$S$-Noetherian generalized power series rings

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Abstract

Let $R$ be a ring with identity, $(M, \leq)$ a commutative positive strictly ordered monoid and $\omega_m$ an automorphism for each $m \in M$. The skew generalized power series ring $R[[M, \omega]]$ is a common generalization of (skew) polynomial rings, (skew) power series rings, (skew) Laurent polynomial rings, (skew) group rings, and Mal'cev Neumann Laurent series rings. If $S \subset R$ is a multiplicative set, then $R$ is called right $S$-Noetherian, if for each ideal $I$ of $R$, $Is \subseteq J \subseteq I$ for some $s \in S$ and some finitely generated right ideal $J$. Unifying and generalizing a number of known results, we study transfers of $S$-Noetherian property to the ring $R[[M, \omega]]$. We also show that the ring $R[[M, \omega]]$ is left Noetherian if and only if $R$ is left Noetherian and $M$ is finitely generated. Generalizing a result of Anderson and Dumitrescu, we show that, when $S \subset R$ is a $\sigma$-anti-Archimedean multiplicative set with $\sigma$ an automorphism of $R$, then $R$ is right $S$-Noetherian if and only if the skew polynomial ring $R[x; \sigma]$ is right $S$-Noetherian.

Keywords: $S$-Noetherian ring, skew generalized power series ring; right archimedean ring; skew Laurent series ring; skew polynomial ring.

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1 Introduction

Throughout this paper, $R$ is a ring (not necessary commutative) with identity. In [3], the authors introduced the concept of “almost finitely generated” to study Querré’s characterization of divisorial ideals in integrally closed polynomial rings. Later, Anderson and Dumitrescu [1] abstracted this notion to any commutative ring and defined a general concept of Noetherian rings. They call $R$ an $S$-Noetherian ring if each ideal of $R$ is $S$-finite, i.e., for each ideal $I$ of $R$, there exist an $s \in S$ and a finitely generated ideal $J$ of $R$ such that $Is \subseteq J \subseteq I$. By [1, Proposition 2(a)], any integral domain $R$ is $(R \setminus \{0\})$-Noetherian; so an $S$-Noetherian ring is not generally Noetherian. Also, $M$ is said to be $S$-finite if there exist an $s \in S$ and a finitely generated $R$-submodule $F$ of $M$ such that $sM \subseteq F$. Also, $M$ is called $S$-Noetherian if each submodule of $M$ is $S$-finite. In [1], the authors gave a number of $S$-variants of well-known results for Noetherian rings: $S$-versions of Cohens result, the Eakin-Nagata theorem, and the Hilbert basis theorem under an additional condition. More precisely, in [1, Propositions 9 and 10], the authors showed that, if $S$ is an anti-Archimedean subset of an $S$-Noetherian ring $R$, then the polynomial ring $R[X_1, \cdots, X_n]$ is also an $S$-Noetherian ring; and if $S$ is an anti-Archimedean subset of an $S$-Noetherian ring $R$ consisting of nonzero divisors, then the power

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series ring \( R[[X_1, \ldots, X_n]] \) is an \( S \)-Noetherian ring. Note that if \( S \) is a set of units of \( R \), then the results above are nothing but the Hilbert basis theorem and a well-known fact that \( R[[X]] \) is Noetherian if \( R \) is Noetherian. In [16, Theorem 2.3], Liu generalized this result to the ring of generalized power series as follows: If \( S \) is an anti-Archimedean subset of a ring \( R \) consisting of nonzero divisors and \( (\Gamma, \leq) \) is a positive strictly ordered monoid (defined in Section 4), then \( R[[M, \leq]] \) is \( S \)-Noetherian if and only if \( R \) is \( S \)-Noetherian and \( \Gamma \) is finitely generated. Note that this recovers the result for the Noetherian case shown in [6, , Theorem 4.3] when \( S \) is a set of units. Also, the authors in [14] study on transfers of the \( S \)-Noetherian property to the skew generalized power series ring \( R \) unify and generalize the above mentioned results, and study transfers of \( S \)-Noetherian rings for noncommutative rings. These considerably strengthen earlier results of Ribenboim [22] and Varadarajan [26], have carried out an extensive study of rings of generalized power series. They investigated conditions under which a ring of generalized power series \( R[[M, \leq]] \) is Noetherian, where \( R \) is a commutative ring with identity and \( (M, \leq) \) is a strictly ordered monoid.

In this paper we obtain results pertaining to Noetherian nature of generalized power series rings. These considerably strengthen earlier results of Ribenboim [22], Varadarajan [26], Brookfield [6], D. D. Anderson, and T. Dumitrescu[1], D. D. Anderson, B. G. Kang, and M. H. Park [2], D. D. Anderson, D. J. Kwak, M. Zafrullah [3] on this topic.

More precisely, we show that, if \( S \) is an \( \sigma \)-anti-Archimedean multiplicative subset of an \( S \)-Noetherian ring \( R \) with an automorphism \( \sigma \), then the skew polynomial ring \( R[x; \sigma] \) is also an \( S \)-Noetherian ring; and if \( (M, \leq) \) is a commutative positively ordered monoid and \( \omega_m \) is an automorphism over \( R \) for every \( m \in M \), then the skew generalized power series ring \( R[[M, \omega]] \) is right Noetherian if and only if \( R \) is right Noetherian and \( M \) is finitely generated. When \( (M, \leq) \) is a commutative positive strictly ordered monoid and \( \omega_m \) is an automorphism for each \( m \in M \), we unify and generalize the above mentioned results, and study transfers of \( S \)-Noetherian property to the skew generalized power series ring \( R[[M, \omega]] \).

## 2 S-Noetherian property on skew polynomial rings

If \( R \) is a commutative ring and \( S \) is a multiplicative subset of \( R \), in [1], the authors proved that the necessary condition for the ring of fractions \( R_S \) to be a Noetherian ring is that \( R \) be an \( S \)-Noetherian ring. In noncommutative rings, the situation is more complicated. In fact, if \( S \) is a right (resp., left) permutable and right (resp., left) reversible (i.e \( S \) is right (resp., left) denominator set), then \( R \) has a ring of fraction \( RS^{-1} \) (resp., \( S^{-1}R \)). In this situation, denominator sets (both left and right denominator sets) act like a multiplicatively closed sets in the commutative case. Our interest in this note is multiplicatively closed subsets (i.e. denominator subsets) in noncommutative rings. First we define the notion of \( S \)-Noetherian rings for noncommutative rings.

