Research minimax and minisum Weber problems on a plane with forbidden zones

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Abstract. Weber problem is a well-known location problem of connected facilities. Two optimality criteria are considered. The first criterion is minimization of the total cost of connections among facilities. The second criterion is minimization of the maximum connection between facilities. This article deals with Weber problems on a plane with rectangular forbidden zones. Location inside in the forbidden zones is not allowed. The sides of forbidden zones are parallel to the coordinate axes. To measure distances rectangular metric is used. An overview of studies for these problems is provided. Some properties of the problems are described. For example, the possibilities of reduction of admissible areas in the search for an optimal solution are presented. Variants of different bounds of the objective functions are provided. Models of integer linear programming with Boolean variables are given. Computational experiments with using the branch and bound algorithms, the constructed models and IBM ILOG CPLEX package were carried out. Usage of the properties of reductions of admissible areas is promising both in solving of the problems by combinatorial methods and using the integer optimization apparatus.

1. Introduction
In practice, it is need to solve problems of optimal location of facilities in a given area [15, 26]. It can be locating service points: fire stations, cellular communication systems, oil refueling stations and others. One of these tasks is the Weber problem. Its wording is as follows. There are facilities that are connected to each other and to the fixed ones. It is need to locate facilities among the fixed ones in a certain area. An example of Weber problem is the location of equipment on production lines. The located facilities are equipment, for example, columns, tanks, refrigerators, pumps, which are connected by some communications, for example, pipelines (see Figure 1). The optimality criterion can be minimization of the total cost of communication among facilities (minisum) or minimization of the maximum communication among facilities (minimax).

The generalization of Weber problem deals with forbidden zones. In the zones the facilities cannot be located. Such zones can be buildings, equipment, sanitary zones, etc. In the literature for different metrics the following problems are considered. Main attention is to locating one facility with one forbidden zone. Moreover, not only a location problem is solved. Also the problem of tracing, i.e. laying connections among facilities by passing forbidden zones (barriers) is solved. Problems with Euclidean metric and a forbidden zone in the form of a circle or polygon (usually convex) are studied in [2, 10, 12]. In [4] the problem with rectangular metric is considered. The competitive problem is investigated in [6].
This article provides an overview of research of Weber problems on a plane with rectangular forbidden zones. Location inside in the forbidden zones is not allowed. The sides of forbidden zones are parallel to the coordinate axes. To measure distances rectangular metric is used. Two optimality criteria are considered. The first criterion is minimization of the total cost of communications among facilities. The second criterion is minimization of the maximum connection between facilities. Some properties of the problems are described. The possibilities of reductions of admissible areas in the search for an optimal solution are presented. So, there is an optimal solution of minimax problem in a rectangular shell. To build a rectangular shell, the solutions of the problem for each of the located facilities singly are used. To find an optimal solution of minisum problem it is sufficient to consider a rectangle of minimal dimensions containing the optimal solution of each facility only relative to the fixed ones. Variants of different bounds of the objective functions for the problems are presented. The models of integer linear programming are given. Computational experiments with using the branch and bound algorithms, the constructed models and IBM ILOG CPLEX package were carried out. It is demonstrated that the application of the properties of reductions of admissible areas is promising both in solving of the problems by combinatorial methods and using the integer optimization apparatus.

2. Problems formulation and summary of research

There are no restrictions on the relative position of facilities on a plane in the classical formulations of minimax or minisum Weber problems. One of the generalizations of such problems is the presence of special areas (forbidden zones) on a plane, in which facilities are not allowed to be located for any reasons.

This section is devoted to a review of the known results of studies of Weber’s problems for point facilities with allowance for forbidden zones.

For simplicity, we present statements of minimax and minisum Weber problems with forbidden zones on a plane. Let $n$ facilities $X_1, \ldots, X_n$ are located on a plane among $m$ fixed facilities $P_1, \ldots, P_m$. Denote by $J = \{1, \ldots, m\}$ and $I = \{1, \ldots, n\}$ sets of numbers of fixed and located facilities; by $d(X_i, P_j)$ and $d(X_i, X_k)$ denote distances between facilities $X_i$, $P_j$ and $X_k$.
In the course of the development of exact algorithms, as well as algorithms for finding approximate solutions, promising areas are the study of the area of acceptable solutions, the possibility of reducing it, —hard. Therefore, the dual to each of which is necessary to determine the maximum cost or the total cost of communications among all facilities was minimal.

