Unstable quantum oscillator

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Abstract. We consider a model for an unstable quantum oscillator. The energy levels, the level widths and wave functions of the unstable states have been explicitly found. The obtained results can be applied for the description of the properties of such systems as quantum wells or series of molecular or nuclear resonances.

1. Introduction
Oscillator [1] is one of the most important systems in physics, because it describes the motion in the vicinity of a stable equilibrium. For this reason one can observe in nature many examples of oscillatory motion, both in the classical and quantum domain. The quantum oscillator appears at all levels: molecular, nuclear and in elementary particles and these systems are usually unstable. For this reason it is important to consider an unstable quantum oscillator. In this paper we will discuss a simple one dimensional model with the oscillator potential for $x \leq 0$ and a free motion for $x > 0$. At $x = 0$ we put a semi-transparent potential wall in the form of the attractive $\delta(x)$ potential with regulated strength. Such a system has a set of (almost) equidistant energy levels with complex eigenvalues that means that they are unstable. One can find the explicit values of the real and imaginary part of the energy and determine how they depend on the parameters of the potential.

The organization of the paper is the following. In Section 2 we introduce the Hamiltonian of our system and give equations for the Green’s function from which we will determine the spectrum and the eigenfunctions. In Section 3 we find explicit solutions for the Green’s function from which in Section 4 we find energy levels and eigenfunctions. We also give a plot of the real and imaginary part of the lowest energy levels. Section 5 contains conclusions.

2. Description of the system
The $-\Omega \delta(x)$ potential has interesting properties. When $\Omega < 0$ then it has exactly one bound state and for either sign of $\Omega$ it is semi transparent and as $\Omega \rightarrow \infty$ it becomes totally opaque.

Let us define the potential $V_b(x)$ of the oscillatory barrier in the following way

$$V_b(x) = \begin{cases} \frac{m\omega^2 x^2}{2} & \text{for } x \leq 0 \\ 0 & \text{for } x > 0. \end{cases}$$ (1)

Obviously there are no bound states for the potential defined in (1).
We now define the Hamiltonian $H$ of the unstable oscillator

$$H = \frac{p^2}{2m} + V_b(x) - \Omega \delta(x). \quad (2)$$

The potential of this oscillator is drawn in Figure 1 from which one can see that it allows the formation of the quasi bound states to the left of the semi-transparent wall produced by the $-\Omega \delta(x)$ part of the potential. In the limit $\Omega \to \infty$ the wall at $x = 0$ becomes totally impenetrable and the system has an infinite number of the oscillator bound states with energies $\hbar \omega (2n + 3)/2$, i.e. the oscillator states that are antisymmetric under the space reflection $x \to -x$.

![Figure 1. The potential of the unstable oscillator.](image)

One can look for the eigenfunctions and the eigenvalues of the Hamiltonian in many possible ways but we will choose the method that is physically the most appealing i.e. by the determination of the Green’s function of our system [3] and then by fixing the positions of the poles of the Green’s function we find the eigenvalues of energy. The eigenfunctions are determined from the behaviour of the Green’s function in the vicinity of the pole.

The boundary conditions for the Green’s function that lead to the determination of the eigenvalues of the quasi bound states are the following

$$G(x, x'; E) \to \begin{cases} 
\text{is bounded} & \text{for } x \to -\infty \\
\sim e^{ikx} & \text{for } x \to +\infty.
\end{cases} \quad (3)$$

We will look for the Green’s function for our system in two steps:

(i) We determine the Green’s function $G_0(x, x'; E)$ of the system without the semi-transparent wall of the pure oscillatory barrier.

(ii) The Green’s function for the full Hamiltonian is given by the following relation

$$G(x, x'; E) = G_0(x, x'; E) + \frac{G_0(x, 0; E)G_0(0, x'; E)}{\Omega - G_0(0, 0; E)}. \quad (4)$$