**Definition 2.1.** Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). An ideal \( I \) of \( R \) is called right \( S \)-finite (resp., \( S \)-principal), if there exists a finitely generated (resp., principal) right ideal \( J \) of \( R \) and some \( s \in S \) such that \( Is \subseteq J \subseteq I \).
A ring $R$ is said to be right $S$-Noetherian (resp., $S$-PRIR), if each right ideal of $R$ is right $S$-finite (resp., $S$-principal). This definition can be done similarly for left side ideals.

Also, we say that an $R$-module $M$ is right (or left) $S$-finite if $Ms \subseteq F$ (resp., $sM \subseteq F$) for some $s \in S$ and a finitely generated submodule $F$ of $M$. A module $M$ is called right (or left) $S$-Noetherian if each submodule of $M$ is a right (or left) $S$-finite module.

The author in [1] justified the definition of $S$-Noetherian for commutative rings by proving some interesting properties of $S$-Noetherian ring. For example, they showed that if $R$ is $S$-Noetherian, then the ring of fractions $RS$ is Noetherian and they found the conditions for the reverse of this proposition.

Given rings $R, T$, an ideal $J$ of $T$ is said to be extended, if there exists an ideal $I$ of $R$ such that $\varphi(I) = J$ where $\varphi : R \to T$ is a ring monomorphism. Also, a ring $R$ is von Neumann regular if for every $a \in R$ there exists an $x$ in $R$ such that $a = axa$. The center of a ring $R$ is denoted by $\text{Cent}(R)$.

**Proposition 2.2.** Let $R$ be a ring, $S \subseteq R$ a multiplicative set and $I$ a right ideal of $R$.

1) If $R$ is von Neumann regular, $S$ a denominator set and $I \cap S \neq \emptyset$, then $I$ is right $S$-principal.

2) If $S \subseteq T$ are right denominator subsets of $R$ and $R$ is right $S$-Noetherian (resp., $S$-PRIR), then $R$ is right $T$-Noetherian (resp., $T$-PRIR).

3) If $R$ is von Neumann regular and $S$ a denominator set, then $R$ is right $S$-Noetherian (resp., $S$-PRIR) if and only if $R$ is right Noetherian (resp., $S$-PRIR).

4) If $S$ is a denominator set and $R$ is right $S$-Noetherian (resp., $S$-PRIR), then $RS^{-1}$ is right Noetherian.

5) If $S$ is central in $R$, then the conditions 1-4 and those of [1, Proposition 2] follow.

**Proof.** 1) Let $S \subseteq R$ be a denominator set, $R$ a von Neumann regular ring and $I$ a right ideal of $R$. Then for each $s \in I \cap S$, one can see that $Is \subseteq Rs = s\frac{1}{s}Rs$, where $\frac{1}{s}$ is the inverse of $s$ in $RS^{-1}$. It is sufficient to see that $\frac{1}{s}Rs \subseteq R$. For each $s \in S$, there exists $a \in R$ such that $sas = s$, so $sa = s\frac{1}{s} = 1$ (in $RS^{-1}$). Thus $sa = 1$ and hence $a = \frac{1}{s}$. Therefore $\frac{1}{s} \in R$ and $Rs \subseteq R$, so $\frac{1}{s}Rs \subseteq R$.

2) Let $S \subseteq T$ be denominator subsets of $R$. If $R$ is right $S$-Noetherian (resp., $S$-PRIR), then for each right ideal of $R$, there exists $s \in S$ such that $Is \subseteq J \subseteq I$ for some finitely generated (resp., principal) right ideal of $R$. Since $s \in S$, $S \subseteq T$, $s \in T$ which means that $R$ is right $T$-Noetherian (resp., $T$-PRIR).

3) Assume that $R$ is a right Noetherian (resp., PRIR) ring. Each right ideal of $R$ is finitely generated (resp., principal). So for each $s \in S$, one can see that $Is \subseteq I$. Hence $R$ is right $S$-Noetherian (resp., $S$-PRIR). On the other hand, assume that $R$ is right $S$-Noetherian (resp., $S$-PRIR), so there exists $s \in S$ such that $Is \subseteq J \subseteq I$ for some finitely generated (resp., principal) right ideal of $R$. Also suppose that $sts = s$ for some $t \in R$. So $Is \subseteq I$. Also, $It \subseteq I$, so $Its \subseteq Is = Ists$. So $Its = Ists$. Hence $It \subseteq Ists$. Also $Is \subseteq I$ yields that $1s = Is = Ists$. Thus $1s = Ists = Is = Ists$. However $Ists = \frac{1}{s}sts = \frac{1}{s}s = I$. So $Is = I$. Thus $I = Is \subseteq J \subseteq I$ and hence $I = J$, and since $J$ is a finitely generated (resp., principal) right ideal of $R$, so is $I$.

4) This proof is an inspiration from [4, proposition 3.11 part (i)]. First, we claim that each ideal of $RS^{-1}$ is extended. Let a right ideal $J$ of ring of fraction $RS^{-1}$ and $\frac{1}{s} = b \in J$. So
\[ \tilde{x} = \frac{2}{3}, \tilde{y} \in J. \] So \( \tilde{x} \in J. \) Hence, \( \varphi^{-1}(\tilde{x}) \in \varphi^{-1}(J) \) which means that \( x \in \varphi^{-1}(J) \). Thus, \( \varphi(x) \in \varphi(\varphi^{-1}(J)) \), so \( \tilde{x} \in \varphi(\varphi^{-1}(J)) \). So \( \tilde{x}, \tilde{y} = \frac{2x}{s}, \frac{2y}{s} = \frac{2x}{s} \in \varphi(\varphi^{-1}(J)) \). Note that \( \varphi(\varphi^{-1}(J)) \) is an ideal of \( RS^{-1} \) and \( s \in U(RS^{-1}) \), so we have
\[
\frac{xs}{s} - \frac{1}{s} = \frac{x}{s} \in \varphi(\varphi^{-1}(J)) \frac{1}{s} \subseteq \varphi(\varphi^{-1}(J)).
\]
So \( b = \frac{x}{s} \in \varphi(\varphi^{-1}(J)) \) which implies \( J \subseteq \varphi(\varphi^{-1}(J)) \). On the other hand, \( \varphi(\varphi^{-1}(J)) \subseteq J \) holds for each ideal of \( RS^{-1} \). Thus \( J = \varphi(\varphi^{-1}(J)) \) and \( J \) is an extended ideal of \( RS^{-1} \).

