The Volume Inside a Black Hole

Brandon DiNunno and Richard A. Matzner

Center for Relativity, University of Texas at Austin, Austin, TX 78712-1081, USA

Abstract

The horizon (the surface) of a black hole is a null surface, defined by those hypothetical “outgoing” light rays that just hover under the influence of the strong gravity at the surface. Because the light rays are orthogonal to the spatial 2-dimensional surface at one instant of time, the surface of the black hole is the same for all observers (i.e. the same for all coordinate definitions of “instant of time”). This value is \(4\pi(2Gm/c^2)^2\) for nonspinning black holes, with \(G\) = Newton’s constant, \(c\) = speed of light, and \(m\) = mass of the black hole.

The 3-dimensional spatial volume inside a black hole, in contrast, depends explicitly on the definition of time, and can even be time dependent, or zero. We give examples of the volume found inside a standard, nonspinning spherical black hole, for several different standard time-coordinate definitions.

Elucidating these results for the volume provides a new pedagogical resource of facts already known in principle to the relativity community, but rarely worked out.
1. INTRODUCTION

The area of the surface of a nonspinning, quiescent (Schwarzschild) black hole is is
\[ 16\pi \left( \frac{Gm}{c^2} \right)^2, \]
which is written \( 16\pi m^2 \) in the usual relativist’s convention in which units are chosen so that both Newton’s constant and the speed of light are set equal to unity. The usual Schwarzschild coordinate \( r \) is defined to be an areal coordinate (spherical area = \( 4\pi r^2 \)), so its value at the black hole surface (the horizon) is \( r = 2m \). Because the black hole is spherical, we simply need to measure the area in a transverse direction. This produces the unique result (\( Area = 16\pi m^2 \)). The uniqueness follows because if we consider a different definition of the 3-space in which we measure the area, we just shift our points in null directions along the (null) horizon. Null directions have zero length and cannot contribute to (or change) the area.

An occasional question to the teacher of relativity is: “...then, what is the volume of a black hole?” The answer is that, unlike the response about the surface, the volume depends on the way that the 3-dimensional “constant-time” space containing the black hole is defined. Gravity, described by General Relativity, is the curvature of space-time, and the implied curvature for the space defined by our choice of constant time depends on how the “now” space is defined.

The simplest black hole is an eternal black hole, one that was not formed by collapsing matter but is a nonlinear “vacuum” solution, with structure anchored in its own gravitational field. Even in this case there are many choices of constant time, and hence many different results for the volume, of the chosen 3-space within the horizon. This article presents a pedagogical description of that (well-known to the expert) fact.

All of the background needed for this paper can be found in Gravitation [1], henceforth MTW. We will also provide original references where appropriate.

2. BACKGROUND

Einstein’s General Relativity describes the gravitational field by giving the spacetime metric. A metric describes the way coordinate increments apply to measurable (“proper”)
space or time increments. The *Special* Relativity metric (describing a spacetime without gravity) is written:

$$ds^2 = -dt^2 + \delta_{ij} \, dx^i \, dx^j \quad (1)$$

Here we use the *summation convention*: repeated indices are summed over through their range; variables $i, j, k, ...$ range through the spatial coordinates $x, y, z$, and $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. (the superscripts $i, j, k, ...$ are indices, not exponents.) In Eq.(1), $s$ is a proper distance, so to make the units match, the first term on the right should have a factor $c^2$, the square of the speed of light. As noted above, relativists simplify expressions by using units in which the speed of light is unity. (For instance: the length unit is the light year and the time unit is the year.) Also, it is useful to rewrite expression (1) using spherical coordinates:

$$ds^2 = -dt^2 + f_{ij} \, dx^i \, dx^j \quad (2)$$

where now $i, j, k, ...$ range through the spatial coordinates $r, \theta, \phi$, and $f_{ij}$ is a diagonal symmetric matrix $f_{ij} = diag[1, r^2, r^2 \sin^2 \theta] = diag[1, (x^1)^2, (x^1)^2 \sin^2(x^2)]$. The metric is an example of a tensor, a geometrical object that is defined independently of any particular coordinate frame, and whose components follow specific rules for expression in different reference frames. Both Eq(1) and Eq(2) present the same geometrical object, the Special Relativity metric tensor, but expressed in different coordinate frames.

