Two-term expansion of the ground state one-body density matrix of a mean-field Bose gas

Phan Thành Nam1 · Marcin Napiórkowski2

Received: 9 October 2020 / Accepted: 17 February 2021 / Published online: 27 April 2021
© The Author(s) 2021

Abstract
We consider the homogeneous Bose gas on a unit torus in the mean-field regime when the interaction strength is proportional to the inverse of the particle number. In the limit when the number of particles becomes large, we derive a two-term expansion of the one-body density matrix of the ground state. The proof is based on a cubic correction to Bogoliubov’s approximation of the ground state energy and the ground state.

MSC code 81Q05 · 47A75

1 Introduction
We consider a homogeneous system of $N$ bosons on the unit torus $\mathbb{T}^d$, for any dimension $d \geq 1$. The system is governed by the mean-field Hamiltonian

$$H_N = \sum_{j=1}^{N} -\Delta x_j + \frac{1}{N-1} \sum_{1 \leq j < k \leq N} w(x_j - x_k)$$

(1)

which acts on the bosonic Hilbert space

$$\mathcal{H}_N^2 = L_\text{sym}^2((\mathbb{T}^d)^N).$$

Here the kinetic operator $-\Delta$ is the usual Laplacian (with periodic boundary conditions). The interaction potential $w$ is a real-valued, even function. We assume that its Fourier transform

Communicated by A. Malchiodi.

1 Phan Thành Nam
   nam@math.lmu.de
2 Marcin Napiórkowski
   marcin.napiorkowski@fuw.edu.pl

1 Department of Mathematics, LMU Munich, Theresienstrasse 39, 80333 Munich, Germany
2 Department of Mathematical Methods in Physics, Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warsaw, Poland
is non-negative and integrable, namely
\[ w(x) = \sum_{p \in 2\pi \mathbb{Z}^d} \hat{w}(p)e^{ip \cdot x} \quad \text{with} \quad 0 \leq \hat{w} \in \ell^1(2\pi \mathbb{Z}^d). \]

In particular, \( w \) is bounded. Since \( w \) is even, \( \hat{w} \) is also even.

Under the above conditions, \( H_N \) is well defined on the core domain of smooth functions. Moreover, it is well-known that \( H_N \) is bounded from below and can be extended to be a self-adjoint operator by Friedrichs’ method. The self-adjoint extension, still denoted by \( H_N \), has a unique ground state \( \Psi_1 \) (up to a complex phase) which solves the variational problem

\[ E_N = \inf_{\|\Psi\|_{\mathcal{H}^N}=1} \langle \Psi, H_N \Psi \rangle. \]

Here \( \langle \cdot, \cdot \rangle \) is the inner product in \( \mathcal{H}^N \).

In the present paper, we are interested in the asymptotic behavior of the ground state \( \Psi_1 \) of \( H_N \) in the limit when \( N \to \infty \). More precisely, we will focus on the one-body density matrix \( \gamma^{(1)}_{\Psi_N} \) which is a trace class operator on \( L^2(\mathbb{T}^d) \) with kernel

\[ \gamma^{(1)}_{\Psi_N}(x, y) = N \int_{\mathbb{T}^d(N-1)} \Psi_N(x, x_1, \ldots, x_N)\overline{\Psi_N(y, x_2, \ldots x_N)}dx_2 \ldots dx_N. \]

Note that \( \gamma^{(1)}_{\Psi_N} \geq 0 \) and \( \text{Tr} \gamma^{(1)}_{\Psi_N} = N \).

1.1 Main result

Our main theorem is

**Theorem 1** (Ground state density matrix) Assume that \( 0 \leq \hat{w} \in \ell^1((2\pi \mathbb{Z})^d) \). Then the ground state \( \Psi_1 \) of the Hamiltonian \( H_N \) in (1) satisfies

\[ \lim_{N \to \infty} \text{Tr} \left| \gamma^{(1)}_{\Psi_N} - \left( N - \sum_{p \neq 0} \gamma^2_p \right) |u_0\rangle \langle u_0| - \sum_{p \neq 0} \gamma^2_p |u_p\rangle \langle u_p| \right| = 0 \]

where

\[ u_p(x) = e^{ip \cdot x}, \quad \gamma_p = \frac{\alpha_p}{\sqrt{1 - \alpha^2_p}}, \quad \alpha_p = \frac{\hat{w}(p)}{p^2 + \hat{w}(p) + \sqrt{p^4 + 2p^2 \hat{w}(p)}}. \]

Here \( |u\rangle \langle u| \) is the orthogonal projection on \( u \). We use the bra-ket notation, where \( |u\rangle = u \) is a vector in the Hilbert space \( \mathcal{H} \) and \( \langle u| \) is an element in the dual space of \( \mathcal{H} \) which maps any vector \( v \in \mathcal{H} \) to the inner product \( \langle u, v \rangle_{\mathcal{H}} \).

To the leading order, our result implies Bose–Einstein condensation, namely

\[ \lim_{N \to \infty} \frac{1}{N} \gamma^{(1)}_{\Psi_N} = |u_0\rangle \langle u_0| \]

in the trace norm. This result is well-known and it follows easily from Onsager’s inequality

\[ \frac{1}{N-1} \sum_{1 \leq i < j \leq N} w(x_i - x_j) \geq \frac{N}{2} \hat{w}(0) - \frac{N}{N - 1} w(0) \]

(see [18]). The significance of Theorem 1 is that it gives the next order correction to \( \gamma^{(1)}_{\Psi_N} \), thus justifying Bogoliubov’s approximation in a rather strong sense as we will explain.

---

1 We use the convention that \( \langle \cdot, \cdot \rangle \) is linear in the second argument and anti-linear in the first.
1.2 Bogoliubov’s approximation

It is convenient to turn to the grand canonical setting. Let us introduce the Fock space

\[ \mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}^n = \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}^2 \oplus \cdots \]

For any Fock space vector \( \Psi = (\Psi_n)_{n=0}^{\infty} \in \mathcal{F} \) with \( \Psi_n \in \mathcal{H}^n \), we define its norm by

\[ \| \Psi \|_{\mathcal{F}}^2 = \sum_{n=0}^{\infty} \| \Psi_n \|_{\mathcal{H}^n}^2. \]

and define the particle number expectation by

\[ \langle \Psi, N \Psi \rangle = \sum_{n=0}^{\infty} n \| \Psi_n \|_{\mathcal{H}^n}^2. \]

In particular, the vacuum state \( |0\rangle = (1, 0, 0, \ldots) \) is a normalized vector on Fock space which has the particle number expectation \( \langle 0|N|0 \rangle = 0 \).

For any \( f \in \mathcal{H} \), the creation operator \( a^*(f) \) on Fock space maps from \( \mathcal{H}^n \) to \( \mathcal{H}^{n+1} \) for every \( n \geq 0 \) and satisfies

\[ (a^*(f)\Psi_n)(x_1, \ldots, x_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} f(x_j)\Psi(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}), \quad \forall \Psi_n \in \mathcal{H}^n. \]

Its adjoint is the annihilation operator \( a(f) \), which maps from \( \mathcal{H}^n \) to \( \mathcal{H}^{n-1} \) for every \( n \geq 0 \) (with convention \( \mathcal{H}^{-1} = \{0\} \)) and satisfies

\[ (a(f)\Psi_n)(x_1, \ldots, x_n-1) = \sqrt{n} \int_{\mathbb{R}^d} f(x_n)\Psi(x_1, \ldots, x_n)dx_n, \quad \forall \Psi_n \in \mathcal{H}^n. \]

We will denote by \( a^*_p \) and \( a_p \) the creation and annihilation operators with momentum \( p \in 2\pi \mathbb{Z}^d \), namely

\[ a^*_p = a^*(u_p), \quad a_p = a(u_p), \quad u_p(x) = e^{ip \cdot x}. \]

They satisfy the canonical commutation relation (CCR)

\[ [a_p, a_q] = [a^*_p, a^*_q] = 0, \quad [a_p, a^*_q] = \delta_{p,q} \] (3)

where \([X, Y] = XY - YX\).

The creation and annihilation operators can be used to express several operators on Fock space. For example, the number operator can be written as

\[ N = \sum_{n=0}^{\infty} n 1_{\mathcal{H}^n} = \sum_{p \in 2\pi \mathbb{Z}^d} a^*_p a_p. \]

Similarly, the Hamiltonian \( H_N \) in (1) can be rewritten as

\[ H_N = \sum_{p \in 2\pi \mathbb{Z}^d} p^2 a^*_p a_p + \frac{1}{2(N-1)} \sum_{p, q, k \in \mathbb{Z}^d} \tilde{w}(k) a^*_{p-k} a^*_{q+k} a_p a_q. \] (4)
The right side of (4) is an operator on Fock space, which coincides with (1) when being restricted to $\mathcal{H}^N$. In the following we will only use the grand–canonical formula (4).

In 1947, Bogoliubov [4] suggested a heuristic argument to compute the low-lying spectrum of the operator $H_N$ by using a perturbation around the condensation. Roughly speaking, he proposed to first substitute all operators $a_0$ and $a_0^*$ in (4) by the scalar number $\sqrt{N}$ (c-number substitution), and then ignore all interaction terms which are coupled with coefficients of order $o(1/N) \to \infty$. All this leads to the formal expression

$$ H_N \approx N^2 \hat{w}(0) + \mathbb{H}_{\text{Bog}} $$

(5)

where

$$ \mathbb{H}_{\text{Bog}} = \sum_{p \neq 0} \left( (p^2 + \hat{w}(p)) a_p^* a_p + \frac{1}{2} \hat{w}(p) (a_p^* a_{-p}^* + a_p a_{-p}) \right). $$

(6)

Note that the expression (5) is formal since $H_N$ acts on the $N$-body Hilbert space $\mathcal{H}^N$ while the Bogoliubov Hamiltonian $\mathbb{H}_{\text{Bog}}$ acts on the excited Fock space $\mathcal{F}^+ = \bigoplus_{n=0}^{\infty} \mathcal{H}_n^+ = \mathbb{C} \oplus \mathcal{H}_+ \oplus \mathcal{H}_2^+ \oplus \cdots$, $\mathcal{H}_+ = Q \mathcal{H}$

where we have introduced the projections

$$ Q = \sum_{p \neq 0} |u_p \rangle \langle u_p| = 1 - P, \quad P = |u_0 \rangle \langle u_0|. $$

In particular, unlike $H_N$, the quadratic Hamiltonian $\mathbb{H}_{\text{Bog}}$ does not preserve the number of particles. Nevertheless, $\mathbb{H}_{\text{Bog}}$ can be diagonalized by the following unitary transformation on $\mathcal{F}^+$

$$ U_B = \exp \left( \sum_{p \neq 0} \beta_p (a_p^* a_{-p}^* - a_p a_{-p}) \right) $$

(7)

where the coefficients $\beta_p > 0$ are determined by

$$ \tanh(2\beta_p) = \alpha_p = \frac{\hat{w}(p)}{p^2 + \hat{w}(p) + \sqrt{p^4 + 2p^2 \hat{w}(p)}}. $$

In fact, by using the CCR (3) it is straightforward to check that

$$ U_B a_p U_B^* = \frac{a_p + \alpha_p a_{-p}^*}{\sqrt{1 - \alpha_p^2}} := \sigma_p a_p + \gamma_p a_{-p}^*, \quad \forall \ p \neq 0 $$

(8)

where

$$ \sigma_p := \frac{1}{\sqrt{1 - \alpha_p^2}} = \cosh(\beta_p), \quad \gamma_p := \frac{\alpha_p}{\sqrt{1 - \alpha_p^2}} = \sinh(\beta_p). $$

Consequently,

$$ U_B \mathbb{H}_{\text{Bog}} U_B^* = E_{\text{Bog}} + \sum_{p \neq 0} e(p) a_p^* a_p, $$

(9)

2 Strictly speaking, for $a_0^* a_0^* a_0 a_0$ we should rewrite it as $(a_0^* a_0)^2 - a_0^* a_0$ before doing the substitution
where
\[ E_{\text{Bog}} = -\frac{1}{2} \sum_{p \neq 0} \left( |p|^2 + \hat{w}(p) - e(p) \right), \quad e_p = \sqrt{|p|^4 + 2|p|^2 \hat{w}(p)}. \]

Note that the assumption \( 0 \leq \hat{w} \in \ell^1(2\pi \mathbb{Z}^d) \) ensures that \( E_{\text{Bog}} \) is finite. Moreover we have the uniform bounds
\[ \sum_{p \neq 0} \gamma_p \leq C, \quad \sup_{p \neq 0} \sigma_p \leq C. \quad (10) \]

Thus Bogoliubov’s approximation predicts that the ground state energy of \( H_N \) is
\[ E_N = \frac{N}{2} \hat{w}(0) + E_{\text{Bog}} + o(1)_{N \to \infty}. \quad (11) \]

In 2011, Seiringer [18] gave the first rigorous proof of (11). He also proved that the low-lying spectrum of \( H_N \) is given approximately by the elementary excitation \( e_p \). These results have been extended to inhomogeneous trapped systems in [11], to more general interaction potentials in [12], to a large volume limit in [9], and to situations of multiple-condensation in [14,17].