Taking into account the entered notations, further mathematical models of minimax and minimum Weber problems are provided. Objective functions for the specified problems have the next form respectively:

\[
\max \{ \max_{i \in I, j \in J} \{ w_{ij}d(X_i, P_j) \}; \max_{i, k \in I, i < k} \{ u_{ik}d(X_i, X_k) \} \} \to \min, \tag{1}
\]

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij}d(X_i, P_j) + \sum_{i=1}^{n-1} \sum_{k=i+1}^{n} u_{ik}d(X_i, X_k) \to \min, \tag{2}
\]

under constraints

\[
X_i \notin \text{Int} \mathcal{B}, \quad i \in I, \tag{3}
\]

where \(d(\cdot, \cdot)\) — some metric, \(\text{Int} K\) — the interior of the set \(K\).

As already noted rectangular metric is used, i.e.

\[
d(X_i, P_j) = |x_i - p_{1j}| + |y_i - p_{2j}|, \quad d(X_i, X_k) = |x_i - x_k| + |y_i - y_k|, \quad i, k \in I, \quad i < k, \quad j \in J.
\]

Weber problem (1) with rectangular metric is sufficiently investigated [6, 8, 18]. In [18] by introducing an additional parameter the problem was removed to a linear programming problem (LP). In [6] problem (1) with restrictions on the maximum admissible distances among facilities was researched. To move from metric \(l_1\) to metric \(l_\infty\) a linear mapping was used. Next the problem was decomposed into two problems along the coordinates \(x\) and \(y\). The dual to each of which is necessary to determine the maximum total cost flow in specific network. Also problem (1) was reduced to finding a shortest path in a network with non-negative arc lengths [8].

Problems (1) and (2) without the condition (3) with rectangular and Euclidean metrics belong to the convex programming problems [9, 21, 27]. Note that the presence of additional restrictions, for example, forbidden zones (3), or conditions of regular location along parallel lines, lead to the fact that most of these problems in general case are \(NP\)–hard. Therefore, promising areas are the study of the area of acceptable solutions, the possibility of reducing it, the development of exact algorithms, as well as algorithms for finding approximate solutions.

In the literature, there are many different formulations of Weber’s problems for point facilities on a plane with forbidden zones or barriers. Forbidden zones are, as a rule, convex sets with a non–empty interior, in which you can not locate facilities, but you can build connections between them through them. As a rule, in such problems the location of facilities is performed. Also the cost of communication between facilities is estimated. Barriers are areas in which location facilities is prohibited. Also lay routes through barriers to establish a connection between facilities is not allowed. Routes should not cross barriers, only bypass or through special passages in them. Usually in problems with barriers, the location of facilities is found, and the problem of tracing routes is solved.

Note that in the literature, more attention is paid to problems with barriers than with forbidden zones. Various approaches to solve problems with barriers are used [2, 3, 7, 14, 15, 16]. For example, heuristic algorithms for a special kind of metrics were described in [3], and genetic algorithms were presented in [2]. In [14], the original nonconvex problem was reduced to a series of polynomial solvable problems of lower dimension. For the case of rectangular metric, a series
of discrete problems was formulated in [16]. The location of one facility with different types of barriers was considered in [15], and such problem for several facility was researched in [7]. An overview of the results of studies of problems with barriers and various metrics was given in [15]. The works [1, 4, 5, 11, 12, 13, 24] were devoted to problems with forbidden zones. Productions differ from each other in the form of zones, the possibility of turning them, changing the position in the location area, etc. Most often, special cases were considered when one facility is located, and there are several zones, or vice versa.

