The Total Green’s Function of a Non-Interacting System

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Despite its centrality in the mathematical structure of perturbative many-body theory, the total Green’s function for the many-body time-dependent Schrodinger equation has been ignored for decades, superseded by single-particle Green’s functions, for which a vast portion of the literature has been devoted. In this paper, we give the first computation the total Green’s function for the time-dependent Schrodinger equation for a non-interacting system of identical particles, setting the stage for a fresh interpretation of perturbative many-body physics.

I. INTRODUCTION

Condensed matter physicists use effective field theories all the time. In trying to engineer emergent phenomena into novel materials at low-energy scales, following [2], it would be nice to be able to define a family of effective Hamiltonians

$$\{H^{\text{eff}}[\Lambda]\}_{\Lambda \geq 0}$$

at every energy-scale $\Lambda$, for a fixed material, and formulate a renormalization group flow naturally in this context, that can interpolate between between a microscopic and effective many-body theory. As it turns out, the total Green’s function

$$G^{\text{eff}}[\Lambda] \equiv (\partial_t + iH^{\text{eff}}[\Lambda])^{-1}$$

is absolutely necessary to implement this program efficiently (See [1]). However, this puts us in an awkward position, because such an object is foreign to the condensed matter literature.

II. THE TOTAL GREEN’S FUNCTION

Time-evolution of a many-body quantum system is given by the time-dependent many-body Schrodinger equation (in units where $\hbar = 1$),

$$(\partial_t + iH)\Psi = 0.$$  

In this paper, we compute the Green’s function of the linear differential equation above, which we’ll call the total Green’s function of our many-body system:

$$G \equiv (\partial_t + iH)^{-1}.$$  

Despite being such a fundamental mathematical quantity, surprisingly no one has actually computed the total Green’s function for a many-body system.

In the case of a complicated interacting system, this computation is intractable. However, we can give the first computations of this function in the case that the dynamics is non-interacting. We can then compute the total Green’s function of a general system via perturbation theory, but we will leave that for another article.

Since our Hamiltonian is non-interacting, it sends each $k$-particle portion of the total Hilbert space to itself: so the total Green’s function $G$ also sends each $k$-particle portion of its Hilbert space to itself:

$$G = \bigoplus_{k \geq 0} G_k$$

The terms on the right-hand-side are the $k$-particle Green’s functions. Therefore, to compute the total Green’s function, it will suffice to compute $G_k$ for all $k \geq 0$.

III. COMPUTATION OF $G_0$ AND $G_1$

Note: For conceptual simplicity, throughout this article, we will assume that the single-particle Hilbert space is finite dimensional, with basis $\{f_i\}$.

The first two terms in the direct sum have already been computed in the literature, and we will not waste any time and just briefly mention the results here:

$$(G_0)_l^{t'} = (\partial_t + iH_0)^{-1}_l^{t'}$$

Since $H_0 \equiv 0$ (see [4]), we get the standard theta-function, the integral kernel of the differential operator $\partial_t$:

$$(G_0)_l^{t'} = (\partial_t)^{-1}_l^{t'} = \theta(t - t')$$

The operator $G_1$ is all over the many-body literature: it is called “the propagator”, or sometimes “the single-particle Green’s function”. It has the following matrix...
follows:

\[(G_1)_{ij}^t = \theta(t-t') \langle [a_\pm(f_i,t), a_\pm^+(f_j,t')] \rangle_{T=0} \]

Computation of $G_1$ is usually given as a trivial exercise in many-body textbooks, such as [3]. Already, with $k = 0, 1$, we can see a pattern forming. We will extrapolate to general values of $k$ in the next section.

**IV. COMPUTATION OF $G_k$ FOR $k \geq 2$**

We now state our main result:

**Theorem IV.1 (The Total Green’s Function)** For $k \geq 2$, define $\tilde{G}_k$ by the following matrix elements:

\[
(\tilde{G}_k)_{i_1\ldots i_k}\l br t' = \theta(t-t') \langle [a_\pm(g_{i_1},t), a_\pm^+(g_{i_2},t')] \rangle \\
\ldots \langle [a_\pm(g_{i_k},t), a_\pm^+(g_{j_1},t')] \rangle_{T=0} \]

Then the $k$-particle Green’s function $G_k$ of the non-interacting system is simply the restriction of $\tilde{G}_k$ to the appropriate (anti-)symmetric subspace.

**Proof.** For a system of non-interacting identical particles, there exists a basis of the single-particle Hilbert space with respect to which the Hamiltonian can be rewritten as

\[
\sum_{ij} A_{ij} a_\pm^+(f_i)a_\pm(g_j) = \sum_i B_i a_\pm^+(g_i)a_\pm(g_i). 
\]

**Lemma:** In the associated basis of our un-(anti)-symmetrized Fock-space, the matrix elements of $\tilde{G}_k$ become

\[
(\tilde{G}_k)_{i_1\ldots i_k}\l br t' = \theta(t-t') \langle [a_\pm(g_{i_1},t), a_\pm^+(g_{i_2},t')] \rangle \ldots \langle [a_\pm(g_{i_k},t), a_\pm^+(g_{j_1},t')] \rangle_{T=0} 
\]

**Proof of Lemma.** Let the change-of-basis be expressed as follows:

\[ g_i = U_i^j f_j \]

We now use the bilinearity of the (anti-)commutator, and the (anti-)linearity of $(a_\pm, a_\pm^\dagger)$:

\[
(\tilde{G}_k)_{i_1\ldots i_k}\l br t' = \theta(t-t') \langle [a_\pm(g_{i_1},t), a_\pm^+(g_{i_2},t')] \rangle \ldots \langle [a_\pm(g_{i_k},t), a_\pm^+(g_{j_1},t')] \rangle_{T=0} 
\]

Which satisfies the formula for induced change-of-basis on the un-(anti-)symmetrized Fock space $\mathcal{F}_k$.

