The Role of the Core Energy in the Vortex Nernst Effect

Gideon Wachtel and Dror Orgad
Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel

We present an analytical study of diamagnetism and transport in a film with superconducting phase fluctuations, formulated in terms of vortex dynamics within the Debye-Hückle approximation. We find that the diamagnetic and Nernst signals decay strongly with temperature in a manner which is dictated by the vortex core energy. Using the theory to interpret Nernst measurements of underdoped La$_{2-x}$Sr$_x$CuO$_4$ above the critical temperature-parameter regime we obtain a considerably better fit to the data than a fit based on Gaussian order-parameter fluctuations. Our results indicate that the core energy in this system scales roughly with the critical temperature and is significantly smaller than expected from BCS theory. Furthermore, it is necessary to assume that the vortex mobility is much larger than the Bardeen-Stephen value in order to reconcile conductivity measurements with the same vortex picture. Therefore, either the Nernst signal is not due to superconducting phase fluctuations, or that vortices in underdoped La$_{2-x}$Sr$_x$CuO$_4$ have highly unconventional properties.

Over the past decade the Nernst effect has become a widely used tool in the study of strongly correlated electronic systems. The Nernst signal $e_N = E_y/(-\partial_x T)$, defined by the ratio between a measured electric field $E_y$ and a transverse applied temperature gradient $\partial_x T$ in an electrically isolated system subjected to an external magnetic field $H_z$, is typically very small in non-magnetic normal metals. Conversely, a much stronger effect may arise in the flux-flow regime of superconductors, owing to the transverse electric fields induced by the motion of vortices down the temperature gradient. Consequently, the observation of a large Nernst signal in the pseudogap state of the cuprates has not been previously justified by an analytical treatment. However, several studies have calculated the Nernst signal arising from superconducting order-parameter fluctuations. The available data imply that both $e_c$ and $T_c$ share a similar doping dependence, with $e_c$ approximately as much larger than the Bardeen-Stephen value.

Experimentally, good agreement with the Gaussian theory was found in amorphous Nb$_{0.15}$Si$_{0.85}$ films and in overdoped, but not underdoped cuprates (see, however, Ref. 17). A different approach, more pertinent to the present study, was taken by Podolsky et al., who built upon the premise that in underdoped cuprates, superconductivity is destroyed at $T_c$ by strong phase fluctuations, whereas pairing correlations survive up to a considerably higher scale $T_p$. Ignoring superconducting amplitude fluctuations the authors calculated the Nernst signal in a stochastic two-dimensional (2D) XY model via numerical simulations and a high-temperature expansion. In addition, they devised a simulation method to calculate the thermoelectric response based on vortex dynamics. In this Letter we aim to bridge the aforementioned theoretical gap and present an analytical study of diamagnetism and transport in an extreme type-II superconducting film that is formulated directly in terms of vortices. We focus on temperatures above $T_c$ where there is a finite density, $n_f$, of free, unbound vortices. Our approach, which treats the vortex interactions within a Debye-Hückle approximation, is inspired by Ambegaokar et al. who considered vortex dynamics in the context of superfluid films. A similar route was taken in the study of the resistive transition of superconducting films by Halperin and Nelson.

Our treatment identifies the vortex core energy $e_c$ as an important energy scale which controls the strong temperature dependence of the fluctuation signals. Using the theory we are able to obtain a fit to the transverse thermoelectric response of underdoped L$\theta_b$-$z$Sr$_2$CuO$_4$ (LSCO) which is superior to the one based on Gaussian fluctuations. Such values are significantly lower than the Fermi energy, which is the expected $e_c$ from BCS theory. Moreover, in order to reconcile the vortex picture with conductivity data, one needs to assume that the vortex mobility is much larger than the Bardeen-Stephen value. Thus, unless the strong Nernst and diamagnetic signals in underdoped LSCO are of non-superconducting origin, it appears that the vortex core is unconventional and plays an important role in this system.

