Quantum modified Regge-Teitelboim cosmology

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(Dated: June 26, 2014)

The canonical quantization of the modified geodetic brane cosmology which is implemented from the Regge-Teitelboim model and the trace of the extrinsic curvature of the brane trajectory, \( K \), is developed. As a second-order derivative model, on the grounds of the Ostrogradski Hamiltonian method and the Dirac’s scheme for constrained systems, we find suitable first- and second-class constraints which allow for a proper quantization. We also find that the first-class constraints obey a sort of truncated Virasoro algebra. The effective quantum potential emerging in our approach is exhaustively studied where it shows that an embryonic epoch is still present. The quantum nucleation is sketched where we observe that it is driven by an effective cosmological constant.

PACS numbers: 04.50.-h, 04.60.Ds, 04.60.Kz, 98.80.Jk

I. INTRODUCTION

The modified geodetic brane gravity (MGBG) [1] is an effective theory consisting of the Regge-Teitelboim model (RT), also named geodetic brane gravity (GBG) [2–6] plus a geometric linear \( K \) term which is, under certain conditions, responsible of mimic some features of the Dvali-Gabadadze-Porrati (DGP) theory [7, 8]. \( K \) denotes the trace of the extrinsic curvature of the codimension one worldvolume swept out by a dynamical brane and, this is a measurement of how the brane elements are oriented in the bulk. Apart from the cosmological constant, the inclusion of this term into the RT model can be regarded as a minimum geometric extension that also leads to second-order equations of motion. In several frameworks such extrinsic curvature term has been studied: in the differential geometry of hypersurfaces [9], in the study of the bending and shape of phospholipid membranes [10] and, in the relativistic context, such a term has been considered to improve the extensible gravitational Dirac model of the electron [11–13] as well as being considered an effective 4D field brane theory with possible applications in cosmology and particle physics [14, 15].

The RT model was originally motivated to describe our Universe in a point- or string-like fashion where our Universe is a (3+1)-dimensional extended object geodesically floating in a fixed higher-dimensional bulk [2]. The associated brane-like cosmology was studied in [16]. Differential geometry aspects discussed in [17] show that to locally embed a metric on a surface, propagating in a \( N \)-dimensional flat background spacetime, the isometric embedding theorems dictate that \( N = n(n + 1)/2 \) dimensions are required. In particular, for \( n = 4 \), a ten-dimensional flat background is necessary. However, if the (3 + 1)-metric on the surface admits some Killing vector fields, \( N \) can be reduced significantly [18]. The above arguments can also be applied when we include such \( K \) term and, in this sense, it is attractive to implement MGBG for cosmology and in particular in quantum cosmology. Geometrically, MGBG is conformed only by the first three Lovelock brane invariants associated to the worldvolume [19]. In fact, the hypersurfaces described by such terms are characterized by a single degree of freedom associated only with the geometric configuration of the system [20]. Relating to this fact there is a linkage with a peculiar set of second-order scalar field theories, free of ghosts, and considered as local modifications of gravity where the scalar degree of freedom \( \pi \), the so-
called *Galileon*, is a type of brane bending mode \[14, 21–29\]. There is thus a strong interest in all these classes of second-order Lagrangians, mainly for their potential applications at the cosmological level.

Within the minisuperspace framework it was shown in \[1\] that the introduction of the linear $K$ term provides an alternative mechanism to contrast the cosmological constant effects into the geodetic brane dynamics thus supplying a dynamical equivalence with the DGP model where the self-(non-self)-accelerated expansion of such brane-like universe is mediated by the sign of the constant $\beta$ accompanying to the $K$ term. Conventionally this quantity is considered as the Gibbons-Hawking-York boundary term but, from the fact that we are not considering the bulk gravity to be dynamical, this second-order term is simply another possible geometrical invariant associated with the worldvolume which also leads to second-order equations of motion. A natural extension of the work developed in \[1\] is the one associated with the quantum approach in order to know some interesting features such as the brane nucleation of this type of universes. In this regards, the quantum theory associated with this brane model involves some technical troubles of considerable complexity where most part of the issues come from the linear dependence on the acceleration of the brane in the Lagrangian. Commonly this fact leads us to identify a divergence term that can be naively neglected without affecting the dynamics of the theory but, getting rid of such a term sometimes results harmful at Hamiltonian level as we cannot obtain constraints quadratic in the momenta in a straightforward way \[1, 2, 4, 5, 30–32, 34\]. To obtain a form which would be appropriate for quantization, a robust prescription consists in maintain the second-order nature of the model and then to use a Hamiltonian development supported by a Dirac's procedure for second-order constrained systems \[35–37\].

This paper provides a companion to \[1\] where the classical aspects of the MGBG within the minisuperspace framework are undertaken. After an Ostrogradski Hamiltonian treatment for constrained systems we find that, to obtain quadratic constraints in the momenta allowing for a canonical quantization, it is necessary to invoke a suitable canonical transformation followed of a gauge fixation. We thus obtain a Wheeler-DeWitt (WDW) type equation where an involved quantum potential emerges. An exhaustive analysis of this potential is done and it is found that a classically disconnected embryonic epoch (a characteristic feature of geodetic brane-like quantum cosmology) is still present. In fact, this embryo exists whenever the conserved energy $\Omega$, which is conjugate to the external time coordinate, is not zero. This quantum treatment paves the way to estimate the probability of creation for this brane-like universe. In this regards we observe that, for negative values of $\beta$ the creation of this type of accelerated universes is more probable, contrary to the case of positive values for $\beta$. Further, the nucleation rate for the particular case of a vanishing energy $\Omega$ is analyzed. It is shown that such probability resembles the one for general relativity by defining an effective cosmological constant in terms of the $\beta$ parameter.

