Cell algebra structures on monoid and twisted monoid algebras

Robert May

1 Introduction

In [10] a class of algebras called cell algebras is defined generalizing the cellular algebras of Graham and Lehrer in [6]. It was shown that, for \( R \) any commutative domain with unit, the semigroup algebra \( R[M] \) is a cell algebra if \( M \) is a transformation semigroup \( T_r \) or a partial transformation semigroup \( P T_r \). In this paper we show that the algebra \( R[M] \) is a cell algebra for any finite monoid \( M \) with the property that for any \( D \)-class \( D \) the group algebra \( R[G_D] \) for the Schützenberger group of \( D \) has a cell algebra structure. We then use the properties of cell algebras given in [10] to derive properties of the algebras \( R[M] \) for such monoids. For example, if \( M \) is a regular semigroup and each cell algebra \( R[G_D] \) has the property that \( (\Lambda_D)_0 = \Lambda_D \) then \( R[M] \) is quasi-hereditary; while if \( M \) is an inverse semigroup, then \( R[M] \) is semi-simple if and only if each cell algebra \( R[G_D] \) is semi-simple. We also show that if \( \pi : M \times M \to R \) is any twisting satisfying a certain simple compatibility condition then the twisted monoid algebra \( R^{\pi}[M] \) is also a cell algebra (with the same sets \( \Lambda, L, R \) and cell basis \( C \) as the cell algebra \( R[M] \)).

In [4], [11], and [8], East, Wilcox, and Guo and Xi have explored analogous results for cellular algebra structures on semigroups and twisted semigroups. Many of the complications in their work arise from the need to construct the involution anti-automorphism \( * \) required by the definition of a cellular algebra. Cell algebras require no such mapping, and the results obtained for such algebras appear to be both considerably simpler and of more general applicability than the corresponding result for cellular algebras. Yet questions such as whether an algebra is quasi-hereditary or semi-simple appear to be not much harder to address in the cell algebra case than in the cellular case.

2 Properties of finite monoids

In this section we review some facts about finite monoids and the standard Green’s relations on a finite monoid \( M \). Most of these results are well-known; see, for example, [11], [5], or [9] for references.
Green’s relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{H}$ and $\mathcal{D}$ are defined, for $x, y \in M$, by

$$x \mathcal{L} y \Leftrightarrow xM = M y, \quad x \mathcal{R} y \Leftrightarrow xM = yM, \quad x \mathcal{J} y \Leftrightarrow xM = yM$$

$$x \mathcal{H} y \Leftrightarrow x \mathcal{L} y \text{ and } x \mathcal{R} y$$

$$x \mathcal{D} y \Leftrightarrow x \mathcal{L} z \text{ and } z \mathcal{R} y \text{ for some } z \in M \Leftrightarrow x \mathcal{R} z \text{ and } z \mathcal{L} y \text{ for some } z \in M.$$  

If $K$ is one of Green’s relations and $m \in M$, denote by $K_m$ the $K$-class containing $m$. Write $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{H}$, or $\mathcal{D}$ for the set of all $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{H}$, or $\mathcal{D}$ classes in $M$.

According to “Green’s lemma”, if $a \mathcal{R} b$ and $s, t \in M$ are such that $a s = b$, $b t = a$, then the map $x \to a x$ is a bijective, $\mathcal{R}$-class preserving map from $L_a$ onto $L_b$ with inverse the map $y \to y t$. There is a dual result when $a \mathcal{L} b$.

For a finite monoid $M$, one has $\mathcal{D} = \mathcal{J}$ (see chapter 5 of [7]) and the set $\mathcal{D}$ of $\mathcal{D}$-classes inherits a partial order defined, for $x, y \in M$, by $D_x \subseteq D_y \Leftrightarrow x \in M y M$. Evidently $D_{xy} \subseteq D_x$ and $D_{x y} \subseteq D_y$ for any $x, y \in M$.

Let $R$ be a commutative domain with unit. For any subset $S \subseteq M$, write $R[S]$ for the free $R$-module with basis $S$. Let $A = R[M]$, the monoid algebra for $M$, and define $A^D = \oplus_{D \subseteq D} R[D]$, $\tilde{A}^D = \oplus_{D' \subset D} R[D']$. Then both $A^D$ and $\tilde{A}^D$ are two sided ideals in $A$.

For a given $D \subseteq M$, choose a “base element” $x, \gamma \in D$ and consider the $\mathcal{H}$-class $H = H_{x, \gamma} \subseteq D$. Let $RT(H) = \{m \in M : Hm \subseteq H\}$ and for $m \in RT(H)$ define the right translation $r_m : H \to H$ by $(h)r_m = hm$, $h \in H$. Each map $r_m$ is a bijection from $H$ to $H$ and $G^R_H = \{r_m : m \in RT(H)\}$ is a group, the (right) Schützenberger group for $H$. (Up to isomorphism, $G^R_H$ depends only on $D$, not the particular $\mathcal{H}$-class $H$ contained in $D$.) Evidently $r_m r_n = r_{mn}$ for any $m, n \in RT(H)$. The map $\phi^R_H : G^R_H \to H$ defined by $\phi^R_H : r_m \mapsto \gamma_D m$ is bijective and extends linearly to a bijective map $\phi^R_H : R[G^R_H] \to R[H]$ of free $R$-modules. For any $m, n \in RT(H)$, $\phi^R_H(r_m)n = (\gamma_D m) \cdot n = \gamma_D(mn) = \phi^R_H(r_{mn}) = \phi^R_H(r_m r_n)$, so in general for any $x \in R[G^R_H]$ we have by linearity $\phi^R_H(x) \cdot n = \phi^R_H(xr_n)$.

In parallel fashion, let $LT(H) = \{m \in M : mH \subseteq H\}$ and for $m \in LT(H)$ define the left translation $l_m : H \to H$ by $l_m(h) = mh$, $h \in H$. Then again each map $l_m$ is a bijection from $H$ to $H$ and $G^L_H = \{l_m : m \in LT(H)\}$ is a group, the (left, or dual) Schützenberger group for $H$. Then $l_ml_n = l_{mn}$ for any $m, n \in LT(H)$, the map $\phi^L_H : G^L_H \to H$ defined by $\phi^L_H : l_m \mapsto m\gamma_D$ is bijective, and the linearly extended map $\phi^L_H : R[G^L_H] \to R[H]$ is a bijection of free $R$-modules. The map $\psi = (\phi^R_H)^{-1} \circ \phi^L_H : G^L_H \to G^R_H$ is then also bijective. Choose an (injective) map $\theta : G^H_H \to RT(H)$ such that $r_{\theta(g)} = g$ for all $g \in G^H_H$. Then for any $m \in LT(H)$ define $\bar{m} \in RT(H)$ by $\bar{m} = \theta \circ \psi(l_m)$, so that $m \gamma_D = \gamma_D \bar{m}$.

For any $m \in LT(H), n \in RT(H)$ we have $m \cdot \phi^R_H(r_n) = m \gamma_D n = m \gamma_D n \cdot n = \phi^R_H(r_m r_n)$. Then for any $x \in R[G^H_H], m \in LT(H)$ we have by linearity $m \phi^R_H(x) = \phi^R_H(r_m x)$.

**Lemma 2.1.** Take any $m \in M$.

i. If there exists an $h \in H$ such that $hm \in H$ then $m \in RT(H)$ and $r_m$ gives a bijective map of $H$ onto $H$. 


ii. If there exists an \( h \in H \) such that \( mh \in H \) then \( m \in \text{LT}(H) \) and \( l_m \) gives a bijective map of \( H \) onto \( H \).

Proof. For i., write \( h = m_1 \cdot \gamma_D \), \( hm = m_2 \cdot \gamma_D \) for some \( m_1, m_2 \in \text{LT}(H) \). Then \( l_{m_1} \in G^H_D \) and we can choose \( \bar{m}_1 \in \text{LT}(H) \) such that \( l_{\bar{m}_1} = (l_{m_1})^{-1} \). Then \( \gamma_D = \bar{m}_1 \cdot h \). Any element of \( H \) has the form \( n \cdot \gamma_D \) for some \( n \in \text{LT}(H) \), and we have \( n \gamma_D \cdot m = n \cdot (\bar{m}_1h) \cdot m = n\bar{m}_1(hm) = n\bar{m}_1(m_2\gamma_D) = (n\bar{m}_1m_2) \cdot \gamma_D \). But \( n\bar{m}_1m_2 \in \text{LT}(H) \) (since \( n, \bar{m}_1, m_2 \) are), so \( n \gamma_D \cdot m = (n\bar{m}_1m_2) \cdot \gamma_D \in H \).

