The complement value problem for a class of second order elliptic integro-differential operators

Wei Sun
Department of Mathematics and Statistics
Concordia University
Montreal, H3G 1M8, Canada
E-mail: wei.sun@concordia.ca

Abstract  We consider the complement value problem for a class of second order elliptic integro-differential operators. Let $D$ be a bounded Lipschitz domain of $\mathbb{R}^d$. Under mild conditions, we show that there exists a unique bounded continuous weak solution to the following equation

\[
\begin{cases}
(\Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla + c + \text{div} \hat{b})u + f = 0 & \text{in } D, \\
u = g & \text{on } D^c.
\end{cases}
\]

Moreover, we give an explicit probabilistic representation of the solution. The recently developed stochastic calculus for Markov processes associated with semi-Dirichlet forms and heat kernel estimates play important roles in our approach.

Keywords  Complement value problem, integro-differential operator, probabilistic representation, semi-Dirichlet form, Fukushima type decomposition, heat kernel estimate.

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1 Introduction and main result

Let $d \geq 1$ and $D$ be a bounded Lipschitz domain of $\mathbb{R}^d$. Suppose that $0 < \alpha < 2$ and $p > d/2$. Let $a > 0$, $b = (b_1, \ldots, b_d)^*$ satisfying $|b| \in L^{2p}(D; dx)$ if $d \geq 2$ and $|b| \in L^\infty(D; dx)$ if $d = 1$, $c \in L^{p \vee 2}(D; dx)$, $\hat{b} = (\hat{b}_1, \ldots, \hat{b}_d)^*$ satisfying $|\hat{b}| \in L^{2(p \vee 1)}(D; dx)$, $c + \text{div} \hat{b} \leq h$ for some $h \in L^{p \vee 1}(D; dx)$ in the distribution sense, $f \in L^4(p \vee 1)(D; dx)$ and $g \in B_0(D^c)$. We consider the complement value
The fractional Laplacian operator $\Delta^{\alpha/2}$ can be written in the form

$$\Delta^{\alpha/2}\phi(x) = \lim_{\varepsilon \to 0} A(d, -\alpha) \int_{|x-y| \geq \varepsilon} \frac{\phi(y) - \phi(x)}{|x-y|^{d+\alpha}} dy, \quad \phi \in C_c^\infty(\mathbb{R}^d),$$

where $A(d, -\alpha) := \alpha 2^{1-\alpha} \pi^{-d/2} \Gamma((d+\alpha)/2) \Gamma(1-\alpha/2)^{-1}$ and $C_c^\infty(\mathbb{R}^d)$ denotes the space of infinitely differentiable functions on $\mathbb{R}^d$ with compact support.

The problem (1.1) is analogue of the Dirichlet problem for second order elliptic integro-differential equations. For these non-local equations, as opposed to the classical local case, the function $g$ should be prescribed not only on the boundary $\partial D$ but also in the whole complement $D^c$. The complement value problem for non-local operators has many applications, for example, in peridynamics [1] [8] [17], particle systems with long range interactions [10], fluid dynamics [7] and image processing [11]. This paper is a continuation of our previous paper [20], which deals with the case that $b \equiv 0$. We should point out that the lower order term $\text{div} \hat{b}$ in (1.1) is just a formal writing since the vector field $\hat{b}$ is merely measurable hence its divergence exists only in the distributional sense. Due to the appearance of the $\text{div} \hat{b}$, all the previous known methods in solving the complement value problems such as those in [2] and [20] ceased to work. In this paper, the recently developed stochastic calculus for Markov processes associated with semi-Dirichlet forms (see [16, 4]) will be applied to overcome the difficulty caused by $\hat{b}$. In particular, the Fukushima type decomposition for semi-Dirichlet forms will play an important role in establishing the existence and representation of the solution to the complement value problem.

Denote $L := \Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla$. By setting $b = 0$ off $D$, we may assume that the operator $L$ is defined on $\mathbb{R}^d$. By [6, Theorem 1.4], the martingale problem for $(L, C_c^\infty(\mathbb{R}^d))$ is well-posed for every initial value $x \in \mathbb{R}^d$. We use $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}^d})$ to denote the strong Markov process associated with $L$. Let $\rho > 0$. Define

$$q_\rho(t, z) = t^{-d/2} \exp \left(-\frac{|z|^2}{t}\right) t^{-d/2} \wedge \frac{t}{|z|^{d+\alpha}}, \quad t > 0, z \in \mathbb{R}^d.$$

By [6, Theorems 1.2-1.4], $X$ has a jointly continuous transition density function $p(t, x, y)$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, and for every $T > 0$ there exist positive constants $C_i, i = 1, 2, 3, 4$ such that

$$C_1 q_{C_2}(t, x-y) \leq p(t, x, y) \leq C_3 q_{C_4}(t, x-y), \quad (t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d.$$

Define

$$\mathcal{E}^0(\phi, \psi) = \int_{\mathbb{R}^d} (\nabla \phi \cdot \nabla \psi) dx + \frac{a^\alpha A(d, -\alpha)}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(\phi(x) - \phi(y))(\psi(x) - \psi(y))}{|x-y|^{d+\alpha}} dx dy$$

$$- \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{b \cdot \nabla \phi \cdot \psi}{|x-y|^{d+\alpha}} dx dy, \quad \phi, \psi \in D(\mathcal{E}^0),$$

$$D(\mathcal{E}^0) = W^{1,2}(\mathbb{R}^d) = \{ u \in L^2(\mathbb{R}^d; dx) : |\nabla u| \in L^2(\mathbb{R}^d; dx) \}.$$

By [20, Lemma 2.1], we know that $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a regular lower-bounded semi-Dirichlet form on $L^2(\mathbb{R}^d; dx)$. Moreover, $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}^d})$ is the Hunt process associated with $(\mathcal{E}^0, D(\mathcal{E}^0))$. By
the assumption on \( b \) and Hölder’s inequality, we find that \( |b|^2 \) belongs to the Kato class. Then, we obtain by [19, Chapter 7, Lemma 7.5] that there exists \( \beta_0 > 0 \) such that

\[
\int_{\mathbb{R}^d} |b|^2 \phi^2 dx \leq \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \phi|^2 dx + \beta_0 \int_{\mathbb{R}^d} |\phi|^2 dx, \quad \forall \phi \in W^{1,2}(\mathbb{R}^d).
\]

For \( \phi \in C_c^{\infty}(\mathbb{R}^d) \), we have \( L \phi \in L^2(\mathbb{R}^d; dx) \) (cf. [20, the proof of Lemm 2.1]). Define

\[
\gamma := \beta_0 + 1,
\]

and

\[
\mathcal{E}^0_\gamma(\phi, \psi) = \mathcal{E}^0_0(\phi, \psi) + \gamma(\phi, \psi), \quad \phi, \psi \in D(\mathcal{E}^0_0).
\]

Hereafter, \((\cdot, \cdot)\) denotes the inner product of \( L^2(\mathbb{R}^d; dx) \). Then, \((\mathcal{E}^0_\gamma, D(\mathcal{E}^0_0))\) is a regular semi-Dirichlet form on \( L^2(\mathbb{R}^d; dx) \).

Although \((\mathcal{E}^0_0, D(\mathcal{E}^0_0))\) is only a lower-bounded semi-Dirichlet form, by replacing 1 and \((\mathcal{E}^0_0, D(\mathcal{E}^0_0))\) with \( \gamma \) and \((\mathcal{E}^0_\gamma, D(\mathcal{E}^0_0))\) respectively in the proof of [16, Theorem 1.4], we can obtain the Fukushima type decomposition for \((\mathcal{E}^0_0, D(\mathcal{E}^0_0))\). Let \( \phi \in W^{1,2}(\mathbb{R}^d) \). We use \( \tilde{\phi} \) to denote an \( \mathcal{E}^0_0 \)-quasi-continuous version of \( \phi \). Note that \( \mu_\phi(dx) := \int_{\mathbb{R}^d} (\phi(x) - \phi(y))^2 dy dx \) is a finite measure on \( \mathbb{R}^d \) and \( X \) has infinite lifetime. Then, \( \phi \) admits a unique Fukushima type decomposition

\[
\tilde{\phi}(X_t) - \tilde{\phi}(X_0) = M_t^\phi + N_t^\phi, \quad t \geq 0,
\]

where \((M_t^\phi)_{t \geq 0}\) is a locally square integrable martingale additive functional and \((N_t^\phi)_{t \geq 0}\) is a continuous additive functional locally of zero quadratic variation.

By the Lax-Milgram theorem, for any vector field \( \eta \in L^2(\mathbb{R}^d; dx) \), there exists a unique \( \eta^H \in W^{1,2}(\mathbb{R}^d) \) such that

\[
\int_{\mathbb{R}^d} \langle \eta, \nabla \phi \rangle dx = \mathcal{E}^0_\gamma(\eta^H, \phi), \quad \forall \phi \in W^{1,2}(\mathbb{R}^d). \tag{1.2}
\]

Similar to [3, Lemma 2.2], we can show that

\[
\eta_n \to \eta \text{ in } L^2(\mathbb{R}^d; dx) \text{ as } n \to \infty \implies \eta_n^H \to \eta^H \text{ in } W^{1,2}(\mathbb{R}^d) \text{ as } n \to \infty. \tag{1.3}
\]

Moreover, for \( \phi \in C_c^{\infty}(\mathbb{R}^d) \), we have

\[
\int_0^t \text{div}(\phi(X_s)) ds = N_t^\phi - \gamma \int_0^t \phi^H(X_s) ds, \quad t \geq 0. \tag{1.4}
\]

Define

\[
e(t) := e^{\int_0^t \text{div}(\phi(X_s)) ds + N_t^\phi - \gamma \int_0^t \phi^H(X_s) ds}, \quad t \geq 0,
\]

and \( \tau := \inf\{t > 0 : X_t \in D^c\} \). Denote

\[
W^{1,2}_0(D) = \{u \in L^2(D; dx) : |\nabla u| \in L^2(D; dx)\},
\]

\[
W^{1,2}_0(D) = \{u \in W^{1,2}(D) : \exists \{u_n\}_{n \in \mathbb{N}} \subset C_c^{\infty}(D) \text{ such that } u_n \to u \text{ in } W^{1,2}(D)\},
\]

and

\[
W^{1,2}_0(D) := \{u : u \phi \in W^{1,2}_0(D) \text{ for any } \phi \in C_c^{\infty}(D)\}.
\]

The main result of this paper is the following theorem.
**Theorem 1.1** There exists $M > 0$ such that if $\|h\|_{L^p} \leq M$, then for any $f \in L^4(D; dx)$ and $g \in B_b(D^c)$, there exists a unique $u \in B_b(\mathbb{R}^d)$ satisfying $u|_D \in W^{1,2}_{loc}(D) \cap C(D)$ and

$$
\begin{cases}
(\Delta + a^\alpha \Delta + b \cdot \nabla + c + \text{div} \hat{b})u + f = 0 \text{ in } D, \\
u = g \text{ on } D^c.
\end{cases}
$$

Moreover, $u$ has the expression

$$
u(x) = E_x \left[ e(\tau)g(X_\tau) + \int_0^\tau e(s)f(X_s)ds \right] \text{ for q.e. } x \in D.
$$

In addition, if $g$ is continuous at $z \in \partial D$ then

$$
\lim_{x \to z} u(x) = u(z).
$$

Hereafter $(\Delta + a^\alpha \Delta + b \cdot \nabla + c + \text{div} \hat{b})u + f = 0$ is understood in the distribution sense: for any $\phi \in C^\infty_c(D)$,

$$
\int_D \langle \nabla u, \nabla \phi \rangle dx - \frac{a^\alpha A(d, -\alpha)}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{d+\alpha}} dxdy
- \int_D \langle b, \nabla u \rangle \phi dx - \int_D cu \phi dx + \int_D \langle \hat{b}, \nabla (u \phi) \rangle dx - \int_D f \phi dx = 0. \quad (1.5)
$$

Note that the double integral appearing in (1.5) is well-defined for any $u \in B_b(\mathbb{R}^d)$ with $u|_D \in W^{1,2}_{loc}(D)$ and $\phi \in C^\infty_c(D)$.

As a direct consequence of Theorem 1.1, we have the following corollary.

