WDVV Equations from Algebra of Forms

A. Marshakov\textsuperscript{1}, A. Mironov\textsuperscript{2}  
Theory Department, P. N. Lebedev Physics Institute, Leninsky prospect 53, Moscow, 117924, Russia  
and ITEP, Moscow 117259, Russia  
and  
A. Morozov\textsuperscript{3}  
ITEP, Moscow 117 259, Russia

Abstract

A class of solutions to the WDVV equations is provided by period matrices of hyperelliptic Riemann surfaces, with or without punctures. The equations themselves reflect associativity of explicitly described multiplicative algebra of (possibly meromorphic) 1-differentials, which holds at least in the hyperelliptic case. This construction is direct generalization of the old one, involving the ring of polynomials factorized over an ideal, and is inspired by the study of the Seiberg-Witten theory. It has potential to be further extended to reveal algebraic structures underlying the theory of quantum cohomologies and the prepotentials in string models with $N = 2$ supersymmetry.

\textsuperscript{1}E-mail address: mars@lpi.ac.ru, andrei@heron.itep.ru, marshakov@nbivms.nbi.dk  
\textsuperscript{2}E-mail address: mironov@lpi.ac.ru, mironov@heron.itep.ru  
\textsuperscript{3}E-mail address: morozov@vxdesy.desy.de
1 Introduction

1.1 WDVV equations

The WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equations state that the third derivatives of the prepotential \( \mathcal{F}(a_i) \) organized in the matrices

\[
(F_i)_{jk} = \frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k},
\]

satisfy

\[
F_i F^{-1}_k F_j = F_j F^{-1}_k F_i \quad \forall i, j, k.
\]

The moduli \( a_i \) are defined up to linear transformations (i.e. define the flat structure on the moduli space) which leave the whole set (2) invariant.

The WDVV equations can be reformulated in the following way. Given an \( y \) metric

\[
G = \sum m g^{(m)} F_m
\]

one may use it to raise up indices and introduce

\[
C_j^{(G)} = G^{-1} F_j,
\]

i.e. \( C_{jk} = (G^{-1})^{im} F_{mj} \), or \( F_{ijk} = G^{-1} C_m^{(m)} \). From now on we omit the superscript \( (G) \) in \( C^{(G)} \) and assume summation over repeated indices. Then the WDVV eqs imply that all matrices \( C \) commute:

\[
C_i C_j = C_j C_i \quad \forall i, j
\]

(and thus can be diagonalized simultaneously). While (2) implies (5), inverse is not true: the WDVV equations are either (2) or the combination of (3), (4) and (5).

The WDVV eqs were first derived in the study of the chiral rings in 2d \( N = 2 \) superconformal topological models, where expresses the associativity of the multiplication of observables \( \phi_i \)

\[
\phi_i \circ \phi_j = C_{jk}^{(G)} \phi_k,
\]

\[
(\phi_i \circ \phi_j) \circ \phi_k = \phi_i \circ (\phi_j \circ \phi_k),
\]

while

\[
\mathcal{F}_{ijk} = \langle \langle \phi_i \phi_j \phi_k \rangle \rangle
\]

are (deformed) 3-point correlation functions on sphere. In this particular context, there is a distinguished observable \( \phi_0 = I \) and associated distinguished metric \( G^{(0)}_{ij} = \langle \langle \phi_i \phi_j I \rangle \rangle = F_{0ij} \).

1.2 Polynomial ring

The basic example of the algebra is the multiplication of polynomials modulo \( dP \) (in the string-theory language this is the case of the Landau-Ginzburg topological models):

\[
\phi_i(\lambda) \phi_j(\lambda) = C_{ij}^{k}(\lambda) \phi_k(\lambda) G'(\lambda) \mod P'(\lambda)
\]
Here $P(\lambda)$ and $G(\lambda)$ are polynomials of $\lambda$, such that their $\lambda$-derivatives $P'(\lambda)$ and $G'(\lambda)$ are co-prime (do not have common divisors), and $\phi_i(\lambda)$ form a complete basis in the linear space of polynomials modulo $P'(\lambda)$. Thus it gives a particular case of the algebra \[9\]

\[\phi_i(\lambda) \circ \phi_j(\lambda) = C_{ij}^k \phi_k(\lambda),\]

and it is associative as a factor of explicitly associative multiplication algebra of polynomials over its ideal $P'(\lambda) = 0$. The structure constants $C_{ij}^k$ depend on the choice of $P(\lambda)$ (the point of the "moduli" space) and $G'(\lambda)$ (the metric).

