ABSTRACT. We show that every Kac-Moody algebra is birationally equivalent to a smash biproduct of two copies of a Weyl algebra together with a polynomial algebra. We also show that the same is true for quantized Kac-Moody algebras where one replaces Weyl algebras with their quantum analogues.

INTRODUCTION

We show that every Kac-Moody algebra is birationally equivalent to a smash biproduct of two copies of an algebra $A_{n-\ell,n} \otimes k[t_1, \ldots, t_\ell]$ defined as a product of a Weyl algebra $A_{n-\ell,n}$ and a polynomial algebra $k[t_1, \ldots, t_\ell]$ where $n$ and $\ell$ are respectively determined by the size and the rank of the underlying generalized Cartan matrix. We also show that the same is true for the Drinfeld-Jimbo quantization of a Kac-Moody algebra where one has to replace the Weyl algebra with its quantum analogue. The result we prove in this article is but a small step towards reformulating the Gel’fand-Kirillov (GK) conjecture for Kac-Moody algebras and their quantizations.

The GK-conjecture states that the universal enveloping algebra $U(g)$ of a finite dimensional Lie algebra $g$ is birationally equivalent to a Weyl algebra $A_n$ for some $n$ \cite{10,11}. The conjecture is known to be false in general \cite{2}, but it is true for $\mathfrak{gl}_n$, $\mathfrak{sl}_n$ and nilpotent Lie algebras \cite{10,11}, for solvable Lie algebras \cite{17,14}, and for every Lie algebra up to dimension 8 \cite{3}. The corresponding conjecture for quantum deformations of Lie algebras and Lie groups is known to be true in many cases, notably for $U_q(\mathfrak{sl}_2)$, $\tilde{U}_q(\mathfrak{sl}_n)$, $\tilde{U}_q^+(\mathfrak{g})$, $O_q(M_n)$, and $O_q(G)$ when $q$ is not a root of unity, or is transcendental over $\mathbb{Q}$ \cite{1,9,18,8,13,4}, and all $U_q^+(\mathfrak{g})$, and therefore, $U_q(g)$ for solvable Lie algebras when $q$ is not a root of unity \cite{19}. A version of the conjecture has also been studied by Colliot-Thélène, Kunyavskiï, Popov and Reichstein in \cite{7} where the authors investigated if algebra of functions on a Lie group $G$ or a Lie algebra $g$ is purely transcendental over its invariant subalgebra.

The birational equivalence we prove in this article transforms the GK-conjecture for Kac-Moody algebras and their quantizations to a similar question on smash biproducts of (quantized) Weyl and polynomial algebras. Inspired by the GK conjecture, in \cite{15} we conjectured that the universal enveloping algebra of a Lie algebra $g$ is birationally equivalent to a smash product of a smooth algebra and a torus whose rank is determined by the rank of the Cartan subalgebra of $g$. In this paper, we verify that our conjecture is true for Borel algebras: any Borel subalgebra of a (quantized) Kac-Moody algebra is birationally equivalent to the smash product of a polynomial algebra and a torus.

Plan of the article. Here is a plan of this article. In Section 1 we define generalized Weyl algebras bound by a generalized Cartan matrix. We then show that when such an algebra is of full rank (See Subsection 2.2) it is birationally equivalent to the product of
an ordinary Weyl algebra and a polynomial algebra. In Theorem 3.1 we prove that the Borel subalgebras of Kac-Moody algebras are birationally equivalent to a generalized Weyl algebra which in turn allows us to show our main result in Corollary 3.2. Then we prove the analogous results for Drinfeld-Jimbo quantizations of Kac-Moody algebras in Section 4.

**Notation, conventions and some background.**

**Ground field.** Throughout the paper we fix a ground field $k$ of characteristic 0. All unadorned tensor products are over $k$.

