On symplectic self-adjointness of Hamiltonian operator matrices

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Abstract Symplectic self-adjointness of Hamiltonian operator matrices is studied, which is important to symplectic elasticity and optimal control. For the cases of diagonal domain and off-diagonal domain, necessary and sufficient conditions are shown. The proofs use Frobenius-Schur factorizations of unbounded operator matrices. Under additional assumptions, sufficient conditions based on perturbation method are obtained. The theory is applied to a problem in symplectic elasticity.

Keywords symplectic elasticity, symplectic self-adjoint, Hamiltonian operator matrix

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1 Introduction
There are a number of very interesting ways that Hamiltonian operator matrices (see Definition 2.3 below) can arise. We mention a few. First, many linear boundary value problems in mathematical physics can be written as the Hamiltonian system (or Hamiltonian equation) \( \dot{u} = Hu + f \), where

\[ H = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \]  

(1.1)
is a Hamiltonian operator matrix acting on the product space \( X \times X \) of some Hilbert space \( X \), so that the solvability of the original boundary value problem is reduced to spectral properties of the Hamiltonian operator matrix \( H \), see [2,13] for ordinary differential equations and [8,20,32] for partial differential equations. This is a typical case in symplectic elasticity. In elasticity, symplectic approach (i.e., Hamiltonian system approach) was first applied in the early 1990s by Zhong [33], see [18,31] and the references therein. The new approach is efficient for solving basic problems in solid mechanics and, moreover, analytical solutions could be obtained by expansion of eigenfunctions. Second, Hamiltonian operator matrices also arise in theory of optimal control. It is well known that the solutions \( K \) to the Riccati equation \( A^*K + KA + KBK - C = 0 \) are in one-to-one correspondence with graph subspaces that are invariant under the operator matrix \( H \) given by (1.1), where \( A, B \) and \( C \) are unbounded linear

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operators and $B, C$ are nonnegative, see [26,30] and the references therein. Therefore, spectral properties of Hamiltonian operator matrices have drawn a lot of interest, see [1,5,6,14–17,21,22,24,28].

The property symplectic self-adjointness (see Definition 2.4 below) is distinctive for certain Hamiltonian operator matrices, since it is only for symplectic self-adjoint Hamiltonian operator matrices that the spectral theorems hold [1] and it is only symplectic self-adjoint Hamiltonian operator matrices that may be invertible which is sometimes important in the investigation of Hamiltonian equations [15].

This paper is devoted to studying methods for proving that Hamiltonian operator matrices are symplectic self-adjoint. For the case of $D(H) = D(A) \times D(A^*)$ or $D(H) = D(B) \times D(D)$, Wu and Chen [28] proved the following basic perturbation result.

**Theorem 1.1.** Let $H = (\begin{array}{cc} A & B \\ C & -A^* \end{array})$ be a Hamiltonian operator matrix. Then $H$ is symplectic self-adjoint if one of the following holds:

1. $C$ is $A$-bounded and $B$ is $A^*$-bounded with both relative bounds less than 1.
2. $A$ is $C$-bounded and $A^*$ is $B$-bounded with both relative bounds less than 1.

In Section 3, by the Frobenius-Schur factorization technique, we shall establish some necessary and sufficient conditions for a Hamiltonian operator matrix to be symplectic self-adjoint which extend Theorem 1.1 to non-perturbation cases. Moreover, we also obtain some perturbation results under the assumptions that are different from or weaker than those in Theorem 1.1. Finally, we shall apply the new results to a problem in symplectic elasticity which seems cannot be resolved by the previously published methods, see Section 4 below. Some lemmas on adjoints are presented in Appendix.

## 2 Preliminaries

Our notion of an operator matrix is taken from [29, p.97], see also [25, Subsection 2.2] for another definition.

