Boundary problems for the one-dimensional kinetic equation with the collisional frequency proportional to the module velocity of molecules

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Abstract

For the one-dimensional linear kinetic equations with collisional frequency of the molecules, proportional to the module velocity of molecules, analytical solutions of problems about temperature jump and weak evaporation (condensation) in rarefied gas are received. Quantities of temperature and concentration jumps are found. Distributions of concentration, mass velocity and temperature are constructed. Necessary numerical calculations and graphic researches are done.

Key words: kinetic equation, collisional frequency, boundary problems, analytical solution, distribution of macroparameters.

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Introduction

In work \cite{1} the linear one-dimensional kinetic equation with collisional integral BGK (Bhatnagar, Gross and Krook) and collisional frequency, affine depending on the module of velocity of molecules has been entered. Preservation laws numerical density (concentration) of molecules, momentum and energy of molecules thus have been used.

In \cite{1} the theorem about structure of common solution of the entered equation has been proved.

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In work [2], being by continuation [1], are received exact solutions of a problem on temperature jump and weak evaporation (condensation) in the rarefied gas. These two problems following [3] we will name the generalized problem of Smolukhovsky.

In work [4] the generalized problem of Smolukhovsky has been solved for special case of affine dependence of collisional frequency on the module of velocity of molecules. Namely, the case, when collisional frequency is constant has been considered.

In the present work other limiting case affine dependence when frequency of collisions is proportional to the module of velocity of molecules is considered. On the basis of the analytical solution of Smolukhovsky’ boundary problem the function of distribution of gas molecules is constructed. Numerical values of jump of temperature and weak evaporation (condensation) coefficients are found. Distributions of concentration, mass velocity and temperature in "half-space" are constructed.

Let us stop on history of exclusively analytical solutions of the generalized Smolukhovsky’ problem.

For the simple (one-atomic) rarefied gas with constant frequency of collisions of molecules the analytical solution of the generalized Smolukhovsky’ problem was received in [5].

In [6] the generalized Smolukhovsky’ problem was analytically solved for the simple rarefied gas with collisional frequency of molecules, linearly depending on the module velocity of molecules.

In [7] the problem about strong evaporation (condensation) with constant frequency of collisions has been analytically solved.

Let us notice, that for the first time the problem about temperature jump with collisional frequency of molecules, linearly depending on the module of molecular velocity, was analytically solved by Cassell and Williams in work [8] in 1972.

Then in works [9, 10, 11] the Smolukhovsky’ problem has been generalized on case of multiatomic (molecular) gases and also the analytical
solution was received.

In works [12, 13, 14] the problem close to the problem about temperature jump for electrons, about behaviour of the quantum Bose-gas at low temperatures is considered. It has been thus used the kinetic equation with fonons excitation agrees N.N. Bogolyubov.

In works [15, 16] the problem about temperature jump for electrons of degenerate plasmas in metal has been solved.

In work [17] the analytical solution of Smolukhovsky’ problem and for quantum gases has been received.

In work of Cercignani and Frezzotti [18] the Smolukhovsky’ problem was considered with use of the one-dimensional kinetic equations. The complete analytical solution of the Smolukhovsky’ problem with use of Cercignani—Frezzotti equation has been received in work [19].

In the present work the analytical solution of the generalized Smolukhovsky’ problem is considered. The case of collisional frequency proportional to the module of molecular velocity, in model of one-dimensional gas is considered. Model of one-dimensional gas gave the good consent with the results devoted to the three-dimensional gas [19].

1. Statement of the problem and basic equations

Let us start with statement of the Smolukhovsky’ problem for the one-dimensional kinetic equation with frequency of collisions, affinne depending on the module of molecular velocity.

Let gas occupies half-space \( x > 0 \). The surface temperature \( T_s \) and concentration of sated steam of a surface \( n_s \) are set. Far from the surface at \( x = 0 \) gas moves with some velocity \( u \), being velocity of evaporation (or condensation), also has the temperature gradient

\[
g_T = \left( \frac{d \ln T}{dx} \right)_{x=+\infty}.
\]

It is necessary to define temperature and concentration jumps depending on velocity \( u \) and temperature gradient \( g_T \).
In the problem about weak evaporation (condensation) is required to define temperature and concentration jumps depending on velocity, including a temperature gradient equal to zero, and velocity of evaporation (condensation) is enough small. The last means, that

\[ u \ll v_T. \]

Here \( v_T \) is the thermal velocity of molecules, having an order of a sound velocity,

\[ v_T = \frac{1}{\sqrt{\beta_s}}, \quad \beta_s = \frac{m}{2k_B T_s}, \]

\( m \) is the mass of molecule, \( k_B \) is the Boltzmann constant.

In the problem about temperature jump is required to define temperature and concentration jumps depending on a temperature gradient, thus velocity evaporation (condensation) is considered equal to zero, and the temperature gradient is considered small. It means, that

\[ lg_T \ll 1, \quad l = \tau v_T, \quad \tau = \frac{1}{\nu_0}, \]

where \( l \) is the mean free path of gas molecules, \( \tau \) is the time of relaxation, i.e. time between two consecutive collisions of molecules.

Let us unite both problems (about weak evaporation (condensation) and temperature jump) in one. We will assume a smallness of gradient temperature (i.e. a smallness of relative difference temperature on length of free path) and a smallness velocity of gas in comparison with velocity of a sound. In this case the problem supposes linearization and function of distribution it is possible to search for in the form

\[ f(x, v) = f_0(v)(1 + h(x, v)), \]

where

\[ f_0(v) = n_s \left( \frac{m}{2\pi k_B T_s} \right)^{1/2} \exp \left[ - \frac{mv^2}{2k_B T_s} \right] \]

is the absolute Maxwellian.
Let us pass to dimensionless velocity

$$\mu = \sqrt{\beta v} = \frac{v}{v_T}$$

and dimensionless coordinate

$$x' = \nu_0 \sqrt{\frac{m}{2k_B T_s}} x = \frac{x}{l}$$

Variable $x'$ we will designate again through $x$.