Within a year of the publication of Einstein’s General Relativity \[2\], Schwarzschild \[3\] obtained the General Relativity metric which is the analog to the simplest Newtonian gravitational field with $\Phi = Gm/r$. (Don’t confuse the coordinate $\phi$ with the potential $\Phi$.) This generalizes Eq.(2) to

$$ds^2 = -(1 - \frac{2\Phi}{c^2}) dt^2 + (1 - \frac{2\Phi}{c^2})^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (3)$$

For large distances from the center, the Schwarzschild form Eq(3) indicates a moderate deviation from the Special Relativity form, and it can be shown that geodesic motion in this spacetime approximates Newtonian motion quite closely. However, if one considers smaller radii, the quantity
can become significant, and apparently cause problems as it approaches unity (the coefficient of \(dr^2\) diverges to infinity; the coefficient of \(dt^2\) goes to zero). This situation was confusing because objects with \(\frac{2Gm}{r} \approx 1\) were obviously extremely compact, so maybe this was a nonphysical configuration; but orbits, particularly expressed in terms of the proper time of the infalling observer, showed this strange surface could be reached in the finite lifetime of the intrepid explorer willing to fall into the center of the field. Only in 1960 did Kruskal and Szekeres recognize that the surface \(r = 2m\) is special but is not singular, and show how to understand this fact. (Here and henceforth we again set both \(c\) and the Newtonian constant \(G\) equal to unity.)

The trick is to realize that the Schwarzschild coordinates are badly behaved near \(r = 2m\), and to introduce new time and radial coordinates (called \(v\), the time coordinate, and \(u\), the new radial coordinate) which behave well (“smoothly”) there. (The angles \(\theta, \phi\) just describe 2-dimensional spheres, and there is no reason to change them.) To make the new coordinates behave well near \(r = 2m\) requires singular transformations \(\{t, r\} \leftrightarrow \{u, v\}\), but this is justified by the fact that all particle and photon orbits remain smooth and continuous when expressed in the new coordinates. Even more usefully, \(u\) and \(v\) are defined so that the radial coordinate speed of light, \(\frac{du}{dv} = \pm 1\) everywhere. The null lines are inclined at 45° just as they are in a flat space diagram. This makes it easy to pick out timelike motion, or 3-spaces of constant time.

The transformations giving \(\{t, r\}\) in terms of \(\{u, v\}\) are:

\[
\left(\frac{r}{2m} - 1\right)e^{\frac{r}{2m}} = u^2 - v^2. \tag{5}
\]

\[
tanh \frac{t}{4m} = \frac{v}{u}, \quad |\frac{v}{u}| \leq 1 \tag{6}
\]

\[
tanh \frac{t}{4m} = \frac{u}{v}, \quad |\frac{v}{u}| \geq 1 \tag{7}
\]

In Eq(5) there is no analytic inverse for the single valued function of \(r\) on the left hand side, but numerical solution is straightforward. The fact that there are two different analytical
expressions relating $t$ to $\{u, v\}$ is partial evidence of the singular coordinate transformation. The inverse transformation giving the Kruskal-Szekeres coordinates $\{u, v\}$ in terms of $\{r, t\}$, can be found in MTW.

What is more useful than the analytic expressions Eqs(5-7) is to graph the lines showing constant Schwarzschild coordinates $t$ and $r$, in a graph whose axes are the Kruskal-Szekeres coordinates $u$ and $v$. See Figure 1.

