Universal Equilibrium Currents in the Quantum Hall Fluid

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The equilibrium current distribution in a quantum Hall fluid that is subjected to a slowly varying confining potential is shown to generally consist of strips or channels of current, which alternate in direction, and which have universal integrated strengths. A measurement of these currents would yield direct independent measurements of the proper quasiparticle and quasihole energies in the fractional quantum Hall states.

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Since the discovery of the integral and fractional quantum Hall effects, a tremendous experimental and theoretical effort has been made to understand the nonequilibrium transport current in a two-dimensional (2D) electron gas subjected to a strong perpendicular magnetic field. It is now known that the essential feature leading to the quantization of the Hall conductance is the existence of stable incompressible states at certain Landau level filling factors. In contrast, relatively little attention has been given to the equilibrium current distribution in a 2D electron gas in the quantum Hall regime, and we shall show here that the existence of incompressible states leads to some remarkable properties of the equilibrium current as well.

Central to the progress in understanding the quantum Hall fluid has been the ability to fabricate semiconductor nanostructures with highly controlled composition and doping, and the ability to subsequently pattern them, or provide them with metal gates, or both. The electron sheet density $\rho$ in a GaAs heterojunction is typically between $10^{11}$ and $10^{12}$ cm$^{-2}$. The quantum Hall regime occurs at low temperatures and at field strengths where the magnetic length $\ell \equiv (\hbar c/eB)^{1/2}$ satisfies $\ell \approx \rho^{-1/2}$. At these high field strengths, the magnetic length is often small compared with the length scale over which the confining potential—produced by the remote donor centers and gates as well as by the electron gas itself—changes by a bulk energy gap. When this condition is satisfied, the confining potential is said to be slowly varying.

In this Letter, we derive a general expression for the low-temperature equilibrium current distribution in a 2D interacting electron gas, subjected to a strong perpendicular magnetic field, and in a slowly varying confining potential. Our expression, which becomes exact in the slowly varying limit, has two components: One is an edge current, which is proportional to the local density gradient, and the other is a bulk current, which is proportional to the gradient of the self-consistent confining potential. At low temperatures, the confined quantum Hall fluid separates into compressible edge channels, where the density varies, and incompressible bulk regions, where the density is uniform. As we shall show, the screening in the compressible regions becomes perfect at zero temperature, in the sense that the self-consistent confining potential, which includes exchange and correlation, becomes constant there. Hence, the edge current term contributes exclusively to the compressible regions, whereas the bulk component contributes only to the incompressible regions. The resulting current distribution therefore consists of strips or channels of current, and we shall show that the direction of these currents generally displays a striking alternating pattern. Furthermore, we shall show that the integrated current in each channel is universal, independent of the width, position, and shape of the channel, as well as the details of the confining potential.

We begin by calculating the current distribution with current-density functional theory, which rigorously accounts for the effects of electron-electron interaction through an exchange-correlation scalar potential $V_{xc}$ and vector potential $A_{xc}$. A study of the equilibrium current distribution that neglects exchange and correlation effects has been reported by us elsewhere, and several other authors also have contributed to the understanding of equilibrium and nonequilibrium current distributions.

I. Equilibrium current distribution.—Let $H$ be the effective single-particle Hamiltonian of current-density functional theory, which contains functionals of the density $\rho$ and current density $j$. We shall use the self-consistent solutions of the corresponding Kohn-Sham equations, $H\psi_{\alpha} = E_{\alpha}\psi_{\alpha}$, to define an effective single-particle Green’s function for the confined quantum Hall fluid,

$$G(r, r', s) \equiv \sum_{\alpha} \frac{\psi_{\alpha}(r')\psi_{\alpha}^*(r)}{s - E_{\alpha}},$$

(1)

where $s$ is a complex energy variable and $\psi$ is a spinor with components $\psi_\sigma (\sigma = \uparrow, \downarrow)$. Although this effective
Green’s function is generally different from the actual single-particle Green’s function of the 2D interacting electron gas, it nevertheless yields the exact equilibrium number density,

\[ \rho(r) = \text{Tr} \int \frac{ds}{2\pi i} f(s) G(r, r, s), \]

and orbital current density,

\[ j(r) = -\frac{e}{m^*} \text{Tr} \int \frac{ds}{2\pi i} f(s) \times \lim_{r' \to r} \text{Re} \left( -i\hbar \nabla + \frac{e}{c} A \right) G(r, r', s), \]

at fixed chemical potential \( \mu \). Here \( e \) is the magnitude of the electron charge, \( m^* \) is the effective mass, and \( A \) is a vector potential associated with the uniform external magnetic field \( B = Be_z \). The contour in the complex energy plane is to be taken in the positive sense around the poles of \( G \), avoiding the poles of the Fermi distribution function \( f(\epsilon) \equiv \left(e^{(\epsilon-\mu)/k_BT} + 1\right)^{-1} \).

