HESSIAN-FREE METHODS FOR CHECKING THE SECOND-ORDER SUFFICIENT CONDITIONS IN EQUALITY-CONSTRAINED OPTIMIZATION AND EQUILIBRIUM PROBLEMS

W. ROSS MORROW

Abstract. Verifying the Second-Order Sufficient Condition (SOSC), thus ensuring a stationary point locally minimizes a given objective function (subject to certain constraints), is an essential component of non-convex computational optimization and equilibrium programming. This article proposes three new “Hessian-free” tests of the SOSC that can be implemented efficiently with gradient evaluations alone and reveal feasible directions of negative curvature when the SOSC fails. The Bordered Hessian Test and a Matrix Inertia test, two classical tests of the SOSC, require explicit knowledge of the Hessian of the Lagrangian and do not reveal feasible directions of negative curvature should the SOSC fail. Computational comparisons of the new methods with classical tests demonstrate the relative efficiency of these new algorithms and the need for careful study of false negatives resulting from accumulation of round-off errors.

1. INTRODUCTION

Verifying the Second-Order Sufficient Condition (SOSC) to certify local optimality of points computed by optimization software is an important but underdeveloped component of computational optimization. Existing optimization solvers compute points satisfying a First-Order or Second-Order Necessary Condition (FONC/SONC), typically without checking the corresponding SOSC. As a result, there remain cases in which computed points are not optimizers such as, for example, Example 1.

While verifying the SOSC is important for non-convex optimization, verifying the SOSC is essential for equilibrium models increasingly being employed in Economics, Operations Research, and Engineering. While much of the theory of computing equilibria relies on assumptions of convexity to ensure the first-order conditions imply optimality rather than just stationarity [26], examples of non-convex games are rapidly appearing in important applications. So long as equilibrium programming methods for general, non-convex games are restricted to solving the combined first-order conditions, algorithms can compute simultaneous first-order points but cannot distinguish equilibria from other types of first-order points; see, e.g., Example 2. Verifying the SOSC is thus fundamental to properly computing equilibria.

At least two ways to test the SOSC have been known for some time. The classical “Bordered Hessian Test” (BHT) [41, 58, 69, 57] can, in principle, be used to verify or reject...
the SOSC at points computed by optimization software. Computationally implementing the BHT relies on a set of nested LU factorizations that can be efficiently taken with LU factorization updating. Determining the inertia of a “KKT matrix” [46, 68] is another “classical” way to verify or reject the SOSC. Efficient implementations of this “Inertia test” are available in the form of existing factorization packages that can compute matrix inertia, and may be easily integrated into SQP solvers for constrained optimization problems.

Both of these “classical” tests require an explicit representation of the Hessian of the Lagrangian, which may not be available. Popular optimization solvers including MINOS, LANCELOT, SNOPT, KNITRO, and MATLAB can utilize gradient evaluations alone through quasi-Newton updates, finite-differences, or automatic differentiation [68]. The Hessian of the Lagrangian will thus not be available for verifying the SOSC when gradient evaluations alone are used. “Hessian-free” algorithms requiring only Hessian-vector products [68] could seamlessly integrate SOSC checks with optimization and equilibrium solvers that do not require users to provide formulas for the second derivatives of the objective or the constraint functions. Such algorithms can fully exploit sparsity patterns in the objective and constraint functions, as well as be implemented using directional finite differences to approximate Hessian-vector products.

This article presents three new Hessian-free algorithms for verifying or rejecting the SOSC at first-order points in smooth equality-constrained optimization or equilibrium problems. The first algorithm is based on Cholesky factorization, the most efficient and stable method for testing Hessian positive-definiteness without constraints [44, 49]. The second algorithm is based on an “oblique” Gram-Schmidt orthogonalization, and the third algorithm is a modification of the Projected Conjugate Gradient algorithm developed for constrained quadratic programming [47, 68]. By being Hessian-free, these algorithms can take full advantage of sparsity patterns in the Hessian of the Lagrangian when it is known and may significantly reduce the number of gradient evaluations necessary to verify or reject the SOSC when the Hessian is not available at all.

Another important feature of an algorithm to verify the SOSC is how easily a feasible direction of negative curvature can be computed should the SOSC fail. Certain optimization algorithms use feasible directions of negative curvature to promote global convergence to second-order necessary points; see, e.g., [18]. Recovery of a feasible direction of negative curvature when an the SOSC fails enables “warm restarts” of optimization algorithms that may not already take advantage of second-order information. The classical BHT and Inertia tests do not provide an obvious path to computing such directions when the SOSC fails. In contrast, the Hessian-free algorithms proposed here make computation of such directions straightforward when the SOSC fails. Indeed, two of the Hessian-free algorithms presented here fail precisely by finding such directions.

Finally, useful algorithms for verifying the SOSC should not return false-positives or false-negatives due to the accumulation of round-off errors. Unfortunately the numerical accuracy of verifying the SOSC with any approach has not yet been addressed. Section 5 provides computational comparisons of the different algorithms including an example that illustrates significant potential for erroneous results due to round-off errors when the constraint gradients are very nearly linearly dependent. In particular, no method can be
2. OPTIMIZATION AND EQUILIBRIUM PROBLEMS

2.1. Optimization Problems. This article considers the equality-constrained, continuous variable optimization problem

\[ \text{minimize} \quad f(x) \]
\[ \text{with respect to} \quad x \in \mathbb{R}^N \]
\[ \text{subject to} \quad c(x) = 0 \]

where \( f : \mathbb{R}^N \to \mathbb{R}, \ c : \mathbb{R}^N \to \mathbb{R}^M \), and \( N, M \in \mathbb{N}, M < N \). The objective \( f \) and constraints \( c \) are nonlinear and twice continuously differentiable functions of \( N \) variables \( x = (x_1, \ldots, x_N) \). The component functions of \( c \) are denoted by \( c_i \), for \( i \in \{1, \ldots, M\} \).

**Theorem 1.** (Optimality Conditions [68, Theorems 12.1, 12.5, 12.6]) Consider Problem (1), and assume the Linear Independence Constraint Qualification (LICQ) is satisfied [68, Def. 12.4]. Define the “Lagrangian” \( \mathcal{L}(x, \lambda) = f(x) - \lambda^\top c(x) \), and let \( A(x) = Dc(x) \in \mathbb{R}^{M \times N} \) denote the Jacobian matrix of the constraint function evaluated at \( x \in \mathbb{R}^N \).

**FONC:** If \( x^* \in \mathbb{R}^N \) is a local solution to Problem (1), then there exists some \( \lambda^* \in \mathbb{R}^M \) such that \( \nabla \mathcal{L}(x^*, \lambda^*) = 0 \). That is, \( \nabla f(x^*) = \sum_{i=1}^M \lambda_i^* \nabla c_i(x^*) \) and \( c(x^*) = 0 \).

**SONC:** Moreover, \( w^\top H(x^*, \lambda^*) w \geq 0 \) for all \( w \in \mathcal{C}(x^*) = \{ h : A(x^*) h = 0 \} \) where \( H(x, \lambda) \) denotes the Hessian of the Lagrangian (or simply “Hessian”),

\[ H(x, \lambda) = \nabla^2_{x,x} \mathcal{L}(x, \lambda) = \nabla^2_{x,x} f(x) - \sum_{i=1}^M \lambda_i \nabla^2_{x,x} c_i(x). \]

**SOSC:** On the other hand, suppose \( x^* \in \mathbb{R}^N \) and \( \lambda^* \in \mathbb{R}^M \) satisfy the FONC and \( w^\top H(x^*, \lambda^*) w > 0 \) for all \( w \in \mathcal{C}(x^*) \). Then \( x^* \) is an isolated local solution to Problem (1).

Note that this SOSC also applies to inequality constrained optimization problems at strictly complementary stationary points [68, Def. 12.5], and as the continuous part of the optimality conditions for mixed-integer nonlinear optimization problems. The remainder of this article assumes that \( x^* \) and \( \lambda^* \) satisfy the FONC and denotes \( H(x^*, \lambda^*), A(x^*), \) and \( \mathcal{C}(x^*) \) by simply \( H, A, \) and \( \mathcal{C} \), respectively. Below the SOSC is denoted compactly using the symbol \( H \succ_c 0 \).

Many existing codes for solving problem (1) solve a variant of the FONC without verifying the SOSC at computed points [65]. Sequential Quadratic Programming (SQP) methods [68, Chapter 18] solve a sequence of local quadratic model problems that, in the equality-constrained case, corresponds to applying Newton’s method to solve the FONC. SQP methods are currently implemented in NPSOL [39], SNOPT [37], filterSQP [30, 29, 31], KNITRO [13, 15], and matlab [48, 73, 72, 40, 28]. Augmented Lagrangian methods [68, Chapter 17], [7] “penalize” the Lagrangian with a measure of the constraint violation and
solve the FONC for a sequence of penalized problems with a variant of Newton’s method. This method is currently implemented in the MINOS [62, 63, 64] and LANCELOT [17, 16] codes. Like Augmented Lagrangian methods, Interior-Point methods [68, Chapter 19] for inequality-constrained problems solve the FONC for a sequence of approximate problems using a variant of Newton’s method. KNITRO [15], LOQO [89, 88], IPOPT [90], and matlab [12, 14] currently contain implementations of interior-point methods.

Obtaining a solution to the FONC is not sufficient to declare the computed point a local solution to (1), as shown in Example 1 below. In practice, sufficient decrease conditions on a merit function (or a filter mechanism) bias existing solvers towards computing constrained minimizers of $f$ [20, 68, 78]. Indeed, this bias towards optimizers is certainly one reason separate codes for verifying the SOSC do not currently exist. Sufficient decrease conditions certainly rule out some types of convergence: for example, these conditions rule out converge to local constrained maximizers of $f$. However, algorithms with sufficient decrease conditions can still converge to saddle points, as in Example 1.

