On the Schrödinger Operator with a Periodic PT-symmetric Matrix Potential

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Abstract
In this article we obtain asymptotic formulas for the Bloch eigenvalues of the operator generated by a system of Schrödinger equations with periodic PT-symmetric complex-valued coefficients. Then using these formulas we classify the spectrum $\sigma(L)$ of $L$ and find a condition on the coefficients for which $\sigma(L)$ contains all half line $[H, \infty)$ for some $H$.

Key Words: Non-self-adjoint differential operator, PT-symmetric coefficients, Periodic matrix potential.
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1 Introduction and Preliminary Facts
Let $L(Q)$ be the differential operator generated in the space $L^m_2(\mathbb{-\infty, \infty})$ of the vector functions by the differential expression

$$-y'' + Qy,$$

where $Q = (q_{i,j})$ is a $m \times m$ matrix with the PT-symmetric $\pi$-periodic locally square integrable entries $q_{i,j}$. In other words,

$$q_{i,j}(-x) = q_{i,j}(x), \quad q_{i,j}(x + \pi) = q_{i,j}(x), \quad q_{i,j} \in L^2_2[0, \pi].$$

It is well-known that $[1, 5, 7]$ the spectrum $\sigma(L(Q))$ of the operator $L(Q)$ is the union of the spectra $\sigma(L_t(Q))$ of the operators $L_t(Q)$ for $t \in (-1, 1]$ generated in $L^m_2[0, \pi]$ by the differential expression (1) and the quasiperiodic conditions

$$y'(\pi) = e^{i\pi t} y'(0), \quad y(\pi) = e^{i\pi t} y(0).$$

Note that $L^m_2(a,b)$ is the set of the vector functions $f = (f_1, f_2, \ldots, f_m)$ with $f_k \in L^2_2(a,b)$ for $k = 1, 2, \ldots, m$. The norm $\|\cdot\|$ and inner product $(\cdot, \cdot)$ in $L^m_2(a,b)$ are defined by

$$\|f\| = \left( \int_{(a,b)} |f(x)|^2 \, dx \right)^{\frac{1}{2}}, \quad (f, g) = \int_{(a,b)} \langle f(x), g(x) \rangle \, dx,$$
where $|\cdot|$ and $\langle \cdot, \cdot \rangle$ are the norm and inner product in $\mathbb{C}^m$. For $t \in (-1, 1]$ the spectra $\sigma(L_t(Q))$ of the operators $L_t(Q)$ consist of the eigenvalues called the Bloch eigenvalues of $L(Q)$. Any eigenfunction $\Psi_{\lambda(t)}$ corresponding to the Bloch eigenvalue $\lambda(t)$ is called the Bloch function. We say that $\Psi_{\lambda(t)}$ is a normalized Bloch Function if its $L_2^m(0, \pi)$ norm is 1.

Let us introduce some preliminary results and describe briefly the scheme of the paper. The eigenvalues of the operator $L_t(Q)$ are the roots of the characteristic determinant

$$
\Delta(\lambda, t) = \det(Y_{2n}^{(\nu-1)}(\pi, \lambda) - \lambda e^{i\pi t}Y_{2n}^{(\nu-1)}(0, \lambda))_{j, \nu=1}^2 = e^{i2m\pi t} + f_1(\lambda)e^{i(2m-1)\pi t} + f_2(\lambda)e^{i(2m-2)\pi t} + \ldots + f_{2m-1}(\lambda)e^{i\pi t} + 1
$$

which is a polynomial of $e^{i\pi t}$ with entire coefficients $f_1(\lambda), f_2(\lambda), \ldots$, where $Y_1(x, \lambda)$ and $Y_2(x, \lambda)$ are the solutions of the matrix equation

$$-Y''(x) + Q(x)Y(x) = \lambda Y(x)
$$

satisfying $Y_1(0, \lambda) = O_m, \ Y_1'(0, \lambda) = I_m$ and $Y_2(0, \lambda) = I_m, \ Y_2'(0, \lambda) = O_m$ (see [6] Chapter 3). Here $O_m$ and $I_m$ are $m \times m$ zero and identity matrices respectively. It is clear that

$$
\varphi_{k,1,t} = \left( \frac{e^{i(2k+1)\nu \pi t}}{\sqrt{\nu}} \right), \quad \varphi_{k,2,t} = \left( \frac{0}{\sqrt{\nu}} \right), \quad \ldots, \quad \varphi_{k,m,t} = \left( \frac{0}{\sqrt{\nu}} \right)
$$

are the normalized eigenfunctions of the operator $L_t(O_m)$ corresponding to the eigenvalue $(2k + t)^2$. If $t \neq 0, 1$, then the multiplicity of the eigenvalue $(2k + t)^2$ is $m$ and the corresponding eigenspace is $E_k(t) = \text{span} \{ \varphi_{k,1,t}, \varphi_{k,2,t}, \ldots, \varphi_{k,m,t} \}$. In the cases $t = 0$ and $t = 1$ the multiplicity of the nonzero eigenvalues $(2k)^2$ and $(2k + 1)^2$ is $2m$ and the corresponding eigenspaces are

$$
E_k(0) = \text{span} \{ \varphi_{n,j,0} : n = k, -k; \ j = 1, 2, \ldots, m \} \\
E_k(\pi) = \text{span} \{ \varphi_{n,j,1} : n = k, -(k+1); \ j = 1, 2, \ldots, m \}
$$

respectively.

The brief scheme of the paper is the following. First we note that (see Theorem 1) if $\lambda$ is an eigenvalue of multiplicity $p$ of the operator $L_t(Q)$, then $\lambda$ is also an eigenvalue of the same multiplicity of $L_t(Q)$. This is the characteristic property of the Schrödinger operator with PT-symmetric potential. Then, in Theorem 2 we prove that there exists a constant $c$ such that the eigenvalues $\lambda(t)$ of the operator $L_t(Q)$ lie on the $c$ neighborhoods of the eigenvalues $(2n + t)^2$ of $L_t(O_m)$ for $n \in \mathbb{Z}$. For this we use the formula

$$
\left( \lambda(t) - (2n + t)^2 \right) (\Psi_{\lambda(t)}, \varphi_{n,s,t}) = (Q\Psi_{\lambda(t)}, \varphi_{n,s,t}),
$$

which can be obtained from the equality $L_t(Q) \Psi_{\lambda(t)} = \lambda(t)\Psi_{\lambda(t)}$ by multiplying both sides by $\varphi_{n,s,t}(x)$ and using $L_t(O_m) \varphi_{n,s,t}(x) = (2n + t)^2 \varphi_{n,s,t}(x)$. In the other words, first we consider the operator $L_t(O_m)$ for an unperturbed operator and the operator of multiplication by $Q$ for a perturbation.
Then to obtain a sharp asymptotic formulas we consider the operator $L_t(Q)$ as perturbation of $L_t(A)$, where

$$A = \int_{(0,\pi)} Q(x) \, dx,$$

by $Q - A$, that is, we take the operator $L_t(A)$ for an unperturbed operator and the operator of multiplication by $Q - A$ for a perturbation. Therefore first we analyze the eigenvalues and eigenfunctions of $L_t(A)$.

