The Indefinite Self-Dual Metrics and Painleve Equations

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Abstract
We classify the $SU(2)$-invariant anti-self-dual metrics with a signature $(+,+,−,−)$. The metrics are specified by a solution of Painlevé VI, V, III or II. Moreover we show the geometric meaning of the metrics specified by each type of Painlevé functions.

1 Introduction

The aim of this paper is to classify the anti-self-dual metrics in real dimension four admitting an isometric action of $SU(2)$ with generically three-dimensional orbits. In this paper, we study not only the definite metrics, but also the indefinite metrics with a signature $(+,+,−,−)$.

Hitchin 
shows that the $SU(2)$-invariant anti-self-dual metric is generically specified by a solution of Painlevé VI with two complex parameters. He used the twistor correspondence to associate the anti-self-dual equation and Painlevé equation. On twistor space, the lifted action of $SU(2)$ determines a pre-homogenous action of $SU(2)$, and it determines a isomonodromic family of connections on $CP^1$, and then we have Painlevé equations. In this framework, Dancer 
shows that the diagonal scalar-flat Kähler metric is specified by a solution of Painlevé III with parameters $(4,4,4−4)$. The author shows that the $SU(2)$-invariant anti-self-dual hermitian metric is specified by a solution of Painlevé III with one complex parameter.

If the metric is definite, then the anti-self-dual equation reduce to either Painlevé VI or III. On the other hand, we show that, if the metric has a a signature $(+,+,−,−)$, then the anti-self-dual equation reduce to not only Painlevé VI or III but also Painlevé V or II. The difference of types of Painlevé is due to the difference of reality on the twistor space.

Painlevé VI is shown to be deformation equations for a linear problem

$$\left( \frac{d}{dz} - B_1 \right) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0,$$

where $B_1$ has four simple poles on $CP^1$. And Painleve II, III, IV, V are degenerated from Painlevé VI:
This is the confluence diagram of poles of $B_1$, where the Roman numerals represent the types of Painlevé, and the parenthesized numbers represent the orders of poles of $B_1$. For example, Painlevé V is shown to be deformation equations for a linear problem with one double and two simple poles. Due to the reality of twistor space, the poles of $B_1$ makes two conjugate pairs. Therefore, the configuration of poles never becomes Painlevé IV type.

The multiple pole of $B_1$ determines a hermitian structure or a structure of null surfaces, and this is the geometric meaning of the metrics specified by each type of Painlevé functions.

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2 The Diagonal Anti-self-dual Equations

In this section, we review the anti-self-dual equations on the $SU(2)$-invariant diagonal metrics.

The $SU(2)$-invariant diagonal metric is written in the following form:

$$g = w_1 w_2 w_3 dt^2 + \frac{w_2 w_3}{w_1} \sigma_1^2 + \frac{w_3 w_1}{w_2} \sigma_2^2 + \frac{w_1 w_2}{w_3} \sigma_3^2.$$  (1)

$w_1$, $w_2$, and $w_3$ are functions of $t$, and $\sigma_1$, $\sigma_2$, $\sigma_3$ are left invariant one-forms on each $SU(2)$-orbit satisfying

$$d\sigma_1 = \sigma_2 \wedge \sigma_3, \quad d\sigma_2 = \sigma_3 \wedge \sigma_1, \quad d\sigma_3 = \sigma_1 \wedge \sigma_2.$$  (2)

Tod [11] showed that the anti-self-dual equations on the $SU(2)$-invariant diagonal metric are given by the following system:

$$\dot{w}_1 = -w_2 w_3 + w_1 (\alpha_2 + \alpha_3),$$
$$\dot{w}_2 = -w_3 w_1 + w_2 (\alpha_3 + \alpha_1),$$
$$\dot{w}_3 = -w_1 w_2 + w_3 (\alpha_1 + \alpha_2),$$
$$\dot{\alpha}_1 = -\alpha_2 \alpha_3 + \alpha_1 (\alpha_2 + \alpha_3),$$
$$\dot{\alpha}_2 = -\alpha_3 \alpha_1 + \alpha_2 (\alpha_3 + \alpha_1),$$
$$\dot{\alpha}_3 = -\alpha_1 \alpha_2 + \alpha_3 (\alpha_1 + \alpha_2),$$  (3)

where $\alpha_1$, $\alpha_2$, $\alpha_3$ are auxiliary functions and the dots denote differentiation with respect to $t$. The anti-self-dual equation (3) has a first integral

$$k = \frac{\alpha_1 (w_2^2 - w_3^2) + \alpha_2 (w_3^2 - w_1^2) + \alpha_3 (w_1^2 - w_2^2)}{8(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)}.$$

