Preconditioners for Saddle Point Problems on Truncated Domains in Phase Separation Modelling

Pawan Kumar*1
1Old Mumbai Highway
1International Institute of Information Technology, Hyderabad
1Hyderabad 500032, India

September 22, 2021

Abstract
The discretization of Cahn-Hilliard equation with obstacle potential leads to a block 2 x 2 non-linear system, where the (1,1) block has a non-linear and non-smooth term. Recently a globally convergent Newton Schur method was proposed for the non-linear Schur complement corresponding to this non-linear system. The solver may be seen as an inexact Uzawa method which has the flavour of an active set method in the sense that the active sets are first identified by solving a quadratic obstacle problem corresponding to the (1,1) block of the block 2 x 2 nonlinear system, and a new decent direction is obtained after discarding the active set region. The problem becomes linear on nonactive set, and corresponds to solving a linear saddle point problem on truncated domains. For solving the quadratic obstacle problem, various optimal multigrid like methods have been proposed. In this paper solvers for the truncated saddle point problem is considered. Three preconditioners are considered, two of them have block diagonal structure, and the third one has block tridiagonal structure. One of the block diagonal preconditioners is obtained by adding certain scaling of stiffness and mass matrices, whereas, the remaining two involves Schur complement. Eigenvalue bound and condition number estimates are derived for the preconditioned untruncated problem. It is shown that the extreme eigenvalues of the preconditioned truncated system remain bounded by the extreme eigenvalues of the preconditioned untruncated system. Numerical experiments confirm the optimality of the solvers.

Keywords: Phase field, Preconditioner, Saddle Point, Newton Schur

1 Introduction
The Cahn-Hilliard equation was first proposed in 1958 by Cahn and Hilliard [6] to study the phase separation process in a binary alloy. Here the term phase stands for the concentration of different components in an alloy. It has been empirically observed that the concentration changes from a given mixed state to a spatially separated two phase state when the alloy under preparation is subjected to a rapid cooling below a critical temperature. This rapid reduction in the temperature the so-called deep quench limit has been found to be modeled efficiently by obstacle potential proposed by Oono and Puri [26, Fig. 7, p. 439] in 1987, and analyzed by Blowey and Elliot [3, p. 237, (1.14)]. The phase separation has been noted to be highly non-linear, and the obstacle potential emulates the nonlinearity and non-smoothness that is empirically observed and much desired in numerical simulations. Consequently, handling the non-smoothness as well as designing robust iterative procedure have been a subject of much active research during last decades. Assuming semi-implicit time discretizations [4] to alleviate the time step restrictions, most of the proposed methods essentially differ in the way the nonlinearity and non-smoothness are handled. There seems to be two main approaches to handle the non-smoothness: regularization around the non-smooth region and subsequently using a variant of smooth solvers, for example, as in [3], or an active set like approach [13], i.e., where one identifies the active sets via a nonlinear solver, subsequently, after discarding the active set nodes, we obtain a reduced (or truncated) problem which is linear. Moreover, the global convergence of the nonlinear solver may be ensured by a proper damping parameter, for example, as done in [13].

*pawan.kumar@iiit.ac.in
The non-linear problem corresponding to Cahn-Hilliard equation with obstacle potential could be written as a non-linear system in block $2 \times 2$ matrix form as follows:

$$
\begin{pmatrix}
F & B^T \\
B & -C
\end{pmatrix}
\begin{pmatrix}
u \\
w
\end{pmatrix} =
\begin{pmatrix}
f \\
g
\end{pmatrix}, \quad u, w \in \mathbb{R}^n,
$$

where $u$ and $w$ are unknowns corresponding to order parameter and chemical potential respectively, $F = A + \partial I_K$, where $I_K$ denotes the indicator functional for $u$ corresponding to the admissible set $K$. We note that $F(\cdot)$ is a set valued mapping due to the presence of set-valued operator $\partial I_K$, hence, we have inclusion in (1) instead of equality. The matrix $A$ corresponds to Laplacian with Neumann boundary conditions perturbed by a rank one term, and is multiplied by a parameter corresponding to interface width. On the other hand, $C$ is also Laplacian with Neumann boundary condition, but multiplied by the time step parameter. Both nonlinearity and non-smoothness are due to $\partial I_K$ in $F$. Various non-linear and nonsmooth solvers have been proposed for [1], [11], [5].

By nonlinear Gaussian elimination of $u$, the system above could be reduced to a nonlinear Schur complement system in $w$ variables [13], where the “negative” nonlinear Schur complement is given by $C + B(F)^{-1}B^T$. Here $(\cdot)^{-1}$ is understood as inversion in the nonlinear sense. In [13], a globally convergent Newton method is proposed for this nonlinear Schur complement system, which is interpreted as a preconditioned Uzawa iteration. To solve the inclusion $F(x) \ni y$ corresponding to the quadratic obstacle problem, many methods have been proposed such as block Gauss-Seidel [2, 9], monotone multigrid method [17, 18, 23], truncated monotone multigrid [14], and truncated Newton multigrid [14].

Once active sets are identified, the corresponding rows and columns are anhilated, we then obtain a reduced linear system as follows

$$
\begin{pmatrix}
\hat{A} & \hat{B}^T \\
\hat{B} & -\hat{C}
\end{pmatrix}
\begin{pmatrix}
\hat{u} \\
\hat{w}
\end{pmatrix} =
\begin{pmatrix}
\hat{f} \\
\hat{g}
\end{pmatrix}, \quad \hat{u}, \hat{w} \in \mathbb{R}^n.
$$

Here a solution to (1) is a new descent direction in the Uzawa iteration. By a choice of appropriate step size along this descent direction, global convergence of the Uzawa method is ensured. As the active sets change during each iteration, the linear system, and hence the preconditioners need to be updated.

In this paper, our goal is to design effective preconditioner and hence an iterative solver for (1) such that the convergence rate is independent of problem parameters and mesh size. There are several classes of preconditioners: multigrid [20, 22], domain decomposition [7, 19], deflation based preconditioners [8, 16, 25, 21]. Three preconditioners are considered; two of them involves Schur complement. Two of these preconditioners have block diagonal structure and they correspond to non-standard norms proposed in [32]. To approximate the Schur complement, we consider an approximation proposed in [5]. It turns out that the building blocks of these preconditioners are same, their analysis is remarkably similar, even though, they may look structurally different from the outset. Eigenvalue bound and condition number estimates are derived for these preconditioners for the untruncated problem. The obtained eigenvalue bounds seem to be tight when compared to numerically computed extreme eigenvalues. Subsequently, it is shown that the extreme eigenvalues of the preconditioned truncated problem are bounded from above and below by the extreme eigenvalues of the corresponding preconditioned untruncated problem. We also verify the effectiveness of these preconditioners numerically for various evolutions.

The rest of this paper is organized as follows. In Section 3, we describe the Cahn-Hilliard model with obstacle potential, we discuss the time and space discretizations and variational formulations. In Section 4, we discuss briefly the solver for Cahn-Hilliard with obstacle problem. The preconditioners for the truncated linear saddle point problem (1), and their eigenvalue analysis are discussed in Section 5. Finally, in Section 6 we show numerical experiments with the proposed preconditioners.

### 2 Notations

Let SPD and SPSD denote symmetric positive definite and symmetric positive semi definite respectively. Let $\kappa(M)$ denote the condition number of SPD matrix $M$. For $x \in \mathbb{R}$, $|x|$ denotes the absolute value of $x$, whereas, for any set $K$, $|K|$ denotes the number of elements in $K$. Let $Id \in \mathbb{R}^{n \times n}$ denote the identity matrix. Let $1$ denote $[1, 1, 1, \ldots, 1]$. For a matrix $Z \in \mathbb{R}^{n \times n}$ with all real eigenvalues, the eigenvalues will be denoted and ordered as follows

$$
\lambda_1(Z) \leq \lambda_2(Z) \leq \cdots \leq \lambda_n(Z).
$$
3 Cahn-Hilliard Problem with Obstacle Potential

3.1 The Model

We will consider a model for phase separation of two components in a binary alloy mixture. Here phase stands for concentration of two components in the mixture. Let $u_1, u_2 \in [0, 1]$ be the concentration of two components in the mixture, then we set $u = u_1 - u_2 \in [-1, 1]$. The phase separation is modelled using Cahn-Hilliard equations, which is obtained by $H^{-1}$ gradient flow of Ginzburg-Landau (GL) energy functional which is given as follows

$$E(u) = \int_{\Omega} \frac{\epsilon}{2} |\nabla u|^2 + \psi(u) \, dx, \quad \Omega = (0, 1) \times (0, 1).$$  \hspace{1cm} (3)

Here the constant $\epsilon$ relates to interfacial thickness, and the obstacle potential $\psi$, which is used to model deep quench phenomena is given as follows

$$\psi(u) = \psi_0(u) + I_{[-1,1]}(u), \quad \text{where} \quad \psi_0(u) = \frac{1}{2}(1 - u^2).$$

Here the subscript $[-1, 1]$ of indicator function $I$ above denotes the range of admissible values of $u$. Here $I_{[-1,1]}(u)$ is defined as follows

$$I_{[-1,1]} = \begin{cases} 0, & \text{if } u(i) \in [-1, 1] \\ \infty, & \text{otherwise.} \end{cases}$$

Moreover, $u_1 + u_2$ is assumed to be conserved. The $H^{-1}$ gradient flow of $E$ leads to the Cahn-Hilliard equation in PDE form

$$\partial_t u = \Delta w, \quad w = -\epsilon \Delta u + \psi'(u) + \mu, \quad \mu \in \partial I_{[-1,1]}(u), \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega.$$  \hspace{1cm} (4)

The unknowns $u$ and $w$ are called order parameter and chemical potential respectively. For a given $\epsilon > 0$, final time $T > 0$, and initial condition $u_0 \in K$, where

$$K = \{ v \in H^1(\Omega) : |v| \leq 1 \},$$

the equivalent initial value problem for Cahn-Hilliard equation with obstacle potential interpreted as variational inequality reads

$$\left\langle \frac{du}{dt}, v \right\rangle + \langle \nabla w, \nabla v \rangle = 0, \quad \forall v \in H^1(\Omega),$$  \hspace{1cm} (5)

where we use the notation $\langle \cdot, \cdot \rangle$ to denote the duality pairing of $H^1(\Omega)$ and $H^1(\Omega)'$. Note that we used the fact that $\psi_0(u) = -u$ in the second term on the left of inequality (3.1) above. The inequalities (3.1) and (3.1) are defined on constrained set $K$, the variational inequality of first kind is also equivalently represented on unconstrained set using the indicator functional [9, p. 2]. The existence and uniqueness of the solution of (3.1) and (3.1) above have been established in Blowey and Elliot [3]. We next consider an appropriate discretization in time and space for (3.1) and (3.1).

3.2 Time and space discretizations

We consider a fixed non-adaptive grid in time interval $(0, T)$ and in space $\Omega$ defined in (3.1). The time step $\tau = T/N$ is kept uniform. We consider the semi-implicit Euler discretization in time and finite element discretization in space as in Barrett et. al. [2] with triangulation $T_h$ with the following spaces

$$S_h = \{ v \in C(\bar{\Omega}) : v|_T \text{ is linear } \forall T \in T_h \},$$

$$P_h = \{ v \in L^2(\Omega) : v|_T \text{ is constant } \forall T \in T \},$$

$$K_h = \{ v \in P_h : |v|_T \leq 1 \quad \forall T \in T_h \} = K \cap S_h \subset K,$$
which leads to the following discrete Cahn-Hilliard problem with obstacle potential:

Find \( u^k_h \in K_h, w^k_h \in S_h \) s.t.

\[
\begin{align*}
\langle u^k_h, v_h \rangle + \tau (\nabla w^k_h, \nabla v_h) &= \langle u^{k-1}_h, v_h \rangle, \quad \forall v_h \in S_h, \\
\epsilon (\nabla u^k_h, \nabla (v_h - u^k_h)) - \langle w^k_h, v_h - u^k_h \rangle &\geq \langle u^{k-1}_h, v_h - u^k_h \rangle, \quad \forall v_h \in K_h
\end{align*}
\]

holds for each \( k = 1, \ldots, N \). The initial solution \( u^0_h \in K_h \) is taken to be the discrete \( L^2 \) projection \( u^0_h, v_h \) s.t.

\[
\begin{align*}
\langle u^0_h, v_h \rangle &= \langle u^0, v_h \rangle, \quad \forall v_h \in S_h.
\end{align*}
\]

Existence and uniqueness of the discrete Cahn-Hilliard equations has been established in \cite{4}. The discrete Cahn-Hilliard equation is equivalent to the set valued saddle point block 2 x 2 nonlinear system (1) with \( F = A + \partial I_{K_h} \) and

\[
\begin{align*}
A &= \epsilon(\langle \lambda_p, 1 \rangle \langle \lambda_p, 1 \rangle + (\nabla \lambda_p, \nabla \lambda_q))_{p,q \in N_h}, \\
B &= (\langle \lambda_p, \lambda_q \rangle)_{p,q \in N_h}, \\
C &= \tau ((\nabla \lambda_p, \nabla \lambda_q))_{p,q \in N_h}.
\end{align*}
\]

We write the above in more compact notations as follows

\[
\begin{align*}
A &= \epsilon(K + mm^T), \\
B &= M, \\
C &= \tau K,
\end{align*}
\]

where \( m = \langle \lambda_p, 1 \rangle, M \) and \( K \) are usual notations for mass and stiffness matrices respectively.

