Gromov Hyperbolicity, Geodesic Defect, and Apparent Pairs in Vietoris–Rips Filtrations

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Abstract
Motivated by computational aspects of persistent homology for Vietoris–Rips filtrations, we generalize a result of Eliyahu Rips on the contractibility of Vietoris–Rips complexes of geodesic spaces for a suitable parameter depending on the hyperbolicity of the space. We consider the notion of geodesic defect to extend this result to general metric spaces in a way that is also compatible with the filtration. We further show that for finite tree metrics the Vietoris–Rips complexes collapse to their corresponding subforests. We relate our result to modern computational methods by showing that these collapses are induced by the apparent pairs gradient, which is used as an algorithmic optimization in Ripser, explaining its particularly strong performance on tree-like metric data.

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1 Introduction
The Vietoris–Rips complex is a fundamental construction in algebraic, geometric, and applied topology. For a metric space $X$ and a threshold $t > 0$, it is defined as the simplicial complex consisting of nonempty and finite subsets of $X$ with diameter at most $t$:

$$\text{Rips}_t(X) = \{ \emptyset \neq S \subseteq X \mid S \text{ finite, } \text{diam } S \leq t \}.$$  

First introduced by Vietoris [30] in order to make homology applicable to general compact metric spaces, it has also found important applications in geometric group theory [18] and topological data analysis [29]. The role of the threshold in these three application areas is notably different. The homology theory defined by Vietoris arises in the limit $t \to 0$. In contrast, the key applications in geometric group theory rely on the fact that the Vietoris–Rips complex of a hyperbolic geodesic space is contractible for a sufficiently large threshold. This observation, originally due to Rips and first published in Gromov’s seminal paper on hyperbolic groups [18], is a fundamental result about the topology of Vietoris–Rips complexes and plays a central role in the theory of hyperbolic groups.

Lemma 1 (Contractibility Lemma; Rips, Gromov [18]). Let $X$ be a $\delta$-hyperbolic geodesic metric space. Then the complex $\text{Rips}_t(X)$ is contractible for every $t > 0$ with $t \geq 4\delta$. 
Here, a metric space \((X, d)\) is called \textit{geodesic} if for any two points \(x, y \in X\) there exists an isometric map \([0, d(x, y)] \to X\) such that \(0 \to x\) and \(d(x, y) \to y\), and it is called \(\delta\)-\textit{hyperbolic} (in the sense of Gromov [13]) for \(\delta \geq 0\) if for any four points \(w, x, y, z \in X\) we have
\[
d(w, x) + d(y, z) \leq \max\{d(w, y) + d(x, z), d(w, z) + d(x, y)\} + 2\delta.
\] (1)

Finally, in applications of Vietoris–Rips complexes to topological data analysis, one is typically interested in the \textit{persistent homology} of the entire filtration of complexes for all possible thresholds. A notable difference to the classical applications is that the metric spaces under consideration are typically finite, and in particular not geodesic. This motivates the interest in a meaningful generalization of the Contractibility Lemma to finite metric spaces. Based on the notion of a discretely \textit{geodesic} space defined by Lang [24], which is the natural setting for hyperbolic groups, and motivated by techniques used in that paper, we consider the following quantitative geometric property (called \(\nu\)-\textit{almost geodesic} in [10, p. 271]).

\begin{definition}
A metric space \(X\) is \(\nu\)-geodesic if for all \(x, y \in X\) and \(r, s \geq 0\) with \(r + s = d(x, y)\) there exists a point \(z \in X\) with \(d(x, z) \leq r + \nu\) and \(d(y, z) \leq s + \nu\). The \textit{geodesic defect} of \(X\), denoted by \(\nu(X)\), is the infimum over all \(\nu\) such that \(X\) is \(\nu\)-geodesic.
\end{definition}

Our first main result is a generalization of the Contractibility Lemma that also applies to non-geodesic spaces using the notion of geodesic defect, and further produces collapses that are compatible with the Vietoris–Rips filtration above the collapsibility threshold.

\begin{theorem}
Let \(X\) be a finite \(\delta\)-hyperbolic \(\nu\)-geodesic metric space. Then there exists a discrete gradient that induces, for every \(u > t \geq 4\delta + 2\nu\), a sequence of collapses
\[
\text{Rips}_u(X) \searrow \text{Rips}_t(X) \searrow \{\}
\]
\end{theorem}

\begin{example}
An important special case is given by a finite \textit{tree metric space} \((V, d)\), where \(V\) is the vertex set of a positively weighted tree \(T = (V, E)\), and where the edge weights are taken as lengths and \(d\) is the associated path length metric, i.e., for two points \(x, y \in V\) their distance is the infimum total weight of any path starting in \(x\) and ending in \(y\). The geodesic defect is \(\nu(V) = \frac{1}{2}\max_{e \in E} l(e)\), where \(l(e)\) is the length of the edge \(e\). Moreover, \((V, d)\) is 0-hyperbolic (see [13] Theorems 3.38 and 3.40) for a characterization of 0-hyperbolic spaces).

This example is of particular relevance in the context of evolutionary biology, where persistent Vietoris–Rips homology has been successfully applied to identify recombinations and recurrent mutations [12, 20, 8]. The metrics arising as genetic distances of aligned RNA or DNA sequences are typically very similar to trees, capturing the phylogeny of the evolution. This motivates our interest in the particular case of tree metrics. These metric spaces are known to have acyclic Vietoris–Rips homology in degree \(>0\), and so any homology is an indication of some evolutionarily relevant phenomenon.

Our second main result is a strengthened version of Theorem 3 for the special case of tree metric spaces that connects the collapses of the Vietoris–Rips filtration to the construction of \textit{apparent pairs}, which play an important role as a computational shortcut in the software Ripser [3]. This result depends on a particular ordering of the vertices: we say that a total order of \(V\) is \textit{compatible} with the tree \(T\) if it extends the unique tree partial order resulting from choosing some arbitrary root vertex as the minimal element.

\begin{theorem}
Let \(V\) be a finite tree metric space for a weighted tree \(T = (V, E)\), whose vertices are totally ordered in a compatible way. Then the \textit{apparent pairs} gradient for the lexicographically refined Vietoris–Rips filtration induces a sequence of collapses
\[
\text{Rips}_u(V) \searrow \text{Rips}_t(V) \searrow T
\]
\end{theorem}
for every \( u > t > 0 \) such that no edge \( e \in E \) has length \( l(e) \in (t, u] \), where \( T_t \) is the subforest with vertices \( V \) and all edges of \( E \) with length at most \( t \). In particular, the persistent homology of the Vietoris–Rips filtration is trivial in degree \( > 0 \).

In the special case of trees with unit edge length, the proofs in [11 Proposition 2.2] and [2 Proposition 3] are similar in spirit to our proof of Theorem 3.4 which is based on discrete Morse theory. Related results about implications of the geometry of a metric space on the homotopy types of the associated Vietoris–Rips complexes can be found in [3, 4, 25].

> Remark 6. Given a vertex order \( \leq \), the lexicographic order on simplices for the reverse vertex order \( \geq \) coincides with the reverse colexicographic order for the original order \( \leq \), which is used for computations in Ripser. As a consequence, when the input is a tree metric vertex order, as an approximation of the reverse tree order for the phylogenetic tree, lead to a huge performance improvement, bringing down the computation time for the persistence barcode from a full day to about 2 minutes.

2 Preliminaries

2.1 Discrete Morse theory and the apparent pairs gradient

A simplicial complex \( K \) on a vertex set \( \text{Vert} \ K \) is a collection of nonempty finite subsets of \( \text{Vert} \ K \) such that for any set \( \sigma \in K \) and any nonempty subset \( \rho \subseteq \sigma \) one has \( \rho \in K \). A set \( \sigma \in K \) is called a simplex, and \( \dim \sigma = \text{card} \sigma - 1 \) is its dimension. Moreover, \( \rho \) is said to be a face of \( \sigma \) and \( \sigma \) a coface of \( \rho \). If \( \dim \rho = \dim \sigma - 1 \), then we call \( \rho \) a facet of \( \sigma \) and \( \sigma \) a cofacet of \( \rho \). The star of \( \sigma \), \( \text{St} \sigma \), is the set of cofaces of \( \sigma \) in \( K \), and the closure of \( \sigma \), \( \text{Cl} \sigma \), is the set of its faces. For a subset \( E \subseteq K \), we write \( \text{St} E = \bigcup_{e \in E} \text{St} e \).

Generalizing the ideas of Forman [16], a function \( f : K \to \mathbb{R} \) is a discrete Morse function if \( f \) is monotonic, i.e., for any \( \sigma, \tau \in K \) with \( \sigma \subseteq \tau \) we have \( f(\sigma) \leq f(\tau) \), and there exists a partition of \( K \) into intervals \( [\rho, \phi] = \{ \psi \in K \mid \rho \subseteq \psi \subseteq \phi \} \) in the face poset such that \( f(\sigma) = f(\tau) \) for any \( \sigma \subseteq \tau \) if and only if \( \sigma \) and \( \tau \) belong to a common interval in the partition. The collection of regular intervals, \( [\rho, \phi] \) with \( \rho \neq \phi \), is called the discrete gradient of \( f \), and any singleton interval \( [\sigma, \sigma] \), as well as the corresponding simplex \( \sigma \), is called critical.

