How much entanglement can be generated between two atoms by detecting photons?

L. Lamata,1,2 J.J. García-Ripoll,2 and J.I. Cirac2

1Instituto de Matemáticas y Física Fundamental, CSIC, Serrano 113-bis, 28006 Madrid, Spain
2Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Strasse 1, 85748 Garching, Germany

In the world of quantum information processing it is widely accepted that, while photons are the ideal candidates for transmitting quantum information, this information is better stored and manipulated using atomic systems. The reason is that, while photons can be moved through long distances with little decoherence, atoms can be easily confined and can preserve quantum information for a long time. Consequently, an ideal design for a quantum network will conceivably be built upon a number of atomic or solid state devices which communicate through photonic quantum channels.

There exist mainly two methods for entangling distant atoms. One is based on emission of photons by the first atom, which afterwards interact with the second atom generating the entanglement. The second method relies on detecting the photons emitted by the two atoms with the subsequent entanglement generation due to interference in the measurement process. Some ingredients of both proposals have been realized experimentally. Most of the experiments with isolated atoms and light experimentally, where two atoms, initially at zero–momentum state, are excited with a very small probability. We consider the state of the atoms after spontaneous emission, when both are in the ground state. The state of the system at the end is given by

\[ |\Psi\rangle \sim \epsilon \int dp a_1^\dagger |\text{vac}\rangle (G_1(p)|-p,0\rangle + G_2(p)|0,-p\rangle) + \epsilon^2 \int dp_1 dp_2 G_1(p_1) G_2(p_2) a_1^\dagger a_2^\dagger |\text{vac}\rangle |p_1,-p_2\rangle + |\text{vac}\rangle |0,0\rangle + O(\epsilon^3). \] (1)

Here, \( p, p_1, p_2 \ldots \) denote the momenta of the emitted photons; \( a_1^\dagger, a_2^\dagger, \) their associated creation operators and \( |\text{vac}\rangle \) the vacuum state of the EM field, and \( \epsilon \ll 1 \) are the excitation probabilities of the atoms. The initial momentum distribution of the emitted photons is given by \( G_i(p) \) for the \( i \)-th atom. As we will see later, we require some uncertainty in the initial momentum in order to generate a large amount of entanglement. Finally, \( |0,0\rangle, |0,-p\rangle \) and \( |-p_1,-p_2\rangle \) denote the recoil momenta of the atoms after emitting the photons. The terms omitted in Eq. (1) correspond to higher order processes where an atom emits more than one photon. These terms will have a very small contribution if the decay time of the atom is longer than the duration of the exciting pulse.

Let us now consider a single detector placed symmetrically below the atoms, as in Fig. 1. If there is one single photon detection, this will amount to a projective measurement onto a single-photon state and out of

\[ ... \]
the state in Eq. (1) only the term on the first row will survive. Since the photons coming from the atoms are indistinguishable, an implicit symmetrization will take place and the final state of the atoms will be of the form \(|\psi_1\rangle|0\rangle + |\psi_2\rangle|0\rangle\), for some motional states \(\psi_1\) and \(\psi_2\). Even though we work with continuous variables, this state can have at most 1 ebit which corresponds to \(\langle \psi_1 | \psi_2 \rangle = \langle \psi_1 | 0 \rangle = \langle \psi_2 | 0 \rangle = 0\).

We are going to show now that with two emitted photons, linear optics and two detectors, we cannot do better than one ebit of entanglement [See Fig. 1b]. The proof generalizes the previous argument with a little bit more care. First of all, linear optics amounts to a linear transformation of the initial momentum modes, \(a_p\), to new operators, \(b_{\gamma(p)} := U_{\gamma}a_pU_{\gamma}^\dagger\). A trivial example of this is a 50% beam splitter, which changes the photons from incident states \(a_{+p}\) and \(a_{-p}\) to \((a_{+p} \pm e^{i\phi}a_{-p})/\sqrt{2}\). Linear optics can be combined with measurements. Without loss of generality, all measurements will take place at the end of the process and they amount to a projection onto the modes \(a_{x_1}\) and \(a_{x_2}\) for the first and second detector, respectively. The state after a projective measurement onto two single-photon detectors reads