Let a right ideal \( K \) of ring of fraction \( RS^{-1} \). Since \( R \) is right \( S \)-Noetherian there exists \( s \in S \) and a finitely generated (resp., principal) right ideal \( W \) of \( R \) such that \( \varphi^{-1}(K)s \subseteq W \subseteq \varphi^{-1}(K) \). So \( \varphi^{-1}(K)s \subseteq W \subseteq \varphi^{-1}(K) \)). We know that \( \varphi(\varphi^{-1}(K)s) = \varphi(\varphi^{-1}(K))\varphi(s) \). Also, \( \varphi(s) \in U(RS^{-1}) \) and \( \varphi^{-1}(K) = K \). So \( K \subseteq W \subseteq K \). Since \( W \) is finitely generated, \( \varphi(W) \) is finitely generated. So \( K \) is finitely generated which means that \( RS^{-1} \) is right Noetherian.

5) The proof is straightforward by [1, Proposition 2].

Now we generalize a theorem of D.D. Anderson and Tiberiu Dumitrescu [1, Proposition 9], for commutative polynomial ring \( R[x] \), in a more general setting. We show that if \( R \) is a right (or left) \( S \)-Noetherian ring with an automorphism \( \sigma \), then \( R[x; \sigma] \) is a right (or left) \( S \)-Noetherian ring.

In [2] the authors defined the notion of anti-Archimedean multiplication set. Now we introduce the notion of \( \sigma \)-anti-Archimedean multiplication set:

**Definition 2.3.** Let \( R \) be a ring with an automorphism \( \sigma \) and \( S \) a multiplicative set. Then \( R \) is called left \( \sigma \)-anti-Archimedean over \( S \), if there exists \( s \in S \), such that
\[
( \bigcap_{i \geq 1, k \geq 0} R\sigma^{k_1}(s)\sigma^{k_2}(s) \cdots \sigma^{k_l}(s)) \cap S \neq \emptyset.
\]

**Theorem 2.4.** Let \( R \) be a ring with an automorphism \( \sigma \) and \( S \subseteq R \) a \( \sigma \)-anti-Archimedean multiplicative set. Then \( R \) is right (or left) \( S \)-Noetherian if and only if \( R[x; \sigma] \) is right (or left) \( S \)-Noetherian.

**Proof.** \((\Rightarrow)\) We prove the theorem for the right version. The proof of left version is similar. First, we claim that if \( D \) is a finitely generated \( R \)-module and \( R \) is a right \( S \)-Noetherian ring, then \( D \) is a right \( S \)-Noetherian module. For this claim, assume that \( D \) is a finitely generated right \( R \)-module. So there exists a finitely generated free right \( R \)-module \( F \) and a surjective homomorphism \( \pi : F \to D \). We show that \( D \) is a right \( S \)-Noetherian \( R \)-module. For this, let \( N := \pi^{-1}(T) \), for a submodule \( T \) of \( D \). We have \( N \cong \bigoplus_{1 \leq i \leq l} I_1 \) for some right ideals \( I_i \) of \( R \), \( 1 \leq i \leq l \). Since \( R \) is a right \( S \)-Noetherian ring, there exists \( s_i \in S \) such that \( I_i s_i \subseteq J_i \) for a finitely generated ideals \( J_i \) of \( R \), \( 1 \leq i \leq l \). Now take \( s' := s_1 s_2 \cdots s_l \in S \), we show that \( N s' \subseteq K \) for a finitely generated \( R \)-submodule \( K \) of \( F \). One can see that \( N s_1 = I_1 s_1 \oplus I_2 s_1 \oplus \cdots \oplus I_l s_1 \). Since \( I_1 \) is a right ideal of \( R \) so we have \( I_i s_1 \subseteq J_i \) for \( i \neq 1 \) and \( I_1 s_1 \subseteq J_1 \), for a finitely generated right ideal \( J_1 \) of \( R \). So we have \( N s_1 \subseteq J_1 \oplus I_2 \oplus \cdots \oplus I_l \). Continuing in this way, \( N s_1 s_2 \cdots s_l \subseteq J_1 \oplus J_2 \oplus \cdots \oplus J_l \simeq K \), where \( J_i \) is a finitely generated right ideal of \( R, 1 \leq i \leq l \), and hence \( K \) is a finitely generated \( R \)-submodule of \( F \). Thus \( N s' \subseteq K \) and hence \( F \) is a right \( S \)-Noetherian \( R \)-module. Next, since \( T = \pi(N) \) and \( N s' \subseteq K \), we have \( \pi(N s') = \pi(N) s' = Ts' \subseteq \pi(K) \). We know that \( K \) is finitely generated in \( F \), so \( \pi(K) \) is finitely generated.
generated $R$-submodule of $D$. Thus, $T s' \subseteq \pi(K)$ which means that $T$ is $S$-finite. Since $T$ is an arbitrary $R$-submodule of $D$, $D$ is a right $S$-Noetherian module.

Now, we prove that $A := R[x; \sigma]$ is a right $S$-Noetherian ring. Let $I$ be right ideal of $A$ and suppose that

$$J = \{ r_i \in R | r_i \text{ is a leading coefficient of any polynomial in } I \} \cup \{ 0 \}.$$ 

It is easy to see that $J$ is a right ideal. Since $R$ is right $S$-Noetherian, $J_s \subseteq (a_1 R + a_2 R + \cdots + a_n R)$ for some $s \in S$ and $a_i \in J$. So there exist polynomials $f_i \in I$ with $f_i = a_{i,n} x^{n_i} + \cdots + a_{0,i}$. Let $d = \max \{ n_i \}$. Assume that $T$ is the set of all polynomials in $I$ with degree less than $d$. Obviously, $T$ is a finitely generated right $R$-submodule of $A$. So by the first claim, $T$ is right $S$-Noetherian. Hence there exist $t \in S$, $g_i \in T$ for $1 \leq i \leq m$ such that $T t \subseteq (g_1 R + g_2 R + \cdots + g_m R)$. Let $h(x) = \sum_{i=1}^n b_i x^i \in I$, so $b_x \in J$ which means that $b_x \in (a_1 R + a_2 R + \cdots + a_n R)$. Thus $h \sigma^{-z}(s)$ can be written as follows:

$$h \sigma^{-z}(s) = v(1) + w(1) + q(1),$$

where $v(1) \in (f_1 A + f_2 A + \cdots + f_n A)$, $w(1) \in \{ f \in A | d + 1 \leq \deg(f) \leq z - 1 \}$ and $q(1) \in T$. Continuing in this way and multiplying $\sigma^{-z+1}(s), \sigma^{-z+2}(s), \ldots, \sigma^{-d-1}(s)$ from right side respectively, so there exists some $v \in (f_1 A + f_2 A + \cdots + f_n A)$, $w \in T$ such that

$$h \sigma^{-z}(s) \sigma^{-z+1}(s) \cdots \sigma^{-d-1}(s) = v + w.$$ 

Assume that $s_i = \sigma^{-z+i}$ and multiplying $t$ from right side, then $h s_1 s_2 \cdots s_{z-d} t = v t + w t$. But $w t \in T t$, so $w t \in (g_1 R + g_2 R + \cdots + g_m R) \subseteq (g_1 A + g_2 A + \cdots + g_m A)$. Hence,

$$h s_1 s_2 \cdots s_{z-d} t \in (f_1 A + f_2 A + \cdots + f_n A + g_1 A + \cdots + g_m A).$$

