Some measure-preserving point transformations on the Wiener space and their ergodicity

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Abstract: Suppose that $T$ is a map of the Wiener space into itself, of the following type: $T = I + u$ where $u$ takes its values in the Cameron-Martin space $H$. Assume also that $u$ is a finite sum of $H$-valued multiple Ito-Wiener integrals. In this work we prove that if $T$ preserves the Wiener measure, then necessarily $u$ is in the first Wiener chaos and the transformation corresponding to it is a rotation in the sense of [9]. Afterwards the ergodicity and mixing of rotations which are second quantizations of the unitary operators on the Cameron-Martin space, are characterized. Finally, the ergodicity of the transformation $dY_t = \gamma(t)dW_t, \quad 0 \leq t \leq 1$ where $W$ is $n$-dimensional Wiener and $\gamma$ is non random is characterized.

1 Introduction

Let $\mu$ be the standard Gaussian measure on $\mathbb{R}^n$, i.e.

$$
\mu \left\{ x : x_i \leq a_i, i = 1, 2, \ldots, n \right\} = \prod_{i=1}^{n} \Phi(a_i)
$$

where

$$
\Phi(a) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{a} e^{-\frac{\eta^2}{2}} d\eta.
$$

Then

(a) The linear point-transformations $T$ on $\mathbb{R}^n$ which leave this measure invariant induce unitary transformations on $L^2(\mu, \mathbb{R}^n)$, which are defined as $O f(x) = f \circ T(x)$.

(b) There are many non-linear transformations on $\mathbb{R}^n$ which leave the measure $\mu$ invariant, too many to characterize without any further restriction.
(c) The transformation $T$ is not ergodic: in fact let $f$ be defined as $f(x) = |x|_{\mathbb{R}^n}$, then

$$f = f \circ T$$

and evidently, $f$ is a non-constant function.

The infinite dimensional extension of this problem leads directly to the formulation of the problem for Wiener processes. Indeed, let $(w_t, t \in [0, 1])$ denote the standard Wiener process and let $(e_i, i \in \mathbb{N})$ be a complete orthonormal basis in the Cameron-Martin space $H$. Denote by $(e_i', i \in \mathbb{N})$ the image of this basis in $L_2([0, 1])$. Define

$$\delta e_i = \int_0^1 e_i'(s) dw(s), \quad (2)$$

then $(\delta e_i, i \in \mathbb{N})$ are i.i.d. $N(0, 1)$-random variables and

$$w_t = \sum_{i=1}^{\infty} \delta e_i e_i(t) = \sum_{i=1}^{\infty} \delta e_i \int_0^t e_i'(s) ds \quad (3)$$

in the sense that

$$\sup_{t \in [0,1]} \left| w_t - \sum_{i=1}^{N} \delta e_i \int_0^t e_i'(s) ds \right| \xrightarrow{a.s.} 0 \quad N \to \infty.$$  

It follows that for any transformation, linear or non-linear, invertible or non-invertible, from \{\delta e_i, i = 1, 2, \cdots\} to another sequence, say $(\eta_i, i \in \mathbb{N})$, of i.i.d. $N(0, 1)$-random variables, defined by

$$\sum_{i=1}^{\infty} \delta e_i e_i \to \sum_{i=1}^{\infty} \eta_i e_i$$

will be a measure invariant transformation of the Wiener space.

A class of transformations which plays an important role in many applications is the shift transformation

$$(Tw)_t = w_t + \int_0^t u_s(w) ds, 0 \leq t \leq 1, \quad (4)$$

where

$$\int_0^1 |u_s|^2 ds < \infty \quad a.s. \quad (5)$$
It is natural to ask for a characterization of the shifts \( u \) for which \( T \) is measure-invariant, i.e. \( Tw \) is also a Wiener process on \( C_0([0,1]) \). In the next section we consider the transformations induced by a finite sum of multiple Wiener-Ito integrals taking values in the Cameron-Martin space and characterize those shifts which induce an invariant measure. We prove in particular their non-ergodicity. In section 3 we study the measure-preserving transformations which are defined via the second quantization of deterministic unitary operators on the Cameron-Martin space which cover also the special kind of shifts presented in the second section. In particular a necessary and sufficient condition for their ergodicity and mixing is proved. Section 4 deals with the special case where

\[
dY_t = \gamma(t) dW_t, \quad 0 \leq t \leq 1, \quad Y_0 = 0,
\]

where \( W \) is a standard \( n \)-dimensional Wiener process and \( \gamma(t) \) is not random and takes values in the group of unitary matrices. The ergodicity of this transformation is characterized.

