I. INTRODUCTION

The Nobel Prize in physics 2016 motivates one to study different aspects of topological properties and topological defects as their related objects. Considering the significant role of the topological defects (especially magnetic strings) in cosmology, here, we will investigate three-dimensional horizonless magnetic solutions in the presence of two generalizations: massive gravity and nonlinear electromagnetic field. The effects of these two generalizations on properties of the solutions and their geometrical structure are investigated. The differences between de Sitter and anti de Sitter solutions are highlighted and conditions regarding the existence of phase transition in geometrical structure of the solutions are studied.

References [1-10]

The cosmic strings in the presence of Maxwell field have been investigated [13]. Furthermore, the superconducting property of these topological defects has been explored in Einstein [14], dilaton [15] and Brans-Dicke [16] theories.
In addition, the QCD applications of the magnetic strings [17] and their roles in quantum theories [18] have been investigated before. The stability of the cosmic strings through quantum fluctuations has been analyzed in Ref. [19]. The limits on the cosmic string tension have been studied by extracting signals of cosmic strings from CMB temperature anisotropy maps [20]. The spectrum of gravitational wave background produced by cosmic strings is obtained in Ref. [21]. For further investigations regarding cosmic strings, we refer the reader to an incomplete list of references [22].

Domain walls and their evolution in de Sitter universe have been studied in [23]. In addition, the gravitational waves produced from decaying domain walls are investigated in Ref. [24]. The localization of the fields on the dynamical domain wall was investigated and it was shown that the chiral spinor can be localized on the domain walls [25]. For further studies regarding this class of topological defects, we refer the reader to Ref. [26].

On the other hand, considering most of physical systems in nature, one finds that they exhibit nonlinear behavior, and therefore, the nonlinear field theories are of importance in physical researches. There are many motivations for studying the nonlinear electrodynamics (NED) such as; (i) These theories are the generalizations of Maxwell field and reduce to linear Maxwell theory in the special cases (weak nonlinearity). (ii) These nonlinear theories can describe the radiation propagation inside specific materials [27]. (iii) Some special NED models can describe the self-interaction of virtual electron-positron pairs [28]. (iv) Theories of NED can remove the problem of point-like charge self-energy. (v) From the standpoint of quantum gravity and its coupling with these nonlinear theories, we can obtain more information and deep insight regarding the nature of gravity [29]. (vi) Compatibility with AdS/CFT correspondence and string theory are other properties of NED theories. (vii) NED theory improves the basic concept of gravitational redshift and its dependency of any background magnetic field as compared to the well-established method introduced by general relativity. (viii) From the perspective of cosmology, it was shown that NED theories can remove both of the big bang and black hole singularities [30]. (ix) From astrophysical point of view, it was found that the effects of NED become quite important in super-strong magnetized compact objects, such as pulsars and particular neutron stars [31].

There are different models of NED, such as Born-Infeld form [32], logarithmic form [33], exponential form [34], arsine-electrodynamics form [35] and etc. One of the interesting branches of the nonlinear electrodynamics is power Maxwell invariant (PMI) theory. The Lagrangian of PMI theory is an arbitrary power of the Maxwell Lagrangian [36] which could reduce to the Maxwell field by choosing the unit power. In addition, the PMI theory has an interesting consequence which distinguishes this NED theory from other theories; this theory enjoys conformal invariancy when the power of Maxwell invariant is a quarter of space-time dimensions (power = dimensions/4). In other words, in this case, the energy-momentum tensor will be a traceless tensor which leads to conformal invariancy and also an inverse square law of the electric field for the point-like charge in arbitrary dimensions [36].

