Geometric Aspects of Confining Strings

M. Cristina Diamantini
Dipartimento di Fisica, Università di Perugia
via A. Pascoli, I-06100 Perugia, Italy

Carlo A. Trugenberger
Département de Physique Théorique, Université de Genève
24, quai E. Ansermet, CH-1211 Genève 4, Switzerland

Confining strings in 4D are effective, thick strings describing the confinement phase of compact $U(1)$ and, possibly, also non-Abelian gauge fields. We show that these strings are dual to the gauge fields, inasmuch their perturbative regime corresponds to the strong coupling ($e \gg 1$) regime of the gauge theory. In this regime they describe smooth surfaces with long-range correlations and Hausdorff dimension two. For lower couplings $e$ and monopole fugacities $z$, a phase transition takes place, beyond which the smooth string picture is lost. On the critical line intrinsic distances on the surface diverge and correlators vanish, indicating that world-sheets become fractal.
1. Introduction

It is an old idea that the confining phase of gauge theories can be formulated as a string theory \cite{1}. The natural relevant term in the action of such a string is the Nambu-Goto term, proportional to the area of the world-sheet. However, this term does not lead to a consistent theory outside the critical dimension 26 \cite{2}.

The Nambu-Goto term describes fundamental strings, which do not have a transverse extension. On the other hand, one can convince oneself on general grounds \cite{3} that the strings describing electric flux tubes in QCD must be thick strings, with a fundamental transverse scale and therefore the theory of these objects is an effective string theory.

In order to take into account the bending rigidity due to the finite width and to cure the problems of the fundamental Nambu-Goto action, Polyakov \cite{4} and Kleinert \cite{5} proposed to add to it a marginal term proportional to the extrinsic curvature of the world-sheet. The so obtained rigid string, however, is plagued by various problems of both geometric and physical nature. From the geometric point of view, the new term turns out to be infrared irrelevant \cite{4,5} and violent fluctuations lead to the formation of a finite correlation length for the normals to the surface and to crumpling \cite{6,7}, which is unacceptable for QCD strings. From the physical point of view, the new term brings about an unphysical ghost pole in the propagator \cite{5} and the spectrum is non-unitary and unbounded by below \cite{8,1}. Moreover, the high-temperature free energy of rigid strings has the opposite sign to the result obtained from large-\(N\) QCD \cite{9}, although the \(\beta\)-dependence comes out correct.

Recently, Polyakov \cite{10} (see also \cite{11}) proposed a new action to describe the confining phase of gauge theories. This confining string theory can be explicitly derived \cite{12} for a 4D compact \(U(1)\) gauge theory in the phase with a condensate of magnetic monopoles \cite{13}. Polyakov \cite{10} conjectured moreover that the only modification for non-Abelian gauge fields should be the inclusion in the string action of a corresponding group factor. In 4D, the confining string is indeed an effective string with a microscopic length scale describing the thickness of the string. The major differences with respect to the rigid string are a non-local interaction between world-sheet elements and a negative stiffness \cite{12}.

A different, but essentially equivalent formulation of the confining string was considered by Kleinert and Chervyakov \cite{14}, who showed that the high-temperature free energy matches the large-\(N\) QCD result also in sign, and that no unphysical ghost pole is present. In \cite{15} we showed that world-sheets of confining strings are characterized by long-range correlations for the normals, due to a non-local “antiferromagnetic” interaction. Moreover,
it is easy to convince oneself that the spectrum of confining strings is perfectly bounded by below \[16\]. So, confining strings are very promising, given that they seem to solve all the problems of rigid strings.

In this paper we address the quantum phase structure and geometric aspects of confining strings. After a review of the confining string action in section 2, in section 3 we compute the one-loop correction \(t_1\) to the classical string tension \(t_0\) in the semiclassical expansion. This allows us to find the combinations of the two dimensionless parameters of the theory, the gauge coupling \(e\) and the monopole fugacity \(z\), for which \(|t_1/t_0| \ll 1\). We show that this perturbative regime corresponds to large coupling \(e\), thereby proving the duality between confining strings and gauge fields. By computing correlators for the normals and estimating the Hausdorff dimension we show that, in the perturbative regime, the world-surfaces are smooth objects with intrinsic dimension two.

In section 4 we describe the modifications induced by the presence of a \(\theta\)-term in the gauge theory; notably we show how duality is modified in this case.

In section 5 we use the formulation of Kleinert and Chervyakov to compute the quantum phase structure of the theory in the large-\(D\) expansion. We show that smooth strings exist only below a critical line in the \((z, 1/e)\) plane: this domain matches the perturbative region found with the one-loop calculation in section 3. The critical line is characterized by the divergence of intrinsic distances on the surface and the vanishing of normals correlators.

Finally, we draw our conclusions in section 6.