Let us recall the approach in [12] which also provides the convergence of the ground state of the mean-field Hamiltonian \( H_N \) in (1). Mathematically, the formal expression (5) can be made rigorous using the unitary operator introduced in [12]
\[ U_N : \mathcal{H}^N \to \mathcal{F}_+^\leq N = 1 \leq N \mathcal{F}_+, \quad 1 \leq N = 1(N_+ \leq N) \]
which is defined by
\[ U_N = \sum_{j=0}^{N} Q^\otimes j \left( \frac{a_{0}^{N-j}}{\sqrt{(N-j)!}} \right), \quad U_N^* = \bigoplus_{j=0}^{N} \left( \frac{(a_{0}^{*})^{N-j}}{\sqrt{(N-j)!}} \right). \quad (12) \]

Recall from [12, Proposition 4.2] that
\[ U_N a_{p}^{*} a_{q} U_N^* = a_{p}^{*} a_{q}, \quad U_N a_{p}^{*} a_{0} U_N^* = a_{p}^{*} \sqrt{N - N_+}, \quad \forall p, q \neq 0 \quad (13) \]
where \( N_+ \) is the number operator on the excited Fock space \( \mathcal{F}_+ \),
\[ N_+ = \sum_{p \neq 0} a_{p}^{*} a_{p}. \]
Thus \( U_N \) implements the c-number substitution in Bogoliubov’s argument because it replaces \( a_{0} \) by \( \sqrt{N - N_+} \approx \sqrt{N} \) (we have \( N_+ \ll N \) due to the condensation). Then the formal expression (5) can be reformulated as
\[ U_N H_N U_N^* \approx \frac{N}{2} \hat{w}(0) + H_{\text{Bog}} \quad (14) \]
which is rigorous since the operators on both sides act on the same excited Fock space. By justifying (14), the authors of [12] recovered the convergence of eigenvalues of \( H_N \) first obtained in [18], and also obtained the convergence of eigenfunctions of \( H_N \) to those of \( H_{\text{Bog}} \). In particular, for the ground state, we have from [12, Theorem 2.2] that
\[ \lim_{N \to \infty} U_N \Psi_N = U_B |0\rangle \quad (15) \]
where $|0\rangle$ is the vacuum in Fock space. The convergence (15) holds strongly in norm of $\mathcal{F}_+$, and also strongly in the norm induced by the quadratic form of $\mathbb{H}_{\text{Bog}}$ in $\mathcal{F}_+$. In particular, this implies the convergence of one-body density matrix

$$\lim_{N \to \infty} Q \gamma_{\Psi_N}^{(1)} Q = \sum_{\mu \neq 0} \gamma_{\mu}^2 |\mu\rangle \langle \mu|$$

in trace class (see (68) for a detailed explanation). Since $\text{Tr} \gamma_{\Psi_N}^{(1)} / \Psi_N = N$, (16) is equivalent to

$$\lim_{N \to \infty} \text{Tr} \left| P \gamma_{\Psi_N}^{(1)} P + Q \gamma_{\Psi_N}^{(1)} Q - \left( N - \sum_{\mu \neq 0} \gamma_{\mu}^2 \right) |u_0\rangle \langle u_0| - \sum_{\mu \neq 0} \gamma_{\mu}^2 |\mu\rangle \langle \mu| \right| = 0.$$  

(17)

Recall that $P = |u_0\rangle \langle u_0| = 1 - Q$. The formula (17) looks similar to the result in Theorem 1, except that the cross term $P \gamma_{\Psi_N}^{(1)} Q + Q \gamma_{\Psi_N}^{(1)} P$ is missing. Putting differently, to get the result in Theorem 1 we have to show that

$$\lim_{N \to \infty} \text{Tr} \left| P \gamma_{\Psi_N}^{(1)} Q + Q \gamma_{\Psi_N}^{(1)} P \right| = 0.$$  

(18)

As explained in [12, Eq. (2.19)], from (16) and the Cauchy–Schwarz inequality one only obtains that the left side of (18) is of order $O(\sqrt{N})$. Moreover, (18) implies that

$$\lim_{N \to \infty} \sqrt{N} (U_N \Psi_N, a_p U_N \Psi_N) = 0, \quad \forall p \neq 0,$$  

(19)

thus answering an open question in [13]. As explained in [13, Section 5], (19) would follow if we could replace $U_N \Psi_N$ by $U_B |0\rangle$ (which is a quasi-free state, and thus satisfies Wick’s Theorem [19, Chapter 10]). However, the norm convergence (15) is not strong enough to justify (19).

### 1.3 Outline of the proof

To prove Theorem 1 we have to extract some information going beyond Bogoliubov’s approximation. Roughly speaking, we will refine (14) by computing exactly the term of order $O(N^{-1})$. Our proof consists of three main steps.

**Step 1 (Excitation Hamiltonian).** After implementing the c-number substitution, instead of ignoring all terms with coefficients of order $o(1)$ as $N \to \infty$, we will keep all terms of order $O(N^{-1})$. More precisely, in Lemma 7 below we show that

$$U_N H_N U_N^* = \frac{N}{2} \hat{w}(0) + G_N + O(N^{-3/2})$$

(20)

in an appropriate sense, where

$$G_N = \mathbb{H}_{\text{Bog}} + \frac{N_+ (1 - N_+)}{2(N - 1)} \hat{w}(0) + \sum_{p \neq 0} \frac{1 - N_+}{N - 1} \hat{w}(p) a_p^* a_p$$

$$+ \left( \frac{1}{2} \sum_{p \neq 0} \hat{w}(p) a_p^* a_{-p}^* \frac{1 - 2N_+}{2N} + \text{h.c.} \right)$$

$$+ \left( \frac{1}{\sqrt{N}} \sum_{\ell, p \neq 0, \ell + p \neq 0} \hat{w}(\ell) a_p^* a_{p+\ell}^* a_{-p} a_{-\ell} + \text{h.c.} \right).$$
\[ + \frac{1}{2(N-1)} \sum_{k,p \neq 0, \ell \neq -p,k} \hat{w}(\ell) a^*_{p+\ell} a^*_{k-\ell} a_p a_k. \]

The formula (20) is obtained by a direct computation using the actions of \( U_N \) as in [12], plus an expansion of \( \sqrt{N-N_+} \) and \( \sqrt{(N-N_+)(N-N_+ - 1)} \) in the regime \( N_+ \ll N \). The advantage of using \( G_N \) is that it is well-defined on the full Fock space \( \mathcal{F}_+ \). This idea has been used to study the norm approximation for the many-body quantum dynamics in [8].

**Step 2 (Quadratic transformation).** Then we conjugate the operator on the right side of (20) by the Bogoliubov transformation \( U_B \) in (7). In Lemma 8 we prove that

\[ U_B G_N U_B^* = \langle 0 | U_B G_N U_B^* | 0 \rangle + \sum_{p \neq 0} e(p) a^*_p a_p + C_N + R_2 \]

where

\[ C_N = \frac{1}{\sqrt{N}} \sum_{p,q \neq 0, p+q \neq 0} \hat{w}(p) \left[ (\sigma_{p+q} \sigma_{-p} \gamma_q + \gamma_{p+q} \gamma_p \sigma_q) a^*_{p+q} a^*_{-p} a^*_{-q} + \text{h.c.} \right] \]

and \( R_2 \) is an error term whose expectation against the ground state is of order \( O(N^{-3/2}) \).

Note that in \( C_N \) we keep only cubic terms with three creation operators or three annihilation operators. These are the most problematic terms. All other cubic terms, as well as all quartic terms, are of lower order and can be estimated by the Cauchy–Schwarz inequality (the quartic terms always come with a factor \( N^{-1} \) instead of \( N^{-1/2} \) and this helps).

As we will see, the energy contribution of the cubic term \( C_N \) is of order \( O(N^{-1}) \). Thus (21) implies that

\[ E_N = \frac{N}{2} \hat{w}(0) + \langle 0 | U_B G_N U_B^* | 0 \rangle + O(N^{-1}) \]

which improves (11). Moreover, for the ground state we have

\[ \langle U_B U_N \Psi_N, \mathcal{N}_+ U_B U_N \Psi_N \rangle \leq C N^{-1} \]

which in turn implies the norm approximation (up to an appropriate choice of the phase factor for \( \Psi_N \))

\[ \| U_N \Psi_N - U_B^* | 0 \|_{\mathcal{F}_+}^2 \leq C N^{-1}. \]

and the following bound on the one-body density matrix

\[ \lim_{N \to \infty} \text{Tr} \left| P_{\gamma_{\Psi_N}}(1) Q + Q_{\gamma_{\Psi_N}}(1) P \right| \leq C. \]

Unfortunately the latter bound is still weaker than (18). Thus the desired result (18) cannot be obtained within Bogoliubov’s theory.

**Step 3 (Cubic transformation).** To factor out the energy contribution of the cubic term \( C_N \) in (21), we will use a cubic transformation. It is given by

\[ U_S = e^S, \quad S = \frac{1}{\sqrt{N}} \sum_{p,q \neq 0, p+q \neq 0} \eta_{p,q} \left( a^*_{p+q} a^*_p a^*_q \mathbb{1}_{\leq N} - \mathbb{1}_{\leq N} a_{p+q} a_{-p} a_{-q} \right) \]
where
\[ \eta_{p,q} = \frac{\hat{w}(p)(\sigma_{p+q}\sigma_p\gamma_q + \gamma_{p+q}\gamma_p\sigma_q)}{e_{p+q} + e_p + e_q}. \] (26)

From the assumption \( \hat{w} \in \ell^1(2\pi \mathbb{Z}^d) \) and the bounds (10) we have the summability
\[ \sum_{p,q \neq 0} |\eta_{p,q}| \leq C. \] (27)

Here we insert the cut-off \( 1 \leq N \) in the definition of \( U_S \) to make sure that it does not change the particle number operator \( N_+ \) too much; see Lemma 5 for details.

The choice of the cubic transformation above can be deduced on an abstract level. Consider an operator of the form
\[ A = A_0 + X \]
where \( X \) stands for some perturbation. Then, in principle, we can remove \( X \) by conjugating \( A \) with \( e^{S} \) provided that
\[ [S, [S, A_0]] = -[S, X] \]
is small in an appropriate sense. This can be seen by the simple expansions
\[ e^S X e^{-S} = X + \int_0^1 e^{sS} [X, S] e^{-sS} ds \]
and
\[ e^S A_0 e^{-S} = A_0 + [S, A_0] + \int_0^1 \int_0^t e^{sS} [S, [S, A_0]] e^{-sY} ds dt. \]

In our situation, \( A_0 = \sum_{p \neq 0} e(p) a_p^* a_p \) and \( X = C_N \), allowing to find \( S \) explicitly in (25).

In Lemma 9 we prove that
\[ U_S U_B G_N U_B^* U_S^* = \left| 0 \right| U_S U_B G_N U_B^* U_S^* \left| 0 \right> + \sum_{p \neq 0} e(p) a_p^* a_p + R_3 \]
with an error term \( R_3 \) whose expectation against the ground state is of order \( O(N^{-3/2}) \). This allows us to obtain the following improvements of (22), (23) and (24).

**Theorem 2** (Refined ground state estimates) Assume that \( 0 \leq \hat{w} \in \ell^1((2\pi \mathbb{Z})^d) \). Then the ground state energy of the Hamiltonian \( H_N \) in (1) satisfies
\[ E_N = \frac{N}{2} \hat{w}(0) + \left| 0 \right| U_S U_B G_N U_B^* U_S^* \left| 0 \right> + O(N^{-3/2}). \]
Moreover, if \( \Psi_N \) is the ground state of \( H_N \), then \( \Phi = U_S U_B \Psi_N \) satisfies
\[ \langle \Phi, N_+ \Phi \rangle \leq CN^{-3/2}. \]

Consequently, we have the norm approximation (up to an appropriate choice of the phase factor for \( \Psi_N \))
\[ \| U_N \Psi_N - U_B^* U_S^* \left| 0 \right> \|^2_{L^2} \leq CN^{-3/2}. \]
As we will explain, Theorem 2 implies (18) and thus justifies Theorem 1.

The idea of using cubic transformations has been developed to handle dilute Bose gases in [1–3,16,20], where the interaction potential has a much shorter range but the interaction strength is much larger in its range. In this case, the contribution of the cubic terms is much bigger, and Bogoliubov’s approximation has to be modified appropriately to capture the short-range scattering effect. Results similar to (11) have been proved recently for the Gross-Pitaevskii limit [2] and for the thermodynamic limit [10,20]. It is unclear to us how to extend Theorem 1 to the dilute regime.

Our work shows that in the mean-field regime, in contrast to the dilute regime, the cubic terms are smaller, and they actually contribute only to the next order correction to Bogoliubov’s approximation (there are also some quadratic and quartic terms which contribute to the same order of the cubic term). On the other hand, it is interesting that the contribution of the cubic terms is not visible in the expansion of the one-body density matrix in Theorem 1; putting differently the approximation in Theorem 1 can be guessed using only Bogoliubov’s theory (although its proof requires more information).

