RESTRICTING CELL MODULES OF PARTITION ALGEBRAS

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ABSTRACT. The restriction of a Specht module to a smaller symmetric group has a filtration by Specht modules of this smaller group. In the cellular structure of the group algebra of the symmetric group, the cell modules are exactly the Specht modules. The partition algebra is a cellular algebra containing the group algebra of the symmetric group. In this article, we study the structure of the restriction of a cell module to the group algebra of a symmetric group (with smaller index) and show that it has Specht filtration if the characteristic of the field is large enough.

1. Introduction

The partition algebra was independently defined by Martin [Mar] and Jones [Jon] to describe the Potts model in statistical mechanics. In representation theory, the partition algebra $P_k(r,\delta)$ arises as a diagram algebra containing the Temperley-Lieb and Brauer algebras. It has nice structural properties such as being cellular [Xi] and quasi-hereditary if and only if $\delta \neq 0$ and $\text{char} k > r$ [KK]. For $r \geq n$, there is Schur-Weyl duality between $k \Sigma_n$ and $P_k(r,n)$, see for example [HR].

An enhanced cellular structure, called cellularly stratified, ensures that the cell modules of $P_k(r,\delta)$ with $\delta \neq 0$, arise from cell modules of the group algebras of symmetric groups with index $n \leq r$ by induction, i.e. they are induced (dual) Specht modules. The same holds for the cell modules of the Brauer algebra. When restricted to a group algebra of a symmetric group, the cell modules of a Brauer algebra admit a filtration by (dual) Specht modules, as shown in [Pag]. The approach used for Brauer algebras is not applicable for the partition algebra, since it is based on the fact that, in a Brauer diagram, each dot is connected to exactly one other dot. For the partition algebra, there is no such regularity, which makes the problem more complex.

In this article, we consider the question of when the restriction of a cell module of the partition algebra $A = P_k(r,\delta)$ to a group algebra of a symmetric group $\Sigma_l$ with $l \leq r$ admits a cell filtration. The main result is the following.

Theorem 1. Let $A = P_k(r,\delta)$ and $n \leq l \leq r$. Let $\text{char} k > \left\lfloor \frac{r-n}{3} \right\rfloor$ or $\text{char} k = 0$ and $X \in k \Sigma_n$−mod. Then the $k \Sigma_l$-module $e_l(A/Ae_{n-1}A)e_{n-1} \otimes X$ admits a dual Specht filtration. In particular, restrictions of cell modules of $A$ to $k \Sigma_l$−mod have dual Specht filtrations.

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where the $e_i$ are special idempotents in the cellularly stratified structure of $A$. They are defined in Subsection 2.2.

The article is organised as follows. Section 2 contains a definition of the partition algebra, some notation and an explanation of the cellular and cellularly stratified structures. Sections 3 and 4 are dedicated to the proof of Theorem 1. Section 3 is a study of the $(k\Sigma_i, k\Sigma_n)$-bimodule $e_1 A e_n / e_1 J_{n-1} e_n = e_1 (A / J_{n-1}) e_n$, the left part of the tensor product from Theorem 1 where $J_{n-1} = Ae_{n-1} A$. We want to show that $e_1 (A / J_{n-1}) e_n$ has a dual Specht filtration as left $k\Sigma_l$-module. A decomposition into direct summands is given in Subsection 3.4. Each summand is defined via the tensor product from Theorem 1, where $\Sigma_n$ is isomorphic to the bimodule $\oplus (k \Sigma_i \otimes k \Sigma_n)$.

Step two is to study the modules $k \Sigma_{l_1} \otimes k \Sigma_{n_1}$ and $k \Sigma_{l_2} \otimes k$ separately. In Subsection 3.4, we show that $k \Sigma_{l_1} \otimes k \Sigma_{n_1}$ is isomorphic to the bimodule $\oplus (k \Sigma_i \otimes k)$, where we define a right $k \Sigma_n$-module structure such that it is a tensor induced module. As left $k \Sigma_l$-module, this is obviously dual Specht filtered, as it is a direct sum of permutation modules. In Subsection 3.4, we show that $k \Sigma_{l_2} \otimes k$ is isomorphic to an induced outer tensor product of Foulkes modules $k \Sigma_{am} \otimes k$. If $\text{char} k > m$ (or $\text{char} k = 0$), we can show that the Foulkes module $k \Sigma_{am} \otimes k$ has a dual Specht filtration. This leads to the assumption $\text{char} k > \lceil \frac{m}{2} \rceil$ (or $\text{char} k = 0$) in Theorem 1. Note that this assumption is sufficient, but potentially not necessary. In fact, Giannelli shows in [Gia, Theorem 1.1] that for $0 < \text{char} k < m$ there is a non-projective summand of $k \Sigma_{am} \otimes k$ which is not a Young module. However, this does not mean that there is no dual Specht filtration of the Foulkes module.

Finally, Section 4 concludes the proof of the theorem by putting together the results from the previous section, the result on Brauer algebras from [Pau] and the characteristic-free version of the Littlewood-Richardson rule [JP].

This article is the first of two articles arising from the author’s PhD thesis [Pau]. The aim of the thesis was to extend the construction of $k \Sigma_l$-permutation modules to permutation modules for a large class of diagram algebras, as it was done by Hartmann and Paget for Brauer algebras [HP]. A list of assumptions which the algebra $A$ has to satisfy was given, including: the cell modules of $A$ admit a dual Specht filtration when restricted to $k \Sigma_l$ - mod. The remaining assumptions are comparatively easy to show for $A = P_k(r, \delta)$, which makes the article at hand the crucial ingredient for the definition of permutation modules for partition algebras.
2. Definition of the Partition Algebra and its Structure

2.1. Definition. Let $k$ be an algebraically closed field of arbitrary characteristic and let $\delta \in k$. Let $r \in \mathbb{N}$.

