Rigorous Analysis of Renormalization Group Pathologies in the 4-State Clock Model

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Abstract

We perform an exact renormalization-group analysis of one-dimensional 4-state clock models with complex interactions. Our aim is to provide a simple explicit illustration of the behavior of the renormalization-group flow in a system exhibiting a rich phase diagram. In particular we study the flow in the vicinity of phase transitions with a first-order character, a matter that has been controversial for years. We observe that the flow is continuous and single-valued, even on the phase transition surface, provided that the renormalized Hamiltonian exist. The characteristics of such a flow are in agreement with the Nienhuis-Nauenberg standard scenario, and in disagreement with the “discontinuity scenario” proposed by some authors and recently disproved by van Enter, Fernández and Sokal for a large class of models (with real interactions). However, there are some points in the space of interactions for which a renormalized Hamiltonian cannot be defined. This pathological behavior is similar, and in some sense complementary, to the one pointed out by Griffiths, Pearce and Israel for Ising models. We explicitly see that if the transformation is truncated so as to preserve a Hamiltonian description, the resulting flow becomes discontinuous and multivalued at some of these points. This suggests a possible explanation for the numerical results that motivated the “discontinuity scenario”.

Keywords: Renormalization Group, Phase Transitions, Clock Models, Non-Gibbsian measures.

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1 Introduction

One of the major features of the Renormalization Group (RG) theory is that it makes possible the description of the singular critical behavior associated to second-order phase transitions in terms of smooth RG transformations on a suitable space of Hamiltonians. By suitable we mean local (in some mathematical sense that must be specified) and translational-invariant Hamiltonians. The thermodynamic singularities associated with critical phenomena arise after an infinite number of RG steps, in the vicinity of a RG fixed point [1, 2].

The smoothness of the RG flow is also expected to hold for other values of the coupling constants. In particular, it was conjectured by Nienhuis and Nauenberg [3] that it can also be smooth around points where first-order phase transitions occur. In their picture (standard scenario) these points are in the domain of attraction of certain fixed points—called discontinuity fixed points—that govern the behavior of the system on the coexistence manifold. These fixed points have relevant directions whose critical exponents are equal to the dimensionality of the model $\gamma = D$. The singularities associated to first-order phase transitions are obtained by an infinite iteration of RG transformations around the discontinuity fixed points [3, 4, 5], in complete analogy with the case of second-order phase transitions.

On the other hand different authors [6, 7, 8, 9, 10] have exhibited numerical evidence and arguments indicating that, in disagreement with the picture advocated by the standard scenario, the RG transformation could be discontinuous at the transition points and could associate different renormalized Hamiltonians to the different phases coexisting at the transition (multivaluedness).

Recently, van Enter-Fernández-Sokal [11, 12] have rigorously shown that the second picture cannot hold for a large class of systems. For classical variables taking values on a compact manifold with real-valued local interactions (e.g. absolutely summable Hamiltonians), they prove that—whenever defined—a renormalization-group map associated to a local (real space) RG prescription is single-valued and continuous on a suitable space of Hamiltonians. However, they have also shown that at—or in the vicinity of—a transition surface, a renormalized local Hamiltonian may not exist at all. This pathology is called non-Gibbsianness, and was previously pointed out by Israel [13] and, in some sense, previously by Griffiths and Pearce [14]. Its occurrence has been rigorously verified [11, 12] for the Ising model in dimensions $D \geq 2$ at sufficient low temperature for some RG prescriptions (wich include decimation, some cases of majority rule, block-averaging and Kadanoff transformations).

These results suggest that the discontinuities observed in Refs. [6, 7, 8] are in fact an artifact of the truncation of the renormalization scheme. If the renormalized local Hamiltonian exists, the size of the observed discontinuity should decrease for smaller truncations. On the other hand, it is plausible that the numerical manifestation of non-Gibbsianness be phase-dependent yielding an apparent discontinuity that persists (possibly acquiring an oscillatory character) for successively smaller truncations [12]. In relation to the validity of these explanations we mention the work of Ref. [13], where truncation errors were estimated for Monte Carlo RG calculations of the two-dimensional Ising model below the critical
temperature. They were found to be of the same order of magnitude as the observed discontinuity.

The observations of the preceding paragraph do not apply to the results of Ref. [10], which are not obtained via a truncation scheme. However, the method used there relies on the hypothesis that there is only one renormalized trajectory flowing away from the discontinuity fixed point [16]. The reported discontinuity of the renormalization flow could hence be a consequence of the fact that this hypothesis is not valid for the two-dimensional Ising model which has two relevant operators: one associated with the temperature ($= 1/\beta$) and the other with an external magnetic field [4, 15].

In this paper we intend to clarify further the above issues by analyzing very simple one-dimensional models that on the one hand exhibit a rich phase diagram and on the other hand admit a RG scheme that can be exactly computed. The models belong to a family of one-dimensional $q$-state clock models with complex-valued Hamiltonians previously introduced by some of us [17]. These models were intended as simplifications of quantum spin models of present interest in which either the complex interactions are present *ab initio* or they appear in the effective Hamiltonian upon integration of fermionic degrees of freedom. Complex couplings arise, for instance, in the classical spin model associated to a quantum Heisenberg chain [18, 19, 20], the effective model for the quantum Hall effect considered in [21, 22], and the chiral Potts model which has been a recent focus of interest [23, 24] and is not unrelated to the model we analyze.

On the other hand, the presence of complex interactions introduces very interesting modifications in the beautiful picture developed for traditional (real-interaction) statistical mechanics—pure phases, ergodic decomposition of states, independence of the boundary conditions for the thermodynamic potentials [25, 26]. This could merit further analysis, given the role played by complex interactions [27, 28, 29, 30, 31] in the study of several phenomena addressed by equilibrium statistical mechanics: Lee-Yang singularity [27, 28, 29, 30], metastability effects [22, 23], high-$T$ analyticity [34, 35, 36] and low-$T$ smoothness [32] of the free energy, and deformations of the phase diagram at low temperature [33]. An important change incorporated by complex interactions is, of course, the enrichment of the phase diagram: the nearest-neighbor one-dimensional models studied here do present phase transitions, a feature totally absent in real-interaction models of comparable simplicity [37].

The models with complex interactions that we consider here have several desirable properties [17]: They possess a Hermitean transfer matrix which is positive-definite for a wide range of coupling constants that includes phase transition points. This last property makes these models useful to generate unitary quantum-mechanical systems in the continuum limit [38, 39]. Although the models have complex interactions the free and internal energies are real for periodic boundary conditions. In addition, these models present a nontrivial phase diagram characterized by a manifold of points where the leading eigenvalues of the transfer matrix cross. At these transition points, the energy density is in general discontinuous, so the transition can be catalogued as first-order; but at the same time for many values of the couplings constants the correlation length diverges with a critical exponent $\nu = 1$. Therefore, the surface formed by these points is also a critical surface. On the other hand, inside the transition manifolds there run
curves where the thermodynamic behavior is, in some sense, even more singular. They correspond to the points of the phase diagram where the partition function is zero for (a sequence of) arbitrarily large volumes, and suitable boundary conditions. For lack of better or established nomenclature, we shall call these points—which have no counterpart in the phase diagrams of real interactions—“Lee-Yang-type” (LYT) points. This type of singularities is familiar to people studying metastability [32, 33]. In principle, they correspond to singularities for the finite-volume free energies and its existence is not enough to infer the non-analyticity of the infinite-volume free energy—it only rules out the usual analyticity proof pioneered by Lee and Yang [27]. For real interactions, the question of whether there is a singularity in the infinite-volume limit is related to the possibility of analytically continuing the (infinite-volume) free energy in the presence of metastable states [32]. In our model, however, the LYT singularities do have infinite-volume consequences: they belong to transition surfaces and in addition, as we shall see, the free energy presents some further singular behavior at these points.