## 4 Iterative solver for Cahn-Hilliard with obstacle potential

In \cite{13}, a nonsmooth Newton Schur method is proposed which is also interpreted as a preconditioned Uzawa iteration. For a given time step \( k \), the Uzawa iteration reads:

\[
\begin{align*}
\begin{align*}
u^{i,k} &= F^{-1}(f^k - B^T w^{i,k}), \\
w^{i+1,k} &= w^{i,k} + \rho^{i,k} \hat{S}^{-1}_{i,k}(Bu^{i,k} - Cw^{i,k} - g^k)
\end{align*}
\end{align*}
\]

for the saddle point problem (1). Here \( i \) denotes the \( i^{th} \) Uzawa step, and \( k \) denotes the \( k^{th} \) time step. Here \( f^k \) and \( g^k \) are defined as follows

\[
\begin{align*}
\langle f, v_h \rangle &= \langle u^{k-1}_h, v_h \rangle, \\
\langle g, v_h \rangle &= -\langle u^{k-1}_h, v_h \rangle.
\end{align*}
\]

The time loop starts with an initial value for \( u^{0,0} \) which can be taken arbitrary as the method is globally convergent, and with the initial value \( w^{0,0} \) obtained from (4). The Uzawa iteration requires three main computations that we describe below.

### 4.1 Computing \( u^{i,k} \)

The first step \( 4 \) corresponds to solving a quadratic obstacle problem interpreted as a minimization problem as follows

\[
\begin{align*}
u^{i,k} &= \arg \min_{v_h \in K_h} \left( \frac{1}{2}\langle Av, v \rangle - \langle f^k - B^T w^{i,k}, v \rangle \right).
\end{align*}
\]

As mentioned in the introduction, this problem has been extensively studied during last decades \cite{2} \cite{14} \cite{17} \cite{18}.

#### 4.1.1 Algebraic Monotone Multigrid for Obstacle Problem

To solve the quadratic obstacle problem (1), we use the monotone multigrid method proposed in \cite{17}. In Algorithm 1, we describe an algebraic variant of the method. The algorithm performs one V-cycle of multigrid; it takes \( u^i \) from the previous iteration, and outputs the improved solution \( u^{i+1} \). The initial set of interpolation operators are constructed using aggregation based coarsening \cite{?}.
4.2 Computing $\hat{S}_{i,k}^{-1}(Bu_i^k - Cw_i^k - g^k)$

The quantity $d_i^k = \hat{S}_{i,k}^{-1}(Bu_i^k - Cw_i^k - g^k)$ in (4) is obtained as a solution of the following reduced linear block $2 \times 2$ system:

\[
\begin{pmatrix}
\hat{A} & \hat{B}^T \\
\hat{B} & -C
\end{pmatrix}
\begin{pmatrix}
\hat{d}_i^k \\
d_i^k
\end{pmatrix} =
\begin{pmatrix}
0 \\
g + Cw_i^k - Bu_i^k
\end{pmatrix},
\]

where

\[
\hat{A} = TAT + \hat{T}, \quad \hat{B} = BT.
\]

Here truncation matrices $T$ and $\hat{T}$ are defined as follows:

\[
T = \diag \begin{pmatrix} 0 & \text{if } u_i^k(j) \in \{-1,1\} \\ 1 & \text{otherwise} \end{pmatrix}, \quad \hat{T} = \diag \begin{pmatrix} 1 & \text{if } T_{jj} = 0 \\ 0 & \text{otherwise} \end{pmatrix}, \quad j = 1, \ldots, |N_k|,
\]

where $u_i^k(j)$ is the $j$th component of $u_i^k$, and $T_{jj}$ is the $j$th diagonal entry of $T$. In words, $\hat{A}$ is the matrix obtained from $A$ by replacing the $i$th row and $i$th column by the unit vector $e_i$ corresponding to the active sets identified by diagonal entries of $T$. Similarly, $\hat{B}$ is the matrix obtained from $B$ by annihilating rows, and $\hat{B}^T$ is the matrix obtained from $B$ by annihilating columns. Rewriting untruncated version of (4.2) in simpler notation as follows

\[
\begin{pmatrix}
\epsilon\hat{K} & M \\
M & -\tau K
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
0 \\
\hat{b}
\end{pmatrix},
\]

where $\hat{K} = K + mm^T$. By a change of variable $y' = y/\epsilon$, we obtain

\[
\begin{pmatrix}
\hat{K} & M \\
M & -\eta K
\end{pmatrix}
\begin{pmatrix}
x \\
y'
\end{pmatrix} =
\begin{pmatrix}
0 \\
\hat{b}
\end{pmatrix},
\]

where $\eta = \epsilon \cdot \tau$. Furthermore, we modify the $(2,2)$ term of the system matrix above as follows

\[-\eta K = -\eta K - \eta mm^T + \eta mm^T = -\eta \hat{K} + (\eta^{1/2}m)(\eta^{1/2}m^T) = -\eta \hat{K} + \tilde{m}\tilde{m}^T,
\]

where $\tilde{m} = \eta^{1/2}$. Now the untruncated system may be rewritten as

\[
\tilde{A} = \begin{pmatrix}
\hat{K} & M \\
M & -\eta K
\end{pmatrix} + \tilde{m}\tilde{m}^T =: \hat{A} + \tilde{m}\tilde{m}^T,
\]

where $\tilde{m} = [0, \tilde{m}^T] \in \mathbb{R}^{2|N_k|}$ is a rank one term with proper extension by zero. Now we are in a position to use Sherman-Woodbury inversion for matrix plus rank-one term to invert $\hat{A}$. In this paper, we shall develop efficient solvers to solve the truncated system

\[
\hat{A}v = z,
\]

where $\hat{A}$ is defined next. We denote

\[
\hat{K} = TKT + \hat{T}, \quad A = \hat{K}, \quad B = M, \quad \hat{A} = TAT + \hat{T}, \quad \hat{B} = TB.
\]

Note that the notation $A$ appearing above has now been redefined. Thus, the truncated system corresponding to (4.2) reads

\[
\hat{A} + \tilde{m}\tilde{m}^T = \begin{pmatrix}
\hat{A} & \hat{B} \\
\hat{B}^T & -\eta A
\end{pmatrix} + \tilde{m}\tilde{m}^T.
\]

Thus, Sherman-Woodbury inversion formula may be used to invert $\hat{A} + \tilde{m}\tilde{m}^T$, and it is enough to find an efficient solver for (4.2).

4.3 Computing step length $\rho^i,k$

The step length $\rho^i,k$ can be computed using a bisection method. We refer the interested reader to [12] p. 88.
Algorithm 1: Monotone Multigrid (MMG) V cycle

Require: Let \( V_1 \subset V_1 \subset V_2 \cdots V_m \) and let \( r_m, b_m \in V_m \).
Require: \( u^*, i > 0 \) solution from previous cycle or \( u^0 \) a given initial solution

1: Compute residual: \( r_m = b_m - A_m u^i \)
2: Compute defect obstacles:
   \[
   \begin{cases}
   \delta_m = \psi - u^i \\
   \delta_m = \psi - u^i
   \end{cases}
   \]
3: for \( \ell = m, \cdots, 2 \) do
4:   Projected Gauss-Seidel Solve using Algorithm 2
   \[
   (D_\ell + L_\ell + \partial I_{K^\ell}) v_\ell = r_\ell,
   \]
   where
   \[
   K^\ell = \{ v \in \mathbb{R}^{n_\ell} \mid \delta_\ell \leq v \leq \bar{\delta}_\ell \}.
   \]
5: Update
   \[
   \begin{cases}
   r_\ell := r_\ell - A_\ell v_\ell \\
   \delta_{\ell-1} := \delta_\ell - v_\ell \\
   \bar{\delta}_{\ell-1} := \bar{\delta}_\ell - v_\ell
   \end{cases}
   \]
6: Restrict and compute new obstacle
   \[
   \begin{cases}
   r_{\ell-1} = P_{\ell-1}^T r_\ell \\
   (\delta_{\ell-1})_i := \max \{ (\delta_{\ell-1})_j \mid \{P_{\ell-1}\}_{ji} \neq 0 \}, i = 1, \ldots, n_{\ell-1} \\
   (\bar{\delta}_{\ell-1})_i := \min \{ (\bar{\delta}_{\ell-1})_j \mid \{P_{\ell-1}\}_{ji} \neq 0 \}, i = 1, \ldots, n_{\ell-1}
   \end{cases}
   \]
7: end for
8: Solve
   \[
   (D_1 + L_1 + \partial I_{K^1}) v_1 = r_1
   \]
9: for \( \ell = 2, \cdots, m \) do
10:   Add corrections
   \[
   v_\ell := v_\ell + P_{\ell-1} r_{\ell-1}
   \]
11: end for
12: Compute
   \[
   u^{i+1} = u^i + v_m
   \]
Ensure: improved solution \( u^{i+1} \)
Algorithm 2: $x^{i+1} \leftarrow \text{PGS}(x^i, A, \psi, \bar{\psi}, b)$

Require: $A \in \mathbb{R}^{n \ell \times n \ell}$, $b, \psi, \bar{\psi} \in \mathbb{R}^{n \ell}$, current iterate $x^i \in \mathbb{R}^{n \ell}$

Ensure: new iterate $x^{i+1} \in \mathbb{R}^{n \ell}$

1: Compute residual:
   
   \[ r := b - Ax^i \]

2: Compute defect obstacles:

   \[ \psi := \psi - x^i \]

   \[ \bar{\psi} := \bar{\psi} - x^i \]

3: for $i = 1 : n \ell$ do

4: for $j = 1 : i$ do

5: Compute $y_i$

   \[ y_i = \begin{cases} \max \left( \min \left( (r_i - A_{ij} y_j) / A_{ii}, \psi_i \right), \bar{\psi}_i \right), & \text{if } A_{ii} \neq 0, \\ 0, & \text{otherwise} \end{cases} \]

6: end for

7: end for

8: $x^{i+1} = x^i + y$

4.4 Mixed Finite Element Formulation of Reduced Linear System

We choose suitable Hilbert spaces for trial and test spaces as follows

\[ \mathcal{V} = \{ v \in H^1(\Omega) : v|_{\Omega_A} = 0 \}, \quad Q = H^1_0(\Omega), \quad \Omega_A = \Omega \setminus \Omega_I, \]

where $\Omega_A = \Omega \setminus \Omega_I$ is the domain where truncation takes place. Indeed, if $\Omega_A$ is empty, then $\Omega_I = \Omega$, and we set $\mathcal{V} = V = H^1(\Omega)$. The weak form of the partial differential equations corresponding to the truncated system (4.2) reads

Find $(u, \lambda) \in \mathcal{V} \times H^1(\Omega) :$

\[ \hat{a}(u, v) + \hat{b}(v, \lambda) = f(v) \quad \text{for all } v \in \mathcal{V}, \]

\[ \hat{b}(u, q) - c(\lambda, q) = g(q) \quad \text{for all } q \in Q, \]

where

\[ \hat{a}(u, v) = \left( (\nabla u, \nabla v) + \int_{\Omega} u \int_{\Omega} v \right) = \left( (\nabla u, \nabla v) + \langle u, 1 \rangle \langle v, 1 \rangle \right), \]

\[ c(\lambda, q) = \eta \left( (\nabla \lambda, \nabla q) + \int_{\Omega} u \int_{\Omega} v \right), \quad \hat{b}(v, \lambda) = \langle v, \lambda \rangle. \]

The mixed variational problem above can also be written as a variational form on product spaces

Find $x \in \mathcal{V} \times Q :$ \[ \hat{\mathcal{B}}(x, y) = \hat{\mathcal{F}}(y) \quad \forall y \in \mathcal{V} \times H^1(\Omega), \]

where $\hat{\mathcal{B}}$ and $\hat{\mathcal{F}}$ are defined as follows

\[ \hat{\mathcal{B}}(z, y) = \hat{a}(w, v) + \hat{b}(v, r) + \hat{b}(w, q) - c(r, q), \quad \hat{\mathcal{F}}(y) = f(v) + g(q) \]

for $y = (v, q) \in \mathcal{V} \times Q$ and $z = (w, r) \in \mathcal{V} \times Q$. The corresponding bilinear form for the untruncated system is given as follows

\[ B(z, y) = a(w, v) + b(v, r) + b(w, q) - c(r, q), \quad \mathcal{F}(y) = f(v) + g(q) \]
for \( y = (v, q) \in V \times Q \) and \( z = (w, r) \in V \times Q \), where \( V = H^1(\Omega) \). The mixed variational problem corresponding to untruncated system now reads

\[
\text{Find } x \in V \times Q : \quad B(x, y) = F(y) \quad \forall y \in V \times H^1(\Omega).
\]  

(14)

In the rest of this paper, we shall consider norms proposed in [32] as follows

\[
\sup_{0 \neq z \in X} \sup_{0 \neq y \in X} \frac{B(z, y)}{\|z\|_X \|y\|_X} \leq \bar{c}_x < \infty.
\]  

(16)

We have the following conjecture for the truncated problem

\[
\sup_{0 \neq z \in X} \sup_{0 \neq y \in X} \frac{\hat{B}(z, y)}{\|z\|_X \|y\|_X} \leq \sup_{0 \neq z \in X} \sup_{0 \neq y \in X} \frac{B(z, y)}{\|z\|_X \|y\|_X} \leq \bar{c}_x < \infty.
\]  

(17)

Similarly, for well-posedness of [4.4], following well known Babuska-Brezzi condition needs to be satisfied

\[
\inf_{0 \neq z \in X} \sup_{0 \neq y \in X} \frac{B(z, y)}{\|z\|_X \|y\|_X} \geq c_x > 0.
\]  

(18)

Similarly, it is not evident whether the following inequality must hold.

\[
c_x \leq \inf_{0 \neq z \in X} \sup_{0 \neq y \in X} \frac{B(z, y)}{\|z\|_X \|y\|_X} \leq \inf_{0 \neq z \in X} \sup_{0 \neq y \in X} \frac{\hat{B}(z, y)}{\|z\|_X \|y\|_X}.
\]  

(19)

We shall call the norms \( \| \cdot \|_X \) and \( \| \cdot \|_X \) optimal if the constants \( \bar{c}_x \) and \( c_x \) remain independent of the problem parameters: \( \tau \) and \( \epsilon \), moreover, in the discrete space, also remains independent of the mesh size \( h \). The reason why we are interested in the inequalities (4.4) and (4.4) is that any optimal norm that is found for untruncated problem shall lead to optimal norm for truncated problem as well. Note that boundary of untruncated problem has certain regularity (for example Lipschitz continuity), but for the truncated problem no such regularity is to be assumed, because the truncations are assumed to be arbitrary. Our plan of attack is to use the approach of [32], which is readily applicable for our untruncated problem. Although, [4.4] and [4.4] are left as conjecture for the moment, we shall try to answer this in the discrete case: we shall show a related result that the extreme eigenvalues of the truncated preconditioned operator are bounded by the extreme eigenvalues of the corresponding untruncated preconditioned operator. Hence, in the following, we first derive optimal preconditioners for the untruncated problem.