Next, we let \( V \) be a slowly varying potential from the remote donor centers and gates, and we write \( H = H^0 + H^1 \), where \( H^0 \equiv \Pi^2/2m^* + \frac{1}{2} g \mu_B \sigma z B, \Pi \equiv p + \frac{e}{c} A \), and

\[ H^1 \equiv \frac{e}{2m^*c} (A_{xc} \cdot \Pi + \Pi \cdot A_{xc}) + V + V_{H} + V_{xc}. \]

Here \( \mu_b \equiv eh/2nc \) is the Bohr magneton, \( V_H \) is the Hartree potential, \( V_{xc} \) is a \( 2 \times 2 \) diagonal matrix with elements \( V_{xc}^{\sigma} \equiv \left( \frac{\partial E_{xc}}{\partial \rho_\sigma} \right)_V \), and

\[ A_{xc} \equiv -\frac{e}{\rho} \nabla \times \left( \frac{\delta E_{xc}}{\delta \rho_\sigma} \right)_{\rho_\sigma}. \]

We have omitted a term in \( H \) proportional to \( \mathbf{j} \cdot A_{xc} \) which is irrelevant in the slowly varying limit. The exchange-correlation energy \( E_{xc}(\rho_\sigma, \mathbf{v}) \) is a functional of \( \rho_\sigma \) and of the gauge-invariant vorticity \( \mathbf{v} \equiv \nabla \times (\mathbf{j}/\rho) \), where \( \mathbf{j} \) is the paramagnetic part of the current density. Then \( H = 0 \) may be written (again suppressing spin indices) as

\[ G(r, r', s) = G^0(r, r', s) + \int d^2r'' G^0(r, r'', s) H^1(r'') G(r'', r', s), \]

where \( G^0 \) is the Green’s function for the unconstrained noninteracting electron gas. For large \( |r - r'| \), the magnitude of \( G^0 \) falls off as \( e^{-|r-r'|^2/4\ell^2} \) except at its poles.

The Dyson equation \( H = 0 \) and the short-ranged nature of \( G^0 \) may be used to evaluate the effective Green’s function by a gradient expansion in the self-consistent confining potential. At each point in the fluid, the confining potential is approximated by a local potential plus a gradient. We sum the local confining potential terms to all orders and the gradients to first order. The orbital current density is found to be \( \mathbf{j} = \mathbf{j}_{\text{edge}} + \mathbf{j}_{\text{bulk}} \), where

\[ \mathbf{j}_{\text{edge}} = -\epsilon \omega_c \ell^2 \sum_n \left( n + \frac{1}{2} \right) \nabla \rho_n \times \mathbf{e}_z + \epsilon \omega_c \rho \frac{A_{xc}}{B}, \]

\[ \mathbf{j}_{\text{bulk}} = -\frac{e}{m^* \omega_c} \text{Tr} \rho_\sigma \nabla V_{\text{eff}}^{\sigma} \times \mathbf{e}_z, \]

and \( \omega_c \equiv eB/m^*c \) is the cyclotron frequency. The electron density is given by

\[ \rho = \frac{1}{2\pi \ell^2} \text{Tr} \sum_n f \left( n + 1 \overline{2} + \frac{1}{2} \gamma \sigma_z \hbar \omega_c + V_{\text{eff}}(r) \right), \]

where \( \gamma \equiv g \mu_B / \hbar \omega_c \) is the dimensionless spin splitting, and \( \rho_n \) is simply the \( n \)th term in \( \mathbf{j} \). These expressions differ from those obtained in \( \text{[2]} \) by the new edge current term proportional to \( A_{xc} \), and by a self-consistent confining potential,

\[ V_{\text{eff}} \equiv V + V_{H} + V_{xc}, \]

which is modified by exchange and correlation. Note that no microscopic wave function was needed to obtain \( \mathbf{j} \) and \( V_{\text{eff}} \). The bulk term \( \mathbf{j}_{\text{bulk}} \) is simply the transverse Hall current responding to a local electric field \( \nabla V_{\text{eff}}/e \). Discontinuities in \( V_{xc} \) will lead to strips of density with fractional filling factor \( \nu \equiv 2\pi \ell^2 \rho \), in the same way that the discontinuities in the chemical potential of the noninteracting system lead to strips at integral \( \nu \). The total conserved current in a magnetic field also includes a spin contribution \( \mathbf{j}_{\text{spin}} = -\frac{1}{2} g \mu_B \nabla (\rho \uparrow - \rho \downarrow) \times \mathbf{e}_z \), which will not be discussed further.

