The Active Bijection

2.a - Decomposition of activities for matroid bases, and Tutte polynomial of a matroid in terms of beta invariants of minors

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Abstract
We introduce and study filtrations of a matroid on a linearly ordered ground set, which are particular sequences of nested sets. A given basis can be decomposed into a uniquely defined sequence of bases of minors, such that these bases have an internal/external activity equal to 1/0 or 0/1 (in the sense of Tutte polynomial activities). This decomposition, which we call the active filtration/partition of the basis, refines the known partition of the ground set into internal and external elements with respect to a given basis. It can be built by a certain closure operator, which we call the active closure. It relies only on the fundamental bipartite graph of the basis and can be expressed also as a decomposition of general bipartite graphs on a linearly ordered set of vertices.

From this, first, structurally, we obtain that the set of all bases can be canonically partitioned and decomposed in terms of such bases of minors induced by filtrations. Second, enumeratively, we derive an expression of the Tutte polynomial of a matroid in terms of beta invariants of minors. This expression refines at the same time the classical expressions in terms of basis activities and orientation activities (if the matroid is oriented), and the well-known convolution formula for the Tutte polynomial. Third, in a companion paper of the same series (No. 2.b), we use this decomposition of matroid bases, along with a similar decomposition of oriented matroids, and along with a bijection in the 1/0 activity case from a previous paper (No. 1), to define the canonical active bijection between orientations/signatures/reorientations and spanning trees/simplices/bases of a graph/real hyperplane arrangement/oriented matroid, as well as various related bijections.

1. Introduction
This paper studies some structural and enumerative properties of matroids on a linearly ordered ground set. We introduce and study filtrations of a matroid on a linearly ordered ground set, which are simple particular sequences of nested subsets of the ground set (Definition 3.1). They induce particular sequences of minors by the following manner: for each subset in the sequence, we consider the minor obtained by restriction to this subset and contraction of the subsets it contains.

A given basis can be decomposed into a uniquely defined sequence of bases of such minors (Theorem 4.22), such that these bases have an internal/external activity equal to 1/0 or 0/1, in the sense of Tutte polynomial activities, as introduced by Tutte in [18]. This decomposition can...
be seen as a partition that refines the known partition of the ground set into internal and external elements with respect to a given basis, as defined by Etienne and Las Vergnas in [3]. We call this unique special filtration/partition the active filtration/partition of the basis.

From a constructive viewpoint, it can be built by applying a certain closure operator, which we call the active closure, to the internally/externally active elements of the basis, by several equivalent possible manners which are detailed in the paper (including notably a simple single pass over the ground set). This construction only relies upon the fundamental bipartite graph of the basis and can be also expressed as a decomposition of bipartite graphs on a linearly ordered set of vertices.

At a global level, we obtain that the set of all bases can be canonically partitioned and decomposed in terms of such uniactive internal/external bases of minors induced by all filtrations, which is the main result of the paper (Theorem 4.25).

As the enumerative counterpart of the above structural decomposition theorem, we derive an expression of the Tutte polynomial of a matroid in terms of beta invariants of minors (Theorem 3.5):

$$t(M; x, y) = \sum_{1 \leq k \leq \epsilon} \left( \prod_{1 \leq k \leq \iota} \beta(M(F_k)/F_{k-1}) \right) \left( \prod_{1 \leq k \leq \varepsilon} \beta^*(M(F'_{k-1})/F'_{k}) \right) x^\iota y^\varepsilon$$

where $\beta^*$ equals $\beta$ of the dual (that is $\beta$ except for an isthmus or a loop), and where the sum is over all (connected) filtrations $\emptyset = F'_0 \subset \ldots \subset F'_i = F_i = F_0 \subset \ldots \subset F_i = E$ of $M$. The beta invariant $\beta(M)$ of a matroid $M$ is equal of the coefficient of $x$ in the Tutte polynomial $t(M; x, y)$. It was specifically considered and so named by Crapo in [1]. In particular, it counts the number of bases having an internal/external activity equal to $1/0$ (or also $0/1$ as soon as the matroid has at least two elements) with respect to any linear ordering of the ground set. It also remarkably counts the number of bounded regions of a real hyperplane arrangement (bipolar orientations in digraphs), as shown by Zaslavsky in [19] and generalized to oriented matroids by Las Vergnas in [14] (see also [10], and see [7, 8] for the connection with bases, or [5, 9] in graphs).

The above expression of the Tutte polynomial in terms of beta invariants of minors thus refines at the same time the following known Tutte polynomial formulas:

- The classical expression of the Tutte polynomial of a matroid in terms of basis activities, given by Tutte in [18] and extended to matroids by Crapo in [2] (recalled in Section 2 as the “enumeration of basis activities” formula). Indeed, by this classical expression, each coefficient of the Tutte polynomial counts the number of bases with given internal/external activity. By the above expression, each coefficient of the Tutte polynomial is decomposed further in terms of numbers of bases of minors with internal/external activity equal to $1/0$ or $0/1$ (see also Theorem 4.25 and the proof of Theorem 3.5 at the very end of the paper).

- The expression of the Tutte polynomial of an oriented matroid in terms of orientation activities, given by Las Vergnas in [15] (recalled in [8, Section 2] as the “enumeration of reorientation activities” formula). Indeed, by this expression, each coefficient of the Tutte polynomial amounts to count the number of reorientations with given dual/primal orientation activity. By the above expression, each coefficient of the Tutte polynomial is decomposed further in terms of numbers of reorientations of minors with dual/primal orientation activity equal to $1/0$ or $0/1$, that is in terms of numbers of bounded regions in minors of the primal and the dual with respect to a topological representation of the oriented matroid. See [8] for details, notably [8, Theorem 4.6 and Remark 4.7].
The convolution formula for the Tutte polynomial, recalled here as Corollary 3.6, so named by Kook, Reiner and Stanton in [11]. This formula was implicit in [3], as it is a direct enumerative corollary of the structural decomposition of the set of bases into bases of minors with internal/external activity equal to zero, given by Etienne and Las Vergnas in [3] (recalled here as Corollaries 4.14 and 4.27). One retrieves this formula from the above by considering only the subsets $F_c$ in the filtrations. It expresses the Tutte polynomial in terms of Tutte polynomials of minors where either the variable $x$ or the variable $y$ is set to zero. By the above expression, each Tutte polynomial of a minor involved in the convolution formula is further decomposed by means of a sequence of minors, thus using only the beta invariant of these minors (that is only the monomials $x$ or $y$ of the Tutte polynomial of these minors).

Let us mention that an algebraic proof of the expression of the Tutte polynomial in terms of beta invariants of minors of Theorem 3.5 could be obtained using the algebra of matroid set functions, a technique introduced by Lass in [12], according to its author [13].

Finally, in the companion paper [8], No. 2.b of the same series, we use the above structural decomposition theorem of matroid bases (Theorem 4.25), along with a similar decomposition of oriented matroids (namely [8, Theorem 4.6]), and along with a bijection in the 1/0 activity case from a previous paper, No. 1 [7] (recalled in [8, Section 5]), to define the canonical active bijection between orientations/signatures/reorientations and spanning trees/simplices/bases of a graph/real hyperplane arrangement/oriented matroid, as well as related bijections.

In brief, the active bijection for graphs, real hyperplane arrangements and oriented matroids (in order of increasing generality) is a framework introduced and studied in a series of papers by the present authors. The canonical active bijection associates an oriented matroid on a linearly ordered ground set with one of its bases. This defines an activity preserving correspondence between reorientations and bases of an oriented matroid, with numerous related bijections, constructions and characterizations. It yields notably a structural and bijective interpretation of the equality of the two expressions of the Tutte polynomial alluded to above: “enumeration of basis activities” by Tutte [18] and “enumeration of reorientation activities” by Las Vergnas [15].

The idea of decomposing matroid bases developed in the present paper has been initiated by an algorithm by Las Vergnas in [16] (given in graphs without proof, and allegedly yielding a correspondence between orientations and spanning trees, different from the active bijection however, see [8, footnote 1]). Most of the main results in this series (including the present paper) were given in the Ph.D. thesis [4] in a preliminary form. A short summary of the whole series (including the above Tutte polynomial formula) has been given in [6]. In the present paper, we will refer only to the journal papers [5, 7, 8] of this series, the reader may see the companion paper [8] for a complete overview and for further references from the authors and from the literature.

The reader primarily interested in graph theory may also read [9], that gives a complete overview of the active bijection in the language of graphs (in contrast with other papers of the series), as well as a proof of the above Tutte polynomial expression in terms of beta invariants of minors by means of decomposing graph orientations (as done in [8] for oriented matroids), instead of decomposing bases/spanning trees (as done in the present paper for matroids). This is possible in graphs since they are orientable, but this is not possible in non-orientable matroids.
2. Preliminaries

**Generalities.**

In the paper, \(\subseteq\) denotes the inclusion, \(\subset\) denotes the strict inclusion, and \(\psi\) (or \(+\)) denotes the disjoint union. Usually, \(M\) denotes a matroid on a finite set \(E\). See [17] for a complete background on matroid theory, notably see [17, Chapter 5] for the translation in terms of graphs, and [17, Chapter 6] for the translation in terms of representable matroids, point configurations or real hyperplane arrangements. A matroid \(M\) on \(E\) can be called ordered when the set \(E\) is linearly ordered. Then, the dual \(M^*\) of \(M\) is ordered by the same ordering on \(E\). A minor \(M/\{e\}\), resp. \(M\{e\}\), for \(e \in E\), can be denoted for short \(M/e\), resp. \(M\cdot e\). A matroid can be called loop, or isthmus, if it has a unique element and this unique element is a loop (\(M = U_{1,1}\)), or an isthmus (\(M = U_{1,0}\)), respectively. An isthmus is also called a coloop in the literature.

Let us first recall some usual matroid notions. A flat \(F\) of \(M\) is a subset of \(E\) such that \(E \setminus F\) is a union of cocircuits; equivalently: if \(C \setminus \{e\} \subseteq F\) for some circuit \(C\) and element \(e\), then \(e \in F\); and equivalently: \(M/F\) has no loop. A dual-flat \(F\) of \(M\) is a subset of \(E\) which is a union of circuits; equivalently: its complement is a flat of the dual matroid \(M^*\); equivalently: if \(D \setminus \{e\} \subseteq E \setminus F\) for some cocircuit \(D\) and element \(e\), then \(e \in E \setminus F\); and equivalently: \(M(F)\) has no isthmus. A cyclic-flat \(F\) of \(M\) is both a flat and a dual-flat of \(M\); equivalently: \(F\) is a flat and \(M(F)\) has no isthmus; or equivalently: \(M/F\) has no loop and \(M(F)\) has no isthmus.

**Activities of matroid bases.**

Let \(M\) be an ordered matroid on \(E\), and let \(B\) be a basis of \(M\). For \(b \in B\), the fundamental cocircuit of \(b\) with respect to \(B\), denoted \(C^*_{M}(B;b)\), or \(C^*(B;b)\) for short, is the unique cocircuit contained in \((E \setminus B) \cup \{b\}\). For \(e \notin B\), the fundamental circuit of \(e\) with respect to \(B\), denoted \(C_{M}(B;e)\), or \(C(B;e)\) for short, is the unique circuit contained in \(B \cup \{e\}\). Let

\[
\text{Int}(B) = \left\{ b \in B \mid b = \min \left( C^*(B;b) \right) \right\},
\]

\[
\text{Ext}(B) = \left\{ e \in E \setminus B \mid e = \min \left( C(B;e) \right) \right\}.
\]

We might add a subscript as \(\text{Int}_M(B)\) or \(\text{Ext}_M(B)\) when necessary. The elements of \(\text{Int}(B)\), resp. \(\text{Ext}(B)\), are called internally active, resp. externally active, with respect to \(B\). The cardinality of \(\text{Int}(B)\), resp. \(\text{Ext}(B)\), is called internal activity, resp. external activity, of \(B\). We might write that a basis is \((i,j)\)-active when its internal and external activities equal \(i\) and \(j\), respectively. Observe that \(\text{Int}(B) \cap \text{Ext}(B) = \emptyset\) and that, for \(p = \min(E)\), we have \(p \in \text{Int}(B) \cup \text{Ext}(B)\).

