Inverse scattering transform of the general coupled Hirota system with nonzero boundary conditions *

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Abstract
The initial value problem for the general coupled Hirota system with nonzero boundary conditions at infinity is solved by reporting a rigorous theory of the inverse scattering transform. With the help of a suitable uniformization variable, both the inverse and the direct problems are analyzed which allows us to develop the inverse scattering transform on the complex \( z \)-plane. Firstly, analyticity of the scattering eigenfunctions and scattering data, properties of the discrete spectrum, symmetries, and asymptotics are discussed in detail. Moreover, the inverse problem is posed as a Riemann-Hilbert problem for the eigenfunctions, and the reconstruction formula of the potential in terms of eigenfunctions and scattering data is presented. Finally, the main characteristics of these obtained soliton solutions are graphically discussed in the 2 \( \times \) 2 self-focusing case. This family of solutions contains novel Akhmediev breather and Kuznetsov-Ma soliton. These results would be of much importance in understanding and enriching breather wave phenomena arising in nonlinear and complex systems, especially in Bose-Einstein condensates.

Key words: Riemann-Hilbert problem; Solitons; Akhmediev breather; Kuznetsov-Ma soliton; Inverse scattering transform.

PACS numbers: 02.30.Ik, 05.45.Yv, 04.20.Jb.

1 Introduction

The solutions of nonlinear wave systems with nonzero boundary conditions (NZBCs) are always physically and mathematically interesting subjects. The inverse scattering transform (IST) was first proposed by Gardner, Greene, Kruskal, and Miura to exactly analyze

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*This work is supported by the National Natural Science Foundation of China under Grant Nos.12201622 and 11975306.
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the initial-value problems for the famous Korteweg-de Vries (KdV) equation with a Lax pair in 1967 [1]. After that, a mass of attempts were presented to develop the application of this approach in other integrable nonlinear wave systems with the so-called Lax pairs [2]. For instance, Zakharov and Shabat investigated the IST of the standard nonlinear Schrödinger (NLS) equation in 1972 [3]. In addition, Ablowitz, Kaup, Newell and Segur (AKNS) presented a class of new integrable systems, called AKNS systems, and found a general framework for their ISTs [4, 5]. Afterwards, many integrable nonlinear wave equations were found to be solved in terms of the IST, such as the sine-Gordon equation [6], modified KdV equation [7, 8], Kadomtsev-Petviashvili equation [9], Camassa-Holm equation [10], Degasperis-Procesi equation [11], and Benjamin-Ono Equation [12], etc.

In recent years, the ISTs of nonlinear systems with NZBCs have been paid much attention based on the solutions of the related Riemann-Hilbert (RH) problem [13]-[34]. However, the IST for the coupled Hirota equation with NZBCs, to the best of the authors’ knowledge, has not been reported before. The main purpose of this paper is to study the IST and soliton solutions for the coupled Hirota equation [35, 36], whose form is

\[ iQ_t + \alpha \left(Q_{xx} - 2\sigma QQ^\dagger Q \right) + i\beta \left(Q_{xxx} - 6\sigma QQ^\dagger Q_x \right) = 0, \]  

(1.1)

where \( Q = Q(x, t) \) is a \( 2 \times 2 \) matrix valued function, and “\(^\dagger\)" denotes the Hermitian conjugate. The choice \( \sigma = +1 \) and \( \sigma = -1 \) distinguishes between the self-defocusing and self-focusing regimes, respectively. Taking the \( 2 \times 2 \) matrix potential \( Q(x, t) \) is the following symmetric matrix

\[
Q = \begin{pmatrix}
q_1 & q_0 \\
q_0 & q_{-1}
\end{pmatrix},
\]

the system (1.1) can be viewed as the matrix Hirota equation. The matrix Hirota equation (1.1) is completely integrable. Particularly, when \( \alpha = 1, \beta = 0 \) system (1.1) can be reduced to the integrable Spin-1 Gross-Pitaevskii equations [37]-[42] which can be used to describe light transmission in bimodal nonlinear optical fibres. Here there are three reasons for choosing the multi-component nonlinear system (1.1) as a model problem: (I): throughout this paper, we allow constants \( \alpha \) and \( \beta \) to be arbitrary, so Eq.(1.1) is quite general and can be used to describe a wide variety of physical processes. (II): dynamics of two or more components with different modes, frequencies, or polarizations in optical fibers can be described by the multi-component NLS systems [43]-[48]. Such systems allow for energy transfer between their additional degrees of freedom and yield rich families of vector solutions. (III): in recent years, many efforts were devoted to studying ISTs for the one component model. However, there are few research studies on ISTs for multi-component nonlinear equations. Therefore, more researches about multi-component nonlinear systems are also inevitable and worthwhile.

In this article we are devoted to extending the IST for Eq.(1.1) in the general case, under NZBCs as \( x \to \pm \infty \), as a technique to deal with the initial-value problem, and also construct soliton solutions as a byproduct of the IST. It is necessary to point out that generally the boundary conditions for \( Q \) must be time\((t)\)-dependent. However, their time\((t)\)-independence can be easily obtained by using the gauge transformation

\[
Q(x, t) = \hat{Q} e^{-2i\sigma k_0^2 t}.
\]
Then Eq. (1.1) reaches to

\[ i Q_t + \alpha \left( Q_{xx} - 2\sigma \left( Q Q^\dagger - k_0^2 I_m \right) Q \right) + i\beta \left( Q_{xxx} - 6\sigma Q Q^\dagger Q_x \right) = 0, \quad (1.2) \]

where \( I_m \) is an \( m \times m \) identity matrix.

In what follows we analyze the matrix Hirota equation (1.2) under constant NZBCs

\[ Q(x, t) \to Q_\pm \quad \text{as} \quad x \to \pm \infty. \quad (1.3) \]

Then we suppose that the following conditions on the boundary conditions hold

\[ Q^\dagger_\pm Q_\pm = Q_\pm Q^\dagger_\pm = k_0^2 I_m, \quad (1.4) \]

where \( k_0 \) is a positive, real constant. As \( x \to \pm \infty \), for a \( 2 \times 2 \) symmetric matrix potential, the latter is similar to the following constraints on the boundary values of the individual entries of \( Q(x, t) \)

\[ |q_{1, \pm}|^2 = |q_{-1, \pm}|^2, \quad q_{1, \pm} q^*_{0, \pm} = k_0^2 - |q_{1, \pm}|^2 \equiv k_0^2 - |q_{-1, \pm}|^2, \quad q_{1, \pm} q^*_{0} + q_{0, \pm} q^*_{-1, \pm} = 0. \quad (1.5) \]

Here, without loss of generality, we notice that the boundary condition \( Q_+ \) can be taken as \( Q_+ = k_0 I_m \).

Recently, there are many investigations on nonlinear wave solutions and long-time asymptotics of nonlinear evolution equations with NZBCs and zero boundary conditions (ZBCs) [49]-[65]. It is also known that the IST is a powerful approach to derive nonlinear wave solutions. However, since system (1.2) admits a \( 4 \times 4 \) matrix spectral problem, the IST for Eq. (1.2) with NZBCs is rather complicated to deal with. The research work in this paper, to our knowledge, has not been conducted so far. The aim of the present paper is to derive the soliton solutions of Eq. (1.2) with NZBCs (1.3) by utilizing IST. Additionally, the main characteristics of these solutions are discussed by controlling suitable parameters.

The paper is organized as follows. In the next section, we investigate the direct scattering problem for Eq. (1.1) in the general case with NZBCs satisfying (1.4). In particular, we derive the uniformization variable, give eigenfunctions and scattering data, and analyze their analyticity as functions of the uniformization variable. In section 3, we study the inverse scattering problem for the eigenfunctions as a RH problem with poles, provide the formal solution of the latter and the reconstruction formula of the potential in view of eigenfunctions and scattering data. In section 4, we construct the soliton solutions for the focusing (\( \sigma = -1 \)) equation in the \( m = 2 \) case. Then the dynamics of the breather wave solutions are analyzed with some graphics. The last section summarizes the main results of this paper.

2 Direct scattering problem

2.1 Riemann surface and uniformization coordinate

Eq. (1.2) is completely integrable. Its Lax pair is

\[ \varphi_x = U \varphi, \quad \varphi_t = V \varphi, \quad (2.1) \]
with

\[
U = -ik\sigma_3 + Q, \quad V = \alpha T_{\text{nls}} + \beta T_{\text{cmKdV}},
\]

\[
T_{\text{nls}} = 2kU + i\sigma_3 \left( Q_x - Q^2 + \sigma k^2_0 I_{2m} \right),
\]

\[
T_{\text{cmKdV}} = 2k \left( T_{\text{nls}} - i\sigma k^2_0 I_{2m} \sigma_3 \right) - [Q, Q_x] + 2Q^3 - Q_{xx},
\]

\[
\sigma_3 = \begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & Q^\dagger \\ \sigma Q & 0 \end{pmatrix}.
\] (2.2)

Throughout this work, 0, 0, 0 are used to represent the \( m \times m \), \( 2m \times 2m \), \( 2m \times m \) zero matrix, respectively.

The asymptotic problem of the Lax pair (2.1) for Eq. (1.1) with NZBCs (1.3) yields

\[
\varphi_x = U_{\pm} \varphi, \quad U_{\pm} = -ik\sigma_3 + Q_{\pm},
\] (2.3)

where \( Q_{\pm} = \begin{pmatrix} 0 & Q_{\pm}^\dagger \\ \sigma Q_{\pm} & 0 \end{pmatrix}. \) Similar to (1.4), we find

\[
Q_{\pm} Q_{\pm}^\dagger = Q_{\pm}^\dagger Q_{\pm} = k_0^2 I_m \iff Q_{\pm} Q_{\dagger\pm} = Q_{\dagger\pm} Q_{\pm} = k_0^2 I_{2m}.
\] (2.4)

It is easy to find that the eigenvalues of \( U_{\pm} \) are \( \pm i\sqrt{k^2 - \sigma k^2_0}. \) In order to further analyze the branching of the eigenvalues, we next consider the two-sheeted Riemann surface given by

\[
\lambda^2 = k^2 - \sigma k^2_0,
\] (2.5)

in which \( \lambda(k) \) represents a single-valued function on this surface.
Figure 1. Transformation relation from \( k \) two-sheeted Riemann surface, \( \lambda \)-plane and \( z \)-plane.

In the \( \sigma = -1 \) case, taking \( k + ik_0 = r_1e^{i\theta_1} \) and \( k - ik_0 = r_2e^{i\theta_2} \), we then set

\[
\begin{align*}
\lambda(k) &= \sqrt{r_1r_2}e^{i(\theta_1+\theta_2)/2}, \quad \text{on sheet } \mathbb{C}_I, \\
\lambda(k) &= -\sqrt{r_1r_2}e^{i(\theta_1+\theta_2)/2}, \quad \text{on sheet } \mathbb{C}_{II},
\end{align*}
\]  

(2.6)

choosing the local angles \( \theta_j \in [-\frac{\pi}{2}, \frac{3\pi}{2}] \) for \( j = 1, 2 \) corresponds to placing the discontinuity of \( \lambda \) on the segment \( ik_0[1,1] \) on the imaginary \( k \)-axis. Then we get the Riemann surface by gluing the two copies of the complex plane along the cut.

A similar analysis used in the \( \sigma = 1 \) case, with the branch cut chosen on \( k \) (real) axis, and exactly for \( k \in (-\infty, k_0) \cup (k_0, +\infty) \). Particularly, one can give the local polar coordinates \( k - k_0 = r_1e^{i\theta_1} \) and \( k + k_0 = e^{i\theta_2} \) on sheet \( I \), with \( r_1, r_2 \) uniquely set by the location of \( k \), and angles \( \theta_1 \in [0, 2\pi) \) and \( \theta_2 \in [-\pi, \pi) \). Additionally, we can also give (2.6). In this situation, \( (\theta_1 + \theta_2)/2 \) varies continuously between 0 and \( \pi \) both in the lower and in the upper planes, with a cut on \( (-\infty, -k_0) \cup (k_0, +\infty) \). The upper branches of the cuts on sheet \( \mathbb{C}_I \) are then glued with the lower branches on sheet \( \mathbb{C}_{II} \), and vice versa, thus \( \lambda(k) \) is again continuous via the cut.

Similar to [13, 17, 31], we have the following uniformization variable

\[ z = k + \lambda, \]  

(2.7)

whose inverse transformation reads

\[ k = \frac{1}{2} \left( z + \sigma \frac{k_0^2}{z} \right), \quad \lambda = \frac{1}{2} \left( z - \sigma \frac{k_0^2}{z} \right). \]  

(2.8)

Summarizing the above results, in \( \sigma = 1 \) case the branch cut on either sheet is mapped to the \( z \) (real) axis, the two sheets (i.e., \( \mathbb{C}_I \) and \( \mathbb{C}_{II} \)) of the Riemann surface are mapped to the lower and upper half-planes of the complex \( z \)-plane, respectively, a neighborhood of \( k = \infty \) on either sheet is mapped onto a neighborhood of \( z = \infty \) (or \( z = 0 \)) relying on the sign of \( \text{Im} k \).

In \( \sigma = -1 \) case the branch cut on either sheet is mapped to the circle \( \mathbb{C}_0; \mathbb{C}_I \) is mapped to the exterior of \( \mathbb{C}_0; \mathbb{C}_{II} \) is mapped to the interior of \( \mathbb{C}_0; \) \( z(\infty_I) = \infty \) and \( z(\infty_{II}) = 0 \).

