The Global Phase Space Structure
of the Wess-Zumino-Witten Model

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ABSTRACT

We present a new parameterisation of the space of solutions of the Wess-Zumino-Witten model on a cylinder, with target space a compact, connected Lie group $G$. Using the covariant canonical approach the phase space of the theory is shown to be the co-tangent bundle of the loop group of the Lie group $G$, in agreement with the result from the Hamiltonian approach. The Poisson brackets in this phase space are derived. Other formulations in the literature are shown to be obtained by locally-valid gauge-fixings in this phase space.
1. Introduction

It has been known for a number of years that the phase space of a classical system can be defined in two different ways. The first is the Hamiltonian definition of the phase space as the space of positions and momenta of the system; we will call this phase space $P_H$. In the second definition, the phase space of a classical system is defined as the space of solutions of its Lagrangian equations of motion (see Ref. [1] for discussions). This ‘covariant canonical’ definition of the phase space gives a space we will call $P_C$. Although it is expected that these two definitions for the phase space of a classical system give spaces that are locally equivalent, globally they may be quite different spaces. However, as we will see below in the case of the Wess-Zumino-Witten (WZW) model, these two phase spaces are the same.

The covariant definition of the phase space is particularly suited to theories for which the general solution of the classical equations of motion is known, and it has been applied extensively to two-dimensional conformal field theories. Much work has been done on the canonical structure of the (WZW) models recently [2-12], and in particular on the covariant approach to the phase space. This has included study of the symplectic structure of the WZW model, defined on the cylinder and with target space a compact, connected Lie group $G$. To construct the phase space $P_C$ of this WZW model, it is necessary to introduce a parameterisation of the space of solutions of the model. Then the phase space $P_C$ is defined as the space of independent parameters necessary to describe these classical solutions. In Ref. [2] the phase space of the theory was taken to be $LG \times LG$, where $LG$ is the loop group of the group $G$. In Ref. [10] the covariant approach led to the phase space $P_C$ being diffeomorphic to $LG \times LG \times G$. An important feature of this parameterisation was the presence of a symmetry in the space of solutions, generated by $G$. However, neither of these phase spaces are diffeomorphic to the phase space of the WZW model defined in the Hamiltonian approach (see Refs. [8, 13]). The latter, as we shall see, is the co-tangent bundle of the loop group $LG$, i.e. $P_H = T^*LG$.

In this paper, we will introduce a new parameterisation of the space of solutions
of the WZW model. The associated phase space $P_C$ is diffeomorphic to $\frac{LG \times LG \times A}{LG}$ where $A$ is the space of connections over the circle. In this parameterisation the solution space symmetry generated by the group $LG$ can be fixed completely; additionally the corresponding Poisson brackets of the WZW model are easily derived. Furthermore, the resulting phase space $P_C$ is diffeomorphic to the phase space $P_H$ given in the Hamiltonian treatment of this model. We will see how phase spaces studied in the literature can be recovered from our phase space by partially gauge fixing the loop group symmetry. However, such gauge-fixings are local and cannot be extended globally because of Gribov-Singer ambiguities. This is why the resulting phase spaces are not diffeomorphic to those obtained by a global gauge-fixing, and the resulting theories are in general different.

This paper has been organised as follows: In Section Two, we will discuss the example of a particle moving upon a group manifold. This proves to have many of the features of the WZW model, in a simpler form. Both covariant and Hamiltonian phase spaces are discussed. The Poisson brackets of the covariant phase space model are calculated by two different methods. In the first method we perform a gauge fixing of the symmetries of the space of solutions of the model and then calculate the Poisson brackets of the theory. In the second method we will enhance the phase space of the model in such a way that the symplectic form becomes invertible, and then we will impose a set of first class constraints. In Section Three, we present the corresponding discussion for the WZW model. We begin by presenting a new parameterisation of the solutions of the WZW model. The resulting phase space is shown to be the co-tangent bundle of the loop group, and we calculate the corresponding Poisson brackets, first by gauge fixing, and second by enhancing the phase space of the theory and imposing constraints, as in the particle model. Finally, in Section Four, we relate our formulation of the phase space of the WZW model to those appearing previously in the literature, and we discuss the global issues which are relevant to this relationship.
2. The Particle on a Group Manifold

