Forced magnetohydrodynamic turbulence in large eddy simulation of compressible fluid

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Abstract. Large eddy simulation technique for studying forced compressible magnetohydrodynamic turbulence is suggested. In our work, this method is based on solution of filtered magnetohydrodynamic equations by finite-difference methods and on a linear representation of driving forces in the momentum conservation and the magnetic induction equations. The emphasis is made on important and so far uninvestigated questions about ability of the large eddy simulation method to reproduce Kolmogorov and Iroshnikov-Kraichnan scale-invariant spectra in compressible magnetohydrodynamic flows.

1. Introduction

Magnetohydrodynamic (MHD) turbulence, similar to hydrodynamic turbulence, possesses scale-similarity properties in the inertial range. A fully developed turbulent state is expected when the integral and dissipation scales are separated by several orders of magnitude.

Large-eddy simulation (LES) method is an effective instrument for the study of complex turbulent fluid flows (Sagaut, 2002). The principal idea of this method involves filtering of governing hydrodynamic equations and subsequent parametrization of the universal part of the turbulent flow. This approach differs essentially from the alternative methods of turbulence study. In contrast to the Reynolds-averaged Navier-Stokes (RANS) technique, LES resolves large eddies and thereby provides information about the statistical and spectral properties of the turbulent field. Moreover for the available computer resources the LES approach allows one to investigate flows with higher values of similarity parameters, since the turbulent flow has many degrees of freedom and the minimal number of mesh points must be so large that the application of direct numerical simulation for the study of turbulent flows with realistic Reynolds numbers is very complicated. The last advantage is particularly important for the investigation of magnetohydrodynamic turbulence of a compressible flow which is characterized by high Reynolds and Mach numbers. Recently, LES approach for study of compressible magnetohydrodynamic flow was developed for the cases of polytropic gas (Chernyshov et al., 2006a, 2007) and heat-conducting fluid (Chernyshov et al., 2006b, 2008) for study of decaying MHD turbulence.

However, validity of the LES method for studies of physical processes in forced compressible MHD turbulence remains unidentified. The problem is that in the case of compressible MHD turbulence for numerical solutions of filtered equations in LES (as well as for the solutions
of governing equations in the DNS approach) the finite-difference and finite-volume schemes in coordinate (or physical) space are the most proper. Moreover, the solutions of the basic equations by finite-difference methods allow to investigate inhomogeneous and non-stationary turbulent flows directly. The traditional way of driving force implementation for the compressible MHD flows relies on studies of incompressible turbulent fluid and is based on the spectral representation of the external force and a subsequent re-calculation of this force in physical space. It should be noted that in compressible MHD there are four types of waves: Alfvén, fast magnetosonic, slow magnetosonic and entropy. In this case the interactions between the aforementioned types of waves can lead to the richer picture of the turbulence spectra in the inertial range. The traditional forms of external forces (based on local spectral representations of a turbulence source) can strongly oversimplify the flow pattern in compressible MHD turbulence and so simplify the description of such turbulence under various similarity parameters. In this case the validation of the LES method requires proving the possibility of reproducing of the scale-similarity Kolmogorov and Iroshnikov-Kraichnan spectra in natural conditions rather than comparing the DNS and LES methods as has been done in our previous works.

For the study of forced compressible MHD turbulence in the inertial range we apply the linear driving force in the form recently introduced in hydrodynamics of neutral incompressible fluid by Lundgren (Lundgren, 2003; Rosales & Meneveau, 2005). In spite of certain advantages of the linear forcing method, it still has not been widely adopted for the studies of turbulence by the DNS and LES methods.

