The periodic steady-state solution for queues with Erlang arrivals and service and time-varying periodic transition rates

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Received: 28 July 2021 / Revised: 13 July 2022 / Accepted: 15 July 2022 / Published online: 10 August 2022
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Abstract
We study a queueing system with Erlang arrivals with \( k \) phases and Erlang service with \( m \) phases. Transition rates among phases vary periodically with time. For these systems, we derive an analytic solution for the asymptotic periodic distribution of the level and phase as a function of time within the period. The asymptotic periodic distribution is analogous to a steady-state distribution for a system with constant rates. If the time within the period is considered part of the state, then it is a steady-state distribution. We also obtain waiting time and busy period distributions. These solutions are expressed as infinite series. We provide bounds for the error of the estimate obtained by truncating the series. Examples are provided comparing the solution of the system of ordinary differential equation with a truncated state space to these asymptotic solutions involving remarkably few terms of the infinite series. The method can be generalized to other level independent quasi-birth-death processes if the singularities of the generating function are known.

Keywords Erlang queues · Time-varying · Waiting time · Matrix analytic methods · Asymptotic periodic solution

Mathematics Subject Classification 60K25 · 05A15 · 65C40 · 60J27

1 Introduction

The goal of this paper is to provide an analytic formula for the asymptotic periodic distribution of \( E_k(t)/E_m(t)/1 \) queue with time-varying periodic transition rates.

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Definition 1 (Asymptotic period distribution) The asymptotic periodic distribution of an ergodic stochastic process with time-varying periodic transition rates is a distribution such that
\[
\lim_{n \to \infty} p(t + nT) = p(t), \quad \text{for } n \in \mathbb{Z}
\]
where \( T \) is the length of the period and \( p(t) \) is a vector giving the probability distribution at time \( t \). In this paper, we take \( T = 1 \).

To find this distribution, we employ a generating function approach that is applicable to any ergodic-level-independent quasi-birth-death (QBD) process with time-varying periodic rates. A QBD is a stochastic process with a two-dimensional state space, \( \{Y(t), J(t)\} \), where \( Y(t) \in \{0, 1, 2, \ldots\} \) represents the level of the process and \( J(t) \in \{0, 1, 2, \ldots, N\} \) the phase. Transitions are permitted only to adjacent levels, but arbitrary transitions among phases are permitted. A QBD is level-independent if transition rates do not depend on the level except possibly for a finite number of boundary states. QBDs are often applied to queueing processes where the level represents the number of customers in the queue and the phase represents a phase of the arrival or service processes or both of these.

Many of the methods described in this paper apply to general level-independent QBDs, though it may be necessary to use numerical inverse transform methods for \( Z \)-transforms for some queueing models depending on the structure of the model. Methods for inverting \( Z \)-transforms are described in a very useful survey paper by Horváth, Mészáros, and Telek [13].

These methods are not needed in the case of Erlang arrivals and departures which is studied in this paper because we can provide explicit formulas for the relevant generating functions and their coefficients. We refer the interested reader to [20] in which we apply the technique employed in this paper to several simple queues and to [21] for an example applying this technique to a QBD which models a two priority queue with finite waiting room for type-2 customers and infinite waiting room for type-1 customers.

In this paper, we explore several quantities related to the Erlang arrival, Erlang service queue with time-varying periodic transition rates. The \( E_k(t)/E_m(t)/1 \) queue is a single server queue. Arrivals occur in \( k \) phases visited sequentially with transitions among phases occurring at rate \( \lambda(t) \). The transition rate is a periodic function of time. Throughout this paper, we take the length of the period to be one as results can be rescaled for periods of arbitrary length. The service process is also Erlang. It consists of \( m \) phases. Transitions among phases occur at rate \( \mu(t) \) (also periodic), with each phase completed in sequence.

This paper is divided into several sections. Section 2 provides an overview of the literature on queues with Erlang arrival and/or service. Section 3 provides some background on generating functions through a series of examples to motivate the method for our analysis of the \( E_k(t)/E_m(t)/1 \) queue. We show the role of roots of unity and fractional powers of the indeterminate in using generating functions to study sequences. We look at scalar examples of generating functions that arise naturally in the study of stochastic processes with time-varying periodic transition rates. We treat the \( M(t)/M(t)/1 \) queue which corresponds to when \( k \) and \( m \) equal one.
for an $E_k(t)/E_m(t)/1$ queue. Each of these ideas plays a role in the analysis of the $E_k(t)/E_m(t)/1$ queue. Section 4 provides a brief overview of results for QBDs with time-varying periodic transition rates. In Sect. 5, we present the main result, an exact formula for the asymptotic periodic probability distribution for the $E_k(t)/E_m(t)/1$ queue with time-varying periodic transition rates. We prove this theorem in Sects. 5 and 6. We employ an eigen decomposition of the matrix generating function for the unbounded process (the process which permits transitions to levels below zero). Roots of unity and fractional powers of the indeterminate $z$ appear in this decomposition. We perform singularity analysis of the generating function for the level distribution in Sect. 6 to complete the proof begun in Sect. 5. In Sect. 7, we provide error bounds for the level probabilities and show that for smooth functions, the truncated series for our exact formulas can be made arbitrarily close. Section 8 gives formulas for the waiting time distribution, and Sect. 9 derives the busy period distribution as the solution of a Volterra equation of the second kind. In Sect. 10, we describe how to compute the boundary probabilities for the process. Section 11 provides a brief summary and discussion of how this approach may be extensible to other models. We also provide an appendix with some basic results related to evolution operators. Evolution operators are generalizations of exponential functions or matrix exponentials. They appear when the infinitesimal generator of the process depends on time as the processes we study here do.

2 Literature review

When service is exponential, the standard deviation of the service time is equal to its expectation. For Erlang-$m$ service (when rates are constant), with parameter $\mu$, the mean is $m\mu$ and the variance is $m\mu^2$. Of course, similar facts hold for Erlang-$k$ arrivals. This is an advantage when modeling processes for which the variance and standard deviation of the service distribution are not equal. Erlang-$m$ service or Erlang-$k$ arrivals also lets us track the stage of service or arrival, respectively, of the customer. These are two advantages cited by Gayon, et al, in choosing Erlang service for modeling a single-item-make-to-stock production system in which items have Erlang production times [8]. Foh and Zukerman [7] used Erlang service to model random access protocols. Jayasuriya, et al [14] use generalized Erlang service to model channel holding times in a mobile environment. Kuo and Wang [34] use an $M/E_m/1$ queue to model a machine repair problem. Maritas and Xirokostas [22] also study a machine repair problem using Erlang service. Their model allows for more than one server. Grassmann [11] provides additional examples of applications of $E_k/E_m/1$ queues.

Many researchers have studied the $E_k/E_m/1$ queue, or the simpler $E_k/M/1$ or $M/E_m/1$ queues with constant transition rates. A traditional approach using generating functions can be found, for example, in Saaty [27], Kleinrock [15] and Medhi [23]. This is the approach that we use in this paper, extending it to queues with time-varying periodic transition rates. This paper extends related work applying this approach to other queues with time-varying periodic transition rates. See [20] when the generating
functions for the queue-length process are scalar, and [21] for QBD processes when the generating functions for the queue-length process are vectors with a component for each phase. In the 2021 paper, we use a two priority queue with finite waiting room for priority 2 customers as an extended example.

Queues with Erlang arrivals, Erlang service or both, have been analyzed by Smith [30], Syski [31] and Takács [32] using Laplace transform techniques. Takács studies the waiting time, queue length and busy period for a queue with Erlang arrivals and general service. Truslove [33] considers this queue with finite waiting room. Leonenko [18] studies the transient solution to the $M/E_k/1$ queue following an approach due to Parthasarathy, [24]. A paper by Griffiths, Leonenko and Williams [12] also provides an exact solution to the transient distribution of the $M/E_k/1$ queue. Arizono, et al [2] use generating functions for the number of minimal lattice paths to find the equilibrium distribution for the $E_k/E_m/1$ queue length distribution.

$E_k/E_m/1$ queues may also be analyzed using matrix analytic methods [16]. Grassmann [11], in his 2011 paper, derives an effective method for finding the characteristic roots of the $k + m$ degree polynomial related to the waiting time distribution that arises from these methods. He builds on the approach due to Syski [31] and Smith [30]. Ivo Adan and Yiqiang Zhao [1] study a $GI/E_m/1$ system and show that for arbitrarily distributed inter-arrival times and Erlang service, the waiting time distribution can be expressed as the finite sum of exponentials which depend on the roots of an equation. They also develop a method for finding these roots. Luh and Liu [19] study the $E_k/E_m/1$ queue and show that the roots of the characteristic polynomial associated with the process are simple if the arrival and service rates are real. They use this result to construct a general solution space of vectors for the stationary solution of the queue length distribution. Poyntz and Jackson [26] find the steady-state solution for the $E_k/E_m/r$ queue, illustrating the method with the $E_k/E_m/2$ queue due to the “tediousness of the algebra.”

In this paper, we study the $E_k(t)/E_m(t)/1$ queue in which the transition rates vary periodically with time. For two fairly recent surveys of research on queueing systems with time-varying parameters, the reader is referred to the papers by Schwarz, et al [28] and Whitt [38].

3 Generating function examples

Why use the generating function approach to solve for the asymptotic periodic solution for the $E_k(t)/E_m(t)/1$? We take this approach because a generating function encodes information about the structure of its corresponding sequence. Essentially, it tells a story. Generating functions have an extensive history in mathematics and probability theory. We will cite a few sources from that rich tradition, but our main inspiration comes from the work of Robert Sedgewick and Philippe Flajolet (based on foundational work of many predecessors). In their texts: Analysis of Algorithms and Analytic Combinatorics, they develop the symbolic method, a systematic approach to the study of generating functions and combinatorial structures. Simple theorems provide a translation mechanism between probability distributions and operations on generating functions ([6] p. 27, [29] chapter 5, pp. 219–256). The structure of the
generating functions can be further exploited to obtain asymptotics of the coefficients which yield our discrete probability distribution.

**Definition 2** The ordinary generating function (OGF) of a sequence \(\{a_n\}_{n=0}^{\infty}\) is the formal power series

\[
A(z) = \sum_{n=0}^{\infty} a_n z^n.
\]

We will be working with probability generating functions (PGF) where the sequence of coefficients represent a discrete probability mass function.

**Example 1** (Poisson random variable) A fundamental building block in the remainder of this paper is the Poisson distribution with parameter \(\lambda\). Let \(p_n = \frac{\lambda^n}{n!} e^{-\lambda}\), then its pgf is given by \(P(z) = \sum_{n=0}^{\infty} p_n z^n = e^{\lambda(z-1)}\).

**Example 2** (Poisson process) For a Poisson process with rate \(\lambda\), the generating function in Example 3 to

\[
\sum_{n=0}^{\infty} p_n(z,t) z^n = e^{\lambda(t) (z-1)}.
\]

**Example 3** (Poisson process, time-varying parameter) Next, we consider a Poisson process with time-varying rate \(\lambda(t)\). In this case, \(p_n(t,s) = \frac{(\int_s^t \lambda(v) dv)^n}{n!} e^{-\int_s^t \lambda(v) dv}\) gives the probability of \(n\) events in the time interval \([s, t]\). Its pgf is given by \(P(z,t,s) = \sum_{n=0}^{\infty} p_n(t,s)z^n = e^{\int_s^t \lambda(v) dv(z-1)}\). \(P(z,t,s)\) is an evolution operator (see the appendix) satisfying the initial value problem:

\[
\frac{\partial u(z, t)}{\partial t} = u(z, t) \lambda(t) (z-1)
\]

\[
u(z, s) = 1.
\]

Observe that \(P(z, t, v) P(z, v, s) = P(z, t, s) = e^{\int_s^t \lambda(v) dv(z-1)}\), and \(P(z, t, t) = 1\). Note that if \(s = 0\) and \(\lambda(t) = \lambda\) is constant, this reduces to example 2.

**Example 4** (Poisson process for steps to the left) Next, we consider a Poisson process with time-varying rate \(\mu(t)\) representing the rate at which a process makes steps to the left. Our probability distribution is given by \(p_{-n}(t,u) = \frac{(\int_u^t \mu(v) dv)^n}{n!} e^{-\int_u^t \mu(v) dv}\) for \(n = 0, 1, 2, \ldots\). The subscript reflects the fact that \(n\) steps to the left is \(-n\) steps to the right. Our generating function for steps to the left is given by \(P_{\text{left}}(z, t, u) = \sum_{n=0}^{\infty} p_{-n}(t, u) z^{-n} = e^{\int_u^t \mu(v) dv(z^{-1}-1)}\). We generalize our generating function definition in the obvious way, to permit negative exponents.

**Example 5** (Random walk with time-varying parameters) We change the labeling of the generating function in Example 3 to

\[
P_{\text{right}}(z, t, u) = \sum_{n=0}^{\infty} p_n(t, u) z^n = e^{\int_u^t \lambda(v) dv(z-1)}
\]

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to emphasize that this is the generating function for steps to the right occurring at rate \( \lambda(t) \). We assume that steps to the right and steps to the left occur independently and we define a new generating function for a random walk with time-varying transition rates \( \lambda(t) \) for steps to the right and \( \mu(t) \) for steps to the left as

\[
P_{\text{RW}}(z, t, u) = P_{\text{right}}(z, t, u)P_{\text{left}}(z, t, u)
= \exp \left\{ \int_u^t \left( \lambda(\nu)(z - 1) + \mu(\nu)(z^{-1} - 1) \right) d\nu \right\}.
\]

We use the notation \([z^n]F(z)\) to represent the coefficient on \(z^n\) of the generating function \(F(z)\). So the probability of \(n\) more steps to the right than to the left is given by

\[
p_{\text{RW}}^n(t, u) = [z^n]P_{\text{RW}}(z, t, u) = e^{-\int_u^t (\lambda(\nu) + \mu(\nu)) d\nu} \left( \int_u^t \lambda(\nu) d\nu \right)^{n/2} \left( \int_u^t \mu(\nu) d\nu \right)^{\ell/2} I_n \left( 2 \sqrt{\left( \int_u^t \lambda(\nu) d\nu \right) \left( \int_u^t \mu(\nu) d\nu \right)} \right).
\]

where \(I_n(x)\) is the modified Bessel function of the first kind defined by

\[
I_n(x) = \left( \frac{x}{2} \right)^{n/2} \sum_{\ell=0}^{\infty} \frac{(x/2)^{2\ell}}{\ell!(n+\ell)!}.
\]

Note that we could reason directly to obtain the random walk probabilities. If there are \(n\) more steps to the right than to the left, then for some integer \(\ell\) there will be \(\ell\) steps to the left and \(n+\ell\) steps to the right. The \(n+\ell\) steps to the right occur with Poisson probability \(\left( \int_u^t \lambda(\nu) d\nu \right)^{n+\ell} / (n+\ell)!\) and the \(\ell\) steps to the left occur with Poisson probability \(\left( \int_u^t \mu(\nu) d\nu \right)^{\ell} / \ell!\), so

\[
p_n(t, u) = e^{-\int_u^t (\lambda(\nu) + \mu(\nu)) d\nu} \sum_{\ell=0}^{\infty} \frac{\left( \int_u^t \lambda(\nu) d\nu \right)^{n+\ell}}{(n+\ell)!} \frac{\left( \int_u^t \mu(\nu) d\nu \right)^{\ell}}{\ell!}.
\]

A little bit of algebra will show that the expressions given for \(p_n(t, u)\) in Eqs. 2 and 3 are equivalent. Expressing the quantity in terms of the modified Bessel function suggests efficient numerics for computation. Expressing it in terms of Poisson probabilities provides a better understanding of how the distribution can be constructed.

\[
\text{Observe that each of the generating functions in Examples 3, 4 and 5 are evolution operators.}
\]
**Example 6** (Transient solution of \(M(t)/M(t)/1\) queue with time-varying parameters) The simplest \(E_k(t)/E_m(t)/1\) queue is the \(M(t)/M(t)/1\) queue when \(k = m = 1\). For this model, we have the system of ordinary differential equations

\[
\dot{p}_0(t) = -\lambda(t)p_0(t) + \mu(t)p_1(t)
\]

\[
\dot{p}_n(t) = \lambda(t)p_{n-1}(t) - (\lambda(t) + \mu(t))p_n(t) + \mu(t)p_{n+1}(t) \quad \text{for } n > 0.
\]

Define the generating function

\[
P(z, t) = \sum_{n=0}^{\infty} p_n(t)z^n.
\]

We obtain an ordinary differential equation for the generating function by multiplying the \(n\)th differential equation in the system given in Eq. (4) by \(z^n\) and summing over all \(n\).

\[
\dot{P}(z, t) = \left(\lambda(t)(z - 1) + \mu(t)(z^{-1} - 1)\right)P(z, t) + \mu(t)(1 - z^{-1})p_0(t)
\]

has solution

\[
P(z, t) = \int_u^t \mu(v)(z^{-1} - 1)p_0(v) \exp \left\{ \int_v^t (\lambda(\xi)(z - 1) + \mu(\xi)(z^{-1} - 1))d\xi \right\} dv + P(z, u) \exp \left\{ \int_u^t (\lambda(\xi)(z - 1) + \mu(\xi)(z^{-1} - 1))d\xi \right\}.
\]

This is the transient solution for the generating function for the single server queue. Notice that the formula tells a story about the behavior of the queueing process. The expression \(\exp \left\{ \int_v^t (\lambda(\xi)(z - 1) + \mu(\xi)(z^{-1} - 1))d\xi \right\}\) is the random walk generating function given in Example 5, so the story of the equation tells is that either the process has hit the boundary at some time \(u \leq v \leq t\) \((p_0(v)\mu(v))\), but not gone below the boundary \((-p_0(v)\mu(v)z^{-1})\), or the process evolved from its initial condition at time \(u\) \((P(z, u))\) according to the random walk process during the time-interval \([t, u]\) \((\exp \left\{ \int_u^t (\lambda(\xi)(z - 1) + \mu(\xi)(z^{-1} - 1))d\xi \right\})\).