The case of the particle moving upon a group manifold has been recently discussed by Alekseev and Faddeev [12]. We would like to rephrase and extend this discussion for our purposes here. It will turn out that this discussion is a close analogue of the corresponding WZW discussion. The action of this particle model is

\[ S = \frac{1}{2} \int dt \text{tr} (\dot{g} g^{-1})^2, \tag{1} \]

where \( \dot{g} = \frac{d}{dt} g \) and \( g \) is a map from the real line \( \mathbb{R} \) into \( G \). \( \text{Lie} G \) is the Lie algebra of the compact, connected Lie group \( G \), with a basis \( \{ t_a \} \) satisfying \([t_a, t_b] = f_{ab}^c t_c \) and \( \text{tr}(t_a t_b) = \delta_{ab} \) (\( a, b, = 1, 2, \ldots, \dim(\text{Lie} G) \)). We have introduced a matrix representation for the group \( G \) and the multiplications in Eqn. (1) are matrix multiplications. The equations of motion of the action (1) are

\[ \frac{d}{dt} (\dot{g} g^{-1}) = 0. \tag{2} \]

The Hamiltonian treatment of this model is straightforward. For convenience we introduce a local parameterisation \( X^i, i = 1, 2, \ldots, \dim G \), of the group \( G \). We have \( \dot{g} g^{-1} = R_t^a \dot{X}^i t_a \), with \( R_t^a \) the right frame of \( G \). Writing the action (1) in these coordinates gives

\[ S = \frac{1}{2} \int dt g_{ij} \dot{X}^i \dot{X}^j, \]

where \( g_{ij} = R_t^a R_j^b \delta_{ab} \) is the bi-invariant metric on the group manifold. Performing the usual Hamiltonian analysis, one finds that the phase space \( P_H \) is the co-tangent bundle \( T^* G \), with co-ordinates \( (X^i, P_j) \), where \( P_j \) is the momentum and \( X^i \) is the position of the particle. The symplectic form \( \omega \) is equal to \( \delta X^i \delta P_i \) and the canonical Hamiltonian is

\[ H = \frac{1}{2} g^{ij} P_i P_j. \]

The Poisson brackets are \( \{ X^i, P_j \} = \delta^i_j \) (note that this fixes our conventions for deriving the Poisson brackets from the symplectic form).

Now we consider the covariant approach to the definition of the phase space of this particle model. The equations of motion (2) are solved by

\[ g = u e^{t_a v}, \tag{3} \]
with $u, v$ time-independent group elements and $a$ a time-independent element of $(\text{Lie}G)^*$. The group action

$$u \rightarrow uh, \quad a \rightarrow h^{-1}ah, \quad v \rightarrow h^{-1}v,$$

with $h \in G$, leaves the solution (3) invariant.

The phase space $P_C$ of the theory is the space of elements $\{u, v, a\}$, modulo the group action of Eqn. (4). $P_C$ is thus the coset space $G \times G \times (\text{Lie}G)^*/G$, which is diffeomorphic to the co-tangent bundle $T^*G = G \times (\text{Lie}G)^*$ of the group $G$, in agreement with the Hamiltonian approach. The symplectic form of $P_C$ in this approach is

$$\omega = \delta X^i \delta (\partial S/\partial \dot{X}^i),$$

giving

$$\omega = \text{tr}\left(\delta g \delta (g^{-1}\dot{g}g^{-1})\right) = \text{tr}\left((g^{-1}\delta g)\frac{d}{dt}(g^{-1}\delta g) + (\delta g g^{-1})\frac{d}{dt}(\delta g g^{-1})\right).$$

(5)

$\omega$ is closed, $\delta \omega = 0$, and time independent, $\dot{\omega} = 0$ (we take $t = 0$ in the following). Now we write $\omega$ in terms of the phase space variables $\{u, v, a\}$, giving

$$\omega = \text{tr}\left(\left((u^{-1}\delta u)^2 + (u^{-1}\delta u) \delta a + (\delta vv^{-1})^2 + (\delta vv^{-1}) \delta a\right)\right).$$

(6)

The Hamiltonian in these coordinates is $H = \frac{1}{2}\text{tr}a^2$. The action (1) is invariant under the action $g \rightarrow lgr^{-1}$, $l, r \in G$, of the group $G$. In terms of the phase space variables $\{u, v, a\}$ the corresponding currents (charges) are $j_l = uau^{-1}$ and $j_r = -v^{-1}av$. These currents are the particle model analogues of the left and right currents of the WZW model.