**ambient algebras.** All algebras are assumed to be unital and associative, but not necessarily commutative or finite dimensional. We also assume that our algebras are domains, i.e. devoid of any zero divisors. There are several classes of algebras that we are going to use frequently in this paper. These are as follows:

1. $k\{t_1, \ldots, t_n\}$ the noncommutative polynomial algebra on $n$ generators,
2. $k[t_1, \ldots, t_n]$ the commutative polynomial algebra on $n$ generators,
3. $\mathbb{C}^n$ the Laurent polynomial algebra $k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, and
4. $\mathcal{A}_{m,n}$ the Weyl algebra given by the presentation

\[
k\{x_1, \ldots, x_m, y_1, \ldots, y_n\} / ([x_i, x_{i'}], [y_j, y_{j'}], [x_i, y_j] - \delta_{ij} \mid 1 \leq i, i' \leq m, 1 \leq j, j' \leq n)\]

where $\delta_{uv}$ is the Kronecker delta.

The classical Weyl algebra $\mathcal{A}_n$ is $\mathcal{A}_{n,n}$, and one has

\[\mathcal{A}_{m,n} \cong \mathcal{A}_{\min(m,n)} \otimes k[t_1, \ldots, t_{|m-n|}]\]

for every $m, n \in \mathbb{N}$.

**Ore sets and localizations.** A multiplicative submonoid $S$ of an algebra $A$ is called a right Ore set if for every $s \in S$ and $u \in A$ there are $s' \in S$ and $u' \in A$ such that $su = u's'$. If $S$ is a right Ore set then one can invert the elements in $S$ to get an algebra $A_S$ and a morphism of algebras $\varphi_S: A \to A_S$ such that $\varphi(S) \subseteq A_S^\times$.

**Birational equivalences.** In classical algebraic geometry, two irreducible algebraic variety $X$ and $Y$ are called birationally equivalent if there is a rational algebraic function $f: X \to Y$ that induces a bijection from a Zariski open subset $U$ of $X$ to a Zariski open subset $V$ of $Y$. An affine (non-commutative) algebraic variety is called irreducible if its coordinate algebra has no idempotents other than 0 and 1. With this notion at hand, we call a morphism of (irreducible) affine non-commutative algebraic varieties $\varphi: A \to A'$ as a birational equivalence if there are suitable localizations $A_S$ and $A'_s$ such that that $\varphi(S) \subseteq S'$ and the extension $\varphi_S: A_S \to A'_S$, is an isomorphism of unital associative algebras.

**Generalized Cartan matrices.** Throughout the article, we work with a fixed generalized symmetrizable Cartan matrix $C = (a_{ij})$ of co-rank $\ell$, i.e. the null space of the matrix has dimension $0 \leq \ell \leq n$. 

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1. Generalized Weyl Algebras

1.1. Distributive laws. Given two algebras $A$ and $B$, and a linear map $\rho: B \otimes A \rightarrow A \otimes B$ is called a distributive law if the vector space $A \otimes B$ is a unital associative algebra with the multiplication $(\mu_A \otimes \mu_B) \circ (id_A \otimes \rho \otimes id_B)$ where $\mu_A$ and $\mu_B$ are multiplication maps of the underlying algebras. The product algebra is called the smash biproduct of $A$ and $B$, and is denoted by $A \#_\rho B$. If the distributive law is clear in the context, we may drop it from the notation.

1.2. The ambient smash product. Let us fix a commutative algebra $A$. For a fixed $n \geq 1$, assume we have a collection of pairwise commuting automorphisms $\sigma_i \in Aut(A)$ for every $1 \leq i \leq n$. Equivalently, we have a smash product algebra $A \#_T^n$ given by the relations $t_i a = \sigma_i(a)t_i$ for $a \in A$ and $1 \leq i \leq n$.

1.3. Generalized Weyl algebras. Let us fix a sequence $b = (b_1, \ldots, b_n)$ of elements from $A$. Then we define the generalized Weyl algebra of rank-$n$ $\mathbb{W}_\sigma(A, b)$ as the subalgebra of $A \#_T^n$ generated by $A$ and elements of the form

\begin{equation}
\tag{1.1}
b_i t_i^{-1} \quad \text{and} \quad -t_i
\end{equation}

for $1 \leq i \leq n$. See [4, 5, 12, 20]. See also [15] and references therein. Now, we extend [15, Thm.2.1] as follows:

**Proposition 1.1.** $\mathbb{W}_\sigma(A, b)$ is birationally equivalent to the smash product algebra $A \#_T^n$.

**Proof.** The subalgebra $\mathbb{W}_\sigma(A, b)$ becomes the whole algebra $A \#_T^n$ when we localize $\mathbb{W}_\sigma(A, b)$ with respect to the Ore set generated by elements $\sigma_i^n(b_j)$ with $m \in \mathbb{Z}$, $n \in \mathbb{N}$ and $i, j = 1, \ldots, n$. \(\square\)

2. Cartan Datum

In this Section we assume $\mathbb{W}_\sigma(A, b)$ is a generalized Weyl algebra.