**Definition 2.1.** Let $X_1$ and $X_2$ be Banach spaces and consider linear operators $A : D(A) \subset X_1 \to X_1$, $B : D(B) \subset X_2 \to X_1$, $C : D(C) \subset X_1 \to X_2$, and $D : D(D) \subset X_2 \to X_2$. Then the matrix $H = (\begin{array}{cc} A & B \\ C & -A^* \end{array})$ is called a block operator matrix on $X_1 \times X_2$. It induces a linear operator on $X_1 \times X_2$ which is also denoted by $H$:

$$D(H) := (D(A) \cap D(C)) \times (D(B) \cap D(D)),$$

$$H \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) := \left( \begin{array}{c} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{array} \right) \quad \text{for} \quad \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \in D(H).$$

The following Frobenius-Schur factorization will play an important role in the proofs of our main theorems.

**Lemma 2.2** (See [25, Subsection 2.2]). Let $H = (\begin{array}{cc} A & B \\ C & -A^* \end{array})$ be a block operator matrix acting on the product space $X \times X$ of some Banach space $X$.

1. Suppose that $D$ is closed with $\rho(D) \neq \emptyset$, and that $D(D) \subset D(B)$. Then for some (and hence for all) $\lambda \in \rho(D)$,

$$H - \lambda = \left( \begin{array}{cc} I & B(D - \lambda)^{-1} \\ 0 & I \end{array} \right) \left( \begin{array}{cc} S_1(\lambda) & 0 \\ 0 & D - \lambda \end{array} \right) \left( \begin{array}{cc} I & 0 \\ (D - \lambda)^{-1}C & I \end{array} \right),$$

where $S_1(\lambda) := A - \lambda - B(D - \lambda)^{-1}C$ is the first Schur complement of $H$ with domain $D(S_1(\lambda)) = D(A) \cap D(C)$.

2. Suppose that $A$ is closed with $\rho(A) \neq \emptyset$, and that $D(A) \subset D(C)$. Then for some (and hence for all) $\lambda \in \rho(A)$,

$$H - \lambda = \left( \begin{array}{cc} I & 0 \\ C(A - \lambda)^{-1} & I \end{array} \right) \left( \begin{array}{cc} A - \lambda & 0 \\ 0 & S_2(\lambda) \end{array} \right) \left( \begin{array}{cc} I & (A - \lambda)^{-1}B \\ 0 & I \end{array} \right),$$

where $S_2(\lambda) := C^{-1}(A - \lambda)^{-1}B - D - \lambda$ is the second Schur complement of $H$ with domain $D(S_2(\lambda)) = D(B) \cap D(D)$. 

where \( S_2(\lambda) := D - \lambda - C(A - \lambda)^{-1}B \) is the second Schur complement of \( H \) with domain \( \mathcal{D}(S_2(\lambda)) = \mathcal{D}(B) \cap \mathcal{D}(D) \).

We use the following definition of a Hamiltonian operator matrix, see also [1,5,30] for other definitions.

**Definition 2.3.** Let \( X \) be a complex Hilbert space. A Hamiltonian operator matrix is a block operator matrix \( H = (\begin{array}{c} A & B \\ C & -A \end{array}) \) acting on \( X \times X \) with closed densely defined operators \( A, B, C \) such that \( B, C \) are self-adjoint and \( H \) is densely defined.

For a Hamiltonian operator matrix \( H \), one readily checks that \( JH \subseteq (JH)^* \), where \( J = (\begin{array}{c} 0 & I \\ -I & 0 \end{array}) \) is the unit symplectic operator matrix \([31, p.11]\).

**Definition 2.4** (See \([28]\)). Let \( H \) be a Hamiltonian operator matrix. If \( JH = (JH)^* \), then \( H \) is called a symplectic self-adjoint Hamiltonian operator matrix.

### 3 Main results

In this section \( H = (\begin{array}{c} A & B \\ C & -A \end{array}) \) will denote a Hamiltonian operator on \( X \times X \) and \( J = (\begin{array}{c} 0 & I \\ -I & 0 \end{array}) \) will denote the unit symplectic operator matrix.

