Ordered spectral statistics in one-dimensional disordered supersymmetric quantum mechanics and Sinai diffusion with dilute absorbers

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Abstract

Some results on the ordered statistics of eigenvalues for one-dimensional random Schrödinger Hamiltonians are reviewed. In the case of supersymmetric quantum mechanics with disorder, the existence of low-energy delocalized states induces eigenvalue correlations and makes the ordered statistics problem non-trivial. The resulting distributions are used to analyze the problem of classical diffusion in a random force field (Sinai problem) in the presence of weakly concentrated absorbers. It is shown that the slowly decaying averaged return probability of the Sinai problem, \( P(x, t|x, 0) \sim \ln^{-2} t \), is converted into a power-law decay, \( P(x, t|x, 0) \sim t^{-\gamma} \), where \( g \) is the strength of the random force field and \( \rho \) the density of absorbers.

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(••• figures may appear in color only in the online journal)

1. Introduction

An ordered (or extreme value) statistics problem can be defined as follows: given a set of \( \mathcal{N} \) ranked random variables \( x_1 < x_2 < \cdots < x_n < \cdots < x_{\mathcal{N}} \), one asks for the probability density \( W_{n,\mathcal{N}}(x) \) for finding \( x_n \) at \( x \). When the random variables are independent and identically distributed (i.i.d) according to a given distribution \( p(x) \), the distributions \( W_{n,\mathcal{N}}(x) \) may be easily related to \( p(x) \). An interesting point emerges when considering the \( \mathcal{N} \to \infty \) limit: in this case, up to a rescaling \( x_n = a_n + b_n y_n \), which depends on the original distribution \( p(x) \), the distribution of the rescaled variable \( y_n \) converges to a universal law \( w_n(y) \). Three universality classes exist, corresponding to the nature of the tail of \( p(x) \), as demonstrated in the pioneering works of Fréchet [7] and Gumbel (see the textbook [10]).

When \( p(x) \) has a tail of exponential type, one obtains the famous generalized Gumbel laws \( w_n(y) = \left( n^{n-1} \right) \exp(n(y - e^n)) \). The study of extreme events is relevant in many areas such as meteorology, finance, computer science, statistical physics, etc, and in general the random variables of interest may not fulfill the hypothesis of statistical independence (see, for example, [18] or [1]). In this case the problem is much more difficult to analyze and no general results are known such as for i.i.d random variables. This situation occurs when considering the set of eigenvalues of a random operator, since eigenvalues are \textit{a priori} correlated. For example, the distribution of the smallest eigenvalue of large random matrices has been obtained in a famous work by Tracy and Widom [27, 28] for the usual Gaussian ensembles; other ensembles have recently been considered in [13, 14, 21, 29].

In this paper, we consider the case when random variables are eigenvalues of a one-dimensional (1D) Schrödinger operator with a random potential.

The paper is organized as follows. In section 2, we define the problem and review some results. In particular, we will focus on the case of disordered \textit{supersymmetric} quantum mechanics for which we obtain a set of non-trivial distributions \( w_n(y) \). Applications of these results are discussed in sections 3 and 4: in the context of 1D disordered quantum mechanics and in the context of classical diffusion in a random force field with absorption.
2. Ordered spectral statistics

Let us consider the Schrödinger operator $H_{\text{scalar}} = -\frac{d^2}{dx^2} + V(x)$ acting on functions defined on the interval $[0, L]$, satisfying the Dirichlet boundary conditions $\psi(0) = \psi(L) = 0$. We denote by $E_1 < E_2 \cdots < E_n < \cdots$ the (infinite) set of eigenvalues. We will be interested in situations where the potential $V(x)$ is random. The section is devoted to the analysis of the probability density for finding the $n$th eigenvalue $E_n$ at level $E$, denoted by $W_n(E; L) = \delta(E - E_n^*),$ where $\cdots$ denotes averaging with respect to the disordered potential $V(x)$. The knowledge of this set of distributions provides some spectral information much more precise than the averaged density of states (DoS) $\varrho(E; L) = \sum_{n=1}^\infty W_n(E; L)$, measuring the probability to find any eigenvalue at level $E$. The determination of the distributions $W_n(E; L)$ has been achieved for different types of random potentials, which we review in the rest of the section. We will consider the limit $L \to \infty$ (in practice $L$ must be larger than a certain scale characteristic of the disorder) analogous to the $N \to \infty$ limit of the introduction.

