Ideals generated by traces or by supertraces in the symplectic reflection algebra $H_{1,\nu}(I_2(2m+1))$ II

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Abstract

The algebra $\mathcal{H} := H_{1,\nu}(I_2(2m+1))$ of observables of the Calogero model based on the root system $I_2(2m+1)$ has an $m$-dimensional space of traces and an $(m+1)$-dimensional space of supertraces. In the preceding paper we found all values of the parameter $\nu$ for which either the space of traces contains a degenerate nonzero trace $tr_\nu$ or the space of supertraces contains a degenerate nonzero supertrace $str_\nu$ and, as a consequence, the algebra $\mathcal{H}$ has two-sided ideals: one consisting of all vectors in the kernel of the form $B_{tr_\nu}(x,y) = tr_\nu(xy)$ or another consisting of all vectors in the kernel of the form $B_{str_\nu}(x,y) = str_\nu(xy)$. We noticed that if $\nu = \frac{z}{2m+1}$, where $z \in \mathbb{Z} \setminus (2m+1)\mathbb{Z}$, then there exist both a degenerate trace and a degenerate supertrace on $\mathcal{H}$.

Here we prove that the ideals determined by these degenerate forms coincide.

Keywords: symplectic reflection algebra, trace, supertrace, ideal, dihedral group

1 Introduction

This paper is a continuation of [5]; we advise the reader to recall [5].

1.1 Definitions

Let $\mathcal{A}$ be an associative $\mathbb{Z}_2$-graded algebra with unit; let $\varepsilon$ denote its parity. All expressions of linear algebra are given for homogenous elements only and are supposed to be extended to inhomogeneous elements via linearity.

A linear complex-valued function $tr$ on $\mathcal{A}$ is called a trace if $tr(fg - gf) = 0$ for all $f, g \in \mathcal{A}$. A linear complex-valued function $str$ on $\mathcal{A}$ is called a supertrace if $str(fg - (-1)^{\varepsilon(f)\varepsilon(g)}gf) = 0$ for all $f, g \in \mathcal{A}$. These two definitions can be unified as follows.

Let $\kappa = \pm 1$. A linear complex-valued function $sp^\kappa$ on $\mathcal{A}$ is called $\kappa$-trace if $sp^\kappa(fg - \kappa^{\varepsilon(f)\varepsilon(g)}gf) = 0$ for all $f, g \in \mathcal{A}$.

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Each nonzero \( \kappa \)-trace \( sp^\kappa \) defines the nonzero symmetric \( \kappa \) bilinear form \( B_{sp^\kappa}(f, g) := sp^\kappa(fg) \).

If \( B_{sp^\kappa} \) is degenerate, then the set of the vectors of its kernel is a proper ideal in \( \mathcal{A} \). We say that the \( \kappa \)-trace \( sp^\kappa \) is degenerate if the bilinear form \( B_{sp^\kappa} \) is degenerate.

1.2 The goal and structure of the paper

The simplicity (or, alternatively, existence of ideals) of Symplectic Reflection Algebras or, briefly, SRA (for definition, see [3]) was investigated in a number of papers, see, e.g., [2], [9]. In particular, it is shown that all SRA \( H_{1,\nu}(G) \) with \( \nu = 0 \) are simple (see [10], [2]).

It follows from [4] and [7] that an associative algebra of observables of the Calogero model with harmonic term in the potential and with coupling constant \( \nu \) is simple (or, alternatively, existence of ideals) of Symplectic Reflection Algebras or, briefly, SRA (for definition, see [3]) was investigated in a number of papers, see, e.g., [2], [9]. In particular, it is shown that all SRA \( H_{1,\nu}(I_2(2m + 1)) \) has an \( m \)-dimensional space of traces and an \( (m + 1) \)-dimensional space of super traces.

We say that the parameter \( \nu \) is singular, if the algebra \( H_{1,\nu}(I_2(n)) \) has a degenerate trace or a degenerate supertrace.

In [5], we found all singular values of \( \nu \) for the algebras \( \mathcal{H} := H_{1,\nu}(I_2(n)) \) in the case of \( n \) odd \( (n = 2m + 1) \) and found the corresponding degenerate traces and super traces; the result is formulated in Theorem 10.1.

We noticed that if \( \nu = \frac{z}{2m+1} \), where \( z \in \mathbb{Z} \setminus (2m+1)\mathbb{Z} \), then there exist both a degenerate trace and a degenerate supertrace on \( H \).

Denote this degenerate trace by \( tr_z \) and the degenerate supertrace by \( str_z \).

Theorem 10.1 proved in [5] implies that if \( z \in \mathbb{Z} \setminus n\mathbb{Z} \), then

(i) the trace given by the formula (10.1) in [5] is degenerate and generates the ideal \( I_{tr_z} \) consisting of all the vectors in the kernel of the degenerate form \( B_{tr_z}(x, y) = tr_z(xy) \),

(ii) the supertrace (10.2) is degenerate and generates the ideal \( I_{str_z} \) consisting of all the vectors in the kernel of the degenerate form \( B_{str_z}(x, y) = str_z(xy) \).

The goal of this paper is Theorem 13.1, which proves

Conjecture 1.1. ([5 Conjecture 9.1]) \( I_{tr_z} = I_{str_z} \).

