Adaptive Simulation of the Heston Model

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Abstract

Recent years have seen an increased level of interest in pricing equity options under a stochastic volatility model such as the Heston model. Often, simulating a Heston model is difficult, as a standard finite difference scheme may lead to significant bias in the simulation result. Reducing the bias to an acceptable level is not only challenging but computationally demanding. In this paper we address this issue by providing an alternative simulation strategy – one that systematically decreases the bias in the simulation. Additionally, our methodology is adaptive and achieves the reduction in bias with “near” minimum computational effort. We illustrate this feature with a numerical example.

Keywords: stochastic volatility models, efficient simulation

1 Introduction

Under the standard Black-Scholes framework, the asset price dynamics is given by the lognormal model

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^S
\]  

where the drift parameter \(\mu\) and the volatility \(\sigma\) are considered constant (or at best piecewise constant). The popularity of this model lies in its convenience and simplicity; however, these features come at a price. The standard Black-Scholes model is not able to capture the volatility smile observed in the trading market. The Heston model assumes that the variance \(V = \sigma^2\) is itself a stochastic process – more specifically a square-root diffusion model of the CIR (Cox-Ingersoll-Ross) type (see [11], [9]). It further allows the variance, \(V\), to be correlated with the stock price \(S\), thereby capturing the volatility smile. Furthermore, the Heston model provides a closed-form solution for pricing European Options, allowing one to fit the model to observed option prices. The Heston model is given by the coupled SDE (stochastic differential equations):

\[
\begin{align*}
\frac{dS_t}{S_t} &= \mu dt + \sqrt{V_t} dW_t^S \\
\frac{dV_t}{V_t} &= \kappa(\theta - V_t) dt + \sigma_V \sqrt{V_t} dW_t^V
\end{align*}
\]  

where the variance \(V\) is modelled by a square-root diffusion process with parameters \(\kappa\) which is the speed at which the process mean reverts to the long term variance \(\theta\), and \(\sigma_V\), the volatility of the variance. We denote by \(\rho\) the instantaneous correlation between the two noise processes: \(d\langle W^S, W^V \rangle = \rho dt\).
By Cholesky factorization, one can rewrite equations (1.2)–(1.3) as:

\[
\begin{align*}
\frac{dS_t}{S_t} &= \mu dt + \sqrt{V_t} \left[ \rho dW_t^{(1)} + \sqrt{1 - \rho^2} dW_t^{(2)} \right] \\
\frac{dV_t}{V_t} &= \kappa(\theta - V_t) dt + \sigma \sqrt{V_t} dW_t
\end{align*}
\]  

(1.4) (1.5)

with independent Brownian motions \(dW_t^{(1)}\) and \(dW_t^{(2)}\). There is a wide range of literature on the simulation of this model. Some of these are based on Euler discretisation; others on the improved finite difference approximations such as higher-order Milstein schemes and Predictor-Corrector methods; and still others based on distributionally exact (5) or approximate (3) simulation methods. In our study we focus mainly on the Exact simulation method as proposed by Broadie and Kaya (5) but we will avoid the numerical inversion of the Laplace Transform which seems to be the most time consuming part of their simulation technique. Doing so, will mean introducing bias in the simulation. We will address this problem by providing an adaptive strategy that systematically controls the bias in the simulation. Furthermore, this is achieved with minimal computational overhead. In most cases, our method should prove faster than that of (5), as there is no costlier inversion to perform.

Here is the layout of the rest of the paper. In Section 2 we recall the general algorithm (cf. Broadie and Kaya (5)) for simulating the Heston model, to emphasise the basic difficulty: the simulation of the integral of the variance process. In Section 3 we focus on the dynamics of the variance process and study its properties by transforming it into a canonical process. We also describe methods for simulating the latter process. In Section 4 we describe a method for simulating bridges corresponding to the latter process, and their use in simulating the integral of the variance process. In Section 5 we relate the integral of the variance process to a weighted integral of the canonical process, and we also calculate some moments of the latter for the canonical bridge process. In Sections 6, 7, we explore adaptive computation of the integral. In Section 8 we describe the results of some numerical experiments on the accuracy and efficiency of our adaptive algorithm. Finally, we provide concluding remarks and future research ideas in Section 9. Lengthy or highly technical proofs are given in Appendices A–C.

2 Simulating the Stock Price

All the details of the simulation algorithm for the equity price governed by the stochastic differential equation in (1.4), are given in (5). We reproduce them here for the sake of completeness and to emphasise the role played by the integral of the variance process, \(V\). Integrating (1.3) between two dates of interest (e.g., coupon/reset dates), \(t_{j-1}\) and \(t_j\), we obtain

\[
S_j = S_{j-1} \exp \left[ \mu \Delta t_j - \frac{1}{2} \int_{t_{j-1}}^{t_j} V_s \, ds + \rho \int_{t_{j-1}}^{t_j} \sqrt{V_s} \, dW_s^{(1)} + \sqrt{1 - \rho^2} \int_{t_{j-1}}^{t_j} \sqrt{V_s} \, dW_s^{(2)} \right]
\]

where \(\Delta t_j = t_j - t_{j-1}\). Similarly, integrating (1.5) we obtain

\[
V_j = V_{j-1} + \kappa \theta \Delta t_j - \kappa \int_{t_{j-1}}^{t_j} V_s \, ds + \sigma \int_{t_{j-1}}^{t_j} \sqrt{V_s} \, dW_s^{(1)}.
\]

(2.1)

In the next section we will see that it is not complicated to simulate \(V_j\) in a manner that is not based on (2.1). We can easily use (2.1) to obtain:

\[
\int_{t_{j-1}}^{t_j} \sqrt{V_s} \, dW_s^{(1)} = \frac{1}{\sigma \sqrt{V_j}} \left( \Delta V_j - \kappa \theta \Delta t_j + \kappa \int_{t_{j-1}}^{t_j} V_s \, ds \right)
\]

(2.2)
where $\Delta V_j = V_j - V_{j-1}$. The only component left is $\int_{t_{j-1}}^{t_j} \sqrt{V_s} dW_s^{(2)}$; but since $V_s$ is independent of the Brownian increments $dW_s^{(2)}$ by construction, then given $V_s, 0 \leq s \leq t_j$,

$$\int_{t_{j-1}}^{t_j} \sqrt{V_s} dW_s^{(2)} \sim N \left( 0, \int_{t_{j-1}}^{t_j} V_s ds \right) .$$

Using this result we have:

$$S_j = S_{j-1} \exp \left[ \mu \Delta t_j - \frac{1}{2} \int_{t_{j-1}}^{t_j} V_s ds + \rho \int_{t_{j-1}}^{t_j} \sqrt{V_s} dW_s^{(1)} + \sqrt{1 - \rho^2} \int_{t_{j-1}}^{t_j} V_s dZ_j \right]$$

where, given $V_s, 0 \leq s \leq t_j$, $Z_j \sim N(0, 1)$ and is conditionally independent of $W_s^{(1)}$. Based on (2.2) and (2.3), it is clear that simulation of the Heston model is straightforward except for the simulation of the integral $\int_{t_{j-1}}^{t_j} V_s ds$. In [5], it is done in an unbiased manner by using the distribution of this integral. The distribution is not available in closed form and is computationally intensive to compute. Our approach is to use a random quadrature with an adaptive control of bias.

3 Squared Bessel Process and its Simulation

Our first task is to be able to simulate equation (1.5) exactly. In order to do so, we cite the following result from [12]:

**Theorem 3.1** Consider the one-dimensional diffusion process with laws $\{\beta Q^d_x, x \geq 0\}$ ($x$ is the initial point; $d \geq 0$ and $\beta$ are fixed parameters, the former called the dimension), with infinitesimal generator

$$2x\frac{D^2}{2x^2} + (2\beta x + d)\frac{D^2}{2}.$$

Then for all real $\beta \neq 0$, $\beta Q^d_x$ is the $Q^d_x$-law of the process,

$$\exp(2\beta t)X \left( \frac{1 - \exp(-2\beta t)}{2\beta} \right)$$

where $Q^d_x$ is the distribution of the $d$-dimensional squared Bessel Process denoted by $BESQ^d$ with infinitesimal generator,

$$2x\frac{D^2}{2x^2} + d\frac{D^2}{2}.$$

**Corollary 3.2** The space-time transformation

$$
V_t = \exp(-\kappa \tau(t))X_{\tau(t)}, \quad V_0 = x_0, \quad (3.1)
$$

$$
\tau(t) = \frac{\sigma^2}{4\kappa} [\exp(\kappa t) - 1], \quad \tau_0 = t_0 = 0 \quad (3.2)
$$

transforms the square-root diffusion process in (1.5) to the $\lambda$-dimensional squared Bessel process:

$$
dX_u = \lambda du + 2\sqrt{X_u}dW_u, \quad X(0) = x_0 = V_0 \quad (3.3)
$$

where $\lambda = \frac{4\kappa \theta}{\sigma^2}$. 

