Abstract

We estimate a general mixture of Markov jump processes. The key novel feature of the proposed mixture is that the transition intensity matrices of the Markov processes comprising the mixture are entirely unconstrained. The Markov processes are mixed with distributions that depend on the initial state of the mixture process. The new mixture is estimated from its continuously observed realizations using the EM algorithm, which provides the maximum likelihood (ML) estimates of the mixture’s parameters. We derive the asymptotic properties of the ML estimators. To obtain estimated standard errors of the ML estimates of the mixture’s parameters, an explicit form of the observed Fisher information matrix is derived. In its new form, the information matrix simplifies the conditional expectation of outer product of the complete-data score function in the Louis (1982) general matrix formula for the observed Fisher information matrix. Simulation study verifies the estimates’ accuracy and confirms the consistency and asymptotic normality of the estimators. The developed methods are applied to a medical dataset, for which the likelihood ratio test rejects the constrained mixture in favor of the proposed unconstrained one. This application exemplifies the usefulness of a new unconstrained mixture for identification and characterization of homogeneous subpopulations in a heterogeneous population.

keywords: mixture of Markov jump processes, EM algorithm, Fisher information matrix, asymptotic distribution, heterogeneous population

1 Introduction

This paper proposes a general mixture of Markov jump processes. This new mixture is an extension of the model in Frydman (2005), which in turn extends the seminal mover-stayer model presented in Blumen, et al. (1955). The key novel feature of the proposed model is that the transition intensity matrices of the Markov processes comprising the mixture are entirely unconstrained: each homogeneous subpopulation evolves according to a Markov process with its own intensity matrix. The mixture’s regime membership distribution is assumed to depend only on the initial state of the process, and may differ between initial states. Because constraining the transition intensities may obscure the identification of
clusters, the proposed unconstrained mixture is particularly suitable for identification and characterization of homogeneous subpopulations in a heterogeneous population. This is illustrated with a medical application in which the likelihood ratio test rejects the constrained mixture from Frydman (2005) in favor of the proposed general mixture.

We obtain the maximum likelihood (ML) estimates of a general mixture (g-mixture) from the data consisting of a set of its continuously observed realizations using the EM algorithm of Dempster, et al. (1977). Using novel methods, we derive the asymptotic properties of the ML estimators thereby extending the classic results from Albert (1962) who considered ML estimation of a single Markov jump process. For a finite sample of realizations, we derive an explicit form of the observed Fisher information matrix to obtain an estimate of the covariance matrix of the MLEs. In its new form, the information matrix simplifies the conditional expectation of outer product of the complete-data score function in Louis’s (1982) general matrix formula for the observed Fisher information matrix. We show through the simulation study that the estimation is accurate and confirms the asymptotic properties of the estimators. We note that Frydman (2005) provided the MLEs of the parameters of a constrained mixture defined below, but hasn’t considered their asymptotic or finite sample properties.

The methods developed here are applied to the ventICU dataset from Cook and Lawless (2018).

The g-mixture is different from the mixture of Markov processes recently considered by Jiang and Cook (2019). There, the regime probabilities depend on covariates through the multinomial logistic regression, while the Markov processes in the mixture are assumed to have the same intensity matrices. In the g-mixture, the regime probabilities depend only on an initial state, while the Markov processes have their own intensity matrices. We observe g-mixture continuously, whereas Jiang and Cook (2019) observe their mixture intermittently.

To define the proposed model, let \( X = \{X_m, 1 \leq m \leq M\} \) be the mixture of \( M \) right-continuous Markov jump processes with the intensity matrices \( Q_m \)'s, and transition matrices \( P_m(t) = \exp(Q_m t) \) defined on the finite state space \( S = E \cup \Delta \), where \( E \) is a set of non-absorbing states and \( \Delta \) a set of absorbing states. There is a separate mixing distribution for each initial state \( i \in E \) of \( X \),

\[
\phi_{i,m} \equiv \mathbb{P}(X = X_m | X_0 = i), \quad 1 \leq m \leq M,
\]

where \( \sum_{m=1}^{M} \phi_{i,m} = 1 \). Let \( D_m = \text{diag}(\phi_{1,m}, \ldots, \phi_{w,m}) \), where \( w \) is the cardinality of \( E \). Then the transition matrix of a mixture process \( X \) is given by

\[
P(t) = \sum_{m=1}^{M} D_m P_m(t), \quad t \geq 0.
\]

In the absence of absorbing states, Frydman (2005) specified the following structure for the intensity matrices of Markov processes comprising the mixture

\[
Q_m = \Gamma_m Q \quad (1 \leq m \leq M),
\]

where \( Q \) is an intensity matrix, \( \Gamma_m = \text{diag}(\gamma_{1,m}, \ldots, \gamma_{w,m}) \), with \( \gamma_{i,m} \geq 0 \), for \( 1 \leq m \leq M - 1 \), and \( \Gamma_M = I \), an identity matrix. Depending on whether \( \gamma_{i,m} = 0 \), \( 0 < \gamma_{i,m} < 1 \), \( \gamma_{i,m} \geq 1 \), the realizations generated by \( Q_m \) do not move out of state \( i \), or move out of state \( i \) at a lower or higher rate than those generated by \( Q \), or at an identical rate. This specification
constrains the transition matrices of Markov chains embedded into Markov processes in the mixture to be all the same. This mixture is restrictive in situations in which the population is heterogeneous not only with respect to exit rates from states, but also with respect to the direction of movement. As illustrated in the medical application in this paper, it is particularly restrictive when the components of the mixture are absorbing Markov processes as in this case the absorption probabilities would be the same for all Markov processes in the mixture. Nevertheless, this mixture has been successfully applied to modeling bond-ratings migration by Frydman and Schuermann (2008), and to clustering of categorical time series by Pamminger and Fruhwirth-Schnatter (2010). Its distributional properties were studied in Surya (2018).

By setting $M = 2$, and $\gamma_{1,1} = ... = \gamma_{w,1} = 0$ in $\Gamma_1$, the transition matrix of $X$ reduces to $D_1 I + (I - D_1)P_2(t)$, a transition matrix of a continuous-time mover-stayer (MS) model. The MS model assumes a simple form of population heterogeneity: there are stayers who never leave their initial states, with $P_1(t) = I$ as their transition matrix, and movers who evolve among the states according to transition matrix $P_2(t) = \exp(tQ)$. The MS model was the first mixture of Markov chains considered in the literature, and its use in Blumen, et al. (1955) to study labor mobility was the first application of stochastic processes in the social sciences. Frydman (1984) obtained the ML estimators of the discrete-time MS model’s parameters by direct maximization of the observed likelihood function, and Fuchs and Greenhouse (1988) did so by using the expectation-maximization (EM) algorithm. In both cases, estimation used the data on independent realizations of the MS model.

Despite capturing a very simple form of population heterogeneity, the MS mixture’s discrete and continuous-time versions have been widely applied in diverse fields, including medicine (Tabar, et al., 1996), labor economics (Fougere and Kamionka, 2003), large data (Cipollini, et al., 2012), farming (Saint-Cyr and Piet, 2017), and credit risk (Frydman and Kadam, 2004, and Ferreti, et al., 2019). In a continuous-time framework, Yi, et al. (2017) estimated an MS model from panel data in the presence of state misclassification. Cook, et al. (2002) developed a generalized MS model, which allows for subject specific absorbing states, and Shen and Cook (2014) considered a dynamic MS model for recurrent events that can be resolved. In a discrete-time framework, Frydman and Matuszyk (2018, 2019) developed an estimation for a discrete-time MS model with covariate effects on stayers’ probability and movers’ transitions.

The paper is organized as follows. Section 2 sets the notation and derives the observed and complete likelihood functions. Section 3 presents the EM algorithm, the derivation of the asymptotic properties of the MLEs and also provides the lower bound for the asymptotic variance of the MLEs. The finite sample covariance matrix of the MLEs is derived in Section 4. Section 5 is devoted to the simulation study and Section 6 applies the developed methods to ventICU data. Section 7 concludes the paper.

2 The observed and complete likelihood functions

2.1 The observed likelihood function

We consider the general continuous-time mixture $X$ with $M$ components defined in the Introduction. Let $X^k = \{X^k_t, 0 \leq t \leq T\}$ denote the $k$'th realization of $X$ on $[0, T]$ where $T$ is the end-of-study time, which can be either fixed, or the absorption time of $X$. Denote
by $X^k$—that realization without an initial state, that is, $X^k = (X^k) \cup X_0$. And, let $R_k$ denote the regime label of the $k$'th realization. To write the observed likelihood of $X^k$, for $1 \leq k \leq K$ and $1 \leq m \leq M$, we define the following quantities associated with $X^k$:

\[
\Phi_{k,m} = \mathcal{I}(R_k = m), \\
B^k_i = \mathcal{I}(X^k_0 = i), i \in E, \\
B_i = \sum_{k=1}^K B^k_i = \# \text{ of realizations with initial state } i, i \in E, \\
N^k_{ij} = \# \text{ of times } X^k \text{ makes an } i \to j \text{ transition, } i \neq j, i \in E, \\
T^k_i = \int_0^T I(X^k_u = i)du = \text{ total time } X^k \text{ spends in state } i, i \in E,
\]

where $\Phi_{k,m}$ is equal to 1 if the $k$'th realization evolves according to the $m$'th Markov process and equal to zero, otherwise. We note that $\Phi_{k,m}$ is unknown, but other quantities in (1) are known. For $1 \leq m \leq M$, let $\theta_m \equiv (q_{ij,m}, i \neq j, \phi_{i,m}, i \in E, j \in S)$ and $\theta = (\theta_m, 1 \leq m \leq M)$ be the mixture’s parameters assumed to live on the compact set $\Theta$ of any positive value of $\theta$. We denote by $\mathbb{P}_\theta$ the probability measure of a complete observation $\{X^k, \Phi_k\}$ of a generic sample path $X^k$ and by $\mathbb{E}_\theta$ the expectation under $\mathbb{P}_\theta$.

The observed likelihood function $L^k(\theta)$ of $X^k, 1 \leq k \leq K$, is

\[
L^k(\theta) = \sum_{m=1}^M \mathbb{P}_\theta(X^k, R_k = m) = \sum_{m=1}^M \prod_{i \in E} \mathbb{P}_\theta(X^k-, X^k_0 = i, R_k = m)^{B^k_i} \\
= \sum_{m=1}^M \prod_{i \in E} \mathbb{P}_\theta(X^k_0 = i)^{B^k_i} \mathbb{P}_\theta(R_k = m|X^k_0 = i)^{B^k_i} \mathbb{P}_\theta(X^k - |X^k_0 = i, R_k = m)^{B^k_i} \\
= \prod_{i \in E} \pi_i^{B^k_i} \sum_{m=1}^M \left\{ \prod_{i \in E} \phi_{i,m}^{B^k_i} \prod_{j \neq i, j \in S} \left( \prod_{i \in E} (q_{ij,m})^{N^k_{ij}} \exp \left[ \left( - \sum_{j \neq i, j \in S} q_{ij,m} T^k_i \right) \right] \right) \right\}
\]

where $\pi_i = \mathbb{P}_\theta(X^k_0 = i)$ with $\sum_{i \in E} \pi_i = 1$, is an initial distribution of the mixture. The loglikelihood function of the $K$ observed sample paths $\mathcal{D} = \bigcup_{k=1}^K X^k$ is

\[
\log L(\theta) = \sum_{k=1}^K \log L^k(\theta) = \sum_{k=1}^K \sum_{i \in E} B^k_i \log \pi_i \\
+ \sum_{k=1}^K \log \sum_{m=1}^M \left\{ \prod_{i \in E} \phi_{i,m}^{B^k_i} \prod_{j \neq i, j \in S} \left( \prod_{i \in E} (q_{ij,m})^{N^k_{ij}} \exp \left[ \left( - \sum_{j \neq i, j \in S} q_{ij,m} T^k_i \right) \right] \right) \right\}
\]

where we can rewrite the first term as $\sum_{i \in E} \sum_{k=1}^K B^k_i \log \pi_i = \sum_{i \in E} B_i \log \pi_i$, to see that, up to a constant, it corresponds to the loglikelihood of the multinomial distribution with parameters $K = \sum_{i \in E} B_i$ and $\pi = (\pi_i, i \in E)$, where $B_i, i \in E$ are multinomial random variables. Hence, as is well known, the MLE of $\pi_i$ is $\hat{\pi}_i = B_i/K$. Therefore, when writing the complete loglikelihood function below, we will omit the term involving $\pi$. We also note that the likelihood function (3) is the same when either the end-of-study $T$ is a fixed time...
in which case the last state occupation time may be right censored, or \( T \) is the first exit time to an absorbing state of the mixture process.

It is in general difficult to obtain the MLE \( \hat{\theta} \) by directly maximizing the observed loglikelihood function \((3)\). To obtain \( \hat{\theta} \), we use the EM algorithm, which requires the loglikelihood function under complete information derived below.

### 2.2 The loglikelihood function under complete information

We now assume that we have complete information \( \bigcup_{k=1}^{K} \{X^k, R_k\} \), that is, we also know \( \Phi_{k,m}(1 \leq k \leq K, 1 \leq m \leq M) \). By \((3)\), the complete loglikelihood of the k’th realization \( X^k \) is

\[
\log L_c^k(\theta) = \log L_c^k(\phi) + \log L_c^k(q),
\]

where

\[
\log L_c^k(\phi) = \sum_{m=1}^{M} \Phi_{k,m} \sum_{i \in E} B_i^k \log \phi_{i,m},
\]

and since \( \sum_{m=1}^{M} \phi_{i,m} = 1 \), it depends only on \( \phi = (\phi_{i,m}, i \in E, 1 \leq m \leq M - 1) \). Therefore we can express it as

\[
\log L_c^k(\phi) = \sum_{m=1}^{M-1} \Phi_{k,m} \sum_{i \in E} B_i^k \log(\phi_{i,m}) + \Phi_{k,m} \sum_{i \in E} B_i^k \log \left(1 - \sum_{m=1}^{M-1} \phi_{i,m}\right),
\]

(4)

Now

\[
\log L_c^k(q) = \sum_{m=1}^{M} \Phi_{k,m} \sum_{i \in E,j \neq i,j \in S} \left(N_{ij}^k \log q_{ij,m} - q_{ij,m}T_i^k\right)
\]

(5)

and depends only on the intensities \( q = \{q_{ij,m}, (i,j), i \in E, j \in S, 1 \leq m \leq M\} \). Then the full information loglikelihood is

\[
\log L_c(\theta) \sim \sum_{k=1}^{K} \log L_c^k(\theta) = \sum_{k=1}^{K} \left[ \log L_c^k(\phi) + \log L_c^k(q) \right]
\]

\[
= \sum_{m=1}^{M} \sum_{i \in E} \left[ \sum_{k=1}^{K} \Phi_{k,m} B_i^k \log \phi_{i,m} + \sum_{k=1}^{K} \Phi_{k,m} \sum_{j \neq i,j \in S} \left(N_{ij}^k \log q_{ij,m} - q_{ij,m}T_i^k\right) \right]
\]

\[
= \sum_{m=1}^{M} \sum_{i \in E} B_i^k \log \phi_{i,m} + \sum_{m=1}^{M} \sum_{i \in E} \sum_{j \neq i,j \in S} \left(N_{ij}^m \log q_{ij,m} - q_{ij,m}T_{i,m}\right),
\]

(6)

where the "wiggle" after \( \log L_c(\theta) \) signifies that we omitted the part of \( \log L_c(\theta) \) which involves \( \pi, B_{i,m} = \sum_{k=1}^{K} \Phi_{k,m} B_i^k \) = the number of regime \( m \) realizations with initial state \( i, N_{ij,m} = \sum_{k=1}^{K} \Phi_{k,m} N_{ij}^k \) = the total number of \( i \rightarrow j, (i \in E, j \in S) \) transitions and \( T_{i,m} = \sum_{k=1}^{K} \Phi_{k,m}T_i^k \) = the total waiting time in state \( i \in E \) for regime \( m \) realizations.