Moreover, let \(B_{\min}\) be the smallest (lexicographic) base of \(M\). Then, as well-known and easy to prove, we have \(\text{Int}(B_{\min}) = B_{\min}\), \(\text{Ext}(B_{\min}) = \emptyset\), and \(\text{Int}(B) \subseteq B_{\min}\) for every base \(B\). Also, let \(B_{\max}\) be the greatest (lexicographic) base of \(M\). Then \(\text{Int}(B_{\max}) = \emptyset\), \(\text{Ext}(B_{\max}) = E \setminus B_{\max}\), and \(\text{Ext}(B) \subseteq E \setminus B_{\max}\) for every base \(B\). Thus, roughly, internal/external activities can be thought of as situating a basis with respect to \(B_{\min}\) and \(B_{\max}\). Finally, we recall that internal and external activities are dual notions:

\[
\text{Int}_M(B) = \text{Ext}_M^*(E \setminus B) \quad \text{and} \quad \text{Ext}_M(B) = \text{Int}_M^*(E \setminus B).
\]

By [18, 2], the Tutte polynomial of \(M\) is

\[
t(M;x,y) = \sum_{i,j} b_{i,j} x^i y^j
\]

(“enumeration of basis activities”)
where \( b_{\iota, \varepsilon} \) is the number of bases of \( M \) with internal activity \( \iota \) and external activity \( \varepsilon \). It does not depend on the linear ordering of \( E \).

Now, given a basis \( B \) of \( M \), if \( \text{Int}(B) = \emptyset \), resp. \( \text{Ext}(B) = \emptyset \), then \( B \) is called \textit{external}, resp. \textit{internal}. If \( \text{Int}(B) \cup \text{Ext}(B) = \{p\} \) then \( B \) is called \textit{uniactive}. Hence, a base with internal activity 1 and external activity 0 can be called uniactive internal, and a base with internal activity 0 and external activity 1 can be called uniactive external. Let us mention that exchanging the two smallest elements of \( E \) yields a canonical bijection between uniactive internal and uniactive external bases, see [7, Proposition 5.1 up to a typing error\(^1\)], see also [5, Section 4] in graphs. See the beginning of Section 4 for a reformulation of the characterization of uniactive internal/external bases (see also [5, Proposition 2] for another characterization, not used in the paper).

In particular, by the above formula, we have that \( b_{1, 0} \) counts the number of uniactive internal bases. This number does not depend on the linear ordering of the element set \( E \). This value

\[
\beta(M) = b_{1, 0}
\]

is known as the \textit{beta invariant} of \( M \) [1]. Assuming \(| E | > 1 \), it is known that \( \beta(M) \neq 0 \) if and only if \( M \) is connected. Let us recall that, for a loopless graph \( G \) with at least three vertices, the associated matroid \( M(G) \) is connected if and only if \( G \) is 2-connected. Also, we have \( \beta(M) = b_{1, 0} = b_{0, 1} = \beta(M^*) \) as soon as \(| E | > 1 \). Note that, assuming \(| E | = 1 \), we have \( \beta(M) = 1 \) if the single element is an isthmus of \( M \), and \( \beta(M) = 0 \) if the single element is a loop of \( M \).

Finally, for our constructions, we need to introduce the following dual slight variation \( \beta^* \) of \( \beta \):

\[
\beta^*(M) = \beta(M^*) = b_{0, 1} = \begin{cases} 
\beta(M) & \text{if } |E| > 1 \\
0 & \text{if } M \text{ is an isthmus} \\
1 & \text{if } M \text{ is a loop.}
\end{cases}
\]

**Fundamental bipartite graph/tableau settings.**

Observe that the above definitions for a basis \( B \) of an ordered matroid \( M \) only rely upon the fundamental circuits/cocircuits of the basis, not on the whole structure \( M \). In the paper, we develop a combinatorial construction that also only depends on this local data, and thus can be naturally expressed in terms of general bipartite graphs on a linearly ordered set of vertices. So let us introduce the following definitions and representations. This is rather formal but necessary.

We call \textit{(fundamental) bipartite graph} \( \mathcal{F} \) on \((B, E \setminus B)\) a bipartite graph on a set of vertices \( E \), which is bipartite w.r.t. a couple of subsets of \( E \) forming a bipartition \( E = B \uplus E \setminus B \). We call \textit{(fundamental) tableau} \( \mathcal{F} \) on \((B, E \setminus B)\) a matrix whose rows and columns are indexed by \( E \), with entries in \( \{\bullet, 0\} \), and such that each diagonal element indexed by \((e, e), e \in E\), is non-zero and, moreover, is the only non-zero entry of its row (when \( e \in B \)), or the only non-zero entry of its column (when \( e \in E \setminus B \)). We use the same notation \( \mathcal{F} \) for a bipartite graph or a tableau since, obviously, bipartite graphs and tableaux are equivalent structures: each non-diagonal entry of the tableau represents an edge of the corresponding bipartite graph. We choose to define both

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\(^1\)Let us correct here an unfortunate typing error in [7, Proposition 5.1 and Theorem 5.3]. The statement has been given under the wrong hypothesis \( B_{\min} = \{p < p' < \ldots\} \) instead of the correct one \( E = \{p < p' < \ldots\} \). Proofs are unchanged (independent typo: in line 10 of the proof of Proposition 5.1, instead of \( B' - f \), read \((E \setminus B') \setminus \{f\}\)). In [5, Section 4], the statement of the same property in graphs is correct.
because graphs are the underlying compact combinatorial structure, whereas tableaux are better for visualization, notably for signs of the fundamental circuits/cocircuits in the oriented matroid case developed in the companion paper [8], and they are consistent with the matrix representation used in the linear programming setting of the active bijection developed in [7]. In what follows (and in [8] too), examples will be illustrated on both representations.

Given a basis $B$ of a matroid $M$ on $E$, the fundamental graph/tableau of $B$ in $M$, denoted $F_M(B)$ is the usual (fundamental) bipartite graph/tableau on $(B, E \setminus B)$ such that, for every $b \in B$, $b$ is adjacent to elements of $C^*(B; b) \setminus \{b\}$, and for every $e \in E \setminus B$, $e$ is adjacent to elements of $C(B; e) \setminus \{e\}$. Recall that

$$e \in C^*(B; b) \text{ if and only if } b \in C(B; e).$$

In fact, every bipartite graph on $(B, E \setminus B)$ is the fundamental graph of some basis $B$ in some matroid $M$ on $E$: one just has to choose $B$ as a vector space basis, and settle elements $e$ of $E \setminus B$ in general position in the subspaces spanned by the elements of $B$ adjacent to $e$ (isthmuses correspond to isolated vertices in $B$ and loops correspond to isolated vertices in $E \setminus B$).

Observe that matroid duality comes down to exchange the roles of $B$ and $E \setminus B$, that is to exchange the two parts of the bipartition of $E$ (in the bipartite graph setting), or to transpose the matrix (in the tableau setting). Precisely, for a bipartite graph/tableau $\mathcal{F}$ on $(B, E \setminus B)$, we define the dual $\mathcal{F}^*$ of $\mathcal{F}$ as the bipartite graph/tableau on $(E \setminus B, B)$ with same edges/transposed values w.r.t. $\mathcal{F}$. Obviously, for a basis of a matroid $M$, we have $\mathcal{F}_{M^*}(E \setminus B) = (\mathcal{F}_M(B))^*$.

Thus, the usual definitions and notations given above can be directly extended to bipartite graphs: for $b \in B$, $C^*(B; b)$ is the set of elements adjacent to $b$, plus $b$; for $e \in E \setminus B$, $C(B; e)$ is the set of elements adjacent to $e$, plus $e$; and, assuming $E$ is linearly ordered, an element is internally, resp. externally, active if it is in $B$ and it is the smallest element of $C^*(B; b)$, resp. the smallest

Figure 1: For the base 256 of the depicted matroid of $K_4$ with ground set $1 < \cdots < 6$, we have $\text{Int}(256) = \emptyset$ and $\text{Ext}(256) = \{1, 3\}$. On the left: a graph representation. In the middle: a hyperplane arrangement representation (we represent $\min(E)$ as a hyperplane at infinity, and we only represent one half of the arrangement, on a given side of $\min(E)$, see [7, Section 2] for more details on such representations); and the vertices associated with fundamental cocircuits of the basis. On the upper right and the bottom right, respectively: the fundamental bipartite graph and the fundamental tableau of the basis (see last part of Section 2).
element of \( C(B; e) \). Similarly, those definitions translate in the tableau setting: for \( b \in B \), \( C^*(B; b) \) is defined by the non-zero entries of the column indexed by \( b \), or column \( b \) for short; for \( e \in E \setminus B \), \( C(B; e) \) is defined by the non-zero entries of the row indexed by \( e \), or row \( e \) for short; and, assuming \( E \) is linearly ordered, an element is internally, resp. externally, active if its corresponding diagonal element is the smallest non-zero entry of its column, resp. its row. Then we can directly extend the notations \( \text{Int} \) and \( \text{Ext} \), and the relative definitions, to those settings.

Finally, for \( A \subseteq E \), we define \( F - A \) as the bipartite graph/tableau obtained by removing all vertices (and their incident edges)/lines in \( A \) from \( F \). For an element \( e \in E \), we can denote \( F - e \) instead of \( F - \{e\} \).

**Example 2.1.** An example of a matroid basis, its internal/external activities, its (fundamental) bipartite graph and its (fundamental) tableau is given in Figure 1. Internal/external activities for all bases of this example are listed in Figure 6 at the end of the paper.

3. Filtrations of an ordered matroid, and Tutte polynomial in terms of beta invariants of minors induced by filtrations

First, we introduce filtrations of a matroid on a linearly ordered ground set, which are particular increasing sequences of subsets of the ground set and which will be continuously used throughout the paper. Then, we introduce a formula for the Tutte polynomial of a matroid in terms of beta invariants of minors induced by filtrations. Its proof will be given at the very end of the paper, as a consequence of the structural decomposition of matroid bases with respect to basis activities, developed in the next section. Let us mention that, in the particular case of oriented matroids (or real hyperplane arrangements, or graphs, whose associated matroids are all orientable), this formula can be equally proved using a decomposition of oriented matroids with respect to orientation activities, using the same filtrations, see [8] (or [9] in graphs).

**Definition 3.1.** Let \( E \) be a linearly ordered finite set. Let \( M \) be a matroid on \( E \). We call filtration of \( M \) (or \( E \)) a sequence \((F'_\varepsilon, \ldots, F'_0, F_c, F_0, \ldots, F_\iota)\) of subsets of \( E \) such that:

- \( \emptyset = F'_\varepsilon \subset \ldots \subset F'_0 = F_c = F_0 \subset \ldots \subset F_\iota = E \);
- the sequence \( \min(F_k \setminus F_{k-1}) \), \( 1 \leq k \leq \iota \) is increasing with \( k \);
- the sequence \( \min(F'_{k-1} \setminus F'_k) \), \( 1 \leq k \leq \varepsilon \), is increasing with \( k \).

The sequence is a connected filtration of \( M \) if, in addition:

- for \( 1 \leq k \leq \iota \), the minor \( M(F_k)/F_{k-1} \) is connected and is not a loop;
- for \( 1 \leq k \leq \varepsilon \), the minor \( M(F'_k)/F'_k \) is connected and is not an isthmus.

In what follows, we can equally use the notations \((F'_\varepsilon, \ldots, F'_0, F_c, F_0, \ldots, F_\iota)\) or \( \emptyset = F'_\varepsilon \subset \ldots \subset F'_0 = F_c = F_0 \subset \ldots \subset F_\iota = E \) to denote a filtration of \( M \). The \( \iota + \varepsilon \) minors involved in Definition 3.1 are said to be associated with or induced by the filtration. The subset \( F_c \) will be called the cyclic-flat of the filtration when it is connected (a term justified by Lemma 3.3 below). Observe that filtrations of \( M \) are equivalent to pairs of partitions of \( M \) formed by a bipartition obtained from the subset \( F_c \) (with possibly one empty part, which is a slight language abuse) and a refinement of this bipartition:

\[
E = F_c \uplus E \setminus F_c,
\]
Indeed, one can retrieve the sequence of nested subsets from the pair of partitions since the subsets in the sequence are unions of parts given by the ordering of the smallest elements of the parts.