Various approaches to solve Weber’s problems with forbidden zones are used [1, 4, 5, 11, 12, 13]. In [11], a heuristic algorithm for solving the minisum problem for Euclidean metric and one forbidden zone in the form of a circle was proposed. A sequence of nonlinear programming problems is solved in this algorithm. The adaptation of the proposed algorithm to the case of several zones in the form of circles and one polygonal zone was described in [12, 13]. A procedure for constructing a special grid whose nodes contain the solution of a problem with an arbitrary number of convex polygonal zones and different types of metrics ($l_p, 1 \leq p \leq 2$) was proposed in [1, 4, 5]. In [24], a property of the problem with a minisum criterion and rectangular forbidden zones was shown. For solution of the problem a combinatorial branch and bounds algorithm was proposed. Models of partial–integer linear programming with Boolean variables for minimax and minisum Weber problems with rectangular metric and rectangular forbidden zones were constructed in [22]. This approach allows you to use the integer optimization apparatus and application software packages.

Optimal location of interconnected facilities on a plane with an arbitrary number of forbidden zones is a much more complex and popular problem. A review of modern literature and Internet sources allows us to conclude that problems with barriers are mainly considered. Problems with rectangular metric, minimax or minisum criterion, and rectangular forbidden zones are insufficiently studied.

This paper provides an overview of studies for minimax and minisum Weber problems with rectangular forbidden zones [22, 24, 25]. Some properties of the problems are described. For example, the possibilities of reductions of admissible areas in the search for an optimal solution are presented. Different variants of lower bounds of the objective functions for the problems are provided. Models of integer linear programming (ILP) are given. Computational experiments with using the branch and bound algorithms, the constructed models and IBM ILOG CPLEX package were carried out. It is demonstrated that the application of the properties of reductions of admissible areas is promising both in solving of the problems by combinatorial methods and using the integer optimization apparatus. This may have practical applications when designing general plans of petrochemical enterprises, it is necessary to provide convenient maintenance of equipment, direct driveways, zoning of the territory.

3. Reductions of admissible areas

Note some properties of problem (1), (3) and problem (2), (3). Let a zone $F_k$ be bounded by a rectangle $[(a_k, c_k); (b_k, d_k)]$, where $(a_k, c_k)$ are the coordinates of its lower–left corner, and $(b_k, d_k)$ are the coordinates of its upper–right corner, $\forall k \in \mathbb{Z}$.

Next, we introduce the notation

$$A = \min_{j \in J, k \in \mathbb{Z}} \{p_{1j}, a_k\}, \quad B = \max_{j \in J, k \in \mathbb{Z}} \{p_{1j}, b_k\},$$
$$C = \min_{j \in J, k \in \mathbb{Z}} \{p_{2j}, c_k\}, \quad D = \max_{j \in J, k \in \mathbb{Z}} \{p_{2j}, d_k\}.$$

To solve the problems, it is sufficient to consider the area $\mathcal{F}$ bounded by a rectangle $[(A, C); (B, D)]$ [18]. If the lower–left corner of $\mathcal{F}$ does not coincide with the origin, then this
can be achieved by a parallel shift. If \( B_F = B - A \) and \( D_F = D - C \), then optimal solutions of the problems are in the rectangle \([(0, 0); (B_F, D_F)]\) (see Figure 2).

Let’s denote by \( R \) the subset of \( F \) in which facilities can be located, i.e. \( R = F \setminus \text{Int} B \) — the area of admissible solutions. In general case, \( R \) is nonconvex and disjoint set.

To solve problem (1), (3) for \( n = 1 \) an algorithm was presented in [22]. Let’s denote by \( X'_1, \ldots, X'_n \) optimal locations of facilities with coordinates \((x'_i, y'_i), i \in I\), which are found by using this algorithm. Let

\[
A' = \min_{i \in I} \{ x'_i \}, \quad B' = \max_{i \in I} \{ x'_i \}, \quad C' = \min_{i \in I} \{ y'_i \}, \quad D' = \max_{i \in I} \{ y'_i \}.
\]

Consider a rectangular area \( F' \), defined by the coordinates \([(A', C'); (B', D')]\) and let \( \partial F' \) is its boundary. Then next proposition is valid.