The precise nature of the compressible regions of the slowly confined quantum Hall fluid as \( T \to 0 \) is made evident by a remarkably simple perfect screening theorem. In a slowly varying system, \( \mathbf{j} \) and \( V_{\text{eff}} \) may be used to obtain a total energy functional \( E[\rho_\sigma] \) of the density only. Minimizing this functional with respect to the \( \rho_\sigma \) \((\sigma = \uparrow, \downarrow)\) under the constraint of fixed total particle number leads to the conditions

\[ \mu_0^\sigma (\rho(\mathbf{r})) + V_{\text{eff}}(\mathbf{r}) = \text{constant}, \]

where \( \mu_0^\sigma \) is the spin-\( \sigma \) chemical potential of a noninteracting Hall fluid of uniform density \( \rho \) in the same field \( \mathbf{B} \). Of course \( \mathbf{j} \) does not apply to incompressible regions because \( \mu_0 \) or \( V_{xc} \) is discontinuous there. In fact, the discontinuity in \( \mu_0 + V_{\text{eff}} \) at an incompressible strip is precisely equal to the electron chemical potential gap \( \Delta \mu \) there. Because \( \mu_0^\sigma (\rho) \) becomes piecewise constant as \( T \to 0 \), the self-consistent potentials \( V_{\text{eff}}(\mathbf{r}) \) in each compressible region must become uniform in this limit. Therefore, there exists a complementarity in the low-temperature phase of the slowly confined quantum Hall fluid: In the incompressible regions, \( \rho \) is uniform and \( V_{\text{eff}} \) varies, whereas in the compressible regions, \( \rho \) varies and \( V_{\text{eff}} \) is uniform.
Because $\rho$ and $\nu$ are slowly varying, (5) may be written as

$$A_{xc} = \left(\frac{m^* c^2}{e^2}\right) \frac{1}{\rho} \nabla \times M_{xc},$$

(12)

where $M_{xc}$ is the exchange-correlation contribution to the orbital magnetization of a uniform 2D electron gas. The second edge current term in (3) may therefore be rewritten as $c \nabla \times M_{xc}$.

The largest contributions to the current density come from the first term in (5) and from (6), and these have opposite signs because $\nabla \rho$ and $\nabla V_{eff}$ are antiparallel. Furthermore, the perfect screening in the edge regions makes the bulk current vanish there, and the incompressibility of the bulk regions causes the density gradient and also $A_{xc}$ to vanish in those regions. Therefore, the current distribution generally consists of a series of strips or channels of distributed current, which follow the equipotentials of the self-consistent confining potential, and which alternate in direction. We shall show that the origin of this striking alternating pattern is the oscillations in the low-temperature magnetization of a 2D electron gas, in a fixed magnetic field, as a function of density. These oscillations are of course caused by the same competition between the energy of a Landau level and its degeneracy that leads to the deHaas–van Alphen effect.

II. Integrated equilibrium currents.—The integrated currents follow straightforwardly from (5) and (6). For example, the magnitude of the integrated current at the edge of the filled Landau level $n$, assuming there is no bulk current present, is

$$I_{edge} = (2n + 1) \frac{e \omega_c}{4\pi} + c \Delta M_{xc},$$

(13)

where $\Delta M_{xc}$ is the change in the $z$ component of $M_{xc}$ across the edge channel. Similarly, the integrated current in a bulk region of filling factor $\nu = 2\pi \ell^2 \rho$ has magnitude

$$I_{bulk} = \nu \frac{e}{h} \Delta \mu = c \Delta M,$$

(14)

where $\Delta \mu$ is the energy gap and $\Delta M$ the discontinuity in the magnetization there. In (14), we have used the fact that, according to the Maxwell relation $\frac{\partial M}{\partial \rho} = -\frac{\partial (\mu \cdot \nabla B)}{\partial \rho}$, the discontinuities in $\mu$ and $M$ are generally related by $\Delta M / \mu_\alpha^c \rho = 2 \Delta \mu / \hbar \omega_c$.

The validity of (13) appears to require that the channel not have any bulk current present from possible incompressible strips at fractional fillings. However, we shall prove now that (13) correctly gives the integrated current at the edge of a filled Landau level, including the bulk currents present from strips at fractional $\nu$.