2. Confining strings

Confining strings have an action which is induced by a Kalb-Ramond antisymmetric tensor field \([17]\). In 4-dimensional Euclidean space it is given by

\[
\exp(-S_{CS}) = \frac{G}{Z(B_{\mu\nu})} \int D B_{\mu\nu} \exp \left\{ -S(B_{\mu\nu}) + i \int d^4 x \, B_{\mu\nu} T_{\mu\nu} \right\},
\]

\[
T_{\mu\nu} = \frac{1}{2} \int d^2 \sigma \, X_{\mu\nu}(\sigma) \, \delta^4(x - x(\sigma)),
\]

\[
X_{\mu\nu} = \epsilon^{ab} \frac{\partial x_\mu}{\partial \sigma^a} \frac{\partial x_\nu}{\partial \sigma^b},
\]

with \(x(\sigma)\) parametrizing the world-sheet and \(G\) the group factor characterizing the underlying gauge group. Given that \(G\) is of no importance for the following, we will henceforth...
set $G$ to its value $G = 1$, valid for a compact $U(1)$ group. At long distances the action for
the Kalb-Ramond field reduces to
\[ S(B_{\mu \nu}) = \int d^4x \frac{1}{12z^2\Lambda^2} H_{\mu \nu \alpha} H_{\mu \nu \alpha} + \frac{1}{4e^2} B_{\mu \nu} B_{\mu \nu}, \]
\[ H_{\mu \nu \alpha} \equiv \partial_{\mu} B_{\nu \alpha} + \partial_{\nu} B_{\alpha \mu} + \partial_{\alpha} B_{\mu \nu}. \]

It depends on a short-distance cutoff $\Lambda$ and on two dimensionless parameters $e$ and $z$. This
action can be explicitly derived \cite{10,12} from a lattice formulation of compact QED in the
phase with condensing magnetic monopoles \cite{13}, and constitutes a special case of a generic
mechanism of $p$-brane confinement proposed in \cite{11}.

In the lattice model $1/\Lambda$ plays the role of the lattice spacing while $z^2$ is the monopole
fugacity; $e$ is the coupling constant of the underlying gauge theory. Note that, for $z^2 \to 0$,
only configurations with $H_{\mu \nu \alpha} = 0$ contribute to the partition function: this means that
the Kalb-Ramond field becomes pure gauge, $B_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$, and we recover
the partition function of QED coupled to point-particles described by the boundaries of the
original world-sheet.

In the continuum formulation above, $\Lambda$ can be viewed as a Higgs mass and, correspon-
dingly, $1/\Lambda$ as a finite thickness of the string. The mass of the Kalb-Ramond field is
given by
\[ m = \Lambda \frac{z}{e}. \]

The dimensionless parameter
\[ \tau \equiv \frac{m}{\Lambda} = \frac{z}{e}, \]
plays thus the same role as the ratio (coherence length/penetration depth) in supercon-
ductivity theory. This close analogy with superconductivity is not surprising when one
realizes that the same action has been derived for magnetic vortices in the framework of
the Abelian Higgs model \cite{18}.

In \cite{15} we have shown that, up to boundary terms (which are of no consequence in
the present paper), the confining string action can be rewritten as
\[ S_{CS} = \Lambda^2 \int d^2\sigma \sqrt{g} \ t_{\mu \nu}(\sigma) \ G \left( z, e, \left( \frac{D}{\Lambda} \right)^2 \right) t_{\mu \nu}(\sigma), \]
where we have introduced the induced metric
\[ g_{ab} \equiv \partial_{a} x_{\mu} \partial_{b} x_{\mu}, \]
\[ g \equiv \det g_{ab}, \]
the tangent tensor
\[ t_{\mu\nu} \equiv \frac{1}{\sqrt{2g}} X_{\mu\nu} , \]  \hspace{1cm} (2.7)

and the covariant Laplacian
\[ D^2 = \frac{1}{\sqrt{g}} \partial_a g^{ab} \sqrt{g} \partial_b . \]  \hspace{1cm} (2.8)

The Green's function \( G \) is defined as the Taylor series obtained from the generating function
\[ G \left( z, e, \left( \frac{D}{\Lambda} \right)^2 \right) = \frac{z^2}{4\pi} K_0 \left( \sqrt{\tau^2 - \left( \frac{D}{\Lambda} \right)^2} \right) , \]  \hspace{1cm} (2.9)

with \( \tau \) defined in (2.4), and \( K \) a modified Bessel function \[19\]. Its first few terms are given by
\[ G \left( z, e, \left( \frac{D}{\Lambda} \right)^2 \right) = t_0 + s \left( \frac{D}{\Lambda} \right)^2 + w \left( \frac{D}{\Lambda} \right)^4 + \ldots , \]
\[ t_0 = \frac{z^2}{4\pi} K_0(\tau) , \]  \hspace{1cm} (2.10)
\[ s = \frac{z^2}{8\pi} \tau^{-1} K_1(\tau) , \]
\[ w = \frac{z^2}{32\pi} \tau^{-2} K_2(\tau) . \]

When compared with the corresponding kernel of rigid strings \[4,5\],
\[ G^{\text{rigid}} = \frac{\mu_0}{\Lambda^2} - \frac{1}{\alpha} \left( \frac{D}{\Lambda} \right)^2 , \]  \hspace{1cm} (2.11)

(with \( \mu_0 \) the bare string tension) this expression exposes best the two crucial aspects of confining strings: a non-local interaction between surface elements and a negative stiffness \( s \). Contrary to the case of rigid strings, where the “local ferromagnetic” interaction (2.11) is not sufficient to prevent crumpling \[3,7\] the “non-local antiferromagnetic” interaction (2.10) does indeed lead to smooth strings \[15\].

3. Perturbative saddle point analysis

From (2.5), (2.9) and (2.10) one would naively conclude that for \( e \to 0 \) one can remove the cutoff \( \Lambda \), so that \( e = 0 \) is an infrared fixed point corresponding to the usual Nambu-Goto string. In the following we show that this is wrong, since the perturbative smooth
regime at strong coupling is separated from the naive Nambu-Goto regime $e \to 0$ by a phase transition.

