Quantum spin liquid (QSL) was first proposed by Anderson as an alternative ground state against long range magnetic order in frustrated magnets. In these systems competing spin exchange interactions result in a large degeneracy of classical ground states, and quantum fluctuations among these states destroy long range symmetry breaking order. A particular kind of quantum spin liquid, the resonant valence bond (RVB) state, has also been proposed to be the key to the high-temperature superconductivity in the cuprate materials. After decades of intense research many numerical evidences of QSL ground states have been found in semi-realistic lattice models, and many artificial parent Hamiltonians for spin liquids have been constructed.

On the other hand, the experimental realization of spin liquids in more than one spatial dimensions remains challenging until several candidate materials have been discovered recently. Two two-dimensional (2D) triangle lattice organic salts Et$_2$MgSb[Pd(dmit)$_2$]$_2$ and κ-(BEDT-TTF)$_2$Cu$_2$(CN)$_3$, the kagome lattice herbertsmithite ZnCu$_3$(OH)$_6$Cl$_2$ and a three-dimensional hyper-kagome lattice Na$_2$Ir$_3$O$_8$ are found to be the most promising candidates of QSL. Despite of structural distinction, they are all Mott insulators with competing interactions and show no magnetic order down to much lower temperature than their exchange interaction. Current measurements of magnetic susceptibility, specific heat, thermal transport and neutron scattering have provided vital information about the properties of these materials. However, there is still no definitive experiment for the identification of quantum spin liquids.

One of the most significant features of quantum spin liquids is that they have exotic excitations called spinons which are uncharged and usually spin-1/2 mobile particles, may obey bosonic or fermionic statistics and may have a gap or not. The fermionic spinons may form Fermi surfaces and are generally accompanied by an emergent gauge field. This “spinon Fermi sea” state has received strong support from the observations of metallic-like specific heat and thermal conductivity in the organic candidates at low temperatures. However these experiments do not provide a direct proof that the mobile and possibly fermionic low energy excitations are spinons. A more reliable proof for the spinon Fermi sea would be the metallic-like spin transport in these Mott insulators.

In this paper, we propose a four-terminal measurement of spin current through a spin liquid material (Mott insulator) as the evidence for the existence of spinons. The proposed four-terminal device consists of a spin liquid material coupled to the left and right normal metal leads and each lead couples to two ferromagnetic (FM) electrodes as shown in Fig. 1. A current source is added between the two right FM electrodes. This is used to create a spin polarized current flowing from one right FM electrode through the right lead to another right FM electrode, leading to a spin-solved chemical potential (i.e. a spin bias $V_R$) in the right lead. Then this spin bias will drive a spin current flowing from the right lead through the spin liquid and finally into the left lead by the spinon-electron spin-exchange interaction at the interfaces between the spin liquid and the leads. At last,
a voltmeter is connected to the two left FM electrodes which are in contact to the left lead. The voltmeter is
used to measure the spin bias $V_L$ created by the spin current in the left lead. Here, the spin bias $V_\alpha$ ($\alpha = L, R$)
is defined as the difference between the spin-up\(\alpha\) chemical potential $\mu_{\alpha\uparrow}$ and the spin-down chemical potential $\mu_{\alpha\downarrow}$ in
the $\alpha$-lead, i.e. $V_\alpha = \mu_{\alpha\uparrow} - \mu_{\alpha\downarrow}$.

In the rest of this paper, we will first establish the general
result of the spin current through the interface between
a Mott insulator in the middle region and a metal-
lic right lead with a spin bias (see Fig. 1) by the nonequi-
librium Green function technique. We will then apply the
general formalism to show that different Mott insulators
can be distinguished by different relations between the
spin current and the spin bias as well as temperature.
In the end we will also discuss relations to experiments and
estimate experimentally relevant parameters.