The symplectic form (6) is degenerate along the directions of the group action (4), and hence is not invertible. To calculate the Poisson brackets of the theory, we may adopt one of two approaches. The first is to gauge fix the symmetry generated by the group action (4). In the second, we enhance the phase space and invert the symplectic form on the enhanced phase space. Then we impose a
first class constraint necessary for the system to have the right number of degrees of freedom. The constraint generates gauge transformations, and one considers gauge-invariant functions on the reduced phase space.

In the gauge-fixing approach, one can simply fix the symmetry (4) by using it to set \( u = e \), where \( e \) is the identity element of the group \( G \) (alternatively one could set \( v = e \), with equivalent results). The gauge choice \( u = e \) is a “good” gauge choice because \( G \) acts freely and transitively on the space of \( u \)’s, \( \{ u \} \). There are thus no residual symmetries. The symplectic form (6) in the gauge \( u = e \) is

\[
\omega = -\frac{1}{2} f_{ab}{}^c R_i^a R_j^b a_c \delta X^i \delta X^j + R_i^b \delta X^i \delta a_b, \tag{7}
\]

where \( X^i \) are the group manifold coordinates corresponding to \( v \). This symplectic form is the standard symplectic form on \( T^*G \) expressed in terms of the right trivialisation of \( T^*G \) and is the same as the symplectic form of the phase space \( P_H \) in the Hamiltonian treatment of the model, with \( P_i = R_i^b a_b \). Similarly the symplectic form of Eqn. (6) in the gauge \( v = e \) is the standard symplectic form of \( T^*G \), this time expressed in terms of the left trivialisation of \( T^*G \). The form in Eqn. (7) is trivial to invert, giving the Poisson brackets

\[
\{ X^i, X^j \} = 0, \quad \{ X^i, a_b \} = R_i^b, \quad \{ a_a, a_b \} = f_{ab}{}^c a_c. \tag{8}
\]

From these brackets it is straightforward to deduce the Poisson brackets of functions of \( X \) and \( a \). For example, in this gauge the Poisson brackets of the currents \( j_l = a \), \( j_r = -v^{-1} a v \), give two commuting copies of the Lie algebra \( \text{Lie}G \), i.e. the Poisson bracket of each current with itself is the same as the Lie bracket of \( \text{Lie}G \) and \( \{ j_l, j_r \} = 0. \)

The other way to invert the symplectic form (4) is to extend the phase space of the particle model, and then to impose constraints. We define an extended model – the ‘lr’ model – to have a phase space \( P_{lr} \) parameterised by \( (u, v, a_l, a_r) \), with
\( u, v \in G \) and \( a_l, a_r \in (\text{Lie}G)^* \), i.e. \( P_{lr} = G \times G \times (\text{Lie}G)^* \times (\text{Lie}G)^* \). The symplectic form is \( \omega_{lr} = \omega_l + \omega_r \), where

\[
\omega_l = \text{tr} \left( (u^{-1}\delta u)^2 a_l + (u^{-1}\delta u) \delta a_l \right), \quad \omega_r = \text{tr} \left( -(\delta vv^{-1})^2 a_r + (\delta vv^{-1}) \delta a_r \right). \quad (9)
\]

These forms are closed and time independent. The Hamiltonian of the system we take to be \( H_{lr} = \frac{1}{4} \text{tr}(a_l^2 + a_r^2) \). The symplectic form \( \omega_{lr} \) is non-degenerate, and hence invertible. The forms \( \omega_l \) and \( \omega_r \) can be inverted separately. In addition the form \( \omega_r \) is the same as the form \( \omega \) of Eqn. (7) and it can be inverted in the same way. Similarly we can invert the form \( \omega_l \). Since the symplectic form \( \omega_{lr} \) factorises, all the Poisson brackets of the \( \{u, a_l\} \) variables with the \( \{v, a_r\} \) variables vanish. To recover the particle model from the \( lr \) model, we impose the constraint \( Q = a_r - a_l = 0 \). This is a first class constraint, whose Poisson bracket with \( H_{lr} \) does not generate additional constraints. The subspace of \( P_{lr} \) satisfying the constraint \( Q = 0 \) is isomorphic to \( G \times G \times (\text{Lie}G)^* \), and the gauge transformations generated by the constraint mod this by the group \( G \), and thus the reduced phase space is isomorphic to \( G \times (\text{Lie}G)^* \). Thus the Hamiltonian approach and the covariant canonical approach (with either gauge-fixing or constraints) yield the same answer for the particle model.