To this end we shall use standard saddle-point techniques to investigate the role of transverse fluctuations $\chi^i(\sigma)$ around a long, straight string configuration parametrized in the Gauss map as

$$x_\mu(\sigma) = (\sigma_0, \sigma_1, \chi^i(\sigma)),$$  

where $-\beta/2 \leq \sigma_0 \leq \beta/2, -R/2 \leq \sigma^1 \leq R/2$.

This analysis is very simple for the confining string action. We start by integrating out the Kalb-Ramond field, which appears quadratically in the induced action (2.1). Up to (here irrelevant) boundary terms we obtain

$$S_{CS} = \frac{z^2 \Lambda^2}{4} \int d^2 \sigma \int d^2 \sigma' \ X_{\mu\nu}(\sigma) \ Y(\mathbf{x}(\sigma) - \mathbf{x}(\sigma')) \ X_{\mu\nu}(\sigma'),$$  

(3.2)

where $Y$ is the 4-dimensional Yukawa Green’s function

$$Y(|\mathbf{x}|) = \frac{m^2}{4\pi^2} \frac{1}{m|\mathbf{x}|} \ K_1(m|\mathbf{x}|),$$  

(3.3)

and $K$ is a modified Bessel function [19]. In order to regulate ultraviolet divergences we shall substitute this Green’s function with

$$Y'(|\mathbf{x}|) = \frac{m^2}{4\pi^2} \frac{1}{m \sqrt{|\mathbf{x}|^2 + 1/\Lambda^2}} \ K_1 \left(m \sqrt{|\mathbf{x}|^2 + 1/\Lambda^2}\right),$$  

(3.4)

so that the potential is cut off on the scale $1/\Lambda$ corresponding to the string thickness. Moreover, for simplicity, we shall use henceforth dimensionless variables by measuring all distances in units of the thickness $1/\Lambda$ and all momenta in units of $\Lambda$:

$$\xi^a \equiv \Lambda \sigma^a,$$

$$\mathbf{r}_\mu \equiv \Lambda \mathbf{x}_\mu,$$

$$r \equiv \Lambda |\mathbf{x}|,$$

$$\phi^i \equiv \Lambda \chi^i.$$  

(3.5)

In the Gauss map (3.1) the components of the tensor $X_{\mu\nu}$ in (2.1) take the following form:

$$X_{01} = -X_{10} = 1,$$

$$X_{0i} = -X_{i0} = \frac{\partial \phi^i}{\partial \xi^1},$$

$$X_{i1} = -X_{1i} = \frac{\partial \phi^i}{\partial \xi^0},$$

$$X_{ij} = -X_{ji} = O\left((\phi)^2\right).$$  

(3.6)
Therefore, retaining only quadratic terms in the action, we are lead to

\[ S_{CS} = S_0 + S_1 , \]  

(3.7)

where \( S_0 \) represents the classical part, which, in the infinite area \( (A = \beta R) \) limit, is given by

\[ S_0 = t_0 \Lambda^2 A . \]  

(3.8)

Here \( t_0 \), given in (2.10), is the classical contribution to the string tension (in dimensionless units). The contribution from transverse fluctuations is

\[ S_1 = \int d^2 \xi \int d^2 \xi' \phi^i(\xi) V(\xi - \xi') \phi^i(\xi') , \]

\[ V(\xi) = V_1(\xi) + V_2(\xi) - \delta^2(\xi) \int d^2 \xi' V_2(\xi') , \]

(3.9)

\[ V_1(\xi) = (-\nabla^2) \frac{z^2 \tau^2}{8\pi^2} \frac{1}{\tau \sqrt{|\xi|^2 + 1}} K_1 \left( \tau \sqrt{|\xi|^2 + 1} \right) , \]

\[ V_2(\xi) = \frac{z^2 \tau^2}{16\pi^2} \frac{1}{|\xi|^2 + 1} K_2 \left( \tau \sqrt{|\xi|^2 + 1} \right) . \]

The term \( V_1 \) arises from keeping linear terms in \( \phi^i \) in \( X_{\mu\nu} \) and setting \( \phi^i = 0 \) in the kernel \( Y \). The second term \( V_2 \), instead, originates from an expansion to second order in \( \phi^i \) of the kernel \( Y \) in (3.2) while keeping only the zeroth-order of \( X_{\mu\nu} \). The \( \delta \)-function subtraction from \( V_2 \) takes into account that \( Y(\mathbf{r}(\xi) - \mathbf{r}(\xi')) \) in (3.2) depends only on differences \( (\phi^i(\xi) - \phi^i(\xi')) \).

At this point we integrate over the two transverse fluctuations to obtain the effective action

\[ S_{CS}^{\text{eff}} = t \Lambda^2 A , \]  

(3.10)

with

\[ t = t_0 + t_1 , \]

\[ t_1 = \frac{1}{2\pi} \int_0^1 dp \ p \ln V(p) , \]  

(3.11)

the renormalized string tension (in dimensionless units). Note that \( p \) is the momentum in units of the short-distance cutoff \( \Lambda \): this is why the integral is cut off at one. The Fourier transform

\[ V(p) = \int d^2 \xi \ V(\xi) \ e^{ip\xi} , \]  

(3.12)
of the fluctuation kernel (3.9) can be computed analytically:

\[ V(p) = V_1(p) + V_2(p), \]

\[ V_1(p) = \frac{z^2}{4\pi} p^2 K_0 \left( \sqrt{\tau^2 + p^2} \right), \]

\[ V_2(p) = \frac{z^2}{8\pi} \left\{ \sqrt{\tau^2 + p^2} K_1 \left( \sqrt{\tau^2 + p^2} \right) - \tau K_1(\tau) \right\}. \]

Note that \( V(p) \propto p^2 \) for \( p \ll 1 \); this shows that, at large distances, the transverse fluctuations are described by 2 free bosons, as expected.