We take the kinetic equation [1]

$$\mu \frac{\partial h}{\partial x} + (1 + \sqrt{\pi a |\mu|}) h(x, \mu) =$$

$$= (1 + \sqrt{\pi a |\mu|}) \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu'^2} (1 + \sqrt{\pi a |\mu'|}) q(\mu, \mu', a) h(x, \mu') d\mu'. \quad (1.1)$$

Here $q(\mu, \mu', a)$ is the kernel of equation,

$$q(\mu, \mu', a) = r_0(a) + r_1(a) \mu \mu' + r_2(a) (\mu^2 - \beta(a)) (\mu'^2 - \beta(a)),$$

$$r_0(a) = \frac{1}{a + 1}, \quad r_1(a) = \frac{2}{2a + 1}, \quad r_2(a) = \frac{4(a + 1)}{4a^2 + 7a + 2},$$

$$\beta = \beta(a) = \frac{2a + 1}{2(a + 1)}.$$

At $a \to 0$ from the equation (1.1) we receive the following kinetic equation with constant frequency of collisions

$$\mu \frac{\partial h}{\partial x} + h(x, \mu) =$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{\mu'^2} \left[ 1 + 2\mu \mu' + 2(\mu^2 - \frac{1}{2}) \left( \mu'^2 - \frac{1}{2} \right) \right] h(x, \mu') d\mu'.$$

Let us deduce the kinetic equation in a second limiting case, when collisional frequency is proportional to the module of the molecular velocity.
Let us consider the second limiting case of the equation (1.1). We will return to expression of frequency of collisions also we will copy it in the form

\[ \nu(\mu) = \nu_0(1 + \sqrt{\pi}a|\mu|) = \nu_0 + \nu_1|\mu|, \]

where

\[ \nu_1 = \sqrt{\pi}\nu_0a. \]

Let us \( \nu_0 \) tends to zero. In this limit the quantity \( a \) tends to \( +\infty \), for

\[ a = \frac{\nu_1}{\sqrt{\pi}\nu_0}. \]

It is easy to see, that in this limit

\[ \lim_{a \to +\infty} (1 + \sqrt{\pi}a|\mu'|)q(\mu, \mu', a) = \sqrt{\pi}|\mu'|q_1(\mu, \mu'), \]

where

\[ q_1(\mu, \mu') = 1 + \mu\mu' + (\mu^2 - 1)(\mu'^2 - 1). \]

Thus the equation (1.1) will be transformed in the form

\[ \frac{\mu}{|\mu|} \frac{\partial h}{\partial x_1} + h(x_1, \mu) = \int_{-\infty}^{\infty} e^{-\mu^2} |\mu'|q_1(\mu, \mu')h(x_1, \mu)d\mu'. \quad (1.2) \]

In this equation

\[ x_1 = \nu_1\sqrt{\beta_s}x = \frac{x}{l_1}, \quad l_1 = v_T\tau_1, \quad \tau_1 = \frac{1}{\nu_1}. \]

This equation is the one-dimensional kinetic equation with collisional frequency proportional to the module of the molecular velocity.

The equation (1.2) it is possible to present in the form

\[ \text{sign } \mu \frac{\partial h}{\partial x_1} + h(x_1, \mu) = \int_{-\infty}^{\infty} e^{-\mu^2} |\mu'|q_1(\mu, \mu')h(x_1, \mu)d\mu'. \quad (1.2') \]

The equation (1.2') contains two equations. One of these equations

\[ + \frac{\partial h}{\partial x_1} + h(x_1, \mu) = \int_{-\infty}^{\infty} e^{-\mu^2} |\mu'|q_1(\mu, \mu')h(x_1, \mu)d\mu'. \quad (1.2'') \]
is defined in the phase quarter-planes \( \mathbb{R}^+ = \{(x, \mu) : x > 0, \mu > 0\} \), and another equation

\[
- \frac{\partial h}{\partial x_1} + h(x_1, \mu) = \int_{-\infty}^{\infty} e^{-\mu^2 |\mu'|} q_1(\mu, \mu') h(x_1, \mu) d\mu'
\]  

\[(1.2'')\]

is defined in other phase quarter-planes \( \mathbb{R}^- = \{(x, \mu) : x > 0, \mu < 0\} \).

Let us solve further the generalized Smolukhovsky’ problem for the equation \((1.2)\). At first it is required to formulate correctly the generalized Smolukhovsky’ problem as a boundary problem of mathematical physics.

2. The kinetic equation with collisional frequency proportional to the module of molecular velocity, and the problem statement

Rectilinear substitution it is possible to check up, that the kinetic equation \((1.2)\) has following four partial solutions

\[
h_0(x, \mu) = 1,
\]

\[
h_1(x, \mu) = \mu,
\]

\[
h_2(x, \mu) = \mu^2,
\]

\[
h_3(x, \mu) = \left(\mu^2 - \frac{3}{2}\right)(x - \text{sign} \ \mu).
\]

Let us consider, that molecules are reflected from a wall purely diffusively, i.e. are reflected from a wall with Maxwell distribution on velocity, i.e.

\[
f(x, v) = f_0(v), \quad v_x > 0.
\]

From here for function \( h(x, \mu) \) we receive

\[
h(0, \mu) = 0, \quad \mu > 0.
\]  

\[(2.1)\]

