Isometric immersions of the hyperbolic plane into the hyperbolic space

Atsufumi Honda
Tokyo Institute of Technology

Jan 27, 2011
Differential Geometry and Tanaka Theory
~ Differential System and Hypersurface Theory ~
Contents

△ Background
  ▶ Non-negative curvature case
  ▶ Developable Surfaces

△ Geometry of the space of oriented geodesics in $H^3$
  ▶ Invariant Metrics
  ▶ Representation Formula

△ Complete Developable Surfaces
  ▶ Behavior at infinity
  ▶ Developable Surfaces of Exponential Type
1.1 Negative & Non-negative curvature

Consider isometric immersions between space forms of same curvature with codim. = 1.

\[ \nabla f : \mathbb{R}^n \xrightarrow{\text{isom. imms.}} \mathbb{R}^{n+1} \iff \text{Cylinder over plane curve} \]
(Hartman-Nirenberg 1959, Massey 1962 (n = 2))

\[ \nabla f : S^n \xrightarrow{\text{isom. imms.}} S^{n+1} \iff \text{Totally geodesic} \]
(O’Neil-Stiel 1963)

\[ \nabla f : H^n \xrightarrow{\text{isom. imms.}} H^{n+1} \rightsquigarrow \exists \text{Non-Trivial Examples} \]
(Nomizu 1973, Abe-Haas 1990)
1.2 Developable Surfaces

Fact

Isom. Imms. $f : H^2 \to H^3$

$\iff$ Complete Developable Surfaces in $H^3$.

- Developable $\overset{\text{def}}{\iff}$ Extrinsically Flat & Ruled.
- Extrinsically Flat $\overset{\text{def}}{\iff}$ product of principal curv. $\equiv 0$.
- Ruled $\overset{\text{def}}{\iff}$ Locus of a motion of geodesics.

- Gauss equation

$$\lambda_1 \lambda_2 = K + 1, \quad (\lambda_1, \lambda_2 : \text{principal curv.})$$

- Proof: an analogue of the method used by Massey for the Euclidean case.
$\mathcal{L}(H^3)$: the Space of Oriented Geodesics in $H^3$.

Ruled Surfaces in $H^3$ $\leftrightarrow$ Curves in $\mathcal{L}(H^3)$.

**Developable** Surfaces in $H^3$ $\leftrightarrow$ ? ? ? Curves in $\mathcal{L}(H^3)$.

---

**Theorem I (H)**

Developables $\leftrightarrow$ Curves in $\mathcal{L}(H^3)$ s.t. \[
\begin{cases}
\text{null w.r.t. } G \\
\text{causal w.r.t. } \hat{G}.
\end{cases}
\]

- $G$, $\hat{G}$: certain neutral metrics on $\mathcal{L}(H^3)$.
- A curve $\alpha : R \longrightarrow (\mathcal{L}(H^3), \langle , \rangle)$: null (resp. causal) \[
\begin{align*}
\text{def} & \quad \langle \alpha', \alpha' \rangle = 0 \quad (\text{resp. } \langle \alpha', \alpha' \rangle \leq 0).
\end{align*}
\]
The space of oriented geodesics in $H^3$

\[ \mathcal{L}(H^3) := \{[\gamma] | \gamma: \text{an unit speed geodesic in } H^3 \} \]

(\text{where, } \gamma_1 \sim \gamma_2 \overset{\text{def}}{\iff} \exists T; \gamma_1(T + \cdot) = \gamma_2(\cdot)).

\[ \mathcal{L}(H^3) = S^2 \times S^2 \setminus \{\text{Diagonal}\}.\]
(µ₁, µ₂) : a (complex) coordinate system of $\mathcal{L}(H^3)$.

$\mathcal{L}(H^3) = (\hat{C} \times \hat{C}) \setminus \Delta, \quad (\Delta = \{1 + \mu_1 \bar{\mu}_2 = 0\}).$

Set

$G := \text{Im} \left[ \frac{4d\mu_1 d\bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} \right], \quad \hat{G} := \text{Re} \left[ \frac{4d\mu_1 d\bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} \right],$

: metrics on $\mathcal{L}(H^3)$ (sgn = (+ + --) : neutral).
2.1 What are $G$ and $\hat{G}$?

$G, \hat{G}$: metrics invariant under $\text{Isom}_0(H^3) \sim L(H^3)$.

