Spontaneous freezing in driven-dissipative quantum systems

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Attaining a deeper knowledge of critical non-equilibrium phenomena is a standing challenge in the fields of open quantum systems and many-body physics. For instance, a comprehensive understanding of how different dissipative phases coexist in multistable systems is a problem of fundamental importance and of practical implications, e.g. in the design of quantum sensors. An interesting question is whether the intermittent switching between states typically reported in multistable systems would manifest in non-ergodic systems. In this work, we address this problem and report a novel non-ergodic phenomenon where a system undergoing dissipative evolution spontaneously selects a state and remains there for the rest of the evolution without any possible intermittency. This “freezing” of the dynamics occurs at the level of individual quantum trajectories and emerges as a consequence of a strong symmetry. Whilst the strong symmetry implies the existence of a conservation law, this law is broken for the individual trajectories. We study this effect through a generalized version of collective resonance fluorescence, where the standard superradiant decay is substituted for a squeezed, collective superoperator.

I. INTRODUCTION

Non-equilibrium systems are present in a wide variety of areas, including physics, life sciences, sociology and finance. In physics, a typical non-equilibrium situation is realized when driving from an external source is compensated by dissipation to the environment. This is the case in numerous examples of many-body and cavity QED systems, such as exciton polaritons [1–3], Rydberg ensembles [4, 5], superconducting circuits [6], trapped atoms [7–9] or in mechanical systems [10–12]. In the ongoing effort to deepen our understanding of out-of-equilibrium phenomena, which typically differ from their classical or quantum counterparts, driven respectively by thermal and quantum fluctuations. There is an important link between DPTs and the spectral properties of the Liouvillian superoperator, \( L \), that governs the dynamics of the density matrix, \( \dot{\rho} = L\rho \). Its largest eigenvalue \( \lambda_0 \) is zero, and the corresponding eigenvector is the steady state \( \rho_0 \). In the usual description of DPTs, a transition occurs when the eigenvalue with the second largest real part \( \lambda_1 \), often called the asymptotic decay rate (ADR), tends to zero, which then implies the existence of several degenerate steady states.

A significant amount of research has been devoted to the definition and characterization of DPTs [19, 20], and to the study of the associated, interrelated phenomena of bistability [3–5, 28–30], hysteresis [2, 31], intermittency [6, 30–34], multimodality [29, 33], metastability [35] and symmetry breaking [36, 37] in open quantum systems. All these phenomena are understood as different manifestations of the coexistence of several non-equilibrium phases.

In the quantum-jump formalism, the dynamics of a driven-dissipative system can also be viewed as a stochastic process, where different realizations of the evolution recover, when ensemble averaged, the predictions of the density matrix [38–40]. From this point of view, the nature of the coexistence between non-equilibrium phases becomes even more intriguing in the particular case of non-ergodic dynamics, in which individual realizations do not necessarily recover, when time averaged, the expected results from ensemble averages [41].

A deeper understanding of this phenomena is not only of a fundamental interest, but also has technological implications. For instance, it has been proven that the multistable behaviour of open quantum systems in the vicinity of a DPT provides enhanced sensitivity to external perturbations [42, 43]. These metrological properties are strongly affected by the intermittent switching between the coexisting phases that has been typically reported in these systems [6, 30–34]. It is therefore a relevant question whether such an intermittency is still present in non-ergodic systems. However, our understanding of non-equilibrium systems and DPTs is hindered by the enormous computational difficulty typically found when dealing with large quantum open systems. It is thus highly desirable to work with exactly solvable systems or at least computationally tractable models that yield an insight about the physics in the thermodynamic limit. Unfortunately the existence of tractable many-body non-equilibrium models is still scarce.

In this work, we address these questions and report a non-ergodic phenomenon where, in a single realization of the dynamics, a system randomly selects one of several possible stable states and remains there for the rest of the evolution. We refer to this effect as spontaneous freezing. To introduce and discuss the phenomenon, we analyse a model that can be solved numerically, yet displays a rich variety of non-ergodic dynamics and provides us with non-trivial information about the thermodynamic limit. Our model consists of a coherently-driven spin ensemble with squeezed, collective

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spin decay. Squeezed decay refers to a quantum jump operator that includes both lowering and raising collective spin operators $S^\pm_N$, with relative weights parametrized by a squeezing angle $\theta$. The amplitude $\Omega$ of the driving field and the squeezing angle $\theta$ are the main tunable parameters. We obtain a phase diagram in the $(\Omega, \theta)$ plane, characterizing the dissipative phases where the system undergoes non-ergodic dynamics. We then identify regions of the phase diagram where spontaneous freezing occurs, in contrast to the intermittency usually reported in bistable systems [6, 30–34]. The emergence of this effect is strongly connected to the existence of a strong symmetry of the dynamics [44]. A strong symmetry implies a conservation law of some operator $L$; a strong symmetry of the dynamics \[ L \phi = 0. \] A strong symmetry of the dynamics is usually reported in bistable systems [6, 30–34]. The emergence of this effect is strongly connected to the existence of a strong symmetry of the dynamics [44]. A strong symmetry implies a conservation law of some operator $L$; a strong symmetry of the dynamics.

The paper is organized as follows. In Sec. II, we introduce the model of squeezed superradiance, describing the phase diagram and steady-state of the system. In Sec. III, we analyse the Liouvillean spectrum of this model and characterize symmetries and regimes of non-ergodicity. In Sec. IV, we describe the phenomenon of spontaneous freezing, and discuss it in the context of thermodynamics of quantum trajectories and phase transitions. Finally, in Sec. V, we analyse signatures of critical, dissipative dynamics in observables of the light emitted by the system.

II. MODEL AND PHASE DIAGRAM

A. Squeezed superradiance: derivation of the spin master equation

The model of squeezed superradiance that we consider in this work is given by the following master equation for the reduced density matrix of an ensemble of $N$ spins ($\hbar = 1$):

\[ \dot{\rho} = -i [H, \rho] + \frac{\Gamma}{2J} L_D \rho, \]

where $L_D [\rho] = 2O\rho O^\dagger - [O^\dagger O, \rho]$ is the usual Lindblad superoperator, and the operator $D_\theta$ describes the quantum jumps undergone by the system

\[ D_\theta \equiv \cos(\theta) S_- + \sin(\theta) S_+. \]

In these equations, $\{S_-, S_+\}$ are collective spin operators obeying angular momentum commutation relations, $\Omega$ is the driving amplitude, $\Gamma$ is the quantum-jump rate, and $J = N/2$ is the total angular momentum, which is conserved in the dynamics. Notably, $D_\theta$ includes both raising and lowering operators, with a weight that we parametrize by the angle $\theta$.

The dynamics in Eq. (1) emerge as the strongly-dissipative limit of the following Hamiltonian

\[ H = \Omega S_x + \frac{g}{\sqrt{N}} \left\{ S_+ \left[ \cos(\theta) a + \sin(\theta) a^\dagger \right] + \text{h.c.} \right\}. \]

This Hamiltonian describes a driven spin ensemble coupled to a single cavity mode in the rotating frame of the driving, with $a$ the bosonic annihilation operator of the cavity and $g$ the spin-cavity coupling rate. Since the total angular momentum $J$ is conserved, the spin ensemble can be described as a single big spin; this can be implemented, for instance, with multi-component atomic condensates [45, 46]. The tunable coupling terms in Eq. (3) can be achieved via cavity-assisted Raman transitions; this approach has been proposed as a way to implement effective Dicke models [47] and used, successfully, to observe alternative forms of the superradiant phase transition [48–51] in atomic condensates [7, 8, 52, 53] and

FIG. 1: Phase diagram and steady state. (a) The phase diagram can be divided into a ferromagnetic (F) and an thermal (T) phase, separated by the critical line $\Omega_c(\theta)$ given by Eq. (5) (white, dashed lines). There is spin-up and spin-down version of each of these phases, separated by the strong-symmetry line $\theta = \pi/4$. (b) Spin Wigner functions of the exact steady states of master equation (1) for a finite system with $N = 50$ at different points $(\Omega, \theta)$, corresponding to: (i) $(0, 0)$, (ii) $(0.5\Gamma, 0)$, (iii) $(0.85\Gamma, 0)$, (iv) $(1.2\Gamma, 0)$, (v) $(0, \pi/8)$, (vi) $(0.6\Gamma, \pi/4)$, (vii) $(0, 0.95\pi/4)$, (viii) $(1.2\Gamma, 0.95\pi/4)$. Together, we plot the vector field of derivatives described by the mean-field equations (45).
thermal atoms [54, 55]. The great control and versatility provided by these schemes has motivated research on generalized non-equilibrium Dicke models [56, 57].

In this work, we are focusing on strongly dissipative versions of these systems—where the fast cavity decay yields an effective, collective spin dissipation—which have attracted interest for their applications to the dissipative generation of spin squeezing and entanglement in the steady state [58, 59]. By taking into account that the cavity experiences dissipation at a rate $\gamma$, the evolution of the system is described by the master equation [60] $\dot{\rho} = -i[H, \rho] + \gamma/2 \mathcal{L}_a[\rho]$. In the limit $\gamma \to \infty$, the bosonic field tends to a stationary vacuum state, and its adiabatic elimination [59] yields the effective dynamics for the spins of Eq. (1), with $\Gamma = 2g^2/\gamma$. It is easy to deduce that the dark state of $D_\theta$ is a spin squeezed state [58], which brings us to refer to the dissipative part of Eq. (1) as a "squeezed decay". Note that, when $\theta = 0$, the model corresponds to the standard case of collective resonance fluorescence [59, 61, 62].