In the sequel, we will consider the following score functions. By \((4)\), the score function of the k’th complete observation with respect to \( \phi_{i,m} \) is

\[
\frac{\partial \log L_c^k(\phi)}{\partial \phi_{i,m}} = \frac{\Phi_{k,m} B_i^k}{\phi_{i,m}} - \frac{\Phi_{k,M} B_i^k}{\phi_{i,M}},
\]

(7)
and by (5), the similar score function with respect to $q_{ij,m}$ is
\[
\frac{\partial \log L_k(q)}{\partial q_{ij,m}} = \frac{\Phi_{k,m} N_{ij}}{q_{ij,m}} - \Phi_{k,m} T_i^k.
\] (8)

Finally, the score function for all complete observations with respect to $\phi_{i,m}$ is
\[
\frac{\partial \log L_c(\phi)}{\partial \phi_{i,m}} = \frac{B_{i,m}}{\phi_{i,m}} - \frac{B_{i,M}}{\phi_{i,M}}, \quad m = 1, \cdots, M - 1.
\] (9)

and with respect to $q_{ij,m}$ is
\[
\frac{\partial \log L_c(\phi)}{\partial q_{ij,m}} = \frac{N_{ij,m}}{q_{ij,m}} - T_{i,m}, \quad m = 1, \cdots, M.
\] (10)

3 Maximum likelihood estimation of the parameter $\theta$

The following result relates the score function of $\bigcup_{k=1}^{K} \{X^k, R_k\}$ and that of incomplete information $D = \bigcup_{k=1}^{K} X^k$. It is used to derive the MLE $\hat{\theta}$, its asymptotic properties and, in Section 4, an explicit form of the observed Fisher information matrix $I(\theta) = -\frac{\partial^2 \log L(\theta)}{\partial \theta^2}$.

Lemma 1 For any $\theta \in \Theta$, the incomplete-information score function of $D$ is given by
\[
\frac{\partial \log L(\theta)}{\partial \theta} = \mathbb{E}_{\theta} \left[ \frac{\partial \log L_c(\phi)}{\partial \theta} \bigg| D \right].
\]

Proof. By the first line in (5),
\[
\mathbb{E}_{\theta} \left[ \frac{\partial \log L_c(\phi)}{\partial \theta} \bigg| D \right] = \sum_{k=1}^{K} \mathbb{E}_{\theta} \left[ \frac{\partial \log L_k^c(\phi)}{\partial \theta} \bigg| X^k \right]
\]
\[
= \sum_{k=1}^{K} \mathbb{E}_{\theta} \left[ \frac{\partial \log \mathbb{P}_\theta(X^k, R_k)}{\partial \theta} \bigg| X^k \right]
\]
\[
= \sum_{k=1}^{K} \sum_{m=1}^{M} \frac{\partial \log \mathbb{P}_\theta(X^k, R_k = m)}{\partial \theta} \mathbb{P}_\theta(R_k = m) \frac{\partial \mathbb{P}_\theta(X^k, R_k = m)}{\partial \theta}
\]
\[
= \sum_{k=1}^{K} \sum_{m=1}^{M} \frac{1}{\mathbb{P}_\theta(X^k)} \sum_{m=1}^{M} \frac{\partial}{\partial \theta} \mathbb{P}_\theta(X^k, R_k = m),
\]

where the last equality follows from applying the Bayes formula for conditional probability. Noting that $\mathbb{P}_\theta(X^k) = \sum_{m=1}^{M} \mathbb{P}_\theta(X^k, R_k = m)$ completes the proof. ■

Define $\hat{B}_{i,m}(\theta) = \mathbb{E}_{\theta} \left[ B_{i,m} \bigg| D \right]$, $\hat{N}_{ij,m}(\theta) = \mathbb{E}_{\theta} \left[ N_{ij,m} \bigg| D \right]$ and $\hat{T}_{i,m}(\theta) = \mathbb{E}_{\theta} \left[ T_{i,m} \bigg| D \right]$. The MLE $\hat{\theta}$ as the solution of the following systems of equations is presented below
\[
\frac{\partial \log L(\theta)}{\partial \theta} = 0.
\]
Proposition 1 For $1 \leq m \leq M$, the maximum likelihood estimates of $\phi_{i,m}$ and $q_{ij,m}$ are

$$\hat{\phi}_{i,m} = \frac{\hat{B}_{i,m}(\theta)}{B_i} \quad (i \in E),$$

$$\hat{q}_{ij,m} = \frac{\hat{N}_{ij,m}(\theta)}{T_{i,m}(\theta)} \quad (j \neq i, i \in E, j \in S).$$

Proof. Using the score function of Frydman, Surya: Statistical inference for a mixture of Markov jump processes

We now use the EM algorithm to obtain the MLE of $\theta$. Following Dempster et al. (1977), the MLE $\hat{\phi}_{i,m}$ is obtained by setting $E_{\theta} \left[ \frac{\partial \log L_c(\theta)}{\partial \phi_{i,m}} \mid D \right] = 0$, which yields $\hat{\phi}_{i,m} = \frac{\hat{B}_{i,m}(\theta)}{\hat{B}_{i,M}(\theta)} \hat{\phi}_{i,M}$. Using the constraint $\sum_{m=1}^{M} \hat{\phi}_{i,m} = 1$ and $\sum_{m=1}^{M} \hat{B}_{i,m}(\theta) = B_i$ for any $\theta \in \Theta$, we have $\hat{\phi}_{i,M} = \frac{\hat{B}_{i,M}(\theta)}{B_i}$ and $\hat{\phi}_{i,m} = \frac{\hat{B}_{i,m}(\theta)}{\hat{B}_{i,M}(\theta)}$. The estimator $\hat{q}_{ij,m}$ is obtained by using the score function for $q_{ij,m}$ from and setting the equation $E_{\theta} \left[ \frac{\partial \log L_c(\theta)}{\partial q_{ij,m}} \mid D \right] = 0$.

In the next section, the EM algorithm is employed to find $\hat{\phi}_{i,m}$ and $\hat{q}_{ij,m}$. The results (11)-(12) will be used in the M-step of the EM-algorithm.

3.1 The EM algorithm

We now use the EM algorithm to obtain the MLE of $\theta = (\theta_m, 1 \leq m \leq M)$ from $K$ realizations. Following Dempster et al. (1977), the MLE of $\theta$ is found iteratively so that the $(p+1)$-th estimate $\theta^{p+1}$ of $\theta$ maximizes the random function

$$E_{\theta^p} \left[ \log L_c(\theta) \mid D \right],$$

where $E_{\theta^p} \left[ \cdot \mid D \right]$ refers to the conditional expectation evaluated using the current estimate $\theta^p$ after $p$ steps of the algorithm. This is the maximization step (M-step) of the algorithm. The evaluation of the above conditional expectation forms the E-step.

Proposition 2 The EM algorithm for the finite g-mixture of Markov jump processes $X = \{X_m, 1 \leq m \leq M\}$ based on $K$ realizations for a fixed $M > 1$ goes as follows.

Step 1 (Initial step) For $1 \leq m \leq M$, choose initial value $\theta^0_m \equiv (q^0_{i,j,m}, i \neq j, \phi^0_{i,m}, i \in E, j \in S)$ of $\theta_m$.

Step 2 (E-step) At the $p$'th iteration ($p = 0, 1, 2...$), for $1 \leq m \leq M$, compute the probability of observing $X^k$ under $\theta^p_m$

$$L^k(\theta^p_m) \sim \phi^p_{i,m} \prod_{i \in E} \prod_{j \neq i, j \in S} (q^p_{i,j,m})^{N^k_{i,j}} \exp \left[ \left. \left( - \sum_{j \neq i} q^p_{i,j,m} \right) T^k_i \right| \right], 1 \leq m \leq M,$$

and then the probability that the path $X^k$ comes from regime $m$

$$E_{\theta^p} \left[ \Phi_{k,m} \mid X^k \right] = \frac{L^k(\theta^p_m)}{\sum_{\ell=1}^{M} L^k(\theta^p_{\ell})}.$$

For $i \in E$, $j \in S$, and $1 \leq m \leq M$, compute

$$E_{\theta^p} \left[ N_{ij,m} \mid D \right] = \sum_{k=1}^{K} N_{ij}^k E_{\theta^p} \left[ \Phi_{k,m} \mid X^k \right], j \neq i.$$
\[
E_{\theta^p}[T_{i,m}|D] \equiv \sum_{k=1}^{K} T_{i}^{k} E_{\theta^p}[\Phi_{k,m}|X^k],
\]
\[
E_{\theta^p}[B_{i,m}|D] \equiv \sum_{k=1}^{K} B_{i}^{k} E_{\theta^p}[\Phi_{k,m}|X^k].
\]

**Step 3 (M-step)** For \(1 \leq m \leq M\), compute, using (11) and (12), the values \(\theta_m^{p+1} \equiv (q_{ij,m}^{p+1}, i \neq j, \phi_{i,m}^{p+1}, i \in E, j \in S)\) by

\[
\phi_{i,m}^{p+1} = \frac{E_{\theta^p}[B_{i,m}|D]}{B_i}, \quad (13)
\]
\[
q_{ij,m}^{p+1} = \frac{E_{\theta^p}[N_{ij,m}|D]}{E_{\theta^p}[T_{i,m}|D]}, \quad (14)
\]

**Step 4** Stop if convergence criterion is achieved, that is, if the Euclidean norm \(||\theta^{p+1} - \theta^p|| < \varepsilon\), where \(\varepsilon\) is a desired small value. Otherwise, return to step 2 and replace \(\theta_m^p\) by \(\theta_m^{p+1}\).

For the choice of mixture model, repeat the EM algorithm for different values of \(M > 1\) until the fitted model has the lowest Akaike information criterion (AIC), see Akaike (1973). This method is applied in Section 6 to the ventICU dataset.

The EM algorithm described above converges to \(\hat{\theta}_m\), the MLE of \(\theta_m\). To be more precise, at convergence, the equations in (13) and (14) take the form (11) and (12) respectively, that is

\[
\hat{\phi}_{i,m} = \frac{E_{\hat{\theta}}[B_{i,m}|D]}{B_i}, \quad \frac{\hat{B}_{i,m}(\hat{\theta})}{B_i},
\]
\[
\hat{q}_{ij,m} = \frac{E_{\hat{\theta}}[N_{ij,m}|D]}{E_{\hat{\theta}}[T_{i,m}|D]} = \frac{\hat{N}_{ij,m}(\hat{\theta})}{\hat{T}_{i,m}(\hat{\theta})}.
\]

**Initialization of the EM algorithm**

Suppose we have \(K\) sample paths from the mixture with \(M\) regimes obtained from the simulation or real data. For the initial distribution \(\pi_i, i \in E\) of the \(g\)-mixture, we use the estimate \(\hat{\pi}_i\). For each \(i \in E\), we set the initial value of \((\phi_{i,m}, m \in M)\) to be a uniform distribution: \((\phi_{i,m}^0, m \in M) = (1/M, \cdots, 1/M)\). The initial value \(Q_0^m\) for the intensity matrix \(Q_m\) is obtained by first randomly dividing \(K\) sample paths into \(M\) subsets with the first \((M - 1)\) subsets having size \([K/M]\) each, and the \(M\)-th subset being of size \(K - (M - 1)[K/M]\). Then, treating the \(m\)'th subset, \(m \in M\), as if it contained realizations from a single Markov process \(X_m\) with intensity matrix \(Q_m\), \(Q_0^m\) was set to be the MLE of the intensity matrix \(Q_m\).

**3.2 Consistency and asymptotic normality of the MLEs**

Below we state the results about consistency and asymptotic distribution of the MLE \(\hat{\theta}\) when the number of sample paths \(K\) increases. The results are valid for a fixed \(T\).
and $T$ being the absorption time. They generalize the results from Theorem 6.1 in Albert (1962) about the asymptotic properties of the MLE of an intensity matrix of a single Markov process. Due to the presence of the term $\hat{\Phi}_{k,m} = E_{\theta_0}[\Phi_{k,m}|X^K]$ in the estimator $\hat{\theta}$, the proofs require a different approach compared to the simpler proofs of the analogous results in Theorem 6.1. For example, consider the estimator $\hat{q}_{ij,m} = \sum_{k=1}^K \hat{\Phi}_{k,m}N_{ij}^k/\sum_{k=1}^K \hat{\Phi}_{k,m}T_i^k$ of $q_{ij,m}$. By iid property of all $K$ realizations $\{X^K\}$, which were generated under the probability measure $P_{\theta_0}$, $\hat{q}_{ij,m}$ converges by LLN as $K \to \infty$ to $E_{\theta_0}[E[\Phi_{k,m}N_{ij}^k|X^K]]/E_{\theta_0}[E[\Phi_{k,m}T_i^k|X^K]]$. However, the law of iterated expectation does not simplify the latter to $E_{\theta_0}[\Phi_{k,m}N_{ij}^k]/E_{\theta_0}[\Phi_{k,m}T_i^k]$, which by Lemma A.1 in SM is equal to $q_{ij,m}$. But, for the estimator $\hat{q}_{ij} = \sum_{k=1}^K N_{ij}^k/\sum_{k=1}^K T_i^k$ of $q_{ij}$ considered in Albert, it immediately follows by LLN that $\hat{q}_{ij}$ converges to $E_{\theta_0}[N_{ij}^k]/E_{\theta_0}[T_i^k]$ as $K \to \infty$.

To establish consistency of $\hat{\theta}$, we use the Shannon-Kolmogorov information inequality, see p. 113 in Ferguson (1996): for any $\theta \neq \theta^0$ and generic paths $X^K$,

$$R(\theta^0, \theta) := E_{\theta_0}\left[\log \left(\frac{L^K(\theta^0)}{L^K(\theta)}\right)\right] > 0,$$

which in turn implies that $\theta^0$ is the global maximum of $M(\theta) := E_{\theta_0}\left[\log L^K(\theta)\right]$.

**Proposition 3 (Consistency of $\hat{\theta}$)** By independence of $\{X^K\}$, $\hat{\theta} \overset{p}{\to} \theta^0$.

**Proof.** We have assumed that the parameter space $\Theta$ is a compact set of any positive values of $\theta$. The MLE $\hat{\theta}$ is defined as the global maximizer of the sample loglikelihood $M_K(\theta) = \frac{1}{K} \sum_{k=1}^K \log L^K(\theta)$. By independence of $\{X^K\}$ and the law of large numbers, $M_K(\theta) \Rightarrow M(\theta) = E_{\theta_0}\left[\log L(\theta)\right]$. Meanwhile, from the Shannon-Kolmogorov information inequality we have $\sup_{\theta \in \Theta} M(\theta) < M(\theta^0) \iff \theta^0 = \arg\max_{\theta \in \Theta} M(\theta)$. Since $\hat{\theta}$ is the global maximizer of $M_K(\theta)$ and the latter converges with probability one to $M(\theta)$, it follows that $\hat{\theta}$ gets closer and closer to the global maximizer $\theta^0$ of $M(\theta)$ as the sample size $K$ increases, which implies that $\hat{\theta} \overset{p}{\to} \theta^0$. 

To establish asymptotic property of the MLE $\hat{\theta}$, the following result is required.

**Lemma 2** For any $\theta \in \Theta$,

$$E_{\theta}\left[\frac{\partial \log L^K(\phi)}{\partial \phi_{i,m}}\right] = 0 \text{ and } E_{\theta}\left[\frac{\partial \log L^K(q)}{\partial q_{ij,m}}\right] = 0.$$

**Proof.** From (7) and $E_{\theta}[\Phi_{k,m}B_i^k] = \phi_{i,m}\pi_i$ for $i \in E$, $1 \leq m \leq M$, we have by Lemma (8) $E_{\theta}\left[\frac{\partial \log L^K(\phi)}{\partial \phi_{i,m}}\right] = E_{\theta}\left[\frac{\partial \log L^K(\phi)}{\partial \phi_{i,m}}|D\right] = 0$, which gives the first result. Similarly, by (8) and Lemma A.1 in SM, $E_{\theta}[\Phi_{k,m}N_{ij}^k] = q_{ij,m} \int_0^T \pi^T D_m e^{Q_m u} e_i du$ and $E_{\theta}[\Phi_{k,m}T_i^k] = \int_0^T \pi^T D_m e^{Q_m u} e_i du$, implying that $E_{\theta}\left[\frac{\partial \log L^K(q)}{\partial q_{ij,m}}\right] = 0$, which is the second result.