From Figure 1 we see that constant Schwarzschild coordinate $t$ is a straight line through the $u, v$ origin; $t = 0$ is the horizontal line coinciding with the line $v = 0$, $t = \infty$ is the $45^\circ$ line $v = u$ (light solid positive-slope line in Figure 1 passing through the origin), $t = -\infty$ is the $45^\circ$ line $v = -u$ (light solid negative-slope line in Figure 1 passing through the origin). From Eq(5), constant Schwarzschild coordinate $r$ defines a hyperbola given by $u^2 - v^2 = const$. If the constant is positive, this defines the situation far from the black hole. However, if one considers smaller values for this constant, corresponding to smaller constant values of $r$, the hyperboloids approach the straight lines $u = \pm v$, which are achieved when $r = 2m$, when the left hand side of Eq(5) is zero. Thus $r = 2m$ overlays (defines the same points as) $t = \pm \infty$. Outside the horizon, $r = const$ is a timelike surface (its tangent is a timelike vector); inside the horizon it is spacelike.

3. VOLUME COMPUTATION: SCHWARZSCHILD COORDINATES

The volume is a three dimensional concept, and it depends only on the $t = constant$ spatial 3-dimensional part of the metric. This means, consider the 4-d metric (Eq (3)) with the differential of the time coordinate set equal to zero ($dt = 0$ in the Schwarzschild coordinates we consider in this section). From the resulting 3-dimensional metric, compute the determinant, $g$, of the matrix of metric components.

$$g = \frac{r^4 \sin^2 \theta}{1 - \frac{2m}{r}}, \quad Schwarzschild \ coordinates.$$ (8)

The volume between two values of $r$, say $r_{inner}$ and $r_{outer}$ is then

$$\int_{r_{inner}}^{r_{outer}} \sqrt{g} d^3x.$$ (9)

We have been asked to compute the volume inside the horizon at a fixed Schwarzschild time $t$. Thus the outer limit in the integral in Eq(9) is $r_{outer} = 2m$. 5
Looking at Figure (1), we see that on no Schwarzschild \( t = \text{constant} \) “slice” does the \( r \) coordinate extend to less than \( r = 2m \). The limits are the same: \( r_{\text{inner}} = r_{\text{outer}} = 2m \), so we expect the integral to vanish. However, the integrand is singular at \( r = 2m \), so we must investigate the integral a little further.

The angular integral just yields \( 4\pi \). The integral in \( r \) has a singularity at \( r = 2m \), but this singularity is in fact integrable. We can investigate this by assuming the value of \( r_{\text{outer}} \) to be just larger than \( 2m \), say \( r_{\text{outer}} = 2m(1 + \epsilon) \), where \( \epsilon \) is small. Then to lowest order in \( \epsilon \), the radial integral is

\[
\int_{r_{\text{inner}}}^{r_{\text{outer}}} \frac{(2m)^{\frac{3}{2}} dr}{\sqrt{r - 2m}}
\]

evaluated at the limits. This integral obviously vanishes as \( \epsilon \) goes to zero, i.e. as \( r_{\text{outer}} \to 2m \).

There is zero volume inside the black hole in any Schwarzschild time slice of a Schwarzschild black hole spacetime.

4. VOLUME COMPUTATION: KERR SCHILD COORDINATES

Convincing oneself of the zero result of Section 3 is made easier by considering a different time-independent form of the 4-dimensional metric describing the Schwarzschild spacetime, which gives a nonzero volume inside the horizon. Kerr-Schild coordinates provide this form.