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Proof A direct proof of this result is deferred to Appendix A.

The squared Bessel process in (3.3) will also be referred to as a squared Bessel process of order \( \nu \equiv \lambda/2 - 1 \). From [4] we know that the boundary point 0 is strongly reflective when \(-1 < \nu < 0\) and is “entrance-not-exit” when \( \nu \geq 0 \). Following [16] and [8] we simulate the squared Bessel process over any interval \((\tau_j, \tau_{j+1}]\) with \( x(\tau_j) \) known (already simulated for \( j > 0 \)), using the randomized Gamma distribution of the first kind, \( \mathcal{G}(\nu + \eta + 1, 2\Delta \tau) \), where \( \Delta \tau = \tau_{j+1} - \tau_j \) and \( \eta \) is sampled from the Poisson distribution \( \mathcal{P}(\mu) \), \( \mu = x(\tau_j) / 2\Delta \tau \):

\[
X(\tau_{j+1}) \sim \mathcal{G}(\nu + \eta + 1, 2\Delta \tau) \quad \text{with} \quad \eta \sim \mathcal{P}\left(\frac{x(\tau_j)}{2\Delta \tau}\right).
\]

Once we obtain the simulated values of \( X \) on the set of points \( \tau_j = \tau(t_j), j = 1, 2, \ldots, N \), we obtain the corresponding \( V_j \) values using the transformation (3.1).

4 Simulation of Bessel Bridge Process

As already emphasised, the most important and difficult piece of our algorithm is the simulation of the integral \( \int_{t_j}^{t_{j+1}} V(s) \, ds \). We propose to do this by recursively applying a Bessel bridge simulation, to fill in the intermediate points in the interval, \([t_j, t_{j+1}]\), for a random quadrature. Methods for selecting the intermediate points are described later in the paper. In this section we describe the method of generation of a Bessel bridge process, and its application to the quadrature.

Suppose that on any generic interval \([\tau_L, \tau_R]\) we need to insert another point \( \tau_M \): \( \tau_L < \tau_M < \tau_R \). The corresponding BESQ\(^A\) value \( x_M \) is simulated using the randomized Gamma distribution of the second kind, \( \mathcal{G}(\cdot, \cdot) \) (see e.g., [16], [8]):

\[
X(\tau_M) \sim \mathcal{G}\left(\nu + \eta_1 + 2\eta_2 + 1, \frac{\Delta \tau}{2\Delta \tau_L \Delta \tau_R}\right)
\]

where \( \Delta \tau = \tau_R - \tau_L, \Delta \tau_L \equiv \tau_M - \tau_L, \Delta \tau_R \equiv \tau_R - \tau_M \), and where \( \eta_1 \) is sampled from a Poisson distribution, \( \mathcal{P}(\cdot) \), and \( \eta_2 \) is sampled from a Bessel distribution, \( \mathcal{B}(\cdot, \cdot) \):

\[
\eta_1 \sim \mathcal{P}\left(1, \frac{\Delta \tau_R}{\Delta \tau_L \Delta \tau_R}x_L + \frac{\Delta \tau_L}{\Delta \tau_L \Delta \tau_R}x_R\right)
, \quad \eta_2 \sim \mathcal{B}\left(\nu, \sqrt{\frac{\Delta \tau_L x_L \Delta \tau_R x_R}{\Delta \tau}}\right) .
\]

Remark 4.1 For simulating the squared Bessel process and the Bessel bridge process we need efficient Poisson and Bessel random variate generators. For generating Poisson variates we refer to [7], [2] and [6]. For generating Bessel variates we refer to Section 2 in [8] and especially to the comments appearing in Section 4 in [16].

Once we have generated intermediate values on \( K - 1 \) intermediate points \( \tau_1, \tau_2, \ldots, \tau_{K-1} \) (\( \tau_0 \) corresponds to the left endpoint and \( \tau_K \) corresponds to the right endpoint of the integration interval,

1. These points may be chosen directly in the \( \tau \)-space, or in the \( t \)-space and mapped to \( \tau \)-space using the one-one mapping. [24].
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\[ \int_{t_j}^{t_{j+1}} V(s) \, ds \approx \sum_{k=1}^{K} \mathbb{E} \left[ \int_{s_{k-1}}^{s_k} V(s) \, ds \mid X(\tau_{k-1}) = x(\tau_{k-1}), X(\tau_k) = x(\tau_k) \right]. \]  

(4.1)

In the next section, we develop a closed form formula for the conditional expectations on the right-hand side of (4.1). For the sake of brevity, when the time interval is fixed, we sometimes denote the conditional expectation operator in (4.1) as \( \mathbb{E}_{X(t),X(\tau)} \), where \( L \) and \( R \) signify the “frozen” left- and right-hand endpoints of the interval, respectively.

We now have the strategy for simulating \( \int_{t_j}^{t_{j+1}} V(s) \, ds \), for some given endpoints \( t_j \) and \( t_{j+1} \), except for the choice of \( K \) and the intermediate points. These topics are discussed in Section 6.

5 Theoretical Results for Integrals of \( V, X \)

In this section, we collect some theoretical results on the integrals of \( V \) and \( X \). The proofs of some of the results are deferred to the Appendices. Our first result expresses the variance integral in the \((V, t)\) space in terms of the one in \((X, \tau)\) space. We work on a general interval, \([t_L, t_R]\) which, in application, could be \([t_j, t_{j+1}]\) or any of its subintervals upon refinement of a partition of \([t_j, t_{j+1}]\).

Set \( \tau_L = \tau(t_L), \tau_R = \tau(t_R) \).

Lemma 5.1 On any interval \([t_L, t_R]\), the integral of the variance \( V \) is described in terms of the canonical BESQ process \( X \) as:

\[ \int_{t_L}^{t_R} V_t \, dt = \int_{t_L}^{t_R} X(u)w_0(u) \, du \]

(5.1)

\[ w_0(u) := a_0[1 + cu]^2, \quad a_0 := 4\sigma_V^{-2}, c := 4\kappa_\sigma_V^{-2}. \]  

(5.2)

Proof Let \( \phi = \tau^{-1} \) so that \( \phi(u) = t \) corresponds to \( \tau(t) = u \), where \( \tau \) is given by (3.2). Now, \( dt = \phi'(u) \, du \) and

\[ \phi'(u) = 1/\tau'(\phi(u)) = \left[ \frac{\sigma_V^2}{4} e^{\kappa\phi(u)} \right]^{-1}. \]

From the relation, \( u = \tau(\phi(u)) \), we have that \( u = \frac{\sigma_V^2}{4\kappa}[e^{\kappa\phi(u)} - 1] \) and so

\[ e^{\kappa\phi(u)} = 1 + \frac{4\kappa}{\sigma_V^2} u, \quad \phi'(u) = \frac{4\kappa}{\sigma_V^2} \left[ 1 + \frac{4\kappa}{\sigma_V^2} u \right]^{-1}. \]

Therefore, by (3.1),

\[ \int_{t_L}^{t_R} V_t \, dt = \int_{t_L}^{t_R} \exp(-\kappa\phi(u))X(u)\phi'(u) \, du \]

\[ = \int_{t_L}^{t_R} X(u) \frac{4}{\sigma_V^2} \left[ 1 + \frac{4\kappa}{\sigma_V^2} u \right]^{-2} \, du \]

\[ \equiv \int_{t_L}^{t_R} X(u)w_0(u) \, du. \]
Although the $\text{BESQ}$ process and corresponding bridge are both temporally translation invariant, the function $w_0$ in (5.2) is not. Therefore, when transforming the $\tau$-space integral to a standard interval, “$[0, \tau]$”, the integrand in (5.3) will be modified, resulting in ($\overset{D}{=} n$ denoting equality in distribution)

$$
\int_{t_L}^{t_R} V_t \, dt \overset{D}{=} \int_0^\tau X(u + \tau_L)w(u) \, du + \int_0^\tau X(u)w(u) \, du, \quad \tau = \tau_R - \tau_L, \quad (5.3)
$$

$$
w(u) \overset{\text{def}}{=} a_1[b_1 + c_1 u]^{-2}; \quad a_1 := 4\sigma_\nu^{-2}, \quad b_1 := 1 + 4\kappa\sigma_\nu^{-2} \tau_L, \quad c_1 := 4\kappa\sigma_\nu^{-2}, \quad (5.4)
$$

where we have suppressed the dependence on $L,R$.