Since $s_i$’s and $t$ are independent from the choice of $h \in I$, we have

$$I s_1 s_2 \cdots s_{z-d} t \subseteq (f_1 A + f_2 A + \cdots + f_n A + g_1 A + \cdots + g_m A).$$

Finally, since $s_1 s_2 \cdots s_{z-d} t \in S$, the ideal $I$ is $S$-finite and because $I$ was chosen an arbitrary right ideal of $A$, hence $A$ is a right $S$-Noetherian ring.

$(\Leftarrow)$ Let $I$ be a right ideal of $R$. Suppose that

$$J = \{ f \in A | \text{ the leading coefficient of } f \text{ is in } I \}.$$ 

Then $J$ is a right ideal of $A$. Since $A$ is right $S$-Noetherian, there exists $s \in S$ such that $J_s \subseteq K \subseteq J$, where $K$ is a finitely generated right ideal of $A$. Suppose that $K = (f_1 A + f_2 A + \cdots + f_l A)$. Let $r \in I$, then there exists some $f \in J$ such that $f s = \sum a_i f_i$. So if $r_i$ is the leading coefficient of $f_i$, $1 \leq i \leq l$, then $r s \in (r_1 R + r_2 R + \cdots + r_l R)$. So $I s \subseteq (r_1 R + r_2 R + \cdots + r_l R)$. Also, $K \subseteq J$, so each leading coefficient of $K$ is a leading coefficient of $J$. So $(r_1 R + r_2 R + \cdots + r_l R) \subseteq I$ and hence $I$ is right $S$-finite and $R$ is right $S$-Noetherian.

We have the following generalization of a theorem of D.D. Anderson and Tiberiu Dumitrescu [1, Proposition 9].

**Corollary 2.5.** Let $R$ be a (not necessarily commutative) ring and $S \subseteq R$ an anti-Archimedean multiplicative set. If $R$ is $S$-Noetherian then so is the polynomial ring $R[X_1, X_2, \ldots, X_n]$. 

5
3 Noetherian Skew Generalized Power Series Rings

Throughout this section, \((M, \leq)\) is assumed to be a strictly ordered commutative monoid. The pair \((M, \leq)\) is called an ordered monoid with order \(\leq\), if for every \(m, m', n \in M\), \(m \leq m'\) implies that \(nm \leq nm'\) and \(mn \leq m'n\). Also, an ordered monoid \((M, \leq)\) is said to be strictly ordered if for every \(m, m', n \in M\), \(m < m'\) implies that \(nm < nm'\) and \(mn < m'n\). Let \((M, \leq)\) be a partially ordered set. The set \((M, \leq)\) is called Artinian if every strictly decreasing sequence of elements of \(M\) stabiized, and also \((M, \leq)\) is called narrow if the number of incomparable elements in every subset of \(M\) is finite. Thus, we can conclude that \((M, \leq)\) is Artinian and narrow if and only if every nonempty subset of \(M\) has at least one but only a finite number of minimal elements.

The author in [24] introduced the ring of generalized power series \(R[[M]]\) for a strictly ordered monoid \(M\) and a ring \(R\) consisting of all functions from \(M\) to \(R\) whose support is Artinian and narrow with the pointwise addition and the convolution multiplication. There are a lot of interesting examples of rings in this form (e.g., Elliott and Ribenboim, [7]; Ribenboim, [23]) and it was extensively studied by many authors, recently.

In [21], the authors defined a “twisted” version of the mentioned construction and study on ascending chain condition for its principal ideals. Now we recall the construction of the skew generalized power series ring introduced in [21]. Let \(R\) be a ring, \((M, \leq)\) a strictly ordered monoid, and \(\omega : M \to \text{End}(R)\) a monoid homomorphism. For \(m \in M\), let \(\omega_m\) denote the image of \(m\) under \(\omega\), that is \(\omega_m = \omega(m)\). Let \(A\) be the set of all functions \(f : M \to R\) such that the support \(\text{supp}(f) = \{m \in M | f(m) \neq 0\}\) is Artinian and narrow. Then for any \(m \in M\) and \(f, g \in A\) the set

\[\chi_m(f, g) = \{(u, v) \in \text{supp}(f) \times \text{supp}(g) : m = uv\}\]

is finite. Thus one can define the product \(fg : M \to R\) of \(f, g \in A\) as follows:

\[fg(m) = \sum_{(u, v) \in \chi_m(f, g)} f(u)\omega_u(g(v)),\]

(by convention, a sum over the empty set is 0). Now, the set \(A\) with pointwise addition and the defined multiplication is a ring, and called the ring of skew generalized power series with coefficients in \(R\) and exponents in \(M\). To simplify, take \(A\) as a formal series \(\sum_{m \in M} r_m x^m\), where \(r_m = f(m) \in R\). This ring can be denoted either by \(R[[M, \omega]]\) or by \(R[[M, \omega]]\) (see [18] and [19]).

For every \(r \in R\) and \(m \in M\) we can defined the maps \(c_r, e_m : M \to R\) by

\[c_r(x) = \begin{cases} r & ; x = 1 \\ 0 & ; \text{Otherwise} \end{cases}, \quad e_m(x) = \begin{cases} 1 & ; x = m \\ 0 & ; \text{Otherwise} \end{cases}(3.1)\]

where \(x \in M\). By way of illustration, \(c_r(x)\) and \(e_m(x)\) are like \(r\) and \(x^m\) in usual polynomial ring \(R[x]\), respectively.

The following proposition which is proved in [11, Theorem 2.1], can characterize all Artinian and narrow sets.
Proposition 3.1. Let \((M, \leq)\) be an ordered set. Then the following conditions are equivalent

1. \((M, \leq)\) is Artinian and narrow.
2. For any sequence \((m_n)_{n \in \mathbb{N}}\) of elements of \(M\) there exist indices \(n_1 < n_2 < n_3 < \cdots\) such that \(m_{n_1} \leq m_{n_2} \leq m_{n_3} \leq \cdots\).
3. For any sequence \((m_n)_{n \in \mathbb{N}}\) of elements of \(M\) there exist indices \(i < j\) such that \(m_i \leq m_j\).

