SYMMETRY AND MONOTONICITY PROPERTIES OF SINGULAR SOLUTIONS TO SOME COOPERATIVE SEMILINEAR ELLIPTIC SYSTEMS INVOLVING CRITICAL NONLINEARITIES

FRANCESCO ESPOSITO

Università della Calabria
Dipartimento di Matematica e Informatica
Via P. Bucci 31B - Rende (CS), 87036, Italy
and
Université de Picardie Jules Verne
LAMFA, CNRS UMR 7352
Rue Saint-Leu 33 - Amiens, 80039, France

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Abstract. We investigate qualitative properties of positive singular solutions of some elliptic systems in bounded and unbounded domains. We deduce symmetry and monotonicity properties via the moving plane procedure. Moreover, in the unbounded case, we study some cooperative elliptic systems involving critical nonlinearities in $\mathbb{R}^n$.

1. Introduction. The aim of this paper is to investigate symmetry and monotonicity properties of singular solutions to some semilinear elliptic systems. In the first part of the paper we start by considering the following semilinear elliptic system

$$\begin{cases}
-\Delta u_i = f_i(u_1, \ldots, u_m) & \text{in } \Omega \setminus \Gamma \\
u_i > 0 & \text{in } \Omega \setminus \Gamma \\
u_i = 0 & \text{on } \partial \Omega,
\end{cases} \quad (1.1)$$

where $\Omega$ is a bounded smooth domain of $\mathbb{R}^n$ with $n \geq 2$ and $i = 1, \ldots, m$ ($m \geq 2$). The technique which is mostly used in this paper is the well-known moving plane method which goes back to the seminal works of Alexandrov [1] and Serrin [39]. See also the celebrated papers of Berestycki-Nirenberg [5] and Gidas-Ni-Nirenberg [23]. Such a technique can be performed in general domains providing partial monotonicity results near the boundary and symmetry when the domain is convex and symmetric. For simplicity of exposition we assume directly in all the paper that $\Omega$ is a convex domain which is symmetric with respect to the hyperplane $\{x_1 = 0\}$. The solution has a possible singularity on the critical set $\Gamma \subset \Omega$. When $m = 1$, system (1.2) reduces to a scalar equation that was already studied in [19, 38]. The moving plane procedure for semilinear elliptic systems has been firstly adapted by Troy in [46] where he considered the cooperative system (1.1) with $\Gamma = \emptyset$ (see

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This technique was adapted in the case of cooperative semilinear systems in the half space by Dancer in [15] and in the whole space by Busca and Sirakov in [10]. For the case of quasilinear elliptic systems in bounded and unbounded domains we suggest [34, 35].

Moreover, motivated by [28], through all the paper, we assume that the following hypotheses (denoted by \((h_{f_i})\) in the sequel) hold:

\[(h_{f_i})\]

(i) \(f_i : \mathbb{R}_+^m \to \mathbb{R}\) are assumed to be \(C^1\) functions for every \(i = 1, \ldots, m\).

(ii) The functions \(f_i (1 \leq i \leq m)\) are assumed to satisfy the monotonicity (also known as cooperative) conditions

\[
\frac{\partial f_i}{\partial t_j}(t_1, \ldots, t_j, \ldots, t_m) \geq 0 \quad \text{for} \quad i \neq j, \quad 1 \leq i, j \leq m.
\]

In this paper the case of singular nonlinearities for systems is not included, while it was considered in the case of scalar equations, see [19]; about these problems we have also to mention the pioneering work of Crandall, Rabinowitz and Tartar [14] and also [8, 12, 21, 27, 44] for the scalar case. In future projects, we would like to consider a more general class of nonlinearities. In particular, it would be interesting to study problems involving singular nonlinearities as in the scalar case, using some techniques developed in [12, 21]. Since we want to consider singular solutions, the natural assumption in our paper is

\[
u_i \in H^1_{loc}(\Omega \setminus \Gamma) \cap C(\overline{\Omega} \setminus \Gamma) \quad \forall i = 1, \ldots, m
\]

and thus the system is understood in the following sense:

\[
\int_{\Omega} \nabla u_i \nabla \varphi_i \, dx = \int_{\Omega} f_i(u_1, u_2, \ldots, u_m) \varphi_i \, dx \quad \forall \varphi_i \in C^1_c(\Omega \setminus \Gamma) \quad (1.2)
\]

for every \(i = 1, \ldots, m\).

Remark 1.1. Note that, by the assumption \((h_{f_i})\), the right hand side in the system \((1.2)\) is locally bounded. Therefore, by standard elliptic regularity theory, it follows that

\[
u_i \in C^1_{loc, \alpha}(\Omega \setminus \Gamma),
\]

where \(0 < \alpha < 1\). We just remark that, in 1968, E. De Giorgi provided a counterexample showing that the scalar case is special and the regularity theory does not work in general for elliptic systems (see [18]); however, in the case of equations involving Laplace operator, Schauder theory is still applicable.

Under the previous assumptions we can prove the following result:

**Theorem 1.2.** Let \(\Omega\) be a convex domain which is symmetric with respect to the hyperplane \(\{x_1 = 0\}\) and let \((u_1, \ldots, u_m)\) be a solution to \((1.1)\), where \(u_i \in H^1_{loc}(\Omega \setminus \Gamma) \cap C(\overline{\Omega} \setminus \Gamma)\) for every \(i = 1, \ldots, m\). Assume that each \(f_i\) fulfills \((h_{f_i})\) and also that \(\Gamma\) is a point if \(n = 2\), while \(\Gamma\) is closed and such that

\[
\text{Cap}_2(\Gamma) = 0,
\]

if \(n \geq 3\). Then, if \(\Gamma \subset \{x_1 = 0\}\), it follows that \(u_i\) is symmetric with respect to the hyperplane \(\{x_1 = 0\}\) and increasing in the \(x_1\)-direction in \(\Omega \cap \{x_1 < 0\}\), for every \(i = 1, \ldots, m\). Furthermore

\[
\partial_{x_1} u_i > 0 \quad \text{in} \quad \Omega \cap \{x_1 < 0\},
\]

for every \(i = 1, \ldots, m\).
The technique developed in the first part of the paper and in [19, 20, 38] (see also [33] for the nonlocal setting) is very powerful and can be adapted to some cooperative systems in \( \mathbb{R}^n \) involving critical nonlinearities. Papers on existence or qualitative properties of solutions to systems with critical growth in \( \mathbb{R}^n \) are very few, due to the lack of compactness given by the Talenti bubbles and the difficulties arising from the lack of good variational methods. We refer the reader to [10, 13, 24, 25, 26, 36] for this kind of systems. The starting point of the second part of the paper is the study of qualitative properties of singular solutions to the following \( m \times m \) system of equations

\[
\begin{cases}
-\Delta u_i = \sum_{j=1}^{m} a_{ij} u_j^{2^* - 1} & \text{in } \mathbb{R}^n \setminus \Gamma \\
u_i > 0 & \text{in } \mathbb{R}^n \setminus \Gamma,
\end{cases}
\]

where \( i = 1, \ldots, m, \) \( m \geq 2, \) \( n \geq 3 \) and the matrix \( A := (a_{ij})_{i,j=1,\ldots,m} \) is symmetric and such that

\[
\sum_{j=1}^{m} a_{ij} = 1 \text{ for every } i = 1, \ldots, m.
\]

These kind of systems, with \( \Gamma = \emptyset, \) was studied by Mitidieri in [31, 32] considering the case \( m = 2, \) \( A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and it is known in the literature as nonlinearity belonging to the critical hyperbola.

If \( m = 1, \) then (1.3) reduces to the classical critical Sobolev equation

\[
\begin{cases}
-\Delta u = u^{2^* - 1} & \text{in } \mathbb{R}^n \setminus \Gamma \\
u > 0 & \text{in } \mathbb{R}^n \setminus \Gamma,
\end{cases}
\]

that can be found in [19, 38]. If \( \Gamma \) reduces to a single point we find the result contained in [45], while if \( \Gamma = \emptyset \) then system (1.5) reduces to the classical Sobolev equation (see [11]). For existence results of radial and nonradial solutions for (1.3), we refer to some interesting papers [24, 25]. We want to remark that in [24, 25] the authors treat the general case of a matrix \( A \) in which its entries \( a_{ij} \) are not necessarily positive and this fact implies that it is not possible to apply the maximum principle. As remarked above the natural assumption is

\[
u_i \in H^1_{\text{loc}}(\mathbb{R}^n \setminus \Gamma) \quad \forall i = 1, \ldots, m
\]

and, thus, the system is understood in the following sense:

\[
\int_{\mathbb{R}^n} \nabla u_i \nabla \varphi_i \, dx = \sum_{j=1}^{m} a_{ij} \int_{\mathbb{R}^n} u_j^{2^* - 1} \varphi_i \, dx \quad \forall \varphi_i \in C^1_c(\mathbb{R}^n \setminus \Gamma)
\]

for every \( i = 1, \ldots, m. \)

What we are going to show is the following result:

**Theorem 1.3.** Let \( n \geq 3 \) and let \((u_1, \ldots, u_m)\) be a solution to (1.3), where \( u_i \in H^1_{\text{loc}}(\mathbb{R}^n \setminus \Gamma) \) for every \( i = 1, \ldots, m. \) Assume that the matrix \( A = (a_{ij})_{i,j=1,\ldots,m}, \) defined above, is symmetric, \( a_{ij} \geq 0 \) for every \( i, j = 1, \ldots, m \) and it satisfies (1.4). Moreover, at least one of the \( u_i \) has a non-removable\(^1 \) singularity in the singular

\(^1\)Here we mean that the solution \((u_1, \ldots, u_m)\) does not admit a smooth extension all over the whole space. Namely it is not possible to find \( \tilde{u}_i \in H^1_{\text{loc}}(\mathbb{R}^n) \) with \( u_i \equiv \tilde{u}_i \) in \( \mathbb{R}^n \setminus \Gamma, \) for some \( i = 1, \ldots, m. \)
set $\Gamma$, where $\Gamma$ is a closed and proper subset of \{ $x_1 = 0$ \} such that
$$\text{Cap}_{\mathbb{R}^n}(\Gamma) = 0.$$ 

Then, all the $u_i$ are symmetric with respect to the hyperplane \{ $x_1 = 0$ \}. The same conclusion is true if \{ $x_1 = 0$ \} is replaced by any affine hyperplane. If at least one of the $u_i$ has only a non-removable singularity at the origin for every $i = 1, \ldots, m$, then each $u_i$ is radially symmetric about the origin and radially decreasing.

Another interesting elliptic system involving Sobolev critical exponents is the following one:
\[
\begin{aligned}
-\Delta u &= u^{2^*-1} + \frac{\alpha}{2^*} u^{\alpha-1} v^{\beta} \quad \text{in } \mathbb{R}^n \setminus \Gamma \\
-\Delta v &= v^{2^*-1} + \frac{\beta}{2^*} u^{\alpha} v^{\beta-1} \quad \text{in } \mathbb{R}^n \setminus \Gamma \\
u, v &> 0 \quad \text{in } \mathbb{R}^n \setminus \Gamma,
\end{aligned}
\]

where $\alpha, \beta > 1$, $\alpha + \beta = 2^* := \frac{2n}{n-2}$ ($n \geq 3$)

The solutions to (1.7) are solitary waves for a system of coupled Gross–Pitaevskii equations. These type of systems arises, e.g., in the Hartree–Fock theory for double condensates, that is, Bose-Einstein condensates of two different hyperfine states which overlap in space. Existence results for these kind of systems are very complicated and the existence of nontrivial solutions is deeply related to the parameters $\alpha, \beta$ and $n$. System (1.7) with $\Gamma = \emptyset$ was studied in [2, 3, 4, 36, 40, 42]. In particular, in [36] the authors show a uniqueness result for least energy solutions, under suitable assumptions on the parameters $\alpha, \beta$ and $n$, while in [13] the authors study also the competitive setting, showing that the system admits infinitely many fully nontrivial solutions, which are not conformally equivalent. Motivated by their physical applications, weakly coupled elliptic systems have received much attention in recent years, and there are many results for the cubic case where $\Gamma = \emptyset$ and $2^*$ is replaced by $4$ in low dimensions $n = 3, 4$ (see e.g. [2, 3, 4, 29, 30, 41, 42]). Since our technique does not work when $1 < \alpha < 2$ or $1 < \beta < 2$, here we study the case $\alpha, \beta \geq 2$ and $n = 3$ or $n = 4$, since we are assuming that $\alpha + \beta = 2^*$.

**Theorem 1.4.** Let $n = 3$ or $n = 4$ and let $(u, v) \in H^1_{\text{loc}}(\mathbb{R}^n \setminus \Gamma) \times H^1_{\text{loc}}(\mathbb{R}^n \setminus \Gamma)$ be a solution to (1.7). Assume that the solution $(u, v)$ has a non-removable\footnote{As above, we mean that the solution $(u, v)$ does not admit a smooth extension all over the whole space. Namely it is not possible to find $(\tilde{u}, \tilde{v}) \in H^1_{\text{loc}}(\mathbb{R}^n) \times H^1_{\text{loc}}(\mathbb{R}^n)$ with $u \equiv \tilde{u}$ or $v \equiv \tilde{v}$ in $\mathbb{R}^n \setminus \Gamma$.} singularity in the singular set $\Gamma$, where $\Gamma$ is a closed and proper subset of \{ $x_1 = 0$ \} such that
$$\text{Cap}_{\mathbb{R}^n}(\Gamma) = 0.$$ 

Moreover, let us assume that $\alpha, \beta \geq 2$ and that holds $\alpha + \beta = 2^*$. Then, $u$ and $v$ are symmetric with respect to the hyperplane \{ $x_1 = 0$ \}. The same conclusion is true if \{ $x_1 = 0$ \} is replaced by any affine hyperplane. If at least one between $u$ and $v$ has only a non-removable singularity at the origin, then $(u, v)$ is radially symmetric about the origin and radially decreasing.

When the paper was completed, we learned that the case of bounded domains was also considered in [7] (see also [6]), obtaining similar results.
2. Notations and preliminary results. We need to fix some notations. For a real number \( \lambda \) we set

\[
\Omega_\lambda = \{ x \in \Omega : x_1 < \lambda \}
\]

\[
x_\lambda = R_\lambda(x) = (2\lambda - x_1, x_2, \ldots, x_n)
\]

which is the reflection through the hyperplane \( T_\lambda := \{ x_1 = \lambda \} \). Moreover, let

\[
a = \inf_{x \in \Omega} x_1.
\]

Since \( \Gamma \) is compact and of zero capacity, \( u_i \) is defined a.e. on \( \Omega \) and Lebesgue measurable on \( \Omega \) for every \( i = 1, \ldots, m \). Therefore the functions

\[
u_{i,\lambda} := u_i \circ R_\lambda
\]

are Lebesgue measurable on \( R_\lambda(\Omega) \). Similarly, \( \nabla u_i \) and \( \nabla v_{i,\lambda} \) are Lebesgue measurable on \( \Omega \) and \( R_\lambda(\Omega) \) respectively.

In the same spirit of [19], we recall some useful properties of the 2-capacity. It is easy to see that, if \( \text{Cap}_{\mathbb{R}^n}^2(\Gamma) = 0 \), then \( \text{Cap}_{\mathbb{R}^n}^2(R_\lambda(\Gamma)) = 0 \). Another consequence of our assumptions is that \( \text{Cap}_{\mathbb{R}^n}^2(B_\lambda^\varepsilon(R_\lambda(\Gamma))) = 0 \) for any open neighborhood \( B_\lambda^\varepsilon \) of \( R_\lambda(\Gamma) \).

Indeed, recalling that \( \Gamma \) is a point if \( n = 2 \) while \( \Gamma \) is closed with \( \text{Cap}_{\mathbb{R}^n}^2(\Gamma) = 0 \) if \( n \geq 3 \) by assumption, it follows that

\[
\text{Cap}_{\mathbb{R}^n}^2(B_\lambda^\varepsilon(R_\lambda(\Gamma))) := \inf \left\{ \int_{B_\lambda^\varepsilon \setminus R_\lambda(\Gamma)} |\nabla \varphi|^2 \, dx < +\infty : \varphi \geq 1 \text{ in } B_\lambda^\varepsilon, \varphi \in C^\infty_c(B_\lambda^\varepsilon) \right\} = 0,
\]

for some neighborhood \( B_\lambda^\varepsilon \subset B_\lambda^\varepsilon \) of \( R_\lambda(\Gamma) \). From this, it follows that there exists \( \varphi_\varepsilon \in C^\infty_c(B_\lambda^\varepsilon) \) such that \( \varphi_\varepsilon \geq 1 \) in \( B_\lambda^\varepsilon \) and \( \int_{B_\lambda^\varepsilon} |\nabla \varphi_\varepsilon|^2 \, dx < \varepsilon \).