Using (2) and the substitution $t = -x$ one can get the equality

$$\int_{0}^{\pi} q_{i,j}(x) \, dx = \int_{0}^{\pi} q_{i,j}(-x) \, dx = -\int_{0}^{-\pi} q_{i,j}(t) \, dt = \int_{0}^{\pi} q_{i,j}(t) \, dt$$

which means that

$$\int_{0}^{\pi} q_{i,j}(x) \, dx \in \mathbb{R}$$

for all $i$ and $j$. Hence the entries of the matrix $A$ are the real numbers. Therefore, the eigenvalues of the matrix $A$ consist of the real eigenvalues and the pairs of the conjugate complex numbers. The distinct eigenvalues of $A$ are denoted by $\mu_1, \mu_2, ..., \mu_p$. Without loss of generality we denote the real distinct eigenvalues by $\mu_1 < \mu_2 < ... < \mu_s$ and the nonreal distinct eigenvalues by $\mu_{s+1}, \mu_{s+2}, ..., \mu_p$. It readily implies that the spectrum of $L(A)$ consists of the real half line $[\mu_1, \infty)$ and nonreal half lines

$$[\mu_j + a : a \in [0, \infty))$$

for $j = s + 1, s + 2, ..., p$.

In Theorem 3 we find a sharp and uniform, with respect to the quasimomenta $t \in (-1, 1]$, asymptotic formula for the Bloch eigenvalues of $L(Q)$ in term of the eigenvalues of the matrix $A$ for any matrix potential $Q$ with locally square integrable entries. This asymptotic formula implies that the spectrum of $L(Q)$ is asymptotically close to the spectrum of the operator $L(A)$. Therefore, if the matrix $A$ has no real eigenvalues then the spectrum of the operator $L(Q)$ with general potential $Q$ approaches the nonreal half lines (7), which implies that the real component $\sigma(L(Q)) \cap \mathbb{R}$ of the spectrum $\sigma(L(Q))$ is contained in a finite interval $[a, b]$ (see Theorem 4). Then we prove that, if the entries of $Q$ are the PT-symmetric functions, and the matrix $A$ has a real eigenvalue with odd multiplicity, then $\sigma(L(Q))$ contains the main part (in sense of (41)) of $[0, \infty)$ (see Theorem 6). For this we consider the multiplicity of the Bloch eigenvalues (see Theorem 5). Note that if $m$ is an odd number, then the matrix $A$ has a real eigenvalue with odd multiplicity, because the total sum of the multiplicities of the eigenvalues $\mu_1, \mu_2, ..., \mu_p$ is $m$ and the nonreal eigenvalues are the pairs of the conjugate complex numbers with the same multiplicity. Therefore Theorem 6 implies that if $m$ is an odd number and (2) holds, then the spectrum of $L(Q)$ contains the main part of $[0, \infty)$ (see Corollary 1). In Theorem 7 we find a condition on the real eigenvalues of $A$ for which the spectrum of $L(Q)$ contains all half line $[H, \infty)$ for some $H$.  

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Finally, note that in the papers (see [4, 9, 11]) we investigated the non-self-adjoint Schrödinger operator with a periodic matrix potential. However, as far as I know, this paper is the first paper about the Schrödinger operator with a PT-symmetric periodic matrix potential. In this paper, we investigate the influence of PT-symmetry on spectrum of the operator $L(Q)$.

2 On the Bloch eigenvalues and Spectrum

First let us note that in the case $m = 1$, it is well-known that if $\lambda \in L_t(Q)$, then $\overline{\lambda} \in L_t(Q)$ (see [3, 8, 12]). Taking into account that this obvious fact was not formulated for arbitrary $m$, we prove the following more general statement.

**Theorem 1.** If $\lambda$ is an eigenvalue of multiplicity $v$ of the operator $L_t(Q)$, then $\overline{\lambda}$ is also an eigenvalue of the same multiplicity of $L_t(Q)$.

**Proof.** First let us note that $\lambda \in \sigma(L_t(Q))$ if and only if there exists a solution $\Psi(\cdot, \lambda)$ of the equation

$$-y''(x) + Q(x)y(x) = \lambda y(x)$$

satisfying the equality

$$\Psi(x + \pi, \lambda) = e^{it}\Psi(x, \lambda)$$

for all $x \in (-\infty, \infty)$. Indeed if $\lambda \in \sigma(L_t(Q))$, then there exists a solution $\Phi(x, \lambda)$ of (8) satisfying (3). Therefore the solution $y(x, \lambda)$ of (8) defined by

$$y(x, \lambda) = \Psi(x + \pi, \lambda) - e^{it}\Psi(x, \lambda)$$

satisfies the initial conditions $y(0, \lambda) = y'(0, \lambda) = 0$. Then by the uniqueness theorem $y(x, \lambda) = 0$ for all $x \in (-\infty, \infty)$, that is, (9) holds.

Now suppose that there exists a solution $\Psi(x, \lambda)$ of (8) satisfying (9). Then it is clear that $\Psi(x, \lambda)$ satisfies (3). It means that $\lambda \in \sigma(L_t(Q))$.

Now we are ready to prove the theorem. If $\lambda$ is an eigenvalue of $L_t(Q)$, then there exists a solution $\Psi(x, \lambda)$ of (8) satisfying (9). Then using (2) one can readily see that the function $\Phi(x)$ defined by $\Phi(x, \lambda) = \Psi(-x, \lambda)$ satisfies the equation

$$-y''(x) + Q(x)y(x) = \overline{\lambda}y(x)$$

and the equality

$$\Phi(x + \pi) = \overline{\Psi(-x - \pi, \lambda)} = e^{-it}\Psi(-x, \lambda) = e^{it}\Phi(x).$$

It means that $\overline{\lambda}$ is also an eigenvalue of $L_t(Q)$ and $\Phi$ is the corresponding eigenfunction.

Instead of the equation (8) using the equations

$$-\Psi_j''(x, \lambda) + Q(x)\Psi_j(x, \lambda) - \lambda\Psi_j(x, \lambda) = \Psi_{j-1}(x, \lambda)$$

(10)
for the associated functions $\Psi_1(x, \lambda), \Psi_2(x, \lambda), \ldots, \Psi_s(x, \lambda)$, where $j = 1, 2, \ldots, s$ and $\Psi_0(x, \lambda) = \Psi(x, \lambda)$ and repeating the above argument we conclude that if $y(x, \lambda)$ is an associated function corresponding to the eigenfunction $\Psi(x, \lambda)$, then $y(-x, \lambda)$ is an associated eigenfunction corresponding to the eigenfunction $\Phi(x, \lambda)$. Therefore the multiplicities of the eigenvalues $\lambda$ and $\bar{\lambda}$ are the same. The theorem is proved.

