Global well-posedness and scattering for the defocusing, $L^2$-critical, nonlinear Schrödinger equation when $d = 1$

Benjamin Dodson

American Journal of Mathematics, Volume 138, Number 2, April 2016, pp. 531-569 (Article)

Published by Johns Hopkins University Press

DOI: https://doi.org/10.1353/ajm.2016.0016

For additional information about this article
https://muse.jhu.edu/article/613794/summary
GLOBAL WELL-POSEDNESS AND SCATTERING FOR THE DEFOCUSING, $L^2$-CRITICAL, NONLINEAR SCHröDINGER EQUATION WHEN $d = 1$

By Benjamin Dodson

Abstract. In this paper we prove global well-posedness and scattering for the defocusing, one dimensional mass-critical nonlinear Schrödinger equation. We make use of a long-time Strichartz estimate and a frequency localized Morawetz estimate. This continues work begun in earlier papers by the author for dimensions $d \geq 3$ and $d = 2$.

1. Introduction. This paper discusses the defocusing, quintic, one dimensional initial value problem

\begin{equation}
\begin{aligned}
  iu_t + \Delta u &= F(u) = |u|^4 u, \\
  u(0, x) &= u_0.
\end{aligned}
\end{equation}

(1.1) is an $L^2$-critical problem. A solution to (1.1) can be rescaled to give a family of solutions. For any $\lambda > 0$, if $u(t, x)$ is a solution to (1.1) on some interval $I$ then

\begin{equation}
\lambda^{1/2} u(\lambda^2 t, \lambda x)
\end{equation}

(1.2)
is a solution to (1.1) on $\frac{L}{\lambda^2} = \{ \frac{t}{\lambda^2} : t \in I \}$ with initial data $\lambda^{1/2} u(0, \lambda x)$. (1.1) is called $L^2$-critical since

\begin{equation}
\|u(0, x)\|_{L^2(R)} = \|\lambda^{1/2} u(0, \lambda x)\|_{L^2(R)}.
\end{equation}

(1.3)

(1.1) is part of a general class of $\dot{H}^{s_c}$-critical problems.

Definition 1.1. ($\dot{H}^{s_c}$-critical) The defocusing initial value problem

\begin{equation}
\begin{aligned}
  iu_t + \Delta u &= F(u) = |u|^p u, \\
  u(0, x) &= u_0,
\end{aligned}
\end{equation}

(1.4)
is called $\dot{H}^{s_c}$-critical if $p = \frac{4}{d - 2s_c}$, where $u : I \times \mathbb{R}^d \to \mathbb{C}$, $0 \in I \subset \mathbb{R}$.

Conjecture 1.1. The mass-critical initial value problem, (1.4) with $p = \frac{4}{d}$, is globally well-posed for any $u_0 \in L^2(\mathbb{R}^d)$, and the solutions scatter forward and backward in time.
Remark. By [17, 18] Conjecture 1.1 is sharp.

[90] confirmed Conjecture 1.1 for \( u_0 \) radial and dimensions \( d \geq 3 \), and [57] for \( u_0 \) radial and \( d = 2 \). [28] removed the radial symmetry condition for dimensions \( d \geq 3 \), and [31] removed the radial symmetry condition for \( d = 2 \). In this paper we complete the affirmation of Conjecture 1.1, proving

**Theorem 1.2.** The initial value problem (1.1) is globally well-posed for any \( u_0 \in L^2(\mathbb{R}) \), and the solutions scatter forward and backward in time.

Remark. [56] combined with this paper gives a profile decomposition for the mass-critical gKdV. [32] exploits this profile decomposition to prove scattering for the defocusing mass-critical gKdV.

This paper is the third in a series of four papers. The fourth paper (see preprint [30]) addresses the focusing initial value problem (\( F(u) = -|u|^{4/d} u \)). There are known counterexamples to global well-posedness and scattering for this problem (see [41, 65, 88, 100, 101, 103]). However, all such counterexamples have \( L^2 \) norm above the \( L^2 \) norm of the ground state. [57, 61] showed that Conjecture 1.1 is true when the \( L^2 \) norm of the initial data is below the \( L^2 \) norm of the ground state for \( d \geq 2 \) and \( u_0 \) radial. [30] removes the symmetry condition and shows that Conjecture 1.1 holds for data with \( L^2 \) norm below the \( L^2 \) norm of the ground state in dimensions \( d \geq 1 \).

**Definition 1.2.** (Well-posedness) A Schrödinger initial value problem is said to be well-posed on \( I \subset \mathbb{R} \) with initial data in a Banach space \( X \) if:

1. A solution exists on \( I \) for all \( u_0 \in X \),
2. The solution satisfies Duhamel’s principle, that is for any \( t \in I \),

\[
(1.5) \quad u(t, x) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-\tau)\Delta} F(u(\tau)) d\tau.
\]

3. The solution is unique.
4. For any compact \( J \subset I \), the map \( u_0 \mapsto u(t, x) \) is continuous from \( X \) to \( Y(J \times \mathbb{R}^d) \), where \( Y \subset L^\infty_t L^2_x \) is also a Banach space.

An initial value problem is said to be globally well-posed if \( I = \mathbb{R} \).

For (1.1) we take \( X = L^2(\mathbb{R}) \), \( Y = L^\infty_t L^2_x \cap L^6_{t,x} \).

**Definition 1.3.** (Scattering) A solution to (1.4) scatters forward in time if there exists \( u_+ \in H^s_{\text{c}}(\mathbb{R}^d) \) such that

\[
(1.6) \quad e^{-it\Delta} u(t, x) \rightarrow u_+,
\]
strongly in $\dot{H}^{s_c}(\mathbb{R}^d)$ as $t \to +\infty$. A solution is said to scatter backward in time if there exists $u_\rightarrow \in \dot{H}^{s_c}(\mathbb{R}^d)$ such that

$$e^{-it\Delta}u(t,x) \longrightarrow u_\rightarrow,$$

strongly in $\dot{H}^{s_c}(\mathbb{R}^d)$ as $t \to -\infty$.

A solution to (1.4) conserves both mass,

$$M(u(t)) = \int |u(t,x)|^2 \, dx = M(u(0)), \quad (1.8)$$

and energy,

$$E(u(t)) = \int \left( \frac{1}{2} |\nabla u(t,x)|^2 + \frac{1}{p+2} |u(t,x)|^{p+2} \right) \, dx = E(u(0)). \quad (1.9)$$

Due to the conserved quantities (1.8) and (1.9), a great deal of research has focused on the mass or $L^2$-critical initial value problem ($p = \frac{4}{d}$) and the $\dot{H}^1$ or energy-critical initial value problem ($p = \frac{4}{d-2}$). See [53, 70, 69] for global well-posedness and scattering results when a solution to (1.4) has an assumed bound on the $\dot{H}^{s_c}$ norm, $0 < s_c < 1$. The author is unaware of any large data scattering results in the absence of a conserved quantity that controls the critical $\dot{H}^{s_c}$ norm or an assumed bound on the $\dot{H}^{s_c}$ bound.

The local theory for (1.4) has been worked out in [12, 13, 14, 15, 16, 89]. To simplify the exposition only the $L^2$-critical results will be presented here.

**Definition 1.4.** (Blowup criterion) Let $I$ be the maximal interval of existence of a solution to (1.4), $p = \frac{4}{d}$. A solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ is said to blow up forward in time if for some $t_0 \in I$,

$$\int_{t_0}^{\sup(I)} \int |u(t,x)|^{\frac{2(d+2)}{d}} \, dx \, dt = \infty. \quad (1.10)$$

Blowing up backward in time is similarly defined.

**Theorem 1.3.** (Local well-posedness) (1) Given $u_0 \in L^2(\mathbb{R}^d)$ there is a solution $u$ to (1.4), $p = \frac{4}{d}$, with initial data $u_0$, on a maximal interval of existence $I$, where $I$ is an open set. Moreover, for any $J \subset I$ compact, the solution map $u(0) \mapsto u(t,x)$, $L^2_x(\mathbb{R}^d) \mapsto L^{\frac{2(d+2)}{d}}_{t,x}(J \times \mathbb{R}^d)$ is continuous in an $L^2$ neighborhood of $u_0$.

(2) If $\sup(I)$ is finite then the solution $u$ blows up forward in time. If $\inf(I)$ is finite then the solution $u$ blows up backward in time.

(3) If $u$ does not blow up forward in time then $u$ scatters forward in time. If $u$ does not blow up backward in time then $u$ scatters backward in time.

(4) If $M(u_0)$ is sufficiently small then the solution $u$ is global and scatters forward and backward in time.
Theorem 1.2 follows from two results:

**Theorem 1.4.** If global well-posedness and scattering for (1.1) fails then there exists a nonzero solution to (1.1) that lies in a compact subset of $L^2(\mathbb{R}^d)$ modulo scaling, translation, and Galilean symmetries for the entire time of its existence.

**Theorem 1.5.** The only solution to (1.1) that lies in a compact subset of $L^2(\mathbb{R})$ modulo scaling, translation, and Galilean symmetries for the entire time of its existence is $u \equiv 0$.

**Definition 1.5.** (Almost periodic solution) A solution that lies in a compact subset of $L^2(\mathbb{R}^d)$ modulo scaling, translation, and Galilean symmetries for the entire time of its existence is called an almost periodic solution.

**Remark.** The argument outlined in Theorems 1.4 and 1.5 is called the concentration compactness method, a method that has proved to be quite useful in many types of partial differential equations. It has been used since at least the 1980’s (see [4, 11]) in elliptic and parabolic partial differential equations. The use of the concentration compactness method to establish global well-posedness and scattering for critical dispersive equations, in the exact form as it is used in this paper, was first introduced and developed in [51] for the focusing, radial, energy-critical Schrödinger equation and in [52] for the focusing, energy-critical wave equation. Historically, progress in dispersive partial differential equations using this method has gone energy-critical focusing NLS, energy-critical focusing wave equation, mass-critical NLS, mass-critical gKdV.

The analysis of defocusing, semilinear, energy-critical wave equation was completed by [2, 36, 42, 43, 47, 74, 75, 76, 81, 86]. These works were followed by [44], which proved that the defocusing, energy-critical Schrödinger problem with three dimensional radial data was globally well-posed. These results were obtained by ruling out energy concentration through the use of Morawetz and dilation identities.

Around the same time [10] utilized an induction on energy argument, proving global well-posedness and scattering for the defocusing, radial, energy-critical nonlinear Schrödinger equation in dimensions $d = 3, 4$. [10] is widely considered to be the seminal result in the study of Schrödinger problems in connection with the induction on energy method. [84] extended this work to higher dimensions.

In a major breakthrough, [24] extended the result of [10] to the nonradial setting in $\mathbb{R}^3$, via the induction on energy method and introducing the interaction Morawetz estimates of [23] to this setting. This work was extended to four dimensions in [73] and to dimensions greater than five in [98]. Global well-posedness
and scattering for the energy-critical defocusing nonradial Schrödinger equation
[24, 59, 73, 97, 98, 99] and for the energy-critical focusing problem [33, 51, 58, 83]
is now complete except for the focusing problem when \( d = 3 \) with nonradial data.

**Proof of Theorem 1.4.** A version of Theorem 1.4 is implicit in [55] for dimensions \( d = 1, 2 \). [3, 66, 92] extended this result to dimensions \( d \geq 1 \). The argument of [92] will be sketched in Section 2.

The proof of Theorem 1.5 will then occupy the remainder of the paper. We prove this by considering two separate cases. Notice that Theorem 1.4 implies that there exists a solution to (1.1) on a maximal interval \( I \subset \mathbb{R} \), along with \( N(t) : I \rightarrow (0, \infty) \), \( \xi(t) : I \rightarrow \mathbb{R}^d \), and \( x(t) : I \rightarrow \mathbb{R}^d \), such that

\[
\left\{ \frac{1}{N(t)^{d/2}} e^{-ix \cdot \xi(t)} u \left( t, \frac{x - x(t)}{N(t)} \right) : t \in I \right\}
\]
is pre compact in \( L^2(\mathbb{R}^d) \). We argue as in [28, 57, 61], but especially [31]. As in [28, 31] it suffices to consider \( N(0) = 1, N(t) \leq 1 \) on \( [0, \infty) \) (see [57] for a proof of this fact). We then consider two scenarios, the rapid frequency cascade scenario,

\[
\int_0^\infty N(t)^3 dt < \infty,
\]
and the quasisoliton,

\[
\int_0^\infty N(t)^3 dt = \infty.
\]

As in [28, 31], we utilize a long time Strichartz estimate. [28] utilized long time Strichartz estimates based on the \( L^2_t L^\infty_x \) Strichartz estimates endpoint results of [50]. In \( d = 2 \), the \( L^2_t \) Strichartz estimates are no longer available so instead [31] utilized the \( U^2_\Delta \) and \( V^2_\Delta \) spaces to construct a long time Strichartz norm. The long time Strichartz spaces of [31] will be recalled in section three. In section four we will make an induction on frequency argument to prove estimates on our long time Strichartz spaces. In fact, the argument in one dimension is substantially simpler than the argument in two dimensions. Then in section five we will prove that there is no nonzero rapid frequency cascade solution. In section six we will prove that there is no nonzero quasisoliton solution.

The exclusion of a quasisoliton solution relies on the frequency localized interaction Morawetz estimates. Morawetz estimates have long proven useful to proving scattering results for dispersive equations (see [8, 39, 64, 67, 71]). The interaction Morawetz estimate has proved to be very useful for nonradial data.
THEOREM 1.6. (Interaction Morawetz estimate) A solution to (1.4) has the bounds
\begin{equation}
\|\nabla \cdot ^{\frac{3-d}{2}} u(t,x)\|^2_{L^2_t L^2_x (I \times \mathbb{R}^d)} \lesssim \|u\|^2_{L^\infty_t L^2_x (I \times \mathbb{R}^d)} \|u\|^2_{L^1_t H^{1/2} (I \times \mathbb{R}^d)},
\end{equation}
for all \(d \geq 1\).

Proof. See [23] when \(d = 3\), [91] when \(d \geq 4\), and [21, 72] when \(d = 1, 2\). [72] proves a Galilean invariant version of (1.14) for \(d \geq 1\). \(\square\)

As in [28, 31], we do not have an a priori bound on \(\|u(t)\|_{H^{1/2} (\mathbb{R})}\), so we truncate to low frequencies. The idea of using frequency truncated interaction Morawetz estimates was introduced in [24] for the energy-critical problem. In that case solutions were truncated in high frequencies since \(\|u(t)\|_{H^1}\) was uniformly bounded, but there was no a priori bound on \(\|u(t)\|_{L^2}\).

Here, as in [28, 31], the errors produced by truncating to low frequencies are successfully estimated using the long time Strichartz estimates. The frequency truncated Morawetz estimates are very closely related to the almost Morawetz estimates of the I-method. See [20, 25, 27] for the almost Morawetz estimates in two dimensions, [26, 29] in one dimension, and [22] for the I-method itself.

Acknowledgment. I am grateful to the anonymous referees for many helpful comments regarding the introduction.

2. Linear estimates and concentration compactness. This section serves to present some results in the theory of Strichartz estimates and concentration compactness.

Definition 2.1. (Admissible pair) A pair \((p,q)\) will be called an admissible pair for \(d = 1\) if \(\frac{2}{p} = \left(\frac{1}{2} - \frac{1}{q}\right)\), and \(p \geq 4\).