**Theorem 1** As the sample size $K$ increases, $\sqrt{K}(\hat{\theta} - \theta^0) \overset{d}{\to} N(0, J^{-1}(\theta^0))$ with

$$J(\theta^0) = E_{\theta}\left[\frac{\partial^2 \log L(\theta)}{\partial \theta^2}\right],$$

being Fisher information matrix.
Proof. From Lemma 1 it is readily verified that the loglikelihood function \( \log L(\theta) \) is twice continuously differentiable. By the Mean Value Theorem applied to the score function \( \frac{\partial \log L(\hat{\theta})}{\partial \theta} \) at the true value \( \theta^0 \), see Feng et al. (2013) and p.20 in Ferguson (1996), we have

\[
0 = \frac{1}{K} \sum_{k=1}^{K} \frac{\partial \log L^k(\theta^0)}{\partial \theta} = \frac{1}{K} \sum_{k=1}^{K} \frac{\partial \log L^k(\theta^0)}{\partial \theta} + \left( \int_0^1 \frac{1}{K} \sum_{k=1}^{K} \frac{\partial^2 \log L^k(\theta_0 + \lambda(\hat{\theta} - \theta^0))}{\partial \theta^2} \, d\lambda \right) (\hat{\theta} - \theta^0).
\]

By consistency of \( \hat{\theta} \) and the fact that \( \theta^0 \) is maximizer of the function \( M(\theta) \) for which \( M''(\theta^0) = \mathbb{E}_{\theta^0} \left[ \frac{\partial^2 \log L^k(\theta^0)}{\partial \theta^2} \right] < 0 \), hence is invertible\(^1\), we have by Slutsky’s and CLT theorem

\[
\sqrt{K}(\hat{\theta} - \theta^0) = \left( - \int_0^1 \frac{1}{K} \sum_{k=1}^{K} \frac{\partial^2 \log L^k(\theta_0 + \lambda(\hat{\theta} - \theta^0))}{\partial \theta^2} \, d\lambda \right)^{-1} \frac{1}{\sqrt{K}} \sum_{k=1}^{K} \left( \frac{\partial \log L^k(\theta^0)}{\partial \theta} - \mathbb{E}_{\theta^0} \left[ \frac{\partial \log L^k(\theta^0)}{\partial \theta} \right] \right) \overset{d}{\rightarrow} N(0, J^{-1}(\theta^0)),
\]

where by Lemma, \( \mathbb{E}_{\theta^0} \left[ \frac{\partial \log L^k(\theta^0)}{\partial \theta} \right] = 0 \), which completes the proof. \( \blacksquare \)

Following the above theorem, estimates of standard errors of the MLE \( \hat{\theta} \) are calculated in Section 3 using the inverse of the observed Fisher information \( I(\theta) \).

However, it is difficult to derive an explicit form of the asymptotic information matrix \( J(\theta) \), see Theorem 3 in Section 4. Instead, in the following section we derive a lower bound for the asymptotic matrix.

### 3.3 Lower bound for the asymptotic variance of the MLEs

To state the corresponding lower bound for asymptotic variance of \( \hat{\theta} \), for \( 1 \leq n \leq M - 1 \), let \( D_n \) be a \((w \times w)\) diagonal matrix with diagonal elements \( \phi_{i,n}, i \in E \), and \( e_i = (0,...1,0) \) a \((w \times 1)\) unit vector with value one on the \( i \)th component and zero otherwise. Denote by \( \theta_0 = (\phi^0, q^0) \) the true value of \( \theta = (\phi, q) \) where \( \phi = (\phi_{i,m}, i \in E, 1 \leq m \leq M - 1) \) and \( q = (q_{i,m}, i,j \neq i, i \in E, j \in S, 1 \leq m \leq M) \), and define an indicator function

\[
\delta_q(z) = \begin{cases} 1, & q = z \\ 0, & \text{otherwise}. \end{cases}
\]

**Theorem 2** Let \( T \) be either a fixed time or the absorption time. Then, as \( K \to \infty \),

\[
J^{-1}(\theta^0) \geq \Sigma(\theta^0),
\]

where \( \Sigma(\theta^0) := \text{Cov}(\hat{\theta} - \theta^0, \hat{\theta}_0 - \theta^0) \) is a \((w(Mw - 1) \times w(Mw - 1))\) block-diagonal matrix with

\[
\Sigma(\theta^0) = \begin{cases} \text{Cov}(\hat{\phi}_{r,n} - \phi^0_{r,n}, \phi_{i,m} - \phi^0_{i,m}) = \frac{\phi^0_{i,m}}{\pi^r} \delta_1(r)(\delta_m(n) - \phi^0_{i,m}), \\
\text{Cov}(\hat{\phi}_{r,n} - \phi^0_{r,n}, \hat{\phi}_{i,m} - q^0_{i,m}) = 0 \\
\text{Cov}(\hat{q}_{rv,n} - q^0_{rv,n}, \hat{q}_{ij,m} - q^0_{ij,m}) = \frac{q^0_{i,m} \delta_m(n) \delta_1(r) \delta_1(v)}{\mathcal{E}_{\theta^0}(\Phi_{r,n}, \Phi^2_{r,n})}, \end{cases}
\]

\(^1\)on account of Theorem 7.2.1 on p. 438 of Horn and Johnson (2013) that every positive definite matrix is invertible and the inverse itself is positive definite.
where for $D^0_n = \text{diag}(\phi_{1,n}, \cdots, \phi_{w,n})$, $Q^0_n = [q_{ij,n}]_{ij}$, and $\pi = (\pi_1, \cdots, \pi_w)$,

$$
\mathbb{E}_{\theta_0}[\Phi_{k,n}^T T^k_i] = \begin{cases} 
\int_0^T \pi^T D^0_n e^{uQ^0_n} e_i du, & \text{for fixed } T \\
\int_0^\infty \pi^T D^0_n e^{uQ^0_n} e_i du, & \text{for absorption time } T.
\end{cases}
$$

**Proof.** See part A in the Supplementary material. □

The inequality (13) corresponds to the resulting information loss presented in incomplete data, see Schervish (1995). In the absence of heterogeneity ($\delta_m(n) = 1$ and $D_m = I$), hence for the case of complete information, the result in (13) coincides with that of Theorem 6.1 in Albert (1962). In the presence of heterogeneity with complete information, estimators of transition rates for different regimes have zero covariances. There is also zero covariance between estimators of regime memberships across different states and transition rates.

4 The finite sample covariance matrix for the MLEs

We will use the following vectors for the computation of the estimated variances of $\hat{\phi}$ and $\hat{q}$ and covariances. Define

$$
\phi_m = (\phi_{1,m}, \cdots, \phi_{w,m}), \quad 1 \leq m \leq M - 1.
$$

We form the row vector $\phi$ of dimension $(1 \times (M - 1)w)$, which combines the vectors $\phi_m$, i.e., $\phi = (\phi_1, \cdots, \phi_{M-1})$. Similarly, let $q_m$ be a $(1 \times w(w - 1))$—vector formed by combining the $E$-row vectors of $Q_m$, with diagonal element $q_{ii,m}$ removed, i.e.,

$$
q_m = (q_{12,m}, \cdots, q_{1w,m}, q_{21,m}, q_{23,m}, \cdots, q_{2w,m}, \cdots, q_{w1,m}, \cdots, q_{w(w-1),m}),
$$

for $1 \leq m \leq M$. Next, we form a row vector $q$ of dimension $(1 \times Mw(w - 1))$, when there are no absorbing states, which combines the vectors $q_m$, i.e., $q = (q_1, \cdots, q_M)$. The g-mixture parameters to be estimated are defined by the $(1 \times w(Mw - 1))$—vector $\theta = (\phi, q)$.

4.1 Observed Fisher information matrix

Based on continuous observation of the sample paths $\mathcal{D} = \bigcup_{k=1}^K X^k$, the observed Fisher information matrix $I(\theta) := - \frac{\partial^2 \log L(\theta)}{\partial \theta \partial \theta'}$ can be derived using the identity of Lemma 11.

**Theorem 3** For any $(\theta_i, \theta_j) \in \Theta$, the $(i, j)$ component $- \frac{\partial^2 \log L(\theta)}{\partial \theta_i \partial \theta_j}$ of $I(\theta)$ is given by

$$
I(\theta_i, \theta_j) = \sum_{k=1}^K \mathbb{E}_{\theta} \left[ \frac{\partial^2 \log L^c_k(\theta)}{\partial \theta_i \partial \theta_j} \right] X^k - \sum_{k=1}^K \mathbb{E}_{\theta} \left[ \left( \frac{\partial \log L^c_k(\theta)}{\partial \theta_i} \right) \left( \frac{\partial \log L^c_k(\theta)}{\partial \theta_j} \right) \right] \mathbb{E}_{\theta} \left[ X^k \right] X^k
$$

$$
+ \sum_{k=1}^K \mathbb{E}_{\theta} \left[ \frac{\partial \log L^c_k(\theta)}{\partial \theta_i} \right] \mathbb{E}_{\theta} \left[ X^k \right] \frac{\partial \log L^c_k(\theta)}{\partial \theta_j} X^k. \tag{17}
$$

Notice that the information matrix (17) is slightly different from Louis (1982) general matrix formula. In its new form, the information matrix (17) simplifies the conditional expectation of outer product of the complete-data score function in the Louis’ matrix formula.
The convergence of $K^{-1}I(\theta_i, \theta_j)$ to the respective element $J(\theta_i, \theta_j)$ of the Fisher information matrix $J(\theta)$ is more immediate from (17) than from the Louis formula. The first term in (17) is the expected complete data observed information and the other two terms combined is the variance of complete data score, conditional on $D$.

**Proof.** Following the third equality in the proof of Lemma 1 we have

$$\frac{\partial \log L(\theta)}{\partial \theta_j} = \sum_{k=1}^{K} \sum_{m=1}^{M} \left( \frac{\partial \log \mathbb{P}_\theta(X^k, R_k = m)}{\partial \theta_j} \right) \mathbb{P}_\theta(R_k = m|X^k).$$

Differentiating with respect to variable $\theta_i$ on both sides of the above equality gives

$$-\frac{\partial^2 \log L(\theta)}{\partial \theta_i \partial \theta_j} = \sum_{k=1}^{K} \sum_{m=1}^{M} \left( -\frac{\partial^2 \log \mathbb{P}_\theta(X^k, R_k = m)}{\partial \theta_i \partial \theta_j} \right) \mathbb{P}_\theta(R_k = m|X^k)
- \sum_{k=1}^{K} \sum_{m=1}^{M} \left( \frac{\partial \log \mathbb{P}_\theta(X^k, R_k = m)}{\partial \theta_j} \right) \frac{\partial \mathbb{P}_\theta(R_k = m|X^k)}{\partial \theta_i}
- \sum_{k=1}^{K} \sum_{m=1}^{M} \left( \frac{\partial \log \mathbb{P}_\theta(X^k, R_k = m)}{\partial \theta_j} \right) \mathbb{P}_\theta(R_k = m|X^k).$$

The final result is obtained after replacing the derivative $\frac{\partial \mathbb{P}_\theta(R_k = m|X^k)}{\partial \theta_i}$ by

$$\mathbb{P}_\theta(R_k = m|X^k) \left( \frac{\partial \log \mathbb{P}_\theta(X^k, R_k = m)}{\partial \theta_i} \right) = -\mathbb{E}_\theta \left[ \frac{\partial \log L^k_c(\theta)}{\partial \theta_i} \right] - \mathbb{E}_\theta \left[ \frac{\partial \log L^k_c(\theta)}{\partial \theta_i} \right] X^k,$$

leading to the information matrix (17). \qed

Using the results of Proposition A.1 in SM, one can show for any $(\theta_i, \theta_j) \in \theta$, $\mathbb{E}_\theta \left[ -\frac{\partial^2 \log L^k_c(\theta)}{\partial \theta_i \partial \theta_j} \right] = \mathbb{E}_\theta \left[ \frac{\partial \log L^k_c(\theta)}{\partial \theta_i} \right] \left( \frac{\partial \log L^k_c(\theta)}{\partial \theta_j} \right)$ which by Lemma 1 and LLN leads to the convergence of $K^{-1}I(\theta_i, \theta_j)$ to $J(\theta_i, \theta_j)$, the $(i,j)$—element of the Fisher information $J(\theta)$.

Below we derive the explicit expressions for $I(\hat{\phi}), I(\hat{q})$ and $I(\hat{\phi}, \hat{q})$.

### 4.2 Elements of the matrices $I(\hat{\phi}), I(\hat{q})$, and $I(\hat{\phi}, \hat{q})$

To derive the expressions for $I(\hat{\phi}), I(\hat{q})$, and $I(\hat{\phi}, \hat{q})$, we consider the score function $\frac{\partial \log L_c(\theta)}{\partial \theta}$ and its derivative, where we let $\hat{A}_{i,j,m}^k = N_{i,j} - \hat{q}_{i,j,m} \tau_i$ and $\hat{\Psi}_{i,m|\Phi}^k = \hat{\Phi}_{k,m} - \hat{\phi}_{i,m} \hat{\Phi}_{k,M}$.

**Proposition 4** From the information matrix (17), we have for $(i,r) \in E$, $(j,v) \in S$, $(i)$ and $n, m = 1, \cdots, M - 1$,

$$I(\hat{\phi}_j, \hat{\phi}_i,m) = \frac{\delta(i)}{\phi_{i,r} \Phi_{j,m}} \sum_{k=1}^{K} \hat{\Psi}_{i,m|\Phi}^k \hat{\Psi}_{j,n|\Phi}^k \hat{B}_{i,m|\Phi}^k.$$

(ii) and for \( n, m = 1, \ldots, M \),
\[
I(\hat{q}_{rv,n}, \hat{q}_{ij,m}) = \frac{\delta_i(r)\delta_j(v)\delta_m(n)}{\hat{q}_{ij,m}q_{rv,n}} \hat{N}_{rv,t} - \frac{1}{\hat{q}_{rv,n}q_{ij,m}} \sum_{k=1}^{K} \Phi_k,n \left( \delta_m(n) - \hat{\Phi}_{k,m} \right) \hat{A}^k_{rv,n} \hat{A}^k_{ij,m},
\]

(iii) and for \( n = 1, \ldots, M; m = 1, \ldots, M - 1 \),
\[
I(\hat{\phi}_{i,m}, \hat{q}_{rv,n}) = -\frac{1}{\phi_{i,m}\hat{q}_{rv,n}} \sum_{k=1}^{K} \Phi_k,n \left( \delta_m(n) - \hat{\phi}_{i,m}\phi_{i,M} \right) \hat{A}^k_{rv,n} B^i.
\]

**Proof.** The proofs of above results are provided in part B of Supplementary material. □

Using expression (7) on p.12 in Magnus and Neudecker (2007) for inversion of block partitioned matrix, inverting the information matrix \( I(\hat{\theta}) \) yields estimated covariance of \( \hat{\theta} \) exhibited in the following proposition.