**Example 7** (Steady-state solution of \(M/M/1\) queue, constant rates) In this paper, we are interested in the asymptotic periodic solution. In the constant rate case, we can assume that the derivatives on the left-hand-side in the system of Eq. (4) are equal to zero. We can then solve directly for the steady-state solution knowing that solutions to linear recurrences with constant coefficients must be a linear combination of terms of the form \(n^j\beta^n\) where \(\beta\) is a root of the characteristic polynomial of the recurrence. Alternatively, we can solve for the generating function and use a partial fractions decomposition of the generating function to find its coefficients. The generating function will be a rational function since linear recurrences with constant...
coefficients always have rational generating functions and rational generating always solve linear recurrences. The generating function in the constant rate case is given by

\[ P(z) = \frac{\mu p_0 (z^{-1} - 1)}{\lambda (z - 1) + \mu (z^{-1} - 1)} = \frac{p_0}{1 - \frac{\lambda}{\mu} z}, \]

so the series solution is given by

\[ P(z) = p_0 \sum_{n=0}^{\infty} \left( \frac{\lambda}{\mu} \right)^n z^n \]

where \( p_0 = \left( 1 - \frac{\lambda}{\mu} \right) \). The generating function for this well-known steady-state solution also tells us about our stochastic process. For any rational generating function, we can read the linear recurrence that defines the sequence in the denominator. In this case, \( p_n = \frac{\lambda}{\mu} p_{n-1} \). Information about the initial term(s) of the sequence are embedded in the numerator of the rational generating function.

If we have a rational generating function for the periodic case, then we can use well-known methods. Poles of the generating function that are inside the unit circle will be zeros of the numerator, providing information that will help find initial terms of the sequence of probabilities, and poles that are outside of the unit circle provide information about the asymptotic behavior of the probabilities, \( p_n \) as \( n \to \infty \). Indeed, we can obtain an explicit formula for the \( p_n \) as a linear combination of powers of the reciprocals of the poles if the poles are all simple. If the poles are not simple, then our solution will also involve polynomial in \( n \). Since we know that a linear recurrence has a rational generating function, we know that the \( p_n \) will have solutions of this form, and we can just compute the solutions directly. This approach assumes that the generating function is rational. However, in the case of time-varying periodic parameters, it is not. In our next example, we look at how to solve this case.

**Example 8** (Asymptotic periodic solution of \( M(t)/M(t)/1 \) queue, time-varying periodic rates) To find the generating function when transition rates are time-varying and periodic (throughout this paper, we assume the period is of length one), we continue from the transient case in Example 6. In Eq. (6), we solved for the transient probability generating function. If the queueing process is ergodic and the rates periodic and the process has run for a long time, then \( P(z, t) = P(z, t-1) \). We replace the generic time \( u \) in Eq. (6) with the time \( t-1 \), that is, the time one period prior to \( t \). Also, note that \( \int_{t-1}^{t} \lambda(v) dv = \bar{\lambda} \), the average value of the parameter over the period. Similarly, \( \int_{t-1}^{t} \mu(v) dv = \bar{\mu} \). This yields the generating function

\[ P(z, t) = \int_{t-1}^{t} \mu(v)(1 - z^{-1}) p_0(v) \exp \left\{ \int_{v}^{t} (\lambda(\xi)(z - 1) + \mu(\xi)(z^{-1} - 1)) d\xi \right\} dv \times \left( 1 - \exp \left\{ \bar{\lambda}(z - 1) + \bar{\mu}(z^{-1} - 1) \right\} \right)^{-1} . \]

(7)
First, notice that the generating function is not rational in $z$. The generating function is meromorphic, that is, it is analytic except at isolated points which are poles of the function. The singularity at $z = 0$ is a removable singularity of the generating function since $\lim_{z\to 0} P(z, t) = p_0(t)$.

Notice that the random walk generating function from example 5 appears twice in the formula for the $M/M/1$ queue with periodic transition rates, so let us examine how to interpret this formula. Consider the expression $(1 - \exp \left\{ \bar{\lambda}(z - 1) + \bar{\mu}(z^{-1} - 1) \right\})^{-1}$. Define

$$P_{\text{SEQ}}(z, t) = (1 - \exp \left\{ \bar{\lambda}(z - 1) + \bar{\mu}(z^{-1} - 1) \right\})^{-1}$$

$$= \sum_{j=0}^{\infty} \exp \left\{ \int_{t}^{t+j} \left( \lambda(\nu)(z - 1) + \mu(\nu)(z^{-1} - 1) \right) d\nu \right\}$$

$$= \sum_{j=0}^{\infty} \exp \left\{ j(\bar{\lambda}(z - 1) + \bar{\mu}(z^{-1} - 1)) \right\}.$$

This is the generating function for a sequence of random walks. This generating function is not a probability generating function. It is easily seen that $P_{\text{SEQ}}(1, t)$, not only is not equal to one, but in fact it diverges.

For concreteness, let us think of each period as a day. The coefficient $[z^n]P_{\text{SEQ}}(z, t) = \sum_{j=0}^{\infty} p_{nRW}(t + j, t)$ gives the expected number of days that the process is $n$ steps to the right of where it began at time $t$ within the initial day when the process is observed at time $t$ within each subsequent day.

The expression

$$P_{\text{SEQ}+}(z, t, u) = \exp \left\{ \int_{t}^{u} \left( \lambda(\xi)(z - 1) + \mu(\xi)(z^{-1} - 1) \right) d\xi \right\}$$

$$\times \left( 1 - \exp \left\{ \bar{\lambda}(z - 1) + \bar{\mu}(z^{-1} - 1) \right\} \right)^{-1}$$

defines a generating function similar to the generating function $P_{\text{SEQ}}(z, t)$. The coefficient $[z^n]P_{\text{SEQ}+}(z, t, u) = \sum_{j=0}^{\infty} p_{nRW}(t + j, u)$ gives the expected number of days that the process is $n$ steps to the right of where it began at time $u$ within the initial day when the process is observed at time $t$ within each subsequent day.

The generating function for the asymptotic periodic queue length distribution for the $M(t)/M(t)/1$ queue given in Eq. (7) then tells the story that there is a last time
that the process hits the boundary at zero \((\mu(v)p_0(v))\), the process then evolves according to a random walk \((P_{SEQ+}(z, t, v))\) making a change of \(n\) more steps to the right than to the left with a correction for those walks that travel below zero \((-z^{-1}\mu(v)p_0(v)P_{SEQ+}(z, t, v))\) and make a net change of \(n\) steps to the right.  

The previous example derives the generating function for the \(M(t)/M(t)/1\) queue with time-varying rates, but does not show how to find the coefficients. To see how to do that, we provide two examples taken from Flajolet and Sedgewick’s *Analytic Combinatorics*, pp. 268-269. We follow their exposition.

The following theorem will be useful in analyzing the generating functions we encounter for processes with time-varying periodic rates.

**Theorem 1** (Expansion of meromorphic functions, Flajolet and Sedgewick, [6, p. 258]) Let \(f(z)\) be a function meromorphic at all points of the closed disc \(|z| \leq R\), with poles at points \(\alpha_1, \alpha_2, \ldots, \alpha_m\). Assume that \(f(z)\) is analytic at all points of \(|z| = R\) and at \(z = 0\). Then there exist \(m\) polynomials \(\{\Pi_j(x)\}_{j=1}^m\) such that:

\[
f_n \equiv [z^n] f(z) = \sum_{j=1}^m \Pi_j(n) \alpha_j^{-n} + O(R^{-n}). \tag{8}
\]

Further, the degree of \(\Pi_j\) is equal to the order of the pole of \(f\) at \(\alpha_j\) minus one.

**Example 9** (Bernoulli numbers) The Bernoulli numbers, written \(B_n\), have the exponential generating function

\[
B(z) = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = \frac{z}{e^z - 1}.
\]

Note that the exponential generating function for the sequence \(\{B_n\}_{n=0}^{\infty}\) is the ordinary generating function for the sequence \(\{\frac{B_n}{n!}\}_{n=0}^{\infty}\). \(B(z)\) has poles at \(\chi_\ell = 2\pi i \ell, \ell \in \mathbb{Z}\backslash\{0\}\). These are simple poles, so the polynomials of Theorem 1 are constants. The residue at \(\chi_\ell\) is equal to \(\chi_\ell\), so

\[
\frac{z}{e^z - 1} \sim \frac{\chi_\ell}{z - \chi_\ell} = \frac{-1}{1 - z/\chi_\ell} \text{ as } z \to \chi_\ell.
\]

Applying Theorem 1 for \(n > 1\), we have that

\[
\frac{B_n}{n!} = \sum_{\ell=1}^{j} \chi_\ell^{-n} + \sum_{\ell=1}^{j} \chi_{-\ell}^{-n} + O(R^{-n})
\]

where \(R\) is chosen so that the contour \(|z| = R\) is between poles, that is, \(2\pi j < R < 2\pi (j + 1)\). As \(j \to \infty\), since the error term \(O(R^{-n})\) goes to zero, we have the exact formula,
\[\frac{B_n}{n!} = \sum_{\ell=1}^{\infty} \chi_\ell^{-n} + \sum_{\ell=1}^{-\infty} \chi_{-\ell}^{-n}, \quad n > 1.\]

Substituting for \(\chi_\ell = 2\pi i \ell\), we have
\[\frac{B_n}{n!} = \frac{1}{(2\pi i)^n} \sum_{\ell=1}^{\infty} \left( \frac{1}{\ell^n} + \frac{1}{(-\ell)^n} \right),\]
which is zero for odd \(n > 1\), so
\[\frac{B_{2n}}{(2n)!} = (-1)^{n-1} 2^{1-2n} \pi^{-2n} \sum_{\ell=1}^{\infty} \frac{1}{\ell^{2n}} = (-1)^{n-1} 2^{1-2n} \pi^{-2n} \zeta(2n)\]
where \(\zeta(n)\) is the Riemann zeta function \(\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}\).

**Example 10** (Surjection numbers) The surjection numbers have exponential generating function
\[R(z) = \sum_{n=0}^{\infty} \frac{R_n}{n!} z^n = \frac{1}{2 - e^z}.\]

There are simple poles at
\[\chi_\ell = \ln 2 + 2\pi i \ell, \quad \ell \in \mathbb{Z}.\]

Hence, reasoning as in Example 9, the exact formula is
\[\frac{R_n}{n!} = \frac{1}{2} \sum_{\ell=-\infty}^{\infty} \chi_\ell^{-n-1}.\]

**Example 11** (Time-varying \(M(t)/M(t)/1\) queue with periodic rates, continued) The poles of the generating function for the time-varying \(M(t)/M(t)/1\) queue with periodic rates given in Eq. (7) are solutions of the quadratic equation
\[\tilde{\lambda} z^2 - (\tilde{\lambda} + \tilde{\mu} + 2\pi i \ell) z + \tilde{\mu} = 0 \text{ for } \ell \in \mathbb{Z}.\]

The poles on or inside the unit circle are roots of the numerator, so we are interested in the poles outside the unit circle. These are given by
\[\chi_\ell = \frac{1}{2\tilde{\lambda}} \left( \tilde{\lambda} + \tilde{\mu} + 2\pi i \ell + \sqrt{(\tilde{\lambda} + \tilde{\mu} + 2\pi i \ell)^2 - 4\tilde{\lambda}\tilde{\mu}} \right).\]

The exact solution is then,
To find $p_0(t)$, we use the fact that for all $t$, $\sum_{n=0}^{\infty} p_n(t) = 1$.  

We are interested in $E_k(t)/E_m(t)$ queues with $k$ and/or $m$ greater than one. Before moving to the fully general problem, we look at generating functions for Erlang–$k$ arrivals. To understand this generating function, we need some background on generating functions and roots of unity.

**Remark 1 (Roots of unity and generating functions)** Define

$$\omega_K = e^{\frac{2\pi i}{K}} = \cos\left(\frac{2\pi}{K}\right) + i \sin\left(\frac{2\pi}{K}\right),$$

a $K$th primitive root of unity. Let $A(z)$ be the ordinary generating function associated with an arbitrary sequence $\{a_n\}_{n=0}^{\infty}$ so that $A(z) = \sum_{n=0}^{\infty} a_n z^n$. Define $A^{(K,j)}(z) = \sum_{n=0}^{\infty} a_{Kn+j} z^n$, the generating function for the sequence $\{a_{Kn+j}\}_{n=0}^{\infty}$. Then

$$A^{(K,j)}(z) = \frac{1}{K} z^{-j} \sum_{\ell=0}^{K-1} A(\omega_K^{\ell} z^{-\frac{1}{K}}) \omega_K^{-j\ell}. \quad (10)$$

For example, the primitive second root of unity is $e^{\pi i} = -1$. For $K = 2$ and $j = 0$, the generating function for the sequence $\{a_{2n}\}_{n=0}^{\infty}$ is

$$A^{(2,0)}(z) = \frac{1}{2} \sum_{\ell=0}^{1} A(\omega_2^{\ell} z^{-\frac{1}{2}}) = \frac{1}{2} \left(A\left(z^{-\frac{1}{2}}\right) + A\left(-z^{-\frac{1}{2}}\right)\right).$$

For $K = 2$ and $j = 1$, the generating function for the sequence $\{a_{2n+1}\}_{n=0}^{\infty}$ is

$$A^{(2,1)}(z) = \frac{1}{2z^\frac{1}{2}} \sum_{\ell=0}^{1} A(\omega_2^{\ell} z^{-\frac{1}{2}}) \omega_2^{-\ell} = \frac{1}{2z^\frac{1}{2}} \left(A\left(z^{-\frac{1}{2}}\right) - A\left(-z^{-\frac{1}{2}}\right)\right).$$

Consider the perhaps less familiar example, for the sequence $\left\{\binom{n}{2}\right\}_{n=0}^{\infty}$, we have the generating function $A(z) = \frac{z^2}{(1-z)}$. The sequence $\left\{\binom{3n}{2}\right\}_{n=0}^{\infty}$ has the generating function

$$A^{(3,0)}(z) = \frac{1}{3} \sum_{\ell=0}^{2} A(\omega_3^{\ell} z^{-\frac{1}{3}}) = \frac{3z(2z+1)}{(1-z)^3}.$$
The sequence \( \left\{ \binom{3n+1}{2} \right\}_{n=0}^{\infty} \) has the generating function

\[
A^{(3,1)}(z) = \frac{1}{3z^2} \sum_{\ell=0}^{2} A(\omega_3^\ell z^\frac{1}{3}) \omega_3^{-\ell} = \frac{3z (z + 2)}{(1 - z)^3}.
\]

For the sequence, \( \left\{ \frac{1}{n!} \right\}_{n=0}^{\infty} \), \( B(z) = e^z \). The generating function for \( \left\{ \frac{1}{(3n)!} \right\}_{n=0}^{\infty} \) is

\[
B^{(3,0)}(z) = \frac{2}{3} e^{-z^{1/3}} \cos \left( \frac{\sqrt{3}}{2} z^{1/3} \right) + \frac{1}{3} e^{z^{1/3}}.
\]

See Herbert Wilf’s text *generatingfunctionology* [39] for more details on generating functions, the role of roots of unity and additional examples.