Coordinates called Kerr-Schild coordinates \([6]\), which we denote \( r_{\text{KS}}, t_{\text{KS}} \), are related (in our spherical case) to the Schwarzschild coordinates by the transformations:

\[
r_{\text{KS}} = r
\]

\[
t_{\text{KS}} = t + 2m \ln\left(\frac{r}{2m} - 1\right)
\]

The form of the 4-dimensional metric in Kerr-Schild coordinates is:

\[
ds^2 = -(1 - \frac{2m}{r_{\text{KS}}})dt_{\text{KS}}^2 + \frac{4m}{r_{\text{KS}}}dt_{\text{KS}}dr_{\text{KS}} + (1 + \frac{2m}{r_{\text{KS}}})dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2.
\]
The Kerr-Schild form of the metric is, like the Schwarzschild form, independent of the time coordinate (here \( t_{KS} \)). It does however contain terms that describe cross terms in the measurement of distance, cross terms between increments in time and increments in radial coordinate. For the 3-d metric, however, these terms are irrelevant. The 3-metric is simply

\[ 3ds_{KS}^2 = (1 + \frac{2m}{r_{KS}})dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \]  

(15)

It is also important to know where the points in the constant \( t_{KS} = \text{constant} \) slice are located with respect to the standard Kruskal-Szekeres picture. Figure 2 shows a series of \( t_{KS} = \text{constant} \) surfaces. It can be seen that these constant \( t_{KS} \) slices differ significantly from the Schwarzschild slices. They penetrate inside \( r = r_{KS} = 2m \) (which the constant Schwarzschild slices do not), and in fact extend inward to \( r = r_{KS} = 0 \). We thus expect a nonzero result from calculation of the volume inside the sphere \( r = r_{KS} = 2m \).

The determinant of the 3-metric Eq(15) is \((1 + \frac{2m}{r_{KS}})r_{KS}^4\sin^2\theta\). Thus the volume at constant \( t_{KS} \) is

\[
\int_0^{2m} \sqrt{(1 + \frac{2m}{r_{KS}}) r_{KS}^4 \sin^2\theta} \, dr_{KS} \, d\theta \, d\phi.
\]

(16)

\[
= \int_0^{2m} \sqrt{(1 + \frac{2m}{r_{KS}}) r_{KS}^2 \sin\theta} \, dr_{KS} \, d\theta \, d\phi.
\]

(17)

Even though the integrand contains the first factor which becomes infinite at \( r_{KS} = 0 \) as \( 1/\sqrt{r_{KS}} \), the factor \( r_{KS}^2 \) moderates the divergence, and the integral is finite. In fact, since we consider the volume for \( r_{KS} \leq 2m \), after doing the angular integrals the integrand for the \( r_{KS} \) integration is clearly between \( 4\pi\sqrt{2m} r_{KS}^{3/2} \) and \( \sqrt{2} \times 4\pi\sqrt{2m} r_{KS}^{3/2} \) so the volume is easily analytically estimated to be between \( \frac{4}{5}4\pi(2m)^3 = (5.026...) \times (2m)^3 \), and \( \sqrt{2} \times \frac{4}{5}4\pi(2m)^3 = (7.108...) \times (2m)^3 \). The complete integral can be done analytically (Mathematica helps), yielding:

\[
Vol_{KS} = \frac{1}{24} [7\sqrt{2} - \frac{3}{2} \ln(\frac{\sqrt{2} - 1}{\sqrt{2} + 1})]4\pi(2m)^3.
\]

(18)

\[
= (6.567...) \times (2m)^3.
\]

(19)
5. VOLUME COMPUTATION: NOVIKOV COORDINATES

Both the Schwarzschild and the Kerr Schild definitions of time are such that the metric (and thus the volume inside the horizon) is static, i.e. independent of the Schwarzschild time $t$, or of the Kerr-Schild time $t_{KS}$ at which it is computed. However it is easy to see that a collection of nearby observers falling into a black hole would recognize a time dependent gravitational field; the tidal force gets stronger as they fall in, getting older. Novikov [9] realized that one could use the proper time $\tau$, the local internal time of each observer, to define a time slicing. He based his coordinates on a collection of observers who at one Schwarzschild instant (Schwarzschild $t = 0$, also labeled $\tau = 0$) are instantaneously each at rest at their own maximum value $r_{\text{max}}$ of the Schwarzschild coordinate $r$. Each observer sets her watch to read zero at this one instant of time. Novikov also introduced a radial coordinate, called a comoving coordinate $R$, defined by: for each observer,