Proposition 2.1. (Strichartz estimates) If \(u(t,x)\) solves the initial value problem
\begin{equation}
iu_t + \Delta u = F(t), \quad u(0,x) = u_0,
\end{equation}
on an interval \(I\), then
\begin{equation}
\|u\|_{L^p_t L^q_x (I \times \mathbb{R})} \lesssim \|u_0\|_{L^2_x (\mathbb{R})} + \|F\|_{L^p_t L^q_x (I \times \mathbb{R})},
\end{equation}
for all admissible pairs \((p,q)\) and \((\tilde{p}, \tilde{q})\). \(\tilde{p}'\) denotes the Lebesgue dual of \(\tilde{p}\).

Proof. See [80] for \(p = q = 6\), [19, 40, 50, 85, 102] for the general result. \(\square\)
PROP. 2.2. (Bilinear Strichartz estimates) If \( \hat{u}_0 \) is supported on \(|\xi| \sim N\), \( \hat{v}_0 \) is supported on \(|\xi| \sim M\), \( M \ll N \),

\[
\left\| \left( e^{it\Delta} u_0 \right) \left( e^{it\Delta} v_0 \right) \right\|_{L^2_{t,x}(\mathbb{R} \times \mathbb{R})} \lesssim N^{-1/2} \left\| u_0 \right\|_{L^2(R)} \left\| v_0 \right\|_{L^2(R)}.
\]

This result also holds in the more general case in which the supports of \( \hat{\nu}_0(\xi) \) and \( \hat{\nu}_0(\xi) \) are separated by a distance \(|\xi| \sim N\).

Proof. The proof in one dimension is much simpler than the higher dimensional results of \([7, 82]\). In fact \([60]\) demonstrates that this result implies the result of \([7]\) for any dimension \( d \geq 1 \).

\[
\left\| \left( e^{it\Delta} u_0 \right) \left( e^{it\Delta} v_0 \right) \right\|_{L^2_{t,x}}^2 = \left\| \left( e^{it\Delta} u_0 \right) \left( e^{it\Delta} v_0 \right) \left( e^{it\Delta} u_0 \right) \left( e^{it\Delta} v_0 \right) \right\|_{L^2_{t,x}}.
\]

Suppose \( \|g(t,x)\|_{L^2_{t,x}} = 1\), \( \hat{g}(t,\xi) \) is the spatial Fourier transform of \( g \), and \( \hat{g}(\tau,\xi) \) the space-time Fourier transform. Then by Parseval’s theorem

\[
\int \langle g(t,x), \left( e^{it\Delta} u_0 \right) \left( e^{it\Delta} v_0 \right) \rangle dt = \iint \hat{g}(t,\xi) \hat{u}_0(\eta)(e^{it(\eta-\xi)^2}) \hat{v}_0(\xi-\eta) d\xi d\eta dt
\]

\[
= \iint \hat{g}(\eta^2 - (\xi - \eta)^2, \xi) \hat{u}_0(\eta) \hat{v}_0(\xi-\eta) d\eta d\xi
\]

\[
= \iint \hat{g}(\xi(\eta - \xi), \xi) \hat{u}_0(\eta) \hat{v}_0(\xi-\eta) d\eta d\xi.
\]

Then \( \eta - (\eta - \xi) = \xi \sim N \), so by a change of variables

\[
\|g\|_{L^2_{t,x}} \left\| u_0 \right\|_{L^2} \left\| v_0 \right\|_{L^2} \lesssim N^{-1/2} \left\| u_0 \right\|_{L^2(R)} \left\| v_0 \right\|_{L^2(R)}.
\]

COROLLARY 2.3.

\[
\left\| \left( e^{it\Delta} u_0 \right) \left( e^{it\Delta} v_0 \right) \right\|_{L^2_{t,x}(\mathbb{R} \times \mathbb{R})} \lesssim \left( \frac{M}{N} \right)^{1/4} \left\| u_0 \right\|_{L^2(R)} \left\| v_0 \right\|_{L^2(R)}.
\]

Proof. By propositions 2.1, 2.2, and Sobolev embedding,

\[
\left\| \left( e^{it\Delta} u_0 \right) \left( e^{it\Delta} v_0 \right) \right\|_{L^2_{t,x}} = \left\| \left( e^{it\Delta} u_0 \right) \left( e^{it\Delta} v_0 \right) \right\|_{L^2_{t,x}}^{1/2} \left\| e^{it\Delta} u_0 \right\|_{L^2_{t,x}}^{1/2} \left\| e^{it\Delta} v_0 \right\|_{L^2_{t,x}}^{1/2} \lesssim \left( \frac{M}{N} \right)^{1/4} \left\| u_0 \right\|_{L^2} \left\| v_0 \right\|_{L^2}.\]

\]
Propositions 2.1 and 2.2 also hold under convolutions with $L^1$ kernels. This will prove to be extremely important since by Galilean invariance a nonsymmetric solution to (1.1) will be localized in frequency around some $\xi(t) \in \mathbb{R}^2$, $\xi(t)$ need not be zero. Suppose $g(t, x - y)$ and $h(t, x - z)$ are convolution kernels with uniform bounds.

(2.10) \[ \| \sup_{t \in \mathbb{R}} |g(t, x)|\|_{L^1(\mathbb{R})}, \quad \| \sup_{t \in \mathbb{R}} |h(t, x)|\|_{L^1(\mathbb{R})} \lesssim 1. \]

Then

(2.11) \[ \| (g(\cdot, \cdot) * e^{it\Delta} u_0)(h(\cdot, \cdot) * e^{it\Delta} v_0)\|_{L^q_t(\mathbb{R} \times \mathbb{R}^d)} = \left\| \int \int g(t, x - y)h(t, x - z)(e^{it\Delta} u_0)(y)(e^{it\Delta} v_0)(z)dydz \right\|_{L^q_t(\mathbb{R} \times \mathbb{R}^d)} \]

(2.12) \[ \lesssim \sup_{y, z} \| (e^{it\Delta} u_0)(y)(e^{it\Delta} v_0)(z)\|_{L^q_t(\mathbb{R} \times \mathbb{R}^d)}. \]

Since Proposition 2.2 is translation invariant, (2.11) and (2.12) can be combined quite nicely with (2.3). The calculation is even easier for the linear Strichartz estimates.

**Sketch of the proof of Theorem 1.4.** Although we only need Theorem 1.4 when $d = 1$, we will go ahead and discuss [92]'s proof of Theorem 1.4 in any dimension. The argument in [55] is almost identical when $d = 1$. It follows from Theorem 1.3 that to prove (1.1) is globally well-posed and scattering it suffices to prove that if $u$ is a solution to the defocusing, mass-critical problem (1.4), $p = \frac{4}{d}$, then there exists a function $A : [0, \infty) \to [0, \infty)$ such that

(2.13) \[ \|u\|_{L^\infty_t(L^2_x(\mathbb{R} \times \mathbb{R}^d))} \leq A(\|u_0\|_{L^2(\mathbb{R}^d)}), \quad d < \infty. \]

During the sketch of the proof we understand that any reference to (1.4) refers to (1.4) with $p = \frac{4}{d}$. Define the function

(2.14) \[ A(m) = \sup \left\{ \frac{\|u\|_{L^\infty_t(L^2_x(\mathbb{R} \times \mathbb{R}^d))}}{(\|u_0\|_{L^2(\mathbb{R}^d)})} : u \text{ solves (1.4), } \|u(0)\|_{L^2(\mathbb{R}^d)} = m \right\}. \]

The stability lemma from [91] implies that $A(m)$ is a continuous function of $m$, which implies that $\{m : A(m) = \infty\}$ is a closed set. Therefore if global well-posedness and scattering does not hold in the defocusing case for all $u_0 \in L^2(\mathbb{R}^d)$, then there must be a minimum $m_0(d)$ with $A(m_0(d)) = \infty$. Moreover, by Theorem 1.3, $m_0(d) > \epsilon(d)$. 

Next, [92] proved that if \( A(m_0) = \infty \), \( A(m) < \infty \) for \( m < m_0 \), then there must exist a blowup solution to (1.4) on \( 0 \in I \subset \mathbb{R} \) satisfying

\[
\| u(t) \|_{L^2(\mathbb{R}^d)} = \| u(0) \|_{L^2(\mathbb{R}^d)} = m_0 \quad \forall t \in I,
\]

\[
\| u \|_{L^{\frac{2(d+1)}{d}}(\inf(I),0) \times \mathbb{R}^d)} = \| u \|_{L^{\frac{2(d+1)}{d}}(\inf(I),0) \times \mathbb{R}^d)} = +\infty,
\]

and \( u(t) \) lying in a compact set modulo the Galilean, translation, and scaling symmetries for all \( t \in I \). It is certainly necessary to take the quotient with respect to this group of symmetries since solutions to (1.1) are invariant under translation. Scaling symmetry has also been discussed in (1.2). Finally, direct calculation verifies that (1.1) is invariant under the Galilean transformation.

**Theorem 2.4.** (Galilean transformation) Suppose \( u(t,x) \) solves

\[
iu_t + \Delta u = F(u),
\]

\[
u(0,x) = u_0.
\]

Then \( v(t,x) = e^{-it|\xi_0|^2}e^{ix \cdot \xi_0}u(t,x - 2\xi_0 t) \) solves the initial value problem

\[
i v_t + \Delta v = F(v),
\]

\[
v(0,x) = e^{ix \cdot \xi_0}u(0,x).
\]

(2.15) and (2.16) are proved by profile decomposition. Shortly after [10], [54] proved a profile decomposition for the energy critical Schrödinger problem in \( \mathbb{R}^3 \) in the same vein as the profile decomposition that [34] proved for the Sobolev embedding and [1] proved for the wave equation.

For the mass-critical problem in two dimensions [66] utilized a result of [68] to prove weak convergence modulo symmetries for a sequence of functions with Strichartz norms uniformly bounded below and mass uniformly bounded above. [55] then proved a profile decomposition for the mass-critical problem in dimensions \( d = 1, 2 \). [3] extended this to higher dimensions using the bilinear result of [82].

[92] then argued that a minimal mass blowup solution to (1.4) must exist entirely on one profile, otherwise if a minimal mass blowup solution were decoupled on two profiles, each with mass below the minimal mass of blowup, one could obtain scattering for each piece separately and then “paste” the two results together to obtain a scattering solution. This completed the proof of Theorem 1.4 for \( d \geq 1 \).

Returning to \( d = 1 \) only, the Arzela-Ascoli theorem implies that \( u(t) \) almost periodic is equivalent to the existence of \( \xi(t) : I \to \mathbb{R}, x(t) : I \to \mathbb{R}, N(t) : I \to (0,\infty), C : (0,\infty) \to (0,\infty) \) such that for all \( t \in I \), where \( I \) is the maximal interval...
of existence, and $\eta > 0$,

\begin{align}
(2.19) \quad & \int_{|x-x(t)| \geq C(\eta) \frac{C(\eta)}{N(t)}} |u(t,x)|^2 dx < \eta, \\
(2.20) \quad & \int_{|\xi-\xi(t)| \geq C(\eta)N(t)} |\hat{u}(t,\xi)|^2 d\xi < \eta.
\end{align}

By the uncertainty principle this is as much as one could possibly hope for.

Now let $u$ be an almost periodic solution to (1.1). If $u$ is symmetric about 0 then $x(t) = \xi(t) = 0$. In the nonradial case $x(t)$ and $\xi(t)$ are free to move around. In this paper we will track $\xi(t)$ and $N(t)$ to gain additional information on an almost periodic solution. As in [28, 31] we will not need to track $x(t)$ since we will use the interaction Morawetz estimates.

**Lemma 2.5.** Suppose $J$ is an interval satisfying

\begin{equation}
(2.21) \quad \|u\|_{L^6_{t,x}(J \times \mathbb{R})} = 1.
\end{equation}

Then for $t_1, t_2 \in J$,

\begin{equation}
(2.22) \quad N(t_1) \sim_{m_0} N(t_2),
\end{equation}

and

\begin{equation}
(2.23) \quad |\xi(t_1) - \xi(t_2)| \lesssim \sup_{t \in J} N(t).
\end{equation}

**Proof.** See lemma 5.18 of [60]. \hfill \Box

Next, since for all $t \in I$, $u(t)$ lies in a compact subset of $L^2(\mathbb{R})$ modulo scaling, translation, and Galilean symmetries,

**Lemma 2.6.** There exists $\delta(m_0) > 0$ such that for any $t_0 \in I$,

\begin{equation}
(2.24) \quad \|u\|_{L^6_{t,x}([t_0, t_0 + \frac{\delta}{N(t_0)^2}] \times \mathbb{R})} \sim \|u\|_{L^6_{t,x}([t_0 - \frac{\delta}{N(t_0)^2}] \times \mathbb{R})} \sim 1.
\end{equation}

**Proof.** Again see lemma 5.18 of [60]. \hfill \Box

If $J$ is an interval with $\|u\|_{L^6_{t,x}(J \times \mathbb{R})} = 1$ then let

\begin{equation}
(2.25) \quad N(J) = \sup_{t \in J} N(t).
\end{equation}

By Lemmas 2.5 and 2.6,

\begin{equation}
(2.26) \quad N(J) \sim \int_J N(t)^3 dt \sim \inf_{t \in J} N(t).
\end{equation}
Combining Lemmas 2.5 and 2.6, we can choose $\xi(t), N(t)$ such that
\begin{equation}
|\xi'(t)| + |N'(t)| \lesssim m_0 N(t)^3.
\end{equation}
Therefore, we have now proved

**Theorem 2.7.** If there exists an almost periodic solution to (1.4) with $0 < \|u(0)\|_{L^2(\mathbb{R})} < \infty$, then there exists an almost periodic solution to (1.1) satisfying (2.19) and (2.20), $0 < \epsilon \leq \|u(0)\|_{L^2(\mathbb{R})} < \infty$,
\begin{equation}
\int_{0}^{\infty} \int |u(t,x)|^6 dx dt = \infty,
\end{equation}

$N(0) = 1, \xi(0) = x(0) = 0$, and $N(t) \leq 1$ on $[0, \infty)$, $|N'(t)| + |\xi'(t)| \lesssim u N(t)^3$.

**Proof:** This was proved in [90, 92], so the argument will merely be sketched here. Theorem 1.3, (2.24), and (2.27) imply that an almost periodic solution with $\|u(0)\|_{L^2} > 0$ implies that $\|u(0)\|_{L^2} \geq \epsilon$, where $\epsilon > 0$ is the small data threshold. Since $u(t)$ lies in a compact set in $L^2(\mathbb{R})$ modulo symmetries, combined with the fact that $N(t)$ is continuous and time reversal symmetry, we can take a limit of $u(t_n), t_n \in I$ under the various symmetries, and obtain $N(t) \leq 1$ for $t \geq 0, N(0) = 1$. By the perturbation theory of [91] the limit of $u(t_n)$ modulo symmetries will be the initial data of an almost periodic $u$ with $x(0) = \xi(0) = 0$. Finally by (2.27) the theorem holds.

At this point, it remains to show that such a solution cannot occur, which will occupy the remainder of the paper. We will now take a solution satisfying Theorem 2.7, and for the rest of the paper any expression of the form $A \lesssim B$ ($A \leq C(u)B$) will be abbreviated $A \lesssim B$.