> Proposition 7 (Hersh [20 Lemma 4.1]; Jonsson [22 Lemma 4.2]). Let \( K \) be a finite simplicial complex, and \( \{ K_{\alpha} \}_{\alpha \in A} \) a set of subcomplexes covering \( K \), each equipped with a discrete gradient \( V_{\alpha} \), such that for any simplex of \( K \):

- there is a unique minimal subcomplex \( K_{\alpha} \) containing that simplex, and
- the simplex is critical for the discrete gradients of all other such subcomplexes.

Then the regular intervals in the \( V_{\alpha} \) are disjoint, and their union is a discrete gradient on \( K \).

An elementary collapse \( K \searrow K \setminus \{ \sigma, \tau \} \) is the removal of a pair of simplices, where \( \sigma \) is a facet of \( \tau \), with \( \tau \) the unique proper coface of \( \sigma \). A collapse \( K \searrow L \) onto a subcomplex \( L \) is a sequence of elementary collapses starting in \( K \) and ending in \( L \). An elementary collapse can
be realized continuously by a strong deformation retraction and therefore collapses preserve the homotopy type. A discrete gradient can encode a collapse.

**Proposition 8** (Forman [16]; see also [23, Theorem 10.9]). Let $K$ be a finite simplicial complex and let $L \subseteq K$ be a subcomplex. Assume that $V$ is a discrete gradient on $K$ such that the complement $K \setminus L$ is the union of intervals in $V$. Then there exists a collapse $K \searrow L$.

Let $f: K \to \mathbb{R}$ be a monotonic function. Assume that the vertices of $K$ are totally ordered. The $f$-lexicographic order is the total order $\leq_f$ on $K$ given by ordering the simplices
- by their value under $f$,
- then by dimension,
- then by the lexicographic order induced by the total vertex order.

We call a pair $(\sigma, \tau)$ of simplices in $K$ a zero persistence pair if $f(\sigma) = f(\tau)$. An apparent pair $(\sigma, \tau)$ with respect to the $f$-lexicographic order is a pair of simplices in $K$ such that $\sigma$ is the maximal facet of $\tau$, and $\tau$ is the minimal cofacet of $\sigma$. The collection of apparent pairs forms a discrete gradient [6, Lemma 3.5], called the apparent pairs gradient.

Assume that $K$ is finite and $f: K \to \mathbb{R}$ a discrete Morse function with discrete gradient $V$. Refine $V$ to another discrete gradient $\tilde{V}$ by doing a minimal vertex refinement on each interval.

**Lemma 9.** The zero persistence apparent pairs with respect to the $f$-lexicographic order are precisely the gradient pairs of $\tilde{V}$.

**Proof.** Let $(\sigma, \tau)$ be a zero persistence apparent pair. Then $f(\sigma) = f(\tau)$, and $\sigma$ and $\tau$ are contained in the same regular interval $I = [\rho, \phi]$ of $V$. Let $v$ be the minimal vertex in $\phi \setminus \rho$. By assumption, $\sigma$ is the maximal facet of $\tau$, and $\tau$ is the minimal cofacet of $\sigma$. Hence, $\sigma$ is lexicographically maximal among all facets of $\tau$ in $I$, and $\tau$ is lexicographically minimal under all cofacets of $\sigma$ in $I$. By the assumption that $(\sigma, \tau)$ forms an apparent pair, we cannot have $v \in \sigma$, as otherwise $\tau \setminus \{v\}$ would be a larger facet of $\tau$ than $\sigma$. Similarly, we cannot have $v \notin \tau$, as otherwise $\sigma \cup \{v\}$ would be a smaller cofacet of $\sigma$ than $\tau$. This means that $\tau = \sigma \cup \{v\}$ and therefore $\{\sigma, \tau\} \in \tilde{V}$.

Conversely, assume that $\{\sigma, \tau\} \in \tilde{V}$ holds. Consider the interval $I = [\rho, \phi]$ of $V$ with $\{\sigma, \tau\} \subseteq I$ and let $v$ be the lexicographically minimal vertex in $\phi \setminus \rho$. By construction of $\tilde{V}$, $\sigma = \tau \setminus \{v\}$ is the lexicographically maximal cofacet of $\tau$ in $I$ and $\tau = \sigma \cup \{v\}$ is the lexicographically minimal cofacet of $\sigma$ in $I$. Therefore, $(\sigma, \tau)$ is a zero persistence apparent pair.

### 2.2 Hyperbolicity and geodesic defect of metric spaces

In this section, we establish some basic facts about the hyperbolicity and the geodesic defect of metric spaces, such as their stability with respect to the Gromov–Hausdorff distance. While this is already known for the hyperbolicity [28], the stability of the geodesic defect, to the best of our knowledge, was not explicitly addressed before. The results shown here are independent of the results in the following sections.

If $X$ is $\delta$-hyperbolic, then it is also $\delta'$-hyperbolic for every $\delta' \geq \delta$. With this in mind, it is natural to consider the infimum over all $\delta$ such that $X$ is $\delta$-hyperbolic, which is called the **hyperbolicity** $\delta(X)$ of $X$. It follows from Equation (1) that we have

$$
\delta(X) = \sup_{w, x, y, z \in X} \left\{ \frac{d(w, x) + d(y, z) - \max\{d(w, y) + d(x, z), d(w, z) + d(x, y)\}}{2} \right\},
$$

Equation (2)
and from this equivalent description of the hyperbolicity it can be seen that $X$ is indeed $\delta(X)$-hyperbolic. Moreover, note that every subspace of a $\delta$-hyperbolic space is also $\delta$-hyperbolic. We now demonstrate that similar spaces have similar hyperbolicity.

A correspondence $C$ between two metric spaces $X$ and $Y$ is a subset $C \subseteq X \times Y$ such that $\text{pr}_X(C) = X$ and $\text{pr}_Y(C) = Y$. The Gromov–Hausdorff distance between $X$ and $Y$ is defined as

$$d_{GH}(X,Y) = \frac{1}{2} \inf \{ \text{dist}(C) \mid C \text{ correspondence between } X \text{ and } Y \},$$

where $\text{dist}(C) = \sup \{|d_X(x,x') - d_Y(y,y')| \mid (x,y),(x',y') \in C\}$ is the distortion of $C$.

Proposition 10 (Mémoli et al. [28]). Let $X$ and $Y$ be metric spaces and $s = d_{GH}(X,Y)$. If $X$ is $\delta$-hyperbolic, then $Y$ is $(\delta + 4s)$-hyperbolic. Hence, $|\delta(X) - \delta(Y)| \leq 4d_{GH}(X,Y)$.

We now turn to the geodesic defect. We start by establishing lower and upper bounds on the geodesic defect, relating it to other familiar geometric quantities.

Proposition 11. For any metric space $X$ we have $\nu(X) \geq \frac{1}{2} \inf_{x \neq y} d(x,y)$.

Proof. If $X$ is not $\nu$-geodesic for any $\nu$, then $\nu(X) = \infty$ and there is nothing to prove. Thus, assume that $X$ is $\nu$-geodesic. Let $\epsilon > 0$ be arbitrary and let $u,w \in X$ be any two points with $u \neq w$ and $d(u,w) - \epsilon \leq I := \inf_{x \neq y} d(x,y)$. Then any other point has distance at least $d(u,w) - \epsilon$ to $u$ and $w$. As $X$ is $\nu$-geodesic, there exists a point $z \in X$ with $d(u,z) \leq \frac{1}{2} d(u,w) + \nu$ and $d(w,z) \leq \frac{1}{2} d(u,w) + \nu$. If $z = w$, then the first inequality implies $\frac{1}{2} I \leq d(u,w) \leq \nu$ and hence $\frac{1}{2} I \leq \nu(X)$, because $\nu(X)$ is the infimum over all $\nu$ such that $X$ is $\nu$-geodesic. If $z \neq w$, then

$$d(u,w) - \epsilon \leq I \leq d(w,z) \leq \frac{1}{2} d(u,w) + \nu$$

and therefore $\frac{1}{2} I - \epsilon \leq \frac{1}{2} d(u,w) - \epsilon \leq \nu$. Letting $\epsilon$ tend to zero implies $\frac{1}{2} I \leq \nu$ and hence $\frac{1}{2} I \leq \nu(X)$, because $\nu(X)$ is the infimum over all $\nu$ such that $X$ is $\nu$-geodesic. ▶

A metric subset $X \subseteq Y$ is $r$-dense if for every $y \in Y$ there exists $x \in X$ with $d(y,x) \leq r$. The following proposition proves an upper bound on the geodesic defect for $r$-dense subsets of a geodesic space. A partial converse for $\delta$-hyperbolic spaces is given by Proposition 18.

Proposition 12. Let $X$ be an $r$-dense subset of a geodesic metric space $Y$. Then $X$ is $r$-geodesic. In particular, $\nu(X) \leq r$.

Proof. Let $x,y \in X$ be any two points and let $t,s \geq 0$ be with $t + s = d(x,y)$. Choose an isometric embedding $\gamma: [0,d(x,y)] \to Y$ with $\gamma(0) = x$ and $\gamma(d(x,y)) = y$. As $X$ is $r$-dense, there exists a point $z \in X$ with $d(\gamma(t),z) \leq r$. By the triangle inequality, we get $d(x,z) \leq d(\gamma(0),\gamma(t)) + d(\gamma(t),z) \leq t + r$ and similarly $d(y,z) \leq s + r$. This proves that $X$ is $r$-geodesic. ▶

A 0-geodesic space is also called metrically convex [14]. If $X$ is a geodesic space, then its geodesic defect is zero, but the converse is not always true. In fact, for any length space [11], i.e., a metric space where the distance between two points is the infimum of lengths of paths connecting them, the geodesic defect is zero. It is worth noting that a complete metric space is a length space if and only if its geodesic defect is zero, and it is a geodesic space if and only if it is 0-geodesic [11] Section 2.4. The punctured unit disk in $\mathbb{R}^2$ is an example for a space that has geodesic defect 0 but that is not 0-geodesic. However, we have the following,
Lemma 13. Let $X$ be a proper metric space, i.e., assume that every closed metric ball is compact. Then $X$ is $\nu(X)$-geodesic.