Example 1. Consider minimizing $f(x) = x^3$ over $\mathbb{R}$. $x^* = 0$ satisfies the FONC ($f'(0) = 0$) and the SONC ($f''(0) = 0$), but does not satisfy the SOSC. Indeed, $f$ is not locally minimized at $x^* = 0$, or at any other finite $x$. Applying SQP [68, Chapter 18.1] to this problem starting at any $x_0 > 0$ generates the sequence $x_n = x_{n-1}/2 = x_0/2^n \to 0$ as $n \to \infty$. Moreover this sequence satisfies the Armijo Condition [68, Eqn. 3.4], a popular sufficient decrease condition. ■

There also exist “second-order” algorithms that converge only to points at which the Hessian is positive semi-definite [18]. Such algorithms make use of feasible directions of negative curvature—vectors $d \in \mathcal{C}$ satisfying $d^T H d < 0$—to promote converge to SONC points. Any such algorithm must (periodically) compute a direction of negative curvature and thus, by definition, contains a check of the SOSC: should no direction be found, the SOSC must hold. While this is certainly sufficient when the Hessian is known explicitly, many practical applications do not have such knowledge. Implementations of second-order algorithms that rely on quasi-Newton approximations to the Hessian only determine whether there is a direction of negative curvature for the approximation, rather than the true Hessian, and thus cannot by themselves verify the SOSC. Many other large-scale Hessian-free codes apply Conjugate-Gradient (CG) type iterations to solve constrained quadratic subproblems [47]. Section 4.3 below, however, demonstrates that convergence of CG methods alone is insufficient to verify the SOSC and thus it is conceivable that CG methods could miss indefiniteness in $H$ (over $\mathcal{C}$) in some exceptional circumstances. A post-convergence verification of the SOSC at computed points appears to be required for proper application of Hessian-free methods for large-scale optimization.

2.2. Equilibrium Problems. Verifying the SOSC is also currently vital for solving non-linear, non-convex equality constrained (generalized Nash) equilibrium problems; that is,
collages of $K \in \{2, 3, \ldots \}$ coupled optimization problems:

$$
\begin{align*}
\text{minimize} & \quad f_k(x_1, \ldots, x_K) \\
\text{with respect to} & \quad x_k \in \mathbb{R}^{N_k} \\
\text{subject to} & \quad c_k(x_1, \ldots, x_K) = 0
\end{align*}
$$

where $f_k : \mathbb{R}^N \to \mathbb{R}$, $c_k : \mathbb{R}^N \to \mathbb{R}^{M_k}$, $N_k, M_k \in \mathbb{N}$, $M_k < N_k$, and $N = N_1 + \cdots + N_K$ for all $k \in \{1, \ldots, K\}$. A (local) equilibrium is a point $(x_1^*, \ldots, x_K^*) \in \mathbb{R}^N$ such that $x_k^* \in \mathbb{R}^{N_k}$ (locally) solves Problem (2) for all $k$. Originating in game theory, such equilibrium models have been used by Economists and Operations Researchers to study electric power [50, 51, 52, 24] and other energy sectors [55, 34], new and used vehicles [5, 42, 43, 84, 71, 55, 6, 3, 53], entertainment goods [45], and food services [8, 66, 67, 83, 85]. Recent “market-systems” research in engineering design is also applying the equilibrium framework [59, 82, 79, 80, 81, 33]. See [26] for a review of similar applications and economic history.

First and second order optimality conditions can be employed in the computation of equilibria. In particular, a FONC for a (local) equilibrium follows from combining the FONC for each underlying optimization problem, resulting in a single nonlinear system (e.g., [60]) or, more generally, Nonlinear Complementarity Problem (NCP) (e.g., [27, 51]); see [26].

“Simultaneously stationary” points that solve combined FONC can often be computed using standard methods for nonlinear systems or NCPs such as trust-region Newton methods [20, 18], non-smooth Newton methods [74, 21, 61], or semi-smooth methods [61]; see, e.g. [2, 51, 60]. However the sufficient decrease conditions that enforce convergence to minimizers or saddle points in optimization problems now apply to a residual norm instead of an objective function, and thus cannot preferentially select equilibria over non-equilibrium stationary points.

Example 2. Morrow & Skerlos [60, Example 10] compute equilibrium prices for a two-firm market with heterogeneous consumers. Both firms set the price for a single “branded” product whose only non-price attribute is “brand”. There are three types of consumers, two of which are brand-loyal and the other is brand-indifferent. Demand within each type is modeled using a Logit model [86, Chapter 3]. The resulting optimal pricing problem is an unconstrained nonlinear optimization problem with a non-convex, multi-modal objective. The combined FONC for equilibrium has nine solutions, with only four of these nine first-order points locally maximizing both firms’ profits. Newton’s method applied to the combined FONC cannot distinguish between any of these nine points; any of the five spurious “solutions” could be mistaken for equilibria if the SOSC were not verified. Again, no general, globally-convergent method for computing only equilibria in this type of non-convex game is currently known.

Some studies compute equilibria using “sequential optimization”, “tattonement”, or “Jacobi/Gauss-Seidel” methods [59, 51]; see the discussion of “practitioners methods” in [26] for algorithmic details. In general, sequential optimization should enable some “filtering” of simultaneously stationary points, as the use of optimization algorithms would
avoid convergence to simultaneously stationary points that are minimizers of some objectives and maximizers of others. However, there are no existing results guaranteeing any sort of convergence behavior from sequential optimization methods in either convex or non-convex equilibrium problems [26]. Because there do exist algorithms proven to converge to solutions of the combined FONC, computing equilibria through solving the combined FONC is currently theoretically preferable to sequential optimization methods. Furthermore, sequential optimization is likely to be efficient only when the optimization problems are weakly coupled; much effort could be wasted when the optimizers for the coupled problems strongly depend on one another.

While strong methods for computing equilibrium points in games with convex objectives and feasible sets exist and are preferable to solving the FONC alone, the alternatives to solving the combined first-order conditions when the players’ objectives and feasible sets are non-convex are currently limited. Until methods guaranteed to compute equilibria in non-convex games are developed, general equilibrium programming must be undertaken with checks of the SOSC.

2.3. Benefits of Hessian-Free Algorithms. A “Hessian-free” algorithm for checking the SOSC will require only matrix-vector products with $H$, rather than requiring $H$ explicitly [68, pg. 170]. Hessian-free algorithms for checking the SOSC have at least two major advantages over algorithms that require the Hessians explicitly.

First, multiplying by $H$ can be more efficient than working directly with the elements of $H$, even when $H$ is known [87, pg. 244]. For example: if $H \in \mathbb{R}^{N \times N}$ is diagonal, then $Hx$ can be computed in $N$ flops rather than the $2N^2$ flops required for arbitrary $H \in \mathbb{R}^{N \times N}$. For very large and sparse $H$ the efficiency gained by algorithms that require only multiplications by $H$ can reduce computational burden by an order of magnitude, without adding the significant overhead required by sparse factorization methods to track entries and maintain sparsity. The benefits of this property is well-known and lauded in numerical linear algebra.

Second, the second-order derivatives of the objective and constraints required to explicitly compute $H$ can be challenging to derive, difficult to program, and computationally intensive to implement for complex optimization and equilibrium problems [65, 54, 68]. Matrix-vector products $Hs$, however, can be obtained with the gradient of the Lagrangian alone using finite-difference approximations [10, 70, 68]:

$$Hs \approx \frac{1}{\sigma||s||} \left( \nabla \mathcal{L}(x + \sigma s, \lambda) - \nabla \mathcal{L}(x, \lambda) \right)$$

for small $\sigma$; see [20, 10] to choose effective scales $\sigma$. In fact, this relationship underlies the effectiveness of some Newton-Krylov solvers for very large and complex nonlinear systems [10, 70].

Of course, an approximation to $H$ itself could be constructed using finite differences with $N$ gradient evaluations. Indeed, this complete approximation is required to use the BHT and Inertia tests when the second derivatives are not explicitly provided; quasi-Newton approximations such as the BFGS approximation cannot be used. The new Hessian-free
algorithms presented below, however, require at most \( L \leq N \) evaluations of the gradient of the Lagrangian. For highly constrained problems \( (L \ll N) \) this represents a significant decrease in function evaluations and thus overall computational burden.

3. THREE TESTS OF THE SOSC

Verifying the SOSC for unconstrained problems requires verifying the positive-definiteness of the Hessian matrix \( H \). Computationally, this is best accomplished by attempting to take a Cholesky factorization of \( H \), a stable, efficient, and symmetry-exploiting algorithm [44, 87, 49]. At least three equivalent tests exist for evaluating the “constrained positive-definiteness” \( H \succ_C 0 \) of the Hessian of the Lagrangian in Problem (1).

The most direct test of the SOSC \( H \succ_C 0 \) is to verify the positive-definiteness of a reduced \( L \times L \) Hessian matrix, where \( L = N - M \):

**SOSC Test 1** (Reduced Matrix Test [57]). \( H \succ_C 0 \) if, and only if, \( W^T H W \in \mathbb{R}^{L \times L} \) is positive-definite for any matrix \( W \in \mathbb{R}^{N \times L} \) whose columns form a basis of \( C \).

The three algorithms described in Sections 4.1, 4.2, and 4.3 below are implementations of this test.

