ON THE COMPLETE cd-INDEX OF A BRUHAT INTERVAL

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Abstract. We study the non-negativity conjecture of the complete cd-index of a Bruhat interval defined by Billera and Brenti. For each cd-monomial $M$ we construct a set of paths, such that if a "flip condition" is satisfied, then the number of these paths is the coefficient of the monomial $M$ in the complete cd-index. When the monomial contains at most one d, then the condition follows from Dyer’s proof of Cellini’s conjecture. Hence the coefficients of these monomials are non-negative. We also relate the flip condition to shelling of Bruhat intervals.

1. Introduction

Let $(W,S)$ be a Coxeter system and $u < v$ two elements in $W$ related in the Bruhat order. Billera and Brenti in [1] define a polynomial $\tilde{\psi}_{u,v}(c,d)$ in the non-commuting variables $c, d$, called the complete cd-index of the interval $[u,v]$. They conjecture that this polynomial has non-negative coefficients. In this article we study the non-negativity conjecture by constructing for each interval $[u,v]$ in the Bruhat order and each cd-monomial $M$ a set of paths $T_M(u,v)$, such that if a condition, called the flip condition is satisfied, then the number of paths in $T_M(u,v)$ is equal to the coefficient of $M$ in the complete cd-index $\tilde{\psi}_{u,v}(c,d)$. We conjecture the flip condition to be true for all intervals and all monomials, which then would imply the non-negativity conjecture.

Using the notation explained in the next section, we briefly describe the flip condition in its different forms and give evidence for it to hold. To construct the set of paths $T_M(u,v)$ we need to fix a reflection order $O$. Let $\overline{T}_M(u,v)$ be the set of paths constructed using the reverse order $\overline{O}$. By induction on the length of $[u,v]$, both sets have the same number of paths, equal to the coefficient of the monomial $M$ in $\tilde{\psi}_{u,v}(c,d)$. Let $F : T_M(u,v) \rightarrow \overline{T}_M(u,v)$ be a bijection, called a flip. The (strong) flip condition states that if $M$ starts with $c$, then one can choose $F$ in such a way that if $F(x) = y$, then the first reflection in $x$ always precedes the first reflection in $y$. (This condition is then used to define $T_M(w,v)$ for longer intervals $[w,v]$, where $w < u < v$.)

As a special case, consider $M = c^n$. Then $T_M(u,v)$ is the set of ascending paths of length $n$ from $u$ to $v$ (ascending with respect to the reflection order $O$), and $\overline{T}_M(u,v)$ is the set of descending paths of length $n$. A result of Dyer [4] states that for any $x \in T_M(u,v)$ and $y \in \overline{T}_M(u,v)$, the first reflection in $x$ always precedes the first reflection in $y$. Hence the flip condition for this $M$ is true for any choice of $F$.

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As we will see below, this result suffices to prove that \( |T_M(u, v)| \) is the coefficient of \( M \) in \( \tilde{\psi}_{u,v}(c, d) \) in case the monomial \( M \) contains at most one \( d \).

The flip condition can be described in an equivalent form as follows. The polynomial \( \tilde{\psi}_{u,v}(c, d) \) is computed by summing the ascent-descent sequences of all paths from \( u \) to \( v \). Let us fix a reflection \( t \) and sum the ascent-descent sequences of all those paths of length \( n \) from \( u \) to \( v \) that have their first reflection \( \leq t \). This sum can be expressed in the form

\[
f_n(c, d) + A g_{n-1}(c, d)
\]

for some homogeneous \( cd \)-polynomials \( f_n, g_{n-1} \) of degree \( n, n - 1 \), respectively. The (strong) flip condition is equivalent to \( g_{n-1} \) having non-negative coefficients.

The second form of the flip condition can be related to the shelling of the Bruhat interval. When \( C \) is a regular \( CW \)-complex that is topologically an \((n - 1)\)-ball or an \((n - 1)\)-sphere, then the \( cd \)-index of of \( C \) can be expressed in the form:

\[
f_n(c, d) + a g_{n-1}(c, d),
\]

for some homogeneous polynomials \( f_n \) and \( g_{n-1} \) with non-negative coefficients \([6]\). The Bruhat order on the interval \([u, v]\) is shellable with respect to the lexicographic ordering of maximal chains \([3]\). This implies that paths of maximal length from \( u \) to \( v \) with first reflection \( \leq t \) are the paths in the poset of a regular \( CW \)-complex \( C \) that is topologically a ball or a sphere. This means that the \( f_n \) and \( g_{n-1} \) in the two formulas above coincide, and in particular that the flip condition holds for paths of maximal length.

We consider the two positive results described above as evidence for the conjecture that the flip condition holds in general.

The approach to computing the \( cd \)-index by counting paths in \( T_M(u, v) \) is motivated by the theory of sheaves on posets \([6]\). One can define a sheaf on an appropriate poset constructed using length \( n \) paths from \( u \) to \( v \) in the Bruhat graph. Then the flip condition states that one can carry out the same operations on this sheaf as in the case of the constant sheaf on a Gorenstein* poset described in \([6]\). The result of these operations is a vector space whose dimension is the coefficient of \( M \) in the \( cd \)-index. However, since the sheaf for the Bruhat graph is constructed from the paths in the graph, the operations reduce to counting paths with a given ascent-descent sequence. Therefore, we only work with paths in the Bruhat graph and do not mention sheaves again.

