A restriction theorem for torsion-free sheaves on some elliptic manifolds

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Abstract

We prove that if $X$ is the total space of an elliptic principal bundle $\pi : X \to B$ which is non-kähler, then the restriction of any torsion-free sheaf on $X$ to the general fiber of $\pi$ is semi-stable.

1 Introduction

In the study of holomorphic vector bundles over a given compact complex manifold $X$, especially in the study of (semi)stable ones, a very useful tool is the study of their restrictions to general members of a given family of subvarieties of $X$. However, the restriction of a (semi)stable vector bundle to a submanifold is not always semistable. Still, under some strong hypothesis, such as $X$ is projective and the family of subvarieties is a family of divisors "ample enough", the restriction of a stable vector bundle to the general member remains (semi)stable: this is "Flenner’s restriction theorem", see [5]. Flenner’s theorem has been extended to the more general context of algebraic varieties in arbitrary characteristic (see e.g [6]), but, to the author’s knowledge, there is no such extension to the case of non-projective manifolds. The present note tackles this case.

2 Notations and basic facts

The context we are working is the following. We fix a compact complex manifold $B$ and an elliptic curve $F$. To every principal elliptic bundle $\pi : X \to B$ one can associate (up to the obvious action of $SL(2, \mathbb{Z})$) a couple of elements

$$(c'_1(\pi), c''_1(\pi)) \in H^2(B, \mathbb{Z}) \times H^2(B, \mathbb{Z})$$
called the Chern classes of the bundle $\pi$ (see e.g. [2]).

If at least one of the Chern classes is non vanishing in $H^2(B, \mathbb{R})$, one can prove by a standard argument using the Leray spectral sequence of the fibration that the homology class of any fiber $[F] \in H_2(X, \mathbb{R})$ vanishes; as the fibers are compact complex submanifolds, this shows that $X$ is not of Kähler type.

We also recollect the notion of stability; since we will use this concept for vector bundles on curves, we will only recall the definition in this case. Hence, a vector bundle $E$ on a smooth projective curve will be called stable (respectively semistable) if for any subbundle $\mathcal{F} \subset E$ with $0 < \text{rank}(\mathcal{F}) < \text{rank}(E)$ one has

$$\frac{\text{deg}(\mathcal{F})}{\text{rank}(\mathcal{F})} < \frac{\text{deg}(E)}{\text{rank}(E)}$$

(resp " $\leq $" for semistability). A vector bundle which is not semistable is called unstable.

Eventually, let us recall a concept which is of relevance only on non-algebraic complex manifolds. If $X$ is compact complex manifold and $\mathcal{F}$ is a coherent sheaf on $X$, then $\mathcal{F}$ is called reducible if there exist a coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ with $0 < \text{rank}(\mathcal{F}') < \text{rank}(\mathcal{F})$; if no such subsheaf exist then $\mathcal{F}$ is called irreducible. Notice that on projective manifolds all coherent sheaves are reducible; still, on general compact complex manifolds this is not always the case, as one can see for instance looking at the tangent bundle of a K3 surface $X$ with $\text{Pic}(X) = 0$ (the general K3 surface is so).

### 3 Some Lemmas

In the following we some lemmas, which are most likely classical and well-known; but since we don’t have any precise reference, we include the proofs here.

**Lemma 1** Let $\pi : X \to B$ be an elliptic principal bundle. If the homology class $[F] \in H_2(X, \mathbb{R})$ vanishes (i.e the Poincaré dual $PD_F$ is zero), then any proper closed analytic subset $Y \subset X, \text{dim}(Y) < \text{dim}(X)$, does not meet the general fiber.

**Proof.** The only non-obvious case is when $Y$ is a hypersurface. But in this case, if $Y$ meets all the fibers, then it meets the general fiber transversely in finitely many points. But then

$$0 < \#(Y \cap F) = \int_X PD_Y \wedge PD_F = 0$$
since $PD_F = 0$ by the assumption that $0 = [F] \in H_2(X, \mathbb{R})$.

**Lemma 2** For $X$ as in the previous Lemma and for any torsion-free sheaf $E$ on $X$ we have

$$\text{deg}(E|_F) = 0$$

for $F =$ general fiber of $\pi$.