Vortex Hamiltonian and dynamics. A 2D superconductor, at temperatures well below $T_p$ where the order parameter amplitude is frozen, can be described by an XY-type Hamiltonian density of a phase field $\theta$ coupled via its charge, $(2\epsilon < 0)$, to an electromagnetic vector potential $A$, and a constant superfluid density $\rho_s$:

$$ H = (1+\psi) \left[ \frac{\rho_s}{2} \left( \nabla \theta - \frac{2\epsilon}{\hbar c} A \right) \right]^{2} + \sum_i \epsilon_c \delta(\mathbf{r} - \mathbf{r}_i) . \quad (1) $$

We assume that only vortices contribute to the otherwise uniform $\nabla \theta$. A vortex $i$ of vorticity $n_i = \pm 1$ at
coordinates \( \mathbf{r}_i = (x_i, y_i) \) contributes

\[
\nabla \theta_i(\mathbf{r}) = n_i \hat{z} \times \nabla \ln \frac{\lvert \mathbf{r} - \mathbf{r}_i \rvert}{r_0} = n_i \frac{\hat{z} \times (\mathbf{r} - \mathbf{r}_i)}{\lvert \mathbf{r} - \mathbf{r}_i \rvert^2},
\]

where \( r_0 \) is the vortex core radius, and \( \hat{z} \) is a unit vector perpendicular to the plane. The continuum model and vortex configuration, Eqs. (12), are valid at scales longer than \( r_0 \). Thus, a region of radius \( r_0 \) around \( \mathbf{r}_i \) is implicitly removed from the first term in Eq. (1). Its energy is given by the vortex core energy \( \epsilon_c \), which we assume to be constant across the sample. Following Luttinger, we have introduced a “gravitational” field \( \psi(\mathbf{r}) \) in order to study the response of the system to a temperature gradient.

For concreteness, we consider a superconducting strip of infinite extent along the \( y \) direction, and of finite width \( L \) in the \( x \) direction. When needed, a constant transverse temperature gradient is applied via \( \psi(\mathbf{r}) = \psi x \), and a uniform electric field \( \mathbf{E} = E_0 \hat{y} \) is applied along the strip. Working in the extreme type-II limit we assume the presence of a uniform perpendicular magnetic field \( B \hat{z} \), and choose the gauge \( \mathbf{A} = \mathbf{A}_0 + \mathbf{A}_E \), where \( \mathbf{A}_0 = Bx \hat{y} \), and \( \mathbf{E} = -\partial_t \mathbf{A}_E / c. \) By symmetry, the average (over vortex' positions) phase gradient \( \langle \nabla \theta \rangle \) is directed along the strip and is independent of the \( y \) coordinate.

We approach the model given by Eq. (1) within a mean-field Debye-Hückle approximation, in which correlations between vortices are ignored. This is possible to show that \( n_f \sim \sqrt{4r_0^{-4}e^{-2e_c/T} + n^2} \),

\[
J_y(x) = \frac{4\pi^2 \rho_c}{\phi_0} (1 + \psi)(u - a).
\]

Thus, the first term in Eq. (3) is just the vortex drift in response to the Magnus force it experiences in an electric current \( J_y \). Note, that all free vortices, and not only those responsible for the excess vorticity, contribute to the vorticity current, Eq. (3), via their response to the Magnus force. As a result, the strong temperature dependence of \( n_f \) is also reflected in the transport coefficients.

\[
M_z = \frac{1}{cA} \int dy \int_0^L dx x J_y \sim -\frac{TB}{\phi_0 n_f},
\]

where \( A \) is the area of the strip. Here, and in the following, we ignore corrections of order \( O(r_s/L) \).
expressions to Eq. [10] were obtained in several previous studies [22,30,31].

Electric conductivity. In order to study the linear response of the system to a weak perturbing field \( E_0(\omega)e^{-i\omega t} \) we need to obtain the dynamics of \( u(x,t) \). By employing translational invariance in the \( y \) direction\,[22] one can show that

\[
\frac{\partial u}{\partial t} = -J_x^e. \tag{11}
\]

This is a local version of the equation used in Refs. [21,22]. Solving it using Eq. (5), we find in the bulk \( u(x,t) = \bar{u}x + u(\omega)e^{-i\omega t} \) where

\[
u(\omega) = \frac{1}{1 - i\omega\tau} \frac{cE_0(\omega)}{\omega\phi_0}, \tag{12}
\]

and where we have introduced the relaxation time \( 1/\tau = 4\pi^2\rho_s\mu_n f \). Eq. (7) then implies an electric conductivity

\[
\sigma_s(\omega) = \frac{4e^2}{h} \frac{1}{\mu_n f} \frac{1}{1 - i\omega\tau}. \tag{13}
\]

This result is identical to the conductivity obtained by Halperin and Nelson\,[22] for temperatures above \( T_c \).

