A CHARACTERISATION OF UNIFORM PRO-$p$ GROUPS

BENJAMIN KLOPSCH AND ILIR SNOPCE

Abstract. Let $p$ be a prime. Uniform pro-$p$ groups play a central role in the theory of $p$-adic Lie groups. Indeed, a topological group admits the structure of a $p$-adic Lie group if and only if it contains an open pro-$p$ subgroup which is uniform. Furthermore, uniform pro-$p$ groups naturally correspond to powerful $\mathbb{Z}_p$-Lie lattices and thus constitute a cornerstone of $p$-adic Lie theory.

In the present paper we propose and supply evidence for the following conjecture, aimed at characterising uniform pro-$p$ groups. Suppose that $p \geq 3$ and let $G$ be a torsion-free pro-$p$ group of finite rank. Then $G$ is uniform if and only if its minimal number of generators is equal to the dimension of $G$ as a $p$-adic manifold, i.e., $d(G) = \dim(G)$. In particular, we prove that the assertion is true whenever $G$ is soluble or $p > \dim(G)$.

1. Introduction

Throughout let $p$ be a prime. Lazard’s seminal paper *Groupes analytiques $p$-adiques* [13], published in 1965, provides a comprehensive treatment of the theory of $p$-adic analytic Lie groups. One of his main results was a solution of the $p$-adic analogue of Hilbert’s 5th problem. More precisely, he obtained the following algebraic characterisation of $p$-adic analytic groups: a topological group is $p$-adic analytic if and only if it contains a finitely generated open pro-$p$ subgroup which is saturable. In the 1980s Lubotzky and Mann introduced the concept of a powerful pro-$p$ group and used this notion to re-interpret the group-theoretic aspects of Lazard’s work, by and large sidestepping the analytic side of the theory. Central to their approach are the uniformly powerful pro-$p$ groups, which play a role similar to the one of saturable pro-$p$ groups in Lazard’s work. A detailed treatment of $p$-adic analytic groups from this point of view and a sample of its manifold applications are given in [1].

A pro-$p$ group $G$ is said to be powerful if $p \geq 3$ and $[G, G] \leq G^p$, or $p = 2$ and $[G, G] \leq G^4$. Here, $[G, G]$ and $G^p$ denote the (closures of the) commutator subgroup and the subgroup generated by all $p$th powers. The definition of a uniformly powerful pro-$p$ group is stated in Section[2]. For the moment it suffices to recall that a pro-$p$ group is uniformly powerful, or uniform for short, if and only if it is finitely generated, powerful and torsion-free. The rank of a pro-$p$ group $G$ is the basic invariant

$$\text{rk}(G) := \sup\{d(H) \mid H \text{ a closed subgroup of } G\},$$

2010 Mathematics Subject Classification. 20E18, 22E20, 20D15.
where \( d(H) \) denotes the minimal cardinality of a topological generating set for \( H \).

The class of \( p \)-adic analytic groups can now be characterised as follows; see [1 Corollary 8.34]. A topological group admits the structure of a \( p \)-adic analytic Lie group if and only if it contains an open pro-\( p \) subgroup of finite rank. Moreover, a pro-\( p \) group has finite rank if and only if it admits a uniform open subgroup.

A key invariant of a \( p \)-adic Lie group \( G \) is its dimension as a \( p \)-adic manifold which we denote by \( \dim(G) \). Algebraically, \( \dim(G) \) can be described as \( d(U) \), where \( U \) is any uniform open pro-\( p \) subgroup of \( G \). In [10] it is shown that, for \( p \geq 3 \), every torsion-free compact \( p \)-adic Lie group \( G \) satisfies \( \text{rk}(G) = \dim(G) \).

1.1. Main results. The purpose of the present paper is to propose a new characterisation of uniform pro-\( p \) groups in terms of their minimal numbers of generators.

It is well known that every uniform pro-\( p \) group \( G \) satisfies \( d(G) = \dim(G) \). We propose and supply evidence for the following conjecture.

**Conjecture 1.1.** Suppose that \( p \geq 3 \) and let \( G \) be a torsion-free pro-\( p \) group of finite rank. Then \( G \) is uniform if and only if \( d(G) = \dim(G) \).

Standard examples show that the assertion of the conjecture cannot extend to \( p = 2 \) without modifications; see Section [2]. It is not quite clear what one could reasonably hope for in this case and for now we shall concentrate on \( p \geq 3 \). Our first results imply that Conjecture 1.1 is indeed true for soluble pro-\( p \) groups. Every profinite group \( G \) of finite rank has a maximal finite normal subgroup. We refer to this subgroup as the periodic radical of \( G \) and denote it by \( \pi(G) \).

**Theorem 1.2.** Suppose that \( p \geq 5 \) and let \( G \) be a soluble pro-\( p \) group of finite rank such that \( \pi(G) = 1 \). If \( d(G) = \dim(G) \) then \( G \) is uniform.

In Section [2] we give an example to show that the assertion of Theorem 1.2 does not extend to \( p = 3 \) without modifications. Nevertheless a separate analysis leads to the following somewhat weaker theorem, still confirming Conjecture 1.1.

**Theorem 1.3.** Let \( G \) be a torsion-free soluble pro-3 group of finite rank. If \( d(G) = \dim(G) \) then \( G \) is uniform.

For a finitely generated nilpotent group \( \Gamma \) we denote by \( \Pi(\Gamma) \) the set of all primes \( q \) such that \( \Gamma \) contains an element of order \( q \). It is known that the torsion elements of \( \Gamma \) form a finite subgroup, hence the set \( \Pi(\Gamma) \) is finite. As a consequence of the above theorems we obtain the following corollary.

**Corollary 1.4.** Let \( \Gamma \) be a finitely generated nilpotent group and let \( h(\Gamma) \) denote the Hirsch length of \( \Gamma \). Suppose that \( p \notin \Pi(\Gamma) \cup \{2\} \). Then the pro-\( p \) completion \( \hat{\Gamma}_p \) of \( \Gamma \) is a uniform pro-\( p \) group if and only if \( \dim_{\mathbb{F}_p}(\Gamma / \Gamma^p[\Gamma, \Gamma]) = h(\Gamma) \).

Our current results for insoluble groups are not quite strong enough to settle Conjecture 1.1 in full generality. However, we are able to confirm the conjecture for a wide range of groups.
From [2] we recall that a subgroup $N$ of a pro-$p$ group $G$ is PF-embedded in $G$ if there exists a central descending series of closed subgroups $N_i$, $i \in \mathbb{N}$, starting at $N_1 = N$ such that $\bigcap_{i \in \mathbb{N}} N_i = 1$ and $[N_i, G, \ldots, G] \leq N_{i+1}^{p-1}$ for all $i \in \mathbb{N}$.

**Theorem 1.5.** Suppose that $p \geq 3$. Let $G$ be a pro-$p$ group of finite rank with an open PF-embedded subgroup and such that the map $G \to G$, $x \mapsto x^p$ is injective. If $d(G) = \dim(G)$ then $G$ is uniform.

