LUSTERNIK-SCHNIRELMANN CATEGORY AND SYSTOLIC CATEGORY OF LOW DIMENSIONAL MANIFOLDS

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Abstract. We show that the geometry of a Riemannian manifold \((M, \mathcal{G})\) is sensitive to the apparently purely homotopy-theoretic invariant of \(M\) known as the Lusternik-Schnirelmann category, denoted \(\text{cat}_{\text{LS}}(M)\). Here we introduce a Riemannian analogue of \(\text{cat}_{\text{LS}}(M)\), called the systolic category of \(M\). It is denoted \(\text{cat}_{\text{sys}}(M)\), and defined in terms of the existence of systolic inequalities satisfied by every metric \(\mathcal{G}\), as initiated by C. Loewner and later developed by M. Gromov. We compare the two categories. In all our examples, the inequality \(\text{cat}_{\text{sys}} M \leq \text{cat}_{\text{LS}} M\) is satisfied, which typically turns out to be an equality, e.g. in dimension 3. We show that a number of existing systolic inequalities can be reinterpreted as special cases of such equality, and that both categories are sensitive to Massey products. The comparison with the value of \(\text{cat}_{\text{LS}}(M)\) leads us to prove or conjecture new systolic inequalities on \(M\).

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In his paper [Ga71], T. Ganea demonstrated the usefulness of the study of numerical (i.e., $\mathbb{Z}$- or $\mathbb{N}$-valued) topological invariants, such as the Lusternik–Schnirelmann category $\text{cat}_{LS}$. For closed smooth manifolds $M$, one can consider additional numerical invariants, such as the minimal number of balls that cover the manifold, or the minimal number of critical points of a smooth function on $M$, etc. Here we introduce (a few versions of) a Riemannian analogue of $\text{cat}_{LS}(M)$, called the systolic category of $M$, as in Definition 2.2. It is defined in terms of the existence of “curvature-free” systolic inequalities satisfied by every metric $G$, as initiated by C. Loewner [Pu52] and developed later by M. Gromov [Gr83].

Given that a dozen or so numerical invariants have by now been studied [CLOT03], one could legitimately ask, why define another one? We feel that systolic category is the only numerical invariant possessing a differential geometric flavor, thus adding some Riemannian spice to an increasingly homotopy-theoretic field. Furthermore, systolic category provides an intuitive point of entry for differential geometers into the field of numerical invariants.

It is natural to compare $\text{cat}_{sys}$ with the other numerical invariants mentioned above. Here we start this program and show that $\text{cat}_{sys}$ is sensitive to $\text{cat}_{LS}$, which is a purely homotopic invariant. We verify the equality $\text{cat}_{sys} = \text{cat}_{LS}$ in dimension 3 based on the results of Gómez-Larrañaga and Gonzáles-Acuña [GG92], see also [OR01]. We show that a number of existing systolic inequalities can be reinterpreted as special cases of such equality, cf. (4.3), (2.4), and (9.5).

We state at the outset that, while a precise Definition 2.2 appears below, it is not entirely clear what the “right” definition of systolic category should be, cf. (2.4), (12.2). We present what we feel
is compelling evidence suggesting the existence of a coherent notion of this sort.

In higher dimensions, no immediate (direct) connections of the two categories are as yet available, but we observe that they have parallel behavior and common points of sensitivity. More specifically, we make the following observations:

- the two categories coincide in dimensions 2 and 3;
- the two categories attain the maximal value (i.e. dimension) simultaneously;
- both categories admit a lower bound in terms of the real cup-length;
- both categories are sensitive to Massey products, cf. section 11.

The comparison with the value of $\text{cat}_{\text{LS}}(M)$ leads us to prove or conjecture new systolic inequalities on $M$, e.g. (7.3), (7.7), (8.2), (9.6). Conversely, the comparison with the value of $\text{cat}_{\text{sys}}$ leads to open questions about $\text{cat}_{\text{LS}}$, cf. (7.6).

All manifolds are assumed to be closed, connected and smooth unless explicitly mentioned otherwise. To the extent that our paper aims to address both a topological and a geometric audience, we attempt to give some indication of proof of pertinent results that may be more familiar to one audience than the other.

1. Systoles

Let $M$ be a (smooth) manifold equipped with a Riemannian metric $G$. We will now define the systolic invariants $\text{sys}_k(M, G)$. In the sequel, we abbreviate $\text{sys}_k(M, G)$ as $\text{sys}_k(M)$, $\text{sys}_k(G)$ or even $\text{sys}_k$, depending on the context.

**Definition 1.1.** The homotopy 1-systole, denoted $\text{sys}_{\pi_1}(M, G)$, is the least length of a non-contractible loop in $M$. The homology 1-systole, denoted $\text{sys}_{h_1}(M, G)$, is defined in a similar way, in terms of loops which are not zero-homologous. Clearly, $\text{sys}_{\pi_1} \leq \text{sys}_{h_1}$. Now let $k \in \mathbb{N}$. Higher homology $k$-systoles $\text{sys}_{h_k}$, with coefficients over a ring $A = \mathbb{Z}$ or $\mathbb{Z}_2$, are defined similarly to $\text{sys}_{h_1}$, as the infimum of $k$-areas of $k$-cycles, with coefficients in $A$, which are not zero-homologous. Note that we adopt the usual convention, convenient for our purposes, that the infimum over an empty set is infinity.

When $k = n$ is the dimension, $\text{sys}_{h_n}(M, G)$ is equal to the total volume $\text{vol}_n(M, G)$ of a compact Riemannian $n$-manifold $(M, G)$. More detailed definitions appear in the survey [CrK03]. We do not consider
higher “homotopy” systoles. All systolic notions can be defined similarly on polyhedra, as well [Gr96, Ba02].

Definition 1.2 (cf. [Fe69, BK03]). Given a class \( \alpha \in H_k(M; \mathbb{Z}) \) of infinite order, we define the stable norm \( \|\alpha\|_r \) by setting

\[
\|\alpha\|_r = \lim_{m \to \infty} m^{-1} \inf_{\alpha(m)} \text{vol}_k(\alpha(m)),
\]

where \( \alpha_r \) denotes the image of \( \alpha \) in real homology, while \( \alpha(m) \) runs over all Lipschitz cycles with integral coefficients representing \( m\alpha \). The stable homology \( k \)-systole, denoted \( \text{stsys}_k(G) \), is defined by minimizing the stable norm \( \|\alpha\|_r \) over all integral \( k \)-homology classes \( \alpha \) of infinite order.

Remark 1.3. All the systolic invariants defined in this section are positive for smooth compact manifolds. For the homology and homotopy 1-systoles, this follows by comparison with the injectivity radius. For stable \( k \)-systoles the positivity follows by a calibration argument. For the \( k \)-systoles (\( k \geq 2 \)), since torsion classes are involved, one cannot use differential forms. An argument proving positivity was outlined in [BCIK04, Section 2].

The stable homology 1-systole \( \text{stsys}_1 \) is easy to compute for orientable surfaces, because it coincides with the ordinary homology 1-systole. In fact this is true for the codimension one systole of any orientable manifold [Fe74, Wh83]. However, the situation is entirely different when the codimension is bigger than 1. For example, every 3-manifold \( M \) with \( b_1(M) \geq 1 \) is (1,2)-systolically free [BabK98], meaning that there exists a sequence of metrics \( G_j \) of volume going to zero as \( j \to \infty \), and yet the product of systoles is big, i.e.

\[
\text{sysh}_1(G_j) \text{sysh}_2(G_j) > 1. \tag{1.1}
\]

Meanwhile, the freedom phenomenon disappears when we replace \( \text{sysh}_1 \) by the stable systole \( \text{stsys}_1 \), cf. inequality (2.3).

2. Systolic categories

Let \( (M^n, G) \) be a Riemannian manifold. Recall that, in our convention, the systolic invariants are infinite when defined over an empty set of loops or cycles.