(=Invariant Metrics)

**Fact (Salvai 2007)**

Any invariant metrics on $L(H^3)$ can be written as

$$aG + b\hat{G} \quad (a, b \in R).$$

What are specific properties of $G$, $\hat{G}$?
Geometric structures $\omega, J, P$ on $\mathcal{L}(H^3)$

\[ \nabla \text{ [Canonical Symplectic Structure } \omega] \]

\[ \omega := \hat{\pi}_*(d\Theta). \]

- $\hat{\pi} : UH^3 \ni (p, v) \mapsto [\gamma_{p,v}] \in \mathcal{L}(H^3)$: geodesic flow.
- $\Theta$: The canonical contact form of $UH^3$ (Liouville form).

\[ \nabla \text{ [Minitwistor Complex Structure } J \text{ (Hitchin 1982)]} \]

- $T_{[\gamma]} \mathcal{L}(H^3) = \mathcal{J}^\perp(\gamma) := \{\text{orthogonal Jacobi field along } \gamma\}.$
- $J_{[\gamma]} : \mathcal{J}^\perp(\gamma) \ni V \mapsto \gamma' \times_\gamma V \in \mathcal{J}^\perp(\gamma)$: rotation by $90^\circ$, (where $\times$: vector product of $H^3$).

\[ \nabla \text{ [para-Complex Structure } P \text{ (Kaneyuki-Kozai, et al.)]} \]

$P_{[\gamma]} : T_{[\gamma]} \mathcal{L}(H^3) \to T_{[\gamma]} \mathcal{L}(H^3)$;

\[ \frac{\partial}{\partial \mu_1} \leftrightarrow -\frac{\partial}{\partial \mu_1}, \quad \frac{\partial}{\partial \mu_2} \leftrightarrow \frac{\partial}{\partial \mu_2}. \]
Characterizations of $G, \hat{G}$

**Proposition (H)**

Let $\omega, J, P$ as above, then

$$G = 2\omega(J \cdot , \cdot), \quad \hat{G} = 2\omega(P \cdot , \cdot).$$

(Other characterizations)

- Conformally flat invariant metrics $\iff \lambda G$ ($\lambda \in \mathbb{R}$)
- Einstein invariant metrics $\iff \lambda \hat{G}$ ($\lambda \in \mathbb{R}$)

Some remarks on $G = G + \sqrt{-1} \hat{G} = \frac{4d\mu_1 d\bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2}$

∇ $-G$: (complex) metric on $\mathcal{L}(H^3) = \text{SL}(2, \mathbb{C})/ \text{GL}(1, \mathbb{C})$.

∇ $(\mathcal{L}(H^3), G) = (\mathbb{P}^1, g_{FS})^C$: complexification.
Remark: General case $\Sigma^3$: a 3-dim. space form.

$$G := 2\omega(J, \cdot, \cdot)$$

: a neutral metric on $\mathcal{L}(\Sigma^3)$.

\[ \nabla \quad \text{[Canonical Symplectic Structure } \omega]\]

$$\omega := \hat{\pi}^* (d\Theta).$$

- $\hat{\pi}: U\Sigma^3 \ni (p, v) \mapsto [\gamma_{p,v}] \in \mathcal{L}(\Sigma^3)$: geodesic flow.
- $\Theta$: The canonical contact form of $U\Sigma^3$ (Liouville form).

\[ \nabla \quad \text{[Minitwistor Complex Structure } J \text{ (Hitchin 1982)]}\]

- $T_{[\gamma]}\mathcal{L}(\Sigma^3) = \mathcal{J}^\perp(\gamma) := \{\text{orthogonal Jacobi field along } \gamma\}.$
- $J_{[\gamma]}: \mathcal{J}^\perp(\gamma) \ni V \mapsto \gamma' \times_\gamma V \in \mathcal{J}^\perp(\gamma)$: rotation by $90^\circ$.
- $J$ is integrable if $\Sigma$ is a space form.

Proposition (H)

Developable surfaces in $\Sigma^3 \implies$ Null curves in $(\mathcal{L}(\Sigma^3), G)$. 
2.2 Representation Formula

Theorem I (H)

Developable surfaces generated by complete geodesics

\[ \leftrightarrow \]

Curves in \( \mathcal{L}(H^3) \) s.t. null w.r.t. \( G \) and causal w.r.t. \( \hat{G} \).