2.1. Twisted differentials. Let us use $D_i$ for the discrete total derivative of the automorphism $\sigma_i$ for every $1 \leq i \leq n$. These are linear operators acting on $A$ as

$$D_i(f) = t_i ft_i^{-1} - f = \sigma_i(f) - f$$

for every $f \in A$ and $1 \leq i \leq n$. Observe that $D_i$ is as a right, or equivalently a left, $\sigma_i$-derivation since

$$D_i(fg) = \sigma_i(fg) - fg = D_i(f)\sigma_i(g) + fD_i(g)$$

for every $f, g \in A$ and $1 \leq i \leq n$. 
2.2. Cartan datum. Let us fix a generalized Cartan matrix \( C = (a_{ij}) \). A generalized Weyl algebra \( \mathbb{W}_\sigma(A, b) \) is said to be bound by \( C \) if the collection \( b = (b_1, \ldots, b_n) \) satisfies the following conditions

\[
\begin{align*}
D_i D_j (b_j) &= a_{ji} \quad \text{for every } i \text{ and } j, \\
D_i^{1-a_{ij}} (b_j) &= 0 \quad \text{for every } i \neq j.
\end{align*}
\]

We will say that a generalized Weyl algebra bound by \( \sigma \) has full rank when the elements \( D_i(b_i) \) are algebraically independent and the subalgebra they generate is birationally equivalent to \( A \). We are going to use \( \mathbb{W}_\sigma(A, C) \) to denote a generalized Weyl algebra \( \mathbb{W}_\sigma(A, b) \) bound by \( C \), and we are going to refer to it as the Cartan datum when the algebra has full rank.

2.3. Cartan datum and ordinary Weyl algebras. Given a linear endomorphism \( T : V \to V \) of a finite dimensional vector space \( V \), one can split \( V \) into direct sum of two spaces \( V_1 \oplus V_2 \) where \( T \) is identically zero on \( V_2 \) and invertible on \( V_1 \). Thus one can write a new invertible endomorphism \( Q : V \to V \) such that \( QT = \text{id}_{V_1} \oplus 0_{V_2} \). We call the matrix \( Q \) as the quasi-inverse of \( T \).

**Proposition 2.1.** Assume \( C \) has co-rank \( \ell \), and let \( \mathbb{W}_\sigma(A, C) \) has full rank. Then \( \mathbb{W}_\sigma(A, C) \) is birationally equivalent to the product algebra \( \mathbb{A}_{n-\ell,n} \otimes k[x_{n-\ell+1}, \ldots, x_n] \).

**Proof.** Assume \( C = (a_{ij}) \) has rank \( n - \ell \) and let \( Q = (e_{ij}) \) be its quasi-inverse. Then we have

\[
\sum_{u=1}^{n} c_{ju} a_{ui} = \begin{cases} 
1 & \text{if } 1 \leq i \leq n - \ell \text{ and } i = j \\
0 & \text{otherwise}.
\end{cases}
\]

Now, we let \( h_i = D_i(b_i) \) and define \( \alpha_i \) in \( \mathbb{W}_\sigma(A, C) \) given by

\[
\alpha_i = \sum_{u=1}^{n} c_{ju} h_u.
\]

These elements satisfy

\[
D_i(\alpha_j) = \sum_u c_{ju} D_i(h_u) = \sum_u c_{ju} a_{ui} = \begin{cases} 
1 & \text{if } 1 \leq i \leq n - \ell \text{ and } i = j \\
0 & \text{otherwise}
\end{cases}
\]

This means \( \alpha_i \) is in the centre when \( n - \ell + 1 \leq i \leq n \). Then we have a morphism of algebras \( \mathbb{A}_{n-\ell,n} \otimes k[x_{n-\ell+1}, \ldots, x_n] \to A^\#T^n \) given by

\[
x_i \mapsto \alpha_i t_i^{-1}, \quad y_i \mapsto -t_i.
\]