2. Linear forcing in compressible MHD turbulence in physical space

The system of equations of compressible magnetohydrodynamic turbulence in the presence of external force is written in the following form:

\[
\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u_j}{\partial x_j}; \\
\frac{\partial \rho u_j}{\partial t} = -\frac{\partial}{\partial x_j} \left( \rho u_i u_j + p \delta_{ij} - \sigma_{ij} + \frac{B^2}{8\pi} \delta_{ij} - \frac{1}{4\pi} B_jB_i \right) + F^u; \\
\frac{\partial B_i}{\partial t} = -\frac{\partial}{\partial x_j} (u_j B_i - u_i B_j) + \eta \frac{\partial^2 B_i}{\partial x_j^2} + F^b,
\]

here, \( \rho \) is the density; \( B_j \) is the magnetic field in direction \( x_j \); \( p \) is the pressure; \( u_j \) is the velocity in direction \( x_j \); \( \sigma_{ij} = 2\mu S_{ij} - \frac{2}{3} \mu S_{kk} \delta_{ij} + \zeta S_{kk} \delta_{ij} \) is the viscous stress tensor; \( S_{ij} = 1/2 (\partial u_i / \partial x_j + \partial u_j / \partial x_i) \) is the strain rate tensor; \( \mu \) is the dynamic viscosity; \( \zeta \) is the bulk viscosity (hereinafter in this work bulk viscosity \( \zeta \) is neglected); \( \eta \) is the magnetic diffusion; \( \delta_{ij} \) is the Kroneker delta.

\( F^u \) and \( F^b \) are driving forces which sustain turbulence and allow us to study statistically stationary flow and to provide a stationary picture of the energy cascade and more statistical sampling. If energy is not injected into a turbulent flow, this flow becomes laminar because of viscosity and diffusion. To sustain a three-dimensional turbulence, a driving force is employed. Recently, "linear forcing" (Lundgren, 2003; Rosales & Meneveau, 2005) was suggested and modeling of incompressible hydrodynamic turbulence of neutral fluid with driving force in physical space was performed. The idea of linear forcing is to add a force proportional to the fluctuating velocity, that resembles a turbulence which forced with a mean velocity gradient, that is, a shear. This force appears as a term in the equation for the fluctuating velocity that corresponds to the production term in the equation of turbulent kinetic energy. In this work, we generalize this approach to the case of compressible MHD flow of electrically conductive fluid.
The equation for the fluctuating part of the velocity in a compressible MHD turbulent flow are written as

\[
\rho \frac{\partial \bar{u}_i}{\partial t} + U_j \frac{\partial \bar{u}_i}{\partial x_j} = - \frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} - \rho \bar{u}_i \frac{\partial U_i}{\partial x_j} - \rho \bar{u}_i \frac{\partial U_i}{\partial x_j} - \rho \bar{u}_i \frac{\partial U_i}{\partial x_j} - \rho \bar{u}_i \frac{\partial U_i}{\partial x_j} \tag{4}
\]

Here the Reynolds decomposition is used: \( u_i = U_i + \bar{u}_i, \), \( B_i = \bar{B}_i + \dot{B}_i, P = P + \dot{p}, \sigma_{ij} = \Sigma_{ij} + \dot{\sigma}_{ij}, \) where \( U_i, \bar{B}_i, \Sigma_{ij}, P \) represent the mean motion, and \( \bar{u}_i, \dot{B}_i, \dot{\sigma}, \dot{\rho} \) are the fluctuating values.

The third term \( \rho \bar{u}_i \dot{B}_i \dot{B}_i/\partial x_j \) on the right-hand side in the equation (4) corresponds to the term with a driving force which is directly proportional to \( \dot{B}_i \dot{B}_i \), \( \dot{B}_i \dot{B}_i \) are the fluctuating values.

\[
\frac{\partial}{\partial x_j} \left( \frac{1}{2} \rho \bar{u}_i \right) U_j + \frac{\partial}{\partial x_j} \left( \frac{1}{2} \rho \bar{u}_i \right) \bar{u}_j - \langle \beta_{ij} \bar{u}_i \rangle = \langle \bar{u}_i \frac{\partial \sigma_{ij}}{\partial x_j} \rangle - \langle \bar{u}_i \frac{\partial p}{\partial x_i} \rangle - \langle \rho \bar{u}_i \dot{B}_i \rangle \frac{\partial U_i}{\partial x_j} - \langle \beta_{ij} \bar{u}_i \rangle \frac{\partial U_j}{\partial x_i} \tag{5}
\]

where \( \beta_{ij} \) is the turbulent magnetic stress expressed in the form:

\[
\beta_{ij} = \frac{\dot{B}_i \dot{B}_j}{4\pi} - \frac{\dot{B}^2}{8\pi} \delta_{ij} \tag{6}
\]