We believe that all these attributes make the clock models a useful laboratory for the study of RG flows. Since the models are one-dimensional and have nearest-neighbor interactions the natural candidate for the RG prescription is decimation. Such transformations have been used for many one-dimensional Ising models with real local couplings, see e.g. [40]. However, these were models without first-order transition points. The first application to a model with this type of transitions has been presented recently [12]. The flow of the decimation transformation is comparatively simple for these models: it involves a finite number of parameters, so it can be computed exactly without any truncation approximation. We present here the results for the 4-state clock model subjected to decimations with blocks of size even. The calculations reveal a number of instructive features which, we think, bear some light on present controversies on the properties of RG transformations. Let us summarize our main observations:

1) Our results are in agreement with the standard scenario of Nienhuis and Nauenberg and the van Enter-Fernández-Sokal theorems (even when, rigorously speaking, our model does not satisfy the hypothesis of these theorems because it has complex interactions): Whenever defined, the flow is continuous and the behavior of the model on the surface of first-order transitions is determined by (three) discontinuity fixed points lying on this surface. One of these fixed points has two relevant directions characterized by the same critical exponent $\gamma = D = 1$, as predicted by the standard scenario. The other two discontinuity fixed points are non-Gibbsian, and hence the flow is singular.

2) In some regions of the phase diagram we observe pathologies similar to those pointed out by Griffiths, Pearce and Israel [13, 14]: Already for the first renormalization step the renormalized Hamiltonian fails to exist. The reason for this non-Gibbsianness is, however, different from the one observed for real interactions. While for the latter the renormalized measure fails to be (quasi)local [11, 12, 13]—roughly speaking it looks as if the coupling constants proliferate in an uncontrollable manner—in the present case the renormalized measure gives zero weight to (open) sets of configurations. This corresponds to

$5^5$Simple example: The singular functions $f_n(x) = \ln \left(1 - \frac{ix}{n}\right)$ converge to the perfectly analytical zero function. We thank Alan Sokal for clarifications regarding this point.
one of the coupling constants attaining the value $+\infty$. In a more abstract language, the renormalized measure fails to satisfy uniform non-nullness. Such a pathology cannot happen in the renormalization of real interactions, except if the renormalization prescription itself excludes some configurations. Moreover, for real interactions quasi-locality and uniform non-nullness are necessary and sufficient conditions for a measure to be Gibbsian \[12\] (in the complex case the matter is much more involved). In this sense, we can say that the pathologies observed in the present example are complementary to the ones discussed in Refs. \[11, 12, 13, 14\]. We observe that if at some of these points the renormalized Hamiltonian is “truncated” by ignoring the coupling constant that acquires an infinite value, the resulting transformation becomes discontinuous and multivalued. In this sense we could say that in this example the non-Gibbsianness produces a “discontinuity scenario” as a result of truncation.

3) We can distinguish two different “degrees” of pathological behavior. In most of the pathological points, the non-Gibbsianness brought by the RG transformation is “recoverable”: a further iteration of the RG transformation restores the Gibbsianness, and this Gibbsianness is preserved under additional iterations. Equivalently, the points are pathological for one particular even-block decimation scheme but not pathological for all the others. In constrast, at the points LYT—and only there—the non-Gibbsianness is not restored by a further renormalization. The points LYT are, therefore, pathological for all even-block decimation schemes. We observe that the LYT singularities correspond to points where all the eigenvalues of the transfer matrix are doubly degenerated (in absolute value). While we do not fully understand the meaning of this observation, we remark that a similar degeneracy seems to be present in the critical curve of the two-dimensional Ising model \[43\].

4) The RG flow exhibits the following characteristics: It involves two families of models, both of them formed by 4-state clock models parametrized by three real numbers denoted $J$, $J_1$ and $\varepsilon$, but differing in the value of a discrete parameter $m$ that gives an imaginary part to the couplings. One family is a (transfer-matrix-Hermiteanness-preserving) complex extension of the other. For each decimation prescription, each family can be divided in two sections: an open region of the parameter space that we call the black-hole section, and its complement, the non-black-hole section. The RG transformation maps models in the non-black-hole section into models of the same family, and models in the black-hole section into models of the other family. The points where the RG transformation has a “recoverable” pathology are precisely those of the boundary between black-hole and non-black-hole sections that are not LYT singularities. These boundaries change with the decimation scheme; the only points common to all of them are curves of LYT singularities.

5) Each family of models exhibits seven different fixed points (plus periodic repetitions), but three of them attract the majority of the models with finite couplings: a “high-temperature” (zero correlation length) fixed point, and the two non-Gibbsian discontinuity fixed points located on the critical surface. Models outside the critical surface are attracted by the high-temperature fixed point—of the same or the other family, depending on which side of the critical surface the initial models are. On the other hand, models on the critical surface are attracted by the non-Gibbsian critical fixed points of the same
or the other family, depending on whether initially the models are inside or outside the black-hole region. There is only a curve of points inside the critical surface that are attracted to the Nienhuis-Nauenberg fixed point mentioned above. Furthermore, we can explain the “terminal” pathologies affecting the points LYT in terms of strongly attracting invariant planes of points: By a single renormalization transformation the points LYT are mapped either into a non-Gibbsian critical fixed point or into an invariant plane of non-Gibbsian models.

The paper is organized as follows. In Section 2 we define the 4-state clock model and study its phase structure in detail. The RG analysis is mainly explained in Section 3. We see that in some regions of the interaction space an analytic continuation of the RG equations is needed (passing to another Riemann sheet). This is performed in Section 4 and consists in adding an additional parameter to the Hamiltonian. The pathologies observed are discussed in Section 5. In Section 6 we present some final comments.

2 Phase Structure of the 4-State Clock Model

A one-dimensional $q$-state clock model is defined by a classical variable $\tilde{s}_n$ ("spin") fluctuating among the $q$ roots of the unity

$$\tilde{s}_n = \left( \cos \left( \frac{2\pi p_n}{q} \right), \sin \left( \frac{2\pi p_n}{q} \right) \right); \quad p_n = 0, \ldots, q - 1$$

(1)

on the sites $n$ of a one-dimensional chain. We consider the Hamiltonian

$$-\beta H = \sum_{n=1}^{N} \sum_{r=1}^{\lfloor \frac{q}{2} \rfloor} \left\{ J_{r-1} \cos \left( \frac{2\pi}{q} (p_n - p_{n+1}) \right) + i \varepsilon_r \sin \left( \frac{2\pi}{q} (p_n - p_{n+1}) \right) \right\},$$

(2)

with $J_r$ and $\varepsilon_r$ real parameters, and where $\lfloor q/2 \rfloor$ denotes the integer part of $q/2$. This type of Hamiltonian is a natural generalization of the Hamiltonians studied in [17], with additional interactions introduced so to have a system of couplings closed under decimation transformations, that is, with as many independent couplings (including the free energy normalization) as RG equations [41]. In the case $q = 4$ the Hamiltonian is

$$-\beta H = \sum_{n=1}^{N} \left\{ J \cos \left( \frac{\pi}{2} (p_n - p_{n+1}) \right) + J_1 \cos(\pi(p_n - p_{n+1})) \right.$$  

$$+ i \varepsilon \sin \left( \frac{\pi}{2} (p_n - p_{n+1}) \right) \right\}. \quad (3)$$

(We have written $J, \varepsilon$ instead of $J_0, \varepsilon_1$ to conform with previous works [17, 41].) The transfer matrix of this model reads

$$T = \begin{pmatrix} e^{J+J_1} & e^{-i\varepsilon-J_1} & e^{-J+J_1} & e^{i\varepsilon-J_1} \\
 e^{i\varepsilon-J_1} & e^{J+J_1} & e^{-i\varepsilon-J_1} & e^{J+J_1} \\
 e^{-J+J_1} & e^{i\varepsilon-J_1} & e^{J+J_1} & e^{-i\varepsilon-J_1} \\
 e^{-i\varepsilon-J_1} & e^{-J+J_1} & e^{i\varepsilon-J_1} & e^{J+J_1} \end{pmatrix}$$

(4)

and its eigenvalues are

$$\lambda_0 = 2e^{J_1} \cosh J + 2e^{-J_1} \cos \varepsilon$$

$$\lambda_2 = 2e^{J_1} \cosh J - 2e^{-J_1} \cos \varepsilon$$

$$\lambda_1 = 2e^{J_1} \sinh J + 2e^{-J_1} \sin \varepsilon$$

$$\lambda_3 = 2e^{J_1} \sinh J - 2e^{-J_1} \sin \varepsilon. \quad (5)$$
The eigenvectors of the transfer matrix are spin waves: the eigenvector for the eigenvalue \( \lambda_k \) has components
\[
\omega_m^{(k)} = e^{i\pi km/2} \quad (m = 0, \ldots, 3). \tag{6}
\]
From (5)–(6) it is easy to obtain explicit expressions for arbitrary powers of the transfer matrix:
\[
(T^n)_{qq'} = (1/4) \sum_{k=0}^{3} (\lambda_k)^n \exp[i\pi k(q - q')/2]. \tag{7}
\]
With this formula we can calculate all the statistical mechanical and thermodynamic properties of the model.

