THE LEVEL STRUCTURE IN QUANTUM $K$-THEORY AND MOCK THETA FUNCTIONS

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Contents
1. Introduction 1
   1.1. Overview 1
   1.2. Mirror theorem and mock theta function 3
   1.3. Plan of the paper 7
   1.4. Acknowledgments 7
2. Level structure 7
   2.1. Determinant line bundles 7
   2.2. Level structure in quasimap theory 8
   2.3. Properties of the level structure in the quasimap theory 10
3. The $K$-theoretic quasimap theory with level structure 16
   3.1. $K$-theoretic quasimap invariants with level structure 16
   3.2. Quasimap graph space and $J^{R,l,\epsilon}$-function 18
   3.3. The permutation-equivariant quasimap $K$-theory with level structure 20
   3.4. The level structure in equivariant quasimap theory and orbifold quasimap theory 22
4. Mirror theorem and mock theta functions 23
   4.1. Wall-crossing 23
   4.2. $I$-function and mock theta function 33
References 36

1. Introduction

1.1. Overview. More than a decade ago, quantum $K$-theory was introduced by Givental and Lee [1, 2] as the $K$-theoretic analog of quantum cohomology. Its recent revival stems partially from a physical interpretation of quantum $K$-theory as a 3D-quantum field theory in the 3-manifold of the form $S^1 \times \Sigma$. Because of this mysterious physical connection, the B-model counterpart of quantum $K$-theory is $q$-hypergeometric series, itself a classical subject. The above connection was recently confirmed by Givental [3] as the mirror of the so-called $J$-function of the permutation-equivariant quantum $K$-theory.

Classically, $K$-theory is more closely associated with representation theory than cohomology. It is natural to revisit quantum $K$-theory from representation theory point of view. In fact, a variant of quantum $K$-theory was already studied by Maulik-Okounkov [4] in relation to the quantum group. One of the predominant features of representation theory is the existence of an additional parameter called the level. A natural question is whether it is possible to extend the current version of quantum $K$-theory to include this notion of level.
In this article, we answer the question affirmatively in the context of the GLSM. This is the first in a sequence of papers to develop the theory of levels in quantum $K$-theory and study its applications.

Our motivating example is an old physical result of Witten [5] in the early 90’s which relates the quantum cohomology ring of the Grassmannian to the Verlinde algebra. Early explicit physical computations [6, 7, 8] indicate that they are isomorphic as algebras, but have different pairings. In [5], Witten gave a conceptual explanation of the isomorphism, by proposing an equivalence between the quantum field theories which govern the Verlinde algebra and the quantum cohomology of the Grassmannian. His physical derivation of the equivalence naturally leads to a mathematical problem that these two objects are conceptually isomorphic (without referring to the detailed computation). A great deal of work has been done by Marian-Oprea [9, 10, 11] and Belkale [12]. However, to the best of our knowledge, a complete conceptual proof of the equivalence is missing. We should emphasise that the numerical invariants or correlators of two theories are different and it would be desirable to correct this defect.

Assuming a basic knowledge of the quantum $K$-theory, a key and yet more or less trivial observation is that Verlinde algebra are $K$-theoretic invariant. To be more precise, suppose that $\mathcal{M}_{g,k}(X, \beta)$ is the moduli space of stable maps. Quantum cohomology studies the integral

$$\int_{[\mathcal{M}_{g,k}(X, \beta)]^{vir}} \alpha$$

where $[\mathcal{M}_{g,k}(X, \beta)]^{vir}$ is the so-called virtual fundamental cycle and $\alpha$ is certain “tautological” cohomology classes. In quantum $K$-theory, we replace the virtual fundamental cycle by the virtual structure sheaf $\mathcal{O}^{vir}_{\mathcal{M}_{g,k}(X, \beta)}$. We also replace the integral by the holomorphic Euler characteristic

$$\chi(\mathcal{M}_{g,k}(X, \beta), E \otimes \mathcal{O}^{vir}_{\mathcal{M}_{g,k}(X, \beta)})$$

where $E$ is some element in the $K$-group of $X$. For the Verlinde algebra, the relevant moduli space is the moduli space of semistable parabolic $U(n)$-bundles $\mathcal{M}_{U(n)}(\alpha_1, \cdots, \alpha_k)$ on a fixed genus $g$ marked curve $(C, p_1, \cdots, p_k)$ with parabolic structure at $p_i$ indexed by $\alpha_i \in V_l(U(n))$ for level $l$ Verlinde algebra $V_l(U(n))$. For simplicity, we assume there are no strictly semistable parabolic vector bundles. In this case, the moduli space is smooth and $\mathcal{O}^{vir}_{\mathcal{M}_{U(n)}}$ is the structure sheaf. A new ingredient is a certain determinant line bundle $det$. The level $l$ Verlinde algebra calculates the holomorphic Euler characteristic

$$\langle \alpha_1, \cdots, \alpha_k \rangle_{g, \text{Verlinde}}^l = \chi(\mathcal{M}_{U(n)}(\alpha_1, \cdots, \alpha_k), det^l).$$

From the above description, the Verlinde algebra is clearly a $K$-theoretic object and we should compare it with the quantum $K$-theory of the Grassmannian (with an appropriate notion of levels). Let $\mathcal{Bun}_G$ be the moduli stack of principal bundle over curves. Let $\pi : \mathcal{C}_{\mathcal{Bun}_{g,k}} \to \mathcal{Bun}_G$ be the universal curve and let $\mathcal{P} \to \mathcal{C}_{\mathcal{Bun}_{g,k}}$ be the universal principal bundle. Given a representation $R$ of $G$, we consider the inverse determinant of cohomology

$$\det_R := det^{-1}(R\pi_* (\mathcal{P} \times_G R)).$$

It is a line bundle over $\mathcal{Bun}_G$. Let $\mathcal{Q}^\epsilon$ be the moduli stack of $\epsilon$-stable quasimaps to a GIT quotient $X = Z // G$ (see Section 2.2). There is a natural forgetful morphism $\mu : \mathcal{Q}^\epsilon \to$
and we simply define the level $l$ determinant line bundle as
\[ \mathcal{D}^{R,l} = \mu^*(\det_R)^l. \]
This definition can be easily generalized to gauged linear sigma models (see Section 2.2).

With the above definition of the level $l$ determinant line bundle $\mathcal{D}^{R,l}$, we can define the level $l$ quantum $K$-theoretic invariants by twisting with $\mathcal{D}^{R,l}$ (see Section 3). The usual quantum $K$-theory corresponds to the case $l = 0$. It is easy to check that the level $l$ quantum $K$-theory satisfies all the axioms of the usual quantum $K$-theory (see Section 2.3).

Let $\langle \alpha'_1, \ldots, \alpha'_k \rangle_{g,\text{quantum}_K}$ be the level $l$ quantum $K$-invariants of $G(n, n + l)$ with a fixed genus $g$ marked curve $(C, p_1, \ldots, p_k)$ and $\alpha'_i \in K(G(n, n + l))$. Motivated by [6], we can formulate the following mathematical conjecture.

**Verlinde Algebra-Grassmanian Correspondence:** There is an identification of basis $H : V_1(U(n)) \to K(G(n, n + l))$ such that
\[ \langle \alpha_1, \ldots, \alpha_k \rangle_{g,\text{Verlinde}} = \langle H(\alpha_1), \ldots, H(\alpha_k) \rangle_{g,\text{quantum}_K}. \]
Keeping with the spirit of [6], we look for a conceptual proof of the above conjecture.

Verlinde algebra is a part of conformal field theory. We certainly expect a much broader correspondence involving the invariants of conformal field theory which is not necessarily interpreted in terms of geometry of stable vector bundle. The above $K$-theoretic conjecture can be viewed as a way to correct the defect of the original statement in terms of quantum cohomology.

### 1.2. Mirror theorem and mock theta function.

The proof of above conjecture will be discussed in a different article. Our main results in this paper are various mirror theorems for the permutation-equivariant version of the level $l$ quantum $K$-theory in the same style of recent work of Givental [3]. Here we see the surprising appearance of Ramanujan’s mock theta function in some of the simplest examples. Let $X = Z//G$ be a GIT quotient. We consider a generating series $J^{R,l,\infty}_X(q, Q)$ of quantum $K$-theory invariants of level $l$. It is called the permutation-equivariant $J$-function of level $l$ and representation $R$. Let $Q$ be the Novikov variable. We fix a $\lambda$-algebra $\Lambda$ which is equipped with Adams operations $\Psi_i$, $i = 1, 2, \ldots$. Let $\{\phi_a\}$ be a basis of $K^0(X) \otimes \mathbb{Q}$ and $\{\phi^a\}$ be the dual basis with respect to the twisted pairing $[3]$. Let $q$ be a formal variable and let $t(q)$ be a Laurent polynomial in $q$ with coefficients in $K^0(X) \otimes \mathbb{Q}$. The definition of $J^{R,l,\infty}_{S_\infty}(t(q), Q)$ is as follows
\[ J^{R,l,\infty}_{S_\infty}(t(q), Q) = 1 + \frac{t(q)}{1 - q} + \sum_a \sum_{\beta \neq 0} Q^\beta \phi^a \langle \frac{\phi_a}{(1 - q)(1 - qL)}, t(L), \ldots, t(L) \rangle_{0, l + 1, k, \beta}. \]
Here $\langle \cdot \rangle_{0, l + 1, k, \beta}^{R,l,S_k}$ denotes the permutation-equivariant quantum $K$ invariants of level $l$ and $L$ denotes the cotangent line bundle. Similarly we can define the generating function $J^{R,l,\epsilon}_{S_\infty}$ for $\epsilon$-stable quasimap theory (see Section 3.3). The permutation-equivariant $J^{R,l,\infty}_{S_\infty}$-function for quantum $K$-theory can be identified with $J^{R,l,\epsilon}_{S_\infty}$ for sufficiently large $\epsilon$. The generating functions $J^{R,l,\epsilon}_{S_\infty}$ can be viewed as elements in the loop space $\mathcal{K}$ defined as
\[ \mathcal{K} := [K(X) \otimes \mathbb{C}(q)] \otimes \mathbb{C}[[Q]], \]
where $\mathbb{C}(q)$ is the field of complex rational functions in $q$. It has a natural Lagrangian polarization $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$, where $\mathcal{K}_+$ consists of Laurent polynomials and $\mathcal{K}_-$ consists of reduced rational functions regular at $q = 0$ and vanishing at $q = \infty$. We denote by
The range of $\mathcal{J}_{S_\infty}^{R,I,\infty}$, i.e., $\mathcal{L}_{S_\infty} = \cup_{t(q)\in \mathcal{K}} (1-q)\mathcal{J}_{S_\infty}^{R,I,\infty}(t(q),Q) \subset \mathcal{K}$. By definition, $(1-q)\mathcal{J}_{S_\infty}^{R,I,\infty}(t(q),Q)$ lies on $\mathcal{L}_{S_\infty}$. Our main theorem extends this property to all $\epsilon$.

**Theorem 1.1** (Corollary 4.12). For any positive $\epsilon \in \mathbb{Q}$, $(1-q)\mathcal{J}_{S_\infty}^{R,I,\epsilon}$ lies on $\mathcal{L}_{S_\infty}$.

When $\epsilon$ is sufficiently close to zero, $\mathcal{J}_{S_\infty}^{R,I,\infty}$ is also denoted as $I^{R,I}$. In particular, Theorem 1.1 implies that $(1-q)I^{R,I}$ lies on $\mathcal{L}_{S_\infty}$. In Theorem 4.14 we give explicit formulas of $I^{R,I}$ for toric varieties. A remarkable phenomenon is the appearance of Ramanujan’s mock theta functions. We first establish some notations. We denote the standard representation of $\mathcal{C}$-equivariant $\mathbb{Z}$-actions by multiplication. The $\mathbb{C}$-action is the standard representation. For the $\mathbb{C}$-equivariant $\mathbb{Z}$-action on $\mathbb{C}^n$ can be explicitly described as

$$\lambda \cdot (x_1, \ldots, x_n) \rightarrow (\lambda^{a_1}x_1, \ldots, \lambda^{a_n}x_n), \text{ where } \lambda \in \mathbb{C}^*.$$ 

In the following propositions, we consider GIT quotients $\mathbb{C}^n // \mathbb{C}^*$ and we refer to the $\mathbb{C}^*$-actions by their associated charge matrices.

**Proposition 1.2.** Consider $X = \mathbb{C} // \mathbb{C}^* = [(\mathbb{C}\setminus\{0\})/\mathbb{C}^*]$ where the $\mathbb{C}^*$-action is the standard action by multiplication. The $\mathbb{C}^*$-equivariant $K$-ring $K_{\mathbb{C}^*}(X)$ is isomorphic to the representation ring $\text{Repr}(\mathbb{C}^*)$. Let $\lambda \in K_{\mathbb{C}^*}(X)$ be the equivariant parameter corresponding to the standard representation. For the $\mathbb{C}^*$-representations $\text{St}^\vee$ and $\text{St}$, we have the following explicit formulas of the equivariant small $I$-functions

$$I_X^{\text{St}^\vee,I}(q,Q) = 1 + \sum_{n \geq 1} \frac{q^{\ln(n-1)}}{(1-\lambda^{-1}q)(1-\lambda^{-1}q^2) \cdots (1-\lambda^{-1}q^n)} Q^n,$$

$$I_X^{\text{St},I}(q,Q) = 1 + \sum_{n \geq 1} \frac{q^{\ln(n-1)}}{(1-\lambda^{-1}q)(1-\lambda^{-1}q^2) \cdots (1-\lambda^{-1}q^n)} Q^n,$$

By choosing certain specializations of the parameters, we obtain Ramanujan’s mock theta functions of order 3

$$I_X^{\text{St}^\vee,I=1}(q^2,Q)|_{\lambda=-1,Q=q} = 1 + \sum_{n \geq 1} \frac{q^{n^2}}{(1+q^2)(1+q^4) \cdots (1+q^{2n})},$$

$$I_X^{\text{St}^\vee,I=1}(q^2,Q)|_{\lambda=q,Q=q} = 1 + \sum_{n \geq 1} \frac{q^{n^2}}{(1-q)(1-q^2) \cdots (1-q^{2n-1})},$$

$$I_X^{\text{St}^\vee,I=1}(q^2,Q)|_{\lambda=-q,Q=1} = 1 + \sum_{n \geq 1} \frac{q^{n(n-1)}}{(1+q)(1+q^3) \cdots (1+q^{2n-1})},$$

and Ramanujan’s mock theta functions of order 5

$$I_X^{\text{St}^\vee,I=2}(q,Q)|_{\lambda=-1,Q=q} = 1 + \sum_{n \geq 1} \frac{q^{n^2}}{(1+q)(1+q^2) \cdots (1+q^n)},$$

$$I_X^{\text{St}^\vee,I=4}(q^2,Q)|_{\lambda=q^{-1},Q=q^2} = 1 + \sum_{n \geq 1} \frac{q^{2n^2}}{(1-q)(1-q^3) \cdots (1-q^{2n-1})},$$
In particular, we have Proposition 1.3. When \( \lambda = -1, Q = 1 \):

\[
I_{C}^{St,l=2}(q, Q)|_{\lambda = -1, Q = 1} = 1 + \sum_{n \geq 1} \frac{q^{n(n+1)}}{(1 + q)(1 + q^2) \cdots (1 + q^n)}.
\]

\[
I_{C}^{St,l=4}(q^2, Q)|_{\lambda = -1, Q = 1} = 1 + \sum_{n \geq 1} \frac{q^{2n^2 + 2n}}{(1 - q)(1 - q^2) \cdots (1 - q^{2n-1})}.
\]

**Proposition 1.3.** Let \( a_1 \) and \( a_2 \) be positive integers. We consider the target \( X_{a_1, a_2} = [(\mathbb{C}^2 \setminus 0) / \mathbb{C}^*] \) with charge matrix \((a_1, a_2)\) and a line bundle \( p = [(\mathbb{C}^2 \setminus 0) \times \mathbb{C}^*] / \mathbb{C}^*\) with charge matrix \((a_1, a_2, 1)\). Let \( \lambda_1 \) and \( \lambda_2 \) be the equivariant parameters. For the \( \mathbb{C}^* \)-representations \( St^V \) and \( St \), we have the following explicit formulas of the equivariant small \( I \)-functions

\[
I_{X_{a_1, a_2}}^{St^V,l}(q, Q) = 1 + \sum_{n \geq 1} \frac{p^{l(n-1)} q^{ln(n-1)}}{(1 - p^{a_1} \lambda_1^{-1} q) \cdots (1 - p^{a_1} \lambda_1^{-1} q^{a_1n})(1 - p^{a_2} \lambda_2^{-1} q^2) \cdots (1 - p^{a_2} \lambda_2^{-1} q^{a_2n})} Q^n;
\]

\[
I_{X_{a_1, a_2}}^{St,l}(q, Q) = 1 + \sum_{n \geq 1} \frac{p^{l(n+1)} q^{ln(n+1)}}{(1 - p^{a_1} \lambda_1^{-1} q) \cdots (1 - p^{a_1} \lambda_1^{-1} q^{a_1n})(1 - p^{a_2} \lambda_2^{-1} q^2) \cdots (1 - p^{a_2} \lambda_2^{-1} q^{a_2n})} Q^n.
\]

When \( l \leq 2 \),

\[
I_{X_{1,1}}^{St^V,l}(q, Q) = I_{X_{1,1}}^{St,l}(q, Q).
\]

In particular, we have

\[
I_{X_{1,1}}^{St^V,l=2}(q^2, Q)|_{p=1, \lambda_1=\lambda_2=-1, Q=q} = 1 + \sum_{n \geq 1} \frac{q^{n^2}}{(1 + q)(1 + q^2) \cdots (1 + q^n)^2},
\]

\[
I_{X_{1,1}}^{St^V,l=2}(q, Q)|_{p=1, \lambda_1=\lambda_2=-1, Q=q} = 1 + \sum_{n \geq 1} \frac{q^{n^2}}{(1 + q^2)(1 + q^3) \cdots (1 + q^n + q^{2n})}.
\]

In general, we have

\[
\frac{1}{(1 - q)^2} I_{X_{1,1}}^{St,l=4}(q^2, Q)|_{p=1, \lambda_1=\lambda_2=-1, Q=q} = \sum_{n \geq 0} \frac{q^{2n^2 + 2n}}{(1 - q)(1 - q^3) \cdots (1 - q^{2n+1})}.
\]

\[
\frac{1}{(1 + q + q^2)} I_{X_{1,1}}^{St,l=4}(q^2, Q)|_{p=1, \lambda_1=-\frac{1 + \sqrt{7}}{2} q^{-1}, \lambda_2=-\frac{1 - \sqrt{7}}{2} q^{-1}, Q=1} = \sum_{n \geq 0} \frac{q^{2n^2 + 2n}}{(1 + q + q^3) \cdots (1 + q^{2n+1} + q^{4n+2})}.
\]

The four functions above are some of Ramanujan’s mock theta functions of order 3.

