On the Simplicity of Eigenvalues of Two Nonhomogeneous Euler-Bernoulli Beams Connected by a Point Mass

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Abstract: In this paper we consider a linear system modeling the vibrations of two nonhomogeneous Euler-Bernoulli beams connected by a point mass. This system is generated by the following equations

\[ \rho(x)y_{tt}(t,x) + (\sigma(x)y_{xx}(t,x))_{xx} - (q(x)y_x(t,x))_x = 0, \quad t > 0, \quad x \in (-1,0) \cup (0,1), \]
\[ My_{tt}(t,0) = (Ty(t,x))_{x=0^-} - (Ty(t,x))_{x=0^+}, \quad t > 0, \]

with hinged boundary conditions at both ends, where \( Ty = (\sigma(x)y_{xx})_x - q(x)y_x \). We prove that all the associated eigenvalues \( \lambda_n \geq 1 \) are algebraically simple, furthermore the corresponding eigenfunctions \( (\phi_n)_{n \geq 1} \) satisfy \( \phi'_nT\phi_n(-1) > 0 \) and \( \phi'_nT\phi_n(1) < 0 \) for all \( n \geq 1 \). These results give a key to the solutions of various control and stability problems related to this system.

Keywords. Euler-Bernoulli beams, point mass, algebraic simplicity, subwronskians.

AMS subject classification. 34A38, 34B08, 34B24, 93B05.

1 Introduction

In the last three decades there has been an increasing interest in the study of the dynamics and control of various hybrid models for systems of rods, strings and beams with attached masses. For the boundary controllability and stability problems related to this type of systems we can refer to [1, 10, 15, 18, 24], see also [13, 22, 27] and references therein. As is well known, the spectral analysis is the key tools for solving these problems.
In this paper, we consider a one-dimensional linear hybrid system which is composed by two nonhomogeneous hinged Euler-Bernoulli beams connected by a point mass. We assume that the first beam occupies the interval $\Omega_1 = (-1,0)$ and the second one occupies the interval $\Omega_2 = (0,1)$. The vibrations of the first and the second beam will be, respectively, presented by the functions 

$$y := y(t,x), \quad (t,x) \in (0,\infty) \times \Omega_1 \quad \text{and} \quad z := z(t,x), \quad (t,x) \in (0,\infty) \times \Omega_2.$$ 

The position of the mass $M > 0$ attached to the beams at the point $x = 0$ is denoted by the function $y := y(t,0)$ for $t > 0$. The dynamic behavior of the system is governed by the following PDE:

$$\begin{align*}
\rho_1(x)y_{tt}(t,x) + (\sigma_1(x)y_{xx}(t,x))_{xx} - (q_1(x)y_x(t,x))_x &= 0, \quad (t,x) \in (0,\infty) \times \Omega_1, \\
\rho_2(x)z_{tt}(t,x) + (\sigma_2(x)z_{xx}(t,x))_{xx} - (q_2(x)z_x(t,x))_x &= 0, \quad (t,x) \in (0,\infty) \times \Omega_2, \\
y(t,-1) = y_{xx}(t,-1) = z(t,1) = z_{xx}(t,1) &= 0, \quad t \in (0,\infty), \quad (1.1) \\
y(t,0) = z(t,0), y_x(t,0) = z_x(t,0), \sigma_1 y_{xx}(t,0) = \sigma_2 z_{xx}(t,0), \quad t \in (0,\infty), \\
M y_{tt}(t,0) = T^1 y(t,0) - T^2 z(t,0), \quad t \in (0,\infty),
\end{align*}$$

where $T^i f(t,x) := (\sigma_i(x) f_{xx}(t,x))_x - q_i(x) f_x(t,x)$, for $t > 0$ and $x \in \Omega_i$ ($i = 1,2$). The coefficients $\rho_i, \sigma_i$ and $q_i$ ($i = 1,2$) of each beam represent, the density, the flexural rigidity and the axial force, respectively, see for instance [21, 26]. By applying separation of variables to System (1.1), we obtain the following spectral problem:

$$\begin{align*}
\left\{ \begin{array}{l}
(\sigma_1(x)u''')'' - (q_1(x)u')' = \lambda \rho_1(x)u, \quad x \in \Omega_1, \\
(\sigma_2(x)v''')'' - (q_2(x)v')' = \lambda \rho_2(x)v, \quad x \in \Omega_2,
\end{array} \right. \quad (1.2)
\end{align*}$$

$$\begin{align*}
u(-1) = u''(-1) = v(1) = v''(1) &= 0, \quad (1.3) \\
u(0) = v(0), \quad u'(0) = v'(0), \quad \sigma_1 u''(0) = \sigma_2 v''(0), \quad (1.4) \\
\left\{ T^1 u(x) - T^2 v(x) \right\}_{x=0} &= -M \lambda u(0), \quad (1.6)
\end{align*}$$

where $T^i f(t,x) := (\sigma_i(x) f''(x))' - q_i(x) f'(x)$ for $x \in \Omega_i$ ($i = 1,2$). Throughout this paper we assume that

$$\rho_i \in C(\Omega_i), \quad \sigma_i \in H^2(\Omega_i), \quad q_i \in H^1(\Omega_i), \quad (1.7)$$

and there exist constants $\rho_0, \sigma_0 > 0$, such that

$$\rho_i(x) \geq \rho_0, \quad \sigma_i(x) \geq \sigma_0, \quad q_i(x) \geq 0, \quad x \in \Omega_i \quad (i = 1,2). \quad (1.8)$$