The second ingredient of the WDVV eqs is the residue formula \[10\],

\[\mathcal{F}_{ijk} = \text{res}_{dP=0} \frac{\phi_i(\lambda)\phi_j(\lambda)\phi_k(\lambda)}{P'(\lambda)} d\lambda\]

In accordance with \[11\],

\[G'(\lambda) = g^{(m)} \phi_m(\lambda) \quad G_{ij} = g^{(m)} \mathcal{F}_{ijm}\]

The last ingredient is the expression of flat moduli $a_i$ in terms of the polynomial $P(\lambda)$ \[12\]:

\[a_i = -\frac{N}{i(N-i)} \text{res} \left( P^i \right) \mod \left( \frac{\partial P}{\partial \lambda_1}, \ldots, \frac{\partial P}{\partial \lambda_n} \right) \]

These formulas have a straightforward generalization to the case of polynomials of several variables, $\phi_i(\bar{\lambda})$:

\[\phi_i(\bar{\lambda})\phi_j(\bar{\lambda}) = C_{ij}^k \phi_k(\bar{\lambda})Q(\bar{\lambda}) \mod \left( \frac{\partial P}{\partial \lambda_1}, \ldots, \frac{\partial P}{\partial \lambda_n} \right),\]

and

\[\mathcal{F}_{ijk} = \text{res}_{dP=0} \frac{\phi_i(\bar{\lambda})\phi_j(\bar{\lambda})\phi_k(\bar{\lambda})}{\prod_{\alpha=1}^n \frac{\partial P}{\partial \lambda_\alpha}} d\lambda_1 \ldots d\lambda_n\]

The algebra \[13\] is always associative, since $dP = \sum_{\alpha=1}^n \frac{\partial P}{\partial \lambda_\alpha} d\lambda_\alpha$ is always an ideal in the space of polynomials. Moreover, one can even take a factor over generic ideal in the space of polynomials, $p_1(\bar{\lambda}) = \ldots = p_n(\bar{\lambda}) = 0$, where polynomials $p_\alpha$ need to be co-prime, but do not need to be derivatives of a single $P(\bar{\lambda})$.

### 1.3 Other examples

The above example of the polynomial ring is very transparent since it is related to an obviously associative algebra of polynomials, and associativity is preserved by factorization over an ideal. Less transparent are the origins of the residue formula and expression for the moduli $a_i$. This problem is, however, resolved in a general framework \[14\], inspired by the Seiberg-Witten theory \[15\].

Before turning to the general situation one should mention that the main stream of study of the WDVV eqs has been so far in another direction. One of the most interesting questions is related to the deformations of the polynomial ring, associated with the Gromov-Witten (GW) classes \[16\], or quantum cohomologies \[17\]. For the rational (coming from rational curves) GW classes, the WDVV eqs \[18\] are still true, but no nice description in terms of ideals of the obviously associative algebra is known yet (or, better, no nice way is yet known to specify the moduli dependence of the ideals). To prove the equations without explicit associative algebra, the sophisticated methods were developed, relating them to the theory of the Frobenius algebras and
Egoroff metrics [3], and to the properties of the moduli spaces $\mathcal{M}_{0,n}$ [1, 10]. Appropriate generalizations of the WDVV eqs to higher genus GW classes and to higher dimensions (from world-sheet instantons inspired by strings to world-volume ones inspired by branes) are difficult to find in such a framework (see, however, [12] and [13] for some results about elliptic case).

Recently, the WDVV equations appeared in a naively different context [7] (see also [14] and [15]): as equations on the prepotentials in the Seiberg-Witten theory [9], describing the low-energy limit of $\mathcal{N} = 2$ supersymmetric Yang-Mills models in 4d. Remarkably, the proof of these equations, suggested in [7, 8], appeared to be actually a return to the approach used in sect. 1.2: the equations are related to an obviously associative multiplication algebra. What happens is that the polynomials (functions on a Riemann sphere) are substituted by the holomorphic 1-differentials on Riemann surfaces (complex curves). They always form a family of closed algebras, parametrized by a triple of holomorphic differentials $dG, dW, d\Lambda$. However, these algebras are not rings in the usual sense of the word, thus they are not immediately associative after factorization over an ideal. Still, associativity is preserved for the classes of hyperelliptic curves appearing in the Seiberg-Witten theory [21].

The purpose of this letter is to give a brief presentation of this construction.

Clearly, it should possess direct generalizations to higher complex dimensions (from holomorphic 1-forms on complex curves to forms on complex manifolds), one can even think that it would provide a new look at the theory of quantum cohomologies. The very fact that the WDVV equations hold for the Seiberg-Witten prepotentials can also imply that there exist universal equations for the prepotentials in string models (which turn into the Seiberg-Witten prepotentials in certain limiting cases). Such equations are not yet known for generic families of Calabi-Yau spaces (the WDVV eqs, inspired by the theory of quantum cohomologies, are empty for Calabi-Yau threefolds). The Picard-Fuchs equations, which are always true, are not universal – they depend strongly on peculiarities of particular family. We shall deliberately ignore further comments on these possible generalizations in this letter, and concentrate on the case of the complex curves.

The construction itself is described in the next section 2, and section 3 lists particular examples (families of hyperelliptic curves and triples $dG, dW, d\Lambda$), which have been analyzed in this framework. Further technical details about these examples can be found in [8].