**Proposition 1.** [25] If \( \partial F' \subseteq R \), then there is an optimal solution of problem (1), (3) in the domain of \( F' \).

Let’s \( Z' \subseteq Z \) is a set of forbidden zone numbers such that for \( j \in Z' \) the condition \( \text{Int}(F'_j \cap F) \neq \emptyset \) is satisfied. Takes locate \( C_i \) and \( D_j \).

**Corollary 1.** [25] If \( Z' \neq \emptyset \) and \( \text{Int}(F'_j \cap F) = \emptyset \) for any \( j \in Z', k \in Z \setminus Z' \), then to find an optimal solution, it is sufficient to consider the area bounded by the contour \( \partial(F' \cup_{j \in Z'} F_j) \).

To construct the specified contour, it can be use the planar sweeping algorithm based on the segment tree [20].

Figure 2. Example of area \( F \), \( A = p_{11}, C = c_2, B = b_3, D = p_{24} \).

Let us consider the possibility of reduction of admissible area of minisum Weber problem (2), (3). For this aim special procedure for constructing the reduced area is used. Let’s describe it briefly. The vertical and horizontal lines passing through the fixed facilities and the sides of forbidden zones designate by \( C_i \) and \( D_j \). Let’s numbers of these vertical lines from left to right are from 1 to \( r \), \( r \leq m + 2z \). Also let’s numbers of horizontal lines from bottom to top are from 1 to \( q \), \( q \leq m + 2z \). The ordered set of points of the intersection of the lines \( C_i \) and \( D_j \),
\[ i = 1, \ldots, r, \ j = 1, \ldots, q, \] located beyond the forbidden zones denote by \( S = \{S_1, \ldots, S_t\} \). Let

their numbers designate by \( T = \{1, \ldots, t\} \). The next proposition is valid.

**Proposition 2.** [19] There exists an optimal solution \( X = (X_1, \ldots, X_n) \) of problem (2), (3), such that \( X_i \in S \), for any \( i \in I \).

**Proposition 3.** [24] Let \( \partial F' \subseteq R \). Then there is an optimal solution of problem (2), (3) in the area \( F' \).

This property allows the original continuous problem (2), (3) is reduced to a discrete one.

Thus, it is possible to reduce the admissible areas for an optimal solution of minimax and minisum Weber problems on a plane with forbidden zones.

4. Models of integer linear programming

In [22] the models of integer linear programming (ILP) for problems (1), (3) and (2), (3) are described. Note that the area \( R \) can be represented as a union of rectangles \( R_k \) (allowed areas) in which facilities can be located, \( k \in G = \{1, \ldots, g\} \). The sides of allowed areas are parallel to the coordinate axes. To build the allowed areas special algorithm from [23] is used.

Let an area \( R_k \) be bounded by a rectangle \([\tilde{a}_k, \tilde{c}_k); (\tilde{b}_k, \tilde{d}_k)]\), \( k \in G \). Introduce boolean variables \( h_{ik} \), \( h_{ik} = 1 \), if \( X_i \in R_k \), otherwise \( h_{ik} = 0 \), \( i \in I \), \( k \in G \). Condition that the facility \( X_i \) belongs to only one allowed area \( R_k \) was written using boolean variables \( h_{ik} \) and has the form:

\[
\begin{align*}
  x_i - \tilde{a}_k h_{ik} &\geq 0 \\
y_i - \tilde{c}_k h_{ik} &\geq 0 \\
-x_i + B_F - h_{ik}(B_F - \tilde{b}_k) &\geq 0 \\
y_i + D_F - h_{ik}(D_F - \tilde{d}_k) &\geq 0 \\
\sum_{k=1}^{G} h_{ik} &= 1 \\
h_{ik} \in \{0, 1\}
\end{align*}
\]

(4)