Now we estimate the Bloch eigenvalues. First we prove the following simple estimation for the Bloch eigenvalues.

**Theorem 2** (a) There exists a constant $M$ such that
\[
\sup_{\lambda \in \sigma(L(Q))} |\Psi(\lambda)| < M, \tag{11}
\]
where $\Psi(\lambda)$ is a normalized Bloch function corresponding to the eigenvalue $\lambda$.

(b) All eigenvalues of the operator $L_t(Q)$ lie on the union of the disks
\[
D \left((2n + t)^2, c\right) := \{ \lambda \in \mathbb{C} : |\lambda - (2n + t)^2| < c \} \tag{12}
\]
for $n \in \mathbb{Z}$, where $t \in (-1, 1]$, $c = MB$ and
\[
B = \left( \int_0^\pi \left( \sum_{i,j=1}^m |q_{i,j}(x)|^2 \right) dx \right)^{1/2}. \tag{13}
\]

**Proof.** (a) Let $\Psi_{\lambda(t)}$ be a normalized eigenfunction (Bloch function) corresponding to the eigenvalue $\lambda(t)$ of the operator $L_t(Q)$. There exists $n$ such that $(2n - 1)^2 \leq \Re \lambda(t) < (2n + 1)^2$. Then it is clear that there exists a constant $C$ such that
\[
\sum_{k \in \mathbb{Z} \setminus B(n)} \frac{1}{|\lambda(t) - (2k + t)^2|} < C \tag{14}
\]
for all $t \in (-1, 1]$, where $B(n) = \{ \pm (n - 1), \pm n, \pm (n + 1) \}$. In the decomposition
\[
\Psi_{\lambda(t)} = \sum_{k \in B(n), s=1,2,\ldots,m} (\Psi_{\lambda(t)}, \varphi_{k,s,t}) \varphi_{k,s,t} + \sum_{k \in (\mathbb{Z} \setminus B(n)), s=1,2,\ldots,m} (\Psi_{\lambda(t)}, \varphi_{k,s,t}) \varphi_{k,s,t}
\]
of $\Psi_{\lambda(t)}$ by the orthonormal basis
\[
\{ \varphi_{k,s,t} : k \in \mathbb{Z}, s = 1, 2, \ldots, m \} \tag{15}
\]
replacing $(\Psi_{\lambda(t)}, \varphi_{k,s,t})$ for $k \in (\mathbb{Z} \setminus B(n))$ by
\[
\frac{(\Psi_{\lambda(t)}, Q^* \varphi_{k,s,t})}{\lambda(t) - (2k + t)^2}
\]
(see (5)) and then using (14), (13) and the obvious inequalities
\[
|(|(\Psi_{\lambda(t)}, Q^* \varphi_{k,s,t})| \leq \|Q^* \varphi_{k,s,t}\| \leq \frac{1}{\sqrt{\pi}} B
\]
we obtain the proof of (11).

(b) Suppose that there exists an eigenvalue $\lambda(t)$ of $L_t(Q)$ lying out of $D((2n + t)^2, c)$ for all $n \in \mathbb{Z}$. Then the inequality

$$|\lambda(t) - (2n + t)^2| \geq c$$

holds for all $n \in \mathbb{Z}$. Using the Parseval’s equality for the orthonormal basis (15) and then (5) we get

$$1 = \sum_{n \in \mathbb{Z}, s=1,2,...,m} |(\Psi_{\lambda(t)}, \varphi_{n,s,t})|^2 \leq \sum_{n \in \mathbb{Z}, s=1,2,...,m} \frac{|(Q\Psi_{\lambda(t)}, \varphi_{n,s,t})|^2}{c^2} = \frac{\|Q\Psi_{\lambda(t)}\|^2}{c^2},$$

where

$$\|Q\Psi_{\lambda(t)}\|^2 = \int_0^\pi |Q(x)\Psi_{\lambda(t)}(x)|^2 dx.$$ 

On the other hand, it follows from (11) that

$$|Q(x)\Psi_{\lambda(t)}(x)|^2 < \left( \sum_{i,j=1}^m |q_{i,j}(x)|^2 \right) M^2.$$ 

Therefore by (13) we have

$$\|Q\Psi_{\lambda(t)}\|^2 < c^2.$$  

Using (17) in (16) we get a contradiction $1 < 1$. The theorem is proved. ■

Now to obtain a sharp and uniform with respect to $t \in (-1,1]$ asymptotic formula we consider the operator $L_t(Q)$ as perturbation of $L_t(A)$ and analyze the spectrum of the operators $L_t(A)$, where $A$ is defined in (6). For this we introduce the following notations. Suppose the matrix $A$ has $p$ distinct eigenvalues $\mu_1, \mu_2, ..., \mu_p$ with multiplicities $m_1, m_2, ..., m_p$ respectively, where $m_1 + m_2 + ... + m_p = m$. Let $u_{j,1}, u_{j,2}, ..., u_{j,s_j}$ be the linearly independent eigenvectors corresponding to the eigenvalue $\mu_j$. Denote by $u_{j,s,1}, u_{j,s,2}, ..., u_{j,s,r_{j,s}}$ the associated vectors corresponding to the eigenvector $u_{j,s}$, such that

$$(A - \mu_j I) u_{j,s,k} = u_{j,s,k-1}$$

for $k = 1, 2, ..., r_{j,s}-1$, where $u_{j,s,0} = u_{j,s}$. Note that $r_{j,s}$ is called the multiplicity of the eigenfunction $u_{j,s}$ and $r_{j,1} + r_{j,2} + ... + r_{j,s_j} = m_j$. The number $r_j$ defined by $r_j = \max_s r_{j,s}$ is a maximum multiplicity of the eigenfunctions corresponding to the eigenvalue $\mu_j$. It is not hard to see that the eigenvalues, eigenfunctions and associated functions of $L_t(A)$ are

$$\mu_{k,j}(t) = (2k + t)^2 + \mu_j, \ \Phi_{k,j,s}(x) = u_{j,s}e^{i(2k+t)x}, \ \Phi_{k,j,s,r}(x) = u_{j,s,r}e^{i(2k+t)x}$$
respectively, since they satisfy the equation obtained from (10) by replacing $Q$ to $A$. Similarly, the eigenvalues, eigenfunctions, and associated functions of $L_t^*(A)$ are $\mu_{k,j}$,

$$
\Phi_{k,j,s}^*(x) = u_{j,s} e^{i(2k+t)x} \quad \Phi_{k,j,s,r}^*(x) = u_{j,s,r} e^{i(2k+t)x},
$$

where $u_{j,s}^* = u_{j,s,0}$ and $u_{j,s,r}^*$ are the eigenvector and associated vector of $A^*$ corresponding to $\mu_{j}$. To obtain the sharp asymptotic formulas we use the formula

$$
(\lambda - \mu_{n,j})^{r+1}(\Psi, \Phi_{n,j,s,r}^*) = \sum_{q=0}^{r} (\lambda - \mu_{n,j})^q ((Q - A)\Psi, \Phi_{n,j,s,q}^*)
$$

of [9] (see (15) of [9]), where $\Psi$ is a normalized eigenfunction of $L_t(Q)$ corresponding to the eigenvalue $\lambda$ and estimate the multiplicands $(\Psi, \Phi_{n,j,s,r}^*)$ and $((Q - A)\Psi, \Phi_{n,j,s,q}^*)$ in the left-hand side and the right-hand side of (20). For this we use the following proposition which can be proved by direct calculations.