In Sections 2–10 we recall the necessary definitions and preliminary facts.

2 The group \( I_2(2m + 1) \)

Hereafter in this paper, \( n = 2m + 1 \).

Definition 2.1. The group \( I_2(n) \) is a finite subgroup of the orthogonal group \( O(2, \mathbb{R}) \) generated by the root system \( I_2(n) \).

The group \( I_2(n) \) is the symmetry group of a flat regular \( n \)-gon; \( I_2(n) \) consists of \( n \) reflections \( R_k \) and \( n \) rotations \( S_k \), where \( k = 0, 1, ..., 2m \). We consider the indices \( k \) as integers modulo \( n \).

These elements \( (R_k \) and \( S_k \) for all \( k \)) satisfy the relations

\[
R_kR_l = S_{k-l}, \quad S_kS_l = S_{k+l}, \quad R_kS_l = R_{k-l}, \quad S_kR_l = R_{k+l}.
\]

The element \( S_0 \) is the unit in the group \( I_2(n) \). Obviously, since \( n \) is odd, all the reflections \( R_k \) are in the same conjugacy class.

\[\text{Initially, we used the term "(super)symmetric bilinear form" currently used by many, e.g., in the paper [1], even in its title. However, in a recent preprint [3], it is explained that the supersymmetry } B(v, w) = (−1)^{p(v)p(w)}B(w, v) \text{ is related with the isomorphism } V \otimes W \simeq W \otimes V \text{ of superspaces and has nothing to do with the (anti)symmetry of the bilinear form } B \text{ on } V = W.\]
The rotations $S_k$ and $S_l$ constitute a conjugacy class if $k + l = n$.

Let
\[ \lambda := \exp \left( \frac{2\pi i}{n} \right). \]

Let
\[ G := \mathbb{C}[I_2(n)] \] (2.1)

be the group algebra of the group $I_2(n)$. In $G$, it is convenient to introduce the following basis
\[ L_p := \frac{1}{n} \sum_{k=0}^{n-1} \lambda^{kp} R_k, \quad Q_p := \frac{1}{n} \sum_{k=0}^{n-1} \lambda^{-kp} S_k. \]

3 Symplectic reflection algebra $H_{1,\nu}(I_2(2m + 1))$

Definition 3.1. The symplectic reflection algebra $H := H_{1,\nu}(I_2(2m + 1))$ is the associative algebra of polynomials in the noncommuting elements $a^\alpha$ and $b^\alpha$, where $\alpha = 0, 1$, with coefficients in $G$ (see Eq. (2.1)), satisfying the relations
\[ \begin{align*}
L_p a^\alpha &= -b^\alpha L_{p+1}, & L_p b^\alpha &= -a^\alpha L_{p-1}, \\
Q_p a^\alpha &= a^\alpha Q_{p+1}, & Q_p b^\alpha &= b^\alpha Q_{p-1}, \\
L_k L_l &= \delta_{k+l} Q_l, & L_k Q_l &= \delta_{k+l} L_l, \\
Q_k L_l &= \delta_{k+l} Q_l, & Q_k Q_l &= \delta_{k+l} Q_l,
\end{align*} \] (3.1)

\[ \begin{align*}
[a^\alpha, b^\beta] &= \varepsilon^{\alpha\beta} (1 + \mu L_0), \\
[a^\alpha, a^\beta] &= \varepsilon^{\alpha\beta} \mu L_1, \\
[b^\alpha, b^\beta] &= \varepsilon^{\alpha\beta} \mu L_{-1},
\end{align*} \]

where $\delta_k := \delta_{k0}$, and $\varepsilon^{\alpha\beta}$ is the skew-symmetric tensor normalized so that $\varepsilon^{01} = 1$, and \[ \mu := n\nu. \]

Defining the parity in $H$ by setting
\[ \varepsilon(a^\alpha) = \varepsilon(b^\alpha) = 1, \quad \varepsilon(R_k) = \varepsilon(S_k) = 0, \]

we turn this algebra into a superalgebra.

The algebra $H_{1,\nu}(I_2(2m + 1))$ depends on one complex parameter $\nu$.

4 Subalgebra of singlets

Consider the elements\footnote{Here the brackets $\{\cdot, \cdot\}$ denote anticommutator.} $T^{\alpha\beta} := \frac{1}{2}(\{a^\alpha, b^\beta\} + \{b^\alpha, a^\beta\})$ of the algebra $H$, and the inner derivations of $H$ they generate:
\[ D^{\alpha\beta} : f \mapsto [f, T^{\alpha\beta}] \text{ for any } f \in H. \]
It is easy to verify that the linear span of these derivations is a Lie algebra isomorphic to $sl_2$.

**Definition 4.1.** A singlet is any element $f \in H$ such that $[f, T^{\alpha\beta}] = 0$ for all $\alpha, \beta$. The subalgebra $H^0 \subset H$ consisting of all singlets of the algebra $H$ is called the subalgebra of singlets.

One can consider the algebra $H$ as an $sl_2$-module and decompose it into the direct sum of irreducible submodules.

Observe, that any $\kappa$-trace is identically zero on all irreducible $sl_2$-submodules of $H$, except for singlets.

