $\mu$, i.e., where the rotation numbers $m$ for both critical points are positive. Since $\mu = \nu^k \lambda$ for some $k$, $F_\mu$ is conjugate to the map $z \mapsto \omega^k F_\lambda(z)$ by the rotation $z \mapsto \eta^k z$ where

$$\eta = \exp(2\pi i/(n + d)(n - 1))$$

(see the paragraphs that follow Symmetry Lemma 3). So there must be a critical point for $\omega^k F_\lambda$ that corresponds to $c_\mu^i$ and that is also rotated by $\omega^m$ when $\omega^k F_\lambda$ is applied to it. But any critical point of $\omega^k F_\lambda$ must also be a critical point for $F_\lambda$. Suppose that $\omega^f c_\lambda^i$ is the critical point for $\omega^k F_\lambda$ that corresponds to $c_\mu^i$. Then we have

$$\omega^k F_\lambda(\omega^f c_\lambda^i) = \omega^m \omega^f c_\lambda^i = \omega^f \omega^m c_\lambda^i = \omega^f F_\lambda(c_\lambda^i).$$

But

$$\omega^k F_\lambda(\omega^f c_\lambda^i) = \omega^{k+\ell n} F_\lambda(c_\lambda^i).$$

Therefore we have $\ell = k + \ell n \mod (n + d)$.

Consequently, $\omega^f F_\lambda(z) = \omega^{k+\ell n} F_\lambda(z)$, and using Symmetry Lemma 2, we obtain

$$\omega^f F_\lambda(z) = \omega^k F_\lambda(\omega^f z)$$

for all $z \in \mathbb{C}$. So $F_\lambda$ is conjugate to $\omega^k F_\lambda$ via the map $z \mapsto \omega^f z$. Therefore $F_\lambda$ is also conjugate to $F_\mu$ via a linear map of the form $z \mapsto \eta \omega^f z$ and $\mu = \nu^k \lambda$ where $k = j(d + 1) \mod (n - 1)$.

The proof for the case where both rotation numbers $m$ are negative is exactly the same.

To prove the second case, the case where the rotations go in opposite directions, we simply conjugate $F_\mu$ to $F_\mu$ by complex conjugation and then invoke the first case. \hfill $\square$

As a consequence of Propositions 3 and 5, this result extends to all parameters in $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_{n-2}$.

**Corollary.** Let $\lambda$ and $\mu$ be any parameters drawn from the main cardioids of any two principal Mandelbrot sets. Then $F_\lambda$ and $F_\mu$ are conjugate on their Julia sets if and only if $\rho(\lambda) = \rho(\mu)$.

Now we can determine exactly which $\mathcal{M}_j$ have conjugate dynamics and the precise number of different conjugacy classes. We write $\mathcal{M}_j \equiv \mathcal{M}_k$ if the parameters at the centers of $\mathcal{M}_j$ and $\mathcal{M}_k$ have conjugate dynamics. Let $g$ be the greatest common divisor of $n - 1$ and $d + 1$. As we proved in this section, the principal Mandelbrot sets with dynamics conjugate to the dynamics of $\mathcal{M}_k$ are those obtained by successive rotations in the parameter plane by $z \mapsto \nu^j g z$ or by these rotations followed by complex conjugation. In particular, we have $\mathcal{M}_0 \equiv \mathcal{M}_{jg}$ for all integers $j$.

**Theorem.** If the greatest common divisor $g$ is even, there are $1 + g/2$ different conjugacy classes among the $\mathcal{M}_j$. If $g$ is odd, there are $(g + 1)/2$ distinct such conjugacy classes.

**Proof:** First suppose that $g = 1$. Then all maps drawn from the $\mathcal{M}_j$ have conjugate dynamics, so we have $1 = (g + 1)/2$ conjugacy classes.
Now suppose \( g > 1 \). We claim that \( \mathcal{M}_k \not\equiv \mathcal{M}_0 \) for any \( k \) with \( 0 < k < g \). If not, then maps at the centers of \( \mathcal{M}_0 \) and some \( \mathcal{M}_k \) would be conjugate by \( z \mapsto \overline{z} \) followed possibly by a rotation. But then \( \mathcal{M}_k \equiv \mathcal{M}_{-k} \) via \( z \mapsto \overline{z} \). Also, \( \mathcal{M}_g \equiv \mathcal{M}_{-g} \) by \( z \mapsto \overline{z} \). Therefore we have \( \mathcal{M}_{-g} \equiv \mathcal{M}_{-k} \) by a rotation, which would imply that the greatest common divisor is smaller than \( g \). So none of the centers of the \( \mathcal{M}_k \) with \( 0 < k < g \) have dynamics conjugate to the center of \( \mathcal{M}_0 \).

If \( g \) is even, we consider \( \mathcal{M}_k \) where \( 0 < k < g/2 \). We have \( \mathcal{M}_k \equiv \mathcal{M}_{-k} \) by complex conjugation. Moreover \( \mathcal{M}_{-k} \equiv \mathcal{M}_{g-k} \) since these sets are symmetric under the rotation \( z \mapsto \nu^g z \), so \( \mathcal{M}_k \equiv \mathcal{M}_{g-k} \). On the other hand, we cannot have \( \mathcal{M}_k \equiv \mathcal{M}_j \) for any other \( j \) with \( 0 < j < g \) via rotation by \( z \mapsto \nu^g z \) or by complex conjugation coupled with a rotation, so the principal Mandelbrot sets with dynamics conjugate to those in \( \mathcal{M}_k \) are just the rotations of \( \mathcal{M}_k \) together with their complex conjugates. The number of such conjugacy classes is \( g/2 - 1 \). We have \( \mathcal{M}_{g/2} \equiv \mathcal{M}_{-g/2} \) by the rotation \( z \mapsto \nu^{-g} z \) as well as by complex conjugation. So \( \mathcal{M}_{g/2} \) lies in a conjugacy class that is distinct from the classes of the \( \mathcal{M}_k \) with \( 0 \leq k \leq g/2 \). The conjugacy class of \( \mathcal{M}_0 \) has not yet been counted. Combining all of these classes, we obtain a total of \( 1 + g/2 \) distinct conjugacy classes.