Thermoelectric coefficients. For systems with particle-hole symmetry or when superconducting fluctuations dominate, the Nernst signal is given by \( e_N = \rho_e u \) \( = -\alpha u \), where \( \alpha \) is defined by \( \alpha = \alpha_{xy} = -\partial_xJ_y = \alpha_{yx}(-\partial_y J_x) \). Luttinger has shown\,[25] that \( \alpha \) can be used to a “gravitational” field \( \psi \) according to the equation

\[
\frac{\partial J_y}{\partial t} = \alpha(x) \frac{\partial \psi}{\partial x}. \tag{16}
\]

This result should be compared with the constant ratio between \( \alpha_{xy} \) and \( cM_z/T \), which was found for high temperatures in Refs. [8,18,20].

Next, we consider the linear response ratio \( \alpha_{xy} \) between an applied electric field and a transverse heat current density, \( J_x^Q = \alpha_{xy}E_y \). We deduce \( J^Q \), which in our model equals the energy current density, from the conservation equation \( \partial_t \mathcal{H} + \nabla \cdot J^Q = \mathcal{J} \cdot \mathbf{E} \). Its source term originates from the explicit time dependence of \( \mathcal{H} \) via \( \mathbf{A} \). The result

\[
J^Q = -\rho_s \frac{\partial \theta}{\partial t} \left( \nabla \theta - \frac{2e}{\hbar c} \mathbf{A} \right) + \sum \epsilon_c J^c, \tag{17}
\]

is consistent with the form used by Ussishkin \textit{et al} \,[26], once modified to include the energy current associated with the vortex cores. If we additionally assume that the long superconducting strip is periodic in the \( y \) direction, then the \( x \) component of the first term in Eq. (17) must vanish by symmetry, and we find that Onsager’s relation \( \alpha_{xy}(-B) = \alpha_{xy}(B) \) is obeyed.

Discussion. Often (see Refs. [11,14] and references therein), a phenomenological quantity called the vortex transport entropy, \( s_0 \), is invoked in order to relate the temperature gradient to the thermal force acting on a vortex, \( \mathbf{f} = -s_0 \nabla T \). Based on Eq. (4) and Luttinger\,[25], we identify \( s_0 = \epsilon_c/T \). For low temperatures where there are no thermally excited vortices and the flux-flow resistivity is the dominant form of damping, one can show by neglecting vortex interactions\,[27] that \( \alpha_{xy} = -cs_0/\phi_0 \). When taken together with the above identification of \( s_0 \), this result is consistent with Eq. (10), since at low temperatures \( \bar{n}_f \phi_0 = 0 \).

As the temperature is raised through \( T_{BKT} \), the density of free vortices, \( n_f \), rapidly increases. Our results, Eqs. (4,14,16), indicate that both \( M_z \) and \( \alpha_{xy} \) should exhibit a consequent strong reduction with temperature, much faster than the 1/\( T \) decay expected from Gaussian fluctuations\,[8,11,12]. To look for such behavior in the cuprates we compare Eq. (16) divided by the LSCO layer separation, \( d = 6.5 \AA, \alpha_{xy}^L \approx \alpha_{xy}/d \) with undoped LSCO data. According to Eq. (4), \( n_f \) is determined by the renormalized vortex core energy \( \epsilon_c \), which reflects fluctuations at distances below \( r_s \) and is temperature dependent. For weak magnetic fields and in the critical regime above \( T_{BKT} \) this renormalization leads to \( n_f \sim \exp(-b/\sqrt{T - T_{BKT}}) \), while at high temperatures \( n_f \sim \exp(-\epsilon_c/(T-b)) \). Here \( b \) and \( \epsilon_c \) are constants and \( \epsilon_c \) is the bare core energy. The lack of detailed knowledge about the \( \epsilon_c \) temperature dependence of \( \epsilon_c \) allows for considerable freedom in the fitting procedure. In order to constrain the fit, and since we are only interested in a rough estimate of \( \epsilon_c \), we choose to consider a constant \( \epsilon_c \) and also set \( \phi_0/2\pi\tau_0^L = 50T \). Furthermore, we concentrate on the limit \( B \to 0 \) and temperatures sufficiently above \( T_c \), where the renormalization effects are expected to be small, but low enough so that vortices are distinct objects, \( \epsilon_c \tau_0^L n_f \ll 1 \). Figure 1 depicts the measured \( B \to 0 \) limit of \( -\epsilon_c^L/B \) for LSCO samples with
\[ x(T_c) = 0.07 \, (11 \, \text{K}), \, 0.10 \, (27.5 \, \text{K}) \, \text{and} \, 0.12 \, (29 \, \text{K}). \]