In the implementation of stopping criteria, the conditional variance of the integral in (5.3) is required, conditional on the endpoint values of the bridge. For that, we state a result on the first two moments of the first kind (see [1]), and the constants $A_i, B_i, C_i, i = 1,2$, are described in Proposition B.4, in Appendix B.

**Theorem 5.2** For the $\text{BESQ}$ process $X$ of order $\nu$, frozen on endpoints, $X(0) = x$ and $X(\tau) = y$ (where $\tau > 0$ is arbitrary), and $w$ as in (5.4), we have

$$
\mathbb{E}_x \left[ \int_0^\tau X(u)w(u) \, du \mid X(\tau) = y \right] = (A_1 + B_1) \left( \nu + 1 + \frac{z}{\tau} R_\nu \left( \frac{z}{\tau} \right) \right) - \frac{(B_1 + C_1)x + (2A_1 + C_1)y}{2\tau} \quad (5.5)
$$

and

$$
\mathbb{E}_x \left[ \left( \int_0^\tau X(u)w(u) \, du \right)^2 \mid X(\tau) = y \right] = 2\left[ A_1^2 + B_1^2 + A_1B_1 - A_2 - B_2 \right] + \left( \frac{A_1^2 + B_1^2 + 2(A_1 + B_1)^2 - 2A_2 - 2B_2}{\nu + (A_1 + B_1)^2} \right) \frac{\tau}{4\nu^2}
$$

$$
+ \left( \frac{B_1 + C_1)x + (2A_1 + B_1)y}{\tau} \right)^2 + \frac{(B_1 + C_1)x + (2A_1 + B_1)y}{\tau} \left( \nu + 1 + \frac{z}{\tau} R_\nu \left( \frac{z}{\tau} \right) \right) \quad (5.6)
$$

where $z = \sqrt{xy}$, $R_\nu(r)$ is the Bessel quotient $I_{\nu+1}(r)/I_\nu(r)$, where $I_\nu$ is the modified Bessel function of the first kind (see [1]), and the constants $A_i, B_i, C_i, i = 1,2$, are described in Proposition B.4 in Appendix B.

**Remark 5.3** It is not evident that the right-hand sides of (5.5) and (5.6) tend to zero as $\tau$ tends to zero. However, they do; e.g., the first moment tends to zero at a linear rate and the variance tends to zero at a quadratic rate. To see this explicitly, we reformulate the moments in terms of the parameters $A,b,c$ (see Proposition B.4 and Corollary B.3 in Appendix B) for which $A \to 0$.
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quadratically fast and . Then, in the expression for \( A_i, B_i, C_i \) \((i = 1, 2)\) we expand,

\[
\log(b + c) = \log b + \log(1 + c/b) = \log b + \frac{c}{b} - \frac{c^2}{2b^2} + \frac{c^3}{3b^3} + \mathcal{O}(\tau^4).
\]

Writing \( c = c_1 \tau \), it is then straightforward to check that

\[
\lim_{\tau \to 0} \frac{A_1}{\tau^2} = \frac{c_1}{b^2}, \\
\lim_{\tau \to 0} \frac{B_1}{\tau^2} = \frac{4c_1}{3b^2}, \\
\lim_{\tau \to 0} \frac{C_1}{\tau^2} = -\frac{2c_1}{b^2}, \\
\lim_{\tau \to 0} \frac{A_2}{\tau^4} = \frac{5c_1^2}{6b^4}, \\
\lim_{\tau \to 0} \frac{B_2}{\tau^4} = \frac{8c_1^2}{15b^4}, \\
\lim_{\tau \to 0} \frac{C_2}{\tau^4} = \frac{4c_1^2}{3b^4}.
\]

6 Adaptive Estimate of the Integral \( \int_{t_j}^{t_{j+1}} V(s) \, ds \)

We return now to the problem stated at the end of Section 4, namely, the selection of the intermediate (quadrature) points for the estimation of the integral, \( \int_{t_j}^{t_{j+1}} V(s) \, ds \).

There are three aspects to the selection of intermediate points, at any stage in the recursion, in an adaptive fashion: (i) the manner of refinement (i.e., the geometric placement of an inserted partition point or points between those already generated); (ii) a choice of stopping criterion, to decide if a subinterval needs to be further refined; (iii) a decision to apply the stopping criteria locally or globally. By definition, a local decision means that a subinterval will be refined if the stopping criterion is not met on that interval, whereas a global decision means that all intervals will be refined if any of them do not meet the stopping criterion.

Regarding (i), we do not force any particular refinement scheme, although for the numerical experiments in Section 8 we use bisection in \( t \)-space. Regarding (iii), we restrict our attention to the class of locally adaptive schemes, which we denote by \( ADAPT \). The main purpose of the current section is to introduce the stopping criteria of aspect (ii).

In general, a stopping criterion on an interval, involves a tolerance, \( \delta \), and a quantity to monitor. The refinement continues as long as the quantity being monitored is not within the tolerance. In most cases of interest, the tolerance depends on the interval being considered; in fact, it may depend on the entire history of refinement that led to that interval. To clarify the tolerance’s dependence on the interval, we make a brief digression on the aspect of tolerance in a refinement scheme.

The initial tolerance, \( \delta_0 \), for the “root” interval \([\tau_j, \tau_{j+1}] \equiv [\tau(t_j), \tau(t_{j+1})] \) will be user-given. For the sake of simplicity we shall relabel this interval as \([\tau_0, \tau_1] \) During the refining of \([\tau_0, \tau_1] \), \( \delta_0 \) is apportioned among the subintervals created by the refinement, leading to the \( \delta \) for each such subinterval.
We next describe one possible and simple set of rules of apportionment; other rules are certainly permitted. In particular, a more complex and efficient set of rules is described in Section 7.

Denote the current subinterval being monitored, by $[\tau_L, \tau_R] \equiv [\tau(t_L), \tau(t_R)]$. If the stopping criterion fails on this interval, then it is partitioned into precisely two subintervals. The typical case is where refinement is by bisection (either in $t$- or $\tau$-space) and the apportionment of tolerance is into equal parts. For example, if the bisection occurs in $\tau$-space, then the interval $[\tau_L, \tau_R]$ inherits the tolerance, $\delta_0(\tau_R - \tau_L)/(\tau_1 - \tau_0)$. Similarly, if the bisection is carried out in $t$-space, the tolerance is given by $\delta_0(t_R - t_L)/(t_1 - t_0)$.

The stopping criterion for our adaptive scheme, hereafter referred to as ADAPT, is based on the computation of the variance of the integral. If the variance of the integral over any interval, as computed from (5.3), (5.5) and (5.6), is below the tolerance, $\delta > 0$, available for the interval, we refrain from further partitioning of that interval: for the interval, $[\tau_L, \tau_R]$,

$$ADAPT\ \text{stopping criterion:}$$

$$\text{Var} \left[ \int_{t_L}^{t_R} V(s) \, ds \mid X_{\tau_L} = x_L, X_{\tau_R} = x_R \right] \equiv \text{Var} \left[ \int_{\tau_L}^{\tau_R} X(u)w_0(u) \, du \mid X_{\tau_L} = x_L, X_{\tau_R} = x_R \right] < \delta. \quad (6.1)$$

This criterion is applied to each subinterval $[\tau_L, \tau_R]$ and, if met, no further partitioning of that interval is done. If the criterion is not met, then the interval is partitioned.

Alternatively, when the criterion is not met, the interval $[t_L, t_R]$ can be partitioned; then the resulting subintervals can be mapped to the corresponding subintervals of $[\tau_L, \tau_R]$, for the generation of intermediate $X$ values.

7 A more efficient adaptive scheme

In the previous section, we introduced some locally adaptive schemes. These schemes are robust in the sense that they recursively refine until the monitored quantity is below the relevant tolerance. The quantity that is monitored is subadditive in the sense that once the adaptivity algorithm has terminated, the sum of the contributions over all the subintervals will be smaller than the initial user-given tolerance, $\delta_0$. However, we did not address the issue that this total over all the elements may be significantly lower than $\delta_0$. In other words, we did not care about the minimality of our refinement. Due to the inherent randomness in the quantity monitored and thus the placement of the intermediate points, it may not be possible to obtain a minimal grid that barely satisfies the adaptivity criterion (6.1), for example.