There have been also remarkable works concerning higher order expansions in powers of $N^{-1}$ in the mean-field regime; see [15] for a study of the ground state, [7] for the low-energy spectrum, and [5,6] for the quantum dynamics. These works are based on perturbative approaches which are very different from ours. Note that the method of Bossmann, Petrat and Seiringer in [7] also gives access to the higher order expansion of the reduced density matrices (see [7, Eq. (3.15)] for a comparison). We hope that our rather explicit strategy complements the previous analysis in [5–7,15] concerning the correction to Bogoliubov’s theory in the mean-field regime.

Organisation of the paper

In Sect. 2 we will derive some useful estimates for the particle number operator $N_+$. Then we analyze the actions of the transformations $U_N, U_B, U_S$ in Sects. 3, 4 and 5, respectively. Finally, we prove Theorem 2 in Sect. 7 and conclude Theorem 1 in Sect. 7.

2 Moment estimates for the particle number operator

In this section we justify the Bose–Einstein condensation by showing that the ground state has a bounded number of excited particles. As explained in [18], the uniform bound on the expectation of $N_+$ follows easily from Onsager’s inequality (2). For our purpose, we will need uniform bounds for higher moments of $N_+$. The following lemma is an extension of [13, Lemma 5].

**Lemma 3** (Number of excited particles) If $\Psi_N$ is the ground state of $H_N$, then

$$\langle \Psi_N, N_+^s \Psi_N \rangle \leq C_s, \quad \forall s \in \mathbb{N}.\quad (29)$$

**Proof** As in [13, Lemma 5], from the operator inequality

$$H_N \geq (2\pi)^2 N_+ + \frac{N^2}{2(N - 1)} \hat{w}(0) - \frac{N}{2(N - 1)} w(0) \quad (28)$$

we obtain

$$|E_N| \leq \frac{N}{2} \hat{w}(0) \quad \text{and} \quad \langle \Psi_N, N_+^s \Psi_N \rangle \leq C \quad (29)$$
for $s = 1, 2, 3$. Let us assume that $s \in \mathbb{N}$ is even. We will show that $\langle \Psi_N, \mathcal{N}_{s+1}^\psi \Psi_N \rangle \leq C$.

Since $\Psi_N$ is a ground state of $H_N$, it solves the Schrödinger equation

$$H_N \Psi_N = E_N \Psi_N.$$ 

Consequently, we get the identity

$$\left\langle \Psi_N, \mathcal{N}_{s+1}^\psi \left( H_N - E_N \right) \mathcal{N}_{s+1}^\psi \Psi_N \right\rangle = \left\langle \Psi_N, \mathcal{N}_{s+1}^\psi \left[ H_N, \mathcal{N}_{s+1}^\psi \right] \Psi_N \right\rangle. \quad (30)$$

The left side of (30) can be estimated using (28) and (29) as

$$\left\langle \Psi_N, \mathcal{N}_{s+1}^\psi \left( H_N - E_N \right) \mathcal{N}_{s+1}^\psi \Psi_N \right\rangle \geq \left\langle \Psi_N, \left( 2\pi \right)^2 \mathcal{N}_{s+1}^\psi - C \mathcal{N}_{s+1}^\psi \right\rangle \Psi_N. \quad (31)$$

For the right side of (30), since

$$[A, B^k] = \sum_{j=0}^{k-1} B^j [A, B] B^{k-j-1},$$

using (4) and the CCR (3) we write

$$\mathcal{N}_{s+1}^\psi \left[ H_N, \mathcal{N}_{s+1}^\psi \right] = \frac{1}{2(N-1)} \sum_{j=0}^{\frac{s}{2}-1} \sum_{\ell \neq 0} \sum_{p-q} \tilde{w}(\ell) \mathcal{N}_{s+1}^\psi \left[ a_{p-\ell} a_{q+\ell} a_p a_q, \mathcal{N}_{s+1}^\psi \right] \mathcal{N}_{s+1}^\psi \Psi_N \Psi_N.$$ 

(32)

Now we take the expectation against $\Psi_N$ and estimate. For the first term on the right side of (32), by the Cauchy–Schwarz inequality, we get for a given $j$

$$\left| \left\langle \Psi_N, \sum_{\ell \neq 0} \tilde{w}(\ell) \mathcal{N}_{s+1}^\psi \left[ a_{p-\ell} a_{q+\ell} a_p a_q, \mathcal{N}_{s+1}^\psi \right] \Psi_N \right\rangle \right|$$

$$= \left| \left\langle \Psi_N, \sum_{\ell \neq 0} \tilde{w}(\ell) \mathcal{N}_{s+1}^\psi \left[ a_{p-\ell} a_p a_{q+\ell} a_q, \mathcal{N}_{s+1}^\psi \right] \Psi_N \right\rangle \right|$$

$$\leq \sum_{\ell \neq 0} \left\| \left( \mathcal{N}_{s+1}^\psi + 1 \right)^{-j} a_p a_q \mathcal{N}_{s+1}^\psi \Psi_N \right\| \left\| \tilde{w}(\ell) \right\| \left( \mathcal{N}_{s+1}^\psi + 1 \right)^j a_{\ell a_{-\ell}} \mathcal{N}_{s+1}^\psi \Psi_N \right\|$$

$$= \sum_{\ell \neq 0} \left\| a_p a_q \mathcal{N}_{s+1}^\psi \Psi_N \right\| \left\| \tilde{w}(\ell) \right\| \left\| a_{\ell a_{-\ell}} \left( \mathcal{N}_{s+1}^\psi + 1 \right)^j \mathcal{N}_{s+1}^\psi \Psi_N \right\|$$

$$\leq \left\| a_p a_q \mathcal{N}_{s+1}^\psi \Psi_N \right\| \left( \sum_{\ell \neq 0} \left\| \tilde{w}(\ell) \right\|^2 \right)^{1/2} \left( \sum_{\ell \neq 0} \left\| a_{\ell a_{-\ell}} \left( \mathcal{N}_{s+1}^\psi + 1 \right)^j \mathcal{N}_{s+1}^\psi \Psi_N \right\|^2 \right)^{1/2}. \quad (32)$$
Here we have used that \( a_0 a_0 \) commutes with \( N_+ \), that \( a_0^* a_0 \leq N \) on \( \mathcal{H}^N \) and that \( \sum |\hat{w}(\ell)|^2 = \|w\|^2 < \infty \). Similarly, for the second term, we have

\[
\left| \left< \Psi_N, \sum_{\ell \neq 0} \hat{w}(\ell) N_+^{\hat{\gamma}+j} a_\ell^* a_0 a_0 a_\ell N_+^{\hat{\gamma}-j-1} \Psi_N \right> \right|
\]

\[
= \left| \left< \Psi_N, \sum_{\ell \neq 0} \hat{w}(\ell) N_+^{\hat{\gamma}+j} a_\ell^* a_0 a_0 a_\ell N_+^{\hat{\gamma}-j-1} \Psi_N \right> \right|
\]

\[
\leq \sum_{\ell \neq 0} \| (N_+ + 1)^{-j} a_\ell a_\ell N_+^{\hat{\gamma}+j} \Psi_N \| \| (N_+ + 1)^{j+1} a_0 a_\ell N_+^{\hat{\gamma}-j-1} \Psi_N \|
\]

\[
\leq C N \langle \Psi_N, (N_+ + 1)^s \Psi_N \rangle
\]

as before. For the third term, we can bound

\[
\left| \left< \Psi_N, \sum_{\ell \neq 0 \neq p \neq \ell} \hat{w}(\ell) N_+^{\hat{\gamma}+j} a_\ell^* a_0 a_0 a_\ell N_+^{\hat{\gamma}-j-1} \Psi_N \right> \right|
\]

\[
\leq \left( \sum_{\ell \neq 0 \neq p \neq \ell} \|a_\ell a_\ell N_+^{\hat{\gamma}} \Psi_N\|^2 \right)^{1/2}
\]

\[
\leq \left( \sum_{\ell \neq 0 \neq p \neq \ell} \| (N_+ - 1)^{-j} a_p a_\ell N_+^{\hat{\gamma}-j-1} \Psi_N \|^2 \right)^{1/2}
\]

\[
\leq C N^{1/2} (\Psi_N, N_+^{\hat{\gamma}} \Psi_N)^{1/2} (\Psi_N, N_+^{\hat{\gamma}+1} \Psi_N)^{1/2}
\]

and proceed similarly for other terms. Thus in summary, from (32) we get

\[
\left| \left< \Psi_N, N_+^{\hat{\gamma}} [H_N, N_+^{\hat{\gamma}}] \Psi_N \right> \right| \leq C \langle \Psi_N, (N_+ + 1)^s \Psi_N \rangle
\]

\[
+ C N^{-1/2} (\Psi_N, N_+^{\hat{\gamma}+1} \Psi_N)^{1/2} (\Psi_N, N_+^{\hat{\gamma}} \Psi_N)^{1/2}.
\]

Inserting (31) and (33) into (30), we obtain

\[
\left< \Psi_N, \left( (2\pi)^2 N_+^{\hat{\gamma}+1} - C N_+^{\hat{\gamma}} \right) \Psi_N \right> \leq C \langle \Psi_N, (N_+ + 1)^s \Psi_N \rangle
\]

\[
+ C N^{-1/2} (\Psi_N, N_+^{\hat{\gamma}+1} \Psi_N)^{1/2} (\Psi_N, N_+^{\hat{\gamma}} \Psi_N)^{1/2}.
\]

By the Cauchy–Schwarz inequality

\[
\langle \Psi_N, N_+^{\hat{\gamma}} \Psi_N \rangle \leq \left( \langle \Psi_N, N_+^{\hat{\gamma}-1} \Psi_N \rangle \right)^{1/2} \left( \langle \Psi_N, N_+^{\hat{\gamma}+1} \Psi_N \rangle \right)^{1/2}
\]

we get

\[
\left< \Psi_N, N_+^{\hat{\gamma}+1} \Psi_N \right> \leq C \left( \langle \Psi_N, N_+^{\hat{\gamma}-1} \Psi_N \rangle \right)^{1/2} \left( \langle \Psi_N, N_+^{\hat{\gamma}+1} \Psi_N \rangle \right)^{1/2}
\]
Lemma 5
Let \( U \in S \)

Proof of Lemma 5
Lemma 4 is similar and simpler.

We can now use

\[
\left\langle \Psi_N, \mathcal{N}_+^{s-1} \Psi_N \right\rangle \leq \left\langle \Psi_N, \mathcal{N}_+^{s-3} \Psi_N \right\rangle^{1/2} \left\langle \Psi_N, \mathcal{N}_+^{s+1} \Psi_N \right\rangle^{1/2}
\]

and obtain

\[
\left\langle \Psi_N, \mathcal{N}_+^{s+1} \Psi_N \right\rangle^{1/4} \leq C \left\langle \Psi_N, \mathcal{N}_+^{s-3} \Psi_N \right\rangle^{1/4}
\]

+ \( CN^{-1/2} \left\langle \Psi_N, \mathcal{N}_+^{s+1} \Psi_N \right\rangle^{1/4} \left\langle \Psi_N, \mathcal{N}_+^{s-1} \Psi_N \right\rangle^{1/4}. \)

Telescoping this inequality and using [13, Lemma 5] we arrive at a bound on \( \left\langle \Psi_N, \mathcal{N}_+^{s+1} \Psi_N \right\rangle \) that is uniform in \( N \). This gives the desired result for odd powers of \( \mathcal{N}_+ \).

In order to put Lemma 3 in a good use, we will also need the fact that the moments of \( \mathcal{N}_+ \) are essentially stable under the actions of the Bogoliubov transformation and the cubic transformation.

Lemma 4
Let \( U_B \) be given in (7). Then

\[
U_B \mathcal{N}_+^k U_B^* \leq C_k (\mathcal{N}_+ + 1)^k, \quad \forall k \in \mathbb{N}.
\]  (35)

Lemma 5
Let \( U_S = e^S \) be given in (25). Then for all \( t \in [-1, 1] \) and \( k \in \mathbb{N} \),

\[
ee^{tS} (\mathcal{N}_+ + 1)^k e^{-tS} \leq C_k (\mathcal{N}_+ + 1)^k. \]  (36)

The results in Lemma 4 and Lemma 5 are well-known. For the completeness, let us quickly explain the proof of Lemma 5, following the strategy in [2, Proposition 4.2] (the proof of Lemma 4 is similar and simpler).

