A Way to the Dark Side of the Universe through Extra Dimensions

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Abstract

As indicated by Einstein’s general relativity, matter and geometry are two faces of a single nature. In our point of view, extra dimensions, as a member of the geometry face, will be treated as a part of the matter face when they are beyond our poor vision, thereby providing dark energy sources effectively. The geometrical structure and the evolution pattern of extra dimensions therefore may play an important role in cosmology. Various possible impacts of extra dimensions on cosmology are investigated. In one way, the evolution of homogeneous extra dimensions may contribute to dark energy, driving the accelerating expansion of the universe. In the other way, both the energy perturbations in the ordinary three-space, combined with homogeneous extra dimensions, and the inhomogeneities in the extra space may contribute to dark matter. In this paper we wish to sketch the basic idea and show how extra dimensions may lead to the dark side of our universe.

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1 Introduction

It is strongly suggested by observational data that our universe has the critical energy density and consists of 1/3 of dark matter and 2/3 of dark energy (see e.g., Ref. [1] and references therein), where “dark” indicates the invisibility. Even though it is generally not an elegant way to explain data via something we cannot see, the avalanche of data, including those from type Ia supernova measurements [2, 3], cosmic microwave anisotropies [4], galactic rotation curves, and surveys of galaxies and clusters (providing the power spectrum of energy density fluctuations), make it more and more convincing. Nevertheless, we accordingly need to ask a question: Why are dark matter and dark energy so dark?

This question reminds us another “dark” stuff, extra dimensions. The existence of extra dimensions is required in various theories beyond the standard model of particle physics, especially in the theories for unifying gravity and other forces, such as superstring theory. Extra dimensions should be “hidden” (or “dark”) for consistency with observations. This common feature, “invisible existence”, of dark energy, dark matter, and extra dimensions provides us a hint that there may be some deep relationship among them.

In this paper we show how extra dimensions may manifest themselves as a source of energy in the ordinary three-space and lead to the dark side of the universe. Basically homogeneous extra dimensions will contribute to dark energy and may also provide some sort of dark matter effectively if combined with the effects of inhomogeneities in the ordinary three-space, and inhomogeneities in the extra space will contribute to dark matter effectively. The basic idea is sketched in the next section, and then we discuss in Sec. 3 how homogeneous extra dimensions provide “effective” dark energy and influence the evolution of the ordinary three-space, especially, producing the accelerating expansion of the universe. The extra dimensions employed throughout this paper are small and compact, as introduced in the Kaluza-Klein theories [8].

2 A Sketch of the Idea

We consider a (3+n+1)-dimensional space-time where n is the number of extra spatial dimensions. The unperturbed metric tensor \( g_{\alpha\beta} \) (\( \alpha, \beta = 0, 1, \ldots, 3 + n \)), which describes a universe with homogeneous, isotropic ordinary three-space and extra space, is defined by

\[
\begin{align*}
\text{d}s^2 &= \text{d}t^2 - a^2(t) \left( \frac{\text{d}r_a^2}{1 - k_r r_a^2} + r_a^2 \text{d}\Omega_a^2 \right) - b^2(t) \left( \frac{\text{d}r_b^2}{1 - k_b r_b^2} + r_b^2 \text{d}\Omega_b^2 \right),
\end{align*}
\]

1 Various scenarios for hidden extra dimensions have been proposed, for example, a brane world with large compact extra dimensions in factorizable geometry proposed by Arkani-Hamed et al. [5, 6], a brane world with extra dimensions in warped nonfactorizable geometry proposed by Randall and Sundrum [7], and small compact extra dimensions in factorizable geometry as introduced in the Kaluza-Klein theories [8].
where $a(t)$ and $b(t)$ are scale factors, and $k_a$ and $k_b$ relate to curvatures of the ordinary 3-space and the extra space, respectively. The value of $r_b$ is set to be within the interval $[0, 1)$ for the compactness of extra dimensions. The perturbed metric describing a lumpy universe is defined by

