Nonlinear error compensation based on the optimization of swing cutter trajectory for five-axis machining

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Abstract
In order to solve the problem of deviation between actual and theoretical machining paths due to the presence of rotation axis in five-axis machining, an interpolation algorithm based on the optimization of swing cutter trajectory and the method of corresponding nonlinear error compensation are proposed. Taking A-C dual rotary table five-axis machine tool as an example, the forward and reverse kinematic model of the machine tool is established according to the kinematic chain of the machine tool. Based on the linear interpolation of rotary axis, the generation mechanism of nonlinear error is analyzed, the modeling methods of cutter center point, and cutter axis vector trajectory are proposed respectively, and the parameterized model of swing cutter trajectory is formed. The formula for the nonlinear error is obtained from the two-dimensional cutter center point trajectory. According to the established model of swing cutter trajectory, the synchronous optimization method of cutter center point trajectory and cutter axis vector trajectory is proposed, and the nonlinear error compensation mechanism is established. First, pre-interpolation is performed on the given cutter location data to obtain a model of the swing cutter trajectory for each interpolated segment. Then, the magnitude of the nonlinear error is calculated based on the parameters of the actual interpolation points during formal interpolation, and the nonlinear error is compensated for the interpolation points where the error exceeds \( |\epsilon| \). In the VERICUT simulation, the maximum machining error was reduced from 50 to 5 \( \mu \text{m} \) by this paper method. In actual machining, the surface roughness of the free-form surface was reduced from 10.5 \( \mu \text{m} \) before compensation to 1.8 \( \mu \text{m} \). The experimental results show that the proposed method can effectively reduce the impact of nonlinear errors on processing, and is of high practical value for improving the accuracy of cutter position and the quality of complex free-form machining in five-axis machining.

Keywords Five-axis machining · Swing cutter trajectory · Free-form surface · Nonlinear error · Error compensation

1 Introduction
Five-axis CNC machine tools have high-speed, high-precision, and complex free-form machining capabilities, widely used in the processing of impellers, blades, engine rotors, and crankshafts and other precision devices [1]. Five-axis CNC machine tools have two more rotary axis compared to three-axis CNC machine tools, with higher machining freedom and more flexible tool orientation, but also due to the rotary motion of the tool brings a new principle error–nonlinear error. With the rapid development of the economy, nonlinear error have become critical research in the field of five-axis machining technology in order to better meet the needs of advanced manufacturing [2, 3].

Although five-axis CNC machine tools are more flexible in tool attitude changes, there is a nonlinear relationship between the position coordinates of the rotary axis and the tool attitude. This makes the actual machining trajectory deviate from the ideal linear machining trajectory, which results in unavoidable nonlinear errors [4–6]. For this reason, many researchers in related fields have conducted a lot of research on the reduction of nonlinear error, among which the most important methods to reduce nonlinear error are improved interpolation algorithms, linear encryption
of cutter location points, cutter location correction method and real-time error compensation. Sang et al. [7] proposed a simplified calculation model for the deviation between ideal cutter trajectory and actual cutter trajectory caused by nonlinear motion of rotary and translational axis, and an improved interpolation algorithm considering geometric deviation and motion constraints, which improves the accuracy and efficiency of machining to a certain extent. Ma et al. [8] considered the nonlinear error of different rotation axes, derived the calculation formula for the change of rotation axis motion angle, and then constructed the feasible domain of high precision tool without interference with nonlinear error as a constraint, which is of great significance to improve the machining quality. Liang et al. [9] proposed a new strategy to control the nonlinear error in five-axis CNC machining by modifying the tool orientation to reduce the nonlinear error based on the machine tool kinematics and machining trajectory without increasing the cutter location data points, but this method cannot guarantee that the nonlinear error are reduced to within the allowed range of high precision machining requirements. Liu et al. [10] proposed a new nonlinear error compensation method to establish a nonlinear error compensation model to obtain a cutter location that satisfies the machining accuracy, which in turn improves the geometric accuracy of the machined part. Tutuena Fatan and Feng [11] proposed a method for accurate error determination different from the conventional string differential method, describing the exact interpolation position law of the tool between two cutter contact points, but did not give a specific method for error compensation. Srijuntongsiri and Makhanov [12] proposed a theory that if the tool trajectory is fixed, the expected trajectory in five-axis machining depends only on the machine type and configuration. Tajima and Sencer [13] investigated a new real-time trajectory generation technique for controlling tool tip and tool attitude errors. Wang et al. [14] reduced the error by densifying the cutter location data. This method is effective in reducing the error, but it leads to multiplying the number of machining program entries. Zhu et al. [15] used the method of modifying the G-code to compensate the geometric deviation of the cutter trajectory, but it is not accurate enough for modeling the cutter trajectory. Fu et al. [16] established a new mathematical model of geometric errors and solved the optimal NC code using particle swarm algorithm, which in turn ensured the machining quality and machining efficiency. Zhong et al. [17] used a multi-body system to model the geometric errors of a large five-axis machining center, and the errors were calculated and compensated by the least squares method to enable the machined parts to meet the design requirements. Makhanov and Munlin [18] proposed an interpolation algorithm based on uniform distribution in angular space to improve machining accuracy by constructing a uniform grid around cutter contacts with large angular variations and did not optimize the machining method from the perspective of nonlinear errors compensation. Wu et al. [19] proposed a machining cutter location point preprocessing method based on NURBS curve fitting technique, which reduces the amount of CNC code and improves the machining efficiency of five-axis CNC machine tools while satisfying the error requirements. Fan et al. [20] proposes a tool axis vector plane interpolation algorithm that can largely reduce the nonlinear error caused by linear interpolation of rotary axis, but it lacks verification of overall free-form surface machining and cannot illustrate the generality of the algorithm. Yang et al. [21] analyzes the position where the deviation angle of the cutter axis obtains the maximum value during the motion of the rotation axis, and proposes a method to limit the angle of the cutter axis between adjacent cutter location point to control the nonlinear error.