The next Lemma 3.2 is used in the Tutte polynomial formula below.

**Lemma 3.2.** Let $M$ be an ordered matroid on $E$. A filtration $\emptyset = F'_{\varepsilon} \subset \ldots \subset F'_0 = F_c = F_0 \subset \ldots \subset F_i = E$ of $M$ is a connected filtration of $M$ if and only if

$$
\left( \prod_{1 \leq k \leq \varepsilon} \beta(M(F_k)/F_{k-1}) \right) \left( \prod_{1 \leq k \leq \varepsilon} \beta^*(M(F'_k)/F'_k) \right) \neq 0.
$$

**Proof.** The result is direct. For a matroid $M$ with at least two elements, we have $\beta(M) \neq 0$ if and only if $M$ is connected, and, according to Section 2, we have $\beta(M) = \beta^*(M)$. Moreover, we have $\beta(M) = 0$ if $M$ is an isthmus, and $\beta(M) = 0$ and $\beta^*(M) = 1$ if $M$ is a loop. 

We give the next Lemma 3.3 for the intuition and information, but it is not practically used thereafter.

**Lemma 3.3.** A connected filtration $(F'_{\varepsilon}, \ldots, F'_0, F_c, F_0, \ldots, F_i)$ of an ordered matroid $M$ satisfies:

- for every $0 \leq k \leq \iota$, the subset $F_k$ is a flat of $M$;
- for every $0 \leq k \leq \varepsilon$, the subset $F'_k$ is a dual-flat of $M$;
- the subset $F_c$ is a cyclic-flat of $M$.

**Proof.** Assume there exists $k, 1 \leq k \leq \iota$, such that $F_{k-1}$ is not a flat. By definition, there exists an element $e$ and a circuit $C$ of $M$ such that $e \not\in F_{k-1}$ and $C \setminus \{e\} \subseteq F_{k-1}$. Let $j$ be the largest integer such that $e \not\in F_{j-1}$. We have $j \geq k$, $C \setminus \{e\} \subseteq F_{j-1}$ since $j \geq k$, $e \not\in F_{j-1}$, and $e \in F_j$ by maximality of $j$. So, $C \setminus F_{j-1} = \{e\}$ is a circuit of $M(F_j)/F_{j-1}$, that is $e$ is a loop of $M(F_j)/F_{j-1}$, contradiction.

Dually, assume there exists $k, 1 \leq k \leq \varepsilon$, such that $F'_{k-1}$ is not a dual-flat. By definition, there exists an element $e$ and a cocircuit $D$ of $M$ such that $e \in F'_{k-1}$ and $D \setminus \{e\} \subseteq E \setminus F'_{k-1}$. Let $j$ be the largest integer such that $e \in F'_j$. We have $j \geq k$, $D \setminus \{e\} \subseteq E \setminus F'_j$ since $j \geq k$, $e \in F'_j$, and $e \not\in F'_j$ by maximality of $j$. So, $D \cap F'_{j-1} = \{e\}$ is a cocircuit of $M'(F_{j-1})/F'_j$, that is $e$ is an isthmus of $M'(F_{j-1})/F'_j$, contradiction.

Finally $F_c = F = 0 = F'_0$ is a cyclic flat as it is both a flat and a dual-flat.

**Observation 3.4.** Let $\emptyset = F'_{\varepsilon} \subset \ldots \subset F'_0 = F_c = F_0 \subset \ldots \subset F_i = E$ be a connected filtration of an ordered matroid $M$. We have the following properties.

- $\emptyset = E \setminus F_i \subset \ldots \subset E \setminus F_0 \subset E \setminus F_0 \subset \ldots \subset E \setminus F'_\varepsilon = E$ is a connected filtration of $M^*$, for the cyclic-flat $E \setminus F_c$ of $M^*$.
- The minors associated with the above filtration of $M^*$ are the duals of the minors associated with the above filtration of $M$. That is, precisely: for every $1 \leq k \leq \iota$,

$$
(M(F_k)/F_{k-1})^* = M^*(E \setminus F_{k-1})/(E \setminus F_k),
$$

and for every $1 \leq k \leq \varepsilon$,

$$
(M(F'_k)/F'_k)^* = M^*(E \setminus F'_k)/(E \setminus F'_{k-1}).
$$
Corollary 3.6. Let $M$ be a matroid on a linearly ordered set $E$. We have

$$t(M; x, y) = \sum \left( \prod_{1 \leq k \leq \ell} \beta(M(F_k)/F_{k-1}) \right) \left( \prod_{1 \leq k \leq \ell} \beta^*(M(F'_k)/F'_{k}) \right) x^t y^\varepsilon$$

where the sum can be equally:

- over all connected filtrations $\emptyset = F'_\varepsilon \subset \ldots \subset F'_0 = F'_e = F_0 \subset \ldots \subset F_e = E$ of $M$;
- or over all filtrations $\emptyset = F''_\varepsilon \subset \ldots \subset F''_0 = F''_e = F_0 \subset \ldots \subset F_e = E$ of $E$.

The fact that the sum in Theorem 3.5 can be equally made over the two types of sequences directly comes from Lemma 3.2: non-zero terms in the second sum correspond to connected filtrations. The proof that the sum yields the Tutte polynomial is postponed at the very end of Section 4, since it is derived from the main result of the paper, namely Theorem 4.25. See the introduction of the paper for comments on how the Tutte polynomial formula given in Theorem 3.5 refines other known formulas. Let us detail in the corollary below how Theorem 3.5 refines the convolution formula for the Tutte polynomial.

Corollary 3.6 ([3, 11]). Let $M$ be a matroid. We have

$$t(M; x, y) = \sum t(M/F_c; x, 0) t(M(F_c); 0, y)$$

where the sum can be either over all subsets $F_c$ of $E$, or over all cyclic flats $F_c$ of $M$.

Proof. By fixing $y = 0$ in Theorem 3.5, we get

$$t(M; x, 0) = \sum \left( \prod_{1 \leq k \leq \ell} \beta(M(F_k)/F_{k-1}) \right) x^t$$

where the sum is over all (connected) filtrations where the subset $F_c$ satisfies $F_c = \emptyset$, that is of the type $\emptyset = F'_\varepsilon = F'_c = F_0 \subset \ldots \subset F_e = E$ of $M$. By fixing $x = 0$, we get

$$t(M; 0, y) = \sum \left( \prod_{1 \leq k \leq \ell} \beta^*(M(F'_k)/F'_{k}) \right) y^\varepsilon$$

where the sum is over all (connected) filtrations where the subset $F_c$ satisfies $F_c = E$, that is of the type $\emptyset = F''_\varepsilon \subset \ldots \subset F''_0 = F''_c = F_0 = E$ of $M$. Then, by decomposing the sum in Theorem 3.5 as $\sum_{F_c} \sum_{i,j} \Pi_{1 \leq k \leq \ell} \Pi_{1 \leq k \leq \ell} \ldots$, and by the fact that connected filtrations of $M/F_c$ and $M(F_c)$ are directly induced by that of $M$, as shown in Observation 3.4, we get the formula $t(M; x, y) = \sum t(M/F_c; x, 0) t(M(F_c); 0, y)$ where the sum is over all cyclic flats $F_c$ of $M$. If $F_c$ is not a cyclic flat, then either $M/F_c$ has a loop or $M(F_c)$ has an isthmus, implying that the corresponding term in the sum equals zero. \qed
4. Decomposition of matroid bases into uniactive internal/external bases of minors (and underlying decomposition of a general bipartite graph)

We begin with giving some properties of the fundamental graph \( F_M(B) \) of a basis \( B \) in a matroid \( M \). Next, we define an active closure operation that can be applied on such a fundamental graph, and in fact on any (fundamental) bipartite graph/tableau (see end of Section 2), as it depends only on this local graph, not on the whole matroid structure. Next, we give a few useful combinatorial lemmas to characterize or to build this operation, they also only rely on the bipartite graph structure. Then, we essentially apply this operation in a matroid setting to build decompositions of a matroid basis. First, we recall and develop a decomposition into two so-called internal and external bases of minors, a construction introduced in [3]. Finally, we build a decomposition, that refines the above one, into a sequence of uniactive internal/external bases of minors, in terms of connected filtrations introduced in Section 3, yielding by the way a proof of Theorem 3.5.

Let us first recall that, from Section 2, given a linearly ordered set \( E \), a bipartite graph/tableau \( F \) on \( (B, E \setminus B) \), or equivalently a basis \( B \) of a matroid \( M \) on \( E \) with fundamental graph \( F = F_M(B) \), is uniactive when the following property holds: for all \( b \in B \setminus \text{min}(E) \), we have \( b \neq \text{min}(C^*_M(B; b)) \), that is \( \text{min}(C^*_M(B; b)) \in E \setminus B \), and, moreover, for all \( e \in (E \setminus B) \setminus \text{min}(E) \), we have \( e \neq \text{min}(C_M(B; e)) \), that is \( \text{min}(C_M(B; e)) \in B \). Then, under these conditions, it is internal, resp. external, if \( \text{min}(E) \) is internally active, that is \( \text{min}(E) \in B \), resp. if \( \text{min}(E) \) is externally active, that is \( \text{min}(E) \in E \setminus B \).

**Property 4.1.** Let \( B \) be a basis of a matroid \( M \) on \( E \). For \( b \in B \), we have

\[
F_M(B) - b = F_{M/b}(B - b).
\]

For \( e \in E \setminus B \), we have

\[
F_M(B) - e = F_{M\setminus e}(B).
\]

**Property 4.2.** Let \( B \) be a basis of a matroid \( M \) on \( E \). Let \( F \subseteq E \). The following properties are equivalent:

(i) \( B \cap F \) is a basis of \( M(F) \);
(ii) \( B \setminus F \) is a basis of \( M/F \);
(iii) for all \( b \in B \setminus F \), we have \( C^*_M(B; b) \cap F = \emptyset \);
(iv) for all \( e \in F \setminus B \), we have \( C_M(B; e) \subseteq F \).

If the above properties are satisfied, we have:

\[
F_M(B) - F = F_{M/F}(B \setminus F);
\]

\[
F_M(B) - (E \setminus F) = F_{M(F)}(B \cap F).
\]

Moreover, if both \( F \subseteq E \) and \( G \subseteq E \) satisfy the above properties, and \( F \subseteq G \), then \( B \cap (G \setminus F) \) is a basis of \( M(G)/F \), and

\[
F_M(B) - ((E \setminus F) \cup G) = F_{M(G)/F}(B \cap (G \setminus F)).
\]
Observation 4.4. As noted previously, the parts $B$ and $E \setminus B$ of $\mathcal{F}$ play dual parts, as well as internally and externally active elements. The definition of the active closure is consistent with this duality as we directly have that: if $X \subseteq \text{Int}(\mathcal{F})$ then $X \subseteq \text{Ext}(\mathcal{F}^*)$ and
\[
\text{acl}_{\mathcal{F}}(X) = \text{acl}_{\mathcal{F}^*}(X).
\]
Note that the lemmas that follow are given in terms of internally active elements, but they can be stated dually as well, for externally active elements. We will focus on internally active elements and simply use duality to extend results.

We give Lemma 4.5 below for consistency with the definition given in [5, Section 5].

**Lemma 4.5.** Let $E$ be a linearly ordered set. Let $F$ be a bipartite graph/tableau on $(B, E \setminus B)$ (or equivalently let $B$ be a basis of a matroid $M$ on $E$ with fundamental graph $F$). For $X \subseteq E$, let

$$\text{acl}(X) = X \cup \left( \bigcup_{b \in X \cap B} C^*(B; b) \right) \cup \left\{ b \in B \setminus X \mid \emptyset \subseteq C^*(B; b) \subseteq X \right\}.$$

Then, for $X \subseteq \text{Int}(F)$, we have

$$\text{acl}(X) = \bigcup_{i \geq 1} \text{acl}^i(X).$$

**Proof.** It is a direct reformulation of Definition 4.3. \hfill \□

The two next lemmas could be used as alternative definitions of the active closure. They are easy reformulations, and useful from a constructive viewpoint.

