A dipolar Gross-Pitaevskii equation with quantum fluctuations: Self-bound states

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Abstract

We prove existence and qualitative properties of standing wave solutions to a generalized nonlocal 3rd-4th order Gross-Pitaevskii equation (GPE), the latter being currently the state-of-the-art model for describing the dynamics of dipolar Bose-Einstein condensates. Using a mountain pass argument on spheres in $L^2$ and constructing appropriately localized Palais-Smale sequences we are able to prove existence of real positive ground states as saddle points of the energy. The analysis is deployed in the set of possible states, thus overcoming the problem that the energy is unbounded below. We also prove a corresponding nonlocal Pohozaev identity with no rest term, a crucial part of the analysis.

1 Introduction

The static and dynamic properties of a Bose-Einstein condensate (BEC) can be studied through an effective mean field equation known as the Gross-Pitaevskii equation (GPE):

$$i\hbar \partial_t \Psi = -\frac{\hbar^2}{2m} \Delta \Psi + g |\Psi|^2 \Psi + V_{\text{ext}} \Psi,$$

a variant of the famous nonlinear Schrödinger equation. Here $\Psi$ is the BEC wavefunction and $V_{\text{ext}}$ an external potential needed to keep the BEC in place (the trapping potential). This is the classical model for BECs; for more details on mean field theory see for example [20] and references therein.

BECs made of dipolar (i.e. highly magnetic) atoms (e.g. chromium, dysprosium, erbium etc) were first created in the mid 2000’s by the group of T. Pfau in Stuttgart. For such gases, a dipole-dipole interaction between the atoms becomes important. This action is long ranged and anisotropic and gives rise to a rich array of new phenomena ([21]). However, recent observations have been made ([18, 28]) during experiments with dysprosium, not accounted for by the standard mean field theory corresponding to (1). These experiments produced a stable droplet crystal, similar to ones observed in classical ferrofluids. In contrast to the

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observation, mean field theory predicted the collapse of these droplets to extremely high densities. Thus it was suggested to modify (1) by adding a nonlocal (convolution integral) term and a nonlinear higher order term, respectively modeling the long range dipole-dipole interactions and the beyond mean field quantum fluctuations; see also [9] and references therein.

This type of pattern formation is a very interesting phenomenon: similar to the so-called Rosensweig instability of ferrofluids (see e.g. [26, 17]), it appears in a system as a stable state. On the other hand, it has been mostly pattern formation at systems driven far from equilibrium (e.g. Rayleigh-Bénard convection, Taylor-Couette flow or current instabilities) that has been the usual case of study ([27]).

Moreover, after experimental observations ([28]) there has been numerical evidence ([4]; concerning the extended GPE theory described above) that the aforementioned patterns remain stable even after the trapping potential is turned off. This is another surprising feature that is not present in the classical GPE theory and is the case of study in this paper.

We shortly describe our results: We rigorously prove existence of “self-bound” (i.e., with an absent trapping potential) standing wave solutions to the extended GPE (equation (2) from next section) containing the long range dipole-dipole interactions and the beyond mean field quantum fluctuations. Our analysis extends the methods of [7, 6]. There, the authors considered a modification of (1) without taking any quantum fluctuation effects into consideration (see also next section). Adding the latter has made the analysis of the problem considerably more complex.

2 Mathematical description of the problem and main results

The extended GPE that has been mentioned above, has the following dimensionless form:

\[
i \partial_t \psi = -\frac{1}{2} \Delta \psi + V_{\text{ext}} \psi + \lambda_1 |\psi|^2 \psi + \lambda_2 (K * |\psi|^2) \psi + \lambda_3 |\psi|^3 \psi, \quad x \in \mathbb{R}^3, \ t > 0,
\]

(2)

where \( V_{\text{ext}} \) is the trapping potential and \( K \) is a convolution kernel that models the dipole-dipole interactions. In particular we consider

\[
K(x) = \frac{1 - 3 \cos^2 \theta(x)}{|x|^5},
\]

where \( \theta(x) \) is the angle between \( x \in \mathbb{R}^3 \) and a given (fixed) dipole axis \( n \in \mathbb{R}^3 \) with \(|n| = 1\), i.e.,

\[
\cos \theta(x) = \frac{x \cdot n}{|x|}.
\]

We assume that the applied magnetic field is parallel to the \( x_3 \)-axis, i.e., \( n := (0, 0, 1) \), so that

\[
K(x) = \frac{x_1^2 + x_2^2 - 2x_3^2}{|x|^5}.
\]

If we use the Fourier transform

\[
\mathcal{F}(f)(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^3} f(x) e^{-ix \cdot \xi} \, dx
\]
on $K$, we get
\begin{equation}
\hat{K}(\xi) = \frac{4\pi}{3} \frac{2\xi_3^2 - \xi_1^2 - \xi_2^2}{|\xi|^2} \in \left[ -\frac{4}{3}\pi, \frac{8}{3}\pi \right];
\end{equation}
see [11] Lemma 2.3]. A typical trap is set with a harmonic potential:
\[ V_{\text{ext}}(x) := \frac{1}{2} \left( x_1^2 + \frac{\omega_2^2}{\omega_1^2} x_2^2 + \frac{\omega_3^2}{\omega_1^2} x_3^2 \right), \]
where $\omega_1, \omega_2, \omega_3$ are the frequencies of the trap, in the $x_1, x_2, x_3$-directions respectively. As already stated in the introduction, we consider only “self-bound” states, i.e.,
\[ V_{\text{ext}} := 0 \]
throughout the paper.
Moreover, $\lambda_1, \lambda_2, \lambda_3$ are constants satisfying
\[ \lambda_1, \lambda_3 \in \mathbb{R}, \lambda_2 \in (0, \infty), \]
and the $\lambda_3$ term models the quantum fluctuations (see e.g. [25, 13]).
Equation (2) possesses a dynamically conserved energy functional (setting $V_{\text{ext}} = 0$)
\begin{equation}
E(u) := \int_{\mathbb{R}^3} \left\{ \frac{1}{2} |\nabla u|^2 + \frac{\lambda_1}{2} |u|^4 + \frac{\lambda_2}{2} (K * |u|^2) |u|^2 + \frac{2}{5} \lambda_3 |u|^5 \right\} \, dx,
\end{equation}
which with the help of Parseval’s identity becomes
\begin{equation}
E(u) = \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (\lambda_1 + \lambda_2 \hat{K}(\xi)) \left| |u|^{12}(\xi) \right|^2 d\xi + \frac{2}{5} \lambda_3 \|u\|^5.
\end{equation}
For an arbitrary $c > 0$, we look for ground states of (1), that is, for functions $u \in H^1(\mathbb{R}; \mathbb{C})$ such that $\|u\|_2^2 = c$, that are critical points of $E$ and study their qualitative properties. Note that a ground or excited state of $E$ corresponds to standing waves for (2) through the Ansatz $\psi(x, t) = e^{-i\beta t} u(x)$; $\beta$ denotes the so-called chemical potential. After making the standing wave Ansatz in (2), the problem reduces into finding a function $u : \mathbb{R}^3 \to \mathbb{C}$ satisfying the side constraint $\|u\|_2^2 = c$ and a number $\beta \in \mathbb{R}$ such that $(u, \beta)$ satisfies the equation
\begin{equation}
-\frac{1}{2} \Delta u + \lambda_1 |u|^2 u + \lambda_2 (K * |u|^2) u + \lambda_3 |u|^3 u + \beta u = 0.
\end{equation}
Here, the number $c > 0$ denotes the number of particles in the condensate.
Existence of minimizers or saddle points depends on the regime the parameters live in. One refers to the so-called stable, respectively unstable regime. Comparing to the classical focusing/defocusing NLS, one generally expects finite time blow-up or global in time existence of solutions; this work focuses in the unstable regime (a precise definition is given below).

**Definition 2.1.** We will make extensive use of the following quantities:
\begin{align*}
A(u) &:= \|\nabla u\|_2^2, \\
B(u) &:= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (\lambda_1 + \lambda_2 \hat{K}(\xi)) \left| |u|^{12}(\xi) \right|^2 d\xi, \\
C(u) &:= \lambda_3 \|u\|^5, \\
Q(u) &:= A(u) + \frac{3}{2} B(u) + \frac{9}{5} C(u).
\end{align*}
Note that with the above definitions the following identity holds:

\[ E(u) = \frac{1}{2} A(u) + \frac{1}{2} B(u) + \frac{2}{5} C(u). \]

**Definition 2.2.** The unstable regime is defined to be the subset of

\[ \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R} \times (0, +\infty) \times \mathbb{R}\}, \]

such that \((\lambda_1, \lambda_2, \lambda_3)\) satisfies at least one of the following conditions:

1. \( \lambda_3 = 0, \lambda_2 > 0 \) and \( \lambda_1 < \frac{4}{3} \pi \lambda_2 \),
2. \( \lambda_3 < 0, \lambda_2 > 0 \) and \( \lambda_1 \leq \frac{4}{3} \pi \lambda_2 \).

**Remark 2.3.** From the above conditions one obtains that \( \max \{B(u), C(u)\} \leq 0 \) for all \( u \in H^1(\mathbb{R}^3; \mathbb{C}) \) and \((B(u), C(u)) \neq (0, 0)\) for \( u \neq 0 \).

**Remark 2.4.** One can formulate more conditions including cases for \( \lambda_2 \leq 0 \), such that the assertion of the above Remark 2.3 as well as the results of this paper remain true. However, \( \lambda_2 \leq 0 \) is not physically accepted (see e.g. [6, 3, 11] and references therein).

We point out that the Laplacian \(-\Delta : H^{s+2}(\mathbb{R}^3) \to H^s(\mathbb{R}^3)\) is well-defined for all \( s \in \mathbb{R} \) (see for instance [11, Theorem 3.41, p.71]). On the other hand, the embedding \( H^1(\mathbb{R}^3) \subset L^p(\mathbb{R}^3) \) for \( p \in [2, 6] \) and the continuity of the convolution operator with kernel \( K \) in \( L^p(\mathbb{R}^3) \) ([11, Lemma 2.1]) implies that \((\lambda_1|u|^2 + \lambda_2(K \ast |u|^2) + \lambda_3|u|^3 + \beta)u\) belongs to \( H^{-1}(\mathbb{R}^3; \mathbb{C}) \).

**Definition 2.5.** We call \((u, \beta) \in H^1(\mathbb{R}^3; \mathbb{C}) \times \mathbb{R}\) a solution of equation (5), if the latter is satisfied in \( H^{-1}(\mathbb{R}^3; \mathbb{C}) \).

### 2.1 Existence of self-bound states

Solutions to (5) will be constructed as critical points of the energy \( E \) in the constraint set

\[ S(c) := \left\{ u \in H^1(\mathbb{R}; \mathbb{C}) : \|u\|^2_2 = c \right\}. \]

(For a more detailed exposition on the geometry of \( S(c) \) as a Finsler manifold we refer to [8] and references therein.) To that end, in the spirit of [6, 7], we give the following definitions:

**Definition 2.6.** For an arbitrary \( c > 0 \), we call \( u_c \in S(c) \) a ground state, if it is a least-energy critical point on \( S(c) \), i.e.,

\[ E(u_c) = \inf \left\{ E(u) : u \in S(c) \text{ and } E|_{S(c)}'(u) = 0 \right\}. \]

where \( E|_{S(c)}'(u) \in T_u^* S(c) \), i.e., \( E|_{S(c)}'(S(c)) \to T^* S(c) \).

**Remark 2.7.** Note that the Lagrange multiplier theorem (see e.g. [2, Corollary 3.5.29]) implies that for any ground state \( u \) exists \( \beta \in \mathbb{R} \) such that \((u, \beta)\) is a solution.

**Definition 2.8** (Mountain pass geometry). Given \( c > 0 \), we say that \( E \) has a mountain pass geometry on \( S(c) \) at level \( \gamma(c) \in \mathbb{R} \), if there exists \( K_c > 0 \), such that

\[ \gamma(c) = \inf_{g \in \Gamma_c} \max_{t \in [0,1]} E(g(t)) > \sup_{g \in \Gamma_c} \max \{ E(g(0)), E(g(1)) \}, \]
where
\[ \Gamma_c := \{ g \in C([0,1], S(c)) : g(0) \in A_{K_c} \text{ and } E(g(1)) < 0 \} \]
and
\[ A_{K_c} := \{ u \in S(c) : \| \nabla u \|_2^2 \leq K_c \}. \]

The main results concerning self-bound standing waves are summarized in the following theorems. Note that a “self-bound” crystal of droplets has been observed in [18] where it was suggested that under specific circumstances it is a good candidate for a ground state. One year later, this suggestion was verified in [30], so that, in contrast to [5], we do not expect that planar radial symmetry to be prominent in ground states. Still, they are positive, which conforms to the custom in physics to pick ground states as the nodeless solutions.