$$ds^2 = [g_{\mu\nu} + \delta g_{\mu\nu}(x)] \, dx^\mu dx^\nu + [1 + \delta_b(x^\mu)]^2 g_{pq} dx^p dx^q,$$

where $g_{\mu\nu}$ and $g_{pq}$ are unperturbed metric tensors, while $\delta g_{\mu\nu}(x)$ and $\delta_b(x^\mu)$ corresponding to perturbations, of the ordinary $(3 + 1)$-dimensional space-time and the extra space, respectively. As a convention, $x^{\mu,\nu}$ and $x^{p,q}$ denote the coordinates of the ordinary space-time and the extra space, respectively, while $x$ denotes all the coordinates. For the sake of simplicity the cross terms $dx^\mu dx^p$ are abandoned by requiring the symmetry with respect to extra space inversion, i.e., $x^p \to -x^p$. We note that the extra space is kept to be homogeneous and isotropic after introducing perturbations, so that we only need $\delta_b(x^\mu)$, the perturbation of the scale factor $b(t)$ as a function of the coordinates of the ordinary space-time, to represent perturbations of the extra space. On the contrary we have no symmetry requirement for the perturbed ordinary space-time, and hence the metric perturbations $\delta g_{\mu\nu}$ in general is a function of all of the coordinates \{x$^\gamma$, $\gamma = 0, 1, \ldots, 3 + n$\}. Assuming that both the metric perturbations $\delta g_{\mu\nu}$ and $\delta_b$ are small, such that the Einstein equations from the perturbed metric can be expanded with respect to these perturbations, we obtain

$$G_{\alpha\beta} = 8\pi \bar{G} T_{\alpha\beta} = 8\pi \bar{G} \left[ T_{\alpha\beta}^{(0)}(t) + \delta T_{\alpha\beta}(x) \right]$$

$$= G_{\alpha\beta}^{(0)} [g_{\mu\nu}(t)] + G_{\alpha\beta}^{(1)} [g_{\mu\nu}(t), \delta g_{\mu\nu}(x)] + G_{\alpha\beta}^{(2)} [g_{\mu\nu}(t), b(t)]$$

$$+ G_{\alpha\beta}^{(3)} [g_{\mu\nu}(t), \delta g_{\mu\nu}(x), b(t)] + G_{\alpha\beta}^{(4)} [g_{\mu\nu}(t), \delta g_{\mu\nu}(x), b(t), \delta_b(x^\mu)],$$

where $\bar{G}$ is the gravitational constant in the higher-dimensional space-time, and $T_{\alpha\beta}$ denotes the energy-momentum tensor, $T_{\alpha\beta}^{(0)}(t)$ the unperturbed, and $\delta T_{\alpha\beta}(x)$ the perturbed one. The first two terms in the above expansion of the Einstein tensor, $G_{\alpha\beta}^{(0)}$ and $G_{\alpha\beta}^{(1)}$, are exactly the unperturbed and the perturbed Einstein tensor, respectively, of the ordinary $(3 + 1)$-dimensional space-time. In contrast, $G_{\alpha\beta}^{(2)}$, $G_{\alpha\beta}^{(3)}$, and $G_{\alpha\beta}^{(4)}$ are additional terms coming from extra dimensions. In our point of view, if observers are too blind to see extra dimensions, these three additional terms will be automatically moved to the right-hand side of the Einstein equations (3) and treated as some sort of energy source, thereby contributing an “effective” part to the energy-momentum tensor. In particular, $G_{\alpha\beta}^{(2)}$ is smoothly distributed in the space and hence contributes to dark energy, while $G_{\alpha\beta}^{(3)}$ and $G_{\alpha\beta}^{(4)}$ have the spatial dependence and contribute to dark matter. In the above discussion we have sketched the main idea. As a demonstration of this idea, we will show in the next section how homogeneous extra dimensions can
lead to “effective” dark energy and consequently change the evolution pattern of a (nonrelativistic-) matter-dominated universe.