The author in [6] introduced the concept of a lower set. A lower set of \(L\) is a subset \(I \subseteq L\) such that \(x \leq y \in I\) implies \(x \in I\) for all \(x, y \in L\), (which we denoted by \(\downarrow L\) for the set of lower sets of \(L\) ordered by inclusion). In this concept, we can ignore the condition narrow by lower set, indeed it is proved that if \(L\) is partially ordered set, then \(\downarrow L\) is strictly increasing map between partially ordered sets, then if \(L\) satisfies Artinian (or Noetherian) property, then so is \(K\). Moreover, if \(\alpha\) is surjective and \(\downarrow K\) satisfies Artinian (or Noetherian) property, then so does \(\downarrow L\).

An ordered monoid \((M, \leq)\) is called positively ordered if \(m \geq 0\) for all \(m \in M\). In this condition, \(m \leq m'\) implies \(m \leq m'\) for all \(m, m' \in M\). Now, according to [6, in section 4] we have

\[
R[[M, \omega, \leq]] = \{ f \in R[[M, \omega]] \mid \downarrow (\text{supp}(f), \leq) \text{ is Artinian} \}.
\]

If \(\downarrow (M, \leq)\) is Artinian, \(R[[M, \omega, \leq]] = R[[M, \omega]]\). For instance, \(\downarrow (\mathbb{F}, \leq)\) and \(\downarrow (\mathbb{F}^n, \leq)\) are Artinian, and so \(R[[\mathbb{F}, \omega, \leq]] = R[[\mathbb{F}, \omega]]\) and \(R[[\mathbb{F}^n, \omega, \leq]] = R[[\mathbb{F}^n, \omega]]\) such that \(\mathbb{F}\) be a free monoid.

Now we give a generalization of a result [6, Theorem 4.3] of G. Brookfield:

Theorem 3.2. Let \(R\) be a ring, \((M, \leq)\) a positive strictly ordered monoid and \(\omega_m\) an automorphism of \(R\) with \(\omega_m \omega_n = \omega_n \omega_m\) for each \(m, n \in M\). Then \(R[[M, \omega]]\) is left Noetherian if and only if \(R\) is left Noetherian and \(M\) is finitely generated.

Proof. \(\iff\) In the first place, we claim that if \(\varphi : (N, \leq) \to (M, \leq)\) is a surjective strict monoid homomorphism, induces a surjective ring homomorphism \(\varphi^* : R[[N, \omega, \leq]] \to R[[M, \omega, \leq]]\). Since \(\varphi\) is strict, \(\varphi^{-1}(x)\) is antichain in \((N, \leq)\), for all \(x \in M\). Thus, if \(f \in R[[N, \omega, \leq]]\) then \(\varphi^{-1}(x) \cap \text{supp}(f)\) is finite and we can define \(\varphi^*(f) = f^*\), where \(f^*(x) = \sum_{x' \in \varphi^{-1}(x)} f(x')\) for \(x \in M\). We show that \(\varphi^*\) is a ring homomorphism. One can see that

\[
(fg)^*(m) = \sum_{m' \in \varphi^{-1}(m)} (fg)(m') = \sum_{xy=m} \sum_{x'y'=m'} \left( f(x')\alpha_x(g(y')) \right).
\]

On the other hand

\[
(f^*g^*)(m) = \sum_{xy=m} \left( \varphi^*(f(x))\varphi^*(g(y)) \right) = \sum_{xy=m} \left( \sum_{x' \in \varphi^{-1}(x)} f(x') \alpha_x \left( \sum_{y' \in \varphi^{-1}(y)} g(y') \right) \right) = \sum_{xy=m} \sum_{x' \in \varphi^{-1}(x)} \sum_{y' \in \varphi^{-1}(y)} f(x')\alpha_x(g(y')).
\]
Since $\varphi^{-1}$ is a homomorphism, $\varphi^{-1}(x)\varphi^{-1}(y) = \varphi^{-1}(xy)$ and so $\varphi^{-1}(m) = \varphi^{-1}(x)\varphi^{-1}(y)$. So

$$ (f^*g^*)(m) = \sum_{xy=m} \sum_{m'=x'y'} \left( f(x')\alpha_{x'}(g(y')) \right). \quad (3.4) $$

By equations 3.3 and 3.4 we see that $(fg^*)(m) = (f^*g^*)(m)$. We have also

$$ (f + g)^*(x) = \sum_{x' \in \varphi^{-1}(x)} (f + g)(x') = \sum_{x' \in \varphi^{-1}(x)} (f(x') + g(x')) $$

$$ = \sum_{x' \in \varphi^{-1}(x)} f(x') + \sum_{x' \in \varphi^{-1}(x)} g(x') = f^*(x) + g^*(x). $$

Thus $\varphi^* : R[[N, \omega, \leq]] \to R[[M, \omega, \leq]]$ is a ring homomorphism. Now, we show that $\varphi^*$ is surjective. Suppose that $f \in R[[M, \omega, \leq]]$, where $\{f(n)\}_{n \in M}$ are the coefficients of $f$ in $R$. For every $n \in M$, the set $\varphi^{-1}(n)$ is nonempty and finite, say $\varphi^{-1}(n) = \{m_1, m_2, \ldots, m_k\}$, where $k$ and all the $m_i$ depends on $n$. We define the function $g \in R[[N, \omega, \leq]]$ as follows

$$ g(m_j) = \begin{cases} f(n) & ; j = 1 \\ 0 & ; \text{otherwise}. \end{cases} \quad (3.5) $$

Notice that $g$ is independent of $n$, since if $n \neq n'$, then $\varphi^{-1}(n) \cap \varphi^{-1}(n') = \emptyset$. Also, for each $n \in M$ we have

$$ \varphi^*(g)(n) = \sum_{m \in \varphi^{-1}(n)} g(m) = \sum_{j=1}^k g(m_j) = g(m_1) = f(n). $$

This means that $\varphi^*(g) = f$, and hence $\varphi^*$ is surjective. So we proved the claim. It is well-known that there is an strict monoid surjection $\varphi : (\mathbb{F}^n, \preceq) \to (M, \preceq)$ for some $n \in \mathbb{N}$. Also, the identity map $(M, \preceq) \to (M, \preceq)$ is a surjection. So the composition of these two maps is a surjection and by [6, Lemma 2.1]. Hence $R[[M, w, \leq]]$ is a homomorphic image of the ring $R[[\mathbb{F}^n, \omega, \leq]]$. Since $R[[\mathbb{F}^n, \omega, \leq]] = R[[\mathbb{F}^n, \omega]]$ and $R[[\mathbb{F}^n, \omega]]$ is Noetherian, its projection $R[[M, w, \leq]]$ is also Noetherian. Moreover, we show that $R[[M, \omega, \leq]] = R[[M, \omega]]$. If $R[[M, \omega, \leq]]$ is left Noetherian, then $\downarrow (M, \preceq)$ is Artinian. By applying [6, Lemma 2.1(2)] to the identity map $(M, \preceq) \to (M, \preceq)$, one can see that $\downarrow (M, \preceq)$ is Artinian. Thus $R[[M, \omega, \leq]] = R[[M, \omega]]$.