Now we construct a function \( \psi_\varepsilon \in C^{0,1}(\mathbb{R}^n, [0,1]) \) such that \( \psi_\varepsilon = 1 \) outside \( B_\varepsilon^\lambda \), \( \psi_\varepsilon = 0 \) in \( B_\delta^\lambda \) and

\[
\int_{\mathbb{R}^n} |\nabla \psi_\varepsilon|^2 \, dx = \int_{B_\lambda^\varepsilon} |\nabla \psi_\varepsilon|^2 \, dx < 4 \varepsilon.
\]

To this end we consider the following Lipschitz continuous function

\[
T_1(s) = \begin{cases} 
1 & \text{if } s \leq 0 \\
-2s + 1 & \text{if } 0 \leq s \leq \frac{1}{2} \\
0 & \text{if } s \geq \frac{1}{2},
\end{cases}
\]

and we set

\[
\psi_\varepsilon := T_1 \circ \varphi_\varepsilon, \quad (2.8)
\]

where we have extended \( \varphi_\varepsilon \) by zero outside \( B_\varepsilon^\lambda \). Clearly \( \psi_\varepsilon \in C^{0,1}(\mathbb{R}^n), 0 \leq \psi_\varepsilon \leq 1 \) and

\[
\int_{B_\lambda^\varepsilon} |\nabla \psi_\varepsilon|^2 \, dx \leq 4 \int_{B_\lambda^\varepsilon} |\nabla \varphi_\varepsilon|^2 \, dx < 4 \varepsilon.
\]

Now we set \( \gamma_\lambda := \partial \Omega \cap T_\lambda \). Recalling that \( \Omega \) is convex, it is easy to deduce that \( \gamma_\lambda \) is made of two points in dimension two. If, instead, \( n \geq 3 \) then it follows that \( \gamma_\lambda \) is a smooth manifold of dimension \( n - 2 \). Note in fact that locally \( \partial \Omega \) is the zero level set of a smooth function \( g(\cdot) \) whose gradient is not parallel to the \( x_1 \)-direction since \( \Omega \) is convex. Then it is sufficient to observe that locally \( \partial \Omega \cap T_\lambda \equiv \{ g(\lambda, x') = 0 \} \) and use the implicit function theorem exploiting the fact that \( \nabla_{x'} g(\lambda, x') \neq 0 \). This
implies that Cap$_{\mathbb{R}^n}(\gamma_\lambda) = 0$ (see e.g. [22]). So, as before, Cap$_\Omega(\gamma_\lambda) = 0$ for any open neighborhood of $\gamma_\lambda$ and then there exists $\varphi_\tau \in C^\infty_c(\Omega_\lambda)$ such that $\varphi_\tau \geq 1$ in a neighborhood $\Omega_\lambda$ with $\gamma_\lambda \subset \Omega_\lambda \subset I_2^\lambda$. As above, we set

$$\phi_\tau := I_1 \circ \varphi_\tau,$$

(2.9)

where we have extended $\varphi_\tau$ by zero outside $I_2^\lambda$. Then $\phi_\tau \in C^{0,1}(\mathbb{R}^n)$, $0 \leq \phi_\tau \leq 1$, $\phi_\tau = 1$ outside $I_\tau^\lambda$, $\phi_\tau = 0$ in $I_\lambda^\lambda$ and

$$\int_{\mathbb{R}^n} |\nabla \phi_\tau|^2 dx = \int_{I_2^\lambda} |\nabla \phi_\tau|^2 dx \leq 4 \int_{I_2^\lambda} |\nabla \phi_\tau|^2 dx < 4\tau.$$

3. Proof of Theorem 1.2. Let us set

$$w_{i,\lambda}^+ = (u_i - u_{i,\lambda})^+,$$

where $i = 1, ..., m$. We will prove the result by showing that, actually, it holds $w_{i,\lambda}^+ \equiv 0$ for $i = 1, ..., m$. To prove this, we have to perform the moving plane method.

In the following, we will exploit the fact that $(u_1, ..., u_1, \lambda)$ is a solution to

$$\int_{\Omega_\lambda} \nabla u_i \nabla \varphi_i dx = \int_{\Omega_\lambda} f_i(u_1, ..., u_{2,\lambda}, ..., u_{m,\lambda}) \varphi_i dx \quad \forall \varphi_i \in C^1_c(\Omega_\lambda \setminus R_\lambda(\Gamma))$$

(3.10)

for every $i = 1, ..., m$, where $\Omega_\lambda := R_\lambda(\Omega)$.

We start by recalling the following helpful lemma, whose proof can be found in [19].

Lemma 3.1 ([19]). Let $\lambda \in (a, 0)$ be such that $R_\lambda(\Gamma) \cap \Omega = \emptyset$ and consider the function

$$\varphi_i := \begin{cases} w_{i,\lambda}^+ \phi_{i,\tau}^2 & \text{in } \Omega_\lambda \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega_\lambda, \end{cases}$$

where $\phi_{i,\tau}$ is as in (2.9), for $i = 1, ..., m$. Then, $\varphi_i \in C^{0,1}_c(\Omega) \cap C^{0,1}_c(R_\lambda(\Omega))$, $\varphi_i$ has compact support contained in $(\Omega \setminus \Gamma) \cap (R_\lambda(\Omega) \setminus R_\lambda(\Gamma)) \cap \{x_1 \leq \lambda\}$ and

$$\nabla \varphi_i = \phi_{i,\tau}^2 \nabla w_{i,\lambda}^+ + 2\phi_{i,\tau} w_{i,\lambda}^+ \nabla \phi_{i,\tau} \quad \text{a.e. on } \Omega \cup R_\lambda(\Omega),$$

for every $i = 1, ..., m$. If $\lambda \in (a, 0)$ is such that $R_\lambda(\Gamma) \cap \Omega \neq \emptyset$, the same conclusions hold true for the function

$$\varphi_i := \begin{cases} w_{i,\lambda}^+ \psi_{i,\tau}^2 \phi_{i,\tau}^2 & \text{in } \Omega_\lambda \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega_\lambda, \end{cases}$$

where $\psi_\tau$ is defined as in (2.8) and $\phi_{i,\tau}$ as in (2.9), for every $i = 1, ..., m$. Furthermore, a.e. on $\Omega \cup R_\lambda(\Omega)$,

$$\nabla \varphi_i = \psi_{i,\tau}^2 \phi_{i,\tau}^2 \nabla w_{i,\lambda}^+ + 2w_{i,\lambda}^+ (\psi_{i,\tau}^2 \phi_{i,\tau} \nabla \phi_{i,\tau} + \psi_{i,\tau} \phi_{i,\tau}^2 \nabla \psi_\tau).$$

(3.11)

In particular, $\varphi_i \in C^{0,1}(\Omega_\lambda)$, $\varphi_i|_{\partial \Omega_\lambda} = 0$ and so $\varphi_i \in H^1_0(\Omega_\lambda)$, for every $i = 1, ..., m$.

Now we are ready to prove an essential tool that we will use to start the moving plane procedure.
Lemma 3.2. Under the assumptions of Theorem 1.2, let \( a < \lambda < 0 \). Then, \( w_{i,\lambda}^{-} \in H_0^1(\Omega) \) for every \( i = 1, \ldots, m \) and

\[
\sum_{i=1}^{m} \int_{\Omega} |\nabla w_{i,\lambda}^{+}|^2 dx \leq \frac{m}{2} \sum_{i=1}^{m} (1 + C_{r}^{2}) \|u_{i}\|_{L^{\infty}(\Omega)}^{2} |\Omega|,
\]

where \( |\Omega| \) denotes the \( n \)-dimensional Lebesgue measure of \( \Omega \) and \( C_{r} \) is a positive constant depending only on \( f_{i} \).

Proof. For \( \psi_{z} \) as in (2.8) and \( \phi_{r} \) as in (2.9), we consider the functions \( \varphi_{i} \) defined in Lemma 3.1. In view of the properties of \( \varphi_{i} \), stated in Lemma 3.1, and a standard density argument, we can use \( \varphi_{i} \) as test function in (1.2) and (3.10) so that, subtracting, we get

\[
\int_{\Omega} |\nabla w_{i,\lambda}^{+}|^2 \psi_{r}^{2} \phi_{r} dx = -2 \int_{\Omega} \nabla w_{i,\lambda}^{+} \nabla \psi_{r} w_{i,\lambda}^{+} \psi_{r}^{2} \phi_{r} dx
\]

\[
-2 \int_{\Omega} \nabla w_{i,\lambda}^{+} \nabla \phi_{r} w_{i,\lambda}^{+} \psi_{r}^{2} \phi_{r} dx
\]

\[
+ \int_{\Omega} [f_{i}(u_{1}, u_{2}, \ldots, u_{m}) - f_{i}(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda})] w_{i,\lambda}^{+} \psi_{r}^{2} \phi_{r} dx.
\]

Exploiting Young’s inequality in the right hand side of (3.13), we get that

\[
\int_{\Omega} |\nabla w_{i,\lambda}^{+}|^2 \psi_{r}^{2} \phi_{r} dx \leq \frac{1}{4} \int_{\Omega} |\nabla w_{i,\lambda}^{+}|^2 \psi_{r}^{2} \phi_{r} dx + 4 \int_{\Omega} |\nabla \psi_{r}|^2 (w_{i,\lambda}^{+})^2 \phi_{r}^2 dx
\]

\[
+ \frac{1}{4} \int_{\Omega} |\nabla w_{i,\lambda}^{+}|^2 \psi_{r}^{2} \phi_{r} dx + 4 \int_{\Omega} |\nabla \phi_{r}|^2 (w_{i,\lambda}^{+})^2 \psi_{r}^2 dx
\]

\[
+ \int_{\Omega} [f_{i}(u_{1}, u_{2}, \ldots, u_{m}) - f_{i}(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda})] w_{i,\lambda}^{+} \psi_{r}^{2} \phi_{r} dx.
\]

The last term of the right hand side of (3.14) can be rewritten as follows

\[
\int_{\Omega} [f_{i}(u_{1}, u_{2}, \ldots, u_{m}) - f_{i}(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda})] w_{i,\lambda}^{+} \psi_{r}^{2} \phi_{r} dx
\]

\[
= \int_{\Omega} [f_{i}(u_{1}, u_{2}, \ldots, u_{m}) \pm f_{i}(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda})] w_{i,\lambda}^{+} \psi_{r}^{2} \phi_{r} dx
\]

\[
= \int_{\Omega} [f_{i}(u_{1}, u_{2}, \ldots, u_{m}) - f_{i}(u_{1,\lambda}, u_{2,\lambda}, u_{3}, \ldots, u_{m})] w_{i,\lambda}^{+} \psi_{r}^{2} \phi_{r} dx
\]

\[
+ f_{i}(u_{1,\lambda}, u_{2,\lambda}, u_{3}, \ldots, u_{m}) - f_{i}(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda})] w_{i,\lambda}^{+} \psi_{r}^{2} \phi_{r} dx.
\]

Using the fact that \( f_{i} \) are \( C^{1} \) functions \( (h_{f_{i}}) - (i) \) and they satisfy \( (h_{f_{i}}) - (ii) \), by (3.15) we have

\[
\int_{\Omega} [f_{i}(u_{1}, u_{2}, \ldots, u_{m}) - f_{i}(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda})] w_{i,\lambda}^{+} \psi_{r}^{2} \phi_{r} dx
\]

\[
\leq \sum_{j=1}^{m} C_{j}(f_{j}) \int_{\Omega} w_{j,\lambda}^{+} w_{i,\lambda}^{+} \psi_{r}^{2} \phi_{r} dx.
\]
Now, compiling all the previous estimates and exploiting Young’s inequality in the right hand side of (3.16), we obtain

\[
\int_{\Omega_\lambda} |\nabla w_{i,\lambda}^+|^2 \psi_e^2 \phi_r^2 \, dx \leq 8 \int_{\Omega_\lambda} |\nabla \psi_e|^2 (w_{i,\lambda}^+)^2 \phi_r^2 \, dx + 8 \int_{\Omega_\lambda} |\nabla \phi_r|^2 (w_{i,\lambda}^+)^2 \psi_e^2 \, dx \\
+ m \int_{\Omega_\lambda} (w_{i,\lambda}^+)^2 \psi_e^2 \phi_r^2 \, dx + \sum_{j=1}^m C_j^2 \int_{\Omega_\lambda} (w_{j,\lambda}^+)^2 \psi_e^2 \phi_r^2 \, dx.
\]

(3.17)

By (3.17) summing with respect to \( i \) we get

\[
\sum_{i=1}^m \int_{\Omega_\lambda} |\nabla w_{i,\lambda}^+|^2 \psi_e^2 \phi_r^2 \, dx \leq 8 \sum_{i=1}^m \int_{\Omega_\lambda} |\nabla \psi_e|^2 (w_{i,\lambda}^+)^2 \phi_r^2 \, dx \\
+ 8 \sum_{i=1}^m \int_{\Omega_\lambda} |\nabla \phi_r|^2 (w_{i,\lambda}^+)^2 \psi_e^2 \, dx \\
+ \frac{m}{2} \sum_{i=1}^m (1 + C_i^2) \int_{\Omega_\lambda} (w_{i,\lambda}^+)^2 \psi_e^2 \phi_r^2 \, dx.
\]

Taking into account the properties of \( \psi_e \) and \( \phi_r \), we see that

\[
\int_{\Omega_\lambda} |\nabla \psi_e|^2 \, dx = \int_{\Omega_\lambda \cap (B^2 \setminus B^1_e)} |\nabla \psi_e|^2 \, dx < 4 \varepsilon,
\]

\[
\int_{\Omega_\lambda} |\nabla \phi_r|^2 \, dx = \int_{\Omega_\lambda \cap (I^2 \setminus I^1_e)} |\nabla \phi_r|^2 \, dx < 4 \tau,
\]

which combined with \( 0 \leq w_{i,\lambda}^+ \leq u_i \), for every \( i = 1, \ldots, m \), immediately lead to

\[
\sum_{i=1}^m \int_{\Omega_\lambda} |\nabla w_{i,\lambda}^+|^2 \psi_e^2 \phi_r^2 \, dx \leq 32 (\varepsilon + \tau) \sum_{i=1}^m \| u_i \|_{L^\infty(\Omega_\lambda)}^2 \\
+ \frac{m}{2} \sum_{i=1}^m (1 + C_i^2) \| u_i \|_{L^\infty(\Omega_\lambda)}^2 |\Omega_\lambda|.
\]

By Fatou Lemma, as \( \varepsilon \) and \( \tau \) tend to zero, we have (3.12). To conclude we note that \( \varphi_i \to w_{i,\lambda}^+ \) in \( L^2(\Omega) \), as \( \varepsilon \) and \( \tau \) tend to zero, by definition of \( \varphi_i \) for every \( i = 1, \ldots, m \). Also, \( \nabla \varphi \to \nabla w_{i,\lambda}^+ \) in \( L^2(\Omega_\lambda) \), by (3.11). Therefore, \( w_{i,\lambda}^+ \) in \( H^0_0(\Omega_\lambda) \), since \( \varphi_i \in H^0_0(\Omega_\lambda) \) again by Lemma 3.1, for every \( i = 1, \ldots, m \), which concludes the proof.