We have separated our results in three parts; the first describes properties of solutions in general:

**Theorem 2.9.** Let $c > 0$. The energy $E$ has a mountain pass geometry at level $\gamma(c) \geq 0$ and possesses no local minimizers on $S(c)$ (with respect to the relative topology). Moreover, let $(u, \beta) \in S(c) \times \mathbb{R}$ be a solution of (5). Then:

1. The phase field $u$ belongs to $W^{3,p}(\mathbb{R}^3; \mathbb{C})$ for all $p \in (1, \infty)$. In particular, $u \in C^2(\mathbb{R}^3; \mathbb{C})$.
2. The chemical potential $\beta$ is positive.
3. There exist constants $L, M > 0$ such that
\[ e^{L|x|} \left( |u(x)| + |\nabla u(x)| \right) \leq M \text{ for all } x \in \mathbb{R}^3. \]
4. The phase field $u$ is a ground state if and only if $E(u) = \gamma(c)$.

The next theorem gives some further properties on the structure of solutions, if we further assume that they are ground states.

**Theorem 2.10.** Let $c > 0$ and $u \in S(c)$ be a ground state (in the sense of Definition 2.6) with chemical potential $\beta > 0$. Then:

1. For all $x \in \mathbb{R}^3$ holds that $|u(x)| > 0$.
2. There exists a constant $\theta \in \mathbb{R}$ such that $u(x) = e^{i\theta}|u(x)|$ for all $x \in \mathbb{R}^3$.
3. $(|u|, \beta)$ is a solution of (5) and $|u|$ is a ground state.

The last result deals with the construction of a ground state at the mountain pass level.

**Theorem 2.11.** Equation (5) possesses a nontrivial solution $(u_c, \beta_c) \in S(c) \times (0, \infty)$ such that $u_c$ is a ground state with $E(u_c) = \gamma(c)$ and $u_c(x) > 0$ for all $x \in \mathbb{R}^3$.

**Remark 2.12.** Theorem 2.10 implies that no vortex structures may exist in ground states.
3 Proofs of the main results

3.1 Proof of Theorem 2.9

3.1.1 Mountain pass geometry on \( S(c) \)–nonexistence of local minimizers

In order to study the geometry of the energy landscape we will use the so-called Cazenave scaling (see for example [12])

\[
 u'(x) := t^{3/2} u(tx) \quad \text{for} \quad t > 0,
\]

under which \( S(c) \) is invariant. One calculates

\[
egin{align*}
 A(u') &= t^2 A(u), \\
 B(u') &= t^3 B(u), \\
 C(u') &= t^{9/2} C(u),
\end{align*}
\]

and therefore

\[
egin{align*}
 E(u') &= \frac{t^2}{2} A(u) + \frac{t^3}{2} B(u) + \frac{2}{5} t^{9/2} C(u), \\
 Q(u') &= t^2 A(u) + \frac{3t^3}{2} B(u) + \frac{9}{5} t^{9/2} C(u).
\end{align*}
\]

Lemma 3.1. Let \( u \in S(c) \). Then:

1. \( A(u'), B(u'), C(u'), E(u'), Q(u') \to 0 \) as \( t \to 0 \);
   \( A(u') \to \infty \) and \( E(u') \to -\infty \) as \( t \to \infty \).
2. If \( E(u) < 0 \) then \( Q(u) < 0 \).
3. There exists \( k_0 > 0 \) not depending on \( u \) such that, if \( A(u) \leq k_0 \) then \( Q(u) > 0 \).

Proof. 1. Due to Remark 2.3 the precise expression of the terms given by (7) to (9) and the fact that the \( B(u) \) and \( C(u) \) terms in (8) and (9) are leading for large \( t \), we obtain the assertion.

2. From the assumptions on \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) (Remark 2.3), we have that \( B(u), C(u) \leq 0 \) and that at least one of them is negative. Moreover, it holds that

\[
\frac{1}{2} Q(u) - E(u) = \frac{1}{4} B(u) + \frac{5}{10} C(u) < 0 \implies Q(u) < 2E(u).
\]

From this we obtain the second statement.

3. We will use the following Gagliardo-Nirenberg inequalities:

\[
\begin{align*}
\| u \|_5 &\leq C \| \nabla u \|_2^{9/10} \| u \|_2^{1/10} = C c^{1/20} A(u)^{9/20}, \\
\| u \|_4 &\leq C \| \nabla u \|_2^{3/4} \| u \|_2^{1/4} = C c^{1/8} A(u)^{3/8},
\end{align*}
\]
where \( C > 0 \) is a given positive constant independent of \( u \). Therefore, since \( \lambda_3 \leq 0 \) is assumed and (3) holds, we can estimate from below as follows:

\[
Q(u) = A(u) + \frac{3}{2}B(u) + \frac{9}{5}C(u)
\geq A(u) + \frac{3}{2} \frac{1}{(2\pi)^3} (\lambda_1 - \frac{4\pi}{3}\lambda_2) \|u\|^4 + C \lambda_3 c^{1/4} A(u)^{9/4}
\geq A(u) + \frac{3}{2} \frac{1}{(2\pi)^3} (\lambda_1 - \frac{4\pi}{3}\lambda_2) C c^{1/2} A(u)^{3/2} + C \lambda_3 c^{1/4} A(u)^{9/4}
= A(u) - C_1 A(u)^{3/2} - C_2 A(u)^{9/4},
\]

with positive constants \( C_1, C_2 \), since \( \|u\|^2 = c \) is constant. From the last inequality we see that \( Q(u) > 0 \) for sufficiently small \( A(u) \). \( \square \)

Now we define the set \( V(c) \) by

\[
V(c) := \{u \in S(c) : Q(u) = 0\}.
\]

**Lemma 3.2.** Let \( u \in S(c) \). Then:

1. \( \frac{\partial}{\partial t} E(u^t) = \frac{Q(u^t)}{t} \), for all \( t > 0 \).
2. There exists a \( t^* > 0 \) such that \( u^{t^*} \in V(c) \).
3. The mapping \( t \rightarrow E(u^t) \) is concave on \( [t^*, \infty) \).
4. We have \( t^*(u) < 1 \) if and only if \( Q(u) < 0 \). Moreover, \( t^*(u) = 1 \) if and only if \( Q(u) = 0 \).
5. The following inequalities hold:

\[
Q(u^t) \begin{cases} > 0, & t \in (0, t^*(u)), \\ < 0, & t \in (t^*(u), \infty). \end{cases}
\]

6. \( E(u^t) < E(u^{t^*}) \) for all \( t > 0 \) with \( t \neq t^* \).

**Proof.** 1. Using (2) and (3), one directly verifies that

\[
\frac{\partial}{\partial t} E(u^t) = tA(u) + \frac{3t^2}{2} B(u) + \frac{9}{5} t^{7/2} C(u) = \frac{1}{t} Q(u^t).
\]

This proves the first statement.

2. Define \( y(t) := \frac{\partial}{\partial t} E(u^t) \) (particularly \( y(t) = \frac{Q(u^t)}{t} \) for \( t > 0 \)). Then

\[
y'(t) = A(u) + 3t B(u) + \frac{63}{10} t^{5/2} C(u),
\]

\[
y''(t) = 3B(u) + \frac{63}{4} t^{3/2} C(u).
\]

Since \( \max\{B(u), C(u)\} \leq 0, (B(u), C(u)) \neq (0, 0) \) and \( y'(0) = A(u) > 0 \) we deduce using the above formulas that \( y'(t) \) is strictly decreasing on \([0, \infty] \), has a unique zero at \( t_0 > 0 \), is positive in \([0, t_0] \) and negative in \((t_0, \infty) \). Together with the expression for \( y(t) \) we obtain that
\textbullet{} \( y(t_0) = \max_{t>0} y(t) \),
\textbullet{} \( y(0) = 0 \), \( \lim_{t \to \infty} y(t) = -\infty \),
\textbullet{} \( y(t) \) is strictly increasing in \([0, t_0)\), strictly decreasing in \((t_0, \infty)\).

From the continuity of \( y(t) \) we deduce that it has a unique zero at \( t^* > 0 \), which shows the second statement.

The third to the fifth statements are also direct consequences of the above claims. Now since \( y(0) = 0 \) we infer that \( y(t) \) is positive on \((0, t^*)\) and negative on \((t^*, \infty)\). Since \( y(t) \) is the derivative of \( E(u^t) \), we obtain the last statement. \( \square \)

**Proposition 3.3.** The energy \( E \) has a mountain pass geometry on \( S(c) \), at level \( \gamma(c) \geq 0 \).

**Proof.** First define for \( k > 0 \) the set

\[
C_k := \{ u \in S(c) : A(u) = k \}
\]

and the numbers

\[
\alpha_k := \sup_{u \in C_k} E(u) \quad \text{and} \quad \beta_k := \inf_{u \in C_k} E(u).
\]

Note that \( C_k \neq \emptyset \), since \( A(u^t) = t^2 A(u) \) for any \( u \in S(c) \). We claim that:

There exists \( k_3 > 0 \) such that for all \( k_2 \in (0, k_3] \) and \( k_1 \in (0, k_2) \) holds that \( \alpha_k \leq \frac{1}{2} \beta_{k_2} \) for all \( k \in [0, k_1] \). \hspace{2cm} (11)

**Proof of the claim.** Let \( k_2 > 0 \) (to be determined) and \( u \in C_{k_2} \). Define the real function

\[
l(s) := \frac{s}{2} - C_1 s^{3/2} - C_2 s^{9/4},
\]

where the positive constants \( C_1, C_2 \) are as in (10). Estimating like (10), we obtain that \( E(u) \geq l(A(u)) \). A direct calculation shows that \( l \) is concave in \([0, \infty)\) and that \( l'(0) = 1/2 \), so that \( l(s) \geq s/4 \) for sufficiently small \( s > 0 \). This, in turn, implies that there exists \( k_3 > 0 \) such that for \( k_2 \in [0, k_3] \) we have \( E(u) \geq k_2/4 \) and, therefore, \( \beta_{k_2} \geq k_2/4 \) for all \( k_2 \in [0, k_3] \). We pick a \( k_2 \) from \((0, k_3]\) and keep it fixed. Now due to Remark 2.3 we get

\[
E(v) = \frac{1}{2} A(v) + \frac{1}{2} B(v) + \frac{2}{5} C(v) \leq \frac{1}{2} A(v),
\]

and taking the supremum over \( v \in C_k \) we get that \( \alpha_k \leq k/2 \) for all \( k > 0 \). In particular, for \( k \in [0, k_2/4] \) we get that \( \alpha_k \leq k_2/8 \). All in all we get that for all \( k \in [0, k_2/4] \):

\[
\alpha_k \leq \frac{11}{24} k_2 \leq \frac{1}{2} \beta_{k_2},
\]

which proves the claim by setting \( k_1 := k_2/4 \).

Now, by construction, pick \( k_2 \leq k_0 \), where \( k_0 \) is from Lemma 3.1–3.4. Take

\[
\Gamma_c = \{ g \in C([0, 1], S(c)) : g(0) \in A_{k_1}, E(g(1)) < 0 \},
\]

where \( k_1 \) is given by (11) and \( A_{k_1} \) is given in Definition 2.8. First we show that \( \Gamma_c \neq \emptyset \). Let \( v \in S(c) \). Recall that \( v'(x) = t^{3/2} v(tx) \) and, in particular, \( A(v') = t^2 A(v) \). Therefore we can
find a sufficiently small $t_1 > 0$ such that $A(v^{t_1}) < k_1$. Moreover, from Lemma 3.1–3.1 we can also pick a sufficiently large $t_2$ such that $E(v^{t_2}) < 0$. Now taking $g(t) := v^{(1-t)t_1 + t_2}$, we see that $g$ is an element of $\Gamma_c$.

Now let $g \in \Gamma_c$. Then $A(g(0)) \leq k_1 < k_2 \leq k_0$, which implies $Q(g(0)) > 0$ (Lemma 3.1–3.1). Now since $E(g(1)) < 0$, we infer from Lemma 3.1–3.2 that $Q(g(1)) < 0$, and therefore, by contraposition, either $A(g(1)) \geq k_2$ or $A(g(1)) \leq 0$. The latter case implies $g(1) = 0$, contradiction to $g(1) \in S(c)$. Also, if $A(g(1)) = k_2$, then $E(g(1)) \geq \beta_{k_2} \geq \frac{1}{2}k_2 > 0$ (from the proof of the claim), a contradiction to $E(g(1)) < 0$. Therefore $A(g(1)) > k_2$. Now define the function

$$\phi : [0, 1] \to [0, \infty), \ t \mapsto A(g(t)).$$

The continuity of $g$ implies the continuity of $\phi$. Since $A(g(0)) < k_2$ and $A(g(1)) > k_2$, we obtain from the intermediate value theorem that there exists a $t_0 \in (0, 1)$ such that $A(g(t_0)) = k_2$ and therefore $E(g(t_0)) \geq \beta_{k_2}$. Then,

$$\max_{t \in [0, 1]} E(g(t)) \geq E(g(t_0)) \geq \beta_{k_2} \geq \frac{1}{2}\beta_{k_2} \geq \alpha_A(g(0)) \geq E(g(0)) \geq \max\{E(g(0)), E(g(1))\},$$

where the last inequality is due to the fact that $E(g(1)) < 0$ and $E(g(0)) \geq 0$, since, if $E(g(0)) < 0$ then Lemma 3.1–3.2 implies that $Q(g(0)) < 0$, a contradiction to Lemma 3.1–3.2 since $A(g(0)) \leq k_0$ and thus $Q(g(0)) > 0$ and $\gamma(c) \geq 0$. Finally, since the left-hand side of (12) is bounded below by $\beta_{k_2}$ and the right-hand side is bounded above by $\frac{1}{2}\beta_{k_2}$, taking respectively the infimum and supremum over $g \in \Gamma_c$ in (12) completes the proof. ■

Proposition 3.4. The energy possesses no local minimizers on $S(c)$.