Therefore, in the \( \sigma = 1 \) case, \( \text{Im} \lambda < 0 \) in the lower-half plane and \( \text{Im} \lambda > 0 \) in the upper-half plane of \( z \)

\[ \sigma = 1: \; D^+ = \{ z \in \mathbb{C} : \text{Im} > 0 \}, \; D^- = \{ z \in \mathbb{C} : \text{Im} < 0 \}. \]  

(2.9)

In the \( \sigma = -1 \) case, \( \text{Im} \lambda \) is not sign-definite in either half-plane; but one has \( \text{Im} \lambda > 0 \) in \( D^+ \) and \( \text{Im} \lambda < 0 \) in \( D^- \), where, for \( \sigma = -1 \)

\[ D^+ = \{ z \in \mathbb{C} : (|z|^2 - k_0^2) \text{Im} z > 0 \}, \]

\[ D^- = \{ z \in \mathbb{C} : (|z|^2 - k_0^2) \text{Im} z < 0 \}. \]  

(2.10)

Then we set the focusing (\( \sigma = -1 \)) equation as a case to demonstrate the transformation between different complex planes in Fig.1. In addition, the two domains \( D^\pm \) and the complex \( z \)-plane are displayed in Fig.2 (\( \sigma = 1 \) case on the below, and \( \sigma = -1 \) case on the
above). As will be analyzed in subsection 2.2, we find that the sign of \( \text{Im}\lambda \) confirms the regions of analyticity of the Jost eigenfunctions. In the following we can rewrite all the \( k \) dependence as rely on \( z \) wherever applicable with some abuse of notation.

Figure 2. The complex \( z \)-plane, showing the regions \( D^\pm \) in which \( \text{Im}\lambda > 0 \) and \( \text{Im}\lambda < 0 \), respectively, in the \( \sigma = \pm 1 \) case. Also seen in the figures are the oriented contours for the RH problem (red), and the symmetries of the discrete spectrum of the scattering problem.
2.2 Jost solutions and analyticity

It follows from [28] that the asymptotic eigenvector matrix yields

\[
X_\pm(k) = \mathcal{I}_{2m} - \frac{i}{k + \lambda} \sigma_3 \mathcal{Q}_\pm \equiv \mathcal{I}_{2m} - \frac{i}{z} \sigma_3 \mathcal{Q}_\pm, \quad U_\pm X_\pm = -i\lambda X_\pm \sigma_3, \tag{2.11}
\]

where \(\mathcal{I}_{2m}\) is a \(2m \times 2m\) identity matrix. Notice that

\[
\det X_\pm(z) = \left(\frac{2\lambda}{\lambda + k}\right)^m = \gamma^m(z), \quad \gamma(z) = 1 - \frac{\sigma k_0^2}{z^2},
\]

\[
X^{-1}_\pm(z) = \frac{1}{\gamma(z)} \left(\mathcal{I}_{2m} + \frac{i}{z} \sigma_3 \mathcal{Q}_\pm\right), \tag{2.12}
\]

where \(X^{-1}_\pm\) are determined for all values of \(z\) and \(\gamma(z) \neq 0\), i.e., away from the branch points \(z \neq \pm ik_0\) in the \(\sigma = -1\) case, and \(z \neq \pm k_0\) in the \(\sigma = 1\) case.

The continuous spectrum \(\Sigma_k\) contain all values of \(k\) such that \(\lambda(k) \in \mathbb{R}\); i.e., \(\Sigma_k = \mathbb{R} \cup ik_0[-1, 1]\) in the \(\sigma = -1\) case, and \(\Sigma_k = \mathbb{R} \setminus k_0(-1, 1)\) in the \(\sigma = 1\) case. The corresponding sets in the complex \(z\)-plane are \(\Sigma_z = \mathbb{R} \cup C_o\) and \(\Sigma_z = \mathbb{R}\), respectively. \(C_o\) being the circle of radius \(k_0\) centered at the origin (see Fig.1). In what follows we omit the subscripts on \(\Sigma\), since the result will be found from the context. For \(\forall z \in \Sigma\), we now consider the Jost eigenfunctions \(\Phi(x, t; z)\) and \(\Psi(x, t; z)\) as the simultaneous solutions of both parts of the Lax pair, we thus have

\[
\begin{align*}
\Phi(x, t; z) &= \left(\phi(x, t; z), \bar{\phi}(x, t; z)\right) = X_-(z)e^{i\theta(x, t; z)\sigma_3} + O(1), \quad x \to -\infty, \\
\Psi(x, t; z) &= \left(\psi(x, t; z), \bar{\psi}(x, t; z)\right) = X_+(z)e^{i\theta(x, t; z)\sigma_3} + O(1), \quad x \to +\infty,
\end{align*}
\]

where

\[
\theta(x, t; z) = \lambda(z) \left\{ -x - \left[\beta \left(4k_0^2 - 2k_0^2\right) + 2\alpha k(z) \right] t \right\}, \tag{2.14}
\]

and \(\phi(x, t; z), \bar{\phi}(x, t; z) (\bar{\psi}(x, t; z), \psi(x, t; z))\) are four \(2m \times m\) matrices which group the last \(m\) column and the first \(m\) vectors of the \(2m \times 2m\) matrix solutions \(\Phi(x, t; z) (\Psi(x, t; z))\). For the sake of convenience, we present the following modified eigenfunctions

\[
\begin{align*}
(M(x, t; z), \bar{M}(x, t; z)) &= \Phi(x, t; z)e^{-i\theta(x, t; z)\sigma_3}, \\
(\bar{N}(x, t; z), N(x, t; z)) &= \Psi(x, t; z)e^{-i\theta(x, t; z)\sigma_3}.
\end{align*}
\]

Similar to [28], the following integral equations can be obtained

\[
\begin{align*}
(M(x, t; z), \bar{M}(x, t; z)) &= X_- + \int_{-\infty}^x X_- e^{i\sigma_3(\xi-x)}X^{-1}_- \left(\mathcal{Q} - \mathcal{Q}_-\right) (M(\xi, t; z), \bar{M}(\xi, t; z)) e^{i\sigma_3(x-\xi)} d\xi, \\
(\bar{N}(x, t; z), N(x, t; z)) &= X_- + \int_{-\infty}^x X_- e^{i\sigma_3(\xi-x)}X^{-1}_- \left(\mathcal{Q} - \mathcal{Q}_+\right) (\bar{N}(\xi, t; z), N(\xi, t; z)) e^{i\sigma_3(x-\xi)} d\xi,
\end{align*}
\]

where the modified eigenfunctions \(\bar{M}(x, t; z)\) and \(\bar{N}(x, t; z)\) can be analytically extended in the complex \(z\)-plane (Im\(\lambda(z) < 0\)), and \(M(x, t; z)\) and \(N(x, t; z)\) can be analytically
extended in the complex z-plane (Im$\lambda(z) > 0$). Let us define by $L^1_s(\mathbb{R})$ the complex Banach space of all measurable functions $f(x)$ for which $(1 + |x|)^s f(x) \in L^1(\mathbb{R})$ for $s = 0, 1$. Then we have the following two theorems (i.e., Theorem 1 and Theorem 2).

**Theorem 1.** Assume that $Q(x, t) - Q_+ \in L^1([x_+, +\infty])$ and $Q(x, t) - Q_- \in L^1([-\infty, x_-])$ hold for $x_\pm \in \mathbb{R}$, all $t \geq 0$, and that the matrix potential function $Q(x, t)$ is the symmetric matrix as well as admits the boundary conditions (2.4). For $x_\pm \in \mathbb{R}$, $D^\pm$ are given by (2.9) and (2.10) for $(\sigma = \pm 1$ (the defocusing/focusing cases, respectively), i.e.,

$$D^+ = \{z \in \mathbb{C} : (|z|^2 + \sigma k_0^2) \operatorname{Im} z > 0\}, \quad D^- = \{z \in \mathbb{C} : (|z|^2 + \sigma k_0^2) \operatorname{Im} z < 0\}. \quad (2.17)$$

Let us suppose that the boundary conditions $Q_\pm$ meet (2.4). Then for the modified eigenfunctions of the problem (2.1) determined by (2.13) and (2.16), we find: $M(x, t; z)$ and $N(x, t; z)$ are analytic functions of $z$ for $z \in D^+$, and they are continuous up to $\partial D^+ \setminus \{\pm \sqrt{\sigma} k_0\}$. Likewise, $\bar{M}(x, t; z)$ and $\bar{N}(x, t; z)$ are analytic functions of $z$ for $z \in D^-$, and they are continuous up to $\partial D^- \setminus \{\pm \sqrt{\sigma} k_0\}$. Similary to [28], the above two theorems can be easily solved by using standard Neumann series representations for the solutions of (2.16).

**2.3 Scattering coefficients**

In view of Jacobi’s formula, we see that any matrix solution $\varphi(x, t; z)$ of (2.1) satisfies

$$\partial_x (\det \varphi) = \varphi_t (\det \varphi) = 0, \quad \text{and} \quad \operatorname{tr}(U) = 0 = \operatorname{tr}(V) = 0 \text{ in } (2.1).$$

Consequently, we have

$$\lim_{x \to -\infty} \Phi(x, t; z) e^{-i\sigma_3} = X_-, \quad \lim_{x \to +\infty} \Psi(x, t; z) e^{-i\sigma_3} = X_+, \quad z \in \Sigma. \quad (2.18)$$

It follows from (2.18) that

$$\det \Phi(x, t; z) = \det \Psi(x, t; z) = \det X_\pm(z) = \gamma^m, \quad x, t \in \mathbb{R}, \quad z \in \Sigma. \quad (2.19)$$

Since $\Sigma_0 = \Sigma \setminus \{\pm \sqrt{\sigma} k_0\}$, we then have that all $z \in \Sigma_0$, both $\Phi$ and $\Psi$ are two fundamental matrix solutions of the scattering problem. As a consequence, the $2m \times 2m$ scattering matrix $S(z)$ admits

$$\Phi(x, t; z) = \Psi(x, t; z) S(z), \quad z \in \Sigma_0, \quad (2.20)$$

with

$$S(z) = \begin{pmatrix} a(z) & \bar{b}(z) \\ b(z) & \bar{a}(z) \end{pmatrix}. \quad (2.21)$$

In view of the analytic groups of columns given in (2.13), one can get

$$\phi(x, t; z) = \psi(x, t; z) b(z) + \bar{\psi}(x, t; z) a(z), \quad \phi(x, t; z) = \psi(x, t; z) b(z) + \bar{\psi}(x, t; z) a(z), \quad (2.22)$$
where \(a(z), b(z), \bar{a}(z), \bar{b}(z)\) represent the \(m \times m\) blocks of the scattering matrix.

Notice that since \(\Phi\) and \(\Psi\) are simultaneous solutions of both parts of the Lax pair, the entries of \(S(z)\) are independent of \(t\). In addition, from Eqs. (2.19) and (2.20), we have
\[
\det S(z) = 1.
\]

It follows from (2.20) that
\[
\det a(z) = \frac{\text{Wr}(\phi, \psi)}{\text{Wr}(\bar{\psi}, \psi)} \equiv \frac{\det(\phi, \psi)}{\gamma^m}, \quad \det \bar{a}(z) = \frac{\text{Wr}(\bar{\phi}, \bar{\psi})}{\text{Wr}(\bar{\psi}, \psi)} \equiv \frac{\det(\bar{\phi}, \bar{\psi})}{\gamma^m},
\]
where \(\text{Wr}(f, g)\) represents the Wronskian determinant of the \(2m \times m\) matrices \(f\) and \(g\). As mentioned in [22], we have the following integral representations for the scattering matrix
\[
S(z) = \int_0^\infty e^{i\lambda(z)\xi}X_+^{-1}(z)\left(Q - Q_+\right)\Phi(\xi, t; z)d\xi
+ X_+^{-1}(z)X_-(z)\left\{I_{2m} + \int_{-\infty}^0 e^{i\lambda(z)\rho_3}X_+^{-1}(z)\left(Q - Q_-\right)\Phi(\xi, t; z)d\xi\right\}.
\]
Then the following theorem 3 holds.

**Theorem 3.** Assume \(Q - Q_+ \in L^1([x_+, +\infty))\) and \(Q - Q_- \in L^1((-\infty, x_-])\) as matrix functions of \(x\) for all \(t \geq 0\), for some \(x_+ \in \mathbb{R}\), and let \(D^\pm\) be defined as in (2.9) and (2.10) for the \(\sigma = \pm 1\) cases. Also suppose that the boundary conditions \(Q_\pm\) admit (2.4). For the scattering matrix \(S(z)\) given in view of the eigenfunctions of the scattering problem by (2.20), we see: the upper diagonal block \(a(z)\) is continuous up to \(\Sigma_0 = \partial D^+ \setminus \{\pm\sqrt{k_0}\}\), and is analytic in \(D^+\), and continuous up to \(\Sigma_0 = \partial D^- \setminus \{\pm\sqrt{k_0}\}\), and the lower diagonal block \(\bar{a}(z)\) is analytic in \(D^-\). The off-diagonal blocks of the scattering matrix \(S(z)\), i.e., \(b(z)\) and \(\bar{b}(z)\), are only expressed for \(z \in \Sigma_0\), in which they are continuous. However, generally they do not admit analytic continuation off \(\Sigma_0\).

Theorem 3 is a direct consequence of Theorem 1 and of the integral representation (2.24). Another proof of the analyticity of \(a(z)\) and \(\bar{a}(z)\) that applies the symmetries in the scattering data will be presented in the next subsection 2.4.