Naturally, the expression (3.11) for the renormalized string tension makes sense only in a perturbative region of parameter space where the ratio

\[ r(z, e) \equiv \left| \frac{t_1(z, e)}{t_0(z, e)} \right| \ll 1, \]

even more so, given that \( t_1 \) is negative for most parameters and it is easy to get a negative overall string tension. To give an idea of the perturbative region of parameter space we choose an arbitrary cutoff at 20% and we plot in fig. 1 the region \( r < 0.2 \) as a function of \( z \) and \( 1/e \).

![Fig. 1: the curve defining the upper boundary of the perturbative region.](image)

We see that the perturbative regime is characterized by large couplings \( e \) and large monopole fugacities \( z \). This is a first important result: confining strings are indeed dual to
compact $U(1)$ gauge fields in the sense that the perturbative regime for the string corresponds to the strong coupling regime for the gauge theory. Moreover, one cannot take the limit $e \to 0$ to remove the cutoff and obtain the Nambu-Goto string. In doing so the renormalized string tension decreases, as shown in fig. 2, until one reaches the non-perturbative region where wild transverse fluctuations destroy the string and a phase transition takes place (see section 5).

![Graph](image)

Fig. 2: the string tension $t$ (dimensionless units) for $z = 10$.

In the following we are going to investigate some geometric properties of the confining string in the perturbative region derived above. First of all we compute the correlation function

$$g_{ab}(\xi - \xi') \equiv \langle \partial_a \phi^i(\xi) \ \partial_b \phi^i(\xi') \rangle ,$$

for the scalar product of the components of the tangent vectors normal to the reference plane $(\xi^0, \xi^1)$ at different points on the surface. This correlation provides a picture of the role of transverse fluctuations as a function of the parameters $z$ and $e$. It is obtained from (3.9) and (3.13) as

$$g_{ab}(\xi - \xi') = \delta_{ab} \frac{1}{(2\pi)^2} \int d^2 p \frac{p^2}{2V(p)} e^{ip(\xi - \xi')} .$$
The requirement on \( g_{ab}(\xi) \) is that its inverse Fourier transform reproduces the correct behaviour

\[
\frac{V(p)}{p^2} = \frac{3}{4} t_0 - \frac{7}{8} s p^2 + O(p^4) ,
\]  

(3.17)

with \( t_0 \) and \( s \) given in (2.10), in the region of small \( p \), where the fluctuations reduce to free bosons.

Using only the expansion (3.17) the correlation function (3.15) can be computed independently of the ultraviolet details of \( V(p) \) in the approximation that higher-order terms provide only a regulator \( 1/R \) for the pole at \( p = \sqrt{6t_0/7s} \) in \( p^2/V(P) \):

\[
g_{ab}(\xi - \xi') \simeq \delta_{ab} \frac{1}{7s} \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\frac{6t_0}{7s}}} \sin \left( \sqrt{\frac{6t_0}{7s}} |\xi - \xi'| - \frac{\pi}{4} \right) .
\]  

(3.18)

To check the correctness of this result it is sufficient to compute backwards its Fourier transform by first noting that (3.18) represents the asymptotic behaviour of the von Neumann function \( 1/(7s) N_0 \left( \sqrt{6t_0/7s} |\xi - \xi'| \right) \). The two-dimensional Fourier transform [19] of this function,

\[
\int d^2 \xi \frac{1}{7s} N_0 \left( \sqrt{\frac{6t_0}{7s}} |\xi| \right) e^{-ip\xi} = \frac{1}{2} \frac{1}{\sqrt{\frac{6t_0}{7s}}} \frac{1}{\frac{3}{4} t_0 - \frac{7}{8} s p^2} , \quad 0 < p < \sqrt{\frac{6t_0}{7s}} ,
\]  

(3.19)

reproduces exactly the small-\( p \) behaviour of the momentum-space correlator (3.17).

The correlation function (3.18) is long-range in the usual sense that \( \int d^2 \xi \ g_{aa}(\xi) \) is infrared divergent. Strictly speaking, the large infrared cutoff \( R \) should be incorporated in the correlation function (3.18). This can be removed to infinity under the integral (3.19) for all \( p \) but \( p = 0 \). So, there is no finite correlation length and \( \int d^2 \xi \ g_{aa}(\xi) \) diverges like \( \sqrt{RA} \), a situation analogous to the “Kosterlitz-Thouless order” in the low-temperature phase of the \( O(2) \) non-linear sigma model [20].