**B. Phase diagram**

The non-equilibrium phases of the system in the $(\Omega, \theta)$ plane are summarized in Fig. 1(a), and the corresponding steady-state observables computed exactly for a finite system ($N = 50$) are depicted in Fig. 2. Fig. 1(b) depicts the steady state of a finite system ($N = 50$) at several points of the phase diagram using the spin Wigner function [63, 64]. Additionally, we plot the vector field of derivatives obtained through a mean field approach (see Appendix I). We can divide the phase diagram into two types of phases:

i) The *ferromagnetic (F) phase* is characterized by a well-defined magnetization (c.f. Fig. 2(a)), a diverging spin-squeezing as we approach the phase transition (Fig. 2(b)), small fluctuations in the counting distributions of quantum jumps (described here by the zero-delay, second-order correlation function of the output field, $g^{(2)} \equiv \langle D_\theta^2 D_\theta^2 \rangle / \langle D_\theta^2 \rangle^2$ (Fig. 2(d))), high purity $\text{Tr}\rho^2$ (not shown) and ergodic dynamics. Any initial state eventually relaxes into a stationary, highly pure gaussian steady-state. In the thermodynamic limit, this phase is well described within a Holstein-Primakoff approximation. 

ii) In the *thermal phase* the steady-state is highly mixed, and close to the infinite-temperature state $\rho \propto \mathbb{1}$. This phase is characterized by a mean zero magnetization (Fig. 2(a)), small purity (not shown), large spin fluctuations, high rate of quantum jumps (activity) (Fig. 2(c)) and large fluctuations in the output field (Fig. 2(d)). As we discuss further below, this phase displays a vanishing asymptotic decay rate (ADR) that leads, in the thermodynamic limit $N \to \infty$, to a closed gap and non-ergodic dynamics, which manifests itself through closed orbits in the mean-field approach (c.f. point (iv) in Fig. 1(b)).

Both phases have a spin-down ($\downarrow$) and spin-up ($\uparrow$) version at each side of the line $\theta = \pi/4$; each of them being a spin-flipped version of the other. Therefore, defining

$$\Gamma_- \equiv \Gamma \cos^2 \theta, \quad \Gamma_+ \equiv \Gamma \sin^2 \theta,$$

all the results and equations obtained for $\theta \leq \pi/4$ are directly applicable in a spin-flipped basis for $\theta \geq \pi/4$ just by exchanging $\Gamma_- \leftrightarrow \Gamma_+$. Hereafter, the analytical results that we provide refer to the spin-down phases ($\theta \leq \pi/4$).

In Appendix I we show that, using a mean-field approach, the transition from the ferromagnetic to the thermal phase oc-
curves at the critical driving:
\[ \Omega_c(\theta) = \Gamma_- - \Gamma_+ = \Gamma(\cos^2 \theta - \sin^2 \theta). \] (5)

C. Spin observables

We consider now the expectation values of the normalized spin operators \( s_i \equiv S_i/J \), \( i \in \{x, y, z\} \) in the steady state. In the ferromagnetic phase, these can be obtained by a displaced Holstein-Primakoff (HP) expansion (see Appendix II); the results are the same as the mean-field predictions, with corrections to order \( 1/J \):

\[
\begin{align*}
\langle s_x \rangle &= M + \mathcal{O}(1/J), \\
\langle s_z \rangle &= 0 + \mathcal{O}(1/J), \\
\langle s_y \rangle &= \sqrt{1 - M^2} + \mathcal{O}(1/J),
\end{align*}
\] (6a, b, c)

where \( M \) is the steady-state magnetization that reads:

\[
M = -\sqrt{1 - \left( \frac{\Omega}{\Gamma_- - \Gamma_+} \right)^2}.
\] (7)

The \( 1/J \) corrections are given by the solution of non-quadratic master equations and therefore analytical expressions are difficult to obtain. It is however possible to get expressions for the spin fluctuations \( \Delta s_{\perp}^2 \) to order \( 1/J \); this is one of the main advantages of using a HP expansion, since it allows to describe the metrological properties of the spin ensemble [65]. In particular, reduced fluctuations along one of the spin directions provides enhanced phase sensitivity in atomic interferometers [66, 67] and greater stability in atomic clocks [68]. States displaying such reduced fluctuations are said to be spin squeezed [65, 69, 70]; the degree of spin squeezing \( \xi_\perp \) along any axis \( \vec{u}_\perp \) perpendicular to the mean spin direction is a popular figure of merit, useful as a witness of entanglement [71] and as a direct measure of the phase sensitivity achievable in interferometry protocols. This quantity can be defined as [69]:

\[
\xi_\perp^2 = \frac{N(\Delta S_\perp)^2}{\langle S^2 \rangle}. \] (8)

According to this definition, a state is spin-squeezed if a direction \( \vec{u}_\perp \) exists such that \( \xi_\perp^2 < 1 \). In our model, the optimal squeezing direction is always the \( u_z \) axis (see Appendix II-D). Using the HP approximation, we find the following expression for the spin squeezing in the ferromagnetic phase:

\[
\xi_\perp^2 = \frac{N(\Delta S_\perp)^2}{\langle S^2 \rangle} = (1 - M) \left( 1 + \frac{\Gamma_+ - \sqrt{\Gamma_- - \Gamma_+}}{\Gamma_- - \Gamma_+} \right). \] (9)

The analytical results in Eq.(6) and (9) are shown in Fig. 2(e-f), compared with numerical calculations for finite system size. Equation (9) shows that, in the thermodynamic limit, spin squeezing diverges (i.e. \( \xi_\perp^2 \to 0 \)) in the vicinity of the critical line, where \( M \to 0 \). This implies a greatly enhanced phase sensitivity and the emergence of many-body correlations, which are general properties associated to second-order phase transitions [19].

![Graph showing the Liouvillian gap for N = 100, Ω = 0.4Γ. In the thermodynamic limit, the gap closes at the critical line Ω_c(θ) (white, dashed). In logarithmic scale, we observe a closing of the gap for finite J at the point θ = π/4 due to the strong symmetry. (b) Liouvillian eigenvalues for a system size J = 10 and Ω = 200Γ.](image)

III. SPECTRAL PROPERTIES OF THE LIOUVILLIAN

Having characterized the phase diagram of the model, we analyse now the spectral properties of the Liouvillian, which contains essential information about the different dissipative phases [19, 20, 35]. In the ferromagnetic phase, we can use the Holstein-Primakoff expansion to obtain an expression for the Liouvillian gap in the thermodynamic limit (see Appendix II):

\[
\lambda = (\Gamma_- - \Gamma_+)M, \] (10)

showing that the gap closes when \( M = 0 \), i.e. at the transition from a ferromagnetic to a thermal phase, in agreement with the usual description of DPTs [19, 20]. Figure 3(a) depicts the exact ADR for a finite system, computed by numerical diagonalization. The ADR in the thermal phase features a small but finite value that, as we prove below, scales with system size as \( 1/J \). Below we focus on this gapless region, which is the most promising in terms of non-ergodic dynamics.

Strong symmetry. Even for a finite system, the ADR closes exactly at the line that separates the \( T_1 \) and \( T_1 \) phase, \( \theta = \pi/4 \), as can be seen from the logarithmic-scale plot.
in Fig. 3(a). The reason for this exact closing, that occurs even at finite system size, is the existence of a strong symmetry at $\theta = \pi/4$. For a general Liouvillian given by $\mathcal{L}\rho = -i[H,\rho] + \sum_\mu (2L_\mu,\rho) + \{L_\mu,\rho\}$, a strong symmetry is defined by a unitary operator $A$ which fulfills

$$[H, A] = 0, \quad (11a)$$

$$[L_\mu, A] = 0. \quad (11b)$$

As demonstrated in Ref. [44], the existence of a strong symmetry implies that, if $A$ has $n_A$ distinct eigenvalues, there are at least $n_A$ distinct steady states of $\mathcal{L}$ with eigenvalue $0$. In the particular case $A = H = L$ (with $L \equiv L_1$ being the only quantum-jump operator), the density matrices $\rho^{(m)} = |m\rangle\langle m|$ are all steady states with $|m\rangle$ being the eigenstates of $A$. In our system, we find $\{H, L\} \propto S_z$ at the strong-symmetry point $\theta = \pi/4$, which means that $S_z$ is a strong symmetry of the Liouvillian and that all its eigenstates are steady-states, explaining the exact closing of the ADR. The existence of a strong symmetry at $\theta = \pi/4$ and its relation to the effect of spontaneous freezing will be a central point in the following discussions.

**Imaginary eigenvalues** A more general analysis of the Liouvillian spectrum in the large driving limit provides further insight into the different ways in which the gap can be closed well within the thermal phase and reveals the existence of eigenstates with purely imaginary values. In the limit $\Omega \gg \Gamma/J$, we can remove counter-rotating terms in the master equation and obtain

$$\dot\rho \approx -i\Omega[S_x, \rho] + \frac{\Gamma_\theta}{2J} S_\theta \rho + \frac{\chi_\theta}{4J} \left(\mathcal{L}_{S_\uparrow} \rho + \mathcal{L}_{S_\downarrow} \rho\right),$$

(12)

where we have defined the ladder operators in the $x$-direction, $S_x^\pm \equiv \frac{1}{2}(S_x \pm iS_y)$, and $\Gamma_\theta \equiv \Gamma(\cos \theta + \sin \theta)^2$, $\chi_\theta \equiv \Gamma(\cos \theta - \sin \theta)^2$. For $\theta \neq \pi/4$, the steady state solution has purely imaginary eigenvalues of the Liouvillian.

**Dynamical symmetries.** Recently, it was shown that absence of a stationary state and the presence of long-time oscillatory dynamics in open quantum systems can be directly implied by the existence of a dynamical symmetry operator $A$ fulfilling [73]:

$$[H, A] = AA,$$

$$[L_\mu, A] = [L_\mu, A] = 0. \quad (15a)$$

In that case, the matrices $\rho^{(m)} = A^n \rho_{\infty}(A^\dagger)^m$, with a form similar to the states that we defined in Eq. (14), are eigenvectors of the Liouvillian with purely imaginary eigenvalues:

$$\mathcal{L}\rho^{(m)} = i(m - n)A^\dagger.'$$

Despite the similarities, in the particular case of our model, the operator $S_x$ does not fulfill the conditions (15) of a dynamical symmetry. However, in the $\Omega/J \gg 1$ limit, where the system is in essence purely Hamiltonian, conditions (15) are immediately satisfied, yielding purely imaginary eigenvalues that are integer multiples of $\Omega$. We note that, in general, this will not happen for any arbitrary dissipative system in the purely Hamiltonian limit. Here, the existence of a dynamical symmetry is a consequence of having a spin Hamiltonian with equally spaced energy levels, preventing the mechanisms of eigenstate thermalization typical of closed many-body systems [76–78].