Recent observations of gravitational waves from a binary black hole merger in LIGO and Virgo collaboration provided a deep insight to general relativity (GR) and existence of massive gravitons [37]. However, gravitons are massless particles with spin 2 in GR which have two degrees of freedom. Since the quantum theory of massless gravitons is non-renormalizable [35], in order to remove this problem, one may modify GR to massive gravity by adding a mass term to the Einstein-Hilbert action. Therefore, considering this action, the graviton will have a mass of \( m \) which in case of \( m \rightarrow 0 \), the effect of massive gravity will be vanished. In other words, massive gravity is a modification of GR that gravitons have mass. Among the motivations of the massive gravity one can mention description of accelerating expansion of universe without considering the cosmological constant [39]. It was shown that the terms of massive gravitons can be equivalent to a cosmological constant [40]. This theory modifies gravity compared with GR which allows the universe to accelerate at the large scale, however at small scale, this theory reduces to GR as well. This theory of gravity may illustrate the dark energy problem [41]. In addition, the existence of massive gravitons provides extra polarization for gravitational waves, and affects the propagation’s speed of the gravitational waves [42], hence, the production of gravitational waves during inflation [43]. By adding the interaction terms to GR, massive gravity with flat background was investigated by Fierz and Pauli [44]. However, this theory suffers a van Dam-Veltman-Zakharov (vDVZ) discontinuity [45]. Generalization of massive theory to curved background was done by Boulware-Deser. This generalization leads to the existence of a typical ghost, the so-called Boulware-Deser ghost [46]. Several models of massive theory were proposed by some authors in order to avoid discontinuity and ghost problems [47]. One of the ghost-free massive theories in three dimensions was introduced by Bergshoeff, Holm and Townsend (new massive gravity (NMG)) [48]. However, NMG has ghost problem in four and higher dimensions. Therefore, in order to resolve ghost problem in diverse dimensions, a new theory of massive gravity was proposed by de Rham, Gabadadze and Tolley (dRGT) in 2011 [49]. The stability of dRGT massive theory was studied and it was shown that such theory enjoys absence of the Boulware-Deser ghost [50]. Black hole and cosmological solutions have been investigated in dRGT massive gravity [51–59]. Also, reentrant phase transitions of higher-dimensional AdS black holes and behavior of quasinormal modes and van der Waals like phase transition of charged AdS black holes in massive gravity have been studied in Refs. [60, 61].

It is notable that in massive gravity theory, the mass terms are produced by consideration of a reference metric.
and the energy-momentum tensor of Eq. (3) is

\[ T_{\mu\nu} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R + m^2 \sum_{i=1}^{4} c_i \mathcal{U}_i (g, f) \right]. \tag{1} \]

where \( R \) is the scalar curvature and \( m^2 \) is related to the mass of gravitons. In addition, \( f \) is a fixed symmetric tensor, \( c_i \)'s are some constants, and \( \mathcal{U}_i \)'s are symmetric polynomials of the eigenvalues of matrix \( K^\mu_\nu = \sqrt{g^{\alpha\beta} F_{\alpha\beta}} \) which are as follow

\[
\mathcal{U}_1 = [K], \quad \mathcal{U}_2 = [K]^2 - [K^2], \quad \mathcal{U}_3 = [K]^3 - 3 [K] [K^2] + 2 [K^3], \quad \mathcal{U}_4 = [K]^4 - 6 [K^2] [K^2] + 8 [K^3] [K] + 3 [K^2]^2 - 6 [K^4].
\]

Charged black hole solutions with (non)linear field and the existence of van der Waals like behavior in extended phase space and also geometrical thermodynamics by considering dRGT massive gravity have been studied [65–67]. Moreover, the hydrostatic equilibrium equation of neutron stars by using this theory of massive gravity was obtained and it was shown that the maximum mass of neutron stars can be about 3.8\(M_\odot\) (where \( M_\odot \) is mass of the sun) [68]. Also, holographic conductivity in this gravity with PMI field has been investigated in Ref. [69]. Besides, the generalization of this theory to include higher derivative gravity [70] and gravity’s rainbow [71] has been done [68].

Varying the action (2) with respect to the gravitational and gauge fields, one can obtain the following field equations

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R - 2\Lambda) + m^2 \chi_{\mu\nu} = T_{\mu\nu}, \tag{3} \]

\[ \partial_{\mu} (\sqrt{-g} \mathcal{L}_F F^{\mu\nu}) = 0, \tag{4} \]

in which \( \mathcal{L}_F = d\mathcal{L}(\mathcal{F})/d\mathcal{F} \) where \( \mathcal{F} = F_{\mu\nu} F^{\mu\nu} \) is the Maxwell invariant, \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) is the Faraday tensor and \( A_\mu \) is the gauge potential. In addition, \( \chi_{\mu\nu} \) is the massive term with the following form

\[ \chi_{\mu\nu} = -\frac{c_1}{2} (4\mathcal{U}_1 g_{\mu\nu} - \mathcal{K}_{\mu\nu}) - \frac{c_2}{2} (4\mathcal{U}_2 g_{\mu\nu} - 2\mathcal{U}_1 K_{\mu\nu} + 2K^2_{\mu\nu}) - \frac{c_3}{2} (4\mathcal{U}_3 g_{\mu\nu} - 3\mathcal{U}_2 K_{\mu\nu} + 6\mathcal{U}_1 K^2_{\mu\nu} - 6K^3_{\mu\nu}) - \frac{c_4}{2} (4\mathcal{U}_4 g_{\mu\nu} - 4\mathcal{U}_3 K_{\mu\nu} + 12\mathcal{U}_2 K^2_{\mu\nu} - 24\mathcal{U}_1 K^3_{\mu\nu} + 24K^4_{\mu\nu}), \tag{5} \]

and the energy-momentum tensor of Eq. (3) is

\[ T_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \mathcal{L}(\mathcal{F}) - 2\mathcal{L}_F F_{\mu\lambda} F^{\lambda\nu}. \tag{6} \]

Here, we want to obtain the magnetic solutions of Eqs. (3) and (4) by considering the Maxwell electromagnetic field (\( \mathcal{L}(\mathcal{F}) = -\mathcal{F} \)).