Given that the vectors \( \partial_a \phi^i \) describe how the surface is growing in and out of the \( (\xi^0, \xi^1) \) plane, the oscillatory behaviour of (3.18) indicates that the surface fluctuates around the reference plane with a wavelength

\[
\ell(\tau) = 2\pi \sqrt{\frac{7s(\tau)}{6t_0(\tau)}} = \sqrt{\frac{7\pi^2}{3} \frac{K_1(\tau)}{\tau K_0(\tau)}} .
\]  

(3.20)

The scale of the amplitude, instead, is set by the parameter

\[
a(z, \tau) = \frac{\sqrt{\ell(\tau)}}{7\pi s(z, \tau)} \propto \frac{1}{z^2} \frac{\tau^{\frac{3}{4}}}{K_1(\tau)^{\frac{3}{4}} K_0(\tau)^{\frac{3}{4}}} .
\]  

(3.21)
Given the behaviour
\[
\begin{align*}
\tau \ll 1 & : \quad a \ll 1 , \\
\tau \gg 1 & : \quad a \gg 1 ,
\end{align*}
\] (3.22)
combined with the fact that \( a \propto 1/z^2 \), we obtain a smooth surface, with waves of small amplitude, for small \( \tau \) and large \( z \): this corresponds to the perturbative domain derived above.

In order to confirm further this result we estimate the Hausdorff dimension of the surface as a function of the parameters \( z \) and \( e \). To this end we follow [7] and compute the ratio \( h(z, e, D) \) between the expectation value of the (squared) distance \( D_E^2 \) of two points on the surface in embedding space and its projection \( D^2 \) on the reference plane \((\xi^0, \xi^1)\).

The (squared) distance in embedding space is the sum of the projection \( D^2 \) on the reference plane and the contribution from normal fluctuations:
\[
\begin{align*}
D_E^2 &= D^2 + D_{\text{perp}}^2 , \\
D^2 &= |\xi - \xi'|^2 , \\
D_{\text{perp}}^2 &= \sum_i \langle |\phi^i(\xi) - \phi^i(\xi')|^2 \rangle ,
\end{align*}
\] (3.23)
so that the ratio \( h \) can be written as
\[
h (z, e, D) = 1 + \frac{D_{\text{perp}}^2 (z, e, D)}{D^2} .
\] (3.24)

The expectation value of the normal fluctuations can be easily computed from (3.9) and (3.13) thereby obtaining
\[
\begin{align*}
h (z, e, D) &= 1 + \frac{1}{\pi D^2} \int_0^1 dp \ (1 - J_0 (pD)) \ \frac{p}{V(p)} .
\end{align*}
\] (3.25)

In fig. 3 we plot the function \( h (z, e, D) \) as a function of \( D \) for \( z = 10 \) and \( 1/e = 0.22 \) and \( 1/e = 0.5 \).
Fig. 3: the function $h$ for $1/e = 0.22$ (lower curve) and $1/e = 0.5$ (upper curve).

The value $1/e = 0.22$ corresponds to the boundary in fig.1, on which the perturbation parameter $r(10, 1/(0.22)) = 0.2$. We see that, at this value of $1/e$, the distance in embedding space still scales with the projected distance on the reference plane, indicating that the Hausdorff dimension of the surface is indeed two. At even lower values of $1/e$, corresponding to the interior of the perturbative region the curve $h(z, e, D)$ is virtually indistinguishable from the constant 1. For purpose of illustration we have also plotted the function $h$ for $z = 10$ and $1/e = 0.5$, to investigate what happens if $1/e$ is increased towards the non-perturbative region. The now strong decrease of $h$ as a function of $D$ shows that the (squared) distance in embedding space begins to scale slower than $D^2$ starting on small to intermediate scales. This indicates that on these scales the surface is loosing its intrinsic dimensionality two, thereby acquiring a higher Hausdorff dimension. Note that the same picture applies to the other values of $z$: for $e$ large enough we have scaling and Hausdorff dimension two; when $e$ is lowered the surface begins to crease on small scales. We conclude thus that, in the perturbative regime, confining strings describe indeed smooth surfaces with intrinsic dimension two.
4. Including a $\theta$-term

When the underlying gauge theory contains a $\theta$-term

$$S^\theta_{\text{gauge}} = \int d^4x \frac{\theta}{64\pi^2} F_{\mu\nu} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta},$$

(4.1)

(in Euclidean space) the Kalb-Ramond action (2.2) is modified as follows [12]:

$$S(B_{\mu\nu}) \rightarrow S(B_{\mu\nu}) + \int d^4x \frac{\theta}{64\pi^2} B_{\mu\nu} \epsilon_{\mu\nu\alpha\beta} B_{\alpha\beta}.$$ 

(4.2)

Note that, for compact U(1) gauge fields, a $\theta$-term is not irrelevant since it produces non-trivial effects, notably it assigns an electric charge $q = e\theta/2\pi$ to elementary magnetic monopoles [21], and these are responsible for the confining string.

This new term in the Kalb-Ramond action (2.2) has two consequences for the non-local formulation of the confining string action (3.2). First of all, there is an additional term,

$$S_{CS} \rightarrow S_{CS} + i \frac{z^2 \Lambda^2 e^2 \theta}{64\pi^2} \int d^2 \sigma \int d^2 \sigma' \epsilon_{\mu\nu\alpha\beta} X_{\mu\nu}(\sigma) Y(x(\sigma) - x(\sigma')) X_{\alpha\beta}(\sigma'),$$

(4.3)

and, secondly, the mass (2.3) characterizing the Yukawa Green’s function $Y$ is modified to

$$\frac{m_\theta}{\Lambda} = \tau_\theta = \frac{ez}{4\pi} \sqrt{\left(\frac{4\pi}{e^2}\right)^2 + t^2},$$

(4.4)

However, it is easy to convince oneself that the additional term in the action does not contribute at all to the saddle-point expansion (3.7) to second order in the transverse fluctuations. Therefore, the only consequence of the inclusion of a $\theta$-term is the modification (4.4) of the parameter $\tau$.