Proof. Assume to the contrary that there exists a relatively open subset $A \subseteq S(c)$ and $v \in A$, such that

$$E(v) = \inf \left\{ E(w) : w \in A \right\}.$$

Recall that $v^t(x) = t^{\frac{3}{2}}v(tx)$, so that $v^t \in A$ for all $t \in (1 - \varepsilon, 1 + \varepsilon)$ for some $\varepsilon > 0$ small enough. Then, since the mapping $t \mapsto E(v^t)$ has a local minimum at $t = 1$, it must hold that $\partial_t(E(v^t)) \big|_{t=1} = 0$ and $\partial_{tt}(E(v^t)) \big|_{t=1} \geq 0$. Recall that

$$\partial_t(E(v^t)) = tA(v) + \frac{3}{2}t^2B(v) + \frac{9}{5}t^2C(v),$$

$$\partial_{tt}(E(v^t)) = A(v) + 3tB(v) + \frac{63}{10}t^2C(v).$$

Evaluating at $t = 1$ and then eliminating $A(v)$, we obtain that $B(v) + 3C(v) \geq 0$, a contradiction to Remark 2.3. ■

3.1.2 Regularity of solutions, a Pohozaev identity and positivity of the chemical potential

The proof of Theorem 2.9–2.1 is given in Proposition 3.7 below. The idea is to apply standard bootstrap arguments involving elliptic regularity theory to get an optimal regularity of the solution. Some specific properties of the nonlocal convolution operator will be essential for our analysis; they are stated in the following lemma.

Lemma 3.5. Let $K$ be the singular integral operator defined by $Ku := K \ast u$. Then
Proposition 3.7. If an arbitrary dimensions \( L \) [13, Lemma 2.1], the latter providing with the assertions for each sum mand in (13).

Remark 3.6. As in [13, Lemma 2.1], Lemma 3.5 can be straightforwardly generalized to arbitrary dimensions \( n \geq 1 \).

Proposition 3.7. If \((u, \beta) \in H^1(\mathbb{R}^3; \mathbb{C}) \times \mathbb{R}\) is a solution of

\[
-\frac{1}{2} \Delta u + \lambda_1 |u|^2 u + \lambda_2 (K * |u|^2) u + \lambda_3 |u|^3 u + \beta u = 0,
\]

then \( u \in W^{3,p}(\mathbb{R}^3; \mathbb{C}) \) for all \( p \in (1, \infty) \). In particular, \( u \in C^{2}(\mathbb{R}^3; \mathbb{C}) \).

Proof. Define

\[
\Sigma u := -2(\lambda_1 |u|^2 u + \lambda_2 (K * |u|^2) u + \lambda_3 |u|^3 u + \beta u),
\]

so that from (5) we get that \( v = u \) is a solution to the linear elliptic equation:

\[
-\Delta v = \Sigma u.
\]

Due to the Sobolev embedding theorem

\[
H^1(\mathbb{R}^3; \mathbb{C}) \hookrightarrow L^p(\mathbb{R}^3; \mathbb{C}) \quad \text{for all} \quad p \in [1, 6],
\]

so that we infer

\[
|u|^2 u, |u|^3 u, u \in L^\frac{2}{3}(\mathbb{R}^3; \mathbb{C}).
\]

In particular we have that \( |u|^2 \in L^3(\mathbb{R}^3; \mathbb{C}) \). From Lemma 3.5 we obtain that \( K * |u|^2 \in L^3(\mathbb{R}^3; \mathbb{C}) \) so that the Cauchy-Schwarz inequality yields \( (K * |u|^2) u \in L^\frac{4}{3}(\mathbb{R}^3; \mathbb{C}) \). Thus

\[
\Sigma u \in L^\frac{4}{3}(\mathbb{R}^3; \mathbb{C}),
\]

and standard elliptic regularity theory (see e.g. [11, Theorem 7.13]) implies that \( u \in W^{2,\frac{4}{3}}(\mathbb{R}^3; \mathbb{C}) \).

But now we conclude from the Sobolev embedding

\[
W^{2,\frac{4}{3}}(\mathbb{R}^3; \mathbb{C}) \hookrightarrow L^p(\mathbb{R}^3; \mathbb{C}) \quad \text{for all} \quad p \in [1, \infty),
\]

that \( \Sigma u \in L^p(\mathbb{R}^3; \mathbb{C}) \) for all \( p \in [1, \infty) \). Again, elliptic regularity yields

\[
u \in W^{2,p}(\mathbb{R}^3; \mathbb{C}) \quad \text{for all} \quad p \in [1, \infty)\)
so that the Sobolev embedding theorem implies that $u \in L^\infty(\mathbb{R}^3; \mathbb{C})$. Thus, using the so-called first Moser inequality (see e.g. [29, Proposition 3.7, p.11]) we get that

$$|u|^2 u, |u|^3 u, |u|^2 u \in W^{1,p}(\mathbb{R}^3; \mathbb{C}) \text{ for all } p \in [1, \infty).$$

Using Lemma 3.5 one more time as before, it follows that

$$(K * |u|^2)u \in W^{1,p}(\mathbb{R}^3; \mathbb{C}) \text{ for all } p \in (1, \infty)$$

and thus $\Sigma u \in W^{1,p}(\mathbb{R}^3; \mathbb{C})$ for all $p \in (1, \infty)$. Finally, we obtain the desired result applying elliptic regularity and Sobolev embedding one last time.

The proof of Theorem 2.9–2.10 is given in Proposition 3.10 below, but before we can proceed, we need the following technical result from [14]:

**Lemma 3.8 ([14] Proposition 5.3).** Let $f \in L^2(\mathbb{R}^3) \cap H_{loc}^1(\mathbb{R}^3)$ be a real valued function. Let $W$ be a real valued even tempered distribution on $\mathbb{R}^3$ such that $f \mapsto W * f$ is continuous from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$, $\hat{W}$ is differentiable a.e. on $\mathbb{R}^3$ and for all $j, k \in \{1, 2, 3\}$, the map $\xi \mapsto \xi_j \partial_k \hat{W}(\xi)$ is bounded and continuous a.e. on $\mathbb{R}^3$. Moreover, let $\psi \in C_c^\infty(\mathbb{R}^3)$ with $\psi(x) = 1$ for $|x| \leq 1$ and $\psi(x) = 0$ for $|x| \geq 2$ and $\psi_n(x) := \psi(|x|^2/n^2)$. Then

$$\lim_{n \to \infty} \left( -\frac{1}{2} \int_{\mathbb{R}^3} (W * f)(x) \psi_n(x) \partial_j f(x) \, dx \right) = \frac{1}{4} \int_{\mathbb{R}^3} (W * f) \, f \, dx - \frac{1}{4}\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \xi_j \partial_j \hat{W}(\xi) |\hat{f}(\xi)|^2 \, d\xi, \text{ for all } j \in \{1, 2, 3\}.$$ 

The next lemma provides with a very important technical tool for our analysis. It is worth noting, that it is the structure of the nonlocal term that allows for a Pohozaev identity to hold. If the nonlocal term fails to have the appropriate structure, non-existence of solutions is possible. For such a case, we refer to [14].

**Lemma 3.9.** If $(u, \beta) \in H^1(\mathbb{R}^3; \mathbb{C}) \times \mathbb{R}$ is a solution of (5), then the following Pohozaev identity holds:

$$\frac{1}{2} A(u) + \frac{3}{2} B(u) + \frac{6}{5} C(u) + 3\beta \|u\|^2_2 = 0.$$  

**Proof.** From Proposition 3.7 we obtain that (5) can be understood in a classical sense so that the divergence theorem can be applied. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. For $i \in \{1, 2, 3\}$ and functions $v_i \in C^\infty_c(\mathbb{R}^3)$, with supp $v_i \subset \Omega$, one obtains using the divergence theorem:

$$\int_{\Omega} v_i |u|^p (u \partial_i \bar{u} + \bar{u} \partial_i u) \, dx = \int_{\Omega} v_i \partial_i \left( \frac{2|u|^{p+2}}{p+2} \right) \, dx = \int_{\Omega} \left( \partial_i (v_i \frac{2|u|^{p+2}}{p+2}) - \partial_i v_i \frac{2|u|^{p+2}}{p+2} \right) \, dx$$

$$= - \int_{\Omega} \partial_i v_i \frac{2|u|^{p+2}}{p+2} \, dx. \tag{15}$$

Moreover, it holds that

$$\int_{\Omega} v_i (K * |u|^2) (u \partial_i \bar{u} + \bar{u} \partial_i u) \, dx = \int_{\mathbb{R}^3} v_i (K * |u|^2) \partial_i (|u|^2) \, dx. \tag{16}$$
Again, setting \( v := (v_1, v_2, v_3) \) and using the divergence theorem, we get that
\[
\int_{\Omega} v \cdot (\Delta u \nabla \bar{v} + \Delta \bar{v} \nabla u) \, dx = \int_{\Omega} (\Delta u v \cdot \nabla \bar{v} + \Delta \bar{v} v \cdot \nabla \bar{v}) \, dx
\]
\[
= \int_{\Omega} 2 \text{Re}(\Delta u v \cdot \nabla \bar{v}) \, dx = \int_{\Omega} 2 \left( \text{div}(\nabla u) v \cdot \nabla \bar{u} \right) \, dx
\]
\[
= \int_{\Omega} -2 \left( \text{Re}(\nabla u \cdot (\nabla v \nabla \bar{v})) + \text{Re}(\nabla u \cdot (\nabla \nabla \bar{u} v)) \right) \, dx
\]
\[
= -2 \int_{\Omega} \left( \text{Re}(\nabla u \cdot (\nabla v \nabla \bar{v})) + \frac{1}{2} \left( \text{div}(|\nabla u|^2 v) - |\nabla u|^2 \text{div} v \right) \right) \, dx
\]
\[
= - \int_{\Omega} (2 \text{Re}(\nabla u \cdot (\nabla v \nabla \bar{v})) - |\nabla u|^2 \text{div} v) \, dx, \tag{17}
\]
where \( \nu_i \) is the \( i \)-th component of the exterior normal vector of \( \partial \Omega \). In order to obtain (\star), we used the following equality:
\[
\text{div}(|\nabla u|^2 v) = |\nabla u|^2 \text{div} v + v \cdot \nabla |\nabla u|^2 = |\nabla u|^2 \text{div} v = \text{div} ((\nabla u \nabla \bar{u} + \nabla \nabla \bar{u} \nabla u). \]

Now let \( \psi \in C_0^\infty(\mathbb{R}^3) \) with \( \psi(x) = 1 \) for \( |x| \leq 1 \) and \( \psi(x) = 0 \) for \( |x| \geq 2 \). Define \( \psi_n(x) := \psi(|x|^2/n^2) \) for \( n \in \mathbb{N} \). We directly obtain that
\[
|\psi_n|, \ |x||\nabla \psi_n| \leq \|\psi\|_\infty,
\]
so that \( v := \psi_n x \) and \( \Omega := B(0, \sqrt{2}n) \), where \( B(0, \sqrt{2}n) \) is the ball in \( \mathbb{R}^3 \) with center 0 and radius \( \sqrt{2}n \), are admissible in (15), (16) and (17).

Define
\[
Fu := -\frac{1}{2} \Delta u + \lambda_1 |u|^2 u + \lambda_2 (K * |u|^2) u + \lambda_3 |u|^3 u + \beta u
\]
and note that, since \( u \) is a solution of (5), the equation
\[
Fu(x) v(x) \cdot \nabla \bar{u}(x) + F\bar{u}(x) v(x) \cdot \nabla u(x) = 0
\]
holds for almost all \( x \in \Omega \). Integrate the above over \( \Omega \) and use (15), (16) and (17) to obtain that
\[
\int_{\Omega} \left\{ \text{Re}(\nabla u \cdot (\nabla v \nabla \bar{u})) - \left( \frac{1}{2} |\nabla u|^2 + \frac{\lambda_1}{2} |u|^4 + \frac{2}{5} \lambda_3 |u|^5 + \beta |u|^2 \right) \text{div} v \right\} \, dx
\]
\[
= -\lambda_2 \int_{\mathbb{R}^3} \sum_{i=1}^3 v_i (K * |u|^2) \partial_i(|u|^2) \, dx.
\]
One directly verifies that \( W = K \) satisfies the conditions of Lemma 3.8. Using the dominated convergence theorem for the left-hand side and Lemma 3.8 for the right-hand side, we can take the limit \( n \to \infty \) to obtain that
\[
\frac{1}{2} A(u) + \frac{3}{2} B(u) + \frac{6}{5} C(u) + 3\beta \|u\|^2_2 = \frac{1}{2} (2\pi)^3 \int_{\mathbb{R}^3} \left( \sum_{j=1}^3 \xi_j \partial_j \widehat{W}(|\xi|) \right) \left| \frac{|\xi|^2}{\xi} \right|^2 d\xi.
\]
Finally, a direct calculation yields that \( \sum_{j=1}^3 \xi_j \partial_j \widehat{W}(|\xi|) = 0 \).
\[ \blacksquare \]
Proposition 3.10. If \((u, \beta) \in H^1(\mathbb{R}^3; \mathbb{C}) \times \mathbb{R}\) is a solution of (5), then \(Q(u) = 0\). Moreover, if \(u \neq 0\), then \(\beta > 0\).