**Corollary 1.2** If $c + \text{div} \hat{b} \leq 0$, then for any $f \in L^4(D; dx)$ and $g \in B_b(D^c)$ satisfying $g$ is continuous on $\partial D$, there exists a unique $u \in B_b(\mathbb{R}^d)$ such that $u$ is continuous on $\overline{D}$, $u|_D \in W^{1,2}_{loc}(D)$, and

$$
\begin{cases}
(\Delta + a^\alpha \Delta + b \cdot \nabla + c + \text{div} \hat{b})u + f = 0 \text{ in } D, \\
u = g \text{ on } D^c.
\end{cases}
$$

Moreover, $u$ has the expression

$$
u(x) = E_x \left[ e(\tau)g(X_\tau) + \int_0^\tau e(s)f(X_s)ds \right] \text{ for q.e. } x \in D.
$$

The proof of Theorem 1.1 will be given in Section 3. In the next section, we first present two lemmas, which will be used to prove the continuity of the weak solution.

### 2 Two lemmas

Throughout this paper, we denote by $C$ (or $C_i$) a generic fixed strictly positive constant, whose value can change from line to line. Define $B_r(x_0) := \{ x \in \mathbb{R}^d : |x - x_0| < r \}$ for $x_0 \in \mathbb{R}^d$ and $r > 0$. 
Lemma 2.1 Suppose that \( u \in B_0(\mathbb{R}^d) \) satisfying \( u|_D \in W^{1,2}_{\text{loc}}(D) \) and \((\Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla + c + \text{div} \hat{b})u + f = 0 \) in \( D \). Then, \( u|_D \) has a locally Hölder continuous version.

Proof. It suffices to assume that \( d \geq 2 \). Set \( \varpi = u + 2\|u\|_\infty, \varrho = b - 2\|u\|_\infty \varpi^{-1} \hat{b}, \vartheta = (1 - 2\|u\|_\infty \varpi^{-1})c + f \varpi^{-1} \) and \( \bar{b} = (1 - 2\|u\|_\infty \varpi^{-1}) \hat{b} \). Then,

\[
(\Delta + a^\alpha \Delta^{\alpha/2} + \bar{b} \cdot \nabla + \vartheta + \text{div} \hat{b})u = 0 \quad \text{in} \quad D.
\]

Thus, to prove Lemma 2.1, we may assume without loss of generality that \( f \equiv 0 \).

The proof given below follows Kassmann \cite{Kassmann}. Denote \( \mathcal{L} := \Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla + c + \text{div} \hat{b} \).

(i) Assume that \( 0 < \rho < r < 1, \ x_0 \in \mathbb{R}^d \) and \( \omega \in L^{q/2}_{\text{loc}}(B_{2r}(x_0)) \) for some \( q > d \). Suppose that \( v \in W^{1,2}(B_{2r}(x_0)) \) is nonnegative in \( \mathbb{R}^d \) and satisfies \(( -\mathcal{L}v, \phi ) \geq (\omega, \phi ) \) for any nonnegative \( \phi \in C_c^\infty(B_{2r}(x_0)) \) and \( v(x) \geq \varepsilon \) for almost all \( x \in B_{2r}(x_0) \) and some \( \varepsilon > 0 \). Let \( \theta \in C_c^\infty(B_{5r/4}(x_0)) \) be a function with \( \text{supp}(\theta) = \overline{B_{r+\rho}(x_0)}, \theta(x) = 1 \) for \( x \in B_r(x_0) \) and \( \theta(x) > 0 \) for \( x \in B_{r+\rho}(x_0) \), \( \|\nabla \theta\|_\infty \leq \iota \rho^{-1} \) for some constant \( \iota > 0 \) and \( \|\theta\|_\infty \leq 1 \).

We have

\[
(\omega, -\theta^2v^{-1}) \geq \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(y) - v(x))(\theta^2v^{-1}(x) - \theta^2v^{-1}(y))}{|x - y|^{d+\alpha}} \ dxdy.
\]

By \cite{Kassmann} (5.10) and \cite{Kassmann} (5.7)-(5.9), there exists a constant \( C > 0 \) which is independent of \( v, x_0, r, \rho, \omega \) and \( \varepsilon \) such that

\[
\int_{\mathbb{R}^d} \theta^2|\nabla (\log v)|^2 \ dx \leq C \rho^{-2} |B_{r+\rho}(x_0)| + \varepsilon^{-1} \|\omega\|_{L^{q/2}(B_{r+\rho}(x_0))} \|1\|_{L^{q/2}(B_{r+\rho}(x_0))}. \quad (2.1)
\]

(ii) Assume that \( 0 < R < 1, \ x_0 \in \mathbb{R}^d \) and \( \omega \in L^{q/2}_{\text{loc}}(B_{5R/4}(x_0)) \) for some \( q > d \). Suppose that \( v \in W^{1,2}(B_{5R/4}(x_0)) \) is nonnegative in \( \mathbb{R}^d \) and satisfies \(( -\mathcal{L}v, \phi ) \geq (\omega, \phi ) \) for any nonnegative \( \phi \in C_c^\infty(B_{5R/4}(x_0)) \) and \( v(x) \geq \varepsilon \) for almost all \( x \in B_{5R/4}(x_0) \) and some \( \varepsilon > \frac{1}{4} R \frac{2(q-d)}{q} \|\omega\|_{L^{q/2}(B_{9R/8}(x_0))} \). By \cite{Kassmann}, similar to \cite{Kassmann} Lemma 5.11, we can show that there exist \( \bar{p} \in (0, 1) \) and \( \iota > 0 \) such that

\[
\left( |B_R(x_0)|^{-1} \int_{B_R(x_0)} v^{\bar{p}} \ dx \right)^{1/\bar{p}} \leq \iota \left( |B_R(x_0)|^{-1} \int_{B_R(x_0)} v^{-\bar{p}} \ dx \right)^{-1/\bar{p}}, \quad (2.2)
\]

where \( \iota \) and \( \bar{p} \) are independent of \( v, x_0, \) \( R, \) and \( \varepsilon \).

(iii) Assume that \( 0 < 8\rho < R < 1 - \rho, \ x_0 \in \mathbb{R}^d \) and \( \omega \in L^{q/2}_{\text{loc}}(B_{5R/4}(x_0)) \) for some \( q > d \). Suppose that \( v \in W^{1,2}(B_{5R/4}(x_0)) \) satisfying \(( -\mathcal{L}v, \phi ) \geq (\omega, \phi ) \) for any nonnegative \( \phi \in C_c^\infty(B_{R}(x_0)) \) and \( v(x) \geq \varepsilon \) for almost all \( x \in B_{R}(x_0) \) and some \( \varepsilon > R \frac{2(q-d)}{q} \|\omega\|_{L^{q/2}(B_{9R/8}(x_0))} \). Let \( \theta \in C_c^\infty(B_{9R/8}(x_0)) \) be a function with \( \text{supp}(\theta) = \overline{B_{r+\rho}(x_0)}, \theta(x) = 1 \) for \( x \in B_{R}(x_0) \) and \( \theta(x) > 0 \) for \( x \in B_{R+\rho}(x_0) \), \( \|\nabla \theta\|_\infty \leq \iota \rho^{-1} \) for some constant \( \iota > 0 \) and \( \|\theta\|_\infty \leq 1 \).
Let \( q > 1 \). We have

\[
(\omega, -\theta^{q+1}v^{-q}) \geq ((\Delta + b \cdot \nabla + c + \text{div} \hat{b})v, \theta^{q+1}v^{-q}) + \frac{\alpha}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (v(y) - v(x)) (\theta^{q+1}v^{-q}(x) - \theta^{q+1}v^{-q}(y)) \frac{dx}{|x-y|^{d+\alpha}}.
\]

By \([21\, (5.10)]\) and \([12\, (5.14)\) and (5.15)], there exists a constant \( C > 0 \) which is independent of \( v, x_0, R, \rho, \rho \) and \( \varepsilon \) such that

\[
\|v^{\varepsilon-1}\|_{L^{q-1}(\mathbb{R}^d)} \leq C (\max\{\rho - 1, (\rho - 1)^2\})^{\rho/2} \|v^{-1}\|_{L^{q-1}(B_{R+\rho}(x_0))},
\]

where \( d^* = d \) if \( d \geq 3 \) and \( 2 < d^* < q \) if \( d = 2 \).

(iv) Assume that \( 0 < R < 1/2, 0 < \mu < 1 < \Theta, x_0 \in \mathbb{R}^d \) and \( \omega \in L^{q/2}_{\text{loc}}(B_{\Theta R}(x_0)) \) for some \( q > d \). Suppose that \( v \in W^{1,2}(B_{\Theta R}(x_0)) \) satisfying \((L_v, \phi) \geq (\omega, \phi)\) for any nonnegative \( \phi \in C_c(\Theta B_{\Theta R}(x_0)) \) and \( v(x) \geq \varepsilon \) for almost all \( x \in B_{\Theta R}(x_0) \) and some \( \varepsilon > (\Theta R)^{2(q-d)/q} \|\omega\|_{L^{q/2}(B_{1+3\Theta R}(x_0))} \).

By \([23\, (2.3)]\), similar to \([12\, \text{Corollary 5.13}]\), we can show that for any \( \rho_0 > 0 \),

\[
\inf_{x \in B_{\mu R}(x_0)} v(x) \geq \iota \left( |B_{R}(x_0)|^{1/q} \int_{B_R(x_0)} v^{-\rho_0} dx \right)^{-1/\rho_0},
\]

where \( \iota > 0 \) is independent of \( v, x_0, R \) and \( \varepsilon \).

(v) By \([22\, (2.4)]\), similar to \([12\, \text{Corollary 5.9}]\), we can establish the following weak Harnack inequality:

There exist positive constants \( \iota_1, \iota_2 \) and \( \rho_0 \) such that for any \( x_0 \in \mathbb{R}^d \), \( R \in (0, 1) \), \( v \in W^{1,2}(B_R(x_0)) \) satisfying \( v \geq 0 \) in \( B_R(x_0) \) and \((L_v, \phi) \geq 0\) for any nonnegative \( \phi \in C_c(\Theta R(x_0)) \), we have

\[
\inf_{B_{R/4}(x_0)} v \geq \iota_1 \left( |B_{R/2}(x_0)|^{-1} \int_{B_{R/2}(x_0)} v^{\rho_0} dx \right)^{1/\rho_0} - \iota_2 R^2 \|v^{-}(z)\|_{L^{q-1}(B_{R/2}(x_0))} \int_{\mathbb{R}^d \setminus B_R(x_0)} \frac{v^{-}(z)}{|x-z|^{d+\alpha}} dz,
\]

where \( v^{-}(z) = -(v(z) \wedge 0) \). The proof is therefore complete by the weak Harnack inequality and \([12\, \text{Corollary 4.2}]\).

Suppose \( d \geq 2 \). We define \( q = \frac{p}{p-1} \). Then \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( 1 < q < \frac{d}{d-2} \). We choose \( \beta \) such that

\[
\frac{d}{2} - 1 < \beta < \frac{d}{2q}.
\]

Let \( M_1 > 0 \) be a constant satisfying

\[
e^{v^q} \geq M_1 |x|^{\beta}, \quad \forall x \in \mathbb{R}^d.
\]

We define \( q^* = \frac{2p}{2p-1} \). Then \( \frac{1}{2p} + \frac{1}{q^*} = 1 \) and \( 1 < q^* < \frac{d}{d-1} \). We choose \( 0 < \gamma < 1 \) such that

\[
q^* < \frac{d}{d-\gamma}.
\]
Let $M_2 > 0$ be a constant satisfying
\[ e^{|x|} \geq M_2|x|^{(d-\gamma)/2}, \quad \forall x \in \mathbb{R}^d. \] (2.6)

We choose $0 < \delta < (2 - \alpha)/2$. Let $M_3 > 0$ be a constant satisfying
\[ e^{|x|} \geq M_3|x|^{1/6}, \quad \forall x \in \mathbb{R}^d. \] (2.7)

Define $\varsigma = \sup_{x \in D} |x|$.

**Lemma 2.2** Let $C, R$ be two positive constants and $\mu$ be a function on $\mathbb{R}^d$ with $\text{supp}[\mu] \subset B_R(0)$.

