A GENERAL DISCRETE WIRTINGER INEQUALITY AND SPECTRA OF DISCRETE LAPLACIANS

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ABSTRACT. We prove an inequality that generalizes the Fan-Taussky-Todd discrete analog of the Wirtinger inequality. It is equivalent to an estimate on the spectral gap of a weighted discrete Laplacian on the circle. The proof uses a geometric construction related to the discrete isoperimetric problem on the surface of a cone.

In higher dimensions, the mixed volumes theory leads to similar results, which allows us to associate a discrete Laplace operator to every geodesic triangulation of the sphere and, by analogy, to every triangulated spherical cone-metric. For a cone-metric with positive singular curvatures, we conjecture an estimate on the spectral gap similar to the Lichnerowicz-Obata theorem.

1. Introduction

1.1. A general discrete Wirtinger inequality. The Wirtinger inequality for $2\pi$-periodic functions says

\[
\int_{\mathbb{S}^1} f \, dt = 0 \implies \int_{\mathbb{S}^1} (f')^2 \, dt \geq \int_{\mathbb{S}^1} f^2 \, dt
\]

The following elegant theorem from [5] can be viewed as its discrete analog.

**Theorem 1** (Fan-Taussky-Todd). For any $x_1, \ldots, x_n \in \mathbb{R}$ such that

\[
\sum_{i=1}^{n} x_i = 0
\]

the following inequality holds:

\[
\sum_{i=1}^{n} (x_i - x_{i+1})^2 \geq 4 \sin^2 \frac{\pi}{n} \sum_{i=1}^{n} x_i^2
\]

(here $x_{n+1} = x_1$). Equality holds if and only if there exist $a, b \in \mathbb{R}$ such that

\[
x_k = a \cos \frac{2\pi k}{n} + b \sin \frac{2\pi k}{n}
\]
In the same article [5], similar inequalities for sequences satisfying the boundary conditions \(x_0 = 0\) or \(x_0 = x_{n+1} = 0\) were proved. Several different proofs and generalizations followed, [13, 9, 11, 4, 1].

In the present article we prove the following generalization of Theorem 1.

**Theorem 2.** For any \(x_1, \ldots, x_n \in \mathbb{R}\) and \(\alpha_1, \ldots, \alpha_n \in (0, \pi)\) such that

\[
\sum_{i=1}^{n} \left( \tan \frac{\alpha_i}{2} + \tan \frac{\alpha_{i+1}}{2} \right) x_i = 0, \quad \sum_{i=1}^{n} \alpha_i \leq 2\pi
\]

the following inequality holds:

\[
\sum_{i=1}^{n} \frac{(x_i - x_{i+1})^2}{\sin \alpha_{i+1}} \geq \sum_{i=1}^{n} \left( \tan \frac{\alpha_i}{2} + \tan \frac{\alpha_{i+1}}{2} \right) x_i^2
\]

Equality holds if and only if \(\sum_{i=1}^{n} \alpha_i = 2\pi\) and there exist \(a, b \in \mathbb{R}\) such that

\[
x_k = a \cos \sum_{i=1}^{k} \alpha_i + b \sin \sum_{i=1}^{k} \alpha_i
\]

If \(\sum_{i=1}^{n} \alpha_i > 2\pi\), then the inequality (3) fails for certain values of \(x_i\).

Theorem 1 is a special case of Theorem 2 for \(\alpha_i = \frac{2\pi}{n}\).

We obtain Theorem 2 as a consequence of the following.

**Theorem 3.** Let \(\alpha_1, \ldots, \alpha_n \in (0, \pi)\). Then the circulant tridiagonal \(n \times n\) matrix

\[
M = \begin{pmatrix}
-\left(\cot \alpha_1 + \cot \alpha_2\right) & \frac{1}{\sin \alpha_2} & \cdots & \frac{1}{\sin \alpha_1} \\
\frac{1}{\sin \alpha_2} & -\left(\cot \alpha_2 + \cot \alpha_3\right) & \ddots & \vdots \\
\vdots & \ddots & \ddots & \frac{1}{\sin \alpha_n} \\
\frac{1}{\sin \alpha_1} & \cdots & \frac{1}{\sin \alpha_n} & -\left(\cot \alpha_n + \cot \alpha_1\right)
\end{pmatrix}
\]

has the signature

\[(2m - 1, 2, n - 2m - 1), \text{ if } \sum_{i=1}^{n} \alpha_i = 2m\pi, \ m \geq 1\]

\[(2m + 1, 0, n - 2m - 1), \text{ if } 2m\pi < \sum_{i=1}^{n} \alpha_i < 2(m + 1)\pi, \ m \geq 0\]