### 3. A long time Strichartz space.

**Definition 3.1.** (Littlewood-Paley decomposition) Let $\phi \in C^\infty_0(\mathbb{R}^2)$ be a radial, decreasing function,
\begin{equation}
\phi(x) = \begin{cases} 
1, & |x| \leq 1; \\
0, & |x| > 2.
\end{cases}
\end{equation}
This gives a partition of unity
\begin{equation}
1 = \phi(x) + \sum_{j=1}^{\infty} [\phi(2^{-j} x) - \phi(2^{-j+1} x)] = \psi_0(x) + \sum_{j=1}^{\infty} \psi_j(x).
\end{equation}
For any integer $j \geq 0$
\begin{equation}
P_j f = \mathcal{F}^{-1} (\psi_j(\xi) \hat{f}(\xi)) = \int K_j(x-y)f(y)dy,
\end{equation}
where $K_j$ is an $L^1$ kernel. Let $P_j f = 0$ when $j$ is an integer less than zero and let

$$P_{j_1 \leq j \leq j_2} f = \sum_{j_1 \leq j \leq j_2} P_j f.$$  

We also define the frequency truncation

$$P_{\leq N} f = \mathcal{F} \left( \phi \left( \frac{\xi}{N} \right) \hat{f}(\xi) \right).$$  

The Littlewood-Paley decomposition respects $L^p$ norms, $1 < p < \infty$.

**Lemma 3.1.** (Littlewood-Paley theorem) For $1 < p < \infty$,

$$\|f\|_{L^p(\mathbb{R})} \sim_p \left\| \left( \sum_{j=0}^{\infty} |P_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})}.$$  

**Proof.** See [78, 79, 94] or [96]. □

Because $\xi(t)$ is free to move around in the nonradial case it is also convenient to define a Littlewood-Paley projection centered around $\xi_0 \in \mathbb{R}$ when $\xi_0 \neq 0$.

**Definition 3.2.** Let $\xi_0 \in \mathbb{R}$. Then define

$$P_{\xi_0,j} u = e^{ix \cdot \xi_0} P_j (e^{-ix \cdot \xi_0} u).$$

Let

$$P_{\xi_0,j} \bar{u} = e^{-ix \cdot \xi_0} P_j (e^{ix \cdot \xi_0} \bar{u}) = \overline{e^{ix \cdot \xi_0} P_j (e^{-ix \cdot \xi_0} u)}.$$  

Also for $1 \leq p \leq \infty$ define the norm

$$\left\| P_{\xi(t),j} f \right\|_{L^p_t L^q_x ([I \times \mathbb{R}])} = \left\| P_{\xi(t),j} f(t) \right\|_{L^q_x ([I \times \mathbb{R}])} \left\| P_{\xi(t),j} f \right\|_{L^p_t ([I])}.$$  

Notice that

$$P_{\xi_0,j} f = \int K_j (x-y) e^{i(x-y) \cdot \xi_0} f(y) dy,$$

which also has an $L^1$ kernel, so if $j$ is fixed in time, we can use (2.11) and (2.12).

Now to construct a function space adapted to the long time Strichartz estimates we utilize a class of function spaces introduced in [93] to study wave maps. [62, 63] applied these spaces to nonlinear Schrödinger problems. See [46] for a general description of these spaces. These spaces are quite useful to critical problems since the $X^{s,b}$ spaces of [5, 6] (see also [35]) are scale invariant when $b = \frac{1}{2}$, which has the same difficulty as the failure of the embedding $\dot{H}^{1/2}(\mathbb{R}) \subset L^\infty(\mathbb{R})$. This section is very similar to section three of [31] so many proofs will be omitted or abridged.
**Definition 3.3.** \((U^p_\Delta\) spaces) Let \(1 \leq p < \infty\). Let \(U^p_\Delta\) be an atomic space whose atoms are piecewise solutions to the linear equation,

\[
U_\lambda = \sum_k 1_{[s_k, t_{k+1})} e^{it\Delta} u_k, \quad \sum_k \|u_k\|_{L^2}^p = 1.
\]

Then for any \(1 \leq p < \infty\),

\[
\|u\|_{U^p_\Delta} = \inf \left\{ \sum_\lambda |c_\lambda| : u = \sum_\lambda c_\lambda u_\lambda, \text{ } u_\lambda \text{ are } U^p_\Delta \text{ atoms} \right\}.
\]

For any \(1 \leq p < \infty\), \(U^p_\Delta \subset L^\infty(L^2)\). Additionally, \(U^p_\Delta\) functions are continuous except at countably many points and right continuous everywhere.

**Definition 3.4.** \((V^p_\Delta\) spaces) Let \(1 \leq p < \infty\). Then \(V^p_\Delta\) is the space of right continuous functions \(u \in L^\infty(L^2)\) such that

\[
\|v\|_{V^p_\Delta} = \|v\|_{L^\infty(L^2)}^p + \sup_{\{t_k\} \uparrow} \sum_k \|e^{-it\Delta} v(t_k) - e^{-it_{k+1}\Delta} v(t_{k+1})\|_{L^2}^p.
\]

The supremum is taken over increasing sequences \(t_k\).

**Theorem 3.2.** The function spaces \(U^p_\Delta\) and \(V^p_\Delta\) obey the embeddings

\[
U^p_\Delta \subset V^p_\Delta \subset U^q_\Delta \subset L^\infty(L^2), \quad p < q.
\]

Let \(DU^p_\Delta\) be the space of functions

\[
DU^p_\Delta = \{(i\partial_t + \Delta)u ; u \in U^p_\Delta\}.
\]

By Duhamel’s formula

\[
\|u\|_{U^p_\Delta} \lesssim \|u(0)\|_{L^2} + \|(i\partial_t + \partial_x^2)u\|_{DU^p_\Delta}.
\]

Finally, there is the duality relation

\[
(DU^p_\Delta)^* = V^p_\Delta.
\]

These spaces are also closed under truncation in time.

\[
\chi I : U^p_\Delta \rightarrow U^p_\Delta, \quad \chi I : V^p_\Delta \rightarrow V^p_\Delta.
\]

**Proof.** See [46].

**Lemma 3.3.** Suppose \(J = I_1 \cup I_2, I_1 = [a, b], I_2 = [b, c], a \leq b \leq c\). Then,

\[
\|u\|_{U^p_\Delta(J \times \mathbb{R})} \leq \|u\|_{U^p_\Delta(I_1 \times \mathbb{R})} + \|u\|_{U^p_\Delta(I_2 \times \mathbb{R})}.
\]
and for \( j = 1, 2 \),

\[
\| u \|^p_{L^p_t L^q_x(I_2 \times \mathbb{R})} \leq \| u \|^p_{U^p_\Delta(J \times \mathbb{R})}.
\]

**Proof.** This follows from the definition of the \( U_\Delta \) spaces. See [31] for a proof. \( \square \)

**Lemma 3.4.** Suppose \( J = \bigcup_{m=1}^k J^m \), where \( J^m = [a_m, b_m] \) are consecutive intervals, and \( a_{m+1} = b_m \). Then for any \( t_0 \in J \),

\[
\left\| \int_0^t e^{i(t-\tau)\Delta} F(\tau) d\tau \right\|_{U^2_\Delta(J \times \mathbb{R})} \lesssim \sum_{m=1}^k \left\| \int_{J^m} e^{-i\tau\Delta} F(\tau) d\tau \right\|_{L^2_t L^2_x(\mathbb{R})}
\]

\[
+ \left( \sum_{m=1}^k \| F \|^2_{D U^2_\Delta(J^m \times \mathbb{R})} \right)^{1/2}.
\]

**Proof.** See [31]. \( \square \)

The \( U^2_\Delta \) spaces respect linear and bilinear Strichartz estimates. Indeed, checking individual atoms shows that if \( p, q \) is an admissible pair then

\[
\| u \|_{L^p_t L^q_x(I \times \mathbb{R})} \lesssim \| u \|_{U^p_\Delta(I \times \mathbb{R})}.
\]

**Lemma 3.5.** Suppose that under some condition on the supports of \( \hat{u}_0(\xi) \) and \( \hat{v}_0(\xi) \),

\[
\| (e^{it\Delta} u_0)(e^{it\Delta} v_0) \|_{L^p_t L^q_x(I \times \mathbb{R})} \lesssim N^{-\alpha} \| u_0 \|_{L^2_t L^2_x(\mathbb{R})} \| v_0 \|_{L^2_t L^2_x(\mathbb{R})},
\]

for some \( 1 \leq p < \infty \). Then if \( \hat{u}(t, \xi) \) and \( \hat{v}(t, \xi) \) satisfy the same conditions,

\[
\| u v \|_{L^p_t L^q_x(I \times \mathbb{R})} \lesssim N^{-\alpha} \| u \|_{U^p_\Delta(I \times \mathbb{R})} \| v \|_{U^p_\Delta(I \times \mathbb{R})}.
\]

**Proof.** By (3.12) it suffices to check individual atoms. Suppose \( u \) is an atom. Then there exists an increasing sequence \( \{ t_k \} \) such that \( u = \sum_k 1_{[t_k, t_{k+1})}(t) e^{it\Delta} u_k \), and

\[
\| u v \|^p_{L^p_t L^q_x(I \times \mathbb{R})} = \sum_k \left\| (e^{it\Delta} u_k) v \right\|^p_{L^p_t L^q_x([t_k, t_{k+1}) \times \mathbb{R})}.
\]

If \( v \) is also an atom then \( v = \sum_l 1_{[t_l, t_{l+1})}(t) e^{it\Delta} v_l \) for an increasing sequence \( \{ t_l \} \), and

\[
\left\| (e^{it\Delta} u_k) v \right\|^p_{L^p_t L^q_x(I \times \mathbb{R})} = \sum_l \left\| (e^{it\Delta} u_k) (e^{it\Delta} v_l) \right\|^p_{L^p_t L^q_x([t_l, t_{l+1}) \times \mathbb{R})}
\]

\[
\lesssim N^{-\alpha} \sum_l \| u_k \|^p_{L^2_t} \| v_l \|^p_{L^2} \lesssim N^{-\alpha} \| u_k \|^p_{L^2}.
\]
This implies
\[ \sum_k \left\| (e^{it\Delta} u_k) v \right\|_{L^p_t L^q_x ([t_k, t_{k+1}] \times \mathbb{R}^2)} \lesssim N^{-\alpha}. \]

Now we can recall the $\tilde{X}_{k_0}$ norm from [31]. Fix three constants,
\[ 0 < \epsilon_3 \ll \epsilon_2 \ll \epsilon_1 < 1. \]

Fix an integer $k_0 \in \mathbb{Z}_{\geq 0}$. Let $[a, b]$ be an interval such that
\[ \int_a^b \int \left| u(t, x) \right|^6 dx dt = 2^{k_0}, \]
and
\[ \int_a^b N(t)^3 dt = \epsilon_3 2^{k_0}. \]
Notice that (3.30) is invariant under (1.2) while (3.31) is not. Therefore, given an interval that satisfies (3.30) it is always possible to rescale so that (3.31) is also satisfied. Choose $\epsilon_1, \epsilon_2, \epsilon_3$ such that
\[ |\xi'(t)| + |N'(t)| \leq 2^{-20} \epsilon_1^{-1/2} N(t)^3, \]
\[ \int_{|x-x(t)| \geq 2^{-20} \epsilon_3^{-1/2} N(t)} |u(t, x)|^2 dx \leq \int_{|\xi-\xi(t)| \geq 2^{-20} \epsilon_3^{-1/2} N(t)} |\hat{u}(t, \xi)|^2 d\xi \leq \epsilon_2^2, \]
and
\[ \epsilon_3 < \epsilon_2^{10}. \]

**Definition 3.5.** (Small intervals) Let $[a, b] = \bigcup_{l=0}^{2k_0-1} J_l$, with $\|u\|_{L^6_t L^6_x (J_l \times \mathbb{R}^2)} = 1$. We will call the intervals $J_l$ the small intervals. Also let $N(J_l) = \sup_{t \in J_l} N(t)$.

**Remark.** Proposition 2.1 combined with conservation of mass implies that for any admissible pair $(p, q)$,
\[ \|u\|_{L^p_t L^q_x (J_l \times \mathbb{R})} \lesssim 1. \]

**Definition 3.6.** ($J^\alpha$ intervals) Let $[a, b] = \bigcup_{\alpha=0}^{2k_0-1} J^\alpha$, $\alpha = 0, \ldots, 2k_0 - 1$, such that
\[ \int_{J^\alpha} (N(t)^3 + \epsilon_3 \|u(t)\|_{L^6_x (\mathbb{R}^2)}^6) dt = 2\epsilon_3. \]
Definition 3.7. (Partition at level $j$) For an integer $0 \leq j < k_0$, $0 \leq k < 2^{k_0-j}$, let

$$G^j_k = \bigcup_{\alpha = k2^j}^{(k+1)2^j-1} J^\alpha.$$  

For $j \geq k_0$ let $G^j_k = [a, b]$.

If $G^j_k = [t_0, t_1]$ let $\xi(G^j_k) = \xi(t_0)$. Define $\xi(J^\alpha)$ and $\xi(J_i)$ in a similar manner.

Remark. By (3.31) and (3.32), for all $t \in G^j_k$,

$$\left| \xi(t) - \xi(G^j_k) \right| \leq \int_{G^j_k} 2^{-20} \epsilon_1^{-1/2} N(t)^3 \, dt \leq 2^{-19} \epsilon_1^{-1/2}.$$  

Therefore, for all $t \in G^j_k$,

$$\left\{ \xi : 2^j-1 \leq |\xi - \xi(t)| \leq 2^{j+1} \right\} \subset \left\{ \xi : 2^{j-2} \leq |\xi - \xi(G^j_k)| \leq 2^{j+2} \right\}$$  

$$\subset \left\{ \xi : 2^{j-3} \leq |\xi - \xi(t)| \leq 2^{j+3} \right\},$$

and

$$\left\{ \xi : |\xi - \xi(t)| \leq 2^{j+1} \right\} \subset \left\{ \xi : |\xi - \xi(G^j_k)| \leq 2^{j+2} \right\}$$  

$$\subset \left\{ \xi : |\xi - \xi(t)| \leq 2^{j+3} \right\}.$$  

Definition 3.8. ($\bar{X}_{k_0}$ spaces) For any $G^j_k \subset [a, b]$, $0 \leq i < j \leq k_0$, let

$$\|P_{\xi(t), i} u\|_{U^2_{\Delta}(G^j_k \times R)}^2 = \sum_{G^2_{\alpha} \subset G^j_k} \|P_{\xi(G^2_{\alpha}), i-2 \leq j \leq i+2} u\|_{U^2_{\Delta}(G^2_{\alpha} \times R)}^2,$$

For $i \geq j$

$$\|P_{\xi(t), i} u\|_{U^2_{\Delta}(G^j_k \times R)}^2 = \|P_{\xi(G^j_k), i-2 \leq j \leq i+2} u\|_{U^2_{\Delta}(G^j_k \times R)}^2,$$

and for $G^j_k \subset [a, b]$ let

$$\|u\|_{X(G^j_k \times R)}^2 = \sum_{0 \leq i < j} 2^{i-j} \|P_{\xi(t), i} u\|_{U^2_{\Delta}(G^j_k \times R)}^2 + \sum_{i \geq j} \|P_{\xi(t), i} u\|_{U^2_{\Delta}(G^j_k \times R)}^2.$$

Then let

$$\|u\|_{X_{k_0} [a, b] \times R}^2 = \sup_{0 \leq j \leq k_0} \sup_{G^j_k \subset [a, b]} \|u\|_{X(G^j_k \times R)}^2.$$  

Also for $0 \leq k_* \leq k_0$, let

$$\|u\|_{X_{k_*} [a, b] \times R}^2 = \sup_{0 \leq j \leq k_*} \sup_{G^j_k \subset [a, b]} \|u\|_{X(G^j_k \times R)}^2.$$
for any $0 \leq j \leq k_*$ is defined in a similar manner.