When $e \ll 1$, the new parameter $t$ is negligible, we regain the original expression $\tau_\theta \simeq z/e$ and we end up in the non-perturbative regime. When $e \gg 1$, $t$ plays a crucial role and we obtain $\tau_\theta \simeq etz/4\pi$. This is the same expression as before with $e \rightarrow 4\pi/et$: since now $e \gg 1$, however, we end up again in the non-perturbative regime. We conclude that, in presence of a $\theta$-term, the smooth string can exist only for intermediate values of $e$. It is worth noting, that these intermediate values of $e$, for which $\tau$ is sufficiently small, constitute exactly the non-perturbative regime of the gauge theory with $\theta$-term, where dyons can condense [22]. Again, we recover the duality between gauge fields and confining strings.

Note also that the restriction to intermediate values of $e$ prevents again removing the cutoff and obtaining the Nambu-Goto string with self-intersection term. This is entirely due to the one-loop corrections, which were not included in [12].
5. Large-$D$ analysis

As we have pointed out in section 2, the two essential characteristics of the confining string action are that it represents a non-local interaction with negative stiffness between surface elements, as best seen in the formulation (2.5). The range of the interaction is determined by the parameter $\tau = z/e$ whereas the overall scale of the interaction depends only on $z$.

The action (2.5), however, has not the best form for the type of analysis we have in mind. Therefore, we shall consider a better-suited action which still incorporates all essential aspects of confining strings. In dimensionless units this is:

$$S = \int d^2 \xi \sqrt{g} \, g^{ab} D_a r^\mu \, W \left( z, e, \left( \frac{D}{\Lambda} \right)^2 \right) D_b r^\mu , \quad (5.1)$$

where $D_a$ denote covariant derivatives along the surface. In this formulation the non-local interaction is written in terms of the tangent vectors $\partial_a r^\mu$ to the surface, instead of the tangent tensor $t^\mu_\nu$, as in (2.5). While the physics of (5.1) is essentially equivalent to (2.5), we do not know which interaction $W$ in (5.1) corresponds exactly to $G$ in (2.5). An exact translation is possible, however, if we are interested only in the first two terms of $W$:

$$W \left( z, e, \left( \frac{D}{\Lambda} \right)^2 \right) = t_0^2 + s \left( \frac{D}{\Lambda} \right)^2 + v \left( \frac{D}{\Lambda} \right)^4 + \ldots , \quad (5.2)$$

with $t_0$ and $s$ given in (2.10) and $(v/s) \ll 1$, $(v/t_0) \ll 1$ for $\tau \gg 1$. The formulation (5.1) is the one used by Kleinert and Chervyakov \[14\] in their computation of the finite temperature free energy.

At this point we use standard large-$D$ techniques along the lines of \[6,7\] . We first introduce a (dimensionless) Lagrange multiplier matrix $\lambda^{ab}$ to enforce the constraint $g_{ab} = \partial_a r^\mu \partial_b r^\mu$,

$$S \rightarrow S + \int d^2 \xi \sqrt{g} \, \lambda^{ab} (\partial_a r^\mu \partial_b r^\mu - g_{ab}) . \quad (5.3)$$

We then parametrize the world-sheet in the Gauss map as

$$r^\mu (\xi) = (\xi_0, \xi_1, \phi^i(\xi)) , \quad i = 2, \ldots, D , \quad (5.4)$$

where $-\Lambda \beta/2 \leq \xi_0 \leq \Lambda \beta/2$, $-\Lambda R/2 \leq \xi^1 \leq \Lambda R/2$ and $\phi^i(\xi)$ describe the $(D-2)$ transverse fluctuations. With the usual isotropy Ansatz

$$g_{ab} = \rho \, \delta_{ab} , \quad \lambda^{ab} = \lambda \, g^{ab} , \quad (5.5)$$
for the metric and the Lagrange multiplier of infinite systems \((\beta, R \to \infty)\) at the saddle point we obtain

\[
S = 2 \int d^2 \xi \left( \frac{t_0}{2} + \lambda (1 - \rho) \right) + \int d^2 \xi \partial_o \phi^i \left( \lambda + W \left( z, e, \left( \frac{D}{\Lambda} \right)^2 \right) \right) \partial_o \phi^i . \tag{5.6}
\]

Integrating over the transverse fluctuations, in the infinite area limit, we get the effective action

\[
S_{\text{eff}} = 2 \Lambda^2 A_{\text{ext}} \left( \frac{t_0}{2} + \lambda (1 - \rho) \right) + \Lambda^2 A_{\text{ext}} \frac{D - 2}{8 \pi^2} \rho \int d^2 p \ln \left\{ p^2 \left( \lambda + W (z, e, p^2) \right) \right\} , \tag{5.7}
\]

where \(A_{\text{ext}} = \beta R\) is the extrinsic, physical area. For large \(D\), the fluctuations of \(\lambda\) and \(\rho\) are suppressed and these variables take their classical values, determined by the two saddle-point equations

\[
\begin{align*}
\lambda &= \frac{D - 2}{8 \pi} \int_0^1 dp \ p \ln \left\{ p^2 \left( \lambda + W (z, e, p^2) \right) \right\} , \\
\frac{\rho - 1}{\rho} &= \frac{D - 2}{8 \pi} \int_0^1 dp \ p \frac{1}{\lambda + W (z, e, p^2)} ,
\end{align*}
\tag{5.8}
\]

where we have introduced again the ultraviolet regularization \(p < 1\). Inserting the first saddle-point equation into (5.7) we get

\[
S_{\text{eff}} = \Lambda^2 \left( t_0 + 2 \lambda \right) A_{\text{ext}} , \tag{5.9}
\]

from where we read off the renormalized string tension

\[
T = \Lambda^2 \ t ,
\]

\[
t \equiv t_0 + 2 \lambda . \tag{5.10}
\]