There exists an extensive mathematical and engineering literature devoted to the spectral analysis for various systems of vibrating beams. The asymptotics, the simplicity of eigenvalues and the oscillations of the eigenfunctions with their derivatives of vibrating beams with point masses have been investigated in [4, 5, 9, 12, 17] for different boundary conditions. These results were extended in a number of works to Euler-Bernoulli beams with end masses, see [2, 3, 5, 11, 25]. However, the spectral proprieties related to a series of beams with interior attached masses have been
considered only in the case of constant physical parameters. Namely, by using a precise spectral analysis together with the theory of non-harmonic Fourier series Castro and Zuazua \[14, 15, 16\] proved the exact controllability for two type of homogenous flexible beams connected by a point masse. Later on, Mercier and Régnier \[23, 24\], extended their results to the case of network of Euler-Bernoulli beams with interior attached masses.

The main result of this paper is the following.

**Theorem 1.1**

(a) The eigenvalues \( \lambda_n \) \( n \geq 1 \) of the spectral problem \((1.2)-(1.6)\) are real, algebraically simple and form an infinitely increasing sequence such that

\[
0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots \to \infty.
\]

(b) The corresponding eigenfunctions \( (\phi_n)_{n \in \mathbb{N}} \) have the following properties:

\[
\phi'_n T^1 \phi_n(-1) < 0 \text{ and } \phi'_n T^2 \phi_n(1) > 0 \text{ for all } n \geq 1. \tag{1.9}
\]

The proof of this Theorem is mainly based on some properties of fourth-order linear differential equations (see \[19\]) and the associated theory of subwronskians (see \[6, 7\]).

This paper is organized as follows. In Section 2, we associate to Problem \((1.2)-(1.6)\) a self-adjoint operator with compact resolvent defined in a well chosen Hilbert space. In Section 3, we establish several lemmas that are used in the proof of Theorem 1.1 and that we believe are of independent interest. Finally in Section 4, we give the proof of Theorem 1.1.

### 2 Operator framework

Let us define the Hilbert space

\[
\mathcal{H} = L^2(-1, 0) \times L^2(0, 1) \times \mathbb{R},
\]

with the scalar product \( \langle ., . \rangle_{\mathcal{H}} \) defined by: for all \( y_i = (u_i, v_i, z_i)^t \in \mathcal{H} \ (i = 1, 2) \), where \(^t\) denotes the transposition, we have

\[
\langle y_1, y_2 \rangle_{\mathcal{H}} = \int_{\Omega_1} u_1 u_2 \rho_1(x) dx + \int_{\Omega_2} v_1 v_2 \rho_2(x) dx + M z_1 z_2.
\]

Let

\[
\mathcal{V} = \{ (u, v) \in H^2(\Omega_1) \times H^2(\Omega_2) : \text{satisfying } (1.4), (1.5) \},
\]

endowed with the norm

\[
\|(u, v)\|_{\mathcal{V}}^2 = \int_{\Omega_1} |u''(x)|^2 dx + \int_{\Omega_2} |v''(x)|^2 dx.
\]
Proof. Let 
\[ y(u, v, z) \in \mathcal{V} \times \mathbb{R} : u(0) = v(0) = z, \]
equipped with the norm \[ \| (u, v, z) \|_{\mathcal{V}}^2 = \| (u, v) \|_{\mathcal{V}}^2. \]
We introduce the operator \( \mathcal{A} \) defined in \( \mathcal{H} \) by setting:
\[
\mathcal{A} y = \left\{ \begin{array}{ll}
\frac{1}{\rho_1(x)} ((\sigma_1(x)u''(x) - q_1(x)u'(x)), \\
\frac{1}{\rho_2(x)} ((\sigma_2(x)v''(x) - q_2(x)v'(x)), \\
- \frac{1}{M} (T^1u(x) - T^2v(x))|_{x=0},
\end{array} \right. (2.1)
\]
where \( y = (u, v, z)^t \) on the domain
\[
\mathcal{D}(\mathcal{A}) = \{(u, v, z) \in \mathcal{W} : (u, v) \in H^4(\Omega_1) \times H^4(\Omega_2)\}
\]
which is dense in \( \mathcal{H} \). Obviously, Problem (1.2)-(1.6) is equivalent to the following spectral problem
\[
\mathcal{A} \phi = \lambda \phi, \quad \phi = (u, v, z)^t \in D(\mathcal{A}),
\]
i.e., the eigenvalues \( \lambda_n \) of the operator \( \mathcal{A} \) and Problem (1.2)-(1.6) coincide together with their multiplicities. Moreover, there is a one-to-one correspondence between the eigenfunctions,
\[
\phi_n(x) = (u_n(x), v_n(x), z_n)^t \leftrightarrow (u_n(x), v_n(x))^t, \quad z_n = u_n(0), \quad n \geq 1.
\]