Proof. Let $x, y \in X$ be two points and $r, s \geq 0$ with $r + s = d(x, y)$. For every natural number $n \in \mathbb{N}$ the space $X$ is $\nu_n$-geodesic, where $\nu_n = \nu(X) + \frac{1}{n}$, and hence there exists a point $z_n \in X$ with $d(x, z_n) \leq r + \nu_n$ and $d(y, z_n) \leq s + \nu_n$. This sequence is contained in the closed ball of radius $r + \nu(X) + 1$ centered at $x$, which is compact by assumption. Hence, there exists a convergent subsequence $z_{n_k} \to z$. The limit point $z \in X$ satisfies $d(x, z) \leq r + \nu(X)$ and $d(y, z) \leq s + \nu(X)$. Therefore, $X$ is $\nu(X)$-geodesic. △

We now show that the geodesic defect, like the hyperbolicity, is a Gromov–Hausdorff stable quantity.

Proposition 14. Let $X$ and $Y$ be metric spaces and $s > d_{\text{GH}}(X, Y)$. If $X$ is $\nu$-geodesic, then $Y$ is $(\nu + 3s)$-geodesic. Hence, $|\nu(X) - \nu(Y)| \leq 3d_{\text{GH}}(X, Y)$.

Proof. Let $C$ be a correspondence between $X$ and $Y$ with $\text{dis} C \leq 2s$. Furthermore, let $y, y' \in Y$ be any two points, and let $r, t \geq 0$ be such that $r + t = d(y, y')$. Choose two corresponding points $x, x' \in X$, with $(x, y), (x', y') \in C$. For $u = r + \text{dis} C/2$ and $w = t + \text{dis} C/2$, we have $u + w \geq d(x, x')$. As $X$ is $\nu$-geodesic, there exists a point $z \in X$ such that $d(x, z) \leq u + \nu$ and $d(x', z) \leq w + \nu$. We can choose a corresponding point $p \in Y$, with $(z, p) \in C$. For this point, we get

$$d(y, p) \leq d(x, z) + \text{dis} C \leq (u + \nu) + 2s = (r + \text{dis} C/2) + 2s \leq r + \nu + 3s$$

and similarly $d(y', p) \leq t + \nu + 3s$. Thus $Y$ is $(\nu + 3s)$-geodesic. △

If $X$ is an $r$-dense subset of a geodesic space $Y$, then $d_{\text{GH}}(X, Y) \leq r$, and the above proposition implies that $X$ is $(3s)$-geodesic for every $s > r$. In particular, $\nu(X) \leq 3r$. Note however that in this case Proposition 12 gives the stronger bound $\nu(X) \leq r$.

The persistent homology of the Vietoris–Rips filtration is commonly used in topological data analysis as a stable multi-scale signature of a finite metric space. For length spaces, it is known that all structure maps in the first persistent homology of the Vietoris–Rips filtration are surjective [13, Corollary 6.2]. This statement generalizes to arbitrary metric spaces using the geodesic defect as follows.

Proposition 15. Let $X$ be a $\nu$-geodesic metric space. Then for every $2\nu < t < u$ the canonical map $H_1(\text{Rips}_u(X)) \to H_1(\text{Rips}_u(X))$ is surjective.

Proof. Let $\{x, y\} \in \text{Rips}_u(X)$ be an edge with length $d(x, y) = u$. As $X$ is $\nu$-geodesic, there exists a point $z \in X$ with $d(x, z) \leq \frac{1}{2}u + \nu$ and $d(y, z) \leq \frac{1}{2}u + \nu$. By assumption, we have $\frac{1}{2}u + \nu = \frac{1}{2}u + \frac{1}{2}t - \left(\frac{1}{2}t - \nu\right) < u - l$, where $l = \left(\frac{1}{2}t - \nu\right)$. Hence, the simplex $\{z, x, y\}$ is contained in $\text{Rips}_u(X)$ and the simplicial chain $[x, y]$ is homologous to $[z, y] - [z, x] \in C_1(\text{Rips}_u(X))$. As every simplicial chain in $C_1(\text{Rips}_u(X))$ is a finite sum of edges and $l > 0$ is a constant, it follows that finitely many reapplications of the argument above yields that this chain is homologous to a chain in $C_1(\text{Rips}_u(X))$, proving the claim. △

### 2.3 Rips’ Contractibility Lemma via the injective hull

In this section, we recall some known facts about embeddings of metric spaces into their injective hull. We adapt these results using the geodesic defect to prove a version of the Contractibility Lemma for finite $\delta$-hyperbolic $\nu$-geodesic metric spaces, following [27].
Let $Y$ be a metric space. The Čech complex of a subspace $X \subseteq Y$ for radius $r > 0$ is the nerve of the collection of closed balls in $Y$ with radius $r$ centered at points in $X$:

$$\check{C}_r(X, Y) = \{ \emptyset \neq S \subseteq X \mid S \text{ finite}, \bigcap_{x \in S} D_r(x) \neq \emptyset \},$$

where $D_r(x) = \{ y \in Y \mid d(x, y) \leq r \}$ denotes the closed ball in $Y$ of radius $r$ centered at $x$.

A metric space is hyperconvex [14] if it is geodesic and if any collection of closed balls has the Helly property, i.e., if any two of these balls have a nonempty intersection, then all balls have a nonempty intersection. The following lemma is a direct consequence of this definition.

**Lemma 16.** If $Y$ is hyperconvex and $X \subseteq Y$ is a subspace, then $\check{C}_r(X, Y) = \text{Rips}_{2r}(X)$.

Let $X$ be a metric space. We describe its injective hull $E(X)$, following Lang [24]. A function $f: X \to \mathbb{R}$ with $f(x) + f(y) \geq d(x, y)$ for all $x, y \in X$ is extremal if $f(x) = \sup_{y \in X} (d(x, y) - f(y))$ for every $x \in X$. The difference between any two extremal functions turns out to be bounded, and so we can equip the set $E(X)$ of extremal functions with the metric induced by the supremum norm, i.e., $d(f, g) = \sup_{x \in X} |f(x) - g(x)|$. We define an isometric embedding $e: X \to E(X)$ by $y \mapsto d_y$, where $d_y(x) = d(y, x)$.

**Remark 17.** $E(X)$ is a hyperconvex space. In particular, $E(X)$ is contractible, and nonempty intersections of closed metric balls are contractible [24] [21]. Moreover, nonempty intersections of open metric balls are also contractible [27 Proposition 2.8 and Lemma 2.15].

The following theorem is essentially due to Lang [24]. Originally, it has been stated for a special case, but the proof applies verbatim to the below statement involving the geodesic defect, which indeed provided the motivation for our definition. Note that the definition of $\delta$-hyperbolic used in [21] differs from the one used here by a factor of 2.

**Proposition 18 (Lang [24 Proposition 1.3]).** Let $X$ be a $\delta$-hyperbolic $\nu$-geodesic metric space. Then the injective hull $E(X)$ is $\delta$-hyperbolic, and every point in $E(X)$ has distance at most $2\delta + \nu$ to $e(X)$.

Now we prove a generalization of the Contractibility Lemma using the injective hull analogously to the proof for geodesic spaces in [27 Corollary 8.4].

**Theorem 19.** Let $X$ be a finite $\delta$-hyperbolic $\nu$-geodesic metric space. Then the complex $\text{Rips}_t(X)$ is contractible for every $t \geq 4\delta + 2\nu$.

**Proof.** By Proposition 18, we know that for $r > \frac{\delta}{2} \geq 2\delta + \nu$ the collection of open balls with radius $r$ centered at the points in $e(X)$ covers $E(X)$. By finiteness of $X$, there exists an $r > \frac{\delta}{2}$ such that the nerve of this cover is isomorphic to $\check{C}_\frac{\delta}{2}(e(X), E(X))$. As $e$ is an isometric embedding, Lemma 16 Remark 17 and the Nerve Theorem 19 Section 4.G imply

$$\text{Rips}_t(X) = \text{Rips}_t(e(X)) = \check{C}_\frac{\delta}{2}(e(X), E(X)) \simeq E(X) \simeq \ast.$$

### 3 Filtered collapsibility of Vietoris–Rips complexes

In this section, we revisit the original proof of the Contractibility Lemma in [18], adapted to the language of discrete Morse theory [16]. Focusing on the finite case, which also constitutes the key part of the original proof, we extend the statement beyond geodesic spaces using the geodesic defect, strengthen the assertion of contractibility to collapsibility, and further extend the result to become compatible with the Vietoris–Rips filtration.
Theorem 20. Let $X$ be a finite $\delta$-hyperbolic $\nu$-geodesic metric space. Then for every $t \geq 4\delta + 2\nu$ there exists a discrete gradient that induces a collapse $\text{Rips}_t(X) \setminus \{p\}$.

