Temperature Dependence of the Chiral Condensate from an Interacting Pion Gas

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Abstract

By exploiting the fact that the chiral condensate is related to the derivative of the free-energy density with respect to the bare quark mass we calculate the temperature dependence of the condensate ratio $\langle \bar{q}q \rangle_T / \langle \bar{q}q \rangle_0$ from an interacting pion gas. When using the Weinberg Lagrangian at the Hartree level we find a depression of the condensate with temperature. For $T < 100$ MeV the results are in good agreement with chiral perturbation theory in the three loop approximation. Near $T_c$, however, there are marked differences due to non-perturbative nature of our approach.

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1 Introduction

The aim of relativistic heavy-ion collisions with cm-energies $E_{cm} \geq 20$ GeV/A is to create a quark-gluon plasma at early stages of the collision. It is predicted that the formation of the plasma is accompanied by a chiral restoration transition which, for two light flavors, is most likely second order \[1\]. Much effort has been devoted to calculate this transition in effective models of QCD, in particular by using pions as effective degrees of freedom. This approach exploits the fact that the chiral $SU_L(2) \times SU_R(2)$ symmetry is spontaneously broken in the non-perturbative vacuum giving rise to nearly massless Goldstone bosons (pions) which dominate the thermodymanics of the hadronic phase for temperatures $T \sim m_\pi$. We adopt the simplified scenario of an interacting pion gas which is assumed to be in thermal as well as chemical equilibrium, $\mu_\pi = 0$, throughout the time evolution of the hadronic phase. This ”black body radiation” will disappear when $T$ has dropped to zero and the pion field has reached its vacuum state.

To obtain an expression for the chiral condensate ratio $\langle \bar{q}q \rangle_T / \langle \bar{q}q \rangle_0$ in terms of pionic degrees of freedom we can use the fact that, according to the Feymann-Hellmann theorem, $\langle \bar{q}q \rangle_T$ is related to the derivative of the free-energy density (the thermodynamic potential) $\hat{\Omega}(T)$ with respect to the bare quark mass $m$ as

$$\langle \bar{q}q \rangle_T = \partial \hat{\Omega}(T) / \partial m$$

(we can ignore the fact that up and down quark masses not the same). Denoting the difference in free energy density as $\Omega(T) = \hat{\Omega}(T) - \hat{\Omega}(0)$ and using the Gell-Mann Oakes Renner relation \[2\]

$$m_\pi^2 f_\pi^2 = -m \partial \hat{\Omega}(0) / \partial m$$

(2)

one immediately derives that

$$\frac{\langle \bar{q}q \rangle_T}{\langle \bar{q}q \rangle_0} = 1 - \frac{1}{f_\pi^2} \frac{\partial \Omega(T)}{\partial m_\pi^2}$$

(3)

This expression is identical to that of ref. \[3\] if one ignores correction to $f_\pi$ and $m_\pi$ of order $m$. Within the current uncertainties their combined effect is only a few % and can
be safely ignored in the present context.

In ref. [3] chiral perturbation theory, including three loops, has been used to evaluate \( \Omega(T) \). As the authors point this is equivalent to a virial expansion in powers of \( T \) up to power \( T^8 \). In the present paper we wish to pursue a different route by calculating \( \Omega(T) \) from a non-pertubative method, well-known for phonon-phonon interactions in condensed matter physics [4, 5]. While less systematic from the point of view of a gradient expansion of the effective QCD Lagrangian it has the advantage of treating the vicinity of the phase transition more properly.

## 2 Free energy of a pion gas

For the evaluation of the free energy \( \Omega(T) \) we shall start from well-known expression for interacting boson systems at finite temperature [4, 5]. In terms of the pion Green’s function [6]

\[
D(\xi, k, T) = (\xi^2 - m^2_\pi - k^2 - \Pi(\xi, k, T))^{-1},
\]

the thermodynamic potential is given by

\[
\Omega(T) = \Omega'(T) - \frac{3}{2} \int \frac{d\xi}{(2\pi)^3} \int \frac{dk}{(2\pi)^3} \text{Im} \left\{ \ln[-D^{-1}(\xi, k, T)] + D(\xi, k, T)\Pi(\xi, k, T) \right\}
\]

where \( \Omega'(T) \) is the sum of all contributions from 'skeleton diagrams' representing the perturbation expansion of \( \Omega \). These are evaluated by using full single-particle Green’s functions rather than bare ones. Furthermore

\[
\frac{\delta \Omega}{\delta \Pi} = 0
\]

and

\[
\frac{\delta \Omega'}{\delta D} = \Pi
\]

(see refs. [4, 5]). Since the physical pion mass enters the propagator \( D \), the self energy \( \Pi(\xi, k, T) \) is the difference

\[
\Pi(\xi, k, T) = \tilde{\Pi}(\xi, k, T) - \tilde{\Pi}(\xi, k, 0),
\]
whereby the infinite contributions to the mass operator $\tilde{\Pi}(\xi, k, 0)$ due to vacuum fluctuations are removed from $\Pi(\xi, k, T)$. Consequently also $\Omega(T)$ is a finite difference $\Omega(T) = \tilde{\Omega}(T) - \tilde{\Omega}(0)$ and no cut-off’s are needed.