## 2 WDVV equations for the families of hyperelliptic curves

### 2.1 WDVV eqs from associativity and residue formula

As we already mentioned, the WDVV equations can be considered as a synthesis of the two ingredients: associativity of algebra and residue formula for prepotential. Namely, imagine that in some context the following statements are true:
1. The holomorphic 1-differentials on the complex curve \( C \) of genus \( g \) form a closed algebra,

\[
d\omega_i(\lambda)d\omega_j(\lambda) = C_{ij}^k d\omega_k(\lambda) dG(\lambda) + D_{ij}^k d\omega_k(\lambda) dW(\lambda) + E_{ij}^k d\omega_k(\lambda) d\Lambda(\lambda) = C_{ij}^k d\omega_k(\lambda) dG(\lambda) \text{ mod } (dW, d\Lambda),
\]

where \( d\omega_i(\lambda), i = 1, \ldots, g \), form a complete basis in the linear space \( \Omega^1 \) (of holomorphic 1-forms), \( dG, dW \) and \( d\Lambda \) are fixed elements of \( \Omega^1 \), e.g. \( dG(\lambda) = \sum_{m=1}^g \eta^{(m)} d\omega_m \).

2. The factor of this algebra over the “ideal” \( dW \oplus d\Lambda \) is associative,

\[
C_i C_j = C_j C_i \quad \forall i, j \text{ at fixed } dG, dW, d\Lambda
\]

(remind that \( (C_i)_j^k \equiv C_{ij}^k \).)

3. The residue formula holds,

\[
\frac{\partial \mathcal{F}}{\partial \omega_i \partial \omega_j \partial \omega_k} = \frac{\text{res}}{dW=0} \frac{d\omega_i d\omega_j d\omega_k}{dW d\Lambda} = - \frac{\text{res}}{d\Lambda=0} \frac{d\omega_i d\omega_j d\omega_k}{dW d\Lambda}
\]

(17)

4. There exists a non-degenerate linear combination of matrices \( \mathcal{F}_i \).

These statements imply the WDVV eqs \( \mathcal{F} \) for the prepotential \( F(a_i) \). Indeed, the substitution of \( (15) \) into \( (17) \) gives

\[
\mathcal{F}_{ijk} = C_{ij}^m G_{mk},
\]

(18)

where

\[
G_{mk} = \frac{\text{res}}{dW=0} \frac{dG d\omega_m d\omega_k}{dW d\Lambda} = \eta^{(l)} F_{lmk},
\]

(19)

and the terms with \( dW \) and \( d\Lambda \) in \( (15) \) drop out from \( \mathcal{F}_{ijk} \) because they cancel \( d\Lambda \) or \( dW \) in the denominator in \( (17) \). Eq.\( (18) \) can be now substituted into \( (16) \) to provide WDVV eqs in the form

\[
\mathcal{F}_i G^{-1} \mathcal{F}_j - \mathcal{F}_j G^{-1} \mathcal{F}_i, \quad G = \eta^{(m)} F_m \quad \forall \left\{ \eta^{(m)} \right\}
\]

(20)

where at least one invertible metric \( G \) exists by requirement (4).

Actually \( (20) \) for all (non-degenerate) \( G \) follows immediately from that for some particular \( G = \hat{G} \). Indeed, if all the \( \hat{C}_i = \hat{G}^{-1} \mathcal{F}_i \) mutually commute, then \( G = \eta^{(m)} F_m = \hat{G} \eta^{(m)} \hat{C}_m \) and

\[
\mathcal{F}_i G^{-1} \mathcal{F}_j = \hat{G} \left( \hat{C}_i \left( \eta^{(m)} \hat{C}_m \right)^{-1} \hat{C}_j \right)
\]

(21)

is obviously symmetric under permutation \( i \leftrightarrow j \) (because of commutativity of matrices \( \hat{C}'s \)).

Thus, we see that the real issue in the study of WDVV eqs is to reveal when the conditions (1)-(4) are true.

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1 Since curves with punctures and the corresponding meromorphic differentials can be obtained by degeneration of smooth curves of higher genera we do not make any distinction between punctured and smooth curves below. We remind that the holomorphic 1-differentials can have at most simple poles at the punctures while quadratic differentials can have certain double poles etc.