**Proposition 1.4.** Let \( a \) and \( b \) be positive integers, we consider the target \( X_{a, -b} = [(\mathbb{C} \setminus 0) \times \mathbb{C}^*] / \mathbb{C}^* \) with charge matrix \((a, -b)\) and a line bundle \( p = [(\mathbb{C} \setminus 0) \times \mathbb{C} \times \mathbb{C}^*] / \mathbb{C}^* \) with charge matrix \((a, -b, 1)\). Let \( \lambda \) and \( \mu \) be the equivariant parameters of the standard \( \mathbb{C}^* \)-action on \( X_{a, -b} \). For the \( \mathbb{C}^* \)-representation \( St^V \), we have the following explicit formula for the equivariant small \( I \)-function

\[
I_{X_{a, -b}}^{St^V,l}(q) = 1 + \sum_{n \geq 1} (-1)^n p^{l(n-1)-bn} q^{\frac{ln(n-1)-bn(ln-1)}{2}} \mu^{-bn} (1 - p^b \mu)(1 - p^b \mu q) \cdots (1 - p^b \mu q^{bn-1}) Q^n.
\]
In particular, we have order 7 mock theta functions
\[ I_{X_{2,-1}}^{\text{St}^\vee, l=3}(q)|_{p=1, \lambda=1, \mu=q, Q=-q^2} = 1 + \sum_{n \geq 1} \frac{q^{2n}}{(1 - q^{n+1}) \cdots (1 - q^{2n})}, \]
\[ \frac{q}{1 - q} I_{X_{2,-1}}^{\text{St}^\vee, l=3}(q)|_{p=1, \lambda=q^{-1}, \mu=q, Q=-q} = \sum_{n \geq 1} \frac{q^{2n}}{(1 - q^n) \cdots (1 - q^{2n-1})}, \]
\[ \frac{1}{1 - q} I_{X_{2,-1}}^{\text{St}^\vee, l=3}(q)|_{p=1, \lambda=q^{-1}, \mu=q, Q=-q} = \sum_{n \geq 1} \frac{q^{2n-n}}{(1 - q^n) \cdots (1 - q^{2n-1})}. \]

Remark 1.5. The targets that we consider in Proposition 1.3 and Proposition 1.4 are in general orbifolds. The full \( I \)-function has components corresponding to the twisted sectors of its (rigidified) inertia stack. However, we only consider its untwisted sector component.

It is interesting that we can recover Ramanujan’s mock theta functions using only very simple targets.

One of the attractive features of quantum \( K \)-theory is the appearance of \( q \)-hypergeometric series as mirrors of \( K \)-theoretic \( J \)-functions. Recall the definition of the \( q \)-Pochhammer symbol
\[ (a; q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) \quad \text{for } n > 0, \]
and \( (a; q)_0 := 1 \). A general \( q \)-hypergeometric series can be written as
\[ r \phi_s = \sum_{n \geq 0} \frac{(\alpha_1; q)_n \cdots (\alpha_r; q)_n}{(\beta_1; q)_n \cdots (\beta_s; q)_n (q; q)_n} (-1)^n q^{\frac{n(n-1)}{2}} z^n. \]
For the quantum \( K \)-theory of level 0, i.e, Givental-Lee’s quantum \( K \)-theory, we only see special \( q \)-hypergeometric series of the form
\[ \sum_{n \geq 0} \frac{(\alpha_1; q)_n \cdots (\alpha_r; q)_n}{(\beta_1; q)_n \cdots (\beta_s; q)_n (q; q)_n} z^n. \]
The level structure naturally introduces the term
\[ (-1)^n q^{\frac{n(n-1)}{2}} \]

Proposition 1.6. Consider the target \( X_{1,-1} := O(-1)^{s+r}_{\mathbb{P}^1} = \{(\mathbb{C}^s \times 0) \times \mathbb{C}^r \}/\mathbb{C}^* \) with the charge vector \((1, 1, \cdots, 1, -1, -1, \cdots, -1)\). Let \( p = [((\mathbb{C}^s \times 0) \times \mathbb{C}^r \times \mathbb{C})]/\mathbb{C}^* \) be a line bundle with charge matrix \((1, \cdots, 1, -1, \cdots, -1, 1, \cdots, 1, -1)\). Let \( \lambda_1, \cdots, \lambda_s, \mu_1, \cdots, \mu_r \) be the equivariant parameters of the standard \((\mathbb{C}^*)^{s+r}\)-action on \( X_{1,-1} \). Then the equivariant small \( I \)-function has the following explicit form
\[ I_{X_{1,-1}}^{\text{St}^\vee, l=1+s}(q) = 1 + \sum_{n \geq 1} (-1)^n \prod_{i=1}^r (p \mu_i)^{-n p^{(1+s)(n-1)}} \frac{(p \mu_1, q)_n \cdots (p \mu_r; q)_n}{(p \lambda_1^{-1} q; q)_n \cdots (p \lambda_s^{-1} q; q)_n} Q^n (q^{\frac{n(n-1)}{2}})^{1+s-r}. \]
Hence we can recover the general \( q \)-hypergeometric series by setting \( p = 1, \lambda_i^{-1} q = \beta_i, \mu_j = \alpha_j, Q = (-1)^{1+s} \prod_{i=1}^r \mu_i. \)

Recall that Gromov-Witten theory (of Calabi-Yau varieties) is related to quasi-modular forms. Mock modular forms are another class of modular objects which are different from the quasi-modular forms. Yet they share some common properties. The above mirror theorems suggest an exciting possibility that the natural geometric home of mock modular forms is
quantum K-theory with a nontrivial level structure. We certainly would like to investigate this surprising connection further.

1.3. Plan of the paper. In Section 2 and Section 3, we introduce the notion of level in the K-theoretic quasimap theory and gauged linear sigma model (GLSM). Furthermore, we establish its main properties. In Section 4, we prove the mirror theorem for quantum K-theory with level structure.

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2. Level structure

As we mentioned in the introduction, the level structure is defined by determinant line bundles. In this section, we recall the definition of determinant line bundles and define level structure in quasimap theory.

2.1. Determinant line bundles. In this section, we briefly review the construction of determinant line bundles.

Let \( X \) be a Deligne-Mumford stack. Let \( E \) be a locally free, finitely generated \( \mathcal{O}_X \) module. We define the determinant line bundle of \( E \) as

\[
\det(E) := \wedge^{\text{rank}(E)} E,
\]

where \( \wedge^i \) denotes the \( i \)-th wedge product. In general, let \( \mathcal{F}^\bullet \) be a complex of coherent sheaves on \( \mathcal{X} \) which has a bounded locally free resolution, i.e., there exists a bounded complex of locally free, finitely generated \( \mathcal{O}_X \) modules \( \mathcal{G}^\bullet \) and a quasi-isomorphism

\[
\mathcal{G}^\bullet \to \mathcal{F}^\bullet.
\]

We define the determinant line bundle associated to \( \mathcal{F}^\bullet \) as

\[
\det(\mathcal{F}^\bullet) := \bigotimes_n \det(\mathcal{F}^n)^{(-1)^n}.
\]

We summarize some basic properties of this construction in the following proposition.

**Proposition 2.1.** Let \( \mathcal{F} \) be a complex of coherent sheaves which has a bounded locally free resolution. Then

1. The construction of \( \det(\mathcal{F}^\bullet) \) does not depend on the locally free resolution.
2. For every short exact sequence of complexes of sheaves which have bounded locally free resolutions

\[
0 \to \mathcal{F}^\bullet \xrightarrow{\alpha} \mathcal{G}^\bullet \xrightarrow{\beta} \mathcal{H}^\bullet \to 0,
\]
we have a functorial isomorphism

\[
i(\alpha, \beta) : \det(\mathcal{F}^\bullet) \otimes \det(\mathcal{H}^\bullet) \xrightarrow{\sim} \det(\mathcal{G}^\bullet).
\]
3. The operator \( \det \) commutes with base change. To be more precise, for every (representable) morphism of Deligne-Mumford stacks \( g : \mathcal{X} \to \mathcal{Y} \),
we have an isomorphism
\[ \det(Lg^*) \sim g^* \det. \]

In the case when \( \mathcal{X} \) is a scheme, the above proposition is proved in \cite{13}. Note that these properties are preserved under flat base change, therefore they hold for stacks as well.

2.2. Level structure in quasimap theory. In this section, we first recall the quasimap theory for nonsingular GIT quotients introduced in \cite{14}. Then we define level structure in quasimap theory and its generalizations in orbifold quasimap theory and the theory of the gauged linear sigma models (GLSMs).

Let \( Z = \text{Spec}(A) \) be a complex affine algebraic variety in \( \mathbb{C}^n \) and let \( G \) be a reductive group acting on it. Let \( \theta : G \rightarrow \mathbb{C}^* \) be a character determining a \( G \)-equivariant line bundle \( L_\theta := Z \times \mathbb{C} \). Let \( Z^s(\theta) \) and \( Z^{ss}(\theta) \) be the stable and semistable loci, respectively. Throughout the paper we assume \( Z^s(\theta) = Z^{ss}(\theta) \) is nonsingular. Furthermore, we assume that \( G \) acts freely on \( Z^s(\theta) \). It follows that the GIT quotient \( Z//G \) is nonsingular and quasi-projective. For simplicity, we drop \( \theta \) from the notation of the GIT quotient. The unstable locus is defined as \( Z_{us} := Z - Z^s(\theta) \).

Recall that we can identify the \( G \)-equivariant Picard group \( \text{Pic}^G(Z) \) with the Picard group \( \text{Pic}([Z/G]) \) of the quotient stack \([Z/G]\) by sending an \( G \)-equivariant line bundle \( L \) to \([L/G]\). Let \( \beta \in \Hom_Z(\text{Pic}^G(Z), \mathbb{Z}) \).

**Definition 2.2** \cite{14}. A quasimap is a tuple \((C, p_1, \ldots, p_k, P, s)\) where

- \((C, p_1, \ldots, p_k)\) is a connected, at most nodal, \( k \)-pointed projective curve of genus \( g \),
- \( P \) is a principal \( G \)-bundle on \( C \),
- \( s \) is a section of the induced fiber bundle \( P \times_G Z \) on \( C \) such that \((P, s)\) is of class \( \beta \),

i.e., the homomorphism
\[ \text{Pic}^G(Z) \rightarrow \mathbb{Z}, \quad L \mapsto \deg_C(s^*(P \times_G L)), \]
is equal to \( \beta \).

We require that there are only finitely many base points, i.e., points \( p \in C \) such that \( s(p) \in Z_{us} \). An element \( \beta \in \Hom_Z(\text{Pic}^G(Z), \mathbb{Z}) \) is called \( L_\theta \)-effective if it can be represented as a finite sum of classes of quasimaps. We denote by \( E \) the semigroup of \( L_\theta \)-effective classes.

A quasimap \((C, p_1, \ldots, p_k, P, s)\) is called **prestable** if the base points are disjoint from the nodes and marked points on \( C \). Given a rational number \( \epsilon > 0 \), a prestable quasimap is called \( \epsilon \)-**stable** if it satisfies the following conditions

1. \( \omega_{C, \log} \otimes L_\theta^\epsilon \) is ample, where \( \omega_{C, \log} := \omega_C(\sum_{i=1}^k p_i) \) is the twisted dualizing sheaf of \( C \) and
   \[ L_\theta := u^*(P \times_G L_\theta) \cong P \times_G \mathcal{O}_\theta. \]
2. \( \epsilon l(x) \leq 1 \) for every point \( x \) in \( C \) where
   \[ l(x) := \text{length}_x(\text{coker}(u^*J) \rightarrow \mathcal{O}_C). \]

Here \( J \) is the ideal sheaf of the closed subscheme \( P \times_G Z_{us} \) of \( P \times_G Z \).

Ciocan-Fontanine-Kim-Maulik showed in \cite{14} that the moduli space of \( \epsilon \)-stable quasimaps \( Q_{g,k}^\epsilon(Z//G, \beta) = \{(C, p_1, \ldots, p_k, P, s)\} \) is a separated Deligne-Mumford stack of finite type and it is proper over the affine quotient \( Z_{aff//G} := \text{Spec}(A^G) \). When \( Z \) has only local complete intersection singularities, the \( \epsilon \)-stable quasimap space \( Q_{g,k}^\epsilon(Z//G, \beta) \) admits a canonical perfect obstruction theory.
Remark 2.3. There are two extreme chambers for the stability parameter $\epsilon$.

(1) $(\epsilon = \infty)$-stable quasimaps. One can check that when $(g, k) \neq (0, 0)$ and $\epsilon > 1$, the quasimap space $\mathcal{Q}_{g,k}^\epsilon(Z \sslash G, \beta)$ is isomorphic to the moduli space of stable maps $\overline{\mathcal{M}}_{g,k}(Z \sslash G, \beta)$. When $(g, k) = (0, 0)$, the same holds with $\epsilon > 2$. Therefore when $\epsilon$ is sufficiently large, we denote the $\epsilon$-stable quasimap space by $\mathcal{Q}_{g,k}^\infty(Z \sslash G, \beta) = \mathcal{M}_{g,k}(Z \sslash G, \beta)$ and refer to it as the $\epsilon = \infty$ theory.

(2) $(\epsilon = 0^+)$-stable quasimaps. Fix $\beta \in \mathcal{E}$. For each $\epsilon \in (0, \beta \theta_{L_\beta}]$, the $\epsilon$-stability is equivalent to the condition that the underlying curve $C$ of a quasimap does not have rational tails and on each rational bridge, the line bundle $L_{\theta}$ has strictly positive degree. Since we need to consider different $\beta$ at the same time, we reformulate the stability condition as

$$\omega_{C, \log} \otimes L_{\theta}^\epsilon$$

is ample for all $\epsilon \in \mathbb{Q}_{>0}$.

Quasimaps which satisfy the above stability condition are referred to as $(\epsilon = 0^+)$-stable quasimaps.

To define the level structure, we introduce some notation first. Let $\mathcal{M}_{g,k}$ be the algebraic stack of pre-stable nodal curves and $\mathcal{B}_{\text{un}} G$ be the relative moduli stack $\mathcal{B}_{\text{un}} G \rightarrow \mathcal{M}_{g,k}$ of principal $G$-bundles on the fibers of the universal curve $\mathcal{C}_{g,k} \rightarrow \mathcal{M}_{g,k}$. The morphism $\phi$ is smooth. There is a forgetful morphism which forgets the section $s$

$$\mathcal{Q}_{g,k}^\epsilon(Z \sslash G, \beta) \rightarrow \mathcal{B}_{\text{un}} G$$

Let $\tilde{\mathcal{P}} : \mathcal{C}_{\mathcal{B}_{\text{un}} g,k} \rightarrow \mathcal{B}_{\text{un}} g,k$ be the universal curve which is the pullback of $\mathcal{C}_{g,k}$ along $\phi$ and let $\mathcal{P} \rightarrow \mathcal{C}_{\mathcal{B}_{\text{un}} g,k}$ be the universal principal $G$-bundle. We denote by $\pi : \mathcal{C}_{g,k} \rightarrow \mathcal{Q}_{g,k}^\epsilon(Z \sslash G, \beta)$ the universal curve on the quasimap space. Let $\mathcal{P} \rightarrow \mathcal{C}_{g,k}$ be the universal principal bundle which is the pullback of $\tilde{\mathcal{P}} \rightarrow \mathcal{C}_{\mathcal{B}_{\text{un}} g,k}$.

Definition 2.4. Given a finite dimensional representation $R$ of $G$, we define the level $l$ determinant line bundle over $\mathcal{Q}_{g,k}^\epsilon(Z \sslash G, \beta)$ as

$$D_{R,l} := \det^{-l} R \pi_* (\mathcal{P} \times_G R).$$

Here $\det^{-l}(\cdot) := \det(\cdot)^{-l}$ denotes the $l$-th power of the inverse of the determinant line bundle. Alternatively, one can define $D_{R,l}$ to be the pullback via $\mu$ of the determinant line bundle $\det^{-l} R \pi_* (\tilde{\mathcal{P}} \times_G R)$ on $\mathcal{B}_{\text{un}} g,k$.

Note $\mathcal{P} \times_G R$ is the pullback of the vector bundle $[Z \times R/G] \rightarrow [Z/G]$ along the evaluation map to the quotient stack $[Z/G]$.

Remark 2.5. The definition mentioned in the introduction is the second one. It is conceptually better in the sense that it does not depend on the different moduli spaces over $\mathcal{B}_{\text{un}} g,k$. In our case, these moduli spaces are the $\epsilon$-stable quasimap spaces $\mathcal{Q}_{g,k}^\epsilon(Z \sslash G, \beta)$ for different $\epsilon$. However, $\mathcal{B}_{\text{un}} g,k$ is an Artin stack and it is technically more difficult to work with it. Formally, we will use the first definition as the working definition.
The above construction can be easily generalized to orbifold quasimap theory and the theory of GLSMs. In orbifold quasimap theory we allow targets to have orbifold structures. To be more precise, suppose the target can be written as \([Z^e/G]\). In orbifold quasimap theory, we do not assume \(G\) acts freely on the stable locus \(Z^e(\theta)\). Therefore \([Z^e/G]\) is in general a Deligne-Mumford stack. The quasimap theory for such orbifold GIT targets is established in [15]. According to [15], Section 2.4.5, we still have the universal curve and the universal principal \(G\)-bundle over the moduli space of \(\epsilon\)-stable orbifold quasimaps. Therefore we can define the level \(l\) determinant line bundle using (1).