**IV. SPONTANEOUS FREEZING OF THE DYNAMICS**

Having completely characterized the dissipative phases of the system and the spectral properties of the Liouvillian, we are ready to describe the dissipative effect of spontaneous freezing. Several manifestations of the coexistence of multiple steady states, such as bistability and intermittency, have attracted a great deal of attention in recent years [3–6, 28–34]. These critical phenomena are typically discussed in contexts in which DPTs take place in the thermodynamic limit. The model of squeezed superradiance differs from these situations, since the gap can be exactly closed even for a finite system. Our motivation is to analyse what are the consequences of this exact closing on the possible multistable behaviour of the system, and in particular, how does the system evolve when...
an initial state is composed of a superposition of these stables states. In this section, we report the effect of spontaneous freezing in individual unravelings of the driven-dissipative dynamics, we obtain a closed form expression for the activity distributions, and then put these results in the context of recent research on the thermodynamics of quantum trajectories.

A. Freezing in individual trajectories

Dissipative evolution of the system density matrix admits an alternative interpretation in terms of individual, stochastic evolution of pure wavefunctions, the so-called quantum-jump or Monte Carlo wavefunction approach [39]. The predictions of the master equation are recovered when one takes an ensemble average over a sufficiently high number of trajectories.

The evolution of a single trajectory can be summarized as follows. At every differential time step $dt$, for each element of the set $\{\gamma_i/2\}L_0,\rho\}$ in the master equation, the wavefunction $\psi(t)$ can randomly undergo a quantum jump with probability $p_i = \gamma_i(t)|O_i|\psi(t)|dt$ that transforms the system, under proper normalization, as

$$|\psi(t + dt)| \propto O_i|\psi(t)|.$$  \hfill (17)

When no jump occurs, the wavefunction evolves under the action of a non-Hermitian Hamiltonian

$$|\psi(t + dt)| \propto (1 - i\tilde{H}\,dt)|\psi(t)|,$$  \hfill (18)

where $\tilde{H} \equiv H - i\sum \gamma_i/2O_i^\dagger O_i$. These trajectories can be physically understood as individual, stochastic realizations of an experiment where quantum jumps are recorded [103]. If the system is ergodic, a time average over a single trajectory also recovers the predictions of the master equation.

In the presence of a strong symmetry, the system is not ergodic, and multiple degenerate steady states can exist [44]. The actual steady state of the system is then composed by a particular superposition of these states, fixed by the initial conditions [20, 35]. However, because the evolution is not necessarily ergodic, it is unclear whether a single trajectory will explore all these states, which is the main assumption behind the notion of intermittency [6, 30–34].

In the particular case $\theta = \pi/4$, the model of squeezed superradiance that we study here represents one of the simplest implementations of a strong symmetry. In this situation, the unraveling of the evolution in individual trajectories features an effect that we term "spontaneous freezing" of the dynamics. The phenomenon is depicted on Fig. 4 (a–c): after initializing the state in a given superposition—in this example, of the $S_x$ eigenstates $|0\rangle$, $|3\rangle$ and $|5\rangle$—the stochastic, dissipative evolution of the wavefunction brings it into one of the eigenstates of $S_x$, with the probability of being in any of the other ones decaying exponentially with time; the evolution is effectively frozen in one eigenstate for an individual realization of the dynamics.

It is indeed easy to show that an eigenstate of a strong symmetry is invariant under this stochastic evolution. To prove this, we consider the general form of any wavefunction undergoing a stochastic, dissipative evolution described by $\tilde{H}$ and the set of quantum jump operators $\{L_\mu\}$. Starting from an initial state $|\psi(0)\rangle$, the wavefunction evolves for a time $t$ experiencing $n$ quantum jumps at times $(t_1, \ldots, t_n) < t$ with jump operators $(L^{(1)}, \ldots, L^{(n)})$, where $L^{(i)} \in \{L_\mu\}$. The form of the wavefunction is then given by a nonunitary evolution $|\psi(t)\rangle = \mathcal{N}^{-1}\tilde{U}(t, t_n, \ldots, t_0)|\psi(0)\rangle$, where $\mathcal{N}$ is a normalizing constant, and $\tilde{U}(t, t_n, \ldots, t_0)$ is an evolution operator given by:

$$\tilde{U}(t, t_n, \ldots, t_0) = e^{-i\tilde{H}(t-t_n)} \prod_{m=1}^n L^{(m)} e^{-i\tilde{H}(t_{m-1}-t_m)},$$  \hfill (19)

with $\prod_{m=0}^n O_m \equiv O_n \cdot O_{n-1} \cdots O_0$. Let us consider a strong symmetry operator $A$; Eqs. (11a, 11b) then imply that $[A, \tilde{U}] = 0$. Therefore, if $|\psi(0)\rangle$ is an eigenstate of a strong symmetry $A|\psi(0)\rangle = \lambda|\psi(0)\rangle$, we obtain

$$A|\psi(t)\rangle = A\tilde{U}(t, t_n, \ldots, t_0)|\psi(0)\rangle = \tilde{U}(t, t_n, \ldots, t_0)A|\psi(0)\rangle = \lambda|\lambda(t)\rangle,$$  \hfill (20)

i.e. an eigenstate of $A$ remains invariant at the level of individual trajectories. This proof can be extended similarly to the eigenstates of any power $A^n$. This fact may suggest that any quantum trajectory could eventually get "trapped" into one of these eigenstates, in a picture somewhat analogue to dark-state cooling [79] or population trapping [80]. However, it is not

![FIG. 4: Three different quantum trajectories at $\theta = \pi/4$ for the same initial state (a superposition of three eigenstates of $S_z$). Panels (a–c) show the three possible types of trajectories that occur with an spontaneously broken symmetry. The inset in (a) shows the exponential decrease of the occupation of non-selected states. Parameters: $J = 5$, $\Omega = 0.8\Gamma$ (irrelevant here).](image-url)
clear whether the system will end up in one eigenstate in the first place, for any possible quantum trajectory or initial state.

Here, we provide an example in which we can prove that this is indeed the case: the master equation \( \dot{\rho} = -i[\hat{A}, \rho] + \Gamma/(2J) \mathcal{L}_A(\rho) \); i.e. dynamics with a single quantum jump \( \hat{L} \) and a general, Hermitian strong-symmetry \( A \propto H \propto L \).

In order to prove the emergence of spontaneous freezing, we set \( t_0 = 0 \) and consider an initial state \( |\psi(0)\rangle = \sum_m c_m(0)|m\rangle \), expanded in the basis of eigenstates of \( \hat{A} \), with eigenvalue \( m \).

For any general quantum trajectory that evolves for a time \( t \) undergoing \( n \) quantum jumps, the probability for the final state to be in an eigenstate of \( |m\rangle \) takes the form (see Appendix III):

\[
p(m; t, n) = \frac{1}{\mathcal{N}} \left( e^{-\frac{\Delta^2}{2}} \right)^{t\Gamma/\hat{J}} |c_m(0)|^2,
\]

with \( \alpha = nJ/(t\Gamma) \) and \( \mathcal{N} \) a normalizing constant. The exponent \( t\Gamma/\hat{J} \) in Eq. (21) tends to enhance the maximum of the function in parenthesis as time increases. Hence, after normalization, \( p(m; t, n) \) tends to zero for all \( m \) except for the optimum value. The function \( e^{-\frac{\Delta^2}{2}} \) has a maximum at \( x = \alpha \).

Therefore, the only values of \( m \) with non-zero \( p(m; t, n) \) for \( t \gg J/\Gamma \) are those \( m \) such that \( \alpha = \hat{m} \) is the eigenvalue of \( A^2 \) closest to \( \alpha^2 \). Equation (21) thus encapsulates the essence of the spontaneous freezing effect and is the main result of this paper: for \( t \gg J/\Gamma \), any general trajectory will be trapped in an eigenstate of \( A^2 \) uniquely determined by the initial conditions and the number of jumps undergone by the system. A quantum-measurement understanding of dissipative dynamics can be invoked to describe this result [81, 82]: the information provided by the quantum jumps makes one of the eigenstates of \( A^2 \) more and more likely and continuously updates the state accordingly. We note, however, that this view applies to any dissipative dynamics and does not necessarily imply the effect that we report here; in fact, in most general cases, the evolution will be ergodic. Here, due to the existence of a strong symmetry \( A \) (which has an associated conservation law), this continuous updating of the wavefunction eventually brings it into an eigenstate of \( A^2 \), inducing a breakdown of the conservation law at the trajectory level.

For the particular case that we study in this paper, \( A = S_x \), \( m = -J, \ldots, J \); therefore the eigenstates of \( A^2 \) are doubly degenerate. The probability distribution for any quantum trajectory is, for \( t \gg J/\Gamma \):

\[
p(m; t, n) = \sum_m \frac{(\delta_{m,\hat{m}} + \delta_{m,-\hat{m}})|c_m(0)|^2}{|c_{\hat{m}}(0)|^2 + |c_{-\hat{m}}(0)|^2},
\]

with \( \hat{m} \) the natural number \( \leq J \) closest to \( \sqrt{nJ/(\Gamma t)} \). The resulting probability distribution versus \( n/t \) is plotted in Fig. 5, for an initial state made of an equal superposition of eigenstates.

**B. Activity distribution**

Now that we have presented the spontaneous freezing effect, it is instructive to analyse it in terms of one of the main observables of interest when discussing multistability; the activity [34, 83]. The activity is defined as the mean number of quantum jumps undergone by the system per unit time; this can be defined through the probability distribution \( p_T(K) \) of counting \( K \) jumps on a time \( T \). Following our previous discussion, we assume the existence of a strong symmetry \( A \) with eigenstates \( |m\rangle \) and only one quantum jump operator, \( \hat{L} = \sqrt{\Gamma/J}A \). We consider an initial state with the form

\[
\rho(0) = \sum_m c_m|m\rangle\langle m|.
\]

This initial state is a steady-state of the system, meaning that its preparation can always be conceived as the long-time limit of another initial state. Other choices of \( \rho(0) \) may involve transient effects that will be irrelevant in the limit \( T \rightarrow \infty \). We can then prove (see Appendix IV) that the photon counting distribution takes the form:

\[
p_T(K) = \sum_m \frac{1}{K!} \left( \frac{T\hat{m}^2}{J} \right)^K e^{-\Gamma\hat{m}^2T/J} c_m,
\]

which is dependent on the initial state. This equation presents the multimodal structure depicted in Fig. 6(a), where we plot it for the particular case of our model, where \( A = S_x \). The physical interpretation is simple: to every eigenstate \( |m\rangle \) of \( A \), there is an associated steady state:

\[
\rho_0^{(m)} = |m\rangle\langle m|,
\]

with a corresponding quantum-jump rate of \( \text{Tr}[\hat{L}^\dagger \hat{L} \rho] = m^2\Gamma/J \). The set of \( \rho_0^{(m)} \)s conform a basis, meaning that any combination of these steady-states is a also steady state. The asymptotic state

\[
\rho_{ss} = \lim_{t \rightarrow \infty} e^{\hat{L}t} \rho(0) = \sum_m \text{Tr}[\rho_0^{(m)}, \rho(0)]\rho_0^{(m)},
\]
is therefore strongly dependent on the initial state and given by its overlap with each of the $\rho_0^{(m)}$. Those $\rho_0^{(m)}$ having a finite overlap with $\rho(0)$ will manifest as a distinct peak in the counting distribution $p_T(K)$, centered at the value $K_m = Tm^2\Gamma/J$.