The methods used in the above literature all reduce the nonlinear errors. However, there is still a problem that the actual machining path and the theoretical machining path deviate too much. To solve these problems in the current research, this paper proposes a method of nonlinear error compensation and control based on the optimization of the swing cutter trajectory, and the literature on nonlinear error compensation from this perspective is relatively scarcity. The parametric model of the swing cutter trajectory is established by the cutter center point and cutter axis vector data, and the corresponding optimization method of the swing cutter trajectory is proposed according to the calculation model of nonlinear error, and then the nonlinear error is compensated and repaired. The optimized trajectory of the swing cutter parameters is used to find the new interpolation point position during practical machining, thus effectively improving the machining position accuracy of the cutter. Compared with previous studies, the study in this paper has the characteristics shown in Table 1.

### 2 The kinematic model of machine tool

The kinematic model of a CNC machine tool is a mathematical model. It is used to describe the relative motion relationship between the workpiece and the cutter and the interference of the motion of the individual motion subsets in CNC machine tool machining [22]. Five-axis CNC machines can be divided into three main categories according to the different configurations of rotary axis movement relative to the workpiece and tool. They are double rotary table, rotary table swing head type and double swing head type. Within each category there are many types of machines based on different combinations of rotary axis. The object of this paper is an A-C dual rotary table five-axis CNC machine, as shown in Fig. 1. The machine tool is composed of X, Y, Z...
three translational axis and A, C two rotation axis, the rotation axis A is a fixed axis, and the machine bed fixed connection, rotary axis C for the dynamic axis, its axis direction with the A axis and change.

### 2.1 The description of machine tool coordinate system

As shown in Fig. 2, the coordinate system of A-C double rotary CNC machine tool is established to describe the motion of the machine. MCS (machine coordinate system) \( O_{M}X_{M}Y_{M}Z_{M} \) is a coordinate system with the machine bed itself solid link, which is the reference coordinate system of all workpiece coordinate system in CNC machine tool, generally not as a programming coordinate system. The A-axis coordinate system \( O_{A}X_{A}Y_{A}Z_{A} \) is fixed to the fixed axis A. In the initial position of the machine, its three main vectors are oriented in the same direction as the machine coordinate system. The C-axis coordinate system \( O_{C}X_{C}Y_{C}Z_{C} \) is fixed to the C-axis, and the axis of C-axis is changed with the rotation of A-axis, and the three main vectors of the initial position are consistent with the A-axis coordinate system. WCS (workpiece coordinate system) \( O_{W}X_{W}Y_{W}Z_{W} \) is fixed to the workpiece and all coordinates for CNC programming are calculated based on this coordinate system. Tool coordinate system \( O_{T}X_{T}Y_{T}Z_{T} \) is fixed to the tool, and the origin of the coordinate system is at the cutter location point.

### 2.2 Kinematics transformation of machine tool

Kinematic transformation of machine tool is the basis for post-processing, toolpath planning, nonlinear error compensation and machine tool precision detection. In the study of MCS to WCS transformation, each axis of the machine tool and the cutter are treated as rigid bodies. The coordinate transformation of a rigid body can be described by the methods of homogeneous coordinate transformation, vector method and quaternion method. Among them, the homogeneous coordinate transformation represents the rotation and translation transformations in a matrix, which has an intuitive geometric meaning.

The homogeneous coordinates of the cutter center point and the cutter axis vector at a certain moment in the WCS are \( P_{W} = (x_{W}, y_{W}, z_{W}) \) and \( T = (i, j, k) \). The coordinates under the WCS correspond to the position coordinates under the MCS as \( P_{M} = (x_{M}, y_{M}, z_{M}, \theta_{A}, \theta_{C}) \).

To simplify the calculation, the axis of the machine tool rotation axis are defined to intersect. And the WCS and the C-axis coordinate system overlap. According to the machine model of A-C double rotary table and the definition of the above coordinate system, the mutual mapping relationship of MCS and WCS are obtained by the homogeneous transformation matrix.
coordinate transformation. The forward transformation of the machine tool kinematics is the coordinate transformation from MCS to WCS, and the transformation equations are Eqs. (1) and (2). The forward transform equation of machine tool kinematics derived the inverse transformation equation of machine tool kinematics from WCS to MCS as Eq. (3).

\[
\begin{align*}
  i &= \sin \theta_A \sin \theta_C \\
  j &= -\sin \theta_A \sin \theta_C \\
  k &= \cos \theta_A \\

  x_W &= x_M \cos \theta_C - y_M \cos \theta_A \sin \theta_C + z_M \sin \theta_A \sin \theta_C \\
  y_W &= x_M \cos \theta_C - y_M \cos \theta_A \cos \theta_C - z_M \sin \theta_A \cos \theta_C \\
  z_W &= x_M \sin \theta_A \sin \theta_C - y_M \sin \theta_A \cos \theta_C + z_M \cos \theta_A \\