Proof of Lemma 5
Take a normalized vector \( \Phi \in \mathcal{F}_+ \) and define

\[
f(t) = \langle \Phi, e^{tS} (\mathcal{N}_+ + 1)^k e^{-tS} \Phi \rangle, \quad \forall t \in [-1, 1].
\]

Then

\[
\partial_t f(t) = \langle \Phi, e^{tS} [S, (\mathcal{N}_+ + 1)^k] e^{-tS} \Phi \rangle
\]

\[
= \frac{2}{\sqrt{N}} \text{Re} \left\langle \Phi, e^{tS} \sum_{p,q \neq 0 \atop p+q \neq 0} \eta_{p,q} a_{p+q}^* a_{-p}^* a_{-q}^* \mathbb{1}_{\leq N} \Theta_k (\mathcal{N}_+) e^{-tS} \Phi \right\rangle
\]

with

\[
\Theta_k (\mathcal{N}_+) = (\mathcal{N}_+ + 1)^k - (\mathcal{N}_+ + 4)^k.
\]

Here we have used

\[
[a_{p+q}^* a_{-p}^* a_{-q}^* \mathbb{1}_{\leq N}, (\mathcal{N}_+ + 1)^k] = a_{p+q}^* a_{-p}^* a_{-q}^* \mathbb{1}_{\leq N} \Theta_k (\mathcal{N}_+).
\]
It is obvious that $|\Theta_k(N_+)| \leq C_k(N_+ + 1)^{k-1}$. Combining with the summability (27) and the Cauchy–Schwarz inequality we obtain

$$
\left| \partial_t f(t) \right| \leq \frac{2}{\sqrt{N}} \left( \sum_{p,q \neq 0 \atop p+q \neq 0} \left\| (N_+ + 1)^{(k-3)/2} a_{p+q} a_{-p-q} e^{-tS} \Phi \right\|^2 \right)^{1/2} 
\times \left( \sum_{p,q \neq 0 \atop p+q \neq 0} \left| \eta_{p,q} \right|^2 \left\| 1^{\leq N} (N_+ + 1)^{(3-k)/2} \Theta_k(N_+) e^{-tS} \Phi \right\|^2 \right)^{1/2}
\leq \frac{C_k}{\sqrt{N}} \left\| N_+^{k/2} e^{-tS} \Phi \right\| \left\| 1^{\leq N} (N_+ + 1)^{(k+1)/2} e^{-tS} \Phi \right\|.
$$

(37)

Thanks to the cut-off, we can bound

$$
\left\| 1^{\leq N} (N_+ + 1)^{(k+1)/2} \right\| \leq \sqrt{N + 1} N_+^{k/2}.
$$

Thus (37) implies that

$$
\left| \partial_t f(t) \right| \leq C_k \left\| (N_+ + 1)^{(k+1)/2} e^{-tS} \Phi \right\|^2 = C_k f(t)
$$

From Grönwall’s lemma, it follows that

$$
f(t) \leq C_k f(0), \quad \forall t \in [-1, 1].
$$

(38)

Since the latter bound is uniform in $\Phi$, we get the desired operator inequality.

We will also need the following refinement of Lemma 5.

**Lemma 6** Let $U_S = e^S$ be given in (25). Then for all $t \in [-1, 1]$ and $k \in \mathbb{N}$,

$$
e^{tS} N_+^k e^{-tS} \leq C_k \left( N_+^k + \frac{(N_+ + 1)^{k+1}}{N} \right).
$$

(39)

**Proof** Take a normalized vector $\Phi \in \mathcal{F}_+$ and define

$$
g(t) = \langle \Phi, e^{tS} N_+^k e^{-tS} \Phi \rangle, \quad \forall t \in [-1, 1].
$$

Then proceeding similarly to (37), we have

$$
\left| \partial_t g(t) \right| \leq \frac{C_k}{\sqrt{N}} \left\| N_+^{k/2} e^{-tS} \Phi \right\| \left\| 1^{\leq N} N_+^{(k+1)/2} e^{-tS} \Phi \right\| \leq \frac{C_k}{\sqrt{N}} \sqrt{g(t) f(t)}
$$

with $f(t)$ being defined in the proof of Lemma 5. Using (38) and the Cauchy–Schwarz inequality we obtain

$$
\left| \partial_t g(t) \right| \leq C_k \left( g(t) + \frac{(N_+ + 1)^{k+1}}{N} \right), \quad \forall t \in [-1, 1].
$$

From Grönwall’s lemma, it follows that

$$
g(t) \leq C_k \left( g(0) + \frac{1}{N} \langle \Phi, (N_+ + 1)^{k+1} \Phi \rangle \right), \quad \forall t \in [-1, 1].
$$

The latter bound is uniform in $\Phi$ and it implies the desired conclusion.
3 Excitation Hamiltonian

In this section, we study the action of the transformation $U_N$ in (12). By conjugating $H_N$ with $U_N$, we can factor out the contribution of the condensation. More precisely, we have

**Lemma 7** We have the operator identity on $\mathcal{F}_{+}^{\leq N}$

$$U_N H_N U_N^* = \frac{N}{2} \hat{w}(0) + \mathbb{1}_{\leq N} \left( G_N + R_1 \right) \mathbb{1}_{\leq N}$$

where

$$G_N = \mathbb{H}_{\text{Bog}} + \frac{N_+ (1 - N_+)}{2(N - 1)} \hat{w}(0) + \sum_{p \neq 0} \frac{1 - N_+}{N - 1} \hat{w}(p) a_p^* a_p$$

$$+ \left( \frac{1}{2} \sum_{p \neq 0} \hat{w}(p) a_p^* a_{-p}^* - \frac{1}{2} \frac{N_+}{N} + \text{h.c.} \right)$$

$$+ \left( \frac{1}{\sqrt{N}} \sum_{\ell, p \neq 0, \ell + p \neq 0} \hat{w}(\ell) a_{p + \ell}^* a_{-\ell}^* a_p + \text{h.c.} \right)$$

$$+ \frac{1}{2(N - 1)} \sum_{k, p \neq 0, \ell \neq -p, k} \hat{w}(\ell) a_{p + \ell}^* a_{k - \ell}^* a_k a_p$$

and the error term $R_1$ satisfies the quadratic form estimate

$$\pm R_1 \leq \frac{C (N_+ + 1)^3}{N^{3/2}}.$$ 

Moreover, we have the operator inequality on $\mathcal{F}_+$

$$\mathbb{1}_{\leq N} U_N H_N U_N^* \mathbb{1}_{\leq N} \leq \frac{N}{2} \hat{w}(0) + G_N + \frac{C (N_+ + 1)^3}{N^{3/2}}.$$

**Proof** A straightforward computation using the relations (12) shows that

$$U_N H_N U_N^* = \frac{N}{2} \hat{w}(0) + \frac{N_+ (1 - N_+)}{2(N - 1)} \hat{w}(0) + \sum_{p \neq 0} \left( p^2 + \frac{N - N_+}{N - 1} \hat{w}(p) \right) a_p^* a_p$$

$$+ \frac{1}{2} \left( \sum_{p \neq 0} \hat{w}(p) a_p^* a_{-p}^* \sqrt{(N - N_+)(N - N_+ - 1)} \frac{1}{N - 1} + \text{h.c.} \right)$$

$$+ \left( \sum_{\ell, p \neq 0, \ell + p \neq 0} \hat{w}(\ell) a_{p + \ell}^* a_{-\ell}^* a_p \sqrt{N - N_+} \frac{1}{N - 1} + \text{h.c.} \right)$$

$$+ \frac{1}{2(N - 1)} \sum_{k, p \neq 0, \ell \neq -p, k} \hat{w}(\ell) a_{p + \ell}^* a_{k - \ell}^* a_k a_p.$$

This operator identity holds on $\mathcal{F}_{+}^{\leq N}$. For further analysis, we will expand $\sqrt{N - N_+}$ and $\sqrt{(N - N_+)(N - N_+ - 1)}$, making the effective expressions well-defined on the whole
By the Cauchy–Schwarz inequality, we have the quadratic form estimates
\[ \left| \frac{\sqrt{N-N_+}}{N-1} \right|^{1/2} \leq \frac{C(N_+ + 1)}{N^{3/2}} \]
and
\[ \left| \frac{(N-N_+)(N-N_+-1)}{N-1} - 1 - \frac{1-2N_+}{2N} \right| \leq \frac{C(N_+ + 1)^2}{N^2}. \]
The operator inequalities (40) and (41) hold on \( \mathcal{F}_+^{\leq N} \). Thus we can write
\[ U_N H_N U_N^* = \frac{N}{2} \hat{w}(0) + \mathbb{H}_{\text{Bog}} + \mathcal{G}_N + R_1 \]
with \( \mathcal{G}_N \) given in the statement of Lemma 7 and with the error term \( R_1 = R_{1a} + R_{1b} \) where
\[ R_{1a} = \frac{1}{2} \sum_{p \neq 0} \hat{w}(p) a_p^* a_{-p} \left( \frac{\sqrt{(N-N_+)(N-N_+-1)}}{N-1} - 1 - \frac{1-2N_+}{2N} \right) + \text{h.c.}, \]
\[ R_{1b} = \sum_{\ell, p \neq 0, \ell+p \neq 0} \hat{w}(\ell) a_p^* a_{-\ell} a_{\ell+p} \left( \frac{\sqrt{N-N_+}}{N-1} - \frac{1}{\sqrt{N}} \right) + \text{h.c.}. \]
By the Cauchy–Schwarz inequality, we have the quadratic form estimates
\[
\pm R_{1a} \leq N^{2} \sum_{p \neq 0} a_p^* a_{-p} (N_+ + 1) a_{-p} a_p + N^2 \sum_{p \neq 0} |\hat{w}(p)|^2 (N_+ + 1)^{-1/2} \times \left( \frac{(N-N_+)(N-N_+-1)}{N-1} - 1 - \frac{1-2N_+}{2N} \right)^2 (N_+ + 1)^{-1/2} \leq \frac{C(N_+ + 1)^3}{N^2}
\]
and
\[
\pm R_{1b} \leq N^{-3/2} \sum_{\ell, p \neq 0, \ell+p \neq 0} a_p^* a_{-\ell} a_{-p} a_{\ell+p} \times \frac{N^{3/2}}{(N+1)^3} + N^3 / 2 \sum_{\ell, p \neq 0, \ell+p \neq 0} |\hat{w}(\ell)|^2 \left( \frac{\sqrt{N-N_+}}{N-1} - \frac{1}{\sqrt{N}} \right) a_p^* a_p \left( \frac{\sqrt{N-N_+}}{N-1} - \frac{1}{\sqrt{N}} \right) \leq \frac{C(N_+ + 1)^3}{N^{3/2}}.
\]
This completes the first part of Lemma 7.

Now let us turn to the operator inequality on the Fock space \( \mathcal{F}_+ \). We have proved that
\[ 1 \leq N U_N H_N U_N^* 1 \leq 1 \leq N \left( \frac{N}{2} \hat{w}(0) + \mathcal{G}_N + \frac{C(N_+ + 1)^3}{N^{3/2}} \right) 1 \leq N. \]
Let us compare the right side of (42) with the corresponding version without the cut-off \( 1 \leq N \). First, consider the terms commuting with \( N_+ \). Since
\[ N \hat{w}(0) + \sum_{p \neq 0} |p|^2 a_p^* a_p + \frac{1}{2(N-1)} \sum_{k, p \neq 0, \ell \neq -p, k} \hat{w}(\ell) a_p^* a_k^* a_\ell a_p a_k + \frac{C(N_+ + 1)^3}{N^{3/2}} \geq 0, \]
this operator is not smaller than its product with the cut-off $\mathbb{1}^{\leq N}$. Moreover, using

$$\mathbb{1}^{>N} = \mathbb{1} - \mathbb{1}^{\leq N} = \mathbb{1}(N_+ > N) \leq \frac{N_+}{N}$$

we have

$$\pm \mathbb{1}^{>N} \left( \frac{N_+(1-N_+)}{2(N-1)} \hat{w}(0) + \sum_{p \neq 0} \frac{1-N_+}{N-1} \hat{w}(p) a_p^* a_p \right) \leq \mathbb{1}^{>N} \frac{C(N_+ + 1)^2}{N} \leq \frac{C(N_+ + 1)^3}{N^2}.$$  

Finally, consider

$$X := \left( \frac{1}{2} \sum_{p \neq 0} \hat{w}(p) a_p^* a_{-p}^* \left( 1 + \frac{1-N_+}{N-1} \right) + \text{h.c.} \right) + \left( \frac{1}{\sqrt{N}} \sum_{\ell, p \neq 0, \ell + p \neq 0} \hat{w}(\ell) a_{p+\ell}^* a_{-\ell-p}^* + \text{h.c.} \right).$$

By the Cauchy–Schwarz inequality

$$\pm X \leq \sum_{p \neq 0} a_p^* a_{-p}(N_+ + 1)^{-1} a_{-p} a_p + \sum_{p \neq 0} |\hat{w}(p)|^2 (N_+ + 1) \left( 1 + \frac{1-N_+}{N-1} \right)^2$$

$$+ \frac{1}{N} \sum_{\ell, p \neq 0, \ell + p \neq 0} a_{p+\ell}^* a_{-\ell} a_{p+\ell} + \sum_{\ell, p \neq 0, \ell + p \neq 0} |\hat{w}(\ell)|^2 a_p^* a_p$$

$$\leq C \left[ (N_+ + 1) + \frac{(N_+ + 1)^2}{N} + \frac{(N_+ + 1)^3}{N^2} \right].$$