The characterization of ergodicity and mixing for real valued Gaussian processes is due to Maruyama (cf. [6]). The results presented in Theorems 2 and Theorem 3 are infinite dimensional extensions of the results of Maruyama and can be derived by starting from Maruyama’s results (by-passing Lemma 3). We preferred, however, the proof presented here as it is more direct and shorter. It is based on the following characterizations (cf. e.g. section 1.7 of [1]). Let \( T \) be an automorphism (invertible, \( T \) and \( T^{-1} \) are measurable and measure preserving) then:

(A) \( T \) is ergodic, if and only if the only eigenfunctions of the induced unitary transformation \( O \) associated with \( \lambda = 1 \) are the constants.

(B) \( T \) is weak mixing if and only if \( O \) has no eigenfunctions other than constants.

(C) Let \( L^2_0(\mu) \) denote the class of real valued square integrable, zero mean Wiener functionals. Set \( a_n(f) = E[(O^n f) \cdot f] \). Then \( T \) is mixing if and only if \( a_n(f) \to 0 \) as \( n \to \infty \) for all \( f \) in \( L^2_0(\mu) \).

Remarks

(a) The results presented here are valid for arbitrary abstract Wiener spaces although the study here is in the setup of the classical Wiener space.

(b) The ergodicity problem considered in this paper deals with invertible transformations. The invertibility however, is not necessary for ergodicity.
Indeed, let \( w \cdot e_i \) be as in equation (3), set
\[
(Tw)_t = \sum_{i=1}^{\infty} \delta e_{i+1} e_i(t)
\]
then it is easily verified that \( T \) is measure preserving and strong mixing.

(c) After this paper was written we learned of the paper [11] by Wiener and Akutowicz which characterizes the mixing properties of transformation discussed in section 3.

2 Shifts induced by multiple Wiener-Ito integrals

In the sequel we denote by \((C_0([0,1]), H, \mu)\) the classical Wiener space, where \( H \) denotes the Cameron-Martin space which consists of absolutely continuous functions on \([0,1]\) with square integrable derivatives and \( \mu \) is the Wiener measure. Recall that one can define a Sobolev derivative on this space respecting the \( \mu \)-equivalence classes (cf. e.g. [3]), whose adjoint, denoted by \( \delta \), called divergence operator, which coincides with the Ito integral of the Lebesgue density of \( H \)-valued functional if the latter is adapted to the filtration of the Wiener process. Let \( \{w_t, t \in [0,1]\} \) be the standard Wiener process on \( C_0([0,1]) \). Assume that \( k_{n+1} \in L^2([0,1]^{n+1}) \) is a kernel which is symmetric in its first \( n \) variables. Let \( I_n(k_{n+1}(s_1, \ldots, s_n, t)) \) or just \( I_n(k_{n+1}(. , t)) \) denote the \( n \)-th order multiple Wiener-Ito integral with respect to \( s_1, \ldots, s_n \) of \( k_{n+1} \). For \( t \in [0,1] \), define

\[
y_t = (Tw)_t = w_t + \sum_{i=1}^{N} \int_0^t I_n(k_{n+1}(\cdot, \eta)) d\eta
\]

for some finite \( N \). Let \( \mu \) be the standard Wiener measure and denote by \( T^* \mu \) the measures induced on \( C_0([0,1]) \) by \( w \to Tw \).

Theorem 1 Let \( Tw \) be as defined by (6), then \( T^* \mu = \mu \) and only if

(a) \( N = 1 \)

(b) and \( (I + K) \) is a unitary operator on \( L^2([0,1]) \), i.e.

\[
(I + K)(I + K)^* = (I + K)^*(I + K) = I,
\]

where \( K \) is defined on \( L^2([0,1]) \) by

\[
Kf(t) = \int_0^1 k_2(t, \tau)f(\tau)d\tau.
\]
Remarks: Condition (b) can be restated as:

(b1) $-1$ is not an eigenvalue of $K$.