**Proof of Theorem 1.2.** We define

\[ \Lambda_0 = \{ a < \lambda < b : u_t \leq u_{t,t} \text{ in } \Omega \setminus R_t(\Gamma) \text{ for all } t \in (a,b) \text{ and for every } i=1,\ldots,m. \} \]

and to start with the moving plane procedure, we have to prove that

**Step 1.** \( \Lambda_0 \neq \emptyset \). Fix \( \lambda_0 \in (a,0) \) such that \( R_{\lambda_0}(\Gamma) \subset \Omega^c \), then for every \( a < \lambda < \lambda_0 \), we also have that \( R_{\lambda}(\Gamma) \subset \Omega^c \). For any \( \lambda \) in this set we consider, on the domain \( \Omega \), the function \( \varphi_i := w_{i,\lambda}^+ \phi_r^2 \chi_{\Omega_\lambda} \), where \( \phi_r \) is as in (2.9) and we proceed as in the proof of Lemma 3.2. Hence, by Lemma 3.1 and a density argument, we can use \( \varphi_i \)
that \( u > \nu < u \) reach a contradiction by proving that 
\[ \int_{\Omega} |\nabla w_{i,\lambda}^+|^2 \phi^2 \ dx = -2 \int_{\Omega} \nabla w_{i,\lambda}^+ \nabla \phi \cdot \nabla w_{i,\lambda}^+ \phi \ dx + \int_{\Omega} [f_i(u_1, u_2, \ldots, u_m) - f_i(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda})] w_{i,\lambda}^+ \phi^2 \ dx. \]

Exploiting Young’s inequality and the assumption \((h_i)\), then we get that 
\[ \int_{\Omega} |\nabla w_{i,\lambda}^+|^2 \phi^2 \ dx \leq \frac{1}{2} \int_{\Omega} |\nabla w_{i,\lambda}^+|^2 \phi^2 \ dx + 2 \int_{\Omega} |\phi \tau|^2 (w_{i,\lambda}^+)^2 \ dx + \sum_{i=1}^{m} C_j \int_{\Omega} w_{j,\lambda}^+ w_{i,\lambda}^+ \phi^2 \ dx. \]

Taking into account the properties of \( \phi \tau \), we see that 
\[ \int_{\Omega} |\nabla \phi \tau|^2 (w_{i,\lambda}^+)^2 \ dx \leq \||u_i||^2_{L^\infty(\Omega)} \int_{\Omega \cap (\mathbb{R}^2 \setminus \mathbb{R}_2)} |\nabla \phi \tau|^2 \ dx \leq 4 ||u_i||^2_{L^\infty(\Omega)} \cdot \tau. \]

We therefore deduce that 
\[ \sum_{i=1}^{m} \int_{\Omega} |\nabla w_{i,\lambda}^+|^2 \phi^2 \ dx \leq 16 \tau \sum_{i=1}^{m} ||u_i||_{L^\infty(\Omega)} + \frac{m}{2} \sum_{i=1}^{m} (1 + C_i^2) \int_{\Omega} (w_{i,\lambda}^+)^2 \phi^2 \ dx. \]

By Fatou Lemma, as \( \tau \) tends to zero, we have 
\[ \sum_{i=1}^{m} \int_{\Omega} |\nabla w_{i,\lambda}^+|^2 \ dx \leq \frac{m}{2} \sum_{i=1}^{m} (1 + C_i^2) \int_{\Omega} (w_{i,\lambda}^+)^2 \ dx \leq \frac{m}{2} \sum_{i=1}^{m} (1 + C_i^2) C_{i,p}(\Omega) \int_{\Omega} |\nabla w_{i,\lambda}^+|^2 \ dx, \tag{3.18} \]

where \( C_{i,p}(\cdot) \) is the Poincaré constant (in the Poincaré inequality in \( H^1_0(\Omega) \)). Since \( C_{i,p}(\Omega) \to 0 \) as \( \lambda \to a \), we can find \( \lambda_1 \in (a, \lambda_0) \), such that 
\[ C_{i,p}(\Omega) < \frac{1}{\sqrt{m(1 + C_i^2)}} \quad \forall \lambda \in (a, \lambda_1) \] and for every \( i = 1, \ldots, m \),

so that by (3.18), we deduce that 
\[ \int_{\Omega} |\nabla w_{i,\lambda}^+|^2 \ dx \leq 0 \quad \forall \lambda \in (a, \lambda_1) \] and for every \( i = 1, \ldots, m \),

proving that \( u_i \leq u_{i,\lambda} \) in \( \Omega_\lambda \setminus R_\lambda(\Gamma) \) for \( \lambda \) close to \( a \), which implies the desired conclusion \( \Lambda_0 \neq \emptyset \).

Now we can set 
\[ \lambda_0 = \sup \Lambda_0. \]

**Step 2.** here we show that \( \lambda_0 = 0. \) To this end we assume that \( \lambda_0 < 0 \) and we reach a contradiction by proving that \( u_i \leq u_{i,\lambda_0+\nu} \) in \( \Omega_{\lambda_0+\nu} \setminus R_{\lambda_0+\nu}(\Gamma) \) for any \( 0 < \nu < \tilde{\nu} \) for some small \( \tilde{\nu} > 0 \) and for every \( i = 1, \ldots, m \). By continuity we know that \( u_i \leq u_{i,\lambda_0} \) in \( \Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma) \) for every \( i = 1, \ldots, m \). Since \( \Omega \) is convex in the \( x_1 \)-direction and the set \( R_{\lambda_0}(\Gamma) \) lies in the hyperplane of equation \( \{x_1 = -2\lambda_0\} \),
we see that $\Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$ is open and connected. Moreover, using $(h_{f_i}) - (ii)$ we have that

$$-\Delta(u_i - u_{i,\lambda_0}) = f(u_1, \ldots, u_m) - f(u_{1,\lambda_0}, \ldots, u_{m,\lambda_0})$$

$$= (f(u_1, \ldots, u_m) - f(u_{1,\lambda_0}, \ldots, u_{m,\lambda_0})) + \cdots$$

$$\cdots + (f(u_{1,\lambda_0}, \ldots, u_m) - f(u_{1,\lambda_0}, \ldots, u_{m,\lambda_0})) \leq 0.$$ 

Therefore, by the strong maximum principle we deduce that $u_i < u_{i,\lambda_0}$ in $\Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$ and for every $i = 1, \ldots, m$.

Now, note that for $K \subset \Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$, there is $\nu = \nu(K, \lambda_0) > 0$, sufficiently small, such that $K \subset \Omega_{\lambda} \setminus R_{\lambda}(\Gamma)$ for every $\lambda \in [\lambda_0, \lambda_0 + \nu]$. Consequently $u_i$ and $u_{i,\lambda}$ are well defined on $K$ for every $\lambda \in [\lambda_0, \lambda_0 + \nu]$ and for every $i = 1, \ldots, m$.

Hence, by the uniform continuity of the functions $g_i(x, \lambda) := u_i(x) - u_i(2\lambda - x_1, \lambda)$ on the compact set $K \times [\lambda_0, \lambda_0 + \nu]$ we can ensure that $K \subset \Omega_{\lambda_0 + \nu} \setminus R_{\lambda_0 + \nu}(\Gamma)$ and $u_i < u_{i,\lambda_0 + \nu}$ in $K$ for any $0 \leq \nu < \tilde{\nu}$, for some $\tilde{\nu} = \tilde{\nu}(K, \lambda_0) > 0$ small. Clearly we can also assume that $\tilde{\nu} < \frac{1}{4}$. 

Let us consider $\psi_\varepsilon$ constructed in such a way that it vanishes in a neighborhood of $R_{\lambda_0 + \nu}(\Gamma)$ and $\phi_\tau$ constructed in such a way it vanishes in a neighborhood of $\gamma_{\lambda_0 + \nu} = \partial \Omega \cap T_{\lambda_0 + \nu}$. As shown in the proof of Lemma 3.2, the functions

$$\varphi_i := \begin{cases} w_{i,\lambda_0 + \nu}^+ \psi_\varepsilon^2 \phi_\tau^2 & \text{in } \Omega_{\lambda_0 + \nu}^+ \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega_{\lambda_0 + \nu}^+ \end{cases}$$

are such that $\varphi_i \to w_{i,\lambda_0 + \nu}^+$ in $H^1_0(\Omega_{\lambda_0 + \nu})$, as $\varepsilon$ and $\tau$ tend to zero. Moreover, $\varphi_i \in C^{0,1}(\Omega_{\lambda_0 + \nu})$ and $\varphi_i|_{\partial \Omega_{\lambda_0 + \nu}} = 0$, by Lemma 3.1, and $\varphi_i = 0$ on an open neighborhood of $K$, by the above argument. Therefore, $\varphi_i \in H^1_0(\Omega_{\lambda_0 + \nu} \setminus K)$ and thus, also $w_{i,\lambda_0 + \nu}^+$ belongs to $H^1_0(\Omega_{\lambda_0 + \nu} \setminus K)$. We also note that $\nabla w_{i,\lambda_0 + \nu}^+ = 0$ on an open neighborhood of $K$.

Now we argue as in Lemma 3.2 and we plug $\varphi_i$ as test function in (1.2) and (3.10) so that, by subtracting, we get

$$\int_{\Omega_{\lambda_0 + \nu}} \left| \nabla w_{i,\lambda_0 + \nu}^+ \right|^2 \psi_\varepsilon^2 \phi_\tau^2 \, dx$$

$$= -2 \int_{\Omega_{\lambda_0 + \nu}} \nabla w_{i,\lambda_0 + \nu}^+ \cdot \nabla \psi_\varepsilon w_{i,\lambda_0 + \nu}^+ \psi_\varepsilon \phi_\tau^2 \, dx$$

$$- 2 \int_{\Omega_{\lambda_0 + \nu}} \nabla \phi_\tau w_{i,\lambda_0 + \nu}^+ \psi_\varepsilon^2 \phi_\tau \, dx$$

$$+ \int_{\Omega_{\lambda_0 + \nu}} [f_i(u_1, \ldots, u_m) - f_i(u_{1,\lambda_0 + \nu}, \ldots, u_{m,\lambda_0 + \nu})] w_{i,\lambda_0 + \nu}^+ \psi_\varepsilon^2 \phi_\tau^2 \, dx.$$ 

Therefore, taking into account the properties of $w_{i,\lambda_0 + \nu}^+$ and $\nabla w_{i,\lambda_0 + \nu}^+$ we also have

$$\int_{\Omega_{\lambda_0 + \nu} \setminus K} \left| \nabla w_{i,\lambda_0 + \nu}^+ \right|^2 \psi_\varepsilon^2 \phi_\tau^2 \, dx$$

$$\leq -2 \int_{\Omega_{\lambda_0 + \nu} \setminus K} \nabla w_{i,\lambda_0 + \nu}^+ \cdot \nabla \psi_\varepsilon w_{i,\lambda_0 + \nu}^+ \psi_\varepsilon \phi_\tau^2 \, dx$$

$$- 2 \int_{\Omega_{\lambda_0 + \nu} \setminus K} \nabla \phi_\tau w_{i,\lambda_0 + \nu}^+ \psi_\varepsilon^2 \phi_\tau \, dx.$$
Now, as in the proof of Lemma 3.2, we use Young’s inequality to deduce that

\[ \int_{\Omega_{\lambda_0+\nu} \setminus K} |\nabla w_i^{+} (\Omega_{\lambda_0+\nu})|^2 \, dx \leq 2 \int_{\Omega_{\lambda_0+\nu} \setminus K} |\nabla \psi \xi | |\nabla \tilde{u}^{+} (\Omega_{\lambda_0+\nu})|^2 \, dx \]

+ \frac{2}{\nu} \int_{\Omega_{\lambda_0+\nu} \setminus K} |\nabla \phi \tau | |\nabla \tilde{u}^{+} (\Omega_{\lambda_0+\nu})|^2 \, dx

+ \sum_{j=1}^{m} C_j (f_j) \int_{\Omega_{\lambda_0+\nu} \setminus K} \tilde{u}^{+} (\Omega_{\lambda_0+\nu}) |\nabla \tilde{u}^{+} (\Omega_{\lambda_0+\nu})|^2 \, dx.

Now, as in the proof of Lemma 3.2, we use Young’s inequality to deduce that

\[ \sum_{i=1}^{m} \int_{\Omega_{\lambda_0+\nu} \setminus K} |\nabla w_i^{+} (\Omega_{\lambda_0+\nu})|^2 \, dx \leq 32 (\epsilon + \tau) \sum_{i=1}^{m} \|u_i\|_{L^\infty(\Omega_{\lambda_0+\nu})}^2 \]

+ \frac{m}{2} \sum_{i=1}^{m} (1 + C^2_i) \int_{\Omega_{\lambda_0+\nu} \setminus K} (w_i^{+} (\Omega_{\lambda_0+\nu})|^2 \, dx.

Passing to the limit, as \((\epsilon, \tau) \to (0, 0)\), in the latter we get

\[ \sum_{i=1}^{m} \int_{\Omega_{\lambda_0+\nu} \setminus K} |\nabla w_i^{+} (\Omega_{\lambda_0+\nu})|^2 \, dx \leq \frac{m}{2} \sum_{i=1}^{m} (1 + C^2_i) \int_{\Omega_{\lambda_0+\nu} \setminus K} (w_i^{+} (\Omega_{\lambda_0+\nu})|^2 \, dx \]

\leq \frac{m}{2} \sum_{i=1}^{m} (1 + C^2_i) C_{i,p} (\Omega_{\lambda_0+\nu} \setminus K) \int_{\Omega_{\lambda_0+\nu} \setminus K} |\nabla w_i^{+} (\Omega_{\lambda_0+\nu})|^2 \, dx,

where \(C_{i,p}(\cdot)\) are the Poincaré constants (in the Poincaré inequalities in \(H^1_0(\Omega_{\lambda_0+\nu} \setminus K)\)). Now we recall that \(C_{i,p} (\Omega_{\lambda_0+\nu} \setminus K) \leq Q(n)|\Omega_{\lambda_0+\nu} \setminus K|^\frac{2}{n} \) for every \(i = 1, \ldots, m\), where \(Q = Q(n)\) is a positive constant depending only on the dimension \(n\), and therefore, by summarizing, we have proved that for every compact set \(K \subset \Omega_{\lambda_0} \setminus \bar{\lambda}_0(\Gamma)\) there is a small \(\bar{\nu} = \bar{\nu}(K, \lambda_0) \in (0, \frac{\lambda_0}{4})\) such that for every \(0 \leq \nu < \bar{\nu}\) we have

\[ \sum_{i=1}^{m} \int_{\Omega_{\lambda_0+\nu} \setminus K} |\nabla w_i^{+} (\Omega_{\lambda_0+\nu})|^2 \, dx \leq \frac{m}{2} \sum_{i=1}^{m} (1 + C^2_i) Q(n)|\Omega_{\lambda_0+\nu} \setminus K|^\frac{2}{n} \int_{\Omega_{\lambda_0+\nu} \setminus K} |\nabla w_i^{+} (\Omega_{\lambda_0+\nu})|^2 \, dx.

(3.19)

Now, we first fix a compact \(K \subset \Omega_{\lambda_0} \setminus \bar{\lambda}_0(\Gamma)\) such that

\[ |\Omega_{\lambda_0} \setminus K|^\frac{2}{n} < [m(1 + C^2_i)Q(n)]^{-1} \]

for every \(i = 1, \ldots, m\),
this is possible since $|R_{\lambda_0}(\Gamma)| = 0$ by the assumption on $\Gamma$, and then we take $\tilde{\nu}_0 < \nu$ such that for every $0 \leq \nu < \tilde{\nu}_0$ we have $|\Omega_{\lambda_0 + \nu} \setminus \Omega_{\lambda_0}|^{\frac{1}{2}} < [4m(1 + C^2)Q(n)]^{-1}$.

Inserting those informations into (3.19), we immediately get that

$$\int_{\Omega_{\lambda_0 + \nu} \setminus K} |\nabla w^+_{i,\lambda_0 + \nu}|^2 \, dx < \frac{1}{2} \int_{\Omega_{\lambda_0 + \nu} \setminus K} |\nabla w^+_{i,\lambda_0 + \nu}|^2 \, dx$$

for every $i = 1, \ldots, m$

and so $\nabla w^+_{i,\lambda_0 + \nu} = 0$ on $\Omega_{\lambda_0 + \nu} \setminus K$ for every $0 \leq \nu < \tilde{\nu}_0$ and $i = 1, \ldots, m$. On the other hand, we recall that $\nabla w^+_{i,\lambda_0 + \nu} = 0$ on an open neighborhood of $K$ for every $0 \leq \nu < \tilde{\nu}$ and $i = 1, \ldots, m$, thus $\nabla w^+_{i,\lambda_0 + \nu} = 0$ on $\Omega_{\lambda_0 + \nu}$ for every $0 \leq \nu < \tilde{\nu}_0$ and $i = 1, \ldots, m$. The latter proves that $u_i \leq u_{i,\lambda_0 + \nu}$ in $\Omega_{\lambda_0 + \nu} \setminus R_{\lambda_0 + \nu}(\Gamma)$ for every $0 < \nu < \tilde{\nu}_0$ and $i = 1, \ldots, m$. Such a contradiction shows that

$$\lambda_0 = 0.$$

**Step 3.** Conclusion. Since the moving plane procedure can be performed in the same way but in the opposite direction, then this proves the desired symmetry result. The fact that the solution is increasing in the $x_1$-direction in $\{x_1 < 0\}$ is implicit in the moving plane procedure. Since $u$ has $C^1$ regularity, the fact that $\partial_{x_1}u_i$ is positive for $x_1 < 0$ follows by the maximum principle, the Hopf lemma and the assumption $(h_f)$.

$\square$

4. **Proof of Theorem 1.3.**

**Proof of Theorem 1.3.** We first note that, thanks to a well-known result of Brezis and Kato [9] and standard elliptic estimates (see also [43]), the solution $(u_1, \ldots, u_m)$ to (1.3) is smooth in $\mathbb{R}^n \setminus \Gamma$. Furthermore, we observe that it is enough to prove the theorem for the special case in which the origin does not belong to $\Gamma$. Indeed, if the result is true in this special case, then we can apply it to the functions $u^{(i)}_z(x) := u_i(x + z)$ for every $i = 1, \ldots, m$, where $z \in \{x_1 = 0\} \setminus \Gamma \neq \emptyset$, which satisfies the system (1.3) with $\Gamma$ replaced by $-z + \Gamma$ (note that $-z + \Gamma$ is a closed and proper subset of $\{x_1 = 0\}$ with $\text{Cap}_2(-z + \Gamma) = 0$ and such that the origin does not belong to it).