The solid color lines are the theoretical fits in the temperature window \( 1.1T_c < T < 3T_c \) with a constant \( \epsilon_c \) as the only free fitting parameter. From these curves we find \( \epsilon_c \approx 58, 114, 143 \, \text{K}, \) for the different doping levels. Comparably, but somewhat larger values, \( \epsilon_c \approx 8T_c \), were found by analyzing penetration depth measurements in underdoped \( \text{Y}_1-x\text{Ca}_x\text{Ba}_2\text{Cu}_3\text{O}_{7-\delta} \) bilayer films. For comparison we also include the best fit to the data based on the theory of Gaussian fluctuations. Clearly, the data exhibits a faster decay than the Gaussian theory above the critical region around \( T_c \). In addition, we fitted the data to the high-\( T \) result \( \alpha_{yx} \propto T^{-4} \) of the stochastic XY model. We obtained a good fit for \( x = 0.12 \), but found overestimation of the data in the range \( 1.1T_c < T < 2T_c \) \( (3T_c) \) for \( x = 0.10 \) \( (0.07) \).

The Nernst effect onset temperature, \( T_{\text{onset}} \), is defined as the temperature for which the Nernst coefficient \( \nu = \epsilon N/B \) goes below a threshold value, typically around \( \nu \approx 4 \, \text{nV/KT} \). Such levels can be reached using Eq. \( (16) \) only if one takes \( r_n^2 n_f \sim 1 \). This, however, is beyond the validity of our theory. Indeed, we find that the experimental data begin to deviate from the theoretical curves at temperatures where \( r_n^2 n_f > 0.35 \), indicated by dashed lines in Fig. 1. Thus, although our theory agrees with the Nernst measurements up to \( T \approx 3T_c \), it cannot account for \( T_{\text{onset}} \), which is probably controlled by a combination of lattice effects and amplitude fluctuations.

The Nernst signal in the cuprate pseudogap regime exhibits a maximum as a function of the magnetic field, which shifts to higher fields with increasing temperature. While we do not have a theory for the maximum we note that Eqs. \( (16) \) imply a crossover, set by the condition \( B/\phi_0 \sim n_f(T, B = 0) \), from a linear-\( B \) dependence of \( \alpha_{yx} \) at weak fields towards saturation at higher fields. Across this scale magnetic field-induced vortexes dominate, screening is reduced and correlation effects are enhanced, leading potentially to the suppression of \( \alpha_{yx} \).

In conclusion, we showed that within the vortex picture of phase fluctuating superconductors, \( \epsilon_c \) plays an essential role in the thermoelectric response. The vortex core energy was also found to be important in determining \( T_c \) of layered superconductors. Uncovering the role played by \( \epsilon_c \) in other phenomena may help in identifying the physics underlying the different temperature scales observed in the cuprates. Equally pertinent is gaining an understanding of the factors which control \( \epsilon_c \) itself. Here we briefly mention the need for a model of “cheap vortexes”, in which vortexes support a state close in energy to the superconducting phase. It seems to us that the checkerboard state observed around vortex cores is a natural candidate.