**Theorem 2.1** The linear operator \( \mathcal{A} \) is positive and self-adjoint such that \( \mathcal{A}^{-1} \) is compact. Moreover, the spectrum of \( \mathcal{A} \) is discrete and consists of a sequence of positive eigenvalues \( \lambda_n, n \in \mathbb{N} \), tending to \( +\infty \):
\[
0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots \xrightarrow{n \to +\infty} +\infty.
\]

**Proof.** Let \( y = (u_1, u_2, z) \in \mathcal{D}(\mathcal{A}) \), then by integration by parts, we have
\[
\langle \mathcal{A} y, y \rangle_{\mathcal{H}} = \sum_{i=1}^{2} \int_{\Omega_i} \left( (\sigma_i(x)u''(x))^2 - (q_i(x)u'(x))^2 \right) u_i dx - (T^1u_1(x) - T^2u_2(x))|_{x=0} z,
\]
\[
= \int_{\Omega_1} (\sigma_1(x)|u_1|^2 + q(x)|u_1'|^2) dx + \int_{\Omega_2} (\sigma_2(x)|u_2|^2 + q(x)|u_2'|^2) dx
\]
Since \( \sigma_i > 0 \) and \( q_i \geq 0 (i = 1, 2) \) then \( \langle \mathcal{A} y, y \rangle_{\mathcal{H}} \geq 0 \), and hence, the linear operator \( \mathcal{A} \) is positive. Furthermore, it is easy to show that \( \text{Ran}(\mathcal{A} - iId) = \mathcal{H} \), and this implies that \( \mathcal{A} \) is selfadjoint. Since the space \( \mathcal{W} \) is continuously and compactly embedded in the space \( \mathcal{H} \), then \( \mathcal{A}^{-1} \) is compact in \( \mathcal{H} \). The proof is complete. \( \square \)
3 Basic Lemmas

In this section, we establish several basic results that will be used frequently in the next section. We consider the linear fourth order differential equation defined on the interval \([a, b], a \geq 0:\)

\[
(\sigma(x)u'')'' - (q(x)u')' - \rho(x)u = 0, \tag{3.1}
\]

where the functions \(\rho(x), \sigma(x)\) are uniformly positive, and \(q(x)\) is nonnegative such that

\[
\rho \in C(a, b), \quad \sigma \in H^2(a, b), \quad q \in H^1(a, b).
\]

We start by mentioning the following lemma due Leighton-Nehari [19].

**Lemma 3.1** [19, Lemma 2.1] Let \(u\) be a nontrivial solution of the differential equation \((3.1)\) for \(q \equiv 0\). If \(u, u', u''\) and \((\sigma u'')'\) are nonnegative at \(x = a\) (but not all zero), they are positive for all \(x > a\). If \(u, -u', u''\) and \(-(\sigma u'')'\) are nonnegative at \(x = b\) (but not all zero) they are positive for all \(x < b\).

The following lemma was stated in [5, Lemma 2.1]. For the reader’s convenience, we propose here a simpler proof.

**Lemma 3.2** Let \(u\) be a nontrivial solution of Equation \((3.1)\). If \(u, u', u''\) and \(Tu = (\sigma(x)u'')' - q(x)u'\) are nonnegative at \(x = a\) (but not all zero), then they are positive for all \(x > a\). If \(u, -u', u''\) and \((-Tu)\) are nonnegative at \(x = b\) (but not all zero), then they are positive for all \(x < b\).

**Proof.** Let \(h\) be the unique solution of the following second order initial value problem:

\[
\begin{align*}
(\sigma(x)h')' - q(x)h &= 0, \quad x \in (a, b) \\
 h(a) &= 1, \quad h'(a) = 0. \tag{3.2}
\end{align*}
\]

It is known, by Sturm comparison theorem [20, Chapter 1] that \(h(x) > 0\) on \([a, b]\). Hence, the following modified Leighton-Nehari substitution [19, Theorem 12.1]

\[
t(x) := \gamma^{-1}(b-a)\int_a^x h(s)ds + a, \quad \gamma = \int_a^b h(s)ds,
\]

transforms Equation \((3.1)\) into

\[
(\tilde{\sigma}(t)\tilde{u})' = \tilde{\rho}(t)\tilde{u}, \quad t \in [a, b], \tag{3.4}
\]

where \(\tilde{\sigma}(t) = (\gamma(b-a)^{-1}h(x(t)))^2\sigma(x(t)), \tilde{\rho}(t) = \gamma(b-a)^{-1}h^{-1}(x(t))\rho(x(t))\) and \(:= \frac{d}{dt}. If \(u\) is a nontrivial solution of \((3.1)\), then \(\tilde{u}(t) \equiv u(x(t))\) is a nontrivial solution of \((3.4)\). Furthermore, we have

\[
\dot{\tilde{u}} = \gamma(b-a)^{-1}h^{-1}u', \quad \gamma^2(b-a)^{-2}h^2\tilde{u} = hu'' - u'h', \quad (\tilde{\sigma}\tilde{u})' = Tu. \tag{3.5}
\]
It is easy to see from (3.3) and (3.5), that \( u, u', u'' \) and \( Tu \) are positive at \( x = a \). Hence, in view of Lemma 3.1

\[
 \hat{u} > 0, \quad \hat{u}' > 0, \quad \hat{u}'' > 0, \quad (\hat{\sigma} \hat{u}'') > 0 \quad \text{in} \quad (a, b]. \tag{3.6}
\]