Firstly, we give a sufficient and necessary condition for a Hamiltonian operator matrix to be symplectic self-adjoint. Throughout the rest of the paper, \( i \) will denote the imaginary unit.

**Proposition 3.1.** \( H \) is symplectic self-adjoint if and only if \( \mathcal{R}(H \pm iJ) = X \times X \).

**Proof.** Since \( JH \subseteq (JH)^* \) and \( J^* = J^{-1} = -J \), the assertion follows from the well-known fact \([23, \text{Theorem VIII.3}]\) that \( JH = (JH)^* \Leftrightarrow \mathcal{R}(JH \mp iI) = X \times X \). \( \square \)

Secondly, we consider the case \( \mathcal{D}(H) = \mathcal{D}(A) \times \mathcal{D}(A^*) \).

**Theorem 3.2.** Suppose that \( \mathcal{D}(H) = \mathcal{D}(A) \times \mathcal{D}(A^*) \), and that \( \rho(A) \neq \emptyset \). Then the following statements are equivalent:

1. \( H \) is symplectic self-adjoint.
2. \( A^* + C(A - \lambda)^{-1}B = (A + B(A^* - \overline{\lambda})^{-1}C)^* \) for some (and hence for all) \( \lambda \in \rho(A) \).
3. \( A + B(A^* - \overline{\lambda})^{-1}C = (A^* + C(A - \lambda)^{-1}B)^* \) for some (and hence for all) \( \lambda \in \rho(A) \).

**Proof.** Take \( \lambda \in \rho(A) \). By applying Lemma 2.2 to \((H - \lambda)\) and then a little calculation we see that

\[
JH - \lambda J = \begin{pmatrix} I & -C(A - \lambda)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} -S_2(\lambda) & 0 \\ 0 & -A + \lambda \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & (A - \lambda)^{-1}B \end{pmatrix}, \tag{3.1}
\]

where \( S_2(\lambda) := -A^* - \lambda - C(A - \lambda)^{-1}B \) is the second Schur complement of \( H \). Note that \((A - \lambda)^{-1}B \) is bounded on its domain \( \mathcal{D}(B) \) since \( \mathcal{D}(B) \supseteq \mathcal{D}(A^*) \) (see \([3, \text{Proposition 3.1}]\)). Then in (3.1) we can replace \((A - \lambda)^{-1}B = (A - \lambda)^{-1}B \mid \mathcal{D}(B) \) by \((A - \lambda)^{-1}B = (B(A^* - \overline{\lambda})^{-1})^* \) since the domain of the middle factor is equal to \( \mathcal{D}(A^*) \times \mathcal{D}(A) \). Thus,

\[
JH - \lambda J = \begin{pmatrix} I & -C(A - \lambda)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} -S_2(\lambda) & 0 \\ 0 & -A + \lambda \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & (B(A^* - \overline{\lambda})^{-1})^* \end{pmatrix}. \tag{3.2}
\]

Similar to the proof of (3.2), we have

\[
JH + \overline{\lambda} J = \begin{pmatrix} 0 & I \\ -I & B(A^* - \overline{\lambda})^{-1} \end{pmatrix} \begin{pmatrix} S_1(-\overline{\lambda}) & 0 \\ 0 & -A^* + \overline{\lambda} \end{pmatrix} \begin{pmatrix} I & 0 \\ (-C(A - \lambda)^{-1})^* & I \end{pmatrix}, \tag{3.3}
\]

where \( S_1(-\overline{\lambda}) := A + \overline{\lambda} + B(A^* - \overline{\lambda})^{-1}C \) is the first Schur complement of \( H \). In the factorization (3.3), the first and last factor are bounded and boundedly invertible, and therefore by Lemmas A.2 and A.3,

\[
(JH)^* - \lambda J = \begin{pmatrix} I & -C(A - \lambda)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} (S_1(-\overline{\lambda}))^* & 0 \\ 0 & -A + \lambda \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & (B(A^* - \overline{\lambda})^{-1})^* \end{pmatrix}. \tag{3.4}
\]
Note that in the factorization (3.2) or (3.4), the first and last factors are bounded and boundedly invertible, so that \( JH = (JH)^* \) if and only if \(-S_2(\lambda) = S_1(-\lambda)^*\). Similarly, \( JH = (JH)^* \) if and only if \( S_1(-\lambda) = (-S_2(\lambda))^*\).