**Proof.** Indeed, as $E$ is torsion-free, we see $\text{Sing}(E)$ has codimension at least two. Let $L = \text{det}(E)^{\vee\vee}$ be the bidual of the determinant of $E$; it is a reflexive sheaf of rank one on $X$, so it is a line bundle (cf e.g. [7]). Moreover, the map $\text{det}(E) \to L$ is an isomorphism outside $\text{sing}(E)$, so if $F$ is any fiber not meeting $\text{Sing}(E)$ we have

$$\text{deg}(E|_F) = \text{deg}(\text{det}(E)|_F) = \text{deg}(L|_F) = i^*(c_1(L))$$

where $i : F \to X$ is the inclusion of the fiber $F$. But as $[F] = 0$ in $H_2(X, \mathbb{R})$ we see $i^*(c_1(L)) = 0$, Q.E.D. Lemma.

**Lemma 3** If $F$ is an elliptic curve and if $E$ is a vector bundle of degree zero on $F$ which is generated by its global sections, then $E$ is trivial.

We use the following argument from L. Ein (cf [4], Proposition 1.1):

"**Lemma.** If $X$ is a compact complex manifold, and $E$ is a globally generated vector bundle on $E$ such that its dual $E^\vee$ has a section, then $E$ splits as $E = \mathcal{O}_X \oplus F$."

We do induction of $\text{rank}(E)$. For $\text{rank}(E) = 1$ the assertion is immediate. If $\text{rank}(E) \geq 2$, letting $K = \text{Ker}(H^0(F, E) \otimes \mathcal{O}_F \to E)$ we get an extension:

$$0 \to K \to H^0(F, E) \otimes \mathcal{O}_F \to E \to 0. \quad (1)$$

Now, either the extension splits (and hence $E$ is trivial), or

$$H^1(F, E^\vee \otimes K) \neq 0.$$  

As $\text{deg}(E) = 0$ we have also $\text{deg}(K) = 0$ so we further get by Riemann-Roch on $F$ that

$$H^0(F, E^\vee \otimes K) \neq 0. \quad (2)$$

Twisting the above extension (1) by $E^\vee$ we get

$$0 \to K \otimes E^\vee \to H^0(F, E) \otimes E^\vee \to E \otimes E^\vee \to 0$$
hence, from (2), we get
\[ H^0(F, E') \neq 0 \]
Applying Ein’s Lemma, we get \( E = O_F \oplus E_1 \). But \( E_1 \) has degree zero and is generated by its global sections too, so by the induction hypothesis, \( E_1 \) is trivial. Consequently, \( E \) is trivial too.

**Lemma 4** Let \( F \) be an elliptic curve and \( L \) a semistable vector bundle on \( F \) such that \( \text{deg}(L) = 0 \). Then there is a Zariski-open subset \( U \subset \text{Pic}_0(F) \) such that \( H^0(F, L \otimes I) = 0 \) for all \( I \in U \).

**Proof.** (See also [3]). Again, we do induction on \( \text{rank}(L) \). For \( \text{rank}(L) = 1 \) the claim is immediate (take \( U = \text{Pic}_0(F) \setminus \{ L' \} \)), so assume \( \text{rank}(L) > 0 \).

In the case \( H^0(F, L) = 0 \), from the existence of the Poincaré bundle and Grauert’s upper continuity theorem we get \( H^0(F, L \otimes I) = 0 \) for all \( I \) in a Zariski neighborhood of \( O_F \).

In the case \( h^0(F, L) > 0 \) take some \( s \in H^0(F, L) \), \( s \neq 0 \); it defines a map
\[ 0 \to O_F \to L \]
We infer that this map has torsion-free cokernel; since otherwise, moding out by the torsion of the cokernel, we would get a nontrivial map into \( L \) from a nontrivial, effective divisor on \( F \), contradicting the hypothesis that \( L \) is semistable. So \( L \) sits in an exact sequence
\[ 0 \to O_F \to L \to L' \to 0 \]
with \( L' \) =torsion-free (hence locally free, as \( F \) is a curve); in particular, \( \text{deg}(L') = 0 \). It is easy to see that \( L' \) is semistable too, so by the induction hypothesis \( H^0(F, L' \otimes I) = 0 \) for all \( I \) is some open subset \( U \subset \text{Pic}_0(F) \). So
\[ H^0(F, L \otimes I) = 0 \]
for all \( I \in U \setminus \{ O_F \} \), Q.E.D. Lemma.