The mathematical theory of the gauged linear sigma model is developed in [16]. We briefly study the pullback and pushforward of the level \(l\) determinant line bundle along some canonical morphisms between the moduli spaces. We work in the stable map setting. The proof

2.3. Properties of the level structure in the quasimap theory. In this section, we study the pullback and pushforward of the level \(l\) determinant line bundle along some canonical morphisms between the moduli spaces. We work in the stable map setting. The proof
for quasimap theory is identical except that “forgetting marked points” is no longer valid. We leave it to the interested reader. To state the properties of the level structure, it is convenient to introduce a combinatorial generalization of \( \overline{M}_{g,k}(X, \beta) \) by incorporating modular graphs \( \tau \) with degree \( \beta \). We briefly recall the definitions here (cf. [21]). Throughout this section we assume \( X = \mathbb{Z}/G \).

A graph \( \tau \) is a quadruple \( (F_\tau, V_\tau, j_\tau, \partial_\tau) \) where \( F_\tau \) and \( V_\tau \) are finite sets of flags and vertices, \( j_\tau : F_\tau \to F_\tau \) is an involution and \( \partial_\tau : F_\tau \to V_\tau \) is a map. Elements in

\[
S_\tau := \{ f \in F_\tau | j_\tau f = f \}
\]

are called marked points and

\[
E_\tau := \{ \{ f_1, f_2 \} \in F_\tau | f_2 = j_\tau f_1, f_1 \neq f_2 \}
\]

is the set of edges. A modular graph is a pair \( (\tau, g) \), where \( \tau \) is a graph and \( g : V_\tau \to \mathbb{Z}_{\geq 0} \) is a map assigning each vertex \( v \) its genus \( g(v) \). A modular graph is stable if

\[
2g(v) + \#F_\tau(v) - 2 > 0
\]

for all vertices \( v \). Here \( F_\tau(v) := \partial_\tau^{-1}(v) \) and \( \#F_\tau(v) \) is the valence of \( v \).

Given a modular graph \( (\tau, g) \), we define the moduli stack of prestable curves of type \( (\tau, g) \) as

\[
\mathcal{M}_{\tau,g} := \prod_{v \in V_\tau} \mathcal{M}_{g(v), \#F_\tau(v)},
\]

where \( \mathcal{M}_{g,n} \) is the usual moduli stack of prestable curves. Similarly, for a stable modular graph \( (\tau, g) \), we define the moduli stack of stable curves \( \overline{M}_{\tau,g} \) by

\[
\overline{M}_{\tau,g} := \prod_{v \in V_\tau} \overline{M}_{g(v), \#F_\tau(v)}.
\]

To describe stable maps, we consider the triple \( (\tau, g, \beta) \), where \( (\tau, g) \) is a modular graph and \( \beta : V_\tau \to H_2(X) \) is the degree map. The triple is called a a stable modular graph with degree \( \beta \) whenever \( g(v) = 0 \) for some vertex \( v \), then \( v \) is a stable vertex, i.e.,

\[
2g(v) + \#F_\tau(v) - 2 > 0
\]

For simplicity, we sometimes use \( \tau \) to denote \( (\tau, g) \) or \( (\tau, g, \beta) \). The moduli stack of stable maps \( \overline{M}(X, \tau, g, \beta) \) is defined by the following three conditions:

1. If \( \tau \) is a graph with only one vertex \( v \), and all flags in \( F_v \) are marked points, then

\[
\overline{M}(X, \tau, g, \beta) := \overline{M}_{g(v), \#F_\tau(v)}(X, \beta)
\]

2. For two stable modular graphs with degree \( \tau_1 \) and \( \tau_2 \), we denote by \( \tau_1 \times \tau_2 \) the disjoint union of two graphs. We define

\[
\overline{M}(X, \tau_1 \times \tau_2) := \overline{M}(X, \tau_1) \times \overline{M}(X, \tau_2)
\]

3. If \( \sigma \) is obtained from \( \tau \) by cutting an edge, then \( \overline{M}(X, \tau) \) is defined by the following cartesian diagram

\[
\begin{array}{ccc}
\overline{M}(X, \tau) & \xrightarrow{M(\phi)} & \overline{M}(X, \sigma) \\
\downarrow & & \downarrow_{ev_i \times ev_j} \\
X & \xrightarrow{\Delta} & X \times X
\end{array}
\]
where $\Delta$ is the diagonal embedding and $ev_i$, $ev_j$ are the evaluation maps corresponding to the marked points where the edge is cut.

Morphisms and operations on $\Mbar_{g,n}(X,\beta)$ can be easily generalized to $\Mbar(X,\tau)$ (see [22] for more details).

Let $\pi : C_\tau \to \Mbar(X,\tau)$ be the universal curve and $P_\tau \to C_\tau$ be the universal principal $G$-bundle. Now we give a list of properties of the level $l$ determinant line bundle (the level structure). To prove these properties, we need to study how the determinant line bundle of cohomology changes with respect to the pushforward and pullback along certain natural morphisms.

2.3.1. Mapping to a point. Assume that $\beta(v) = 0$ for all $v \in V_\tau$. Then the image of any curve is a point and the morphism $\Mbar(X,\tau) \xrightarrow{\text{stab} \times ev} \Mbar_\tau \times X$ is an isomorphism. Here $\text{stab} : \Mbar(X,\tau) \to \Mbar_\tau$ denotes the stabilization morphism of the source curve of the stable map. Let $P$ be the $G$-bundle $Z^* \to X = Z/\!\!/G$.

**Lemma 2.6.** The universal bundle $P_\tau$ over the universal curve $C_{\pi_2^*} = C_\tau \times X$ is equal to $\pi_2^*P$, where $C_{\pi_2}$ is the universal curve over $\Mbar_\tau$ and $\pi_2 : C_\tau \times X \to X$ is the second projection.

**Proof.** In general, there is an evaluation map from the universal curve $C_\tau$ to the quotient stack $[Z/G]$ and $P_\tau$ is the pullback of $P$ along this map. The lemma follows from the observation that the evaluation map is given by the second projection $\pi_2$ in this case. □

**Corollary 2.7.** Let $R := P \times_G R$ be the associated vector bundle on $X$ and let $\pi : C_{\pi_2} \to \Mbar_\tau$ be the canonical morphism. We have

$$D_{R,l}^{\Mbar}(X,\tau \times \sigma) = D_{R,l}^{\Mbar}(X,\sigma) \times D_{R,l}^{\Mbar}(X,\tau).$$

**Proof.** By Lemma 2.6, we have $P_\tau \times_G R = \pi_2^*(R)$. Therefore the pushforward $R \pi_* (\pi_2^*(R))$ is equal to $\mathcal{O}_{C_{\pi_2}} \boxtimes R - R^1 \pi_* \mathcal{O}_{C_{\pi_2}} \boxtimes R$ via projection formula. Note $\text{rk}(R^1 \pi_* \mathcal{O}_{C_{\pi_2}}) = g$. □

2.3.2. Products. Let $\sigma$ and $\tau$ be stable modular graphs. By definition we have

$$\Mbar(X,\tau \times \sigma) = \Mbar(X,\tau) \times \Mbar(X,\sigma).$$

It is easy to check that

$$D_{R,l}^{\Mbar}(X,\tau \times \sigma) = D_{R,l}^{\Mbar}(X,\tau) \boxtimes D_{R,l}^{\Mbar}(X,\sigma).$$

2.3.3. Cutting edges. Let $\sigma$ be a modular graph obtained from $\tau$ by cutting an edge. By definition, we have $M_{\sigma} = M_{\tau} = M$. Let $C' = C(X,\sigma,\beta)$ and $C = C(X,\tau,\beta)$ be the universal curves over $\Mbar(X,\sigma)$ and $\Mbar(X,\tau)$, respectively. Suppose $i,j$ are the markings created by the cut. They give rise to the sections $x_i : \Mbar(X,\sigma) \to C$ for $i = 1,2$. By definition, we have the cartesian diagram

$$\begin{array}{ccc}
\Mbar(X,\tau) & \xrightarrow{M(\Phi)} & \Mbar(X,\sigma) \\
\downarrow & & \downarrow ev_i \times ev_j \\
M \times X & \xrightarrow{\Delta} & M \times X \times X.
\end{array}$$
Proposition 2.8. Let \( x : \overline{M}(X, \tau) \to C \) be the section corresponding to the node. Then we have
\[
(2) \quad \Delta^! \mathcal{D}_{\overline{M}(X, \sigma)}^{R, l} = \mathcal{D}_{\overline{M}(X, \tau)}^{R, l} \otimes \det^{-l}(x^* (\mathcal{P} \times_G R)).
\]

Proof. For simplicity, we denote \( \mathcal{D}_{\overline{M}(X, \sigma)}^{R, l} \) and \( \mathcal{D}_{\overline{M}(X, \tau)}^{R, l} \) by \( \mathcal{D}'_R \) and \( \mathcal{D}_R \), respectively. Note that \( \Delta^!(\mathcal{D}'_R) = \overline{M}(\Phi)^*(\mathcal{D}'_R) \). Let \( \mathcal{P} \) and \( \mathcal{P}' \) be the universal \( G \)-bundles over \( C \) and \( C' \), respectively. Notice that \( C \) is obtained by gluing along two sections \( x_1 \) and \( x_2 \) of \( \overline{M}(\Phi)^* C' \). We have the following commutative diagram
\[
\begin{array}{ccc}
\overline{M}(\Phi)^* C' & \xrightarrow{p} & C \\
\downarrow \pi' & & \downarrow \pi \\
\overline{M}(X, \sigma) & & \overline{M}(X, \tau).
\end{array}
\]

For any locally free sheaf \( F \) on \( C \), we have a short exact sequence (the normalization exact sequence)
\[
0 \to F \to p_* p^* F \to x_* x^* F \to 0.
\]

It induces the following natural isomorphism
\[
(3) \quad \det^{-1}(R\pi_*(p_* p^* F)) \cong \det^{-1}(R\pi_*(F)) \otimes \det^{-1}(R\pi_*(x_* x^* F)).
\]

Notice that
\[
\det^{-1}(R\pi_*(p_* p^* F)) = \det^{-1}(R\pi_*(p^* F)) \quad \text{and} \quad \det^{-1}(R\pi_*(x_* x^* F)) = \det^{-1}(x^* F).
\]

Now take \( F = \mathcal{P} \times_G R \) and \( F' = \mathcal{P}' \times_G R \). The lemma follows from the fact that \( p^* F \cong \overline{M}(\Phi)^* F' \) and equation (3). \( \square \)

2.3.4. Forgetting marked points.

Proposition 2.9. Let \( \tau \) be obtained from \( \sigma \) by forgetting a tail and let \( \overline{M}(\Phi) : \overline{M}(X, \sigma) \to \overline{M}(X, \tau) \) be the forgetful morphism. Then
\[
\overline{M}(\Phi)^* \mathcal{D}_{\overline{M}(X, \tau)}^{R, l} = \mathcal{D}_{\overline{M}(X, \sigma)}^{R, l}.
\]

Proof. Consider the natural morphism \( p : \mathcal{C}(X, \sigma) \to \overline{M}(\Phi)^* \mathcal{C}(X, \tau) \) which fits into the following commutative diagram
\[
\begin{array}{ccc}
\mathcal{C}(X, \sigma) & \xrightarrow{p} & \overline{M}(\Phi)^* \mathcal{C}(X, \tau) \\
\downarrow \pi & & \downarrow \pi' \\
\overline{M}(X, \sigma) & & \overline{M}(X, \tau).
\end{array}
\]

For any locally free sheaf \( F \) on \( \overline{M}(\Phi)^* \mathcal{C}(X, \tau) \), we have a canonical isomorphism \( F \to p_* p^* F \). This follows from the fact that \( p \) is the stabilization morphism and the components it contracts in the fiber curves are rational. Let \( \mathcal{P} \) and \( \mathcal{P}' \) be the universal principal bundle over \( \mathcal{C}(X, \sigma) \) and \( \mathcal{C}(X, \tau) \), respectively. By taking \( F = \overline{M}(\Phi)^* (\mathcal{P}' \times_G R) \), we have an isomorphism
\[
\overline{M}(\Phi)^* (\mathcal{P}' \times_G R) \cong p_* p^* \overline{M}(\Phi)^* (\mathcal{P}' \times_G R).
\]
After applying the functor $\det^{-1}R\pi'_s$ to the above isomorphism and simplifying both sides, we get an isomorphism

$$M(\Phi)^*([\det^{-1}R\pi''_{s}(\mathcal{P}' \times_G R)]) \tilde{\rightarrow} \det^{-1}R\pi_s(*_{\mathcal{M}(\Phi)}(\mathcal{P}' \times_G R)), $$

where $\pi'' : \mathcal{C}(X, \tau) \to \mathcal{M}(X, \tau)$ is the universal curve. Note that

$$p^*\mathcal{M}(\Phi)^*(\mathcal{P} \times_G R) = \mathcal{P} \times_G R.$$ 

This is because the morphism $\mathcal{M}(\Phi) \circ p : \mathcal{C}(X, \sigma) \to \mathcal{C}(X, \tau)$ contracts genus-zero unstable components of the source curves and $\mathcal{P}$ is trivial over those components. The lemma follows from the isomorphism (4). □

Let $\sigma$ be obtained from $\tau$ by forgetting a tail. We can also consider the commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}(X, \tau) & \xrightarrow{\Phi} & \mathcal{M}(X, \sigma) \\
\downarrow & & \downarrow \\
\mathcal{M}_\tau & \xrightarrow{\Phi} & \mathcal{M}_\sigma \\
\end{array}
$$

It induces a morphism

$$\Psi : \mathcal{M}(X, \tau) \to \mathcal{M}_\tau \times_{\mathcal{M}_\sigma} \mathcal{M}(X, \sigma).$$

**Proposition 2.10.** We have

$$\Psi_*(\mathcal{D}_{\mathcal{M}(X, \tau)}) = \Phi^! \mathcal{D}_{\mathcal{M}(X, \sigma)}. $$

**Proof.** The proposition follows essentially from Proposition 2.9 and the argument in [2, Prop. 9]. □

2.3.5. **Isomorphisms.** Suppose that $\sigma$ is isomorphic to $\tau$ and let $\Psi : \mathcal{M}(X, \tau) \to \mathcal{M}(X, \sigma)$ be the induced isomorphism. Then it is clear that we have

$$\Psi^*(\mathcal{D}_{\mathcal{M}(X, \sigma)}) = \mathcal{D}_{\mathcal{M}(X, \tau)} \quad \text{and} \quad \Psi_*(\mathcal{D}_{\mathcal{M}(X, \tau)}) = \mathcal{D}_{\mathcal{M}(X, \sigma)}. $$

2.3.6. **Contractions.** Let $\phi : \tau \to \sigma$ be a contraction of stable modular graphs, contracting one edge or one loop $e$ in $\tau$. Let $(\sigma', \beta)$ be a stable modular graph with degree where $\sigma'$ is a fixed modular graph obtained from $\sigma$ by adding $k$ marked points.

Consider the stable modular graphs with degrees $(\tau_i^i, \beta_i^i)$ such that $\tau_i^i$ are obtained from $\tau$ by adding $k$ tails in ways compatible with $\sigma' \to \sigma$ and $\beta_i^i$ are the degrees on $\tau_i^i$ compatible with $\beta$. We have the following commutative diagram

$$
\begin{array}{ccc}
\tau_i^i & \xrightarrow{\phi} & \sigma' \\
\downarrow & & \downarrow \\
\tau & \xrightarrow{\phi} & \sigma \\
\end{array}
$$

where the two horizontal arrows contract $e$ and vertical arrows forget the $k$ new marked points.