By maintaining the additional parameter \( x_0 \geq 0 \), problem (1) was reduced to the equivalent linear programming problem (LP) [18]. Taking into account the conditions (4), we obtain the following model for minimax problem:

\[
\begin{align*}
x_0 &\to \text{min} \\
x_i + y_i + \frac{a_{ij}}{w_{ij}} &\geq p_{ij} + p_{2j} \\
x_i - y_i + \frac{a_{ij}}{w_{ij}} &\geq -p_{ij} + p_{2j} \\
x_i + y_i + \frac{a_{ij}}{w_{ij}} &\geq p_{ij} - p_{2j} \\
x_i - y_i + \frac{a_{ij}}{w_{ij}} &\geq -p_{ij} - p_{2j}
\end{align*}
\]

(5), \( i \in I, \ j \in J \)

\[
\begin{align*}
x_i - x_k + y_i - y_k + \frac{a_{ik}}{u_{ik}} &\geq 0 \\
x_i + x_k + y_i - y_k + \frac{a_{ik}}{u_{ik}} &\geq 0 \\
x_i - x_k - y_i + y_k + \frac{a_{ik}}{u_{ik}} &\geq 0 \\
x_i + x_k - y_i + y_k + \frac{a_{ik}}{u_{ik}} &\geq 0
\end{align*}
\]

(6), \( i, k \in I, \ i < k \)

Note that if there is a subset of \( G' \subseteq G \) such that \( F' \subseteq \bigcup_{k \in G'} R_k \), then problem (1), (3) is polynomial solvable.

Further let’s write the ILP model for problem (2), (3). Note, that function (2) without the condition (3) can be represented as two independent LP problems with additional variables [1]: \( s = (s_{ij}), \ l = (l_{ij}), \ i \in I, \ j \in J; \ t = (t_{ik}), \ u = (u_{ik}), \ i, k \in I, \ i < k \). The condition (3) for minisum problem it is written in the same way (4) that for minimax problem. Then the ILP
model for the problem (2), (3) is [24]:

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij} (s_{ij} + l_{ij}) + \sum_{i=1}^{n-1} \sum_{k=i+1}^{n} u_{ik} (t_{ik} + v_{ik}) \rightarrow \min, \quad (8)
\]

\[
-s_{ij} \leq x_i - p_{1j} \leq s_{ij}, \quad i \in I, j \in J, \quad (9)
\]

\[
-t_{ik} \leq x_i - x_k \leq t_{ik}, \quad i, k \in I, i < k, \quad (10)
\]

\[
-l_{ij} \leq y_i - p_{2j} \leq l_{ij}, \quad i \in I, j \in J, \quad (11)
\]

\[
-v_{ik} \leq y_i - y_k \leq v_{ik}, \quad i, k \in I, i < k, \quad (12)
\]

\[
A' \leq x_i \leq B', i \in I, \quad (13)
\]

\[
C' \leq y_i \leq D', i \in I, \quad (14)
\]

\[
\sum_{k=1}^{g} h_{ik} = 1, \quad h_{ik} \in \{0, 1\} \quad (15)
\]

5. Branch and bound algorithms

Consider the branch and bound algorithms for minimax (1), (3) and minisum (2), (3) problems [25, 24]. In any branch and bound algorithm it is important to construct lower bounds of the objective function’s value. Also it is need to divide a set of admissible solutions into subsets (branching).

For minimax problem branching is performed by sequentially fixing facilities in the specified order in the allowed areas. Several variants for constructing lower bounds of objective function’s value for the location of facilities in allowed areas are proposed in [25]. Let \( G' \) is the set of the numbers of the areas in which the located facilities are fixed.

In the first variant, \( X_i \) is fixed in \( R_k \). Let \( I(R_s) \) is the set of located facilities in area \( R_s, s \in G' \subseteq G \). Then lower bound of value of function (1) when the facility \( X_i \) in the area \( R_k \) is calculated by the formula:

\[
\max \{ \max_{j \in J} (w_{ij} d(R_k, P_j)), \max_{s \in G'} (u_{it(s)} d(R_k, R_s)) \},
\]

where \( u_{it(s)} = \max_{r \in I(R_s)} u_{ir}, s \in G' \) and distances between sets are defined by a standard way.