In the end, for \(z \in \Sigma_0\), it follows from (2.15) and (2.22) that
\[
M(x, t; z)a^{-1}(z) = \bar{N}(x, t; z) + e^{-2i\theta(x, t; z)}N(x, t; z)\rho(z),
\]
\[
\bar{M}(x, t; z)\bar{a}^{-1}(z) = N(x, t; z) + e^{2i\theta(x, t; z)}\bar{N}(x, t; z)\bar{\rho}(z),
\]
where \(M(x, t; z)a^{-1}(z)\) and \(\bar{M}(x, t; z)\bar{a}^{-1}(z)\) are meromorphic in \(D^+\) and \(D^-\), and we now introduce reflection coefficients
\[
\rho(z) = b(z)a^{-1}(z), \quad \bar{\rho}(z) = \bar{b}(z)\bar{a}^{-1}(z), \quad z \in \Sigma_0.
\]

### 2.4 Symmetries

If the IST can be used to solve an initial-value problem, we must analyze the symmetry of the potential function. This is because the symmetries of the eigenfunctions can be constructed by making use of the symmetries of the potential function. The symmetries
for the IST with NZBCs are more complicated, since $\lambda(k)$ changes sign from one sheet of the Riemann surface to the other, i.e., $\lambda_{II}(k) = -\lambda_I(k)$. According to the uniformization variable $z$, we need know the following results:

1: (same sheet) $z \mapsto z^*$ means $(k, \lambda) \mapsto (k^*, \lambda^*)$;
2: (opposite sheets) $z \mapsto \sigma k_0^2 / z$ (outside/inside $\mathbb{C}_o$) means $(k, \lambda) \mapsto (k, -\lambda)$.

Both these transformations correspond to symmetries of the scattering problem. The previous one is the conjugate symmetry in the potential, $Q^\dagger = \sigma Q$. Then the second one is a straightforward consequence of the branching of the scattering parameter $k$-plane. Besides, we must introduce a third symmetry that corresponds to assuming $Q^T = Q$, which in view of $Q$ admits

$$Q = -\sigma_2 Q^T \sigma_2, \quad \sigma_2 = \begin{pmatrix} 0 & iI_m \\ -iI_m & 0 \end{pmatrix},$$

(2.27)$\sigma_2$ being a $2m \times 2m$ generalization of the $2 \times 2$ Pauli matrix $\sigma_2$.

### 2.4.1 First symmetry

Following the same process as in [31, 70] we consider the relationship of the scattering data and eigenfunctions for the matrix equation with ZBCs when the above involution is investigated. Let us introduce for $z \in \Sigma$

$$f(x, t; z) = \Phi^\dagger(x, t; z^*) J_\sigma \Phi(x, t; z), \quad g(x, t; z) = \Psi^\dagger(x, t; z^*) J_\sigma \Psi(x, t; z),$$

with

$$J_\sigma = \begin{pmatrix} I_m & 0 \\ 0 & -\sigma I_m \end{pmatrix},$$

(2.28)$J_\sigma$ is the $2m \times 2m$ identity in the $\sigma = -1$ case. Since $\Phi, \Psi$ are solutions of the scattering problem (2.24), it is not hard to check that $f, g$ are independent of $x$. Then by evaluating the limits as $x \to \pm \infty$, we find

$$\Phi^\dagger(x, t; z^*) J_\sigma \Phi(x, t; z) = \Psi^\dagger(x, t; z^*) J_\sigma \Psi(x, t; z) = \gamma(z) J_\sigma.$$  

(2.29)

On the one hand, we write the above relations as

$$\Psi^{-1}(x, t; z) = \frac{1}{\gamma(z)} J_\sigma \Psi^\dagger(x, t; z^*) J_\sigma, \quad \Phi^{-1}(x, t; z) = \frac{1}{\gamma(z)} J_\sigma \Phi^\dagger(x, t; z^*) J_\sigma.$$  

(2.30)

Then the following representation for the scattering matrix can be obtained

$$S(z) = \Psi^{-1}(x, t; z) \Phi(x, t; z) = \frac{1}{\gamma(z)} J_\sigma \Psi^\dagger(x, t; z^*) J_\sigma \Phi(x, t; z).$$

(2.31)

For the sake of convenience, we introduce the following notation for the upper/lower blocks of the eigenfunctions

$$\Phi(x, t; z) = \begin{pmatrix} \phi_{\text{up}} & \phi_{\text{dn}} \\ \bar{\phi}_{\text{up}} & \bar{\phi}_{\text{dn}} \end{pmatrix}, \quad \Psi(x, t; z) = \begin{pmatrix} \bar{\psi}_{\text{up}} & \psi_{\text{up}} \\ \bar{\psi}_{\text{dn}} & \psi_{\text{dn}} \end{pmatrix},$$
where each block $\up, \dn$ represents an $m \times m$ matrix. Then solving the $m \times m$ blocks of $S(z)$ in (2.31) and comparing with (2.20) yields

\[
\begin{align*}
\gamma(z)a(z) &= \left(\psi_{\up}(x, t; z^*)^\dagger \phi_{\up}(x, t; z) - \sigma \left(\psi_{\dn}(x, t; z^*)^\dagger \phi_{\dn}(x, t; z)\right)\right), \\
\gamma(z)a(z) &= \left(\psi_{\dn}(x, t; z^*)^\dagger \phi_{\dn}(x, t; z) - \sigma \left(\psi_{\up}(x, t; z^*)^\dagger \phi_{\up}(x, t; z)\right)\right), \\
\gamma(z)b(z) &= \left(\psi_{\dn}(x, t; z^*)^\dagger \phi_{\dn}(x, t; z) - \sigma \left(\psi_{\up}(x, t; z^*)^\dagger \phi_{\up}(x, t; z)\right)\right), \\
\gamma(z)b(z) &= \left(\psi_{\up}(x, t; z^*)^\dagger \phi_{\up}(x, t; z) - \sigma \left(\psi_{\dn}(x, t; z^*)^\dagger \phi_{\dn}(x, t; z)\right)\right). \\
\end{align*}
\]

(2.32)

From the above expressions, we find that $a(z)$ can be analytically continued in $D^+$, and $\bar{a}(z)$ can be analytically continued in $D^-$. Consequently, the following theorem can be easily established.

**Theorem 4.** Assume that $\mathcal{Q} - \mathcal{Q}_+ \in L^{1,1}([x_+, +\infty))$ and $\mathcal{Q} - \mathcal{Q}_- \in L^{1,1}([-\infty, x_-))$ as matrix functions of $x$ for all $t \geq 0$, for some $x_+ \in \mathbb{R}$, then $\gamma(z)S(z)$ is continuous for all $z \in \Sigma$, containing the branch points. The functions $a(z), \bar{a}(z), b(z), \bar{b}(z)$ have simple poles at the branch points $z = \pm \sqrt{\sigma}k_0$, the following residue conditions can be obtained

\[
\begin{align*}
\text{Res}_{z = \pm ik_0} a(z) &= \pm \frac{k_0}{2} \left[\left(\psi_{\up}(x, t; \mp k_0)\right)^\dagger \phi_{\up}(x, t; \pm k_0) - \left(\psi_{\dn}(x, t; \mp k_0)\right)^\dagger \phi_{\dn}(x, t; \pm k_0)\right], \\
\text{Res}_{z = \pm ik_0} \bar{a}(z) &= \pm \frac{k_0}{2} \left[\left(\psi_{\dn}(x, t; \mp k_0)\right)^\dagger \phi_{\dn}(x, t; \pm k_0) - \left(\psi_{\up}(x, t; \mp k_0)\right)^\dagger \phi_{\up}(x, t; \pm k_0)\right], \\
\lim_{z \to \pm ik_0} (z \mp ik_0)b(z) &= \pm \frac{k_0}{2} \left[\left(\psi_{\up}(x, t; \mp k_0)\right)^\dagger \phi_{\up}(x, t; \pm k_0) - \left(\psi_{\dn}(x, t; \mp k_0)\right)^\dagger \phi_{\dn}(x, t; \pm k_0)\right], \\
\lim_{z \to \mp ik_0} (z \mp ik_0)\bar{b}(z) &= \pm \frac{k_0}{2} \left[\left(\psi_{\dn}(x, t; \mp k_0)\right)^\dagger \phi_{\dn}(x, t; \pm k_0) - \left(\psi_{\up}(x, t; \mp k_0)\right)^\dagger \phi_{\up}(x, t; \pm k_0)\right], \\
\end{align*}
\]
in the $\sigma = -1$ case, and
\[
\begin{align*}
\text{Res}_{z=\pm k_0} a(z) &= \\
&\pm \frac{k_0}{2} \left[ (\psi^{\text{up}}(x, t; \pm k_0))^\dagger \phi^{\text{up}}(x, t; \pm k_0) - (\psi^{\text{dn}}(x, t; \pm k_0))^\dagger \phi^{\text{dn}}(x, t; \pm k_0) \right], \\
\text{Res}_{z=\pm k_0} \tilde{a}(z) &= \\
&\pm \frac{k_0}{2} \left[ (\psi^{\text{dn}}(x, t; \pm k_0))^\dagger \phi^{\text{dn}}(x, t; \pm k_0) - (\psi^{\text{up}}(x, t; \pm k_0))^\dagger \phi^{\text{up}}(x, t; \pm k_0) \right], \\
\lim_{z \to \pm k_0} (z \mp k_0) b(z) &= \\
&\pm \frac{k_0}{2} \left[ (\psi^{\text{up}}(x, t; \pm k_0))^\dagger \phi^{\text{up}}(x, t; \pm k_0) - (\psi^{\text{dn}}(x, t; \pm k_0))^\dagger \phi^{\text{dn}}(x, t; \pm k_0) \right], \\
\lim_{z \to \pm k_0} (z \mp k_0) \tilde{b}(z) &= \\
&\pm \frac{k_0}{2} \left[ (\psi^{\text{up}}(x, t; \pm k_0))^\dagger \phi^{\text{up}}(x, t; \pm k_0) - (\psi^{\text{dn}}(x, t; \pm k_0))^\dagger \phi^{\text{dn}}(x, t; \pm k_0) \right],
\end{align*}
\]
in the $\sigma = 1$ case. Additionally, under the condition $\det a(z) \neq 0$, $\det \tilde{a}(z) \neq 0$ for $z \in \Sigma$, the reflection coefficients $\rho(z)$ and $\tilde{\rho}(z)$ in (2.26) are a removable singularity at the branch points $z \pm \sqrt{\sigma k_0}$ for all $z \in \Sigma$.

Eq. (2.29) claims that the off-diagonal blocks of
\[\Phi^\dagger(z^*) \mathcal{J}_\sigma \Phi(z) = \Psi^\dagger(z^*) \mathcal{J}_\sigma \Psi(z)\]
are zero, it then follows from (2.31) that
\[
\begin{align*}
(\phi^{\text{up}}(x, t; z^*))^\dagger \phi^{\text{up}}(x, t; z) &= \sigma(\phi^{\text{dn}}(x, t; z^*))^\dagger \phi^{\text{dn}}(x, t; z), \\
(\psi^{\text{up}}(x, t; z^*))^\dagger \psi^{\text{up}}(x, t; z) &= \sigma(\psi^{\text{dn}}(x, t; z^*))^\dagger \psi^{\text{dn}}(x, t; z),
\end{align*}
\]
In addition, Eq. (2.29) implies
\[
\mathcal{S}^\dagger(z^*) \mathcal{J}_\sigma \mathcal{S}(z) = \mathcal{J}_\sigma, \quad z \in \Sigma.
\]
Precisely, in view of $\mathcal{S}(z)$ in (2.20) one has the same symmetries as in the case of ZBCs
\[
\begin{align*}
a^\dagger(z^*) a(z) - \sigma b^\dagger(z^*) b(z) &= \mathcal{I}_m, \\
a^\dagger(z^*) \tilde{b}(z) - \sigma b^\dagger(z^*) \tilde{a}(z) &= 0, \\
b^\dagger(z^*) a(z) - \sigma \tilde{a}^\dagger(z^*) b(z) &= 0, \\
b^\dagger(z^*) \tilde{b}(z) - \sigma \tilde{a}^\dagger(z^*) \tilde{a}(z) &= \mathcal{I}_m,
\end{align*}
\]
and consequently also the following symmetry
\[
\tilde{\rho}(z) = \sigma \rho^\dagger(z^*), \quad z \in \Sigma,
\]
as well as
\[
\begin{align*}
a(z)a^\dagger(z^*) &= \left[ \mathcal{I}_m - \sigma \rho^\dagger(z^*) \rho(z) \right]^{-1}, \\
\tilde{a}(z)\tilde{a}^\dagger(z^*) &= \left[ \mathcal{I}_m - \sigma \tilde{\rho}^\dagger(z^*) \tilde{\rho}(z) \right]^{-1}.
\end{align*}
\]
It follows from (2.34) that

\[ S^{-1}(z) = J_{\sigma} S^\dagger(z^*) J_{\sigma}, \quad S^{-1}(z) = \begin{pmatrix} \bar{c}(z) & d(z) \\ d(z) & c(z) \end{pmatrix}, \] (2.38)

which indicates a relationship between those of its inverse and the blocks of \( S(z) \) for \( z \in \Sigma \), i.e.,

\[ \bar{c}(z) = a^\dagger(z^*), \quad c(z) = \bar{a}^\dagger(z^*), \quad d(z) = -\sigma b^\dagger(z^*), \quad \bar{d}(z) = -\sigma b(z^*). \] (2.39)

\( \bar{c}(z) = a^\dagger(z^*), \ c(z) = \bar{a}^\dagger(z^*) \) can be extended to \( D^+ \) and \( D^- \) by Schwarz reflection principle, respectively, while the \( d(z) = -\sigma b^\dagger(z^*), \ \bar{d}(z) = -\sigma b(z^*) \) are generally only valid for \( z \in \Sigma \).