Definition 2.1. Given \( k \in \mathbb{N}, k > 1 \) we set

\[
\text{sys}_k(G) = \min \{ \text{sysh}_k(G, \mathbb{Z}), \text{sysh}_k(G, \mathbb{Z}_2), \text{stsys}_k(G) \}.
\]

Furthermore, we define

\[
\text{sys}_1(G) = \min \{ \text{sys}^1_1(G), \text{sysh}_1(G, \mathbb{Z}), \text{sysh}_1(G, \mathbb{Z}_2), \text{stsys}_1(G) \}.
\]
Let $d \geq 2$ be an integer. Consider a partition

$$n = k_1 + \ldots + k_d,$$

(2.1)

where $k_i \geq 1$ for all $i = 1, \ldots, d$. We will consider scale-invariant inequalities “of length $d$” of the following type:

$$\text{sys}_{k_1}(\mathcal{G}) \text{sys}_{k_2}(\mathcal{G}) \ldots \text{sys}_{k_d}(\mathcal{G}) \leq C(M) \text{vol}_n(\mathcal{G}),$$

(2.2)

satisfied by all metrics $\mathcal{G}$ on $M$, where the constant $C(M)$ depends only on the topological type of $M$ but not on the metric $\mathcal{G}$. Here $\text{sys}_k$ denotes the minimum of all non-vanishing systolic invariants in dimension $k$ as defined above.

**Definition 2.2.** Systolic category of $M$, denoted $\text{cat}_{\text{sys}}(M)$, is the largest integer $d$ such that there exists a partition (2.1) with

$$\prod_{i=1}^{d} \text{sys}_{k_i}(M, \mathcal{G}) \leq C(M) \text{vol}_n(M, \mathcal{G})$$

for all metrics $\mathcal{G}$ on $M$.

In particular, $\text{cat}_{\text{sys}} M \leq \dim M$.

**Example 2.3.** Every orientable $n$-manifold $N$ with positive Betti number $b_k(N) \geq 1$ satisfies Gromov’s stable systolic inequality

$$\text{stsys}_k(N) \text{stsys}_{n-k}(N) < C(b_k(N)) \text{vol}_n(N),$$

(2.3)

In our notation, inequality (2.3) implies the following bound for the stable systolic category:

$$\text{cat}_{\text{sys}}(N) \geq 2 \text{ if } b_k(N) \geq 1 \text{ for some } 1 \leq k \leq n - 1,$$

(2.4)

see (9.5) for a generalisation in terms of real cuplength. The constant in (2.3) depends on the Betti number, and was studied in [Heb86, BK03]. The constant can be optimal in certain cases, such as Gromov’s inequality for the stable 2-systole of complex projective spaces [9.4], as well as for $k = 1$ [BK03, BK04].

In general, systolic inequalities involving any higher $k$-systole are notoriously hard to come by, in view of the widespread phenomenon of systolic freedom, see ([1.1], Gr99 Systolic reminiscences, p. 271), as well as [Gr99 Appendix D], [Ba02, Ka02].

**Example 2.4.** It is still unknown whether $\mathbb{R}P^3$ admits a $(1, 2)$-systolic inequality, or whether on the contrary it is $(1, 2)$-systolically free. Here, of course, the systoles are over $\mathbb{Z}_2$. The analogous question for the
3-manifold $S^1 \times S^2$ was resolved, in favor of freedom, by M. Freedman [Fr99]. The case of $\mathbb{R}P^3$ is not interesting from the vantage point of category, since it is essential and therefore

$$\text{cat}_{LS}(\mathbb{R}P^3) = \text{cat}_{sys}(\mathbb{R}P^3) = 3$$

and not 2, see Theorem 4.1. Thus, the question of the existence of a $(1,2)$-systolic inequality does not affect the value of the categories. On the other hand, the case of $\mathbb{R}P^2 \times S^2$ is a simple case that’s completely open, cf. Example 7.4. Smale’s spin rational homology 5-sphere $M_k$ (see [Sm62]) of Example 8.4 satisfies $\text{cat}_{LS}(M_k) = 2$ by Theorem 5.1 but its systolic category is inaccessible with the techniques available.

We express the hope that by exhibiting a connection with Lusternik-Schnirelmann category as well as some motivated conjectures, we will stimulate further research in the direction of new systolic inequalities. We therefore conclude with the following questions.

**Question 2.5.** To what extent can Gromov’s filling techniques [Gr83] be generalized in the direction of proving systolic inequalities involving higher $k$-systoles?

**Question 2.6.** It is clear that $\text{cat}_{sys}(M \times N) \leq \text{cat}_{sys}(M) + \text{cat}_{sys}(N)$. How far can the inequality be from equality? Are there examples with $\text{cat}_{sys}(M \times S^k) = \text{cat}_{sys} M$?

### 3. Categories agree in dimension 2

Every compact surface $M$ (orientable or not) with infinite fundamental group satisfies Gromov’s inequality [Gr83]

$$\text{sys} \pi_1(M)^2 \leq \frac{4}{3} \text{area}(M).$$

For the projective plane $\mathbb{R}P^2$, we have Pu’s optimal inequality [Pu52]

$$\left(\text{sys} \pi_1(\mathbb{R}P^2)\right)^2 \leq \frac{\pi}{2} \text{area}(\mathbb{R}P^2). \quad (3.1)$$

Tighter estimates are available as Euler characteristic becomes unbounded [Gr83, KS04], but they are irrelevant here. The conclusion is that

$$\text{cat}_{LS} M = \text{cat}_{sys} M \quad (3.2)$$

for all closed surfaces $M$. Namely, we have $\text{cat}_{LS} M = 1$ for $M = S^2$ and $\text{cat}_{LS} M = 2$ for all $M \neq S^2$. However, one cannot expect (3.2) to hold in all dimensions, cf. Example 8.5.
4. Essential manifolds and detecting elements

A closed $n$-manifold $M$ is called essential if there exist a group $\pi$, a (possibly twisted) coefficient system $A$ on the Eilenberg–Mac Lane space $K(\pi, 1)$, and a map $f : M \to K(\pi, 1)$, such that the homomorphism
\[ f^* : H^n(K(\pi, 1); A) \to H^n(M; f^*(A)) \]
is non-zero. Here without loss of generality we can assume that $\pi = \pi_1(M)$. When we want to fix the twisted system $A$, we say that $M$ is $A$-essential. In the language of [OR01, Ru99], $M$ is $A$-essential if and only if the fundamental cohomology class in $H^n(M; A)$ is a detecting element of category weight $n$. Clearly, $M$ is essential if it admits a map to a space $K(\pi, 1)$ such that the induced homomorphism in $n$-dimensional homology sends the $A$-fundamental class to a nonzero class.

Theorem 4.1. Let $M$ be a manifold of dimension $n$. The following three conditions are equivalent:

1. the manifold $M$ is essential;
2. we have $\text{cat}_{LS} M = n = \dim M$;
3. if a map $f : M \to K(\pi_1(M), 1)$ induces an isomorphism of fundamental groups, then $f(M)$ is not contained in the $(n-1)$-skeleton of $K(\pi_1(M), 1)$.

Proof. The equivalence of (1) and (2) is proved in [Ber76], see also [CLOT03, Theorem 2.51]. In fact, I. Berstein [Ber76] proved that $\text{cat}_{LS} M = \dim M$ if and only if $M$ is essential in a particular sheaf, namely the tensor product $I \otimes \ldots \otimes I$ of the augmentation ideal $I$ of the group ring of the fundamental group. In greater detail, one has an element $a \in H^1(M; I)$, the characteristic class of the universal cover, and $\text{cat}_{LS} M = \dim M = n$ if and only if $a^n \neq 0$.