2 We are indebted to A. Rosly for this simple argument.
2.2 Algebra of holomorphic (1,0)-forms

Existence of the multiplication algebra \([13]\) is a rather general feature of compact complex manifolds. Indeed, there are \(g\) holomorphic 1-differentials on the complex curve of genus \(g\). However, their products \(d\omega_i d\omega_j\) are not linearly independent: they belong to the \(3g - 3\)-dimensional space \(\Omega^2\) of the holomorphic quadratic differentials. Given three holomorphic 1-differentials \(dG, dW, d\Lambda\), one can make an identification

\[
\Omega^1 \cdot \Omega^1 \equiv \Omega^1 \cdot (dG \oplus dW \oplus d\Lambda) \tag{22}
\]

which in particular basis is exactly \([13]\). For given \(i, j\) there are \(3g\) adjustment parameters \(C^k_{ij}, D^k_{ij}\) and \(E^k_{ij}\) at the r.h.s. of \([13]\), with 3 “zero modes” – in the directions \(dGdW, dGd\Lambda\) and \(dWd\Lambda\) (i.e. one can add \(dW\) to \(C^k_{ij}d\omega_k\) and simultaneously subtract \(dG\) from \(D^k_{ij}d\omega_k\)). Thus we get exactly \(3g - 3\) parameters to match the l.h.s. of \([13]\) – this makes decomposition \([13]\) existing and unique.

2.3 Associativity

Thus we found that the existence of the closed algebra \([13]\) is a general feature, in particular it does not make any restrictions on the choice of Riemann surfaces. However, this algebra is not a ring: it maps the square of \(\Omega^1\) into another space: \(\Omega^1 \otimes \Omega^1 \rightarrow \Omega^2 \neq \Omega^1\). Thus, its factor over the condition \(dW = d\Lambda = 0\) is not guaranteed to have all properties of the ring. In particular the factor-algebra

\[
d\omega_i \circ d\omega_j = C^k_{ij}d\omega_k \tag{23}
\]

does not need to be associative, i.e. the matrices \(C\) alone (neglecting \(D\) and \(E\)) do not necessarily commute.

However, the associativity would follow if the expansion of \(\Omega^3\) (the space of the holomorphic 3-differentials containing the result of triple multiplication \(\Omega^1 \cdot \Omega^1 \cdot \Omega^1\)),

\[
\Omega^3 = \Omega^1 \cdot dG \cdot dG \oplus \Omega^2 \cdot dW \oplus \Omega^2 \cdot d\Lambda \tag{24}
\]

is unique. Then it is obvious that

\[
0 = (d\omega_i d\omega_j) d\omega_k - d\omega_i (d\omega_j d\omega_k) = (C^l_{ij} C^m_{lk} - C^m_{il} C^l_{jk}) d\omega_m dG^2 \mod(dW, d\Lambda) \tag{25}
\]

would imply \([C_i, C_k] = 0\). However, the dimension of \(\Omega^3\) is \(5g - 5\), while the number of adjustment parameters at the r.h.s. of \([24]\) is \(g + 2(3g - 3) = 7g - 6\), modulo only \(g + 2\) zero modes (lying in \(\Omega^1 \cdot dW d\Lambda, \Omega^1 \cdot dW dG^2\) and \(\Omega^1 \cdot d\Lambda dG^2\)). For \(g > 3\) there is no match: \(5g - 5 < 6g - 8\), the expansion \([24]\) is not unique, and associativity can (and does) \([3]\) break down unless there is some special reason for it to survive.

This special reason can exist if the curve \(C\) has specific symmetries. The most important example is the set of curves with an involution \(\sigma: C \rightarrow C, \sigma^2 = 1\), such that all \(\sigma(d\omega_i) = -d\omega_i\), while \(\sigma(dW) = -dW, \sigma(d\Lambda) = +d\Lambda\). To have \(d\Lambda\) different from all \(d\omega_i\) one should actually take it away from \(\Omega^1\), e.g. allow it to be meromorphic. This also requires some reexamination of our reasoning in the sect.2.2 we are now going to turn to.

\(^3\)See \([8]\) for an explicit example of non-associativity (actually, this happens in the important Calogero model).
2.4 Associative algebra of holomorphic 1-forms on hyperelliptic surfaces

The hyperelliptic curves are described by the equation

\[ Y^2 = \text{Pol}_{2g+2}(\lambda), \]

and the involution is \( \sigma : (Y, \lambda) \rightarrow (-Y, \lambda) \). The space of holomorphic differentials is \( \Omega^1 = \text{Span}\left\{ \frac{\lambda^\alpha d\lambda}{Y(\lambda)^\beta} \right\} \), \( \alpha = 0, \ldots, g-1 \). This space is odd under \( \sigma \), \( \sigma(\Omega^1) = -\Omega^1 \), and an example of the (meromorphic) 1-differential which is even is

\[ d\Lambda = \lambda^r d\lambda, \]

\[ \sigma(d\Lambda) = +d\Lambda. \]

We will assume that \( dG\) and \( dW \) still belong to \( \Omega^1 \) and thus are \( \sigma \)-odd. In the case of hyperelliptic curves with punctures, \( \Omega^1 \) can include also \( \sigma \)-even holomorphic 1-differentials (like \( \frac{d\Lambda}{(\lambda-\alpha_1)\cdots(\lambda-\alpha_2)} \) or just \( d\Lambda \)), in such cases we consider the algebra \( \Omega^1 \) of the \( \sigma \)-odd holomorphic differentials \( \Omega^1_\perp \), and assume that \( d\omega_i \), \( dG \) and \( dW \) belong to \( \Omega^1_\perp \), while \( d\Lambda \in \Omega^1_+ \).