Here \((p, q, r)\) means \(p\) positive, \(q\) zero, and \(r\) negative eigenvalues.

The vector \(1 = (1, 1, \ldots, 1)\) is always a positive vector for the associated quadratic form:

\[
\langle M1, 1 \rangle > 0
\]

If \(\sum_{i=1}^{n} \alpha_i \equiv 0(\text{mod } 2\pi)\), then \(\ker M\) consists of all vectors of the form (4).

The relation between Theorems 2 and 3 is the same as between the Wirtinger inequality (1) and the spectral gap of the Laplacian on \(S^1\). Thus we can interpret the matrix \(M\) in Theorem 3 as (the weak form) of the operator \(\Delta + \text{id}\).
1.2. The discrete isoperimetric problem: a generalization of the L’Huilier theorem. About two hundred years ago L’Huilier proved that a circumscribed polygon has the greatest area among all polygons with the same side directions and the same perimeter. Theorems 2 and 3 are related to a certain generalization of the L’Huilier theorem. Again, this imitates the smooth case, as the Wirtinger inequality first appeared in [2] in connection with the isoperimetric problem in the plane.

Define the euclidean cone of angle \( \omega > 0 \) as the space \( C_\omega \) resulting from gluing isometrically the sides of an infinite angular region of size \( \omega \). (If \( \omega > 2\pi \), then paste together several smaller angles, or cut the infinite cyclic branched cover of \( \mathbb{R}^2 \).)

Figure 1. The discrete isoperimetric problem on a cone.

**Theorem 4.** If \( \omega \leq 2\pi \), then every polygon with the sides tangent to a circle centered at the apex of \( C_\omega \) encloses the largest area among all polygons that have the same side directions and the same perimeter.

- If \( \omega < 2\pi \), then the optimal polygon is unique.
- If \( \omega = 2\pi \), then the optimal polygon is unique up to translation.
- If \( \omega > 2\pi \), then the circumscribed polygon is not optimal.

1.3. The discrete Wirtinger inequality with Dirichlet boundary conditions. In a similar way we generalize another inequality from [5].

**Theorem 5** (Fan-Taussky-Todd). For any \( x_1, \ldots, x_n \in \mathbb{R} \) the following inequality holds:

\[
\sum_{i=0}^{n} (x_i - x_{i+1})^2 \geq 4 \sin^2 \frac{\pi}{2(n+1)} \sum_{i=0}^{n} x_i^2
\]

where \( x_0 = x_{n+1} = 0 \). Equality holds if and only if there is \( a \in \mathbb{R} \) such that

\[
x_k = a \sin \frac{k\pi}{n+1}
\]

**Theorem 6.** For any \( x_0, \ldots, x_{n+1} \in \mathbb{R} \) and \( \alpha_1, \ldots, \alpha_{n+1} \in (0, \pi) \) such that

\[
x_0 = x_{n+1} = 0, \quad \sum_{i=1}^{n} \alpha_i \leq \pi
\]
the following inequality holds:

\[
\sum_{i=0}^{n} \frac{(x_i - x_{i+1})^2}{\sin \alpha_{i+1}} \geq \sum_{i=1}^{n} \left( \tan \frac{\alpha_i}{2} + \tan \frac{\alpha_{i+1}}{2} \right) x_i^2
\]

Equality holds if and only if \(\sum_{i=1}^{n+1} \alpha_i = \pi\) and there is \(a \in \mathbb{R}\) such that

\[
x_k = a \sin \sum_{i=1}^{k} \alpha_i
\]

If \(\sum_{i=1}^{n+1} \alpha_i > \pi\), then the inequality (3) fails for certain values of \(x_i\).