In turn, similar to (2.23), we have

\[
\begin{align*}
\det c(z) &= \frac{\text{Wr}(\phi, \psi)}{\text{Wr}(\phi, \bar{\phi})} \equiv \frac{\det(\phi, \psi)}{\gamma^m}, \\
\det \bar{c}(z) &= \frac{\text{Wr}(\bar{\phi}, \bar{\psi})}{\text{Wr}(\phi, \bar{\phi})} \equiv \frac{\det(\bar{\phi}, \bar{\psi})}{\gamma^m},
\end{align*}
\] (2.40)

which allow us to conclude that

\[
\det a(z) = \det c(z) \quad \text{for} \quad z \in D^+, \quad \det \bar{a}(z) = \det \bar{c}(z) \quad \text{for} \quad z \in D^-.
\] (2.41)

Finally, it follows from (2.29) that

\[
\det \bar{a}(z) = \det a^\dagger(z^*) \equiv (\det a(z^*))^* \quad \text{for} \quad z \in D^-.
\] (2.42)

**2.4.2 Second symmetry**

Actually, it is easy to check that

\[
X_\pm(z) = -\frac{i}{z} X_\pm (\sigma k_0^2/z) \sigma_3 Q_\pm,
\]

which, in view of that \( \theta(\sigma k_0^2/z) = -\theta(z) \) and \( Q_\pm e^{i\theta(z)\sigma_3} = e^{-i\theta(z)\sigma_3} Q_\mp \), yields

\[
\begin{align*}
\Phi(x, t; z) &= \Phi \left( x, t; \sigma k_0^2/z \right) \sigma_3 Q_-/iz, \\
\Psi(x, t; z) &= \Psi \left( x, t; \sigma k_0^2/z \right) \sigma_3 Q_+/iz,
\end{align*}
\] (2.43)

with \( z \in \Sigma \). The \( 2m \times m \) blocks of columns means

\[
\begin{align*}
\phi(x, t; z) &= \frac{i\sigma}{z} \phi \left( x, t; \sigma k_0^2/z \right) Q_-^\dagger, \quad \bar{\phi}(x, t; z) = -\frac{i}{z} \phi \left( x, t; \sigma k_0^2/z \right) Q_-, \\
\bar{\psi}(x, t; z) &= \frac{i\sigma}{z} \bar{\psi} \left( x, t; \sigma k_0^2/z \right) Q_+^\dagger, \quad \psi(x, t; z) = -\frac{i}{z} \bar{\psi} \left( x, t; \sigma k_0^2/z \right) Q_+,
\end{align*}
\] (2.44)

and for all \( z \in \Sigma \), it follows from (2.20) and (2.23) that

\[
S(\sigma k_0^2/z) = \sigma_3 Q_+ S(z) Q_-^{-1} \sigma_3 \equiv \frac{\sigma}{\kappa_0} \sigma_3 Q_+ S(z) Q_- \sigma_3.
\] (2.45)
In view of (2.20) and (2.43), we then have
\[ a\left(\sigma k_0^2/z\right) = \frac{1}{k_0^2} Q_+ a(z) Q_+^\dagger, \quad \bar{a}\left(\sigma k_0^2/z\right) = \frac{1}{k_0^2} Q_+^\dagger a(z) Q_- \]
\[ b\left(\sigma k_0^2/z\right) = -\sigma k_0^2 Q_+ b(z) Q_+^\dagger, \quad \bar{b}\left(\sigma k_0^2/z\right) = -\sigma k_0^2 Q_+^\dagger b(z) Q_- \]
Finally, we obtain the corresponding symmetry
\[ \rho\left(\sigma k_0^2/z\right) = -\sigma Q_+^\dagger \bar{\rho}(z) Q_+^{-1} \equiv -\sigma/k_0^2 Q_+^\dagger \bar{\rho}(z) Q_+^\dagger, \quad \forall z \in \Sigma. \] (2.47)

2.4.3 Third symmetry
Similar to the first symmetry in the potential \( Q = -\sigma_2 Q^T \sigma_2 \), corresponding to \( Q^T = Q \), let us next introduce
\[ \bar{f}(x, t; z) = \Phi^T(x, t; z) \sigma_2 \Phi(x, t; z), \quad \bar{g}(x, t; z) = \Psi^T(x, t; z) \sigma_2 \Psi(x, t; z) \]
for \( z \in \Sigma \), where \( \sigma_2 \) is given by (2.27). In addition, we can easily check that \( \bar{f}, \bar{g} \) are independent of \( x \). For example, it follows from the scattering problem (2.41) that
\[ \partial_x \bar{f} = \Phi^T \left( -k_0^2 \sigma_3 \sigma_2 + Q^T \sigma_2 - ik \sigma_2 \sigma_3 + \sigma_2 Q \right) \Phi = 0, \]
since \( \sigma_3 \sigma_2 = -\sigma_2 \sigma_3 \) and \( Q^T \sigma_2 = -\sigma_2 Q \). Evaluating the limits as \( x \to \pm \infty \), one gets
\[ \Phi^T(x, t; z) \sigma_2 \Phi(x, t; z) = \Psi^T(x, t; z) \sigma_2 \Psi(x, t; z) = \gamma(z) \sigma_2, \] (2.48)
which indicates
\[ S^T(z) \sigma_2 S(z) = \sigma_2, \quad z \in \Sigma. \] (2.49)
It follows from the blocks of the scattering matrix that the latter gives
\[ b^T(z) a(z) = a^T(z) b(z), \quad \bar{b}^T(z) \bar{a}(z) = \bar{a}^T(z) \bar{b}(z), \quad a^T(z) \bar{a}(z) - b^T(z) \bar{b}(z) = \mathcal{I}_m, \]
which then particularly mean
\[ \rho^T(z) = \rho(z), \quad \bar{\rho}^T(z) = \bar{\rho}(z), \] (2.50)
showing that the reflection coefficients should be symmetric matrices themselves, as well as
\[ a(z) a^T(z) = \left( \mathcal{I}_m - \bar{\rho}(z) \rho(z) \right)^{-1}, \quad z \in \Sigma. \] (2.51)
At last, it also follows from (2.39) that \( S^{-1}(z) = \sigma_2 S^T(z) \sigma_2 \) for \( z \in \Sigma \), i.e.,
\[ \bar{c}(z) = \bar{a}^T(z), \quad c(z) = a^T(z), \quad d(z) = -\bar{b}^T(z), \quad \bar{d}(z) = -\bar{b}^T(z), \] (2.52)
which, in view of (2.39), reach to
\[ \bar{a}(z) = a^*(z^*), \quad \bar{b}(z) = \sigma b^*(z^*), \quad z \in \Sigma. \] (2.53)
Summarizing the results of Section 2.4, we have following proposition.
**Proposition 5.** If \( Q - Q_+ \in L^1([x_+, +\infty)) \) and \( Q - Q_- \in L^1((-\infty, x_-]) \) as matrix functions of \( x \) for all \( t \geq 0 \), for some \( x_\pm \in \mathbb{R} \). For all \( z \in \Sigma_0 \), the coefficients \( \rho(z) \) and \( \tilde{\rho}(z) \) given in view of the blocks of the scattering matrix \( S(z) \) by (2.26) admit
\[
\rho(z) = \sigma \rho^*(z^*), \quad \rho \left( \frac{\sigma k_0^2}{z} \right) = -\frac{\sigma}{k_0^2} Q_+^\dagger \tilde{\rho}(z) Q_+^\dagger.
\]
Furthermore, for \( z \in D^- \bigcup \Sigma_0 \), the diagonal blocks of the scattering matrix \( a(z) \) and \( \tilde{a}(z) \) satisfy
\[
\det \tilde{a}(z) = \det a^1(z^*),
\]
\[
a \left( \frac{\sigma k_0^2}{z} \right) = \frac{1}{k_0^2} Q_+ \tilde{a}(z) Q_+^\dagger \Rightarrow \det \tilde{a}(z) = \frac{k_0^{2m}}{\det Q_+ (\det Q_-)^t} \det a \left( \frac{\sigma k_0^2}{z} \right).
\]
If, moreover, \( Q(x, t) \) is a symmetric matrix, we then have
\[
\rho^T(z) = \rho(z), \quad \tilde{\rho}^T(z) = \tilde{\rho}(z), \quad z \in \Sigma_0,
\]
and
\[
\tilde{a}(z) = a^*(z^*), \quad z \in D^- \bigcup \Sigma_0.
\]
Assume that \( Q - Q_+ \in ([x_+, +\infty)) \) and \( Q - Q_- \in L^{1,1}((-\infty, x_-]) \) as matrix functions of \( x \) for all \( t \geq 0 \), for some \( x_\pm \in \mathbb{R} \), the above three symmetries also extend to contain the branch points, and consequently are valid for \( z \in \Sigma \), and \( z \in D^- \bigcup \Sigma \), respectively.

### 2.5 Discrete spectrum and residue conditions

Similar to [29], these discrete spectral points are the zeros of the functions \( \det a(z) \) and \( \det \tilde{a}(z) \) in \( D^+ \) and \( D^- \), respectively. Suppose that \( \det a(z) \) admits a finite number \( \mathcal{N} \) of simple zeros \( z_1, \ldots, z_N \) in \( D^+ \bigcap \{ z \in \mathbb{C} : \text{Im} z > 0 \} \). That is to say, let \( \det a(z_n) = 0 \) and \( (\det a)'(z_n) \neq 0 \), with \( |z_n| > k_0 \) and \( \text{Im} z_n > 0 \) for \( n = 1, 2, \ldots, \mathcal{N} \), and in which the prime represents differentiation with respect to \( z \). From the symmetries (2.42) and (2.46), it follows that
\[
\det a(z_n) = 0 \iff \det \tilde{a}(z_n^*) = 0 \iff \det \tilde{a} \left( \frac{\sigma k_0^2}{z_n} \right) = 0 \iff \det a \left( \frac{\sigma k_0^2}{z_n} \right) = 0.
\]
(2.54)

For each \( n = 1, 2, \ldots, \mathcal{N} \), we thus have a quartet of discrete eigenvalues, which indicates that the discrete spectrum is expressed by the set
\[
Z = \left\{ z_n, z_n^*, \frac{\sigma k_0^2}{z_n}, \frac{\sigma k_0^2}{z_n^*} \right\}_{n=1}^{\mathcal{N}}.
\]
(2.55)

Then, the discrete spectrum is expressed by
\[
\sigma = -1 \text{ (focusing case)} : \quad Z = \left\{ z_n, -\frac{k_0^2}{z_n^*}, z_n^*, -\frac{k_0^2}{z_n} \right\}_{1}^{\mathcal{N}},
\]
\[
\sigma = 1 \text{ (defocusing case)} : \quad Z = \left\{ \zeta_n, \zeta_n^* \right\}_{1}^{\mathcal{N}},
\]
(2.56)
where in the defocusing (\( \sigma = 1 \)) case the eigenvalues are on the circle \( C_\sigma \); in the focusing (\( \sigma = -1 \)) case each first pair is in \( D^+ \) and each second pair is in \( D^- \);
Now suppose that \( \det a(z) \) admits \( N \) simple zeros \( z_n \) \( (n = 1, 2, \ldots, N) \), namely, \( \det a(z_n) = 0 \), which implies that from Eq.\( (2.23) \) the Jost eigenfunctions \( \psi(x, t; z_n) \) and \( \phi(x, t; z_n) \) are linearly dependent. Therefore there is a nonzero constant \( b_n \) that admits the following equation

\[
\phi(x, t; z_n) = \psi(x, t; z_n) b_n, \quad \bar{\phi}(x, t; \bar{z}_n^*) = \bar{\psi}(x, t; \bar{z}_n^*) \bar{b}_n,
\]

where \( b_n, \bar{b}_n \) are \( m \times m \) non-zero constant matrices.

In the following we construct the residue conditions that will be required for the inverse problem. In view of \( (2.57) \), we have \( M(x, t; z) = e^{2i\theta(x, t; z_n)} N(x, t; z_n) b_n \). As a result, we have the following residue condition in the context of a simple zero of \( \det a(z) \)

\[
\text{Res}_{z=z_n} [M(x, t; z) a^{-1}(z)] = e^{-2i\theta(x, t; z_n)} N(x, t; z_n) C_n, \quad C_n = \frac{b_n \alpha(z_n)}{(\det a)'(z_n)},
\]

where \( \alpha(z) := \text{cof}(a(z)) \) is the cofactor (or adjugate) matrix of \( a(z) \). Following a similar way, if \( z_n^* \in D^- \) is a simple zero of \( \det \bar{a}(z) \) we also get

\[
\text{Res}_{z=z_n^*} [M(x, t; z) \bar{a}^{-1}(z)] = e^{2i\theta(x, t; z_n^*)} \bar{N}(x, t; z_n^*) \bar{C}_n, \quad \bar{C}_n = \frac{b_n \alpha(z_n^*)}{(\det \bar{a})'(z_n^*)},
\]

where \( \alpha(z) \) denotes the cofactor matrix of \( \bar{a}(z) \).

As is well known, for an \( m \times m \) matrix \( A \), one obtains \( \det(\text{cof}A) = (\det A)^{m-1} \), generally, so

\[
\det \alpha(z) = (\det a(z))^{m-1},
\]

which, particularly, implies

\[
\det \alpha(z) = \det a(z),
\]

for the special case \( m = 2 \). As a result, \( \det \alpha(z) \) admits a zero of the same order as \( \det a(z) \) for each \( z_n \in D^+ \cap Z \) in the physically relevant case. The same of course holds for \( \det \bar{a}(z) \), which will have a zero of the same order as \( \det \bar{a}(z) \) for each \( z_n^* \in D^- \cap Z \).

For the simple eigenvalues, we also have

\[
\tau_n := \text{Res}_{z=z_n} a^{-1}(z) = \frac{\alpha(z)}{(\det a)'(z_n)}, \quad \bar{\tau}_n := \text{Res}_{z=z_n^*} \bar{a}^{-1}(z) = \frac{\bar{\alpha}(z)}{(\det \bar{a})'(z_n^*)},
\]

and \( \det \tau_n = \det \bar{\tau}_n = 0 \), so in the case of simple eigenvalues, the residues are always rank \( m-1 \) matrices. We then show the norming constants presented in \( (2.58) \) and \( (2.59) \) in view of the above residues

\[
C_n = b_n \tau_n, \quad \bar{C}_n = \bar{b}_n \bar{\tau}_n,
\]

and we think that for simple discrete eigenvalues one knows

\[
\det C_n = \det \bar{C}_n = 0,
\]

so for simple zeros of \( \det a(z) \), the norming constants are rank \( m-1 \) matrices.