2.1. Fully localized spectrum

The distributions $W_n(E; L)$ were determined when $V(x)$ is a Gaussian white noise: the distribution of the ground state $\{\xi_n \}$ is the integrated DoS $\varrho(E; L) = \int_0^L dE \varrho(E; L)$, measuring the probability to find any eigenvalue at level $E$. The change of variable $\xi \to \xi^*$ corresponds to unfolding the spectrum: it relates a set of random variables $\{E_n\}$ distributed with a non-uniform density $\varrho(E; L)$ to $L N(E_n)$, to random variables $\{\xi_n\}$ distributed with a uniform density equal to unity. The fact that the latter variables are distributed according to a Poisson law $\sigma_n(\xi) = \frac{1}{(n-1)!}\xi^{n-1}e^{-\xi}$ demonstrates the absence of energy level correlations [20], since $\sigma_n(\xi) = \frac{1}{n!}$ the probability that $n - 1$ eigenvalues lie in the interval $[0, \xi]$.

2.3. Supersymmetric case

The results reviewed in the previous subsection apply to the generic situation where all eigenstates are localised. However this does not exhaust all possible scenarios, as there exist one-dimensional disordered models exhibiting delocalized states. Such an example is provided by the supersymmetric quantum Hamiltonian

$$H_{\text{susy}} = -\frac{d^2}{dx^2} + \phi(x)^2 + \phi'(x) = Q^\dagger Q,$$

where $Q = \frac{d}{dx} + \phi(x)$. This Hamiltonian appears in many interesting contexts (see [4, 5, 25] for reviews). When $\phi(x)$ has short-range correlations and $\phi(0) = 0$, we obtain low-energy properties opposite of the ones mentioned previously for $H_{\text{scalar}}$: (i) the disorder increases the low-energy DoS (Dyson singularity) and (ii) the states get delocalized near the spectrum boundary at $E = 0$ (this was demonstrated in [2] when $\phi(x)$ is a Gaussian white noise). The delocalization of the first eigenstates is responsible for energy level correlations and the distributions $W_n(E; L)$ are no longer expected to coincide with the generalized Gumbel laws [23]. We now explain the principle of the method, introduced in [23], allowing for the determination of these distributions, when $\phi(x)$ is a Gaussian white noise such that $\varphi(0) = 0$ and $\varphi(x)\varphi(x') = g \delta(x - x')$.

The starting point consists in converting the Sturm–Liouville (spectral) problem, $H_{\text{susy}}\psi(x) = E\psi(x)$ with boundary conditions $\psi(0) = \psi(L) = 0$, into a Cauchy problem, $H_{\text{susy}}\psi(x; E) = E\psi(x; E)$ with the boundary conditions $\psi(0; E) = 0$ and $\psi(0; E) = 1$. The former admits solutions only for energy in a discrete set $E \in \{E_n\}$, while the latter has solutions $\psi \in \mathbb{R}$. We introduce the notation $\ell_1, \ell_2, \ldots$ for the lengths between the consecutive nodes of the wave function ($1$), and $P(\ell)$ their common distribution, which will be obtained by studying the statistical properties of $\psi(x; E)$, or rather of some related Riccati variable. Then the probability for the eigenenergy $E_n$ to be at level $E$ coincides with the probability for the $n$th zero to coincide with the boundary $\sum_{n=1}^\infty \ell_n = L$, i.e. $W_n(E; L) \propto (P \ast \cdots \ast P)(L)$, as illustrated in figure 1. This is the essence of the node counting method, also called phase formalism in the context of disordered systems [16] or the Dyson–Schmidt method for discrete models [17].