The implication (3) $\Rightarrow$ (1) results from an obstruction theoretic argument, cf. [Ba93, Lemma 8.5], while the implication (1) $\Rightarrow$ (3) is obvious. □

Theorem 4.2 ([Gr83]). Every essential Riemannian manifold $M$ satisfies the inequality
\[ \text{sys}_{\pi_1}(M)^n \leq C_n \text{vol}_n(M). \] (4.1)

□

Here the constant $C_n$ is on the order of $n^{2n^2}$. In other words, the quotient
\[ \frac{\text{vol}_n}{(\text{sys}_{\pi_1})^n} > 0 \] (4.2)
is bounded away from zero. In our terminology, this implies that homotopy systolic category is the maximal possible:

$$\text{cat}_{\text{sys}}(M) = n$$

for essential $M$ of dimension $n$. (4.3)

Better bounds are available as the topological complexity of $M$ increases [KS04], but they are irrelevant here for the moment.

Remark 4.3. In the appendix to [Gr83], Gromov proved an inequality of type (4.1) for Riemannian $n$-dimensional polyhedra that satisfy condition (3) of Theorem 4.1. The converse was proved in [Ba93]. Namely, systolic category is less than $n$ for inessential $n$-manifolds, cf. Section 5. Combined with Theorem 4.1, this yields the following theorem, in harmony with equality (3.2).

Theorem 4.4. Let $M$ be a $n$-manifold. Then $\text{cat}_{\text{sys}}(M) = n$ if and only if $\text{cat}_{\text{LS}}(M) = n$. □

5. INESSENTIAL MANIFOLDS AND PULLBACK METRICS

We prove a compression result valid for any of our systolic notions, including the homotopy 1-systole.

Theorem 5.1 (Compression theorem). Let $n = k_1 + \ldots + k_d$. Let $f : M^n \to K$ be map to a polyhedron such that

1. $f(M) \subset K^{(n-1)}$;
2. $\text{sys}_{k_i}(K) > 0$ for each $k_i < n$, $i = 1, \ldots, d$;
3. the induced homomorphism $f_*$ is injective in $k_i$-dimensional homology for each $k_i < n$, $i = 1, \ldots, d$ (or in $\pi_1$ if we are working with $\text{sys}_{\pi_1}$).

Then the inequality

$$\prod_i \text{sys}_{k_i} \leq C \text{vol}_n$$

is violated, for any $C < \infty$, by a suitable metric on $M$.

Proof. Without loss of generality we can assume that $M$ and $K$ are polyhedra and $f$ is linear on each simplex, and on each top dimensional simplex $f$ is a projection to a face of positive codimension. Choose a fixed piecewise linear metric $G_{PL}$ on $K^{(n-1)}$. This amounts to a choice of a positive quadratic form on each simplex, such that the forms of neighboring simplices agree on their common face. Then the pullback $f^*(G_{PL})$ is a positive quadratic form on $M$ which is of rank at most $(n-1)$ at every point of $M$. Nonetheless, volume of a Lipschitz simplex can be defined as usual with respect to this form. Thus
its \( k \)-systole can be defined. Since \( f \) induces a monomorphism \( f_* \) by hypothesis, the systole is positive by construction:

\[
\text{sys}_k (M, f^*(\mathcal{G}_{PL})) \geq \text{sys}_k (K^{(n-1)}, \mathcal{G}_{PL}) > 0.
\]

Then the inequality (5.1) is certainly violated, to the extent that the left hand side is positive, whereas the right hand side vanishes. The “metric” \( f^*(\mathcal{G}_{PL}) \) has two shortcomings. First, it is only defined on the full tangent space at \( x \in M \) when \( x \notin M^{(n-1)} \), i.e. outside the codimension 1 skeleton of \( M \). Furthermore, the positive quadratic form is not definite, even in the interior of a top dimensional cell. Both shortcomings are overcome by the metric

\[
\phi_{M^{(n-1)}} f^*(\mathcal{G}_{PL}) + (1 - \phi_{M^{(n-1)}}) \mathcal{G}_0,
\]

where \( \mathcal{G}_0 \) is a fixed smooth “background” metric on \( M \), while \( \phi_{M^{(n-1)}} \) is a function of “bump” type, which equals 1 outside of an \( \varepsilon \)-neighborhood of the codimension 1 skeleton, and vanishes in an \( \varepsilon/2 \)-neighborhood. The volume becomes arbitrarily small when \( \varepsilon \) tends to zero, thus violating (5.1), cf. \[Ba93\], \[BabK98, section 6\].

The following corollary, which is a converse of Theorem 4.2, is due to I. Babenko \[Ba93\].

**Corollary 5.2.** If \( M \) is not essential, then it fails to satisfy Gromov’s inequality (4.1) for the homotopy 1-systole.

**Proof.** Indeed, if \( M^n \) is inessential then, by Theorem 4.1, there exists a map \( f : M \to K = K(\pi_1(M), 1) \) that induces an isomorphism of fundamental groups and such that \( f(M) \subset K^{(n-1)} \), and we apply our compression theorem to the homotopy 1-systole. \[\square\]

6. **Manifolds of dimension 3**

The main result of the paper \[GG92\], cf. also \[OR01\], is the following theorem.

**Theorem 6.1.** Let \( M \) be a 3-manifold. Then \( \text{cat}_{1LS} M = 1 \) if \( M \) is simply connected, \( \text{cat}_{1LS} M = 2 \) if \( \pi_1(M) \) is a free non-trivial group, and \( \text{cat}_{1LS} M = 3 \) otherwise. \[\square\]

The proof, in the orientable case, goes roughly as follows. Decompose a 3-manifold \( M \) as a connected sum. If \( \pi_1(M) \) is not free, then at least one of the summands has finite fundamental group or is aspherical \[Hem76\]. Let \( X \) be such a summand. If \( X \) is aspherical, then \( M \) is essential. If \( \pi_1(X) = \pi \) is finite, we have \( \pi_2(X) = 0 \), since \( X \) is covered by the homotopy 3-sphere. Hence we have the Hopf exact sequence

\[
\pi_3(X) \to H_3(X) \to H_3(\pi) \to 0,
\]
and the degree of the first (Hurewicz) map is equal to the order of fundamental group.

In the case of free fundamental group, the theorem follows from the fact that the manifold is homotopy equivalent to a connected sum of 2-sphere bundles over the circle.

**Corollary 6.2.** Let \( M \) be a 3-manifold. Then:

(i) if \( M \) is simply connected then \( \text{cat}_{LS} M = \text{cat}_{sys} M = 1 \);
(ii) if \( \pi_1(M) \) is not a free group then \( \text{cat}_{LS} M = \text{cat}_{sys} M = 3 \);
(iii) if \( M \) is orientable and \( \pi_1(M) \) is a free non-trivial group, then \( \text{cat}_{LS} M = \text{cat}_{sys} M = 2 \).

For non-orientable 3-manifold with free fundamental group we only prove that \( 1 \leq \text{cat}_{sys} M \leq \text{cat}_{LS} M = 2 \). In fact, it turns out that in this case we also have \( \text{cat}_{sys} M = 2 \) [KR05].

**Proof.** (i) is obvious.

(ii) The equality \( \text{cat}_{LS} M = 3 \) follows from Theorem 6.1, and we apply Theorem 4.4.