The equilibrium density is stationary, so $\nabla \cdot j = 0$, and we may write the orbital current as

$$j = c \nabla \times M e_z,$$

(15)

where $M$ is the $z$ component of the local thermodynamic magnetization. Therefore, between any two regions in the 2D electron gas having slow density variation, the integrated equilibrium current is simply

$$I = -e \Delta M.$$  

(16)

This result is valid regardless of whether there are additional incompressible channels present, and regardless of whether the system is slowly varying in the intermediate region.

Let $M = M_0 + M_{xc}$, where $M_0$ is the kinetic contribution. The ground-state kinetic energy per unit area, ignoring spin, is $\mathcal{E}_0 = f_0(\nu) \left(\hbar \omega_c / 2\pi \ell^2\right)$, where $f_0(\nu) \equiv \frac{1}{2} \hat{\nu}^2 + \frac{1}{4} \left(\nu + \frac{1}{2}\right)(\nu - |\nu|)$, and where $|\nu|$ denotes the integer part of $\nu$. The kinetic chemical potential $\mu_0 \equiv (\partial \mathcal{E}_0 / \partial \rho)_B$ is discontinuous at all integral fillings by an amount $\hbar \omega_c$. At $T = 0$,

$$M_0 = \left(\nu - (2|\nu| + 1)(\nu - |\nu|)\right) \frac{\mu_0^c}{2\pi \ell^2},$$

(17)

where $\mu_0^c = (m/m^*) \mu_0$. Note the discontinuity in $M_0 / \mu_0^c \rho$ at integral filling factors, which is equal to twice the discontinuity in $\mu_0 / \hbar \omega_c$. The change in $M_0$ across the edge of a filled Landau level leads to the first term in (13), and the change in $M_{xc}$ leads to the second term. Therefore, (13) is correct regardless of whether there are fractional incompressible strips present in the edge region. We shall return to this point below.

The interaction energy per area of a quantum Hall fluid generally depends on $\rho$ and $B$ separately, but with the assumption of negligible Landau level mixing by the interactions, we may write $\mathcal{E}_{xc} = f_{xc}(\nu) \left(e^2 / 2\pi \kappa \ell^2\right)$, where $\kappa$ is the bulk dielectric constant. In terms of $f_{xc}$,

$$M_{xc} = \alpha \left(-3 f_{xc} + 2 \nu f'_{xc}\right) \frac{\mu_0^c}{2\pi \ell^2},$$

(18)

where $\alpha \equiv (e^2 / \kappa \hbar \omega_c)$ is the dimensionless Coulomb interaction strength.

We now calculate the integrated current at the edge of the lowest spin-polarized Landau level ($\nu = 0$), between $\nu = 1$ and $\nu = 0$, to order $\alpha$. We shall consider for convenience a long Hall bar oriented in the $y$ direction, with a confining potential $V(x)$ that varies in the $x$ direction only (except near the two ends of the Hall bar, which we avoid). The current is then directed along the Hall bar in the $y$ direction. Near $\nu = 0$, the ground-state energy is expected to be close to that of a Wigner crystal, so $f_{xc} \propto -\nu^2$ there. Particle-hole symmetry implies $f_{xc}(1 - \nu) = f_{xc}(\nu) + (1 - 2\nu) f_{xc}(1)$, where $f_{xc}(1) = -\left(\pi / 8\right) \hat{\nu}$. Hence, we find $M_{xc}(1^-) = -\alpha(\pi / 8)^2 \frac{\mu_0^c}{2\pi \ell^2}$, and therefore

$$I_{01} = -\left(1 + \alpha \frac{\pi}{8} \right) \frac{e \omega_c}{4\pi}.$$  

(19)
The discontinuity in the quasihole and quasiparticle energies defined in [5]. Expressions for the integrated currents at the edge of higher filled Landau levels, obtained by a similar analysis, are presented in Table I.