The model Hamiltonian and formulation. In our theo-
retic analysis, we consider the model of a Mott insulator
(a spin liquid or a collinear antiferromagnet) coupled to
two normal leads under a spin bias on the right lead. The
Hamiltonian of the system is given by $H = H_0 + H_M + H_I$,
where $H_0$, $H_M$, and $H_I$ are the Hamiltonians of the leads
(Metal), the middle region (Mott insulator), and the inter-
faces respectively. $H_0$ and $H_I$ are assumed to be

$$H_0 = \sum_{\alpha=L,R} \sum_{k,\sigma} \varepsilon_{\alpha k\sigma} c_{\alpha k\sigma}^\dagger c_{\alpha k\sigma},$$

and

$$H_I = J_I \sum_{r_0} S_R(r_0) \cdot S_M(r_0) + J_I \sum_{r_1} S_L(r_1) \cdot S_M(r_1),$$

where

$$S_\alpha(r) = \frac{1}{2} \sum_{\mu,\mu'} \sigma_{\mu \mu'} c_{\alpha \mu}^\dagger(r) c_{\alpha \mu'}(r),$$

with $S_\alpha(r)$ the (dimensionless) spin operator of the mid-
dle region at the position $r$. $c_{\alpha k\sigma}^\dagger(r)$ is creation
(annihilation) of the spin-$\sigma$ electron in the $\alpha=\mathrm{L,R}$-lead.
$\varepsilon_{\alpha k\sigma} = \varepsilon_{\alpha k} - \mu_{\alpha\sigma}$, where $\varepsilon_{\alpha k}$ is the electron dispersion relation and $\mu_{\alpha\sigma}$ is the spin dependent chemical
potential in the $\alpha$-lead. The spin exchange interaction constants $J_I$ are determined by the interface properties
of Mott insulator and metal.\[3.1\] $\sum_{r_0}$ means integral over the interface. $H_M$ depends on the type of the Mott insulator
we consider and will be specified later. We emphasize that there is no single electron tunneling term in $H_I
because the middle region is a Mott insulator.

Due to the spin exchange interaction in $H_I$, the spin
current can flow from the normal lead to Mott insulator
and vice versa. When the right lead is under a spin bias,
the spin current $I_s$ flowing into the middle region from
the right lead is,

$$I_s = - \frac{d}{dt} S_\uparrow \equiv i \langle [S_\uparrow, H(t)] \rangle$$

$$= i J_I \sum_{r_0} \langle [S_\uparrow, S_R(r_0) \cdot S_M(r_0)] \rangle$$

$$= J_I \sum_{r_0} \text{Re} \langle \Gamma \rangle,$$

with $S_\uparrow = (h/2)(N^L_\uparrow - N^R\uparrow) = (h/2) \sum_k \{c_{R\uparrow}^\dagger c_{R\uparrow} \}$
and defined $\Gamma = i S^\dagger_\uparrow(r_0, t, t) > 0 \cdot S^\uparrow_\uparrow(r_0) = S^\uparrow_\uparrow(r_0) - i S^\dagger_\uparrow(r_0)$.

In order to solve the Keldysh Green functions above, we first apply equation of motion technique to solve
$\Gamma = i \langle r_0, k, k', \tau, \tau' \rangle = -i < T_c \{ S^\dagger_\uparrow(r_0, \sigma) c_{R\uparrow}(\tau) \} c_{R\uparrow}(\tau') >$.

By keeping the lowest order terms of $J_I$ we have

$$\Gamma = \frac{2J_I S_\uparrow}{2N_R} \int \frac{d\tau_1}{\Delta} \chi^\dagger(k, \tau) \chi(k, \tau_1)$$

$$\times g_{R\uparrow}(k', \tau, \tau') \exp[-i(k - k') \cdot r_0],$$

where $\chi^\dagger(k, \tau', \tau, \tau_1) = -i T_c \{ S^\dagger_\uparrow(r_0, \tau) S^\uparrow_\uparrow(r_0, \tau_1) \}$ and
$g_{R\uparrow}(k', \tau, \tau') = -i T_c \{ c_{R\uparrow}(\tau) c_{R\uparrow}(\tau') \}$ are contour-order Green functions for spin operator in middle
dregion and free electrons in right lead, respectively.