Multimodality (as a signature of multistability) has been recently associated with dynamical phase transitions [34] that feature the coexistence of two phases in time, with a stochastic switching between these phases that has been observed experimentally on multiple occasions [6, 30–33]. While we obtain a clear multimodal structure for the activity distribution, our results on the spontaneous freezing do not match this notion of intermittency. Let us therefore put our results in the context of the theory used in Ref [34]: the thermodynamics of quantum trajectories.

C. Thermodynamics of quantum trajectories: Dynamical phase transition

1. Brief introduction to thermodynamics of quantum trajectories

Recently, several works [34, 83–85] have approached the questions of multimodality and intermittency from the perspective of the thermodynamics of quantum trajectories. This approach regards the set of quantum trajectories in which the dynamics can be unraveled as a statistical ensemble that can be analysed using the tools of statistical mechanics. In the following, we briefly outline this theory (a comprehensive description can be found in Refs. [34, 83]) and discuss its implications in systems, such as the one we report here, where spontaneous freezing of the dynamics occurs.

Let us consider a system governed by the master equation $\dot{\rho} = \mathcal{L}\rho = -i[H, \rho] + L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\}$. The evolution of $\rho$ can be unraveled as a set of quantum trajectories [38–40] by which a conditional density matrix $\rho_K(t)$ can be built from the ensemble average of all the trajectories of duration $t$ having $K$ quantum jumps. The activity distribution is then given by $p_K(t) = \text{Tr}\rho_K(t)$. We can define a generating function $Z = \langle e^{sK}\rangle$:

$$Z = \sum_{K=0}^{\infty} e^{sK}p_K(t) = \text{Tr} \sum_{K=0}^{\infty} e^{sK}\rho_K(t) = \text{Tr}\rho_s(t), \quad (27)$$

where $\rho_s(t) = \sum_{K=0}^{\infty} e^{sK}\rho_K(t)$ is a Laplace transformed density matrix that evolves according a tilted master equation:

$$\mathcal{W}_s\rho_s = \dot{\rho}_s = -i[H, \rho_s] + e^sL\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\}, \quad (28)$$

and the “counting field” $s$ is a variable conjugate to $K$. For $s = 0$, Eq. (28) corresponds to the normal master equation, $\mathcal{W}_s = \mathcal{L}$. For $s \neq 0$, Eq. (28) is not a physical trace-preserving master equation, and describes a class of dynamics in which the quantum jumps are biased by the factor $e^s$. Despite $\mathcal{W}_s$ being unphysical, its spectral properties contain valuable information about the fluctuations of the ensemble of trajectories. In particular, the partition function acquires, in the long time limit, a large deviation form $Z \simeq e^{s\lambda(s)}$, with $\lambda(s)$ the eigenvalue of $\mathcal{W}_s$ with the largest real part. This allows us to write the activity or mean emission rate as:

$$\langle k \rangle = \langle K \rangle/t = \left. \frac{1}{t} \frac{\partial Z}{\partial s} \right|_{s=0} = \left. \frac{\partial \lambda(s)}{\partial s} \right|_{s=0}. \quad (29)$$

This suggests the definition of a $s$-dependent emission rate $\langle k \rangle_s = \partial \lambda/\partial s(s)$. Equivalently, fluctuations in the activity can be described by Mandel’s $Q$ parameter, $Q = (\langle K^2 \rangle - \langle K \rangle^2)/\langle K \rangle - 1$, given by:

$$Q = \left. \frac{\partial^2 \lambda/\partial s^2}{\partial \lambda/\partial s} \right|_{s=0}. \quad (30)$$

To sum up, the behavior of $\lambda(s)$ around the vicinity of $s = 0$ characterizes the fluctuations of the ensemble of quantum trajectories.

The connection to thermodynamics put forward in [83] can be made by assuming that, in the long-time limit, $p_K(t)$ also acquires a large deviation form

$$p_K(t) \simeq e^{-t\varphi(K/t)}. \quad (31)$$

If $p_K(t)$ describes the probability distribution of an statistical ensemble, then the rate function $\varphi(K/t) = -\ln(p_K(t)/t)$ plays the role of an entropy density [86]. By plugging Eq. (31)
into Eq. (27), we obtain directly that \( \varphi(k = K/t) \) and \( \lambda(s) \) are related by a Legendre transformation:

\[
\lambda(s) = \max_k [ks - \varphi(k)],
\]

meaning that \( \lambda(s) \) has the properties of a free energy. The inverse transformation

\[
\varphi(k) = \max_s [ks - \lambda(s)]
\]

is a useful relation that allows us to obtain \( \varphi(k) \) from the knowledge of \( \lambda(s) \), which can in turn be computed from the eigenvalues of \( \mathcal{W}_s \). However, this relation follows from the Gärtner-Ellis Theorem [86], that requires \( \lambda(s) \) to be differentiable for all \( s \in \mathbb{R} \) or, equivalently, \( \varphi(k) \) to be concave for all \( k \in \mathbb{R} \). These are precisely the conditions that are violated when a phase transition occurs.

2. Multistability and breaking of the intermittency: connection to known models

In Ref. [34], the coexistence of dynamical phases was linked to a discontinuity in the \( s \)-dependent order parameter \( \langle k \rangle_s \) at the physical point \( s = 0 \), i.e. a first-order phase transition. Based on this temporal coexistence between phases, such a first-order phase transition was then referred to as a dynamical phase transition. As we show in Fig. 7(a), where we plot a numerical calculation of \( \langle k \rangle_s \) versus \( \theta \), the closing of the gap at the strong symmetry point gives rise to such a discontinuity; the limit \( s \to 0^+ \) features a bright phase characterized by a high activity, whereas for \( s \to 0^- \) we find a dark phase with virtually no quantum jumps. The discontinuity turns into a continuous crossover as we depart from the point \( \theta = \pi/4 \).

We argue that, in our model, intermittency is linked to the existence of such a crossover. We will show that intermittency is replaced by the effect of spontaneous freezing at the point \( \theta = \pi/4 \), where the crossover becomes a first-order phase transition.

Let us first prove that a strong symmetry implies that \( \langle k \rangle_s \) is discontinuous (a similar analysis was performed in Ref. [36]). Following our previous discussions, we focus on the case where a strong symmetry \( \mathcal{A} \) is present, and \( L = \sqrt{\Gamma/J} \mathcal{A} \).

We can immediately see that the steady states \( \rho_0^{(m)} \) in Eq. (25) are also eigenstates of \( \mathcal{W}_s \), with eigenvalues:

\[
\lambda^{(m)}(s) = \frac{\Gamma}{J} m^2 (e^s - 1).
\]

Since these are the largest eigenvalues for \( s = 0 \), they must also be in the vicinity of that point. Therefore, we can write \( \lambda(s) \) around \( s = 0 \) as:

\[
\lambda(s) = \begin{cases} 
(\Gamma/J) m_{\text{min}}^2 (e^s - 1) & s < 0 \\
(\Gamma/J) m_{\text{max}}^2 (e^s - 1) & s > 0,
\end{cases}
\]

with \( m_{\text{min}} \) and \( m_{\text{max}} \) the minimum/maximum eigenvalues of \( \mathcal{A} \). If \( m_{\text{min}} \neq m_{\text{max}} \), it is clear that \( \lambda(s) \) has a discontinuity at \( s = 0 \). Contrary to the situations typically considered, the discontinuity does not become a crossover when the system has a finite size, since its origin is the exact closing of the Liouvillian gap due to the strong symmetry (see Fig. 3). If we were to try to find \( \varphi(k) \) by blindly applying Eq. (33) with a generic expression for \( \lambda(s) = (\Gamma/J)m^2(e^s - 1) \), we would
find that the value of $s$ that maximizes $ks - \lambda(s)$ is given by:

$$
\begin{align*}
    s &= \begin{cases} 
        \ln \left[ Jk/(\Gamma m_{\min}^2) \right] & k < \frac{1}{7}m_{\min}^2, \\
        \ln \left[ Jk/(\Gamma m_{\max}^2) \right] & k > \frac{1}{7}m_{\max}^2, \\
        0 & \frac{1}{7}m_{\min}^2 < k < \frac{1}{7}m_{\max}^2,
    \end{cases}
\end{align*}
$$

(36)

This yields the rate function shown in dashed-red in Fig. 6(b): the non-concave regions of $\varphi(K/t)$ associated to multimodality translate into a nonphysical flat plateau when one tries to use the inverse Legendre transformation in Eq. (33). This result connects back to standard thermodynamics, where phase transitions are associated with non-concavities in the underlying fundamental equations for the thermodynamic potentials. A multimodal distribution $p_T(k)$ as we obtained in Eq. (24) will always yield a discontinuous $\lambda(s)$ and will therefore be linked to a first-order phase transition.

To summarize, we have discussed the notions of spontaneous freezing (21), multimodal activity distributions (24) and first order phase transitions at the trajectory level (36). We conclude that these phenomena are linked, since all of them emerge from the existence of a strong symmetry that yields a perfect closing of the Liouvillian gap for any system size. Intermittency is therefore a consequence of the finite system size; it implies a “smoothening” of the phase transition that allows to make use of Eq. (33), but that gives in turn a unimodal probability distribution: i.e. in the long time limit, intermittency destroys multimodality. Spontaneous freezing can therefore be alternatively described as the survival of multimodality in the long-time limit. In quantum metrology, this has strong implications for the scaling in time of the Fisher information [42].