Proof. Without loss of generality, assume that $\delta > 0$; if $X$ is 0-hyperbolic, then it is also $\epsilon$-hyperbolic for any $\epsilon > 0$, and for sufficiently small $\epsilon > 0$ we have $\text{Rips}_{t+2\nu}(X) = \text{Rips}_{t}(X)$.

Choose a reference point $p \in X$ and order the points according to their distance to $p$, choosing a total order $p = x_1 < \cdots < x_n$ on $X$ such that $x_i < x_j$ implies $d(x_i, p) \leq d(x_j, p)$. Let $t \geq 4\delta + 2\nu$ and consider the filtration

$$\{p\} = K_1 \subseteq \cdots \subseteq K_n = \text{Rips}_t(X),$$

where $K_i = \text{Rips}_t(X_i)$ for $X_i := \{x_1, \ldots, x_i\}$. We prove that for $i \in \{2, \ldots, n\}$ there exists a discrete gradient $V_i$ on $K_i$ inducing a collapse $K_i \setminus K_{i-1}$.

First assume $d(x_i, p) < t$. Then for any vertex $x_k$ of $K_i$ we have $k \leq i$ and $d(x_k, p) \leq d(x_i, p) < t$, so $K_i$ is a simplicial cone with apex $p$. Pairing the simplices containing $p$ with those not containing $p$, we obtain a discrete gradient inducing a collapse $K_i \setminus K_{i-1}$:

$$V_i = \{(\sigma \setminus \{p\}, \sigma \cup \{p\}) : \sigma \in K_i \setminus K_{i-1}\}.$$

Now assume $d(x_i, p) \geq t$. We show that there exists a point $z \in X_{i-1}$ such that for every simplex $\sigma \in K_i \setminus K_{i-1}$, the union $\sigma \cup \{z\}$ is also a simplex in $K_i \setminus K_{i-1}$. To this end, we show that any vertex $y$ of $\sigma$ has distance $d(y, z) \leq t$ to $z$. For $r = d(x_i, p) - 2\delta - \nu$ and $s = 2\delta + \nu$ we have $r + s = d(x_i, p)$, and therefore, by the assumption that $X$ is a $\nu$-geodesic space, there exists a point $z \in X$ with $d(z, p) \leq r + \nu = d(x_i, p) - 2\delta$, implying $z \leq x_i$, and $d(z, x_i) \leq s + \nu = 2\delta + 2\nu$. By assumption $t \geq 4\delta + 2\nu$, and thus we get $d(z, x_i) \leq t - 2\delta$. Note that $y \in X_i$ implies $d(y, p) \leq d(x_i, p)$, and $y, x_i \in \sigma$ implies $d(y, x_i) \leq \text{diam} \sigma \leq t$. The four-point condition (1) now yields

$$d(y, z) \leq \max\{d(y, x_i) + d(z, p), d(y, p) + d(z, x_i)\} + 2\delta - d(x_i, p)$$

$$= \max\{d(y, x_i) + d(z, p) - d(x_i, p), d(y, p) - d(x_i, p) + d(z, x_i)\} + 2\delta \leq t.$$ (3)

Similarly to the above, pairing the simplices containing $z$ with those not containing $z$ yields a discrete gradient inducing a collapse $K_i \setminus K_{i-1}$:

$$V_i = \{(\sigma \setminus \{z\}, \sigma \cup \{z\}) : \sigma \in K_i \setminus K_{i-1}\}.$$

Finally, by Proposition 7 the union $V = \bigcup_i V_i$ is a discrete gradient on $\text{Rips}_t(X)$ and by Proposition 8 it induces a collapse $\text{Rips}_t(X) \setminus \{p\}$. ▷

Remark 21. For a simplicial complex $K$, a particular type of simplicial collapse called an elementary strong collapse from $K$ to $K \setminus \text{St} v$ is defined in [b] for the case where the link of the vertex $v$ is a simplicial cone. The proof of Theorem 20 actually shows that for $t \geq 4\delta + 2\nu$ there exists a sequence of elementary strong collapses from $\text{Rips}_t(X)$ to $\{p\}$.

We can now extend the proof strategy of Theorem 20 to obtain a filtration-compatible strengthening of the Contractibility Lemma.

Proof of Theorem 3. As in the proof of Theorem 20 we can assume that $\delta > 0$, and order the points in $X$ according to their distance to a chosen reference point $p = x_1 < \cdots < x_n$.

As $X$ is finite, we can enumerate the values of pairwise distances by $0 = r_0 < \cdots < r_n$. For every $r_m > 4\delta + 2\nu$ we construct a discrete gradient $W_m$ inducing a collapse $\text{Rips}_{r_m}(X) \setminus \{p\}$.
Rips_{r_{m-1}}(X)$. This will prove the theorem, because it follows from Theorem 20 that there exists a discrete gradient $V$ that induces a collapse $\text{Rips}_{4\delta+2\nu}(X) \setminus \{\ast\}$, and an application of Proposition 7 assembles these gradients into a single gradient $W = V \cup \bigcup_n W_n$ on $\text{Cl}(X)$ inducing collapses $\text{Rips}_u(X) \setminus \{\ast\}$ for every $u > t \geq 4\delta + 2\nu$.

Let $m$ be arbitrary such that $r_m > 4\delta + 2\nu$. Consider the filtration

$$\text{Rips}_{r_{m-1}}(X) = K_1 \subseteq \cdots \subseteq K_n = \text{Rips}_r(X),$$

where $K_i = \text{Rips}_{r_{m-1}}(X) \cup \text{Rips}_r(X)$ for $X_i := \{x_1, \ldots, x_i\}$. We prove that for $i \in \{2, \ldots, n\}$ there exists a discrete gradient $V_i$ on $K_i$ inducing a collapse $K_i \setminus K_{i-1}$. Note that $K_1 \setminus K_{i-1}$ consists of all simplices of diameter $r_m$ that contain $x_i$ as the maximal vertex.

First assume $d(x_i, p) < r_m$. Let $\sigma \in K_1 \setminus K_{i-1}$. As $x_i$ is the maximal vertex of $\sigma$, we have $d(v, p) \leq d(x_i, p) < r_m$ for all $v \in \sigma$. Since $\sigma$ has diameter $r_m$, this implies that $\sigma \cup \{p\}$ also has diameter $r_m$. Moreover, this implies that there exists an edge $e = \sigma \setminus \{p\} \subseteq \sigma$ not containing $p$ with diam $e = r_m$. Therefore, $\sigma \setminus \{p\}$ also has diameter $r_m$. As $p < x_i$, both simplices $\sigma \setminus \{p\}$ and $\sigma \cup \{p\}$ contain $x_i$ as the maximal vertex and are thus contained in $K_1 \setminus K_{i-1}$. Pairing the simplices containing $p$ with those not containing $p$, we obtain a discrete gradient inducing a collapse $K_i \setminus K_{i-1}$:

$$V_i = \{ (\sigma \setminus \{p\}, \sigma \cup \{p\}) \mid \sigma \in K_i \setminus K_{i-1} \}.$$

Now assume $d(x_i, p) \geq r_m$. We show that there exists a point $z \in X_{i-1}$ such that for every simplex $\sigma \in K_1 \setminus K_{i-1}$, the simplices $\sigma \setminus \{z\}$ and $\sigma \cup \{z\}$ are also contained in $K_1 \setminus K_{i-1}$. To this end, we show first that any vertex $y$ of $\sigma$ has distance $d(y, z) \leq r_m$ to $z$. As in the proof of Theorem 20 there exists a point $z \in X$ with $d(z, p) \leq d(x_i, p) - 2\delta$, implying $z < x_i$, and $d(z, x_i) \leq 2\delta + 2\nu$. By assumption $r_m > 4\delta + 2\nu$, and thus we get $d(z, x_i) < r_m - 2\delta$.

Similar to Equation 3, we have the following estimate

$$d(y, z) \leq \max\{d(y, x_i) + d(x_i, p), d(y, p) - d(x_i, p), d(z, x_i)\} \leq r_m,$$

and if $d(y, x_i) < r_m$, then $d(y, z) < r_m$. Hence, $\text{diam}(\sigma \cup \{z\}) = r_m$, and $\text{diam} \sigma = r_m$ implies $\text{diam} \sigma \setminus \{z\} = r_m$, by an argument similar to the above. As $z < x_i$, both simplices $\sigma \setminus \{z\}$ and $\sigma \cup \{z\}$ contain $x_i$ as the maximal vertex and are thus contained in $K_1 \setminus K_{i-1}$. Pairing the simplices containing $z$ with those not containing $z$, we obtain a discrete gradient inducing a collapse $K_i \setminus K_{i-1}$:

$$V_i = \{ (\sigma \setminus \{z\}, \sigma \cup \{z\}) \mid \sigma \in K_i \setminus K_{i-1} \}.$$

By Proposition 7 the union $W_m = \bigcup V_i$ is a discrete gradient on $\text{Rips}_r(X)$, and by Proposition 8 it induces a collapse $\text{Rips}_r(X) \setminus \text{Rips}_{r_{m-1}}(X)$. 

\begin{remark}
We do not know whether this bound is tight for spaces with positive hyperbolicity. However, it is sharp for every finite tree metric, as can be deduced from Example 4.
\end{remark}

### 4 Collapsing Vietoris–Rips complexes of trees by apparent pairs

In this section, we analyze the Vietoris–Rips filtration of a tree metric space $(V, d)$ for a positively weighted finite tree $T = (V, E)$, with the goal of proving the collapses in Theorem 5 using the apparent pairs gradient. To this end, we introduce two other discrete gradients: the canonical gradient, which is independent of any choices, and the perturbed gradient, which
coarsens the canonical gradient and can be interpreted as a gradient that arises through a symbolic perturbation of the edge lengths. We then show that the intervals in the perturbed gradient are refined by apparent pairs of the lexicographically refined Vietoris–Rips filtration, with respect to a particular total order on the vertices.