The physical meaning of the terms in the equation (5) is following: \( \frac{1}{2} \rho \bar{u}_i \) \( U_j \) is the flux of turbulent kinetic energy associated with the mean velocity, \( \frac{1}{2} \rho \bar{u}_i \bar{u}_j \) is the diffusive turbulent flux of turbulent kinetic energy, \( \frac{\partial}{\partial x_i} \frac{\partial \sigma_{ij}}{\partial x_j} \) is the dissipation of turbulent kinetic energy due to viscosity, \( \frac{\partial \sigma_{ij}}{\partial x_j} \) is the pressure-strain rate term. The last terms on the right-hand side and on the left-hand side of the equation (5) describe connection turbulent kinetic energy and magnetic energy, i.e. the influence of work of magnetic forces on the turbulent flow. The term \( \langle \rho \bar{u}_i \dot{B}_i \dot{B}_j \rangle \frac{\partial \sigma_{ij}}{\partial x_j} \) is the production of turbulent energy per unit volume per unit time resulting from the interaction between the Reynolds stress and the mean shear. In the equation (4), it corresponds to the term with a driving force which is directly proportional to \( \dot{v} \). Thus, this term for isotropic homogeneous turbulence is appropriate to force a stationary flow with a driving term proportional to the velocity

\[
F_i^u = \Theta \rho \bar{u}_i \tag{7}
\]

where \( \Theta \) is the coefficient which is determined from the balance of kinetic energy for a statistically stationary state taking into consideration that the gradient of mean velocity is zero:

\[
\Theta = \frac{1}{3\langle \rho \rangle} u_{rms}^2 \left[ \langle u_j \frac{\partial \sigma_{ij}}{\partial x_j} \rangle + \varepsilon \right] = \frac{1}{8\pi} \langle u_j \frac{\partial p}{\partial x_j} \rangle + \frac{B^2}{8\pi} \delta_{ij} \tag{8}
\]

where \( \varepsilon = -\langle u_j \frac{\partial \sigma_{ij}}{\partial x_j} \rangle \) is the mean dissipation rate of turbulent energy into heat. Also it is taken into account in (8) that 

\[
1/\langle \rho u^2 \rangle = 1/(3\langle \rho \rangle u_{rms}^2),
\]

because \( u_{rms}^2 = \langle (\rho u^2) \rangle / (3\langle \rho \rangle) \) is a mass-averaged root-mean-square velocity. Note that in compressible homogenous turbulence
the term \( \langle u_j (\partial p / \partial x_j) \rangle \) = \(-\langle p (\partial u_j / \partial x_j) \rangle \). It is worth noting that the coefficient \( \Theta \) in (7) can be both constant and recalculated during a simulation (Rosales & Meneveau, 2005).

The forcing function \( F_i^u = \Theta \rho u_i \) in the physical space is equivalent to force all the Fourier modes in the spectral space. It is in fact the only difference from the standard spectral forcing when energy is added in to system only in the range of small wave numbers (wavenumber shell), that is, in integrated (large) scale of turbulence.

The determination of the driving force \( F^b_j \) in the magnetic induction equation is similar. The equation for the fluctuating part of the magnetic field in a compressible MHD turbulence is written as:

\[
\frac{\partial \hat{B}_i}{\partial t} + U_j \frac{\partial \hat{B}_i}{\partial x_j} = \hat{B}_j \frac{\partial U_i}{\partial x_j} - \hat{B}_j \frac{\partial U_j}{\partial x_j} - \hat{B}_j \frac{\partial \hat{u}_i}{\partial x_j} + \eta \frac{\partial^2 \hat{B}_i}{\partial x_j^2} + \left[ \hat{B}_j \frac{\partial \hat{u}_i}{\partial x_j} - \langle \hat{B}_j \frac{\partial \hat{u}_i}{\partial x_j} \rangle \right] - \langle \hat{B}_j \frac{\partial \hat{u}_i}{\partial x_j} \rangle 
\]