(i) Suppose $d \geq 2$. Then, there exist positive constants $C_1, C_2$ which are independent of $\mu$ such that for any $t > 0$ and $x \in D$,
\[
\int_0^t \int_{y \in \mathbb{R}^d} \left( s^{-d/2} \exp \left( -\frac{C|x-y|^2}{s} \right) + s^{-d/2} \wedge \frac{s}{|x-y|^{d+\alpha}} \right) |\mu(y)| dy ds \\
\leq C_1 \left( t^{\delta} + t^{d+1/2} \right) \left( \int_{y \in \mathbb{R}^d} |\mu(y)|^p dy \right)^{1/p}
\]
and
\[
\int_0^t \int_{y \in \mathbb{R}^d} \left( s^{-(d+1)/2} \exp \left( -\frac{C|x-y|^2}{s} \right) + s^{-(d+1)/2} \wedge \frac{s}{|x-y|^{d+1+\alpha}} \right) |\mu(y)| dy ds \\
\leq C_2 \left( t^{1-\gamma)/2} + t^{\delta} \right) \left( \int_{y \in \mathbb{R}^d} |\mu(y)|^{2p} dy \right)^{1/(2p)}.
\]

(ii) Suppose $d = 1$. Then, for any $t > 0$ and $x \in D$,
\[
\int_0^t \int_{-\infty}^{\infty} \left( s^{-1/2} \exp \left( -\frac{C|x-y|^2}{s} \right) + s^{-1/2} \wedge \frac{s}{|x-y|^{1+\alpha}} \right) |\mu(y)| dy ds \\
\leq 4t^{1/2} \int_{-\infty}^{\infty} |\mu(y)| dy,
\] (2.8)

and there exists a positive constant $C_3$ which is independent of $\mu$ such that for any $t > 0$ and $x \in D$,
\[
\int_0^t \int_{-\infty}^{\infty} \left( s^{-1} \exp \left( -\frac{C|x-y|^2}{s} \right) + s^{-1} \wedge \frac{s}{|x-y|^{2+\alpha}} \right) |\mu(y)| dy ds \\
\leq C_3 \left( t^{1/6} + t^{\delta} \right) \left( \int_{-\infty}^{\infty} |\mu(y)|^2 dy \right)^{1/2}.
\]

**Proof.** We first consider the case that $d \geq 2$. We have
\[
s^{-d/2} \wedge \frac{s}{|x-y|^{d+\alpha}} \leq s^{-d/2} \leq e^{C} s^{-d/2} \exp \left( -\frac{C|x-y|^2}{s} \right) \quad \text{if} \ |x-y|^2 < s. \] (2.9)
By (2.5) and (2.9), we get
\[
\int_0^t \int_{y \in \mathbb{R}^d} \left( s^{-d/2} \exp \left( - \frac{C |x - y|^2}{s} \right) + s^{-d/2} \wedge \frac{s}{|x - y|^{d+\alpha}} \right) |\mu(y)| dy ds \\
\leq \int_0^t \int_{y \in \mathbb{R}^d} (1 + e^C) s^{-d/2} \exp \left( - \frac{C |x - y|^2}{s} \right) |\mu(y)| dy ds \\
+ \int_0^t \int_{|x - y| \geq \sqrt{3}} \frac{s}{|x - y|^{d+\alpha}} |\mu(y)| dy ds
\]
\[
\leq \int_0^t \int_{y \in \mathbb{R}^d} \frac{1 + e^C}{M_1 s^{d/2}} (C|x - y|^2/s)^{\beta} |\mu(y)| dy ds + t^\delta \int_{y \in \mathbb{R}^d} \int_0^{1/\delta} s^{1-\delta} ds \frac{1}{|x - y|^{d+\alpha}} |\mu(y)| dy \\
= \int_0^t \frac{1 + e^C}{C^\beta M_1 s^{d/2 - \beta}} ds \left( \int_{y \in B_R(0)} \frac{1}{|x - y|^{2\beta_\eta}} dy \right)^{1/q} \left( \int_{y \in \mathbb{R}^d} |\mu(y)|^p dy \right)^{1/p} \\
+ \frac{t^\delta}{2 - \delta} \left( \int_{y \in B_R(0)} \frac{1}{|x - y|^{(d+\alpha-4+2\delta)q}} dy \right)^{1/q} \left( \int_{y \in \mathbb{R}^d} |\mu(y)|^p dy \right)^{1/p} \\
= \frac{(1 + e^C) t^{\beta+1-d/2}}{C^\beta M_1 (\beta + 1 - d/2)} \left( C_1 \int_0^{R+\epsilon} s^{-2\beta_\eta - 1} ds \right)^{1/q} \left( \int_{y \in \mathbb{R}^d} |\mu(y)|^p dy \right)^{1/p} \\
+ \frac{t^\delta}{2 - \delta} \left( C_2 \int_0^{R+\epsilon} s^{-(d+\alpha-4+2\delta)q - 1} ds \right)^{1/q} \left( \int_{y \in \mathbb{R}^d} |\mu(y)|^p dy \right)^{1/p} \\
:\leq C_1 (t^{\beta+1-d/2} + t^\delta) \left( \int_{y \in \mathbb{R}^d} |\mu(y)|^p dy \right)^{1/p} 
\]

By (2.6) and (2.9), we get
\[
\int_0^t \int_{y \in \mathbb{R}^d} \left( s^{-(d+1)/2} \exp \left( - \frac{C |x - y|^2}{s} \right) + s^{-(d+1)/2} \wedge \frac{s}{|x - y|^{d+1+\alpha}} \right) |\mu(y)| dy ds \\
\leq \int_0^t \int_{y \in \mathbb{R}^d} (1 + e^C) s^{-(d+1)/2} \exp \left( - \frac{C |x - y|^2}{s} \right) |\mu(y)| dy ds \\
+ \int_0^t \int_{|x - y| \geq \sqrt{3}} \frac{s}{|x - y|^{d+1+\alpha}} |\mu(y)| dy ds
\]
\[
\leq \int_0^t \int_{y \in \mathbb{R}^d} \frac{1 + e^C}{M_2 s^{(d+1)/2}} (C|x - y|^2/s)^{(d-\gamma)/2} |\mu(y)| dy ds + t^\delta \int_{y \in \mathbb{R}^d} \int_0^{1/\delta} s^{1-\delta} ds \frac{1}{|x - y|^{d+\alpha}} |\mu(y)| dy \\
= \int_0^t \frac{1 + e^C}{C^{(d-\gamma)/2} M_2 s^{(1+\gamma)/2}} ds \left( \int_{y \in \mathbb{R}^d} |\mu(y)| \right)^{1/q} \left( \int_{y \in \mathbb{R}^d} |\mu(y)|^{2p} dy \right)^{1/(2p)} \\
+ \frac{t^\delta}{2 - \delta} \left( \int_{y \in B_R(0)} \frac{1}{|x - y|^{(d+\alpha-3+2\delta)q}} dy \right)^{1/q} \left( \int_{y \in \mathbb{R}^d} |\mu(y)|^{2p} dy \right)^{1/(2p)} 
\]
\[\begin{align*}
2(1 + e^C) & \cdot \frac{(1 - \gamma)/2}{C(d-\gamma)/2 M_2(1 - \gamma)} \left( C_3 \int_0^{R + \epsilon} r^{d - (d - \gamma)q - 1} dr \right)^{1/q} \left( \int_{y \in \mathbb{R}^d} |\mu(y)|^2 dy \right)^{1/(2p)} \\
& + \frac{t^\delta}{2 - \delta} \left( \int_0^{R + \epsilon} r^{d + (d + \epsilon - 3 - 2\delta)q - 1} dr \right)^{1/q} \left( C_4 \int_{y \in \mathbb{R}^d} |\mu(y)|^2 dy \right)^{1/(2p)} \\
& := C_5(t (1 - \gamma)/2 + t^\delta) \left( \int_{y \in \mathbb{R}^d} |\mu(y)|^2 dy \right)^{1/(2p)}.
\end{align*}\]

We now consider the case that \(d = 1\). It is easy to see that (2.8) holds. By (2.7) and (2.9), we get

\[\begin{align*}
& \int_0^t \int_{-\infty}^\infty \left( s^{-1} \exp \left( -\frac{C|x - y|^2}{s} \right) + s^{-1} \wedge \frac{s}{|x - y|^{2+\alpha}} \right) |\mu(y)| dy ds \\
\leq & \int_0^t \int_{-\infty}^\infty (1 + e^C) s^{-1} \exp \left( -\frac{C|x - y|^2}{s} \right) |\mu(y)| dy ds \\
& + \int_0^t \int_{|x-y| \geq \sqrt{s}} \frac{s}{|x - y|^{2+\alpha}} |\mu(y)| dy ds \\
\leq & \int_0^t \int_{-\infty}^\infty M_3 s(C|x - y|^2/|s|)^{1/6} |\mu(y)| dy ds + t^\delta \int_{-\infty}^\infty \int_0^{s^{-\delta} ds} \frac{1}{|x - y|^{2+\alpha}} |\mu(y)| dy \\
& = \int_0^t \int_{-\infty}^\infty M_3 s(C|x - y|^2/|s|)^{1/6} |\mu(y)| dy ds + t^\delta \int_{-\infty}^\infty \left( 2 - \delta \right) |x - y|^{\alpha - 2 + 2\delta} |\mu(y)| dy \\
\leq & \frac{6(1 + e^C)^{1/6}}{C^{1/6} M_3} \left( \int_{-R}^{R} \frac{1}{|x - y|^{2/3}} dy \right)^{1/2} \left( \int_{-\infty}^\infty |\mu(y)|^2 dy \right)^{1/2} \\
& + \frac{t^\delta}{2 - \delta} \left( \int_{y \in B_R(0)} \frac{1}{|x - y|^{2\alpha(2 + 2\delta)}} dy \right)^{1/2} \left( \int_{-\infty}^\infty |\mu(y)|^2 dy \right)^{1/2} \\
& = \frac{6(1 + e^C)^{1/6}}{C^{1/6} M_3} \left( C_5 \int_0^{R + \epsilon} r^{-2/3} dr \right)^{1/2} \left( \int_{-\infty}^\infty |\mu(y)|^2 dy \right)^{1/2} \\
& + \frac{t^\delta}{2 - \delta} \left( C_6 \int_0^{R + \epsilon} r^{2(2 + 2\delta)} dr \right)^{1/q} \left( \int_{-\infty}^\infty |\mu(y)|^2 dy \right)^{1/2} \\
& := C_3(t^{1/6} + t^\delta) \left( \int_{-\infty}^\infty |\mu(y)|^2 dy \right)^{1/2}.
\end{align*}\]
3 Proof of Theorem 1.1

3.1 Existence of weak solution

Define
\[ u^*(x) = E_x \left[ e(\tau)g(X_\tau) + \int_0^\tau e(s)f(X_s)ds \right] \quad \text{for } x \in \mathbb{R}^d. \] (3.1)

By [20, Lemma 2.5], there exists \( C > 0 \) such that
\[ \sup_{x \in D} E_x \left[ \int_0^\tau v(X_s)ds \right] \leq C \|v\|_{L^{p\vee 1}}, \quad \forall v \in L^{p\vee 1}(D). \] (3.2)

By Khasminskii’s inequality and (3.2), we find that there exist positive constants \( M \) and \( \Upsilon \) such that for any \( v \in L^{p\vee 1}(D) \) satisfying \( \|v\|_{L^{p\vee 1}} \leq M \), we have
\[ \sup_{x \in D} E_x \left[ e^{\int_0^\tau 8v(X_s)ds} \right] \leq \Upsilon. \] (3.3)

In particular, this implies that there exists \( \nu > 0 \) such that
\[ \sup_{x \in D} E_x \left[ e^{\nu \tau} \right] < \infty. \] (3.4)

Define
\[ J(x) = \frac{1_{\{|x|<1\}}e^{-\frac{1}{1-|x|^2}}}{\int_{\{|y|<1\}} e^{-\frac{1}{1-|y|^2}} dy}, \quad x \in \mathbb{R}^d. \]

For \( k \in \mathbb{N} \) and \( x \in \mathbb{R}^d \), set
\begin{align*}
J_k(x) &= k^d J(kx), \\
\hat{b}_k(x) &= \int_{\mathbb{R}^d} \hat{b}(x-y)J_k(y)dy, \\
c_k(x) &= \int_{\mathbb{R}^d} c(x-y)J_k(y)dy, \\
h_k(x) &= \int_{\mathbb{R}^d} h(x-y)J_k(y)dy.
\end{align*}

We have
\[ \hat{b}_k \to \hat{b} \quad \text{in } L^2(\mathbb{R}^d;dx) \quad \text{as } k \to \infty, \] (3.5)
and
\[ c_k \to c \quad \text{in } L^2(\mathbb{R}^d;dx) \quad \text{as } k \to \infty. \] (3.6)