Eventually, we recollect a fact which is true more generally

**Lemma 5** Let \( F \) be an elliptic curve and
\[ 0 \to L \to M \to R \to 0 \]
an exact sequence of vector bundles of \( F \) with
\[ \text{deg}(L) = \text{deg}(R) = 0. \]
If \( L \) and \( R \) are semistable, then \( M \) is semistable too.
Proof. Using Lemma 4 we get a line bundle $I \in \text{Pic}_0(F)$ such that

$$H^0(F, R \otimes I) = H^0(F, L \otimes I) = 0;$$

this implies $H^0(F, M \otimes I) = 0$ as well.

So, replacing $M$ by $M \otimes I$ we can further assume $H^0(F, M) = 0$. Now, if $M$ would be unstable, we would get a destabilizing vector subbundle $D \subset M$ with $\deg(D) > 0$. But $\deg(D) > 0$ implies $H^0(F, D) \neq 0$; so $H^0(F, M) \neq 0$ as well, contradiction, Q.E.D. Lemma.

4 The main result

We are now in position to state and prove the main result.

Theorem 1 Let $\pi : X \to B$ be an elliptic principal bundle with at least one of the Chern classes non-vanishing in $H^2(B, \mathbb{R})$ (in particular, $X$ is nonK"ahler). Then the restriction of any torsion-free sheaf $E$ on $X$ to the general fiber of $\pi$ is semi-stable.

Before proving it, let us make a small comment. As one can see, the theorem gives the semi-stability of the restriction of $E$ to the general fiber of $\pi$ with no apriori assumptions like (semi)stability for $E$. This is not completely surprising; in the non-projective context, more exactly on non-projective surfaces, the "Bogomolov inequality" $\Delta(E) \geq 0$, holds similarly for all torsion-free sheaves $E$ (cf [1], or [3] for a simpler proof), in contrast to the projective case, when it holds mainly for stable vector bundles.

Proof of the theorem. We do induction on the rank $r = rk(E)$. For $r = 1$ there is nothing to prove, so we assume $r \geq 2$.

Case 1: $E$ is reducible. That is, $E$ sits in an exact sequence

$$0 \to L \to E \to R \to 0$$

By the Lemma 2 we see that for a general fiber $F$ of $\pi$, $L|_F, R|_F$ are locally free of degree zero. More, by the induction hypothesis, both $L|_F, R|_F$ are also semistable, so $E|_F$ is semistable too, by Lemma 5.

Case 2: $E$ is irreducible. We distinguish again two subcases:

Subcase 2.1: $\pi_*(E) = 0$. In this case, $H^0(F, E|_F) = 0$ for the general fiber. But as also $\deg(E|_F) = 0$ for the general fiber $F$, we see at once that $E|_F$ is semistable. Indeed, if this is not the case, then a destabilizing subsheaf $D \subset E$ would have $\deg(D) > 0$; but then $h^0(F, D) > 0$ so $h^0(F, E|_F) > 0$ too, contradiction.
Subcase 2.2: $\pi_*(E) \neq 0$. Let $\alpha : \pi^*\pi_*(E) \to E$ be the canonical morphism and let $F = \text{Im}(\alpha)$. As $E$ is irreducible and as $\alpha$ is non-trivial, we see we have

$$\text{rank}(F) = \text{rank}(E).$$

Let $Y = \text{Supp}(E/F)$; by Lemma [1] $Y$ cannot meet all the fibers of $\pi$ so for the general fiber $F$ we have $F|_F = E|_F$; more, by Lemma [2] we can assume $\deg(E|_F) = 0$.

So, for the general fiber $F$ we have a surjection

$$\pi^*\pi_*(E)|_F \to E|_F.$$

But

$$\pi^*\pi_*(E)|_F$$

is trivial, so $E|_F$ is spanned by its global sections. As it is also of degree zero, it follows by Lemma [3] that $E|_F$ is trivial, in particular semi-stable.

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