3. The WZW Model

There is a direct correspondence between the particle model described in the previous section, and the WZW model. This will lead us to a new parameterisation of the space of solutions of the WZW model, with a corresponding definition of the phase space, and a derivation of the Poisson brackets.

The equations of motion of the WZW model are

\[
\partial_-(\partial_+ gg^{-1}) = 0. \quad (10)
\]

where \( g \) is a map from a cylinder \( S^1 \times \mathbb{R} \) to a compact, connected Lie group \( G \). The pairs \((x, t) : 0 \leq x < 1, -\infty < t < \infty \) are the co-ordinates of \( S^1 \times \mathbb{R} \) and we
take $x^\pm = x \pm t, \partial_\pm = \frac{1}{2} (\partial_x \pm \partial_t)$ (note that these conventions differ from those of Ref. [11]). The semilocal transformations $g \to l(x^+) g r^{-1}(x^-)$, with $l, r$ maps from $S^1$ into $G$, are symmetries of the theory; the corresponding currents are

$$J_+ = -\frac{\kappa}{4\pi} \partial_+ gg^{-1}, \quad J_- = \frac{\kappa}{4\pi} g^{-1} \partial_- g.$$ (11)

where $\kappa$ is the coupling constant of the WZW model.

In the Hamiltonian approach, one may consider the WZW model as a two dimensional non-linear sigma model with Wess-Zumino term, whose target space is the manifold of the group $G$. Applying the usual Hamiltonian analysis to this sigma model action, one finds directly that the phase space of the WZW model is the co-tangent bundle of its configuration space $LG$, i.e. $P_H = T^* LG$.

Next we consider the covariant approach to the phase space. The symplectic form of the WZW model is (see Ref. [10], for example)

$$\Omega = \frac{\kappa}{8\pi} \int_0^1 dx \, \text{tr} \left( (g^{-1} \delta g) \partial_+ (g^{-1} \delta g) - (\delta g g^{-1}) \partial_- (\delta g g^{-1}) \right).$$ (12)

This symplectic form is closed and time independent (we take $t = 0$ in the following). The key step we take at this juncture is to parameterise the space of solutions to the field equations (10) in the following manner:

$$g(x, t) = U(x^+) \mathcal{W}(A; x^+, x^-) V(x^-),$$

$$\mathcal{W}(A; x^+, x^-) = P \exp \int_{x^-}^{x^+} A(s) ds,$$ (13)

where $U$ and $V$ are maps from $S^1$ to the group $G$, and the field $A$ in the path-ordered exponential is a Lie$G$-valued connection* over $S^1$. The fields $U, V$ and $A$

* to be precise, $A$ is a (Lie$G$)*-valued periodic one-form on the real line, but for simplicity we have identified Lie$G$ with its dual using the invariant metric on Lie$G$
are thus periodic in \(x\). The expression for \(g(x, t)\) in Eqn. (13) is then periodic in \(x\) and solves the field equations (10). The latter result follows immediately from the fact that \(W\) satisfies the parallel transport equation

\[
\partial_s W(A; s, x^-) = A(s) W(A; s, x^-).
\]  

(14)

This equation implies that \(\partial_+ W = A(x^+) W, \partial_- W = -W A(x^-)\). Inserting the solution (13) into the symplectic form (12) gives

\[
\Omega = -\frac{\kappa}{8\pi} \int_0^1 dx \operatorname{tr} \left( (U^{-1} \delta U) \partial_x (U^{-1} \delta U) + 2(U^{-1} \delta U)^2 A + 2(U^{-1} \delta U) \delta A 
- (\delta V V^{-1}) \partial_x (\delta V V^{-1}) - 2(\delta V V^{-1})^2 A + 2(\delta V V^{-1}) \delta A \right).
\]  

(15)

The solution \(g\) of the WZW equations of motion given in the parameterisation (13) is invariant under the transformations

\[
U(x) \rightarrow U(x) h(x), \quad V(x) \rightarrow h^{-1}(x) V(x),
A(x) \rightarrow -h^{-1}(x) \partial_x h(x) + h^{-1}(x) A(x) h(x),
\]  

(16)

where \(h \in LG\). To prove this, we observe that under these transformations \(W(A; x^+, x^-) \rightarrow h^{-1}(x^+) W(A; x^+, x^-) h(x^-)\). The phase space \(P_G\) of the WZW model is then the space of fields \(\{U, V, A\}\), modulo the transformations (16). This is \(\frac{LG \times LG \times A}{LG}\) where \(A\) is the space of \(G\)-connections over the circle. This is diffeomorphic to \(T^* LG\), \(i.e.\) it is the same as the phase space \(P_H\) derived from the Hamiltonian treatment of the theory.