These ideas are supported by Fig. 7(b-c), where we show $p_T(k)$ computed from sets of quantum trajectories, for time windows of the order of the inverse Liouvillian gap, $T \approx (2|\lambda_1|)^{-1}$ The value of $\Omega = 0.8 \Gamma$ is such that we can observe the transition from the ferromagnetic to the thermal phase at $\theta_c \approx 0.4$. When this transition is crossed, fluctuations start increasing as $\theta \rightarrow \pi/4$ and the unimodal distribution is strongly distorted. This characteristic of the thermal phase is the consequence of the increased asymmetry on $\langle k \rangle_s$ at $s = 0$—see panel (a)—related to the closing of the Liouvillian gap. As we get close to $\pi = \pi/4$, where $\lambda_1 = 0$, it becomes impossible to simulate times of the order of $|\lambda_1|^{-1}$. In the plot, this is identified by the emergence of several peaks in $p_T(k)$: the crossover in $\langle k \rangle_s$ gives rise to a multi-peaked structure that would merge into a single peak were $T$ long enough. Since this multimodality does not correspond to the long-time limit, the large-deviations approach is unable to describe it; this is the situation in which intermittency occurs. On the other hand, the strong symmetry point features a multimodal $p_T(k)$ for any $T$; that survival of the multimodal structure is the signature of spontaneous freezing of the dynamics.

Finally, we note that the closing of the Liouvillian gap in the thermodynamic limit of the thermal phase—c.f. Eq. (13)—also yields a crossover in $\langle k \rangle_s$ (see Fig. 8). Since this closing is of a different nature (associated with eigenvalues with imaginary part), it offers the interesting prospect of studying multistability and intermittency between phases displaying coherent, oscillatory dynamics in the long-time limit. This could be done, for instance, by analysing the time correlations between the spectral features of the different phases, as we discuss in the following section.

V. SIGNATURES OF CRITICAL DYNAMICS IN THE EMITTED LIGHT

Many of the essential features of critical dissipative dynamics are encoded in the spectral properties of the Liouvillian. Stationary observables of the form $\langle O(t)O(t+\tau) \rangle$ contain a limited amount of information about these properties, since they depend only on the lowest eigenvalue of $\mathcal{L}$. However, observables involving two-time correlators of the form $\langle O(t)O(t+\tau) \rangle$ require a knowledge not only of $\rho_0$, but also of the Liouvillian $\mathcal{L}$. Consequently, they carry information about the dynamics of the system that is not present $\rho_0$, and can provide valuable data about the Liouvillian, such as its spectral properties, in an experimentally accessible way. To illustrate this point, we focus here on the case of the spectrum of emission, providing a closed-form expression in terms of the Liouvillian eigenvalues and right and left eigenstates.

We define the (unnormalized) spectrum of emission as

$$
S(\omega) = \lim_{t \rightarrow \infty} \frac{1}{\pi} \text{Re} \int_0^\infty d\tau \, e^{i\omega \tau} \langle a^\dagger(t)a(t+\tau) \rangle,
$$

(37)

where, generally, $a$ is some system operator linked to the bath output operator by input-output relations (in our case, $a = D_0$). By applying the quantum regression theorem [87], we obtain:

$$
S(\omega) = \lim_{t \rightarrow \infty} \frac{1}{\pi} \text{Re} \int_0^\infty d\tau \, e^{i\omega \tau} \text{Tr} \left[ a e^{\mathcal{L} \tau} \rho(t) a^\dagger \right].
$$

(38)

Note that, typically, the limit $t \rightarrow \infty$ will imply that $\rho(t)$ is simply $\rho_0$, the steady state of the system. In most systems this steady state is unique, but here we want to take into account the possibility of multiple steady states (i.e. multiple eigenstates of $\mathcal{L}$ with eigenvalues with zero real part), meaning that $\rho(t)$ can be any superposition of these steady states, defined by the initial state. Therefore, we take the limit $t \rightarrow \infty$ and substitute $\rho(t)$ by an arbitrary superposition of steady states, $\rho_{\text{ss}}$, determined by the initial state. We can perform a spectral
FIG. 9: (a) Liouvillian eigenvalues $\lambda_\mu$ weighted by $L_\mu$. This illustrates the set of eigenvalues that are experimentally accessible by the measurement of the spectrum of emission. (b) Spectrum of emission versus $\Omega$, for $\theta = 0$. (c) Spectrum versus $\theta$, for $\Omega = 0.8\Gamma$. At the strong symmetry point, the gap closes exactly, and the spectrum features an extreme line-narrowing. White, dashed lines indicate where a phase transition occurs; the signature of the phase transition is the emergence of sideband peaks.

decomposition of the Liouvillian to write, for any $\rho$:

$$ e^{Lt} \rho = \sum_\mu e^{\lambda_\mu t} \text{Tr}[\rho L_\mu] \rho R_\mu, $$

where $\rho L/R_\mu$ is the left/right eigenstate of $L$ with eigenvalue $\lambda_\mu$. This allows us to write

$$ S(\omega) = \frac{1}{\pi} \text{Re} \int_0^\infty d\tau \sum_\mu e^{(i\omega + \lambda_\mu)\tau} \times \text{Tr}[a \rho R_\mu] \text{Tr}[a^\dagger \rho L_\mu \rho_{ss}], $$

By defining

$$ \omega_\mu \equiv \text{Im}\{\lambda_\mu\} $$

$$ \gamma_\mu/2 \equiv -\text{Re}\{\lambda_\mu\} $$

$$ L_\mu \equiv \text{Re}\{\text{Tr}[a \rho R_\mu] \text{Tr}[a^\dagger \rho L_\mu \rho_{ss}]\} $$

$$ K_\mu \equiv \text{Im}\{\text{Tr}[a \rho R_\mu] \text{Tr}[a^\dagger \rho L_\mu \rho_{ss}]\}, $$

we can formally integrate Eq. (40) and obtain

$$ S(\omega) = \frac{1}{\pi} \sum_{\mu, \text{Re}\{\lambda_\mu\} \neq 0} \left(\frac{\gamma_\mu/2}{(\gamma_\mu/2)^2 + (\omega + \omega_\mu)^2}\right) L_\mu \delta(\omega + \omega_\mu) + \frac{K_\mu}{\pi} \text{P.V.} \left(\frac{1}{\omega + \omega_\mu}\right). $$

Note that terms with $\text{Re}\{\lambda_\mu\} = 0$ give rise to a series of $\delta$-peaks in the spectrum, positioned at frequencies that are given by the imaginary part of the eigenvalues with zero real part. The last term means that the principal value integral of $1/\omega$ should be computed when integrating that distribution. That term never appears in the case of a unique steady state ($\mu = 0$), since in that case $\rho L,0 = 1$ and $K_0 = \text{Im}\{|\langle a_\text{ss} \rangle|^2\} = 0$. All the terms proportional to $K_\mu$ in Eq. (42) are dispersive lineshapes that break the symmetry of the corresponding Lorentzians (proportional to $L_\mu$). Although they may appear unphysical (since they can yield negative values) they give a physical result once the sum is performed.

Equation (42) tells us that the spectrum of emission can be used to probe the Liouvillian spectrum and also infer, indirectly, information about the right and left eigenvectors. Similar formal integrations of Eq. (38) have been presented before [88–90]; ours differ from these in that they make explicit use of the left and right eigenvectors of $L$. In particular, we...
see that the existence of eigenvalues with zero real part and finite imaginary part translates into the presence of measurable δ-peaks in the spectrum. These turn into peaks with a finite width when the linewidth of the detectors and/or other unavoidable losses to different channels are included in the description. Figure 9 illustrates the information about the Liouvillian eigenvalues provided by the spectrum: panel (a) shows distribution of eigenvalues for \( N = 5, 10 \), weighted by their value of \( L_n \). This way, features like the emergence of imaginary eigenvalues with vanishing real part in the thermodynamic limit can be directly addressed in the laboratory. We show this in panel (b), where the ferromagnetic-thermal DPT is shown to be accompanied by the emergence of sideband peaks in the fluorescence spectrum; this is the well known generalization of the Mollow triplet to the case of collective resonance fluorescence \[91\]. The result that the Liouvillian gap closes in this phase as \( 1/N \) can be confirmed experimentally: as shown in Fig. 10, it can be measured directly as decrease in the linewidth of the spectral peaks. Finally, Fig. 9(c) shows the emergence of sideband peaks when \( \theta \) is varied so as to enter in the thermal phase, and the observation of extreme line-narrowing as the gap is closed exactly at the strong-symmetry point.

These results open the intriguing possibility of exploring the notions of ergodicity, intermittency and spontaneous freezing in systems with Liouvillian eigenvalues with vanishing real part and finite imaginary part by studying temporal correlations between different spectral windows \[89, 92–98\]. This will be a topic of study for future works.

### VI. CONCLUSIONS

We have reported the phenomenon of spontaneous freezing in individual realizations of dissipative dynamics. To study this effect, we have analysed the minimal model of squeezed superradiance, which displays non-trivial critical behaviour and admits exact numerical solutions. In order to identify the relevant regimes of non-ergodic dynamics, we have completely characterized the phase diagram of the system, its metrological properties, and its Liouvillian spectrum. We have connected the phenomenon of spontaneous freezing with the theory of thermodynamics of quantum trajectories, showing that this effect manifests as a multimodal structure of the activity distribution that survives in the long-time limit, which may have strong implications in quantum metrology. The phenomenon is linked to the existence of a strong symmetry, that guarantees a closing of the Liouvillian gap even for finite system size.

Our work sheds new light on the critical behaviour of open systems with finite system size, and might provide new routes in the development of sensors based on driven-dissipative quantum systems.