It is also worth to point out that since for any \( z \in D^+ \setminus Z \), one has

\[
a^{-1}(z) = \alpha(z) / (\det a(z)).
\]
Because $\alpha(z)$ is analytic in $D^+$, we find that $a^{-1}(z)$ will be meromorphic in $D^+$, with poles at each of the discrete eigenvalues, and the order of the pole at each $z_n$ is at most equal to the order of $z_n$ as a zero of $\det a(z)$. Obviously, the same holds for $\tilde{a}^{-1}(z)$ in $D^{-1}$.

If $z_n$ is a second order zero of $\det a(z)$, then $\det \alpha(z)$ admits zero of order $2(m-1)$ at $z_n$. However, in a neighborhood of $z_n$ one has

$$a^{-1}(z) = \frac{1}{(z-z_n)^2} \tau_{n,2} + \frac{1}{z-z_n} \tau_{n,1} + \tilde{a}(z),$$

where $\tilde{a}(z)$ is analytic at $z_n$, and

$$\tau_{n,2} = \lim_{z \to z_n} (z-z_n)^2 a^{-1}(z) \equiv \frac{2}{(\det a)^\sigma(z_n)},$$

$$\tau_{n,1} = \lim_{z \to z_n} \frac{d}{dz} \left[ (z-z_n)^2 a^{-1}(z) \right] = \frac{2}{(\det a)^\sigma(z_n)} \alpha'(z_n) - \frac{2}{3} \frac{\det a''(z_n)}{\left( (\det a)^\sigma(z_n) \right)^2} \alpha(z_n).$$

Since $\tau_{n,2} = 0_m$, $a^{-1}(z)$ admits only a pole of first order at $z_n$, and (2.58), Eq.(2.59) reads

$$\text{Res}_{z=z_n} [M(x,t;z) a^{-1}(z)] = e^{2i\theta(x,t;z_n)} N(x,t;z_n) C_n, \quad C_n = \frac{2}{(\det a)^\sigma(z_n)} b_n \alpha'(z_n),$$

$$\text{Res}_{z=z_n} [\hat{M}(x,t;z) \hat{a}^{-1}(z)] = e^{-2i\theta(x,t;z_n^*)} N(x,t;z_n^*) \hat{C}_n, \quad \hat{C}_n = \frac{2}{(\det \hat{a})^\sigma(z_n^*)} b_n \hat{\alpha}'(z_n^*).$$

The norming constants are related by the above symmetries. In the following, we analyze the symmetry relationship between these norming constants $C_n$ and $\hat{C}_n$,

$$\hat{C}_n = \sigma C_n^T.$$

Furthermore, the third symmetry also claims that $C_n$ and $\hat{C}_n$ be symmetric matrices

$$C_n = C_n^T, \quad \hat{C}_n = \hat{C}_n^T.$$

In the $(\sigma = -1)$ focusing case we should discuss the remaining two points of the eigenvalue quartet. Similar to (2.57), we introduce

$$\phi(x,t;\hat{z}_n) = \psi(x,t;\hat{z}_n) \hat{b}_n, \quad \hat{z}_n = \sigma k_0^2 / z_n^*,$$

$$\phi(x,t;\hat{z}_n^*) = \psi(x,t;\hat{z}_n^*) \hat{b}_n, \quad \hat{z}_n^* = \sigma k_0^2 / z_n^*.$$  

This second symmetry for the discrete eigenvalues and associated norming constants only applies to the $\sigma = -1$ (focusing) case, thus for the rest of this section one choose $\sigma = -1$.

Using (2.44) and (2.67), we find

$$\phi(x,t;z_n) = \frac{i\sigma}{z_n} \bar{\phi}(x,t;\hat{z}_n^*) Q_n^\dagger = \frac{i\sigma}{z_n} \bar{\psi}(x,t;\hat{z}_n^*) b_n Q_n^\dagger.$$

Following the similar idea, we then have

$$\phi(x,t;z_n) = \psi(x,t;z_n) b_n = -\frac{i}{z_n} \bar{\psi}(x,t;\hat{z}_n^*) Q_n b_n.$$
From above two expression, we finally get
\[ \hat{b}_n = -\sigma Q_+^\dagger b_n \left( Q_-^\dagger \right)^{-1} = -\frac{\sigma}{k_0^2} Q_+^\dagger b_n Q_- \quad (2.68) \]

In analogous of (2.68), it follows from (2.44) and (2.57) that
\[ \check{b}_n = -\sigma \sqrt{\gamma} \left( Q_-^\dagger \right)^{-1} = -\frac{\sigma}{k_0^2} \sqrt{\gamma} \check{b}_n Q_-^\dagger \quad (2.69) \]

Besides, differentiating (2.46) with respect to \( z \) and evaluating at \( z = z_n \) (or \( z = z_n^* \)), we find
\[
\left( \det \alpha \right) \left( \alpha k_0^2/z_n^* \right) = -\sigma \left( \frac{z_n^*}{k_0} \right)^2 \frac{\det Q_- \det Q_+^\dagger}{k_0^2} \left( \det \alpha \right) \left( z_n^* \right),
\]
\[
\left( \det \check{\alpha} \right) \left( \alpha k_0^2/z_n \right) = -\sigma \left( \frac{z}{k_0} \right)^2 \frac{\det Q_- \det Q_+^\dagger}{k_0^2} \left( \det \check{\alpha} \right) \left( z \right). \quad (2.70)
\]

It also follows from (2.46) that
\[
\alpha (\sigma k_0^2/z_n^*) = \frac{1}{k_0^2} \text{cof} \left( Q_-^\dagger \right) \check{\alpha} (z_n^* \text{cof} \left( Q_+ \right)),
\]
\[
\check{\alpha} (\sigma k_0^2/z_n) = \frac{1}{k_0^2} \text{cof} \left( Q_+^\dagger \right) \alpha (z_n \text{cof} \left( Q_- \right)). \quad (2.71)
\]

Summarizing these relations, we then obtain
\[
\text{Res}_{z = \tilde{z}_n = \sigma k_0^2/z_n^*} \left[ M(x, t; z)a^{-1}(z) \right] = e^{-2i\theta(x, t; \tilde{z}_n)} N(x, t; \tilde{z}_n) \hat{C}_n,
\]
\[
\text{Res}_{z = \tilde{z}_n = \sigma k_0^2/z_n} \left[ \tilde{M}(x, t; z)\tilde{a}^{-1}(z) \right] = e^{2i\theta(x, t; \tilde{z}_n)} \tilde{N}(x, t; \tilde{z}_n) \check{C}_n, \quad (2.72)
\]

where the norming constants \( \hat{C}_n \) admit the following relations
\[
\hat{C}_n = \frac{1}{(z_n^*)} Q_+^\dagger \check{C}_n Q_+^\dagger, \quad \check{C}_n = \frac{1}{z_n^2} Q_+^\dagger \check{C}_n Q_+. \quad (2.73)
\]

Note that \( \hat{C}_n = \sigma \check{C}_n^\dagger \).

Summarizing the results of subsection 2.5 regarding the discrete scattering data, the following proposition holds.

**Proposition 6.** The discrete spectrum of the scattering problem (2.1) is defined by
\[
\sigma = -1 : \quad Z = \left\{ z_n, -k_0^2/z_n^*, z_n^*, -k_0^2/z_n \right\}_{n=1}^N,
\]
\[
\sigma = 1 : \quad Z = \left\{ \zeta_n, \zeta_n^* \right\}_{n=1}^N.
\]

where in the \( \sigma = 1 \) case each eigenvalue is on the circle \( \mathbb{C}_o \). In the \( \sigma = -1 \) case each first pair is in \( D^+ \), and \( \text{Im} z_n > 0 \) and each second pair is in \( D^- \). In the \( \sigma = 1 \) case the eigenvalues are on the circle \( \mathbb{C}_o \) (Fig.2).
Discrete eigenvalues that correspond to simple poles of $a^{-1}(z)$ in $D^+$ and $\tilde{a}^{-1}(z)$ in $D^-$, one considers a pair or a quartet of norming constants such that

$$
\text{Res}_{z=z_n} \left[ M(x, t; z) a^{-1}(z) \right] = e^{-2i\theta(x, t; z_n)} N(x, t; z_n) C_n,
$$

$$
\text{Res}_{z=z_n} \left[ \tilde{M}(x, t; z) \tilde{a}^{-1}(z) \right] = e^{2i\theta(x, t; z_n)} \tilde{N}(x, t; z_n^*) \tilde{C}_n,
$$

$$
\text{Res}_{z=\sigma k_0^\dagger / z_n} \left[ M(x, t; z) a^{-1}(z) \right] = e^{-2i\theta(x, t; \tilde{z}_n)} N(x, t; \tilde{z}_n) \tilde{C}_n, \quad \tilde{C}_n = \frac{1}{(z_n^*)^2} Q_+^1 C_n Q_+^1,
$$

$$
\text{Res}_{z=\sigma k_0^\dagger / z_n} \left[ \tilde{M}(x, t; z) \tilde{a}^{-1}(z) \right] = e^{2i\theta(x, t; z_n^*)} \tilde{N}(x, t; z_n^*) \tilde{C}_n, \quad \tilde{C}_n = \frac{1}{z_n^*} Q_+^1 C_n Q_+.
$$

In the $\sigma = 1$ case only the first two equations are discussed. Generally, when the discrete eigenvalues are simple zeros, for $m = 2$, which indicates that the norming constants are a matrix with rank 1.

### 2.6 Generalized norming constants

Let us first introduce the $2m \times 2m$ matrix solutions of (2.1)

$$
P(x, t; z) = (\phi(x, t; z), \psi(x, t; z)), \quad \bar{P}(x, t; z) = (\bar{\phi}(x, t; z), \bar{\psi}(x, t; z)).
$$

where $P(x, t; z)$ is analytic for $z \in D^+$ and $\bar{P}(x, t; z)$ is analytic for $z \in D^-$. One then discuss the bilinear combinations $A_\sigma(z) = \bar{P}^\dagger(x, t; z^*) J_\sigma P(x, t; z)$ and $A_\sigma^\dagger(z^*) = P^\dagger(x, t; z^*) J_\sigma \bar{P}(x, t; z)$, which are analytic in $D^+$ and $D^-$, respectively, and independent of $x$. Computing the $m \times m$ blocks of $P^\dagger(x, t; z^*) J_\sigma P(x, t; z)$ and $\bar{P}^\dagger(x, t; z^*) J_\sigma \bar{P}(x, t; z)$ in view of the blocks of the eigenfunctions, and it follows from (2.40) and (2.33) that

$$
A_\sigma(z) = \bar{P}^\dagger(x, t; z^*) J_\sigma P(x, t; z) \equiv \begin{pmatrix} \gamma(z) a(z) & 0 \\ 0 & -\sigma \gamma^*(z^*) \bar{a}^\dagger(z^*) \end{pmatrix}, \quad z \in D^+, \quad A_\sigma^\dagger(z^*) = P^\dagger(x, t; z^*) J_\sigma \bar{P}(x, t; z) \equiv \begin{pmatrix} \gamma^*(z^*) \bar{a}^\dagger(z^*) & 0 \\ 0 & -\sigma \gamma(z) \bar{a}(z) \end{pmatrix}, \quad z \in D^-.
$$

Next we discuss that $z_n \in D^+$ is the simple zero of the det $a(z)$, we easily obtain the det $\tilde{a}(z_n^*) = 0$ with $(\det \tilde{a})(z_n^*) \neq 0$ from the first symmetry. Now for free $m$ let $\chi_n \in \mathbb{C}^m \setminus \{0\}$ be a right null vector of $P(x, t; z_n)$, i.e., $\chi_n \in \ker P(x, t; z_n)$. If we show

$$
\chi_n = \begin{pmatrix} \chi_n^\up \chi_n^\dn \end{pmatrix}, \quad \chi_n^\up, \chi_n^\dn \in \mathbb{C}^m,
$$

then it follows from the definition (2.21) of $P(x, t; z_n)$ that

$$
\phi(x, t; z_n) \chi_n^\up + \psi(x, t; z_n) \chi_n^\dn = 0_{2m \times m},
$$

showing that a right null vector of $P(x, t; z_n)$ arrives at

$$
\phi(x, t; z_n) \eta_n = \psi(x, t; z_n) \xi_n,
$$

(2.76)
with \( \eta_n = \chi_n^{\text{up}} \) and \( \xi_n = -\chi_n^{\text{dn}} \). Following the similar way, at \( z_n^* \in D^- \cap Z \)

\[
\tilde{\phi}(x, t; z_n^*) \tilde{\eta}_n = \tilde{\psi}(x, t; z_n^*) \tilde{\xi}_n,
\]

for some \( \tilde{\xi}_n, \tilde{\eta}_n \in \mathbb{C}^m \setminus \{0\} \). It is obvious that \( \tilde{\xi}_n, \tilde{\eta}_n \) and \( \xi_n, \eta_n \) are not uniquely given.

Since the first \( m \) columns of \( P(x, t; z_n) \) are linearly independent, and so are the last \( m \) columns, necessarily \( \eta_n = \chi_n^{\text{up}} \neq 0 \) and \( \xi_n = -\chi_n^{\text{dn}} \neq 0 \). Given \( \xi_n \) and \( \eta_n \) as in (2.76), the \( 2m \times 1 \) vector \( \chi_n = (\eta_n, -\xi_n)^T \) is a right null vector of \( P(x, t; z_n) \). Apparently, the analog of all the above discussions can be proved for \( z_n^* \in D^- \cap Z \) and \( P(x, t; z) \).