The spaces \( \Omega^2 \) and \( \Omega^3 \) also split into \( \sigma \)-even and \( \sigma \)-odd parts: \( \Omega^2 = \Omega^2_+ \oplus \Omega^2_- \) and \( \Omega^3 = \Omega^3_+ \oplus \Omega^3_- \). Multiplication algebra maps \( \Omega^1_\perp \) into \( \Omega^2_+ \) and further into \( \Omega^3_+ \), which have dimensions \( 2g-1+2n \) and \( 3g-2+3n \) respectively. Here \( n \) enumerates the punctures, where holomorphic 1-differentials are allowed to have simple poles, while quadratic and the cubic ones have at most second- and third-order poles respectively. For our purposes we assume that punctures on the hyperelliptic curves enter in pairs: every puncture is accompanied by its \( \sigma \)-image. Parameter \( n \) is the number of these pairs, and the dimension of \( \Omega^1_\perp \) is \( g+n \).

Obviously, if all the \( d\omega_i \) in \( \Omega^1_\perp \) are from \( \Omega^1_\perp \), then all \( E_{ij}^k = 0 \), i.e. we actually deal with the decomposition

\[ \Omega^2_+ = \Omega^1_\perp \cdot dG + \Omega^1_+ \cdot dW \]

Parameter count now gives: \( 2g-1+2n = 2(g+n) - 1 \) where \( -1 \) is for the zero mode \( dGdW \). Thus, the hyperelliptic reduction of the algebra \( \Omega^3 \) does exist.

Moreover, it is associative, as follows from consideration of the decomposition

\[ \Omega^3 = \Omega^1_\perp \cdot dG^2 + \Omega^2_+ \cdot dW \]

Of crucial importance is that now there is no need to include \( d\Lambda \) in this decomposition, since it does not appear at the r.h.s. of the algebra itself. Parameter count is now: \( 3g-2+3n = (g+n) + (2g-1+2n) - 1 \) (there is the unique zero mode \( dWdG^2 \)). Thus, we see that this time decomposition \( \Omega^1_\perp \) is unique, and our algebra is indeed associative.

In fact, one could come to the same conclusions much easier just noting that all elements of \( \Omega^1_\perp \) are of the form

\[ d\omega_i = \frac{\phi_i(\lambda)d\lambda}{YQ(\lambda)}, \]

where all \( \phi_i(\lambda) \) are polynomials and \( Q(\lambda) = \prod_{i=1}^n (\lambda - m_i) \) is some new polynomial, which takes into account the possible singularities at punctures \( (m_i, \pm Y(m_i)) \). Then our algebra is just the one of the polynomials \( \phi_i(\lambda) \) and it is existing and associative just for the reasons discussed in sect. 2.2. The reasoning in this section can be easily modified in the case when hyperelliptic curve possesses an extra involution. The families of such curves appear in the Seiberg-Witten context for the groups \( SO(N) \) and \( Sp(N) \): the extra involution in these cases is \( \rho : \lambda \rightarrow -\lambda \). Then one considers \( \Omega^1_{\perp\perp} \) instead of just \( \Omega^1_\perp \) (see \[8\] for further details).
2.5 Residue formula

Let us now forget for a while about the hyperelliptic curves and discuss the most general raison d’etre of the residue formula. It is essentially implied \[8\] by the Seiberg-Witten theory \[9\], the Dubrovin-Krichever-Novikov theory of Whitham hierarchies \[6, 16\] and Hitchin’s description of integrable models \[17\].

Namely, imagine that we consider an integrable model with a Lax operator \(L(w)\), which is a \(N \times N\) matrix-valued function on a bare spectral curve \(E, w \in E\), which is usually torus or sphere. Then one can introduce a family of complex curves, defined by the spectral equation

\[ C : \quad \det (L(w) - \lambda) = 0 \] \hspace{1cm} (31)

The family is parametrized by the moduli that in this context are values of the \(N\) Hamiltonians of the system (since Hamiltonians commute with each other, these are actually c-numbers). We obtain this family in a peculiar parametrization, which represents the full spectral curves \(C\) as the ramified \(N\)-sheet coverings over the bare curve \(E\),

\[ P(\lambda; w) = 0, \] \hspace{1cm} (32)

where \(P\) is a polynomial of degree \(N\) in \(\lambda\).

The fact that we started from a Hamiltonian (integrable) system provides us with additional structure: the symplectic form on the “bundle” \(C \to M\) (\(M\) is the moduli space). It defines a “generating” form \(dS = \Lambda dW\) on every \(C\), which possesses the property:

\[ \frac{\partial dS}{\partial \text{moduli}} \in \Omega^1, \] \hspace{1cm} (33)

i.e. every variation of \(dS\) with the change of moduli is a holomorphic differential on \(C\) (normally, even if differential is holomorphic, its moduli-derivative is not).