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Note that our definition differs from the standard one by the sign of the last equation, which means that the angle Θ is defined with respect to the negative z-axis in order to make the lowest eigenstate of $S_z$ correspond to $Θ = 0$. The dynamical equations for the spherical angles are:

\[
\dot{Θ} = Ω \sin Φ - (Γ_+ - Γ_−) \sin Θ \tag{45a}
\]
\[
\dot{Φ} = Ω \cos Φ \cot Θ \tag{45b}
\]

The previous equations define a vector field of derivatives on the Bloch sphere; these field lines are sketched in Fig. 1(b) for different values of $(Ω/Γ, Θ)$, together with the spin Wigner function \[63, 64\] of the exact steady state on a finite system $(N = 50)$. A mean field approach does not necessarily offer a faithful description of the dynamics \[28\]; in our model, it assumes a classical, point-like state on the Bloch sphere, therefore failing to describe spin fluctuations. Despite this, it is interesting to notice that, in a finite system, the shape of the fluctuations in the Bloch sphere actually bears some similarities with the vector field of derivatives predicted by the mean-field \[100\]. This is observed in Fig. 1, where it is clearly seen that the asymmetry in the density of field lines at both sides of the steady-state (moving along the meridian) is replicated as an asymmetry in the corresponding Wigner function.

We move now into analyzing the steady solutions of these dynamical equations. Regarding the angle $Φ$, Eq. (45b) always has a stationary solution at $Φ = ±π/2$. It is instructive to consider the dynamics of $Θ$ for $Φ = π/2$, which reduces to:

\[
\dot{Θ} = Ω - (Γ_+ − Γ_−) \sin Θ. \tag{46}
\]

One can picture this as a dynamical equation for a pendulum, driven by the first term and damped by the second. The steady state solution $Θ_0$ is determined by setting (46) to zero, which, from Eq. (44), yields the magnetization $M \equiv s_z(t \rightarrow ∞)$ given by Eq. (7). In general, the stationary solutions of Eqs. (43a–43c) read:

\[
\langle s_z \rangle = M, \tag{47a}
\]
\[
\langle s_z \rangle = 0, \tag{47b}
\]
\[
\langle s_y \rangle = \sqrt{1 − M^2}, \tag{47c}
\]

There are two situations in which these solutions do not hold.

1. At the point $Γ_− − Γ_+ = 0$, where $S_z$ becomes a strong symmetry, Eq. (46) does not have a stationary solution except for the trivial case $Ω = 0$. In particular, looking back at Eqs. (43a–43c), we see that at this point the evolution corresponds to a circular motion on a plane of constant $s_x$, with $s_y = (1 − s_z^2) \cos(Ωt)$, $s_z = (1 − s_z^2) \sin(Ωt)$.

2. At the critical value

\[
Ω_c = Γ_− − Γ_+ = Γ(\cos^2 θ − \sin^2 θ), \tag{48}
\]

we have $M = 0$, which means that the energy supplied by $Ω$ is enough to reach the equator of the Bloch sphere, where the drag is maximum.

**APPENDIX I: MEAN FIELD EQUATIONS**

The study of mean-field equations provides insight into the system dynamics and the different dissipative phases in the thermodynamic limit, $J \rightarrow ∞$. In that case, writing the commutator between the normalized angular momentum operators $s_i = S_i/J, i \in \{x, y, z\}$, yields a value $s_i, s_j = iε^{ijk} s_k/J$ (with $ε^{ijk}$ the Levy-Civita symbol) that tends to zero. One thus obtains the set of equations:

\[
s_x = (Γ_− − Γ_+) s_x s_z, \tag{43a}
\]
\[
s_y = −Ω s_x + (Γ_− − Γ_+) s_y s_z, \tag{43b}
\]
\[
s_z = Ω s_y − (Γ_− − Γ_+) (s_x^2 + s_y^2), \tag{43c}
\]

where $Γ_±$ are given by Eqs. (4a, 4b). At the level of description of the mean-field equations, the role of the squeezing angle $θ$ is therefore to renormalize the decay rate $Γ$ by the factor $(\cos^2 θ − \sin^2 θ)$, since $Γ_− − Γ_+ = Γ(\cos^2 θ − \sin^2 θ)$. Given that these equations conserve the total norm $N = s_x^2 + s_y^2 + s_z^2$, we can write them as a reduced set of dynamical equations in terms of the polar angles $Θ ∈ [0, π], Φ ∈ [−π, π]$, related to the cartesian coordinates as:

\[
s_x = \sin Θ \cos Φ, \tag{44a}
\]
\[
s_y = \sin Θ \sin Φ, \tag{44b}
\]
\[
s_z = −\cos Θ. \tag{44c}
\]
Therefore, for values \( \Omega > \Omega_c \), the pendulum is able to go beyond the equator, with a driving that now is large enough for it to engage in a perpetual oscillation across the Bloch sphere. This is reflected on the fact that Eq. (46) has no stationary solution and in the unphysical imaginary value of \( M \) predicted by Eq. (7) for \( \Omega > \Omega_c \). The emergence of initial-state-dependent closed trajectories at \( \Omega > \Omega_c \) is represented on points (iv) and (viii) of Fig. 1(b). This transition to a phase with time-periodic steady states corresponds, in the case \( \theta = 0 \), to the well-studied second order DPT of collective resonance fluorescence [59, 61, 62], it is related to the existence of steady states with imaginary eigenvalues [73] and it was the subject of a recent work [74, 75] where similar models have been used to describe dissipative time crystals.

In general, we observe that the role of the squeezed decay parametrized by \( \theta \) is to lower the value of critical driving towards \( \Omega_c \to 0 \) as \( \theta \to \pi/4 \) (and \( \Gamma_+ \to \Gamma_0 \)). Note that such an apparent non-ergodic dynamics does not survive in the full quantum solution for a finite system, which does reach stationarity on a time that, however, diverges with the system size (as predicted by the eigenvalue equation Eq. (13)). The stationary oscillations predicted by the mean-field equations are, therefore, the thermodynamic limit of a transient phenomena.

**APPENDIX II: SPIN OBSERVABLES**

### A. Holstein-Primakoff approximation

Hewe we use a Holstein-Primakoff (HP) approximation [101] to obtain analytical expressions for spin mean values and fluctuations, which can be linked to the Liouvillian gap in the ferromagnetic phase. The exact HP transformation writes the angular momentum operator in terms of a bosonic place operator:

\[
S_+ = (\sqrt{2J-b\beta})b \\
S_z = b\beta - J
\]  

(49)

The HP approximation, consisting of a truncated series expansion of the square root in Eq. (49), is based on the premise that the upper levels of the finite ladder of eigenstates of \( S_z \) are not occupied. Therefore, the nonlinear features that distinguish such a finite ladder from the infinite one of an harmonic oscillator are negligible, and \( S_- \) is accurately described by the bosonic operator \( b \). We will use the equations (49) for \( \theta < \pi/4 \) (where we know they are a better description since the system tends to be polarized towards the negative \( z \) direction) and assume the same result applies for \( \theta > \pi/4 \) by flipping the spin and changing parameters \( \Gamma_+ \leftrightarrow \Gamma_- \).

Following the approach outlined in Ref. [19], we use a displaced operator:

\[b \to b + \sqrt{J}\beta\]

(50)

that accounts for the mean polarization of the system. Using the renormalized operators \( s_- \equiv S_-/J \) and \( s_z \equiv S_z/J \), the corresponding HP expression expanded in terms of \( \epsilon = 1/\sqrt{J} \) reads:

\[
s_- = \sqrt{k}\sqrt{1 - \epsilon \beta b^2 + \epsilon^2} - \epsilon^2 \frac{b^2}{k} (\beta + \epsilon b) = \sum_i e^i s_-^{(i)}.
\]

(51)

with \( k = 2 - |\beta|^2 \). Up to first order in \( \epsilon \), we have:

\[
s_-^{(0)} = \sqrt{k}\beta,
\]

(52a)

\[
s_-^{(1)} = \frac{1}{2\sqrt{k}}[(2k - |\beta|^2)b - \beta^2 b^\dagger].
\]

(52b)

For the \( s_z \) operator, we have \( s_z = \sum_i e^i s_z^{(i)} \), with:

\[
s_z^{(0)} = |\beta|^2 - 1,
\]

(53a)

\[
s_z^{(1)} = \beta b^\dagger + \beta^* b.
\]

(53b)

It is useful to expand equation (1) as:

\[
\dot{\rho} = -i[\Omega S_x, \rho] + \frac{\Gamma_+}{2J} L_{S-}\rho + \frac{\Gamma_-}{2J} L_{S+}\rho + \frac{\chi}{2J}(2S_-\rho S_- - \{S_z^2, \rho\} + 2S_+\rho S_+ - \{S_+^2, \rho\}),
\]

(54)

where \( \Gamma_\pm \) are defined by Eqs. (4a, 4b), \( \chi \equiv \Gamma \sin \theta \cos \theta \), and we defined the Lindblad operators \( \mathcal{L}_O(\rho) = 2O\rho O^\dagger - O^\dagger O\rho - \rho O^\dagger O \). Then, we obtain

\[
\frac{1}{\sqrt{J}}\dot{\rho} = -i[\Omega S_x, \rho] + \frac{\Gamma_+}{2} L_{S-}\rho + \frac{\Gamma_-}{2} L_{S+}\rho + \frac{\chi}{2}(2S_-\rho S_- - \{S_z^2, \rho\} + 2S_+\rho S_+ - \{S_+^2, \rho\})
\]

\[
= \left[\mathcal{L}_O^{(0)} + \epsilon \mathcal{L}^{(1)} + \epsilon^2 \mathcal{L}^{(2)} + O(\epsilon^3)\right] \rho.
\]

(55)

From Eq. (53) we immediately obtain \( \mathcal{L}_O^{(0)} = 0 \). To all orders in the expansion, the Hamiltonian term describing coherent driving can be grouped together with a term coming from the dissipative part, in the following form:

\[
-\frac{i}{\sqrt{J}} \left[ s_z^{(n)} \left( \Omega - i s_z^{(0)} (\Gamma_+ - \Gamma_-) \right) \right] + \text{h.c.}, \rho
\]

(56)

We can therefore simplify the dynamics by eliminating the driving terms to all orders if we choose a proper value for the displacement \( \beta \), such that

\[
\Omega - is_z^{(0)} (\Gamma_+ - \Gamma_-) = \Omega - i\sqrt{2 - |\beta|^2} \beta (\Gamma_+ - \Gamma_-) = 0.
\]

(57)

This equation has three solutions that, written in terms of \( r \) and \( \phi \) as \( \beta_\epsilon = r e^{i\phi}_\epsilon \), read:

\[
r_1 = \sqrt{1 + \frac{1}{Q}} \quad \phi_1 = -\pi/2,
\]