The physics of confining strings in the large-\(D\) limit is determined thus by the two saddle-point equations (5.8). The first of these equations requires the vanishing of the “saddle-function”

\[
f(z, e, \lambda) \equiv \lambda - \frac{D - 2}{8 \pi} \int_0^1 dp \ p \ln \left\{ p^2 \left( \lambda + W (z, e, p^2) \right) \right\} , \tag{5.11}
\]

and determines the Lagrange multiplier \(\lambda\) as the solution of a transcendental equation. The second saddle-point equation determines then the metric, once the Lagrange multiplier has been found, and can be written simply as

\[
\rho = \frac{1}{f'(z, e, \lambda)} , \tag{5.12}
\]
where a prime denotes the derivative with respect to $\lambda$.

We start our analysis of the saddle-point equations by examining the weak coupling case $e \ll z$ for fixed $z = 0(1)$. In this case we have $\tau \gg 1$ and we can expand the potential $V$ in (5.1) as in (5.2). Keeping only the first, dominant term gives

$$f(z, e, \lambda) = \lambda - \frac{D - 2}{16\pi} \left\{ \ln \left( \frac{\lambda + t_0}{2} \right) - 1 \right\}. \quad (5.13)$$

This function has a global minimum at

$$\lambda^* = -\frac{t_0}{2} + \frac{D - 2}{16\pi}, \quad (5.14)$$

at which the function takes the value

$$f(z, e, \lambda^*) = -\frac{t_0}{2} + \frac{D - 2}{16\pi} \left( 2 - \ln \frac{D - 2}{16\pi} \right). \quad (5.15)$$

For $\tau$ sufficiently large this expression is positive for $D = 4$. This shows, independently of the details of the interaction $W$, that for sufficiently weak coupling $e \ll 1$ (and fixed $z$) there is no solution to the saddle-point equations. The dominant large-$D$ approximation admits a smooth confining string only for large coupling $e$, in full agreement with the results of the perturbative analysis of section 3.

Things become harder in the case $e \gg 1$. In order to shed light on the complete quantum phase structure of confining strings we shall resort to a toy model, by choosing the specific interaction

$$W \left( z, e, \left( \frac{D}{\Lambda} \right)^2 \right) = \tilde{W} \left( z, e, \left( \frac{D}{\Lambda} \right)^2 \right) = \frac{z^2}{\tau^2 - \left( \frac{D}{\Lambda} \right)^2}, \quad (5.16)$$

which is essentially the model of Kleinert and Chervyakov [14]. As in (2.5) and (2.9) we have two mass scales in addition to the ultraviolet cutoff $\Lambda$. The mass $z\Lambda$ determines the overall scale of the interaction between surface elements, whereas the mass $z\Lambda/e$ determines the range of this interaction. In this case we get

$$f(z, e, \lambda) = \lambda - \frac{D - 2}{16\pi} \left\{ -1 - \tau^2 \ln \left( 1 + \frac{1}{\tau^2} \right) - \frac{\tau^2 \lambda + z^2}{\lambda} \ln \left( \frac{\tau^2 \lambda + z^2}{1 + \tau^2} \right) \right\}$$

$$- \frac{D - 2}{16\pi} \left\{ \frac{\tau^2 \lambda + z^2 + \lambda}{\lambda} \ln \left( \lambda + \frac{z^2}{1 + \tau^2} \right) \right\}. \quad (5.17)$$
This function has the following limiting values:

\[ \lim_{\lambda \to \infty} f(z, e, \lambda) = \infty, \]

\[ \lim_{\lambda \to \lambda_{\min}} = -z^2 - \frac{D - 2}{16\pi} \left( -1 + \ln z^2 \right) + O(\tau^2 \ln \tau^2), \quad \tau \ll 1, \quad (5.18) \]

\[ \lambda_{\min} = -\frac{z^2}{1 + \tau^2}. \]

For \( z \) sufficiently large we have \( \lim_{\lambda \to \lambda_{\min}} < 0 \) and there exists at least one solution to the saddle-point equation \( f(z, e, \lambda) = 0. \)

The derivative of the saddle-function \( f \),

\[ f'(z, e, \lambda) = 1 - \frac{D - 2}{16\pi} \left\{ \frac{1}{\lambda} - \frac{z^2}{\lambda^2} \left( \ln \left( 1 + \frac{1}{\tau^2} \right) + \ln \frac{\lambda + \frac{z^2}{1 + \tau^2}}{\lambda + \frac{z^2}{\tau^2}} \right) \right\}, \quad (5.19) \]

determines the metric element \( \rho \) via (5.12). Given that

\[ \lim_{\lambda \to \infty} f'(z, e, \lambda) = 1, \]

\[ \lim_{\lambda \to \lambda_{\min}} f'(z, e, \lambda) = -\infty, \quad (5.20) \]

the saddle-function \( f \) must have an odd number of extrema. Our numerical analysis shows that it has exactly one minimum: when this minimum lies above zero, the saddle-point equations have no solutions. When the minimum lies below zero the saddle-point equation \( f(z, e, \lambda) = 0 \) has two solutions. Only the largest of these two solutions, however, is physical since at the smallest one we have \( f'(z, e, \lambda) = 1/\rho < 0 \). So, in this case we have exactly one physical solution of the saddle-point equations.