Magnetic branes (or horizonless solution) are interesting objects which have been investigated by many authors [72–80]. Our main motivation here is to understand the effects of two generalizations on the magnetic horizonless solutions with interpretation of topological defects. These two generalizations include massive gravity and PMI electromagnetic field. Considering the applications of topological defects in dark matter, CMB, gravitational waves, large scale structure and etc., it is necessary to investigate the effects of the massive gravitons on the structure and formation of topological defects. Here, we intend to show how generalization to massive gravity would modify geometrical structure of the magnetic solutions. To do so, we apply the massive gravity generalization and investigate
geometrical properties such as deficit angle. Considering the electromagnetically charged aspect of the objects of interest in this paper (magnetic solutions), we will take two cases of linear and nonlinear electromagnetic fields into account. Here, we would investigate the effects of Maxwell and PMI electromagnetic fields on the deficit angle, hence, geometrical structure of the topological defects known as horizonless magnetic solutions. The combinations of massive gravity and PMI theory is another subject of interest which would be addressed. It is notable that such magnetic source was interpreted as a kind of magnetic monopole reminiscent of a Nielsen-Olesen vortex solution [81], while Dias and Lemos interpreted it as a composition of two symmetric and superposed electric charges [13]. In other words, one of the mentioned electric charges is at rest and the other is rotating, and therefore, there is no electric field since the total electric charge is zero, but angular electric current produces a magnetic field.

Now, we use the new metric of three dimensional spacetime with (− + +) signature which was introduced in Ref. [80]

\[
ds^2 = -\frac{l^2}{\rho^2} dt^2 + \frac{d\rho^2}{g(\rho)} + \rho^2 d\varphi^2,
\]

where \(g(\rho)\) is an arbitrary function of radial coordinate \(\rho\) which should be determined. The scale length factor \(l\) is related to the cosmological constant \(\Lambda\), and the angular coordinate \(\varphi\) is dimensionless as usual and ranges in \([0, 2\pi]\). The motivation of considering the metric gauge \([g_{\mu\nu} \propto -\rho^2\) and \((g_{\rho\rho})^{-1} \propto g_{\varphi\varphi}\)] instead of the usual Schwarzschild like gauge \([(g_{\rho\rho})^{-1} \propto g_{tt}\) and \(g_{\varphi\varphi} \propto \rho^2\)] comes from the fact that we are looking for magnetic solutions without curvature singularity. It is easy to show that using a suitable transformation, the metric \(\text{[7]}\) can be mapped to 3-dimensional Schwarzschild like spacetime locally, but not globally [80].

In order to obtain exact solutions, we should make a choice for the reference metric. We consider the following ansatz metric

\[
f_{\mu\nu} = \text{diag}(\frac{c^2}{l^2}, 0, 0),
\]

where in the above equation \(c\) is a positive constant. Using the metric ansatz \([5]\), \(U_i\)'s are \([65, 72]\]

\[
U_1 = \frac{c}{\rho}, \quad U_2 = U_3 = U_4 = 0,
\]

which indicate that the only contribution of massive gravity comes from \(U_1\) in three dimensions. Before proceeding we give a reason for such choice of the reference metric \([5]\). For three dimensional black holes, the spacetime metric with (−, +, +) signature has the following explicit form

\[
ds^2 = -g(\rho) dt^2 + \frac{d\rho^2}{g(\rho)} + \rho^2 d\varphi^2.
\]

In order to obtain exact black hole solutions, we consider the ansatz metric as \(f_{\mu\nu} = \text{diag}(0, 0, c^2)\) (see Refs. [63], [65] and [66], for more details). Here, the metric function \((g(\rho))\) is factors of radial and spatial coordinates in magnetic spacetime metric (Eq. \([7]\)). In order to have exact solutions in an axially symmetric spacetime with the form \(\text{[7]}\), it is necessary to consider the reference metric as \(f_{\mu\nu} = \text{diag}(\frac{c^2}{l^2}, 0, 0)\). This form of reference metric is expectable. Comparing black hole metric, Eq. \([10]\), with magnetic spacetime, Eq. \(\text{[7]}\), we find that Eq. \(\text{[7]}\) can be reproduced from Eq. \([10]\) by the following local transformations:

\[
t \rightarrow il\varphi \quad \text{and} \quad \varphi \rightarrow it/l.
\]

Since we changed the role of \(t\) and \(\varphi\) coordinates, the nonzero component of the reference metric should be changed accordingly.