**Corollary 3.3.** Suppose that \( \mathcal{D}(H) = \mathcal{D}(A) \times \mathcal{D}(A^*) \), and that \( \rho(A) \neq \emptyset \). Then \( JH = (JH)^* \) if one of the following holds:

1. \( C \) is \( A \)-bounded with relative bound 0.
2. \( B \) is \( A^* \)-bounded with relative bound 0.

**Proof.** We prove the claim in Case (1); the proof of Case (2) is analogous. Taking \( \lambda \in \rho(A) \). It is enough to prove that \( (A + B(A^* - \lambda)^{-1}C)^* = A^* + C(A - \lambda)^{-1}B \). First, we note that \( B(A^* - \lambda)^{-1}C \) is \( A \)-bounded with relative bound 0 since \( B(A^* - \lambda)^{-1} \) is bounded (note that it is a closed everywhere defined operator) and \( C \) is \( A \)-bounded with relative bound 0. Next, we prove \( (B(A^* - \lambda)^{-1}C)^* = A^* \)-bounded with relative bound 0. To this end, we first claim that \( C(A - \lambda)^{-1}B \) is \( A^* \)-bounded with relative bound 0. In fact, by the assumptions, for every \( \varepsilon > 0 \) there is a real number \( b(\varepsilon) \) such that for all \( x \in \mathcal{D}(A), \| Cx \| \leq \varepsilon \| (A - \lambda)x \| + b(\varepsilon) \| x \| \), and for some \( a \) and \( b \) in \( \mathbb{R} \) and all \( x \in \mathcal{D}(A^*), \| Bx \| \leq a \| A^*x \| + b \| x \| \), so for all \( x \in \mathcal{D}(A^*), \| C(A - \lambda)^{-1}Bx \| \leq \varepsilon \| Bx \| + b(\varepsilon) \| (A - \lambda)^{-1}Bx \| \leq a \| A^*x \| + b(\varepsilon, \lambda) \| x \| \), since \( (A - \lambda)^{-1}B \) is bounded on \( \mathcal{D}(B) \). It follows that \( C(A - \lambda)^{-1}B \) is \( A^* \)-bounded with relative bound 0.

For the definition and properties of a maximal accretive operator in the following corollary, see [19, Subsection IV.4].

**Corollary 3.4.** Suppose that \( \mathcal{D}(H) = \mathcal{D}(A) \times \mathcal{D}(A^*), \) and that \( A - \lambda \) is maximal accretive. Then \( JH = (JH)^* \) if one of the following holds:

1. \( C \) is \( A \)-bounded with relative bound \( < 1 \) and \( B \) is \( A^* \)-bounded with relative bound \( \leq 1 \).
2. \( C \) is \( A \)-bounded with relative bound \( \leq 1 \) and \( B \) is \( A^* \)-bounded with relative bound \( < 1 \).