  \theta_A &= \arccos k, \quad \text{arbitrary value, when } k = \pm 1 \\
  \theta_C &= \begin{cases} 
    \pi/2, & \text{when } k \neq \pm 1, j = 0, i > 0 \\
    -\pi/2, & \text{when } k \neq \pm 1, j = 0, i < 0 \\
    \arctan(-i/j), & \text{other value}
  \end{cases} \\

  x_M &= x_W \cos \theta_C + y_W \sin \theta_C \\
  y_M &= -x_W \cos \theta_A \sin \theta_C + y_W \cos \theta_A \cos \theta_C + z_W \cos \theta_A \\
  z_M &= x_W \sin \theta_A \sin \theta_C - y_W \sin \theta_A \cos \theta_C + z_W \cos \theta_A \\

\end{align*}
\]

(1) (2) (3)

3 Swing cutter trajectory model

3.1 Linear interpolation of rotation axis

Rotary axis linear interpolation is an angular linear interpolation in the MCS, where the angle of the rotary axis is uniformly varied within one interpolation period. Under the WCS, the starting and ending cutter position information is as follows.

The coordinates of the starting cutter center point is \( P_{Wi} = (x_{W_i}, y_{W_i}, z_{W_i}) \), the starting cutter axis vector is \( T_s = (i_s, j_s, k_s) \), the coordinates of the ending cutter center point is \( P_{We} = (x_{W_e}, y_{W_e}, z_{W_e}) \), and the ending cutter axis vector is \( T_e = (i_e, j_e, k_e) \). According to the machine tool kinematic inverse transformation formula Eq. (2), the starting and ending cutter position information under the WCS is transformed into the position information under the MCS as \( P_{M_i} = (x_{M_i}, y_{M_i}, z_{M_i}, \theta_{A_i}, \theta_{C_i}) \) and \( P_{M_e} = (x_{M_e}, y_{M_e}, z_{M_e}, \theta_{A_e}, \theta_{C_e}) \).

Suppose the interpolation period of the CNC system is \( \tau \), and the real-time feed rate is \( f \). Then, the formula for the number of interpolation segments \( n \) in one interpolation cycle is Eq. (4). If the swing cutter length is \( L \), the coordinates of the beginning and end cutter axis points in the WCS can be calculated according to Eq. (5). According to the machine tool interpolation method, the formula for the \( i \)th interpolation cutter axis point is Eq. (6).

\[
n = \left\lfloor \frac{\sqrt{(x_{We} - x_{Ws})^2 + (y_{We} - y_{Ws})^2 + (z_{We} - z_{Ws})^2}}{f \cdot \tau} \right\rfloor
\]

where \( \lfloor \cdot \rfloor \) means rounding operation.

\[
Q = P + L \cdot T
\]

\[
Q_{Ws} = (1 - \lambda)Q_{Ws} + \lambda Q_{We}
\]

where \( \lambda = i/n \).

The schematic diagram of linear interpolation of the rotation axis angle is shown in Fig. 3, and the rotation axis angle of the \( i \)th interpolation point is obtained according to Eq. (7) [23]. Substituting this angle into the kinematic positive transformation Eqs. (1) and (2), the interpolated cutter axis vector \( T_i \) in the WCS is obtained. After the cutter axis vector and cutter axis point of the \( i \)th interpolation point are calculated, the interpolation cutter center point \( P_{Wi} \) is back-calculated from Eq. (5) as shown in Eq. (8).

\[
\begin{align*}
  \theta_{A_i} &= (1 - \lambda)\theta_{A_s} + \lambda \theta_{A_e} \\
  \theta_{C_i} &= (1 - \lambda)\theta_{C_s} + \lambda \theta_{C_e} \\

  x_{Wi} &= (1 - \lambda)(x_{Ws} + L \cdot i_s) + \lambda(x_{We} + L \cdot i_e) - L \cdot i_i \\
  y_{Wi} &= (1 - \lambda)(y_{Ws} + L \cdot j_s) + \lambda(y_{We} + L \cdot j_e) - L \cdot j_i \\
  z_{Wi} &= (1 - \lambda)(z_{Ws} + L \cdot k_s) + \lambda(z_{We} + L \cdot k_e) - L \cdot k_i
\end{align*}
\]

(7) (8)

3.2 Swing cutter trajectory model

The swing cutter trajectory model is a synthesis of the cutter center point trajectory model and the cutter axis vector trajectory model. As shown in Fig. 4, according to the parameter \( t \), the cutter center point and cutter axis vector of the swing cutter at position \( t = t_i \) during the machining
The calculated plane equation coefficients $A$, $B$, $C$ and $D$ all contain a factor $L$, indicating that the plane equation is independent of the swing cutter length $L$.