Then \( m \in \text{RT}(H) \) as desired. The proof of ii. is parallel. \( \square \)

We now investigate general products \( am \), \( ma \) where \( a \in D \in \mathbb{D} \) and \( m \in M \).

As mentioned above, we have \( D_{am} \leq D_a \) and \( D_{ma} \leq D_a \). We need to study the cases when \( D_{am} = D_a \) or \( D_{ma} = D_a \). The first tool is the following result (see e.g. chapter 5 of [7]):

**Lemma 2.2.** For any finite monoid \( M \) and \( a, m \in M \),

i. If \( D_a = D_{am} \), then \( R_a = R_{am} \).

ii. If \( D_a = D_{ma} \), then \( L_a = L_{ma} \).

Proof. We use the facts (again, see chapter 5 of [7]) that in any finite monoid \( D = \mathcal{F} \) and that for any element \( x \) in a finite monoid some power \( x^n \) is idempotent. For i., assume \( D_a = D_{am} \) for some \( a, m \in M \).

To prove \( R_a = R_{am} \), it suffices to check that \( a \in amM \) and \( am \in aM \). Since \( am \in aM \) is obvious, we need show only \( a \in amM \). Now \( D_a = D_{am} \Rightarrow J_a = J_{am} \Rightarrow a \in MamM \Rightarrow a = x(am)y = xa(my) \) for some \( x, y \in M \). Then by repeated substitutions we have \( a = x^ka(my)^k \) for any positive integer \( k \). Choose \( k = n \) so that \( x^n \) is an idempotent \( e \). Then \( a = e(a(my)^n) = e \cdot e(a(my)^n) = ea(my)^n = a \). But then \( a = (a(my))^n = am \cdot (my)^{n-1} \in amM \) as desired. The proof of ii. is similar. \( \square \)

We also use the “egg-box” picture of a given fixed class \( D \in \mathbb{D} \):

Picture the \( H \)-classes contained in \( D \) in a rectangular array where the \( H \)-classes in a given column are all in the same \( L \)-class and the \( H \)-classes in a given row are all in the same \( R \)-class. So the number of columns in the array is the number \( n(D, L) \) of distinct \( L \)-classes in \( D \), while the number of rows is the number \( n(D, R) \) of distinct \( R \)-classes in \( D \). Write \( iH_j \) for the \( H \)-class in row \( i \), column \( j \). We can assume that \( iH_1 \) is our “base class” \( H_\gamma \).

The following result follows from Green’s lemma and its dual:

**Lemma 2.3.** For each \( i \in \{1, 2, \ldots, n(D, R)\} \), \( j \in \{1, 2, \ldots, n(D, L)\} \) there exist elements \( a_i, \bar{a}_i, b_j, \bar{b}_j \in M \) such that

i. For each column \( j \), the left translation \( h \mapsto a_jh \) defines a bijective map \( l_i : iH_j \to iH_j \) with inverse \( l_i^{-1} : iH_j \to iH_j \) defined by left translation \( l_i^{-1} : h \mapsto \bar{a}_jh \).
ii. For each row $i$, the right translation $h \mapsto hb_j$ defines a bijective map $r_j : iH_j \to iH_j$ with inverse $r_j^{-1} : iH_j \to iH_1$ defined by right translation $r_j^{-1} : h \mapsto h\bar{b}_j$.

Then for any row $i$, column $j$, and $h \in H_i$, $h \mapsto a_ihb_j$ gives a bijective map $i\psi_j : H_i \to iH_j$ of $\mathcal{H}$-classes which extends linearly to a bijection $i\psi_j : R[H_i] \to R[H_j]$.

We can now state the main result of this section.

Proposition 2.1. Suppose that for some $d \in D, m \in M$ we have $dm \in D$. For some $i,j$, this $d \in iH_j = i\psi_j(H_i) = a_iH_jb_j$ and so $d = a_i\phi^R_H(gd)b_j$ for some $gd \in G^R_H$. Then:

i. $dm \in iH_k$ for some $k$. Define $m^* = b_jmb_k$.

ii. $m^* \equiv b_jmb_k \in RT(H)$.

iii. The right translation $r_m : h \mapsto hm$ gives a bijection $r_m : iH_j \to iH_k$.

iv. If $h = a_i\phi^R_H(g)b_j \in iH_j$, then $hm = (h)r_m = a_i\phi^R_H(g)rm^*b_k$.

v. For any $x \in R[iH_j]$, write $x = a_i\phi^R_H(y)b_j$ for some $y \in R[G^R_H]$. Then $xm = a_i\phi^R_H(yrm^*)b_k$.

Proof. Since $D_d = D_{dm} = D$, we have $R_d = R_{dm}$ by lemma 2.2. So $H_d$ and $H_{dm}$ lie in the same $\mathcal{R}$-class and are in the same row of the “egg-box” for $D$. So if $d \in iH_j$, then $dm \in iH_k$ for some $k$, proving i. Write $d = a_i\phi^R_H(gd)b_j$ and $dm = a_i\phi^R_H(gdm)b_k$ for some $gd,gdm \in G^R_H$. Then $dm = a_i\phi^R_H(gd)b_jm = a_i\phi^R_H(gdm)bk$. Multiplying by $a_i$ on the left and by $b_k$ on the left gives $\phi^R_H(gd)b_jmb_k = \phi^R_H(gdm) \in H$. So by lemma 2.1, $m^* \equiv b_jmb_k \in RT(H)$ proving ii. Then $r_m^* : H \to H$ is a bijection $\phi^R_H(g)m^* = \phi^R_H(grm^*)$ for any $g \in G^R_H$. Now take any $h = a_i\phi^R_H(g)b_j \in iH_j$ and compute $hm^*b_k = a_i\phi^R_H(g)b_jmb_k = a_i\phi^R_H(g)m^*b_k$. Multiplying on the right by $b_k$ then proves iv., and v. then follows by linearity. For iii., observe that $r_m$ can be written as a composition of bijections: $r_m = i\psi_k \circ \phi^R_H \circ i\psi_m \circ (\phi^R_H)^{-1} \circ (i\psi_j)^{-1}$ where $\bar{r}_m^* : G^R_H \to G^R_H$ is the bijection $g \mapsto grm^*$.

As an immediate corollary we have

Corollary 2.1. For any $\mathcal{H}$-class $iH_j \subseteq D$ and any $m \in M$ either

i. $R[iH_j] \cdot m \subseteq \bar{A}D = \oplus_{D' \subset D} R[D']$ or

ii. $R[iH_j] \cdot m = R[iH_k]$ for some $k$ and if $x = a_i\phi^R_H(y)b_j$ for some $y \in R[G^R_H]$, then $xm = a_i\phi^R_H(yrm^*)b_k$, where $m^* = b_jmb_k \in RT(H)$.

We obtain the dual versions of proposition 2.1 and corollary 2.1 given next by using part ii. of lemmas 2.1 and 2.2 along with the formula $m\phi^R_H(g) = \phi^R_H(r\bar{m}g)$ for $g \in G^R_H$, where $\bar{m} = \vartheta \circ \psi(l_m) \in RT(H)$.
Proposition 2.2. Suppose that for some \( d \in D, m \in M \) we have \( md \in D \). For some \( i, j \), this \( d \in H_j = i \psi_j (H_\gamma) = a_i H_\gamma b_j \) and so \( d = a_i \phi^R_H (g_d) b_j \) for some \( g_d \in G^R_H \). Then:

i. \( md \in k H_j \) for some \( k \). Define \( m^* = \bar{a}_k m a_j \).

ii. \( m^* \equiv \bar{a}_k m a_j \in LT (H) \), so \( m^* = \emptyset \circ \psi (l_{m^*}) \in RT (H) \).

iii. The left translation \( l_m : h \mapsto mh \) gives a bijection \( l_m : h H_j \rightarrow k H_j \).

iv. If \( h = a_i \phi^R_H (g) b_j \in H_j \), then \( mh = l_m (h) = a_k \phi^R_H (r_{\bar{m}^*} g) b_j \).

v. For any \( x \in R [H_j] \), write \( x = a_i \phi^R_H (y) b_j \) for some \( y \in R [G^R_H] \). Then \( mx = a_k \phi^R_H (r_{\bar{m}^*} y) b_j \).