The symplectic form (15) is degenerate along the directions of the action (16) of the loop group \(LG\). We may deal with this by gauge-fixing or by imposing constraints.

We first consider the gauge-fixing approach and analogously to the case of the particle model we may choose as a gauge fixing condition \(U = e\) where \(e\) is the
identity element of the loop group $LG$. This is a good gauge choice, as $LG$ acts freely and transitively on the space of $U$’s, $\{U\}$. The symplectic form (15) then becomes

$$\Omega = -\frac{\kappa}{8\pi} \int_0^1 dx \text{tr} \left( -(\delta V V^{-1}) \partial_x (\delta V V^{-1}) - 2(\delta V V^{-1})^2 A + 2(\delta V V^{-1})\delta A \right).$$  \hspace{1cm} (17)

This symplectic form is not degenerate and is invertible. The simplest way to invert it is to first rewrite it in terms of a local parameterisation $X^i(x)$ for the maps $V$ ($V = V(X)$). This gives

$$\Omega = -\frac{\kappa}{8\pi} \int_0^1 dx \left( -(R^a_i \delta X^i) \partial_x (R^a_j \delta X^j) - f_{ab}^c R^a_i R^b_j A_c \delta X^i \delta X^j + 2R^a_i \delta X^i \delta A_a \right).$$  \hspace{1cm} (18)

The remarkable feature of this expression for the form $\Omega$ is that one does not need to invert any differential operator in order to invert the form (c.f. Refs. [10,11], where in order to invert the symplectic form it was necessary to find the inverse of the operator $\partial_x$ on the circle). Like the case of the particle model, the gauge $U = e$ parameterises the symplectic form on $T^*LG$ in terms of the right trivialisation of $T^*LG$ and the gauge $V = e$ parameterises the same symplectic form in terms of the left trivialisation. The inversion of the form (18) is straightforwardly carried out, and leads to the Poisson brackets ($\beta = -\frac{4\pi}{\kappa}$)

$$\{X^i(x), X^j(y)\} = 0,$$

$$\{X^i(x), A_a(y)\} = \beta R^i_a [X(x)] \delta(x,y),$$  \hspace{1cm} (19)

$$\{A_a(x), A_b(y)\} = \beta \left( \delta_{ab} \partial_x + f_{ab}^c A_c(x) \right) \delta(x,y),$$

where $\delta(x,y)$ is the delta function on $S^1$. The brackets (19) are the Poisson brackets on the co-tangent bundle of the loop group which one would expect – here we have derived them from the WZW model, using the corresponding
symplectic form in the parameterisation (13). Using Eqn. (19), we can calculate Poisson brackets involving $V$ and $A$—for example $\{V(x) \otimes V(y)\} = 0$, $\{V(x), A_a(y)\} = \beta V(x) t_a \delta(x, y)$. In this gauge, the WZW currents (11) become $J_+ = -\frac{\kappa}{4\pi} A$, $J_- = \frac{\kappa}{4\pi} (V^{-1} \partial_x V - V^{-1} A V)$, and it can be verified by a straightforward calculation that their Poisson bracket algebra is isomorphic to two commuting copies of a Kac-Moody algebra with a central extension.

In the constraint approach to the degeneracy of the form (15), we introduce an ‘LR’ model, with phase space $P_{LR} = LG \times LG \times A \times A$, with coordinates $(U, V, A_L, A_R)$. The symplectic form on $P_{LR}$ is defined to be $\Omega_{LR} = \Omega_L + \Omega_R$, where

$$\Omega_L = -\frac{\kappa}{8\pi} \int_0^1 dx \text{tr} \left( (U^{-1} \delta U) \partial_x (U^{-1} \delta U) + 2( U^{-1} \delta U)^2 A^L + 2 (U^{-1} \delta U) \delta A^L \right),$$

$$\Omega_R = -\frac{\kappa}{8\pi} \int_0^1 dx \text{tr} \left( - (\delta V V^{-1}) \partial_x (\delta V V^{-1}) - 2(\delta V V^{-1})^2 A^R + 2 (\delta V V^{-1}) \delta A^R \right).$$