Suppose that \( \|h\|_{L^{p\vee 1}} \leq M \). Since \( c + \text{div}\hat{b} \leq h \) implies that \( c_k + \text{div}\hat{b}_k \leq h_k \) for \( k \in \mathbb{N} \), we obtain by (3.3) that
\[ \sup_{k \in \mathbb{N}} \sup_{x \in D} E_x \left[ e^{\int_0^\tau 8(c_k+\text{div}\hat{b}_k)(X_s)ds} \right] \leq \sup_{k \in \mathbb{N}} \sup_{x \in D} E_x \left[ e^{\int_0^\tau 8h_k(X_s)ds} \right] \leq \Upsilon. \] (3.7)
By (1.3) and (3.5), we get
\[ \hat{b}^H_k \to \hat{b}^H \text{ in } W^{1,2}(\mathbb{R}^d) \text{ as } k \to \infty. \] (3.8)

Further, by [16, Theorem 1.15] and [13] Lemma A.6, Theorem A.8 and Lemma A.9, similar to [9, Corollary 5.2.1] we can show that there exists a subsequence \( \{k_l\} \) such that for q.e. \( x \in \mathbb{R}^d \),
\[ P_x \left\{ \lim_{l \to \infty} N^k_t = N^\tau_t \text{ uniformly on any finite interval of } t \right\} = 1. \] (3.9)

For simplicity, we still use \( \{k\} \) to denote the subsequence \( \{k_l\} \). By (1.4), (3.2), (3.6)–(3.9) and Fatou’s lemma, we find that there exists an \( \mathcal{E}^0 \)-exceptional set \( F \subset D \) such that for \( x \in D \setminus F \),
\[ E_x [e^2(\tau)] \leq \liminf_{k \to \infty} E_x \left[ e^{2 \left( \int_0^\tau c_k(X_s) ds + N^k_t - \gamma \int_0^\tau \hat{b}^H(X_s) ds \right)} \right] \]
\[ = \liminf_{k \to \infty} E_x \left[ e^{\int_0^\tau 2(c_k + \text{div} \hat{b}_k)(X_s) ds} \right] \]
\[ \leq \liminf_{k \to \infty} E_x \left[ e^{\int_0^\tau 2(h_k)(X_s) ds} \right] \]
\[ \leq \varpi^{1/4}, \] (3.10)

and
\[ E_x \left[ \left( \int_0^\tau e(s)f(X_s) ds \right)^2 \right] \]
\[ \leq \liminf_{k \to \infty} E_x \left[ \left( \int_0^\tau e^{\int_0^\tau (c_k + \text{div} \hat{b}_k)(X_t) dt} f(X_s) ds \right)^2 \right] \]
\[ \leq \liminf_{k \to \infty} \left( E_x \left[ \left( \int_0^\tau (\tau \vee 1) e^{\int_0^\tau 2(c_k + \text{div} \hat{b}_k)(X_t) dt} ds \right)^2 \right] \right)^{1/2} \left( E_x \left[ \left( \int_0^\tau \frac{1}{\tau \vee 1} f^2(X_s) ds \right)^2 \right] \right)^{1/2} \]
\[ \leq \liminf_{k \to \infty} E_x \left[ e^{\int_0^\tau \text{div} \hat{b}_k(X_s) ds} \cdot (\tau \vee 1)^2 \right]^{1/2} \left( E_x \left[ \int_0^\tau f^4(X_s) ds \right] \right)^{1/2} \]
\[ \leq C^{1/2} \liminf_{k \to \infty} \left( E_x \left[ e^{\int_0^\tau \text{div} \hat{b}_k(X_s) ds} \right] \right)^{1/4} \left( E_x \left[ (\tau^4 (\tau \vee 1)^4) \right] \right)^{1/4} \|f^4\|_{L^{p \vee 1}}^{1/2} \]
\[ \leq C^{1/2} \varpi^{1/4} \left( E_x \left[ (\tau^4 (\tau \vee 1)^4) \right] \right)^{1/4} \|f^4\|_{L^{p \vee 1}}^{1/2}. \] (3.11)

By (3.4), (3.10) and (3.11), we know that there exists \( M > 0 \) such that if \( \|h\|_{L^{p \vee 1}} \leq M \), then for any \( f \in L^{4(p \vee 1)}(D; dx) \) and \( g \in B^p_0(D^c) \), \( u^* \) is bounded on \( \mathbb{R}^d \) except for an \( \mathcal{E}^0 \)-exceptional set.

For \( k \in \mathbb{N} \), define
\[ e_k(t) := e^{\int_0^t (c_k + \text{div} \hat{b}_k)(X_s) ds}, \quad t \geq 0. \]

By [20, Theorem 1.1], we know that the unique bounded continuous weak solution to the complement value problem
\[ \begin{cases}
(\Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla + c_k + \text{div} \hat{b}_k)u_k + f = 0 & \text{in } D \\
\quad u_k = g & \text{on } D^c \end{cases} \] (3.12)
is given by

\[ u_k(x) = E_x \left[ e_k(\tau)g(X_\tau) + \int_0^T e_k(s)f(X_s)ds \right], \quad x \in \mathbb{R}^d. \]

By (1.4), (3.6) and (3.8)–(3.11), we obtain that

\[ \lim_{k \to \infty} u_k(x) = u^*(x) \text{ for q.e. } x \in D. \]  \hspace{1cm} (3.13)

We now show that \( u^* \) is a weak solution to the complement value problem (1.1). By (3.10) and (3.11), we find that \( \{u_k\}_D \) is a sequence of uniformly bounded functions on \( D \). Let \( y \in D \) and \( 0 < r < R \) such that \( B_R(y) \subset D \). Let \( \gamma : [0, \infty) \mapsto [0, \infty) \) be defined as

\[
\begin{cases}
\gamma(s) = 1, & 0 \leq s \leq r, \\
\frac{R-s}{R-r}, & r < s < R, \\
0, & s > R
\end{cases}
\]

and let \( \psi(x) = \gamma(|x - y|) \). Then \( \phi = \psi^2(u_k) \in W^{1,2}_0(D) \) with compact support in \( D \). For \( \varepsilon > 0 \), we obtain by (3.12) that

\[
\int_D \psi^2 |\nabla u_k|^2 dx
= -2 \int_D \psi_k \langle \nabla u_k, \nabla \psi \rangle dx + \int_D \langle \nabla u_k, \nabla (u_k \psi^2) \rangle dx
= -2 \int_D \psi_k \langle \nabla u_k, \nabla \psi \rangle dx - \frac{\alpha A(d, -\alpha)}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u_k(x) - u_k(y))(u_k^2(x) - u_k^2(y))}{|x - y|^{d+\alpha}} dx dy
+ \int_D \langle b, \nabla u_k \rangle u_k \psi^2 dx + \int_D c_k u_k^2 \psi^2 dx - \int_D \langle \hat{b}_k, \nabla (u_k \psi^2) \rangle dx + \int_D f u_k \psi^2 dx
= -2 \int_D \psi_k \langle \nabla u_k, \nabla \psi \rangle dx
- \frac{\alpha A(d, -\alpha)}{2} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u_k^2(x) - u_k^2(y))^2}{|x - y|^{d+\alpha}} dx dy \right)
+ \int_D \langle b, \nabla u_k \rangle u_k \psi^2 dx + \int_D c_k u_k^2 \psi^2 dx - \int_D \langle \hat{b}_k, \nabla (u_k \psi^2) \rangle dx + \int_D f u_k \psi^2 dx
\leq \varepsilon \int_D \psi^2 |\nabla u_k|^2 dx + \frac{1}{\varepsilon} \int_D u_k^2 |\nabla \psi|^2 dx - \frac{\alpha A(d, -\alpha)}{2} \|u_k\|_\infty^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(\psi(x) - \psi(y))^2}{|x - y|^{d+\alpha}} dx dy
+ \frac{\varepsilon}{2} \int_D \psi^2 |\nabla u_k|^2 dx + \frac{1}{2\varepsilon} \int_D |b|^2 u_k^2 \psi^2 dx + \int_D |c_k| u_k^2 \psi^2 dx
+ \varepsilon \int_D u_k^2 |\nabla \psi|^2 dx + \frac{1}{\varepsilon} \int_D |\hat{b}_k|^2 u_k^2 \psi^2 dx + \varepsilon \int_D \psi^2 |\nabla u_k|^2 dx + \frac{1}{\varepsilon} \int_D \psi^2 |\nabla u_k|^2 dx + \int_D |f||u_k|\psi^2 dx
\leq \frac{5\varepsilon}{2} \int_D \psi^2 |\nabla u_k|^2 dx + \frac{\|u_k\|_\infty^2}{2\varepsilon} \int_D (2|\nabla \psi|^2 + |b|^2 + 4|\hat{b}_k|^2) dx
+ \|u_k\|^2 \left( \int_D |c_k|\psi^2 dx + \varepsilon \int_D |\nabla \psi|^2 dx \right) + \|u_k\|_\infty \int_D |f| dx
+ \frac{\alpha A(d, -\alpha)}{2} \|u_k\|_\infty^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(\psi(x) - \psi(y))^2}{|x - y|^{d+\alpha}} dx dy. \]  \hspace{1cm} (3.14)
Let $\varepsilon = 1/5$. Then, we obtain by the uniform boundedness of $\{u_k|D\}$ and (3.14) that

$$\sup_{k \in \mathbb{N}} \int_{B_{r}(y)} |\nabla u_k|^2 \, dx < \infty.$$ 

Since $y$ and $r$ are arbitrary, we obtain by (3.13) that $u^*|D \in W^{1,2}_{loc}(D)$.

By taking a subsequence if necessary, we may assume that for any $\psi \in C_c^\infty(D)$, $u_k \psi \rightharpoonup u^* \psi$ weakly in $W^{1,2}(D)$ as $k \to \infty$ and that

$$u'_k \psi := \frac{1}{k} \sum_{l=1}^{k} u_l \psi \rightharpoonup u^* \psi \text{ in } W^{1,2}(D) \text{ as } k \to \infty. \quad (3.15)$$

Let $\phi \in C_c^\infty(D)$. Suppose that $\text{supp}[\phi] \subset U \subset \overline{U} \subset D$ for some open set $U$. We choose $\varrho \in C_c^\infty(D)$ satisfying $\varrho \equiv 1$ on $U$. Then, there exists a positive constant $C$ which is independent of $k$ such that

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u'_k(x) - u'_k(y))((\phi(x) - \phi(y)))}{|x-y|^{d+\alpha}} \, dx \, dy - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))((\phi(x) - \phi(y)))}{|x-y|^{d+\alpha}} \, dx \, dy \right|$$

$$\leq \int_{U} \int_{U} \frac{|((u'_k \varrho - u^* \varrho)(x) - (u'_k \varrho - u^* \varrho)(y))\phi(x) - \phi(y))|}{|x-y|^{d+\alpha}} \, dx \, dy$$

$$+ \int_{\text{supp}[\phi]} \int_{U^c} \frac{|((u'_k - u^*)(x) - (u'_k - u^*)(y))\phi(y)|}{|x-y|^{d+\alpha}} \, dx \, dy$$

$$+ \int_{U^c} \int_{\text{supp}[\phi]} \frac{|((u'_k - u^*)(x) - (u'_k - u^*)(y))\phi(x)|}{|x-y|^{d+\alpha}} \, dx \, dy$$

$$\leq C \|u'_k \varrho - u \varrho\|_{W^{1,2}(D)} \|\phi\|_{W^{1,2}(D)}$$

$$+ \int_{\text{supp}[\phi]} \int_{U^c} \frac{2 \left( \sup_{k \in \mathbb{N}} \|u'_k\|_\infty + \|u^*\|_\infty \right) \|\phi\|_\infty}{|x-y|^{d+\alpha}} \, dx \, dy$$

$$+ \int_{U^c} \int_{\text{supp}[\phi]} \frac{2 \left( \sup_{k \in \mathbb{N}} \|u'_k\|_\infty + \|u^*\|_\infty \right) \|\phi\|_\infty}{|x-y|^{d+\alpha}} \, dx \, dy.$$

Therefore, we obtain by (3.13) and the dominated convergence theorem that

$$\lim_{k \to \infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u'_k(x) - u'_k(y))((\phi(x) - \phi(y)))}{|x-y|^{d+\alpha}} \, dx \, dy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u^*(x) - u^*(y))((\phi(x) - \phi(y)))}{|x-y|^{d+\alpha}} \, dx \, dy. \quad (3.16)$$

Note that

$$\lim_{k \to \infty} \int_D \langle \nabla u_k', \nabla \phi \rangle \, dx = \int_D \langle \nabla u^*, \nabla \phi \rangle \, dx, \quad (3.17)$$

$$\lim_{k \to \infty} \int_D \langle b, \nabla u_k' \rangle \phi \, dx = \int_D \langle b, \nabla u^* \rangle \phi \, dx, \quad (3.18)$$

$$\lim_{k \to \infty} \int_D c_k u'_k \phi \, dx = \int_D c u^* \phi \, dx, \quad (3.19)$$

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and
\[ \lim_{k \to \infty} \int_D \langle \hat{b}_k, \nabla(u'_k\phi) \rangle dx = \int_D \langle \hat{b}, \nabla(u^*\phi) \rangle dx. \] (3.20)

Therefore, we obtain by (3.12) and (3.16)–(3.20) that \( u^* \) is a weak solution to the complement value problem (1.1).