If \(\alpha_i = \frac{\pi}{n+1}\) for all \(i\), then this becomes a Fan-Taussky-Todd inequality. Similarly to the above, the inequality follows from a theorem about the signature of a tridiagonal (this time non-circulant) matrix, see Section 4. It is related to a discrete version of the Dido isoperimetric problem.

1.4. Related work. Milovanović and Milovanović [9] studied the question of finding optimal constants \(A\) and \(B\) in the inequalities

\[
A \sum_{i=0}^{n} p_i x_i^2 \leq \sum_{i=0}^{n} r_i (x_i - x_{i+1})^2 \leq B \sum_{i=0}^{n} p_i x_i^2
\]

for given sequences \((p_i)\) and \((r_i)\). They dealt only with the Dirichlet boundary conditions \(x_0 = 0\) or \(x_0 = x_{n+1} = 0\), and the answer is rather implicit: \(A\) and \(B\) are the minimum and the maximum zeros of a recursively defined polynomial (the characteristic polynomial of the corresponding quadratic form).

There is a partial generalization of Theorems 2 and 3 to higher dimensions. Instead of the angles \(\alpha_1, \ldots, \alpha_n\), one fixes a geodesic Delaunay triangulation of \(S^{d-1}\), and the matrix \(M\) is defined as the Hessian of the volume of polytopes whose normal fan is the given triangulation. The signature of \(M\) follows from the Minkowski inequality for mixed volumes. A full generalization would deal with a Delaunay triangulated spherical cone-metric on \(S^{d-1}\) with positive singular curvatures, and would be a discrete analog of the Lichnerowicz theorem on the spectral gap for metrics with Ricci curvature bounded below. See [7] and Section 5 below for details.

The spectral gap of the Laplacian on “short circles” plays a crucial role in the rigidity theorems for hyperbolic cone-manifolds with positive singular curvatures [6, 8, 15] based on Cheeger’s extension of the Hodge theory to singular spaces [3]. As elementary as it is, Theorem 2 could provide a basis for spectral estimates for natural discrete Laplacians, and in particular an alternative approach to the rigidity of cone-manifolds.

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2. Wirtinger, Laplace, and isoperimetry in the smooth case

2.1. Wirtinger’s inequality and the spectral gap.

**Theorem 7** (Wirtinger’s lemma). Let $f: \mathbb{S}^1 \to \mathbb{R}$ be a $C^\infty$-function with zero average:

$$\int_{\mathbb{S}^1} f(t) \, dt = 0$$

Then

$$\int_{\mathbb{S}^1} f^2(t) \, dt \leq \int_{\mathbb{S}^1} (f')^2 \, dt$$

Equality holds if and only if

$$f(t) = a \cos t + b \sin t$$

for some $a, b \in \mathbb{R}$.

**Theorem 8** (Spectrum of the Laplacian). The spectrum of the Laplace operator

$$\Delta f = f''$$

for $f \in C^\infty(\mathbb{S}^1)$ is $\{-k^2 \mid k \in \mathbb{Z}\}$. The zero eigenvalue consists of the constant functions; the eigenvalue $-1$ is double, and the associated eigenspace consists of the functions of the form (6).

Theorem 7 is equivalent to the fact that the spectral gap of the Laplace operator equals 1. Indeed, the zero average condition can be rewritten as

$$\langle f, 1 \rangle_{L^2} = 0$$

that is $f$ is $L^2$-orthogonal to the kernel of the Laplacian. This implies

$$\int_{\mathbb{S}^1} (f')^2 \, dt = -\int_{\mathbb{S}^1} f'' \cdot f \, dt = -\langle \Delta f, f \rangle_{L^2} \geq \lambda_1 \| f \|^2$$

which is the Wirtinger inequality since $\lambda_1 = 1$. Equality holds only for the eigenfunctions of $\lambda_1$.

2.2. Wirtinger’s inequality and the isoperimetric problem. Blaschke used Wirtinger’s inequality in 1916 to prove Minkowski’s inequality in the plane, and by means of it the isoperimetric inequality [2, §23]. For historic references, see [10].