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Furthermore, if we set
\[ x = \frac{\alpha_2 - \alpha_1}{\alpha_2 - \alpha_3}, \]
\[ q = \frac{w_2(\alpha_1 - \alpha_2)(w_2^2 - w_3^2) + 2\sqrt{2k} w_1 w_3(\alpha_1 - \alpha_3))}{w_1^2(w_2^2 - w_3^2)\alpha_1 + w_2^2(w_3^2 - w_1^2)\alpha_2 + w_3^2(w_1^2 - w_2^2)\alpha_3}, \]
then the system (3) generically reduces to a family of Painlevé VI with special parameters
\[ (\alpha, \beta, \gamma, \delta) = \left( \frac{(\sqrt{2k} - 1)^2}{2}, k, k, \frac{1 + 2k}{2} \right). \]
We will review the Painlevé equations in the appendix.

3 The Non-diagonal Anti-self-dual Equations
We can write a $SU(2)$-invariant metric in the form
\[ g = f(\tau)d\tau^2 + \sum_{l,m=1}^{3} h_{lm}(\tau) \sigma_l \sigma_m. \]
Using the Killing form, we can diagonalize the metric $g$ on each $SU(2)$-orbit. Then we can express the metric as follows:
\[ g = (abc)^2 dt^2 + a^2 d\hat{\sigma}_1^2 + b^2 d\hat{\sigma}_2^2 + c^2 d\hat{\sigma}_3^2, \]
for some $t = t(\tau)$, $a = a(t)$, $b = b(t)$, $c = c(t)$ and
\[ \begin{pmatrix} \hat{\sigma}_1 \\ \hat{\sigma}_2 \\ \hat{\sigma}_3 \end{pmatrix} = R(t) \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}, \]
where $R(t)$ is $SO(3)$-valued function.

Since $\dot{RR}^{-1} \in \mathfrak{so}(3)$, we can write
\[ d \begin{pmatrix} \hat{\sigma}_1 \\ \hat{\sigma}_2 \\ \hat{\sigma}_3 \end{pmatrix} = R(t) \begin{pmatrix} \sigma_2 \wedge \sigma_3 \\ \sigma_3 \wedge \sigma_1 \\ \sigma_1 \wedge \sigma_2 \end{pmatrix} + \dot{R} dt \wedge \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} \]
\[ = \begin{pmatrix} \sigma_2 \wedge \sigma_3 \\ \sigma_3 \wedge \sigma_1 \\ \sigma_1 \wedge \sigma_2 \end{pmatrix} + \begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix} dt \wedge \begin{pmatrix} \hat{\sigma}_1 \\ \hat{\sigma}_2 \\ \hat{\sigma}_3 \end{pmatrix}, \]
for some $\xi_1 = \xi_1(t)$, $\xi_2 = \xi_2(t)$, $\xi_3 = \xi_3(t)$.

If $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = 0$, then the matrix $(h_{lm})$ can be chosen to be diagonal for all $\tau$, and then we say that $g$ is a diagonal metric.

In the following, we mainly study the non-diagonal case.

We set $w_1 = bc$, $w_2 = ca$, $w_3 = ab$ and determine $\alpha_1, \alpha_2, \alpha_3$ by
\[ \dot{w}_1 = -w_2 w_3 + w_1 (\alpha_2 + \alpha_3), \]
\[ \dot{w}_2 = -w_3 w_1 + w_2 (\alpha_3 + \alpha_1), \]
\[ \dot{w}_3 = -w_1 w_2 + w_3 (\alpha_1 + \alpha_2). \] (4)
Then the anti-self-dual equations are as follows [8]:

\[\begin{align*}
\dot{\alpha}_1 &= -\alpha_2\alpha_3 + \alpha_1(\alpha_2 + \alpha_3) + \frac{1}{4}(w_2^2 - w_3^2)^2 \left( \frac{\xi_1}{w_2w_3} \right)^2 \\
&\quad + \frac{1}{4}(w_3^2 - w_1^2)(3w_1^2 + w_3^2) \left( \frac{\xi_2}{w_3w_1} \right)^2 \\
&\quad + \frac{1}{4}(w_2^2 - w_1^2)(3w_1^2 + w_2^2) \left( \frac{\xi_3}{w_1w_2} \right)^2,
\end{align*}\]

\[\begin{align*}
\dot{\alpha}_2 &= -\alpha_3\alpha_1 + \alpha_2(\alpha_3 + \alpha_1) + \frac{1}{4}(w_3^2 - w_1^2)^2 \left( \frac{\xi_1}{w_3w_1} \right)^2 \\
&\quad + \frac{1}{4}(w_1^2 - w_2^2)(3w_2^2 + w_1^2) \left( \frac{\xi_2}{w_1w_2} \right)^2 \\
&\quad + \frac{1}{4}(w_2^2 - w_3^2)(3w_3^2 + w_2^2) \left( \frac{\xi_3}{w_2w_3} \right)^2,
\end{align*}\]