Under this assumption, we consider the map $K : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}$ defined by $K(x) := \frac{x}{|x|^2}$. Given $(u_1, \ldots, u_m)$ solution to (1.3), the Kelvin transform of $u_i$ is given by

$$\hat{u}_i(x) := \frac{1}{|x|^{n-2}} u_i \left( \frac{x}{|x|^2} \right), \quad x \in \mathbb{R}^n \setminus \{\Gamma^* \cup \{0\} \},$$

(4.20)

where $\Gamma^* = K(\Gamma)$ and $i = 1, \ldots, m$. It follows that $(\hat{u}_1, \ldots, \hat{u}_m)$ weakly satisfies (1.3) in $\mathbb{R}^n \setminus \{\Gamma^* \cup \{0\} \}$ (i.e. in the sense that it satisfies (1.6)) and that $\Gamma^* \subset \{x_1 = 0\}$ since, by assumption, $\Gamma \subset \{x_1 = 0\}$. Moreover, we also have that $\Gamma^*$ is bounded (not necessarily closed) since we assumed that $0 \notin \Gamma$.

To proceed further we recall some useful lemma whose proofs are contained in [19].

**Lemma 4.1** ([19]). Let $F : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}$ be a $C^1$-diffeomorphism and let $A$ be a bounded open set of $\mathbb{R}^n \setminus \{0\}$. If $C \subset A$ is a compact set such that

$$\text{Cap}_2(C) = 0,$$

then

$$\text{Cap}_2(F(C)) = 0.$$
Lemma 4.2 ([19]). Let \( \Gamma \) be a closed subset of \( \mathbb{R}^n \), with \( n \geq 3 \). Moreover, let us assume that \( 0 \notin \Gamma \) and

\[
\text{Cap}_2(\Gamma) = 0.
\]

Then

\[
\text{Cap}_2(\Gamma^*) = 0.
\]

Let us now fix some notations. We set

\[
\Sigma_{\lambda} = \{ x \in \mathbb{R}^n : x_1 < \lambda \}.
\]

As above \( x_{\lambda} = (2\lambda - x_1, x_2, \ldots, x_n) \) is the reflection of \( x \) through the hyperplane \( T_{\lambda} = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = \lambda \} \). Finally, we consider the Kelvin transform \((\hat{u}_1, \ldots, \hat{u}_m)\) of \((u_1, \ldots, u_m)\) defined in (4.20) and we set

\[
\hat{w}_{i,\lambda}^+ = (\hat{u}_i - \hat{u}_{i,\lambda})^+,
\]

where \( i = 1, \ldots, m \). Note that \((\hat{u}_1, \ldots, \hat{u}_m)\) weakly solves

\[
\int_{\mathbb{R}^n} \nabla \hat{u}_i \nabla \varphi_i \, dx = \sum_{j=1}^m a_{ij} \int_{\mathbb{R}^n} \hat{u}_j^{2^* - 1} \varphi_i \, dx \quad \forall \varphi_i \in C_0^1(\mathbb{R}^n \setminus \Gamma^* \cup \{0\}) \quad (4.21)
\]

and \((\hat{u}_{1,\lambda}, \ldots, \hat{u}_{m,\lambda})\) weakly solves

\[
\int_{\mathbb{R}^n} \nabla \hat{u}_{i,\lambda} \nabla \varphi_i \, dx = \sum_{j=1}^m a_{ij} \int_{\mathbb{R}^n} \hat{u}_j^{2^* - 1} \varphi_i \, dx \quad \forall \varphi_i \in C_0^1(\mathbb{R}^n \setminus \hat{\Gamma}_{\lambda}(\Gamma^* \cup \{0\})) \quad (4.22)
\]

where \( i = 1, \ldots, m \). The properties of the Kelvin transform, the fact that \( 0 \notin \Gamma \) and the regularity of \( u_i \) imply that \(|\hat{u}_i(x)| \leq C|x|^{2-n}\), for every \( x \in \mathbb{R}^n \) and \( i = 1, \ldots, m \) such that \(|x| \geq R\), where \( C \) and \( R \) are positive constants (depending on \( u_i \)).

In particular, for every \( \lambda < 0 \), we have

\[
\hat{u}_i \in L^{2^*}(\Sigma_{\lambda}) \cap \text{L}^{\infty}(\Sigma_{\lambda}) \cap C^0(\overline{\Sigma}_{\lambda}),
\]

for every \( i = 1, \ldots, m \). We will prove the result by showing that, actually, it holds \( \hat{w}_{i,\lambda}^+ \equiv 0 \) for every \( i = 1, \ldots, m \). To prove this, we have to perform the moving plane method.

**Lemma 4.3.** Under the assumption of Theorem 1.3, for every \( \lambda < 0 \), we have that \( \hat{w}_{i,\lambda}^+ \in L^2(\Sigma_{\lambda}), \nabla \hat{w}_{i,\lambda}^+ \in L^2(\Sigma_{\lambda}) \) and

\[
\sum_{i=1}^m \|w_{i,\lambda}^+\|_{L^{2^*}(\Sigma_{\lambda})}^2 \leq \sum_{i=1}^m C_{i,S}^2 \int_{\Sigma_{\lambda}} |\nabla \hat{w}_{i,\lambda}^+|^2 \, dx \leq 2^{n+2} \sum_{i,j=1}^m a_{ij} C_{i,S}^2 \|\hat{u}_j\|_{L^{2^*}(\Sigma_{\lambda})}^2 \|\hat{u}_i\|_{L^{2^*}(\Sigma_{\lambda})},
\]

where \( C_{i,S} \) are the best constants in Sobolev embeddings.

**Proof.** We immediately see that \( w_{i,\lambda}^+ \in L^2(\Sigma_{\lambda}) \), since \( 0 \leq w_{i,\lambda}^+ \leq \hat{u}_i \in L^2(\Sigma_{\lambda}) \) for every \( i = 1, \ldots, m \). The rest of the proof follows the lines of the one of Lemma 3.2. Arguing as in section 2, for every \( \varepsilon > 0 \), we can find a function \( \psi_\varepsilon \in C^{0,1}(\mathbb{R}^n \setminus [0, 1]) \) such that

\[
\int_{\Sigma_{\lambda}} |\nabla \psi_\varepsilon|^2 < 4\varepsilon
\]

and \( \psi_\varepsilon = 0 \) in an open neighborhood \( B_\varepsilon \) of \( R_{\lambda}(\{\Gamma^* \cup \{0\}\}) \), with \( B_\varepsilon \subset \Sigma_{\lambda} \).
Fix $R_0 > 0$ such that $R_\lambda(\{\Gamma^* \cup \{0\}\}) \subset B_{R_0}$ and, for every $R > R_0$, let $\eta_R$ be
a standard cut off function such that $0 \leq \eta_R \leq 1$ on $\mathbb{R}^n$, $\eta_R = 1$ in $B_R$, $\eta_R = 0$
outside $B_{2R}$ with $|\nabla \eta_R| \leq 2/R$, and consider
\[
\varphi_i := \begin{cases} 
  w^{+}_{i,\lambda} \psi^2 \eta_R^2 & \text{in } \Sigma_\lambda \\
  0 & \text{in } \mathbb{R}^n \setminus \Sigma_\lambda,
\end{cases}
\]
for every $i = 1, \ldots, m$. Now, as in Lemma 3.1 we see that $\varphi_i \in C^{0,1}_c(\mathbb{R}^n)$ with
$\text{supp}(\varphi_i)$ contained in $\Sigma_\lambda \cap B_{2R} \setminus R_\lambda(\{\Gamma^* \cup \{0\}\})$ and
\[
\nabla \varphi_i = \psi^2 \eta_R \nabla w^{+}_{i,\lambda} + 2w^{+}_{i,\lambda}(\psi^2 \eta_R \nabla \eta_R + \psi \eta^2_R \nabla \psi).
\] (4.24)
Therefore, by a standard density argument, we can use $\varphi_i$ as test functions respectively in (4.21) and in (4.22) so that, subtracting we get
\[
\int_{\Sigma_\lambda} |\nabla w^{+}_{i,\lambda}|^2 \psi^2 \eta_R^2 \, dx = -2 \int_{\Sigma_\lambda} \nabla w^{+}_{i,\lambda} \nabla \psi \psi w^{+}_{i,\lambda} \psi \eta_R \, dx
\]
\[
-2 \int_{\Sigma_\lambda} \nabla w^{+}_{i,\lambda} \nabla \eta_R w^{+}_{i,\lambda} \eta \, dx
\]
\[
+ \sum_{i=1}^m a_{ij} \int_{\Sigma_\lambda} (\hat{\varphi}_j^{2r-1} - \hat{\varphi}_j^{2r-1}) w^{+}_{i,\lambda} \psi \eta_R^2 \, dx
\]
\[
=: I_1 + I_2 + I_3.
\]
Exploiting also Young’s inequality and recalling that $0 \leq w^{+}_{i,\lambda} \leq \hat{u}_i$, we get that
\[
|I_1| \leq \frac{1}{4} \int_{\Sigma_\lambda} |\nabla w^{+}_{i,\lambda}|^2 \psi^2 \eta_R^2 \, dx + 4 \int_{\Sigma_\lambda} |\nabla \psi|^2 (w^{+}_{i,\lambda})^2 \psi^2 \eta_R^2 \, dx
\]
\[
\leq \frac{1}{4} \int_{\Sigma_\lambda} |\nabla w^{+}_{i,\lambda}|^2 \psi^2 \eta_R^2 \, dx + 16 \epsilon \|\hat{u}_i\|^2_{L^\infty(\Sigma_\lambda)}. 
\] (4.26)
Furthermore we have that
\[
|I_2| \leq \frac{1}{4} \int_{\Sigma_\lambda} |\nabla w^{+}_{i,\lambda}|^2 \psi^2 \eta_R^2 \, dx + 4 \int_{\Sigma_\lambda \cap (B_{2R} \setminus B_R)} |\nabla \eta_R|^2 (w^{+}_{i,\lambda})^2 \psi^2 \, dx
\]
\[
\leq \frac{1}{4} \int_{\Sigma_\lambda} |\nabla w^{+}_{i,\lambda}|^2 \psi^2 \eta_R^2 \, dx
\]
\[
+ 4 \left( \int_{\Sigma_\lambda \cap (B_{2R} \setminus B_R)} |\nabla \eta_R|^n \, dx \right)^{\frac{2}{n}} \left( \int_{\Sigma_\lambda \cap (B_{2R} \setminus B_R)} \hat{u}_i^{2r} \, dx \right)^{\frac{n-2}{n}} (4.27)
\]
\[
\leq \frac{1}{4} \int_{\Sigma_\lambda} |\nabla w^{+}_{i,\lambda}|^2 \psi^2 \eta_R^2 \, dx + c(n) \left( \int_{\Sigma_\lambda \cap (B_{2R} \setminus B_R)} \hat{u}_i^{2r} \, dx \right)^{\frac{n-2}{n}},
\]
where $c(n)$ is a positive constant depending only on the dimension $n$. Let us now estimate $I_3$. Since $\hat{u}_i(x), \hat{u}_{i,\lambda}(x) > 0$, by the convexity of $t \to t^{2r-1}$, for $t > 0$, we obtain
\[
\hat{u}_i^{2r-1}(x) - \hat{u}_{i,\lambda}^{2r-1}(x) \leq \frac{n+2}{n-2} \hat{u}_i^{2r-2}(x)(\hat{u}_i - \hat{u}_{i,\lambda}(x))
\]
for every $x \in \Sigma_\lambda$ and $i = 1, \ldots, m$. Thus, by making use of the monotonicity of $t \to t^{2r-2}$, for $t > 0$ and the definition of $w^{+}_{i,\lambda}$, we get
\[
(\hat{u}_i^{2r-1} - \hat{u}_{i,\lambda}^{2r-1}) w^{+}_{i,\lambda} \leq \frac{n+2}{n-2} \hat{u}_i^{2r-2}(\hat{u}_i - \hat{u}_{i,\lambda}) w^{+}_{i,\lambda} \leq \frac{n+2}{n-2} \hat{u}_i^{2r-2}(w^{+}_{i,\lambda})^2
\]
for every $i = 1, \ldots, m$. Therefore

$$
|I_3| \leq \frac{n+2}{n-2} \sum_{j=1}^{m} a_{ij} \int_{\Sigma_{\lambda}} \hat{u}_{ij}^{q_i-2} w_{i,\lambda}^+ \psi_x^2 \eta_R^2 \, dx
$$

$$
\leq \frac{n+2}{n-2} \sum_{j=1}^{m} a_{ij} \int_{\Sigma_{\lambda}} \hat{u}_{ij}^{q_i-2} \hat{u}_{ij} \, dx = \frac{n+2}{n-2} \sum_{j=1}^{m} a_{ij} \int_{\Sigma_{\lambda}} \hat{u}_{ij}^{q_i-1} \, dx
$$

$$
= \frac{n+2}{n-2} \left( a_{ii} \|\hat{u}_i\|_{L^{2^*}(\Sigma_{\lambda})}^{q_i} + \sum_{j \neq i} a_{ij} \int_{\Sigma_{\lambda}} \hat{u}_{ij}^{q_i-1} \, dx \right)
$$

$$
\leq \frac{n+2}{n-2} \left( a_{ii} \|\hat{u}_i\|_{L^{2^*}(\Sigma_{\lambda})}^{q_i} + \sum_{j \neq i} a_{ij} \left( \int_{\Sigma_{\lambda}} \hat{u}_{ij}^{q_i} \, dx \right)^{\frac{n+2}{2q_i}} \left( \int_{\Sigma_{\lambda}} \hat{u}_i^{q_i} \, dx \right)^{\frac{2}{q_i}} \right)
$$

$$
= \frac{n+2}{n-2} \sum_{j=1}^{m} a_{ij} \|\hat{u}_j\|_{L^{2^*}(\Sigma_{\lambda})}^{q_i-1} \|\hat{u}_i\|_{L^{2^*}(\Sigma_{\lambda})},
$$

where we also used that $0 \leq w_{i,\lambda}^+ \leq \hat{u}_i$ for every $i = 1, \ldots, m$ and Hölder inequality.

Taking into account the estimates on $I_1$, $I_2$ and $I_3$, by (4.25) we deduce that

$$
\int_{\Sigma_{\lambda}} |\nabla \psi_x^2 \eta_R^2 | \, dx \leq 32 \varepsilon \|\hat{u}_i\|_{L^\infty(\Sigma_{\lambda})}^{q_i} + 2c(n) \left( \int_{\Sigma_{\lambda} \cap (B_{2R} \setminus B_R)} \hat{u}_i^{2^*} \, dx \right)^{\frac{n+2}{2}}
$$

$$
+ 2 \frac{n+2}{n-2} \sum_{j=1}^{m} a_{ij} \|\hat{u}_j\|_{L^{2^*}(\Sigma_{\lambda})}^{q_i-1} \|\hat{u}_i\|_{L^{2^*}(\Sigma_{\lambda})},
$$

which in turns yields

$$
\sum_{i=1}^{m} \int_{\Sigma_{\lambda}} |\nabla \psi_x^2 \eta_R^2 | \, dx \leq 32 \varepsilon \sum_{i=1}^{m} \|\hat{u}_i\|_{L^\infty(\Sigma_{\lambda})}^{q_i} + 2c(n) \sum_{i=1}^{m} \|\hat{u}_i\|_{L^{2^*}(\Sigma_{\lambda} \cap (B_{2R} \setminus B_R))}^{2}
$$

$$
+ 2 \frac{n+2}{n-2} \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} \|\hat{u}_j\|_{L^{2^*}(\Sigma_{\lambda})}^{q_i-1} \|\hat{u}_i\|_{L^{2^*}(\Sigma_{\lambda})}.
$$

$$
(4.29)
$$

By Fatou Lemma, as $\varepsilon$ tends to zero and $R$ tends to infinity, we deduce that

$$
\nabla \psi_x^2 \eta_R^2 \in L^2(\Sigma_{\lambda})
$$

by definition of $\varphi_i$, and that $\nabla \varphi_i \rightarrow \nabla \psi_x^2 \eta_R^2$ in $L^2(\Sigma_{\lambda})$, by (4.24) and the fact that $w_{i,\lambda}^+ \in L^2(\Sigma_{\lambda})$ for every $i = 1, \ldots, m$. Therefore, by (4.29) we have

$$
\sum_{i=1}^{m} \int_{\Sigma_{\lambda}} |\nabla \psi_x^2 \eta_R^2 | \, dx \leq 2 \frac{n+2}{n-2} \sum_{i,j=1}^{m} a_{ij} \|\hat{u}_j\|_{L^{2^*}(\Sigma_{\lambda})}^{q_i-1} \|\hat{u}_i\|_{L^{2^*}(\Sigma_{\lambda})}.
$$

$$
(4.30)
$$

Now we apply the Sobolev embedding theorem to (4.30), to deduce (4.23).