By (3.22), (3.39) and (3.40), for $i < j$, $(p, q)$ an admissible pair,

$$
\| P_{\xi(t),i} u \|_{L_t^p L_x^q(G_k^j \times \mathbb{R})} \lesssim 2^{\frac{(j-i)}{p}} \| u \|_{X(G_k^j \times \mathbb{R})}.
$$

Also, by (3.6), if $q < \infty$

$$
\| P_{\xi(t),i} u \|_{L_t^p L_x^q(G_k^j \times \mathbb{R})} \sim \left( \sum_{l \geq j} \| P_{\xi(t),i} u \|_{L_t^2 L_x^q(G_k^j \times \mathbb{R})}^2 \right)^{1/2} \lesssim \| u \|_{X(G_k^j \times \mathbb{R})}.
$$

4. Long time Strichartz estimate.

**Theorem 4.1.** (Long time Strichartz estimate) If $u$ is an almost periodic solution to (1.1) then for any positive integer $k_0$, $\epsilon_1$, $\epsilon_2$, $\epsilon_3$ satisfying (3.32)–(3.34),

$$
\int_0^T N(t)^3 dt = 2^{k_0} \epsilon_3, \text{ and } \int_0^T \int |u(t,x)|^6 dx dt = 2^{k_0},
$$

(4.1)

$$
\| u \|_{X_{k_0}(0,T \times \mathbb{R})} \lesssim 1.
$$

**Remark.** Throughout this section the implicit constant depends only on $u$ and not on $k_0$, or $\epsilon_1$, $\epsilon_2$, $\epsilon_3$.

**Proof.** It is necessary to prove that for any $0 \leq j \leq k_0$, $G_k^j \subset [0,T]$,

$$
\sum_{0 \leq i \leq j} 2^{i-j} \sum_{G_{\alpha}^i \subset G_k^j} \| P_{\xi(G_{\alpha}^i),i-2 \leq i \leq i+2} u \|_{U_{\Delta}^2(G_{\alpha}^i \times \mathbb{R})}^2

+ \sum_{i > j} \| P_{\xi(G_k^j),i-2 \leq i \leq i+2} u \|_{U_{\Delta}^2(G_k^j \times \mathbb{R})}^2 \sim 1.
$$

(4.2)

Fix $k_0$, $0 \leq j \leq k_0$, and $G_k^j \subset [0,T]$. For $0 \leq i \leq j$, Duhamel’s principle implies

$$
\| P_{\xi(G_{\alpha}^i),i-2 \leq i \leq i+2} u \|_{U_{\Delta}^2(G_k^j \times \mathbb{R})}^2

\lesssim \| P_{\xi(G_{\alpha}^i),i-2 \leq i \leq i+2} u(t_{\alpha}^i) \|_{L_t^2(\mathbb{R})}^2

+ \left\| \int_{t_{\alpha}^i}^t e^{i(t-\tau)\Delta} P_{\xi(G_{\alpha}^i),i-2 \leq i \leq i+2} F(u(\tau)) d\tau \right\|_{U_{\Delta}^2(G_k^j \times \mathbb{R})}.
$$

(4.3)

For any $0 \leq i < j$, $G_{\alpha}^i \subset G_k^j$, choose $t_{\alpha}^i$ such that

$$
\| P_{\xi(G_{\alpha}^i),i-2 \leq i \leq i+2} u(t_{\alpha}^i) \|_{L_t^2(\mathbb{R})} = \inf_{t \in G_{\alpha}^i} \| P_{\xi(G_{\alpha}^i),i-2 \leq i \leq i+2} u(t) \|_{L_t^2(\mathbb{R})}.
$$

(4.4)
Then by (3.36), (3.39), and (3.40),

\[
(4.5) \sum_{0 \leq i < j} 2^{i-j} \sum_{G_{i_k} \subset G_{j_k}} \| P_{\xi(G_{i_k}),i-2 \leq i+2 u(t_{\alpha}^i)} \|_{L^2_\alpha(R)}^2 \\
\lesssim 2^{-j} \varepsilon_3^{-1} \int_{G_{i_k}} [(\varepsilon_3 \|u(t)\|_{L^6_\alpha(R)}^6 + N(t)^3) \sum_{0 \leq i < j} \| P_{\xi(t),i-3 \leq i+3 u(t)} \|_{L^2_\alpha(R)}^2] dt \\
\lesssim 2^{-j} \varepsilon_3^{-1} \|u\|_{L^2_\alpha([0,T] \times R)}^2 \int_{G_{i_k}} (N(t)^3 + \varepsilon_3 \|u(t)\|_{L^6_\alpha(R)}) dt \lesssim 1.
\]

(4.6)

For \( i \geq j \) simply take \( t_{\alpha}^i = t_0 \), where \( t_0 \) is a fixed element of \( G_{i_k} \), say the left endpoint. Then

\[
(4.7) \sum_{i \geq j} \| P_{\xi(G_{i_k}),i-2 \leq i+2 u(t_{\alpha}^i)} \|_{L^2_\alpha(R)}^2 \lesssim \|u(t_0)\|_{L^2_\alpha(R)}^2 \lesssim 1.
\]

Therefore,

\[
(4.8) \sum_{0 \leq i < j} 2^{i-j} \sum_{G_{i_k} \subset G_{j_k}} \| P_{\xi(G_{i_k}),i-2 \leq i+2 u(t_{\alpha}^i)} \|_{L^2_\alpha(R)}^2 \\
+ \sum_{i > j} \| P_{\xi(G_{i_k}),i-2 \leq i+2 u(t_{\alpha}^i)} \|_{L^2_\alpha(R)}^2 \lesssim 1.
\]

(4.8) implies

\[
\|u\|_{X(G_{i_k} \times R)}^2 \lesssim 1 + \sum_{i \geq j} \left| \int_{t_{\alpha}}^{t} e^{i(t-\tau)\Delta} P_{\xi(G_{i_k}),i-2 \leq i+2 F(u(\tau)) d\tau} \right|^2 \|U_{\Delta}^2(G_{i_k} \times R)} \\
+ \sum_{0 \leq i < j} 2^{i-j} \sum_{G_{i_k} \subset G_{j_k}} \left| \int_{t_{\alpha}}^{t} e^{i(t-\tau)\Delta} P_{\xi(G_{i_k}),i-2 \leq i+2 F(u(\tau)) d\tau} \right|^2 \|U_{\Delta}^2(G_{i_k} \times R)}.
\]

(4.9)

Now take an interval \( G_{i_k} \subset G_{j_k} \) with \( N(G_{i_k}) \geq \varepsilon_3^{1/2}2^i-5 \). (3.32) implies that \( N(t) \geq \varepsilon_3^{1/2}2^i-6 \) for all \( t \in G_{i_k} \). We compute

\[
(4.10) \sum_{0 \leq i < j} 2^{i-j} \sum_{G_{i_k} \subset G_{j_k} : N(G_{i_k}) \geq \varepsilon_3^{1/2}2^i-5} \|F(u)\|_{L^2_\alpha L^2_\alpha(G_{i_k} \times R)}^2.
\]

There are at most two small intervals, call them \( J_1 \) and \( J_2 \), that overlap \( G_{i_k} \) but do not intersect \( G_{j_k} \). Now by (3.35), \( \|u\|_{L^2_\alpha L^2_\alpha(J_1 \times R)} \lesssim 1 \). Therefore,

\[
(4.11) \sum_{0 \leq i < j} 2^{i-j} \sum_{G_{i_k} \subset G_{j_k}} \|F(u)\|_{L^2_\alpha L^2_\alpha(G_{i_k} \cap (J_1 \cup J_2) \times R)}^2 \lesssim 1.
\]
implies

\[ \sum_{0 \leq i < j} 2^{i-j} \sum_{J_l \subset G^j_i; N(J_l) \geq 2^{i-6} \varepsilon^j_3} \| F(u) \|_{L^5_t L^3_x (J_l \times \mathbb{R})}^2 \]

\[ = 1 + \sum_{J_l \subset G^j_i} \sum_{0 \leq i < j} 2^{i-j} \lesssim 2^{-j} \varepsilon^j_3 \sum_{J_l \subset G^j_i} N(J_l) \lesssim 1. \]

The last inequality follows from (2.25) and (3.36). Now suppose \( N(G^j_k) \geq 2^{j-5} \varepsilon^j_3 \). By (2.26), \( \int_{G^j_k} N(t)^3 dt \lesssim N(G^j_k) \). Then \( N(t) \geq 2^{j-5} \varepsilon^j_3 \) for all \( t \in G^j_k \), implies \( \int_{G^j_k} N(t)^2 dt \lesssim 1 \), so by (2.24),

\[ \sum_{i \geq j} \| P_{\xi(G^j_k), i-2 \leq \cdot \leq i+2} F(u) \|_{L^5_t L^3_x (G^j_k \times \mathbb{R})}^2 \lesssim \| u \|_{L^5_t L^3_x (G^j_k \times \mathbb{R})}^2 \lesssim 1. \]

It only remains to estimate

\[ \sum_{i \geq j; N(G^j_k) \leq 2^{i-5} \varepsilon^j_3} \left\| \int_{t^\alpha}^{t} e^{i(t-\tau) \Delta} P_{\xi(G^j_k), i-2 \leq \cdot \leq i+2} F(u(\tau)) d\tau \right\|_{U^3_{g^j_k} (G^j_k \times \mathbb{R})}^2 + \sum_{0 \leq i \leq j} 2^{i-j} \]

\[ \times \sum_{G^j_\alpha \subset G^j_k; N(G^j_\alpha) \leq 2^{i-5} \varepsilon^j_3} \left\| \int_{t^\alpha}^{t} e^{i(t-\tau) \Delta} P_{\xi(G^j_\alpha), i-2 \leq \cdot \leq i+2} F(u(\tau)) d\tau \right\|_{U^3_{g^j_\alpha} (G^j_\alpha \times \mathbb{R})}^2. \]

Since \( \| u \|_{L^5_t L^3_x (J_l \times \mathbb{R})} \lesssim 1 \), by (3.17), (3.22), \( \| u \|_{U^3_{g^j_k} (J_l \times \mathbb{R}^2)} \lesssim 1 \) on each small interval \( J_l \), so by Definition 3.8,

\[ \| u \|_{\mathcal{X}_0 ([0,T] \times \mathbb{R})} \leq C(u). \]

Moreover it is clear from Definition 3.8 that for any \( 0 \leq k_* < k_0 \),

\[ \| u \|_{\mathcal{X}_{k_*+1} ([0,T] \times \mathbb{R})} \leq 2 \| u \|_{\mathcal{X}_{k_*} ([0,T] \times \mathbb{R})}. \]

Therefore it suffices to prove:
\textbf{Theorem 4.2.}

(4.18) \[\sum_{i \geq j : N(G_{k}^{j}) \leq 2^{i-5} \epsilon_{1}/2} \left\| \int_{t_{\alpha}}^{t} e^{i(t-\tau)\Delta} P_{\xi(G_{k}^{j}), i-2 \leq i+2} F(u(\tau)) d\tau \right\|_{L_{x}^{2}(G_{k}^{j} \times \mathbb{R})}^{2} \]
+ \sum_{0 \leq i \leq j} 2^{i-j} \[\sum_{G^{i}_{k} \subset G_{k}^{j} : N(G_{k}^{i}) \leq 2^{i-5} \epsilon_{1}/2} \left\| \int_{t_{\alpha}}^{t} e^{i(t-\tau)\Delta} P_{\xi(G_{k}^{i}), i-2 \leq i+2} F(u(\tau)) d\tau \right\|_{L_{x}^{2}(G_{k}^{i} \times \mathbb{R})}^{2} \]
\[\lesssim \epsilon_{2}^{1/2} (1 + \|u\|_{\dot{X}_{j}([0,T] \times \mathbb{R})})^{4} \]

Indeed, suppose

(4.19) \[\|u\|_{\dot{X}_{k+1}([0,T] \times \mathbb{R}^{2})} \leq 2C_{0}, \]

Then by (4.9)–(4.14) and (4.18),

(4.20) \[\|u\|_{\dot{X}_{k+1}([0,T] \times \mathbb{R})} \leq C(u) (1 + \epsilon_{2}^{1/2} C_{0}^{4}). \]

Taking \(C_{0} = 2C(u), \epsilon_{2} > 0\) sufficiently small closes the bootstrap and implies

(4.21) \[\|u\|_{\dot{X}_{k+1}([0,T] \times \mathbb{R}^{2})} \leq C_{0}. \]

Then Theorem 4.1 follows by (4.16) and induction on \(k_{\alpha}. \)

\[\square\]

\textbf{Proof of Theorem 4.2.} Start with a bilinear estimate. By the Sobolev embedding theorem, Proposition 2.2, (2.11), (2.12), (3.38), and Definition 3.8, for \(l_{0} > i-5, \)

\[\| (P_{\xi(G_{k}^{i}), l_{0}} u)(P_{\xi(t), \leq i-10} u) \|_{L_{t,x}^{2}(G_{k}^{i} \times \mathbb{R})}^{2} \]
\[\lesssim \sum_{0 \leq l_{0} \leq l_{1} \leq l_{2} \leq l_{3} \leq i-10} \| (P_{\xi(G_{k}^{i}), l_{0}} u)(P_{\xi(t), l_{1}} u) \|_{L_{t,x}^{2}}^{2} \times \| (P_{\xi(G_{k}^{i}), l_{0}} u)(P_{\xi(t), l_{2}} u) \|_{L_{t,x}^{2}} \times \| P_{\xi(t), l_{3}} u \|_{L_{t,x}^{2}} \times \| P_{\xi(t), l_{4}} u \|_{L_{t,x}^{2}} \]
\[\lesssim \| P_{\xi(G_{k}^{i}), l_{0}} u \|_{L_{t,x}^{2}}^{2} \times \sum_{0 \leq l_{2} \leq l_{1} \leq l_{0} \leq l_{3} \leq i-10} 2^{l_{2}-l_{0}} \| P_{\xi(t), l_{2}} u \|_{L_{t,x}^{2}} \| P_{\xi(t), l_{1}} u \|_{L_{t,x}^{2}} \]
\[\| P_{\xi(t), l_{3}} u \|_{L_{t,x}^{2}} \| P_{\xi(t), l_{4}} u \|_{L_{t,x}^{2}} \]
\[\lesssim 2^{i-l_{0}} \| P_{\xi(G_{k}^{i}), l_{0}} u \|_{L_{t,x}^{2}}^{2} \times \| u \|_{\dot{X}(G_{k}^{i} \times \mathbb{R})}^{2}. \]
Now

\begin{equation}
\left\| \int_{t_0}^{t} e^{i(t-\tau)\Delta} P_{\xi(G_{\alpha}^i),i-2\leq i+2} F(u(\tau)) d\tau \right\|_{U_\Delta^2(G_{\alpha}^i \times \mathbb{R})}
\end{equation}

(4.24)

\begin{equation}
\lesssim \left\| \int_{t_0}^{t} e^{i(t-\tau)\Delta} P_{\xi(G_{\alpha}^i),i-2\leq i+2} (P_{\xi(G_{\alpha}^i),i-5} u)^2 u^3 d\tau \right\|_{U_\Delta^2(G_{\alpha}^i \times \mathbb{R})}
\end{equation}

(4.25)

\begin{equation}
+ \left\| \int_{t_0}^{t} e^{i(t-\tau)\Delta} P_{\xi(G_{\alpha}^i),i-2\leq i+2} (P_{\xi(G_{\alpha}^i),i-5\leq i+5} u) (P_{\xi(G_{\alpha}^i),i\leq i-5})^4 d\tau \right\|_{U_\Delta^2(G_{\alpha}^i \times \mathbb{R})}
\end{equation}

