Strongly coupled electron fluids in the Poiseuille regime

Johanna Erdmenger, Ioannis Matthaiakakis, René Meyer, David Rodriguez Fernández

1Institute for Theoretical Physics and Astrophysics, Julius-Maximilians-Universität Würzburg, 97074 Würzburg, Germany

(Dated: June 29, 2018)

In the context of describing electrons in solids as a fluid in the hydrodynamic regime, we consider a flow of electrons in a channel of finite width, i.e. a Poiseuille flow. The electrons are accelerated by a constant electric field. We develop the appropriate relativistic hydrodynamic formalism in 2+1 dimensions and show that the fluid has a finite dc conductivity due to boundary-induced momentum relaxation, even in the absence of impurities. We use methods involving the AdS/CFT correspondence to examine the system in the strong-coupling regime. We calculate and study velocity profiles across the channel, from which we obtain the differential resistance $dV/dI$. We find that $dV/dI$ decreases with increasing current $I$ as expected for a Poiseuille flow, also at strong coupling. We also vary the coupling strength by varying $\eta/s$, the ratio of shear viscosity over entropy density. We find that $dV/dI$ decreases when the coupling is increased. Moreover, we find that strongly coupled fluids are more likely to become ultra-relativistic and turbulent. These conclusions are insensitive to the presence of impurities. In particular, we predict that in channels which are clearly in the hydrodynamic regime already at small currents, the DC channel resistance strongly depends on $\eta/s$.

I. INTRODUCTION

In recent years, hydrodynamic behavior in electron systems has received considerable interest from both theoretical and experimental condensed matter physics.1–4 Experimentally, the hydrodynamic regime was first reached in AlGaAs high mobility wires more than two decades ago when the Gurzhi effect was observed,5,6 and more recently in several other materials including graphene.7–9 The Gurzhi effect in such wires is a crossover between boundary-dominated scattering (the Knudsen regime) at low densities and hydrodynamic behavior at higher densities (the Poiseuille regime). It manifests itself as a crossover in the channel resistance as a function of the applied current. The channel resistance rises in the Knudsen regime, while it falls in the Poiseuille regime. In spite of the experimental successes mentioned, observing clear signs of viscous hydrodynamic transport or exactly measuring the value of the viscosity turns out to be challenging in electronic systems in general.10 One key reason is that the interpretation of averaged observables such as e.g. voltages in nonlocal measurements (proposed e.g. as a signature of hydrodynamic whirling)12 depend crucially on the boundary conditions of the measurement setup. Nevertheless, theoretical proposals exist to measure the viscosity using non-local transport in Hall bars,13 rheometers,14 and AC transport.15 Also, in the presence of impurity scattering, the interesting hydrodynamic regime may actually be restricted to a narrow range of parameters, as can e.g. be seen from the weak coupling phase diagram of the relativistic Gurzhi effect derived from kinetic theory.16

In order to make progress towards resolving some of the issues mentioned, in the present paper we investigate hydrodynamic behavior as realized by a Poiseuille flow in a strongly coupled system. This is conveniently done by using methods of the AdS/CFT correspondence.15–20 In particular, we present a model where we are able to investigate the dependence of the flow on the coupling strength starting from very large couplings by varying only one parameter, which is the ratio of shear viscosity over entropy density.

The AdS/CFT conjecture provides universal predictions for observables of strongly coupled and correlated systems.21,22 One such prediction is that the low-energy excitations of strongly coupled gapless systems described via AdS/CFT are governed by the laws of hydrodynamics. The hydrodynamic equations of motion are conservation laws for the only long-lived excitations expected in a strongly interacting system. These are the long-wavelength, low-energy excitations of conserved quantities such as e.g. energy, momentum, or charge. This is not surprising since a very strongly coupled gapless system at low energies is expected to have a very short equilibration time and mean free path beyond which the system behaves hydrodynamically. In the absence of additional conserved quantities or other parametric suppressions by ratios of scales,23 the equilibration time will be of order

$$\tau_{eq} \sim \frac{\hbar}{k_B T}.$$  

(1.1)

In particular, (1.1) holds in all rotationally invariant holographic models of electronic transport with a stable ground state. In these models, the approach to local equilibrium is governed by the most long-lived quasinormal mode of the dual black brane, which has a lifetime of order \(\tau_{eq}\).24 However, an equilibration time of order (1.1) is not a necessary condition for hydrodynamic behaviour. The exact conditions of applicability of hydrodynamics as a low-energy effective theory will be discussed in sec. IIA.

The AdS/CFT correspondence allows not only to derive hydrodynamics from perturbations of black branes in
AdS space-time but also the values of the transport coefficients (viscosities, conductivities, etc.). Calculating the shear viscosity from AdS/CFT for rotationally invariant systems, the ratio of shear viscosity to entropy density takes a universal value

$$\frac{\eta}{s} = \frac{1}{4\pi} \frac{\hbar}{k_B} ,$$  \hspace{1cm} (I.2)

independent of the other parameters of the system such as e.g. temperature or chemical potential. It is conjectured that constitutes a universal minimal bound for all interacting quantum systems.

$$\eta/s \geq \frac{1}{4\pi} \frac{\hbar}{k_B} .$$  \hspace{1cm} (I.3)

This so-called Kovtun-Son-Starinets (KSS) bound can be motivated by e.g. scaling arguments in quantum critical phases, and ultimately should be related to fundamental properties of transport in interacting quantum systems.

It is of great interest to experimentally test the predictions. In heavy-ion collision experiments, $\eta/s$ was experimentally found to be of the same order as 3–5,17 albeit with large error bars. The main issue in extracting $\eta/s$ in heavy-ion collisions is the very indirect dependence of the final state particle distributions measured in the detector on the value of the hydrodynamic transport coefficients. The transport properties of strongly coupled and correlated electron systems in the hydrodynamic regime could hence provide a more direct window into the strong coupling regime of interacting quantum systems. In particular, two-dimensional Dirac materials such as graphene, or the surface states of topological insulators such as e.g. HgTe, as well as Weyl and Dirac semimetals in three spatial dimensions are of interest here due to their relativistic band structures. If electron-electron interactions are the fastest way for electrons to redistribute their energy and momentum and equilibrate, the electrons should form a relativistic fluid governed by the equations of relativistic hydrodynamics.

The ratio $\eta/s$ depends explicitly on the coupling constant, and hence serves as a measure of the interaction strength itself. In the weak coupling regime, $\eta/s$ can be calculated in perturbation theory. Increasing the interactions, one expects the system to enter the strongly coupled regime and saturate the bound (I.3). Kinetic theory calculations of $\eta/s$ for Coulomb interacting two-dimensional Dirac materials support this picture: At charge neutrality and to first order in the effective temperature dependent Coulomb coupling $\alpha(T)$, $\eta/s$ was found to be\(^{34,35}\)

$$\frac{\eta}{s} = \frac{C_\eta \pi}{9c(3)} \frac{1}{\alpha^2(T)} \frac{\hbar}{k_B} \approx 1.64 \frac{1}{4\pi} \frac{\hbar}{\alpha^2(T)} \frac{1}{k_B} ,$$  \hspace{1cm} (I.4)

where $C_\eta \approx 0.449$ is a numerical constant. As $\alpha$ enters the strong coupling regime, $\alpha \approx 1$, $\eta/s$ is expected to approach the holographic value\(^{2,3,10}\). Hence, if the hydrodynamic regime can be reached experimentally in materials with large Coulomb coupling, and if observables sensitive to $\eta/s$ can be found, it will become possible to experimentally test the AdS/CFT prediction in hydrodynamic electron systems.

In view of further progress in detecting hydrodynamic behaviour, the channel setup\(^{3,31,32}\) showing the two-dimensional Gurzhi effect described above is probably the simplest setup possible. In hydrodynamical terms, this corresponds to the simplest flow geometry in which a controlled hydrodynamic flow can be achieved, such that it may be easiest to find observables sensitive to changes of the value of the ratio $\eta/s$ alone. The channel setup consists of two parallel boundaries with prescribed boundary conditions, and a laminar flow in between in the presence of an electric field along the channel direction. Two different choices of boundary conditions are common\(^{3,32}\). The boundary condition of vanishing velocity at the channel wall gives rise to the Poiseuille flow, which is depicted in Fig. 1. The choice\(^{32}\) of vanishing stress at the boundaries leads to an Ohmic flow. In our work, we take the channel setup as a starting point and provide a detailed study of parametric effects for a relativistic charged Poiseuille flow in the absence of parity or time reversal breaking. We in particular study the dependence on the value of $\eta/s$ of the Poiseuille flow and observables derived from it such as the differential wire resistance. We vary $\eta/s$ independently of the impurity relaxation time, channel width, chemical potential, temperature and external electrical field. Varying the value of $\eta/s$ from (I.2) to larger values allows us in particular to interpolate from the strong to the weak coupling regime, and in this way to search for physical signatures of strongly coupled fluids.

We first analyze the Poiseuille flow in the absence of momentum-relaxing impurities. We find that $\eta/s$ has a strong effect on the fluid flows, and in particular that strongly coupled holographic fluids satisfying flow faster compared to their weakly coupled counterparts.
The reason is that in the absence of impurities, the only way for the fluid to lose momentum is through the boundaries, which absorb the momentum density. The rate of momentum transfer between adjacent fluid layers is controlled by $\eta/s$, and hence the momentum transfer to the boundaries becomes as inefficient as it can be for fluids satisfying (1.2). The rate of momentum loss through the boundaries is characterized by an associated wall relaxation time scale $\tau_w$. We calculate the wall relaxation time scale and find that it is longer for strongly coupled holographic fluids than for weakly coupled ones. We furthermore calculate the differential channel resistance $dV/dI$, and find as expected that it decreases with decreasing $\eta/s$. Holographic fluids restricted to channels hence show lower channel resistance than weakly coupled ones.

We also consider the effect of momentum relaxing impurities on the Poiseuille flow. In microscopic descriptions of real-world condensed matter systems, the physics of impurities in general depends on the nature of the scattering potential (extended, hard, soft, long or short range etc.). A similar model dependence is also inherent in holographic models, which depend on the precise mechanism used to break translation invariance and hence momentum conservation. Nevertheless, in both cases the relaxation time approximation, i.e. the assumption of a constant relaxation time, often proves to be a good and universal approximation to the weak impurity limit. In our setup, we dial the impurity relaxation time independently of the other parameters of the system, in order to distinguish impurity effects from hydrodynamic ones. Hydrodynamics is only a good description if the impurity relaxation scale is a slow scale in the system, slower than the equilibration time (I.1) and the time-scale of hydrodynamic fluctuations. The latter is given by the gradients of hydrodynamic fields velocity, temperature, and chemical potential. If the momentum relaxation time scale is short compared to the momentum transport time set by $\eta/s$, we find a crossover to an impurity-dominated Ohmic regime even for the zero velocity boundary conditions characterizing the Poiseuille flow. The crossover happens if the ratio

$$\Pi = \frac{1}{T}\frac{\eta}{s}$$  \hspace{1cm} (I.5)

is of $O(1)$. The Poiseuille regime occurs for $\Pi \ll 1$. In the Ohmic regime $\Pi \gg 1$, hydrodynamics is no longer valid as a low-energy effective theory.

This paper is organized as follows: In sec. II we review the equations of relativistic hydrodynamics (sec. II.A) as well as the pieces of the AdS/CFT correspondence necessary to understand our analysis (sec. II.B). In sec. III we then discuss the effect of the finite channel width leading to the the Poiseuille flow (sec. III.A), the effect of momentum relaxation through the walls (sec. III.B), as well as the time scale for energy relaxation (sec. III.C). Sec. IV is devoted to the analysis of the flow velocity profiles and the differential channel resistance in the absence of impurities. Sec. V then includes momentum relaxing impurities into the analysis. In section VI we present preliminary results on the onset of turbulence in Gurzhi-type channel setups. Finally, in sec. VII we discuss our results and give an outlook to possible future research. We estimate in app. B the Reynolds number both in the nonrelativistic and ultrarelativistic limits.

II. RELATIVISTIC HYDRODYNAMICS AND THE ADS/CFT CORRESPONDENCE

A. Relativistic hydrodynamics

In this section we review elements of hydrodynamics relevant to our analysis. Relativistic hydrodynamics is the effective field theory of long-wavelength low-energy fluctuations of matter in local thermal equilibrium. Its dynamical equations are the conservation laws of energy, momentum and charge. We neglect the imbalance current in relativistic systems, which is approximately conserved at weak coupling and close to charge neutrality. This additional conserved current does not couple to the external electromagnetic field, and hence does not contribute to electric transport. It will however contribute to thermal and thermoelectric transport.