\[\begin{align*}
\dot{\alpha}_3 &= -\alpha_1\alpha_2 + \alpha_3(\alpha_1 + \alpha_2) + \frac{1}{4}(w_1^2 - w_2^2)^2 \left( \frac{\xi_1}{w_1w_2} \right)^2 \\
&\quad + \frac{1}{4}(w_2^2 - w_3^2)(3w_3^2 + w_2^2) \left( \frac{\xi_2}{w_2w_3} \right)^2 \\
&\quad + \frac{1}{4}(w_1^2 - w_3^2)(3w_3^2 + w_1^2) \left( \frac{\xi_3}{w_3w_1} \right)^2,
\end{align*}\]

and

\[\begin{align*}
(w_2^2 - w_3^2) \frac{d}{dt} \left( \frac{\xi_1}{w_2w_3} \right) &= \frac{\xi_2}{w_3w_1w_1w_2} \left( -2w_2^2w_3^2 + w_2^2w_1^2 + w_1^2w_3^2 \right) \\
&\quad + \frac{\xi_1}{w_2w_3} (\alpha_2w_2^2 - 3\alpha_2w_1^2 + 3\alpha_2w_3^2),
\end{align*}\]

\[\begin{align*}
(w_3^2 - w_1^2) \frac{d}{dt} \left( \frac{\xi_2}{w_3w_1} \right) &= \frac{\xi_3}{w_1w_2w_2w_3} \left( -2w_3^2w_1^2 + w_1^2w_3^2 + w_2^2w_3^2 \right) \\
&\quad + \frac{\xi_1}{w_3w_1} (\alpha_3w_3^2 - \alpha_1w_1^2 + 3\alpha_3w_1^2 + 3\alpha_1w_3^2),
\end{align*}\]

\[\begin{align*}
(w_1^2 - w_2^2) \frac{d}{dt} \left( \frac{\xi_3}{w_1w_2} \right) &= \frac{\xi_1}{w_1w_3w_3w_1} \left( -2w_1^2w_3^2 + w_2^2w_3^2 + w_1^2w_3^2 \right) \\
&\quad + \frac{\xi_3}{w_1w_2} (\alpha_1w_1^2 - \alpha_2w_2^2 + 3\alpha_1w_2^2 + 3\alpha_2w_1^2).
\end{align*}\]

**Remark 3.1** If \(\xi_1 = 0, \xi_2 = 0\) and \(\xi_3 = 0\) then the system of equations (5), (6) and (7) reduces to a sixth-order system (8) given by Tod [7]. Furthermore, if \(\alpha_1 = w_1, \alpha_2 = w_2, \alpha_3 = w_3\) then (4), (5), (6) reduce to a third-order system which determines Atiyah-Hitchin family [4], and if \(\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0\) then the system reduces to a third-order system which determines BGG family [4].

**Remark 3.2** If \(w_2 = w_3\), then we can set \(\xi_1 = 0, \xi_2 = 0\) and \(\xi_3 = 0\) by taking another flame. This is also a diagonal case. Therefore we assume \((w_2 - w_3)(w_3 - w_1)(w_1 - w_2) \neq 0\).
4 The Isomonodromic Deformations

Let \((M, g)\) be an oriented Riemannian four manifold. We define a manifold \(Z\) to be the unit sphere bundle in the bundle of self-dual two-forms, and let \(\pi : Z \rightarrow M\) denote the projection. Each point \(z\) in the fiber over \(\pi(z)\) defines a complex structure on the tangent space \(T_{\pi(z)}M\), compatible with the metric and its orientation.

Using the Levi-Civita connection, we can split the tangent space \(T_zZ\) into horizontal and vertical spaces, and the projection \(\pi\) identifies the horizontal space with \(T_{\pi(z)}M\). This space has a complex structure defined by \(z\) and the vertical space is the tangent space of the fiber \(S^2 \cong \mathbb{C}P^1\) which has its natural complex structure.

The almost complex structure on \(Z\) is integrable if and only if the metric is anti-self-dual \([2, 10]\). In this situation \(Z\) is called the twistor space of \((M, g)\) and the fibers are called the real twistor lines.