Before discussing the phase diagram we observe the following symmetries of the eigenvalues (5):
\[
\begin{align*}
\lambda_0, 2 &= \lambda_2, 0 (\pi/2 - \varepsilon) = \lambda_1, 3 (\pi/2 - \varepsilon) = \lambda_1, 3 (\varepsilon + \pi/2) \tag{8} \\
\lambda_0, 2 &= \lambda_0, 2 (\varepsilon) = \lambda_1, 3 (-\varepsilon) = \lambda_3, 1 (\varepsilon) \tag{9} \\
\lambda_0, 2 &= \lambda_0, 2 (J) = \lambda_1, 3 (-J) = -\lambda_3, 1 (J) \tag{10}
\end{align*}
\]
Due to them, we only need and will describe the phase diagram in the region \( 0 \leq \varepsilon \leq \pi/2, J \geq 0 \). The diagram on \( \pi/2 \leq \varepsilon \leq \pi \) can be obtained by reflections on the plane \( \varepsilon = \pi/2 \) together with the interchange \( \lambda_0 \leftrightarrow \lambda_2 \) (symmetry (8)); the diagram on \( -\pi \leq \varepsilon \leq 0 \) by reflections on the plane \( \varepsilon = 0 \) plus the interchange \( \lambda_1 \leftrightarrow \lambda_3 \) (symmetry (8)); and the diagram for other values of \( \varepsilon \) follows from the 2\( \pi \)-periodicity. The diagram for \( J \leq 0 \) is obtained by reflections on the plane \( J = 0 \) followed by the interchange \( \lambda_1 \leftrightarrow -\lambda_3 \) (symmetry (10)). As an example, Fig. 1 shows the dependence of those eigenvalues on the coupling \( \varepsilon \) of the imaginary term for some choice of \( J \) and \( J_1 \).

In the thermodynamic limit \( (N \to \infty) \) the behavior of the system is regulated by the leading eigenvalues. By “leading” we mean largest in absolute value. For those values of the parameters for which there is only one leading eigenvalue, say \( \lambda_{k_0} \), we can see from (7) that the free energy density for even chains takes the value
\[
6 \log |\lambda_{k_0}| \tag{6}
\]
and all the expectations are independent of the boundary conditions. By all accounts, these correspond to regions with only one phase present. Crossing points of two leading eigenvalues correspond to transition points, (see Fig. 1).

In the region \( 0 \leq \varepsilon \leq \pi/2, J \geq 0 \), the transition surface is composed of three pieces (Fig. 3):

- The surface \( C_{0,1} \) defined by the condition \( \lambda_0 = \lambda_1 > \max (\lambda_2, |\lambda_3|) \):
\[
C_{0,1} = \left\{ (J, J_1, \varepsilon) : 0 \leq \varepsilon \leq \pi/2, J \geq 0 \text{ and } \varepsilon = \frac{\pi}{4} + \arcsin \left( \frac{e^{-J+2J_1}}{\sqrt{2}} \right) \right\}. \tag{11}
\]

- The surface \( C_{1,-3} \) defined by the condition \( \lambda_1 = -\lambda_3 > \max (\lambda_0, \lambda_2) \):
\[
C_{1,-3} = \left\{ (J, J_1, \varepsilon) : \pi/4 \leq \varepsilon \leq \pi/2, J = 0 \text{ and } \varepsilon \geq \frac{\pi}{4} + \arcsin \left( \frac{e^{2J_1}}{\sqrt{2}} \right) \right\}. \tag{12}
\]

\footnote{This also holds for odd chains with an appropriate branch choice in the logarithm definition.}
• The surface \( C_{0,2} \) defined by the condition \( \lambda_0 = \lambda_2 > \max (|\lambda_1|, |\lambda_3|) \):

\[
C_{0,2} = \{(J, J_1, \varepsilon) : \varepsilon = \pi/2 \text{ and } 0 \leq J \leq 2J_1\}.
\]  

(13)

In addition, the degenerated planes \( J = \infty \) and \( J_1 = \infty \) can also be considered transition surfaces, but we are more interested in the properties of models with finite couplings.

It is trivial to see that the energy density is discontinuous at the transition points, which implies that the system undergoes a first-order phase transition. However, the correlation length also diverges at the points of the surface \( C_{0,1} \) and this transition surface can also be considered as a critical surface. This can be seen, for instance, by computing the correlation of two spin variables

\[
\langle \vec{s}_n \cdot \vec{s}_{n+m} \rangle = \frac{1}{2} \left( \left( \frac{\lambda_{k_0}-1}{\lambda_{k_0}} \right)^m + \left( \frac{\lambda_{k_0}+1}{\lambda_{k_0}} \right)^m \right),
\]

(14)

where \( \lambda_{k_0} \) is the leading eigenvalue. Eq. (14) implies that the correlation length diverges at the transition points \( C_{0,1} \) as \( \xi(\beta) = (\beta^{-1} - \beta_c^{-1})^{-1} \) and hence one obtains critical exponents \( \nu = \eta = 1 \). The correlation length also diverges when \( J \) goes to infinity and, in fact, in the limit \( J \to \infty \) the discontinuity of the energy density goes to zero and the transition becomes purely second order.

An important feature brought by the use of complex couplings is the presence of “Lee-Yang-type” (LYT) singularities. We so denominate those values of the couplings for which there exists a sequence of volumes and boundary conditions giving a partition function equal to zero. For the present models this condition requires the existence of a sequence of powers of the transfer matrix with some entry (which could be different for different powers) equal to zero. Therefore, the locus of the points LYT can easily be obtained from (7) together with the form (5) of the eigenvalues \( \lambda_k \). The conclusion is that, for finite couplings, there are only two curves of points LYT (again, for the region \( 0 \leq \varepsilon \leq \pi/2, \; J \geq 0 \); for the rest of the phase diagram one must proceed as commented below (10)):

• The line defined by the equations \( \lambda_0 = \lambda_2 \) and \( \lambda_1 = -\lambda_3 \); that is:

\[
\text{LYT1} = \{(J, J_1, \varepsilon) : J = 0, \varepsilon = \pi/2\}.
\]

(15)

For couplings on this line,

\[
(T^n)_{qq'} = 0 \iff n \text{ is even and } |q - q'| = 1, 3.
\]

(16)

• The curve defined by the equations \( \lambda_0 = \lambda_1 \) and \( \lambda_2 = -\lambda_3 \); that is:

\[
\text{LYT2} = \{(J, J_1, \varepsilon) \in C_{0,1} : \cosh 2J \cos 2\varepsilon = -1\}
\]

(17)

On this curve,

\[
(T^n)_{qq'} = 0 \iff n \text{ is even and } |q - q'| = 2
\]

(18)
We remark that these are the only solutions if we search for a sequence of volumes with zero partition functions. In addition there are whole surfaces in the coupling-constant space formed by points for which the partition function is zero for some volume and boundary condition. These points are precisely those of the boundary $\partial B_{2k}$ of the black-hole region (see next section) for some even-block decimation. We note, in passing, that the diagonal entries $(T^{2n})_{qq}$ are always nonvanishing, in fact strictly positive (for finite couplings).