(iii) If the fundamental group of \( M \) is free, then \( \text{cat}_{LS} M = 2 \) by Theorem 6.1. Since \( b_1(M) \geq 1 \), we have \( \text{cat}_{sys} M \geq 2 \) by (2.3), and \( \text{cat}_{sys} M \leq 2 \) by Theorem 4.4. \( \square \)

**Remark 6.3.** Let \( f : M \to K(\pi_1(M), 1) \) be a map that induces an isomorphism of fundamental groups. An interesting role in dimension 3 is played by the lowest dimension \( d_{\min} \) of the skeleton of \( K(\pi_1(M), 1) \) that \( f \) can be deformed into. For 3-manifolds, this dimension determines both categories. Namely, if \( d_{\min} = 1 \) then both categories are 2, whereas if \( d_{\min} = 3 \), both categories are also 2. Here \( d_{\min} \) cannot be equal to 2. (Indeed, if \( f \) can be deformed into the 2-skeleton then \( M \) is inessential, and so \( \text{cat}_{LS} M \leq 2 \), and so \( \pi_1(M) \) is free, and so \( K(\pi_1(M), 1) \) is homotopy equivalent to a wedge of circles.) The case of maximal \( d_{\min} = n = \text{dim}(M) \) also characterizes the case when both categories equal \( n \), by Theorem 4.4. Could one say something in general when this dimension is smaller than \( n \)?

**7. Manifolds with category smaller than dimension**

Assume \( b_1(M) = b \geq 1 \). An optimal systolic inequality (7.5) was studied in [IK04], cf. [BCIK04, BCIK05, KL04]. Denote by \( J_1(M) \) the \( b \)-dimensional (Jacobi) torus of \( M \). Consider the homomorphism \( \varphi : \pi_1(M) \to H_1(M; \mathbb{Z}) \to \mathbb{Z}^b \), where the first homomorphism is the abelianisation and the second one is the quotient homomorphism over the torsion subgroup, and let

\[
\mathcal{A} = \mathcal{A}_M : M \to J_1(M) \tag{7.1}
\]
be a map that induces the homomorphism \( \varphi \) on fundamental groups (the so-called Abel-Jacobi map, cf. [Li69]).

We have the pull-back diagram

\[
\begin{array}{ccc}
\overline{M} & \xrightarrow{\pi} & \mathbb{R}^b \\
\downarrow & & \downarrow^{p} \\
M & \xrightarrow{A} & J_1(M)
\end{array}
\] (7.2)

where the map \( p \) is the universal cover. A typical fiber \( \overline{F_M} \) of \( \overline{A} \) projects diffeomorphically to a typical fiber \( F_M \) of \( A \). Denote by \( \overline{[F_M]} \in H_{n-b}(\overline{M}; G) \) the homology class of \( \overline{F_M} \) in \( \overline{M} \) where \( G = \mathbb{Z} \) if \( M \) is orientable and \( G = \mathbb{Z}_2 \) if \( M \) is non-orientable).

Let \( G \) be a metric on \( M \). Following Gromov [Gr83], denote by \( \deg(A) \) the least \( G \)-area of an \((n-b)\)-cycle representing the homology class \( \overline{[F_M]} \).

Note that this notion of degree is not a topological invariant, unless of course \( n = b \).

**Theorem 7.1.** Let \( M \) be an \( n \)-manifold with \( b_1(M) = b \). Then the following lower bound for the systolic category holds:

\[
\text{if } \overline{[F_M]} \neq 0 \text{ then } \text{cat}_{\text{sys}}(M) \geq b + 1,
\] (7.3)

where we allow a systole of \((\overline{M}, \overline{G})\) to participate in the definition of systolic category (2.2).

**Proof.** The Jacobi torus is equipped with the stable norm \( \| \| \) associated with \( G \). Equip the Jacobi torus with the Euclidean norm \( \| \|_E \) defined by the ellipsoid of largest volume inscribed in the unit ball of the stable norm \( \| \| \). It was proved in [IK04] that the homotopy class of \( A \) contains a map

\[
f : (M, G) \to (J_1(M), \| \|_E),
\] (7.4)

where the map \( f \) may not be distance decreasing, but is nonetheless non-increasing on \( b \)-dimensional areas. It follows from the coarea formula that \( M \) satisfies an optimal inequality

\[
\text{stsys}_1(G)^b \deg(A) \leq (\gamma_b)^{b/2} \text{vol}_n(G),
\] (7.5)

where \( \gamma_b \) is the Hermite constant, i.e. the maximum, over all lattices of unit covolume in \( \mathbb{R}^n \), of the least square-length of a nonzero vector in the lattice. By hypothesis, the class of the fiber is nontrivial, and therefore \( \text{sysh}_{n-b}(\overline{M}, \overline{G}) \leq \deg(A) \). It follows that its systolic category is at least \( b + 1 \). \( \square \)
Remark 7.2. If $[F_M] \neq 0$, then, by Poincaré duality, we have
\[ \text{cat}_{LS} M \geq \text{cup-length}(M) \geq b + 1, \]
\[ \text{cf. (9.5)}, \]
in harmony with (3.2). Here we must use Poincaré duality with $\mathbb{Z}_2$-coefficients in the non-orientable case. Notice, however, that the condition $[F_M] \neq 0$ is more general than $[F_M] \neq 0$, and in particular can be satisfied even if the cup product is trivial on $M$, as in the case of the NIL geometry on a compact 3-manifold.

Question 7.3. Does a manifold with $[F_M] \neq 0$ satisfy
\[ \text{cat}_{LS}(M) \geq b_1(M) + 1 \quad ? \] (7.6)

Example 7.4. Let $\Sigma$ be a surface different from $S^2$. Let $M = \Sigma \times S^n$. Then the cup-length of $M$ is 3 for a suitable choice of coefficients. Hence Lusternik-Schnirelmann category is 3, and one can ask if the systolic category of $\Sigma \times S^n$ is equal 3. This is true in the orientable case since real cup-length is a lower bound for systolic category (9.5). Now consider $M = \mathbb{RP}^2 \times S^n$. We conjecture that $M$ satisfies an inequality of type
\[ \text{sys}_1 \text{sys}_1 \text{sys}_n < C(M) \text{vol}_{n+2}, \] (7.7)
where perhaps $\text{sys}_n$ should be replaced by either the stable systole, or systole with $\mathbb{Z}_2$ coefficients. See also (9.10). How does this fit in with the results of M. Freedman [Fr99] ?

8. Category of simply connected manifolds

We first recall the following general result.

Theorem 8.1 ([CLOT03, Theorem 1.50]). For every $(k - 1)$-connected CW-space $X$ we have $\text{cat}_{LS} X \leq \dim X/k$. \[ \square \]

Corollary 8.2. If $M^n, n = 4, 5$ is a simply connected manifold which is not homotopy equivalent to the sphere $S^n$, then $\text{cat}_{LS} M = 2$.

Proof. Since $M$ is not a homotopy sphere, then $H^2(M; \mathbb{Z}_p) \neq 0$ for some prime $p$. Indeed, otherwise, by Poincaré duality, $H^{n-2}(M; \mathbb{Z}_p) = 0$ for all prime $p$. In other words, $H^i(M; \mathbb{Z}_p) = 0$ for all primes $p$ and all $i \neq 0, n$. By the Universal Coefficient Theorem, we have $H_i(M) = 0$ for $i \neq 0, n$, and so $M$ is a homotopy sphere because it is simply connected.

Let $x \in H^2(M; \mathbb{Z}_p) \setminus \{0\}$. By Poincaré duality, there exists an element $y \in H^{n-2}(M; \mathbb{Z}_p)$ such that $xy \neq 0$. Now the cup-length estimate implies $\text{cat}_{LS} M \geq 2$, and the claim follows from Theorem 8.1. \[ \square \]
Example 8.3 (Four-manifolds with category 2). If $M^4$ is simply connected and is not homotopy equivalent to $S^4$, then the second Betti number is positive, and $M$ satisfies the stable systolic inequality in middle dimension:

$$\text{stsyst}_2(M)^2 \leq C(b_2(M)) \text{vol}_4(M), \quad (8.1)$$

*cf.* inequalities (2.3) and (9.4), and hence $\text{catsyst}(M) = 2$. On the other hand, $\text{cat}_{LS} M = 2$ by Corollary 8.2 in harmony with equally (3.2).