Corollary 2.2. For any \( H \)-class \( H_j \subseteq D \) and any \( m \in M \) either

i. \( m \cdot R [H_j] \subseteq \bar{A} D = \oplus_{D' < D} R [D'] \) or

ii. \( m \cdot R [H_j] = R [k H_j] \) for some \( k \) and if \( x = a_i \phi^R_H (y) b_j \) for some \( y \in R [G^R_H] \), then \( mx = a_k \phi^R_H (r_{\bar{m}^*} y) b_j \), where \( m^* \equiv \bar{a}_k m a_j \in LT (H) \) and \( m^* = \emptyset \circ \psi (l_{m^*}) \in RT (H) \).

3 Cell algebra structures on monoid algebras

In [10], a class of algebras called cell algebras is defined which generalize the cellular algebras of Graham and Lehrer [6]. These algebras share many of the nice properties of cellular algebras. We will give conditions for a monoid algebra to be a cell algebra. We first review the definition.

Let \( R \) be a commutative integral domain with unit 1 and let \( A \) be an associative, unital \( R \)-algebra. Let \( \Lambda \) be a finite set with a partial order \( \geq \) and for each \( \lambda \in \Lambda \) let \( L(\lambda), R(\lambda) \) be finite sets of “left indices” and “right indices”. Assume that for each \( \lambda \in \Lambda, s \in L(\lambda), \) and \( t \in R(\lambda) \) there is an element \( c_{\lambda}^s \in A \) such that the map \( (\lambda, s, t) \mapsto c_{\lambda}^s \) is injective and \( C = \{ c_{\lambda}^s : \lambda \in \Lambda, s \in L(\lambda), t \in R(\lambda) \} \) is a free \( R \)-basis for \( A \). Define \( R \)-submodules of \( A \) by \( A^\lambda = \text{span of} \ \{ c_{\mu}^s : \mu \in \Lambda, \mu \geq \lambda, s \in L(\mu), t \in R(\mu) \} \) and \( \bar{A}^\lambda = \text{span of} \ \{ c_{\mu}^s : \mu \in \Lambda, \mu > \lambda, s \in L(\mu), t \in R(\mu) \} \).

Definition 3.1. Given \((A, \Lambda, C)\), \( A \) is a cell algebra with poset \( \Lambda \) and cell basis \( C \) if

i. For any \( a \in A, \lambda \in \Lambda, \) and \( s, s' \in L(\lambda) \), there exists \( r_L = r_L (a, \lambda, s, s') \in R \) such that, for any \( t \in R(\lambda) \), \( a \cdot s c_{\lambda}^s = \sum_{s' \in L(\lambda)} r_L \cdot s' c_{\lambda}^{s'} \mod \bar{A}^\lambda \) and

ii. For any \( a \in A, \lambda \in \Lambda, \) and \( t, t' \in R(\lambda) \), there exists \( r_R = r_R (a, \lambda, t, t') \in R \) such that, for any \( s \in L(\lambda) \), \( s c_{\lambda}^s \cdot a = \sum_{t' \in R(\lambda)} r_R \cdot s' c_{\lambda}^{s'} \mod A^\lambda \).
Now for each $D$-class $D$ in a finite monoid $M$, choose a base element $\gamma_D$ and base $H$-class $H = H_\gamma$ as above. Then define the Schützenberger group $G_D$ of $D$ to be $G_D = G_H^R$. (Up to isomorphism, this is independent of the choice of base class $H$.) Let $R[G_D]$ be the group algebra of $G_D$ over $R$.

**Definition 3.2.** A monoid $M$ satisfies the R.C.A. condition for a given domain $R$ if $R[G_D]$ has a cell algebra structure for every $D$-class $D$ in $M$.

If an $H$-class $H$ contains an idempotent, then $H$ is actually a subgroup of $M$. In fact, the maximal subgroups of $M$ are just the $H$-classes which contain an idempotent. In this case the group $H$ is isomorphic to the Schützenberger group $G_H^R$. If $M$ is a regular semigroup, then every $D$-class $D$ contains an idempotent $e$, so $G_D$ is isomorphic to a maximal subgroup $H_e$ of $M$. Then for regular semigroups $M$ the R.C.A. condition is equivalent to requiring that $R[G]$ have a cell algebra structure for every maximal subgroup $G$ of $M$.

Our main result is the following theorem.

**Theorem 3.1.** Let $M$ be a finite monoid satisfying the R.C.A. condition for a domain $R$. Then $A = R[M]$ is a cell algebra.

**Proof.** For a given $D \in D$, put $A_D = R[G_D]$ and assume $\Lambda_D, L_D, R_D$ define a cell algebra structure on $A_D$ with cell basis

$$C_D = \left\{ sC_{\lambda}^D : \lambda \in \Lambda_D, s \in L_D(\lambda), t \in R_D(\lambda) \right\}.$$

Define a poset $\Lambda$ to consist of all pairs $(D, \lambda)$ where $D$ is a $D$-class in $M$ and $\lambda \in \Lambda_D$. Define the partial order by $(D_1, \lambda_1) > (D_2, \lambda_2)$ if $D_1 < D_2$ or $D_1 = D_2$ and $\lambda_1 > \lambda_2$ in $\Lambda_D$. For $(D, \lambda) \in \Lambda$, define $L(D, \lambda)$ to be all pairs $(R, s)$ where $R$ is an $R$-class contained in $D$ and $s \in L_D(\lambda)$. Similarly, define $R(D, \lambda)$ to be all pairs $(L, t)$ where $L$ is an $L$-class contained in $D$ and $t \in R_D(\lambda)$. Finally, given $(D, \lambda) \in \Lambda$, $(R, s) \in L(D, \lambda)$, $(L, t) \in R(D, \lambda)$, assume $R$ corresponds to row $i$ and $L$ corresponds to column $j$ in the “egg-box” for $D$. Then define

$$(r,s)C^{(D,\lambda)}_{(L,t)} = i\psi_j \left( \phi_H^R \left( sC_{\lambda}^i \right) \right) = a_i \phi_H^R \left( sC_{\lambda}^i \right) b_j.$$

For fixed $D, R, L$, since $\left\{ sC_{\lambda}^D : \lambda \in \Lambda_D, s \in L_D(\lambda), t \in R_D(\lambda) \right\}$ is a basis for $R[G_D]$ by assumption and $i\psi_j$ and $\phi_H^R$ are bijective,

$$\left\{ (r,s)C^{(D,\lambda)}_{(L,t)} : \lambda \in \Lambda_D, s \in L_D(\lambda), t \in R_D(\lambda) \right\}$$

will give a basis for $R[H]$. Then since $A = R[M]$ is the direct sum of the free submodules $R[H]$ as $H$ varies over all $H$-classes in $M$,

$$C = \left\{ (r,s)C^{(D,\lambda)}_{(L,t)} : (D, \lambda) \in \Lambda, (R, s) \in L(D, \lambda), (L, t) \in R(D, \lambda) \right\}$$

is a basis for $A$. To show $C$ is a cell basis, we must check the cell conditions (i) and (ii). Note first that any element $m$ in a $D$-class $D_m$ will be in the span of
\[ \left\{ (r,s)C_{(l,t)}^{(D,m)} : \mu \in \Lambda_{D,m}, (R,s) \in L(D_m,\mu), (L,t) \in R(D_m,\mu) \right\}. \] Then if \( D_m < D \) for some \( D \)-class \( D \), we have \((D_m,\mu) > (D,\lambda)\) for any \( \mu \in \Lambda_{D,m} \) and \( \lambda \in \Lambda_D \), so \( m \in A^{(D,\lambda)} \). So \( \otimes_{D' < D} R[D'] = A^D \subseteq A^{(D,\lambda)} \). To prove (i), we can assume \( a \) is a basis element \( m \in M \). Take \((D,\lambda) \in \Lambda \), \((R,s) \in L(D,\lambda) \), \((L,t) \in R(D,\lambda) \) and assume \( R \) corresponds to row \( i \) and \( L \) corresponds to column \( j \) in the “egg-box” for \( D \). Then \((r,s)C_{(l,t)}^{(D,\lambda)} = i\psi_j (\phi^R_H (sC^\lambda_j)) = a_i\phi^R_H (sC^\lambda_j) b_j \in R[i][H_j] \). By corollary 2.2 there are two cases to consider.