Since we are going to study the linearly magnetic solutions, we choose the Lagrangian of Maxwell field \(\mathcal{L}(\mathcal{F}) = -\mathcal{F}\) for Eqs. \([2]\), \([4]\), and \([6]\). It is well-known that the electric field is associated with the time component of the vector potential \(A_t\), while the magnetic field is associated with the angular component \(A_\varphi\). Due to our interest to investigate the magnetic solutions, we assume the vector potential as

\[
A_\mu = h(\rho) \delta_\mu^\varphi.
\]

Using the Maxwell equation \([4]\) with \(\mathcal{L}(\mathcal{F}) = -\mathcal{F}\), and the metric \([7]\), one finds the following differential equation

\[
F_{\varphi\rho} + \rho F'_{\varphi\rho} = 0,
\]
where \( F_{\varphi \rho} = h'(\rho) \) in which the prime denotes differentiation with respect to \( \rho \). Equation (13) has the following solution

\[
F_{\varphi \rho} = \frac{q}{\rho},
\]

where \( q \) is an integration constant. To find the metric function \( g(\rho) \), one may insert Eq. (14) in the field equation (3) by considering the metric (7). After some calculations, one can obtain the following differential equations

\[
\begin{align*}
\frac{g''(\rho)}{\rho} + 2\Lambda - 2\frac{q^2 l^2}{m^2} & = 0, \\
\frac{g'(\rho)}{\rho} + 2\Lambda + 2\frac{q l}{m} & = 0,
\end{align*}
\]

where the double prime is the second derivative versus \( \rho \). It is straightforward to show that these equations have the following solution

\[
g(\rho) = m_0 - \Lambda \rho^2 + \frac{2q^2}{l^2} \ln \left( \frac{\rho}{l} \right) + cc_1 m^2 \rho,
\]

which \( m_0 \) is an integration constant which is related to the mass parameter, and \( l \) is an arbitrary constant with length dimension which is coming from the fact that the logarithmic arguments should be dimensionless. As one can see, the massive parameter appears in the metric function as a factor for the linear function of \( \rho \). We should note that the obtained metric function (16) satisfies all components of the field equation (3), simultaneously. In addition, the asymptotical behavior of the solution (16) is adS or dS provided \( \Lambda < 0 \) or \( \Lambda > 0 \). Also, it is worthwhile to mention that in the absence of massive parameter \( (m = 0) \), the metric function (16) reduces to the result of Ref. [80] for \( s = 1 \).

A: Energy Conditions

Now, we examine the energy conditions to find physical solutions. To do so, we consider the orthonormal contravariant basis vectors, and then we obtain the three dimensional energy momentum tensor as \( T^{\mu \nu} = \text{diag}(\mu, p_r, p_t) \) in which \( \mu, p_r, \) and \( p_t \) are the energy density, the radial pressure and the tangential pressure, respectively. Having the energy momentum tensor at hand, we are in a position to investigate the energy conditions. We use the following known constraints in three dimensions

\[
\begin{align*}
p_r + \mu & \geq 0, & \text{for null energy condition (NEC)} \\
p_t + \mu & \geq 0, & \text{for weak energy condition (WEC)} \\
\mu & \geq 0, & \text{for dominant energy condition (DEC)} \\
-\mu & \leq p_r \leq \mu, & \text{for dominant energy condition (DEC)} \\
-\mu & \leq p_t \leq \mu, & \text{for strong energy condition (SEC)} \\
p_r + \mu & \geq 0, & \text{for strong energy condition (SEC)} \\
p_t + \mu & \geq 0, & \text{for strong energy condition (SEC)} \\
\mu + p_r + p_t & \geq 0
\end{align*}
\]

Table (1): Energy conditions criteria

In order to simplify the mathematics and physical interpretations, we use the following orthonormal contravariant (hatted) basis vectors for diagonal static metric (7)

\[
\begin{align*}
e_t = \frac{l}{\rho} \frac{\partial}{\partial t}, & \quad e_\rho = \sqrt{g} \frac{\partial}{\partial \rho}, & \quad e_\varphi = \frac{1}{l \sqrt{g}} \frac{\partial}{\partial \varphi}.
\end{align*}
\]

It is a matter of straightforward calculations to show that the nonzero components of stress-energy tensor are

\[
T_{tt} = T_{\varphi \varphi} = T_{\varphi \varphi} = \left( \frac{F_{\varphi \rho}}{l} \right)^2.
\]
All components of stress-energy tensor are the same and positive and it is easy to find that NEC, WEC, DEC and SEC are satisfied, simultaneously.