Nevertheless, if the Nernst signal in underdoped cuprates is, in fact, due to thermally excited vortexes, one must also understand why experiments do not show signatures of fluctuation enhanced conductivity over a similar temperature range. More specifically, if the vortex mobility is given by the Bardeen-Stephen result, \( \mu \approx 8\pi e^2 r_n^2/h^2 \sigma_n \), then Eq. \( (13) \) gives a fluctuation contribution \( \sigma_{\alpha} = \sigma_n/2\pi r_n^2 n_f \), where \( \sigma_n \) is the normal state conductivity. This would imply, using our estimate \( \epsilon_c \approx 4-5T_c \), from fitting the LSCO Nernst data, and Eq. \( (16) \), that \( \sigma_{\alpha} > \sigma_n \) for \( T < 2T_c \), in contradiction to experiments. To avoid such a contradiction within our model, we must therefore assume that \( \mu \) is much larger than the Bardeen-Stephen value, thereby reducing \( \sigma_{\alpha} \) while not affecting \( M_z \) and \( \alpha_{yx} \). A similar conclusion regarding \( \mu \) was reached based on THz time-domain spectroscopy in LSCO. The above discussion further indicates that understanding the vortex core in the cuprates may call for physics beyond standard BCS theory.

We would like to thank Daniel Podolsky for helpful discussions. This research was supported by the Israel Science Foundation (Grant No. 585/13).

---

1. For a review of earlier results see, R. P. Huebener, Supercond. Sci. Technol. 8, 189 (1995).
2. Z. A. Xu, N. P. Ong, Y. Wang, T. Kakeshita, and S. Uchida, Nature (London) 406, 486 (2000).
3. Y. Wang, Z. A. Xu, T. Kakeshita, S. Uchida, S. Ono, Y. Ando, and N. P. Ong, Phys. Rev. B 64, 224519 (2001).
4. Y. Wang, L. Li, and N. P. Ong, Phys. Rev. B 73, 024510 (2006).
\[\psi(t) = \frac{\sqrt{\pi}}{\sqrt{2\mu}} \exp\left(-\frac{r^2}{2\mu}\right)\]

The effect of a temperature gradient on a vortex enters this Fokker-Planck equation via the "gravitational" field \(\psi\) in the Hamiltonian. Alternatively, it is possible to introduce a position dependent temperature, \(T(r)\), directly into the Fokker-Planck equation, in a manner which depends on the underlying microscopic dynamics. Assuming a vortex behaves like a Brownian particle, the corresponding probability current density is

\[J_i(r, t) = -\mu P \langle 0 | \nabla_i H | 0 \rangle - \mu \nabla_i \langle 0 | T(r) P(r, t) | 0 \rangle \]

This approach reproduces the same results we get by calculating the response to \(\psi\), assuming that the density of free vortices is at local equilibrium, \(n_f(r) \sim \exp(-\epsilon_i/T(r))\).
I. DEBYE-HÜCKLE APPROXIMATION IN EQUILIBRIUM

At high temperatures, it is possible to study the vortex Hamiltonian within the Debye-Hückle approximation, which is best formulated using a variational mean-field approach. Assume that the state of the system is defined by the vorticity at each lattice site, \( n_r = 0, -1, +1 \). In the variational mean-field ansatz the density matrix is factored into a product of local probabilities,

\[
\rho = \prod_r \rho_r(n_r),
\]

with the effect that the entropy is given by

\[
S = -\text{Tr} \rho \ln \rho = -\sum_r \sum_{n_r} \rho_r(n_r) \ln \rho_r(n_r).
\]

Additionally, one approximate the average Hamiltonian by

\[
\langle H \rangle \approx \frac{1}{2} \rho_s \int d^2 r (1 + \psi) \left( (\nabla \theta) - \frac{2e}{\hbar c} A \right)^2 + e_c \sum_r (1 + \psi(r)) \langle |n_r| \rangle,
\]

while ignoring the contribution coming from fluctuations in \( \nabla \theta \),

\[
\langle H_{\text{fluc.}} \rangle = \frac{1}{2} \rho_s \int d^2 r (1 + \psi) \left( (\nabla \theta)^2 - \langle \nabla \theta \rangle^2 \right).
\]

\( \langle \nabla \theta \rangle \) is given by

\[
\langle \nabla \theta(r) \rangle = \nabla \theta + \sum_{r'} \langle n_{r'} \rangle \begin{pmatrix} \hat{z} \times (r - r') \end{pmatrix} \frac{(r' - r)}{(r' - r)^2},
\]

where \( \nabla \theta \) is the uniform part of \( \nabla \theta(r) \), which does not rise from vortices,

\[
\langle n_r \rangle = \sum_{n_r} \rho_r(n_r)n_r,
\]

and

\[
\langle |n_r| \rangle = \sum_{n_r} \rho_r(n_r)|n_r|.
\]