Since \( \sigma h'(x) = \int_a^x q_h \rho(x) \, dx \), then \( h'(x) > 0 \) on \( (a, b] \). Therefore, combining (3.5) and (3.6), one gets

\[
 u > 0, \quad u' > 0, \quad u'' > 0, \quad Tu > 0 \quad \text{in} \quad (a, b].
\]

For the proof of the second statement it is sufficient to replace the initial conditions (3.3) by

\[
 h(b) = 1, \quad h'(b) = 0.
\]

By Sturm comparison Theorem, \( h > 0 \) on the interval \([a, b]\). The Lemma is proved. \( \square \)

Using Lemma 3.2, we can establish the following lemma.

**Lemma 3.3**

1. Let \( E_u \) be the space of solutions of Equation (1.2) for \( \lambda > 0 \), satisfying one of the following sets of boundary conditions :

\[
\begin{align*}
 u(-1) &= u''(-1) = 0, \quad \alpha_1 u'(0) = \beta_1 u''(0), \tag{3.7} \\
 u(-1) &= u''(-1) = 0, \quad \alpha_1 Tu(0) = \beta_1 u(0), \tag{3.8}
\end{align*}
\]

where \( (\alpha_1, \beta_1) \in \mathbb{R}^2 \setminus \{(0, 0)\} \) and \( \alpha_1 \beta_1 \leq 0 \). Then \( \text{Dim} E_u = 1 \).

2. Let \( E_v \) be the space of solutions of Equation (1.3) for \( \lambda > 0 \), satisfying one of the following sets of boundary conditions :

\[
\begin{align*}
 v(1) &= v''(1) = 0, \quad \alpha_2 v'(0) = \beta_2 v''(0), \tag{3.9} \\
 v(1) &= v''(1) = 0, \quad \alpha_2 Tu(0) = \beta_2 v(0), \tag{3.10}
\end{align*}
\]

where \( (\alpha_2, \beta_2) \in \mathbb{R}^2 \setminus \{(0, 0)\} \) and \( \alpha_2 \beta_2 \geq 0 \). Then \( \text{Dim} E_v = 1 \).

**Proof.** Suppose that there exist two linearly independent solutions \( u_i \) (\( i = 1, 2 \)) of Problem (1.2)-(3.7) . Both \( u_i'(-1) \) and \( u_i''(-1) \) must be different from zero since otherwise it would follow from Lemma 3.2 that \( u_i' u''(0) > 0 \) (\( i = 1, 2 \)) and this is in contradiction with the last boundary condition in (3.7). In view of the assumptions about \( u_1 \) and \( u_2 \), the solution

\[
 u(x) = u'_2(-1) u_1(x) - u'_1(-1) u_2(x)
\]

satisfies \( u(-1) = u'(-1) = u''(-1) = 0 \) and \( u' u''(0) \leq 0 \). This again contradicts Lemma 3.2 unless \( u \equiv 0 \). The other statements of the Lemma can be proved in a same way. \( \square \)
Lemma 3.4 Every solution \( \phi_0(x) \) of the regular problem (1.2) for \( M = 0 \) has only simple zeros in \((-1,1)\).

Proof. Without loss of generality, assume that there exists \( x_0 \in \Omega_1 \) such that \( \phi_0(x_0) = \phi_0'(x_0) = 0 \). If \( \phi_0'' \mathcal{T}^1 \phi_0(x_0) < 0 \), then the second statement of Lemma 3.2 yields a contradiction with the boundary condition \( \phi_0(-1) = 0 \). Now, if \( \phi_0'' \mathcal{T}^1 \phi_0(0) \geq 0 \), and hence from (1.5) with \( M = 0 \), we have \( \phi_0'' \mathcal{T}^2 \phi_0(0) \geq 0 \). Therefore by the first statement of Lemma 3.2 \( \phi_0(1) \neq 0 \), a contradiction. The proof is complete. \( \square \)

It is known that any solution of Equation (1.2) which satisfies the initial conditions \( u(-1) = u''(-1) = 0 \) may be expressed as a linear combination of \( y_1(x) \) and \( y_2(x) \), where \( y_i \) (\( i = 1,2 \)), are the fundamental solutions of (1.2) satisfying the initial conditions:

\[
\begin{align*}
y_1(-1) &= y_1'(1) = \mathcal{T}^1 y_1(-1) = 0, \quad y_1'(1) = 1, \\
y_2(-1) &= y_2'(1) = y_2''(1) = 0, \quad \mathcal{T}^1 y_2(-1) = 1.
\end{align*}
\tag{3.11, 3.12}
\]

In view of Lemma 3.2 \( y_1, y_2, y''_1 \) and \( \mathcal{T}^1 y_i \) (\( i = 1,2 \)) are positive in \( \Omega_1 \cup \{0\} \). We introduce the following subwronskians (see [6, 7]):

\[
\begin{align*}
\sigma_1 &= y_1 y_2' - y_2 y_1', \quad \sigma_1' = y_1 y_2'' - y_2 y_1'', \quad \sigma_1 = y_1 \mathcal{T}^1 y_2 - y_2 \mathcal{T}^1 y_1. 
\end{align*}
\tag{3.13}
\]