The almost complex structure on \(Z\) can be determined by the following \((1, 0)\)-forms:

\[
\begin{align*}
\Theta_1 &= z(e^1 + \sqrt{-1}e^2) - (e^0 + \sqrt{-1}e^3), \\
\Theta_2 &= z(e^0 - \sqrt{-1}e^3) + (e^1 - \sqrt{-1}e^2), \\
\Theta_3 &= dz + \frac{1}{2}z^2(\omega_0^0 - \omega_1^1 + \sqrt{-1}(\omega_0^1 - \omega_1^0)) \\
&\quad - \sqrt{-1}z(\omega_2^0 - \omega_3^1) + \frac{1}{2}(\omega_1^0 - \omega_2^1 - \sqrt{-1}(\omega_1^0 - \omega_2^1)),
\end{align*}
\]

(7)

where \(\{e^0, e^1, e^2, e^3\}\) is an orthonormal frame, and \(\omega_j^i\) are the connection forms determined by \(de^i + \omega_j^i \wedge e^j = 0\) and \(\omega_j^i + \omega_i^j = 0\). Then the anti-self-dual condition is

\[
d\Theta_1 \equiv 0, \quad d\Theta_2 \equiv 0, \quad d\Theta_3 \equiv 0 \quad (\text{mod } \Theta_1, \Theta_2, \Theta_3).
\]

(8)

**Theorem 4.1** If the metric is positive definite, then the Pfaffian is invariant under conjugate action and \(z \rightarrow -1/z\) \([3]\). If the metric has a signature \((+, +, -, -)\), then the Pfaffian is invariant under conjugate action and \(z \rightarrow \bar{z}\).

If the metric is \(SU(2)\) invariant, we can write

\[
\begin{pmatrix}
\Theta_1 \\
\Theta_2 \\
\Theta_3
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 1 \\
0 & v_1 & v_2 \\
v_3 & v_2 & 1
\end{pmatrix} dt + A \begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{pmatrix},
\]

(9)

where \(v_1 = v_1(z, t), v_2 = v_2(z, t), v_3 = v_3(z, t); A = (a_{ij}(z, t))_{i,j=1,2,3}\).

If \(\det A \equiv 0\), then metric turns to be diagonal, and the metric is in the BGPP family \([3]\).

If \(\det A \neq 0\), then we can write

\[
\begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{pmatrix} \equiv -A^{-1} \begin{pmatrix}
0 & 0 & 1 \\
0 & v_1 & v_2 \\
v_3 & v_2 & 1
\end{pmatrix} dt, \quad \text{mod } \Theta_1, \Theta_2, \Theta_3.
\]

(10)
If we set
\[
\begin{pmatrix}
s_1 \\
s_2 \\
s_3
\end{pmatrix} := -A^{-1} \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} \, dz + \begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix} \, dt,
\] (11)
then
\[
d \begin{pmatrix}
s_1 \\
s_2 \\
s_3
\end{pmatrix} \equiv \begin{pmatrix}
s_2 \wedge s_3 \\
s_3 \wedge s_1 \\
s_1 \wedge s_2
\end{pmatrix}, \quad \text{mod } \Theta_1, \Theta_2, \Theta_3.
\] (12)
Since \(s_1, s_2, s_3\) are one-forms on \((z, t)\)-plane, the congruency equation (12) turns to be a plain equation:
\[
d \begin{pmatrix}
s_1 \\
s_2 \\
s_3
\end{pmatrix} = \begin{pmatrix}
s_2 \wedge s_3 \\
s_3 \wedge s_1 \\
s_1 \wedge s_2
\end{pmatrix}.
\] (13)
If the metric is positive definite, then \(s_1, s_2, s_3\) are invariant under conjugate action and \(z \to -1/\bar{z}\) by theorem 4.1. And if the metric has a signature \((+, +, -, -)\), then \(s_1, s_2, s_3\) are invariant under conjugate action and \(z \to \bar{z}\).

If we set
\[
\Sigma = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{1}s_1 \\
\sqrt{-1}s_2 \\
-s_3 + \sqrt{-1}s_2 \\
-s_3 + \sqrt{1}s_1
\end{pmatrix} \quad \text{(14)}
\]
\[
=: -B_1 \, dz - B_2 \, dt, \quad \text{(15)}
\]
then
\[
d\Sigma + \Sigma \wedge \Sigma = 0. \quad \text{(16)}
\]
This is the isomonodromy condition for the following linear problem [5]
\[
\begin{pmatrix}
\frac{d}{dz} - B_1
\end{pmatrix} \begin{pmatrix}
y_1 \\
y_2
\end{pmatrix} = 0. \quad \text{(17)}
\]

**Lemma 4.2** The components of \(B_1\) are rational functions of \(z\),
\[
B_1 = \frac{F(z)}{G(z)},
\]
where \(F(z)\) is degree 2 and \(G(z)\) is degree 4. If the metric is positive definite, then \(B_1 \to -\mathbf{e}^1 B_1\) under conjugate action and \(z \to -1/\bar{z}\). And if the metric has a signature \((+, +, -, -)\), then \(B_1 \to -\mathbf{e}^1 B_1\) under conjugate action and \(z \to \bar{z}\).