We shall provide equivalent conditions as in [32] for [4.4] and [4.4] that lead to deriving the optimal norms, i.e., optimal preconditioners. But first we introduce some notations for operators corresponding to bilinear forms. It is easy to see that \( V \times Q \) is a Hilbert space itself as \( V \) and \( H^1(\Omega) \) are themselves Hilbert spaces. It is convenient to associate linear operators for the bilinear forms \( a, b, \) and \( c \) as follows

\[
\langle Aw, v \rangle = a(w, v), \quad A \in L(V, V^*),
\]

\[
\langle Bw, q \rangle = b(w, q), \quad B \in L(V, Q^*),
\]

\[
\langle Cr, q \rangle = c(r, q), \quad C \in L(Q, Q^*),
\]

\[
\langle B^*r, v \rangle = \langle Bv, r \rangle, \quad B^* \in L(Q, V^*).
\]  

(20)

Consequently, the operator corresponding to mixed bilinear form \( A \), and the right hand side \( F \) (reusing the notation) in operator notation are given as follows

\[
A = \begin{pmatrix} A & B^* \\ B & -C \end{pmatrix}, \quad F = \begin{pmatrix} f \\ g \end{pmatrix}, \quad x = \begin{pmatrix} u \\ p \end{pmatrix}.
\]

The untruncated problem is denoted as follows

\[
Ax = F,
\]  

(21)
and the corresponding truncated problem reads
\[ \hat{A}x = \hat{f}, \]
where \( \hat{A} \) is given as follows
\[
\hat{A} = \begin{pmatrix} \hat{A} & \hat{B}^* \\ \hat{B} & -C \end{pmatrix}, \quad \hat{f} = \begin{pmatrix} \hat{f} \\ \hat{g} \end{pmatrix}, \quad \hat{x} = \begin{pmatrix} \hat{u} \\ \hat{p} \end{pmatrix},
\]
where analogous to (4.4), we have following definitions for truncated operators
\[
\langle \hat{A}w, v \rangle = \hat{a}(w, v), \quad \hat{A} \in L(\hat{V}, \hat{V}^*),
\langle \hat{B}w, q \rangle = \hat{b}(w, q), \quad \hat{B} \in L(\hat{V}, \hat{Q}^*),
\langle \hat{B}^* r, v \rangle = \langle \hat{B}v, r \rangle, \quad \hat{B}^* \in L(\hat{Q}, \hat{V}^*),
\]
where \( C \) is defined as in (4.4). In [32], starting from the abstract theory on Hilbert spaces that lead to representation of isometries, a preconditioner is proposed; it is based on non-standard norms, or isometries that correspond to block diagonal preconditioner of the following form
\[ \mathcal{B} = \begin{pmatrix} I_V & I_Q \\ \end{pmatrix}. \]
In the next section, our goal is to determine \( I_Q \) and \( I_V \).

4.5 Choice of norm: a brief introduction to Zulehner’s idea

Before we move further, we introduce some notations. The duality pairing \( \langle \cdot, \cdot \rangle_H \) on \( H^* \times H \) is defined as follows
\[ \langle \ell, x \rangle_H = \ell(x) \quad \text{for all } \ell \in H^*, \ x \in H. \]
Let \( \mathcal{I}_H : H \to H^* \) be an isometric isomorphism defined as follows
\[ \langle \mathcal{I}_H x, y \rangle = \langle x, y \rangle_H. \]
The inverse \( \mathcal{R}_H = \mathcal{I}_H^{-1} \) is Riesz-isomorphism, by which functionals in \( H^* \) can be identified with elements in \( H \) and we have
\[ \langle \ell, x \rangle_H = \langle \mathcal{R}_H \ell, x \rangle_H. \]
We already chose the type of norm in (4.4), we now look for explicit representation of isometries or norms in terms of operators defined in (4.4). The main ingredient is the following theorem.

**Theorem 4.1.** [32][p. 543, Th. 2.6] If there are constants \( \gamma_v, \tilde{\gamma}_v, \gamma_q, \tilde{\gamma}_q > 0 \) such that
\[ \gamma_v \|w\|_V^2 \leq a(w, w) + \|Bw\|_{Q^*}^2 \leq \tilde{\gamma}_v \|w\|_V^2, \quad \forall w \in V \tag{22} \]
and
\[ \gamma_q \|r\|_Q^2 \leq c(r, r) + \|B^* r\|_{V^*}^2 \leq \tilde{\gamma}_q \|r\|_Q^2, \quad \forall r \in Q \tag{23} \]
then
\[ \epsilon_x \|z\|_X \leq \|Az\|_{X^*} \leq \tilde{\epsilon}_x \|z\|_X, \quad \forall z \in X \tag{24} \]
is satisfied with constants \( \epsilon_x, \tilde{\epsilon}_x > 0 \) that depend only on \( \gamma_v, \tilde{\gamma}_v, \gamma_q, \tilde{\gamma}_q \) and on \( \tilde{\gamma}_q. \) And, vice versa, if the estimates (4.1) are satisfied with constants \( \epsilon_x, \tilde{\epsilon}_x > 0, \) then the estimates (4.1) and (4.1) are satisfied.
Similarly, as conjectured for (4.4) and (4.4), and recalling that \( \hat{X} = \hat{V} \times Q \subset V \times Q = X \), we may ask whether the following bounds hold for truncated system

\[
\inf_{z \in X^*} \| \hat{A} \hat{z} \|_{X^*} \geq \inf_{z \in X^*} \| Az \|_{X^*}, \quad \text{sup} \| \hat{A} \hat{z} \|_{X^*} \leq \sup_{z \in X^*} \| Az \|_{X^*}.
\]

However, we shall show a similar result in finite dimension using Fischer’s theorem in Lemma 5.2. In [32], the terms \( \| Bw \|_{Q^*} \) and \( \| B^* r \|_{r^*} \) in (4.1) and (4.1) respectively are defined using isometries \( I_V \) and \( I_Q \) as follows:

\[
\| Bw \|_{Q^*}^2 = \langle B^* I_Q^{-1} Bw, w \rangle, \quad \| B^* r \|_{r^*}^2 = \langle BL_V^{-1} B^* r, r \rangle.
\]

Using (4.5), the equations (4.1) and (4.1) are equivalently written as follows

\[
\gamma_q \langle I_V w, w \rangle \leq \langle (A + B^* T_{Q}^{-1} B) w, w \rangle \leq \gamma_q \langle I_V w, w \rangle, \quad \text{for all} \ w \in V,
\]

\[
\gamma_q \langle I_Q r, r \rangle \leq \langle (C + B I_V^{-1} B^*) r, r \rangle \leq \gamma_q \langle I_Q r, r \rangle, \quad \text{for all} \ r \in Q.
\]

In short, in new notation \( \sim \) meaning “spectrally similar,” we obtain the following equivalent conditions for isometries \( I_V \) and \( I_Q \):

\[
I_V \sim A + B^* T_{Q}^{-1} B \quad \text{and} \quad I_Q \sim C + B I_V^{-1} B^* \quad \sim \quad I_V \sim A + B^* (C + B I_V^{-1} B^*)^{-1} B \quad \text{and} \quad I_Q \sim C + B I_V^{-1} B^* \quad \sim \quad I_Q \sim C + B (A + B^* T_{Q}^{-1} B)^{-1} B \quad \text{and} \quad I_V \sim A + B^* T_{Q}^{-1} B
\]

Let \( M \) and \( N \) be any SPD matrices, consequently, they define inner products and a Hilbert space structure in \( \mathbb{R}^n \). Moreover, the intermediate Hilbert spaces between \( M \) and \( N \) are given as follows

\[
[M, N]_{\theta} = M^{1/2}(NM^{1/2})^{\theta}N^{1/2}, \quad \theta \in [0, 1].
\]

Continuing from above, when \( A \) and \( C \) are non singular, the generic form of the norms are given by the following lemma.

**Lemma 4.1.** Let \( A \), consequently, \( C = \eta A, \eta > 0 \) be nonsingular. Then

\[
I_V = A + [A, B^T C^{-1} B]_{\theta}, \quad I_Q = C + [C, B A^{-1} B^T]_{1-\theta}, \quad \theta \in [0, 1]. \tag{26}
\]

**Proof.** See [32] p. 547-548.

The isometries \( I_V \) and \( I_Q \) above provide a general template for obtaining a variety of preconditioners. Obviously, our goal is to find those that are easier to compute with numerically. Before we propose preconditioners, we shall need some properties of the \((1,1)\) block of \( A \), and that for the negative Schur complement \( S = C + B A^{-1} B^T \). Such properties will be useful in developing preconditioners using \( I_V \) and \( I_Q \).

### 4.6 Properties of the system matrix and Schur complement

An important property that we shall need shortly when analyzing preconditioners is that the eigenvalues of the truncated matrix is bounded from above and below by the eigenvalues of the untruncated matrix.

**Lemma 4.2.** The operator \( A \) is symmetric and indefinite.

**Proof.** Symmetry is obvious. Indefiniteness follows from below:

\[
x^T Ax = \begin{bmatrix} u^T & v^T \end{bmatrix} \begin{bmatrix} \hat{K} & M \\ M & -\eta \hat{K} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \| u \|_{\hat{K}}^2 - \eta \| v \|_{\hat{K}}^2 + 2Re[(Mu, v)].
\]

For the choice of \([u^T, v^T] = [0^T, v^T] \), \( x^T Ax = -\eta \| v \|_{\hat{K}}^2 \leq 0 \).

**Lemma 4.3.** \( A, A + B \) is SPD.

**Proof.** From (4.2), we recall that \( A = \hat{K} = K + mm^T \). Here \( K \) being a stiffness matrix corresponding to natural boundary condition is SPD except on the span of vector \( \mathbf{1} = [1, 1, 1, \ldots, 1]^T \), which is in the kernel of \( K \), but \( \langle mm^T \mathbf{1}, \mathbf{1} \rangle > 0 \). Also, \( B = M \) being a mass matrix is SPD, \( A + B \) is SPD.
Fact 4.1 (Permutation preserves eigenvalues). Let $P \in \mathbb{Z}^{n \times n}$ be a permutation matrix, then $P^T \hat{A}P$ and $\hat{A}$ are similar.

Proof. $P$ being a permutation matrix, $P^T P = Id$, hence the proof.

Lemma 4.4 (Poincare separation theorem for eigenvalues). Let $Z \in \mathbb{R}^{n \times n}$ be any symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and let $P$ be a semi-orthogonal $n \times k$ matrix such that $P^T P = Id \in \mathbb{R}^{k \times k}$. Then the eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-k+1}$ of $P^T Z P$ are separated by the eigenvalues of $Z$ as follows

$$\lambda_i \leq \mu_1 \leq \lambda_{n-k+i}.$$ 

Proof. The theorem is proved in [28, p. 337].

Lemma 4.5 (Eigenvalues of the truncated (1,1) block). Let $n = |\mathcal{N}_h|$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $A$, and let $\hat{\lambda}_1 \leq \hat{\lambda}_2 \leq \cdots \leq \hat{\lambda}_n$ be the eigenvalues of truncated matrix $\hat{A}$. Let $k = \sum_{i=1}^{n} T(i,i)$ be the number of untruncated rows in $\hat{A}$. Let $\hat{\lambda}_{n_1} \leq \hat{\lambda}_{n_2} \leq \cdots \leq \hat{\lambda}_{n_k}$ be the eigenvalues of $\hat{A}$ excluding the $n-k$ trivial eigenvalues one of $\hat{A}$ that appear due to addition of $T$ in (4.2). Then the eigenvalues of truncated and untruncated matrices are related as follows

$$\lambda_i \preceq \hat{\lambda}_{n_i} \preceq \lambda_{n-k+i}, \quad 1 \leq i \leq k.$$ 

Proof. The proof follows by an application of Poincare separation theorem. For this, it is convenient to permute the matrix into truncated and untruncated rows and columns. Let $P$ be a permutation matrix that renumbers the rows and columns such that the truncated rows and columns are numbered first, i.e.,

$$P^T \hat{A}P = \begin{pmatrix} I & R^T P^T \hat{A}P \end{pmatrix},$$

where $R \in \mathbb{R}^{n \times k}$ is the restriction matrix defined as follows

$$R = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{n-k \times k}.$$ 

Clearly $R^T R = Id \in \mathbb{R}^{k \times k}$. From Lemma 4.1 $P^T \hat{A}P$ and $\hat{A}$ are similar and $P^T AP$ and $A$ are similar. Applying Lemma 4.4 to $P^T AP$ and $R^T (P^T AP) R$, we have the proof.