### 3.2 Continuity of weak solution

In this subsection, we consider the continuity of the weak solution \( u^* \) given by (3.1). By Lemma 2.1, we know that \( u^*|_D \) has a continuous version, which is denoted by \( u|_D \). By (3.13) and (3.15), we find that \( u^*|_D \) is quasi-continuous and hence \( u(x) = u^*(x) \) for q.e. \( x \in D \). We will show below that if \( g \) is continuous at \( z \in \partial D \) then
\[ \lim_{x \to z} u(x) = g(z). \]

Note that for \( x \in \mathbb{R}^d \),
\[ u^*(x) = E_x \left[ g(X_\tau) + \int_0^\tau f(X_s)ds \right] + E_x \left[ (e(\tau) - 1)g(X_\tau) + \int_0^\tau (e(s) - 1)f(X_s)ds \right]. \]

Suppose that \( g \) is continuous at \( z \in \partial D \). By [20, Theorem 1.1], to prove \( \lim_{x \to z} u(x) = g(z) \), it suffices to show that there exists an \( \mathcal{E}^0 \)-exceptional set \( F \subset D \) such that
\[ \lim_{x \to z} E_x[(e(\tau) - 1)g(X_\tau)] = 0 \] (3.21)
and
\[ \lim_{x \to z} E_x \left[ \int_0^\tau (e(s) - 1)f(X_s)ds \right] = 0. \] (3.22)

For \( t > 0 \), we have
\[ E_x[(e(\tau) - 1)g(X_\tau)] = E_x[(e(\tau) - 1)g(X_\tau); \tau \leq t] + E_x[(e(\tau) - 1)g(X_\tau); \tau > t]. \]

By (3.10), there exists an \( \mathcal{E}^0 \)-exceptional set \( F \subset D \) such that for \( x \in D \setminus F \),
\[ |E_x[(e(\tau) - 1)g(X_\tau); \tau > t]| \leq \|g\|_\infty \{P_x(\tau > t) + E_x[e(\tau); \tau > t]\} \leq \|g\|_\infty \{P_x(\tau > t) + (E_x[e^2(\tau)])^{1/2}(P_x(\tau > t))^{1/2}\} \leq \|g\|_\infty \{P_x(\tau > t) + \gamma^{1/8}(P_x(\tau > t))^{1/2}\}. \] (3.23)

By [20, Lemma 2.3(3) and Lemma 2.7], we have
\[ \limsup_{x \to z} P_x(\tau > t) = 0. \] (3.24)
Then, we obtain by (3.23) and (3.24) that for every $t > 0$,
\[
\lim_{x \rightarrow z} \sup_{x \in D \setminus F} |E_x[(e(\tau) - 1)g(X_\tau); \tau > t]| = 0.
\]

Hence, to prove (3.21), it suffices to show that
\[
\lim_{t \downarrow 0} \sup_{x \in D \setminus F} |E_x[(e(\tau) - 1)g(X_\tau); \tau \leq t]| = 0. \tag{3.25}
\]

For $x \in D \setminus F$ and $t > 0$, we obtain by Fatou’s lemma that
\[
|E_x[(e(\tau) - 1)g(X_\tau); \tau \leq t]| \
\leq \|g\|_\infty \liminf_{k \rightarrow \infty} E_x \left[ \left| e^{\int_0^t (c_k + \text{div} \hat{b}_k)(X_s)ds} - 1 \right|; \tau \leq t \right] \
\leq \|g\|_\infty \left\{ \sup_{x \in D \setminus F} \limsup_{k \rightarrow \infty} E_x \left[ e^{\int_0^t h_k(X_s)ds} - 1; \tau \leq t \right] \
+ \limsup_{k \rightarrow \infty} E_x \left[ \left( 1 - e^{\int_0^t (c_k + \text{div} \hat{b}_k - h_k)(X_s)ds} \right) ; \tau \leq t \right] \right\} \
\leq \|g\|_\infty \left\{ \limsup_{k \rightarrow \infty} E_x \left[ e^{\int_0^t h_k(X_s)ds} - 1 \right] \
+ E_x \left[ \left( 1 - e^{\int_0^t (c_k + \text{div} \hat{b}_k - h_k)(X_s)ds} \right) \right] \right\}.
\]

By Lemma 2.2 and Khasminskii’s inequality, we get
\[
\lim_{t \downarrow 0} \sup_{x \in D \setminus F} E_x \left[ e^{\int_0^t h_k(X_s)ds} \right] = 1.
\]

Hence, to prove (3.25), we need only show that
\[
\liminf_{t \downarrow 0} \inf_{x \in D \setminus F} E_x \left[ e^{\int_0^t (c_k + \text{div} \hat{b}_k)(X_s)ds} \right] \geq 1.
\]

Further, by Jensen’s inequality, we need only show that
\[
\limsup_{t \downarrow 0} \sup_{x \in D \setminus F} E_x \left[ \int_0^t (h_k - c_k - \text{div} \hat{b}_k)(X_s)ds \right] = 0.
\]

By Lemma 2.2 and [6] Theorems 2.1, 2.2 and 4.3, we get
\[
\sup_{x \in D \setminus F} E_x \left[ \int_0^t (h_k - c_k - \text{div} \hat{b}_k)(X_s)ds \right] \
= \sup_{x \in D \setminus F} \int_0^t \int_{y \in D} p_D(s, x, y)(h_k - c_k - \text{div} \hat{b}_k)(y)dyds \
\leq \sup_{x \in D \setminus F} \int_0^t \int_{y \in \mathbb{R}^d} p(s, x, y)(h_k - c_k - \text{div} \hat{b}_k)(y)dyds \
\leq \sup_{x \in D \setminus F} C_1 \int_0^t \int_{y \in \mathbb{R}^d} \left( s^{-d/2} \exp \left( -\frac{C_2|x - y|^2}{s} \right) + s^{-d/2} \wedge \frac{s}{|x - y|^{d + \alpha}} \right) (|h_k| + |c_k|)(y)dyds \
+ \sup_{x \in D \setminus F} C_3 \int_0^t \int_{y \in \mathbb{R}^d} \left( s^{-(d+1)/2} \exp \left( -\frac{C_4|x - y|^2}{s} \right) + s^{-(d+1)/2} \wedge \frac{s}{|x - y|^{d+1+\alpha}} \right) (\hat{b}_k)(y)dyds \
\rightarrow 0 \text{ as } t \downarrow 0.
\]
where $p^D(t, x, y)$ is the transition density function of the part process $((X^D_t)_{t \geq 0}, (P_x)_{x \in D})$ and $C_i$, $i = 1, 2, 3, 4$, are positive constants. Then (3.25) holds and therefore (3.24) holds.

For $t > 0$, we have
\[
E_x \left[ \int_0^t (e(s) - 1)f(X_s)ds \right] = E_x \left[ \int_0^t (e(s) - 1)f(X_s)ds; \tau \leq t \right] + E_x \left[ \int_0^t (e(s) - 1)f(X_s)ds; \tau > t \right].
\]

By (3.2) and (3.11), we obtain that for $x \in D \setminus F$,
\[
\left| E_x \left[ \int_0^t (e(s) - 1)f(X_s)ds; \tau > t \right] \right| \\
\leq E_x \left[ \int_0^t e(s)f(X_s)ds; \tau > t \right] + E_x \left[ \int_0^t f(X_s)ds; \tau > t \right] \\
\leq \left( E_x \left[ \left( \int_0^t e(s)f(X_s)ds \right)^2 \right] \right)^{1/2} (P(\tau > t))^{1/2} \\
+ (E[\tau; \tau > t])^{1/2} \left( E_x \left[ \int_0^t f^2(X_s)ds \right] \right)^{1/2} \\
\leq C^{1/2} \tau^{1/8} (E_x \left[ \tau^4(\tau \vee 1)^4 \right])^{1/8} \| f^4 \|^{1/4}_{L^p(1)} (P(\tau > t))^{1/2} \\
+ C^{1/2}(E[\tau^2])^{1/4} (P(\tau > t))^{1/4} \| f^2 \|^{1/2}_{L^p(1)} \\
\leq C^{1/2}(\tau^{1/8} + 1) (E_x \left[ \tau^4(\tau \vee 1)^4 \right])^{1/8} \| f^4 \|^{1/4}_{L^p(1)} (P(\tau > t))^{1/4}.
\]

Then, we obtain by (3.4) and (3.24) that
\[
\limsup_{x \to x} \left| E_x \left[ \int_0^t (e(s) - 1)f(X_s)ds; \tau > t \right] \right| = 0.
\]
Hence, to prove (3.22), it suffices to show that
\[
\limsup_{t \downarrow 0} \sup_{x \in D \setminus F} \left| E_x \left[ \int_0^t (e(s) - 1)f(X_s)ds; \tau \leq t \right] \right| = 0. \tag{3.26}
\]

For $x \in D \setminus F$ and $t > 0$, we obtain by (3.2) and (3.10) that
\[
\left| E_x \left[ \int_0^t (e(s) - 1)f(X_s)ds; \tau \leq t \right] \right| \\
\leq E_x \left[ \int_0^{t \wedge \tau} e(s)|f|(X_s)ds \right] + E_x \left[ \int_0^{t \wedge \tau} |f|(X_s)ds \right] \\
\leq \left( E_x \left[ \int_0^{t \wedge \tau} e^2(s)ds \right] \right)^{1/2} \left( E_x \left[ \int_0^{t \wedge \tau} f^2(X_s)ds \right] \right)^{1/2} + E_x \left[ \int_0^{t \wedge \tau} |f|(X_s)ds \right] \\
\leq t^{1/2}\tau^{1/8} \left( E_x \left[ \int_0^t f^2(X_s)ds \right] \right)^{1/2} + t^{1/2} \left( E_x \left[ \int_0^t f^2(X_s)ds \right] \right)^{1/2} \\
\leq C^{1/2}(\tau^{1/8} + 1) \| f^2 \|^{1/2}_{L^p(1)} t^{1/2}.
\]
Then (3.26) holds and therefore (3.22) holds.