As one can see, the massive parameter do not contribute to the energy-momentum tensor, so the energy conditions are independent of the massive parameter. In order to investigation the effects of charge on the energy density of the spacetime, we plot the $T^{tt}$ versus $\rho$. Considering Fig. 1, one can find that the energy density of the spacetime is positive everywhere, and increasing the charge parameter leads to increasing the concentration of energy density.

**B: Geometric Properties**

Now, we want to study the properties of spacetime described by Eq. (7) with obtained metric function (16). At first, we calculate $R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}$ for examination of existence of curvature singularity

$$R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} = 12\Lambda^2 - \frac{8\Lambda q^2}{l^2\rho^2} + \frac{2cc_1m^2(4q^2 + cc_1m^2l^2\rho - 4\Lambda l^2\rho^2)}{l^2\rho^3} + \frac{12q^4}{l^4\rho^4}. \quad (19)$$

Considering Eq. (19), the Kretschmann scalar reduces to $12\Lambda^2$ for $\rho \to \infty$, which confirms that the asymptotical behavior of this spacetime is (a)dS. It is also obvious that the Kretschmann scalar diverges at $\rho = 0$, and therefore one might think that there is a curvature singularity located at $\rho = 0$. But as we will see, the spacetime will never achieve $\rho = 0$. There are two possible cases for the metric function: the metric function has no real positive root which is interpreted as naked singularity (this case is not of interest here), or metric function has at least one real positive root. If one considers $r_0$ as the largest root of metric function, it is clear that for $\rho < r_0$ there will be a change in signature of metric (see Fig. 2). In other words, for $\rho < r_0$ the metric function is negative, hence metric signature is $(-, -, -)$, and for $\rho > r_0$ the metric function is positive, therefore metric signature is legal $(-, +, +)$. This change in the metric signature results into a conclusion: it is not possible to extend spacetime to $\rho < r_0$. In order to exclude the forbidden zone ($\rho < r_0$), we introduce a new radial coordinate $r$ as

$$r^2 = \rho^2 - r_0^2 \implies d\rho^2 = \frac{r^2}{r^2 + r_0^2} dr^2, \quad (20)$$

where for the allowed region, $\rho \geq r_0$, leads to $r \geq 0$ in the new coordinate system. Applying this coordinate transformation, the metric (7) should be written as

$$ds^2 = \frac{r^2 + r_0^2}{l^2} dt^2 + \frac{r^2}{(r^2 + r_0^2) g(r)} dr^2 + l^2 g(r) d\varphi^2, \quad (21)$$

in which the coordinate $r$ assumes the values $0 \leq r < \infty$, and obtained $g(r)$ (Eq. (16)) is now given by

$$g(r) = m_0 - \Lambda (r^2 + r_0^2) + \frac{q^2}{l^2} \ln \left( \frac{r^2 + r_0^2}{l^2} \right) + cc_1m^2 \sqrt{r^2 + r_0^2}. \quad (22)$$

**FIG. 1:** $T^{tt}$ versus $\rho$ for $l = 1$ and $q = 0.5$ (dashed line), $q = 1.0$ (dotted line) and $q = 1.5$ (continuous line).
FIG. 2: $g(\rho)$ versus $\rho$ for $l = 1$, $q = 1$, $\Lambda = -1$, $c = 1$, $c_1 = 1$, $m = 0.5$ and $m_0 = 1.5$.

The nonzero component of electromagnetic field in the new coordinate can be given by

$$F_{\varphi r} = \frac{q}{\sqrt{r^2 + r_0^2}},$$  \hspace{1cm} (23)$$

One can show that all curvature invariants do not diverge in the range $0 \leq r < \infty$, and $g(r)$ (Eq. (22)) is positive definite for $0 \leq r < \infty$. It is evident that for having singular solutions both $r$ and $r_0$ must be zero whereas this case is never reached due to considering nonzero value for $r_0$. So, this spacetime has no curvature singularity and horizon. Due to the fact that the limit of the ratio "circumference/radius" is not $2\pi$, the spacetime [24] has a conic geometry and therefore the spacetime has a conical singularity at $r = 0$

$$\lim_{r \to 0} \frac{1}{r} \sqrt{\frac{g_{\varphi \varphi}}{g_{rr}}} \neq 1.$$  \hspace{1cm} (24)$$

On the other hand, the conical singularity can be removed if one exchanges the coordinate $\varphi$ with the following period

$$Period_\varphi = 2\pi \left(\lim_{r \to 0} \frac{1}{r} \sqrt{\frac{g_{\varphi \varphi}}{g_{rr}}} \right)^{-1} = 2\pi \left(1 - 4\mu\right),$$  \hspace{1cm} (25)$$

in which the deficit angle is defined as $\delta \varphi = 8\pi\mu$, where $\mu$ is given by

$$\mu = \frac{1}{4} + \frac{1}{4lr_0^2},$$  \hspace{1cm} (26)$$

where $\Omega$ is

$$\Omega = \Lambda - \frac{q^2}{l^2r_0^2} - \frac{cc_1m^2}{2r_0}.$$  \hspace{1cm} (27)$$