One can check that it is well-defined since defining relations are satisfied. Since the \( Q \) is invertible, the image subalgebra is isomorphic to the subalgebra generated by \( h_i t_i^{-1} \) and \( t_i \). Moreover, since the algebra has full rank, the elements \( h_i \) birationally generate \( A \) as a polynomial algebra. If we localize \( A^\#T^n \) on the Ore set generated by elements \( h_i \), we get the desired birational equivalence between \( \mathbb{A}_{n-\ell,n} \otimes k[x_{n-\ell+1}, \ldots, x_n] \) and \( A^\#T^n \).

Now, we use Proposition [1,1]. The result follows. \( \square \)
3. **Birational Equivalence for Kac-Moody Algebras**

### 3.1. Kac-Moody algebra of a generalized Cartan matrix

After [16 Defn. 3.17], we define the Kac-Moody Lie algebra \( g(C) \) associated with the generalized Cartan matrix \( C = (a_{ij}) \) as the Lie algebra generated by vectors \( E_i, F_i, H_i \) for \( 1 \leq i \leq n \) subject to the relations

\[
\begin{align*}
[H_i, H_j] &= 0, & [E_i, F_j] &= \delta_{ij} H_j, \\
[H_i, E_j] &= a_{ij} E_j, & [H_i, F_j] &= -a_{ij} F_j,
\end{align*}
\]

for every \( 1 \leq i, j \leq n \), and

\[
\begin{align*}
\text{ad}(E_i)^{1-a_{ij}}(E_j) &= 0 & \text{ad}(F_i)^{1-a_{ij}}(F_j) &= 0
\end{align*}
\]

for every \( i \neq j \) where \( \text{ad}(x)(y) \) is defined to be \( xy - yx \) for every \( x, y \in g(C) \).

The Kac-Moody algebra of a generalized Cartan matrix \( C \) is the universal enveloping algebra of the Kac-Moody Lie algebra \( g(C) \). We will use \( U(g) \) for the algebra \( U(g(C)) \) dropping \( C \) from the notation.

### 3.2. Merging subalgebras

The following subalgebras of \( U(g) \) are going to be used in the sequel:

1. The subalgebra \( U^0(g) \) generated by \( H_i \) for \( 1 \leq i \leq n \).
2. The subalgebra \( U^{>0}(g) \) generated by \( E_i \) and \( H_i \) for \( 1 \leq i \leq n \).
3. The subalgebra \( U^{<0}(g) \) generated by \( F_i \) and \( H_i \) for \( 1 \leq i \leq n \).

Consider the algebras \( U^{>0}(g) \) and \( U^{<0}(g) \), and define a distributive law of the form

\[
\omega: U^{<0}(g) \otimes U^0(g) U^{>0}(g) \to U^{>0}(g) \otimes U^0(g) U^{<0}(g)
\]

given by

\[
\omega(F_j E_i) = E_i F_j - \delta_{ij} H_i
\]

for every \( 1 \leq i, j \leq n \). Since we have a Poincare-Birkhoff-Witt basis, one can easily see that the resulting product is exactly \( U(g) \).

### 3.3. The canonical Cartan datum of a Kac-Moody algebra

Let \( A \) be the polynomial algebra \( k[h_1, \ldots, h_n] \) and let us define

\[
\sigma_i(h_j) = h_j + a_{ji}
\]

for every \( 1 \leq i, j \leq n \). Notice that with this choice we get \( D_i(h_j) = a_{ji} \) for every \( i \) and \( j \). Let us recall the elements \( \alpha_i \) from (2.3) and we choose

\[
b_i = \frac{1}{4} h_i(h_i - 2) + \beta_i
\]

for every \( 1 \leq i \leq n \) where \( \beta_i \in k[h_1, \ldots, h_n] \) is an undetermined element in the subalgebra generated by \( \alpha_j \) with \( j \neq i \) that we constructed in the proof of Proposition 2.1.