3.3 Uniqueness of solution

In this subsection, we prove the uniqueness of solution. To this end, we will show that there exists $M > 0$ such that if $\|h\|_{L^p} \leq M$, then $v \equiv 0$ is the unique function in $B_b(\mathbb{R}^d)$ satisfying $v|_{D} \in W^{1,2}_{loc}(D) \cap C(D)$ and

$$
\begin{cases}
  (\Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla + c + \text{div} \hat{b})v = 0 & \text{in } D, \\
  v = 0 & \text{on } D^c.
\end{cases}
$$

(3.27)

Denote by $((X_t)_{t \geq 0}, (P_x^t)_{x \in \mathbb{R}^d})$ the Hunt process associated with $(E_t^0, W^{1,2}(\mathbb{R}^d))$. For $\phi \in W^{1,2}(\mathbb{R}^d)$, we obtain by [16, Theorem 1.4] that $\phi$ admits a unique Fukushima type decomposition w.r.t. $((X_t)_{t \geq 0}, (P_x^t)_{x \in \mathbb{R}^d})$:

$$
\tilde{\phi}(X_t) - \tilde{\phi}(X_0) = M_t^{\gamma, \phi} + N_t^{\gamma, \phi}, \quad t \geq 0,
$$

where $(M_t^{\gamma, \phi})_{t \geq 0}$ is a locally square integrable martingale additive functional on $I(\zeta)$ and $(N_t^{\gamma, \phi})_{t \geq 0}$ is a continuous additive functional locally of zero quadratic variation. Hereafter $\zeta$ denotes the lifetime of $((X_t)_{t \geq 0}, (P_x^t)_{x \in \mathbb{R}^d})$ with the totally inaccessible part $\zeta_i$ and $I(\zeta) = [[0, \zeta[[[\zeta]]]],$ which is a predictable set of interval type. Similar to [3, Lemma 2.2], we can show that for $\phi \in C_c^\infty(\mathbb{R}^d)$,

$$
\int_0^t \text{div} \phi(X_s)ds = N_t^{\gamma, \phi}, \quad t \geq 0.
$$

(3.28)

Suppose that $v \in B_b(\mathbb{R}^d)$ satisfying $v|_{D} \in W^{1,2}_{loc}(D) \cap C(D)$ and (3.27). Let $\{D_n\}_{n \in \mathbb{N}}$ be a sequence of relatively compact open subsets of $D$ such that $D_n \subset D_{n+1}$ and $D = \cup_{n=1}^\infty D_n$, and $\{\chi_n\}_{n \in \mathbb{N}}$ be a sequence of functions in $C_c^\infty(D)$ such that $0 \leq \chi_n \leq 1$ and $\chi_n|_{D_n} = 1$. We have $v\chi_n \in W^{1,2}_0(D)$. Note that

$$
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|(v(y) - v(x))(\chi_n(y) - \chi_n(x))|}{|x - y|^{d+\alpha}} dy dx < \infty.
$$

(3.29)

Let $\phi \in W^{1,2}_0(D)$. By (3.27) and (3.29), we get

$$
E_t^0(v\chi_n, \phi) = \int_{\mathbb{R}^d} \langle \nabla(v\chi_n), \nabla \phi \rangle dx - \int_{\mathbb{R}^d} \langle b, \nabla(v\chi_n) \rangle \phi dx + \frac{\alpha \mathcal{A}(d, -\alpha)}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v\chi_n)(x) - (v\chi_n)(y))(\phi(x) - \phi(y))}{|x - y|^{d+\alpha}} dx dy + \langle \gamma, v\chi_n \phi \rangle
$$

$$
= E_t^0(v, \chi_n \phi) - \int_{\mathbb{R}^d} (L\chi_n) v \phi dx - 2 \int_{\mathbb{R}^d} \langle \nabla v, \nabla\chi_n \rangle \phi dx - a^\alpha \mathcal{A}(d, -\alpha) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(y) - v(x))(\chi_n(y) - \chi_n(x))}{|x - y|^{d+\alpha}} dy \phi(x) dx + \langle \gamma, v\chi_n \phi \rangle
$$
\[
\begin{align*}
&= \left( (c + \gamma)v\chi_n - (L\chi_n)v - 2\langle \nabla v, \nabla \chi_n \rangle - a^\alpha A(d, -\alpha) \int_{\mathbb{R}^d} \frac{(v(y) - v(\cdot))(\chi_n(y) - \chi_n(\cdot))}{|\cdot - y|^{d+\alpha}} dy, \phi \right) \\
&\quad - \int_{\mathbb{R}^d} \langle \hat{b}, \nabla (v\chi_n) \rangle dx \\
&:= (\theta_n, \phi) - \int_{\mathbb{R}^d} \langle \hat{b}, \nabla (v\chi_n) \rangle dx. \quad (3.30)
\end{align*}
\]

Let \( n > m \) and \( \phi \in C_c^\infty(D_m) \). By (3.30), we get
\[
(\theta_n, \phi) - \int_{\mathbb{R}^d} \langle \hat{b}, \nabla (v\chi_n) \rangle dx \\
= \mathcal{E}^0(v, \phi) + a^\alpha A(d, -\alpha) \int_{D_m} \int_{D_n} \frac{v(y)(1 - \chi_n(y))}{|x - y|^{d+\alpha}} dy \phi(x) dx + (\gamma, v\phi) \\
= \left( (c + \gamma)v + a^\alpha A(d, -\alpha) \right) \int_{D_m \cap D_n} \frac{v(y)(1 - \chi_n(y))}{|x - y|^{d+\alpha}} dy, \phi \right) - \int_{\mathbb{R}^d} \langle \hat{b}, \nabla (v\phi) \rangle dx. \quad (3.31)
\]

Since \( \phi \in C_c^\infty(D_m) \) is arbitrary, by (3.31), we find that for \( n > m \),
\[
\theta_n(x) = (c(x) + \gamma)v(x) + a^\alpha A(d, -\alpha) \int_{D_m \cap D_n} \frac{v(y)(1 - \chi_n(y))}{|x - y|^{d+\alpha}} dy, \quad x \in D_m.
\]

Then,
\[
\theta_n \text{ converges to } (c + \gamma)v \text{ uniformly on any compact subset of } D \text{ as } n \to \infty. \quad (3.32)
\]

We fix a function \( \xi \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d; dx) \) with \( 0 < \xi \leq 1 \). Set \( \mathcal{G} = G_1^\gamma \xi \) and \( \hat{\mathcal{G}} = \hat{G}_1^\gamma \xi \), where \( (G_1^\gamma)_{\gamma \in (0, 1)} \) and \( (\hat{G}_1^\gamma)_{\gamma \in (0, 1)} \) denote the resolvent and co-resolvent associated with \( (\mathcal{E}_\gamma, W^{1,2}(\mathbb{R}^d)) \), respectively. Define
\[
\Theta := \{\{V_n\} | V_n \text{ is } \mathcal{E}^0_\gamma\text{-quasi-open}, V_n \subset V_{n+1} \text{ } \mathcal{E}^0_\gamma\text{-q.e.} \}
\forall \ n \in \mathbb{N}, \text{ and } E = \bigcup_{n=1}^{\infty} V_n \text{ } \mathcal{E}^0_\gamma\text{-q.e.}.\}
\]

By [14] [16], there exists a \( \{V_l\} \in \Theta \) such that \( \hat{\mathcal{G}}, \hat{\mathcal{G}}^H, \text{ and } \hat{\mathcal{G}}_k^H, \ k \in \mathbb{N}, \) are all bounded on each \( V_l \). By [13] proposition 2.18, we may assume without loss of generality that there exists a sequence \( \{\phi_n\} \subset C_c^\infty(\mathbb{R}^d) \) satisfying
\[
\phi_n \to \hat{b}^H \text{ in } W^{1,2}(\mathbb{R}^d) \text{ and } \phi_n \to \hat{\mathcal{G}}^H \text{ uniformly on each } V_l \text{ as } n \to \infty. \quad (3.33)
\]

Let \( V \) be a quasi-open set of \( \mathbb{R}^d \). We denote \( W^{1,2}_0(V) := \{ \phi \in W^{1,2}(\mathbb{R}^d) : \phi = 0 \text{ on } V^c \} \). For \( l \in \mathbb{N} \), we define \( E_l = \{ x \in \mathbb{R}^d : \hat{G}_1^\nu \xi(x) > 1/l \} \), where \( (\hat{G}_1^\nu)_{\nu > 0} \) is the co-resolvent associated with the part Dirichlet form \( (\mathcal{E}_\gamma, W^{1,2}_0(V)) \). Then \( \{ E_l \} \in \Theta \) satisfying \( \overline{E_l}^\mathcal{E}_\gamma \subset E_{l+1} \text{ q.e. and } E_l \subset V_l \text{ q.e. for each } l \in \mathbb{N} \). Here \( \overline{E_l}^\mathcal{E}_\gamma \) denotes the \( \mathcal{E}_\gamma \) quasi-closure of \( E_l \). Define \( \omega_l = lG_1^\nu \xi \land 1 \). Then
\[ \omega_t \in B_b(\mathbb{R}^d) \cap W^{1,2}(\mathbb{R}^d), \omega_1 = 1 \text{ on } E_t \text{ and } \omega_t = 0 \text{ on } V_t^c. \] For \( \phi, \psi \in W^{1,2}_{loc}(\mathbb{R}^d), \) we define the stochastic integral \( \int_0^t \phi(X_s) dA_s^\psi \) as in [4].

By [16] Theorems 1.4 and 1.15, \( (M_t^{\gamma, bH})_{t \geq 0} \) is a \( P_2 \)-square-integrable martingale for q.e. \( x \in \mathbb{R}^d \) and \( l \in \mathbb{N} . \) For \( \phi \in W^{1,2}(\mathbb{R}^d), \) we define

\[ A_t^\phi = \tilde{\phi}(X_t) - \tilde{\phi}(X_0), \quad t \geq 0. \]

For \( l \in \mathbb{N}, n > m, t \geq 0 \) and \( x \in E_l \cap D_m, \) we define

\[ B_{t}^{l,m,n} := \int_0^{t \wedge \tau_l \cap D_m} \theta_n(X_s) ds + \int_0^{t \wedge \tau_l \cap D_m} v(X_s) dA_k^H \]

and

\[ c_{t,s}^{l,m,n}(x) := E_x^{\gamma,E}[B_{t}^{l,m,n}]. \]

Denote by \( (p_t^{\gamma,E_l \cap D_m}(t,x,y))_{t \geq 0} \) and \( (p_t^{\gamma,E_l \cap D_m})(t,x,y) \) the transition density function and the transition semigroup of the part process \( (X_t^{E_l \cap D_m})_{t \geq 0}, (P_x^{\gamma})_{x \in E_l \cap D_m} \), respectively. By the continuity of the function \( t \mapsto p_t^{\gamma,E_l \cap D_m}(t,x,y) \), which can be proved similar to [20] Lemma 2.3(4), we know that the function \( t \mapsto c_{t,s}^{l,m,n}(x) \) is continuous for \( dx \)-a.e. \( x \in E_l \cap D_m. \) We have \( c_{t,s}^{l,m,n} \in L^2(E_l \cap D_m; dx) \) for \( t \geq 0 \) and

\[ c_{t+s}^{l,m,n}(x) = c_t^{l,m,n}(x) + p_t^{\gamma,E_l \cap D_m} c_{s}^{l,m,n}(x), \quad t, s \geq 0. \tag{3.34} \]

Denote by \( (\hat{T}_t^{\gamma,E_l \cap D_m})_{t \geq 0} \) and \( (\hat{G}_t^{\gamma,E_l \cap D_m})_{t \geq 0} \) the co-semigroup and the co-resolvent of \( (\mathcal{E}_0^{\gamma}, W_0^{1,2}(E_l \cap D_m)) \) on \( L^2(E_l \cap D_m; dx) \), respectively. Let \( \phi \in C_0^\infty(E_l \cap D_m) \). Since \( \hat{G} \) is bounded on \( V_t \), both \( \phi \) and \( \hat{G}_t^{\gamma,E_l \cap D_m} |\phi| \) are bounded on \( E_l \cap D_m \). Hence \( \hat{T}_t^{\gamma,E_l \cap D_m} \hat{G}_t^{\gamma,E_l \cap D_m} |\phi| \) is bounded on \( E_l \cap D_m \) and \( v \hat{T}_s^{\gamma,E_l \cap D_m} \hat{G}_t^{\gamma,E_l \cap D_m} |\phi| \in W^{1,2}(\mathbb{R}^d) \) for all \( s \geq 0 \). Further, we have

\[ \sup_{0 \leq s \leq t} \| \hat{T}_s^{\gamma,E_l \cap D_m} \hat{G}_t^{\gamma,E_l \cap D_m} |\phi| \|_\infty \]
\[ \leq e^t \| G_1^{\gamma,E_l \cap D_m} |\phi| \|_\infty < \infty, \tag{3.35} \]

\[ \lim_{t \downarrow 0} \sup_{0 \leq s \leq t} \| \hat{T}_s^{\gamma,E_l \cap D_m} \hat{G}_t^{\gamma,E_l \cap D_m} |\phi| - \hat{G}_1^{\gamma,E_l \cap D_m} |\phi| \|_\infty \]
\[ = \lim_{t \downarrow 0} \sup_{0 \leq s \leq t} \| e^{-s \hat{T}_s^{\gamma,E_l \cap D_m} G_1^{\gamma,E_l \cap D_m} |\phi| - \hat{G}_1^{\gamma,E_l \cap D_m} |\phi| \|_\infty \]
\[ = \lim_{t \downarrow 0} \| \| \int_0^s e^{-w \hat{T}_w^{\gamma,E_l \cap D_m} |\phi|} dw \| \|_\infty \]
\[ \leq \lim_{t \downarrow 0} t \| |\phi| \|_\infty \]
\[ = 0, \tag{3.36} \]

\[ \sup_{0 \leq s \leq t} \mathcal{E}_0^{\gamma+1}(\hat{T}_s^{\gamma,E_l \cap D_m} \hat{G}_t^{\gamma,E_l \cap D_m} |\phi|, \hat{T}_s^{\gamma,E_l \cap D_m} \hat{G}_t^{\gamma,E_l \cap D_m} |\phi|) \]
\[ = \sup_{0 \leq s \leq t} (\hat{T}_s^{\gamma,E_l \cap D_m} \hat{G}_t^{\gamma,E_l \cap D_m} |\phi|, \hat{T}_s^{\gamma,E_l \cap D_m} |\phi|) \]
\[ < \infty, \tag{3.37} \]
and

\[
\lim_{t \to 0} \sup_{0 \leq s \leq t} \mathcal{E}^0_{\gamma+1}(T_s^{\gamma}, E_l \cap D_m \phi) = \lim_{t \to 0} \sup_{0 \leq s \leq t} (T_s^{\gamma}, E_l \cap D_m \phi) = 0.
\] (3.38)