$\Rightarrow$ The method of this part is inspired from [6, Theorem 4.3]. The trivial case of $M$ is obvious. By [6, Lemmas 3.1 and 3.2], $M$ is strict and $\preceq$ is a partial order on $M$.

Suppose $T = R[[M, \omega, \leq]]$ is left Noetherian. One can see that $M$ is finitely generated similar to the proof of [6, Theorem 4.3]. Hence we have to prove that $R$ is Noetherian similar to the proof of ([25, Theorem 5.2(i)], [26, Theorem 3.1(i)]). Let $I_T = \{ f \in T \mid \omega_x(f(y)) \in I \mid x, y \in M \}$. It is easy to see that $I_T$ is a left ideal of $T$. So for each ideal $I$ of $R$, there is a correspondent ideal in $T$. Also if $I \subset J$, then $I_T \subset J_T$. Hence if there exists a nonstabilized ascending chain in $R$, then there is one in $T$. But this is impossible, so $R$ is left Noetherian.

In Theorem 3.2 if we set $\sigma$ the identity homomorphism then we have:

**Corollary 3.3.** [6, Theorem 4.3] Let $R$ be a ring and $(M, \leq)$ a positive strictly ordered monoid. Then $R[[M, \leq]]$ is left Noetherian if and only if $R$ is left Noetherian and $M$ is finitely generated.
Finally, we conclude the following result which connects the results of previous sections.

**Corollary 3.4.** Let $R$ be an $S$-Noetherian von Neumann regular ring and $S$ a denominator set. Assume that $(M, \leq)$ is a finitely generated positive strictly ordered monoid and $\omega_m$ an automorphism of $R$ with $\omega_m \omega_n = \omega_n \omega_m$ for each $m, n \in M$. Then $(S^{-1}R)[[M, \omega]]$ is a left Noetherian ring.

**Proof.** The ring $S^{-1}R$ is Noetherian by Theorem 2.2. Since $(M, \leq)$ is a positive strictly ordered monoid and $\omega_m$ is an automorphism for all $m \in M$, $(S^{-1}R)[[M, \omega]]$ is a Noetherian ring by Theorem 3.2. \qed

## 4 S-Noetherian property of generalized skew power series rings

Recall that a ring is called right duo (resp., left duo) if all of its right (resp., left) ideals are two-sided. Also, a right and left duo ring is called a duo ring. We know that if a ring is duo, then every prime ideal is completely prime. It is known that a power series ring over a duo ring need not be duo (on either side).

**Lemma 4.1.** Let $R$ be a duo ring and $S \subseteq R$ a denominator set. If $s \in S$, $r \in R$ then there exists $s_1 \in S$ such that $sr s_1 = rs s_1$.

**Proof.** Let $s \in S$ and $r \in R$. Since $R$ is duo, there exist $s' \in S$ such that $sr = rs'$, so $\frac{1}{s'} \frac{sr}{\frac{1}{s'} t} = \frac{rs'}{\frac{1}{s'} t}$. Hence $\frac{r}{t} = \frac{rs'}{s'} \frac{1}{\frac{1}{s'} t}$. Thus $\frac{r}{t}(1 - \frac{s'}{s}) = 0$, which means that $\frac{r(s - s')}{s} = 0_{S^{-1}R}$. So $r(s - s')s_1 = 0_R$. So $rss_1 = rs's_1$ and since $rs' = sr$ we have $sr s_1 = rs s_1$. \qed

In the previous result, it is easy to see that if $s \in S$, $r \in R$, then there exists $s_1 \in S$ such that $s_1 sr = s_1 rs$. We will use this point in the proposition below.

**Proposition 4.2.** Let $R$ be a duo ring, $S \subseteq R$ a denominator set and $M$ an $S$-finite $R$-module. Then $M$ is $S$-Noetherian if and only if $PM$ is an $S$-finite submodule, for each $S$-disjoint prime ideal $P$ of $R$.

**Proof.** The “only if” part is clear. For the converse, assume that $PM$ is $S$-finite for each $P$ prime ideal of $R$ with $P \cap S = \emptyset$. Since $M$ is $S$-finite, $wM \subseteq F$ for some $w \in S$ and some finitely generated submodule $F$. If $M$ is not $S$-Noetherian, the set $\mathcal{F}$ of all non-$S$-finite submodules of $M$ is not empty. So $\mathcal{F}$ has a maximal element like $N$ by Zorn’s lemma. We claim that $P = [N : M] := \{r \in R \mid rM \subseteq N\}$ is a prime ideal of $R$ and is disjoint from $S$. Suppose to the contrary that $P \cap S \neq \emptyset$ and $s \in P \cap S$. Then we have

$$swN \subseteq swM \subseteq sF \subseteq sM \subseteq N.$$  

So $swN \subseteq sF \subseteq N$ and $N$ becomes $S$-finite. This contradiction shows that $P \cap S = \emptyset$. Now suppose that $P$ is not a prime ideal of $R$. So $P$ is not completely prime. So there exist $a, b \in R \setminus P$ and $ab \in P$. So $N + aM$ is $S$-finite, hence $s(N + aM) \subseteq (R(n_1 + am_1) + \cdots + R(n_p + am_p))$ for some $s \in S$, $n_i \in N$ and $m_i \in M$. Also $[N : a]$ is $S$-finite. So $t(N : a) \subseteq (Rq_1 + Rq_2 + \cdots + Rq_k)$ for some $t \in S$ and $q_j \in [N : a]$. Since $R$ is duo and $S$ is a denominator set in $R$, there exists $s'' \in S$ such that $s'' at = s'' ta$ by Theorem 4.1. Also $s(N + aM) \subseteq (R(n_1 + am_1) + \cdots + R(n_p + am_p))$. Thus $sx = \sum r_i n_i + r_i a m_i$. This means that $sx = \sum r_i n_i + a \sum r'_i m_i$ for some $r'_i \in R$. Since $sx, \sum r_i n_i \in N$, we have $\sum r'_i m_i \in [N : a]$. So

$$s'' t s x = s'' t \sum r_i n_i + s'' t a r'_i m_i = \sum s'' t r_i n_i + s'' a s' t \sum r'_i m_i = \sum s'' t r_i n_i + s'' a \sum c_j q_j.$$  

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Assume that sending \( x \) to zero and \( \sigma(f_0) = a \). Thus, for each \( a \in R \) there exists \( f_0 \in R \) such that \( a = \sigma(f_0) \).