Consider a two-dimensional electron system in a wire geometry of width $W$ and length $l \gg W$. Local thermalization is dictated by the average time between electron-electron collisions $\tau_{ee}$. It is possible to describe the electron gas by means of hydrodynamics if the electron-electron scattering time is the shortest timescale present, and if all other external time-scales are much longer than the time-scale of hydrodynamic fluctuations. Within our setup, we have two additional relevant timescales: the time between electron-impurity collisions, i.e. the momentum relaxation time-scale $\tau_{imp}$, and $W/v_F$. The corresponding length scales are obtained by multiplying with the (constant) Fermi velocity $v_F$. Therefore, the two-dimensional electron gas will behave as a fluid if, in natural units,

$$\ell_{ee} \ll ||\partial_\mu (T, \mu, u^\mu)|| \ll \ell_{imp}, W.$$  \hspace{1cm} (II.1)

This is the regime we consider in this work.

If the electrons have a relativistic dispersion, the local thermal equilibrium will be described on scales exceeding $\ell_{ee} = v_F\tau_{ee}$ by a local temperature $T(x)$, a local chemical potential $\mu(x)$, and a local relativistic velocity $u^\mu(x)$, where e.g. $x^\mu = (t, x, y)$ for a planar 2+1-dimensional system. The velocity field must be time-like and normalized to the speed of light relevant for the relativistic fluid, which e.g. in case of graphene is the Fermi velocity $v_F$ appearing in the dispersion relation of the Dirac particles. We focus on 2+1-dimensional hydrodynamics, as it is relevant to two-dimensional electron
systems. In terms of the velocities \((\beta_x(x^\mu), \beta_y(x^\mu))\),

\[
  w^\mu = \gamma (v_F, \beta_x, \beta_y), \quad \gamma = \frac{1}{\sqrt{1 - \beta_x^2 - \beta_y^2}}, \quad (II.2)
\]

\[
  w^\mu u_\mu = u^\mu u^\nu \eta_{\mu\nu} = -v_F^2, \quad \eta_{\mu\nu} = \text{diag}(-1,1,1,1). \quad (II.3)
\]

The equations of relativistic hydrodynamics are the conservation equations for the Lorentz covariant energy-momentum tensor \(T_{\mu\nu}\) and the relativistic charge current \(j^\mu\). In absence of impurities, the covariant energy-momentum and current conservation equations for a fluid in the presence of an external electromagnetic field are

\[
  \partial_\mu T^{\mu\nu} = \frac{1}{e^c} j_\alpha D^{\alpha\nu}, \quad (II.4)
\]

\[
  \partial_\mu j^\mu = 0. \quad (II.5)
\]

We assume the background metric to be the metric of flat Minkowski space-time, \(g_{\mu\nu} = \eta_{\mu\nu}\) \(D^{\alpha\beta} = \) the electromagnetic displacement tensor, which in 2+1 dimensions takes the form

\[
  D^{\alpha\beta} = \begin{pmatrix}
  0 & E_x \epsilon_0 \epsilon_r c & E_y \epsilon_0 \epsilon_r c \\
  -E_x \epsilon_0 \epsilon_r c & 0 & -B/\mu_0 \mu_F \\
  E_y \epsilon_0 \epsilon_r c & B/\mu_0 \mu_F & 0
  \end{pmatrix}, \quad (II.6)
\]

with \(D_{\nu\rho} = \eta_{\alpha\beta} \eta_{\mu\lambda} D^{\alpha\beta}\). The electric field, given by \(E^\alpha = h/(c^2) u_\alpha D^{\alpha\nu}\), fulfills \(u_{\alpha} E^\alpha = 0\). The charge density to which the electric field couples is then \(v_F^2 e\rho = -\gamma^0 u_\alpha\). Note that while the speed of light relevant for the Lorentz symmetry of the fluid is \(c_F\), the speed of light relevant for the coupling to the external electromagnetic field is the usual one, \(c\). \(\epsilon_0, \mu_0, \epsilon_r\), and \(\mu_r\) are the dielectric constant of the vacuum, the vacuum permeability, the relative permittivity, and the relative permeability in medium.

In order for (II.4)–(II.5) to form a set of equations for the hydrodynamic variables \((T, \mu, w^\mu)\), we need to specify the stress-energy tensor \(T_{\mu\nu}\) and the charge current \(j^\mu\) in terms of these variables. This is formulated as a derivative expansion around global equilibrium, characterized by constant \(T\) and \(\mu\), and vanishing velocity in the rest frame, \(w^\mu = (v_F, 0, 0, 0)\). We decompose the stress-energy tensor and charge current in terms of tensor structures parallel and perpendicular to the field \(w^\mu(x)\),

\[
  T_{\mu\nu} = \mathcal{E} u_\mu u_\nu + \mathcal{P} \Delta_{\mu\nu} + (\mathcal{q}_\mu w_\nu + \mathcal{q}_\nu w_\mu) + \eta_{\mu\nu}, \quad (II.7a)
\]

\[
  j_\mu = N u_\mu + v_F^\prime, \quad (II.7b)
\]

where \(\Delta_{\mu\nu} = w^\mu w^\nu/v_F^2 + \eta_{\mu\nu}\) is the projector onto the space orthogonal to the field. \(\mathcal{E}, \mathcal{P},\) and \(\mathcal{N}\) are Lorentz scalars, \(q_\mu, l_\mu,\) and \(j_\mu\) are transverse, \(u_{\alpha} q^\alpha = 0,\ u_{\alpha} l^\alpha = 0,\ u_{\alpha} v_F^\prime = 0,\) and \(v_F^\prime\) is symmetric and traceless. As for any effective field theory, redefinitions of the fields \((T, \mu, w^\mu)\) can mix the different terms in (II.7). It is hence important to specify conditions that fix the freedom to redefine the hydrodynamical fields, i.e. to choose a hydrodynamical frame.

The common frame is the Landau frame \(\mathcal{L}\) in Landau frame, the local energy density in equilibrium \(\epsilon\) is identified with \(\mathcal{E}\) via the condition \(T^{\mu\nu} u_\mu = -\epsilon u^\nu\), and the density of conserved charge \(e\rho\) is identified with \(\mathcal{N}\) via \(j^\mu u_\mu = -e\rho u_F^\mu\). The fluid is comoving to the energy flow, amounting to \(q^\alpha = 0\) in (II.7a).

We exclusively work in Landau frame in this paper.

Having fixed the frame, the expansion up to first order in derivatives of \(w^\mu\) for \(T^{\mu\nu}\) and \(j^\mu\) for a rotationally invariant charged fluid in 2+1 dimensions is

\[
  T^{\mu\nu} = T^{\mu\nu}_{\text{(ideal)}} + T^{\mu\nu}_{\text{(viscous)}}, \quad (II.8)
\]

\[
  T^{\mu\nu}_{\text{(ideal)}} = \varepsilon u^\mu u^\nu/v_F^2 + p D^{\mu\nu}, \quad (II.9)
\]

\[
  T^{\mu\nu}_{\text{(viscous)}} = -\eta \Delta_{\alpha\beta} (\partial_\alpha u_\beta - \Delta_{\alpha\beta} \partial_\gamma u^\gamma) - \xi \Delta_{\alpha\beta} \partial_\gamma \partial_\beta u^\gamma, \quad (II.10)
\]

\[
  j^\mu = e \rho u^\mu + \sigma_Q \left( E^\mu - T \partial_\nu \Delta_{\mu\nu} (\rho/\varepsilon) \right), \quad (II.11)
\]

where \(\varepsilon, p, \rho\) are energy density, pressure and number density of the fluid in local equilibrium, \(\sigma_Q\) is the intrinsic quantum critical conductivity due to the counterflow of electrons and holes, \(\eta\) and \(\xi\) are the shear and bulk viscosities. For application to Dirac materials such as e.g. graphene, which are approximately scale invariant, we neglect the bulk viscosity, \(\xi = 0\).

In the presence of momentum relaxing impurities, the classical Drude model alters the right hand side of the Newton equation \(\dot{m} = m\varepsilon\) by adding a friction term,

\[
  \dot{m} = m\dot{\varepsilon} + \frac{m}{\tau_{\text{imp}}} \varepsilon, \quad (II.12)
\]

We first project into the direction of time given by the velocity field \(w^\mu\) and perpendicular to it by multiplying with \(w^\mu\) and \(\Delta_{\mu\nu}\), respectively,

\[
  u_\mu \partial_\mu T^{\mu\nu} = \frac{h}{e^c} u_\nu j_\alpha D^{\alpha\nu}, \quad (II.13)
\]

\[
  \Delta_{\mu\nu} \partial_\mu T^{\mu\nu} = \frac{h}{e^c} \Delta_{\rho\sigma} j_\alpha D^{\alpha\nu}. \quad (II.14)
\]

The first equation enforces the conservation of energy, whereas the second equation is the relativistic version of the Navier-Stokes equation. We add a loss term to the right-hand side of the Navier-Stokes equations, \(u_\mu \partial_\mu T^{\mu\nu}\),

\[
  u_\mu \partial_\mu T^{\mu\nu} = \frac{h}{e^c} u_\nu j_\alpha D^{\alpha\nu}, \quad (II.15)
\]

\[
  \Delta_{\rho\sigma} \partial_\mu T^{\mu\nu} = \frac{h}{e^c} \Delta_{\rho\sigma} j_\alpha D^{\alpha\nu} - \Delta_{\rho\sigma} \frac{T^{\mu\nu}}{v_F^\prime T_{\text{imp}}}. \quad (II.16)
\]

The momentum relaxation time \(\tau_{\text{imp}}\) can be thought of arising from electron-impurity scattering in the relaxation time approximation. We henceforth assume that this is the only relaxation time entering the relativistic hydrodynamic equations. We in particular assume that energy relaxation is sufficiently fast such that in the non-equilibrium steady states considered, the non-equilibrium energy density is close to the energy density in local thermal equilibrium at the electron temperature. This assumption is supported by quantum criticality arguments, as well as measurements of scattering
times in materials with strong correlations.\cite{22,23} Assuming a quantum critical relation of the form (I.1) for the energy relaxation time, we estimate the nonequilibrium energy density in sec. III C, and show that it indeed is close to the equilibrium energy density for realistic parameter choices.

B. Strongly coupled fluids in AdS/CFT

In order to solve the hydrodynamic equations \cite{(II.15)} and \cite{(II.16)}, we need to specify the thermodynamic variables that enter the constitutive relations \cite{(II.9)}, \cite{(II.10)}, \cite{(II.11)}, as well as the shear viscosity \(\eta\) and the quantum critical conductivity \(\sigma_Q\). For strongly coupled fluids, it is natural to take these values from the AdS/CFT correspondence. In this section we review these facts from the AdS/CFT correspondence. Since we are interested in planar 2+1 dimensional systems, we will work in 3+1 bulk dimensions, i.e. consider a gravity model with an AdS4 ground state.