The key quantity in eq. (5) is the self energy $\Pi$. To calculate it we use the Weinberg Lagrangian. To lowest order in the coupling constant $1/f_\pi$ it takes the form

$$L = L^0 + L'$$

$$L^0 = \frac{1}{2} \partial_\mu \pi \partial^\mu \pi + \frac{1}{2} \pi^2 m_\pi^2$$

$$L' = \lambda (- \partial_\mu \pi \partial^\mu \pi + \frac{1}{2} \pi^2 m_\pi^2) \pi^2$$

(9)

where $\lambda = f_\pi^{-2}/4 (f_\pi = 93 \text{ MeV})$. Here $L^0$ is free pion Lagrangian and $L'$ denotes the $\pi\pi$–interaction. The second term in $L'$, which violates chiral symmetry for physical pions, is introduced according to ref. [8]. The Weinberg Lagrangian reproduces the current algebra results for the $\pi\pi$ scattering lengthts, and has been successful in describing low-energy $\pi\pi$ scattering [9], which is sufficient in our case. The interaction Lagrangian $L'$ is irreducible and describes a point-like pion-pion interaction. Hence, by definition, it suffers no medium modification.

Given eq. (8), the pion self energy $\tilde{\Pi}$ contains single-pion and three-pion contributions

$$\tilde{\Pi}(\xi, k, T) = \tilde{\Pi}(\xi, k, T)^{1\pi} + \tilde{\Pi}(\xi, k, T)^{3\pi}$$

(10)

represented by the diagrams given in Fig. 1. The lines represent the full propagator $D$. The small dots denote the irreducible $\pi\pi$ vertex while the heavy dot indicates the full $\pi\pi$ vertex [9]. Because of eq. (8), the difference $\Pi(\xi, k, T)$ only includes intermediate virtual states with energies $E \leq T$. Because of resonances, the three-pion contribution should reduce to diagrams involving heavy mesons in the intermediate states (Fig. 2). Since their masses are larger than the temperature range of interest we shall only include the first part of eq. (10) which is equivalent to the Hartree approximation.

In this case, the quantity $\Pi(\xi, k, T)$ takes the form, see ref. [10]

$$\Pi^{1\pi}(\omega, k, T) = -10d\lambda m_\pi^2 + 6d\lambda(\omega^2 - k^2) + 6d\lambda \tilde{m}_\pi^2$$

(11)
and pion propagator is given by

\[ D(\omega, k, T) = \gamma (\omega^2 - k^2 - \tilde{m}_\pi^2)^{-1}, \quad (12) \]

It contains a temperature dependent effective mass

\[ \tilde{m}_\pi^2 = \frac{1 - 10\lambda d(T)}{1 - 12\lambda d(T)} m_\pi \quad (13) \]

and a residue \( \gamma = (1 - 6\lambda d(T))^{-1} \). The quantity \( d(T) \) is determined by

\[ d(T) = \frac{1}{2\pi^2} \int_0^\infty \frac{dk k^2}{\omega_k} \frac{\chi(\omega_k)}{1 - 6\lambda d(T)}; \chi(\omega_k) = \left[ \exp(\omega_k/T) - 1 \right]^{-1} \quad (14) \]

with

\[ \omega_k^2 = k^2 + \tilde{m}_\pi^2; \quad (15) \]

It should be noted that, for any temperature, \( d(T) \) never reaches the value \( 1/12\lambda \). In the Hartree approximation, \( \Omega' \) is given by the diagram displayed in Fig. 3 and takes the form

\[ \Omega' = \frac{3\lambda}{4} \int \frac{dk}{(2\pi)^3} \int \frac{dp}{(2\pi)^3} \int_{-\infty}^\infty \frac{d\xi}{\pi} \chi(\xi) \text{Im} D(\xi, k, T) \int_{-\infty}^\infty \frac{d\eta}{\pi} \chi(\eta) \text{Im} D(\eta, p, T) \left( -10m_\pi^2 + 6(\xi^2 - k^2) + 6(\eta^2 - p^2) \right). \quad (16) \]

The expressions in eq. (5), eqs. (11-15) and eq. (16) determine \( \Omega(T) \) in the Hartree approximation and can be used to find the chiral condensate ratio via eq. (3). This is done in the next section.