(b2)

\[
k_2(s, t) + k_2(t, s) + \int_0^1 k_2(\theta, s)k_2(\theta, t)d\theta = 0
\]

or equivalently

\[
k_2(s, t) + k_2(t, s) + \int_0^1 k_2(s, \theta)k_2(t, \theta)d\theta = 0.
\]

for any $(s, t) \in [0, 1]^2$, $ds \times dt$ almost surely.

Proof: To show necessity, let $h(t)$ be in $L^2([0, 1])$. If $T^*\mu = \mu$ then

\[
\int_0^1 h(s)dy_s = \int_0^1 h(s)dw_s + \sum_{n=1}^N \int_0^1 h(\eta)I_n\left(k_{n+1}(\cdot, \eta)\right)d\eta
\]

is a zero mean Gaussian random variable. By a standard convergence argument, the order of integration can be interchanged and it holds that

\[
\int_0^1 h(s)dy_s = \int_0^1 h(s)dw_s + \sum_{n=1}^N \left(\int_0^1 k_{n+1}(\cdot, \eta)h(\eta)d\eta\right). \quad (7)
\]

The term on the left hand side is Gaussian and for $n \geq 2$, $I_n(\cdot)$ is non-Gaussian. Moreover, a result of McKean (cf. section 8 of [7]) states that if $f_k(s_1, \cdots s_k)$ are non-zero elements of $L^2([0, 1]^k)$ then for some positive $\alpha$ and $\beta$ and for $x$ large enough

\[
\exp -\alpha x^{2/N} \leq \text{Prob}\left(\left|\sum_{k=1}^N I_k(f_k)\right| > x\right) \leq \exp -\beta x^{2/N}.
\]

Since there can be no cancellation between the terms in (7), we must have $N = 1$, and (7) becomes

\[
\int_0^1 h(s)dy_s = \int_0^1 h(s)dw_s + \int_0^1 \left(\int_0^1 k_2(s, \theta)h(\theta)d\theta\right)dw_s. \quad (8)
\]

The operator $K$ corresponding to the kernel $k_2$ is Hilbert-Schmidt on $L^2[0, 1]$, hence it has a discrete spectrum. If $\lambda = -1$ is an eigenvalue of $K$ and $h$ is a corresponding eigenfunction then, almost surely, $\int_0^1 h(s)dy_s = 0$ which
contradicts the assumption that $w \to y(w)$ is Wiener, this yields condition (b1). Furthermore, if $w \to y(w)$ is Wiener then

$$E\left[\int_0^1 g_1(s)dy_s \int_0^1 g_2(s)dy_s\right] = \int_0^1 g_1(s)g_2(s)ds.$$ 

Hence, for any $h, \alpha \in H$, by (8)

$$E[(\delta h \circ T) (\delta \alpha \circ T)] = (h, \alpha)_H + \left(K^*h, K^*\alpha\right)_H + \left(K^*hh, \alpha\right)_H + \left(K^*h, \alpha\right)_H$$

hence

$$\left(KK^*h + K^*h + Kh, \alpha\right)_H = 0$$

therefore (b) and (b2) follow.

\[\square\]

**Corollary 1** Under the hypothesis of Theorem 1, the mapping $T$ is almost surely invertible and we have

$$|\det_2(I_H + K)| \exp\left\{-I_2(k_2) - 1/2|\delta K|_H^2\right\} = 1$$

and

$$|\det_2(I_H + K)| = 1,$$

where $\det_2(I_H + K)$ denotes the modified Carleman-Fredholm determinant (cf. [3]).

**Proof:** The hypothesis implies that $T$ is invertible. Indeed, $w \to T^{-1}(w)$ is given by

$$T^{-1}(w) = w - \int_0^1 I_1(\beta(t, \cdot))dt,$$

where $\beta(s, t)$ is the symmetric kernel associated to the Hilbert-Schmidt operator $(I_H + K)^{-1}K$ (cf. [2]). By the change of variables formula, for any continuous and bounded function $f$ on the Wiener space, we have ([6], [10]) $T^*\mu \sim \mu$ and

$$E[f \circ T | \Lambda] = E[f],$$

\[1\] This means the existence of a measurable map $S : W \to W$ such that $\mu(T \circ S = S \circ T = I_W) = 1$, cf. [11].