The simplest holographic set-up which accommodates matter at finite charge density and temperature is the field theory dual to Einstein-Maxwell theory,

\[
S = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} \left[ R - 2\Lambda - L^2 F_{\mu\nu} F^{\mu\nu} \right]. \tag{II.17}
\]

Here \(R\) is the Ricci curvature scalar, \(\Lambda = -\frac{2}{L^2}\) is the negative cosmological constant necessary to generate an Anti de Sitter space-time of curvature radius \(L\) as the ground state of \cite{(II.17)}, and \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) the field strength for the \(U(1)\) gauge connection \(A_\mu\). Indices are raised and lowered with the space-time metric \(g_{\mu\nu}\) and its inverse \(g^{\mu\nu}\). The line element of the AdS-Reissner-Nordström black brane\cite{24}\ is

\[
ds^2 = \frac{L^2}{r^2 f(r)} dr^2 + \frac{r^2}{L^2} \left[-f(r)v_F^2 dt^2 + dx^2\right]. \tag{II.18}
\]

The black brane encloses a charge density inside the horizon, sourcing a Coulomb potential \(A_N dx^\alpha = A_0(r) dx^\alpha\) varying in the holographic direction \(r\). The blackening factor \(f(r)\) and Coulomb potential \(A_0(r)\) are

\[
f(r) = 1 - \frac{r_H^2}{r^2} - \frac{1}{4\hbar^2 v_F^2} \left[1 - \frac{r_H^2}{r^2}\right],
\]

\[
A_0 = \frac{\mu}{\hbar v_F} \left(1 - \frac{r_H^2}{r^2}\right). \tag{II.19}
\]

The metric \cite{(II.18)} asymptotes to an AdS4 space of radius \(L\) for \(r \to \infty\). The Fermi velocity \(v_F\) replaces the vacuum speed of light \(c\) since we want to relate to the hydrodynamic fluid of relativistic electrons described in the previous section. The position of the event horizon \(r_H\) is fixed in terms of the chemical potential and the Hawking temperature,

\[
r_H = \frac{1}{k_B T} \left[\frac{6\pi}{L^2} + \frac{3\mu^2}{k_B T^2}\right]. \tag{II.20}
\]

The thermodynamic variables read

\[
\epsilon = \frac{L^2}{8\pi G_4} \hbar v_F r_H^3, \quad p = \frac{L^2}{16\pi G_4} \hbar v_F r_H^3, \quad \rho = \frac{L^2}{64\pi G_4} \left(\frac{r_H}{L^2}\right) \frac{\mu}{\hbar v_F}, \quad s = \frac{k_B r_H^2}{4G_4 L^2}. \tag{II.21}
\]

They satisfy the Gibbs-Duhem relation

\[
\epsilon + p = \mu \rho + T s. \tag{II.22}
\]

The dimensionless ratio of the AdS4 radius \(L\) and the gravity coupling \(G_4\) in \cite{(II.21)} is related to the rank of the gauge group \(N\) in the dual quantum field theory\cite{60}.

\[
\frac{L^2}{G_4} = \frac{1}{3} (2N)^{3/2}. \tag{II.23}
\]

In order to compare our results with perturbative results like \cite{(I.4)} as well as ultimately with experiment, we fix the rank \(N\) as follows: The experiment\cite{61} are performed in the limit of the chemical potential being much larger than the temperature, \(\mu \gg k_B T\). The renormalization of the Fermi velocity \(v_F\) is negligible at most temperatures except exponentially small ones\cite{24}. We fix \cite{(II.23)} by identifying the holographic number density that appears in \cite{(II.21)} with the Fermi-Dirac distribution,

\[
\rho_{\text{Dirac}} = n_h - n_h^+ = \frac{g}{(2\pi)^2} \int d^2p \frac{1}{e^{\left(\epsilon_F - p\right)/T} + 1} - \frac{1}{e^{\left(\epsilon_F + p\right)/T} + 1}. \tag{II.24}
\]

where \(g\) stands for the degeneracy. For massless fermions, \(E = v_F |p| = \hbar v_F |k|\). Integrating \cite{(II.24)} yields

\[
\rho_{\text{Dirac}} = -g \left(\frac{k_B T}{\hbar v_F}\right)^2 \left[\frac{1}{2\pi} \text{Li}_2 \left(-e^{\mu/k_B T}\right) + \frac{\pi}{24}\right]. \tag{II.25}
\]
with $\text{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$. At leading order in $\mu/k_B T \gg 1$ we obtain

$$\rho_{\text{Dirac}} \bigg|_{\mu \gg k_B T} = \frac{g}{4\pi} \frac{\mu^2}{h^2 v_F^2}. \tag{II.26}$$

In the same limit, the number density (II.21) becomes

$$\rho_{\text{ADS}} \bigg|_{\mu \gg k_B T} = \frac{1}{128\pi \sqrt{3}} \frac{L^2 \mu^2}{G_4 h^2 v_F^2}. \tag{II.27}$$

Identifying (II.26) with (II.27) in the limit $\mu/k_B T \gg 1$, the ratio (II.28) can be fixed to

$$\frac{L^2}{G_4} = 32\sqrt{3} \gamma. \tag{II.28}$$

For $g = 2$, we find $N \approx 24$, compatible with the large $N$-limit in holography. We will use this value of $N$ for numerical estimates. Figure 2 shows the ratio between the charge densities from the Fermi-Dirac distribution and from holography. The difference is small for $\mu/k_B T > 10$, and not too large for values closer to the charge neutrality point. This is due to the coinciding $\mu/k_B T$ dependence of both densities for $\mu > k_B T$. The dependence on $\mu/(k_B T)$ also drops out in this ratio for $\mu/(k_B T) \ll 1$. We hence could have matched the two densities at the charge neutrality point as well, obtaining a slightly smaller value of $N \approx 5$. Experience from applications of AdS/CFT to QCD suggests that this value of $N$ is also sufficiently large to suppress $1/N$ corrections.

### III. Finite size effects

#### A. Poiseuille flow

Presently, we investigate the physical effects that arise from the motion of a fluid in a channel of finite size. Considering an incompressible fluid moving along a wire of width $W$ and length $l$. We may think of the fluid as a superposition of fluid layers along the width of the channel. The fluid’s flow is called laminar, if these layers never mix. A particular kind of laminar flow, is the Poiseuille flow characterized by the no-slip boundary conditions,

$$u'(y) \big|_{y=W} = 0, \quad u'(y) \big|_{y=0} = 0. \tag{III.1}$$

Simply stated, the velocity vanishes at the boundaries. A parametrization of the velocity $u^\rho$ is given by

$$u^\rho = \gamma(y) (v_F, \beta_x(y), 0), \quad \gamma = \frac{1}{\sqrt{1 - \frac{\beta_x(y)^2}{v_F^2}}} \tag{III.2}$$

This Ansatz is incompressible,

$$\partial_y u^\rho = \partial_0 u^\rho + \partial_i u^i = \frac{1}{v_F} \partial_y u^\rho + \partial_i u^i = 0. \tag{III.3}$$

Moreover, the boundary condition (III.1) is equivalent to

$$\beta_x(y) \big|_{y=W} = 0, \quad \beta_x(y) \big|_{y=0} = 0. \tag{III.4}$$

Now we derive the hydrodynamic equations in terms of $\beta_x$ (II.16). Focusing on the case of a homogeneous external electric field pointing along the $x$-direction, the current density (II.11) reads

$$j^\mu = ev_F \gamma \left( \rho + \frac{\hbar \epsilon_0}{e^2} \frac{\beta_x}{v_F} \epsilon_Q E^x, \rho \frac{\beta_x}{v_F} + \frac{\hbar \epsilon_0}{e^2} \epsilon_Q E^x, 0 \right). \tag{III.5}$$

Here $\epsilon_0 \epsilon_r$ is the electric permittivity of the fluid, $\rho$ its number density, $v_F$ its Fermi velocity, and $\sigma_Q$ its quantum critical conductivity. Decomposing (II.16) into components, we obtain two independent equations

$$0 = \eta \left( \beta_x' + \frac{2}{v_F} \gamma^2 \beta_x \beta_x' \right) + \frac{1}{\gamma} \left( v_F \epsilon_r \frac{\hbar \epsilon_0}{e^2} \epsilon_Q E^x \rho \frac{\beta_x}{v_F} - \rho + \frac{\epsilon}{\gamma} \beta_x \right) \tag{III.6} \frac{\rho + \epsilon_0 \beta_x}{v_F} \frac{E^x}{v_F} \left( \beta_x \right), \tag{III.7}$$

We find a dynamical equation for the velocity profile (III.2), as well as a relativistic form of the Hagen-Poiseuille equation for the pressure. For strongly coupled theories, the ratio $\eta/s$ is fixed to a constant value, which is why we divided (III.6) by the entropy density $s$. The solution to the pressure equation (III.7) is given by

$$p(y) = p_0 + \frac{\eta}{\tau_{\text{imp}}} \gamma(y). \tag{III.8}$$

Note that the ratio $\frac{\eta}{\tau_{\text{imp}}}$ controlling the pressure shift will be small in the parameter regime of applicability of hydrodynamics (II.1). The non-equilibrium fluid pressure (III.8) will hence be close to the thermodynamic one as e.g. derived from the AdS solution, (II.20), $p \approx p_0(\mu, T)$. Therefore, we may use the homogeneous thermodynamic variables (II.21) even in the presence of impurities, as long as we restrict ourselves to small electric fields and non-relativistic flows. In that limit, the pressure shift
becomes constant in the direction $y$ perpendicular to the channel,
\[ p = p_0 + \frac{\eta}{\tau_{\text{imp}}} \gamma, \quad \text{(III.9)} \]
since $\gamma \sim 1$. The pressure shift may however become large and strongly $y$-dependent for relativistic flows, and possibly observable.

**B. Momentum relaxation through the walls**

In this subsection we will show that the Gurzhi channel setup of Fig. 4 allows for a novel mechanism of momentum relaxation, momentum relaxation through the boundaries. We will also introduce and calculate the relaxation time scale $\tau_w$ associated to this new mechanism of relaxing momentum.

For a stationary flow, if we suddenly turn off the electric field, momentum will diffuse through the walls, defined at $y = 0$ and $y = W$. Therefore, the $x$ component of the fluid momentum along the velocity direction changes due to momentum loss through the boundaries and through collisions with impurities, i.e.
\[ \frac{1}{v_F} \partial_t T^{tx} = \partial_y T^{wx} - \frac{T^{tx}}{v_F \tau_{\text{imp}}}. \quad \text{(III.10)} \]

We parametrize the rate of momentum loss through the boundaries by $\tau_w$. Analogously to $\tau_{\text{imp}}$, the impurity scattering rate, $\tau_w$ is also related to momentum loss. However, this particular form of momentum diffusion appears only from finite size effects, unlike $\tau_{\text{imp}}$ which can also be present in bulk samples.

If we approximate the time derivative that appears in (III.10) by
\[ \partial_t T^{tx} \sim \frac{T^{tx}}{\tau_w}, \quad \text{(III.11)} \]
and integrate along the $y$ direction, we obtain
\[ \frac{1}{v_F} \int_0^W dy T^{tx} \left( \frac{1}{\tau_w} + \frac{1}{\tau_{\text{imp}}} \right) = T^{xy} \bigg|_0^W. \quad \text{(III.12)} \]

From the no-slip conditions (III.4), we have that
\[ T^{xy} \bigg|_0^W = -\eta \beta_x'(y) \bigg|_0^W. \quad \text{(III.13)} \]

Notice that the integral in the left-hand side of (III.12) is non-divergent, precisely because we are integrating over the width of a finite channel. Substituting $T^{0x}$’s components, we obtain
\[ \int_0^W dy \frac{\beta_x}{v_F} (p + \epsilon) \gamma^2 \left( \frac{1}{\tau_w} + \frac{1}{\tau_{\text{imp}}} \right) = -\eta \beta_x'(y) \bigg|_0^W. \quad \text{(III.14)} \]

Equation (III.14) can be solved for $\tau_w$ once the velocity profile and thermodynamic variables are specified.

**C. Energy relaxation**

Similarly to the momentum of the fluid, the energy of the fluid is subject to relaxation effects. We discuss those effects in this subsection.

If a charged fluid initially at rest at time $t = 0$ is accelerated by an electric field, it will reach a steady state when the gain of momentum due to the electric field is balanced by the loss of momentum through momentum relaxing mechanisms. A similar process occurs for the fluid’s energy density, $\epsilon$. The value of $\epsilon$ for a fluid at rest is given by the thermodynamic energy density. However, the energy density in the nonequilibrium steady state may be different due to additional contributions from Joule heating and viscous effects. We label the time required to attain the steady state value by $\tau_\epsilon$. Assuming a time-dependent energy density, $\epsilon = \epsilon(t)$, for the fluid profile given by (III.2), (III.13) gives the energy evolution equation
\[ \partial_t \epsilon = \frac{1}{\gamma} \frac{\hbar^2 v_F^2 e^2}{e^4} \epsilon^2 \sigma Q E_z^2 + \gamma^3 \eta \beta_x^2 = \dot{\epsilon}_{\text{JH}} + \dot{\epsilon}_V, \quad \text{(III.15)} \]
where the first term, $\dot{\epsilon}_{\text{JH}}$, is due to intrinsic Joule heating, while the second term, labeled as $\dot{\epsilon}_V$, is due to viscous effects. In the small velocity approximation ($\beta_x \ll v_F$), $\dot{\epsilon}_V$ is suppressed, while
\[ \partial_t \epsilon \sim E_z^2 \sigma Q. \quad \text{(III.16)} \]
is the standard expression for the Joule heating. Since the small velocity approximation also implies that the electric field is small, the Joule heating term, $\dot{\epsilon}_{\text{JH}}$, becomes negligible. On the other hand, if the velocity is large, $\beta_x \sim v_F$ and $\gamma \to \infty$ (see (III.2)), the $\gamma$-dependence on both terms ensures the Joule heating term $\dot{\epsilon}_{\text{JH}}$ to be subleading compared to the viscous contribution $\dot{\epsilon}_V$.