**Proposition 5** The estimated covariance matrix of the MLEs \( \hat{\theta} \) is given by
\[
\widehat{\text{Var}}(\hat{\theta}) = \begin{pmatrix}
\widehat{\text{Var}}(\hat{\phi}) & \widehat{\text{Cov}}(\hat{\phi}, \hat{q}) \\
\widehat{\text{Cov}}^\top(\hat{\phi}, \hat{q}) & \widehat{\text{Var}}(\hat{q})
\end{pmatrix},
\]
where the estimated variances for \( \hat{q}, \hat{\phi} \), and their covariance are defined by
\[
\widehat{\text{Var}}(\hat{q}) = \left[ I(\hat{q}) - I(\hat{q}, \hat{\phi}) I^{-1}(\hat{\phi}) I(\hat{\phi}, \hat{q}) \right]^{-1},
\]
\[
\widehat{\text{Cov}}(\hat{\phi}, \hat{q}) = -I^{-1}(\hat{\phi}) I(\hat{\phi}, \hat{q}) [I(\hat{q}) - I(\hat{q}, \hat{\phi}) I^{-1}(\hat{\phi}) I(\hat{\phi}, \hat{q})]^{-1},
\]
\[
\widehat{\text{Var}}(\hat{\phi}) = I^{-1}(\hat{\phi}) + \widehat{\text{Cov}}(\hat{\phi}, \hat{q}) \left[ \widehat{\text{Var}}(\hat{q}) \right]^{-1} \widehat{\text{Cov}}^\top(\hat{\phi}, \hat{q}).
\]
Thus, if \( I(\hat{\phi}, \hat{q}) = 0 \), \( \widehat{\text{Cov}}(\hat{\phi}, \hat{q}) = 0 \), \( \widehat{\text{Var}}(\hat{\phi}) = I^{-1}(\hat{\phi}) \), and \( \widehat{\text{Var}}(\hat{q}) = I^{-1}(\hat{q}) \).

**Remark 1** Note that in order to prevent having singularity in estimating the variance of \( \hat{\theta} \) by inverting the information matrix \( I(\hat{\theta}) \), we exclude the estimators \( \hat{\phi}_{i,m} \) and \( \hat{q}_{ij,m} \) whose values are (very close) to zero.

### 4.3 Computation of \( I(\hat{\theta}) \) for a two-regime mixture model

We specialize the general results from Proposition 4 to the case of two-regime mixture model defined on state space \( S = E \cup \Delta \), where \( E = \{1,2\} \) is a set of transient states and \( \Delta = \{3,4\} \) is a set of two absorbing states corresponding to the ventICU dataset used in the application section with the state diagram described by Figure 1. Since \( \hat{\phi}_{i,2} = 1 - \hat{\phi}_{i,1} \), we have \( \text{Var}(\hat{\phi}_{i,2}) = \text{Var}(\hat{\phi}_{i,1}) \) for each \( i \in E \), and thus consider only the vector \( \phi = (\phi_{1,1}, \phi_{2,1})^\top \). The \( I(\hat{\phi}) \) is a \( (2 \times 2) \)--diagonal matrix with the entries given, using Proposition 4 by
\[
I_{11} = \sum_{k=1}^{K} \left( \frac{\Psi_{i,1,2}^k}{\phi_{i,1}} \right)^2 B^i, \quad i \in E.
\]
To simplify the presentation of the matrix $I(\tilde{q})$, we split it into block partitioned matrix:

$$I(\tilde{q}) = \begin{pmatrix} I(\tilde{q}_1) & I(\tilde{q}_1, \tilde{q}_2) \\ I(\tilde{q}_2) & I(\tilde{q}_2) \end{pmatrix}.$$  

Note that by the symmetry property, $I(\tilde{q}_2, \tilde{q}_1) = I^\top(\tilde{q}_1, \tilde{q}_2)$. To write elements of the matrix $I(\tilde{q}_m)_{6 \times 6}$, with $m = 1, 2$, we use the sequence $\tilde{q}_m = (\tilde{q}_{1m}, \tilde{q}_{13m}, \tilde{q}_{14m}, \tilde{q}_{21m}, \tilde{q}_{23m}, \tilde{q}_{24m})$ and number its components from 1 to 6, so that it is read as $\tilde{q}_m = (\tilde{q}_{11}, ..., \tilde{q}_{66})$. Similarly, we label the $\ell$–th component of the vectors $(\tilde{N}_{12m}, \tilde{N}_{13m}, \tilde{N}_{14m}, \tilde{N}_{21m}, \tilde{N}_{23m}, \tilde{N}_{24m})$ and $(\tilde{A}_{12m}^k, \tilde{A}_{13m}^k, \tilde{A}_{14m}^k, \tilde{A}_{21m}^k, \tilde{A}_{23m}^k, \tilde{A}_{24m}^k)$ by $\tilde{N}_{\ell, m}$ and $\tilde{A}_{\ell, m}$.

Then, for $1 \leq \ell \leq 6$, the diagonal elements of the matrix $I(\tilde{q}_m)$ are

$$I_{\ell\ell}(\tilde{q}_m) = \frac{\tilde{N}_{\ell, m}}{\tilde{q}_{\ell, m}^2} - \frac{1}{\tilde{q}_{\ell, m}^2} \sum_{k=1}^{K} \tilde{\Phi}_{k, m}(1 - \tilde{\Phi}_{k, m}) \left( \tilde{A}_{\ell, m}^k \right)^2,$$

while, for $1 \leq \ell \neq \nu \leq 6$, the off-diagonal elements are

$$I_{\ell\nu}(\tilde{q}_m) = -\frac{1}{\tilde{q}_{\ell, m} \tilde{q}_{\nu, m}} \sum_{k=1}^{K} \tilde{\Phi}_{k, m}(1 - \tilde{\Phi}_{k, m}) \tilde{A}_{\ell, m}^k \tilde{A}_{\nu, m}^k.$$

Moreover, the $(\ell, \nu)$–element of the matrix $I(\tilde{q}_1, \tilde{q}_2)$ is

$$I_{\ell\nu}(\tilde{q}_1, \tilde{q}_2) = \frac{1}{\tilde{q}_{\ell, 1} \tilde{q}_{\nu, 2}} \sum_{k=1}^{K} \tilde{\Phi}_{k, 1} \tilde{\Phi}_{k, 2} \tilde{A}_{\ell, 1}^k \tilde{A}_{\nu, 2}^k.$$

For convenience, we write $I(\tilde{\phi}, \tilde{q}) = [I(\tilde{\phi}, \tilde{q}_1), I(\tilde{\phi}, \tilde{q}_2)]$, where for $m = 1, 2$, each $I(\tilde{\phi}, \tilde{q}_m)$ is a $(2 \times 6)$–matrix whose $(\ell, \nu)$–element is

$$I_{\ell\nu}(\tilde{\phi}, \tilde{q}_m) = \begin{cases} -\frac{1}{\phi_{\ell, 1} \phi_{\nu, 2}} \sum_{k=1}^{K} \tilde{\Phi}_{k, 1}(1 - \tilde{\Phi}_{k, 1}) \tilde{B}_{\ell, 1}^k, & m = 1, \\ \frac{1}{\phi_{\ell, 1} \phi_{\nu, 2}} \sum_{k=1}^{K} \tilde{\Phi}_{k, 1} \left( \phi_{\ell, 2} + \tilde{\Phi}_{k, 1} \right) \tilde{B}_{\ell, 2}^k, & m = 2, \end{cases}$$

for $\ell = 1, 2$ and $1 \leq \nu \leq 6$.

## 5 Simulation Study

We consider a mixture of two continuous-time Markov jump processes $X_m$, $m = 1, 2$, with the intensity matrices $Q_m$, $m = 1, 2$ on state space $\{1, 2, 3\}$. For the purpose of simulation, we express $Q_m$ as

$$Q_m = \text{diag}(q_{1m}, q_{2m}, q_{3m})(P_m - I), m = 1, 2$$

where $\text{diag}(q_{1m}, q_{2m}, q_{3m})$ is the diagonal matrix with $q_{iim} = -q_{ii,m}$ being the exit rate of $X_m$ from state $i$, $P_m$ the transition matrix of a discrete time Markov chain $Z^m$ embedded in the Markov process $X_m$ and $I$ an identity matrix.

For the true values of the mixture’s parameters, we chose uniform initial distribution $\pi = (1/3, 1/3, 1/3)$, and the regime 1 and 2 probabilities as $\phi_1 = (\phi_{1,1}, \phi_{2,1}, \phi_{3,1}) = \ldots$
(0.5, 0.25, 1/5) and \( \phi_2 = (1, 1, 1) - \phi_1 \), respectively. Furthermore, we chose the true transition matrices of the embedded Markov chains \( Z^m, m = 1, 2 \), to be

\[
P_1 = \begin{pmatrix}
0 & 0.6 & 0.4 \\
0.5 & 0 & 0.5 \\
0.4 & 0.6 & 0
\end{pmatrix}
\quad \text{and} \quad
P_2 = \begin{pmatrix}
0 & 0.8 & 0.2 \\
0.5 & 0 & 0.5 \\
0.2 & 0.8 & 0
\end{pmatrix}
\]

and the true exit rates from states as \( q_1 = (1/3, 2/5, 1/5) \) and \( q_2 = (1/2, 1/4, 3/4) \) for regime 1 and 2 respectively. The elements in the true intensity matrices can be found in Table 1 below.

### 5.1 Simulating a sample path of a two-regime mixture

We simulate a sample path of a two-regime mixture on time interval \((0, T) = (0, 30)\). The simulation uses Sigman’s (2017) method for simulating a sample path of a discrete-time Markov chain. In the simulation, \( W_i^m \) denotes a waiting time of \( X_m \) in state \( i \). And we write \( W_i^m \sim \exp(q_{i,m}) \) to say that \( W_i^m \) has an exponential distribution with parameter \( q_{i,m} \). We also note that \( Z_0^m = X_m(0) \) for \( m = 1, 2 \).

**Step 1** Draw at random an initial state \( i_0 \) from a uniform distribution on states 1, 2, 3.

**Step 2** Given \( i_0 \), draw the regime indicator \( m \) from the Bernoulli distribution with success probability equal to \( \phi_{i_0,1} \), where success corresponds to regime \( Q_1 \).

**Step 3** Set \( j = 1 \)

**Step 4** For a chosen \( m \) in Step 2, set \( Z_{j-1}^m = i_{j-1} \), simulate waiting time \( W_{i_{j-1}}^m \sim \exp(q_{i_{j-1},m}) \) and compute \( S_{j-1}^m = \sum_{k=0}^{j-1} W_{i_k}^m \). Stop if \( S_{j-1}^m > T \). If not, go to Step 5

**Step 5** Simulate \( Z_j^m = i_j \), conditioning on state \( i_{j-1} \)

- if \( i_{j-1} = 1 \) and \( U_j \leq p_{12,m} \), set \( Z_j^m = 2 \)
- if \( i_{j-1} = 1 \) and \( U_j > p_{12,m} \), set \( Z_j^m = 3 \)
- if \( i_{j-1} = 2 \) and \( U_j \leq p_{21,m} \), set \( Z_j^m = 1 \)
- if \( i_{j-1} = 2 \) and \( U_j > p_{21,m} \), set \( Z_j^m = 3 \)
- if \( i_{j-1} = 3 \) and \( U_1 \leq p_{31,m} \), set \( Z_j^m = 1 \)
- if \( i_{j-1} = 3 \) and \( U_1 > p_{31,m} \), set \( Z_j^m = 2 \)

where \( U_j \) is drawn, independently from previous draws, from \( U(0, 1) \), a uniform distribution on \([0, 1]\). Increase \( j \) by one and go to Step 4.

Let \( J \equiv \min(j : S_{j-1}^m > T) \). Then \( J \) is the iteration at which the simulation stops.

The resulting sample path is \( \{Z_0^m = i_0, W_{i_0}^m, Z_1^m = i_1, W_{i_1}^m, ..., Z_{J-1}^m = i_{j-1}, W_{i_{j-1}}^{m,C}\} \), where \( W_{i_{j-1}}^{m,C} \) is a right-censored waiting time in state \( i_{j-1} \).

### 5.2 Simulation results

We independently obtained \( N = 200 \) sets of \( K = 800, 1200 \) and \( 2000 \) sample paths with each sample path generated in the way described in Section 5.1. We chose \( \phi^0 = (1/2, 1/2, 1/2) \) to
be the initial value of \((\phi_{1,1}, \phi_{2,1}, \phi_{3,1})\), and chose the initial values of \(Q_1\) and \(Q_2\) as described in Section 3.1. The EM algorithm was run until convergence criterion \(||\theta^{p+1} - \theta^p|| < 10^{-4}\) was achieved and was repeated 200 times for each \(K\) sample paths.

The simulation results were evaluated using Bias and the Root Mean Squared Error (RMSE) and are reported in Table 1. We observe that as the sample size \(K\) increases both Bias and RMSE decrease confirming consistency of the EM estimates. We see from Table 2 that the standard errors obtained from the simulation are close to the theoretical standard errors i.e., the ones obtained from the inverse of Fisher information. We next confirmed that for \(K = 2000\) and \(N = 200\), the distribution of \(\hat{\theta}_K\) is approximately normal using the Kolmogorov-Smirnov (KS) test as follows. For each element of \(\theta\), the simulation yields a random sample of 200 biases, with each bias obtained from a set of sample paths of size 2000. We then standardized the biases by dividing them by their standard error. The fit of the standardized biases to the standard normal cdf was assessed using KS test. The p-value of the KS test, reported for each element in Table 2, confirms the large sample normality of its estimator.

### 6 Application to ventICU dataset

#### 6.1 Data and the choice of the mixture model

This section applies the methods developed in the paper to the ventICU dataset from Appendix D in Cook and Lawless (2018). The ventICU dataset comes from a prospective cohort study of patients in an intensive care unit (ICU), and contains information on the...
occurrence of infections and the need for mechanical ventilation along with discharge and death times. Cook and Lawless suggest that the multi-state model in Figure 1 would be suitable for examining the relation between mechanical ventilation status and risk of death or discharge for the sample of 747 patients, but do not provide any further analysis of the ventICU dataset. In Figure 1, state 1 represents being off mechanical ventilator; state 2 being on mechanical ventilator; state 3 discharge and state 4 death. The numbers of transitions between states are exhibited in Table 4 from which we see that at the end of the study 733 patients were either discharged or died, and 14 were still in the hospital. Thus, these 14 patients have right-censored times of stay in the hospital. The focus of our analysis is on identifying the subgroups of patients characterized by different relations between mechanical ventilation status and risk of death or discharge.