Similarly, we consider the stable modular graphs with degrees $(\tau_m^i, \beta_m^i)$ which are obtained from $\tau$ by first replacing $e$ with a chain $c_m$ of $m$ edges and $m - 1$ vertices of genus zero.
Then we add $k$ marked points to this new graph in ways compatible with $\sigma' \to \sigma$. We have the commutative diagrams

$$
\begin{array}{cccc}
\tau_m & \xrightarrow{p_m} & \sigma' \\
\downarrow{a_m} & & \downarrow{} \\
\tau & \xrightarrow{m} & \sigma
\end{array}
$$

and

$$
\begin{array}{cccc}
\overline{\mathcal{M}}(X, \tau_m^i, \beta_m^j) & \xrightarrow{\phi_m} & \overline{\mathcal{M}}(X, \sigma', \beta) \\
\downarrow{a_m} & & \downarrow{t} \\
\overline{\mathcal{M}}_{\tau} & \xrightarrow{\phi} & \overline{\mathcal{M}}_{\sigma}
\end{array}
$$

The above commutative diagram induces a morphism

$$
\Psi_m : \bigsqcup \overline{\mathcal{M}}(X, \tau_m^i, \beta_m^j) \to \overline{\mathcal{M}}_{\tau} \times_{\overline{\mathcal{M}}_{\sigma}} \overline{\mathcal{M}}(X, \sigma', \beta).
$$

**Proposition 2.11.** We have

$$
\sum_m (-1)^{m+1} \Psi_m \sum_{i,j} \mathcal{D}_{\mathcal{P}(X, \tau_m^i, \beta_m^j)}^{R,l} = \Phi^*(\mathcal{D}_{\mathcal{P}(X, \sigma', \beta)}^{R,l}).
$$

**Proof.** Consider the cartesian diagram

$$
\begin{array}{cccc}
\mathcal{P}_{\tau_m^i, \beta_m^j} & \xrightarrow{\phi_m} & \mathcal{P}_{\sigma', \beta} \\
\downarrow & & \downarrow \\
\mathcal{C}_{\tau_m^i, \beta_m^j} & \xrightarrow{\phi_m} & \mathcal{C}_{\sigma', \beta} \\
\downarrow & & \downarrow \\
\overline{\mathcal{M}}(X, \tau_m^i, \beta_m^j) & \xrightarrow{\phi_m} & \overline{\mathcal{M}}(X, \sigma', \beta) \\
\downarrow{w} & & \downarrow{t} \\
\overline{\mathcal{M}}_{\tau_m^i, \beta_m^j} & \xrightarrow{\phi_m} & \overline{\mathcal{M}}_{\sigma'}
\end{array}
$$

where $\mathcal{C}$ denotes the universal curve and $\mathcal{P}$ denote the universal principal $G$-bundle over the corresponding moduli space. Let $\mathcal{D}_{\mathcal{P}}^{\tau,\beta}$ and $\mathcal{D}_{\mathcal{P}_{\tau_m^i, \beta_m^j}}^{\sigma', \beta}$ be the inverse determinant line bundle of cohomology over $\overline{\mathcal{M}}(X, \tau_m^i, \beta_m^j)$ and $\overline{\mathcal{M}}(X, \sigma', \beta)$, respectively. It follows from the above cartesian diagram that

$$
(5) \quad w^* \mathcal{D}_{\mathcal{P}}^{\tau, \beta} = \phi_m^* \mathcal{D}_{\mathcal{P}_{\tau_m^i, \beta_m^j}}^{\sigma', \beta} = \mathcal{D}_{\mathcal{P}_{\tau_m^i, \beta_m^j}}^{\tau_m^i, \beta_m^j}.
$$
We consider the following diagram:

\[
\xymatrix{
\bigsqcup \overline{\mathcal{M}}(X, \tau_{m_i}, \beta_{m_i}) \ar[r]^{\psi_m} \ar[d] & \overline{\mathcal{M}}(X, \sigma', \beta) \ar[d] \\
\bigsqcup \overline{\mathcal{M}}_{\tau_{m_i}} \ar[r]^{\psi_m} \ar[d] & \overline{\mathcal{M}}_{\sigma'} \ar[d] \\
\overline{\mathcal{M}}_{\tau} \ar[r]^{\Phi} & \overline{\mathcal{M}}_{\sigma'}
}
\]

According to [2, Prop. 11], we have

\[
\sum_{m,i} (-1)^{m+1} \psi_{m,*} O_{\mathcal{M}_{\tau_{m_i}}} = \Phi^{*}(O_{\mathcal{M}_{\tau_{m}}}).
\]

Now we calculate the Gysin pullback of level \(l\) determinant line bundle as follows

\[
\Phi^{*}(D_{R,l}^{\mathcal{M}(X,\sigma',\beta)}) = a^{*} s^{*} \Phi^{*}(D_{l}^{\mathcal{M}(X,\sigma',\beta)}) = \sum_{m} (-1)^{m+1} \psi_{m,*} (m \circ \psi_{m})^{*} D_{l}^{\mathcal{M}(X,\sigma',\beta)}
\]

The last equation follows from (5). \(\square\)

3. THE \(K\)-THEORETIC QUASIMAP THEORY WITH LEVEL STRUCTURE

In this section, we first define the \(K\)-theoretic quasimap invariants with level structure and their permutation-equivariant version. For most of the discussion, we assume the GIT quotient \(Z//G\) is nonsingular and projective. The cases when the target \([Z^*//G]\) is noncompact or an orbifold are discussed at the end of this section.

3.1. \(K\)-THEORETIC QUASIMAP INVARIANTS WITH LEVEL STRUCTURE. In this section, we first briefly recall Givental-Lee’s quantum \(K\)-theory. Then we define \(K\)-theoretic quasimap invariants with level structure.

The quantum \(K\)-theory or \(K\)-theoretic Gromov-Witten theory was introduced by Givental-Lee [1, 2]. Let \(X\) be a smooth projective variety and let \(\overline{\mathcal{M}}_{g,k}(X, \beta)\) be the moduli space of stable maps to \(X\). The moduli space is known to be a proper Deligne-Mumford stack (see for example [21]). In particular, for any coherent sheaf \(\mathcal{E}\) on \(\overline{\mathcal{M}}_{g,k}(X, \beta)\), we can consider its \(K\)-theoretic pushforward to the point \(\text{Spec} (\mathbb{C})\), i.e., we can take its Euler characteristic

\[
\chi(\mathcal{E}) = \sum_{i} (-1)^{i} h^{i}(\mathcal{E}),
\]

where \(h^{i}(\mathcal{E}) := \dim_{\mathbb{C}} H^{i}(\overline{\mathcal{M}}_{g,k}(X, \beta), \mathcal{E})\).

From the perfect obstruction theory, Lee [2] constructs a virtual structure sheaf \(O_{\mathcal{M}}^{\text{vir}} \in K_{0}(\overline{\mathcal{M}}_{g,k}(X, \beta))\), where \(K_{0}(\overline{\mathcal{M}}_{g,k}(X, \beta))\) denotes the Grothendieck group of coherent sheaves on \(\overline{\mathcal{M}}_{g,k}(X, \beta)\). The virtual structure sheaf \(O_{\mathcal{M}}^{\text{vir}}\) has the following properties:
(1) If the obstruction sheaf is trivial and hence $\overline{\mathcal{M}}_{g,k}(X, \beta)$ is smooth, then $\mathcal{O}^\text{vir}$ is the structure sheaf of $\overline{\mathcal{M}}_{g,k}(X, \beta)$.

(2) If the obstruction sheaf $\text{Obs}$ is locally free, then $\mathcal{O}^\text{vir} = \sum (-1)^i \wedge^i \text{Obs}$. Here $\wedge^i \text{Obs}$ denotes the $i$-th wedge product of the dual of the obstruction bundle.

Since we assume $X$ to be smooth, the Grothendieck group of locally free sheaves on $X$, denoted by $K^0(X)$, is isomorphic to the Grothendieck group of coherent sheaves $K_0(X)$. We refer the reader to [2] for more details.

We define the quantum $K$-potential of genus 0 as

$$F(t, Q) := \frac{1}{2} \langle t, t \rangle + \sum_{k=0}^{\infty} \sum_{\beta \in E} \frac{Q^k}{k!} \langle t, \ldots, t \rangle_{0,k,\beta},$$

where $t \in K(X)_Q := K(X) \otimes \mathbb{Q}$ and $(t, t) := \chi(t \otimes t)$ is the $K$-theoretic Poincaré pairing. Let $\phi_0 = \mathcal{O}_X, \phi_1, \phi_2, \ldots$ be a basis of $K(X)_Q$. One can define the “quantized” pairing on $K(X)_Q$ by

$$((\phi_i, \phi_j)) := F_{ij} = \partial_{i\bar{j}} F(t, Q).$$

It was shown in [2] that quantum $K$-theory satisfies all the usual axioms of cohomological Gromov-Witten theory except the flat identity axiom. We refer the reader to [2] for more details.

In the following discussion we assume $X$ can be represented as a GIT quotient $Z / \! / G$. As mentioned before, the moduli space of stable maps $\overline{\mathcal{M}}_{g,k}(Z / \! / G, \beta)$ can be identified with $Q^\epsilon_{g,k}(Z / \! / G, \beta)$ for large $\epsilon$. According to [14], for general $\epsilon$, the $\epsilon$-stable quasimap space $Q^\epsilon_{g,k}(Z / \! / G, \beta)$ is proper and admits a two-term perfect obstruction theory, assuming $Z$ has only l.c.i singularities. Hence by the result in [2], one can construct a virtual structure sheaf $\mathcal{O}^\text{vir}$ on $Q^\epsilon_{g,k}(Z / \! / G, \beta)$.

**Definition 3.1.** The $K$-theoretic quasimap invariants of level $l$ are defined by

$$\langle E_1 L_{1}^{l_1}, \ldots, E_k L_{k}^{l_k} \rangle^{Z / \! / G, R, l, \epsilon}_{g,k,\beta} = \chi(Q^\epsilon_{g,k}(Z / \! / G, \beta), \prod \text{ev}_i^* E_i \otimes L_i^l \otimes \mathcal{O}^\text{vir} \otimes \mathcal{D}^{R,l}) \in \mathbb{Z},$$

where $E_i$ is in $K(Z / \! / G)$. We shall usually suppress $Z / \! / G$ from the notation if no confusion can arise. Notice that these invariants are all integers.

**Remark 3.2.** In the GLSM, if all the insertions are of compact type, we have a cosection localized virtual structure sheaf on the moduli space of LG quasimaps. Therefore we can define the $K$-theoretic invariants of level $l$ for insertions of compact type. However if we consider broad insertions, the full ordinary $K$-theoretic GLSM requires the setting of matrix factorization and it has not been developed yet. Once we have a definition of the virtual structure sheaf (or matrix factorization) in the theory of the GLSM, it is immediate to define the $K$-theoretic invariants with level structure.
3.2. Quasimap graph space and $J^{R,l,e}$-function. In this section, we first recall the definition and properties of the $\epsilon$-stable quasimap graph space. Then we define an important generating series $J^{R,l,e}$ of the $K$-theoretical quasimap invariants of level $l$.

Given a rational number $\epsilon > 0$, the quasimap graph space, denoted by $\mathcal{QG}_{g,k}^\epsilon (Z \parallel G, \beta)$, is introduced in [14]. It is the moduli space of the tuples $((C, x_1, \ldots, x_k), P, u, \varphi)$, where $((C, x_1, \ldots, x_k), P, u)$ is a prestable quasimap, satisfying $e\ell(x) < 1$ for every point $x$ on $C$, and the new data $\varphi$ is a degree 1 morphism from $C$ to $\mathbb{P}^1$. The curve $C$ has a unique rational component $C_0$ such that $\varphi|_{C_0} : C_0 \to \mathbb{P}^1$ is an isomorphism and the complement $C/C_0$ is contracted by $\varphi$. The ampleness condition imposed on the tuples is modified to:

$$\omega_{C\setminus C_0}(\sum x_i + \sum y_j) \otimes L_0$$

is ample,

where $x_i$ are marked points on $C \setminus C_0$ and $y_i$ are the nodes $C \setminus C_0 \setminus C_0$. It is shown in [14] that the quasimap graph space is also a separated Deligne-Mumford stack which is proper over the affine quotient. Moreover, when $Z$ has only lci singularities, the canonical obstruction theory on the graph space is perfect. Similarly, we can define the level $l$ determinant line bundle $\mathcal{D}_{QG}$ on $\mathcal{QG}_{g,k}^\epsilon (Z \parallel G, \beta)$ using the universal principal $G$-bundles over its universal curve.

There is a natural $\mathbb{C}^*$-action on the graph spaces. Let $[x_0, x_1]$ be homogeneous coordinates on $\mathbb{P}^1$ and set $0 := [1, 0]$ and $\infty := [0, 1]$. We consider the standard $\mathbb{C}^*$-action on $\mathbb{P}^1$:

$$t \cdot [x_0, x_1] = [tx_0, x_1], \quad \forall t \in \mathbb{C}^*.$$ 

It induces an action on the $\epsilon$-stable quasimap graph space $\mathcal{QG}_{g,k}^\epsilon (Z \parallel G, \beta)$ by rescaling the parametrized rational component. According to [23] §4.1, the $\mathbb{C}^*$-fixed locus can be described as

$$(\mathcal{QG}_{g,k}^\epsilon (Z \parallel G, \beta))^\mathbb{C}^* = \bigsqcup_{\beta_1} \mathcal{F}_{g_1,k_1,\beta_1}^{g_1,k_1,\beta_1},$$

where the disjoint union is over all possible splittings

$$g = g_1 + g_2, \quad k = k_1 + k_2, \quad \beta = \beta_1 + \beta_2,$$

with $g_i, k_i \geq 0$ and $\beta_i$ effective. In the stable cases, an $\epsilon$-stable parametrized quasimap $((C, x_1, \ldots, x_k), P, u, \varphi) \in \mathcal{F}_{g_1,k_1,\beta_1}^{g_1,k_1,\beta_1}$ is obtained by gluing two $\epsilon$-stable quasimaps of types $(g_1, k_1, \beta_1)$ and $(g_2, k_2, \beta_2)$ to a constant map $\mathbb{P}^1 \to p \in Z \parallel G$ at 0 and $\infty$, respectively. Therefore, the component $\mathcal{F}_{g_1,k_1,\beta_1}^{g_1,k_1,\beta_1}$ is isomorphic to the fiber product

$$\mathcal{Q}_{g_1,k_1,\bullet}^\epsilon (Z \parallel G, \beta_1) \times_{Z \parallel G} \mathcal{Q}_{g_2,k_2,\bullet}^\epsilon (Z \parallel G, \beta_2)$$

over the evaluation maps at the special marked points $\bullet$. When one of the components at 0 or $\infty$ is unstable, we use the following conventions.

1. For the unstable cases $(g_1, k_1, \beta_1) = (0, 0, 0)$ or $(1, 0, 0)$ (and likewise for $(g_2, k_2, \beta_2)$), we define

$$\mathcal{Q}_{0,0,\bullet}^\epsilon (Z \parallel G, 0) := Z \parallel G, \quad \mathcal{Q}_{0,1,\bullet}^\epsilon (Z \parallel G, 0) := Z \parallel G, \quad \text{ev}_\bullet = \text{Id}_{Z \parallel G}.$$ 

2. For the unstable cases $(g_1, k_1, \beta_1) = (0, 0, \beta_1)$ with $\beta_1 \neq 0$ and $\epsilon \leq \frac{1}{\beta_1(L_0)}$, we denote by

$$\mathcal{Q}_{0,0,\bullet}^\epsilon (Z \parallel G, \beta_1)$$
the moduli space of quasimaps \((C = \mathbb{P}^1, P, u)\) such that \(u(x) \in P \times_G Z^s\) for \(x \neq 0 \in \mathbb{P}^1\) and \(0 \in \mathbb{P}^1\) is a base point of length \(\beta_1(L_0)\). Similarly, we define \(\mathcal{Q}_{0,0+}(Z \parallel G, \beta_2)_\infty\) to be the moduli space of quasimaps whose only base point is of length \(\beta_2(L_0)\) and located at \(\infty\). Using these definitions we have

\[
F_{g,k,\beta_1}^{0,0,\beta_2} \simeq \mathcal{Q}_{g,k+\bullet}(Z \parallel G, \beta_1) \times_{Z/G} \mathcal{Q}_{0,0+}(Z \parallel G, \beta_2)_\infty
\]

for \(k \geq 1\) and \(\epsilon \leq \frac{1}{\beta_2(L_0)}\). Similarly we have

\[
F_{g,k,\beta_2}^{0,0,\beta_1} \simeq \mathcal{Q}_{0,0+}(Z \parallel G, \beta_1) \times_{Z/G} \mathcal{Q}_{g,k+\bullet}(Z \parallel G, \beta_2)_\infty
\]

for \(k \geq 1\) and \(\epsilon \leq \frac{1}{\beta_1(L_0)}\). When \(g = k = 0\) and \(\epsilon \leq \min\{\frac{1}{\beta_1(L_0)}, \frac{1}{\beta_2(L_0)}\}\), we have

\[
F_{0,0,\beta_1}^{0,0,\beta_2} \simeq \mathcal{Q}_{0,0+}(Z \parallel G, \beta_1) \times_{Z/G} \mathcal{Q}_{0,0+}(Z \parallel G, \beta_2)_\infty.
\]

We denote by \(\mathcal{R}\) the vector bundle \(Z \times_G R \to [Z/G]\) and its restriction to \(Z \parallel G\). We define the twisted pairing on \(K(Z \parallel G)_Q\) by

\[
(u, v)^{R,l} := \chi(u \otimes v \otimes \det^{-l}(\mathcal{R})) \quad \text{where} \quad u, v \in K(Z \parallel G)_Q.
\]

Let \(\{\phi_a\}\) be a basis of \(K(Z \parallel G)_Q\) and let \(\{\phi_a^*\}\) be the dual basis with respect to the above twisted pairing \((\cdot, \cdot)^{R,l}\). Let \(t = \sum_i t^i \phi_i \in K(Z \parallel G)_Q\). We define the \(\mathcal{J}^{R,l,\epsilon}\)-function of level \(l\) to be

\[
\mathcal{J}^{R,l,\epsilon}(t, Q) = 1 + \frac{t}{1 - q} + \sum_{a, (k, \beta) \neq (0,0),(1,0)} Q^\beta \frac{k!}{(1 - q)(1 - qL)} \phi_a^* \phi_a \langle t, \ldots, t \rangle^{R,l,\epsilon}_{0,k+1,\beta}.
\]

In the summation above, the quasimap moduli spaces are empty when \(k = 0, \beta \neq 0, \beta(L_0) \leq 1/\epsilon\) and the unstable terms are defined by \(C^*\)-localization on the graph space \(\mathcal{Q}_G^\epsilon(0,0)(Z \parallel G, \beta)\). To be more precise, we consider the fixed point locus \(F_{0,\beta} := \mathcal{Q}_{0,0+}(Z \parallel G, \beta)_0\) of the \(C^*\)-action. The unstable terms in (9) are defined to be

\[
\sum_{a, \beta \neq 0, \beta(L_0) \leq 1/\epsilon} Q^\beta \chi \left(F_{0,\beta}, \mathcal{O}^\text{vir}_{F_{0,\beta}} \otimes \text{ev}^* (\phi_a) \otimes \left(\frac{\text{tr}_{C^*} D^{R,l}_{C^*}}{\text{tr}_{C^*}} \wedge^* N_{F_{0,\beta}}^\vee\right)\right) \phi_a^*.
\]

where \(N_{F_{0,\beta}}\) is the virtual normal bundle of the fixed locus \(F_{0,\beta}\) in \(\mathcal{Q}_G^\epsilon(0,0)(Z \parallel G, \beta)\) and \(\wedge^* N_{F_{0,\beta}}^\vee := \sum_i (-1)^i \wedge^i N_{F_{0,\beta}}^\vee\) is the K-theoretic Euler class of \(N_{F_{0,\beta}}\). Here the trace of a \(C^*\)-equivariant bundle \(V\) when restricted to the fixed point locus is a virtual bundle defined by the eigenspace decomposition with respect to the \(C^*\)-action, i.e., we have

\[
\text{tr}_{C^*}(V) := \sum_i q^i V(i),
\]

where \(t \in C^*\) acts on \(V(i)\) as multiplication by \(t^i\).