Here the first term \( \hat{B}_i (\partial U_i / \partial x_j) \) on the right-hand side in (9) corresponds to a production term in the equation of turbulent magnetic energy. The equation for the turbulent magnetic energy obtained by averaging and has the following form:

\[
\frac{\partial}{\partial t} \left( \frac{\hat{B}_i^2}{8\pi} \right) + \frac{\partial}{\partial x_j} \left( \frac{\hat{B}_i^2}{8\pi} U_j \right) + \left( \frac{\hat{B}_i \hat{B}_j}{4\pi} \right) \frac{\partial U_i}{\partial x_j} = \left( \frac{\hat{B}_i \hat{B}_j}{4\pi} \right) \frac{\partial U_i}{\partial x_j} + \eta \frac{\partial^2 \hat{B}_i}{\partial x_j^2} + \frac{\eta}{4\pi} \left( \frac{\hat{B}_i^2}{\partial x_j^2} \right) \]

The terms in (10) describe transport, generation, and dissipation of turbulent magnetic energy in electrically conductive gas. The \( \langle \hat{B}_i^2 / 8\pi \rangle U_j \) is advection of the turbulent magnetic energy, \( \langle \hat{B}_i^2 / 8\pi \rangle U_j \) is turbulent diffusion of the turbulent magnetic energy, \( \langle \hat{B}_i \hat{B}_j \rangle / 4\pi \) represents the interaction between the turbulent magnetic energy and fluctuating components of the mean shear, \( \eta / 4\pi \left( \frac{\hat{B}_i^2}{\partial x_j^2} \right) \) is resistive dissipation of the magnetic turbulent energy due to presence of magnetic diffusion. The first term \( \langle \hat{B}_i \hat{B}_j \rangle / 4\pi \frac{\partial \hat{u}_i}{\partial x_j} \) on the right-hand side of equation (10) is a source term for the production of the magnetic turbulent energy in a consequence of interaction between the magnetic field and the mean fluid shear. Notice that the given term corresponds to the term \(-\langle \rho \hat{u}_i \hat{u}_j \rangle \frac{\partial \hat{u}_i}{\partial x_j} \) in the turbulent kinetic energy equation (5). We can suggest that driving force in the magnetic induction equation is directly proportional to magnetic field. Therefore, we define the driving force \( F^b_j \) as:

\[
F^b_j = \Psi B_i \]

where \( \Psi \) is the coefficient. Similar the above evaluation of \( \Theta \), the coefficient \( \Psi \) is determined from the balance of magnetic energy for the statistically stationary state (when the time derivative is zero)

\[
\Psi = \frac{\chi}{3B_{rms}^2} \]

where \( \chi = \langle \eta B_i (\partial^2 B_j / \partial x_j^2) \rangle \) is resistive dissipation of the turbulent magnetic energy in MHD turbulence and \( B_{rms}^2 = \langle B^2 \rangle / 3 \) is the root-mean-square magnetic field. Similarly to the parameter \( \Theta \) in (7), the coefficient \( \Psi \) in formula (11) can be either constant as recalculated on each time step during modeling of MHD turbulence with driving force.
3. LES of forced compressible MHD turbulence

The large-eddy simulation method was suggested for the study of compressible MHD turbulence in our previous works where its validity for the investigation of decaying MHD turbulence was shown. In LES technique every physical parameter is expanded into large-scale \( f \) and small-scale \( f' \) components, that is \( f = \bar{f} + f' \). Thus, the large-scale effects are computed directly, while the small-scale ones are modeled. Filtering procedure is applied to the governing MHD equations. The filtered (or large-scale) part \( \bar{f}(x_i) \) is defined by the following way: \( \bar{f}(x_i) = \int_{\Omega} f(x_i)G(x_i, \xi; \Delta)dx_i \), where \( G \) is the filter function, \( f \) is a flow parameter, \( \Theta \) is the flow domain, \( \Delta \) is the filter-width and \( x_j = (x, y, z) \) are axes of Cartesian coordinate system. Filter \( G \) satisfies the normalization property: \( \int_{\Omega} G(x_i, \xi; \Delta)dx_j = 1 \) for any \( x \in \Theta \). In order to simplify the equations describing turbulent MHD flow with variable density, we use the Favre filtering (it is also called the mass-weighted filtering) to avoid additional subgrid-scale (SGS) terms associated with inconstant density.