We can integrate (III.15) between $t \in [0, \tau_\epsilon]$, giving
\[ \epsilon(\tau_\epsilon, \mu, T) = \epsilon_0(\mu, T) + \tau_\epsilon \left[ \frac{1}{\gamma} \frac{\hbar^2 v_F^2 e^2}{e^4} \epsilon^2 \sigma Q E_z^2 + \gamma^3 \eta \beta_x^2 \right]. \quad \text{(III.17)} \]

The interpretation of this equation is that both the electric field and viscous effects introduce work into the system, shifting the energy from its initial value $\epsilon_0(\mu, T)$ to a final value $\epsilon(\tau_\epsilon, \mu, T)$. In order to estimate how large such a deviation is, we need to estimate $\tau_\epsilon$. In the $\mu \gg k_B T$ regime, the only low energy scale is $T$. Therefore, the energy relaxation time $\tau_\epsilon$ can be approximated by the Planckian dissipation time, given by
\[ \tau_\epsilon \sim \frac{\hbar}{k_B T} \sim 10^{-11} \frac{T}{\text{K}} \cdot \text{s}. \quad \text{(III.18)} \]
Using the input parameters from sec. IV, in particular the holographic value of $\sigma_Q$ from (IV.12), we verified that the energy is approximately unchanged, i.e. $(\epsilon - \epsilon_0)/\epsilon_0 \sim 10^{-9}$. This is true for all the calculations presented hereafter.

IV. FLOW AND DIFFERENTIAL RESISTANCE IN THE ABSENCE OF IMPURITIES

A. Velocity profiles

We now proceed to the calculation of the velocity profile $\beta_x$ of the fluid as a function of the coordinate $y$ perpendicular to the fluid motion. To do this, we numerically integrate the Navier-Stokes equation (III.6) with the boundary conditions (III.4). This will allow us to determine how the fluid profile depends on the applied electric field, the temperature, the chemical potential and the width of the wire. Moreover, from the velocity profile we will calculate the differential resistance $dV/dI$. In this section, we will consider the case where impurities are absent ($\tau_{\text{imp}} \to \infty$), leaving the inclusion of impurity effects to sec. V. For the sake of completeness, we shall not impose any restriction on the maximal fluid velocity, which can be arbitrarily close to the Fermi velocity. Therefore, it is necessary to solve the fully relativistic equation (III.6) numerically. First, it is convenient to define the reduced variables

$$ u = \frac{k_BT}{h v_F} y, \quad \mathcal{E}^x = \frac{E_x v_F^2 h^2 e \epsilon_0 \epsilon_r}{T^2 e k_B^2}, \quad w = \frac{k_BT}{h v_F} W. \quad (IV.1) $$

The Fermi velocity $v_F$ and relative permittivity $\epsilon_r$, which are intrinsic properties of the material, need to be fixed. For concreteness, we take values typical for the surface states of the topological insulator, such as HgTe(61),

$$ v_F = 10^5 \text{m/s}, \quad \epsilon_r \sim 3 - 5. \quad (IV.2) $$

The choice of parameters (IV.2) coincides with having a rather strong electron-electron Coulomb interaction. Indeed, the effective fine-structure constant in a medium with the input parameters (IV.2) is rather large,

$$ \alpha_{\text{eff}} = \frac{\alpha_0 c}{v_F \epsilon_r} \approx 4.4 \gg \alpha_0, \quad \alpha_0 \approx \frac{1}{137}. \quad (IV.3) $$

$\epsilon_r$ is similar in graphene(2) but $v_F = 10^6$ m/s. For the Fermi velocity and relative permittivity (IV.2), $\mathcal{E}^x = 1$ is equivalent to $E_x \approx 1.5 \times 10^{-3} \text{ V/\mu m}$ whereas $w = 1$ to $W \approx 5 \text{ \mu m}$. Then, (III.6) becomes

$$ \frac{\eta}{s} \left( \bar{\beta}_x + \frac{2\gamma^2}{v_F^2 \beta_x \beta_x^2} \right) + \hbar \frac{v_F \mathcal{E}^x}{\gamma^2} \frac{\rho}{s} = 0, \quad \beta_x = \frac{\partial \bar{\beta}_x}{\partial u}. \quad (IV.4) $$

At this point, we remark the high sensitivity of the profile $\beta_x$ on the set of external values $\{\eta/s, \mathcal{E}^x, \mu/k_BT\}$. In particular, from the first term in (IV.4) it is clear that reducing $\eta/s$ while keeping the electric field fixed, one can reach higher maximal velocities in the middle of the channel. This is due to the fact that $\eta/s$ controls the momentum transfer between adjacent fluid layers. In terms of the reduced variable $u$, the boundary conditions (III.4) are rewritten as

$$ \beta_x(u) \bigg|_{u=0} = \beta_x(u) \bigg|_{u=w} = 0. \quad (IV.5) $$

The set of variables

$$ \{w, \mathcal{E}^x, \mu, T, \eta/s\} \quad (IV.6) $$

defines the parameter space of possible fluid velocities. Hereafter, we will refer to this set as input parameters. Throughout the present work, we will often draw our attention to the ratio $\eta/s$, which controls important physical properties of the fluid. It has been discussed both in high energy physics(12,62,63) as well as in the context of condensed matter physics(20) that this ratio depends strongly on the coupling and, hence, is a measure of the coupling strength itself. In the extreme strong coupling limit, it is conjectured to attain the universal AdS/CFT value (12). For a phenomenological investigation of the dependence of the velocity profile and the differential resistance on the coupling strength, we will vary the value of $\eta/s$, increasing it from the AdS/CFT value. This will correspond to moving to smaller couplings beginning from the extreme strong coupling limit. To avoid large corrections, we will remain close to the holographic bound (12), i.e. increase $\eta/s$ by at most a factor of 20.

We now list the results for the velocity profiles and conclusions obtained from this analysis. Each plot is obtained by varying one parameter while keeping the others fixed.

- First, from figure 3 we infer that the fluid attains a higher maximal velocity in the middle of the channel as the width $w$ is increased. The reason is that for wider
The velocity of the fluid $\beta_x^u$ for the Poiseuille flow as a function of the dimensionless variable $u$ at $E^x = 1, w = 1$ and different values of $\mu/k_B T$. From top to bottom, $\mu/k_B T = 100, 50, 20, 10, 5, 1$. If we assume constant $T$, as the density increases, so does the velocity of the fluid.

Increasing the electric field implies that the fluid velocity increases.

Increasing the external electric field $E^x$ increases the velocity of the fluid (see figure 5), since a stronger electric field transfers more momentum to the system.

Finally, from (IV.4), we see that if $\eta/s$ increases, the kinetic term needs to become smaller in order for the equation to be satisfied at a fixed applied electric field. We confirm this from figure 6, where we display the velocity profile as function of increasing $\eta/s$. We find that strongly coupled fluids move faster than their counterparts at weaker coupling, i.e. at larger $\eta/s$. This is one of the main results of our work.

We now determine the range of values for the parameters $\{\eta/s, E^x, \mu/k_B T\}$ for which the fluid flows relativistically, i.e. for which the velocity is not small any more compared to the Fermi velocity. From the final point above, we expect the fluid to flow fastest in the strong coupling limit, and hence we take $\eta/s$ to be the holographic value (I.2). From figure 7, we see that at $E^x \approx 5$ (corresponding to $E_0 \approx 7.5 \text{ mV/} \mu\text{m}$), $\mu/k_B T \approx 0.5$, and the width $w = 1$, the velocity becomes about 10% of the Fermi velocity. Hence, for holographic fluids satisfying (I.2), the relativistic regime can be reached for small electric fields and chemical potentials.
FIG. 7. Keeping the ratio $\eta/s$ fixed to $\hbar/4\pi k_B$, we plot the maximum speed of the fluid $\beta_x$ as a function of the ratio $\mu/k_B T$ at different values of the electric field $E^x = 5, 1, 0.1, 0.01, 10^{-3}$. For illustrative purposes, we have set $w = 1$. In order to present all possible curves into a single plot, we have employed a logarithmic scale, both on the $x$-axis and $y$-axis. Note also that $\beta_x$ never reaches the Fermi velocity $v_F$.

B. Momentum relaxation

The wall relaxation time $\tau_w$ introduced in sec. IIIB characterizes the rate of momentum outflow through the boundaries. It is expected to be finite due to the finite size $w$ of the channel. Nevertheless, the magnitude of this time-scale relative to the problem (c.f. II.1) is a priori unclear. In this section we compute $\tau_w$ as a function of the input parameters $\mu/k_B T, \varepsilon_x, w$ and $\eta/s$ in the absence of impurities. The relevant parameter to compare $\tau_w$ to is the energy relaxation time $\tau_{\varepsilon}^{E\parallel}$ which was related to the Planckian dissipation time in (III.18). Again, we resort to numerics in order to evaluate (III.14) on the profile solution $\beta_x(y)$.

In figure 8 we plot $\tau_w$ as a respective function of one of the input parameters $\mu/k_B T, \varepsilon_x, w$ and $\eta/s$, while keeping fixed all other input parameters. From these plots, we infer that $\tau_w$ is closely related to the speed of the hydrodynamic fluid. Combinations of input parameters that lead to higher velocities will also unavoidably lead to a larger $\tau_w$. The underlying reason is that $\tau_w$ is a measure for the time needed to lose all the momentum in the flow through the walls. This process naturally takes longer if there is more total momentum in the flow, i.e. if the velocity $\beta_x(y)$ is larger. By looking at the slope $\tau_w$ as a function of $\eta/s$ in the lower right plot of fig. 8 we find a scaling law of the wall relaxation time as a function of $\eta/s$,

$$\tau_w \propto (\eta/s)^{-1}. \quad \text{(IV.7)}$$

From 8 we see that the typical window of values for $\tau_w$ is

$$\tau_w \in [10^{-9} - 10^{-4}] \text{ s}, \quad \text{(IV.8)}$$

larger than the energy relaxation time $\tau_{\varepsilon} \sim 10^{-11} - 10^{-8} \text{ s}$, as defined in (III.18). Having $\tau_w \gg \tau_{\varepsilon}$ means that energy relaxation occurs much faster than momentum relaxation through the walls. We should also point out that what we computed in (III.11) is the instantaneous rate of momentum loss after instantaneously switching off the electric field $E_x$. In other words, we assumed that the profile $\beta_x(y)$ after switching off the electric field is still the steady state Poiseuille flow. It is expected that the process of momentum loss will slow down as the flow profile $\beta_x(y, t)$ evolves non-linearly as a function of time. Hence we expect that the obtained $\tau_w$ is a lower bound to the $\tau_{\varepsilon}$ that would be calculated from solving the time-dependent hydrodynamic equations (II.13) and (II.14).

C. Differential wire resistance

In this section we determine the differential wire resistance $\mathcal{R}$, defined by the inverse of the derivative of the current $I$ through the wire with respect to the voltage $V$,

$$\mathcal{R}(I) = \frac{dV}{dI}. \quad \text{(IV.9)}$$

Using (II.11), the total current $I$ that flows across the channel is given by

$$I = \int_0^W j^x(y)dy = \int_0^W \left[ j^x_{(\text{fluid})} + j^x_{(Q)} \right] dy, \quad \text{(IV.10)}$$

with

$$j^x_{(\text{fluid})} = e\mu w^x, \quad j^x_{(Q)} = \sigma_Q \frac{h}{e^2 c} u_{\alpha} D^{x\alpha}. \quad \text{(IV.11)}$$

From (IV.10) and (III.5), we observe that there are two different types of contributions to $I$: The first contribution, $j^x_{(\text{fluid})} \propto u^x$, is the flow of the particles along the transverse section of the channel. The second contribution, labeled as $j^x_{(Q)}$, is related to the quantum critical conductivity $\sigma_Q$ and it accounts for the counterflow of electrons and holes. For the charged black brane model (II.17), $\sigma_Q$ is given by

$$\sigma_Q = \left( \frac{sT}{\varepsilon + p} \right)^2 \frac{L^2}{G_4} \frac{e^2}{2\hbar}, \quad \text{(IV.12)}$$

where, as explained in sec. IIIB $L^2/G_4$ is fixed via (II.28).