It is not hard to see that every closed subgroup of a finitely generated saturable pro-$p$ group has an open PF-embedded subgroup. Since it is one of the built-in features of a saturable pro-$p$ group $G$ that the map $G \to G$, $x \mapsto x^p$ is injective, we obtain immediately the following corollary.

**Corollary 1.6.** Suppose that $p \geq 3$ and let $G$ be a closed subgroup of a finitely generated saturable pro-$p$ group. If $d(G) = \dim(G)$ then $G$ is uniform.

We recall that every uniform pro-$p$ group is saturable, albeit the converse is not true; see [9]. The next corollary can be used to produce easily examples of saturable groups which are not uniform.

**Corollary 1.7.** Suppose that $p \geq 3$ and let $G$ be a finitely generated saturable pro-$p$ group. Then $G$ is uniform if and only if $d(G) = \dim(G)$.

Furthermore, a result of González-Sánchez and Klopsch in [7] allows us to derive the following consequence.

**Corollary 1.8.** Let $G$ be a torsion-free pro-$p$ group of finite rank such that $p > \dim(G)$. If $d(G) = \dim(G)$ then $G$ is uniform.

Finally, it is natural to ask whether there is an analogue of Conjecture 1.1 for finite $p$-groups. Currently, there is perhaps not enough evidence to support a ‘formal’ conjecture, but we can raise the following problem. For a finite $p$-group $G$ the subgroup $\Omega_1(G)$ is the group generated by all elements of order $p$ in $G$.

**Question 1.9.** Suppose that $p \geq 3$ and let $G$ be a finite $p$-group. Is it true that $G$ is powerful if and only if $d(G) = \log_p|\Omega_1(G)|$?

Similarly as for Conjecture 1.1 the forward implication in Question 1.9 is known to be true. Suppose that $p \geq 3$ and let $G$ be a powerful finite $p$-group. Then $|G : G^p| = |\Omega_1(G)|$, by [20] Theorem 3.1, and from $[G, G] \subseteq G^p$ we conclude that $d(G) = |G : G^p| = |\Omega_1(G)|$. The actual problem is whether the implication in the other direction is true. Our next result shows that a positive answer to this question would resolve Conjecture 1.1.

**Proposition 1.10.** Conjecture 1.1 is true if the answer to Question 1.9 is ‘yes’.

Section 2 contains a variation of Question 1.9 and a short discussion, showing that the answer is indeed ‘yes’ for certain special classes of groups, namely for finite $p$-groups which are regular, potent or $p$-central.
1.2. **Some applications.** The property of being powerful is readily inherited by factor groups and by direct products, but straightforward examples show that often it is not inherited by subgroups. We say that a pro-$p$ group $G$ is *hereditarily powerful* if every open subgroup of $G$ is powerful. Similarly, we say that $G$ is *hereditarily uniform* if every open subgroup of $G$ is uniform. In [15] Lubotzky and Mann proved that a finite $p$-group is hereditarily powerful if and only if it is modular and, if $p = 2$, not Hamiltonian. From their result we deduce the following.

**Theorem 1.11** (Lubotzky and Mann). Let $G$ be a finitely generated pro-$p$ group. Then $G$ is hereditarily powerful if and only if there exist an abelian normal subgroup $A$ of $G$, an element $b \in G$ and $s \in \mathbb{N} \cup \{\infty\}$, with $s \geq 2$ if $p = 2$, such that
\[ G = \langle b \rangle A, \]
where the group $A$ is written additively and $b$ acts as multiplication by $1 + p^s$.

**Corollary 1.12.** Let $G$ be a finitely generated pro-$p$ group. Then $G$ is hereditarily uniform if and only if one of the following holds:

1. $G \simeq \mathbb{Z}_p^d$ is abelian for some $d \in \{0\} \cup \mathbb{N}$;
2. $G \simeq \langle b \rangle \rtimes A$ for $\langle b \rangle \simeq \mathbb{Z}_p$ and $A \simeq \mathbb{Z}_p^{d-1}$, where $d \geq 2$ and $b$ acts on $A$ as multiplication by $1 + p^s$ for some $s \in \mathbb{N}$, with $s \geq 2$, if $p = 2$.

In [11] the authors of the present paper used Lie ring methods to classify all finitely generated pro-$p$ groups with constant generating number on open subgroups; see Theorem 3.1. In Section 3 we indicate how Corollary 1.12 yields an alternative proof in the case $p \geq 3$, which does not require Lie ring techniques.

1.3. **Notation.** Throughout the paper, $p$ denotes a prime. The $p$-adic integers and $p$-adic numbers are denoted by $\mathbb{Z}_p$ and $\mathbb{Q}_p$. We write $C_p$ to refer to a cyclic group of order $p$.

Subgroups $H$ of a topological group $G$ are tacitly taken to be closed and by generators we mean topological generators as appropriate. The minimal number of generators of a group $G$ is denoted by $d(G)$. Likewise the minimal cardinality of a generating set for a module $M$ over a ring $R$ is denoted by $d_R(M)$.

A pro-$p$ group $G$ is said to be *just-infinite* if it is infinite and every non-trivial normal subgroup of $G$ has finite index in $G$. Precise descriptions of uniform and saturable pro-$p$ groups are given in Section 2.

### 2. Uniform pro-$p$ groups

2.1. Let $G$ be a pro-$p$ group. The lower central $p$-series of $G$ is defined as follows:
\[ P_1(G) = G \text{ and } P_{i+1}(G) = P_i(G)^p[P_i(G), G] \text{ for } i \in \mathbb{N}. \]
We recall the definition of a uniformly powerful pro-$p$ group.

**Definition 2.1.** A pro-$p$ group $G$ is *uniformly powerful*, or *uniform* for short, if

(i) $G$ is finitely generated;
Proof of Theorem 1.2. Suppose that $[G, G] \subseteq G^p$ if $p \geq 3$, and $[G, G] \subseteq G^4$ if $p = 2$; furthermore, we provide examples to explain the restrictions that we impose on $G$ (Theorem 4.5) and Theorem 2.2. We now prove the results about soluble pro-$p$ groups.

A useful characterisation of uniform pro-$p$ groups is the following.

**Theorem 2.2** ([1, Theorem 4.5]). A pro-$p$ group is uniform if and only if it is finitely generated, torsion-free and powerful.

If a finitely generated pro-$p$ group $G$ is powerful, then it has rank $\text{rk}(G) = d(G)$, but the converse does not generally hold; see [1, Theorem 3.8]. In view of [10, Theorem 1.3], we can rephrase Conjecture 1.1 as follows.

**Conjecture.** Suppose that $p \geq 3$ and let $G$ be a pro-$p$ group. Then $G$ is uniform if and only if it is finitely generated, torsion-free and $\text{rk}(G) = d(G)$.