If \( \xi_n, \eta_n \in \mathbb{C}^m \setminus \{0\} \) admit (2.76), then \( \chi_n = (\eta_n, -\xi_n)^T \) is a right null vector of \( A_\sigma(z_n) = \tilde{P}_\sigma^1(x, t; z_n^*)J_\sigma P(x, t; z_n) \)

it then follows from (2.75) that

\[
a(z_n) \eta_n = \tilde{0}, \quad \tilde{a}^\dagger(z_n^*) \eta_n = \tilde{0},
\]

indicating that \( \eta_n \) has to be in the right null space of \( a(z_n) \), and \( \xi_n \) has to be in the right null space of \( \tilde{a}^\dagger(z_n^*) \). On the contrary, right null vectors of \( a(z_n) \) and \( \tilde{a}^\dagger(z_n^*) \) present vectors that satisfy (2.76). Repeating the same process, we can demonstrate that the same holds for \( \xi_n, \tilde{\eta}_n \)

\[
a(z_n) \tilde{\xi}_n = \tilde{0}, \quad \tilde{a}^\dagger(z_n^*) \tilde{\eta}_n = \tilde{0},
\]

so that \( \tilde{\eta}_n \) is in the right null space of \( a(z_n^*) \) and \( \tilde{\xi}_n \) is in the right null space of \( \tilde{a}^\dagger(z_n) \).

As shown in [29], when \( m = 2 \) we take two right null vectors of \( P(x, t; z_n) \) such that the first two components of each vector coincide with the first and the second columns of \( \alpha(z_n) \), let \( -C_n \) denote the \( 2 \times 2 \) matrix that collects column wise the remaining two components of said null vectors

\[
\tilde{0} = P(x, t; z_n) \begin{pmatrix} \alpha(z_n) \\ -C_n \end{pmatrix} \iff \phi(x, t; z_n) \alpha(z_n) = \psi(x, t; z_n) C_n.
\]

If the right null space of \( P(x, t; z_n) \) is 1-dimensional, which happens if \( z_n \) is a simple zero of \( \det a(z) \), then the two columns of the matrix multiplying \( P(x, t; z_n) \) have to be proportional to each other, which then means \( C_n \) is a rank 1 matrix, as well as, since \( \alpha(z) = a^{-1}(z) / \det a(z) \), in the context of a simple zero the above equation can be written as

\[
\text{Res}_{z = z_n} \frac{\phi(x, t; z) \alpha(z)}{\det a(z)} = \psi(x, t; z_n), \quad \det C_n = 0,
\]

which provides the definition of the norming constant \( C_n \) for a simple discrete eigenvalue \( z_n \). Following the same way, we also have

\[
\text{Res}_{z = z_n} \frac{\tilde{\phi}(x, t; z) \tilde{\alpha}(z)}{\det \tilde{a}(z)} = \tilde{\psi}(x, t; z_n^*), \quad \det \tilde{C}_n = 0.
\]
2.7 Asymptotics as $z \to \infty$ and $z \to 0$

The asymptotic properties of the eigenfunctions and the scattering matrix are used to define the inverse problem. In addition, we reconstruct the potential from the solution of the RH problem in terms of the asymptotic behavior of the eigenfunctions.

It is worth mentioning that both limits (the limit $k \to \infty$ corresponds to $z \to \infty$ in $\mathbb{C}_I$ and to $z \to \infty$ in $\mathbb{C}_{II}$). In view of the uniformization variable the asymptotic expansion of the eigenfunctions can be found via the Wentzel-Kramers-Brillouin (WKB) expansions. Obviously, the eigenfunctions $\mu = \varphi e^{i\sigma_3}$ admit

$$\mu_x = (-i\kappa + Q) \mu + i\lambda \sigma_3,$$

which we can write in terms of the uniformization variable $z$ with the help of (2.8). Then, it follows from $\Phi(x, t; z) e^{i\theta(x)} = (M(x, t; z), \bar{M}(x, t; z))$ that

$$M_{x}^{\text{up}} = -\left(\frac{i\sigma k_{0}^{2}}{z}\right) M^{\text{up}} + Q M^{\text{dn}}, \quad M_{x}^{\text{dn}} = \sigma Q^{\dagger} M^{\text{up}} + i z M^{\text{dn}},$$

$$\bar{M}_{x}^{\text{dn}} = \left(\frac{i\sigma k_{0}^{2}}{z}\right) \bar{M}^{\text{dn}} + \sigma Q^{\dagger} M_{x}^{\text{up}}, \quad \bar{M}_{x}^{\text{up}} = Q^{\dagger} \bar{M}^{\text{dn}} - i z \bar{M}_{x}^{\text{up}}.$$

We can then write the WKB expansion as

$$M^{\text{up}} = I_{m} + A_{1} \frac{1}{z} + \text{h.o.t.}, \quad M^{\text{dn}} = B_{1} \frac{1}{z} + \text{h.o.t.}$$

(Here and in the following $h.o.t.$ represents higher order terms) where $A_{1}, B_{1}, \ldots$ are $m \times m$ matrix functions of $x$, $t$ to be known. Plugging the WKB ansatz into the above differential equations, and matching equal powers of $z$ leads to $B_{1} = i\sigma Q^{\dagger}$ and $A_{1,x} = i\sigma (Q Q^{\dagger} - k_{0}^{2} I_{m})$, which in turn yields

$$M(x, t; z) = \left(I_{m} + \frac{i\alpha}{z} \int_{-\infty}^{z} (Q(x', t) Q^{\dagger}(x', t) - k_{0}^{2} I_{m}) \, dx'\right) + O\left(\frac{1}{z^{2}}\right), \quad z \to \infty, \quad z \in D^{+}, \quad (2.77)$$

where we have taken the boundary conditions for $M$ as $x \to -\infty$ into account.

Following a similar way one can see the asymptotic expansion for $\bar{M}$, as well as $N, \bar{N}$ as $z \to \infty$ in the suitable region of analyticity

$$M(x, t; z) = \left(I_{m} - \frac{i\alpha}{z} \int_{-\infty}^{x} \left(\frac{-i}{z} Q(x, t) \int_{-\infty}^{x} (Q(x', t) Q^{\dagger}(x', t) - k_{0}^{2} I_{m}) \, dx'\right) + O\left(\frac{1}{z^{2}}\right), \quad z \to \infty, \quad z \in D^{-}, \quad (2.78)$$

and

$$\bar{N}(x, t; z) = \left(I_{m} + \frac{i\alpha}{z} \int_{-\infty}^{x} \left(\frac{-i}{z} Q(x, t) \int_{-\infty}^{x} (Q(x', t) Q^{\dagger}(x', t) - k_{0}^{2} I_{m}) \, dx'\right) + O\left(\frac{1}{z^{2}}\right), \quad z \to \infty, \quad z \in D^{-},$$

$$N(x, t; z) = \left(I_{m} - \frac{i\alpha}{z} \int_{-\infty}^{x} \left(\frac{-i}{z} Q(x, t) \int_{-\infty}^{x} (Q(x', t) Q^{\dagger}(x', t) - k_{0}^{2} I_{m}) \, dx'\right) + O\left(\frac{1}{z^{2}}\right), \quad z \to \infty, \quad z \in D^{+}. \quad (2.79)$$
Then asymptotics as $z \to \infty$ in the appropriate regions $D^\pm$ reach to

\[
M(x,t;z) = \left( \begin{array}{c} Q \sigma_{z} / k_0^2 + O(z) \\ i \sigma_{z} / z + O(1) \end{array} \right), \quad M(x,t;z) = \left( \begin{array}{c} -iQ / z + O(1) \\ \sigma_{z} / k_0^2 + O(z) \end{array} \right),
\]

\[
N(x,t;z) = \left( \begin{array}{c} Q \sigma_{z} / k_0^2 + O(z) \\ i \sigma_{z} / z + O(1) \end{array} \right), \quad N(x,t;z) = \left( \begin{array}{c} -iQ / z + O(1) \\ \sigma_{z} / k_0^2 + O(z) \end{array} \right).
\]

The above expressions will help us to derive the scattering potential $Q(x,t)$ from the solution of the inverse problem for the eigenfunctions.

In the end, substituting the above asymptotic expansions into (2.20), one can see that, as $z \to \infty$ in the proper regions of the complex $z$-plane

\[
S(z) = I_{2m} + O(1/z).
\]

The above asymptotics can obtained with $\text{Im}z \geq 0$ and $\text{Im}z \leq 0$ for $a(z)$ and $\bar{a}(z)$, respectively, and with $z \in \Sigma$ for $b(z)$ and $\bar{b}(z)$. In a similar way, one also finds that as $z \to \infty$

\[
S(z) = \frac{1}{k_0} \left( \begin{array}{cc} Q_+ Q_+^\dagger & 0_m \\ 0_m & Q_- Q_-^\dagger \end{array} \right) + O(z),
\]

where the asymptotics for the off-diagonal blocks hold for $z \in \Sigma$, while the asymptotics for the block diagonal entries of $S(z)$ can be extended to $D^+$ for $a(z)$ and $D^-$ for $\bar{a}(z)$.

### 3 Inverse problem

#### 3.1 Riemann-Hilbert problem

As usual, the inverse scattering problem is formulated in terms of a suitable RH problem. As mentioned above, the starting point for the formulation of the inverse problem is (2.25), which we now regard as a relationship between eigenfunctions analytic in $D^-$ and those analytic in $D^+$.

We first introduce the sectionally meromorphic matrices

\[
\mu^+(x,t;z) = (Ma^{-1},N), \quad \mu^-(x,t;z) = (N,\bar{M}a^{-1}),
\]

where superscripts $\pm$ differentiate between analyticity in $D^+$ and $D^-$, respectively. It follows from (2.25) that

\[
\mu^-(x,t;z) = \mu^+(x,t;z) (I_{2m} - G(x,t;z)), \quad z \in \Sigma,
\]

where

\[
G(x,t;z) = \begin{pmatrix} 0 & -e^{2i\theta(x,t;z)} \bar{\rho}(z) \\ e^{-2i\theta(x,t;z)} \rho(z) & \rho(z) \bar{\rho}(z) \end{pmatrix}.
\]

Eqs. 3.1-3.3 give a matrix, multiplicative, homogeneous RH problem. In order to finish the formulation of the RH problem one needs a normalization condition, which is the
asymptotic behavior of $\mu^\pm$ as $z \to \infty$. According to the asymptotic behavior of the Jost eigenfunctions and scattering coefficients, we find

$$\mu^\pm = \mathcal{I}_{2m} + O\left(\frac{1}{z}\right), \quad z \to \infty.$$ 

In addition,

$$\mu^\pm = -\frac{i}{z} \sigma_3 Q_+ + O(1), \quad z \to 0.$$ 

To solve the RH problem, one needs to regularize it by subtracting out the asymptotic behavior and the pole contributions, which below we will suppose corresponding to poles of order 1. Review that in the $\sigma = -1$ focusing case discrete eigenvalues come in symmetric quartets. It is convenient to give $\zeta_n = z_n$ and $\zeta_{n+N} = \sigma k_0^2/z_n^2$ for $n = 1, 2, \ldots, N$, as well as $C_{n+N} = \hat{C}_n$ and $\bar{C}_{n+N} = \hat{C}_n$ for $n = 1, 2, \ldots, N$ and rewrite (3.2) as

$$\mu^- - \mathcal{I}_{2m} + (i/z) \sigma_3 Q_+ - \sum_{n=1}^{2N} \left( \text{Res} \frac{\mu^-}{\zeta_n} \right) / (z - \zeta_n^*) =$$

$$\mu^+ - \mathcal{I}_{2m} + (i/z) \sigma_3 Q_+ - \sum_{n=1}^{2N} \left( \text{Res} \frac{\mu^+}{\zeta_n} \right) / (z - \zeta_n) - \sum_{n=1}^{2N} \left( \text{Res} \frac{\mu^-}{\zeta_n} \right) / (z - \zeta_n^*) - \mu^+ G.$$  \hspace{1cm} (3.4)

The left-hand side of (3.4) is now analytic in $D^-$ and is $O(1/z)$ as $z \to \infty$ from $D^-$, while the sum of the first four terms of the right-hand side is analytic in $D^+$ and is $O(1/z)$ as $z \to \infty$ from $D^+$. In the end, the asymptotic behavior of the off-diagonal scattering coefficients implies that $G(x, t; z)$ is $O(1/z)$ as $z \to \pm \infty$ and $O(z)$ as $z \to \infty$ along the real axis. We then introduce the analog of Cauchy projectors $P^\pm$ over $\Sigma$

$$P_{\pm}[f](z) = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\zeta)}{\zeta - (z \pm i0)} d\zeta,$$

where $\int_{\Sigma}$ represents the integral along the oriented contours displayed in Fig.2, and the notation $z \pm i0$ means that, when $z \in \Sigma$, the limit is taken from the left/right of it. Now refer to Plemelj's formulas: if $f^\pm$ are analytic in $D^\pm$ and are $O(1/z)$ as $z \to \infty$, one obtains $P^\pm f^\pm = \pm f^\pm$ and $P^+ f^- = P^- f^+ = 0$. Applying $P^+$ and $P^-$ to (3.4) we then obtain

$$\mu(x, t; z) = \mathcal{I}_{2m} - \left( \frac{i}{z} \right) \sigma_3 Q_+ + \sum_{n=1}^{2N} \frac{\text{Res} \frac{\mu^+}{\zeta_n}}{z - \zeta_n} + \sum_{n=1}^{2N} \frac{\text{Res} \frac{\mu^-}{\zeta_n}}{z - \zeta_n^*} + \frac{1}{2\pi i} \int_{\Sigma} \frac{\mu^+(x, t; z)}{\zeta - z} G(x, t; \zeta) d\zeta, \quad z \in \mathbb{C} \setminus \Sigma. \hspace{1cm} (3.5)$$

The expressions for $\mu^+$ and $\mu^-$ are formally identical, except for the conclusion that the integral appearing on the right-hand side is a $P^+$ and a $P^-$ projector, respectively. In addition, in the $\sigma = 1$ (defocusing) case the sums are only for $n = 1, 2, \ldots, N$. 