Using the mass-weighted filtration operation, we rewrite the MHD equations for compressible flow in dimensionless form using the standard procedure where \( Re = \rho_0 u_0 L_0 / \mu_0 \) is the Reynolds number, \( Re_m = u_0 L_0 / \eta_0 \) is the magnetic Reynolds number. \( M_s = u_0 / c_s \) is the Mach number, where \( c_s \) is the velocity of sound defined by the relation: \( c_s = \sqrt{\gamma p_0 / \rho_0} \), and \( M_a = u_0 / a_0 \) is the magnetic Mach number, where \( a_0 = B_0 / (\sqrt{4\pi \rho_0}) \) is the Alfvén velocity.

\[
\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \bar{\rho} \bar{u}_i}{\partial x_j} = 0; \tag{13}
\]

\[
\frac{\partial \bar{\rho} \bar{u}_i}{\partial t} + \frac{\partial \bar{\rho} \bar{u}_j}{\partial x_j} \left( \bar{\rho} \bar{u}_i \bar{u}_j + \frac{\bar{\rho}^2}{\gamma M_s^2} \delta_{ij} - \frac{1}{Re} \sigma_{ij} + \frac{\bar{B}^2}{2M_s^2} \delta_{ij} - \frac{1}{M_a^2} \bar{B}_i \bar{B}_j \right) = -\frac{\partial \bar{\tau}_{ij}^a}{\partial x_j} + \bar{F}_i^a; \tag{14}
\]

\[
\frac{\partial \bar{B}_i}{\partial t} + \frac{\partial \bar{u}_j \bar{B}_i - \bar{u}_i \bar{B}_j}{\partial x_j} - \frac{1}{Re_m} \frac{\partial^2 \bar{B}_i}{\partial x_j^2} = -\frac{\partial \bar{\tau}_{ij}^b}{\partial x_j} + \bar{F}_i^b; \tag{15}
\]

The first terms in right-hand sides in the equations (14) - (15) contain turbulent tensors \( \tau_{ij}^a \) and \( \tau_{ij}^b \) that designate influence of subgrid-scale terms on the filtered part: \( \tau_{ij}^a = \bar{\rho} (\bar{u}_i \bar{u}_j - \bar{u}_i \bar{u}_j) - \frac{1}{M^2} \left( \bar{B}_i \bar{B}_j - \bar{B}_i \bar{B}_j \right) \) and \( \tau_{ij}^b = (\bar{u}_i \bar{B}_j - \bar{u}_i \bar{B}_j) - (\bar{B}_i \bar{u}_j - \bar{B}_i \bar{u}_j) \).

We use extended Smagorinsky model for compressible MHD case for subgrid-scale parametrisation. The Smagorinsky model for compressible MHD turbulence showed accurate results in a wide range of similarity numbers:

\[
\tau_{ij}^a - \frac{1}{3} \tau_{kk} \delta_{ij} = -2C_1 \bar{\rho} \bar{\Delta}^2 |\bar{S}| \left( \bar{S}_{ij} - \frac{1}{3} \bar{S}_{kk} \delta_{ij} \right), \tag{16}
\]

\[
\tau_{ij}^b = -2D_1 \bar{\Delta}^2 |\bar{J}| \bar{J}_{ij}, \tag{17}
\]

\[
\tau_{kk}^b = 2Y_1 \bar{\rho} \bar{\Delta}^2 |\bar{S}|^2 \tag{18}
\]

The parameters \( C_1, Y_1 \) and \( D_1 \) in the equations (16) - (18) are model constants.