We can now complete the proof of Theorem 1.3. As for the proof of Theorem 1.2, we split the proof into three steps and we start with

**Step 1.** there exists $M > 1$ such that $\hat{u}_i \leq \hat{u}_{i,\lambda}$ in $\Sigma_{\lambda} \setminus R_{\lambda}(\Gamma^* \cup \{0\})$, for all $\lambda < -M$ and $i = 1, \ldots, m$. 

\[\square\]
Arguing as in the proof of Lemma 4.3 and using the same notations and the same construction for $\psi_\varepsilon, \eta_R$ and $\varphi_i$, we get

\[
\int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^2 \eta_R^2 \, dx = -2 \int_{\Sigma_\lambda} \nabla w_{i,\lambda}^+ \nabla \psi_\varepsilon w_{i,\lambda}^+ \psi_\varepsilon \eta_R^2 \, dx
- 2 \int_{\Sigma_\lambda} \nabla w_{i,\lambda}^+ \nabla \eta_R w_{i,\lambda}^+ \eta_R \psi_\varepsilon^2 \, dx
+ \sum_{i=1}^m a_{ij} \int_{\Sigma_\lambda} (\hat{u}_{j}^{2\varepsilon - 1} - \hat{u}_{j,\lambda}^{2\varepsilon - 1}) w_{i,\lambda}^+ \psi_\varepsilon \eta_R^2 \, dx
=: I_1 + I_2 + I_3,
\]

where $I_1, I_2$ and $I_3$ can be estimated exactly as in (4.26), (4.27) and (4.28). The latter yield

\[
\sum_{i=1}^m \int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^2 \eta_R^2 \, dx \leq 32\varepsilon \sum_{i=1}^m \|\hat{u}_i\|_{L^\infty(\Sigma_\lambda)}^2 + 2c(n) \sum_{i=1}^m \|\hat{u}_i\|_{L^{2\varepsilon}(\Sigma_\lambda \cap (B_{2R} \setminus B_R))}^2
+ \frac{2n+2}{n-2} \sum_{i=1}^m \sum_{j=1}^m a_{ij} \int_{\Sigma_\lambda} \hat{u}_j^{2\varepsilon - 2} w_{j,\lambda}^+ w_{i,\lambda}^+ \psi_\varepsilon^2 \eta_R^2 \, dx.
\]

Taking the limit in the latter, as $\varepsilon$ tends to zero and $R$ tends to infinity, leads to

\[
\sum_{i=1}^m \int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2 \, dx \leq \frac{2n+2}{n-2} \sum_{i=1}^m \sum_{j=1}^m a_{ij} \int_{\Sigma_\lambda} \hat{u}_j^{2\varepsilon - 2} w_{j,\lambda}^+ w_{i,\lambda}^+ \, dx \leq +\infty,
\]

which combined with Lemma 4.3 gives

\[
\sum_{i=1}^m \int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2 \, dx \leq \frac{2n+2}{n-2} \sum_{i=1}^m \sum_{j=1}^m a_{ij} \int_{\Sigma_\lambda} \hat{u}_j^{2\varepsilon - 2} (w_{j,\lambda}^+)^2 \, dx + \int_{\Sigma_\lambda} \hat{u}_j^{2\varepsilon - 2} (w_{i,\lambda}^+)^2 \, dx
\]
\[
\leq \frac{2n+2}{n-2} \sum_{i=1}^m \sum_{j=1}^m a_{ij} \left( \int_{\Sigma_\lambda} \hat{u}_j^{2\varepsilon - 2} \, dx \right) \left( \int_{\Sigma_\lambda} (w_{j,\lambda}^+)^2 \, dx \right)^{\frac{2}{n-2}}
\]
\[
\leq \frac{2n+2}{n-2} \sum_{i=1}^m \sum_{j=1}^m a_{ij} \left( \int_{\Sigma_\lambda} \hat{u}_j^{2\varepsilon - 2} \, dx \right)^{\frac{2}{n-2}} \left( \int_{\Sigma_\lambda} (w_{i,\lambda}^+)^2 \, dx \right)^{\frac{n-2}{n-2}}
\]
\[
\leq \frac{n+2}{n-2} \sum_{i=1}^m \sum_{j=1}^m a_{ij} \|\hat{u}_j\|_{L^{2\varepsilon}(\Sigma_\lambda)}^{2\varepsilon - 2} \left( C_{j,S}^2 \int_{\Sigma_\lambda} |\nabla w_{j,\lambda}^+|^2 \, dx + C_{i,S}^2 \int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2 \, dx \right)
\]
\[
= \sum_{i=1}^m \frac{n+2}{n-2} \sum_{j=1}^m a_{ij} \left( 2\delta_{ij} C_{i,S}^2 \|\hat{u}_i\|_{L^{2\varepsilon}(\Sigma_\lambda)}^{2\varepsilon - 2}
+ (1 - \delta_{ij}) C_{j,S}^2 \|\hat{u}_j\|_{L^{2\varepsilon}(\Sigma_\lambda)}^{2\varepsilon - 2} \right) \int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2 \, dx,
\]

where

\[
\delta_{ij} := \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j.
\end{cases}
\]

(4.31)
Recalling that \( \hat{u}_i, \hat{u}_j \in L^2(\Sigma_\lambda) \) for every \( i, j = 1, \ldots, m \), we deduce the existence of \( M > 1 \) such that

\[
\frac{n+2}{n-2} \sum_{j=1}^{m} a_{ij} \left( 2 \delta_i \| \hat{u}_i \|_{L^2(\Sigma_\lambda)}^2 + (1 - \delta_i) \| \hat{u}_j \|_{L^2(\Sigma_\lambda)}^2 \right) < 1
\]

for every \( \lambda < -M \) and \( i = 1, \ldots, m \). The latter and (5.50) lead to

\[
\int_{\Sigma_\lambda} |\nabla w_{i,\lambda}^+|^2 \, dx = 0.
\]

This implies that for every \( i = 1, \ldots, m \) we have \( w_{i,\lambda}^+ = 0 \) by Lemma 4.3 and the claim is proved.

To proceed further we define

\[
\Lambda_0 := \{ \lambda < 0 : \hat{u}_i \leq \hat{u}_{i,t} \text{ in } \Sigma_t \backslash R_t(\Gamma^* \cup \{0\}) \text{ for all } t \in (a, \lambda] \text{ and } i = 1, \ldots, m. \}
\]

and

\[
\lambda_0 := \sup \Lambda_0.
\]

**Step 2.** we have that \( \lambda_0 = 0 \). We argue by contradiction and suppose that \( \lambda_0 < 0 \). By continuity we know that \( \hat{u}_i \leq \hat{u}_{i,\lambda_0} \in \Sigma_{\lambda_0} \backslash R_{\lambda_0}(\Gamma^* \cup \{0\}) \) for every \( i = 1, \ldots, m \). Applying the strong maximum principle we deduce that \( \hat{u}_i < \hat{u}_{i,\lambda_0} \in \Sigma_{\lambda_0} \backslash R_{\lambda_0}(\Gamma^* \cup \{0\}) \) for every \( i = 1, \ldots, m \). Indeed, \( \hat{u}_i = \hat{u}_{i,\lambda_0} \) in \( \Sigma_{\lambda_0} \backslash R_{\lambda_0}(\Gamma^* \cup \{0\}) \) is not possible if \( \lambda_0 < 0 \), since in this case each \( \hat{u}_i \) would be singular somewhere on \( R_{\lambda_0}(\Gamma^* \cup \{0\}) \). Now, for some \( \bar{\tau} > 0 \), that will be fixed later on, and for any \( 0 < \tau < \bar{\tau} \) we show that \( \hat{u}_i \leq \hat{u}_{i,\lambda_0+\tau} \in \Sigma_{\lambda_0+\tau} \backslash R_{\lambda_0+\tau}(\Gamma^* \cup \{0\}) \) obtaining a contradiction with the definition of \( \lambda_0 \) and thus proving the claim. To this end we are going to show that, for every \( \delta > 0 \) there are \( \bar{\tau}(\delta, \lambda_0) > 0 \) and a compact set \( K \) (depending on \( \delta \) and \( \lambda_0 \)) such that

\[
K \subset \Sigma_\lambda \backslash R_\lambda(\Gamma^* \cup \{0\}), \quad \int_{\Sigma_\lambda \backslash K} \hat{u}_i^{2\tau} < \delta, \quad \forall \lambda \in [\lambda_0, \lambda_0 + \bar{\tau}] \text{ and } i = 1, \ldots, m.
\]

To see this, we note that for every \( \delta > 0 \) there are \( \tau_1(\delta, \lambda_0) > 0 \) and a compact set \( K \) (depending on \( \delta \) and \( \lambda_0 \)) such that \( \int_{\Sigma_\lambda \backslash K} \hat{u}_i^{2\tau} < \frac{\delta}{2} \) for every \( i = 1, \ldots, m \) and

\[
K \subset \Sigma_\lambda \backslash R_\lambda(\Gamma^* \cup \{0\}) \text{ for every } \lambda \in [\lambda_0, \lambda_0 + \tau_1].
\]

Consequently \( \hat{u}_i \) and \( \hat{u}_{i,\lambda} \) are well defined on \( K \) for every \( \lambda \in [\lambda_0, \lambda_0 + \tau_1] \). Hence, by the uniform continuity of the functions \( g_i(x, \lambda) := \hat{u}_i(x) - \hat{u}_i(2\lambda - x, 1, x) \) on the compact set \( K \times [\lambda_0, \lambda_0 + \tau_1] \) we can ensure that \( K \subset \Sigma_{\lambda_0+\tau} \backslash R_{\lambda_0+\tau}(\Gamma^* \cup \{0\}) \) and \( \hat{u}_i < \hat{u}_{i,\lambda_0+\tau} \in K \) for any \( 0 \leq \tau < \tau_2 \), for some \( \tau_2 = \tau(\delta, \lambda_0) \in (0, \tau_1) \). Clearly we can also assume that \( \tau_2 < \frac{|\lambda_0|}{4} \). Finally,

\[
\int_{\Sigma_\lambda \backslash K} \hat{u}_i^{2\tau} < \frac{\delta}{2}
\]

for each \( i = 1, \ldots, m \), we obtain the existence of \( \bar{\tau} \in (0, \tau_2) \) such that \( \int_{\Sigma_\lambda \backslash K} \hat{u}_i^{2\tau} < \delta \) for all \( \lambda \in [\lambda_0, \lambda_0 + \bar{\tau}] \) and \( i = 1, \ldots, m \).

Now we repeat verbatim the arguments used in the proof of Lemma 4.3, but using the test functions

\[
\varphi_i := \begin{cases} w_{i,\lambda_0+\tau}^{+} \psi_i^{2\tau} \eta^2_R & \text{in } \Sigma_{\lambda_0+\tau} \\ 0 & \text{in } \mathbb{R}^n \backslash \Sigma_{\lambda_0+\tau} \end{cases}
\]
Thus we recover the last inequality in (4.31), which immediately gives, for any $0 \leq \tau < \bar{\tau}$

$$
\sum_{i=1}^{m} \int_{\Sigma_{\lambda_0+\tau} \setminus K} |\nabla w_{i,\lambda_0+\tau}^+|^2 \\
\leq \frac{n+2}{n-2} \sum_{i,j=1}^{m} a_{ij} \left[ 2\delta_{ij} C_{i,S}^2 \|\tilde{u}_i\|_{L^2(\Sigma_{\lambda_0+\tau} \setminus K)}^{2\tau-2} \\
+ (1 - \delta_{ij}) C_{j,S}^2 \|\tilde{u}_j\|_{L^2(\Sigma_{\lambda_0+\tau} \setminus K)}^{2\tau-2} \right] \int_{\Sigma_{\lambda_0+\tau} \setminus K} |\nabla w_{i,\lambda_0+\tau}^+|^2 dx,
$$

since $w_{i,\lambda_0+\tau}^+$ and $\nabla w_{i,\lambda_0+\tau}^+$ are zero in a neighborhood of $K$, by the above construction for every $i = 1, \ldots, m$. Now we fix $\delta > 0$ such that for every $i = 1, \ldots, m$ we have

$$
\frac{n+2}{n-2} \sum_{i,j=1}^{m} a_{ij} \left[ 2\delta_{ij} C_{i,S}^2 \|\tilde{u}_i\|_{L^2(\Sigma_{\lambda_0+\tau} \setminus K)}^{2\tau-2} \\
+ (1 - \delta_{ij}) C_{j,S}^2 \|\tilde{u}_j\|_{L^2(\Sigma_{\lambda_0+\tau} \setminus K)}^{2\tau-2} \right] < \frac{1}{2},
$$

for all $0 \leq \tau < \bar{\tau}$, which plugged into (4.32) implies that

$$
\int_{\Sigma_{\lambda_0+\tau} \setminus K} |\nabla w_{i,\lambda_0+\tau}^+|^2 dx = 0
$$

for every $0 \leq \tau < \bar{\tau}$ and $i = 1, \ldots, m$. Hence

$$
\int_{\Sigma_{\lambda_0+\tau}} |\nabla w_{i,\lambda_0+\tau}^+|^2 dx = 0
$$

for every $0 \leq \tau < \bar{\tau}$, since $\nabla w_{i,\lambda_0+\tau}^+$ are zero in a neighborhood of $K$. The latter and Lemma 4.3 imply that $w_{i,\lambda_0+\tau}^+$ are zero on $\Sigma_{\lambda_0+\tau}$ for every $0 \leq \tau < \bar{\tau}$ and $i = 1, \ldots, m$, thus $\tilde{u}_i = w_{i,\lambda_0+\tau}^+$ in $\Sigma_{\lambda_0+\tau} \setminus R_{\lambda_0+\tau}(\Gamma^* \cup \{0\})$ for every $0 \leq \tau < \bar{\tau}$ and $i = 1, \ldots, m$, which proves the claim of Step 2.

**Step 3.** conclusion. The symmetry of the Kelvin transform $(\tilde{u}_1, \ldots, \tilde{u}_m)$ follows now performing the moving plane method in the opposite direction. The fact that every $\tilde{u}_i$ is symmetric w.r.t. the hyperplane $\{x_1 = 0\}$ implies the symmetry of the solution $(u_1, \ldots, u_m)$ w.r.t. the hyperplane $\{x_1 = 0\}$. The last claim then follows by the invariance of the considered problem with respect to isometries (translations and rotations).

5. **Proof of Theorem 1.4.**

**Proof of Theorem 1.4.** As we observed in the proof of Theorem 1.3, thanks to a well-known result of Brezis and Kato [9] and standard elliptic estimates (see also [43]), the solution $(u, v)$ is smooth in $\mathbb{R}^n \setminus \Gamma$. Furthermore, we recall that it is enough to prove the theorem for the special case in which the origin does not belong to $\Gamma$.

Under this assumption, we consider the map $K : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ defined by $K = K(x) := \frac{x}{|x|^2}$. Given $(u, v)$ solution to (1.7), its Kelvin transform is given by

$$
(\tilde{u}(x), \tilde{v}(x)) := \left( \frac{1}{|x|^{n-2}} u \left( \frac{x}{|x|^2} \right), \frac{1}{|x|^{n-2}} v \left( \frac{x}{|x|^2} \right) \right) \quad x \in \mathbb{R}^n \setminus \{\Gamma^* \cup \{0\}\},
$$

(5.33)
where $\Gamma^* = K(\Gamma)$. It follows that $(\check{u}, \check{v})$ weakly satisfies (1.7) in $\mathbb{R}^n \setminus \{\Gamma^* \cup \{0\}\}$ and that $\Gamma^* \subset \{x_1 = 0\}$ since, by assumption, $\Gamma \subset \{x_1 = 0\}$. Furthermore, we also have that $\Gamma^*$ is bounded (not necessarily closed) since we assumed that $0 \notin \Gamma$.