\( \rho_r(n_r) \) itself is determined by minimizing the free energy \( F = \langle H \rangle - TS \), with the constraint

\[
\sum_{n_r} \rho_r(n_r) = 1,
\]

\[
\frac{\partial F}{\partial \rho_r(n_r)} = \rho_s \int d^2 r' \left( (\nabla' \theta(r')) - \frac{2e}{\hbar c} A(r') \right) \cdot \begin{pmatrix} \hat{z} \times (r' - r) \end{pmatrix} \frac{\hat{z} \times (r' - r)}{(r' - r)^2} \rho_r(n_r) + e_c(1 + \psi(r))|n_r| + T \ln \rho_r(n_r) = \alpha.
\]

Solving for \( \rho_r \) we find

\[
\rho_r(n_r) = \frac{1}{z_r} e^{-\beta e_c |n_r| - \beta \varphi(r)n_r},
\]

where

\[
\varphi(r) = \rho_s \int d^2 r' \left( (\nabla' \theta(r')) - \frac{2e}{\hbar c} A(r') \right) \cdot \begin{pmatrix} \hat{z} \times (r' - r) \end{pmatrix} \frac{\hat{z} \times (r' - r)}{(r' - r)^2}.
\]
and
\[ z_r = 1 + e^{-\beta \epsilon} \cosh \beta \varphi(r). \] (29)

For small \( e^{-\beta \epsilon} \) we find
\[ \langle n_r \rangle \approx e^{-\beta \epsilon} \cosh \beta \varphi(r), \] (30)
and
\[ \langle n_r \rangle \approx -e^{-\beta \epsilon} \sinh \beta \varphi(r). \] (31)
Eliminating \( \varphi \) gives
\[ \langle |n_r| \rangle = \sqrt{4e^{-2\beta \epsilon} + \langle n_r \rangle^2}, \] (32)
which, after dividing through by \( r_0^2 \), reads
\[ n_f = \sqrt{4r_0^{-4} e^{-2\beta \epsilon} + n^2}. \] (33)

II. VORTEX DYNAMICS

A. Mean-Field Fokker-Planck equations

In order to formulate dynamics of the vortices in our model, we assume that the number of vortices is the same as in equilibrium, and that their vorticity is fixed. Events of vortex-anti-vortex creation and annihilation are important for non-linear response at \( T_c \), but have a negligible effect on linear response, and are therefore ignored. Thus, it is possible to formulate vortex dynamics using a Fokker-Planck equation for the positions of all vortices, \( \{r_i\} \), each with a given vorticity \( \{n_i = \pm 1\} \):
\[ \frac{\partial P(\{r_i\}, t)}{\partial t} = \sum_i \left\{ \mu \nabla_i \cdot [P(\{r_i\}, t) \nabla_i H] + \mu T \nabla_i^2 P(\{r_i\}, t) \right\}, \] (34)
where \( \mu \) is the vortex mobility, \( \nabla_i \) is the gradient with respect to \( r_i \), and \( k_B = 1 \) is used throughout. This is a complicated equation to solve, but it can be treated approximately, in a manner similar to the Debye-Hückle approximation in equilibrium, by factoring the probability density into a product of single vortex probabilities,
\[ P(\{r_i\}, t) = \prod_i P_i(r_i, t). \] (35)

Integrating the left side of Eq. (34) over the positions of all vortices aside from the position of the \( i \)th gives
\[
\prod_{j \neq i} \int d^2 r_j \frac{\partial P(\{r_i\})}{\partial t} = \prod_{j \neq i} \int d^2 r_j \sum_k \prod_{i \neq k} P_i(\{r_i\}, t) \frac{\partial P_k(\{r_k\}, t)}{\partial t} \\
= P_i(\{r_i\}, t) \sum_{k \neq i} \prod_{j \neq i, k} \left( \int d^2 r_j P_j(\{r_j\}, t) \right) \left( \int d^2 r_k \frac{\partial P_k(\{r_k\}, t)}{\partial t} \right) + \prod_{j \neq i} \left( \int d^2 r_j P_j(\{r_j\}, t) \right) \\
= \frac{\partial P_i(\{r_i\}, t)}{\partial t},
\] (36)
where we demand that the single vortex probabilities are normalized,
\[ \int d^2 r_j P_j(\{r_j\}, t) = 1. \] (37)