In the next section we recall the definition of the complete \( cd \)-index in terms of a reflection order. We then construct the sets \( T_M(u, v) \) and give the condition for these sets to count the coefficient of \( M \) in the complete \( cd \)-index.

2. The complete \( cd \)-index.

We fix a Coxeter system \((W, S)\) (see \([5, 2]\)) and a reflection order \( \mathcal{O} \) (see \([3]\)). The latter is a total order on the set of reflections of \((W, S)\), satisfying a condition on dihedral subgroups. The reverse of the order \( \mathcal{O} \) is also a reflection order. We denote it by \( \overline{\mathcal{O}} \).
Let \( l(x) \) be the length function on \( W \). We write \( u \prec v \) if \( l(u) < l(v) \) and \( u^{-1}v \) is a reflection. The relation \( \prec \) generates the Bruhat order on \( W \). The Bruhat graph has vertex set \( W \) and an edge from \( u \) to \( v \) if \( u \prec v \).

Let \( u < v \) in the Bruhat order. A path of length \( n \) from \( u \) to \( v \) in the Bruhat graph is a sequence

\[
x = (u = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_n \prec x_{n+1} = v).
\]

(Note a slightly unusual convention for the length. For example, the path \((u < v)\) has length 0.) We let \( B_n(u, v) \) be the set of all paths of length \( n \) from \( u \) to \( v \), and \( B(u, v) = \cup_n B_n(u, v) \). We label an edge \( x_i < x_{i+1} \) with the reflection \( t_i = x_i^{-1}x_{i+1} \).

The ascent-descent sequence of the path is

\[
w(x) = \beta_1\beta_2\ldots\beta_n,
\]

where

\[
\beta_i = \begin{cases} A & \text{if } t_{i-1} < t_i, \\ D & \text{if } t_{i-1} > t_i. \end{cases}
\]

The reflections \( t_i \) here are related by the reflection order \( \mathcal{O} \).

**Example 2.1.** Figure 1 shows the Bruhat graph of the interval \([2134, 4321]\) in the Coxeter system where the group is the symmetric group \( S_4 \) generated by transpositions \((12), (23), (34)\). The full Bruhat graph of this system can be found in [2].
reflections here are the transpositions in $S_4$ and they are ordered as follows:

$$(12) < (13) < (14) < (23) < (24) < (34).$$

We number the reflections so that $(12)$ has number 1, $(13)$ has number 2, and so on. The edges in the Bruhat graph are then labeled with the numbers of the corresponding reflections. For example, the path $2134 < 2143 < 4123 < 4132 < 4312 < 4321$ has labels 62646, hence its ascent-descent sequence is $DADA$. The path $2134 < 3124 < 3421 < 4321$ has labels 251 and ascent-descent sequence $AD$.

Let $\mathbb{Z}\langle A, D \rangle$ be the polynomial ring in non-commuting variables $A$ and $D$. Summing the ascent-descent sequences of all paths from $u$ to $v$ gives a polynomial in $A$ and $D$:

$$\tilde{\phi}_{u,v}(A, D) = \sum_{x \in B(u,v)} w(x).$$

The complete $cd$-index is obtained from this polynomial by a change of variable. Let $c = A + D$ and $d = AD + DA$. This gives an inclusion of rings

$$\mathbb{Z}\langle c, d \rangle \subset \mathbb{Z}\langle A, D \rangle.$$ 

Billera and Brenti [1] prove that the polynomial $\tilde{\phi}_{u,v}(A, D)$ lies in this subring, hence can be expressed in terms of $c$ and $d$:

$$\tilde{\phi}_{u,v}(A, D) = \tilde{\psi}_{u,v}(c, d).$$

The polynomial $\tilde{\psi}_{u,v}(c, d)$ is the complete $cd$-index of the interval $[u, v]$. It does not depend on the chosen reflection order $\mathcal{O}$.

The rings $\mathbb{Z}\langle c, d \rangle$ and $\mathbb{Z}\langle A, D \rangle$ are graded so that $A, D, c$ have degree 1 and $d$ has degree 2. We define the involution $f \mapsto \overline{f}$ in the ring $\mathbb{Z}\langle A, D \rangle$ by $\overline{f(A, D)} = f(D, A)$. Elements of $\mathbb{Z}\langle c, d \rangle$ are invariant by this involution.

We will consider below homogeneous polynomials $p(A, D) \in \mathbb{Z}\langle A, D \rangle$ that can be expressed in the form $f_n(c, d) + g_{n-1}(c, d)D$ for some homogeneous $cd$-polynomials $f_n$ and $g_{n-1}$. If such an expression exists, then it is unique. We can recover $g_{n-1}$ by computing $p(A, D) - \overline{p(A, D)} = g_{n-1}(c, d)(D-A)$, and then subtracting $g_{n-1}(c, d)D$, we recover $f_n$. More generally, every homogeneous $p(A, D) \in \mathbb{Z}\langle A, D \rangle$ of degree $n$ can be expressed in a unique way as

$$p(A, D) = f_n(c, d) + f_{n-1}(c, d)D + f_{n-2}(c, d)D^2 + \cdots + f_0D^n$$

for some homogeneous $cd$-polynomials $f_i$.