Let the skew-symmetric tensor $\epsilon_{\alpha\beta}$ be normalized so that $\epsilon_{01} = 1$ and so $\sum_{\alpha} \epsilon_{\alpha\beta} \epsilon_{\alpha\gamma} = \delta_{\beta}^{\gamma}$. We set

$$s := \frac{1}{4i} \sum_{\alpha,\beta=0,1} \left( \{a^\alpha, b^\beta\} - \{b^\alpha, a^\beta\} \right) \epsilon_{\alpha\beta}.$$

**Proposition 4.2.** ([5, Proposition 4.2]) The subalgebra of singlets $H^0$ is the algebra of polynomials in the element $s$ with coefficients in the group algebra $\mathbb{C}[I_2(2m+1)]$.

The commutation relations of the singlet $s$ with generators of the algebra $H$ have the form:

$$[s, Q_p] = [s, S_k] = [T^{\alpha\beta}, s] = 0,$$

$$sL_p = -L_p s,$$

$$sR_k = -R_k s,$$

$$(s - i \mu L_0)a^\alpha = a^\alpha(s + i + i \mu L_0).$$

5 The form of ideals in $H$ and in $H^0$

**Theorem 5.1.** ([5, Theorem 4.3]) Let $\mathcal{I}$ be a proper ideal in the algebra $H$, and $\mathcal{I}_0 := \mathcal{I} \cap H^0$. Then, there exist nonzero polynomials $\phi^0_k \in \mathbb{C}[s]$, where $k = 0, \ldots, n-1$, such that $\mathcal{I}_0$ is the span over $\mathbb{C}[s]$ of the elements

$$\phi^0_k(s)Q_k, \quad \phi^0_{n-k}L_k, \text{ where } k = 0, \ldots, n-1 \text{ and } \phi^0_n := \phi^0_0.$$

**Proposition 5.2.** ([5, Proposition 4.4]) If $\mathcal{I} \subset H$ is a proper ideal, then $\mathcal{I}_0 = \mathcal{I} \cap H^0$ is a proper ideal in $H^0$.

**Definition 5.3.** For each $p = 0, \ldots, 2m$, we define the ideals $\mathcal{J}_p$ and $\mathcal{J}^p$ in the algebra $\mathbb{C}[s]$ by setting

$$\mathcal{J}_p := \{ f \in \mathbb{C}[s] \mid f(s)Q_p \in \mathcal{I} \}, \quad \mathcal{J}^p := \{ f \in \mathbb{C}[s] \mid f(s)L_p \in \mathcal{I} \}.$$

**Proposition 5.4.** ([5, Proposition 4.7]). We have $\mathcal{J}_p = \mathcal{J}^{-p}$.

**Proposition 5.5.** ([5, Proposition 4.8]). We have $\mathcal{J}_p \neq 0$ for any $p = 0, \ldots, 2m$.

Since $\mathbb{C}[s]$ is a principal ideal ring, we have the following statement:

**Corollary 5.6.** For any $p = 0, \ldots, 2m$, there exists a nonzero polynomial $\phi^0_p \in \mathbb{C}[s]$ such that $\mathcal{J}_p = \phi^0_p \mathbb{C}[s]$.

Theorem 5.1 evidently follows from Corollary 5.6.

6 Generating functions of $\kappa$-traces

For each $\kappa$-trace $sp^\kappa$ on $H$, one can define the following set of generating functions which allow one to calculate the $\kappa$-trace of arbitrary element in $H^0$ via finding the values of the derivatives of these
functions with respect to parameter $t$ at zero:

$$F_p^\kappa(t) := sp^\kappa(\exp(t(s - i\mu L_0))Q_p), \quad \Psi_p^\kappa(t) := sp^\kappa(\exp(ts)L_p), \quad \text{where } p = 0, \ldots, 2m. \quad (6.1)$$

Since $L_0Q_p = 0$ for any $p \neq 0$, it follows from the definition (6.1) that

$$F_p^\kappa(t) = sp^\kappa(\exp(ts)Q_p) \quad \text{if } p \neq 0, \quad F_0^\kappa(t) = sp^\kappa(\exp(t(s - i\mu L_0))Q_0). \quad (6.2)$$

It is easy to find $\Psi_p^\kappa$ for $p \neq 0$. Since $sL_q = -L_q s$ for any $q = 0, \ldots, 2m$, we have

$$\Psi_q^\kappa(t) = sp^\kappa(\exp(ts)L_q) = sp^\kappa(L_q). \quad (6.3)$$

The form of generating functions is related with (non)degeneracy of the form $B_{sp^\kappa}$ as described in Proposition 7.1 below.

7 Degeneracy conditions for the $\kappa$-trace

**Proposition 7.1.** ([5, Proposition 6.1]). The $\kappa$-trace on the algebra $\mathcal{H}$ is degenerate if and only if the generating functions $F_p^\kappa$ defined by formula (6.1) have the following form

$$F_p^\kappa(t) = \sum_{j=1}^{j_p} \exp(t\omega_{j,p})\varphi_{j,p}(t), \quad (7.1)$$

where $\omega_{j,p} \in \mathbb{C}$ and $\varphi_{j,p} \in \mathbb{C}[t]$ might depend on $\kappa$.