First, the free-form surface model is built in UG, as shown in Fig. 5. Then, use UG’s machining module for toolpath planning of the surface part to generate a cutter location file with the suffix “.cls.” The code “GOTO/” in the cutter location file is followed by the tool position information under WCS. A set of adjacent cutter location point data is taken to verify that the interpolated cutter center point lies on a defined plane. The data of cutter location point are as follows:

$$\begin{align*}
P_{Wi} &= (33.0720, 28.0273, 43.5392), \\
T_i &= (0.017727, -0.6584962, 0.7523741), \\
P_{We} &= (31.5662, 28.0503, 43.5951), \\
T_e &= (0.0184790, -0.6376429, 0.7701104).
\end{align*}$$

Taken $n = 10$, $L = 75$ mm. The beginning and end cutter location point data is obtained by kinematic inverse transformation Eq. (3) to obtain the rotation axis angle information in the MCS, and then the rotation axis angle of each interpolation point is obtained according to Eq. (7). The interpolation angle data is brought into the machine tool kinematic positive transformation Eqs. (1) and (2) to obtain the interpolated cutter center location information in the WCS as shown in Table 2. When $i = 5$, the cutter location point data and the calculated $T_i$ are brought into Eq. (8) to obtain the coefficients of the plane equation determined for points $P_{Wi}$, $P_{Wi}$ and $P_{Wi}$ as $A = 0.0356$, $B = 0.7635$, $C = 0.6448$, and $D = -50.5614$, respectively.

The distance of the next point $P_{Wi}$ to the plane is calculated according to the distance Eq. (11) of the point to the plane, and the determination Eq. (12) is used to determine whether the point lies in the plane. The magnitude of $d$ can be approximated as zero when the distance $d$ and the chord length $l$ differ in magnitude by five orders of magnitude, so take $|\delta| = 1 \times 10^{-3}$. 

Fig. 5 Complex free-form model
\[ l = \sqrt{(x_{Wc} - x_{Ws})^2 + (y_{Wc} - y_{Ws})^2 + (z_{Wc} - z_{Ws})^2}. \]  

Bring \( P_{W6} \) into the Eqs. (9)–(11) to get the 6th interpolation point to the cutter center plane distance \( 6.4185 \times 10^{-6} \), and this interpolation period of \( [\delta] \) is equal to \( 1.507 \times 10^{-5} \), from the judgment Eq. (10) know that the point is located in the last cutter center plane. In fact, it is verified that all interpolated cutter center points satisfy the condition of being in the plane, as shown in Fig. 6.

The interpolated cutter center points within an interpolation cycle lie in the same plane, which provides a theoretical basis for simplifying the cutter center point trajectory, which used to be considered in space. This can now be studied by reducing the dimensionality to the two-dimensional plane, which brings many advantages. For example, the cutter center point trajectory model in two-dimensional plane is much simpler and more efficient to compute.

In order to determine a more accurate plane of cutter center points, considering all discrete cutter center points of the interpolation period, the problem becomes one of fitting an optimal plane to spatially discrete points. The optimal plane is the one in which the sum of the distances from these discrete points to the optimal plane is minimized. According to the SVD transformation of the covariance matrix, the singular vector corresponding to the minimum singular value is the normal vector \( n = (A,B,C) \) of the optimal plane. In turn, the general equation of the plane is derived to prepare for the subsequent construction of the two-dimensional swing cutter trajectory.

In statistics and probability theory, each element of the covariance matrix is the covariance between the individual vector elements. The dimension of the covariance matrix in the three-dimensional coordinate system is 3. As shown in Eqs. (13), (14), and (15), arrange the coordinates of all the discrete cutter center points in rows to form a matrix \( \Phi \). After finding the average value of each column, subtract the elements of each column of matrix \( \Phi \) from the corresponding average value to obtain the new matrix \( \Psi \). Then, the covariance matrix \( COV \) is obtained, the matrix \( COV \) is a symmetric matrix. The SVD decomposition of the \( COV \) matrix yields the three matrices multiplied by the matrix shown in Eq. (16), where the third column of the \( V \) matrix constitutes the column vector that is the normal vector \( n \) of the plane.

![Fig. 6 Interpolation point to cutter center point plane distance map](image-url)
The benefits of polynomial parametric curves are good fitting, fast computation, and easy storage of expressions. Take the previous cutter location point data as an example, after finding out the three-dimensional interpolation cutter center points are transformed to the WCS and cutter center plane coordinate system, all the interpolated cutter center points are transformed to the three-dimensional coordinates in the WCS by Eq. (19).

After knowing the transformation relationship between WCS and cutter center plane coordinate system, all the interpolated cutter center points are transformed to the cutter center plane, and the third-order polynomial parametric curve is used to fit the cutter center point trajectory. The obtained cutter center point trajectory is a two-dimensional curve.

Assume that the two bases of the found cutter center point coordinate system are \( \mathbf{e}_X = (e_{x1}, e_{x2}, e_{x3}) \) and \( \mathbf{e}_Y = (e_{y1}, e_{y2}, e_{y3}) \), respectively, and the point \( \mathbf{P}_{Wm} \) is a coordinate under the WCS, and its corresponding coordinate under the cutter center point plane coordinate system is \( \mathbf{P}_{Wm}^\parallel = (x_{Wm}^\parallel, y_{Wm}^\parallel) \). According to the transformation Eq. (18), the three-dimensional coordinates under the WCS can be transformed into two-dimensional coordinates under the cutter center point plane. The two-dimensional coordinates of the cutter center point plane are transformed into three-dimensional coordinates in the WCS by Eq. (19).