Both curves LYT are contained in transition surfaces: LYT2 is in $C_{0,1}$ and LYT1 is in $C_{0,2}$ for $J_1 \geq 0$ and in $C_{1,-3}$ for $J_1 \leq 0$. In fact, the curves correspond to those points of the transition manifold where all the eigenvalues become pairwise degenerate (in absolute value). The curves intersect at the point

$$P_{LYT} = (J = 0, J_1 = 0, \varepsilon = \pi/2)$$

which correspond to all the eigenvalues having the same absolute value (maximum degeneration). This point is extremely singular for the flow of the decimation transformations (Section 5).

Within the critical manifold $C_{0,1}$ we observe some thermodynamic features very different from those of real interactions: the infinite-volume free energy acquires a dependence on the boundary conditions, and no boundary condition produces truncated correlations with decaying behaviour. Let us denote $f_{q,q'}$ the free energy for boundary conditions $q$ on the left and $q'$ on the right. From (19) we obtain, for example, that in the surface $C_{0,1}$, $f_{q,q'} = \log \lambda_0$ (=$\log \lambda_1$) for all boundary conditions $q,q'$ except if $q' = q \pm 2$. In that case the free energy takes the values $\log \lambda_2$ or $\log |\lambda_3|$ depending on which is the subleading eigenvalue. The transition between these two values takes place at the curve LYT2—both non-leading eigenvalues become equal in absolute value—where $f_{q,q\pm2}$ can only be defined by considering volumes of odd size.

For the construction of quantum-mechanical systems via scaling limit, only the region of parameters for which the model satisfies the reflection-positivity condition is of interest. For odd chains, this condition is satisfied for all values of the parameters because the transfer matrix is Hermitean \[17\]. For even chains the reflection-positivity holds only if all the eigenvalues of the transfer matrix are non-negative, that is, for $J \geq J_{RP}(J_1, \varepsilon)$, with

$$J_{RP}(J_1, \varepsilon) = \max \left\{ \arccosh(e^{-2J_1}|\cos \varepsilon|), \arcsinh(e^{-2J_1}|\sin \varepsilon|) \right\}$$

(Figs. 3 and 4). In this work we do not restrict ourselves to the reflection-positive region; if we did we would miss the curves LYT which are the most interesting regions from the point of view of the pathologies of the decimation transformation.
3 Renormalization Group Flow

A single one-spin decimation transformation in the system described above yields the following Renormalization Group equations

\[
\begin{align*}
J' &= \frac{1}{2} \log \left\{ \frac{1 + e^{4J_1} \cosh 2J}{e^{4J_1} + \cos 2\varepsilon} \right\} \\
J_1' &= \frac{1}{4} \log \left\{ \frac{(1 + e^{-4J_1} \cos 2\varepsilon)(1 + e^{4J_1} \cosh 2J)}{2(\cosh 2J + \cos 2\varepsilon)} \right\} \\
\varepsilon' &= \frac{1}{2} \arccos \left\{ \frac{1 + \cosh 2J \cos 2\varepsilon}{\cosh 2J + \cos 2\varepsilon} \right\}
\end{align*}
\]  

(21)

where the primes denote the renormalized quantities. To decide the quadrant of \(\varepsilon'\), the last equation must be complemented with

\[
\tan \varepsilon' = \tan \varepsilon \tanh J.
\]  

(22)

This is the complete exact RG flow except that, as usual, we have omitted the renormalization of the coupling associated with the identity operator. This corresponds to a spin-independent constant added to the Hamiltonian, which is therefore irrelevant for the analysis of correlation functions and thermodynamic potentials. By iteration of (21) we can understand the flow of all decimation transformations of even blocks (decimation of an odd number of spins = even powers of \(T\)).

Starting with the coarser features of the flow, we first observe that all such transformations map the region \(J < J_{RP}(J_1, \varepsilon)\) (non-reflection-positive region) onto the complementary region \(J > J_{RP}(J_1, \varepsilon)\) by a single RG transformation \([41]\). This is due to the fact that the matrix \(T^2\) is always reflection positive. In particular all the fixed points lie in the reflection-positive region. We then notice that the flow defined by (21)–(22) exhibits a periodicity of \(\pi\) in \(\varepsilon\) and very distinct symmetries with respect to the change of sign of \(J\) and \(\varepsilon\). We conclude that it is enough to study the flow in the region of Fig. 2 — \((J \geq 0 \text{ and } 0 \leq \varepsilon \leq \pi/2)\) — and so we will in the sequel. The flow for \(J \geq 0\) and \(-\pi/2 \leq \varepsilon \leq 0\) is an identical copy, while the points with \(J \leq 0\) change quadrant: their renormalized couplings are the same as those for \(|J|\), but with opposite sign for \(\varepsilon'\). Having said this, we can forget about the quadrant-fixing equation (22).

Another conspicuous feature of the RG transformation (21) is the presence of a large region in the coupling constant space where the renormalized couplings are not real numbers. We shall call this region the black hole \(B_2\) (see Figs. 3 and 4):

\[
B_2 = \{ (J, J_1, \varepsilon) : e^{4J_1} < - \cos 2\varepsilon, \varepsilon > \frac{\pi}{4}\}
\]  

(23)

Notice that the intersection of the black hole region with the transition surface is non-empty, even in the region where reflection positivity holds (Fig. 3). While (23) corresponds to the decimation of alternated spins, decimations of chains of \(k\) spins also exhibit a corresponding black-hole region \(B_{2k}\). In Section 4 we discuss how this region changes with the order \(k\) of the decimation. In the black-hole region we must extend analytically (21) into the complex space \((J, J_1, \varepsilon) \in \mathbb{C}^3\). However, this extension must be performed so as to preserve a physically highly desirable property that motivated the initial restriction.
to real \((J, J_1, \varepsilon)\): a real free energy. For this we extend (21) in such a way that the transfer matrix \(T\) remains Hermitean. This extension will be done in detail in Section 4.

The boundary
\[
\partial B_2 = \left\{ (J, J_1, \varepsilon) : e^{4J_1} = -\cos 2\varepsilon, \quad \varepsilon > \frac{\pi}{4} \right\}
\] (24)
of the black hole is a pathological region for the RG flow: at these points the renormalized couplings take the value \(J' = +\infty, J'_1 = -\infty\). As we discuss in more detail in Section 3, this implies that the renormalized measure gives zero weight to some sets of configurations, a property incompatible with a Boltzman probability weight (exponentials are never zero). The renormalized Hamiltonian simply does not exist. There is another pathological region, namely the line LYT1 \((J = 0, \varepsilon = \pi/2)\), where \(J'_1 = +\infty\) and \(\varepsilon'\) is undefined. In Section 3 we shall discuss more extensively these pathologies.

In the rest of the coupling constant space (including the points on the transition surface not in \(\partial B_2\) or LYT1), the flow is well defined: to every point the transformation (21) associates a unique renormalized Hamiltonian. Such a non-pathological flow will be described in the remaining part of this section.

Due to the special features of the decimation procedure we can easily locate RG invariant surfaces, that is, surfaces that are mapped onto themselves by a RG transformation. One of them is the reflection positivity interface defined by (20) which is defined by the condition that the smallest eigenvalue of the transfer matrix be equal to zero. This condition is invariant under a decimation transformation, a fact that can be more easily seen if we write this transformation in the form
\[
\lambda_k^2(J, J_1, \varepsilon) = \lambda_k(J', J'_1, \varepsilon') L(J', J'_1, \varepsilon'),
\] (25)
where \(L\) is the (strictly positive) overall factor associated to the renormalization of the free energy. The same expression (25) shows that equalities and inequalities among the absolute values of eigenvalues of \(T\) are also preserved by decimations of any even block, and, therefore, that the reflection-positive part of the transition surfaces are invariant under these transformations. In particular, the surfaces \(C_{0,1}, C_{0,2}\), \(J = \infty\) and \(J_1 = \infty\) are RG invariant. The intersections of two invariant surfaces are RG invariant curves, that is, renormalized trajectories. Some of them are drawn in Fig. 5a.