8 Equations for the generating functions $F_p^\kappa$

In [5, Eq. (7.1)], the following system of differential equations for the generating functions is obtained:

$$\frac{d}{dt}F_p^\kappa - \kappa e^{it}F_{p+1}^\kappa = iF_p^\kappa + \kappa i e^{it}F_{p+1}^\kappa + 2\kappa i \frac{d}{dt} \left(e^{it}\Delta_{p+1}^\kappa\right). \quad (8.1)$$
The initial conditions for this system are:

\[ F_p^{sp}(0) = sp^\kappa(Q_p). \]

To solve the system (8.1), we consider its Fourier transform. Let

\[ \lambda := e^{2\pi i/(2m+1)}, \]

\[ G_k^{sp} := \sum_{p=0}^{2m} \lambda^{kp} F_p^{sp}, \quad \text{where} \quad k = 0, \ldots, 2m, \]

\[ \tilde{\Delta}_k^{sp} := \sum_{p=0}^{2m} \lambda^{kp} \Delta_{p+1}^{sp} = \lambda^{-k} (\sin(\mu t)sp^\kappa(L_0)), \quad \text{where} \quad k = 0, \ldots, 2m. \]

For the functions \( G_k^{sp} \), we then obtain the equations

\[ \frac{d}{dt} G_k^{sp} = \left( \frac{\lambda^k + \kappa e^{it}}{\lambda^k - \kappa e^{it}} G_k^{sp} \right) \frac{2i\lambda^k}{\lambda^k - \kappa e^{it}} \frac{d}{dt} \left( e^{it} \tilde{\Delta}_k^{sp} \right) \]

with the initial conditions

\[ G_k^{sp}(0) = sp^\kappa(S_k). \]

We choose the following form of the solution of the system (8.3):

\[ G_k^{sp}(t) = \frac{\kappa e^{it}}{(e^{it} - \lambda^k)^2} \lambda^k g_k^{sp}(t), \]

where

\[ g_k^{sp}(t) = \left( \frac{2}{\mu} (\cos(\mu t) - 1) + 2i(1 - e^{it}) \sin(t \mu) \right) sp^\kappa(L_0) + \kappa \lambda^{-k}(\kappa - \lambda^k)^2 sp^\kappa(S_k). \]

Evidently, this solution satisfies the initial condition (8.4) for each \( \kappa \) and \( k \), except for the case where \( \kappa = +1 \) and \( k = 0 \).

If \( \kappa = +1 \) and \( k = 0 \), then the expression (8.5) for \( G_0^{tr} \) has a removable singularity at \( t = 0 \). In this case, instead of the condition \( G_0^{tr}(0) = tr(S_0) \), we consider the condition \( \lim_{t \to 0} G_0^{tr}(t) = tr(S_0) \).

When \( \kappa = +1 \) the solution (8.3) – (8.6) gives

\[ G_0^{tr}(t) = \frac{e^{it}}{(e^{it} - 1)^2} \left( \frac{2}{\mu} (\cos(\mu t) - 1) + 2i(1 - e^{it}) \sin(t \mu) \right) tr(L_0), \]

and one can easily see that

\[ \lim_{t \to 0} G_0^{tr}(t) = -\mu tr(L_0). \]

It is shown in Subsection 9.1 that if \( \kappa = +1 \), then

\[ tr(S_0) = -\mu tr(L_0) \]

for any trace \( tr \) on \( \mathcal{H} \).

So, \( G_0^{tr}(t) \) satisfies the initial conditions (8.4) also.

In the case where \( \kappa = -1 \), the \( \kappa \)-trace is a supertrace (see [4]). In this case, the \( m + 1 \) values \( str(S_k) = str(S_{2m+1-k}) \) for \( k = 0, \ldots, m \) completely define the supertrace on \( \mathcal{H} \) (see [7]).

In the case where \( \kappa = +1 \), the \( \kappa \)-trace is a trace (see [4]). In this case, the \( m \) values \( tr(S_k) = tr(S_{2m+1-k}) \) for \( k = 1, \ldots, m \) completely define the trace on \( \mathcal{H} \) (see [7]). The value \( tr(S_0) \) linearly depends on parameters \( tr(S_k) \), where \( k = 1, \ldots, m \), and this value is found in Subsection 9.1 (see Eqs. (9.4) – (9.5)).
9 Values of the $\kappa$-trace on $\mathbb{C}[I_2(2m+1)]$

From [5] we have

\[ sp^\kappa(R_k) = -\frac{2\mu}{2m+1} \left( \frac{1+\kappa}{2} X^{tr} + \frac{1-\kappa}{2} Y^{str} \right) , \]  

(9.1)

where

\[ X^{tr} := \sum_{r=1}^{2m} \sin^2 \left( \frac{\pi r}{2m+1} \right) tr(S_r) , \]  

(9.2)

\[ Y^{str} := \sum_{r=0}^{2m} \cos^2 \left( \frac{\pi r}{2m+1} \right) str(S_r) . \]  

(9.3)

Below we consider these values for the traces and supertraces separately.

9.1 Values of the traces ($\kappa = +1$) on $\mathbb{C}[I_2(2m+1)]$

The group $I_2(2m+1)$ has $m$ conjugacy classes without the eigenvalue +1 in the spectrum: \{$S_p, S_{2m+1-p}$\}, where $p = 1, ..., m$.