(20)

The symplectic form (20) is similar to the symplectic form of Eqn. (15) but with different connections in the $U$ and $V$ sectors. It is straightforward to invert the forms in Eqn. (20). Indeed, the form $\Omega_R$ is the same as the symplectic form of Eqn. (17) which we have already inverted in the gauge-fixing method. The symplectic form $\Omega_L$ can be treated in a similar way. Poisson brackets of variables of the $U$ sector with variables of the $V$ sector are zero, because the symplectic form $\Omega_{LR}$ factorises. For completeness we give the Poisson brackets of the $U$ sector:

$$\{Y^i(x), Y^j(y)\} = 0, \quad \{Y^i(x), A^L_a(y)\} = \beta L^i_a [Y(x)] \delta(x, y),$$

$$\{A^L_a(x), A^L_b(y)\} = \beta \left( - \delta_{ab} \partial_x - f_{abc} A^L_c(x) \right) \delta(x, y),$$

(21)

where $U = U(Y)$, with $Y$ a parameterisation of $LG$ in terms of local co-ordinates on the group manifold $G$. The Hamiltonian of the $LR$ model is taken to be $H = \frac{1}{4} \text{tr} \int_0^1 dx (J^2_L + J^2_R)$, where $J_L = -\frac{\kappa}{4\pi} (\partial_x U U^{-1} + U A^L U^{-1})$ and $J_R = \frac{\kappa}{4\pi} (V^{-1} \partial_x V - \cdots$
\( V^{-1}A^R V \). To recover the WZW model we introduce the constraint \( Q = A^R - A^L \) in the phase space \( P_{LR} \) of the LR model. Using the above Poisson brackets, it follows that this constraint is first class and its Poisson bracket with \( H_{LR} \) does not induce any other constraints. The Poisson bracket algebra of the constraints is a Kac-Moody algebra without a central extension. The reduced phase space that we get by factoring out the transformations on \( P_{LR} \) generated by the constraints \( Q \) is the same as \( T^*LG \), thus giving agreement with both the gauge-fixed covariant approach and the Hamiltonian approach.

4. Discussion

We would now like to discuss how our results relate to other work in the literature. The symmetries of the space of solutions of the particle model (Eqn. (4)) and the WZW model (Eqn. (16)) can be treated by choosing the gauge-fixing conditions to be different from those considered in Sections Two and Three above. Using these other gauge-fixing conditions, we can make contact with the parameterisations of the spaces of solutions of these models in Refs. [10,12]. However, these gauge-fixings suffer from Gribov-Singer-type ambiguities. Because of this, the resulting spaces of parameters are topologically different from those obtained from the Hamiltonian treatments of these models. In the following for simplicity we assume that the group \( G \) is simply connected.

For the particle model, the symmetry Eqn. (4) can be partially gauge-fixed by putting \( a \) in the Cartan subalgebra \( \mathfrak{h} \subset \text{Lie}G \). There is a residual symmetry associated with this gauge fixing. This is \( u \rightarrow uT, v \rightarrow T^{-1}v, a \rightarrow a (a \in \mathfrak{h}) \) where \( T \) is in a maximal torus \( T \) of the group \( G \). This parameterisation of the particle model was studied in Ref. [12]. Apart from the residual symmetry which must still be fixed, there is a Gribov-Singer ambiguity associated with this gauge fixing. One way to see this is to observe that the space of independent parameters that describes the solutions of the particle model, after introducing the above gauge-fixing, is \( \frac{G \times G \times \mathfrak{h}}{T} \). This space is not diffeomorphic to the phase space \( T^*G \) of the
particle model – for example, $\pi_2(T^*G) = 0 \neq \pi_2(G \times G \times h)$. This results from the fact that this gauge-fixing condition is local, and cannot be extended globally. This is in contrast to the gauge-fixing which we used in Section Two.