Corollary 4.1. From Theorem 4.5 we have

$$\lambda_{\min}(\hat{A}) \geq \lambda_{\min}(A) > 0,$$

$$\lambda_{\max}(\hat{A}) \leq \lambda_{\max}(A),$$

hence $\hat{A}$ is SPD since $A$ is SPD from Lemma 4.3. Moreover, $\operatorname{cond}(\hat{A}) \leq \operatorname{cond}(A)$.

Remark 4.1. From (4.2) and (4.3), we have

$$\hat{A} = \begin{pmatrix} T & Id \end{pmatrix} \begin{pmatrix} A & B \\ B^T & -\eta A \end{pmatrix} \begin{pmatrix} T & Id \end{pmatrix}.$$ 

Using similar argument as in Lemma 4.5 and Cor. 4.1 we have $\lambda_{\min}(\hat{A}) \geq \lambda_{\min}(A) > 0$ and $\lambda_{\max}(\hat{A}) \leq \lambda_{\max}(A)$.

We know that the matrix $M$ is SPD, and $K$ is a SPSD. In the following, we observe the properties of truncated matrices.

Definition 4.1. Let $G(A) = (V,E)$ be the adjacency graph of a matrix $A \in \mathbb{R}^{N \times N}$. The matrix $A$ is called irreducible if any vertex $i \in V$ is connected to any vertex $j \in V$. Otherwise, $A$ is called reducible.
Definition 4.2. A matrix $A \in \mathbb{R}^{N \times N}$ is called an $M$-matrix if it satisfies the following three properties: $a_{ii} > 0$ for $i = 1, \ldots, N$, $a_{ij} \leq 0$ for $i \neq j, i, j = 1, \ldots, N$, and $A$ is non-singular and $A^{-1} \succeq 0$.

Definition 4.3. A square matrix $A$ is strictly diagonally dominant if the following holds
\[ |a_{ii}| > \sum_{j \neq i} |a_{ij}|, \quad i = 1, \ldots, N, \]
and it is called irreducibly diagonally dominant if $A$ is irreducible and the following holds
\[ |a_{ii}| \geq \sum_{j \neq i} |a_{ij}|, \quad i = 1, \ldots, N, \] (27)
where strict inequality holds for at least one $i$.

A simpler criteria for $M$-matrix property is then given by the following theorem.

Lemma 4.6. If the coefficient matrix $A$ is strictly or irreducibly diagonally dominant and satisfies the following conditions
1. $a_{ii} > 0$ for $i = 1, \ldots, N$
2. $a_{ij} \leq 0$ for $i \neq j, i, j = 1, \ldots, N$
then $A$ is an $M$-matrix.

Remark 4.2. Note that $K$ is not an $M$-matrix because $K \cdot 1 = 0$, hence, the condition of (4.3) that strict inequality must hold for at least one row is not satisfied. Moreover, mass matrix $M$ has positive off-diagonal entries, hence, it does not satisfy the hypothesis of Lemma 4.6. Thus, we cannot conclude that $M$ is an $M$-matrix either.

Although, from Lemma 4.2, $K$ is not an $M$-matrix, the truncated matrix $\tilde{K}$ defined in (4.2) with at least one truncated row and column is an $M$-matrix. Let the set of truncated nodes be defined by $N_h^* = \{ i : T(i, i) = 0 \}$.

Lemma 4.7. Let $|N_h^*| \geq 1$, then $\bar{K}$, $P^T \bar{K} P$, and $R^T P^T \bar{K} P R$ are $M$-matrices.

Proof. Since $|N_h^*| \geq 1$, for all rows corresponding to truncated set $N_h^*$, it is trivial that we have strict diagonal dominance:
\[ \hat{k}_{ii} = 1 = |\hat{k}_{ii}| > 0 = \sum_{j \neq i} \hat{k}_{ij}, \quad \forall i \in N_h^*, \quad j = 1, \ldots, |N_h|, \] (28)
where as, for rows corresponding to untruncated set $N_h \setminus N_h^*$, we have
\[ \hat{k}_{ii} = k_{ii} = |\hat{k}_{ii}| \geq \sum_{j \neq i} |k_{ij}| \geq \sum_{j \neq i} |\hat{k}_{ij}|, \quad \forall i \in N_h \setminus N_h^*, \quad j = 1, \ldots, |N_h|. \] (29)
Moreover, we have
\[ \hat{k}_{ij} = \begin{cases} 1, & \forall i \in N_h^*, \\ k_{ij} > 0, & \forall i \in N_h \setminus N_h^*, \\ k_{ij} < 0, & \forall i \in N_h. \end{cases} \] (30)
The sufficient conditions of Lemma 4.6 are now satisfied: from (4.6) and (4.6), we conclude that $\tilde{K}$ is irreducibly diagonally dominant, and (4.6) satisfies hypothesis 1. and 2. of Lemma 4.6. Hence, $\tilde{K}$ is an $M$-matrix. $P^T \tilde{K} P$ being the symmetric permutation of rows and columns of $\tilde{K}$ remains an $M$-matrix. Lastly, $R^T P^T \tilde{K} P R$ being a principle submatrix of $P^T \tilde{K} P$ is also an $M$-matrix, see proof in [13] [p. 114].

Remark 4.3. To solve with $\hat{A}$, we use the Sherman-Woodbury formula
\[ \hat{A}^+ = (\hat{K} + \tilde{m} \tilde{m}^T)^+ = \hat{K}^+ - \frac{\hat{K}^+ \tilde{m} \tilde{m}^T \hat{K}^+}{1 + \tilde{m}^T \hat{K}^+ \tilde{m}}. \]
Here $\hat{K}^+$ denotes pseudo-inverse of $\hat{K}$, however, $\hat{K}$ is a non-singular $M$-matrix for $|N_h^*| \geq 1$, thus, in this case, we may replace $\hat{K}^+$ by $\hat{K}^{-1}$. Since $\tilde{K}$ is an $M$-matrix from Lemma 4.7 above for $|N_h^*| \geq 1$, algebraic multigrid, or incomplete Cholesky (which is as stable as exact Cholesky factorization, [24][Theorem 3.2]) may be used as a preconditioner to solve with $\hat{K}$ inexactly.
Before we define a preconditioner involving Schur complement, it is essential to know whether \( S \) is nonsingular.

In the following, we provide a slightly different proof then in [13], where similar result is shown for continuous Schur complement.

**Theorem 4.2.** The negative Schur complement \( S = C + \hat{B} \hat{A}^{-1} \hat{B}^T \) is non-singular, in particular, SPD if and only if \( |N_h^\bullet| < |\mathcal{N}_h| \).

**Proof.** If \( |N_h^\bullet| = |\mathcal{N}_h| \), then \( \hat{B} \) is the zero matrix, consequently, \( S = C = \eta K \) is singular since \( K \) corresponds to stiffness matrix with pure Neumann boundary condition. For other implication, we recall from (4.2) that \( \hat{B}^T = \hat{M}^T = TM \), where \( T \) is defined in (4.2). The \((i,j)^{th}\) entry of element mass matrix is given as follows

\[
M^K_{ij} = \int_K \phi_i \phi_j dx = \frac{1}{12} (1 + \delta_{ij} |K|) \quad i, j = 1, 2, 3,
\]

where \( \delta_{ij} \) is the Kronecker symbol, that is, it is equal to 1 if \( i = j \), and 0 if \( i \neq j \). Here \( \phi_1, \phi_2, \) and \( \phi_3 \) are hat functions on triangular element \( K \) with local numbering, and \( |K| \) is the area of triangle element \( K \). From (4.6), it is easy to see that

\[
M^K = \frac{1}{12} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.
\]

Evidently, entries of global mass matrix \( M = \sum_K M^K \) are also all positive, hence all entries of truncated mass matrix \( \hat{M} \) remain non-negative. In particular, due to our hypothesis \( |N^\bullet| > 0 \), there is at least one untruncated column, hence, at least few positive entries. Consequently, \( M \mathbf{1} \neq 0 \), i.e., \( \mathbf{1} \) or span\{\( \mathbf{1} \)\} is neither in kernel of \( M \), nor in the kernel of \( \hat{M} \), in particular, \( \mathbf{1}^T \hat{M}^T \mathbf{1} > 0 \). The proof of the theorem then follows since \( C \) is SPD except on \( \mathbf{1} \) for which \( \hat{B}^T \mathbf{1} \) is non-zero, and the fact that \( \hat{A} \) is SPD yields

\[
\left\langle \hat{B} \hat{A}^{-1} \hat{B}^T \mathbf{1}, \mathbf{1} \right\rangle = \left\langle \hat{A}^{-1} (\hat{B}^T \mathbf{1}), (\hat{B}^T \mathbf{1}) \right\rangle = \left\langle \hat{A}^{-1} (-\hat{M}^T \mathbf{1}), (-\hat{M}^T \mathbf{1}) \right\rangle > 0.
\]

\( \square \)

**Remark 4.4.** The negative Schur complement \( S = \eta \hat{A} + \hat{B}^T \hat{A}^{-1} \hat{B} \) with \( \hat{A} \) defined in (4.2) is nonsingular even for \( |N_h| = |\mathcal{N}| \).

**Theorem 4.3** (Condition number of the truncated Schur complement). Following holds

- \( \lambda_{\min}(\hat{S}) > \lambda_{\min}(\hat{A}) \) and \( \lambda_{\max}(\hat{S}) < \lambda_{\max}(\hat{A}) \)

**Proof.** From [30] p. 111], following holds

\[
\lambda_{\min}(\hat{S}) > \lambda_{\min}(\hat{A}), \quad \lambda_{\max}(\hat{S}) < \lambda_{\max}(\hat{A}),
\]

and from Poincare separation theorem 4.5 we have

\[
\lambda_{\min}(\hat{A}) > \lambda_{\min}(\hat{A}), \quad \lambda_{\max}(\hat{A}) < \lambda_{\max}(\hat{A}).
\]

\( \square \)

## 5 Preconditioner for the Linear System

In this section, we propose preconditioners for the linear system for the untruncated system, and we propose the related truncated preconditioners for the truncated system.

### 5.1 Block Diagonal Preconditioner (BD)

Since \( A \), hence, \( C = \eta A, \eta > 0 \) are non-singular, assumptions of Lemma 4.1 are satisfied. Specifically, \( \theta = 1/2 \) yields

\[
\mathcal{I}_V = A + \eta^{-1/2} [A, BA^{-1} B]_{1/2}, \quad \mathcal{I}_Q = C + \eta^{1/2} [A, BA^{-1} B]_{1/2}.
\]
But \([A, BA^{-1}B]_{1/2} = B\), thus, further simplification yields
\[
\mathcal{I}_V = A + \eta^{-1/2}B, \quad \mathcal{I}_Q = \eta A + \eta^{1/2}B.
\] (32)

Choice of \(\theta = 0, 1\) in \((4.1)\) in Lemma 4.1 brings back Schur complements, but, we have avoided it. However, later we shall consider the case \(\theta = 0\). Any other intermediate value of \(\theta\) does not look interesting or useful. For large problems, it won’t be feasible to solve with \(\mathcal{I}_V\) and \(\mathcal{I}_Q\) in \((5.1)\) exactly, or not even up to double precision using prohibitively expensive direct methods such as QR or LU factorizations [11].

**Remark 5.1 (Ensuring M-matrix property of the preconditioner).** For existence and subsequent application of fast inexact solvers for \(\mathcal{I}_V\) and \(\mathcal{I}_Q\), an important property to look for is M-matrix property, but it must be pointed out that this property is not guaranteed in \((5.1)\), consequently, the diagonal dominance of \(\mathcal{I}_V\) or \(\mathcal{I}_Q\) may be lost for certain values of \(\eta\). To sketch the proof for \(\mathcal{I}_Q\), we observe that
\[
A^K_{ij} = \left( \int_K \nabla \phi_i \cdot \nabla \phi_j dx + \int_K \phi_i \phi_j dx \int_K \phi_j dx \right), \quad i, j = 1, 2, 3,
\]
\[
= (b_ib_j + c_i c_j) \int_K dx + mm^T = (b_ib_j + c_i c_j)|K| + mm^T, \quad i, j = 1, 2, 3,
\]
where \(|K|\) is the area of triangle element \(K\), and
\[
b_i = \frac{x^k_i - x^k_j}{2|K|}, \quad c_i = \frac{x^k_j - x^k_i}{2|K|}, \quad \{j, k\} \in \{1, 2, 3\},
\]
where \((x^k_i, x^k_j), i = 1, 2, 3\) are coordinates of the three vertices of element \(K\). The \((i, j)\)th entry of element mass matrix is given in \((4.6)\). Evidently, entries of global mass matrix \(M = \sum_K M^K\) are also all positive. We have
\[
\eta A^K_{ij} + \eta^{1/2}B^K_{ij} = \eta A^K_{ij} + \eta^{1/2}M^K_{ij} = \eta(b_ib_j + c_i c_j)|K| + \eta m^T + \eta^{1/2} \frac{1}{12} (1 + \delta_{ij})|K|.
\]
Thus, the off-diagonal entries of \(\mathcal{I}_Q\) may become positive, due to addition of the mass matrix \(M\) for certain values of \(\eta\), thereby violating the sufficient condition of Lemma 4.6 for \(\mathcal{I}_Q\) to be an M-matrix. However, the M-matrix property of \(\mathcal{I}_V\) is ensured by lumping the mass matrix: we proved earlier in Lemma 4.7 that the truncated matrix \(\bar{K}\) is an M-matrix if there is at least one truncated node, addition of lumped mass matrix further enhances the diagonal dominance of \(\mathcal{I}_Q\), and does not violate sufficient condition of Lemma 4.6. Similarly, \(\mathcal{I}_V = \eta \mathcal{I}_Q\) can be kept M-matrix. Hence, algebraic multigrid may be used to solve with \(\mathcal{I}_V\) and \(\mathcal{I}_Q\).