$$\Gamma = i \langle r_0, k, k', \tau, \tau' \rangle = -i < S^\dagger_\uparrow(r_0, \sigma) c_{R\uparrow}(\tau) c_{R\uparrow}(\tau') >$$

can be obtained by analytic continuation from $\Gamma$, and a
Fourier transform then produces $\Gamma \langle r_0, t, t \rangle$. Plug the result into Eq. 3 we have

$$I_s = \frac{J_I^2 N_\uparrow S_\uparrow}{4N_R^2 N_M^2} \sum_{q, k, k'} A_M(q, \xi_{R\uparrow} - \xi_{R\uparrow} + \Delta)$$

$$\times \{ [1 + n_B(\xi_{R\uparrow} - \xi_{R\uparrow} + \Delta)] n_F(\xi_{R\uparrow} + \Delta) n_F(\xi_{R\uparrow} + \Delta)$$

$$- n_B(\xi_{R\uparrow} - \xi_{R\uparrow} + \Delta) n_F(\xi_{R\uparrow} - \xi_{R\uparrow} + \Delta) \}$$

where $N_M$ and $N_R$ are the number of unit cells in the
middle region and right lead, respectively. Here $N_\uparrow$ is
the number of transverse mode(parallel to the interface)
and $V = \mu_{\uparrow} - \mu_{\downarrow}$ is the spin bias in the
right lead. $n_B(\omega)$ and $n_F(\omega)$ are the Bose and
Fermi distribution functions, respectively. $\delta_{q_{\perp} + k_{\perp} - k'_{\perp}}$ indicates transverse(parallel to the interface)
momentum conservation. The power spectrum $A_M$ are
defined as $A_M(q, \omega) = \int dt < S^\uparrow_\uparrow(-q, t) S^\uparrow_\uparrow(q, 0) >$
and $\delta_{q_{\perp} + k_{\perp} - k'_{\perp}}$ is the Fourier transform of

$$S^\dagger_\uparrow(r, t) = S^\dagger_\uparrow(r, t) \pm S^\dagger_\dagger_\uparrow(r, t).$$

We note that the behavior of the spin current is mainly
determined by the power spectrum of the middle
region. Under appropriate conditions the momentum
integrations over $q, k, k'$ can be approximately sepa-
rated, and the transverse momentum conservation fac-
tor $\delta_{q_{\perp} + k_{\perp} - k'_{\perp}}$ will provide only a constant factor.32 The
Fermi (Bose) function will be treated exactly at zero temperature and expanded in series of \( V/(k_BT) \) at finite temperature. In the following parts of this paper we will apply Eq. (1) and analyze several kinds of Mott insulators as the middle region, including several spin liquids.

Spin liquids. Spinons in spin liquids may or may not be gapped. The gapped spin liquids will have exponentially vanishing spin transport at low temperature and small spin bias. We therefore restrict ourselves to the type of QSLs with gapless fermionic spinons. We describe such spin liquids by the following mean-field Hamiltonian

\[
H_M = \sum_{k,\sigma} \zeta_k f_{k\sigma}^\dagger f_{k\sigma},
\]

where \( f \) are fermionic spinons and \( \zeta_k = \epsilon_k - \mu_s \) with \( \mu_s \) the spinon chemical potential. The spinon dispersion \( \epsilon_k \) may have a Fermi surface (the “spinon Fermi sea” state) or Dirac points at Fermi level.

For illustration purpose we first consider a one-dimensional (1D) spinon Fermi sea state. In 1D case the transverse momentum conservation factor in Eq. (1) does not exist, and the momentum integrations can be done separately. At \( T = 0 \) and \( \mu_s > > \omega \), since \( \epsilon_q = \hbar^2 q^2/(2m_s) \) with \( m_s \) the spinon effective mass, the power spectrum \( A_M \) divided by \( N_s^2 \) and integrated over momentum \( q \) (the “density of states of spin excitations”) is

\[
\frac{1}{N_s^2} \sum_q A_M(q) \propto \omega.
\]

Replace it in Eq. (1) we find spin current \( I_s \propto V^3 \) at \( T = 0 \), and \( I_s \propto (k_BT)^2 V \) at \( T > 0 \) with \( V << k_BT \).

Now we come to the two dimensional spinon Fermi sea case, which is the most relevant to the 2D organic spin liquid candidate materials. The spinon dispersion is \( \epsilon_q = \hbar^2 q^2/(2m_s) \) with \( m_s \) the spinon effective mass, the density of states of spin excitations is

\[
\frac{1}{N_s^2} \sum_q A_M(q) = 2\pi N_s^2(E_F^s) \omega \propto \omega.
\]

Thus the spin current \( I_s \propto V^3 \) at \( T = 0 \) (see Fig. 2), and \( I_s \propto (k_BT)^2 V \) at \( T > 0 \) with \( V << k_BT \) (see Fig. 3).