The building block is now the distribution $P(\ell)$ and we explain how it can be calculated. Introducing the Riccati variable $z = \psi'/\psi - \phi$, we map the Schrödinger equation onto a Langevin-like equation

$$\frac{dz}{dx} = -E - z(x)^2 - 2z(x)\phi(x).$$

In this language the distance $\ell_n$ separates two consecutive divergencies of the Riccati variable, e.g. $z(0) = +\infty$ and $z(\ell_1) = -\infty$. The study of $P(\ell)$ is therefore mapped onto a first passage problem for a stochastic Markovian process, a problem solved by well-known techniques. We introduce the auxiliary ‘time’ $t_1$ needed by the process $z(x)$ in order to
reach $-\infty$ for the first time, given that it has started at $z$. The characteristic function $h(z; \alpha) = e^{-\alpha z^2}$ obeys a backward Fokker–Planck equation

$$B_z h(z; \alpha) = \alpha h(z; \alpha) \tag{3}$$

for boundary conditions $h(-\infty; \alpha) = 1$ and $h(+\infty; \alpha) = 0$, where $B_z = 2g_z d^2 \delta - (E + z^2) \frac{d}{dz}$ is the generator of the diffusion. Finally, the distribution $P(\ell)$ can be obtained by an inverse Laplace transform

$$h(+\infty; \alpha) = \int_0^\infty d\ell P(\ell) e^{-\alpha \ell}. \tag{4}$$

An approximation scheme was proposed in [23] in order to solve equation (3) in the low-energy regime $E \ll g^2$. This supposes that the distributions $W_n(E, L)$ are mostly concentrated on such energy scales, which is expected to occur when $L \to -\infty$. As a result, it was shown that

$$h(+\infty; \alpha) \simeq \frac{1}{\cosh^2 \sqrt{\alpha/N_{\text{asy}}}(E)}. \tag{5}$$

where $1/\mathcal{T} = N_{\text{asy}}(E) = 2g^2/\ln^3(g^2/E)$ is the IDoSpul of the supersymmetric Hamiltonian (1) in bulk ($L = \infty$). After spectrum unfolding ($E \ll g^2$), the distributions of the eigenvalues $W_n(E, L) = N_{\text{asy}}(E)$ are given by

$$\sigma_n(\xi) = \int_{\mathcal{B}} \frac{dq}{2\pi} \frac{e^{i\xi}}{\cosh(q\sqrt{\alpha})}, \tag{6}$$

where $\mathcal{B}$ is a Bromwich contour. The fact that these distributions deviate from the Poisson distribution (i.e. $W_n(E, L)$ differ from the generalized Gumbel laws) signals spectral correlations induced by delocalization. These distributions were explicitly calculated in [25] by a generating function method:

$$\sigma_n(\xi) = \frac{2^{2n}}{\sqrt{\pi} \xi^{3/2}} \sum_{m=n}^{\infty} (-1)^{m+n} m \left( \frac{m+n-1}{m-n} \right) e^{-m^2/\xi}. \tag{7}$$

The large $\xi$ behavior is $\sigma_n(\xi) \simeq \frac{\pi}{\sqrt{2m \xi}} 2^{2n-1} \exp -\frac{\pi^2}{4} \xi$. The first distributions are presented in figure 2.