If $M(c, d)$ is a $cd$-monomial, consider the $AD$-monomial $M(A, DA)$. This correspondence gives a bijection between $cd$-monomials and $AD$-monomials in which every $D$ is followed by an $A$. Below we will often use the letter $M$ to denote either the $cd$-monomial $M(c, d)$ or the $AD$-monomial $M(A, DA)$, with the distinction being clear for the context. For example, we define $T_{M(c, d)}(u, v) = T_{M(A, DA)}(u, v)$.

3. COEFFICIENTS OF THE COMPLETE $cd$-INDEX

Let $u < v$ in the Bruhat order and let $M(c, d)$ be a $cd$-monomial of degree $n$. We wish to express the coefficient of $M$ in $\tilde{\psi}_{u,v}(c, d)$ as a number of certain paths in
\(B_n(u,v)\). We start by defining a number \(s_M(x)\) for every path \(x \in B_n(u,v)\), giving the contribution of \(x\) to the coefficient of \(M\). The numbers \(s_M(x)\) are in the set \{-1,0,1\}. We then study the case when \(s_M(x)\) is non-negative for every \(x\) and call it the flip condition. If the flip condition is satisfied, the number of paths \(x\) with \(s_M(x) = 1\) is the coefficient of \(M\) in \(\tilde{\psi}_{u,v}(c,d)\).

For any \(AD\)-monomial \(w\), the number of paths \(x \in B(u,v)\) with \(w(x) = w\) is equal to the number of paths \(y \in B(u,v)\) with \(w(y) = \overline{w}\). Let \(F = F_{u,v} : B(u,v) \to B(u,v)\) be an involution, such that \(w(F(x)) = w(x)\). We fix one such \(F_{u,v}\) for every \(u < v\) and call it a flip.

Let \(x = (u < x_1 < x_2 < \cdots < x_n < v) \in B_n(u,v)\), and let \(1 \leq m \leq n\). We apply the flip \(F_{x_m,v}\) to the tail of \(x\) to get \(y = (u < x_1 < \cdots < x_m < y_{m+1} < \cdots < y_n < v)\). If \(w(x) = \beta_1 \cdots \beta_m \cdots \beta_n\), then \(w(y) = \beta_1 \cdots \beta_{m-1} \alpha_m \overline{\beta_{m+1}} \cdots \overline{\beta_n}\), where \(\alpha_m\) could be either \(A\) or \(D\). Define:

\[
s_{m,A}(x) = \begin{cases} 
1 & \text{if } \beta_m = A \\
0 & \text{otherwise},
\end{cases}
\]

\[
s_{m,D}(x) = \begin{cases} 
1 & \text{if } \beta_m = D, \alpha_m = A \\
-1 & \text{if } \beta_m = A, \alpha_m = D \\
0 & \text{otherwise}.
\end{cases}
\]

Now let \(M(A,DA)\) be the \(AD\)-monomial \(\gamma_1 \cdots \gamma_n\) and define

\[
s_M(x) = s_{M(A,DA)}(x) = \prod_{m=1}^{n} s_{m,\gamma_m}(x).
\]

Let \(x \in B_n(u,v)\), and let \(y = F_{u,v}(x)\). Then the ascent-descent sequence of \(y\) when computed using the reverse reflection order \(\overline{\mathcal{O}}\) is the same as the ascent-descent sequence of \(x\) computed using the order \(\mathcal{O}\). Let us denote by \(\overline{s}_M(y)\) the number \(s_M(y)\) computed as above, but using the order \(\overline{\mathcal{O}}\). We say that \(F\) is compatible with the reflection order \(\mathcal{O}\) if \(s_M(x) = \overline{s}_M(y)\) for any \(u < v\), \(M\) and \(x\).

**Theorem 3.1.** Assume that \(F\) is compatible with the reflection order \(\mathcal{O}\). For any \(cd\)-monomial \(M(c,d)\) of degree \(n\), the coefficient of \(M\) in \(\tilde{\psi}_{u,v}(c,d)\) is equal to

\[
\sum_{x \in B_n(u,v)} s_M(x).
\]

**Proof.** Write \(N(A,D) = M(A,DA)\) and for \(0 \leq m \leq n\) let \(N = N_mN_{n-m}\), where \(N_m, N_{n-m}\) are \(AD\)-monomials of degree \(m, n - m\), respectively. Define

\[
P_m = \sum_{x \in B_n(u,v)} w(x_1 < \cdots < x_{m+1}) \cdot s_{N_{n-m}}(x_m < x_{m+1} < \cdots < x_n < v).
\]

Note that \(P_n\) is the degree \(n\) part of \(\tilde{\phi}_{u,v}\) and hence can be expressed as a homogeneous \(cd\)-polynomial of degree \(n\). The statement of the theorem is that \(P_0\) is the coefficient of \(M\) in \(P_n\).
Lemma 3.2. For $0 \leq m \leq n$ there exist homogeneous cd-polynomials $f_m$ and $g_{m-1}$ of degree $m$ and $m-1$, respectively, such that

$$P_m = f_m(c, d) + g_{m-1}(c, d)D.$$ 

Moreover, $P_{m-1}$ can be computed from $P_m$ as follows.

1. If $N_m$ ends with $A$, then $P_{m-1} = f_{m-1}(c, d) + g_{m-2}(c, d)D$, where $f_m = f_{m-1}c + g_{m-2}d$.
2. If $N_m$ ends with $D$, then $P_{m-1} = g_{m-1}(c, d)$.