Condition \((2.1)\) is the first boundary condition to the equation \((1.2)\).
For asymptotics Chapmen—Enskog distribution we will search in the form of the linear combination of its partial solutions with unknown coefficients

\[ h_{as}(x, \mu) = A_0 + A_1 \mu + A_2 \left( \mu^2 - \frac{1}{2} \right) + A_3 \left[ \left( \mu^2 - \frac{3}{2} \right)(x - \text{sign } \mu) - \frac{1}{\sqrt{\pi} \mu} \right]. \tag{2.2} \]

Let us notice, that in (2.2) velocity mode is orthogonal to thermal mode, i.e.

\[
\int_{-\infty}^{\infty} e^{-\mu^2} \mu \left[ \left( \mu^2 - \frac{3}{2} \right)(x - \text{sign } \mu) - \frac{1}{\sqrt{\pi} \mu} \right] d\mu = 0.
\]

Besides, constant mode is orthogonal to temperature mode

\[
\int_{-\infty}^{\infty} e^{-\mu^2} \left( \mu^2 - \frac{1}{2} \right) d\mu = 0.
\]

For definition of four constants \( A_0, A_1, A_2, A_3 \) let us take advantage of definitions of macroparameters of gas: concentration, mass velocity, temperature and jumps of temperature and concentration (numerical density).

Let us consider distribution of numerical density

\[ n(x) = \int_{-\infty}^{\infty} f(x, v)dv = \int_{-\infty}^{\infty} f_0(v)(1 + h(x, v))dv = n_0 + \delta n(x). \]

Here

\[ n_0 = \int_{-\infty}^{\infty} f_0(v)dv, \quad \delta n(x) = \int_{-\infty}^{\infty} f_0(v)h(x, v)dv. \]

From here we find that

\[ \frac{\delta n(x)}{n_0} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} h(x, \mu)d\mu. \]
We denote
\[ n_e = n_0 \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} (1 + h_{as}(x = 0, \mu)) d\mu. \]

From here we receive that
\[ \varepsilon_n \equiv \frac{n_e - n_0}{n_0} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} h_{as}(x = 0, \mu) d\mu. \] \tag{2.3}

The quantity \( \varepsilon_n \) is the required quantity of concentration jump.
Substituting (2.2) in (2.3), we find that
\[ \varepsilon_n = A_0. \] \tag{2.4}

From definition of dimensional velocity of gas
\[ u(x) = \frac{1}{n(x)} \int_{-\infty}^{\infty} f(x, v) v d\mu \]
we receive that in linear approximation dimensional mass velocity equals
\[ U(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} h(x, \mu) \mu d\mu. \]

Setting "far from a wall" velocity of evaporation (condensation), let us write
\[ U = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} h_{as}(x, \mu) \mu d\mu. \] \tag{2.5}

Substituting in (2.5) distribution (2.2), we receive, that
\[ U = \frac{\sqrt{\pi}}{2} A_1. \] \tag{2.6}

We consider the temperature distribution
\[ T(x) = \frac{2}{kn(x)} \int_{-\infty}^{\infty} \frac{m}{2} (v - u_0(x))^2 f(x, v) dv. \]
From here we find that

\[
\frac{\delta T(x)}{T_0} = -\frac{\delta n(x)}{n_0} + \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} h(x, \mu) \mu^2 d\mu =
\]

\[
= \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} h(x, \mu)(\mu^2 - \frac{1}{2}) d\mu.
\]

Now from here follows, that at \( x \to +\infty \) asymptotic distribution is equal

\[
\frac{\delta T_{as}(x)}{T_0} = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} h_{as}(x, \mu)(\mu^2 - \frac{1}{2}) d\mu. \tag{2.7}
\]

Definition of a gradient of temperature far from a wall means, that distribution of temperature looks like

\[
T(x) = T_e + \left( \frac{dT}{dx} \right)_{x=+\infty} \cdot x = T_e + G_T x,
\]

where

\[
G_T = \left( \frac{dT}{dx} \right)_{x=+\infty}.
\]

This distribution we will present in the form

\[
T(x) = T_s \left( \frac{T_e}{T_s} + g_T x \right) = T_s \left( 1 + \frac{T_e - T_s}{T_s} + g_T x \right), \quad x \to +\infty,
\]

where

\[
g_T = \left( \frac{d \ln T}{dx} \right)_{x=+\infty},
\]

or

\[
T(x) = T_s (1 + \varepsilon_T + g_T x), \quad x \to +\infty, \tag{2.8}
\]

where

\[
\varepsilon_T = \frac{T_e - T_s}{T_s}
\]

is the required quantity of temperature jump.

From expression (2.8) it is visible, that relative temperature change far from a wall is described by linear function

\[
\frac{\delta T_{as}(x)}{T_s} = \frac{T(x) - T_s}{T_s} = \varepsilon_T + g_T x, \quad x \to +\infty. \tag{2.9}
\]
Substituting (2.2) in (2.7), we receive, that
\[ \frac{\delta T_{as}(x)}{T_s} = A_2 + A_3 x. \]  
(2.10)

Comparing (2.9) and (2.10), we find
\[ A_2 = \varepsilon_T, \quad A_3 = g_T. \]

So, asymptotic distribution function of Chapmen—Enskog is constructed
\[ h_{as}(x, \mu) = \varepsilon_n + (2U - g_T) \frac{\mu}{\sqrt{\pi}} + \]
\[ + \varepsilon_T \left( \mu^2 - \frac{1}{2} \right) + g_T \left( \mu^2 - \frac{3}{2} \right) (x - \text{sign } \mu). \]  
(2.11)

Now we will formulate the second boundary condition to the equation (1.2)
\[ h(x, \mu) = h_{as}(x, \mu) + o(1), \quad x \to +\infty. \]  
(2.12)

Now we will formulate the basic boundary problem, named "generalized Smolukhovsky' problem". This problem consists in finding of the such solution of the kinetic equation (1.2), which satisfies to boundary conditions (2.1) and (2.12), and in (2.12) asymptotic function of Chapmen—Enskog distribution $h_{as}(x, \mu)$ is defined by equality (2.11).