This structure allows one to define the holomorphic differentials in a rather explicit form. Let \(s_I\) denote some coordinates on the moduli space \(M\). Then

\[ \frac{\partial dS}{\partial s_I} \cong \frac{\partial \Lambda}{\partial s_I} dW = -\frac{\partial P}{\partial s_I} \frac{dW}{P'} \equiv dv_I, \] \hspace{1cm} (34)

and \(dv_I\) provide a set of holomorphic differentials on \(C\). It is easy to see that they are indeed holomorphic – the variation of (32) at constant moduli gives:

\[ P' d\Lambda + \frac{\partial P}{\partial w} dw = 0, \] \hspace{1cm} (35)

i.e. the zeroes of \(P'\) are always the zeroes of \(dw\). Note that prime denotes the derivative with respect to \(\Lambda\), which can be different from the \(\lambda\)-derivatives.

The set of \(dv_I\) is not necessarily the same as \(\Omega^1\), it can be both smaller and bigger (in the latter case some \(dv_I\) are linearly dependent). It is a special requirement (standard in the context of integrable theories) that the family (32) and generating differential \(dS\) give rise to \(dv_I\)'s forming a complete basis in \(\Omega^1\) (or in \(\Omega^1\)). The finite-gap and Hitchin-like integrable systems provide a large class of examples when this is true.
The prepotential $F(a_I)$ for the Hamiltonian system is defined in terms of the cohomological class of $dS$:

$$ a_I = \oint_{A_I} dS, $$

$$ \frac{\partial F}{\partial a_I} = \oint_{B_I} dS, $$

$$ A_I \circ B_J = \delta_{IJ} $$

(36)

The cycles $A_I$ include the $A_i$’s wrapping around the handles of $C$ and $A_\iota$’s going around the punctures. The conjugate contours $B_I$ include the cycles $B_i$ and the non-closed contours $B_\iota$ ending in the singularities of $dS$ (see sect.5 of [18] for more details).

The self-consistency of the definition (36) of $F$, i.e. the symmetricity of the period matrix

$$ \frac{\partial^2 F}{\partial a_I \partial a_J} = T_{IJ} $$

(38)

is guaranteed by the following reasoning. Let us differentiate equations (36) with respect to moduli $s_K$ and use (34). Then we get:

$$ \int_{B_I} dv_K = \sum_J T_{IJ} \oint_{A_J} dv_K. $$

(37)

where the second derivative

$$ \frac{\partial^2 F}{\partial a_I \partial a_J} = T_{IJ} $$

(38)

is the period matrix of the (punctured) Riemann surface $C$. As any period matrix, it is symmetric

$$ \sum_{IJ} (T_{IJ} - T_{JI}) \oint_{A_I} dv_K \oint_{A_J} dv_L = \sum_I \left( \oint_{A_I} dv_K \int_{B_I} dv_L - \int_{B_I} dv_K \oint_{A_I} dv_L \right) = \text{res} (v_K dv_L) = 0 $$

(39)

Note also that the holomorphic differentials $dv_I$, associated with the flat moduli $a_I$ are canonical $d\omega_I$ such that

$$ \oint_{A_I} d\omega_J = \delta_{IJ} $$

$$ \oint_{B_I} d\omega_J = T_{IJ}. $$

(40)

In order to derive the residue formula one should now consider the moduli derivatives of the period matrix. It is easy to get:

$$ \sum_{IJ} \frac{\partial T_{IJ}}{\partial s_M} \oint_{A_I} dv_K \oint_{A_J} dv_L = \sum_I \left( \oint_{A_I} dv_K \int_{B_I} \frac{\partial dv_L}{\partial s_M} - \int_{B_I} dv_K \oint_{A_I} \frac{\partial dv_L}{\partial s_M} \right) = \text{res} \left( v_K \frac{\partial dv_L}{\partial s_M} \right) $$

(41)

The r.h.s. is non-vanishing, since differentiation w.r.t. moduli produces new singularities. From (34)

$$ - \frac{\partial dv_L}{\partial s_M} = \frac{\partial^2 P}{\partial s_L \partial s_M} \frac{dW}{P'} + \left( \frac{\partial P}{\partial s_M} \right)' \frac{dW}{P} - \frac{\partial P}{\partial s_M} \frac{dW}{P} + \frac{\partial P}{\partial s_L} \frac{dW}{P} + \frac{\partial P}{\partial s_M} \frac{dW}{P} \frac{dW}{P} + \frac{\partial^2 P}{\partial s_L \partial s_M} \frac{dW}{P} \frac{dW}{P} $$