Since we apply a constant electric field along the $x$ direction, the voltage is proportional to the applied electric field, $V = E^x l$, with $l$ the length of the channel, assumed much larger than the width $W$. Both Eqs. (IV.10) and (IV.9) have to be evaluated numerically, since the velocity profile is obtained from a numerical integration. We recast both the $I$ and the differential resistance $\mathcal{R}$ in terms of the reduced variables (IV.1),

$$I = \int_0^w du \gamma(u) \left[ e\beta_x(u) + \sigma_Q E^x \frac{T^2 k_B^2}{e^2 h v_F} \right], \quad \text{(IV.13)}$$

$$\mathcal{R} = \frac{ek_B^2}{h^2 \varepsilon_0 v_F c} T^2 \left( \frac{dI}{dE} \right)^{-1}. \quad \text{(IV.13)}$$
In sec. [IV.A], we focused on deriving the qualitative features of the Poiseuille flow at strong coupling. The aim of this subsection is to obtain a realistic prediction for the value of the resistance. Therefore, we need to use a physically viable combination of input parameters. In experiments, the temperatures are in the range of a few Kelvin, whereas the width and length are of order of a few micrometers. We will hereafter assume $T = 2K$, $W = 4\mu m$ and $l = 20\mu m$. (IV.14)

The chemical potential can be found from the number density $\rho$ after solving (II.21) for $\mu$. Taking for instance the typical value of the density in experiments, $\rho \simeq 10^{11} \text{cm}^{-2}$, and a spin degeneracy of $g = 2$ as well as (IV.14), we find $\mu \simeq 4.5 \text{meV}$. (IV.15)

As shown in fig. 2 for this particular estimate there is not a large difference between the holographic number density (II.21) and the Dirac number density (II.25), since we matched the two in the low-temperature regime ($\mu/k_B T > 20$) in sec. [II.B]. The Fermi velocity and relative permittivity are set according to (IV.2).

In fig. 9 we display the differential resistance as a function of $I$ at different values of the width $W$. It starts with a finite value at $I = 0$, and decreases with increasing current until it asymptotes to zero. Increasing the width of the channel leads to a decrease on the differential resistance, as expected for a Poiseuille flow. One might expect that at low $I$, or equivalently $\beta_x \ll v_F$, the differential resistance should coincide with the quantum critical resistance, $R_Q \sim l/(W \sigma_Q)$, since at low $I$ there is no macroscopic charge transfer and the wire should act as an Ohmic resistor with an intrinsic conductivity $\sigma_Q$ given by (IV.12). However, this expectation does not bear out. Consider the solution for $\beta_x$ as obtained from (III.6) restricted to small values of $I$. In this limit the applied electric field must be small and $\beta_x \ll v_F$. The convection term that appears in the right hand side of (III.6) can be neglected and we obtain a linear differential equation

$$0 = \frac{h \epsilon_0 v_F E_x x \rho}{e \eta} + \beta''_x(y),$$

where its solution, consistent also with the boundary conditions (III.4), is given by the usual parabolic profile of the non-relativistic Poiseuille flow

$$\beta_x = \epsilon_0 \frac{h \epsilon_0 v_F E_x x \rho}{e \eta} \frac{y(W - y)}{2\eta}.$$  (IV.17)

Inserting (IV.17) into (IV.10) and integrating over $y$, we obtain the differential resistance

$$\frac{1}{R} = \frac{h \epsilon_0 v_F W}{l} \left( \frac{\sigma_Q}{e^2} + \frac{W^2 \rho^2}{12\eta} \right) = \frac{1}{R_Q} + \frac{1}{R_\eta}. \quad (IV.18)$$

Equation (IV.18) contains two contributions of different physical origin: the first comes from the quantum critical resistance $R_Q$, while the second, $R_\eta$, is the viscous
resistance generated by viscous effects alone. At fixed chemical potential and temperature, the quantum critical conductance [IV.12] as well as the number density $\rho$ defined in [II.21] are both fixed and

$$ \mathcal{R}_Q^{-1} \propto W, \quad \mathcal{R}_n^{-1} \propto \frac{W^3}{\eta}. \quad (IV.19) $$

Therefore, $\mathcal{R} \simeq \mathcal{R}_Q$ only if $W$ is small, or $\eta$ large. Finally, we see that the two contributions to $\mathcal{R}_Q$, as given by (III.16) for finite $\tau$ and the role of $\mathcal{R}_n^{-1}$ is played by $\mathcal{R}_Q$ and the role of $\mathcal{R}_2^{-1}$ is played by $\mathcal{R}_n$.

The asymptotic form of $\mathcal{R}(I)$ in Fig.9 is related to the asymptotic behavior of $\beta_x$ when approaching the Fermi velocity. For large enough electric field, the velocity profile approaches the Fermi velocity, and the first piece of $\mathcal{R}_Q^{-1}$ asymptotes to a constant value, while the second contribution is still linear in $E_x \propto V$. Taking the derivative with respect to $V$, only the second contribution survives, and the resistance asymptotes to zero due to the $\gamma$ factor. This asymptotic behavior is universal, in the sense that it is not affected by the choice of input parameters within the regime of applicability of hydrodynamics.

We now analyze the dependence of the resistance $\mathcal{R}$ on the viscosity over entropy density ratio $\eta/s$. This provides important information for two reasons: First, as explained in sec. $\eta/s$ can be understood as a measure for the coupling strength. Second, if the remaining input parameters are known and two different samples are well in the hydrodynamic regime, it will be possible to infer the relative value of $\eta/s$ through a measurement of the wire resistance at small current. This will be of interest from the experimental point of view. In figure 10 we plot the wire resistance $\mathcal{R}$ as a function of the integrated current $I$ at different ratios of $\eta/s$. We find that $dV/dI$ increases with increasing $\eta/s$. This behavior is related to the decrease of the maximal velocity $\beta_x$, and hence of the integral $\mathcal{R}$, as $\eta/s$ increases. This is the second main result of this work.

V. FLOW AND DIFFERENTIAL RESISTANCE IN THE PRESENCE OF IMPURITIES

We now examine the effect of impurities on the results of section IV as is essential in view of comparison with experiments. In this case, the AdS/CFT results [II.21] are no longer directly applicable since the pressure becomes $y$-dependent, as given by (III.7) for finite $\tau_{\text{imp}}$. In the non-relativistic limit, however, where $\gamma \to 1$, this $y$-dependence drops out again from all the thermodynamic variables. In this limit, the Joule heating term, which according to (III.16) is the only one relevant for the time derivative of the energy density $\dot{\varepsilon}$, becomes negligible, such that the energy density remains constant.

We thus consider the non-relativistic limit in this section. This restricts us to small values of the applied electric field. On the other hand, we are able to obtain analytical results in this regime, and to perform a general hydrodynamical analysis applicable to any value of the electron-electron coupling.

A. Velocity profiles

We begin by examining the velocity profile of the fluid. As discussed above, we consider the non-relativistic limit, in which the non-linear terms that enter the Navier-Stokes equations [III.6] are suppressed and it is possible
to find an analytic solution for $\beta_x$. Linearizing \(\text{III.6}\) in $\beta_x$ gives
\[
\eta s \left[ \beta_x + \frac{\beta_x}{\tau_{\text{imp}} v_F^2} \right] + E_x q v_F \frac{\rho}{s} \left( \frac{\hbar \epsilon_0}{e} \right) - \frac{\beta_x}{\tau_{\text{imp}} v_F^2} \frac{\epsilon + p}{s} = 0,
\]
where we made use of \(\text{III.9}\); $p = p(\mu, T) + \eta/\tau_{\text{imp}}$. Imposing $\beta_x(0) = \beta_x(W) = 0$, we obtain the profile as a function of the $y$ coordinate,
\[
\beta_x(y) = v_F q v_F \frac{E_x \hbar \epsilon_0}{\eta s} \frac{\rho}{\eta y^2} \times \left[ 1 - \cosh (\mathcal{G} y) + \sinh (\mathcal{G} y) \tanh \left( \frac{\mathcal{G} W}{2} \right) \right],
\]
with
\[
\mathcal{G} = \frac{1}{v_F} \sqrt{\frac{\epsilon + p}{\tau_{\text{imp}}} + \frac{1}{\tau_{\text{imp}}}^2}, \quad |\mathcal{G}| = \text{m}^{-1}.
\]
Note that the result for the velocity profile is analytical in the limit considered here, while the relativistic result of the preceding section is numerical. We also note that $\beta_x$ given by \(\text{V.2}\) reduces to the parabolic flow \(\text{IV.17}\) in the limit $\tau_{\text{imp}} \to \infty$, as expected. In Fig. 11 we display the velocity profile of \(\text{V.2}\) as function of the impurity scattering time. We observe that when increasing the impurity density while keeping all other input parameters fixed, the velocity decreases.

If the density of impurities is such that $\tau_{\text{imp}}$ becomes the shortest time scale of the system, shorter than the electron-electron scattering time, then impurity collision effects are dominant. In this case, the Poiseuille hydrodynamic behavior is suppressed and a standard Ohmic conductivity law is expected instead. Indeed, the Poiseuille flow connects smoothly to an Ohmic regime as may be expected. In Fig. 11, we display the velocity profile of \(\text{V.2}\) as function of the impurity scattering time. We observe that when increasing the impurity density while keeping all other input parameters fixed, the velocity decreases.

for the input parameters given around \(\text{IV.2}\) and \(\text{IV.15}\). This implies that the quantum critical conductivity plays a more significant role in the current $j^*$ of \(\text{V.5}\), although both are of the same order of magnitude.

This shows that at strong coupling, the electron-hole scattering effect that gives rise to the quantum critical conductivity cannot be neglected compared to impurity scattering: At weak coupling, the quantum critical resistivity drops to zero as $\alpha^2(T)$ and the Drude resistivity is expected to approach a constant. At strong coupling on the other hand, the quantum critical resistivity is expected to saturate as well, to a universal value $\sim \frac{\hbar}{2\pi}$. Hence, while at weak coupling the ratio \(\text{V.7}\) is expected to approach zero, it is expected to asymptote to a constant for strongly interacting electron systems. We will further elaborate on the differences of strong and weakly coupled electron systems in sec. \(\text{VII}\).

In conclusion, we observe that when increasing the impurity density while keeping all other input parameters fixed, the fluid velocity decreases. Moreover, if $\tau_{\text{imp}} < \tau_{\text{ee}}$, the flow will no longer be of Poiseuille form, since the impurities absorb momentum. If the density of impurities is large enough, impurity effects dominate and the electron flow becomes Ohmic. Since the transition between the Poiseuille and Ohmic flows is determined by the interplay of two scales, the electron-electron scattering rate and the impurity scattering rate, this transition is a cross-over.

### B. Momentum relaxation

Here we calculate the wall relaxation time $\tau_{\text{w}}$ within hydrodynamics for the Poiseuille flow \(\text{V.2}\). The result is valid for any value of the electron-electron coupling. This is particularly interesting from the AdS/CFT point of view since wall momentum relaxation has not yet been
analyzed in this context. Moreover, we aim at retrieving the physics associated to the interplay of impurity and finite size effects, both of relevance in experiments.

Since the velocity profile $\beta_x$ of (V.2) is known analytically, $\tau_w$ may also be computed analytically. We expand (III.14) at small $\beta_x$,

$$\int_0^W dy \frac{\beta_x}{v_F} (p + \varepsilon) \left( \frac{1}{\tau_w} + \frac{1}{\tau_{\text{imp}}} \right) = \eta \beta_x(y)|_{y=W}^y=0, \quad (V.8)$$

and insert the solution (V.2) into (V.8) to find

$$\tau_w = \frac{\tau_{\text{imp}}}{2} \left[ \frac{W G}{4 \tanh \left( \frac{W^2}{2} \right) - W G} - 1 \right]. \quad (V.9)$$

As we see from (V.3), $G$ is an involved function of the thermodynamic variables, the shear viscosity $\eta$ and the impurity scattering rate $\tau_{\text{imp}}$.