2.2. We now prove the results about soluble pro-$p$ groups stated in Section [11]. Furthermore, we provide examples to explain the restrictions that we impose on $p$.

**Proof of Theorem 2.2.** Suppose that $d(G) = \dim(G)$. We need to prove that $G$ is powerful and torsion-free. If $G$ is the trivial group there is nothing further to do. Hence suppose that $G \neq 1$.

First we show that $G$ is powerful. Choose a normal subgroup $N$ of $G$ such that $H := G/N$ is just-infinite. Note that both $\pi(N)$ and $\pi(H)$ are trivial. By [10, Theorem 1.3] we have $d(N) \leq \dim(N)$ and $d(H) \leq \dim(H)$. Thus we deduce from [1, Theorem 4.8] that

$$\dim(G) = d(G) \leq d(H) + d(N) \leq \dim(H) + \dim(N) = \dim(G),$$

which implies $d(H) = \dim(H)$ and $d(N) = \dim(N)$. Since $\dim(N) < \dim(G)$, it follows by induction that $N$ is powerful. We observe that in order to show that $G$ is powerful it suffices to show that $H$ is powerful: if $H$ is powerful then

$$|G : G^p| = |G : G^pN||N : N \cap G^p| \leq |H : H^p||N : N^p| = p^{d(H) + d(N)} = p^{\dim(H) + \dim(N)} = p^{\dim(G)} = |G : G^p[G, G]| \leq |G : G^p|,$$

and we obtain $[G, G] \leq G^p$.

Therefore we may assume that $G = H$ is just-infinite. Since $G$ is soluble, we deduce that $G$ is virtually abelian; see [14, Ch. 12]. Put $d = d(G) = \dim(G)$ and choose an open normal subgroup $B \leq G$ such that $B \cong \mathbb{Z}_p^d$. Let $A := C_G(B) \leq G$, the centraliser of $B$ in $G$, and write $Z(A)$ for the centre of $A$. Then $|A : Z(A)| \leq |A : B| < \infty$, and hence $[A, A]$ is finite by Schur’s theorem. Since $G$ is just-infinite we must have $[A, A] = 1$. Hence $A$ is abelian and self-centralising in $G$. Since $\pi(G) = 1$, we conclude that $A$ is torsion-free. The group $\bar{G} := G/A$ is finite and acts faithfully on $A \cong \mathbb{Z}_p^d$. In this way we obtain an embedding $\bar{G} \hookrightarrow \text{GL}(A) \cong \text{GL}_d(\mathbb{Z}_p)$. If $G$ is trivial then $G = A$ is abelian, hence powerful.

For a contradiction, we now assume that $\bar{G} \neq 1$. Let $C = \langle x \rangle A$ be a subgroup of $G$ such that $\bar{C} := C/A = \langle \bar{x} \rangle$ is cyclic of order $p$ and contained in the centre.
Example 2.3. Consider the pro-3 group $\mathbb{Z}(\bar{G})$ of $\bar{G}$. According to [4, Theorem 2.6], there are three indecomposable types of $\mathbb{Z}_p\bar{C}$-modules which are free and of finite rank as $\mathbb{Z}_p$-modules:

(i) the trivial module $I = \mathbb{Z}_p$ of $\mathbb{Z}_p$-dimension 1,

(ii) the module $J = \mathbb{Z}_p\bar{C}/(\Phi(x))$ of $\mathbb{Z}_p$-dimension $p - 1$, where $\Phi(X) = 1 + X + \ldots + X^{p-1}$ denotes the $p$th cyclotomic polynomial,

(iii) the free module $K = \mathbb{Z}_p\bar{C}$ of $\mathbb{Z}_p$-dimension $p$.

Hence the $\mathbb{Z}_p\bar{C}$-module $A$, which is free and of finite rank as a $\mathbb{Z}_p$-module, decomposes as a direct sum of indecomposable submodules

$$A = (\oplus_{i=1}^{m_1} I_i) \oplus (\oplus_{j=1}^{m_2} J_j) \oplus (\oplus_{k=1}^{m_3} K_k),$$

where $m_1, m_2, m_3 \in \{0\} \cup \mathbb{N}$ and $I_i \cong I, J_j \cong J, K_k \cong K$ for all indices $i, j, k$.

Put $A_1 = \oplus_{i=1}^{m_1} I_i, A_2 = \oplus_{j=1}^{m_2} J_j$ and $A_3 = \oplus_{k=1}^{m_3} K_k$. Since $\bar{C}$ is central in $\bar{G}$, the decomposition $A = A_1 \oplus A_2 \oplus A_3$ is $\bar{G}$-invariant. Since $G$ is just-infinite we conclude that $A = A_i$ for precisely one $i \in \{1, 2, 3\}$. We cannot have $A = A_1$, because $\bar{C}$ acts faithfully on $A$. Thus either $A = A_2$ or $A = A_3$, and consequently $d_{z,C}(A) \leq \max\{m_2, m_3\} \leq \left\lfloor \frac{d}{p-1} \right\rfloor$. Moreover, [10, Proposition 3.5] shows that $d(\bar{G}) \leq \left\lfloor \frac{d}{p-1} \right\rfloor$. Hence from $p \geq 5$ we obtain

$$d(G) \leq d(\bar{G}) + d_{z,C}(A) \leq 2\left\lfloor \frac{d}{p-1} \right\rfloor < d = \dim(G)$$

in contradiction to $d(G) = \dim(G)$. This concludes the proof that $G$ is powerful.

It remains to show that $G$ is torsion-free. By [1, Theorem 4.20], the collection of all elements of finite order in $G$ forms a characteristic subgroup $T$ of $G$. Clearly, $T \subseteq \pi(G) = 1$ implies that $G$ is torsion-free.

The following example illustrates that the assertion of Theorem 1.2 does not extend without modifications to $p = 3$.

Example 2.3. Consider the pro-3 group $G = \langle z \rangle \rtimes \mathbb{Z}_3[\xi]$, where $\langle z \rangle \cong C_3$, $\mathbb{Z}_3[\xi] = \mathbb{Z}_3 + \mathbb{Z}_3\xi \cong \mathbb{Z}_3^2$ for a primitive 3rd root of unity $\xi$ and where $z$ acts on $\mathbb{Z}_3[\xi]$ as multiplication by $\xi$. One easily verifies that $G$ is not powerful, even though $G$ is soluble, $\pi(G) = 1$ and $d(G) = 2 = \dim(G)$.

In order to prove Theorem 1.3 we need to analyse more carefully the case $p = 3$.

Lemma 2.4. Suppose that $p \geq 3$ and let $G$ be a just-infinite soluble pro-$p$ group of finite rank such that $\pi(G) = 1$. If $d(G) = \dim(G)$, then $\dim(G) \leq 2$.