In the two dimensional Dirac spin liquid case, the low energy spinon dispersion is \( \epsilon_q = \pm \hbar v_F q - q_F \), with \( v_F \) the Fermi velocity and \( q_F \) the Fermi vector.

At \( T = 0 \), \( \sum_q A_M(q) \propto \omega^3 \), the spin current \( I_s \propto V^5 \). At \( T > 0 \), \( V < < k_BT \), expanding \( I_s \) in the series of \( V/(k_BT) \) directly, we find \( I_s \propto (k_BT)^4 V \).

Collinear antiferromagnetic Néel order. Antiferromagnetic (AFM) order is a common competitor for quantum spin liquids. Let us now consider the simplest AFM ordered state, the collinear AFM Néel order on a bipartite lattice, and show that it has different spin transport behavior compared to the previous spin liquid cases. We describe the spin excitations by a linearized spin wave Hamiltonian,

\[
H_M = \sum_k E_0 (a_k \dagger a_{-k} + b_k \dagger b_{-k} + \gamma_k a_k b_{-k} + \gamma_k a_{-k} b_{k})
\]

where \( E_0 = 2ZS|J| \) and \( \gamma_k = \sum_{\delta} \cos(k \cdot \delta)/Z \) with \( \delta \) sums over the nearest-neighbor lattice sites. Here \( Z \) is the coordination number, \( S \) is spin quantum number and \( J \) is
TABLE I: Behavior of the spin current through the interface between different Mott insulators(rows) and a metallic lead with respect to the spin bias $V$ and temperature $T$.

| Insulator Type                  | $T = 0K$ | $T = 1K$ | $T = 10K$ |
|---------------------------------|-----------|-----------|-----------|
| 1-d spinon Fermi sea           | $0.3nV$   | $10nV$    | $1000nV$  |
| 2-d spinon Fermi sea           | $0.07nV$  | $3nV$     | $300nV$   |
| 2-d Dirac spin liquid          | $2 \times 10^{-3}nV$ | $5 \times 10^{-3}nV$ | $3nV$     |

TABLE II: Numerical estimates of the induced spin bias in the left lead with different kinds of Mott insulator middle regions(rows) at three different temperatures $T = 0K$, $1K$, and $10K$. Other parameters used are given in the main text.

| Insulator Type                  | $T = 0K$ | $T = 1K$ | $T = 10K$ |
|---------------------------------|-----------|-----------|-----------|
| 1-d spinon Fermi sea           | $0.3nV$   | $10nV$    | $1000nV$  |
| 2-d spinon Fermi sea           | $0.07nV$  | $3nV$     | $300nV$   |
| 2-d Dirac spin liquid          | $2 \times 10^{-3}nV$ | $5 \times 10^{-3}nV$ | $3nV$     |

the spin exchange constant, $b_k$ ($b_k^\dagger$) and $a_k$ ($a_k^\dagger$), are the annihilation (creation) of Holstein-Primakoff bosons on B-sublattice and A-sublattice, respectively.

Since spin wave dispersions is $E_q = 4\sqrt{2S} |J | q a$ with $a$ the lattice constant on square lattice, the density of states of spin excitations is

$$\frac{1}{N_R} \sum_q A_M(q, \omega) \propto \omega^2.$$  

The spin current $I_s \propto V|V|^3$ and $I_s \propto (kT)^3V$ at $T=0$, and $T > 0$ with $V << kT$, respectively.

All the cases we have discussed above are summarized in Table I.

**Numerical estimates of the spin current.** We use the following estimates of the parameters, with the interface exchange coupling $J_s = 10meV$ and spin bias $V = 1kT \approx 0.1meV$. The metallic conductivity $\rho = 10^{-8}Q.m$, lattice constant $a = 3nm$, and Fermi level $E_F = 1eV$. The effective spinon mass $m_s \approx 10m_e$ and Fermi level $E_F = 10meV$. The bias induced in the left lead is evaluated by $V_L \approx (2e/h)(I_s/N_L) \ast \rho \ast I_s$ with $N_L$ the number of transverse mode (see Table II).