Now we resume our proof. Since the Hamiltonian is diagonal in the basis $\{g_i\}$, it may be verified that

\[
a_\pm^+(g_j,t') = e^{-i(t-t')B_j} a_\pm^+(g_j,t) 
\]

Therefore, we can begin to simplify the matrix elements of $\tilde{G}_k$ as follows:

\[
(\tilde{G}_k)_{i_1\ldots i_k}\l br t' = \theta(t-t') \langle e^{-i(t-t') B_{i_1}} [a_\pm(g_{i_1},t), a_\pm^+(g_{i_2},t)] \rangle \ldots \langle e^{-i(t-t') B_{i_k}} [a_\pm(g_{i_k},t), a_\pm^+(g_{j_1},t)] \rangle_{T=0} 
\]

Using the equal-time (anti-)commutation relations

\[
[a_\pm(f,t), a_\pm(g,t)] = (f,g),
\]

we get

\[
(\tilde{G}_k)_{i_1\ldots i_k}\l br t' = \theta(t-t') \delta^{i_1\ldots i_k}_{j_1\ldots j_k} e^{-i(t-t') (B_{i_1} + \ldots + B_{i_k})}. 
\]

We now compute the time-derivative of the above expression:

\[
(\partial_t \tilde{G}_k)_{i_1\ldots i_k}\l br t' = \delta^{i_1\ldots i_k}_{j_1\ldots j_k} e^{-i(t-t') (B_{i_1} + \ldots + B_{i_k})} 
\]

Since the first term in the above expression vanishes for $t = t'$, we can eliminate the phase-factor which multiplies it, yielding

\[
(\partial_t \tilde{G}_k)_{i_1\ldots i_k}\l br t' = \delta^{i_1\ldots i_k}_{j_1\ldots j_k} 
\]

Recall that we can extend a non-interacting Hamiltonian to the un-(anti)-symmetrized Fock space by letting

\[
\tilde{H}_k(f_1 \otimes \ldots \otimes f_k) = \sum_i f_i \otimes \ldots \otimes H_1 f_1 \otimes \ldots \otimes f_k, 
\]

where $H_1$ is our associated single-particle Hamiltonian (see the first page for the definition of $H_1$). Therefore, acting on our Green’s function with $i\tilde{H}_k$, we get

\[
(i\tilde{H}_k \tilde{G}_k)_{i_1\ldots i_k}\l br t' = i(B_{i_1} + \ldots + B_{i_k}) \tilde{G}_k_{i_1\ldots i_k}\l br t' 
\]

Where we get the simple factor because we are implicitly in an eigenbasis of $\tilde{H}$, and so the action of $\tilde{H}$ is diagonal. Therefore, putting it all together, we get

\[
((\partial_t + i\tilde{H}_k) \circ (\tilde{G}_k)_{i_1\ldots i_k}\l br t') = \delta(t-t')\delta^{i_1\ldots i_k}_{j_1\ldots j_k} 
\]

In basis-independent language, this is the simple identity $(\partial_t + i\tilde{H}) \circ \tilde{G}_k = I$, i.e., we have verified that, on the un-(anti-)symmetrized Fock space, \[
\tilde{G}_k = (\partial_t + i\tilde{H}_k)^{-1}. 
\]

Therefore, restricting this operator expression to the (anti-)symmetrized Fock space $\mathcal{F}_k$ yields our desired identity:

\[
G_k = (\partial_t + i\tilde{H}_k)^{-1}. \quad \square
\]
V. CONCLUDING REMARKS

In this paper, we computed the Green’s function

\[ G = (\partial_t + iH)^{-1} \]

depending on the time-dependent Schrödinger equation, in the case of non-interacting identical particles, by computing each term in the direct-sum decomposition. The final result was

\[ G = \bigoplus_{k \geq 0} P_\pm \tilde{G}_k P_\pm, \]

where \( P_\pm \) is the (anti-)symmetrization operator, and i.e., the \( k \)-particle Green’s function \( G_k \) of the non-interacting system is simply the restriction of \( \tilde{G}_k \) to the appropriate (anti-)symmetric subspace. **Example:** for a non-interacting system of identical spinless fermions, and in traditional notation,

\[
\begin{align*}
\tilde{G}_0(t,t') &= \theta(t-t') \\
\tilde{G}_1(x,x',t,t') &= \theta(t-t') \langle \{ \Psi(x,t), \Psi^\dagger(x',t') \} \rangle_{T=0} \\
G_2(x,x',y,y',t,t') &= \theta(t-t') \langle \{ \Psi(x,t), \Psi^\dagger(x',t') \} \rangle_{T=0} \\
&\quad \cdot \langle \{ \Psi(y,t), \Psi^\dagger(y',t') \} \rangle_{T=0} 
\end{align*}
\]

This work paves the way for a reformulation of perturbation and renormalization theory in terms of the full many-body Green’s function.

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[1] But there is no Hamiltonian, Jan Derezinski [http://www.fuw.edu.pl/~derezins/nohamiltonian.pdf](http://www.fuw.edu.pl/~derezins/nohamiltonian.pdf)

[2] K. Costello, Renormalization and effective field theory, (2010). Available at [http://www.math.northwestern.edu/~costello/](http://www.math.northwestern.edu/~costello/)

[3] Bruus, H., and K. Flensberg, 2004, Many-Body Quantum Theory in Condensed Matter Physics: An Introduction (Oxford University Press, Oxford).

[4] O. Bratteli and D.W. Robinson. Operator algebras and quantum statistical mechanics I, II. Springer, New York, 1979-1981. [There is a 2nd ed. (1997) of vol. II].