These renormalized trajectories intersect at the RG fixed points (see Figs. 4 and 5a). They are listed in Table 1. From (14) it can be seen that the correlation length diverges at \(F_c, F^*_c, F^\pm_0\) and \(F^\mp_0\).
Table 1: Fixed points for even-block decimations.

| Point   | Position \((J, J_1, \varepsilon)\) | Type     |
|---------|-----------------------------------|----------|
| \(F_\infty\) | \((0, 0, 0)\)                     | stable   |
| \(F_\infty^\dagger\) | \((0, \infty, 0)\)               | saddle point |
| \(F_c\)  | \((\frac{1}{2} \log 3, \frac{1}{2} \log 3, \pi/2)\) | saddle point |
| \(F_*\)  | \((\infty, -\infty, \pi/4)\)      | saddle point |
| \(F_{c\dagger}\) | \((0, \infty, \pi/2)\)         | saddle point |
| \(F_{*\dagger}\) | \((\infty, \infty, \varepsilon) ; \varepsilon \in [0, \pi/4]\) | unstable |
| \(F_{0*}\) | \((\infty, \frac{1}{4} \log \cos 2\varepsilon, \varepsilon) ; \varepsilon \in [0, \pi/4]\) | saddle point |

The linearization of the RG equations (21) around the different fixed points yields the matrices

\[
L_{F_\infty} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \quad L_{F_c} = \begin{pmatrix}
\frac{2}{3} & -\frac{4}{3} & 0 \\
\frac{2}{3} & \frac{4}{3} & 0 \\
0 & 0 & 2 \\
\end{pmatrix}
\]

\[
L_{F_{c\dagger}} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \quad L_{F_{0*}} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

\[
L_{F_{0*}} = \begin{pmatrix}
1 & 1 & \frac{1}{2} \tan 2\varepsilon \\
1 & \frac{1}{2} & -\frac{1}{4} \tan 2\varepsilon \\
0 & 0 & 1 \\
\end{pmatrix},
\]

whose eigenvalues \((\zeta_i)\) and eigenvectors determine the critical exponents \((y_i = \log \zeta_i / \log 2)\) and the main properties of the renormalized trajectories.

The fixed point \(F_\infty\) is completely attractive in all directions. The point \(F_{c\dagger}\) behaves as an attractor with respect to perturbations along the \(J\)-axis and the \(\varepsilon\)-axis, but has a marginal direction \((y = 0)\) along the \(J_1\)-axis. The latter is actually a relevant direction, as can be checked numerically, and flows toward the infinite-temperature fixed point \(F_\infty\).

There are three fixed points on the transition manifold: \(F_c\), \(F_{c\dagger}\) and \(F_{*c}\). The first one has features predicted by the standard scenario: It is a saddle point with one irrelevant and two relevant directions, both with critical exponents equal to \(y = D = 1\). Of the relevant directions, one corresponds to perturbations along the \(\varepsilon\)-axis with the flow escaping towards the fixed point \(F_\infty\), while the other is tangent to the renormalized trajectory \(J = J_{RP}(J_1, \pi/2)\) where the flow moves towards \(F_{c\dagger}\) and \(F_{*c}\). The flow towards \(F_{*c}\) is along the renormalized trajectory defined by the intersection between the reflection-positivity interface and the transition surface \(C_{0,1}\) (given by (20) and (11) respectively); the flow towards \(F_{c\dagger}\) is along the line \(J = J_{RP}(J_1, \pi/2)\). The irrelevant direction is along the RG trajectory \(J = 2J_1, \varepsilon = \pi/2\).

The other two fixed points on the transition surface correspond to singularities of the flow (they are non-Gibbsian) and therefore the RG transformations can not be linearized around them. We have
analyzed the flow numerically and we conclude that $F_{0 \uparrow}$ is attractive for those transition points satisfying $J < 2J_1$ (triangular-looking cusp in the surface of Fig. 5a), while $F_{0 \uparrow}^*$ attracts the rest of points of the transition surface. In both fixed points there is one relevant direction that flows away from the transition surface, whose critical exponent is not well defined.

Finally we have two (“zero-temperature”) fixed-point lines $F_{0 \uparrow}$ and $F_0^*$. There are three marginal directions at each point of $F_{0 \uparrow}$, but only the one tangent to the line $F_{0 \uparrow}$ itself is truly marginal. The other two are actually relevant and the fixed points become unstable. Each of the points of $F_0^*$ has one irrelevant perturbation ($y = -1$) along the $(2, -1, 0)$ direction and two marginal ones: one, truly marginal, tangent to the line $F_0^*$ itself, and the other given by the lines $J_1 = \infty$, $\varepsilon \in [0, \pi/4]$, which are RG trajectories flowing from $F_{0 \uparrow}$ to $F_0^*$. At these fixed points one can take the continuum limit. The existence of these lines of fixed points leads to different quantum mechanical systems in the scaling limit [38].

We now analyze the flow outside of the black-hole section $B_2$, to follow the points inside the black hole we need an analytic extension to be discussed in Section 4 (See Figs. 5a, 5b, and 6b). For reading the rest of this section we recommend to look at the Figures 4, 5a, 6a and 6b as many times as needed. We start with the flow inside the invariant planes. There are four of them, corresponding to $\varepsilon = 0$, $\varepsilon = \pi/2$, $J_1 = \infty$ and $J = \infty$. From (21) we see that $F_\infty$ attracts all points in the invariant plane $\varepsilon = 0$, except those sitting at the line $J_1 = \infty$ which flow to $F_{0 \uparrow}\infty$. The fixed points $F_{0 \uparrow}\infty$ are completely unstable. In the plane $\varepsilon = \pi/2$ the situation is less simple: $F_\infty$ attracts the transition points on the renormalized trajectory $J = 2J_1$ and $F_{0 \uparrow}$ is attractive for all transition points above this line ($J < 2J_1$). The remaining points in this plane flow toward the black hole region in a finite number of RG steps, except those of the boundary $\partial B_2$ which go to the plane $J = \infty$ (at its intersection with $J_1 = -\infty$) in one renormalization step. In the plane $J_1 = \infty$ all points are attracted by $F_{0 \uparrow}$, except those belonging to the line $\varepsilon = \pi/2$ which flow toward $F_{0 \uparrow}$. We remark that the points lying on the line $J = 0$ are mapped to $F_{\infty \uparrow}$ in a single RG step.

Finally, in the plane $J = \infty$ every line $\varepsilon = $ constant is a trajectory of the renormalization transformation. When $\varepsilon \in [\pi/4, \pi/2]$ they flow towards the black hole region, while the points with $\varepsilon \in [0, \pi/4]$ are attracted by $F_0^*$ and repelled by $F_{0 \uparrow}^*$.

In the rest of the space the flow is as follows. The points above the critical surface $C_{0,1}$ (see Fig. 5a) are driven by the RG transformation to the infinite-temperature fixed point $F_\infty$. They will reach this point after an infinite number of RG steps. In particular, the points at the plane $J = 0$ are mapped into the plane $\varepsilon = 0$ in a single renormalization step, where they remain during their migration to $F_\infty$. (The points with $J < 0$ go to the previous quadrant $-\pi/2 \leq \varepsilon < 0$.) The points below $C_{0,1}$ are swallowed up by the black hole region in a finite number of RG steps. That means that the part of the black-hole region $B_{2k}$ located below the surface $C_{0,1}$ grows with the order $k$ of the decimation procedure. For instance, the corresponding black-hole region for the transformation $T \rightarrow T^3$ (double decimation) contains, besides all the points belonging to $B_2$, also the points which are mapped into the black hole $B_2$ by a single
decimation. (In Section 4 we shall see that the part of $B_{2k}$ above the surface $C_{0,1}$ decreases in size with $k$.) Finally, the points on the transition surface $C_{0,1}$ remain there after renormalization. There are three domains of attraction: a) if $J = 2J_1 (\varepsilon = \pi/2)$ the flow goes towards $F_c$; b) if $J < 2J_1 (\varepsilon = \pi/2)$, towards $F_{c1}$; and c) if $J > 2J_1$, toward $F_{c*}$. In all cases, an infinite number of RG steps are needed to arrive at the corresponding fixed point.