Proof. Suppose that $d := d(G) = \dim(G)$. If $G$ is abelian, then $G \cong \mathbb{Z}_p$ and $d = 1$. Now suppose that $G$ is not abelian. Arguing as in the proof of Theorem 1.2 we find an open normal subgroup $A \leq G$ such that the quotient $\bar{G} = G/A \neq 1$ acts faithfully on $A \cong \mathbb{Z}_p^d$. Moreover, there is a central cyclic subgroup $\langle \bar{x} \rangle = \bar{C} \leq \bar{G}$ of order $p$ such that $A$, regarded as a $\mathbb{Z}_p\bar{C}$-module, decomposes into a $\bar{G}$-invariant homogeneous direct sum of pairwise isomorphic indecomposable submodules:

$$A = \oplus_{j=1}^{m} J_j \quad \text{or} \quad A = \oplus_{k=1}^{m} K_k \quad \text{with} \quad d = m(p - 1) \quad \text{or} \quad d = mp,$$
just as in the proof of Theorem 1.2. If \( p \geq 5 \), then (2.1) yields a contradiction.

Hence we have \( p = 3 \). Then \( d(\bar{G}) \leq \lfloor d/2 \rfloor \) by [10] Proposition 3.5 and, if \( A = \bigoplus_{k=1}^{m} K_{k} \) with \( d = 3m \), then

\[
d(\bar{G}) \leq d(\bar{G}) + d_{\bar{G}}(A) \leq \lfloor d/2 \rfloor + d/3 < d = \dim(G),
\]
yielding a contradiction. Hence we obtain \( A = \bigoplus_{j=1}^{m} J_{j} \) with \( d = 2m \). We can regard \( A \) as a free module of rank \( m \) over the ring \( R = \mathbb{Z}_p[\bar{G}]/(\bar{x}^2 + \bar{x} + 1) \). This ring is naturally isomorphic to the valuation ring \( \mathbb{Z}_3[\xi] \) of the totally ramified extension \( \mathbb{Q}_3(\xi) \) of \( \mathbb{Q}_3 \) obtained by adjoining a primitive 3rd root of unity \( \xi \). The element \( \pi \) represented by \( \bar{x} - 1 \) is a uniformiser of \( R \). We obtain an embedding \( \eta: \bar{G} \hookrightarrow \text{GL}(A) \cong \text{GL}_m(R) \). The Sylow-3 subgroups of \( \text{GL}_m(R) \) are all conjugate to one another and we may assume that \( \bar{G} \) consists of matrices which are upper uni-triangular modulo \( \pi \). If \( \bar{G} \) is not contained in \( \text{GL}_m(R) = \ker(\text{GL}_m(R) \to \text{GL}_m(R/\pi R)) \) then

\[
d(\bar{G}) \leq d(\bar{G}) + d_{\bar{G}}(A) \leq d/2 + (m - 1) = d - 1 < d = \dim(G)
\]
yields a contradiction.

Hence we obtain \( \bar{G} \subseteq \text{GL}_m^1(R) \). We observe that \( \text{GL}_m^2(R) = \ker(\text{GL}_m(R) \to \text{GL}_m(R/\pi^2 R)) \) is torsion-free. Since \( \bar{G} \) is finite, this implies that \( \bar{G} \) embeds into \( \text{GL}_m^1(R) / \text{GL}_m^2(R) \). But the latter group is elementary abelian, hence \( \bar{G} \) is elementary abelian. Let \( F \cong \mathbb{Q}_3(\xi) \) denote the field of fractions of \( R \). Since \( G \) is just-infinite, \( V := F \otimes_{\mathbb{R}} A \) is an irreducible \( \bar{G} \)-module. Since \( \bar{G} \) is abelian of exponent 3 and \( F \) contains a primitive 3rd root of unity, we deduce that \( m = \dim_{\mathbb{F}}(V) = 1 \) and \( d = 2 \).

We remark that Lemma 2.4 has no analogue for insoluble groups, because there are just-infinite uniform pro-\( p \) groups of arbitrarily large dimension.

**Lemma 2.5.** Let \( G \) be a pro-\( p \) group with a powerful open normal subgroup \( N \trianglelefteq G \), and suppose that \( z \in G \setminus N \). If \( G \) is torsion-free, then \( \langle z \rangle \cap (N \setminus N^p) \neq \emptyset \).

**Proof.** Suppose that \( G \) is torsion-free. Without loss of generality we may assume that \( z^p \in N \) and, for a contradiction, we assume that \( z^p \notin N^p \). Note that \( N \) is uniform so that \( z^p = a^p \) for \( a \in N \). Furthermore, \( N \) admits the structure of a \( \mathbb{Z}_p(z) \)-module which is free and of finite rank over \( \mathbb{Z}_p \); see [11] Theorem 4.17. Thus \( a^p \in C_N(z) \) implies \( a \in C_N(z) \) so that \( (za^{-1})^p = 1 \). Since \( za^{-1} \neq 1 \), this yields a contradiction.

**Proof of Theorem 1.3.** Suppose that \( d(\bar{G}) = \dim(G) \). If \( G \) is trivial, there is nothing to prove. Now suppose that \( G \neq 1 \) and choose a normal subgroup \( N \trianglelefteq G \) such that \( H := G/N \) is just-infinite. Proceeding as in the proof of Theorem 1.2 we conclude that \( d(N) = \dim(N) \) and \( d(H) = \dim(H) \). By induction, \( N \) is uniform. Moreover, since \( \pi(H) = 1 \), Lemma 2.4 implies that \( \dim(H) \leq 2 \).

For a contradiction, assume that \( \dim(H) = 2 \).
Hence Proposition 2.6. Suppose that it follows that $G$ is torsion-free, the powerful group $G_z$ which is a contradiction. This proves the claim.

Proof of the claim. We analyse $H$ as before. Let $A \cong \mathbb{Z}_3^2$ be a self-centralising normal subgroup of $H$. Then $H/A$ acts faithfully on $A$ so that we obtain an embedding $\bar{H} = H/A \hookrightarrow \text{GL}_2(\mathbb{Z}_3)$. Inspection of the Sylow pro-3 subgroup of $\text{GL}_2(\mathbb{Z}_3)$ shows that $\bar{H} \cong C_3$ is acting fixed-point-freely on $A$. Thus $H = \langle z \rangle \rtimes A$, where $z$ has order 3 and acts on $A \cong \mathbb{Z}_3[\xi]$ as multiplication by a 3rd root of 1. This proves the claim.

In particular, $d(H) = 2$ and we can complement $z$ to a generating pair for $H$. Choose a pre-image $\tilde{z} \in G$ of $z$ with respect to the quotient map $G \to G/N$. Since $G$ is torsion-free, the powerful group $N$ is uniform and Lemma 2.5 implies that $\tilde{z}^3 \in N \setminus N^3 = N \setminus N^3[N, N]$. This implies that

$$d(G) \leq d(H) + (d(N) - 1) = \dim(H) + \dim(N) - 1 < \dim(G),$$

which is a contradiction.