**Lemma 4.6.** Let $E = e_1 < \ldots < e_n$ be a linearly ordered set. Let $F$ be a bipartite graph/tableau on $(B, E \setminus B)$ (or equivalently let $B$ be a basis of a matroid $M$ on $E$ with fundamental graph $F$). Let $X \subseteq \text{Int}(F)$. Then $\text{acl}(X)$ is given by the following definition (yielding a linear algorithm).

For all $1 \leq i \leq n$:

- if $e_i \in X$ then $e_i \in \text{acl}(X)$;
- if $e_i \in B$ is not internally active
  - and if all $c \in C^*(B; e_i)$ with $c < e_i$ satisfies $c \in \text{acl}(X)$, then $e_i \in \text{acl}(X)$;
  - if $e_i \not\in B$ and there exists $c \in C(B; e_i)$ with $c < e_i$ and $c \in \text{acl}(X)$ then $e_i \in \text{acl}(X)$;
- in every other case, $e_i \not\in \text{acl}(X)$.

**Proof.** Let $1 \leq i \leq n$. We analyze under which condition the element $e_i$ belongs to $\text{acl}(X)$. If $e_i \in X$ then $e_i \in \text{acl}(X)$ directly by definition. Let $e_i \in B \setminus X$. If $e_i$ is internally active, then $C^*(B; e_i) = \emptyset$, hence $e_i \not\in \text{acl}(X)$, by definition. Assume $e_i$ is not internally active. We have $e_i \in \text{acl}(X)$ if and only if $C^*(B; e_i) \subseteq \text{acl}(X)$, that is if and only if, for all $c \in C^*(B; e_i)$, we have $c \in \text{acl}(X)$, which is the condition given in the algorithm. Now let $e_i \not\in B$. Using the definition given in Lemma 4.5, we have $e_i \in \text{acl}(X)$ if and only if $e_i$ is added to $\text{acl}(X)$ by $\text{acl}^j(X)$ for some (minimal) $j$, $e_i$ being an element of $C^*(B; c)$ for some $c \in \text{acl}^{j-1}(X) \cap B$. Such a $c$ satisfies $c < e$, since $C^*(B; c) \subseteq \text{acl}^{j-1}(X)$ and $e_i \not\in \text{acl}^{j-1}(X)$. And it satisfies $c \in C(B; e_i)$, as this property is equivalent to $e_i \in C^*(B; c)$. So we have that $e_i \in \text{acl}(X)$ if and only if there exists $c \in C(B; e_i)$, with $c < e_i$ and $c \in \text{acl}(X)$. \hfill \□

**Lemma 4.7.** Let $E$ be a linearly ordered set. Let $F$ be a bipartite graph/tableau on $(B, E \setminus B)$ (or equivalently let $B$ be a basis of a matroid $M$ on $E$ with fundamental graph $F$). Assume $E = e_1 < \cdots < e_n$. Let $X \subseteq \text{Int}(F)$. Then $\text{acl}(X)$ is given by the following algorithmic definition.

Initialize $\text{acl}(X) := \emptyset$.
For $i$ from 1 to $n$ do:
- if $b_i \in X$ or $b_i$ satisfies $\emptyset \subseteq C^*(B; b_i) \subseteq \text{acl}(X)$ then $\text{acl}(X) := \text{acl}(X) \cup C^*(B; b_i)$. 

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Proof. This alternative formulation for a definition of acl is intermediate between the ones given in Definition 4.3 and Lemma 4.6. The proof is straightforward. □

Lemma 4.8. Let $E$ be a linearly ordered set. Let $\mathcal{F}$ be a bipartite graph/tableau on $(B, E \setminus B)$ (or equivalently let $B$ be a basis of a matroid $M$ on $E$ with fundamental graph $\mathcal{F}$). Let $X \subseteq \text{Int}(\mathcal{F})$. We have

$$\text{acl}(X) \cap (\text{Int}(\mathcal{F}) \cup \text{Ext}(\mathcal{F})) = X.$$  

In particular, if $\text{acl}(X) = E$ then $X = \text{Int}(\mathcal{F})$, and if $\text{acl}({x}) = E$ for $x \in \text{Int}(\mathcal{F})$ then $\mathcal{F}$ is uniactive internal.

Proof. Direct by Lemma 4.6: if $e_i$ is internally active and $e_i \not\in X$ then $e_i \not\in \text{acl}(X)$; and if $e_i$ is externally active then there exist no $c < e_i$ with $c \in C(B; e_i)$, and then $e_i \not\in \text{acl}(X)$. □

We give Lemma 4.9 below for practical purpose. It notably shows that the active closure of $X \subseteq \text{Int}(\mathcal{F})$ can be computed using active closures of its elements, successively in any order, while deleting successively the results from $\mathcal{F}$.

Lemma 4.9. Let $E$ be a linearly ordered set. Let $\mathcal{F}$ be a bipartite graph/tableau on $(B, E \setminus B)$ (or equivalently let $B$ be a basis of a matroid $M$ on $E$ with fundamental graph $\mathcal{F}$). Let $X \subseteq \text{Int}(\mathcal{F})$ and let $Y, Z$ such that $X = Y \uplus Z$. We have

$$\text{acl}_\mathcal{F}(X) = \text{acl}_\mathcal{F}(Y) \uplus \text{acl}_{\mathcal{F}-\text{acl}_\mathcal{F}(Y)}(Z).$$

Proof. Let us denote $\mathcal{F}_Y = \mathcal{F} - \text{acl}_\mathcal{F}(Y)$. Assume $E = e_1 < \cdots < e_n$. We prove the result by induction. We assume that $\text{acl}_\mathcal{F}(X) \cap \{e_1, \ldots, e_{i-1}\} = (\text{acl}_\mathcal{F}(Y) \cup \text{acl}_{\mathcal{F}_Y}(Z)) \cap \{e_1, \ldots, e_{i-1}\}$. And we apply the definition given in Lemma 4.6. If $e_i \in Y$ then $e_i \in \text{acl}(X)$ and $e_i \in \text{acl}(Y)$. If $e_i \in Z$ then $e_i \in \text{acl}(X)$ and $e_i \in \text{acl}_{\mathcal{F}_Y}(Z)$. If $\emptyset \subset C^*(B; e_i) \subset \text{acl}(X)$, then $e_i \not\in \text{acl}(X)$. Moreover, in this case, we have $\emptyset \subset C^*(B; e_i) \subset \text{acl}_\mathcal{F}(Y) \cup \text{acl}_{\mathcal{F}_Y}(Z)$ by induction hypothesis, then: either $\emptyset \subset C^*(B; e_i) \subset \text{acl}(Y)$, and in this case $e_i \in \text{acl}(Y)$; or $\emptyset \subset C^*(B; e_i) \subset \text{acl}_\mathcal{F}(Y) \subseteq \text{acl}_{\mathcal{F}_Y}(Z)$, and in this case $e_i \in \text{acl}_{\mathcal{F}_Y}(Z)$. If $e_i \not\in B$ and there exists $c \in C(B; e_i)$ with $c < e_i$ and $c \in \text{acl}(X)$ then $e_i \in \text{acl}(X)$. Moreover, in this case, by induction hypothesis, we have: either there exists $c \in C(B; e_i)$ with $c < e_i$ and $c \in \text{acl}(Y)$, and in this case $e_i \in \text{acl}(Y)$; or there exist no $c \in C(B; e_i)$ with $c < e_i$ and $c \in \text{acl}(Y)$, and then there exists $c \in C(B; e_i) \setminus \text{acl}(Y)$ with $c < e_i$ and $c \in \text{acl}_{\mathcal{F}_Y}(Z)$, and in this case $e_i \in \text{acl}_{\mathcal{F}_Y}(Z)$. In every other case, $e_i \not\in \text{acl}(X)$, $e_i \not\in \text{acl}(Y)$ and $e_i \not\in \text{acl}_{\mathcal{F}_Y}(Z)$.

Finally, we have shown that, in every case, $e_i \in \text{acl}(X)$ if and only if $e_i \not\in \text{acl}(Y)$ or $e_i \not\in \text{acl}_{\mathcal{F}_Y}(Z)$, which achieves the proof by induction. Observe that the resulting union is a disjoint union since $\text{acl}_{\mathcal{F}_Y}(Z) \cap \text{acl}_{\mathcal{F}_Y}(Z) = \emptyset$ by definition of $\mathcal{F}_Y$. □

Proposition 4.10 (in terms of bipartite graphs/tableaux) and Proposition 4.11 (the same result rephrased more specifically in terms of matroids) below provide a general characterization of the active closure in terms of activities of fundamental graphs induced in minors. Hence it could be used to provide various decompositions of activities for (fundamental) bipartite graphs/tableaux. In what follows, it will be practically used in a restricted form, essentially when $X$ is the set of internally active elements greater than a given one.

Proposition 4.10. Let $E$ be a linearly ordered set. Let $\mathcal{F}$ be a bipartite graph/tableau on $(B, E \setminus B)$. Let $X \subseteq \text{Int}(\mathcal{F})$. The set $\text{acl}(X)$ is the unique subset $A$ of $E$ satisfying the following properties:
(i) for all $e \in E \setminus (A \cup B)$, we have $C(B; e) \cap A = \emptyset$; or equivalently: for all $b \in B \cap A$, we have $C^*(B; b) \subseteq A$;
(ii) $\text{Int}(F - A) = \text{Int}(F) \setminus X$;
(iii) $\text{Ext}(F - A) = \text{Ext}(F)$;
(iv) $\text{Int}(F - E \setminus A) = X$;
(v) $\text{Ext}(F - E \setminus A) = \emptyset$.

Proof. First we verify that the two properties stated in (i) are equivalent. Indeed the first property can be written equivalently: for all $e \in E \setminus B$, if $C(B; e) \cap A \neq \emptyset$ then $e \not\in A$; that is: for all $e \in E \setminus B$, if $e \in C^*(B; b)$ for some $b \in A \cap B$ then $e \not\in A$; that is: for all $e \in E \setminus B$, for all $b \in A$, if $e \in C^*(B; b)$ and $b \in A$ then $e \not\in A$; that is: for all $b \in B$, if $b \in A$ then $C^*(B; b) \subseteq A$.

Next, we show that $\text{acl}(X)$ satisfies the five properties (i)-(v).

(i) Let $e \not\in \text{acl}(X) \cup B$. If $C(B; e) = \{e\}$ then $C(B; e) \cap \text{acl}(X) = \emptyset$. Otherwise, let $b \in B \cap C(B; e)$, which implies $e \in C^*(B; b)$. If $b \in \text{acl}(X)$ then $C^*(B; b) \subseteq \text{acl}(X)$ by definition of $\text{acl}(X)$, so $e \in \text{acl}(X)$, which is a contradiction. Hence $C(B; e) \cap \text{acl}(X) = \emptyset$.

(ii) Let $b \in \text{Int}(F - \text{acl}(X))$. By definition of $\text{Int}$, we have $b \in B \setminus \text{acl}(X)$ and $b = \min(C^*(B; b) \setminus \text{acl}(X))$. Then $C^*(B; b)< \subseteq \text{acl}(X)$, so $C^*(B; b)< = \emptyset$ otherwise $b \in \text{acl}(X)$ by definition of $\text{acl}(X)$, so $b = \min(C^*(B; b))$, that is $b \in \text{Int}(F)$. Since $b \not\in \text{acl}(X)$, we get $b \in \text{Int}(F) \setminus X$. Conversely, let $b \in \text{Int}(F) \setminus X$. Since $b \not\in X$ and $b = \min(C^*(B; b))$ then $b \not\in \text{acl}(X)$, by definition of $\text{acl}(X)$. So $b = \min(C^*(B; b) \setminus \text{acl}(X))$, that is $b \in \text{Int}(F - \text{acl}(X))$.

(iii) Let $e \in \text{Ext}(F - \text{acl}(X))$. By definition of $\text{Ext}$, we have $e \in (E \setminus B) \setminus \text{acl}(X)$ and $e = \min(C(B; e) \setminus \text{acl}(X))$. Then there exist no $c \in C(B; e) \cap \text{acl}(X)$ with $c < e$, otherwise $e \in \text{acl}(X)$ (by Lemma 4.6). So $e = \min(C(B; e))$, that is $e \in \text{Ext}(F)$. Conversely, let $e \in \text{Ext}(F)$. We have $e \not\in B$. Since $e = \min(C(B; e))$, we have $e \not\in \text{acl}(X)$ (by Lemma 4.6). So $e = \min(C(B; e) \setminus \text{acl}(X))$, that is $e \in \text{Ext}(F - \text{acl}(X))$.