We write $D_r(x) = \{ y \in V \mid d(x, y) \leq r \}$ and $S_r(x) = \{ y \in V \mid d(x, y) = r \}$.

**Lemma 23.** Let $x, y \in V$ be two distinct points at distance $d(x, y) = r$. Then we have $\text{diam } D_r(x) \cap D_r(y) = r$. Furthermore, if $a, b \in D_r(x) \cap D_r(y)$ are points with $d(a, b) = r$, then these points are contained in the union $S_r(x) \cup S_r(y)$.

**Proof.** We start by showing the first claim. Let $a, b \in D_r(x) \cap D_r(y)$ be any two points. We show that $d(a, b) \leq r$ holds, implying $\text{diam } D_r(x) \cap D_r(y) \leq r$. Because $x, y \in D_r(x) \cap D_r(y)$ we also have $\text{diam } D_r(x) \cap D_r(y) \geq r$, proving equality.

Write $[n] = \{1, \ldots, n\}$ and let $\gamma : ([n], \{\{i, i+1\} \mid i \in [n-1]\}) \to T$ be the unique shortest path $x \rightsquigarrow y$. Moreover, let $\Psi_a$ and $\Psi_b$ be the unique shortest paths $x \rightsquigarrow a$ and $x \rightsquigarrow b$, respectively. Consider the largest numbers $t_a, t_b \in [n]$ with $\gamma(t_a) = \Psi_a(t_a)$ and $\gamma(t_b) = \Psi_b(t_b)$ and assume without loss of generality $t_a \leq t_b$. Note that the unique shortest path $a \rightsquigarrow b$ is then given by the concatenation $a \rightsquigarrow \gamma(t_a) \rightsquigarrow \gamma(t_b) \rightsquigarrow b$, where $\gamma(t_a) \rightsquigarrow \gamma(t_b)$ is the restricted path $\gamma|_{[t_a, t_b]}$. By assumption, we have $d(a, y) \leq r$ and this implies the inequality

$$d(a, \gamma(t_a)) + d(\gamma(t_a), y) = d(a, y) \leq r = d(x, y) = d(x, \gamma(t_a)) + d(\gamma(t_a), y),$$

which is equivalent to $d(a, \gamma(t_a)) \leq d(x, \gamma(t_a))$. Similarly, the assumption $d(x, b) \leq r$ implies $d(\gamma(t_b), b) \leq d(\gamma(t_b), y)$. Thus, the distance $d(a, b)$ satisfies

$$d(a, b) = d(a, \gamma(t_a)) + d(\gamma(t_a), \gamma(t_b)) + d(\gamma(t_b), b) \leq d(x, \gamma(t_a)) + d(\gamma(t_a), \gamma(t_b)) + d(\gamma(t_b), y) = d(x, y) = r,$$

which finishes the proof of the first claim.

We now show the second claim; assume $d(a, b) = r$. From the inequalities $d(a, \gamma(t_a)) \leq d(x, \gamma(t_a))$, $d(\gamma(t_a), \gamma(t_b)) \leq d(\gamma(t_b), y)$ together with the assumption $d(a, b) = r$, we deduce the equalities $d(a, \gamma(t_a)) = d(x, \gamma(t_a))$ and $d(\gamma(t_b), y) = d(\gamma(t_b), y)$. Hence,

$$d(a, y) = d(a, \gamma(t_a)) + d(\gamma(t_a), y) = d(x, \gamma(t_a)) + d(\gamma(t_a), y) = d(x, y) = r$$

and similarly $d(x, b) = r$, proving the second claim.

Enumerate the values of pairwise distances by $0 = r_0 < \cdots < r_l = \text{diam } V$. Let $K_m : = \text{Rips}_{r_{m-1}}(V) \cup T_{r_m}$. We show that the complement $C_m : = \text{Rips}_{r_m}(V) \setminus K_m$ is the set of all cofaces of non-tree edges of length $r_m$. We further show that it is partitioned into regular intervals in the face poset, and that this constitutes a discrete gradient.

**Lemma 24.** Every edge $e \in \text{Rips}_{r_m}(V) \setminus \text{Rips}_{r_{m-1}}(V)$ is contained in a unique maximal simplex $\Delta_e \in \text{Rips}_{r_m}(V) \setminus \text{Rips}_{r_{m-1}}(V)$. Moreover, if $e$ is a tree edge of length $r_m$, then $\Delta_e = e$, and if $e \in C_m$, then $\Delta_e \subset C_m$ and $e \subset \Delta_e$.

**Proof.** By definition, $e$ corresponds to two points $x, y \in V$ at distance $d(x, y) = r_m$. If $e$ is contained in the simplex $\Delta \in \text{Rips}_{r_m}(V)$, then the points in $\Delta$ lie in the intersection $D_{r_m}(x) \cap D_{r_m}(y)$, which has diameter $r_m$ by Lemma 23. Hence, the maximal simplex $\Delta_e$ is spanned by all the points in $D_{r_m}(x) \cap D_{r_m}(y)$.

If $e$ is a tree edge of length $r_m$, then this intersection only contains $x$ and $y$, and hence $\Delta_e = e$. If $e \in C_m$, then this intersection contains at least one vertex different from $x$ and $y$ that lies on the unique shortest path $x \rightsquigarrow y$. This implies $e \subset \Delta_e$. \hfill ☐
For every maximal simplex $\Delta \in C_m \subseteq \text{Rips}_{r_m}(V)$, we write $E_\Delta$ for the set of edges $e \in C_m$ with $\Delta_e = \Delta$. Note that $E_\Delta$ is the set of non-tree edges of length $r_m$ contained in $\Delta$.

### 4.1 Generic tree metrics

Before dealing with the general case, let us focus on the special case where the metric space $(V, d)$ is generic, meaning that the pairwise distances are distinct. In this case, Lemma 24 implies that the diameter function $d_{\text{diam}} : \text{Cl}(V) \to \mathbb{R}$ is a discrete Morse function, defined on the full simplicial complex on $V$, with discrete gradient

$\{[e, \Delta_e] \mid \text{non-tree edge } e \subseteq \text{Cl}(V)\},$

which we call the generic gradient, and only the vertices $V$ and the tree edges $E$ are critical.

▶ **Example 25.** Consider the following weighted tree with vertex set $V = \{a, b, c, d\}$:

```
a  \(\overset{1}{\longrightarrow}\) b  \(\overset{2}{\longrightarrow}\) c  \(\overset{4}{\longrightarrow}\) d
```

The generic gradient is given by $\{[[a, c]], [[a, b, c]], [[a, d]], [[a, b, d]], [[c, d]], [[a, b, c, d]]\}$. These intervals are the preimages under the diameter function of the non-tree distances $3, 5, \text{ and } 6$, respectively.

Together with Proposition 8, this yields the following theorem.

▶ **Theorem 26.** If the tree metric space $(V, d)$ is generic, then the generic gradient induces, for every $m \in \{1, \ldots, l\}$, a sequence of collapses

$\text{Rips}_{r_m}(V) \downarrow (\text{Rips}_{r_{m-1}}(V) \cup T_{r_m}) \downarrow T_{r_m}.$

Moreover, it follows from Lemma 9 that for the Vietoris–Rips filtration, refined lexicographically with respect to an arbitrary total order on the vertices, the zero persistence apparent pairs refine the generic gradient, and therefore also induce the above collapses.

▶ **Theorem 27.** If the tree metric space $(V, d)$ is generic, then the apparent pairs gradient induces, for every $m \in \{1, \ldots, l\}$, a sequence of collapses

$\text{Rips}_{r_m}(V) \downarrow (\text{Rips}_{r_{m-1}}(V) \cup T_{r_m}) \downarrow T_{r_m}.$

### 4.2 Arbitrary tree metrics

We now turn to the general case, where Lemma 9 is not directly applicable anymore, as the diameter function is not necessarily a discrete Morse function. Nevertheless, we show that Theorem 27 is still true without the genericity assumption, if the vertices $V$ are ordered in a compatible way. Let $\Delta$ be a maximal simplex $\Delta \in C_m \subseteq \text{Rips}_{r_m}(V)$.

▶ **Lemma 28.** We have $\text{St} E_\Delta = C_m \cap \text{Cl} \Delta$.

**Proof.** The inclusion $\text{St} E_\Delta \subseteq C_m \cap \text{Cl} \Delta$ holds by definition of $E_\Delta$. To show the inclusion $\text{St} E_\Delta \supseteq C_m \cap \text{Cl} \Delta$, let $\sigma \in C_m \cap \text{Cl} \Delta$ be any simplex. As the Vietoris–Rips complex is a clique complex, there exists an edge $e \subseteq \sigma \subseteq \Delta$ with $\text{diam} e = r_m$. By Lemma 24, this edge can not be a tree edge end hence $e \in C_m$. Therefore, $e \in E_\Delta$ and $\sigma \in \text{St} e \subseteq \text{St} E_\Delta$. ▶
Lemma 29. If two distinct maximal simplices \( \Delta, \Delta' \in C_m = \text{Rips}_r(V) \setminus K_m \) intersect in a common face \( \Delta \cap \Delta' \), then this face is contained in \( K_m \).