Now, from (4) we see that the zeros of the denominator as the function of energy

$$\frac{1}{\Omega} - G_0(0, 0; E) = 0 \quad (5)$$

are the positions of the poles of the Green’s function $G(x, x'; E)$ and thus correspond to the eigenvalues of the energy of the Hamiltonian.
3. Green’s function
We will determine the energy levels of our unstable oscillator defined in (2) from the Green’s function which fulfills the equation

$$(H - E)G(x, x'; E) = \delta(x - x').$$

As explained earlier we have to find first the Green’s function of the Schrödinger equation without the semi-transparent barrier $G_0(x, x'; E)$ from the equation

$$(H_0 - E)G_0(x, x'; E) = \left(\frac{p^2}{2m} + V_0(x) - E\right)G_0(x, x'; E) = \delta(x - x')$$

which fulfills the same boundary condition (3) as $G(x, x'; E)$. In order to determine the function $G_0(x, x'; E)$ one has to find two solutions of the Schrödinger equation

$$(H_0 - E)\psi^{\pm\infty}(x) = 0$$

with the following behaviour at $\pm\infty$

$$\psi^{-\infty}(x) \text{ is bounded for } x \to -\infty$$
$$\psi^{+\infty}(x) \sim e^{ikx} \text{ for } x \to +\infty$$

and then the function $G_0(x, x'; E)$ is equal

$$G_0(x, x'; E) = C_0 \begin{cases} 
\psi^{-\infty}(x)\psi^{+\infty}(x') & \text{for } x \leq x' \\
\psi^{+\infty}(x)\psi^{-\infty}(x') & \text{for } x \geq x',
\end{cases}$$

where $C_0$ is a suitable constant which can be determined from the Wronskian of the two solutions $\psi^{\pm\infty}(x)$.

Equation (8) is of the oscillator type for $x \leq 0$ and for $x \geq 0$ it is the equation for the free motion. Both types of equations have known solutions and thus the functions $\psi^{\pm\infty}(x)$ are equal

$$\psi^{-\infty}(x) = \begin{cases} 
e^{-\frac{i}{2}\xi^2} & 
\frac{1}{2} \left(1 - \frac{2i}{k} \sqrt{\frac{m\omega}{\hbar}} \frac{\Gamma \left(\frac{3-2\epsilon}{4}\right)}{\Gamma \left(\frac{1-2\epsilon}{4}\right)} \right) e^{ikx} + \text{c.c.}
\end{cases}$$

for $x \leq 0$

$$(11)$$

and

$$\psi^{+\infty}(x) = \begin{cases} 
e^{-\frac{i}{2}\xi^2} & 
\frac{1}{2} \left(1 - \frac{2i}{k} \sqrt{\frac{m\omega}{\hbar}} \frac{\Gamma \left(\frac{3-2\epsilon}{4}\right)}{\Gamma \left(\frac{1-2\epsilon}{4}\right)} \right) e^{ikx}
\end{cases}$$

for $x \geq 0$.

Here $1F_1(\alpha, \beta; z)$ is the confluent hypergeometric function, $\Gamma(z)$ is the Euler’s gamma function and $\xi, \epsilon$ and $k$ are equal

$$\xi = \sqrt{\frac{m\omega}{\hbar} \frac{k}{2}}, \quad \epsilon = \frac{E}{\hbar\omega}, \quad k = \sqrt{2mE \hbar} = \sqrt{\frac{m\omega}{\hbar} \frac{k}{2\epsilon}}.$$
Finally the constant \( C_0 \) is calculated to be
\[
C_0 = i \frac{2m}{\hbar^2} \frac{1}{k + 2i \sqrt{\frac{m \omega}{\hbar} \Gamma(\frac{3}{2} - 2\epsilon) \Gamma(\frac{1}{2} - 2\epsilon)}}
\]  
and this completes the calculation of the Green’s function.

4. Energy levels and eigenfunctions

The poles on the real axis of the complex energy plane of the Green’s function correspond to the energies of the bound states of the system and the continuum of states produces a discontinuity on the real energy axis. From the Green’s function \( G_0(x, x'; E) \) in (10) and the dependence on energy \( E \) of the constant \( C_0 \) in (13) we conclude that the system without the semi-transparent wall has a continuum of states on the positive real axis of energy \( E \), because \( k \sim \sqrt{E} \). The full Green’s function is given in (4) and from this it follows that the spectrum of the unstable oscillator will consist of the continuum \( E \geq 0 \) and the complex or real poles obtained from the zeros of the denominator, equation (5).