Clearly, if for some \( \lambda > 0 \) and \( x_0 \in \Omega_1 \cup \{0\} \), \( \sigma_1(x_0, \lambda) = 0 \), then \( \lambda \) is an eigenvalue and \( u(x) = y_1(x_0) y_2(x) - y_2(x_0) y_1(x) \) is the corresponding eigenfunction of the problem determined by (1.2) and the boundary conditions

\[
u(-1) = u''(-1) = u(x_0) = u'(x_0) = 0.
\tag{3.14}
\]

Similar conclusions for the other subwronskians \( \sigma_1' \) and \( \sigma_1'' \).

Analogously, we introduce the following subwronskians associated with Equation (1.3):

\[
\begin{align*}
\sigma_2 &= z_1 z_2' - z_2 z_1', \quad \sigma_2' = z_1 z_2'' - z_2 z_1'', \quad \sigma_2 = z_1 \mathcal{T}^2 z_2 - z_2 \mathcal{T}^2 z_1,
\end{align*}
\tag{3.15}
\]

where \( z_1 \) and \( z_2 \) are two linearly independent solutions of (1.2) which satisfy the initial conditions:

\[
\begin{align*}
z_1(1) &= z_1'(1) = \mathcal{T}^2 z_1(1) = 0, \quad z_1'(1) = -1, \\
z_2(1) &= z_2'(1) = z_2''(1) = 0, \quad \mathcal{T}^2 z_2(1) = -1.
\end{align*}
\tag{3.16, 3.17}
\]

Obviously, we have

\[
z_i > 0, \quad z_i'' > 0, \quad z_i' < 0, \quad \mathcal{T}^2 z_i < 0, \quad i = 1,2.
\tag{3.18}
\]

If one of the subwronskians \( \sigma_2, \sigma_2' \) and \( \sigma_2'' \) vanishes for some \( \lambda > 0 \) and \( x_0 \in \Omega_2 \cup \{0\} \), then \( \lambda \) is an eigenvalue of the problem determined by (1.3) and the boundary conditions

\[
v(x_0) = v'(x_0) = v(1) = v''(1) = 0.
\tag{3.19}
\]

\[7\]
Lemma 3.5 The following formulas hold:
\[ \tau_1 = y_1'\sigma_1 y_2'' - y_2'\sigma_1 y_2'' \quad \text{and} \quad \tau_2 = z_1'\sigma_2 z_2'' - z_2'\sigma_2 z_2''. \] (3.20)

Proof. By multiplying Equation (1.2) (for \( u = y_1 \)) by \( y_2 \), and twice integrating by parts from \(-1\) to \( x \), yields
\[ y_1 T^1 y_2(x) - \sigma_1 y_1'' y_2'(x) = \lambda \int_{-1}^{x} y_1 y_2 \rho_1(x) dx - \int_{-1}^{x} (\sigma_1 y_1'' y_2'(x) + q_1 y_1' y_2'(x)) dx. \]
Similarly,
\[ y_2 T^1 y_1(x) - \sigma_1 y_2'' y_1'(x) = \lambda \int_{-1}^{x} y_1 y_2 \rho_1(x) dx - \int_{-1}^{x} (\sigma_1 y_1'' y_2'(x) + q_1 y_1' y_2'(x)) dx. \]
Subtracting these equalities together with (3.13), we get the expression (3.20). The proof of the second expression in (3.20) is similar. \( \square \)

Lemma 3.6 Let \( \lambda > 0 \) and fixed \( x_0 \in \Omega_i \cup \{0\} \) (\( i = 1, 2 \)). If one of the subwronskians \( \sigma_i, \sigma_i', \) and \( \tau_i \) \( (i = 1, 2) \) vanishes at \( (x_0, \lambda) \), then all the other subwronskians are different from zero at \( (x_0, \lambda) \).

Proof. Without loss of generality, suppose that \( \sigma_1(x_0, \lambda) = \sigma_1'(x_0, \lambda) = 0 \), for some \( x_0 \in \Omega_1 \cup \{0\} \) and \( \lambda > 0 \). Then there exist two eigenfunctions \( u_1 \) and \( u_2 \) of the problems determined by Equation (1.2) and the boundary conditions (3.14) and
\[ u(-1) = u''(-1) = u(0) = u''(0) = 0, \] (3.21)
respectively. Since \( \varphi_1 \) and \( \varphi_2 \) satisfy in common \( u(-1) = u''(-1) = u(0) = 0 \), then by Lemma 3.3, they are colinear, i.e., \( \varphi_1 = c\varphi_2, \ c \in \mathbb{R} \). This means that \( \varphi_1(x_0) = \varphi_1'(x_0) = \varphi_1''(x_0) = 0 \), and this is in contradiction with the second statement of Lemma 3.2. The proof is complete. \( \square \)