We use Akaike information criterion (AIC) to decide on the choice of the g-mixture for the ventICU data. We estimate six g-mixtures for the ventICU data with the number of regimes $M$ ranging from 1 (single Markov process) to 6. Table 5 shows that $\text{AIC}_M = 2|\hat{\theta}_M| - 2\log L(\hat{\theta}_M)$, where $|\hat{\theta}_M|$ is the number of parameters in the $M$'th model, is the smallest for $M = 2$, and thus we use the 2-regime mixture in the further analysis of the ventICU data.

| $\theta$ | True Value | Estimate $\hat{\theta}_K$ | RMSE | $\sqrt{\text{I}^{-1}(\hat{\theta}_K)}$ | $\sqrt{\Sigma(\theta_0)}$ | KS |
|----------|------------|--------------------------|------|--------------------------|--------------------------|-----|
| $\phi_{1,1}$ | 0.5000 | 0.4990 | 4.1612 | 4.1559 | 1.9365 | 0.4340 |
| $\phi_{2,1}$ | 0.2500 | 0.2514 | 3.8424 | 3.8082 | 1.6771 | 0.3701 |
| $\phi_{3,1}$ | 0.7500 | 0.7484 | 3.8237 | 3.8113 | 1.6771 | 0.9209 |
| $q_{12,1}$ | 0.2000 | 0.1997 | 0.6322 | 0.6429 | 0.4230 | 0.4340 |
| $q_{13,1}$ | 0.1333 | 0.1336 | 0.4339 | 0.4153 | 0.3455 | 0.8198 |
| $q_{31,1}$ | 0.2000 | 0.2004 | 0.8134 | 0.8057 | 0.5043 | 0.8240 |
| $q_{32,1}$ | 0.3000 | 0.3007 | 0.7916 | 0.7717 | 0.6176 | 0.8679 |
| $q_{12,2}$ | 0.4000 | 0.3999 | 1.2999 | 1.2841 | 0.7618 | 0.5306 |
| $q_{13,2}$ | 0.1000 | 0.0996 | 0.5158 | 0.5153 | 0.3809 | 0.1062 |
| $q_{21,2}$ | 0.2000 | 0.1998 | 0.4977 | 0.4731 | 0.3854 | 0.2508 |
| $q_{23,2}$ | 0.2000 | 0.2005 | 0.4461 | 0.4744 | 0.3854 | 0.2041 |
| $q_{31,2}$ | 0.0667 | 0.0666 | 0.4181 | 0.4310 | 0.2629 | 0.8180 |
| $q_{32,2}$ | 0.2667 | 0.2670 | 0.6234 | 0.6465 | 0.5258 | 0.3770 |

Table 2: Estimated standard errors of the MLEs $\hat{\theta}_K = \frac{1}{N} \sum_{n=1}^{N} \hat{\theta}_{n,K}$ using $\text{RMSE}(\hat{\theta}_K) = \left[ \frac{1}{N} \sum_{n=1}^{N} (\hat{\theta}_{n,K} - \theta)^2 \right]^{1/2}$ and the inverse of Fisher information $\text{I}(\hat{\theta}_K) = \frac{1}{N} \sum_{n=1}^{N} \text{I}(\hat{\theta}_{n,K})$, where each matrix $\text{I}(\hat{\theta}_{n,K})$ is computed using the result of Proposition 4, for $K = 2000$ and $N = 200$, compared to the lower bound $\sqrt{\Sigma(\theta_0)}$ of the covariance matrix $\sqrt{\text{I}^{-1}(\hat{\theta}_K)}$. The last column lists the p-value of Kolmogorov-Smirnov statistic for goodness-of-fit between empirical CDF of standardized biases and $\text{N}(0,1)$ CDF.
Figure 1: State diagram for the m-th regime of ventICU mixture model.

|   | 1 | 2 | 3 | 4 | Censored |
|---|---|---|---|---|----------|
| 1 | 0 | 75| 585| 21| 5        |
| 2 | 319| 0 | 72 | 55| 9        |

Table 3: Observed transitions among different states and the numbers of patients right-censored in states 1 and 2.

| Model       | AIC   | log $\mathcal{L}(\theta)$ |
|-------------|-------|---------------------------|
| Markov      | 9648.185 | -4817.092                |
| 2 Mixture   | 9611.868 | -4790.934                |
| 3 Mixture   | 9611.991 | -4782.996                |
| 4 Mixture   | 9618.506 | -4778.253                |
| 5 Mixture   | 9628.735 | -4775.367                |
| 6 Mixture   | 9641.453 | -4773.727                |

Table 4: Summary of model statistics AIC and log $\mathcal{L}(\theta)$.

6.2 Parameter estimates of the two-regime g-mixture model

From the EM algorithm, the estimated numbers $\hat{B}_{i,m}$ of patients starting in the state $i$ and making transitions according to Markov process $X_m, m = 1, 2$, and the regime probabilities $\hat{\phi}_{i,m} = \hat{B}_{i,m}/B_i$ with their standard errors are reported in Table 5. We will refer to the patients estimated to evolve according to $X_m$ as $X_m$ patients. We see from Table 5 that initially there were 367 patients in state 1 and 380 in state 2. Using this information together with $\hat{D}_m = \text{diag}(\hat{\phi}_{1,m}, \hat{\phi}_{2,m}), m = 1, 2$, we obtain the estimated total number of $X_1$ patients, by summing the elements of the vector $\hat{C}_1 \equiv (367, 380)\hat{D}_1 = (230.75, 209.16)$ to be approximately 440. Similarly summing the elements of $\hat{C}_2 \equiv (367, 380)\hat{D}_2 = (136.25, 170.84)$, we
Table 5: Estimates of $B_{i,m}'s$ and $\phi_{i,m}'s$ with their standard errors for the ventICU dataset.

| State(i) | $\hat{B}_{i,1}$ | $\hat{B}_{i,2}$ | $\hat{\phi}_{i,1}(SE)$ | $\hat{\phi}_{i,2}(SE)$ |
|----------|-----------------|-----------------|----------------------|----------------------|
| 1        | 367             | 230.75          | 136.25               | 0.6287 (.1674)       |
|          |                 | 163.25          | 0.3713 (.1674)       |                      |
| 2        | 380             | 209.16          | 170.84               | 0.5504 (.1391)       |
|          |                 | 170.84          | 0.4496 (.1391)       |                      |

get the estimated total number of $X_2$ patients to be approximately 307. The estimates of the intensity matrices $Q_1$ and $Q_2$ of the Markov processes $X_1$ and $X_2$ with the standard errors (in parentheses), computed using the results in Section 4.2, are

$$\hat{Q}_1 = \begin{pmatrix}
-0.16112 & 0.01657 & 0.14455 & 0.00000 \\
(0.00370) & (0.01045) & & \\
0.12071 & -0.13832 & 0.01405 & 0.00356 \\
(0.03143) & (0.00558) & (0.00532) & \\
\end{pmatrix},$$

and

$$\hat{Q}_2 = \begin{pmatrix}
-0.11594 & 0.01441 & 0.09102 & 0.01051 \\
(0.00470) & (0.01600) & (0.00416) & \\
0.02309 & -0.04550 & 0.01094 & 0.01147 \\
(0.00599) & (0.00286) & (0.00248) & \\
\end{pmatrix},$$

respectively, where we omitted the last two rows of zeros, which correspond to the two absorbing states. To prevent having singularity in the inverse of $I(\hat{\theta})$, we excluded $\hat{q}_{14,1}$ from the vector $\hat{q}_1$ in the estimation of the standard errors, and assumed that the true value of $q_{14,1} = 0$. We see that the regime 2 death intensities are much higher from both initial states than the regime 1 death intensities: in fact regime 1 death intensity for patients who were initially not on ventilator is zero. We also see that regime 1 discharge intensities are higher from both initial states compared to the similar regime 2 intensities. In both regimes, not being initially on ventilator results in a larger discharge intensity compared to the death intensity.

### 6.2.1 Absorption probabilities

To gain a better understanding of the differences between the two regimes, we compare their absorption probabilities $\hat{f}_{i,j,m}$, $m = 1, 2$, from state $i = 1, 2$ into states $j = 3, 4$. The standard equations for these probabilities are derived based on the transition matrices of the discrete time Markov chains embedded into the Markov processes $X_1$ and $X_2$, and are given in part C of Supplementary material. Here we just state the results: the absorption probability matrices for the two regimes, denoted by $\hat{F}_1$ and $\hat{F}_2$, with entries $\hat{f}_{i,j,1}$ and $\hat{f}_{i,j,2}$, ($i = 1, 2, (j = 3, 4)$, are

$$\hat{F}_1 = \frac{1}{2} \begin{pmatrix}
0.9971 & 0.0029 \\
0.9717 & 0.0283 \\
\end{pmatrix} \quad \text{and} \quad \hat{F}_2 = \frac{1}{2} \begin{pmatrix}
0.8698 & 0.1302 \\
0.6818 & 0.3182 \\
\end{pmatrix},$$

respectively. Comparing $\hat{F}_1$ with $\hat{F}_2$, we see a striking difference in the estimated probability of eventual death in the two regimes. In regime 1, the probability of death for a patient initially in state 1 (not on ventilator) is very small (0.0029), whereas this probability for
a patient initially in state 1 in regime 2 is large (0.1302). For a patient initially in state 2 (on ventilator), the probability of death is about 11 times larger in regime 2 compared to regime 1. As a result of very large differences in the death probabilities between two regimes, we also observe large differences in their eventual discharge probabilities.

To translate above results into those involving patients’ absorption frequencies in each regime, we would want to pre-multiply \( \hat{F}_m \) matrix by the row vector \( \hat{C}_m \) to obtain a row vector showing the estimated number of \( X_m \) patients who were eventually discharged, or died. By doing so we would overestimate the absorption frequencies by 14 patients who were right-censored at the end of the study. Among those patients, 5 were initially in state 1 and 9 died. By doing so we would overestimate the absorption frequencies by 14 patients who were right-censored at the end of the study. Among those patients, 5 were initially in state 1 and 9 died. By doing so we would overestimate the absorption frequencies by 14 patients who were right-censored at the end of the study. Among those patients, 5 were initially in state 1 and 9 died.

To compute absorption frequencies, we have to subtract the 14 patients from the \( \hat{C}_m \) vectors, which we do as follows. According to \( \hat{D}_m \) matrices, the 5 patients with initial state 1, contribute \( 5(\hat{e}_{11}) = 5(0.6287) \) patients to regime 1 patients and \( 5(\hat{e}_{12}) = 5(0.3713) \) to regime 2 patients. Similarly, the patients initially in state 2, contribute \( 9(\hat{e}_{21}) = 9(0.5504) \) patients to regime 1 patients and \( 9(\hat{e}_{22}) = 9(0.4496) \) to regime 2 patients. Hence, vector \( \hat{C}_1 = (230.75, 209.16) \) has to be modified to become vector \( \hat{C}_{1,U} \) describing the regime 1 uncensored patients’ absorption frequencies by initial state:

\[
\hat{C}_{1,U} = \hat{C}_1 - [5(0.6287), 9(0.5504)] = (227.6065, 204.2064),
\]

and vector \( \hat{C}_2 = (136.25, 170.84) \) has to be modified to become vector \( \hat{C}_{2,U} \) describing the regime 2 uncensored patients’ absorption frequencies by initial state:

\[
\hat{C}_{2,U} = \hat{C}_2 - [5(0.3713), 9(0.4496)] = (134.3935, 166.7936)
\]

We can now compute the regime 1 absorption frequencies:

\[
\hat{C}_{1,U}\hat{F}_1 \approx (226.95, 0.66) + (198.43, 5.78) = (425.38, 6.44),
\]

where the first vector shows that absorption frequencies from state 1 and the second those frequencies from state 2 among estimated \( X_1 \) patients. Summing the two vectors tells us that in regime 1, about 425 patients were eventually discharged and about 6 died. For regime 2, we have

\[
\hat{C}_{2,U}\hat{F}_2 \approx (116.9, 17.5) + (113.72, 53.07) = (230.62, 70.57),
\]

where the interpretation of the vectors is analogous to the one for the regime 1. We see that the estimated total number of deaths is 77.01 \( \approx \) 77 which is one more than the observed number of deaths, see Table 5, and the estimated total number of discharges is about 656, which is one less than the observed number of discharges. This difference is likely due to the rounding errors. The regime 1 death rate is 6.44/431.82 \( \approx \) 0.015 or 1.5% whereas the regime 2 death rate is about 70.57/301.2 = 0.234 or 23.4%. Consequently, the rate of discharge from both initial states is much larger in regime 1 compared to regime 2. In both regimes most of the patients who died were initially on ventilator (state 2). The proportion of deaths when initially on ventilator is 53.07/70.57 or about 0.75 in regime 2 and 5.78/6.44 \( \approx \) 0.9 in regime 1.

Thus, the g-mixture model has identified two regimes corresponding to high risk (regime 2) and low (regime 1) risk of death, with the patients in the high risk regime having about 15 times higher death rate than those in the low risk regime.
6.3 The likelihood ratio test of the constrained vs unconstrained mixture

We want to see, if for ventICU data, there is a benefit in using the general mixture proposed in this paper over the constrained mixture considered by Frydman (2005). In the two-regime c-mixture, the two transition intensity matrices \( Q_1 \) and \( Q_2 \) are constrained by assuming that \( Q_1 = \Gamma Q_2 \), where \( \Gamma = \text{diag}(\gamma_{1,1}, \gamma_{2,1}) \). This constraint implies that the two regimes have the same embedded Markov chains and thus also the same absorption probabilities.

The test of c-mixture vs g-mixture can be formulated as \( H_0 : Q_1 = \Gamma Q_2 \) vs \( H_a : Q_1 \) and \( Q_2 \) are unrestricted. To carry out the test, we use the likelihood ratio statistic, \(-2 \log \Lambda :\)

\[
\Lambda = \frac{L_{c-\text{Mixture}}(\hat{\theta}_c)}{L_{g-\text{Mixture}}(\hat{\theta}_g)}
\]

where \( \hat{\theta}_c \) and \( \hat{\theta}_g \) are the MLEs of the vector parameters \( \theta_c = \{\phi_{1,1}^c, \phi_{2,1}^c\} \cup \{q_{ij}^c, i \in E, j \in S, i \neq j\} \cup \{\gamma_{1,1}, \gamma_{2,1}\} \) and \( \theta_g = \{\phi_{1,1}, \phi_{2,1}\} \cup \{q_{ij}, 1 \leq m \leq 2, i \in E, j \in S, i \neq j\} \), respectively, where \( E = \{1, 2\} \) and \( \Delta = \{3, 4\} \). We see that a general mixture has 14 parameters while the constraint one has 10, which means that, under \( H_0 \), \(-2 \log \Lambda \) has \( \chi^2 \) distribution with 4 degrees of freedom. To evaluate \(-2 \log \Lambda \), we note that the likelihood function of a general mixture, \( L_{g-\text{Mixture}}(\theta_g) \), is given in (3) and the likelihood function of the constrained mixture is of the form

\[
L_{c-\text{Mixture}}(\theta_c) = \left( \prod_{k=1}^{K_{ij}} \prod_{i \in E} \pi_i^{B_k} \right) \prod_{k=1}^{K} \sum_{m=1}^{M} \left( \phi_{i,m}^{B_k} \right) \prod_{i \in E} \left( \gamma_{i,m} q_{ij}^{N_k} \right) \exp \left( - \sum_{j \neq i, j \in S} \gamma_{i,m} q_{ij} T_i^k \right)
\]

where the factor in parentheses involving \( \pi_i^j \)'s is the same as the factor involving \( \pi_i^j \)'s in \( L_{g-\text{Mixture}}(\theta_g) \) and thus \( \pi_i^j \)'s play no role in the evaluation of \(-2 \log \Lambda \). From Table 4, \( \log(L_{g-\text{Mixture}}(\hat{\theta}_g)) = -4790.934 \), and from the EM algorithm applied to fit the c-mixture to ventICU data, \( \log(L_{c-\text{Mixture}}(\hat{\theta}_c)) = -4797.926 \). Thus, \(-2 \log \Lambda = 13.984 \) with the p-value of 0.00735 show that we can reject c-mixture in favor of g-mixture at \( \alpha = 0.01 \).

7 Concluding remarks

We proposed and estimated a new unconstrained mixture of Markov processes. We showed the consistency and asymptotic normality of the estimators of the mixture’s parameters and obtained the finite sample standard errors of the estimates. The simulation study verified that the estimation was accurate and confirmed the asymptotic properties of the estimators. The application of the proposed mixture to VenICU illustrated its usefulness in identifying subpopulations and its dominance over the constrained mixture. We believe that the unconstrained mixture will dominate the constrained one in many other data sets arising from heterogeneous populations. We intend to extend the proposed general mixture in a number of ways which include incorporation of covariates into our continuous observation time framework, and developing estimation from observing the mixture at discrete time points in the presence of covariates. The estimation of a discretely observed Markov jump
process without covariates has been considered by Bladt and Sørensen (2005, 2009), Inamura (2006), Mostel et al. (2020), and Pfeuffer et. al. (2019), among others. The mixture proposed here and the mixtures of finite-state continuous-time Markov processes arising from the above potential extensions should be useful in a variety of contexts in which modeling of the population heterogeneity is important.

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Supplementary material for:
Statistical inference for a mixture of Markov jump processes

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A Proof of Theorem 2

A.1 Unconditional moments of $B_{i,m}$, $N_{ij,m}$ and $T_{i,m}$

For the assertion of the theorem, we need the unconditional moments of $B_{i,m}$, $N_{ij,m}$ and $T_{i,m}$.