The external driving forces \( F_i^a \) and \( F_i^b \) are on the right-hand sides of equations (14) - (15) respectively. These forces are defined by means of the theory of linear forcing in (7) and (11), and have the following dimensionless form:

\[
\bar{F}_i^a = \frac{1}{3(\bar{\rho}) \bar{u}_{rms}^2} [\bar{\epsilon} + \langle \bar{u}_j \frac{\partial \bar{\rho}}{\partial x_j} \bar{\rho} \delta_{ij} \rangle + \langle \bar{u}_j \frac{\partial \bar{B}^2}{\partial x_j} \delta_{ij} \rangle \bar{\rho} \bar{u}_i], \tag{19}
\]

\[
\bar{F}_i^b = \frac{1}{3B_{rms}^2} [\langle \frac{1}{Re_m} \bar{B}_i \frac{\partial^2 \bar{B}_i}{\partial x_j} \rangle \bar{B}_i] \tag{20}
\]
where $\tilde{\varepsilon} = -\left(\frac{\partial \hat{u}_i}{\partial x_j} \frac{\partial \hat{p}}{\partial x_j}\right)$.

It is necessary to notice that additional term associated with subgrid-scale tensor can arise in an energy balance when defining a coefficient in the expression for the driving force. However, we apply a dynamic procedure for definition of model constants $C_1$, $Y_1$ and $D_1$ in the Smagorinsky model for MHD case. Therefore, their values are self-consistently computed during run time using the dynamic procedure that has been described in details and was applied for MHD turbulence (Müller & Carati, 2002; Chernyshov et al., 2006a). In the dynamic procedure, model constants are chosen to minimize (applying least-squares method) the dependence of turbulent energy balance when defining a coefficient in the expression for the driving force. However, we stress tensor defined through the following subgrid terms on the right-hand side of equations: the SGS turbulent diffusion $Q_j= -\frac{1}{\gamma M_s^2} \tilde{\rho} \left(\hat{u}_j T - \hat{u}_i \hat{T}\right)$; SGS heat flux $Q_j= \rho \left(\hat{u}_j T - \hat{u}_i \hat{T}\right)$; SGS magnetic energy flux $V_j = \left(\hat{B}_k \hat{B}_l \hat{u}_j - \hat{B}_k \hat{B}_l \hat{u}_j\right)$; SGS energy of the interaction between the magnetic tension and velocity $G_j = \left(\hat{B}_k \hat{B}_l \hat{u}_j - \hat{u}_k \hat{B}_l \hat{B}_j\right)$.

We use Smagorinsky model for MHD case (16) - (18) for parametrization of the SGS stress tensor and the magnetic SGS stress tensor.

The eddy diffusivity model is used for the closure of the subgrid-scale heat flux $Q_j = \rho \left(\hat{u}_j T - \hat{u}_i \hat{T}\right)$:

$$Q_j = -C_s \frac{\tilde{\Delta}^2 \tilde{\rho} \tilde{S}_v}{Pr_T} \frac{\partial \hat{T}}{\partial x_j},$$

where $C_s$ is the model coefficient used above in the Smagorinsky model for MHD case. Constant $C_s$ is computed using the dynamic procedure. $Pr_T$ in expression (23) is the turbulent Prandtl number which also is calculated dynamically.

A model for the subgrid-scale turbulent diffusion $J_j = \rho \left(\hat{u}_j \hat{u}_k \hat{u}_k - \hat{u}_j \hat{u}_k \hat{u}_k\right)$. It is based on an analogy with Reynolds-averaged Navier-Stokes equations and on the assumption that $\hat{u}_i \approx \tilde{u}_i$. Then, the model for $J_j$ is written as:
\[ J_j \simeq \tilde{u}_k \tau_{jk}^u, \quad (24) \]

where SGS tensor \( \tau_{jk}^u \) was found above.