4 Proof of Theorem 1.1
In this section we prove Theorem 1.1

Proof of Assertion (a). Let \( \lambda \) be an eigenvalue of Problem (1.2)-(1.6) and let \( E_\lambda \) be the corresponding eigenspace (i.e., \( E_\lambda = \text{Ker} (A - \lambda I) \)). We shall prove that \( \text{Dim} E_\lambda = 1 \). To this end, let \( \varphi_1, \varphi_2 \) and \( \varphi_3 \) be three solutions of Equation (1.2), satisfying the boundary conditions
\[ \varphi_1(-1) = \varphi_1''(-1) = \varphi_1(0) = 0, \] (4.1)
\[ \varphi_2(-1) = \varphi_2''(-1) = \varphi_2(0) = 0, \] (4.2)
\( \phi_3(-1) = \phi'_3(0) = \phi''_3(0) = 0, \) \hspace{1cm} (4.3)

respectively. By virtue of Lemma 3.3, \( \phi_1, \phi_2 \) and \( \phi_3 \), are the unique solutions, up to a multiplicative constant, of Equation (1.2) satisfying the boundary conditions (4.1), (4.2) and (4.3), respectively. Similarly, let \( \psi_i \) \( (i = 1, 2, 3) \) be the unique solutions of Equation (1.3), satisfying the boundary conditions

\[
\psi_1(0) = \psi_1(1) = \psi''_1(1) = 0, \hspace{1cm} (4.4)
\]

\[
\psi'_2(0) = \psi'_2(1) = \psi''_2(1) = 0, \hspace{1cm} (4.5)
\]

and

\[
\psi''_3(0) = \psi_3(1) = \psi''_3(1) = 0, \hspace{1cm} (4.6)
\]

respectively. It is easy to see that \( \phi_i \) and \( \psi_i \) \( (i = 1, 2, 3) \), can be written as follows :

\[
\phi_1(x) = y_1(0)y_2(x) - y_2(0)y_1(x), \hspace{1cm} (4.7)
\]

\[
\phi_2(x) = y'_1(0)y_2(x) - y'_2(0)y_1(x), \hspace{1cm} (4.8)
\]

\[
\phi_3(x) = y''_1(0)y_2(x) - y''_2(0)y_1(x), \hspace{1cm} (4.9)
\]

for \( x \in [-1, 0], \) and

\[
\psi_1(x) = z_1(0)z_2(x) - z_2(0)z_1(x), \hspace{1cm} (4.10)
\]

\[
\psi_2(x) = z'_1(0)z_2(x) - z'_2(0)z_1(x), \hspace{1cm} (4.11)
\]

\[
\psi_3(x) = z''_1(0)z_2(x) - z''_2(0)z_1(x), \hspace{1cm} (4.12)
\]

for \( x \in [0, 1]. \) Let \( \phi_1(x, \lambda) \) and \( \phi_2(x, \lambda) \) be two solutions of Equation (1.2) and (1.3) which satisfy the initial conditions \( u(-1) = u''(-1) = 0 \) and \( v(1) = v''(1) = 0, \) respectively. For clarity, the rest of the proof is divided into three steps.

STEP 1. Assume that the subwronskians \( \Sigma_i(0, \lambda) \neq 0 \) for \( i = 1, 2. \)

This implies that \( \phi'_1(0) \neq 0 \) and \( \phi_2(0) \neq 0 \) (resp. \( \psi'_1(0) \neq 0 \) and \( \psi_2(0) \neq 0 \)). Under this assumption \( \phi_i \) (resp. \( \psi_i \), \( i = 1, 2, \) are linearly independent solutions of Equation (1.2) (resp. Equation (1.3)). Therefore, there exist constants \( a, b, c \) and \( d \) such that

\[
\phi_1(x, \lambda) = a\phi_1(x, \lambda) + b\phi_2(x, \lambda) \quad \text{and} \quad \phi_2(x, \lambda) = c\psi_1(x, \lambda) + d\psi_2(x, \lambda).
\]

From the first two conditions of (1.5), we have

\[
c = \frac{\phi'_1(0)}{\psi'_1(0)}a \quad \text{and} \quad d = \frac{\phi'_2(0)}{\psi'_2(0)}b. \hspace{1cm} (4.13)
\]

Using this together with the last condition of (1.5), we get

\[
a \left( \sigma_1\phi''_1(0) - \frac{\sigma_2\phi'_1(0)}{\psi'_1(0)} \right) + b \left( \sigma_1\phi''_2(0) - \frac{\sigma_2\phi'_2(0)}{\psi'_2(0)} \right) = 0. \hspace{1cm} (4.14)
\]
It is obvious that if \((\sigma_1 \psi_1' \phi''(0) - \sigma_2 \phi_1' \psi_1''(0)) \neq 0\) or \((\sigma_1 \psi_2' \phi''(0) - \sigma_2 \phi_2' \psi_2''(0)) \neq 0\), then \(\text{Dim} E_\lambda = 1\). Assume now the alternative case, i.e.,

\[
\sigma_1 \psi_1' \phi''(0) - \sigma_2 \phi_1' \psi_1''(0) = 0 \quad \text{and} \quad \sigma_1 \psi_2' \phi''(0) - \sigma_2 \phi_2' \psi_2''(0) = 0, \tag{4.15}
\]
then \(a, b \in \mathbb{R}\). From (1.6) and (4.13), one gets

\[
a \left( \mathcal{T}^1 \phi_1(0) - \frac{\phi_1' \mathcal{T}^2 \psi_1(0)}{\psi_1'(0)} \right) + b \left( \mathcal{T}^1 \phi_2(0) - \frac{\phi_2' \mathcal{T}^2 \psi_2(0)}{\psi_2'(0)} \right) = -M \lambda b \phi_2(0). \tag{4.16}
\]