The following eigenvalue bound is similar to the one in [32]. Our system matrix is different in that in place of \(K\) we have \(\bar{K}\) and in place of \(M\) we have \(\bar{K}\). Consequently, for our system, the bound is slightly tighter in the sense that the eigenvalues lie in the open interval as shown below, whereas, in [32] they lie in the closed interval.

**Theorem 5.1 (Eigenvalue bound for \(B_{\infty}^{-1}A\)).** There holds
\[
\lambda \left( \begin{bmatrix} \bar{K} + \eta^{-1/2}M & 0 \\ 0 & \eta\bar{K} + \eta^{1/2}M \end{bmatrix} ^{-1} \begin{bmatrix} \bar{K} & M \\ M & -\eta\bar{K} \end{bmatrix} \right) \in \left( -1, -\frac{1}{\sqrt{2}} \right) \cup \left( \frac{1}{\sqrt{2}}, 1 \right).
\]

*Proof.* We first consider the generalized eigenvalue problem
\[
\bar{K}z = \mu(\bar{K} + \eta^{-1/2}M)z. \tag{33}
\]
Since \(\bar{K}\) and \(\bar{K} + \eta^{-1/2}M\) are SPD, there is a basis \(e_1, e_2, \ldots,\) of eigenvectors \(e_i\) with corresponding eigenvalues \(\mu_i \in (0, 1)\), which are orthonormal with respect to the \(\bar{K} + \eta^{-1/2}M\) inner product. This is easily seen by looking at the Rayleigh quotient
\[
0 < \frac{\langle x, x \rangle_{\bar{K}}} {\langle x, x \rangle_{\bar{K} + \eta^{-1/2}M}} = \frac{x^T \bar{K}x} {x^T \bar{K}x + \eta^{-1/2}x^T Mx} < 1 \quad \forall x \text{ s.t. } \|x\|_{\bar{K} + \eta^{-1/2}M} = 1,
\]
since \(x^T \bar{K}x + \eta^{-1/2}x^T Mx \geq x^T \bar{K}x, \forall x\). We now look at the following generalized eigenvalue problem
\[
\begin{bmatrix} \bar{K} & M \\ M & -\eta\bar{K} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} \bar{K} + \eta^{-1/2}M & 0 \\ 0 & \eta\bar{K} + \eta^{1/2}M \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{34}
\]
Proof. Follows from Theorem 5.1.

Corollary 5.1 (Condition number estimate for \( B_{bd}^{-1}A \)). The condition number is given as follows

\[
\kappa \left( \begin{bmatrix} \hat{K} + \eta^{-1/2}M & 0 \\ 0 & \eta\hat{K} + \eta^{1/2}M \end{bmatrix} \right)^{-1} \begin{bmatrix} \hat{K} & M \\ M & -\eta\hat{K} \end{bmatrix} \right) < \sqrt{2}.
\]

Proof. Follows from Theorem 5.1. \( \square \)

Our goal is to solve truncated problem. We want to bound the extreme eigenvalues of the preconditioned truncated matrix by those for the preconditioned untruncated matrix. To this end, following theorem is useful.
Lemma 5.1. (Fischer) [p. 281, [31]] Let \( X = \mathbb{R}^n \), where \( n \) is some positive integer. Let \( A, B \in \mathbb{R}^{n \times n} \) be any two Hermitian matrices and let \( B \) be SPD. Let the eigenvalues of \( B^{-1}A \) be ordered as follows \( \lambda_{\max} = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = \lambda_{\min} \) (Note: such an ordering is possible because the eigenvalues of \( B^{-1}A \) are real, since \( B^{-1}A \) is similar to a symmetric matrix \( B^{-1/2}AB^{-1/2} \)). Then

\[
\lambda_i = \max_{\dim(X) = i} \min_{x \neq 0} \frac{x^T Ax}{x^T Bx},
\]

and

\[
\lambda_i = \max_{\dim(X) = n-i+1} \min_{x \neq 0} \frac{x^T Ax}{x^T Bx}.
\]

In particular, there holds

\[
\lambda_{\min}(B^{-1}A) = \min_{x \neq 0} \frac{x^T Ax}{x^T Bx}, \quad \lambda_{\max}(B^{-1}A) = \max_{x \neq 0} \frac{x^T Ax}{x^T Bx}.
\]

Lemma 5.2 (Bound on extreme eigenvalues of the preconditioned truncated matrix). The non-zero extreme eigenvalues of the preconditioned truncated operator with at least one truncation are bounded from above and below by the eigenvalues of the preconditioned untruncated operator.

Proof. Let \( P \) be a permutation matrix that permutes the rows and columns such that the truncated nodes are numbered first. Let \( T \) be the truncation matrix as in (4.2), and let \( R \) be a restriction operator as in (4.4) that compresses the matrix to untruncated nodes. We use the following notation for compressed matrices

\[
\tilde{A} = R^T TP^T APTR, \quad \tilde{B}_{bd} = R^T TP^T B_{bd} PTR.
\]

Let \( Z = PTR \). To use Lemma 5.1, we note that \( B_{bd} \) is SPD, hence from Poincaré separation theorem, i.e., from Lemma 4.4, \( \tilde{B}_{bd} \) is SPD. Alternatively, \( \tilde{B}_{bd} \) being a principle submatrix of \( P^T B_{bd} P \) is SPD. Since \( \dim(\text{Range}(TR)) \leq |N_h| \), we clearly have

\[
\lambda_{\min}(\tilde{B}^{-1}_{bd} \tilde{A}) = \min_{x, z \neq 0, x \in \mathbb{R}^{N_h}} \frac{x^T \tilde{A} x}{x^T \tilde{B}_{bd} x} = \min_{x, z \neq 0, x \in \mathbb{R}^{N_h}} \frac{(Px)^T A(Px)}{(Px)^T \tilde{B}_{bd} (Px)} = \min_{x \neq 0, x \in \mathbb{R}^{N_h}} \frac{x^T Ax}{x^T \tilde{B}_{bd} x} = \lambda_{\min}(B_{bd}^{-1} A).
\]

Similarly, we have

\[
\lambda_{\max}(\tilde{B}^{-1}_{bd} \tilde{A}) = \max_{x, z \neq 0, x \in \mathbb{R}^{N_h}} \frac{x^T \tilde{A} x}{x^T \tilde{B}_{bd} x} = \max_{x, z \neq 0, x \in \mathbb{R}^{N_h}} \frac{(Px)^T A(Px)}{(Px)^T \tilde{B}_{bd} (Px)} = \max_{x \neq 0, x \in \mathbb{R}^{N_h}} \frac{x^T Ax}{x^T \tilde{B}_{bd} x} = \lambda_{\max}(B_{bd}^{-1} A).
\]

Remark 5.2. Due to Lemma 5.2 above, an optimal preconditioner \( \tilde{B}_{bd} \) for the truncated system \( \tilde{A} \) is given as follows

\[
\tilde{B}_{bd} = \begin{pmatrix} \hat{K} + \eta^{-1/2} \hat{M} & \eta \hat{K} + \eta^{1/2} \hat{M} \\ \eta \hat{K} + \eta^{1/2} \hat{M} & \eta \hat{K} + \eta^{1/2} \hat{M} \end{pmatrix},
\]

where \( \hat{M} = TM \), where \( T \) is defined in (4.2), and \( \hat{K} \) and \( \hat{K} \) are defined in (4.2).

For comparison, we consider block triangular preconditioners of the form used in Bosch et. al. [3]. In the following, we briefly describe this preconditioner in our notation.
5.2 Block Tridiagonal Schur Complement Preconditioner (BTDSC)

In Bosch et. al. [5], a preconditioner is proposed in the framework of a semi-smooth Newton method combined with Moreau-Yosida regularization for the same problem. However, the preconditioner was constructed for a linear system which is different from the one we consider here in (4.2). The preconditioner proposed in [5] has the following block lower triangular form

\[ \mathcal{P}_{\text{btdsc}} = \begin{pmatrix} \bar{K} & 0 \\ M & -S \end{pmatrix}, \tag{39} \]

where \( S = \eta \bar{K} + M \bar{K}^{-1} M^T \) is the negative Schur complement. From Lemma 4.3, \( \bar{K} \) is SPD, hence, invertible and from Remark 4.4, \( S \) is also invertible. Hence by block 2 \times 2 inversion formula, we have

\[ \mathcal{P}_{\text{btdsc}}^{-1} = \begin{pmatrix} \bar{K} & 0 \\ M & -S \end{pmatrix}^{-1} = \begin{pmatrix} \bar{K}^{-1} & 0 \\ S^{-1} M^T \bar{K}^{-1} & -S^{-1} \end{pmatrix}. \]

Let \( S_{\text{pre}} \) be an approximation of Schur complement \( S \) in \( \mathcal{P}_{\text{btdsc}} \) in (5.2), then the new preconditioner \( \mathcal{B}_{\text{btdsc}} \), and the corresponding preconditioned operator \( \mathcal{B}_{\text{btdsc}}^{-1} \mathcal{A} \) are given as follows

\[ \mathcal{B}_{\text{btdsc}} = \begin{pmatrix} \bar{K} & 0 \\ M & -S_{\text{pre}} \end{pmatrix}, \quad \mathcal{B}_{\text{btdsc}}^{-1} \mathcal{A} = \begin{pmatrix} I & \bar{K}^{-1} M^T \\ 0 & S_{\text{pre}}^{-1} \end{pmatrix}. \tag{40} \]

In this paper, we choose a preconditioner \( S_{\text{pre}} \) for \( S \) as follows

\[ S_{\text{pre}} = (M + \sqrt{\eta} \bar{K}) \bar{K}^{-1} (M + \sqrt{\eta} \bar{K}) = (\eta \bar{K} + M \bar{K}^{-1} M) + 2 \sqrt{\eta} M = S + 2 \sqrt{\eta} M. \tag{41} \]

Such preconditioners had been used before for example, in [5, 27]. We note the following trivial result.

**Lemma 5.3.** \( S_{\text{pre}} \) is SPD.

**Proof.** Follows from (5.2) and from Theorem 4.4 that \( M \) and \( S \) are SPD. \qed

In view of (5.2), the following fact follows.

**Fact 5.1.** Let \( \mathcal{B}_{\text{btdsc}} \) be defined as in (5.2), then there are \( |\mathcal{N}_h| \) eigenvalues of \( \mathcal{B}_{\text{btdsc}}^{-1} \mathcal{A} \) equal to one, and the rest are the eigenvalues of the preconditioned Schur complement \( S_{\text{pre}}^{-1} S \).

In view of Fact 5.1, it is sufficient to estimate eigenvalues of the preconditioned Schur complement. Using (5.2) and the fact that both \( S_{\text{pre}} \) and \( S \) are SPD from Lemma 5.3 and from Lemma 4.4 respectively, looking at the Rayleigh quotient with \( v^T v = 1, v \in \mathbb{R}^{\mathcal{N}_h} \), and using the fact that \( \bar{K} \) and \( M \) are SPD, consequently, \( \eta \bar{K} + M \bar{K}^{-1} M \) is SPD, we have

\[ \frac{v^T (S) v}{v^T (S_{\text{pre}}) v} = \frac{v^T (\eta \bar{K} + M \bar{K}^{-1} M) v}{v^T (\eta \bar{K} + M \bar{K}^{-1} M) v + 2 \sqrt{\eta} v^T M v} = \frac{1}{1 + Z}, \]

where

\[ Z = \frac{2 v^T \sqrt{\eta} M v}{v^T (\eta \bar{K} + M \bar{K}^{-1} M) v}. \]

We have

\[ \min_v Z = \min_v \frac{2 \sqrt{\eta} \cdot v^T M v}{v^T (\eta \bar{K} + M \bar{K}^{-1} M) v} = \min_v \frac{2 \sqrt{\eta}}{\eta v^T M^{-1} K v + v^T K^{-1} M v} \]

\[ = \min_v \frac{2 \sqrt{\eta}}{K^{1/2} \cdot \eta v^T M^{-1} K v + v^T K^{-1} M v} \cdot K^{-1/2} \]

\[ = \min_v \frac{2 \sqrt{\eta}}{\eta v^T K^{1/2} M^{-1} K^{1/2} v + v^T K^{-1/2} M K^{-1/2} M v} \]

\[ = \min_v \frac{2 u^T w}{u^T u + w^T w}, \]

for \( u = K^{1/2} v, w = K^{-1/2} M v \).
where \( u = \sqrt{\eta}M^{-1/2}\tilde{K}^{-1/2}v \) and \( w = M^{1/2}\tilde{K}^{-1/2}v \). Similarly,

\[
\max_v Z = \max_v \frac{2u^T w}{u^T u + w^T w}.
\]

Since \( \eta > 0 \), \((u - w)^T(u - w) \geq 0\), and that \( u^T u + w^T w > 0\), we clearly have

\[
0 \leq \frac{2u^T w}{u^T u + w^T w} \leq 1,
\]

which leads to the following bounds

\[
\frac{1}{2} \leq \lambda_{\min}(S_{\text{pre}}^{-1}S) = \min_{v \neq 0} \frac{v^T(-S)v}{v^T(-S_{\text{pre}})v} \leq \max_{v \neq 0} \frac{v^T(-S)v}{v^T(-S_{\text{pre}})v} = \lambda_{\max}(S_{\text{pre}}^{-1}S) < 1.
\]

We note this result as theorem below.