In fig. 4 we plot the critical line in parameter space below which there exists one solution to the saddle-point equations for \( D = 4. \)
As expected, the region bounded by the critical line matches essentially the perturbative region found with the one-loop calculation of section 3. Note, however, that, in our toy model, there is no phase transition for large $z$ and fixed (large) $e$. In this region the critical line $[1/e] (z)$ becomes a constant.

We now compute the correlation function

$$g_{ab}(\xi - \xi') = \left\langle \partial_a \phi^i(\xi) \partial_b \phi^i(\xi') \right\rangle = \delta_{ab} \frac{1}{(2\pi)^2} \int d^2p \frac{1}{2 \left( \lambda + W(z, e, p^2) \right)} e^{i\sqrt{\rho_p}(\xi - \xi')},$$

(5.21)

where $W(z, e, p^2)$ is the Fourier transform of the interaction in (5.1). We start by rewriting this correlation function as

$$g_{ab} = \delta_{ab} \frac{1}{(2\pi)^2} \int d^2p \left[ \frac{1}{2\lambda} + \frac{1}{\delta (\tilde{t} - 2\tilde{s}p^2)} \right] e^{i\sqrt{\rho_p}(\xi - \xi')},$$

(5.22)

where

$$\tilde{t} \equiv 2 \left( \lambda + \frac{z^2}{\tau^2} \right),$$

$$\tilde{s} \equiv \frac{z^2}{\tau^4},$$

(5.23)

are the (dimensionless) tension and stiffness, respectively and we have introduced the new parameter $\delta \equiv |\lambda|\tau^2/z^2$. Also, we have used the fact that the saddle-point solution for $\lambda$ is
negative. The first, constant term in (5.22) corresponds to a δ-function contribution which we shall drop from now on. The second term, instead, can be treated with a computation completely analogous to the one leading to (3.18):

\[ g_{ab}(\xi - \xi') \simeq \delta_{ab} \frac{1}{8\delta^2} \sqrt{\frac{2}{\pi \sqrt{\frac{\tau}{2\delta \bar{s}}} |\xi - \xi'|}} \sin \left( \sqrt{\frac{\tau}{2\delta \bar{s}}} |\xi - \xi'| - \frac{\pi}{4} \right), \tag{5.24} \]

where, as before, this form is valid up to a large infrared cutoff \( R \) such that \( 1/R \) regulates the pole in (5.22). Again we find long-range correlations for the normal components of tangent vectors to the world-sheet, indicating a smooth surface.

Let us now examine what happens when the critical line of fig. 4 is approached from below. In this case the minimum value of the saddle-function \( f(z, e, \lambda) \) approaches zero. As a consequence, the solution of the saddle-point equation \( f(z, e, \lambda) = 0 \) coincides with the value \( \lambda^* \) where the function \( f \) takes the minimum value. At this value \( \lambda^* \), however, we have also \( f'(z, e, \lambda^*) = 0 \), so that, due to (5.12) the metric element \( \rho \) diverges. We conclude that

\[ \lim_{(z,e) \to (z_c, e_c)} \rho = \infty. \tag{5.25} \]

This means that, approaching the critical line, the ratio \( A_{\text{int}}/A_{\text{ext}} \) of the intrinsic to the extrinsic (\( \beta R \)) area of the surface diverges. Moreover, since \( \rho \to \infty \), the correlations \( \langle \partial_a \phi^i(\xi) \partial_b \phi^i(\xi') \rangle \) vanish for all \( \xi \neq \xi' \).

We don’t know the exact nature of the phase transition occurring on the critical line depicted in fig. 4. The diverging intrinsic distances and the vanishing correlators, however, indicate that surely the surface looses its intrinsic dimensionality two and becomes at best a fractal object. Note, however, that it is not a crumpling transition: in a crumpled phase we should still have a solution to the saddle-point equations, even if the correlations (5.21) are short-range. It is not clear to us whether a generic solution of the saddle-point equations, not restricted by the ansatz (5.5), exists beyond the critical line. Such a solution would correspond to a fractal phase (possibly a branched-polymer phase [23]) of the string. The other possibility is that no solutions exist at all beyond the critical line in which case all kind of strings would simply be suppressed.
6. Conclusions

We have shown that confining strings have smooth world-sheets with long-range correlations in a perturbative region characterized by strong gauge coupling $e$ and large monopole fugacity $z$. Decreasing the coupling $e$ (at fixed fugacity $z$) or decreasing fugacity (at fixed coupling $e$) a phase transition takes place at which the world-sheets become at best fractal objects. Together with the facts that the confining string theory describes a compact $U(1)$ gauge theory at strong coupling [12] and that its finite temperature free energy matches the large-$N$ QCD result both in temperature dependence and sign [14], our results make confining strings indeed very good candidates to describe the strong coupling phase of gauge theories.
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