It can be easily verified from (3.13), (3.15), (4.7)-(4.8) and (4.10)-(4.11), that

\[
\begin{align*}
\sigma_1 \phi''(0) &= \tau_1(0) = \mathcal{T}^1 \phi_1(0) \quad \text{and} \quad \sigma_2 \psi_2''(0) = \tau_2(0) = \mathcal{T}^2 \psi_1(0). & \tag{4.17}
\end{align*}
\]

Using this and (4.15), we get

\[
\left( \mathcal{T}^1 \phi_1(0) - \frac{\phi_1' \mathcal{T}^2 \psi_1(0)}{\psi_1'(0)} \right) = \left( \sigma_1 \phi''(0) - \frac{\sigma_2 \phi_2' \psi_2''(0)}{\psi_2'(0)} \right) = 0. \tag{4.18}
\]

Now, if \(b \neq 0\), then by (4.18), Equality (4.16) takes the form

\[
\frac{\mathcal{T}^1 \phi_2(0)}{\phi_2(0)} - \frac{\mathcal{T}^2 \psi_2(0)}{\psi_2'(0)} = -M \lambda. \tag{4.19}
\]

On the other hand, in view of Lemma 3.2, \(\phi_2\) satisfies one of the following properties:

**Case 1.** \(\phi_2(0) > 0, \phi_2''(0) \geq 0\) and \(\mathcal{T}^1 \phi_2(0) \geq 0\).

Then the first inequality in (4.18) together with Lemma 3.2 imply that \(\psi_2 \psi_2''(0) \geq 0\), and \(\phi_2 \mathcal{T}^2 \psi_2(0) < 0\). As consequence, the left hand in (4.19) is nonnegative, a contradiction.

**Case 2.** \(\phi_2(0) < 0, \phi_2''(0) \leq 0\) and \(\mathcal{T}^1 \phi_2(0) \leq 0\).

The Case 2 can be handled in a same way.

**Case 3.** \(\phi_2 \phi_2''(0) < 0\).

Then from (3.13) and (4.7)-(4.8) together with (4.17), we obtain

\[
0 > \sigma_1 \phi_2 \phi_2''(0) = -\tau_1 \phi_1(0) = -\phi_1' \mathcal{T}^1 \phi_1(0),
\]
whence, \(\phi_1' \mathcal{T}^1 \phi_1(0) > 0\). Again by Lemma 3.2 we have \(\phi_1' \phi_1'(0) > 0\). According to (4.15) and (4.18), \(\phi_1' \mathcal{T}^2 \psi_1(0) > 0\) and \(\psi_1' \phi_1''(0) > 0\). This is in contradiction with Lemma 3.2 and the condition (1.4).

Therefore \(b = 0\), and hence, from (4.13) we deduce that \(\text{Dim} E_\lambda = 1\).

**STEP 2.** Assume that \(\tau_i(0) = 0\) for \(i = 1, 2\).

Under this assumption together with Lemma 3.3 we have \(\phi_1'(0) = \psi_1'(0) = 0, \phi_1''(0) \neq 0\) and \(\psi_1''(0) \neq 0\). On the other hand by Lemma 3.6 \(\tau_i'(0) \neq 0\) for \(i = 1, 2\). This means that all of the functions \(\phi_3, \phi_3', \psi_3\) and \(\psi_3'\) does not vanish at \(x = 0\). Therefore, \(\phi_1\) and \(\phi_3\) (resp. \(\psi_1\) and \(\psi_3\)) are linearly independent solutions of (1.2) (resp. of (1.3)). Consequently, there exist constants \(a, b, c\) and \(d\) such that

\[
\phi_1(x, \lambda) = a \phi_1(x, \lambda) + b \phi_3(x, \lambda) \quad \text{and} \quad \phi_2(x, \lambda) = c \psi_1(x, \lambda) + d \psi_3(x, \lambda).
\]
Substituting these expressions into the condition (1.5), one gets

\[
\begin{align*}
    b\frac{\varphi_3(0)}{\psi_3(0)} &= d, \quad b\frac{\varphi'_3(0)}{\psi'_3(0)} = d \quad \text{and} \quad c = \frac{\sigma_1\varphi''_3(0)}{\sigma_2\psi''_1(0)}a. 
\end{align*}
\]

This implies that

\[
b \left( \frac{\varphi_3(0)}{\psi_3(0)} - \frac{\varphi'_3(0)}{\psi'_3(0)} \right) = 0.
\]