**Theorem 5.2.** The eigenvalues of the preconditioned untruncated system \( S_{\text{pre}}^{-1}S \) satisfies

\[
\lambda(S_{\text{pre}}^{-1}S) \in [1/2, 1).
\]

**Corollary 5.2.** The condition number is bounded as follows

\[
\kappa(S_{\text{pre}}^{-1}S) < 2.
\]

**Remark 5.3.** When using GMRES [27], right preconditioning is preferred. As in Theorem 5.1, similar estimate for the right preconditioned matrix \( SS_{\text{pre}}^{-1} \) holds, because both \( S_{\text{pre}}^{-1}S \) and \( S_{\text{pre}}^{-1}S_{\text{pre}} \) are similar to a symmetric matrix \( S_{\text{pre}}^{-1/2}SS_{\text{pre}}^{-1/2} \).

Let \( x = [x_1, x_2], b = [b_1, b_2] \). The preconditioned system \( B_{\text{btdec}}^{-1}Ax = B_{\text{btdec}}^{-1}b \) is given as follows

\[
\begin{pmatrix}
I & \tilde{K}^{-1}M^T \\
0 & S_{\text{pre}}^{-1}S
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
=
\begin{pmatrix}
\tilde{K}^{-1} \\
S_{\text{pre}}^{-1}M^T\tilde{K}^{-1}
\end{pmatrix}
\begin{pmatrix}
b_1 \\
b_2
\end{pmatrix}
\]

from which we obtain the following set of equations

\[
x_1 + \tilde{K}^{-1}M^T x_2 = \tilde{K}^{-1}b_1, \quad S_{\text{pre}}^{-1}S x_2 = S_{\text{pre}}^{-1}(M^T\tilde{K}^{-1}b_1 - b_2).
\]

**Algorithm 5.1.** Objective: Solve \( B_{\text{btdec}}^{-1}Ax = B^{-1}b \)

1. Solve for \( x_2 : S_{\text{pre}}^{-1}S x_2 = S_{\text{pre}}^{-1}(M^T\tilde{K}^{-1}b_1 - b_2) \)
2. Set \( x_1 = \tilde{K}^{-1}(b_1 - M^T x_2) \)

Here if Krylov subspace method is used to solve for \( x_2 \), then matrix vector product with \( S \) and a solve with \( S_{\text{pre}} \) is needed. However, when the problem size, i.e., \( |N| \) is large, it won’t be feasible to do exact solve with \( \tilde{K} \), and we need to solve it inexactly, for example, using algebraic multigrid methods. In the later case, the decoupling of \( x_1 \) and \( x_2 \) as in Algorithm 5.1 is not possible; then we use GMRES [29] p. 269 preconditioned by \( B_{\text{btdec}} \).

In view of Fact 5.1 and Theorem 5.2, we already have eigenvalue estimates for \( B_{\text{btdec}}^{-1}A \), however, as before, we can derive the eigenvalue bound and condition number estimate for \( B_{\text{btdec}}^{-1}A \) directly without explicitly reducing it to Schur complement system. To this end, we consider again the related generalized eigenvalue problem

\[
\begin{pmatrix}
\tilde{K} & M \\
M & -\eta\tilde{K}
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
=
\lambda
\begin{pmatrix}
\tilde{K} \\
M
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}.
\]

Note that we have rewritten \( S_{\text{pre}} \) in (5.2) as follows

\[
S_{\text{pre}} = -\eta(\tilde{K} + \eta^{-1/2}M)\tilde{K}^{-1}(\tilde{K} + \eta^{-1/2}M).
\]

From (5.2), we have

\[
\tilde{K}u + Mv = \lambda\tilde{K}u
\]

\[
Mu - \eta\tilde{K}v = \lambda(Mu - \eta(\tilde{K} + \eta^{-1/2}M)\tilde{K}^{-1}(\tilde{K} + \eta^{-1/2}M)v).
\]
As before, we consider the eigenvalue problem \([5.1]\) with the eigenbasis \([e_1, e_2, \ldots]\) which are orthonormal w.r.t. \(K + \eta^{-1/2}M\) inner product. Expanding \(u\) and \(v\) in eigenbasis \([e_1, e_2, \ldots]\) as in \([5.1]\), and looking at the \(i\)th columns of these equations, we get
\[
\hat{u}_i Ke_i + \hat{v}_i Me_i = \lambda \hat{u}_i Ke_i
\]
\[
\hat{u}_i Me_i - \eta \hat{v}_i Ke_i = \lambda (\hat{u}_i Me_i - \eta \hat{v}_i (K + \eta^{-1/2}M) \hat{K}^{-1}(K + \eta^{-1/2}M)e_i).
\]  
Again from \([5.1]\)
\[
Me_i = \eta^{1/2}(1 - \mu_i)\mu_i^{-1} Ke_i.
\]
Substituting \(Me_i\) from above in two equations of \([5.2]\), we have
\[
\hat{u}_i Ke_i + \hat{v}_i \eta^{1/2}(1 - \mu_i)\mu_i^{-1} \hat{K} e_i = \lambda \hat{u}_i Ke_i
\]
\[
\hat{u}_i \eta^{1/2}(1 - \mu_i)\mu_i^{-1} \hat{K} e_i - \eta \hat{v}_i \hat{K} e_i = \lambda \hat{u}_i \eta^{1/2}(1 - \mu_i)\mu_i^{-1} \hat{K} e_i - \lambda \eta \mu_i^{-2} \hat{K} e_i \hat{v}_i.
\]
Multiplying by \(e_i^T\) from the left and dividing by \(\hat{e}_i^T \hat{K} e_i \neq 0\) throughout, we have
\[
\hat{u}_i + \eta^{1/2}(1 - \mu_i)\mu_i^{-1} \hat{v}_i = \lambda \hat{u}_i
\]
\[
\eta^{1/2}(1 - \mu_i)\mu_i^{-1} \hat{u}_i - \eta \hat{v}_i = \lambda \eta^{1/2}(1 - \mu_i)\mu_i^{-1} \hat{u}_i - \lambda \eta \mu_i^{-2} \hat{v}_i.
\]
Rearranging above,
\[
\begin{pmatrix}
1 \\
\eta^{1/2}(1 - \mu_i)\mu_i^{-1}(1 - \lambda) \\
\eta(1 - \mu_i)^2 \mu_i^{-2}
\end{pmatrix}
\begin{pmatrix}
\hat{u}_i \\
\hat{v}_i
\end{pmatrix}
= \lambda
\begin{pmatrix}
1 \\
\eta^{1/2}(1 - \mu_i)\mu_i^{-1} \\
\eta(1 - \mu_i^2)
\end{pmatrix}
\begin{pmatrix}
\hat{u}_i \\
\hat{v}_i
\end{pmatrix}.
\]
There exists at least one \(i\) such that
\[
\det
\begin{pmatrix}
1 - \lambda \\
\eta^{1/2}(1 - \mu_i)\mu_i^{-1}(1 - \lambda) \\
\eta(1 - \mu_i^2)
\end{pmatrix}
= 0,
\]
which implies
\[
(1 - \lambda \mu_i^{-2}) + (1 - \mu_i)^2 \mu_i^{-2} = 0
\]
\[
\Rightarrow \lambda = \mu_i^2 + (1 - \mu_i)^2.
\]
The function \(f(\mu_i) = \mu_i^2 + (1 - \mu_i)^2\) has a critical point at \(\mu_i = 1/2\), and \(f(\cdot)\) monotonically decreases from 1 to 1/2 for \(\mu_i \in (0, 1/2]\), and monotonically increases from 1/2 to 1 for \(\mu_i \in [1/2, 1]\). All this leads to the following bound.

**Theorem 5.3.** There holds
\[
\lambda(B_{\text{pre}}^{-1} A) \in [1/2, 1].
\]

**Corollary 5.3.** The condition number satisfies the following bound
\[
\kappa(B_{\text{pre}}^{-1} A) < 2.
\]

**Remark 5.4 (Relation between eigenvalues of truncated and untruncated system).** We have two cases

1. (1, 1) block is solved inexactly: as mentioned before, in this case, the preconditioner is block tridiagonal hence unsymmetric, consequently, Fischer theorem cannot be used to show relation between truncated and untruncated system

2. (1, 1) block is solved exactly: in this case, the problem reduces to Schur complement system, and due to Lemma \([4.4]\), the truncated Schur complement remains SPD. The preconditioner for truncated Schur complement \(\hat{S}\) is defined below
\[
\hat{S}_{\text{pre}} = (\hat{M} + \sqrt{\eta \hat{K}}) \hat{K}^{-1} (\hat{M} + \sqrt{\eta \hat{K}}) = \eta \hat{K} + \hat{M} \hat{K}^{-1} \hat{M}^T + \sqrt{\eta \hat{M}} + \sqrt{\eta \hat{K}} \hat{K}^{-1} \hat{M}^T
\]
\[
= \hat{S} + \sqrt{\eta} (\hat{M} + \hat{K} \hat{K}^{-1} \hat{M}^T).
\]
First, it is not evident whether \(\hat{S}_{\text{pre}}\) is similar to a symmetric matrix. If it is, then we want to know whether the following holds
\[
\lambda_{\max}(\hat{S}_{\text{pre}}^{-1} \hat{S}) \leq \lambda_{\max}(S_{\text{pre}}^{-1} S), \quad \lambda_{\min}(\hat{S}_{\text{pre}}^{-1} \hat{S}) \geq \lambda_{\min}(S_{\text{pre}}^{-1} S).
\]
We leave this as a subject of future work. Since \(S_{\text{pre}}\) may be unsymmetric, we shall use \([2]\) with GMRES that allows unsymmetric preconditioners.
5.3 Block Diagonal Schur Complement Preconditioner (BDSC)

Substituting $\theta = 0$ in (4.1), we obtain a block diagonal preconditioner involving Schur complement as follows

$$B_{\text{bdsc}} = \begin{pmatrix} 2\bar{K} & \bar{S} \\ \bar{S} & K \end{pmatrix} \sim \begin{pmatrix} \bar{K} & \bar{S} \\ \bar{S} & S \end{pmatrix},$$

where $S = \eta\bar{K} + M\bar{K}^{-1}M$. Once again $S$ is approximated by $S_{\text{pre}}$ as before.

**Remark 5.5.** As in Lemma 5.2, we have

$$\lambda_{\min}(\bar{B}_{\text{bdsc}}^{-1}\bar{A}) \geq \lambda_{\min}(B_{\text{bdsc}}^{-1}A), \quad \lambda_{\max}(\bar{B}_{\text{bdsc}}^{-1}\bar{A}) \geq \lambda_{\max}(B_{\text{bdsc}}^{-1}A),$$

which suggests the following optimal preconditioner for $\bar{A}$

$$\hat{B}_{\text{bdsc}} = \begin{pmatrix} \bar{K} \\ S \end{pmatrix},$$

moreover, due to the spectral equivalence of $S_{\text{pre}}$ and $S$ established in (5.2), we propose the following preconditioner using same notation

$$\hat{B}_{\text{bdsc}} = \begin{pmatrix} \bar{K} \\ S_{\text{pre}} \end{pmatrix}.$$

In practice, we shall replace $S_{\text{pre}}$ by $\hat{S}_{\text{pre}}$ defined in (2).