6
where

\[
\Lambda = \det_2(I_H + K) \exp \left\{-I_2(k_2) - \frac{1}{2} \int_0^1 \left( \int_0^1 k_2(s,t)dw_s \right)^2 dt \right\}.
\]

Since \(E[|\Lambda|] = 1\), in order to show that \(|\Lambda| = 1\) it suffices to show that

\[-I_2(k_2) - \frac{1}{2} \int_0^1 \left( \int_0^1 k_2(s,t)dw_s \right)^2 dt\]

is independent of \(w\). Now, by Ito’s rule

\[
I_2(k_2) + \frac{1}{2} \int_0^1 \left( \int_0^1 k_2(s,t)dw_s \right)^2 dt
= I_2(k_2) + I_2 \left( \int_0^1 k_2(s,t)k_2(\theta,t)dt \right) + \frac{1}{2} \int_0^1 \int_0^1 k_2^2(s,t)ds dt
\]

and the \(I_2\) terms must vanish since from (b2), we have

\[
k_2(s,\theta) + k_2(\theta,s) + \int_0^1 k_2(s,t)k_2(\theta,t)dt = 0
\]

and the proof follows.

\[\square\]

**Corollary 2** The class of transformations \(T\) satisfying the conditions of the Theorem 1 form a subgroup of the group of transformations

\[
Tw = w + \int_0^t a_s(w)ds,
\]

with \(\int_0^1 |a_s(w)|^2 ds < \infty\) a.s. for which \(T^*\mu = \mu\).

**Proof:** Setting

\[
T_1w(t) = w_t + \int_0^t \int_0^1 k(s,\theta)dw_s d\theta
\]

\[
T_2w(t) = w_t + \int_0^t \int_0^1 q(s,\theta)dw_s d\theta
\]

7
and assuming that $k$ and $q$ satisfy the conditions of the theorem then 
\((T_2T_1)^*\mu = \mu\). Now,

\[
T_2(T_1w)(t) = w_t + \int_0^t \int_0^1 k(s, \theta)dw_s d\theta \\
+ \int_0^t \left[ \int_0^1 q(s, \theta)dw_s + \int_0^1 k(\rho, \theta)dw_\rho \cdot ds \right] d\theta
\]

\[
= w_t + \int_0^t \int_0^1 k(s, \theta)dw_s d\theta + \int_0^t \int_0^1 q(s, \theta)dw_s d\theta \\
+ \int_0^t \left( \int_0^1 q(\theta, \eta)k(s, \eta) \right) dw_s d\theta
\]

and the result follows since $q$ and $k$ are Hilbert-Schmidt kernels, so is \(\int_0^1 q(\cdot, \eta)k(\cdot, \eta) d\eta\).

\[\square\]

Such a transformation is never ergodic as it is proven in the following

**Proposition 1** Any transformation of the Wiener space satisfying the conditions of Theorem 3 is non-ergodic.

**Proof:** Assume that $\lambda$ is an eigenvalue of $K$, with the corresponding eigenfunction $h$. Then $I_1(h)$ is an eigenfunction of $O$ with the eigenvalue $1 + \lambda$. Since $O$ is an isometry, we should have necessarily $|1 + \lambda| = 1$, moreover

\[
O|I_1(h)| = |I_1(h) \circ T|
\]

\[
= |1 + \lambda||I_1(h)|
\]

Consequently, $|I_1(h)|$ is a non-trivial invariant function, hence $f \to f \circ T$ can not be ergodic.

\[\square\]

### 3 Ergodicity of transformations induced by rotations

Let $w$ denote, as before, the standard Wiener path and let $R$ be a non-random, unitary transformation of the Cameron-Martin space $H$. Let $(e_i, i \in \mathbb{N})$
be a complete, orthonormal basis of $H$ whose image in $L^2([0,2\pi])$ will be denoted by $(\epsilon_i^\prime)$. Set

$$Tw = \sum_{i=1}^{\infty} \delta(Re_i) \epsilon_i. \quad (9)$$

Since $\delta(Re_i)$ are i.i.d. and $N(0,1)$, $Tw$ is also a Wiener path (cf. [3] or [4] for more general cases).