$$R^2 + 1 = \frac{r_{\text{max}}}{2m}, \quad (20)$$

At any later Novikov time $\tau$, each particle still has the same value of $R$ but has fallen to a smaller value of $r$. For general values of $\tau \neq 0$ one has only an implicit functional relation between $r$ and $R$:

$$\frac{\tau}{2m} = (R^2 + 1)[\frac{r}{2m} - (\frac{r/2m}{R^2 + 1})^{1/2} + (R^2 + 1)^{3/2}\arccos[(\frac{r/2m}{R^2 + 1})^{1/2}]] . \quad (21)$$

For each value of $\tau$ this is an implicit function giving $r(R)$, and $R(r)$ in that particular $\tau = \text{const}$ Novikov 3-space. Eq(21) holds for positive $\tau$. For negative $\tau$, use the fact that the relation between $r$ and $R$ is even in $\tau$: $r(R, \tau) = r(R, -\tau)$, and $R(r, \tau) = R(r, -\tau)$. Once we have determined this relation, we can (by taking the differential of Eq (21) and setting $d\tau = 0$) evaluate $\partial r/\partial R$ and $\partial R/\partial r$ at any time $\tau$. Figure 3 shows the relation between Novikov and Kruskal-Szekeres coordinates.

The Novikov coordinate 4-metric is:

$$ds^2 = -d\tau^2 + (\frac{R^2 + 1}{R^2})(\frac{\partial r}{\partial R})^2 dR^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (22)$$

The appearances of $r$ in this equation should be eliminated so that only $R$ and $\tau$ appear (using Eq(21)), but we keep the compact symbol $r$ to mean the function $r(R, \tau)$.  

8
The determinant of the 3-metric is
\[
\left( \frac{R^2 + 1}{R^2} \right) \left( \frac{\partial r}{\partial R} \right)^2 r^4 \sin^2 \theta.
\]
(23)

Thus the volume to be evaluated is
\[
\int_{R_{inner}}^{R_{outer}} \sqrt{\left( \frac{R^2 + 1}{R^2} \right) \left| \frac{\partial r}{\partial R} \right|} r^2 \sin \theta \, dR \, d\theta \, d\phi
\]
(24)

The integrand is time dependent, because the factor \( |\partial r/\partial R| \) depends on the Novikov time \( \tau \). But, additionally, the limits of the integral are time dependent. For instance larger and larger values of \( R \) fall toward the horizon as \( \tau \) increases. Hence the value \( R_{outer} \) increases monotonically as \( \tau \) increases. Because we have access to the factor \( (\partial r/\partial R) \), we can freely transform the integral between one expressed in coordinate \( R \), and one in coordinate \( r \). The 3-space \( \tau = 0 \) is identical to the Schwarzschild 3-space \( t = 0 \), so the volume evaluated for \( \tau = 0 \) is the same as found for the standard Schwarschild coordinate case: \( Vol = 0 \). This is consistent with the result from the integral (Eq[24]) evaluated in terms of Schwarschild coordinate \( r \). The upper limit in every case is \( r_{outer} = 2m \). The lower limit on the initial \((\tau = 0)\) space is \( r_{inner} = 2m \) also, because this is the minimum \( r \) reached in the 3-space \( \tau = 0 \). At later \( \tau \), since we are looking for the volume contained inside the horizon, \( r_{outer} = 2m \) continues to hold. However, \( r_{inner} \) decreases, because the observer originally at \( r = 2m \) (i.e. at \( R = 0 \)) falls inward. That person falls inward for a time of \( \pi m \), whereupon she reaches \( r = 0 \) and her world line terminates (she is destroyed) because of the arbitrarily large tidal forces at \( r = 0 \). She is the first observer to reach \( r = 0 \). The \( R \) for each infalling observer stays fixed at its initial value, and in particular the first observer’s Novikov coordinate stays at \( R = 0 \).