In order to have a better insight of the behavior of deficit angle, we calculate the root and divergence points of the deficit angle as

$$r_0|_{\delta \varphi = 0} = \left\{ \begin{array}{ll} \frac{1}{4\Lambda} \left(cc_1m^2l - 2 \pm \sqrt{c^2c_1^2m^4l^2 - 4 (cc_1m^2l - 1 - 4\Lambda q^2)} \right) \\ \frac{1}{4\Lambda} \left(-cc_1m^2l - 2 \pm \sqrt{c^2c_1^2m^4l^2 - 4 (cc_1m^2l - 1 - 4\Lambda q^2)} \right) \end{array} \right.,$$  \hspace{1cm} (28)$$
Here, we see that the roots are functions of the cosmological constant, massive gravity and electric charge. Existence of the real valued root is restricted to following condition

$$c^2 c_1^2 m^4 l^2 - 4 (c c_1 m^2 l - 1 - 4 q^2) \geq 0.$$  

(30)

The effects of the massive gravity and electric charge are only observed in numerator of the roots while the effects of the cosmological constant could be observed in both numerator and denominator of the roots. The electric charge is coupled with cosmological constant. While such coupling is not observed for the massive gravity.

As for the divergencies of the deficit angle, one can observe that its existence is also restricted to satisfaction of specific condition in the following form

$$c^2 c_1^2 m^4 l^2 + 16 \Lambda q^2 \geq 0.$$  

(31)

In the absence of the massive gravity, only for dS spacetime divergencies are observable for deficit angle. Generalization to massive gravity provides the possibility of the divergencies for deficit angle in adS spacetime under certain circumstances. This highlights the effects of the massive gravity. Here, similar to the case of roots, a coupling between cosmological constant and electric charge is observed while such coupling could not be seen for massive gravity.

II. GENERALIZATION OF ACHIEVEMENTS TO THE CASE OF NONLINEAR ELECTRODYNAMICS: PMI THEORY

In this section, we are going to obtain the solutions in presence of PMI source and investigate the properties. We start with the following PMI Lagrangian

$$\mathcal{L}_\text{PMI}(\mathcal{F}) = (-\kappa \mathcal{F})^s,$$  

(32)

where $\kappa$ and $s$ are coupling and power constants, respectively. Obviously, the PMI Lagrangian (32) reduces to the standard Maxwell Lagrangian ($\mathcal{L}_\text{Maxwell}(\mathcal{F}) = -\mathcal{F}$) for $s = 1$ and $\kappa = 1$ which we have investigated before.

Following the method of previous section and considering Eqs. (4), (7) and (32), one can obtain the following differential equation for nonzero component of Faraday tensor

$$F_{\varphi \rho} + (2s - 1) \rho F_{\varphi \rho} = 0,$$  

(33)

with the following solution

$$F_{\varphi \rho} = q \rho^{1/(1 - 2s)},$$  

(34)

in which $q$ is an integration constant. In order to have a physical asymptotical behavior, we should consider $s > 1/2$. On the other hand, one can easily show that the vector potential $A_\varphi$, is

$$A_\varphi = q \rho^{2(s - 1)/2s - 1},$$  

(35)

the electromagnetic gauge potential should be finite at infinity ($\rho \to \infty$), therefore, one should impose following restriction to have this property, so we have

$$\frac{2(s - 1)}{2s - 1} < 0.$$  

(36)

The above equation leads to the following restriction on the range of $s$, as

$$\frac{1}{2} < s < 1.$$  

(37)
Here, one can insert Eq. (34) in the gravitational field equation (3) by considering the metric (7) to obtain the metric function $g(\rho)$ as

$$g(\rho) = m_0 - \Lambda \rho^2 + c c_1 m^2 \rho + \frac{(2s-1)^2 \rho^2}{2(s-1)} \chi(\rho),$$ \hspace{1cm} (38)

where

$$\chi(\rho) = \left(\frac{2q^2}{l^2} \rho^{2/(1-2s)}\right)^s.$$ \hspace{1cm} (39)

It is notable that, the obtained metric function in Eq. (38) is related to $s \neq 1$. Also, $m_0$ is an integration constant which is related to the mass of solutions.