The sum of the distributions characterizes the averaged DoS

$$D(E, L) = \sum_{n=1}^\infty W_n(E, L) \tag{8}$$

for a finite interval. It coincides with the bulk DoS $\sigma(E, L) \simeq L N_{\text{asy}}(E)$ when eigenstates are localized on scales smaller than $L$ and do not feel the boundaries. A log-normal depletion, $\sigma(E, L) \sim \frac{1}{\sqrt{2\pigL}} \exp -\frac{1}{2gL} \ln^2(g^2/E)$, is however obtained at low energy $E \ll g^2 \exp -\sqrt{2gL}$, illustrating the sensitivity of low-energy delocalized states to the boundaries. The DoS is presented in figure 2 (dashed line) after spectrum unfolding. Conversely, the boundary sensitivity of the averaged DoS $\bar{\sigma}(E, L)$ may be used as a localization criterion, which gives the energy dependence of the localization length $\xi_E \simeq \frac{1}{2} \ln^2(g^2/E)$.

Details of the results reviewed in this section can be found in [23, 25].

### 3. Supersymmetry broken by disorder

As we have mentioned, the random Hamiltonians $H_{\text{scalar}}$ and $H_{\text{asy}}$ present opposite spectral and localization properties. This observation has led us to question the interplay between the two types of disorder and consider the mixed case

$$H = -\frac{d^2}{dx^2} + \phi(x)^2 + \phi'(x) + V(x). \tag{9}$$

The case when $\phi$ and $V$ are two Gaussian white noises was analyzed in [11]; however, it turns out that the case when the potential $V$ describes a random superposition of repulsive scatterers at random positions, $V(x) = \sum_{i} \alpha_i \delta(x-x_i)$, can be related to an interesting problem in the context of classical diffusion (discussed in the next section). In the latter case, the spectral density of the Hamiltonian (9) may be obtained from the distributions $W_n(E, L)$ obtained in the previous section by using a Lifshits-like argument [24]. Let us denote by $N(E)$ the IDoSpul of the Hamiltonian (9). In the limit of large weights $\alpha_i \to \infty$ (in practice $\alpha_i \gg \rho$ and $g$) the impurities decouple the intervals free of impurity and impose Dirichlet boundary conditions at their locations. Thus the spectrum of (9) is given the addition of spectra of $H_{\text{asy}}$ on all intervals and we obtain the DoSpul $N'(E)$ for the mixed disorders by averaging the DoS (8) of the supersymmetric Hamiltonian for a
finite length
\[ N'(E) \simeq \rho \langle \phi(E; L) \rangle, \quad (10) \]
where \( \langle \cdot \cdot \cdot \rangle_L \) denotes averaging with respect to the length of the interval, namely with an exponential law \( \rho e^{-\rho L} \) characterizing the uncorrelated positions \( x_i \). Using (6) and (8), we obtain
\[ N(E) \simeq \frac{\rho}{\sin^2 \frac{\rho}{\sqrt{N_{\text{sys}}(E)}}}. \quad (11) \]

Above the threshold energy \( E_t = g^2 \exp -\frac{2g}{\rho} \), the DoS coincides with the one of the supersymmetric Hamiltonian. Below the threshold, the Dyson singularity is transformed into a power-law singularity
\[ N(E) \simeq 4\rho \left( \frac{E}{g^2} \right)^{\frac{2g}{\rho}}. \quad (12) \]
This power-law behavior has been very well confirmed by numerical calculations [24]. Moreover, the numerics has shown that the exponent \( \sqrt{2g/\rho} \) seems to be more general than the range of applications of the Lifshits argument suggests; we will comment on this in the last section.

4. Sinai diffusion with weakly concentrated absorbers

The study of classical diffusion in a random environment has attracted much attention in various contexts ranging from mathematics, statistical physics (glassy dynamics or polymer physics) to even finance. It may be described within the framework of the Fokker–Planck equation \( \partial_t P(x; t) = \partial_x \left( \partial_x - 2\phi(x) \right) P(x; t) \), where \( \phi(x) \) is chosen to be a Gaussian white noise with \( \langle \phi(x) \rangle = \mu g \) and \( \phi(x) \phi(x') = g \delta(x-x') \). The model exhibits a rich dynamics as a function of the drift \( \mu \) [2]. In the absence of drift, for \( \mu = 0 \), the diffusion is controlled by overcoming potential barriers and is extremely slow, \( x(t) \sim \frac{1}{g} \ln^2 (g/t) \) (Sinai diffusion [22]).