**Example 12** (Erlang–\( k \) arrival process) For this arrival process, arrivals occur after the completion of \( k \) sequential phases, with each phase distributed exponentially with time-varying parameter, \( \lambda(t) \). Starting from a system with the current arrival either not yet started, or not having completed its first phase, the probability of exactly \( \ell \) arrivals during the time interval \([u, t)\) is

\[
P^{k-\text{Erlang}}(z, t, u) = e^{-\int_u^t \lambda(v)dv} \sum_{\ell=0}^{\infty} \frac{\left( \int_u^t \lambda(v)dv \right)^{k\ell}}{(k\ell)!} z^{\ell}.
\]

When \( k = 3 \), we have an expression similar to that given in Eq. (11). Writing \( \int_u^t \lambda(v)dv \) as just \( \lambda \), we have

\[
P^{3-\text{Erlang}}(z, t, u) = \left( \frac{2}{3} e^{-z^{1/3}} \lambda \cos \left( \frac{\sqrt{3}}{2} z^{1/3} \lambda \right) + \frac{1}{3} e^{z^{1/3} \lambda} \right) e^{-\lambda}.
\]

In general applying formula (10),

\[
P^{k-\text{Erlang}}(z, t, u) = \sum_{j=0}^{k-1} \exp \left\{ \lambda \binom{z^{1/k}}{\omega_k^j} - 1 \right\} = \sum_{j=0}^{k-1} P^{\text{right}} \left( \binom{z^{1/k}}{\omega_k^j}, t, u \right)
\]

where \( \omega_k \) is the \( k \)th primitive root of unity given in formula (9). In Sect. 5, we see \( k \)-Erlang arrival generating functions where we observe the process in phase \( a_1 \) and then again in phase \( a_2 \) so that if \( \ell \) arrivals are completed, then the generating function becomes

\[
P^{k-\text{Erlang}, a_2-a_1}(z, t, u) = e^{-\int_u^t \lambda(v)dv} \sum_{\ell=0}^{\infty} \frac{\left( \int_u^t \lambda(v)dv \right)^{k\ell+a_2-a_1}}{(k\ell + a_2 - a_1)!} z^{\ell}.
\]
We may employ formula (10) to write this in terms of the generating function for Poisson arrivals with time-varying rates given in Example 3.

The generating function for \(m\)-Erlang departures is similar, but \(z\) will have a negative exponent, the parameter \(\lambda\) is replaced with \(\mu\) and the process has \(m\) phases rather than \(k\). 

\[ \begin{align*}
\text{Example 13} \quad \text{(Unbounded Erlang arrival and departure process)} \quad & \text{We assume that the arrival and departure processes are independent.} \\
& \text{The generating function for the unbounded process (in which negative levels are permitted) for which the coefficient on } z^n \text{ represents the probability of } n \text{ more arrivals than departures during the time interval } [u, t), \text{ a shift from arrival phase } a_1 \text{ to arrival } a_2 \text{ and from service phase } s_1 \text{ to } s_2 \text{ is given by the product of the corresponding generating functions for } k\text{-Erlang arrivals and } m\text{-Erlang departures.}
\end{align*} \]

\[ \begin{align*}
\text{4 Review of results for quasi-birth-death (QBDs) processes with time-varying periodic rates} \\
\text{We study the asymptotic periodic solution of ergodic queues with Erlang arrivals and service and time-varying periodic transition rates. } E_k(t)/E_m(t)/1 \text{ queues are level-independent QBDs. In this section, we recap one of the main results from [21] for these QBDs. Solutions for these systems are expressed in terms of an integral over a single period. The integrals involve the idle probabilities for the system.}
\end{align*} \]

\[ \begin{align*}
The \text{infinitesimal generator for a (QBD) with time-varying periodic transition rates:} \\
Q(t) &= \begin{bmatrix}
B(t) & A_1(t) \\
A_{-1}(t) & A_0(t) & A_1(t) \\
A_{-1}(t) & A_0(t) & A_1(t) & \ddots & \ddots
\end{bmatrix}.
\end{align*} \]

This leads to the system of differential equations:

\[ \dot{p}_0(t) = p_0(t)B(t) + p_1(t)A_{-1}(t) \]

\[ \dot{p}_n(t) = p_{n-1}(t)A_1(t) + p_n(t)A_0(t) + p_{n+1}(t)A_{-1}(t), \quad n \in \mathbb{N}\setminus\{0\}, \quad (12) \]

where \(p_n(t)\) is a \(K\)-element row vector whose \(j\)th component reflects the probability of being in phase \(j\) and level \(n\) at time \(t\). The \(A_m(t), m = -1, 0, 1\) are \(K \times K\) matrices reflecting transitions among phases and within the current level or to an adjacent level.

We can use the system of ordinary differential equations given in (12) to solve for the generating function for the asymptotic periodic distribution (see Breuer [4] for more details). The asymptotic periodic distribution is the limiting distribution at time \(t\) within the period as the number of periods tends to infinity. Such a limit will exist if the process is ergodic. To obtain Eq. (14), we have assumed that \(p_n(t) = p_n(t - 1)\), so

\[ P(z, t) = \sum_{n=0}^{\infty} p_n(t)z^n = P(z, t - 1). \]

Note that \(P(z, t)\) is a row vector of generating functions.
We multiply the derivative $\dot{p}_n(t)$ by $z^n$ and sum over all $n$ to obtain the differential equation

$$\dot{P}(z, t) = P(z, t)(A_1(t)z + A_0(t) + A_{-1}(t)z^{-1}) + p_0(t)(B(t) - A_0(t) - z^{-1}A_{-1}(t)).$$

This differential equation has the solution

$$P(z, t) = \int_v^t p_0(u)(B(u) - A_0(u) - z^{-1}A_{-1}(u))\Phi(z, t, u)du + P(z, v)\Phi(z, t, v)$$

where $v \leq t$ is an arbitrary initial time. The properties of $\Phi(z, u, t)$ are discussed below. Equation (13) gives the generating function for the transient solution. Letting, $v = t - 1$ and imposing the periodic boundary condition, we have

$$P(z, t) = \int_{t-1}^t p_0(u)(B(u) - A_0(u) - z^{-1}A_{-1}(u))\Phi(z, t, u)du$$

$$\quad + P(z, t - 1)\Phi(z, t, t - 1)$$

$$= \int_{t-1}^t p_0(u)(B(u) - A_0(u) - z^{-1}A_{-1}(u))\Phi(z, t, u)du$$

$$\quad + P(z, t)\Phi(z, t, t - 1).$$

Solving this equation for $P(z, t)$ gives the key equation for the generating function:

$$P(z, t) = \sum_{j=0}^{\infty} p_j(t)z^j = \int_{t-1}^t p_0(u)\left(B(u) - A_0(u) - z^{-1}A_{-1}(u)\right)\Phi(z, t, u)du$$

$$\quad \times (I - \Phi(z, t, t - 1))^{-1},$$

where $\Phi(z, t, u)$ is the generating function for the unbounded process, that is, the process that permits negative levels. $\Phi(z, t, u)$ is a linear operator in two variables (the indeterminate $z$ is a parameter) that satisfies two properties:

$$\Phi(z, t, t) = I,$$

and

$$\Phi(z, u, s)\Phi(z, t, u) = \Phi(z, t, s).$$

It satisfies the differential equations

$$\frac{\partial}{\partial t}\Phi(z, t, u) = \Phi(z, t, u)A(z, t),$$

and

$$\frac{\partial}{\partial u}\Phi(z, t, u) = -A(z, u)\Phi(z, t, u).$$
where

$$A(z, t) = A_1(t)z + A_0(t) + A_{-1}(t)z^{-1}. \quad (19)$$

When $A(z, t) = A(z)$ does not depend on time, then

$$
\Phi(z, t, 0) = \exp\{A(z)t\}, \quad (20)
$$
is just a matrix exponential. If the matrices $A(z, t)$ and $A(z, u)$ commute for arbitrary $t$ and $u$, then

$$
\Phi(z, t, u) = \exp\left\{\int_u^t A(z, v)dv\right\}. \quad (21)
$$
The result in Eq. (21) does not hold for general time-varying QBDs, but we will show in Sect. 5 that it does hold for $E_k(t)/E_m(t)/1$ queues.

For further details about QBDs with periodic transition rates, see [21]. $\Phi(z, t, u)$ is an evolution operator. For some background on evolution operators, refer to the appendix.

The coefficient of $z^n$ of the $j$th component in the generating function $P(z, t)$ gives the asymptotic periodic probability of being in level $n$ and phase $j$ at time $t$ within the period.

Notice that the generating function for the asymptotic periodic distribution of the level independent QBD with time-varying periodic transition rates given in Eq. (14) is not a rational function. Instead of an expression involving $-A^{-1}_z$ as we would have in the constant rate case we have $(I - \Phi(z, t, t - 1))^{-1}$. As in the scalar example for the $M(t)/M(t)/1$ queue, Example 8, this generating function is meromorphic, having only isolated singularities. As in that example, the singularity at $z = 0$ is removable, since we have from the definition of $P(z, t)$ that $P(0, t) = p_0(t)$.

5 Erlang arrivals and service, the $E_k(t)/E_m(t)/1$ queue

The Erlang arrival and service queue with time-varying periodic transition rates, that is, the $E_k(t)/E_m(t)/1$ queue, can be modeled with a three-dimensional state space $\{X(t), K(t), J(t)\}$ in which $X(t) \in \{0, 1, \ldots\}$ represents the level at time $t$, $K(t) \in \{0, 1, \ldots, k - 1\}$ is the arrival phase, and $J(t) \in \{0, 1, \ldots, m - 1\}$ is the service phase. Arrivals are $k$-Erlang with time-varying periodic transition rate $\lambda(t)$ among arrival phases. Service is $m$-Erlang with time-varying periodic transition rate $\mu(t)$ among service phases.

We define a $km$ component vector $p_j(t)$ for each level $X(t) = j \geq 1$, and each combination of the $k$ arrival phases and $m$ service phases. The probability vector $p_0(t)$ has $k$ components, one for each arrival phase. The states are arranged in lexicographic order with the arrival phase first and then the service phase. $p_{q,a,s}(t)$ gives the asymptotic periodic probability of being in state $\{X(t) = q, K(t) = a, J(t) = s\}$ at time $t$ within the period.

We will prove the following result:
Theorem 2 (The asymptotic periodic distribution of the $E_k(t)/E_m(t)/1$ queue) The asymptotic periodic distribution of the $E_k(t)/E_m(t)/1$ queue with time-varying periodic arrival rates $\lambda(t)$ and service rates $\mu(t)$ and period length $1$ may be expressed in terms of the poles of its generating function,

$$P(z, t) = \sum_{j=1}^{\infty} p(t)z^j,$$

that are outside the unit circle. These poles, denoted $\chi_{\ell,n}$, may be obtained from the solutions to the polynomial equation

$$\tilde{\lambda}y^{k+m} - (\bar{\lambda} + \bar{\mu} + 2\pi in)y^k + \bar{\mu} = 0, \quad n \in \mathbb{Z}. \quad (23)$$

Equation (23) has $m$ roots outside the unit circle and $k$ roots on or inside the unit circle. The generating function pole, $\chi_{\ell,n}$, for $\ell = 1, 2, \ldots, m$ and for fixed $n \in \mathbb{Z}$ is one of the $m$ solutions to Eq. (23) raised to the $km$ power. The vector of probabilities the system is in level $j$ and in one of the $km$ possible phases is given by

$$p_j(t) = \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^{m} f(\chi_{\ell,n}, t)\chi_{\ell,n}^{-j} \left[ \begin{array}{c} 1 \\ \chi_{\ell,n}^{-1/k} \\ \vdots \\ \chi_{\ell,n}^{(1-k)/k} \\ \vdots \\ \chi_{\ell,n}^{(m-1)/m} \end{array} \right] \otimes \left[ \begin{array}{c} 1 \\ \chi_{\ell,n}^{1/m} \\ \vdots \\ \chi_{\ell,n}^{(m-1)/m} \end{array} \right], \quad (24)$$

where

$$f(x, t) = \int_{t-1}^{t} \frac{e^{\int_{v}^{t}(\lambda(v)(x^{1/k} - 1) + \mu(v)(x^{-1/m} - 1))dv}}{m\tilde{\lambda}x^{1/k} - k\bar{\mu}x^{-1/m}} \times \left( p_{0,k-1}(u)x\lambda(u) - \mu(u) \sum_{q=0}^{k-1} p_{1,q,m-1}(u)x^{q/k} \right) du. \quad (25)$$

The proof of the theorem proceeds as follows. First we find the infinitesimal generator for the process using standard techniques from matrix analytic methods [16] so that we may analyze it as a QBD. Next we find $P(z, t)$, the generating function for the $p_j(t)$. This generating function will be a vector function of $z$ and $t$. We express $P(z, t)$ in terms of an integral equation involving the generating function for the unbounded process $\Phi(z, t, u)$ and the boundary probabilities $p_0(t)$ and $p_1(t)$. We use eigen decomposition to facilitate computations involving this generating function. In the next section, we use the singularities of the generating function $P(z, t)$ to find an exact formula for the probability vectors $p_j(t)$.

**Modeling the $E_k(t)/E_m(t)/1$ queue as a QBD.** This process can be modeled as a QBD. The infinitesimal generator for this process is defined in terms of the Erlang arrival and departure processes which we define here, followed by the infinitesimal generator for the $E_k(t)/E_m(t)/1$ queue.

The arrival process is expressed in terms of the transition rate matrices $D_0(t)$ and $D_1(t)$, and the departure process is defined in terms of the transition rate matrices $C_0(t)$ and $C_1(t)$. These are given by:
The matrices $D_0(t)$ and $D_1(t)$ are $k \times k$, and the matrices $C_0(t)$ and $C_1(t)$ are $m \times m$ reflecting transitions among arrival and service phases, respectively. Let $e_1$ represent an appropriately dimensioned row vector with a one in the first position, and zeros elsewhere. The inter-arrival arrival distribution in the constant rate case is given by

$$F_{T_a}(t) = 1 - e_1 \exp(D_0 t) 1_k = 1 - e^{-\lambda t} \sum_{j=0}^{k-1} \frac{\lambda^j t^j}{j!}$$

where $1_k$ is a $k$ column vector of ones. In the time-varying case, we have

$$F_{T_{a,u}}(t) = 1 - e_1 \Lambda(u + t, u) 1_k = 1 - e^{-\int_{t}^{t+u} \lambda(v)dv} \sum_{j=0}^{k-1} \frac{(\int_{u}^{t+u} \lambda(v)dv)^j}{j!} .$$

An explicit formula for $\Lambda(u + t, u)$ is

$$\Lambda(u + t, u) = e^{-\int_{u}^{t+u} \lambda(v)dv} \begin{bmatrix} 1 & \int_{u}^{t+u} \lambda(v)dv & \cdots & \frac{(\int_{u}^{t+u} \lambda(v)dv)^{k-1}}{(k-1)!} \\ 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & 1 \end{bmatrix} .$$

Note that $\Lambda(t, u)$ is an evolution operator satisfying

$$\frac{d}{dt} \Lambda(t, u) = \Lambda(t, u) D_0(t),$$
\[
\frac{d}{du} \Lambda(t, u) = -D_0(u) \Lambda(t, u),
\]

and

\[
\Lambda(t, t) = I.
\]

The departure process is similarly defined with

\[
F_{T_d}(t) = 1 - e_1 \exp(C_0 t) \mathbf{1}_k = 1 - e^{-\mu t} \sum_{j=0}^{k-1} \frac{\mu^j t^j}{j!}
\]
in the constant rate case and

\[
F_{T_d, u}(t) = 1 - e_1 M(u + t, u) \mathbf{1}_k = 1 - e^{-\int_u^{t+u} \mu(v) dv} \sum_{j=0}^{m-1} \frac{\left(e^{\int_u^{t+u} \mu(v) dv}ight)^j}{j!}
\]
when rates are time-varying. The matrix function \(M(u + t, u)\) is given by

\[
M(u + t, u) = e^{-\int_u^{t+u} \mu(v) dv} \begin{bmatrix}
1 & \int_u^{t+u} \mu(v) dv & \cdots & \frac{\left(\int_u^{t+u} \mu(v) dv\right)^{m-2}}{(m-2)!}
0 & \ddots & \cdots & \ddots
\vdots & \ddots & \ddots & \ddots
0 & \cdots & 0 & 1
\end{bmatrix}
\]

It is also an evolution operator.

We number the \(k\) arrival phases from phase zero to phase \(k - 1\) and the \(m\) departure phases from zero to \(m - 1\). We arrange states in lexicographic order, i.e., \((0, 0), (0, 1), \ldots, (0, k - 1), (1, 0, 0), (1, 0, 1), \ldots, (1, 0, m - 1), (1, 1, 0), \ldots, (1, 1, m - 1), \ldots\).

The infinitesimal generator for the \(E_k(t)/E_m(t)/1\) queue, \(Q(t)\), is given by

\[
Q(t) = \begin{bmatrix}
D_0 & Q_{0,1} & D_0 \otimes I_m
Q_{1,0} & D_0 \otimes C_0 & D_1 \otimes I_m
I_k \otimes C_1 & D_0 \otimes C_0 & D_1 \otimes I_m
D_0 \otimes I_m & D_1 \otimes C_0 & D_1 \otimes I_m
\vdots & \ddots & \ddots
\vdots & \ddots & \ddots
\end{bmatrix},
\]

where

\[
Q_{0,1}(t) = \begin{bmatrix}
\lambda(t)
\end{bmatrix}
\]
is a $k \times mk$ matrix,

$$Q_{1,0}(t) = I_k \otimes \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mu(t) \end{bmatrix},$$

$I_k$ is a $k \times k$ identity matrix, $\otimes$ and $\oplus$ represent the Kronecker product and Kronecker sum, respectively. For definitions of the Kronecker product and Kronecker sum see, for example, the textbook by Alan Laub [17], or MathWorld [36, 37]. The dependence of $Q(t)$ on $t$ is suppressed in the notation in Eq. (27).