The partition algebra $P_k(r, \delta)$ is the associative $k$-algebra with basis consisting of set partitions of $\{1, \ldots, r, 1', \ldots, r'\}$. A set partition of a set $X$ is a collection of pairwise disjoint subsets $X_i \subseteq X$, such that $\bigsqcup X_i = X$. Regarding $P_k(r, \delta)$ as a diagram algebra, this means that the basis consists of diagrams with two rows of $r$ dots each (top row labelled by $1, \ldots, r$ and bottom row labelled by $1', \ldots, r'$), where dots which belong to the same part of the partition are connected transitively. Note that this description is not unique. For example, the set partition

$$\{\{1, 2', 3'\}, \{2\}, \{3, 4, 5, 5', 6'\}, \{6, 4'\}, \{1'\}\}$$

corresponds, among others, to the diagram

\[ \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1' & 2' & 3' & 4' & 5' & 6'
\end{array} \]

as well as to the diagram

\[ \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1' & 2' & 3' & 4' & 5' & 6'
\end{array} \]

Multiplication is given by concatenation of diagrams, i.e. writing one diagram on top of the other, identifying the bottom row of the upper with the top row of the lower diagram and following the lines from top to bottom or within one row. Parts which have no dot in top or bottom row are replaced by a factor $\delta$. This multiplication is independent of the choice of diagram. We usually omit the labels $1, \ldots, r, 1', \ldots, r'$.

Example. Let $x = \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{array}$

and $y = \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{array}$

Then $xy = \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{array}$

We choose to write all diagrams as follows: First, connect dots of the top row belonging to the same part from left to right. Do the same in the bottom row. Parts which contain both top and bottom row dots will be connected via the respective leftmost dots. Parts connecting top and bottom row are often called propagating parts in the literature. The number $\#_p(d)$ of propagating parts of a diagram $d$ is called propagating number. We call the actual line connecting a top and a bottom row dot propagating line. We denote the top row of a diagram $d$ by $\text{top}(d)$, its bottom row by $\text{bottom}(d)$ and the permutation induced by the propagating
lines by \( \Pi(d) \). Note that multiplication of diagrams cannot increase the propagating number, since a propagating part of \( x \cdot y \) connects top(\( x \)) to bottom(\( y \)) via bottom(\( x \)) = top(\( y \))\), hence \( \#_p(x \cdot y) \leq \min(\#_p(x), \#_p(y)) \). The unit element of \( P_k(r, \delta) \) is given by the set partition \( \{\{1, 1'\}, \{2, 2'\}, \ldots, \{r, r'\}\} \) = \( \bullet \quad \bullet \quad \ldots \quad \bullet \).

A diagram consisting of only one row with \( r \) dots and arbitrary connections is called partial diagram. We have to distinguish certain parts from others; we say they are labelled and write the dots as empty circles \( \circ \) instead of dots \( \bullet \). We count the labelled parts from left to right, according to the leftmost dot of the part. Let \( V_n \) be the vector space with basis all partial diagrams with exactly \( n \) labelled parts (and possibly further unlabelled parts). For example, \( \circ \circ \circ \circ \circ \circ \circ \circ \circ \bullet \) is the second labelled part from left to right, according to the leftmost dot of the part. Let \( V_n \) be the vector space with basis all partial diagrams with exactly \( n \) labelled parts (and possibly further unlabelled parts). For example, \( \circ \circ \circ \circ \circ \circ \circ \circ \circ \bullet \) is a labelled singleton \( \circ \) is the first labelled part, \( \circ \circ \circ \circ \circ \circ \circ \circ \circ \bullet \) is the second.

### 2.2. Structural Properties

The group algebra \( k\Sigma_r \) is a unitary subalgebra of \( P_k(r, \delta) \), where a permutation \( \pi \in \Sigma_r \) corresponds to a diagram where all parts are of size 2 and propagating, i.e. each dot of the top row is connected to exactly one dot of the bottom row. For \( l < r \), we have different embeddings of \( k\Sigma_l \) into \( P_k(r, \delta) \)

\[
\begin{array}{ccc}
k\Sigma_r & \subset & P_k(r, \delta) \\
k\Sigma_l & \subset & P_k(l, \delta) \\
\end{array}
\]

where a smaller partition algebra \( P_k(l, \delta) \) is embedded into \( P_k(r, \delta) \) by adding dots \( l + 1, \ldots, r \) in the top row and \( (l + 1)', \ldots, r' \) in the bottom row, and attaching the new dots of the top and bottom row, respectively, to the \( l \)th, respectively \( l' \)th, dot of the diagram in \( P_k(l, \delta) \).

Xi showed that \( P_k(r, \delta) \) is cellular by considering it as iterated inflation of group algebras of symmetric groups.

**Theorem 2** ([Xi] Theorem 4.1). The partition algebra \( P_k(r, \delta) \) is cellular as an iterated inflation of the form \( \bigotimes_{n=0}^r k\Sigma_n \otimes_k V_n \otimes_k V_n \), with respect to the involution \( i \) turning a diagram upside down.

In [HHPK], the partition algebra is one of the main examples for cellularly stratified algebras. For the cellularly stratified structure, we need the existence of idempotents \( e_n = 1_{\Sigma_n} \otimes u_n \otimes v_n \) such that \( e_n e_m = e_m e_n \) for \( m \leq n \). Let \( \delta \neq 0 \) and set

\[
e_0 := \frac{1}{\delta} \quad \bullet \quad \bullet \quad \ldots \quad \bullet \quad r \quad e_n := \frac{1}{\delta} \quad \bullet \quad \ldots \quad \bullet \quad \bullet \quad \ldots \quad \bullet \quad \ldots \quad \bullet \quad \ldots \quad \bullet \quad r \quad \quad \text{for } n \geq 1
\]

**Theorem 3** ([HHPK] Proposition 2.6). The partition algebra \( P_k(r, \delta) \) is cellularly stratified with stratification data \( (k, V_0, k, V_1, k\Sigma_2, V_2, \ldots, k\Sigma_r, V_r) \) and idempotents \( e_n \) for all parameters \( \delta \in k \setminus \{0\} \).
Visually speaking, a cellular algebra is cellularly stratified, if there is a chain of two-sided ideals $0 = J_0 \subseteq J_1 \subseteq \ldots \subseteq J_r = A$ such that each subquotient $J_i/J_{i-1}$ is an algebra without unit of the form $B_l \otimes V_n \otimes V_n$ for some smaller cellular algebra $B_l$ and vector space $V_n$. We call these subquotients layers of the algebra, and multiplication cannot move to a higher layer.