The question I would like to discuss now is: how does the introduction of absorbing sites affect the Sinai dynamics? Let us go back for a moment to the free diffusion: in this case the return probability decays in time as a power law \( P(x, t|x, 0) \sim 1/\sqrt{4\pi t} \). The dynamics is slowed down by the introduction of a random force field, which is reflected in the return probability by an extremely slow decay \( P(x, t|x, 0) \sim g \ln^2 (g^2t) \), related to the aforementioned behavior. On the other hand, when a weak concentration \( \rho \) of efficient absorbers with large absorption rates \( \alpha \gg \rho \) is introduced, the free power-law decay is transformed into an exponential decay \( P(x, t|x, 0) \sim \exp -\frac{3}{2} \rho^2 \frac{2}{\rho^2} t/3 \) (Lifshits tail). The question is: what is the behavior of the return probability when both a random force field and absorbers are present? Is the return probability increased or decreased?

The answer to this question can be obtained by a simple mapping to the model analyzed in the previous section. The effect of absorbers on the Sinai diffusion may be accounted for by adding to the Fokker–Planck equation a term:
\[ \partial_t P(x; t) = \partial_x \left( \partial_x - 2\phi(x) \right) P(x; t) - V(x) P(x; t), \quad (13) \]
where \( V(x) = \sum_i \alpha_i \delta(x - x_i) \) describes absorbers located at a set of positions \( \{x_i\} \). The positive coefficient \( \alpha_i \) measures the efficiency of the absorber located at \( x_i \). The Fokker–Planck equation (13) may be mapped onto the Schrödinger equation \( -\partial_x \psi(x; t) = H \psi(x; t) \) for the Hamiltonian (9) thanks to the transformation \( P(x; t) = \psi(x; t) \exp \int_0^t dx' \phi(x') \). It follows that the return probability can be related to the spectral density (DoSpul) \( \rho(E) = N'(E) \) of Hamiltonian (9) by
\[ P(x, t|x, 0) = \int_0^\infty dE \rho(E) \exp ^{-Et}. \quad (14) \]
We immediately deduce the long-time behavior of the return probability from (12):
\[ P(x, t|x, 0) \sim t^{-\frac{1}{2}} \exp \frac{\sqrt{2g}}{\rho} \ln^2 (g^2t) \quad \text{for} \quad t \gg t_c, \quad (15) \]
where the scale \( t_c = 1/E_c = \frac{g^2}{\rho} \exp \frac{2g}{\rho} \) is the time needed by the particle moving in the random force field to reach the closest absorber \( x(t_c) \sim 1/\rho \), where \( x(t) \sim \frac{1}{g} \ln^2 (g^2t) \). The power law can be easily explained by analyzing the lowest absorption rate \( E_{1}[\phi(x), L] \) in a finite interval: the fact that its distribution presents a log-normal behavior \( W(E; L) \sim \frac{1}{\sqrt{2\pi}} \exp -\frac{1}{2} \ln^2 (g^2/E) \) for \( E \to 0 \) explains the exponent. The behavior (15) was first obtained in [24] and later confirmed in [15] by the real space renormalization group method. The return probability decays more slowly than in the free case for a sufficiently weak concentration of absorbers \( \rho < g/\rho \), whereas it decays faster for \( \rho > g/\rho \). Note, however, that the origin of the decay in the present model is mostly due to absorption, as demonstrated in [15] by analyzing the survival probability, which was shown to present the same power-law decay up to a logarithmic correction \( \delta_{\text{DoSpul}}(t) = \int dx P(x, t|x, 0) \sim (\ln t) t^{-\frac{1}{2} \rho^2} \).