4 Analytic Extensions of the 4-State Clock Model

As explained in the last section, at every point belonging to the region $B_2$ one has to extend (21) analytically to the complex $(J, J_1, \varepsilon)$ space. However, to have a real free energy this extension must be performed in such a way that the transfer matrix remains Hermitian. This amounts to a suitable definition of the complex logarithm. Alternatively, we can ask ourselves how many hermitean extensions of a matrix $T$ of the form (4) can be constructed. After some straightforward algebra we conclude that there are two of them, which can be labelled by an integer $m$ taking the values 0 and 1. The transfer matrix now reads

$$T(m) = \begin{pmatrix}
    e^{J+J_1} & e^{-i\varepsilon-J_1} & (-1)^m e^{-J+J_1} & e^{i\varepsilon-J_1} \\
    e^{i\varepsilon-J_1} & e^{J+J_1} & e^{-i\varepsilon-J_1} & (-1)^m e^{-J+J_1} \\
    (-1)^m e^{-J+J_1} & e^{i\varepsilon-J_1} & e^{J+J_1} & e^{-i\varepsilon-J_1} \\
    e^{-i\varepsilon-J_1} & (-1)^m e^{-J+J_1} & e^{i\varepsilon-J_1} & e^{J+J_1}
\end{pmatrix}$$

(27)

where $(J, J_1, \varepsilon) \in \mathbb{R}^3$. This implies that the couplings which appear in the Hamiltonian (3) must be replaced by the following quantities

$$J \rightarrow J - \frac{im\pi}{2}, \quad J_1 \rightarrow J_1 + \frac{m\pi}{4}, \quad \varepsilon \rightarrow \varepsilon, \quad A \rightarrow A + \frac{m\pi}{4}$$

(28)

where $A$ is the real coupling associated to the identity operator. Notice that $\varepsilon$ always remains real. The case $m = 0$ corresponds to the original clock model (3), and $m = 1$ to its analytic continuation with coupling constants carrying an imaginary part. This last model will be called the extended clock model and quantities related with it will be denoted with a bar. In Ref. [41] the parameter $m$ was not considered.

In order to study the phase structure of this extended model, one has to compute the eigenvalues of the new transfer matrix $\overline{T} = T(1)$ [given in (27)]:

$$\overline{\lambda}_0 = 2e^{J_1} \sinh J + 2e^{-J_1} \cos \varepsilon$$
$$\overline{\lambda}_2 = 2e^{J_1} \sinh J - 2e^{-J_1} \cos \varepsilon$$
$$\overline{\lambda}_1 = 2e^{J_1} \cosh J + 2e^{-J_1} \sin \varepsilon$$
$$\overline{\lambda}_3 = 2e^{J_1} \cosh J + 2e^{-J_1} \sin \varepsilon.$$  

(29)

We can relate easily these ones with those given by (3)

$$\overline{\lambda}_m(J, J_1, \varepsilon) = \lambda_{m+1}(J, J_1, \varepsilon + \pi/2).$$  

(30)

---

7 We thank Alexei Morozov for pointing out the interest of this question.
This relation implies that the phase structure of the extended model can be obtained from that of the original one \((m = 0)\) by reflecting Fig. 5a with respect to the plane \(\varepsilon = \pi/4\). Thus, the RG flow in the extended model is the same as in the original one, except for a reflection (Fig. 5b). In particular, to each fixed point located at \((J, J_1, \varepsilon)\) in the original model (listed in Section 3), corresponds an analogous fixed point but located at \((J, J_1, \pi/2 - \varepsilon)\). For the same reason there exists another black hole

\[ B_2 = \{(J, J_1, \varepsilon) : e^{4J_1} < \cos 2\varepsilon, \varepsilon < \frac{\pi}{4}; m = 1\} \quad (31) \]

where an analytic extension of the corresponding RG equations is needed. The curves of LYT singularities are now

\[ \text{LYT1} = \{(J, J_1, \varepsilon) : J = 0 = \varepsilon\} \quad (32) \]

and

\[ \text{LYT2} = \{(J, J_1, \varepsilon) \in C_{0,1} : \cosh 2J \cos 2\varepsilon = 1\}, \quad (33) \]

which intersect at

\[ \text{PLYT} = (J = 0, J_1 = 0, \varepsilon = 0). \quad (34) \]

The new feature is the flow of points inside the black-hole regions \(B_2\) and \(\overline{B}_2\) from one model to the other. To study it we need the RG equations in the slightly higher-dimensional space \((J, J_1, \varepsilon, m)\), for a transfer matrix \(T(m)\) of the form (27). These are

\[
J' = \frac{1}{2} \log \left\{ (-1)^{m+m'} \frac{1+e^{4J_1} \cosh 2J}{e^{4J_1} + (-1)^m \cos 2\varepsilon} \right\}
\]

\[
J'_1 = \frac{1}{4} \log \left\{ (-1)^{m+m'} \frac{(1+(-1)^m e^{-4J_1} \cos 2\varepsilon)(1+e^{4J_1} \cosh 2J)}{2(\cosh 2J + (-1)^m \cos 2\varepsilon)} \right\} \quad (35)
\]

\[
\varepsilon' = \frac{1}{2} \arccos \left\{ \frac{(-1)^m \cosh 2J \cos 2\varepsilon}{\cosh 2J + (-1)^m \cos 2\varepsilon} \right\},
\]

where the renormalized integer \(m'\) must be chosen in such a way that there exist real numbers \(J'\) and \(J'_1\) satisfying (35). This choice is unique.

Consider the original clock model \((m = 0)\). If \((J, J_1, \varepsilon)\) does not belong to the black hole \(B_2\), then it follows from (35) that \(m' = 0\). The flow in this region was described in detail in Section 3. However if \((J, J_1, \varepsilon) \in B_2\) then one must choose \(m' = 1\); so the renormalized Hamiltonian belongs to the extended model. By the same arguments, if \(m = 1\) the renormalized Hamiltonian will have \(m' = 1\) whenever we start at a point not belonging to \(\overline{B}_2\), but \(m' = 0\) if \((J, J_1, \varepsilon) \in \overline{B}_2\). Thus, the analytic extension of the extended model is the original one.

Furthermore, the decimation transformation sends the points inside a black hole to a different side of the critical surface: A point \((J, J_1, \varepsilon, m = 0) \in B_2\) placed below \(C_{0,1}\) is mapped to a point \((J', J'_1, \varepsilon', m' = 1) \notin \overline{B}_2\) located above \(\overline{C}_{0,1}\). Such a point is eventually attracted by the fixed point \(\mathcal{F}_{\infty}\) except if it belongs to the line \(J = \infty\) in which case is attracted by the line \(\mathcal{F}_{0}^\varepsilon\) (see Figs. 6a and 6b). On the other hand, if \((J, J_1, \varepsilon, m = 0)\) belongs to the part of the black hole on top of \(C_{0,1}\), it is mapped to a point outside \(\overline{B}_2\) and below \(\overline{C}_{0,1}\). Such a point ends up, after a finite number of iterations, inside \(\overline{B}_2\) and below.
\(C_{0,1}\), so it is returned to the \(m = 0\) model outside the black hole \(B_2\) and above \(C_{0,1}\). The attractor for such a point is therefore \(F_\infty\).