The structure of the paper is as follows. In Sec. \[II\] we present a brief review of the modified geodetic brane gravity in order to set the physical stage. We specialize to a Friedmann-Robertson-Walker (FRW) metric on the brane embedded in a flat background. This minimal embedding calls for only one extra dimension. Then we obtain an effective Lagrangian. In Secs. \[III\] and \[IV\] we have succeeded in showing that by using an Ostrogradski Hamiltonian formulation besides a unique canonical transformation it is possible to obtain quadratic constraints in the physical momenta in order to pave the way to a naive canonical quantization. Further, we find a truncated Virasoro structure in the first-class constraint algebra. We establish a WDW equation in Sec. \[V\] where the emerging quantum potential is analyzed. In addition, the nucleation probability for this brane-like universe is calculated for a special case in Sec. \[VI\]. We finish in Sec. \[VII\] with some conclusions of the work.

**II. MODIFIED GEODETIC BRANE GRAVITY**

Consider a three-dimensional dynamical brane. The $(3 + 1)$-dimensional worldvolume $m$, the brane-like universe, is embedded in a $(4 + 1)$-dimensional Minkowski background spacetime with metric $\eta_{\mu \nu} (\mu, \nu = 0, 1, \ldots, 4)$. We will assume that the dynamical variables are the embedding functions of $m$, $X^\alpha (x^a)$, where $x^a$ are the worldvolume coordinates $(a, b = 0, 1, 2, 3)$. We construct the induced metric $g_{ab} = \eta_{\mu \nu} e^\mu_a e^\nu_b := e_a \cdot e_b$ and the extrinsic curvature $K_{ab} = -\eta_{\mu \nu} n^\mu \partial_a e^\nu_b$ where $e^\mu_a = \partial_a X^\mu$ are the tangent vectors to $m$ and $n^\mu$ is the normal vector defined uniquely (up to a sign) by $e_a \cdot n = 0$ and $n \cdot n = 1$.

Under these geometric conditions, the MGBG theory for a three-dimensional brane is defined as \[1\]

$$ S[X] = \int_m d^4 x \sqrt{-g} \left( \frac{\alpha}{2} R - \Lambda + \beta K \right), \quad (1) $$

where $R$ and $K = g^{ab} K_{ab}$ denote to the Ricci scalar and the mean extrinsic curvature of $m$, respectively. Here, $g := \det(g_{ab})$. In addition, $\alpha$ and $\beta$ are constants of dimensions $[L]^{-2}$ and $[L]^{-1}$ in Planck units, respectively, and $\Lambda$ is a positive cosmological constant defined on $m$. It is possible to consider some matter Lagrangians into the action \[1\]. Once matter is included, the form of the equations of motion is not affected \[1, 5\]. In this work we will only consider the cosmological constant effects, for simplicity. The MGBG possesses as a main symmetry the invariance under reparametrizations of $m$. A variational procedure yields the equation of motion \[1\]

$$ \alpha G_{ab} K^{ab} - \beta R + \Lambda K = 0, \quad (2) $$

where $G_{ab}$ is the worldvolume Einstein tensor. This compact geometrically form represents a single second-order
differential equation in derivatives of $X^\mu$ because of the presence of the extrinsic curvature tensor. This is so even though we have the presence of second-order derivative quantities in the action through the scalars $R$ and $K$. Within a cosmological scenario the integration of the Eq. (2) gives rise to an important integration constant $\Omega$, which is nothing but the conserved bulk energy $\Omega$. For our purposes below, we embed a closed FRW universe in a Minkowski bulk $ds^2 = -d\tau^2 + a^2d\Omega_3^2$ where $d\Omega_3^2 = d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi \sin^2\theta d\phi^2$ is the unit three-sphere. By considering

$$X^\mu(x^a) = (t(\tau), a(\tau), \chi, \theta, \phi),$$

(3)

the induced metric is the FRW one

$$ds^2 = -N^2d\tau^2 + a^2d\Omega_3^2,$$

(4)

where $N^2 = \dot{a}^2 - \ddot{a}^2$ and $a(\tau)$ being the scale factor. An overdot denotes differentiation with respect to $\tau$. Moreover, the unit normalized vector to $m$ is given by

$$n^\mu = \frac{1}{N}(\dot{a}, \dot{t}, 0, 0, 0).$$

(5)

This geometric configuration leads to

$$R = \frac{6\dot{t}}{N^3a^2}(\dot{a}\dot{t} - \dot{a}\dot{t} + N^2\dot{t}),$$

(6)

$$K = \frac{1}{N^3}(\dot{t}\dot{a} - \dot{a}\dot{t}) + \frac{3\dot{t}}{aN}.$$ 

(7)

From Eq. (2) we have the equation of motion

$$\frac{d}{d\tau}\left(\frac{\dot{u}}{t}\right) + \frac{N^2(\dot{t}^2 - 3\Lambda N^2a^2 + 6\beta Na\dot{t})}{at(3t^2 - \Lambda N^2a^2 + 6\beta Na\dot{t})} = 0,$$

(8)

where we have introduced the notation $\bar{\Lambda}^2 := \Lambda/3\alpha$ and $\beta := \beta/3\alpha$. In order to write down the action $S$ in analogy with analytical mechanics, we substitute first (6) and (7) into (11), then after an integration over the spatial coordinates the action reduces to $S = 6\pi^2 \int d\tau L$ where the Lagrangian function reads

$$L = \frac{\dot{a}t}{N^3}(\dot{a}\dot{t} - \dot{a}\dot{t} + N^2\dot{t}) - Na^3\bar{\Lambda}^2 + \frac{3a^2\beta}{N^2}(\dot{t}\dot{a} - \dot{a}\dot{t}) + 3a^2\beta \dot{t}.$$ 