Moreover, since $X$ changes the number of particles by at most 2, we have

$$X + \mathbb{1}^{>N} X \mathbb{1}^{>N} - \mathbb{1}^{\leq N} X \mathbb{1}^{\leq N} = \mathbb{1}^{>N} X + X \mathbb{1}^{>N} = \mathbb{1}^{>N} X \mathbb{1}^{>N-2} + \mathbb{1}^{>N-2} X \mathbb{1}^{>N}.$$  

Hence, combining with the above bound on $\pm X$ we find that

$$\pm (X - \mathbb{1}^{\leq N} X \mathbb{1}^{\leq N}) = \pm \left( \mathbb{1}^{>N} X \mathbb{1}^{>N-2} + \mathbb{1}^{>N-2} X \mathbb{1}^{>N} - \mathbb{1}^{>N} X \mathbb{1}^{>N} \right)$$

$$\leq C \left[ (N_+ + 1) + \frac{(N_+ + 1)^2}{N} + \frac{(N_+ + 1)^3}{N^2} \right] \mathbb{1}^{>N-2}$$

$$\leq \frac{C(N_+ + 1)^3}{N^2}.$$  

This completes the proof of the operator inequality on $\mathcal{F}_+$ in Lemma 7. □

### 4 Quadratic transformation

Recall that the Bogoliubov transformation $U_B$ in (7) diagonalizes $\mathbb{H}_{\text{Bog}}$ as in (9). In this section, we will study the action of $U_N$ on the operator $\mathcal{G}_N$. We have
Lemma 8 Let $G_N$ be given in Lemma 7. Then we have the operator identity on $\mathcal{F}_+$

$$U_B G_N U_B^* = \langle 0 | U_B G_N U_B^* | 0 \rangle + \sum_{p \neq 0} e(p) a_p^* a_p + C_N + R_2$$

where

$$\mathcal{C}_N = \frac{1}{\sqrt{N}} \sum_{p,q \neq 0, p+q \neq 0} \hat{w}(p) \left( (\sigma_{p+q} \sigma_{-p} \gamma_q + \gamma_{p+q} \gamma_p \sigma_q) a_p^* a_{-p}^* a_q^* a_{-q}^* + \text{h.c.} \right)$$

and the error term $R_2$ satisfies

$$\pm R_2 \leq \frac{C}{\sqrt{N}} N^2 + \frac{C(N_+ + 1)^3}{N^{3/2}}.$$

Proof Let us decompose

$$G_N - \mathcal{H}_{\text{Bog}} = \mathcal{D}_N + \mathcal{C}_N$$

where

$$\mathcal{C}_N = \frac{1}{\sqrt{N}} \sum_{\ell, p \neq 0, \ell + p \neq 0} \hat{w}(\ell) a_p^* a_{-p}^* + \text{h.c.}$$

Non-cubic terms. Let us prove that $U_B \mathcal{D}_N U_B^* - \langle 0 | U_B \mathcal{D}_N U_B^* | 0 \rangle$ contains only the terms of the form

$$\sum_{m_1, \ldots, m_s, n_1, \ldots, n_t \neq 0} A_{m_1, \ldots, m_s, n_1, \ldots, n_t} a_{m_1}^* \ldots a_{m_s}^* a_{n_1} \ldots a_{n_t}$$

(43)

with $1 \leq s + t \leq 4$ and the coefficients $A_{m_1, \ldots, m_s, n_1, \ldots, n_t}$ satisfy

$$\sup_{m_1, \ldots, m_s \neq 0} \sum_{n_1, \ldots, n_t \neq 0} |A_{m_1, \ldots, m_s, n_1, \ldots, n_t}| \leq \frac{C}{N}, \quad \sup_{n_1, \ldots, n_t \neq 0} \sum_{m_1, \ldots, m_s \neq 0} |A_{m_1, \ldots, m_s, n_1, \ldots, n_t}| \leq \frac{C}{N^2}.$$

(44)

Let us start with the quadratic terms involving $a_p^* a_{-p}^*$. Using (8) and the CCR (3) we have

$$U_B \left( \frac{1}{4N} \sum_{p \neq 0} \hat{w}(p) a_p^* a_{-p}^* \right) U_B^* = \frac{1}{4N} \sum_{p \neq 0} \hat{w}(p) (\sigma_p a_p^* + \gamma_p a_{-p}^*) (\sigma_p a_p^* + \gamma_p a_{-p}^*)$$

$$= \frac{1}{4N} \sum_{p \neq 0} \hat{w}(p) \left[ \sigma_p^2 a_p^* a_{-p}^* + 2\sigma_p \gamma_p a_p^* a_{-p}^* + \gamma_p^2 a_p^* a_{-p}^* + \sigma_p \gamma_p a_{-p}^* a_p^* \right].$$

(45)

Obviously the constant in (45) satisfies

$$\frac{1}{4N} \sum_{p \neq 0} \hat{w}(p) \sigma_p \gamma_p = \langle 0 | U_B \left( \frac{1}{4N} \sum_{p \neq 0} \hat{w}(p) a_p^* a_{-p}^* \right) U_B^* | 0 \rangle.$$
\[
\frac{1}{4N} \sum_{p \neq 0} \hat{a}(p) \gamma^2_p a_{-p} a_p = \frac{1}{4N} \sum_{p, q} \hat{a}(p) \gamma^2_p \delta_p = -q a_p a_q.
\] (48)

All of the sums in (46), (47), (48) are of the general form (43)–(44), thanks to the uniform bounds (10). The quadratic terms involving \(a^*_p a_p\) can be treated similarly.

Next, consider

\[
U_B \left( \frac{1}{2N} \sum_{p \neq 0} \hat{w}(p) a^*_p a^*_p N_+ \right) U_B^* = U_B \left( \frac{1}{2N} \sum_{p, q \neq 0} \hat{w}(p) a^*_p a^*_p a_q a_q \right) U_B^*
\]

\[
= \frac{1}{2N} \sum_{p, q \neq 0} \hat{w}(p) (\sigma^2_p a^*_p a^*_p + 2 \sigma^*_p \gamma_p a^*_p a_p + \gamma^2_p a_p a_p + \gamma_p \gamma_p) \times
\]

\[
\times \left[ (\sigma^2_q + \gamma^2_q) a^*_q a_q + \sigma_q \gamma_q (a^*_q a^*_q + a_{-q} a_q) + \gamma^2_q \right]
\]

\[
= \frac{1}{2N} \sum_{p, q \neq 0} \hat{w}(p) [\sigma^2_p a^*_p a^*_p + \sigma_p \gamma_p] \left[ (\sigma^2_q + \gamma^2_q) a^*_q a_q + \sigma_q \gamma_q (a^*_q a^*_q + a_{-q} a_q) + \gamma^2_q \right]
\]

\[
\]

It is straightforward to see that, except the constant

\[
\frac{1}{2N} \sum_{p, q \neq 0} \hat{w}(p) \sigma_p \gamma_p \gamma^2_q + \frac{1}{2N} \sum_{p, q \neq 0} \hat{w}(p) \gamma^2_p \sigma_q \gamma_q (\delta_{p, q} + \delta_{p, -q})
\]

\[
= \left\{ 0 \left| U_B \left( \frac{1}{2N} \sum_{p} \hat{w}(p) a^*_p a^*_p N_+ \right) U_B^* \right| 0 \right\}.
\]

all other terms in (49) can be written as in (43), with the corresponding bound (44) following from (10). By the same argument, we can show that the terms involving \(a^*_p a_p a_{N_+} a_{N_+ - 1}\) are of the general form (43) and (44).

Next, let us bound the terms of the general form (43) and (44). We consider the case \(s \geq t\) (the other case is treated similarly). By the Cauchy–Schwarz inequality

\[
Y^* Z + Z^* Y \leq Y^* Y + Z^* Z
\]

we have

\[
\pm \left( \sum_{m_1, ..., m_s, n_1, ..., n_t} A_{m_1, ..., m_s, n_1, ..., n_t} a_{m_1}^* \cdots a_{m_s}^* a_{n_1} \cdots a_{n_t} \right).
\]
\[ \leq \varepsilon^{-1} \sum_{m_1, \ldots, m_n, n_1, \ldots, n_t} |A_{m_1, \ldots, m_n, n_1, \ldots, n_t}| a_{m_1}^* \cdots a_{m_n}^* (N_+ + 5)^{1-s} a_{m_1} \cdots a_{m_n} + \varepsilon \sum_{m_1, \ldots, m_n, n_1, \ldots, n_t} |A_{m_1, \ldots, m_n, n_1, \ldots, n_t}| a_{n_1}^* \cdots a_{n_t}^* (N_+ + 5)^{s-1} a_{n_1} \cdots a_{n_t} \]

\[ \leq \varepsilon^{-1} \left( \sup_{m_1, \ldots, m_n, n_1, \ldots, n_t} \sum_{m_1, \ldots, m_n} |A_{m_1, \ldots, m_n, n_1, \ldots, n_t}| \right) \sum_{m_1, \ldots, m_n} a_{m_1}^* \cdots a_{m_n}^* (N_+ + 5)^{1-s} a_{m_1} \cdots a_{m_n} + \varepsilon \left( \sup_{n_1, \ldots, n_t, m_1, \ldots, m_n} \sum_{m_1, \ldots, m_n} |A_{m_1, \ldots, m_n, n_1, \ldots, n_t}| \right) \sum_{n_1, \ldots, n_t} a_{n_1}^* \cdots a_{n_t}^* (N_+ + 5)^{s-1} a_{n_1} \cdots a_{n_t} \]

\[ \leq \varepsilon^{-1} \frac{C}{N} N_+ + \varepsilon \frac{C}{N} (N_+ + 1)^{t+s-1} \] (50)

for all \( \varepsilon > 0 \). Note that if \( \min(t, s) \geq 1 \), then on the right side of (50) we can replace \((N_+ + 1)^{t+s-1}\) by \( N_+^{t+s-1} \).

In particular, for the non-cubic term \( \tilde{D}_N \), using (50) with \( \varepsilon = N^{-1/2} \) and \( t + s \leq 4 \) we get

\[ \pm \left( U_B \tilde{D}_N U_B^* - (0|U_B \tilde{D}_N U_B^*|0) \right) \leq \frac{C}{\sqrt{N}} N_+ + \frac{C(N_+ + 1)^3}{N^{3/2}}. \] (51)

**Cubic terms.** By using (8) we have

\[ U_B \left( \frac{1}{\sqrt{N}} \sum_{\ell, p \neq 0, \ell+p \neq 0} \hat{w}(\ell) a_{p+\ell}^* a_{-\ell}^* a_p \right) U_B^* \]

\[ = \frac{1}{\sqrt{N}} \sum_{\ell, p \neq 0, \ell+p \neq 0} \hat{w}(\ell)(\sigma_{p+\ell} a_{p+\ell}^* + \gamma_{p+\ell} a_{-p-\ell}^*)(\sigma_{-\ell} a_{-\ell}^* + \gamma_{-\ell} a_p) + \gamma_{p+\ell} a_{-p-\ell} a_p \]

\[ = \frac{1}{\sqrt{N}} \sum_{\ell, p \neq 0, \ell+p \neq 0} \hat{w}(\ell) \left( \sigma_{p+\ell} \sigma_{-\ell} a_{p+\ell}^* a_{-\ell}^* + \gamma_{p+\ell} \gamma_{-\ell} a_{p+\ell} a_{-\ell}^* \right) \]

\[ + \frac{1}{\sqrt{N}} \sum_{\ell, p \neq 0, \ell+p \neq 0} \left( \hat{w}(\ell) \sigma_{p+\ell} \gamma_p a_{p+\ell} + \hat{w}(p) a_{p+\ell} \right) \gamma_{p+\ell} \gamma_{-\ell} a_{p+\ell} a_{-\ell} \]

\[ + \frac{1}{\sqrt{N}} \sum_{\ell, p \neq 0, \ell+p \neq 0} \left( \hat{w}(\ell) \sigma_{p+\ell} \gamma_p a_{p+\ell} + \hat{w}(p) a_{p+\ell} \right) \gamma_{p+\ell} \gamma_{-\ell} a_{p+\ell} a_{-\ell}. \] (52)

By using (10), we can write the last sum of (52) as

\[ \sum_{p, q, r} \tilde{A}_{p, q, r} a_{p}^* a_q a_r \]

with

\[ \sup_p \sum_{q, r} |\tilde{A}_{p, q, r}| \leq \frac{C}{\sqrt{N}}, \quad \sup_{q, r} \sum_p |\tilde{A}_{p, q, r}| \leq \frac{C}{\sqrt{N}}. \]

Using (50) with \( \varepsilon = 1, t = 1, s = 2 \), we get

\[ \pm \left( \sum_{p, q, r} \tilde{A}_{p, q, r} a_{p}^* a_q a_r + \text{h.c.} \right) \leq \frac{C}{N} (N_+ + N_+^2) \leq \frac{CN_+^2}{N}. \] (53)
Here $N_+ \leq N_+^2$ since the spectrum of $N_+$ is $\{0, 1, 2, \ldots\}$. The second sum on the right side of (52) can be treated by the same way. Thus from (52) and its adjoint, we have
\[
\pm \left( U_B \tilde{C}_N U_B^* - \frac{1}{\sqrt{N}} \sum_{\ell, \ell' \neq 0} \tilde{w}(\ell) \left[ (\sigma_{\ell+p} \sigma_{\ell} \gamma_{\ell} a_{\ell+p}^* a_{\ell}^* - a_{\ell+p} a_{\ell}^* \gamma_{\ell} \right] + \gamma_{\ell+p} \gamma_{\ell} \sigma_{\ell} a_{\ell-p} a_{\ell} \right) + \text{h.c.} \right) \right) 
\leq C N_+^2 \frac{N}{N_+}
\]
which is equivalent to
\[
\pm \left( U_B \tilde{C}_N U_B^* - C_N \right) \leq C N_+^2 \frac{N}{N_+}. \tag{54}
\]
In particular, (54) implies that
\[
\langle 0 | U_B \tilde{C}_N U_B^* | 0 \rangle = 0.
\]
Therefore, from (9), (51) and (54) we obtain the desired conclusion of Lemma 8.