This forces $\dim(H) = 1$ so that $H \cong \mathbb{Z}_3$ is powerful. From

$$3^{d(G)} = 3^d(G) \leq \frac{|G : G^3|}{3|N : N^3|} = 3^{\dim(G) + \dim(N)} = 3^{\dim(G)}$$

it follows that $|G : G^3[G, G]| = 3^d(G) = |G : G^3|$. We deduce that $[G, G] \subseteq G^3$. Hence $G$ is powerful and, because $G$ is torsion-free, $G$ is uniform.

Implicitly the proofs of Theorems 1.2 and 1.3 also yield the following fact.

**Proposition 2.6.** Suppose that $p \geq 3$ and let $G$ be a soluble uniform pro-$p$ group. Then $G$ is poly-$\mathbb{Z}_p$, i.e., there exists a finite chain of subgroups $G = G_1 \supseteq \ldots \supseteq G_{r+1} = 1$ such that $G_{i+1} \subseteq G_i$ and $G_i/G_i + 1 \cong \mathbb{Z}_p$ for $1 \leq i \leq r$.

**Example 2.7.** Consider the metabelian group $G := \langle y \rangle \rtimes A$, where $\langle y \rangle \cong \mathbb{Z}_2$, $A \cong \mathbb{Z}_2^{d-1}$ with $d \geq 2$ and $y$ acts on $A$ as scalar multiplication by $-1$. Note that $G$ is not powerful, even though it is torsion-free and $d(G) = \dim(G) = \text{rk}(G)$.

**Proof of Corollary 1.4.** By Proposition 16.4.2(v) the pro-$p$ completion $\hat{\Gamma}_p$ of $\Gamma$ is a torsion free pro-$p$ group of finite rank and $\dim(\hat{\Gamma}_p) = h(\Gamma)$. Clearly, $\hat{\Gamma}_p$ is soluble. Now, using Theorems 1.2 and 1.3 we deduce that $\hat{\Gamma}_p$ is a uniform pro-$p$ group if and only if $d(\hat{\Gamma}_p) = \dim(\hat{\Gamma}_p)$. Since

$$d(\hat{\Gamma}_p) = \dim_{\mathbb{F}_p}(\hat{\Gamma}_p/(\hat{\Gamma}_p)^p[\hat{\Gamma}_p, \hat{\Gamma}_p]) = \dim_{\mathbb{F}_p}(\Gamma/\Gamma^p[\Gamma, \Gamma]),$$

the assertion follows.

Let $\Gamma$ be a finitely generated nilpotent group. If $\Gamma/\langle \Gamma, \Gamma \rangle$ and $\Gamma$ have the same Hirsch length, i.e., $h(\Gamma/\langle \Gamma, \Gamma \rangle) = h(\Gamma)$, then $[\Gamma, \Gamma]$ is finite and $\Gamma$ is an FC-group, i.e., a group with finite conjugacy classes, so that $|\Gamma : Z(\Gamma)| < \infty$. In this case $\hat{\Gamma}_p$ is uniform for almost all primes $p$. On the other hand, if $h(\Gamma/\langle \Gamma, \Gamma \rangle) < h(\Gamma)$ then $\hat{\Gamma}_p$ is uniform for at most finitely many primes $p$: the group fails to be uniform for all $p$ with $p \notin \Pi(\Gamma/\langle \Gamma, \Gamma \rangle)$. This should be contrasted with the fact that in any
case \( \hat{\Gamma}_p \) is saturable for almost all primes \( p \), because it is so for all \( p \not\in \Pi(\Gamma) \) with \( p > h(\Gamma) \), by [7, Theorem A].

2.3. Next we prove the assertions in Section 1.4 about pro-\( p \) groups \( G \) of finite rank which are not necessarily soluble. We recall the following concepts from [2].

**Definition 2.8.** Let \( G \) be a pro-\( p \) group. A **potent filtration** in \( G \) is a descending series \( N_i, i \in \mathbb{N} \), of subgroups of \( G \) satisfying the following conditions:

(i) \( \bigcap_{i \in \mathbb{N}} N_i = 1 \),

(ii) \([N_i, G] \leq N_{i+1}\) for all \( i \in \mathbb{N} \),

(iii) \([N_i, G, \ldots, G] \leq N_{i+1}^{p-1}\) for all \( i \in \mathbb{N} \).

We say that a subgroup \( N \) of \( G \) is **PF-embedded** in \( G \) if there is a potent filtration in \( G \) starting at \( N \). The pro-\( p \) group \( G \) is called a **PF-group** if \( G \) is PF-embedded in itself.

Note that if \( N_i, i \in \mathbb{N} \), is a potent filtration in a pro-\( p \) group \( G \) then for each \( k \in \mathbb{N} \) the series \( N_i, i \geq k \), is a potent filtration starting at \( N_k \). In particular, each \( N_k \) is a PF-group.

A finitely generated pro-\( p \) group is saturable if it admits a certain type of ‘valuation map’; for precise details we refer to [9]. For instance, if \( G \) is a uniform pro-\( p \) group then one can show that \( G \) is saturable by considering the valuation map

\[ \omega: G \to \mathbb{R}_{>0} \cup \{\infty\}, \quad x \mapsto \sup\{k \mid k \geq 1 \text{ and } x \in G^{p^{k-1}}\}. \]

In [5], González-Sánchez proved that a finitely generated pro-\( p \) group \( G \) is saturable if and only if it is a torsion-free PF-group.

**Proof of Theorem 1.5.** Suppose that \( d(G) = \dim(G) \). Clearly, \( G \) is torsion-free and it suffices to prove that \( G \) is powerful. Let \( N \) be an open PF-embedded subgroup of \( G \) and let \( N_i, i \in \mathbb{N} \), be a potent filtration in \( G \) starting at \( N_1 = N \).

For any subset \( X \subseteq G \) we denote by \( X^{(p)} \) the collection of all \( p \)th powers \( x^p \) of elements \( x \in X \). Since \( N \) is a saturable pro-\( p \) group, we have \( N^p = N^{(p)} \).

**Claim.** For every \( x \in G \) we have \( (xN)^{(p)} = x^pN^p \).

**Proof of the claim.** P. Hall’s collection formula shows that for all \( x \in G \) and \( u \in N \),

\[ (xu)^p \equiv x^p u^p \mod \gamma_2(\langle x, u \rangle)^p \gamma_p(\langle x, u \rangle). \]

Since \( N_i, i \in \mathbb{N} \), is a potent filtration in \( G \) it follows that for all \( x \in G \) and \( w \in N_i \) we have \( \gamma_2(\langle x, w \rangle)^p \subseteq [N_i, G]^p \subseteq N_{i+1}^{p-1} \) and \( \gamma_p(\langle x, w \rangle) \subseteq [N_i, G, \ldots, G] \subseteq N_{i+1}^{p-1} \).