Proof. We use induction on $m$. When $m = n$, then $P_m$ is a homogeneous cd-polynomial of degree $n$. Assume that $P_m = f_m(c, d) + g_{m-1}(c, d)D$ and let us prove the "moreover" statement.

If $N_m$ ends with $A$, let $x \in B_n(u, v)$ with $w(x) = \beta_1 \cdots \beta_m \cdots \beta_n$. Then

$$s_{AN_{n-m}}(x_{m-1} \prec x_m \prec \cdots \prec x_n \prec v) = \begin{cases} s_{N_{n-m}}(x_m \prec \cdots \prec x_n \prec v) & \text{if } \beta_m = A \\ 0 & \text{otherwise} \end{cases}$$

Thus, to compute $P_{m-1}$ from $P_m$, we consider only those monomials that end with $A$ and then delete this last $A$. When contracting $P_m = f_m(c, d) + g_{m-1}(c, d)D$ with $A$ from the right, we get $f_{m-1}(c, d) + g_{m-2}(c, d)D$, where $f_m = f_{m-1}c + g_{m-2}d$.

Now suppose $N_m$ ends with $D$. By induction on $m$, the polynomials $f_m$ and $g_{m-1}$ depend only on $u < v$ and monomial $M$, not on the reflection order or the flip $F$. Let us denote by $\overline{w}$ and $\overline{s}_{N_{n-m}}$ the quantities computed using the same flip $F$, but with the reverse reflection order $\overline{O}$. This does not change the polynomial $P_m$. Then $\overline{w}(x)$ is obtained from $w(x)$ by switching $A$ and $D$. If

$$(x_m \prec y_{m+1} \prec \cdots \prec y_n \prec v) = F_{x_m, u}(x_m \prec x_{m+1} \prec \cdots \prec x_n \prec v),$$

then

$$\overline{s}_{N_{n-m}}(x_m \prec y_{m+1} \prec \cdots \prec y_n \prec v) = s_{N_{n-m}}(x_m \prec x_{m+1} \prec \cdots \prec x_n \prec v)$$

by the compatibility condition on $F$.

Let $F_m : B_n(u, v) \rightarrow B_n(u, v)$ be the involution that flips the tail of a path:

$$F_m(u \prec x_1 \prec \cdots \prec x_n \prec v) = (u \prec x_1 \prec \cdots \prec x_m \prec y_{m+1} \prec \cdots \prec y_n \prec v).$$

Since this is a bijection, we may compute $P_m$ with respect to $\overline{O}$ by summing over $F_m(x)$. We call this new polynomial $Q_m$. By the previous discussion $Q_m = P_m$.

$$Q_m = \sum_{x \in B_n(u, v)} \overline{w}(u \prec x_1 \prec \cdots \prec y_{m+1}) \cdot \overline{s}_{N_{n-m}}(x_m \prec y_{m+1} \prec \cdots \prec y_n \prec v)$$

$$= \sum_{x \in B_n(u, v)} \overline{w}(u \prec x_1 \prec \cdots \prec y_{m+1}) \cdot s_{N_{n-m}}(x_m \prec x_{m+1} \prec \cdots \prec x_n \prec v).$$

Let us now compute $P_m - Q_m = g_{m-1}(c, d) : (D - A)$:

$$\sum_{x \in B_n(u, v)} (w(u \prec x_1 \prec \cdots \prec x_{m+1}) - w(u \prec x_1 \prec \cdots \prec y_{m+1})) \cdot s_{N_{n-m}}(x_m \prec \cdots \prec x_n \prec v).$$
For $x \in B_n(u, v)$, let $w(u < x_1 < \cdots < x_{m+1}) = \beta_1 \cdots \beta_m$ and $w(u < x_1 < \cdots < y_{m+1}) = \beta_1 \cdots \beta_{m-1}\alpha_m$. Then $x$ contributes to this sum if and only if $\beta_m \neq \alpha_m$. The contribution is
\[
\pm w(u < x_1 < \cdots < x_m)(D - A)s_{N_{n-m}}(x_m < \cdots < x_n < v),
\]
where the sign is positive if $\beta_m = D, \alpha_m = A$ and negative otherwise. Notice that, with the same sign,
\[
\pm s_{N_{n-m}}(x_m < \cdots < x_n < v) = s_{DN_{n-m}}(x_{m-1} < x_m < \cdots < x_n < v).
\]
This means that $P_{m-1} = g_{m-1}$. \hfill \Box

Now suppose $m$ is such that $M(c, d) = M_m(c, d) \cdot M_{n-m}(c, d)$ where $M_m, M_{n-m}$ are $cd$-monomials of degree $m, n - m$, respectively. Then the inductive computation of $P_{m-1}$ from $P_m$ in the lemma can be restated as follows. Let $f_m = f_{m-1}c + g_{m-2}d$.