**Theorem 9** (Isoperimetric problem in the plane). Among all convex closed $C^2$-curves in the plane with the total length $2\pi$, the unit circle encloses the largest area.

Below is Blaschke-Wirtinger’s argument, with a shortcut avoiding the more general Minkowski inequality.

Let $\Gamma$ be a convex closed curve in $\mathbb{R}^2$. Define the **support function** of $\Gamma$ as

$$h: \mathbb{S}^1 \to \mathbb{R}, \quad h(t) = \max\{\langle x, t \rangle \mid x \in \Gamma\}$$
(Here $\mathbb{S}^1$ is viewed as the set of unit vectors in $\mathbb{R}^2$.) If $\Gamma$ is strictly convex and of class $C^2$, then the Gauss map $\Gamma \to \mathbb{S}^1$ is a diffeomorphism. The corresponding parametrization $\gamma: \mathbb{S}^1 \to \Gamma$ by its normal has the form $\gamma(t) = ht + \nabla h$

The perimeter of $\Gamma$ and the area of the enclosed region can be computed as

$$L(\Gamma) = \int_{\mathbb{S}^1} h \, dt, \quad A(\Gamma) = \frac{1}{2} \int_{\mathbb{S}^1} h(h + h'') \, dt = \frac{1}{2} \int_{\mathbb{S}^1} (h^2 - (h')^2) \, dt$$

Now assume $L(\Gamma) = 2\pi$ and put $f(t) = h(t) - 1$. We have

$$\int_{\mathbb{S}^1} f(t) \, dt = L(\Gamma) - 2\pi = 0$$

It follows that

$$\int_{\mathbb{S}^1} h^2(t) \, dt = \int_{\mathbb{S}^1} (1 + f(t))^2 \, dt = 2\pi + \int_{\mathbb{S}^1} f^2(t) \, dt$$

Hence

$$A(\Gamma) = \frac{1}{2} \int_{\mathbb{S}^1} (h^2(t) - (h'(t))^2) \, dt = 2\pi + \frac{1}{2} \int_{\mathbb{S}^1} (f^2(t) - (f'(t))^2) \, dt \geq 2\pi$$

by the Wirtinger inequality.

It is also possible to derive Wirtinger’s inequality from the isoperimetric one: start with a twice differentiable function $f$ and choose $\varepsilon > 0$ small enough so that $1 + \varepsilon f$ is the support function of a convex curve.

See [12] for the general theory of convex bodies, and [14] for a nice survey on the isoperimetry and Minkowski theory.

3. WIRTINGER, LAPLACE, AND ISOPERIMETRY IN THE DISCRETE CASE

Since we will use geometric objects in our proof of Theorem 3, let us start with geometry.

3.1. The geometric setup. Take $n$ infinite angular regions $A_1, \ldots, A_n$ of angles $\alpha_1, \ldots, \alpha_n \in (0, \pi)$ respectively and glue them along their sides in this cyclic order. This results in a cone $C_\omega$ with $\omega = \sum_{i=1}^n \alpha_i$. Let $R_i$ be the ray separating $A_i$ from $A_{i+1}$, and let $\nu_i$ be the unit vector along $R_i$ pointing away from the apex. See Figure 2, left.

![Figure 2. The geometric setup for the isoperimetric problem on the cone.](image)
Develop the angle $A_i \cup A_{i+1}$ into the plane, choose $x_{i-1}, x_i, x_{i+1} \in \mathbb{R}$ and draw the lines

$$L_j = \{ p \in \mathbb{R}^2 \mid \langle p, \nu_i \rangle = x_j \}, j = i - 1, i, i + 1$$

Orient the line $L_i$ as pointing from $A_i$ into $A_{i+1}$ and denote by $\ell_i$ the signed length of the segment with the endpoints $L_i \cap L_{i-1}$ and $L_i \cap L_{i+1}$. A simple computation yields

$$\ell_i = \frac{x_{i-1} - x_i \cos \alpha_i}{\sin \alpha_i} + \frac{x_{i+1} - x_i \cos \alpha_{i+1}}{\sin \alpha_{i+1}} \tag{7}$$

This defines a linear operator $\ell : \mathbb{R}^n \to \mathbb{R}^n$. It turns out that $\ell(x) = Mx$, where $M$ is the matrix from Theorem 3.