Let us now fix some notations. We set

$$\Sigma_\lambda = \{ x \in \mathbb{R}^n : x_1 < \lambda \}.$$

As above $x_\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$ is the reflection of $x$ through the hyperplane $T_\lambda = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 = \lambda \}$. Finally we consider the Kelvin transform $(\check{u}, \check{v})$ of $(u,v)$ defined in (5.33) and we set

$$\xi_\lambda(x) = \check{u}(x) - \check{u}_\lambda(x) = \check{u}(x) - \check{u}(x_\lambda),$$
$$\zeta_\lambda(x) = \check{v}(x) - \check{v}_\lambda(x) = \check{v}(x) - \check{v}(x_\lambda).$$

Note that $(\check{u}, \check{v})$ weakly solves

$$\int_{\mathbb{R}^n} \nabla \check{u} \nabla \varphi \, dx = \int_{\mathbb{R}^n} \check{u}^{2^* - 1} \varphi \, dx + \frac{\alpha}{2^*} \int_{\mathbb{R}^n} \check{u}^{\alpha - 1} \check{v}^\beta \varphi \, dx \quad \forall \varphi \in C^1_c(\mathbb{R}^n \setminus \Gamma^* \cup \{0\}),$$
$$\int_{\mathbb{R}^n} \nabla \check{v} \nabla \psi \, dx = \int_{\mathbb{R}^n} \check{v}^{2^* - 1} \psi \, dx + \frac{\beta}{2^*} \int_{\mathbb{R}^n} \check{u}^{\alpha} \check{v}^{\beta - 1} \psi \, dx \quad \forall \psi \in C^1_c(\mathbb{R}^n \setminus \Gamma^* \cup \{0\}),$$

and $(\check{u}_\lambda, \check{v}_\lambda)$ weakly solves

$$\int_{\mathbb{R}^n} \nabla \check{u}_\lambda \nabla \varphi \, dx = \int_{\mathbb{R}^n} \check{u}_\lambda^{2^* - 1} \varphi \, dx + \frac{\alpha}{2^*} \int_{\mathbb{R}^n} \check{u}_\lambda^{\alpha - 1} \check{v}_\lambda^\beta \varphi \, dx \quad \forall \varphi \in C^1_c(\mathbb{R}^n \setminus \Gamma^* \cup \{0\}),$$
$$\int_{\mathbb{R}^n} \nabla \check{v}_\lambda \nabla \psi \, dx = \int_{\mathbb{R}^n} \check{v}_\lambda^{2^* - 1} \psi \, dx + \frac{\beta}{2^*} \int_{\mathbb{R}^n} \check{u}_\lambda^{\alpha} \check{v}_\lambda^{\beta - 1} \psi \, dx \quad \forall \psi \in C^1_c(\mathbb{R}^n \setminus \Gamma^* \cup \{0\}).$$

The properties of the Kelvin transform, the fact that $0 \notin \Gamma$ and the regularity of $u, v$ imply that $|\check{u}(x)| \leq C_u |x|^{2 - n}$ and $|\check{v}(x)| \leq C_v |x|^{2 - n}$ and for every $x \in \mathbb{R}^n$ such that $|x| \geq R$, where $C_u, C_v$ and $R$ are positive constants (depending on $u$ and $v$).

In particular, for every $\lambda < 0$, we have

$$\check{u}, \check{v} \in L^2(\Sigma_\lambda) \cap L^\infty(\Sigma_\lambda) \cap C^0(\Sigma_\lambda).$$

**Lemma 5.1.** Under the assumption of Theorem 1.3, for every $\lambda < 0$, we have that $\xi_\lambda^+, \zeta_\lambda^+ \in L^2(\Sigma_\lambda), \nabla \xi_\lambda^+, \nabla \zeta_\lambda^+ \in L^2(\Sigma_\lambda)$ and

$$\int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \, dx + \int_{\Sigma_\lambda} |\nabla \zeta_\lambda^+|^2 \, dx \leq \frac{2}{n - 2} \left[ (1 + \alpha) ||\check{u}||^2_{L^2(\Sigma_\lambda)} + (1 + \beta) ||\check{v}||^2_{L^2(\Sigma_\lambda)} \right].$$

(5.36)

**Proof.** We immediately see that $\xi_\lambda^+, \zeta_\lambda^+ \in L^2(\Sigma_\lambda)$, since $0 \leq \xi_\lambda^+ \leq \check{u} \in L^2(\Sigma_\lambda)$ and $0 \leq \zeta_\lambda^+ \leq \check{v} \in L^2(\Sigma_\lambda)$. The rest of the proof follows the lines of the one of Lemma 3.2. Arguing as in Section 2, for every $\varepsilon > 0$, we can find a function $\psi_\varepsilon \in C^{0,1}(\mathbb{R}^n, [0, 1])$ such that

$$\int_{\Sigma_\lambda} |\nabla \psi_\varepsilon|^2 < 4\varepsilon$$

and $\psi_\varepsilon = 0$ in an open neighborhood $B_\varepsilon$ of $R_\lambda(\{\Gamma^* \cup \{0\}\})$, with $B_\varepsilon \subset \Sigma_\lambda$.

Fix $R_0 > 0$ such that $R_\lambda(\{\Gamma^* \cup \{0\}\}) \subset B_{R_0}$ and, for every $R > R_0$, let $\eta_R$ be a standard cut off function such that $0 \leq \eta_R \leq 1$ on $\mathbb{R}^n$, $\eta_R = 1$ in $B_R$, $\eta_R = 0$
outside $B_{2R}$ with $|\nabla \eta_R| \leq 2/R$, and consider

$$\Phi := \begin{cases} \xi^+_\lambda \psi^2 \eta^2_R & \text{in } \Sigma_{\lambda}, \\ 0 & \text{in } \mathbb{R}^n \setminus \Sigma_{\lambda} \end{cases} \quad \text{and} \quad \Psi := \begin{cases} \zeta^+ \psi^2 \eta^2_R & \text{in } \Sigma_{\lambda}, \\ 0 & \text{in } \mathbb{R}^n \setminus \Sigma_{\lambda}. \end{cases}$$

Now, as in Lemma 3.1 we see that $\Phi, \Psi \in C^{0,1}_c(\mathbb{R}^n)$ with $\text{supp}(\Phi)$ and $\text{supp}(\Psi)$ contained in $\Sigma_{\lambda} \cap B_{2R} \setminus R_{\lambda}(\{\Gamma^* \cup \{0\}\})$ and

$$\nabla \Phi = \psi^2 \eta^2_R \nabla \xi^+_\lambda + 2 \xi^+_\lambda (\psi^2 \eta_R \nabla \eta_R + \psi^2 \eta^2_R \nabla \psi), \quad (5.37)$$

$$\nabla \Psi = \psi^2 \eta^2_R \nabla \zeta^+_\lambda + 2 \zeta^+_\lambda (\psi \eta_R \nabla \eta_R + \psi \eta^2_R \nabla \psi). \quad (5.38)$$

Therefore, by a standard density argument, we can use $\Phi$ and $\Psi$ as test functions respectively in (5.34) and in (5.35) so that, subtracting we get

$$\int_{\Sigma_{\lambda}} |\nabla \xi^+_\lambda|^2 \psi^2 \eta^2_R \, dx = -2 \int_{\Sigma_{\lambda}} \nabla \xi^+_\lambda \nabla \psi \xi^+_\lambda \psi \eta^2_R \, dx - 2 \int_{\Sigma_{\lambda}} \nabla \xi^+_\lambda \nabla \eta_R \xi^+_\lambda \eta \psi^2 \, dx$$

$$+ \int_{\Sigma_{\lambda}} (\hat{u}^2 - \hat{u}^2 \xi^+_\lambda) \xi^+_\lambda \psi^2 \eta^2_R \, dx$$

$$+ \frac{\alpha}{2 \sigma} \int_{\Sigma_{\lambda}} (\hat{u} (\hat{u} - \hat{u} \xi^+_\lambda) \xi^+_\lambda \psi^2 \eta^2_R \, dx$$

$$=: I_1 + I_2 + I_3 + I_4, \quad (5.39)$$

$$\int_{\Sigma_{\lambda}} |\nabla \zeta^+_\lambda|^2 \psi^2 \eta^2_R \, dx = -2 \int_{\Sigma_{\lambda}} \nabla \zeta^+_\lambda \nabla \psi \zeta^+_\lambda \psi \eta^2_R \, dx - 2 \int_{\Sigma_{\lambda}} \nabla \zeta^+_\lambda \nabla \eta_R \zeta^+_\lambda \eta \psi^2 \, dx$$

$$+ \int_{\Sigma_{\lambda}} (\hat{v}^2 - \hat{v}^2 \zeta^+_\lambda) \zeta^+_\lambda \psi^2 \eta^2_R \, dx$$

$$+ \frac{\beta}{2 \sigma} \int_{\Sigma_{\lambda}} (\hat{v} (\hat{v} - \hat{v} \zeta^+_\lambda) \zeta^+_\lambda \psi^2 \eta^2_R \, dx$$

$$=: E_1 + E_2 + E_3 + E_4. \quad (5.40)$$

Exploiting also Young's inequality and recalling that $0 \leq \xi^+_\lambda \leq \hat{u}$ and $0 \leq \zeta^+_\lambda \leq \hat{v}$, we get that

$$|I_1| \leq \frac{1}{4} \int_{\Sigma_{\lambda}} |\nabla \xi^+_\lambda|^2 \psi^2 \eta^2_R \, dx + 4 \int_{\Sigma_{\lambda}} |\nabla \psi| \xi^+_\lambda \eta \psi^2 \, dx$$

$$\leq \frac{1}{4} \int_{\Sigma_{\lambda}} |\nabla \xi^+_\lambda|^2 \psi^2 \eta^2_R \, dx + 16 \varepsilon \|\hat{u}\| L^\infty(\Sigma_{\lambda}), \quad (5.41)$$

$$|E_1| \leq \frac{1}{4} \int_{\Sigma_{\lambda}} |\nabla \zeta^+_\lambda|^2 \psi^2 \eta^2_R \, dx + 4 \int_{\Sigma_{\lambda}} |\nabla \psi| \zeta^+_\lambda \eta \psi^2 \, dx$$

$$\leq \frac{1}{4} \int_{\Sigma_{\lambda}} |\nabla \zeta^+_\lambda|^2 \psi^2 \eta^2_R \, dx + 16 \varepsilon \|\hat{v}\| L^\infty(\Sigma_{\lambda}). \quad (5.42)$$

Furthermore we have that

$$|I_2| \leq \frac{1}{4} \int_{\Sigma_{\lambda}} |\nabla \xi^+_\lambda|^2 \psi^2 \eta^2_R \, dx + \int_{\Sigma_{\lambda} \cap (B_{2R} \setminus B_R)} |\nabla \eta_R| \xi^+_\lambda \eta \psi^2 \, dx$$

$$\leq \frac{1}{4} \int_{\Sigma_{\lambda}} |\nabla \xi^+_\lambda|^2 \psi^2 \eta^2_R \, dx$$
where $c$ and the definition of $\xi^+$ for every $n$

Let us now estimate $I_3$ and $E_3$. Since $\hat{u}(x), \hat{u}_\lambda(x), \hat{v}(x), \hat{v}_\lambda(x) > 0$, by the convexity of $t \to t^{2^*-1}$, for $t > 0$, we obtain

$$\hat{u}^{2^*-1}(x) - \hat{u}^{2^*-1}_\lambda(x) \leq \frac{n+2}{n-2} \hat{u}_\lambda^{2^*-2}(x)(\hat{u}(x) - \hat{u}_\lambda(x))$$

and

$$\hat{v}^{2^*-1}(x) - \hat{v}^{2^*-1}_\lambda(x) \leq \frac{n+2}{n-2} \hat{v}_\lambda^{2^*-2}(x)(\hat{v}(x) - \hat{v}_\lambda(x)),$$

for every $x \in \Sigma_\lambda$. Thus, by making use of the monotonicity of $t \to t^{2^*-2}$, for $t > 0$ and the definition of $\xi^+$ and $\zeta^+$ we get

$$(\hat{u}^{2^*-1} - \hat{u}^{2^*-1}_\lambda)\xi^+ \leq \frac{n+2}{n-2} \hat{u}_\lambda^{2^*-2}(\hat{u} - \hat{u}_\lambda)\xi^+ \leq \frac{n+2}{n-2} \hat{u}_\lambda^{2^*-2}(\xi^+)^2$$

and

$$(\hat{v}^{2^*-1} - \hat{v}^{2^*-1}_\lambda)\zeta^+ \leq \frac{n+2}{n-2} \hat{v}_\lambda^{2^*-2}(\hat{v} - \hat{v}_\lambda)\zeta^+ \leq \frac{n+2}{n-2} \hat{v}_\lambda^{2^*-2}(\zeta^+)^2.$$

Therefore

$$|I_3| \leq \frac{n+2}{n-2} \int_{\Sigma_\lambda} \hat{u}^{2^*-2}(\xi^+)^2 \psi_\epsilon^2 \eta_R^2 dx$$

$$\leq \frac{n+2}{n-2} \int_{\Sigma_\lambda} \hat{u}^{2^*-2} \hat{u}^2 dx \leq \frac{n+2}{n-2} \int_{\Sigma_\lambda} \hat{u}^{2^*} dx = \frac{n+2}{n-2} \|\hat{u}\|_{L^{2^*}(\Sigma_\lambda)}^2,$$ (5.45)

$$|E_3| \leq \frac{n+2}{n-2} \int_{\Sigma_\lambda} \hat{v}^{2^*-2}(\zeta^+)^2 \psi_\epsilon^2 \eta_R^2 dx$$

$$\leq \frac{n+2}{n-2} \int_{\Sigma_\lambda} \hat{v}^{2^*-2} \hat{v}^2 dx \leq \frac{n+2}{n-2} \int_{\Sigma_\lambda} \hat{v}^{2^*} dx = \frac{n+2}{n-2} \|\hat{v}\|_{L^{2^*}(\Sigma_\lambda)}^2,$$ (5.46)

where we also used that $0 \leq \xi^+ \leq \hat{u}$ and $0 \leq \zeta^+ \leq \hat{v}$.
Finally we have to estimate $I_4$ and $E_4$. Since $\dot{u}(x), \dot{u}(x), \dot{v}(x), \dot{v}(x) > 0$, by the convexity of the functions $t \to t^\alpha, t \to t^{\alpha - 1}, t \to t^\beta, t \to t^{\beta - 1}$ for $t > 0$, we obtain

$$
\begin{align*}
\dot{u}^\alpha(x) - \dot{u}_\lambda^\alpha(x) &\leq \alpha \dot{u}_\lambda^{\alpha - 1}(x)(\dot{u}(x) - \dot{u}_\lambda(x)), \\
\dot{u}^{\alpha - 1}(x) - \dot{u}_\lambda^{\alpha - 1}(x) &\leq (\alpha - 1) \dot{u}_\lambda^{\alpha - 2}(x)(\dot{u}(x) - \dot{u}_\lambda(x)), \\
\dot{v}^\beta(x) - \dot{v}_\lambda^\beta(x) &\leq \beta \dot{v}_\lambda^{\beta - 1}(x)(\dot{v}(x) - \dot{v}_\lambda(x)), \\
\dot{v}^{\beta - 1}(x) - \dot{v}_\lambda^{\beta - 1}(x) &\leq (\beta - 1) \dot{v}_\lambda^{\beta - 2}(x)(\dot{v}(x) - \dot{v}_\lambda(x)),
\end{align*}
$$

for every $x \in \Sigma$. By the monotonicity of $t \to t^\alpha, t \to t^{\alpha - 1}, t \to t^\beta, t \to t^{\beta - 1}$ for $t > 0$ and the definition of $\xi_\lambda^\alpha$ and $\zeta_\lambda^\beta$ we get

$$(\dot{u}^\alpha(x) - \dot{u}_\lambda^\alpha(x))\xi_\lambda^\alpha \leq \alpha \dot{u}_\lambda^{\alpha - 2}(\dot{u} - \dot{u}_\lambda)\xi_\lambda^\alpha \leq \alpha \dot{u}_\lambda^{\alpha - 2}(\xi_\lambda^\alpha)^2,$$

$$(\dot{u}^{\alpha - 1}(x) - \dot{u}_\lambda^{\alpha - 1}(x))\xi_\lambda^\alpha \leq (\alpha - 1) \dot{u}_\lambda^{\alpha - 2}(\dot{u} - \dot{u}_\lambda)\xi_\lambda^\alpha \leq (\alpha - 1) \dot{u}_\lambda^{\alpha - 2}(\xi_\lambda^\alpha)^2,$$

$$(\dot{u}^\beta(x) - \dot{u}_\lambda^\beta(x))\zeta_\lambda^\beta \leq \beta \dot{u}_\lambda^{\beta - 2}(\dot{u} - \dot{u}_\lambda)\zeta_\lambda^\beta \leq \beta \dot{u}_\lambda^{\beta - 2}(\zeta_\lambda^\beta)^2,$$

$$(\dot{u}^{\beta - 1}(x) - \dot{u}_\lambda^{\beta - 1}(x))\zeta_\lambda^\beta \leq (\beta - 1) \dot{u}_\lambda^{\beta - 2}(\dot{u} - \dot{u}_\lambda)\zeta_\lambda^\beta \leq (\beta - 1) \dot{u}_\lambda^{\beta - 2}(\zeta_\lambda^\beta)^2.$$