After establishing the cutter center plane coordinate system, the discrete cutter center point coordinates under the WCS are transformed into the coordinates under the cutter center plane coordinate system by coordinate transformation, and then the cutter center point trajectory is fitted. As shown in Fig. 8, the obtained cutter center point trajectory is a two-dimensional curve.

As shown in Fig. 7, the discrete cutter center points in one interpolation cycle lie in the same plane, then the cutter center point trajectory is also a plane curve. In order to better describe the cutter center point trajectory, a two-dimensional plane coordinate system is established in the cutter center point plane to simplify the cutter center point trajectory model and improve the efficiency of subsequent error compensation. The X-axis of the plane coordinate system is the line connecting the start and end cutter center points, and its positive direction is from the start cutter center point to the end cutter center point. The Y-axis is determined by the normal vector of the plane of the cutter center point and the X-axis according to the right-hand rule. The two bases \( \mathbf{e}_X \) and \( \mathbf{e}_Y \) of the plane coordinate system in the WCS are shown in Eq. (17).

After establishing the cutter center plane coordinate system, the discrete cutter center point coordinates under the WCS are transformed into the coordinates under the cutter center plane coordinate system by coordinate transformation, and then the cutter center point trajectory is fitted. As shown in Fig. 8, the obtained cutter center point trajectory is a two-dimensional curve.
The parameter $t$ is taken to have a range of values $[0,1]$ for uniform parameterization. Since the parametric curve passes through the origin, the constant term of the parametric curve in both horizontal scale and vertical coordinates is zero. Therefore, the parametric equation is set as Eq. (20), so that only 4 coefficients need to be stored when storing, and the unknown coefficients of the parametric equation are solved according to the two-dimensional coordinate values corresponding to the parameter nodes.

\[
\begin{align*}
  x(t) &= a_1 t^2 + a_2 t \\ 0 \leq t \leq 1 \\
y(t) &= b_1 t^2 + b_2 t \\ 0 \leq t \leq 1
\end{align*}
\]

(20)

After solving for the unknown coefficients, the parametric cutter center point trajectory for this interpolation period is obtained as shown in Fig. 9. There is a certain deviation between the actual cutter center point trajectory and the fitted cutter center point trajectory, which is evaluated by calculating the MSE (mean square error) and RMSE (root mean square error) of the deviation. The results show that the fit is very good. Then Eqs. (17) and (18) are combined to obtain the parameter curve of the cutter center point trajectory in the WCS, and the subsequent nonlinear error model can be modeled according to the cutter center point trajectory.

The interpolated cutter axis vector obtained from the pre-interpolation is used to solve the cutter axis vector trajectory parameter curve coefficients, and then the cutter axis vector trajectory parameter curve is obtained.

\[
T = A(t)i + B(t)j + C(t)k
\]

(21)

\[
\begin{align*}
  A(t) &= A_1 t^2 + A_2 t + A_3 \\
  B(t) &= B_1 t^2 + B_2 t + B_3 \\
  C(t) &= C_1 t^2 + C_2 t + C_3
\end{align*}
\]

(22)

\[
\begin{bmatrix}
  A_1 & B_1 & C_1 \\
  A_2 & B_2 & C_2 \\
  A_3 & B_3 & C_3
\end{bmatrix} =
\begin{bmatrix}
  t_i^2 & t_i & 1 \\
  \vdots & \vdots & \vdots \\
  t_{n+1}^2 & t_{n+1} & 1
\end{bmatrix}^+ 
\begin{bmatrix}
  i_1 & j_1 & k_1 \\
  \vdots & \vdots & \vdots \\
  i_{n+1} & j_{n+1} & k_{n+1}
\end{bmatrix}
\]

(23)

where $t_i = \frac{i-1}{n}$, \(i = 1, 2, \ldots, n, n + 1\).
4 Nonlinear error model

4.1 Analysis of the mechanism of nonlinear error generation

Linear interpolation is commonly used in CNC machines, which has the advantages of simple programming, easy implementation and smooth machining. In a three-axis CNC machine tool, there are no nonlinear error because there are only three translational axis. The five-axis machine tool has two additional rotary axis in comparison, and the tool movement in space is more flexible. It can be used to machine complex free-form surfaces, but it brings inevitable nonlinear error. The trajectory of the swing cutter in five-axis CNC machine tool machining is determined by the cutter center point and the cutter axis vector together, and their unintended deviation in the machining process will lead to nonlinear error. The mechanism of nonlinear error generation is shown in Fig. 11. Between adjacent cutter locations \((P_s, T_s)\) and \((P_e, T_e)\), the \(i\)th interpolation point cutter location is \((P_i, T_i)\). Ideally, the cutter follows the green path in Fig. 11. However, the tool motion of the CNC machine depends on the angle of the rotation axis, and the actual cutter location of the \(i\)th interpolation point is \((P_i', T_i')\). The starting cutter shaft point \(Q_s\) is linearly interpolated to the ending cutter shaft point \(Q_e\) according to the black straight line, and the tool deviates between the actual machining trajectory and the theoretical machining trajectory due to swing, thus generating a nonlinear error \(\varepsilon\) in space. This nonlinear error is the principle error caused by the use of linear interpolation of the rotary axis of the machine, which cannot be completely eliminated but can be reduced through error compensation to reduce its impact on the quality of machining.