(4.26)

First take (4.25). Since \(N(G_{\alpha}^i) \leq 2^{-i} \epsilon_3^{1/2}\), (4.23), (3.32)–(3.34) imply

\begin{equation}
\left\| (P_{\xi(G_{\alpha}^i),i-5} u)^2 (P_{\xi(t),i-10} u)^3 \right\|_{L_t^{4/3} L_x^2(G_{\alpha}^i \times \mathbb{R})}
\end{equation}

(4.27)

\begin{equation}
\lesssim \left\| P_{\xi(G_{\alpha}^i),i-5} u \right\|_{L_t^{1/2} L_x^2}^{1/2} \left\| (P_{\xi(G_{\alpha}^i),i-5} u) (P_{\xi(t),i-10} u)^2 \right\|_{L_t^{3/2} L_x^3}^{3/2}
\end{equation}

Now take \(\hat{\delta}(\xi)\) supported on \(|\xi - \xi(G_{\alpha}^i)| \sim 2^j\), \(\|v\|_{V_\Delta^2(G_{\alpha}^i \times \mathbb{R})} = 1\). By Corollary 2.3, \(V_\Delta^2 \subset U_\Delta^3\), (3.46), (3.47), which implies \(\|P_{\xi(t),i-10} u\|_{L_t^{4/3} L_x^2(G_{\alpha}^i \times \mathbb{R})} \lesssim \|u\|_{X(G_{\alpha}^i \times \mathbb{R})}\),

\begin{equation}
\int_{G_{\alpha}^i} \left\langle v, (P_{\xi(G_{\alpha}^i),i-5} u)^2 (P_{\xi(t),i-10} u)^3 \right\rangle dt
\end{equation}

(4.28)

\begin{equation}
\lesssim \left\| P_{\xi(t),i-5} u \right\|_{L_t^{3/2} L_x^3} \left\| (P_{\xi(t),i-10} u)^3 \right\|_{L_t^{3/2} L_x^3} \left\| v(P_{\xi(G_{\alpha}^i),i-5} u) \right\|_{L_t^1 L_x^2}
\end{equation}

(4.29)

\textbf{THEOREM 4.3.} For a fixed \(G_{\beta}^i \subset [0,T]\),

\begin{equation}
\sum_{0 \leq i < j} 2^{i-j} \sum_{G_{\alpha} \subset G_{\beta}^i} \left\| \int_{t_0}^{t} e^{i(t-\tau)\Delta} P_{\xi(G_{\alpha}^i),i-2\leq i+2} ((P_{\xi(G_{\alpha}^i),i-5} u)^2 u^3) d\tau \right\|_{U_\Delta^2(G_{\alpha}^i \times \mathbb{R})}
\end{equation}

(4.24)

\begin{equation}
\lesssim \epsilon_2 \|u\|_{X_j([0,T] \times \mathbb{R})}^{6} + \epsilon_3 \|u\|_{X_j([0,T] \times \mathbb{R})}^{8}.
\end{equation}

(4.25)
Proof. For any \(0 \leq l \leq j\), \(G_k^i\) overlaps \(2^{j-l}\) intervals \(G^l_j\) and for \(0 \leq i \leq l\), each \(G^l_j\) overlaps \(2^{l-i}\) intervals \(G^i_\alpha\). Additionally, each \(G^i_\alpha\) is the subset of one \(G^l_j\). By (4.27), (4.28), and interpolation,

\[
(4.29) \lesssim \left( e_2 \|u\|_{X(G^0_\alpha \times \mathbb{R})}^6 + e_2^2 \|u\|_{X(G^8_\alpha \times \mathbb{R})}^8 \right) \quad \times \quad \sum_{0 \leq i \leq j} 2^{i-j} \sum_{G^j_\alpha \subset G^i_\alpha} \sum_{l \geq l-10} 2^{(i-l)/4} \|P_{\xi(G^l_\alpha),l} u\|_{U^2_{\Delta}(G^i_\alpha \times \mathbb{R})}^2
\]

\[
(4.30) \quad \sum_{0 \leq i \leq j} 2^{i-j} \sum_{G^j_\alpha \subset G^i_\alpha} \sum_{l \geq l-10} 2^{(i-l)/4} \|P_{\xi(G^l_\alpha),l} u\|_{U^2_{\Delta}(G^i_\alpha \times \mathbb{R})}^2
\]

\[
(4.31) \quad \lesssim \sum_{0 \leq i \leq j} 2^{i-j} \sum_{G^j_\beta \subset G^i_\alpha} \sum_{i-10 \leq l \leq i} 2^{(i-l)/4} \|P_{\xi(G^l_\beta),l} u\|_{U^2_{\Delta}(G^i_\alpha \times \mathbb{R})}^2
\]

\[
(4.32) \quad + \sum_{0 \leq l \leq j} 2^{l-j} \sum_{G^j_\beta \subset G^i_\alpha} \|P_{\xi(G^l_\beta),l} u\|_{U^2_{\Delta}(G^i_\alpha \times \mathbb{R})}^2 \left( \sum_{0 \leq i \leq l} 2^{(i-l)/4} \right)
\]

\[
(4.33) \quad + \sum_{0 \leq i \leq j} \sum_{l > j} 2^{(i-l)/4} \|P_{\xi(G^l_\alpha),l} u\|_{U^2_{\Delta}(G^i_\alpha \times \mathbb{R})}^2 \lesssim \|u\|_{\tilde{X}_j([0,T] \times \mathbb{R})}^2
\]

and

\[
(4.34) \quad \sum_{i \geq j} \sum_{l \geq l-10} 2^{(i-l)/4} \|P_{\xi(G^l_\alpha),l} u\|_{U^2_{\Delta}(G^i_\alpha \times \mathbb{R})}^2
\]

\[
(4.35) \quad \lesssim \sum_{i \geq j} \sum_{l \geq l-10} \sum_{l < j} \sum_{G^l_\beta \subset G^i_\alpha} 2^{(i-l)/4} \|P_{\xi(G^l_\beta),l} u\|_{U^2_{\Delta}(G^i_\alpha \times \mathbb{R})}^2
\]

\[
(4.36) \quad + \sum_{i > j} \sum_{l \geq l-10} \sum_{l \geq l} 2^{(i-l)/4} \|P_{\xi(G^l_\alpha),l} u\|_{U^2_{\Delta}(G^i_\alpha \times \mathbb{R})}^2 \lesssim \|u\|_{\tilde{X}_j([0,T] \times \mathbb{R})}^2.
\]

This proves Theorem 4.3. \(\square\)

To estimate (4.26) we sharpen (4.22). For a given \(G^i_\alpha\) there are at most two small intervals \(J_1, J_2\) that overlap \(G^i_\alpha\) but are not contained in \(G^i_\alpha\). Let \(\tilde{G}^i_\alpha = G^i_\alpha \setminus (J_1 \cup J_2)\). By (3.32)–(3.34) and the Sobolev embedding theorem,

\[
(4.37) \quad \|P_{\xi(t),l} u\|_{L^p_{\tau,w}(J_1 \times \mathbb{R})} \lesssim 2^{l^1/2} e_2 + 2^{l^1/4} N(J_l)^{1/4} \epsilon_3^{-1/8}.
\]
Making a calculation similar to (4.22),

\[
\sum_{J_l \subseteq G_t} \sum_{0 \leq l_4 \leq l_3 \leq l_2 \leq l_1 \leq i-10} \| (P_{\xi(G_t^i)}, i-5 \leq i+5 u)(P_{\xi(t), l_1 u}) \|_{L^2_{t,x}(J_l \times \mathbb{R})}^2
\times \| (P_{\xi(G_t^i)}, i-5 \leq i+5 u)(P_{\xi(t), l_2 u}) \|_{L^2_{t,x}(J_l \times \mathbb{R})}
\]

\[
\lesssim \sum_{J_l \subseteq G_t} \sum_{0 \leq l_4 \leq l_3 \leq l_2 \leq l_1 \leq i-10} \| (P_{\xi(G_t^i)}, i-5 \leq i+5 u)(P_{\xi(t), l_1 u}) \|_{L^2_{t,x}(J_l \times \mathbb{R})}
\times \| (P_{\xi(G_t^i)}, i-5 \leq i+5 u)(P_{\xi(t), l_2 u}) \|_{L^2_{t,x}(J_l \times \mathbb{R})}
\]

\[
\lesssim \epsilon_2^2 \| u \|_{X(G_t^i \times \mathbb{R})}^2 \| P_{\xi(G_t^i), i-5 \leq i+5 u} \|_{U^2_{\Delta}(G_t^i \times \mathbb{R})}^2
\times 2^{-i} \sum_{0 \leq l_4 \leq l_3 \leq l_2 \leq l_1 \leq i-10} 2^{(l_3+l_4)/4} \left( \sum_{J_l \subseteq G_t} \epsilon_3^{-1/2} N(J_l) \right)^{1/2}
\]

In the last estimate we used \( \| u \|_{U^2_{\Delta}(J_l \times \mathbb{R})} \lesssim 1, J_l \subseteq G_t^i \), and the bilinear estimate

\[
\| (P_{\xi(G_t^i)}, i-5 \leq i+5 u)(P_{\xi(t), l_1 u}) \|_{L^2_{t,x}(J_l \times \mathbb{R})}
\lesssim 2^{-i/2} \| u \|_{X(G_t^i \times \mathbb{R})} \| P_{\xi(G_t^i), i-5 \leq i+5 u} \|_{U^2_{\Delta}(G_t^i \times \mathbb{R})},
\]

which implies

\[
2^{-i} \epsilon_3^{-1/4} \sum_{0 \leq l_4 \leq l_3 \leq l_2 \leq l_1 \leq i-10} 2^{(l_3+l_4)/4} \left( \sum_{J_l \subseteq G_t} \epsilon_3^{-1/2} N(J_l) \right)^{1/2}
\times \| (P_{\xi(G_t^i), i-5 \leq i+5 u})(P_{\xi(t), l_1 u}) \|_{L^2_{t,x}(G_t^i \times \mathbb{R})}
\lesssim \epsilon_3^{1/4} \| P_{\xi(G_t^i), i-5 \leq i+5 u} \|_{U^2_{\Delta}(G_t^i \times \mathbb{R})} \| u \|_{X(G_t^i \times \mathbb{R})}.
\]

Therefore,

\[
\| (P_{\xi(G_t^i), i-5 \leq i+5 u})(P_{\xi(t), i-10 u}) \|_{L^2_{t,x}(G_t^i \times \mathbb{R})}
\lesssim \epsilon_2 \| P_{\xi(G_t^i), i-5 \leq i+5 u} \|_{U^2_{\Delta}(G_t^i \times \mathbb{R})} \| u \|_{X(G_t^i \times \mathbb{R})}.
\]
Next, since $N(t) \leq \epsilon_3^{1/2} 2^{i-5}$, by (3.32)–(3.34), the fact that $J_1$ is a small interval, the Sobolev embedding theorem, and a bilinear Strichartz estimate, we have

\begin{equation}
\left\| \left( P_{\xi(G_{\alpha}^i)}, i-5 \leq i \leq i+5 u \right) \left( P_{\xi(t)}, i-10 u \right) \right\|^2_{L_{t,x}^2(J_1 \times \mathbb{R})} \lesssim \left\| \left( P_{\xi(G_{\alpha}^i)}, i-5 \leq i \leq i+5 u \right) \left( P_{\xi(t)}, i-10 u \right) \right\|^2_{L_{t,x}^2(J_1 \times \mathbb{R})} \\
\times \left\| P_{\xi(t)}, i-\epsilon_3^{-1/4} N(t) u \right\|_{L_{t,x}^2(J_1 \times \mathbb{R})}^2 \times \left\| P_{\xi(t)}, i+5 u \right\|_{L_{t,x}^2(J_1 \times \mathbb{R})}^2 \times \left\| P_{\xi(t)}, i-10 u \right\|_{L_{t,x}^2(J_1 \times \mathbb{R})}^2 \\
+ \left\| P_{\xi(G_{\alpha}^i)}, i-5 \leq i \leq i+5 u \right\|_{L_{t,x}^2(J_1 \times \mathbb{R})}^2 \times \left\| P_{\xi(t)}, i-10 u \right\|_{L_{t,x}^2(J_1 \times \mathbb{R})}^2 \\
\lesssim \epsilon_2 \left\| P_{\xi(G_{\alpha}^i)}, i-5 \leq i \leq i+5 u \right\|^2_{U_{\Delta}^2(G_{\alpha}^i \times \mathbb{R})}.
\end{equation}

(4.44)

Since the same estimate holds for $J_2$,

\begin{equation}
\left\| \left( P_{\xi(G_{\alpha}^i)}, i-5 \leq i \leq i+5 u \right) \left( P_{\xi(t)}, i-10 u \right) \right\|^2_{L_{t,x}^2(G_{\alpha}^i \times \mathbb{R})} \lesssim \epsilon_2 \left\| P_{\xi(G_{\alpha}^i)}, i-5 \leq i \leq i+5 u \right\|^2_{U_{\Delta}^2(G_{\alpha}^i \times \mathbb{R})} \left( \| u \|_{X(G_{\alpha}^i \times \mathbb{R})} + 1 \right).
\end{equation}

(4.45)

**Theorem 4.4.**

\begin{equation}
\sum_{0 \leq i \leq j} 2^{i-j} \sum_{0 \leq l_1 \leq l_2 \leq l_3 \leq l_4 \leq i-10} \left\| \int_{t_1}^t e^{i(t-\tau)\Delta} P_{\xi(G_{\alpha}^i), i-5 \leq i \leq i+5 u} \starskip P_{\xi(t), l_1 u} \starskip P_{\xi(t), l_2 u} \starskip P_{\xi(t), l_3 u} \starskip P_{\xi(t), l_4 u} \right\|^2_{U_{\Delta}^2(G_{\alpha}^i \times \mathbb{R})} \lesssim \epsilon_2 \left\| u \right\|^4_{X_j([0,T] \times \mathbb{R})}.
\end{equation}

(4.46)

**Proof.** Use Lemma 3.4. Partition $G_{\alpha}^i$ into sets $G_{\alpha}^{l_2}$. (If $l_2 \geq j$ then since $i \geq l_2$, $G_{\alpha}^i = G_{k}^j = G_{\beta}^{l_2}$). Suppose $\partial_{l_2}^{l_3} (\xi)$ are supported on $|\xi - \xi(G_{\alpha}^i)| \sim 2^i$, $\| v_{l_2}^i \|_{L_{t,x}^2(\mathbb{R})}$. Then by Proposition 2.2 and Definition 3.8,

\begin{equation}
\sum_{G_{\alpha}^{l_2} \subset G_{\alpha}^i} \| e^{i \Delta v_{l_2}^i} (P_{\xi(t), l_2 u}) \|_{L_{t,x}^2(G_{\alpha}^i \times \mathbb{R})} \lesssim 2^{-i/2} \| P_{\xi(t), l_2 u} \|_{U_{\Delta}^2(G_{\alpha}^i \times \mathbb{R})}.
\end{equation}