Proof. Testing (5) with \(\bar{u}\), we deduce that
\[
\frac{1}{2} A(u) + B(u) + C(u) + \beta \|u\|^2_2 = 0. 
\] (18)
Multiplying the above with 3 and subtracting from the Pohozaev identity (14) we get that
\[
Q(u) = A(u) + \frac{3}{2} B(u) + \frac{9}{5} C(u) = 0.
\]
Again, subtracting (14) from (18) we get
\[
\beta \|u\|^2_2 = -\left(\frac{1}{4} B(u) + \frac{1}{10} C(u)\right) > 0
\]
for \(u \neq 0\), which ends the proof. 

3.1.3 Exponential decay at infinity

The proof of Theorem 2.9–3. is given in the following proposition.

Proposition 3.11. Let \((u, \beta) \in S(c) \times (0, \infty)\) be a solution of (5). Then

1. \(|u(x)|, |\nabla u(x)|, \left| (K * |u|^2)(x) \right| \to 0\) for \(|x| \to \infty\).

2. There exist constants \(L, M > 0\) such that
\[
e^{L|x|} \left( |u(x)| + |\nabla u(x)| \right) \leq M \quad \text{for all} \quad x \in \mathbb{R}^3.
\]

Proof. The proof follows closely the lines of [13, Theorem 2.4]; we include it here for the sake of completeness. From Proposition 3.5 and the Sobolev embedding theorem we know that \(u, \nabla u\) and \((K * |u|^2)u\) are Lipschitz continuous. Since they are also integrable on the whole \(\mathbb{R}^3\), we obtain the first statement.

Note that the rescaling \(v(x) := u((2\beta)^{-1/2}x)\) transforms equation (5) to
\[
-\frac{1}{2} \Delta v + \frac{\lambda_1}{2\beta} |v|^2 v + \frac{\lambda_2}{2\beta} (K * |v|^2)v + \frac{\lambda_3}{2\beta} |v|^3 v + \frac{1}{2} v = 0.
\]
Thus without loss of generality, since we still stay in the unstable regime when we rescale as above, we can assume \(\beta = 1/2\).

Now for \(\varepsilon > 0\) define \(\theta_{\varepsilon}(x) = \exp(|x|/\varepsilon |1|)\). Then \(\theta_{\varepsilon}\) is bounded, Lipschitz continuous and satisfies \(|\nabla \theta_{\varepsilon}| \leq \theta_{\varepsilon}\) in \(\mathbb{R}^3 \setminus \{0\}\). We then multiply (5) by \(\theta_{\varepsilon} \bar{u}\), and using the triangle inequality we obtain that
\[
\int_{\mathbb{R}^3} \left( \text{Re} (\nabla u \cdot \nabla (\theta_{\varepsilon} \bar{u})) + \theta_{\varepsilon} |u|^2 \right) dx \\
\leq 2|\lambda_1| \int_{\mathbb{R}^3} \theta_{\varepsilon} |u|^4 dx + 2\lambda_2 \int_{\mathbb{R}^3} \theta_{\varepsilon} |u|^2 dx + 2|\lambda_3| \int_{\mathbb{R}^3} \theta_{\varepsilon} |u|^5 dx.
\]
Since
\[
\text{Re} (\nabla u \cdot \nabla (\theta_{\varepsilon} \bar{u})) \geq \theta_{\varepsilon} |\nabla u|^2 - \theta_{\varepsilon} |u||\nabla u|,
\]
we have
\[ \int_{\mathbb{R}^3} \left( \theta \varepsilon |\nabla u|^2 - \theta \varepsilon |u| \nabla u + \theta \varepsilon |u|^2 \right) \, dx \]
\[ \leq 2|\lambda_1| \int_{\mathbb{R}^3} \theta \varepsilon |u|^4 \, dx + 2\lambda_2 \int_{\mathbb{R}^3} \theta \varepsilon |u|^2 \, K * |u|^2 \, dx + 2|\lambda_3| \int_{\mathbb{R}^3} \theta \varepsilon |u|^5 \, dx. \]

Now choose \( \delta \in (0, \frac{1}{8|\lambda_1| + 2\lambda_2 + |\lambda_3|}) \). From the first statement we know that there exists some \( R_1 > 0 \) such that
\[ |(K * |u|^2)(x)|, |u(x)|^2, |u(x)|^3 < \delta \]
for all \( |x| \geq R_1 \). Using the Cauchy-Schwarz inequality, we obtain that
\[ \frac{1}{2} \int_{\mathbb{R}^3} \left( \theta \varepsilon |\nabla u|^2 + \theta \varepsilon |u|^2 \right) \, dx \]
\[ \leq 2 \int_{|x| \leq R_1} e^{\varepsilon |x|} |u|^2 \left( |\lambda_1| |u|^2 + \lambda_2 K * |u|^2 + |\lambda_3| |u|^3 \right) \, dx + 2(|\lambda_1| + \lambda_2 + |\lambda_3|) \delta \int_{|x| \geq R_1} \theta \varepsilon |u|^2 \, dx \]
\leq 2 \int_{|x| \leq R_1} e^{\varepsilon |x|} |u|^2 \left( |\lambda_1| |u|^2 + \lambda_2 K * |u|^2 + |\lambda_3| |u|^3 \right) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \theta \varepsilon |u|^2 \, dx, \]
from which it follows
\[ \frac{1}{4} \int_{\mathbb{R}^3} \left( \theta \varepsilon |\nabla u|^2 + \theta \varepsilon |u|^2 \right) \, dx \leq 2 \int_{|x| \leq R_1} e^{\varepsilon |x|} \left( |\lambda_1| |u|^2 + \lambda_2 K * |u|^2 + |\lambda_3| |u|^3 \right) \, dx. \]

But the right-hand side of the above equation is bounded, since the integral is taken over a bounded ball. Thus we conclude that
\[ \int_{\mathbb{R}^3} \left( \theta \varepsilon |\nabla u|^2 + \theta \varepsilon |u|^2 \right) \, dx \leq M_1 \]
for some positive constant \( M_1 \). Letting \( \varepsilon \downarrow 0 \) we obtain that
\[ \int_{\mathbb{R}^3} e^{\varepsilon |x|} \left( |\nabla u|^2 + |u|^2 \right) \, dx \leq M_1. \quad (19) \]

Now to show the second statement, one only needs to consider sufficiently large \( |x| \) and conclude due to continuity. Due to the first statement, there exists some \( R_2 > 0 \) such that
\[ |u(x)| + |\nabla u(x)| < 1 \quad \text{if} \ |x| \geq R_2. \]

Fix an arbitrary \( |x| \geq R_2 \). Since \( u \) and \( \nabla u \) are Lipschitz continuous (see beginning of the proof), there exists \( L_1 > 0 \) such that
\[ |u(x)|^2 + |\nabla u(x)|^2 \leq 2 \left( |u(y)|^2 + |\nabla u(y)|^2 + L_1^2 |x - y|^2 \right) \quad \text{for all} \ y \in \mathbb{R}^3. \quad (20) \]

Let
\[ \rho := \frac{1}{2L_1} \left( |u(x)|^2 + |\nabla u(x)|^2 \right)^{\frac{1}{2}} \leq \frac{1}{2L_1} \quad (21) \]
and multiply \((20)\) by \( e^{\varepsilon |x|} \) to obtain
\[ e^{\varepsilon |x|} \left( |u(x)|^2 + |\nabla u(x)|^2 \right) \leq 4e^{\varepsilon |x|} \left( |u(y)|^2 + |\nabla u(y)|^2 \right) \quad \text{for all} \ y \in B(x, \rho). \quad (22) \]
Since \( y \in B(x, \rho) \) and \( \rho \) is bounded by \( 1/2L_1 \) due to (21), it follows

\[ |x| \leq |y| + \frac{1}{2L_1} \quad \text{for all} \quad y \in B(x, \rho), \]

and thus

\[ e^{|x|} \leq e^{\frac{1}{2L_1} |y|} \quad \text{for all} \quad y \in B(x, \rho). \]

Therefore, the estimate

\[ e^{|x|} \left( |u(x)|^2 + |\nabla u(x)|^2 \right) \leq 4e^{\frac{1}{2L_1} |y|} \left( |u(y)|^2 + |\nabla u(y)|^2 \right) \quad (23) \]

follows from (22) for all \( y \in B(x, \rho) \). Integrating (23) over \( B(x, \rho) \), we obtain that

\[ \omega_3 \rho^3 e^{|x|} \left( |u(x)|^2 + |\nabla u(x)|^2 \right) \leq 4e^{\frac{1}{2L_1}} \int_{B(x, \rho)} e^{|y|} \left( |u(y)|^2 + |\nabla u(y)|^2 \right) \, dy, \quad (24) \]

where \( \omega_3 \) is the area of the unit ball of \( \mathbb{R}^3 \). Evaluating (24) by inserting the precise value of \( \rho \), we finally obtain that

\[ \omega_3 e^{|x|} \left( |u(x)|^2 + |\nabla u(x)|^2 \right)^{\frac{2}{3}} \leq 4e^{\frac{1}{2L_1}} \int_{B(x, \rho)} e^{|y|} \left( |u(y)|^2 + |\nabla u(y)|^2 \right) \, dy \leq 4e^{\frac{1}{2L_1}} M_1, \]

where the last inequality comes from (19). This completes the proof. ■

### 3.1.4 The energy level of ground states

In this section we complete the proof of Theorem 2.9 by showing that ground states may exist only at the mountain pass level. We first state a useful characterization of the latter.

**Lemma 3.12.** It holds that \( \gamma(c) = \inf_{u \in V(c)} E(u) \).

**Proof.** Let \( v \in V(c) \) so that \( Q(v) = 0 \). Therefore from Lemma 3.2.4, we conclude that \( t^*(v) = 1 \). Due to equations (7) and (8), we can find \( 0 < t_1 < 1 < t_2 \) such that \( v^{t_1} \in A_{k_1} \) and \( E(v^{t_2}) < 0 \). Now define

\[ g(t) := v^{(1-t) t_1 + t t_2} \in \Gamma_c = \{ g \in C([0, 1], S(c)), g(0) \in A_{k_1}, E(g(1)) < 0 \}. \]

Using Lemma 3.2.6, we get that

\[ \gamma(c) \leq \max_{t \in [0,1]} E(g(t)) = E(v^1) = E(v) \]

and therefore \( \gamma(c) \leq \inf_{u \in V(c)} E(u) \). On the other hand, for a path \( g \in \Gamma_c \), we obtain that \( Q(g(0)) > 0 \) and \( Q(g(1)) < 0 \) (as in the proof of Proposition 3.3). Therefore, from the intermediate value theorem, any path in \( \Gamma_c \) will cross \( V(c) \). Thus

\[ \max_{t \in [0,1]} E(g(t)) \geq \inf_{u \in V(c)} E(u). \]

This completes the proof. ■
Proof of Theorem 2.9–3.1. Since, due to Proposition 3.10, the set of solutions is a subset of \( V(c) \), it follows from the Lagrange multiplier theorem (see e.g. [2] Corollary 3.5.29) that

\[
\{ v \in S(c) : E|_{S(c)}(v) = 0 \} \subseteq V(c).
\]

This, using Lemma 3.12, implies

\[
\inf \{ E(v) : v \in S(c) \text{ and } E|_{S(c)}(v) = 0 \} \geq \inf_{v \in V(c)} E(v) = \gamma(c).
\]

(25)

If \((u, \beta)\) is a solution, it holds that

\[
E(u) \geq \inf \{ E(v) : v \in S(c) \text{ and } E(v)|_{S(c)} = 0 \}.
\]

(26)

Now, if \( E(u) = \gamma(c) \), all inequalities in (25) and (26) become equalities and it follows that \( u \) is a ground state.

On the other hand, let \( v \) be a ground state. Then \( E(u) = E(v) \) for all ground states \( u \). We claim that there exists a ground state \( u \) such that \( E(u) = \gamma(c) \), which finishes the proof. Proving the claim is far from trivial. It relies on the construction of a Palais-Smale sequence at the specific level, which will be done in the next section. To be more precise, Proposition 3.22 yields the existence of a solution \((u, \beta)\) such that \( E(u) = \gamma(c) \) and its proof is independent from the proofs of the Theorems 2.9 and 2.10. But then, the previous argumentation implies that \( u \) is a ground state.

3.2 Proof of Theorem 2.10

3.2.1 The structure of solutions at mountain pass level

The proof of the theorem follows the lines of [6, Lemma 1.1, 1.2]; it relies on the observation that \(|u|\) will be a minimizer on \( V(c) \).

Lemma 3.13. Let \( c > 0 \) and \((u, \beta) \in S(c) \times \mathbb{R} \) be a solution of (5) such that \( E(u) = \gamma(c) \). Then \( A(|u|) = A(u) \) and in particular, \( E(|u|) = E(u) \) and \( Q(|u|) = Q(u) \).