3. The general solution of the one-dimensional kinetic equation

Let us notice, that the continuous spectrum of the characteristic equation, answering to the initial equation (1.2), represents the empty set (see [2] and [4]). This fact speaks that the equation (1.2) does not contain convection derivative.

Therefore for the solution of the initial equation (1.2) we will search in the form of a polynom on velocity variable. We search for the solution in the form its linear combination of invariants collisions with unknown coefficients depending from "spatial" variable
\[ h(x, \mu) = a_0(x) + a_1(x) \mu + a_2(x) (\mu^2 - 1) + \]
\[ + \text{sign } \mu[b_0(x) + b_1(x)\mu + b_2(x)(\mu^2 - 1)]. \quad (3.1) \]

Distribution function (3.1) contains two distribution functions. One function
\[ h^+(x, \mu) = a_0(x) + a_1(x)\mu + a_2(x)(\mu^2 - 1) + \]
\[ + b_0(x) + b_1(x)\mu + b_2(x)(\mu^2 - 1), \quad \mu > 0. \quad (3.1') \]
describes the molecules flying to the wall.

The second function
\[ h^-(x, \mu) = a_0(x) + a_1(x)\mu + a_2(x)(\mu^2 - 1) - \]
\[ - [b_0(x) + b_1(x)\mu + b_2(x)(\mu^2 - 1)], \quad \mu < 0. \quad (3.1'') \]
describes the molecules reflected from a wall.

The left part of the equation (1.2) is equal to the sum of expressions
\[
\text{sign } \mu \frac{\partial h^\pm}{\partial x} = \begin{cases} 
  a'_0 + b'_0 + (a'_1 + b'_1)\mu + (a'_2 + b'_2)(\mu^2 - 1), \quad \mu > 0, \\
  -(a'_0 - b'_0) - (a'_1 - b'_1)\mu - (a'_2 - b'_2)(\mu^2 - 1), \quad \mu < 0,
\end{cases}
\]
and
\[ h^\pm(x, \mu) = (a_0 \pm b_0) + (a_1 \pm b_1)\mu + (a_2 \pm b_2)(\mu^2 - 1), \quad \pm \mu > 0. \]

Thus, at \( \mu > 0 \) the left part of the equation (1.2) for the molecules reflected from a wall is equal
\[ \frac{\partial h^+}{\partial x} + h^+(x, \mu) = [a_0(x) + b_0(x) + a'_0(x) + b'_0(x)] + \]
\[ + \mu[a_1(x) + b_1(x) + a'_1(x) + b'_1(x)] + \]
\[ + (\mu^2 - 1)[a_2(x) + b_2(x) + a'_2(x) + b'_2(x)]. \]

At \( \mu < 0 \) the right part of the equation (1.2) for flying to a wall molecules is equal
\[ -\frac{\partial h^-}{\partial x} + h^-(x, \mu) = [a_0(x) - b_0(x) - a'_0(x) + b'_0(x)] + \]
\[ + \mu[a_1(x) - b_1(x) - a'_1(x) + b'_1(x)] + \]
\[ (\mu^2 - 1)[a_2(x) - b_2(x) - a'_2(x) + b'_2(x)]. \]

The right part of the equation (1.2) is equal
\[ R[h(x, \mu)] = \int_0^\infty e^{-\mu'^2} q(\mu, \mu') h^+(x, \mu') d\mu' + \]
\[ + \int_0^\infty e^{-\mu'^2} q(\mu, -\mu') h^-(x, -\mu') d\mu'. \]

Let us substitute in this right part distribution function
\[ h^+(x, \mu) = (a_0 + b_0) + (a_1 + b_1)\mu + (a_2 + b_2)(\mu^2 - 1) \]
and
\[ h^-(x, -\mu) = (a_0 - b_0) - (a_1 - b_1)\mu + (a_2 - b_2)(\mu^2 - 1). \]

We receive, that the right part is equal
\[ R[h] = \frac{1}{2}[a_0(x) + b_0(x)] + \frac{\sqrt{\pi}}{4}[a_1(x) + b_1(x)] + \]
\[ + \mu \left\{ \frac{\sqrt{\pi}}{4}[a_0(x) + b_0(x)] + \frac{1}{2}[a_1(x) + b_1(x)] + \frac{\sqrt{\pi}}{8}[a_2(x) + b_2(x)] \right\} + \]
\[ + (\mu^2 - 1) \left\{ \frac{\sqrt{\pi}}{8}[a_1(x) + b_1(x)] + \frac{1}{2}[a_2(x) + b_2(x)] \right\} + \]
\[ + \frac{1}{2}[a_0(x) - b_0(x)] - \frac{\sqrt{\pi}}{4}[a_1(x) - b_1(x)] + \]
\[ - \mu \left\{ \frac{\sqrt{\pi}}{4}[a_0(x) - b_0(x)] - \frac{1}{2}[a_1(x) - b_1(x)] + \frac{\sqrt{\pi}}{8}[a_2(x) - b_2(x)] \right\} + \]
\[ + (\mu^2 - 1) \left\{ -\frac{\sqrt{\pi}}{8}[a_1(x) - b_1(x)] + \frac{1}{2}[a_2(x) - b_2(x)] \right\}. \]