4.2 Nonlinear error calculation model

Nonlinear errors are defined in a number of ways, such as distance representation, angular representation and cutter axis vector deviation [24]. As shown in Fig. 12, the maximum deviation between the theoretical machining path and the actual machining path is defined in this paper as the nonlinear error \(\varepsilon\) of a machining interpolation segment. The theoretical machining path is the line between the start and end cutter center points, denoted as \(L(t)\). The actual machining path is the parametric trajectory of the cutter center point \(P(t)\). Thus, the nonlinear error \(\varepsilon\) can be expressed as Eq. (24).

\[
\varepsilon = \max_{0 \leq t \leq 1} \{|P(t) - L(t)|\}
\]  

(24)

The accurate nonlinear error calculation formula can be further obtained from the previously established swing cutter parameter trajectory. In the cutter center point plane, the nonlinear error is the maximum value of the height
difference between the two-dimensional cutter center point trajectory and the transverse coordinate. Thus, the problem of solving the nonlinear error is transformed into the problem of finding the highest point of the parametric trajectory of the cutter center point. Since the highest point of the parabola is located on the symmetry axis, the formula for the nonlinear error obtained according to the two-dimensional cutter center point parameter trajectory is Eq. (25). The nonlinear error equation thus obtained is calculated quickly and satisfies the condition of real-time error compensation.

$$\varepsilon = \max_{0 \leq t \leq 1} \left\{ \left| y(t) \right| \right\} = \frac{b_2^2}{4|b_1|}$$  \hspace{1cm} (25)

### 5 Nonlinear error compensation mechanism

The nonlinear error is caused by the unintended deviation of the cutter center point and cutter axis vector in the swing cutter trajectory model during machining, so the compensation method for the nonlinear error is also studied from the swing cutter trajectory. A compensation method is proposed to optimize the swing cutter trajectory based on the previously established swing cutter trajectory model. The process of nonlinear error compensation is shown in Fig. 13. A complex free-form surface part is modeled by CAD software and then tool path for machining are generated by CAM software. Export the cutter location file containing the cutter center point \( P_w(t) \) and cutter axis vector \( T(t) \) from the CAM software, and build the swing cutter trajectory model by the pre-interpolation model established in the second part. The swing cutter trajectory model includes cutter center point trajectory and cutter axis vector trajectory. The nonlinear error is calculated based on the cutter center point trajectory, and the error compensation is performed for the cutter location segments that do not meet the error limit \( \varepsilon \) to meet the requirements, thus ensuring the surface quality of the machined parts. The process of swing cutter trajectory optimization is the optimization of cutter center point and cutter axis vector. Optimize the adjustment on the basis of the original swing cutter trajectory, so as to ensure that the nonlinear error of all cutter location segments meet the given error requirements during the formal interpolation.

#### 5.1 Cutter center point trajectory optimization

The optimization of the cutter center point trajectory is carried out in the established cutter center point plane, so that the optimized cutter center point trajectory still lies in the cutter center point plane. Therefore, the reduction of nonlinear error is in the direction of measuring the nonlinear error, and the curve tool fitted to the original cutter center point trajectory is used to express the optimized cutter center point trajectory. There are two constraints on the optimization of the cutter center point trajectory, one is the error limit constraint, that is, the nonlinear error generated by the optimized cutter center point trajectory should be less than the set error limit \( \varepsilon \). The other is the cutter axis point constraint, that the offset of the cutter axis point within one interpolation cycle should not be too large, it will lead to discontinuous machining and reduce the machining quality of the part. By optimizing the cutter center point trajectory with the above two constraints, the optimized two-dimensional curve equation is changed into the parametric equation shown in Eq. (26).

$$\begin{align*}
\hat{x}(t) &= x(t) \\
\hat{y}(t) &= \hat{b}_1 t^2 + \hat{b}_2 t (0 \leq t \leq 1)
\end{align*}$$  \hspace{1cm} (26)

where \( \hat{b}_2 = \eta \cdot |\varepsilon|, \hat{b}_1 = -\hat{b}_2, \eta = 0.9 \).

Substituting the optimized two-dimensional cutter center point trajectory parameter equation into Eq. (19), we obtain the cutter center point trajectory parameter equation in three dimensions. The result of cutter center point trajectory optimization is shown in Fig. 14. Figure 13a shows the result of cutter center point trajectory optimization in two-dimensional coordinates, and Fig. 13b shows the result of cutter center point trajectory optimization in the coordinates of WCS. The purpose of nonlinear error compensation
is achieved by the optimization of the cutter center point trajectory.

5.2 Cutter axis vector trajectory optimization

The optimization of the cutter axis vector trajectory is based on the idea of the shortest distance between two points on the sphere, called the “shortest navigation” principle. This principle means that the shortest trajectory between two points on the sphere is an inferior arc path through the great circle of these two points. We optimize the trajectory of the cutter axis vector as an inferior arc trajectory of a great circle, which can make the interpolated cutter axis vector pass through the shortest distance on the sphere. This way all cutter axis vectors in one interpolation cycle will not deviate too much, and therefore the swing cutter trajectory will not produce too much deviation, thus effectively reducing nonlinear errors. The schematic diagram of the cutter axis vector optimization is shown in Fig. 15. The starting cutter axis vector is interpolated along the blue trajectory to reach the ending cutter axis vector, which is the cutter axis vector trajectory before optimization. Within the unit ball, the beginning and end cutter axis vector can determine a cutter axis vector plane that passes through the center of the unit ball and intersects a great circle trajectory with it, as shown by the green line in the figure. The inferior arc (red line) of the starting cutter axis vector and the ending cutter axis vector on the great circle of this section is the optimized cutter axis vector trajectory.