Proof. From the diamagnetic inequality (see e.g. [24, 7.21 Theorem, p. 193]) it follows that \( A(|u|) \leq A(u) \). Set \( v := |u| \) and assume that \( A(v) < A(u) \). Then \( Q(v) < 0 \) due to the definition of \( Q(u) \) (see Definition 2.1), since \( B(|u|) = B(u) \) and \( C(|u|) = C(u) \). From Lemma 3.2, we can thus find a \( t^\ast \in (0, 1) \) such that \( Q(v^t^\ast) = 0 \). Since \( v^t = |u|^t \), we obtain from the diamagnetic inequality that \( A(v^t^\ast) \leq A(u^t^\ast) \), and thus \( E(v^t^\ast) \leq E(u^t^\ast) \) (using similar arguments as for the fact \( Q(v) < 0 \)). Now since \( Q(u) = 0 \), \( u = u^1 \) is the maximizer of \( t \mapsto E(u^t) \), due to Lemma 3.2. Therefore \( E(v^t^\ast) \leq E(u^t^\ast) \leq E(u) \). But \( u \) is also a minimizer of \( E \) constrained on \( V(c) \). Since \( v^t \) is also an element of \( V(c) \), we have \( E(u) \leq E(v^t) \). Thus \( E(u) = E(v^t^\ast) = \gamma(c) \). But then we must have \( E(u) = E(u^t^\ast) = E(v^t^\ast) \) and the only possibility is \( t^\ast = 1 \). This contradicts the fact that \( t^\ast \in (0, 1) \). Thus \( A(|u|) = A(u) \). □

Proof of Theorem 2.10. Proposition 3.10 together with Lemma 3.13 implies that \(|u| \in V(c)\), so that, since \( E(|u|) = E(u) \) from Lemma 3.13, \(|u| \) is also minimizer of \( E|_{V(c)} \). From the Lagrange multiplier theorem [19, Proposition 14.3] we obtain that there exist numbers \( \mu_1 \) and \( \mu_2 \) such that

\[
E'(|u|) - \mu_1 Q'(|u|) - 2\mu_2 |u| = 0,
\]

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from which we deduce that
\[
(1 - 2\mu_1) \left( -\Delta |u| \right) + 2(1 - 3\mu_1) \left( \lambda_1 |u|^3 + (K \ast |u|^2) |u| \right) + (2 - 9\mu_1) \lambda_3 |u|^4 - 2\mu_2 |u| = 0.
\]
(27)

To show that \(|u|\) is a solution of (27), it suffices to show that \(\mu_1 = 0\), i.e., it is a critical point of \(E|_{S(c)}\) (see e.g. [2, Corollary 3.5.29]). Multiplying (27) with \(|u|\) and integrating we obtain that
\[
(1 - 2\mu_1) A(|u|) + 2(1 - 3\mu_1) B(|u|) + (2 - 9\mu_1) C(|u|) - 2\mu_2 \|u\|_2^2 = 0.
\]
(28)

Analogous to the proof of Lemma 3.9 we obtain the following Pohozaev identity corresponding to (27):
\[
\frac{1}{2}(1 - 2\mu_1) A(|u|) + \frac{3}{2}(1 - 3\mu_1) B(|u|) + \frac{3}{5}(2 - 9\mu_1) C(|u|) - 3\mu_2 \|u\|_2^2 = 0.
\]
(29)

Eliminating \(\|u\|_2^2\) from (28) and (29) we obtain that
\[
(1 - 2\mu_1) A(|u|) + \frac{3}{2}(1 - 3\mu_1) B(|u|) + \frac{9}{10}(2 - 9\mu_1) C(|u|) = 0.
\]
(30)

Now since \(|u| \in V(c)\) we know that
\[
Q(|u|) = A(|u|) + \frac{3}{2} B(|u|) + \frac{9}{5} C(|u|) = 0.
\]
(31)

Eliminating \(A(|u|)\) from (30) and (31) we have
\[
\frac{3\mu_1}{2} B(|u|) + \frac{9\mu_1}{2} C(|u|) = 0.
\]

Since \(B(|u|) + 3C(|u|)\) is negative, we must have \(\mu_1 = 0\) and therefore \((|u|, -\mu_2)\) is a solution of (5) and from Theorem 2.9–3 it is also a ground state. From the Pohozaev identity and Lemma 3.13 we then get that \(-\mu_3 = \beta\), which completes the proof of Theorem 2.10–3.

Next we show that \(u\) is nowhere zero, which equivalently means that \(|u|\) is everywhere positive and thus proves Theorem 2.10–1. Define
\[
h(x) := 2\lambda_1 |u|^2 + 2K \ast |u|^2 + 2\lambda_3 |u|^3 + \beta,
\]
then (27) implies that \(v = |u|\) is a solution of the linear elliptic equation
\[
\Delta v - h^+ v = -h^- v \leq 0,
\]
where \(h^+ = \max\{0, h\}\) and \(h^- = -\min\{0, h\}\). Notice that \(h^+, h^-, v\) are nonnegative and \(h\) is uniformly bounded, since \(u\) and \(K \ast |u|^2\) are of class \(L^\infty\) due to Proposition 3.7. Thus if there exists some \(x \in \mathbb{R}^3\) with \(v(x) = 0\), then from the strong maximum principle (see e.g. [16, Theorem 8.19]; note that (8.6) in [16], one of the conditions of [16, Theorem 8.19], is satisfied, since \(h\) is uniformly bounded) it must follow that \(v\) is constant and zero in \(\mathbb{R}^3\), a contradiction. Thus \(v\) is positive. Theorem 2.10–2 is then obtained directly from [5, Theorem 5].
3.3 Proof of Theorem 2.11

3.3.1 Construction of a Palais-Smale sequence

In the following we use the idea given in [7] to construct a bounded “localized” Palais-Smale sequence \( \{u_n\}_{n \in \mathbb{N}} \), in the sense that \( \text{dist}(u_n, V(c)) = o(1) \). Let us consider the set

\[
L = \{ u \in V(c) : E(u) \leq \gamma(c) + 1 \}.
\]

The set \( L \) is a bounded set in \( H^1(\mathbb{R}^3; \mathbb{C}) \), since for \( u \in L \) it follows \( Q(u) = 0 \), and therefore

\[
\gamma(c) + 1 \geq E(u) = E(u) - \frac{1}{3} Q(u) = \frac{1}{6} A(u) - \frac{1}{5} C(u) \geq \frac{1}{6} A(u) = \frac{1}{6} \| \nabla u \|_2^2 > 0.
\]

Together with the fact that \( u \in S(c) \) we obtain the boundedness of \( L \). Now let \( R_0 > 0 \) such that \( L \subset B(0, R_0) \), where \( B(0, R_0) \) is the ball in \( H^1(\mathbb{R}^3; \mathbb{C}) \) with center 0 and radius \( R_0 \).

**Lemma 3.14.** Let

\[
J_\mu := \{ u \in S(c) : |E(u) - \gamma(c)| \leq \mu, \; \text{dist}(u, V(c)) \leq 2\mu \; \text{and} \; \| E'[S(c)](u) \|_{T^*S(c)} \leq 2\mu \}.
\]

Then for any \( \mu > 0 \), \( J_\mu \cap B(0, 3R_0) \neq \emptyset \).

**Proof.** Since \( E[S(c)] : S(c) \to \mathbb{R} \) is a \( C^1 \) functional, applying [8, Lemma 4], we obtain the existence of a locally Lipschitz pseudo-gradient vector field \( Y \in C^1(\tilde{S}(c); TS(c)) \), satisfying

\[
\| Y(u) \|_{H^1} \leq 2 \| E'[S(c)](u) \|_{T^*S(c)},
\]

\[
\langle E'[S(c)](u), Y(u) \rangle_{T^*S(c), TS(c)} \geq \| E'[S(c)](u) \|_{T^*S(c)}^2
\]

for all \( u \in \tilde{S}(c) \), where

\[
\tilde{S}(c) := \{ u \in S(c) : E'[S(c)](u) \neq 0 \}
\]

and \( TS(c) \) is the tangent bundle. In particular, (33) implies that \( \| Y(u) \|_{H^1} \neq 0 \) for \( u \in \tilde{S}(c) \). Now define

\[
\tilde{N}_\mu := \{ u \in S(c) : |E(u) - \gamma(c)| \leq \mu, \; \text{dist}(u, V(c)) \leq 2\mu \; \text{and} \; \| Y(u) \|_{H^1} \geq 2\mu \},
\]

\[
N_\mu := \{ u \in S(c) : |E(u) - \gamma(c)| < 2\mu \} \subset \tilde{N}_\mu.
\]

Since \( E(u^t) \to -\infty \) as \( t \to \infty \) (Lemma 3.1-1), we conclude that \( N_\mu \neq \emptyset \) for any \( \mu > 0 \), where the complement is taken in \( H^1(\mathbb{R}^3; \mathbb{C}) \). We need only to consider \( \tilde{N}_\mu \neq \emptyset \), since this will be the case in our contradiction argument. Then, one can define the function \( h : S(c) \to [0, 1] \) by

\[
h(u) = \frac{\text{dist}(u, N_\mu^c)}{\text{dist}(u, \tilde{N}_\mu) + \text{dist}(u, N_\mu^c)}.
\]

One verifies that \( h \) is a bounded and locally Lipschitz function with

\[
h(u) = \begin{cases} 1, & u \in \tilde{N}_\mu, \\ 0, & u \in N_\mu^c. \end{cases}
\]

Now define on \( S(c) \) the vector field

\[
W(u) := \begin{cases} -h(u) \frac{Y(u)}{\| Y(u) \|_{H^1}}, & u \in \tilde{S}(c), \\ 0, & u \in S(c) \setminus \tilde{S}(c) \end{cases}
\]
and consider the ODE
\[
\begin{aligned}
\begin{cases}
\partial_t \eta_u = W(\eta_u), \\
\eta_u(0) = u.
\end{cases}
\end{aligned}
\tag{36}
\]

Due to \cite[Lemma 6]{8}, the ODE (36) has a unique solution \( \eta_u : \mathbb{R} \to S(c) \) for all \( u \in S(c) \). In particular, \( \eta(t, u) := \eta_u(t) \) has the following properties:

- If \( |E(u) - \gamma(c)| \geq 2\mu \), then \( \eta(t, u) = u \) for all \( t \in \mathbb{R} \).
- For all \( t \in \mathbb{R} \) and \( u \in S(c) \) holds that \( \frac{d}{dt} E(\eta(t, u)) \leq 0 \).

Now define
\[
\Lambda_\mu := \{ u \in S(c) : |E(u) - \gamma(c)| \leq \mu, \text{ dist } (u, V(c)) \leq 2\mu \}.
\]

Suppose that the lemma does not hold. Then there exists a \( \tilde{\mu} > 0 \) such that
\[
u \in \Lambda_{\tilde{\mu}} \cap B(0, 3R_0) \implies \|E'(\eta(u))\|_{T \cdot S(c)} > 2\tilde{\mu}.
\]

From the definition of \( \gamma(c) \) we deduce that
\[
\sup_{u \in A_{K_c}} E(u) < \kappa \gamma(c)
\]
for some \( \kappa \in (0, 1) \), where \( A_{K_c} \) is given in Definition \cite{2,8} \( K_c \) has been given in the proof of Proposition \cite{3.3}. Since \( J_{\mu_1} \subset J_{\mu_2} \) for \( \mu_1 < \mu_2 \), we can assume without loss of generality that \( \tilde{\mu} \in (0, \frac{(1-\kappa)\gamma(c)}{2}) \). Then from (34) it follows
\[
u \in \Lambda_{\tilde{\mu}} \cap B(0, 3R_0) \implies \nu \in \tilde{N}_{\tilde{\mu}}.
\tag{37}
\]

From the mean value theorem we get the existence of \( s(t) \in (0, t) \) such that
\[
\eta(t, u) = \eta(0, u) + \partial_t \eta(t, u)\big|_{t=s(t)} t = u + \partial_t \eta(t, u)\big|_{t=s(t)} t.
\]

We also notice that
\[
\|\partial_t \eta(t, u)\|_{H^1} = \|W(\eta(t, u))\|_{H^1} \leq 1.
\]

Thus there exists some \( s_0 > 0 \) such that for all \( s \in (0, s_0) \)
\[
u \in \Lambda_{\tilde{\mu}} \cap B(0, 2R_0) \implies \eta(s, u) \in B(0, 3R_0) \land \text{ dist } (\eta(s, u), V(c)) \leq 2\tilde{\mu}.
\tag{38}
\]

In the end of the proof of the lemma we will show that for all sufficiently small \( \varepsilon > 0 \) we can construct a path \( g_{\varepsilon} \in \Gamma_c \) satisfying:

- \( g_{\varepsilon}(t) = u^{(1-t)\lambda_1+t\lambda_2} \) for some \( u \in V(c) \) and \( 0 < \lambda_1 < 1 < \lambda_2 < \infty \),
- \( \max_{t \in [0,1]} E(g_{\varepsilon}(t)) \leq \gamma(c) + \varepsilon \) and
- \( E(g_{\varepsilon}(t)) \geq \gamma(c) \implies g_{\varepsilon}(t) \in \Lambda_{\tilde{\mu}} \cap B(0, 2R_0) \).