**Lemma 1** The definition of $Tw$ is independent of the choice of the basis $(\epsilon_i, i \in \mathbb{N})$. Hence we have also

$$Tw = \sum_{i=1}^{\infty} \delta\epsilon_i \cdot (R^{-1}\epsilon_i). \quad (10)$$

Moreover, for any $h \in H$, one has

$$\exp\{\delta h - 1/2|h|^2_H\} \circ T = \exp\{\delta(Rh) - 1/2|h|^2_H\}.$$  

In particular, if $F \in L^2(\mu)$ has the Wiener chaos representation as

$$F = E[F] + \sum_{i=1}^{\infty} I_n(f_n),$$

then

$$F \circ T = E[F] + \sum_{i=1}^{\infty} I_n(R^{\otimes n}(f_n)), \quad (11)$$

where $R^{\otimes n}$ denotes $n$-th tensor power of the operator $R$.

**Proof:** Let $\alpha$ be an element of the continuous dual $C([0,1])^*$ of $C([0,1])$, i.e., a bounded Borel measure on $[0,1]$. Let $\tilde{\alpha}$ denote its image under the injection $C([0,1])^* \hookrightarrow H$. Then it is easy to see that $\tilde{\alpha}(t) = \int_0^t \alpha([s,1])ds$. We have

$$<Tw, \alpha> = \sum_{i=1}^{\infty} \delta(Re_i) \int_0^1 \epsilon_i(s)d\alpha(s)$$

$$= \sum_{i=1}^{\infty} \delta(Re_i)(\tilde{\alpha}, \epsilon_i)_H$$

$$= \sum_{i=1}^{\infty} \delta(Re_i)(R\tilde{\alpha}, Re_i)_H$$

$$= \delta(R\tilde{\alpha})$$
since \((R_{ei}, i \in \mathbb{N})\) is a complete, orthonormal basis of \(H\). By the density of \(C([0, 1])^*\) in \(H\), we obtain that \(\delta h \circ T = \delta(Rh)\) for any \(h \in H\), the second claim is now obvious and the identity \((11)\) follows from it.

Remarks: (i) since \(R\) is unitary, it possesses the spectral representation
\[
R = \int_0^{2\pi} e^{i\theta} d\theta \Pi_\theta
\]
where \((\Pi_\theta, \theta \in [0, 2\pi])\) is a resolution of the identity. It follows by standard arguments that
\[
\delta h \circ T^n(w) = \int_0^{2\pi} e^{in\theta} d\theta \delta(\Pi_\theta h),
\]
where the integral at the right hand side is to be interpreted as a stochastic integral with respect to the martingale \(\theta \to \delta \Pi_\theta h\) (cf. \([10]\)). Hence
\[
E[\delta h_1 \circ T^n \delta h_2] = \int_0^{2\pi} e^{in\theta} d\theta (\Pi_\theta h_1, h_2).
\]

(ii) The transformation studied in the previous section (cf. Theorem \([3]\)) is a particular case of this one defined by \((9)\) since \(I_H + K\) is a unitary operator on \(H\).

Before proceeding further let us prove a technical result which will be useful in the sequel:

Lemma 2 Let \((\Pi_\theta, \theta \in [0, 2\pi])\) be a resolution of identity on \(H\). Assume that \(\theta \to (\Pi_\theta h, k)_H\) is continuous for any \(h, k \in H\). Then for any \(f, g \in H^{\otimes n}\) (i.e. the symmetric tensor product of order \(n\)),
\[
(\theta_1, \ldots, \theta_n) \to (\Pi_{\theta_1} \otimes \ldots \otimes \Pi_{\theta_n}) f, g)_{H^{\otimes n}}
\]
is continuous on \([0, 2\pi]^n\). Moreover, for any \(f \in H^{\otimes n}\),
\[
(\theta_1, \ldots, \theta_n) \to d((\Pi_{\theta_1} \otimes \ldots \Pi_{\theta_n}) f, f)_{H^{\otimes n}}
\]
is a \(\sigma\)-additive and atomless measure on \([0, 2\pi]^n\).