As a consequence of the cellularly stratified structure, we have that $k\Sigma_r$ is also a quotient of $P_k(r, \delta)$ by the ideal generated by all diagrams with propagating number at most $r - 1$.

From now on, let $A := P_k(r, \delta)$. By abuse of notation, we write $e_n(A/J_{n-1})$ for the $(e_nAe_n, A)$-bimodule $e_nA/e_nJ_{n-1}$, where $J_{n-1} = Ae_{n-1}A$ is the two-sided ideal generated by all diagrams $d$ with $\#_p(d) \leq n - 1$. We regard $e_nA$ as left $k\Sigma_n$-module via the embedding $k\Sigma_n \to e_nAe_n \simeq P_k(n, \delta)$.

The group algebra $k\Sigma_n$ is cellular; we choose as cell modules the dual Specht modules $S_\lambda$. Let

$$\mathcal{F}_n(S) = \left\{ M \in k\Sigma_n - \text{mod} \mid M = M_s \supset M_{s-1} \supset \ldots \supset M_1 \supset M_0 = 0, M_i/M_{i-1} \simeq S_{\lambda_i} \text{ for some partition } \lambda_i \text{ of } n \right\}$$

denote the category of $k\Sigma_n$-modules admitting a dual Specht filtration.

3. The bimodule $e_l(A/J_{n-1})e_n$

In this section, we study the bimodule $e_l(A/J_{n-1})e_n$, which appears in the restriction of the cell module $(A/J_{n-1})e_n \otimes_{\Sigma_l} S_\nu$ to $k\Sigma_l - \text{mod}$. By giving multiple isomorphic descriptions for the summands as $(k\Sigma_l, k\Sigma_n)$-bimodule in this section, we are able to show (in the next section) that the left $k\Sigma_l$-module $e_l(A/J_{n-1})e_n$ has a dual Specht filtration.

3.1. Notation. Let $n \leq l \leq r$ and let $V_n^l$ be the subspace of $V_n$ generated by all partial diagrams with $n$ labelled parts, where the last $r - l + 1$ dots lie in the same part. We regard $k\Sigma_l$ as the subalgebra of $P_k(r, \delta)$ with basis consisting of all diagrams with top and bottom row consisting of $l - 1$ labelled dots followed by one labelled part of size $r - l + 1$, and $l$ propagating lines connecting the $l$ parts of the top row with the $l$ parts of the bottom row. Let $v, w \in V_n^l$. We say that $v$ is equivalent to $w$, $v \sim w$, if and only if there is a $\pi \in \Sigma_l$ such that $\pi v = w$, where $\pi v$ is defined as follows: Write the diagram $\pi$ on top of $v$ and identify $\text{bottom}(\pi)$ with $v$. Then $\pi v$ is the top row of this diagram, where a part is labelled if and only if it contains at least one labelled part. In diagrams, this means that $v$ and $w$ are equivalent, if and only if, for each size, the number of labelled parts and the number of unlabelled parts of $v$ and $w$ coincide, where the last $r - l + 1$ dots count as one.

Example. Let $r = 7, l = 6, n = 2, \pi = (56) \in \Sigma_6$ and

$$v = \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ$$

Then

$$\pi v = \text{top} \left( \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ
\end{array} \right) = \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ$$

For $v \in V_n^l$, we define $d_v$ to be the diagram with $\text{top}(d_v) = v$, $\text{bottom}(d_v) = \text{bottom}(e_n)$ and $\Pi(d_v) = 1_{k\Sigma_n}$. Let $b \in e_l(A/J_{n-1})e_n$ be a diagram with $\text{top}(b) \sim v$. 


By definition, there is a $\pi \in \Sigma_l$ such that $\top(b) = \pi v$. Then $b = \pi d_v \Pi(\pi d_v)^{-1} \Pi(b)$, so we have

\[(3.1)\] Any diagram in $e_1(A/J_{n-1})e_n$ with top row in the equivalence class of $v$ equals $\tau d_v \eta$ for some $\tau \in \Sigma_l, \eta \in \Sigma_n$.

**Example.** Let $r = 7, l = 6, n = 3, v = \circ \circ \circ \circ \circ \circ \circ$ and $b = \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ$.

Then $\top(b) = (2354)v$ and $\Pi(b) = (132)$. In particular,

\[
(b) = (2354)d_v \Pi((2354)d_v)^{-1} \Pi(b) = \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ
\]

Let $U_v$ be the $(k \Sigma_l, k \Sigma_n)$-bimodule generated by $d_v$.

\[(3.2)\] For $w \in V^l_n$ with $w \sim v$, we have $U_w \cap U_v = \emptyset$

because a diagram in the intersection would have top row equivalent to $v$ and to $w$ simultaneously. Therefore, every diagram in $e_1(A/J_{n-1})e_n$ lies in exactly one of the $U_v$’s, see (3.1) and (3.2), and every diagram of $U_v$ is a diagram in $e_1(A/J_{n-1})e_n$ by definition. Hence, the $(k \Sigma_l, k \Sigma_n)$-bimodule $e_1(A/J_{n-1})e_n$ decomposes into a direct sum $\bigoplus_{v \in V^l_n} U_v$.

Fix a partial diagram $v \in V^l_n$ and set $d := d_v$. Let $\alpha_i$ be the number of labelled parts of size $i$ and $\beta_i$ the number of unlabelled parts of size $i$ of $v$, where the last $r - l + 1$ dots count as one dot. Then $\sum_i (\alpha_i \cdot i) + \sum_i (\beta_i \cdot i) = l$ and $\sum_i \alpha_i = n$. Without loss of generality, assume that the parts of $v$ are ordered as follows: the labelled parts are on the left hand side, the unlabelled parts on the right hand side. The parts are then ordered increasingly from left to right. Let $S^l_i \subseteq \{1, \ldots, l\}$ be the set of dots of $v$ belonging to the $j$th labelled part of size $i$ and let $T^l_i \subseteq \{1, \ldots, l\}$ be the set of dots of $v$ belonging to the $j$th unlabelled part of size $i$. Then $\Pi_{\alpha} := \prod_{i \geq 1, \alpha_i \neq 0} (\Sigma_{\alpha_i} \times \ldots \times \Sigma_{\alpha_1})$ is the stabilizer of the labelled parts of $v$ and $\Pi_{\beta} := \prod_{i \geq 1, \beta_i \neq 0} ((\Sigma_{\beta_i} \times \ldots \times \Sigma_{\beta_1}) \times \Sigma_{\beta_i})$ is the stabilizer of the unlabelled parts of $v$.