3 Dark Energy from Homogeneous Extra Dimensions

We consider in this section the case of a (3+n+1)-dimensional space-time described by the unperturbed metric defined in Eq. (1), i.e., both the ordinary three-space and the extra space are homogeneous and isotropic. Assuming that the matter content in this higher-dimensional space is a perfect fluid with the energy-momentum tensor

\[ T_{\alpha\beta} = \text{diag}(\bar{\rho}, -\bar{p}_a, \ldots, -\bar{p}_b, \ldots), \]

we can write the Einstein equations as

\[ 3 \left[ \left( \frac{\dot{a}}{a} \right)^2 + k_3 \right] + \frac{n(n-1)}{2} \left[ \left( \frac{\dot{b}}{b} \right)^2 + k_3 \right] + 3\frac{\ddot{a} \dot{a}}{a b} = 8\pi\bar{G}\bar{\rho}, \]

\[ 2\frac{\ddot{a}}{a} + n\frac{\ddot{b}}{b} + \left[ \left( \frac{\dot{a}}{a} \right)^2 + k_3 \right] + \frac{n(n-1)}{2} \left[ \left( \frac{\dot{b}}{b} \right)^2 + k_3 \right]
+ 2n \left( \frac{\dot{a}}{a} \right) \left( \frac{\dot{b}}{b} \right) = -8\pi\bar{G}\bar{p}_a, \]

\[ 3\frac{\ddot{a}}{a} + (n-1)\frac{\ddot{b}}{b} + 3 \left[ \left( \frac{\dot{a}}{a} \right)^2 + k_3 \right] + \frac{(n-1)(n-2)}{2} \left[ \left( \frac{\dot{b}}{b} \right)^2 + k_3 \right]
+ 3(n-1) \left( \frac{\dot{a}}{a} \right) \left( \frac{\dot{b}}{b} \right) = -8\pi\bar{G}\bar{p}_b, \]

where \( \bar{\rho} \) is and the energy density in the higher-dimensional world, and \( \bar{p}_a \) and \( \bar{p}_b \) are the pressures in the ordinary three-space and the extra space, respectively.

In the previous work by Gu and Hwang [9], the case with \( k_a = k_b = 0 \) was considered, in which the accelerating expansion of the present (nonrelativistic-) matter-dominated universe was proposed to be generated along with the evolution of extra dimensions. Here we also focus on a matter-dominated universe, setting \( \bar{p}_a \) and \( \bar{p}_b \) to

\(^2\)The part of “effective” dark matter originated from extra dimensions is currently under investigation, and will not be discussed in detail in the rest of this paper.
zero accordingly, but consider a more general case in which only \( k_b = 0 \) is assumed while \( k_a \) is treated as a free parameter. In this case Eqs. (7) and (8) can be rearranged to become

\[
(n + 2) \ddot{a} + (2n + 1) \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{k_a}{a^2} \right] + n(n - 1) \frac{\dot{a} \dot{b}}{a b} - \frac{n(n - 1)}{2} \left( \frac{\dot{b}}{b} \right)^2 = 0, \tag{9}
\]

\[
(n + 2) \ddot{b} - 3 \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{k_a}{a^2} \right] + 6 \frac{\dot{a} \dot{b}}{a b} + \frac{(n - 1)(n + 4)}{2} \left( \frac{\dot{b}}{b} \right)^2 = 0, \tag{10}
\]

and then we can rewrite the Einstein equations, using new variables \( u(t) \equiv \dot{a}/a \) and \( v(t) \equiv \dot{b}/b \), as

\[
3 \left( u^2 + \frac{k_a}{a^2} \right) + 3nuv + \frac{n(n - 1)}{2}v^2 = 8\pi \tilde{G}\tilde{\rho}, \tag{11}
\]

\[
(n + 2) \dot{u} + 3(n + 1)u^2 + (2n + 1) \frac{k_a}{a^2} + n(n - 1)uv - \frac{n(n - 1)}{2}v^2 = 0, \tag{12}
\]

\[
(n + 2) \dot{v} - 3 \left( u^2 + \frac{k_a}{a^2} \right) + 6uv + \frac{n(n + 5)}{2}v^2 = 0. \tag{13}
\]