In order to derive the SGS terms arising from the presence of the magnetic field, an approach based on generalized central moments was proposed by Chernyshov et al. (Chernyshov et al., 2008, 2006b). The sum of SGS magnetic energy flux \( V_j \) and the SGS energy of the interaction between the magnetic tension and velocity \( G_j \) is:

\[ \frac{1}{2} V_j - G_j \simeq \bar{B}_k \tau_{jk}^b. \quad (25) \]

The general expression of driving forces for complete system of equations of heat-conducting fluid will be the same as in expressions (7) and (11) used for the momentum conservation and the magnetic induction equations. When external forces are specified with the help of linear forcing, the difference from the polytropic case is that the pressure is defined via an equation of state and depends on temperature in examining a heat-conducting plasma. Meanwhile, the dependence of pressure on the density in the form \( p = \rho^\gamma \) (where \( \gamma \) is a polytropic index) is assumed in a polytropic gas. Thus, the driving force \( F_{iu} \) for heat-conducting charged fluid is:

\[
F_{iu} = \frac{1}{3(\bar{\rho})\tilde{u}_{rms}^3} \left[ -\left( \frac{\bar{u}_j}{Re} \frac{\partial \bar{\sigma}}{\partial x_j} \right) + \frac{1}{\gamma M^2_\sigma} \left( \bar{u}_j \frac{\partial \bar{T}}{\partial x_j} - \bar{\rho} \delta_{ij} \right) + \frac{1}{2M^2_\sigma} \right] \frac{\partial \bar{\rho}}{\partial x_j} \bar{B}^2 \delta_{ij} \bar{\rho} \tilde{u}_i \quad (26)
\]

4. Numerical results and discussion

We perform three-dimensional numerical simulation of forced compressible MHD turbulence and the numerical code of the fourth order accuracy for MHD equations in the conservative form is used. The skew-symmetric form for nonlinear terms is applied to reduce discretization errors when finite difference scheme is employed for modeling of turbulent flow. The third order low-storage Runge-Kutta method is applied for time integration. The explicit LES method is used in this work. To separate large and small eddy components of the turbulent flow, Gaussian filter of the fourth order of accuracy is applied. Periodic boundary conditions for all the three dimensions are applied. The uniform mesh with \( 64^3 \) grid cells is used for obtaining LES results. The simulation domain is a cube \( \pi \times \pi \times \pi \). The initial isotropic turbulent spectrum close to \( k^{-2} \) with random amplitudes and phases in all three directions was chosen for kinetic and magnetic energies in Fourier space. The choice of such spectrum as initial conditions is due to velocity perturbations with an initial power spectrum in Fourier space similar to that

![Figure 1. Time evolution of the kinetic energy for the first case.](image1.png)

![Figure 2. Time evolution of the magnetic energy for the first case.](image2.png)

![Figure 3. Time dynamics of the mean density for the first case.](image3.png)
of developed turbulence. This $k^{-2}$ spectrum corresponds to spectrum of Burgers turbulence. Initial conditions for the velocity and the magnetic field have been obtained in the physical space using inverse Fourier transform.

Initially, we consider polytropic electrically conductive fluid. For the first case similarity numbers are: $Re = 1500$, $Re_M = 800$, $M_s = 0.89$, $\gamma = 1.7$ and $E_k \gg E_M$, that is, the value of kinetic energy is initially much higher of magnetic energy. The temporal evolution of the kinetic energy and the magnetic energy is presented for this case in Fig.1 and Fig.2. It is visible from the numerical results that statistically stationary turbulence arises after an initial time interval when large fluctuations are observed and the values of $E_K$ and $E_M$ practically do not vary with time. It means that the balance between dissipation and energy injected in the system is occurred. It is interesting to note that $E_K$ reaches a stationary regime slightly faster than $E_M$. Time dynamics of mean density is shown in Fig.3. The density fluctuations decrease after an initial time interval when strong fluctuations are observed. In the steady-state stationary regime the density weakly fluctuates around the mean value.