The remaining two tests involve two \( (N + M) \times (N + M) \) matrices:

\[
B = \begin{bmatrix} 0 & A \\ A^T & H \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} H & A^T \\ A & 0 \end{bmatrix}
\]

where in both cases \( 0 \in \mathbb{R}^{M \times M} \). \( B \) is the well-known “Bordered Hessian” [58, 57, 69]. \( K \) appears in the FONC for equality-constrained quadratic programs and is thus often referred to as a “KKT matrix” [68, Chapter 16]. More generally, \( K \) is an example of a saddle-point matrix or equilibrium system; for general information see the extensive review in [4].

The “Bordered Hessian Test” (BHT) is a classical test of \( H \succ_C 0 \) particularly popular in economics that uses \( B \):

**SOSC Test 2** (Bordered Hessian Test [41, 58, 57]). \( H \succ_C 0 \) if, and only if, the last \( L \) leading principle minors of \( B \) all have sign \((-1)^M\); specifically, \( \text{sign} (\det (B_i)) = (-1)^M \) for all \( i = 1, \ldots, L \), where \( B_i \) is the submatrix of \( B \) formed by taking the first \( 2M + i \) rows and columns.

Note that the BHT requires \( L \) determinant calculations, and thus \( L \) LU-factorizations. Section 4.4 below outlines an efficient procedure for verifying the BHT using updated LU factorization.

A symmetry-exploiting, single-factorization test based on the KKT matrix \( K \) can also be derived. The *inertia* of a matrix is a triple containing the number of positive, negative, and zero eigenvalues [44, 68]. Gould [46] proves the following equation:

**Lemma 1.** \((46)\) \( \text{inertia}(K) = \text{inertia}(W^T H W) + (M, M, 0) \) for any basis \( W \) of \( C \).

Because \( H \) is positive-definite over \( C \) if, and only if, \( \text{inertia}(W^T H W) = (L, 0, 0) \) — i.e., all of \( (W^T H W) \)’s eigenvalues are positive — Lemma 1 establishes the following test:
Table 1. Several methods for verifying the SOSC elaborated on in Section 4. “Hessian-Free” refers to the ability of an algorithm to operate with Hessian-vector products, rather than the actual elements of the Hessian. “Basis of $C$?” refers to the requirement that an algorithm start by computing a basis of $C$. “Dir. of Neg. Curvature” refers to the ability of an algorithm to reveal or compute a feasible direction of negative curvature when the SOSC fails.

| Test | Algorithm | Hessian-Free? | Basis of $C$? | Dir. of Neg. Curvature |
|------|-----------|---------------|---------------|------------------------|
| 1    | Alg. 2 (Sec. 4.1) | Yes           | Yes           | Yes                    |
| 1    | Alg. 4 (Sec. 4.2) | Yes           | Yes           | Yes                    |
| 1    | Alg. 5 (Sec. 4.3) | Yes           | No            | Yes                    |
| 2    | Sec. 4.4   | No            | No            | No                     |
| 3    | Sec. 4.5   | No            | No            | No                     |

**SOSC Test 3** (([The Inertia Test [46]])). $H >_C 0$ if, and only if, $\text{inertia}(K) = \text{inertia}(B) = (N, M, 0)$.

Section 4.5 describes how this test can be implemented using an inertia-revealing symmetric-indefinite factorization to compute the inertia of $K$ [11, 46, 1, 23].

## 4. FIVE ALGORITHMS

This section derives several algorithms for verifying the SOSC; see Table 1. The focus is on three new Hessian-free algorithms for Test 1, (Sections 4.1, 4.2, and 4.3) rather than the existing factorization-based BHT and Inertia tests (Section 4.4 and 4.5). Appendix A discusses ways to compute a basis $W$ of $C$, needed for the algorithms described in Sections 4.1 and 4.2; Algorithm 5 provides a Hessian-free method that does not require computing a basis of $C$.

### 4.1. An Implicit, Projected Cholesky Factorization for Test (1)

The most direct way to verify $H >_C 0$ given a basis $W$ of $C$ is to attempt to take a Cholesky factorization of $W^T HW$. In the $n^{th}$ step the Cholesky factorization derived in [87, Lecture 23],

$$W^T HW = L_1 \cdots L_{n-1} \begin{bmatrix} I_{n-1} & 0 & 0 \\ 0 & W_{n:L}^T H_n W_{n:L} & L_{n-1}^T \cdots L_1^T \end{bmatrix}$$

where

$$L_n = \begin{bmatrix} I_{n-1} & 0 & 0 \\ 0 & W_{n+1:L}^T H_n w_n / \sqrt{\alpha_n} & 0 \\ 0 & W_{n+1:L}^T H_n w_n / \sqrt{\alpha_n} & I_{L-n-1} \end{bmatrix},$$

$H_1 = H$,

$$H_n = H_{n-1} - H_{n-1} w_{n-1} w_{n-1}^T H_{n-1} / \alpha_n,$$

and $\alpha_n = w_n^T H_n w_n$ for all $n \in \{1, \ldots, L\}$. $H >_C 0$ if, and only if, $\alpha_1, \ldots, \alpha_L > 0$. 


Algorithm 1 ("Classical" Implicit Cholesky).
1. \( V \leftarrow HW \)
2. for \( n = 1, \ldots, L \),
3. for \( m = 1, \ldots, n - 1 \),
4. \( \beta \leftarrow v_m^T w_n / \alpha_m \)
5. \( v_n \leftarrow v_n - \beta v_m \)
6. \( \alpha_n \leftarrow w_n^T v_n \)
7. if \( \alpha_n \leq 0 \), fail,

Algorithm 2 ("Modified" Implicit Cholesky).
1. \( V \leftarrow HW \)
2. for \( n = 1, \ldots, L \),
3. \( \alpha \leftarrow w_n^T v_n \)
4. if \( \alpha \leq 0 \), fail,
5. for \( m = n + 1, \ldots, L \),
6. \( \beta \leftarrow v_m^T w_m / \alpha \)
7. \( v_m \leftarrow v_m - \beta v_n \)

Figure 1. "Classical" (Left) and "Modified" (Right) Gram-Schmidt style Implicit Cholesky Algorithms.

Observing that the Cholesky factors \( L_k \) do not need to be explicitly computed to compute the \( \alpha \) numbers determining whether the Cholesky process succeeds or fails, the Cholesky process can be reduce to the following:

Lemma 2. For any \( n \in \{1, \ldots, L\} \) such that \( \alpha_m = w_m^T v_m > 0 \) for all \( m < n \), \( \alpha_n = w_n^T v_n \) where

\[
(v_n) = H_n w_n = H w_n - \sum_{m=1}^{n-1} \left( \frac{v_m^T w_n}{\alpha_m} \right) v_m.
\]

Proof. The formula for \( \alpha_n \) is its definition accepting \( v_n = H_n w_n \); the second formula for \( v_n \) follows from Eqn. (3) by induction.

Two "implicit" Cholesky algorithms that implement Eqn. (4) are given in Algs. 1 and 2. While \( W^T HW \) is not explicitly formed, Algs. 1 and 2 implicitly form the upper (or lower) triangle of \( W^T HW \). Note that Algs. 1 and 2 only need to compute the products \( H w_n \), rather than work with the elements of \( H \) explicitly, and is thus Hessian-free. Finally, feasible directions of negative curvature can be computed easily should either algorithm reject the SOSC:

Lemma 3. Suppose Algorithm 1 or 2 fails in the \( n \)th step, for some \( n \in \{1, \ldots, L\} \), with \( \alpha_n < 0 \). Then \( d = s_1 w_1 + \cdots + s_n w_n \) is a feasible direction of negative curvature, where \( s_n = 1 \) and

\[
s_m = s_{m+1}(v_m^T w_{m+1}) + \cdots + s_n(v_m^T w_n) / \alpha_m
\]

for all \( m \in \{1, \ldots, n - 1\} \).

Implementing this formula requires storing the \( \alpha \) values computed in Alg. 2 and would benefit from storing the triangular array of inner products \( v_m^T w_{m+k} \) that are computed as part of the Implicit Cholesky process, rather than re-computing them to find a direction of negative curvature.
Algorithm 3 ("Classical" Diagonalization).

1. \( V \leftarrow W \)
2. \( \text{for } n = 1, \ldots, L, \)
3. \( \text{for } m = 1, \ldots, n-1, \)
4. \( \beta \leftarrow z_m^T v_n / \alpha_m \)
5. \( v_n \leftarrow v_n - \beta v_m \)
6. \( z_n \leftarrow Hv_n \)
7. \( \alpha_n \leftarrow v_n^T z_n \)
8. \( \text{if } \alpha_n \leq 0, \text{ fail}, \)

Algorithm 4 ("Modified" Diagonalization).

1. \( V \leftarrow W \)
2. \( \text{for } n = 1, \ldots, L, \)
3. \( z \leftarrow Hv_n \)
4. \( \alpha_n \leftarrow v_n^T z \)
5. \( \text{if } \alpha_n \leq 0, \text{ fail}, \)
6. \( \text{for } m = n+1, \ldots, L, \)
7. \( \beta \leftarrow w_m^T z / \alpha_m \)
8. \( v_m \leftarrow v_m - \beta v_n \)

Figure 2. "Classical" (Left) and "Modified" (Right) Gram-Schmidt style Diagonalization algorithms. Note that the vectors \( z_1, \ldots, z_L \) can overwrite \( w_1, \ldots, w_L \).