\section{Cubic transformation}

To factor out the cubic term $C_N$ in Lemma 8, we will use a cubic renormalization. We will prove

\begin{lemma}
Let $C_N$ be the cubic term in Lemma 8 and let $U_S$ be given in (25). Then we have the operator identity on Fock space $\mathcal{F}_+$
\[
U_S U_B \tilde{G}_N U_B^* U_S^* = \left( 0 | U_B \tilde{G}_N U_B^* U_S^* | 0 \right) + \sum_{p \neq 0} e(p) a_p^* a_p + R_3
\]
where
\[
\pm R_3 \leq \frac{C N_+^2}{\sqrt{N}} + \frac{C (N_+ + 1)^4}{N^{3/2}}.
\]
\end{lemma}

\begin{proof}
Recall that from Lemma 8 we have
\[
U_S U_B \tilde{G}_N U_B^* U_S^* = \left( 0 | U_B \tilde{G}_N U_B^* | 0 \right) + U_S \left( d\Gamma(\xi) + C_N \right) U_S^* + U_S R_2 U_S^* \tag{55}
\]
with
\[
d\Gamma(\xi) = \sum_{p \neq 0} e(p) a_p^* a_p, \quad \pm R_2 \leq \frac{C}{\sqrt{N}} N_+^2 + \frac{C (N_+ + 1)^3}{N^{3/2}}.
\]
Thanks to Lemmas 5 and 6, we find that
\[
\pm U_S R_2 U_S^* \leq \frac{C}{\sqrt{N}} N_+^2 + \frac{C (N_+ + 1)^3}{N^{3/2}}.
\]
Thus this error term is part of $R_3$.

\square
For the main term, we use $U_S = e^S$ and the Duhamel formula
\[ e^X Y e^{-X} = Y + \int_0^1 e^{tX} [X, Y] e^{-tX} dt \] (56)
we can write
\[
e^S \left( d\Gamma(\xi) + C_N \right) e^{-S} = d\Gamma(\xi) + C_N + \int_0^1 e^{tS} \left( [S, d\Gamma(\xi)] + [S, C_N] \right) e^{-tS} dt
\]
\[
= d\Gamma(\xi) + \int_0^1 e^{tS} \left( C_N + [S, d\Gamma(\xi)] + [S, C_N] \right) e^{-tS} dt - \int_0^1 \int_0^t e^{sS} [S, C_N] e^{-sS} ds dt.
\] (57)

**Controlling** $C_N + [S, d\Gamma(\xi)]$. Since $d\Gamma(\xi)$ commutes with $\mathcal{N}_+$ and
\[
[a^*_k a_k, a^*_{p+q} a^*_{-p} a^*_{-q}] = (\delta_{k,p+q} + \delta_{k,-p} + \delta_{k,-q})a^*_p a^*_q a^*_{-p} a^*_{-q}
\]
we find that
\[
[d\Gamma(\xi), S] = \frac{1}{\sqrt{N}} \sum_{\substack{p, q \neq 0 \\ p+q \neq 0 \\ k \neq 0}} e(k) \eta_{p,q} [a^*_k a_k, a^*_{p+q} a^*_{-p} a^*_{-q}] \mathbb{1} \leq N + \text{h.c.}
\]
\[
= \frac{1}{\sqrt{N}} \sum_{\substack{p, q \neq 0 \\ p+q \neq 0}} (e(p + q) + e(p) + e(q)) \eta_{p,q} a^*_p a^*_q a^*_{-p} a^*_{-q} \mathbb{1} \leq N + \text{h.c.}
\]
\[
= \frac{1}{\sqrt{N}} \sum_{\substack{p, q \neq 0 \\ p+q \neq 0}} \tilde{w}(p) (\sigma_{p+q} \sigma_p \gamma_q + \gamma_{p+q} \gamma_p \sigma_q) a^*_p a^*_q a^*_{-p} a^*_{-q} \mathbb{1} \leq N + \text{h.c.}
\]
which is equivalent to
\[
C_N + [S, d\Gamma(\xi)] = \frac{1}{\sqrt{N}} \sum_{\substack{p, q \neq 0 \\ p+q \neq 0}} \tilde{w}(p) (\sigma_{p+q} \sigma_p \gamma_q + \gamma_{p+q} \gamma_p \sigma_q) a^*_p a^*_q a^*_{-p} a^*_{-q} \mathbb{1} > N + \text{h.c.}
\]
where $\mathbb{1} > N = \mathbb{1} - \mathbb{1} \leq N = \mathbb{1} (\mathcal{N}_+ > N)$. Thanks to the summability (10), we can use the Cauchy–Schwarz inequality similarly to (50) (with $\varepsilon = 1$) to get
\[
\pm \left( C_N + [S, d\Gamma(\xi)] \right) \leq \frac{C}{\sqrt{N}} \mathbb{1}_{N+} + \frac{C(N_+ + 1)^2}{\sqrt{N}} \mathbb{1}_{>N} \leq \frac{C}{\sqrt{N}} \mathbb{1}_{N+} + \frac{C(N_+ + 1)^3}{N^{3/2}}
\]
Combining with Lemmas 5 and 6 we obtain
\[
e^{tS} \left( C_N + [S, d\Gamma(\xi)] \right) e^{-tS} \leq \frac{C}{\sqrt{N}} \mathbb{1}_{N+} + \frac{C(N_+ + 1)^3}{N^{3/2}}, \ \forall t \in [-1, 1].
\] (58)

**Controlling** $[S, C_N]$. Let us decompose $S = \tilde{S} - S^>$ where
\[
\tilde{S} = \frac{1}{\sqrt{N}} \sum_{\substack{p, q \neq 0 \\ p+q \neq 0}} \eta_{p,q} a^*_p a^*_q a^*_{-p} a^*_{-q} - \text{h.c.},
\]
\[
S^> = \frac{1}{\sqrt{N}} \sum_{\substack{p, q \neq 0 \\ p+q \neq 0}} \eta_{p,q} a^*_p a^*_q a^*_{-p} a^*_{-q} \mathbb{1} > N - \text{h.c.}
\]
The main contribution comes from
\[ [C_N, \tilde{S}] = \frac{1}{N} \sum_{p,q,r \neq 0} \sum_{p',q',r' \neq 0} \delta_{p+q+r=0} \delta_{p'+q'+r'=0} \eta_{p',q'} \hat{w}(p)(\sigma_r \sigma_p \gamma_q + \gamma_r \gamma_p \sigma_q) \times \]
\[ a_r a_p q, a_r^* a_p^* q, a_r^* a_p a_r^* q, + h.c. \]
\[ = \frac{1}{N} \sum_{p,q,r \neq 0} \sum_{p',q',r' \neq 0} \delta_{p+q+r=0} \delta_{p'+q'+r'=0} \eta_{p',q'} \hat{w}(p)(\sigma_r \sigma_p \gamma_q + \gamma_r \gamma_p \sigma_q) \times \]
\[ \left( \delta_{r=r'} a_r a_p q, a_r^* a_p^* q + \delta_{r=r'} a_r a_p q, a_r^* a_p^* q + \delta_{r=q} a_r a_p q, a_r^* a_p^* q + \delta_{r=q} a_r a_p q, a_r^* a_p a_r \right) + h.c. \]

By using the CCR (3) as in (49), we can write
\[ [\tilde{S}, C_N] = \langle 0 | [\tilde{S}, C_N] | 0 \rangle + \sum_{p,q \neq 0} A_{pq} a_p^* a_q + \sum_{p,q,r,k \neq 0} B_{pqrk} a_p^* a_q a_r a_k \]
where
\[ \sup_{q \neq 0} \sum_{p \neq 0} |A_{pq}| \leq \frac{C}{N} \]
\[ \sup_{p \neq 0} \sum_{q \neq 0} |A_{pq}| \leq \frac{C}{N} \]
\[ \sup_{p,q,r \neq 0} \sum_{r,k \neq 0} |B_{pqrk}| \leq \frac{C}{N} \]
\[ \sup_{r,k \neq 0} \sum_{p,q \neq 0} |B_{pqrk}| \leq \frac{C}{N} \]

By the Cauchy–Schwarz inequality as in (50), we get
\[ \pm \left( [\tilde{S}, C_N] - \langle 0 | [\tilde{S}, C_N] | 0 \rangle \right) \leq C \frac{\sqrt{N}}{\sqrt{N}} + \frac{C(N_+ + 1)^3}{N^{3/2}}. \] (59)

It remains to bound \([S^>, C_N]\). From the explicit form of \(S^>\) and \(C_N\), it is straightforward to check that
\[ \pm [S^>, C_N] = \pm \left( (S^>) C_N + C_N (S^>)^* \right) \leq (S^>) (S^>)^* + C_N \leq C \frac{N (N_+ + 1)^3}{N} \]

On the other hand, we observe that
\[ S^> = 1^{>N-4} S^> 1^{>N-4} \]
and that \(C_N\) does not change the number of particles more than 3. Therefore,
\[ \pm [S^>, C_N] = \pm 1^{>N-7} [S^>, C_N] 1^{>N-7} \leq C \frac{N (N_+ + 1)^3}{N^2} 1^{>N-7} \leq C \frac{N (N_+ + 1)^4}{N^2}. \] (60)

Moreover, it is obvious that
\[ \langle 0 | [S^>, C_N] | 0 \rangle = 0 \]
for \(N \geq 10\). Thus from (59) and (60) we obtain
\[ \pm \left( [S, C_N] - \langle 0 | [S, C_N] | 0 \rangle \right) \leq C \frac{\sqrt{N}}{\sqrt{N}} + \frac{C(N_+ + 1)^3}{N^{3/2}}. \]
Combining with Lemma 5 we conclude that
\[ \pm e^{tS} \left( [S, C_N] - \langle 0 | [S, C_N] | 0 \rangle \right) e^{-tS} \leq C \frac{N_+^4}{\sqrt{N}} + \frac{C(N_+ + 1)^4}{N^{3/2}}, \quad \forall t \in [-1, 1]. \] (61)

**Conclusion.** Inserting (58) and (61) in (57) we find that
\[ \pm \left( e^{S\left(d\Gamma(\xi) + C_N\right)} e^{-S} - \frac{1}{2} \langle 0 | [S, C_N] | 0 \rangle \right) \leq C \frac{N_+^2}{\sqrt{N}} + \frac{C(N_+ + 1)^4}{N^{3/2}}. \]

Combining with (55) we deduce that
\[ \pm \left( \langle 0 | U_S U_B G_N U_B^* U_S^* | 0 \rangle - \langle 0 | U_B G_N U_B^* | 0 \rangle - \frac{1}{2} \langle 0 | [S, C_N] | 0 \rangle \right) \leq C \frac{N_+^2}{\sqrt{N}} + \frac{C(N_+ + 1)^4}{N^{3/2}}. \]

Taking the expectation of the latter bound again the vacuum, we find that
\[ \pm \left( \langle 0 | U_S U_B G_N U_B^* U_S^* | 0 \rangle - \langle 0 | U_B G_N U_B^* | 0 \rangle - \frac{1}{2} \langle 0 | [S, C_N] | 0 \rangle \right) \leq C \frac{N^3}{N^3}. \]

Thus we obtain the desired conclusion
\[ \pm \left( \langle 0 | U_S U_B G_N U_B^* U_S^* | 0 \rangle - \langle 0 | U_B G_N U_B^* | 0 \rangle - \frac{1}{2} \langle 0 | [S, C_N] | 0 \rangle \right) \leq C \frac{N_+^2}{\sqrt{N}} + \frac{C(N_+ + 1)^4}{N^{3/2}}. \]
This completes the proof of Lemma 9. \( \square \)

### 6 Proof of Theorem 2

**Proof** We will prove the ground state energy estimate
\[ E_N = \frac{N}{2} \tilde{w}(0) + \langle 0 | U_S U_B G_N U_B^* U_S^* | 0 \rangle + O(N^{-3/2}). \]