**Theorem 4.4.** Let \( R \) be a ring, \( S \subseteq R \) a \( \sigma \)-anti-Archimedean denominator set (consisting nonzero devisors) and \( \sigma_1, \ldots, \sigma_n \) are monomorphisms of \( R \) with \( \sigma_i \sigma_j = \sigma_j \sigma_i \), for each \( i, j \). Assume that \( R[[X_1, \ldots, X_n; \sigma_1, \ldots, \sigma_n]] \) is a duo ring. If \( R \) is \( S \)-Noetherian, then the ring \( R[[X_1, \ldots, X_n; \sigma_1, \ldots, \sigma_n]] \) is also \( S \)-Noetherian.

**Proof.** We use the method in [1, Proposition 10] employed by Anderson and Dumitrescu. As \( S \) is \( \sigma \)-anti-Archimedean in every ring containing \( R \) as a subring, we shall prove the case \( n = 1 \), so we assume that \( T = R[[x; \sigma]] \) is duo and \( \sigma \) is an automorphism of \( R \). It is enough to prove that every prime ideal \( P \) of \( T \) is \( S \)-finite. Let \( \pi : T \to R \) the \( R \)-algebra homomorphism sending \( x \) to zero and \( P' = \pi(P) \). Since \( R \) is \( S \)-Noetherian, there exists \( s \in S \) such that \( sP' \subseteq (Rg_1(0) + Rg_2(0) + \cdots + Rg_k(0)) \) for some \( g_i \in P \). If \( x \in P \), then \( P = (TP' + Tx) \). If \( g_i(x) = \sum a_i x^i \), then \( g_i(x) = \sum x^i \sigma^{-i}(a_i) \in (TP' + Tx) \). So \( sP \subseteq (TP' + Tx) = (Tg_1 + \cdots + Tg_k) \subseteq P \). This means that \( P \) is \( S \)-finite. Let \( x \notin P \) and \( f \in P \). So \( sf(0) = \sum d_{0,j}g_j(0) \) for some \( d_{0,j} \in R \). So \( xf_1 = sf - \sum d_{0,j}g_j \in P \) for some \( f_1 \in T \). Considering \( x \notin P \), \( f_1 \in P \). So \( sf_1 = \sum d_{1,j}g_j + x f_2 \) for some \( f_2 \in T \). Hence \( \sigma(s)sf = \sum \sigma(s)d_{0,j}g_j + x \sum d_{1,j}g_j + x^2 f_2 \). Also \( f_2 \in P \), since \( x \notin P \) and \( sf_1 - \sum d_{1,j}g_j \in P \). In this way, one can see that for each \( L \geq 0 \),

\[
(\prod_{l=0}^{L} \sigma^l(s))f = \sum_{i=0}^{L} x^i \sum_{j=1}^{k} (\prod_{l=0}^{i-1} \sigma^l(s))d_{i,j}g_j + x^{L+1} f_{L+1}.
\]

Since \( S \cap \bigcap_{1 \leq i,j \leq n \cup \{0\}} \sigma^{i_1}(s) \cdots \sigma^{i_k}(s)R \neq \emptyset \), there exists \( t \in R \) such that \( \frac{t}{\sigma_{i_1}(s) \cdots \sigma_{i_k}(s)} \in R \) for each \( i_j \in \mathbb{N} \cup \{0\}, k \in \mathbb{N} \). Moreover

\[
tf = \sum_j \sum_i \left( \frac{t \sigma^{-i}(d_{j})}{\prod_i \sigma^i(s)} \right) x^i g_j.
\]

So \( tf = \sum_j h_j g_j \) where \( h_j = \sum_i \frac{t \sigma^{-i}(d_{j})}{\prod_i \sigma^i(s)} x^i \). So \( tf \in (Tg_1 + Tg_2 + \cdots + Tg_k) \). Hence \( tP \subseteq (Tg_1 + Tg_2 + \cdots + Tg_k) \). Since \( g_i \in P \), \( (Tg_1 + Tg_2 + \cdots + Tg_k) \subseteq P \). Thus \( R[[x; \sigma]] \) is an \( S \)-Noetherian ring.
The following proposition which is proved in [1], is the corollary of the above theorem.

**Corollary 4.5.** [1, Proposition 10] Let \( R \) be a commutative ring and \( S \subseteq R \) an anti-Archimedean multiplicative set of \( R \). If \( R \) is \( S \)-Noetherian, then so is \( R[[X_1, \ldots, X_n]] \).

A ring \( R \) is called strongly regular if every principal right (or left) ideal is generated by a central idempotent. A ring is said to be left self injective if it is injective as a left module over itself. Hirano in [12, Theorem 4] shows that if \( R \) is a self-injective strongly regular ring, then \( R[[x]] \) is a duo ring.

We have the following generalization of a theorem of D.D. Anderson and Tiberiu Dumitrescu [1, Proposition 10].

**Theorem 4.6.** Let \( R \) be a duo ring with an automorphism \( \sigma \) and \( S \subseteq R \) a \( \sigma \)-anti-Archimedean denominator set (consisting nonzero devisors). If \( R \) is \( S \)-Noetherian, then so is the skew power series ring \( R[[x; \sigma]] \).

**Proof.** We can prove this theorem in a similar way as in Theorem 4.4. Consider the notations in the proof of Theorem 4.4. Let \( x \in P \). Since \( \sigma \) is bijective, \( P \) is \( S \)-finite. Let \( x \not\in P \) and \( f \in P \), so \( xf_1 = sf - \sum d_ig_i \in P \). Note that for each \( h \in R[[x; \sigma]] \) and \( I \) is a left ideal of \( R[[x; \sigma]] \), \( xh \in I \) yields that \( xR[[x; \sigma]]h \in I \). So \( f_1 \in P \). The rest of the proof is similar to what we did in Theorem 4.4. \( \square \)

The following corollary is a generalization of the case \( n = 1 \) in [1, Proposition 10] for the category of duo rings.

**Corollary 4.7.** Let \( R \) be a duo ring and \( S \subseteq R \) an anti-Archimedean denominator set (consisting nonzero devisors) of \( R \). If \( R \) is \( S \)-Noetherian, then so is the power series ring \( R[[x]] \).