In the second variant of bound, the maximum weighted distance from \( X_i \) to fixed facilities \( P_j, j \in J \) is estimated. To solve problem (1), (3) for \( n = 1 \) the algorithm from [22] is used.

In the third variant of bound, restrictions on the minimum admissible distances among facilities are added. Facilities are fixed in allowed areas. Restrictions are determined by the minimum distances among the corresponding allowed areas. Next the solution of LP problem is found.

In [24] branch and bound algorithm for minisum problem (2), (3) was described. The main idea of the combinatorial branch and bound algorithm is to locate facilities at the points of the set \( S \) (introduced in paragraph 3). At each iteration all vertices of the branching tree are considered and lower bounds of values of function (2) for these vertices are calculated. The algorithm terminates as soon as a subset with a minimum lower bound is found and estimation is equal to the value of the objective function (2) in the admissible solution.
We find the optimal solutions $X'_1, \ldots, X'_n$, for each arrangeable facility relative only to fixed facilities (introduced in paragraph 3). Suppose that the record $\text{Rec}$ is equal to the value $f(X'_1, \ldots, X'_n)$. Next, there is following process of the modification for the solution. In the specified order, one by one, we release the next $X_i$ and locate it relative to $P_1, \ldots, P_m$ and $X'_{i-1}, X'_{i+1}, \ldots, X'_n$, $\forall i \in I$. Thus we find the new location $\bar{X}_i$ for facility $X_i$. After performing this procedure for each facility, we get a new solution $\bar{X}_1, \ldots, \bar{X}_n$. If $f(\bar{X}_1, \ldots, \bar{X}_n) < \text{Rec}$, then $\text{Rec} := f(\bar{X}_1, \ldots, \bar{X}_n)$. As a result, we get a new improved initial solution $\bar{X}_1, \ldots, \bar{X}_n$ of problem (2), (3). Calculate the following values:

$$\bar{f}(\bar{X}_i) = \sum_{j=1}^{m} w_{ij} d(\bar{X}_i, P_j), \quad \forall i \in I.$$ 

Determine the lower bound of the objective function’s value on the set $S$:

$$\xi(S) = \sum_{i=1}^{n} \bar{f}(\bar{X}_i).$$

If $\text{Rec} = \xi(S)$, then $\bar{X}_1, \ldots, \bar{X}_n$ is an optimal solution of problem (2), (3). Otherwise, move on to the next stage branching and building a lower bound.

The set of all valid solutions $S$ is divided into subsets $S_1, \ldots, S_l$. Where $S_{i_1}$ is a subset of solutions in which the first facility is located at a point with the number $i_1$, $i_1 \in T$. Denote by $\bar{X}_i$ the location facility $X_i$ at the point from set $T$. Each subset $S_i$ we divide into subsets $S_{i_1}, \ldots, S_{i_l}$. Where $S_{i_1}$ is a subset of solutions, in which the first facility is located at the point with the number $i_1$ and the second is located at the point with number $i_2$, $i_1, i_2 \in T$, etc. So, all facilities are located. As a result we get subsets $S_{i_1, i_2, \ldots, i_l}$ such that the $X_i$ is located at the point with the number $i_l$, $i_l \in T$, $l \in I$.

Consider the subset $S_{i_1, i_2, \ldots, i_l}$ of the tree of variants, $l \leq n$. The lower bound of the objective function’s value on this subset is calculated as follows:

$$\xi(S_{i_1, i_2, \ldots, i_l}) = \sum_{i=1}^{l} \sum_{j=1}^{m} w_{ij} d(\bar{X}_i, P_j) + \sum_{i=1}^{l-1} \sum_{k=i+1}^{l} u_{ik} d(\bar{X}_i, \bar{X}_k) + \sum_{i=l+1}^{n} \bar{f}(\bar{X}_i).$$

In addition, the objective function’s value is calculated on the subset $S_{i_1, i_2, \ldots, i_l}$:

$$f(S_{i_1, i_2, \ldots, i_l}) = f(\bar{X}_1, \ldots, \bar{X}_i, \bar{X}_{i+1}, \ldots, \bar{X}_n).$$

If $f(S_{i_1, i_2, \ldots, i_l}) < \text{Rec}$, then $\text{Rec} := f(S_{i_1, i_2, \ldots, i_l})$, otherwise the record remains unchanged.