\[
|\Psi_{\text{at}}^{(2)}\rangle = \int dp_1 dp_2 G_1(p_1)G_2(p_2) \times \langle a_{x_1} a_{x_2} b_{\gamma_1(p_1)}^\dagger b_{\gamma_2(p_2)}^\dagger |\text{vac}\rangle |p_1, -p_2\rangle. \tag{2}
\]

Note that the modes \(a_{x_1}\) and \(a_{x_2}\) detected by the first and second detector are expressed on an orthonormal basis different from that of the \(a_p\) or \(b_\gamma\) operators. We enclose this information, plus the initial wavefunction of the photon in the following \(c\)-numbers

\[
f_j(x_i, p_j) := G_j(p_j)[a_{x_i}, b_{\gamma_j(p_j)}^\dagger]. \tag{3}
\]

Using these wavefunctions we define the motional states

\[
|\psi_{ij}\rangle := \int dp f_j(x_i, p) |p\rangle. \tag{4}
\]

The expectation value in Eq. (2) can be written in terms of the \(f_j(x_i, p_j)\). We thus arrive to the following expression for the atomic state after the measurement

\[
|\Psi_{\text{at}}^{(2)}\rangle \propto |\psi_{11}\rangle |\psi_{22}\rangle + |\psi_{12}\rangle |\psi_{21}\rangle. \tag{5}
\]

This state cannot have more than 1 ebit of entanglement, which happens when all the states \(\psi_{11}, \psi_{12}, \psi_{21}\) and \(\psi_{22}\) are orthogonal to each other.

We must make several remarks. First of all, adding more detectors does not improve the outcome. Second, our proof is valid independently of the number of beam splitters, prisms, lenses and even polarizers we use. In particular, attenuating elements such as polarizers and filters can be treated as a linear operation plus a measurement and are covered by the previous formalism.

We propose now to use an ETPD to obtain an arbitrary degree of entanglement between the two atoms. An ETPD is by definition a device that clicks whenever two photons arrive simultaneously and with their momenta satisfying a certain constraint. An example would be a parametric up-conversion crystal, in which pairs of photons with momenta \(p_1\) and \(p_2\) are converted with a certain probability into a new photon with momentum \(p = p_1 + p_2\). One imposes a constraint on the initial state by post-selecting a window of final momenta. For example, restricting the measurement to photons with transverse momentum \(p_\perp = 0\), then the initial contributing momenta must be those satisfying \(p_{1\perp} + p_{2\perp} = 0\) [Fig. 1b]. In this example the ETPD ideally projects the initial two photon product state \(|\Psi_{\text{ph}}^0\rangle = \int dp_1 dp_2 G_1(p_1)G_2(p_2)a_{p_1}^\dagger a_{p_2}^\dagger |\text{vac}\rangle\) onto the probably entangled state

\[
|\Psi_{\text{ETPD}}\rangle := \int dp_1 dp_2 g(p_{\text{ph}}) a_{p_1}^\dagger a_{p_2}^\dagger |\text{vac}\rangle. \tag{6}
\]

Here \(g(p_{\text{ph}})\) is the acceptance function of the detector or, equivalently, the constraint that the final detected momenta \(p_{\text{ph}}\) obey.