**Proposition 1** Let $A(k, t)$ be $\{k\}$, $\{\pm k\}$, $\{k, -k - 1\}$ and $\{k, -k + 1\}$ respectively if $t \in ((-2/3, -1/3) \cup (1/3, 2/3))$, $t \in (-1/3, 1/3)$, $t \in (2/3, 1)$ and $t \in (-1, -2/3)$. If $|k| > 1$, then

$$
\left| (2n + t)^2 - (2k + t)^2 \right| \geq \frac{4}{3} (2 |k| - 1)
$$

for all $n \in (Z \setminus A(k, t))$.

In the following estimations we use the positive constants denoted by $c_k$ for $k = 1, 2, ..., $ independent on $t$, whose exact values are inessential.

**Lemma 1** Let $\Psi$ be a normalized eigenfunction of $L_t(Q)$ corresponding to the eigenvalue $\lambda$ lying in the disk $D((2k+t)^2, c)$, where

$$
|k| > 1, \frac{4}{3} (2 |k| - 1) > 3c
$$

and $c$ is defined in (12). Then there exists $n \in A(k, t)$ such that

$$
\left| (\Psi, \Phi_{n,j,s,r}^*) \right| \geq c_1
$$

for some $j, s, r$ and

$$
\left| ((Q - A)\Psi, \Phi_{n,j,s,q}^*) \right| \leq c_2 \left( \frac{1}{|k|} + q_k \right),
$$

for all $j, s, q$, where

$$
q_k = \max\{ |q_{i,j,s}| : i, j = 1, 2, ..., m; s \in \{ \pm 2k, \pm (2k + 1), \pm (2k - 1) \}\},
$$
\[ q_{i,j,s} = \frac{1}{\sqrt{\pi}} \int_0^\pi q_{i,j}(x) e^{-2isx} \, dx, \]

and \( q_{i,j}(x) \) is the entry of the matrix \( Q(x) \).

**Proof.** By Proposition 1 we have

\[ \left| \lambda - (2n + t)^2 \right| \geq \frac{4}{3} (2|k| - 1) - c > 2c \]  \hspace{1cm} (24)

for all \( t \in (-1, 1] \) and \( n \in (\mathbb{Z} \setminus A(k,t)) \) if \( \lambda \in D((2k + t)^2, c) \) and \( k \) satisfies conditions (21). Therefore using (5) Bessel inequality and (17) we obtain

\[
\sum_{n \in (\mathbb{Z} \setminus A(k,t))} \left| (\Psi, \varphi_{n,s,t}) \right|^2 = \sum_{n \in (\mathbb{Z} \setminus A(k,t))} \left| (\Psi_Q, \varphi_{n,s,t}) \right| \frac{\sqrt{\lambda - (2\pi n + t)^2}}{4c^2} \leq \frac{1}{4},
\]

It with the Parsaval’s equality implies that there exist \( n \in A(k,t) \) and \( i \in \{1, 2, ..., m\} \) such that

\[ \left| (\Psi, \varphi_{n,i,t}) \right| > c_3. \]

Since the system of the root vectors of the matrix \( A^* \) is a basis of \( \mathbb{C}^m \) and (19) holds, the last inequality implies that (22) holds for some \( j,s,r \).

Now to prove (23) we estimate the term \( ((Q - A)\Psi, \varphi_{n,i,t}) \) and take (19) into account. Using the decomposition

\[ \Psi = \sum_{v \in \mathbb{Z}, \ s=1,2,\ldots,m} (\Psi, \varphi_{v,s,t}) \varphi_{v,s,t} \]

of \( \Psi \) by the orthonormal basis (15) we obtain

\[ ((Q - A)\Psi, \varphi_{n,i,t}) = \sum_{v \in (\mathbb{Z} \setminus \{n\})} q_{i,s,n-v} (\Psi, \varphi_{v,s,t}) . \]  \hspace{1cm} (25)

The right-hand side of (25) is the sum of

\[ S_1 =: \sum_{v \in A(k,t) \setminus \{n\}; \ s=1,2,\ldots,m} q_{i,s,n-v} (\Psi, \varphi_{v,s,t}) \]

and

\[ S_2 =: \sum_{v \in (\mathbb{Z} \setminus A(k,t)); \ s=1,2,\ldots,m} q_{i,s,n-v} (\Psi, \varphi_{v,s,t}) . \]

Here \( \Psi \) and \( \varphi_{v,s,t} \) are the normalized eigenfunctions and the set \( A(k,t) \setminus \{n\} \) consist of at most one number Moreover, it follows from the definition of \( A(k,t) \) that

\[ (n - v) \in \{\pm 2k, \pm (2k + 1), \pm (2k - 1)\} \]

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for all $n \in A(k, t)$ and $v \in (A(k, t) \setminus \{n\})$. Therefore, using the definition of $q_k$ we obtain

$$|S_1| \leq mq_k. \quad (26)$$

Now let us estimate $S_2$. By (5) and (24) we have

$$|S_2| \leq c_4 \frac{1}{|k|} \sum_{v \in (Z \setminus A(k, t))} |q_s,i,n - v| \cdot |(Q\Psi, \varphi_{v,s,t})|. \quad (27)$$

Now using Schr"{o}dinger inequality of the space $l_2$ and taking (17) into account we obtain

$$|S_2| \leq c_5 \frac{1}{|k|}. \quad (27)$$

Thus (23) follows from (26), (27) and (19).

Now we are ready to consider the Bloch eigenvalues in detail. The following theorem follows from (20) and Lemma 1.

**Theorem 3** If (21) holds, then the eigenvalues of the operator $L_t(Q)$ lying in $D((2k + t)^2, c)$ are contained in $\varepsilon_k$ neighborhood $D(\mu_{n,j}(t), \varepsilon_k)$ of the eigenvalues $\mu_{n,j}(t)$ of $L_t(A)$ for $j = 1, 2, ..., m$ and $n \in A(k, t)$, where $\varepsilon_k \leq c_6(|\frac{1}{k}| + q_k)^{1/r_j}$.