For a fluid propagating through the channel, we expect that the wall momentum relaxation time is larger in presence than in absence of impurities, $\tau_w(\tau_{\text{imp}} \to \infty) > \tau_w(\tau_{\text{imp}} \to \infty)$. This is due to impurity momentum relaxation competing with wall momentum absorption. Indeed, if we consider a small density of impurities or equivalently large $\tau_{\text{imp}}$, we find from expanding (V.9) that the wall momentum relaxation time in the presence of impurities is always larger than in a completely clean sample,

$$\tau_w(\tau_{\text{imp}} < \infty) \sim \frac{\tau_w(\tau_{\text{imp}} \to \infty)}{\tau_w(\tau_{\text{imp}} \to \infty) + \frac{W^2(p + \varepsilon)^2}{180 \tau_{\text{imp}} v_F^4}}. \quad (V.10)$$

with

$$\tau_w(\tau_{\text{imp}} \to \infty) = \frac{W^2(p + \varepsilon)}{12 \eta v_F^2}. \quad (V.11)$$

This behaviour is expected from the balance equation (III.14), in which the r.h.s. is determined by the velocity profile at fixed $\eta$, whereas the l.h.s. is proportional to the sum of $\tau_w^{-1}$ and $\tau_{\text{imp}}^{-1}$. Therefore, for a fixed velocity profile $\beta_x$, $\tau_w$ becomes larger if $\tau_{\text{imp}}$ decreases. Equivalently, momentum diffusion through the boundaries is impeded by the presence of impurities, even though both mechanisms lead to momentum absorption.

C. Differential resistance

Finally we compute the differential resistance of our channel in the presence of impurities. We use again the same formulae for the current and differential resistance, (IV.10) and (IV.9). As before, for a constant energy density even in the presence of impurities, we have to consider the non-relativistic limit. This in turn leads to analytical results.

We insert the velocity profile (V.2) into the current (IV.10) and find

$$I = \frac{V}{t} v_F \varepsilon_r (\hbar \varepsilon_r) \left\{ \frac{W^2}{2} C_{\text{Q}} + \frac{\rho^2}{G^2 \eta} \left[ W G - 2 \tanh \left( \frac{W G}{2} \right) \right] \right\}. \quad (V.12)$$

Here, $W$ is the dimensionful width of the channel, $l$ is its length, $G$ is given by (V.3), $\sigma_Q$ is the quantum critical conductivity, $\rho$ is the number density and $\eta$ the shear viscosity. (V.12) is linear in $V$, so that

$$R^{-1} = \frac{I}{V} = v_F \hbar \varepsilon_r (\hbar \varepsilon_r) \frac{1}{l} \times \left\{ \frac{W^2}{2} C_{\text{Q}} + \frac{\rho^2}{G^2 \eta} \left[ W G - 2 \tanh \left( \frac{W G}{2} \right) \right] \right\}. \quad (V.13)$$

The differential resistance $R$ given by (V.13) is independent of the current $I$. This follows from the non-relativistic limit in which $I$ is linear in $V \propto E_x$. Terms of higher order in $E_x^2$ contributing to the equations of motion (V.1) and to the velocity profile solution (V.2) would signal relativistic effects, as we may see from the analysis of the velocity profile when impurities are absent (Fig.7). This means that the relation between the current $I$ and the quantities on the r.h.s. of (V.12) is valid only for small current $I$, in consistency with the approximation of small $E_x$. Nevertheless, (V.12) can be used to predict the value of the maximum of the differential resistance in a hydrodynamic Poiseuille flow once the input parameters and thermodynamic variables are fixed.

Viscous and impurity effects cannot be disentangled in the expression for $R$ (recall that $G$, given by (V.3), in (V.13), depends both on $\eta$ and $\tau_{\text{imp}}$). This follows from the fact that the $j_{\text{fluid}}^I$ contribution to the current density (IV.11) inherits a non-trivial dependence on $\eta$ and $\tau_{\text{imp}}$ from the velocity profile (V.2). Viscous and impurity effects may not even be disentangled in the limit of vanishing $\tau_{\text{imp}} \to \infty$. This may be seen by expanding (V.13) at large $\tau_{\text{imp}},$

$$R^{-1}(\tau_{\text{imp}} < \infty) \sim R^{-1}(\tau_{\text{imp}} \to \infty) = \frac{W^5}{120 \eta^2 \tau_{\text{imp}}^2} \frac{h}{v_F} \ell_0 \varepsilon_r \rho^2 (p + \varepsilon), \quad (V.14)$$

with $R^{-1}(\tau_{\text{imp}} \to \infty)$ as defined in (IV.18). The second piece on the left-hand side of (V.14) depends on both $\eta$ and $\tau_{\text{imp}}$.

VI. REYNOLDS NUMBER FOR CHANNEL FLOWS

One of the main results of this work is that strongly coupled fluids flow fastest. For sufficiently large fluid velocities, the fluid can become turbulent. Between the laminar regime and fully developed turbulence, there is a transitional preturbulent regime in which both laminar and turbulent flows coexist. The Reynolds number $Re$ is
used to quantitatively determine the transition between laminar and turbulent flows. In appendix [B] we provide a derivation of the Reynolds number, both for ultra-relativistic and non-relativistic viscous fluids. We find that in the ultra-relativistic case the Reynolds number is 3/2 times larger that in the non-relativistic case. This implies that the turbulent regime is reached more easily in the ultra-relativistic limit. Nevertheless, since this is a relatively small enhancement, it is still reasonable that the flow of consideration in our set-up is laminar. The non-relativistic Reynolds number is shown in (B.4).

For Newtonian fluids moving through a 2+1-dimensional channel, a Reynolds number \( Re < W \) is associated with clearly laminar flows, while highly turbulent flows occur for \( Re \gg 100^{61,62} \). Using the KSS bound for \( \eta/s \) \([3]\), an upper bound on the Reynolds number \([3,4]\),

\[
Re \leq \frac{4\pi k_B (\varepsilon + p) W}{\hbar s v_{\text{max}}} \left( \frac{v_{\text{max}}}{v_F} \right)^2,
\]

(VI.1)
can be derived. Based on this, for the same maximal flow velocity \( v_{\text{max}} \) and width \( W \), it is reasonable to assume that a strongly coupled fluid is more likely to become turbulent than its weakly coupled counterpart. From (II.21), in the near Fermi liquid limit we obtain

\[
\varepsilon + p = \frac{\sqrt{3} \mu}{16\pi k_B}.
\]

(VI.2)

Hence, large chemical potentials enhance the transition to the turbulent regime. Estimating the right hand side of (VI.1) with input parameters (IV.14)-(IV.15), we find \( Re \leq 50 \). Therefore, for our channel widths, the flow is clearly laminar\([63,64]\), but if wider channels could be synthesized in the hydrodynamic regime, higher Reynolds numbers can be achieved since \( Re \propto W \). Also, following\([65,66]\) inserting an obstacle into the channel or considering flows through a constriction may allow for pre-turbulent physics such as vortex shedding.

However, impurities should be accounted for in this analysis. We can accomplish this by introducing the Gurzhi length \( \lambda_G \), defined as

\[
\lambda_G = v_F \sqrt{\kappa \tau_{\text{imp}}}, \quad \kappa = \frac{\eta}{\varepsilon + p},
\]

(VI.3)

where \( \kappa \) is the kinematic viscosity of the fluid.\([21]\) Using \( \lambda_G \) we may define two inequivalent regimes: First, if \( W < \lambda_G \), it is likely that the fluid will propagate according to a Poiseuille flow, with a large speed in the middle of the channel. By using the definition of Reynolds number in the ultra-relativistic limit \([3,6]\) derived in appendix [B], we find

\[
Re(W<\lambda_G) = \frac{\tau_{\text{imp}} v_F}{W} \left( \frac{W}{\lambda_G} \right)^2 \left( \frac{v_F}{v_{\text{max}}} \right)^3.
\]

(VI.4)

Notice that the condition \( W < \lambda_G \) introduces a bound on \( \tau_{\text{imp}} \). This bound is given by

\[
\tau_{\text{imp}} > \frac{W^2}{v_F \kappa}.
\]

(VI.5)

Since \( v_{\text{max}} \leq v_F \), even though they are of similar magnitude, using (VI.5) we find that the Reynolds number is bounded from below by

\[
Re > \frac{W}{v_F \kappa} \left( \frac{W}{\lambda_G} \right)^2.
\]

(VI.6)

Second, when \( W > \lambda_G \), the fluid motion signals the onset of an Ohmic flow although the no-slip conditions at the boundary are still satisfied. Qualitatively speaking, instead of a Poiseuille-type velocity profile, one should expect a plateau around the middle of the channel. We can confirm this behavior from Fig.11 wherein we plotted \( \beta_x \) at different \( \eta_G \) (mind that \( \eta, \) the temperature and the chemical potential were fixed, which fixes \( \kappa \)). We expect the fluid velocity not to be large, \( v_{\text{max}} \ll v_F \), in the \( W > \lambda_G \) regime. The condition \( W > \lambda_G \) is translated then into an upper bound for the impurity scattering rate

\[
\tau_{\text{imp}} < \frac{W^2}{v_F \kappa}.
\]

(VI.7)

The (non-relativistic) Reynolds number in this case is given by

\[
Re(W>\lambda_G) = \frac{2}{3} \frac{v_F}{W} \left( \frac{W}{\lambda_G} \right)^2 v_{\text{max}}.
\]

(VI.8)

Inserting (VI.7) and (VI.3) into (VI.8), we see that the Reynolds number is bounded from above. The bound is given by

\[
Re < \frac{2}{3} \frac{W}{v_F \kappa} \left( \frac{W}{\lambda_G} \right)^2 v_{\text{max}}.
\]

(VI.9)

It is also worthwhile to mention that

\[
Re(W<\lambda_G) > Re(W>\lambda_G),
\]

(VI.10)

since \( \tau_{\text{imp}} \), when \( W < \lambda_G \) needs to be large for consistency. Based on this last result (VI.10), we can conclude that the onset of turbulence is more likely to appear in clean samples.

VII. DISCUSSION AND OUTLOOK

In this work, we considered the motion of strongly coupled electron fluids propagating along a channel under an applied electric field. Due to the incompressibility of the flow as well as the expected near-conformality of relativistic electron systems such as graphen\([22]\), or HgTe\([23]\), transport is characterized by a single transport coefficient, the ratio of shear viscosity to entropy density \( \eta/s \). We analyzed the qualitative dependence of the fluid flow on \( \eta/s \) in clean systems in sec. [V] and including impurities in sec. [V]. We varied \( \eta/s \) from its strong coupling value \([24]\) predicted by AdS/CFT towards the perturbative regime in which \( \eta/s \) is large\([25]\). We generically find that keeping the other input parameters such as e.g. electric field...
and impurities fixed, strongly coupled holographic fluids satisfying \( \eta/s \) flow fastest.

A very important observable for hydrodynamic behavior in such channels, also called high-mobility wires, is the differential resistance. It is sensitive to the ballistic-to-hydrodynamic crossover. Our hydrodynamic simulations show that the differential resistance has the form expected for a Poiseuille flow, and depends sensitively on the value of \( \eta/s \). In particular, we found that the differential resistance becomes minimal for strongly coupled fluids satisfying \( \eta/s \). This is due to the dependence of the current defined in \( \text{IV.10} \) on the velocity profile, which itself depends strongly on \( \eta/s \). Thus, keeping all other parameters fixed, we expect the holographic fluids to exhibit the smallest resistance. From the qualitative behavior of the channel resistance \( dV/dI \) as a function of \( \eta/s \) obtained from our hydrodynamic simulations (c.f. Fig. 10) we conclude that if the equilibration length associated with \( \eta/w \) is of the order of the channel width \( w \), the position of the Gurzhi maximum indicative of the Knudsen-Poiseuille crossover strongly depends on \( \eta/s \).

Moreover, we calculated the wall momentum-relaxation timescale \( \tau_w \), which describes how fast momentum is lost through the walls. We found it to be largest for holographic fluids. We interpret this as follows: The shear viscosity \( \eta \) is a measure of the momentum transfer between adjacent fluid layers. The entropy density \( s \) is a measure of the number of degrees of freedom in each layer, at a given temperature and chemical potential. Hence, \( \eta/s \) can be interpreted as the rate of momentum transfer between adjacent fluid layers per effective degree of freedom. This implies that the momentum transfer between layers is less efficient for small \( \eta/s \) and hence \( \tau_w \) is larger in this case. Note also that \( \eta/s \) is related to the relativistic analogue of the kinematic viscosity by means of the Gibbs-Duhem relation \( \text{II.22} \).

Furthermore, we found that boundary-induced momentum relaxation is not independent of momentum loss through impurities: Decreasing the impurity density leads to an increased total momentum of the flow, which in turn decreases wall momentum relaxation (c.f. \( \text{V.8} \)). We derived the exact relationship between wall momentum relaxation \( \tau_w \) and impurity momentum relaxation \( \tau_{\text{imp}} \) for non-relativistic flows in \( \text{V.9} \). Experimentally verifying the consequences of wall momentum relaxation requires to synthesize clean enough samples with small \( \eta/s \) such that \( \tau_w \) is comparable to \( \tau_{\text{imp}} \) (c.f. \( \text{IV.8} \)).