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3.2 Residue conditions and reconstruction formula

Eq. (3.5) is an integral equation for $\mu^\pm(x, t; z)$ with $z \in D^\pm$ which also relies on the residues of $\mu^\pm(x, t; z)$ at its poles in $D^\pm$. The residues appearing on the right-hand side of (3.5) are proportional to the values of the analytic columns of $\mu^\pm(x, t; z)$ at the discrete eigenvalues, and therefore the analytic columns of (3.5) reduce to a system of linear algebraic-integral equations. In fact, it follows from the definition (3.1) that only the first $m$ columns of $\mu^+$ admit a pole at $z = z_0$ and $z = \sigma k_0^2/\zeta_n^*$ in $D^+$, and only the last $m$ columns of $\mu^-$ have a pole at $z = z^*$ and $z = \sigma k_0^2/\zeta_n$ in $D^-$. By using the residue relations (2.58) and (2.72), we have the following relationships

$$\begin{align}
\text{Res}_{\zeta_n} \mu^+ &= \left( e^{-2i\theta(x, t; \zeta_n)} N(x, t; \zeta_n) C_n, \hat{0} \right), \quad n = 1, 2, \ldots, 2N, \\
\text{Res}_{\zeta_n} \mu^+ &= \left( \hat{0}, e^{2i\theta(x, t; \zeta_n^*)} \bar{N}(x, t; \zeta_n^*) \bar{C}_n \right), \quad n = 1, 2, \ldots, 2N. \tag{3.6}
\end{align}$$

As a result, we can evaluate the last $m$ columns of (3.5) at $z = z_n$ and at $z = \sigma k_0^2/\zeta_n^*$, obtaining

$$N(x, t; \zeta_n^*) = \left( -\frac{iQ_+}{\mathcal{I}_m} \zeta_n \right) + \sum_{j=1}^{2N} \frac{e^{2i\theta(x, t; \zeta_n^*)}}{\zeta_n - \zeta_n^*} \bar{N}(x, t; \zeta_j) \bar{C}_j + \frac{1}{2\pi i} \int_{\Sigma} \frac{(\mu^+ G)_2(x, t; \zeta) \bar{\rho}(\zeta)}{\zeta - \zeta_n} d\zeta$$

for $n = 1, 2, \ldots, 2N$, and where the subscript 2 in $\mu^+ G$ denotes the last $m$ columns of the product, i.e.,

$$(\mu^+ G)_2(x, t; \zeta) = -e^{2i\theta(x, t; \zeta)} \bar{N}(x, t; \zeta) \bar{\rho}(\zeta).$$

Similarly, the first $m$ columns of (3.5) at $z = z_n^*$ can arrive at

$$\bar{N}(x, t; \zeta_n^*) = \left( \frac{\mathcal{I}_m}{i\sigma Q_+^1/\zeta_n^*} \right) + \sum_{j=1}^{2N} \frac{e^{-2i\theta(x, t; \zeta_j)}}{\zeta_n^* - \zeta_j} \bar{N}(x, t; \zeta_j) \bar{C}_j + \frac{1}{2\pi i} \int_{\Sigma} \frac{(\mu^+ G)_1(x, t; \zeta)}{\zeta - \zeta_n^*} d\zeta,$$

where $n = 1, 2, \ldots, 2N$, and the subscript 1 in $\mu^+ G$ represents the first $m$ columns of the product, i.e.,

$$(\mu^+ G)_1(x, t; \zeta) = e^{-2i\theta(x, t; \zeta)} N(x, t; \zeta) \rho(\zeta).$$
In the end, evaluating the first $m$ columns of $\mu^-$ and the last $m$ columns of $\mu^+(x, t; z)$ through (3.5) for $z \in \Sigma$ we get

\[
N(x, t; z) = \left( -i \frac{Q_+}{I_m} \right) + \sum_{j=1}^{2\nu} e^{2i\theta(x, t, \zeta_j)} \tilde{N}(x, t; \zeta_j) \tilde{C}_j
\]

\[
- \frac{1}{2\pi i} \int_{\Sigma} \frac{e^{2i\theta(x, t, \zeta)} \tilde{N}(x, t; \zeta) \tilde{\rho}(\zeta)}{\zeta - (z + i0)} d\zeta,
\]

\[
\tilde{N}(x, t; z) = \left( \frac{I_m}{i \sigma Q_+^\dagger} \right) + \sum_{j=1}^{2\nu} e^{-2i\theta(x, t, \zeta_j)} N(x, t; \zeta_j) C_j
\]

\[
- \frac{1}{2\pi i} \int_{\Sigma} \frac{e^{-2i\theta(x, t, \zeta)} N(x, t; \zeta) \rho(\zeta)}{\zeta - (z - i0)} d\zeta,
\]

(3.7)

which, together with equations (3.7), lead to a closed system of linear algebraic-integral equations for the solution of the RH problem.

The last task is to derive the potential from the solution of the RH problem. From (3.5), one gets the asymptotic behavior of $\mu^\pm(x, t; z)$ as $z \to \infty$

\[
\mu^\pm(x, t; z) = I_{2m} + O \left( \frac{1}{z^2} \right) + \frac{1}{z} \left\{ -i\sigma_3 Q_+ + \sum_{n=1}^{2\nu} \left[ \text{Res} \mu^+ \zeta_n + \text{Res} \mu^- \zeta_n \right] - \frac{1}{2\pi i} \int_{\Sigma} \mu^+(x, t; \zeta) G(x, t; \zeta) d\zeta \right\},
\]

(3.8)

where the residues are expressed by (3.6). Taking $\mu = \mu^+$ and comparing the upper right $m \times m$ block of (3.8) of this expression with (2.79) yields

\[
Q(x, t) = Q_+ + i \sum_{n=1}^{2\nu} e^{2i\theta(x, t, \zeta_n^*)} N^{\text{up}}(x, t; \zeta_n^*) \tilde{C}_n + \frac{1}{2\pi} \int_{\Sigma} e^{2i\theta(x, t; \zeta)} N^{\text{up}}(x, t; \zeta) \tilde{\rho}(\zeta) d\zeta.
\]

(3.9)

Following the similar way, taking $\mu = \mu^-$ and comparing the lower left $m \times m$ block of (3.8) of this expression with (2.79), then we get the reconstruction formula for the potential

\[
Q^\dagger(x, t) = Q_+^\dagger - i\sigma \sum_{n=1}^{2\nu} e^{-2i\theta(x, t, \zeta_n)} N^{\text{dn}}(x, t; \zeta_n) C_n + \frac{\sigma}{2\pi} \int_{\Sigma} e^{-2i\theta(x, t; \zeta)} N^{\text{dn}}(x, t; \zeta) \rho(\zeta) d\zeta.
\]

(3.10)

The above formulas help us to determine the symmetries of the norming constants. Actually, considering that $N^{\text{dn}}(x, t; z) \sim I_m$ as $x \to \infty$ for any $z \in D^+$, and $N^{\text{up}}(x, t; z) \sim I_m$ as $x \to \infty$ for any $z \in D^-$, comparing the two equations in (3.9) and (3.10) gives

\[
\tilde{C}_n = \sigma C_n^\dagger, \quad n = 1, 2, \ldots, 2\nu.
\]

Because of $Q^T = Q$ the norming constants must admit the same symmetry, i.e.,

\[
C_n^T = C_n, \quad \tilde{C}_n^T = \tilde{C}_n, \quad n = 1, 2, \ldots, 2\nu.
\]
3.3 Focusing reflectionless potentials

In this section, we first discuss potentials \( Q(x,t) \) for which the reflection coefficient \( \rho(z) \) fades away identically for \( z \in \Sigma \). In this case there is no jump from \( \mu^+ \) to \( \mu^- \) across the continuous spectrum, and the inverse problem thus can be reduced to an algebraic system, whose solution gives the soliton solutions of (1.2).

In the following we investigate solutions of the focusing equations \( \sigma = -1 \). In this focusing case discrete eigenvalues happen in quartets, with \( \zeta_{N+j} = -k_j^2/z_j^* \) and \( C_{N+j} = Q_j^\dagger \tilde{C}_j Q_j^\dagger/(z^*)^2 \) for all \( j = 1, 2, \ldots, N \). Furthermore, one can also easily find that \( \theta(x,t;z^*) = \theta^*(x,t;z) \). According to (2.65) one has \( \tilde{C}_j = -C_j^\dagger \) for all \( j = 1, 2, \ldots, 2N \). For convenience, we introduce the quantities

\[
\text{c}_j(x,t;z) = \frac{C_j}{z - \zeta_j} e^{-2i\theta(x,t;\zeta_j)}, \quad j = 1, 2, \ldots, 2N.
\]

Then the algebraic systems constructed from the inverse problem for said upper blocks are expressed as

\[
N_{\text{up}}^{\text{up}}(\zeta_j) = -\frac{i}{\zeta_j} Q_+ - \sum_{\ell=1}^{2N} \tilde{N}_{\text{up}}^{\text{up}}(\zeta_\ell^*) c_\ell^\dagger(\zeta_\ell^*), \quad j = 1, 2, \ldots, 2N,
\]

and substituting (3.11) reaches to

\[
\tilde{N}_{\text{up}}^{\text{up}}(\zeta_n^*) = I_m + \sum_{j=1}^{2N} N_{\text{up}}^{\text{up}}(\zeta_j) c_j(\zeta_n^*), \quad n = 1, 2, \ldots, 2N, \quad (3.11)
\]

and defining the block matrix \( \Gamma = (\Gamma_{n,\ell}) \), in which

\[
\Gamma_{n,\ell} = \sum_{j=1}^{2N} c_j(\zeta_\ell^*) c_j(\zeta_n^*), \quad n, \ell = 1, 2, \ldots, 2N,
\]

the system (3.12) yields simply

\[
AX = B, \quad A = I_p + \Gamma,
\]

where \( I_p \) is the identity matrix of size \( p = 2mN \). Summarizing the above results, we obtain the following theorem.
Theorem 5. Substituting $X_1,\ldots,X_{2N}$ into the reconstruction formula (3.9) and (3.10), one gets the corresponding $N$ soliton solution for $Q(x,t)$

$$Q(x,t) = Q_+ + i \sum_{n=1}^{2N} e^{2i\theta(x,t;z_n^*)} X_n \bar{C}_n,$$

where $X_n$ and $\bar{C}_n$ are given by (3.13)-(3.15).

4 Soliton solutions for the focusing model

The focusing and defocusing Hirota equation with NZBCs admit a rich family of soliton solutions [31], and in this section we will construct their counterpart in the symmetric matrix Hirota equation (1.2) with $m = 2$. Let us start by constructing the one-soliton solution in the $\sigma = -1$ (focusing) case, with one quartet of discrete eigenvalues. $X_j = \bar{N}_j^{up}(x,t;\zeta_j^*)$ for $j = 1,2$, and utilizing the symmetries (3.11), we obtain

$$X_1D_1 = I_2 - \frac{i}{\zeta_1} X_2 Q_+ c_1(\zeta_1^*), \quad X_2D_2 = I_2 + \frac{i\zeta_2^*}{k_0^2} X_1 Q_+ c_2(\zeta_2^*),$$

where

$$D_1 = I_2 + \frac{i}{(\zeta_1^*)^2 + k_0^2} C_1^\dagger Q_+^\dagger e^{2i\theta(\zeta_1^*)}, \quad D_2 = D_1^\dagger,$$

where we have used the expression of the matrices $c_j(\zeta_j^*) = C_j e^{2i\theta(\zeta_j^*)}/(\zeta_j^* - \zeta_j)$, as well as $\zeta_2 = -k_0^2/\zeta_1^*$, and the two symmetries for the norming constant $C_2 = -Q_+^\dagger C_1^\dagger Q_+/(\zeta_1^*)^2$.

This is formally a linear algebraic system of two expressions in the two unknowns $X_1$, $X_2$, but both the unknowns and the coefficients are $2 \times 2$ matrices. We solve the expression by back substitution, and get

$$X_1 = \left[ I_2 - \frac{i}{\zeta_1} D_2^{-1} Q_+ c_1(\zeta_1^*) \right] \left[ D_1 - \frac{\zeta_1^*}{k_0^2} Q_+ c_2(\zeta_2^*) D_2^{-1} Q_+ c_1(\zeta_1^*) \right]^{-1},$$

$$X_2 = \left[ I_2 + \frac{i\zeta_2^*}{k_0^2} D_1^{-1} Q_+ c_2(\zeta_2^*) \right] \left[ D_2 - \frac{\zeta_1^*}{k_0^2} Q_+ c_1(\zeta_1^*) D_1^{-1} Q_+ c_2(\zeta_2^*) \right]^{-1},$$

where

$$c_1(\zeta_1^*) = \frac{C_1 e^{-2i\theta(x,t;\zeta_1)}}{\zeta_1^* - \zeta_1}, \quad c_2(\zeta_2^*) = \frac{\zeta_1^*}{\zeta_1^* k_0^2 (\zeta_1^* - \zeta_1)} Q_+^\dagger C_1^\dagger Q_+ e^{2i\theta(\zeta_1^*;x,t)}. $$

We know that in the above expression the eigenvalue $\zeta_1 \in D^+$ is set to be located in the upper half plane, i.e., such that $|\zeta_1| > k_0$, $\text{Im} \zeta_1 > 0$, and the associated norming constant $C_1$ is a free $2 \times 2$ symmetric matrix with $\det C_1 = 0$. Here we can also investigate the case when $C_1$ is not rank 1, corresponding to having $a(\zeta_1) = 0$. Generally, all entries of $C_1$ can be complex, and thus the norming constant is determined in view of two free complex numbers in the rank 1 case, and three in the rank 2 case:

$$C_1 = \begin{pmatrix} \gamma_1 & \gamma_0 \\ \gamma_0 & \gamma_{-1} \end{pmatrix},$$

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with \( \gamma_j \in \mathbb{C} \) for \( j = 1, 0, -1 \). In terms of \( X_1, X_2 \) via (3.9) and (3.10), we have the following one-soliton solution

\[
Q_{[1]} = Q_+ - iX_1 e^{2i\theta(x,t;\zeta_1)} C_1^\dagger + iX_2 e^{-2i\theta(x,t;\zeta_1)} Q_+ C_1/Q_1^2.
\]

(4.1)

The above results can then be used to obtain the various soliton solutions. In the following, Figs. 3-9 present the various breather waves by varying the suitable parameters, which are useful for understanding the dynamical behaviors of the soliton solutions.