- Nullity for \( G \) = Extrinsically Flatness.
- Causality for \( \hat{G} \) = Regularity (of Surface).
\[\text{[Representation formula for developables]}\]

A curve \(\alpha = (\mu_1(s), \mu_2(s)) : \mathbb{R} \to \mathcal{L}(H^3) \cong (\mathring{C} \times \mathring{C}) \setminus \Delta:\]

\[
\begin{align*}
\text{s.t.} \quad & \left\{ \begin{array}{l}
\text{Im} \frac{4\mu'_1(s)\bar{\mu}'_2(s)}{(1+\mu_1(s)\bar{\mu}_2(s))^2} = 0 \quad (\text{i.e., } G(\alpha', \alpha') = 0), \\
\text{Re} \frac{4\mu'_1(s)\bar{\mu}'_2(s)}{(1+\mu_1(s)\bar{\mu}_2(s))^2} \leq 0 \quad (\text{i.e., } \hat{G}(\alpha', \alpha') \leq 0). \\
\end{array} \right.
\end{align*}
\]

\[
\Rightarrow f(s, t) = \frac{1}{2|1 + \mu_1\bar{\mu}_2|} \begin{pmatrix}
(\bar{\mu}_2(s)^2 + 1)e^t + (\mu_1(s)^2 + 1)e^{-t} \\
2(e^t \text{ Re } \mu_2(s) - e^{-t} \text{ Re } \mu_1(s)) \\
2(e^t \text{ Im } \mu_2(s) - e^{-t} \text{ Im } \mu_1(s)) \\
-(\bar{\mu}_2(s)^2 - 1)e^t + (\mu_1(s)^2 - 1)e^{-t}
\end{pmatrix} \in H^3
\]

gives a developable surface, where

\[
H^3 = \left\{ \mathbf{x} = ^t(x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1 \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1, \ x_0 > 0 \right\},
\]

\((\mathbb{R}^4_1: \text{the Lorentz-Minkowski 4-space}).\)
(Ex. 1) [Totally Geodesic]

- \( \mu_1(s) = -\tanh s, \quad \mu_2(s) = \tanh s. \)
- \( G(\alpha', \alpha') = \frac{4\mu_1'(s)\bar{\mu}_2'(s)}{(1+\mu_1(s)\bar{\mu}_2(s))^2} = -4. \)

\[ \therefore G(\alpha', \alpha') = 0 \quad \& \quad \hat{G}(\alpha', \alpha') < 0. \]

**Figure:** Totally geodesic.
(Ex. 2) [Hyperbolic analogues of Cylinders]

\[ \mu_1(s) = -\zeta(s), \quad \mu_2(s) = \zeta(s) \]

(where, \( \zeta(s) : \mathbb{R} \rightarrow \mathbb{D} \subset \mathbb{C} \) : regular curve).

\[ G(\alpha', \alpha') = -\frac{4|\zeta'(s)|^2}{(1+|\zeta(s)|^2)^2} < 0. \]

\[ \therefore \quad G(\alpha', \alpha') = 0 \quad \& \quad \hat{G}(\alpha', \alpha') < 0. \]

Figure: \( \zeta(s) = e^{is}/3. \)
(Ex. 3) [Ideal Cones]

- $\mu_1(s) = \text{const.}, \quad \mu_2(s) = \mu(s)$
  (where, $\mu(s) : \mathbb{R} \rightarrow \mathbb{C} : \text{regular curve}$).
- $G(\alpha', \alpha') = 0$.

$\therefore G(\alpha', \alpha') = 0 \quad \& \quad \hat{G}(\alpha', \alpha') = 0.$

\textbf{Figure:} const. $= 0, \; \mu(s) = e^{is}/2.$
(Ex. 4) [Rectifying Developables of Helices]

$\mu_1(s) := \kappa \frac{4 \sqrt{2} \sqrt{\kappa^2 + \tau^2} i + 4\tau A_-}{(\sqrt{2} \sqrt{\kappa^2 + \tau^2} i + 4\tau A_+)(a_+ + a_-)^2 + 4\kappa A_-} \exp\left(\frac{A_+ + iA_-}{\sqrt{2}} s\right),$ 

$\mu_2(s) := \frac{1}{\kappa} \frac{4 \sqrt{2} \sqrt{\kappa^2 + \tau^2} - \tau A_+)(a_+ + a_-)^2 - 4\kappa A_-}{4 \sqrt{2} \sqrt{\kappa^2 + \tau^2} i + 4\tau A_+ - (a_+ + a_-)^2 A_+} \exp\left(\frac{-A_+ + iA_-}{\sqrt{2}} s\right)$

(where, $\kappa, \tau \in \mathbb{R} \setminus \{0\},$

$a_\pm := \sqrt{(\kappa \pm 1)^2 + \tau^2}, A_\pm := \sqrt{\pm(1 - \kappa^2 - \tau^2) + a_+a_-}.$

Figure: $\kappa = \tau = 1.$
### 3.1 Behavior at infinity: Ideal Cones

**Proposition**

Complete developables corresp. to curves null w.r.t. $G, \hat{G}$ have an end asymptotic to a point in $\partial H^3$.