(58a)

\[
r_2 = \sqrt{1 - \frac{1}{Q}} \quad \phi_2 = -\pi/2,
\]

(58b)

\[
r_3 = \sqrt{1 + \frac{M}{Q}} \quad \phi_3 = \pi,
\]

(58c)

where \( M \) is given by Eq. (7) and we defined:

\[
Q \equiv \sqrt{1 + \left( \frac{\Omega}{\Gamma_+ - \Gamma_-} \right)^2}.
\]

(59)
The dissipative phase transition.

order in $\epsilon$ of the Liouvillian \cite{19} with highest real part, to the lowest
are cancelled, we have $L^{(1)} = 0$. We define $A \equiv (2k - |\beta|^2)/(2\sqrt{\mathcal{E}})$ and $B \equiv -\beta^2/(2\sqrt{\mathcal{E}})$, so that $s_n^{(1)} = Ab + Bb^\dagger$, and expand the density matrix $\rho(t) = \sum_n \epsilon^n \rho^{(n)}(t)$. By equating powers of $\epsilon$, Eq. (55) yields a master equation for the lowest order density matrix, $\rho^{(0)}(t)$:

$$\rho^{(0)}(t) = L^{(2)}\rho^{(0)}(t) + \frac{\eta}{2}(2bp^{(0)}b - \{bb, \rho^{(0)}\} + 2b^\dagger\rho^{(0)}b^\dagger - \{b^\dagger b, \rho^{(0)}\}),$$

where $\gamma_\pm = \Gamma_\pm A^2 + \Gamma_\mp B^2 + 2\chi AB$, $\gamma_\mp \equiv \Gamma_\pm A^2 + \Gamma_\mp B^2 + 2\chi AB$, $\eta \equiv AB(\Gamma_\pm + \Gamma_\mp + \chi (A^2 + B^2))$ are all real quantities (since $\beta = -i\epsilon_1$ is purely imaginary). The dynamics for $(b)$ and $(b^\dagger)$ is given by the equation $\dot{\rho} = \mathcal{W}\rho$, with $\mathcal{W} = (\{b, (b^\dagger)\})^T$ and

$$\mathcal{W} = \frac{1}{2} \begin{pmatrix} \gamma_+ - \gamma_- & 0 \\ 0 & \gamma_+ - \gamma_- \end{pmatrix}.$$  

The eigenvalues of $\mathcal{W}$ describe the energy excitation spectrum of the Liouvillian \cite{19} with highest real part, to the lowest order in $\epsilon$. We therefore find that the gap in the Liouvillian

$$\lambda = (\gamma_+ - \gamma_-)/2$$

where $\gamma_\pm = \Gamma_\pm A^2 + \Gamma_\mp B^2$ is purely real:

$$\lambda = \frac{\Gamma_+ - \Gamma_-}{2} (A^2 - B^2) = - (\Gamma_+ - \Gamma_-)(1 - |\beta|^2).$$

From the three values of $\beta_i = r_i e^{i\phi_i}$ we get:

$$\lambda_1 = (\Gamma_+ - \Gamma_-)M,$$

$$\lambda_2 = - (\Gamma_+ - \Gamma_-)M,$$

$$\lambda_3 = (\Gamma_+ - \Gamma_-)Q.$$

These three solutions are shown in Fig. 11. Only $\lambda_1$ has a negative real part in the region $\Omega < \Gamma_+ - \Gamma_-$ where these solutions are valid, and therefore the only valid choice of displacement is

$$\beta = e^{-i\pi/2}\sqrt{1 + M}.$$  

The other choices give $\gamma_+ > \gamma_-$, which clearly yield unstable equations of motion for the bosonic mode, since the effective pumping is larger than the losses and observables diverge; this is related to the instability of the corresponding steady mean-field solutions. The point where the gap closes $\gamma_+ = \gamma_-$ is therefore associated with this instability in the equations of motion of the bosonic mode; this indicates that fluctuations in the spin become comparable to $J$ and indicates the onset of the dissipative phase transition.

![FIG. 11: Eigenvalues as a function of the normalized driving amplitude $\Omega/\Gamma_-,\Gamma_+$, assuming $\Gamma_+ > \Gamma_-$. Lines: analytical solutions given by Eq. (63). Markers: numerical solutions for finite systems. Below $\Omega/\Gamma_+ = 1$, the only valid solution is $\lambda_1$.](image)

B. Spin polarization

We can now compute spin observables in the ferromagnetic phase, where the HP expansion holds. In order to expand spin mean values $\langle s_{z/\pm} \rangle$ in powers of $\epsilon$, we must take into account both the HP expansions [Eqs. (52) and (53)] and the expansion of $\rho(t)$. Doing so, one obtains, to order $\epsilon^2$:

$$\langle s_{z/\pm}(t) \rangle = \text{Tr}[s_{z/\pm} \rho^{(0)}(0)] + \epsilon \left\{ \text{Tr}[s_{z/\pm} \rho^{(1)}(0)] + \text{Tr}[s_{z/\pm} \rho^{(1)}(0)] \right\}$$

$$+ \epsilon^2 \left\{ \text{Tr}[s_{z/\pm}^2 \rho^{(0)}] + \text{Tr}[s_{z/\pm} \rho^{(2)}(1)] + \text{Tr}[s_{z/\pm} \rho^{(2)}(2)] \right\} + O(\epsilon^3),$$

where we omitted the time dependence of the $\rho^{(n)}(t)$ for simplicity. Noting that $s_{z/\pm}^0$ is a c-number, and that, by definition, $\text{Tr}[\rho^{(1)}] = \text{Tr}[\rho^{(2)}] = 0$, the terms $\text{Tr}[s_{z/\pm} \rho^{(1)}]$ and $\text{Tr}[s_{z/\pm}^2 \rho^{(2)}(2)]$ in Eq. (65) are equal to zero.

However, there are non-vanishing terms proportional to $\epsilon^2$ that depend on $\rho^{(1)}$. Since the effective master equation $\dot{\rho}^{(1)}(t) = L^{(2)}\rho^{(1)} + L^{(3)}\rho^{(0)}$ is no longer quadratic, these terms prevent us to obtain a closed expression for $\langle s_{z/\pm}(t) \rangle$ at order $\epsilon^2 = 1/J$.

Let us define the correlators to zeroth order in $\rho$ as $\langle O \rangle_0 \equiv \text{Tr}[O \rho_0]$. In order to evaluate the first-order terms $\langle s_{z/\pm}(t) \rangle_0$ in Eq. (65), we must use Eq. (60) to obtain correlators of the form $\langle b \rangle_0$. In general, the dynamics of any arbitrary correlator $\langle b^m b^n \rangle_0$ will be given by:

$$\frac{d}{dt} \langle b^m b^n \rangle_0 = \frac{\gamma_+ - \gamma_-}{2}(n + m)\langle b^m b^n \rangle_0$$

$$+ \gamma_+ mn \langle b^m b^n \rangle_0 - \frac{\eta}{2} m(m - 1)\langle b^{m-2} b^n \rangle_0 - \frac{\eta}{2} n(n - 1)\langle b^m b^{n-2} \rangle_0.$$  

In particular, we are interested in the stationary limit $t \to \infty$, where the density matrix fulfils $L^{(2)} \rho^{(0)} = 0$ (in the following, the notation $\langle . \rangle_0$ and $\rho^{(n)}$ will refer to stationary values).
We obtain steady state values of the correlators by setting the derivatives of Eq. (66) to zero. This way, we get, for the case \(m = 0, n = 1\):

\[
\langle b \rangle_0 = 0. \tag{67}
\]

Since \(\langle s_z^{(1)} \rangle_0\) and \(\langle s_z^{(1)} \rangle_0\) are proportional to \(\langle b \rangle_0\) and \(\langle b \rangle_0\), we find that they are all zero and, therefore, conclude that \(\langle s_{z/\pm} \rangle\) has no first-order dependence on \(\epsilon\). Therefore, using Eqs. (53), (52) and (64), we find that the stationary expectation values \(\langle s_z \rangle\), \(\langle s_x \rangle\) and \(\langle s_y \rangle\) are given, with corrections to second order in \(\epsilon\), by the zeroth-order terms \(\text{Tr}[s_z^0 \rho^0]\), which coincide with the solutions of the mean-field equations (47):

\[
\begin{align*}
\langle s_z \rangle &= M = -R + O(\epsilon^2) \tag{68a} \\
\langle s_x \rangle &= 0 + O(\epsilon^2) \tag{68b} \\
\langle s_y \rangle &= \frac{\Omega_{+} - \Omega_{-}}{\Gamma_{+} - \Gamma_{-}} + O(\epsilon^2) \tag{68c}
\end{align*}
\]

C. Spin fluctuations: spin squeezing

Our lack of an analytical expression of \(\rho^{(n)}\) for \(n > 0\) prevents us from obtaining closed-form expressions for the second-order corrections to the mean spin. However, it is possible to get expressions for the fluctuations \(\Delta s_{z/\pm}\) to second order, which is the lowest in their expansion. In particular, it is easy to prove that:

\[
\Delta s_{z/\pm}^2 = \langle s_{z/\pm}^2 \rangle_0 + O(\epsilon^3). \tag{69}
\]

The mean spin direction in the thermodynamic limit, obtained from Eqs. (68a–68c), can be written as:

\[
u_m = \frac{\langle s_z \rangle \mathbf{u}_x + \langle s_y \rangle \mathbf{u}_y + \langle s_z \rangle \mathbf{u}_z}{\sqrt{\langle s^2 \rangle}} = \sqrt{1 - M^2} \mathbf{u}_y + M \mathbf{u}_z \tag{70}
\]

We are interested in the squeezing along some direction in the plane perpendicular to \(\mathbf{u}_m = \mathbf{u}_m(\varphi) \equiv \cos(\varphi) \mathbf{u}_x - \sin(\varphi) [\mathbf{M} \mathbf{u}_y + \sqrt{1 - M^2} \mathbf{u}_z]\); this direction is determined by finding \(\varphi\) that maximizes the squeezing. As we prove in Appendix V ID, \(\mathbf{u}_x\) is always the preferential direction of squeezing. In order to compute \(\xi_{\perp}\), it is useful to obtain, from the solution of Eq. (60), the expression for the mean quadratic correlators:

\[
\begin{align*}
\langle b^2 \rangle &= \frac{\gamma_{+}}{\gamma_{-} - \gamma_{+}}, \tag{71a} \\
\langle b^2 \rangle &= \frac{\gamma_{+}}{\gamma_{-} - \gamma_{+}}. \tag{71b}
\end{align*}
\]

Using these, we can write the following expression for the variance:

\[
\Delta s_x^2 = k \left( \frac{\langle b^2 \rangle - \langle b^2 \rangle + \frac{1}{2}}{+ O(\epsilon^3)} \right) \tag{72}
\]

and from there, obtain the expression for the spin squeezing:

\[
\xi_{\perp}^2 = \frac{N(\Delta S_x)^2}{\langle S \rangle^2} = k \left( \frac{\gamma_{+} + \gamma_{-}}{\gamma_{-} - \gamma_{+} + \frac{1}{2}} + O(\epsilon) \right) \tag{73}
\]

which can be rewritten in the form shown in Eq. (9).