### 3 Chiral Condensate

The thermodynamic potential \( \Omega \) depends on the free pion mass \( m_\pi \), both, explicitly and through the self energy \( \Pi \). Due to the stationarity condition (7) the \( m_\pi \)-dependence of \( \Pi \) does not influence the calculation of \( \partial\Omega/\partial m_\pi^2 \) and we obtain

\[ \frac{\partial\Omega}{\partial m_\pi^2} = \frac{3}{2} \int \frac{dk}{(2\pi)^3} \int_{-\infty}^\infty \frac{d\xi}{\pi} \chi(\xi) \text{Im} D(\xi, k, T) \]

\[ -\frac{30\lambda}{4} \int \frac{dk}{(2\pi)^3} \int \frac{dp}{(2\pi)^3} \int_{-\infty}^\infty \frac{d\xi}{\pi} \chi(\xi) \text{Im} D(\xi, k, T) \int_{-\infty}^\infty \frac{d\eta}{\pi} \chi(\eta) \text{Im} D(\eta, p, T) \quad (17) \]
which can be expressed in terms of $d(T)$ as

$$\frac{\partial \Omega}{\partial m^2_{\pi}} = \frac{3}{2}d(T) - \frac{30\lambda}{4}d^2(T).$$

where $d(T)$ is the solution of the integral equation (14). The result is displayed in Fig. 4. As expected, one observes a decrease of $\langle \bar{q}q \rangle_T / \langle \bar{q}q \rangle$ with temperature which, for the physical values of $m_\pi$ MeV and $\lambda$, reaches, however, a constant asymptotic value of $\sim 0.7$ at high temperature. This is easily understood from eq. (18), recalling the behaviour of the quantity $d(T)$ for $T \to \infty$, as mentioned above. Our result is quite different from the chiral perturbation theory [3]. In that work the condensate ratio monotonically drops with temperature. One should recall here that our calculations are carried out in the Hartree approximation summing a selected, but infinite, number of loops. In such a self-consistent scheme the strong pion-pion interaction first enhances $\partial \Omega/\partial m^2_{\pi}$ with $T$ growth, but then it prevents a further increase due to non-linearities.

To demonstrate the non-perturbative effects we have repeated the calculation artificially increasing $f_\pi$ by a factor of two, i.e. reducing the coupling constant by a factor of 4. In this case perturbation theory should be more appropriate. One finds that, for low values of $T$ the reduction, is the same as with the full coupling, but then continues to higher $T$. Perhaps accidentally these results are very close to those of ref. [3].

True chiral restoration only takes place in the chiral limit $m_\pi \to 0$. We have studied this limit as well and the result is also shown in Fig. 4. In this case the temperature-dependent effective mass $\tilde{m}_\pi(T)$ remains zero and the pion propagator becomes

$$D_0 = \gamma(\omega^2 - k^2)^{-1}. \quad (19)$$

with the residue $\gamma$ being of the same form as for $m_\pi \neq 0$. The solution for $d(T)$ in eq. (14) now reduces to the very simple form

$$d_0(T) = \frac{1 - \sqrt{1 - 24\lambda j(T)}}{12\lambda},$$

$$j(T) = \int_0^{\infty} \frac{k^2 dk \chi(\omega^2_k)}{2\pi^2 \omega^2_k}, \quad \omega^2_k = k. \quad (20)$$
One concludes immediately that, for temperatures where \( j(T) > 1/24\lambda \) there no longer exists a real solution for \( d_0(T) \), i.e. the residue \( \gamma \) becomes imaginary. Physically this means that, above the critical temperature, \( T_c \), where \( j(T_c) = 1/24 \) no stable single-pion states exist. This we interprete as the signature of the chiral phase transition. At this point the chiral condensate remains finite, however, due to non-pertubative effects. Reducing the coupling strength to \( \lambda/4 \) (dashed line) the condensate drops to zero again.

4 Summary and Conclusions

From the thermodynamic potential \( \Omega(T) \) of an interacting pion gas we have calculated the temperature dependence of the chiral condensate in the Hartree approximation. As from other effective \( QCD \) models, we find a reduction with increasing temperature. In the chiral limit we identify the critical temperature, \( T_c \), for the chiral phase transition as the point where the residue of the single-pion propagator becomes purely imaginary. To within the numerical accuracy the well-known result, \( T_c = \sqrt{2}f_\pi \), the well-known result from the linear \( \sigma \) model. At this point, the chiral condensate ratio is non-zero, however, with \( \langle \bar{q}q \rangle_T \sim 0.7\langle \bar{q}q \rangle_0 \) as a result of non-perturbative effects. By reducing the coupling constant \( \lambda \) in the chiral Lagrangian, we have shown, that perturbation theory is valid at low temperatures \( T < m_\pi \) but breaks down near \( T_c \). For the physical pion mass \( \langle \bar{q}q \rangle_T/\langle \bar{q}q \rangle_0 \) drops more slowly with \( T \) and eventually reaches a constant value due to explicit chiral symmetry breaking. This value is rather large.

For an improved description of the condensate ratio one should include resonances such a the \( \rho \)-meson in the evaluation of the pion self energy. Furthermore a thorough treatment of the phase transition demands proper account of the order-parameter fluctuations near the critical point, as is well known. Work in this direction is in progress.

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Figure Captions

**Fig. 1:** The pion self energy including 1-π and 3-π intermediate states.

**Fig. 2:** Self-energy diagrams from heavy-meson exchanges.

**Fig. 3:** Contribution to $\Omega'(T)$ in the Hartree approximation.

**Fig. 4:** Temperature dependence of the chiral condensate from an interacting pion gas. The full lines denote the results with $\lambda = f_\pi^2/4$ both for the physical pion mass and in the chiral limit. The dashed lines give the corresponding results with $\lambda/4$. 
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