\( 19 \)
Fix $\varepsilon \in (0, \min\{1, \bar{\mu}, \frac{\bar{\mu}}{2}\})$ such that (39) to (41) hold. Therefore the function $u$ given by (39) satisfies $\text{dist} \ (u, V(c)) = 0$ and

$$|E(u) - \gamma(c)| \leq \varepsilon \leq \min \{1, \bar{\mu}, \frac{\bar{\mu}s_0}{2}\} \leq \bar{\mu}.$$ 

Thus $u \in A_\bar{\mu} \cap B(0, 3R_0)$ and $\|Y(u)\| \geq 2\bar{\mu}$ due to (37). From

$$g_\varepsilon(0) \in A_{\kappa_\varepsilon} \Rightarrow E(g_\varepsilon(0)) - \gamma(c) < (\kappa - 1)\gamma(c) < 0$$

$$\Rightarrow |E(g_\varepsilon(0)) - \gamma(c)| > (1 - \kappa)\gamma(c) > 2\bar{\mu}$$

and

$$E(g_\varepsilon(1)) < 0 \Rightarrow E(g_\varepsilon(1)) - \gamma(c) < -\gamma(c) < 0$$

$$\Rightarrow |E(g_\varepsilon(1)) - \gamma(c)| > \gamma(c) > 2\bar{\mu}$$

we can infer that $\eta(s, g_\varepsilon(\cdot))$ is in $\Gamma_\varepsilon$ for all $s > 0$, since $\eta(s, u) = u$ for all $u$ satisfying $|E(u) - \gamma(c)| \geq 2\mu$. Now let $s^* := \frac{2\bar{\mu}}{\mu} < s_0$. We claim that

$$\max_{t \in [0,1]} E\left(\eta(s^*, g_\varepsilon(t))\right) < \gamma(c).$$

(42)

Indeed, if $E(g_\varepsilon(t)) < \gamma(c)$, then using that $\frac{d}{dt}E(\eta(t, u)) \leq 0$ for all $t \in \mathbb{R}$ and $u \in S(c)$ (i.e., $E(\eta(t, u))$ is decreasing) we obtain

$$E(\eta(s^*, g_\varepsilon(t))) \leq E(\eta(0, g_\varepsilon(t))) = E(g_\varepsilon(t)) < \gamma(c).$$

Otherwise let $E(g_\varepsilon(t)) \geq \gamma(c)$. We assume by contradiction that $E(\eta(s^*, g_\varepsilon(t))) \geq \gamma(c)$. Then $E(\eta(s, g_\varepsilon(t))) \geq \gamma(c)$ for all $s \in [0, s^*]$. Also, from (40),

$$E(\eta(s, g_\varepsilon(t))) \leq E(\eta(0, g_\varepsilon(t))) = E(g_\varepsilon(t)) \leq \gamma(c) + \varepsilon,$$

which implies

$$\left|E(\eta(s, g_\varepsilon(t))) - \gamma(c)\right| \leq \varepsilon \leq \bar{\mu}$$

for all $s \in [0, s^*]$. Therefore, together with (37), (38) and (41) it follows $\eta(s, g_\varepsilon(t)) \in \tilde{N}_\mu$ and $\|Y(\eta(s, g_\varepsilon(t)))\| \geq 2\bar{\mu}$ for all $s \in [0, s^*]$. From (35), (36) and chain rule it follows

$$\partial_s E(\eta(s, g_\varepsilon(t))) = \left\langle E'_{\eta(s, g_\varepsilon(t))} - \frac{Y(\eta(s, g_\varepsilon(t)))}{\|Y(\eta(s, g_\varepsilon(t)))\|_{H^1}}\right\rangle_{T^sS(c), TS(c)}.$$

Integrating both sides of the equation over $[0, s^*]$ and using (33) and (34), we obtain that

$$E(\eta(s^*, g_\varepsilon(t))) \leq E(g_\varepsilon(t)) - s^\varepsilon \leq \gamma(c) + \varepsilon - 2\varepsilon = \gamma(c) - \varepsilon,$$

but this is a contradiction to the assumption $E(\eta(s^*, g_\varepsilon(t))) \geq \gamma(c)$. Therefore (42) is valid, but this is again a contradiction to the definition of $\gamma(c)$. This completes the proof of the lemma.

Construction of $g_\varepsilon$. Let $u \in V(c)$ with $E(u) \leq \gamma(c) + \varepsilon$ (which is valid, since $\gamma(c) = \inf_{u \in V(c)} E(u)$) and let $0 < \lambda_1 < 1 < \lambda_2 < \infty$ be chosen such that $u^{\lambda_1} \in A_{\kappa_\varepsilon}, E(u^{\lambda_2}) < 0$ (see Lemma 3.1–7). We define $g_\varepsilon(t)$ by

$$g_\varepsilon(t) := u^{(1-t)\lambda_1 + t\lambda_2}.$$
From Lemma 3.2–4 and 3.2–6, it follows that

\[
\max_{t \in [0,1]} E(g_c(t)) \leq \gamma(c) + \varepsilon. \tag{43}
\]

Recall that

\[
E(u^t) = \frac{t^2}{2} A(u) + \frac{t^3}{2} B(u) + \frac{2}{5} t^{9/2} C(u),
\]
\[
Q(u^t) = t^2 A(u) + \frac{3t^3}{2} B(u) + \frac{9}{5} t^{9/2} C(u).
\]

Let \(m(t) = (1 - t)\lambda_1 + t\lambda_2\). We calculate

\[
\frac{d^2}{dt^2} E(g_c(t)) = (\lambda_2 - \lambda_1)^2 \left( A(u) + 3m(t) B(u) + \frac{63}{10} m(t)^{5/2} C(u) \right),
\]
\[
\frac{d^3}{dt^3} E(g_c(t)) = (\lambda_2 - \lambda_1)^2 \left( 3B(u) + \frac{63}{4} m(t)^{3/2} C(u) \right) < 0.
\]

Now let \(t_\varepsilon := \frac{1 - \lambda_1}{\lambda_2 - \lambda_1} \in (0, 1)\), so that \(m(t_\varepsilon) = 1\). Then

\[
\frac{d^2}{dt^2} E(g_c(t)) \bigg|_{t = t_\varepsilon} = (\lambda_2 - \lambda_1)^2 \left( A(u) + 3B(u) + \frac{63}{10} C(u) \right)
\]
\[
= (\lambda_2 - \lambda_1)^2 \left( Q(u) + \frac{3}{2} B(u) + \frac{9}{2} C(u) \right)
\]
\[
= (\lambda_2 - \lambda_1)^2 \left( \frac{3}{2} B(u) + \frac{9}{2} C(u) \right) := (\lambda_2 - \lambda_1)^2 (-\zeta < 0),
\]

since \(u \in V(c)\).

Now let \(t \in (0, 1)\) with \(E(g_c(t)) \geq \gamma(c)\). We first consider the case \(t = t_\varepsilon - h\) with \(h > 0\). Since \(u \in V(c)\), Lemma 3.2–1 implies \(\frac{d}{dt} E(g_c(t)) \bigg|_{t = t_\varepsilon} = 0\). Thus using a Taylor expansion we see that there exists some \(s \in [t, t_\varepsilon]\) such that

\[
\gamma(c) \leq E(g_c(t)) = E(g_c(t_\varepsilon)) + \frac{1}{2} h^2 \frac{d^2}{dt^2} E(g_c(t)) \bigg|_{t = t_\varepsilon} + \frac{1}{6} h^3 \frac{d^3}{dt^3} E(g_c(t)) \bigg|_{t = s}
\]
\[
\leq \gamma(c) + \varepsilon - \frac{h^2}{2} (\lambda_2 - \lambda_1)^2 \zeta - \frac{1}{6} h^3 \frac{d^3}{dt^3} E(g_c(t)) \bigg|_{t = s}. \tag{44}
\]

Now since \(h \in (0, t_\varepsilon) = \left( 0, \frac{1 - \lambda_1}{\lambda_2 - \lambda_1} \right)\), \((m(s) \in [\lambda_1, 1] \text{ and } B(u), C(u) \leq 0\) we infer that

\[
\left| -\frac{1}{6} h^3 \frac{d^3}{dt^3} E(g_c(t)) \bigg|_{t = s} \right| \leq \frac{1}{6} (1 - \lambda_1)^3 \left( -3B(u) - \frac{63}{4} C(u) \right)
\]
\[
\leq -\frac{1}{6} \left( 3B(u) + \frac{63}{4} C(u) \right) =: \zeta > 0.
\]

From (44) it follows

\[
h^2 \leq \frac{2(\varepsilon + \zeta)}{(\lambda_2 - \lambda_1)^2 \zeta}.
\]
Since $\lambda_2$ can be chosen arbitrary large, we pick a $\lambda_2$ with $(\lambda_2 - \lambda_1)^2 \geq \frac{2(\epsilon + \tilde{\epsilon})}{\epsilon \zeta}$, thus

$$0 < h \leq \epsilon.$$  

Next we deal with the case $t = t_\epsilon + h$ with $h > 0$: Doing a Taylor expansion as in (44) (notice the third order term in (44) has now positive sign) we obtain

$$\gamma(c) \leq E(g_\epsilon(t)) \leq \gamma(c) + \epsilon - \frac{h^2}{2}(\lambda_2 - \lambda_1)^2 \zeta.$$  

Thus if $(\lambda_2 - \lambda_1)^2 \geq \frac{2}{\epsilon \zeta}$, then $0 < h \leq \epsilon$. Therefore, we infer that picking $\lambda_2$ with

$$(\lambda_2 - \lambda_1)^2 = \max \left\{ \frac{2}{\epsilon \zeta}, 2(\epsilon + \tilde{\epsilon}) \right\},$$

implies

$$\{ t \in [0, 1] : E(g_\epsilon(t)) \geq \gamma(c) \} \subset (t_\epsilon - \epsilon, t_\epsilon + \epsilon).$$

Now, if $E(g_\epsilon(t)) \geq \gamma(c)$, then (13) implies that $|E(g_\epsilon(t)) - \gamma(c)| \leq \epsilon$ and for $\epsilon < \tilde{\mu}/2$ we get that $|E(g_\epsilon(t)) - \gamma(c)| < \tilde{\mu}/2$.

Moreover, for $\epsilon$ small enough, we get that $g_\epsilon(t) \in L$, where $L$ is given in (32). Thus $g_\epsilon(t) \in B(0, 2R_0)$.

Finally, from (15), we get that $(\lambda_2 - \lambda_1) \leq C/\sqrt{\epsilon}$, for some positive constant $C$. Let $v \in C_0^\infty(\mathbb{R}^3; \mathbb{C})$ such that $\|u - v\|_{H^1} \leq \lambda_2 \tilde{\mu}/4$. Since $v$ and its derivatives are Lipschitz continuous, there exists a constant $C_\tilde{\mu}$ such that

$$\|v^{t_1} - v^{t_2}\|_{H^1} \leq C_\tilde{\mu} |t_1 - t_2|, \text{ for all } t_1, t_2 \in [0, 1].$$

Due to the fact that the $L^2$-norm is invariant with respect to the scaling (3), that the $L^2$-norm of the gradient rescales 1-homogeneously (see (7)) and that $m(t_\epsilon) = 1$, we estimate

$$\text{dist}(g_\epsilon(t), V(c)) \leq \left\| g_\epsilon(t) - g_\epsilon(t_\epsilon) \right\|_{H^1} = \left\| u^{m(t)} - u^{m(t_\epsilon)} \right\|_{H^1} \leq \left\| u^{m(t)} - u^{m(t_\epsilon)} \right\|_{H^1} + \left\| u^{m(t_\epsilon)} - u^{m(t_\epsilon)} \right\|_{H^1} \leq 2 \max\{1, m(t)\} \|u - v\|_{H^1} + C_\tilde{\mu} |t - t_\epsilon| |\lambda_2 - \lambda_1| \leq \tilde{\mu} + C C_\tilde{\mu} \sqrt{\epsilon} \leq \tilde{\mu}$$

for $\epsilon \leq \frac{1}{4} \left( \frac{\tilde{\mu}}{C C_\tilde{\mu}} \right)^2$. All in all, we have shown (11). \[\blacksquare\]

**Proposition 3.15.** There exists a bounded Palais-Smale sequence $\{u_n\}_{n \in N}$ in $S(c)$, i.e., there exits a $H^1$-bounded sequence $\{u_n\}_{n \in N} \subset S(c)$ such that $Q(u_n) = o(1)$, $E(u_n) = \gamma(c) + o(1)$ and $\|E|_{T^*S(c)}(u_n)\|_{T^*S(c)} = o(1)$.