(iv) Let $b \in X$. Since $X \subseteq \text{Int}(F)$ and $X \subseteq \text{acl}(X)$, $b$ is internally active in $F$ and $b \in \text{acl}(X)$. So $b = \min(C^*(B; b)) = \min(C^*(B; b) \cap \text{acl}(X))$, so $b \in \text{Int}(F - E \setminus \text{acl}(X))$. Conversely, let $b \in \text{Int}(F - E \setminus \text{acl}(X))$. Then $b = \min(C^*(B; b) \cap \text{acl}(X))$ by definition of $\text{Int}$. So $C^*(B; b)< \subseteq \text{acl}(X)$. By Lemma 4.6, $b \in \text{acl}(X)$ implies $b \in X$ or $C^*(B; b)< \subseteq \text{acl}(X)$. So we have $b \in X$.

(v) Assume $e \in \text{Ext}(F - E \setminus \text{acl}(X))$. We have $e \in \text{acl}(X)$, so, by Lemma 4.6, there exists $c < e$ in $C(B; e) \cap \text{acl}(X)$, so $e \neq \min(C(B; e) \cap \text{acl}(X))$, a contradiction with the definition of $\text{Ext}$. Hence $\text{Ext}(F - E \setminus A) = \emptyset$.

Now, let $A \subseteq E$ satisfying these five properties. We show that $A$ satisfies Definition 4.3 of $\text{acl}(X)$. The property (iv) implies $X \subseteq A$, which is the first property to satisfy in Definition 4.3. As shown above, the property (i) can be stated: for all $b \in B$, if $b \in A$ then $C^*(B; b) \subseteq A$, which is the second property to satisfy in Definition 4.3. Finally, assume that there exists $b \in B$ such that $\emptyset \subset C^*(B; b) \subseteq A$ and $b \not\in A$. Then $b \not\in \text{Int}(F)$ as $C^*(B; b)< \not= \emptyset$. And $b = \min(C^*(B; b) \setminus A)$ as $C^*(B; e)< \subseteq A$, that is: $b \in \text{Int}(F - A)$. So $b \in \text{Int}(F - A) \setminus \text{Int}(F)$ which is a contradiction with property (ii). So $A$ satisfies the third property in Definition 4.3. Since $A$ satisfies the three properties in Definition 4.3, and $\text{acl}(X)$ is the smallest set satisfying those three properties, we have shown $\text{acl}(X) \subseteq A$.

To conclude, let us assume that there exists $e \in A \setminus \text{acl}(X)$, in a first case, let us assume that $e \in B$. Then $C^*(B; e) \subseteq A$ by property (i). If $e = \min(C^*(B; e))$ then we have $e \in \text{Int}(F - E \setminus A)$ (by
definition of Int, since $e \in A$), which implies $e \in X$ by property (iv), which is a contradiction with $e \not\in \text{acl}(X)$. So there exists $f < e$ in $C^*(B; e) \setminus \text{acl}(X)$ (otherwise $\emptyset \subset C^*(B; e) \subseteq \text{acl}(X)$, which implies $e \in \text{acl}(X)$ by definition of $\text{acl}(X)$). So there exists $f < e$ with $f \notin A \setminus \text{acl}(X)$. In a second case, let us assume that $e \notin B$. Then, by property (v), there exists $f < e$ with $f \in C(B; e) \cap A$ (otherwise $e$ is externally active in $F \setminus E \setminus A$). By assumption we have $e \in E \setminus (\text{acl}(X) \cup B)$, so, by property (i) satisfied by $\text{acl}(X)$, we have $C(B; e) \subseteq E \setminus \text{acl}(X)$. So we have $f \in A \setminus \text{acl}(X)$. In any case, the existence of $e$ in $A \setminus \text{acl}(X)$ implies the existence of $f < e$ in $A \setminus \text{acl}(X)$, which is impossible. So we have proved $A = \text{acl}(X)$.

**Proposition 4.11** (equivalent to Proposition 4.10). Let $E$ be a linearly ordered set. Let $B$ be a basis of a matroid $M$ on $E$ with fundamental graph $\mathcal{F}$. Let $X \subseteq \text{Int}_M(B)$. The set $F = E \setminus \text{acl}(X)$ is the unique subset of $E$ satisfying the following properties:

(i) $B \cap F$ is a basis of $M(F)$, and, equivalently, $B \setminus F$ is a basis of $M/F$

(ii) $\text{Int}_{M(F)}(B \cap F) = \text{Int}_M(B) \setminus X$,

(iii) $\text{Ext}_{M(F)}(B \cap F) = \text{Ext}_M(B)$,

(iv) $\text{Int}_{M/F}(B \setminus F) = X$,

(v) $\text{Ext}_{M/F}(B \setminus F) = \emptyset$.

**Proof.** This proposition is essentially a reformulation of Proposition 4.10 in the language of matroids, using Property 4.2. Let $F = E \setminus \text{acl}(X)$. By Proposition 4.10, $E \setminus F$ is the unique subset of $E$ satisfying properties (i)-(v) stated in Proposition 4.10. Observe that property (i) is stated as: for all $e \in F \setminus ((E \setminus F) \cup B)$, we have $C(B; e) \cap (E \setminus F) = \emptyset$. That is, equivalently: for all $e \in F \setminus B$, we have $C(B; e) \subseteq F$. That is, equivalently, by Property 4.2: $B \cap F$ is a basis of $M(F)$. Now, by Property 4.2, properties (ii)-(v) of Proposition 4.10 translate directly to properties (ii)-(v) of the present result.

**Proposition 4.12.** Let $E$ be a linearly ordered set. Let $\mathcal{F}$ be a bipartite graph/tableau on $(B, E \setminus B)$ (or equivalently let $B$ be a basis of a matroid $M$ on $E$ with fundamental graph $\mathcal{F}$). We have

$$E = \text{acl}\left(\text{Int}(\mathcal{F})\right) \cup \text{acl}\left(\text{Ext}(\mathcal{F})\right).$$

**Proof.** By Proposition 4.10, $\text{acl}(\text{Int}(\mathcal{F}))$ is the unique subset $A \subseteq E$ such that:

- for all $b \in B \cap A$, we have $C^*(B; b) \subseteq A$;
- $\text{Int}(\mathcal{F} - A) = \emptyset$;
- $\text{Ext}(\mathcal{F} - A) = \text{Ext}(\mathcal{F})$;
- $\text{Int}(\mathcal{F} - E \setminus A) = \text{Int}(\mathcal{F})$;
- $\text{Ext}(\mathcal{F} - E \setminus A) = \emptyset$.

Now we apply Proposition 4.10 to $\mathcal{F}^*$, bipartite graph on $(E \setminus B, B)$, with $X = \text{Ext}(\mathcal{F}) = \text{Int}(\mathcal{F}^*)$. We get that $E \setminus \text{acl}(\text{Ext}(\mathcal{F}))$ is the unique subset $E \setminus A'$ of $E$ such that the following properties hold, where we replace the statements of properties of $\mathcal{F}^*$ with equivalent statements for $\mathcal{F}$:

- for all $e \in E \setminus ((E \setminus A') \cup (E \setminus B))$, we have $C_{\mathcal{F}^*}(B; e) \cap (E \setminus A') = \emptyset$;
- that is equivalently: for all $e \in A' \cap B$, we have $C_{\mathcal{F}^*}(B; e) \subseteq A'$;
- $\text{Ext}(\mathcal{F} - E \setminus A') = \emptyset$;
- $\text{Int}(\mathcal{F} - E \setminus A') = \text{Int}(\mathcal{F})$.
- Ext(\(F - A'\)) = Ext(\(F\));
- Int(\(F - A'\)) = \(\emptyset\).

Finally, the properties satisfied by \(A\) and by \(A'\) are exactly the same, hence \(A = A'\) by uniqueness in Proposition 4.10, that is acl(Int(\(F\))) = \(E \setminus \text{acl}(\text{Ext}(\(F\)))\).

**Definition 4.13.** Let \(E\) be a linearly ordered set. Let \(F\) be a bipartite graph/tableau on \((B, E \setminus B)\) (or equivalently let \(B\) be a basis of a matroid \(M\) on \(E\) with fundamental graph \(F\)). The set \(F = \text{acl}(\text{Ext}(\(F\)))\) is called the external part of \(E\) w.r.t. \(F\), and the set \(E \setminus F = \text{acl}(\text{Int}(\(F\)))\) is called the internal part of \(E\) w.r.t. \(F\). Observe that, in the case where \(B\) is a basis of a matroid \(M\), \(F_c\) is a cyclic flat of \(M\) (as \(\text{acl}(\text{Ext}(\(F\)))\), resp. \(\text{acl}(\text{Int}(\(F\)))\), is a union of circuits, resp. cocircuits).

From Proposition 4.12, using the formulation used in Proposition 4.11, we directly retrieve the following result from [3] (in an equivalent form). Let us mention that we complete it with a practical characterization in Corollary 4.15 below.

**Corollary 4.14** ([3]). Let \(B\) be a basis of a matroid \(M\) on a linearly ordered set \(E\) with fundamental graph \(F\). Let \(F_c\) be the external part of \(E\) w.r.t. \(F\). The subset \(F_c\) is the unique subset (or cyclic flat) \(F\) of \(M\) such that:

(i) \(B \cap F\) is a basis of \(M(F)\), and \(B \setminus F\) is a basis of \(M/F\),

(ii) \(\text{Int}_{M(F)}(B \cap F) = \emptyset\),

(iii) \(\text{Ext}_{M(F)}(B \cap F) = \text{Ext}_M(B)\),

(iv) \(\text{Int}_{M/F}(B \setminus F) = \text{Int}_M(B)\),

(v) \(\text{Ext}_{M/F}(B \setminus F) = \emptyset\).

**Corollary 4.15.** Let \(E\) be a linearly ordered set. Let \(F\) be a bipartite graph/tableau on \((B, E \setminus B)\) (or equivalently let \(B\) be a basis of a matroid \(M\) on \(E\) with fundamental graph \(F\)). The partition of \(E\) into internal and external parts w.r.t. \(F\) is given by the following definition (yielding a linear algorithm by a single pass over \(E\) in increasing order).

If \(e \in B\):
- if there exists \(c < e\) external in \(C^*(B; e)\) then \(e\) is external
- otherwise \(e\) is internal

If \(e \notin B\):
- if there exists \(c < e\) internal in \(C(B; e)\) then \(e\) is internal,
- otherwise \(e\) is external

**Proof.** Observe that if \(e\) is internally, resp. externally, active then \(C^*(B; e)^< = \emptyset\), resp. \(C(B; e)^< = \emptyset\), and then \(e\) is internal, resp. external. Then, the computation of the internal part comes directly from Lemma 4.6 applied to \(X = \text{Int}(F)\). The other cases, where \(e\) is not internal, imply that \(e\) is external, equivalently either by duality (the cases are dual), or by Proposition 4.12.

**Lemma 4.16.** Let \(E\) be a linearly ordered set. Let \(F\) be a bipartite graph/tableau on \((B, E \setminus B)\) (or equivalently let \(B\) be a basis of a matroid \(M\) on \(E\) with fundamental graph \(F\)). Let \(X \subseteq \text{Int}(F)\). Let \(F_c\) be the external part of \(E\) w.r.t. \(F\). We have

\[
\text{acl}_F(X) = \text{acl}_{F - F_c}(X).
\]

Moreover, the external part of \(E \setminus \text{acl}_F(X)\) w.r.t. \(F - \text{acl}_F(X)\) is also \(F_c\).
Proof. We have \( \text{acl}(\text{Int}(\mathcal{F})) \cap \text{acl}(\text{Ext}(\mathcal{F})) = \emptyset \) (Proposition 4.12), hence \( \text{acl}(X) \cap F_c = \emptyset \). Then, first, one sees directly that the computation of \( \text{acl}(X) \) given by Lemma 4.7 yields the same result as if it is applied to \( \mathcal{F} - F_c \). So \( \text{acl}_\mathcal{F}(X) = \text{acl}_{\mathcal{F}-F_c}(X) \). And, second, for the same reason, the computation of \( \text{acl}(\text{Ext}(\mathcal{F})) = \text{acl}(\text{Int}(\mathcal{F}^*)) \) given by Lemma 4.7 applied to \( \mathcal{F}^* \) yields the same result as if it is applied to \( \mathcal{F}^* - \text{acl}(X) \). So \( \text{acl}_\mathcal{F}(\text{Ext}(\mathcal{F})) = \text{acl}_{\mathcal{F}-\text{acl}(X)}(\text{Ext}(\mathcal{F} - \text{acl}(X))) \). 