Proof. The idea is straightforward: if \( d = Ws \) where \( s \) solves \( L_n^\top \cdots L_1^\top s = e_n \), then \( d \in \text{range}(W) = C \) and \( d^\top Hd = \alpha_n < 0 \). Eqn. (5) follows from applying back substitution to solve for this \( s \). \( \square \)

4.2. A Diagonalization Method for Test (1). The following facts furnish a different version of Test (1):

Lemma 4. (i) Let the columns of \( W \in \mathbb{R}^{N \times L} \) be any basis for \( C \), let \( S \in \mathbb{R}^{L \times L} \) be any nonsingular matrix, and set \( V = WS \). \( W^\top HW \) is positive-definite if, and only if, \( V^\top HV \) is positive-definite. (ii) Let \( V \) be any basis of \( C \) such that \( V^\top HV \) is diagonal. \( H \succ C 0 \) if, and only if, all of \( V^\top HV \)'s diagonal entries are positive.

Proof. Claim (i) follows from Sylvester’s Law of Inertia and claim (ii) is trivial. \( \square \)

Thus, given any basis \( W \) of \( C \), the definiteness of \( H \) over \( C \) can be revealed by finding a nonsingular \( S \) such that \( V = WS \) and \( V^\top HV \) is diagonal. An “oblique” Gram-Schmidt process makes this possible:

Lemma 5. Set \( v_1 = w_1 \) and recursively define

\[
(6) \quad v_n = w_n - \sum_{k=1}^{n-1} \left( \frac{v_k^\top Hv_n}{\alpha_k} \right) v_k
\]

for any \( n \in \{2, \ldots, L\} \), so long as \( \alpha_k = v_k^\top Hv_k \neq 0 \) for all \( k \in \{1, \ldots, n-1\} \). For \( v_1, \ldots, v_n \) so defined, \( v_m^\top Hv_k = 0 \) for all \( m, k \in \{1, \ldots, n\} \), \( m \neq k \).

Proof. The proof is a straightforward induction. \( \square \)

Again, two algorithms for Test 1 based on Eqn. (6) are provided in Algs. 3 and 4. Note that Algs. 3 and 4 require exactly the same linear-algebraic operations as Algs. 1 and 2,
but on different quantities. In particular, Algs. 3 and 4 are also Hessian-free. As written, Algs. 3 and 4 require an additional $N \times L$ matrix of storage for $z_1, \ldots, z_L$, although these vectors can be written over $w_1, \ldots, w_L$ if the basis of $C$ is not needed after the SOSC check. Note, however, no additional computation is required to extract a feasible direction of negative curvature from Algs. 3 or 4 when the SOSC fails:

**Lemma 6.** Suppose Algorithm 3 or 4 fails in step $n \leq L$ with $\alpha < 0$. Then $v_n$ is a feasible direction of negative curvature.

Thus, Alg. 3 or 4 is a useful re-organization of Alg. 1 or 2 (respectively) if directions of negative curvature are important.

### 4.3. Projected Conjugate Gradients for Test (1)

The Conjugate Gradient (CG) algorithm is a widely-used iterative method for solving symmetric positive-definite linear systems [44, 87]. The Projected Conjugate Gradient (PCG) algorithm is an equivalent algorithm for solving the FONC for equality-constrained quadratic programming problems [47, 68]. Verifying the SOSC with PCG is based on the following converse question:

Can (P)CG verify that a matrix is (constrained) positive-definite?

This section describes a Hessian-free approach for checking Test (1) based on existing PCG algorithms. For simplicity, the majority of the derivation neglects constraints, and considers how CG can be adapted to verify or reject the positive-definiteness of a matrix $H \in \mathbb{R}^{N \times N}$. The extension to the constrained case is a straightforward adaptation of this discussion and existing PCG methods as described in, e.g. [68, Chapter 16]. CG applied to the system $Hx = b$, $b \neq 0$ is denoted by $\text{CG}(H,b)$; see [44, pg. 527] or [68, pg. 112] for a formal algorithm. Because CG started at $x_0 \neq 0$ is equivalent to $\text{CG}(H,b')$ started at $0$, where $b' = b - Hx_0$, it is assumed that $\text{CG}(H,b)$ always starts at $0$.

While $\text{CG}(H,b)$ converges if $H$ is definite [44, 87, 68], convergence of $\text{CG}(H,b)$ alone is not an indicator of definiteness:

**Example 3.** Let $H = \text{diag}(1,-1)$ and $b = (1,2)$. $\text{CG}(H,b)$ converges in two steps, despite being applied to an indefinite matrix. However if $b = (1,1)$, then $\text{CG}(H,b)$ breaks down in the first step.

This example shows that convergence (or divergence) of CG alone cannot be used as a basis for verifying positive-definiteness.

The foundation for using $\text{CG}(H,b)$ as a positive-definiteness check is, in fact, Lemma 4(ii). In exact arithmetic $\text{CG}(H,b)$ constructs a set of $N' \leq N$ linearly independent, $H$-conjugate vectors $p_i$ and evaluates $p_i^\top Hp_i$ for all $i \in \{1, \ldots, N'\}$ [44, 87]. Verifying that $p_i^\top Hp_i > 0$ for all $i$ verifies positive-definiteness, by Lemma 4(ii), assuming that $N' = N$.

**Example 4.** Again let $H = \text{diag}(1,-1)$ and $b = (1,2)$. $\text{CG}(H,b)$ encounters $p_2^\top Hp_2 < 0$ in the second step, and thus discovers indefiniteness in $H$.

Ensuring that $p_i^\top Hp_i > 0$ for all $i \in \{1, \ldots, N'\}$ is already a component of some CG codes [68].
Unfortunately CG(H, b) may converge for some N' < N prior to building a basis of \( \mathbb{R}^N \) and thus cannot always determine whether H is positive-definite. Indeed this “early convergence” is the central benefit of CG for solving large linear systems.

**Example 5.** Consider again \( H = \text{diag}(1,-1) \), and take \( b = (1,0) \). CG(H, b) converges in a single step (1 = N' < N = 2), and cannot identify that H is indefinite. ■

In fact, in exact arithmetic, CG(H, b) always converges with N' < N when b lies in a proper invariant subspace of H. (A proper invariant subspace of H is a proper subspace W of \( \mathbb{R}^N \) such that \( Hw \in W \) for all \( w \in W \).) The same issue presents in finite-precision arithmetic, where CG(H, b) “converges” with N' < N when the residual \( Hx - b \) or the search directions become “small”.

When CG(H, b) converges with N' < N, the search can be continued consistent with Lemma 4(ii) by “restarting” with a new right-hand-side \( b' \) that is H-conjugate to all previous search directions. The continued search should proceed until (a) positive-definiteness has been rejected, (b) the restarted process itself converges, or (c) the remaining dimensions of the space have been searched without rejecting positive-definiteness.

**Example 6.** To continue the case in Example 5 above choose \( b' = (0,1) \). \( b' \perp Hb \) and \( (b')^\top Hb' < 0 \); thus the “restarted” CG(H, b') would identify indefiniteness. ■

Figure 3 gives a formal algorithm implementing the continued PCG approach—with constraints—to verify (or reject) \( H >_C 0 \). Algorithm 5 is Hessian-free and, like Algorithm 4, does not require any additional computation to extract a feasible direction of negative curvature:

**Lemma 7.** Suppose Algorithm 5 fails in step \( j \leq L \) with \( \eta < 0 \). Then \( p_j \) is a feasible direction of negative curvature.

Algorithm 5, being modeled on existing PCG algorithms [47, 68], allows flexibility in choosing an operator \( \text{proj}_C : \mathbb{R}^N \to C \) that orthogonally projects vectors from \( \mathbb{R}^N \) onto \( C \subset \mathbb{R}^N \). This projection can always be done using a basis \( W \) of \( C \) by setting \( \text{proj}_C(r) = W(W^\top W)^{-1}W^\top r \), a formula that is particularly simple when \( W \) is orthonormal. However, projection onto \( C \) can also be accomplished without computing a basis for \( C \) [47]. This is an important distinguishing feature of Algorithm 5 relative to Algorithms 2 and 4: In principle, PCG provides a Hessian-free way to verify \( H >_C 0 \) if it is impractical to project onto \( C \) using a basis of \( C \) computed from the constraint gradients.

Gould et al [47] study two projectors in the traditional PCG algorithm (in the context of finite-precision arithmetic). The first sets \( \text{proj}_C r = r - A^\top v \) where \( A A^\top v = Ar \) can be solved using a Cholesky factorization of \( A A^\top \). A QR factorization or SVD of \( A^\top \) could also be used to solve \( \min_{v \in \mathbb{R}^M} \| A^\top v - r \|_2 \), the equivalent least-squares problem defining \( v \). The second method considered in [47] solves an “augmented system” with symmetric-indefinite factorization; see [47, 68, 32]. Any necessary factorizations need only be taken once, prior to executing Algorithm 5. Continuation, however, requires projecting onto the intersection of \( C \) with the image under \( H \) of the previous search directions. Computationally, this amounts to appending certain rows to \( A \) or, equivalently, columns to \( A^\top \). Using Householder QR.
Algorithm 5 ((Continued Projected Conjugate Gradients)).

1. choose $b \in C$
2. $i \leftarrow 0$
3. while $i \leq L$,
4.  
5.  
6.  
7.  
8.  
9.  

Algorithm 6 ((Projected Conjugate Gradient Step)).

1. $r \leftarrow b$, $\omega \leftarrow r^\top r$, $r \leftarrow r/\sqrt{\omega}$, $p \leftarrow r$
2. for $j = i + 1, \ldots, L$
3.  
4.  
5.  
6.  
7.  
8.  
9.  