However, one can do slightly better than the plain adaptivity schemes described in the previous section. One can approach, what we choose to describe as a near-minimal grid by introducing a global reservoir of tolerance, $\delta_R$. In essence, we allow the cross-subordination of the tolerance assigned to various intermediate elements. When an element passes the adaptivity criteria, it releases the excess tolerance that it has over the monitored quantity, to the reservoir. In the testing of subsequent elements, the monitored quantity is compared against a more lenient tolerance – one that is the sum of the inherited tolerance level, $\delta$ (as usual), plus the reservoir tolerance, $\delta_R$.

Here is the precise scheme, in algorithmic form. Suppose we need to evaluate the integral of the Variance process on an interval $[t_0, t_1]$ and further suppose the tolerance level is set to $\delta_0$. Our algorithm consists of the following steps:
Step 1: (Initialisation) Create a reserve tolerance, $\delta_R$, (whose role will be explained in Steps 3, 4) and initialise it to zero ($\delta_R = 0$). Prepare an empty stack of 5-tuples $(T_L, T_R, X_L, X_R, \delta)$ and push the initial data, $(t_0, t_1, X(\tau(t_0)), X(\tau(t_1)), \delta_0)$, onto the stack, where $X$ is the $BESQ^\lambda$ process.

Step 2: If the stack is empty go to Step 6.

Step 3: Pop the top element from the stack. Extract the values and check if $\Delta < 0$, where

$$\Delta := \text{Var} \left[ \int_{T_L}^{T_R} V(s) \, ds \mid X(\tau(T_L)) = X_L, X(\tau(T_R)) = X_R \right] - (\delta + \delta_R)$$

If satisfied, go to Step 4; otherwise go to Step 5. (Notice the way in which we use the reserve tolerance $\delta_R$ to facilitate passing of the stopping criterion for other “more needy” elements).

Step 4: Set $\delta_R = -\Delta$. Go to Step 2.

Step 5: Compute the midpoint $T_M = (T_L + T_R)/2$. Sample $X_M = X(\tau(T_M))$ from the $BESQ^\lambda$ process. Push the elements $(T_L, T_M, X_L, X_M, \delta/2)$ and $(T_M, T_R, X_M, X_R, \delta/2)$ onto the stack and go to Step 3. (Notice the way in which we distribute the tolerance in equal parts to the newly spawned intervals).

Step 6: Exit.

As can be seen from this algorithm, we have incorporated the following features that allow us to claim that the algorithm will result in a “near minimum” number of degrees of freedom necessary:

- Our strategy is adaptive. An interval only gets subdivided if it fails the test; in the event of a failure, it gives out its acquired delta (from its parent interval) to the newly spawned intervals. Thus there is no loss (or waste) of $\delta$.
- In the event the test is successful on any interval, it only consumes the amount of acquired delta that is necessary to pass the test, and releases the excess to the tolerance reservoir. The accumulated tolerance in the reservoir can be used later to help pass the test on the remaining intervals.

Due to the inherent randomness in the above strategy, an adaptive algorithm that passes under an absolute minimum number of degrees of freedom, may be difficult to find.

8 Numerical Experiments

In this section we support the ideas introduced in this paper by a series of numerical experiments. To compare the results of our method with other methods available, such as the finite difference method, we choose the same test problem as the one appearing in [5] (Table I, Section 4). For ease of reference, the input parameter values are: $S = 100$, $K = 100$, $V_0 = 0.010201$, $\kappa = 6.21$, $\theta = 0.019$, $\sigma_V = 0.61$, $\rho = -0.7$, $r = 3.19\%$, $T = 1$ year. Also, as a benchmark, the true option price = 6.8061.

In our tests we compare our methodology with a variant of the finite difference method which uses a predictor-corrector step for better convergence. Also, we have chosen to carry out the refinement scheme in the $t$-space and the refinement involves simple bisection of the interval in question.

Since our method is based on discretisation, a bias in the numerical method is expected. However, in contrast to the other methods that we know of, our method allows a systematic control of this bias by means of relating the bias to another numerically observable quantity: the variance of the integral of the variance process over the interval in question. Furthermore, we put great effort in ensuring that the computational cost of achieving the limit on the variance of the integral (and
therefore indirectly on the bias in our method) is kept to a near minimum as explained in Section 7. To this end, it is clear that the very first set of tests should demonstrate the ability of our algorithm to consume as little computational resources as possible. We do this in Tables 1 and 2. For our test problem, we note that the dimension of the BESQ process is given by $\lambda = 1.2684$ and the order is $\nu = -0.3658$. The origin is accessible and is also reflective. The moderately high value of the volatility of the Variance may also result in many paths taking very high values. Based on this observation we decided to capture the properties of our algorithm by banding the endpoint value. As expected, we see from Table 1 that for a path that starts at a moderate value and reaches a fairly high value, the number of the intermediate points that need to be inserted, to reduce the variance of the integral below a given tolerance, is also very high. It appears that the number of intermediate points is proportional to the absolute difference between the left and the right endpoint values. We also observe that the distribution of the number of intermediate points is tighter for lower endpoint values compared to higher endpoint values.

In Table 2, we demonstrate the effect of cross subordination of tolerance which forms an essential part of our algorithm. What we show in Table 2 is the tolerance level that was wasted by the algorithm; i.e., for the interval in question, the difference between the given tolerance by the user, and the sum of the actual variances of the integrals over the subintervals. We again study this in terms of bands of endpoint values. As to be expected, the wasted tolerance decreases significantly when more and more intermediate points are inserted (as can be seen with the distribution’s very short left tail, for high endpoint values). This effect is clear because, when inserting more points, more iterations are made and hence more use is made of the reserve tolerance. This is an important feature of our algorithm as it shows that the wastage is minimal when there is highest demand for refinement.

Figures 1 and 2 illustrate the expected fact that the bias is reduced in both the predictor-corrector method and our adaptive algorithm, as more and more intermediate points are introduced. In the predictor-corrector method the independent variable is directly the number of intermediate partitions, whereas in our algorithm, the independent variable is the tolerance provided by the user. Due to this mismatch in the independent variable it is not obvious how to compare the relative performance of these methods. So in this sense, Figures 1 and 2 may be considered just a sanity check of the expected way in which these algorithms are supposed to work.

Our next task is to compare the predictor-corrector method with our algorithm. For this we introduce the quantification of accuracy, which is defined as the inverse of the absolute relative bias. Furthermore, we saw from Figures 1 and 2 that using the predictor-corrector method with smaller interval size has the same directional effect as reducing the tolerance level in our algorithm, and both these actions result in increasing the time spent in simulation. Therefore the most ideal way of comparing the two methods is by plotting the accuracy, as defined above, versus the time spent in simulation. This is what is shown in Figure 3. As a byproduct of this analysis, we make a very interesting observation regarding our method. It is exponentially rewarding in the initial part with a much steeper slope than the predictor-corrector method. Note that the accuracy is plotted on a logarithmic scale. We also note that both the methods taper off as we move to the right, with diminishing rewards. This may be attributed to the fact that reducing the bias substantially below the simulation error is fruitless. This last observation brings us to the final figure of this section, Figure 4. In most risk management work, the computational budget associated with pricing a financial derivative is limited. Based on that constraint, only a small number of simulation paths (typically between 1000–5000) are used for pricing. Of course, this results in a large value for the error associated with the Monte Carlo method. It is clearly pointless to control the bias to any order of magnitude below this error. Figure 4 allows us to demonstrate that controlling the bias with a
Table 1: Number of intervals used in the calculation of the integral of $V$ (tolerance $= 0.000001$).