Proof. Assume for a contradiction that \( \emptyset \neq \Delta \cap \Delta' \not\subseteq K_m \), implying \( \Delta \cap \Delta' \subseteq C_m \). By Lemma 28 there exists an edge \( e \in \Delta \subseteq C_m \) with \( e \subseteq \Delta \cap \Delta' \), and therefore \( \Delta = \Delta' \) by uniqueness of the maximal simplex containing \( e \) (Lemma 24), a contradiction.

We denote by \( L_\Delta \) the set of all vertices of \( \Delta \) that are not contained in any edge in \( E_\Delta \).

Lemma 30. Let \( e = \{u, w\} \in E_\Delta \) be an edge. Then any point \( x \in V \setminus \{u, w\} \) on the unique shortest path \( u \sim w \) of length \( r_m \) in \( T \) is contained in \( L_\Delta \). In particular, \( L_\Delta \) is nonempty.

Proof. By assumption, we have \( d(u, x) < r_m \), \( d(w, x) < r_m \) and \( d(u, w) = r_m \). Therefore, \( \text{diam}\{u, w, x\} = r_m \) and \( x \in \{u, w, x\} \subseteq \Delta \). Assume for a contradiction that \( x \) is contained in an edge in \( E_\Delta \). Then it follows from Lemma 23 that we have \( d(u, x) = r_m \) or \( d(w, x) = r_m \), contradicting the above. We conclude that \( x \in L_\Delta \).

4.2.1 The canonical gradient

We now describe a discrete gradient that is compatible with the diameter function and induces the same collapses as in Theorem 26 even if the tree metric is not generic. This construction is canonical in the sense that it does not depend on the choice of an order on the vertices, in contrast to the subsequent constructions.

Lemma 31. For any two edges \( f, e \in E_\Delta \) and any vertex \( v \in f \) there exists a vertex \( z \in e \) such that \( \{v, z\} \in E_\Delta \) is an edge in \( E_\Delta \).

Proof. Let \( f = \{v, w\}, e = \{x, y\} \); note that \( d(v, w) = d(x, y) = r_m \). Since \( f \) and \( e \) are both contained in the maximal simplex \( \Delta \), we have \( v, w \in \text{Dr}_r(x) \cap \text{Dr}_r(y) \). Both \( \{v, x\} \) and \( \{v, y\} \) are contained in \( \{v, x, y\} \subseteq \Delta \) and Lemma 23 implies that at least one of these two edges is contained in \( \text{Rips}_r(V) \setminus \text{Rips}_{r-1}(V) \); call this edge \( e_v \). It follows from Lemma 24 that \( e_v \) is not a tree edge, and therefore \( e_v \in E_\Delta \).

Lemma 32. The set \( \text{St} E_\Delta = C_m \cap \text{Cl} \Delta \) is partitioned by the intervals

\[
W_\Delta = \{[\cup S, (\cup S) \cup L_\Delta] \mid \emptyset \neq S \subseteq E_\Delta \},
\]

and these form a discrete gradient on \( \text{Cl} \Delta \) inducing a collapse \( \text{Cl} \Delta \setminus (K_m \cap \text{Cl} \Delta) \).

Proof. The intervals in \( W_\Delta \) are disjoint and contained in \( \text{St} E_\Delta \) by construction. They are regular, because \( L_\Delta \) is nonempty (by Lemma 30). By Proposition 8 it remains to show that the intervals in \( W_\Delta \) partition \( \text{St} E_\Delta = \text{Cl} \Delta \setminus (K_m \cap \text{Cl} \Delta) \) and that \( W_\Delta \) is a discrete gradient.

To show the first claim, it suffices to prove that any simplex \( \sigma \in \text{St} E_\Delta \) is contained in a regular interval of \( W_\Delta \). Consider the simplex \( \tau = \sigma \setminus L_\Delta \subseteq \sigma \). As \( \sigma \in \text{St} E_\Delta \), there exists an edge \( e \in E_\Delta \) with \( \emptyset \subseteq e \subseteq \sigma \). By the definition of \( L_\Delta \), we have \( e \subseteq \sigma \setminus L_\Delta = \tau \). Any other vertex \( v \in \tau \setminus e \) is also contained in one of the edges \( E_\Delta \). By Lemma 31 there exists an edge \( e_v = \{v, w\} \in E_\Delta \), where \( w \in e \). Then \( e = \cup \{v, w\} \subseteq e_v \) and \( \sigma \in \tau \cup U \) \subseteq W_\Delta \).

The second claim now follows from the observation that the function

\[
\sigma \mapsto \begin{cases} 
\dim(\sigma \cup L_\Delta) & \sigma \in \text{St} E_\Delta \\
\dim \sigma & \sigma \notin \text{St} E_\Delta 
\end{cases}
\]

is a discrete Morse function with discrete gradient \( W_\Delta \).
Assume that \( a \) and \( e \) are \( 4 \)-perturbed. Consider a maximal simplex \( \sigma \) of \( \Delta = V = \{a, b, c, d\} \) and let \( r \) be the increasing sequence \( r \) of \( 4 \)-perturbations of \( \sigma \). We have \( E_\Delta = \{\{a, c\}, \{a, d\}, \{c, d\}\} \) and the set \( L_\Delta \) only contains the vertex \{b\}. Therefore, we get
\[
W_\Delta = \{(\{a, c\}, \{a, b, c\}), (\{a, d\}, \{a, b, d\}, (\{c, d\}, \{b, c, d\}), (\{a, c, d\}, \{a, b, c, d\})\}.
\]

Consider the union \( W_m = \bigcup_\Delta W_\Delta \), where \( \Delta \) runs over all maximal simplices in \( C_m \) and \( W_\Delta \) is as in \([5]\). We call \( W = \bigcup_m W_m \) the canonical gradient.

**Example 33.** Consider the following tree with vertex set \( V = \{a, b, c, d\} \), whose edges all have length one:
\[
\begin{array}{c}
\text{a} \\
\downarrow \\
\text{b} \\
\downarrow \\
\text{c} \\
\downarrow \\
\text{d}
\end{array}
\]

The complex \( \text{Rips}_V(V) \) is the full simplicial complex \( \text{Cl} V \) with maximal simplex \( \Delta = V = \{a, b, c, d\} \). We have \( E_\Delta = \{\{a, c\}, \{a, d\}, \{c, d\}\} \) and the set \( L_\Delta \) only contains the vertex \{b\}. Hence, by Proposition 8, it induces a collapse \( \text{Cl} \sigma \Delta \) on the full subcomplex \( \Delta \). By construction of the \( \text{Rips}_V(V) \), the canonical gradient is a discrete gradient on \( \text{Cl} V \).

**Theorem 34.** The canonical gradient is a discrete gradient on \( \text{Cl}(V) \). For every \( m \in \{1, \ldots, l\} \), it induces a sequence of collapses
\[
\text{Rips}_{r_m}(V) \searrow \text{Rips}_{r_{m-1}}(V) \cup T_{r_m} \searrow T_{r_m}.
\]

**Proof.** Let \( \Delta \) be a maximal simplex in \( \Delta \in C_m = \text{Rips}_{r_m}(V) \setminus K_m \), where \( K_m = \text{Rips}_{r_{m-1}}(V) \cup T_{r_m} \). It follows from Lemma 29 that the set \( W_\Delta \) is a discrete gradient on the full subcomplex \( \text{Cl} \Delta \subseteq \text{Rips}_{r_m}(V) \) that partitions \( \text{St} E_\Delta = \text{Cl} \Delta \setminus (K_m \cap \text{Cl} \Delta) \) and that induces a collapse \( \text{Cl} \Delta \searrow (K_m \cap \text{Cl} \Delta) \).

It follows directly from Lemma 29 and Proposition 7 that the union \( W_m = \bigcup_\Delta W_\Delta \) is a discrete gradient on \( \text{Rips}_{r_m}(V) \). Again by Proposition 7, the union \( W = \bigcup_m W_m \) is a discrete gradient on \( \text{Cl}(V) \).

By construction of the \( W_\Delta \), the union \( W_m \) partitions the complement \( \text{Rips}_{r_m}(V) \setminus K_m \).

Hence, by Proposition 8, it induces a collapse \( \text{Rips}_{r_m}(V) \searrow K_m = \text{Rips}_{r_{m-1}}(V) \cup T_{r_m} \). Since only the vertices and the tree edges are critical for \( W_m \), this also yields the collapse to \( T_{r_m} \).

## 4.2.2 The perturbed gradient

Assume that \( V \) is totally ordered. We construct a coarsening of the canonical gradient to the perturbed gradient, such that under a specific total order of \( V \) the perturbed gradient is refined by the zero persistence apparent pairs of the diam-lexicographic order \( \prec \) on simplices.

Consider a maximal simplex \( \Delta \in C_m \), where \( m \in \{1, \ldots, l\} \). Note that all edges in \( E_\Delta \) have length \( r_m \) and thus are ordered lexicographically. Enumerate them as \( e_1 < \cdots < e_q \).

Every simplex \( \sigma \in C_m \cap \text{Cl} \Delta \) contains a maximal edge \( e_\sigma \in \text{Cl} \sigma \cap E_\Delta \).

**Lemma 35.** For every edge \( e_i \in E_\Delta \) the union \( \Sigma_i = \bigcup_{e_\sigma = e_i} \sigma \subseteq \Delta \) is a simplex in \( C_m \) and the maximal edge among \( \text{Cl} \Sigma_i \cap E_\Delta \) is \( e_i \).