Figure 3. Formal algorithm for the continued PCG method for verifying (or rejecting) $H \succ C 0$.

factorization of $A^\top$ provides an efficient and stable way to applying and updating this projector.

4.4. Updated LU Factorizations for the BHT, Test (2). The BHT requires computing the sign of $L$ determinants, each typically computed using LU factorization; see [44, pg. 97] or [49, pg. 279]. Explicitly forming the $L$ LU factorizations to compute the leading principal minor determinants for the BHT, however, requires an unacceptable and unnecessary amount of work asymptotically proportional to $L(N + M)^3$.

An efficient LU updating scheme for computing the associated determinant signs follows from a recursive definition of the leading principle submatrices of $B$. The matrices $B_i$, $i = 2, \ldots, L$ (defined in Test 2) satisfy the recursion

$$B_i = \begin{bmatrix} B_{i-1} & b_i \\ b_i^\top & \gamma_i \end{bmatrix}$$

for some $b_i \in \mathbb{R}^{2M+i-1}$, $\gamma_i \in \mathbb{R}$. 
An LU factorization of $B_i$ can then be constructed by applying the pivots and row eliminations from earlier LU factorizations to $b_i$ and then zeroing out $b_i^\top$ with $2M + i - 1$ new row eliminations [44, Section 3.2].

For brevity the full LU updating process is not described here; deriving a formal statement of this update is straightforward, if a bit tedious. The first LU factorization can be specially designed to account for the leading $M \times M$ submatrix of zeros in $B$ and standard pivoting strategies can be used to enhance stability. Existing codes such as LAPACK [56], LUSOL [36, 75] PARDISO [76], or HSL’s LA15 [19] can be exploited to take the first LU factorization as well as, in some cases, implement the factorization updates. While computing determinants can be inaccurate due to numerical over- and under-flow [49, pg. 279], computing the sign of the determinant is exact for the computed LU factorization. In any case, the BHT ultimately requires factoring the $L$ matrices $B_i$ and may thus be inappropriate for large scale problems without significant sparsity. Furthermore, the BHT does not provide an obvious route to computing a direction of negative curvature should the SOSC fail.

4.5. **Block Symmetric-Indefinite (LDL) Factorization for the Inertia Test (3).** Implementing the Inertia Test (3) is, in principle, rather straightforward. The inertia of $K$ can be computed using a stable symmetric-indefinite block “LDL” factorization [11, 1, 49, 23] of the form $PKP^\top = LDL^\top$ where $P$ is a permutation matrix, $L$ is unit lower triangular, and $D$ is a block-diagonal matrix with $1 \times 1$ and $2 \times 2$ blocks; see [49, Chapter 11]. By Sylvester’s Law of Inertia, $\text{inertia}(K) = \text{inertia}(D)$ [44, pg. 403] and computing the inertia of the computed $D$ is efficient and exact even in inexact arithmetic; see [49, Problem 11.2]. Indeed, symmetric-indefinite codes [23, 77] already have options to compute inertia, and the matrix inertia is used in some optimization solvers to promote global convergence [38, 15, 78]. However accuracy of the Inertia test in inexact arithmetic depends on whether the inertia of the $D$ computed using finite-precision arithmetic actually equals the inertia of $K$. The Inertia test requires factoring a single $(N + M) \times (N + M)$ matrix and may thus be applicable to large scale problems only when there is significant sparsity. There are, however, extremely efficient codes for forming this factorization. Like the BHT, it is not obvious how to compute a direction of negative curvature from $K$ when the Inertia Test fails.

5. **EXAMPLES**

This section presents several computational examples. Sections 5.3 and 5.4 illustrate the relative performance characteristics of the different algorithms discussed above on dense test problems for which $H$, $A$, and the truth of $H \succ C \mathbf{0}$ is known explicitly. Section 5.5 demonstrates the advantages of the Hessian-free properties of Cholesky and Diagonalization using a test problem from the COPS collection [22].

5.1. **Computational Details.** Algs. 2, 4, 5, the updated LU BHT, and the Inertia test have been implemented in C making extensive use of BLAS [9] routines and LAPACK’s routines for QR ($\text{dgeqrf/dormqr}$), LU ($\text{dgetrf}$), and symmetric-indefinite block LDL ($\text{dsytrf}$)
factorizations [56]. The basis $W$ of $C$ used in Algorithms 2 and 4 is computed using a QR factorization of $A^\mathsf{T}$, primarily for consistency with the updated QR approach to the PCG algorithm. Using the SVD would probably provide more stable (though also more expensive) computations of $W$. The updated LU BHT described above was verified using a “naive” BHT that computes $\text{sign}\left(\text{det}(B_i)\right)$ via $L$ independent LU factorizations. The time savings from updating LU factorizations ranged from 10-90% depending on $N$ and $M$. All computations were undertaken on an Apple MacPro tower with dual quad-core 2.26 GHz “Nehalem” processors (each with an 8 MB cache) and 32 GB of RAM running Mac OS X 10.6.6 and Apple’s implementations of BLAS and LAPACK.

5.2. Generating Dense Random Test Problems. Random test problems were obtained as follows: Let $N \in \{2, 3, \ldots\}$, $M \in \{1, \ldots, N - 1\}$, and $P \in \{0, \ldots, N\}$. Choose symmetric positive-definite $\Lambda_+ \in \mathbb{R}^{P \times P}$, symmetric negative-definite $\Lambda_- \in \mathbb{R}^{(N-P) \times (N-P)}$, orthogonal $Q \in \mathbb{R}^{N \times N}$, upper-triangular $R \in \mathbb{R}^{M \times M}$, and set

$$(7)\quad H = Q \begin{bmatrix} \Lambda_+ & 0 \\ 0 & \Lambda_- \end{bmatrix} Q^\mathsf{T} \quad \text{and} \quad A^\mathsf{T} = Q \begin{bmatrix} 0 \\ R \end{bmatrix}$$

Whether $H \succ C 0$ for such $H$ and $A$ is known analytically:

Lemma 8. Let $H$ and $A$ be defined as in Eqn. (7) and let $C = \text{null}(A)$. (i) $H \succ C 0$ if, and only if, $L = N - M \leq P$. (ii) For uniformly drawn $M \in \{1, \ldots, N - 1\}$ and $P \in \{0, \ldots, N\}$, $\mathbb{P}(H \succ C 0) = (N + 2)/(2(N + 1))$.

The techniques described by Higham [49, pg. 517-518] are used to draw random orthogonal matrices ($Q$) and random symmetric matrices with known eigenvalues ($\Lambda_+, \Lambda_-$). Note that any such construction is likely to be dense, and thus does not take advantage of the Hessian-free characteristics of the Cholesky, Diagonalization, or PCG algorithms.

5.3. A Well-Conditioned Example.

Example 7. Let $A$ be defined by an $R$ with off diagonal elements $r_{i,j}$, $j > i$, drawn from a standard normal distribution and diagonal elements $r_{i,i}$ drawn from a normal distribution with mean zero and variance $(M - i)^2$. ■

Over 20,000 numerical tests were run for Ex. 7. 90 distinct values of $N$ were drawn from $\{10, …, 5000\}$ and, for each $N$, at least 200 $M, P$ pairs were drawn. Slightly more than half ($\sim 50.3\%$) of these trials have $H \succ C 0$, with $H \not\succ C 0$ in the remaining trials; see Lemma 8(ii). Every Hessian matrix $H$ drawn has eigenvalues $\lambda$ satisfying $0.1 \leq |\lambda| \leq 100$. The tolerance for convergence in the PCG approach (Alg. 5) was $10^{-10}$.

In Ex. 7, all methods correctly verified or rejected $H \succ C 0$ in most tests. No method returned a false positive in any trial, and only the Inertia test returned false negatives. Indeed, the performance of the Inertia test degrades when $H \succ C 0$ as $N$ grows, as shown in Table 5.3. For $N \approx 5,000$, more than 3% of the SOSO checks using the Inertia test gives a false negative. No other algorithm returned a single false negative.

Figs. 4 and 5 compare the time required by each method to computationally verify $H \succ C 0$ for Ex. 7. The Inertia test, generally the fastest method, is used as a benchmark.
Table 2. Percent of trials in Example 7 for which $H \succ_C 0$ yet the Inertia Test incorrectly declares $H \not\succ_C 0$; i.e. false negatives (FN).

| N    | 77  | 242 | 478 | 679 | 899 | 908 | 1153 | 1161 |
|------|-----|-----|-----|-----|-----|-----|------|------|
| FN   | 1.0%| 0.5%| 2.0%| 1.1%| 1.0%| 1.0%| 1.0%  | 2.1% |

| N    | 1530| 1538| 1718| 2306| 2460| 3687| 4637 | 4903 |
|------|-----|-----|-----|-----|-----|-----|------|------|
| FN   | 1.0%| 1.0%| 2.9%| 2.0%| 1.1%| 2.7%| 1.9%  | 3.3% |

Table 3. Percent of trials in Example 7 for which $H \succ_C 0$ and PCG was continued at least once.

| All $N$ | $N \geq 20$ | $N \geq 50$ | $N \geq 100$ | $N \geq 1,000$ |
|---------|-------------|-------------|-------------|----------------|
| 68%     | 76%         | 88%         | 94%         | 99%           |

for comparison; see Fig. 4. Cholesky (Alg. 2) and Diagonalization (Alg. 4) are as fast or faster than the Inertia test for highly constrained problems with $M \geq 0.75N$. However if $M$ is small relative to $N$, Algs. 2 and 4 can take 10-20 times longer than the Inertia test. Alg. 4 is slightly slower than Alg. 2 in part because the Hessian-vector products $Hw$ are done one-by-one in Alg. 4, instead of “all-at-once” in Alg. 2. When $H$ is known the level-3 BLAS routines used to form $HW$ optimize efficient cache memory usage. Generally speaking these results are more encouraging than they appear: *dsytrf*, the code for the Inertia test, is highly optimized while the implementations of Algs. 2 and 4 have not yet been optimized. “Blocked” variants of Algs. 2 and 4 that fully exploit memory traffic efficiencies in the level-3 BLAS are conceptually easy to derive, and will be even more competitive with the Inertia test from the perspective of compute time.