3.2. Proof of the signature theorem.

**Lemma 3.1.** The corank of the matrix $M$ from Theorem 3 is as follows.

$$\dim \ker M = \begin{cases} 0, & \text{if } \sum_{i=1}^n \alpha_i \equiv 0 \pmod{2\pi} \\ 2, & \text{if } \sum_{i=1}^n \alpha_i \not\equiv 0 \pmod{2\pi} \end{cases}$$

**Proof.** We will show that the elements of $\ker M$ are in a one-to-one correspondence with parallel 1-forms on $C_\omega \setminus \{0\}$. If $\omega \not\equiv 0 \pmod{2\pi}$, then every parallel form vanishes. If $\omega \equiv 0 \pmod{2\pi}$, then all of them are pullbacks of parallel forms on $\mathbb{R}^2$ via the developing map, and thus $\ker M$ has dimension 2. This will imply the statement of the lemma.

With any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ associate a family of 1-forms $\xi_i \in \Omega^1(A_i)$ where each $\xi_i$ is parallel on $A_i$ and is determined by

$$\xi_i(\nu_{i-1}) = x_{i-1}, \quad \xi_i(\nu_i) = x_i$$

Here $\nu_i$ denotes, by abuse of notation, the extension of the vector $\nu_i$ to a parallel vector field on $A_i \cup A_{i+1}$. We claim that $x \in \ker M$ if and only if the form $\xi_i$ is parallel to $\xi_{i+1}$ for all $i$.

![Figure 3. Vectors $X_i$ and $X_{i+1}$ dual to the forms $\xi_i$ and $\xi_{i+1}$.](image)

To compare the forms $\xi_i$ and $\xi_{i+1}$, develop the angle $A_i \cup A_{i+1}$ on the plane. We have

$$\xi_i(v) = \langle X_i, v \rangle,$$
where $X_i \in \mathbb{R}^2$ is the vector whose projections to the rays $R_{i-1}$ and $R_i$ have lengths $x_{i-1}$ and $x_i$, respectively, see Figure 3. Thus $\xi_i$ is parallel to $\xi_{i+1}$ if and only if $X_i = X_{i+1}$. On the other hand, by Section 3.1 we have

$$\|X_{i+1} - X_i\| = |\ell_i(x)|$$

Hence 1-forms $\xi_i$ define a parallel form on $C_\omega \setminus \{0\}$ if and only if $Mx = 0$. □

Proof of Theorem 3. Put $\omega = \sum_{i=1}^n \alpha_i$ and define

$$\alpha_i(t) = (1 - t)\alpha_i + t\frac{\omega}{n}, \quad t \in [0, 1]$$

For all $t$ we have $\alpha_i(t) \in (0, \pi)$ and $\sum_{i=1}^n \alpha_i(t) = \omega$. Hence, by Lemma 3.1 the matrix $M_t$ constructed from the angles $\alpha_i(t)$ has a constant rank for all $t$. Therefore its signature does not depend on $t$. It remains to determine the signature of the matrix $M_1$. After scaling by a positive factor $M_1$ becomes

$$
\begin{pmatrix}
-2 \cos \frac{\omega}{n} & 1 & \ldots & 1 \\
1 & -2 \cos \frac{\omega}{n} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
1 & \ldots & 1 & -2 \cos \frac{\omega}{n}
\end{pmatrix}
$$

The eigenvalues of this matrix are

$$\left\{ 2 \cos \frac{2\pi k}{n} - 2 \cos \frac{\omega}{n} \bigg| k = 1, 2, \ldots, n \right\}$$

If $\omega = 2\pi m$, then exactly two of these eigenvalues are zero (the ones with $k = m$ and $k = n - m$). For $\omega > 2\pi m$ there are exactly $2m + 1$ positive eigenvalues. The theorem is proved. □

3.3. Proof of the general discrete Wirtinger inequality. Let us show that Theorem 3 implies Theorem 2.

The key point is that inequality (3) is equivalent to $\langle Mx, x \rangle \leq 0$ and that

$$\sum_{i=1}^n \left( \tan \frac{\alpha_i}{2} + \tan \frac{\alpha_{i+1}}{2} \right) x_i = \langle Mx, 1 \rangle$$

Assume first $\sum_{i=1}^n \alpha_i \leq 2\pi$. By Theorem 3 the quadratic form $M$ has positive index 1 and takes a positive value on the vector $1$. Hence it is negative semidefinite on the orthogonal complement to $1$:

$$\langle Mx, 1 \rangle = 1 \Rightarrow \langle Mx, x \rangle \leq 0$$

This proves the first statement of Theorem 2.