**Proof.** Dividing (20) by $(\Psi, \Phi^*_{n,j,s,r})$, and using (22) and (23) we obtain

$$(\lambda - \mu_{n,i}(t))^{r_i+1} = \sum_{q=0}^{r_i} (\lambda - \mu_{n,i}(t))^{q}O \left( \frac{1}{|k|} + q_k \right),$$

where $r_i + 1 = r_j$, from which we obtain the proof of the theorem.

Now using these results we classify the spectrum of $L(Q)$. Theorems 2 and 3 imply that the spectrum of $L(Q)$ is asymptotically close to the spectrum of the operator $L(A)$, since $\varepsilon_k \to 0$ as $k \to \infty$. It is clear that if the Fourier coefficients $q_{i,j,k}$ of the entries $q_{i,j}$ of $Q$ is $O(1/k)$, for example if the entries are the continuous functions, then $\varepsilon_k = O(1/k)$. The following theorem immediately follows from Theorems 2 and 3.

**Theorem 4** If the matrix $A$ has no real eigenvalues, then the real component of the spectrum of $L(Q)$ is contained in a finite interval $[a, b]$.

**Proof.** If the eigenvalues $\mu_1, \mu_2, ..., \mu_p$ of the matrix $A$ are nonreal numbers then there exists $c_7$ such that

$$\varepsilon_k < \min_{j=1,2,...,p} |\text{Im} \mu_j| \quad (28)$$

and (21) holds for $|k| > c_7$. Inequality (28) implies that the disks $D(\mu_{n,j}(t), \varepsilon_k)$ for $|k| > c_7$, $n \in \mathbb{Z}$ and $j = 1, 2, ..., p$ have no intersection points with the real lines. Therefore using Theorems 2 and 3 we obtain that the real component of
the spectrum of $L(Q)$ is contained in the bounded set $\bigcup_{|k| \leq c} D((2k + t)^2, c)$. It implies the proof of the theorem. ■

Now we consider the cases when the matrix $A$ has the real eigenvalues and investigate the real component of the spectrum of $L(q)$. Recall that (see the end of the introduction) the real and nonreal distinct eigenvalues are denoted respectively by $\mu_1 < \mu_2 < \ldots < \mu_s$ and $\mu_{s+1}, \mu_{s+2}, \ldots, \mu_p$.

To investigate the real spectrum of $L(Q)$, we use Theorems 2 and 3 and find the conditions on $k$ and $t$ such that the boundary of the closed disk $D(\mu, \varepsilon)$ for $j \leq s$ belong to the resolvent set of the operator $L_i(Q)$. Since $\mu_{k,i}(t) = \mu_{-k,i}(-t)$ (see (18)), it is enough to study the disks $D(\mu_{k,i}(t), \varepsilon)$ for $k \geq 0$ and $t \in (-1, 1]$. We consider the closed disks $D(\mu_{n,j}(t), \varepsilon)$ for $j \leq s, n \in A(k,t)$ and $k \geq N_1$, where $N_1$ is a positive integer such that if $k > N_1$, then

$$ \delta_k < c, \quad (29) $$
$$ \frac{4}{3} (2k - 1) > 3c + |\mu_j|, \quad (30) $$
$$ \delta_k < \min_{j=s+1,s+2,\ldots,p} |\text{Im} \mu_j|, \quad \delta_k < |\mu_j - \mu_i| \quad (31) $$

for $j \leq s, i \leq s$ and $i \neq j$, where $\delta_k = 2 \max \{\varepsilon_k, \varepsilon_{-k}, \varepsilon_{-k-1}, \varepsilon_{-k+1}\}$

**Lemma 2** Suppose that $k > N_1, 1 \leq j \leq s$ and $t \in (-1, 1]$. Then

(a) The closed disk $D(\mu_{k,j}(t), \varepsilon)$ has no common points with the disks $D((2n + t)^2, c)$ and $D(\mu_{k,i}(t), \varepsilon)$ for $n \notin A(k,t)$ and $i \neq j$.

(b) The disk $D(\mu_{k,j}(t), \varepsilon)$ has no common points with the disks $D(\mu_{n,i}(t), \varepsilon)$ for $n \in A(k,t)$ and $s + 1 \leq i \leq p$ for all $t \in (-1, 1]$. The disk $D(\mu_{k,j}(t), \varepsilon)$ has no common points with the disks $D(\mu_{n,i}(t), \varepsilon)$ for $n \in A(k,t)$ and $1 \leq i \leq s$ if

$$ t \in (-1, 1) \setminus U(j,k,\delta_k), \quad (32) $$

where $U(j,k,\delta_k) = U(j,k,i,-k,\delta_k) \cup U(j,k,i,-k-1,\delta_k) \cup U(j,k,i,-k+1,\delta_k)$,

$$ U(j,k,i,-k,\delta_k) = \left( \frac{\mu_i - \mu_j - \delta_k}{8k}, \frac{\mu_i - \mu_j + \delta_k}{8k} \right), $$

$$ U(j,k,i,-k-1,\delta_k) = \left( 1 + \frac{\mu_i - \mu_j - \delta_k}{4(2k + 1)}, 1 + \frac{\mu_i - \mu_j + \delta_k}{4(2k + 1)} \right), $$

and

$$ U(j,k,i,-k+1,\delta_k) = \left( -1 + \frac{\mu_i - \mu_j - \delta_k}{4(2k + 1)}, -1 + \frac{\mu_i - \mu_j + \delta_k}{4(2k + 1)} \right). $$

**Proof.** (a) Using Proposition 1 and (30) one can easily verify that the distance between the centres of the disks $D(\mu_{k,j}(t), \varepsilon)$ and $D((2n + t)^2, c)$ for $n \notin A(k,t)$ is greater than $3c$. On the other hand, by (29) the total sum of the radii of these disks is less than $2c$. Therefore these disks have no common points.
It follows from (31) that the distance \(|\mu_j - \mu_i|\) between the centres of the disks \(D(\mu_{k,j}(t), \varepsilon_k)\) \(D(\mu_{k,i}(t), \varepsilon_k)\) is greater than the total sum of their radii. That is why they also have no common point.

(b) The proof of the first statement follows from the first inequality of (31).