Example 8.4. By [Sm62], a simply connected spin 5-manifold which is a rational homology sphere is a connected sum of manifolds $M_k$, $k \geq 2$, where

$$H_2(M_k) = \mathbb{Z}_k + \mathbb{Z}_k.$$  

By Corollary 8.2, all these manifolds have Lusternik-Schnirelmann category equal to 2. It is unknown whether any systolic inequalities are satisfied by $M_k$. Based on the value of its $\mathbb{Z}_k$-cup-length, we could conjecture the existence of a systolic inequality

$$\text{sys}_2 \text{sys}_3 \leq C(M_k) \text{vol}_5, \quad (8.2)$$

where the systoles are calculated over the cycles with $\mathbb{Z}_k$ coefficients which are not zero-homologous, *cf.* (9.6). An interesting family of metrics on $M_k$ was studied in [PP03, Theorem 8.2]. Does it satisfy inequality (8.2)?

Example 8.5. There are examples of $S^2$-bundles $M$ over spheres satisfying $\text{cat}_{LS}(M) = 3$ [Si78, §3]. Such a bundle $M^{16} \to S^{14}$ can be induced by an element of $\pi_{14}(S^4)$ from the bundle $\mathbb{CP}^3 \to S^4$ obtained as a circle quotient of the quaternionic Hopf bundle $S^7 \to S^4$, *cf.* [Iw02]. Let $u \in H^2(M^{16}, \mathbb{Z}) = \mathbb{Z}$ be a generator, and let

$$f : M \to K(\mathbb{Z}, 2) = \mathbb{CP}^2 \infty$$

classify $u$. Because of the construction of $M^{16}$, the map $f$ factors through a skeleton $\mathbb{CP}^3 \subset K(\mathbb{Z}, 2)$.

Hence $M^{16}$ admits a positive quadratic form of zero “volume” (as well as zero 14-systole), but positive 2-systole, *cf.* Section 6. By compression theorem 5.1 there is no inequality of type $\text{sys}_2^2 \leq C \text{vol}_{16}$. By Poincaré duality and (2.4), the manifold $M^{16}$ has stable systolic category 2. Since there is no torsion in homology, no higher value could be expected for systolic category, in contrast with (8.2).
9. Gromov’s calculation for stable systoles

In this section, we present Gromov’s proof of the optimal stable systolic inequality (9.3) for complex projective space \(\mathbb{CP}^n\), cf. [Gr99, Theorem 4.36], based on the cup product decomposition of its fundamental class, cf. (9.6). In Section 11 we will adapt this calculation to a situation where cup product is trivial. Following Gromov’s notation, let \(\alpha \in H_2(\mathbb{CP}^n; \mathbb{Z})\) be a generator in homology, and \(\omega \in H^2(\mathbb{CP}^n; \mathbb{Z})\) the dual generator in cohomology. Then \(\omega^n\) is a generator of \(H^{2n}(\mathbb{CP}^n, \mathbb{Z})\). Let \(\eta \in \omega\) be a closed differential 2-form. Then

\[
1 = \int_{\mathbb{CP}^n} \eta^n. \quad (9.1)
\]

Now let \(\mathcal{G}\) be a metric on \(\mathbb{CP}^n\). Then

\[
1 \leq n! (\|\eta\|_\infty)^n \text{vol}_{2n}(\mathcal{G}), \quad (9.2)
\]

where \(\|\|_\infty\) is the comass norm on forms (see [Gr99] for a discussion of the constant). Here the comass norm of a differential \(k\)-form is the supremum of pointwise comass norms. The pointwise comass norm for decomposable linear \(k\)-forms coincides with the natural Euclidean norm on \(k\)-forms associated with \(\mathcal{G}\). In general it can be defined by evaluating on \(k\)-tuples of unit vectors and taking the maximal value, cf. [Fe69, BK03]. Therefore

\[
1 \leq n! (\|\omega\|^*\|^\alpha\|^*_{\|\|}) \text{vol}_{2n}(\mathcal{G}), \quad (9.3)
\]

where \(\|\|_{\text{vol}}\) is the comass norm in cohomology, obtained by minimizing the norm \(\|\|_{\infty}\) over all closed forms representing the given cohomology class. Denote by \(\|\|\) the stable norm in homology. Recall that the normed lattices \((H_2(M, \mathbb{Z}), \|\|)\) and \((H^2(M, \mathbb{Z}), \|\|^*)\) are dual to each other [Fe69, BK03]. Therefore

\[
\|\alpha\| = \frac{1}{\|\omega\|^*},
\]

and hence

\[
\text{stsys}_2(\mathcal{G})^n = \|\alpha\|^n \leq n! \text{vol}_{2n}(\mathcal{G}). \quad (9.4)
\]

Moreover the inequality so obtained is sharp (equality is attained by the 2-point homogeneous Fubini-Study metric). In our terminology, this calculation can be summarized by writing

\[
\text{cat}_{\text{sys}}(\mathbb{CP}^n) = n,
\]

which agrees with \(\text{cat}_{\text{L},S}(\mathbb{CP}^n)\), in harmony with equality (3.2).

Gromov also proved a stable systolic inequality associated with any decomposition of the real fundamental cohomology class of \(M\) as a cup
product, cf. [Gr83, BK03]. This can be restated in terms of category as follows:
\[
\text{cat}_{\text{sys}}(M) \geq \text{cup-length}_R(M).
\] (9.5)
Pu’s inequality (3.1) for $\mathbb{R}P^2$ as well as its generalisations by Gromov lend support to the following conjecture:
\[
\text{cat}_{\text{sys}}(M) \geq \text{cup-length}(M),
\] (9.6)
(for the appropriate choice of systoles), which would be implied by a corresponding equality in (3.2). Interesting test cases are (7.7), (8.2).

10. Massey products via DGA algebras

Let $(A, d)$ be a differential graded associative (=DGA) algebra with a differential $d$ of degree 1, and let $H = \ker d/\text{im} d$ be the homology algebra of $(A, d)$. We assume that the induced product in the graded algebra $H$ is skew-commutative. Given (homogeneous) $u, v, w \in H$ with $uv = 0 = vw$, the triple Massey product
\[
\langle u, v, w \rangle = \langle u, v, w \rangle_A \subset H
\]
is defined as follows. Let $a, b, c$ be elements of $A$ whose homology classes are $u, v, w$ respectively. Then $dx = ab, dy = bc$ for suitable $x, y \in A$, and $\langle u, v, w \rangle$ is the set of elements of the form
\[
x c - (-1)^{|u|} a y,
\]
see [Ma69, RT00] for more details. The set $\langle u, v, w \rangle$ is a coset with respect to the so-called indeterminacy subgroup $\text{Indet} \subset H^{\geq |u|+|v|+|w|} - 1$,
\[
\text{Indet} = u H^{\geq |v|+|w|} - 1 + H^{\geq |u|+|v|} - 1 w.
\]

Proposition 10.1. Let $X$ be a finite CW-space. Let $C^\ast(X; \mathbb{Z})$ and $C^\ast(X; \mathbb{R})$ be the singular cochain complexes with coefficients, respectively, in $\mathbb{Z}$ and $\mathbb{R}$. We equip these chain complexes with the Alexander–Whitey product. Given $u, v, w \in H^\ast(X; \mathbb{Z})$, suppose that the Massey product $\langle u, v, w \rangle \subset H^\ast(M; \mathbb{Z})$ is defined and let $\text{Indet}$ be its indeterminacy subgroup. Then the Massey product
\[
\langle u_R, v_R, w_R \rangle \subset H^\ast(M; \mathbb{R})
\]
is defined, and we have
\[
\langle u_R, v_R, w_R \rangle = \langle u, v, w \rangle_R + \text{Indet} \otimes \mathbb{R}.
\]

Proof. Clearly, if $x \in \langle u, v, w \rangle$ then $x_R \in \langle u_R, v_R, w_R \rangle$. Furthermore, the indeterminacy subgroup for $\langle u_R, v_R, w_R \rangle$ is the subgroup
\[
u_R H^{\geq |v|+|w|} - 1 (X; \mathbb{R}) + H^{\geq |u|+|v|} - 1 (X; \mathbb{R}) w_R,
\]
and the result follows.
It is clear that if \( f : A \to A' \) is a morphism of DGA algebras such that \( f_* : H \to H' \) is an isomorphism, then \( f_* : H \to H' \) induces an isomorphism of Massey products. However, one can relax the requirement for \( f \) to be a ring homomorphism.