This implies that

\[ (xw)^p \equiv x^p w^p \mod N_{i+1}^p \]

for all \( x \in G \) and \( w \in N_i \). Now since \( N = N_1 \), we deduce that \( (xN)^{(p)} \subseteq x^pN^p \).
For the reverse inclusion, we need to show that, given \( x \in G \) and \( u \in N \), there exists \( v \in N \) such that \((xv)^p = xu^p\). Note that by (2.2) we have \((xu)^pu^p = x^pu^p\) for some \( u_2 \in N_2 \). Applying once more (2.2), we obtain \((xu_2)^pu^p = x^pu^p\) for some \( u_3 \in N_3 \). Proceeding in this way, we construct inductively a sequence \((u_i)^{\infty}_{i=1}\), with \( u_1 = u \), such that \( u_i \in N_i \) and \((xu_1u_2 \cdots u_i)^pu^p = x^pu^p\) for every \( i \in \mathbb{N} \). Writing \( v_i = u_1u_2 \cdots u_i \), we see that \((v_i)^{\infty}_{i=1}\) is a Cauchy sequence and, taking limits, we obtain \((xv)^p = xu^p\), where \( v = \lim_{i \to \infty} v_i \in N \). This proves the claim.

Now suppose that \( x, y \in G \) with \( x^pN^p = y^pN^p \). Then there exists \( u \in N \) such that \((xu)^p = y^p\). Since the \( p \)-power map is injective on \( G \), we conclude that \( xu = y \) so that \( xN = yN \). Hence \( G \to G, x \mapsto x^p \) induces a bijective correspondence between the cosets of \( N \) in \( G \) and their \( p \)-th powers, i.e., a bijection \( \{xN \mid x \in G\} \to \{x^pN^p \mid x \in G\} \), \( xN \mapsto (xN)^{(p)} = x^pN^p \).

Let \( \mu \) be the normalised Haar measure on the compact group \( G \) so that \( \mu(G) = 1 \). Since \( N \) is a saturable open subgroup of \( G \), note that \( |N : N^p| = p^{\dim(N)} = p^{\dim(G)} \).

For every \( x \in G \) we see that
\[
\mu(x^pN^p) = \mu(N^p) = |N : N^p|^{-1}\mu(N) = p^{-\dim(G)}|G : N|^{-1},
\]
and thus we conclude that
\[
\mu(G^{(p)}) = |G : N|\left(p^{-\dim(G)}|G : N|^{-1}\right) = p^{-\dim(G)}.
\]

Using \( d(G) = \dim(G) \), this yields
\[
p^{d(G)} = |G : G^{[G,G]}| \leq |G : G^p| = (\mu(G^p))^{-1} \leq (\mu(G^{(p)}))^{-1} = p^{\dim(G)} = p^{d(G)}.
\]

Hence \([G,G] \subseteq G^p \), and \( G \) is powerful. \( \square \)

**Proof of Corollary 1.6.** Suppose that \( G \) is a subgroup of a finitely generated saturable pro-\( p \) group \( S \). Since \( S \) is saturable, the map \( S \to S, x \mapsto x^p \) is injective and therefore its restriction to \( G \) is also injective. By Theorem 1.5 it suffices to show that \( G \) has an open PF-embedded subgroup.

The saturable closure of \( G \) in \( S \), denoted by \( \text{sat}_S(G) \), is a saturable subgroup of \( S \) which contains \( G \) as an open subgroup and is the smallest saturable subgroup with this property; see [13] or [9]. Replacing \( S \) by \( \text{sat}_S(G) \), we may assume that \( |S : G| \) is finite. Since \( S \) is saturable, it is a PF-group. Let \( S_i, i \in \mathbb{N} \), be a potent filtration of \( S \) starting at \( S_1 = S \). Since \( |S : G| < \infty \), there exists a positive integer \( k \) such that \( S^{p^k} \subseteq G \). Then \( S^{p^k} \) is an open subgroup of \( G \) and [2] Proposition 3.2(iii)] implies that \( S_i^{p^k}, i \in \mathbb{N} \), is a potent filtration of \( S \). In particular, \( S_i^{p^k} \) is an open PF-embedded subgroup of \( G \). \( \square \)

As mentioned in the introduction, saturable pro-\( p \) groups need not be uniform. Corollary 1.7, which provides a practical criterion for deciding whether a saturable group is uniform, is a direct consequence of Corollary 1.6.
Proof of Corollary 1.8. In [7] González-Sánchez and Klopsch proved that every torsion-free pro-$p$ group of dimension less than $p$ is saturable. Thus the assertion follows from Corollary 1.7. □

Suppose that $p \geq 3$. We recall that a subgroup $N$ of $G$ is powerfully embedded in $G$ if $[N,G] \subseteq N^p$. If $N$ is powerfully embedded in $G$ then $N^{p^{i+1}}$, $i \in \mathbb{N}$, is a potent filtration in $G$; in particular, $N$ is PF-embedded in $G$. Thus we obtain from Theorem 1.5 also the following corollary.

**Corollary 2.9.** Suppose that $p \geq 3$. Let $G$ be a pro-$p$ group of finite rank with an open powerfully embedded subgroup and such that the map $G \to G$, $x \mapsto x^p$ is injective. If $d(G) = \dim(G)$ then $G$ is uniform.

2.4. Finally, we explore a possible analogue of Conjecture 1.1 for finite $p$-groups, as suggested toward the end of Section 1.1.

**Proof of Proposition 1.10.** Let $p \geq 3$ and suppose that the answer to Question 1.9 is ‘yes’. Let $G$ be a torsion-free pro-$p$ group of finite rank such that $d(G) = \dim(G)$. We need to prove that $G$ is powerful.

Let $U$ be a uniform open normal subgroup of $G$. The open subgroups $U^n$, $n \in \mathbb{N}$ form a base for the neighbourhoods of 1 in $G$. By Lemma 2.5 we have that \{ $x \in G$ | $x^p \in U^n$ \} $\subseteq U$. Since $U$ is uniform, this implies that for all $n \in \mathbb{N}$,

\[
(2.3) \quad \Omega_1(G/U^n) = \langle xU^n \in G/U^n \mid x^p \in U^n \rangle = U^{p^{n-1}}/U^n
\]

has size $p^{\dim(U)}$. Since $p \geq 3$, we conclude from a result of Laffey [12, Corollary 2] that

\[
\dim(G) = d(G) = \limsup_{n \in \mathbb{N}} d(G/U^n) \leq \limsup_{n \in \mathbb{N}} \log_p|\Omega_1(G/U^n)| = \dim(G).
\]

This implies that $d(G/U^n) = \log_p|\Omega_1(G/U^n)| = d(G)$ for infinitely many $n \in \mathbb{N}$. Let $k$ be the smallest positive integer such that $d(G/U^k) = d(G)$. Then for each $n \geq k$ we have $d(G) \geq d(G/U^n) \geq d(G/U^k) = d(G)$, and consequently, $d(G/U^n) = d(G) = \log_p|\Omega_1(G/U^n)|$. The positive answer to Question 1.9 implies that $G/U^n$ is a powerful finite $p$-group for each $n \geq k$. Since $G$ is the inverse limit of the groups $G/U^n$, $n \geq k$, we deduce from [11, Corollary 3.3] that $G$ is powerful. □

We remark that the group $\Omega_1(G/U^n) = U^{p^{n-1}}/U^n$ in (2.3) is an elementary abelian $p$-group. The proof of Proposition 1.10 shows that Conjecture 1.1 is true whenever the following ‘weaker’ version of Question 1.9 has a positive answer.