(4.47)
Replacing
\[
\| (P_{\xi(G_\alpha)}, i-5 \leq i \leq i+5 u) (P_{\xi(t), l_2} u) \|_{L_{t,x}^2(G_\alpha \times \mathbb{R})} \\
\lesssim 2^{-i/2} \| P_{\xi(G_\alpha), i-5 \leq i \leq i+5 u} \|_{U_{\Delta}^2(G_\alpha \times \mathbb{R})} \| P_{\xi(t), l_2} u \|_{U_{\Delta}^2(G_\alpha \times \mathbb{R})}
\]
in (4.38) with (4.47) implies
\[
\sum_{0 \leq l_4 \leq l_3 \leq l_2 \leq l_1 \leq i-10} \sum_{G_{l_2}^{l_2}} \left\| \int_{G_{l_2}^{l_2}} e^{-i t \Delta} (P_{\xi(G_\alpha), i-5 \leq i \leq i+5 u}) \times (P_{\xi(t), l_1} u) (P_{\xi(t), l_4} u) \right\|_{L_{t}^2(\mathbb{R})} dt \\
\lesssim \epsilon_2^2 \| P_{\xi(G_\alpha), i-5 \leq i \leq i+5 u} \|_{U_{\Delta}^2(G_\alpha \times \mathbb{R})} (1 + \| u \|_{X(G_\alpha \times \mathbb{R})})^2.
\]
This takes care of the first term in (3.21). Now for each \( G_{l_2}^{l_2} \) take \( v_{l_2}^{l_2} \) such that
\[
\| v_{l_2}^{l_2} \|_{V_{\Delta}^2(G_{l_2}^{l_2} \times \mathbb{R})} = 1 \quad \text{and} \quad \hat{v}_{l_2}^{l_2} \text{ is supported on } |\xi - \xi(G_{l_2}^{l_2})| \approx 2^i.
\]
\[
\sum_{G_{l_2}^{l_2} \subset G_\alpha} \left\| (v_{l_2}^{l_2}) (P_{\xi(\tau), l_2} u) (P_{\xi(t), l_4} u) \right\|_{L_{t,x}^2(G_{l_2}^{l_2} \times \mathbb{R})}^2 \\
\times \left\| (P_{\xi(G_\alpha), i-5 \leq i \leq i+5 u}) (P_{\xi(\tau), l_1} u) (P_{\xi(t), l_3} u) \right\|_{L_{t,x}^2(G_{l_2}^{l_2} \times \mathbb{R})}^2 \\
\lesssim \left( \sup_{G_{l_2}^{l_2} \subset G_\alpha} \left\| (v_{l_2}^{l_2}) (P_{\xi(\tau), l_2} u) (P_{\xi(t), l_4} u) \right\|_{L_{t,x}^2(G_{l_2}^{l_2} \times \mathbb{R})}^2 \\
\times \left\| (P_{\xi(G_\alpha), i-5 \leq i \leq i+5 u}) (P_{\xi(\tau), l_1} u) (P_{\xi(t), l_3} u) \right\|_{L_{t,x}^2(G_{l_2}^{l_2} \times \mathbb{R})}^2 \right)
\]
By (4.45),
\[
\lesssim \epsilon_2^2 \| P_{\xi(G_\alpha), i-5 \leq i \leq i+5 u} \|_{U_{\Delta}^2(G_\alpha \times \mathbb{R})} (1 + \| u \|_{X(G_\alpha \times \mathbb{R})})^2 \\
\times \left( \sup_{G_{l_2}^{l_2} \subset G_\alpha} \left\| (v_{l_2}^{l_2}) (P_{\xi(\tau), l_2} u) (P_{\xi(t), l_4} u) \right\|_{L_{t,x}^2(G_{l_2}^{l_2} \times \mathbb{R})}^2 \right)
\]
Since \( V_{\Delta}^2 \subset U_{\Delta}^{5/2} \), interpolating Proposition 2.2 and Corollary 2.3, (3.46), and the Sobolev embedding theorem,
\[
\| (v_{l_2}^{l_2}) (P_{\xi(t), l_2} u) \|_{L_{t,x}^{12/5}(G_{l_2}^{l_2} \times \mathbb{R})} \| P_{\xi(t), l_4} u \|_{L_{t,x}^{12/5}(G_{l_2}^{l_2} \times \mathbb{R})} \\
\lesssim 2^{l_4/4 l_2/8 - 3 i/8} 2^{(i-l_4)/12} \| u \|_{X_{j, (0,T] \times \mathbb{R})}^2.
\]
\[ \begin{align*}
&\sum_{0\leq l_1\leq l_2\leq l_3\leq l_4\leq l_5} 2^{(l_4-i)/2} 2^{(l_2-i)/4} \epsilon_2 \| P_{\xi(G_{a_i}),j-5\leq i+5\leq l_1\leq l_2\leq l_3\leq l_4\leq l_5} \|^2_{U_0^2 (G_{a_i} \times \mathbb{R})} \\
&\quad \times (1 + \| u \|_{X_j([0,T] \times \mathbb{R})})^6 \\
&\quad \lessapprox \epsilon_2^2 \| P_{\xi(G_{a_i}),j-5\leq i+5\leq l_1\leq l_2\leq l_3\leq l_4\leq l_5} \|^2_{U_0^2 (G_{a_i} \times \mathbb{R})} (1 + \| u \|_{X_j([0,T] \times \mathbb{R})})^6.
\end{align*} \]

This completes the proof of the theorem. \( \square \)

Plugging this into (4.30)–(4.36), it only remains to prove,

\[ \begin{align*}
&\| (P_{\xi(G_{a_i}),j}, [0,T] \times \mathbb{R}) \|^2_{U_0^2 (G_{a_i} \times \mathbb{R})} \\
&\quad \lessapprox \epsilon_2 \| P_{\xi(G_{a_i}),j-5\leq i+5\leq l_1\leq l_2\leq l_3\leq l_4\leq l_5} \|^3_{X(G_{a_i} \times \mathbb{R})}.
\end{align*} \]

However this follows from (3.32)–(3.34), (3.46), and (3.47). The proof of Theorem 4.2 is now complete. \( \square \)

5. Rapid frequency cascade. Now consider the scenario described in Theorem 2.7 with \( \int_0^\infty N(t)^3 dt = K < \infty \).

**Theorem 5.1.** If \( u \) is an almost periodic solution to (1.1) in the form of Theorem 2.7 and \( \int_0^\infty N(t)^3 dt = K < \infty \), then

\[ \| u(t) \|_{H^5(\mathbb{R})} \lessapprox K^5. \]

**Proof.** Suppose \([0,T]\) is an interval such that

\[ \int_0^T \int |u(t,x)|^6 dx dt = 2^{k_0} \]

for some \( k_0 \in \mathbb{Z}_+ \). Let \( u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x) \), \( \lambda = \frac{\epsilon_2^{2k_0}}{K} \). Then

\[ \int_0^{\frac{T}{\lambda^2}} N_\lambda(t)^3 dt = \epsilon_2 2^{k_0}, \]

so by Theorem 4.1,

\[ \| u_\lambda \|_{X_{k_0}([0,\frac{T}{\lambda^2}] \times \mathbb{R})} \leq C_0. \]

By (3.32), \( |\xi(t)| \leq 2^{-20} \epsilon_3 \epsilon_1^{-1/2} 2^{k_0} \) for all \( t \in [0, \frac{T}{\lambda^2}] \), (2.26) and scaling implies \( N(t) \leq \epsilon_3 2^{k_0} \).

**Remark.** \( N(0) = 1 \) so we only need to be concerned about \( K \geq 1 \). Following (3.5),

\[ \| P_{>N} u_\lambda \|_{U_0^2([0,\frac{T}{\lambda^2}] \times \mathbb{R})}^2 = \sum_{j:2^j \geq N} \| P_j u \|_{U_0^2([0,\frac{T}{\lambda^2}] \times \mathbb{R})}^2. \]
Duhamel’s principle combined with Theorems 4.1, 4.3, 4.4, and (4.54) implies that for $N \geq 2^{k_0}$,

$$
\left\| P_{> N} u \right\|_{L^2_\Delta([0, \frac{T}{\lambda^2}] \times \mathbb{R})} \lesssim \inf_{t \in [0, \frac{T}{\lambda^2}]} \left\| P_{> N} u(t) \right\|_{L^2_\Delta(\mathbb{R}^2)} + \varepsilon_2^{1/2} C_0^3 \left\| P_{> \|N\|} u \right\|_{L^2_\Delta([0, \frac{T}{\lambda^2}] \times \mathbb{R})}.
$$

(5.6)

Since the $U^2_\Delta$ and $L^2$ norms are scale invariant, (5.6) implies that for $N \geq K \varepsilon_3^{-1}$,

$$
\left\| P_{> N} u(t) \right\|_{U^2_\Delta([0, T] \times \mathbb{R})} \lesssim \inf_{t \in [0, T]} \left\| P_{> N} u(t) \right\|_{L^2_\Delta(\mathbb{R})} + \varepsilon_2^{1/2} C_0^3 \left\| P_{> \|N\|} u \right\|_{U^2_\Delta([0, T] \times \mathbb{R})}.
$$

(5.7)

By Theorem 4.1,

$$
\limsup_{T \to +\infty} \left\| P_{> K \varepsilon_3^2} u(t) \right\|_{U^2_\Delta([0, T] \times \mathbb{R})} \leq C_0.
$$

(5.8)

Also, $\int_0^\infty N(t)^3 \, dt = K < \infty$ combined with (3.32) implies that $\lim_{t \to \infty} N(t) = 0$ and $|\xi(t)| \leq 2^{-20} \varepsilon_3^{-1/2} K$, so for $N \geq \varepsilon_3^{-1} K$,

$$
\lim_{t \to \infty} \left\| P_{> N} u(t) \right\|_{L^2_\Delta(\mathbb{R})} = 0.
$$

(5.9)

Choosing $\varepsilon_2, \varepsilon_3 > 0$ sufficiently small and satisfying (3.29), in particular $\varepsilon_2^{1/2} C(u) C_0^3 \ll 1$, (5.7) implies that for $N \geq K \varepsilon_3^{-1}$,

$$
\limsup_{T \to +\infty} \left\| P_{> N} u(t) \right\|_{U^2_\Delta([0, T] \times \mathbb{R})} \leq 2^{-30} \left( \limsup_{T \to +\infty} \left\| P_{> \|N\|} u \right\|_{U^2_\Delta([0, T] \times \mathbb{R})} \right).
$$

(5.10)

Therefore, by conservation of mass, (5.8), induction on $N$, and $U^2_\Delta \subset L^\infty T L^2_\Delta$,

$$
\left\| u(t) \right\|_{H^5(\mathbb{R})} \lesssim \varepsilon_3^{-5} K^5.
$$

(5.11)

**Theorem 5.2.** *The only almost periodic solution to (1.1) in the form of Theorem 2.7 with $\int_0^\infty N(t)^3 \, dt = K < \infty$ is $u \equiv 0$.*

**Proof.** Let $\xi_\infty = \lim_{t \to +\infty} \xi(t)$. (3.32) implies $|\xi_\infty| \leq 2^{-20} \varepsilon_1^{-1/2} K$, so after making a Galilean transformation that shifts $\xi_\infty$ to the origin,

$$
\left\| u(t) \right\|_{H^5(\mathbb{R})} \lesssim \varepsilon_3^{-5} K^5.
$$

(5.12)

$\xi(t) \to 0$, $N(t) \to 0$, and (2.20) implies that

$$
\lim_{t \to \infty} \left\| P_{\xi(t), \leq C(\eta) N(t)} u(t) \right\|_{H^5(\mathbb{R})} = 0.
$$

(5.13)

Interpolating (5.12) with (2.20) also implies

$$
\left\| P_{\xi(t), \geq C(\eta) N(t)} u(t) \right\|_{H^1(\mathbb{R})} \lesssim \eta^{2/5}.
$$

(5.14)
Since $\eta > 0$ can be made arbitrarily small (5.13) and (5.14) imply
\begin{equation}
\lim_{t \to +\infty} \|u\|_{\dot{H}^1(R)} = 0.
\end{equation}
By Sobolev embedding this implies
\begin{equation}
\lim_{t \to +\infty} E(u(t)) = 0,
\end{equation}
which by conservation of energy proves that $E(u(t)) = 0$, which implies that $u \equiv 0$. \qed

6. The quasi-soliton. Now we exclude the scenario described in Theorem 2.7 with $\int_0^\infty N(t)^3 dt = \infty$. To do this we modify the proof in [72] of
\begin{equation}
\| \partial_x |u(t, x)|^2 \|_{L^2_t L^2_x([0, T] \times R)} \lesssim \|u(t)\|_{L^\infty_t \dot{H}^1([0, T] \times R)} \|u(t)\|^{3}_3 L^2_{t,x}([0, T] \times R),
\end{equation}
For this particular almost periodic solution we will not prove any additional regularity of the minimal mass blowup solution. Instead, as in [28, 31] we will truncate to low frequencies and use Theorem 4.1 to estimate the errors. This estimate will imply $u \equiv 0$.

**Theorem 6.1.** Suppose $u(t, x)$ is the minimal mass blowup solution of Theorem 2.7, for some $k_0 \in \mathbb{Z}_+$
\begin{equation}
\int_0^T \int |u(t, x)|^6 dx dt = 2^{k_0},
\end{equation}
and
\begin{equation}
\int_0^T N(t)^3 dt = \epsilon_3 K.
\end{equation}
Make the frequency truncation
\begin{equation}
\hat{I}u(t, \xi) = \phi\left(\frac{\xi}{K}\right) \hat{u}(t, \xi),
\end{equation}
$\phi \in C_c^\infty(R)$, $\phi$ is symmetric, $\phi \equiv 1$ for $|x| \leq 1$, $\phi \equiv 0$ for $|x| > 2$. Then
\begin{equation}
\| \partial_x |Iu(t, x)|^2 \|_{L^2_t L^2_x([0, T] \times R)} \lesssim \sup_{[0,T]} M^I(t) + o(K),
\end{equation}
where $M^I(t)$ is a modification of the Morawetz action in [72] (see (6.7)).

**Proof.** Let
\begin{equation}
a(x) = \frac{x}{|x|}.
\end{equation}
Define the Morawetz action

\[ M(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} a(x - y) |u(t, y)|^2 \text{Im}[\bar{u}(t, x) \partial_x u(t, x)] dx dy, \]  

(6.7)

Integrating by parts, [72] proved

\[ \int_0^T \int |\partial_x |u(t, x)|^2|^2 dx dt \lesssim \int_0^T \partial_t M(t) \lesssim \sup_{[0,T]} |M(t)|. \]  

(6.8)

Now take the frequency truncated action

\[ M_I(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} a(x - y) |Iu(t, y)|^2 \text{Im}[\bar{Iu}(t, x) \partial_x Iu(t, x)] dx dy. \]  

(6.9)

\[ \partial_t(Iu) = i\Delta(Iu) - iF(Iu) + iF(Iu) - iIF(u). \]  

(6.10)

If we simply had

\[ \partial_t(Iu) = i\Delta(Iu) - iF(Iu), \]  

(6.11)

then we could copy the arguments in [72] and prove

\[ \int_0^T \int |\partial_x |Iu(t, x)|^2|^2 dx dt \lesssim \int_0^T \partial_t M(t) \lesssim \sup_{[0,T]} |M_I(t)|. \]  

(6.12)

Instead,

\[ \int_0^T \int |\partial_x |Iu(t, x)|^2|^2 dx dt \lesssim \int_0^T \partial_t M(t) + E \lesssim \sup_{[0,T]} |M(t)| + E, \]  

(6.13)

where \( E \) is the error arising from the Fourier truncation of \( u \),

\[ E = 2 \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} a(x - y) \text{Im}[\bar{Tu} IF(u)](t, y) \text{Im}[\bar{Tu} \partial_x Iu](t, x) dx dy dt \]  

(6.14)

\[ + \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} a(x - y) |Iu(t, y)|^2 \text{Re}[(IF(\bar{u}) - F(\bar{Iu})) \partial_x Iu](t, x) dx dy dt \]  

(6.15)

\[ + \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} a(x - y) |Iu(t, y)|^2 \text{Re}[\bar{Tu} \partial_x (F(Iu) - IF(u))](t, x) dx dy dt. \]  

(6.16)
We prove that $\mathcal{E} \leq o(K)$, where $\lim_{K \to \infty} \frac{o(K)}{K} = 0$. To prove this we use a fact well exploited by [72], namely that $a(x - y)$ odd implies that $\mathcal{E}$ is Galilean invariant.