As before, we consider the eigenvalue problem (5.1) with the eigenbasis $\{e_1, \ldots, \}$ which are orthonormal w.r.t. $\bar{K} + \eta^{-1/2}M$ inner product. Consider the following generalized eigenvalue problem

$$\begin{pmatrix} \bar{K} & M \\ M & -\bar{K} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} \eta(\bar{K} + \eta^{-1/2}M)\bar{K}^{-1}(\bar{K} + \eta^{-1/2}M) \\ \eta(\bar{K} + \eta^{-1/2}M)\bar{K}^{-1}(\bar{K} + \eta^{-1/2}M) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

which leads to

$$\bar{K}u + Mu = \lambda\bar{K}u$$

$$Mu - \eta\bar{K}v = \lambda\eta(\bar{K} + \eta^{-1/2}M)\bar{K}^{-1}(\bar{K} + \eta^{-1/2}M)v.$$  \hspace{1cm} (48)

As before, expanding $u$ and $v$ in eigenbasis $\{e_1, e_2, \ldots, \}$:

$$u = \sum_i \hat{u}_ie_i, \quad v = \sum_i \hat{v}_ie_i,$$

and substituting $u$ and $v$ from above in (5.3), and looking at the $i$th rows of both equations, we have

$$\hat{u}_i\bar{K}e_i + \hat{v}_iMe_i = \lambda\hat{u}_i\bar{K}e_i$$

$$\hat{u}_iMe_i - \eta\hat{v}_i\bar{K}e_i = \lambda\eta(\bar{K} + \eta^{-1/2}M)\bar{K}^{-1}\hat{v}_i(\bar{K} + \eta^{-1/2}M)e_i.$$  \hspace{1cm} (49)

Again from (5.1)

$$Me_i = \eta^{1/2}(1 - \mu_i)\mu_i^{-1}\bar{K}e_i.$$  \hspace{1cm} (50)

Substituting $Me_i$ from above in (5.3), we have

$$\hat{u}_i\bar{K}e_i + \hat{v}_i\eta^{1/2}(1 - \mu_i)\mu_i^{-1}\bar{K}e_i = \lambda\hat{u}_i\bar{K}e_i$$

$$\hat{u}_i\eta^{1/2}(1 - \mu_i)\mu_i^{-1}\bar{K}e_i - \eta\hat{v}_i\bar{K}e_i = \lambda\eta(\bar{K} + \eta^{-1/2}M)\bar{K}^{-1}\hat{v}_i(\bar{K}e_i + \eta^{-1/2}\eta^{1/2}(1 - \mu_i)\mu_i^{-1}\bar{K}e_i)$$

$$= \lambda\eta(\bar{K} + \eta^{-1/2}M)(e_i + (1 - \mu_i)\mu_i^{-1}e_i)\hat{v}_i$$

$$= \lambda\eta(\bar{K} + \eta^{-1/2}M)e_i(1 + (1 - \mu_i)\mu_i^{-1})\hat{v}_i$$

$$= \lambda\eta(1 + (1 - \mu_i)\mu_i^{-1})\mu_i^{-1}\bar{K}e_i\hat{v}_i, \quad \text{from (5.1)}.$$  \hspace{1cm} (51)
The equation (5.3) has two roots as follows
\[ \hat{u}_i + \hat{v}_i \eta^{1/2} (1 - \mu_i) \mu_i^{-1} = \lambda \hat{u}_i \]
\[ \hat{u}_i \eta^{1/2} (1 - \mu_i) \mu_i - \eta \mu_i^2 \hat{v}_i = \lambda \hat{v}_i , \]
writing in matrix form, we obtain
\[
\begin{pmatrix}
1 & \eta^{1/2} (1 - \mu_i) \mu_i^{-1} \\
\eta^{1/2} (1 - \mu_i) \mu_i & -\eta \mu_i^2
\end{pmatrix} = \lambda
\begin{pmatrix}
1 \\
\eta
\end{pmatrix}
\begin{pmatrix}
\hat{u}_i \\
\hat{v}_i
\end{pmatrix} .
\]
Since \( u \) and \( v \) are eigenvectors, there exists at least one \( i \) such that following holds
\[
\det\left( \begin{pmatrix}
1 & \eta^{1/2} (1 - \mu_i) \mu_i^{-1} \\
\eta^{1/2} (1 - \mu_i) \mu_i & -\eta \mu_i^2
\end{pmatrix} - \lambda \begin{pmatrix}
1 \\
\eta
\end{pmatrix} \right) = 0
\]
\[
\Rightarrow -\eta (1 - \lambda) (\mu_i^2 + \lambda) - \eta (1 - \mu_i)^2 = 0
\]
\[
\Rightarrow (1 - \lambda) (\mu_i^2 + \lambda) + (1 - \mu_i)^2 = 0
\]
\[
\Rightarrow \mu_i^2 + \lambda - \mu_i^2 - \lambda^2 + 1 + \mu_i^2 - 2 \mu_i = 0
\]
\[
\Rightarrow -\lambda^2 + \lambda (1 - \mu_i^2) + 2 \mu_i^2 - 2 \mu_i + 1 = 0 .
\] (52)

The equation (5.3) has two roots as follows
\[
\lambda_1 (\mu_i) = \frac{1 - \mu_i^2}{2} + \frac{\mu_i^4 + 6 \mu_i^2 - 8 \mu_i + 5}{2} ,
\lambda_2 (\mu_i) = \frac{1 - \mu_i^2}{2} - \frac{\mu_i^4 + 6 \mu_i^2 - 8 \mu_i + 5}{2} ,
\] (53)

with the constraints that \( \mu_i \in (0, 1] \). The critical points of the first equation in (5.3) is given by the roots of the following equation
\[
\frac{d \lambda_1}{d \mu_i} = (4 \mu_i^3 + 12 \mu_i - 8) / 4 (\mu_i^4 + 6 \mu_i^2 - 8 \mu_i + 5)^{1/2} - \mu_i = 0.
\]

The roots are \( \mu_i = 1, -\sqrt{2} - 1 \), where the last one is discarded since it is outside the constraint interval \( (0, 1] \). Since only the boundary points are critical points, \( \lambda_1 \) is either monotonically increasing or monotonically decreasing, but by checking, we have \( \lambda_1 (0) = (\sqrt{5} + 1) / 2, \lambda_1 (1) = 1 \), thus, \( \lambda_1 \) is monotonically decreasing for \( \mu_i \in (0, 1] \). Thus \( \lambda_1 \in [1, (\sqrt{5} + 1) / 2] \). Similarly, we now consider the second root \( \lambda_2 \) in (5.3) whose critical points are given by the roots of
\[
\frac{d \lambda_2}{d \mu_i} = -\mu_i - (4 \mu_i^3 + 12 \mu_i - 8) / (4 (\mu_i^4 + 6 \mu_i^2 - 8 \mu_i + 5)^{1/2}) = 0 ,
\]
and it has repeated roots \( \mu_i = \sqrt{2} - 1 \). To determine whether it is a maxima or minima, we consider
\[
\frac{d^2 \lambda_2}{d \mu_i^2} = (4 \mu_i^3 + 12 \mu_i - 8) / (8 (\mu_i^4 + 6 \mu_i^2 - 8 \mu_i + 5)^{3/2}) - (12 \mu_i^2 + 12) / (4 (\mu_i^4 + 6 \mu_i^2 - 8 \mu_i + 5)^{1/2}) - 1 ,
\]
which is negative for \( \mu_i = \sqrt{2} - 1 \), thus, it is a maxima for which \( \lambda_2 \) attains the value \( 1 - \sqrt{2} \). Since there are no other critical points, the minima must occur at one of the two boundaries of \( (0, 1] \). For \( \mu_i = 0, \lambda_2 = (1 - \sqrt{5}) / 2 \), and for \( \mu_i = 1, \lambda_2 = -1 \). Thus we have the following bound for eigenvalues.

**Theorem 5.4 (Eigenvalue bounds of \( B_{bdsc}^{-1} A \)).** There holds
\[
\lambda (B_{bdsc}^{-1} A) \in [-1, 1 - \sqrt{2}] \cup [1, (\sqrt{5} + 1) / 2] .
\]

The condition number estimate then follows.

**Corollary 5.4 (Condition number of \( B_{bdsc}^{-1} A \)).** There holds
\[
\kappa (B_{bdsc}^{-1} A) < \frac{\sqrt{5} + 1}{2 (\sqrt{2} - 1)} \approx 3.90.
\]
6 Numerical Experiments

All the experiments were performed in double precision arithmetic in MATLAB. A fixed number of 12 Uzawa iterations per time step is executed. The obstacle problem is solved using monotone multigrid. For the linear subproblem, the Krylov solver used was restarted GMRES with inner subspace dimension of 60, and maximum number of iterations allowed was 300. The iteration was stopped as soon as the relative residual was below the tolerance of $10^{-7}$. The local sub-blocks of the preconditioner was solved using aggregation based AMG; the stopping criteria for AMG was decrease of relative residual below $10^{-7}$. Three test cases are considered

- Evolution of square
- Evolution of randomly mixed phases
- Randomly truncated systems

We describe the numerical experiments with each of these test cases below.

6.1 Experiments with Various Evolutions

In both the evolution problems, we chose $\epsilon = 2 \times 10^{-2}$ and $\tau = 10^{-5}$. We consider the mesh sizes $h = 1/256, 1/400$ with 66049 and 160801 nodes respectively. In the Tables 1, 2, and 3, we show the number of truncations denoted by $\#trunc$, and percentage of truncations denoted by $\%trunc$ during evolutions. We recall from (4.2), that we need to solve twice, since, we use Sherman-Woodbury inversion [10] (2.1.5), p. 65): in the tables, the iteration counts for the first solve is denoted by it1, and those for the second solve is denoted by it2. The time in the table denotes the total time in seconds for both these solves. We compare three preconditioners: bd, bdsc, and btdsc.

6.1.1 Evolution of Randomly Mixed Phases

In this test case, we take initial solution $u$ to have random values between -0.3 and 0.5 except for two pure phases of $u(1) = 1$ and $u(\text{end}) = -1$. In Figure 1(a), we show the initial active set configuration. The evolution for various time steps are shown in Figures 1(a) to 1(j). For this test case, already at time step $\tau = 80$, about half of the nodes are truncated; suggesting fast separation initially. The iteration counts for btdsc is the least. Except for $\#trunc=2$, btdsc has the least CPU time of all three preconditioners. Although, bdsc has slightly less iterations than bd, the CPU times are large compared to that for bd, especially, initially when the number of truncations are less. The larger CPU times are attributed to the fact that bdsc requires three elliptic solves and one matrix vector product, whereas, bd requires only two elliptic solves. Being a block tridiagonal preconditioner, btdsc has more costs compared to bd and bdsc, but since the iteration counts for btdsc is almost half of those for bd and bdsc, it converges significantly faster. For this evolution, although truncations increase, the iteration counts remain steady during various time steps for all three preconditioners. We observe that initial fast dynamics of phase separation later slows down after about $\tau = 120$, when we do not see any significant increase in truncations. This suggests that the system remains structurally and spectrally similar, this is suggested by the iteration count that remains almost constant after $\tau = 120$ for all three methods.

6.1.2 Evolution of Square

In this test case, we consider evolution of a square with a diffuse interface. The initial active set configuration in Figure 2(a) is obtained by two squares; the innermost square is prescribed by the lower left and upper right diagonal ends with coordinates (0.25, 0.25) and (0.75, 0.75), and the outermost square is defined by the coordinates of the diagonal ends joining (0.25 $- 10h^2$, 0.25 $- 10h^2$) and (0.75 $+ 10h^2$, 0.75 $+ 10h^2$). Thus the diffuse interface has a thickness of roughly $10h^2$. In the diffuse interface region, we consider mixed phases with random values in $[-0.3, 0.5]$ and outside the diffuse region we prescribe pure phases of $+1$ (pink region) and $-1$ (light blue region). In Table 2, we show active set configurations for time steps $\tau = 1, 20, 40, 60, 80, 100, 120, 140, 160, 180, 200$. We observe that for this test case, the number of truncations remain very high at above 85%. As for previous test case, we see significant changes until about $\tau = 120$, after which it evolves very slowly. In Table 2, we compare three preconditioners for various time steps. Here again btdsc is the best: it has least iteration count and small CPU times compared to bd and bdsc. Comparing bd and bdsc, we find that although bdsc has less iteration count compared to bd, bd has smaller CPU time. The reason for this has been explained above. As before, for all three methods, the number of iterations remain almost constant for various time steps with time step $\tau \geq 20$. For $h = 1/256$, bd is slightly faster compared to bdsc, and for $h = 1/400$, bd is significantly faster compared to bdsc.
Figure 1: Evolution of Random Initial Active Set configuration

Table 1: Initial Random Active Set Configuration

| $1/h$ | #tstp | #trunc | %trunc | bd | bdsc | btdsc |
|-------|-------|--------|--------|----|------|-------|
|       |       |        |        | it1 | it2  | time  |
| 256   | 1     | 2      | 0.00   | 17  | 15   | 18.3  |
|       | 20    | 13865  | 20.99  | 23  | 22   | 29.0  |
|       | 40    | 25696  | 38.90  | 23  | 21   | 25.8  |
|       | 60    | 31109  | 47.09  | 23  | 21   | 25.0  |
|       | 80    | 34907  | 52.85  | 23  | 21   | 24.8  |
|       | 100   | 37336  | 56.52  | 23  | 21   | 24.9  |
|       | 120   | 39922  | 60.44  | 22  | 19   | 21.5  |
|       | 140   | 40357  | 61.10  | 21  | 19   | 21.1  |
|       | 160   | 40861  | 61.86  | 21  | 19   | 20.2  |
|       | 180   | 41215  | 62.40  | 21  | 19   | 20.7  |
|       | 200   | 41490  | 62.81  | 21  | 19   | 20.6  |

| 400   | 1     | 2      | 0.00   | 17  | 15   | 46.1  |
|       | 20    | 16136  | 10.03  | 22  | 22   | 71.3  |
|       | 40    | 55886  | 34.75  | 23  | 23   | 69.6  |
|       | 60    | 72514  | 45.09  | 23  | 21   | 62.9  |
|       | 80    | 85496  | 53.16  | 23  | 21   | 57.4  |
|       | 100   | 92787  | 57.70  | 21  | 21   | 53.3  |
|       | 120   | 95995  | 59.69  | 21  | 19   | 49.0  |
|       | 140   | 98593  | 61.31  | 21  | 19   | 50.4  |
|       | 160   | 100733 | 62.64  | 21  | 19   | 50.4  |
|       | 180   | 102625 | 63.82  | 21  | 19   | 49.6  |
|       | 200   | 104522 | 65.00  | 21  | 19   | 48.3  |

(b) $\tau = 20$

(c) $\tau = 40$

(d) $\tau = 60$

(e) $\tau = 80$

(f) $\tau = 100$

(g) $\tau = 120$

(h) $\tau = 160$

(i) $\tau = 180$

(j) $\tau = 200$
6.1.3 Artificial Randomly Truncated System

This is a non-evolution example. Here we choose $\epsilon = \tau$, where we study the effectiveness of the solver for various values of $\epsilon$. We artificially create truncations. In Table 3, we show experiments with this test case, and compare the iterations, and CPU time for iterative solve. We notice that for each mesh sizes, we observe a slight increase in the iteration count from $\epsilon = 10^{-2}$ to $\epsilon = 10^{-5}$, then decreases again for $\epsilon = 10^{-8}$. The iteration counts for $h = 1/400$ are comparable to those for $h = 1/256$. As before, btdsc remains the fastest, except in some cases, when there are small truncations when $bd$ converges faster. In particular, for $\epsilon = 10^{-8}$, $bd$ is fastest in most cases.