Non penetrable wall

Let us first discuss the limiting case \( \Omega \to +\infty \), i.e. when the wall becomes opaque. In such a case, following (5), the poles of the Green’s function are obtained from the equation
\[
G_0(0, 0; E) = C_0 \psi^{-\infty}(0) \psi^{+\infty}(0) = i \frac{2m}{\hbar^2} \frac{1}{k + 2i \sqrt{m \omega \Gamma(\frac{3}{2} - 2\epsilon) \Gamma(\frac{1}{2} - 2\epsilon)}} = 0.
\]
which can be easily solved, using the fact that the gamma function \( \Gamma(z) \) has simple poles for negative integers. From this we get that the energies of the bound states are
\[
\frac{3 - 2\epsilon}{4} = -n \quad \Rightarrow \quad \epsilon = \frac{3}{2} + 2n \quad \Rightarrow \quad E = \hbar \omega \left( \frac{1}{2} + (2n + 1) \right).
\]
The energies in equation (15) correspond to the odd states of the harmonic oscillator. This result was expected from the elementary analysis of our system and the limiting property of the \( \delta(x) \) potential.

Unstable states

For finite values of \( \Omega \) the \( \delta(x) \) potential is semi-transparent and the states within the potential well are not stable. The pole position in energy \( E \) of the full Green’s function determined from (5) gives the energy and width of unstable state. Explicitly the equation for the pole positions is the following
\[
\frac{1}{\Omega} - i \frac{2m}{\hbar^2} \frac{1}{k + 2i \sqrt{m \omega \Gamma(\frac{3}{2} - 2\epsilon) \Gamma(\frac{1}{2} - 2\epsilon)}} = 0.
\]
which written using the dimensionless variables has the form
\[
\frac{1}{\Omega \sqrt{\frac{m}{\omega \hbar^2}}} - \frac{2i}{\sqrt{2\epsilon + 2i \Gamma(\frac{1}{2} - 2\epsilon) \Gamma(\frac{3}{2} - 2\epsilon)}} = 0.
\]
Equation (17) has a series of solutions and for the dimensionless parameter \( \Omega \sqrt{\frac{m}{\omega \hbar^2}} \) large, the real part of the solution \( \epsilon \) is close to the value of \( \epsilon_n \) for the opaque wall, given in (15). The positions of the lowest complex poles for \( \Omega \sqrt{\frac{m}{\omega \hbar^2}} = 10 \) are given in Figure 2. From this figure one can see that the width of the oscillator grows with the value of the quantum number \( n \).
**Figure 2.** Positions of the poles of an unstable oscillator with the strength of the $\delta(x)$ wall equal to $\Omega \sqrt{\frac{m}{\omega \hbar}} = 10$.

**Approximate formula for the positions of the poles**

One can find approximate roots of equation (17) using the asymptotic behaviour of the Euler’s gamma function. We will be looking for a solution in the vicinity of the $n$-th pole and denote the position of the pole by

$$\epsilon_n = \frac{3}{2} + 2n + \Delta \epsilon_n.$$  \hfill (18)

We then have

$$\frac{\Gamma \left(\frac{3-2\epsilon_n}{4}\right)}{\Gamma \left(\frac{1-2\epsilon_n}{4}\right)} \approx \frac{\Gamma \left(-n - \frac{\Delta \epsilon_n}{2}\right)}{\Gamma \left(-n - \frac{1}{2} - \frac{\Delta \epsilon_n}{2}\right)} = \frac{\cos \left(\frac{n\Delta \epsilon_n}{2}\right)}{\sin \left(\frac{n\Delta \epsilon_n}{2}\right)} \left( n + \frac{3}{4} + \frac{\Delta \epsilon_n}{2} \right)^{\frac{1}{2}} \times \left[ 1 + O \left( \frac{1}{n^2} \right) \right]$$

for large $n$.  \hfill (19)

After inserting (19) into equation (17) one obtains

$$\frac{1}{\Omega \sqrt{\frac{m}{\omega \hbar}}} - \frac{2i}{\sqrt{3 + 4n + 2\Delta \epsilon_n \left( 1 + i \cot \frac{\Delta \epsilon_n}{2} \right)}} = 0$$

(20)

from which one obtains an approximate expression for the $n$-th pole position

$$\epsilon_n = \frac{3}{2} + 2n + \Delta \epsilon_n \approx \frac{3}{2} + 2n + \frac{2}{\pi} \arctan \left( \frac{1}{i + 2\Omega \sqrt{\frac{m}{\omega \hbar}} \sqrt{3 + 4n}} \right).$$  \hfill (21)