In addition, a point on the critical surface \(C_{0,1}\) and inside \(B_2\) goes to a point on the critical surface \(\overline{C}_{0,1}\) of the extended model. This image point is outside \(\overline{B}_2\), so it is eventually attracted by \(\overline{F}_c\).

In particular, all this implies that when the order \(k\) of the decimation increases, the part of the black hole \(B_{2k}\) on top of \(C_{0,1}\) decreases in size, with its boundary approaching \(C_{0,1}\). Complementarily, the part of \(B_{2k}\) located below \(C_{0,1}\) increases in size, with the boundary also approaching \(C_{0,1}\) for large \(k\). Therefore, in the limit of decimations of large order \(k\), the black-hole region occupies the whole region situated below \(C_{0,1}\), and the boundary \(\partial B_{2k}\) asymptotically coincides with this critical surface. All these boundaries \(\partial B_{2k}\) contain the curve \(\text{LYT}_2\) defined in (17), and they “turn” around it when they approach \(C_{0,1}\) as \(k \to \infty\). Moreover, the curve \(\text{LYT}_2\) is the intersection of each \(\partial B_{2k}\) with \(C_{0,1}\) and it divides \(\partial B_{2k}\) in two parts: a part inside the black hole \(B_{2k}\) (above the curve \(\text{LYT}_2\) in Fig. 5a), and a part outside it. In Section 5 we discuss other important features of the RG transformation at this curve.

Analogous considerations hold for the extended \((m = 1)\) model. In particular, points in \(\overline{B}_2\) and below \(\overline{C}_{0,1}\) are attracted by the fixed points \(\overline{F}_\infty\) or \(\overline{F}_c^\varepsilon\) of the original model, while points in \(\overline{B}_2\) and above \(\overline{C}_{0,1}\) are attracted by \(\overline{F}_c\). The points in \(\overline{B}_2 \cap \overline{C}_{0,1}\) are attracted by \(\overline{F}_c^\ast\).

Finally, the points on the transition surface \(C_{1,-3}\) are attracted by the fixed point \(\overline{F}_c^\dagger\), except those belonging to the curve \(C_{1,-3} \cap C_{0,1}\) which flow to \(F_c\).

This concludes the analysis of the flow at the non-pathological points, both in the original and the extended model. We observe no trace of ambiguities: to each Hamiltonian there corresponds a unique renormalized Hamiltonian given by (35). Besides, all fixed points are reached after an infinite number of RG steps, except \(\overline{F}_\infty^\dagger\) and \(\overline{F}_\infty^\ast\) which act as very strong attractors (attract in one RG step) for the points on the lines \(J = 0\), \(J_1 = \infty\), \(m = 0\) and \(J = 0\), \(J_1 = \infty\), \(m = 1\) respectively.

What is left is the study of the pathological regions: the boundaries \(\partial B_2\) and \(\partial \overline{B}_2\)—or, more generally, \(\partial B_{2k}\) and \(\partial \overline{B}_{2k}\)—of the black-hole regions, and the line \(\text{LYT}_1\). This is the subject of next section.

5 Pathologies in the 4-State Clock Model

Griffiths and Pearce [14] were, to our knowledge, the first to point out possible “peculiarities” for RG transformations at a first-order phase transition. This pioneer call of attention was followed by numerical results and tentative arguments [1, 2, 3, 4, 5] that seemed to indicate the possibility of discontinuity and multivaluedness of the flow at the coexistence points. The theorems proved in [12] have ruled out such lack of smoothness for real interactions and compact single-spin space, and have left the more subtle phenomenon of non-Gibbsianness [11, 12, 13] as the main pathology for real-space renormalization.

For real interactions, the class of Gibbsian measures is exactly characterized by the class of quasilocal and uniformly non-null measures [12]. Roughly speaking, quasilocality means that the direct influence of far away spins on a given spin \(s_{x_0}\) is small. That is, if we take a set \(\Lambda\) centered in \(x_0\) and fix the spins
\( \vec{s}_x \in \Lambda, \ x \neq x_0, \) then the change of the expected value of \( \vec{s}_{x_0} \) with the boundary conditions outside \( \Lambda \) is vanishingly small as the diameter of \( \Lambda \) goes to infinity. On the other hand, a necessary condition for a measure to be uniformly non-null is that every open set of configurations have nonzero measure. For example, the measures obtained for spin systems in the zero-temperature limit are typically not uniformly non-null as they are concentrated on the configurations which minimize the energy of the system. For general complex interactions the theory of Gibbs measures is not well developed, but in any case the exponential form of the Boltzmann weights implies that uniform non-nullness is also a necessary condition for Gibbiansness.

The non-Gibbiansness of the renormalized measure exhibited in \([11, 12, 13]\) for Ising models at or close to a first-order phase transition, is a consequence of lack of quasilocality. In the present case, however, the origin of the pathologies is rather related to the loss of uniform non-nullness. The pathologies occur when there are renormalized models for which some matrix elements of the transfer matrix \([27]\) vanishes. For instance, if the transfer matrix is diagonal it would imply that the corresponding measure is concentrated on constant configurations. In terms of coupling constants, measures that are not uniformly-non-null correspond to manifolds where \( J \) or \( J_1 \) take an infinite value, and the pathological points are those mapped into such manifolds by a finite number of iterations of the RG transformation.

We remark that the “pathological” character of a RG transformation at these points only appears if we insist on finding a renormalized Hamiltonian. The renormalized transfer matrix is always well defined. This remark is equivalent to the fact, emphasized in \([11, 12]\), that renormalization transformations (for real interactions) are always well-defined as maps between probability measures. It is only at the level of Hamiltonians that the induced transformation can become sick.

To see at which points the single-spin decimation is pathological, we have to look at the matrix elements of \( T^2 \) obtained from \([27]\). These are:

\[
(T^2)_{q,q} = 2 \left( e^{2J_1 \cosh 2J + e^{-2J_1}} \right) \\
(T^2)_{q,q+1} = 2 \left( e^{J_1 - i\varepsilon} + e^{-J_1 + i\varepsilon} \right) = (T^2)_{q,q+3}^* \\
(T^2)_{q,q+2} = 2 \left( (-1)^m e^{2J_1} + e^{-2J_1 \cos 2\varepsilon} \right).
\]

Zeros of any of these elements determine points that are pathological for a single RG map. The diagonal elements are always non-vanishing; however, the off-diagonal ones can vanish at some points of the coupling constant space. On the one hand, \( (T^2)_{q,q+2} = 0 \) at the boundaries \( \partial B_2 \) and \( \partial \bar{B}_2 \) of the black hole region, and \( (T^2)_{q,q+1} = (T^2)_{q,q+3} = 0 \) at the lines LYT1 and \( \bar{\text{LYT1}} \). The matrix \( T^2 \) is diagonal at \( P_{LYT} \) and \( \bar{P}_{LYT} \).

It is interesting to follow the flow of the points at these pathological regions. For concreteness we analyze the original \( (m = 0) \) model; the discussion for the extended model is analogous. We see that all the points in \( \partial B_2 \) are mapped into the line \( J_1' = \infty, J_1 = -\infty \): The points above LYT2 go onto the segment \( 0 < \varepsilon' < \pi/4 \) (i.e. outside the black-hole region \( B_2 \)), while those below LYT2 end up at the complementary segment \( \pi/4 < \varepsilon' < \pi/2 \) (deep inside \( B_2 \)). All the points in LYT2 are mapped to
the point with $\varepsilon' = \pi/4$, i.e. to the non-Gibbsian fixed point $F^*_c$. Applying a further iteration of the transformation one can see (using, for instance, the expression for $T^4$ obtained via (4)), that the points with $\varepsilon' \neq \pi/4$ recover finitely-valued couplings: Those with $0 < \varepsilon' < \pi/4$ are mapped to points above $C_{0,1}$ and flow towards $F_\infty$; while those with $\pi/4 < \varepsilon' < \pi/2$ are mapped to points above $C_{0,1}$ ($m = 1$ model) and flow towards $\overline{F}_\infty$. On the other hand, the points in LYT2 remain in $F^*_c$ (it is a fixed point!) and so they lead always to non-Gibbsian renormalized measures. This non-Gibbsianness is the result of the strong attraction that a non-Gibbsian fixed point has on the curve LYT2 (in the sense of attracting it in finitely many—in fact one—steps).