**Proof.** Since \( -H \) is also a Hamiltonian operator, it is enough to prove the case that \( A \) is maximal accretive. We prove the claim in Case (2); the proof of Case (1) is analogous. Note that the operators \( A \) and \( A^* \) can be maximal accretive only simultaneously. To prove \( JH = (JH)^* \), we have to show that for some \( \lambda > 0, \)

\[
(A + B(A^* + \lambda^{-1}C)^* = A^* + C(A + \lambda)^{-1}B. \tag{3.5}
\]

**Step 1.** We start from the claim that for \( \lambda > 0 \) large enough, \( B(A^* + \lambda)^{-1}C \) is \( A \)-bounded with relative bound \( < 1 \). Since \( C \) is \( A \)-bounded with relative bound \( \leq 1 \), it is enough to prove

\[
\| B(A^* + \lambda)^{-1} \| < 1, \tag{3.6}
\]

for \( \lambda > 0 \) large enough. We observe that for \( x \in \mathcal{D}(A^*), \)

\[
\| (A^* + \lambda)x \|^2 = \| A^*x \|^2 + 2\lambda \text{Re}(A^*x, x) + \lambda^2 \| x \|^2 \geq \| A^*x \|^2 + \lambda^2 \| x \|^2, \tag{3.7}
\]

where the last inequality follows from the fact that \( A^* \) is maximal accretive. By the assumption that \( B \) is \( A^* \)-bounded with relative bound \( < 1 \), there are real numbers \( a < 1 \) and \( b \) such that for \( x \in \mathcal{D}(A^*), \)

\[
\| Bx \| \leq a \| A^*x \| + b \| x \|, \tag{3.8}
\]

so that for \( x \in \mathcal{D}(A^*), \) we have, using (3.7) twice and then (3.8), \( \| Bx \| \leq (a + \frac{b}{\lambda}) \| (A^* + \lambda)x \|. \) It is enough to choose \( \lambda > 0 \) large enough such that \( a + \frac{b}{\lambda} < 1. \)
Step 2. In this step, we show that for $\lambda > 0$ large enough, $(B(A^* + \lambda)^{-1}C)^*$ is $A^*$-bounded with relative bound $< 1$. Noting that
\[
(B(A^* + \lambda)^{-1}C)^* = C^*(B(A^* + \lambda)^{-1})^* = C(A + \lambda)^{-1}B
\]
and that $\mathcal{D}(B) \supset \mathcal{D}(A^*)$, it is enough to prove for $\lambda > 0$ large enough, $C(A + \lambda)^{-1}B$ is $A^*$-bounded with relative bound $< 1$. Since $C$ is $A$-bounded with relative bound $\leq 1$ and the operator $A$ is maximal accretive, we have, with arguments similar to the ones used in the proof of Theorem 3.2, for every $\varepsilon > 0$ there is an $N(\varepsilon) > 0$ such that for $\lambda > N(\varepsilon)$,
\[
\|C(A + \lambda)^{-1}\| < 1 + \varepsilon.
\]
(3.9)
It follows from (3.8) and (3.9) that for $\lambda > N(\varepsilon)$ and $x \in \mathcal{D}(A^*)$, $\|C(A + \lambda)^{-1}Bx\| \leq (1 + \varepsilon)a\|A^*x\|$ + $(1 + \varepsilon)b\|x\|$. It is enough to choose $\varepsilon > 0$ small enough such that $(1 + \varepsilon)a < 1$.

Step 3. Now (3.5) follows from Steps 1 and 2 by applying Lemma A.1.

Corollary 3.5. Suppose that $\mathcal{D}(H) = \mathcal{D}(A) \times \mathcal{D}(A^*)$, and that $A$ is self-adjoint. Then $JH = (JH)^*$ if one of the following holds:

(1) $C$ is $A$-bounded with relative bound $< 1$ and $B$ is $A^*$-bounded with relative bound $\leq 1$.

(2) $C$ is $A$-bounded with relative bound $\leq 1$ and $B$ is $A^*$-bounded with relative bound $< 1$.

Proof. Similar to the proof of Corollary 3.4, we have $(A + B(A^* + i\lambda)^{-1}C)^* = A^* + C(A - i\lambda)^{-1}B$, for some $\lambda > 0$.

Finally, we consider the case $\mathcal{D}(H) = \mathcal{D}(C) \times \mathcal{D}(B)$.