**Definition 4.17.** Let \( E \) be a linearly ordered set. Let \( \mathcal{F} \) be a bipartite graph/tableau on \((B,E\setminus B)\), (or equivalently let \( B \) be a basis of a matroid \( M \) on \( E \) with fundamental graph \( \mathcal{F} \)), with \( \iota \) internally active elements \( a_1 < \ldots < a_\iota \) and \( \varepsilon \) externally active elements \( a'_1 < \ldots < a'_\varepsilon \). The active filtration of \( \mathcal{F} \) (or \( B \)) is the sequence of subsets \((F'_\varepsilon, \ldots, F'_0, F_c, F_0, \ldots, F_\iota)\) of \( E \) defined by the following:

\[
F_c = \text{acl}(\text{Ext}(\mathcal{F})) = E \setminus \text{acl}(\text{Int}(\mathcal{F}));
\]

\( F_\iota = E \), and for every \( 0 \leq k \leq \iota - 1 \),

\[
F_k = E \setminus \text{acl}\{a_{k+1}, \ldots, a_{\iota}\};
\]

\( F'_\varepsilon = \emptyset \), and for every \( 0 \leq k \leq \varepsilon - 1 \),

\[
F'_k = \text{acl}\{a'_{k+1}, \ldots, a'_{\varepsilon}\}.
\]

**Lemma 4.18.** Using the above notations, we have

\[
\emptyset = F'_\varepsilon \subset \ldots \subset F'_0 = F_c = F_0 \subset \ldots \subset F_\iota = E.
\]

The active filtration of \( \mathcal{F} \) is a filtration of \( E \) (Definition 3.1). Moreover, we have, for \( 1 \leq k \leq \iota \),

\[
F_k \setminus F_{k-1} = \text{acl}\{a_{k}, \ldots, a_{\iota}\} \setminus \text{acl}\{a_{k+1}, \ldots, a_{\iota}\}
= \text{acl}_{\mathcal{F}(E\setminus F_k)}(\{a_{k}\}),
\]

\[
\min(F_k \setminus F_{k-1}) = a_k,
\]

and, for \( 1 \leq k \leq \varepsilon \),

\[
F'_{k-1} \setminus F'_k = \text{acl}\{a'_{k}, \ldots, a'_{\varepsilon}\} \setminus \text{acl}\{a'_{k+1}, \ldots, a'_{\varepsilon}\}
= \text{acl}_{\mathcal{F}(E\setminus F'_k)}(\{a'_{k}\}),
\]

\[
\min(F'_{k-1} \setminus F'_k) = a'_k.
\]

Moreover, in the case where \( B \) is a basis of a matroid \( M \), we have:

- for \( 0 \leq k \leq \iota \), \( F_k \) satisfies the properties of Property 4.2, and \( F_k \) is a flat of \( M \);
- for \( 0 \leq k \leq \varepsilon \), \( F'_k \) satisfies the properties of Property 4.2, and \( F'_k \) is a dual-flat of \( M \).

In particular, \( F_0 = F_c = F'_0 \) is a cyclic-flat of \( M \).

**Proof.** Since \( \text{acl} \) is increasing, for \( 1 \leq k \leq \iota \), we have \( \text{acl}\{a_{k+1}, \ldots, a_{\iota}\} \subseteq \text{acl}\{a_{k}, \ldots, a_{\iota}\} \subseteq \text{acl}(\text{Int}(\mathcal{F})) \), and, by definition of \( \text{acl} \), \( a_k \in \text{acl}\{a_{k}, \ldots, a_{\iota}\} \setminus \text{acl}\{a_{k+1}, \ldots, a_{\iota}\} \). So \( F_c \subseteq F_{k-1} \subseteq F_k \). And dually, we have, for \( 1 \leq k \leq \varepsilon \), \( F'_k \subset F_{k-1} \subseteq F_c \). So we have \( \emptyset = F'_\varepsilon \subset \ldots \subset F'_0 = F_c = F_0 \subset \ldots \subset F_\iota = E \).
Moreover \( a_k = \min(\text{acl}(\{a_k, \ldots, a_i\})) \) so \( a_k = \min(F_k \setminus F_{k-1}) \), \( 1 \leq k \leq \iota \), which is increasing with \( k \) by hypothesis. And, dually, we have \( a'_k = \min(F'_{k-1} \setminus F'_k) \), \( 1 \leq k \leq \varepsilon \), which is increasing with \( k \) by hypothesis. So the active filtration is a filtration of \( E \), according to Definition 3.1.

Moreover, by Lemma 4.9, we have \( \text{acl}(\{a_k, \ldots, a_i\}) = \text{acl}(\{a_{k+1}, \ldots, a_i\}) \cup \text{acl}_{F-(E \setminus F_k)}(a_k) \). And by Lemma 4.9 applied to \( F^* \), we have \( \text{acl}(\{a'_k, \ldots, a'_i\}) = \text{acl}(\{a'_{k+1}, \ldots, a'_i\}) \cup \text{acl}_{F-F'_k}(a'_k) \).

For \( 0 \leq k \leq \varepsilon \), by Proposition 4.10, we have that \( F_k \) satisfies: for all \( b \in B \cap (E \setminus F_k) \), we have \( C^*(B; b) \subseteq (E \setminus F_k) \). That is: for all \( b \in B \setminus F_k \), \( C^*_M(B; b) \cap F_k = \emptyset \). Hence \( F_k \) satisfies properties of Property 4.2. And \( F_k \) is a flat of \( M \) as its complement is a union of cocircuits.

For \( 0 \leq k \leq \varepsilon \), by Proposition 4.10 applied to the dual \( F^* \) (or the basis \( E \setminus B \) of \( M^* \)), we have that \( F'_k \) satisfies: for all \( e \in (E \setminus B) \cap F'_k \), we have \( C^*_M(E \setminus B; e) \subseteq F'_k \). That is: for all \( e \in F'_k \setminus B \), we have \( C_M(B; e) \subseteq F'_k \). Hence \( F'_k \) satisfies properties of Property 4.2. And \( F'_k \) is a dual-flat of \( M \) as it is a union of circuits. □

**Definition 4.19.** Using the above notations, the active filtration of \( F \) (or \( B \)), induces a partition of the ground set \( E \), which we call the active partition of \( F \) (or \( B \)):

\[
E = (F_{\varepsilon-1}^c \setminus F_\varepsilon^c) \cup \ldots \cup (F_0^c \setminus F_1^c) \cup (F_1 \setminus F_0) \cup \ldots \cup (F_\iota \setminus F_{\iota-1}).
\]

Also, we call active minors w.r.t. \( F \) (or \( B \)) the minors induced by the active filtration of \( F \) (or \( B \)), that is the minors \( M(F_k)/F_{k-1} \) for \( 1 \leq k \leq \iota \), and the minors \( M(F'_k)/F'_k \) for \( 1 \leq k \leq \varepsilon \).

**Observation 4.20.** The active partition of \( F \) (or \( B \)) determines the active filtration of \( F \) (or \( B \)), hence it is an equivalent notion. Precisely, using the above notations, knowing only the subsets forming the active partition of \( F \) (or \( B \)) allows us to build:

- the subset \( F_c \) of the active filtration of \( F \) (or \( B \)), since the smallest element of a part is in \( B \) if and only if this part is of type \( F_k \setminus F_{k-1} \) for some \( 1 \leq k \leq \iota \);
- the active filtration of \( F \) (or \( B \)), since the sequence \( \min(F_k \setminus F_{k-1}) \), \( 1 \leq k \leq \iota \), is increasing with \( k \), and the sequence \( \min(F'_{k-1} \setminus F'_k) \), \( 1 \leq k \leq \varepsilon \), is increasing with \( k \), so that the position of each part of the active partition with respect to the active filtration is identified.

From a constructive viewpoint, let us remark that, by Lemma 4.18, and more generally by Lemma 4.9, the active partition of \( F \) can be computed directly from \( F \), or also from the successive subgraphs of \( F \) induced by the active filtration of \( F \), computing the active closure of active elements one by one (or also from successive corresponding minors in a matroid setting, by Property 4.2, as made explicit in next Theorem 4.22).

Moreover, and more practically, Proposition 4.29 (postponed at the end of the paper) gives a direct construction of the active partition by a linear algorithm consisting in a single pass over \( E \).

Finally, let us notice that, in the definitions that precede and the results that follow, the particular case of internal fundamental graphs (or internal bases) is addressed as the case where \( F_c = \emptyset \), and case of external fundamental graphs (or external bases) is addressed as the case where \( F_c = E \). Those cases are dual to each other. Let us deepen this with the next observation, which comes directly from Observation 4.4 (for duality), and from Lemma 4.16 (for restriction to \( F-(E \setminus F_c) \) or dually to \( F-F_c \)). It will be deepened again in Observation 4.23.

**Observation 4.21.** Using the above notations, let \( \emptyset = F_{\varepsilon-1}^c \subseteq \ldots \subseteq F_0^c = F_c = F_0 \subseteq \ldots \subseteq F_\iota = E \) be the active filtration of \( F \) (or \( B \)), with external part \( F_c \). We have:
1. \( \emptyset = E \setminus F_i \subset ... \subset E \setminus F_0 = E \setminus F_c = E \setminus F'_0 \subset ... \subset E \setminus F'_c = E \) is the active filtration of \( \mathcal{F}^* \) (or of the basis \( E \setminus B \) of \( M^* \)), with external part \( E \setminus F_c \);

2. \( \emptyset = F'_c \subset ... \subset F'_0 = F_c = F \) is the active filtration of \( \mathcal{F} - (E \setminus F_c) \) (or of the external base \( B \setminus F_c \) of \( M(F_c) \), with external part \( F_c \));

3. \( \emptyset = \emptyset = F_0 \setminus F_c \subset ... \subset F_0 \setminus F_c = E \setminus F_c \) is the active filtration of \( \mathcal{F} - F_c \) (or of the internal base \( B \setminus F_c \) of \( M/F_c \)), with external part \( \emptyset \).

For the sake of concision, we state the following Theorem 4.22 in terms of matroids (it is technically the main result of this section), but it could be equally stated in terms of bipartite graphs/tableaux as a decomposition into particular uniactive bipartite graphs/tableaux (using Property 4.2 as previously for the translation).

**Theorem 4.22.** Let \( E \) be a linearly ordered set. Let \( B \) be a basis of a matroid \( M \) on \( E \) with fundamental graph \( \mathcal{F} \). The active filtration of \( \mathcal{F} \) is the unique (connected) filtration \( \emptyset = F'_c \subset ... \subset F'_0 = F_c = F_0 \subset ... \subset F_i = E \) of \( E \) (or \( M \)) such that:

- for all \( 1 \leq k \leq \iota \), the set
  \[
  B_k = B \cap F_k \setminus F_{k-1}
  \]
  is a uniactive internal basis of the minor
  \[
  M_k = M(F_k)/F_{k-1};
  \]

- for all \( 1 \leq k \leq \varepsilon \), the set
  \[
  B'_k = B \cap F'_{k-1} \setminus F'_k
  \]
  is a uniactive external basis of the minor
  \[
  M'_k = M(F'_{k-1})/F'_k.
  \]

Notice that the active filtration of \( \mathcal{F} \) is actually a connected filtration of \( M \) (Definition 3.1). Notice also that, for \( 1 \leq k \leq \iota \), if \( M(F_k)/F_{k-1} \) is an isthmus, then \( B_k \) equals this isthmus, and that, for \( 1 \leq k \leq \varepsilon \), if \( M(F'_{k-1})/F'_k \) is a loop, then \( B'_k = \emptyset \).