Before getting numerical solutions, we use simple analytical operations to extract, from the above Einstein equations, essential features of these equations and the evolution patterns governed by them. We first obtain, from Eq. (9), conditions for the accelerating and the decelerating expansion:

\[
\begin{align*}
\ddot{a} &< 0, \quad v/u < J_- \\
\ddot{a} &> 0, \quad J_- < v/u < J_+,
\end{align*}
\tag{14}
\]

where

\[
J_{\pm} \equiv 1 \pm \sqrt{\frac{(n + 1)(n + 2) + 2(2n + 1)k_a/(a^2u^2)}{n(n - 1)}}. \tag{15}
\]

We then read off from Eq. (11) that the condition for positive energy density \( \tilde{\rho} \) is

\[
v/u > K_+ \quad \text{or} \quad v/u < K_-, \tag{16}
\]

where

\[
K_{\pm} \equiv -\frac{3}{n - 1} \pm \sqrt{\frac{3}{n(n - 1)} \left( \frac{n + 2}{n - 1} - \frac{2k_a}{a^2u^2} \right)}. \tag{17}
\]

Observing Eqs. (14)–(17), we notice that variables \( v/u \) and \( k_a/(a^2u^2) \) play essential roles in the above expressions of these conditions. These two essential variables can also be recognized from Eqs. (11)–(13), which tell us that values of all the quantities
in them are determined, up to an overall factor related to the initial value of \( u \), once the values of \( v/u \) and \( k_a/(a^2 u^2) \) are given. It is therefore a good way to analyze the evolution of the universe governed by Eqs. (11)–(13) via a two-dimensional diagram described by \( v/u \) and \( k_a/(a^2 u^2) \).

Figure 1: Conditions for various signs of energy density \( \rho \) and acceleration \( \ddot{a} \) are illustrated, where the number of extra dimensions \( n \) is specified to be three.

Conditions in Eqs. (14)–(17) are summarized in Fig. 1, where the number of extra dimensions \( n \) is specified to be three as an example. The grey area denoted by “\( \rho < 0 \)” is a forbidden region if positive energy density is required. In addition, flow vectors in \((k_a/(a^2 u^2), v/u)\)-diagram, as determined by Eqs. (12) and (13), are plotted in Fig. 2 (where \( n = 3 \)). There are two “attractors” denoted by grey dots in the flow diagram: one at \((k_a/(a^2 u^2), v/u) = (-1, 0)\), and the other at \((k_a/(a^2 u^2), v/u) = (0, -\left[3 + \sqrt{3(n+2)/n}\right]/(n-1))\). The attractor at \((-1, 0)\) is on the margin of the forbidden region (i.e., indicating \( \rho = 0 \)) and corresponding to a state of the higher-dimensional universe entailing stable extra dimensions and vanishing \( \ddot{a} \). We note that the existence of solutions corresponding to stable extra dimensions is a good feature for building models in a higher-dimensional space-time. The other attractor is also on the margin, with zero energy density, of the forbidden region, entailing collapsing extra dimensions and positive acceleration.

For a concrete illustration, we now solve Eqs. (11)–(13) numerically for the case of \( n = 3 \). We plot in Fig. 3 four trajectories corresponding to four numerical solutions.