**Upper bound.** We use the following \( N \)-body trial state
\[ \tilde{\Psi}_N = \frac{1}{\| U_N^* 1 \leq N U_B^* U_S^* | 0 \|} U_N^* 1 \leq N U_B^* U_S^* | 0 \). \]

Then by the variational principle and the operator inequality on \( \mathcal{F}_+ \) in Lemma 7 we have
\[
E_N \leq \langle \tilde{\Psi}_N, H_N \tilde{\Psi}_N \rangle = \frac{1}{\| U_N^* 1 \leq N U_B^* U_S^* | 0 \|} \| U_N^* 1 \leq N U_B^* U_S^* | 0 \| \langle 0 | U_S U_B 1 \leq N U_B^* U_S^* | 0 \rangle \langle 0 | U_B U_B^* (\frac{N \tilde{w}(0)}{2} + G_N + \frac{C(N_+ + 1)^3}{N^{3/2}}) U_B^* U_S^* | 0 \rangle.
\]

By Lemmas 4 and 5 we know that
\[ \langle 0 | U_S U_B (N_+ + 1)^3 U_B^* U_S^* | 0 \rangle \leq C. \]

Consequently,
\[
\| U_N^* 1 \leq N U_B^* U_S^* | 0 \| = 1 - \langle 0 | U_S U_B 1 \leq N U_B^* U_S^* | 0 \rangle \geq 1 - \langle 0 | U_S U_B (N^3 / N^3) U_B^* U_S^* | 0 \rangle \geq 1 - CN^{-3}.
\]
Combining with Lemma 9 we find that

\[ E_N \leq \frac{1}{\|U^*_N \|^2} \left( \frac{N \hat{\omega}(0)}{2} + \left| \langle 0 | U_S U_B G_N U_B^* U^*_S | 0 \rangle \right| + \frac{C}{N^{3/2}} \right) \]

\[ \leq \frac{N \hat{\omega}(0)}{2} + \left| \langle 0 | U_S U_B G_N U_B^* U^*_S | 0 \rangle \right| + \frac{C}{N^{3/2}}. \]

In the last estimate, we have also used the simple upper bound

\[ \left| \langle 0 | U_S U_B G_N U_B^* U^*_S | 0 \rangle \right| \leq C \] (62)

which will be justified below.

**Lower bound.** Let \( \Psi_N \) be the ground state of \( H_N \) and denote \( \Phi := U_S U_B U_N \Psi_N \in \mathcal{F}_+ \). By Lemmas 3, 35 and 36, we have

\[ \langle \Phi, (N_+ + 1)^4 \Phi \rangle \leq C. \]

Then from the operator identity on \( \mathcal{F}^{-N}_+ \) in Lemma 7 it follows that

\[ E_N = \langle \Psi_N, H_N \Psi_N \rangle = \langle U_N \Psi_N, U_N H_N U^*_N U_N \Psi_N \rangle \]

\[ \geq \langle U_N \Psi_N, \left( \frac{N \hat{\omega}(0)}{2} + G_N - C \frac{(N_+ + 1)^3}{N^{3/2}} \right) U_N \Psi_N \rangle \]

\[ \geq \frac{N \hat{\omega}(0)}{2} + \langle \Phi, U_S U_B G_N U_B^* U^*_S \Phi \rangle - C N^{-3/2}. \]

Next, using Lemma 9 together with two simple estimates:

\[ \sum_{p \neq 0} e(p) a^*_p a_p \geq \left( \inf_{q \neq 0} e(q) \right) \sum_{p \neq 0} a^*_p a_p \geq (2\pi)^2 N_+ \]

and

\[ \frac{N_+^2}{\sqrt{N}} \leq \varepsilon N_+ + \frac{N_+^2}{\sqrt{N}} \mathbb{E}(N_+ > \varepsilon \sqrt{N}) \leq \varepsilon N_+ + C_{\varepsilon} N_+^4 N^{-3/2} \]

for \( \varepsilon > 0 \) small (but independent of \( N \)), we obtain

\[ U_S U_B G_N U_B^* U^*_S \geq \langle 0 | U_S U_B G_N U_B^* U^*_S | 0 \rangle + N_+ - C \frac{(N_+ + 1)^4}{N^{3/2}}. \]

Therefore,

\[ \langle \Phi, U_S U_B G_N U_B^* U^*_S \Phi \rangle \geq \langle 0 | U_S U_B G_N U_B^* U^*_S | 0 \rangle + \langle \Phi, N_+ \Phi \rangle - C N^{-3/2}. \]

Thus

\[ E_N \geq \frac{N \hat{\omega}(0)}{2} + \langle 0 | U_S U_B G_N U_B^* U^*_S | 0 \rangle + \langle \Phi, N_+ \Phi \rangle - C N^{-3/2}. \] (63)

From (63), since \( \langle \Phi, N_+ \Phi \rangle \geq 0 \) we obtain the desired energy lower bound

\[ E_N \geq \frac{N \hat{\omega}(0)}{2} + \langle 0 | U_S U_B G_N U_B^* U^*_S | 0 \rangle + O(N^{-3/2}). \]

This and the obvious upper bound \( E_N \leq \hat{\omega}(0)(N/2) \) imply the simple estimate (62). Thus the matching energy upper bound is valid, and hence we conclude that

\[ E_N = \frac{N \hat{\omega}(0)}{2} + \langle 0 | U_S U_B G_N U_B^* U^*_S | 0 \rangle + O(N^{-3/2}). \] (64)
Ground state estimates. By comparing the ground state energy expansion \((64)\) with the lower bound \((63)\) we deduce that
\[
\langle \Phi, \mathcal{N}_+ \Phi \rangle \leq CN^{-3/2}.
\] (65)

Let us write \(\Phi = (\Phi_j)_{j=0}^\infty\) with \(\Phi_j \in \mathcal{H}_+^j\). We can choose a phase factor for \(\Psi_N\) such that \(\Phi_0 \geq 0\). Then
\[
\|\Phi - |0\rangle\|^2 = |\Phi_0 - 1|^2 \leq 1 - |\Phi_0|^2 = \sum_{j \geq 1} |\Phi_j|^2 \leq \langle \Phi, \mathcal{N}_+ \Phi \rangle \leq CN^{-3/2}.
\]

Putting back the definition \(\Phi = U_S U_B U_N \Psi_N\) we obtain the norm approximation
\[
\|U_N \Psi_N - U_B^* U_S^* |0\rangle\| = \|\Phi - |0\rangle\|^2 \leq CN^{-3/2}.
\]

This completes the proof of Theorem 2. \(\square\)

7 Proof of Theorem 1

Proof Let \(\Psi_N\) be the ground state for \(H_N\). As explained in the introduction, we will decompose
\[
\gamma^{(1)}_{\Psi_N} = P \gamma^{(1)}_{\Psi_N} P + Q \gamma^{(1)}_{\Psi_N} Q + P \gamma^{(1)}_{\Psi_N} Q + Q \gamma^{(1)}_{\Psi_N} P.
\]

Diagonal terms. For \(Q \gamma^{(1)}_{\Psi_N} Q\), recall from [12, Theorem 2.2 (iii)] that
\[
U_N \Psi_N \rightarrow U_B^* |0\rangle
\]
strongly in the quadratic form of \(\mathbb{H}_{\text{Bog}}\) on \(\mathcal{F}_+\). Moreover, it is easy to see that
\[
\mathbb{H}_{\text{Bog}} \geq \frac{1}{2} \sum_{p \neq 0} |p|^2 a_p^* a_p - C \geq \mathcal{N}_+ - C
\]
(see e.g. [13, Proof of Theorem 1]). Therefore, in the limit \(N \rightarrow \infty\),
\[
\text{Tr} Q \gamma^{(1)}_{\Psi_N} Q = \langle U_N \Psi_N, \mathcal{N}_+ U_N \Psi_N \rangle \rightarrow \langle 0| U_B \mathcal{N}_+ U_B^* |0\rangle
\]
\[
= \left| 0 \right\rangle \sum_{p \neq 0} (\gamma p a_p^* + \gamma p a_{-p}^*)(\gamma p a_p + \gamma p a_{-p}^*) \left| 0 \right\rangle = \sum_{p \neq 0} \gamma_p^2.
\] (66)

Here we have used Bogoliubov’s transformation \((8)\). Similarly, for any \(p, q \neq 0\) we have
\[
\langle u_p, Q \gamma^{(1)}_{\Psi_N} Q u_q \rangle = \langle U_N \Psi_N, a_p^* a_q U_N \Psi_N \rangle \rightarrow \langle 0| U_B (a_p^* a_q) U_B^* |0\rangle
\]
\[
= \left| 0 \right\rangle (\gamma p a_p^* + \gamma p a_{-p})(\gamma q a_q + \gamma q a_{-q}) \left| 0 \right\rangle = \gamma_p^2 \delta_{p,q}.
\] (67)

From \((66)\) and \((67)\), we conclude that
\[
Q \gamma^{(1)}_{\Psi_N} Q \rightarrow \sum_{p \neq 0} \gamma_p^2 |u_p \rangle \langle u_p| \quad \text{(68)}
\]
strongly in trace class. Consequently,
\[
\text{Tr}(P \gamma^{(1)}_{\Psi_N} P) = N - \text{Tr} Q \gamma^{(1)}_{\Psi_N} Q = N - \sum_{p \neq 0} \gamma_p^2
\]
and hence
\[ \text{Tr} \left| P \gamma_{\Psi N}^{(1)} P + Q \gamma_{\Psi N}^{(1)} Q - \left( N - \sum_{p \neq 0} \gamma_p^2 \right) |u_0\rangle \langle u_0| - \sum_{p \neq 0} \gamma_p^2 |u_p\rangle \langle u_p| \right| \to 0. \]

**Off-diagonal terms.** Let us prove that
\[ \text{Tr} \left| P \gamma_{\Psi N}^{(1)} P + Q \gamma_{\Psi N}^{(1)} Q \right| \leq C N^{-1/4}. \tag{69} \]

By using \( P = |u_0\rangle \langle u_0| \) and the Cauchy–Schwarz inequality, it suffices to show that
\[ \| Q \gamma_{\Psi N}^{(1)} u_0 \|^2 \leq C N^{-1/2}. \]

Since \( \{u_p\}_{p \neq 0} \) is an orthonormal basis for \( \mathcal{H}_+ \), we have
\[ \| Q \gamma_{\Psi N}^{(1)} u_0 \|^2 = \sum_{p \neq 0} |\langle u_p, \gamma_{\Psi N}^{(1)} u_0 \rangle|^2 = \sum_{p \neq 0} |\langle \Psi_N, a_0^* a_p \Psi_N \rangle|^2. \]

Using the excitation map \( U_N \) and the relations (12) we can decompose
\[ \langle \Psi_N, a_0^* a_p \Psi_N \rangle = \langle U_N \Psi_N, \sqrt{N - N_+} a_p U_N \Psi_N \rangle \]
\[ = \sqrt{N} \langle U_N \Psi_N, a_p U_N \Psi_N \rangle + \langle U_N \Psi_N, (\sqrt{N - N_+} - \sqrt{N}) a_p U_N \Psi_N \rangle. \]

Therefore, by the Cauchy–Schwarz inequality
\[ \| Q \gamma_{\Psi N}^{(1)} u_0 \|^2 \leq 2N \sum_{p \neq 0} |\langle U_N \Psi_N, a_p U_N \Psi_N \rangle|^2 \]
\[ + 2 \sum_{p \neq 0} |\langle U_N \Psi_N, (\sqrt{N - N_+} - \sqrt{N}) a_p U_N \Psi_N \rangle|^2. \tag{70} \]

For the second sum in (70), using the Cauchy–Schwarz inequality, the simple bound
\[ \left( \sqrt{N - N_+} - \sqrt{N} \right)^2 = \left( \frac{N_+}{\sqrt{N} + \sqrt{N_+}} \right)^2 \leq \frac{N^2}{N} \]
and Lemma 3, we find that
\[ \sum_{p \neq 0} |\langle U_N \Psi_N, (\sqrt{N - N_+} - \sqrt{N}) a_p U_N \Psi_N \rangle|^2 \]
\[ \leq \sum_{p \neq 0} \| (\sqrt{N - N_+} - \sqrt{N}) U_N \Psi_N \|^2 \| a_p U_N \Psi_N \|^2 \]
\[ \leq N^{-1} \langle U_N \Psi_N, N_+^2 U_N \Psi_N \rangle \langle U_N \Psi_N, N_+ U_N \Psi_N \rangle \]
\[ = N^{-1} \langle \Psi_N, N_+^2 \Psi_N \rangle \langle \Psi_N, N_+ \Psi_N \rangle \leq C N^{-1}. \tag{71} \]

To control the first sum in (70), we will use the bound from Theorem 2:
\[ \langle \Phi, N_+ \Phi \rangle \leq C N^{-3/2}, \quad \text{with} \quad \Phi = U_S U_B U_N \Psi_N. \tag{72} \]