If $\sum_{i=1}^n \alpha_i < 2\pi$, then $M$ is negative definite on the complement to $1$, hence equality holds in (3) only for $x = 0$. If $\sum_{i=1}^n \alpha_i = 2\pi$, then equality holds only if $Mx = 0$ (all isotropic vectors of a semidefinite quadratic form lie in its kernel). We have $Mx = 0$ if and only if all vectors $X_i$ on Figure 3 are equal, that is iff $x_i = \langle X, \nu_i \rangle$ for some $X \in \mathbb{R}^2$. This proves the second statement of Theorem 2.
Finally, under the assumption \( \sum_{i=1}^{n} \alpha_i > 2\pi \) the quadratic form \( \langle Mx, x \rangle \) is indefinite on the orthogonal complement to \( 1 \), hence the inequality (3) fails for some \( x \).

### 3.4. Proof of the isoperimetric inequality.

First we have to define a convex polygon on \( C_\omega \) with given side directions. Let \( C_\omega \) be assembled from the angular regions \( A_i \) as in Section 3.1 and let \( x_1, \ldots, x_n > 0 \). Then we can draw the lines \( L_i \) as described in Section 3.1 directly on \( C_\omega \). If \( \omega < 2\pi \), then \( L_i-1 \) and \( L_i \) may intersect in more than one point, but their lifts to the universal branched cover have only one point in common. Denote the projection of this point to \( C_\omega \) by \( p_i \). We obtain a closed polygonal line \( p_1 \ldots p_n \) with sides lying on \( L_i \). If \( \ell_i(x) > 0 \), then we call this line a convex polygon on \( C_\omega \) with the exterior normals \( \nu_1, \ldots, \nu_n \) and support numbers \( x_1, \ldots, x_n \).

The polygon with the support numbers \( 1 \) is circumscribed about the unit circle centered at the apex.

**Proof of Theorem 4.** The perimeter and the area of a convex polygon with the support numbers \( h \) are computed as follows.

\[
L(h) = \sum_{i=1}^{n} \ell_i(h) = \langle Mh, 1 \rangle
\]

\[
A(h) = \frac{1}{2} \sum_{i=1}^{n} h_i \ell_i(h) = \frac{1}{2} \langle Mh, h \rangle
\]

It suffices to prove the theorem in the special case of a polygon circumscribed about the unit circle, that is we need to show

\[
L(h) = L(1) \Rightarrow A(h) \leq A(1)
\]

Put \( f = h - 1 \in \mathbb{R}^n \). Due to the assumption \( L(h) = L(1) \) we have

\[
\langle Mf, 1 \rangle = 0
\]

Hence by Theorem 3 we have \( \langle Mf, f \rangle \leq 0 \), so that

\[
A(h) = \frac{1}{2} \langle M(1 + f), 1 + f \rangle = \frac{1}{2} \langle M1, 1 \rangle + \frac{1}{2} \langle Mf, 1 \rangle + \frac{1}{2} \langle Mf, f \rangle = \frac{1}{2} \langle Mf, f \rangle \leq A(1)
\]

The statements on the uniqueness and optimality follow from the facts about the signature of \( M \) and the values of \( M \) on the vectors \( 1 \).

\( \square \)

### 4. The Wirtinger inequality with boundary conditions

For functions vanishing at the endpoints of an interval we have the following.
Theorem 10. Let $f: [0, \pi] \to \mathbb{R}$ be a $C^\infty$-function such that $f(0) = f(\pi) = 0$. Then
\[
\int_0^\pi f^2(t) \, dt \leq \int_0^\pi (f'(t))^2 \, dt
\]
Equality holds if and only if $x = a \sin t$.