**Definition 10.2 (cf. [Ma69, BG76]).** We say that an additive chain map \( f : A \to A' \) is a ring map up to higher homotopies if there exists a family of linear maps \( f_i : A_i \to A_i' \) of degree \( 1 - i \), where \( f_0 = f \) and

\[
d \circ f_i + (-1)^i f_i \circ d = \sum_{j=1}^{i-1} (-1)^j (\mu(f_j \otimes f_{i-j} - f_{i-1}(1^{j-1} \otimes \mu \otimes 1^{i-j-1})))
\]

(10.1)

Here \( d \) and \( \mu \) are the differential and the multiplication in \( A' \), respectively, and \( 1^{\otimes k} \) is the identity map of \( A^{\otimes k} \).

**Theorem 10.3.** If \( f : A \to A' \) is a ring map up to higher homotopies, then the map \( f_* : H \to H' \) is a ring homomorphism. Moreover, if a Massey triple product \( \langle u, v, w \rangle \) is defined for some \( u, v, w \in H \), then the Massey triple product \( \langle f_*u, f_*v, f_*w \rangle \) is defined, and

\[
f_*(\langle u, v, w \rangle) \subset \langle f_*u, f_*v, f_*w \rangle.
\]

(10.2)

Finally, if \( f_* : H \to H' \) is an isomorphism then

\[
f_*(\langle u, v, w \rangle) = \langle f_*u, f_*v, f_*w \rangle.
\]

**Proof.** Since \( df_1 - f_1d = 0 \) and \( df_2 + f_2d = f_1 \mu - \mu(f_1 \otimes f_1) \), we conclude that \( f \) induces a ring homomorphism \( H \to H' \). The claims on Massey products can be proved directly but a bit tediously, cf. [Ma69, Theorem 1.5] where more general results are proved.

**Remark 10.4.** As it is clear from the proof, the existence of \( f_0, f_1, f_2 \) already implies that \( f_* \) is a ring homomorphism. Moreover, the existence of \( f_0, f_1, f_2 \) yields formula (10.2). In fact, one can \( n \)-tuple Massey products, and an analog of (10.2) holds provided the maps \( f_1, \ldots, f_n \) as in (10.1) exist [Ma69, Theorem 1.5].

Given a manifold \( M \), let \( \Omega^*(M) \) be the de Rham algebra of differential forms on \( M \), and let \( C^*(M; \mathbb{R}) \) be the singular cochain with the Alexander–Whitney product. Let

\[
\rho : \Omega^*(M) \to C^*(M; \mathbb{R}) = \text{Hom}(C_*(M; \mathbb{Z}), \mathbb{R})
\]

be the map given by integration of a form over a chain. (To be rigorous, we must consider chains in \( C_*(M) \) generated by smooth simplices, but one can prove that the corresponding chain complex yields the standard
homology group.) By the de Rham Theorem, the map \( \rho \) induces an isomorphism of homology of these two complexes, i.e. an isomorphism \( H^*_{dR}(M) \to H^*(M; \mathbb{R}) \).

Clearly, the map \( \rho \) is not a ring homomorphism (since the algebra \( C^*(M; \mathbb{R}) \) is not commutative), but it is a ring map up to higher homotopies \([BG76, \text{Proposition 3.3}]\). Therefore, by Theorem 10.3, the map \( \rho_* : H^*_{dR}(M) \to H^*(M; \mathbb{R}) \) induces a bijection of Massey products.

Consider the map

\[
r : H^*(M; \mathbb{Z}) \to H^*(M; \mathbb{R}) \to H^*_{dR}(M)
\]

where the first map has the form

\[
H^*(M; \mathbb{Z}) \to H^*(M; \mathbb{Z}) \otimes \mathbb{R} = H^*(M; \mathbb{R}), \quad a \mapsto a \otimes 1.
\]

**Definition 10.5.** We say that a de Rham cohomology class is **integral** if it belongs to the image of the map \( r \) as in \((10.3)\).

**Lemma 10.6.** Let \( u, v, w \in H^*_{dR}(M) \) be integral classes and assume that the groups \( H^{[u]+[v]}(M; \mathbb{Z}) \) and \( H^{[v]+[w]}(M; \mathbb{Z}) \) are torsion free. If the Massey product \( \langle u, v, w \rangle \subset H^*_{dR}(M) \) is defined, then it contains an integral class.

**Proof.** Let \( u, v, w \in H^*(M; \mathbb{Z}) \) be elements such that \( r(\alpha) = x \) for \( x = u, v, w \). By our torsion hypotheses, we conclude that \( \overline{u} \overline{v} = 0 = \overline{v} \overline{w} \), and therefore the Massey product \( \langle u, v, w \rangle \subset H^*_{dR}(M) \) is defined. But, by Proposition 10.1 and Theorem 10.3 we have the inclusion

\[
r(\langle u, v, w \rangle) \subset \langle u, v, w \rangle,
\]

and the lemma follows from the fact that the set \( r(\langle u, v, w \rangle) \) consists of integral classes. \(\square\)

11. Gromov’s calculation in the presence of a Massey product

We will follow Gromov’s suggestion \([Gr83, \text{7.4.C}]\) of exploiting nontrivial Massey products in combination with isoperimetric quotients, so as to obtain geometric inequalities. Some examples appear in \([DR03]\). We will use in an essential way the identification of two distinct Massey product theories, cf. \((11.3)\).

Given a metric \( \mathcal{G} \) on \( M \), let \( \text{IQ}(M, \mathcal{G}) \) be its isoperimetric quotient, defined as the maximin of quotients of the comass norm of a \((k-1)\)-dimensional differential form which is a primitive, by the comass norm of an exact \( k \)-form on \( M \), over all \( k \). Namely,

\[
\text{IQ}(\mathcal{G}) = \max_k \sup_{\alpha \in \Omega^k} \inf_{\beta \in \Omega^k} \left\{ \frac{\|\beta\|}{\|\alpha\|} \middle| d\beta = \alpha \right\}.
\]
Theorem 11.1. Let $M$ be an orientable $n$-manifold and $p_1, p_2$ positive integers. Let $\omega_i \in H_{dR}^{p_i}(M), i = 1, 2$ be the image of an integral generator under the map $r$, as in (10.3). Suppose the Massey triple product $\langle \omega_1, \omega_1, \omega_2 \rangle$ is defined and nontrivial. Let

$$p_3 = n - (2p_1 + p_2 - 1),$$

and assume $b_{p_1} = b_{p_2} = b_{p_3} = 1$. Assume furthermore that the group $H^{p_1 + p_2}(M; \mathbb{Z}), j = 1, 2$ is torsion free. Set

$$A_j = \frac{n!}{p_j! (p_1 + p_j)! p_1!} (p_1 + p_j)^j, j = 1, 2.$$

Then every metric $G$ on $M$ satisfies the inequality

$$\text{stsys}_{p_1}(G)^2 \text{stsys}_{p_2}(G) \text{stsys}_{p_3}(G) \leq (A_1 + A_2) \text{IQ}(G) \text{vol}_n(G).$$

Proof. Throughout the proof, we will denote the de Rham cohomology group $H_{dR}^*(M)$ by $H^*(M)$. We have $b_{n-p_3} = b_{p_3} = 1$ by Poincaré duality. Recall that the indeterminacy of the Massey product $\langle \omega_1, \omega_1, \omega_2 \rangle$ is an $\mathbb{R}$-vector subspace of $H^{n-p_3}(M; \mathbb{R}) = \mathbb{R}$. Since the Massey product is assumed nontrivial, it must have zero indeterminacy.