$\mathcal{E} = 2 \int_0^T \int \int a(x - y) \text{Im}[(\overline{u} I F(u))](t, y) \text{Im}[\bar{u}(\partial_x - i\xi(t))u](t, x) dx dy dt$

(6.18)

\begin{align*}
+2 \int_0^T \int \int a(x - y) \text{Im}[(\overline{u} I F(u))](t, y) |I u(t, x)|^2 \xi(t) dx dy dt
\end{align*}

(6.19)

\begin{align*}
+ \int_0^T \int \int a(x - y) |I u(t, y)|^2 Re[(I F(\bar{u}) - F(\overline{u}))(\partial_x - i\xi(t))I u](t, x) dx dy dt
\end{align*}

(6.20)

\begin{align*}
- \int_0^T \int \int a(x - y) |I u(t, y)|^2 \xi(t) \text{Im}[(I F(\bar{u}) - F(\overline{u}))I u](t, x) dx dy dt
\end{align*}

(6.21)

\begin{align*}
+ \int_0^T \int \int a(x - y) |I u(t, y)|^2 Re[I u(\partial_x - i\xi(t))(F(I u) - I F(u))](t, x) dx dy dt
\end{align*}

(6.22)

\begin{align*}
- \int_0^T \int \int a(x - y) |I u(t, y)|^2 \xi(t) \text{Im}[I u(F(I u) - I F(u))](t, x) dx dy dt.
\end{align*}

The fact that $a(x - y)$ is odd implies that (6.18) + (6.20) + (6.22) = 0.

We will prove that (6.17), (6.19), (6.21) $\leq o(K)$ for a very general $a(x - y)$.

**Theorem 6.2.** Suppose $a(t, x)$ is an odd function of $x$ such that for for each $t$ there is some $C$ such that

(6.23)

$|a(t, x)| \leq C,$

and

(6.24)

$\partial_x a_j(t, x) = f(t, x),$

with $\|f(t, x)\|_{L^1(\mathbb{R})}$ uniformly bounded in time. If $u$ is an almost periodic solution to (1.1), and $F(u) = \pm |u|^4 u$, then (6.17), (6.19), and (6.21) $\leq Co(K)$.

**Remark.** If $a(x) = \frac{x}{|x|}$ then $\partial_x a$ is not an $L^1$ function but simply a finite measure. However, since we approximate $u_0$ with a sequence of Schwartz functions, we can make the same computation as we would if $\partial_x a(x)$ were an $L^1$ function. See [72] for more information. One can also adopt the method of [21] and approximate $a(x)$ by a smoothed out version that approaches $a(x)$ in the limit.

**Remark.** The interaction Morawetz estimates of [21], [72] are not positive definite when $F(u) = -|u|^4 u$, which prevents us from obtaining an estimate of the
By the fundamental theorem of calculus, if (6.27) the last inequality follows from (2.20), (3.46), and Theorem 4.1. Now let $u$ (6.25) in this case, (6.28) is defined. Remark. Since $E$ is clearly linear in $a$, without loss of generality take $C = 1$.

Now let $\lambda = \frac{2k_0}{K}$ and rescale, letting $I_\lambda$ be the Fourier multiplier $\phi(\frac{\xi}{2k_0}).$

\[
\left\| \left( \partial_x - i\xi(t) \right) Iu \right\|_{L_t^4 L_x^\infty([0,T] \times \mathbb{R})} = 2^{-k_0} K \left\| \left( \partial_x - i\lambda\xi(t) \right) I\lambda u_\lambda \right\|_{L_t^4 L_x^\infty([0,T] \times \mathbb{R})} \\
\lesssim 2^{-k_0} K \sum_{j=0}^{k_0+2} 2^j \left\| P\lambda\xi(t,j) u_\lambda \right\|_{L_t^4 L_x^\infty([0,T] \times \mathbb{R})} \lesssim K.
\]

The last inequality follows from (2.20), (3.46), and Theorem 4.1. Now let $u_l = P_{\leq \frac{K}{2\delta}} u$, $u_l + u_h = u$. (6.26)

\[
I(\|u_l\|^4 u_l) - |I u_l|^4 (I u_l) \equiv 0.
\]

Next, (6.27)

\[
I(\|u_l\|^4 u_h) - |I u_l|^4 (I u_h) = I(\|u_l\|^4 u_h) - |u_l|^4 (I u_h).
\]

By the fundamental theorem of calculus, if $|\xi_1| \sim N_1$, $|\xi_2| \sim N_2$, $N_2 \ll N_1$, so

(6.28)

\[
|m(\xi_2 + \xi_1) - m(\xi_1)| \lesssim N_2 \sup_{|\xi| \sim N_1} |\partial_x m(\xi)|.
\]

In this case, $m(\xi) = \phi(\frac{\xi}{K})$, $N_1 \sim K$, so

\[
\left\| I(\|u_l\|^4 u_h) - |u_l|^4 (I u_h) \right\|_{L_t^{4/3} L_x^4([0,T] \times \mathbb{R})} \\
\lesssim \frac{1}{K} \left\| I u_h |\partial_x| u_l|^4 \right\|_{L_t^{4/3} L_x^4([0,T] \times \mathbb{R})} \\
\lesssim \frac{1}{K} \left\| u_h u_l^2 \right\|_{L_t^4 L_x^4([0,T] \times \mathbb{R})} \left\| \partial_x |u_l|^2 \right\|_{L_t^4 L_x^4([0,T] \times \mathbb{R})} \\
\lesssim \frac{1}{K} \left\| \partial_x |u_l|^2 \right\|_{L_t^4 L_x^4([0,T] \times \mathbb{R})}.
\]

The last inequality follows from (4.23).

(6.30)

\[
\left\| \partial_x |u_l|^2 \right\|_{L_t^4 L_x^4([0,T] \times \mathbb{R})} = \left\| \partial_x \left( (e^{-ix\cdot\xi(t)} u_l)(e^{ix\cdot\xi(t)} \bar{u}_l) \right) \right\|_{L_t^4 L_x^4([0,T] \times \mathbb{R})}.
\]
By (2.20) and (6.25)

\[(6.31)\quad \| \left( P_{\xi(t) \geq C(\eta)N(t)} u \right) ((\partial_x - i\xi(t))u) \|_{L^4_t L^2_x([0,T] \times \mathbb{R})} \lesssim \eta K.\]

We assume \( C(\eta) \gg \varepsilon_1^{-1} \). By (2.20),

\[(6.32)\quad \sum_{10C(\eta) \leq 2^j \leq \frac{K}{C(\eta)}} 2^j N(J) \| (P_{\xi(J), \leq 2C(\eta)N(J)} u) (P_{\xi(J), \geq 2C(\eta)N(J)} u) \|_{L^4_t L^2_x(J \times \mathbb{R})}^{1/2} \times \| P_{\xi(J), \geq 2C(\eta)N(J)} u \|_{L^4_t L^2_x(J \times \mathbb{R})}^{1/2} \lesssim o(K^{3/4}) N(J)^{1/4} C(\eta)^{1/4}.\]

Using (6.3) to sum this up in \( l^4 \) over \( J_l \subset [0,T] \), along with

\[(6.33)\quad 10C(\eta) N(J) \| P_{\xi(J), \leq 2C(\eta)N(J)} \|_{L^4_t L^2_x(J \times \mathbb{R})} \| P_{\xi(J), \geq 10C(\eta)N(J)} u \|_{L^4_t L^2_x(J \times \mathbb{R})} \lesssim C(\eta) N(J),\]

\[(6.34)\quad \| \partial_x |u| \|^2 \|_{L^2_t L^2_x([0,T] \times \mathbb{R})} \lesssim \eta K + C(\eta) K^{1/4} + C(\eta) o(K).\]

Then taking \( \eta(K) \gg 0 \) as \( K \to \infty \), possibly very slowly,

\[(6.35)\quad \| \partial_x |u| \|^2 \|_{L^2_t L^2_x([0,T] \times \mathbb{R})} \lesssim o(K).\]

Therefore, (6.35) combined with (6.29) implies

\[(6.36)\quad \| I(|u|^4 u_h) - |u|^4 (I u_h) \|_{L_t^{4/3} L_x^1([0,T] \times \mathbb{R})} \lesssim o(1).\]

The same holds for \( I(|u|^2 u_h^2) - |u|^2 u_h^2 (I u_h) \). Next, by Theorem 4.1, (2.20), and (4.22),

\[(6.37)\quad \| u_h^3 u_l^3 + u_h^5 \|_{L_t^{4/3} L_x^1([0,T] \times \mathbb{R})} \lesssim \| (u_h) u_l^2 \|_{L_t^2 L_x^2([0,T] \times \mathbb{R})}^{3/2} \| u_h \|_{L_t^2 L_x^2([0,T] \times \mathbb{R})}^{1/2} + \| u_h \|_{L_t^2 L_x^2([0,T] \times \mathbb{R})} \| u_h \|_{L_t^2 L_x^2([0,T] \times \mathbb{R})} \lesssim o(1).\]

Therefore, by (6.25), (6.37),

\[(6.38)\quad (6.19) \lesssim o(K).\]
Next, integrating by parts in space,

\begin{equation}
(6.39) = - \int_0^T \int \int a(x-y) \text{Re}[(\partial_x + i\xi(t)) \overline{u}(t,x)](F(Iu) - IF(u))(t,x)\,dx\,dy\,dt
\end{equation}

\begin{equation}
(6.40) = \int_0^T \int \partial_x a(x-y) |Iu(t,y)|^2 \overline{u}(t,x)(F(Iu) - IF(u))(t,x)\,dx\,dy\,dt.
\end{equation}

(6.39) = (6.19) \lesssim o(K). By (6.24), \( \partial_x a(x-y) \) is an \( L^1 \) function so by Young’s inequality (see [77]),

\begin{equation}
(6.41) \lesssim \|Iu\|_{L^6_t L^3_x([0,T] \times \mathbb{R})}^3 \|I(u^4 - Iu^4)\|_{L^{4/3}_t L^2_x([0,T] \times \mathbb{R})}.
\end{equation}

Interpolating (6.25), conservation of mass, along with (6.36) and (6.37), (6.21) \( \lesssim o(K) \).

Finally we turn to (6.17).

\begin{equation}
(6.42) \quad \text{Im}[(\overline{Iu})IF(u)] = \text{Im}[(\overline{Iu})(IF(u) - F(Iu))].
\end{equation}

By (6.25), (6.36), and (6.37),

\begin{equation}
(6.43) \quad \|(\overline{Iu})I(u^4 - Iu^4)\|_{L^1_{t,x}([0,T] \times \mathbb{R})} \lesssim o(1).
\end{equation}

By (4.22), Theorem 4.1, and the fact that \( U_\Delta^2 \) is invariant under the scaling \( \lambda = \frac{2k_0}{K} \),

\begin{equation}
(6.44) \quad \|u_h^2 u_t^3 + u_h^5\|_{L^1_{t,x}([0,T] \times \mathbb{R})} \lesssim \|u_h u_t^2\|_{L^2_{t,x}([0,T] \times \mathbb{R})}^2
\end{equation}

\begin{equation}
+ \|u_h\|_{L^4_t L^\infty_x([0,T] \times \mathbb{R})}^4 \|u\|_{L^2_t L^1_x([0,T] \times \mathbb{R})}^2 \lesssim 1.
\end{equation}

Finally,

\begin{equation}
(6.45) \quad I(u^4_t u_h) - u^4_t (Iu_h) = I(u^4_t (P_{> \frac{K}{2}} u)) - u^4_t (P_{> \frac{K}{2}} u),
\end{equation}

which is supported on \( |\xi| \geq \frac{K}{2} \). Because \( u_t \) is supported on \( |\xi| \leq \frac{K}{2} \),

\begin{equation}
(6.46) \quad (\overline{u_t}) (I(u^4_t u_h) - u^4_t (Iu_h))
\end{equation}
is supported on $|\xi| \geq \frac{K}{4}$. Integrating by parts,

\begin{equation}
\int_0^T \int \int a(x-y) \frac{\partial^2}{\partial y^2} (P > \frac{K}{2} (u^S u_h)(t,y)) \overline{\text{Im}[Iu(\partial_x - i\xi(t))Iu]}(t,x) \, dx \, dy \, dt
\end{equation}

\begin{equation}
= \frac{1}{K} \|u_h u^I_1\|_{L^2_{t,x}([0,T] \times \mathbb{R})} \|(\partial_x - i\xi(t))Iu\|_{L^2_{t,x}([0,T] \times \mathbb{R})} \|Iu\|^4_{L^8_{t,x}([0,T] \times \mathbb{R})}
\end{equation}

Sobolev embedding combined with (3.46) implies

\begin{equation}
\frac{1}{K} \|Iu\|^4_{L^8_{t,x}([0,T] \times \mathbb{R})} \lesssim 1.
\end{equation}

Finally since $N(t) \leq 1$, (2.20) implies that

\begin{equation}
\|(\partial_x - i\xi(t))Iu\|_{L^2_{x}([0,T] \times \mathbb{R})} \lesssim o(K).
\end{equation}

Combining (6.43), (6.44), (6.47), (6.48), (6.49), and conservation of mass,

\begin{equation}
(6.17) \lesssim o(K).
\end{equation}

This concludes the proof of Theorem 6.2, which in turn proves Theorem 6.1. \hfill \Box

**Theorem 6.3.** There does not exist an almost periodic solution with $N(t) \leq 1$, $\int_0^\infty N(t)^3 \, dt = \infty$, $\|u(0)\|_{L^2} = m_0 > 0$.

**Proof.** Suppose there did exist an almost periodic solution with positive mass, $N(t) \leq 1$ and $\int_0^\infty N(t)^3 \, dt = \infty$. By Theorem 6.1, (6.49), and conservation of mass,

\begin{equation}
\|\partial_x |Iu(tx)|^2\|_{L^1_{t,x}([0,T] \times \mathbb{R}^2)} \lesssim \sup_{[0,T]} M^I(t) + Co(K).
\end{equation}

\begin{equation}
\int\int a(x-y) \overline{Iu(t,x)Iu(t,y)} (\partial_x - i\xi(t))(Iu(t,x)Iu(t,y)) \, dx \, dy
\end{equation}

\begin{equation}
\lesssim C \|Iu\|_{L^2_{t,x}([0,T] \times \mathbb{R})}^3 \|(\partial_x - i\xi(t))Iu\|_{L^2_{t,x}([0,T] \times \mathbb{R})} \lesssim Co(K).
\end{equation}

Also,

\begin{equation}
\int\int a(x-y) \text{Im}[Iu(t,x)Iu(t,y)i\xi(t)Iu(t,x)Iu(t,y)] \, dx \, dy
\end{equation}

\begin{equation}
= \int\int a(x-y) |Iu(t,x)|^2 |Iu(t,y)|^2 \xi_j(t) \, dx \, dy.
\end{equation}
Since $|Iu(t,x)|^2|Iu(t,y)|^2$ is symmetric in $x$ and $y$ and $a(x-y)$ is anti-symmetric in $x$ and $y$, (6.53) = 0. Therefore,

$$\|\partial_x |Iu(t,x)|^2\|_{L^2_{t,x}([0,T] \times \mathbb{R})}^2 \lesssim o(K).$$

By the Sobolev embedding theorem (see [96, Chapter 13, Proposition 8.4] for a proof),

$$\int_0^T \|\|Iu(t,x)|^2\|_{\dot{C}^{1/2}([0,T] \times \mathbb{R})}^2 \lesssim o(K).$$

**Definition 6.1 (Hölder continuity).** For any $0 < s < 1$, $f \in \dot{C}^{1/2}$ if and only if

$$|f(x) - f(y)| \leq \|f\|_{\dot{C}^s} |x - y|^s$$

for all $x, y \in \mathbb{R}^d$. This is equivalent to

$$\|P_k f\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^{-sk} \|f\|_{\dot{C}^s(\mathbb{R}^d)}$$

for all $k \in \mathbb{Z}$.