Proof: If \(f = a_1 \otimes \ldots \otimes a_n\) and \(g = h_1 \otimes \ldots \otimes h_n\), with \(a_i, h_j \in H\) (we say in this case that \(f\) and \(g\) are pure vectors), then, denoting the vector \((\theta_1, \ldots, \theta_n)\) by \(\theta\),
\[
(\Pi_{\theta}^{\otimes n} f, g)_{H^{\otimes n}}
\]
will be a finite linear combination of the terms \((a_i, \Pi_\theta h_k)_H\), hence the scalar product in \(H^\otimes n\) will be continuous with respect to \(\theta \in [0, 2\pi]^n\). Assume now that \((f_n)\) is a sequence of finite linear combinations of pure vectors converging to \(f\) in \(H^\otimes n\). Then

\[
\sup_{\theta \in [0,2\pi]^n} \left| \left( f_k - f_l, \Pi_\theta \otimes (h_1 \otimes \ldots \otimes h_n) \right)_{H^\otimes n} \right| \to 0,
\]

hence the limit is uniform, and this proves the continuity when \(g\) is a finite linear combination of pure vectors. Assume now that \(g\) is also a general symmetric tensor, then it can be approximated, as \(f\), by a sequence \((g_n)\) whose elements are the finite linear combinations of pure vectors. Then we have again the following result

\[
\sup_{\theta \in [0,1]^n} \left| \left( f, \Pi_\theta (g_k - g_l) \right)_{H^\otimes n} \right| 
\leq \|f\|_{H^\otimes n} \|g_k - g_l\|_{H^\otimes n} \to 0,
\]

which implies the uniform convergence with respect to \(\theta\). The last claim is obvious when \(f\) is a finite linear combination of pure vectors. A general \(f\) can be approximated with such vectors, say \((f_k, k \in \mathbb{N})\). Then, for any \(x \in \mathbb{R}^n\),

\[
\int_{[0,2\pi]^n} e^{i(x,\theta)}_{\mathbb{R}^n} d(\Pi_\theta \otimes f_k, f_k) = \left( (R_{x_1} \otimes \ldots \otimes R_{x_n}) f_k, f_k \right)_{H^\otimes n}
\]

and this converges, as \(k \to \infty\), to the map

\[
(x_1, \ldots, x_n) \rightarrow \left( (R_{x_1} \otimes \ldots \otimes R_{x_n}) f, f \right)_{H^\otimes n},
\]

which is a continuous function on \(\mathbb{R}^n\) at \(x = 0\) by the spectral representation of \(R\). Then the claim follows from the theorem of Paul Lévy about the characterization of the weak convergence of measures via the convergence of the characteristic functions. \(\square\)

We give now the main result of this section:

**Theorem 2** Let \(R\) be a unitary operator on the Cameron-Martin space \(H\) whose resolution of identity is denoted by \((\Pi_\theta, \theta \in [0, 2\pi])\). Then the corresponding (measure preserving) transformation \(T\) is ergodic if and only if \(\theta \rightarrow (\Pi_\theta h, k)_H\) is continuous on \([0, 2\pi]\) for any \(h, k \in H\). Moreover, if \(T\) is ergodic, it is also weak mixing.
Proof: Let us first prove the necessity: assume that the resolution of identity is not continuous. Then, from Hahn-Banach theorem, there exists an $h \in H$ and some $\tau \in (0,1)$ such that

$$
(\Pi_{\tau^+}h - \Pi_{\tau^-}h, k)_H = \lim_{\varepsilon \to 0} (\Pi_{\tau^+\varepsilon}h - \Pi_{\tau^-\varepsilon}h, k)_H \\
\neq 0
$$

for some $k \in H$. Let $z_\tau$ denote $\Pi_{\tau^+}h - \Pi_{\tau^-}h$. Note that we can represent $z_\tau$ as

$$
z_\tau = \int_{[0,2\pi]} 1_{\{\tau\}}(t) d\Pi_t h.
$$

For any $k \in H$, the spectral representation of $R$ gives

$$
(Rz_\tau, k)_H = \lim_{\varepsilon \to 0} \int_{[0,2\pi]} e^{i\theta} d(\Pi_{\theta \wedge (\tau+\varepsilon)}h - \Pi_{\theta \wedge (\tau-\varepsilon)}h, k) \\
= e^{i\tau}(z_\tau, k)_H.
$$