Now we prove the second statement. By the definition of \(A(k,t)\) it is enough to prove that the disks \(\overline{D(\mu_{k,j}(t), \varepsilon_k)}\) and \(\overline{D(\mu_{n,i}(t), \varepsilon_n)}\) have no common points for \(n = -k, n = -k - 1\) and \(n = -k + 1\) respectively if \(t \in ((-1, 1) \setminus U(j, k, i, -k, \delta_k))\), \(t \in ((-1, 1) \setminus U(j, k, i, -k - 1, \delta_k))\) and \(t \in ((-1, 1) \setminus U(j, k, i, -k + 1, \delta_k))\), where \(i = 1, 2, \ldots, s\). This can be easily verified by the direct calculations of the differences \(\mu_{k,j}(t) - \mu_{-k,i}(t), \mu_{k,j}(t) - \mu_{-k-1,i}(t)\) and \(\mu_{k,j}(t) - \mu_{-k+1,i}(t)\) and using the definitions of the sets
\[
U(j, k, i, -k, \delta_k), U(j, k, i, -k - 1, \delta_k)\text{ and } U(j, k, i, -k + 1, \delta_k).
\]
The lemma is proved. ■

Now using this lemma we prove the following theorem which plays a crucial role in the investigation of the real component of the spectrum of \(L(Q)\).

**Theorem 5** Suppose that \(k > N_1, j \leq s\) and (32) hold.

(a) If \(\mu_j\) is an eigenvalue of \(A\) of multiplicity \(v\) then the number of the eigenvalues (counting multiplicity) of \(L_t(Q)\) lying in \(D(\mu_{k,j}(t), \varepsilon_k)\) is \(v\). Moreover, if \(\lambda\) is an eigenvalue of multiplicity \(v\) of the operator \(L_t(Q)\) lying in \(D(\mu_{k,j}, \varepsilon_k)\), then \(\lambda\) is also an eigenvalue of the same multiplicity of \(L_t(Q)\) lying in \(D(\mu_{k,j}, \varepsilon_k)\).

(b) If \(\mu_j\) is a simple eigenvalue of \(A\), then the eigenvalue of \(L_t(Q)\) lying in \(U(\mu_{k,j}, \varepsilon_k)\) is a simple and real eigenvalue.

**Proof.** (a) First we prove that the boundary of \(D(\mu_{k,j}, \varepsilon_k)\) lies in the resolvent set of \(L_t(Q)\). By Theorem 2 the eigenvalues of \(L_t(Q)\) lie in the disks \(D((2n + t)^2, c)\) for \(n \in \mathbb{Z}\). By Lemma 2(a) the eigenvalues lying \(D((2n + t)^2, c)\) for \(n \in (\mathbb{Z} \setminus A(k,t))\) do not lie in the boundary of \(D(\mu_{k,j}, \varepsilon_k)\). It remains to prove the following statement. The eigenvalues lying \(D((2n + t)^2, c)\) for \(n \in A(k,t)\) do not lie in the boundary of \(D(\mu_{k,j}, \varepsilon_k)\).

To prove this statement we consider the following three cases separately: \(t \in ((-2/3, -1/3) \cup (1/3, 2/3))\), \(t \in (-1/3, 1/3)\) and \(t \in (-1, -2/3)\).

If \(t \in ((-2/3, -1/3) \cup (1/3, 2/3))\) then \(A(k,t) = \{k\}\) (see definition of \(A(k,t)\) in Proposition 1). Therefore the proof of the statement follows from Lemma 2(a).

If \(t \in (-1/3, 1/3)\), then \(A(k,t) = \{\pm k\}\). The case \(n = k\) follows from Lemma 2(a). Let us consider the case \(n = -k\). By Theorem 3 the eigenvalues of \(L_t(Q)\) lying in the disks \(D((-2k + t)^2, c)\) are contained in \(D(\mu_{n,j}, \varepsilon_k)\) for \(j = 1, 2, \ldots, p\) and \(n \in A(-k, t)\). Since \(A(-k, t) = \{\pm k\}\) we need to consider the disks \(D(\mu_{k,j}, \varepsilon_k)\) and \(D(\mu_{-k,j}, \varepsilon_k)\). The proof of the statement for \(D(\mu_{k,j}, \varepsilon_k)\) and \(D(\mu_{-k,j}, \varepsilon_k)\) follows from Lemma 2(a) and Lemma 2(b) respectively.

If \(t \in (2/3, 1)\) and \(t \in (-1, -2/3)\) then \(A(k,t) = \{k, -k - 1\}\) and \(A(k,t) = \{k, -k + 1\}\) respectively. Moreover \(A(-k - 1, t) = \{-k - 1, k\}\) and \(A(-k + 1, t) = \{-k + 1, k\}\). Therefore repeating the proof of the case \(t \in (-1/3, 1/3)\) we get the proof of this statement for the cases \(t \in (2/3, 1)\) and \(t \in (-1, -2/3)\).
Thus the circle \( \{ \lambda \in \mathbb{C} : |\lambda - \mu_{k,j}| = \varepsilon_k \} \) lies in the resolvent set of \( L_\varepsilon(Q) \). Consider the following family of operators

\[
L_\varepsilon = L_\varepsilon(A) + \varepsilon(Q - A), \quad 0 \leq \varepsilon \leq 1.
\]

Repeating the proof of the case \( \varepsilon = 1 \), one can easily verify that the circle \( \{ \lambda \in \mathbb{C} : |\lambda - \mu_{k,j}| = \varepsilon_k \} \) lies in the resolvent set of \( L_\varepsilon \) for \( \varepsilon \in [0,1] \). Therefore, taking into account that the family \( L_\varepsilon \) is holomorphic (in the sense of [2]) with respect to \( \varepsilon \), we obtain that the number of the eigenvalues of \( L_\varepsilon \) lying inside of \( \{ \lambda \in \mathbb{C} : |\lambda - \mu_{k,j}| = \varepsilon_k \} \) are the same for all \( \varepsilon \in [0,1] \). Since the operator \( L_0 = L_\varepsilon(A) \) has \( v \) eigenvalues (counting the multiplicity) inside \( \{ \lambda \in \mathbb{C} : |\lambda - \mu_{k,j}| = \varepsilon_k \} \) the operator \( L_\varepsilon(Q) \) has also \( v \) eigenvalues inside of this circle. Since \( \mu_j \) for \( j \leq s \) is a real number, if \( \lambda \in D(\mu_{k,j}, \varepsilon_k) \) then \( X \in D(\mu_{k,j}, \varepsilon_k) \) too. Therefore, the last statement of (a) follows from Theorem 1.

(b) It follows from (a) that if \( \mu_1 \) is a simple eigenvalue of \( A \), then the operator \( L_\varepsilon(Q) \) has a unique eigenvalue (counting multiplicity) in the disk \( D(\mu_{k,1}(t), \varepsilon_k) \). It means that \( \lambda_{k,1}(t) \) is a simple eigenvalue. Moreover, if \( \mu_j \) is a real number and \( \lambda_{k,j}(t) \) is a nonreal eigenvalue of \( L_\varepsilon(Q) \) lying in \( D(\mu_{k,j}(t), \varepsilon_k) \) then \( \lambda_{k,j}(t) \) is also an eigenvalue of \( L_\varepsilon(Q) \) lying in \( D(\mu_{k,j}(t), \varepsilon_k) \), which contradicts to the above uniqueness. Note also that if \( \mu_j \) is a simple eigenvalue of \( A \), then \( r_j = 1 \) and hence \( \varepsilon_k \leq c_0(\frac{1}{k} + q_k) \) (see Theorem 3). The theorem is proved.