By Theorem 2.3 in [4], the values of the trace on these conjugacy classes

\[ s_k := tr(S_k) , \]

where $s_{2m+1-k} = s_k , \quad k = 1, ..., m,$

are arbitrary and completely define the trace on the algebra $\mathcal{H}$. Therefore, the dimension of the space of traces is equal to $m$.

Further, the group $I_2(2m+1)$ has one conjugacy class with one eigenvalue +1 in its spectrum: \{$R_1, ..., R_{2m+1}$\}. The value of $tr(R_k)$ is expressed via $s_k$ by formula (9.1).

Besides, the group $I_2(2m+1)$ has one conjugacy class with two eigenvalues +1 in its spectrum: \{$S_0$\}.

The traces on conjugacy classes with two eigenvalues +1 in the spectrum is calculated in [5] using Ground Level Conditions (for their definition, see [4]):

\[ tr(S_0) = 2\nu^2(2m+1)X^{tr} . \]  

(9.4)

We also note that

\[ tr(L_0) = -\frac{2\mu}{2m+1} X^{tr} , \quad tr(L_p) = 0 \text{ for } p \neq 0 , \quad tr(S_0) = -\mu tr(L_0) . \]  

(9.5)

9.2 Values of the supertraces ($\kappa = -1$) on $\mathbb{C}[I_2(2m+1)]$

The group $I_2(2m+1)$ has $m + 1$ conjugacy classes without the eigenvalue $-1$ in the spectrum:

\{$S_0$, $S_p, S_{2m+1-p}$\}, where $p = 1, ..., m$.

By [4] Theorem 2.3, the values of the supertrace on these conjugacy classes

\[ u_k := str(S_k) = str(S_{2m+1-k}) , \]

where $k = 0, ..., m,$

are arbitrary parameters that completely define the supertrace $str$ on the algebra $\mathcal{H}$, and therefore the dimension of the space of supertraces is equal to $m + 1$.

Besides, the group $I_2(2m+1)$ has one conjugacy class with one eigenvalue $-1$ in the spectrum: \{$R_1, ..., R_{2m+1}$\}.

The supertraces of the conjugacy class with eigenvalue $-1$ in its spectrum are given by Eq. (9.1):

\[ str(R_k) = -2\nu Y^{str} , \text{ where } k = 0, 1, ..., 2m , \text{ and where } Y^{str} \text{ is defined by Eq (9.3)} . \]
10 Singular values of the parameter $\mu$

The solution Eq. [8.5]-[8.6] determines the generation functions of traces and supertraces on $H^0$ for any trace and any supertrace on $\mathcal{H}$. Generally speaking, $G^{sp_\kappa}_k$ is a meromorphic function on $t$, but if $\mu$ and $sp_\kappa$ are such that the form $B_{sp_\kappa}$ is degenerate, then $G^{sp_\kappa}_k$ is an integer function on $t$ for each $k$. The complete list of such pairs of $\mu$ and $sp_\kappa$ is given in Theorem 10.1. For these values of $\mu$ and $sp_\kappa$, the functions $G^{sp_\kappa}_k$ are Laurent polynomials in $\exp(it)$.

**Theorem 10.1. (5 Theorem 9.1)**. Let $m \in \mathbb{Z}$, where $m \geq 1$, and $n = 2m + 1$. Then

1) The associative algebra $H_{1,\nu}(I_2(n))$ has a 1-parameter set of nonzero traces $tr_z$ such that the symmetric invariant bilinear form $B_{tr_z}(x,y) = tr_z(xy)$ is degenerate if and only if $\nu = \frac{z}{n}$, where $z \in \mathbb{Z} \setminus n\mathbb{Z}$. These traces are completely defined by their values at $S_k$ for $k = 1, \ldots, m$:

$$tr_z(S_k) = \frac{\tau}{n\sin^2 \frac{\pi k}{n}} \left(1 - \cos \frac{2\pi k z}{n}\right), \quad \text{where} \quad \tau \in \mathbb{C}, \quad \tau \neq 0. \quad (10.1)$$

Here $\tau$ is an arbitrary parameter specifying the trace in 1-dimensional space of traces.

2) The associative superalgebra $H_{1,\nu}(I_2(n))$ has a 1-parameter set of nonzero supertraces $str_z$ such that the symmetric invariant bilinear form $B_{str_z}(x,y) = str_z(xy)$ is degenerate if $\nu = \frac{z}{n}$, where $z \in \mathbb{Z} \setminus n\mathbb{Z}$. These supertraces are completely defined by their values at $S_k$ for $k = 0, \ldots, m$:

$$str_z(S_k) = \frac{\tau}{n\cos^2 \frac{\pi k}{n}} \left(1 - (-1)^z \cos \frac{2\pi k z}{n}\right), \quad \text{where} \quad \tau \in \mathbb{C}, \quad \tau \neq 0. \quad (10.2)$$

Here $\tau$ is an arbitrary parameter specifying the supertrace in 1-dimensional space of supertraces.

3) The associative superalgebra $H_{1,\nu}(I_2(n))$ has a 1-parameter set of nonzero supertraces $str_{1/2}$ such that the symmetric invariant bilinear form $B_{str_{1/2}}(x,y) = str_{1/2}(xy)$ is degenerate if $\nu = z + \frac{1}{2}$, where $z \in \mathbb{Z}$. These supertraces are completely defined by their values at $S_k$ for $k = 0, \ldots, m$:

$$str_{1/2}(S_k) = \frac{\tau}{n\cos^2 \frac{\pi k}{n}}, \quad \text{where} \quad \tau \in \mathbb{C}, \quad \tau \neq 0.$$

Here $\tau$ is an arbitrary parameter specifying the supertrace in 1-dimensional space of supertraces.