Complete developables corresp. to curves null w.r.t. $G, \hat{G}$ def $\leftrightarrow$: Ideal Cones.
3.2 Developables of Exponential Type

Lem (Hyperbolic Massey’s lemma)

\[ f : M^2 \rightarrow H^3: \text{ extrinsically flat surface.} \]

\( H \): mean curvature.

\( l \): an asymptotic curve in non umbilic point set.

\( t \): arc-length parameter of \( l \).

\[ \Rightarrow \quad \frac{\partial^2}{\partial t^2} \left( \frac{1}{H} \right) = \frac{1}{H} \quad \text{on } l. \]

∇ Thus, \[ \frac{1}{H} = P \cosh t + Q \sinh t: \]
\[
\frac{1}{H} = \begin{cases} 
A \cosh (t + B) \\
A e^{\pm t} \\
A \sinh (t + B)
\end{cases}
\]

\(\nabla\) For a **complete** developables, \(1/H\) never vanishes. \(\implies\) third case does not occur. Thus,

(c) \(H = a(s)/\cosh(t + b(s))\) or

(e) \(H = \rho(s)e^t\) holds.

\(\nabla\) Complete dev. with (e) \(\iff\) : **Exponential Type**.

\(\nabla\) **Example:** Ideal cones \(\Rightarrow\) Exponential Type.

(Converse?)
Theorem II (H)

Real analytic developable surfaces of exponential type are ideal cones.

\[ \nabla \text{ Rem. } \exists \text{ non-real-analytic exponential dev. which is not an ideal cone.} \]
(Review.)

\[ \textbf{Lem 1.} \] Two unit speed geodesics in \( H^3 \)

\[ \alpha(t) = (\cosh t)p + (\sinh t)v, \quad \beta(t) = (\cosh t)q + (\sinh t)w \]

are asymptotic if and only if

\[ \langle p + v, q + w \rangle = 0. \]

\[ \textbf{Lem 2.} \text{(Frenet-Serret formula)} \] Let \( \mathcal{F} = (\gamma, e = \gamma', n, b) \)

be the Frenet frame for a curve \( \gamma \) in \( H^3 \)

\[ \Rightarrow \quad \gamma'' = \gamma + \kappa n, \quad n' = -\kappa e + \tau b, \quad b' = -\tau n. \]
Sketch of the proof

Let \( f \) be a real analytic exponential developable

\[
    f(s, t) = (\cosh t)\gamma(s) + (\sinh t)\xi(s) \quad \left( \in H^3 \subset \mathbb{R}^4_1 \right),
\]

such that \( H(s, t) = \delta(s)e^t \).

By Lem 1, it suffices to prove that

\[
    (\varphi(s) := ) \langle \gamma(s) + \xi(s), \gamma(s_0) + \xi(s_0) \rangle \equiv 0 \quad (1)
\]

for some \( s_0 \in \mathbb{R} \).

To show (1), we shall determine \( \xi \).
(To determine $\xi$)

1. Determine the geodesic foliation of $H^2$ induced by $f$
   \[ F(s, t) = (\cosh t)c(s) + (\sinh t)v(s): \text{Geod. Foli. of } H^2 \]
   **Codazzi equation** $\Rightarrow c(s): \text{horocycle.}$

2. Represent $\xi$ in terms of $e, n, b$
   **Gauss equation** $\Rightarrow$
   \[ \xi = f_*(v(s)) = \frac{n(s) + \delta(s)b(s)}{\kappa(s)}. \]

Finally, applying Lem2 (Frenet-Serret), it holds that
\[
\varphi'(s) = \left(\gamma(s) + \frac{n(s) + \delta(s)b(s)}{\kappa(s)}, \gamma(s_0) + \xi(s_0)\right)' \equiv 0.
\]
\[\boxrule\]