**Proof.** Note that \( \Sigma_i \subseteq \Delta \in \text{Rips}_{r_m}(V) \) is a simplex and it is contained in \( C_m \), because it is a coface of the non-tree edge \( e_i \) of length \( r_m \).

To prove the second claim, let \( e_j \in \text{Cl} \Sigma_i \cap E_\Delta \) be any edge. Write \( e_i = \{x, y\} \) with \( x < y \) and \( e_j = \{a, b\} \) with \( a < b \). By construction of \( \Sigma_i \), there exist simplices \( \sigma_a, \sigma_b \in C_m \cap \text{Cl} \Delta \) with \( a \in \sigma_a, b \in \sigma_b \) and \( e_\sigma_a = e_\sigma_b = e_i \). Note that \( \{x, y, a\} \subseteq \sigma_a \) and \( \{x, y, b\} \subseteq \sigma_b \).

By Lemma 23 we have \( x, y \in S_m(a) \cup S_m(b) \) and therefore \( d(a, y) = r_m \) (implying \( a \neq y \)) or \( d(b, y) = r_m \) (implying \( b \neq y \)). As \( \{a, y\} \subseteq \sigma_a \) and \( \{b, y\} \subseteq \sigma_b \), this implies \( \{a, y\} \leq e_\sigma_a = e_i = \{x, y\} \) or \( \{b, y\} \leq e_\sigma_a = e_i = \{x, y\} \), respectively. In particular, we have \( a \leq x \) or \( a < b \leq x \), and if \( a = x \), then \( e_j \subseteq \sigma_b \). In any case, \( e_j < e_i = e_\sigma_b \) as claimed. ▶
This lemma implies that $N_{\Delta} = \{[e_i, \Sigma_i]\}_{i=1}^q$ is a collection of disjoint intervals. It follows from Lemma 32 that for each $j \in \{1, \ldots, q\}$ the interval $[e_j, \Sigma_j]$ is the union

$$[e_j, \Sigma_j] = \bigcup\{|\cup S, (\cup S) \cup L_{\Delta}| S \subseteq E_{\Delta\!}, e_j \text{ maximal element of } \text{Cl}(\cup S \cap E_{\Delta})\}$$

and that $N_{\Delta}$ partitions $C_m \cap \text{Cl} \Delta$. Moreover, it is the discrete gradient of the function

$$f_{\Delta}: \text{Cl} \Delta \to \mathbb{R}, \sigma \mapsto \begin{cases} i & \sigma \in [e_i, \Sigma_i] \\ \dim \sigma - \dim \Delta & \sigma \in K_m \end{cases}$$

and the intervals are regular, because $L_{\Delta}$ is nonempty (Lemma 30). By Proposition 8, $N_{\Delta}$ induces a collapse $\text{Cl} \Delta \searrow K_m \cap \text{Cl} \Delta$. Therefore, the total order on $V$ induces a symbolic perturbation scheme on the edges, establishing the situation of a generic tree metric as in Section 4.1.

**Example 36.** Recalling the tree metric from Example 33, we get

$$N_{\Delta} = \{\{a, c\}, \{a, b, c\}, \{a, d\}, \{a, b, d\}, \{c, d\}, \{a, b, c, d\}\}.$$  

Note that this gradient is different from $W_{\Delta}$. 

Consider the union $N_m = \bigcup_{\Delta} N_{\Delta}$, where $\Delta$ runs over all maximal simplices in $C_m$. We call $N = \bigcup_m N_m$ the **perturbed gradient**. By (6), the perturbed gradient $N$ coarsens the canonical gradient $W$. Analogously to Theorem 34, we obtain the following result.

**Theorem 37.** The perturbed gradient is a discrete gradient on $\text{Cl}(V)$. For every $m \in \{1, \ldots, l\}$, it induces a sequence of collapses

$$\text{Rips}_{r_m}(V) \searrow \text{Rips}_{r_{m-1}}(V) \cup T_{r_m} \searrow T_{r_m}.$$  

**Remark 38.** As the lower bounds of the intervals in the perturbed gradient are edges, it follows from Theorem 37 that these collapses can be expressed as edge collapses [9], a notion that is similar to the elementary strong collapses described in Remark 21.

### 4.2.3 The apparent pairs gradient

Finally, we show that for a specific total order of $V$, which we describe next, the perturbed gradient is refined by the zero persistence **apparent pairs** of the diam-lexicographic order.

From now on, assume that the tree $T$ is rooted at an arbitrary vertex and orient every edge away from this point. Let $\leq_{V}$ be the partial order on $V$ where $u$ is smaller than $w$ if there exists an oriented path $u \rightarrow w$. In particular, we have the identity path $id: u \rightarrow u$.

Note that for any two vertices $u, w \in V$ the unique shortest unoriented path $u \rightsquigarrow w$ can be written uniquely as a zig-zag $u \overset{\gamma}{\rightsquigarrow} z \overset{\eta}{\rightsquigarrow} w$, where $z$ is the greatest point with $z \leq_{V} u, z \leq_{V} w$, and $\gamma, \eta$ are oriented paths in $T$ that intersect only in $z$. If $w \rightsquigarrow p$ is another unique shortest unoriented path with the zig-zag $w \overset{\varphi}{\rightsquigarrow} z' \overset{\mu}{\rightsquigarrow} p$, then we can form the following diagram

$$
\begin{aligned}
&z'' \quad \xi \quad \lambda \quad \eta \quad \varphi \quad \mu \\
&\quad \gamma \quad z \quad \eta \quad z' \quad z' \\
&u \quad w \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
Remark 39. Note that in general the oriented paths $z'' \rightarrow u$ and $z'' \rightarrow p$ can intersect in a point different from $z''$. In particular, the zig-zag $u \rightarrow z'' \rightarrow p$ is not necessarily a decomposition of the unique shortest unoriented path $u \leftrightarrow p$.

Extend the partial order $\preceq_V$ on $V$ to a total order $\prec$ and consider the diam-lexicographic order on simplices. As this total order on the simplices extends $\prec$ under the identification $v \mapsto \{v\}$, we will also denote it by $\prec$. The following lemma directly implies Theorem 5.

Lemma 40. The intervals in the perturbed gradient $N$ are refined by apparent pairs with respect to $\prec$. For every $m \in \{1, \ldots, l\}$, the zero persistence apparent pairs induce the collapse

$$Rips_{r_m}(V) \setminus Rips_{r_{m-1}}(V) \cup T_{r_m}.$$ 

Proof. Consider a maximal simplex $\Delta \in C_m$. Recall that $N_\Delta$ is the discrete gradient of the function $f_\Delta : \text{Cl} \Delta \rightarrow \mathbb{R}$ defined in (7), using the same vertex order as above. By Lemma 9, the zero persistence apparent pairs with respect to the $f_\Delta$-lexicographic order $\prec_{f_\Delta}$ are precisely the gradient pairs of the minimal vertex refinement of $N_\Delta$.

We next show that each apparent pair $(\sigma, \tau = \sigma \cup \{v\}) \subseteq [e_i, \Sigma_i]$ with respect to $\prec_{f_\Delta}$, where $v$ is the minimal vertex in $\Sigma_i \setminus e_i$, is an apparent pair with respect to $\prec$. Clearly, these pairs have persistence zero with respect to the diameter function, as they appear in the same interval of the perturbed gradient. As the apparent pairs of $\prec_{f_\Delta}$, taken over all $\Delta$, yield a partition of $C_m = Rips_{r_m}(V) \setminus (Rips_{r_{m-1}}(V) \cup T_{r_m})$, the same is then true for the apparent pairs of $\prec$. Thus, by Proposition 8, the apparent pairs gradient induces a collapse

$$Rips_{r_m}(V) \setminus Rips_{r_{m-1}}(V) \cup T_{r_m}.$$ 

First, let $\sigma \cup \{p\} \in C_m$ be a cofacet of $\sigma$ not equal to $\tau$. We show that we must have $\tau \prec \sigma \cup \{p\}$, proving that $\tau$ is the minimal cofacet of $\sigma$ with respect to $\prec$: If $p \in \Sigma_i$, then $p \in \Sigma_i \setminus e_i$, and the statement is true by minimality of $v$ in the minimal vertex refinement. Now assume that $p \notin \Sigma_i$ and write $e_i = \{u, w\}$ with $u < w$. By (6), we have $L_\Delta \subseteq \Sigma_i$ and hence it follows that $p \notin L_\Delta$ and that the point $p$ is contained in an edge in $E_\Delta$, by definition of $L_\Delta$. It follows from Lemma 23 that $p$ together with at least one vertex of $e_i$ forms an edge in $E_\Delta$. Call this edge $g$; if there are two such edges, consider the larger one, and call it $g$. From $\{u, w, p\} \subseteq \Delta$ and $p \notin \Sigma_i$ we get $e_i < g$: The edge $e_i$ is not the maximal edge of the two simplex $\{u, w, p\}$, since otherwise $p$ would be contained in $\Sigma_i$. Hence, one of the two edges is maximal, and that edge is $g$ by definition. Considering the two possible cases $q = \{u, p\}$ and $g = \{w, p\}$, we must have $u < p$. We will argue that $v \prec p$ holds, which proves $\tau = \sigma \cup \{v\} < \sigma \cup \{p\}$.

Consider the diagram (5). If $\gamma \neq \text{id}$, then it follows from the fact that $e_i = \{u, w\}$ is not a tree edge that along the unique shortest path $u \leftrightarrow w$ there exists a vertex $x$ distinct from $u$ and $w$ with $x < u < p$. Then $x \in L_\Delta \subseteq \Sigma_i \setminus e_i$ by Lemma 30 and as $v$ is the minimal element in $\Sigma_i \setminus e_i$, we get $v \leq x \prec p$.