Lemma A.1 For $i \in E$, $j \in S$, and $1 \leq m \leq M$, we have for a fixed $T > 0$,

\[
\mathbb{E}_\theta [\Phi_{k,m} N_{ij}^k] = q_{ij,m} \int_0^T \pi^\top D_m e^{Q_m u} e_i du,
\]
\[
\mathbb{E}_\theta [\Phi_{k,m} T_i^k] = \int_0^T \pi^\top D_m e^{Q_m u} e_i du,
\]

while for $T$ being the absorption time of $X$,

\[
\mathbb{E}_\theta [\Phi_{k,m} N_{ij}^k] = q_{ij,m} \int_0^\infty \pi^\top D_m e^{Q_m u} e_i du,
\]
\[
\mathbb{E}_\theta [\Phi_{k,m} T_i^k] = \int_0^\infty \pi^\top D_m e^{Q_m u} e_i du.
\]

For a fixed time $T > 0$

The proofs are based on adapting similar arguments to the proof of Theorem 5.1 in Albert (1962) by dividing the interval $[0, T]$ into $n$ equal parts of length $h = T/n$. Then, by dominated convergence theorem,

\[
\mathbb{E}_\theta [\Phi_{k,m} N_{ij}^k] = \mathbb{E}_\theta \left[ \sum_{\ell=0}^\infty 1\{\Phi_{k,m}=1, X^k_{\ell h}=i, X^k_{(\ell+1)h}=j, (\ell+1)h \leq T\} \right]
\]

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\[
\begin{align*}
&= \sum_{\ell=0}^{n-1} \sum_{v \in S} P_\theta\{X^k_{(\ell+1)h} = i, X^k_{\ell h} = j, \Phi_{k,m} = 1, X^k_0 = v\} \\
&= \sum_{\ell=0}^{n-1} \sum_{v \in S} P_\theta\{X^k_{\ell h} = i\} P_\theta\{\Phi_{k,m} = 1, X^k_{\ell h} = j, X^k_0 = v\} \\
&\quad \times P_\theta\{X^k_{\ell h} = i, X^k_{(\ell+1)h} = j, \Phi_{k,m} = 1, X^k_0 = v\}
\end{align*}
\]

For the second statement, let \( Y^k_i(u) = 1_{\{X^k_u = i\}} \). Then by Fubini’s theorem,

\[
\mathbb{E}_\theta[\Phi_{k,m} T^k_i] = \mathbb{E}_\theta\left[ \int_0^T \Phi_{k,m} Y^k_i(u) du \right]
\]

\[
= \int_0^T \mathbb{P}_\theta\{X^k_u = i, \Phi_{k,m} = 1\} du
\]

\[
= \int_0^T \sum_{v \in S} \mathbb{P}_\theta\{X^k_u = i, \Phi_{k,m} = 1, X^k_0 = v\} du
\]

\[
= \int_0^T \sum_{v \in S} \mathbb{P}_\theta\{X^k_0 = v\} \mathbb{P}_\theta\{\Phi_{k,m} = 1, X^k_0 = v\} \\
\quad \times \mathbb{P}_\theta\{X^k_{\ell h} = i, X^k_{(\ell+1)h} = j, \Phi_{k,m} = 1, X^k_0 = v\}
\]

\[
= \int_0^T \sum_{v \in S} \pi_v \phi_{v,m} e^\top v e Q_m e_i q_{ij,m} h
\]

\[
= \int_0^T \pi^\top D_m e Q_m e_i q_{ij,m} h \rightarrow 0 \sum_{\ell=0}^{[T/h]} \pi^\top D_m e Q_m e_i du
\]

which completes the proof of the claim for fixed \( T > 0 \).  

**For an absorption time \( T \)**

Since \( T \) is the absorption time of \( X \), there are no \( i \rightarrow j \) transitions from state \( i \in E \) to \( j \in S \) after \( T \). Therefore, for a given \( h > 0 \), we have

\[
\mathbb{E}_\theta[\Phi_{k,m} N^k_{ij}] = \mathbb{E}_\theta\left[ \sum_{\ell=0}^{[T/h]-1} \Phi_{k,m} X^k_{\ell} (i,j) \right]
\]

\[
= \mathbb{E}_\theta\left[ \sum_{\ell=0}^{[T/h]-1} \Phi_{k,m} X^k_{\ell} (i,j) \right] + \mathbb{E}_\theta\left[ \sum_{\ell=[T/h]}^{\infty} \Phi_{k,m} X^k_{\ell} (i,j) \right]
\]
\[
\begin{align*}
\mathbb{E}_\theta \left[ \sum_{\ell=0}^{\infty} \Phi_{k,m} X_k^\ell(i,j) \right] &= \\
= \sum_{\ell=0}^{\infty} \mathbb{P}_\theta \{ \Phi_{k,m} = 1, X_{\ell h}^k = i, X_{(\ell+1)h}^k = j \} &= \\
= \sum_{\ell=0}^{\infty} \sum_{v \in S} \mathbb{P}_\theta \{ X_0^k = v, \Phi_{k,m} = 1, X_{\ell h}^k = i, X_{(\ell+1)h}^k = j \} \\
= \sum_{\ell=0}^{\infty} \sum_{v \in S} \mathbb{P}_\theta \{ X_0^k = v \} \mathbb{P}_\theta \{ \Phi_{k,m} = 1 | X_0^k = v \} \\
\times \mathbb{P}_\theta \{ X_{\ell h}^k = i | \Phi_{k,m} = 1, X_0^k = v \} \\
\times \mathbb{P}_\theta \{ X_{(\ell+1)h}^k = j | \Phi_{k,m} = 1, X_{\ell h}^k = i, X_0^k = v \} \\
= \sum_{\ell=0}^{\infty} \sum_{v \in S} \pi_v \Phi_{v,m} e_i Q_m \ell \mathbb{P}_\theta \{ X_k^0 = v \} \\
\pi^T D_m e_i e_i Q_m h \rightarrow q_{ij,m} \int_0^\infty \pi^T D_m e_i e_i Q_m u \mathbb{P}_\theta \{ \Phi_{k,m} = 1 | X_k^0 = v \} \\
\times \mathbb{P}_\theta \{ X_k^\ell h = i | \Phi_{k,m} = 1, X_k^0 = v \} \\
\times \mathbb{P}_\theta \{ X_k^{(\ell+1)h} = j | \Phi_{k,m} = 1, X_k^\ell h = i, X_k^0 = v \} \\
&= \int_0^\infty \pi^T D_m e_i e_i Q_m u \mathbb{P}_\theta \{ \Phi_{k,m} = 1 | X_k^0 = v \} \\
\times \mathbb{P}_\theta \{ X_k^\ell h = i | \Phi_{k,m} = 1, X_k^0 = v \} \\
\times \mathbb{P}_\theta \{ X_k^{(\ell+1)h} = j | \Phi_{k,m} = 1, X_k^\ell h = i, X_k^0 = v \} \\
&= \int_0^\infty \pi^T D_m e_i e_i Q_m u \mathbb{P}_\theta \{ \Phi_{k,m} = 1 | X_k^0 = v \} \\
&= \int_0^\infty \pi^T D_m e_i e_i Q_m u \mathbb{P}_\theta \{ \Phi_{k,m} = 1 | X_k^0 = v \} \\

\text{which establishes the second claim for absorption time.} \quad \blacksquare
\end{align*}
\]

**A.2 Unconditional second moments of \( N_{ij,m}, T_{i,m} \) and \( B_{i,m} \)**

This part will have a proposition with its proof. It generalizes the results of Theorem 5.1 of Albert (1962) to a mixture of Markov jump processes.
Proposition A.1 Let $D_m$ be a $(w \times w)$ diagonal matrix with diagonal elements $\phi_{i,m}$, $i \in E$. The unconditional moments of $B_i^k$, $N_{ij}^k$, and $T_i^k$ are

\[
\mathbb{E}_\theta[\Phi_{k,m}B_i^kN^k_{ij}] = q_{ij,m} \pi_i \pi_{i',m} e_i^T \int_0^T e^{Q_m u} e_{i} du, \tag{a}
\]
\[
\mathbb{E}_\theta[\Phi_{k,m}B_i^kT_i^k] = \pi_i \pi_{i',m} e_i^T \int_0^T e^{Q_m u} e_{i} du, \tag{b}
\]
\[
\mathbb{E}_\theta[\Phi_{k,m}N^k_{ij}N^k_{i'j'}] = q_{ij,m} \delta_i(i') \delta_j(j') \pi_i \pi_{i',m} e_i^T \int_0^T e^{Q_m u} du \sum_k \sum_{\ell} e_{i} e_{j} e_{i'} e_{j'} e^{Q_m (u-t)} e_{i} e_{i'} du,
\]
\[
\mathbb{E}_\theta[\Phi_{k,m}T_i^kT_i'^k] = \pi_i \pi_{i',m} e_i^T \int_0^T e^{Q_m u} e_i e_i' \ e^{Q_m (u-t)} e_i e_i' du \sum_k \sum_{\ell} e_{i},
\]
\[
\mathbb{E}_\theta[\Phi_{k,m}N^k_{ij}T_i^k] = q_{ij,m} \pi_i \pi_{i',m} e_i^T \int_0^T e^{Q_m u} e_i e_i' \ e^{Q_m (u-t)} e_i du \sum_k \sum_{\ell} e_{i},
\]

where we have denoted by $e_i = (0, \cdots, 1, \cdots, 0)$ a $(w \times 1)$ unit vector with value one on the $i$th element and zero otherwise.

Proof of (31)

The proofs are based on adapting similar arguments to that of Theorem 5.1 in Albert (1962) by dividing the interval $[0, T]$ into $n$ equal parts of length $h = T/n$. Then, by dominated convergence theorem, $N_{ij}^k = \sum_{\ell=0}^{n-1} X_{i,j}^k$ with $X_{i,j}^k = I(X_{i,j}^k = i)$, $(i,j) \in S$. Thus,

\[
\mathbb{E}_\theta[\Phi_{k,m}B_i^kN^k_{ij}] = \mathbb{E}_\theta \left[ \sum_{\ell=0}^{n-1} \Phi_{k,m}B_i^kX_{i,j}^k \right]
\]
\[
= \sum_{\ell=0}^{n-1} \mathbb{P}_\theta \left( X_{i,j}^k = i, X_{\ell+1}^k = j, X_{0}^k = i', \Phi_{k,m} = 1 \right)
\]
\[
= \sum_{\ell=0}^{n-1} \mathbb{P}_\theta \left( X_{\ell+1}^k = i' \right) \mathbb{P}_\theta \left( \Phi_{k,m} = 1 \mid X_{0}^k = i' \right) \mathbb{P}_\theta \left( X_{\ell+1}^k = i \mid X_{0}^k = i' \right)
\]
\[
\times \mathbb{P}_\theta \left( X_{\ell+1}^k = j \mid \Phi_{k,m} = 1, X_{\ell}^k = i, X_{0}^k = i' \right)
\]
which converges by dominated convergence to the one claimed. ■

**Proof of (b)**

By Fubini’s theorem, Bayes’ formula, and Markov property of $X_m$,

$$
\mathbb{E}_\theta[\Phi_{k,m} B^k_i T^k_i] = \mathbb{E}_\theta[\Phi_{k,m} B^k_i \int_0^T 1_{X^k_b = i} du]
$$

$$= \int_0^T \mathbb{P}_\theta \{X^k_u = i, X^k_0 = i', \Phi_{k,m} = 0\} du
$$

$$= \int_0^T \mathbb{P}_\theta \{X^k_0 = i'\} \mathbb{P}_\theta \{\Phi_{k,m} = 1 \mid X^k_0 = i'\} \mathbb{P}_\theta \{X^k_i = i \mid \Phi_{k,m} = 1, X^k_0 = i'\} du,$$

which indeed gives the second statement. ■

**Proof of (c)**

Using representation $N^k_{ij} = \sum_{\ell=0}^{n-1} X^k_\ell(i, j)$, one can write

$$
\Phi_{k,m} N^k_{ij} N^k_{i'j'} = \sum_{\ell=0}^{n-1} \Phi_{k,m} X^k_\ell(i, j) X^k_\ell(i', j')
$$

$$+ \sum_{\ell=1}^{n-1} \sum_{r<\ell} \Phi_{k,m} X^k_\ell(i, j) X^k_r(i', j') + \sum_{\ell=0}^{n-2} \sum_{r>\ell} \Phi_{k,m} X^k_\ell(i, j) X^k_r(i', j')
$$

$$= \sum_{\ell=0}^{n-1} \Phi_{k,m} X^k_\ell(i, j) X^k_\ell(i', j')
$$

$$+ \sum_{\ell=1}^{n-1} \sum_{r=0}^{\ell-1} \Phi_{k,m} X^k_\ell(i, j) X^k_r(i', j') + \sum_{r=1}^{n-1} \sum_{\ell=0}^{r-1} \Phi_{k,m} X^k_\ell(i, j) X^k_r(i', j').$$

Therefore,

$$
\mathbb{E}_\theta[\Phi_{k,m} N^k_{ij} N^k_{i'j'}] = \mathbb{E}_\theta \left[ \sum_{\ell=0}^{n-1} \Phi_{k,m} X^k_\ell(i, j) X^k_\ell(i', j') \right]
$$

$$+ \mathbb{E}_\theta \left[ \sum_{\ell=1}^{n-1} \sum_{r=0}^{\ell-1} \Phi_{k,m} X^k_\ell(i, j) X^k_r(i', j') \right] + \mathbb{E}_\theta \left[ \sum_{r=1}^{n-1} \sum_{\ell=0}^{r-1} \Phi_{k,m} X^k_\ell(i, j) X^k_r(i', j') \right]. \quad (1)
$$

Since an initial state is chosen randomly using probability $\pi$, the first expectation is obtained using Bayes’ formula and the law of total probability,

$$
\mathbb{E}_\theta \left[ \sum_{\ell=0}^{n-1} \Phi_{k,m} X^k_\ell(i, j) X^k_\ell(i', j') \right] = \sum_{\ell=0}^{n-1} \mathbb{E}_\theta \left[ \Phi_{k,m} X^k_\ell(i, j) X^k_\ell(i', j') \right]
$$

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\[
\sum_{\ell=0}^{n-1} \mathbb{P}_{\theta}\{ X_{i_{\ell+1}}^k = i, X_{(\ell+1)h}^k = j, X_{(\ell+1)h}^k = j', \Phi_{k,m} = 1 \} = \delta_i(i') \delta_j(j') \sum_{\ell=0}^{n-1} \mathbb{P}_{\theta}\{ X_{i_{\ell+1}}^k = i', X_{(\ell+1)h}^k = j', \Phi_{k,m} = 1 \}
\]

which by dominated convergence theorem the last sum converges:

\[
q_{i'i'\ell,m} \delta_i(i') \delta_j(j') \sum_{\ell=0}^{n-1} \pi^\top D_m e^{Q_m^\ell h} e_i e_{i'} \int_0^T \pi^\top D_m e^{Q_m^u} e_i du.
\]

The second sum in (\(\Pi\)) can be worked out as follows.

\[
\mathbb{E}_{\theta}\left[ \sum_{\ell=1}^{n-1} \sum_{r=0}^{\ell-1} \Phi_{k,m} X_{i_{\ell+1}}^k (i,j) X_{r_{\ell+1}}^k (i',j') \right] = \sum_{\ell=1}^{n-1} \sum_{r=0}^{\ell-1} \mathbb{E}_{\theta}\left[ \Phi_{k,m} X_{i_{\ell+1}}^k (i,j) X_{r_{\ell+1}}^k (i',j') \right]
\]

\[
= \sum_{\ell=1}^{n-1} \sum_{r=0}^{\ell-1} \mathbb{P}_{\theta}\{ X_{i_{\ell+1}}^k = i, X_{(\ell+1)h}^k = j, X_{r_{\ell+1}}^k = i', X_{(\ell+1)h}^k = j', \Phi_{k,m} = 1 \}
\]

\[
= \sum_{\ell=1}^{n-1} \sum_{r=0}^{\ell-1} \sum_{v \in S} \mathbb{P}_{\theta}\{ X_{i_{\ell+1}}^k = i, X_{(\ell+1)h}^k = j, X_{r_{\ell+1}}^k = i', X_{(r_{\ell+1})h}^k = j', \Phi_{k,m} = 1, X_0^k = v \}
\]

\[
= \sum_{\ell=1}^{n-1} \sum_{r=0}^{\ell-1} \sum_{v \in S} \mathbb{P}_{\theta}\{ X_{i_{\ell+1}}^k = i, X_{r_{\ell+1}}^k = v \} \mathbb{P}_{\theta}\{ \Phi_{k,m} = 1 | X_0^k = v \} \mathbb{P}_{\theta}\{ X_{r_{\ell+1}}^k = i' | \Phi_{k,m} = 1, X_0^k = v \}
\]

\[
\times \mathbb{P}_{\theta}\{ X_{r_{\ell+1}}^k = j' | \Phi_{k,m} = 1, X_{r_{\ell+1}}^k = i', X_0^k = v \}
\]

\[
\times \mathbb{P}_{\theta}\{ X_{r_{\ell+1}}^k = j | \Phi_{k,m} = 1, X_{r_{\ell+1}}^k = j', X_0^k = v \}
\]

\[
= \sum_{\ell=1}^{n-1} \sum_{r=0}^{\ell-1} \pi_v \phi_v, m e_v^\top e Q_m^r h e_i e_{i'} e_j e_{j'} e Q_m^0 (h+(r+1)) e_i q_{ij,m} h
\]