For \(1 < \epsilon \leq \infty\), i.e., the \((\epsilon = \infty)\)-theory, (9) defines the \(\mathcal{J}\)-function in the quantum \(K\)-theory of level \(l\). In this case, we use \(\langle \cdot \rangle^{R,l,\infty}\) or simply \(\langle \cdot \rangle^{R,l}\) to denote the quantum \(K\)-invariants of level \(l\). Following Givental-Tonita [24], we introduce the symplectic loop space formalism. Recall that \(E\) denotes the semigroup ring of the \(L_0\)-effective curve classes on \(Z \parallel G\). The Novikov ring \(\mathbb{C}[[Q]]\) is defined as

\[
\mathbb{C}[[Q]] := \{\sum_{\beta \in E} c_\beta Q^\beta | c_\beta \in \mathbb{C}\}.
\]
Here the completion is taken with respect to the \( m \)-adic topology where \( m \) is the maximal ideal generated by nonzero elements of \( E \). We define the loop space as
\[
\mathcal{K} := [K(Z \sslash G) \otimes \mathbb{C}(q)] \otimes \mathbb{C}[[Q]],
\]
where \( \mathbb{C}(q) \) is the field of complex rational functions in \( q \). By viewing the elements in \( \mathbb{C}(q) \otimes \mathbb{C}[[Q]] \) as the coefficients, we extend the twisted pairing \( (\cdot, \cdot)^{R,l} \) to \( \mathcal{K} \) via linearity. There is a natural symplectic form \( \Omega \) on \( \mathcal{K} \) defined by
\[
\Omega(f, g) := [\text{Res}_{q=0} + \text{Res}_{q=\infty}](f(q), g(q^{-1}))^{R,l} \frac{dq}{q}, \quad \text{where } f, q \in \mathcal{K}.
\]

With respect to \( \Omega \), there is a Lagrangian polarization \( \mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_- \) where
\[
\mathcal{K}_+ = [K(Z \sslash G) \otimes \mathbb{C}[q, q^{-1}]] \otimes \mathbb{C}[[Q]] \quad \text{and} \quad \mathcal{K}_- = \{ f \in \mathcal{K} | f(0) \neq \infty, f(\infty) = 0 \}.
\]

As before, let \( \{ \phi_a \} \) be a basis of \( K(Z \sslash G)_Q \) and let \( \{ \phi^a \} \) be the dual basis with respect to the twisted pairing \( (\cdot, \cdot)^{R,l} \). Let \( t(q) = \sum_{i,j} t_{ij}^a \phi_i q^j \in \mathcal{K}_+ \). We define the big \( J \)-function of level \( l \) to be the function \( J^{R,l}(t(q), Q) : \mathcal{K}_+ \to \mathcal{K} \) given by
\[
J^{R,l}(t(q), Q) = 1 + \frac{t(q)}{1-q} + \sum_a \sum_{(k, \beta) \neq (0, 0), (1, 0)} \frac{Q^\beta}{k!} \phi_a \langle \frac{\phi_a}{(1-q)(1-qL)}, t(L), \ldots, t(L) \rangle_{0,k+1, \beta}^{R,l,\infty}.
\]

We define the genus-0 \( K \)-theoretic descendant potential of level \( l \) as
\[
F^{R,l}(t, Q) := \sum_{k,\beta} \frac{Q^\beta}{k!} \langle t(L), \ldots, t(L) \rangle_{0,k, \beta}^{R,l,\infty}.
\]

We identify the cotangent bundle \( T^*\mathcal{K}_+ \) with the symplectic loop space \( \mathcal{K} \) via the Lagrangian polarization and the dilaton shift \( f \to f + (1-q) \). Then \( (1-q)J^{R,l} \) coincides with the differential of the descendant potential up to the dilaton shift, i.e., we have
\[
(1-q)J^{R,l} = 1 + t(q) + dt J^{R,l}(t, Q).
\]

In the case when \( l = 0 \), the above fact is proved in [24, §2]. The same argument works for arbitrary \( l \).

For \( (\epsilon = 0+) \)-stable quasimap theory, the definition [5] gives the \( I \)-function of level \( l \) of \( Z \sslash G \):
\[
I^{R,l}(t, Q) := J^{R,l,0+}(t, Q) = 1 + \frac{t}{1-q} + \sum_a \sum_{\beta \neq 0} Q^\beta \chi_{\mathbb{C}^*} \left( F_{0,\beta}, \mathcal{O}_{F_{0,\beta}}^{\text{vir}} \otimes \mathcal{O}_0^{\text{vir}}(\phi_a) \otimes \left( \frac{\text{tr}_{\mathcal{C}_1} D^{R,l}_{\mathcal{C}_1}}{\text{tr}_{\mathcal{C}_1} \Lambda^* N^{\mathcal{C}_1}_{F_{0,\beta}}} \right) \right) \phi_a^a
\]
\[
+ \sum_a \sum_{k \geq 0, (k, \beta) \neq (1, 0)} \frac{Q^\beta}{k!} \phi_a \langle \frac{\phi_a}{(1-q)(1-qL)}, t, \ldots, t \rangle_{0,k+1, \beta}^{R,l,\epsilon=0+},
\]
where \( t \in K(Z \sslash G)_Q \).

3.3. The permutation-equivariant quasimap \( K \)-theory with level structure. Getzler [3] introduced the permutation-equivariant quantum \( K \)-theory, which takes into account the \( S_n \)-action on the moduli spaces of stable maps by permuting the marked points. The definition can be easily generalized to incorporate the level structure.

Let \( \Lambda \) be a \( \lambda \)-algebra, i.e. an algebra over \( \mathbb{Q} \) equipped with abstract Adams operations \( \Psi^k, k = 1, 2, \ldots \). Here \( \Psi^k : \Lambda \to \Lambda \) are ring homomorphisms which satisfy \( \Psi^r \Psi^s = \Psi^{rs} \) and \( \Psi^1 = \text{id} \). We assume that \( \Lambda \) includes the Novikov variables, the algebra of symmetric polynomials in a given number of variables and the torus equivariant \( K \)-ring of a point. We
also assume that $\Lambda$ has a maximal ideal $\Lambda_+$ and is equipped with $\Lambda_+$-adic topology. For example, we can choose

$$\Lambda = \mathbb{Q}[[N_1, N_2, \ldots]][[Q]][\Lambda_0^\pm, \ldots, \Lambda_N^\pm],$$

where $N_i$ are the Newton polynomials (in infinitely or finitely many variables) and $Q$ denotes the Novikov variable(s). The parameters $\Lambda_i$ denote the torus-equivariant parameters. The Adams operations $\Psi^r$ act on $N_m$ and $Q$ by $\Psi^r(N_m) = N_{rm}$ and $\Psi^r(Q^\beta) = Q^{r\beta}$, respectively. Their actions on the torus-equivariant parameters are trivial.

Similar to the “ordinary” quasimap $K$-theory with level structure, we define the loop space as

$$\mathcal{K} := [K(Z \sslash G) \otimes \Lambda] \otimes \mathbb{C}(q).$$

As before, it is equipped with a symplectic form defined by (10) and it has a Lagrangian polarization

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-,$$

where $\mathcal{K}_+$ is the subspace of Laurent polynomials in $q$ and $\mathcal{K}_-$ is the subspace of rational functions which are regular at $q = 0$ and vanish at $q = \infty$.

Consider the natural $S_k$-action on the quasimap moduli space $Q_{g,k}^e(Z \sslash G, \beta)$ by permuting the $k$ marked points. Notice that the virtual structure sheaf $O_{Q_{g,k}^e(Z \sslash G, \beta)}$ and the determinant line bundle $\mathcal{D}^{R,l}_i$ are invariant under this action. Therefore we have the following $S_k$-module

$$[t(L), \ldots, t(L)]_{g,k,\beta} := \sum_m (-1)^m H^m(O_{Q_{g,k}^e(Z \sslash G, \beta)}^\text{vir} \otimes \mathcal{D}^{R,l}_i \otimes_{i=1}^k t(L_i)),$$

for $t(q) \in \mathcal{K}_+$.

**Definition 3.3.** The correlators of the permutation-equivariant quasimap $K$-theory of level $l$ are defined as

$$\langle t(L), \ldots, t(L) \rangle_{g,k,\beta}^{R,l,e,S_k} := \pi_* (O_{Q_{g,k}^e(Z \sslash G, \beta)}^\text{vir} \otimes \mathcal{D}^{R,l}_i \otimes_{i=1}^k t(L_i)),$$

where $\pi_*$ is the $K$-theoretic pushforward along the projection

$$\pi : [Q_{g,k}^e(Z \sslash G, \beta)/S_k] \to [pt].$$

For the permutation-equivariant quasimap $K$-theory, we also consider the $J^e$-function and define the cone $\mathcal{L}_{S_k}$ to be the range of the $J^\infty$-function.

**Definition 3.4.** The permutation-equivariant $K$-theoretic $J^e$-function of $Z \sslash G$ of level $l$ is defined as

$$(12) J_{S_k}^{R,l,e}(t(q), Q) := 1 + \frac{t(q)}{1 - q} + \sum_a \sum_{(k,\beta) \neq (0,0),(1,0)} Q^\beta \frac{\phi_a}{(1-q)(1-qL)} \langle t(L), \ldots, t(L) \rangle_{0,k+1,\beta}^{R,l,e,S_k} \phi^a,$$

where the unstable terms in the summation are the same as those in (9).

**Definition 3.5.** We define the Givental’s cone $\mathcal{L}_{S_k}$ as the range of $J_{S_k}^{R,l,\infty}$, i.e.,

$$\mathcal{L}_{S_k} := \bigcup_{t(q) \in \mathcal{K}_+} (1-q) J_{S_k}^{R,l,\infty}(t(q), Q) \subset \mathcal{K}.$$
Remark 3.6. In the ordinary quantum $K$-theory, the range of the $\mathcal{J}$-function is a cone which coincides with the differential of the descendant potential (up to the dilaton shift). Therefore the range of the ordinary $K$-theoretic $\mathcal{J}$-function is a Lagrangian cone in the loop space $\mathcal{K}$. However, in the permutation-equivariant theory, it is explained in [3] VII that the cone $\mathcal{L}_{S_\infty}$ is not Lagrangian.

3.4. The level structure in equivariant quasimap theory and orbifold quasimap theory. When $Z \sslash G$ is not proper, one can still define the equivariant quasimap invariants if $Z \sslash G$ has an additional torus action such that the the fixed point loci in the quasimap moduli spaces are proper. It is explained in [14] §6.3 how to define the cohomological quasimap invariants via virtual localization. Similarly, one can define the equivariant $K$-theoretic quasimap invariants (with level structure) for noncompact GIT targets using the $K$-theoretic virtual localization formula (see Section 4.1.1). With this understood, we define the $\mathcal{J}^{R,l,\epsilon}$-function of the equivariant $K$-theoretic quasimap invariants of level $l$ using (9). Its permutation-equivariant generalization is straightforward.

If we do not assume $G$ acts freely on the stable locus $Z^s$, then the target $X := [Z^s/G]$ is naturally an orbifold. For such orbifold GIT targets, a quasimap is a tuple $((C, x_1, \ldots, x_k), [u])$ where $(C, x_1, \ldots, x_k)$ is a $k$-pointed, genus $g$ twisted curve (see [25] §4) and $[u]$ is a representable morphism from $(C, x_1, \ldots, x_k)$ to $X$. We refer the reader to [15] §2.3 for the details of the $\epsilon$-stability imposed on those tuples. We denote by $Q^\epsilon_{g,k}(X, \beta)$ the moduli stack of $\epsilon$-stable quasimaps to the orbifold $X$. It is shown in [15] Thm. 2.7] that this moduli stack is Deligne-Mumford and proper over the affine quotient. Furthermore if $Z$ only has l.c.i. singularities, then $Q^\epsilon_{g,k}(X, \beta)$ has a canonical perfect obstruction theory. Let $\mathcal{C} \to Q^\epsilon_{g,k}(X, \beta)$ be the universal curve. The universal principal $G$-bundle $\mathcal{P} \to \mathcal{C}$ is defined by the pullback of the principal $G$-bundle $Z \to [Z/G]$ via the universal morphism $[u] : \mathcal{C} \to [Z/G]$. In the orbifold setting, we can still define the level $l$ determinant line bundle $D^{R,l}$ using (11).

According to [15] §2.5.1, there are natural evaluation morphisms

$$ev_i : Q^\epsilon_{g,k}(X, \beta) \to \bar{I}_\mu X, ((C, x_1, \ldots, x_k), [u]) \mapsto [u]_{x_i}, \quad \text{for } i = 1, \ldots, k.$$ 

Here $\bar{I}_\mu X$ denotes the rigidified cyclotomic inertia stack of $X$ and it parameterizes representable maps from gerbes banded by finite cyclic groups to $X$. Let $L_i$ be the universal cotangent line bundle whose fiber at $((C, x_1, \ldots, x_k), [u])$ is the cotangent space of the coarse curve $C$ of $C$ at the $i$-th marked point $x_i$. For non-negative integers $l_i$ and classes $E_i \in K^0(\bar{I}_\mu X) \otimes \mathbb{Q}$, we define the $K$-theoretic quasimap invariants of level $l$ as

$$\langle E_1 L_1^{l_1}, \ldots, E_k L_k^{l_k} \rangle^{X,R,l,\epsilon}_{g,k,\beta} = \chi(\mathcal{Q}^\epsilon_{g,k}(X, \beta) \prod_i ev_i^* E_i \otimes L_i \otimes \mathcal{O}^{vir} \otimes D^{R,l}).$$

When $\epsilon = \infty$, this recovers the $K$-theoretic Gromov-Witten invariants of $X$ defined in [26].

In the orbifold setting, one can still define the quasimap graph space $Q^\epsilon_{g,k}(X, \beta)$ (see [15] §2.5.3]). The definition of the determinant line bundle $D^{R,l}$ over the graph space is straightforward. We choose a basis $\{\phi_a\}$ of $K^0(\bar{I}_\mu X) \otimes \mathbb{Q}$. Let $\{\phi^a\}$ be the dual basis with respect to the twisted pairing $\langle \cdot, \cdot \rangle^{R,l}$ on $K^0(\bar{I}_\mu X) \otimes \mathbb{Q}$ given by

$$(u, v)^{R,l} := \chi(\bar{I}_\mu X, u \otimes \iota^*v \otimes \det^{-l}(\bar{I}_\mu \mathcal{R})).$$

Here $\iota$ is the involution induced by $(x, g) \mapsto (x, g^{-1})$ and $I_\mu \mathcal{R}$ is a vector bundle over $I_\mu X$ such that the fiber over $(x, H)$, with $H \subset \text{Aut } x$, is the $H$-fixed subspace of $\mathcal{R}_x$. With all the
notations understood, it is straightforward to adapt the definition of cohomological orbifold quasimap $J^\epsilon$-function [15] Def. 3.1 to the (permutation-equivariant) $K$-theoretic setting.

4. Mirror theorem and mock theta functions

In this section we prove a wall-crossing result relating the generating functions of $K$-theoretic quasimap invariants for different $\epsilon$. We also compute the equivariant small $I$-function of any toric variety. In some cases, we recover Ramanujan’s mock theta functions.

4.1. Wall-crossing. Tseng-You [27] established a wall-crossing result relating the genus-0 permutation-equivariant $K$-theoretic quasimap invariants without the level structure for different stability parameters. In this section, we generalize their results to $K$-theoretic quasimap theory with level structure. Their strategy applies here essentially because the determinant line bundle splits “nicely” over nodal strata in the localization computation. As a corollary, we obtain a mirror theorem for quantum $K$-theory with level structure. In the following proof, we focus on the case when $Z \parallel G$ is a smooth GIT quotient. In Remark 4.13 we discuss the case when the target is an orbifold. Throughout this section, we assume that $Z$ has an action by the algebraic torus $T$ that commutes with the $G$-action. Then the torus action descends to $Z \parallel G$ and we assume it has isolated fixed points and isolated 1-dimensional orbits in $Z \parallel G$.

4.1.1. Virtual Lefschetz-Kawasaki’s Riemann-Roch formula and $K$-theoretic localization formula. To understand the poles of the generating series of the permutation-equivariant quasimap invariants, we recall Lefschetz-Kawasaki’s Riemann-Roch formula in [3, IX].

Let $h$ be a finite order automorphism of a holomorphic orbibundle $E$ over a compact smooth orbifold $\mathcal{M}$. The (super)trace of $h$ on the sheaf cohomology $H^*(\mathcal{M}, E)$ can be computed as an integral over the $h$-fixed point locus $I\mathcal{M}^h$ in the inertia orbifold $I\mathcal{M}$:

$$
(13) \quad \text{tr}_h H^*(\mathcal{M}, E) = \chi^{fake}(I\mathcal{M}^h, \frac{\text{tr}_h E}{\text{tr}_h \wedge^* N^\vee_{I\mathcal{M}^h}}) := \int_{[I\mathcal{M}^h]} \text{td}(T_{I\mathcal{M}^h}) \text{ch} \left( \frac{\text{tr}_h E}{\text{tr}_h \wedge^* N^\vee_{I\mathcal{M}^h}} \right).
$$

By definition, we can choose an atlas of local charts $U \to U/G(x)$ of $\mathcal{M}$. The local description of the inertia orbifold $I\mathcal{M}$ near $x \in \mathcal{M}$ is given by $[\prod_{g \in G(x)} U^g/G(x)]$, where $U^g \subset U$ is the fixed point locus of $g$. The automorphism $h$ can be lifted to an automorphism $\tilde{h}$ of the chart $U^g$. We denote by $(U^g)^\tilde{h}$ the fixed point locus of $\tilde{h}$ in $U^g$. Then the local description of the orbifold $I\mathcal{M}^h$ is given by $[\prod_{g}(U^g)^\tilde{h}/G(x)]$. We refer to the connected components of $I\mathcal{M}^h$ as Kawasaki strata. Near a point $(x, [g]) \in I\mathcal{M}^h$, the tangent and normal orbifold bundles $T_{I\mathcal{M}^h}$ and $N^\vee_{I\mathcal{M}^h}$ are identified with the tangent bundle and normal bundle to $(U^g)^\tilde{h}$ in $U$, respectively. The trace bundle $\text{tr}_h F$ is the virtual orbifold bundle defined as

$$
\text{tr}_h F := \sum \lambda F_\lambda,
$$

where $F_\lambda$ are the eigenspaces of $h$ corresponding to the eigenvalues $\lambda$. Finally td and ch denote the Todd class and Chern character.