| right bin boundaries | 0.000001 | 0.0001 | 0.01 | 0.04 | 0.09 | 0.16 | 0.25 | 0.36 | 0.49 | 0.64 | 0.81 | 1 |
|----------------------|----------|--------|------|------|------|------|------|------|------|------|------|---|
| 8                    | 0        | 0      | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0  |
| 16                   | 2539     | 2507   | 1715 | 874  | 353  | 77   | 16   | 0    | 0    | 0    | 0    | 0  |
| 24                   | 5098     | 5134   | 4978 | 4329 | 3295 | 1937 | 730  | 188  | 36   | 0    | 0    | 0  |
| 32                   | 1886     | 1857   | 2490 | 3241 | 3732 | 3780 | 3008 | 1656 | 651  | 180  | 40   | 0  |
| 40                   | 366      | 385    | 594  | 1016 | 1658 | 2437 | 3037 | 3027 | 2255 | 1364 | 608  | 156|
| 48                   | 96       | 97     | 183  | 382  | 637  | 1123 | 1887 | 2663 | 3032 | 2761 | 1986 | 1057|
| 56                   | 12       | 19     | 36   | 132  | 244  | 463  | 908  | 1517 | 2279 | 2801 | 2999 | 2453|
| 64                   | 3        | 0      | 4    | 22   | 70   | 141  | 298  | 646  | 1124 | 1744 | 2342 | 2820|
| 72                   | 0        | 1      | 0    | 2    | 8    | 33   | 85   | 203  | 419  | 726  | 1266 | 1973|
| 80                   | 0        | 0      | 0    | 2    | 1    | 6    | 21   | 59   | 139  | 293  | 536  | 1016|
| 88                   | 0        | 0      | 0    | 0    | 2    | 3    | 10   | 41   | 65   | 128  | 223  | 518 |

Table 2: Unused tolerance (0.000001) in the calculation of the integral of $V$.

| right bin boundaries | 0.000001 | 0.0001 | 0.01 | 0.04 | 0.09 | 0.16 | 0.25 | 0.36 | 0.49 | 0.64 | 0.81 | 1 |
|----------------------|----------|--------|------|------|------|------|------|------|------|------|------|---|
| 8                    | 0        | 0      | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0  |
| 16                   | 2539     | 2507   | 1715 | 874  | 353  | 77   | 16   | 0    | 0    | 0    | 0    | 0  |
| 24                   | 5098     | 5134   | 4978 | 4329 | 3295 | 1937 | 730  | 188  | 36   | 0    | 0    | 0  |
| 32                   | 1886     | 1857   | 2490 | 3241 | 3732 | 3780 | 3008 | 1656 | 651  | 180  | 40   | 0  |
| 40                   | 366      | 385    | 594  | 1016 | 1658 | 2437 | 3037 | 3027 | 2255 | 1364 | 608  | 156|
| 48                   | 96       | 97     | 183  | 382  | 637  | 1123 | 1887 | 2663 | 3032 | 2761 | 1986 | 1057|
| 56                   | 12       | 19     | 36   | 132  | 244  | 463  | 908  | 1517 | 2279 | 2801 | 2999 | 2453|
| 64                   | 3        | 0      | 4    | 22   | 70   | 141  | 298  | 646  | 1124 | 1744 | 2342 | 2820|
| 72                   | 0        | 1      | 0    | 2    | 8    | 33   | 85   | 203  | 419  | 726  | 1266 | 1973|
| 80                   | 0        | 0      | 0    | 2    | 1    | 6    | 21   | 59   | 139  | 293  | 536  | 1016|
| 88                   | 0        | 0      | 0    | 0    | 2    | 3    | 10   | 41   | 65   | 128  | 223  | 518 |

tolerance below $1.56e-06$, say, has diminishing returns.
Figure 1: Estimation of bias in the predictor-corrector method (1000 trials of 10000 samples).

Figure 2: Estimation of bias in the ADAPT method (1000 trials of 10000 samples).
Figure 3: Comparison of the predictor-corrector against ADAPT algorithms: accuracy versus time; logarithmic scale on vertical axis.

Figure 4: Accuracy versus time trade-off in the ADAPT method (based on 10000 trials with 1000 samples). Node labels on graph are the tolerances logarithmic scale on vertical axis.
9 Conclusion

In this paper we explored a variant of the methodology proposed by Broadie and Kaya in [5], to simulate the dynamics of the Heston model. As our method relies on the numerical computation of the integral of the variance process, it is subject to bias. The adaptive nature of our method allows the efficient, practical control of this bias.

It is expected that for options on instruments with a large number of reset dates, the adaptive method of this paper should outperform the exact method of Broadie and Kaya.

In summary, our method provides the following benefits:

1. Unlike finite-difference methods, our method cannot generate negative values for the Bessel process and associated bridge.
2. Bias is efficiently controlled, as shown numerically by Figures 2 and 4.
3. It allows a much greater degree of flexibility than any of the other methods we have seen. This is advantageous where one can increase the tolerance level for middle-office risk-management work and reduce the tolerance level for front-office pricing. In other words, the tolerance level is a function of computational budget that one has at one’s disposal.
4. Perhaps the most interesting feature of our method is a “near-invariance” of the number of partitions (intermediate reset/coupon dates). Based on the observation that the most demanding part of our algorithm is the adaptive computation of the integral, we expect that the number of intermediate points we require for one big step of $T$ years is roughly equal to the total number of intermediate points required, had we decided to take $n$ steps of length $T/n$. In the latter case our method should perform much better than the Broadie-Kaya method.
5. As our algorithm derives, in essence, from the Broadie-Kaya algorithm we can safely assume that all the extensions to jump diffusion models presented in [5] should work with our algorithm as well. For the sake of brevity we do not reproduce the details in our paper.
6. Our method is straightforward to implement because many of the generators are now readily available in standard libraries.

The most important extension that awaits investigation, is to higher dimensional systems, for pricing options on equity baskets.

Appendices

Appendix A: Proof of Corollary 3.2

We will actually give the derivation in the opposite direction, from $X$ to $V$. Let $\phi = \tau^{-1}$ so that $u = \tau(t)$ corresponds to $t = \phi(u)$; $u_0 = \tau(t_0)$. Also, for a constant $c > 0$, to be determined, denote $Y^c_t = e^{ct}X_{\tau(t)} \equiv e^{c\phi(u)}X_u$; so that $Y^0_t = X_u$. Now, by (3.3),

$$X_u - X_{u_0} = \lambda(u - u_0) + 2\int_{u_0}^{u} \sqrt{X_s} \, dW_s$$

so

$$X_{\tau(t)} - X_{\tau(t_0)} = \lambda[\tau(t) - \tau(t_0)] + 2\int_{\tau(t_0)}^{\tau(t)} \sqrt{X_{\tau(\phi(s))}} \, dW_{\tau(\phi(s))}$$

$$= \lambda[\tau(t) - \tau(t_0)] + 2\int_{\tau(t_0)}^{t} \sqrt{X_{\tau(r)}} \, dW_{\tau(r)}$$
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where we have used a general time-substitution result for stochastic integrals (see Proposition (30.10) in Chapter IV of [14]) to transform the integral in the second equality. Therefore

\[ dY_t^c = cY_t^c dt + e^{ct} \left[ \lambda d\tau(t) + 2 \sqrt{X_{\tau(t)}} dW_{\tau(t)} \right] \]

Therefore

\[ dY_t^c = cY_t^c dt + \sigma^2 \frac{d}{4} dt + 2e^{ct/2} \sqrt{Y_t} dW_t. \] (A.1)

Let \( Z \) be a given Brownian motion and take for \( W \), the process defined by

\[ W_u = \frac{\sigma_V}{2} \int_{\phi(u_0)}^{\phi(u)} e^{-cs/2} dZ_s, \]

so that

\[ W_{\tau(t)} = \frac{\sigma_V}{2} \int_{t_0}^{t} e^{-cs/2} dZ_s \] (A.2)

and, by Itô’s formula,

\[ dw^2_{\tau(t)} = \frac{\sigma_V^2}{4} \left[ \frac{4}{\sigma_V} W_{\tau(t)} e^{-ct/2} dZ_t + e^{-ct} dt \right]. \]

Thus

\[ W_{\tau(t)}^2 - \frac{\sigma_V^2}{4} \int_{t_0}^{t} e^{-cs} dt \equiv W_{\tau(t)}^2 - \frac{\sigma_V^2}{4} \left[ \exp(-ct) - \exp(-ct_0) \right]/c \]

is a martingale. With the choice \( c = -\kappa \), we obtain that \( W_{\tau(t)}^2 - \tau(t) \) is a martingale and changing variables back to \( u \), that \( W_u^2 - u \) is a martingale (with respect to a different filtration, of course). Since \( W \) itself is clearly a continuous martingale, we conclude from Lévy’s theorem that \( W \) is a Brownian motion on \([t_0, \infty)\).