If $\gamma = \text{id}$, then $u = z$, and it follows from $d(w, p) \leq r_m$ and $p \notin e_i = \{u, w\}$ that we must have $\lambda \neq \text{id}$ and $\xi = \text{id}$: Otherwise $\lambda = \text{id}$ and $u = z$ lies on $\varphi$. Therefore, $u$ lies on the unique shortest path from $w$ to $p$ and $d(w, p) = d(w, u) + d(u, p) = r_m + d(u, p) > r_m$, yielding a contradiction. Thus, the unique shortest path $(u = z) \leftrightarrow p$ decomposes as $u \leftrightarrow z' \leftrightarrow p$, where $u \leftrightarrow z'$ is contained in $u \leftrightarrow z' \leftrightarrow w$. Note that $u \neq z'$, because $\lambda \neq \text{id}$. Hence, as $e_i$ is not a tree edge, the immediate successor $x$ of $u$ on the path $u \leftrightarrow w$ is distinct from $u$ and $w$ with $x \leq z'$. This point satisfies $x \leq z' \leq p$, and it follows from Lemma 30 that we have $x \in L_\Delta \subseteq \Sigma_i \setminus e_i$. Because $p \notin L_\Delta$ we even have $x < p$. Therefore, as $v$ is the minimal vertex in $\Sigma_i \setminus e_i$, it follows that $v \leq x \prec p$.

It remains to prove that $\sigma$ is the maximal facet of $\tau$ with respect to $\prec$. We write $e_i = \{u, w\}$ with $u < w$ and $\tau = \{b_0, \ldots, b_{\dim \tau}\}$ with $b_0 < \cdots < b_{\dim \tau}$. As $e_i \subseteq \tau$, there are indices
As a weighted graph:

\[ \begin{align*}
\text{Remark 41.} & \quad \text{The preceding Lemma} \ref{lem:collapse} \text{also implies Theorem} \ref{thm:apparent_pairs} \text{in the special case of tree metrics: if} u > t \geq 2e \nu(v) = \max_{e \in E} \nu(e) \text{are real numbers, then} T_i = T \text{is the entire tree, and we obtain collapses} \text{Rips}_1(V) \setminus \text{Rips}_i(V) \setminus T \setminus \{\ast\}. \text{If all edges of} T \text{have the same length, it turns out that the collapse} T \setminus \{\ast\} \text{is also induced by the apparent pairs gradient for the same order} <. \\
\text{Remark 42.} & \quad \text{For metrics other than tree metrics, the collapse} \text{Rips}_u(X) \setminus \text{Rips}_i(X) \text{from Theorem} \ref{thm:apparent_pairs} \text{is not always achieved by the apparent pairs gradient. Consider the following weighted graph:}
\end{align*} \]

\[
\text{For this graph metric, the hyperbolicity is} \delta(X) = 1 \text{and the geodesic defect is} \nu(X) = 5. \text{Therefore, we have} 4\delta(X) + 2\nu(X) = 14. \text{The maximal Vietoris–Rips complex has 31 simplices in total. For the apparent pairs gradient only the simplices} \{b, c\} \text{and} \{b, d, e\} \text{are critical, and both have diameter 15. Thus, the collapse} \text{Rips}_{15}(X) \setminus \text{Rips}_{14}(X) \text{is not induced by the apparent pairs gradient.} \\
\text{Example 43.} & \quad \text{Revisiting the tree metric from Example} \ref{ex:tree_metric} \text{once more, we see that the apparent pairs of the lexicographically refined Vietoris–Rips filtration with diameter two are} \\
& \quad \{(a, c), \{a, b, c\}, \{(a, d), \{a, b, d\}, \{(c, d), \{a, c, d\}, \{(b, c, d), \{a, b, c, d\}. \}
\]

Note that together with Example \ref{ex:apparent_pairs} this shows that the canonical gradient, the perturbed gradient, and the apparent pairs gradient can all be different in general.

\section*{References}
\begin{enumerate}
\item Michał Adamaszek. \textit{Clique complexes and graph powers.} \textit{Israel J. Math.}, 196(1):295–319, 2013. \url{doi:10.1007/s11856-012-0166-1}
\item Michał Adamaszek, Henry Adams, Ellen Gasparovic, Maria Gommel, Emilie Purvine, Radmila Sadanovic, Bei Wang, Yusu Wang, and Lori Ziegelmeier. \textit{On homotopy types of Vietoris–Rips complexes of metric gluings.} \textit{J. Appl. Comput. Topol.}, 4(3):425–454, 2020. \url{doi:10.1007/s41468-020-00054-y}
\end{enumerate}
Dominique Attali, André Lieutier, and David Salinas. Vietoris-Rips complexes also provide topologically correct reconstructions of sampled shapes. *Comput. Geom.*, 46(4):448–465, 2013. doi:10.1016/j.comgeo.2012.02.009

Dominique Attali, André Lieutier, and David Salinas. When convexity helps collapsing complexes. In *35th International Symposium on Computational Geometry*, volume 129 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 11, 15. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2019. doi:10.4230/LIPIcs.SoCG.2019.11

Jonathan Ariel Barmak and Elias Gabriel Minian. Strong homotopy types, nerves and collapses. *Discrete Comput. Geom.*, 47(2):301–328, 2012. doi:10.1007/s00454-011-9357-5

Ulrich Bauer. Ripser: efficient computation of Vietoris-Rips persistence barcodes. *J. Appl. Comput. Topol.*, 5(3):391–423, 2021. doi:10.1007/s41468-021-00071-5

Ulrich Bauer and Herbert Edelsbrunner. The Morse theory of Čech and Delaunay complexes. *Trans. Amer. Math. Soc.*, 369(5):3741–3762, 2017. doi:10.1090/tran/6991

Michael Bieker, Lukas Hahn, Juan Angel Patino-Galindo, Mathieu Carriere, Ulrich Bauer, Raul Rabadan, and Andreas Ott. Topology identifies emerging adaptive mutations in SARS-CoV-2. Preprint, 2021. arXiv:2106.07292

Jean-Daniel Boissonnat and Siddharth Pritam. Edge collapse and persistence of flag complexes. In *36th International Symposium on Computational Geometry*, volume 164 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 19, 15. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2020. doi:10.4230/LIPIcs.SoCG.2020.19

M. Bonk and O. Schramm. Embeddings of Gromov hyperbolic spaces. In *Selected works of Oded Schramm. Volume 1, 2*, Sel. Works Probab. Stat., pages 243–284. Springer, New York, 2011. With a correction by Bonk. doi:10.1007/978-1-4419-9675-6_10

Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001.

Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

Patricia Hersh. On optimizing discrete Morse functions. *Adv. in Appl. Math.*, 35(3):294–322, 2005. doi:10.1016/j.aam.2005.04.001

Robin Forman. Morse theory for cell complexes. *Adv. Math.*, 134(1):90–145, 1998. doi:10.1006/aima.1997.1650

Ragnar Freij. Equivariant discrete Morse theory. *Discrete Math.*, 309(12):3821–3829, 2009. doi:10.1016/j.disc.2008.10.029

M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987. doi:10.1007/978-1-4613-9586-7_3

Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

Patricia Hersh. On optimizing discrete Morse functions. *Adv. in Appl. Math.*, 35(3):294–322, 2005. doi:10.1016/j.aam.2005.04.001

J. R. Isbell. Six theorems about injective metric spaces. *Comment. Math. Helv.*, 39:65–76, 1964. doi:10.1007/BF02566944

Jakob Jonsson. *Simplicial complexes of graphs*, volume 1928 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2008. doi:10.1007/978-3-540-75859-4

Dmitry N. Kozlov. *Organized collapse: an introduction to discrete Morse theory*, volume 207 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2020.
24 Urs Lang. Injective hulls of certain discrete metric spaces and groups. *J. Topol. Anal.*, 5(3):297–331, 2013. doi:10.1142/S1793525313500118

25 Janko Latschev. Vietoris-Rips complexes of metric spaces near a closed Riemannian manifold. *Arch. Math. (Basel)*, 77(6):522–528, 2001. doi:10.1007/PL00000526

26 Michael Lesnick, Raúl Rabadán, and Daniel I. S. Rosenbloom. Quantifying genetic innovation: mathematical foundations for the topological study of reticulate evolution. *SIAM J. Appl. Algebra Geom.*, 4(1):141–184, 2020. doi:10.1137/18M118150X

27 Sunhyuk Lim, Facundo Memoli, and Osman Berat Okutan. Vietoris-Rips Persistent Homology, Injective Metric Spaces, and The Filling Radius. Preprint, 2020. arXiv:2001.07588

28 Facundo Memoli, Osman Berat Okutan, and Qingsong Wang. Metric Graph Approximations of Geodesic Spaces. Preprint, 2018. arXiv:1809.05566

29 Vin de Silva and Gunnar Carlsson. Topological estimation using witness complexes. In Markus Gross, Hanspeter Pfister, Marc Alexa, and Szymon Rusinkiewicz, editors, *SPBG’04 Symposium on Point - Based Graphics 2004*. The Eurographics Association, 2004. doi:10.2312/SPBG/SPBG04/157-166

30 Leopold Vietoris. Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen. *Math. Ann.*, 97(1):454–472, 1927. doi:10.1007/BF01447877