**Discussions.** Many factors ignored by our analysis may affect the results. First we have assumed the conservation of the z-component of spin in the entire system, so the spin current is well-defined. However in the real materials spin-orbit coupling(SOC) will generically be present. We hope our results can still be applied to such systems if the linear dimensions of the sample are much smaller than the inverse of SOC. For the same reason we did not considered non-collinear AFM orders, e.g. the 120° order on triangular lattice.

Secondly we have ignored the emergent $U(1)$ gauge field in the spinon Fermi sea and Dirac spin liquid cases. It is well-known that coupling to this gauge field can significantly change the low energy behaviors of the (spinon) Fermi sea. However such effects have not been found in the specific heat and thermal conductivity measurements of the organic spin liquid candidates, the conventional Fermi liquid behaviors were observed instead. We therefore believe our results are still valid in these materials. The effect of $U(1)$ gauge field is an interesting theoretical question and will be left for future studies.

Finally we have assumed a clean and free spinon or magnon system in the middle region, without any scattering of spinons or magnons by interactions among themselves or impurities. We think this is not a serious problem for experiments, according to the large value of $1m$ of the experimentally estimated spinon mean free path.

In summary, we propose to experimentally identify the spinons by measuring the spin current flowing through the spin liquid candidate materials, which would be a direct test for the existence of spin-carrying mobile excitations. By the nonequilibrium Green function technique we evaluate the spin current through the interface between a Mott insulator and a metal under a spin bias. It is found that different kinds of Mott insulators, including quantum spin liquids, can be distinguished by different relations between the spin current spin bias as well as temperature. We hope our results can stimulate more experimental studies of the spin liquid candidate materials and further promote the exchange of ideas between different fields (e.g. spintronics and strongly correlated electrons) in condensed matter physics.

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1. P. W. Anderson, Mater. Res. Bull. 8, 153 (1973).
2. L. Balents, Nature (London) 464, 199 (2010).
3. P. W. Anderson, Science 235, 1196 (1987).
4. P. A. Lee, N. Nagaosa, and X. G. Wen, Rev. Mod. Phys. 78, 17 (2006).
5. G. Misguich, B. Bernu, C. Lhuillier, and C. Waldtmann, Phys. Rev. Lett. 81, 1098 (1998).
6. M. Q. Wen, D. N. Sheng, Z. Y. Weng, R. J. Bursill, Phys. Rev. B 74, 012407 (2006).
7. S. Yunoki, S. Sorella, Phys. Rev. B 74, 014408 (2006).
8. D.N. Sheng, O. I. Motrunich, and M.P.A. Fisher, Phys. Rev. B 79, 205112 (2009).
9. H.-Y. Yang, A. M. Läuchli, F. Mila, and K. P. Schmidt, Phys. Rev. Lett. 105, 267204 (2010).
10. P. Lecheminant, B. Bernu, C. Lhuillier, L. Pierre, and P. Sindzingre, Phys. Rev. B 56, 2521 (1997).
H.C. Jiang, Z.Y. Weng, and D.N. Sheng, Phys. Rev. Lett. 101, 117203 (2008).

Simeng Yan, D. A. Huse, and S. R. White, Science 332, 1173 (2011).

H.-C. Jiang, H. Yao, and L. Balents, Phys. Rev. B 86, 024424 (2012).

H. Morita, S. Watanabe, and M. Imada, J. Phys. Soc. Jpn. 71, 2109 (2002).

Z. Y. Meng, T. C. Lang, S. Wessel, F. F. Assaad, and A. Muramatsu, Nature 464, 847 (2010).

D.J. Klein, J. Phys. A. Math. Gen. 15, 661 (1982).

Y. Okamoto, M. Nohara, H. Aruga-Katori, and H. Takagi, Phys. Rev. Lett. 99, 107204 (2007).

T. Itou, A. Oyamada, S. Maegawa, M. Tamura, R. Kato, Phys. Rev. B 77, 104413 (2008).

M. Yamashita, N. Nakata, Y. Kasahara, T. Sasaki, N. Yoneyama, N. Kobayashi, S. Fujimoto, T. Shibauchi, and Y. Matsuda, Nature Phys. 10, 44 (2008).