In particular, $\Pi_{\beta}$ stabilizes $d$, while $\Pi_{\alpha}$ can rearrange the propagating lines of $d$. Note that $\Pi_{\alpha} \simeq \prod_{i \geq 1, \alpha_i \neq 0} (\Sigma_{\alpha_i} \times \Sigma_{\alpha_i})$ and $\Pi_{\beta} \simeq \prod_{i \geq 1, \beta_i \neq 0} (\Sigma_{\beta_i} \times \Sigma_{\beta_i})$, where $\Sigma$ denotes the wreath product.

**Example.** Let $r = 12, l = 11, n = 3$ and $v = \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ$.

Then $\alpha = (0, 3), \beta = (1, 2)$ and

$$\Pi_{\alpha} = (\Sigma_{(1,2)} \times \Sigma_{(3,4)} \times \Sigma_{(5,6)}) \times \Sigma_{\circ} \simeq \Sigma_{2} \times \Sigma_{3}$$

and
3.2. Isomorphic descriptions of the summands. Consider the \((k\Sigma_l, k\Sigma_n)\)-bimodule \(k\Sigma_l \otimes_{k\Pi_l \ast k\Pi_n} k\Sigma_n\), where \(\Pi_\beta\) acts trivially on \(k\Sigma_n\) and the action of \(\Pi_\alpha\) on \(k\Sigma_n\) is given by \(\zeta \cdot \eta := \Pi(\zeta \eta)\eta\) for \(\zeta \in \Pi_\alpha, \eta \in \Sigma_n\). We have \(\text{top}(\zeta \eta) = \text{top}(d)\), so \(\zeta d = d\Pi(\zeta \eta)\) for \(\zeta \in \Pi_\alpha\).

**Lemma 1.** The map

\[
\Psi : k\Sigma_l \otimes_{k\Pi_l \ast k\Pi_n} k\Sigma_n \to U_v \\
\tau \otimes \eta \quad \mapsto \quad \tau d\eta
\]

is an isomorphism of \((k\Sigma_l, k\Sigma_n)\)-bimodules.

**Proof.** Let \(x \in \Pi_\alpha\) and \(y \in \Pi_\beta\). Then \(yd = d\) and \(xd = d\Pi(xd)\). Consider the map \(\Psi : k\Sigma_l \times k\Sigma_n \to U_v\) given by \(\Psi(\tau, \eta) = \tau d\eta\). Then \(\Psi(\tau xy, \eta) = \tau xyd\eta = \tau d\Pi(xd)\eta = \Psi(\tau, \Pi(xd)\eta) = \Psi(\tau, x y, \eta)\). This shows that \(\Psi\) is well-defined.

Let \(\tau, \tau' \in \Sigma_l\) and \(\eta, \eta' \in \Sigma_n\). Then \(\psi(\tau' \tau, \eta \eta') = \tau' \tau d\eta \eta' = \gamma' \psi(\tau, \eta)\eta'\), so \(\psi\) is a bimodule-homomorphism.

The inverse map is given by

\[
\check{\psi} : U_v \to k\Sigma_l \otimes_{k\Pi_l \ast k\Pi_n} k\Sigma_n \\
b \mapsto \tau \otimes \Pi(\tau d)^{-1}\Pi(b) \quad \text{if top}(b) = \tau v.
\]

We show that \(\check{\psi}\) is well-defined. If \(\text{top}(b) = \tau_1 v = \tau_2 v\), there are \(x \in \Pi_\alpha, y \in \Pi_\beta\) such that \(\tau_1 = \tau_2 xy\). Then

\[
\tau_1 \otimes \Pi(\tau_1 d)^{-1}\Pi(b) = \tau_2 xy \otimes \Pi(\tau_2 xyd)^{-1}\Pi(b)
\]

\[
= \tau_2 \otimes \Pi(xd)\Pi(\tau_2 d\Pi(xd))^{-1}\Pi(b)
\]

\[
= \tau_2 \otimes \Pi(xd)(\Pi(\tau_2 d)\Pi(xd))^{-1}\Pi(b)
\]

\[
= \tau_2 \otimes \Pi(xd)(\Pi(\tau_2 d)\Pi(xd))^{-1}(\Pi(\tau_2 d)^{-1}\Pi(b))
\]

\[
= \tau_2 \otimes \Pi(\tau_2 d)^{-1}\Pi(b).
\]

\(\square\)

Set \(l_1 := \sum_i \alpha_i \cdot i\) and \(l_2 := \sum_i \beta_i \cdot i\), so \(l = l_1 + l_2\). Fix \(\Pi_\alpha \subset \Sigma_{l_1}\) and \(\Sigma_{l_2} \simeq \Sigma_{l_1 + \Sigma_{l_1'}} \cap \Pi_\beta\). Fix coset representatives \(\omega_1, \ldots, \omega_l\) of \(k\Sigma_l/k\Sigma_{l_1, l_2}\). Denote by \(X \boxtimes Y \in k\Sigma_{l_1, l_2}\) - mod the exterior tensor product of \(X \in k\Sigma_{l_1}\) - mod and \(Y \in k\Sigma_{l_2}\) - mod given by

\[
(\tau_1, \tau_2) \cdot (x \boxtimes y) = \tau_1 x \boxtimes \tau_2 y
\]

for \(\tau_1 \in \Sigma_{l_1}, \tau_2 \in \Sigma_{l_2}, x \in X, y \in Y\).