Now, having in mind all these estimates, we need a fine analysis in view of the cooperativity of the system. Since $\alpha + \beta = 2^* = \frac{2n}{n-2}$ and $\alpha, \beta \geq 2$ we have to split

$$
|I_4| \leq \frac{\alpha}{2} \int_{\Sigma} \left| \dot{u}^{\alpha - 1} \dot{v}^\beta - \dot{u}_\lambda^{\alpha - 1} \dot{v}_\lambda^\beta \right| \eta_\lambda^2 \eta_\lambda^2 dx + \frac{\alpha}{2} \int_{\Sigma} \left| \dot{u}^{\alpha - 1} \dot{v}_\lambda^\beta - \dot{u}_\lambda^{\alpha - 1} \dot{v}_\lambda^\beta \right| \eta_\lambda^2 \eta_\lambda^2 dx
$$

$$
\leq \frac{\alpha \beta}{2^*} \int_{\Sigma} \dot{u}^{\alpha - 1} \dot{v}_\lambda^\beta \xi_\lambda^\alpha \psi_\lambda^2 \eta_\lambda^2 dx + \frac{\alpha(\alpha - 1)}{2^*} \int_{\Sigma} \dot{u}_\lambda^{\alpha - 2} \dot{v}_\lambda^\beta \eta_\lambda^2 \eta_\lambda^2 dx
$$

$$
\leq \frac{\alpha \beta}{2^*} \int_{\Sigma} \dot{u}^{\alpha - 1} \dot{v}_\lambda^\beta \xi_\lambda^\alpha \psi_\lambda^2 \eta_\lambda^2 dx + \frac{\alpha(\alpha - 1)}{2^*} \int_{\Sigma} \dot{u}_\lambda^{\alpha - 2} \dot{v}_\lambda^\beta \psi_\lambda^2 \eta_\lambda^2 dx
$$

$$
\leq \frac{\alpha \beta}{2^*} \int_{\Sigma} \dot{u}^{\alpha - 1} \dot{v}_\lambda^\beta \xi_\lambda^\alpha \psi_\lambda^2 \eta_\lambda^2 dx + \frac{\alpha(\alpha - 1)}{2^*} \int_{\Sigma} \dot{u}_\lambda^{\alpha - 2} \dot{v}_\lambda^\beta \psi_\lambda^2 \eta_\lambda^2 dx
$$

$$
= \frac{\alpha(\alpha - 1)}{2^*} \int_{\Sigma} \dot{u}^{\alpha - 1} \dot{v}_\lambda^\beta dx,
$$

(5.47)

$$
|E_4| \leq \frac{\beta}{2^*} \int_{\Sigma} \left| \dot{u}^{\alpha - 1} \dot{v}^\beta - \dot{u}_\lambda^{\alpha - 1} \dot{v}_\lambda^\beta \right| \eta_\lambda^2 \eta_\lambda^2 dx + \frac{\beta}{2^*} \int_{\Sigma} \left| \dot{u}_\lambda^{\alpha - 1} \dot{v}_\lambda^\beta - \dot{u}_\lambda^{\alpha - 1} \dot{v}_\lambda^\beta \right| \eta_\lambda^2 \eta_\lambda^2 dx
$$

$$
\leq \frac{\alpha \beta}{2^*} \int_{\Sigma} \dot{u}\dot{v}_\lambda^\beta \xi_\lambda^\alpha \psi_\lambda^2 \eta_\lambda^2 dx + \frac{\beta(\beta - 1)}{2^*} \int_{\Sigma} \dot{u}_\lambda^{\alpha - 2} \dot{v}_\lambda^\beta \eta_\lambda^2 \eta_\lambda^2 dx
$$

$$
\leq \frac{\alpha \beta}{2^*} \int_{\Sigma} \dot{u}\dot{v}_\lambda^\beta \xi_\lambda^\alpha \psi_\lambda^2 \eta_\lambda^2 dx + \frac{\beta(\beta - 1)}{2^*} \int_{\Sigma} \dot{u}_\lambda^{\alpha - 2} \dot{v}_\lambda^\beta \eta_\lambda^2 \eta_\lambda^2 dx
$$

$$
\leq \frac{\alpha \beta}{2^*} \int_{\Sigma} \dot{u}\dot{v}_\lambda^\beta dx + \frac{\alpha(\alpha - 1)}{2^*} \int_{\Sigma} \dot{u}\dot{v}_\lambda^\beta dx
$$

(5.48)

$$
= \frac{\beta(\alpha - 1)}{2^*} \int_{\Sigma} \dot{u}\dot{v}_\lambda^\beta dx.
$$
Hence, by applying Hölder inequality with exponents \( \left( \frac{\alpha}{2^n}, \frac{\beta}{2^n} \right) \), it follows that

\[
|I_4| + |E_4| \leq (2^* - 1) \int_{\Sigma_\lambda} \hat{u}^\alpha \hat{v}^\beta \, dx \leq (2^* - 1) \| \hat{u} \|_{L^{2^*}(\Sigma_\lambda)} \| \hat{v} \|_{L^{2^*}(\Sigma_\lambda)}.
\]

Taking into account the estimates on \( I_1, I_2, I_3, I_4, E_1, E_2, E_3 \) and \( E_4 \), by adding (5.39) and (5.40), we deduce that

\[
\int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \psi_e \eta_R^2 \, dx + \int_{\Sigma_\lambda} |\nabla \tilde{\xi}_\lambda^+|^2 \psi_e \eta_R^2 \, dx
\]

\[
\leq 32 \varepsilon \left( \| \hat{u} \|_{L^{\infty}(\Sigma_\lambda)} + \| \hat{v} \|_{L^{\infty}(\Sigma_\lambda)} \right) + 2c(n) \left( \int_{\Sigma_\lambda \cap (B_2 \setminus B_1)} \hat{u}^{2^*} \, dx \right)^{\frac{n-2}{n}} + 2c(n) \left( \int_{\Sigma_\lambda \cap (B_2 \setminus B_1)} \hat{v}^{2^*} \, dx \right)^{\frac{n-2}{n}} + 2(2^* - 1) \| \hat{u} \|_{L^{2^*}(\Sigma_\lambda)} \| \hat{v} \|_{L^{2^*}(\Sigma_\lambda)}.
\]

By Fatou Lemma, as \( \varepsilon \) tends to zero and \( R \) tends to infinity, we deduce that \( \nabla \xi_\lambda^+, \nabla \tilde{\xi}_\lambda^+ \in L^2(\Sigma_\lambda) \). We also note that \( \Phi \to \xi_\lambda^+ \) and \( \Psi \to \xi_\lambda^+ \) in \( L^2(\Sigma_\lambda) \), by definition of \( \Phi \) and \( \Psi \), and that \( \nabla \Phi \to \nabla \xi_\lambda^+ \) and \( \nabla \Psi \to \nabla \tilde{\xi}_\lambda^+ \) in \( L^2(\Sigma_\lambda) \), by (5.37), (5.38) and the fact that \( \xi_\lambda^+, \tilde{\xi}_\lambda^+ \in L^2(\Sigma_\lambda) \). Therefore

\[
\int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \, dx + \int_{\Sigma_\lambda} |\nabla \tilde{\xi}_\lambda^+|^2 \, dx \leq 2 \left( \frac{n+2}{n-2} \left( \| \hat{u} \|_{L^{2^*}(\Sigma_\lambda)} + \| \hat{v} \|_{L^{2^*}(\Sigma_\lambda)} \right) + 2(2^* - 1) \| \hat{u} \|_{L^{2^*}(\Sigma_\lambda)} \| \hat{v} \|_{L^{2^*}(\Sigma_\lambda)} \right).
\]

Exploiting Young inequality in the right hand side of (5.49), with conjugate exponents \( \left( \frac{\alpha}{2^n}, \frac{\beta}{2^n} \right) \), we obtain (5.36).

We can now complete the proof of Theorem 1.4. As for the proof of Theorem 1.2 and Theorem 1.3, we split the proof into three steps and we start with

**Step 1.** there exists \( M > 1 \) such that \( \hat{u} \leq \hat{u}_\lambda \) and \( \hat{v} \leq \hat{v}_\lambda \) in \( \Sigma_\lambda \setminus R_\lambda(\Gamma^* \cup \{0\}) \), for all \( \lambda < -M \).

Arguing as in the proof of Lemma 5.1 and using the same notations and the same construction for \( \psi_e, \eta_R, \varphi \) and \( \psi \), we get

\[
\int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \psi_e \eta_R^2 \, dx = -2 \int_{\Sigma_\lambda} \nabla \xi_\lambda^+ \nabla \psi_e \xi_\lambda^+ \psi_e \eta_R^2 \, dx - 2 \int_{\Sigma_\lambda} \nabla \xi_\lambda^+ \nabla \eta_R \xi_\lambda^+ \eta_R \psi_e^2 \, dx
\]

\[
+ \int_{\Sigma_\lambda} (\hat{u}^{2^*-1} - \hat{u}_\lambda^{2^*-1}) \xi_\lambda^+ \psi_e^2 \eta_R^2 \, dx
\]

\[
+ \frac{\alpha}{2^n} \int_{\Sigma_\lambda} (\hat{u}^{a-1} \hat{v}^\beta - \hat{u}_\lambda^{a-1} \hat{v}_\lambda^\beta) \xi_\lambda^+ \psi_e^2 \eta_R^2 \, dx
\]

\[
=: I_1 + I_2 + I_3 + I_4.
\]
\[
\int_{\Sigma}\left|\nabla\zeta_+^2\right|^2\psi_+^2\eta_R^2\,dx = -2\int_{\Sigma}\nabla\psi_+^2\nabla\zeta_+^2\psi_+^2\eta_R^2\,dx - 2\int_{\Sigma}\nabla\zeta_+^2\nabla\eta_R\zeta_+^2\eta_R\psi_+^2\,dx
\]
\[
+ \int_{\Sigma} (\ddot{u}_\alpha^2 - \ddot{u}_\lambda^2)\zeta_+^2\psi_+^2\eta_R^2\,dx
\]
\[
+ \frac{\beta}{2}\int_{\Sigma} (\ddot{u}_\alpha \dot{v}_\beta^2 - \ddot{u}_\lambda \dot{v}_\lambda^2)\zeta_+^2\psi_+^2\eta_R^2\,dx
\]
\[
=: E_1 + E_2 + E_3 + E_4,
\]
where \(I_1, I_2, I_3, I_4, E_3, I_4 \) and \(E_4 \) can be estimated exactly as in (5.41), (5.42), (5.43), (5.44), (5.45), (5.46), (5.47) and (5.48). The latter yield
\[
\int_{\Sigma}\left|\nabla\zeta_+^2\right|^2\psi_+^2\eta_R^2\,dx
\]
\[
\leq 32\varepsilon \left(\|\ddot{u}\|_{L^\infty(\Sigma)}^2 + \|\ddot{v}\|_{L^\infty(\Sigma)}^2\right) + 2c(n) \left(\int_{\Sigma\cap(B_{2R}\setminus B_R)} \ddot{u}_\alpha^2\,dx\right)^{2s}
\]
\[
+ 2c(n) \left(\int_{\Sigma\cap(B_{2R}\setminus B_R)} \ddot{v}_\beta^2\,dx\right)^{2s} + \frac{n + 2}{n - 2} \int_{\Sigma} \ddot{u}_\alpha^2(\zeta_+^2)^2\psi_+^2\eta_R^2\,dx
\]
\[
+ \frac{n + 2}{n - 2} \int_{\Sigma} \ddot{v}_\beta^2(\zeta_+^2)^2\psi_+^2\eta_R^2\,dx + 4\frac{\alpha\beta}{2s} \int_{\Sigma} \ddot{u}_\alpha \dot{v}_\beta(\zeta_+^2)^2\psi_+^2\eta_R^2\,dx
\]
\[
+ \frac{\alpha(\alpha - 1)}{2s} \int_{\Sigma} \ddot{u}_\alpha \dot{v}_\beta(\zeta_+^2)^2\psi_+^2\eta_R^2\,dx + \frac{\beta(\beta - 1)}{2s} \int_{\Sigma} \ddot{u}_\alpha \dot{v}_\beta(\zeta_+^2)^2\psi_+^2\eta_R^2\,dx.
\]
Passing to the limit in the latter, as \(\varepsilon\) tends to zero and \(R\) tends to infinity, we obtain
\[
\int_{\Sigma}\left|\nabla\zeta_+^2\right|^2\,dx + \int_{\Sigma}\left|\nabla\zeta_+^2\right|^2\,dx
\]
\[
\leq 2\frac{n + 2}{n - 2} \left(\int_{\Sigma} \ddot{u}_\alpha^2(\zeta_+^2)^2\,dx + \int_{\Sigma} \ddot{v}_\beta^2(\zeta_+^2)^2\,dx\right)
\]
\[
+ 4\frac{\alpha\beta}{2s} \int_{\Sigma} \ddot{u}_\alpha \dot{v}_\beta(\zeta_+^2)^2\,dx
\]
\[
+ \frac{\alpha(\alpha - 1)}{2s} \int_{\Sigma} \ddot{u}_\alpha \dot{v}_\beta(\zeta_+^2)^2\,dx
\]
\[
+ \frac{\beta(\beta - 1)}{2s} \int_{\Sigma} \ddot{u}_\alpha \dot{v}_\beta(\zeta_+^2)^2\,dx < +\infty,
\]
which combined with Young inequality gives
\[
\int_{\Sigma}\left|\nabla\zeta_+^2\right|^2\,dx + \int_{\Sigma}\left|\nabla\zeta_+^2\right|^2\,dx
\]
\[
\leq 2\frac{n + 2}{n - 2} \left(\int_{\Sigma} \ddot{u}_\alpha^2(\zeta_+^2)^2\,dx + \int_{\Sigma} \ddot{v}_\beta^2(\zeta_+^2)^2\,dx\right)
\]
\[
+ \frac{\alpha(2s + \beta - 1)}{2s} \int_{\Sigma} \ddot{u}_\alpha \dot{v}_\beta(\zeta_+^2)^2\,dx
\]
\[
+ \frac{\beta(2s + \beta - 1)}{2s} \int_{\Sigma} \ddot{u}_\alpha \dot{v}_\beta(\zeta_+^2)^2\,dx
\]
\[
=: A_1 + A_2 + A_3.
\]
Exploiting Hölder inequality with conjugate exponents \( \left( \frac{2^*}{2^* - 2}, \frac{2^*}{2} \right) \) we obtain

\[
|A_1| \leq \frac{2n+2}{n-2} \left( \int_{\Sigma_\lambda} \hat{u}^{2^*} \, dx \right)^{\frac{2}{2^*}} \left( \int_{\Sigma_\lambda} (\xi_\lambda^+)^{2^*} \, dx \right)^{\frac{2^*}{2^*}} + \frac{2n+2}{n-2} \left( \int_{\Sigma_\lambda} \hat{v}^{2^*} \, dx \right)^{\frac{2}{2^*}} \left( \int_{\Sigma_\lambda} (\xi_\lambda^+)^{2^*} \, dx \right)^{\frac{2^*}{2^*}}.
\] (5.51)

Exploiting Hölder inequality with conjugate exponents \( \left( \frac{2^*}{\alpha - 2}, \frac{2^*}{\beta} \right) \) (we note that if \( \alpha = 2 \) we have \( \beta = 2 \) and the conjugate exponents would be \( \left( \frac{2^*}{2}, \frac{2^*}{2} \right) \)) we obtain

\[
|A_2| \leq \alpha \left( \frac{2^* + \beta - 1}{2^*} \right) \left( \int_{\Sigma_\lambda} \hat{u}^{2^*} \, dx \right)^{\frac{2}{2^*}} \left( \int_{\Sigma_\lambda} \hat{v}^{2^*} \, dx \right)^{\frac{2^*}{2}} \left( \int_{\Sigma_\lambda} (\xi_\lambda^+)^{2^*} \, dx \right)^{\frac{2^*}{2^*}}.
\] (5.52)

Exploiting Hölder inequality with conjugate exponents \( \left( \frac{2^*}{\alpha - 2}, \frac{2^*}{\beta - 2} \right) \) (we note that if \( \beta = 2 \) we have \( \alpha = 2 \) and the conjugate exponents would be \( \left( \frac{2^*}{2}, \frac{2^*}{2} \right) \)) we obtain

\[
|A_3| \leq \beta \left( \frac{2^* + \alpha - 1}{2^*} \right) \left( \int_{\Sigma_\lambda} \hat{u}^{2^*} \, dx \right)^{\frac{2}{2^*}} \left( \int_{\Sigma_\lambda} \hat{v}^{2^*} \, dx \right)^{\frac{2^*}{2}} \left( \int_{\Sigma_\lambda} (\xi_\lambda^+)^{2^*} \, dx \right)^{\frac{2^*}{2^*}}.
\] (5.53)