(41)

and new singularities (second order poles) are at zeroes of $P'$ (i.e. at those of $dW$). Note that the contributions from the singularities of $\partial P/\partial s_L$, if any, are already taken into account in the l.h.s. of (41). Picking up the coefficient at the leading singularity, we obtain:

$$ \text{res} v_K \frac{\partial dv_L}{\partial s_M} = - \text{res} \left. \frac{dW}{dW=0} \frac{\partial P}{\partial s_K} \frac{\partial P}{\partial s_L} \frac{\partial P}{\partial s_M} \frac{dW^2}{(P')^3 d\Lambda} \right|_{dW=0} \frac{dv_K dv_L dv_M}{dW d\Lambda} $$

(42)
The integrals at the l.h.s. of (40) serve to convert the differentials \(dv_I\) into canonical \(d\omega_I\). The same matrix \(\oint A_I dv_J\) relates the derivative w.r.t. the moduli \(s_I\) and the periods \(a_I\). Putting all together we obtain (see also [16]):

\[
\frac{\partial T_{IJ}}{\partial s^K} = \text{res}_{dW=0} \frac{d\omega_J d\omega_J dv_K}{dW d\Lambda} = \text{res}_{dW=0} \frac{d\omega_J d\omega_J d\omega_K}{dW d\Lambda}.
\]

(43)

Note that these formulas essentially depend only on the symplectic structure \(dW \wedge d\Lambda\): e.g. if one makes an infinitesimal shift of \(dW\) by \(d\Lambda\), then \((dW d\Lambda)^{-1}\) is shifted by \(-(dW)^{-2}\), i.e. the shift does not contain poles at \(d\Lambda = 0\) and thus does not contribute to the residue formula.

### 2.6 Summary

We described the rather general origins of associative algebra of holomorphic 1-forms and residue formulas. We saw that associativity requires restriction to particular families of the complex curves, for example, hyperelliptic ones. In their turn, residue formulas need the Hitchin-like families of curves, which are peculiar ramified coverings. All these requirements are satisfied simultaneously for the families of hyperelliptic curves, associated with certain integrable systems: exactly the ones relevant for most examples discussed in the Seiberg-Witten theory. Some of them will be mentioned in the next sect.3.

The main example, when the WDVV eqs are not true [8], is the elliptic (Calogero) case, when the bare spectral curve is elliptic (so that \(\mathcal{C}\) is no longer hyperelliptic). In this case the WDVV eqs require substantial modifications, which are yet to be discovered (presumably, it can be related to the theory of elliptic Gromov-Witten classes). One of the things that happens in such examples, is the violation of condition (4) from sect.2.1: if all the moduli are taken into account, all the possible metrics \(G\) become degenerate (as the corollary of conformal invariance of the 4d model). Instead, new non-trivial moduli enter the game, like the parameter \(\tau\) of elliptic curve \(E\) (the remnant of the dilaton of the heterotic string). Further analysis of such examples should shed new light on the interplay between Seiberg-Witten theory, integrability, field theory and quantum geometry.

### 3 Examples

#### 3.1 Holomorphic differentials on a punctured sphere

If Riemann sphere has punctures at the points \(\lambda_i, \ i = 1, \ldots, N\), then the canonical basis in the space \(\Omega^1\) is:

\[
d\omega_i = \frac{(\lambda_i - \lambda_N) d\lambda}{(\lambda - \lambda_i)(\lambda - \lambda_N)}, \quad i = 1, \ldots, N - 1
\]

(44)

We assumed that the \(A_i\) cycles wrap around the points \(\lambda_i\), while their conjugated \(B_i\) connect \(\lambda_i\) with the reference puncture \(\lambda_N\). Multiplication algebra of \(d\omega_i\)'s is defined modulo

\[
dW = d \log P_N(\lambda) = \frac{dP_N(\lambda)}{P_N(\lambda)},
\]

(45)

\(P_N(\lambda) = \prod_{i=1}^{N}(\lambda - \lambda_i)\), and it is obviously associative.
The periods \( a_i \) depend on the choice of the generating differential \( dS = \Lambda dW \). There are two essentially different choices \( \Lambda = \lambda \) and \( \Lambda = \log \lambda \), i.e.

\[
dS^{(4)} = \lambda \, d\log P_N(\lambda) \quad \text{and} \quad dS^{(5)} = \log \lambda \, d\log P_N(\lambda)
\]

In order to fulfill the requirement (33) one should assume that \( \sum_{i=1}^{N} \lambda_i = 0 \) in the case of \( dS^{(4)} \), while \( \prod_{i=1}^{N} \lambda_i = 1 \) in the case of \( dS^{(5)} \). Since \( A_i \) cycle just wraps around the point \( \lambda = \lambda_i \), the \( A_i \)-periods of such \( dS \) are

\[
a_i^{(4)} = \oint_{\lambda_i} dS^{(4)} = \lambda_i, \\
a_i^{(5)} = \oint_{\lambda_i} dS^{(5)} = \log \lambda_i
\]