Finding the generating function for the $E_k(t)/E_m(t)/1$ queue. Next, we find the generating function for the system. We have the following differential equations corresponding to the infinitesimal generator in Eq. (27):

$$\dot{p}_0(t) = p_0(t)D_0(t) + p_1(t)Q_{1,0}(t)$$
$$\dot{p}_1(t) = p_0(t)Q_{0,1}(t) + p_1(t)\left(D_1(t) \otimes I_m\right) + p_2(t)\left(I_k \otimes C_1(t)\right)$$
$$\dot{p}_j(t) = p_{j-1}(t)\left(D_1(t) \otimes I_m\right) + p_j(t)\left(D_0(t) \otimes C_0(t)\right)$$
$$\quad + p_{j+1}(t)\left(I_k \otimes C_1(t)\right) \quad j \geq 2.$$  \hspace{1cm} (28) \hspace{1cm} (29) \hspace{1cm} (30)

For the $E_k(t)/E_m(t)/1$ QBD, the matrices $A_i(t)$, $i = -1, 0, 1$ described in Sect. 4 are given by

$$A_{-1}(t) = I_k \otimes C_1(t),$$
$$A_0(t) = D_0(t) \otimes I_m + I_k \otimes C_0(t)$$

and

$$A_1(t) = D_1(t) \otimes I_m.$$  \hspace{1cm}

With $A(z, t) = z^{-1}A_{-1} + A_0(t) + zA_1(t)$ as in Eq. (19), then the function $\Phi(z, t, u)$ is an evolution operator which satisfies Eqs. (15)–(18). For each of these $km \times km$ matrices, we reference the components of the matrix as $((a_1, s_1)(a_2, s_2))$ where $(a_1, s_1)$ refer to the arrival and service phases of the row and $(a_2, s_2)$ give the arrival and service phases of the column.

The function $\Phi(z, t, u)$ is a Laurent series in $z$ with $km \times km$ matrix coefficients where the $((a_1, s_1), (a_2, s_2))$ entry of the coefficient on $z^\ell$ represents the probability of a net change of $\ell$ levels during the time interval $[u, t)$ and a sequence of transitions that begin in arrival phase $a_1$ and service phase $s_1$ at time $u$ and end in arrival phase $a_2$ and service phase $s_2$ at time $t$.

The key equation gives the generating function for this QBD in terms of an integral over a single time period. See [21] and the brief summary in Sect. 4 for the general case for QBDs. For the $E_k(t)/E_m(t)/1$ system with periodic transition rates, the key equation is given by
\[ P(z, t) = \sum_{j=1}^{\infty} p_j(t)z^j = \int_{t-1}^{t} \left( (p_0(u)zQ_{0,1}(u) - p_1(u)A_{-1}(u)) \Phi(z, t, u) \right) du \times (I - \Phi(z, t, t-1))^{-1}. \]  

Equation (31) depends on the unknown boundary probabilities \( p_0(t) \) and \( p_1(t) \). An approach for computing these probabilities is discussed in Sect. 10.

Note that \( A(z, t) \) may be expressed in terms of a Kronecker sum as

\[ A(z, t) = (D_0(t) + zD_1(t)) \oplus (C_0(t) + z^{-1}C_1(t)). \]  

**Eigen decomposition.** We can find an eigen decomposition of \( A(z, t) \). Finding such a decomposition allows us to do many computations as though we were working in the scalar setting by performing those computations on the eigenvalues of the matrix. In some settings, an eigen decomposition may not be advantageous because the eigenvalues are difficult to compute or the numerical accuracy is poor. That is not a problem with the \( E_k(t)/E_m(t)/1 \) queue because explicit formulas are available. These formulas do depend on the numerical calculation of roots of the polynomial Eq. (23) but these roots are easily found. There is more discussion of the roots in Sect. 6 on singularities of the generating function.

The eigen decomposition of \( A(z, t) \) allows us to find an explicit formula for the evolution operator \( \Phi(z, t, u) \). We can verify that this is the correct formula using probabilistic arguments.

The eigenvalues of \( A(z, t) \) are the sum of eigenvalues of the arrival matrix \((D_0(t) + zD_1(t))\) and the departure matrix \((C_0(t) + z^{-1}C_1(t))\) and the eigenvectors are the Kronecker product of the corresponding eigenvectors.

The primitive roots of unity defined in equation (9) appear in these expressions. The matrix \( D_0(t) + zD_1(t) \) has eigenvalues

\[ \xi_\ell(z, t) = \lambda(t)(\omega^\ell z^{1/k} - 1), \quad \ell = 0, \ldots, k-1, \]  

and corresponding eigenvectors

\[ v_\ell = \frac{1}{\sqrt{k}} \begin{bmatrix} z^{(1-k)/k} \omega^0_k \\ z^{(2-k)/k} \omega^1_k \\ z^{(3-k)/k} \omega^2_k \\ \vdots \\ z^{0} \omega^{(k-1)\ell}_k \end{bmatrix}. \]  

The matrix \( C_0(t) + \frac{1}{z}C_1(t) \) has eigenvalues

\[ \epsilon_j(z, t) = \mu(t)(\omega^j_m z^{-1/m} - 1), \quad j = 0, \ldots, m-1, \]
and corresponding eigenvectors

\[
 u_j = \frac{1}{\sqrt{m}} \begin{bmatrix}
 z^{(m-1)/m} \omega_0^0 \\
 z^{(m-2)/m} \omega_0^j \\
 z^{(m-3)/m} \omega_0^j m \\
 \vdots \\
 z^{(m-1)/m} \omega_0^j m \\
 \end{bmatrix}.
\] (36)

The proof of the eigen decompositions for \( D_0(t) + zD_1(t) \) and \( C_0(t) + zC_1(t) \) is straightforward but tedious. Simply perform the matrix multiplication for a generic element of the matrix. The details are omitted.

We can now compute the eigenvalues and eigenvectors of 

\[
 \begin{bmatrix}
 D_0(t) + zD_1(t) \\
 C_0(t) + zC_1(t) \\
 \end{bmatrix}
\] .

The eigenvalues of a Kronecker sum are the sum of the eigenvalues so the eigenvalues of 

\[
 A(z, t) = \begin{bmatrix}
 D_0(t) + zD_1(t) \\
 C_0(t) + zC_1(t) \\
 \end{bmatrix}
\] are

\[
 \xi_\ell(z, t) + \epsilon_j(z, t), \quad j = 0, \ldots, m - 1, \quad \ell = 0, \ldots, k - 1.
\]

The eigenvector for 

\[
 \begin{bmatrix}
 D_0(t) + zD_1(t) \\
 C_0(t) + zC_1(t) \\
 \end{bmatrix}
\] corresponding to the eigenvalue \( \xi_\ell(z, t) + \epsilon_j(z, t) \) is

\[
 \bar{v}_\ell \otimes u_j \quad j = 0, \ldots, m - 1, \quad \ell = 0, \ldots, k - 1.
\] (37)

Note that while the eigenvalues depend on \( t \), the eigenvectors do not. The eigenvalue and eigenvector computations for \( A(z, t) \) follow from theorems for Kronecker products [17].

Because the eigenvectors do not depend on \( t \), the matrices \( A(z, t_1) \) and \( A(z, t_2) \) commute for all times \( t_1 \) and \( t_2 \), so we have the following lemma:

**Lemma 1** *The evolution operator \( \Phi(z, t, u) \) corresponding to the \( E_k/E_m/1 \) queue which satisfies Eqs. (16)–(20) for \( A(z, t) \) as defined in Eq. (32) is given by

\[
 \Phi(z, t, u) = e^{\int_u^t A(z, v)dv}.
\]

This enables us to easily compute the eigenvalues for the matrices: \( \Phi(z, t, u), (I - \Phi(z, t, t - 1))^{-1} \) and \( \Phi(z, t, u) (I - \Phi(z, t, t - 1))^{-1} \). The eigenvectors for each of these matrices are those given in Eq. (37), the same as for \( A(z, t) \).

Let

\[
 \bar{\xi}_\ell(z) = \int_0^1 \lambda(u)(\omega_0^\ell z^{1/k} - 1)du = \bar{\lambda}(\omega_0^\ell z^{1/k} - 1)
\]

and

\[
 \bar{\epsilon}_j(z) = \int_0^1 \mu(u)(\omega_0^j z^{-1/m} - 1)du = \bar{\mu}(\omega_0^j z^{-1/m} - 1)
\]

give the average value of the eigenvalues for the arrival and departure processes, respectively, over a single time-period. We have defined \( \bar{\lambda} = \int_{t-1}^t \lambda(u)du \), the average
Table 1  Four matrix functions which share the eigenvectors $v_\ell \otimes u_j$

| Matrix | Eigen value |
|--------|-------------|
| $A(z, t)$ | $\xi_\ell(z, t) + \epsilon_j(z, t)$ |
| $\Phi(z, t, u)$ | $\exp \left[ \int_0^u (\xi_\ell(z, \nu) + \epsilon_j(z, \nu))d\nu \right]$ |
| $(I - \Phi(z, t, t - 1))^{-1}$ | $(1 - \exp(\tilde{\xi}_\ell(z) + \tilde{\epsilon}_j(z)))^{-1}$ |
| $\Phi(z, t, u)(I - \Phi(z, t, t - 1))^{-1}$ | $\exp \left[ \int_0^u (\xi_\ell(z, \nu) + \epsilon_j(z, \nu))d\nu \right] \times (1 - \exp(\tilde{\xi}_\ell(z) + \tilde{\epsilon}_j(z)))^{-1}$ |

value of $\lambda(t)$ over a single time period, and $\bar{\mu} = \int_{t-1}^t \mu(u)du$, the average value of $\mu(t)$ over a single time period. Then the eigenvalues for the four matrices with common eigenvectors are as given in the following lemma.

Define the matrices

$$\mathbf{\Omega}_K = \begin{bmatrix} \omega_0^0 & \omega_0^0 & \cdots & \omega_0^0 \\ \omega_0^1 & \omega_1^1 & \cdots & \omega_1^1 \\ \vdots & \vdots & \ddots & \vdots \\ \omega_K^0 & \omega_K^K & \cdots & \omega_K^{(K-1)^2} \end{bmatrix}$$

and $\overline{\mathbf{\Omega}_K}$, its complex conjugate. Let

$$\mathbf{H}_C \mathbf{D}_C \mathbf{H}_C^{-1} = \mathbf{C}_0 + z^{-1}\mathbf{C}_1$$

be an eigen decomposition of $\mathbf{C}_0 + z^{-1}\mathbf{C}_1$ where $\mathbf{D}_C$ is a diagonal matrix of the eigenvalues, $\epsilon_j(z, t)$, of $\mathbf{C}_0 + z^{-1}\mathbf{C}_1$ and $\mathbf{H}_C$ is a matrix whose columns are the eigenvectors, $u_j$, of $\mathbf{C}_0 + z^{-1}\mathbf{C}_1$. The matrix

$$\mathbf{H}_C = \frac{1}{\sqrt{m}} \text{diag}[z^{(m-1)/m} z^{(m-2)/m} \cdots 1] \mathbf{\Omega}_m.$$  

Similarly, let

$$\mathbf{H}_D \mathbf{D}_D \mathbf{H}_D^{-1} = \mathbf{D}_0 + z\mathbf{D}_1$$

be an eigen decomposition of $\mathbf{D}_0 + z^{-1}\mathbf{D}_1$ where $\mathbf{D}_D$ is a diagonal matrix of the eigenvalues, $\bar{\xi}_\ell(z, t)$, of $\mathbf{D}_0 + z\mathbf{D}_1$ and $\mathbf{H}_D$ is a matrix whose columns are the eigenvectors, $v_\ell$, of $\mathbf{D}_0 + z\mathbf{D}_1$. The matrix

$$\mathbf{H}_D = \frac{1}{\sqrt{k}} \text{diag}[z^{(1-k)/k} z^{(2-k)/k} \cdots 1] \mathbf{\Omega}_k.$$
Then

\[ H_D \otimes H_C = \frac{1}{\sqrt{km}} \left( \text{diag}[z^{(1-k)/k} \ z^{(2-k)/k} \ldots \ 1] \otimes \text{diag}[z^{(m-1)/m} \ z^{(m-2)/m} \ldots \ 1] \right) \times (\Omega_k \otimes \Omega_m) . \]

Similarly,

\[ H_D^{-1} \otimes H_C^{-1} = \frac{1}{\sqrt{km}} \left( \text{diag}[z^{(k-1)/k} \ z^{(k-2)/k} \ldots \ 1] \otimes \text{diag}[z^{(1-m)/m} \ z^{(2-m)/m} \ldots \ 1] \right) . \]

We will check some of the entries in Table 1. Define

\[ H = H_D \otimes H_C . \tag{38} \]

Suppose that \( D_{\Phi} \) is the diagonal matrix made up of the eigenvalues of \( \Phi \) and \( D_A \) is a diagonal matrix of eigenvalues of \( A(z, t) \). Then \( \Phi = HD_{\Phi}H^{-1} \). Only the eigenvalues of \( \Phi \) depend on \( t \), so

\[
\frac{\partial}{\partial t} \Phi(z, t, u) = \frac{\partial}{\partial t} \left( HD_{\Phi}H^{-1} \right) \\
= H \frac{\partial}{\partial t} D_{\Phi}H^{-1} \\
= HD_{\Phi}D_AH^{-1} \\
= HD_{\Phi}H^{-1}HD_AH^{-1} \\
= \Phi A .
\]

This is consistent with Eq. (17). Equations (15), (16) and (18) can also be easily checked.

We will also show that if \( \Phi \) has the eigen decomposition given above, then \( (I - \Phi)^{-1} \) has the eigen decomposition indicated in Table 1. \( (I - \Phi)^{-1} \) can be expressed as a geometric series so we have

\[
(I - \Phi)^{-1} = \sum_{n=0}^{\infty} \Phi^n \\
= \sum_{n=0}^{\infty} (HD_{\Phi}H^{-1})^n \\
= H \sum_{n=0}^{\infty} (D_{\Phi})^n H^{-1} .
\]
\[
= HD(I - \Phi)^{-1}H^{-1}
\]

and it is easily seen that the matrix \(D(I - \Phi)^{-1} = \sum_{n=0}^{\infty} (D\Phi)^n\) where \(D(I - \Phi)^{-1}\) is a diagonal matrix with the eigenvalues for \((I - \Phi)^{-1}\) given in Table 1 as its diagonal elements.

For any of the four matrix functions in Table 1: \(A(z, t), \Phi(z, t, u), (I - \Phi(z, t, t - 1))^{-1}\) or \(\Phi(z, t, u)(I - \Phi(z, t, t - 1))^{-1}\), we can use the eigen decomposition to find an expression for an element of the matrix in terms of the corresponding eigenvalues \(d_{\ell,i}\). We call the generic matrix \(\Xi\) and compute

\[
\Xi_{(a_1, s_1)(a_2, s_2)} = \frac{1}{km} z^{s_2-s_1/m} + \frac{a_1-a_2}{k} \sum_{i=0}^{m-1} \omega_m^i(s_1-s_2) \sum_{\ell=0}^{k-1} d_{\ell,i}\omega_k^\ell(a_1-a_2).
\]

We apply the roots of unity formula, Eq. (10), twice to

\[
\frac{1}{km} z^{s_2-s_1/m} + \frac{a_1-a_2}{k} \sum_{i=0}^{m-1} \omega_m^i(s_1-s_2) \sum_{\ell=0}^{k-1} d_{\ell,i}\omega_k^\ell(a_1-a_2)
\]

to obtain explicit formulas for \([z^n]\Xi_{(a_1, s_1)(a_2, s_2)}\), the components of the coefficient matrices.

Note that the Poisson and random walk generating functions we saw in Examples 3, 4 and 5 appear in three of our \(\Xi\) matrices. We work out in detail, the simplest of these. When \(\Xi = \Phi(z, t, u)\) with \(s = s_2 - s_1, a = a_2 - a_1, \lambda = f_u^t \lambda(v)dv\) and \(\mu = f_u^t \mu(v)dv\), we have

\[
[\Phi(z, t, u)]_{(a_1, s_1)(a_2, s_2)} = \\
\frac{1}{km} \frac{z^m}{m} \sum_{i=0}^{m-1} \omega_m^i s z^{-a/k} \sum_{k=1}^{k-1} e^{k(\omega_m^iz^{1/m}-1)+\mu(\omega_m^is^{-1/m}-1)} \omega_k^{-\ell a}
\]

(factoring exponentials)

\[
= \frac{z^m}{m} \sum_{i=0}^{m-1} \omega_m^i(\omega_m^iz^{1/m}-1) \omega_m^i s z^{-a/k} k \sum_{k=1}^{k-1} e^{k(\omega_m^iz^{1/m}-1)} \omega_k^{-\ell a}
\]

(applying equation (10))

\[
= \frac{z^m}{m} \sum_{i=0}^{m-1} \omega_m^i(\omega_m^iz^{1/m}-1) \omega_m^i s z^{-a/k} \lambda^{\ell a} \sum_{n=0}^{\infty} \frac{\lambda^{nk+a}z^n}{(nk+a)!}
\]

(applying equation (10))

\[
= e^{-\lambda - \mu} \sum_{\ell=0}^{\infty} \frac{\lambda^{nk+a}z^n}{(nk+a)!} (m\ell + s)!(n + \ell)!(n + \ell)(n + \ell)(n + \ell)k + a
\]

(coefficient on \(z^n\)).