Combining (5.51), (5.52) and (5.53) and applying Sobolev inequality to (5.50)

\[
\int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \, dx + \int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \, dx \leq C_1 \int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \, dx + C_2 \int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \, dx,
\]

where

\[
C_1 := \left[ \frac{2n+2}{n-2} \left\| \hat{u} \right\|_{L^{2^*}(\Sigma_\lambda)}^{\frac{2}{2^*}} + \frac{\alpha(2^* + \beta - 1)}{2^*} \left\| \hat{v} \right\|_{L^{2^*}(\Sigma_\lambda)}^{\frac{2^*}{2^*}} \right] C_{u,S} \left\| \hat{u} \right\|_{L^{2^*}(\Sigma_\lambda)}^{\frac{\alpha-2}{2}} \]

\[
C_2 := \left[ \frac{2n+2}{n-2} \left\| \hat{v} \right\|_{L^{2^*}(\Sigma_\lambda)}^{\frac{2}{2^*}} + \frac{\beta(2^* + \alpha - 1)}{2^*} \left\| \hat{u} \right\|_{L^{2^*}(\Sigma_\lambda)}^{\frac{\alpha-2}{2}} \right] C_{v,S} \left\| \hat{v} \right\|_{L^{2^*}(\Sigma_\lambda)}^{\frac{\beta-2}{2}}
\]

\( C_{u,S} \) and \( C_{v,S} \) are the Sobolev constants. Recalling that \( \hat{u}, \hat{v} \in L^{2^*}(\Sigma_\lambda) \), we deduce the existence of \( M > 1 \) such that

\[
C_1 := \left[ \frac{2n+2}{n-2} \left\| \hat{u} \right\|_{L^{2^*}(\Sigma_\lambda)}^{\frac{2}{2^*}} + \frac{\alpha(2^* + \beta - 1)}{2^*} \left\| \hat{v} \right\|_{L^{2^*}(\Sigma_\lambda)}^{\frac{2^*}{2^*}} \right] C_{u,S} \left\| \hat{u} \right\|_{L^{2^*}(\Sigma_\lambda)}^{\frac{\alpha-2}{2}} < 1
\]

and

\[
C_2 := \left[ \frac{2n+2}{n-2} \left\| \hat{v} \right\|_{L^{2^*}(\Sigma_\lambda)}^{\frac{2}{2^*}} + \frac{\beta(2^* + \alpha - 1)}{2^*} \left\| \hat{u} \right\|_{L^{2^*}(\Sigma_\lambda)}^{\frac{\alpha-2}{2}} \right] C_{v,S} \left\| \hat{v} \right\|_{L^{2^*}(\Sigma_\lambda)}^{\frac{\beta-2}{2}} < 1
\]

for every \( \lambda < -M \). The latter and (5.50) lead to

\[
\int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \, dx = 0 \quad \text{and} \quad \int_{\Sigma_\lambda} |\nabla \xi_\lambda^+|^2 \, dx = 0.
\]

This implies that \( \xi_\lambda^+ = \xi_\lambda^+ = 0 \) by Lemma 5.1 and the claim is proved.

To proceed further we define

\[
\Lambda_0 = \{ \lambda < 0 : \hat{u} \leq \hat{u}_t \text{ and } \hat{v} \leq \hat{v}_t \text{ in } \Sigma_t \setminus R_t(\Gamma^* \cup \{0\}) \text{ for all } t \in (-\infty, \lambda) \}.
\]
and

\[ \lambda_0 = \sup \Lambda_0. \]

**Step 2.** We have that \( \lambda_0 = 0 \). We argue by contradiction and suppose that \( \lambda_0 < 0 \). By strong maximum principle we deduce that \( \hat{u} < \hat{u}_{\lambda_0} \) and \( \hat{v} < \hat{v}_{\lambda_0} \) in \( \Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma^* \cup \{0\}) \). By the definition of \( \lambda_0 \) and proving thus the claim. To this end we recall that, repeating verbatim the argument used in the roof of Theorem 1.3, it is possible to prove that for every \( \delta > 0 \) there are \( \bar{\tau}(\delta, \lambda_0) > 0 \) and compact set \( K \) (depending on \( \delta \) and \( \lambda_0 \)) such that

\[ K \subset \Sigma_{\lambda} \setminus R_{\lambda}(\Gamma^* \cup \{0\}), \quad \int_{\Sigma_{\lambda_0} \setminus K} \hat{u}^2 < \delta \quad \text{and} \quad \int_{\Sigma_{\lambda_0} \setminus K} \hat{v}^2 < \delta \quad \forall \lambda \in [\lambda_0, \lambda_0 + \bar{\tau}]. \tag{5.54} \]

Now we repeat verbatim the arguments used in the proof of Lemma 5.1 but using the test function

\[ \Phi := \left\{ \begin{array}{ll} \xi_{\lambda_0 + \tau}^+ \psi^2_n R & \text{in } \Sigma_{\lambda_0 + \tau} \setminus \Sigma_{\lambda_0 + \tau} \text{ and } \Psi := \left\{ \begin{array}{ll} \xi_{\lambda_0 + \tau}^+ \psi^2_n R & \text{in } \Sigma_{\lambda_0 + \tau} \setminus \Sigma_{\lambda_0 + \tau} \end{array} \right. \end{array} \right. \]

Thus we recover the first inequality in (5.50), and repeating verbatim the arguments used in (5.51), (5.52) and (5.53) which immediately gives, for any \( 0 < \tau < \bar{\tau} \)

\[ \int_{\Sigma_{\lambda_0 + \tau} \setminus K} |\nabla \xi_{\lambda_0 + \tau}^+|^2 \, dx + \int_{\Sigma_{\lambda_0 + \tau} \setminus K} |\nabla \xi_{\lambda_0 + \tau}^+|^2 \, dx \leq C_1 C_{u,s} \|\hat{u}\|_{L^{2^*}(\Sigma_{\lambda_0 + \tau} \setminus K)}^{2^* - 2} \int_{\Sigma_{\lambda_0 + \tau} \setminus K} |\nabla \xi_{\lambda_0 + \tau}^+|^2 \, dx + C_2 C_{v,s} \|\hat{v}\|_{L^{2^*}(\Sigma_{\lambda_0 + \tau} \setminus K)}^{2^* - 2} \int_{\Sigma_{\lambda_0 + \tau} \setminus K} |\nabla \xi_{\lambda_0 + \tau}^+|^2 \, dx, \tag{5.55} \]

where

\[ C_1 := \frac{2n + 2}{n - 2} \|\hat{u}\|_{L^{2^*}(\Sigma_{\lambda_0 + \tau} \setminus K)}^2 + \frac{\alpha(2^* + \beta - 1)}{2^*} \|\hat{v}\|_{L^{2^*}(\Sigma_{\lambda_0 + \tau} \setminus K)}^{2^* - 2}, \]

\[ C_2 := \frac{2n + 2}{n - 2} \|\hat{v}\|_{L^{2^*}(\Sigma_{\lambda_0 + \tau} \setminus K)}^2 + \frac{\beta(2^* + \beta - 1)}{2^*} \|\hat{v}\|_{L^{2^*}(\Sigma_{\lambda_0 + \tau} \setminus K)}^{2^* - 2}, \]

\( C_{u,s} \) and \( C_{v,s} \) are the Sobolev constants. Now taking the compact set \( K \) sufficiently large and thanks to (5.54), we can fix \( \delta > 0 \) such that

\[ \delta < \min \{C_1 C_{u,s} \|\hat{u}\|_{L^{2^*}(\Sigma_{\lambda_0 + \tau} \setminus K)}^{2^* - 2}, C_2 C_{v,s} \|\hat{v}\|_{L^{2^*}(\Sigma_{\lambda_0 + \tau} \setminus K)}^{2^* - 2}\} \]

and we observe that, thanks to (5.54), with this choice we have

\[ C_1 C_{u,s} \|\hat{u}\|_{L^{2^*}(\Sigma_{\lambda_0 + \tau} \setminus K)}^{2^* - 2} \leq 1 \quad \text{and} \quad C_2 C_{v,s} \|\hat{v}\|_{L^{2^*}(\Sigma_{\lambda_0 + \tau} \setminus K)}^{2^* - 2} < 1, \quad \forall \ 0 \leq \tau < \bar{\tau} \]

which plugged into (5.55) implies that

\[ \int_{\Sigma_{\lambda_0 + \tau} \setminus K} |\nabla \xi_{\lambda_0 + \tau}^+|^2 \, dx = \int_{\Sigma_{\lambda_0 + \tau} \setminus K} |\nabla \xi_{\lambda_0 + \tau}^+|^2 \, dx = 0. \]
for every $0 \leq \tau < \bar{\tau}$. Hence
\[
\int_{\Sigma_{\lambda_0+\tau}} |\nabla \xi_{\lambda_0+\tau}|^2 \, dx = \int_{\Sigma_{\lambda_0+\tau}} |\nabla \zeta_{\lambda_0+\tau}|^2 \, dx = 0
\]
for every $0 \leq \tau < \bar{\tau}$, since $\nabla \xi_{\lambda_0+\tau}$ and $\nabla \zeta_{\lambda_0+\tau}$ are zero in a neighbourhood of $K$. The latter and Lemma 5.1 imply that $\xi_{\lambda_0+\tau} = 0$ and $\zeta_{\lambda_0+\tau} = 0$ on $\Sigma_{\lambda_0+\tau}$ for every $0 \leq \tau < \bar{\tau}$, which proves the claim of Step 2.

**Step 3.** conclusion. The symmetry of the Kelvin transform $v$ follows now performing the moving plane method in the opposite direction. The fact that $\hat{u}$ and $\hat{v}$ are symmetric w.r.t. the hyperplane $\{x_1 = 0\}$ implies the symmetry of the solution $(u,v)$ w.r.t. the hyperplane $\{x_1 = 0\}$. The last claim then follows by the invariance of the considered problem with respect to isometries (translations and rotations).

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**REFERENCES**

[1] A. D. Alexandrov, A characteristic property of the spheres, Ann. Mat. Pura Appl., 58 (1962), 303–354.

[2] A. Ambrosetti and E. Colorado, Standing waves of some coupled nonlinear Schrödinger equations, J. Lond. Math. Soc., 75 (2007), 67–82.

[3] T. Bartsch, N. Dancer and Z. Q. Wang, A Liouville theorem, a-priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system, Calc. Var. Partial Differ. Equ., 37 (2010), 345–361.

[4] T. Bartsch, Z. Q. Wang and J. Wei, Bound states for a coupled Schrödinger system, J. Fixed Point Theory Appl., 2 (2007), 353–367.

[5] H. Berestycki and L. Nirenberg, On the method of moving planes and the sliding method, Bulletin Soc. Brasil. de Mat Nova Ser, 22 (1991), 1–37.

[6] S. Biagi, E. Valdinoci and E. Vecchi, A symmetry result for elliptic systems in punctured domains, Commun. Pure Appl. Anal., 18 (2019), 2819–2833.

[7] S. Biagi, E. Valdinoci and E. Vecchi, A symmetry result for cooperative elliptic systems with singularities, arXiv:1904.02003.

[8] L. Boccardo and L. Orsina, Semilinear elliptic equations with singular nonlinearities, Calc. Var. Partial Differential Equations, 37 (2010), 363–380.

[9] H. Brezis and T. Kato, Remarks on the Schrödinger operator with singular complex potentials, J. Math. Pures Appl., 58 (1979), 137–151.

[10] J. Busca and B. Sirakov, Symmetry results for semilinear elliptic systems in the whole space, J. Differential Equations, 163 (2000), 41–56.

[11] L. A. Caffarelli, B. Gidas and J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, Comm. on Pure and Appl. Mat., 42 (1989), 271–297.

[12] A. Canino, F. Esposito and B. Sciunzi, On the Höフ boundary lemma for singular semilinear elliptic equations, J. of Differential Equations, 266 (2019), 5488–5499.

[13] M. Clapp and A. Pistoia, Existence and phase separation of entire solutions to a pure critical competitive elliptic system, Calc. Var. Partial Differential Equations, 57 (2018), Art. 23, 20 pp.

[14] M. G. Crandall, P. H. Rabinowitz and L. Tartar, On a Dirichlet problem with a singular nonlinearity, Comm. P.D.E., 2 (1977), 193–222.

[15] E. N. Dancer, Moving plane methods for systems on half spaces, Math. Ann., 342 (2008), 245–254.
D. G. De Figueiredo and J. Yang, Decay, symmetry and existence of solutions of semilinear elliptic systems, *Nonlinear Anal.*, 33 (1998), 211–234.

E. De Giorgi, Un esempio di estremali discontinue per un problema variazionale di tipo ellittico, *Boll. Un. Mat. Ital.*, 1 (1968), 135–137.

F. Esposito, A. Farina and B. Sciunzi, Qualitative properties of singular solutions to semilinear elliptic problems, *J. of Differential Equations*, 265 (2018), 1962–1983.

F. Esposito, L. Montoro and B. Sciunzi, Monotonicity and symmetry of singular solutions to quasilinear problems, *J. Math. Pure Appl.*, 126 (2019), 214–231.

F. Esposito and B. Sciunzi, On the Höpfboundary lemma for quasilinear problems involving singular nonlinearities and applications, arXiv:1810.13294.

L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, 1992.

B. Gidas, W. M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.*, 68 (1979), 209–243.

F. Gladiali, M. Grossi and C. Troesler, A non-variational system involving the critical Sobolev exponent. The radial case, *J. Anal. Math.*, 138 (2019), 643–671, arXiv:1603.05641.

F. Gladiali, M. Grossi and C. Troesler, Entire radial and nonradial solutions for systems with critical growth, *Calc. Var. Partial Differential Equations*, 57 (2018), Art. 53, 26 pp.

Y. Guo and J. Liu, Liouville type theorems for positive solutions of elliptic system in $\mathbb{R}^n$, *Comm. Partial Differential Equations*, 33 (2008), 263–284.

A. C. Lazer and P. J. McKenna, On a singular nonlinear elliptic boundary-value problem, *Proc. AMS*, 111 (1991), 721–730.

G. Leoni and M. Morini, Necessary and sufficient conditions for the chain rule in $W^{1,1}_{loc}(\mathbb{R}^n;\mathbb{R}^d)$ and $BV_{loc}(\mathbb{R}^n;\mathbb{R}^d)$, *J. Eur. Math. Soc.*, 9 (2007), 219–252.

T. C. Lin and J. Wei, Ground state of $N$ coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \leq 3$, *Commun. Math. Phys.*, 255 (2005), 629–653.

T. C. Lin and J. Wei, Spikes in two coupled nonlinear Schrödinger equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 22 (2005), 403–439.

E. Mitidieri, A Rellich type identity and applications, *Comm. Partial Differential Equations*, 18 (1993), 125–151.

E. Mitidieri, Nonexistence of positive solutions of semilinear elliptic systems in $\mathbb{R}^n$, *Differential Integral Equations*, 9 (1996), 465–479.

L. Montoro, F. Punzo and B. Sciunzi, Qualitative properties of singular solutions to nonlocal problems, *Ann. Mat. Pura Appl.* (4), 197 (2018), 941–964.

L. Montoro, G. Riey and B. Sciunzi, Qualitative properties of positive solutions to systems of quasilinear elliptic equations, *Adv. Differential Equations*, 20 (2015), 717–740.

L. Montoro, B. Sciunzi and M. Squassina, Symmetry results for nonvariational quasi-linear elliptic systems, *Adv. Nonlinear Stud.*, 10 (2010), 939–955.

S. Peng, Y. F. Peng and Z. O. Wang, On elliptic systems with Sobolev critical growth, *Calc. Var. Partial Differential Equations*, 55 (2016), Art. 142, 30 pp.

W. Reichel and H. Zou, Non-existence results for semilinear cooperative elliptic systems via moving spheres, *J. Differential Equations*, 161 (2000), 219–243.

B. Sciunzi, On the moving Plane Method for singular solutions to semilinear elliptic equations, *J. Math. Pures Appl.*, 108 (2017), 111–123.

J. Serrin, A symmetry problem in potential theory, *Arch. Rational Mech. Anal.*, 43 (1971), 304–318.

B. Sirakov, Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$, *Commun. Math. Phys.*, 271 (2007), 199–221.

N. Soave, On existence and phase separation of solitary waves for nonlinear Schrödinger systems modelling simultaneous cooperation and competition, *Calc. Var. Partial Differ. Equ.*, 53 (2015), 689–718.

N. Soave and H. Tavares, New existence and symmetry results for least energy positive solutions of Schrödinger systems with mixed competition and cooperation terms, *J. Differential Equations*, 261 (2016), 505–537.

M. Struwe, *Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. Fourth Edition*, Springer-Verlag, Berlin, 2008.
[44] C. A. Stuart, Existence and approximation of solutions of nonlinear elliptic equations, *Math. Z.*, 147 (1976), 53–63.

[45] S. Terracini, On positive entire solutions to a class of equations with a singular coefficient and critical exponent, *Adv. Differential Equations*, 1 (1996), 241–264.

[46] W. C. Troy, Symmetry properties in systems of semilinear elliptic equations, *J. of Differential Equations*, 42 (1981), 400–413.

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E-mail address: esposito@mat.unical.it