4) For all other values of $\nu$, all nonzero traces and supertraces are nondegenerate.

11 Generating functions $F^{sp_\kappa}_p$ for the degenerate $\kappa$-trace

Let $\mu \in \mathbb{Z} \setminus n\mathbb{Z}$. Substitute the solutions (10.1) for the case $\kappa = +1$ and (10.2) for the case $\kappa = -1$ to Eqs. (8.5)-Eq. (8.6). We obtain the formula for both values of $\kappa$

$$g^{sp_\kappa}_k = -\frac{4\tau}{n} \left[\cos(t\mu) + i\mu \lambda^{-k}(\lambda^k - \kappa e^{it})\sin(t\mu) - \kappa^\mu \cos \frac{2\pi k \mu}{n}\right]. \quad (11.1)$$

Introducing the new variable $y$ instead of $t$

$$y := \kappa e^{it} \quad (11.2)$$

we can rewrite Eq. (11.1) in the form

$$g^{sp_\kappa}_k = -\frac{2\tau}{n} \kappa^\mu \left[(y^\mu + y^{-\mu}) + \mu \lambda^{-k}(\lambda^k - y)(y^\mu - y^{-\mu}) - 2 \cos \frac{2\pi k \mu}{n}\right].$$
and Eq. (8.5) in the form
\[ G_k^{sp\kappa} = \frac{\lambda^k y - \lambda^k y^2}{(y - \lambda^k)^2} g_k^{sp\kappa}. \] (11.3)

Now we see that \( G_k^{sp\kappa} \) are the Laurent polynomials in \( y \) with the highest degree \( \leq |\mu| \) and the lowest degree \( \geq 1 - |\mu| \).

Note, that the expressions (11.3) are even functions of the parameter \( \mu \), so we can assume that \( \mu \) is a positive integer.

Let \( \mu > 0 \) in what follows.

Thus, \( G_k^{sp\kappa} \) can be expressed in the form
\[ G_k^{sp\kappa} = \kappa^{1-\mu} \sum_{\ell=\mu}^{1-\mu} \beta_k^\ell y^\ell, \] (11.4)

where the \( \beta_k^\ell \) are constants not depending on \( \kappa \) and not all of them equal to zero.

Eq. (11.4) implies that
\[ \beta_k^\mu = \frac{2\tau \mu}{n}. \] (11.5)

Further, Eq. (8.2) implies
\[ F_p^{sp\kappa} = \frac{1}{n} \sum_{k=0}^{2m} \lambda^{-kp} G_k^{sp\kappa} \]

and the generating functions \( F_p^{sp\kappa} \) have the form
\[ F_p^{sp\kappa} = \kappa^{1-\mu} \sum_{\ell=\mu}^{1-\mu} \alpha_k^p y^\ell, \] (11.6)

where the \( \alpha_k^\ell \) are constants not depending on \( \kappa \). Observe that \( F_p^{sp\kappa} \) can be equal to zero for some \( p \neq 0 \) (e.g., if \( \mu = 1 \), then \( F_p^{sp\kappa} = 0 \) for each \( p \neq 0 \)), but \( F_0^{sp\kappa} \neq 0 \) since Eq. (11.3) implies \( \alpha_0^\mu = \frac{2\tau \mu}{n} \neq 0 \). Eq. (11.5) implies also that \( \alpha_{-\mu}^0 = 0 \).

12 The generating function \( F^{sp\kappa} = sp\kappa (\exp(\tau s)Q_0) \) for the degenerate \( \kappa \)-trace

Let \( \mu \in \mathbb{Z} \setminus n\mathbb{Z} \) and \( \kappa \)-trace be defined by Eq. (10.1) in the case \( \kappa = +1 \) and by Eq. (10.2) in the case \( \kappa = -1 \).

In this section we introduce the function
\[ F^{sp\kappa} := sp\kappa (\exp(\tau s)Q_0) \]

and express it via \( F_0^{sp\kappa} \).

**Proposition 12.1.** \( F^{sp\kappa} \) is an even function of \( t \): \[ F^{sp\kappa} = sp\kappa (\cosh(\tau s)Q_0). \] (12.1)
Indeed, $F^{sp^x} = sp^x(\cosh(ts)Q_0 + \sinh(ts)Q_0)$ and $sp^x(\sinh(ts)Q_0) = 0$ since

$$sp^x(\sinh(ts)Q_0) = sp^x((\sinh(ts)L_0)L_0) = sp^x(L_0(\sinh(ts)L_0)) =$$

$$= sp^x((L_0\sinh(ts))L_0) = sp^x((-\sinh(ts)L_0)L_0) = sp^x(-\sinh(ts)Q_0)$$

Now, decompose $F_0^{sp^x}$:

$$F_0^{sp^x} = sp^x(e^{t(s-\imath \mu L_0)}Q_0) = F_{even} + F_{odd}$$ where

$$F_{even} = sp^x\left(\sum_{s=0}^{\infty} \frac{1}{(2s)!}(t(s - \imath \mu L_0))^{2s}Q_0\right) = sp^x\left(\sum_{s=0}^{\infty} \frac{1}{(2s)!}t^{2s}(s^2 - \mu^2)^sQ_0\right), \quad (12.2)$$

$$F_{odd} = sp^x\left(\sum_{s=0}^{\infty} \frac{1}{(2s+1)!}(t(s - \imath \mu L_0))^{2s+1}Q_0\right) =$$

$$= sp^x\left(\sum_{s=0}^{\infty} \frac{1}{(2s+1)!}t^{2s+1}(s^2 - \mu^2)^s(s - \imath \mu L_0)Q_0\right) =$$

$$= sp^x\left(\sum_{s=0}^{\infty} \frac{1}{(2s+1)!}t^{2s+1}(s^2 - \mu^2)^s(-\imath \mu L_0)Q_0\right) =$$

$$= \sum_{s=0}^{\infty} \frac{1}{(2s+1)!}t^{2s+1}(-\mu^2)^s(-\imath \mu)sp^xL_0 =$$

$$= \sinh(-\imath \mu)sp^xL_0 = -\frac{\zeta^\mu}{2}(y^\mu - y^{-\mu})sp^xL_0. \quad (12.3)$$

Eq. (11.6) implies that

$$F_{odd} = \frac{\zeta^\mu}{2}\left(\sum_{\ell=\mu}^{\frac{-\mu}{2}} \alpha_0^\ell y^\ell - \sum_{\ell=-\mu}^{\frac{-\mu}{2}} \alpha_0^{-\ell} y^{-\ell}\right). \quad (12.4)$$

Comparing Eq. (12.4) with Eq. (12.3) implies

$$\alpha_0^\ell = \alpha_0^{-\ell}, \text{ if } \ell \neq \mu, \ell \neq -\mu,$$

$$\alpha_0^\mu - \alpha_0^{-\mu} = -sp^xL_0, \quad (12.5)$$

and

$$F_{even} = \frac{\zeta^\mu}{2}\alpha_0^\mu(y^\mu + y^{-\mu}) + \frac{\zeta^\mu}{2}\sum_{\ell=0}^{\mu-1} \alpha_0^\ell(y^\ell + y^{-\ell}) = \alpha_0^\mu \cosh(it\mu) + \zeta^\mu\sum_{\ell=0}^{\mu-1} \zeta^\ell \alpha_0^\ell \cosh(it\ell). \quad (12.6)$$

**Proposition 12.2.**

$$F^{sp^x}(t) = \alpha_0^\mu + \zeta^\mu\sum_{\ell=0}^{\mu-1} \zeta^\ell \alpha_0^\ell \cosh(t\sqrt{\mu^2 - \ell^2}).$$

**Proof.** Taking Proposition 12.1 into account let us decompose Eq (12.1) into the Taylor series:

$$F^{sp^x}(t) = \sum_{s=0}^{\infty} a_{2s}(\frac{t^{2s}}{(2s)!}),$$
where \( a_{2s} := sp^\kappa(s^{2s}Q_0) \) for \( s = 0, 1, 2, \ldots \).

Eq (12.2) implies
\[
a_{2s} = \left( \frac{d^2}{dt^2} + \mu^2 \right)^s F_{even}|_{t=0},
\]
and Eq (12.6) implies
\[
a_{2s} = \begin{cases} a^0_\mu + \kappa^\mu \sum_{\ell=0}^{\mu-1} \kappa^\ell a^0_\ell & \text{if } s = 0 \\ \kappa^\mu \sum_{\ell=0}^{\mu-1} \kappa^\ell a^0(\ell^2 + \mu^2)^s & \text{if } s \neq 0. \end{cases}
\]
So
\[
F_{sp^\kappa}(t) = \sum_{s=0}^{\infty} a_{2s} (2s)! = a^0_\mu + \kappa^\mu \sum_{\ell=0}^{\mu-1} \kappa^\ell \alpha^0_\ell \cosh \left( t\sqrt{\mu^2 - \ell^2} \right).
\]

\[\square\]

13 Ideals generated by degenerate \( \kappa \)-traces

Let \( \mu = z \in \mathbb{Z} \setminus n\mathbb{Z} \) and the \( \kappa \)-trace be defined by Eq (10.1) for the case \( \kappa = +1 \) and by Eq (10.2) for the case \( \kappa = -1 \).

These degenerate \( \kappa \)-traces are denoted in Theorem 10.1 by \( tr_z \) and \( str_z \). Denote the ideals generated by these traces \( sp^\kappa \) by \( I^\kappa \); in \( H^0 \), consider the ideals \( I^\kappa_0 := I^\kappa \cap H^0 \).