Theorem 3.6. Suppose $\mathcal{D}(H) = \mathcal{D}(C) \times \mathcal{D}(B)$. Then the following statements are equivalent:

(1) $H$ is symplectic self-adjoint.

(2) $C + A^*(B - \lambda)^{-1}A = (C + A^*(B - \overline{\lambda})^{-1}A)^*$ for some (and hence for all) $\lambda \in \rho(B)$.

(3) $B + A(C - \lambda)^{-1}A^* = (B + A(C - \overline{\lambda})^{-1}A^*)^*$ for some (and hence for all) $\lambda \in \rho(C)$.

Proof. Let $\lambda \in \rho(C)$. By applying Lemma 2.2 to $(JH - \lambda)$ and $(\lambda J - \overline{\lambda})$, respectively, we have, with arguments similar to the ones used in the proof of Theorem 3.2,
\[
JH - \lambda = \begin{pmatrix}
I & 0 \\
-A(C - \lambda)^{-1} & I
\end{pmatrix}
\begin{pmatrix}
C - \lambda & 0 \\
0 & S_2(\lambda)
\end{pmatrix}
\begin{pmatrix}
I & (A(C - \overline{\lambda})^{-1})* \\
0 & I
\end{pmatrix},
\]
\[
(JH)^* - \lambda = \begin{pmatrix}
I & 0 \\
-A(C - \overline{\lambda})^{-1} & I
\end{pmatrix}
\begin{pmatrix}
C - \lambda & 0 \\
0 & S_2(\overline{\lambda})*
\end{pmatrix}
\begin{pmatrix}
I & (A(C - \overline{\lambda})^{-1})* \\
0 & I
\end{pmatrix},
\]
where $S_2(\lambda) := -B - A(C - \lambda)^{-1}A^*$ is the second Schur complement of $JH$, so that $JH = (JH)^*$ if and only $S_2(\lambda) = S_2(\overline{\lambda})^*$. Similarly, $JH = (JH)^*$ if and only if $S_1(\lambda) = S_1(\overline{\lambda})^*$ for some (and hence for all) $\lambda \in \rho(B)$, where $S_1(\lambda) := C + \lambda + A^*(B - \lambda)^{-1}A$ is the first Schur complement of $JH$.

Corollary 3.7. Suppose $\mathcal{D}(H) = \mathcal{D}(C) \times \mathcal{D}(B)$. Then $JH = (JH)^*$ if one of the following holds:

(1) $A$ is $C$-bounded with relative bound $< 1$ and $A^*$ is $B$-bounded with relative bound $\leq 1$.

(2) $A$ is $C$-bounded with relative bound $\leq 1$ and $A^*$ is $B$-bounded with relative bound $< 1$.

Proof. Similar to the proof of Corollary 3.4, we have $(B + A(C + i\lambda)^{-1}A^*)^* = B + A(C - i\lambda)^{-1}A^*$, for some $\lambda > 0$.

The following example shows that the relative bounds in the assumptions of Corollary 3.4 (Corollaries 3.5 and 3.7, respectively) cannot be improved.

Example 3.8. Let $A$ be a nonnegative unbounded self-adjoint operator on $X$. Note that $A$ is also maximal accretive. Consider the Hamiltonian operator $H := \left( \begin{array}{cc} A & A \\ -A & A \end{array} \right)$. It is not difficult to see that $H$ is not closed since $A$ is unbounded, so that $JH \neq (JH)^*$. 