Case i: \( m \cdot R[i][H_j] \subseteq \otimes_{D' < D} R[D'] \). Then \( m \cdot (r,s)C_{(l,t)}^{(D,\lambda)} \subseteq m \cdot R[i][H_j] \subseteq \otimes_{D' < D} R[D'] \subseteq A^{(D,\lambda)} \) and we can satisfy (i) by taking all coefficients \( r_L \) to be zero.

Case ii: \( m \cdot R[i][H_j] = R[k][H_j] \) for some \( k \). By corollary 2.2, since \((r,s)C_{(l,t)}^{(D,\lambda)} = a_i\phi^R_H (sC^\lambda_j) b_j \in R[i][H_j] \), then \( m \cdot (r,s)C_{(l,t)}^{(D,\lambda)} = a_k\phi^R_H (m^* sC^\lambda_j) b_j \), where \( m^* = \tilde{a}_km \) and \( m^* \in \tilde{A}_D \), is the cell algebra property (i) for \( R[G_D] \) gives \( g \cdot sC^\lambda_j = \sum_{s' \in L_D(\lambda)} r_L \cdot s'C^\lambda_j \) in \( A^{(D,\lambda)} \), where \( r_L \) depends on \( s, s', \lambda, m \), but is independent of \( t \). Then

\[
m \cdot (r,s)C_{(l,t)}^{(D,\lambda)} = a_k\phi^R_H (g sC^\lambda_j) b_j = \sum_{s' \in L_D(\lambda)} r_L \cdot a_k\phi^R_H (s'C^\lambda_j) b_j \text{ modulo } a_k\phi^R_H \left( \tilde{A}_D \right) b_j
\]

where

\[
a_k\phi^R_H \left( \tilde{A}_D \right) b_j \subseteq \text{span} \left\{ (r,s)C_{(l,t)}^{(D,\mu)} : \mu > \lambda, s \in L_D(\mu), t \in R_D(\mu) \right\} \subseteq A^{(D,\lambda)}.
\]

But \( a_k\phi^R_H (sC^\lambda_j) b_j = (r',s')C_{(l,t)}^{(D,\lambda)} \) where \( R' \) is the \( R \)-class corresponding to row \( k \) of the “egg-box”. Then \( m \cdot (r,s)C_{(l,t)}^{(D,\lambda)} = \sum_{s' \in L_D(\lambda)} r_L \cdot a_k\phi^R_H (s'C^\lambda_j) b_j = \sum_{s' \in L_D(\lambda)} r_L \cdot (r',s')C_{(l,t)}^{(D,\lambda)} \) in \( A^{(D,\lambda)} \). Since \( r_L \) is independent of \( L \) and \( t \), this yields property (i) for this case ii.

The proof of condition (ii) is parallel. Thus \( C \) is a cell basis and \( A = R[M] \) is a cell algebra.

If \( M \) is a finite monoid satisfying the R-C.A. condition, we will assume a fixed cell algebra structure is given to each \( R[G_D] \). We will then call the cell algebra structure obtained in the proof of theorem 3.4 the standard cell algebra structure on \( R[M] \).

In [4], [11], and [8], East, Wilcox, and Guo and Xi worked with finite regular semigroups with cellular algebra (hence cell algebra) structure on \( R[G] \) for maximal subgroups \( G \) of \( M \). Their examples therefore satisfy the R-C.A. condition and can be seen to be cell algebras without considering the complicated involution requirements involved in showing a cellular algebra structure. In [4], East gives examples of inverse semigroups with cellular \( R[G] \) for all maximal \( G \).
which lack an appropriate involution and therefore do not have cellular \( R[M] \). These examples would be cell algebras by theorem 3.1.

More typical examples of cell algebras that are not cellular are the algebras \( R[M] \) for \( M \) a transformation semigroup \( T_r \) or partial transformation semigroup \( P^rT_r \). These were shown to be cell algebras in [10]. For these examples, a \( D \)-class \( D \) consists of mappings of a given rank \( i \) and the group \( G_D \) is the symmetric group \( S_i \). Since the symmetric groups have cellular (hence cell) structures on their group algebras, theorem 3.1 applies and provides a cell algebra structure on \( M \).

4 Properties of the cell algebra \( A = R[M] \)

In this section we assume that \( M \) is a finite monoid satisfying the R-C.A. condition, that is, such that for every \( D \in \mathbb{D} \) the group algebra \( R[G_D] \) of the Schützenberger group for \( D \) is a cell algebra. Then by Theorem 3.1 \( A = R[M] \) is a cell algebra and we can apply the results in [10] to \( A \) with the standard cell algebra structure.

For \( (D, \lambda) \in \Lambda \), the left cell module \( L^{C(D, \lambda)} \) is a left \( A \)-module which is a free \( R \)-module with basis \( \{(R,s)C^{(D, \lambda)}: (R,s) \in L(D, \lambda)\} \). Similarly, the right cell module \( C^{(D, \lambda)} \) for \( (D, \lambda) \) is a right \( A \)-module and a free \( R \)-module with basis \( \{C^{(D, \lambda)}(L,t): (L,t) \in R(D, \lambda)\} \). For each \( (D, \lambda) \in \Lambda \) there is an \( R \)-bilinear map \( \langle -,- \rangle : (C^{(D,\lambda)}R, L^{C(D,\lambda)}) \to R \) defined by the property \( \langle (R',s')C^{(D,\lambda)}(L',t'), (R,s)C^{(D,\lambda)}(L,t) \rangle \equiv A^{(D,\lambda)} \) for any choice of \( R', s', L', t' \).

Right and left radicals are defined by

\[
\text{rad} \left( C^{(D,\lambda)}_R \right) = \left\{ x \in C^{(D,\lambda)}_R : \langle x, y \rangle = 0 \text{ for all } y \in L^{C(D,\lambda)} \right\} \\
\text{rad} \left( L^{C(D,\lambda)} \right) = \left\{ y \in L^{C(D,\lambda)} : \langle x, y \rangle = 0 \text{ for all } x \in C^{(D,\lambda)}_R \right\}.
\]

Then define \( D^{(D, \lambda)}_R = \frac{C^{(D, \lambda)}_R}{\text{rad}(C^{(D, \lambda)}_R)} \) and \( L^{D(D, \lambda)} = \frac{C^{(D, \lambda)}_L}{\text{rad}(L^{C(D, \lambda)})} \). Finally, define \( \Lambda_0 = \left\{ (D, \lambda) \in \Lambda : \langle x, y \rangle \neq 0 \text{ for some } x \in C^{(D, \lambda)}_R, y \in L^{C(D, \lambda)} \right\} \). Evidently, \( \lambda \in \Lambda_0 \iff D^{(D, \lambda)}_R \neq 0 \iff L^{D(D, \lambda)} \neq 0 \). A major result of [10] is

**Theorem 4.4.** Assume \( R = k \) is a field. Then

(a) \( \left\{ D^{(D, \mu)}_R : (D, \mu) \in \Lambda_0 \right\} \) is a complete set of pairwise inequivalent irreducible right \( A \)-modules and

(b) \( \left\{ L^{D(D, \mu)} : (D, \mu) \in \Lambda_0 \right\} \) is a complete set of pairwise inequivalent irreducible left \( A \)-modules.