For this lemma, generically \(B_1\) has four simple poles. In this case, the deformation equation of (17) is Painlevé VI.

**Theorem 4.3** The anti-self-dual equations on \(SU(2)\)-invariant metrics generically reduce to Painlevé VI.
The idea of Hitchin \[7\] is that the lifted action of $SU(2)$ on the twistor space $Z$ gives a homomorphism of vector bundles $\alpha : Z \times su(2) \rightarrow TZ$, and the inverse of $\alpha$ gives a flat meromorphic $SL(2, \mathbb{C})$-connection, which determine isomonodromic deformations. Since one-forms $\Theta_1, \Theta_2, \Theta_3$ on $Z$ can be considered as are infinitesimal variations, we can identify $\Sigma$ with $\alpha^{-1}$.

First, we review the positive definite metric. By lemma 4.2 the poles of $B_1$ make antipodal pairs $\zeta_0, -1/\bar{\zeta}_0$, and $\zeta_1, -1/\bar{\zeta}_1$ on $\mathbb{CP}^1$.

Therefore, we have two types of configuration of poles of $B_1$:

(a) $B_1$ has four simple poles $\zeta_0, \bar{\zeta}_0, \zeta_1, \bar{\zeta}_1$ on $\mathbb{C} \setminus \mathbb{R}$.

$$B_1 = \frac{A_0}{z - \zeta_0} + \frac{-t \bar{A}_0}{z + 1/\zeta_0} + \frac{A_1}{z - \zeta_1} + \frac{-t \bar{A}_1}{z + 1/\zeta_1}. $$

The deformation equation is Painlevé VI with parameters,

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{1}{2}(\theta_0 - 1)^2, \frac{1}{2}\theta_0^2, -\frac{1}{2}\theta_1^2, \frac{1}{2}(1 + \bar{\theta}_1^2)\right),$$

where $\theta_0^2 = 2 \text{tr} A_0^2, \theta_1^2 = 2 \text{tr} A_1^2$.

(b) $B_1$ has two double poles $\zeta, \bar{\zeta}$ on $\mathbb{C} \setminus \mathbb{R}$.

$$B_1 = \frac{A_2}{(z - \zeta)^2} + \frac{-t \bar{A}_2}{(z + 1/\zeta)^2} + \frac{\sqrt{-C}}{z - \zeta} + \frac{-\sqrt{-C}}{z + 1/\zeta} + \frac{-t \bar{A}_2/\bar{C}^2}{(z + 1/\zeta)^2},$$

where $C = -\bar{C}$. The deformation equation is Painlevé III with parameters,

$$(\alpha, \beta, \gamma, \delta) = (4\theta, 4(1 + \bar{\theta}), 4, -4),$$

where $\theta^2 = 2(\text{tr}(A_3C))^2/\text{tr}C^2$.

**Theorem 4.4** If the metric is positive definite, the anti-self-dual equations reduce to the following two Painlevé equations:

(a) A family of Painlevé VI with two complex parameters,

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{1}{2}(\theta_0 - 1)^2, \frac{1}{2}\theta_0^2, -\frac{1}{2}\theta_1^2, \frac{1}{2}(1 + \bar{\theta}_1^2)\right),$$

(b) A family of Painlevé III with one complex parameter,

$$(\alpha, \beta, \gamma, \delta) = (4\theta, 4(1 + \bar{\theta}), 4, -4).$$

**Remark 4.5** Hitchin shows the anti-self-dual equations reduce to Painlevé VI with the parameters above (\[7\], P.50 (14)). Dancer \[4\] shows the scalar-flat diagonal Kähler metric is specified by a solution of Painlevé III with parameters $(\alpha, \beta, \gamma, \delta) = (4, 4, 4, -4)$. Now, theorem 4.4 (b) is a generalization of Dancer’s result.

From now on, we will classify the anti-self-dual metrics with a signature $(+, +, -,-)$. By lemma 4.2 the poles of $B_1$ make conjugate pairs $\zeta_0, \bar{\zeta}_0, \zeta_1, \bar{\zeta}_1$ in $\mathbb{CP}^1$. Therefore we have five types of metrics corresponding to configuration of poles of $B_1$.
**Theorem 4.6** If the metric has a signature \((+,+,−,−)\), the anti-self-dual equations reduce to the following five Painlevé equations:

(a) Painlevé VI with two complex parameters

\[
(\alpha, \beta, \gamma, \delta) = \left( \frac{1}{2}(\theta_0 - 1)^2, \frac{1}{2}\theta_0^2, -\frac{1}{2}\theta_1^2, \frac{1}{2}(1 + \bar{\theta}_1^2) \right),
\]

(b) Painlevé V with one real and one complex parameters

\[
(\alpha, \beta, \gamma, \delta) = \left( \frac{1}{2}(\theta_0 + \bar{\theta}_0 + \theta_\infty)^2, -\frac{1}{2}(\theta_0 + \bar{\theta}_0 - \theta_\infty)^2, 1 - \theta_0 + \bar{\theta}_0, \frac{1}{2} \right),
\]

where \(\theta_\infty \in \mathbb{R}\).