The PCG approach (Alg. 5) and the BHT are as fast or faster than the Inertia test only for $M \approx N$, and can take more than 100 times longer than the Inertia test on problems with $N \geq 1000$ and small $M$. Though it is not shown in the plots, continuation is an important part of the PCG approach; see Table 5.3. Alg. 5 converged at least once in over 68% of the trials for which $H \succ_C 0$, and would thus “fail” in more than 68% of our cases if it were not continued. Moreover as $N$ grows, Alg. 5 tends to converge more often. Based on the trials undertaken for Ex. 7, we would “expect” Alg. 5 to converge in $\sim N^{0.85}$ cases. Specifically, Alg. 5 converged no more than $N^{0.9}$ times as $N$ grew in the trials undertaken for Ex. 7, and converged at least $\sim N^{0.8}$ times in more than 50% of the trials. While these predictions should not be extrapolated beyond this example, they clearly demonstrate the necessity of continuation in the PCG approach to verifying the SOSC.

5.4. A Poorly Conditioned Example.

**Example 8.** Let $A$ be defined by an $R$ with all elements drawn from a standard normal distribution. □
Figure 4. Compute times for the Inertia test applied in Example 7 in trials for which $H \succ_C 0$ is true.

Figure 5. Compute times for Example 7 in trials for which $H \succ_C 0$ is true, relative to the Inertia test. Solid black lines give the maximum and minimum ratios or times for a given $N$ over the values of $M$ tested. Thin grey lines in the four relative plots represent the maximum ratio of times for subsets of trials such that $M \geq 0.25N$ (lightest), $M \geq 0.5N$ (midtone), and $M \geq 0.75N$ (darkest).

Again, over 20,000 numerical tests were run for Ex. 8 with the same character as for Ex. 7, discussed above.

Ex. 8 demonstrates that numerical accuracy is far from guaranteed for computational SOSC checks. Fig. 6 illustrates the fraction of correct tests results and thus also of false negatives. The Cholesky, Diagonalization, PCG, and BHT tests appear more stable than the Inertia test. However all methods have a false negative rate close to 70% for problems with as few as $N \approx 100$ variables. With $N \approx 1,000$ variables, each test is so overcome by roundoff error that virtually no correct results are obtained. As with Ex. 7, there were no false positives.

The relatively poorer performance of the Inertia test in both examples deserves some discussion. Weyl’s Eigenvalue Perturbation Theorem provides one way to control numerical error in the Inertia test:
Problem Size

Fraction of Correct Results

Fraction of False Negatives

5000
Algs. 1, 2, 3, and BHT

Inertia Test

Figure 6. Fraction of correct test results (left axis) and, symmetrically, false negatives (right axis), over all trials in Example 8 for which $H \succ C 0$.

Theorem 2. (Weyl’s Theorem [25, Theorem 4.1]) If $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_n$ are the eigenvalues of $X$ and $X + E$, respectively, for any $n \times n$ symmetric matrix $X$ and any $n \times n$ matrix $E$, then $|\lambda_i - \mu_i| \leq ||E||_2$ for all $i$.

Corollary 1. Let $P(K + E)P^T = LDL^T$ be the computed LDL factorization of $K$ where errors are accumulated into a perturbation $E$ of $K$; see [49, Chapter 11]. If $|\mu|_{\min} = \min\{|\mu| : \mu \in \lambda(K + E)\} > ||E||_2$ then $\text{inertia}(K) = \text{inertia}(D)$.

Proof. The Weyl Theorem states that $\lambda_i = \mu_i + \epsilon_i$ for some $|\epsilon_i| \leq ||E||_2$ for all $i$. If $|\mu_i| > ||E||_2$ for all $i$, then also $\text{sign}(\lambda_i) = \text{sign}(\mu_i)$. Hence $\text{inertia}(K) = \text{inertia}(K + E) = \text{inertia}(D)$. \qed

If, however, $|\mu|_{\min} \leq ||E||_2$ then $\text{inertia}(K + E) = \text{inertia}(D)$ may not be equal to $\text{inertia}(K)$. That is, if the magnitude of any eigenvalue of $K + E$ is less than the norm of the accumulated round-off errors, this eigenvalue of the perturbed KKT matrix may have the wrong sign relative to the true eigenvalue. As $N + M$ grows, the likelihood of both a small $|\mu|_{\min}$ and large $||E||_2$ increases.

Note that $K$ can have small eigenvalues even if $H$ does not. Specifically, $K$ has small eigenvalues whenever $A$ has nearly linearly dependent columns, as the simple example in Appendix B shows.

Using the “exact” basis $Q_1:L$ of $C$ in the Cholesky and Diagonalization tests eliminated the false negatives seen in Fig. 6. This suggests that the numerical errors in these two tests, as well as perhaps the PCG test, are entirely a consequence of numerical errors in the QR factorization of $A^T$. Two remarks along these lines must be made: First, using $Q_1:L$ is a device of the artificial numerical examples. In practice, some factorization must be used compute a basis for $C$ from $A$. Second, the effect of error in the computed basis of $C$ will depend on the spectrum of $H$. For example if $H$ is positive-definite these representational errors in would be irrelevant to the accuracy of the test. On the other hand, false negatives can be obtained only if the numerical errors in the representation of $C$
magnify the contribution of $H$’s negative eigenvalues to the quadratic forms. Understanding when the construction of a basis of $C$ is sufficiently accurate relative to $H$ will be essential to understanding when SOSC checks using the Cholesky and Diagonalization approaches are themselves numerically accurate.

Unfortunately there will also be numerical errors associated with the Cholesky and Diagonalization approaches themselves for some Hessians $H$, independent of what errors are made in the construction of a representation of $C$. What specific properties of $H$ to monitor and control in this respect are not yet known and will be the subject of future investigations.

5.5. A Hessian-Free Example from the COPS Collection. The “Thomson Problem” of finding the minimal energy configuration of $K \in \{2, 3, \ldots\}$ points on a sphere is a large-scale equality-constrained optimization problem from the COPS collection [22]:

\[
\min f(x_1, \ldots, x_K) = \sum_{k=1}^{K-1} \sum_{m=k+1}^{K} \frac{1}{||x_k - x_m||_2}
\]

with respect to $x_1, \ldots, x_K \in \mathbb{R}^3$

subject to $||x_k||_2^2 = 1$ for all $n \in \{1, \ldots, K\}$

Prob. (8) depends only on the Euclidean norms of $K$ vectors in $\mathbb{R}^3$, which are invariant over orthogonal transformations of $\mathbb{R}^3$. Thus, Prob. (8) as stated has infinitely many solutions: specifically, if $(x_1, \ldots, x_K)$ is a solution then so is $(Qx_1, \ldots, Qx_K)$ for each rotation or reflection $Q$ of $\mathbb{R}^3$. An “orthogonally-invariant” Thomson problem can be obtained by restricting the locations of the first and second points in a manner inspired by Householder QR factorization:

\[
\min f(x_1, \ldots, x_K) = \sum_{k=1}^{K-1} \sum_{m=k+1}^{K} \frac{1}{||x_k - x_m||_2}
\]

with respect to $x_1, \ldots, x_K \in \mathbb{R}^3$

subject to $||x_k||_2^2/2 = 1/2$ for all $n \in \{1, \ldots, K\}$

$x_{1,2} = x_{1,3} = x_{2,3} = 0$

Problem (9) has $N = 3K$ variables and $M = K + 3$ constraints.

Prob. (9) was solved for specific values of $K$ between 3 and 333 using Matlab’s Hessian-free interior-point algorithm. The SOSC was then verified at the computed $(x_1, \ldots, x_K)$ and $\lambda$ using our C implementations of Cholesky, Diagonalization, and the Inertia test. The Hessian-free nature of the Cholesky and Diagonalization algorithms was exploited by using directional finite differences to approximate Hessian-vector products $Hs$. For the Inertia test the full Hessian $H$ was approximated with finite-differences. The PCG and BHT algorithms were not used; the results in Section 5.3 suggest they are not competitive.

In this case the Hessian-free SOSC checks provided by the Cholesky and Diagonalization algorithms tended to reduced time to verify the SOSC relative to the Inertia test by just
Figure 7. Left: Computed solution to Prob. (9) for $K = 100$. Right: Percent reduction in time to computationally verify $H \succcurlyeq 0$ for various instances of Prob. (9). When only the black dot is visible, the gray and black dots coincide. Note that $N = 3K$, where $K$ is the number of points distributed over the surface of the unit sphere.

over 20%; see Fig. 7. Recall that the Inertia test was the fastest method on the dense test problems above with $\sim 30\%$ of the variables constrained. Thus the Hessian-free application of the Cholesky and Diagonalization algorithms results in significant computational savings.