Therefore $z_\tau$ is an eigenfunction of $R$ with the corresponding eigenvalue $e^{i\tau}$. Let $f(w) = |\delta z_\tau(w)|$. It is easy to see that $f \circ T = f$ almost surely, hence $T$ can not be ergodic and this contradiction proves the necessity. To prove the sufficiency, let $F$ be Wiener functional such that $F \circ T = F$ almost surely. Without loss of generality we may assume that $F$ is bounded. From Lemma 1, if we represent $F$ as $E[F] + \sum_n I_n(f_n)$, then $I_n(f_n) \circ T = I_n(R^{\otimes n}(f_n)) = I_n(f_n)$ for any $n \geq 1$. Consequently

$$
0 = E \left[ |I_n(f_n) - I_n(R^{\otimes n}(f_n))|^2 \right] \\
= n! |f_n - R^{\otimes n}f_n|_{H^{\otimes n}}^2 \\
= n! \int_{[0,2\pi]^n} \left| 1 - e^{i \sum_{k=1}^n \theta_k} \right|^2 d(\Pi_{\theta_1} \otimes \ldots \otimes \Pi_{\theta_n}) f_n, f_n)_{H^{\otimes n}}
$$

this result implies that the positive measure $d((\Pi_{\theta_1} \otimes \ldots \otimes \Pi_{\theta_n}) f_n, f_n)_{H^{\otimes n}}$ is concentrated on the set $\{ \theta \in [0,2\pi]^n : \exp i \sum_{1 \leq k \leq n} \theta_k = 1 \}$, which is in contradiction with the fact that it does not have atoms due to Lemma 2. The proof of weak mixing is similar with some obvious modifications. \hfill \Box

The mixing property of $T$ is straight forward.

Theorem 3 The transformation $T$ defined by (3) is mixing if and only if

$$
\lim_{n \to \infty} (R^n h, h)_H = 0,
$$

for any $h \in H$.  

12
Proof: By a density argument, $T$ is mixing if and only if
\[
\lim_{n \to \infty} E \left[ \rho(\delta h) \circ T^n \rho(\delta h) \right] = 1
\]
for any $h \in H$, where
\[
\rho(\delta h) = \exp \left\{ \delta h - \frac{1}{2} |h|_H^2 \right\}.
\]
We have
\[
\rho(\delta h) \circ T^n \rho(\delta h) = \exp \left\{ \delta (R^n h + h) - |h|_H^2 \right\} = \rho(\delta (R^n h + h)) \exp \left\{ 1/2 |R^n h + h|_H^2 - |h|_H^2 \right\} = \rho(\delta (R^n h + h)) \exp (R^n h, h)_H.
\]
Hence
\[
\lim_{n \to \infty} E \left[ \rho(\delta h) \circ T^n \rho(\delta h) \right] = 1
\]
if and only if $(R^n h, h)_H \to 0$ as $n \to \infty$. \hfill \Box

We say that a sequence of random variables $(\eta_n, n \in \mathbb{Z})$ is ergodic or mixing if the shift transformation is ergodic or mixing respectively. We have now the following corollary:

Corollary 3 The transformation $T$ is ergodic or mixing if and only if, for any $h \in H$, the sequence $(\delta R^n h, n \in \mathbb{Z})$ is ergodic or mixing respectively.

Proof: The necessity is evident, for the sufficiency it suffices to remark that, Maruyama theorem implies the continuity of the spectral measure associated to the sequence $(\delta R^n h, n \in \mathbb{Z})$, which is nothing but the measure $\theta \to d(\Pi \theta h, h)_H$, whose continuity for any $h$ implies the ergodicity of $T$ by Theorem 2. For the mixing we proceed similarly. \hfill \Box

4 An example

Let
\[
dY_t = \gamma(t)dW_t; \quad Y_0 = 0, t \in [0, 1]
\]
where $W$ is a standard $n$-dimensional Brownian motion and $\gamma(t), t \in [0, 1]$ is a $n \times n$ unitary matrix, the elements of $\gamma(t)$ will be assumed to be non-random and Lebesgue measurable. The ergodicity of $Y = T(w)$ will be
discussed in this section. Let $e^{i\psi_j(t)}, 0 < \psi_j \leq 2\pi$, denote the eigenvalues of the unitary matrix $\gamma$ and let $u(\cdot)$ denote the unit step function $t$

$$u(\alpha) = \begin{cases} 1 & , \alpha \geq 0 \\ 0 & , \alpha < 0 \end{cases}$$

Then we have

**Theorem 4** A necessary and sufficient condition for the ergodicity of $T$ is the continuity of $\int_0^1 u(\theta - \psi_j(t)) \, dt \in [0, 2\pi]$ for all $i = 1, \cdots, n$. Otherwise stated $T$ is ergodic iff the Lebesgue measure of $C_j(\theta) = \{ t : \psi_j(t,w) = \theta \}$ is zero for $\theta$ and all $j$.