We immediately get that \( D_i(b_i) = h_i \) for every \( 1 \leq i \leq n \). Moreover, we see that

\[
D_i D_j(b_j) = D_i(h_j) = a_{ji}.
\]

Since in the basis \( \alpha_i \)'s the twisted differentials \( D_i \)'s behave like ordinary differential operators, one can solve the system of differential equations \( D^{1-a_{ij}}(b_j) \) for \( b_i \)'s and determine \( \beta_i \)'s. This means the generalized Weyl algebra \( \mathbb{W}_\sigma(A, b) \) is bound by \( C \) and has full-rank, i.e. we have a Cartan datum \( \mathbb{W}_\sigma(A, C) \).
3.4. Birational equivalence for Kac-Moody algebras.

**Theorem 3.1.** The algebras $U^>_{\geq 0}(\mathfrak{g})$ and $U^<_{\leq 0}(\mathfrak{g})$ are birationally equivalent to the product $A_{n-\ell,n} \otimes k[t_1, \ldots, t_\ell]$ when the underlying Cartan matrix has co-rank $\ell$.

**Proof.** We will give the proof for $U^>_{\geq 0}(\mathfrak{g})$. The proof for $U^<_{\leq 0}(\mathfrak{g})$ is similar.

We have a well-defined morphism of algebras of the form $U^>_{\geq 0}(\mathfrak{g}) \to \mathbb{S}_\sigma(k[h_1, \ldots, h_n], C)$ given by

$$H_i \mapsto h_i \quad E_i \mapsto \left(\frac{1}{4} h_i(h_i - 2) + \beta_i\right) t_i^{-1}$$

for $1 \leq i \leq n$ which is a birational equivalence when we invert $H_i$ and $h_i$‘s and $E_i$ and $e_i$’s. On the other hand Proposition 2.1 tells us that $\mathbb{W}_\sigma(k[h_1, \ldots, h_n], C)$ is birationally equivalent to $A_{n-\ell,n} \otimes k[x_{n-\ell+1}, \ldots, x_n]$ since $\mathbb{W}_\sigma(k[h_1, \ldots, h_n], C)$ has full rank. The result follows. \hfill \Box

**Corollary 3.2.** Let $C$ be a generalized Cartan matrix of co-rank $\ell$ and let $\mathbb{W}_\sigma(A, C)$ be a Cartan datum. The Kac-Moody algebra associated with $C$ is birationally equivalent to the smash biproduct of two copies of the algebra $\mathbb{W}_\sigma(A, C)$, or equivalently smash biproduct of two copies of the algebra $A_{n-\ell,n} \otimes k[t_1, \ldots, t_\ell]$.

**Proof.** Follows Proposition 2.1 and Theorem 3.1. \hfill \Box

4. Birational Equivalence for Quantized Kac-Moody Algebras

As before, let us fix an algebra $A$, a set of commuting automorphisms $\sigma_i \in \text{Aut}(A)$ and elements $b_i \in A$ for $1 \leq i \leq n$. This time we assume that the Cartan matrix is symmetrizable, i.e. there are scalars $d_i \in k^\times$ such that $d_i a_{ij} = d_j a_{ij}$ for $1 \leq i, j \leq n$.

4.1. Quantized Cartan datum. A generalized Weyl algebra $\mathbb{W}_\sigma(A, b)$ is called the quantum generalized Weyl algebra bound by a generalized Cartan matrix $C = (a_{ij})$ if the elements $b_1, \ldots, b_n$ satisfy the following conditions

$$\sigma_i(b_j) = q^{d_{i,j}} b_j \quad \text{for every } i \text{ and } j,$n$$

$$D_i^{[1-a_{ij}]}(b_j) = 0 \quad \text{for every } i \neq j$$

where we define

$$D_i^{[m]} = \prod_{\ell=0}^{m-1} (\sigma_i - q^{2\ell d_i})$$

for every $m \geq 0$ and $1 \leq i \leq n$. As before, we say $\mathbb{W}_\sigma(A, C)$ is of full rank if the subalgebra generated by $b_i$’s are algebraically independent and birationally generate $A$. We are going to refer such an algebra as the quantum Cartan datum, and denote it by $\mathbb{W}_\sigma(A, C)$.