(9)

Notice a linear dependence in the accelerations of the coordinates $a(\tau)$ and $t(\tau)$. In the fashion (2) we infer that the configuration space is spanned by $\{t, a, \dot{t}, \dot{a}\}$. Certainly, $L$ can be split as $L = L_b + L_d$ where

$$L_b = \frac{d}{d\tau}\left[\frac{\dot{a}^2}{N} + a^2\beta \text{arctanh}\left(\frac{\dot{a}}{t}\right)\right],$$

(10)

and

$$L_d = -\frac{\dot{a}^2}{N} + aN(1 - a^2\bar{\Lambda}^2) + 3a^2\beta \left[t - \dot{a}\text{arctanh}\left(\frac{\dot{a}}{t}\right)\right].$$

(11)

$L_b$ denotes a boundary Lagrangian term which produces no dynamics so that we can neglect it without affecting the equations of motion. To Hamiltonian purposes this strategy sometimes is not suitable if we want to obtain quadratic constraints in the momenta unless we introduce auxiliary field variables which extends the canonical analysis. Fortunately, from a second-order derivative viewpoint, a robust prescription lies in to maintain intact the Lagrangian followed by an Ostrogradski Hamiltonian approach as we will see shortly.

### III. OSTROGRADSKI HAMILTONIAN APPROACH

Given the second-order Lagrangian (9), we must note that due to its linear dependence on the acceleration, this is degenerate but stable, as we will discuss below by using the Dirac’s framework needed to deal with constrained systems. First, by following the Ostrogradski construction we identify that the phase space is spanned by $\{t, p_t, a, p_a; \dot{t}, P_t, \dot{a}, P_a\}$ where the conjugate momenta to the velocities $\{t, \dot{a}\}$ are given by

$$P_t = \frac{\partial L}{\partial \dot{t}} = -\frac{a^2\dot{a}}{N^3}(t + Na\beta),$$

(12a)

$$P_a = \frac{\partial L}{\partial \dot{a}} = \frac{a^2\dot{t}}{N^3}(t + Na\beta),$$

(12b)

and

$$p_t = \frac{\partial L}{\partial \dot{t}} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{t}}\right) = \frac{at}{N^3}\dot{a}^2 + N^2(1 - a^2\bar{\Lambda}^2) + 3\beta Na\dot{t} =: -\Omega(13a)$$

$$p_a = \frac{\partial L}{\partial \dot{a}} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{a}}\right) = -\frac{a\dot{a}}{N^3}(a^2 + N^2(1 - a^2\bar{\Lambda}^2) + 3\beta Na\dot{t}),$$

(13b)

being the conjugate momenta to the position variables $\{t, a\}$. It is worthwhile to mention that $p_t$ is not affected by the surface Lagrangian term because it is nothing but the conserved bulk energy $\Omega$. In the Einstein limit whenever $\beta \rightarrow 0$. With regards the momentum $p_a$, it is composed by two contributions, $\tilde{p}_a = p_{a_1} + p_{a_2}$. The momentum $p_{a_2}$ is associated to the equivalent dynamical theory defined by whereas $p_{a_1}$ is related to the boundary Lagrangian term. Explicitly, they are given by

$$p_\alpha = -\frac{a\dot{a}}{N^3}[\dot{a}^2 + N^2(3 - a^2\bar{\Lambda}^2)]$$

$$- 3a^2\beta \left[\dot{t} + \frac{\dot{a}}{t} + \text{arcosh}\left(\frac{\dot{a}}{t}\right)\right],$$

(14)

and

$$p_\alpha = \frac{2a\dot{a}}{N^3} + 3a^2\beta \text{arctanh}\left(\frac{\dot{a}}{t}\right).$$

(15)
In this sense, \( p_t = \mathbf{p}_t \). For our analysis below, it is crucial to maintain \( p_a \) in terms of the two pieces, \([13]\) and \([15]\).

The canonical Hamiltonian which defines the appropriate phase space is provided by the Ostrogradski construction \([37, 38]\):

\[
H_0 = P \cdot \dot{X} + p \cdot \dot{X} - L,
\]

\[
= p \cdot \dot{X} + N \left( a^3 \bar{\Lambda}^2 - \frac{1}{a^3} N^2 \dot{P}^2 + \beta^2 a^3 \right) - \beta \bar{\Lambda}^2 \frac{\dot{t}}{N},
\]

The definition of the momenta \([12a]\) and \([12b]\) gives rise to two primary linear constraints in the momenta

\[
\phi_1 = P t + \frac{a^2 \dot{t}}{N^3} (t + Na \beta) \approx 0, \tag{17}
\]

\[
\phi_2 = P a - \frac{a^2 \dot{t}}{N^3} (t + Na \beta) \approx 0, \tag{18}
\]

which can be collected in the compact form \( \phi_{\mu} = P_{\mu} - \frac{a^2 (t + Na \beta)}{N^2} n_{\mu} \). Here, \( \approx \) stands for weak equality in the Dirac’s scheme for constrained systems. By projecting \( \phi_{\mu} \) along the velocity vector as well as the unit normal vector to \( m \) at a fixed time, we can obtain a more suitable set of primary constraints \([51]\):

\[
\varphi_1 = P t + P a \dot{a} = P \cdot \dot{X} \approx 0, \tag{19a}
\]

\[
\varphi_2 = N (P \cdot n) - \frac{a^2}{N} (t + \beta Na) \approx 0, \tag{19b}
\]

so that the total Hamiltonian is \( H_T = H_0 + u^1 \varphi_1 + u^2 \varphi_2 \) where \( u^{1,2} \) are Lagrange multipliers enforcing \([19a]\) and \([19b]\). Apparently, in \( H_0 \) the linear dependence in the momentum \( p_a \) leads to the so-called Ostrogradski linear instability \([39]\) which force to the manifestation of ghost degrees of freedom but this appearance is however deceptive as we will show later on.