4 An application to symplectic elasticity

Consider the rectangular thin plate bending problem with two opposite edges simply supported. The basic governing equation in terms of displacement is

\[ D \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 w = q, \quad \text{for} \quad 0 < x < h \quad \text{and} \quad 0 < y < 1, \quad (4.1) \]

the boundary conditions for simply supported edges are

\[ w = 0, \quad \frac{\partial^2 w}{\partial y^2} = 0, \quad \text{for} \quad y = 0 \quad \text{or} \quad y = 1, \quad (4.2) \]

and the boundary conditions for the other two edges are

\[ w, \quad \frac{\partial w}{\partial x} = \text{given functions}, \quad \text{for} \quad x = 0 \quad \text{or} \quad x = h, \quad (4.3) \]

see [31, Subsection 8.1]. To obtain the analytical solution of the above boundary value problem, the key step is rewriting (4.1) and (4.2) into an operator equation, see [33]. Let

\[ u_1 = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}, \quad u_2 = \frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3}, \quad u_3 = \frac{\partial^3 w}{\partial x^2 \partial y^2} + \frac{\partial^3 w}{\partial x \partial y^3}, \quad u_4 = - \frac{\partial^3 w}{\partial x^2 \partial y} - \frac{\partial^3 w}{\partial y^3}. \]

Then the boundary value problem (4.1) with (4.2) becomes [27, Example 6.3.1]

\[ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\partial}{\partial y} & 1 & -\frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial y} & 1 \\ 0 & 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & -\frac{\partial}{\partial y} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{f}{y} \end{bmatrix}, \quad (4.4) \]

\[ u_1 = u_3 = 0 \quad \text{for} \quad y = 0 \quad \text{or} \quad y = 1. \]

Next, we write (4.4) as an operator equation in a Hilbert space. Let

\[ D(T_0) := \{ g \in L^2(0, 1) \ | \ g \in AC[0, 1], \ g' \in L^2(0, 1), \ g(0) = g(1) = 0 \}, \quad T_0 g := g' \quad \text{for} \quad g \in D(T_0). \]

Then \( T_0 \) is a closed densely defined linear operator on the Hilbert space \( L^2(0, 1) \) and, furthermore, its adjoint operator is determined by [12, Example III.5.31]

\[ D(T_0^*) := \{ g \in L^2(0, 1) \ | \ g \in AC[0, 1], \ g' \in L^2(0, 1) \}, \quad T_0^* g := -g' \quad \text{for} \quad g \in D(T_0^*). \]

Let \( A := \begin{pmatrix} 0 & T_0 \\ T_0 & 0 \end{pmatrix} \). Then (4.4) becomes \( \dot{u} = Hu + f \), where

\[ H := \begin{pmatrix} A & A + 1 \\ 0 & -A \end{pmatrix} \quad (4.5) \]

is a Hamiltonian operator matrix on the Hilbert space \( (L^2(0, 1))^4 \) and \( u := (u_1 u_2 u_3 u_4)^t, \ f := (0 0 0 f)^t \), so that the spectral properties of the operator \( H \) are essential for us to get the analytical solution to the boundary value problem (4.1)–(4.3). In the proof of the following proposition, the essential spectrum of a closed operator \( T \) is defined as [25, Definition 2.1.9], \( \sigma_{\text{ess}}(T) := \{ \lambda \in \mathbb{C} \mid \text{(T - \lambda) is not Fredholm} \} \).

**Proposition 4.1.** For the Hamiltonian operator matrix \( H \) given by (4.5), we have:

1. \( H \) is symplectic self-adjoint.
2. \( \sigma(H) = \sigma_p(H) = \{ k\pi, k \in \mathbb{Z} \} \) consists of eigenvalues of finite algebraic multiplicity.
symplectic self-adjoint, so \(\sigma\) eigenvalues of finite algebraic multiplicity since \(\rho\).

It has been proved that the root vector system of \(\sigma\).