4.2. Quantized Cartan datum and quantized Weyl algebras. Let $Q = (c_{ij})$ be the quasi-inverse of the Cartan matrix, and let $g_i$ be the smallest positive integer such that $c_{ij} g_j$ is an integer for every $1 \leq i, j \leq n$. We define the quantum Weyl algebra $A^q_{m,n}$ with the presentation

$$k\{x_1, \ldots, x_m, y_1, \ldots, y_n\} \langle [x_i, x_{i'}], [y_j, y_{j'}], y_j x_i - g^{q \delta_{ij}} x_i y_j \mid i, i' = 1, \ldots, m, \; j, j' = 1, \ldots, n \rangle$$
Proposition 4.1. Assume $C$ has co-rank $\ell$. Then the quantized Cartan datum $\mathbb{W}_q^\ell(A, C)$ is birationally equivalent to $\mathbb{K}^q_{n-\ell,n} \otimes k[t_1, \ldots, t_\ell]$.

Proof. Let $S$ be the Ore set in $A \# \mathbb{T}^n$ generated by the elements $b_i$ for $1 \leq i \leq n$. For each $1 \leq i \leq n$ we consider

$$\omega_i := \prod_{u=1}^n b_u^{a_i u_{gi}}.$$  

in $(A \# \mathbb{T}^n)_S$. Now, observe that

$$\sigma_j(\omega_i) = \prod_u \sigma_j(b_u^{a_i u_{gi}}) = \prod_u q^{a_j u_{gi}} b_u^{a_i u_{gi}} = \begin{cases} q^{a_i} \omega_i & \text{if } 1 \leq i = j \leq n - \ell \\ \omega_i & \text{otherwise.} \end{cases}$$

The elements $\omega_i$ need not be elements in $A$ since we may have negative exponents. This problem is solved by using our suitable localization. Now, one can define an algebra map after such a localization $\mathbb{K}^q_{n-\ell,n} \rightarrow (A \# \mathbb{T}^n)_S$ by

$$x_i \mapsto \omega_i t_i^{-1}, \quad y_i \mapsto t_i$$

for $1 \leq i \leq n$. One can check that the defining relations are satisfied. Now, determining whether this embedding is a birational equivalence boils down to checking if the embedding of the subalgebra of $A$ generated by $\omega_1, \ldots, \omega_n$ is a birational equivalence. The result follows. \qed

4.3. Quantized Kac-Moody algebra. We fix a $q \in k^\times$ and let $d_1, \ldots, d_n$ be the set of positive integers such that $d_ia_{ij} = d_ia_{ji}$ for every $1 \leq i, j \leq n$. We define the quantum Kac-Moody algebra $U_q(g)$ associated with the generalized Cartan matrix as the algebra generated by non-commuting indeterminates $E_i, F_i, K_i^{\pm 1}$ for $1 \leq i \leq n$ subject to the following relations

$$K_i K_j = K_j K_i, \quad [E_i, F_j] = \delta_{ij} K_i - K_i^{-1}$$

$$E_i K_i = q^{-d_ia_{ij}} K_i E_i, \quad F_i K_i = q^{d_ia_{ij}} K_i F_i,$$

for every $1 \leq i, j \leq n$ and

$$\text{ad}_q(E_i)^{1-a_{ij}}(E_j) = 0, \quad \text{ad}_q(F_i)^{1-a_{ij}}(F_j) = 0$$

where $\text{ad}_q(E_i)(E_j)$ and $\text{ad}_q(F_i)(F_j)$ are defined to be

$$\text{ad}_q(E_i)(E_j) = E_i E_j - q^{d_ia_{ij}} E_j E_i \quad \text{and} \quad \text{ad}_q(F_i)(F_j) = F_i F_j - q^{-d_ia_{ij}} F_j F_i$$

for every $i \neq j$.