Now, one can calculate the nonzero components of stress-energy tensor by using the introduced basis vectors in Eq. (17) as

$$T^{\hat{t}\hat{t}} = \frac{1}{2} \left(2 \sqrt{F_{\phi \phi}} \frac{\partial \phi}{\partial \rho}\right)^s,$$ \hspace{1cm} (40)

$$T^{\hat{\rho}\hat{\rho}} = T^{\hat{\phi}\hat{\phi}} = \left(s - \frac{1}{2}\right) \left(2 \sqrt{F_{\phi \phi}} \frac{\partial \phi}{\partial \rho}\right)^s.$$ \hspace{1cm} (41)

According to the above equation, $\mu (T^{\hat{t}\hat{t}})$ is positive, and so the NEC, WEC, and SEC are satisfied, simultaneously. In addition, in order to satisfy the DEC, the parameter of PMI ($s$) must be in the range $\frac{1}{2} < s < 1$. As we have mentioned before, the energy conditions do not depend on the massive parameter. Here, we want to investigate the effects of PMI parameter ($s$) and electrical charge ($q$) on the energy conditions, so we plot $T^{\hat{t}\hat{t}}$ versus $\rho$ in Fig. 3.

As one can see, increasing the parameter of PMI theory and electrical charge leads to increasing the concentration of energy density.

One can show that the metric (7) with the metric function (38) has a singularity at $\rho = 0$ by calculating the Kretschmann scalar as

$$R_{\mu \nu \lambda \kappa} R^{\mu \nu \lambda \kappa} = 12 \Lambda^2 - \frac{4 cc_1 m^2}{\rho} \left[2 \Lambda - (2s - 1) \chi_{q,\rho,s} - \frac{cc_1 m^2}{2 \rho}\right]$$

$$+ \left[(8s^2 - 8s + 3) \chi_{q,\rho,s} - 4(4s - 3) \Lambda\right] \chi_{q,\rho,s}.$$ \hspace{1cm} (42)

From Eq. (42), it is obvious that the Kretschmann scalar reduces to $12 \Lambda^2$ for $\rho \to \infty$ and diverges at $\rho = 0$. On the other hand, as we mentioned before, it is not possible to extend spacetime to $\rho < r_0$ because of signature
changing. Also, one can apply the coordinate transformation (20) to the metric (7) and find the metric function as

\[ g(r) = m_0 - \Lambda \left(r^2 + r_0^2\right) + cc_1m^2\left(r^2 + r_0^2\right)^{1/2} + \frac{(2s - 1)^2 \left(r^2 + r_0^2\right)}{2(s - 1)} \chi(r), \]

where

\[ \chi(r) = \left(\frac{2q^2}{r^2}\right)^{\frac{1}{2}} \left(r^2 + r_0^2\right)^{1/2(1-2s)} \]

and the electromagnetic field in the new coordinate is

\[ F_{r\varphi} = q \left(r^2 + r_0^2\right)^{1/2(1-2s)}. \]

Due to complexity of obtained relation in Eq. (46), it is not possible to calculate the root and divergence points of deficit angle analytically, therefore, we study them in some graphs in next section.

III. DEFICIT ANGLE DIAGRAMS

In order to study the effects of different parameters on the properties of deficit angle for the Maxwell and PMI cases, we have plotted various diagrams (Figs. 4-10 for Maxwell case and Figs. 11-15 for PMI case). The left panels are dedicated to adS spacetime while the right ones are related to dS spacetime. In Ref. [82], it was pointed out that in cases, we have plotted various diagrams (Figs. 4-6 for Maxwell case and Figs. 7-10 for PMI case). The left panels are given by Eq. (26) and \( \Omega \) has the following form

\[ \Omega = \Lambda - \frac{cc_1m^2}{2r_0} - \frac{(2s - 1)}{2} \left(\frac{2q^2}{r_0^2}\right)^{2/(1-2s)} \]

Due to complexity of obtained relation in Eq. (46), it is not possible to calculate the root and divergence points of deficit angle analytically, therefore, we study them in some graphs in next section.
FIG. 4: Maxwell solutions: $\delta \varphi$ versus $r_0$ for $q = 0.1$, $c = 1$ and $c_1 = 2$, $m = 0$ (continuous line), $m = 0.5$ (dotted line), $m = 0.9$ (dashed line) and $m = 1.2$ (dashed-dotted line).
Left diagram: $\Lambda = -1$; Right diagram: $\Lambda = 1$.