By using the extended Poisson bracket (PB) between two phase space functions, \( f \) and \( g \),

\[
\{f, g\} = \frac{\partial f}{\partial t} \frac{\partial g}{\partial p_t} + \frac{\partial f}{\partial \mathbf{p}_t} \frac{\partial g}{\partial \mathbf{p}_t} + \frac{\partial f}{\partial \mathbf{p}_a} \frac{\partial g}{\partial \mathbf{p}_a} + \frac{\partial f}{\partial \dot{t}} \frac{\partial g}{\partial \dot{t}} + \frac{\partial f}{\partial \dot{a}} \frac{\partial g}{\partial \dot{a}} + \frac{\partial f}{\partial \dot{p}_t} \frac{\partial g}{\partial \dot{p}_t} + \frac{\partial f}{\partial \dot{p}_a} \frac{\partial g}{\partial \dot{p}_a},
\]

as befits a second-order derivative theory, we obtain that secondary constraints are generated by the consistency relations \( \varphi_{1,2} = \{ \varphi_{1,2}, H_T \} \approx 0 \). Thus, we obtain two secondary constraints

\[
\varphi_3 = H_0 \approx 0, \tag{21a}
\]

\[
\varphi_4 = p a \dot{a} + p a \dot{t} = N (p \cdot n) \approx 0. \tag{21b}
\]

There are not tertiary constraints. The relevant physical information is obtained when primary and secondary constraints are separated into first- and second-class constraints \( F \)'s and \( S \)'s, respectively. For our case we have

\[
\mathcal{F}_1 = P \cdot \dot{X} \approx 0, \tag{22a}
\]

\[
\mathcal{F}_2 = \left( \frac{N \Theta}{a^2 \Phi} \right) \varphi_2 + H_0 \approx 0, \tag{22b}
\]

\[
S_1 = \varphi_2 \approx 0, \tag{22c}
\]

\[
S_2 = \varphi_4 \approx 0. \tag{22d}
\]

In fact, the evolution predicted by \( H_T \) and \( H \) is the same \([36]\). To complement our canonical approach we must replace the PB with the Dirac bracket (DB) defined by

\[
\{f, g\}^* := \{ f, g \} - \{ f, S_i \} S^{-1}_{ij} \{ S_j, g \}, \tag{24}
\]

where \( S^{-1}_{ij} \) denotes the inverse elements of the second-class constraint matrix \( S_{ij} := \{ S_i, S_j \}, (i, j = 1, 2) \). Explicitly

\[
(S_{ij}) = \frac{a \Phi}{N} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{25}
\]

In view of the Dirac’s constraint method, we must consider the second-class constraints to vanish strongly which helps to eliminate the part proportional to \( \varphi_2 \) in \([22b]\) leading thus to a simplified expression for \( \mathcal{F}_2 \).

The counting of the physical degrees of freedom (dof) is straightforward \([31,32]\): dof = \([8 - 2 \times 2 - 2]/2 = 1 \). This geometrical dof is the one that account for the brane bending mode in our approach and related to \( a(t) \).

As for the first-class constraint algebra, the DB between the \( \mathcal{F}_1 \) and the reduced \( \mathcal{F}_2 \) reads

\[
\{ \mathcal{F}_i, \mathcal{F}_j \}^* = -\epsilon_{ij} \mathcal{F}_2, \quad i, j = 1, 2, \tag{26}
\]

with \( \epsilon_{ij} \) being the Levi-Civita symbol such that \( \epsilon_{12} = 1 \). Now, this expression suggests to introduce the notation \( L_0 := \mathcal{F}_1 \) and \( L_1 := \mathcal{F}_2 \). The relation \([20]\) transforms then into

\[
\{ L_m, L_n \}^* = (m - n) L_{m+n}, \quad m = 0, n = 1; \tag{27}
\]

which characterizes to a truncated Virasoro algebra \([31,32,34]\). We claim, based on some models recently studied \([11,12,30]\), that this is a symmetry inherited by all Lovelock brane Lagrangians characterized by a linear dependence in the acceleration of the brane. It will be reported elsewhere. In summary, we have two first-class constraints \([22a]\) and \([22b]\) reflecting the invariance under reparametrizations of the worldvolume \( m \) and obeying a truncated Virasoro algebra \([24]\). On the other hand, we have two second-class constraints \([21a]\) and \([21b]\) that signal the fact that the velocities and their conjugate momenta are not physical fields.
IV. CANONICAL TRANSFORMATION AND GAUGE FIXING

In order to get quadratic constraints in the momenta we need to re-express the set of constraints (22a-22d) in a convenient way. To do this, we consider the following canonical transformation (CT) [12,30]

\[
N := \sqrt{P^2 - \dot{a}^2},
\]

(28)

\[
\Pi_N := \frac{1}{N}(P \cdot \dot{X}),
\]

(29)

\[
v := -\left[ N(P \cdot n) - \frac{a^2}{N}(\dot{t} + \beta Na) \right],
\]

(30)

\[
\Pi_v := \text{arctanh}\left(\frac{\dot{a}}{t}\right),
\]

(31)

together with the transformation \(X^\mu = X^\mu\) and \(p_a = P_a - p_a\). This canonical transformation preserves the Poisson bracket structure in the sense that

\[
\{N, \Pi_N\} = 1 = \{v, \Pi_v\} \quad \text{and} \quad \{X^\mu, p_v\} = \delta^\mu_v. \quad (32)
\]