6. CONCLUSIONS

This article has presented three novel Hessian-free algorithms for verifying (or rejecting) the SOSC for constrained continuous optimization. These algorithms also make computation of feasible directions of negative curvature easy when the SOSC fails, a feature not available in classical tests. Numerical trials have demonstrated (1) the inefficiency of the Bordered Hessian Test, (2) the relative speed of the Inertia test, (3) the computational efficiency of the new algorithms, especially when their Hessian-free properties can be exploited, and (4) the potential for significant loss of accuracy due to round-off error using any method, even on small problems. Future work will optimize implementations of the new algorithms and undertake a detailed mathematical analysis of round-off errors to determine computable certificates of test accuracy.

References

1. Cleve Ashcraft, Roger G. Grimes, and John G. Lewis, Accurate symmetric indefinite linear equation solvers, SIAM journal of matrix analysis and applications 20 (1998), no. 2, 513–561.
2. Hande Y. Benson, Arun Sen, David F. Shanno, and Robert J. Vanderbei, Interior-point algorithms, penalty methods, and equilibrium problems, Computational Optimization and Applications 34 (2006), no. 2, 155–182.
3. Antonio M. Bento, Lawrence H. Goulder, Mark R. Jacobsen, and Roger H. von Haefen, *Distributional and efficiency impacts of increased us gasoline taxes*, American Economic Review **99** (2009), no. 3, 667–699.
4. Michele Benzi, Gene H Golub, and Jorg Leisen, *Numerical solution of saddle point problems*, Acta Numerica **14** (2005), 1–137.
5. Steven Berry, James Levinsohn, and Ariel Pakes, *Automobile prices in market equilibrium*, Econometrica **63** (1995), no. 4, 841–890.
6. ______., *Differentiated products demand systems from a combination of micro and macro data: The new car market*, Journal of Political Economy **112** (2004), no. 1, 68–105.
7. Dimitri P. Bertsekas, *Nonlinear programming*, Athena Scientific, 1999.
8. David Besanko, Sachin Gupta, and Dipak Jain, *Logit demand estimation under competitive pricing behavior: An equilibrium framework*, Management Science **44** (1998), no. 11, 1533–1547.
9. BLAS, *Basic linear algebra subprograms*, 2010.
10. Peter N. Brown and Youcef Saad, *Hybrid krylov methods for nonlinear systems of equations*, SIAM Journal of Scientific and Statistical Computing **11** (1990), no. 3, 450–481.
11. James R. Bunch and Linda Kaufman, *Some stable methods for calculating inertia and solving symmetric linear systems*, Mathematics of Computation **31** (1977), no. 137, 163–179.
12. Richard H. Byrd, Jean Charles Gilbert, and Jorge Nocedal, *A trust region method based on interior point techniques for nonlinear programming*, Mathematical Programming, Series A **89** (2000), no. 1, 149–185.
13. Richard H. Byrd, Nicholas I. M. Gould, Jorge Nocedal, and Richard A. Waltz, *An algorithm for nonlinear optimization using linear programming and quality constrained subproblems*, Mathematical Programming, Series B **100** (2004), no. 1, 27–48.
14. Richard H. Byrd, Mary E. Hribar, and Jorge Nocedal, *An interior point algorithm for large-scale nonlinear programming*, SIAM Journal on Optimization **9** (1999), no. 4, 877–900.
15. Richard H. Byrd, Jorge Nocedal, and Richard A. Waltz, *KNITRO: An integrated package for nonlinear optimization*, Tech. report, Ziena Inc., 2006.
16. Andrew R. Conn, Nicholas I. M. Gould, and Philippe L. Toint, *Lancelot.*
17. ______., *LANCELOT: A fortran package for large-scale nonlinear optimization (release a)*, Springer Series in Computational Mathematics, vol. 17, Springer-Verlag, 1992.
18. ______., *Trust region methods*, SIAM, 2000.
19. Science & Technology Facilities Council, *HSL mathematical software library catalogue: LA15 v 1.2.0*, Tech. report, Research Councils UK, 2010.
20. John E. Dennis and Robert B. Schnabel, *Numerical methods for unconstrained optimization and nonlinear equations*, SIAM, 1996.
21. Steven P. Dirkse and Michael C. Ferris, *The PATH solver: A non-monotone stabilization scheme for mixed complementarity problems*, Optimization Methods and Software **5** (1995), 123–156.
22. Elizabeth D. Dolan, Jorge J. More, and Todd S. Munson, *Benchmarking optimization software with COPS 3.0*, Tech. Report ANL/MCS-273, Argonne National Laboratory, February 2004.
23. Iain S. Duff, *MA57 - a code for the solution of sparse symmetric definite and indefinite systems*, ACM Transactions on Mathematical Software **30** (2004), no. 2, 118–144.
24. Andreas Ehrenmann and Karsten Neuhoff, *A comparison of electricity market designs in networks*, Operations Research **57** (2009), no. 2, 274–286.
25. Stanley C. Eisenstat and Ilse C. F. Ipsen, *Relative perturbation results for eigenvalues and eigenvectors of diagonalisable matrices*, SIAM Journal of Matrix Analysis and Applications **20** (1998), no. 1, 149–158.
26. Francisco Facchinei and Christian Kanzow, *Generalized nash equilibrium problems*, Annals of Operations Research **175** (2009), 177–211.
27. Michael C. Ferris and Jong-Shi Pang, *Engineering and economic applications of complementarity problems*, SIAM Review **39** (1997), no. 4, 669–713.
28. Roger Fletcher, *Practical methods of optimization*, Wiley and Sons, 1987.
29. Roger Fletcher, Nicholas I. M. Gould, Sven Leyffer, Philippe L. Toint, and Andreas Wachter, *Global convergence of a trust-region sqp-filter algorithm for general nonlinear programming*, SIAM Journal on Optimization 13 (2002), no. 2, 635–659.

30. Roger Fletcher and Sven Leyffer, *Nonlinear programming without a penalty function*, Mathematical Programming, Series A 91 (2002), no. 2, 239–269.

31. Roger Fletcher, Sven Leyffer, and Philippe L. Toint, *On the global convergence of a filter-SQP algorithm*, SIAM Journal on Optimization 13 (2002), no. 1, 44–59.

32. Anders Forsgren, Philip E. Gill, and Joshua D. Griffin, *Iterative solution of augmented systems arising in interior methods*, SIAM Journal on Optimization 18 (2007), no. 2, 666–690.

33. Bart D. Frischknecht, Katie Whitefoot, and Panos Y. Papalambros, *On the Suitability of Econometric Demand Models in Design for Market Systems*, ASME Journal of Mechanical Design 132 (2010), no. 121007, 1–11.

34. Steven A. Gabriel, Supat Kiet, and Jifang Zhuang, *A mixed complementarity-based equilibrium model of natural gas markets*, Operations Research 53 (2005), no. 5, 799–818.

35. Steven A. Gabriel, Andy S. Kydes, and Peter Whitman, *The national energy modeling system: A large-scale energy-economic equilibrium model*, Operations Research 49 (2001), no. 1, 14–25.

36. P. E. Gill, W. Murray, M. A. Saunders, and M. H. Wright, *Maintaining LU factors of a general sparse matrix*, Linear Algebra and its Applications 88/89 (1987), 239–270.

37. Philip E. Gill, Walter Murray, and Michael A. Saunders, *SNOPT: An SQP algorithm for large-scale constrained optimization*, SIAM Review 47 (2005), no. 1, 99–131.

38. Philip E. Gill, Walter Murray, Michael A. Saunders, and Margaret H. Wright, *Inertia-controlling methods for general quadratic programming*, SIAM Review 33 (1991), no. 1, 1–36.

39. , *User’s guide for NPSOL 5.0: A FORTRAN package for nonlinear programming*, Tech. Report SOL 86-6, Stanford University, 2001.

40. Philip E. Gill, Walter Murray, and Margaret H. Wright, *Practical optimization*, Academic Press, 1981.

41. Robert Pollack Gillespie, *Partial differentiation*, Interscience Publishers, 1951.

42. Pinelopi K. Goldberg, *Product differentiation and oligopoly in international markets: The case of the u.s. automobile industry*, Econometrica 63 (1995), no. 4, 891–951.

43. , *The effects of the corporate average fuel efficiency standards in the US*, The Journal of Industrial Economics 46 (1998), no. 1, 1–33.

44. Gene H Golub and Charles F Van Loan, *Matrix computations*, The Johns Hopkins University Press, 1996.

45. Austan Goolsbee and Amil Petrin, *The consumer gains from direct broadcast satellites and the competition with cable tv*, Econometrica 72 (2004), no. 2, 351–381.

46. Nicholas I. M. Gould, *On practical conditions for the existence and uniqueness of solutions to the general equality quadratic programming problems*, Mathematical Programming 32 (1985), 90–99.

47. Nicholas I. M. Gould, Mary E. Hribar, and Jorge Nocedal, *On the solution of equality constrained quadratic programming problems arising in optimization*, SIAM Journal of Scientific Computing 23 (2001), no. 4, 1376–1395.

48. S. P. Han, *A globally convergent method for nonlinear programming*, Journal of Optimization Theory and Applications 22 (1977), no. 3, 297–309.

49. Nicholas J. Higham, *Accuracy and stability of numerical algorithms*, 2nd edition ed., SIAM, 2002.

50. Benjamin F. Hobbs and J. S. Pang, *Nash-cournot equilibria in electric power markets with piecewise linear demand functions and joint constraints*, Operations Research 55 (2007), no. 1, 113–127.

51. Xinmin Hu and Daniel Ralph, *Using EPECs to model bilevel games in restructured electricity markets with locational prices*, Operations Research 55 (2007), no. 5, 809–827.

52. Xinmin Hu, Daniel Ralph, Eric K. Ralph, Peter Bardsley, and Michael C. Ferris, *Electricity generation with looped transmission networks: Bidding to an iso*, 2007.