Thus, between \( \tau = 0 \) and \( \tau = \pi m \), the inner boundary for the integral is \( r(R = 0, \tau) \), a value of \( r \) that has to be evaluated numerically, but which decreases monotonically from \( r = 2m \) to zero. We thus expect the volume inside the horizon to increase during this time.

Note also (most easily seen from the diagram) that \( r_{inner} \) is the smallest value of \( r \) reached, but that the space inside the horizon on the left contains an equal volume, and is connected on this 3-space at \( r_{inner} \). Hence between \( \tau = 0 \) and \( \tau = \pi m \) the correct integration contains a factor of 2: \( 2 \times \int_{r_{inner}}^{2m} \).

For \( \tau > \pi m \) the singularity at \( r = 0 \) intrudes to reduce the volume to that on only one side of \( R = 0 \). Hence there is an instantaneous drop by half in the volume, at the time \( \tau = \pi m \).
Subsequently to $\tau = \pi m$ the limits in terms of $r$ remain fixed at 0, $2m$. However the volume is still a function of time because the quantities involving $R$ in the integrand have different dependences on $r$, for different values of $\tau$.

The resulting $\tau$-dependent volume is plotted in Figure 4.

6. VOLUME COMPUTATION: KRUSKAL-SZEKERES COORDINATES

The 4-metric for the Scharzschild spacetime expressed in Kruskal-Szekeres coordinates is:

$$ds^2_K = \frac{32m^3}{r} e^{-\frac{r}{2m}} (-dv^2 + du^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

(25)

where one must view the appearance of the function $r$ as shorthand for the function $r(u,v)$ defined by Eq(5). We can investigate the time dependent (i.e. $v$-dependent) volume inside the horizon in a $v = \text{const}$ 3-space in close analogy to the Novikov analysis above.

In fact, Figure 1 shows that the 3-spaces $v = \text{const}$ of the Kruskal-Szekeres coordinates are like straightened versions of the Novikov 3-spaces $\tau = \text{const}$. Completely analogously to the Novikov case, an observer initially at $r = 2m$, i.e. $u = 0$, falls inward for a time of $v = 1$, whereupon she reaches $r = 0$ and her world line terminates (she is destroyed) because of the arbitrarily large tidal forces at $r = 0$. She stays at the Kruskal-Szekeres coordinate $u = 0$ as she falls.

Rather than compute by using the transformation to the coordinate $r$ as we did for the Novikov case, we find it simpler to compute the volume directly in the Kruskal-Szekeres coordinates, where the limits on the radial integration are $u_{\text{inner}} = 0$ and $u_{\text{outer}} = v$ so long as $v \leq 1$, and $u_{\text{inner}} = \sqrt{v^2 - 1}$, $u_{\text{outer}} = v$ when $v > 1$.

The determinant of the $v = \text{const}$ 3-metric is

$$\left(\frac{32m^3}{r} e^{-\frac{r}{2m}}\right)r^4\sin^2\theta.$$ 

(26)

Thus the volume to be evaluated is:

$$\int_{u_{\text{inner}}}^{u_{\text{outer}}} \sqrt{\frac{32m^3}{r} e^{-\frac{r}{2m}}} r^2\sin\theta \, du \, d\theta \, d\phi.$$ 

(27)

Again as before, the initial ($v = 0$) slice corresponds to the Schwarzschild coordinate 3-space, so the volume is zero. (This can also be seen because it is an integration with finite
integrands from $u = 0$ to $u = v$ when $v$ is zero.) For somewhat larger $v$, the limits expand, and the integral is nonzero. For spaces before the singularity appears, we include a factor 2 because the space extends to the horizon “on the left”; at the time $v = 1$ the singularity appears in the space (the 3-space touches the singularity; the 3-space evolves a singularity) and the volume drops by half. Subsequently the evolution is over limits $\sqrt{v^2 - 1}$, $v$, and the volume continues to evolve. Figure 5 presents the time evolution of the volume.