1. If $M_m$ ends with $c$, then
   \[
P_{m-1} = f_{m-1} + g_{m-2} \cdot D.
   \]

2. If $M_m$ ends with $d$, then
   \[
P_{m-2} = g_{m-2}.
   \]

If we only consider the degree $m$ term $f_m$ of $P_m$, then in the first case $f_{m-1}$ is obtained from $f_m$ by contracting with $c$ from the right. In the second case $f_{m-2}$ is obtained from $f_m$ by contracting with $d$ from the right. It follows that $P_0$ is the number that is obtained from $P_n$ by contracting with the monomial $M$. In other words, $P_0$ is the coefficient of $M$ in $P_n$. \hfill \Box

4. Non-negativity of the complete $cd$ index.

There are two problems with computing the coefficients of the complete $cd$-index as described in the previous section. The first is that the formula involves negative signs. The second problem is that it is not clear how to define a flip $F$ that is compatible with the reflection order.

In this section we define the ”flip condition” requiring that all terms $s_M(x)$ that go into the computation of the coefficient of $M$ in the complete $cd$-index are non-negative. In this case we define a set of paths $T_M(u, v) \subset B(u, v)$, such that $\abs{T_M(u, v)}$ is the coefficient of $M$ in $\psi_{u,v}(c, d)$. It also turns out that the flip condition gives an optimal way of defining the flip $F_{u,v}$. The flip condition for the interval $[u,v]$ only involves the flips $F_{w,v}$ where $u < w < v$, hence this gives an inductive procedure for defining $F$, checking the flip condition and constructing the set $T_M(u,v)$.

Let $u < v$ in the Bruhat order, and let $M(c, d)$ be a $cd$-monomial of degree $n$. Let $M(A, DA)$ be the $AD$-monomial $\gamma_1 \cdots \gamma_n$.

**Definition 4.1.** Let
\[
T_M(u, v) = T_{\gamma_1 \cdots \gamma_n}(u, v) = \{ x \in B_n(u, v) \mid s_{m, \gamma_m}(x) = 1 \text{ for all } 1 \leq m \leq n \}.
\]
Using the definition of $s_{m, \gamma_m}$, a path $x$ lies in $T_M(u, v)$ if and only if
ascent-descent sequence. Hence three paths are the paths 462, then the resulting path must have ascent-descent sequence AD.

To find all paths \( u \) consists of all ascending paths of length \( M \) sequence check the ascent-descent sequences after applying the flips are 6 paths with ascent-descent sequence DADA. These, only the third and the fifth have the required ascent descent sequence.

First need to construct the sets \( T \)

This reduces the candidate paths to two: 63416 and 41516. To check the flip condition holds for the interval \([u,v]\) and monomial \( M \) if for every \( x \in B_n(u,v) \) the following is satisfied. If \( s_{m,\gamma m}(x) = -1 \) for some \( m \), then there exists \( l > m \) such that \( s_{l,\gamma l}(x) = 0 \).

This condition can be re-written using the definition of \( s_{m,\gamma m} \) by saying that the flip condition is violated for some \( x \in B_n(u,v) \) if there exists \( m \) such that

(1) \((x_m < \cdots < x_n < v) \in T_{\gamma m+1} \cdots \gamma m(x_m, v). \) (Equivalently, \( s_{l,\gamma l}(x) = 1 \) for \( l > m \).)

(2) \( \gamma m = D \) and if

\[
(x_m < y_{m+1} < \cdots < y_n < v) = F_{x_m,v}(x_m < \cdots < x_n < v),
\]

then \( w(x_{m-1} < x_m < x_{m+1}) = A \) and \( w(x_{m-1} < x_m < y_{m+1}) = D. \) (Equivalently, \( s_{m,\gamma m}(x) = -1. \))

From Theorem 3.1 we now get:

**Corollary 4.3.** Assume that \( F \) is compatible with the reflection order \( \mathcal{O} \). If the flip condition holds for the interval \([u,v]\) and monomial \( M \), then \(|T_M(u,v)|\) is the coefficient of \( M \) in \( \hat{\psi}_{u,v}(c,d) \). \( \square \)

**Example 4.4.** Consider the Bruhat interval \([u,v] = [2134, 4321]\) shown in Figure 1. We compute the sets \( T_M(u,v) \) for different monomials. When \( M = c^a \), then \( T_M(u,v) \) consists of all ascending paths of length \( n \) from \( u \) to \( v \). Thus, \( T_{c^2}(u,v) = \{346, 235\} \) and \( T_{c^3}(u,v) = \{23456\} \). When \( M = d \), we consider paths with ascent-descent sequence \( M(A, DA) = DA \). There are three such paths: 436, 514, 625. We need to check that when we flip a path \((u < x_1 < y_2 < v)\) to \((u < x_1 < y_2 < v)\), then the resulting path must have ascent-descent sequence \( AD \). The flips of the three paths are the paths 462, 521, 652. Only the first one of these has the correct ascent-descent sequence. Hence \( T_d(u,v) = \{436\} \).

For a slightly longer computation, let us find the set \( T_{d^2}(u,v) \). For this we need to find all paths \((u < x_1 < y_2 < v)\) with ascent-descent sequence \( DADA \) and check the ascent-descent sequences after applying the flips \( F_{x_3,v} \) and \( F_{x_1,v} \). There are 6 paths with ascent-descent sequence \( DADA \),

\[
62646, 64614, 63416, 63524, 41516, 41624.
\]

Applying the flip \( F_{x_3,v} \) we get paths 62654, 64621, 63461, 63541, 41561, 41641. Among these, only the third and the fifth have the required ascent descent sequence \( DAAD \). This reduces the candidate paths to two: 63416 and 41516. To check the flip condition, we first need to construct the sets \( T_{ADAD}(x_1,v) \) and \( T_{ADAD}(x_1,v) \). For the first path 63416 we find that \( T_{ADAD}(2143, v) = \{3416\} \) and \( T_{ADAD}(2143, v) = \{4361\} \). Thus, applying
the flip to the path 63416 gives 64361. This path does not have the required ascent-descent sequence ADAD. For the second path 41516, we find $T_{\text{AD}(2314,v)} = \{1516\}$ and $T_{\overline{\text{AD}(2314,v)}} = \{5361\}$. The result of applying the flip to 41516 is 45361 with the required ascent-descent sequence ADAD. Thus, $T_{d^2}(u,v) = \{41516\}$.