Choose a representative $\eta_i \in \omega_i$. By hypothesis, the $(p_1 + p_j)$-form $\eta_1 \wedge \eta_j, j = 1, 2$ is exact. Choose a primitive $\eta_j$ satisfying $d\eta_j = \eta_1 \wedge \eta_j$ of minimal comass norm, so that

$$\frac{\|\eta_{1j}\|_{\infty}}{\|\eta_1 \wedge \eta_j\|_{\infty}} \leq \text{IQ}(M; G).$$

(11.2)

Notice that the cohomology class $\omega$ of the closed form

$$\eta_{11} \wedge \eta_2 - (-1)^{p_3} \eta_1 \wedge \eta_{12}$$

is equal to the Massey product $\langle \omega_1, \omega_1, \omega_2 \rangle_{dR} \in H^{n-p_3}(M; \mathbb{R})$. By Poincaré duality, there is a class $\omega_3 \in H^{p_3}(M; \mathbb{R}) = \mathbb{R}$ which is the image of an integral generator, such that $(\omega_3 \cup \omega)[M] > 0$. By Lemma 10.6, the class $\omega$ is integral, and hence the class $\omega_3 \cup \omega$ is integral, so that

$$(\omega_3 \cup \omega)[M] \geq 1.$$  

(11.3)

Choose a $p_3$-form $\eta_3$ in the cohomology class $\omega_3$. Then

$$1 \leq \int_M (\eta_{11} \wedge \eta_2 - (-1)^{p_1} \eta_1 \wedge \eta_{12}) \wedge \eta_3.$$  

(11.4)

Similarly to (9.2) and exploiting (11.2), we obtain

$$1 \leq \sum_{j=1}^{2} \frac{n!}{p_j! (p_1 + p_j)!} (p_1 + p_j)^j \|\eta_j\|_{\infty} \|\eta_{1j}\|_{\infty} \|\eta_3\|_{\infty} \text{vol}_n(G)$$

$$\leq (A_1 + A_2) \text{IQ}(G) \|\eta_1\|_{\infty} \|\eta_1\|_{\infty} \|\eta_2\|_{\infty} \|\eta_3\|_{\infty} \text{vol}_n(G).$$
Hence, similarly to (9.3), we obtain
\[ \leq (A_1 + A_2) \text{IQ}(\mathcal{G}) \left( \|\omega_1\|^* \right)^2 \|\omega_2\|^* \|\omega_3\|^* \text{vol}_n(\mathcal{G}). \]
Now let \( \alpha_i \in H_{p_i}(M; \mathbb{R}) \) be the generator dual to \( \omega_i \). By duality of comass and stable norm, and since \( b_{p_i} = 1 \), we have \( \|\alpha_i\| = \frac{1}{\|\omega_i\|^*}. \) Thus, similarly to (9.4) we obtain the inequality
\[ \text{stsys}_{p_1}(\mathcal{G})^2 \text{stsys}_{p_2}(\mathcal{G}) \text{stsys}_{p_3}(\mathcal{G}) \leq (A_1 + A_2) \text{IQ} \circ \text{vol}_n(\mathcal{G}), \]
as required.

12. A HOMOGENEOUS EXAMPLE

Consider the homogeneous manifold
\[ M^{n^2+2n-16} = SU(n+1)/SU(3) \times SU(3), \]
where \( n \geq 5 \) \[\text{T93}\], \[\text{GHV76}\] Chapter 11]. Denote by \( \omega_i \) the generator of \( H^1(M) \), for \( i = 4, 6 \). The Massey triple product
\[ \langle \omega_4, \omega_4, \omega_6 \rangle \in H^13(M, \mathbb{Z}) \]
is nontrivial. For \( n = 5 \), we obtain that every metric \( \mathcal{G} \) on the manifold \( SU(6)/SU(3) \times SU(3) \) satisfies
\[ \frac{\text{stsys}_4(\mathcal{G})^2 \text{stsys}_6(\mathcal{G})^2}{\text{IQ}(\mathcal{G})} \leq 19! \text{vol}_{19}(\mathcal{G}). \] (12.1)

Remark 12.1. This inequality is not exactly of the form envisioned in (2.2), since the constant is metric-dependent via the IQ factor. Applying this technique to the standard 3-dimensional nilmanifold, we get a lower bound given by a product of four systoles, for the quantity \( \text{IQ}(\mathcal{G}) \circ \text{vol}(\mathcal{G}) \). Since we expect the category of a 3-manifold to be at most 3, it is natural to define an analog of systolic category in the presence of a Massey product, by subtracting the number of IQ factors, as in (12.2) below, appearing as the result of integrations involved in defining a Massey product. This would be consistent with the lower bound for \( \text{cat}_{\text{LS}}(M) \) in terms of weights of Massey products. Thus, the inequality can be restated as a lower bound for an IQ-modified systolic category:
\[ \text{cat}_{\text{sys}}^{\text{IQvol}}(M^{19}) \geq 4 - 1 = 3. \] (12.2)

On the LS side, it can be shown that \( 3 \leq \text{cat}_{\text{LS}} M^{19} \leq 4 \), and we expect that that \( \text{cat}_{\text{LS}} = 4 \). Meanwhile, John Oprea (private communication) proved the following proposition. The definition and properties of rational Lusternik–Schnirelmann category can be found in \[\text{CLOT03}\].

Proposition 12.2. The rational Lusternik–Schnirelmann category of the manifold \( M^{19} \) is equal to 3.
Proof. For every simply connected manifold $X$, the rational Lusternik–Schnirelmann category of $X$ is equal to its rational Toomer invariant $e_0(X)$, see [FHL98]. To compute $e_0(X)$, notice that the Sullivan model of the space $SU(6)/(SU(3) \times SU(3))$ has the form

$$\Lambda(x_4, x_6, y_7, y_9, y_{11})$$

with differential

$$dx_i = 0, \ dy_7 = x_4^2, \ dy_9 = x_4x_6, \ dy_{11} = x_6^2.$$ 

Therefore a top class is given by

$$x_4^2y_{11} - x_4x_6y_9, \quad (12.3)$$

and so $e_0(M^{19}) = 3$ by [CLOT03] Prop. 5.23. □

13. ACKNOWLEDGMENTS

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REFERENCES

[Ba93] Babenko, I.: Asymptotic invariants of smooth manifolds. Russian Acad. Sci. Izv. Math. 41 (1993), 1–38.

[Ba02] Babenko, I.: Forte souplesse intersystolique de variétés fermées et de polyèdres. Annales de l’Institut Fourier 52, 4 (2002), 1259-1284.

[BabK98] Babenko, I.; Katz, M. Systolic freedom of orientable manifolds. Ann. Sci. Ecole Norm. Sup. (Sér. 4) 31 (1998), 787–809.

[BCIK04] Bangert, V; Croke, C.; Ivanov, S.; Katz, M.: Boundary case of equality in optimal Loewner-type inequalities, Trans. A.M.S., to appear. See arXiv:math.DG/0406008

[BCIK05] Bangert, V; Croke, C.; Ivanov, S.; Katz, M.: Filling area conjecture and ovalless real hyperelliptic surfaces, Geometric and Functional Analysis (GAFA), to appear. See arXiv:math.DG/0405583

[BK03] Bangert, V.; Katz, M.: Stable systolic inequalities and cohomology products, Comm. Pure Appl. Math. 56 (2003), 979-997. Available at arXiv:math.DG/0204181

[BK04] Bangert, V; Katz, M.: An optimal Loewner-type systolic inequality and harmonic one-forms of constant norm. Comm. Anal. Geom. 12 (2004), number 3, 703-732. See arXiv:math.DG/0304494

[Ber76] Berstein, I.: On the Lusternik–Schnirelmann category of Grassmannians, Proc. Camb. Phil. Soc. 79 (1976), 129-134.