Consider the \((k\Sigma_l, k\Sigma_n)-\)bimodule \(k\Sigma_l \otimes_{k\Sigma_{l_1, l_2}} ((k\Sigma_{l_1} \otimes_{k\Pi_{l_1}} k\Sigma_n) \otimes (k\Sigma_{l_2} \otimes_{k\Pi_{l_2}} k))\) with right \(k\Sigma_n\)-module structure given by

\[
(\omega \otimes ((\tau_1 \otimes \eta) \otimes (\tau_2 \otimes 1))) \cdot \eta' := \omega \otimes ((\tau_1 \otimes \eta \eta') \otimes (\tau_2 \otimes 1))
\]

for \(\omega \otimes ((\tau_1 \otimes \eta) \otimes (\tau_2 \otimes 1)) \in k\Sigma_l \otimes_{k\Sigma_{l_1, l_2}} ((k\Sigma_{l_1} \otimes_{k\Pi_{l_1}} k\Sigma_n) \otimes (k\Sigma_{l_2} \otimes_{k\Pi_{l_2}} k))\) and \(\eta' \in \Sigma_n\).
Lemma 2. \( k\Sigma_l \otimes k\Sigma_n \) and \( k\Sigma_l \otimes (k\Sigma_l \otimes k\Sigma_n) \) are isomorphic as \((k\Sigma_l, k\Sigma_n)\)-bimodules.

Proof. Define
\[
\Theta : k\Sigma_l \times k\Sigma_n \to k\Sigma_l \otimes ((k\Sigma_l \otimes k\Sigma_n) \otimes (k\Sigma_l \otimes k) \otimes (k\Sigma_l \otimes k))
\]
\[
(\tau, \eta) \mapsto \omega_1 \otimes ((\tau_1 \otimes \eta) \otimes (\tau_2 \otimes 1)) \quad \text{if } \tau = \omega_1 \tau_1 \tau_2
\]

where \( \tau_1 \in \Sigma_{l_1} \) and \( \tau_2 \in \Sigma_{l_2} \), and let \( x \in \prod_{i} \). Then \( x \in \Sigma_{l_1} \times \) \( \{0\} \subset \Sigma_{l_2} \), so \( \tau_2 x = x \tau_2 \) for \( \tau_2 \in \Sigma_{l_2} \). Thus, \( \Theta(\tau xy, \eta) = \Theta(\omega_1 \tau_1 x \tau_2 y, \eta) = \omega_1 \otimes ((\tau_1 x \otimes \eta) \otimes (\tau_2 y \otimes 1)) = \omega_1 \otimes ((\tau_1 \otimes \Pi(x(\eta y)) \otimes (\tau_2 \otimes 1)) = \Theta(\tau, xy \otimes y). \)

Hence, the map
\[
\Theta : k\Sigma_l \otimes k\Sigma_n \to k\Sigma_l \otimes ((k\Sigma_l \otimes k\Sigma_n) \otimes (k\Sigma_l \otimes k))
\]
\[
\tau \otimes \eta \mapsto \omega_1 \otimes ((\tau_1 \otimes \eta) \otimes (\tau_2 \otimes 1)) \quad \text{if } \tau = \omega_1 \tau_1 \tau_2
\]
is well-defined.

Let \( \tau' \in \Sigma_l \) such that \( \tau' \omega_1 = \omega_2 \tau'_1 \tau'_2 \) and let \( \eta' \in \Sigma_n \). Then
\[
\theta(\tau' \otimes \eta' \tau) = \theta(\omega_2 \tau'_1 \tau'_2 \tau_1 \tau_2 \otimes \eta' \tau) = \theta(\omega_2 \tau'_1 \tau'_2 \tau_1 \otimes \eta' \tau) = \omega_2 \otimes ((\tau'_1 \otimes \eta' \tau) \otimes (\tau'_2 \otimes 1)) = \omega_2 \otimes (\tau_1 \otimes \eta' \tau) = \omega_2 \otimes (\tau_1 \otimes \eta' \tau) = \tau' \omega_1 \otimes ((\tau_1 \otimes \eta' \tau) \otimes (\tau_2 \otimes 1)) = \tau' \omega_1 \otimes (\tau_1 \otimes \eta' \tau) = \tau' \omega_1 \otimes (\tau_1 \otimes \eta' \tau) = \tau' \omega_1 \otimes (\tau_1 \otimes \eta' \tau) = \tau' \omega_1 \otimes (\tau_1 \otimes \eta' \tau) = \tau' \omega_1 \otimes (\tau_1 \otimes \eta' \tau)
\]
so \( \theta \) is a homomorphism of \((k\Sigma_l, k\Sigma_n)\)-bimodules.

The inverse is given by
\[
\theta^{-1} : k\Sigma_l \otimes ((k\Sigma_l \otimes k\Sigma_n) \otimes (k\Sigma_l \otimes k)) \to k\Sigma_l \otimes k\Sigma_n
\]
\[
\tau \otimes ((\theta \otimes \eta) \otimes (\nu \otimes 1)) \mapsto (\tau \theta \nu \otimes \eta)
\]

3.3. The tensor induced module. In this subsection, we find an isomorphic description for \( k\Sigma_l \otimes k\Sigma_n \), which shows that it has a dual Specht filtration as left \( k\Sigma_l \)-module. Let \( \gamma \) be the composition \( (1^a_1, 2^a_2, \ldots) \) of \( l_1 \). Then \( \Sigma_\gamma = \bigcap_{l \in l_1} \Sigma_{l_1} \) and \( \Sigma_\gamma \times \prod \) is the stabilizer of \( d \). Fix coset representatives \( \sigma_1, \ldots, \sigma_s \) of \( \Sigma_\alpha \) \( \Sigma_\gamma \) and define a right \( k\Sigma_n \)-module structure on \( \bigoplus_{l=1}^n (k\Sigma_l \otimes k) \) by
\[
(\tau \otimes 1)^{(i)} \otimes \eta := (\tau \theta \otimes 1)^{(i)} \quad \text{if } \sigma_i, \eta = \theta \sigma_j \text{ and } \Pi(\theta d) = \theta
\]
for \( \tau \in \Sigma_{l_1}, \eta \in \Sigma_n, \theta \in \Sigma_\alpha \) and \( \theta \in \Sigma_{l_1} \), where \( (\tau \otimes 1)^{(i)} \) denotes the element \((0, \ldots, 0, \tau \otimes 1, 0, \ldots, 0)\) with non-zero entry in the \( i \)-th position. With this module structure, \( \bigoplus_{l=1}^n (k\Sigma_l \otimes k) \) is called tensor induced module, cf. [CR §13].
Lemma 3. The map

\[ \varphi : k\Sigma_{\lambda_1} \otimes k\Sigma_{\lambda_n} \rightarrow \bigoplus_{\lambda_i} (k\Sigma_{\lambda_i} \otimes k) \]

\[ \tau \otimes \eta \mapsto (\tau \zeta \otimes 1)^{(i)} \]

where \( \zeta = \Pi(\hat{\zeta}d) \) for some \( \zeta \in \prod_\alpha \).