It follows from the discussion below and in the introduction that the interval $[u,v]$ in this example satisfies the flip condition for any monomial (either the paths have maximal length or the monomial contains at most one $d$). This implies that the complete cd-index $\overline{\psi}_{u,v}(c,d)$ has non-negative coefficients. By the computation above, $\overline{\psi}_{u,v}(c,d) = 2c^2 + d + c^4 + xc^2d + yedc + zdc^2 + d^2$ for some $x,y,z \geq 0$.

Let us now turn to the definition of the flip $F_{u,v}$. Note that in the definition of $T_M(u,v)$ and the flip condition we only need to apply the flip $F_{x_m,v}$ to paths $(x_m \prec \cdots \prec x_n \prec v)$ that lie in $T_{\gamma_{m+1} \cdots \gamma_n}(x_m,v)$. Since $F$ is compatible with the reflection order, the result lies in $T_{\overline{\gamma}_{m+1} \cdots \gamma_n}(x_m,v)$, where $\overline{T}_M$ denotes the same set $T_M$ constructed using the reverse reflection order $\overline{O}$. Assuming the flip condition on $[u,v]$, these two sets have the same number of elements, hence $F_{x_m,v} : T_{\gamma_{m+1} \cdots \gamma_n}(x_m,v) \rightarrow \overline{T}_{\overline{\gamma}_{m+1} \cdots \gamma_n}(x_m,v)$ is a bijection. Thus, to define $F_{u,v}$, we only need a bijection $T_M(u,v) \rightarrow \overline{T}_M(u,v)$ (and we can extend it to an involution on $B_n(u,v)$ in an arbitrary way if we so wished). Note that such $F_{u,v}$ automatically satisfies the compatibility condition because $s_M(x) = \overline{s}_M(F_{u,v}(x)) = 1$ for any $x \in T_M(u,v)$ (and $s_M(x) = \overline{s}_M(F_{u,v}(x)) = 0$ for any $x \notin T_M(u,v)$).

**Definition 4.5.** Assume that the flip condition holds for the interval $[u,v]$ and monomial $M$. Define the flip $F_{u,v} : T_M(u,v) \rightarrow \overline{T}_M(u,v)$ as the bijection that preserves the lexicographic ordering of paths. (The path with the smallest first reflection maps to a path with the smallest first reflection, using the order $O$ on both sides.)

The flip $F_{u,v}$ is optimal in the following sense. Consider a path $(u \prec u \prec x_1 \prec \cdots \prec x_n \prec v)$ and the AD-monomial $\gamma_0 \gamma_1 \cdots \gamma_n$. We claim that if the path $z$ violates the flip condition for $m = 0$ and for the flip $F_{u,v}$ defined above, then it violates the flip condition for any $F$. Equivalently, if the flip condition holds for some $F$ then it holds for the $F_{u,v}$ defined above. Indeed, the path violates the flip condition for $m = 0$ when $x = (u \prec x_1 \prec \cdots \prec x_n \prec v) \in T_M(u,v)$ and after applying $F_{u,v}$ we get the path $(u \prec y_1 \prec \cdots \prec y_n \prec v)$, such that $w(z \prec u \prec x_1) = A$ and $w(z \prec u \prec y_1) = D$. This implies that $u^{-1}y_1 < z^{-1}u < u^{-1}x_1$.

Here $u^{-1}x_1$ is the first reflection in $(u \prec x_1 \prec \cdots \prec v) \in T_M(u,v)$ and $u^{-1}y_1$ is the first reflection in $(u \prec y_1 \prec \cdots \prec v) \in \overline{T}_M(u,v)$. Since $F_{u,v}$ preserves ordering by first reflection, it follows that $T_M(u,v)$ has fewer paths with first reflection less than $z^{-1}u$ than does $\overline{T}_M(u,v)$. Thus, no matter how the flip $F$ is chosen, some $x \in T_M(u,v)$ with first reflection greater than $z^{-1}u$ maps to $y \in \overline{T}_M(u,v)$ with first reflection smaller than $z^{-1}u$. Hence the flip condition is violated for any $F$. (Note that in the argument above $\gamma_0 = D$, hence for $\gamma_0 \cdots \gamma_n$ to come from a cd-monomial, we need $\gamma_1 = A$ and thus $M = cM'$. Then the path $(z \prec u \prec x_1 \prec \cdots \prec x_n \prec v)$ violates the flip condition for the interval $[z,v]$ and monomial $dM'$.)
Let us say that the **strong flip condition** holds if for any \( x \in T_M(u, v) \) and \( y = F_{u,v}(x) \),
\[
u^{-1}x_1 \leq u^{-1}y_1
\]
whenever \( M \) starts with \( c \), \( M = cM' \). This condition is stronger than the flip condition because the flip condition allows \( u^{-1}y_1 < u^{-1}x_1 \) as long as there is no \( z^{-1}u \) between them for some \( z < u \).