Next, we shall assume that there is an incompressible strip at an odd-denominator filling factor \( \nu_0 = \frac{1}{2} \) present in this edge channel. At filling factors very close to \( \nu_0 \), the ground state is expected to be a Laughlin state plus a Wigner crystal of fractionally charged quasiparticles or quasiholes. Let \( \epsilon_L \) be the interaction energy per electron in the Laughlin state at \( \nu_0 \), in units of \( e^2/\kappa \ell_0 \), and let \( \epsilon_{qp} \) and \( \epsilon_{qh} \) be the associated gross quasiparticle and quasihole energies in the same units. Then close to \( \nu_0 \),

\[
f_{xc}(\nu) = \left\{ \begin{array}{ll} \frac{\epsilon_L}{q} + q\epsilon_{qh}(\nu_0 - \nu) + \cdots & \text{for } \nu \leq \nu_0 \\ \frac{\epsilon_L}{q} + q\epsilon_{qp}(\nu - \nu_0) + \cdots & \text{for } \nu \geq \nu_0 \end{array} \right. \tag{20}
\]

For example, Morf and Halperin [3] have evaluated \( \epsilon_L \), \( \epsilon_{qp} \), and \( \epsilon_{qh} \) using trial wave functions at \( \nu_0 = \frac{1}{2} \); they find \( \epsilon_L = -0.410 \), \( \epsilon_{qp} = -0.132 \), and \( \epsilon_{qh} = 0.231 \). According to (20), \( \mu_{xc}(q\epsilon_L/\partial \rho)_\mu \) is discontinuous by an amount \( \Delta \mu = qE_{\text{gap}} \), where \( E_{\text{gap}} \equiv \epsilon_{qh} + \epsilon_{qp} \) is the energy required to create a single well-separated quasiparticle-quasihole pair. The interaction contribution to the orbital magnetization near \( \nu_0 \) is therefore

\[
M_{xc} = \left\{ \begin{array}{ll} -2\alpha \tilde{\epsilon}_{qh}(\mu_0^+/2\pi \ell_0^2) & \text{for } \nu = \nu_0^- \\ 2\alpha \tilde{\epsilon}_{qp}(\mu_0^-/2\pi \ell_0^2) & \text{for } \nu = \nu_0^+ \end{array} \right., \tag{21}
\]

where \( \tilde{\epsilon}_{qh} = \epsilon_{qh} + 3\epsilon_L/2q \) and \( \tilde{\epsilon}_{qp} = \epsilon_{qp} - 3\epsilon_L/2q \) are the proper quasiholes and quasiparticle energies defined in [3]. The discontinuity in \( M_{xc}/\mu_0^+ \rho \) at \( \nu_0 \) is equal to twice the discontinuity in \( \mu_{xc}/\rho \), as expected.

Let \( I_1 \) be the integrated current in the edge channel between \( \nu = 1^- \) and \( \nu = \nu_0^+ \), \( I_2 \) be the current in the incompressible strip at \( \nu = \nu_0 \), and \( I_3 \) be the current in the edge channel between \( \nu = \nu_0^- \) and \( \nu = 0 \). According to (21) or (22),

\[
I_1 = -\left[ 1 - \frac{1}{q} + \alpha \left( \frac{\pi}{8} + 2\tilde{\epsilon}_{qp} \right) \right] \frac{e\omega_c}{4\pi}. \tag{22}
\]

Similarly, \( I_2 \) is given by \( 2\alpha E_{\text{gap}}(e\omega_c/4\pi) \). Restoring units to \( E_{\text{gap}} \) leads to

\[
I_2 = \frac{e}{h} E_{\text{gap}} = \nu_0 \frac{e}{h} \Delta \mu, \tag{23}
\]

as in (21). The magnitude of the integrated equilibrium current in any incompressible strip is \( \nu(e/h) \Delta \mu \), where \( \Delta \mu \) is the electron chemical potential gap in the uniform quantum Hall fluid at filling factor \( \nu \). Finally,

\[
I_3 = -\left[ \frac{1}{q} + 2\alpha \tilde{\epsilon}_{qh} \right] \frac{e\omega_c}{4\pi}. \tag{24}
\]

Note the alternating signs of the integrated currents, and that their sum agrees with (15), even though there is now an incompressible strip at \( \nu_0 \). Furthermore, a measurement of the currents (22-24) would provide direct independent measurements of the fundamental quantities \( \tilde{\epsilon}_{ap} \), \( \epsilon_{qh} \), and, of course, \( E_{\text{gap}} \).

In conclusion, we have derived an expression for the low-temperature equilibrium current distribution in a confined quantum Hall fluid. The current distribution has two components, (9) and (10), which contribute exclusively to the compressible and incompressible regions respectively, and have opposite signs. The current distribution therefore consists of strips or channels of current, which alternate in direction. The integrated currents in the channels are also shown to be universal, and it is noted that their measurement would yield direct independent measurements of the proper quasiparticle and quasihole energies in the fractional quantum Hall states. Several experimental groups are exploring the possibility of directly imaging the current distribution in a 2D electron gas, and preliminary results have already been reported [6].

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