is an isomorphism of \((k\Sigma_1, k\Sigma_n)\)-bimodules. In particular, \( k\Sigma_{\lambda_1} \otimes k\Sigma_{\lambda_n} \cong \bigoplus_{i=1}^{\lambda} M^\gamma \) as left \( k\Sigma_{\lambda_1} \)-modules, so \( k\Sigma_{\lambda_1} \otimes k\Sigma_{\lambda_n} \in \mathcal{F}_{\lambda_1}(S) \).

Proof. We show that \( \varphi \) is independent of the choice of \( \hat{\zeta} \). Let \( \hat{\zeta}, \zeta \in \prod_\alpha \) such that \( \Pi(\hat{\zeta}d) = \Pi(\hat{\zeta}d) = \zeta \in \prod_\alpha \). Since \( \prod_\alpha \times \prod_\beta \) is the stabilizer of top\( (d) \), we have top\( (\hat{\zeta}d) = \text{top}(d) = \text{top}(\hat{\zeta}d) \), and so \( \hat{\zeta}d = \hat{\zeta}d \). In particular, there is \( (\vartheta, \vartheta') \in \Sigma_\gamma \times \Sigma_\beta \), the stabilizer of \( d \), such that \( \hat{\zeta} = \hat{\zeta} \vartheta' \). But \( \hat{\zeta} = \hat{\zeta} \vartheta' \) and \( \vartheta' \) are elements of \( \Pi_\alpha \) while \( \vartheta' \in \Pi_\beta \), so \( \vartheta' = 1 \). Hence, \( (\tau \zeta \otimes 1)^{(i)} = (\tau \zeta \vartheta \otimes 1)^{(i)} = (\tau \zeta \vartheta \otimes 1)^{(i)} \) and \( \varphi \) is independent of the choice of \( \hat{\zeta} \).

Let \( \Phi : k\Sigma_{\lambda_1} \times k\Sigma_{\lambda_n} \rightarrow \bigoplus_{\lambda_i} (k\Sigma_{\lambda_i} \otimes k) \) with \( \Phi(\tau, \eta) = (\tau \zeta \otimes 1)^{(i)} \) for \( \eta = \zeta \sigma_i, \Pi(\hat{\zeta}d) = \zeta \) and let \( \lambda_i \in \prod_\alpha \). Then \( \Phi(\tau \xi, \eta) = \Phi(\tau \xi, \sigma_i) = (\tau \xi \zeta \otimes 1)^{(i)} \) and \( \Phi(\tau, \Pi(\hat{\zeta}d) \eta) = \Phi(\tau, \Pi(\hat{\zeta}d) \sigma_i) = (\tau \xi \zeta \otimes 1)^{(i)} \), since \( \Pi(\zeta \zeta \zeta d) = \Pi(\zeta \zeta \zeta d) = \Pi(\zeta \zeta \zeta d) = \Pi(\zeta \zeta \zeta d) \). So \( \varphi \) is well-defined.

Let \( \tau, \tau' \in \Sigma_{\lambda_1} \) and \( \eta, \eta' \in \Sigma_{\lambda_n} \) such that \( \eta = \zeta \sigma_i \) and \( \sigma_i \eta' = \zeta' \sigma_j \). Then \( \varphi(\tau' \tau \otimes \eta \eta') = \varphi(\tau' \tau \otimes \zeta' \zeta \sigma_j) = (\tau' \tau \zeta \otimes 1)^{(i)} \) where \( \Pi(\hat{\zeta}d) = \zeta' \zeta \). On the other hand,

\[ \tau' \varphi(\tau \otimes \eta) \eta' = \tau' (\tau \zeta \otimes 1)^{(i)} \eta' \]

with \( \Pi(\hat{\zeta}d) = \zeta \)

\[ = (\tau' \tau \zeta \otimes 1)^{(i)} \]

with \( \Pi(\hat{\zeta}d) = \zeta' \).

\[ \Pi(\hat{\zeta}d) = \Pi(\hat{\zeta}d) \Pi(\hat{\zeta}d) = \zeta' = \Pi(\hat{\zeta}d) \], so there is a \( \vartheta \in \Sigma_\gamma \) such that \( \hat{\zeta} = \hat{\zeta} \vartheta \). Hence \( \tau' \tau \zeta \hat{\zeta} \otimes 1 = \tau' \tau \zeta \vartheta' \otimes 1 = \tau' \tau \zeta \otimes 1 \) and \( \varphi \) is a homomorphism of \((k\Sigma_1, k\Sigma_n)\)-bimodules.

The inverse is given by

\[ \bigoplus_{\lambda_i} (k\Sigma_{\lambda_i} \otimes k) \rightarrow k\Sigma_{\lambda_1} \otimes k\Sigma_{\lambda_n} \]

\[ (\tau \otimes 1)^{(i)} \mapsto \tau \otimes \sigma_i. \]

\[
\]

3.4. The Foulkes module. We will now study the module \( k\Sigma_{\lambda_2} \otimes k \). Let \( t \) be the maximal size of an unlabelled part of \( v \) and set \( \Sigma_{\gamma_i} := \bigcup_{j=1}^{\gamma} \Sigma_i \) for \( i = 1, ..., t \) and \( \Sigma_{\gamma} := \bigcup_{i=1}^{\gamma} \Sigma_{\gamma_i} \). Then \( \Pi_\beta \subset \Sigma_{\gamma} \subset \Sigma_{\lambda_2} \).