[BG76] Bousfield, A.; Gugenheim, V.: On PL de Rham theory and rational homotopy type. Mem. Amer. Math. Soc. 8, 179, Providence, R.I., 1976.

[BI94] Burago, D.; Ivanov, S.: Riemannian tori without conjugate points are flat. Geom. Funct. Anal. 4 (1994), no. 3, 259–269.

[BI95] Burago, D.; Ivanov, S.: On asymptotic volume of tori. Geom. Funct. Anal. 5 (1995), no. 5, 800–808.

[CoS94] Conway, J.; Sloane, N.: On lattices equivalent to their duals. J. Number Theory 48 (1994), no. 3, 373–382.

[CLOT03] Cornea, O.; Lupton, G.; Oprea, J.; Tanrée, D.: Lusternik-Schnirelmann category. Mathematical Surveys and Monographs, 103. American Mathematical Society, Providence, RI, 2003.

[CrK03] Croke, C.; Katz, M.: Universal volume bounds in Riemannian manifolds, Surveys in Differential Geometry VIII (2003), 109-137. Available at arXiv:math.DG/0302248

[DR03] Dranishnikov, A.; Rudyak, Y.: Examples of non-formal closed simply connected manifolds of dimensions 7 and more. Proc. Amer. Math. Soc. (2004), in press. See arXiv:math.AT/0306299.

[Fe69] Federer, H.: Geometric measure theory. Grundlehren der mathematischen Wissenschaften, 153. Springer-Verlag, Berlin, 1969.

[Fe74] Federer, H. Real flat chains, cochains, and variational problems. Indiana Univ. Math. J. 24 (1974), 351–407.

[FHL98] Felix, Y.; Halperin, S.; Lemaire, J.-M.: The rational LS category of products and of Poincaré duality complexes. Topology 37 (1998), no. 4, 749–756.

[Fr99] Freedman, M.: $Z_2$-Systolic-Freedom. Geometry and Topology Monographs, 2, Proceedings of the Kirbyfest (J. Hass, M. Scharlemann eds.), 113–123, Geometry & Topology, Coventry, 1999.
[Ga71] Ganea, T.: Some problems on numerical homotopy invariants. Symposium on Algebraic Topology (Battelle Seattle Res. Center, Seattle Wash., 1971), pp. 23–30. Lecture Notes in Math., 249, Springer, Berlin, 1971.

[GG92] Gómez-Larrañaga, J.; González-Acuña, F.: Lusternik-Schnirel’mann category of 3-manifolds. Topology 31 (1992), no. 4, 791–800.

[Gr83] Gromov, M.: Filling Riemannian manifolds, J. Diff. Geom. 18 (1983), 1-147.

[Gr96] Gromov, M.: Systoles and intersystolic inequalities. Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), 291–362, Sémin. Congr., vol. 1, Soc. Math. France, Paris, 1996.

[Gr99] Gromov, M.: Metric structures for Riemannian and non-Riemannian spaces. Progr. in Mathematics 152, Birkhäuser, Boston, 1999.

[GHV76] Greub, W.; Halperin, S.; Vanstone, R.: Connections, curvature, and cohomology. Vol. III: Cohomology of principal bundles and homogeneous spaces, Pure and Applied Math., 47, Academic Press, New York-London, 1976.

[Heb86] Hebda, J.: The collars of a Riemannian manifold and stable isosystolic inequalities. Pacific J. Math. 121 (1986), 339–356.

[Hem76] Hempel, J.: 3-Manifolds. Ann. of Math. Studies 86, Princeton Univ. Press, Princeton, New Jersey 1976.

[IK04] Ivanov, S.; Katz, M.: Generalized degree and optimal Loewner-type inequalities. Israel J. Math. 141 (2004), 221-233. arXiv:math.DG/0405019

[Iw02] Iwase, N.: $A_\infty$-method in Lusternik-Schnirelmann category. Topology 41 (2002), no. 4, 695–723.

[Ka02] Katz, M.: Local calibration of mass and systolic geometry. Geometric and Functional Analysis 12 (2002), 598-621.

[Ka03] Katz, M.: Four-manifold systoles and surjectivity of period map, Comment. Math. Helv. 78 (2003), 772-786. arXiv:math.DG/0302306

[KL04] Katz, M.; Lescop, C.: Filling area conjecture, optimal systolic inequalities, and the fiber class in abelian covers. Proceedings of conference and workshop in memory of R. Brooks, held at the Technion, Israel Mathematical Conference Proceedings (IMCP), Contemporary Math., Amer. Math. Soc., Providence, R.I. (to appear). See arXiv:math.DG/0412011

[KR05] Katz, M.; Rudyak, Y.: Systolic category of nonorientable 3-manifolds, in preparation.

[KS1] Katz, M.; Sabourau, S.: Hyperelliptic surfaces are Loewner, Proc. Amer. Math. Soc., to appear. See arXiv:math.DG/0407009

[KS04] Katz, M.; Sabourau, S.: Entropy of systolically extremal surfaces and asymptotic bounds, Ergodic Theory and Dynamical Systems, to appear. See arXiv:math.DG/0410312

[KS05] Katz, M.; Sabourau, S.: An optimal systolic inequality for CAT(0) metrics in genus two, preprint.

[Li69] Lichnerowicz, A.: Applications harmoniques dans un tore. C.R. Acad. Sci., Sér. A, 269 (1969), 912–916.

[Ma69] May, J.: Matric Massey products. J. Algebra 12 (1969) 533–568.

[OR01] Oprea, J.; Rudyak, Y.: Detecting elements and Lusternik-Schnirelmann category of 3-manifolds. Lusternik-Schnirelmann category and related
topics (South Hadley, MA, 2001), 181–191, *Contemp. Math.*, 316, Amer. Math. Soc., Providence, RI, 2002. arXiv:math.AT/0111181

[PP03] Paternain, G. P.; Petean, J.: Minimal entropy and collapsing with curvature bounded from below. *Invent. Math.* 151 (2003), no. 2, 415–450.

[Pu52] Pu, P.M.: Some inequalities in certain nonorientable Riemannian manifolds. *Pacific J. Math.* 2 (1952), 55–71.

[Ru99] Rudyak, Y. B.: On category weight and its applications. *Topology* 38, 1, (1999), 37–55.

[RT00] Rudyak, Y. B.; Tralle, A.: On Thom spaces, Massey products, and nonformal symplectic manifolds. *Internat. Math. Res. Notices* 10 (2000), 495–513.

[Sa04] Sabourau, S.: Systoles des surfaces plates singulières de genre deux, *Math. Zeitschrift* 247 (2004), no. 4, 693–709.

[Sa05] Sabourau, S.: Entropy and systoles on surfaces, preprint.

[Sa06] Sabourau, S.: Systolic volume and minimal entropy of aspherical manifolds, preprint.

[Si78] Singhof, W.: Generalized higher order cohomology operations induced by the diagonal mapping. *Math. Z.* 162, 1, (1978), 7–26.

[Sm62] Smale, S.: On the structure of 5-manifolds. *Ann. of Math.* (2) 75 (1962), 38–46.

[Tr93] Tralle, A.: On compact homogeneous spaces with nonvanishing Massey products, in *Differential geometry and its applications (Opava, 1992)*, pp. 47–50, Math. Publ. 1, Silesian Univ. Opava, 1993.

[Wh83] White, B.: Regularity of area-minimizing hypersurfaces at boundaries with multiplicity. *Seminar on minimal submanifolds (E. Bombieri ed.)*, 293-301. Annals of Mathematics Studies 103. Princeton University Press, Princeton N.J., 1983.

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