Clearly, if \((\psi_3\varphi'_3(0) - \varphi_3\psi'_3(0)) \neq 0\), then \(b = 0\), and hence, \(\text{Dim}E_\lambda = 1\). Now, suppose that

\[
\psi_3\varphi'_3(0) = \varphi_3\psi'_3(0), \quad (4.21)
\]

then \(a, b \in \mathbb{R}\). A combination of (3.13), (3.15), (3.20) and (4.7)-(4.12) yields

\[
\begin{align*}
    \varphi''_1(0) &= \sigma'_1(0) = -\varphi_3(0), \quad T^1\varphi_1(0) = \tau_1(0) = -\sigma_1\varphi'_3(0), \\
    \psi''_1(0) &= \sigma'_2(0) = -\psi_3(0), \quad T^2\psi_1(0) = \tau_2(0) = -\sigma_2\psi'_3(0).
\end{align*}
\]

Using these relations together with (4.21), one has

\[
\sigma_2\psi''_1(0)T^1\varphi_1(0) - \sigma_1\varphi''_1(0)T^2\psi_1(0) = 0. \quad (4.22)
\]

Since \(\varphi_1, \varphi'_1, \psi_1,\) and \(\psi'_1\) vanish at \(x = 0\), then by (1.5) and (4.22), the function

\[
\phi_0(x) = \begin{cases} 
    \sigma_2(0)\psi''_1(0)\varphi_1(x), & x \in [-1, 0], \\
    \sigma_1(0)\varphi''_1(0)\psi_1(x), & x \in [0, 1], 
\end{cases}
\]

is a solution of the regular problem (1.2)-(1.6) for \(M = 0\), which satisfies \(\phi_0(0) = \phi'_0(0) = 0\), and this is in contradiction with Lemma 3.4. Therefore, \(b = 0\), and hence by (4.20), we deduce that \(\text{Dim}E_\lambda = 1\).

**STEP 3.** Assume that \(\tau_1(0) = 0\) and \(\tau_2(0) \neq 0\) (or conversely).

Then \(\varphi_1(0) = \varphi'_1(0) = 0\). Let us recall from Step 1, that \(\psi_1\) and \(\psi_2\) are linearly independent solutions of (1.3). Furthermore, from Step 2, \(\varphi_1\) and \(\varphi_3\) are linearly independent solutions of (1.2). Hence there exist constants \(a, b, c\) and \(d\) such that

\[
\phi_1(x, \lambda) = a\varphi_1(x, \lambda) + b\varphi_3(x, \lambda) \quad \text{and} \quad \phi_2(x, \lambda) = c\psi_1(x, \lambda) + d\psi_2(x, \lambda).
\]

Substituting these expressions into the condition (1.5), we find

\[
\begin{align*}
    d &= \frac{\varphi_3(0)}{\psi_3(0)}b, \quad c = \frac{\varphi'_3(0)}{\psi'_3(0)}b, \\
    a\sigma_1\varphi''_1(0) &= b \left( \frac{\sigma_2\varphi'_3\psi''_1(0)}{\psi'_1(0)} + \frac{\sigma_2\varphi_3\psi''_2(0)}{\psi'_2(0)} \right). 
\end{align*}
\]

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Obviously, $\varphi''_1(0) \neq 0$ since otherwise $\sigma'_1(0) = 0$, and this is in contradiction with Lemma 3.6. Hence from (4.23) and (4.24), $E_\lambda$ is generated by an unique solution, i.e., $\text{Dim} E_\lambda = 1$.

From the above, we deduce that the geometric multiplicity of the eigenvalue $\lambda$ is equal to one. On the other hand, by Theorem 2.1, the linear operator $A$ is self-adjoint in $\mathcal{H}$, then $\lambda$ is algebraically simple. The proof is complete. \qed

Proof of Assertion (b). It should be noted that by (1.6), $\mathcal{T}^1u(0)$ and $\mathcal{T}^2v(0)$ do not necessarily coincide. Then it is not possible to apply directly Lemma 3.2. Let $\lambda_n$, $(n \geq 1)$ be an eigenvalue of Problem (1.2)-(1.6) and $\phi_n$ be the corresponding eigenfunction. Obviously, $\phi_n$ can be written, up to a multiplicative constant, in the unique form
\[
\phi_n(x, \lambda_n) = \begin{cases} u_n(x) = u(x, \lambda_n), & x \in [-1, 0], \\ v_n(x) = v(x, \lambda_n), & x \in [0, 1]. \end{cases}
\]

Suppose that
\[
\phi'_n \mathcal{T}^1 \phi_n(-1) = u'_n \mathcal{T}^1 u_n(-1) \geq 0 \text{ for some } n \geq 1,
\]
say $u'_n \geq 0$ and $\mathcal{T}^1 u_n(-1) \geq 0$. Since $u_n(-1) = u'_n(-1) = 0$, then by the first statement of Lemma 3.2, $u_n$, $u'_n$, $u''_n$ and $\mathcal{T}^1 u_n$ are positive at $x = 0$. Thus from (1.5), one has
\[
v_n(0) > 0, \quad v'_n(0) > 0 \quad \text{and} \quad v''_n(0) > 0.
\]
Since $\lambda_n > 0$, it follow from (1.6), that $\mathcal{T}^2 v_n(0) > 0$. Again the first statement of Lemma 3.2 yields a contradiction with the boundary condition $v_n(1) = 0$. The proof of the second inequality in (1.9) is similar. \qed

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