**Proof.** First, let us directly check that the active filtration \( (F'_c, ..., F'_0, F_c, F_0, ..., F_i) \) satisfies the given properties. The basis \( B \) has \( \iota \geq 0 \) internally active elements, which we denote \( a_1 < ... < a_\iota \), and \( \varepsilon \geq 0 \) externally active elements, which we denote \( a'_1 < ... < a'_\varepsilon \). Let \( 1 \leq k \leq \iota \). By Lemma 4.18, we have \( a_k = \min(F_k \setminus F_{k-1}) \) and we have \( F_k \setminus F_{k-1} = acl_{\mathcal{F} - (E \setminus F_k)}(\{a_k\}) \). Obviously, since \( acl_{\mathcal{F} - (E \setminus F_k)}(\{a_k\}) \subseteq E \setminus F_{k-1} \), we have in fact \( F_k \setminus F_{k-1} = acl_{\mathcal{F} - (E \setminus F_k)}(\{a_k\}) = acl_{\mathcal{F} - (F_k \setminus F_{k-1}) \cup (E \setminus F_k)}(\{a_k\}) \). By Property 4.2, we have that \( B_k = B \cap F_k \setminus F_{k-1} \) is a basis of \( M_k = M(F_k)/F_{k-1} \). Let us denot \( F_k = \mathcal{F} \setminus (F_k \setminus F_{k-1} \cup (E \setminus F_k)) = F_{M_k}(B_k) \). Since \( F_k \setminus F_{k-1} = acl_{F_k}(\{a_k\}) \), we have that \( a_k \) is internally active in \( F_k \). Moreover, by Lemma 4.8, \( F_k \setminus F_{k-1} = acl_{F_k}(\{a_k\}) \) implies that \( F_k \) is uniactive internal. Dually, let \( 1 \leq k \leq \varepsilon \). By Lemma 4.18 and Property 4.2, we have similarly that \( F'_{k-1} \setminus F'_k = acl_{\mathcal{F} - F'_{k-1}}(a'_k) \), that \( a'_k = \min(F'_{k-1} \setminus F'_k) \), that \( B'_k = B \cap F'_{k-1} \setminus F'_k \) is a basis of \( M'_k = M(F'_{k-1})/F'_k \), and that \( F'_k \) is uniactive external. So, we have proved that the active filtration satisfies the given properties.

Now, notice that each involved minor \( M_k, 1 \leq k \leq \iota \), or \( M'_k, 1 \leq k \leq \varepsilon \), has a uniactive internal or a uniactive external basis, which implies that this minor is an isthmus (in this case the basis
equals this isthmus), or a loop (in this case, the basis is the empty set), or a connected matroid (since $\beta(M) \neq 0$ and $|E| > 1$). This proves that the active filtration of $F$ is a connected filtration of $M$ (Definition 3.1).

It remains to prove the uniqueness property. Assume that a filtration $S = (F'_e, \ldots, F'_0, F_e, F_0, \ldots, F_1)$ satisfies the properties stated in the proposition. Let us denote $a_k = \min(F_k \setminus F_{k-1})$, $1 \leq k \leq \iota$, and $a'_k = \min(F'_k \setminus F'_0)$, $1 \leq k \leq \varepsilon$.

First, recall that in any matroid $M$, for every set $F$, the union of a basis of $M/F$ and a basis of $M(F)$ is basis of $M$. Hence, since $B_k$ is a basis of $M_k$, $1 \leq k \leq \iota$, and $B'_k$ is a basis of $M'_k$, $1 \leq k \leq \varepsilon$, we have that, for any $0 \leq k \leq \iota$, the set $B \cap F_k$, resp. $B \setminus F_k$, is a basis of $M(F_k)$, resp. $M/F_k$, as it is obtained by union of some of these former bases.

Second, let us prove that $\text{Int}(\mathcal{F}) = \{a_1, \ldots, a_\iota\}$.

Let $b \in \text{Int}(\mathcal{F})$. By definition, $b \in B$ and $b = \min(C^*(B;b))$. By assumption on the sequence $S$, $b$ is an element of a minor of $M$ induced by this sequence $S$: either $N = M_k$ for some $1 \leq k \leq \iota$ or $N = M'_k$ for some $1 \leq k \leq \varepsilon$. In any case, $b$ is an element of the basis $B_N$ induced by $B$ in $N$: either $B_N = B_k$ if $N = M_k$, or $B_N = B'_k$ if $N = M'_k$. Moreover, since $N$ is of type $M(G)/F$ and its basis $B_N$ of type $B \cap (G \setminus F)$, we have by Property 4.2 that $C'_N(B_N;b)$ is obtained from $C_M(B;b)$ by removing elements not in the ground set of $N$. So $b = \min(C'_N(B_N;b))$, so $b$ is internally active in $N$. By assumption on the sequence $S$ this implies that $b = a_k$ for some $1 \leq k \leq \iota$. Hence $\text{Int}(\mathcal{F}) \subseteq \{a_1, \ldots, a_\iota\}$.

Conversely, let $1 \leq k \leq \iota$. By assumption on the sequence $S$, we have $a_k = \min(C^*_M(B;b;a_k))$. As above, by Property 4.2, we have $C^*_M(B_k;a_k) = C^*_M(B;b_k) \cap (F_k \setminus F_{k-1})$. Let $e = \min(C^*_M(B;b_k))$ and assume that $e < a_k$. Since $S$ is a filtration, by Definition 3.1, the sequence $a_j = e$ is increasing with $j$. Hence $a_k = \min(E \setminus F_{k-1})$. On the other hand, by properties of matroid contraction, since $B \setminus F_{k-1}$ is a basis of $M/F_{k-1}$, we have $C^*_M(B;b_k) \cap F_{k-1} = \emptyset$, which is a contradiction with $e \in F_{k-1}$. So we have $e = a_k$. So $a_k \in \text{Int}(\mathcal{F})$ and we have proved $\text{Int}(\mathcal{F}) \supseteq \{a_1, \ldots, a_\iota\}$. Finally, we have proved $\text{Int}(\mathcal{F}) = \{a_1, \ldots, a_\iota\}$.

Third, let us prove that for every $k$, $0 \leq k \leq \iota$, we have $\text{Int}(M(F_k))(B \cap F_k) = \{a_1, \ldots, a_k\}$, resp. $\text{Int}(M(F_k))(B \setminus F_k) = \{a_{k+1}, \ldots, a_\iota\}$.

We obtain this result by directly applying the above result (that is $\text{Int}_M(B) = \{a_1, \ldots, a_\iota\}$) in the minor $M(F_k)$, resp. $M/F_k$, of $M$. Precisely, let $0 \leq k \leq \iota$. As noticed above, the set $B \cap F_k$, resp. $B \setminus F_k$, is a basis of $M(F_k)$, resp. $M/F_k$. Obviously, by Definition 3.1, we have that $\emptyset = F'_e \subset \ldots \subset F'_0 = F_e = F_0 \subset \ldots \subset F_k$, resp. $\emptyset = F_c \subset F_{k+1} \setminus F_k \subset \ldots \subset F_1 \setminus F_0$ is a filtration of $F_k$, resp. $E \setminus F_k$, and that it satisfies the properties given in the proposition statement (as the induced minors are minors also induced by $S$, that is by $\emptyset = F'_e \subset \ldots \subset F'_0 = F_e = F_0 \subset \ldots \subset F_i$). The set of smallest elements of successive differences of sets in the sequence is $\{a_1, \ldots, a_k\}$, resp. $\{a_{k+1}, \ldots, a_\iota\}$. So we can apply the same reasoning as above to the minor $M(F_k)$, resp. $M/F_k$, of $M$, and we obtain the same result.

Fourth, let us prove that, for every $k$, $0 \leq k \leq \iota$, we have $\text{Ext}(M(F_k))(B \cap F_k) = \text{Ext}(M)(B)$, resp. $\text{Ext}(M(F_k))(B \setminus F_k) = \emptyset$.

Applying the above result (that is $\text{Int}_M(B) = \{a_1, \ldots, a_\iota\}$) in the dual $M^*$ of $M$, we directly have $\text{Ext}_M(B) = \{a'_1, \ldots, a'_\iota\}$. Now, as above, let us apply this last result (that is $\text{Ext}_M(B) = \{a'_1, \ldots, a'_\iota\}$) in the minor $M(F_k)$, to the filtration $\emptyset = F'_e \subset \ldots \subset F'_0 = F_e = F_0 \subset \ldots \subset F_k$ of $F_k$. We obtain $\text{Ext}(M(F_k))(B \cap F_k) = \{a'_1, \ldots, a'_k\} = \text{Ext}(M)(B)$. And let us apply the same result in the minor $M/F_k$ $\emptyset = F_c \subset F_{k+1} \setminus F_k \subset \ldots \subset F_1 \setminus F_k$, to the filtration of $E \setminus F_k$. We obtain $\text{Ext}(M(F_k))(B \setminus F_k) = \emptyset$.  

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Finally, we have proved that, for every $k$, $0 \leq k \leq \iota$, and denoting $X = \{a_1, \ldots, a_r\}$, the following conditions are satisfied:

(i) $B \cap F_k$ is a basis of $M(F_k)$
(ii) $\text{Int}_{M(F_k)}(B \cap F_k) = \text{Int}_M(B) \setminus X$,
(iii) $\text{Ext}_{M(F_k)}(B \cap F_k) = \text{Ext}_M(B)$,
(iv) $\text{Int}_{M(F_k)}(B \setminus F_k) = X$,
(v) $\text{Ext}_{M(F_k)}(B \setminus F_k) = \emptyset$.

By uniqueness in Proposition 4.11, this implies $F_k = E \setminus \text{acl}(X) = E \setminus \text{acl}((a_{k+1}, \ldots, a_\iota))$, which matches Definition 4.17 of the active filtration.

At last, by duality, we also have, for every $k$, $0 \leq k \leq \varepsilon$, denoting $X = \{a'_1, \ldots, a'_\varepsilon\}$, that $F'_k = \text{acl}((a_{k+1}, \ldots, a_\varepsilon))$, which matches Definition 4.17 of the active filtration (notice that, in particular, $F_0 = F'_0$). So finally the filtration $(F'_\varepsilon, \ldots, F'_0, F_c, F_0, \ldots, F_1)$ is the active filtration of $F$.

**Observation 4.23.** Let us continue and refine Observation 4.21. Let $\emptyset = F'_\varepsilon \subset \cdots \subset F'_0 = F_c = F_0 \subset \cdots \subset F_1 = E$ be the active filtration of the basis $B$ of $M$. And let $F$ and $G$ be two subsets in this sequence such that $F \subseteq G$. Then, by Theorem 4.22, the active filtration of the basis $B \cap G \setminus F$ of $M(G)/F$ is obtained from the subsequence with extremities $F$ and $G$ (i.e. $F \subset \cdots \subset G$) of the active filtration of $B$ by subtracting $F$ from each subset of the subsequence (with $F_c \setminus F$ as cyclic flat). In particular, the subsequence ending with $F$ (i.e. $\emptyset \subset \cdots \subset F$) yields the active filtration of $B \cap F$ in $M(F)$, and the subsequence beginning with $F$ (i.e. $F \subset \cdots \subset E$) yields the active filtration of $B \setminus F$ in $M/F$ by subtracting $F$ from each subset.

**Example 4.24.** Figures 2, 3, 4, 5 show active decompositions/partitions of some fundamental graphs/tableaux. They illustrate also bases of $K_4$ from Example 2.1 and Figure 1. In the graphs:
The full circles and full squares show the internal/external active elements, and the bold paths of edges connected to these elements show the active partition (restricting the fundamental graph to the subsets of edges forming these parts yield uniaactive fundamental graphs); and the light edges are not involved in the construction. In the tableaux: the full circles and full squares show the active partition (restricting the fundamental tableau to the subsets of entries forming these parts yield uniaactive fundamental tableaux); and the circled crosses and the little squares are not involved in the construction (circled crosses disappear when restricting to tableaux induced by the active partition). The fundamental circuits and cocircuits are also indicated at the beginning of concerned rows and columns of the tableaux.