Lemma 4. There is an isomorphism

\[ k\Sigma_{\lambda_2} \otimes k \cong k\Sigma_{\lambda_2} \otimes (k \otimes (k \otimes \Sigma_{\gamma_2}) \otimes (k \otimes \Sigma_{\gamma_3}) \otimes \ldots \otimes (k \otimes \Sigma_{\gamma_t}) \otimes k) \]

of left \( k\Sigma_{\lambda_2} \)-modules.
Proof. Let $\epsilon_1, \ldots, \epsilon_a$ be coset representatives of $\Sigma_i / \Sigma_\tilde{i}$ and $\tau = \epsilon_i \tau_2 \ldots \tau_t$ with $\tau_j \in \Sigma_{\tilde{i}j}$. Then the assignment

$$\tau \otimes 1 \mapsto \epsilon_i \otimes (1 \otimes (\tau_2 \otimes 1) \otimes \cdots (\tau_t \otimes 1))$$

defines the isomorphism, like in Lemma 2. □

The module $H(a^m) := k\Sigma_{am} \otimes_{k(\Sigma_a \wr \Sigma_m)} k$ is called Foulkes module. If the characteristic of the field $k$ is strictly greater than $m$, or zero, the Foulkes module is isomorphic to a direct summand of the permutation module $M(a^m)$, as mentioned in [Gia]. We will give a proof of this statement in Lemma 5. In smaller positive characteristic, this is not true. In general, it is not known whether or not a Foulkes module $H(a^m)$ has a Specht filtration in the case $0 < \mathrm{char} k \leq m$ and $a > 3$. The case $a = 2$ was solved in [Pag] for arbitrary characteristic of the field.

Lemma 5. If $\mathrm{char} k = 0$ or $\mathrm{char} k > m$, the Foulkes module

$$H(a^m) = k\Sigma_{am} \otimes_{k(\Sigma_a \wr \Sigma_m)} k$$

is isomorphic to a direct summand of the permutation module $M(a^m)$.

Proof. We came across the wreath product $\Sigma_a \wr \Sigma_m$ as the stabilizer of the $m$ (unlabelled) parts of size $a$ of a partial diagram. The Foulkes module $k\Sigma_{am} \otimes_{k(\Sigma_a \wr \Sigma_m)} k$ has a vector space basis indexed by left cosets $\Sigma_{am}/(\Sigma_a \wr \Sigma_m)$. Such a coset decides which dots belong to the same part. Thus, $k\Sigma_{am} \otimes_{k(\Sigma_a \wr \Sigma_m)} k$ has a vector space basis of set partitions of the form

$$\{\{x_1, \ldots, x_a\}, \ldots, \{x_{(m-1)a+1}, \ldots, x_{ma}\}\}$$

with $x_i \in \{1, \ldots, am\}, x_i \neq x_j$ for $i \neq j$.

Recall that the permutation module $M(a^m)$ has a basis of $(a^m)$-tabloids. Define maps $M(a^m) \xrightarrow{\Phi} H(a^m)$ with

$$\begin{array}{c}
\begin{array}{cccc}
1 & \ldots & x_a & \\
\vdots & & & \\
x_{(m-1)a+1} & \ldots & x_{ma} & \\
\end{array}
\end{array} \xrightarrow{\Phi} \begin{array}{c}
\begin{array}{cccc}
\{x_1, \ldots, x_a\} & \ldots & \{x_{(m-1)a+1}, \ldots, x_{ma}\} & \\
\end{array}
\end{array}$$

where the $\ast$-action of $\Sigma_m$ permutes the rows of a tabloid. The $\ast$ and $\cdot$ actions commute: Let $\sigma \in \Sigma_m, \tau \in \Sigma_a$ and $x_i$ in row $k$ of the tabloid $x$. If $\sigma(k) = l$, then $x_i$ is in row $l$ of $\sigma \ast x$, so $x_{\tau(i)}$ is in row $l$ of $\tau \cdot (\sigma \ast x)$. On the other hand, $x_{\tau(i)}$ is in row $k$ of $\tau \cdot x$ and therefore in row $l$ of $\sigma \ast (\tau \cdot x)$.
We have
\[
\Phi \left( \tau \cdot \begin{array}{cccc}
    x_1 & \ldots & x_a \\
    \vdots & & \\
    x_{(m-1)a+1} & \ldots & x_{ma}
\end{array} \right) = \Phi \left( \begin{array}{cccc}
    x_{\tau(1)} & \ldots & x_{\tau(a)} \\
    \vdots & & \\
    x_{\tau((m-1)a+1)} & \ldots & x_{\tau(ma)}
\end{array} \right)
\]
= \left\{ \{x_{\tau(1)}, \ldots, x_{\tau(a)}\}, \ldots, \{x_{\tau((m-1)a+1)}, \ldots, x_{\tau(ma)}\} \right\}

Corollary 6. If \( \text{char } k = 0 \) or \( \text{char } k > m \), then the indecomposable direct summands of the Foulkes module \( H(\alpha^m) = k\Sigma_{am} \otimes k \) are Young modules. In particular, \( H(\alpha^m) \) is both Specht and dual Specht filtered.

Corollary 7. If \( \text{char } k = 0 \) or \( \text{char } k > \max \beta_i \), then \( k\Sigma_{l_2} \otimes k \in \mathcal{F}_{l_2}(S) \).

Proof. By Lemma 4, \( k\Sigma_{l_2} \otimes k \) is induced from an exterior tensor product of Foulkes modules. Corollary 6 shows that the Foulkes modules are dual Specht filtered, provided the characteristic of the field is large enough. The characteristic-free version of the Littlewood-Richardson rule [JP] then says that the exterior tensor product of Foulkes modules has a dual Specht filtration.