Now using Theorem 5 we investigate the real component of the spectrum of \( L(Q) \). This investigation is based on the following idea. First we consider the set

\[
\{ \mu_{k,j}(t) : t \in ((-1,1) \setminus U(j,k,\delta_k)) \},
\]  

where \( U(j,k,\delta_k) \) is defined in (32). By the definition of \( \mu_{k,j}(t) \) the set

\[
\{ \mu_{k,j}(t) : t \in (-1,1) \} \text{ is the interval } (\mu_j + (2k - 1)^2, \mu_j + (2k + 1)^2].
\]

The union \( \{ \mu_{k,j}(t) : k > N_1, t \in (-1,1] \} \) is an increasing set. Since the set \( U(j,k,\delta_k) \) is the union of the small intervals for large \( k \), the union of the sets (33) for \( k > N_1 \) contains the main part of the real line \( \mu_j + (2N_1 + 1)^2, \infty \). Note that \( U(j,k,\delta_k) \) consists of the intervals defined in Lemma 2(b). Since \( \delta_k \to 0 \) as \( k \to \infty \), there exists \( N(j) > N_1 \) such that if \( k \geq N(j) \), then these intervals are pairwise disjoint. Thus we have

\[
(-1,1) \setminus U(j,k,\delta_k) = [a_1, b_1] \cup [a_2, b_2] \cup \ldots \cup [a_l, b_l],
\]  

for \( k \geq N(j) \), where \( a_i < b_i < a_2 < b_2 < \ldots \) and the sum of the length of these intervals is asymptotically close to the length of \((-1,1)\). Namely

\[
2 - \sum_{i=1}^{l} (b_i - a_i) < c_0 \frac{\delta_k}{k}.
\]  

Note that if \( a_1 = -1 \), then in (34) the interval \([a_1, b_1]\) should be replaced by \((a_1, b_1]\). This replacement does not change the investigation. Therefore, without loss of generality, we assume that (34) holds. Since \( \mu_{k,j} \) is an increasing
function on \((-1, 1]\), the image \(\mu_{k,j}([a_i, b_i])\) of the interval \([a_i, b_i]\) is the interval 

\[
\mu_{k,j}((-1, 1]\setminus U(j, k, \delta_k)) = \bigcup_{i=1,2,\ldots,l} [\mu_{k,j}(a_i), \mu_{k,j}(b_i)]
\]  

(36)

for \(k \geq N(j)\). Using (35) and definition of the function \(\mu_{k,j}\) we see that the length of the intervals in (36) is asymptotically close to the length of \((\mu_j + (2k - 1)^2, \mu_j + (2k + 1)^2]\) in the sense that

\[
8k - \sum_{i=1,2,\ldots,l} (\mu_{k,j}(b_i) - \mu_{k,j}(a_i)) < c_{10}\delta_k \to 0
\]  

(37)

as \(k \to \infty\). Thus the set \(\bigcup_{k \geq N(j)} \mu_{k,j}((-1, 1]\setminus U(j, k, \delta_k))\) contains the main part of the half line \((\mu_j + (2N(j) - 1)^2, \infty)\).

Now, using Theorem 5 and (37) we prove that the spectrum of \(L(Q)\) contains the main part of the half line \([0, \infty)\). First let us explain it in the simplest case, when \(\mu_j\) is a simple eigenvalue of \(A\). Then by Theorem 5(b) the eigenvalue \(\lambda_{k,j}(t)\) of \(L_t(Q)\) lying in \(U(\mu_{k,j}, \varepsilon_k)\) is a simple and real eigenvalue. Moreover, the simplicity of the eigenvalue \(\lambda_{k,j}(t)\) implies that it continuously depend on \(t\). As a result the set

\[
\{\lambda_{k,j}(t) : t \in ((-1, 1]\setminus U(j, k, \delta_k))\}
\]  

(38)

contains the main part of the set (33). Therefore the union of the sets (38) for \(k \geq N(j)\) contain the main part of the half line \((\mu_j + (2N(j) - 1)^2, \infty)\).

Now let us discuss it in more general case when \(\mu_j\) is a real eigenvalue of odd multiplicity \(v\). Then by Theorem 5(a) the operator \(L_t(Q)\) has \(v\) eigenvalues in the neighborhood of \(\mu_{k,j}(t)\). All these eigenvalues can not be nonreal, since they are pairwise conjugate number and \(v\) is an odd number. Thus the operator \(L_t(Q)\) has a real eigenvalue. In the following theorem we prove that these real eigenvalues fill the main part of the half line \([0, \infty)\).

**Theorem 6** Suppose that the matrix \(A\) has a real eigenvalue \(\mu_j\) of odd multiplicity.

(a) Then the real component of the spectrum of \(L(Q)\) contains the subintervals

\[
\Gamma_{k,j,i} = [\mu_{k,j}(a_i) + \delta_k, \mu_{k,j}(b_i) - \delta_k]
\]  

(39)

for \(i = 1, 2, \ldots, l\) of the interval \(\Gamma_{k,j} = (\mu_j + (2k - 1)^2, \mu_j + (2k + 1)^2]\), where \(k \geq N(j)\). These subintervals are pairwise disjoint and satisfy the inequality

\[
\mu(\Gamma_{k,j}) - \sum_{i=1,2,\ldots,l} \mu(\Gamma_{k,j,i}) < c_{11}\delta_k,
\]  

(40)

where \(\mu(E)\) denotes the measure of the set \(E\) and \(\delta_k = O(\frac{1}{r} + q_k)^{1/r}\).

(b) The spectrum of \(L(Q)\) contains the main part of \([0, \infty)\) in the sense that

\[
\lim_{n \to \infty} \frac{\mu([0, n]\setminus \sigma(L(Q)))}{\mu(\sigma(L(Q)) \cap [0, n])} = 0.
\]  

(41)
Proof. Let \( \mu_j \) be a real eigenvalue of multiplicity \( v \) of the matrix \( A \), where \( v \) is an odd number. Then by Theorem 5 the disk \( D(\mu_{k,j}(t),\delta_k) \) for \( t \in [a_i,b_i] \) and \( i = 1,2,...,l \) contains \( v \) eigenvalues (counting the multiplicity) of \( L_t(Q) \), where \([a_i,b_i]\) is defined in (34). Let us denote they by \( \lambda_{k,1}(t), \lambda_{k,2}(t),...,\lambda_{k,v}(t) \). Consider the unordered \( v \)-tuple \( \{\lambda_{k,1}(t), \lambda_{k,2}(t),...,\lambda_{k,v}(t)\} \). It follows from (4) that this unordered \( v \)-tuple depend continuously (in sense of Theorem 5.2 of [2]) on the parameter \( t \in [a_i,b_i] \). Then by Theorem 5.2 of [2] there exist \( p \) single-valued continuous functions \( \lambda_1(t), \lambda_2(t),...,\lambda_v(t) \) the value of which constitute the \( v \)-tuple \( \{\lambda_{k,1}(t), \lambda_{k,2}(t),...,\lambda_{k,v}(t)\} \) for \( t \in [a_i,b_i] \).