(c) Painlevé III with one complex parameter

\[
(\alpha, \beta, \gamma, \delta) = (4\theta, 4(1 + \bar{\theta}), 4, -4).
\]

(d) Painlevé III with with two real parameters

\[
(\alpha, \beta, \gamma, \delta) = (4\theta_1, 4(1 + \theta_2), 4, -4).
\]

(e) Painlevé II with one real parameter \(\alpha\).

**Proof.**

Since the poles of \(B_1\) make conjugate pairs \(\zeta_0, \bar{\zeta}_0\) and \(\zeta_1, \bar{\zeta}_1\), we have five types of configuration of poles of \(B_1\). In each case, we can calculate local exponents at singularities. These local exponents corresponding to parameters of Painlevé equations (see [6]).

(a) Generically, \(B_1\) has four simple poles \(\zeta_0, \bar{\zeta}_0, \zeta_1, \bar{\zeta}_1\) on \(\mathbb{C} \setminus \mathbb{R}\).

\[
B_1 = \frac{A_0}{z - \zeta_0} + \frac{-t^*A_0}{z - \bar{\zeta}_0} + \frac{A_1}{z - \zeta_1} + \frac{-t^*A_1}{z - \bar{\zeta}_1}.
\]

The deformation equations are Painlevé VI with parameters,

\[
(\alpha, \beta, \gamma, \delta) = \left( \frac{1}{2}(\theta_0 - 1)^2, \frac{1}{2}\theta_0^2, -\frac{1}{2}\theta_1^2, \frac{1}{2}(1 + \bar{\theta}_1^2) \right),
\]

where \(\theta_0^2 = 2 \text{tr}A_0^2, \theta_1^2 = 2 \text{tr}A_1^2\).

(b) If \(\zeta_0 = \bar{\zeta}_0(= \eta)\), then \(B_1\) has two simple poles \(\zeta_1, \bar{\zeta}_1\) on \(\mathbb{C} \setminus \mathbb{R}\), and one double pole \(\eta\) on \(\mathbb{R}\).

\[
B_1 = \frac{C}{(z - \eta)^2} + \frac{-A_2 + t^*A_2}{z - \eta} + \frac{A_2}{z - \zeta_1} + \frac{-t^*A_2}{z - \bar{\zeta}_1},
\]

where \(C = -t^*\tilde{C}\). The deformation equation is Painlevé V with parameters,

\[
(\alpha, \beta, \gamma, \delta) = \left( \frac{1}{2}(\theta_0 + \bar{\theta}_0 + \theta_\infty)^2, -\frac{1}{2}(\theta_0 + \bar{\theta}_0 - \theta_\infty)^2, 1 - \theta_0 + \bar{\theta}_0, \frac{1}{2} \right),
\]

where \(\theta_0^2 = 2 \text{tr}A_0^2, \theta_\infty^2 = 2 (\text{tr} (A_2 - t^*A_2) C)^2 / \text{tr}C^2\).
(c) If \( \zeta_0 = \zeta_1 (= \zeta) \), then \( B_1 \) has two double poles \( \zeta, \bar{\zeta} \) on \( \mathbb{C} \setminus \mathbb{R} \).

\[
B_1 = \frac{A_3}{(z - \zeta)^2} + \frac{\sqrt{-1}C}{z - \zeta} + \frac{-\sqrt{-1}C}{z - \bar{\zeta}} + \frac{-tA_3}{(z - \zeta)^2},
\]
where \( C = -t\bar{C} \). The deformation equation is Painlevé III with parameters,

\[
(\alpha, \beta, \gamma, \delta) = (4\theta, 4(1 + \bar{\theta}), 4, -4),
\]
where \( \theta^2 = 2(\text{tr}\ A_1C)^2/\text{tr}C^2 \).