Denote \( L^\gamma := \Delta + a^\alpha \Delta^\alpha/2 + (b - \hat{b}) \cdot \nabla - \gamma \). By [5, Proposition 4.1.10] (note that the assertion holds true also in the setting of semi-Dirichlet forms), [11] (22) and Theorem 2.8 with its proof, [33], [35], [37], [15] I. Corollary 4.15 and the equivalence of the \( \mathcal{E}^0_\gamma \)-norm and the \( W^{1,2} \)-norm, we get

\[
\mathcal{E}^\gamma_{G_1, E_l \cap D_m} \left( \int_0^{t \wedge T_{E_l \cap D_m}} v(X_{s-}) dA_s^H \right) = \lim_{n \to \infty} \mathcal{E}^\gamma_{G_1, E_l \cap D_m} \left( \int_0^{t \wedge T_{E_l \cap D_m}} v(X_{s-}) dA_s^{\phi_n} \right) = \lim_{n \to \infty} \int_0^t \mathcal{E}^0_{\gamma} (\dot{\theta}_n, v T_s^{\gamma}, E_l \cap D_m \hat{G}_1^{\gamma}, E_l \cap D_m \phi) ds = \int_0^t \mathcal{E}^0_{\gamma} (b^H, v \hat{T}_s^{\gamma}, E_l \cap D_m \hat{G}_1^{\gamma}, E_l \cap D_m \phi) ds.
\] (3.39)

Then, we obtain by (1.2), (3.30), (3.36), (3.38), (3.39), [15] I. Corollary 4.15 and the equivalence of the \( \mathcal{E}^0_\gamma \)-norm and the \( W^{1,2} \)-norm that

\[
\lim_{t \to 0} \frac{1}{t} \mathcal{E}^\gamma_{G_1, E_l \cap D_m} \left( B^l_{t,m,n} \right) = \lim_{t \to 0} \frac{1}{t} \int_0^t (p_s^{\gamma, E_l \cap D_m} \theta_n, \hat{G}_1^{\gamma}, E_l \cap D_m \phi) ds + \lim_{t \to 0} \frac{1}{t} \mathcal{E}^\gamma_{G_1, E_l \cap D_m} \left( \int_0^{t \wedge T_{E_l \cap D_m}} v(X_{s-}) dA_s^{\phi_n} \right) = (\theta_n, \hat{G}_1^{\gamma}, E_l \cap D_m \phi) - \mathcal{E}^0_{\gamma} (b^H, v \hat{G}_1^{\gamma}, E_l \cap D_m \phi) = (\theta_n, \hat{G}_1^{\gamma}, E_l \cap D_m \phi) - \int_{\mathbb{R}^d} \langle \dot{b}, \nabla (v \hat{G}_1^{\gamma}, E_l \cap D_m \phi) \rangle dx = \mathcal{E}^0_{\gamma} (v \chi_n, \hat{G}_1^{\gamma}, E_l \cap D_m \phi).
\] (3.40)

Define

\[ \eta_{l,m,n}(x) = E^\gamma_x [v \chi_n (X_{t \wedge E_l \cap D_m})], \quad x \in \mathbb{R}^d. \]

We have

\[ \eta_{l,m,n}(x) = E^\gamma_x [\eta_{l,m,n} (X_{t \wedge E_l \cap D_m})], \quad t \geq 0, \quad x \in E_l \cap D_m, \]

and \( \eta_{l,m,n}(x) = v \chi_n(x) \) for q.e. \( x \in (E_l \cap D_m)^c \). By [18] Theorem 3.5.1, we get

\[ \mathcal{E}^0_{\gamma} (v \chi_n, \hat{G}_1^{\gamma}, E_l \cap D_m \phi) = \mathcal{E}^0_{\gamma} (v \chi_n - \eta_{l,m,n}, \hat{G}_1^{\gamma}, E_l \cap D_m \phi), \quad \forall \phi \in \hat{G}_1^{\gamma}, E_l \cap D_m (B_b(E_l \cap D_m)) \] (3.42)
Define
\[ \hat{S}_{t,m}^l := \int_0^t \hat{T}_{s}^{\gamma,E_{1} \cap D_m} d s, \quad t \geq 0. \] 

(3.43)

Similar to [9] (1.5.5), page 39, we can show that
\[ \mathcal{E}_0^t(\varpi, \hat{S}_{t,m}^l) = (\varpi, \rho - \hat{T}_{t}^{\beta,D_m} \rho), \quad \forall \varpi \in W^{1,2}_0(E_{t} \cap D_m), \quad \rho \in L^2(E_{t} \cap D_m; d x). \] 

(3.44)

Then, we obtain by (3.31), (3.40) and (3.42) (3.44) that for \( \phi \in \hat{G}_{1}^{\gamma,E_{1} \cap D_m}(B_{b}(E_{t} \cap D_m)) \) and \( t, r > 0 \),
\[
\begin{align*}
&\left( c_{t,m,n}^{l}, \hat{G}_{1}^{\gamma,E_{1} \cap D_m} \phi - \hat{T}_{t}^{\gamma,E_{1} \cap D_m} \hat{G}_{1}^{\gamma,E_{1} \cap D_m} \phi \right) \\
&= \lim_{s \to 0} \frac{1}{s} \left( c_{t,m,n}^{l}, \hat{S}_{t,m}^{l} \hat{G}_{1}^{\gamma,E_{1} \cap D_m} \phi - \hat{T}_{t}^{\gamma,E_{1} \cap D_m} \hat{S}_{t,m}^{l} \hat{G}_{1}^{\gamma,E_{1} \cap D_m} \phi \right) \\
&= \lim_{s \to 0} \frac{1}{s} \left( c_{s,m,n}^{l}, \hat{S}_{s,m}^{l} \hat{G}_{1}^{\gamma,E_{1} \cap D_m} \phi - \hat{T}_{s}^{\gamma,E_{1} \cap D_m} \hat{S}_{s,m}^{l} \hat{G}_{1}^{\gamma,E_{1} \cap D_m} \phi \right) \\
&= \mathcal{E}_0^{t}(v_{x_{n}}, \hat{S}_{t,m}^{l} \hat{G}_{1}^{\gamma,E_{1} \cap D_m} \phi - \hat{T}_{t}^{\gamma,E_{1} \cap D_m} \hat{S}_{t,m}^{l} \hat{G}_{1}^{\gamma,E_{1} \cap D_m} \phi) \\
&= \mathcal{E}_0^{t}(v_{x_{n}} - \eta_{l,m,n}, \hat{S}_{t,m}^{l} \hat{G}_{1}^{\gamma,E_{1} \cap D_m} \phi - \hat{T}_{t}^{\gamma,E_{1} \cap D_m} \hat{S}_{t,m}^{l} \hat{G}_{1}^{\gamma,E_{1} \cap D_m} \phi) \\
&= \left( v_{x_{n}} - \eta_{l,m,n}, \hat{G}_{1}^{\gamma,E_{1} \cap D_m} \phi - \hat{T}_{t}^{\gamma,E_{1} \cap D_m} \hat{G}_{1}^{\gamma,E_{1} \cap D_m} \phi \\
&- \hat{T}_{t}^{\gamma,E_{1} \cap D_m} \hat{G}_{1}^{\gamma,E_{1} \cap D_m} \phi + \hat{T}_{t+r}^{\gamma,E_{1} \cap D_m} \hat{G}_{1}^{\gamma,E_{1} \cap D_m} \phi \right) \\
&= \left( v_{x_{n}} - \eta_{l,m,n} - p_{t}^{\gamma,E_{1} \cap D_m} (v_{x_{n}} - \eta_{l,m,n}), \hat{G}_{1}^{\gamma,E_{1} \cap D_m} \phi - \hat{T}_{t}^{\gamma,E_{1} \cap D_m} \hat{G}_{1}^{\gamma,E_{1} \cap D_m} \phi \right).
\end{align*}
\]

Hence \( \kappa_{t}^{l,m,n} := \left( c_{t,m,n}^{l} - (v_{x_{n}} - \eta_{l,m,n}) \right) + p_{t}^{E_{1},E_{1} \cap D_m} (v_{x_{n}} - \eta_{l,m,n}), \hat{G}_{1}^{\gamma,E_{1} \cap D_m} \phi \) satisfies the linear equation \( \kappa_{t}^{l,m,n} = \kappa_{t+r}^{l,m,n} - \kappa_{r}^{l,m,n} \). By (3.40) and (3.42), we get \( \lim_{t \to 0} \kappa_{t}^{l,m,n}/t = 0 \). Then, \( \kappa_{t}^{l,m,n} = 0 \). Since \( \phi \in \hat{G}_{1}^{\gamma,E_{1} \cap D_m}(B_{b}(E_{t} \cap D_m)) \) is arbitrary, we obtain by the continuity of the function \( t \mapsto p_{t}^{E_{1},E_{1} \cap D_m}(t, x, y) \) and the continuity of the function \( t \mapsto c_{t}^{l,m,n}(x) \) that for \( dx \)-a.e. \( x \in E_{t} \cap D_m \),
\[
\begin{align*}
(v_{x_{n}} - \eta_{l,m,n})(x) &= E_{x}^{\gamma}[v_{x_{n}} - \eta_{l,m,n}(X_{t \wedge \gamma,E_{1} \cap D_m})] + E_{x}^{\gamma} \int_{0}^{t \wedge \gamma,E_{1} \cap D_m} \theta_{n}(X_{s}) d s \\
&+ E_{x}^{\gamma} \int_{0}^{t \wedge \gamma,E_{1} \cap D_m} v(X_{s-}) d A_{s}^{1,H} \\
&= E_{x}^{\gamma}[v_{x_{n}} - \eta_{l,m,n}(X_{t \wedge \gamma,E_{1} \cap D_m})] + E_{x}^{\gamma} \int_{0}^{t \wedge \gamma,E_{1} \cap D_m} \theta_{n}(X_{s}) d s \\
&+ E_{x}^{\gamma} \int_{0}^{t \wedge \gamma,E_{1} \cap D_m} v(X_{s-}) d N_{s}^{1,H}, \quad \forall t \geq 0.
\end{align*}
\]

By (3.41), we obtain that for \( dx \)-a.e. \( x \in E_{t} \cap D_m \),
\[
\begin{align*}
(v_{x_{n}})(x) &= E_{x}^{\gamma}[v_{x_{n}}(X_{t \wedge \gamma,E_{1} \cap D_m})] + E_{x}^{\gamma} \int_{0}^{t \wedge \gamma,E_{1} \cap D_m} \theta_{n}(X_{s}) d s \\
&+ E_{x}^{\gamma} \int_{0}^{t \wedge \gamma,E_{1} \cap D_m} v(X_{s-}) d N_{s}^{1,H}, \quad \forall t \geq 0.
\end{align*}
\]

(3.45)

Note that \( v \in B_{b}(\mathbb{R}^{d}) \) and \( v = 0 \) on \( D^{c} \). Letting \( n \to \infty \), we obtain by (3.32) and (3.45) that for \( dx \)-a.e. \( x \in E_{t} \cap D_m \),
\[
v(x) = E_{x}^{\gamma}[v(X_{t \wedge \gamma,E_{1} \cap D_m})] + E_{x}^{\gamma} \int_{0}^{t \wedge \gamma,E_{1} \cap D_m} ((c + \gamma)v)(X_{s}) d s
\]

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Letting $m \to \infty$, we obtain by (3.46) that for $dx$-a.e. $x \in E_l \cap D$,

$$v(x) = E_x^\gamma [v(X_t 1_{\tau_{E_l \cap D} > t})] + E_x^\gamma \left[ \int_0^{\tau_{E_l \cap D}} ((c + \gamma)v)(X_s) ds \right] + E_x^\gamma \left[ \int_0^{\tau_{E_l \cap D}} v(X_s) dN_s^{\gamma, b^H} \right], \quad \forall t \geq 0. \tag{3.47}$$