In addition, this CT dictates that the velocity vector can be written as

\[
\dot{X}^\mu = N(\cosh \Pi_v, \sinh \Pi_v, 0, 0, 0),
\]

(33)

while the momenta (12a) and (12b) become

\[
p_t = a[\sinh^2 \Pi_v + (1 - a^2 \bar{A}^2) + 3\beta a \cosh \Pi_v] \cosh \Pi_v = -\Omega,
\]

(34)

\[
p_a = -a[\sinh^2 \Pi_v + (1 - a^2 \bar{A}^2) + 3\beta a \cosh \Pi_v] \sinh \Pi_v = \Omega \tanh \Pi_v,
\]

(35)

or, in a more compact form

\[
p_\mu = \Omega(-1, \tanh \Pi_v, 0, 0, 0). \quad (36)
\]

With regards to the momenta (13a) and (13b) we have

\[
p_a = \left\{ -a \left[ \sinh^2 \Pi_v + (3 - a^2 \bar{A}^2) \right] - 3\beta a^2 \cosh \Pi_v \right\} \times \\
\sinh \Pi_v - 3\beta a^2 \Pi_v,
\]

(37)

\[
p_a = 2a \sinh \Pi_v + 3\beta a^2 \Pi_v.
\]

(38)

\[
\mathcal{F}_1 = N\Pi_N,
\]

\[
\mathcal{F}_2 = N \left[ p_t \cosh \Pi_v + (p_a + p_a) \sinh \Pi_v + a^3 \bar{A}^2 + \frac{1}{a^3} N^2 \Pi_N^2 - a \cosh^2 \Pi_v - 3\beta a^2 \cosh \Pi_v \right],
\]

(40a)

\[
S_1 = v, \quad (41)
\]

\[
S_2 = N(p_a - 2a \sinh \Pi_v - 3\beta a^2 \Pi_v) \cosh \Pi_v. \quad (42)
\]

We impose the so-called cosmic gauge

\[
C_1 = N - 1 \approx 0, \quad (43)
\]

and

\[
C_2 = \cosh \Pi_v - \sqrt{\gamma} a \bar{A} \approx 0, \quad (44)
\]

where \(\gamma = \gamma(a)\). From the expression \(C_2\) and the definition of the momenta \(p_t\), Eq. (13a), we see that \(\gamma\) must obey the rather involved equation

\[
\gamma \left( \gamma - 1 + 3\sqrt{\frac{\beta}{\Lambda}} \right)^2 = \frac{\Omega^2}{a^8 \bar{A}^6}. \quad (45)
\]
Inclusion of the function $\gamma(a)$ will be helpful in order to introduce the conserved energy $\Omega$ within our quantum approach. This gauge condition is totally equivalent to the expression $\sqrt{a^2 + N^2} - \sqrt{N}a\bar{\Lambda} = 0$ where we have used the time component of (33) and the new canonical variable $N$ given by (28). The relations (43) and (44) completely fix the gauge freedom associated to the invariance under reparametrizations. These gauge conditions are good enough since the square matrix $(\gamma_1, \gamma_2)$ results nondegenerate in the constraint surface. Indeed, taking advantage that the symplectic structure as defined in Eq. (19) holds when evaluated with respect to the new canonical variables (28,31) together with $X^\mu$ and $p_\mu$, we have

$$\{C_1, F_1\} = C_1 + 1, \quad \{C_1, F_2\} = 0, \quad \{C_2, F_1\} = 0, \quad \{C_2, F_2\} = G(a, N, v, \Pi_v),(47)$$

where $G$ is a nonvanishing function (51). Hence, the condition det $(\{C_1, F_2\}) \neq 0$ with $i, j = 1, 2$, is fulfilled.

The key point now is to express the physical momenta, $p_t$ and $p_a$, in terms of the gauge fixing conditions. From Eqs. (34), (37) we have

$$p_t = a(\sinh^2 \Pi_v + (1 - a^2\bar{\Lambda}^2) + 3\beta a \cosh \Pi_v) \cosh \Pi_v, \quad (48)$$

$$- (p_a + 3\beta a^2 \Pi_v) = a(\cosh^2 \Pi_v - a^2\bar{\Lambda}^2 + 2 + 3\beta a \cosh \Pi_v) \sinh \Pi_v. \quad (49)$$

Now, by considering the gauge condition (44) we have

$$\cosh \Pi_v = \frac{p_t}{a(\gamma - 1)a^2\bar{\Lambda}^2 + 3\beta \sqrt{\gamma}a^2\bar{\Lambda}}, \quad (50)$$

$$\sinh \Pi_v = \frac{(p_a + 3\beta a^2 \Pi_v)}{a(\gamma - 1)a^2\bar{\Lambda}^2 + 2 + 3\beta \sqrt{\gamma}a^2\bar{\Lambda}}. \quad (51)$$

$$\chi_1 = N\Pi_N, \quad \chi_2 = -\frac{N}{a(\gamma - 1)a^2\bar{\Lambda}^2 + 2 + 3\beta \sqrt{\gamma}a^2\bar{\Lambda}} \left\{ (p_a + 3\beta a^2 \Pi_v)^2 + \frac{p_t^2}{a(\gamma - 1)a^2\bar{\Lambda}^2 + 3\beta \sqrt{\gamma}a^2\bar{\Lambda}} + \frac{1}{a^2}N^2 \Pi_N^2 + 2a(\gamma a^2\bar{\Lambda}^2 - 1) - (\gamma - 1)a^2\bar{\Lambda}^2 - 3\beta a^3 \sqrt{\gamma}a \right\}, \quad (52)$$

where $\chi_2$ is reflected a quadratic dependence in the momenta of the theory. Thus, following the Dirac’s formalism for constrained systems, once we fix the gauge freedom we are left with a pure second-class system ($\chi_1, \chi_2, \chi_3 := S_1, \chi_4 := S_2$). These second-class constraints are regarded as simple identities expressing some dynamical variables in terms of others and all the equations of the theory are formulated in terms of the DB. We have learned then that a canonical transformation resolves the conflict of obtaining an appropriate form for quantization as remarked in the Introduction. As a byproduct, note that we have removed the Ostrogradski linear instability by removing structures associated to higher order terms.