5. Concluding remarks

In this brief review, I have analyzed the ordered statistics (extreme value statistics) for eigenvalues of a random supersymmetric Hamiltonian. The fact that this quantum Hamiltonian possesses delocalized low-energy eigenstates makes the problem non-trivial and escape from the universal generalized Gumbel laws obtained for generic 1D random Hamiltonians with a full spectrum of localized states.

These results have some utility in analyzing the effect of dilute absorbers on the classical diffusion in a random environment (Sinai diffusion). We have obtained that the return probability presents a power-law decay, mostly due to absorption. This conclusion relies on the analysis of the spectral density for a mixed random Hamiltonian with a ‘supersymmetric part’ \( \phi^2 + \phi' \), where \( \phi(x) \) is a Gaussian white noise, and a ‘scalar part’ \( V(x) = \sum \alpha_i \delta(x - x_i) \). Using a Lifshits-like argument, it was shown that the DoS of this Hamiltonian presents a power-law singularity at low energy \( \rho(E) = N'(E) \sim E^{-\frac{1}{2} \rho^2} \). The result is interesting by itself since only a few power-law spectral singularities are known in the context of 1D Anderson localization: (i) the Halperin singularity for randomly dropped attractive impurities of fixed weights \( \alpha_i = -v \) [12, 16]; \( \rho(E) \sim |E + \frac{1}{2} v^2|^{-1+2/v} \) for \( E \sim -\frac{1}{4} v^2 \); (ii) the power-law behavior for the supersymmetric Hamiltonian \( H_{\text{DoSpul}} \) for a finite \( \phi(x) = \mu g \neq 0 \) for which \( \rho(E) \sim E^{-1+\mu} \) [2].
Surprisingly, numerical simulations of the model (9) revealed that the exponent appearing in the IDoSpul (12) is much more robust than one would have expected on the basis of the heuristic Lifshitz argument. Examining this observation has led us to a solvable version of the model by considering random weights $\alpha_i$. Let us consider Hamiltonian (9) for a finite drift $\phi(x) = \mu g \neq 0$ and with exponentially distributed positive weights $p(\alpha_i) = \frac{1}{\pi} \exp -\frac{\alpha_i}{\mu}$. This model becomes solvable for a particular value of the parameter $v = \overline{\theta}$; if $v = 2g$ the effect of the ‘scalar term’ $V(x)$ can be absorbed in a redefinition of the parameter $\mu$ [26]

$$\mu \rightarrow v = \sqrt{\mu^2 + \frac{2g}{\mu}}. \quad (16)$$

The IDos of (9) can be obtained by performing this substitution in the IDos of $H_{\text{asy}}$ (that can be found in [2]); therefore

$$N(E) = \frac{2g}{\pi^2} J_0(\sqrt{E}/g)^2 + N_v(\sqrt{E}/g)^2, \quad (17)$$

where $J_0(x)$ and $N_v(x)$ are the Bessel functions [8]. Using their asymptotic behaviors we obtain the low-energy IDos $N(E) \approx \frac{2g}{\pi^2} \left( \frac{E}{g^2} \right)^{1/2}$ for $E \ll g^2$. Therefore we have recovered the power law exponent $\sqrt{2g/\mu}$ (setting $\mu = 0$) within a solvable model, confirming the robustness of this exponent valid for an arbitrary value of the ratio $\mu/g$. It is worth emphasizing that the solvable model applies to a regime with weights $\alpha_i \sim g$, while the analysis of the previous sections relied rather on the hypothesis that $\alpha_i \gg g, \rho$. This result also generalizes the analysis to a finite drift $\mu \neq 0$ (note that the same exponent $\sqrt{\mu^2 + 2g/\mu}$ was obtained by Le Doussal [15] by another method, in a different regime $\alpha_i \rightarrow \infty$). Details of this analysis will be published in a forthcoming paper [26].