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which by dominated convergence theorem the last sum converges:

\[ q_{ij,m}q_{i'j',m} \sum_{\ell=1}^{n-1} \sum_{r=0}^{\ell-1} \pi^{\top} D_{m} e^{Q_{m+r}^h e_{i'}} h e_{j'} e^{Q_{m+(r+1)h} e_{i} h} \]

Similarly, following the same arguments, one can show that the third sum in (11)

\[
\mathbb{E}_{\theta} \left[ \sum_{r=1}^{n-1} \sum_{\ell=0}^{r-1} \Phi_{k,m} X_k^i(i,j) X_k^{i'}(i',j') \right] \\
\xrightarrow{h \to 0} q_{ij,m}q_{i'j',m} \int_{0}^{T} \int_{0}^{u} \pi^{\top} D_{m} e^{Q_{m+t} e_{i} e_{j}^\top e^{Q_{m(u-t)} e_{i'} dt} du.}
\]

Putting the three limiting integrals together yields the third statement. ■

**Proof of (d)**

Applying Fubini’s theorem we get

\[
\mathbb{E}_{\theta}\left[ \Phi_{k,m} T_i^{k} T_{i'}^{k} \right] = \int_{0}^{T} \int_{0}^{T} \mathbb{P}_{\theta}\{ X_t^k = i, X_u^k = i', \Phi_{k,m} = 1 \} dt du
\]

\[
= \int_{0}^{T} \int_{0}^{u} \mathbb{P}_{\theta}\{ X_t^k = i, X_u^k = i', \Phi_{k,m} = 1 \} dt du
\]

\[
+ \int_{0}^{T} \int_{u}^{T} \mathbb{P}_{\theta}\{ X_t^k = i, X_u^k = i', \Phi_{k,m} = 1 \} dt du.
\]

The first double integral can be worked out as follows

\[
\int_{0}^{T} \int_{0}^{u} \mathbb{P}_{\theta}\{ X_t^k = i, X_u^k = i', \Phi_{k,m} = 1 \} dt du
\]

\[
= \int_{0}^{T} \int_{0}^{u} \sum_{v \in S} \mathbb{P}_{\theta}\{ X_t^k = i, X_u^k = i', \Phi_{k,m} = 1, X_0^k = v \} dt du
\]

\[
= \int_{0}^{T} \int_{0}^{u} \sum_{v \in S} \mathbb{P}_{\theta}\{ X_0^k = v \} \mathbb{P}_{\theta}\{ \Phi_{k,m} = 1 | X_0^k = v \} \mathbb{P}_{\theta}\{ X_t^k = i | \Phi_{k,m} = 1, X_0^k = v \} dt du
\]

\[
\times \mathbb{P}_{\theta}\{ X_u^k = i' | \Phi_{k,m} = 1, X_t^k = i, X_0^k = v \} dt du
\]

\[
= \int_{0}^{T} \int_{0}^{u} \sum_{v \in S} \pi_{\theta} \phi_{v,m} e_{i} e^{Q_{m} e_{j} e^{Q_{m(u-t)} e_{i'} dt} du}
\]

\[
= \int_{0}^{T} \int_{0}^{u} \pi^{\top} D_{m} e^{Q_{m+t} e_{i} e_{j}^\top e^{Q_{m(u-t)} e_{i'} dt} du.
\]
By the same lines of arguments, we get
\[
\int_0^T \int_0^T \mathbb{P}_\theta\{X_t^k = i, X_u^k = i', \Phi_{k,m} = 1\} dt du
\]
\[
= \int_0^T \int_0^T \pi^T D_m e^{Q_m t} e_i e_j' e^{Q_m (u-t)} e_i' dt du
\]
\[
= \int_0^T \int_0^t \pi^T D_m e^{Q_m u} e_i e_j' e^{Q_m (t-u)} e_i' du dt,
\]
where the last integral was due to using change of variable. Thus, the claim on the fourth statement is established. \(\blacksquare\)

**Proof of (e)**

For notational convenience, define for \(j \in S, u > 0, Y^k_j(u) = 1_{\{X_t^k = j\}}.\)

\[
\mathbb{E}_\theta [\Phi_{k,m} N^k_{ij} T_i^k] = \mathbb{E}_\theta \left[ \sum_{\ell=0}^{n-1} \int_0^T \Phi_{k,m} Y^k_{ij}(u) X^k_{\ell}(i, j) du \right]
\]
\[
= \sum_{\ell=0}^{n-1} \int_0^T \mathbb{P}_\theta \{X^k_{\ell h} = i, X^k_{(\ell+1)h} = j, X^k_u = i', \Phi_{k,m} = 1\} du
\]
\[
= \sum_{\ell=0}^{n-1} \int_0^T \sum_{v \in S} \mathbb{P}_\theta \{X^k_{\ell h} = i, X^k_{(\ell+1)h} = j, X^k_u = i', \Phi_{k,m} = 1, X^k_0 = v\} du
\]
\[
= \sum_{\ell=0}^{n-1} \int_0^{\ell h} \sum_{v \in S} \mathbb{P}_\theta \{X^k_{\ell h} = i, X^k_{(\ell+1)h} = j, X^k_u = i', \Phi_{k,m} = 1, X^k_0 = v\} du
\]
\[
+ \sum_{\ell=0}^{n-1} \int_{\ell h}^{(\ell+1)h} \sum_{v \in S} \mathbb{P}_\theta \{X^k_{\ell h} = i, X^k_{(\ell+1)h} = j, X^k_u = i', \Phi_{k,m} = 1, X^k_0 = v\} du
\]
\[
+ \sum_{\ell=0}^{n-1} \int_{(\ell+1)h}^T \sum_{v \in S} \mathbb{P}_\theta \{X^k_{\ell h} = i, X^k_{(\ell+1)h} = j, X^k_u = i', \Phi_{k,m} = 1, X^k_0 = v\} du.
\]

We evaluate each term in the last equality one by one. For the first term,

\[
\sum_{\ell=0}^{n-1} \int_0^{\ell h} \sum_{v \in S} \mathbb{P}_\theta \{X^k_{\ell h} = i, X^k_{(\ell+1)h} = j, X^k_u = i', \Phi_{k,m} = 1, X^k_0 = v\} du
\]
\[
= \sum_{\ell=0}^{n-1} \int_0^{\ell h} \sum_{v \in S} \mathbb{P}_\theta \{X^k_0 = v\} \mathbb{P}_\theta \{\Phi_{k,m} = 1|X^k_0 = v\} \mathbb{P}_\theta \{X^k_u = i'|\Phi_{k,m} = 1, X^k_0 = v\}
\]
\[
\times \mathbb{P}_\theta \{X^k_{\ell h} = i|\Phi_{k,m} = 1, X^k_u = i', X^k_0 = v\}
\]
\[
\times \mathbb{P}_\theta \{X^k_{(\ell+1)h} = j|\Phi_{k,m} = 1, X^k_{\ell h} = i, X^k_u = i', X^k_0 = v\}
\]
\[
= \sum_{\ell=0}^{n-1} \int_0^{\ell h} \sum_{v \in S} \pi_v \phi_{u,v,m} e_i e_{i'} e_j e_{i'} e^{Q_m (u-\ell h)} e_i q_{ij,m} du
\]
\[= \sum_{\ell=0}^{n-1} \int_{0}^{\ell h} \pi^T D_m e^{Q_m u} e_i^T e^{Q_m(\ell h-u)} e_j q_{ij,m} h du\]

\[\rightarrow q_{ij,m} \int_{0}^{T} \int_{0}^{t} \pi^T D_m e^{Q_m u} e_i^T e^{Q_m(t-u)} e_j dudt,\]

where the convergence is due to dominated convergence theorem. Furthermore,

\[= \sum_{\ell=0}^{n-1} \int_{0}^{(\ell+1)h} \sum_{v \in S} \mathbb{P}_\theta \{X^{k}_{\ell h} = i, X^{k}_{(\ell+1)h} = j, X^k_u = i', \Phi_{k,m} = 1, X^k_0 = v\} du\]

\[= \sum_{\ell=0}^{n-1} \int_{0}^{(\ell+1)h} \sum_{v \in S} \mathbb{P}_\theta \{X^k_0 = v\} \mathbb{P}_\theta \{\Phi_{k,m} = 1 | X^k_0 = v\}\]

\[\times \mathbb{P}_\theta \{X^{k}_{\ell h} = i | \Phi_{k,m} = 1, X^k_0 = i\}\]

\[\times \mathbb{P}_\theta \{X^k_u = i' | \Phi_{k,m} = 1, X^k_0 = i\}\]

\[\times \mathbb{P}_\theta \{X^{k}_{(\ell+1)h} = j | \Phi_{k,m} = 1, X^k_0 = j, X^k_0 = i\}\]

\[= \sum_{\ell=0}^{n-1} \int_{0}^{(\ell+1)h} \sum_{v \in S} \pi_v \phi_{v,m} e_i^T e^{Q_m \ell h} e_i^T e^{Q_m(u-(\ell+1)h)} e_j dudt\]

\[= \sum_{\ell=0}^{n-1} \int_{0}^{(\ell+1)h} \pi^T D_m e^{Q_m \ell h} e_i^T e^{Q_m(u-(\ell+1)h)} e_j dudt\]

\[= \mathcal{O}(h).\]

The last equality is due to fact that for \(\ell h \leq u < (\ell + 1)h\), \(e_i^T e^{Q_m(u-(\ell+1)h)} e_j \approx e_i^T [I + Q_m h] e_j = q_{ij,m} h\) for small \(h > 0\). Similarly, \(e_i^T e^{Q_m((\ell+1)h-u)} e_j \approx q_{ij,m} h\). Thus, the contribution of the second term in the decomposition of \(\mathbb{E}_\theta[\Phi_{k,m} N^{k}_{ij} T^{k}_{ij}]\) is negligible. Finally, the third term is evaluated as follows.

\[= \sum_{\ell=0}^{n-1} \int_{0}^{T} \sum_{v \in S} \mathbb{P}_\theta \{X^{k}_{\ell h} = i, X^{k}_{(\ell+1)h} = j, X^k_u = i', \Phi_{k,m} = 1, X^k_0 = v\} du\]

\[= \sum_{\ell=0}^{n-1} \int_{0}^{T} \sum_{v \in S} \mathbb{P}_\theta \{X^k_0 = v\} \mathbb{P}_\theta \{\Phi_{k,m} = 1 | X^k_0 = v\}\]

\[\times \mathbb{P}_\theta \{X^{k}_{\ell h} = i | \Phi_{k,m} = 1, X^k_0 = i\}\]

\[\times \mathbb{P}_\theta \{X^k_u = i' | \Phi_{k,m} = 1, X^k_0 = i\}\]

\[\times \mathbb{P}_\theta \{X^{k}_{(\ell+1)h} = j | \Phi_{k,m} = 1, X^k_0 = j, X^k_0 = i\}\]

\[= \sum_{\ell=0}^{n-1} \int_{0}^{T} \sum_{v \in S} \pi_v \phi_{v,m} e_i^T e^{Q_m \ell h} e_i^T e^{Q_m(u-(\ell+1)h)} e_j dudt\]

\[= q_{ij,m} \sum_{\ell=0}^{n-1} \int_{0}^{T} \pi^T D_m e^{Q_m \ell h} e_j^T e^{Q_m(u-(\ell+1)h)} e_i dudt\]

\[\rightarrow q_{ij,m} \int_{0}^{T} \pi^T D_m e^{Q_m t} e_j^T e^{Q_m(u-t)} e_i dudt\]
$$= q_{ij,m} \int_0^T \int_0^u \pi^\top D_m e^{Q_m t} e_i e_j^\top e^{Q_m(u-t)} e_i' dt du,$$

which in turn establishes the claim for the fifth statement. ■

### A.3 Derivation of the covariance matrix \(\Sigma(\theta^0)\)

If the EM estimator \(\hat{\theta}\) converges to the true value \(\theta\), it follows from Chernoff (1956) and Cramer (1946) p. 254 that for each \((i,j) \in S\) and \(1 \leq m \leq M\) the random variable \(\widehat{N}_{ij,m}/K - N_{ij,m}/K\), which is equal to \(\frac{1}{K} \sum_{k=1}^K \hat{\Phi}_{k,m}^i N_{ij,k}^k - \frac{1}{K} \sum_{k=1}^K \Phi_{k,m} N_{ij}^k\), converges to zero in probability as \(K \to \infty\). It follows, by independence of \(\{X^k\}\), and above references, \(\frac{1}{K} \sum_{k=1}^K \mathbb{E}_\theta[\Phi_{k,m}^i X^k] N_{ij}^k = \frac{1}{K} \sum_{k=1}^K \mathbb{E}_\theta[\Phi_{k,m} N_{ij}^k X^k] \xrightarrow{p} \mathbb{E}_\theta[\Phi_{k,m} N_{ij}^k]\) which is the same convergence as \(\frac{1}{K} \sum_{k=1}^K \Phi_{k,m} N_{ij}^k\). This implies that \(\widehat{N}_{ij,m}/K\) and \(N_{ij,m}/K\) have the same distributional convergence. See van der Vaart (2000), Ch. 2 on stochastic convergence of random variables. The same reasoning applies to \(\widehat{T}_{i,m}/K - T_{i,m}/K\) and \(\widehat{B}_{i,m}/K - B_{i,m}/K\). As \(\hat{\theta} \to \theta\), it follows from the statement above that

$$\sqrt{K}(\widehat{q}_{ij,m} - q_{ij,m}) = \frac{\sqrt{K}}{\widehat{T}_{i,m}/K} \left(\widehat{N}_{ij,m}/K - q_{ij,m} \widehat{T}_{i,m}/K\right)$$

has the same asymptotic distribution as the random variable

$$\frac{\sqrt{K}}{\mathbb{E}_\theta[\Phi_{k,m} T_{i}^k]} \left(N_{ij,m}/K - q_{ij,m} T_{i,m}/K\right).$$

By the multivariate central limit theorem, the above has asymptotic multivariate normal distribution with mean zero and the covariance matrix

$$\mathbb{E}_\theta \left[ (\Phi_{k,n} N_{i'j'}^k - q_{i'j',n} \Phi_{k,n} T_{i'}^k) (\Phi_{k,m} N_{ij}^k - q_{ij,m} \Phi_{k,m} T_{i}^k) \right] \mathbb{E}_\theta[\Phi_{k,n} T_{i'}^k] \mathbb{E}_\theta[\Phi_{k,m} T_{i}^k].$$

Notice that we have used the fact that \(\mathbb{E}_\theta[\Phi_{k,n} N_{i'j'}^k - q_{i'j',n} \Phi_{k,n} T_{i'}^k] = 0\), see Lemma A.1. By Proposition [A.1] we have

$$\mathbb{E}_\theta \left[ (\Phi_{k,n} N_{i'j'}^k - q_{i'j',n} \Phi_{k,n} T_{i'}^k) (\Phi_{k,m} N_{ij}^k - q_{ij,m} \Phi_{k,m} T_{i}^k) \right]$$

$$= \delta_m(n) \mathbb{E}_\theta[\Phi_{k,m} N_{ij}^k N_{i'j'}^k] - \delta_{m,n}(q_{i'j',n}) \mathbb{E}_\theta[\Phi_{k,n} N_{i'j'}^k T_{i'}^k]$$

$$+ \delta_{m,n}(q_{ij,m}) \mathbb{E}_\theta[\Phi_{k,m} N_{i'j'}^k T_{i}^k] + q_{ij,m} q_{i'j',n} \delta_{m,n} \mathbb{E}_\theta[\Phi_{k,n} T_{i'}^k T_{i}^k]$$

$$= \delta_{m,n}(q) \delta_{i}(i') \delta_{j}(j') q_{ij,n} \int_0^T \pi^\top D_n e^{Q_n u} e_i' du,$$

which establishes the assertion about \(\text{Cov}(\widehat{q}_{i'j',n} - q_{i'j',n}, \widehat{q}_{ij,m} - q_{ij,m})\) noting that

$$\mathbb{E}_\theta[\Phi_{k,n} T_{i}^k] = \int_0^T \pi^\top D_n e^{Q_n u} e_i' du.$$
By similar arguments, one can show subsequently using Proposition A.1

\[
\text{Cov}(\hat{\phi}_i, n - \phi_i, n, \hat{q}_{ij, m} - q_{ij, m}) = \frac{E_\theta \left[ (\Phi_{k,n} B_{ij}^k - \phi_{ij, n} B_{ij}^k) (\Phi_{k,m} \delta_{ij, m} - q_{ij, m} \Phi_{k,m} T_i^k) \right]}{E_\theta [B_{ij}^k] E_\theta [\Phi_{k,m} T_i^k]} = 0,
\]

and

\[
\text{Cov}(\hat{\phi}_i, n - \phi_i, n, \hat{\phi}_{i,m} - \phi_{i,m}) = \frac{E_\theta \left[ (\Phi_{k,n} B_{ij}^k - \phi_{ij, n} B_{ij}^k) (\Phi_{k,m} B_{ij}^k - \phi_{i,m} B_{ij}^k) \right]}{E_\theta [B_{ij}^k] E_\theta [B_{ij}^k]} = \frac{\phi_{ij, n} \delta_i (\delta_m (n) - \phi_{i,m})}{\pi_i^2},
\]

which complete the proof of the theorem. ■

The proof for absorption time \( T \) follows very similar arguments to those in the proof for fixed \( T \).