By Hölder’s inequality, (2.19), and the definition of Hölder continuity,

$$\int_{|x-x(t)| \leq \frac{c(m_0^2)}{N(t)}} |Iu(t,x)|^2 \, dx \lesssim \int_{|x-x(t)| > \frac{c(m_0^2)}{N(t)}} |Iu(t,x)|^2 \, dx + \left(\frac{C(m_0^2)}{N(t)}\right)^{3/2} \|\|Iu(t,x)|^2\|_{\dot{C}^{1/2}(\mathbb{R})}$$

$$\lesssim \frac{m_0^2}{1000} + \left(\frac{C(m_0^2)}{N(t)}\right)^{3/2} \|\partial_x |Iu| |^2\|_{L^2(\mathbb{R})}.$$  

Now for $K > C(\frac{m_0^2}{1000})$,

$$\frac{m_0^2}{2} < \int_{|x-x(t)| \leq \frac{c(m_0^2)}{N(t)}} |Iu(t,x)|^2 \, dx.$$ 

Therefore,

$$\int_0^T m_0^4 N(t)^3 \, dt \lesssim \int_{-T}^T N(t)^3 \left(\int_{|x-x(t)| \leq \frac{c(m_0^2)}{N(t)}} |Iu(t,x)|^2 \, dx \right)^2 \, dt$$

$$\lesssim \|\partial_x |Iu(t,x)|^2\|_{L^2_{t,x}([0,T] \times \mathbb{R})}^2 \lesssim Co(K).$$

But then $m_0^4 \lesssim o(K)$. Taking $K \to \infty$ implies $u \equiv 0$. □
This completes the proof of Theorem 1.5, which in turn completes the proof of Theorem 1.2.

DEPARTMENT OF MATHEMATICS, THE JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 21218
E-mail: dodson@math.jhu.edu

REFERENCES

[1] H. Bahouri and P. Gérard, High frequency approximation of solutions to critical nonlinear wave equations, *Amer. J. Math.* 121 (1999), no. 1, 131–175.
[2] H. Bahouri and J. Shatah, Decay estimates for the critical semilinear wave equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 15 (1998), no. 6, 783–789.
[3] P. Bégout and A. Vargas, Mass concentration phenomena for the $L^2$-critical nonlinear Schrödinger equation, *Trans. Amer. Math. Soc.* 359 (2007), no. 11, 5257–5282.
[4] H. Berestycki and P.-L. Lions, Existence d’ondes solitaires dans des problèmes nonlinéaires du type Klein-Gordon, *C. R. Acad. Sci. Paris Sér. A-B* 288 (1979), no. 7, A395–A398.
[5] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations, *Geom. Funct. Anal.* 3 (1993), no. 2, 107–156.
[6] ________, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation, *Geom. Funct. Anal.* 3 (1993), no. 3, 209–262.
[7] ________, Refinements of Strichartz’ inequality and applications to 2D-NLS with critical nonlinearity, *Int. Math. Res. Not. IMRN* 1998 (1998), no. 5, 253–283.
[8] ________, Scattering in the energy space and below for 3D NLS, *J. Anal. Math.* 75 (1998), 267–297.
[9] ________, Global Solutions of Nonlinear Schrödinger Equations, *Amer. Math. Soc. Colloq. Publ.*, vol. 46, Amer. Math. Soc., Providence, RI, 1999.
[10] ________, Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case, *J. Amer. Math. Soc.* 12 (1999), no. 1, 145–171.
[11] H. Brezis and J.-M. Coron, Convergence of solutions of $H$-systems or how to blow bubbles, *Arch. Ration. Mech. Anal.* 89 (1985), no. 1, 21–56.
[12] T. Cazenave, *An Introduction to Nonlinear Schrödinger Equations*, Textos de Métodos Matemáticos, vol. 26, Instituto de Matemática, Rio de Janeiro, 1993.
[13] ________, *Semilinear Schrödinger Equations*, Courant Lect. Notes Math., vol. 10, New York University, Courant Institute of Mathematical Sciences, New York, 2003.
[14] T. Cazenave and F. B. Weissler, The Cauchy problem for the nonlinear Schrödinger equation in $H^1$, *Manuscripta Math.* 61 (1988), no. 4, 477–494.
[15] ________, Some remarks on the nonlinear Schrödinger equation in the subcritical case, *New Methods and Results in Nonlinear Field Equations (Bielefeld, 1987)*, Lecture Notes in Phys., vol. 347, Springer-Verlag, Berlin, 1989, pp. 59–69.
[16] ________, The Cauchy problem for the critical nonlinear Schrödinger equation in $H^s$, *Nonlinear Anal.* 14 (1990), no. 10, 807–836.
[17] M. Christ, J. Colliander, and T. Tao, Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations, *Amer. J. Math.* 125 (2003), no. 6, 1235–1293.
[18] ________, A priori bounds and weak solutions for the nonlinear Schrödinger equation in Sobolev spaces of negative order, *J. Funct. Anal.* 254 (2008), no. 2, 368–395.
[19] M. Christ and A. Kiselev, Maximal functions associated to filtrations, *J. Funct. Anal.* 179 (2001), no. 2, 409–425.
[20] J. Colliander, M. Grillakis, and N. Tzirakis, Improved interaction Morawetz inequalities for the cubic nonlinear Schrödinger equation on $\mathbb{R}^2$, *Int. Math. Res. Not. IMRN* 2007 (2007), no. 23, Art. ID rnm090, 30.
[21] Tensor products and correlation estimates with applications to nonlinear Schrödinger equations, *Comm. Pure Appl. Math.* 62 (2009), no. 7, 920–968.

[22] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation, *Math. Res. Lett.* 9 (2002), no. 5-6, 659–682.

[23] Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on $\mathbb{R}^1$, *Comm. Pure Appl. Math.* 57 (2004), no. 8, 987–1014.

[24] Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^d$, *Ann. of Math.* (2) 167 (2008), no. 3, 767–865.

[25] J. Colliander and T. Roy, Bootstrapped Morawetz estimates and resonant decomposition for low regularity global solutions of cubic NLS on $\mathbb{R}^2$, *Commun. Pure Appl. Anal.* 10 (2011), no. 2, 397–414.

[26] D. de Silva, N. Pavlović, G. Staffilani, and N. Tzirakis, Global well-posedness and polynomial bounds for the defocusing $L^2$-critical nonlinear Schrödinger equation, *Comm. Partial Differential Equations* 33 (2008), no. 7-9, 1395–1429.

[27] B. Dodson, Improved almost Morawetz estimates for the cubic nonlinear Schrödinger equation, *Commun. Pure Appl. Anal.* 10 (2011), no. 1, 127–140.

[28] Global well-posedness and scattering for the defocusing, $L^2$-critical nonlinear Schrödinger equation when $d \geq 3$, *J. Amer. Math. Soc.* 25 (2012), no. 2, 429–463.

[29] Global well-posedness for the defocusing, quintic nonlinear Schrödinger equation in one dimension for low regularity data, *Int. Math. Res. Not.* IMRN 2012 (2012), no. 4, 870–893.

[30] Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state, *Adv. Math.* 285 (2015), 1589–1618.

[31] Global well-posedness and scattering for the focusing, $L^2$-critical nonlinear Schrödinger equation when $d = 2$, preprint, http://arxiv.org/abs/1006.1375v2.

[32] Global well-posedness and scattering for the defocusing, mass-critical generalized KdV equation, preprint, http://arxiv.org/abs/1304.8025.

[33] Global well-posedness and scattering for the focusing, energy-critical nonlinear Schrödinger problem in dimension $d = 4$ for initial data below a ground state threshold, preprint, http://arxiv.org/abs/1409.1950.

[34] P. Gérard, Description du défaut de compacité de l’injection de Sobolev, *ESAIM Control Optim. Calc. Var.* 3 (1998), 213–233.

[35] J. Ginibre, Le problème de Cauchy pour des EDP semi-linéaires périodiques en variables d’espace (d’après Bourgain). Séminaire Bourbaki, Vol. 1994/95, Astérisque 237 (1996), Exp. No. 796, 163–187.

[36] J. Ginibre, A. Soffer, and G. Velo, The global Cauchy problem for the critical nonlinear wave equation, *J. Funct. Anal.* 110 (1992), no. 1, 96–130.

[37] J. Ginibre and G. Velo, On the global Cauchy problem for some nonlinear Schrödinger equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 1 (1984), no. 4, 309–323.

[38] , The global Cauchy problem for the nonlinear Schrödinger equation revisited, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 2 (1985), no. 4, 309–327.

[39] Scattering theory in the energy space for a class of nonlinear Schrödinger equations, *J. Math. Pures Appl.* (9) 64 (1985), no. 4, 363–401.

[40] Smoothing properties and retarded estimates for some dispersive evolution equations, *Comm. Math. Phys.* 144 (1992), no. 1, 163–188.

[41] R. T. Glassey, On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations, *J. Math. Phys.* 18 (1977), no. 9, 1794–1797.

[42] M. G. Grillakis, Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity, *Ann. of Math.* (2) 132 (1990), no. 3, 485–509.

[43] Regularity for the wave equation with a critical nonlinearity, *Comm. Pure Appl. Math.* 45 (1992), no. 6, 749–774.

[44] On nonlinear Schrödinger equations, *Comm. Partial Differential Equations* 25 (2000), no. 9-10, 1827–1844.

[45] E. P. Gross, Hydrodynamics of a superfluid condensate, *J. Math. Phys.* 4 (1963), no. 2, 195–207; see also E. P. Gross, Structure of a quantized vortex in boson systems, *Nuovo Cimento* (10) 20 (1961), 454–477.

[46] M. Hadac, S. Herr, and H. Koch, Well-posedness and scattering for the KP-II equation in a critical space, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 26 (2009), no. 3, 917–941.

[47] L. Kapitanski, Global and unique weak solutions of nonlinear wave equations, *Math. Res. Lett.* 1 (1994), no. 2, 211–223.
[48] T. Kato, On nonlinear Schrödinger equations, *Ann. Inst. H. Poincaré Phys. Théor.* 46 (1987), no. 1, 113–129.

[49] ———, On nonlinear Schrödinger equations. II. $H^s$-solutions and unconditional well-posedness, *J. Anal. Math.* 67 (1995), 281–306, correction: *J. Anal. Math.* 68 (1996), 305.

[50] M. Keel and T. Tao, Endpoint Strichartz estimates, *Amer. J. Math.* 120 (1998), no. 5, 955–980.

[51] C. E. Kenig and F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case, *Invent. Math.* 166 (2006), no. 3, 645–675.

[52] ———, Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation, *Acta Math.* 201 (2008), no. 2, 147–212.

[53] ———, Scattering for $H^{1/2}$ bounded solutions to the cubic, defocusing NLS in 3 dimensions, *Trans. Amer. Math. Soc.* 362 (2010), no. 4, 1937–1962.

[54] S. Keraani, On the defect of compactness for the Strichartz estimates of the Schrödinger equations, *J. Differential Equations* 175 (2001), no. 2, 353–392.

[55] ———, On the blow up phenomenon of the critical nonlinear Schrödinger equation, *J. Funct. Anal.* 235 (2006), no. 1, 171–192.

[56] R. Killip, S. Kwon, S. Shao, and M. Visan, On the mass-critical generalized KdV equation, *Discrete Contin. Dyn. Syst.* 32 (2012), no. 1, 191–221.

[57] R. Killip, T. Tao, and M. Visan, The cubic nonlinear Schrödinger equation in two dimensions with radial data, *J. Eur. Math. Soc. (JEMS)* 11 (2009), no. 6, 1203–1258.

[58] R. Killip and M. Visan, The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher, *Amer. J. Math.* 132 (2010), no. 2, 361–424.

[59] ———, Global well-posedness and scattering for the defocusing quintic NLS in three dimensions, *Anal. PDE* 5 (2012), no. 4, 855–885.

[60] ———, Nonlinear Schrödinger equations at critical regularity, *Evolution Equations, Clay Math. Proc.*, vol. 17, Amer. Math. Soc., Providence, RI, 2013, pp. 325–437.

[61] R. Killip, M. Visan, and X. Zhang, The mass-critical nonlinear Schrödinger equation with radial data in dimensions three and higher, *Anal. PDE* 1 (2008), no. 2, 229–266.

[62] H. Koch and D. Tataru, A priori bounds for the 1D cubic NLS in negative Sobolev spaces, *Int. Math. Res. Not. IMRN* 2007 (2007), no. 16, Art. ID rnm053, 36.

[63] ———, Energy and local energy bounds for the 1-d cubic NLS equation in $H^{-1/4}$, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 29 (2012), no. 6, 955–988.

[64] J. E. Lin and W. A. Strauss, Decay and scattering of solutions of a nonlinear Schrödinger equation, *J. Func. Anal.* 30 (1978), no. 2, 245–263.

[65] F. Merle, Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power, *Duke Math. J.* 69 (1993), no. 2, 427–454.

[66] F. Merle and L. Vega, Compactness at blow-up time for $L^2$ solutions of the critical nonlinear Schrödinger equation in 2D, *Int. Math. Res. Not. IMRN* 1998 (1998), no. 8, 399–425.

[67] C. S. Morawetz, Time decay for the nonlinear Klein-Gordon equations, *Proc. Roy. Soc. Ser. A* 306 (1968), 291–296.

[68] A. Moyua, A. Vargas, and L. Vega, Restriction theorems and maximal operators related to oscillatory integrals in $\mathbb{R}^3$, *Duke Math. J.* 96 (1999), no. 3, 547–574.

[69] J. Murphy, The defocusing $H^{1/2}$-critical NLS in high dimensions, *Discrete Contin. Dyn. Syst.* 34 (2014), no. 2, 733–748.

[70] ———, Intercritical NLS: critical $H^s$-bounds imply scattering, *SIAM J. Math. Anal.* 46 (2014), no. 1, 939–997.

[71] K. Nakanishi, Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions 1 and 2, *J. Funct. Anal.* 169 (1999), no. 1, 201–225.

[72] F. Planchon and L. Vega, Bilinear virial identities and applications, *Ann. Sci. École Norm. Sup. (4)* 42 (2009), no. 2, 261–290.

[73] E. Ryckman and M. Visan, Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$, *Amer. J. Math.* 129 (2007), no. 1, 1–60.

[74] J. Shatah and M. Struwe, Regularity results for nonlinear wave equations, *Ann. of Math. (2)* 138 (1993), no. 3, 503–518.

[75] ———, Well-posedness in the energy space for semilinear wave equations with critical growth, *Int. Math. Res. Not. IMRN* 1994 (1994), no. 7, 303–309.

[76] ———, Geometric Wave Equations, *Clay Math. Proc.*, vol. 2, New York University, Courant Institute of Mathematical Sciences, New York, 1998.