Returning to (A.1) and substituting the differential form of (A.2), we obtain, with \( V \equiv Y^{-\kappa} \) and \( \lambda = 4\kappa \theta \sigma_y^2 \),

\[ dV_t = -\kappa V_t dt + \kappa \theta dt + \sigma_V \sqrt{V_t} dZ_t. \]

Finally, we identify \( Z \) with \( W^{(1)} \). \( \square \)

Appendix B: Proof of Theorem 5.2

Our starting point is the following representation taken from [13] (see Theorem 3.2 and its proof on pages 442–443, therein) for the Laplace functional of the integral of a BESQ bridge, \( X \), which starts at \( x \) and ends at \( y \):

\[ E_{xy} \left[ \exp \left\{ - \int_0^1 X(u) d\mu(u) \right\} \right] \equiv E_x \left[ \exp \left\{ - \int_0^1 X(u) d\mu(u) \right\} \left| X(1) = y \right. \right], \quad x, y \geq 0 \]

where \( \mu \) is a Radon measure on \([0, \infty)\) with support in \([0, 1]\).

2. Note that we have replaced the arbitrary measure \( \mu \) by \( 2\mu \) and corrected two typographic errors on page 443: \( \delta \) should be divided by 2 and the subscript \( \rho^2(1) \) on \( q \) should be \( \sigma^2(1) \).
Then the constant value is in the classical sense thereon. Of course, where
\( \phi \) is constant on \((1, \infty)\) and being continuous everywhere, the constant value is \( \phi(1) \). We now show that \( \phi \) satisfies a Neumann boundary condition at \( u = 1 \).

**Theorem B.1** Let \( \mu \) be a Radon measure on \([0, \infty)\) with support in \([0, 1]\), and set
\[
F(x, y, \mu) := E_{x,y} \left[ \exp \left\{ - \int_0^1 X(u) \, d\mu(u) \right\} \right].
\]
Then
\[
F(x, y, \mu) = \left[ \phi(1) \int_0^1 \phi(u)^{-2} \, du \right]^{-1} \cdot I_\nu \left( \phi(1) \int_0^1 \phi(u)^{-2} \, du \right) \cdot \exp \left\{ \frac{x}{2} \left[ \phi'(0) - \left( \int_0^1 \phi(u)^{-2} \, du \right)^{-1} + 1 \right] \right\} \cdot \exp \left\{ \frac{y}{2} \left[ 1 - \left( \phi(1)^2 \int_0^1 \phi(u)^{-2} \, du \right)^{-1} \right] \right\}
\]
where \( I_\nu \) is the modified Bessel function of order \( \nu \) and \( \phi \) is the unique solution (in the sense of generalised functions) of the ODE
\[
\phi'' = 2\mu \cdot \phi, \quad \phi(0) = 1, \quad \phi \geq 0, \quad \phi \text{ nonincreasing on } [0, \infty).
\]
Consequently, \( \phi \) is convex and right-differentiable with a right-continuous, nonpositive right-derivative.

This result can be transferred to general interval \([0, \tau]\), by using the following result (see \cite{12} or \cite{13}): If \( Q_{x,y}^\tau \) denotes the law of \( X \) under which it is a Bessel bridge, starting at \( x \) and ending at \( y \) (at time \( \tau \)), then \( Q_{x,y}^\tau \) is also the \( Q_{x/\tau,y/\tau}^1 \)-law of \( \tau X(u/\tau) \), on \([0, \tau]\). The only case of interest to us, is when \( \mu \) has a density with respect to Lebesgue measure, necessarily of the form, \( m(u)1_{[0,\tau]}(u) \), \( 0 \leq u < \infty \). With a slight abuse of notation, we write \( F(x, y, m) \) instead of \( F(x, y, \mu) \) in this setting. The details of the transformation are as follows. Given the density \( m \), we set \( m_\tau(u) := \tau^2 m(\tau u) \), \( 0 \leq u \leq 1 \); \( m_\tau(u) := 0 \), for \( u > 1 \). Using the cited equivalence of laws and then making the change of variables, \( u \mapsto \tau u \), yields (with \( E_{x,y}^\tau \) denoting the \( Q_{x,y}^\tau \) expectation):
\[
E_{x,y}^\tau \left[ \exp \left\{ - \int_0^\tau X(u) m(u) \, du \right\} \right] = E_{x/\tau,y/\tau}^1 \left[ \exp \left\{ - \int_0^\tau X(u/\tau) m(\tau u) \, du \right\} \right] = E_{x/\tau,y/\tau}^1 \left[ \exp \left\{ - \int_0^1 X(u) m_\tau(u) \, du \right\} \right] = F \left( \frac{x}{\tau}, \frac{y}{\tau}, m_\tau \right).
\]

There is one more technical result which is needed to completely localise the problem to the support of \( \mu \). Again we restrict attention to the case where \( \mu \) has a density \( m \) which we assume is continuous on its support, the interval \([0, 1]\). In that case, the ODE is
\[
\phi'' = 2m \cdot \phi, \quad \phi(0) = 1, \quad \phi \geq 0, \quad \phi \text{ nonincreasing}.
\]

Standard regularity theory yields that \( \phi \) is smooth \((C^2)\) on \([0, 1]\) and satisfies the equation, \( \phi'' = 2m \phi \) in the classical sense thereon. Of course, \( \phi \) is constant on \((1, \infty)\) and being continuous everywhere, the constant value is \( \phi(1) \). We now show that \( \phi \) satisfies a Neumann boundary condition at \( u = 1 \).

**Lemma B.2** The left-hand derivative, \( \phi'(1^-) = 0 \), so that \( \phi \) is \( C^1 \) smooth across \( u = 1 \).
Proof Let $g \in C^\infty_0((0, \infty))$ be a test function. The ODE (aside from the boundary and side conditions) means that

$$\frac{1}{2} \int_0^\infty \phi(u) g''(u) \, du = \int_0^\infty m(u) \mathbb{1}_{[0,1]}(u) \phi(u) g(u) \, du = \int_0^1 m(u) \phi(u) g(u) \, du.$$ 

Also,

$$\frac{1}{2} \int_0^\infty \phi(u) g''(u) \, du = \int_0^1 \phi(u) g''(u) \, du + \int_1^\infty \phi(u) g''(u) \, du$$

$$= \int_0^1 \phi(u) g''(u) \, du - \phi(1) g'(1)$$

$$= -\phi'(1^-) g(1) + \int_0^1 \phi''(u) g(u) \, du$$

$$= -\phi'(1^-) g(1) + \int_0^1 m(u) \phi(u) g(u) \, du.$$ 

Therefore $\phi'(1^-) g(1) = 0$ for all $g \in C^\infty_0((0, \infty))$, which implies that $\phi'(1^-) = 0$. 

Thus we arrive at the final formulation of the required Laplace transform:

Corollary B.3 Let $m_\tau = a[b + cu]^{-2}$, $0 \leq u \leq 1$, where

$$a = \frac{8\theta \tau^2}{\sigma_V^2}, \quad b = 1 + \frac{4\kappa \tau L}{\sigma_V^2}, \quad c = \frac{4\kappa \tau}{\sigma_V^2}$$

($m_\tau = 2\theta \tau^2 w(\tau u)$, where $w$ was introduced at (5.4)) and set

$$L(\theta) := E_{xy}\left[ \exp \left\{ -\theta \int_0^\tau X(u) w(u) \, du \right\} \right].$$

Then

$$L(\theta) = \left[ \phi(1) \int_0^1 \phi(u)^{-2} \, du \right]^{-1} \cdot \frac{I_{\nu} \left( \frac{\sqrt{xy}}{\tau} \right)}{I_{\nu} \left( \frac{\sqrt{xy}}{\tau} \right)}$$

$$\cdot \exp \left\{ \frac{x}{2\tau} \left[ \phi'(0) - \left( \int_0^1 \phi(u)^{-2} \, du \right)^{-1} + 1 \right] \right\}$$

$$\cdot \exp \left\{ \frac{y}{2\tau} \left[ 1 - \left( \phi(1)^2 \int_0^1 \phi(u)^{-2} \, du \right)^{-1} \right] \right\}$$

(B.1)
where \( I_\nu \) is the modified Bessel function of order \( \nu \) and \( \phi \) is the unique solution to the BVP:

\[
\phi'' = m_r(u) \cdot \phi, \quad \phi(0) = 1, \quad \phi'(1) = 0. \tag{B.2}
\]

For the first two moments, we are interested in the coefficients of \(-\theta \) and \( \theta^2/2 \) in the Taylor expansion of \( L(\theta) \) about \( \theta = 0 \), since \( e^{-r} = 1 - r + r^2/2 + \cdots \) and thus the left-hand side of (B.1) equals

\[
1 - \theta \cdot E_{xy} \left[ \int_0^\tau X(u)w(u) \, du \right] + \frac{\theta^2}{2} \cdot E_{xy} \left[ \left( \int_0^\tau X(u)w(u) \, du \right)^2 \right] + O(\theta^3).
\]

Accordingly, we work out the Taylor expansion of the right-hand side of (B.1) to second order.