The coefficient on \(z^n\) in this Laurent series reflects the probability of \(n\) more arrivals (which require completion of \(k\) phases at rate \(\lambda(t)\)) than service completions (which
require completion of $m$ service phases at rate $\mu(t)$ and a transition from arrival phase $a_1$ to arrival phase $a_2$ (net change $a = a_2 - a_1$) and from service phase $s_1$ to $s_2$ (net change $s = s_2 - s_1$) occurring during the time interval from $u$ to $t$.

The $((a_1, s_1), (a_2, s_2))$ component of the matrix coefficient on $z^n$ of the function $\Phi(z, t, u) (I - \Phi(z, t, t - 1))^{-1}$ gives the expected number of times $t$ within the period that the process has made a net change of $n$ levels and is in arrival phase $a_2$ and service phase $s_2$ having started at phases $(a_1, s_1)$ at time $u$ within an earlier period. This coefficient does not count the expected number of visits, but rather the expected number of periods that the process is in a given state at time $t$ within the period.

The components of the matrix function $\Phi(z, t, u) (I - \Phi(z, t, t - 1))^{-1}$ are linear combinations of generating functions of the form

$$
eq \sum_{n=-\infty}^{\infty} z^n \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} e^{-f_u^{t+j}(\lambda(v)+\mu(v))} \left( f_u^{t+j} \mu(v) dv \right) \frac{(\ell + n)! \ell!}{(\ell + n)! (\ell + n)! (\ell m + s)!}.
$$

evaluated at $k$th and $m$th roots of the indeterminate $z$ times a root of unity. An exact formula for the coefficient on $z^n$ of the $((a_1, s_1)(a_2, s_2))$ component is given by

$$[z^n] \left[ \Phi(z, t, u) (I - \Phi(z, t, t - 1))^{-1} \right]_{(a_1, s_1)(a_2, s_2)} = \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} e^{-f_u^{t+j}(\lambda(v)+\mu(v))} \left( f_u^{t+j} \lambda(v) dv \right) \frac{(\ell + n)! \ell!}{(\ell + n)! (\ell m + s)!}.
$$

For the $((a_1, s_1)(a_2, s_2))$ component of $(I - \Phi(z, t, t - 1))^{-1}$, we have

$$[z^n] \left[ (I - \Phi(z, t, t - 1))^{-1} \right]_{(a_1, s_1)(a_2, s_2)} = \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} e^{-f_u^{t+j}(\lambda(v)+\mu(v))} \left( f_u^{t+j} \mu(v) dv \right) \frac{(\ell + n)! \ell!}{(\ell + n)! (\ell m + s)!}.
$$

These formulas, while exact, are not conducive to computation.

We substitute the eigen decomposition for $\Phi(z, t, u) (I - \Phi(z, t, t - 1))^{-1}$ into Eq. (31) to obtain

$$P(z, t) = \int_{t-1}^{t} (p_0(u)zQ_{0,1}(u) - p_1(u)A_{-1}(u))$$

$$\times H \text{ diag} \left[ \exp \left\{ \int_u^t (\xi_{\ell}(z, v) + \epsilon_j(z, v)) dv \right\} \right] H^{-1} du.
$$
where the matrix $H$ is the matrix of eigenvectors of $\Phi(z, t, u) (I - \Phi(z, t, t - 1))^{-1}$ defined in Eq. (38).

We now have an expression for the generating function in terms of an integral equation over a single time period and the unknown boundary probabilities. Our next step is to exploit the singularities of the generating function to find formulas for the asymptotic periodic probabilities $p_j(t)$.

### 6 Singularity analysis

Following the approach of Sedgewick and Flajolet [6], we note that the singularities of the generating function are reflected in the coefficients. In this section, we explore the zeros of the denominator of the generating function, $P(z, t)$. Note that the generating function $P(z, t)$ has singularities wherever

$$1 - \exp\left\{\bar{\lambda}z^{1/k} - 1\right\} + \bar{\mu}(z^{-1/m} - 1) = 0.$$  

This occurs for $z$ such that

$$\bar{\lambda}(\bar{\omega}z^{1/k} - 1) + \bar{\mu}(z^{-1/m} - 1) = 2\pi in, \quad n \in \mathbb{Z}. \quad (40)$$

Let $y = z^{1/k}$ when $\ell = j = 0$, then Eq. (40) becomes the polynomial equation (23) in Theorem 2.

Figure 1 shows the roots of

$$1 - e^{\bar{\lambda}y^{1/k} + \bar{\mu}(y^{-1/m} - 1)}.$$  

Using Rouchè’s theorem, we can show that the polynomial given in Eq. (23) has $k$ roots on or inside the unit circle and $m$ roots outside of the unit circle. Provided that the $m$ solutions to $y^{m} = \frac{k(\bar{\lambda} + \bar{\mu} + 2\pi in)}{\bar{\lambda}(k + m)}$ are not solutions to (23), the roots are distinct.

We can substitute

$$y^* = \left(\frac{k(\bar{\lambda} + \bar{\mu} + 2\pi in)}{\bar{\lambda}(k + m)}\right)^{\frac{1}{m}}$$

into Eq. (23) to show that if $y^*$ solves (23), then

$$\left(\frac{\bar{\lambda} + \bar{\mu} + n\pi in}{k + m}\right)^{k+m} = \left(\frac{\bar{\lambda}}{k}\right)^{k} \left(\frac{\bar{\mu}}{m}\right)^{m}.$$  

When $n = 0$,

$$\left(\frac{\bar{\lambda} + \bar{\mu}}{k + m}\right) \geq \left(\frac{\bar{\lambda}}{k}\right)^{k} \left(\frac{\bar{\mu}}{m}\right)^{m}$$

by Young’s inequality. Equality holds only if $\frac{\bar{\lambda}}{k} = \frac{\bar{\mu}}{m}$, or if $k$ or $m$ is zero. We have assumed ergodicity, so $\frac{\bar{\lambda}}{k} < \frac{\bar{\mu}}{m}$; that is, the mean arrival rate must be less than the
Fig. 1 These figures were produced using MATLAB code written by Elias Wegert [35]. Both plots show the region from $-2-2i$ to $2+2i$. The expression given in Eq. (41) is plotted in the complex plane. Shading shows contour lines. The colors represent the argument, so points where multiple colors come together are zeros of the function. In this example, $k = 7$, $m = 4$, $\bar{\lambda} = 3$ and $\bar{\mu} = 5$. Note that there are $k = 7$ petals in the rose inside the unit circle and $m = 4$ inverted petals outside the unit circle.

mean service rate. Neither $k$ nor $m$ equals zero since the arrival and service processes must have at least one phase. When $n \neq 0$, it is clear that the two sides of the equation are not equal because the real and imaginary parts are not equal.

Therefore, the roots of Eq. (23) are distinct. In fact, we can use the following contraction mappings to find the $k + m$ roots for each fixed $n$. To find the $m$ roots outside the unit circle, we may use the iteration:

$$y_0^{(q)} = e^{\frac{2\pi i q}{m}} \left(\frac{1}{\bar{\lambda}} \left(2\pi i n + \bar{\lambda} + \bar{\mu}\right)\right)^{1/m}, \quad q = 0, \ldots, m - 1$$

with

$$y_{n+1}^{(q)} = e^{\frac{2\pi i q}{m}} \left(\frac{1}{\bar{\lambda}} \left(2\pi i n + \bar{\lambda} + \bar{\mu} \left(1 - (y_n^{(q)})^{-k}\right)\right)\right)^{1/m}, \quad q = 0, \ldots, m - 1,$$

though the roots command from MATLAB, for example, works perfectly well. To find the $k$ roots on or in the unit circle, we may use the iteration:

$$y_0^{(q)} = e^{\frac{2\pi i q}{k}} \left(\frac{\bar{\mu}}{2\pi i n + \bar{\lambda} + \bar{\mu}}\right)^{1/k}, \quad q = 0, \ldots, k - 1.$$
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Fig. 2 Zeros of the real part of equation (41) are asymptotic to solutions to $r^k = \frac{\bar{\lambda}}{y_0} \cos(k\theta)$ (shown in blue inside the unit circle) as $r \to 0$ and asymptotic to $r^m = \frac{\bar{\lambda} + \bar{\mu}}{k} \sec(m\theta)$ (shown in red outside the unit circle) as $r \to \infty$. Zeros of the real part of equation (41) are shown as dashed black lines.

(b) Zeros of the real part of equation (41) are shown in white for an $E_3(t)/E_5(t)/1$ system; zeros of the imaginary part for $n = -7$ are shown in cyan. The $m = 5$ intersections of the cyan and white curves outside of the unit circle show the $m$ roots for $n = -7$ outside the unit circle. The intersection of the white and cyan three petal roses show the $k = 3$ roots corresponding to $n = -7$ inside the unit circle.

(a) Zeros of the real part of equation (41) are asymptotic to solutions to $r^k = \frac{\bar{\mu}}{\bar{\lambda} + \bar{\mu}} \cos(k\theta)$ (shown in blue inside the unit circle) as $r \to 0$ and asymptotic to $r^m = \frac{\bar{\lambda} + \bar{\mu}}{k} \sec(m\theta)$ (shown in red outside the unit circle) as $r \to \infty$. Zeros of the real part of equation (41) are shown as dashed black lines.

Fig. 2 Zeros of the real part of the denominator of the generating function for an $E_7(t)/E_4(t)/1$ queueing system are shown on the left, and zeros of the denominator of the generating function for an $E_3(t)/E_5(t)/1$ queueing system are shown on the right. Note that the petals inside the unit circle correspond to the number of arrival phases.

with

$$y_{n+1}^{(q)} = e^{2\pi i q/k} \left( \frac{\bar{\mu}}{2\pi in + \bar{\lambda}(1 - (y_n^{(q)})^m)) + \bar{\mu}} \right)^{1/k}, \quad q = 0, \ldots, k - 1.$$  

Because $P(z, t)$ is a generating function for an ergodic process, it must converge for all complex $|z| < 1$. This means that zeros of the denominator inside the unit circle are also zeros of the numerator of the generating function. We focus our attention on the $m$ roots of Eq. (23) outside of the unit circle. We label these roots, $\chi_{\ell, n}$, $\ell = 1, \ldots, m$ and $n \in \mathbb{Z}$.

We consider examples where $k$ and $m$ are relatively prime. Suppose $\alpha$ is a root of the polynomial (23), then $\omega_{km}^j \alpha$ is a root of

$$\bar{\lambda}\omega_{k}^{j+m} y^{k+m} - (\bar{\lambda} + \bar{\mu} + 2\pi in)y^k + \bar{\mu}\omega_{m}^q \alpha = 0, \quad n \in \mathbb{Z},$$

where $x$ is the minimum non-negative integer such that $mj + mx \equiv 0 \mod mk$ and $kq - kx \equiv 0 \mod mk$. Note that $\alpha$ is the $km$th root of $\chi_{j, n}$ for some $j \in \{1, \ldots, m\}$.  

$\square$ Springer
More generally, we have

\[
\lim_{y \to \omega_{km}} \left( 1 - \frac{y}{\omega_{km}} \right) e^{f'_u \epsilon_q (y^{km}, v) + \xi_j (y^{km}, v)} = \frac{e^{f'_u (\lambda(v)(\alpha^m - 1) + \mu(v)(\alpha^{-k} - 1))} \lambda \alpha^m - k \bar{\mu} \alpha^{-k}}{m\lambda \alpha^m - k \bar{\mu} \alpha^{-k}} \quad (42)
\]

independent of the indices \( j \) and \( q \). (If \( k \) and \( m \) are not relatively prime, the approach in this paper can still be used, but the limit given in Eq. (42) would not be independent of \( j \) and \( q \). We would need to find the roots of more than one equation for each \( n \).)

So, we approximate \( e^{f'_u \epsilon_q (y^{km}, v) + \xi_j (y^{km}, v)} \) with the series

\[
\frac{e^{f'_u (\lambda(v)(\alpha^m - 1) + \mu(v)(\alpha^{-k} - 1))} \lambda \alpha^m - k \bar{\mu} \alpha^{-k}}{m\lambda \alpha^m - k \bar{\mu} \alpha^{-k}} \sum_{n=0}^{\infty} \frac{y^n}{\omega_{km}^n} \alpha^n
\]

near the pole at \( \alpha \).

Define

\[
D_\chi = \begin{bmatrix}
1 & \chi^{-1/k} & \ldots & \chi^{(1-1/k)/k} \\
\chi^{1/k} & 1 & \ldots & \chi^{(2-1/k)/k} \\
\vdots & \ddots & \ddots & \vdots \\
\chi^{(k-1)/k} & \chi^{(k-2)/k} & \ldots & 1
\end{bmatrix}
\]

and

\[
C_\chi = \begin{bmatrix}
1 & \chi^{1/m} & \ldots & \chi^{(m-1)/m} \\
\chi^{-1/m} & 1 & \ldots & \chi^{(m-2)/m} \\
\vdots & \ddots & \ddots & \vdots \\
\chi^{(1-m)/m} & \chi^{(2-m)/m} & \ldots & 1
\end{bmatrix}
\]

Then we may express the generating function given in Eq. (31), as

\[
P(z, t) = \sum_{j=1}^{\infty} p_{j}(t) z^{j} = \sum_{j=1}^{\infty} \int_{t-1}^{t} \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^{m} e^{f'_u (\lambda(v)(\chi^{1/k}_{\ell,n} - 1) + \mu(v)(\chi^{-1/k}_{\ell,n} - 1))} \frac{m\lambda \chi^{1/k}_{\ell,n} - k \bar{\mu} \chi^{-1/k}_{\ell,n}}{m\lambda \chi^{1/k}_{\ell,n} - k \bar{\mu} \chi^{-1/k}_{\ell,n}} \nu \left( p_{0,k-1}(u) \chi_{\ell,n} \lambda(u) - \mu(u) \sum_{q=0}^{k-1} \nu_{1,q,m-1}(u) \chi^{q/k}_{\ell,n} \right) du
\]

\[
\times \left[ 1 \chi^{-1/k}_{\ell,n} \ldots \chi^{(1-1/k)/k}_{\ell,n} \right] \otimes \left[ 1 \chi^{1/m}_{\ell,n} \ldots \chi^{(m-1)/m}_{\ell,n} \right] \left( \frac{z^{j}}{\chi^{j}_{\ell,n}} \right) . \quad (43)
\]
With \( f(x, t) \) as defined in Eq. (25), the probability vector for level \( j \) is \([z^n]P(z, t)\) from Eq. (43), yields Eq. (24)

\[
p_j(t) = \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^{m} f(\chi_{\ell,n}, t) \chi_{\ell,n}^{-j} \left[ 1 \chi_{\ell,n}^{-1/k} \cdots \chi_{\ell,n}^{(1-k)/k} \right] \otimes \left[ 1 \chi_{\ell,n}^{1/m} \cdots \chi_{\ell,n}^{(m-1)/m} \right].
\]

This expression is exact. See [21] for more details. This concludes the proof of Theorem 2.

To illustrate the method, we consider an example of an \( E_7(t)/E_4(t)/1 \) queue with

\[
\lambda(t) = 3 - 2 \sin(2\pi t)
\]

and

\[
\mu(t) = 5 + 4 \sin(2\pi t).
\]

We approximate the distribution with

\[
p^{(q)}_j(t) = \sum_{n=-q}^{q} \sum_{\ell=1}^{m} f(\chi_{\ell,n}, t) \chi_{\ell,n}^{-j} \left[ 1 \chi_{\ell,n}^{-1/k} \cdots \chi_{\ell,n}^{(1-k)/k} \right] \otimes \left[ 1 \chi_{\ell,n}^{1/m} \cdots \chi_{\ell,n}^{(m-1)/m} \right].
\]

(44)

The plots in Figs. 3 and 4 show convergence of the asymptotic estimates \( p^{(q)}_j(t) \) to the level probabilities \( p_j(t) \) as the number of terms in the estimate increases. In Sect. 7, we provide error bounds for the asymptotic estimates of the level probabilities.