53. Mark R. Jacobson, *Evaluating u. s. fuel economy standards in a model with producer and household heterogeneity*, Working Paper, Stanford University, January 2010.
54. Kenneth L. Judd, *Numerical methods in economics*, MIT Press, 1998.
55. Andrew Kleit, *Impacts of long-range increases in the corporate average fuel economy (cafe) standard*, Economic Inquiry 42 (2004), no. 2, 279–294.
56. LAPACK, *Lapack - linear algebra package*, 2010.
57. David G. Luenberger and Yinyu Ye, *Linear and nonlinear programming*, third edition ed., Springer, 2009.
58. Daniel L. McFadden, *Appendix A: Definite quadratic forms subject to constraint*, Production Economics: A Dual Approach to Theory and Applications (Melvyn Fuss and Daniel L. McFadden, eds.), vol. I: The Theory of Production, Amsterdam: North-Holland, 1978.
59. Jeremy J. Michalek, Panos Y. Papalambros, and Steven J. Skerlos, *A Study of Fuel Efficiency and Emission Policy Impact on Optimal Vehicle Design Decisions*, ASME Journal of Mechanical Design 126 (2004), 1062–1070.
60. W. Ross Morrow and Steven J. Skerlos, *Fixed-point approaches to computing bertrand-nash equilibrium prices under mixed-logit demand*, Operations Research (Forthcoming).
61. Todd S. Munson, *Algorithms and environments for complementarity*, Ph.D. thesis, University of Wisconsin, Madison, 2000.
62. Bruce A. Murtagh and Michael A. Saunders, *Large-scale linearly constrained optimization*, Mathematical Programming 14 (1978), no. 1, 41–72.
63. ______, *A projected lagrangian algorithm and its implementation for sparse nonlinear constraints*, Mathematical Programming Study 16 (1982), 84–117.
64. ______, *MINOS 5.5 user’s guide*, Tech. Report SOL 83-20R, Stanford University, 1998.
65. Stephen G. Nash, *Nonlinear Programming*, ORMS Today (1998).
66. Aviv Nevo, *Mergers with differentiated products: The case of the ready-to-eat cereal industry*, The RAND Journal of Economics 31 (2000), no. 3, 395–421.
67. ______, *Measuring market power in the ready-to-eat cereal industry*, Econometrica 69 (2001), no. 2, 307–342.
68. Jorge Nocedal and Stephen J. Wright, *Numerical optimization*, Springer-Verlag, 2006.
69. Panos Y. Papalambros and Douglas J. Wilde, *Principles of optimal design: Modeling and computation*, Cambridge University Press, 2000.
70. Michael Pernice and Homer F. Walker, *NITSOL: A newton iterative solver for nonlinear systems*, SIAM Journal of Scientific Computing 19 (1998), no. 1, 302–318.
71. Amil Petrin, *Quantifying the benefits of new products: The case of the minivan*, Journal of Political Economy 110 (2002), no. 4, 705–729.
72. M. J. D. Powell, *The convergence of variable metric methods for nonlinearly constrained optimization calculations*, Nonlinear Programming (O. L. Mangasarian, R.R. Meyer, and S.M. Robinson, eds.), vol. 3, Academic Press, 1978.
73. ______, *A fast algorithm for nonlinearly constrained optimization calculations*, Numerical Analysis (G. A. Watson, ed.), Lecture Notes in Mathematics, vol. 630, Springer-Verlag, 1978.
74. Daniel Ralph, *Global convergence of damped newton’s method for nonsmooth equations, via the path search*, Mathematics of Operations Research 19 (1994), 352–389.
75. Michael A. Saunders, *LUSOL: Sparse LU for Ax = b*.
76. Olaf Schenk and Klaus Gartner, *Solving unsymmetric sparse systems of linear equations with PARISOS*, Journal of Future Generation Computer Systems 20 (2004), no. 3, 475–487.
77. ______, *On fast factorization pivoting methods for symmetric indefinite systems*, Electronic Transactions on Numerical Analysis 23 (2006), 158–179.
78. Olaf Schenk, Andreas Wächter, and Martin Weiser, *Inertia-revealing preconditioning for large-scale nonconvex constrained optimization*, SIAM Journal of Scientific Computing 31 (2008), no. 2, 939–960.
79. Ching-Shin Norman Shiau and Jeremy J. Michalek, *Optimal Product Design Under Price Competition*, ASME Journal of Mechanical Design 131 (2009), no. 071003, 1–10.
80. , , Should Designers Worry About Market Structure?, ASME Journal of Mechanical Design 131 (2009), no. 011011, 1–9.
81. Ching-Shin Norman Shiau, Jeremy J. Michalek, and Chris T. Hendrickson, A Structural Analysis of Vehicle Design Responses to Corporate Average Fuel Economy Policy, Transportation Research A: Policy and Practice 43 (2009), 814–828.
82. Steven J. Skerlos, Jeremy J. Michalek, and W. Ross Morrow, Sustainable Design Engineering and Science: Selected Challenges and Case Studies, Sustainability Science and Engineering, Volume 1: Defining Principles (M. A. Abraham, ed.), Elsevier Science, 2005.
83. Howard Smith, Supermarket choice and supermarket competition in market equilibrium, The Review of Economic Studies 71 (2004), 235–263.
84. K Sudhir, Competitive pricing behavior in the auto market: A structural analysis, Marketing Science 20 (2001), no. 1, 42–60.
85. Raphael Thomadsen, The effect of ownership structure on prices in geographically differentiated markets, The RAND Journal of Economics 36 (2005), no. 4, 908–929.
86. Kenneth Train, Discrete choice methods with simulation, Cambridge University Press, 2003.
87. Lloyd N. Trefethen and David Bau, Numerical linear algebra, SIAM, 1997.
88. Robert J. Vanderbei, Loqo user’s manual - version 4.05, Tech. Report ORFE-99-??, Princeton University, 2006.
89. Robert J. Vanderbei and David F. Shanno, An interior point algorithm for nonconvex nonlinear programming, Computational Optimization and Applications 13 (1999), no. 1-3, 231–252.
90. Andreas Wachter and Lorenz T. Biegler, On the implementation of an interior-point filter line search algorithm for large-scale nonlinear programming, Mathematical Programming 106 (2006), no. 1, 25–57.

APPENDIX A. COMPUTING A BASIS FOR \( C \) GIVEN \( A \)

Both Algorithm 2 and 4 require a basis \( W \) for \( C \), rather than the constraint gradients \( A \). Implicitly, each method considers the equation \( AW = 0 \) and uses some factorization of \( A \) (or \( A^T \)) to find a formula for \( W \). Several examples follow:

- **SVD of \( A \):** If \( A = U[ \Sigma \ 0 ]V^T \) is a full Singular Value Decomposition (SVD; \cite{87, Lecture 4}) of \( A \), then \( W = V_{M+1:N} \) is an orthonormal basis for \( C \).
- **QR factorization of \( A^T \):** Similarly if \( A^T = QR \) then \( W = Q_{M+1:N} \) is an orthonormal basis for \( C \).
- **QR or LU factorization of \( A \):** If \( A = Q[ R \ S ] \) for upper-triangular \( R \), or if \( PA = L[ U \ S ] \) for upper-triangular \( U \) and some permutation matrix \( P \), then

\[
W = \begin{bmatrix} R^{-1}S \\ I \end{bmatrix} \quad \text{or} \quad W = \begin{bmatrix} U^{-1}S \\ I \end{bmatrix}
\]

are bases for \( C \). In both cases, \( I \in \mathbb{R}^{L \times L} \).

Stability considerations typically suggest that either of the first two approaches are preferable to the third.

APPENDIX B. \( K \) HAS SMALL EIGENVALUES WHEN \( A \) IS NEARLY RANK DEFICIENT

Let \( H \in \mathbb{R}^{N \times N} \) and \( A' \in \mathbb{R}^{(M-1) \times N} \) be arbitrary (but full-rank). Let \( a_m \in \mathbb{R}^N \) denote the \( m^{th} \) row of \( A' \), for \( m \in \{1, \ldots, M - 1\} \). Choose \( \beta_1, \ldots, \beta_{M-1} \) (not all zero), \( \epsilon \in \mathbb{R}^N \)
with small 2-norm, and set $a_M = \sum_{m=1}^{M-1} \beta_m a_m + \epsilon$. The $\beta$'s and $\epsilon$ can be chosen so that $||a_M||_2 = 1$, even if $||\epsilon||_2$ is very small. Define

$$K = \begin{bmatrix} H & (A')^T & a_M \\ A' & 0 & 0 \\ a_M^T & 0 & 0 \end{bmatrix}$$

and

$$F = \begin{bmatrix} 0 & 0 & -\epsilon \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

noting that $K + F$ is singular. Specifically,

$$\begin{pmatrix} K + \begin{bmatrix} 0 & 0 & -\epsilon \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{pmatrix} \begin{bmatrix} 0 \\ -\beta \\ 0 \end{bmatrix} = \begin{bmatrix} H & (A')^T & a_M - \epsilon \\ A' & 0 & 0 \\ a_M^T & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -\beta \\ 0 \end{bmatrix} = \begin{bmatrix} -\sum_{m=1}^{M-1} \beta_m a_m + a_M - \epsilon \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, $K + F$ has a zero eigenvalue and Weyl’s Theorem states that $K$ has an eigenvalue $\lambda$ with $|\lambda| \leq ||F||_2 = ||\epsilon||_2$.

**Departments of Mechanical Engineering and Economics, Iowa State University, Ames, Iowa 50011**

_E-mail address: wrmorrow@iastate.edu_