If \( g \) is odd, we count in exactly the same way except that we do not have a conjugacy class that corresponds to \( \mathcal{M}_{g/2} \) in this case.

See Figure 9 for the three orbit diagrams that arise if \( n = 13 \) and \( d = 7 \). In Figure 10, we consider the case where \( n = 11 \) and \( d = 4 \), and group the Mandelbrot sets whose centers have conjugate dynamics.

![Fig. 9. If \( n = 13 \) and \( d = 7 \), then \( g = 4 \), and consequently, there are three conjugacy classes. This figure contains one orbit diagram for each of the three classes.](image)

### 4. A Group Action

Since the conjugacies among the \( \mathcal{M}_k \) arise from reflective and rotational symmetries, we can count the number of conjugacy classes by viewing them as orbits of the action of a dihedral group on the set \( \{\mathcal{M}_k\} \), viewed as the vertices of a regular \((n - 1)\)-gon.

Let \( a = (n - 1)/g \). We claim that the natural group that produces these orbits is \( D_{2a} \), the group of symmetries of a regular \( a \)-gon. Let \( s \) be the generator of \( D_{2a} \) corresponding to reflection and \( r \) be the
If $n = 11$ and $d = 4$, then $g = 5$, and consequently, there are three conjugacy classes. The parameters with conjugate dynamics are connected by segments of the same color, e.g., the four Mandelbrot sets connected by white segments all have conjugate dynamics.

generator corresponding to rotation. We define the action of $D_{2a}$ on $\set{M_k}$ by

\begin{align*}
sM_k &= M_{-k \mod n-1} \\
rM_k &= M_{k+g \mod n-1}
\end{align*}

These rules produce a well-defined $D_{2a}$ action, and since the actions on $\set{M_k}$ by $s$ and $r$ are exactly complex conjugation and rotation by $z \mapsto \nu^g z$, respectively, the orbits of this action correspond exactly to the conjugacy classes among the $M_k$.

By Burnside's Lemma, the number of orbits is

$$\frac{1}{|D_{2a}|} \sum_{x \in D_{2a}} |\text{fix}(x)|$$

where $\text{fix}(x) = \{M_i \in \set{M_k} : xM_i = M_i\}$ (see [Gallian, 2002]).

The group $D_{2a}$ has $2a$ elements, and each can be written as $r^j$ or $sr^j$ with $0 \leq j < a$. The identity fixes all $n-1$ elements of $\set{M_k}$, and $r^j$ fixes none for $0 < j < a$. Thus the number of orbits is

$$\frac{1}{2a} \left( n - 1 + \sum_{j=0}^{a-1} |\text{fix}(sr^j)| \right)$$

An element of the form $sr^j$ rotates each $M_k$ by $z \mapsto \nu^{jg} z$ and then reflects it about the real axis. Equivalently, it reflects the $M_k$ through some axis of symmetry of the set viewed as a regular $(n-1)$-gon. Thus, if $n - 1$ is odd, every such axis passes through exactly one of the $M_k$, and thus $|\text{fix}(sr^j)| = 1$ for all $j$. The formula above then shows the number of conjugacy classes is $(n - 1 + a)/2a = (g + 1)/2$.

If $n - 1$ is even, half of the axes of symmetry pass through two of the $M_k$, and half pass through none. Thus $sr^j$ fixes either two or zero of the $M_k$. There exists a $j$ such that $sr^j$ fixes none of the $M_k$ if and only if there is some $i$ such that $sr^i M_i = M_{i+1 \mod n-1}$, i.e., the axis of reflection passes
between $\mathcal{M}_i$ and $\mathcal{M}_{i+1 \mod n-1}$ for some $i$. For such an $i$, $r^j \mathcal{M}_i = s^{-1} \mathcal{M}_{i+1 \mod n-1} = \mathcal{M}_{-i-1 \mod n-1}$ which equals $\mathcal{M}_{i+jg \mod n-1}$ by the definition of the action of $r$. Thus $-i-1 \equiv i+jg \mod n-1$, and hence, $2i+jg+1 \equiv 0 \mod n-1$. If either $j$ or $g$ is even, this equality is impossible since $n-1$ is even, and therefore, $sr^j$ must fix two of the $\mathcal{M}_k$. If $j$ and $g$ are both odd, however, any $i$ with $i \equiv (-jg-1)/2 \mod n-1$ satisfies the congruence, and thus $sr^j$ fixes none of the $\mathcal{M}_k$.

Therefore, if $n-1$ and $g$ are even, then $|\text{fix}(sr^j)| = 2$ for all $j$, and the number of conjugacy classes is $(n-1+2a)/2a = 1+g/2$. If $n-1$ is even and $g$ is odd, $|\text{fix}(sr^j)|$ equals $2$ if $j$ is even, and $0$ if $j$ is odd. Hence, there are $(n-1+a)/2a = (g+1)/2$ conjugacy classes.

Finally, if $n-1$ is odd, $g$ must be odd, so the possible cases really depend only on the parity of $g$ and not of $n-1$. Hence, the number of conjugacy classes is $(g+1)/2$ if $g$ is odd and $1+g/2$ if $g$ is even.

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