Now we can prove Conjecture 1.1 ([5, Conjecture 9.1]):

**Theorem 13.1.** \( I^+_1 = I^{-1} \).

To prove Theorem 13.1, we use Theorem 4.2 from [6] which in our case implies

**Theorem 13.2.** ([6, Theorem 4.2]) \( I^+_1 = I^{-1} \) if and only if \( I^+_1 \cap H^0 = I^{-1} \cap H^0 \).

So, Theorem 13.1 follows from

**Theorem 13.3.** \( I^+_1 \cap H^0 = I^{-1} \cap H^0 \).

**Proof.** For degenerate \( sp^\kappa \), we established the following facts:

\[
F_p^{sp^\kappa}(t) = sp^\kappa(e^{tz}Q_p) = \kappa^\mu \sum_{\ell=\mu}^{\mu-1} \alpha^p_\ell \kappa^\ell e^{i\ell t} \quad \text{for } p = 1, 2, \ldots, n - 1,
\]

\[
F_{sp^\kappa}(t) = sp^\kappa(e^{tz}Q_0) = \alpha^0_\mu + \kappa^\mu \sum_{\ell=0}^{\mu-1} \kappa^\ell \alpha^0_\ell \cosh \left( t\sqrt{\mu^2 - \ell^2} \right) \quad \text{where } \alpha^0_\mu \neq 0,
\]

and where the \( \alpha \)-s do not depend on \( \kappa \).

For any \( p = 1, \ldots, n \), it is easy to find the lowest degree polynomial differential operators with constant coefficients \( D_p^\kappa(d/dt) \) such that \( D_p^\kappa(d/dt)F_p^{sp^\kappa}(t) = 0 \):

\[
D_p^\kappa \left( \frac{d}{dt} \right) = \begin{cases} \prod_{\ell=-\mu: \alpha^p_\ell \neq 0}^{\mu} \left( \frac{d}{dt} - i\ell \right) & \text{if } F_p^{sp^\kappa} \neq 0, \\ 1 & \text{if } F_p^{sp^\kappa} = 0, \end{cases}
\]

and \( D_0^\kappa(d/dt) \) such that \( D_0^\kappa(d/dt)F_{sp^\kappa}(t) = 0 \):

\[
D_0^\kappa \left( \frac{d}{dt} \right) = \frac{d}{dt} \prod_{\ell=0: \alpha^0_\ell \neq 0}^{\mu-1} \left( \frac{d^2}{dt^2} - \mu^2 + \ell^2 \right).
\]
Further, it is a simple exercise to prove that
\[ D_p^\varepsilon(s)Q_p, \ D_p^\varepsilon(s)L_{-p} \in \mathcal{I}_0^\varepsilon \] for any \( p = 0, \ldots, n - 1 \),
namely,
\[ B_{sp^\varepsilon}(D_p^\varepsilon(s)Q_p, f) = B_{sp^\varepsilon}(D_p^\varepsilon(s)L_{-p}, f) = 0 \] for any \( f \in H^0 \) and \( p = 0, \ldots, n - 1 \).

Consider, for example, \( B_{sp^\varepsilon}(D_0^\varepsilon(s)Q_0, f) \) for \( f = g(s)Q_p \) and \( f = g(s)L_p \):
\[ sp^\varepsilon(D_0^\varepsilon(s)Q_0 g(s)Q_p) = sp^\varepsilon(D_0^\varepsilon(s)Q_0 g(s)L_p) = 0 \] for \( p \neq 0 \),
since \( Q_0 Q_p = Q_0 L_p = 0 \) for \( p \neq 0 \),
\[ sp^\varepsilon(D_0^\varepsilon(s)Q_0 g(s)Q_0) = sp^\varepsilon(D_0^\varepsilon(s)g(s)Q_0) = \left. D_0^\varepsilon \left( \frac{d}{dt} \right) g \left( \frac{d}{dt} \right) sp^\varepsilon(e^{ts}Q_0) \right|_{t=0} = = g \left( \frac{d}{dt} \right) D_0^\varepsilon \left( \frac{d}{dt} \right) F_{sp^\varepsilon}(t) \big|_{t=0} = 0, \]
\[ sp^\varepsilon(D_0^\varepsilon(s)Q_0 g(s)L_0) = sp^\varepsilon(D_0^\varepsilon(s)g(s)L_0) = \left. D_0^\varepsilon \left( \frac{d}{dt} \right) g \left( \frac{d}{dt} \right) sp^\varepsilon(e^{ts}L_0) \right|_{t=0} = = g \left( \frac{d}{dt} \right) D_0^\varepsilon \left( \frac{d}{dt} \right) sp^\varepsilon(L_0) = 0 \]
due to Eq (6.2) and since the operator \( D_0^\varepsilon \left( \frac{d}{dt} \right) \) contains the factor \( \frac{d}{dt} \).

Further, it is easy to see that for each of the ideals \( \mathcal{I}_0^\varepsilon \), where \( \varepsilon = \pm 1 \), the polynomials \( \phi_p^0(s) \in \mathbb{C}[s] \) defined in Corollary 5.6 satisfy the relations \( \phi_p^0(s) = D_p^\varepsilon(s) \) for \( p = 0, \ldots, n - 1 \).

So, Theorem 5.1 implies that the \( \mathbb{C}[s] \)-span of the \( D_p^\varepsilon(s)Q_p \) and \( D_p^\varepsilon(s)L_{-p} \) for \( p = 0, \ldots, n - 1 \) is \( \mathcal{I}_0^\varepsilon \).

Since \( D_p^{+1} = D_p^{-1} \), we have \( \mathcal{I}_0^{+1} = \mathcal{I}_0^{-1} \), and as result, \( \mathcal{I}^{+1} = \mathcal{I}^{-1} \). \( \square \)

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