From the equation of the sphere and the equation of the vector plane of the cutter axis, the equation of the great circle curve of the section is determined as in Eq. (27). The parametric equation of the cutter axis vector trajectory is obtained from Eq. (27) as Eq. (28), and the parameter $\theta$ in this parametric equation is from the angle $\theta_s$ corresponding to the starting cutter axis vector to the angle $\theta_e$ corresponding to the ending cutter axis vector. In order to correspond to the parameter interval of the parametric equation of the cutter center point trajectory, so Eq. (28) is normalized to transform the parameters to the interval $[0,1]$. First take the new parameter $t = (u - u_s) / (u_e - u_s)$ and perform the best quadratic approximation to the sine and cosine function in Eq. (28) to obtain the approximation result as Eqs. (29), (30) and (31). Then, bring Eq. (29), (30) and (31) into Eq. (28) to obtain the parameter equation of the cutter axis vector trajectory with the optimized parameter interval $[0,1]$.

\[
\begin{align*}
E_x + F_y + G_z &= 0 \\
x^2 + y^2 + z^2 &= 1
\end{align*}
\]  

(27)

\[
\begin{align*}
x &= \frac{F \sin u}{H_2} - \frac{E \cos u}{H_2 H_3} \\
y &= \frac{E \sin u}{H_2} - \frac{F \cos u}{H_2 H_3} \\
z &= \frac{H_2}{H_3} \cos u
\end{align*}
\]

(28)

where $H_2 = \sqrt{F^2 + G^2}$, $H_3 = \sqrt{E^2 + F^2 + G^2}$.

\[
\begin{align*}
\sin[(u_e - u_s)u + u_s] &= m_1 t^2 + m_2 t + m_3 \\
\cos[(u_e - u_s)t + u_s] &= n_1 t^2 + n_2 t + n_3 \\
(0 \leq t \leq 1)
\end{align*}
\]

(29)

\[
\begin{bmatrix}
m_1 \\
m_2 \\
m_3
\end{bmatrix} = \begin{bmatrix}
9 & -36 & 30 \\
-36 & 192 & -180 \\
30 & -180 & 180
\end{bmatrix} \begin{bmatrix}
-\Delta C_{21} \Delta u \\
\Delta S_{21} \Delta u^2 - C_2 \Delta u \\
2 \Delta S_{21} \Delta u^2 + 2 \Delta C_{21} \Delta u^3 - C_2 \Delta u
\end{bmatrix}
\]

(30)

\[
\begin{bmatrix}
n_1 \\
n_2 \\
n_3
\end{bmatrix} = \begin{bmatrix}
9 & -36 & 30 \\
-36 & 192 & -180 \\
30 & -180 & 180
\end{bmatrix} \begin{bmatrix}
\Delta S_{21} \Delta u \\
\Delta C_{21} \Delta u^2 - S_2 \Delta u \\
2 \Delta C_{21} \Delta u^2 - 2 \Delta S_{21} \Delta u^3 - S_2 \Delta u
\end{bmatrix}
\]

(31)

where $\Delta \theta = 1 / (u_e - u_s)$, $S_1 = \sin u_s$, $S_2 = \sin u_e$, $C_1 = \cos u_s$, $C_2 = \cos u_e$, $\Delta S_{21} = S_2 - S_1$, $\Delta C_{21} = C_2 - C_1$. 
5.3 Calculation of interpolation points for formal interpolation

The new cutter center point and cutter axis vector trajectory are obtained after the optimization of the swing cutter trajectory, and the new interpolation point coordinates are calculated according to the optimized parameter trajectory during the actual interpolation, so as to ensure the accuracy of the cutter location. For the parameterized trajectory just calculate the parameter \( t \) corresponding to the interpolation point to find out the cutter center point and cutter axis vector coordinates of the corresponding position, and the cutter location of the new interpolation point is precisely determined.

In the formal interpolation, after first calculating the trajectory parameter \( t \) corresponding to the actual interpolation point, the magnitude of the nonlinear error is then calculated, and then error compensation is performed for the interpolation points with large errors. After the nonlinear error compensation, the cutter center point trajectory is \( \hat{P}(t) \) and the cutter axis vector trajectory is \( \hat{T}(t) \). After bringing the parameter \( t \) into \( \hat{P}(t) \) and \( \hat{T}(t) \), solve for the position of the new interpolation point. Combine with Eq. (4) to obtain the new interpolation tool axis point \( \hat{Q}(t) \).

6 Simulation and experimental validation

6.1 Simulation results

Through the nonlinear error compensation method proposed in Sect. 3, the validity and real-time performance of the method is verified in matlab using the complex free-form surface shown in Fig. 5 as an example. A section of tool path information is intercepted from the cutter location file for verification of the nonlinear error control method, and
finally the machine tool simulation software VERICUT is used to verify the overall machining of the part. The CNC system used for the simulation is FANUC_5X_AC, and the interpolation period $\tau$ is 0.04 s.