4. Restriction of Cell Modules

We are now able to put the results about tensor induced and Foulkes modules together to show that the restriction of a cell module of \( P_k(\tau, \delta) \) to a group algebra of a symmetric group with index \( l \leq r \) is dual Specht filtered.
Proof. By Lemmas 1 and 2, the summands of $k\Sigma_{\beta_2} \otimes k$ is the stabilizer of unlabelled parts of size 2 and it is dual Specht filtered by Proposition 8. For $i > 2$, the factor $k\Sigma_{\beta_i} \otimes \beta_i$ is dual Specht filtered if $\text{char} k = 0$ or $\text{char} k > \beta_i$ by Corollary 6. The maximal amount of unlabelled parts of a certain size greater than two occurs in the summands $U_v$, where $v$ consists of $n$ labelled singletons and $\lfloor \frac{\lfloor n \rfloor}{3} \rceil$ unlabelled parts of size 3. The remaining 0, 1 or 2 dots form additional unlabelled parts.

**Proposition 8.** Let $\text{char} k = 0$ or $\text{char} k > \lfloor \frac{\lfloor n \rfloor}{3} \rceil$. Then $e_1(A/J_{n-1})e_n \in \mathcal{F}_l(S)$.

**Proof.** By Lemmas 1 and 2 the summands of $e_1(A/J_{n-1})e_n$ are of the form

$$k\Sigma_i \otimes (k\Sigma_{\beta_1} \otimes k\Sigma_{\beta_2} \otimes k),$$

where $\prod_\alpha \approx k \prod_i (\Sigma_i \otimes \Sigma_{\alpha_i})$ and $\prod_\beta \approx k \prod_i (\Sigma_i \otimes \Sigma_{\beta_i})$, $l_1 = \sum \alpha_i \cdot i$ and $l_2 = \sum \beta_i \cdot i$. Lemma 3 shows that $k\Sigma_i \otimes k\Sigma_{\beta_i} \in \mathcal{F}_l(S)$ and by Corollary 7, we have that $k\Sigma_i \otimes k \in \mathcal{F}_l(S)$ in case $\text{char} k > \beta_i$ or zero. Since we are looking at the whole bimodule $e_1(A/J_{n-1})e_n$ and not just its summands $U_v$, we have to consider all possible top rows. By the above arguments, we have that the maximal amount of unlabelled parts of size $\geq 3$ is $\lfloor \frac{\lfloor n \rfloor}{3} \rceil$. Hence, we have to assume $\text{char} k > \lfloor \frac{\lfloor n \rfloor}{3} \rceil$. The characteristic-free version of the Littlewood-Richardson rule [JP] then says that $k\Sigma_i \otimes k\Sigma_{\beta_i} \otimes (k\Sigma_{\beta_1} \otimes k\Sigma_{\beta_2} \otimes k) \in \mathcal{F}_l(S)$.

**Theorem** (Theorem 1). Let $A$ be the partition algebra $P_k(r, \delta)$ and $n \leq l \leq r$. Let $\text{char} k = 0$ or $\text{char} k > \lfloor \frac{\lfloor n \rfloor}{3} \rceil$ and $X \in k\Sigma_n - \text{mod}$. Then the $k\Sigma_i$-module $e_1(A/J_{n-1})e_n \otimes X$ is in $\mathcal{F}_l(S)$. In particular, restrictions of cell modules to $k\Sigma_i - \text{mod}$ are dual Specht filtered.

**Proof.** By Lemmas 1, 2, 3 and 4, $e_1(A/J_{n-1})e_n$ decomposes as $(k\Sigma_i, k\Sigma_n)$-bimodule into a direct sum of modules of the form

$$k\Sigma_i \otimes \bigotimes_{\alpha=1}^s (k\Sigma_{\beta_i} \otimes k\Sigma_{\beta_j}) \otimes (k\Sigma_{\beta_1} \otimes k\Sigma_{\beta_2} \otimes k\Sigma_{\beta_3} \otimes k) \otimes X.$$

Hence, an element of $e_1(A/J_{n-1})e_n \otimes X \simeq e_1(A/J_{n-1})e_n \otimes X$ is of the form

$$\omega \otimes (\sum_{i=1}^s (\pi_i \otimes 1)^{(1)} \otimes (v \otimes (1 \otimes (\tau_2 \otimes 1) \otimes \ldots \otimes (\tau_l \otimes 1)))) \otimes x$$

$$= \omega \otimes (\sum_{i=1}^s (\pi_i \otimes 1)^{(1)} \otimes (v \otimes (1 \otimes (\tau_2 \otimes 1) \otimes \ldots \otimes (\tau_l \otimes 1)))) \otimes \sigma_i x$$

$$= \omega \otimes (\sum_{i=1}^s (\pi_i \otimes 1)^{(1)} \otimes (v \otimes (1 \otimes (\tau_2 \otimes 1) \otimes \ldots \otimes (\tau_l \otimes 1)))) \otimes \sigma_i x$$

$$= \omega \otimes (\sum_{i=1}^s (\pi_i \otimes 1)^{(1)} \otimes (v \otimes (1 \otimes (\tau_2 \otimes 1) \otimes \ldots \otimes (\tau_l \otimes 1)))) \otimes \sigma_i x$$

with $\omega \in k\Sigma_i$, $\pi_i \in k\Sigma_i$, $v \in k\Sigma_2$ and $\tau_i \in k\Sigma_{\beta_i}$. Hence the summands of $e_1(A/J_{n-1})e_n \otimes X$ are isomorphic to

$$Y := Z \otimes (k \otimes k\Sigma_n \otimes X),$$
where $Z$ is the module $k\Sigma_{l_1} \otimes \left((k\Sigma_{l_2} \otimes k)(k\Sigma_{l_3} \otimes k)\right)$. The factor $k\Sigma_{l_1} \otimes k$ is clearly dual $k\Sigma_{l_1}$-Specht filtered, as it is a permutation module for $k\Sigma_{l_1}$. The factor $k\Sigma_{l_2} \otimes k$ is dual $k\Sigma_{l_2}$-Specht filtered by Corollary 7 if $\text{char } k > \max \beta_i$ or $\text{char } k = 0$.

Now we can apply the characteristic-free version of the Littlewood-Richardson rule again to have $Z \in \mathcal{F}_l(S)$. Then the left $k\Sigma_l$-module $Y = \bigoplus_{i=1}^h Z$ is in $\mathcal{F}_l(S)$, where $h = \dim(k \otimes X)$.

\[\Box\]

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