**Question 2.10.** Suppose that $p \geq 3$ and let $G$ be a finite $p$-group such that $\Omega_1(G)$ is an elementary abelian $p$-group. Is it true that $G$ is powerful if and only if $d(G) = \log_p|\Omega_1(G)|$?
Using [12, Corollary 2], it is easy to see that Question 1.9 has positive answer for all finite $p$-groups $G$ with the property $|G : G^p| = |\Omega_1(G)|$. In the 1930s P. Hall showed that this condition is satisfied for regular $p$-groups; see [3]. It was proved more recently in [6] that for $p \geq 3$ the same property is shared by every finite $p$-group $G$ which is potent, i.e., which satisfies $\gamma_p(G) \leq G^p$p; indeed, in this situation it is true that $|N : N^p| = |\Omega_1(N)|$ for every normal subgroup $N$ of $G$.

Suppose that $p \geq 3$ and consider a finite $p$-group $G$ with $\Omega_1(G) \leq Z(G)$; such a group is called a $p$-central group. Suppose that $d(G) = \log_p|\Omega_1(G)|$. Since $G$ is $p$-central, by [17, Proposition 4], we have $|G : G^p| \leq |\Omega_1(G)|$. Now from

$$|\Omega_1(G)| = p^{d(G)} = |G : G^p[G, G]| \leq |G : G^p| \leq |\Omega_1(G)|$$

we deduce that $[G, G] \subseteq G^p$, which means that $G$ is powerful. Hence for $p$-central groups the answer to Question 1.9 is positive.

2.5. We conclude the section with two observations. In [10] Klopsch proved for $p \geq 3$ that every pro-$p$ group $G$ of finite rank with $\pi(G) = 1$ satisfies $\text{rk}(G) = \dim(G)$. This allows us to replace $\dim(G)$ by $\text{rk}(G)$ in all the results of this section. For instance we obtain the following consequence.

**Corollary 2.11.** Suppose that $p \geq 3$. Let $G$ be a finitely generated pro-$p$ group with an open PF-embedded subgroup and such that the map $G \rightarrow G$, $x \mapsto x^p$ is injective. If $d(G) = \text{rk}(G)$ then $G$ is uniform.

Secondly, we record a straightforward, but useful result, which can be regarded as a modification of [1, Proposition 4.4].

**Proposition 2.12.** Let $G$ be a finitely generated powerful pro-$p$ group such that $d(G) = \dim(G)$. Then $G$ is uniform.

**Proof.** Let $G_i = P_i(G)$, $i \in \mathbb{N}$, denote the terms of the lower central $p$-series of $G$, and put $d_i := \log_p|G_i : G_{i+1}|$. By [1, Theorem 3.6(iv)], we have $d_1 \geq d_2 \geq \ldots$, hence there exists $k \in \mathbb{N}$ such that $d_i = d_k$ for all $i \geq k$. Now, by [1, Theorem 3.6(ii)], we obtain $P_j(G_k) = G_{k+j-1} = G_k^p = P_{j+k-1}(G)$ for all $j \in \mathbb{N}$, and [1, Theorem 3.6(i)] shows that $G_k$ is powerful. Moreover, we have $|P_j(G_k) : P_{j+1}(G_k)| = |G_k : P_2(G_k)|$ for each $j \in \mathbb{N}$, which means that $G_k$ is uniform. Now we have

$$d(G) = d_1 \geq d_2 \geq \ldots \geq d_k = d(G_k) = \dim(G) = d(G),$$

and consequently $d_i = d_1$ for all $i \in \mathbb{N}$. Hence $G$ is uniform. \qed

3. Hereditarily powerful pro-$p$ groups

In the present section we prove the assertions in Section 1.2.

**Proof of Theorem 1.11.** Suppose that $G$ is hereditarily powerful. Then $G$ is the inverse limit of hereditarily powerful finite $p$-groups $G_i$, $i \in I$, with respect to
connecting homomorphisms \( \varphi_{ij} : G_i \to G_j \) for \( i \geq j \), where \( I \) is a suitable directed set. By \cite{15} Theorems 3.1 and 4.3.1, a finite \( p \)-group is hereditarily powerful if and only if it is modular and, if \( p = 2 \), not Hamiltonian. (Thus excluded are all direct products of the quaternion group \( Q_8 \) with elementary abelian \( 2 \)-groups.)

Finite modular groups were classified by Iwasawa; see \cite{2} or \cite{13} Theorem 2.3.1.

According to this classification, every finite \( p \)-group \( H \) which is hereditarily powerful is of the following form: \( H \) contains an abelian normal subgroup \( K \) such that \( H/K \) is cyclic and there exist an element \( h \in H \) with \( H = \langle h \rangle K \) and a positive integer \( s \) such that \( h^{-1}kh = k^{1+p^s} \) for all \( k \in K \), with \( s \geq 2 \) in case \( p = 2 \). Hence each \( G_i, i \in I \), is of this form and we denote by \( X_i \) the non-empty, finite set consisting of all triples \((A_{i\lambda}, b_{i\lambda}, s_{i\lambda})\) such that \( G_i = \langle b_{i\lambda} \rangle A_{i\lambda} \) with \( A_{i\lambda} \trianglelefteq G_i \) abelian, \( s_{i\lambda} \in \{1, 2, \ldots, \log_p |G_i| \} \) and \( b_{i\lambda} \in G_i \) acting on \( A_{i\lambda} \) as multiplication by \( 1 + p^{s_{i\lambda}} \) (if \( p = 2 \) we also require \( s_{i\lambda} \geq 2 \)).

We consider the inverse system of the \( X_i, i \in I \), with respect to the connecting maps \( X_i \to X_j \) for \( i \geq j \) induced by the homomorphisms \( \varphi_{ij} : G_i \to G_j \). By compactness, the inverse limit \( X = \varprojlim_{i \in I} X_i \) is non-empty and any \( x = (A, b, s) \in X \) yields \( A \leq G \) abelian, \( b \in G \) and \( s \in \mathbb{N} \cup \{\infty\} \) such that \( G = \langle b \rangle A \) with \( b \) acting on \( A \) as multiplication by \( 1 + p^s \). Moreover, we obtain \( s \geq 2 \) for \( p = 2 \).

The proof that groups of the described shape are hereditarily powerful is straightforward. \[\square\]

**Proof of Corollary 1.12.** By Theorem 2.2, a finitely generated pro-\( p \)-group is hereditarily uniform if and only if it is hereditarily powerful and torsion-free.