To find \( \Lambda_0 \) for \( A \) we need to relate the bracket \( \langle -,- \rangle \) on \( A \) to the brackets \( \langle -,- \rangle_D \) on the cell algebras \( A_D = R[G_D] \). For a given \( D \)-class \( D \), write \( R_i \)
for the $\mathcal{R}$-class corresponding to row $i$ of the “egg-box” and $L_j$ for the $\mathcal{L}$-class corresponding to column $j$.

**Definition 4.1.** $R_i, L_j$ are matched if there exist $x \in L_j$, $y \in R_i$ such that $xy \in D$. $R_i, L_j$ are unmatched if $L_j R_i \cap D = \emptyset$.

**Lemma 4.1.** Assume $R_i, L_j$ are matched. Write $x = a_i \phi_H^R(\xi) b_j \in R[L_j]$ and $y = a_i \phi_H^R(\eta) b_j' \in R[R_i]$ for any $\xi, \eta \in R[G_D]$. Then $xy = a_i \phi_H^R(\xi r_{m(i,j)}(\eta)) b_j'$ where $m(i,j) \equiv b_j a_i \gamma \in RT(H)$.

**Proof.** If $R_i, L_j$ are matched, then for some choice of $i', j'$ and some elements $x \in \nu H_{j'} \subseteq L_j$, $y \in H_{j'} \subseteq R_i$ we have $xy \in \nu H_{j'} \subseteq D$. But then (for the given $i', j'$) we have $xy \in \nu H_{j'} \subseteq D$ for any $x \in \nu H_{j}$, $y \in \nu H_{j'}$ by propositions 2.1 and 2.2. In particular, for $x = a_i \gamma b_j \in \nu H_j$, $y = a_i \gamma b_{j'} \in \nu H_{j'}$ we get $xy = a_i \gamma b_j a_i \gamma b_{j'} \in \nu H_{j'}$. Then multiplying by $\bar{a}_j'$ on the left and $b_{j'}$ on the right gives $\gamma b_j a_i \gamma \in H$. But then $m(i,j) \equiv b_j a_i \gamma \in RT(H)$ by lemma 2.1.

Now consider any $i', j'$ and any $g = r_m, h = r_n \in G_D$ and let

$$x = a_i \phi_H^R(g) b_j \in R[L_j], \quad y = a_i \phi_H^R(h) b_{j'} \in R[R_i].$$

Then

$$xy = (a_i \phi_H^R(g) b_j)(a_i \phi_H^R(h) b_{j'}) = a_i \gamma b_j a_i \gamma b_{j'} = a_i \gamma m \cdot m(i,j) \cdot b_{j'} = a_i \phi_H^R(r_m r_{m(i,j)} r_n) b_{j'} = a_i \phi_H^R(gr_{m(i,j)}h) b_{j'}.$$ 

Then by linearity of $\phi_H^R$, $xy = a_i \phi_H^R(\xi r_{m(i,j)}(\eta)) b_{j'}$ for arbitrary $\xi, \eta \in R[G_D]$ when $x = a_i \phi_H^R(\xi) b_j \in R[L_j], \quad y = a_i \phi_H^R(\eta) b_{j'} \in R[R_i]$, proving the lemma. □

For $i \in \{1, 2, \ldots, n(D,R)\}$, define $(L C^{(D,\lambda)})_i$ to be the free $R$-module with basis $\{ (R_i,s) C^{(D,\lambda)} : \lambda \in L_D, s \in L_D(\lambda) \}$, so $L C^{(D,\lambda)} = \bigoplus_i (L C^{(D,\lambda)})_i$. Similarly, for $j \in \{1, 2, \ldots, n(D,L)\}$, define $(C^{(D,\lambda)})_j$ to be the free $R$-module with basis $\{ C^{(D,\lambda)}(L_j, t) : \lambda \in L_D, t \in R_D(\lambda) \}$, so $C^{(D,\lambda)} = \bigoplus_j (C^{(D,\lambda)})_j$. Notice that for each $i$, $\phi_i : (R_i,s) C^{(D,\lambda)} \rightarrow C^{\Lambda}$ gives an isomorphism (of $R$-modules) $\phi_i : (L C^{(D,\lambda)})_i \rightarrow L C^{\Lambda}$. Similarly, $\phi_j : C^{(D,\lambda)}(L_j, t) \rightarrow C^{\Lambda}$ gives an isomorphism $\phi_j : (C^{(D,\lambda)})_j \rightarrow C^{\Lambda}$.

**Proposition 4.1.** Take $X \in (C^{(D,\lambda)})_j$, $Y \in (L C^{(D,\lambda)})_i$. Then

(a) If $R_i, L_j$ are not matched, then $\langle X, Y \rangle = 0$,

(b) If $R_i, L_j$ are matched, then

$$\langle X, Y \rangle = \langle \phi_j(X), r_{m(i,j)} \phi_i(Y) \rangle_D = \langle r_{m(i,j)} \phi_j(X), \phi_i(Y) \rangle_D$$

where $m(i,j) \equiv b_j a_i \gamma \in RT(H)$. 

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Proof. It suffices to check (a) and (b) for basis elements $X, Y$, so assume $X = \Lambda_{(L,j,t)}^{(D,\lambda)}$, $Y = (R_i,s) \Lambda^{(D,\lambda)}$ and put $x = (R_{i'},s') \Lambda^{(D,\lambda)}$ and $y = (R_i,s) \Lambda^{(D,\lambda)}$ = \Lambda^{(D,\lambda)} b_j \in L_j$ and $y = (R_i,s) \Lambda^{(D,\lambda)} = a_{i'} \theta_{i'}^R (s C_\lambda^\lambda b_j' \in R_i$. Then $xy = \langle X, Y \rangle_{(R_i,s')} (R_{i',s'}) \Lambda^{(D,\lambda)}$ mod $\hat{A}^{(D,\lambda)}$.

If $R_i$, $L_j$ are not matched, then $xy \in \hat{A}^{D} \subseteq \hat{A}^{(D,\lambda)}$, so $\langle X, Y \rangle = 0$, proving (a).

If $R_i, L_j$ are matched, then, by lemma 4.1,

$$xy = a_{i'} \theta_i^R (s C_\lambda^\lambda \cdot r_{m(i,j)} \cdot C_\lambda^\lambda b_j'$$

$$= a_{i'} \theta_i^R \left( (C_\lambda^\lambda, r_{m(i,j)} \cdot C_\lambda^\lambda) \cdot C_\lambda^\lambda b_j' \mod \hat{A}^{(D,\lambda)} \right)$$

$$= \left\langle C_\lambda^\lambda, r_{m(i,j)} \cdot C_\lambda^\lambda \cdot a_{i'} \theta_i^R (s C_\lambda^\lambda b_j') \mod \hat{A}^{(D,\lambda)} \right\rangle$$

$$= \left\langle \phi_j(X), r_{m(i,j)} \cdot \phi_i(Y) \right\rangle_{\hat{A}^{D}} (R_{i',s'}) \Lambda^{(D,\lambda)} \mod \hat{A}^{(D,\lambda)}.$$ This gives $\langle X, Y \rangle = \langle \phi_j(X), r_{m(i,j)} \cdot \phi_i(Y) \rangle_D = \langle r_{m(i,j)} \cdot \phi_j(X), \phi_i(Y) \rangle_D$, proving part (b). \qed

We note the following corollary for future use.