Define

$$\mathcal{I}_t^l = v(X_t) 1_{\{\tau_{E_l \cap D} > t\}} + \int_0^{\tau_{E_l \cap D}} ((c + \gamma)v)(X_s) ds + \int_0^{\tau_{E_l \cap D}} v(X_s) dN_s^{\gamma, b^H}. \tag{3.48}$$

By (3.47), we find that $(\mathcal{I}_t^l)_{t \geq 0}$ is a locally bounded martingale under $P_x^\gamma$ for $dx$-a.e. $x \in E_l \cap D$. For $k \in \mathbb{N}$ and $t \geq 0$, define

$$e^k_\gamma(t) := e^l_{0, (c_k + \text{div} \hat{b}_k + \gamma)(X_s)} ds.$$

The integration by parts formula for semi-martingales implies that

$$e^k_\gamma(t) \mathcal{I}_t^l - v(x) = \int_0^t \mathcal{I}_s^l e^k_\gamma(s) + \int_0^t e^k_\gamma(s) d\mathcal{I}_s^l.$$

By (3.28), we have

$$N_t^{\gamma, \phi_n - \hat{b}^H_k} = \int_0^t (L^\gamma \phi_n - \text{div} \hat{b}_k)(X_s) ds, \quad t \geq 0. \tag{3.49}$$

By (3.33), (3.48), (3.49), [14, Theorem 2.8 and (22)] and [16, (11) and (12)], we obtain that for $dx$-a.e. $x \in E_l \cap D$ and $t \geq 0$,

\begin{align*}
v(x) & + \int_0^t e^k_\gamma(s) d\mathcal{I}_s^l \\
& = e^k_\gamma(t) \mathcal{I}_t^l - \int_0^t \mathcal{I}_s^l e^k_\gamma(s) \\
& = e^k_\gamma(t) v(X_t) 1_{\{\tau_{E_l \cap D} > t\}} + e^k_\gamma(t) \int_0^{\tau_{E_l \cap D}} ((c + \gamma)v)(X_s) ds \\
& \quad + e^k_\gamma(t) \left( \lim_{n \to \infty} \int_0^{\tau_{E_l \cap D}} v(X_{s-}) dN_s^{\gamma, \phi_n} \right) \\
& - \int_0^t v(X_s) 1_{\{\tau_{E_l \cap D} > s\}} e^k_\gamma(s) - \int_0^t \int_0^{s \wedge \tau_{E_l \cap D}} ((c + \gamma)v)(X_w) dw e^k_\gamma(s) \\
& - \lim_{n \to \infty} \int_0^t \int_0^{s \wedge \tau_{E_l \cap D}} v(X_{w-}) dN_w^{\gamma, \phi_n} e^k_\gamma(s) \\
& = e^k_\gamma(t) v(X_t) 1_{\{\tau_{E_l \cap D} > t\}} + e^k_\gamma(t) \int_0^{\tau_{E_l \cap D}} ((c + \gamma)v)(X_s) ds
\end{align*}
\[+ e_{\gamma}^k(t) \left( \lim_{n \to \infty} \int_0^{\mathcal{T}_{E_{(\cap D)}}} v(X_{s-}) dN_{s, \phi_n}^{\gamma} \right)\]

\[- \int_0^t e_{\gamma}^k(s) ((c_k + \text{div } \hat{b}_k + \gamma)v)(X_s) 1_{\{\mathcal{T}_{E_{(\cap D)}} > s\}} ds \]

\[- \int_0^t ((c + \gamma)v)(X_w) 1_{\{\mathcal{T}_{E_{(\cap D)}} \geq w\}} \left( \int_0^t e_{\gamma}^k(s) (c_k + \text{div } \hat{b}_k + \gamma)(X_s) ds \right) dw \]

\[- \lim_{n \to \infty} \int_0^t v(X_{w-}) 1_{\{\mathcal{T}_{E_{(\cap D)}} \geq w\}} \left( \int_0^t e_{\gamma}^k(s) (c_k + \text{div } \hat{b}_k + \gamma)(X_s) ds \right) dN_{w, \phi_n}^{\gamma} \]

\[= e_{\gamma}^k(t) v(X_t) 1_{\{\mathcal{T}_{E_{(\cap D)}} > t\}} + \int_0^t e_{\gamma}^k(s) ((c - c_k - \text{div } \hat{b}_k)v)(X_s) 1_{\{\mathcal{T}_{E_{(\cap D)}} > s\}} ds \]

\[+ \lim_{n \to \infty} \int_0^t e_{\gamma}^k(s) v(X_{s-}) 1_{\{\mathcal{T}_{E_{(\cap D)}} > s\}} dN_{s, \phi_n}^{\gamma} \]

\[= e_{\gamma}^k(t) v(X_t) 1_{\{\mathcal{T}_{E_{(\cap D)}} > t\}} + \int_0^t e_{\gamma}^k(s) ((c - c_k)v)(X_s) 1_{\{\mathcal{T}_{E_{(\cap D)}} > s\}} ds \]

\[+ \lim_{n \to \infty} \int_0^t e_{\gamma}^k(s) v(X_{s-}) 1_{\{\mathcal{T}_{E_{(\cap D)}} > s\}} dN_{s, \phi_n - \hat{b}_k^H}^{\gamma} \]

\[= e_{\gamma}^k(t) v(X_t) 1_{\{\mathcal{T}_{E_{(\cap D)}} > t\}} + \int_0^{\mathcal{T}_{E_{(\cap D)}}} e_{\gamma}^k(s) ((c - c_k)v)(X_s) ds \]

\[+ \lim_{n \to \infty} e_{\gamma}^k(t \land \mathcal{T}_{E_{(\cap D)}}) \int_0^{\mathcal{T}_{E_{(\cap D)}}} v(X_{s-}) dN_{s, \phi_n - \hat{b}_k^H}^{\gamma} \]

\[= e_{\gamma}^k(t) v(X_t) 1_{\{\mathcal{T}_{E_{(\cap D)}} > t\}} + \int_0^{\mathcal{T}_{E_{(\cap D)}}} e_{\gamma}^k(s) ((c - c_k)v)(X_s) \]

\[+ \lim_{n \to \infty} e_{\gamma}^k(t \land \mathcal{T}_{E_{(\cap D)}}) \int_0^{\mathcal{T}_{E_{(\cap D)}}} v(X_{s-}) dN_{s, \phi_n - \hat{b}_k^H}^{\gamma} \]

\[= e_{\gamma}^k(t) v(X_t) 1_{\{\mathcal{T}_{E_{(\cap D)}} > t\}} + \int_0^{\mathcal{T}_{E_{(\cap D)}}} e_{\gamma}^k(s) ((c - c_k)v)(X_s) \]

\[+ \lim_{n \to \infty} e_{\gamma}^k(t \land \mathcal{T}_{E_{(\cap D)}}) \int_0^{\mathcal{T}_{E_{(\cap D)}}} v(X_{s-}) dN_{s, \phi_n - \hat{b}_k^H}^{\gamma} \]

\[= e_{\gamma}^k(t) v(X_t) 1_{\{\mathcal{T}_{E_{(\cap D)}} > t\}} + \int_0^{\mathcal{T}_{E_{(\cap D)}}} e_{\gamma}^k(s) ((c - c_k)v)(X_s) \]

\[+ \lim_{n \to \infty} e_{\gamma}^k(t \land \mathcal{T}_{E_{(\cap D)}}) \int_0^{\mathcal{T}_{E_{(\cap D)}}} v(X_{s-}) dN_{s, \phi_n - \hat{b}_k^H}^{\gamma} \]

\[+ \int_0^{\mathcal{T}_{E_{(\cap D)}}} \left( \int_0^s v(X_{w-}) dN_{w, \phi_n - \hat{b}_k^H}^{\gamma} \right) e_{\gamma}^k(s) (h_k - c_k - \text{div } \hat{b}_k)(X_s) ds \]

\[- \lim_{n \to \infty} \int_0^{\mathcal{T}_{E_{(\cap D)}}} \left( \int_0^s v(X_{w-}) dN_{w, \phi_n - \hat{b}_k^H}^{\gamma} \right) e_{\gamma}^k(s) (h_k + \gamma)(X_s) ds \]

\[= e_{\gamma}^k(t) v(X_t) 1_{\{\mathcal{T}_{E_{(\cap D)}} > t\}} + \int_0^{\mathcal{T}_{E_{(\cap D)}}} e_{\gamma}^k(s) ((c - c_k)v)(X_s) \]

\[+ \lim_{n \to \infty} e_{\gamma}^k(t \land \mathcal{T}_{E_{(\cap D)}}) \int_0^{\mathcal{T}_{E_{(\cap D)}}} v(X_{s-}) dN_{s, \phi_n - \hat{b}_k^H}^{\gamma} \]

\[+ \int_0^{\mathcal{T}_{E_{(\cap D)}}} \left( \int_0^s v(X_{w-}) dN_{w, \phi_n - \hat{b}_k^H}^{\gamma} \right) e_{\gamma}^k(s) (h_k - c_k - \text{div } \hat{b}_k)(X_s) ds \]

\[- \lim_{n \to \infty} \int_0^{\mathcal{T}_{E_{(\cap D)}}} \left( \int_0^s v(X_{w-}) dN_{w, \phi_n - \hat{b}_k^H}^{\gamma} \right) e_{\gamma}^k(s) (h_k + \gamma)(X_s) ds. \] (3.50)
Note that
\[
\begin{align*}
&\left|\int_0^{t \wedge \tau_{E_l \cap D}} \left( \int_0^s v(X_{w-})dN_w^{\gamma,\partial H} \right) e^k_s(h_k - c_k - \text{div} \tilde{b}_k)(X_s)ds \right| \\
&\quad + \left|\int_0^{t \wedge \tau_{E_l \cap D}} \left( \int_0^s v(X_{w-})dN_w^{\gamma,\partial H} \right) e^k_s(h_k + \gamma)(X_s)ds \right| \\
&\quad \leq \left( \sup_{0 \leq s \leq t \wedge \tau_{E_l \cap D}} \left| \int_0^s v(X_{w-})dN_w^{\gamma,\partial H} \right| \right) e^{t \wedge \tau_{E_l \cap D}}(h_k + \gamma)(X_s)ds \\
&\quad \cdot \left( \int_0^{t \wedge \tau_{E_l \cap D}} (2h_k - c_k + \gamma)(X_s)ds - N_t^{\gamma,\partial H} \right). \tag{3.51}
\end{align*}
\]

Define
\[
e_\gamma(t) := e^{\int_0^t (c + \gamma)(X_s)ds + N_t^{\gamma,\partial H} ds}, \quad t \geq 0,
\]
and
\[
\mathcal{J}_t^l := e_\gamma(t)v(X_t)1_{\{\tau_{E_l \cap D} > t\}}.
\]

By [16, Theorem 1.15] and [13, Lemma A.6, Theorem A.8 and Lemma A.9], similar to [9, Corollary 5.2.1] we can show that there exists a subsequence \(\{k_l\}\) such that for q.e. \(x \in \mathbb{R}^d\),
\[
P_x \left\{ \lim_{l \to \infty} N_t^{\gamma,\partial H} = N_t^{\gamma,\partial H} \text{ uniformly on any finite interval of } t \right\} = 1. \tag{3.52}
\]

For simplicity, we still use \(\{k\}\) to denote the subsequence \(\{k_l\}\). Letting \(k \to \infty\), we obtain by (3.2), (3.6)–(3.8), (3.50)–(3.52) and [4, Theorem 2.8 and (22)] that there exists a sequence of stopping times \(\omega_n \uparrow \infty\) such that \((\mathcal{J}_t^l \cap \omega_n)_{t \geq 0}\) is a martingale under \(P_x^\gamma\) for \(dx\)-a.e. \(x \in E_l \cap D\) and \(n \in \mathbb{N}\). Then, for \(dx\)-a.e. \(x \in E_l \cap D\) and \(n \in \mathbb{N}\), we have
\[
v(x) = E_x^\gamma[e_\gamma(t \wedge \omega_n)v(X_{t \wedge \omega_n})1_{\{\tau_{E_l \cap D} > t \wedge \omega_n\}}]. \tag{3.53}
\]

Letting \(t, n \to \infty\), we obtain by (3.13), (3.54) and [20, Lemma 2.3(1)] that \(v(x) = 0\) for \(dx\)-a.e. \(x \in E_l \cap D\). Since \(l \in \mathbb{N}\) is arbitrary and \(v|D \in C(D)\), we obtain that \(v \equiv 0\) on \(\mathbb{R}^d\). The proof is complete. \(\Box\)

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