We claim now that with two emitted photons, linear operations and an ETPD, there is no limit to the attainable entanglement. To prove it we consider that after projection of the photon part of state in Eq. (1) into \(|\Psi_{\text{ETPD}}\rangle\) the resulting atomic state will take the form

\[
|\Psi_{\text{at}}\rangle := \int dp_1 dp_2 g(p_{\text{ph}}) |\Psi(p_{\text{ph}})\rangle. \tag{7}
\]
with the already entangled state

$$\left| \Psi(p_a, p_b) \right> := \int dp_1 dp_2 \left[ f_1(p_a, p_1) f_2(p_b, p_2) + f_1(p_a, p_1) f_2(p_b, p_2) \right] | - p_1, - p_2 \rangle. \quad (8)$$

Depending on the specific shape of the functions $g(p_a, p_b)$ and $f_i(p, p_i)$ $l = a, b$, $i = 1, 2$, the corresponding state may reach an unbounded degree of entanglement. For example, let us consider that the photons evolve freely in space without any linear optics elements, $f_i(p, p_i) = G_i(p_i) \delta(p - p_i)$, and assume that the detector has a very narrow acceptance function $g(p_a, p_b) = \delta(p_a + p_b)$. The wider the initial momentum widths of the two photons, the larger the resulting bipartite atomic entanglement, because a higher uncertainty in the wave packets allows for higher nonlocal correlations, which are not bounded from above. Indeed, in this ideal case the outcome will be much like the EPR pairs from the seminal paper Ref. [22].

Current Kerr media are too inefficient to practically implement the ETPD introduced here. Motivated by this we have designed another protocol that simulates the outcome of an ETPD using linear optics, additional photons and postselection. As shown in the KLM proposal [21], any highly entangling quantum gate can be performed this way, though a lot of care is needed to reduce the number of gates. Our proposal starts up from the two atoms after having emitted two photons which are combined with $N - 2$ additional ancillary photons,

$$\left| \Psi(0) \right> = \int dp_1 dp_2 \cdots dp_N G_1(p_1) G_2(p_2) \cdots G_N(p_N) \times a_{p_1}^\dagger a_{p_2}^\dagger \cdots a_{p_N}^\dagger | \text{vac} \rangle \otimes | - p_1, - p_2 \rangle. \quad (9)$$

The resulting state after linear operations on the $N$ photons, and $N$-fold coincidence count on the $N$ detectors, will be, analogously to the two-photon and two-detector case [Eqs. (2)-(5)]

$$\left| \psi_{\text{at}}^{(N)} \right> = \sum_{(i_1, \ldots, i_N) \in \Pi_N} \int dp_1 \cdots dp_N \prod_k f_k(x_{i_k}, p_k) \times$$

$$\times | - p_1, - p_2 \rangle, \quad (10)$$

where $\Pi_N$ denotes the set of permutations of $N$ elements. This state may contain much more than one ebit of entanglement. In fact, an upper bound to the degree of attainable entanglement is $S = \log_2 N$ ebits. We will show afterwards that this bound is indeed saturated.

As a clarifying example we consider the setup in Fig. [1]: with three photons and three detectors. Photons $P_1$ and $P_2$ come from their respective atoms, we introduce a single auxiliary photon, $P_3$ and we place three detectors symmetrically to the atoms, $X_1$, $X_2$, $X_3$. The final state for the two atoms, considering that all the three detectors are excited by the three photons, and fixing relative phases equal to 1 for simplicity photons, will be

$$\left| \Psi_{\text{at}}^{(3)} \right> = \frac{1}{\sqrt{6}} (|1, 2 \rangle + |2, 3 \rangle + |3, 1 \rangle)$$

where we denote with $|i, j\rangle$ the atomic state associated to detection of $P_1$ in $X_i$, and $P_2$ in $X_j$. In Fig. [2] we show the $N! = 6$ processes that contribute coherently to the two-atom final entangled state. This procedure gives an entanglement of $S = 1.25$ ebits.