**Corollary 4.1.** Let $M$ be a finite monoid satisfying the R-C.A. condition and place the standard cell algebra structure on $R[M]$. Then for any $\lambda \in \Lambda_D$, $D \in \mathbb{D}$:

1. If $\text{rad}_D \left( L \Lambda^{(D,\lambda)} \right) \neq 0$, then $\text{rad} \left( L \Lambda^{(D,\lambda)} \right) \neq 0$,

2. If $\text{rad}_D \left( D_R^{(D,\lambda)} \right) \neq 0$, then $\text{rad} \left( D_R^{(D,\lambda)} \right) \neq 0$.

**Proof.** Assume $\text{rad}_D \left( L \Lambda^{(D,\lambda)} \right) \neq 0$ and take a $y \neq 0$ in $\text{rad}_D \left( L \Lambda^{(D,\lambda)} \right)$. Write $y = \sum_{s \in L(\lambda)} c(s) \cdot s \Lambda^{\lambda}$ and put $Y = \sum_{s \in L(\lambda)} c(s) (R_i,s) \Lambda^{(D,\lambda)} \in \left( L \Lambda^{(D,\lambda)} \right)_D \subset L \Lambda^{(D,\lambda)}$ for some $R_i \subseteq D$. Then $Y \neq 0$ and we claim $Y \in \text{rad} \left( L \Lambda^{(D,\lambda)} \right)$: Take any $X \in \left( L \Lambda^{(D,\lambda)} \right)_0$. Then by the proposition, if $R_i$, $L_j$ are not matched we have $\langle X, Y \rangle = 0$, while if $R_i$, $L_j$ are matched, then

$$\langle X, Y \rangle = \langle r_{m(i,j)} \cdot \phi_j(X), \phi_i(Y) \rangle_D = \langle r_{m(i,j)} \cdot \phi_j(X), y \rangle_D = 0$$

since $y \in \text{rad}_D \left( L \Lambda^{\lambda} \right)$. Then $\langle X, Y \rangle = 0$ for any $X \in C^{(D,\lambda)}_R$ and $Y \in \text{rad} \left( L \Lambda^{(D,\lambda)} \right)$ as claimed. The proof of ii. is parallel. \qed

Write $D^2 = \{ xy : x, y \in D \}$ and recall that $\hat{A}^D = \oplus_{D' \subset D} R[D'] \subseteq \hat{A}^{(D,\lambda)}$ for any $\lambda \in \Lambda_D$.

**Corollary 4.2.** For any $D$-class $D$,

(a) If $D^2 \subseteq \hat{A}^D$, then $(D, \lambda) \notin \Lambda_0$ for any $\lambda \in \Lambda_D$

(b) If $D^2 \notin \hat{A}^D$, then $(D, \lambda) \in \Lambda_0$ $\iff \lambda \in (\Lambda_D)_0$.  

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Proof. $D^2 \subseteq \hat{A}D$ if and only if no pair $R_i, L_j$ are matched.

(a) $D^2 \subseteq \hat{A}D$ implies no pair $R_i, L_j$ is matched, so by proposition 4.1, $(X,Y) = 0$ for every $X \in C_R^{(D,\lambda)}$, $Y \in L C^{(D,\lambda)}$. Then $\text{rad} \left(C_R^{(D,\lambda)}\right) = C_R^{(D,\lambda)}$ and $(D,\lambda) \notin \Lambda_0$.

(b) If $D^2 \not\subset \hat{A}D$ then $R_i, L_j$ are matched for at least one pair $i, j$.

For $\Rightarrow$: If $(D,\lambda) \in \Lambda_0$, then $(X,Y) \neq 0$ for some $X \in \left( C_R^{(D,\lambda)} \right)_j$, $Y \in \left( L C^{(D,\lambda)} \right)_j$, where $i,j$ must be matched. Then by proposition 4.1 $(X,Y) = \langle \phi_j(X), r_{m(i,j)} \phi_i(Y) \rangle_D \neq 0$, where $\phi_j(X) \in C_R^{\lambda}$, $r_{m(i,j)} \phi_i(Y) \in L C^{\lambda}$. So $\text{rad}_D \left(C_R^{(D,\lambda)}\right) \neq C_R^{(D,\lambda)}$ and $(D,\lambda) \notin (\Lambda_D)_0$.

For $\Leftarrow$: If $\lambda \in (\Lambda_D)_0$, then there exist $\xi \in C_R^{\lambda}$, $\eta \in L C^{\lambda}$ with $(\xi,\eta)_D \neq 0$. Choose $i, j$ with $R_i$, $L_j$ matched and let $X = \phi_j^{-1}(\xi) \in \left( C_R^{(D,\lambda)} \right)_j$, $Y = \phi_i^{-1}(r_{m(i,j)}^{-1} \cdot \eta) \in \left( L C^{(D,\lambda)} \right)_i$. (Note that $r_{m(i,j)} \in G_D$, so $r_{m(i,j)}^{-1} \in G_D$ is well defined.) Then by proposition 4.1 $(X,Y) = \langle \phi_j(X), r_{m(i,j)} \phi_i(Y) \rangle_D = \langle \xi, \eta \rangle_D \neq 0$. Then $\text{rad} \left(C_R^{(D,\lambda)}\right) \neq C_R^{(D,\lambda)}$ and $(D,\lambda) \in \Lambda_0$. □

Consider the question of whether a monoid algebra $k[M]$ is quasi-hereditary (as defined by [2]). The following was proved in [10]:

**Proposition 4.2.** Let $k$ be a field and $A$ be a cell algebra over $k$ such that $\Lambda_0 = \Lambda$. Then $A$ is quasi-hereditary.

Then by corollary 4.2 we have

**Corollary 4.3.** Let $k$ be a field and $M$ a finite monoid such that for every class $D \in \mathcal{D}$, $D^2 \not\subset \hat{A}D$ and the group algebra $k[G_D]$ of the Schützenberger group for $D$ is a cell algebra with $(\Lambda_D)_0 = \Lambda_D$. Then $A = k[M]$ is quasi-hereditary.

Recall that a semigroup $M$ is regular if for every $x \in M$ there is a $y \in M$ such that $x = xyx$. In a regular semigroup, every $D$-class $D$ contains an idempotent $\rho$. Then $\rho^2 = \rho^2 \in D \cap D^2$, so $D^2 \not\subset \hat{A}D$. Thus

**Corollary 4.4.** Let $k$ be a field and $M$ a regular finite monoid such that for every class $D \in \mathcal{D}$, the group algebra $k[G_D]$ of the Schützenberger group for $D$ is a cell algebra with $(\Lambda_D)_0 = \Lambda_D$. Then $A = k[M]$ is quasi-hereditary.

Note that in a regular semi-group the base $H$-class $H$ for any class $D$ can be chosen to contain an idempotent. Then $H$ is a (maximal) subgroup of $M$ with $H$ isomorphic to $G_D$. So the condition in corollary 4.4 can be replaced by the requirement that for any maximal subgroup $G$ of $M$, the group algebra $k[G]$ must be a cell algebra with $(\Lambda_G)_0 = \Lambda_G$.

Now consider the question of semi-simplicity for a monoid algebra $k[M]$. The following criterion for semi-simplicity of a cell algebra follows from results in [10].

**Proposition 4.3.** A cell algebra $A$ over a field $k$ is semi-simple if and only if for every $\lambda \in \Lambda$ we have $L C^{\lambda} = L D^{\lambda}$ and $C_R^{\lambda} = D_R^{\lambda}$. 

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Proof. Assume $A$ is semi-simple. Then for every $\lambda \in \Lambda_0$ we have $L^{P^{\lambda}} = L^{D^{\lambda}}$, where $L^{P^{\lambda}}$ is the principal indecomposable left module corresponding to the irreducible $L^{D^{\lambda}}$. But since $L^{P^{\lambda}}$ always has a filtration with $L^{C^{\lambda}}$ as the “top” quotient, we must have $L^{P^{\lambda}} = L^{C^{\lambda}} = L^{D^{\lambda}}$. As in \cite{10}, for $\mu \in \Lambda_0$, $\lambda \in \Lambda$, let $Rd_{\lambda\mu} = [C^{\lambda}_R : D^{\mu}_R]$ be the multiplicity of the irreducible $D^{\mu}_R$ as a composition factor in $C^{\lambda}_R$, let $Ld_{\lambda\mu} = [L^{C^{\lambda}} : L^{D^{\mu}}]$ be the multiplicity of $L^{D^{\mu}}$ in $L^{C^{\lambda}}$, and write $RD = \sum Ld_{\lambda\mu}$ and $LD = \sum Rd_{\lambda\mu}$ as the decomposition matrices of $A$. Then $RD$ and $LD$ are both square identity matrices. The right Cantor matrix, $RC$, and left Cantor matrix, $LC$, are the square $|\Lambda_0| \times |\Lambda_0|$ matrices where for $\lambda, \mu \in \Lambda_0$, $\mu \in \Lambda_0, RC_{\lambda\mu} = [P^{\lambda}_R : D^{\mu}_R]$ and $LC_{\lambda\mu} = [L^{P^{\lambda}} : L^{D^{\mu}}]$. As shown in \cite{10}, we have $RC = LD^T \cdot RD$ and $LC = RD^T \cdot LD$ (where $T$ denotes the transpose matrix). Then $RC$ and $LC$ are also identity matrices and we must have $P^{\lambda}_R = D^{\lambda}_R$ and $L^{P^{\lambda}} = L^{D^{\lambda}}$ for every $\lambda \in \Lambda = \Lambda_0$. Since $A$ is a direct sum of principle indecomposable left modules isomorphic to the various $L^{P^{\lambda}} = L^{D^{\lambda}}$, it is a direct sum of irreducible modules and therefore semi-simple. \hfill $\Box$

Consider a finite monoid $M$ satisfying the R.C.A. condition and place the standard cell algebra structure on $M$.