The points LYT1 are also victims of a strong attraction: In one decimation transformation the half-line $J = 0, J_1 > 0, \varepsilon = \pi/2$ is mapped into the plane $J_1 = \infty$, while the other half-line—$J = 0, J_1 < 0, \varepsilon = \pi/2$—is mapped into the plane $\overline{J}_1 = \infty$. As these planes are RG-invariant and formed by non-Gibbsian models, those points remain non-Gibbsian under iterations of renormalization group transformations. It is suggestive to observe that, at the level of coupling constants, the RG transformations exhibit discontinuities and multivaluedness at LYT1. For example, points arbitrarily close to the line LYT1 and contained in the plane $J = 0$ are renormalized into points with $\varepsilon' = 0$, while equally close points but contained in the plane $\varepsilon = \pi/2$ remain with $\varepsilon' = \pi/2$. This shows an explicit discontinuity. Moreover, the value of $\varepsilon'$ given in (21) is undefined at the line LYT1: By approaching this line ("preparing the system") from different directions $J(\varepsilon)$ one can obtain any value of $\varepsilon'$. This is an instance of multivaluedness. In this sense, the "most" pathological point is $P_{LYT} = LYT1 \cap LYT2$. Its renormalized transfer matrix is diagonal and, moreover, both $\varepsilon'$ and $J'_1$ are undefined for this point, and can be made to take any value by approaching it in different ways.

Again we emphasize that this discontinuity and multivaluedness appears because we are trying to follow finitely-valued couplings, and “truncating” the infinitely-valued ones. There is no ambiguity or lack of smoothness at the level of renormalized transfer matrices. Analogously, non-Gibbsianness prevents us from linearizing the transformation as a function of the couplings around the critical fixed points $F^*_c$ and $F^+_c$, and hence the matrices $L_{F^*_c}$ and $L_{F^+_c}$ are not defined. The linearization is perfectly possible for the transformation as a function of the entries of the transfer matrix.

Other pathologies of the single-spin decimation show up if we look for zeroes among the entries of higher powers of $T$. Such pathologies correspond to the loss of Gibbsianness after a finite number of renormalizations. They coincide with single-renormalization pathologies for decimations of higher order, and they occur at the successive boundaries $\partial B_{2k}$. Such pathologies have the same features as those discussed above: except at the points LYT2—contained in all such boundaries—the Gibbsianness is recovered for good after a further iteration of the transformation ($\partial B_{2k} \cap \partial B_{2k'} = LYT2$ if $k \neq k'$). On the other hand the above pathological points depend very much on the renormalization group prescription and in this sense they are not universal. However their existence in the espace of local Hamiltonians
cannot be avoided by the choice of a suitable choice of a renormalization group prescription. In fact, the analysis in a general decimation prescription can be obtained from the study of the continuous flow defined by the infinitesimal transformation $T \rightarrow T^t$ ($t \in \mathbb{R}$). This is given (for $m = 0$) by the following differential equations

\begin{align*}
\dot{j} &= (\lambda_1 \log \lambda_1 + \lambda_3 \log \lambda_3) \cosh J 4e^{Jt} - (\lambda_0 \log \lambda_0 + \lambda_2 \log \lambda_2) \sinh J 4e^{Jt} \\
\dot{j}_1 &= \frac{(\lambda_0 \log \lambda_0 + \lambda_2 \log \lambda_2)}{8e^{Jt} \cosh J} - \frac{(\lambda_0 \log \lambda_0 - \lambda_2 \log \lambda_2)}{8e^{-Jt} \cos \varepsilon} - \frac{\varepsilon \tan \varepsilon}{2} - j \frac{\tanh J}{2} \\
\dot{\varepsilon} &= \frac{e^{Jt}}{4} \left\{ (\lambda_1 \log \lambda_1 - \lambda_3 \log \lambda_3) \cos \varepsilon - (\lambda_0 \log \lambda_0 - \lambda_2 \log \lambda_2) \sin \varepsilon \right\}
\end{align*}

(37)

which are valid only in the reflection-positive region. These equations are well defined for every point in this domain which contains some points of $\partial B_2$. The only pathology of the points of $B_2$ is that they are attracted by the line $J = \infty, J_1 = -\infty, \varepsilon \in (\pi/4, \pi/2]$ in a finite “time” $t < 2$. This feature explains the especial behavior of the points of $B_2$ under the action of $T^2$-renormalization group transformations: they are mapped into the $m = 1$ domain. The singularity arises also in this infinitesimal scheme when we integrate up to a finite renormalization scale $t$. For each $t$ there is a black hole region $B_t$ that changes as we increase the scale $t$. In fact, as we have seen, every point below the transition surface belongs to a region $B_t$ for $t$ large enough because they are attracted by the line $J = \infty, J_1 = -\infty, \varepsilon \in (\pi/4, \pi/2)$. This implies that the flow defined by the differential equations (37) does not generate a one-parameter group of global transformations.

## 6 Conclusions

We believe that the model studied here provides an instructive illustration of the behavior of real-space renormalization transformations in the vicinity of coexistence manifolds. Admittedly, some of the features observed could be specifically due to the use in particular of decimation procedures, but we expect others to be indications of phenomena of more general validity.

The present example explicitly exhibits a picture that takes considerable more work to verify for real interactions: the reconciliation of the standard scenario and the presence of pathologies, and non-Gibbsianness as a source of an observed discontinuity and multivaluedness in the flow of the coupling constants. In addition, we consider singularly suggestive the observations that the worst type of pathologies takes place at points where all the eigenvalues of the transfer matrix are doubly degenerate, and where there are singularities of the type linked, in some studies, to metastability effects.

The analysis performed here can be extended to $q$-state clock models of higher $q$. We do not expect any new phenomenon except, perhaps, in the limit $q \rightarrow \infty$ (chain of plane rotors with complex interactions). But of course, the challenge is to find a model with real interactions exhibiting, with comparable

\[8\text{In some cases a description of the renormalization group transformation in terms of non-local variables might lead to a continuous flow [44].}\]
explicitness, non-Gibbsianness of renormalized measures and its consequences for computational schemes (an aspect left unresolved in previous work [11, 13, 14]). Such an example, however, may not be easy to construct given the rather subtle character of non-quasilocality.

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Figure Captions

Figure 1: Eigenvalues $\lambda_k$ ($k = 0, \ldots, 3$) of the transfer matrix (5) as a functions of $\varepsilon$ for $J = 1.5$ and $J_1 = 0.5$.

Figure 2: Transition surfaces and LYT points for the model in the region $J \geq 0, 0 \leq \varepsilon \leq \pi/2$. The intersection with the reflection-positivity interface of Fig. 3 is also shown.

Figure 3: Reflection positivity interface (20) for the model. We have also drawn some of the fixed points, the renormalized trajectory joining $F_c$ and $F_c^*$ and the black hole region (23).

Figure 4: Transition surface outside the black hole region (23). The contour of the reflection positivity interface is also drawn for clarity.

Figure 5: Renormalization group flow in the reflection positive region: (a) for the original model, and (b) for the extended one. Fixed points and some of the renormalized trajectories are also depicted. The black hole region is depicted in the right side of (a) and in the left one of (b).

Figure 6: Universality classes (a) for $\varepsilon \in (0, \pi/2)$, and (b) $\varepsilon = \pi/2$. The fixed point which attracts all points belonging to a region is written inside it. Notice that the surface $C_{0,-3}$, defined by the condition $\lambda_0 = -\lambda_3 > \max(|\lambda_1|,|\lambda_2|)$, is a critical one in the region $J < 0$ (See Eq. (10)). The dashed line represents the boundary $\partial B_2$ and the dash-dotted line the LYT1 points.