F. L. Pratt, P. J. Baker, S. J. Blundell, T. Lancaster, S. Ohira-Kawamura, C. Baines, Y. Shimizu, K. Kanoda, I. Watanabe, G. Saito, Nature (London) 471, 612 (2011).

S. Yamashita, Y. Nakazawa, M. Oguni, Y. Oshima, H. Nojiri, Y. Shimizu, K. Miyagawa, and K. Kanoda, Nature Phys. 4, 459 (2008).

S. Yamashita, T. Yamamoto, Y. Nakazawa, M. Tamura, R. Kato, Nature Commun. 2, 275 (2011).

O. I. Motrunich, Phys. Rev. B 72, 045105 (2005).

S.-S. Lee and P. A. Lee, Phys. Rev. Lett. 95, 036403 (2005).

M. Yamashita, N. Nakata, Y. Senshu, M. Nagata, H. M. Yamamoto, R. Kato, T. Shibauchi, and Y. Matsuda, Science 328, 1246 (2010).

N. Tombros, C. Jozsa, M. Popinciuc, H. T. Jonkman, and B. J. van Wees, Nature (London) 448, 571 (2007).

D.-K. Wang, Q.-F. Sun, and H. Guo, Phys. Rev. B 69, 205312 (2004).

H.-Z. Lu and S.-Q. Shen, Phys. Rev. B 77, 235309 (2008); Q.-F. Sun, Y. Xing, and S.-Q. Shen, ibid. 77, 195313 (2008).

M. R. Norman, and T. Micklitz, Phys. Rev. Lett. 102, 067204 (2009); P. Bruno, Phys. Rev. B 52, 411 (1995); M. D. Stiles, ibid. 48, 7238 (1993); 54, 14679 (1996).

H. Haug and A.-P. Jauho, Quantum Kinetics in Transport and Optics of Semiconductors, 2nd ed. (Springer, Berlin, 2008).

Gerald D. Mahan, Many-Particle Physics (Plenum Press, New York, 1981).

See supplemental material for the derivation of this fact.

Y. Ran, M. Hermle, P. A. Lee, and X.-G. Wen, Phys. Rev. Lett. 98, 117205 (2007).

T. Holstein, and H. Primakoff, Phys. Rev. 58, 1908 (1940).

B. I. Halperin, P. A. Lee, and N. Read, Phys. Rev. B 47, 7312 (1993).

C. P. Nave, P. A. Lee, Phys. Rev. B 76, 235124 (2007).
supplementary material of detect spinons via spin current

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I. SPIN CURRENT

In the main text, we have introduced

\[ I_s = \frac{J_s^2 N_\perp}{4 N_R^2 N_M^2} \sum_{q,k,k'} A_M(q,\xi_{Rk\uparrow} - \xi_{Rk\downarrow} + V) \delta_{q_{\perp} + k_{\perp} - k'_{\perp}} \]

× \{[1 + n_B(\xi_{Rk\uparrow} - \xi_{Rk\downarrow} + V)]n_F(-\xi_{Rk\downarrow})n_F(\xi_{Rk\uparrow})

\[-n_B(\xi_{Rk\uparrow} - \xi_{Rk\downarrow} + V)n_F(\xi_{Rk\downarrow})n_F(-\xi_{Rk\uparrow})\} \]

where \(N_M\) and \(N_R\) is the number of unit cells in the middle region and right lead, respectively. Here \(N_\perp\) is the number of transverse mode (parallel to the interface) and \(V = \mu_{R\uparrow} - \mu_{R\downarrow}\) is the spin bias in the right lead. \(n_B(\omega)\) and \(n_F(\xi_{Rk\sigma})\) are the Bose and Fermi distribution function, respectively. \(\delta_{q_{\perp} + k_{\perp} - k'_{\perp}}\) indicates transverse (parallel to the interface) momentum conservation. The power spectrum \(A_M\) are defined as \(A_M(q,\omega) = \int dt < S_M^-(q,t)S_M^+(q,0) > \) \(\exp(i\omega t)/n_B(\omega)\), where \(S_M^\pm(q,t)\) is the Fourier form of \(S_M^\pm(r,t) = S_M^\pm(r,0) \pm S_M^\pm(r,t)\).