Suppose that \( G = \langle b \rangle A \) is as described in Theorem 1.11 and, in addition, torsion-free. Put \( d = \dim(G) \). If \( G \) is abelian then \( G \cong \mathbb{Z}_p^d \). If \( G \) is non-abelian then, because \( A \) has elements of infinite order, \( \langle b \rangle \cap A = 1 \) so that \( G \) is a semidirect product of \( \langle b \rangle \cong \mathbb{Z}_p \) and \( A \cong \mathbb{Z}_p^{d-1} \). \[\square\]

In \cite{11} Theorem 1.1 we classified finitely generated pro-\( p \)-groups with constant generating number on open subgroups, that is, pro-\( p \)-groups \( G \) with the property \( d(H) = d(G) \) for every open subgroup \( H \leq G \); see also \cite{19}.

**Theorem 3.1** (Klopsch and Snopce). Let \( G \) be a finitely generated pro-\( p \)-group and let \( d := d(G) \). Then \( G \) has constant generating number on open subgroups if and only if it is isomorphic to one of the groups in the following list:

1. the abelian group \( \mathbb{Z}_p^d \), for \( d \geq 0 \);
2. the metabelian group \( \langle y \rangle \rtimes A \), for \( d \geq 2 \), where \( \langle y \rangle \cong \mathbb{Z}_p \), \( A \cong \mathbb{Z}_p^{d-1} \) and \( y \) acts on \( A \) as scalar multiplication by \( \lambda \), with \( \lambda = 1 + p^s \) for some \( s \geq 1 \), if \( p > 2 \), and \( \lambda = \pm (1 + 2^s) \) for some \( s \geq 2 \), if \( p = 2 \);
3. the group \( \langle w \rangle \rtimes B \) of maximal class, for \( p = 3 \) and \( d = 2 \), where \( \langle w \rangle \cong C_3 \), \( B = \mathbb{Z}_3 + \mathbb{Z}_3 \omega \cong \mathbb{Z}_3^2 \) for a primitive 3rd root of unity \( \omega \) and where \( w \) acts on \( B \) as multiplication by \( \omega \);
the metabelian group $\langle y \rangle \rtimes A$, for $p = 2$ and $d \geq 2$, where $\langle y \rangle \cong \mathbb{Z}_2$, $A \cong \mathbb{Z}^{d-1}_2$ and $y$ acts on $A$ as scalar multiplication by $-1$.

Note that, if $G$ is a hereditarily uniform pro-$p$ group, then $d(H) = d(G)$ for all open subgroups $H \leq G$. Hence, Corollary 1.12 can also be regarded as a consequence of Theorem 3.1.

Conversely, Corollary 1.12 can be used to give a new proof of Theorem 3.1 at least in the case $p \geq 3$. Indeed, the argument given in [11] proceeds by induction on the index of a saturable open normal subgroup of a given group $G$ with constant generating number on open subgroups. The induction base, when $G$ itself is saturable, was established using Lie ring methods. For $p \geq 3$ we can use the results in the present paper to give a new proof of the base step as follows. Suppose that $G$ is saturable and $d(H) = d(G)$ for all open subgroups $H \leq G$. Then $d(H) = \dim(H)$ for all open subgroups $H \leq G$ and, by Corollary 1.6, the group $G$ is hereditarily uniform. Hence $G$ is one of the groups described in Corollary 1.12. This establishes [11, Corollary 2.4]; for the induction step one proceeds in the same way as in [11].

ACKNOWLEDGEMENT The first author gratefully acknowledges support through a grant of the London Mathematical Society as well as the support and hospitality of the Instituto de Matemática of the Universidade Federal do Rio de Janeiro in 2012.

REFERENCES

[1] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal, Analytic pro-$p$ groups, Cambridge Studies in Advanced Mathematics 61, Cambridge University Press, Cambridge, second edition, 1999.
[2] G. A. Fernández-Alcober, J. González-Sánchez and A. Jaikin-Zapirain, Omega subgroups of pro-$p$ groups, Israel J. Math. 166 (2008), 393–412.
[3] P. Hall, A contribution to the theory of groups of prime-power order, Proc. London Math. Soc. 36 (1933), 29–95.
[4] A. Heller and I. Reiner, Representations of cyclic groups in rings of integers I., Ann. of Math. 76 (1962), 73–92.
[5] J. González-Sánchez, On $p$-saturable groups, J. Algebra 315 (2007), 809–823.
[6] J. González-Sánchez and A. Jaikin-Zapirain, On the structure of normal subgroups of potent $p$-groups, J. Algebra 276 (2004), 193–209.
[7] J. González-Sánchez and B. Klopsch, Analytic pro-$p$ groups of small dimensions, J. Group Theory 12 (2009), 711–734.
[8] K. Iwasawa, Über die endlichen Gruppen und die Verbände ihrer Untergruppen, J. Univ. Tokyo 4 (1941), 171–199.
[9] B. Klopsch, On the Lie theory of $p$-adic analytic groups, Math. Z. 249 (2005), 713–730.
[10] B. Klopsch, On the rank of compact $p$-adic Lie groups, Arch. Math. 96 (2011), 321–333.
[11] B. Klopsch and I. Snopce, Pro-$p$ groups with constant generating number on open subgroups, J. Algebra 331 (2011), 263–270.
[12] T. J. Laffey, *The minimum number of generators of a finite p-group*, Bull. London Math. Soc. 5 (1973), 288–290.
[13] M. Lazard, *Groupes analytiques p-adiques.*, Publ. Math. IHÉS 26 (1965), 389–603.
[14] C. R. Leedham-Green and S. McKay, *The structure of groups of prime power order*, Oxford University Press, Oxford, 2002.
[15] A. Lubotzky and A. Mann, *Powerful p-groups. I. Finite Groups*, J. Algebra 105 (1987), 484–505.
[16] A. Lubotzky and D. Segal, *Subgroup Growth*, volume 212, Birkhäuser, 2003.
[17] A. Mann, *The power structure of p-groups II*, J. Algebra 318 (2007), 953–956.
[18] R. Schmidt, Subgroup lattices of groups, de Gruyter Expositions in Mathematics 14, Walter de Gruyter & Co., Berlin, 1994.
[19] I. Snopce, *Pro-p groups of rank 3 and the question of Iwasawa*, Arch. Math. (Basel) 92 (2009), 19–25.
[20] L. Wilson, *On the power structure of powerful p-groups*, J. Group Theory 5 (2002), 129–144.

Department of Mathematics, Royal Holloway, University of London, Egham TW20 0EX, UK
*Current address:* Institut für Algebra und Geometrie, Mathematische Fakultät, Otto-von-Guericke-Universität Magdeburg, 39016 Magdeburg, Germany
*E-mail address:* Benjamin.Klopsch@rhul.ac.uk

Universidade Federal do Rio de Janeiro, Instituto de Matemática, 20785-050 Rio de Janeiro, RJ, Brasil
*E-mail address:* ilir@im.ufrj.br