### 7 Error bound

Our goal is to estimate the error

\[
\left\| p_j(t) - p^{(q)}_j(t) \right\|_{\infty}
\]

where \( p^{(q)}_j(t) \) is defined in Eq. (44). Our first bound applies for \( j \geq 3 \). We do this by finding bounds for

(a) the modulus of the roots \( |\chi_{\ell,n}| \),

(b) \( f(\chi_{\ell,n}, t) \), defined in Eq. (25), and on

(c) \( \left\| \left[ 1 \chi_{\ell,n}^{-1/k} \cdots \chi_{\ell,n}^{(1-k)/k} \right] \otimes \left[ 1 \chi_{\ell,n}^{1/m} \cdots \chi_{\ell,n}^{(m-1)/m} \right] \right\|_{\infty} \).

Our asymptotic estimates for the \( p_j(t) \) are governed by the singularities of the generating function \( P(z, t) \) and the function \( f(x, t) \) given in Eq. (25). The \( km \)th roots of these are the zeros of

\[
1 - e^{\bar{\lambda}(y^m-1) + \bar{\mu}(y^{-k}-1)}.
\]
Fig. 3 These graphs compare the asymptotic estimate given by $p_{1,0}^{(1)}(t)$ for the probability of being in level 1 and the specified arrival and service phases for the $E_{7}(t)/E_{4}(t)/1$ system. See Eq. (44)
These graphs compare the asymptotic estimate given by $p_{1,1,0}(t)$ for the probability of being in level 1, arrival phase 1 and service phase 0 for the $E_7(t)/E_4(t)/1$ system. The asymptotic estimate is shown with the blue dashed line and the solution from a system of ordinary differential equations, truncated at 50 levels is shown in red. The graphs are for three different values of $q$, with estimate given by $p_{1,1,0}^{(q)}(t)$ as defined in Eq. (44).

We examine the asymptotic behavior of the roots which are outside the unit circle. Return again to Eq. (40). Write $z = re^{i\theta}$ in polar form and consider the limit of $z^{1/k}$ as $n \to \infty$. Assume $r > 1$ and that $r^{1/k}$ is also positive. From Eq. (40), the roots of the singularities of the generating function satisfy

$$\tilde{\lambda} z^{1/k} \omega_k^\ell = \tilde{\lambda} - \bar{\mu}(\omega_m^{1/m} z^{-1/m} - 1) + 2\pi in, \quad n \in \mathbb{Z}. \tag{45}$$

Dividing both sides by $\tilde{\lambda} \omega_k^\ell e^{i\theta/k} n$,

$$\lim_{n \to \pm \infty} \frac{r^{1/k}}{n} = \lim_{n \to \pm \infty} \left( \frac{\tilde{\lambda}}{\lambda n} - \frac{\bar{\mu}(\omega_m^{1/m} e^{-i\theta/m} - 1)}{\lambda n} + \frac{2\pi in}{\lambda n} \right) e^{-i\theta/k} \omega_k^{-\ell} \tag{46}$$

where the last equality follows from the fact that $r^{1/k}$ is real and positive. This, in turn, implies that the limiting value of $\theta$, $\theta^*$, as $n \to \pm \infty$ is such that

$$e^{-i\theta^*/k-2\pi i \ell/k + \pi i/2} = 1$$

or

$$\theta^* = \frac{k\pi}{2}.$$ 

Hence, for $r > 1$,

$$r \sim \left( \frac{2\pi n}{\lambda} \right)^k.$$
Similarly, if \( r < 1 \), then

\[
\lim_{n \to \pm \infty} \frac{r^{-1/m}}{n} = \frac{2\pi}{\bar{\mu}}
\]

and the limiting value of \( \theta \) is \(-\frac{m\pi}{2}\).

From the preceding analysis, we see that the modulus of the \( k \)th root of the singularity is bounded by

\[
\frac{2\pi |n|}{\bar{\lambda}} - \frac{\bar{\lambda} + 2\bar{\mu}}{\sqrt{2\bar{\lambda}}} < \left| \chi_{\ell,n}^{1/k} \right| < \frac{2\pi |n|}{\bar{\lambda}} + \frac{\bar{\lambda} + 2\bar{\mu}}{\sqrt{2\bar{\lambda}}}. \tag{46}
\]

These bounds are independent of \( \ell = 0, \ldots, m - 1 \).

Next we consider the function \( f(x, t) \) given in Eq. (25). We consider three expressions separately:

\[
f_1(x, t, u) = e^{\int_u^t (\lambda(v)(x^{1/k} - 1) + \mu(v)(x^{-1/m} - 1)) dv},
\]

\[
f_2(x) = m\bar{\lambda}x^{1/k} - k\bar{\mu}x^{-1/m}
\]

and

\[
f_3(x, u) = \left( p_{0,k-1}(u)x\lambda(u) - \mu(u) \sum_{q=0}^{k-1} p_{1,q,m-1}(u)x^{q/k} \right)
\]

so that

\[
f(x, t) = \int_{t-1}^t \frac{f_1(x, t, u)}{f_2(x)} f_3(x, u) du.
\]

We can compute the following bound for \( |f_3(\chi, u)| \), with \( |\chi| \geq 1 \):

\[
\left| p_{0,k-1}(u)\chi\lambda(u) - \mu(u) \sum_{q=0}^{k-1} p_{1,q,m-1}(u)\chi^{q/k} \right|
\leq p_{0,k-1}(u) |\chi| \lambda(u) + \mu(u) \sum_{q=0}^{k-1} p_{1,q,m-1}(u) \left| \chi^{q/k} \right|
\leq \lambda(u) |\chi| + \mu(u) \sum_{q=0}^{k-1} p_{1,q,m-1}(u) \left| \chi^{q/k} \right|
\leq (\lambda(u) + \mu(u)) |\chi|,
\]

where we have used the fact that the phase transition rates are real and non-negative, as are probabilities.
We can find a lower bound for \(|f_2(\chi)|, |\chi| > 1\). \(\chi_{\ell,n}\) is a root of Eq. (45). Because of the limit (42), we may take the exponents \(j\) and \(\ell\) equal to zero, so

\[
\chi_{\ell,n}^{1/k} = \frac{1}{k} \left( \bar{\lambda} + \bar{\mu} - \bar{\mu} \chi_{\ell,n}^{-1/m} + 2\pi i n \right).
\]

Then

\[
|f_2(\chi_{\ell,n})| = \left| m \tilde{\lambda} \chi_{\ell,n}^{1/k} - k \bar{\mu} \chi_{\ell,n}^{-1/m} \right|
\]

\[
= \left| m \left( \tilde{\lambda} + \bar{\mu} - \tilde{\mu} \chi_{\ell,n}^{-1/m} + 2\pi i n \right) - k \bar{\mu} \chi_{\ell,n}^{-1/m} \right| \quad \text{substituting for } \chi_{\ell,n}^{1/k}
\]

\[
= \left| m \left( \tilde{\lambda} + \bar{\mu} + 2\pi i n \right) - (k + m) \bar{\mu} \chi_{\ell,n}^{-1/m} \right| \quad \text{collect terms}
\]

\[
\geq \left| m \left( |\tilde{\lambda} + \bar{\mu} + 2\pi i n| - (k + m) \bar{\mu} \right) \chi_{\ell,n}^{-1/m} \right| \quad k, m \text{ and } \tilde{\lambda} \text{ are positive and } |a - b| \geq |a| - |b|
\]

\[
\geq \left| m \left( |\tilde{\lambda} + \bar{\mu} + 2\pi i n| - (k + m) \bar{\mu} \right) \chi_{\ell,n}^{-1/m} \right| < 1
\]

\[
= m \left( |\tilde{\lambda} + \bar{\mu} + 2\pi i n| - (k + m) \bar{\mu} \right) \chi_{\ell,n}^{-1/m} \geq (k + m) \bar{\mu}
\]

for \(\frac{m}{\tilde{\lambda}} > \frac{k}{\bar{\mu}}\) (our ergodicity condition).

Now consider \(|f_1(\chi_{\ell,n}, t, u)|\).

\[
|f_1(\chi_{\ell,n}, t, u)| = \left| e^{\int_u^t (\lambda(v)(\chi_{\ell,n}^{1/k} - 1) + \mu(v)(\chi_{\ell,n}^{-1/m} - 1))dv} \right|
\]

\[
= \left| e^{\int_u^t \left( \frac{\lambda(v)}{\chi} \left[ \bar{\mu} (1 - \chi_{\ell,n}^{-1/m}) + 2\pi i n \right] - \mu(v) \left( \bar{\mu} \chi_{\ell,n}^{-1/m} - \mu(v) \right) \right)dv} \right| \quad \text{substitution for } \chi_{\ell,n}^{1/k}
\]

\[
= \left| e^{\int_u^t \left( \frac{\lambda(v)}{\chi} \bar{\mu} + \left( \mu(v) - \frac{\lambda(v) \bar{\mu}}{\chi} \right) \chi_{\ell,n}^{-1/m} - \mu(v) \right)dv} \right| \quad \text{simplification}
\]

\[
= \left| e^{\int_u^t \left( \frac{\lambda(v) \bar{\mu}}{\chi} - \mu(v) + \mu(v) \bar{\mu} \chi_{\ell,n}^{-1/m} \right)dv} \right| \leq \left| e^{\int_u^t \frac{\lambda(v) \bar{\mu}}{\chi} dv} \right| \quad |\chi_{\ell,n}^{-1/m}| < 1
\]

Then, putting these inequalities all together,

\[
|f(\chi_{\ell,n}, t)| \leq \frac{|\chi_{\ell,n}|}{m \sqrt{(\tilde{\lambda} + \bar{\mu})^2 + 4\pi^2 n^2} - (k + m) \bar{\mu}} \int_{t-1}^t (\lambda(u) + \mu(u)) e^{\bar{\mu} \int_u^t \lambda(v)dv} du
\]

\[
= |\chi_{\ell,n}| C_n,
\]

where

\[
C_n = \frac{\int_{t-1}^t (\lambda(u) + \mu(u)) e^{\bar{\mu} \int_u^t \lambda(v)dv} du}{m \sqrt{(\tilde{\lambda} + \bar{\mu})^2 + 4\pi^2 n^2} - (k + m) \bar{\mu}}.
\]
The $C_n$ form a decreasing sequence.
A bound on $\left\| \left[ 1 \chi_{\ell,n}^{-1/k} \cdots \chi_{\ell,n}^{(1-k)/k} \right] \otimes \left[ 1 \chi_{\ell,n}^{1/m} \cdots \chi_{\ell,n}^{(m-1)/m} \right] \right\|_\infty$ is $|\chi_{\ell,n}|$.

Applying the lower bound (because the exponent is negative) for $\left\| \chi_{\ell,n}^{1/k} \right\|$ given in inequality (46), we have

$$\left\| p_j(t) - p_j^{(q)}(t) \right\|_\infty \leq 2mC_q \sum_{n=q+1}^{\infty} \left( \frac{2\pi n}{\bar{\lambda}} - \frac{\bar{\lambda} + 2\bar{\mu}}{\bar{\lambda}\sqrt{2}} \right)^{-kj+2k}$$

$$\leq 2mC_q \int_q^{\infty} \left( \frac{2\pi x}{\bar{\lambda}} - \frac{\bar{\lambda} + 2\bar{\mu}}{\bar{\lambda}\sqrt{2}} \right)^{-kj+2k} dx$$

$$= \frac{mC_q}{\pi} \left( \frac{2\pi q - \frac{1}{\sqrt{2}} (\bar{\lambda} + 2\bar{\mu})}{k(j-2) - 1}) \bar{\lambda}^{k(2-j)} \right)$$

The leading coefficient $2m$ is for $m$ roots for each fixed $n$, and two tails of the sum over $n$. We also employ a bound on $|\chi_{\ell,n}|$ in this step.
For a monotone decreasing function, the given integral is greater than the sum.
This bound goes to zero as $q \to \infty$ for $j \geq 3$.

The plots in Figs. 3 and 4 show rapid convergence even for level one. We explore why this is so in Sect. 7.1.

### 7.1 A Riemann–Lebesgue-type lemma

The functions $f(\chi_{\ell,n}, t)$ defined in Eq. (25), for fixed $t$, are not Fourier coefficients, but they behave somewhat similarly. We have

$$f(\chi_{\ell,n}, t) = \int_{t-1}^{t} e^{\int_{u}^{\lambda} \left( \frac{\bar{\lambda} - 2\bar{\mu} + 2\pi m}{\lambda} \right) du} \left( \frac{\mu(\nu)}{m(\lambda + \bar{\mu} + 2\pi m)} - (k+m)\bar{\mu} \chi_{\ell,n}^{-1/m} \right) \chi_{\ell,n}^{q/k} du.$$  

$$\times \left( p_{0,k-1}(u) \chi_{\ell,n} \lambda(u) - \mu(u) \sum_{q=0}^{k-1} p_{1,q,m-1}(u) \chi_{\ell,n}^{q/k} \right) du.$$  

We perform a change of variables. For fixed $t$, let

$$x = \int_{u}^{t} \frac{\lambda(\nu)}{\lambda} d\nu = g(u).$$  

$$\begin{align*}
\end{align*}$$
$g(u)$ is a decreasing function. Hence, it has an inverse. The differential

$$\text{d}x = -\frac{\lambda(u)}{\lambda} \text{d}u,$$

so

$$\text{d}u = -\frac{\lambda}{\lambda(g^{-1}(x))} \text{d}x.$$

$$f(\chi_{\ell,n}, t) = \int_0^1 \frac{e^{2\pi i nx + \tilde{\mu}x - \tilde{\mu}x^{-1/m}(g^{-1}(x))}}{m(\bar{\lambda} + \tilde{\mu} + 2\pi in) - (k + m)\tilde{\mu}x^{-1/m}}$$

$$\times \left( p_{0,k-1}(g^{-1}(x)) \chi_{\ell,n} \lambda(g^{-1}(x)) - \mu(g^{-1}(x)) \sum_{q=0}^{k-1} p_{1,q,m-1}(g^{-1}(x)) \chi_{\ell,n}^{q/k} \right)$$

$$\times \frac{\bar{\lambda}}{\lambda(g^{-1}(x))} \text{d}x$$

As $|n| \to \infty$,

$$\chi_{\ell,n}^{-1/m} \approx \frac{k}{m} (\bar{\lambda} + \tilde{\mu} + 2\pi in)^{-k/m}.$$

Define

$$h_{\ell,n}(x) = \frac{e^{\tilde{\mu}x - \tilde{\mu}x^{-1/m} + \int_{g^{-1}(x)}^{x} \mu(v)(\chi_{\ell,n}^{-1/m} - 1)\text{d}v}}{m(\bar{\lambda} + \tilde{\mu} + 2\pi in) - (k + m)\tilde{\mu}x^{-1/m}}$$

$$\times \left( p_{0,k-1}(g^{-1}(x)) \chi_{\ell,n} \lambda(g^{-1}(x)) - \mu(g^{-1}(x)) \sum_{q=0}^{k-1} p_{1,q,m-1}(g^{-1}(x)) \chi_{\ell,n}^{q/k} \right)$$

$$\times \frac{\bar{\lambda}}{\lambda(g^{-1}(x))}$$

so that

$$f(\chi_{\ell,n}, t) = \int_0^1 e^{2\pi i nx} h_{\ell,n}(x) \text{d}x.$$

Let $\mathbb{T} = \{0, 1\}$. If $h_{\ell,n}(x)$ is a continuous $N$ times differentiable periodic function ($h_n(x) \in C^N(\mathbb{T})$ with $h_{\ell,n}^{(k)}(0) = h_{\ell,n}^{(k)}(1)$ for $0 \leq k \leq N$), then repeated applications of integration by parts will yield

$$f(\chi_{\ell,n}, t) = \left( \frac{-1}{2\pi in} \right)^N \int_0^1 h_{\ell,n}^{(N)}(x) e^{2\pi i nx} \text{d}x.$$
where $h_{\ell,n}^{(N)}(x)$ is the $N$th derivative of $h_{\ell,n}(x)$. Then

$$\left| f(\chi_{\ell,n}, t) \right| \leq \left( \frac{1}{2 \pi |n|} \right)^N \int_0^1 \left| h_{\ell,n}^{(N)}(x) \right| \, dx.$$ 

Note that

$$\chi_{\ell,n} = \left( \frac{1}{\lambda} \left[ \lambda + \mu \left( 1 - \chi_{\ell,n}^{-1/m} \right) + 2\pi in \right] \right)^k$$

so that $h_{\ell,n}(x) \sim C(x)n^{k-1}$ for some function $C(x)$ that does not depend on $n$. The contribution from $\chi_{\ell,n}^{-1/m} \to 0$ as $|n|$ increases. However, so long as $h_{\ell,n}(x)$ is sufficiently smooth, the integral

$$f(\chi_{\ell,n}, t) = \int_0^1 e^{2\pi inx} h_{\ell,n}(x) \, dx \to 0$$

as $n \to \infty$. This happens because the rapid oscillations introduced by the factor $e^{2\pi inx}$ cause the integral to go to zero. Figure 5 shows a graph of $e^{2\pi inx} h_{\ell,25}(x)$ for $t = 0.25$ over a single time period to illustrate this idea. See Loukas Grafakos text *Classical Fourier Analysis* [10], Theorem 3.3.9, p. 196 for a similar result for Fourier coefficients.