When \( M = c^n \), then the strong flip condition was proved by Dyer [4]. Since to check the flip condition, we only need to check the flip for each occurrence of \( d \) in \( M \), it follows from this that the flip condition holds for any interval and any monomial \( M \) that contains at most one \( d \). Thus, the coefficients of such monomials are non-negative in any \( \tilde{\psi}_{u,v}(c,d) \).

5. **Shelling of the Bruhat interval**

In this section we give an equivalent formulation of the flip condition that is related to shelling of Bruhat intervals.

Let \( t \) be a reflection. Denote
\[
B_n(u,v)\leq t = \{(u < x_1 < \cdots < x_n < v) \in B_n(u,v) | u^{-1}x_1 \leq t\}.
\]
Also let
\[
T_M(u,v)\leq t = T_M(u,v) \cap B_n(u,v)\leq t,
\]
\[
\overline{T_M(u,v)\leq t} = T_M(u,v) \cap B_n(u,v)\leq t.
\]

**Theorem 5.1.** The \( AD \)-polynomial
\[
\tilde{\phi}_{u,v}^{\leq t} = \sum_{x \in B_n(u,v)\leq t} w(x)
\]
can be expressed in the form \( f_n(c,d) + Ag_{n-1}(c,d) \) for some homogeneous \( cd \)-polynomials \( f_n, g_{n-1} \) of degree \( n, n-1 \), respectively. Assuming that the flip condition holds for the interval \([u,v]\) and monomial \( M \), then \( |\overline{T_M(u,v)\leq t}| \) is the coefficient of \( M \) in \( f_n \) and \( |T_M(u,v)\leq t| \) is the coefficient of \( M \) in \( f_n + cg_{n-1} \).

Before we prove this theorem, let us derive an equivalent form of the flip condition from it. Suppose the flip condition holds for the interval \([u,v]\) and monomial \( M = cM' \), but is violated for some \((z < u < x_1 < \cdots < x_n < v)\) and monomial \( dM' \). Let \( t = z^{-1}u \). Then, as in the previous section, we must have
\[
|T_M(u,v)\leq t| < |\overline{T_M(u,v)\leq t}|.
\]
By the theorem, the difference between the two numbers is the coefficient of \( M \) in \( cg_{n-1} \). Clearly this argument can also be reversed to get an equivalent condition. For simplicity we will state it without specifying the intervals and monomials.

**Corollary 5.2.** The flip condition holds for all intervals and all monomials if and only if in the expression \( \tilde{\phi}_{u,v}^{\leq t} = f_n(c,d) + Ag_{n-1}(c,d) \) the polynomial \( g_{n-1}(c,d) \) has non-negative coefficients for all intervals \([u,v]\) and all reflections \( t = z^{-1}u \), where \( z < u \). \(\square\)
The strong flip condition defined at the end of previous section is equivalent to \( g_{n-1} \) having non-negative coefficients for any interval \([u, v]\) and any reflection \( t \).

By the theorem, the flip condition also implies that the polynomial \( f_n(c, d) \) has non-negative coefficients.

**Proof.** To prove the first statement of the theorem, it suffice to show that the \( AD \)-polynomial

\[
\sum_{w^{-1}x_1 = t} w(x),
\]

where the sum runs over all \( x \in B_n(u, v) \) having \( t \) as its first reflection, has the stated form. This sum can be written as

\[
A \sum_{y \in B_{n-1}(x_1, v) \leq t} w(y) + D \sum_{y \in B_{n-1}(x_1, v) > t} w(y)
\]

\[
= A \sum_{y \in B_{n-1}(x_1, v) \leq t} w(y) + D( \sum_{y \in B_{n-1}(x_1, v)} w(y) - \sum_{y \in B_{n-1}(x_1, v) \leq t} w(y))
\]

\[
= (A - D) \sum_{y \in B_{n-1}(x_1, v) \leq t} w(y) + D \sum_{y \in B_{n-1}(x_1, v)} w(y).
\]

Here the subscript \( > t \) has similar meaning to \( \leq t \). Using induction, we can write this as

\[
(A - D)(f_{n-1}(c, d) + Ag_{n-2}(c, d)) + Dh_{n-1}(c, d)
\]

\[
= (2A - A - D)f_{n-1} + (A^2 + AD - AD - DA)g_{n-2} + (A + D - A)h_{n-1}
\]

\[
= (-c_{f_{n-1}} - dg_{n-2} + ch_{n-1}) + A(2f_{n-1} + cg_{n-2} - h_{n-1}),
\]

for some \( cd \)-polynomials \( f_{n-1}, g_{n-2}, h_{n-1} \).

The proof of the second statement is very similar to the proof of Theorem 3.1, so we only sketch it.