7. CONCLUSIONS

The area of a Schwarzschild black hole is unique, and can be defined by an idealized transverse measurement of a particular spherical surface. In contrast, the volume inside a black hole requires a definition of the particular 3-space in which the volume is computed, which may be explicitly time dependent, and an understanding of the (possibly time dependent) limits of the integral required to compute this volume. Understanding these points and computing the volumes as we have done here introduces and uses a number of concepts and techniques of general use in relativistic calculation, and can be a useful pedagogical tool.

Acknowledgments

This work was supported by NSF grant PHY 0354842, and by NASA grant NNG04GL37G. We thank Michael Salamon for bringing this problem to our attention.
[1] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W.H. Freeman, New York, 1970).

[2] A. Einstein, *Preuss. Akad. Wiss. Berlin, Sitzber.* 778-786 (1915), ibid. 799-801 (1915), ibid. 844-847 (1915).

[3] K. Schwarzschild, *Sitzber. Deut. Akad. Wiss. Berlin, Kl. Math.-Phys. Tech.*, 189-196 (1916)

[4] M. D. Kruskal, *Phys. Rev.* 1119 1743-1745 (1960).

[5] G. Szekeres, *Publ. Mat. Debrecen* 7 285-301 (1960).

[6] Roy Kerr, A. Schild, in *IV Centenario Della Nascita di Galileo Galilei, 1564-1964,* p.222 G. Barbera, editor; Pubblicazioni del Comitato Nazionale per le Manifestazioni Celebrative, Firenze (1965). For the spherical case we consider, these coordinates were previously discovered by Eddington [7] and rediscovered by Finkelstein [8], so the spherical coordinates are often called “Eddington-Finkelstein” coordinates.

[7] A. S. Eddington, *Nature* 113 192 (1924).

[8] D. Finkelstein, *Phys. Rev.* 110 965-967 (1958).

[9] I. D. Novikov, Doctoral Dissertation, Shternberg Astronomical Institute, Moscow (1963).
FIG. 1: The lines where the Schwarzschild coordinate $t = constant$, plotted in the Kruskal $\{u,v\}$ plane (straight “double-dot” lines passing through the origin); and the hyperbolae where the Schwarzschild radial coordinate $r = constant$ (“double-dot” curves). We also show several values of Kruskal coordinate $v = constant$ (heavy “dot-dash” horizontal lines. The spacelike hyperbolae $v^2 - u^2 = 1$ (heavy curves) are the locations where the Schwarzschild $r = 0$. The curvature tensor is singular at $r = 0$. 
FIG. 2: The curves ("double-dot" curves) where the Kerr-Schild coordinate $t_{KS} = \text{constant}$, plotted in the Kruskal \{u,v\} plane. The radial coordinate in the Kerr-Schild system is the same as the standard Schwarzschild coordinate.
FIG. 3: The surfaces $\tau = \text{constant}$ (curved, roughly horizontal lines) plotted in the Kruskal diagram. $\tau = 0$ coincides with $v = 0$, and the figure is reflection symmetric across $v = 0$. $\tau$ is the proper time of infalling observers, and is used as the time coordinate in Novikov coordinates.
FIG. 4: The volume inside the horizon in Novikov coordinate 3-spaces, as a function of the Novikov time $\tau$, which is also the proper time of the infalling observers defining the coordinates.
FIG. 5: The volume inside the horizon on a constant Kruskal-Szekeres time 3-space $v = constant$ as a function of $v$. The factor-of-two drop at $v = 1$ arises because “the throat pinches off” at that instant, separating two halves of the volume inside the horizon.