7 Conclusion

For the solution of large scale linear saddle point problems on truncated domains, we studied and compared three preconditioners. We also derived eigenvalue bounds and condition number estimates for untruncated problem, and related those bounds to the related truncated problem whenever possible. The numerical experiments suggest that these are effective preconditioners for such problems. The work is in progress to extend these solvers to three space dimensions, and to multicomponent phase field models. Note that eigenvalue bounds and condition number estimates are independent of space dimensions and should essentially hold for higher dimensions for appropriate discretizations.

8 Acknowledgement

This research was partially carried out at IIIT, Hyderabad and at Einstein Foundation, Berlin.

References

[1] L. Banas. A Multigrid Method for the Cahn-Hilliard Equation with Obstacle Potential. *Applied Mathematics and Computation*, 213(2):290–303, 2014.

[2] J.W. Barrett, R. Nurnberg, and V. Styles. Finite element approximation of a phase field model for void electromigration. *SIAM J. Numer. Anal.*, 42(2):738–772, 2004.

[3] J. F. Blowey and C. M. Elliott. The Cahn-Hilliard gradient theory for phase separation with non-smooth free energy Part I: Numerical analysis. *European J. Appl. Math.*, 2(2):233–280, 1991.
Table 2: Initial Square Active Set Configuration

| \(1/h\) | \#tstp | \#trunc | %trunc | |it1 | it2 | time| |it1 | it2 | time| |it1 | it2 | time|
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 256 | 1 | 0 | 0.00 |25 | 18 | 18.1 |11 | 14 | 15.8 |12 | 12 | 16.3 |
| 20 | 57176 | 86.56 | 21 | 17 | 17.6 |16 | 14 | 18.3 |9 | 8 | 10.8 |
| 40 | 57358 | 86.84 | 21 | 17 | 18.3 |16 | 13 | 18.6 |9 | 8 | 10.8 |
| 60 | 57368 | 86.85 | 21 | 16 | 17.8 |16 | 13 | 17.7 |9 | 8 | 10.9 |
| 80 | 57447 | 86.97 | 21 | 17 | 17.3 |16 | 13 | 18.7 |9 | 8 | 10.8 |
| 100 | 57426 | 86.94 | 21 | 16 | 17.3 |16 | 13 | 17.6 |9 | 8 | 11.1 |
| 120 | 57362 | 86.84 | 21 | 16 | 16.8 |16 | 13 | 17.2 |9 | 8 | 10.8 |
| 140 | 57346 | 86.82 | 21 | 16 | 17.0 |16 | 13 | 17.3 |9 | 8 | 10.6 |
| 160 | 57366 | 86.85 | 21 | 16 | 17.7 |16 | 13 | 17.7 |9 | 8 | 10.9 |
| 180 | 57312 | 86.77 | 21 | 16 | 16.6 |16 | 13 | 17.2 |9 | 8 | 10.7 |
| 200 | 57313 | 86.77 | 21 | 16 | 17.8 |16 | 12 | 16.9 |9 | 8 | 11.2 |
| 400 | 1 | 0 | 0.00 |27 | 16 | 59.0 |14 | 9 | 52.9 |15 | 9 | 54.9 |
| 20 | 139237 | 86.58 | 21 | 16 | 50.3 |15 | 16 | 64.9 |9 | 9 | 35.4 |
| 40 | 139647 | 86.84 | 21 | 16 | 49.6 |15 | 16 | 66.9 |9 | 7 | 49.4 |
| 60 | 139839 | 86.96 | 20 | 16 | 60.9 |16 | 16 | 67.2 |9 | 7 | 46.0 |
| 80 | 139823 | 86.95 | 21 | 16 | 52.5 |16 | 16 | 67.4 |9 | 7 | 45.8 |
| 100 | 139858 | 86.97 | 21 | 16 | 50.4 |16 | 14 | 58.1 |9 | 12 | 45.9 |
| 120 | 139788 | 86.93 | 21 | 17 | 52.5 |16 | 14 | 60.7 |9 | 9 | 35.5 |
| 140 | 139731 | 86.89 | 21 | 16 | 50.7 |16 | 16 | 64.5 |9 | 8 | 32.1 |
| 160 | 139735 | 86.89 | 21 | 16 | 52.0 |16 | 15 | 66.2 |9 | 12 | 47.3 |
| 180 | 139739 | 86.90 | 21 | 16 | 51.2 |16 | 13 | 56.9 |9 | 12 | 49.4 |
| 200 | 139720 | 86.89 | 19 | 16 | 50.6 |16 | 13 | 53.9 |9 | 9 | 34.7 |

[4] J. F. Blowey and C. M. Elliott. The Cahn-Hilliard gradient theory for phase separation with non-smooth free energy Part II: Numerical analysis. *European J. Appl. Math.*, 2(3), 1992.

[5] Jessica Bosch, Martin Stoll, and Peter Benner. Fast solution of Cahn-Hilliard variational inequalities using implicit time discretization and finite elements. *Journal of Computational Physics*, 262:38–57, 2014.

[6] John W Cahn and John E Hilliard. Free Energy of a Nonuniform System. I. Interfacial Free Energy. *The Journal of Chemical Physics*, 28(2), 1958.

[7] Shrutimoy Das, Siddhant Katyan, and Pawan Kumar. Domain decomposition based preconditioned solver for bundle adjustment. In R. Venkatesh Babu, Mahadeva Prasanna, and Vinay P. Namboodiri, editors, *Computer Vision, Pattern Recognition, Image Processing, and Graphics*, pages 64–75, Singapore, 2020. Springer Singapore.

[8] Shrutimoy Das, Siddhant Katyan, and Pawan Kumar. A deflation based fast and robust preconditioner for bundle adjustment. In *Proceedings of the IEEE/CVF Winter Conference on Applications of Computer Vision (WACV)*, pages 1782–1789, January 2021.

[9] R. Glowinski. *Numerical Methods for Nonlinear Variational Problems*. Springer Verlag Berlin Heidelberg, 2008.

[10] Gene H. Golub and Charles F. van Loan. *Matrix Computations*. The John Hopkins University Press, 2013.

[11] Gene H. Golub and Loan Charles F. Van. *Matrix Computations*. Johns Hopkins University Press, 1996.

[12] Carsten Graeser. *Convex Minimization and Phase Field Model*. PhD thesis, FU Berlin, 2011.

[13] Carsten Graeser and Ralf Kornhuber. Nonsmooth newton methods for set-valued saddle point problems. *SIAM Journal on Numerical Analysis*, 47(2):1251–1273, 2009.

[14] Carsten Graser and Ralf Kornhuber. Multigrid Methods for Obstacle Problems. *Journal of Computational Mathematics*, 27(1):1–44, 2009.
| $1/h$ | $\epsilon$ | $\%\text{trun}$ | $\text{it1}$ | $\text{it2}$ | $\text{time}$ | $\text{it1}$ | $\text{it2}$ | $\text{time}$ | $\text{it1}$ | $\text{it2}$ | $\text{time}$ |
|-------|--------|---------------|-------------|-------------|--------------|-------------|-------------|--------------|-------------|-------------|--------------|
| $10^{-2}$ | 0.00  | 16 | 16 | 27.9 | 17 | 20 | 47.4 | 9 | 14 | 31.5 |
|       | 19.86 | 26 | 24 | 31.4 | 16 | 17 | 32.2 | 16 | 17 | 30.8 |
|       | 67.11 | 19 | 18 | 17.6 | 14 | 13 | 16.2 | 14 | 13 | 13.3 |
|       | 98.73 | 8  | 6  | 6.7  | 5  | 5  | 5.6  | 5  | 5  | 5.6  |
| 256   | 0.00  | 18 | 13 | 8.6  | 16 | 29 | 32.1 | 7 | 10 | 11.9 |
|       | 19.86 | 22 | 21 | 13.1 | 23 | 23 | 26.9 | 9 | 10 | 11.4 |
|       | 67.11 | 26 | 24 | 19.2 | 19 | 18 | 17.5 | 10 | 10 | 9.9  |
|       | 98.73 | 25 | 22 | 29.6 | 13 | 11 | 16.9 | 10 | 9  | 13.9 |
| $10^{-5}$ | 0.00  | 6  | 4  | 1.4  | 3  | 6  | 10.5 | 2 | 3  | 4.0  |
|       | 19.86 | 6  | 4  | 1.9  | 7  | 5  | 6.3  | 4 | 3  | 4.1  |
|       | 67.11 | 6  | 4  | 3.4  | 7  | 5  | 5.5  | 4 | 3  | 3.9  |
|       | 98.73 | 6  | 4  | 6.7  | 7  | 3  | 7.5  | 4 | 2  | 5.6  |
| $10^{-8}$ | 0.00  | 16 | 16 | 65.3 | 17 | 16 | 111.2 | 8 | 9  | 71.4 |
|       | 19.7  | 24 | 23 | 69.9 | 17 | 16 | 72.9 | 16 | 17 | 69.9 |
|       | 66.6  | 17 | 16 | 36.9 | 12 | 12 | 32.9 | 13 | 11 | 32.6 |
|       | 98.7  | 6  | 6  | 13.7 | 4  | 6  | 13.1 | 5 | 5  | 12.4 |

[15] R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, 1991.

[16] Siddhant Katyan, Shrutimoy Das, and Pawan Kumar. Two-grid preconditioned solver for bundle adjustment. In *2020 IEEE Winter Conference on Applications of Computer Vision (WACV)*, pages 3588–3595, 2020.

[17] Ralf Kornhuber. Monotone multigrid methods for elliptic variational inequalities I. *Numerische Mathematik*, 2(69):167–184, 1994.

[18] Ralf Kornhuber. Monotone multigrid methods for elliptic variational inequalities II. *Numerische Mathematik*, 2(72):481–499, 1996.

[19] Pawan Kumar. Purely algebraic domain decomposition methods for the incompressible navier-stokes equations, 2011.

[20] Pawan Kumar. Aggregation based on graph matching and inexact coarse grid solve for algebraic two grid. *International Journal of Computer Mathematics*, 91(5):1251–1273, 2014.

[21] Pawan Kumar. Multithreaded direction preserving preconditioners. In *2014 IEEE 13th International Symposium on Parallel and Distributed Computing*, pages 148–155, 2014.

[22] Pawan Kumar, Stefano Markidis, Giovanni Lapenta, Karl Meerbergen, and Dirk Roose. High performance solvers for implicit particle in cell simulation. *Procedia Computer Science*, 18:2251–2258, 2013. 2013 International Conference on Computational Science.

[23] Jan Mandel. A Multilevel Iterative Method for Symmetric, Positive Definite Linear Complementarity Problems. *applied mathematics and optimization*, 11:77–95, 1984.

[24] J.A. Meijerink and H.A. van der Vorst. An iterative solution method for linear system of which the coefficient matrix is a symmetric M-matrix. *Math. Comp.*, 31:148–162, 1977.
[25] Qiang Niu, Grigori L., Kumar P., and F. Nataf. Modified tangential frequency filtering decomposition and its fourier analysis. *International Journal of Computer Mathematics*, 116(5):123–148, 2010.

[26] Y Oono and S Puri. Study of phase-separation dynamics by use of cell dynamical systems. I. Modeling. *Physical Review A*, 38(1), 1987.

[27] J. Pearson and A. Wathen. A new approximation of the Schur complement in preconditioners for PDE-constrained optimization. *Numerical Linear Algebra with Applications*, 19:816–829, 2012.

[28] C. R. Rao and M. B. Rao. *Matrix Algebra and Its Applications to Statistics and Econometrics*. World Scientific, 1998.

[29] Yousef Saad. *Iterative Methods for Sparse Linear Systems*. SIAM, Philadelphia, 2 edition, 2003.

[30] B. Smith, P. Bjorstad, and W. Gropp. *Domain Decomposition-Parallel Multilevel Methods for Elliptic Partial Differential Equations*. Cambridge University Press, 1996.

[31] G. Stewart and J. Sun. *Matrix Perturbation Theory*. Academic Press, 1990.

[32] Walter Zulehner. Nonstandard norms and robust estimates for saddle point problems. *SIAM Journal on Matrix Analysis and Applications*, 32(2):536–560, 2011.