D. Preferential direction of squeezing

To complete our previous discussion, we demonstrate here that \(\mathbf{u}_x\) is the direction with minimum fluctuations finding the angle \(\varphi\) that minimizes spin fluctuations along the general direction \(\mathbf{u}_m(\varphi)\). We define the a short notation for the following quantities with the properties of sine and cosines, \(c \equiv \cos(\varphi), s \equiv \sin(\varphi), \tilde{c} \equiv M\) and \(\tilde{s} \equiv \sqrt{1 - M^2}\), and define the covariance \(\text{cov}[X, Y] \equiv \langle (X - \langle X \rangle)(Y - \langle Y \rangle) \rangle\). Then, we get, for the fluctuations along a general direction perpendicular to the mean spin:

\[
\begin{align*}
\frac{(\Delta S_{\perp})^2}{J^2} &= \frac{1}{J^2} \left\{ s^2 \left[ e^{2} 2 \text{cov}[S_{+}, S_{-}] - (\Delta S_{+})^2 - (\Delta S_{-})^2 - 2(S_z) + \tilde{s}^2 (\Delta S_z)^2 \right] \\
&\quad - i \tilde{c} \left[ (\Delta S_{+})^2 - (\Delta S_{-})^2 + \langle S_{+} \rangle + \langle S_{-} \rangle \right] \right\} + e^{2} \left[ 2 \text{cov}[S_{+}, S_{-}] + (\Delta S_{+})^2 + (\Delta S_{-})^2 - 2(S_z) \right] \\
&\quad + \text{sc} \left[ \frac{i \tilde{c} (\Delta S_{+})^2 - (\Delta S_{-})^2}{2} - \tilde{s} \left( \text{cov}[S_{+}, S_{z}] + \text{cov}[S_{-}, S_{z}] + \langle S_{+} \rangle - \langle S_{-} \rangle \right) \right] \right\} \tag{74}
\end{align*}
\]

that we can express, grouping the coefficients of \(s^2, e^2\) and \(sc\) into three parameters \(\kappa, \lambda\) and \(\mu\) respectively, as:

\[
\begin{align*}
\frac{(\Delta S_{\perp})^2}{J^2} &= \frac{1}{J} \left[ \kappa \sin(\varphi)^2 + \lambda \cos(\varphi)^2 + \mu \cos(\varphi) \sin(\varphi) \right] \\
&= \frac{1}{2J} \left[ (\lambda - \kappa) \cos(2\varphi) + \mu \sin(2\varphi) + \kappa + \lambda \right]. \tag{75}
\end{align*}
\]
ing the following solution for \( \varphi \):

\[
2\varphi = \arctan \left( \frac{\mu}{\lambda - \kappa} \right). \tag{76}
\]

This function is usually treated as a single-valued function by restricting the domain of \( \tan(x) \) to \( x \in [-\pi/2, \pi/2] \). We know from numerical calculations that indeed \( \varphi \approx 0 \), so we use this single-valued definition of \( \arctan(x) \). In that case, we can use the properties:

\[
\cos[\arctan(x)] = \frac{1}{\sqrt{x^2 + 1}}, \tag{77a}
\]

\[
\sin[\arctan(x)] = \frac{x}{\sqrt{x^2 + 1}}, \tag{77b}
\]

and then get:

\[
\frac{(\Delta S_s)^2}{J^2} = \frac{1}{2J} \left[ \kappa + \lambda + (\lambda - \kappa) \sqrt{1 + \left( \frac{\mu}{\lambda - \kappa} \right)^2} \right]. \tag{78}
\]

We are now left to compute the values of \( \kappa, \lambda \) and \( \mu \). To do so, let us observe that, to order \( \epsilon^2 \):

\[
\frac{\langle S_z^2 \rangle}{J^2} = \langle s_0^2 \rangle + \frac{1}{J} \left[ \langle s_1 \rangle + \langle s_0 \rangle (\langle s_2 \rangle) + \langle s_2 \rangle \langle s_0 \rangle \right]. \tag{79}
\]

and, since \( s_0 \) is a c-number, we have that \( (\Delta S_s)^2/J^2 = \langle s_1^2 \rangle / J \). By following the same argument to express the rest of variances and covariances present in the equation in terms of the \( s_{\pm, \pm} \), we can write down the following values of \( \kappa, \lambda \) and \( \mu \), to zero order in \( \epsilon \):

\[
\kappa = \epsilon^2 \frac{2\langle s_1 \rangle - \langle s_0 \rangle + 2\langle s_2 \rangle}{4} + \epsilon^2 \left[ \frac{\langle s_1 \rangle - \langle s_0 \rangle}{2} \right], \tag{80}
\]

\[
\lambda = \frac{2\langle s_1 \rangle + \langle s_0 \rangle - 2\langle s_2 \rangle}{4}, \tag{81}
\]

\[
\mu = i\epsilon \frac{\langle s_1 \rangle - \langle s_0 \rangle}{2} - s \left( \frac{\langle s_1 \rangle + \langle s_0 \rangle}{2} \right). \tag{82}
\]

Taking into account that \( \langle b^2 \rangle = \langle b^4 \rangle \), and \( s_0 = A b + B b^\dagger \), we can write the expressions of the correlators appearing in the equations:

\[
\langle s_{\pm}^2 \rangle = \langle b^2 \rangle (A^2 + B^2) + 2AB \langle b b^\dagger \rangle + AB \tag{83a}
\]

\[
\langle s_0 s_- \rangle = \langle b b^\dagger \rangle (A + B) + 2AB \langle b \rangle + B^2 \tag{83b}
\]

\[
\langle s_0^2 \rangle = |\beta|^2 \left[ 2 \left( \langle b^2 \rangle - \langle b \rangle^2 \right) + 1 \right] \tag{83c}
\]

\[
\langle s_0 s_\pm \rangle = i |\beta| (A - B) \langle b \rangle (b^\dagger + b^\dagger b^\dagger) + B \beta \tag{83d}
\]

\[
\langle s_\pm s_\pm \rangle = i |\beta| (A - B) (b^\dagger - b^\dagger b^\dagger) + A \beta \tag{83e}
\]

We know \( \langle s_0 \rangle = \sqrt{\kappa} \beta = -\langle s_1 \rangle \), and from Eq. (83e) and (83e) we have that \( \langle s_0 \rangle + \langle s_0 \rangle = \beta (A + B) = -i |\beta| = -i \beta. \) Also, \( \langle s_+ s_+ \rangle = \langle s_- s_- \rangle \). It is then easy to see that

\[
\mu = 0 \rightarrow \varphi = 0 \tag{84}
\]

proving that, in the thermodynamic limit, \( u_x \) is always the preferential direction for squeezing.

### APPENDIX III: PROBABILITY AMPLITUDES OF GENERAL MONTE CARLO TRAJECTORIES

In this section demonstrate Eq. (21) of the main text. By expanding the wavefunction in eigenstates \( |m\rangle \) of the strong symmetry, we find, for a trajectory with jumps at times \( (t_1, \ldots, t_n) < t \):
\[ |\psi(t)\rangle \propto e^{-i\tilde{H}(t-t_n)A}\psi(T_n)\rangle \propto \sum_m e^{-i\hat{H}(t-t_n)mc_m(t_n)}|m\rangle \propto \sum_m e^{-i\tilde{H}(t-t_n)}me^{-i\hat{H}(t_n-t_{n-1})mc_m(t_{n-1})}|m\rangle \]
\[ \propto \ldots \propto e^{-i\hat{H}t_n^n}c_m(0)|m\rangle \quad (85) \]

**APPENDIX IV: EXACT EXPRESSION FOR THE ACTIVITY DISTRIBUTION**

In this section, we demonstrate Eq. (24) of the main text. Defining the quantum-jump superoperator \( J\{\cdot\} \equiv L\{\cdot\}L^\dagger \) and the no-jump part of the Liouvillian \( \mathcal{S} = \mathcal{L} - J \), the probability for the system to experience \( K \) quantum jumps on a time \( T \), starting at the state \( \rho(0) \), is given by \([38, 102]\):

\[ p_T(K) = \int_0^T dK_\varepsilon \int_0^{t_{K-1}} dK_\varepsilon \ldots \int_0^{t_2} dK_\varepsilon \int_0^{t_1} \text{Tr} \left[ e^{\mathcal{S}(T-t_1)}J e^{\mathcal{S}(t_1-t_{K-1})} \ldots \mathcal{J} e^{\mathcal{S}(t_{K-2})} \ldots \mathcal{J} e^{\mathcal{S}(t_2)} \rho(0) \right] . \quad (88) \]

From Eqs. (11a, 11b), we find:

\[ e^{\mathcal{S}t}|m\rangle\langle m| = e^{-\Gamma m^2 t/J}|m\rangle\langle m| \quad (89a) \]
\[ J|m\rangle\langle m| = \frac{\Gamma}{J} m^2|m\rangle\langle m| \quad (89b) \]

and therefore:

\[ p_T(K) = \int_0^T dt_K \ldots \int_0^{t_2} dt_1 \sum_m c_m \left( \frac{\Gamma}{J} m^2 \right)^K e^{-\Gamma m^2 T/J} \]
\[ = \sum_m \frac{1}{K!} \left( \frac{TT m^2}{J} \right)^K e^{-\Gamma m^2 T/J} c_m . \quad (90) \]