**Proof:** Assume first that for a.a. $t \in [0, 1]$, $\gamma$ possesses $n$-distant eigenvalues. Also, assume that $\psi_{j+1} > \psi_j$. Since $\gamma(t)$ is unitary it has the representation

$$\gamma(t) = A(t) \cdot \text{diag} e^{i\psi_j(t)} \cdot A^{-1}(t) \quad (13)$$

where $A(t) = [a_1(t), \cdots a_n(t)]$ is unique and $\{a_j(t) \} = 1, 2, \cdots, n$ are orthogonal $n$-vectors $\gamma(t) \cdot a_i(t) = e^{i\psi_j(t)}a_j(t)$. Let $h = \int_0^1 \hat{h}(s) \, ds$ where $\hat{h}$ takes values in $\mathbb{R}^n$ and let $(\cdot, \cdot)$ denote the scalar product in $\mathbb{R}^n$ then

$$\gamma(t) \hat{h}(t) = \sum_j \left( a_j(t), h'(t) \right) e^{i\psi_j(t)} a_j(t)$$

$$= \int_0^{2\pi} e^{i\theta} \sum_{j=1}^n u(\theta - \psi_j(t)) \left( a_j(t), h'(t) \right) \cdot a_j(t)$$

Hence

$$Rh = \int_0^{2\pi} e^{i\theta} d\theta \Pi_\theta h$$

where

$$\Pi_\theta h = \int_0^1 \sum_j u\left( \theta - \psi_j(t) \right) \left( a_j(t), h'(t) \right) a_j(t) \, dt$$

and

$$|\Pi_\theta h|^2 = \int_0^{2\pi} \sum_j u\left( \theta - \psi_j(t) \right) \left| \left( a_j(t), h'(t) \right) \right|^2 \, dt \quad (14)$$

if, as $\varepsilon \to 0$, $u(\theta + \varepsilon - \psi_i(t)) \to u(\theta - \psi_i(t))$ for almost (Lebesgue) all $t$ in $[0, 2\pi]$ then by monotone convergence $|\Pi_\theta h|^2$ is continuous in $\theta$. Conversely, if $|\Pi_\theta h|^2$ is discontinuous at $\theta = \theta_0$

$$\lim_{\varepsilon \to 0} \text{Leb} \left\{ t : u(\theta_0 + \varepsilon - \psi_j(t)) - u(\theta_0 - \varepsilon, \psi_j(t)) \neq 0 \right\} > 0$$

14
Hence $\int_0^{2\pi} u(\theta - \psi_j(t))dt$ is discontinuous at $\theta = \theta_0$. This proves the theorem for the case where there are $n$ distinct eigenvalues. If $\gamma$ possesses only $m < n$ distinct eigenvalues then $A(t)$ in (13) still holds with $\psi_j \leq \psi_{j+1}$ but is no longer unique. This, however, can be overcome by constructing a measurable selection which will provide a unique and measurable representation for $A(t)$. The rest of the proof remains unchanged.

**Corollary 4** If $n$ is odd then $T$ is non ergodic.

**Proof:** If $n$ is odd then at least one of the eigenvalues of $\gamma(t)$ is either 1 or -1. Now if $\psi_i(t) = \pi$ on a set of positive measure, then obviously (14) is discontinuous at $\theta_0 = \pi$. For $\lambda_i = 1$ on a set of positive $t$ measure, set $\psi_j = 2\pi$ (since $\Pi_\theta$ is, by definition continuous as $\theta_2 \searrow \theta_1$ and $\pi_0 = 0$) and $\int_0^1 u(\theta - \psi_j(t))dt$ must be discontinuous at $\theta_0 = 2\pi$. \qed

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