There is an obvious relation to the Dirichlet spectrum of the Laplacian.

In a way similar to this and to the argument in Section 3.3, Theorem 6 is implied by the following.

Theorem 11. Let $\alpha_1, \ldots, \alpha_{n+1} \in (0, \pi)$. Then the tridiagonal $n \times n$ matrix
\[
M = \begin{pmatrix}
-(\cot \alpha_1 + \cot \alpha_2) & \frac{1}{\sin \alpha_2} & \cdots & 0 \\
\frac{1}{\sin \alpha_2} & -(\cot \alpha_2 + \cot \alpha_3) & \ddots & \vdots \\
\vdots & \ddots & \ddots & \frac{1}{\sin \alpha_n} \\
0 & \cdots & \frac{1}{\sin \alpha_n} & -(\cot \alpha_n + \cot \alpha_{n+1})
\end{pmatrix}
\]
has the signature
\[
(m - 1, 1, n - m), \text{ if } \sum_{i=1}^{n+1} \alpha_i = m\pi, m \geq 1
\]
\[
(m, 0, n - m), \text{ if } m\pi < \sum_{i=1}^{n+1} \alpha_i < (m + 1)\pi, m \geq 0
\]
Here $(p, q, r)$ means $p$ positive, $q$ zero, and $r$ negative eigenvalues.

If $\sum_{i=1}^{n+1} \alpha_i = m\pi$, then $\ker M$ consists of the vectors of the form
\[
x_k = a \sin \sum_{i=1}^k \alpha_i
\]

Proof. Similarly to Section 3.2, consider the angular region $A_\omega$ glued out of $n$ regions $A_i$ of the angles $\alpha_i$.

First show that $\dim \ker M = 1$ if $\sum_{i=1}^{n+1} \alpha_i = m\pi$ and $\dim \ker M = 0$ otherwise. For this, associate as in Section 3.2 with every element of the kernel a parallel 1-form $\xi$ on $A_\omega$ such that $\xi(\nu_0) = \xi(\nu_{n+1}) = 0$. Since the angle between $\nu_0$ and $\nu_{n+1}$ is $\omega$, such a form exists only if $\omega = m\pi$.

Then deform the angles $\alpha_i$, while keeping their sum fixed, to $\alpha_i = \frac{\omega}{n+1}$ and use the fact that the matrix
\[
\begin{pmatrix}
-2 \cos \frac{\omega}{n+1} & 1 & \cdots & 0 \\
1 & -2 \cos \frac{\omega}{n+1} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & 1 & -2 \cos \frac{\omega}{n+1}
\end{pmatrix}
\]
has the spectrum
\[
\left\{ 2 \cos \frac{\pi k}{n+1} - 2 \cos \frac{\omega}{n+1} \bigg| k = 1, 2, \ldots, n \right\}
\]
It follows that the signature of $M$ is as stated in the theorem. □

5. Higher Dimensions

5.1. The quermassintegrals.

**Definition 5.1.** The $i$-th quermassintegral $W_i(K)$ of a convex body $K \subset \mathbb{R}^n$ is the coefficient in the expansion

$$\text{vol}_n(K_t) = \sum_{i=0}^{n} t^i \binom{n}{i} W_i(K)$$

where $K_t = \{ x \in \mathbb{R}^n \mid \text{dist}(x, K) \leq t \}$ is the $t$-neighborhood of $K$.

In particular,

$$W_0(K) = \text{vol}_n(K), \quad W_1(K) = \frac{1}{n} \text{vol}_{n-1} \partial K, \quad W_n(K) = \text{vol}_n(B^n)$$

Also, $W_i$ is proportional to the mean volume of the projections of $K$ to $(n-i)$-dimensional subspaces, as well as to the integral of the $(i-1)$-st homogeneous polynomial in the principal curvatures (provided $\partial K$ is smooth):

$$W_i(K) = c_{n,i} \int_{\text{Gr}(n,n-i)} \text{vol}_{n-i}(\text{pr}_x(K)) \, d\xi = c'_{n,i} \int_{\partial K} \sigma_{i-1} \, dx$$

We will need the following expressions for $W_{n-1}$ and $W_{n-2}$ in terms of the support function $h: S^{n-1} \to \mathbb{R}$:

$$W_{n-1}(K) = \frac{1}{n} \int_{S^{n-1}} h \, d\nu, \quad W_{n-2}(K) = \frac{1}{n} \int_{S^{n-1}} h((n-1)h + \Delta h) \, d\nu$$

See e.g. [12] for a proof.