V. MODIFIED GEODETIC BRANE QUANTUM COSMOLOGY

The transition to the quantum mechanical scheme is carried out in the standard way. The structure of the DB is replaced with that of a commutator. Therefore, the
correspondence rule $i\{A,B\} = [\hat{A},\hat{B}]$ for two quantum operators $\hat{A}$ and $\hat{B}$ (modulo factor ordering and, $\hbar = 1$) with $v$ and $\Pi_v$ replaced by the zero operator, yield a satisfactory theory in which only the canonical pairs $(N,\Pi_N)$, $(t,p_t)$ and $(a,p_a)$ are realized as nontrivial quantum operators. Hence, we are now equipped to canonically quantize our model and according to the usual procedure we claim first that in a coordinate representation

$$p_t \to \hat{p}_t = -i \frac{\partial}{\partial t}, \quad (55)$$

$$p_a \to \hat{p}_a = -i \frac{\partial}{\partial a}, \quad (56)$$

$$\Pi_N \to \hat{\Pi}_N = -i \frac{\partial}{\partial N}. \quad (57)$$

With this prescription we can consistently enforce our constraints as operator equations. The Hamiltonian $\hat{H}$, composed now by the second-class constraints $\chi_1$ and $\chi_2$, is the one which is to be quantized. Thus, the physical states, $\Psi$, for our constrained system are those annihilated by the operator equations

$$\hat{\chi}_1 \Psi = 0, \quad (58)$$

$$\hat{\chi}_2 \Psi = 0. \quad (59)$$

Here, for simplicity we will choose a trivial factor ordering which allow us to get rid of the denominator in (55) (see the discussion, for example, in [12]). Thus, by inserting (55-57) into (52) and (53), acting on $\Psi$, we obtain the differential equations

$$\hat{\chi}_1 \Psi = -iN \frac{\partial \Psi}{\partial N} = 0, \quad (60)$$

$$\hat{\chi}_2 \Psi = -\frac{N}{a \left[ (\gamma - 1)a^2 \Lambda^2 + 2 + 3\beta \sqrt{\gamma a^2 \Lambda} \right]} \left\{ -\left( \frac{\partial^2}{\partial a^2} \right) - \frac{1}{a^3} N^2 \left( \frac{\partial^2}{\partial N^2} \right) + 2a(\gamma a^2 \Lambda^2 - 1) - (\gamma - 1)a^3 \Lambda^2 - 3\beta a^3 \sqrt{\gamma a^2 \Lambda} \right\} \Psi = 0. \quad (61)$$

Eq. (60) entails that the physical states $\Psi$ have not a $N$ dependence. Consequently, we are left with equation (61) which results to be the Schrödinger-like equation that we are looking for, as it was expected.

We assume then that $\Psi$ is represented in the usual manner as $\Psi(a,t) := \psi(a)e^{-i\Omega t}$ in agreement with the classical definition of $\Omega$. Substituting $\Psi$ in (61) followed of a lengthy but straightforward computation, we find after removal of the exponential term that $\psi(a)$ satisfies the WDW type equation

$$\left[ -\frac{\partial^2}{\partial a^2} + U(a) \right] \psi(a) = 0, \quad (62)$$

which looks like a zero-energy Schrödinger equation with the quantum potential

$$U(a) = a^2 \left[ (\gamma - 1)a^2 \Lambda^2 + 2 + 3\beta \sqrt{\gamma a^2 \Lambda} \right] - \left( 1 - \gamma a^2 \Lambda^2 \right), \quad (63)$$

where the $\gamma$ function is obtained from (45). The geodetic brane limit is approached when $\beta \to 0$, which was deeply studied in [16]. Also, the Einstein limit is approached as $\Omega \to 0$ and $\beta \to 0$, which is equivalent to $\gamma \to 1$ and $\beta \to 0$.

The parameter $\beta$ which marks the presence of MGBG is still arbitrary at this stage. For $\gamma$ real, this potential is well defined for all values of $a$ and it exhibits a global maximum in the intermediate region. In fact,
this potential has a barrier provided \((\Omega \Lambda)^2 - \left(\frac{2}{3 \sqrt{\Omega}}\right)^2 \leq \left(\frac{2}{5}\right) \left[4\Omega \Lambda \left(\frac{2}{5}\right)^2 + \frac{1}{3} \left(\frac{2}{5}\right) + 2\Omega \Lambda\right]\) where the barrier is stretched between \(a_l < a < a_r\) with \(a_l, a_r\) being the turning points which are the roots of \(\Lambda^2 a^3 - 3\beta a^2 - a + \Omega = 0\). For the interesting case \(\Omega \Lambda \ll 1\) we have that

\[
a_l \approx \Omega, \quad a_r \approx \frac{1}{\Lambda} \left[\left(\frac{3\beta}{2\Lambda}\right) + \sqrt{\left(\frac{3\beta}{2\Lambda}\right)^2 + 1}\right] - \frac{(\Omega/2)}{1 + \left(\frac{3\beta}{2\Lambda}\right) + \sqrt{\left(\frac{3\beta}{2\Lambda}\right)^2 + 1}}. \tag{65}\]