Similar comments apply to the WZW model. In our parameterisation (13) one can gauge-fix the connection $A$ so that it is a constant connection over the circle. The residual transformations for this gauge-fixing are the constant gauge transformations. The constant gauge transformations are parameterised by the elements of the group $G$ and they act on the parameters of the solutions as $U \rightarrow Uk$, $V \rightarrow k^{-1}V$ and $A \rightarrow k^{-1}Ak$ where $k \in G$ and $A$ is a constant connection. This parameterisation of the space of solutions is that of Ref. [10], and the resulting phase space of the theory is $LG \times LG \times \text{Lie}_G$. This phase space is not diffeomorphic to the phase space $T^*LG$ of the WZW model, which we obtained in the discussion above (for example, the second homotopy groups differ). The reason for this difference is that there is again a Gribov-Singer ambiguity associated with this gauge fixing; note that this is the one-dimensional analogue of the four-dimensional Yang-Mills Gribov-Singer ambiguity [14]. The $k$-symmetry just mentioned can be further gauge fixed by choosing $A$ to be in the Cartan subalgebra $\mathfrak{h}$ of $\text{Lie}_G$, and this parameterisation was used together with a version of the $LR$ model to calculate the Poisson brackets of this theory in Refs. [10, 11].

A requirement upon any choice of a space of solutions for a model is that it should correspond to the Cauchy data for the model. A solution of the equations of motion of the WZW model can be specified in a neighbourhood of a Cauchy surface $S^1$ (say the Cauchy surface $t=0$) by the Cauchy data $g(x, 0) = f(x)$ and $(g^{-1}\partial_t g)(x, 0) = w(x)$, where $f$ and $w$ are independent functions. If $u$ and $v$ have monodromy (as in Ref. [10]), then the solution $g(x, t) = u(x^+)v(x^-)$ of the WZW model has unconstrained Cauchy data (our solution (13) similarly has unconstrained Cauchy data). However, if one requires that $u$ and $v$ are periodic (as in Ref.[2]), then the Cauchy data carried by $f$ and $w$ is constrained. This constraint is that the holonomy of the connection $\frac{1}{2}(f^{-1}\partial_x f - w)$ on the circle $S^1$ must be the identity group.
In conclusion, the parameterisation of the solutions of the WZW model given in Section Three (Eqn. (13)) is general in the sense that it is invariant under a larger symmetry than other parameterisations considered in the literature, and the latter can be thought of as locally-valid gauge-fixed versions of it. In our parameterisation, the covariant canonical phase space of the WZW model is the same as the Hamiltonian phase space of the theory, and the calculation of the Poisson brackets is straightforward.

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REFERENCES

1. R. Abraham and J.E. Marsden, *Foundations of Mechanics*, Benjamin / Cummings Publishing Company, 1978;
   G. Zuckerman, in *Mathematical Aspects of String Theory*, ed. S.-T. Yau, World Scientific, Singapore 1987;
   Č. Crnković and E. Witten, in *Three Hundred Years of Gravitation*, eds. S.W. Hawking and W. Israel, C.U.P. Cambridge, 1987.

2. E. Witten, Commun. Math. Phys. 92 (1984) 455.

3. B. Blok, Phys. Lett. 233B (1989) 359.

4. A.Yu. Alekseev and S. Shatashvili, Commun. Math. Phys. 128 (1990) 197;
   133 (1990) 353.

5. L.D. Faddeev, Commun. Math. Phys. 132 (1990) 131.

6. J. Balog, L. Dąbrowski and L. Fehér, Phys. Lett. 244B (1990) 227.

7. G. Felder, K. Gawędski and A. Kupiainen, Nucl. Phys. B299 (1988) 355;
   Commun. Math. Phys. 117 (1988) 127.

8. K. Gawędski, Commun. Math. Phys. 139 (1991) 201.

9. G. Bimonte, P. Salomonson, A. Simoni and A. Stern, *Poisson Bracket Algebra for Chiral Group Elements in WZNW Model*, preprint UAHEP 9114.
10. M.F. Chu, P. Goddard, I. Halliday, D. Olive and A. Schwimmer, Phys. Lett. B266 (1991) 71.

11. G. Papadopoulos and B. Spence, *The Canonical Structure of Wess-Zumino-Witten Models*, preprint Imperial 91-92/19, QMW 92/2.

12. A.Yu. Alekseev and L.D. Faddeev, *(T*(G)), : A Toy Model for Conformal Field Theory*, Commun. Math. Phys. to appear;
   L.D. Faddeev, *Quantum Symmetry in Conformal Field Theory by Hamiltonian Methods*, Cargese Lectures 1991.

13. I. Bakas and D. McMullan, Phys. Lett. B189 (1987) 141.

14. V.N. Gribov, Nucl. Phys. B139 (1978) 1;
   I. Singer, Commun. Math. Phys. 60 (1978) 7.