Corollary 4.5. If $k[M]$ is semi-simple, then $k[G_D]$ is semi-simple for every $D \in \mathbb{D}$.

Proof. If $k[G_D]$ is not semi-simple for some $D$, then by the proposition either $\text{rad}_D (L^{C^{\lambda}}) \neq 0$ or $\text{rad}_D (D^{\lambda}_R) \neq 0$. Assume $\text{rad}_D (L^{C^{\lambda}}) \neq 0$ (if $\text{rad}_D (D^{\lambda}_R) \neq 0$ the proof is similar). Then by corollary 4.4 we have $\text{rad} (L^{C(D,\lambda)}) = 0$. But then $k[M]$ is not semi-simple by proposition 4.3. \hfill $\Box$

If all the algebras $k[G_D]$ are semi-simple, we would like a condition guaranteeing that $k[M]$ itself is semi-simple. For any $D$-class $D$, let $\mathbb{L}(D)$ be the set of all $L$-classes contained in $D$ and $\mathbb{R}(D)$ be the set of all $R$-classes contained in $D$. Define the “bijection condition” on $D$:

**Bijection condition:** There exists a bijection $F : \mathbb{L}(D) \rightarrow \mathbb{R}(D)$ such that $L$ and $F(L)$ are matched while $L$ and $R$ are not matched if $R \neq F(L)$.
Corollary 4.6. Let $k$ be a field and $M$ a monoid such that for every $D$-class $D$

1. $k[G_D]$ is a semi-simple cell algebra and

2. the bijection condition is satisfied for $D$.

Then $k[M]$ is a semi-simple cell algebra.

Proof. Place the standard cell algebra structure on $k[M]$ as in theorem 3.4. We will use proposition 4.4 to show semi-simplicity. Take $(D, \lambda) \in \Lambda$. We will show $\text{rad} \left( L C^{(D, \lambda)} \right) = 0$ and therefore $L C^{(D, \lambda)} = L D^{(D, \lambda)}$. Take any $Y = \sum_k Y_k \in \text{rad} \left( L C^{(D, \lambda)} \right)$ where $Y_k \in \left( L C^{(D, \lambda)} \right)_k$. Then $\langle X, Y \rangle = 0$ for any $X \in C^{(D, \lambda)}$. If $X \in \left( C^{(D, \lambda)}_R \right)_i$, then $\langle X, Y_k \rangle = 0$ whenever $L_j \neq F(R_k)$ by proposition 4.3. Then we must also have $\langle X, Y_k \rangle = 0$ when $L_j = F(R_k)$. But when $L_j, R_k$ are matched, proposition 4.1 gives $\langle X, Y_k \rangle = \langle \phi_j(X), r_{m(i,j)} \phi_i(Y_k) \rangle_D = 0$. Since $\phi_j(X)$ is an arbitrary basis element in $C^{\lambda}_R$, we have $r_{m(i,j)} \phi_i(Y_k) \in \text{rad}_D \left( L C^{\lambda} \right)$. But since $k[G_D]$ is semi-simple, $\text{rad}_D \left( L C^{\lambda} \right) = 0$ by proposition 4.3. Then $r_{m(i,j)} \phi_i(Y_k) = 0$, and since $r_{m(i,j)} \in G_D$ is invertible, $Y_k = 0$. Since this is true for each $k$, we have $Y = 0$. So $\text{rad} \left( L C^{(D, \lambda)} \right) = 0$. A parallel argument shows that $\text{rad} \left( C^{(D, \lambda)}_R \right) = 0$, so $k[M]$ is semi-simple by proposition 4.3.

Recall that an inverse of an element $a$ in a semi-group is an element $a^{-1}$ such that $aa^{-1}a = a$ and $a^{-1}a^{-1} = a^{-1}$. An inverse semi-group is a semi-group in which each element has a unique inverse. A standard result in semi-group theory is that any inverse semi-group satisfies the bijection condition. (In fact each $L$-class and each $R$-class contains a unique idempotent. Given an $L$-class $L$, the unique matching $R$-class $R = F(L)$ is the class containing the same idempotent as $L$.) Then proposition 4.4 and corollary 4.5 yield

Corollary 4.6. Let $k$ be a field and $M$ a finite monoid which is an inverse semi-group and satisfies the $k$-C.A. condition. Then $k[M]$ is semi-simple if and only if $k[G_D]$ is semi-simple for every $D$-class $D$.

For an inverse semi-group the base $H$-class $H$ for any class $D$ can be chosen to be a (maximal) subgroup of $M$ with $H$ isomorphic to $G_D$. So the condition in corollary 4.4 can be replaced by the requirement that for any maximal subgroup $G$ of $M$ the group algebra $k[G]$ must be a semi-simple cell algebra.

5 Twisted monoid algebras

A twisting on a monoid $M$ (with values in a commutative domain $R$ with unit 1) is a map $\pi : M \times M \to R$ such that (i) for all $x, y, z \in M$,

$$\pi(x, y) \pi(xy, z) = \pi(x, yz) \pi(y, z)$$
and (ii) for all \( x \in M \),
\[
\pi(x, id) = 1 = \pi(id, x)
\]
(where \( id \) is the identity in \( M \)).

Given a twisting \( \pi \) on \( M \), define an algebra \( R^\pi [M] \) to be the free \( R \)-module with basis \( M \) and multiplication \( x \circ y = \pi(x, y) xy \) for \( x, y \in M \). Then \( R^\pi [M] \) is an associative \( R \)-algebra with unit 1.

We would like conditions under which \( R^\pi [M] \) will be a cell algebra. In [11] and [8], Wilcox and Guo and Xi have investigated when \( R^\pi [M] \) can be a cellular algebra. Much of the difficulty in their analyses involves defining the involution anti-isomorphism \( * \) required for a cellular algebra. Since cell algebras don’t require such a map, the corresponding results are both simpler to obtain and of more general applicability. We require one “compatibility” condition for our twisting:

**Definition 5.1.** A twisting \( \pi \) on a monoid \( M \) is compatible if for all \( a, x \in M \)

1. If \( axDx \), then \( \pi(a, y) = \pi(a, x) \) whenever \( y \in H_x \).
2. If \( xaDx \), then \( \pi(y, a) = \pi(x, a) \) whenever \( y \in H_x \).

In [6], \( \pi \) is defined to be an \( LR \)-twisting if \( xLy \Rightarrow \pi(x, z) = \pi(y, z) \) and \( yRz \Rightarrow \pi(x, y) = \pi(x, z) \) for all \( x, y, z \in M \). Clearly any \( LR \)-twisting is compatible.

**Theorem 5.1.** Let \( M \) be a finite monoid satisfying the \( R \)-C.A. condition. Let \( \pi \) be a compatible twisting on \( M \). Then \( R^\pi [M] \) has a cell algebra structure with the same \( \Lambda, R, L \) and cell basis \( C \) as for the standard cell algebra structure on \( R[M] \).

**Proof.** Since as \( R \)-modules \( R^\pi [M] \) and \( R[M] \) are identical, \( C \) is an \( R \)-basis for \( R^\pi [M] \) and we need only check that conditions (i) and (ii) for a cell algebra are satisfied for the new multiplication.