5.2. The smooth case. The following three theorems generalize those from Section 2. Again, Theorem 13 implies the other two, where for Theorem 14 one needs the formulas for $W_{n-1}$ and $W_{n-2}$ from the previous section.

**Theorem 12.** Let $f: S^{n-1} \to \mathbb{R}$ be a $C^1$-function with the zero average:

$$\int_{S^{n-1}} f(x) \, dx = 0$$

Then

$$\int_{S^{n-1}} f^2(x) \, dx \leq \frac{1}{n-1} \int_{S^{n-1}} \| \nabla f \|^2 \, dx$$

Equality holds if and only if $f$ is a spherical harmonic of order 1, that is a restriction to $S^{n-1}$ of a linear function on $\mathbb{R}^n$.

**Theorem 13.** The spectrum of the Laplace-Beltrami operator

$$\Delta f = \text{tr} \nabla^2 f$$

on the unit sphere $S^{n-1}$ is $\{-k(k+n-2) \mid k \in \mathbb{Z}\}$. The zero eigenspace consists of the constant functions; the eigenvalue $-(n-1)$ has multiplicity $n$, and the associated eigenspace consists of the restrictions of linear functions on $\mathbb{R}^n$. 

Theorem 14. Among all convex bodies in $\mathbb{R}^n$ with smooth boundary and with the average width 2, the unit ball has the largest average projection area to the 2-dimensional subspaces.

5.3. The discrete Laplacian and the discrete Lichnerowicz conjecture. The quermassintegrals are defined for all convex bodies, and in particular for convex polyhedra. For a convex polyhedron $P(h)$ with fixed outward unit facet normals $\nu_1, \ldots, \nu_n$ and varying support numbers $h_1, \ldots, h_n$, the quermassintegral $W_{n-2}(h)$ is a quadratic form in $h$, provided that the combinatorial type of $P(h)$ does not change.

Definition 5.2. Let $\nu_1, \ldots, \nu_n$ be in general position, so that all polyhedra $P(h)$ with $h$ close to 1 have the same combinatorics. Denote by $M = M(\nu)$ the symmetric $n \times n$-matrix such that

$$W_{n-2}(h) = \langle Mh, h \rangle$$

Due to the last formula from Section 5.1, the self-adjoint operator $M$ is the discrete analog of the operator $(n - 1)\text{id} + \Delta$.

Theorem 15. If $x \in \mathbb{R}^n$ is such that $\langle Mx, 1 \rangle = 0$, then $\langle Mx, x \rangle \leq 0$. Equality holds if and only if there is $X \in \mathbb{R}^n$ such that $x_i = \langle X, \nu_i \rangle$ for all $i$.

It is possible to define the matrix $M$ for any triangulation of $S^{n-1}$ whose simplices are equipped with a spherical metric.

Conjecture 5.3. If all cone angles in a spherical cone metric on $S^{n-1}$ are less that $2\pi$, and the triangulation is Delaunay, then

$$\langle Mx, 1 \rangle = 0 \Rightarrow \langle Mx, x \rangle \leq 0$$

This conjecture is true for $n = 3$. One applies the argument from the proof of Theorem 15 to show that $\langle Mx, x \rangle$ has largest possible rank. See [7] for details. This argument uses the negative semidefiniteness of the quadratic forms of the links of vertices (Theorem 2), thus the positive curvature condition is essential. Then one deforms the spherical cone metric on $S^2$ to the nonsingular spherical metric, keeping all curvatures non-negative. This last step seems difficult to perform in higher dimensions. Ideally one would like to derive a “discrete Weitzenboeck formula” in analogy to the proof of the Lichnerowicz theorem.

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