\(^3\text{Attractors are stable fixed points toward which the nearby points (or "state") tend to flow.}\)
Figure 2: Flow vectors in \( (k_a / (a^2 u^2), v/u) \)-diagram are plotted. Two grey dots denote two “attractors” at (-1,0) and \( (0, - \left[ 3 + \sqrt{3(n+2)/n} \right] /(n-1)) \) (where \( n = 3 \)), respectively.

with respect to initial conditions, \( (k_a / (a^2 u^2), v/u) = (a) (-0.0001, 4), (b) (-0.001, 0), (c) (0.0001, 4), and (d) (1.3, -1.4) \). These four trajectories represent four different kinds of evolution path:

a. \textbf{acceleration \rightarrow deceleration \rightarrow acceleration}, eventually approaching the attractor at \((-1, 0)\) with stable extra dimensions and zero acceleration, possessing negative spatial curvature.

b. \textbf{deceleration \rightarrow acceleration}, eventually merging to the trajectory (a) and approaching the attractor at \((-1, 0)\) with stable extra dimensions and zero acceleration, possessing negative spatial curvature.

c. \textbf{eternal deceleration}, possessing increasing positive curvature contribution.

d. \textbf{deceleration \rightarrow acceleration}, eventually approaching the attractor at \( (0, - \left[ 3 + \sqrt{3(n+2)/n} \right] /(n-1)) \) with collapsing extra dimensions, possessing decreasing positive curvature contribution.

It is therefore indicated that there are many possibilities of evolution patterns in this higher-dimensional universe, in contrast to the unique manner of evolution, eternally decelerating expansion, for a matter-dominated universe in the standard cosmology without extra dimensions.
Figure 3: Four trajectories corresponding to four numerical solutions with respect to initial conditions, \( \left( \frac{k_a}{a^2u^2}, \frac{v}{u} \right) = (a) \left( -0.0001, 4 \right), (b) \left( -0.001, 0 \right), (c) \left( 0.0001, 4 \right), \) and (d) \( (1.3, -1.4) \), are plotted, where the black dot at one end of each trajectory denotes the initial position. (As in Fig. 2, two grey dots are “attractors” and \( n = 3 \).)

4 Discussion and summary

In this paper we make a point that there may be a deep relationship between “hidden” (or “dark”) extra dimensions and the dark side of the universe, i.e., dark matter and dark energy. This conjecture is based on Einstein’s general relativity, which indicates an important aspect that matter (with energy and momentum) and geometrical structures of a space-time are two faces of a single nature, to be called matter face and geometry face, respectively. In our point of view, if there exists a part of the geometry face which is beyond our poor vision, this missing part will be treated as a member of the matter face, and consequently provide mysterious, dark, “effective” energy sources. A possible missing part of the geometry face we consider in this paper is the existence of extra dimensions. This idea is sketched in Sec. 2 via analyzing the Einstein equations, including perturbations of both the metric tensor and the energy-momentum tensor, for a higher-dimensional world. We conclude that extra dimensions may manifest themselves as a source of energy in the ordinary three-space, such as “effective” dark energy, under the consideration of homogeneous extra dimensions, and “effective” dark matter, as contributed by inhomogeneities in the extra space or the ordinary three-space.

As a particular demonstration of the general idea, we consider in Sec. 3 a (non-
relativistic-) matter-dominated universe with homogeneous extra dimensions and show that the evolution of homogeneous extra dimensions can lead to “effective” dark energy and consequently change the evolution pattern of the universe. There are many possibilities of evolution patterns in this higher-dimensional universe, in contrast to the unique way of evolution, eternally decelerating expansion, for a matter-dominated universe in the standard cosmology without extra dimensions. It needs further detailed studies to determine which evolution pattern can appropriately describe our universe. In addition, there are various possible realizations of this idea worthy of further quests, and some are currently under our investigation.

As mentioned in Sec. 1, this work is motivated by a fundamental question: Why are dark matter and dark energy so dark? Through the preliminary studies of the general idea discussed in this paper, here comes up a possible answer: Dark matter and dark energy are generated from the extra dimensions, a nature of geometry we are too blind to see. This simple answer indicates an intriguing possibility of unifying these two kinds of dark entities, extra dimensions and dark energy sources, into one.
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