When $\mathcal{M}$ is no longer smooth, Tonita [28] proved a virtual Kawasaki formula: under the assumption that $\mathcal{M}$ has a perfect obstruction theory and admits an embedding into a smooth orbifold, Kawasaki’s formula still holds true if we replace the structure sheaves, tangent and normal bundles in the formula by their virtual counterparts. According to [14] §6.3 and [15] §2.5.4, the moduli stacks of $\epsilon$-stable quasimaps to (orbifold) GIT targets satisfy the
Proposition 4.1. The operator assumptions of Tonita’s theorem. If we choose the transformation \( h \) to be the automorphism induced by permuting the marked points, then the Kawasaki strata parametrize quasimaps with prescribed automorphisms.

Based on Lefschetz’s fixed point formula and virtual Lefschetz-Kawasaki’s Riemann-Roch formula, Givental showed in [3] that the permutation-equivariant quantum \( K \)-theory of the point target space emerges as a necessary ingredient in the fixed point localization in quantum \( K \)-theory. In particular, Givental’s analysis shows that the generating functions of permutation-equivariant quantum \( K \)-invariants or more generally, \( K \)-theoretic quasimap invariants, factorize among nodal strata in the fixed locus. This factorization property will be used constantly in the following proof of the mirror theorem.

4.1.2. The \( S \)-operator and \( P \)-series. We use double brackets to denote the generating function
\[
\langle \langle \gamma_1 L_1^{a_1}, \ldots, \gamma_k L_k^{a_k} \rangle \rangle_{R,l}^{a,b} := \sum_{n \geq 0, \beta \geq 0} Q^d \langle \gamma_1 L_1^{a_1}, \ldots, \gamma_k L_k^{a_k}, t, \ldots, t \rangle_{0,k+n,\beta}^{R,l, S_n}
\]
where \( t \in K(Z \sslash G)_\mathbb{Q} \otimes \Lambda \). Let \( \{ \phi_a \} \) be a basis of \( K(Z \sslash G)_\mathbb{Q} \). We define the permutation-equivariant twisted quantum \( K \)-metric as
\[
G_{a,b}^{R,l} := (\phi_a, \phi_b)^{R,l} + \langle \langle \phi_a, \phi_b \rangle \rangle_{0,2}^{R,l}.
\]
The inverse is given by
\[
G_{R,l}^{a,b} = (\phi^a, \phi^b)^{R,l} - \langle \langle \phi^a, \phi^b \rangle \rangle_{0,2}^{R,l} + \langle \langle \phi^a, \phi^c \rangle \rangle_{0,2}^{R,l} \langle \langle \phi^c, \phi^b \rangle \rangle_{0,2}^{R,l} - \cdots
\]
where \( \{ \phi^a \} \) is the dual basis with respect to the twisted pairing \( \langle , \rangle_{R,l} \).

Let \( t \in K(Z \sslash G)_\mathbb{Q} \otimes \Lambda \). We define the operator \( S_t^{R,l,\epsilon} : \mathcal{K} \rightarrow \mathcal{K} \) as
\[
(S_t^{R,l,\epsilon})(q)(\gamma) = \sum_a \left( (\phi_a, \gamma)^{R,l} + \sum_{(k,\beta) \neq (0,0)} Q^b \langle \frac{\phi_a}{1 - qL, \gamma, t, \ldots, t} \rangle_{0,k+1,\beta}^{R,l, S_n} \right) \phi^a,
\]
and \( (S_t^{R,l,\epsilon})^* : \mathcal{K} \rightarrow \mathcal{K} \) as
\[
(S_t^{R,l,\epsilon})^*(q)(\gamma) = \sum_{a,b} \left( (\gamma, \phi_a)^{R,l} + \sum_{(k,\beta) \neq (0,0)} Q^b \langle \frac{\gamma}{1 - qL, \phi_a, t, \ldots, t} \rangle_{0,k+1,\beta}^{R,l, S_n} \right) G_{R,l}^{a,b} \phi_b.
\]

**Proposition 4.1.** The operator \( S_t^{R,l,\epsilon} \) is unitary. To be more precise, we have
\[
(S_t^{R,l,\epsilon})^*(q) \circ (S_t^{R,l,\epsilon})(1/q) = Id.
\]

**Proof.** Following [29], we consider the following generating series of invariants on quasimap graph space
\[
(14) \quad \sum_{k \geq 0, \beta \geq 0} Q^\beta \langle \gamma(1 - p_0), t, \ldots, t, \delta(1 - p_\infty) \rangle_{0,k+1,\beta}^{QG^*, R,l, S_n},
\]
where \( \gamma, \delta \in K(Z \sslash G)_\mathbb{Q} \) and \( p_0, p_\infty \in K_{\mathbb{C}^*}(\mathbb{P}^1) \) are defined by the restriction to the fixed points:
\[
p_0|_0 = q, p_0|_\infty = 1, \quad \text{and} \quad p_\infty|_0 = 1, p_\infty|_\infty = 1/q.
\]
There are three types of \( \mathbb{C}^* \)-fixed loci:
1. \( F_{k_1+1, \beta_1}^{q_1, k_1+1, \beta_2} = Q_{k_1+1, \beta_1+1}(Z \sslash G, \beta_1) \times_{Z \sslash G} Q_{0,k_2+1, \beta_2}^{q_2, k_2+1, \beta_2}(Z \sslash G, \beta_2), \)
2. \( F_{k+1, \beta} \simeq Z \sslash G \times_{Z \sslash G} Q_{0,k+1, \beta}(Z \sslash G, \beta) \) and \( F_{1,0}^{q_1, 1} \simeq Q_{0,k+1, \beta}(Z \sslash G, \beta) \times_{Z \sslash G} Z \sslash G, \)
3. \( F_{1,0}^{q_1, 1} \simeq Z \sslash G. \)
In the following, we show that in case (1), the level structure \( \mathcal{D}^{R,l} \) splits “nicely”. The proof for the second case is similar. For simplicity, we denote by \( \mathcal{Q}_0 \) and \( \mathcal{Q}_\infty \) the moduli spaces of quasimaps at 0 and \( \infty \), respectively. The fixed point loci \( F := F_{k_1+1, \beta_1}^{k_2+1, \beta_2} \) can be written as \( \mathcal{Q}_0 \times_{\mathbb{Z}/G} \mathcal{Q}_1 \times_{\mathbb{Z}/G} \mathcal{Q}_\infty \), where \( \mathcal{Q}_1 \cong (Z \sslash G) \). Let \( C \) be the underline curve of a quasimap with parametrized rational component \( C_1 \) in strata (1). We denote by \( C_0 \) and \( C_\infty \) the two connected components of \( \overline{C \setminus C_1} \) attaching to 0 and \( \infty \), respectively. Then we have the normalization exact sequence

\[
0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_{C_0} \oplus \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_\infty} \rightarrow \mathcal{O}_{C_0 \cap C_1} \oplus \mathcal{O}_{C_1 \cap C_\infty} \rightarrow 0.
\]

Let \( \mathcal{D}^{R,l}_i, i = 0, 1, \infty \) be the determinant line bundles over the corresponding moduli spaces and let \( \text{ev} : F \rightarrow Z \sslash G \) be the projection onto the second factor \( \mathcal{Q}_1 \). It follows from the normalization exact sequence that we have the following identity in \( K^0(F) \):

\[
\mathcal{D}^{R,l} = \mathcal{D}^{R,l}_0 + \mathcal{D}^{R,l}_1 + \mathcal{D}^{R,l}_\infty - 2 \cdot \text{ev}^*(\det^{-1}(\mathcal{R}))
\]

\[
= \mathcal{D}^{R,l}_0 + \mathcal{D}^{R,l}_\infty - \text{ev}^*(\det^{-1}(\mathcal{R})).
\]

Here we use the fact that \( \mathcal{D}^{R,l}_1 \cong \text{ev}^*(\det^{-1}(\mathcal{R})) \). Notice that the third summand is absorbed by the (inverse) twisted pairing at the node. Therefore by \( \mathbb{C}^* \)-localization, the above generating series is equal to

\[
\sum_a \left( (\phi_a, \gamma)^{R,l} + \langle \frac{\phi_a}{1 - qL} \gamma \rangle_{0,2}^{R,l,\epsilon} \right) \left( (\delta, \phi^a)^{R,l} + \langle \frac{\phi^a}{1 - L/q} \delta \rangle_{0,2}^{R,l,\epsilon} \right) \\
= \sum_a (\phi_a, \gamma)^{R,l} \left( (\delta, \phi^a)^{R,l} + \langle \frac{\phi^a}{1 - L/q} \delta \rangle_{0,2}^{R,l,\epsilon} \right) + f^*(t, q, Q) \\
= (\delta, \gamma)^{R,l} + \langle \frac{\phi^a}{1 - L/q} \delta \rangle_{0,2}^{R,l,\epsilon} + f^*(t, q, Q),
\]

where \( f^*(t, q, Q) \in \mathcal{K}_- \) is a sum of (reduced) rational functions in \( q \) with coefficients in \( K(Z \sslash G) \otimes \Lambda[[t]] \) and the rational functions have possible poles only at roots of unity.\(^1\)

The first equation comes from expanding \( 1/(1 - qL) \) and \( 1/(1 - L/q) \) over different Kawasaki strata. To be more precise, a stratum is represented by a quasimap with a prescribed automorphism (possibly trivial). The automorphism acts on the cotangent space at the marked point with an eigenvalue \( \xi \), where \( \xi \) is a (primitive) root of unity. Therefore on the right hand side of the Kawasaki-Riemann-Roch formula \( \text{(13)} \), the content in the “fake” holomorphic Euler characteristic over this stratum has a factor \( 1/(1 - \xi L) \) or \( 1/(1 - L \xi/q) \).

Since \( 1 - L \) is nilpotent in the \( \text{K}-\text{group} \) of this stratum, we have the following expansions

\[
\frac{1}{1 - q\xi L} = \sum_{i \geq 0} \frac{(q\xi)^i(L - 1)^i}{(1 - q\xi)^{i+1}} = O(\frac{1}{1 - q\xi})
\]

and

\[
\frac{1}{1 - L\xi/q} = 1 + \frac{\xi}{q - \xi} + \sum_{i \geq 1} \frac{q^i\xi^i(L - 1)^i}{(q - \xi)^{i+1}} = 1 + O(\frac{1}{1 - q\xi^{-1}}).
\]

\(^1\)The argument in \( \text{(29)} \) Prop. 3.1] is slightly incorrect. The authors ignore that \( f^*(t, q, Q) \) have possible poles at roots of unity other than 1.
Notice that the generating series (14) is regular at roots of unity. Therefore
\[
\sum_a \left( (\phi_a, \gamma)^{R,l} + \langle\langle (\phi_a, \gamma) \rangle \rangle_{0,2}^{R,l,\epsilon} \right) \left( (\delta, \phi^a)^{R,l} + \langle\langle (\delta, \phi^a) \rangle \rangle_{0,2}^{R,l,\epsilon} \right) = (\delta, \gamma)^{R,l} + \langle\langle (\delta, \gamma) \rangle \rangle_{0,2}^{R,l,\epsilon}.
\]

Finally, the proposition follows from the following calculation:
\[
(S_t^{R,l,\epsilon})^* (q) \circ (S_t^{R,l,\epsilon})(1/q)
= \sum_{a,b} G_{R,l,\epsilon}^{ab} \phi_b \sum_c \left( (\phi_c, \phi_a)^{R,l} + \langle\langle (\phi_c, \phi_a) \rangle \rangle_{0,2}^{R,l,\epsilon} \right) \left( \gamma, (\phi^c)^{R,l} + \langle\langle \gamma, (\phi^c) \rangle \rangle_{0,2}^{R,l,\epsilon} \right)
= \sum_{a,b} \phi_b (\phi^b, \gamma)^{R,l}
= \gamma.
\]

We define the \(P\)-series of level \(l\) by
\[
P^{R,l}(t, q) := \sum_{a,b} \phi_b G_{R,l,\epsilon}^{ab} \langle\langle (\phi_a(1 - p_\infty)) \rangle \rangle_{0,1}^{Q^c, R,l},
\]
where \(p_\infty \in K_{C^*} (\mathbb{P}^1)\) is defined by
\[
p_\infty|_0 = 1, \quad p_\infty|_\infty = 1/q.
\]

Similar to the case of \(K\)-theoretic quasimap invariants without level structure, we have the following “Birkhoff factorization” formula.

**Proposition 4.2.** For every \(\epsilon \geq 0+\), we have \(J_{S_{\infty}}^{R,l,\epsilon} = S_t^{R,l,\epsilon}(q)(P^{R,l,\epsilon}(t, q))\).

**Proof.** The same calculation used in the proof of [29, Prop. 3.2] works here. \(\square\)

One corollary of Proposition [14,2] is the following identity for the \(J\)-function of quantum \(K\)-theory:
\[
J_{S_{\infty}}^{R,l} = S_t^{R,l,\infty}(q)(1).
\]

To prove it, we consider the expansions
\[
J_{S_{\infty}}^{R,l} = 1 + f(t, q, Q)
\]
and
\[
(S_t^{R,l,\infty})^* (1/q)(\gamma) = \sum_{a,b} G_{R,l,\infty}^{ab} \phi_b \left( (\phi_a, \gamma)^{R,l} + \langle\langle (\phi_a, \gamma) \rangle \rangle_{0,2}^{R,l,\epsilon} \right) + g_\gamma(t, q, Q)
= \gamma + g_\gamma(t, q, Q)
\]
where \(f(t, q, Q), g_\gamma(t, q, Q) \in K_{C}\) are sums of (reduced) rational functions in \(q\) with coefficients in \(K(Z/G)Q \otimes \Lambda[[t]]\) and poles only possible at roots of unity. Therefore
\[
P^{R,l,\infty}(t, q) = (S_t^{R,l,\infty})^* (1/q)(J_{S_{\infty}}^{R,l}) = 1 + h(t, q, Q),
\]
where \( h(t, q, Q) \) is a rational function in \( q \) with poles only possible at roots of unity. The corollary follows from the fact that \( P_{R_l,\infty}(t, q) \) is regular at roots of unity and hence \( P_{R_l,\infty}(t, q) = 1 \).

Let \( \gamma = 1 + O(Q) \in K(Z \sslash G)_Q \otimes \Lambda \) be an invertible element and let \( \{ t_\alpha \} \) be the coordinates of \( t \) in the basis \( \{ \phi_\alpha \} \). We have the expansion

\[
(1 - q)S^{R_l, \epsilon}_{t}(q)(\gamma) = (1 - q)\gamma + \tau^{R_l, \epsilon}_{t}(t) + r^{\epsilon}_{t}(t, q, Q)
\]

where \( r^{\epsilon}_{t}(t, q, Q) \) is a sum of reduced rational functions in \( q \) with coefficients in \( K(Z \sslash G)_Q \otimes \Lambda[[t_\alpha]] \) and poles only possible at roots of unity (including 1). We also have

\[
(1 - q)S^{R_l, \epsilon}_{t}(q)(\gamma) \equiv (1 - q)S^{R_l, \infty}_{t}(q)(1) \quad \text{mod } Q
\]

and hence \( \tau^{R_l, \epsilon}_{t}(t) = t + O(Q) \)

and hence \( \tau^{R_l, \epsilon}_{t}(t) \) is an invertible transformation on \( K(Z \sslash G)_Q \otimes \Lambda \). For \( 0^+ \leq \epsilon_1 \leq \epsilon_2 \leq \infty \), we define the generalized string transformation as

\[
\tau^{\epsilon_1, \epsilon_2}_{t}(t) := (\tau^{R_l, \epsilon_1}_{t})^{-1} \circ \tau^{R_l, \epsilon_2}_{t}(t).
\]

Then we have

\[
(1 - q)S^{R_l, \epsilon}_{t}(q)(\gamma) = (1 - q)S^{R_l, \epsilon_1}_{t}(q)(\gamma) + r^{\epsilon_1, \epsilon_2}_{t}(q),
\]

where \( r^{\epsilon_1, \epsilon_2}_{t}(q) \) is a sum of reduced rational functions in \( q \) with possible poles only at roots of unity. According to \cite{29} Lem. 3.3, we have the following lemma\footnote{The statement of \cite{29} Lem. 3.3 is slightly incorrect because the authors ignore possible poles at roots of unity other than 1.}. For completeness, we include the proof here.

**Lemma 4.3** (\cite{29}). For every \( \epsilon \geq 0^+ \), there exists a uniquely determined element \( P^{\infty, \epsilon}(t, q) \in K_+ \) convergent in the \( Q \)-adic topology for each \( t \), and a uniquely determined map \( t \mapsto \tau^{\infty, \epsilon}(t) \) on \( K^0(Z \sslash G) \otimes \Lambda \) satisfying the following properties:

1. \( \tau^{\infty, \epsilon}(t) = t \mod Q \);
2. \( P^{\infty, \epsilon}(t, q) = 1 \mod Q \);
3. \( (1 - q)S^{R_l, \epsilon}_{t}(q)(P^{R_l, \epsilon}(t, q)) = (1 - q)S^{R_l, \infty}_{t}(q)(P^{\infty, \epsilon}(\tau^{\infty, \epsilon}(t), q)) + r_{\epsilon, \infty}(t, q, Q) \), where \( r_{\epsilon, \infty}(t, q, Q) \in K_- \) is a sum of (reduced) rational functions in \( q \) with possible poles only at roots of unity.