Let us simplify the previous expression
\[ R[h] = a_0(x) + \frac{\sqrt{\pi}}{2} b_1(x) + \mu \left[ \frac{\sqrt{\pi}}{2} b_0(x) + a_1(x) + \frac{\sqrt{\pi}}{4} b_2(x) \right] + \]
\[ + (\mu^2 - 1) \left[ \frac{\sqrt{\pi}}{4} b_1(x) + a_2(x) \right]. \]
Let us equate the left and right parts of the equation (1.2). We will receive system, consisting of six equations

\[
\begin{align*}
    a_0' + b_0' + b_0 &= \frac{\sqrt{\pi}}{4} b_1, \\
    a_0' - b_0' + b_0 &= -\frac{\sqrt{\pi}}{4} b_1, \\
    a_1' + b_1' + b_1 &= \frac{\sqrt{\pi}}{2} b_0 + \frac{\sqrt{\pi}}{4} b_2, \\
    -a_1' + b_1' - b_1 &= \frac{\sqrt{\pi}}{2} b_0 + \frac{\sqrt{\pi}}{4} b_2, \\
    a_2' + b_2' + b_2 &= \frac{\sqrt{\pi}}{4} b_1, \\
    -a_2' + b_2' - b_2 &= \frac{\sqrt{\pi}}{4} b_1.
\end{align*}
\]

Adding the first equation with the second, the third with the fourth, the fifth with the sixth, and then subtracting, we will simplify this system

\[
\begin{align*}
    a_0'(x) + b_0(x) &= 0, \quad (3.2) \\
    b_0'(x) &= \frac{\sqrt{\pi}}{2} b_1(x), \quad (3.3) \\
    b_1'(x) &= \frac{\sqrt{\pi}}{2} b_0(x) + \frac{\sqrt{\pi}}{4} b_2(x), \quad (3.4) \\
    a_1'(x) + b_1(x) &= 0, \quad (3.5) \\
    b_2'(x) &= \frac{\sqrt{\pi}}{4} b_1(x), \quad (3.6) \\
    a_2'(x) + b_2(x) &= 0. \quad (3.7)
\end{align*}
\]

We differentiate the equation (3.4) and we will take advantage of the equations (3.3) and (3.6). We receive the equation

\[
    b_1''(x) = \frac{5\pi}{16} b_1(x),
\]

whence we find

\[
    b_1(x) = B_1 e^{-\gamma_0 x}, \quad (3.8)
\]
where $B_1$ is the arbitrary constant, and

$$\gamma_0 = \frac{\sqrt{5\pi}}{4} \approx 0.9908.$$  

From the equations (3.6) and (3.3) by means of (3.8) we receive

$$b_2(x) = -\frac{2}{\sqrt{5}} B_1 e^{-\gamma_0 x} + B_2,$$  

$$b_0(x) = -\frac{2}{\sqrt{5}} B_1 e^{-\gamma_0 x} + B_0,$$  

where $B_0$ and $B_2$ are arbitrary constants.

From the equation (3.5) by means of (3.8) it is found

$$a_1(x) = -\frac{1}{\gamma_0} B_1 e^{-\gamma_0 x} + A_1,$$  

where $A_1$ is the arbitrary constant.

From the equation (3.2) by means of (3.11) it is found

$$a_0(x) = -\frac{8}{5\sqrt{\pi}} B_1 e^{-\gamma_0 x} - B_0 x + A_0,$$  

where $A_0$ is the arbitrary constant.

At last, from the equation (3.7) by means of (3.9) it is found

$$a_2(x) = -\frac{8}{5\sqrt{\pi}} B_1 e^{-\gamma_0 x} - B_2 x + A_2,$$  

where $A_2$ is the arbitrary constant.

Let us write out on the basis of equalities (3.8) – (3.13) general solution of equation (1.2) in the explicit form

$$h(x, \mu) = -\frac{2}{\sqrt{5}} B_1 e^{-\gamma_0 x} - B_0 x + A_0 + \mu \left[ -\frac{1}{\gamma_0} B_1 e^{-\gamma_0 x} + A_1 \right] +$$

$$+ (\mu^2 - 1) \left[ -\frac{8}{5\sqrt{\pi}} B_1 e^{-\gamma_0 x} - B_2 x + A_2 \right] + \text{sign} \mu \left\{ -\frac{2}{\sqrt{5}} B_1 e^{-\gamma_0 x} + B_0 +$$

$$+ \mu B_1 e^{-\gamma_0 x} + (\mu^2 - 1) \left[ -\frac{2}{\sqrt{5}} B_1 e^{-\gamma_0 x} + B_2 \right] \right\}. \quad (3.14)$$
Let us allocate in this decision (3.14) exponential decreasing and polynomial solutions

\[ h(x, \mu) = B_1 e^{-\gamma_0 x} \left( \mu - \frac{\sqrt{\pi}}{2\gamma_0} \mu^2 \right) \left( \frac{1}{\gamma_0} + \text{sign } \mu \right) + \]

\[ + A_0 + A_1 \mu + A_2 (\mu^2 - 1) + (\text{sign } \mu - x) [B_0 + B_2 (\mu^2 - 1)]. \quad (3.15) \]

4. The solution of the generalized Smolukhovsky’ problem

In this item we will prove the theorem about the analytical solution of the basic boundary problem (1.2), (2.1) and (2.11).