### 8 Waiting time distribution

If a customer enters the system when there are already $j$ customers ahead of him and the customer being served is in service phase $s$, then at least $m - s + m(j - 1)$
additional service phases must be completed before he begins service and \( m - s + mj \) must be completed before his service is finished and he leaves the queue. Let \( W_q(u) \) represent the waiting time until a customer arriving at time \( u \) reaches the front of the queue and \( W(u) \) represent the waiting time including service for that customer. The waiting time distributions, given that \( X(u) = j \), \( J(u) = s \) (\( s \in \{1, 2, \ldots, m\} \)) and \( j \geq 1 \) are

\[
P\{W_q(u) \leq t | X(u) = j, J(u) = s\} = \sum_{q=mj-s}^{\infty} \frac{\left(\int_u^{u+t} \mu(v)dv\right)^q}{q!} e^{-\int_u^{u+t} \mu(v)dv}
\]

and

\[
P\{W(u) \leq t | X(u) = j, J(u) = s\} = \sum_{q=mj-s+m}^{\infty} \frac{\left(\int_u^{u+t} \mu(v)dv\right)^q}{q!} e^{-\int_u^{u+t} \mu(v)dv}.
\]

From Eq. (24),

\[
P\{X(u) = j, J(u) = s\} = \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^{m} f(\chi_{\ell,n}, u) \left( \sum_{a=0}^{k-1} \chi_{\ell,n}^{-a/k} \right) \chi_{\ell,n}^{-j/s/m}.
\]

This yields the following theorem:

**Theorem 3** (Waiting time distributions) The waiting time distributions for an ergodic \( E_k(t)/E_m(t)/1 \) queue with periodic transition rates with period one and \( \lambda(t) \) for arrival phases and \( \mu(t) \) for departure phases for a customer entering at time \( u \) within the period are given by

\[
P\{W_q(u) \leq t\} = \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^{m} f(\chi_{\ell,n}, u) \left( \sum_{a=0}^{k-1} \chi_{\ell,n}^{-a/k} \right) \chi_{\ell,n}^{-1/m} \times \left(1 - e^{-\int_u^{u+t} \mu(v)(\chi_{\ell,n}^{-1/m} - 1)dv}\right) + \sum_{\ell=0}^{k-1} p_{0,\ell}(u)
\]

and

\[
P\{W(u) \leq t\} = \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^{m} f(\chi_{\ell,n}, u) \left( \sum_{a=0}^{k-1} \chi_{\ell,n}^{-a/k} \right) \chi_{\ell,n}^{-1/m} \times \left(1 - e^{-\int_u^{u+t} \mu(v)dv}\right) \sum_{q=0}^{m} \frac{\left(\int_u^{u+t} \mu(v)dv\right)^q}{q!} e^{-\int_u^{u+t} \mu(v)dv} \left(\chi_{\ell,n}^{-1/m} \int_u^{u+t} \mu(v)dv - \sum_{a=0}^{m} \left(\chi_{\ell,n}^{-1/m} \int_u^{u+t} \mu(v)dv\right)^a \chi_{\ell,n}^{-a/k} \int_u^{u+t} \mu(v)dv \right)\right)
\]
These graphs compare the asymptotic estimate for the waiting time distribution to the ODE estimate for the distribution for the specified arrival and service phases for the $E_7(t)/E_4(t)/1$ system. See Eq. (47)

$$
\begin{align*}
+ \sum_{\ell=0}^{k-1} p_{0,\ell}(u) & \left( 1 - \sum_{q=1}^{m} \frac{\left( \int_{u}^{u+t} \mu(v) \, dv \right)^q}{q!} e^{-\int_{u}^{u+t} \mu(v) \, dv} \right).
\end{align*}
$$

We estimate the waiting time distribution with the expression

$$
P(W_{q}^{(q)}(u) \leq t) = \sum_{n=-q}^{q} \sum_{\ell=1}^{m} f(\chi_{\ell,n}, u) \left( \sum_{a=0}^{k-1} \chi_{\ell,n}^{-a/k} \right) \frac{\chi_{\ell,n}^{-1/m}}{1 - \chi_{\ell,n}^{-1/m}} \times \left( 1 - e^{\int_{u}^{u+t} \mu(v) \, dv} \right) + \sum_{\ell=0}^{k-1} p_{0,\ell}(u).
$$

An example appears in Fig. 6.
9 Busy period distribution

In this section, we follow the approach of Baek, Moon and Lee [3] and apply it to the case of time-varying periodic parameters to find the busy period in terms of a Volterra equation of the second kind. Let us define the first passage time

\[ \tau_j = \inf \{ t > u, N(t) = 0 | N(u) = j, J(u) = q \}. \]

We note that \( \tau_j \) is the length of a busy period that starts with \( j \) customers in the system and with an arriving customer in phase \( q \).

Let us define the following probabilities:

\[
Q^{(j)}_n(t) = P\{ N(t) = n, \tau_j > t | N(u) = j, J(u) = q \},
\]

\[
Q^{(j)}_0(t) = P\{ \tau_j < t | N(u) = j, J(u) = q \}.
\]

We find the busy time distribution. \( \{ N(t), J(t) \} \) is a continuous time Markov chain with absorbing boundary at \( N(\tau_j) = 0 \). We set up the following system of ordinary differential equations:

\[
\begin{align*}
\frac{d}{dt} Q^{(j)}_0(t) &= Q^{(j)}_1(t)A_{-1}(t) \\
\frac{d}{dt} Q^{(j)}_1(t) &= Q^{(j)}_1(t)A_0(t) + Q^{(j)}_2(t)A_{-1}(t) \\
\frac{d}{dt} Q^{(j)}_n(t) &= Q^{(j)}_{n-1}(t)A_1(t) + Q^{(j)}_n(t)A_0(t) + Q^{(j)}_{n+1}(t)A_{-1}(t) \\
\end{align*}
\]

(48)

To solve the system of differential equations (48), we define the generating function

\[ G^{(j)}(z, t, u) = \sum_{n=0}^{\infty} Q^{(j)}_n(t)z^n. \]

The differential equation for the generating function is

\[ \frac{d}{dt} G^{(j)}(z, t, u) = G^{(j)}(z, t, u)A(z, t) - Q^{(j)}_0(t, u)A(z, t) \]

with solution

\[ G^{(j)}(z, t, u) = G^{(j)}(z, u, u)\Phi(z, t, u) - \int_{u}^{t} Q^{(j)}_0(v)A(z, v)\Phi(z, t, v)dv \]

where \( G(z, u, u) = z^j e_q \) and \( e_q \) is a row vector with a one at component \( q \) and zeros elsewhere. Since \( Q^{(j)}_0(t) = [z^0] G^{(j)}(z, t, u) \), \( Q^{(j)}_0(t) \) solves the Volterra equation of
the second kind:

\[
Q_0^{(j)}(t) = e_q[z^{-j}][\Phi(z, t, u) - \int_u^t Q_0^{(j)}(\nu) A_{-1}(\nu)[z^1]\Phi(z, t, \nu) + A_0(\nu)[z^0]\Phi(z, t, \nu) + A_1(\nu)[z^{-1}]\Phi(z, t, \nu)] d\nu.
\]

(49)

The matrix coefficient on \(z^n\) in the generating function for the unbounded process that appears (several times) in Eq. (49) is given in Eq. (39). For example,

\[
[z^{-j}][\Phi(z, t, u)](a_1, s_1)(a_2, s_2)
\]

\[
e^{-\int_u^t (\lambda(\nu) + \mu(\nu)) d\nu} \sum_{\ell = j} \left( \int_u^t \mu(\nu) d\nu \right)^{\ell m + s} \left( \int_u^t \lambda(\nu) d\nu \right)^{(-j + \ell)k + a} \frac{(\ell m + s)! \cdot (-j + \ell)k + a)!}{a_a = a_2 - a_1, \quad s = s_2 - s_1 \quad \text{and} \quad a \geq 0, \quad s \geq 0.}
\]

10 Finding the boundary probabilities

In Theorem 2, we derive that probability distribution for the level probability vectors \(p_j(t)\) in terms of an integral function involving the boundary probabilities \(p_0, k-1(t)\) and \(p_1, q, m-1(t)\) for \(q = 0, \ldots, k - 1\). In this section, we describe how to compute those probabilities.

We solve Eq. (28) for \(p_0(t)\) in terms of \(p_1(t)\). We have

\[
p_0(t) = \int_{t-1}^t p_1(u)Q_{1,0}(u)\Lambda(t, u)du \left( I - \Lambda(t, t - 1) \right)^{-1}
\]

(50)

where \(\Lambda(t, u)\) is the evolution operator defined in Eq. (26). The expression \(\left( I - \Lambda(t, t - 1) \right)^{-1}\) depends only on the average value of \(\lambda(t)\) over a single time period, and not on the value of \(t\), so we define

\[
\bar{\Lambda} = \Lambda(t, t - 1).
\]

The lack of dependence on \(t\) means the quantity need only be computed once. We also define

\[
p_1^{(m-1)}(t) = \left[ p_{1,0,m-1}(t) \quad p_{1,1,m-1}(t) \quad \cdots \quad p_{1,k-1,m-1}(t) \right].
\]

(51)

The vector \(p_1^{(m-1)}(t)\) gives the probability that the process is in level 1, service phase \(m - 1\) and in one of \(k\) arrival phases. This allows us to write Eq. (50) more succinctly as

\[
p_0(t) = \int_{t-1}^t p_1^{(m-1)}(u)\mu(u)\Lambda(t, u)du \left( I - \bar{\Lambda} \right)^{-1}.
\]

(52)
We use Eq. (24) in Theorem 2 to express $p_1(t)$ in terms of $p_0(t)$ and itself:

$$p_1(t) = \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^{m} f(\chi_{\ell,n}, t) \chi_{\ell,n}^{-1} \left[ 1 - \chi_{\ell,n}^{-1/k} \cdots \chi_{\ell,n}^{(1-k)/k} \right] \otimes \left[ 1 - \chi_{\ell,n}^{1/m} \cdots \chi_{\ell,n}^{(m-1)/m} \right]$$

where $f(x, t)$ is as defined in Eq. (25). The probability vector $p_1^{(m-1)}(t)$ defined in Eq. (51) is given by

$$p_1^{(m-1)}(t) = \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^{m} f(\chi_{\ell,n}, t) \chi_{\ell,n}^{-1/m} \left[ 1 - \chi_{\ell,n}^{-1/k} \cdots \chi_{\ell,n}^{(1-k)/k} \right].$$

Define

$$g(x, t, u) = \frac{\exp_{\mu}(\lambda(v)(x^{1/k}-1) + \mu(v)(x^{-1/m}-1)) dv}{m\lambda x^{1/k} - k\lambda x^{1/m}},$$

then

$$f(x, t) = \int_{t-1}^{t} g(x, t, u) \left( p_{0,k-1}(u)x(\lambda(u) - \mu(u)) \sum_{q=0}^{k-1} p_{1,q,m-1}(u) x^{q/k} \right) du.$$

From Eq. (52),

$$p_{0,k-1}(t) = \int_{t-1}^{t} p_1^{(m-1)}(u) \mu(u) \Lambda(t, u) du (I - \Lambda(t, t-1))^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \quad (53)$$

So

$$f(x, t) = \int_{t-1}^{t} p_1^{(m-1)}(u) \mu(u) \int_{u}^{t} \Lambda(v, u)(I - \bar{\Lambda})^{-1}\lambda(v) g(x, t, v) dv x - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} x - \begin{bmatrix} 1 \\ x^{1/k} \\ \vdots \\ x^{(k-1)/k} \end{bmatrix} g(x, t, u) du.$$

Let

$$B(x, t, u) = \int_{u}^{t} \Lambda(v, u)(I - \bar{\Lambda})^{-1}\lambda(v) g(x, t, v) dv \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} x - \begin{bmatrix} 1 \\ x^{1/k} \\ \vdots \\ x^{(k-1)/k} \end{bmatrix} g(x, t, u).$$
For each triple \((x, t, u)\), \(B(x, t, u)\) is a \(k \times 1\) vector. The probability vector \(p^{(m-1)}_1(t)\) may then be expressed in terms of itself as

\[
p^{(m-1)}_1(t) = \int_{t-1}^t p^{(m-1)}_1(u) \mu(u) \times 
\sum_{n=-\infty}^{\infty} \sum_{\ell=1}^m B(\chi_{\ell,n}, t, u) du \chi_{\ell,n}^{-1/m} \left[ \begin{array}{c} 1 \chi_{\ell,n}^{-1/k} \cdots \chi_{\ell,n}^{(1-k)/k} \end{array} \right].
\]

This equation may be rewritten as

\[
p^{(m-1)}_1(t) = \int_0^t p^{(m-1)}_1(u) \mu(u) \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^m B(\chi_{\ell,n}, t, u) du \chi_{\ell,n}^{-1/m} \left[ \begin{array}{c} 1 \chi_{\ell,n}^{-1/k} \cdots \chi_{\ell,n}^{(1-k)/k} \end{array} \right] + \int_t^1 p^{(m-1)}_1(u) \mu(u) 
\times \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^m B(\chi_{\ell,n}, 1+t, u) du \chi_{\ell,n}^{-1/m} \left[ \begin{array}{c} 1 \chi_{\ell,n}^{-1/k} \cdots \chi_{\ell,n}^{(1-k)/k} \end{array} \right]
\]

\quad (54)

using periodicity. The purpose of breaking the integral into the two intervals \([0, t]\) and \([t, 1]\) will become evident when we discretize the integral Eq. (54) and solve it numerically.

We can solve the integral Eq. (54) for \(p^{(m-1)}_1(t)\) up to a constant multiple numerically. To complete the calculation, we use the fact that

\[
p_0(t) 1_k + \sum_{j=1}^{\infty} p_j(t) 1_{km} = 1
\]

for all \(t\). Rewriting Eq. (24) using this notation

\[
p_j(t) = \int_{t-1}^t p^{(m-1)}_1(u) \mu(u) \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^m B(\chi_{\ell,n}, t, u) du \chi_{\ell,n}^{-j} 
\times \left[ \begin{array}{c} 1 \chi_{\ell,n}^{-1/k} \cdots \chi_{\ell,n}^{(1-k)/k} \end{array} \right] \otimes \left[ \begin{array}{c} 1 \chi_{\ell,n}^{1/m} \cdots \chi_{\ell,n}^{(m-1)/m} \end{array} \right]
\]

so

\[
1 = p_0(t) 1_k + \sum_{j=1}^{\infty} p_j(t) 1_{km} = \int_{t-1}^t p^{(m-1)}_1(u) \mu(u) \left\{ \Lambda(t, u) (1 - \bar{\Lambda})^{-1} 1_k 
+ \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^m B(\chi_{\ell,n}, t, u) \frac{1 - \chi_{\ell,n}^{-1}}{\chi_{\ell,n}^{1/m} - 1} \frac{1 - \chi_{\ell,n}^{-1/k}}{\chi_{\ell,n}^{1/k} - 1} \right\} du
\]

\quad (55)
for all $t$. To obtain Eq. (55), we have used the fact that

$$
\sum_{j=1}^{\infty} \chi_{\ell,n}^{-j} \left[ 1 \ 1/\chi_{\ell,n}^{-1/k} \cdots \chi_{\ell,n}^{(1-k)/k} \right] \otimes [1 \ 1/\chi_{\ell,n}^{-1/k} \cdots \chi_{\ell,n}^{(m-1)/m}] \mathbf{1}_{km}
$$

is a $k \times k$ matrix. For $t_i \leq t_j$, define

$$
\mathbf{B}_{i,j}^{(r,a)} = \left[ \frac{\mu(t_i)}{N} \sum_{m=-q}^{q} \sum_{\ell=1}^{m} \mathbf{B}(\chi_{\ell,n},t_j,t_i) \chi_{\ell,n}^{-1/m} \left[ 1 \ 1/\chi_{\ell,n}^{-1/k} \cdots \chi_{\ell,n}^{(1-k)/k} \right] \left[ \mathbf{1} \ 1/\chi_{\ell,n}^{-1/k} \cdots \chi_{\ell,n}^{(m-1)/m} \right] \right]_{r,v}
$$

for each ordered pair $(t_i, t_j)$. If $t_i > t_j$, then

$$
\mathbf{B}_{i,j}^{(r,a)} = \left[ \frac{\mu(t_i)}{N} \sum_{m=-q}^{q} \sum_{\ell=1}^{m} \mathbf{B}(\chi_{\ell,n},t_j+1,t_i) \chi_{\ell,n}^{-1/m} \left[ 1 \ 1/\chi_{\ell,n}^{-1/k} \cdots \chi_{\ell,n}^{(1-k)/k} \right] \left[ \mathbf{1} \ 1/\chi_{\ell,n}^{-1/k} \cdots \chi_{\ell,n}^{(m-1)/m} \right] \right]_{r,v}
$$

The matrix $\mathbf{B}$ is then built from the $k^2 N \times N$ blocks $\mathbf{B}^{(r,a)}$, that is, $\mathbf{B}$ is a $kN \times kN$ matrix with entries in lexicographic order with phase first and then time within the period. $\mathbf{B}_{i,j}^{(r,a)}$ is the row $rN+i+1$, column $vN+j+1$ component of the matrix $\mathbf{B}$ for $r, v = 0, \ldots, k-1$ and $i, j = 0, \ldots, N-1$. The fraction $1/N$ that is part of the definition of the matrix $\mathbf{B}$ and in the matrix $\mathbf{C}$ that follows is a numerical integration weight for the trapezoidal method. For the trapezoidal method, the end weights are $1/2N$. For any periodic function $a(t)$ with period 1, $a(t) = a(t+1)$, so $1/2N a(t) + 1/2N a(t+1) = 1/N a(t)$. For this reason, the factor $1/2N$ does not appear in the calculations.