In the nonrelativistic limit we derived a closed form for the differential resistance \( R \), given in \( \text{V.13} \). For all other input parameters held fixed, \( R \) is a monotonically decreasing function of \( \tau_{\text{imp}} \), as expected. We stress that the expression \( \text{V.13} \) for \( R \) is only valid for small values of the current \( I \). Therefore, provided that the system behaves hydrodynamically, \( \text{V.13} \) gives the maximal resistance of the channel. In the present work, we have assumed that the system is well in the hydrodynamic regime \( \text{II.1} \). In addition, we found that \( R \) satisfies an inverse Matthiessen’s rule, which in the absence of impurities is given by \( \text{IV.18} \). In the presence of impurities, \( R_{\text{imp}}^{-1} \) in \( \text{IV.18} \) is replaced by the second term in \( \text{V.13} \). We see that quantum critical conductance effects are clearly separable from viscous and impurity effects. The reason is that in the constitutive relation for the current \( \text{II.11} \), they enter as two independent terms. This will also hold for flows with other boundary conditions \( \text{II.2} \).

This is reminiscent of situations in AdS/CFT models of strange metallic physics, where the quantum critical part and the Drude part of the conductivity also follow an inverse Matthiessen rule \( \text{II.3} \) or even more nonlinear relations \( \text{II.4} \). We emphasize that the expression \( \text{V.13} \) for the wire resistance was derived without any particular assumption about the equation of state or the value of \( \eta/s \). It is hence valid at weak coupling as well, as long as the conditions for hydrodynamics \( \text{II.1} \) apply.

Throughout our calculations, we used the expression \( \text{IV.12} \) for the quantum critical conductivity \( \sigma_Q \). This choice is most natural at strong coupling for the following reason: The quantum critical conductivity can be calculated at weak coupling \( \text{II.2} \) to leading order in \( \alpha(T) \). The result is similar to \( \text{I.4} \) for \( \eta/s \),

\[
\sigma_Q = 0.76 < e^2 \frac{1}{h \alpha^2(T)}. \tag{VII.1}
\]

Following the discussion of \( \eta/s \) in sec. I and extrapolating this result to strong coupling, the natural value for the quantum critical conductivity at strong coupling is of order

\[
\sigma_Q \sim \frac{e^2}{h}. \tag{VII.2}
\]

From \( \text{V.7} \), we see that at strong coupling, the quantum critical resistivity is of the same order as the Drude resistivity for realistic values of the impurity density. Hence, the quantum critical part of the resistivity cannot be neglected, contrary to the situation at weak coupling. The normalization of the \( F^2 \) term in our holographic model \( \text{II.17} \) was chosen to bear out this expectation \( \text{II.23} \) to yield \( \text{IV.13} \).

In view of the discussion on the onset of turbulence in sec. \( \text{VI} \), it will be highly interesting to perform fully space-time dependent hydrodynamic simulations in \( 2+1 \) dimensional charged fluids. This is also interesting from the point of view of the viscosity and impurity induced pressure drop \( \text{III.9} \) found as a solution to the Hagen-Poiseuille equation \( \text{III.7} \). The classical Hagen-Poiseuille law relates the pressure drop along a laminar flow in a pipe, in our case the wire, to the viscosity of the flowing fluid. This effect is present as long as the viscosity is non-vanishing, independently of the presence of impurities. If we allow for \( x \)-dependence in our hydrodynamic simulations, we will recover that well-known effect, as well as \( \text{III.9} \). In any fluid, pressure gradients translate into gradients in energy density via the equation of state \( \epsilon(p) \), and hence into a spatially dependent temperature. It will be very interesting to investigate this and other space-time dependent effects further.
Another interesting question is the physical significance of the relative ordering of the channel with $W$ versus the impurity mean free path $\ell_{\text{imp}}$ in \( \text{(I.1)} \). The relevant length scale to compare $W$ to is the Gurzhi length $\lambda_G = v_F \sqrt{\kappa \ell_{\text{imp}}}$, with $\kappa$ the kinematic viscosity. The Gurzhi length is the scale on which the viscous drag in the channel is efficient. In the Poiseuille regime $W \leq \lambda$, the viscous drag from the boundaries permeates the entire channel. In the opposite regime $W \geq \lambda$, the viscous drag is important up to a distance of order $\lambda_G$ away from the walls, but in the center of the channel, the physics is effectively 2+1-dimensional, and the flow is Ohmic. The latter point is noteworthy as it represents an Ohmic electric response in the regime of applicability of hydrodynamics \( \text{(II.1)} \).

The results presented in this work assume the validity of hydrodynamics \( \text{(II.1)} \) at all electron temperatures and input parameters. The hydrodynamic regime is reached if the electron-electron mean free path $\ell_{\text{ee}}$ is the smallest length scale present. This is expected to be the case in particular in systems with strong electron-electron correlations. Realizing such strongly correlated electron systems will hence be an important avenue for further development.

Finally, it will be interesting to investigate the effect of parity and time reversal breaking \( 2 \mathbb{Z}_2 \mathbb{Z}_2 \) in the setup considered, as well as thermal and thermoelectric transport.

**ACKNOWLEDGMENTS**

We thank Hartmut Buhmann, Ewelina Hankiewicz, Carlos Hoyos, Oleksiy Kashuba, Laurens Molenkamp, Valentin Müller and Björn Trauzettel for useful discussions. We gratefully acknowledge support from the DFG via SFB 1170 ‘Topological and Correlated Electronics at Surfaces and Interfaces’.

**Appendix A: Holographic preliminaries**

In this appendix we compute the thermodynamic variables derived from the holographic duality. In order to describe a QFT at non-zero temperature, we need to consider a black brane in the gravity dual. The radial coordinate spans from the horizon position $r_H$ to the boundary, that is $r \in [r_H, \infty)$. The Bekenstein-Hawking entropy density is given by

\[ s = \frac{A_H k_B}{4 l_s^2}, \quad l_s^2 = \frac{\hbar G_4}{v_F}, \quad (A.1) \]

where stands for the effective Planck length, with $v_F$ instead of $c$ as the speed of light. The area density, $A_H$, is

\[ A_H = \sqrt{g_{xx} g_{yy}} \bigg|_{r \rightarrow r_H}, \quad (A.2) \]

For the metric element \( \text{(II.18)} \),

\[ ds^2 = \frac{L^2}{r^2 f(r)} dr^2 + \frac{r^2}{L^2} \left[ -f(r)v_F^2 dt^2 + dx^2 \right], \quad (A.3) \]

we get

\[ A_H = \frac{r_H^2}{L^2}. \quad (A.4) \]

A finite density in the dual QFT is obtained by turning on a dynamical $U(1)$ gauge field,

\[ A_\mu dx^\mu = A_0 dt. \quad (A.5) \]

There is a non-trivial relation between the bulk fields $f$ and $A_0$ and the chemical potential and temperature. The chemical potential is defined as the boundary value of the time component of the bulk gauge $U(1)$ field,\n
\[ \mu = A_0(r) \bigg|_{r \rightarrow \infty}. \quad (A.6) \]

The temperature of the dual QFT is the Hawking temperature of the black brane, $T = T_H$. The Hawking temperature $T_H$ is obtained from the surface gravity $\kappa_H$, defined as

\[ 2\pi \frac{v_F k_B}{\hbar} T_H = \kappa_H = \sqrt{\left( \nabla A \zeta_B \right) \left( \nabla A \zeta_B \right)} \bigg|_{r = r_H}. \quad (A.7) \]

For a static black brane, one Killing vector is simply $\zeta = (\zeta^t, \zeta^1, \zeta^2, \zeta^3) = (0, 1, 0, 0)$. $\nabla_a$ stands for the covariant derivative, defined from the bulk metric \( \text{(II.1)} \). With this,\n
\[ T_H = \frac{h v_F}{k_B} \frac{r_H}{4 \pi L^2 f'(r_H)} e^{A(r_H)} , \quad (A.8) \]

which, for the functions defined in \( \text{(II.19)} \), gives\n
\[ T_H = \frac{1}{h v_F k_B} \frac{\mu^2 L^2}{16 \pi T_H} - \frac{h v_F}{k_B} \frac{3 r_H}{4 \pi L^2} . \quad (A.9) \]

The holographic duality establishes that the partition function of the QFT and of the gravity theory are identified. Taking variations of the on-shell gravity action with respect to the metric or the gauge field will give the corresponding $n$-point correlation functions. The on-shell action is divergent and we need to renormalize it. This constitutes the holographic renormalization prescription.\n
We introduce a cutoff in the radial direction $r_A$ such that the divergences of the bulk action \( \text{(II.17)} \) are regulated. The volume divergence is canceled by adding the counterterm

\[ S_{\text{c.t.}} = - \int_{r = r_A} d^3 x \mathcal{L}_{\text{c.t.}}, \quad \mathcal{L}_{\text{c.t.}} = \sqrt{-\gamma} \frac{6}{L}, \quad (A.10) \]

Further divergences, subleading with respect to the volume divergence are remove if in addition to $S_{\text{c.t.}}$, one adds the Gibbons-Hawking term

\[ S_{\text{GH}} = \int_{r = r_A} d^3 x \sqrt{-\gamma} K. \quad (A.11) \]
The on-shell action together with the counterterms $S_{\text{c.t.}}$ and $S_{\text{GH}}$ guarantee a well-defined variational principle. We assume that the form of the metric is

$$ds^2 = N^2 dr^2 + \gamma_{\mu\nu} dx^\mu dx^\nu,$$  \hspace{1cm} (A.12)

and define the extrinsic curvature and Brown-York tensors in the usual way

$$K_{\mu\nu} = \frac{1}{2N} \partial_{\gamma} \gamma_{\mu\nu}, \quad K = \gamma_{\mu\nu} K_{\mu\nu}, \quad \Pi_{\mu\nu}^{BY} = (K_{\mu\nu} - \gamma_{\mu\nu} K).$$  \hspace{1cm} (A.13)

In particular, there is no need for adding a counterterm for the gauge field. The renormalized action is then

$$S_{\text{ren}} = \lim_{r_A \to \infty} [S_{\text{bulk}} + S_{\text{GH}} + S_{\text{c.t.}}].$$  \hspace{1cm} (A.14)

Renormalized expectation values of the stress tensor and current are computed from variations of the action with respect to the metric $g_{\mu\nu}$ and gauge field $A_\mu$,

$$\langle T^{\mu\nu} \rangle = \frac{1}{8\pi G_4} \int \lim_{r_A \to \infty} \left[ \frac{\gamma^{\mu\nu}}{L^2} \left[ -\sqrt{-\gamma} \Pi_{\mu\nu}^{BY} + \frac{\delta L_{\text{c.t.}}}{\delta \gamma_{\mu\nu}} \right] \right] d^4 x,$$

$$\langle J^\mu \rangle = \frac{1}{16\pi G_4} \int \lim_{r_A \to \infty} \left[ -4L^2 \sqrt{-\gamma} g^{\mu\nu} g^{\rho\sigma} F_{\rho\sigma} \right] d^4 x,$$  \hspace{1cm} (A.15)

which gives the thermodynamic variables (II.21).

Finally, we comment on the quantum critical conductivity (IV.12) from AdS/CFT models such as (II.17). One reason for the naturalness of (VII.2) is the electromagnetic self-duality of the holographic action (II.17) if the Maxwell term is normalized canonically to $-\frac{1}{2} F_{\mu\nu}^2$. This emergent discrete symmetry fixes $\sigma_Q = \frac{\gamma}{2\pi}$ in the Dirac regime $\mu/(k_B T) \ll 1$. Finite coupling corrections which break the self-duality can be calculated from universal higher derivative terms on the gravity side of the AdS/CFT correspondence. In summary, there is ample evidence that (VII.2) is the strong coupling limit of the perturbative quantum critical conductance (VII.2), which motivates our choice of normalization of the $F^2$ term in (II.17).

**Appendix B: Reynolds number for relativistic hydrodynamics**

In the present appendix we derive the Reynolds number, $Re$, for relativistic flows. In non-relativistic hydrodynamics, $Re$ appears in the dimensionless Navier-Stokes equations (NSe) as

$$\frac{D\vec{u}}{Dt} = -\nabla P + \frac{1}{Re_{\text{NS}}} \nu^2 \vec{u},$$  \hspace{1cm} (B.1)

where $\frac{D}{Dt} \equiv \partial_t + \vec{v} \cdot \nabla$ is the convective derivative.