In Figure 11 we have depicted this potential function. This clearly displays that the negative values of the parameter \(\beta\) facilitate the creation of an expanding universe as the hill of the potential barrier and the turning points are smaller in comparison with those obtained by considering the corresponding positive values of \(\beta\). This is in fully agreement with the results obtained at classical level reported in [1], where the self-accelerated expansion of this type of universe is owing to \(\beta < 0\). There, this parameter plays the role of the crossover scale \(r_c\) in the self-accelerated branch of the DGP model. In addition, at short scale factors the \(a \to 0\) limit implies \(\gamma \to \infty\). This gives rise to assume that at early times the function can be approximated as \(\gamma \simeq \Omega^{2/3} / (a^{8/3} \Lambda^2)\) so that the potential becomes

\[
U(a \leq \Omega) \simeq -\Omega^2 - 3\Omega^{4/3} a^{2/3} + 4a^2, \tag{66}\]

which proves the presence of an embryonic epoch. Note that this expression is insensitive to the value of \(\beta\) and it is similar to the GBG case [16]. This is related to the order of approximation that we have used. On the other hand, at long distances the potential becomes

\[
U(a \gg \Omega) \simeq 4a^2 \left\{1 - a^2 \Lambda^2 \gamma_0 - \Omega \Lambda \sqrt{\gamma_0} - \left(\frac{3\beta}{2\Lambda}\right) \frac{\Omega}{a^2 \Lambda \sqrt{\gamma_0} \left(\sqrt{\gamma_0} + \left(\frac{3\beta}{2\Lambda}\right)\right)}\right\}, \tag{67}\]

where we have introduced

\[
\gamma_0 := 1 + 2 \left(\frac{3\beta}{2\Lambda}\right) - \sqrt{\left(\frac{3\beta}{2\Lambda}\right)^2 + 1}. \tag{68}\]

In fact, \(\gamma_0\) is the solution to the Eq. (15) when \(\Omega\) vanishes. Clearly, for \(\Omega \to 0\) and \(\beta \to 0\) the potential (67) approaches to the usual GR quantum potential [16]. It is immediately to note that we can rewrite the potential (67) in terms of an effective cosmological constant as

\[
U(a \gg \Omega) \simeq 4a^2 \left(1 - \Lambda_{\text{eff}} \Omega - \Lambda_{\text{eff}}^2 a^2\right) + U(\Omega, \Lambda, \beta), \tag{69}\]

where

\[
\Lambda_{\text{eff}}(\Lambda, \beta) := \sqrt{\gamma_0} \Lambda, \tag{70}\]

and

\[
U(\Omega, \Lambda, \beta) = \frac{4 \left(\Lambda_{\text{eff}}\right)}{1 + \left(\frac{3\beta}{2\Lambda_{\text{eff}}}\right)}. \tag{71}\]

In particular, we have that \(\Lambda_{\text{eff}}(0, \beta) = 0\) and \(\Lambda_{\text{eff}}(\Lambda, 0) = \Lambda\).

In a like manner, for a vanishing cosmological constant, by a similar development we find a potential given by

\[
U(a) = a^2 \left(\gamma a^2 + 2 + 3\beta \sqrt{\gamma}\right)^2 (1 - \gamma a^2), \tag{72}\]

where now the \(\gamma\) function satisfies the algebraic equation \(\gamma(\gamma + 3\beta \sqrt{\gamma})^2 = \Omega^2 / \Lambda^8\). This potential is depicted in Figure 2 for positive values of \(\beta\) where, in addition, it is compared with those cases where the cosmological constant is non-zero.

It is expected that the potential function (72) may arise from a quantum version of the model for an accelerated universe without cosmological constant reported in [41]. For this brane-like universe we see that the creation from nothing to a region of unbounded expansion is possible and it is privileged whenever we consider a cosmological constant on the brane. Moreover, for small values of the parameter \(\beta\) and \(\Lambda = 0\) the potential barrier grows rapidly making harder the analysis of the tunneling effects. We further observe, as long as \(\Omega \neq 0\), for the range of small values for \(a\) we still have an embryonic epoch because in such regions the Universe can exist classically.

In fact, the embryonic epoch takes place whenever the brane energy \(\Omega \neq 0\) which is the main element of the unified brane gravity [42]. In this regard, Fig. 3 shows that the embryonic region is bigger for small values of \(\beta\) and large values of \(\Omega\).
VI. NUCLEATION RATE

In the quantum cosmology framework the whole universe is described by a wavefunction. The question of the right boundary conditions for the wavefunction is hard to answer because, unlike ordinary quantum mechanics where boundary conditions for the wavefunction are fixed by the physical set-up external to the system, in 4D quantum cosmology there is nothing external giving as a consequence that this question does not have a clear resolution [43]. In our case, the existing embedding spacetime makes the main difference. This is so because the presence of the bulk space gives, without ambiguity, the following interpretation: the Hartle-Hawking and Linde boundary conditions include parts that correspond to expanding and contracting universe whereas the tunneling boundary condition only includes an expanding component for the Universe (see the discussion, for example, in [44]).

We opt to think that this brane-like universe was a small nearly spherical brane nucleating in a Minkowski background spacetime and we choose the tunneling boundary condition as the right boundary condition because it corresponds to the idea that the tunneling mechanism was the process involved in the nucleation of this universe. For our case, by a WKB approximation it is possible to calculate the nucleation probability considering the tunneling boundary condition driven by the involved potential [43] as follows [45–47]

\[
P \sim \exp \left( -2 \int_{a_l}^{a_r} |\sqrt{U(a)}| da \right), \tag{73}
\]

where, \(a_l\) and \(a_r\) are the turning points of the potential. Clearly, this expression is hard to work out.