The previous example is suboptimal. The maximal amount of entanglement of $S = \log_2 N$ ebits is reachable for some of the states in Eq. [10]. To prove it we consider a very symmetric configuration in which the detectors are located along a circle, equidistant to both atoms [Fig. [1]]. We will assume for simplicity that the two emitted photons and the $N-2$ auxiliary ones are in $s$-wave states and arrive with equal probability and phase to every detector. In a similar fashion as in Eq. [11], the final bipartite atomic state will take the form

$$\left| \psi_{\text{sym}} \right> = \sum_{ij} C_{ij} |i, j\rangle \propto \sum_{ij} (1 - \delta_{ij}) |i, j\rangle, \quad (12)$$

where $|i, j\rangle$ is the final bipartite atomic state after detection of photon $P_1$ in detector $X_i$, and photon $P_2$ in detector $X_j$. In matrix form, the coefficients $C_{ij}$ are

$$C_{ij} \propto N \tilde{e}_i (\tilde{e}_j)^T - \mathbb{1}_{N \times N}, \quad (13)$$

FIG. 2: Outcome for an experiment with two atoms and three photons, as shown in Eq. [11].

FIG. 3: Quantum circuit for saturating the bound of $\log_2 N$ ebits as described in the text.

+ $|1, 3\rangle + |3, 2\rangle + |2, 1\rangle$, \quad (11)
where \( \epsilon_i^T := 1/\sqrt{N(1,1,...,1)} \). Both the reduced density matrix of one atom and the Schmidt rank can be obtained from this matrix. The previous state can be rewritten in the form

\[
C_{ij} \propto (N-1)\epsilon_i^T - \sum_{i=2}^{N} \epsilon_i^T \epsilon_i^T,
\]

where \( \{\epsilon_i\}, i = 2,...,N \) is a completion of \( \epsilon_1 \) to an orthonormal basis in \( C^N \). From here it is obvious that the density matrix has full-rank and we can with local operations obtain a maximally entangled state of the form, up to local phases, \( C_{ij} \propto I_{N \times N} \). To do so we must reduce the contribution of the term \( \epsilon_1 \). As shown in Ref. [23], a network of beam-splitters and phase-shifters can be used to perform a unitary operation, \( U_{\epsilon_1} \), that maps the mode \( \alpha_{\epsilon_1} \propto \sum_{i=1}^{N} \alpha_{\epsilon_i} \) to a single optical port. If, as shown in Fig. 3, we place on that port a filter \( F \) that decreases its amplitude by a factor \( N-1 \), when the \( N \) detectors click simultaneously the atoms will get projected onto a maximally entangled state with \( C_{ij} = \epsilon_i^T \epsilon_j^T - \sum_{i=2}^{N} \epsilon_i^T \epsilon_i^T \). The proof is cumbersome and involves studying how all the photon modes in Eq. (9) transform under the nonunitary operation given by the network in Fig. 3 and then ensuring that the detection of \( N \) photons does indeed give rise to the maximally entangled state.

Summing up, in this paper we have demonstrated that it is possible to achieve an arbitrary amount of entanglement in the motional state of two atoms by using spontaneous emitted photons, linear optics and projective measurements. The resulting states can be used to study violation of Bell inequalities and also as a resource for quantum information processing. We expect that similar ideas can be used to entangle atomic clouds, replacing the photons with atoms, because in this case it is easy to build a two-atom detector.

Regarding the implementation, the ideas shown here can be tested easily in current experiments. We would suggest using two trapped ions as target atoms. The ions should be either on a very weak trap, or released right before excitation. The entanglement in the momentum will translate into an entanglement in the position of the atoms after a short time of flight. In practice, with only one additional photon, 1.58 ebits can be produced, and we expect a value of 2 ebits to be both experimentally achievable and realistic. At the cost of a slightly lower fidelity, one can use \( N \) independent attenuated coherent beams instead of true single-photon sources. Clearly, even though there is not a fundamental limit, both the requirement of having good single-photon sources and the detector efficiency will make it very difficult to scale this last scheme to larger \( N \) and more ebits. L.L. acknowledges hospitality at the Max-Planck Institute for Quantum Optics and support from Spanish MEC FPU grant No. AP2003-0014 and project No. FIS2005-05304. J.I.C. acknowledges support from EU projects SCALA and DFG-Forschungsgruppe 635.

*lamata@imaff.cfmac.csic.es

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