For (i), assume \( a \) is a basis element, \( a \in M \), and take any \( \pi((r,s)C^{(D,\lambda)}_{(L,\iota)}) \in C \).

Then \( \pi((r,s)C^{(D,\lambda)}_{(L,\iota)}) \in R[H] \) is an \( R \)-linear combination of elements in an \( H \)-class \( H = L \cap R \subseteq D \). By corollary 2.2 we have two possibilities: I. \( a \cdot R[H] \subseteq \hat{A}^D = \oplus_{D'<D} R[D'] \) or II. \( a \cdot R[H] \subseteq D \). If I holds, \( a \circ (r,s)C^{(D,\lambda)}_{(L,\iota)} \subseteq \hat{A}^D \subseteq \hat{A}^{(D,\lambda)} \) and we can take the coefficients \( r_L \) required in (i) to be all 0. So assume II. holds. Take an element \( x \in H \) and let \( c = \pi(a, x) \). Since \( ax \in D = D_x \), compatibility gives \( \pi(a, y) = c \) for any \( y \in H \). But then, by linearity, \( a \circ (r,s)C^{(D,\lambda)}_{(L,\iota)} = c \cdot a \circ (r,s)C^{(D,\lambda)}_{(L,\iota)} \). We can then take the coefficients \( r_L \) required in (i) to be just \( c \) times the corresponding coefficients in the cell algebra \( R[M] \).

The proof of condition (ii) is parallel. \( \square \)

Let \( M \) be a finite monoid satisfying the \( R \)-C.A. condition and place the standard cell algebra structure on \( M \). Suppose \( R_i, L_j \) are matched classes in a \( D \)-class \( D \) of \( M \). Then by definition there exist \( x \in L_j, y \in R_i \) such that
For any Proposition 5.2. $xy \in D$. If $\pi$ is a compatible twisting on $M$ define $c(j,i) = \pi(x,y)$. Then $\pi(a,b) = c(j,i)$ for any $a \in L_j, b \in R_i$, so for any $X \in R[L_j], Y \in R[R_i]$ we have $X \circ Y = c(j,i)XY$. It follows that for $X \in (C^{(D,\lambda)}_R)_j, Y \in (LC^{(D,\lambda)})_i$, we have $(X,Y)^\pi = c(j,i)(X,Y)$ where $(X,Y)^\pi$ is the bracket in the cell algebra $R^\pi[M]$.

**Definition 5.2.** A twisting $\pi$ on a monoid $M$ is strongly compatible if for all $a,x \in M$

1. If $axDx$, then $\pi(a,y) = \pi(a,x) \neq 0$ whenever $y \in H_x$,
2. If $xaDx$, then $\pi(y,a) = \pi(x,a) \neq 0$ whenever $y \in H_x$.

So for a strongly compatible twisting we have $c(j,i) \neq 0$ whenever $R_i, L_j$ are matched classes. But $c(j,i) \neq 0$ in the domain $R$ yields $(X,Y)^\pi = 0 \Leftrightarrow (X,Y) = 0$. Using this observation, it is easy to modify the proofs to obtain the following generalizations to twisted monoid algebras of the results in section 4.

In the following, assume $R$ is a domain, $M$ a finite monoid satisfying the $R$-C.A. condition, and $\pi$ a strongly compatible twisting from $M$ to $\pi$. Put the standard cell algebra structures on $R[M]$ and $R^\pi[M]$ as given by theorems 3.1 and 5.1. Let $\Lambda, \Lambda_0, \langle -,- \rangle, \text{rad}$, etc., refer to the cell algebra $R[M]$ and $\Lambda^\pi, \Lambda_0^\pi, \langle -,- \rangle^\pi \text{rad}^\pi$, etc., refer to the cell algebra $R^\pi(M)$. Recall that $\Lambda^\pi = \Lambda$.

**Proposition 5.1.** For any $\lambda \in \Lambda_D$, $D \in \mathbb{D}$:

1. If $\text{rad}_D(LC^\lambda) \neq 0$, then $\text{rad}^\pi(LC^{(D,\lambda)}) \neq 0$,
2. If $\text{rad}_D(D^\lambda_R) \neq 0$, then $\text{rad}^\pi(D^{(D,\lambda)}_R) \neq 0$.

**Proposition 5.2.** For any $D$-class $D$,

(a) If $D^2 \subseteq \hat{A}^D$, then $(D,\lambda) \not\in \Lambda_0^\pi$ for any $\lambda \in \Lambda_D$.
(b) If $D^2 \not\subseteq \hat{A}^D$, then $(D,\lambda) \in \Lambda_0^\pi \Leftrightarrow \lambda \in (\Lambda_D)_0$.

**Proposition 5.3.** Let $R = k$ be a field. Assume that for every class $D \in \mathbb{D}$, $D^2 \not\subseteq \hat{A}^D$ and $k[G_D]$ is a cell algebra with $(\Lambda_D)_0 = \Lambda_D$. Then $A^\pi = k^\pi[M]$ is quasi-hereditary.

**Proposition 5.4.** Let $R = k$ be a field. Assume that $M$ is regular and that for every class $D \in \mathbb{D}$, $k[G_D]$ is a cell algebra with $(\Lambda_D)_0 = \Lambda_D$. Then $A^\pi = k^\pi[M]$ is quasi-hereditary.

**Proposition 5.5.** If $R = k$ is a field and $k^\pi[M]$ is semi-simple, then $k[G_D]$ is semi-simple for every $D \in \mathbb{D}$.

**Proposition 5.6.** Let $R = k$ be a field. Assume that for every $D$-class $D$

1. the cell algebra $k[G_D]$ is a semi-simple and
2. the bijection condition is satisfied for $D$.
Then the cell algebra $k^\pi[M]$ is semi-simple.

**Proposition 5.7.** Let $R = k$ be a field. Assume the monoid $M$ is also an inverse semi-group. Then $k^\pi[M]$ is semi-simple if and only if $k[G_D]$ is semi-simple for every $D$-class $D$.

The various examples such as Brauer algebras, Temperly-Lieb algebras, and other partition algebras which were studied and shown to be cellular in [11] and [8] are all twisted monoid algebras with a compatible twisting on a monoid satisfying the $R$-C.A. condition. Thus they can be seen to be cell algebras by Theorem 5.1 without constructing the anti-isomorphism needed for the cellular structure. Related algebras which lack the anti-isomorphism $\ast$, and hence are not cellular, could also be shown to be cell algebras by Theorem 5.1. We note again that questions such as whether an algebra is quasi-hereditary or semi-simple are not much harder to answer for cell algebras than for cellular algebras.

**References**

[1] Clifford, A.H. and Preston, G.B., The Algebraic Theory of Semigroups, vol. 1, Math. Surveys, vol.7, American Mathematical Society, Providence, 1961.

[2] Cline, E., Parshall, B., and Scott, L., Finite dimensional algebras and highest weight categories, Math. Ann.,259(1982)153-199.

[3] Curtis, Charles W. and Reiner, Irving, Representation theory of finite groups and associative algebras. Reprint of the 1962 original. AMS Chelsea Publishing, Providence, RI, 2006.

[4] East, James, Cellular algebras and inverse semigroups, J. Algebra 296 (2006), no. 2, 505-519.

[5] Ganyushkin, O. and Mazorchuk, V., Classical finite transformation semigroups. An introduction. Algebra and Applications, 9. Springer-Verlag London, Ltd., London, 2009.

[6] Graham, J.J. and Lehrer, G.I., Cellular algebras, Invent. Math. 123(1996) 1-34.

[7] Grillet, P.A., Semigroups, an introduction to the structure theory, vol 193, Monog. Textbooks in Pure and Appl. Math.,Marcel Dekker, NY, 1995.

[8] Guo, Xiaojiang and Xi, Changchang, Cellularity of twisted semigroup algebras, J. Pure Appl. Algebra 213(2009),no.1,71-86.

[9] Howie, John M., Fundamentals of semigroup theory, London Mathematical Society Monographs, New Series 12. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[10] May, Robert, Cell algebras, J. Algebra 425(2015)107-132.
[11] Wilcox, Stewart, Cellularity of diagram algebras as twisted semigroup algebras, J. Algebra 309 (2007), no.1, 10-31.