### 6. Results of experiments

To compare the results of solving problems with and without the found property of reduction of admissible area (propositions) computational experiments are carried out. Using the branch and bounds algorithms and the LP models and the IBM ILOG CPLEX package solutions of the problems were found.

We briefly describe the results of computational experiments for the minimax problem (1), (3). The coordinates of forbidden zones and the costs of communication between facilities were generated randomly. To solve the problem, the branch and bounds algorithm with the first bound of the objective function’s value is implemented. The results of the experiments are given in Tables 1 and 2. Let’s $t^*_B$ and $t_B$ are the running time (in seconds) of the algorithm with and without the property of reduction of admissible area respectively. Similarly, $t^*_\text{CPLEX}$ and $t_{\text{CPLEX}}$ are the running time of the CPLEX package with and without the property of reduction
Table 1. Solving minimax problem by the branch and bounds algorithm

| N  | n  | m  | g  | \( g^* \) | \( t_B \) | \( t_B^* \) |
|----|----|----|----|---------|--------|--------|
| 1  | 3  | 2  | 3  | 1       | 0,202  | 0,140  |
| 2  | 5  | 5  | 8  | 3       | 1,076  | 0,562  |
| 3  | 6  | 7  | 5  | 2       | 2,371  | 1,155  |
| 4  | 6  | 7  | 4  | 2       | 0,873  | 0,561  |
| 5  | 5  | 9  | 10 | 3       | 3,712  | 1,373  |
| 6  | 8  | 2  | 5  | 2       | 0,889  | 0,499  |
| 7  | 10 | 10 | 10 | 3       | 4,586  | 0,843  |
| 8  | 10 | 15 | 10 | 3       | 5,928  | 1,762  |
| 9  | 16 | 19 | 7  | 3       | 9,142  | 6,614  |
| 10 | 20 | 5  | 7  | 3       | 5,897  | 3,135  |

Table 2. Solving minimax problem by CPLEX

| n  | m  | g  | \( g^* \) | \( t_{CPLEX} \) | \( t_{CPLEX}^* \) |
|----|----|----|---------|-------------|-------------|
| 1  | 3  | 2  | 3       | 0,12        | 0,01        |
| 2  | 5  | 5  | 8       | 0,44        | 0,11        |
| 3  | 6  | 7  | 5       | 0,13        | 0,01        |
| 4  | 6  | 7  | 4       | 0,22        | 0,2         |
| 5  | 5  | 9  | 10      | 0,36        | 0,05        |
| 6  | 8  | 2  | 5       | 0,33        | 0,22        |

of admissible area. The number of allowed areas in the problem with and without the property denoted by \( g^* \) and \( g \), respectively.

As a result of experiments, it can be concluded that the proven property reduces the time to solve the problem both when solving it by the algorithm and when using the package. Note that similar experiments were performed for the minisum problem (2), (3) in work [24]. As a result, it is experimentally confirmed that taking into account the found properties makes it possible to reduce the time for solving the problem.

7. Conclusion

Minimax and minisum Weber problems on a plane with rectangular metric and rectangular forbidden zones were investigated. An overview of studies for these problems was provided. Some properties of the problems are described. The possibilities of reductions of admissible areas in the search for an optimal solution were presented. Different variants of lower bounds of the objective functions for the problems are provided. Models of integer linear programming are given. Computational experiments with using the branch and bound algorithms, the constructed models and IBM ILOG CPLEX package were carried out. Usage of the properties of reductions of admissible areas is promising both in solving of the problems by combinatorial methods and using the integer optimization apparatus. This may have practical applications when designing general plans of petrochemical enterprises, it is necessary to provide convenient maintenance of equipment, direct driveways, zoning of the territory.

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