\textbf{Theorem.} The boundary problem (1.2), (2.1) and (2.11) has the unique solution, representable in the form of the sum exponential decreasing and polynomial solutions

\[ h(x, \mu) = -(2U - g_T) \frac{e^{-\gamma_0 x}}{\sqrt{\pi}} \frac{1 + \gamma_0 \text{sign } \mu}{1 + \gamma_0} \left( \mu - \frac{2\pi}{\sqrt{\pi} \gamma_0^2} \right) + \]

\[ + \varepsilon_n + \varepsilon_T + (2U - g_T) \frac{\mu}{\sqrt{\pi}} + \left( \mu^2 - \frac{3}{2} \right) [\varepsilon_T + g_T (x - \text{sign } \mu)], \quad (4.1) \]

and quantities of temperature jump \( \varepsilon_T \) and concentration jump \( \varepsilon_n \) are given by equalities

\[ \varepsilon_T = \left( 1 + \frac{1}{2\gamma_0} \right) g_T - \frac{1}{2\gamma_0} (2U), \quad (4.2) \]

and

\[ \varepsilon_n = -\left( 1 - \frac{1}{4\gamma_0} \right) g_T - \frac{1}{4\gamma_0} (2U). \quad (4.3) \]

The solution (4.1) contains solutions of two problems: problem about temperature jump (when \( U = 0 \)) (see fig. 1)

\[ \frac{h^T(x, \mu)}{g_T} = \frac{e^{-\gamma_0 x}}{\sqrt{\pi}} \frac{1 + \gamma_0 \text{sign } \mu}{1 + \gamma_0} \left( \mu - \frac{2\mu^2}{\sqrt{5}} \right) - \]

\[- \left( 1 - \frac{1}{4\gamma_0} \right) - \frac{\mu}{\sqrt{\pi}} + \left( 1 + \frac{1}{\gamma_0} \right) \left( \mu^2 - \frac{1}{2} \right) + (x - \text{sign } \mu) \left( \mu^2 - \frac{3}{2} \right), \]
and problem about weak evaporation (when \( g_T = 0 \)) (see fig. 2)

\[
\frac{hU(x, \mu)}{2U} = -\frac{e^{-\gamma_0 x}}{\sqrt{\pi}} \frac{1 + \gamma_0 \text{sign} \mu}{1 + \gamma_0} \left( \mu - \frac{2\mu^2}{\sqrt{5}} \right) + \frac{1}{4\gamma_0} + \frac{\mu}{\sqrt{\pi}} - \frac{\mu^2}{2\gamma_0}.
\]

**Proof.** Let us take advantage of boundary condition "far from a wall" (2.12). We will substitute in condition (2.12) the decomposition (3.15). We receive following equation

\[
A_0 + A_1\mu + A_2(\mu^2 - 1) - (x - \text{sign} \mu)[B_0 + B_2(\mu^2 - 1)] =
\]

\[
= \varepsilon_n + (2U - g_T)\frac{\mu}{\sqrt{\pi}} + \varepsilon_T\left( \mu^2 - \frac{1}{2} \right) + g_T\left( \mu^2 - \frac{3}{2} \right)(x - \text{sign} \mu).
\]

From here at once we find

\[
A_1 = \frac{2U}{\sqrt{\pi}} - \frac{g_T}{\sqrt{\pi}},
\]

\[
B_2 = -g_T,
\]

\[
B_0 = \frac{g_T}{2},
\]

\[
A_2 = \varepsilon_T,
\]

\[
A_0 = \varepsilon_n + \frac{\varepsilon_T}{2}.
\]

Let us take advantage of the boundary condition reflection of molecules from a wall. Let us substitute decomposition (3.15) in the boundary condition (2.1). We receive the algebraic equation

\[
B_1\left( \mu - \frac{2}{\sqrt{5}}\mu^2 \right)\left( \frac{1}{\gamma_0} + 1 \right) + \varepsilon_n +
\]

\[
+ \frac{2U - g_T}{\sqrt{\pi}}\mu + \varepsilon_T\left( \mu^2 - \frac{1}{2} \right) - g_T\left( \mu^2 - \frac{3}{2} \right) = 0.
\]

From here we receive system from three equations

\[
\varepsilon_n - \frac{\varepsilon_T}{2} + \frac{3}{2}g_T = 0,
\]

\[
B_1\left(1 + \frac{1}{\gamma_0} \right) + (2U - g_T)\frac{1}{\sqrt{\pi}} = 0,
\]
\[-B_1 \frac{2}{\sqrt{5}} \left(1 + \frac{1}{\gamma_0}\right) + \varepsilon_T - g_T = 0.\]

From these equations we find the constant $B_1$

$$B_1 = -\frac{2U - g_T}{\sqrt{\pi} (1 + 1/\gamma_0)}.$$ 

and also quantities of temperature and concentration jumps:

$$\varepsilon_T = \left(1 + \frac{1}{2\gamma_0}\right) g_T - \frac{1}{2\gamma_0} (2U), \quad (4.4)$$

$$\varepsilon_n = -\left(1 - \frac{1}{4\gamma_0}\right) g_T - \frac{1}{4\gamma_0} (2U). \quad (4.5)$$

Formulas (4.4) and (4.5) in accuracy coincide with formulas of temperatures and concentration jumps (4.2) and (4.3).

Thus, the solution of the boundary problem is constructed and has the following form

$$h(x, \mu) = -\frac{(2U - g_T)e^{-\gamma_0 x}}{\sqrt{\pi} \left(1 + \frac{1}{\gamma_0}\right)} \left(\mu - \frac{2\mu^2}{\sqrt{5}}\right) \left(\text{sign } \mu + \frac{1}{\gamma_0}\right) +$$

$$+ \varepsilon_n + (2U - g_T) \frac{\mu}{\sqrt{\pi}} + \varepsilon_T \left(\mu^2 - \frac{1}{2}\right) + \left(x - \text{sign } \mu\right) \left(\mu^2 - \frac{3}{2}\right) g_T. \quad (4.6)$$

Expansion (4.6) in accuracy coincides with expansion (4.1), if to consider expressions (4.2) and (4.3) for quantities of temperatures and concentration jumps. The theorem is proved.