The $kN$ component row vector $\mathbf{p}^{(m-1)}_1$ is formed by discretizing the $k$ component vector function $\mathbf{p}^{(m-1)}_1(t)$, also in lexicographic order, first phase and then time. We solve numerically for the vectors $\mathbf{p}^{(m-1)}_{1,i}(t_i)$ for $i = 0, \ldots, N-1, j = 0, \ldots, k-1$ by evaluating the expression

$$
\mathbf{P}^{(m-1)}_1 = \mathbf{p}^{(m-1)}_1 \mathbf{B}
$$

(56)
with the additional requirement that \( \sum_i p_1^{(m-1)}(t_i) 1_k = c \) for a constant \( c \) whose value is to be determined.

Define the \( kN \times N \) matrix \( \mathbf{C} \) so that

\[
\mathbf{C}_{i,j}^{(r)} = \left[ \frac{\mu(t_i)}{N} \left\{ \lambda(t_j, t_i)(\mathbf{I} - \mathbf{\bar{A}})^{-1} 1_k \right. \right.
\]
\[
+ \sum_{n=-q}^{q} \sum_{\ell=1}^{m} B(\chi_{\ell,n}, t_j, t_i) \frac{1 - \chi_{\ell,n}^{-1}}{(\chi_{\ell,n}^{1/m} - 1)(1 - \chi_{\ell,n}^{-1}/k)} \left. \right\} \right]_r
\]

for \( t_i \leq t_j \) and

\[
\mathbf{C}_{i,j}^{(r)} = \left[ \frac{\mu(t_i)}{N} \left\{ \lambda(t_j + 1, t_i)(\mathbf{I} - \mathbf{\bar{A}})^{-1} 1_k \right. \right.
\]
\[
+ \sum_{n=-q}^{q} \sum_{\ell=1}^{m} B(\chi_{\ell,n}, t_j + 1, t_i) \frac{1 - \chi_{\ell,n}^{-1}}{(\chi_{\ell,n}^{1/m} - 1)(1 - \chi_{\ell,n}^{-1}/k)} \left. \right\} \right]_r
\]

for \( t_j > t_i \). \( \mathbf{C}_{i,j}^{(r)} \) is the row \( rN + i + 1 \), column \( j + 1 \) component of the matrix \( \mathbf{C} \) for \( r = 0, \ldots, k-1, i, j = 0, \ldots, N-1 \). The matrix \( \mathbf{C} \) is composed of \( k \) blocks with the \( N \times N \) block \( \mathbf{C}_{i,j}^{(r)} \) corresponding to phase \( r \), \( r = 0, \ldots, k-1 \).

Then the constant \( c \) is chosen so that \( p_1^{(m-1)} \) solves the equation

\[
1_k^T = p_1^{(m-1)} \mathbf{C}.
\]

Equations (56) and (57) give a numerical solution for \( p_1^{(m-1)}(t) \) which can in turn be used to compute \( p_0(t), p_1(t) \) and then all of the other quantities that depend on these boundary probabilities.

### 11 Conclusion

In this paper, we developed a method for computing the asymptotic periodic distribution of the level and phase probabilities for a queue with \( k \) Erlang arrival phases and \( m \) Erlang service phases. We also showed how to compute the waiting time distribution seen by a customer arriving at any time within the period assuming that the system is in its asymptotic periodic “steady state.” This calculation requires computing an integral over a single time-period. We provide exact Fourier like expansions, but require only finitely many of these terms to compute the level probabilities to arbitrary accuracy. We compare our results to those obtained by solving a truncated version of the infinite system of differential equations and letting the system run until asymptotic periodic distribution is achieved.

The computations require the asymptotic periodic solution for the queue being idle or having a single customer. These boundary probabilities can be computed by writing \( p_1(t) \) in terms of itself and using the fact that for each time \( t \) within the period, the
probabilities must sum to one. We also express the busy period as a solution of a Volterra equation of the second kind.

The method is applicable in principle to any ergodic-level independent QBD process with time-varying periodic transition rates. For many such models, analytic solutions of the components of the generating function may not be as readily found. In those cases, there is some hope from advances in numerical inversion of $Z$-transforms.

**Supplementary Information** The online version contains supplementary material available at https://doi.org/10.1007/s11134-022-09851-x.

**Acknowledgements** The author would like to thank two anonymous referees and the editor of this journal for helpful comments and suggestions.

**Funding** Not applicable.

**Code Availability** Not applicable.

**Declarations**

**Conflict of interest** The authors declare that they have no conflicts of interest.

**A Appendix: Evolution operators**

A Banach space is a complete normed vector space. Let $X$ be a Banach space. For every $t$, $0 \leq t \leq T$ let $A(t): D(A(t)) \subset X \to X$ be a linear operator in $X$. Consider the initial value problem

$$\begin{align*}
\dot{u}(t) &= u(t)A(t) \\
u(s) &= x.
\end{align*}$$  

(58)

A $X$ valued function $u: [s, T] \to X$ is a classical solution of (58) if $u$ is a continuous function on $[s, T]$, $u(t) \in D(A(t))$, for $s < t \leq T$, $u$ is continuously differentiable on $s < t \leq T$ and satisfies (58).

**Theorem 4** (Pazy) [25] Let $X$ be a Banach space for every $t$, $0 \leq t \leq T$. Let $A(t)$ be a bounded linear operator on $X$. If the function $t \to A(t)$ is continuous in the uniform operator topology then for every $x \in X$ the initial value problem (58) has a unique classical solution $u$.

**Definition 3** We define the evolution operator of the initial value Problem (58) by

$$xU(t, s) = u(t)$$

for $0 \leq s \leq t \leq T$ where $u$ is the solution of (58).

The evolution operator is also called a solution operator or propagator in the literature. If we meet the conditions in theorem 4, then our evolution operator has the following properties:

**Theorem 5** (Evolution operator properties) [25] For every $0 \leq s \leq t \leq T$, $U(t, s)$ is a bounded linear operator and
1. \[ ||U(t, s)|| \leq \exp \left\{ \int_s^t A(\tau) \, d\tau \right\}. \]
2. \[ U(t, t) = I, \quad U(t, s) = U(t, r)U(r, s) \text{ for } 0 \leq s \leq r \leq t \leq T. \]
3. \((t, s) \rightarrow U(t, s)\) is continuous in the uniform operator topology for \(0 \leq s \leq t \leq T.\)
4. \[ \frac{\partial U(t, s)}{\partial s} = U(t, s)A(t) \text{ for } 0 \leq s \leq t \leq T. \]
5. \[ \frac{\partial U(t, s)}{\partial t} = -A(s)U(t, s) \text{ for } 0 \leq s \leq t \leq T. \]

**Definition 4** A two parameter family of bounded linear operators \(U(t, s), 0 \leq s \leq t \leq T,\) on \(X\) is called an evolution system if the following two conditions are met:
1. \[ U(s, s) = I, \quad U(t, r)U(r, s) = U(t, s) \text{ for } 0 \leq s \leq r \leq t \leq T. \]
2. \((t, s) \rightarrow U(t, s)\) is strongly continuous for \(0 \leq s \leq t \leq T.\)

In this paper, we observe evolution operators in one of two contexts: 1) the \(u(t)\) are probability vectors, \(x\) is the probability distribution at some initial time \(s,\) and \(A(t)\) is an infinite-dimensional matrix with bounded non-negative entries; or 2) \(u(z, t)\) are vectors of probability generating functions and \(A(z, t)\) are finite dimensional matrices with parameter \(z\) defined in Eq. (19).

The evolution operator is a generalization of the exponential function. If \(x\) and \(A\) are scalars, \(U(t, s)\) is a scalar exponential function. If \(x\) is a vector and \(A(t) = A\) an appropriately dimensioned matrix, then \(U(t, s)\) is the matrix exponential \(e^{A(t-s)}.\) The matrix exponential may be defined in terms of its Taylor series expansion:

\[
e^{At} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n. \tag{59}\]

When \(A(t)\) depends on time, we have the Peano series representation for \(U(t, s).\) Define

\[ \mathcal{I}_n(t, s) = \int_s^t A(\tau_1) \int_s^{\tau_1} A(\tau_2) \cdots \int_s^{\tau_{n-1}} A(\tau_n) d\tau_1 \cdots d\tau_n, \]

with \(\mathcal{I}_0 = I,\) then:

\[ U(t, s) = \sum_{n=0}^{\infty} \mathcal{I}_n(t, s). \tag{60}\]

In the event that \(A(t) = A\) is a constant matrix, the Peano series given in Eq. (60) yields the matrix exponential \(e^{A(t-s)}.\)

\(U(t, s)\) solves the Volterra equation

\[ U(t, s) = I + \int_s^t U(\tau, s)A(\tau) \, d\tau. \]

If the matrix \(A(t)\) is finite, then \(U(t, s)\) is given by the product integral

\[ U(t, s) = \lim_{\Delta\tau_i \to 0} \prod_i e^{A(\tau_i)} \Delta\tau_i. \]
This product integral is sometimes called the time-ordered exponential from \( s \) to \( t \), or the multiplicative integral. See Dollard and Friedman [5] or Gill and Johansen [9]. There are several examples in the body of the paper.

References

1. Adan, I., Zhao, Y.: Analyzing \( GI/E_r/1 \) queues. Oper. Res. Lett. 19(4), 183–190 (1996). https://doi.org/10.1016/0167-6377(96)00024-7
2. Arizono, I., Ohta, H., Deutsch, S., Wang, C.C.: An analysis of the \( E_l/E_k/1 \) queueing system by restricted minimal lattice paths. J. Oper. Res. Soc. 46(2), 245–253 (1995). https://doi.org/10.1057/jors.1995.29
3. Baek, J., Moon, S., Lee, H.: A time-dependent busy period queue length formula for the \( M/E_k/1 \) queue. Statist. Probab. Lett. 78, 98–104 (2014). https://doi.org/10.1016/j.spl.2014.01.004
4. Breuer, L.: The periodic \( BMAP/PH/c \) queue. Queueing Syst. 38, 67–76 (2001). https://doi.org/10.1023/A:10108721028919
5. Dollard, J., Friedman, C.: Product Integration with Applications to Differential Equations (with an appendix by P.R. Masani), Encyclopedia of Mathematics and its Applications, vol. 10. Addison-Wesley (1979)
6. Flajolet, P., Sedgewick, R.: Analytic Combinatorics. Cambridge University Press, Cambridge (2009). https://doi.org/10.1017/CBO9780511801655
7. Foh, C.H., Zukerman, M.: A new technique for performance evaluation of random access protocols. In: Proceedings of the 2002 IEEE International Conference on Communications, vol. 3, pp. 2284–2288 (2002). https://doi.org/10.1109/ICC.2002.997253
8. Gayon, J.P., de Véricourt, F., Karaesmen, F.: Stock rationing in an \( m/e_r/1 \) multi-class make-to-stock queue with backorders. IIE Trans. 41(12), 1096–1109 (2009). https://doi.org/10.1080/07408170902800279
9. Gill, R., Johansen, S.: A survey of product-integration with a view toward application in survival analysis. Ann. Stat. 18(4), 1501–1555 (1990)
10. Grafakos, L.: Classical Fourier Analysis. Springer (2014). https://doi.org/10.1007/978-1-4939-1194-3
11. Grassmann, W.K.: A new method for finding the characteristic roots of \( E_n/E_m/1 \) queues. Methodol. Comput. Appl. Probab. 13, 873–886 (2011). https://doi.org/10.1007/s11009-010-9199-2
12. Griffiths, J., Leonenko, G., Williams, J.: The transient solution to \( M/E_k/1 \) queue. Oper. Res. Lett. 34, 349–354 (2006). https://doi.org/10.1016/J.ORL.2005.05.010
13. Horváth, I., Mészáros, A., Telek, M.: Numerical inverse transformation methods for z-transform. Mathematics (2020). https://doi.org/10.3390/math8040556
14. Jayasuriya, A., Green, D., Asenstorfer, J.: Modelling service time distribution in cellular networks using phase-type service distributions. In: Proceedings of the 2001 IEEE International Conference on Communications, vol. 2, pp. 440–444. ICC2001 (2001)
15. Kleinrock, L.: Queueing Systems: Theory, vol. I. Wiley (1975)
16. Latouche, G., Ramaswami, V.: Introduction to Matrix Analytic Methods in Stochastic Modelling, 1st edition. ASA SIAM (1999). https://doi.org/10.1137/1.9780898719734
17. Laub, A.J.: Matrix Analysis for Scientists and Engineers. Society for Industrial and Applied Mathematics, Philadelphia (2005). https://doi.org/10.5555/1062366
18. Leonenko, G.: A new formula for the transient solution of the erlang queueing model. Stat. Probab. Lett. 79(3), 400–406 (2009). https://doi.org/10.1016/j.spl.2008.09.014
19. Luh, H., Liu, H.Y.: A note on simple eigenvalues of matrix polynomials in queueing models with Erlang distributions. J. Appl. Math. Comput. 21, 57–67 (2006). https://doi.org/10.1007/BF02896388
20. Margolius, B.: Asymptotic Estimates for Queueing Systems with Time-Varying Periodic Transition Rates, pp. 307–326. Springer, Cham (2019). https://doi.org/10.1007/978-3-030-11102-1_14
21. Maranov, B.H.: Eulerian polynomials and quasi-birth-death processes with time-varying-periodic rates. In: Contemporary Mathematics, vol. 774, pp. 175–193. American Mathematical Society (2021)
23. Medhi, J.: Stochastic Models in Queueing Theory, 2nd edn. Academic Press (2003). https://doi.org/10.1016/B978-0-12-487462-6.X5000-0
24. Parthasarathy, P.: A transient solution to an $M/M/1$ queue: a simple approach. Adv. Appl. Probab. 19, 997–998 (1987). https://doi.org/10.2307/1427113
25. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer (1983)
26. Poyntz, C., Jackson, R.: The steady-state solution for the queueing process $E_k/E_m/r$. Oper. Res. Q. 24(4), 615–625 (1973). https://doi.org/10.1057/S0305004100028620
27. Saaty, T.L.: Elements of Queueing Theory with Applications. McGraw-Hill Book Company, Inc. (1961)
28. Schwarz, J.A., Selinka, G., Stolletz, R.: Performance analysis of time-dependent queueing systems: survey and classification. Omega 63, 170–189 (2016). https://doi.org/10.1016/j.omega.2015.10.013
29. Sedgewick, R., Flajolet, P.: An Introduction to the Analysis of Algorithms: Introdu Analysi Algorithms. Addison-Wesley (2013)
30. Smith, W.L.: On the distribution of queueing times. Math. Proc. Cambridge Philos. Soc. 49(3), 449–461 (1953). https://doi.org/10.1017/S0305004100028620
31. Syski, R.: Introduction to Congestion Theory in Telephone Systems. Oliver and Boyd Ltd (1960)
32. Takács, L.: Transient behavior of queueing processes with Erlang input. Trans. Am. Math. Soc. 100(1), 1–28 (1961). https://doi.org/10.1090/S0002-9947-1961-0181024-9
33. Truslove, A.: Length for the $E_k/G/1$ queue with finite waiting room. Adv. Appl. Probab. 7, 215–226 (1975). https://doi.org/10.2307/1425861
34. Wang, K.H., Kuo, M.Y.: Profit analysis of the $M/E_r/1$ machine repair problem with a non-reliable service station. Comput. Ind. Eng. 32(3), 587–594 (1997). https://doi.org/10.1016/S0360-8352(96)00313-0
35. Wegert, E.: Complex Function Explorer. https://www.mathworks.com/matlabcentral/fileexchange/45464-complex-function-explorer (2021)
36. Weisstein, E.W.: Kronecker Product. From Mathworld—A Wolfram Web Resource. https://mathworld.wolfram.com/KroneckerProduct.html (2021)
37. Weisstein, E.W.: Kronecker Sum. From Mathworld—A Wolfram Web Resource. https://mathworld.wolfram.com/KroneckerSum.html (2021)
38. Whitt, W.: Time-varying queues. Queue. Models Service Manag. 1(2), 79–164 (2018)
39. Wilf, H.: Generating Functionology. https://www2.math.upenn.edu/~wilf/DownldGF.html (2004)

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