Also, from Lemmas 3, 4 and 5 it follows that
\[ \langle \Phi, (N_+ + 1)^4 \Phi \rangle \leq C. \tag{73} \]
Using the action of Bogoliubov’s transformation in (8) and the uniform bounds (10) we obtain

\[
\sum_{p \neq 0} |\langle U^*_N \Psi_N, a_p U_N \Psi_N \rangle|^2 = \sum_{p \neq 0} |\langle \Phi, U_S U_B a_p U_B^* U_S^* \Phi \rangle|^2
\]

\[
= \sum_{p \neq 0} |\langle \Phi, U_S (\sigma_p a_p + \gamma_p a_p^* U_S^* \Phi) \rangle|^2 \leq C \sum_{p \neq 0} |\langle \Phi, U_S a_p U_S^* \Phi \rangle|^2. \tag{74}
\]

To estimate further the right side of (74), we use the Duhamel formula

\[
U_S a_p U_S^* = a_p + \int_0^1 e^{tS} [S, a_p] e^{-tS} dt
\]

and the Cauchy–Schwarz inequality to get

\[
\sum_{p \neq 0} |\langle \Phi, U_S a_p U_S^* \Phi \rangle|^2 \leq 2 \sum_{p \neq 0} |\langle \Phi, a_p \Phi \rangle|^2 + 2 \sum_{p \neq 0} \int_0^1 |\langle \Phi, e^{tS} [S, a_p] e^{-tS} \Phi \rangle|^2 dt. \tag{75}
\]

Thanks to (72) we can bound

\[
\sum_{p \neq 0} |\langle \Phi, a_p \Phi \rangle|^2 \leq \sum_{p \neq 0} \|a_p \Phi\|^2 = \langle \Phi, N_+ \Phi \rangle \leq CN^{-3/2}. \tag{76}
\]

It remains to handle the term involving the commutator \([S, a_p]\) in (75). Using the CCR (3) and the identity \([a_p, \mathbb{1} \leq N]\) = \(-\mathbb{1}(N_+ = N)a_p\) we can decompose

\[
[a_p, S] = \frac{1}{\sqrt{N}} \sum_{m,n \neq 0, m+n \neq 0} \eta_{m,n} \left[ a_p, a_{m+n}^* a_{-m}^* a_{-n}^* \mathbb{1}_N \mathbb{1}_{\leq N}^{m+n} a_{m+n} a_{-m} a_{-n} \right]
\]

\[
= \frac{1}{\sqrt{N}} \sum_{m,n \neq 0, m+n \neq 0} \eta_{m,n} (\delta_{p, m+n} a_{-m}^* a_{-n}^* + \delta_{p, -m} a_{m+n}^* a_{-n}^* + \delta_{p, -n} a_{m+n}^* a_{m}^*) \mathbb{1}_N
\]

\[
- \frac{1}{\sqrt{N}} \sum_{m,n \neq 0, m+n \neq 0} \eta_{m,n} a_{m+n}^* a_{-m}^* a_{-n}^* \mathbb{1}(N_+ = N)a_p
\]

\[
+ \frac{1}{\sqrt{N}} \sum_{m,n \neq 0, m+n \neq 0} \eta_{m,n} \mathbb{1}(N_+ = N)a_p a_{m+n} a_{-m} a_{-n}
\]

\(=: I_1(p) + I_2(p) + I_3(p).\)

Hence, by the Cauchy–Schwarz inequality we have for all \(t \in [0, 1],\)

\[
\sum_{p \neq 0} |\langle \Phi, e^{tS} [a_p, S] e^{-tS} \Phi \rangle|^2 \leq 3 \sum_{k=1}^3 \sum_{p \neq 0} |\langle \Phi, e^{tS} I_k(p) e^{-tS} \Phi \rangle|^2. \tag{77}
\]

The right side of (77) can be bounded using the Cauchy–Schwarz inequality, the summability (27), Lemmas 5, 6, (72) and (73). For the terms involving \(I_1(p),\) we have

\[
\sum_{p \neq 0} |\langle \Phi, e^{tS} I_1(p) e^{-tS} \Phi \rangle|^2
\]
\[ \leq \frac{C}{N} \left( \sum_{m,n,p \neq 0 \atop m+n \neq 0} |\eta_{m,n}|^2 \delta_{p,m+n} \right) \left( \sum_{m,n \neq 0 \atop m+n \neq 0} \|(N_+ + 3)^{-1/2} a_m a_n e^{-tS} \Phi\|^2 \right) \]

\[ \| (N_+ + 3)^{1/2} I \leq N e^{-tS} \Phi \|^2 \].

\[ + \frac{C}{N} \left( \sum_{m,n,p \neq 0 \atop m+n \neq 0} |\eta_{m,n}|^2 \delta_{p,-m} \right) \left( \sum_{m,n \neq 0 \atop m+n \neq 0} \|(N_+ + 3)^{-1/2} a_m a_{-m} e^{-tS} \Phi\|^2 \right) \]

\[ \| (N_+ + 3)^{1/2} I \leq N e^{-tS} \Phi \|^2 \].

\[ + \frac{C}{N} \left( \sum_{m,n,p \neq 0 \atop m+n \neq 0} |\eta_{m,n}|^2 \delta_{p,-n} \right) \left( \sum_{m,n \neq 0 \atop m+n \neq 0} \|(N_+ + 3)^{-1/2} a_m a_{-n} e^{-tS} \Phi\|^2 \right) \]

\[ \| (N_+ + 3)^{1/2} I \leq N e^{-tS} \Phi \|^2 \].

\[ \leq \frac{C}{N} \left( \Phi, e^{tS}, N_+ e^{-tS} \Phi \right) \left( \Phi, e^{tS} (N_+ + 3) e^{-tS} \Phi \right) \]

\[ \leq \frac{C}{N} \left( \Phi, (N_+ + 1)^2 \Phi \right) \left( \int \Phi, (N_+ + 1)^2 \Phi \right) \leq \frac{C}{N^2}. \]

Similarly, the terms involving \( I_2(p) \) are bounded by

\[ \sum_{p \neq 0} \left| \langle \Phi, e^{tS} I_2(p) e^{-tS} \Phi \rangle \right|^2 \]

\[ \leq \frac{C}{N} \left( \sum_{m,n \neq 0 \atop m+n \neq 0} |\eta_{m,n}|^2 \right) \left( \sum_{m,n \neq 0 \atop m+n \neq 0} \| \mathbb{I} (N_+ = N) a_m a_{-m} a_n e^{-tS} \Phi \|^2 \right) \left( \sum_{p \neq 0} \| a_p e^{-tS} \Phi \|^2 \right) \]

\[ \leq \frac{C}{N} \left( \Phi, e^{tS}, N_+^3 e^{-tS} \Phi \right) \left( \Phi, e^{tS} N_+ e^{-tS} \Phi \right) \]

\[ \leq \frac{C}{N} \left( \Phi, (N_+ + 1)^3 \Phi \right) \left( \int \Phi, (N_+ + 1)^2 \Phi \right) \leq \frac{C}{N^2}. \]

Finally, for the terms involving \( I_3(p) \), using

\[ \mathbb{I} (N_+ = N) \leq (N_+/N)^4 \]

we have

\[ \sum_{p \neq 0} \left| \langle \Phi, e^{tS} I_3(p) e^{-tS} \Phi \rangle \right|^2 \]

\[ \leq \frac{C}{N} \left( \sum_{m,n \neq 0 \atop m+n \neq 0} |\eta_{m,n}|^2 \right) \left( \sum_{m,n \neq 0 \atop m+n \neq 0} \| \mathbb{I} (N_+ = N) e^{-tS} \Phi \|^2 \right) \left( \sum_{m,n,p \neq 0 \atop m+n \neq 0} \| a_m a_{-m} a_n a_p e^{-tS} \Phi \|^2 \right) \]

\[ \leq \frac{C}{N} \left( \Phi, e^{tS} (N_+/N)^4 e^{-tS} \Phi \right) \left( \Phi, e^{tS} N_+^4 e^{-tS} \Phi \right) \]

\[ \leq \frac{C}{N^5} \left( \Phi, (N_+ + 1)^4 \Phi \right)^2 \leq \frac{C}{N^5}. \]

Thus we conclude from (77) that

\[ \sum_{p \neq 0} \left| \langle \Phi, e^{tS} [S, a_p] e^{-tS} \Phi \rangle \right|^2 \leq \frac{C}{N^2}, \quad \forall t \in [0, 1]. \tag{78} \]
Consequently,
\[
\sum_{p \neq 0} \int_0^1 \left| \langle \Phi, e^{iS} [S, a_p] e^{-iS} \Phi \rangle \right|^2 dt \leq \frac{C}{N^2}.
\]

Inserting (79) and (76) in (75) and (74) we obtain
\[
\sum_{p \neq 0} \left| \langle U_N \Psi_N, a_p U_N \Psi_N \rangle \right|^2 \leq C \sum_{p \neq 0} \left| \langle \Phi, U_S a_p U_S^* \Phi \rangle \right|^2 \leq CN^{-3/2}.
\]

Using the latter bound and (71), we deduce from (70) that
\[
\| Q^{(1)}_{\Psi_N} u_0 \| \leq CN^{-1/2}.
\]
This implies (69) and completes the proof of Theorem 1.

Acknowledgements We thank Robert Seiringer and Nicolas Rougerie for helpful discussions. The research is funded by the Polish-German Beethoven Classic 3 project “Mathematics of many-body quantum systems”. PTN acknowledges the support from the Deutsche Forschungsgemeinschaft (DFG Project Nr. 426365943). MN acknowledges the support from the National Science Centre (NCN Project Nr. 2018/31/G/ST1/01166).

Funding Open Access funding enabled and organized by Projekt DEAL.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Adhikari, A., Brennecke, C., Schlein, B.: Bose–Einstein condensation beyond the Gross–Pitaevskii regime. Annales Henri Poincaré 22, 1163–1233 (2021)
2. Boccato, C., Brennecke, C., Cenatiempo, S., Schlein, B.: Bogoliubov theory in the Gross–Pitaevskii limit. Acta Math. 222, 219–335 (2019)
3. Boccato, C., Brennecke, C., Cenatiempo, S., Schlein, B.: Optimal rate for Bose–Einstein condensation in the Gross–Pitaevskii regime. Commun. Math. Phys. 376, 1311–1395 (2020)
4. Bogoliubov, N.N.: On the theory of superfluidity. J. Phys. (USSR) 11, 23 (1947)
5. Boßmann, L., Pavliotić, N., Pickl, P., Soffer, A.: Higher order corrections to the mean-field description of the dynamics of interacting bosons. J. Stat. Phys. 178, 1362–1396 (2020)
6. Boßmann, L., Petrat, S., Pickl, P., Soffer, A.: Beyond bogoliubov dynamics, preprint (2019). arXiv:1912.11004
7. Boßmann, L., Petrat, S., Seiringer, R.: Asymptotic expansion of the low-energy excitation spectrum for weakly interacting bosons. Preprint (2020). arXiv:2006.09825
8. Brennecke, C., Nam, P.T., Napiórkowski, M., Schlein, B.: Fluctuations of N-particle quantum dynamics around the nonlinear Schrödinger equation. Annales de l’Institut Henri Poincaré (C) Non-Linear Analysis 36, 1201–1235 (2019)
9. Dereziński, J., Napiórkowski, M.: Excitation spectrum of interacting bosons in the mean-field infinite-volume limit. Ann. Henri Poincaré 15, 2409–2439 (2014)
10. Fournais, S., Solovej, J.P.: The energy of dilute Bose gases. Ann. of Math. (to appear). arXiv:1904.06164
11. Grech, P., Seiringer, R.: The excitation spectrum for weakly interacting bosons in a trap. Commun. Math. Phys. 322, 559–591 (2013)
12. Lewin, M., Nam, P.T., Serfaty, S., Solovej, J.P.: Bogoliubov spectrum of interacting Bose gases. Commun. Pure Appl. Math. 68, 413–471 (2015)
13. Nam, P.T.: Binding energy of homogeneous Bose gases. Lett. Math. Phys. 108, 141–159 (2018)
14. Nam, P.T., Seiringer, R.: Collective excitations of Bose gases in the mean-field regime. Arch. Ration. Mech. Anal. 215, 381–417 (2015)
15. Pizzo, A.: Bose particles in a box III. A convergent expansion of the ground state of the Hamiltonian in the mean field limiting regime, preprint (2015). arXiv:1511.07026
16. Rademacher, S., Schlein, B.: Central limit theorem for Bose–Einstein condensates. J. Math. Phys. 60, 071902 (2019)
17. Rougerie, N., Spehner, D.: Interacting bosons in a double-well potential: Localization regime. Commun. Math. Phys. 361, 737–786 (2018)
18. Seiringer, R.: The excitation spectrum for weakly interacting bosons. Commun. Math. Phys. 306, 565–578 (2011)
19. Solovej, J. P.: Many-body quantum physics. Lecture notes at Erwin Schrödinger Institute, (2014). http://web.math.ku.dk/~solovej/MANYBODY/
20. Yau, H.-T., Yin, J.: The second order upper bound for the ground energy of a Bose gas. J. Stat. Phys. 136, 453–503 (2009)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.