Preforming the same integral on the right side of the Fokker-Planck equation gives
\[
\frac{\partial P_i(r_i, t)}{\partial t} = \prod_{j \neq i} \int d^2 r_j \mu \sum_k \nabla_k \cdot \left[ P(\{r_k\}, t) \nabla_k H(\{r_k\}) + T \nabla_k P(\{r_k\}) \right] \\
= P_i(\{r_i\}, t) \mu \sum_{k \neq i} \int d^2 r_k \nabla_k \cdot \left[ P_k(\{r_k\}, t) \nabla_k H(\{r_k\}) + T \nabla_k P_k(\{r_k\}) \right] + \mu \nabla_i \cdot \left[ P_i(\{r_i\}, t) (\nabla_i H) + T \nabla_i P_i(\{r_i\}) \right],
\] (38)
where

\[
\langle \nabla_i H \rangle_i = \prod_{j \neq i} \left( \int d^2 r_j P_j(r_j, t) \right) \nabla_i H, \tag{39}
\]

and

\[
\langle \nabla_k H \rangle_{ik} = \prod_{j \neq i, k} \left( \int d^2 r_j P_j(r_i, t) \right) \nabla_k H. \tag{40}
\]

\(\langle \nabla_k H \rangle_{ik}\) is similar to \(\langle \nabla_k H \rangle_k\) except for an interaction term \(H_{ik}\) between vortex \(k\) and vortex \(i\):

\[
\langle \nabla_k H \rangle_{ik} = \langle \nabla_k H \rangle_k - \langle \nabla_k H \rangle_{ik} + \nabla_k H_{ik}. \tag{41}
\]

Substituting Eq. 41 into Eq. 38 we find that the single vortex Fokker-Planck equation is

\[
\frac{\partial P_i(r_i, t)}{\partial t} = \mu \nabla_i \cdot \left[ P_i(r_i, t) \langle \nabla_i H \rangle_i + T \nabla^2_i P_i(r_i, t) \right], \tag{42}
\]

provided that

\[
\sum_{k \neq i} \int d^2 r_k \nabla \cdot \left[ P_k(r_k, t) \nabla_k H_{ik} - P_k(r_k, t) \langle \nabla_k H \rangle_{ik} \right] = 0. \tag{43}
\]

This can be shown to be the case on our strip where there is translational invariance in the \(y\) direction.

**B. Derivation of the vorticity current**

As shown above, the Fokker-Planck equation can be separated into single vortex equations,

\[
\frac{\partial P_i(r_i, t)}{\partial t} = \mu \nabla_i \cdot \left[ P_i(r_i, t) \langle \nabla_i H \rangle_i + T \nabla^2_i P_i(r_i, t) \right], \tag{44}
\]

where \(\langle -\nabla_i H \rangle_i\) is the force on vortex \(i\), averaged over the position of all other vortices.

\[
\langle \nabla_i H \rangle_i = \prod_{j \neq i} \left( \int d^2 r_j P_j(r_j, t) \right) \nabla_i H(\{r\}) = \nabla_i \frac{\delta \langle H \rangle}{\delta P_i(r_i)}. \tag{45}
\]

Various average quantities can be calculated using the single vortex probability density

\[
P_i(r, t) = \langle \delta(r - r_i(t)) \rangle, \tag{46}
\]

and the probability current density

\[
J_i(r, t) = \langle \delta(r - r_i(t)) \dot{r}_i(t) \rangle. \tag{47}
\]

Interpreting the single vortex Fokker-Planck equation as a probability conservation condition, it is evident that

\[
J_i(r_i, t) = -\mu P_i(r_i, t) \langle \nabla_i H \rangle_i - \mu T \nabla_i P_i(r_i, t). \tag{48}
\]