FIG. 5: Maxwell solutions: $\delta \varphi$ versus $r_0$ for $m = 1$, $c = 1$ and $c_1 = 2$, $q = 0$ (continuous line), $q = 0.5$ (dotted line), $q = 1$ (dashed line) and $q = 1.5$ (dashed-dotted line).
Left diagram: $\Lambda = -1$; Right diagram: $\Lambda = 1$.

with cutting an arbitrary slice and sewing together the edges. The singular point is located at the apex of cone. Now, considering this concept, one can see that positive values of the deficit angle represent missing segment of the 2-dimensional plane (Fig. 11). On the contrary, the negative values of the deficit angle represent the additional part that we can add to the mentioned plane (Fig. 12). Therefore, the positivity/negativity of the deficit angle plays a crucial role in the topological structure of the solutions. Here, we see that depending on choices of different parameters, it is possible to obtain negative and positive values of the deficit angle. The roots of deficit angle could be interpreted as transition points in which the total shape of the object is modified. On the other hand, the existence of divergencies for deficit angle marks the possibility of the absence of magnetic solutions which was observed for both the dS and adS spacetimes. Previously, through several studies, it was shown that existence of deficit/surplus angle enables one to regard the cosmological constant problem [83]. The main motivation of this paper was understanding the effects of massive gravity and PMI theory on the magnetic solutions. The variation in deficit angle shows that the total structure of the solutions depends on contributions of these two generalizations. Specially, we observed that generalization to massive gravity provided the possibility of existence of divergence points for adS spacetime. It is worthwhile to mention that for adS case, between two divergencies, the values of deficit angle are within prohibited range. This indicates that there is no acceptable deficit angle between the divergencies in adS case.
FIG. 6: Maxwell solutions: $\delta \varphi$ versus $r_0$ for $q = 0.1$, $c = 1$ and $m = 1$, $c_1 = -5$ (continuous line), $c_1 = -4$ (dotted line), $c_1 = -3$ (dashed line), $c_1 = -1.95$ (dashed-dotted line) and $c_1 = -1$ (bold line).
Left diagram: $\Lambda = -1$; Right diagram: $\Lambda = 1$.

FIG. 7: PMI solutions: $\delta \varphi$ versus $r_0$ for $q = 0.1$, $c = 1$, $c_1 = 2$ and $s = 0.9$, $m = 0$ (continuous line), $m = 0.5$ (dotted line), $m = 0.85$ (dashed line) and $m = 1.2$ (dashed-dotted line).
Left diagram: $\Lambda = -1$; Right diagram: $\Lambda = 1$.

IV. CONCLUSIONS

In this paper, we have considered magnetic solutions which contain a conical singularity without any event horizon and curvature singularity. The set up for the gravity and energy momentum tensor were consideration of two generalizations: massive gravity and PMI nonlinear electromagnetic field.

The geometrical properties of the solutions were obtained and deficit angle for the two cases of Maxwell-massive and PMI-massive were extracted. It was shown that the general structure of the solutions depends on choices of different parameters through positivity and negativity of the deficit angle. Existence of root and divergency were reported and it was shown that these properties of the solutions depend on the choices of different parameters, such as massive gravity and nonlinearity parameter. In addition, it was shown that depending on the nature of background (being dS or adS), deficit angle, hence geometrical structure of the solutions would be different. The difference was highlighted analytically and numerically through several diagrams.

The existence of root and divergency for deficit angle was reported which indicates that under certain conditions, suitable choices of different parameters, topological defects known as magnetic solutions would enjoy geometrical phase transition. The dependency of geometrical phase transition on nonlinearity parameter and massive gravity highlighted the importance and roles of massive gravity and also nonlinear electromagnetic field generalizations. Especially, the
existence of divergency for AdS spacetime in the presence of massive gravity could be pointed out.

The existence of deficit and surplus angles results into two completely different astrophysical objects which essentially requires different methods for detection (see Figs. 11 and 12). In fact, when we are talking about deficit angle, it means that the geometrical structure of the solutions enjoys a positive tension in their structures. On the contrary, existence of the surplus angle corresponds to presence of the negative tension [62]. In this paper, we showed that depending on choices of different parameter, the possibility of both are provided for our magnetic solutions. In fact, in some cases, the existence of discontinuity, hence phase transition between deficit angle and surplus angle was reported for our solutions. Considering the important applications of the deficit/surplus angle in the context of cosmology and cosmological constant problem, one can employ the results of present paper to understand the roles of massive gravity and nonlinear electromagnetic field on these applications and their corresponding results. We leave these matters for future works.
FIG. 10: PMI solutions: $\delta \varphi$ versus $r_0$ for $q = 0.1$, $c = 1$, $c_1 = 2$ and $m = 1$, $s = 0.6$ (continuous line), $s = 0.7$ (dotted line), $s = 0.8$ (dashed line) and $s = 0.9$ (dashed-dotted line).

Left diagram: $\Lambda = -1$; Right diagram: $\Lambda = 1$.

FIG. 11: deficit angle: by sewing the two edges together (left panel), a cone is formed (right panel).

FIG. 12: surplus angle: by adding additional angle (middle panel) to a circle (left panel), we obtain a new figure (right panel).
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