In the main text, we have assumed that the transverse momentum conservation term can be separated properly. This will be explained it in detail in the rest of this material.

II. SPIN LIQUID

We describe gapless spin liquids by the following mean-field Hamiltonian

\[ H_M = \sum_{k,\sigma} \zeta_k f_{k\sigma}^\dagger f_{k\sigma} \]

where \(f\) are fermionic spinons and \(\zeta_k = \epsilon_k - \mu_s\) with \(\mu_s\) the spinon chemical potential. In this case, \(S_M(r) = \frac{1}{2} \sum_{\mu,\mu'} \sigma_{\mu\mu'} J^{\dagger}_{\mu}(r) f_{\mu'}(r)\), with \(\sigma\) the Pauli matrix and \(\mu = \uparrow, \downarrow\). The spinon dispersion \(\epsilon_k\) may have a Fermi surface (the "spinon Fermi sea" state) or Dirac points at Fermi level.

Because

\[ < S_M^-(q,t)S_M^+(q,0) > = \sum_{p,p'} f_{p-q\downarrow}^\dagger(t) f_{p-\uparrow}^\dagger(t) f_{p'\uparrow}^\dagger f_{p'\downarrow}(0) > \]

\[ = \sum_p < f_{p-q\downarrow}^\dagger(t) f_{p-\uparrow}(0) > \]

we get

\[ \sum_q A_M(q,\omega) \delta_{q_{\perp} + k_{\perp} - k'_{\perp}} \]

\[ = \sum_q \delta_{q_{\perp} + k_{\perp} - k'_{\perp}} \frac{1}{n_B(\omega)} \int dt \exp(i\omega t) < S_M^-(q,t)S_M^+(q,0) > \]

\[ = \sum_q \delta_{q_{\perp} + k_{\perp} - k'_{\perp}} \frac{n_F(\xi_{p-q}) n_F(-\zeta_p) \delta(\omega - \zeta_p + \zeta_{p-q})}{n_B(\omega)} \]

\[ = \sum_{p'p} \delta_{p_{\perp} - p'_{\perp} + k_{\perp} - k'_{\perp}} \frac{n_F(\zeta_{p'}) n_F(-\zeta_p) \delta(\omega - \zeta_p + \zeta_{p'})}{n_B(\omega)} .\]
Now we come to the two dimensional spinon Fermi sea case, which is the most relevant to the 2D organic spin liquid candidate materials. If the energy of spinon excitations $\epsilon_k$ and spin-$\uparrow$ ($\downarrow$) electron excitations $\xi_{Rk\uparrow}$ ($\xi_{Rk\downarrow}$) are much smaller spinon Fermi level $\mu_s$ and spin-$\uparrow$ ($\downarrow$) electron Fermi level $\mu_{R\uparrow(\downarrow)}$,

$$p_\perp - p'_\perp + k_\perp - k'_\perp = p \cos \theta_1 - p' \cos \theta'_1 + k \cos \theta_2 - k' \cos \theta'_2 \approx k'_{F,\perp} \cos \theta_1 - k'_{F,\perp} \cos \theta'_1 + k_{F,\perp} \cos \theta_2 - k_{F,\perp} \cos \theta'_2.$$  

In the two dimensional Dirac spin liquid case, the low energy spinon dispersion is $\epsilon_q = \pm\hbar v_F |q - q_F|$, with $v_F$ the Fermi velocity and $q_F$ the Fermi vector,

$$p_\perp - p'_\perp + k_\perp - k'_\perp = p \cos \theta_1 - p' \cos \theta'_1 + k \cos \theta_2 - k' \cos \theta'_2 \approx k'_{F,\perp} \cos \theta_2 - k'_{F,\perp} \cos \theta'_2.$$  