Translational invariance in the \(y\) direction (along the strip) requires that \(P_i\) and \(J_i\) are independent of the \(y\) coordinate. For example, the vorticity can be written as

\[
\partial_x u(x, t) = \sum_i \langle n_i \delta(r - r_i(t)) \rangle = \sum_i n_i P_i(x, t), \tag{49}
\]

the free vortex density is

\[
n_f(x, t) = \sum_i \langle \delta(r - r_i(t)) \rangle = \sum_i P_i(x, t), \tag{50}
\]
and the vorticity current is given by

\[ J^v_x(x,t) = \sum_i \langle n_i \delta(\mathbf{r} - \mathbf{r}_i(t)) \dot{x}_i \rangle = \sum_i n_i J_{i,x}(x,t). \]  

(51)

Ignoring the same fluctuation term in \( \langle H \rangle \) as in Eq. 20, we find

\[
\frac{\partial \langle H \rangle}{\partial x_i} = \frac{\partial}{\partial x_i} \delta \langle H \rangle \\
\approx n_i \rho_s \frac{\partial}{\partial x_i} \int d^2 r' [1 + \psi(x')] \left( \mathbf{\nabla} \psi(x') \right) \frac{2 e}{\hbar c} \mathbf{A}(r') \cdot \hat{z} \times (r' - \mathbf{r}_i) + e \varepsilon \frac{\partial}{\partial x_i} \psi(r_i) \\
= n_i \rho_s \frac{\partial}{\partial x_i} \int dx' 2\pi [1 + \psi(x')] [u(x') - a(x')] \int dy' \frac{x' - x_i}{(x' - x_i)^2 + (y' - y_i)^2} + e \varepsilon \frac{\partial}{\partial x_i} \psi(x_i) \\
= n_i \rho_s \frac{\partial}{\partial x_i} \int dx' 2\pi [1 + \psi(x')] [u(x') - a(x')] \pi \text{sign}(x' - x_i) + e \varepsilon \frac{\partial}{\partial x_i} \psi(x_i) \\
= -n_i 4\pi^2 \rho_s [1 + \psi(x')][u(x_i) - a(x_i)] + e \varepsilon \frac{\partial}{\partial x_i} \psi(x_i). 
\]

(52)

Therefore, the vorticity current density is

\[
J^v_x(x,t) = \sum_i n_i J_{i,x}(x,t) \\
= \sum_i \left[ - \mu P_i(x,t) \frac{\partial}{\partial x_i} \langle H \rangle \right]_{x_i = x} \\
= \sum_i \left[ \mu P_i(x,t) n_i 4\pi^2 \rho_s [1 + \psi(x')][u(x_i) - a(x_i)] - \mu P_i(x,t) e \varepsilon \frac{\partial}{\partial x_i} \psi(x_i) \right]_{x_i = x} \\
= \sum_i \left[ 4\pi^2 \rho_s \mu P_i(x,t) [1 + \psi(x)][u(x) - a(x)] - \mu e \varepsilon \partial_x \psi \partial_x u - \mu T n_i \partial_x P_i(x,t) \right], 
\]

(53)

which finally gives

\[ J^v_x = 4\pi^2 \rho_s \mu n_f (1 + \psi)(u - a) - \mu e \varepsilon \partial_x \psi \partial_x u - \mu T \partial^2_x u. \]  

(54)

III. DYNAMIC EQUATION FOR \( u \)

In order to study the linear response of the system to weak, time dependent, perturbing fields \( \mathbf{E} \) and \( \nabla \psi \), we must obtain the dynamics of the field \( u(x,t) \).

\[
\frac{\partial u}{\partial t} = \frac{1}{2\pi} \left\langle \sum_i \dot{x}_i \frac{\partial}{\partial x_i} \partial_\theta \right\rangle \\
= \frac{1}{2\pi} \left\langle \sum_i \dot{x}_i \frac{\partial}{\partial x_i} \right\rangle \\
= \frac{1}{2\pi} \left\langle \sum_i \dot{x}_i \frac{\partial}{\partial x_i} n_i \pi \text{sign}(x - x_i) \right\rangle \\
= -\left\langle \sum_i \dot{x}_i n_i \delta(x - x_i) \right\rangle = -\int dy J^v_x. 
\]

(55)

By translational invariance in the \( y \) direction we find

\[ \frac{\partial u}{\partial t} = -J^v_x. \]  

(56)