With reference to (B.2), the function \( \phi \) depends implicitly on \( \theta \) through the constant, \( a \), which appears in the definition of the function \( m_r \). We make this dependence explicit in our notation, by writing \( \phi(u; \theta) \). Clearly \( \phi(\cdot; 0) \equiv 1 \); so \( \phi'(0; 0) = 0 \). (Differentiation with respect to \( u \) will continue to be denoted by a prime (‘) superscript; differentiation with respect to \( \theta \) will be written explicitly; e.g., as a partial derivative, \( \partial/\partial \theta \).) Thus we set

\[
\phi(1; \theta) = 1 + A_1 \theta + A_2 \theta^2 + O(\theta^3)
\]

\[
\int_0^1 \phi(u; \theta)^{-2} \, du = 1 + B_1 \theta + B_2 \theta^2 + O(\theta^3)
\]

\[
\phi'(0; \theta) = C_1 \theta + C_2 \theta^2 + O(\theta^3)
\]

leaving the determination of the coefficients, \( A_i, B_i, C_i \) (\( i=1,2 \)) for later.

Expansion of the right-hand side of (B.1) in powers of \( \theta \), can be effected in a few stages. The terms involving a multiplicative inverse, like \( \left( \int_0^\tau \phi(u)^{-2} \, du \right)^{-1} \), can be expanded using the expansion for \( \phi \) and the geometric series expansion, \( r^{-1} = (1 - [1 - r])^{-1} = 1 + [1 - r] + [1 - r]^2 + O([1 - r]^3) \), for \( r \) close to 1. The two exponentials can be combined and then handled with the usual expansion, \( e^r = 1 + r + r^2 + O(r^3) \), for \( r \) close to 0. The ratio of modified Bessel functions can be handled, using the following two identities for modified Bessel functions (see 9.6.1, 9.6.26 in [1]):

\[
I_\nu''(r) = \left( \frac{r}{\nu} \right)^2 I_\nu(r) - \frac{1}{r} I_\nu'(r) \tag{B.3}
\]

\[
I_\nu'(r) = I_{\nu+1}(r) + \frac{\nu}{r} I_\nu(r). \tag{B.4}
\]

We can substitute (B.3) into (B.3) and divide both identities by \( I_\nu(r) \) to obtain the following ones in terms of the so-called Bessel quotient function, \( R_\nu = I_{\nu+1}(r)/I_\nu(r) \):

\[
\frac{I_\nu''(r)}{I_\nu(r)} = 1 + \frac{\nu^2 - \nu}{r^2} - \frac{1}{r} R_\nu(r)
\]

\[
\frac{I_\nu'(r)}{I_\nu(r)} = \frac{\nu}{r} + R_\nu(r).
\]

We can apply these identities to (B.1) by writing \( r_0 = \sqrt{xy}/\tau \) and \( r = r_0/\phi(1) \int_0^1 \phi(u)^{-2} \, du \), and expressing the ratio of Bessel functions in (B.1), in the form

\[
\frac{I_\nu(r)}{I_\nu(r_0)} = \frac{I_\nu(r_0 + [r - r_0])}{I_\nu(r_0)} = \frac{I_\nu(r_0) + I_\nu'(r_0) [r - r_0] + I_\nu''(r_0) [r - r_0]^2/2 + O([r - r_0]^3)}{I_\nu(r_0)}
\]

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and noting that

\[ r - r_0 = r_0[(\phi(1) \int_0^1 \phi(u)^{-2} du)^{-1} - 1] \]

has an expansion in \( \theta \), without constant term.

The remaining details are straightforward but tedious algebra which is omitted. The final result is the expression in Theorem 5.2.

We now turn to the calculation of the constants, \( A_i, B_i, C_i \) (i=1,2), in terms of the constants \( a, b, c \).

Denoting

\[ \Delta \theta(u) := \frac{\partial \phi(u; \theta)}{\partial \theta}, \quad \Delta(u) := \Delta_0(u) \]

\[ \Gamma \theta(u) := \frac{\partial^2 \phi(u; \theta)}{\partial \theta^2}, \quad \Gamma(u) := \Gamma_0(u) \]

we then have

\[ A_1 = \Delta(1), \quad A_2 = \frac{1}{2} \Gamma(1); \quad C_1 = \Delta'(0), \quad C_2 = \frac{1}{2} \Gamma'(0). \]  

(B.5)

Also, for \( B_1, B_2 \), note that

\[ \frac{d}{d \theta} \int_0^1 \phi(u; \theta)^{-2} du = \int_0^1 -2\phi(u; \theta)^{-3} \frac{\partial \phi(u; \theta)}{\partial \theta} du \]

\[ \frac{d^2}{d \theta^2} \int_0^1 \phi(u; \theta)^{-2} du = \int_0^1 6\phi(u; \theta)^{-4} \left[ \frac{\partial \phi(u; \theta)}{\partial \theta} \right]^2 - 2\phi(u; \theta)^{-3} \frac{\partial^2 \phi(u; \theta)}{\partial \theta^2} du. \]

Evaluating these results at \( \theta = 0^+ \), we obtain

\[ B_1 = \left. \frac{d}{d \theta} \right|_{\theta=0^+} \int_0^1 \phi(u; \theta)^{-2} du = -2 \int_0^1 \Delta(u) du, \]  

(B.6)

\[ B_2 = \frac{1}{2} \left. \frac{d^2}{d \theta^2} \right|_{\theta=0^+} \int_0^1 \phi(u; \theta)^{-2} du = \int_0^1 3\Delta(u)^2 - \Gamma(u) du. \]  

(B.7)

The functions \( \Delta \) and \( \Gamma \) can be found by differentiating \( \left( \frac{\partial}{\partial \theta} \right) \) the ODE and boundary conditions for \( \phi \) and then solving the resulting, very simple problems at \( \theta = 0^+ \). To that end, we bring out the \( \theta \)-dependence of \( m_\tau \) explicitly by writing

\[ m_\tau \equiv \theta M_\tau \]

with \( M_\tau \) independent of the parameter, \( \theta \). Then,

\[ \Delta'' = \frac{\partial}{\partial \theta} \phi'' = \frac{\partial}{\partial \theta} (\theta M_\phi) = M_\tau \phi + \theta M_\tau \frac{\partial \phi}{\partial \theta} \Rightarrow \Delta'' = M_\tau; \quad \text{also } \Delta(0) = 0, \ \Delta'(1) = 0; \]

\[ \Gamma'' = \frac{\partial^2}{\partial \theta^2} \phi'' = 2M_\tau \frac{\partial \phi}{\partial \theta} + \theta M_\tau \frac{\partial^2 \phi}{\partial \theta^2} \Rightarrow \Gamma'' = 2M_\tau \Delta; \quad \text{also } \Gamma(0) = 0, \ \Gamma'(1) = 0. \]

Integrating and using the boundary conditions, we obtain:

\[ \Delta'(u) = - \int_u^1 M_\tau(v) dv, \quad \Delta(u) = - \int_0^u \Delta'(v) dv; \]

\[ \Gamma'(u) = -2 \int_u^1 M_\tau(v) \Delta(v) dv, \quad \Gamma(u) = \int_0^u \Gamma'(v) dv. \]  

(B.8)

(B.9)
Proposition B.4 With \(a, b, c\) as in Corollary B.3, \(A := a/\theta\), and \(\log^2\) denoting the square of the log function,

\[
A_1 = \frac{A}{c} \left[ \frac{1}{b+c} + \frac{1}{c} \log[b/(b+c)] \right] 
\]

\[
A_2 = \frac{A^2}{c^3(b+c)} \left[ (b+c) \log^2[b/(b+c)] - 3b \log[b/(b+c)] - \frac{c(3b + 2c)}{b+c} \right] 
\]

\[
B_1 = -\frac{2A}{c^2} \left[ \frac{2b + 3c}{2(b+c)} + \frac{b+c}{c} \log[b/(b+c)] \right] 
\]

\[
B_2 = \frac{2A}{c^3(b+c)} \left[ 2(b+c)^2 \log^2[b/(b+c)] - 2(3b^2 + 8bc + 4c^2) \log[b/(b+c)] - c(6b + 11c) \right] 
\]

\[
C_1 = -\frac{A}{b(b+c)} 
\]

\[
C_2 = \frac{A^2}{c^3(b+c)} \left[ \frac{c(2b+c)}{b(b+c)} + 2 \log[b/(b+c)] \right] 
\]