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Appendix. Diffusion constant of a single $d$-dimensional Brownian curve

In [25], the distribution of the maximal height of a Brownian excursion was shown to be related to the distributions (6), (7), or more precisely to their sum $\sum_{\alpha=1}^{\infty} \sigma_\alpha(\xi)$. In this appendix, we point out that the distributions

$$\sigma_\alpha(\xi) = \frac{d}{4} \frac{\xi^{1/2}}{\cosh^{1/2} \xi}, \quad (A.1)$$

generalizing $\sigma_0(\xi) = \sigma_\infty(\xi)$ and considered in [25], are relevant for characterizing a certain property of the $d$-dimensional Brownian motion.

Let us consider a $d$-dimensional Brownian curve $\bar{r}(\tau)$ with $\tau \in [0, t]$ starting from the origin $\bar{r}(0) = 0$. This curve is weighted according to the Wiener measure

$$D\bar{r}(\tau) e^{-\int_0^t \frac{\bar{r}(\tau)^2}{2d} \, d\tau^2}, \quad (A.2)$$

where $D$ is the diffusion constant, usually defined by $D \overset{df}{=} \lim_{t \to \infty} \frac{1}{2t} E(\bar{r}(t)^2)$, where $E(\cdot)$ denotes the expectation with respect to the Wiener measure (A.2). We now ask the question: is it possible to obtain an estimation of the diffusion constant from a given realization of the Brownian path by a time average? For this purpose we introduce the ‘trajectory-dependent diffusion constant’ $D(\bar{r}) \overset{df}{=} \frac{1}{t} \int_0^t \frac{\bar{r}(\tau)^2}{2d} d\tau$, or preferably the dimensionless Brownian functional

$$\chi[\bar{r}(\tau)] \overset{df}{=} D(\bar{r}) = \frac{1}{D(\bar{r})} \int_0^t \frac{\bar{r}(\tau)^2}{2d} \, d\tau. \quad (A.3)$$

Note that this problem was also considered in [3], starting from a different definition of the trajectory-dependent diffusion constant $\int_0^t \frac{\bar{r}(\tau)^2}{2d} \, d\tau$. The characteristic function of the functional $\chi[\bar{r}]$ may be written thanks to a path integral

$$\Gamma(p) \overset{df}{=} \int \mathcal{D}\bar{r} \int_0^t \frac{\bar{r}(\tau)^2}{2d} \, d\tau \mathcal{D}\bar{r}(\tau) e^{-\int_0^t \frac{\bar{r}(\tau)^2}{2d} \, d\tau} \left[ \frac{\bar{r}(\tau)^2}{2d} + \frac{\xi}{4} \right]. \quad (A.4)$$

We recognize the integral of the imaginary time propagator $\langle R e^{-iH} \rangle_0$ for a quantum harmonic oscillator with mass $m \to 1/(2D)$ and pulsation $\omega \to \frac{1}{2} \sqrt{\frac{d}{2}}$. Using its well-known expression [6], we straightforwardly obtain

$$\Gamma(p) = \frac{1}{\cosh^d \left( \frac{1}{2} \sqrt{p/d} \right)}. \quad (A.5)$$

We check that $\mathbb{E}(\chi[\bar{r}]) = 1$ as it should. The variance is given by $Var(\chi[\bar{r}]) = \frac{1}{4}$ and vanishes in the limit of large dimension, as expected from the central limit theorem. In the case of even dimensions, we recognize the Laplace transform of the distribution (A.1). Therefore we deduce that the distribution of the functional $\chi[\bar{r}]$ is

$$P_d(x) = \frac{d}{4} x^{d/4} \left( \frac{d}{4} x \right)^{d/4} \quad \text{for dimension } d \text{ even.} \quad (A.6)$$

Expressions of the distributions $\sigma_\alpha(\xi)$ can be found in [25].

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