**Proof.** We construct \( \tau^{\infty, \epsilon}(t) \) and \( P^{\infty, \epsilon}(t, q) \) by an inductive procedure in the degree \( d = \beta(L_0) \). We write

\[
\tau^{\infty, \epsilon}(t) = t + \sum_{d' \geq 1} \tau^{d'}_d Q^{d'} \quad \text{and} \quad P^{\infty, \epsilon}(t, q) = 1 + \sum_{i, d' \geq 1} P_{i, d'} q^i Q^{d'}
\]
We assume that for all $d' < d$ and $i$, the coefficients $\tau_{d'}$ and $P_{d',i}$ are uniquely determined from properties (1)-(3). Using the definition of the $S$-operator, it is not difficult to see that the degree-$d$ part of $(1-q)S_{\tau,\psi,0}^{R,1,\infty}(P^{\infty,\psi,0}(\tau,\psi,0(t),q))$ is the sum of

$$
(1-q) \left( \sum_i P_{d,i}q^i + \sum_{k \geq 1, a} \langle \phi_a, 1-qL \sum_{i,d \geq 1} P_{d,i}q^i, t, \ldots, t \rangle_{0, k+2, 0} R, 1, \psi, k, 0 + 2 \phi^a \right) Q^d 
$$

and terms that are determined from the induction. Note that we have

$$Q_{0,k+2}(Z \parallel G, 0) \cong \overline{M}_{0,k+2} \times Z \parallel G.$$ 

Using the above isomorphism, Corollary [2.7] and the string equation (see [3, VII]), we can simplify the first two terms of (16) as follows

$$\sum_i P_{d,i}q^i + \sum_{k \geq 1, a} \langle \phi_a, 1-qL \sum_{i,d \geq 1} P_{d,i}q^i, t, \ldots, t \rangle_{0, k+2, 0} R, 1, \psi, k, 0 + 2 \phi^a = \sum_i P_{d,i}q^i + \sum_{k \geq 1, a} \phi^a \phi_a \sum_{i,d \geq 1} P_{d,i}q^i \otimes t^k \langle R, 1, 1, 1, \ldots, 1 \rangle_{0, k+2}$$

$$= \sum_i P_{d,i}q^i \cdot \left( 1 + \frac{t}{1-q} + \sum_{k \geq 2} \frac{q}{1-q} t^k \langle 1, 1, \ldots, 1 \rangle_{0, k+1} \right)$$

$$= \sum_i P_{d,i}q^i \cdot \left( 1 + \frac{t}{1-q} + \sum_{k \geq 2} \frac{q}{1-q} \frac{1}{(1-q)(1-qL)} \langle 1, 1, \ldots, 1 \rangle_{0, k+1} t^k \right)$$

Here $\langle 1, 1, \ldots, 1 \rangle_{0, k+1}$ denote the permutation-equivariant quantum $K$-invariants of the point target. We define the small $J$-function for $X = pt$ as

$$J_{pt}(t) := 1 + \frac{t}{1-q} + \sum_{k \geq 2} \frac{q}{(1-q)(1-qL)} \langle 1, 1, \ldots, 1 \rangle_{0, k+1} t^k.$$ 

Then according to [3, 1], $J_{pt}(t)$ has the following explicit form

$$J_{pt}(t) = e^{\sum_{k>0} t^k/(1-q^k)}.$$ 

Similarly, we can simplify the third term in (16) and obtain

$$\tau_d \frac{1}{1-q} e^{\sum_{k>0} t^k/(1-q^k)}.$$ 

In sum, the formula (16) equals

$$((1-q) \sum_i P_{d,i}q^i + \tau_d) e^{\sum_{k>0} t^k/(1-q^k)}.$$ 

Since $e^{\sum_{k>0} t^k/(1-q^k)} \in K_-$ is invertible, the coefficients $P_{d,i}$ and $\tau_d$ are uniquely determined. □
Now to prove the mirror theorem, we consider the torus action on $Z \text{ e}_G$. Let $\{\phi_\mu\}_\mu$ be the $T$-fixed point basis of $K(Z \text{ e}_G)_T$ and let $\mu \in (W \text{ e}_G)^T$ be a fixed point. We consider the restriction of the $T$-equivariant $S$-operator $S_{t,T}^{R,l,\epsilon}$ of level $l$ to $\mu$, i.e.,

$$\mu^* S_{t,T}^{R,l,\epsilon}(q)(\gamma) = (\phi_\mu, \gamma)^{R,l} + \sum_{(k,d) \neq (0,0)} Q^d (\phi_\mu, 1 - qL, \gamma, t, \ldots, t)_{0,k+2,d,\epsilon}^{R,l,\epsilon,S_t}.$$

Let $M$ be a connected component of the fixed locus $Q_{0,n+2}(X,d)^T$. Then following [23], we say $M$ is of initial type if the first marking is on a contracted irreducible component of the domain curve, and $M$ is of recursion type if the first marking is on a non-contracted irreducible component. Using the same argument as in the proof of [29, Lem. 4.1], we obtain the following description of the poles of $\mu^* S_{t,T}^{R,l,\epsilon}$.

**Lemma 4.4** (Poles of $\mu^* S_{t,T}^{R,l,\epsilon}$). For each fixed $d$ and each tuple $(k_i)$ of nonnegative integers, the coefficient of the monomial $Q^d \prod_i t_i^{k_i}$ in $\mu^* S_{t,T}^{R,l,\epsilon}$ is a rational function of $q$ with possible poles only at $0, \infty$, roots of unity and at $\lambda(\mu, \nu)^{1/m}$, where $m \in \mathbb{Z}_{>0}$ and $\lambda(\mu, \nu)$ is the character of the torus action on the tangent line at the fixed point $\mu$ corresponding to the 1-dimensional orbit connecting the fixed points $\mu$ and $\nu$.

The following lemma can be proved using the same argument as in [29, Lem. 4.2] due to the splitting property (see Proposition [2,3]) of the determinant line bundle.

**Lemma 4.5** (Recursion relation). The series $\mu^* S_{t,T}^{R,l,\epsilon}$ satisfies the recursion relation

$$\mu^* S_{t,T}^{R,l,\epsilon}(q) = I_{\mu}^{R,l,\epsilon}(q) + \sum_{\nu \in o(\mu)} \sum_{m=1}^{\infty} Q^m d(\mu, \nu) \frac{\phi_\mu}{m} C_{\mu,\nu,1} \frac{1}{1 - \lambda(\mu, \nu)^{1/m} q} \mu^* S_{t,T}^{R,l,\epsilon}(\lambda(\mu, \nu)^{1/m}),$$

where

- $I_{\mu}^{R,l,\epsilon}(q)$ is the sum of the contributions of all the components of initial type. Each $(Q, \{t_i\})$ coefficient of $I_{\mu}^{R,l,\epsilon}(q)$ is of the form
  $$\sum_{\xi: \text{root of unity}} \sum_{i \geq 0} c_{i,\xi}(\xi q)^i/(1 - \xi q)^{i+1}.$$ 
  In particular, $I_{\mu}^{R,l,\epsilon}(q)$ has possible poles only at roots of unity.
- $o(\mu)$ is the set of all fixed points $\nu$ connected to $\mu$ by a one-dimensional $T$-orbit, $d(\mu, \nu)$ is the homology class of the orbit, and $\lambda(\mu, \nu)$ is the character of the torus representation on the tangent line at $\mu$ corresponding to the orbit.
- The recursion coefficient $C_{\mu,\nu,1}$ is the $T$-equivariant K-theoretic Euler class of the virtual cotangent space to the moduli space $\overline{M}_{0,2}(Z \text{ e}_G, m)$ at the corresponding fixed point. In particular, this recursion coefficient does not depend on $\epsilon$.

**Remark 4.6.** Using the same argument as in [29, Rem. 4.3 ], one can show that the series $S_{t}^{R,l,\epsilon}(q)(P_{t,\epsilon}(t, q))$ and $S_{t,0}^{R,l,\epsilon}(P_{\infty,\epsilon}(t, q))$ satisfy the same recursive relation (17).

The following lemma shows that $S_{\mu}^{R,l,\epsilon}$ satisfies a polynomiality property and it can be proved by using a simple generalization of the argument in [29, Lem. 4.7]. For completeness, we include the proof here.
Lemma 4.7. For each torus fixed point $\mu \in (Z / / G)^T$, the series 
\[ D_\mu(S^{R,l,\epsilon}_\mu) := S^{R,l,\epsilon}_\mu(Q, t, q)S^{R,l,\epsilon}_\mu(Qq^{aL_\mu}, t, 1/q) \]
has no poles at roots of unity, where $a \in \mathbb{Z}$ and $(Qq^{aL_\mu})^\beta = Q^\beta q^\beta L_\mu$. Furthermore, for $r \in \mathbb{Z}_{>0}$, the series $\Psi^r(D_\mu(S^{R,l,\epsilon}_\mu))$ have no pole at roots of unity, where $a \in \mathbb{Z}[T]$ and the Adams operations $\Psi^r$ are extended to $\Lambda[[q, q^{-1}]]$ by setting $\Psi^r(q) = q^r$.

Proof. For effective classes $\beta_1, \beta_2$, we denote by $U_{\beta_1, \beta_2}(L_\theta)$ the universal $C^*$-equivariant line bundle on $\mathcal{Q}G^{\epsilon}_{0,2+m,\beta}$ defined in [23, §3.3]. According to [23], we have
\[ U_{\beta_1, \beta_2}(L_\theta)_{|_{k_1, \beta_1}} = \text{ev}^*_{\lambda}(\mathcal{O}(\theta)) \boxtimes C_{\beta_2}(L_\theta), \]
where $\mathcal{O}(\theta)$ is the canonical polarization on $Z / / G$. Let $\mathcal{Q}G^{\epsilon}_{0,2+m,\beta}(Z / / G)_\mu$ be the $T$-fixed locus consisting of quasimaps with the parametrized rational component contracted to the point $\mu$. We define
\[ \text{Res}_\mu \mathcal{O}^\text{vir}_{\mathcal{Q}G^{\epsilon}_{0,2+m,\beta}(Z / / G)_\mu} := t_* \left( \frac{\mathcal{O}^\text{vir}_{\mathcal{Q}G^{\epsilon}_{0,2+m,\beta}(Z / / G)_\mu}}{\text{tr}_T \wedge^*(N^\text{vir})^\vee} \right) \]
where $t : \mathcal{Q}G^{\epsilon}_{0,2+m,\beta}(Z / / G)_\mu \rightarrow \mathcal{Q}G^{\epsilon}_{0,2+m,\beta}(Z / / G)$ is the inclusion map. Let $\gamma = 1 + \sum_{\beta \neq 0} Q^\beta \chi(\mathcal{Q}G^{\epsilon}_{0,2+m,\beta}(Z / / G)/S_m, \text{Res}_\mu \mathcal{O}^\text{vir}_{\mathcal{Q}G^{\epsilon}_{0,2+m,\beta}(Z / / G)} \otimes (U_{\beta_1, \beta_2}(L_\theta))^\alpha) \otimes \mathcal{D}^{R,l} \otimes \text{ev}^*_1(\gamma) \otimes \text{ev}^*_2(\gamma) \prod_{i=3}^m \text{ev}^*_i(t)).$

On the one hand, it is defined without $C^*$-localization and hence it is regular at roots of unity. On the other hand, by $C^*$-localization, we have
\[ \langle \gamma(1 - p_0), \gamma(1 - p_\infty); U(L_\theta) \rangle_{\mathcal{Q}G^{\epsilon}_{0,2+m,\beta}} = \lambda_\theta^a \omega_{R,l,\mu} D(S^{R,l,\epsilon}_\mu), \]
where $\omega_{R,l,\mu}$ and $\lambda_\theta$ are the $T$-weights on the fibers of $\mathcal{O}(\theta)$ and $\text{det}^{-1}(\mathcal{R})$ at $\mu$, respectively.

In general, if $a \in \mathbb{Z}[T]$, the series $\Psi^r(D_\mu(S^{R,l,\epsilon}_\mu))$ equals the generating series obtained by applying $\Psi^r$ to the right side of equation (18). It should be understood in terms of Adams-Riemann-Roch theorem (see [30] and [31]). Since it is defined without $C^*$-localization, it is regular at roots of unity. \hfill $\square$

Remark 4.8. Since the series $P^{R,l,\epsilon}(t, q)$ and $P^{\infty,\epsilon}(\tau^{\infty,\epsilon}(t), q)$ in Theorem 4.11 are regular at roots of unity, the above proof also applies to the components of $S^{R,l,\epsilon}_\mu(Q^{R,l,\epsilon}(t, q))$ and $S^{\infty,\epsilon}_\mu(Q^{\infty,\epsilon}(\tau^{\infty,\epsilon}(t), q))$, after appropriate modification of the generating series.

The following uniqueness lemma is a slight modification of [23, Lem. 4.9].

Lemma 4.9. Let $\{S_1, \mu \in (Z / / G)^T\}$ and $\{S_2, \mu \in (Z / / G)^T\}$ be two systems of power series in $\Lambda[[t_1]] \otimes \mathbb{C}(q)$ which satisfy the following properties:

(1) For all $\mu \in (Z / / G)^T$, the power series $S_{1, \mu}$ and $S_{2, \mu}$ are rational functions in $q$ with possible poles only at $0, \infty$, or roots of unity; and at most simple poles at $q = (\lambda(\mu), \nu)^{1/m}$, where $m \in \mathbb{Z}_+$ and $\lambda(\mu, \nu)$ is the character of the $T$-action on the
tangent line at \( \mu \) corresponding to the 1-dimensional orbit connecting the fixed points \( \mu \) and \( \nu \).

(2) The systems \( \{S_{1,\mu}\}_{\mu \in (Z/\mathbb{G})^T} \) and \( \{S_{2,\mu}\}_{\mu \in (Z/\mathbb{G})^T} \) both satisfy the recursion relation \( (17) \).

(3) For all \( \mu \in (Z \parallel G)^T \), the series

\[
D(S_{1,\mu}) := S_{1,\mu}(Q, t, q)S_{1,\mu}(Qq^{aL_\theta}, t, 1/q)
\]

and

\[
D(S_{2,\mu}) := S_{2,\mu}(Q, t, q)S_{2,\mu}(Qq^{aL_\theta}, t, 1/q)
\]

are regular at roots of unity for arbitrary \( a \in \mathbb{Z} \). Furthermore, for \( r \in \mathbb{Z}_{>0} \) and \( a \in \mathbb{Z}[1/r] \), the series \( \Psi^r(D(S_{1,\mu})) \) and \( \Psi^r(D(S_{2,\mu})) \) have no pole at roots of unity.

(4) For all \( \mu \in (Z \parallel G)^T \),

\[
(1 - q)S_{1,\mu} = (1 - q)S_{2,\mu} + r_{1,2}(Q, t, q),
\]

where \( r_{1,2}(Q, t, q) \in \mathcal{K}_- \) is a sum of reduced rational functions in \( q \) with possible poles only at roots of unity.

(5) For all \( \mu \in (Z \parallel G)^T \),

\[
S_{1,\mu} = S_{2,\mu} \mod Q.
\]

Then \( S_{1,\mu} = S_{2,\mu} \) for all \( \mu \in (Z \parallel G)^T \).

Proof. We prove this lemma by slightly modifying the argument of [29, Lem. 4.9]. We write

\[
S_{1,\mu} = \sum_{\beta} Q^\beta \sum_{k} c_{1,\mu,\beta,k}(q)\prod_i t_i^{k_i} \quad \text{and} \quad S_{2,\mu} = \sum_{\beta} Q^\beta \sum_{k} c_{2,\mu,\beta,k}(q)\prod_i t_i^{k_i},
\]

where \( k := (k_i)_i \) are tuples of nonnegative integers and \( \beta \) are effective classes. We define the bi-degree of \( Q^\beta \prod_i t_i^{k_i} \) to be \( (\sum_i k_i, \beta(L_\theta)) \). Let \( S_{1,\mu}^{(m,d)} \) and \( S_{2,\mu}^{(m,d)} \) be the part of bi-degree \((m, d)\) of \( S_{1,\mu} \) and \( S_{2,\mu} \), respectively. To prove the lemma, we show that for all \( \mu \in (Z \parallel G)^T \) and all \((m, d) \in \mathbb{N} \times \mathbb{N} \), we have

\[
S_{1,\mu}^{(m,d)} = S_{2,\mu}^{(m,d)}.
\]

The proof is by induction on \((m, d)\), using the lexicographic order. By property (5), the base case when \( d = 0 \) is true. For \( d \geq 1 \), we assume that \( (19) \) is true for all \((m', d') < (m, d)\) and all fixed points \( \mu \). We consider the difference \( D(S_{1,\mu}) - D(S_{2,\mu}) \) and denote its part of bi-degree \((m, d)\) by \( D^{m,d} \). By the induction assumption, we have

\[
D^{m,d} = S_{1,\mu}^{(m,d)}(q) - S_{2,\mu}^{(m,d)}(q) + q^{ad}(S_{1,\mu}^{(m,d)}(1/q) - S_{2,\mu}^{(m,d)}(1/q)).
\]

By property (2), the induction assumption and property (1), the difference \( S_{1,\mu}^{(m,d)}(q) - S_{2,\mu}^{(m,d)}(q) \) is the sum of rational functions in \( q \) with possible poles only at 0, \( \infty \) and roots of unity. Given a primitive root of unity \( \xi \), we denote by

\[
(S_{1,\mu}^{(m,d)}(q) - S_{2,\mu}^{(m,d)}(q))_\xi
\]

the sum of terms of \( S_{1,\mu}^{(m,d)}(q) - S_{2,\mu}^{(m,d)}(q) \) with poles at \( \xi \). Similarly, we denote by \( (D^{m,d})_\xi \) the sum of terms of \( D^{m,d} \) with poles at \( \xi \).

There are three cases. First when \( \xi = 1 \), we write \( (21) \) as

\[
(1 - q)^{-2n}
\]

\[
\left(\frac{A}{1 - q} + B + O(1 - q)\right)
\]
with \( n \geq 0 \) and \( A, B \in \Lambda[[t]] \) homogeneous of bi-degree \((m, d)\). Then \((D^{m,d})_1\) can be simplified as

\[
(1 - q)^{-2n}(2n + 1 + ad)A + 2B + O(1 - q).
\]

Since \( a \) can be any positive integer, it follows from the regularity of \((20)\) at \( q = 1 \) that \( A = B = 0 \). By descending induction we obtain \( n = 0 \).