Now (4.7) and (4.9) imply that

\[
\sigma(H^2) = \sigma(A^2) = \sigma(T_0^*T_0) \cup \sigma(T_0T_0^*), \quad \sigma_{ess}(H^2) = \sigma_{ess}(A^2) = \sigma_{ess}(T_0^*T_0) \cup \sigma_{ess}(T_0T_0^*).
\]

From [12, Example V.3.25] we see that the self-adjoint operators \(T_0^*T_0, T_0T_0^*\) are determined by

\[
D(T_0^*T_0) := \{g \in L^2(0, 1) \mid g, g' \in AC[0, 1], g'' \in L^2(0, 1), g(0) = g(1) = 0\},
\]

\[
T_0^*T_0g := -g'' \quad \text{for} \quad g \in D(T_0^*T_0),
\]

\[
D(T_0T_0^*) := \{g \in L^2(0, 1) \mid g, g' \in AC[0, 1], g'' \in L^2(0, 1), g'(0) = g'(1) = 0\},
\]

\[
T_0T_0^*g := -g'' \quad \text{for} \quad g \in D(T_0T_0^*),
\]

and so \(\sigma(T_0^*T_0) = \{(n\pi)^2, n = 1, 2, 3, \ldots\}, \sigma(T_0T_0^*) = \{(n\pi)^2, n = 0, 1, 2, \ldots\}, \sigma_{ess}(T_0^*T_0) = \sigma_{ess}(T_0T_0^*) = \emptyset,\) so that

\[
\sigma(H^2) = \sigma(T_0^*T_0) = \{(n\pi)^2, n = 0, 1, 2, \ldots\},
\]

\[
\sigma_{ess}(H^2) = \emptyset.
\]

Moreover, \(H^2 + 1\) is bijective since \(H^2\) is a nonnegative self-adjoint operator, so it follows from \(H^2 + 1 = (H - i)(H + i) = (H + i)(H - i)\) that \((H - i)\) is bijective, and hence \(i \in \rho(H)\). Then, by [9, Theorem VII.9.10] and [7, Lemma 2], respectively,

\[
\sigma(H^2) = \{\lambda^2 \mid \lambda \in \sigma(H)\},
\]

\[
\sigma_{ess}(H^2) = \{\lambda^2 \mid \lambda \in \sigma_{ess}(H)\}.
\]

Now (4.7) and (4.9) imply that \(\sigma_{ess}(H) = \emptyset\) and, furthermore, \(\sigma(H)\) consists of countably many isolated eigenvalues of finite algebraic multiplicity since \(\rho(H) \neq \emptyset\) (see [10, Theorem XVII.2.1]). But \(H\) is symplectic self-adjoint, so \(\sigma_p(H) = \sigma_p(H) \cup \sigma_c(H)\) is symmetric with respect to the imaginary axis (see [1, Theorem 3.6]), and therefore \(\sigma(H) = \sigma_p(H) = \{k\pi, k \in \mathbb{Z}\}\) by (4.6) and (4.8).

**Remark 4.2.** The assertion that \(H\) is symplectic self-adjoint with \(\sigma_c(H) \cup \sigma_e(H) = \emptyset\) is new.

**Remark 4.3.** It has been proved that the root vector system of \(H\) forms a Schauder basis with parentheses for \((L^2(0, 1))^4\), so the analytical solution to (4.4) could be obtained by expansion of root vectors, see [27, Example 6.3.1] for details.

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A Appendix: Some lemmas on adjoints

Lemma A.1 (See [11, Corollary 1]). Let $T$ be a closed densely defined linear operator on a Hilbert space. Suppose $S$ is a $T$-bounded operator such that $S^*$ is $T^*$-bounded, with both relative bounds < 1. Then $S + T$ is closed and $(S + T)^* = S^* + T^*$.

Lemma A.2 (See [12, Problem III.5.26]). Let $S$ be a bounded everywhere defined operator and $T$ be a densely defined operator on a Hilbert space. Then $(ST)^* = T^*S^*$.

Lemma A.3 (See [4, Theorem 4.3]). Let $S$ and $T$ be densely defined operators on a Hilbert space. If $T$ is closed and $R(T)$ is closed and has finite codimension, then $ST$ is a densely defined operator and $(ST)^* = T^*S^*$. 