4.4. Localization of a quantized Kac-Moody algebra. If we localize $U_q(g)$ with respect to $E_i$ and $F_i$ for $1 \leq i \leq n$, the defining relation (4.8) manifests itself as

$$\text{Ad}(K_j^{-1} E_j)(K_i) = q^{-d_ia_{ij}} K_i \quad \text{and} \quad \text{Ad}(K_j^{-1} F_j)(K_i) = q^{d_ia_{ij}} K_i$$

and the Chevalley-Serre relations become

$$\prod_{\ell=0}^{a_{ij}} (\text{Ad}(K_i^{-1} E_i) - q^{2d_i})(E_j) = 0 \quad \prod_{\ell=0}^{a_{ij}} (\text{Ad}(K_i^{-1} F_i) - q^{2d_i})(F_j) = 0$$

for every $1 \leq i, j \leq n$. 
4.5. Merging quantized Borel subalgebras. As in the case of Kac-Moody algebras, we have following subalgebras.

(1) The Cartan subalgebra \( U^0_q(\mathfrak{g}) \) generated by \( K_i^{\pm 1} \) for \( 1 \leq i \leq n \).

(2) The Borel subalgebra \( U^{\geq 0}_q(\mathfrak{g}) \) generated by \( E_i \) and \( K_i^{\pm 1} \) for \( 1 \leq i \leq n \).

(3) The Borel subalgebra \( U^{\leq 0}_q(\mathfrak{g}) \) generated by \( F_i \) and \( K_i^{\pm 1} \) for \( 1 \leq i \leq n \).

Again as before, one can merge the Borel subalgebras \( U^{\geq 0}_q(\mathfrak{g}) \) and \( U^{\leq 0}_q(\mathfrak{g}) \) via a suitable distributive law:

\[
\omega_q: U^{\leq 0}_q(\mathfrak{g}) \otimes U^{\geq 0}_q(\mathfrak{g}) \to U^{\geq 0}_q(\mathfrak{g}) \otimes U^{\leq 0}_q(\mathfrak{g})
\]
given by

\[
\omega(F_i E_j) = E_j F_i - \delta_{i\ell} \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}
\]
for every \( 1 \leq i, j \leq n \). Since we have a Poincare-Birkhoff-Witt basis, one can easily see that the resulting product is exactly \( U_q(\mathfrak{g}) \).

4.6. Birational equivalences for quantum Kac-Moody algebras.

Theorem 4.2. The quantum Borel algebras \( U^{\geq 0}_q(\mathfrak{g}) \) and \( U^{\leq 0}_q(\mathfrak{g}) \) are birationally equivalent to the product algebra \( K^{n-\ell,n}_q \otimes k[t_1, \ldots, t_{\ell}] \) when the underlying Cartan matrix has co-rank \( \ell \).

Proof. Let \( A \) be the torus \( T^n \) generated by \( K_1, \ldots, K_n \) and let

\[
\sigma_i(K_j) = q^{-d_i a_{ij}} K_j
\]
for every \( 1 \leq i, j \leq n \). If we let \( b_i = K_i^{-1} \) then the defining relation \( (4.1) \) is satisfied in its localized form \( (4.11) \) for every \( 1 \leq i, j \leq n \). The defining equation \( (4.2) \) is satisfied because we have \( (4.12) \). This means the algebra isomorphism \( U^{\geq 0}_q(\mathfrak{g}) \to \mathcal{W}_q(T^n, C) \) given by

\[
K_i \mapsto K_i, \quad E_i \mapsto K_i^{-1} t_i^{-1}
\]
for every \( 1 \leq i \leq n \) is well-defined after a suitable localization. One can also see that \( \mathcal{W}_q(T^n, C) \) is full-rank since we use \( b_i = K_i \) for \( 1 \leq i \leq n \). So, we obtain a birational equivalence of the form \( U^{\geq 0}_q(\mathfrak{g}) \to \mathcal{W}_q(T^n, C) \). We also have another birational equivalence of the form \( \mathcal{A}_q^{n-\ell,n} \otimes k[x_{n-\ell+1}, \ldots, x_n] \to \mathcal{W}_q(T^n, C) \) by Proposition 4.1. The result follows.

Corollary 4.3. The quantized Kac-Moody algebra \( U_q(\mathfrak{g}) \) is birationally equivalent to a smash biproduct of two copies of the product algebra \( \mathcal{A}_q^{n-\ell,n} \otimes k[t_1, \ldots, t_{\ell}] \) when the underlying Cartan matrix has co-rank \( \ell \).

Proof. Follows from Proposition 4.1, Theorem 4.2, and Section 4.5

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