Let us notice, that polynomial "tail" of solution (4.1) is asymptotic Chapman–Enskog expansion. It means, that solution (4.1) it is possible to present in the form

$$h(x, \mu) = -\frac{(2U - g_T)e^{-\gamma_0 x}}{\sqrt{\pi} \left(1 + \frac{1}{\gamma_0}\right)} h^*(\mu) + h_{as}(x, \mu),$$
where
\[ h^*(\mu) = (\mu - \frac{2\mu^2}{\sqrt{5}}) \left( \text{sign} \mu + \frac{1}{\gamma_0} \right). \]

Let us transform Chapman–Enskog decomposition
\[ h_{as}(x, \mu) = \]
\[ = \varepsilon_n + (2U - g_T) \frac{\mu}{\sqrt{\pi}} + \varepsilon_T \left( \mu^2 - \frac{1}{2} \right) + g_T \left( \mu^2 - \frac{3}{2} \right) (x - \text{sign} \mu) \]
by means of equalities for temperature and concentration jumps. As a result we receive, that
\[ h_{as}(x, \mu) = \]
\[ = g_T \left[ - \left( 1 - \frac{1}{4\gamma_0} \right) - \frac{\mu}{\sqrt{\pi}} + \left( 1 + \frac{1}{2\gamma_0} \right) \left( \mu^2 - \frac{1}{2} \right) + (x - \text{sign} \mu) \left( \mu^2 - \frac{3}{2} \right) \right] + \]
\[ + (2U) \left[ \frac{1}{4\gamma_0} + \frac{\mu}{\sqrt{\pi}} - \frac{\mu^2}{2\gamma_0} \right]. \quad (4.7) \]

Thus, definitively distribution function in the generalized Smolukhovsky’ problem is equal
\[ h(x, \mu) = -\frac{e^{-\gamma_0 x}}{\sqrt{\pi}} (2U - g_T) \left( \mu - \frac{2\mu^2}{\sqrt{5}} \right) \frac{1 + \gamma_0 \text{sign} \mu}{1 + \gamma_0} + h_{as}(x, \mu), \quad (4.8) \]
and Chapmen–Enskog decomposition \( h_{as}(x, \mu) \) is defined by equality (4.7).

Expansion (4.8) in accuracy coincides with expansion (4.1).

**REMARK 4.1.** Expression (4.8) contains distribution function of reflected molecules from wall:
\[ h^+(x, \mu) = -\frac{e^{-\gamma_0 x}}{\sqrt{\pi}} (2U - g_T) \left( \mu - \frac{2\mu^2}{\sqrt{5}} \right) + \]
\[ + \varepsilon_n + \varepsilon_T + \frac{2U - g_T}{\sqrt{\pi}} \mu + \left( \mu^2 - \frac{3}{2} \right) [\varepsilon_T + g_T(x - 1)]. \]
and also distribution function of molecules flying to the wall, which it is expressed by Chapmen–Enskog distribution
\[ h^-(x, \mu) = -\frac{e^{-\gamma_0 x}}{\sqrt{\pi}} (2U - g_T) \frac{1 - \gamma_0}{1 + \gamma_0} \left( \mu - \frac{2\mu^2}{\sqrt{5}} \right) + \]
\[ +\varepsilon_n + \varepsilon_T + \frac{2U - g_T}{\sqrt{\pi}} \mu + \left( \mu^2 - \frac{3}{2} \right) [\varepsilon_T + g_T(x + 1)]. \]

Fig. 1. Distribution function in problem about temperature jump. Curves 1, 2, 3 correspond to values \( x = 0, 1, 2 \).
Fig. 2. Distribution function in problem about weak evaporation. Curves 1, 2, 3 correspond to values \( x = 0, 1, 2 \).
Fig. 3. Distribution function in problem about weak evaporation. Curves 1, 2, 3, 4 correspond to values $x = 0, 0.05, 0.1, 0.2$. 
Remark 4.2. Expansion (4.1) contains solutions of two problems—solution of problem about temperature jump

\[
\frac{h^T(x, \mu)}{g_T} = e^{-\gamma_0 x} h^*(\mu) + h^T_{as}(x, \mu)
\]

and solution of the problem about weak evaporation

\[
\frac{h^U(x, \mu)}{g_T} = -e^{-\gamma_0 x} h^*(\mu) + h^U_{as}(x, \mu).
\]

In these equalities are entered designations

\[
h^*(\mu) = \left( \mu - \frac{2\mu^2}{\sqrt{5}} \right) \frac{1 + \gamma_0 \text{sign} \mu}{\sqrt{\pi}(1 + \gamma_0)},
\]

\[
h^T_{as}(x, \mu) = -\left(1 - \frac{1}{4\gamma_0}\right) - \frac{\mu}{\sqrt{\pi}} + \left(1 + \frac{1}{2\gamma_0}\right) \left(\mu^2 - \frac{1}{2}\right) + (x - \text{sign} \mu) \left(\mu^2 - \frac{3}{2}\right),
\]

\[
h^U_{as}(x, \mu) = \frac{1}{4\gamma_0} + \frac{\mu}{\sqrt{\pi}} - \frac{\mu^2}{2\gamma_0}.
\]

Remark 4.3. From resulted above equalities it is visible, that distribution function in the problem about temperature jump is discontinuous in a point \( \mu = 0 \), and distribution function in the problem about weak evaporation is continuous on all real axis, including in the point \( \mu = 0 \).