**B  Proof of Proposition 4**

Since the observed Fisher information matrix \( I(\theta) \) presented in (17) of the paper, simplifies the conditional expectation of outer product of the complete-data score function, the elements \( I(\theta_i, \theta_j) \) of the matrix \( I(\theta) \) can be derived in straightforward way.

**Elements of the intensity matrix \( I(\phi_{j,n}, \phi_{i,m}) \)**

To obtain the elements \( I(\phi_{j,n}, \phi_{i,m}) \) of \( I(\phi) \), we compute the first and second order derivatives of \( \log L_k^k(\phi) \) for \((i, j) \in E \) and \( 1 \leq m, n \leq M - 1 \):

\[
\mathbb{E}_\theta \left[ -\frac{\partial^2 \log L_k^k(\phi)}{\partial \phi_{j,n} \partial \phi_{i,m}} \bigg| X^k \right] = \mathbb{E}_\theta \left[ \left( \Phi_{k,n} \delta_{m}(n) + \Phi_{k,M} \frac{B_{j,M}^k}{\phi_{j,M}^2} \right) B_{j}^k \delta_i (j) \bigg| X^k \right]
= \left( \frac{\Phi_{k,n} B_{j,n}^k}{\phi_{j,n}^2} \delta_{m}(n) + \frac{\Phi_{k,M} B_{j,M}^k}{\phi_{j,M}^2} \right) \delta_i (j).
\]

Using the score function with respect to \( \phi_{i,m} \), see (7) of the paper, and since \( B_{i}^k B_{j}^k = B_{j}^k B_{i}^k \), \( \Phi_{k,m} \Phi_{k,n} = \Phi_{k,n} \delta_{m}(n) \), and \( \delta_M (m) = 0 \) for \( m \neq M \),

\[
\mathbb{E}_\theta \left[ \left( \frac{\partial \log L_k^k(\phi)}{\partial \phi_{j,n}} \right) \left( \frac{\partial \log L_k^k(\phi)}{\partial \phi_{i,m}} \right) \bigg| X^k \right]
= \mathbb{E}_\theta \left[ \left( \Phi_{k,n} B_{j,n}^k \phi_{i,m} - \Phi_{k,M} B_{j,M}^k \phi_{i,m} \right) \left( \Phi_{k,n} B_{i,n}^k \phi_{j,n} - \Phi_{k,M} B_{i,M}^k \phi_{j,n} \right) \bigg| X^k \right]
= \mathbb{E}_\theta \left[ \Phi_{k,n} B_{j,n}^k \phi_{i,m} \phi_{j,n} \delta_i (j) \delta_{m}(n) + \Phi_{k,M} B_{j,M}^k \phi_{i,m} \phi_{j,M} \delta_i (j) \bigg| X^k \right]
\]
Following the score function for \( \phi_{i,m} \), for \( i \in E \) and \( 1 \leq m \leq M - 1 \), we have

\[
\mathbb{E}_\theta \left[ \frac{\partial \log L_c^k(\phi)}{\partial \phi_{i,m}} \bigg| X^k \right] = \mathbb{E}_\theta \left[ \frac{\Phi_{k,m}B_j^k}{\phi_{i,m}^\phi_{j,m}} - \frac{\Phi_{k,M}B_j^k}{\phi_{i,M}^\phi_{j,M}} \delta_i(j) \delta_m(n) \bigg| X^k \right] = \frac{\hat{\Phi}_{k,m}B_j^k}{\phi_{i,m}} - \frac{\hat{\Phi}_{k,M}B_j^k}{\phi_{i,M}}.
\]

(2)

Replacing the parameters \( \phi_{i,m} \) and \( \phi_{j,n} \) by their respective estimates \( \hat{\phi}_{i,m} \) and \( \hat{\phi}_{j,n} \), the elements \( I(\hat{\phi}_{i,m}, \hat{\phi}_{j,n}) \) is obtained by taking the sum \( \sum_{k=1}^K \) of all the three pieces, given by

\[
I(\hat{\phi}_{i,m}, \hat{\phi}_{j,n}) = \frac{\delta_i(j)}{\hat{\phi}_{j,n}^\phi_{j,m}} \sum_{k=1}^K \hat{\Psi}_{i,m|k}\hat{\Psi}_{i,m|k}B_j^k.
\]

This completes the proof. ■

Elements of the intensity matrix \( I(q_{rv,n}, \hat{q}_{ij,m}) \)

Since the first order partial derivative of the log-likelihood \( \log L_c^k(q) \) with respect to the parameter \( q_{ij,m} \) is given by \( \frac{\partial \log L_c^k(q)}{\partial q_{ij,m}} = \frac{\Phi_{k,m}N_{ij,m}^k}{q_{ij,m}} - \Phi_{k,m}T_i,m \), we have

\[
\mathbb{E}_\theta \left[ - \frac{\partial^2 \log L_c^k(q)}{\partial q_{ij,m} \partial q_{rv,n}} \bigg| X^k \right] = \mathbb{E}_\theta \left[ \frac{\Phi_{k,m}N_{ij,m}^k}{q_{ij,m}q_{rv,n}} \delta_i(r) \delta_j(v) \delta_m(n) \bigg| X^k \right] = \frac{\hat{\Phi}_{k,m}N_{ij,m}^k}{q_{ij,m}q_{rv,n}} \delta_i(r) \delta_j(v) \delta_m(n).
\]

Furthermore, since \( A_{rv,n}^k := N_{rv,n}^k - q_{rv,n}T_{rv,n}^k \) for \( (r,v) \in S, 1 \leq n \leq M \),

\[
\mathbb{E}_\theta \left[ \left( \frac{\partial \log L_c^k(q)}{\partial q_{rv,n}} \right) \left( \frac{\partial \log L_c^k(q)}{\partial q_{ij,m}} \right) \bigg| X^k \right] = \mathbb{E}_\theta \left[ \left( \frac{\Phi_{k,m}A_{ij,m}^k}{q_{ij,m}} \right) \left( \frac{\Phi_{k,n}A_{rv,n}^k}{q_{rv,n}} \right) \bigg| X^k \right] = \frac{\delta_m(n)}{q_{ij,m}q_{rv,n}} \hat{\Phi}_{k,n}A_{ij,m}^kA_{rv,n}^k.
\]

Using the score function for \( q_{ij,m} \), for \( i \in E, j \in S \) and \( 1 \leq m \leq M \), we obtain

\[
\mathbb{E}_\theta \left[ \frac{\partial \log L_c^k(q)}{\partial q_{ij,m}} \bigg| X^k \right] = \frac{\hat{\Phi}_{k,m}A_{ij,m}^k}{q_{ij,m}}
\]

(3)

from which the elements of information matrix \( I(q_{rv,n}, q_{ij,m}) \) are obtained as

\[
I(\hat{q}_{ij,m}, \hat{q}_{rv,n}) = \frac{\delta_i(r) \delta_j(v) \delta_m(n)}{\hat{q}_{ij,m}\hat{q}_{rv,n}} \hat{N}_{rv,n} - \frac{1}{\hat{q}_{ij,m}\hat{q}_{rv,n}} \sum_{k=1}^K \hat{\Phi}_{k,n}(\delta_m(n) - \hat{\Phi}_{k,m})\hat{A}_{ij,m}^k\hat{A}_{rv,n}^k,
\]

which completes the proof. ■
Elements of the intensity matrix $I(\hat{\phi}_{i,m}, \hat{q}_{rv,n})$

It is straightforward to check that

$$\mathbb{E}_\theta \left[ \frac{\partial \log L_k^c(\theta)}{\partial \phi_{i,m}} \frac{\partial \log L_k^c(\theta)}{\partial q_{rv,n}} \right] X^k = 0.$$

Using the complete-information score functions of $\phi_{i,m}$ and $q_{rv,n}$, we have for $i \in E, (r, v) \in S, 1 \leq m \leq M - 1$ and $1 \leq n \leq M$,

$$\mathbb{E}_\theta \left[ \frac{\partial \log L_k^c(\theta)}{\partial \phi_{i,m}} \frac{\partial \log L_k^c(\theta)}{\partial q_{rv,n}} \right] X^k = \mathbb{E}_\theta \left[ \frac{\Phi_{k,m} B_i^k}{\phi_{i,m} q_{rv,n}} \Phi_{k,n} A_{rv,n}^k \right] X^k$$

$$= \mathbb{E}_\theta \left[ \frac{\delta_m(n) \Phi_{k,n} B_i^k A_{rv,n}^k}{\phi_{i,m} q_{rv,n}} - \frac{\delta_M(n) \Phi_{k,n} B_i^k A_{rv,n}^k}{\phi_{i,m} q_{rv,n}} \right] X^k$$

$$= \frac{\delta_m(n) \Phi_{k,n} B_i^k A_{rv,n}^k}{\phi_{i,m} q_{rv,n}} - \frac{\delta_M(n) \Phi_{k,n} B_i^k A_{rv,n}^k}{\phi_{i,m} q_{rv,n}}.$$

Following (2) and (3), we obtain

$$\mathbb{E}_\theta \left[ \frac{\partial \log L_k^c(\theta)}{\partial \phi_{i,m}} X^k \right] \mathbb{E}_\theta \left[ \frac{\partial \log L_k^c(q)}{\partial q_{rv,n}} X^k \right] = \left( \frac{\Phi_{k,m} B_i^k}{\phi_{i,m} q_{rv,n}} - \frac{\Phi_{k,M} B_i^k}{\phi_{i,M} q_{rv,n}} \right) \Phi_{k,n} A_{rv,n}^k$$

$$= \frac{\delta_m(n) \Phi_{k,n} B_i^k A_{rv,n}^k}{\phi_{i,m} q_{rv,n}} - \frac{\delta_M(n) \Phi_{k,n} B_i^k A_{rv,n}^k}{\phi_{i,m} q_{rv,n}}.$$

Adding the three pieces together and replacing $\phi_{i,m}$ and $q_{rv,n}$ by their respective ML estimates, we obtain

$$I(\hat{\phi}_{i,m}, \hat{q}_{rv,n}) = \sum_{k=1}^{K} \frac{\Phi_{k,n} B_i^k A_{rv,n}^k}{\phi_{i,m} q_{rv,n}} \left[ \delta_m(n) - \frac{\hat{\phi}_{i,m}}{\hat{\phi}_{i,M}} \delta_M(n) + \frac{\hat{\phi}_{i,m}}{\hat{\phi}_{i,M}} \hat{\Phi}_{k,M} - \hat{\Phi}_{k,m} \right],$$

which ends the proof. ■

C Absorption probabilities for the two-regime g-mixture

The absorption probabilities $f_{ij,m}, m = 1, 2$, from state $i \in E = \{1, 2\}$ into states $j \in \Delta = \{3, 4\}$, are computed using the transition matrix of an $m$th discrete Markov chain embedded into an $m$th Markov jump process, see Theorem 11.6 in Ch.11 of Grinstead and Snell (1997). The transition matrix of the $m$th embedded Markov chain, denoted by $P_m$, is obtained from the intensity matrix $Q_m$ of an $m$th Markov jump process by

$$p_{ij,m} = \frac{q_{ij,m}}{q_{ii,m}}, i \in E, j \in S, j \neq i,$$
where \( q_{ij,m} \) is an (i,j)th entry in the intensity matrix \( Q_m \). To estimate \( f_{ij,m}, m = 1, 2 \), we use \( \hat{Q}_m \) from Section 6.3 to first obtain \( \hat{P}_m \), the estimate of \( P_m \):

\[
\hat{P}_1 = \frac{1}{2} \begin{pmatrix}
0 & 0.10283 & 0.89717 & 0.00000 \\
0.87269 & 0 & 0.10155 & 0.02576
\end{pmatrix},
\]

and

\[
\hat{P}_2 = \frac{1}{2} \begin{pmatrix}
0 & 0.12431 & 0.78503 & 0.09066 \\
0.50746 & 0 & 0.24046 & 0.25208
\end{pmatrix},
\]

where for notational convenience, the last two rows of zeros were deleted. We then obtain the equations for the estimated absorption probabilities \( \hat{f}_{ij,m} \) in terms of \( \hat{P}_m, m = 1, 2 \), see Exercise 34 in Ch.11 of Grinstead and Snell (1997):

\[
\hat{f}_{13,m} = \hat{p}_{13,m} + \hat{p}_{12,m} \hat{f}_{23,m},
\]

\[
\hat{f}_{14,m} = \hat{p}_{14,m} + \hat{p}_{12,m} \hat{f}_{24,m},
\]

and

\[
\hat{f}_{23,m} = \hat{p}_{23,m} + \hat{p}_{21,m} \hat{f}_{13,m},
\]

\[
\hat{f}_{24,m} = \hat{p}_{24,m} + \hat{p}_{21,m} \hat{f}_{14,m},
\]

from which we obtain \( \hat{f}_{ij,m}, m = 1, 2 \), explicitly as

\[
\hat{f}_{13,m} = \frac{\hat{p}_{13,m} + \hat{p}_{12,m} \hat{p}_{23,m}}{1 - \hat{p}_{12,m} \hat{p}_{21,m}},
\]

\[
\hat{f}_{14,m} = \frac{\hat{p}_{14,m} + \hat{p}_{12,m} \hat{p}_{24,m}}{1 - \hat{p}_{12,m} \hat{p}_{21,m}},
\]

and

\[
\hat{f}_{23,m} = \frac{\hat{p}_{23,m} + \hat{p}_{21,m} \hat{p}_{13,m}}{1 - \hat{p}_{12,m} \hat{p}_{21,m}},
\]

\[
\hat{f}_{24,m} = \frac{\hat{p}_{24,m} + \hat{p}_{21,m} \hat{p}_{14,m}}{1 - \hat{p}_{12,m} \hat{p}_{21,m}}.
\]

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