**Proof.** Due to Lemma 3.14 a bounded sequence $\{u_n\}_{n \in N} \subset S(c)$ which satisfies $E(u_n) = \gamma(c) + o(1)$ and $\|E|_{T^*S(c)}(u_n)\|_{T^*S(c)} = o(1)$ exists. In particular, $\text{dist}(u_n, V(c)) = o(1)$. It also holds that $\|dQ\|_{H^{-1}}$ is bounded on bounded sets of $H^1(\mathbb{R}^3; \mathbb{C})$. Now for $w \in V(c)$ we obtain that

$$Q(u_n) = Q(w) + dQ[a(u_n)] + (1 - a)w[(u_n - w)$$

$$dQ[a(u_n)] + (1 - a)w][u_n - w]$$

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for some $a \in [0, 1]$, since $Q(w) = 0$. Now choose $\{w_n\} \subset V(c)$ such that
\[
\|u_n - w_n\|_{H^1} \leq 2 \text{ dist } (u_n, V(c)).
\]
Since dist $(u_n, V(c)) = o(1)$, the sequence $\{w_n\}_{n\in \mathbb{N}}$ is bounded. Therefore
\[
|Q(u_n)| \leq \max_{a \in [0, 1]} \|dQ[au_n + (1 - a)w_n]\|_{H^{-1}} \|u_n - w_n\|_{H^1}
\leq C \text{ dist } (u_n, V(c)) = o(1).
\]
This completes the proof. □

### 3.3.2 Construction of a solution at the mountain pass level

In this section we finish the proof of Theorem 2.9. What is left to show, is that the localized Palais-Smale sequence constructed in the previous section converges to a solution of the equation (5), that this solution satisfies the side constraint and that its energy equals $\gamma(c)$. The first assertion is proven in Proposition 3.17 below; for its proof we need the following straightforward variant of [8, Lemma 3]:

**Lemma 3.16.** Let $H$ be a (real or complex) Hilbert space and $X$ a (real or complex) Banach space, such that $X \hookrightarrow H \cong H^* \hookrightarrow X^*$ with continuous embeddings and, without loss of generality, that $\|\cdot\|_H \leq \|\cdot\|_X$. Let $c > 0$ and define $M := \{u \in X : \|u\|^2_H = c\}$. Let $E: X \to \mathbb{R}$ be a $C^1$ functional and let $\{u_n\}_{n \in \mathbb{N}} \subset M$ be a bounded sequence in $X$. Then the following are equivalent:

1. $\|E'(M(u_n))\|_{T^*M} \to 0$

2. $E'(u_n) - c^{-1}\langle E'(u_n), u_n\rangle_{X^*, X} u_n \to 0$ in $X^*$.

**Proof.** The proof follows the lines of [8, Lemma 3] and is provided here for the sake of completeness: Fix a $u \in M$. Each $v \in X$ can be decomposed into $v = c^{-1}(u, v)_H u + p_u v$, where $p_u v \in T_u M$ is the projection of $v$ onto $T_u M$ with respect to the inner product in $H$, i.e., $p_u v := v - c^{-1}(u, v)_H u$. We have that $\|\langle v, w\rangle_H\| \leq \sqrt{c} \|v\|_H \leq \sqrt{c} \|v\|_X$, which implies
\[
\|p_u v\|_X \leq (1 + c^{-1/2} \|u\|_X) \|v\|_X.
\]
Define $J(v) := E'(v) - c^{-1}\langle E'(v), v\rangle_{X^*, X} v$ and note that $J(v) \in X^*$ and that for all $w \in T_u M$ holds that
\[
\langle J(u), w\rangle_{X^*, X} = \langle E'_M(u), w\rangle_{T^*M, TM},
\]
i.e., $\|E'_M(u)\|_{T^*M} \leq \|J(u)\|_{X^*}$. Since $u$ was arbitrary we obtain that $2. \implies 1$.

On the other hand, we have that
\[
\langle J(u), v\rangle_{X^*, X} = \langle E'(u), p_u v\rangle_{X^*, X} = \langle E'_M(u), p_u v\rangle_{T^*M, TM},
\]
since $p_u v \in T_u M$, which implies
\[
|\langle J(u_n), v\rangle_{X^*, X}| \leq \|E'_M(u_n)\|_{T^*M} (1 + c^{-1/2} \|u_n\|_X) \|v\|_X,
\]
that is, $1. \implies 2$. □
Proposition 3.17. Let \( \{u_n\}_{n \in \mathbb{N}} \subset S(c) \) be the bounded Palais-Smale sequence constructed in Proposition 3.16. Then there exist \( u \in H^1(\mathbb{R}^3, \mathbb{C}) \), \( \beta \in \mathbb{R} \) and a (sub)sequence \( \{\beta_n\} \subset \mathbb{R} \) such that:

1. \( u_n \rightharpoonup u \) in \( H^1(\mathbb{R}^3; \mathbb{C}) \).
2. \( \beta_n \to \beta \) in \( \mathbb{R} \).
3. \( -\frac{1}{2} \Delta u_n + \beta_n u_n + \lambda_1 |u_n|^2 u_n + 2(K \ast |u_n|^2) u_n \to 0 \) in \( H^{-1}(\mathbb{R}^3; \mathbb{C}) \).
4. \( -\frac{1}{2} \Delta u + \beta u + \lambda_1 |u|^2 u + 2(K \ast |u|^2) u \to 0 \) in \( H^{-1}(\mathbb{R}^3; \mathbb{C}) \).

Proof. Assertion 1. follows from standard arguments. Now set

\[
\beta_n := -\frac{\langle E'(u_n), u_n \rangle_{H^{-1}, H^1}}{\langle u_n, u_n \rangle_{H^{-1}, H^1}} = -\frac{1}{\|u_n\|_{H^1}^2} \left( \frac{1}{2} \|\nabla u_n\|_2^2 + \lambda_1 \|u_n\|_4^4 + \lambda_2 \int_{\mathbb{R}^3} (K \ast |u_n|^2) |u_n|^2 \, dx + \lambda_3 \|u_n\|_3^6 \right).
\]

The boundedness of \( \{\beta_n\}_{n \in \mathbb{N}} \) can be obtained from the Sobolev’s embedding theorem and the fact that \( u_n \in S(c) \), so that we obtain 2.

By construction we have that

\[
\langle E'(u_n), u_n \rangle_{H^{-1}, H^1} u_n = -\beta_n \|u_n\|_2^2 u_n = -c \beta_n u_n.
\] (46)

Due to Lemma 3.16

\[
\|E'_{S(c)}(u_n)\|_{T^*S(c)} \to 0 \iff E'(u_n) - c^{-1} \langle E'(u_n), u_n \rangle_{H^{-1}, H^1} u_n \to 0 \text{ in } H^{-1}(\mathbb{R}^3; \mathbb{C}),
\]

so that, together with Proposition 3.15 and equation (46) we get that

\[
E'(u_n) + \beta_n u_n \to 0 \text{ in } H^{-1}(\mathbb{R}^3; \mathbb{C}),
\]

i.e., we have proven 3.

Now 4. follows immediately from 2. and 3.

Finally, to show 5., we note that assertion 4. implies that \( \langle E(u_n) + \beta u_n, v \rangle_{H^{-1}, H^1} \to 0 \) for all \( v \in H^1(\mathbb{R}^3; \mathbb{C}) \). We will show that \( \langle E(u_n) + \beta u_n, v \rangle_{H^{-1}, H^1} \to \langle E(u) + \beta u, v \rangle_{H^{-1}, H^1} \) for all \( v \in H^1(\mathbb{R}^3; \mathbb{C}) \): The weak convergence of the first two summands of the expression in 4. follows from the weak convergence of \( u_n \) to \( u \) in \( H^1(\mathbb{R}^3; \mathbb{C}) \). Concerning the rest three summands, it suffices to show the convergence of the last one; the weak convergence of the third and fourth one can be shown analogously. Notice that 1. implies the weak convergence \( u_n \rightharpoonup u \) in \( H^1(B(0, R), \mathbb{C}) \) for any ball \( B(0, R) \subset \mathbb{R}^3 \) with arbitrary radius \( R > 0 \). But from Rellich’s compact embedding theorem we obtain in fact the strong convergence \( u_n \to u \) in \( L^p(B(0, R), \mathbb{C}) \) for any \( p \in [1, 6) \). Therefore we obtain \( |u_n|^3 u_n \to |u|^3 u \) strongly in \( L^{5/4}(B(0, R), \mathbb{C}) \). In particular, since \( R > 0 \) was arbitrary, we conclude that

\[
\int_{\mathbb{R}^3} |u_n|^3 u_n \varphi \, dx \to \int_{\mathbb{R}^3} |u|^3 u \varphi \, dx
\]

for all \( \varphi \in C_c^\infty(\mathbb{R}^3; \mathbb{C}) \) and, by density, for all \( \varphi \in H^1(\mathbb{R}^3; \mathbb{C}) \). This completes the proof. \( \blacksquare \)
The next lemma ensures that the limit function $u$ that was found in Proposition 3.17 is nontrivial.

**Lemma 3.18.** Let $\{u_n\}_{n \in \mathbb{N}} \subset S(c)$ be a bounded sequence in $H^1(\mathbb{R}^3; \mathbb{C})$, with $Q(u_n) = o(1)$ and $E(u_n) \to \gamma(c) > 0$. Then up to a subsequence and up to translation, there exists $u \in H^1(\mathbb{R}^3; \mathbb{C})$ with $u \neq 0$ and $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3; \mathbb{C})$.

**Proof.** If either $B(u_n)$ or $C(u_n)$ is not equal to $o(1)$, then the claim follows from the Gagliardo-Nirenberg inequality, the pqr-lemma (see e.g. [23, Lemma 2.1]) and the Lieb translation [22, Lemma 6]. Otherwise $B(u_n) = o(1)$ and $C(u_n) = o(1)$. Recall that

$$E(u) = \frac{1}{2} A(u) + \frac{1}{2} B(u) + \frac{2}{5} C(u),$$

$$Q(u) = A(u) + \frac{3}{2} B(u) + \frac{9}{5} C(u).$$

It follows

$$E(u_n) - \frac{1}{2} Q(u_n) = -\frac{1}{4} B(u_n) - \frac{1}{2} C(u_n).$$

Thus $E(u_n) = o(1)$, contradiction to $E(u_n) \to \gamma(c) > 0$. This completes the proof. ■

**Remark 3.19.** Due to the proof of Proposition 3.17, one can identify the function $u$ from Lemma 3.18 with the one from Proposition 3.17.

Before we can finish the proof of the theorem we need a couple of technical tools.

**Lemma 3.20.** Let

$$f(a, b, c) := \max_{t > 0} \{at^2 - bt^3 - ct^{9/2}\}$$

for $a > 0$, $b \geq 0$, $c \geq 0$ and $(b, c) \neq (0, 0)$. Then $f$ is continuous in $(0, \infty) \times [0, \infty) \times [0, \infty) \setminus (0, \infty) \times \{0\} \times \{0\}$.

**Proof.** Let $g(a, b, c, t) = at^2 - bt^3 - ct^{9/2}$. Then

$$\partial_t g(a, b, c, t) = 2at - 3bt^2 - \frac{9}{2}t^{7/2},$$

$$\partial_{tt} g(a, b, c, t) = 2a - 6bt - \frac{63}{4} t^{5/2}.$$

One directly verifies that for each $a_0 > 0$, $b_0 \geq 0$, $c_0 \geq 0$ and $(b_0, c_0) \neq (0, 0)$ there exists a unique $t_0 > 0$ such that $\partial_t g(a_0, b_0, c_0, t_0) = 0$ and $\partial_{tt} g(a_0, b_0, c_0, t_0) < 0$. Then using the implicit function theorem as in [7, Lemma 5.2] we obtain the result. ■

**Lemma 3.21.** The function $c \mapsto \gamma(c)$ is non-increasing for $c > 0$.

**Proof.** Let $0 < c_1 < c_2$. To show the claim it suffices to show that for arbitrary $\varepsilon > 0$ we have

$$\gamma(c_2) \leq \gamma(c_1) + \varepsilon.$$

For any $u_1 \in V(c_1)$, one obtains from Lemma 3.20 that

$$E(u_1) = \max_{t > 0} E(u_1^t).$$
Moreover, from Lemma 3.12 we can find a $u_1 \in V(c_1)$ such that $E(u_1) \leq \gamma(c_1) + \varepsilon/2$. Let $\eta \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function with $\eta(x) = 1$ for $|x| \leq 1$, $\eta(x) = 0$ for $|x| \geq 2$ and $\eta \in [0,1]$ for $|x| \in (1,2)$. For $\delta > 0$, define

$$\tilde{u}_{1,\delta}(x) := \eta(\delta x) \cdot u_1(x).$$

Then $\tilde{u}_{1,\delta} \to u_1$ in $H^1(\mathbb{R}^3; \mathbb{C})$ as $\delta \to 0$. Therefore,

$$A(\tilde{u}_{1,\delta}) \to A(u_1),$$
$$B(\tilde{u}_{1,\delta}) \to B(u_1),$$
$$C(\tilde{u}_{1,\delta}) \to C(u_1)$$

as $\delta \to 0$. Using the continuity property provided by Lemma 3.20, we conclude that there exists sufficiently small $\delta > 0$ such that

$$\max_{t>0} E(\tilde{u}_{1,\delta}^t) = \max_{t>0} \left\{ \frac{t^2}{2} A(\tilde{u}_{1,\delta}) + \frac{t^3}{2} B(\tilde{u}_{1,\delta}) + \frac{2}{5} t^{9/2} C(\tilde{u}_{1,\delta}) \right\}$$

$$\leq \max_{t>0} \left\{ \frac{t^2}{2} A(u_1) + \frac{t^3}{2} B(u_1) + \frac{2}{5} t^{9/2} C(u_1) \right\} + \frac{\varepsilon}{4}$$

$$= \max_{t>0} E(u_1^t) + \frac{\varepsilon}{4}.$$ 

Now let $v \in C_0^\infty(\mathbb{R}^3)$ with $\text{supp}(v) \subset \mathbb{R}^3 \setminus B(0,2/\delta)$ and define

$$v_0 := c_2 - \frac{\|\tilde{u}_{1,\delta}\|^2}{\|v\|^2} \cdot v.$$ 

We have $\|v_0\|^2 = c_2 - \|\tilde{u}_{1,\delta}\|^2$ and for $\lambda \in (0,1)$,

$$A(v_0^\lambda) = \lambda^2 A(v_0),$$
$$B(v_0^\lambda) = \lambda^3 B(v_0),$$
$$C(v_0^\lambda) = \lambda^{9/2} C(v_0).$$