**Proof** All of the integrals in (B.8) and (B.9) are straightforward to evaluate, as well as the integrals (B.6) and (B.7), for \(B_1\) and \(B_2\). We omit the elementary calculus and just state the end results, as they can be easily verified by differentiation and checking a boundary condition. Then one can use (B.5) to obtain \(A_1\), \(A_2\), \(C_1\), and \(C_2\).
\[
\int_0^u 3\Delta(v)^2 - \Gamma(v) \, dv
= \frac{A^2 u^3}{c^2(b + c)^2} + \frac{2A^2}{c^2(b + c)^2} [(2c^3 + 3bc^2 - (b + c)c^2 \log b + 2(b + c)c^2 \log[b + c]) u^2
+ 2A^2 b(3b + 2c) \log b \, u + \frac{2A^2}{c^4(b + c)^2} \frac{-4(b + c)b^2 + (5b + 3c)b^2 \log b - 6bc(b + c)}{c^3(b + c)^2}
+ \frac{A^2}{c^2(b + c)^2} (b + cu)^2 [2 \log[b + cu] - \log[b + cu] + 2]
+ \frac{A^2}{c^2(b + c)^2} (b + cu) [2(b + c)^2 \log^2[b + cu] - 2(b^2 - 3cb - 3c^2) \log[b + cu] - 2b(2b + c) \log[b + cu] + 2(b + 3c)(b + c)].
\]

Appendix C: Explicit Laplace transform

In the previous appendix, we avoided solving the BVP \([C.2]\) for \(\phi\) and then calculating \(\int_1^0 \phi(u)^{-2} \, du\). In order to extricate the moments of \(\int_1^0 X(u)w(u) \, du\) from its Laplace transform, it was sufficient to apply the method of variation of parameters, which led to expressions for the moments in terms of \(\phi\) and its derivatives with respect to the Laplace parameter, \(\theta\), at \(\theta = 0\). This method could work for more general weight functions than our specific \(w\).

However, to distributionally validate the \(ADAPT\) schemes, it is most convenient to have an explicit form for the Laplace transform \([B.1]\), and for that we now solve the BVP.

**Proposition C.1** The solution to the BVP

\[
\phi'' = m_\tau(u) \cdot \phi, \quad \phi(0) = 1, \quad \phi'(1) = 0
\]

is

\[
\phi(u) = eb^p(b + cu)^{-p} + (1 - e)b^q(b + cu)^{-q}
\]

\[
p = \frac{-1 + \sqrt{c^2 + 4a/c}}{2}, \quad q = \frac{-1 - \sqrt{c^2 + 4a/c}}{2}
\]

\[
\epsilon = -qb^\lbrack/[pb^s(b + c)] - qb^\lbrack].
\]

where \(a, b, c\) were defined in Corollary \([B.3]\).

**Proof** It is a simple matter to verify that the function described by \([C.2] - [C.4]\) satisfies the ODE and boundary conditions \([C.1]\). A sketch of the method of solution is as follows. We set \(\psi(u) = \phi(u/\sqrt{a})\) and then \(\psi(u) = f(r)\) where \(r = [b + \delta u]^{-1}\) and \(\delta = c/\sqrt{a}\). This leads to the following BVP for \(f\) on the interval \([r_e, b^{-1}]), r_e \equiv [b + c]^{-1}:

\[
r^2 \frac{d^2 f(r)}{dr^2} + 2r \frac{df(r)}{dr} = \delta^{-2} f(r), \quad f(b^{-1}) = 1, \quad f'(r_e) = 0.
\]
Seeking a solution of the ODE alone, in the form $r^p$, leads to the condition

$$p^2 + p - \delta^2 = 0$$

which has the two solutions $$(\text{C.3}).$$ The function

$$f(r) = \epsilon b^p r^p + (1 - \epsilon) b^q r^q$$

then satisfies the boundary condition, $f(b^{-1}) = 1$, for any $\epsilon$, while the choice, $$(\text{C.4}),$$ guarantees that the other boundary condition, $f'(r_e) = 0$, is satisfied. Unwinding the definitions from $f$ back to $\phi$ yields $$(\text{C.2}).$$

**Corollary C.2** With $p, q$ defined at $$(\text{C.3}),$$

$$\phi'(0) = -a b^q - b^p (b + c)^{1/2} (p - q) - (b + c)^2 (p - q) - (b + c)^2 (p - q)$$

$$\phi(1) = \frac{1}{b} \frac{p(b + c)^2 - q(b + c)^2}{b^p (b + c)^{1/2} (p - q) - q b^q}$$

$$\int_0^1 \phi(u)^{-2} du = \frac{(b(b + c))^{p - q} e^2}{8 (c^2 + 4a\theta)^{3/2} ((c^2 + 2a\theta) (b(b + c))^{p - q} + a (b^2 (p - q) + (b + c)^2 (p - q) \theta) \cdot \left( (p - q) b^{p - q} + (p - q)^2 + (b + c)^{p - q} \sqrt{b} + (b + c)^2 (p - q) \right)^2 \cdot \left( 2c^2 (p - q) (b(b + c))^{p - q} + (b + c)^2 (p - q) e^2 + 4a\theta \right) - b^2 (p - q) (4a\theta + c + c(p - q)) - e(b + c)^2 (p - q) e(p - q)}}$$

**Proof** The results $$(\text{C.5})$$ and $$(\text{C.6})$$ are straightforward calculations. On the other hand, the derivation of $$(\text{C.7})$$ is not straightforward; it was effected with the aid of Mathematica. 

**References**

[1] Abramowitz, M., Stegun, I. (Eds.): Handbook of Mathematical Functions. Dover Publications, New York (1970)

[2] Ahrens, J. H., Dieter, U.: Computer generation of Poisson deviates from modified normal distributions. ACM Trans. Math. Software, 8, No. 2, 163–179 (1982)

[3] Andersen, L.: Efficient simulation of the Heston stochastic volatility model. Journal of Computational Finance, Vol. 11, No. 3, 1–42 (2008)

[4] Borodin, A., Salminen, P.: Handbook of Brownian Motion – Facts and Formulae. Birkhäuser, Basel (1996)

[5] Broadie, M., Kaya, O.: Exact simulation of stochastic volatility and other affine jump processes. Operations Research, Vol. 54, Issue 2, 217–231 (2006)

[6] Hörmann, W.: The transformed rejection method for generating Poisson random variables. Insurance, Mathematics and Economics, 12, 39–45 (1993)

[7] Kemp, C. D., Kemp, A. W.: Poisson random variate generation. Appl. Statist., 40, No. 1, 143–158 (1991)
[8] Campolieti, G. Makarov, R.: Pricing path-dependent options on state-dependent volatility models with a Bessel bridge. JTAFA Vol. 10, Issue 01, 51–88 (2007)
[9] Cox, J.C., Ingersoll, J.E., Ross, S.A.: A theory of the term structure of interest rates. Econometrica, Vol. 53, No. 2, 385–407 (1985)
[10] Gatheral, J.: The Volatility Surface: A Practitioner’s Guide. John Wiley & Sons, Hoboken N.J. (2006)
[11] Heston, S.: A closed-form solution of options on assets with stochastic volatility with applications to bond and currency options. The Review of Financial Studies, Vol. 6, No. 2, 327–343 (1993)
[12] Pitman, J., Yor, M.: A decomposition of Bessel bridges. Z. W. verw. Gebiete 59, 425–457 (1982)
[13] Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion. (2nd edition) Springer-Verlag (1994)
[14] Rogers, L.C.G., Williams, D.: Diffusions, Markov Processes and Martingales: Volume 2, Itô Calculus (2nd edition). Cambridge University Press (2000)
[15] Soni, R. P.: On an inequality for modified Bessel functions. J. Math. Phys. 44, 406–407 (1965)
[16] Yuan, L., Kalbfleisch, J.: On the Bessel distribution and related problems. Ann. Inst. Statist. Math. Vol. 52, No. 3, 438–447 (2000)