Second, when \( \xi = -1 \), we write \((21)\)

\[
(1 + q)^{-2n}\left(\frac{A}{1 + q} + B + O(1 + q)\right)
\]

with \( n \geq 0 \) and \( A, B \in \Lambda[[t]] \) homogeneous of bi-degree \((m, d)\). We can simplify \((D^{m,d})_{-1}\) to be

\[
(1 + q)^{-(2n+1)}\left(1 - (-1)^{ad}\right)A + (1 + q)\left(\left((-1)^{ad}\right)2n + 1 + ad\right) + \left(1 + (-1)^{ad}\right)B + O(1 + q).
\]

If we choose \( a = 2d'/d, d' \in \mathbb{Z} \), then

\[
\Psi^d((D^{m,d})_{-1}) = (1 + q^d)^{-2n}(2n + 1 + ad) + 2B + O(1 + q).
\]

By descending induction, we have \( n = 0 \). If \( n = 0 \), we set \( a = 1/d \) and obtain

\[
\Psi^d((D^{m,d})_{-1}) = (1 + q)^{-1}\left(2A + (1 + q)(-2A) + O(1 + q)\right).
\]

It follows from the regularity that \( A = 0 \).

For the third case where \( \xi \neq 1, -1 \), one can prove similarly that \((D^{m,d})_{-\xi} + (D^{m,d})_{\xi - 1}\) is regular at \( \xi \). In sum, we show that the difference \( S_{1,\mu}^{(m,d)}(q) - S_{2,\mu}^{(m,d)}(q) = \frac{A}{1 - q} + O(1) \) for some \( A \). Then the lemma follows from property \((4)\).

\[
\square
\]

It follows from Lemma \([4.4]\), Lemma \([4.5]\), Lemma \([4.7]\) and \([15]\) that \( \{S_{\xi_1^{e_1} \xi_2^{e_2} (t)}^{R,l}(q)(\gamma)\} \) and \( \{S_{\xi_1^{e_1} \xi_2^{e_2} (t)}^{R,l}(q)(\gamma)\} \) satisfy the properties in the uniqueness lemma. As a corollary, we have the following theorem.

**Theorem 4.10.** Assume the torus \( T \)-action on \( \mathbb{Z} \parallel G \) has isolated fixed points and isolated 1-dimensional orbits. Let \( 0+ \leq \epsilon_1 < \epsilon_2 \leq \infty \). Assume \( \gamma \in K_T(\mathbb{Z} \parallel G) \otimes \Lambda \) is invertible of the form \( 1 + O(Q) \). Then

\[
S_{\xi_1^{e_1} \xi_2^{e_2} (t)}^{R,l}(q)(\gamma) = S_{\xi_1^{e_1} \xi_2^{e_2} (t)}^{R,l}(q)(\gamma).
\]

Similarly, it follows from Lemma \([4.5]\), Remark \([4.6]\) and Remark \([4.8]\) that \( \{S_{\xi_1^{e_1} \xi_2^{e_2} (t)}^{R,l}(P^{R,l}(t, q))\} \) and \( \{S_{\xi_1^{e_1} \xi_2^{e_2} (t)}^{R,l}(P^{\infty}(t, q))\} \) satisfy the required properties in the uniqueness lemma. Therefore we have the following mirror theorem.

**Theorem 4.11.** Assume the torus \( T \)-action on \( \mathbb{Z} \parallel G \) has isolated fixed points and isolated 1-dimensional orbits, then for all \( \epsilon \geq 0+ \),

\[
J^{R,l,\epsilon}(t, q) = S_{\xi_1^{e_1} \xi_2^{e_2} (t)}^{R,l,\epsilon}(q)(P^{R,l}(t, q)) = S_{\xi_1^{e_1} \xi_2^{e_2} (t)}^{R,l,\infty}(P^{\infty}(t, q)),
\]

**Corollary 4.12.** \((1 - q)J^{R,l,\epsilon}(t, q)\) lies on \( \mathcal{L}_{S_{\infty}} \). In particular, up to the multiplication by \((1 - q)\), the \( I \)-function of level \( l \) lies on \( \mathcal{L}_{S_{\infty}} \).
Remark 4.13. When the target is an orbifold, the determinant line bundle still has the splitting property among nodal strata. As in [15], with appropriate modifications, the above proof also works for orbifold targets. Notice that in the orbifold case, the $J^R{,l,\epsilon}$-function is a generating series with coefficients in $K^0(I_\mu X)$. To recover examples of mock modular forms, we only consider the untwisted sector component of $I$-functions for the orbifold targets mentioned in the introduction.

4.2. $I$-function and mock theta function. In this section, we explicitly compute the small $I$-functions with level structure for toric varieties. A remarkable phenomenon is the appearance of Ramanujan’s mock theta functions.

Let $M \cong \mathbb{Z}^n$ be a $n$-dimensional lattice and let $N$ be its dual lattice. For every complete nonsingular fan $\Sigma \subset N_\mathbb{R}$, we can associate a $n$-dimensional smooth projective variety $X_\Sigma$. We denote by $\Sigma(1)$ the set of 1-dimensional cones in $\Sigma$. Let $m = |\Sigma(1)|$. Each $\rho \in \Sigma(1)$ determines a Weil divisor $D_\rho$ on $X_\Sigma$ and the Picard group of $X_\Sigma$ is determined by the following short exact sequence:

\[(22)\quad 0 \to M \to \mathbb{Z}^{\Sigma(1)} \to \text{Pic}(X_\Sigma) \to 0.\]

Here the inclusion is defined by $m \mapsto \sum_\rho (m, \rho)D_\rho$. Now let us describe the quotient construction of $X_\Sigma$. Since $\text{Pic}(X_\Sigma)$ is torsion free, we choose an integral basis $\{L_1, \ldots, L_s\}$ of it, where $s = m - n$. Then the inclusion map in (22) is given by an integral $s \times n$ matrix $Q = (Q_{ap})$ which is called the charge matrix of $X_\Sigma$. Applying $\text{Hom}(-, \mathbb{C}^\ast)$ to the exact sequence (22), we get an exact sequence.

\[1 \to G \to (\mathbb{C}^\ast)^{\Sigma(1)} \to N \otimes \mathbb{C}^\ast \to 1,\]

where $G := \text{Hom}(\text{Pic}(X_\Sigma), \mathbb{C}^\ast) \cong (\mathbb{C}^\ast)^\ast$. The first map in the above short exact sequence defines the following $G$-action on $\mathbb{C}^{\Sigma(1)}$

\[(23)\quad t \cdot (z_{\rho_1}, \ldots, z_{\rho_m}) = \left( \prod_{a=1}^s t_a^{Q_{a\rho_1}} z_{\rho_1}, \ldots, \prod_{a=1}^s t_a^{Q_{a\rho_m}} z_{\rho_m} \right),\]

where $t = (t_1, \ldots, t_s) \in (\mathbb{C}^\ast)^\ast$. By choosing an appropriate linearization of the trivial line bundle on $\mathbb{C}^{\Sigma(1)}$ (see e.g., [32, Chapter 12]), the semistable and stable loci are equal. We denote this linearized trivial line bundle by $L_\Sigma$ and the stable loci by $U(\Sigma)$. Let $z_\rho$ be the coordinates in $\mathbb{C}^{\Sigma(1)}$. We define a subvariety

\[Z(\Sigma) = \{(z_\rho) \in \mathbb{C}^{\Sigma(1)} | \prod_{\rho \not\in \sigma} z_\rho = 0, \sigma \in \Sigma\}.\]

Then we have

\[U(\Sigma) = \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma).\]

The toric variety $X_\Sigma$ is the geometric quotient $U(\Sigma)/G$. Let $P$ be the principal $G$-bundle $\mathbb{C}^{\Sigma(1)} \to [\mathbb{C}^{\Sigma(1)}/G]$. Let $\pi_i : G \to \mathbb{C}^\ast$ be the projection to the $i$-th component and let $R_j$ be the characters given by $t = (t_1, \ldots, t_s) \mapsto \prod_{a=1}^s t_a^{Q_{a\rho_j}}$ for $1 \leq j \leq m$. Then the line bundles $L_i$ and $\mathcal{O}(D_{\rho_j})$ are the restrictions of the associated line bundles of $P$ with the characters $\pi_i$ and $R_j$, respectively to $X_\Sigma$.

Note that $X_\Sigma$ admits a $(\mathbb{C}^\ast)^{\Sigma(1)}$-action. We denote by $P_\rho$ and $U_\rho$ the equivariant line bundles corresponding to $L_i$ and $\mathcal{O}(D_{\rho_j})$. In the $G$-equivariant $K$-group $K_G(X_\Sigma) \otimes \mathbb{Q}$, we
have the following multiplicative relation:

\[ U_\rho = \prod_{i=1}^s P_i^{\otimes Q_{i\rho}} \Lambda_\rho, \text{ where } \Lambda_\rho \text{ are generators of } \text{Repr}(\mathbb{C}^*)^{\Sigma(1)}. \]

Now let us compute the small $I$-function of $X_\Sigma$ with level structures. Let $\beta \in \text{Hom}_\mathbb{Z}(\text{Pic}^G(\mathbb{C}^*)^{\Sigma(1)}, \mathbb{Z})$ be an $L_\Sigma$-effective class. According to [33], Definition 7.2.1, a point in the quasimap graph space $QG_{0,k}^{L_\Sigma}(X_\Sigma, \beta)$ is specified by the following data

\[ ((C, p_1, \ldots, p_k), \{P_i|i = 1, \ldots, s\}, \{u_{i\rho}\}_{\rho \in \Sigma(1)}, \varphi), \]

where

- $(C, p_1, \ldots, p_s)$ is a connected, at most nodal, curve of genus 0 and $p_i$ are distinct nonsingular points of $C$,
- $P_i$ are line bundles on $C$ of degree $f_i := \beta(L_i)$,
- $u_{i\rho} \in \Gamma(C, L_{i\rho})$ where $L_{i\rho}$ is defined by
  \[ L_{i\rho} := \otimes_{\rho = 1}^s P_i^{\otimes Q_{i\rho}}, \]
- $\varphi : C \to \mathbb{P}^1$ is a regular map such that $\varphi_*[C] = [\mathbb{P}^1]$.

The stability conditions are discussed in Section 3.2. In the case when $(g, k) = (0, 0)$, we have $C \cong \mathbb{P}^1$ and $P_i \cong \mathcal{O}_{\mathbb{P}^1}(f_i)$. The line bundles $L_{i\rho}$ are isomorphic to $\mathcal{O}_{\mathbb{P}^1}(\sum_{i=1}^s f_i Q_{i\rho}) = \mathcal{O}_{\mathbb{P}^1}(\beta_{i\rho})$, where $\beta_{i\rho} := \beta(\mathcal{O}(D_{i\rho}))$. Therefore a point on $QG_{0,k}^{L_\Sigma}(X_\Sigma, \beta)$ is specified by sections $\{u_{i\rho} \in \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\beta_{i\rho}))|\rho \in \Sigma(1)\}$. We choose coordinates $[x_0, x_1]$ on $\mathbb{P}^1$ and consider the standard action $\mathbb{C}^*$-action defined by (7). Let $F_0$ be the distinguished fixed point locus for which the degree $\beta$ is concentrated only at 0. We have an identification

\[ F_0 \cong \bigcap_{\{\beta_{i\rho} < 0\}} D_{i\rho} \subset X_\Sigma \]

(24)

\[ (z_0 x_0^{\beta_{i\rho}}) \to (z_0), \]

where $(z_0)$ are the coordinates on $X_\Sigma$.

Let $R$ be a character of $G = (\mathbb{C}^*)^s$ defined by $t \cdot z = \prod_{i=1}^s t_i^{r_i} z$ where $r_i \in \mathbb{Z}$. Recall that the small $I$-function of $X_\Sigma$ of level $l$ and representation $R$ is defined by

\[ I_{R,l}(q) = 1 + \sum_{a} \sum_{\beta \neq 0} Q^\beta \chi_{D_{\rho}}(F_0, ev^*(\phi_a) \otimes \frac{\text{tr}_{\mathbb{C}^*}D_{R,l}}{\text{tr}_{\mathbb{C}^*}D_{R,h}}(N_{F_0/QG}^{\text{vir}}))^{\phi_{a}}, \]

It is not difficult to check that under the identification (24), we can identify the virtual normal bundle $N_{F_0/QG}^{\text{vir}}$ in $K^0(F_0)$ with

\[ N_{F_0/QG}^{\text{vir}} = \sum_{\{\beta_{i\rho} > 0\}} \sum_{i=1}^{\beta_{i\rho}} \mathcal{O}(D_{i\rho})|_{F_0} \otimes \mathbb{C}_{-i} - \sum_{\{\beta_{i\rho} < 0\}} \sum_{i=1}^{\beta_{i\rho}-1} \mathcal{O}(D_{i\rho})|_{F_0} \otimes \mathbb{C}_{i}, \]

where $\mathcal{C}_{a}$ denotes the representation of $\mathbb{C}^*$ on $\mathbb{C}$ with weight $a \in \mathbb{Z}$. Let $P$ be the universal principal $G$-bundle on $F_0 \times \mathbb{P}^1 \subset X_\Sigma \times \mathbb{P}^1$. Then the associated line bundle $P \times_G R$ can be identified with $\otimes_{i=1}^s L_i^{r_i} \otimes \mathcal{O}_{\mathbb{P}^1}(\beta_R)$, where $\beta_R := \sum_{i=1}^s r_i f_i$. Let $\pi : F_0 \times \mathbb{P}^1 \to F_0$ be the
Theorem 4.14. The small $I$-function of a toric variety $X_{\Sigma}$ of level $l$ and with representation $R$ is given by

$$I^{R,l}(q) = 1 + \sum_{\beta \neq 0} Q^\beta (\otimes_{i=1}^s \mathcal{L}_i^\ast)^{-l(\beta_R+1)} \otimes q^{l_\beta_R(\beta_R+1)/2} \prod_{\rho \in \Sigma(1)} \prod_{j=-\infty}^{\rho_0} (1 - O(-D_{\rho})q^j)$$

and its equivariant version is given by

$$I^{R,l,G}(q) = 1 + \sum_{\beta \neq 0} Q^\beta (\otimes_{i=1}^s \mathcal{L}_i^\ast)^{-l(\beta_R+1)} \otimes q^{l_\beta_R(\beta_R+1)/2} \prod_{\rho \in \Sigma(1)} \prod_{j=-\infty}^{\rho_0} (1 - U_{\rho}q^j)$$

where $P_i$ and $U_{\rho}$ are the equivariant line bundles corresponding to $\mathcal{L}_i$ and $O(D_{\rho})$, respectively.

We denote by $\text{St}$ and $\text{St}^\vee$ the standard representation and its dual representation of a torus. As corollaries of Theorem 4.14, we give proofs for Propositions 1.2, 1.4, and 1.6.

4.2.1. Proof of Proposition 1.2. Let the target be $X = (\mathbb{C} - 0)/\mathbb{C}^*$ where the action is the standard action. Then the Proposition follows directly from the Theorem 4.14.

4.2.2. Proof of Proposition 1.3. Let the target be $X_{a_1,a_2} = (\mathbb{C}^2 - \{(0,0)\})/\mathbb{C}^*$ with charge matrix $(a_1,a_2)$. As mentioned in Remark 1.5 and Remark 4.13, we only consider the untwisted component of the orbifold $I$-function and its formula is given by Theorem 4.14 (see [15] §5). Now let us prove the claim that the $I$-function equals the $J$-function when $(a_1,a_2) = (1,1)$ and $l \leq 2$. In this case, the target is $\mathbb{P}^1$. Notice that each coefficient of $Q^n$ is a rational function in $q$. When $l \leq 2$, the degree of the denominator minus the degree of the numerator is greater than 1. Since $I^{\text{St},l}_{X_{1,1}}$ is equal to $J^{\text{St},l}_{X_{1,1}}(t)$ for some $t$, this implies that $t = 0$. Therefore the $I$-function equals the $J$-function in this case.

4.2.3. Proof of Proposition 1.4. For positive integers $a, b$, we consider $X_{a,-b} = ((\mathbb{C} - 0) \times \mathbb{C})/\mathbb{C}^*$ with charge matrix $(a, -b)$. Let $\lambda, \mu$ be the generators of $\text{Repr}(\mathbb{C}^2)$ corresponding to the first and second projections of $((\mathbb{C}^*)^2$ onto its factors. From Theorem 4.14, the untwisted component of the orbifold $I$-function is given by

$$I^{\text{St},l}_{X_{a,-b}}(q) = 1 + \sum_{n \geq 1} p^{\binom{n-1}{2}} q^{\binom{n(n-1)}{2}} (1 - p\mu) (1 - pq_0) \cdots (1 - p\mu q_0) Q^n$$

$$= 1 + \sum_{n \geq 1} (-1)^bn p^{\binom{n(n-1)}{2}} q^{\binom{n(n-1)-bn}{2}} \mu^{-bn} (1 - p\mu) (1 - p\mu q_0) \cdots (1 - p\mu q_0 q_n) Q^n.$$
4.2.4. Proof of Proposition 1.5. We consider the target $O(-1)^{\oplus r}_{p^{-1}} = X_{1,-1} = \{(\mathbb{C}^* - 0) \times \mathbb{C}^r\} / \mathbb{C}^*$ with the charge vector $(1,1,\cdots,1,-1,-1,\cdots,-1)$. It follows from Theorem 4.14 that

$$I_{X_{1,-1}}^{\hat{S}^1, l = 1+s}(q) = 1 + \sum_{n \geq 1} Q^n p(l(n-1))q^{\frac{n(n-1)}{2}} \prod_{i=1}^r (p\mu_i)^{-n} \left(\frac{(p^{-1}\mu_1^{-1}; q)_n \cdots (p^{-1}\mu_r^{-1}; q)_n}{(p\lambda_1^{-1}; q)_n \cdots (p\lambda_s^{-1}; q)_n}\right)^{Q^n(q^{\frac{n(n-1)}{2}})^{1+s-r}}.$$

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