A portion of the tool path from the exported cutter location file is used for the verification of the theoretical method proposed earlier. Pre-interpolation is carried out first according to the theory of the swing cutter trajectory model established in Sect. 2, and the acceleration and deceleration control of the actual interpolation is not required to be considered in the pre-interpolation. The interpolation points are uniformly interpolated between adjacent cutter axis points, and then the uniformized data are brought into the swing cutter trajectory model proposed in Sect. 2, which in turn leads to the equation of the swing cutter trajectory for each segment. The nonlinear error is also calculated for each interpolation cycle. As shown in Fig. 16, the interpolation points with excessive nonlinear errors are reduced to below the error limits after the theoretical compensation and control proposed in third part of this paper, so that the machining quality requirements are ensured.

The interpolation algorithm needs to be guaranteed in real time when the curved part is formally machined on the machine. Therefore, the real-time performance of the interpolation algorithm needs to be verified, that is the computational time consumed when interpolating according to the new interpolation algorithm should meet the real-time requirements of the CNC system. As shown in Fig. 17, when

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Fig. 20  Machining process of free-form surface. a VMC-850 vertical machining center. b Blank and fixture. c Before processing. d In process. e After processing

![Fig. 20](image_url)

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Fig. 21  Free-form surface machining results

![Fig. 21](image_url)
interpolation is performed according to the above interpolation theory, the computational time consumed at the interpolation points requiring nonlinear error compensation is significantly more than that at the interpolation points not requiring compensation. The average calculation time of the interpolation points that need compensation is about 4.5 μs, and the average calculation time of the interpolation points that do not need compensation is about 0.401 μs. Comparison with the interpolation period of the CNC, which can well meet the real-time requirements of the CNC system.

The proposed method of nonlinear error compensation above is used for the overall simulation machining of free-form parts. The complex free-form surface parts established by using A-C dual-table five-axis CNC machine in the machine tool simulation software VERICUT were simulated [25], and the comparison experiments before and after nonlinear error compensation were made. Compare and analyze the change of error before and after compensation, and then illustrate the effectiveness of the compensation method. The four-dimensional diagrams of the errors before and after the compensation are shown in Figs. 18 and 19. The maximum machining error was reduced from 50 μm before compensation to 5 μm after compensation, and most of the surface errors of the compensated free-form surfaces were reduced to less than 2.5 μm, which greatly improved the machining accuracy of the parts and improved the machining quality. This shows that the nonlinear error compensation theory proposed in this paper has a good effect on the control of nonlinear errors.

### 6.2 Experiment results

In order to further verify the nonlinear error compensation method proposed in this paper, the established free-form surface model is actually machined on a CNC machining center. The CNC machine used in the experiment is a model VMC-850 vertical machining center, as shown in Fig. 20a, which is equipped with the FANUC-0i MD CNC system. As shown in Fig. 20b, the material of the blank part is aluminum 6061 with size 60 × 60 × 40 mm. The whole process is divided into three parts: roughing, semi-finish machining and finish machining. The tool used for roughing is a 10 mm diameter cylindrical cutter. The tool used for semi-finish machining is a cylindrical cutter, and the tool used for the

### 7 Conclusions

In this paper, an error compensation method based on the optimization of the swing cutter trajectory is proposed to reduce the nonlinear errors. This paper makes the following major findings and significant intellectual contributions.

1. The kinematic model of AC double rotary table is established, and the principle of nonlinear error generation is explained. The theory of coplanarity of interpolation points between adjacent cutter center points is proposed, and the modeling method of swing cutter trajectory is given, which includes the parametric trajectory equations of cutter center points and cutter axis vectors.

2. A nonlinear error compensation mechanism is given. The cutter center point trajectory is optimized according to the nonlinear error and the constraints of the tool axis points. The cutter axis vector trajectory is optimized according to the “shortest navigation” principle.

3. Finally, the simulation of the nonlinear error compensation method and the verification of the actual machining were carried out on a five-axis CNC machine. VERICUT simulation results and actual machining verifi-
cation show that the proposed method can effectively reduce the impact of nonlinear errors on the machining quality of complex free-form surfaces. To provide a reference for controlling nonlinear errors in the field of five-axis machining.

Although the research in this paper has made some progress, the nonlinear error model considers fewer factors and ignores factors such as thermal errors, geometric errors of the machine tool and dynamic errors in actual machining. Moreover, the curve trajectory used to model the swing cutter trajectory is relatively simple, while the actual trajectory is much more complex. Therefore, in future research, we should further consider more complex models without reducing the effectiveness of error control methods.

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Author contribution The overarching research goals were developed by Liangji Chen and Jinmeng Tang. Liangji Chen and Jinmeng Tang established the models and calculated the predicted consequence. Liangji Chen, Jinmeng Tang, and Wenyi Wu analyzed the calculated results. The initial draft of the manuscript was written by Liangji Chen, Jinmeng Tang, Wenyi Wu, and Zisen Wei.

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Availability of data and material The datasets used or analyzed during the current study are available from the corresponding author on reasonable request.

Code availability The codes used or analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Consent to participate Not applicable.

Consent for publication Not applicable.

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