A similar definition for $Re$ does not exist for relativistic hydrodynamics; we amend this below.

Our first derivation of $Re$ mimics the derivation of $Re_{\text{NS}}$: we’ll bring the relativistic equations of motion (EOM), $\partial_{\mu} T_{\mu\nu} = 0$, into a form similar to the NSe, and read off $Re$. Thus, we have

$$\frac{1}{\nu_F^2} \frac{D\vec{u}}{D\tau} = - \left( \frac{\vec{u}}{\nu_F^2} \frac{D}{D\tau} + \nabla \right) \ln(\varepsilon + p) - \frac{\eta}{\varepsilon + p} \vec{\Sigma}$$  \hspace{1cm} (B.2)

Equation (B.2) is clearly the relativistic counterpart of the NSe, (B.1). We bring (B.2) into a dimensionless form by applying the transformation $\vec{u} \to \vec{u}/v_{\text{max}}$, $x^\mu \to x^\mu/W$, where $v_{\text{max}} \equiv |\vec{u}|(y = W/2), W$ are the characteristic velocity and length scales of our flows (see Figure 1). The result is

$$\frac{D\vec{u}}{D\tau} = - \left( \frac{\vec{u}}{v_{\text{max}}^2} \frac{D}{D\tau} + \left( \frac{v_F}{v_{\text{max}}} \right)^2 \nabla \right) \ln(\varepsilon + p)$$

$$- \left( \frac{v_F}{v_{\text{max}}} \right)^2 \frac{\eta v_{\text{max}}^2}{W(\varepsilon + p)} \vec{\Sigma},$$  \hspace{1cm} (B.3)

where $\vec{\Sigma}$ is the rescaled version of $\vec{\Sigma}$.

Now we can use (B.3) in order to read off $Re$. Unfortunately, there is not a single dimensionless number multiplying the second term in the r.h.s. of (B.3), because of $\vec{\Sigma}$. Thus there is no natural definition of $Re$ in the relativistic case. We can bypass this problem by considering the non-relativistic(NR) and the ultra-relativistic(UR) limits of (B.3).

- **non-relativistic limit:** In this case, $v_F/v_{\text{max}} \gg 1$, and the non-relativistic $Re$ is

$$Re_{\text{NR}} = \frac{(\varepsilon + p)W}{\eta v_{\text{max}}} \left( \frac{v_{\text{max}}}{v_F} \right)^2.$$  \hspace{1cm} (B.4)

$Re_{\text{NR}}$ is proportional to the Navier-Stokes $Re$. We see this if we substitute $\varepsilon = \rho v_F^2$ and $p = \frac{1}{2} \rho v_F^2$, where $\rho$ is the fluid mass density, into (B.4) i.e.

$$Re_{\text{NR}} = \frac{3}{2} Re_{\text{NS}} = \frac{3}{2} \frac{v_{\text{max}}^2 W}{\eta}.$$  \hspace{1cm} (B.5)

- **ultra-relativistic limit:** In this case, we keep the terms proportional to $(v_{\text{max}}/v_F)^4$ in $\vec{\Sigma}$ and obtain

$$Re_{\text{UR}} = \frac{(\varepsilon + p)W}{\eta} \left( \frac{v_F}{v_{\text{max}}} \right)^2.$$  \hspace{1cm} (B.6)
Our second derivation of $Re$ is based on the physical meaning of $Re$, which is defined as the ratio of inertial forces to the ratio of viscous forces. As usual, the inertial and viscous forces are given by

\[ F_{in} = \partial_i T^{0i}, \quad F_v = - \eta \partial_j \sigma^{ji}, \quad i = 1, 2 \]  \hspace{1cm} (B.7)

Therefore $Re$ can be defined through the ratio $\vec{F}_{in}/F_v$. Computing this ratio leads us, as expected, to equations (B.5) and (B.6).

Equations (B.5) and (B.6) above summarize our derivation of $Re$ for non-relativistic and ultra-relativistic fluids.
In the strong coupling regime, higher orders in $\alpha(T)$ are not negligible. Presumably only string theories including all order $\alpha$ corrections are causal, but $\eta/s$ has not been calculated in such a toy model. Similar remarks apply to finite $N$ corrections which also correspond to higher derivative terms in the dual gravitational theory. In summary, there is good evidence that \([1.2]\) holds in the stable ground states of all consistent models.

L. V. Delacrétaz, T. Hartman, S. A. Hartnoll, and R. Mahajan, Phys. Rev. Lett. 119, 141601 (2017) [arXiv:1706.00019 [hep-th]].

T. Hartman, S. A. Hartnoll, and R. Mahajan, Phys. Rev. Lett. 119, 141601 (2017) [arXiv:1706.00019 [hep-th]].

H. Song, S. A. Bass, U. Heinz, T. Hirano, and C. Shen, Physical Review Letters 106, 192301 (2011).

For electrons in solids we have direct access to the hydrodynamic regime. In QCD, we need to reconstruct the hydrodynamic regime from the data collected by particle detectors back through the hadronization crossover.

K. S. Novoselov, A. K. Geim, S. Morozov, D. Jiang, M. Katsnelson, I. Grigorieva, S. Dubonos, Firsov, and AA, nature 438, 197 (2005).

M. Königs, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, and S.-C. Zhang, Science 318, 766 (2007).

F. R. Klinkhammer and G. E. Volovik, Int. J. Mod. Phys. A20, 2795 (2005) [arXiv:hep-th/0403037 [hep-th]].

G. E. Volovik, Int. J. Mod. Phys. A11, 219 (2006).

X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, Phys. Rev. B 83, 205101 (2011).

L. Fritz, J. Schmalian, M. Müller, and S. Sachdev, Physical Review B 78, 085116 (2008) [arXiv:0802.4299].

M. Mueller, J. Schmalian, and L. Fritz, Physical Review Letters 103, 025301 (2009) [arXiv:0903.4178 [cond-mat.mes-hall]].

In the strong coupling regime, higher orders in $\alpha(T)$ will become as important as the lowest order term in \([1.4]\). Nevertheless, since all $\eta/s$ has the interpretation of the rate of diffusive momentum transfer between adjacent fluid layers, we expect higher orders only to enhance the effect and hence decrease $\eta/s$, i.e. we expect $\eta/s$ to be a monotonic increasing function of $\alpha$. Also, unitarity as well as the local second law of thermodynamics \([43]\) restricts $\eta \geq 0$.

In experiment, the channels are typically prepared by etching them out of a thin layer of material. During that process, the channel edges are sufficiently disordered, such that zero velocity boundary conditions are applicable.

In this work, we assume that we can vary $\eta/s$ a priori physically independent of $s$. This amounts to varying the coupling strength independently of the temperature, in which case, the entropy density $s$ at fixed temperature is a constant in thermal and chemical equilibrium. We leave the investigation on the possible experimental realization for the future.

L. Landau and E. Lifshitz, Fluid Mechanics, 2nd ed., Course of Theoretical Physics S (Butterworth-Heinemann, 1987).

P. Romatschke, International Journal of Modern Physics E 19, 1 (2010).

O. Z. Luciano Rezzolla, Relativistic Hydrodynamics (Oxford University Press, 2013).

Introducing a nontrivial metric is useful when e.g. considering perturbations that correspond to a temperature gradient acting on the fluid \([29]\), and can be achieved by replacing partial derivatives in \([1.4]\)–\([1.5]\) by the covariant derivative with an appropriately chosen connection, which most of the time will be the metric compatible Christoffel connection.

We normalized the electric field by means of the fundamental constants $e, c$ and in such a way that $[E^\mu] = V/m$, $[j^\mu] = A/m$ and $[\mathbf{D}^\mu] = A S^\mu/m$.

Another common frame is Eckart frame, in which the fluid is comoving w.r.t. the current fluid, amounting to $j^\mu = 0$ in \([1.17]\). Another example can be found in \([29]\) where the $MB$ in the presence of a constant magnetic field can either be considered zeroth order in derivatives due to the constancy of $B$ and incorporated into the pressure, or, if $B$ is small, to be of first order due to the fact that the magnetic field is the derivative of the vector potential. It is possible to switch between both frames through field transformations \([29]\).

Scale invariance implies vanishing stress-energy tensor trace, $T^\mu_{\mu} = 0$, and hence $\xi = 0$. Furthermore, due to renormalization effects the Fermi velocity runs logarithmically at low temperatures $v_F = v_F(T/F).$ Another reason for neglecting the bulk viscosity term is that we are working with incompressible liquid flows, i.e. $\partial_i u^i = 0.$ The conservation equation for the current $\mathbf{j}$ is trivially satisfied for the flow we will elaborate on in sec. III and hence we do not discuss it hereafter.

J. Bruin, H. Sakai, R. Perry, and A. Mackenzie, Science 339, 804 (2013).

J. Gooth, F. Menges, C. Shekhar, V. Süss, N. Kumar, Y. Sun, U. Drechsler, R. Zierold, C. Feber, and B. Gotsmann, Phys. Rev. X 4, 021012 (2014). Another reader also pointed out the $\eta/s - \kappa$ correspondence in \([99,96]\) correctly.

A black brane is a black hole with planar horizon topology, in this case \([92]\).

J. Erdmenger, D. Fernandez, P. Goulart, and P. Witkowski, JHEP 03, 147 (2017) [arXiv:1611.09381 [hep-th]].

C. Brüne, C. Thielen, M. Stubler, J. Böttcher, H. Buhmann, E. G. Novik, C.-X. Liu, E. M. Hankiewicz, and L. W. Molenkamp, Physical Review X 4, 025301 (2014) [arXiv:1407.6537 [cond-mat.mes-hall]].

A. Majumder, B. Müller, and X.-N. Wang, Phys. Rev. Lett. 99, 192301 (2007) [arXiv:hep-ph/0703082 [hep-ph]].

A. Buchel, J. T. Liu, and A. O. Starinets, Nucl. Phys. B 707, 56 (2005) [arXiv:hep-th/0406264 [hep-th]].

We remind the reader that both $\tau_T$ and $\tau_{\kappa_\alpha}$ are a priori physically independent to the electron-electron scattering rate $\tau_{ee}.$

S. A. Hartnoll, P. K. Kovtun, M. Muller, and S. Sachdev, Phys. Rev. B76, 144502 (2007) [arXiv:0706.3215 [cond-mat.str-el]].

S. A. Hartnoll and C. P. Herzog, Phys. Rev. D76, 106012 (2007) [arXiv:0706.4228 [hep-th]].

The quantum critical resistance is defined via Ohm’s law, $R = \frac{i}{\sigma \xi^2 \alpha^4}$ with $\alpha$ the cross-sectional area of the wire. Here $\alpha = W.$

R. W. Hanks, AIChE Journal 9, 45 (1963).

R. W. Hanks, AIChE Journal 15, 25 (1969).

M. Mendoza, H. J. Herrmann, and S.ucci, Physical Review Letters 106, 156501 (2011) [arXiv:1201.6590 [cond-mat.mes-hall]].

We define the kinematic viscosity $\kappa$ as the ratio of viscosity and energy density. There exists another kinematic viscosity, $\nu$, defined as the ratio of viscosity and mass density. The two viscosities are related by $\nu = v_F \kappa.$

E. I. Kiselev and J. Schmalian, arXiv.org (2018), 1806.03933v2.
In particular in the Dirac limit $\mu/(k_B T) \ll 1$, $\varepsilon + p \approx sT$, and the dependence on thermodynamic variables drops out of (IV.12).

In fact, the center flow can be understood as an effective channel flow with no-stress boundary conditions. These conditions are imposed at the edge of a layer of width $\lambda_{G}$. These boundary conditions then lead to Ohmic hydrodynamic flow at the center of the channel.\[2\]

Recall that the fluid in question is conformal. This is in contrast to common non-relativistic fluids, for which $p \ll \varepsilon$. In this case, $Re_{NR} = Re_{NS}$.

For example, $\vec{v} = \vec{u}/v_c$, where $\vec{u}, v_c$ is the fluid and characteristic velocity, respectively.

Without loss of generality, we have set the external fields to zero and taken $\partial_\mu \varepsilon = 0 = \partial_\mu u^\mu$.

Recall that the fluid in question is conformal. This is in contrast to common non-relativistic fluids, for which $p << \varepsilon$. In this case, $Re_{NR} = Re_{NS}$.