Inertial range properties are defined as time averages over periods of stationary turbulence conditions. Kolmogorov-like spectrum $-5/3$ is observed when MHD turbulence is examined and magnetic energy is much less than kinetic energy, that is, nonlinear interactions are much more considerable than magnetic ones and the fluid is practically the neutral hydrodynamic. In Fig.4, the spectra of the kinetic and the magnetic energies are shown (the solid line is the kinetic energy spectrum and the dashed line is the magnetic energy spectrum). The spectra are normalized by factor $k^{5/3}$. The dot line represents Kolmogorov scaling. The spectra of the kinetic and magnetic energy are obtained after time averaging in the statistically stationary regime and therefore they clearly mark inertial turbulent range with the Kolmogorov-like spectrum $k^{-5/3}$ for both kinetic energy and magnetic energy as follows from Fig.4. The normalized (that is, the product $E_{T}^{k} k^{5/3}$, where $E_{T}^{k}$ is the total energy in Fourier space and $k$ is the wave vector) and the smoothed spectrum of the total energy $E_{T} = E_M + E_K$ is shown in Fig.5. It follows from Fig.5, that in the first case Kolmogorov-like spectrum is occurred. This result supports the theoretical expectations. The residual energy spectrum that is determined as $E_{R}^{K} = |E_{M}^{K} - E_{K}^{K}|$ is shown for the first case. This spectrum is interesting because it gives an insight into the spectral interplay of kinetic and magnetic energies and exhibits self-similar scaling. Fig.6 demonstrates the normalized smoothed spectrum of residual energy and $E_{R}^{K} \sim k^{-7/3}$ in inertial range of turbulence, spectrum which was theoretically obtained and was numerically confirmed for incompressible MHD turbulence.
The second case corresponds to the numerical computations when the similarity numbers are $Re = 200$, $Re_M = 230$, $M_s = 0.81$, $\gamma = 1.7$ and $E_k < E_M$. In Fig.7 and Fig.8, time dynamics of the kinetic energy $E_k$ and the magnetic energy $E_M$ respectively is shown. Note that rapid increase of the kinetic energy takes place while sharp reduction of the magnetic field values is observed. Then, kinetic energy and magnetic energy attain to a stationary regime. Finally, the mean density fluctuations become negligibly small. The normalized and smoothed spectrum of the total energy $E^3_T$ (multiplied by $k^{3/2}$) is shown in Fig.9. It is evident from Fig.9 that clearly defined inertial range of MHD turbulence exists and Iroshnikov-Kraichnan spectrum with power exponent $k^{-3/2}$ is observed in agreement with theoretical studies. In this case, fluctuations in the form of Alfvén waves take place in electrically conducting fluid and magnetic interactions play the noticeable role in turbulent energy cascade that results in Iroshnikov-Kraichnan spectrum.

Also, we consider a heat-conducting case of compressible MHD turbulence because it is necessary to have information about time and spatial dynamics of thermodynamic quantities (that is, temperature, internal energy) in inertial range. The third case of compressible MHD turbulence of heat-conducting plasma is discussed when the similarity numbers are $Re = 120$, $Re_M = 40$, $M_s = 0.64$, $\gamma = 1.5$, $Pr = 1.0$ and $E_k \sim E_M$. In Fig.10, time evolution of the magnetic energy is demonstrated in Fig.11. The magnetic energy reaches a stationary regime of compressible MHD turbulence faster than a stationary kinetic energy regime is established. The internal energy fluctuates with larger amplitudes as the temperature is increased as it can be seen in Fig.12. Decrease of the kinetic energy and the magnetic energy with time is accompanied by simultaneous increasing of the internal energy because dissipation of the kinetic energy transforms into heat.

5. Conclusions

LES method for the study of compressible MHD turbulence in the inertial range by means of an external force is suggested. The expressions of the external force, which allows to obtain a statistically stationary regime of turbulence, are derived. The formulas used for the formulation of large-eddy simulation approach are obtained. The potential possibilities of the LES method to reproduce physics of the flow under investigation in a stationary regime both for polytropic and for heat-conducting charged gases are studied. It is shown that for the case when the kinetic energy of the flow is initially much larger than the magnetic energy, a Kolmogorov-like spectrum $-5/3$ is obtained. When the magnetic energy is initially larger than the kinetic energy, the Iroshnikov-Kraichnan spectrum $k^{-3/2}$ occurs. Thus, we have demonstrated the efficiency of the LES method for studies of scale-invariant properties of compressible MHD turbulence. The
Figure 10. Time dynamics of the kinetic energy for the third case.

Figure 11. Time dynamics of the magnetic energy for the third case.

Figure 12. Time dynamics of the internal energy for the third case.

linear representation of the driving force developed in this work for the LES method may also be of use in the DNS approach.

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