When the discrete eigenvalue \( \zeta_1 \in D^+ \) is purely imaginary, the corresponding soliton solutions are stationary, and a solution (analog of Kuznetsov-Ma (KM) breather of the focusing NLS) that is homoclinic in \( x \) and periodic in \( t \), shown in Fig.3 for the polar state (rank 2 norming constant). By comparing Figs.3(a)-(c) and Figs.3(d)-(f), we find, when the free parameter \( \beta \) become large, the direction of the wave propagation can be changed, but its shape remain unchanged. On the other hand, in the \( |q_0| \) component, the breather wave emerges without valleys, and Fig.3(b) and Fig.4(b) reveal that two peaks of the breather wave without valley merge into one peak, since \( \gamma_0 \) is chosen as 1 in Fig.3 instead of 2 in Fig.4. If one considers the limit as the discrete eigenvalue \( \zeta_1 \) approaches the circle \( C_0 \), one obtains the analog of the Akhmediev breather (AB). The corresponding polar state is plotted in Fig.5, namely, Fig.5 is plotted for the breather waves with suitable parameters, which is not the time-periodic breather but the space-periodic breather, thus revealing the usual AB features. Note that these solutions are periodic in \( x \) and homoclinic in \( t \).

A one-soliton solution (analog of the Tajiri-Watanabe soliton for the scalar focusing NLS) corresponding to \( \zeta_1 \in D^+ \) in generic position is displayed in Fig.6, and when the norming constant is a symmetric matrix (rank 2). From Fig.7, we can easily see that \( |q_1| \) (or \( q_{-1} \)) and \( |q_0| \) have different structures. In the \( |q_1| \) (or \( q_{-1} \)) component, the dark-breather wave is displayed in Fig.7(a) and Fig.7(a). In addition, Fig.6(a) and Fig.7(a) reveal that bright-breather wave turns into the dark-breather wave, since \( \gamma_j \) is chosen as \( \gamma_1 = i, \gamma_0 = 1 + i, \gamma_{-1} = i \) (complex) in Fig.7 instead of \( \gamma_1 = 1, \gamma_0 = 1, \gamma_{-1} = 2 \) (real) in Fig.6. Note that Fig.8 and Fig.9 are plotted for the breather waves with suitable parameters, which are homoclinic in \( x \) and periodic in \( t \), thus revealing the usual KM breather features. Besides, we also exhibit a range of interesting and complicated dynamics, obtained by varying the available parameters (see Figs.10 and 11). To best of our knowledge, the most types of dynamic patterns presented in Figs.3-11 have never emerged in standard NLS equation so far.
Figure 3. (Color online) Breather wave via solution (4.1) with parameters \( \alpha = 1, Q_+ = I_2, \zeta_1 = 2i, k_0 = 1, \gamma_1 = 1, \gamma_0 = 1, \gamma_{-1} = 1 \) (a, b, c): \( \beta = 0.1 \); (d, e, f): \( \beta = 1 \).

Figure 4. (Color online) Breather wave via solution (4.1) with parameters \( \alpha = -1, \beta = 0.01, Q_+ = I_2, \zeta_1 = 2i, k_0 = 1, \gamma_1 = 1, \gamma_0 = 2, \gamma_{-1} = 1 \).

Figure 5. (Color online) Breather wave via solution (4.1) with parameters \( \alpha = -1, \beta = 0.01, Q_+ = I_2, \zeta_1 = 0.5 + 0.8i, k_0 = 1, \gamma_1 = 0, \gamma_0 = 1, \gamma_{-1} = 0 \).

Figure 6. (Color online) Breather wave via solution (4.1) with parameters \( \alpha = 1, \beta = 0.01, Q_+ = I_2, \zeta_1 = 1 + 2i, k_0 = 1, \gamma_1 = 1, \gamma_0 = 1, \gamma_{-1} = 2 \).
Figure 7. (Color online) Breather wave via solution (4.1) with parameters $\alpha = 1, \beta = 0.1, Q_+ = I_2, \zeta_1 = 1 + 2i, k_0 = 1, \gamma_1 = i, \gamma_0 = 1 + i, \gamma_{-1} = i$.

Figure 8. (Color online) Breather wave via solution (4.1) with parameters $\alpha = -1, \beta = 0.1, Q_+ = I_2, \zeta_1 = 2i, k_0 = 1, \gamma_1 = 2i, \gamma_0 = i, \gamma_{-1} = 2i$.

Figure 9. (Color online) Breather wave via solution (4.1) with parameters $\alpha = -1, \beta = 0.1, Q_+ = I_2, \zeta_1 = 2i, k_0 = 1, \gamma_1 = 1, \gamma_0 = i, \gamma_{-1} = 1$. 
Figure 10. (Color online) Breather wave via solution (4.1) with parameters $\alpha = 1$, $Q_+ = I_2$, $\zeta_1 = 2 \imath$, $k_0 = 1$, $\gamma_1 = 1$, $\gamma_0 = 2$, $\gamma_{-1} = 4$. (a, b, c): $\beta = 0.1$; (d, e, f): $\beta = 1$.

Figure 11. (Color online) Breather wave via solution (4.1) with parameters $\alpha = 1$, $\beta = 0.1$, $Q_+ = I_2$, $\zeta_1 = \frac{1}{2} + \frac{\sqrt{3}}{2} \imath$, $k_0 = 1$, $\gamma_1 = \imath$, $\gamma_0 = 2$, $\gamma_{-1} = -4 \imath$.

5 Conclusions and discussions

In this paper, the IST with NZBCs at infinity is developed for the general coupled Hirota system (1.2) with higher-order effects, and we have provided that the problem is significantly more complicated than the scalar case. Moreover, the exact solutions of the general coupled Hirota system are presented. In particular, we have found that these solutions provided in this paper possess a rich family of soliton solutions. We expect the content of this work to be useful in characterizing recent experiments in BEC [67, 68] and nonlinear optics [70, 70, 71]. More importantly, these new rational solutions show the potential rich dynamics in breather wave solutions, and promote our understanding of breather wave phenomena. In addition, the matrix Hirota system (1.2) we investigated in this work is fairly more general as it admits the free constants $\alpha$, $\beta$. Consequently, the solutions of the integrable spin-1 Gross-Pitaevskii equations and the modified matrix Korteweg-de Vries equation can be respectively constructed by reducing the solutions of the the matrix Hirota system (1.2).

Appendix: Trace formula

As reported in [16, 28, 72], the trace formula also offers a relationship between the scattering data and the asymptotic phase difference of the potential under NZBCs. In what follows we will construct the trace formula for $\det a(z)$, from which $\det \bar{a}(z)$ can be obtained by symmetry, in the $\sigma = 1$ case. This will also offer a weak version of the $\theta$-condition, establishing a relationship between the asymptotic phases of $\det Q_+$ and $\det Q_+$ and the spectral data. Following a similar way, the $\sigma = 1$ case can also be obtained.

Simple zeros: Let us investigate the case where all discrete eigenvalues are simple zeros of $\det a(z)$. In particular, the corresponding norming constants are of rank one. $z = z_n$ and $z = -k_0^2/z_n^*$ are the simple zeros of $\det a(z)$ (are analytic in $D^+$), and $z = z_n^*$ and
can be written as

\[ \hat{\alpha}^+(z) = \det a(z) \prod_{n=1}^{N} \frac{(z - z_n^*) (z + k_0^2/z_n)}{(z - z_n) (z + k_0^2/z_n^*)}, \quad z \in D^+, \]

\[ \hat{\alpha}^-(z) = \det a(z) \prod_{n=1}^{N} \frac{(z - z_n) (z + k_0^2/z_n^*)}{(z - z_n^*) (z + k_0^2/z_n)}, \quad z \in D^-, \]  

where \( \hat{\alpha}^\pm(z) \) are analytic in \( D^\pm \), respectively, and they do not have zeros. Furthermore, it follows from (2.81) that \( \hat{\alpha}^\pm(z) \to 1 \) as \( z \to \infty \) in the suitable region, and from (2.36) and (2.37) for \( \sigma = -1 \) it follows that

\[ \hat{\alpha}^+(z) \hat{\alpha}^-(z) = \det \left( I_m + \rho^i(z^*) \rho(z) \right)^{-1}, \quad z \in \Sigma. \]  

Using the Cauchy projectors reported in Section 3.1 the formal solution of the RH problem can be written as

\[ \log \hat{\alpha}^\pm(z) = \mp \frac{1}{2\pi i} \int_\Sigma \log det \left( I_m + \rho^i(\zeta^*) \rho(\zeta) \right) \frac{d\zeta}{\zeta - z}, \quad z \in D^\pm, \]

So, a weak form of the trace formula is also obtained

\[ \det a(z) = \exp \left\{ -\frac{1}{2\pi i} \int_\Sigma \log det \left( I_2 + \rho^i(\zeta^*) \rho(\zeta) \right) \frac{d\zeta}{\zeta - z} \right\} \prod_{n=1}^{N} \frac{(z - z_n) (z + k_0^2/z_n^*)}{(z - z_n^*) (z + k_0^2/z_n)}. \]

In view of (2.81) one can compute the behavior of \( \det a(z) \) as \( z \to 0 \)

\[ \det a(z) \sim \frac{1}{k_0^{2m}} \det Q_+ \det Q_+^-, \quad z \to 0. \]

It follows from above expressions that

\[ \det Q_+ \det Q_-^m = k_0^{2m} \exp \left\{ -\frac{1}{2\pi i} \int_\Sigma \log det \left( I_m + \rho^i(\zeta^*) \rho(\zeta) \right) \frac{d\zeta}{\zeta} \right\} \prod_{n=1}^{N} e^{4i\delta_n}, \]

where \( \delta_n \) represents the phase of the discrete eigenvalue \( z_n \) (\( z_n = |z_n| e^{i\delta_n} \)).

From the constraint (2.4) on the boundary conditions, we find

\[ \det Q_+ = k_0^m e^{i\theta_+}, \quad \det Q_- = k_0^m e^{i\theta_-}, \]

then a weak form of the \( \theta \)-condition can be derived

\[ \theta_+ - \theta_- = \frac{1}{2\pi} \int_\Sigma \log det \left( I_m + \rho^i(\zeta^*) \rho(\zeta) \right) \frac{d\zeta}{\zeta} + 4 \sum_{n=1}^{N} \delta_n. \]

**Double zeros:** In this part, we investigate the case in which all discrete eigenvalues are double zeros. Similar to the simple zeros, we first introduce

\[ \hat{\alpha}^+(z) = \det a(z) \prod_{n=1}^{N} \frac{(z - z_n^2) (z + k_0^2/z_n^2)}{(z - z_n^2) (z + k_0^2/z_n^2)}, \quad z \in D^+, \]

\[ \hat{\alpha}^-(z) = \det a(z) \prod_{n=1}^{N} \frac{(z - z_n^2) (z + k_0^2/z_n^2)}{(z - z_n^2) (z + k_0^2/z_n^2)}, \quad z \in D^-, \]
where $\tilde{\alpha}^{\pm}$ do not have zeros. Proceeding as before we obtain

$$
\det a(z) = \exp \left\{ -\frac{1}{2\pi i} \int_{\Sigma} \log \det (I_m + \rho^i(\zeta^*)\rho(\zeta)) \frac{d\zeta}{\zeta - z} \right\} \prod_{n=1}^{N} \frac{(z - z_n)^2 (z + k_0^2/z_n^*)^2}{(z - z_n^*)^2 (z + k_0^2/z_n)^2},
$$

and

$$
\theta_+ - \theta_- = \frac{1}{2\pi} \int_{\Sigma} \log \det (I_m + \rho^i(\zeta^*)\rho(\zeta)) \frac{d\zeta}{\zeta} - 8 \sum_{n=1}^{N} \delta_n.
$$

Let $\{z_n\}_{n=1}^{N_1}$ be all the simple zeros of $\det a(z)$, and Let $\{\tilde{z}_n\}_{n=1}^{N_2}$ be all the simple zeros of $\det a(z)$. Then we can write the trace formula and the $\theta$-condition as

$$
\det a(z) = \exp \left\{ -\frac{1}{2\pi i} \int_{\Sigma} \log \det (I_m + \rho^i(\zeta^*)\rho(\zeta)) \frac{d\zeta}{\zeta - z} \right\} \prod_{n=1}^{N_1} \frac{(z - z_n)^2 (z + k_0^2/z_n^*)^2}{(z - z_n^*)^2 (z + k_0^2/z_n)^2} \prod_{n=1}^{N_2} \frac{(z - \tilde{z}_n)^2 (z + k_0^2/\tilde{z}_n^*)^2}{(z - \tilde{z}_n^*)^2 (z + k_0^2/\tilde{z}_n)^2},
$$

$$
\theta_+ - \theta_- = \frac{1}{2\pi} \int_{\Sigma} \log \det (I_m + \rho^i(\zeta^*)\rho(\zeta)) \frac{d\zeta}{\zeta} - 4 \sum_{n=1}^{N_1} \delta_n - 8 \sum_{n=1}^{N_2} \tilde{\delta}_n,
$$

where $z_n = |z_n| e^{i\delta_n}$ and $\tilde{z}_n = |\tilde{z}_n| e^{i\tilde{\delta}_n}$.

Acknowledgments

This work is supported by the National Natural Science Foundation of China under Grant Nos. 12201622 and 11975306.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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