(d) If \( \zeta_0 = \bar{\zeta}_0 (= \eta_0), \zeta_1 = \bar{\zeta}_1 (= \eta_1) \), then \( B_1 \) has two double poles \( \eta_0, \eta_1 \) on \( \mathbb{R} \).

\[
B_1 = \frac{C_1}{(z - \eta)^2} + \frac{C_2}{z - \eta_0} + \frac{-C_2}{z - \eta_1} + \frac{C_3}{(z - \eta)^2},
\]
where \( C_i = -t\bar{C}_i \) \((i = 1, 2, 3)\). The deformation equation is Painlevé III with parameters,

\[
(\alpha, \beta, \gamma, \delta) = (4\theta_1, 4(1 + \theta_2), 4, -4),
\]
where \( \theta_1^2 = 2(\text{tr}C_1C_2)^2/\text{tr}C_2^2, \theta_2^2 = 2(\text{tr}C_2C_3)^2/\text{tr}C_3^2 \).

(e) If \( \zeta_0 = \bar{\zeta}_0 = \zeta_1 = \bar{\zeta}_1 (= \eta) \), then \( B_1 \) has one quadruple pole \( \eta \) on \( \mathbb{R} \).

\[
B_1 = \frac{C_1}{(z - \eta)^4} + \frac{C_2}{(z - \eta)^3} + \frac{C_3}{(z - \eta)^2},
\]
where \( C_i = -t\bar{C}_i \) \((i = 1, 2, 3)\). If \( \det C_1 \neq 0 \), then the deformation equation is Painlevé II with a parameter,

\[
\alpha = \frac{1}{2}(1 + \text{tr}C_2C_3).
\]

If \( \det C_1 = 0 \), then the deformation equation is Painlevé I, but since \( C_1 = -t\bar{C}_1 \), this never occurs.

5 Hermitian Structure and Null Surfaces

In this section we will consider the geometric meaning of the metrics corresponding with each type of Painlevé functions.

Lemma 5.1 Let \( g \) be an anti-self-dual metric. If \( B_1 \) has a multiple pole, then the Pfaffian \( \Theta_1|_{z = \zeta(t)} \), \( \Theta_2|_{z = \zeta(t)} \) is integrable. Conversely, if the Pfaffian \( \Theta_1|_{z = \zeta(t)} \), \( \Theta_2|_{z = \zeta(t)} \) is integrable for some \( z = \zeta(t) \), then \( B_1 \) has a multiple pole on \( z = \zeta(t) \).

Proof.

If \( z = \zeta(t) \) is a multiple zero of \( G(z) \), then \( G|_{z = \zeta(t)} \) and \( dG|_{z = \zeta(t)} \) must vanish. Furthermore, \( \Theta_3 \equiv 0 \) \((\text{mod} \ 1, \Theta_1, \Theta_2, G, dG) \). Therefore, the Pfaffian \( \Theta_1|_{z = \zeta(t)}, \Theta_2|_{z = \zeta(t)} \) is integrable.

Conversely, if the Pfaffian \( \Theta_1|_{z = \zeta(t)}, \Theta_2|_{z = \zeta(t)} \) is integrable for some \( z = \zeta(t) \), then \( \Theta_3|_{z = \zeta(t)} \equiv 0 \) \((\text{mod} \ 1, \Theta_2) \). Therefore the denominator \( G \) of \( B_1 \) has zero on \( z = \zeta(t) \). Furthermore, \( \Theta_3|_{z = \zeta(t)} \equiv 0 \) \((\text{mod} \ 1, \Theta_2, G) \) is equivalent to \( dG|_{z = \zeta(t)} = 0 \), therefore \( z = \zeta(t) \) is a double pole of \( B_1 \).
In [8] we show that if a positive definite SU(2) invariant anti-self-dual metric is corresponding with Painlevé III with one complex parameter $(4\theta, 4(1 + \bar{\theta}), 4, -4)$, then the Pfaffian $\Theta_1|_{z=\zeta(t)}, \Theta_2|_{z=\zeta(t)}$ ($z = \zeta(t)$ is a double pole of $B_1$) determines a SU(2) invariant hermitian structure.

In the same way, we have the following theorem for the metric with a signature $(+, +, -, -)$.

**Theorem 5.2** If the anti-self-dual equations reduce to Painlevé III with one complex parameter $(4\theta, 4(1 + \bar{\theta}), 4, -4)$, then there exists a SU(2)-invariant hermitian structure. Conversely, if there exists a SU(2)-invariant hermitian structure, then the anti-self-dual equations reduce to Painlevé III with parameters above.

**Definition 5.3** If $g(X, X) = 0$, then $X \in TM$ is said to be a null direction.

The Pfaffian $\Theta_1|_{z=\eta(t)}, \Theta_2|_{z=\eta(t)}(\eta(t) \in \mathbb{R})$ determines two dimensional null directions on $TM$.

**Definition 5.4** Let $N$ be a two dimensional subspace of $M$. If $g(X, X) = 0$ for any $X \in TN$, then $N$ is called a null surface.