A special case focused on the very early Universe is contained in the case \(\Omega = 0\). From Eq. (45) the \(\gamma\) function reduces to \(\gamma_0\). Under this condition the effective quantum potential [43] takes the form

\[
U(a) = 4a^2 \left[ 1 - a^2 \left( \sqrt{\gamma_0 \bar{\Lambda}} \right)^2 \right]. \tag{74}
\]

The turning points becomes

\[
a_l = 0, \tag{75}
\]

\[
a_r \approx \frac{1}{\bar{\Lambda}} \left[ \frac{3 \beta^2}{2 \bar{\Lambda}} + \sqrt{\frac{3 \beta^2}{2 \bar{\Lambda}}}^2 + 1 \right]. \tag{76}
\]

Thus, from (73) we may estimate the tunneling probability

\[
P \sim e^{-\frac{4}{\Lambda_\text{eff} \Lambda}}. \tag{77}
\]

Notice that both the potential (74) and the nucleation probability (77) resemble the standard GR case with an effective cosmological constant defined in (70),

\[
\Lambda_{\text{eff}} = \sqrt{\gamma_0 \Lambda} = \sqrt{\bar{\Lambda}^2 + \left( \frac{3 \beta^2}{2} \right)^2 - \frac{3 \beta^2}{2}}. \tag{78}
\]

From expression (77), we also infer that it is more probable to create universes of this type with a value of \(\bar{\Lambda}\) greater than the usual situation of GR but with the main difference that this effect can be increased by considering negative values of the parameter \(\beta\). The opposite situation occurs whenever \(\beta > 0\).

Associated with this case, when \(\Lambda = 0\) and \(\Omega = 0\), and by using the potential (74) we have that

\[
U(a) = 4a^2 \left( 1 - 9 \beta^2 a^2 \right). \tag{79}
\]
From this expression we may also identify another effective cosmological constant given by $\Lambda_{\text{eff}} := -3\beta$. These results together suggest that in general, at quantum level, the parameter $\beta$ still continues to modify the cosmological constant, as it was elucidated at a classical level in [1].

VII. CONCLUDING REMARKS

In this paper we have canonically quantized the modified Regge-Teitelboim brane model within a minisuperspace framework. The associated brane-like universe also constitutes a controlled deviation from Einstein limit provided the bulk energy $\Omega$ and the $\beta$ parameter vanish. By means of an Ostrogradski Hamiltonian procedure besides the introduction of a suitable canonical transformation followed of a gauge fixing procedure, we have succeeded in finding constraints quadratic in the momenta. The canonical quantization scheme is possible once the Dirac brackets enter the game. The resulting WDW type equation allows to identify a quantum potential. We calculated then the nucleation probability using the WKB approximation for the simple case $\Omega = 0$. For this case, the higher probability for the nucleation of the brane universe is obtained by considering the highest value of the resulting effective cosmological constant that is constructed with the cosmological constant $\Lambda$ and the $\beta$ parameter. For the case of positive $\beta$ there is a relation between $\beta$ and $\Lambda$ in order to get the maximum probability. For $\beta < 0$ the maximum probability is achieved for the largest absolute values of $\beta$ and $\Lambda$. For a non-zero brane cosmological constant, the parameter $\beta$ which plays akin role to the crossover scale $r_c$ in the self-accelerated branch of the DGP model, is similar to a cosmological constant at a quantum level. In some manner, this is consistent with the results reported in [1], where the self-accelerated expansion on this type of universes is due to a negative value of the parameter $\beta$. Based on the previous results, it is worth to mention that exist the possibility that the nucleation probability for the DGP model follows a similar pattern.

Another way to extend our work resides in the direction of trying to extract some physical observable consequences from the nucleation rate. For example, we can use an inflaton field on the brane, within a particular inflationary model, with the purpose to calculate some of most probable observable cosmological parameters and to investigate its corresponding physical implications. This will be the subject of future work.

We believe also that this model will serve as precursor to obtain more enriched geometrical theories. In this sense, Lovelock brane models [13] are guesses at alternative physical theories that might underlie the cosmic acceleration that deserve a more detailed exploration. We will report elsewhere the cosmological implications of consider, for example, a cubic correction term in the extrinsic curvature to the RT model through the so-called Gibbons-Hawking-York-Myers term, $K_{GB}$ [48 $\Omega$].

Acknowledgments

The authors are grateful to Alexander Vilenkin for helpful discussions. AM acknowledges support from PROMEP UASLP-PTC-402. ER acknowledges partial support from grant PROMEP, CA-UV: Algebra, Geometría y Gravitación. MC was supported by PUCV through Proyecto DI Postdoctorado 2014. Also, ER and MC acknowledge partial support from the grant CONACyT CB-2012-01-177519. This work was partially supported by SNI (México). RC also acknowledges support from EDI, COFAA-IPN, SIP-20131541 and SIP-20144150.

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This fact is supported by using an existing completeness relation in the geometry of deformations for branes, namely
\[ \eta^{\mu \nu} = n^\mu n^\nu - \eta^A \eta^B + h^{AB} \epsilon^\mu_A \epsilon^\nu_B. \]
See Ref. [30].

This function is given by the relation
\[ G = \left[ \frac{1}{a^2} \left( v - a^2 \cosh \varPi - \overline{\beta} a^3 \right) - \overline{\Lambda} \left( \sqrt{\gamma} + \frac{1}{a} \frac{\partial}{\partial a} \right) \right] \sinh \varPi. \]