Really, in case of the problem about temperature jump for right-hand and left-hand limits in the point \( \mu = 0 \) we have

\[
h^T_+(x, +0) = g_T(x - 1) \left( -\frac{3}{2} \right),
\]

\[
h^T_-(x, -0) = g_T(x + 1) \left( -\frac{3}{2} \right).
\]

Hence, quantity of jump of distribution function of the reflected from a wall and molecules flying to a wall in the point \( \mu = 0 \) it is equal

\[
h^T_+(x, +0) - h^T_-(x, -0) = 3g_T.
\]

Let us notice, that the quantity of this jump does not depend from spatial variable \( x \), i.e. it is identical at all \( x > 0 \).
In case of the problem about weak evaporation it is obvious, that

\[ h^U_+(x, +0) - h^U_-(x, -0) = 0. \]

**Remark 4.4.** Let us notice, that in the problem about weak evaporation distribution function can be transformed to the following form

\[ \frac{h^U(x, \mu)}{2U} = \frac{1}{4\gamma_0} + \frac{1}{\sqrt{\pi}} \left( \mu - \frac{2\mu^2}{\sqrt{5}} \right) \left[ 1 - \frac{1 + \gamma_0 \text{sign} \mu}{1 + \gamma_0} e^{-\gamma_0 \mu} \right]. \]

From this formula, in particular, follows, that on border \( x = 0 \) distribution function of the reflected molecules is constant (see fig. 2)

\[ \frac{h^U_+(0, \mu)}{2U} = \frac{1}{4\gamma_0}, \quad \mu > 0. \]

**5. Temperature jump and weak evaporation (condensation).**

**Distribution of macroparameters of gas**

Numerical calculations of coefficients of temperature and concentration jump result in the following

\[ \varepsilon_T = 1.5046g_T - 0.5046(2U), \]

\[ \varepsilon_n = -0.7477g_T - 0.2523(2U). \]

For comparison we will bring coefficients of temperature and concentration jumps found by means of the one-dimensional kinetic equations with constant frequency of collisions \[4\]

\[ \varepsilon_T = 1.3068g_T - 0.4443(2U), \]

\[ \varepsilon_n = -3.3207g_T - 0.8958(2U). \]

Let us consider distribution of concentration, mass velocity and temperature depending on coordinate \( x \).
Let us begin with concentration distribution (numerical density). On
to definition it is received
\[
\frac{\delta n(x)}{n_0} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} h(x, \mu) d\mu =
\]
\[
= \varepsilon_n - g_T x - \frac{\gamma_0}{\sqrt{\pi}} - \frac{\sqrt{\pi}}{4\gamma_0} e^{-\gamma_0 x} =
\]
\[
= \varepsilon_n - g_T x - 0.0317(2U - g_T)e^{-\gamma_0 x} =
\]
\[
= -g_T N_T(x) - (2U) N_U(x),
\]
where
\[
N_T(x) = 0.7477 + x - 0.0317 e^{-\gamma_0 x},
\]
\[
N_U(x) = 0.2523 + 0.0317 e^{-\gamma_0 x}.
\]
Distribution of mass velocity at \(x > 0\) is trivial
\[
U(x) \equiv U.
\]
Really, having taken advantage of the solution (4.1), easy check up,
that
\[
U(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} \mu h(x, \mu) d\mu \equiv U.
\]
This fact is quite obvious, and follows from the equation continuity.
Let us consider temperature distribution. By definition we receive
\[
\frac{\delta T(x)}{T_0} = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} \left( \mu^2 - \frac{1}{2} \right) h(x, \mu) d\mu =
\]
\[
= \varepsilon_T + g_T x + \frac{\sqrt{\pi}}{8} - \frac{1}{\sqrt{5}} e^{-\gamma_0 x} (2U - g_T) =
\]
\[
= \varepsilon_T + g_T x - 0.0639 e^{-\gamma_0 x} (2U - g_T) =
\]
Fig. 4. Behaviour of kinetic coefficient $N_T(x)$ (curve 1). The curve 2 is the asymptotic $N_{Tas}(x) = 0.7477 + x$.

$$N_T(x) = T_T(x) g_T - T_U(x)(2U),$$

where

$$T_T(x) = 1.5046 + x + 0.0639e^{-\gamma_0 x},$$
$$T_U(x) = 0.5046 + 0.0639e^{-\gamma_0 x}.$$
Fig. 5. Behaviour of kinetic coefficient $N_T(x)$ (curve 1). The curve 2 is the asymptotic $N_{Uas}(x) = 0.2523$. 
Fig. 6. Behaviour of kinetic coefficient $N_T(x)$ (curve 1). The curve 2 is the asymptotic $T_{Tas}(x) = 1.5046 + x$. 
Fig. 7. Behaviour of kinetic coefficient $N_T(x)$ (curve 1). The curve 2 is the asymptotic $T_{Uas}(x) = 0.5046$. 
6. Conclusion

In the present work the analytical solution of boundary problems for the one-dimensional kinetic equation with collisional frequency of molecules proportional to the module molecular velocity is considered. This equation is the limiting case of affine dependence of collisional frequency of molecules on the module of their velocity.

The analytical solution of generalized Smolukhovsky problem (about temperature jump and weak evaporation (condensation)) is considered.

Formulas for calculation of temperature and concentration jumps are deduced. Distribution function of gas molecules in explicit form, and also distributions of concentration and temperature in half-space $x > 0$ are received.

It has appeared, that distribution function in problem about temperature jump is discontinuous in the point $\mu = 0$, and distribution function in problem about weak evaporation is continuous at all velocities of molecules. All necessary numerical calculations are done. It is spent graphic research of distribution function of reflected molecules and molecules flying to the wall, and also all kinetic coefficients are investigated.

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