From lemma [3.3], if $z = \eta(t) \in \mathbb{R}$ is a multiple pole of $B_1$, then the Pfaffian $\Theta_1|_{z=\eta(t)}, \Theta_2|_{z=\eta(t)}$ is integrable, and then for any $x \in M$ there exists a SU(2)-invariant null surface passing through $x$. Conversely, for any $x \in M$, if there exists a SU(2)-invariant null surface passing through $x$, then the null surface is represented by Pfaffian $\Theta_1|_{z=\eta(t)}, \Theta_2|_{z=\eta(t)}$ for some $\eta(t) \in \mathbb{R}$, and then $z = \eta(t)$ is a multiple pole of $B_1$.

**Theorem 5.5** If the anti-self-dual equations reduce to Painlevé V with one real and one complex parameters $(\frac{1}{2}(\theta_0 + \bar{\theta}_0 + \theta_\infty)^2, -\frac{1}{2}(\theta_0 + \bar{\theta}_0 - \theta_\infty)^2, 1 - \theta_0 + \bar{\theta}_0, \frac{1}{2})$, where $\theta_\infty \in \mathbb{R}$, or Painlevé II with one real parameter $\alpha$, then for any $x \in M$ there exists one SU(2)-invariant null surface passing through $x$. Conversely, for any $x \in M$ there exists one SU(2)-invariant null surface passing through $x$, then the anti-self-dual reduce to Painlevé V or II with parameters above.

If the anti-self-dual equations reduce to Painlevé III with two real parameters $(4\theta_1, 4(1 - \theta_2), 4, -4)$, then for any $x \in M$ there exist two SU(2)-invariant null surfaces passing through $x$. Conversely, for any $x \in M$ there exist two SU(2)-invariant null surfaces passing through $x$, then the anti-self-dual reduce to Painlevé III with parameters above.

6 Summary

We classified the SU(2)-invariant anti-self-dual metric with a signature $(+, +, -, -)$ into the five cases (a)–(e) (theorem [4.0]). The meaning of the types of Painlevé equations are as follows:

1. Generically, the anti-self-dual metric are specified by a solution of Painlevé VI with two complex parameters.
2. If the anti-self-dual metric is specified by a solution of Painlevé I with two complex parameters, then there exist a $SU(2)$-invariant hermitian structure.

3. If the anti-self-dual metric is specified by a solution of Painlevé V with one real and two complex parameters, or Painlevé II with one real parameter, then for any $x \in M$ there exists one real null surface passing through $x$.

4. If the anti-self-dual metric is specified by a solution of Painlevé III with two real parameters, then for any $x \in M$ there exist two real null surfaces passing through $x$.

7 Appendix

We review the Painlevé equations, second order nonlinear differential equations without moving critical points. We list six equations classified by Painlevé and Gambier, where $\alpha, \beta, \gamma, \delta$ are parameters [9].

1. Painlevé I

$$\frac{d^2 q}{dx^2} = 6q^2 + x.$$

2. Painlevé II

$$\frac{d^2 q}{dx^2} = 2q^3 + xq + \alpha.$$

3. Painlevé III

$$\frac{d^2 q}{dx^2} = \frac{1}{q} \left( \frac{dq}{dx} \right)^2 - \frac{1}{x} \frac{dq}{dx} + \frac{1}{x} (\alpha q^2 + \beta) + \gamma q^3 + \frac{\delta}{q}.$$

4. Painlevé IV

$$\frac{d^2 q}{dx^2} = \frac{1}{2q} \left( \frac{dq}{dx} \right)^2 + \frac{3}{2} q^3 + 4xq^2 + 2(x^2 - \alpha)q + \frac{\beta}{q}.$$

5. Painlevé V

$$\frac{d^2 q}{dx^2} = \left( \frac{1}{q} + \frac{1}{q - 1} \right) \left( \frac{dq}{dx} \right)^2 - \frac{1}{x} \frac{dq}{dx} + \frac{(q - 1)^2}{x^2} \left( \alpha q + \frac{\beta}{q} \right) + \frac{\gamma q}{x} + \frac{\delta q(q + 1)}{q - 1}.$$

6. Painlevé VI

$$\frac{d^2 q}{dx^2} = \frac{1}{2} \left( \frac{1}{q} + \frac{1}{q - 1} + \frac{1}{q - x} \right) \left( \frac{dq}{dx} \right)^2 - \left( \frac{1}{x} + \frac{1}{x - 1} + \frac{1}{q - x} \right) \frac{dq}{dx}$$

$$+ \frac{q(q - 1)(q - x)}{x^2 (x - 1)^2} \left\{ \alpha + \beta \frac{x}{q^2} + \gamma \frac{x}{(q - 1)^2} + \delta \frac{x(x - 1)}{(q - x)^2} \right\}.$$
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