Let $w_\lambda := \tilde{u}_{1,\delta} + v_0^\lambda$. Then $\text{supp} \tilde{u}_{1,\delta} \cap \text{supp} v_0^\lambda = \emptyset$. Thus

$$\|w_\lambda\|_2 = \|	ilde{u}_{1,\delta}\|_2 + \|v_0^\lambda\|_2,$$

$$A(w_\lambda) = A(\tilde{u}_{1,\delta}) + A(v_0^\lambda),$$

$$C(w_\lambda) = C(\tilde{u}_{1,\delta}) + C(v_0^\lambda),$$

and the first equation above implies that $w_\lambda \in S(c_2)$. From Plancherel’s identity, Hölder’s inequality and the boundedness of $\hat{K}$, we also obtain

$$|B(w_\lambda) - B(\tilde{u}_{1,\delta}) - B(v_0^\lambda)|$$

$$\leq C \int_{\mathbb{R}^3} \left( \|\tilde{u}_{1,\delta}\|_2 \|v_0^\lambda\|_2^2 + \|\tilde{u}_{1,\delta}\|_3 \|v_0^\lambda\|_3 + \|\tilde{u}_{1,\delta}\|_3 \|v_0^\lambda\|_9/2 \right)$$

$$= C \left( \lambda^{3/2} \|	ilde{u}_{1,\delta}\|_2 \|v_0\|_4^2 + \lambda^{1/2} \|	ilde{u}_{1,\delta}\|_3 \|v_0\|_3 + \lambda^{5/2} \|	ilde{u}_{1,\delta}\|_9/2 \right) \to 0$$
as \( \lambda \to 0 \). To sum up,
\[
A(w_\lambda) \to A(\tilde{u}_{1,\delta}), \\
B(w_\lambda) \to B(\tilde{u}_{1,\delta}), \\
C(w_\lambda) \to C(\tilde{u}_{1,\delta})
\]
as \( \lambda \to 0 \). Again using Lemma 3.20 we get that
\[
\max_{t>0} E(w^t_\lambda) \leq \max_{t>0} E(\tilde{u}^t_{1,\delta}) + \frac{\varepsilon}{4}
\]
for sufficiently small \( \lambda > 0 \). Finally, we calculate
\[
\gamma(c_2) \leq \max_{t>0} E(w^t_\lambda) \leq \max_{t>0} E(\tilde{u}^t_{1,\delta}) + \frac{\varepsilon}{4} \leq \max_{t>0} E(u^t_1) + \frac{\varepsilon}{2} = E(u_1) + \frac{\varepsilon}{2} \leq \gamma(c_1) + \varepsilon,
\]
which completes the proof. ■

With the next proposition we can finalize the proof of Theorem 2.9.

**Proposition 3.22.** For an arbitrary \( c > 0 \), let \( \beta \in \mathbb{R} \) and \( u \in H^1(\mathbb{R}^3; \mathbb{C}) \) be given from Proposition 3.17. Then \((u, \beta) \in S(c) \times (0, \infty) \) and \( E(u) = \gamma(c) \).

**Proof.** First note that due to Lemma 3.18 we can assume that \( u \not\equiv 0 \). Truncating outside balls of increasingly large radius, applying Rellich’s compact embedding theorem and deploying a diagonal argument we obtain a subsequence of \( \{u_n\}_{n \in \mathbb{N}} \) such that \( u_n \) converges to \( u \) almost everywhere. Then the splitting property given by [10, Theorem 1] implies that
\[
B(u_n - u) + B(u) = B(u_n) + o(1), \\
C(u_n - u) + C(u) = C(u_n) + o(1), \\
D(u_n - u) + D(u) = D(u_n) + o(1),
\]
where \( D(v) := \|v\|_2^2 \). For \( a, b \) elements of an arbitrary Hilbert space \( H \), we have
\[
(a - b, a - b)_H + (b, b)_H = (a, a)_H + (b, b - a)_H + (b - a, b)_H.
\]
If \( a = b_n \to b \), it follows that
\[
(b_n - b, b_n - b)_H + (b, b)_H = (b_n, b_n)_H + o(1).
\]
Thus we also have
\[
A(u_n - u) + A(u) = A(u_n) + o(1).
\]
Moreover \( E(u) = \frac{1}{2} A(u) + \frac{1}{2} B(u) + \frac{2}{3} C(u) \) implies that
\[
E(u_n - u) + E(u) = E(u_n) + o(1).
\]
Since \( u \not\equiv 0 \), the lower semi continuity of the \( L^2 \)-norm implies that \( u \in V(c_1) \) for some \( c_1 \in (0, c] \). It holds that \( \gamma(c_1) = \inf_{u \in V(c_1)} E(u) \) and \( \gamma(c) = E(u_n) + o(1) \). We infer that
\[
E(u_n - u) + \gamma(c_1) \leq \gamma(c) + o(1).
\] (47)
On the other hand, recall that

$$E(v) - \frac{1}{2} Q(v) = -\frac{1}{4} B(v) - \frac{1}{2} C(v)$$  \hspace{1cm} (48)$$

for all $v \in H^1(\mathbb{R}^3; \mathbb{C})$. Since $u \in V(c_1)$ it holds that $Q(u) = 0$. Thus

$$Q(u_n - u) = Q(u_n - u) + Q(u) = Q(u_n) + o(1) = o(1).$$

Inserting this into (48), we conclude that $E(u_n - u) \geq o(1)$, since the right-hand side of (48) is always non-negative. From Lemma 3.21 we know that $\gamma(c_1) \geq \gamma(c)$, therefore it follows from (47) that $E(u_n - u) \leq o(1)$. Thus $E(u_n - u) = o(1)$. From this and (48) we infer that

$$B(u_n - u) = o(1),$$

$$C(u_n - u) = o(1).$$

But then since $E(u_n - u)$ is a linear combination of $A(u_n - u), B(u_n - u)$ and $C(u_n - u)$, it follows also $A(u_n - u) = o(1)$. Now the Proposition 3.17–4. and 3.17–5. imply

$$\frac{1}{2} A(u_n) + \beta D(u_n) + B(u_n) + C(u_n) = \frac{1}{2} A(u) + \beta D(u) + B(u) + C(u) + o(1).$$

Using the previous splitting properties one has that

$$\beta D(u_n) = \beta D(u) + o(1)$$

$$= \beta\left(-D(u_n - u) + D(u_n)\right) + o(1).$$  \hspace{1cm} (49)$$

From this we infer that $\beta D(u_n - u) = o(1)$. But from Proposition 3.10, $\beta > 0$. Thus $D(u_n - u) = o(1)$ and $A(u_n - u) = o(1)$, which means $u_n \rightarrow u$ in $H^1(\mathbb{R}^3; \mathbb{C})$ and $u \in S(c)$. From the splitting property of $E$ and the fact that $E(u_n - u) = o(1)$, $E(u_n) = \gamma(c) + o(1)$ conclude that $E(u) = \gamma(c)$. \hfill \blacksquare

Now we can conclude the proof of Theorem 2.11.

**Proof of Theorem 2.11.** From Proposition 3.22 we obtain a solution $(u, \beta)$ with $E(u) = \gamma(c)$. Theorem 2.9–3. yields that it is a ground state. We conclude the proof using Theorem 2.10 with $u_c := |u|$. \hfill \blacksquare

**References**

[1] H. Abels. *Pseudodifferential and singular integral operators*. De Gruyter Graduate Lectures. De Gruyter, Berlin, 2012. An introduction with applications.

[2] R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, tensor analysis, and applications*, volume 75 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1988.

[3] P. Antonelli and C. Sparber. Existence of solitary waves in dipolar quantum gases. *Physica D: Nonlinear Phenomena*, 240(4):426 – 431, 2011.

[4] D. Baillie, R. M. Wilson, R. N. Bisset, and P. B. Blakie. Self-bound dipolar droplet: A localized matter wave in free space. *Phys. Rev. A*, 94:021602, Aug 2016.
[5] J. Bellazzini, N. Boussaïd, L. Jeanjean, and N. Visciglia. Existence and stability of standing waves for supercritical NLS with a partial confinement. *Comm. Math. Phys.*, 353(1):229–251, 2017.

[6] J. Bellazzini and L. Jeanjean. On dipolar quantum gases in the unstable regime. *SIAM J. Math. Anal.*, 48(3):2028–2058, 2016.

[7] J. Bellazzini, L. Jeanjean, and T. Luo. Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson equations. *Proc. Lond. Math. Soc. (3)*, 107(2):303–339, 2013.

[8] H. Berestycki and P.-L. Lions. Nonlinear scalar field equations. II. Existence of infinitely many solutions. *Arch. Rational Mech. Anal.*, 82(4):347–375, 1983.

[9] R. N. Bisset, R. M. Wilson, D. Baillie, and P. B. Blakie. Ground-state phase diagram of a dipolar condensate with quantum fluctuations. *Phys. Rev. A*, 94:033619, Sep 2016.

[10] H. Brézis and E. Lieb. A relation between pointwise convergence of functions and convergence of functionals. *Proc. Amer. Math. Soc.*, 88(3):486–490, 1983.

[11] R. Carles, P. A. Markowich, and C. Sparber. On the Gross-Pitaevskii equation for trapped dipolar quantum gases. *Nonlinearity*, 21(11):2569–2590, 2008.

[12] T. Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[13] R. Cipolatti. On the existence of standing waves for a Davey-Stewartson system. *Comm. Partial Differential Equations*, 17(5-6):967–988, 1992.

[14] A. de Laire. Nonexistence of traveling waves for a nonlocal Gross-Pitaevskii equation. *Indiana Univ. Math. J.*, 61(4):1451–1484, 2012.

[15] I. Ferrier-Barbut, H. Kadau, M. Schmitt, M. Wenzel, and T. Pfau. Observation of quantum droplets in a strongly dipolar bose gas. *Phys. Rev. Lett.*, 116:215301, May 2016.

[16] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[17] M. D. Groves, D. J. B. Lloyd, and A. Stylianou. Pattern formation on the free surface of a ferrofluid: Spatial dynamics and homoclinic bifurcation. *Physica D: Nonlinear Phenomena*, 350:1–12, 2017.

[18] H. Kadau, M. Schmitt, M. Wenzel, C. Wink, T. Maier, I. Ferrier-Barbut, and T. Pfau. Observing the Rosensweig instability of a quantum ferrofluid. *Nature*, 530(7589):194–197, Feb 2016. Letter.

[19] O. Kavian. *Introduction à la théorie des points critiques et applications aux problèmes elliptiques*, volume 13 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Paris, 1993.
[20] P. G. Kevrekidis, D. J. Frantzeskakis, and R. Carretero-González. *The defocusing nonlinear Schrödinger equation*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2015. From dark solitons to vortices and vortex rings.

[21] T. Lahaye, C. Menotti, L. Santos, M. Lewenstein, and T. Pfau. The physics of dipolar bosonic quantum gases. *Reports on Progress in Physics*, 72(12):126401, 2009.

[22] E. H. Lieb. On the lowest eigenvalue of the Laplacian for the intersection of two domains. *Invent. Math.*, 74(3):441–448, 1983.

[23] E. H. Lieb and M. Loss. Stability of Coulomb systems with magnetic fields. II. The many-electron atom and the one-electron molecule. *Comm. Math. Phys.*, 104(2):271–282, 1986.

[24] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.

[25] A. R. P. Lima and A. Pelster. Beyond mean-field low-lying excitations of dipolar bose gases. *Phys. Rev. A*, 86:063609, Dec 2012.

[26] E. Parini and A. Stylianou. A free boundary approach to the Rosensweig instability of ferrofluids. *Z. Angew. Math. Phys.*, 69(2):32, Feb 2018.

[27] R. Richter and A. Lange. Surface instabilities of ferrofluids. In S. Odenbach, editor, *Colloidal Magnetic Fluids*, volume 763 of *Lecture Notes in Physics*, pages 1–91. Springer Berlin Heidelberg, 2009.

[28] M. Schmitt, M. Wenzel, F. Böttcher, I. Ferrier-Barbut, and T. Pfau. Self-bound droplets of a dilute magnetic quantum liquid. *Nature*, 539:259–262, Nov 2016.

[29] M. E. Taylor. *Partial differential equations III. Nonlinear equations*, volume 117 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2011.

[30] M. Wenzel, F. Böttcher, T. Langen, I. Ferrier-Barbut, and T. Pfau. Striped states in a many-body system of tilted dipoles. *Phys. Rev. A*, 96:053630, Nov 2017.