Formula (21) is rather accurate even for low values of $n$, e.g. for $n=5$, the precise value obtained from (17) is $\Delta \epsilon_5 = (0.1431 - i 0.0335)$, while the one obtained from (21) is $\Delta \epsilon_5 = (0.1423 - i 0.0329)$. For larger values of $n$ the approximation becomes better.
5. Single bound state pole

This unstable harmonic oscillator has one stable state that has its origin in a single stable state of the $\delta(x)$ well. To find the energy of this state we analyse equation (17) for large negative values of $\epsilon$. In such a case we have

$$k = \sqrt{\frac{m\omega}{\hbar}} \sqrt{2\epsilon} = i \sqrt{\frac{m\omega}{\hbar}} \sqrt{2|\epsilon|}$$

and

$$\frac{\Gamma\left(\frac{3-2\epsilon}{4}\right)}{\Gamma\left(\frac{1-2\epsilon}{4}\right)} \sim \sqrt{-\epsilon} = \sqrt{|\epsilon|}$$

and after inserting these relations into (17) we obtain the position of the stable pole

$$|\epsilon| = \frac{1}{2} \frac{m\Omega^2}{\omega\hbar^2}$$

which gives the following value of the energy of the bound state

$$E = -\frac{1}{2} \frac{m\Omega^2}{\hbar^2}.$$  \hspace{1cm} (22)

It is worth to note that the energy of this stable state coincides with the one of the pure $\delta(x)$ well without the oscillator potential.

**Wave function of unstable state**

The generic form of the Green’s function near the pole $E = E_0$ of a bound or unstable state has the form

$$\psi(x) \psi(x') = \frac{\psi(x) \psi(x')}{E - E_0}.$$  \hspace{1cm} (23)

From (4) one can see that the part of the Green’s function of our oscillator that develops the pole has the form

$$\frac{G_0(x, 0; E) G_0(0, x'; E)}{\frac{1}{\Omega} - G_0(0, 0; E)}$$  \hspace{1cm} (24)

so it means that the wave function of an unstable state whose pole position is $E_n$ has the form

$$\psi_n(x) \sim G_0(x, 0; E_n)$$  \hspace{1cm} (25)

and what remains is the calculation of the normalization of the wave function. The denominator of equation (24) in the vicinity of the pole $E = E_n$ has the form

$$\frac{1}{\Omega} - G_0(0, 0; E) \approx N_n^2 (E - E_n),$$  \hspace{1cm} (26)

where $N_n$ is the normalization of the wave function. From (26) one immediately obtains

$$N_n^2 = \left. \frac{\partial}{\partial E} \left( \frac{1}{\Omega} - G_0(0, 0; E) \right) \right|_{E=E_n} = - \left. \frac{\partial}{\partial E} (G_0(0, 0; E)) \right|_{E=E_n}. \hspace{1cm} (27)$$

Thus the normalization $N_n$ of the wave function $\psi_n(x)$ is equal

$$N_n^2 = \frac{d\epsilon}{dE} \frac{\partial}{\partial \epsilon} \left( 2i \sqrt{\frac{m}{\omega\hbar^2}} \frac{1}{\sqrt{2\epsilon}} \right) \left( \frac{\Gamma\left(\frac{3-2\epsilon}{4}\right)}{\Gamma\left(\frac{1-2\epsilon}{4}\right)} \right) \bigg|_{E_n}$$  \hspace{1cm} (28)

and the wave function of the unstable state at $E_n$ is equal

$$\psi_n(x) = \frac{G_0(x, 0; E_n)}{N_n}. \hspace{1cm} (29)$$
6. Conclusions
We have presented a model of an unstable quantum oscillator. The spectrum of such a model consists of a continuum \( E \geq 0 \) and a series of unstable levels that are in correspondence with the energy levels of an oscillator. Additionally this model has one stable state that has its origin in the stable state of the \( \delta(x) \) potential that is the source of the semi-transparent wall. The wave functions of the unstable states were also explicitly obtained with proper normalization. This normalization cannot be obtained from the direct method of the determination of the wave functions as the eigenfunctions of the Schrödinger equation.

An unstable oscillator can have application in nuclear physics in the study of the equidistant series of nuclear resonances and also in molecular physics, where appear series of unstable levels. An interesting application can also be in the study of quantum wells, where appear semi-transparent parabolic walls.

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