Now we prove that

\[
[\mu_{k,j}(a_i) + \delta_k, \mu_{k,j}(b_i) - \delta_k] \cap (\bigcup_{s=1}^{v} \gamma_s) \subset \sigma(L(Q)),
\]

where \( \gamma_s \) is the curve \( \{\lambda_s(t) : t \in [a_i,b_i]\} \). Assume the converse. Then there exists

\[
a \in [\mu_{k,j}(a_i) + \delta_k, \mu_{k,j}(b_i) - \delta_k] \setminus (\bigcup_{s=1}^{v} \gamma_s).
\]

It means that the continuous curves \( \gamma_s \) extend from \( \lambda_s(a_i) \in U(\mu_{k,j}(a_i),\delta_k) \) to \( \lambda_s(b_i) \in U(\mu_{k,j}(b_i),\delta_k) \) pas above or below of the point \( a \), since \( \text{Re} \lambda_s(a_i) < a < \text{Re} \lambda_s(b_i) \). Moreover by Theorem 5(a) if \( \gamma_s \) passes above of \( a \) then there exist \( j \in \{1,2,...,v\} \) such that \( j \neq s \) and \( \gamma_j \) passes below of \( a \). It implies that the number \( v \) of the curves \( \gamma_s \) is an even number. It contradicts to the assumption that \( v \) is an odd number. Thus (42) is proved and the subintervals \( \Gamma_{k,j,i} \) defined in (39) are the subsets of the spectrum of \( L(Q) \). Estimation (40) follows from (37). Note that \( N(j) \) can be chosen so that \( \mu_{k,j}(b_i) - \mu_{k,j}(a_i) - 2\delta_k > 0 \) for \( k \geq N(j) \).

The proof of (41) follows from (a). \( \blacksquare \)

Corollary 1 If \( m \) is an odd number, then the real component of the spectrum of \( L(Q) \) contains the main part of \([0, \infty)\) and (41) holds.

Now we find a condition on the eigenvalues of the matrix \( A \) for which the the real component of the spectrum of \( L(Q) \) contains a half line \([H, \infty)\) for some \( H \).

Theorem 7 If the matrix \( A \) has at least three real eigenvalues \( \mu_{j_1}, \mu_{j_2} \) and \( \mu_{j_3} \) of odd multiplicities such that

\[
\min_{i_1,i_2,i_3} (\text{diam}(\{\mu_{j_1} + \mu_{i_1}, \mu_{j_2} + \mu_{i_2}, \mu_{j_3} + \mu_{i_3}\})) = d \neq 0,
\]

where \( i_k = 1,2,...,s \) for \( k = 1,2,3 \) and

\[
\text{diam}(E) = \sup_{x,y \in E} |x - y|,
\]

then there exists a number \( H \) such that \([H, \infty) \in \sigma(L(Q))\).
Proof. By Theorem 6(a) we have

\[ \sigma(L(Q)) \supset (S(j_1, N(j_1)) \cup S(j_2, N(j_2)) \cup S(j_3, N(j_3))) \]  

where

\[ S(j, N) = \bigcup_{i=1,2,\ldots} \{ \mu_{k,j}(a_i) + \delta_k, \mu_{k,j}(b_i) - \delta_k \} \]  

Using the definition of \( U(j, k, \delta_k) \) one can easily verify that there exists \( \alpha_k \) such that \( \alpha_k \to 0 \) as \( k \to \infty \) and

\[ \mu_{k,j}(U(j, k, \delta_k)) \subset \bigcup_{n=-1,0,1; i=1,2,\ldots,s} C(k, j, i, \alpha_k, n), \]  

where

\[ C(k, j, i, a, n) = \{ x \in \mathbb{R} : | x - ((2k + n))^2 - \frac{\mu_i + \mu_j}{2} | < a \}. \]  

Then we have

\[ \bigcup_{k \geq N(j_p)} \{ \mu_{k,j_p}(t) : t \in (-1, 1) \setminus U(j_p, k, \alpha_k) \} \subset \bigcup_{n=-1,0,1; k \geq N; i=1,2,\ldots,s} C(k, j_p, i, \alpha_k, n), \]  

where \( h = (2N(j_p) - 1)^2 + \mu_{j_p} \) and \( p = 1, 2, 3 \). Therefore using (44)-(47) we obtain that there exist \( H > h \) and \( \beta_k \) such that \( \beta_k \to 0 \) as \( k \to \infty \) and

\[ \sigma(L(q)) \supset (H, \infty) \setminus \bigcup_{n=-1,0,1; k \geq N; i=1,2,\ldots,s} C(k, j_p, i, \beta_k, n) \]  

for all \( p = 1, 2, 3 \), where \( N \geq \max \{ N(j_1), N(j_2), N(j_3) \} \). Now we prove that

\[ (H, \infty) \subset \sigma(L(q)). \]  

By (48) to prove (49) it is enough to show that the set

\[ \bigcap_{p=1,2,3} \left( \bigcup_{n=-1,0,1; k \geq N; i=1,2,\ldots,s} C(k, j_p, i, \beta_k, n) \right) \]  

is empty. Since \( \beta_k \to 0 \) the number \( N \) can be chosen so that \( 4\beta_k < d \) for \( k \geq N \). If the set (50) contains an element \( x \), then

\[ x \in \bigcup_{n=-1,0,1; k \geq N; i=1,2,\ldots,s} C(k, j_p, i, \beta_k, n) \]  

for all \( p = 1, 2, 3 \). Using this and the definition of \( C(k, j_p, i, a, n) \) (see (47)), we obtain that there exist \( k \geq N; n \in \{-1,0,1\} \) and \( i_p \in \{1,2,\ldots,s\} \) such that

\[ | x - ((2k + n))^2 - \frac{\mu_{j_p} + \mu_i}{2} | < \beta_k \]
for all $p = 1, 2, 3$. This implies that
\[
\left| \mu_{j_p} + \mu_{q_p} - (\mu_{j_q} + \mu_{q_q}) \right| < 4 \beta_k < d
\]
for all $p, q = 1, 2, 3$, where $p \neq q$. It contradicts the condition (43). Theorem is proved. \[\Box\]

Note that in paper [10] we proved Theorem 7 for the self-adjoint operator $L(Q)$ under additional condition that $\mu_{j_1}, \mu_{j_2}$ and $\mu_{j_3}$ are the simple eigenvalues. In this paper we prove it for the non-self-adjoint operator $L(Q)$ and without the simplicity condition.

3 Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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