Since $p_\perp - p'_\perp + k_\perp - k'_\perp$ is the only angular dependent term, we can express the spin current as

$$I_s = \frac{J_z^2}{4N_R N_M} \sum_{q,k,k'} A_M(q, \xi_{Rk\uparrow} - \xi_{Rk\downarrow} + V) \times \left[ (1 + n_B(\xi_{Rk\uparrow} - \xi_{Rk\downarrow} + V)) n_F(-\xi_{Rk\downarrow}) n_F(\xi_{Rk\uparrow}) - n_B(\xi_{Rk\uparrow} - \xi_{Rk\downarrow} + V) n_F(\xi_{Rk\uparrow}) n_F(-\xi_{Rk\uparrow}) \right],$$

with $\Theta_q = \int d\theta_1 d\theta'_1 d\theta_2 d\theta'_2 (\cos \theta_1 - \cos \theta'_1 + k'_{F,\perp} \cos \theta_2 - k'_{F,\perp} \cos \theta'_2)/(2\pi)^4$ for spinon Fermi sea case and $\Theta_q = \int d\theta_2 d\theta'_2 (\cos \theta_2 - k'_{F,\perp}/k'_{F,\perp} \cos \theta'_2)/(2\pi)^4$ for U(1) Dirac spin liquid, respectively.

Thus the transverse momentum conservation term is separated independently.

We note that in one dimension case, $\Theta_q = 1$ and the number transverse mode $N_0 = 1$.

### III. COLLINEAR ANTI FERROMAGNETIC NÉEL ORDER.

Let us now consider the simplest AFM ordered state, the collinear AFM Néel order on a bipartite lattice, and show that it has different spin transport behavior compared to the previous spin liquid cases. We describe the spin excitations by a linearized spin wave Hamiltonian,

$$H_M = \sum_k E_0 (a_k^\dagger a_k + b_k^\dagger b_k + \gamma_k a_k^\dagger b_{-k} + \gamma_k a_k b_{-k}^\dagger)$$

where $E_0 = 2ZS|J|$ and $\gamma_k = \frac{1}{2} \sum_\delta \cos(\mathbf{k} \cdot \mathbf{\delta})$ with $\delta$ sums over the nearest-neighbor lattice sites. Here $Z$ is the coordination number, $S$ is spin quantum number and $J$ is the spin exchange constant. In this case, $S_M^+(\mathbf{r}) = S_M^+(\mathbf{r}) + iS_M^-(\mathbf{r}) = a_k^\dagger + b_k^\dagger$, $S_M^- = S_M^-(\mathbf{r}) - iS_M^+(\mathbf{r}) = a_k - b_k$ and $S_M^z = a_k^\dagger a_k - b_k^\dagger b_k$, where $a_k^\dagger$ and $b_k^\dagger$ are the Fourier form of $a_k$ and $b_k$, respectively. Here, $b_k$ and $a_k$ are the annihilation creation of Holstein-Primakoff bosons on B-sublattice and A-sublattice, respectively.

If the wave vector of spin wave is much smaller than spin-$\uparrow$ ($\downarrow$) electron Fermi vector, i.e. $k_1 << k'_{F,\perp}/k_{F,\perp}$, then

$$q_\perp - k_\perp + k'_\perp = q \cos \theta_1 - k \cos \theta_2 + k' \cos \theta_3 \approx k'_{F,\perp} \cos \theta_2 - k_{F,\perp} \cos \theta_3.$$  

The spin current is

$$I_s = \frac{J_z^2}{4N_R N_M} \sum_{q,k,k'} A_M(q, \xi_{Rk\uparrow} - \xi_{Rk\downarrow} + V) \times \left[ (1 + n_B(\xi_{Rk\uparrow} - \xi_{Rk\downarrow} + V)) n_F(-\xi_{Rk\downarrow}) n_F(\xi_{Rk\uparrow}) - n_B(\xi_{Rk\uparrow} - \xi_{Rk\downarrow} + V) n_F(\xi_{Rk\uparrow}) n_F(-\xi_{Rk\uparrow}) \right],$$

with $\Theta_q = \int d\theta_2 d\theta'_2 (\cos \theta_2 - k'_{F,\perp}/k'_{F,\perp} \cos \theta'_2)/(2\pi)^3$. Thus the transverse momentum conservation term is separated.

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1. O. I. Motrunich, Phys. Rev. B 72, 045105 (2005).
2. S.-S. Lee and P. A. Lee, Phys. Rev. Lett. 95, 036403 (2005).
3. Y. Ran, M. Hermele, P. A. Lee, and X.-G. Wen, Phys. Rev. Lett. 98, 117205 (2007).
4. T. Holstein, and H. Primakoff, Phys. Rev. 58, 1908 (1940).