**Theorem 4.25.** Let $M$ be a matroid on a linearly ordered set $E$.

$$\{ \text{bases of } M \} = \bigcup_{\emptyset = F_0 \subset \cdots \subset F_\varepsilon = F_\iota} \{ B'_1 \uplus \cdots \uplus B'_\varepsilon \uplus B_1 \uplus \cdots \uplus B_\iota \mid \text{connected filtration of } M$$

for all $1 \leq k \leq \varepsilon$, $B'_k$ base of $M(F'_{k-1})/F'_k$ with $\iota(B'_k) = 0$ and $\varepsilon(B'_k) = 1$,

for all $1 \leq k \leq \iota$, $B_k$ base of $M(F_k)/F_{k-1}$ with $\iota(B_k) = 1$ and $\varepsilon(B_k) = 0$.

With the above notations and $B = B'_1 \uplus \cdots \uplus B'_\varepsilon \uplus B_1 \uplus \cdots \uplus B_\iota$, we then have:

$$\text{Int}(B) = \bigcup_{1 \leq k \leq \varepsilon} \min(F_k \setminus F_{k-1}) = \bigcup_{1 \leq k \leq \iota} \text{Int}(B_k),$$

$$\text{Ext}(B) = \bigcup_{1 \leq k \leq \varepsilon} \min(F'_{k-1} \setminus F'_k) = \bigcup_{1 \leq k \leq \iota} \text{Ext}(B'_k).$$

Moreover, the connected filtration associated to the basis $B$ in the right-hand side of the equality is the active filtration of (the fundamental graph of) $B$.  

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Proof. This theorem simply consists in applying Theorem 4.22 to all bases at the same time. Let \( B \) be a basis of \( M \) with fundamental graph \( F \). By Theorem 4.22, the active filtration of \( F \) induces exactly the partition \( B = B'_1 \uplus ... \uplus B'_\epsilon \uplus B_1 \uplus ... \uplus B_\iota \) as stated in the present theorem. By properties of these bases, we have \( \cup_{1 \leq k \leq \iota} \min(F_k \setminus F_{k-1}) = \cup_{1 \leq k \leq \iota} \text{Int}(B_k) \) and \( \cup_{1 \leq k \leq \epsilon} \min(F'_{k-1} \setminus F'_k) = \cup_{1 \leq k \leq \epsilon} \text{Ext}(B'_k) \). And, by definition of the active filtration and Lemma 4.18, we have \( \text{Int}(B) = \cup_{1 \leq k \leq \iota} \min(F_k \setminus F_{k-1}) \) and \( \text{Ext}(B) = \cup_{1 \leq k \leq \epsilon} \min(F'_{k-1} \setminus F'_k) \).

Conversely, let \( B'_1, ..., B'_\epsilon, B_1, ..., B_\iota \) as stated in the present theorem for a given connected filtration of \( M \). Obviously, and as shown in the proof of Theorem 4.22, we have that \( B = B'_1 \uplus ... \uplus B'_\epsilon \uplus B_1 \uplus ... \uplus B_\iota \) is a basis of \( M \). Furthermore, by uniqueness property in Theorem 4.22, we have that the filtration is the active filtration of the fundamental graph of \( B \), which implies as above that \( \text{Int}(B) = \cup_{1 \leq k \leq \iota} \min(F_k \setminus F_{k-1}) = \cup_{1 \leq k \leq \iota} \text{Int}(B_k) \), and \( \text{Ext}(B) = \cup_{1 \leq k \leq \epsilon} \min(F'_{k-1} \setminus F'_k) = \cup_{1 \leq k \leq \epsilon} \text{Ext}(B'_k) \). \( \square \)

**Remark 4.26.** One sees how the uniqueness result in Theorem 4.22 is important. The easier result, without uniqueness, contained in this theorem just states that the bases induced in the active minors induced by the active filtration are uniactive internal/external. From this weaker result, one could derive a weaker version of Theorem 4.25 above with a union instead of a disjoint union, and then a weaker version of the Tutte polynomial formula in Theorem 3.5 with an inequality instead of an equality. It is the uniqueness that allows to state Theorems 4.25 and 3.5 as they are.

**Corollary 4.27** ([3]). Let \( M \) be a matroid on a linearly ordered set \( E \).

\[
\{ \text{bases of } M \} = \biguplus_{F_c \text{ cyclic flat of } M} \{ \text{bases of } M(F_c) \text{ with internal activity } 0, \text{ base of } M/F_c \text{ with external activity } 0 \}
\]

**Proof.** Direct by Observation 4.21 and Theorem 4.25 applied to decompose the set of bases of \( M \), the set of external bases of \( M(F_c) \), and the set of internal bases of \( M/F_c \), for all cyclic flats \( F_c \) of \( M \). \( \square \)

**Example 4.28.** Figure 6 shows the decomposition of bases of \( K_4 \), provided by Theorem 4.25, completing Example 2.1, Example 4.24, and Figures 1, 2, 3, 4, 5.
Proposition 4.29 (Single-pass computation of the active partition of a matroid basis or a fundamental graph/tableau). Let $M$ be a matroid on a linearly ordered set of elements $E = e_1 < \ldots < e_n$. Let $B$ be a base of $M$. The algorithm below computes the active partition of $B$ as a mapping, denoted $\text{Part}$, from $E$ to $\text{Int}(B) \cup \text{Ext}(B)$, that maps an element onto the smallest element of its part in the active partition of $B$. An element is called internal, resp. external, if its image is in $\text{Int}(B)$, resp. $\text{Ext}(B)$. Hence the active partition of $B$ is

$$\bigcup_{e \in \text{Int}(B) \cup \text{Ext}(B)} \text{Part}^{-1}(e),$$

with external part given by $\text{Part}^{-1}(\text{Ext}(B))$. The algorithm consists in a single pass over $E$. It only relies upon the fundamental graph/tableau (and can be equally applied to decompose a fundamental graph/tableau). Note that the rules when $e_k \in B$ are dual to the rules when $e_k \notin B$, and that the rules when $e_k$ is internal are dual to the rules when $e_k$ is external.

For $k$ from 1 to $n$ do

if $e_k \in B$ then

if $e_k$ is internally active w.r.t. $B$ then

$e_k$ is internal

let $\text{Part}(e_k) := e_k$

otherwise

it there exists $c < e_k$ external in $C^*(B; e_k)$ then

$e_k$ is external
Proof. Let us denote elements of algorithm. Not interfere in each other, since they consist in refinement of the two separate outputs of the first algorithms, each consisting in a single pass over formal proof, let us mention that this algorithm simply consists in a direct combination of the partitions that Part(c)

First, assume that e \in acl(a_1, \ldots, a_i), resp. a'_1, \ldots, a'_i, the set of internally, resp. externally, active elements of B, and (F'_1, \ldots, F'_i) the active filtration of F_M(B). Before giving a formal proof, let us mention that this algorithm simply consists in a direct combination of the following algorithms, each consisting in a single pass over E. The second and third algorithms do not interfere in each other, since they consist in refinements of the two separate outputs of the first algorithm.

- The algorithm of Corollary 4.14 that computes the external/internal partition.
- The algorithm of Lemma 4.6, applied in priority to X = \{a_i\}, then to X = \{a_{i-1}, a_i\}, etc., then to X = \{a_1, \ldots, a_i\}. By this manner, an element e belonging to the internal part is mapped onto a_i, where i is the greatest possible such that e \in acl(a_i, \ldots, a_i) = E \setminus F_{i-1}, in order to have e \in F_i \setminus F_{i-1}, consistently with the definition of the active partition.
- The algorithm of Lemma 4.6 applied in the dual, and in priority to X = \{a'_i\}, then to X = \{a'_i, a'_i\}, etc., then to X = \{a'_1, \ldots, a'_i\}. By this manner, an element e belonging to the external part is mapped onto a'_i, where i is the greatest possible such that e \in acl(a'_i, \ldots, a'_i) = F'_i, in order to have e \in F'_i \setminus F'_i, consistently with the definition of the active partition.

Now, let us verify precisely the assignments given in the algorithm statement. Let 1 \leq k \leq n. First, assume that e_k \in B and e_k is internally active, then, obviously, Part(e_k) = e_k.

Second, assume that e_k \in B, e_k is not internally active, and every c in C^*(B; e_k) is internal. Then, by Corollary 4.14, e_k is internal. Then, by Lemma 4.6, we have e_k \in acl(a_i, \ldots, a_i) for all i such that C^*(B; e_k) \subseteq acl(a_i, \ldots, a_i). Let i be the greatest possible with this property. By definition of the active partition, we have Part(c) = a_i, as we have e_k \in acl(a_i, \ldots, a_i) \setminus acl(a_{i+1}, \ldots, a_i) = F_i \setminus F_{i-1}. Let c \in C^*(B; e_k). We also have by definition of the active partition that Part(c) = a_j where j is the greatest possible such that c \in acl(a_j, \ldots, a_i). Since c \in
$C^*(B; e_k)^< \subseteq \text{acl}(a_i, \ldots, a_i)$ by definition of $i$, we have $i \leq j$. Assume now that $c \not\in \text{acl}(a_{i+1}, \ldots, a_i)$ (such a $c$ exists by definition of $i$). In this case, we have $i = j$, by definition of $j$. We have proved that $a_i = \text{Part}(e_k)$ is the smallest possible $a_j = \text{Part}(c)$ over all $c \in C^*(B; e_k)^<$, which is exactly the assignment given in the algorithm.

Third, let us assume that assume that $e_k \notin B$, $e_k$ is not externally active, and there exists $c \in C(B; e_k)^<$ which is internal. Then, by Corollary 4.14, $e_k$ is internal. Then, by Lemma 4.6, we have $e_k \in \text{acl}(a_1, \ldots, a_i)$ for all $i$ such that there exists $c \in C(B; e_k)^< \cap \text{acl}(a_1, \ldots, a_i)$. Let $i$ be the greatest possible with this property. By definition of the active partition, we have $\text{Part}(e_k) = a_i$ (as above). By definition of $c$, we have also $c \in \text{acl}(a_1, \ldots, a_i) \setminus \text{acl}(a_{i+1}, \ldots, a_i) = F_i \setminus F_{i-1}$, that is $\text{Part}(c) = a_i = \text{Part}(a_k)$. We have proved that $a_i = \text{Part}(e_k)$ is the greatest possible $a_j = \text{Part}(c)$ over all $c \in C(B; e_k)^< \cap \text{acl}(a_1, \ldots, a_i)$, which is exactly the assignment given in the algorithm.

The three other cases (where $e_k$ is external) are dual to the three above cases, which completes the proof.

Proof of Theorem 3.5. First, the fact that the two sets of sequences can be equally used directly comes from Lemma 3.2. Now, let us focus on the sum over connected filtrations of $M$. Recall that:

- for a matroid $M$ with at least two elements, there exists a uniactive internal basis, and equivalently a uniactive external basis, of $M$ if and only if $\beta(M) \neq 0$, and equivalently $\beta^*(M) \neq 0$;
- for a matroid $M$ with one element, $\beta(M) \neq 0$ if and only if $M$ is an isthmus (which is an internal basis);
- for a matroid $M$ with one element, $\beta^*(M) \neq 0$ if and only if $M$ is a loop (which is an external basis).

So we have that $\beta(M) \neq 0$, resp. $\beta^*(M) \neq 0$, if and only if $M$ has a uniactive internal, resp. external, basis. Then the formula given in the theorem is exactly the enumerative translation of Theorem 4.25.

More precisely, consider the set of bases of $M$ with internal activity $\iota$ and external activity $\varepsilon$, whose cardinality is $b_{\iota, \varepsilon}$. By Theorem 4.25, using the same notations, this set bijectively corresponds to the set $\bigcup \{B \mid 1 \leq k \leq \iota, B_k \text{ uniactive internal in } M_k, \text{ and for } 1 \leq k \leq \varepsilon, B_k' \text{ uniactive external in } M_k'\}$ where the union is over all connected filtrations of $M$ with fixed $\iota$ and $\varepsilon$. The cardinality of each part of this set is obviously $\left(\prod_{1 \leq k \leq \iota} \beta(M_k)\right) \left(\prod_{1 \leq k \leq \varepsilon} \beta^*(M_k')\right)$ since $\beta$, resp. $\beta^*$, counts the number of uniactive internal, resp. external, bases. By construction, the sum is the number of bases with internal activity $\iota$ and external activity $\varepsilon$, that is the coefficient $b_{\iota, \varepsilon}$ of $x^\iota y^\varepsilon$ in the Tutte polynomial, hence the result.

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