Let \( N(A, D) = M(A, DA) \) and for \( 0 \leq m \leq n \) write \( N = N_mN_{n-m} \), where \( N_m, N_{n-m} \) are \( AD \)-monomials of degree \( m, n - m \), respectively. Define

\[
P_{m}^{\leq t} = \sum_{x \in B_n(u, v) \leq t} w(u < x_1 < \cdots < x_{m+1}) \cdot s_{N_{m}}(x_m < x_{m+1} < \cdots < x_n < v).
\]

\[
Q_{m}^{\leq t} = \sum_{x \in B_n(u, v) \leq t} w(u < x_1 < \cdots < x_{m+1}) \cdot s_{N_{m}}(x_m < x_{m+1} < \cdots < x_n < v).
\]

Note that

\[
P_{n}^{\leq t} = \phi_{n, v}^{\leq t} = f_n(c, d) + Ag_{n-1}(c, d),
\]

\[
Q_{n}^{\leq t} = T_{n}^{\leq t} = f_n(c, d) + Dg_{n-1}(c, d).
\]

On the other hand,

\[
P_{0}^{\leq t} = |T_M(u, v)_{\leq t}|, \quad Q_{0}^{\leq t} = |T_M(u, v)_{\leq t}|.
\]
Lemma 5.3. There exist $cd$-polynomials $f_m, g_{m-1}, h_{m-1}, l_{m-2}$ of degree $m, m - 1, m - 1, m - 2$, respectively, such that

$$P_{m}^{\leq t} = (f_m + A g_{m-1}) + (h_{m-1} + A l_{m-2}) D.$$ 

For $m > 0$,

$$Q_{m}^{\leq t} = (f_m + D g_{m-1}) + (h_{m-1} + D l_{m-2}) D.$$ 

Moreover, $P_{m-1}^{\leq t}, Q_{m-1}^{\leq t}$ can be computed from $P_{m}^{\leq t}, Q_{m}^{\leq t}$ as follows.

1. If $N_m$ ends with $A$ and $m \geq 1$, then

   $$P_{m-1}^{\leq t} = (f_{m-1} + A g_{m-2}) + (h_{m-2} + A l_{m-3}) D,$$

   where $f_m = f_{m-1} c + h_{m-2} d$ and $g_{m-1} = g_{m-2} c + l_{m-3} d$. 

2. If $N_m$ ends with $A$ and $m = 1$, let $P_{1}^{\leq t} = \alpha c + A \beta + \gamma D$ for $\alpha, \beta, \gamma \in \mathbb{Z}$. Then $P_{0}^{\leq t} = \alpha + \beta$ and $Q_{0}^{\leq t} = \alpha$. 

3. If $N_m$ ends with $D$, then

   $$P_{m-1}^{\leq t} = h_{m-1} + A l_{m-2}.$$ 

Proof. If $N_m$ ends with $A$, we contract $P_{m}^{\leq t}$ and $Q_{m}^{\leq t}$ with $A$ from the right to get $P_{m-1}^{\leq t}$ and $Q_{m-1}^{\leq t}$.

If $N_m$ ends with $D$, then as before,

$$P_{m-1}^{\leq t}(D - A) = P_{m}^{\leq t} - Q_{m}^{\leq t} = h_{m-1}(D - A) + A l_{m-2}(D - A),$$

$$Q_{m-1}^{\leq t}(D - A) = Q_{m}^{\leq t} - P_{m}^{\leq t} = h_{m-1}(D - A) + D l_{m-2}(D - A).$$

Let $m$ be such that $M = M_m M_{n-m}$, where $M_m, M_{n-m}$ are $cd$-monomials of degree $m, n - m$, respectively. The lemma then implies:

1. If $M_m$ ends with $c$ and $m > 1$, then $f_{m-1} + A g_{m-2}$ is obtained by contracting $f_m + A g_{m-1}$ with $c$ from the right.
2. If $M_m$ ends with $c$ and $m = 1$, then $P_0$ is obtained by contracting $f_1 + c g_0$ with $c$ from the right and $Q_0$ is obtained by contracting $f_1$ with $c$ from the right.
3. If $M_m$ ends with $d$, then $f_{m-2} + A g_{m-3}$ is obtained by contracting $f_m + A g_{m-1}$ with $d$ from the right.

It follows from this that if $P_{n}^{\leq t} = f_n + A g_{n-1}$, then $P_{0}^{\leq t}$ is obtained from $f_n + c g_{n-1}$ by contracting with $M$. Thus, $P_{0}^{\leq t} = |T_M(u, v)_{\leq t}|$ is the coefficient of $M$ in $f_n + c g_{n-1}$. Similarly, $Q_{0}^{\leq t}$ is obtained from $f_n$ by contracting with $M$, hence $Q_{0}^{\leq t} = |T_M(u, v)_{\leq t}|$ is the coefficient of $M$ in $f_n$. □
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REFERENCES

[1] L. J. Billera and F. Brenti. Quasisymmetric functions and Kazhdan-Lusztig polynomials. ArXiv e-prints, October 2007.
[2] A. Björner and F. Brenti. Combinatorics of Coxeter groups, volume 231 of Graduate Texts in Mathematics. Springer, New York, 2005.
[3] M. J. Dyer. Hecke algebras and shellings of Bruhat intervals. Compositio Math., 89(1):91–115, 1993.
[4] M. J. Dyer. Proof of Cellini’s conjecture on self-avoiding paths in Coxeter groups. Preprint, February 2011.
[5] J. E. Humphreys. Reflection groups and Coxeter groups, volume 29 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1990.
[6] K. Karu. The cd-index of fans and posets. Compos. Math., 142(3):701–718, 2006.

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