Now we extend the last result for the skew generalized power series ring \( R[[M, \omega]] \).

**Theorem 4.8.** Let \( R \) be a duo ring, \((M, \leq)\) a positive strictly ordered commutative monoid and \( \omega_m \) a monomorphism of \( R \) with \( \omega_m \omega_n = \omega_n \omega_m \) for each \( m, n \in M \). Assume that \( S \subseteq R \) is an \( \omega_m \)-anti-Archimedean denominator set (consisting nonzero devisors) of \( R \) and \( R[[M, \omega]] \) be a duo ring. Then \( R[[M, \omega]] \) is left (or right) \( S \)-Noetherian if and only if \( R \) is left (or right) \( S \)-Noetherian and \( M \) is finitely generated.

**Proof.** (\( \Rightarrow \)) We use the method of G. Brookfeild employed in [6]. We know that the surjective homomorphism \( \varphi : F^n \to M \) (where \( F \) is a free monoid) induces a projection

\[
\varphi^* : R[[F^n, (\omega, \leq)]] \to R[[M, (\omega, \leq)]]
\]

and \( R[[M, (\omega, \leq)]] = R[[M, \omega]] \) by [6, Theorem 4.3]. Moreover, since \( R[[F^n, (\omega, \leq)]] \) is \( S \)-Noetherian, so is \( R[[M, \omega]] \) by [16, Lemma 2.2] for noncommutative version.

(\( \Leftarrow \)) Let \( A := R[[M, \omega]] \) be \( S \)-Noetherian. Let \( \{m_n | n \in N\} \) be an infinite sequence in \( M \). Let \( I = (Ae_{m_1} + Ae_{m_2} + \cdots) \). Since \( A \) is \( S \)-Noetherian, there exists \( s \in S \) such that \( csI \subseteq J \subseteq I \) for \( J \) finitely generated ideal of \( A \). So \( csI \subseteq (Ae_{m_1} + Ae_{m_2} + \cdots + Ae_{m_k}) \) for some \( k \in N \). So \( cse_{m_l} = \sum_{i=0}^{k} f_i e_{m_i} \) for some \( l \neq i \). So \( m_l \in \bigcup_{i=0}^{k} \text{supp}(f_i e_{m_i}) \) for each \( m \in M \). Since \( \sigma(e_{m_l})(m) = \sum_{m' | m'' = m} f_l(m') \omega_{m'}(e_{m'_l}(m'')) \). So \( m_l \in \bigcup_{l=0}^{k} \{ \text{supp}(f_l) + \text{supp}(\omega_{m'}(e_{m'_l}(m''))) \} \). There exists \( m_1 \in M \) such that \( m_1 m_i = m \) for some \( 0 \leq l \leq L \). So

\[
(f_l e_{m_i})(m) = f_l(m_1) \omega_{m_1}(e_{m_1}(m_1)) = f_l(m_1).
\]
Thus $m_t \in \text{supp}(f_t)$ and $m_t m_i \in \text{supp}(f_t e_{m_i})$ for some $0 \leq t \leq L$. So for each $m \in \text{supp}(\omega_m'(e_{m_t}(m^t)))$, $m_t \leq m$ for some $0 \leq t \leq L$. Since $m_t \in \text{supp}(f_t e_{m_i})$, $m_{ii} \leq m_t$ for some $0 \leq t \leq L$. Since $M$ is positive strictly ordered monoid, $M$ is finitely generated by \cite[Lemma 3.3]{Anderson5}.

Let $I$ be an ideal of $R$, so $AI$ is an ideal of $A$. So there exists $s \in S$ such that $c_s AI \subseteq J \subseteq AI$ for some $J$ finitely generated ideal of $A$. Set

$$T = \{f(\pi(f)) | f \in c_s AI\}.$$ 

We claim that $T = sI$. Let $t \in T$, so $t = h(\pi(h))$ and $h = c_s g$ for some $g \in AI$. So $t = sg(\pi(\pi(g)))$. This means that $t \in sI$ considering the fact that

$$I = \{f(\pi(f)) | f \in AI\}.$$ 

So $T \subseteq sI$. Now let $i \in I$, so $i \in AI$. Since $si(m) = 0$ for $m \neq 1$, $si \in T$. Thus $sI \subseteq T$. Hence $sI = T$. But $sI = T \subseteq J' \subseteq I$ where $J' = \{f(\pi(f)) | f \in J\}$. Let $J = (A_{j_1} + A_{j_2} + \cdots + A_{j_p})$. So it is easy to show that $J' = (R_{j_1}(\pi(j_1)) + R_{j_2}(\pi(j_2)) + \cdots + R_{j_p}(\pi(j_p)))$. So $J'$ is finitely generated in $R$. Hence $I$ is $S$-finite and $R$ is left $S$-Noetherian.

Recall from \cite{Anderson4}, that a ring $R$ is right (left) $\aleph_0$-injective provided any homomorphism from a countably generated right (left) ideal of $R$ into $R$ extends to a right (left) $R$-module endomorphism of $R$. By an $\aleph_0$-injective ring we mean a right and left $\aleph_0$-injective ring.

**Corollary 4.9.** Let $R$ be an strongly regular and an $\aleph_0$-injective ring with automorphisms $\sigma_1, \sigma_2$ such that $\sigma_1 \sigma_2 = \sigma_2 \sigma_1$. Assume that $S \subseteq R$ is an $\sigma_1, \sigma_2$ anti-Archimedean denominator set (consisting nonzero devisors). If $R$ is left (or right) $S$-Noetherian, then so is $R[[x,y;\sigma_1,\sigma_2]]$.

**Proof.** Assume that $R$ is an strongly regular and $\aleph_0$-injective ring. Then by \cite{Anderson3}, $A = R[[x;\sigma_1]]$ is duo ring and $S$-left Noetherian ring. Then $A[[y;\sigma_2]]$ is left $S$-Noetherian ring.

The following corollary is a generalization of the case $n = 2$ of in \cite[Proposition 10]{Anderson1} for the category of duo rings.

**Corollary 4.10.** Let $R$ be an strongly regular self-injective ring and $S \subseteq R$ an anti-Archimedean denominator set (consisting nonzero devisors) of $R$. If $R$ is left (or right) $S$-Noetherian, then so is $R[[x,y]]$.

**Corollary 4.11.** Let $R$ be a duo ring, $S \subseteq R$ an anti-Archimedean denominator set (consisting nonzero devisors) of $R$. Assume that $R[[M]]$ is a duo ring. Then $R[[M]]$ is left (or right) $S$-Noetherian if and only if $R$ is left (or right) $S$-Noetherian and $M$ is finite generated.

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