The corresponding residue formulas are

\[
\mathcal{F}^{(4)}_{ijk} = \sum_{m=1}^{N} \frac{\text{res}_{\lambda_m} \frac{d\omega_i d\omega_j d\omega_k}{d\lambda d\log P_N}}{d\lambda d\log P_N}, \\
\mathcal{F}^{(5)}_{ijk} = \sum_{m=1}^{N} \frac{\text{res}_{\lambda_m} \frac{d\omega_i d\omega_j d\omega_k}{d\lambda d\log P_N}}{d\lambda d\log P_N},
\]

and they both provide solutions to the WDVV equations \[8\]. The prepotentials are:

\[
\mathcal{F}^{(4)}(a_i) = \frac{1}{2} \sum_{1 \leq i < j \leq N} (a_i - a_j)^2 \log(a_i - a_j), \quad \sum_{i=1}^{N} a_i = 0,
\]

and

\[
\mathcal{F}^{(5)}(a_i) = \sum_{1 \leq i < j \leq N} \bar{L}_3(\bar{e}^{a_i - a_j}) - \frac{N}{2} \sum_{1 \leq i < j < k \leq N} a_ia_ja_k, \quad \sum_{i=1}^{N} a_i = 0,
\]

\[
\partial^2_x \bar{L}_3(\bar{e}^x) \equiv \log 2 \sinh x, \quad \bar{L}_3(\bar{e}^x) = \frac{1}{6} x^3 - \frac{1}{4} \bar{L}_3(e^{-2x})
\]

They describe the perturbative limit of the \( N = 2 \) supersymmetric \( SU(N) \) gauge models in 4d \[19\] and 5d \[20\] respectively.

If the punctures \( \lambda_i \) are not all independent, the same formulas provide solutions to the WDVV equations, associated with the other simple groups: \( SO(N) \), \( Sp(N) \), \( F_4 \) and \( E_{6,7,8} \) (\( G_2 \) does not have enough moduli to provide non-trivial solutions to the WDVV eqs). If \( P_N \) is substituted by

\[
P_N \rightarrow \frac{P_N}{Q_N^{1/2}} = \frac{\prod_{i=1}^{N} (\lambda - \lambda_i)}{\prod_{i=1}^{Nf} (\lambda - m_i)^{1/2}},
\]

one gets solutions, interpreted as (perturbative limits of) the gauge models with matter supermultiplets in the first fundamental representation. Inclusion of matter in other representations seems to destroy the WDVV equations, at least, generically; note that such models do not arise in a natural way from string compactifications, and there are no known curves associated with them in the Seiberg-Witten theory (see \[8\] for details).

### 3.2 Holomorphic differentials on hyperelliptic curves

Non-perturbative deformations of the above prepotentials arise when the punctures on Riemann sphere are
blown up to form handles of the hyperelliptic curve:

\[
w + \frac{1}{w} = 2 \frac{P_N(\lambda)}{Q(\lambda)^{1/2}} , \\
w - \frac{1}{w} = 2 \frac{Y(\lambda)}{Q(\lambda)^{1/2}} ,
\]

(52)

\[Y^2(\lambda) = P_N^2(\lambda) - Q_{N_f}(\lambda)\]

These curves, together with the corresponding differentials \(dS\)

\[
dS^{(4)} = \lambda \frac{dw}{w}, \quad dS^{(5)} = \log \lambda \frac{dw}{w},
\]

(53)

(i.e. \(dW = \frac{dw}{w}\) and \(d\Lambda^{(4)} = d\lambda, \ d\Lambda^{(5)} = \frac{d\lambda}{\lambda}\)) are implied by integrable models of the Toda-chain family

[21, 22, 23, 20]. Together with the residue formula (17) these provide the solution to the WDVV equations. Further details about these examples can be found in [4, 8].

### 3.3 Other examples

The very natural question is what happens with the WDVV equations for Toda chain models, associated with the exceptional groups. The problem is that the associated spectral curves are not hyperelliptic – at least naively. Still they have enough symmetries to make our general reasoning working, but this requires a special investigation.

The number of examples can be essentially increased by the study of various integrable hierarchies, peculiar configurations of punctures etc. In recent paper [24] it was actually suggested that – at least in peculiar models – \(dS\) can be expressed through the Baker-Akhiezer function: \(dS = A d \log \Psi\). The study of such examples involves generic expressions for the prepotentials, sometimes with the higher time-variables included, see [3, 21, 23, 18].

Of more importance should be development in another direction. We mentioned above that transition from 4\(d\) to 5\(d\) models [20] includes just the change of parametrization of punctured Riemann sphere: from “plane” parametrization to the “annulus” one (\(\lambda \rightarrow \log \lambda\)). The crucially interesting lift to 6\(d\) models requires interpretation of \(\lambda\) as a coordinate on elliptic curve. This is rather straightforward, and it is extremely interesting to know, if this transition breaks the WDVV equations (as happens in case of elliptization of the dual variable \(w\)). See [22] for preliminary discussion of the relevant elliptic XYZ model.

The main goal of these studies – as mentioned in the Introduction – can be better understanding of quantum cohomologies and structures behind the prepotentials of the string models.

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